A LANDSCAPE OF CONTACT MANIFOLDS VIA RATIONAL SFT

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Abstract. We define a hierarchy functor from the exact symplectic cobordism category to a totally ordered set from a $BL_\infty$ (Bi-Lie) formalism of the rational symplectic field theory (RSFT). The hierarchy functor consists of three levels of structures, namely algebraic planar torsion, order of semi-dilation and planarity, all taking values in $\mathbb{N} \cup \{\infty\}$, where algebraic planar torsion can be understood as the analogue of Latschev-Wendl’s algebraic torsion [48] in the context of RSFT. The hierarchy functor is well-defined through a partial construction of RSFT and is within the scope of established virtual techniques. We develop computation tools for those functors and prove all three of them are surjective. In particular, the planarity functor is surjective in all dimension $\geq 3$. Then we use the hierarchy functor to study the existence of exact cobordisms. We discuss examples including iterated planar open books, spinal open books, Bourgeois contact structures, affine varieties with uniruled compactification and links of singularities.

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1. Introduction

One central subject in symplectic and contact topology is the study of symplectic cobordisms. Unlike the usual cobordism relation in differential topology, a symplectic cobordism is asymmetric; the collection of such cobordisms endows the collection of contact manifolds with a structure similar to a partial order. The fundamental dichotomy between overtwisted contact structures and tight contact structures discovered by Eliashberg in dimension 3 [29] and Borman-Eliashberg-Murphy in higher dimensions [6] is reflected by the fact that overtwisted contact structures behave like minimal elements. Namely, there is always an exact cobordism from an overtwisted contact manifold to any other contact manifold in dimension 3 [32] and the same holds for higher dimensions when the obvious topological obstructions vanish [30]. To explore the realm of the more mysterious class of tight contact structures, the hierarchy imposed by the existence of symplectic cobordisms is a useful guiding principle, as the complexity of contact topology should not decrease in a cobordism. In dimension 3, a further hierarchy in the world of tight contact manifolds was...
discovered by Giroux [41] and Wendl [73]. In higher dimensions, the notion of Giroux torsion was generalized by Massot-Niederkrüger-Wendl [53].

On the other hand, since contact manifolds and (exact) symplectic cobordisms form a natural category, which we will refer to as the (exact) symplectic cobordism category \( \mathcal{C} \), one natural approach to study \( \mathcal{C} \) is by understanding functors from \( \mathcal{C} \) to some algebraic category, a.k.a. a field theory. Symplectic field theory (SFT), as proposed by Eliashberg-Givental-Hofer in [28], is a very general framework for defining such functors, and many invariants of contact manifolds and symplectic cobordisms can be defined via suitable counts of punctured holomorphic curves which approach Reeb orbits at their punctures. The formidable algebraic richness of the general theory, together with the serious technical difficulties arising in building its analytical foundations, conspire to make explicit computations a complicated matter. Therefore, rather than focusing on computing the full SFT invariant, one could focus on extracting simpler invariants from the general theory whose computation is in principle approachable via currently available techniques. An example of this philosophy is the notion of algebraic torsion introduced by Latschev-Wendl in [48], which associates to every contact manifold a number in \( \mathbb{N} \cup \{\infty\} \) and can be viewed as the algebraic interpretation of the geometric concept of planar torsion defined by Wendl [73].

In this paper, we follow the same methodology of Latschev-Wendl to study the structure of \( \mathcal{C} \). Instead of the full SFT, we use the rational SFT (RSFT), i.e. we only consider genus 0 curves, to construct a functor from \( \mathcal{C} \) to a totally ordered set. Our main theorem is the following.

**Theorem A.** There is a covariant monoidal functor \( H_{\text{cx}} \) from \( \mathcal{C} \) to \( \mathcal{H} \), where \( \mathcal{H} \) is the totally ordered set
\[
\{0^{\text{APT}} < 1^{\text{APT}} < \ldots < \infty^{\text{APT}} < 0^{\text{SD}} < 1^{\text{SD}} < \ldots < \infty^{\text{SD}} < 2^{\text{P}} < \ldots < \infty^{\text{P}}\}.
\]

Here, functoriality of \( H_{\text{cx}} \) means that \( H_{\text{cx}}(Y) \leq H_{\text{cx}}(Y') \) whenever there exists an exact cobordism from \( Y \) to \( Y' \). The monoidal structure on \( \mathcal{C} \) is given by disjoint union and the monoidal structure on \( \mathcal{H} \) is more involved; it will be explained in §3 and §4. In particular, one can compute \( H_{\text{cx}} \) of a disjoint union from its components. The hierarchy functor \( H_{\text{cx}} \) of contact complexity is assembled from three functors: algebraic planar torsion \( \text{APT} \), planarity \( \text{P} \) and order of semi-dilation \( \text{SD} \), all of them taking values in \( \mathbb{N} \cup \{\infty\} \). There are overlaps between those invariants: when \( \text{APT} \) is finite, it is necessary to have \( \text{P} = 0 \); and if \( \text{SD} \) is finite, it is necessary to have \( \text{P} = 1 \). Therefore in the definition of the totally ordered set \( \mathcal{H} \), \( k^{\text{APT}} \) stands for the case where \( \text{APT} = k \) and \( \text{P} = 0 \), \( k^{\text{SD}} \) stands for the case where \( \text{SD} = k \) and \( \text{P} = 1 \), and \( k^{\text{P}} \) stands for the case where \( \text{P} = k \). APT can be viewed as the analogue of algebraic torsion in the context of RSFT. In particular, finiteness of \( \text{APT}(Y) \) implies that \( Y \) has no strong filling just like algebraic torsion. However, \( H_{\text{cx}} \) goes well beyond non-fillable contact manifolds, i.e. \( \text{SD}, \text{P} \) provide measurements for fillable contact manifolds. Roughly speaking, \( \text{APT} \) looks for rational curves without negative punctures and \( \text{P} \) looks for rational curves with a point constraint in symplectizations. And \( \text{SD} \) is defined using the \( \mathbb{Q}[u] \)-module structure on linearized contact homology introduced by Bourgeois-Oancea [12]. \( \text{APT} \) measures the obstruction to augmentations of RSFT, while \( \text{SD, P} \) can be phrased in the linearized theory, hence require the existence of augmentations. To make \( \text{SD, P} \) independent of the augmentation, we need to define \( \text{SD} \) and \( \text{P} \) via traversing the set of all possible augmentations of the RSFT.

1.1. Rational SFT. The original algebraic formalism of SFT in [28] packaged the full SFT into a super Weyl algebra with a distinguished odd degree Hamiltonian \( H \) such that \( H \ast H = 0 \). Cieliebak-Latschev reformulated the algebra into a \( BV_{\infty} \) algebra [22], which was used in the definition of algebraic torsion.
The BV∞ algebra structure was further refined to an IBL∞ (Involutive Bi-Lie infinity) algebra by Cieliebak-Fukaya-Latschev [21], which roughly speaking, is precisely the boundary combinatorics for the SFT compactification [9]. For rational SFT, the original algebraic formalism was a Poisson algebra with a distinguished odd degree Hamiltonian h such that \{h, h\} = 0. Analogous reformulations of the algebraic structure of RSFT can be found in Hutchings’ “q-variable only RSFT” [45], and an L∞ formalism of RSFT by Siegel [70]. In this paper, we introduce a notion of BL∞ (Bi-Lie infinity) algebra to describe RSFT, which precisely describes the boundary combinatorics for rational curves in the SFT compactification and is a specialization of the IBL∞ formalism. By building functors from the category of BL∞ algebras to totally ordered sets, we can build the hierarchy functor in Theorem A by a composition

\[ H_{\text{cx}} : \mathcal{C} \circ \text{con}^{\text{RSFT}} \to \mathcal{B}L_{\infty} \text{ (with additional structures up to homotopy)} \to \mathcal{H}. \]

On the other hand, the general holomorphic curve theory in manifolds with contact boundaries faces serious analytical challenges, which makes a complete construction of the first functor in (1.1) a difficult task. To obtain a construction of SFT/RSFT, one needs to deploy more powerful virtual techniques, e.g., either polyfold approaches [33, 44] by Fish-Hofer and Hofer-Wysocki-Zehnder, implicit atlases and virtual fundamental cycles by Pardon [65, 66], or Kuranishi approaches by Ishikawa [47]. However, for the purpose of defining H_{\text{cx}}, it is sufficient to build RSFT partially. In particular, we do not need to discuss compositions and homotopies for BL∞ algebras as \mathcal{H} is a totally ordered set, where there is no ambiguity for compositions and homotopy equivalences. This greatly simplifies our demands for virtual machinery, as homotopies in SFT is a subtle subject. Moreover, the combinatorics for a BL∞ algebra is “tree-like”, which is very similar to the combinatorics for contact homology. As a consequence, we can use Pardon’s construction of contact homology [66] to provide all the analytic foundation of the functor H_{\text{cx}}. In particular, Theorem A is well-posed without any hidden hypotheses on virtual machinery. Moreover, it is expected that any other virtual technique will suffice for Theorem A. We will also explain how to obtain another construction of H_{\text{cx}} from a small part of the polyfold construction of SFT [33].

In general, a full computation of RSFT and SFT is very difficult, as we need to understand many moduli spaces. On the other hand, the hierarchy functor H_{\text{cx}} extracts partial information from BL∞ algebras, so that only partial knowledge of the moduli spaces are needed. In particular, H_{\text{cx}} is relatively computable. It is a nontrivial question whether H_{\text{cx}} is independent of the choice of virtual technique. However, since every virtual technique has the property that we can count a compactified moduli space geometrically if it is cut out transversely in the classical sense, the following theorem does not depend on the choice of virtual technique:

**Theorem B.** The functors above have the following properties.

1. If Y has planar k-torsion [73], then APT(Y) ≤ k.
2. If Y is overtwisted then APT(Y) = 0.
3. If Y has (higher dimensional) Giroux torsion [53], then APT(Y) ≤ 1.
4. If APT(Y) < ∞, then Y is not strongly fillable. If Y admits an exact filling then P(Y) ≥ 1.
5. If Y is an iterated planar open book [2] where the initial page has k-punctures, then P(Y) ≤ k.
6. If Y has an exact filling that is not k-uniruled [55], then P(Y) ≥ k + 1.
7. APT, SD, P are all surjective. In particular, P is surjective in all odd dimension ≥ 3.

The relation between APT and algebraic torsion AT [48] is not direct. In fact, they are both implied by a stronger notion of torsion, which is implied by planar k-torsion [73], defined through an alternative representation of the IBL∞ algebra of the full SFT. There is a grid of torsions serving as obstructions to
strong fillings, where algebraic planar torsion and algebraic torsion are two axes. The only common ground is that 0-algebraic planar torsion is equivalent to 0-algebraic torsion and algebraically overtwistedness \([11]\), which is implied by overtwistedness \([15, 77]\). Moreover, there are 5-dimensional examples with underlying smooth manifold \(Y = S^*X \times \Sigma_g\) (where \(S^*X\) is the unit cotangent bundle of a hyperbolic surface \(X\) and \(\Sigma_g\) is the orientable surface of genus \(g \geq 1\)), considered originally in \([58]\), which we conjecturally expect to have \(\APT(Y) > 1\) but \(\AT(Y) = 1\); see \([58, \text{Sec. 6.5}]\). This would provide concrete examples on which the two notions of torsion strictly differ.

1.2. Applications. Since \(H_{\text{cx}}\) is a measurement of the complexity of contact topology, the main application of \(H_{\text{cx}}\) is obstructing the existence of exact cobordisms. The following theorem answers a conjecture of Wendl \([73]\) affirmatively, although the invariant we use is \(P\) instead of algebraic torsion (as opposed to the original conjecture).

**Theorem C.** For any dimension \(\geq 3\), there exists an infinite sequence of contact manifolds \(Y_1, Y_2, \ldots\), such that there is an exact cobordism from \(Y_i\) to \(Y_{i+1}\), but there is no exact cobordism from \(Y_{i+1}\) to \(Y_i\).

The above result was obtained in dimension 3 in \([48]\). In fact, there are many examples of \(Y_i\), the simplest example being the boundary of the product of \(n\) copies of a \(k\)-punctured sphere \(S^k\), as we will show in §6 that \(P(\partial(S_k)^n) = k\) for \(n \geq 2\). There are many more examples for Theorem C to hold; see e.g. Theorem L below.

Following the definition of \(P\), it is easy to see that if \(Y\) admits a contact structure without Reeb orbits, then \(P(Y) = \infty\). Therefore, as a corollary, we have the following.

**Corollary D.** If \(P(Y) < \infty\), then the Weinstein conjecture holds for \(Y\).

In other words, counterexamples to the Weinstein conjecture (if any) should be looked for in the highest complexity level \(\infty^P\). In particular, the combination of (5) in Theorem B and Corollary D yields a proof of the Weinstein conjecture for iterated planar open books, which was previously obtained for dimension 3 in \([1]\) and higher dimensions in \([2, 5]\). In some sense, the proof of (5) of Theorem B endows the ruling holomorphic curve in \([1, 2, 5]\) with a homological meaning, i.e. the ruling curve defines a map that is visible on homology; in particular, such curve can not be eliminated by perturbing the contact form. On the other hand, not every contact manifold with finite planarity is iterated planar: for example \(P(T^3, \xi_{\text{std}}) = 2\) by Corollary 6.9, while \((T^3, \xi_{\text{std}})\) is not supported by a planar open book by \([31]\) (it is, however, supported by a planar spinal open book \([72, 52]\)). By functoriality, if there is an exact cobordism from \(Y\) to \(Y'\) with \(P(Y') < \infty\), then the Weinstein conjecture holds for \(Y\).

The study of planar open book in dimension 3 has a very long history, since they enjoy nice properties like equivalence of weak fillability and Weinstein fillability \([62, 72]\). We refer readers to the introduction of \([3]\) for a comprehensive summary on the subject. Obstructions to planar open book structures were obtained in \([31, 64]\). In higher dimensions, obstructions to supporting an iterated planar open book were found in \([5]\). By (5) of Theorem B, infinite planarity is an obstruction to an iterated planar structure. In particular, we answer \([4, \text{Question 1.14}]\) negatively by the following general result.

**Corollary E.** In all dimension \(\geq 5\), consider \((Y, J)\) an almost contact manifold which has an exactly fillable contact representative \((Y, \xi)\). Then there is a contact structure \(\xi'\) in the homotopy class of \(J\), such that \((Y, \xi')\) is not iterated planar.

In particular, since every simply connected almost contact 5-manifold is almost Weinstein fillable \([39]\), there is a contact structure in each homotopy class of almost complex structures that is not iterated planar for every simply connected 5-manifold.
1.3. Examples. In addition to Theorem B, there are many situations where we can compute or estimate $H_\text{cx}$. By (6) of Theorem B, it is natural to look at affine varieties with a uniruled projective compactification. One special case is affine varieties with a $\mathbb{C}\mathbb{P}^n$ compactification.

**Theorem F.** Let $D$ be $k$ generic hyperplanes in $\mathbb{C}\mathbb{P}^n$ for $n \geq 2$, then we have the following.

1. $P(\partial D^c) \geq k + 1 - n$ for $k > n + 1$.
2. $P(\partial D^c) = k + 1 - n$ for $n + 1 < k < \frac{3n-1}{2}$ and $n$ odd.
3. $P(\partial D^c) = 2$ for $k = n + 1$.
4. $H_\text{cx}(\partial D^c) = 0$ for $k \leq n$.

The condition on $n$ being odd (also for Theorem H, I below) is not essential. We use it to obtain automatic closedness of a chain in the computation of planarity for any augmentation. In Remark 7.20, we explain how one can drop this condition using polyfold techniques in [82]. On the other hand, the role of $k < \frac{3n-1}{2}$ is more mysterious. Although it is unlikely to be optimal, whether an upper bound is necessary is unclear. One difficulty of computing $P$ and obtaining cobordism obstructions is that we need to carry out computation for all hypothetical “fillings”, i.e. augmentations. Indeed, different choice of augmentation will affect the computation dramatically. For example, there exists affine varieties with a $\mathbb{C}\mathbb{P}^n$ compactification whose contact boundary has infinite planarity, cf. Theorem 7.12. However, if we use the augmentation from the affine variety, then the planarity is finite.

On the other hand, $D_k^c$ embeds exactly into $D_{k+1}^c$, which follows from a general construction, as follows. Let $\mathcal{L}$ be a very ample line bundle over a smooth projective variety $X$. Then for any nonzero holomorphic section $s \in \mathbb{P}H^0(\mathcal{L})$, $X \setminus s^{-1}(0)$ is an affine variety whose contact boundary is denoted by $Y_s$. The projective space $\mathbb{P}H^0(\mathcal{L})$ should be stratified by the singularity type of $s^{-1}(0)^2$, with the top stratum corresponding to the case where $s^{-1}(0)$ is smooth with multiplicity 1. We say that there is a morphism from stratum $A$ to stratum $B$, if we can change $s^{-1}(0)$ from $A$ to $B$ by an arbitrarily small perturbation of the section $s$, i.e. $A$ is contained in the closure of $B$. Moreover, one obtains an exact cobordism from $Y_s$ to $Y_{s'}$, where $s'$ is the perturbed section. Then we have a natural functor from the category of strata to $\mathfrak{Con}$. As a concrete example, consider $\mathcal{L} = \mathcal{O}(2)$ on $\mathbb{C}\mathbb{P}^2$, then the category of stratification is the graph $A_1 \to A_2 \to A_3$, where $A_1, A_2, A_3$ correspond to a double line, two generic lines and a smooth quadratic curve as the divisor, respectively. The corresponding affine varieties as exact domains are $\mathbb{C}^2, \mathbb{C} \times T^*S^1, T^*\mathbb{R}\mathbb{P}^2$, which clearly have the exact embedding relations as claimed.

In view of this, when we view one of the $k$ hyperplanes in $D_k$ as having multiplicity 2 (a “double” hyperplane), we can get an exact cobordism from $\partial D_k^c$ to $\partial D_{k+1}^c$, by perturbing the double hyperplane to two distinct hyperplanes. Then Theorem F asserts that a reversed exact cobordism can not be found if $n \leq k < \frac{3n-1}{2}$ and $n$ is odd. Note that the natural inclusion $D_{k+1}^c \subset D_k^c$ is symplectic, hence we always have a strong cobordism from $\partial D_{k+1}^c$ to $\partial D_k^c$, which shows the essential difference between these two notions of cobordisms and the obstruction from $P$ is not topological. When $k \leq n$, $D_k^c$ is in fact subcritical and they can be embedded exactly into each other regardless of $k$.

As a concept closely related to $\mathfrak{Con}$, we introduce $\mathfrak{Con}_*$ as the under category of $\mathfrak{Con}$ under $\emptyset$, i.e. the objects of $\mathfrak{Con}_*$ are pairs of contact manifolds with exact fillings and morphisms are exact embeddings. Then $\text{SD}, P$ can be defined on $\mathfrak{Con}_*$ using the augmentation from the given exact filling. Moreover, we

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2It is quite a nontrivial task to make this stratification precise, as in general we do not have a classification of the possible singularities of the divisor.

3The functorial property of those two functors requires a full construction of RSFT, including compositions and homotopies. In particular, it makes more demands for virtual constructions than explained in this paper.
recall another functor $U$, called the order of uniruledness, which is defined to be the minimal $k$ such that an exact domain $W$ is $k$-uniruled in the sense of McLean [55]. That $U$ is a functor from $\text{Con}_* \to \mathbb{N}_+ \cup \{\infty\}$ was proven in [55]. By (6) of Theorem B, $P(Y)$ is bounded below by $U(W)$ for an exact filling $W$ of $Y$. The functor $U$ measures the complexity of exact domains and serves as an exact embedding obstruction. An interesting aspect of $U$ is that the well-definedness and basic properties of $U$ do not depend on any Floer theory. As a byproduct of the proof of Theorem F, we have following for any $n \geq 1$.

**Theorem G.** Let $D_k$ denote the divisor of $k$ generic hyperplanes in $\mathbb{CP}^n$ for $n \geq 1$ and $D^*_k$ denote the complement affine variety. Then $U(D^*_k) = \max\{1, k+1-n\}$. In particular, $D^*_{k+1}$ cannot be embedded into $D^*_k$ exactly for $k \geq 1$.

**Remark 1.1.** The same embedding question is studied independently by Ganatra and Siegel [37], where more general normal crossing divisors in $\mathbb{CP}^n$ are studied. The planarity for exact domains mentioned above is equivalent to $G(p)$ in [37]. The authors of [37] also consider holomorphic curves with local tangent constraints to define functors $G(T^np)$ on $\text{Con}_*$, which is briefly explained in §3.7 in our context. In view of the local tangent constraints, one can define an analogous order of uniruledness with local tangent conditions, the well-definedness and functorial property of such invariants follows from the same argument of [55]. Such functor can also serve as an embedding obstruction as $U$ in Theorem G. It is an interesting question on whether those geometric invariants are the same as the algebraic invariants (defined via RSFT in [37]), which is the case for Theorem G.

In view of (6) of Theorem B, one can also consider affine varieties with uniruled compactification, in particular those affine varieties with Fano hypersurfaces as compactification. In general, we have the following.

**Theorem H.** Let $X$ be a smooth degree $m$ hypersurface in $\mathbb{CP}^{n+1}$ for $2 \leq m \leq n$ and $D$ be $k \geq n$ generic hyperplanes, i.e. $D = (H_1 \cup \ldots \cup H_k) \cap X$ for $H_i$ is a hyperplane in $\mathbb{CP}^{n+1}$ in generic position with each other and $X$, then $P(\partial D^*_k) = k + m - n$ for $n$ odd and $k + m < \frac{3n+1}{2}$.

When $m = n + 1$, $X$ is again uniruled and results with similar nature should hold. However, $X$ is not uniruled by the degree 1 curves, while our proof uses somewhere injectivity of degree 1 curves to obtain transversality in various places, hence is only applicable to $m \leq n$. A more systematic way to study $H_{\text{cx}}$ is deriving formulas for RSFT of affine varieties with normal crossing divisor complement using log/relative Gromov-Witten invariants similar to the formula for symplectic cohomology in [27].

The following results provide affine variety examples with nontrivial SD.

**Theorem I.** Assume $D_s$ is a smooth degree $2 \leq k < \frac{n+1}{2}$ hypersurface in $\mathbb{CP}^n$ for $n \geq 3$, then $P(\partial D^*_k) = 1$ and $H_{\text{cx}}(\partial D^*_k) \leq (k^2 - 1)^{\text{SD}}$. When $n$ is odd, then same holds for $2 \leq k < n$, and moreover we have $H_{\text{cx}}(\partial D^*_k) \geq (k-1)^{\text{SD}}$.

Another rich class of contact manifolds comes from links of isolated singularities. They provide examples with every order of semi-dilation.

**Theorem J.** Let $LB(k, n)$ denote the contact link of the Brieskorn singularity $x_0^k + \ldots + x_n^k = 0$. Then $H_{\text{cx}}(LB(k, n))$ is

1. $(k-1)^{\text{SD}}$ if $k < n$;
2. $\geq (k-1)^{\text{SD}}$ if $k = n$ and $> 1^P$ if $k = n + 1$;
3. $\propto$, if $k > n + 1$. 

Another type of singularity is the quotient singularity, whose contact links are not exact fillable in many cases [79]. In fact, the symplectic aspect of the proof in [79] can be restated as a computation of $H_{\text{cx}}$ as follows.

**Theorem K.** Let $Y$ be the quotient $(S^{2n-1}/\mathbb{Z}_k, \xi_{\text{std}})$ by the diagonal action of $e^{\frac{2\pi i}{k}}$ for $n \geq 2$.

1. If $n > k$, we have $H_{\text{cx}}(Y) = 0^{SD}$.
2. If $n \leq k$, we have $0^{SD} \leq H_{\text{cx}}(Y) \leq (n-1)^{SD}$. When $n = k$, we have $H_{\text{cx}}(Y) \geq 1^{SD}$.

The second case of the above theorem is another situation where the computation depends on the augmentation. Roughly speaking, $H_{\text{cx}}(Y) = 0^{SD}$ means that any exact filling of $Y$ has vanishing symplectic cohomology. And if there is a (possibly strong) filling with vanishing symplectic cohomology, then the order of semi-dilation using the induced augmentation from the filling is 0.\(^4\) The natural prequantization bundle filling provides augmentations such that the symplectic cohomology vanishes [67]. On the other hand, there are other augmentations with positive orders of semi-dilation. For example the exact filling $T^*S^2$ of $(\mathbb{R}P^3, \xi_{\text{std}})$ has order of semi-dilation 1, such phenomenon was also explained in [79, Remark 2.16].

**Theorem L.** Let $V$ be an exact domain and $S_k$ be the $k$-punctured sphere. Then

1. $P(\partial(V \times S_k)) \leq k$.
2. If $V$ is an affine variety that is not $(k-1)$-uniruled, then $P(\partial(V \times S_k)) = k$.
3. $H_{\text{cx}}(\partial(V \times \mathbb{D})) = 0^{SD}$.

In particular, (2) in Theorem L provides many examples to Theorem C and (3) is a reformulation of the symplectic step in [80] to obtain uniqueness results on fillings of $\partial(V \times \mathbb{D})$. We also discuss Bourgeois contact structures and cosphere bundles in §6 and §7.

**Organization of the paper.** We introduce the concept of $BL_\infty$ algebra in §2 and then define algebraic planar torsion as well as planarity at the level of algebra. In §3, we implement Pardon’s VFC [66] to define APT and $P$. We then discuss their properties and generalizations including the related formalism on the full SFT. We recall in §4 the $\mathbb{Q}[u]$ module structure on linearized contact homology following [12] to define SD and finish the proof of Theorem A. We give a lower bound for $P$ in §5 and an upper bound for $P$ in §6. We discuss examples, applications, and finish the proof of Theorem B in §7.

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## 2. $L_\infty$ Algebras and $BL_\infty$ Algebras

In this section, we recall the basics of $L_\infty$ algebras and introduce $BL_\infty$ algebras, which serve as the underlying algebraic structures for rational symplectic field theory. The algebraic formalism here is essentially the $q$-variable only reformulation in [45] and the $L_\infty$ algebra formalism on contact homology algebra in [70], but we make the compatibility of the algebraic structure on the contact homology algebra with the $L_\infty$ structure more precise and define such an object as a $BL_\infty$ algebra, which is a specialization of the $IBL_\infty$ algebra in [21] and the homotopic version of bi-Lie algebras (with curvature). The algebraic relations in

\(^4\)Assuming the filling is monotone, so that we can evaluate $T = 1$ in the Novikov coefficient.
$BL_\infty$ algebra are precisely the boundary combinatorics of the moduli spaces of rational curves in the SFT compactification. We then introduce algebraic planar torsion and planarity at the algebraic level.

2.1. $L_\infty$ algebras. Throughout this section, we assume $k$ is a field with characteristic 0. Let $V$ be a $\mathbb{Z}_2$-graded $k$-vector space. Then we have the $\mathbb{Z}_2$-graded symmetric algebra $SV := \oplus_{k \geq 0} S^k V$ and the non-unital symmetric algebra $\overline{SV} = \oplus_{k \geq 1} S^k V$, where $S^k V = \otimes^k V / \text{Sym}_k$ in the graded sense. In particular, we have

$$ab = (-1)^{|a||b|}ba$$

for homogeneous elements $a, b$ in $SV, \overline{SV}$. Therefore $S^k V$ is spanned by vectors of the form $v_1 \ldots v_k$ for $v_i \in V$. However, to introduce the $L_\infty$ algebra, we will view $SV, \overline{SV}$ as coalgebras by the following coproduct operation:

$$\Delta(v_1 \ldots v_k) = \sum_{i=1}^{k-1} \sum_{\sigma \in Sh(i,k-i)} (-1)^\sigma (v_{\sigma(1)} \ldots v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \ldots v_{\sigma(k)}),$$

where $Sh(i,k-i)$ is the subset of permutations $\sigma$ such that $\sigma(1) < \ldots < \sigma(i)$ and $\sigma(i+1) < \ldots < \sigma(k)$ and

$$\phi = \sum_{1 \leq i < j \leq k} |v_i| |v_j|.$$ 

Then both $SV$ and $\overline{SV}$ satisfy the coassociativity property $(\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta$, and the cocommutativity property $R \circ \Delta = \Delta$, where $R : SV \otimes SV \to SV \otimes SV$ is given by $R(x \otimes y) = (-1)^{|x||y|} y \otimes x$ for homogeneous elements $x, y$. A coderivation of the coalgebra $(SV, \Delta)$ is a $k$-linear map $\delta : SV \to SV$ satisfying the coLeibniz rule $\Delta \circ \delta = (\delta \otimes \text{id}) \circ \Delta + (\text{id} \otimes \delta) \circ \Delta$. Here we use the Koszul-Quillen sign convention that $(f \otimes g)(x \otimes y) = (-1)^{|x||g|} f(x) \otimes g(y)$, for $x, y \in V, W$ and $f, g \in V^\vee, W^\vee$.

Definition 2.1. We use $sV$ to denote $V[1]$. An $L_\infty$ algebra on $V$ is a degree 1 coderivation $\ell$ on $\overline{SV}$ satisfying $\ell^2 = 0$.

Note that we have a well-defined degree $-1$ map $s : V \to sV$. The coderivation property of $\ell$ implies that it is determined by maps $\ell^k : S^k sV \to sV$ defined by the composition $S^k sV \hookrightarrow \overline{SV} \xrightarrow{\ell} \overline{SV} \twoheadrightarrow sV$, where the first map is the natural inclusion and the last map is the natural projection, satisfying the quadratic relation

$$(\overline{SV}, \ell)$$

is called the reduced bar complex. The word length filtration $\overline{B}^1 sV \subset \overline{B}^2 sV \subset \ldots \subset \overline{SV}$ is compatible with the differential, where $\overline{B}^k sV := \bigoplus_{j=1}^k S^j sV$.

Definition 2.2. An $L_\infty$ homomorphism from $(V, \ell)$ to $(V', \ell')$ is a degree 0 coalgebra map $\phi : \overline{SV} \to \overline{SV}'$ such that $\phi \circ \ell = \ell' \circ \phi$. 

Given \( \phi_i : S^{k_i} sV \rightarrow sV', 1 \leq i \leq n \) and \( m = \sum_{i=1}^{n} k_i \), we define \( \phi_1 \ldots \phi_n : S^m sV \rightarrow S^m sV' \) by sending \( sv_1 \ldots sv_m \) to
\[
\pi \left( \sum_{\sigma} \frac{(-1)^{\sigma'}}{k_1! \ldots k_n!} (\phi_1 \otimes \ldots \otimes \phi_n)((sv_{\sigma(1)} \ldots sv_{\sigma(k_1)}) \otimes \ldots \otimes (sv_{\sigma(m-k_n+1)} \ldots sv_{\sigma(m)})) \right).
\]
Here \( \pi \) is the natural map \( \otimes^k sV \rightarrow S^k sV \). By the coalgebra property, if \( \hat{\phi} \) is an \( L_\infty \) morphism, we know that \( \hat{\phi} \) is determined by \( \{ \phi^k : S^k sV \rightarrow \mathbb{S}sV \xrightarrow{\hat{\phi}} \mathbb{S}sV' \rightarrow sV' \}_{k \geq 1} \). More explicitly, \( \hat{\phi} \) is defined by the following formula,
\[
\hat{\phi}(sv_1 \ldots sv_n) = \sum_{k \geq 1} \frac{1}{k!} (\psi^{i_1} \ldots \psi^{i_k})(sv_1 \ldots sv_n).
\]
The relation \( \hat{\phi} \circ \hat{\ell} = \hat{\phi} \circ \hat{\psi} \) can be written as
\[
\sum_{p+q=n+1} \sum_{\sigma \in Sh(q,n-q)} (-1)^{\sigma'} \phi^p(\ell^q(sv_{\sigma(1)} \ldots sv_{\sigma(q)}))sv_{\sigma(q+1)} \ldots sv_{\sigma(n)} = \sum_{k \geq 1} \frac{1}{k!} \ell^k(\psi^{i_1} \ldots \psi^{i_k})(sv_1 \ldots sv_n).
\]
In particular, \( \hat{\phi} \) preserves the word length filtration. The composition of \( L_\infty \) homomorphism is the naive composition \( \hat{\phi} \circ \hat{\psi} \), which is clearly a coalgebra chain map. Unwrapping the definition, we have
\[
(\phi \circ \psi)^n = \sum_{k \geq 1} \frac{1}{k!} \phi^k(\psi^{i_1} \ldots \psi^{i_k})(sv_1 \ldots sv_n).
\]

2.2. \( BL_\infty \) algebras. In this section, we define the \( BL_\infty \) (bi-Lie-infinity) algebra on a \( \mathbb{Z}_2 \) graded vector space \( V \), which will govern the rational symplectic field theory. Let \( EV \) denote \( \mathbb{S}sV \). Given a linear operator \( p^{k,l} : S^k V \rightarrow S^l V \) for \( k \geq 1, l \geq 1 \), we will define a map \( \tilde{p}^{k,l} : S^k sV \rightarrow sV \). To emphasize the differences between products on two symmetric algebras, we use \( \odot \) for the product on the outside symmetric product \( \mathbb{S} \) and \( * \) for the product on the inside symmetric product \( S \) when it can not be abbreviated. We will first describe the definition using formulas and then introduce a graph description, which is very convenient to describe \( BL_\infty \) algebras as well as various related structures and also governs all the signs and coefficients. Let \( w_1, \ldots, w_k \in SV \), then \( \tilde{p}^{k,l} \) is defined by the following properties.

1. \( \tilde{p}^{k,l} \mid_{\odot^k V \subset S^k SV} \) is defined by \( p^{k,l} \).
2. If \( w_i \in \mathbf{k} \), then \( \tilde{p}^{k,l}(w_1 \odot \ldots \odot w_k) = 0 \).
3. \( \tilde{p}^{k,l} \) satisfies the Leibniz rule in each argument, i.e. we have
\[
\tilde{p}^{k,l}(w_1 \odot \ldots \odot w_k) = \sum_{j=1}^{m} (-1)^{\square} v_1 \ldots v_{j-1} p^{k,l}(w_1 \odot \ldots \odot v_j \odot \ldots \odot w_k) v_{j+1} \ldots v_m.
\]
Here \( w_i = v_1 \ldots v_m \) and
\[
\square = \sum_{s=1}^{i-1} |w_s| \cdot \sum_{s=1}^{j-1} |v_s| + \sum_{s=1}^{j-1} |v_s| p^{k,l} + \sum_{s=i+1}^{n} |w_s| \cdot \sum_{s=j+1}^{m} |v_s|.
\]
It is clear from the definition that \( \hat{p}^{k,l} \) is determined uniquely by the above three conditions. More explicitly, \( \hat{p}^{k,l} \) is defined by the following:

\[
  w_1 \circ \ldots \circ w_k \mapsto \sum_{(i_1, \ldots, i_k) \atop 1 \leq i_j \leq n_j} (-1)^{\sum p^{k,l}(v^1_{i_1} \circ \ldots \circ v^k_{i_k})} \hat{w}_1 \ldots \hat{w}_k,
\]

where \( w_j = v^j_{i_1} \ldots v^j_{i_{n_j}}, \hat{w}_j = v^j_{i_1} \ldots \hat{v}^j_{i_{n_j}} \) and \( w_1 \ldots w_k = (-1)^{\sum v^1_{i_1} \ldots v^k_{i_k}} \hat{w}_1 \ldots \hat{w}_k \). Then we define \( \hat{p}^k : S^k S V \to SV \) by \( \bigoplus_{l \geq 0} \hat{p}^{k,l} \). To assure it is well-defined, we need to assume for any \( v_1, \ldots, v_k \in V \), there are at most finite many \( l \) such that \( p^{k,l}(v_1 \circ \ldots \circ v_k) \neq 0 \). Then we can define \( \hat{p} : EV \to EV \) by

\[
  w_1 \circ \ldots \circ w_n \mapsto \sum_{k=1}^n \sum_{\sigma \in Sh(k, n-k)} (-1)^{\hat{p}^k(w_{\sigma(1)} \circ \ldots \circ w_{\sigma(k)})} \circ w_{\sigma(k+1)} \circ \ldots \circ w_{\sigma(n)},
\]

i.e. following the same rule of \( \hat{e}^k \) from \( \hat{e}^k \).

**Definition 2.3.** \( (V, \{p^{k,l}\}_{k \geq 1, l \geq 0}) \) is a \( BL_\infty \) algebra if \( \hat{p} \circ \hat{p} = 0 \) and \( |\hat{p}| = 1 \).

To explain the terminology, assume \( p^{1,0} = 0, p^{2,0} = 0 \). Then \( p^{1,1} \) defines a Lie bracket on the homology of \( (V, l^{1,1}) \) and \( p^{1,2} \) defines a Lie cobracket on the homology. The compatibility is that \( p^{1,2} \circ p^{2,1} = 0 \) on the homology level. The main difference with the \( IBL_\infty \) algebra [21] is that we will not consider the compatibility condition on \( p^{2,1} \circ p^{1,2} \), which will increase genus\(^5\). A direct consequence of the definition is that \( (SV, \hat{p}^1) \) is a chain complex and the \( \hat{p}^k \) define an \( L_\infty \) structure on \( (SV)[-1] \). As noted in \[70, Remark 3.12\], \( SV \) carries a natural commutative algebra structure, the Leibniz rule in the definition of \( \hat{p}^{k,i} \) implies the \( L_\infty \) structure is compatible with the algebra structure, and \( (SV)[-1] \) should be some version of a \( G_\infty \) algebra. Definition 2.3 can be viewed as making the compatibility precise.

**Remark 2.4.** \( BL_\infty \) algebra is not a “direct” specialization of the \( IBL_\infty \) algebra as introduced in [21]. However, there is an equivalent reformulation of the \( IBL_\infty \) relations, from which one can see that an \( IBL_\infty \) algebra contains a \( BL_\infty \) algebra, as well as algebras with any genus upper bound, see §3.8.

A useful way to explain the combinatorics of operations is the following description using graphs, which appeared in [70, §3.4.2]. The combinatorics is also relevant in the virtual technique setup, see §3.6. We represent a word \( w \in S^k V \) by a graph with \( k+1 \) vertices, where \( k \) vertices are connected to the remaining vertex. Those \( k \) vertices are placeholders for elements from \( V \) and are not ordered, which reflects that the input is from \( S^k V \) rather than from \( \otimes^k V \). We use \( \bullet \) to represent those placeholders and the other vertex will not be emphasized, whose only role is connecting all other vertices. To represent \( p^{k,l} \), we use a graph with \( k+l+1 \) vertices, \( k \) top vertices representing the placeholders for \( k \) inputs and \( l \) bottom vertices representing the placeholders for \( l \) outputs and one middle vertex \( \bigcirc \) representing the operation type. And the \( k \) input vertices and \( l \) output vertices are connected to \( \bigcirc \). To represent \( \hat{p} \) on \( S^{i_1} V \circ \ldots \circ S^{i_n} V \), we first place the \( n \) clusters on the top row representing \( S^{i_1} V, \ldots, S^{i_n} V \), then we add on second level a graph representing \( p^{k,l} \) with \( k \) inputs glued to \( k \) placeholder vertices on the top row, such that the glued graph has no cycle. Then we add a vertical dashed edge to every top row placeholder that is not glued to represent the identity map. Then each connected component of the glued graph represents a \( \bigcirc \)-component in the output, and each bottom row vertex represent a placeholder for a \( * \)-component in the corresponding \( \bigcirc \)-component represented.

---

\(^5\)The other difference is that the \( IBL_\infty \) algebra in [21] describes the algebra for linearized SFT, where \( p^{k,0,g} = 0 \) for any number of positive punctures \( k \) and genus \( g \).
by the connected component. Then \( \hat{p} \) is the sum of all such glued graphs. In particular, \( \hat{p}^{k,l}_{2} \) is the case of attaching \( p^{k,l} \) to get a connected graph. To compute \( \hat{p} \), we just plug in vectors in \( V \) at the placeholders with a sign by the Koszul-Quillen convention.

Example 2.5. If we use \((v_1v_2v_3) \circ (v_4v_5v_6) \circ (v_7v_8)\) as an input into the configuration in Figure 1, then the output is

\[
\pi \left( (\otimes^2 \text{id} \otimes (p^{2,3} \circ \pi) \otimes \text{id})(v_1 \otimes \ldots \otimes v_8) \right) = (-1)^{|v_1|+|v_2|}(v_1v_2p^{2,3}(v_3v_4)v_5v_6) \circ (v_7v_8),
\]

where \( \pi \) is the projection from tensor product to symmetric product. It is easy to check that, because of the sign convention, when we “reorder” the same graph the output does not depend on the order. For example, the following two presentations of the same graph will give the same output up to a sign \((-1)^{\circ}\), where \( v_1 \ldots v_8 = (-1)^{\circ} v_8 \ldots v_1 \), which is the sign difference from reordering the input.

We use \( p^{k,l}_{2} : S^k V \to S^l V \) for \( k \geq 1, l \geq 0 \) to denote the sum of all connected graphs with two levels of \( \circ \) vertices, \( k \) input vertices and \( l \) output vertices. Note that \( p^{k,l}_{2} \) can be viewed as the codimension 1 boundary of the rational SFT moduli space. The following proposition shows that the \( BL_{\infty} \) algebra structure captures exactly such combinatorics.

Proposition 2.6. \( \{p^{k,l}\}_{k \geq 1, l \geq 0} \) forms a \( BL_{\infty} \) algebra iff \( p^{k,l}_{2} = 0 \) for \( k \geq 1, l \geq 0 \).

Proof. Let \( \pi_{1,l} \) denote the projection \( EV \to S^1 SV \to S^l V \), then we have \( p^{k,l}_{2} = \pi_{1,l} \circ \hat{p}^{2}_{\circ k V} \). Therefore if \( \{p^{k,l}\}_{k \geq 1, l \geq 0} \) forms a \( BL_{\infty} \) algebra then \( p^{k,l}_{2} = 0 \) for \( k \geq 1, l \geq 0 \). Now assume \( p^{k,l}_{2} = 0 \) for \( k \geq 1, l \geq 0 \), then we have

\[
\hat{p}^{2}_{\circ k V} = \sum_{i=1}^{k} \sum_{l=0}^{\infty} p^{i,l}_{2} \otimes \text{id} = 0.
\]

Then we will argue inductively on \( i_1, \ldots, i_k \) and \( k \) such that \( \hat{p}^{2}_{\circ i_1, \ldots, \circ i_k V} = 0 \). When we consider \( \hat{p}^{2}_{\circ i_1, \ldots, \circ i_k V}(ab) \circ w_2 \circ \ldots \circ w_n \), if the two \( p^{*,*} \) does not connect to the cluster representing \( ab \), then it reduces a
Figure 2. A component of \( \widehat{\phi} \) from \( S^3V \odot S^3V \odot S^2V \) to \( S^6V' \)

In terms of formulas, we define \( \widehat{\phi}^k := \pi_1 \circ \widehat{\phi}|_{S^kSV} : S^kSV \rightarrow SV' \), which is represented by all glued graph that is connected and has 4 components in the top row. In particular, it is determined by the following.

1. \( \widehat{\phi}^{k+1}(w_1 \odot \ldots \odot w_k \odot 1) = 0 \) for \( k \geq 1 \) and \( \widehat{\phi}^1(1) = 1 \).
2. \( \widehat{\phi}^k : \odot^kV \subset S^kSV \rightarrow SV' \) is defined by \( \sum_{l \geq 0} \phi^{k,l} \).
3. Let \( \{i_j\}_{1 \leq j \leq k} \) be a sequence of positive integers. We define \( N := \sum_{j=1}^k i_j \) and \( N_i := \sum_{j=1}^{i_j} i_j \). Let \( w_i = v_{N_{i-1}+1} \ldots v_{N_i} \). The following sum is over all partitions \( J_1 \sqcup \ldots \sqcup J_b = \{1, \ldots, N\} \), such that the graph with \( k+b+N \) vertices \( A_1, \ldots, A_k, B_1, \ldots, B_b, v_1, \ldots, v_N \) with \( A_i \) connected to \( v_{N_{i-1}+1}, \ldots, v_{N_i} \) and \( B_i \) connected to \( v_j \) if \( j \in J_i \), has no circle:

\[
\widehat{\phi}^k(w_1 \odot \ldots \odot w_k) = \sum_{\text{admissible partitions} \ J_1 \sqcup \ldots \sqcup J_b} \frac{(-1)^\circ}{b!} \sum_{l=0}^\infty \phi^{J_1\vdash l}(w^{J_1}) \odot \ldots \odot \sum_{l=0}^\infty \phi^{J_b\vdash l}(v^{J_b}),
\]

where \( w_1 \ldots w_k = (-1)^\circ v^{J_1} \ldots v^{J_b} \).

The reduction by \( b! \) is a consequence of the fact that we are counting over different graphs with unordered vertices. Then we define \( \phi \) from \( \widehat{\phi}^k \) just like the \( L_\infty \) morphism \( \widehat{\phi} \) built from \( \phi^k \).
Figure 3. An admissible partition

Definition 2.7. \( \{ \phi_{k,l} \}_{k \geq 1, l \geq 0} \) is a \( BL_\infty \) morphism from \((V, p)\) to \((V', p')\) if \( \hat{\phi} \circ \hat{p} = \hat{p'} \circ \hat{\phi} \) and \( |\hat{\phi}| = 0 \).

The composition of \( \phi : V \to V' \) and \( \psi : V' \to V'' \) is defined by the following. Let \( I = \{ 1, \ldots, k \} \) and \( I_1 \sqcup \ldots \sqcup I_a \) be an admissible partition of \( I \) as above, then

\[
(\psi \circ \phi)^{k,l}(v_1 \ldots v_k) = \pi_l \left( \sum_{\text{admissible partitions } I_1 \sqcup \ldots \sqcup I_a} \hat{\psi}^a \left( \frac{(-1)^{\bigcirc}}{a!} \sum_{l=0}^{\infty} \phi_{|I_1|,l} (v^{I_1}) \circ \ldots \circ \sum_{l=0}^{\infty} \phi_{|I_a|,l} (v^{I_a}) \right) \right),
\]

where \( v_1 \ldots v_k = (-1)^{\bigcirc} v^{I_1} \ldots v^{I_a} \) and \( \pi_l \) is the projection \( SV'' \to S^l V'' \). It is clear that the graph representing \( \hat{\psi} \circ \hat{\phi} \) has no cycle. Then \( (\psi \circ \phi)^{k,l} \) is represented by connected graphs without cycles glued from one level from \( \phi \) and one level from \( \psi \). It is clear that \( \hat{\psi} \circ \hat{\phi} = \hat{\psi} \circ \hat{\phi} \) by construction.

Remark 2.8. An \( L_\infty \) algebra can be described by special graphs such that each cluster has one placeholder and we only have \( p^{k,1} \). Similarly for \( L_\infty \) morphisms and their compositions.

Remark 2.9. There are different notions of homotopies between \( BL_\infty \) morphisms if we wish to define notions of homotopy equivalences of \( BL_\infty \) algebras. In practice, we can not associate a conical \( BL_\infty \) algebra to a contact manifold but one depends on various choices and is only well-defined up to homotopy. However, for the purpose of this paper, we are constructing functors from \( \text{Con} \) to a totally ordered set, homotopy invariance is not needed. Nevertheless, we have the following brief remarks on homotopy.

1. One can define a notion of homotopy, which is a homotopy on the bar/cobar complex. That is one can define a map by counting rigid but disconnected curves in a one-parameter family. One advantage of such definition is that it is easier to construct as we will neglect the structures from each connected component. Any homological structure on the level of bar/cobar complex will be an invariant. For example, the contact homology in [66] used this notion of homotopy.

2. Another notion of homotopy is defined through the notion of path objects, e.g. [21, Definition 4.1], see also [70, Definition 2.9] for the homotopy in the \( L_\infty \) context with a specific path object model. This definition is the right one to discuss linearized theory but is more involved to get in the construction of SFT. In particular, homotopic augmentations give rise to homotopic linearized theories with such notion of homotopy. Such homotopy is expected to be derived from the homotopy used in [28].

However, from the curve counting point of view, such construction is more subtle.

A detailed construction of RSFT in terms of \( BL_\infty \) algebras up to homotopy will appear in a future work.

2.3. Augmentations. When \( V = \{ 0 \} \), it has a unique trivial \( BL_\infty \) algebra structure by \( p^{k,l} = 0 \). We use \( 0 \) to denote this trivial \( BL_\infty \) algebra. Note that \( 0 \) is the initial object in the category of \( BL_\infty \) algebras, with \( 0 \to V \) defined by \( \phi^{k,l} = 0 \).
Definition 2.10. A $BL_{\infty}$ augmentation is a $BL_{\infty}$ morphism $\epsilon : V \to 0$, i.e. a family of operators $\epsilon^k : S^k V \to k$ satisfying Definition 2.7.

For a $BL_{\infty}$ algebra $V$, we define $E^k V = \overline{B}^k SV$, which is a filtration on $EV$ compatible with the differential $\overline{p}$. Note that $E0 = k \oplus S2k \oplus \ldots$ with $\overline{p} = 0$, we have $H_*(E0) = E0$. Similarly we have $H_*(E^k 0) = E^k 0$ for all $k \geq 1$. We define $1_0$ be the generator in $E^1 0$, then $1_0 \neq 0 \in H_*(E^k 0)$ for all $k \geq 1$. Then we define $1_V \in H_*(E^k V)$ to be the image of $1_0$ under the chain map $E^k 0 \to E^k V$ induced by the trivial $BL_{\infty}$ morphism $0 \to V$.

Proposition 2.11. If there exists $k \geq 1$, such that $1_V \in H_*(E^k V)$ is zero, then $V$ has no $BL_{\infty}$ augmentation.

Proof. If there is an augmentation $\epsilon : V \to 0$, then the sequences of $BL_{\infty}$ morphisms $0 \to V \overset{\epsilon}{\to} 0$ induce a chain morphism $E^k 0 \to E^k V \to E^k 0$. It is direct to check the composition is identity by definition. If $1_V \in H^*(E^k V)$ is zero, then we have a contradiction since $1_0 \neq 0 \in H^*(E^k 0)$.

Definition 2.12. We define the torsion of a $BL_{\infty}$ algebra $V$ to be

$$T(V) := \min\{k - 1 | 1_V = 0 \in H^*(E^k V), k \geq 1\}.$$

Here the minimum of an empty set is defined to be $\infty$.

By definition, we have that $T(V) = 0$ iff $1_V \in H^*(SV, \overline{p}^1)$ is zero. Since $H^*(SV, \overline{p}^1)$ is an algebra with $1_V$ a unit, we have $H^*(SV, \overline{p}^1) = 0$. In the context of SFT, $T(V) = 0$ iff the contact homology vanishes, i.e. algebraically overtwisted [11].

Since a $BL_{\infty}$ morphism preserves the word filtration on the bar complex, we know that if there is a $BL_{\infty}$ morphism from $V$ to $V'$ then $T(V) \geq T(V')$. Therefore we have the following obvious property, which is crucial for the invariant property for our applications in symplectic topology.

Proposition 2.13. If there are $BL_{\infty}$ morphisms between $V, V'$ in both directions, then we have $T(V) = T(V')$.

Given a $BL_{\infty}$ augmentation $\epsilon$, we can linearize w.r.t. $\epsilon$ by the following procedure. More precisely, there is a change of coordinate to kill off all constant terms $p^{k,0}$. We define $F^{1,1}_\epsilon = \text{id}_V$ and $F^{k,0}_\epsilon = \epsilon^k$ and all other $F^{k,l}_\epsilon = 0$. Then following the recipe of constructing $\hat{\phi}$ from $\phi^{k,l}$, we can define $\hat{F}_\epsilon$ on $EV$. Then $\hat{F}_\epsilon$ preserves the word length filtration and on the diagonal $\pi_k \circ \hat{F}_\epsilon|_{S^k SV}$ is $\circ^k \hat{F}^{1,1}_\epsilon$, where $\hat{F}^{1,1}_\epsilon$ is an algebra isomorphism determined by $\hat{F}^{1,1}_\epsilon(x) = x + \epsilon^1(x)$ and $\pi_k$ is the projection $EV \to S^k SV$. Indeed the inverse is given by the following proposition.

Proposition 2.14. Let $\hat{F}_{-\epsilon}$ denote the map on $EV$ defined by $F^{1,1}_{-\epsilon} = \text{id}_V$ and $F^{k,0}_{-\epsilon} = -\epsilon^{k,0}$ and all other $F^{k,l}_{-\epsilon} = 0$, then $\hat{F}_{-\epsilon}$ is the inverse of $\hat{F}_\epsilon$.

Proof. We use the graph representation to prove the claim. Note that in the composition $\hat{F}_{-\epsilon} \circ \hat{F}_\epsilon$, we can find pairs of configuration as follows as long as there are some components from $\epsilon$. 
Figure 4. An example with $\epsilon^2$

It is clear those pairs will cancel with each other. Hence the only remaining term is those only has identity map component. Hence $\tilde{F}_{-\epsilon} \circ \tilde{F}_{\epsilon} = \text{id}$, similarly, we have $\tilde{F}_{\epsilon} \circ \tilde{F}_{-\epsilon} = \text{id}$.

We use $\tilde{F}_{\epsilon}$ as a change of coordinate on $EV$ and consider $\hat{p}_\epsilon := \tilde{F}_{\epsilon} \circ \tilde{p} \circ \tilde{F}_{-\epsilon} : EV \to EV$, then $\hat{p}_\epsilon^2 = 0$. We can define

$$p_{\epsilon}^{k,l} := \pi_{1,l} \circ \tilde{F}_{\epsilon} \circ \tilde{p} \circ \tilde{F}_{-\epsilon}|_{\cdot kV},$$

where $\pi_{1,l}$ is the projection $EV \to S^1S^lV$. Since $\tilde{p}(1 \circ \ldots) = 0$, we have $p_{\epsilon}^{k,l} = \pi_{1,l} \circ \tilde{F}_{\epsilon} \circ \tilde{p}|_{\cdot kV}$ and it is represented by the sum of following connected graphs without cycle with $k$ inputs, $l$ outputs, one $\bigcirc$ component and possibly several components from $\epsilon$.

Figure 5. A component of $p_{\epsilon}^{4,1}$

Proposition 2.15. $\hat{p}$ is determined by $p_{\epsilon}^{k,l}$ following the same recipe for $\hat{p}$ from $p^{k,l}$. Moreover, we have $p_{\epsilon}^{k,0} = 0$ for all $k$.

Proof. In the graph configuration of $\tilde{F}_{\epsilon} \circ \tilde{p} \circ \tilde{F}_{-\epsilon}$ on $S^{i_1}V \circ \ldots \circ S^{i_k}V$, there is exactly one component containing a $p_{\epsilon}^{k,l}$ as a subgraph. In the connected component containing this subgraph, we might have other components containing $\pm \epsilon$ and we have other connected components that are not purely identities, i.e. containing $\pm \epsilon$. To prove that $\hat{p}$ is determined by $p_{\epsilon}^{k,l}$, we need to rule out all those other components containing $\pm \epsilon$. This follows from the same argument in Proposition 2.14, as in both case they will pair up.
and cancel each other. Because \( \epsilon \) is an augmentation, we have \( \pi^{1,0} \circ \tilde{F}_\epsilon \circ \tilde{p} = \tilde{\epsilon} \circ \tilde{p} = 0 \). Therefore \( \tilde{p}^{k,0} = 0 \) for all \( k \).

As a corollary of Proposition 2.15, \( \ell^k_\epsilon \) defines an \( L_\infty \) structure on \( V[-1] \). Next we introduce the structure which will be relevant to the definition of planarity. Let \( \hat{p}^{k,l}_\epsilon : S^k V \to S^l V, k \geq 1, l \geq 0 \) be a family of linear maps, we can define \( \hat{p}^{k,l}_\epsilon \) and \( \hat{p}_\epsilon \) just like \( \hat{p}^{k,l}_\epsilon \) and \( \hat{p} \) by component-wise Leibniz rule and coLeibniz rule with the modification that \( |p^{k,l}_\epsilon| \) is not necessarily \( 1 \in \mathbb{Z}_2 \).

**Definition 2.16.** We say \( \{p^{k,l}_\epsilon\} \) is a pointed map, iff \( \hat{p}_\epsilon \circ \hat{p} = (-1)^{|\hat{p}^{k,l}_\epsilon|} \hat{p} \circ \hat{p}_\epsilon \).

In applications, \( p^{k,l}_\epsilon \) will come from counting holomorphic curves with one interior marked point subject to a constrain from \( H_*(Y) \). The degree of \( p_* \) is same as the degree of the constraint. Typically we will only consider a point constraint, then the degree is 0. Note that it does not define \( BL_\infty \) morphisms as the combinatorics for packaging \( \hat{p}_\epsilon \) is different from \( \hat{p} \). Nevertheless, \( \hat{p}_\epsilon \) still defines a morphism on the bar complex and preserves the word length filtration.

Then by the same argument in Proposition 2.15, we can define \( \hat{p}^{k,0}_\epsilon := \hat{F}_\epsilon \circ \hat{p}^{k,0}_\epsilon \circ \hat{F}^{-1}_\epsilon \) and \( \hat{p}^{k,0}_\epsilon \) is determined by \( p^{k,0}_\epsilon \), which is defined similarly to \( p^{k,0}_\epsilon \). Note that we also have \( \hat{p}_\epsilon \circ \hat{p}^{k,0}_\epsilon = \hat{p}^{k,0}_\epsilon \circ \hat{p}_\epsilon \). However, it is not necessarily true that \( \hat{p}^{k,0}_\epsilon = 0 \). In fact, the failure of this property on homological level will be another hierarchy that we are interested in. We define \( \ell^{k,0}_\epsilon \) by \( p^{k,0}_\epsilon \). Then \( \ell^{k,0}_\epsilon := \sum_{k \geq 0} \hat{p}^{k,0}_\epsilon \) defines a chain morphism \( (\overline{SV}, \ell_\epsilon) \to k \).

That \( \hat{p}^{k,0}_\epsilon \) commutes with \( \ell_\epsilon \) follows from \( \pi_k \circ \hat{p}_\epsilon \circ \hat{p}^{k,0}_\epsilon = \pi_k \circ \hat{p}^{k,0}_\epsilon \circ \hat{p}_\epsilon \) restricted to \( \overline{SV} = \overline{SS}^1 V \subset EV \) and \( \pi_k \) is the projection from \( EV \) to \( k \subset \mathbb{S}^1 \mathbb{S}^1 V \subset EV \).

**Definition 2.17.** Given a \( BL_\infty \) augmentation and a pointed map \( p_* \), the \( (\epsilon, p_*) \) order of \( V \) is defined to be

\[
O(V, \epsilon, p_*) := \min \left\{ k \left| 1 \in \im \ell^{k,0}_\epsilon \right|_{H_*(\overline{SV}, \ell_\epsilon)} \right\},
\]

where the minimum of an empty set is defined to be \( \infty \).

There is another notion of order, which is related to \( O(V, \epsilon, p_*) \). We use \( \overline{EV} \) to denote \( \overline{SS} \), then \( EV \) has a splitting

\[
EV = (\overline{EV} \oplus \mathbb{C} k) \oplus ((\overline{EV} \oplus k) \oplus \mathbb{C} 2) \oplus ((\overline{EV} \oplus k) \oplus \mathbb{C} 3) \oplus \ldots
\]

\[
= \oplus_{k=1}^{\infty} (\overline{EV} \oplus k)
\]

Since \( \hat{p}^{k,0}_\epsilon = 0 \) for all \( k \), we know that \( \hat{p} \) is a differential on \( \overline{EV} \). Moreover, note that \( \hat{p}_\epsilon (k \circ \ldots) = 0 \) by definition, we have the homology of \( EV \) respects the splitting, i.e. \( H_*(\overline{EV}, \hat{p}) = \oplus_{k=1}^{\infty} (H_*(\overline{EV}, \hat{p}) \oplus k) \). On the other hand, \( \hat{p}_\epsilon \) does not respect such splitting. In particular, one interesting portion is the restriction of \( \pi_k \circ \hat{p}^{k,l}_\epsilon \) to the first copy of \( H_*(\overline{EV}, \hat{p}) \), where \( \pi_k \) is the projection to the first \( k \) in the splitting \( H_*(\overline{EV}, \hat{p}) = \oplus_{k=1}^{\infty} (H_*(\overline{EV}, \hat{p}) \oplus k) \).

**Definition 2.18.** Given a \( BL_\infty \) augmentation and a pointed map \( p_* \), we define

\[
\hat{O}(V, \epsilon, p_*) := \min \left\{ k \left| 1 \in \pi_k \circ \hat{p}^{k,l}_\epsilon \right|_{H_*(\overline{SV}, \hat{p}_\epsilon)} \right\}.
\]

**Proposition 2.19.** \( O(V, \epsilon, p_*) \leq \hat{O}(V, \epsilon, p_*) \).

**Proof.** We use \( B_2^k \subset \overline{B}^k \overline{SV} \) to denote the subspace \( \left( \oplus_{k=2}^{\infty} \overline{SV} \right) \oplus k^{-1} \overline{SV} \). Since \( \hat{p}^{k,0}_\epsilon = 0 \), we have \( B_2^k \) is a subcomplex. Moreover, the quotient complex \( \overline{B}^k \overline{SV}/B_2^k \) is exactly \( (\overline{B}^k V, \ell_\epsilon) \). Since \( \pi_k \circ \hat{p}^{k,l}_\epsilon \) on \( B_2^k \) is zero. We know that \( 1 \in \pi_k \circ \hat{p}^{k,l}_\epsilon \) on \( H_*(\overline{SV}, \hat{p}_\epsilon) \) implies that \( 1 \in \im \ell^{k,0}_\epsilon \) on \( H_*(\overline{SV}, \hat{p}_\epsilon) \). Hence the claim follows. \( \square \)
The inequality in Proposition 2.19 is likely to be necessary. The main issue is that a closed class in $H_* (\overline{B}^k V)$ may not be closed in $H_* (\overline{S}^* V)$. For the definition of planarity, we will use $O(V, \epsilon, p_*)$, which has the benefit of the existence of another hierarchy when $O(V, \epsilon, p_*) = 1$.

**Remark 2.20.** We can similarly define a more generalized order as
\[
\min \left\{ k \left\vert 1 \in \pi_k \circ (\hat{\rho}_*, \epsilon) \right\vert_{H_* (\overline{B}^k SV, \hat{\rho}_*)} \right\}.
\]

It it easy to show that the number is non-decreasing w.r.t. $l$, where the $l = 1$ case is $\hat{O}(V, \epsilon, p_*)$. See §3.7 for this invariant in the context of RSFT and its relation to Siegel’s higher symplectic capacities with multiple point constraints [70].

Next we need to compare the construction under $BL_\infty$ morphisms. Given a $BL_\infty$ morphism $\phi : (V, p) \to (V', q)$ and a family of morphisms $\phi^{k,l}_* : S^k V \to S^l V$, then we can define $\hat{\phi}_* : EV \to EV'$ by the same rule of $\hat{\phi}$ with exactly one $\phi^{k,l}_*$ component and all the others are $\phi^{k,l}_*$ components with the exception that $\hat{\phi}_* (k \circ \ldots) = 0$.

**Definition 2.21.** Assume $p_*, q_*$ are two pointed maps of $(V, p), (V', q)$ respectively of the same degree. We say $p_*, q_*, \phi$ are compatible, if there is a family of $\phi^{k,l}_*$ such that $\hat{\phi}_* \circ \hat{\phi} = (-1)^{k_1} \hat{\phi}_* \circ \hat{\phi}_* = q_\circ \hat{\phi}_* - (-1)^{k_1} \hat{\phi}_* \circ \hat{\phi}_*$ and $|\hat{\phi}_*| = |\hat{\phi}_*| + 1$.

In practice, $\phi^{k,l}_*$ is defined by counting connected rational holomorphic curves in the cobordism with a marked point passing through a cobordism between the constraints in the definition of $p_*, q_*$. In our typical case of point constraint, the cobordism will be a path connecting the point constraints, where we have $|\hat{\phi}_*| = 1$. In principle, we can consider the category consists of pairs $(p, p_*)$ with morphisms given by pairs $(\phi, \phi_*)$ with a suitable definition of composition. Then the definition of orders is functorial. For our purpose, we only need the following property without the precise definition of a composition.

**Proposition 2.22.** Assume $\phi$ is a $BL_\infty$-morphism from $(V, p)$ to $(V', q)$ with pointed maps $p_*, q_*$ of degree 0 respectively, such that $p_*, q_*, \phi$ are compatible. Then for any $BL_\infty$ augmentation $\epsilon$ of $V'$, we have $O(V, \epsilon \circ \phi, p_*) \geq O(V', \epsilon, q_*)$ and $O(V', \epsilon \circ \phi, p_*) \geq O(V', \epsilon, q_*)$.

**Proof.** By the definition of compatibility, we have
\[
\hat{\phi}_* \circ \hat{\phi} \circ \hat{\phi}_* \circ \hat{\phi} = \hat{\phi}_* \circ \hat{\phi} \circ \hat{\phi} \circ \hat{\phi} = \hat{\phi}_* \circ \hat{\phi} \circ \hat{\phi} \circ \hat{\phi} = \hat{\phi}_* \circ \hat{\phi} \circ \hat{\phi} \circ \hat{\phi}.
\]

From $\hat{\phi} \circ \hat{\phi} = \hat{\phi} \circ \hat{\phi}$, we have $\hat{\phi}_* \circ \hat{\phi} \circ \hat{\phi} \circ \hat{\phi} \circ \hat{\phi} \circ \hat{\phi} = \hat{\phi}_* \circ \hat{\phi} \circ \hat{\phi} \circ \hat{\phi}$. Therefore we have the following commutative diagram of complexes up to homotopy, where we define $\phi_\epsilon = \hat{\phi}_\epsilon \circ \hat{\phi} \circ \hat{\phi}_\epsilon \circ \hat{\phi}$.

\[
\begin{array}{ccc}
EV & \xrightarrow{\hat{\rho}_*, \epsilon \circ \phi} & EV' \\
| \hat{\phi}_\epsilon \downarrow & & \hat{\phi}_\epsilon \downarrow \\
EV & \xrightarrow{\hat{\phi}_*, \epsilon} & EV'
\end{array}
\]

Similar to Proposition 2.14, $\hat{\phi}_\epsilon$ is determined by $\phi^{k,l}_\epsilon$ following a similar rule to $\hat{\phi}$ and $\phi^{k,l}_\epsilon$, where $\phi^{k,l}_\epsilon$ is defined similar to $p^{k,l}_\epsilon$. Moreover, $\phi^{k,0}_\epsilon = 0$, as nontrivial contributions pair up and cancel each other similar to Proposition 2.14. As a consequence, $\hat{\phi}_\epsilon$ preserves the splitting in the definition of $\hat{O}$. Since all the maps including the homotopy preserve the word length filtration, if $1 \in \pi_k \circ \hat{\rho}_*, \epsilon \circ \phi \mid_{H_* (\overline{B}^k SV, \hat{\rho}_*)}$, then
1 ∈ \text{im} \pi_k \circ \tilde{\gamma}_\bullet \vert_{H_* (B^k SV', \hat{\phi}_c)}. Therefore \( \tilde{O}(V, \epsilon \circ \phi, p_\bullet) \geq \tilde{O}(V', \epsilon, q_\bullet) \). In order to prove the other inequality, we have a diagram

\[
\begin{array}{ccc}
SV & \xrightarrow{\tilde{\gamma}_\bullet \circ \phi} & k \\
\downarrow \tilde{\phi}_\epsilon & & \downarrow \text{id} \\
SV' & \xrightarrow{\tilde{\gamma}_\epsilon} & k
\end{array}
\]

where \( \tilde{\phi}_\epsilon \) is determined by the \( L_\infty \) morphism \( \phi^{k, 1}_\epsilon \) since \( \phi^{k, 0}_\epsilon = 0 \). The diagram is in fact commutative up to homotopy \( \tilde{\phi}^0_\bullet, \epsilon : SV \to k \) defined by \( \sum_{k \geq 1} \phi^{k, 0}_\epsilon \). Therefore \( 1 \in \text{im} \tilde{\gamma}_\bullet \circ \phi \vert_{H_* (B^k V', \hat{\phi}_\epsilon)} \) implies that

\[
1 \in \text{im} \tilde{\gamma}_\bullet \vert_{H_* (B^k V', \hat{\phi}_c)}. \]  

Hence \( O(V, \epsilon \circ \phi, p_\bullet) \geq O(V', \epsilon, q_\bullet) \). \( \square \)

3. Rational symplectic field theory

In this section, we explain the construction of rational symplectic field theory (RSFT) as \( BL_\infty \) algebras. RSFT was original packaged into a Poisson algebra with a distinguished odd degree class \( h \) such that \( \{ h, h \} = 0 \) in [28]. However for the purpose of building hierarchy functors from contact manifolds, it is useful to reformulate RSFT as \( BL_\infty \) algebras. It is important to note that we will use same moduli spaces of holomorphic curves as the original RSFT but reinterpret the relations as other algebraic structures.

3.1. Notations on symplectic topology. We first briefly recall the basics of symplectic and contact topology. A (co-oriented) contact manifold \((Y, \xi)\) is a \( 2n - 1 \) dimensional manifold with a (co-oriented) hyperplane distribution \( \xi \) such that there is a one form \( \alpha \) with \( \xi = \text{ker} \alpha \) and \( \alpha \wedge (d\alpha)^{n-1} \neq 0 \). Such one form \( \alpha \) is called a contact form and we will call \((Y, \alpha)\) a strict contact manifold. Given a contact form \( \alpha \), the Reeb vector field \( R_\alpha \) is characterized by \( \alpha(R_\alpha) = 0, \iota_{R_\alpha} d\alpha = 0 \). We say a contact form \( \alpha \) is non-degenerate iff all Reeb orbits are non-degenerate. Any contact form can be perturbed into a non-degenerate contact form and in particular, every contact manifold admits non-degenerate contact forms. Throughout this paper \((Y, \alpha)\) is always assumed to be a strict contact manifold with a non-degenerate contact form unless specified otherwise.

**Definition 3.1.** A symplectic manifold \((X, \omega)\) with \( \partial W = Y^- \cup Y^+\) is

1. a strong cobordism from \((Y^-, \xi^-)\) to \((Y^+, \xi^+)\) iff \( \omega = d\lambda \) near \( Y^- \) with \( \xi^\pm = \text{ker} \lambda^\pm \) such that we define \( V^\pm \) by \( \iota_{V^\pm} \omega = \lambda^\pm \), then \( V^\pm \) points out on \( Y^+ \) and \( V^- \) points in on \( Y^- \);
2. an exact cobordism from \((Y^-, \xi^-)\) to \((Y^+, \xi^+)\) if moreover \( \omega = d\lambda \) on \( X \). The vector field \( V \) defined by \( \iota_V \omega = \lambda \) is called the Liouville vector field;
3. a Weinstein cobordism from \((Y^-, \xi^-)\) to \((Y^+, \xi^+)\) if moreover the Liouville vector field is gradient like for some Morse function with \( Y^\pm \) as the regular level sets of maximum/minimum.

We say a cobordism \((X, \omega)\) from \((Y^-, \alpha^-)\) to \((Y^+, \alpha^+)\) is strict iff \( \lambda^\pm |_{Y^\pm} = \alpha^\pm \). It is clear from definition that we can glue strict cobordisms to get a strict cobordism. In general, given two exact cobordisms \( W_1, W_2 \) from \( Y_1, Y_2 \) to \( Y_2, Y_3 \) respectively, the composition \( W_2 \circ W_1 \) from \( Y_1 \) to \( Y_3 \) is not uniquely defined, but up to homotopies of Liouville structures [20, \S 11.2], it is well-defined. The central geometric object of our interests is the following cobordism category.

**Definition 3.2.** The exact cobordism category of contact manifolds \( \text{Con} \) is defined to be the category whose objects are contact manifolds and morphisms are exact cobordisms up to homotopy. The composition is given by gluing cobordisms. We will use \( \text{Con}^{2k-1} \) to denote the subcategory of \( 2k - 1 \) dimensional contact manifolds.
Similarly, we use $\text{Con}_W$ to denote the Weinstein cobordism category and $\text{Con}_S$ to denote the strong cobordism category.

All of the categories above have monoidal structures given by the disjoint union. It is clear that we have natural functors $\text{Con}_W \rightarrow \text{Con} \rightarrow \text{Con}_S$, which are identities on the object level.

**Remark 3.3.** There is a forgetful functor from $\text{Con}$ to the cobordism category of almost contact manifolds, where the cobordisms are almost symplectic cobordisms. In the case of $\text{Con}_W$, there is a forgetful map to the almost Weinstein cobordism category of almost contact manifolds. These are purely topological objects, the latter was studied thoroughly in [17, 18].

Roughly speaking, the principle in the symplectic cobordism category is that the complexity of contact geometry increases in the direction of cobordism. In view of this, we can introduce the following category, which only remembers if there exists a cobordism.

**Definition 3.4.** We define $\text{Con}_{\leq}$ to be the category of contact manifolds, such that there is at most one arrow between two contact manifolds and the arrow exists iff there is an exact cobordism. Similarly, we can define $\text{Con}_{\leq,W}$ and $\text{Con}_{\leq,S}$.

It is a natural question to ask whether $\text{Con}_{\leq}$ is a poset. It clear that we only need to prove that $Y_1 \leq Y_2, Y_2 \leq Y_1$ implies that $Y_1 = Y_2$. Unfortunately, this is not the case, as we may take $Y_1,Y_2$ as two different 3-dimensional overtwisted contact manifolds [32] or suitable flexible fillable contact manifolds. One extreme case is that $Y_1$ can be different from $Y_2$ even if the cobordisms are inverse to each other [26]. However, we can modulo out this ambiguity to get a poset. It is clear that the existence of (exact) cobordisms between $Y_1,Y_2$ in both directions define an equivalence relation.

**Definition 3.5.** We define $\overline{\text{Con}}_{\leq}$ to be the poset, such that the object is an equivalence class of contact manifolds and there is a morphism $[Y_1] \leq [Y_2]$ iff there is an exact cobordism from $Y_1$ to $Y_2$. Similarly, we can define the posets $\overline{\text{Con}}_{\leq,W}$ and $\overline{\text{Con}}_{\leq,S}$.

Under this condition, all overtwisted contact structure becomes the same minimal object in $\overline{\text{Con}}_{\leq}^3 [32]$. In the higher dimensions, overtwisted contact manifolds are minimal objects up to topological constrains [30].

It is clear that we have functors $\text{Con} \rightarrow \overline{\text{Con}}_{\leq} \rightarrow \overline{\text{Con}}_{\leq}$. The theme of this paper is constructing functors from $\text{Con}$ to some totally ordered set. Since it always descends to $\overline{\text{Con}}_{\leq}$, results in this paper can be understood as some structures on the poset $\overline{\text{Con}}_{\leq}$. It is also an interesting question on whether $\overline{\text{Con}}_{\leq}$ is a totally ordered set, see §7.5 for more discussions on this as well as the differences between $\overline{\text{Con}}_{\leq}, \overline{\text{Con}}_{\leq,W}, \overline{\text{Con}}_{\leq,S}$.

An exact/Weinstein/strong cobordism from $\emptyset$ to $Y$ is called an exact/Weinstein/strong filling of $Y$. We also introduce a category $\text{Con}_*$ as the over category over the empty set.

**Definition 3.6.** The objects of $\text{Con}_*$ are pairs $(Y,W)$, where $W$ is an exact filling of $Y$ up to homotopy. A morphism from $(Y_1,W_1)$ to $(Y_2,W_2)$ is an exact cobordism $X$ from $Y_1$ to $Y_2$ such that $X \circ W_1 = W_2$ up to homotopy, or equivalently an exact embedding of $W_1$ into $W_2$ up to homotopy.

**Example 3.7.** The symplectic cohomology is a functor from $\text{Con}_*$ to the category of BV algebras, where the functoriality follows from the Viterbo transfer map. The $S^1$-equivariant symplectic cochain complex is also a functor $\text{Con}_*$ to the homotopy category of $S^1$-cochain complexes. The order of dilation and the order of semi-dilation in [78] are functors from $\text{Con}_*$ to $\mathbb{N} \cup \{\infty\}$.

For the total order, we will not consider $\emptyset$ as an object in $\overline{\text{Con}}_{\leq}$, as overtwisted contact manifolds and $\emptyset$ are obviously not comparable in $\overline{\text{Con}}_{\leq}$.
3.2. Geometric setups for holomorphic curves. As usual, an almost complex structure $J$ on the symplectization $(\mathbb{R} \times Y, d(e^{s} \alpha))$ is said to be admissible iff

1. $J$ is invariant under the $s$-translation and restricts to a tame almost complex structure on $(\xi = \text{ker } \alpha, d\alpha)$,
2. $J$ sends $\partial_s$ to the Reeb vector $R_\alpha$.

Let $(W, \lambda)$ be an exact filling and $(X, \lambda)$ an exact cobordism. An almost complex structure $J$ on completions $(\hat{W}, \hat{\lambda})$ or $(\hat{X}, \hat{\lambda})$ is admissible iff

1. $J$ is tame for $d\hat{\lambda}$
2. $J$ is admissible on cylindrical ends.

Occasionally, we will also consider strong fillings $(W, \omega)$, where the definition of admissible almost complex structure on $\hat{W}$ is similar. For each Reeb orbit $\gamma$, we can fix a basepoint $b_s$ on the image. Now fix an admissible $J$, and consider two collections of Reeb orbits $\gamma_1^+, \ldots, \gamma_s^+$ and $\gamma_1^-, \ldots, \gamma_s^-$, possibly with duplicates. A pseudoholomorphic map in the symplectization $\mathbb{R} \times Y$ or completions $\hat{W}, \hat{X}$ with positive asymptotics $\gamma_1^+, \ldots, \gamma_s^+$ and negative asymptotics $\gamma_1^-, \ldots, \gamma_s^-$ consists of:

1. a sphere $\Sigma$, with a complex structure denoted by $\{\}$,
2. a collection of pairwise distinct points $z_1^+, \ldots, z_s^+, z_1^-, \ldots, z_s^-$ in $\Sigma$, each equipped with an asymptotic marker, i.e. a direction in the tangent sphere bundle $S_{z_i} \Sigma$,
3. a map $\hat{\Sigma} \to \mathbb{R} \times Y, \hat{\Sigma}$ satisfying $d\hat{u} \circ j = J \circ du$, where $\hat{\Sigma}$ denotes the punctured Riemann surface $\Sigma \setminus \{z_1^+, \ldots, z_s^+, z_1^-, \ldots, z_s^–\}$,
4. for each $z_i^+$ with corresponding polar coordinates $(r, \theta)$ around $z_i^+$ such that the asymptotic marker corresponding to $\theta = 0$, we have

$$\lim_{r \to 0} (\pi_Y \circ u)(re^{i\theta}) = +\infty, \quad u_{z_i^+}(\theta) := \lim_{r \to 0} (\pi_Y \circ u)(re^{i\theta}) = \gamma_i^+ (\frac{1}{2\pi} T_i^+ \theta)$$

where $T_i^+$ is the period of the parameterized orbit $\gamma_i^+$ and $\gamma_i^+(0) = b_i^+$,

5. for each $z_i^-$ with corresponding polar coordinates $(r, \theta)$ compatible with asymptotic marker, we have

$$\lim_{r \to 0} (\pi_Y \circ u)(re^{i\theta}) = -\infty, \quad u_{z_i^-}(\theta) := \lim_{r \to 0} (\pi_Y \circ u)(re^{-i\theta}) = \gamma_i^- (-\frac{1}{2\pi} T_i^- \theta)$$

where $T_i^-$ is the parameterized orbit $\gamma_i^-$ and $\gamma_i^-(0) = b_i^-$. 

A holomorphic curve is an equivalence of holomorphic maps modulo biholomorphisms of $\Sigma$ commuting with all the data. Throughout this paper, we will work with $\mathbb{Z}_2$-grading unless specified otherwise. Let $\Gamma^+ = \{\gamma_1^+, \ldots, \gamma_s^+\}$, $\Gamma^- = \{\gamma_1^-, \ldots, \gamma_s^-\}$ be two ordered sets of Reeb orbits possible with duplicates. Choosing trivializations of $\xi$ over orbits in $\Gamma^+, \Gamma^-$, we can assign the Conley-Zehnder index $\mu_{CZ}(\gamma_i^\pm)$ to each orbit. With such trivialization, we have a relative first Chern class

$$c_1 : H_2(Y, \Gamma^+ \cup \Gamma^-; \mathbb{Z}) \to \mathbb{Z},$$

similarly for $W$ and $X$. Let $A$ be a relative homology representing the curve $u$, we have the Fredholm index of the Cauchy Riemann operator at $u$ minus the dimension of automorphism group (i.e. biholomorphism of $\Sigma$ commuting with all the data) is the following,

$$\text{ind}(u) = (n - 3)(2 - s^+ - s^-) + \sum_{i=1}^{s^+} \mu_{CZ}(\gamma_i^+) - \sum_{i=1}^{s^-} \mu_{CZ}(\gamma_i^-) + 2c_1(A).$$
In this paper, we will consider the following moduli spaces.

1. \(\mathcal{M}_{Y,A}(\Gamma^+,\Gamma^-)\) is the moduli space of rational holomorphic curves in the symplectization, modulo automorphism and the \(\mathbb{R}\) translation. The expected dimension is \(\text{ind}(u) - 1\).

2. \(\mathcal{M}_{W,A}(\Gamma^+,\emptyset)\) and \(\mathcal{M}_{X,A}(\Gamma^+,\Gamma^-)\) are the moduli spaces of rational holomorphic curves in the filling, respectively cobordism, modulo automorphism. The expected dimension is \(\text{ind}(u)\).

3. \(\mathcal{M}_{Y,A,o}(\Gamma^+,\Gamma^-)\) is the moduli space of rational holomorphic with one interior marked point in the symplectization modulo automorphism. And the marked point is required to be mapped to \((0,o)\in \mathbb{R} \times Y\) for a point \(o\in Y\). The expected dimension is \(\text{ind}(u) + 2 - 2n\).

4. \(\mathcal{M}_{W,A,o}(\Gamma^+,\emptyset)\) is the moduli space of rational holomorphic with one interior marked point in the filling modulo automorphism. And the marked point is required to be mapped to \(o\in W\). The expected dimension is \(\text{ind}(u) + 2 - 2n\).

5. \(\mathcal{M}_{X,A,o}(\Gamma^+,\Gamma^-)\) is the moduli space of rational holomorphic with one interior marked point in the exact cobordism \(X\) modulo automorphism. The marked point is required to go through a path \(\gamma\), which is the completion of a path \(\gamma\) from a point in \(Y_+\) to a point in \(Y_-\), i.e. extension by constant maps in each slice in the cylindrical ends. The expected dimension is \(\text{ind}(u) + 1 - 2n\).

Another fact that is important for our later proof is that

\[\int_{\Gamma^+} \alpha - \int_{\Gamma^-} \alpha \geq 0,\]

whenever \(\mathcal{M}_{Y,A}(\Gamma^+,\Gamma^-)\) or \(\mathcal{M}_{Y,A,o}(\Gamma^+,\Gamma^-)\) are not empty. All of the moduli spaces above have a SFT building compactification by \([9]\) denoted by \(\overline{\mathcal{M}}\). The orientation convention follows \([10]\), and we need to require that all asymptotic Reeb orbits are good \([76, \text{Definition 11.6}]\). One property of this convention is that if we switch two orbits \(\gamma_1, \gamma_2\) that are next to each other in \(\Gamma^+\) or \(\Gamma^-\), the induced orientation is changed by \((-1)^{\mu_{\mathbb{Z}}(\gamma_1) + n - 3} |\mu_{\mathbb{Z}}(\gamma_2) + n - 3|\) \([76, \text{§11.2}]\). In the following, we will count zero dimensional moduli spaces to define coefficients in the structure maps. First of all, this requires an orientation, hence we can only count when all asymptotic Reeb orbits are good\(^7\). Next we need transversality, where the count is a honest count of orbifold points, or a virtual machinery, where the count is a count of weighted orbifold points in perturbed moduli spaces \([33, 47]\) or an algebraic count after fixing some auxiliary data \([66]\). For simplicity, we will just use \#\(\overline{\mathcal{M}}\) to denote the count.

### 3.3. Contact homology algebra.

We will first recall the definition of the contact homology algebra. There are two possible choices of coefficient, the rational number field \(\mathbb{Q}\) and the Novikov field \(\Lambda\) = \(\{\sum a_i T_i^b | a_i \in \mathbb{Q}, \lim b_i \in \mathbb{R} \to \infty\}\). Since the only non-exact symplectic manifold we will consider is strong fillings, therefore the only place we really need to use the Novikov field is augmentations from strong fillings. In the following, we will state the formula using \(\Lambda\), but we can always set \(T = 1\) to go back to \(\mathbb{Q}\) whenever the underlying symplectic manifold is exact.

Let \(V_\alpha\) denote the free \(\Lambda\)-module generated by formal variable \(q_\gamma\) for each good orbit \(\gamma\) of \((Y, \alpha)\). \(q_\gamma\) is graded as \(\mu_{\mathbb{Z}}(\gamma) + n - 3\), which should be understood as a well-defined \(\mathbb{Z}_2\) grading in general. The contact homology algebra \(\text{CHA}(Y)\) is the free symmetric algebra \(SV_\alpha\). The differential is defined as follows.

\[\partial_t(q_\gamma) = \sum_{||\Gamma|| = l} \#\overline{\mathcal{M}}_{Y,A}(\{\gamma\}, \Gamma) \frac{T_f A^d \alpha}{\mu_{\mathbb{R}}(\Gamma)} q_\Gamma.\]

\(^7\)Alternatively, the count is evaluated in the fixed space of an orientation line with a group action, the appearance of a bad orbit is exactly when the group action is not trivial, see \([66]\) for details.
The sum is over all multiset $|\Gamma|$, i.e. sets with duplicates, of size $l$. And $\Gamma$ is an ordered representation of $|\Gamma|$, e.g. $\Gamma = \{\eta_1, \ldots, \eta_i, \ldots, \eta_m, \ldots, \eta_m\}$ is an ordered set of good orbits with $\eta_i \neq \eta_j$ for $i \neq j$ and $\sum i_j = l$. We write $\mu_\Gamma = i_1! \cdot \cdots \cdot i_m!$ and $\kappa_\Gamma = \kappa_1^{i_1} \cdot \cdots \cdot \kappa_m^{i_m}$ is the product of multiplicities, and $q^\Gamma = q_{\eta_1} \cdots q_{\eta_m}$.

We modulo out $\mu_\Gamma$ as we should count holomorphic curves with unordered punctures and modulo out $\kappa_\Gamma$ to compensate that we have $\kappa_\gamma$ different ways to glue when we have a breaking at $\gamma$. The orientation property of $\overline{M}_{Y,A}(\{\gamma\}, \Gamma)$ implies that (3.4) is independent of the representative $\Gamma$. The differential on a single generator is defined by

$$\partial(q_\gamma) = \sum_{l=0}^{\infty} \partial_l(q_\gamma),$$

which is always a finite sum by (3.3). Then the differential on $CHA(Y)$ is defined by the Leibniz rule

$$\partial(q_{\gamma_1} \cdots q_{\gamma_l}) = \sum_{j=1}^{l} (-1)^{|q_{\gamma_1}| + \cdots + |q_{\gamma_j-1}|} q_{\gamma_1} \cdots q_{\gamma_{j-1}} \partial(q_{\gamma_j}) q_{\gamma_{j+1}} \cdots q_{\gamma_l}.$$

The relation $\partial^2 = 0$ follows from the boundary configuration of $\overline{M}_{Y,A}(\{\gamma\}, \Gamma)$ with virtual dimension 1.

Given an exact cobordism $(X, \lambda)$ from $Y_-$ to $Y_+$, then we have an algebra map $\phi$ from $CHA(Y_+)$ to $CHA(Y_-)$, which on generator is defined by

$$\phi(q_\gamma) = \sum_{l=0}^{\infty} \sum_{|\Gamma| = l} \#\overline{M}_{X,A}(\{\gamma\}, \Gamma) \frac{Tf_A^\omega}{\mu_\Gamma \kappa_\Gamma} q^\Gamma,$$

where $\Gamma$ is a collection of good orbits of $Y_-$. The boundary configuration of $\overline{M}_{X,A}(\{\gamma\}, \Gamma)$ with virtual dimension 1 gives the relation $\partial \circ \phi = \phi \circ \partial$. Then we have a functor from $\mathfrak{Con}$ to the category of supercommutative differential graded algebras.

**Theorem 3.8 ([66]).** The homology $H_*(CHA(Y))$ above realized in VFC gives a monoidal functor from $\mathfrak{Con}$ to the category of (super)commutative differential graded algebras.

### 3.4. Rational SFT as $BL_\infty$ algebras.

To assign strict contact manifolds with $BL_\infty$ algebras, we need to consider moduli spaces with multiple inputs and multiple outputs. We use $\overline{M}_{Y,A}(\Gamma^+, \Gamma^-)$ to denote the compactified moduli space of rational curves in class $A$ with positive asymptotics $\Gamma^+$ and negative asymptotics $\Gamma^-$ in the symplectization $\mathbb{R} \times Y$. Then we can define $p^{kl}$ by

$$p^{kl}(q^{\Gamma^+}) = \sum_{|\Gamma^-| = l} \#\overline{M}_{Y,A}(\Gamma^+, \Gamma^-) \frac{Tf_A^\omega}{\mu_{\Gamma^+} \mu_{\Gamma^-} \kappa_{\Gamma^-}} q^{\Gamma^-}. \quad (3.5)$$

Here $|\Gamma^-|$ is a multiset with $\Gamma^-$ an ordered set representative and $|\Gamma^+| = k$. In particular, the orientation property of $\overline{M}_{Y,A}(\Gamma^+, \Gamma^-)$ implies that $p^{kl}$ is map from $S^k V_\alpha$ to $S^l V_\alpha$. Then the boundary of the 1-dimensional moduli spaces $\overline{M}_{Y,A}(\Gamma^+, \Gamma^-)$ would yield that $\{p^{kl}\}$ is a $BL_\infty$ algebra $RSFT(Y)$, see Theorem 3.9 for details.
A LANDSCAPE OF CONTACT MANIFOLDS VIA RATIONAL SFT

Figure 6. $\hat{p} \circ \hat{p} = 0$, where $T$ stands for a trivial cylinder.

Similarly for a strict exact cobordism $X$ from $Y_-$ to $Y_+$, by considering the moduli spaces $\overline{\mathcal{M}}_{X,A}(\Gamma^+,\Gamma^-)$ of rational curves in $X$, we can define a $BL_\infty$ morphism from $RSFT(Y_+)$ to $RSFT(Y_-)$ by following,

$$\phi^{k,l}(q^{\Gamma^+}) = \sum_{||\Gamma^-||=l} \#\overline{\mathcal{M}}_{X,A}(\Gamma^+,\Gamma^-) \frac{T_{\alpha_{\Gamma^+}}^{\alpha_{\Gamma^-}}}{\mu_{\Gamma^+} \mu_{\Gamma^-} \kappa_{\Gamma^-}} q^{\Gamma^-},$$

where $|\Gamma^+| = k$. Then the boundary of the 1-dimensional moduli spaces $\overline{\mathcal{M}}_{X,A}(\Gamma^+,\Gamma^-)$ would yield that $\{\phi^{k,l}\}$ is a $BL_\infty$ morphism $RSFT(Y_+) \to RSFT(Y_-)$. In the following figure we indicate the cobordism level with 'C'.

Figure 7. $\hat{\phi} \circ \hat{p} = \hat{p} \circ \hat{\phi}$

If we fix a point $o$ in $Y$, by consider moduli spaces $\overline{\mathcal{M}}_{Y,A,o}(\Gamma^+,\Gamma^-)$, we can define a pointed morphism $p_\bullet$ by

$$p^{k,l}_\bullet(q^{\Gamma^+}) = \sum_{||\Gamma^-||=l} \#\overline{\mathcal{M}}_{Y,A,o}(\Gamma^+,\Gamma^-) \frac{T_{\alpha_{\Gamma^+}}^{\alpha_{\Gamma^-}}}{\mu_{\Gamma^+} \mu_{\Gamma^-} \kappa_{\Gamma^-}} q^{\Gamma^-}.$$
Then the boundary of the 1-dimensional moduli spaces $\mathcal{M}_{Y,A,o}(\Gamma^+, \Gamma^-)$ would yield that $\{p_{k,l}^*\}$ is a pointed morphism of degree 0. Note that $\mathcal{M}_{Y,A,o}(\Gamma^+, \Gamma^-)$ counts holomorphic curves with a point constraint in the symplectization with a $s$-independent $J$, therefore in the level containing $\mathcal{M}_{Y,A,o}(\Gamma^+, \Gamma^-)$ in a rigid breaking, there is only one nontrivial component.

For a strict exact cobordism $X$ from $Y_-$ to $Y_+$, if we choose a path $\gamma$ from $o_- \in Y_-$ to $o_+ \in Y_+$, then we can complete the path $\gamma$ to a proper $\tilde{\gamma}$ path in $\tilde{X}$ by constants in the cylindrical ends. Then we claim that the pointed morphisms $p_{\gamma+,\gamma-}$ determined by $o_-, o_+$ and $B\mathcal{L}_\infty$ morphism $\phi$ from $X$ are compatible, with $\phi_\bullet$ given by

$$\phi^{k,l}_\bullet(q^{\Gamma^+}) = \sum_{|\Gamma^-| = l} #\mathcal{M}_{X,A,\gamma}(\Gamma^+, \Gamma^-) \frac{T_{\Lambda} \omega}{\mu_\Gamma + \mu_{\Gamma^-} - \kappa_{\Gamma^-}} q^{\Gamma^-}.$$
The main theorem of this section is that after fixing auxiliary choices depending on the choice of virtual machinery, we almost have a functor from strict contact cobordism category (with auxiliary choice) to the category of $BL_\infty$ algebras$^8$.

**Theorem 3.9.** Let $(Y, \alpha)$ be a strict contact manifold with a non-degenerate contact form, then we have the following.

1. There exists a non-empty set of auxiliary data $\Theta$, such that for each $\theta \in \Theta$ we have a $BL_\infty$ algebra $p_\theta$ on $V_\alpha$.
2. For any point $o \in Y$, there exists a set of auxiliary data $\Theta_o$ with a surjective map $\Theta_o \to \Theta$, such that for any $\theta_o \in \Theta_o$, we have a pointed map $p_{\bullet, \theta_o}$ for $p_\theta$, where $\theta$ is the image of $\theta_o$ in $\Theta_o \to \Theta$.
3. When there is a strict exact cobordism $X$ from $(Y', \alpha')$ to $(Y, \alpha)$. Let $\Theta, \Theta'$ be the sets of auxiliary data for $\alpha, \alpha'$, there exist a set of auxiliary data $\Xi$ with a surjective map $\Xi \to \Theta \times \Theta'$, such that for $\xi \in \Xi$, there is a $BL_\infty$ morphism $\phi_\xi$ from $(V_\alpha, p_\theta)$ to $(V_{\alpha'}, p_{\theta'})$, where $(\theta, \theta')$ is the image of $\xi$ under $\Xi \to \Theta \times \Theta'$.
4. Assume in addition, we fix a point $o' \in Y'$ that is in the same component of $o$ in $X$, then for any compatible auxiliary data $\theta, \theta', \theta_o, \theta_{o'}, \xi$, we have $p_{\bullet, \theta_o, \theta_{o'}, \xi}$ are compatible.

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*The composition is not discussed, nor is needed for our application.
(5) For compatible auxiliary data $\theta, \theta_0$, there exists compatible auxiliary data $k\theta, k\theta_0$ for $(Y, k\alpha)$ for $k \in \mathbb{R}_+$, such that $p_{k\theta}, p_{k\theta_0}$ are identified with $p_\theta, p_{\theta_0}$ by the canonical identification between $V_\alpha$ and $V_{k\alpha}$.

To make sense of $\overline{\mathcal{M}}$, we need to fix a choice of virtual machinery, the meaning of auxiliary data also depends on the choice. If one adopts the perturbative scheme in [33, 47], Theorem 3.9 is a special case of their main constructions. On the other hand, since we only consider rational curves, the combinatorics is not essentially different from the construction of differentials and morphisms in [66]. In particular, Pardon’s VFC works in a verbatim account. We will explain the VFC construction to prove Theorem 3.9 and discuss other virtual techniques in §3.6.

3.5. Augmentations and linearized theories. We start this section with following definition.

Definition 3.10. For a strict contact manifold $(Y, \alpha)$, we fix an auxiliary choice $\theta \in \Theta$, then we define algebraic planar torsion $\text{APT}(Y, \alpha, \theta)$ to be the torsion of the $BL_\infty$ algebra $(V_\alpha, p_\theta)$ over $\mathbb{Q}$.

As a consequence of Proposition 2.13 and Theorem 3.9, we have $\text{APT}(Y, \alpha, \theta)$ is an invariant for $Y$ in the following sense.

Proposition 3.11. $\text{APT}(Y, \alpha, \theta)$ is independent of $\alpha, \theta$, hence can be abbreviated as $\text{APT}(Y)$. Moreover, $\text{APT} : \text{Cons} \to \mathbb{N} \cup \{\infty\}$ is monoidal functor, where the monoidal structure on $\mathbb{N} \cup \{\infty\}$ is given by $a \otimes b = \min\{a, b\}$.

Proof. By (5) of Theorem 3.9, we have $(V_\alpha, p_\theta) = (V_{k\alpha}, p_{k\theta})$ for any $k \in \mathbb{R}_+$. Let $\alpha'$ be another contact form, $\theta'$ is an auxiliary data. Then there exists $k_1, k_2$, such that there are strict cobordisms from $(Y, k_1\alpha), (Y, \alpha')$ to $(Y, \alpha'), (Y, k_2\alpha)$ respectively. Then by (2) of Theorem 3.9 and Proposition 2.13, we have $\text{APT}(Y, \alpha, \theta) = \text{APT}(Y, \alpha', \theta')$. For $(Y_1, \alpha_1, \theta_1), (Y_2, \alpha_2, \theta_2)$, the $BL_\infty$ algebra for the disjoint union $(Y_1 \sqcup Y_2, \alpha_1 \sqcup \alpha_2, \theta_1 \times \theta_2)$ is given by $(V_{\alpha_1} \oplus V_{\alpha_2}, \{p_{\theta_1} \oplus p_{\theta_2}\})$, i.e. there are no mixed structure maps. Then it is clear that $\text{APT}(V_{\alpha_1} \oplus V_{\alpha_2}, \{p_{\theta_1} \oplus p_{\theta_2}\}) = \text{APT}(V_{\alpha_1}, p_{\theta_1}), \text{APT}(V_{\alpha_2}, p_{\theta_2})$. That $\text{APT}$ is a functor follows from (2) of Theorem 3.9.

When $\text{APT}(Y) = 0$, it is equivalent to that $H_4 \text{CHA}(Y) = 0$, which is also known as algebraically overtwisted [11] or 0-algebraic torsion [48], and is implied by overtwistedness [15, 77]. Similarly, we can define $\text{APT}_A(Y)$ to be the order of torsion for RSFT$(Y)$ over the Novikov field $\Lambda$.

Proposition 3.12. $\text{APT}_A(Y) = \text{APT}(Y)$.

Proof. To differentiate the two vector spaces, we use $V_\alpha$ to denote the $\mathbb{Q}$-space and $V_\alpha^A$ to denote the $\Lambda$ space. For an element $q \in V_\alpha^A$, we define the weight $w(q) := \int \gamma^* \alpha$ and $w(T^A) := A$. Then we define the weight (uniquely) on $SV_\alpha^A$ and $EV_\alpha^A$ by the following properties $w(xy) = w(x) + w(y), w(x \circ y) = w(x) + w(y)$ and $w(x + y) \leq \max\{w(x), w(y)\}$. Then by (3.5), $\tilde{p}$ preserves the weight on $EV_\alpha^A$. Now assume $\text{APT}(Y) = k < \infty$, we have $1 = \tilde{p}(x)$ for $x \in EV_\alpha$. Then $x$ induces an element $\tilde{x}$ in $EV_\alpha^A$ by sending each $q \gamma$ to $T^{-f} \gamma^* \alpha q \gamma$. Therefore $\tilde{x}$ has pure weight 0, i.e. $\tilde{x}$ is a sum of monomials with weight 0. We know that $\tilde{p}(\tilde{x})$ also has pure weight 0. Since $\tilde{p}(\tilde{x})|_{T=1} = \tilde{p}(x) = 1$, we must have $\tilde{p}(\tilde{x}) = 1$. This proves that $\text{APT}_A(Y) \leq k$, in particular, $\text{APT}_A(Y) \leq \text{APT}(Y)$. On the other hand, assume $\text{APT}_A(Y) = k < \infty$, i.e. $1 = \tilde{p}(\tilde{x})$ for $\tilde{x} \in EV_\alpha^A$. Then we can assume $\tilde{x}$ has pure weight 0. Since $\tilde{x}$ is in $\overline{\mathcal{S}}SV_\alpha^A$, $\tilde{x}$ is written as finite linear combination of terms in the form of $w_1 \circ \ldots \circ w_k$ with $w_i = q_{\gamma_1} \ldots q_{\gamma_j}$. Because $\tilde{x}$ has pure weight, we must have the coefficient of each those terms is a monomial $T^A$. In particular, it makes sense to define $x \in EV_\alpha$ by $\tilde{x}|_{T=1}$ and $\tilde{p}(x) = 1$. Therefore $\text{APT}(Y) \leq \text{APT}_A(Y)$, which finishes the proof.
Since finite order of torsion is an obstruction to augmentations, finite algebraic planar torsion is an obstruction to symplectic fillings in view of the following.

**Proposition 3.13.** Let \((Y, \alpha)\) be a strict contact manifold with an auxiliary data \(\theta\).

1. If \((W, d\lambda)\) is a strict exact filling, then there is a \(BL_\infty\) augmentation to \((V_\alpha, p_\theta)\) over \(\mathbb{Q}\).
2. If \((W, \omega)\) is a strict strong filling, then there is a \(BL_\infty\) augmentation to \((V_\alpha, p_\theta)\) over \(\Lambda\).

**Proof.** The augmentation is defined by

\[
e^k(q^{\Gamma^+}) = \sum_A \# \mathcal{M}_{W,A}(\Gamma^+, \emptyset) \frac{\mathcal{T}_{f_\lambda} \omega}{\mu_{\Gamma^+}},
\]

where \(|\Gamma^+| = k\). The remaining follows from the same argument in Theorem 3.9. In the exact case, it is a special case of the cobordism case of Theorem 3.9.

**Remark 3.14.** The theory with \(\Lambda\) coefficient considered in this paper is a naive version, and can be transferred to and recovered from the \(\mathbb{Q}\) coefficient version. The more correct version for \(\Lambda\) coefficient theory should be completions with respect to the weight considered in the proof of Proposition 3.12. For strong symplectic cobordisms with non-empty negative boundary, it is necessary to use the completion to describe Maurer-Cartan elements [22]. If we use \(\bar{E}V^A_\alpha\) to denote the completion of \(E^A_\alpha\) and \(\bar{E}kV^A_\alpha\) to denote the completion of \(E^kV^A_\alpha\). Then it is no longer true that \(1 = 0 \in H_*(\bar{E}kV^A_\alpha)\) implies that \(1 = 0 \in H_*(E^kV_\alpha)\). Moreover, \(1 = 0 \in H_*(\bar{E}V^A_\alpha)\) does not imply that there is a \(k > 0\) such that \(1 = 0 \in H_*(\bar{E}kV^A_\alpha)\).

**Corollary 3.15.** If \(APT(Y) < \infty\), then \(Y\) has no strong filling.

**Proof.** If \(APT(Y) = APT_\Lambda(Y) < \infty\), then there is no \(BL_\infty\) augmentation over \(\Lambda\) by Proposition 2.11. Then proposition 3.13 implies the claim.

Roughly speaking the algebraic planar torsion looks at rigid curves with multiple positive punctures and no negative puncture. One particular situation, where we can infer information of planar algebraic torsion, is the planar torsion introduced by Wendl [73], which generalizes overtwisted contact structures and the Giroux torsion in dimension 3. The following two results were essentially proven in [48].

**Theorem 3.16.** If \(Y\) is a 3-dimensional contact manifold with planar torsion of order \(k\), then \(APT(Y) \leq k\).

**Proof.** This follows from the same argument of [48, Theorem 6] based on a precise description of low energy curves in [48, Proposition 3.6], see also [73]. In fact, we do not need the genus > 0 assertion from the fifth property of [48, Proposition 3.6], as we do not consider higher genus curves.

**Theorem 3.17.** For any \(k \in \mathbb{N}\), there exists a 3-dimensional contact manifold \(Y\) with \(APT(Y) = k\).

**Proof.** This follows from the same argument of [48, Theorem 4]. In fact, we only need the genus 0 part of [48, Lemma 4.15] to get a lower bound.

**Remark 3.18.** Following from [48, Corollary 1], there are examples \(Y_i\) with planar torsion of order \(k\), such that there is exact cobordism from \(Y_i\) to \(Y_{i+1}\) but no exact cobordism from \(Y_{i+1} \rightarrow Y_i\). On the other hand, there is always a strong cobordism from \(Y_{i+1}\) to \(Y_i\) by [74, Theorem 1]. We will see similar phenomena in higher dimensions in §6 and §7.

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9Note that \(p_\theta\) in \(\mathbb{Q}\) coefficient and \(\Lambda\) coefficient are different. But we can recover one from the other as the weight is determined by contact action.
Remark 3.19. It is an interesting question to understand the relations between the algebraic planar torsion and algebraic torsion. First we recall the BV\(_\infty\) description of SFT and the definition of algebraic torsion from [48]. Let \(SV_\alpha[[h]]\) be the algebra of power series in \(h\) with coefficients in \(SV_\alpha\). Then we have a full SFT differential defined as follows

\[
D_{SFT}q^\Gamma = \sum_{g=0}^{\infty} \sum_{A} \sum_{\Gamma'} \sum_{k=1}^{\lvert \Gamma \rvert} h^{g+k-1} n_{A,g}(\Gamma, \Gamma'; k) q^{\Gamma'}
\]

where \(n_{A,g}(\Gamma, \Gamma'; k)\) is the count of holomorphic curves, which possibly have disconnected components but only one nontrivial component of \(k\) positive punctures, genus \(g\), homology class \(A\), and positive/negative asymptotics \(\Gamma\) and \(\Gamma'\). Then we say \(Y\) admits a \(k\)-algebraic torsion iff \(h^k\) is 0 in the homology of \((SV_\alpha[[h]], D_{SFT})\).

Let’s consider the simplest case with an algebraic planar \(1\)-torsion, i.e., there are two generators \(q_1, q_2\) such that \(p^{2l}(q_1, q_2) = 1, p^{2l}(q_1, q_2) = 0\) for all \(l > 0\), and \(p^{1l}(p_1) = 0\) for all \(l \geq 0\) and \(i = 1, 2\). That is we know \(\sum_A n_{A,g}(\Gamma, \Gamma'; k)\) for \(g = 0\) and \(\Gamma = \{\gamma_1, \gamma_2\}\). The natural candidate for algebraic torsion is \(q_1 q_2\), and we compute

\[
D_{SFT}q_1 q_2 = \sum_{g=0}^{\infty} \sum_{A} \sum_{\Gamma'} \sum_{k=1}^{2} h^{g+k-1} n_{A,g}(\{\gamma_1, \gamma_2\}, \Gamma'; k) q^{\Gamma'}
\]

\[
= h^{+} + \sum_{g=1}^{\infty} \sum_{A} \sum_{\Gamma'} \sum_{k=1}^{2} h^{g+k-1} n_{A,g}(\{\gamma_1, \gamma_2\}, \Gamma'; k) q^{\Gamma'}
\]

Since we have no knowledge of \(n_{A,g}\) for \(g > 0\) in RSFT, one should not expect that \(q_1 q_2\) is a primitive of \(h\) in \(D_{SFT}\). We note here the above consideration is a very special case and in general an algebraic planar \(k\)-torsion is not equivalent to that \(h^k\) is the image of the genus 0 term of \(D_{SFT}\). In fact, the algebraic torsion and algebraic planar torsion can be viewed as two “independent” axes in a grid of torsions, see §3.8 for detail.

Remark 3.20. One can define \(BL_\infty\) algebras over group rings as in [48] for stable Hamiltonian fillings and weak symplectic fillings. Then the finiteness of algebraic planar torsion in this setup is an obstruction to stable/weak fillings. One example with finite algebraic planar torsion in the group ring setup is those with the fully separating planar \(k\)-torsions [73, Definition 1.3], where the finiteness follows from the same proof of [48, Theorem 6 (2)].

Giroux was generalized to higher dimensions in [53], the following theorem is a reformulation of [58, Theorem 1.7].

Theorem 3.21. If \(Y\) has Giroux torsion, then \(\text{APT}(Y) \leq 1\).

Now we assume \((V_\alpha, p_\alpha)\) does have a \(BL_\infty\) augmentation \(\epsilon\) over \(\mathbb{Q}\), then \(\text{APT}(Y)\) is \(\infty\). In view of §2 and Theorem 3.9, for a point \(o \in Y\) and an auxiliary data \(\theta_o\), which give rise to a pointed morphism \(p_{\bullet, \theta_o}\). Hence we can define the order \(O(V_\alpha, \epsilon, p_{\bullet, \theta_o})\). In the following, we use \(\text{Aug}_\mathbb{Q}(V_\alpha)\) to denote the set of \(BL_\infty\) augmentations over \(\mathbb{Q}\).

Definition 3.22. For a strict contact manifold \((Y, \alpha)\) with auxiliary data \(\theta\), we define

\[
O(Y, \alpha, \theta) := \max \{ O(V_\alpha, \epsilon, p_{\bullet, \theta_o}) \mid \forall \epsilon \in \text{Aug}_\mathbb{Q}(V_\alpha), o \in Y, \theta_o \in \Theta_o \}
\]

where the maximum of an empty set is defined to be zero.
Proposition 3.23. \( O(Y, \alpha, \theta) \) is independent of \( \alpha \) and \( \theta \), hence will be abbreviated as \( P(Y) \) and is called the planarity of \( Y \). Moreover, \( P : \mathcal{C} \to \mathbb{N} \cup \{ \infty \} \) is a monoidal functor, where the monoidal structure on \( \mathbb{N} \cup \{ \infty \} \) is given by \( 0 \otimes a = 0, \forall a \) and \( a \otimes b = \max\{a, b\}, \forall a, b \geq 1 \).

Proof. We first show that if there is a strict exact cobordism \( X \) from \( (Y_-, \alpha_-) \) to \( (Y_+, \alpha_+) \), then \( O(Y_+, \alpha_+, \theta_+) \geq O(Y_-, \alpha_-, \theta_-) \) for any \( \theta_+, \theta_- \). For any \( \alpha_+ \in Y_+ \), there exists a path in \( X \) connecting \( \alpha_+ \) and \( \alpha_- \). Then by (4) of Theorem 3.9 and Proposition 2.22, for any augmentation \( \epsilon \) to \( V_{\alpha_-} \) and auxiliary data \( \theta_{\alpha_-} \), there exists an auxiliary data \( \xi \in \Xi \) and \( \theta_{\alpha_+} \), such that \( O(V_{\alpha_+}, \epsilon \circ \phi_{\xi, p, \theta_{\alpha_+}}) \geq O(V_{\alpha_-}, \epsilon, \rho_{\theta_{\alpha_-}}) \). Hence \( O(Y, \alpha_+, \theta_+) \geq O(Y, \alpha_-, \theta_-) \). Then by the same argument in Proposition 3.11, \( O(Y, \alpha, \theta) \) is independent of \( \alpha \) and \( \theta \). Then we have \( P(Y) \leq O(Y) \) by Proposition 2.19.

Remark 3.24. Similarly, we can define another invariant \( \overline{P}(Y) \) using the maximum of \( O(V, \epsilon, p\ast) \). Then we have \( P(Y) \leq \overline{P}(Y) \) by Proposition 2.19.

Similarly, we define \( P_\Lambda(Y) \) using augmentations over \( \Lambda \), it is not clear to us whether \( P(Y) = P_\Lambda(Y) \) except the obvious relation \( P(Y) \leq P_\Lambda(Y) \). Since any augmentation over \( \mathbb{Q} \) will induce an augmentation over \( \Lambda \), where the degree in \( T \) is declared to the contact action of the positive asymptotics. However, a \( \Lambda \)-augmentation may not induce a \( \mathbb{Q} \)-augmentation. Since finite algebraic planar torsion is an obstruction to \( BL_{\infty} \) augmentation over both \( \mathbb{Q} \) and \( \Lambda \), we have that \( A\text{PT}(Y) \leq \infty \) implies that \( P(Y)/P_\Lambda(Y) = 0 \). Since \( P(Y)/P_\Lambda(Y) = 0 \) is precisely those without augmentations, the algebraic planar torsion is the inner hierarchy inside \( P(Y)/P_\Lambda(Y) = 0 \). However it is still possible (at least on the algebraic level) that \( P(Y)/P_\Lambda(Y) = 0 \) but \( A\text{PT}(Y) = \infty \), i.e. there is no augmentation nor finite torsion.

3.6. Implementation of virtual techniques. In the following, we will explain how to get the algebraic count of moduli spaces in Theorem 3.9 using virtual techniques. Any choice of virtual machinery should give a construction of \( P \) and \( A\text{PT} \) with the claimed properties, although it is not clear whether different virtual techniques give rise to the same \( P \) and \( A\text{PT} \). However, the geometric results, examples and applications in this paper, do not depend on the choice, as we have the following axiom for virtual machinery, which holds for any one of the virtual techniques mentioned in this paper.
Axiom 3.25. A virtual implementation of a holomorphic curve theory has the property that the virtual count of a compactified moduli space equals to the geometric count, when transversality holds for that moduli space.

In the following, we will finish the proof of Theorem 3.9 by implementing Pardon’s implicit atlas and virtual fundamental cycles [66]. The construction is essentially the constructions of contact homology algebra and morphisms in [66]. As explained in [65, §1.8], the only difference that one needs to pay attention to is the underlying combinatorics for holomorphic curves. One needs to show that implicit atlas with cell-like stratification still exists for RSFT, in particular the space of gluing parameters has a cell-like stratification. However, the combinatorics for RSFT is also “tree-like” like contact homology, hence the construction is a verbatim account of [66]. In the following, we give a brief description of the construction.

3.6.1. $\mathcal{R}$ modules. We first introduce a category $\mathcal{R}$ which will play the same role of $\mathcal{S}_I$ in [66] to govern the combinatorics of rational holomorphic curves in the symplectization. The objects of $\mathcal{R}$ are connected non-empty directed graphs without cycles, such that each vertex has at least one incoming edge. Edges with missing source, i.e. input edges, and edges with missing sink, i.e. output edges are allowed. Those edges are called external edges and all other edges are called interior edges. The graph $T$ is equipped with decorations as follows.

1. For each edge $e \in E(T)$, a Reeb orbit $\gamma_e$.
2. For each vertices $v \in V(T)$, a relative homology class $\beta_v \in H_2(Y, \{\gamma_{e^+}\}_{e^+ \in E^+(v)} \sqcup \{\gamma_{e^-}\}_{e^- \in E^-(v)})$, where we denote by $E^+(v)$ the set of incoming edges at $v$ and $E^-(v)$ the set of outgoing edges at $v$, which can be empty.
3. For each external edge $e \in E^{ext}(T)$, a basepoint $b_e \in \text{im} \gamma_e$.

A morphism $\pi : T \to T'$ in $\mathcal{R}$ consists first of a contraction of the underlying graph of $T$ to $T'$ by collapsing some of the interior edges of $T$. The decorations have the following property.

1. For each non-contracted edge $e \in E(T)$, we have $\gamma_{\pi(e)} = e$.
2. For each vertex $v' \in V(T')$, we have $\beta'_{v'} = \#_{v' \to v} \beta_v$.

Finally, we specify for each external edge $e \in E^{ext}(T) = E^{ext}(T')$ a path along $\text{im} \gamma_e$ between the basepoints $b_e$ and $b'_e$ modulo the relation that identifies such two paths iff their difference lift to $\gamma_e$. In particular, there are exactly $\kappa_{\gamma_e}$ different equivalences of paths. Then automorphism group of $T$ with a single vertex is a product of cyclic groups and symmetric groups with cardinality $\mu_I + \mu_{I^-} - \kappa_T + \kappa_{I^-}$. For $T' \to T$, we use $\text{Aut}(T'/T)$ denote the subgroup of $\text{Aut}(T')$ compatible with $T' \to T$.

A concatenation in $\mathcal{R}$ consists of a finite non-empty collection of objects $T_i \in \mathcal{R}$ along a matching between some pairs of output edges and input edges with matching orbit label, such that the resulting gluing is a directed graph without cycles, along with a choice of paths between the basepoints for each pair of matching edges. Given a concatenation $\{T_i\}_i$ in $\mathcal{R}$, there is a resulting object $\#_i T_i \in \mathcal{R}$. A morphism of concatenations $\{T_i\}_i \to \{T'_i\}_i$ means a collection of morphisms $T_i \to T'_i$ covering a bijection of index sets. Then a morphism $\{T_i\}_i \to \{T'_i\}_i$ induces a morphism $\#_i T_i \to \#_i T'_i$. If $\{T_i\}_i$ is a concatenation and $T_i = \#_j T_{ij}$ for some concatenation $\{T_{ij}\}_{ij}$, then there is a resulting composite concatenation $\{T_{ij}\}_{ij}$ with natural isomorphisms $\#_j T_{ij} = \#_i \#_j T_{ij} = \#_i T_i$. We use $\text{Aut}(\{T_i\}_i/\#_i T_i)$ to represent the group of automorphism of $\{T_i\}_i$ acting trivially on $\#_i T_i$, i.e. the product $\prod_e \mathbb{Z}_{\kappa_{\gamma_e}}$ over junction edges.

The key concept to organize the moduli spaces, implicit atlases, and virtual fundamental cycles is the following $\mathcal{R}$-module.

Definition 3.26 ([66, Definition 4.5]). A $\mathcal{R}$-module $X$ valued in a symmetric monoidal category $\mathcal{C}^\otimes$ consists of the following data.
(1) A functor $X : \mathcal{R} \to \mathcal{C}$.
(2) For every concatenation $\{T_i\}_i$ in $\mathcal{R}$, a morphism
$$\otimes_i X(T_i) \to X(\#_i T_i),$$
such that the following diagrams commute:
$$\begin{align*}
\otimes_i X(T_i) & \longrightarrow X(\#_i T_i) \\
\otimes_{i,j} X(T_{ij}) & \longrightarrow X(\#_{ij} T_{ij})
\end{align*}$$
for any morphism of concatenations and composition of concatenations.

**Example 3.27.** A holomorphic building of type $T \in \mathcal{R}$ consists of the following data.

(1) For every vertex $v$, a closed, connected nodal Riemann surface of genus zero $C_v$, along with distinct points $\{p_{v,e} \in C_v\}_e$ indexed by the edges incident at $v$ and a $J$ holomorphic map $u_v : C_v \setminus \{p_{v,e}\}_e \to \mathbb{R} \times Y$ up to the $\mathbb{R}$ translation.
(2) $u_v$ converges to $\gamma_e^+$ near $p_{v,e}^+$ in the sense of (3.1) for $e^+ \in E^+(v)$ and converges to $\gamma_e^-$ near $p_{v,e}^-$ in the sense of (3.2). We use $(u_v)_{p_{v,e}} : S^1 \to Y$ to denote the $Y$-component of the limit map near punctures.
(3) For every input/output edge $e$, an asymptotic marker $L_e \in S_{p_{v,e}} C_v$ which is mapped to the basepoint $b_e$ by $(u_v)_{p_{v,e}}$.
(4) For every interior edge $v \to v'$, a matching isomorphism $m_e : S_{p_{v,e}} C_v \to S_{p_{v',e}} C_{v'}$ intertwining $(u_v)_{p_{v,e}}$ and $(u_{v'})_{p_{v',e}}$.

An isomorphism between two buildings is a collections of isomorphisms between $C_v$ commuting with all the data. Then we define $\mathcal{M}(T)$ to be the set of isomorphism classes of holomorphic buildings of type $T$. Note that $\text{Aut}(T)$ acts on $\mathcal{M}(T)$ by changing markings. Then we define
$$\overline{\mathcal{M}}(T) := \bigcup_{T' \sim T} \mathcal{M}(T') / \text{Aut}(T'/T).$$

The union is over the set of isomorphism classes in the over category $\mathcal{R}/T$. Moreover, $\overline{\mathcal{M}}(T)$ is endowed with the Gromov topology and is a compact Hausdorff space [66, §2.9, 2.10]. Note that here for each $v \in V(T)$, we view $u_v$ as a curve in its own copy of symplectization. In particular, we have no level structure and the topology is slightly different from the buildings in [9] by forgetting all trivial cylinders. However this poses no difference for the compactness. In particular, there is a surjective map from the compactification in [9] to $\overline{\mathcal{M}}(T)$ by collapsing the boundary containing levels with multiple disconnected nontrivial curves into corners. The functor $\overline{\mathcal{M}}$ is a $\mathcal{R}$ module in the category of compact Hausdorff spaces with disjoint union as the monoidal structure. The natural map $\overline{\mathcal{M}}(T) \to \overline{\mathcal{R}}/T$ is a stratification in the sense of [66, Definition 2.14]. We define $\text{vdim}(T)$ as $\sum_{v \in V(T)} (\text{ind}(u_v) - 1)$ and $\text{codim}(T'/T)$ is the number of interior edges collapsed in $T' \to T$. Then we have $\text{codim}(T'/T) + \text{vdim}(T') = \text{vdim}(T)$.

**Example 3.28.** For each non-degenerate Reeb orbit $\gamma$ (good or bad) and a basepoint $b \in \text{im} \, \gamma$, [66, Definition 2.46] constructs a canonical $\mathbb{Z}_2$ graded line $\mathfrak{o}_{\gamma,b}$ with grading $\mu_{CZ}(\gamma) + n - 3 \mod 2$. Any path $b \to b'$ gives rise to a functorial isomorphism $\mathfrak{o}_{\gamma,b} \to \mathfrak{o}_{\gamma,b'}$, two paths induces the same isomorphism if the difference can be lift to $\gamma$. As a consequence, $\mathbb{Z}_{\kappa_{\gamma}}$ acts on $\mathfrak{o}_{\gamma,b}$. Then $\gamma$ is good iff the action is trivial. Let $T$ be a tree, then we have the determinant line $\mathfrak{o}_{T}$ of the linearized Cauchy-Riemann operator at the vertices and conical
isomorphism from $\mathcal{O}_T^\circ$ to $\otimes_{e^+ \in \mathcal{E}^+(v)} \mathcal{O}_{e^+} \otimes_{e^- \in \mathcal{E}^-(v)} \mathcal{O}_{e^-}^\vee$. Moreover, $\mathcal{O}_T^\circ$ is a $\mathcal{R}$-module [65, Example 4.7]. We define $\mathcal{O}_T$ by $\mathcal{O}_T^\circ \otimes (\mathcal{O}_T^\circ)^{\vee}(T)$. Moreover, for $T' \to T$, there is an induced isomorphism $\mathcal{O}_{T'} \to \mathcal{O}_T$ by [66, (2.61)].

An object $T \in \mathcal{R}$ is called effective iff $\mathcal{M}(T) \neq \emptyset$. Then for any morphism $T \to T'$, if $T$ is effective, so is $T'$. For any concatenation $\{T_i\}_i$, every $T_i$ is effective iff $\#_i T_i$ is effective. In the following, $\mathcal{R}$ will mean the full subcategory spanned by effective objects, which depends on $J$. Then $\mathcal{R}$ has the following properties, which allows one to apply inductive constructions.

1. Every $T$ can be written as a concatenation of maximal elements $\#_i T_i$, an element $T_i$ is maximal iff there is only isomorphism mapping out of $T_i$. That is $T_i$ has only one interior vertex.
2. Let $T, T' \in \mathcal{R}$, we say $T' \preceq T$ iff there is a morphism $\#_i T \to T$ with some $T_i$ isomorphic to $T'$. Then there is no infinite strictly decreasing sequences. This is a consequence of compactness or positivity of contact energy (3.3) in the exact cobordism setting of this paper.

As a consequence, a lot of the constructions can be built inductively from the minimal elements in $(\mathcal{R}, \preceq)$. Note that maximal $T$ is not necessarily maximal in $\preceq$, but minimal elements of $(\mathcal{R}, \preceq)$ are necessarily maximal.

**Example 3.29.** Thickening datum defined in [66, Definition 3.9] works verbatim for our purpose. Then we have the set of thickening datums $A(T)$. Then

$$\mathbb{A}(T) := \bigsqcup_{T' \preceq T} A(T'),$$

where the disjoint union is over all connected subgraphs that are in $\mathcal{R}$. Then clearly $\mathbb{A}$ is a $\mathcal{R}^{op}$-module to the category of sets.

**Proposition 3.30.** $\mathcal{M}(T)$ is equipped with an implicit atlas $\mathbb{A}(T)$ with oriented cell-like stratification.

**Proof.** First of all, we have the space of gluing parameters $G_{T/I}$ that associates to each interior edge a number in $(0, \infty]$. Since there is no cycles in $T$, there is no relations among those gluing parameters. In particular $G_{T/I}$ has a cell-like stratification like $(G_I)_{T/I}$ in [66, Lemma 3.5]. Then the claim follows from the same proof of [66, Theorem 3.23]. The analogues of [66, Theorem 3.31, 3.32] hold for our setup since we only glue one puncture at a time, hence the gluing analysis in [66, §5] applies in a verbatim way. \hfill $\Box$

With the existence of implicit atlas with cell-like stratification, the machinery of virtual fundamental cycles induces a pushforward map

$$C_{\ast + \text{vdim} (T)}^\ast (\mathcal{M}(T) \rel \partial; \mathbb{A}(T)) \to C_{\ast - \ast} (E; \mathbb{A}(T)), \tag{3.9}$$

where $E$ is part of the datum in $\mathbb{A}(T)$. The remaining of the construction is hinged on purely combinatorial properties.

Following the same procedure of [66, Definition 4.19], there is a canonical construction of $\mathcal{R}$ module $C_{\ast + \text{vdim} (T)}^\ast (\mathcal{M}(T) \rel \partial)$ by homotopy colimit in the category of cochain complexes such that $C_{\ast + \text{vdim} (T)}^\ast (\mathcal{M}(T) \rel \partial)$ is quasi-isomorphic to $C_{\ast + \text{vdim} (\mathcal{M}(T) \rel \partial; \mathbb{A}(T))}^\ast$. Similarly, by the homotopy colimit as in [66, Definition 4.20], there is a $\mathcal{R}$ module $C_{\ast} (E)$ and (3.9) leads to a canonical map of $\mathcal{R}$-modules $C_{\ast + \text{vdim} (\mathcal{M}(T) \rel \partial)}^\ast \to C_{\ast - \ast} (E)$. Similar to [66, Definition 4.14], there is $\mathcal{R}$ module $\mathbb{Q}[\mathcal{R}]$ governing the boundary information,

$$\mathbb{Q}[\mathcal{R}](T) := \mathbb{Q}[\mathcal{R}/T] = \bigoplus_{T' \to T} \mathcal{O}_{T'}^\circ \text{vdim}(T')$$
with the differential given by the sum of all codimension one maps \( T'' \to T' \) in \( R/T \) of boundary map \( \partial_T \to \partial_{T''} \) in Example 3.28. Then \( \mathbb{Q}[\mathcal{R}] \) is again cofibrant in the sense of [66, Definition 4.24] by the same argument of [66, Lemma 4.26]. There is a cofibrant replacement \( C^\text{cof}_s(E) \) with a quasi-isomorphism \( C^\text{cof}_s(E) \to C_s(E) \), which is surjective on maximal \( T \) by induction on the partial order \( \preceq \) by [66, Definition 4.28].

**Definition 3.31.** Given \( \alpha, J \), an element of \( \Theta(\alpha, J) \) consists a commuting diagram of \( R \)-modules

\[
\begin{array}{c}
\mathbb{Q}[\mathcal{R}] \\
\xrightarrow{w_*} C^\text{cof}_s(E) \\
\xrightarrow{\sim} C^+_{\text{vir}}(\mathcal{M}\text{ rel } \partial) \\
\xrightarrow{p_*} \mathbb{Q}
\end{array}
\]

satisfying the following properties.

1. \( p_* \) induces the canonical isomorphism \( H^*_s(E) = H_s(E) = \mathbb{Q} \).
2. \( w_* \) satisfies the property that for any \( T \in \mathcal{R}, w_* \in \text{Hom}_{\mathcal{R}/T}(\mathbb{Q}[\mathcal{R}], C^+_{\text{vir}}(\mathcal{M}\text{ rel } \partial)) \) on cohomology level represents the constant function \( 1 \in \check{H}^0(\mathcal{M}(T)) \) under the identification in [66, Lemma 4.23].

**Proof of (1) of Theorem 3.9.** In the context of VFC, \( \Theta(\alpha) = \sqcup \Theta(J, \Theta(\alpha, J)) \). Moreover \( \Theta(\alpha, J) \) is not empty. The existence of \( p_* \) follows from [66, Lemma 4.30], the existence of \( w_* \) follows from [66, Lemma 4.31], the existence of lifting \( w_* \) follows from [66, Proposition 4.34], as the induction on \( \preceq \) can be applied.

Given a diagram (3.10), we have a \( \mathcal{R} \) module map \( p_* \circ \tilde{w}_*: \mathbb{Q}[\mathcal{R}] \to \mathbb{Q} \), which is assigning each \( T \) with \( \text{vdim}(T) = 0 \) an element \( \#\mathcal{M}(T)_{\text{vir}} \in (\mathcal{D}_T)^{\text{Aut}(T)} \), which after fixing a trivialization of \( \partial \gamma_h \) for every Reeb orbits is a rational number. If an exterior edge of \( T \) is labeled by a bad orbit, then being \( \text{Aut}(T) \) invariant implies that \( \#\mathcal{M}(T)_{\text{vir}} = 0 \). Finally, being a \( \mathcal{R} \) module implies that

\[
\begin{align*}
0 &= \sum_{\text{codim}(T'/T) = 1} \frac{1}{|\text{Aut}(T'/T)|} \#\mathcal{M}(T')_{\text{vir}} \\
\#\mathcal{M}(_{\text{vir}}) &= \frac{1}{|\text{Aut}(\{T_i\}_i/\#T_i)|} \prod_i \#\mathcal{M}(T_i)_{\text{vir}}
\end{align*}
\]

(3.12)

Let \( T \) be a tree with one interior vertex, \( k \) input edges labeled by \( \Gamma^+ \), and \( l \) output edges labeled by \( \Gamma^- \). If \( \text{vdim}(T) = 0 \), then we define \( q^{\Gamma^-} \) coefficient of \( p^{k,l}(q^{\Gamma^+}) \), i.e. \( \langle p^{k,l}(q^{\Gamma^+}), q^{\Gamma^-} \rangle \), by \( \frac{1}{\text{Aut}(T) \#\mathcal{M}(T)} \). In view of Proposition 2.6, we need to prove that \( \langle p^{k,l}_2(q^{\Gamma^+}), q^{\Gamma^-} \rangle \) is zero for any multisets \( \Gamma^+,\Gamma^- \) with \( |\Gamma^+| = k, |\Gamma^-| = l \). We claim that

\[
\langle p^{k,l}_2(q^{\Gamma^+}), q^{\Gamma^-} \rangle = \frac{1}{\mu_{\Gamma^+} \mu_{\Gamma^-} \kappa_{\Gamma^-}} \sum_{\text{codim}(T'/T) = 1} \#\mathcal{M}(T')_{\text{vir}} = 0
\]

for maximal \( T \) with \( \text{vdim}(T) = 1 \). Assume

\[
\Gamma^+ := \{\gamma_1, \ldots, \gamma_1, \ldots, \gamma_m, \ldots, \gamma_m\}, \Gamma^- := \{\eta_1, \ldots, \eta_1, \ldots, \eta_s, \ldots, \eta_s\}
\]

for \( \sum_i k_i = k, \sum_i l_i = l \) and \( \gamma_i \neq \gamma_j, \eta_i \neq \eta_j \) for \( i \neq j \). For \( T' \) with \( \text{codim}(T'/T) = 1 \), then \( T' \) has two vertices with one connecting interior edge. We can cut out the interior edge to obtain \( T'_1, T'_2 \) with the interior edge turning into an output edge for \( T'_1 \), i.e. \( T' = T'_1 \# T'_2 \). Assume the input edges
of $T'_1$ is labeled by $\tilde{\Gamma}^+ := \{\gamma_1, \ldots, \gamma_{k'_1}, \ldots, \gamma_m, \ldots, \gamma_{k'_m}\}$ for $k'_i \geq 0$ and output edges of $T'_2$ is labeled by $\tilde{\Gamma}^- := \{\eta_1, \ldots, \eta_{l'_1}, \ldots, \eta_{s'_1}, \ldots, \eta_{l'_s}\}$ for $l'_i \geq 0$, Then there are $\prod_{i=1}^{m} \binom{k'_i}{k_i} \prod_{i=1}^{s} \binom{l'_i}{l_i}$ many codimension one $T'$ with such labels. Since $\text{Aut}(T'/T) = 1$, by (3.12), the contribution to (3.11) from $T'$ with such labels multiplying $\frac{1}{\mu_{T^+} \mu_{T^-} \mu_{\tilde{\Gamma}^-}}$ is

$$\sum_{\gamma} d_\gamma^- \cdot d_\gamma^+ \cdot \langle p^{\tilde{\Gamma}^+}_1 (|\tilde{\Gamma}^-| + 1) (q^{\tilde{\Gamma}^-}_0), q^{\tilde{\Gamma}^-} \cup \{\gamma\} \rangle \cdot \langle p^{\tilde{\Gamma}^+}_1 (|\tilde{\Gamma}^-| + 1) (q^{\tilde{\Gamma}^-}_0) \cup \{\gamma\}, q^{\tilde{\Gamma}^+} \rangle$$

where $\gamma$ is the label on the interior edge of $T'$ with $d_\gamma^-$ is the number of $\gamma$ in $(\tilde{\Gamma}^-)^c \cup \{\gamma\}$ and $d_\gamma^+$ is the number of $\gamma$ in $(\tilde{\Gamma}^+)^c \cup \{\gamma\}$. Then (3.13) is the part of $\langle p^{k_1}_2(q^{\tilde{\Gamma}^+}), q^{\tilde{\Gamma}^-} \rangle$, as we have $d^-_\gamma \cdot d^+_\gamma$ many ways of gluing that arise in $p^{k_1}_2$. Then the sum of all $T'$ yields that $\langle p^{k_1}_2(q^{\tilde{\Gamma}^+}), q^{\tilde{\Gamma}^-} \rangle = 0$. Hence $p^{k_1}$ gives a $BL_\infty$ structure.

The next proposition follows from [66, Proposition 4.33]. It is Axiom 3.25 in the context of VFC.

**Proposition 3.32.** If $\overline{\mathcal{M}}(T)$ is cut out transversely with $\text{vdim}(T) = 0$, then $\# \overline{\mathcal{M}}(T)^{\text{vir}} = \# \overline{\mathcal{M}}(T) = \# \mathcal{M}(T)$ for any $\theta \in \Theta(\alpha, J)$.

### 3.6.2. $\mathcal{R}_{II}, \mathcal{R}^*,$ and $\mathcal{R}^*_{II}$ modules.

In the following, we introduce $\mathcal{R}_{II}, \mathcal{R}^*$ and $\mathcal{R}^*_{II}$ to govern moduli spaces as well virtual fundamental cycles for $BL_\infty$ morphisms, pointed maps and the homotopy in Definition 2.21.

1. The category $\mathcal{R}_{II}$ is the analogue of $S_{II}$ in [66]. The objects of $\mathcal{R}_{II}$ are graphs without cycles as before, but now each edge $e \in E(T)$ is labeled with a symbol $*(e) \in \{0, 1\}$ such that all input edges are labeled with 0 and all output edges are labeled with 1. For each vertex $v \in V(T)$, we associate it with a pair of symbols $*(e) \in \{0, 1\}$ such that $*^+(v) \leq *^-(v)$ and $*(e^\pm(v)) = *^\pm(v)$. If $*^+(v) = *^-1(v)$, then $v$ is called a symplectization vertex and if $*^+(v) < *^-1(v)$, then $v$ is called a cobordism vertex. Given an exact cobordism $X$ from $Y_-$ to $Y_+$, for every $T \in \mathcal{R}_{II}$, we can similarly define the moduli space $\mathcal{M}_{II}(T)$, where the curve attached to a symplectization vertex $v$ with $*^\pm(v) = 0$ is a holomorphic curve in $\mathbb{R} \times Y_+$ modulo $\mathbb{R}$-translation, the curve attached to a symplectization vertex $v$ with $*^\pm(v) = 1$ is a holomorphic curve in $\mathbb{R} \times Y_+$ modulo $\mathbb{R}$-translation, the curve attached to a cobordism vertex $v$ is a holomorphic curve in $\tilde{X}$. Then we have the analogous compactification $\overline{\mathcal{M}}_{II}(T)$ using the over category over $T$, which is a $\mathcal{R}^*_{II}$ module.

2. The category $\mathcal{R}^*$ is similar to $\mathcal{R}$ but with exactly one vertex labeled by $*$. The morphism in $\mathcal{R}^*$ is again contractions of graphs such that the $*$ vertex is mapped to the $*$ vertex. For every $T \in \mathcal{R}^*$, we can associate a moduli space $\mathcal{M}^*(T)$, which is defined similar to $\mathcal{M}(T)$ but the map associated to $*$ vertex is holomorphic curve with a marked point mapped to the fixed point $(0, o) \in \mathbb{R} \times Y$. We can similarly define the compactified moduli spaces $\overline{\mathcal{M}}^*(T)$, which is $\mathcal{R}^*$ module.

3. The category $\mathcal{R}^*_{II}$ is the combination of $\mathcal{R}_{II}$ and $\mathcal{R}^*$, i.e. the objects are the same as $\mathcal{R}_{II}$ with one of the vertex is marked with $*$. In the definition of $\mathcal{M}^*_{II}(T)$, the curve attached to the $*$ vertex is a curve in the symplectization with a point constraint if the vertex is a symplectization vertex and is a curve in the cobordism with the path constraint if the vertex is a cobordism vertex.

**Proof of the rest of Theorem 3.9.** We need to argue that $\overline{\mathcal{M}}_{II}(T), \overline{\mathcal{M}}^*(T), \overline{\mathcal{M}}^*_{II}(T)$ are equipped with implicit atlases with oriented cell-like stratification. For this, we only need to argue that the gluing parameter
spaces are cell-like like, the remaining of the argument is the same as [66, Theorem 3.23]. The gluing parameter space \((G^*)_{T_f}\) for \(\mathcal{R}^*_{T_f}\) is same as the \(G_{T_f}\), i.e. \((0, \infty)^{E^{int}(T)}\), since there is no relations among gluing parameters. The gluing parameter space \((G_{II})_{T_f}\) for \((\mathcal{R}_{II})_{T_f}\) is defined as a subset of

\[
\left\{ (\{g_e\}_e, \{g_v\}_v) \in (0, \infty)^{E^{int,0}(T)} \times [-\infty, 0)^{E^{int,1}(T)} \times (0, \infty)^{V_{00}(T)} \times [-\infty, 0)^{V_{11}(T)} \right\},
\]

subject to the constraints

\[ g_v = g_e + g_{e'}, \text{ for } v \xrightarrow{e} v' \text{ with } * (e) = 0, \quad g_{e'} = g_e + g_v, \text{ for } v \xrightarrow{e} v' \text{ with } * (e) = 0. \]

where \(g_v\) is interpreted as 0 if \(v \in V_{01}(T)\), \(V_{ij}(T)\) is the set of vertices with \(*^+(v) = i, *^-(V) = j\) and \(E^{int,i}(T)\) is the set of interior edges \(e\) such that \(*(e) = i\). Then \(g_v\) can be viewed as the height of the vertex \(v\) for \(v \in V_{ij}(T)\), where the heights of all cobordism vertices are 0, as all of them are placed in the same level. Following the argument of [66, Lemma 3.6], it is sufficient to prove \((G_{II})_{T_f}\) is a topological manifold with boundary. We can perform the the same change of coordinates \(h = e^{-\theta} \in [0, 1)\) for \(v \in V_{00}(T), e \in E^{int,0}(T),\) and \(h = e^\theta \in [0, 1)\) for \(v \in V_{11}(T), e \in E^{int,1}(T)\). We allow \(h \in [0, \infty)\) for convenience. Then the relation becomes \(h_v = h_e h_{e'}\) for \(* (e) = 0\) and \(h_{e'} = h_e h_v\) for \(* (e) = 1\). Now the difference with [66, Lemma 3.6] is that we do not have \(r_{\text{max}}\), which in contact homology corresponds to the vertex with the input edge. In our case, the subgraph generated \(V_{00}(T)\) is a disjoint union of graphs \(\{T^0_i\}_{i \in I^1}\) and the subgraph generated \(V_{11}(T)\) is a disjoint union of graphs \(\{T^1_i\}_{i \in I^1}\). We pick a vertex \(v_i^0, v_i^1\) in \(T_i^0, T_i^1\) respectively. Since \(T_i^0\) has no cycles and we can view \(v_i^0\) as a root, we can parameterize the gluing parameters associated to \(T_i^0\) by \(h_{i^0} \in [0, \infty), g_e = h_e^2 - h_{e'} \in \mathbb{R}\) if \(e\) is in the same direction with the direction pointed away from the root \(v_i^0\), and \(h_e \in [0, \infty)\) if \(e\) is the opposite direction with the tree direction. In this case, \(h_{i^0} \in [0, \infty), g_e = h_e^2 - h_{e'} \in \mathbb{R}\) determine \(h_e, h_{e'} \in [0, \infty)\) as in [66, Lemma 3.6]. It is clear that such change of coordinate parameterize the gluing parameters by \([0, \infty) \times \mathbb{R}^{\{e \text{ in same direction}\}} \times [0, \infty)^{\{e \text{ in opposite direction}\}}\). Similarly we parameterize the gluing parameters on \(T_i^1\) by \(h_{i^1} \in [0, \infty), h_e \in [0, \infty)\) if \(e\) is in the same direction with the tree direction, and \(g_e = h_e - h_{e'} \in \mathbb{R}\) if \(e\) is the opposite direction with the tree direction. As a consequence \((G_{II})_{T_f}\) is a topological manifold with boundary, and the top stratum corresponds to the interior. The gluing parameter space \((G^*_{II})_{T_f}\) is same as \((G_{II})_{T_f}\). Therefore \(\mathcal{M}_{II}(T), \mathcal{M}^*(T), \mathcal{M}^*_{II}(T)\) are equipped with implicit atlases with oriented cell-like stratification.

The virtual fundamental cycles for \(BL_{\infty}\) morphisms, pointed maps and homotopies are module morphisms \(\mathbb{Q}[\mathcal{R}_{II}] \rightarrow \mathbb{Q}, \mathbb{Q}[\mathcal{R}^*] \rightarrow \mathbb{Q}\) and \(\mathbb{Q}[\mathcal{R}_{II}^*] \rightarrow \mathbb{Q}\) respectively that are derived from diagrams like (3.10). The non-emptiness of such diagrams and surjectivity of the projections of admissible auxiliary data follows from [66, Proposition 4.34]. That module morphisms \(\mathbb{Q}[\mathcal{R}_{II}] \rightarrow \mathbb{Q}, \mathbb{Q}[\mathcal{R}^*] \rightarrow \mathbb{Q}\) and \(\mathbb{Q}[\mathcal{R}_{II}^*] \rightarrow \mathbb{Q}\) give rise counts to \(BL_{\infty}\) morphisms, pointed maps and homotopies follows from the same proof of (1) of Theorem 3.9. For (5) of Theorem 3.9, it is clear the whole construction for \(a\) can be identified with the construction for \(ka\) as long as we use the same admissible almost complex structure \(J\) for \(k > 0\).

### 3.6.3. Polyfold approach

The polyfold construction of SFT [33], which is described in [34], will imply Theorem 3.9 as well. However, we can not use the “tree-like” compactification as in \(\mathcal{M}(T)\) because the gluing parameter space is only topological manifold with boundary. For the analytic requirement in polyfold, it is important to use the building compactification in [9] so that all gluing parameters are independent and form a smooth manifold with boundary and corner. To implement the polyfold construction for our purpose, it is sufficient to build polyfold strong bundles with sc-Fredholm sections for the SFT building compactification, which is sketched in [34].
Since we will not need to discuss more subtle cases like neck-stretching and homotopies, the abstract theory of polyfold developed in [44] suffices to provide transverse perturbations by the similar induction on \((\mathcal{R}, \mathfrak{s})\) starting from minimal elements in \(\mathfrak{s}\), which are polyfolds without boundaries. The non-empty set \(\Theta\) in Theorem 3.9 now consists pairs \((J, \sigma)\), where \(J\) is an admissible almost complex structure and \(\sigma\) is a family of compatible \(sc^+\)-multisections in general position. Then (3.11) and (3.12) follows from the Stokes' theorem in [44], where the coefficients can be explained to be the discrepancies of isotropy among polyfolds with their boundary polyfolds and boundary polyfolds with product polyfolds.

To verify Axiom 3.25, we first note that classical transversality implies polyfold transversality by definition. If \(\overline{\mathcal{M}}(T)\) is cut out transversely for \(\text{vdim}(T) = 0\), we may still need to perturb the \(sc\)-Fredholm section on the associated polyfold, because we construct perturbations by induction. Even though we know that the section is transverse on the boundary polyfolds, but the section can be non-transverse on some factor of the boundary, which will be perturbed before we construct perturbations for \(\overline{\mathcal{M}}(T)\). However, we can choose our perturbations small enough to get the local invariance of \(#\overline{\mathcal{M}}(T)\) when \(\text{vdim}(T) = 0\). In other words, Axiom 3.25 holds if we choose sufficiently small perturbations. This matches with Proposition 3.32, as \(\Theta(\alpha, J)\) in VFC can be understood as “infinitesimal” perturbations.

**Remark 3.33 (Kuranishi approach).** The Kuranishi approach of SFT [47] would also imply Theorem 3.9. Axiom 3.25 should follow from the same argument above for small enough perturbation in a reasonable measurement.

### 3.7. Generalized constraints.

There are few directions where one can generalize the constructions above. In the following, we will briefly describe one of such generalizations by considering more general constraints.

#### 3.7.1. General constraints from \(Y\). We can pick a closed submanifold of \(Y\) or more generally a closed singular chain \(C\) of \(Y\) to construct a pointed map \(p_C\) by considering rational holomorphic curves in \(\mathbb{R} \times Y\) with one marked point mapped to \(\{0\} \times C\). Then given a \(BL_\infty\) augmentation \(\epsilon\), we have an induced chain map \(\tilde{\ell}_{C, \epsilon} : (\overline{\mathcal{B}}^k V, \tilde{\ell}_\epsilon) \to \mathbb{Q}\) defined using \(p_C^{k,0}\). When \(X\) is an exact cobordism from \(Y_-\) to \(Y_+\), let \(C_-, C_+\) be two closed singular chains in \(Y_-\), \(Y_+\), which define two pointed map \(p_{C_-}, p_{C_+}\). If \(C_-\) and \(C_+\) are homologous in \(X\), then one can show that \(p_{C_-}, p_{C_+}\) and the \(BL_\infty\) morphisms induced from \(X\) are compatible in the sense of Definition 2.21 by counting holomorphic curves in \(\hat{X}\) with a point passing through the singular chain whose boundary is \(-C_- \cup C_+\). As a consequence of Proposition 2.22, \(\tilde{\ell}_{C, \epsilon}\) up to homotopy only depends on the homology class \([C]\). Therefore for any \(k \geq 1\), we have a linear map

\[
\delta^\epsilon_k : H_* (\overline{\mathcal{B}}^k V, \tilde{\ell}_\epsilon) \otimes H_* (Y) \to \mathbb{Q}, \quad (x, [C]) \mapsto \tilde{\ell}_{C, \epsilon}(x).
\]

Or equivalently, we can write it as \(\delta^\epsilon : H_* (\overline{\mathcal{B}}^k V, \tilde{\ell}_\epsilon) \to H^* (Y, \mathbb{Q})\). Combining the method in [12] and the argument in §5, one can show that \(H_* (\overline{\mathcal{B}}^k V, \tilde{\ell}_\epsilon) \to H^* (Y, \mathbb{Q})\) is isomorphic to the map \(SH^*_{+, si} (W; \mathbb{Q}) \to H^*_{s1} (W; \mathbb{Q}) := H^{*+1} (W; \mathbb{Q}) \otimes \mathbb{Q} [u, u^{-1}] / [u] \to H^{*+1} (Y; \mathbb{Q})\), where \(W\) is an exact filling and \(\epsilon\) is the augmentation induced from the filling \(W^{10}\). This fact was used in [83, 78] (but not phrased in SFT) to define obstructions to Weinstein fillings. In principle, \(H_* (\overline{\mathcal{B}}^k V, \tilde{\ell}_\epsilon) \to H^* (Y, \mathbb{Q})\) can be used to obstructed Weinstein fillings if the image contains an element of degree \(> \frac{\dim Y + 1}{2}\) for all possible augmentations that could be from a Weinstein filling, e.g. \(\mathbb{Z}\)-graded augmentations and \(c_1 (Y) = 0\) and \(\dim Y \geq 5\).

In the following, we will explain the functorial part of \(\delta^\epsilon\) without proof. Given an exact cobordism \(X\) from \(Y_-\) to \(Y_+\), let \(C\) be a closed chain in \(X\). Then by counting rational holomorphic curves in \(X\) with

\[\text{dim} Y + 1 \leq \frac{\dim Y + 1}{2}\]Note that symplectic cohomology in [78] is graded by \(n - \mu C Z\), this explains the discrepancy of parity.
a marked point mapped to $C$, we obtain a family of maps $\phi_{k,l}^{}: S^k V_{\alpha^+} \to S^l V_{\alpha^-}$. Along with the $BL_{\infty}$ morphism $\phi_{k,l}^{}$ from $X$, we can construct a map $\hat{\phi}_C : EV_{\alpha^+} \to EV_{\alpha^-}$ from $\phi_{k,l}^{}$, $\phi_{k,l}^{}$ by the same rule of $\hat{\phi}_\star$. Now that $C$ is closed, we have $\hat{\phi}_C \circ \hat{p}_{+} = \hat{p}_{-} \circ \hat{\phi}_C$. Then given an augmentation $\epsilon$ of $V_{\alpha^-}$, we have a linearized relation $\hat{\phi}_{C,\epsilon} \circ \hat{p}_{+,\epsilon} = \hat{p}_{-,\epsilon} \circ \hat{\phi}_{C,\epsilon}$. In particular, $\hat{\phi}_{C,\epsilon}^{k,0}$ defines a chain map $(\hat{B}^k V_{\alpha^+}, \hat{\ell}_{\epsilon_0}) \to \mathbb{Q}$. By the similar argument as before, such construction yields a map

$$\delta_{X,\epsilon}^{}: H_\star(\hat{B}^k V_{\alpha^+}, \hat{\ell}_{\epsilon_0}) \otimes H_\star(X; \mathbb{Q}) \to \mathbb{Q}$$

where the dual version $H_\star(\hat{B}^k V_{\alpha^+}, \hat{\ell}_{\epsilon_0}) \to H^\star(X; \mathbb{Q})$ is denoted by $\delta_{X,\epsilon}$. If $C$ comes from $H_\star(Y_+)$, then the map coincides with $\delta_{X,\epsilon}^\prime$ by the same argument of Proposition 5.14. If $C$ comes from $Y_-$, then $\delta_{X,\epsilon}^\prime$ factors through $\hat{\phi}_\epsilon : H_\star(\hat{B}^k V_{\alpha^+}, \hat{\ell}_{\epsilon_0}) \to H_\star(\hat{B}^k V_{\alpha^-}, \hat{\ell}_{\epsilon})$ by an argument similar to Proposition 5.14. Dualizing those properties, we have the following commutative diagram,

$$\begin{array}{ccc}
H_\star(\hat{B}^k V_{\alpha^+}, \hat{\ell}_{\epsilon_0}) & \xrightarrow{\delta_{X,\epsilon}} & H^\star(X; \mathbb{Q}) \\
\downarrow \hat{\phi}_\epsilon & & \downarrow \\
H_\star(\hat{B}^k V_{\alpha^-}, \hat{\ell}_{\epsilon}) & \xrightarrow{\delta_{X,\epsilon}} & H^\star(Y_-; \mathbb{Q})
\end{array}$$

3.7.2. Tangency conditions. Another type of generalization is considering point constraint with tangency conditions, i.e we consider curves in the symplectization passing through a fixed point $p$ and tangent to a local divisor near $p$ with order $m$. Such holomorphic curves were considered in [23, 70]. Those curves also give rise to pointed maps, hence can be used to define a new order. In many cases, if we have a holomorphic with a point constraint without tangent conditions, then multiple covers of it might have tangent properties. One can show that the order with tangent condition for $(S^{2n-1}, \xi_{std}), n \geq 2$ is always 1 no matter what is the order of tangency. The curve is easy to find in an very thin ellipsoid with one positive puncture asymptotic to a multiple cover of the shortest Reeb orbit. Then one can use a neck-stretching argument to prove that it is independent of the augmentation. The invariants with local tangent constraints for exact domains are defined and computed in [37].

3.7.3. Multiple point constraints. It is natural to consider generalizations of pointed maps of $BL_{\infty}$ algebras to maps induced from counting curves with multiple constraints. For example, we can consider rational holomorphic curves with 2 marked points passing through two fixed points in the contact manifold, where the curve can be disconnected. More specifically, we have three families of maps from $S^k V$ to $S^l V$ that are $p_{*,*}^{k,l}$ coming from counting connected holomorphic curves with two marked points, $p_{*,1}^{k,l}$, $p_{*,2}^{k,l}$ coming from counting connected holomorphic curves with each one of the point constraints respectively. Then we can assemble them to $\hat{p}_{*,*}$ by the same rule of $\hat{p}_\star$ except the middle level consists of one $p_{*,*}$ or both $p_{*,1}^{k,l}$, $p_{*,2}^{k,l}$. Note that the combinatorics behind $\hat{p}_{*,*}$ is slightly different from $\hat{p}_{*,1}^{k,l}$ and $\hat{p}_{*,2}^{k,l}$. Namely, $\hat{p}_{*,*}$ does not satisfy the component-wise Leibniz rule on each $SV$. More precisely, the component from $p_{*,*}$ satisfies the Leibniz rule, while the component from $p_{*,1}^{k,l}$ and $p_{*,2}^{k,l}$ is a second order differential operator. Nevertheless, we have $\hat{p}_{*,*} \circ \hat{p} = \hat{p} \circ \hat{p}_{*,*}$ and given a $BL_{\infty}$ augmentation, we can similarly define $\hat{p}_{*,*,\epsilon}$ which is assembled from $p_{*,*,\epsilon}, p_{*,1,\epsilon}$ and $p_{*,2,\epsilon}$. Then we can define $\hat{O}(V, \epsilon, p_{*,*})$ by the same recipe for $\hat{O}(V, \epsilon, p_{*,*})$. By moving the
two point constraints towards infinity in the opposite directions similar to the proof of Proposition 5.14, we obtain the relation $p^2 = p_0 \sim p$ up to homotopy on $EV$. Similarly, we have $p^2_0,\epsilon$ and $p_{\epsilon,p}$ are homotopic. Therefore it is direct to see that $\tilde{O}(V,\epsilon,p) \geq \tilde{O}(V,\epsilon,p)$. It is not clear whether the inequality can be strict. In general, we can consider disconnected rational homomorphic curves with $k$ marked points passing through $k$ point constraints, which give rise to an operator $\tilde{p}_k$ on $EV$. $\tilde{p}_k$ is homotopic to $\hat{p}_k$ and we can define an order $\tilde{O}(V,\epsilon,p_k)$ for a $BL_\infty$ augmentation $\epsilon$.

Given a strict exact filling $(W,\lambda)$ of $(Y,\alpha)$, assume $\tilde{O}(V,\epsilon,p_k) < \infty$ for the $BL_\infty$ augmentation $\epsilon$ coming from $W$. Then we can define the spectral invariant for $l \geq \tilde{O}(V,\epsilon,p_k)$,

$$\tau^{<\ell}(p,\ldots,p) := \inf_k \left\{ a : T^a \in \im \pi_{\Lambda \geq 0} \circ \hat{p}_k,\epsilon | H_*(\overline{B^k SV_\alpha^{\geq 0},\alpha}) \right\} < \infty,$$

where $V_\alpha^{>0}$ is the free $\Lambda \geq 0 := \{ \sum a_i T^{b_i} | b_i \geq 0, \lim b_i = \infty, a_i \in \mathbb{Q} \}$ module generated by good Reeb orbits of $\alpha$. This is exactly the higher symplectic capacity $\tau^{<\ell}(p,\ldots,p)$ defined by Siegel in [70, §6]. In [70], the author considered disconnected holomorphic curves in $W$ with $k$ point constraints, viewed them as a chain map from $\overline{S}\, \text{CHA}(Y) = EV_\alpha$ to $\Lambda$. Then the capacity is defined to be the infimum of $a$ such that there is a closed class in $x \in B^k SV_\alpha^{>0}$ such that $x$ is mapped to $T^a$ by the chain map. The equivalence of these two definition can be seen from the same argument of Proposition 5.14. Similarly, combining the tangency conditions and multiple point constraints, we can define the analogous orders, whose spectral invariant is again equivalent to the higher capacity $\tau^{<\ell}(T^{m_1}p,\ldots,T^{m_k}p)$ in [70, §6]. Similarly, the spectral invariant for $O(V,\epsilon,p)$ is equivalent to $g^{<\ell}$ and the spectral invariant for the analogous version of pointed map with tangency conditions of order $m$ is $g^{<\ell}_m$ in [70].

**Example 3.34.** Let $W$ be an irrational ellipsoid with $\alpha$ the contact form on $\partial W$. Then we have $\tilde{O}(V,\epsilon,w,p_k) = 1$ for all $k \geq 1$. To see this, we let $\gamma$ denote the shortest Reeb orbits, then for a generic point in $W$ and a generic admissible almost complex structure, there is one holomorphic plane in $\overline{W}$ asymptotic to $\gamma$ and passing through the point. Since $\mu_{CA}(\gamma') + n - 3 = 0 \mod 2$ for any Reeb orbit $\gamma'$, we have $q_{\gamma} \in B^k SV_\alpha$ is a closed class and $\pi_{Q} \circ \hat{p}_{k,\epsilon}(q_{\gamma}) = 1$ by the holomorphic curve above. In general, we have $k q_{\gamma} \in B^k SV_\alpha$ is closed by degree reasons. One the other hand, $\pi_{Q} \circ \hat{p}_{k,\epsilon}(q_{\gamma})$ must only use disconnected curves with $k$ components and each component has one positive puncture, for otherwise, genus has to be created. As a consequence, we have $\pi_{Q} \circ \hat{p}_{k,\epsilon}(q_{\gamma}) = 1$ by the disconnected $k$ copies of the curve above. Then one can show such argument is independent of the augmentation, as the curve above is contained in the symplectization after a neck-stretching, see [83, 79].

Although there is nothing interesting happening for the order $\tilde{O}(V,\epsilon,w,p_k)$. We know that the spectral invariant $\tau^{<\ell}(p,\ldots,p)$ is defined for all $k \geq 1$ and $l \geq 1 = \tilde{O}(V,\epsilon,w,p_k)$. Those numerical invariants are very sensitive to the shape of $W$ and are powerful tools to study embedding problems, see [70] for details.

Finally, we give the analogue of $O(V,\epsilon,p)$ when we have multiple point constraints. Let $v \in B^k SV_\alpha$, we define the width $w(v)$ to be the maximal number $k$ such that $v$ has a component in the form of $(v_1 \ldots v_k) \circ \ldots$. We define $w(0) = \infty$. Since $p_k^{k,0} = 0$ for $k \geq 1$, we have $w(\tilde{p}(v)) \geq w(v)$. For $m \geq 1$, let $\pi_m$ denote the projection $B^k SV_\alpha \rightarrow B^k B^m V_\alpha$, then the kernel of $\pi_m$ is exactly those elements with width $> m$. Therefore

\[\text{More precisely, the algebraic count of such curves is 1.}\]
\( \hat{p}_\epsilon^m := \pi_m \circ \hat{p}_\epsilon|_{\mathcal{B}^m \mathcal{V}_\alpha} \) squares to zero. Then \( \hat{p}_\epsilon^m \) uses the knowledge of \( \hat{p}_{k,l} \) for \( l \leq m \) and \( \hat{p}_\epsilon^1 = \hat{\ell}_\epsilon \). In particular, we have \( \pi_m \) is a chain map. Moreover, we have the following commutative diagram,

\[
\begin{array}{ccc}
\mathcal{B}^k \mathcal{B}^m \mathcal{V}_\alpha & \xrightarrow{\pi_m \circ \hat{p}_{m,\epsilon}|_{\mathcal{B}^m \mathcal{V}_\alpha}} & \mathbb{Q} \\
\pi_m & \downarrow & \\
\mathcal{B}^k \mathcal{V}_\alpha & \xrightarrow{\pi_m \circ \hat{p}_{m,\epsilon}} & \mathbb{Q}
\end{array}
\]

This is because \( \hat{p}_{m,\epsilon} \) can have at most \( m \) nontrivial connected components. Therefore if \( w(v) \geq m + 1 \) then \( w(\hat{p}_{m,\epsilon}(v)) \geq 1 \), in particular, \( \pi_Q \circ \hat{p}_{m,\epsilon}(v) = 0 \).

**Definition 3.35.** We define \( O(V, \epsilon, p_m) := \min \left\{ k \mid 1 \in \text{im}\pi_Q \circ \hat{p}_{m,\epsilon}|_{H_*\left(\mathcal{B}^m \mathcal{V}_\alpha \mathcal{B}^m \mathcal{V}_\alpha \right)} \right\} \)

Then (3.14) implies the following.

**Proposition 3.36.** \( O(V, \epsilon, p_k) \leq \hat{O}(V, \epsilon, p_k) \).

**Remark 3.37.** The spectral invariants for \( O(V, \epsilon, p_k) \) provide a new family of higher symplectic capacities.

### 3.8. Higher genera.

Another natural direction to generalize is by increasing the genus of holomorphic curves, i.e. considering the full SFT. Originally, the full SFT was phrased as a differential Weyl algebra with a distinguished odd degree Hamiltonian \( H \) such that \( H \ast H = 0 \) [28]. There are two closely related ways to view the full SFT as a functor from \( \mathfrak{C}on \) in a more convenient way, namely the \( BV_\infty \) formulation [22] and the \( IBL_\infty \) formulation [21]. In the following, we will first briefly recall the definitions of them. In view of the \( BL_\infty \) formalism in this paper, we will give a slightly different but equivalent definition of \( IBL_\infty \) algebras. Then we will make some speculations assuming the analytical foundation of the full SFT is completed.

#### 3.8.1. \( IBL_\infty \) algebras and \( BV_\infty \) algebras.

The original definition of \( IBL_\infty \) algebra on a \( \mathbb{Z}_2 \) graded vector space involves taking a suspension as the formalism for \( L_\infty \) algebras in §2.1. To align with the notation of \( BL_\infty \) algebras, we will not take the suspension. In particular, if \( V \) is an \( IBL_\infty \) algebra in Definition 3.38, then \( V[-1] \) is an \( IBL_\infty \) algebra in the sense of [21]. Let \( \phi : S^k V \to S^l V \), we can define \( \hat{\phi} : \mathfrak{S} V \to \mathfrak{S} V \) by \( \hat{\phi} = 0 \) on \( S^m V \) with \( m < k \) and

\[
\hat{\phi}(v_1 \ldots v_k) = \sum_{\sigma \in S(k, m-k)} \frac{(-1)^{\sigma}}{k!(m-k)!} \phi(v_{\sigma(1)} \ldots v_{\sigma(k)})v_{\sigma(k+1)} \ldots v_{\sigma(m)},
\]

for \( m > k \).

**Definition 3.38 ([21, Definition 2.3]).** Let \( V \) be a \( \mathbb{Z}_2 \) graded vector space over \( k \), an \( IBL_\infty \) structure on \( V \) is a family of operators \( p^{k,l,g} : S^k V \to S^l V \) for \( k, l \geq 1 \) and \( g \geq 0 \), such that

\[
\hat{p} := \sum_{k,l=1}^{\infty} \sum_{g=0}^{\infty} \hat{p}^{k,l,g} \hat{p}^{k+g-1, l+1+2g-2} : \mathfrak{S} V[[h, \tau]] \to \mathfrak{S} V[[h, \tau]]
\]

satisfies that \( \hat{p} \circ \hat{p} = 0 \) and \( |\hat{p}| = 1 \). Here \( |h| = 0 \) and \( |\tau| = 0 \)

In the case of \( V \) being \( \mathbb{Z} \) graded, then one can define \( IBL_\infty \) structures of degree \( d \) by requiring \( |h| = 2d \) and \( |\hat{p}| = -1 \). In the case of SFT, \( d = n - 3 \) for a \( 2n-1 \) dimensional contact manifold \((Y, \xi)\) with \( c_1(\xi) = 0 \). If moreover we know that for any \( v_1, \ldots, v_k \in V \) and \( g \geq 0 \) there exists at most finitely many \( l \geq 1 \), such
that $p^{k,l,g}(v_1 \ldots v_k) \neq 0$. Then $\hat{p}$ is well-defined on $SV[[h]]$ by setting $\tau = 1$. Note that we only consider $p^{k,l,g}$ with $l \geq 1$ in Definition 3.38, in the context of SFT, this is equivalent to that there is no holomorphic curve with no negative punctures in the symplectization, which is in general not true. Indeed, Definition 3.38 is used to model the linearized theory, i.e. SFT associated to an exact domain. In general, we will need the following (curved) $IBL_\infty$ algebras.

**Definition 3.39.** Let $V$ be a $\mathbb{Z}_2$ graded vector space over $k$, a (curved) $IBL_\infty$ structure on $V$ is a family of operators $p^{k,l,g} : S^k V \to S^l V$ for $k \geq 1, l, g \geq 0$, such that

1. for $v_1, \ldots, v_k \in V, g \geq 0$, there are finitely many $l$ such that $p^{k,l,g}(v_1 \ldots v_k) \neq 0$,

2. $\hat{p} := \sum_{k=1}^{\infty} \sum_{l=g}^{\infty} \sum_{l=0}^{g-1} p^{k,l,g} h^{k+g-1} : SV[[h]] \to SV[[h]]$ satisfies $\hat{p} \circ \hat{p} = 0$ and $|\hat{p}| = 1$, where $\hat{p}(1)$ is defined to be 0.

Note that Definition 3.39 gives rise to a $BV_\infty$ algebra introduced in [22], which was used to define algebraic torsions in [48], c.f. Remark 3.19. For the translations between $BV_\infty$ algebras and differential Weyl algebras with distinguished Hamiltonians, see [22] for details. In the following, we give an alternative description of Definition 3.39. Assume we are given a family of operators operators $p^{k,l,g} : S^k V \to S^l V$ for $k \geq 1, l, g \geq 0$, such that (1) of Definition 3.39 is satisfied. We use $EV[[h]]$ to denote $S(SV[[h]])$, where $S$ is the symmetric product over $k[[h]]$. Then we define $\hat{p} : EV[[h]] \to EV[[h]]$ by the following graph description. We use a cluster with $k + 1$ vertices with a label $g \geq 0$ to represent a class $h^g v_1 \ldots v_k \in SV[[h]]$. Then we use the graph with $\circ$ to represent operators $p^{k,l,g}$ as before, but we label the $\circ$ with a number $g \geq 0$. To define $\hat{p}$, we represent a class in $EV[[h]]$ by a row of clusters with labels, then we glue in one graph representing $p^{k,l,g}$ and dashed vertical lines representing the identity map. Here we allow cycles being created. The rule for the output is the same as before with the degree of $h$ is determined by the sum of the labels $g$ in the connected component of the glued graph with the number of (independent) cycles in that component. Then $\hat{p}$ is the sum of all such glued graphs.

![Figure 11](image)

**Figure 11.** A component of $\hat{p}$ from $h^{g_1} S^3 V \odot h^{g_2} S^3 V \odot h^{g_3} S^2 V$ to $h^{g_1+g_2+g_3+1} S^6 V \odot h^{g_3} S^2 V$ using $p^{3,3,3}.$

One can think of the cluster representing a class in $SV[[h]]$ as a genus $g$ surface with $k$ negative punctures and $p^{k,l,g}$ is represented by a genus $g$ surface with $k$ positive punctures and $l$ negative punctures. Then a glued graph represented a possibly disconnected surface with only negative punctures, which represents a class in $EV[[h]]$ with the degree of $h$ in each $\odot$ component is represented by the genus of the connected component.

**Definition 3.40.** $(V, p^{k,l,g})$ is a (curved) $IBL_\infty$ algebra iff $\hat{p} : EV[[h]] \to EV[[h]]$ defined above satisfies $\hat{p} \circ \hat{p} = 0$ and $|\hat{p}| = 1$. 


Proposition 3.41. Definition 3.39 and 3.40 are equivalent.

Proof. Note that \( \tilde{p} : SV[[h]] \subset EV[[h]] \to SV[[h]] \) as we can not increase the number of connected components in the graph description of \( \tilde{p} \). However \( \tilde{p}|_{SV[[h]]} \) is exactly the \( \tilde{p} \) in Definition 3.39 as the \( h^{k-1} \) is exactly the number of cycles in the glued graph. Therefore Definition 3.40 implies Definition 3.39.

On the other hand, if \( \tilde{p} : SV[[h]] \to SV[[h]] \) squares to zero. If we consider the \( h^{N}S^{m}V \) part of \( \tilde{p} \circ \tilde{p}(v_{1} \ldots v_{n}) \), which should be zero. We need to consider disconnected graphs with \( n \) input vertices and \( m \) output vertices and two \( \circ \) vertices such that if we glue all input vertices together by adding a new vertices, the resulted graph has \( N \) cycles. Then there are two cases, (1) the two \( \circ \) vertices are in different components, then all of components such that the glued graph with signature \( (n,m, \ldots ) \) should be created and the rule for the order of \( \pi \) is same as \( \tilde{p} \). Or equivalently, \( p_{a,b,k} \) is defined by all possible two-level breaking of a the graph with two interior vertices and signature \( (a,b,k) \). Then the vanishing of the \( h^{N}S^{m}V \) part of \( \tilde{p} \circ \tilde{p}(v_{1} \ldots v_{n}) \) implies

\[
\sum_{a=1}^{n} p_{a,m-n+a,N-a+1}^{n,m,N} \neq 0 \text{ is exactly the } \hat{SV} \text{ part of } \tilde{p} \text{ and } \hat{SV}(x) = 0.
\]

for all \( n \geq 1, m, N \geq 0 \). By setting \( n = 1 \), we have \( p_{1,m,N}^{1,m,N} = 0 \) for \( m, N \geq 0 \). Then by setting \( n = 2 \) and \( p_{2,m,N}^{1,m,N} = 0, \) we have \( p_{2,m,N}^{2,m,N} = 0 \). Similarly we have \( p_{n,m,N}^{2,m,N} = 0 \) for all \( n \geq 1, m, N \geq 0 \). This is exactly describing all maps from two-level breaking of a connected graph with signature \( (n,m,N) \) should sum up to zero. It implies that \( \hat{p} \) is zero on \( EV[[h]] \) by the same argument in Proposition 2.6

Remark 3.42. From the proof of Proposition 3.41. Both Definition 3.39 and Definition 3.40 are equivalent to \( p_{2,m,N}^{n,m,N} = 0 \), which in the SFT world, corresponding to the algebraic counts of all rigid codimension 1 breaking of connected holomorphic curves with \( n \) positive punctures and \( m \) negative punctures and genus \( N \) sum up to zero.

Note that \( SV \subset SV[[h]] \) induces an inclusion \( EV \subset EV[[h]] \), and we use \( \pi_{0} \) to denote the natural projection \( EV[[h]] \to EV \). It is easy to check that if \( x \in \ker \pi_{0} \), then \( \tilde{p}(x) \in \ker \pi_{0} \). As a consequence, \( \pi_{0} \circ \tilde{p}_{|EV} : EV \to EV \) squares to zero. Moreover, \( \pi_{0} \circ \tilde{p}_{|EV} \) is assembled from \( p_{k,l,0}^{k,l,0} \). Then we have the following instant corollary.

Corollary 3.43. Let \( (V,p_{k,l,g}) \) be a (curved) \( IBL_{\infty} \) algebra, then \( (V,p_{k,l} := p_{k,l,0}) \) is a \( BL_{\infty} \) algebra.

3.8.2. A grid of torsions. Given a family of operators \( \phi_{k,l,g} : S^{k}V \to S^{l}V \) for \( k \geq 1, l, g \geq 0 \), assume that for any \( g \geq 0 \) and \( v_{1}, \ldots , v_{k} \in V \), there are at most finitely many \( l \) such that \( \phi_{k,l,g}(v_{1} \ldots v_{k}) \neq 0 \). Then we can assemble \( \tilde{\phi} : EV[[h]] \to EV[[h]] \) by the same rule for \( BL_{\infty} \) morphisms except that cycles are allowed to be created and the rule for the order of \( \pi \) is same as \( \tilde{p} \).

Definition 3.44. The family of operators is an \( IBL_{\infty} \) morphism from \( (V,p_{k,l,g}) \) to \( (V^\prime,q_{k,l,g}) \) if \( \hat{q} \circ \tilde{\phi} = \tilde{\phi} \circ \hat{p} \).

Then we have a trivial \( IBL_{\infty} \) algebra \( 0 \) := \{ 0 \} with \( p_{k,l,g} = 0 \). \( 0 \) is an initial object in the category of \( IBL_{\infty} \) algebras with \( \phi_{k,l,g} = 0 \). Then an \( IBL_{\infty} \) augmentation of \( V \) is an \( IBL_{\infty} \) morphism from \( V \) to \( 0 \). \( IBL_{\infty} \) augmentation may not always exist. One obstruction is torsion. Unlike the torsions for \( BL_{\infty} \) algebras, there are many more torsions for \( IBL_{\infty} \) algebras. Let \( E^{k}V[[h]] := \overline{B}^{k}SV[[h]] \).
Definition 3.45. For \( n, m \geq 0 \), we say \( V \) has a \((n, m)\) torsion if \( [h^n] = 0 \in H_\ast(E^{m+1}V[[h]]) \).

Then the algebraic torsion in \([48]\) is the \((n, 0)\) torsion of the \( IBL_\infty \) algebra associated to a contact manifold by SFT.

Proposition 3.46. If \( V \) has \((n, m)\) torsion, then \( V \) has \((n + 1, m - 1)\) torsion.

Proof. We use \( h^{-1}EV[[h]] \) to denote \( EV[[h]] \otimes_{k[[h]]} h^{-1}k[[h]], h^{-1}k[[h]] \) is the \( k[[h]] \) module generated by \( h^{-1} \). Then we have a map \( c_k : \odot^k SV[[h]] \to h^{-1} \odot^k SV[[h]] \) for \( k \geq 2 \) induced by

\[
 w_1 \odot \ldots \odot w_k \mapsto \sum_{i < j} (-1)^{i}h^{-1}(w_i w_j) \odot w_1 \odot \ldots \odot \hat{w}_i \odot \ldots \odot \hat{w}_j \odot \ldots \odot w_k,
\]

where \( w_1 \ldots w_k = (-1)^{\ell}w_i w_j w_1 \ldots \hat{w}_i \ldots \hat{w}_j \ldots w_k \) for homogeneous \( w_i \). Then we have the following commutative diagram

\[
 \begin{array}{ccc}
 E^{m+1}V[[h]] & \xrightarrow{\hat{c}} & h^{-1}E^{m}V[[h]] \\
 \downarrow \hat{\beta} & & \downarrow \hat{\beta} \\
 E^{m+1}V[[h]] & \xrightarrow{c} & h^{-1}E^{m}V[[h]] \\
 \end{array}
\]

where \( \hat{c} = \sum_{i=2}^{m+1} \epsilon_i + \sum_{i=1}^{m} \epsilon_i \), where \( \epsilon_i \) is the obvious inclusion \( S^i SV[[h]] \to h^{-1} S^i SV[[h]] \). If \( x \in E^{m+1}V[[h]] \), such that \( \hat{c}(x) = h^n \). Since \( c(h^n) = h^n \), we have \( \hat{\beta} \circ \hat{c}(x) = h^n \). Since \( \hat{\beta} \) is \( Q[[h]] \) linear, we have \( h\hat{c}(x) \in E^{m}V[[h]] \) and \( \hat{\beta}(h\hat{c}(x)) = h^{n+1} \). This finishes the proof.

However the torsion for the associated \( BL_\infty \) algebra is not necessary a \((0, m)\) torsion unless \( m = 0 \) \([11]\). We use \( p_0 \) to denote the genus 0 part of \( p \), i.e. the induced \( BL_\infty \) algebra on \( V \). Indeed if \( 1 = \hat{\beta}_0(x) \) for \( x \in E^{m+1}V \), then we only have \( \hat{\beta}(x) = 1 + O(h) \), where \( O(h) \) is in \( \ker \pi_0 \) with \( O(h) \) potentially comes from holomorphic curves with higher genera. On the other hand, as we will see later that a \((0, m)\) torsion always implies a \( m \) torsion for the \( BL_\infty \) algebra. To explain the relations, we will explain the following procedure which increases genus one at a time.

Let \( (V, p^{k,l,g}) \) be an \( IBL_\infty \) algebra. Let \( v \in SV[[h]] \), we define the \( h \)-width \( w_h(v) \in N \cup \{ \infty \} \) to be the maximal \( k \), such there exists \( v' \in SV[[h]] \) with \( h^k v' = v \). In particular, \( w_h(0) = \infty \). For \( v \in EV[[h]] \), \( w_h(v) \) is uniquely characterized by \( w_h(v_1 \odot v_2) = w_h(v_1) + w_h(v_2) \) and \( w_h(v_1 + v_2) \geq \max\{w_h(v_1), w_h(v_2)\} \). Then we have \( w_h(\beta(v)) \geq w_h(v) \) for all \( v \in EV[[h]] \). We define \( EV[[h]]_m := EV[[h]] \otimes_{k[[h]]} \odot (k[[h]]/h^{m+1}) \), then we consider the projection \( \pi_m : EV[[h]] \to EV[[h]]_m \). Then \( \ker \pi_m \) is exactly those elements with \( w_h > m \). We can view \( Q[[h]]/[h^{m+1}] \) as polynomials of \( h \) of degree at most \( m \), we can view \( EV[[h]]_m \subset EV[[h]] \). As a consequence we have the following commutative diagram,

\[
 \begin{array}{ccc}
 EV[[h]] & \xrightarrow{\hat{\beta}} & EV[[h]] \\
 \downarrow \pi_m & & \downarrow \pi_m \\
 EV[[h]]_m & \xrightarrow{\hat{\beta}_m := \pi_m \circ \hat{\beta}} & EV[[h]]_m \\
 \end{array}
\]

Then we have \( \pi_m \circ \hat{\beta}_m = 0 \) with \( \hat{\beta}_0 \) is the associated \( BL_\infty \) structure. If we unwrap the definition, then we know that \( \hat{\beta}_m \) uses \( p^{k,l,0} \) for \( g \leq m \). Take \( m = 1 \) as an example, \( \hat{\beta}_1 \) uses both \( p^{k,l,1}, p^{k,l,0} \). However, to get an output in \( EV[[h]]_1 \), the \( k \) inputs of \( p^{k,l,1} \) must glue to \( k \) different clusters while at most 2 of the \( k \) inputs of \( p^{k,l,0} \) can glue to the same cluster. We can similarly define \( E^k V[[h]]_m \) and we have analogous diagrams.
Definition 3.47. We say $(V, p^{k,l,g})$ has an $(n, m)_k$ torsion iff $[h^n] = 0 \in H^*(E^{m+1}V[[h]])_k$.

Then by definition, we always have $(n, m)_k$ torsion for $n > k$. Moreover, $(0, m)_0$ torsion is the $m$-torsion for $BL_\infty$ algebras. The $(n, m)$ torsion can be viewed as $(n, m)_\infty$ torsion. We summarize the basic properties of those torsion in the following.

Proposition 3.48. The torsions have the following properties.

1. If $V$ has $(n, m)_k$ torsion then $V$ has $(n, m)_{k-1}$, $(n+1, m)_k$ and $(n, m+1)_k$ torsions. In particular if $V$ has $(n, m)$ torsion, then $V$ has $(n, m)_k$ torsion for any $k \geq 0$.
2. If $V$ has $(n, m)_k$ torsion, then $V$ has $(n+1, m-1)_k$ torsion.

Proof. (1) follows from (3.16), the filtration $E^kV[[h]]_m$ and the linear property with respect to $h$. (2) follows from the $EV[[h]]_m$ version of (3.15). □

As a corollary, the existence of $(0, n)$ torsion implies both $n$-algebraic torsion and $n$-algebraic planar torsion. Contact manifolds in Theorem 3.16 and 3.17 actually has $(0, k)$ torsion by the same argument in [48]. Therefore it implies both algebraic torsion and algebraic planar torsion. Roughly speaking, different torsions make different requirements on holomorphic curves. For $(n, 0)$ torsion, we can have higher genus curves without negative puncture to contribute to the torsion, while all higher genus curves with the same positive punctures and non-empty negative punctures must sum up to zero. For $(0, n)$ torsion, we can only have rational curves without negative puncture to contribute to the torsion, while all higher genus curves with the same positive punctures and non-empty negative punctures must sum up to zero. For $(0, n)_0$ torsion, we can only have rational curves without negative puncture to contribute to the torsion, and all rational curves with the same positive punctures and non-empty negative punctures must sum up to zero.

Let $2^N^3$ denote the category of subsets of $N^2 \times (N \cup \{\infty\})$, the arrow from $V$ to $W$ is an inclusion $W \subset V$. $2^N^3$ is a monoidal category where the monoidal structure is given by taking union. We use $SFT(Y)$ to denote the full SFT as an $IBL_\infty$ algebra in the sense of Definition 3.40 for a contact manifold $Y$.

Theorem 3.49. Let $Y$ be a contact manifold, we define $T(V) := \{(m, n, k) | SFT(Y) has (m, n)_k torsion\}$. Then $T : \mathbb{C}on \rightarrow 2^N^3$ is a covariant monoidal functor.

Remark 3.50. For the proof of Theorem 3.49, we only need to construct $SFT(Y)$ to the same extent as in Theorem 3.9, which is expected from [33, 47]. It is also expected that one can generalize the construction in [66], in particular prove the existence of implicit atlases with cell-like stratification. One of the main differences is that gluing parameters are subject to more relations if we allow cycles to be created, and we may not be able to find a “free basis” to build a topological manifold with boundary like $(G_{II})_T/\partial T$ in the proof of Theorem 3.9.

3.8.3. Analogues of planarity. Given an $IBL_\infty$ augmentation $\epsilon$, then we have similar constructions of linearization to construct another $IBL_\infty$ structure $p^{k,l,g}_\epsilon$ such that $p^{k,l,g}_\epsilon = 0$ whenever $l = 0$. As a consequence, we arrive at an $IBL_\infty$ structure in the sense of Definition 3.38. Then we can introduce the analogue of pointed maps and the analogue of orders in the context of $IBL_\infty$ algebras, which in the SFT case considers holomorphic curves passing through a fixed point in the symplectization. Moreover, we will have a grid of orders as the case of torsions above. However, it is a much harder task to find examples with holomorphic curves with higher genus. In fact, we do not know any nontrivial examples with such structures except for punctured Riemann surfaces as exact fillings of a disjoint union of contact circles.
4. Semi-dilations

In this section, we introduce an inner hierarchy called the order of semi-dilation for the \( P = 1 \) case. Note that if \( P(Y) = 1 \), then \( \text{RSFT}(Y) \) admits \( BL_\infty \) augmentations and for any \( BL_\infty \) augmentation \( \epsilon \), we have the order is 1 for any point in \( Y \). Note that \( (\tilde{B}^1 V_\alpha, \ell_\epsilon) \) is the chain complex \((V_\alpha, \ell_\epsilon^i)\) for the linearized contact homology. Since \( P(Y) = 1 \), for any point in \( Y \), we have an class \( x \in H_*(V_\alpha, \ell_\epsilon^1) \) such that \( \ell_\epsilon^1(x) = 1 \).

If the augmentation \( \epsilon_W \) is from an exact filling \( W \), then by \([12, 14]\), the linearized contact homology \( \text{LCH}_s(Y, \epsilon_W) = \text{LCH}_s(W) := H_*(V_\alpha, \ell_\epsilon^1) \) is isomorphic to the equivariant symplectic (co)homology when transversality holds, which as a \( S^1 \)-equivariant theory carries a \( H^*(BS_1) = \mathbb{Q}[u] \)-module structure and fits into the following Gysin sequences,

\[
\cdots \longrightarrow \text{SH}^{2n-3-k}_+(W) \longrightarrow \text{LCH}_k(W) \xrightarrow{u} \text{LCH}_{k-2}(W) \longrightarrow \text{SH}^{2n-2-k}_+(W) \longrightarrow \cdots \nabla
\]

\[
\cdots \longrightarrow \text{SH}^{2n-3-k}_+(W) \longrightarrow \text{SH}^{2n-3-k}_+(W) \longrightarrow \text{SH}^{2n-1-k}_+(W) \longrightarrow \text{SH}^{2n-2-k}_+(W) \longrightarrow \cdots \nabla
\]

As we use homological convention in this paper, \( u \) has degree \(-2\) in the case with a \( \mathbb{Z} \) grading for LCH. And \( u \) has degree 2 for the \( S^1 \)-equivariant symplectic cohomology, which is graded by \( n - \mu_{CZ} \). In fact, the map \( u \) is defined on the linearized contact homology \( H_*(V_\alpha, \ell_\epsilon^1) \) for any augmentation \( \epsilon \). And for any element \( x \in H_*(V_\alpha, \ell_\epsilon^1) \) there exists \( k \in \mathbb{N}_+ \) such that \( u^k(x) = 0 \). In the following, we first recall on the definition of \( u \) for linearized contact homology.

4.1. \( H_*(V_\alpha, \ell_\epsilon^1) \) as a \( \mathbb{Q}[u] \) module. To explain the \( u \)-map, we recall the following two moduli spaces from \([12, \S 7.2]\). In some sense, the following moduli spaces should be viewed as a version of cascades moduli spaces.

4.1.1. \( \mathcal{M}^1_{Y,A}(\gamma^+, \gamma^-, \Gamma^-) \). Let \( \gamma^+, \gamma^- \) be two good Reeb orbits and \( \Gamma^- \) be an ordered multiset of good Reeb orbits of cardinality \( k \geq 0 \). Then an element in \( \mathcal{M}^1_{Y,A}(\gamma^+, \gamma^-, \Gamma^-) \) consists of the following data.

1. A sphere \((\Sigma, j)\), with one positive puncture \( z^+ \) and \( 1+k \) negative punctures \( z^-, z_1^-, \ldots, z_k^- \). We pick an asymptotic marker on \( z^+ \), then by choosing a global polar coordinate on \( \Sigma \backslash \{z^+, z^-\} \), there is a conically induced asymptotic marker on \( z^- \) by requiring it having the same angle as the asymptotic marker at \( z^+ \) in the polar coordinate. We also pick free asymptotic markers on \( z_i^- \) for \( 1 \leq i \leq k \).
2. A map \( u : \Sigma \rightarrow \mathbb{R} \times Y \) such that \( du \circ j = J \circ du \) and \([u] = A\) modulo automorphism and the \( \mathbb{R} \)-translation, where \( \Sigma \) is the \( 2+k \) punctured sphere.
3. \( u \) is asymptotic to \( \gamma^+, \gamma^-, \Gamma^- \) near \( z^+, z^- \) and \( \{z_i^-\}_i \).

Then for an exact cobordism \( X \), we can similarly define \( \mathcal{M}^1_{X,A}(\gamma^+, \gamma^-, \Gamma^-) \) where we do not modulo the \( \mathbb{R} \) translation.

4.1.2. \( \mathcal{M}^2_{Y,A}(\gamma^+, \gamma^-, \Gamma_1^-, \Gamma_2^-) \). Let \( \gamma^+, \gamma^-, \) be three good Reeb orbits and \( \Gamma_1^-, \Gamma_2^- \) be two ordered multisets of good Reeb orbits of cardinality \( k_1, k_2 \geq 0 \). Then an element in \( \mathcal{M}^2_{Y,A}(\gamma^+, \gamma^-, \Gamma_1^-, \Gamma_2^-) \) consists of the following data.

1. Two spheres \((\Sigma, j), (\bar{\Sigma}, j)\), each with one positive puncture \( z^+, \bar{z}^+ \), and \( 1+k_1 \) negative punctures \( z^-, z_1^- , \ldots, z_{k_1}^- \), \( 1+k_2 \) negative punctures \( \bar{z}^-, \bar{z}_1^- , \ldots, \bar{z}_{k_2}^- \) respectively. Each puncture is equipped with an asymptotic marker.
(2) Two holomorphic curves \( u, \tilde{u} \) from \( \Sigma, \tilde{\Sigma} \) to \( \mathbb{R} \times Y \mod \) \( \mathbb{R} \)-translations, such that \( [u] \#_\gamma [\tilde{u}] = A \).

(3) \( u \) is asymptotic to \( \gamma^+, \gamma, \Gamma^1 \) and \( \tilde{u} \) is asymptotic to \( \gamma, \gamma^-, \Gamma^- \).

(4) Let \( L_- \) and \( L_+ \) be two asymptotic markers on \( z^- \) and \( \tilde{z}^+ \) that are induced from the chosen asymptotic markers on \( z^+ \) and \( \tilde{z}^- \) by global polar coordinates\(^{12}\). Then we can define \( ev_{L_-}(u) \), \( ev_{L_+}(\tilde{u}) \) to be the limit point in the \( Y \) component evaluated along the asymptotic markers \( L_-, L_+ \). Then we require \((b_\gamma, ev_{L_-}(u), ev_{L_+}(\tilde{u}))\) is the natural order on im \( \gamma \), where \( b_\gamma \) is the marked point on im \( \gamma \).

We can similarly define \( \mathcal{M}_{X,A}^{2,\downarrow}(\gamma^+, \gamma, \gamma^-, \Gamma^1, \Gamma^-) \) and \( \mathcal{M}_{X,A}^{2,\downarrow}(\gamma^+, \gamma, \gamma^-, \Gamma^-) \) for an exact cobordism \( X \).

The difference is that the former one has \( u \) in \( \tilde{X} \) and the latter one has \( \tilde{u} \) in \( \tilde{X} \). We use \( \overline{\mathcal{M}}^1 \) and \( \overline{\mathcal{M}}^2 \) to denote their compactification. Note that in the case of \( \overline{\mathcal{M}}^2 \), we need to add in the stratum corresponding to the collision of \((b_\gamma, ev_{L_-}(u), ev_{L_+}(\tilde{u}))\) in addition to usual building structures.

Given an dga augmentation \( \epsilon \) to \( \text{CHA}(Y) \), i.e. a map \( \epsilon : V_\alpha \to \mathbb{Q} \), which extends to an algebra map \( \tilde{\epsilon} : \text{CHA}(Y) \to \mathbb{Q} \) such that \( \tilde{\epsilon} \circ \tilde{\partial}^1 = 0 \). Then \( u : V_\alpha \to V_\alpha \) is defined by

\[
\begin{align*}
    u(q_{\gamma^+}) &= \sum_{\gamma^- : [\Gamma^-]} \frac{1}{\kappa_{\gamma^-}} \frac{1}{\mu_{\Gamma^-}} \frac{1}{\kappa_{\Gamma^-}} \#_\gamma \mathcal{M}_{Y,A}^1(\gamma^+, \gamma^-, \Gamma^-) \prod_{\gamma' \in \Gamma^-} \epsilon(\gamma') q_{\gamma^-} \\
                 &+ \sum_{\gamma^- : [\Gamma^-]} \frac{1}{\kappa_{\gamma^-}} \frac{1}{\mu_{\Gamma^-}} \frac{1}{\kappa_{\Gamma^-}} \#_\gamma \mathcal{M}_{Y,A}^2(\gamma^+, \gamma^-, \gamma^-, \Gamma^1, \Gamma^-) \prod_{\gamma' \in \Gamma^- \cup \Gamma^-} \epsilon(\gamma') q_{\gamma^-} \\
\end{align*}
\]

(4.1)

Remark 4.1. The \( \mathcal{M}_{Y,A}^2 \) in [12, \S 7.2] requires modulo an equivalence \((L_-, L_+) \simeq (L_-, \frac{2\pi}{\kappa_\gamma}, L_+, \frac{2\pi}{\kappa_\gamma}) \), i.e. the moduli space should be thought as glued two-level buildings. Here we do not introduce the equivalence, the discrepancy is just the extra \( \frac{1}{\kappa_\gamma} \) in (4.1) compared to [12, (85)]\(^{13}\).

The reason that \( u \) is a chain map from \((V_\alpha, \ell^1_\epsilon) \) to itself follows from the boundary of 1-dimensional \( \overline{\mathcal{M}}_{Y,A}^1 \) and \( \overline{\mathcal{M}}_{Y,A}^2 \). More precisely, the codimension 1 boundary of \( \overline{\mathcal{M}}_{Y,A}^1 \) consists of (1) a level breaking where the lower level does not contain \( z_- \) and (2) a level breaking where the lower level contains \( z_- \). For case one, such contribution is zero when capping \( \Gamma^- \) off with \( \epsilon \) by the relation \( \tilde{\epsilon} \circ \tilde{\partial}^1 = 0 \). For case two, the contribution will cancel with the codimension 1 boundary part of \( \overline{\mathcal{M}}_{Y,A}^2 \) corresponding to the collision of \( ev_{L_-}(u), ev_{L_+}(\tilde{u}) \).

The other parts of codimension 1 boundary of \( \overline{\mathcal{M}}_{Y,A}^2 \) consists of (1) a level breaking of \( u \) where the lower level does not contain \( z_- \), this is again killed by the capping off with \( \epsilon \); (2) A level breaking of \( u \) where the lower level contains \( z_- \), this corresponds to a component of \( \tilde{\ell}_\epsilon^1 \circ u \), where the \( u \) part is contributed by a \( \overline{\mathcal{M}}_{Y,A}^2 \); (3) Similar level breakdowns for \( \tilde{u} \); (4) The collision of \( b_\gamma \) and \( ev_{L_-}(u) \), this corresponds to a component of \( \ell^1_\epsilon \circ u \), where the \( u \) part is contributed by a \( \overline{\mathcal{M}}_{Y,A}^1 \), similarly, the collision of \( b_\gamma \) and \( ev_{L_+}(\tilde{u}) \) is the remaining part of \( u \circ \ell^1_\epsilon \).

Similarly, given an exact cobordism, we can show that chain morphism \( \phi^{1,1}_\epsilon : V_\alpha \to V_{\alpha'} \) is commutative with \( u \) up to homotopy, where the homotopy is defined by \( \overline{\mathcal{M}}_{X,A}^1 \), \( \overline{\mathcal{M}}_{X,A}^{2,\downarrow} \), and \( \overline{\mathcal{M}}_{X,A}^{2,\downarrow} \) by a similar formula to (4.1) with a similar argument.

\(^{12}\)In particular, they may be different from the chosen asymptotic markers on \( z^- \) and \( \tilde{z}^+ \)

\(^{13}\)The extra coefficient \( \mu_\Gamma \) comes from that we consider \( \Gamma \) as an ordered set, and \( \mu_\Gamma \) is the size of the isotropy coming from permutation.
The following proposition asserts that the counts of moduli spaces above can be defined after appropriate setup of virtual machinery. In particular, it follows from the same argument of Theorem 3.9.

**Proposition 4.2.** Let $(Y, \alpha)$ be a non-degenerate contact manifold and $\theta$ be an auxiliary data which is used in defining a $BL_\infty$ structure $p_\theta$.

1. There is an auxiliary data $\theta_u$ for the definition of $u$, such that for any $BL_\infty$ augmentation $\epsilon$ of $(V_\alpha, p_\epsilon)$, we have a map $u_{\theta_u} : H_\ast(V_\alpha, \ell^1_\epsilon) \to H_{\ast-2}(V_\alpha, \ell^1_\epsilon)$ and for any $x \in H_\ast(V_\alpha, \ell^1_\epsilon)$ there exists $k$ such that $u_{\theta_u}^k(x) = 0$.

2. When there is a strict exact cobordism $X$ from $(Y', \alpha')$ to $(Y, \alpha)$ with admissible auxiliary data $\theta, \theta'$, $\theta_u, \theta'_u$ for $\alpha, \alpha'$ and their u-maps respectively, then there exists auxiliary data $\xi$, such that the $\phi^{1,1}_{\xi,\epsilon} : H_\ast(V_\alpha, \ell^1_{\epsilon_0\phi\xi}) \to H_\ast(V_{\alpha'}, \ell^1_\epsilon)$ commutes with the u-maps for any $BL_\infty$ augmentation $\epsilon$ for $(V_{\alpha'}, p_{\epsilon'})$.

3. For any $k \in \mathbb{R}_+$, there exists $k\theta_u$, such that $u_{k\theta_u}$ is canonical identified with $u_{\theta_u}$.

That $u_{\theta_u}^k(x) = 0$ follows from that $u$ strictly decreases the contact action for non-degenerate $\alpha$.

**Definition 4.3.** Let $Y$ be a contact manifold with $P(Y) = 1$, then we define the order of semi-dilation $SD(Y)$ as follows,

$$SD(Y) := \max \left\{ \min \left\{ k \left| u^{k+1}(x) = 0, x \in H_\ast(V_\alpha, \ell^1_\epsilon), \ell^1_{\ast,\epsilon}(x) = 1 \right. \right\} \mid \theta \in Y, \epsilon \in Aug_{Q}(V_\alpha) \right\}$$

$SD$ can be defined on all of $\mathcal{C}on$, by declaring $SD(Y) = \infty$ if $P(Y) \geq 2$ and $SD(Y) = -1$ for $P(Y) = 0$.

**Proposition 4.4.** For those contact manifolds $Y$ with $P(Y) = 1$, the assignment of $SD(Y)$ is a monoidal functor from the full subcategory of $\mathcal{C}on$ to $\mathbb{N} \cup \{\infty\}$, where the monoidal structure on $\mathbb{N}$ is defined by $a \otimes b = \max\{a, b\}$.

**Proof.** That $SD(Y)$ is independent of all choices follows from the same argument of Proposition 3.11. The monoidal structure follows from $H_\ast(V \oplus V', \ell^1_{\epsilon'} \oplus \ell^1_{\epsilon}) = H_\ast(V', \ell^1_{\epsilon'}) \oplus H_\ast(V', \ell^1_{\epsilon})$ as $Q[u]$-modules.

$k$-dilation and $k$-semi-dilation were introduced in [78] as structures on $S^1$-equivariant symplectic cohomology, which are generalizations of symplectic dilation of Seidel-Solomon [69]. More precisely, an exact domain $W$ carries a $k$-dilation, if there is a class $x \in SH^s_{+,s1}(W)$ such that $x$ is sent to 1 by $SH^s_{+,s1}(W) \to H^s_{S^1}(W)$ and $u^{k+1}(x) = 0$, where $H^s_{S^1}(W) := H^s(W) \otimes_{Q[u]} (Q[u, u^{-1}]/[u])$. $W$ carries a $k$-semi-dilation iff $x$ is sent to 1 in $SH^s_{+,s1}(W) \to H^s_{S^1}(W) \to H^s_{S^1}(\partial W)$ and $u^{k+1}(x) = 1$. Under the isomorphism $SH^s_{+,s1}(W) = LCH_{2n-3-s}(W) [12, 14]$, the element we are looking for in Definition 4.3 is $x \in SH^s_{+,s1}(W)$ that is sent to one in $SH^s_{+,s1}(W) \to H^s_{S^1}(W) \to H^0(W)$ with $u^{k+1}(x) = 0$, where the last map is the natural projection. It is natural to expect that examples with nontrivial $k$-dilation found in [78] and we will show in §7 that $SD$ is surjective.

**Remark 4.5.** A priori, the semi-dilation used in this paper is weaker than the semi-dilation in [78] for exact fillings. However, we do not know any example justifying that there are differences between those two definitions.

**Proof of Theorem A.** It follows from Proposition 3.11, Proposition 3.23, and Proposition 4.4, with the monoidal stricture explained therein.
Remark 4.6. In fact, $\infty^{APT}$, $\infty^{SD}$ are the only two elements in $\mathcal{H}$ that we do not know if it is in the image of $H_{cx}$. $H_{cx}(Y) = \infty^{APT}$ corresponds to that RSFT($Y$) has no $BL_{\infty}$ augmentation while RSFT($Y$) has infinite torsion. Note that a $BL_{\infty}$ augmentation is essentially a solution to a family of algebraic equations (in an infinite dimensional space). Using that $\mathbb{Q}$ is not algebraically closed, it is easy to define a (finite dimensional) $BL_{\infty}$ algebra over $\mathbb{Q}$ with no augmentation and infinite algebraic planar torsion. However, it is unclear how to construct a geometric example. It is also an interesting question on obstructions to $BL_{\infty}$ augmentations beyond torsion besides using that $\mathbb{Q}$ is not algebraically closed.

4.2. Planarity and semi-dilation for fillings. Since fillings of a contact manifold give rise to $BL_{\infty}$ augmentations, one can define planarity and order of semi-dilation for fillings as follows.

Definition 4.7. Let $W$ be an exact domain, we define $P(W)$ as

$$P(W) := \max \{ O(V, p, \varepsilon_W) \mid o \in Y, \alpha, \varepsilon_W \},$$

the maximal is taken over all non-degenerate contact forms $\alpha$ and all $BL_{\infty}$ augmentations $\varepsilon_W$ from the filling (i.e. for all choices of auxiliary data). Similarly, we define $SD(W)$ in the case of $P(W) = 1$,

$$SD(W) := \max \{ \min \left\{ k \left| u^{k+1}(x) = 0, x \in H'(V, \ell_{\varepsilon_W}), \ell_{\varepsilon_W}(x) = 1 \right\} \mid o \in Y, \alpha, \varepsilon_W \} \}.$$

We define $SD(W) = \infty$ if $P(W) > 1$.

Remark 4.8. In fact, the choice of $\alpha, \varepsilon_W$ is redundant. $P(W)$ and $SD(W)$ can be computed using just one $\alpha$ and $\varepsilon_W$. However this requires introducing the notation of homotopy between $BL_{\infty}$ augmentations for linearized theories, which will be carried out in the future. The trick in Proposition 3.11, 3.23, 4.4 cannot not help dropping the dependence on $\alpha, \varepsilon_W$, since it requires that the composition of the morphism from an exact cobordism $X$ and the augmentation from an exact filling $W$ is (homotopic to) an augmentation from $X \circ W$. However this involves neck-stretching and is essentially a $BL_{\infty}$ homotopy.

Planarity $P(W)$ can also be computed from (5.1) by Proposition 5.14.

Claim 4.9. $P$ and $SD$ are functors from $\mathcal{C}_{un}$ to $\mathbb{N}_+$. $SD(W) \leq k$ if and only there exists $x \in SH^*_{+, S^1}(W)$ that is mapped to 1 in $SH^*_{+, S^1}(W) \to H^{k+1}(S^1) \to H^0(W)$ and $u^{k+1}(x) = 0$. In particular, if $W$ carries a $k$-semi-dilation, then $SD(W) \leq k$.

We leave it as a claim, because the proof of functoriality of $P$ and $SD$ requires building the full package RSFT to discuss linearized theory up to homotopy, and the second claim requires proving the isomorphism $SH^*_{+, S^1}(W) = LCH_{2n-3-\varepsilon}(W)$ for any exact domain $W$, i.e. implementing virtual machinery for [12, 14].

5. Lower bounds for planarity

As explained in §3, the curve responsible for finiteness of planarity is a curve with multiple positive punctures and a point constraint. Since planarity does not depend on the choice of the point, one should expect that finiteness of planarity implies uniruledness. In this section, we will prove such implication and a lower bound for planarity. We first recall the notion of uniruledness from [55].

5.1. Order of uniruledness.

Definition 5.1 ([55, §2]). Let $(W, \lambda)$ be an exact domain. A $d\lambda$-compatible almost complex structure $J$ on $W$ is convex iff there is a function $\phi$ such that

1. $\phi$ attains its maximum on $\partial W$ and $\partial W$ is a regular level set,
(2) \( \lambda \circ J = d\phi \) near \( \partial W \).

**Definition 5.2** ([55, Definition 2.2]). Let \( k > 0 \) be an integer and \( \Lambda > 0 \) a real number. We say that an exact domain \((W, \lambda)\) is \((k, \Lambda)\) uniruled if, for every convex almost complex structure \( J \) on \( W \) and every \( p \in W^\circ \) (the interior of \( W \)) where \( J \) is integrable near \( p \), there is a proper \( J \)-holomorphic map \( u : S \rightarrow W^\circ \) passing through \( p \) and the following holds,

\[
\begin{align*}
(1) \quad & \text{\( S \) is a genus 0 Riemann surface and the rank of } H_1(S; \mathbb{Q}) \leq k - 1, \\
(2) \quad & \int_S u^*d\lambda \leq \Lambda.
\end{align*}
\]

We say \( W \) is \( k \)-uniruled if \( W \) is \((k, \Lambda)\) uniruled for some \( \Lambda > 0 \).

The number \( \Lambda \) depends on the Liouville form \( \lambda \) which is not relevant for our purpose. However the number \( k \) only depends on the Liouville structure up to homotopy.

**Definition 5.3.** Let \( W \) be an exact domain, we define the order of uniruledness

\[
U(W) := \min\{k|W \text{ is } k \text{ uniruled.}\}
\]

The following was proven by McLean [55].

**Proposition 5.4.** \( U \) is a functor from \( \mathcal{C}on_* \) to \( \mathbb{N}_+ \cup \{\infty\} \).

**Proof.** Let \( V \subset W \) be an exact subdomain, then \( U(V) \leq U(W) \) by [55, Proposition 3.1]. It is clear from definition that \( U(V, \lambda) = U(V, t\lambda) \) for \( t > 0 \). Since for any Liouville structure \( \theta \) on \( V \) that is homotopic to \( \lambda \), we have exact embeddings \((V, t^{-1}\lambda) \subset (V, \theta) \subset (V, t\lambda) \) for \( t \gg 0 \), therefore \( U \) is a well-defined functor on \( \mathcal{C}on_* \). \( \square \)

**Remark 5.5.** A worth noting point is that the definition and functorial property of \( U \) do not depend on any Floer theory. However \( U \) gives a measurement of “complexity” of exact domains. By [78, Theorem 3.23], the exist of \( k \)-(semi)-dilation implies that the order of uniruledness is 1. Hence the order of (semi)-dilation in [78, Corollary D] is a refined hierarchy in \( U = 1 \).

For an affine variety \( V \), we define the order of algebraically uniruledness \( AU(V) \) be the minimal number \( k \) such that \( V \) is algebraically \( k \) uniruled, i.e. through every generic \( p \in V \) there is a polynomial map \( S \rightarrow A \) passing through \( p \) with \( S \) is a punctured \( \mathbb{C}P^1 \) with at most \( k \) punctures.

**Proposition 5.6** ([55, Theorem 2.5]). Let \( V \) be an affine variety then \( U(V) \geq AU(V) \).

**Example 5.7.** Let \( S_k \) be the \( k \)-punctured sphere. Then \( U(S_k) = k \). Let \( \Sigma_{g,k} \) be the \( k \)-punctured genus \( g \geq 1 \) surface, then \( U(\Sigma_{g,k}) = \infty \). It is clear that \( S_k \) embeds exactly into \( S_{k+1} \). However \( S_{k+1} \) can only be embedded in \( S_k \) symplectically but not exactly.

In general we have the following.

**Theorem 5.8.** We have \( U((S_k)^n) = k \) and \( U((\Sigma_{g,k})^n) = \infty \) for \( g \geq 1 \). In particular, \( U \) is a surjective functor in any dimension \( \geq 2 \).

**Proof.** Note that \( S_k^n \) has a projective compactification \((\mathbb{C}P^1)^n \), using the fact for any compatible almost complex structure \( J \) there is a holomorphic curve passing any fixed point in the class \([\mathbb{C}P^1 \times \{pt\} \times \ldots \times \{pt\}]\) and intersecting each divisor \( \{p_i\} \times (\mathbb{C}P^1)^{n-1} \) exactly once for \( 1 \leq i \leq k \), where \( p_i \) is the \( i \)th puncture. We may assume \( J \) is an extension of a convex almost complex structure on \( S_k^n \). Therefore \( U((S_k)^n) \leq k \), by neck-stretching.
On the other hand, when we view \((S_k)^n\) and \((\Sigma_{g,k})^n\) as affine varieties with the product complex structure. We know a rational algebraic curve in \((S_k)^n\) and \((\Sigma_{g,k})^n\) projects to each factor a rational algebraic curve. Then every rational curve must have at least \(k\) punctures. Therefore \(AU((S_k)^n) = k\) and \(AU((\Sigma_{g,k})^n) = \infty\), and the claim follows from Proposition 5.6.

As a consequence, we find in each dimension a nested sequence of exact domains \(V_1 \subset V_2 \ldots\), such that \(V_i\) cannot be embedded into \(V_j\) exactly if \(i > j\). Sequences with such property in \(\dim \geq 10\) were also obtained in [50, Corollary 1.5].

**Remark 5.9.** In §6, we will show that \(P(\partial(S_k)^n) = k\) if \(n \geq 2\). Therefore not only there is no exact embedding from \((S_{k+1})^n\) to \((S_k)^n\), but also there is no exact cobordism from \(\partial(S_{k+1})^n\) to \(\partial(S_k)^n\).

**Remark 5.10.** From \(U\) on \(\mathcal{C}on_{\Lambda}\), we can build a functor \(U_\partial\) on \(\mathcal{C}on\) as follows
\[
U_\partial(Y) := \max\{U(W) \mid W \text{ is an exact filling of } Y\}.
\]

Then Corollary 5.15 below implies that \(U_\partial \leq P\). The equality does not always hold. For example, \(U_\partial(\mathbb{P}^{2n-1}, \xi_{std}) = 0\) for \(n \neq 2^k\) by [79], but \(P(\mathbb{P}^{2n-1}, \xi_{std}) = 1\) when \(n \geq 3\) by Theorem 7.28. Those discrepancies come from the difference between fillings and augmentations. It is possible to generalize the notion of order of uniruledness \(U, U_\partial\) to strong fillings or even weak fillings, but we will not pursue this in this paper.

In the following, we introduce an alternative definition of \(k\)-uniruledness.

**Definition 5.11.** Let \((W, \lambda)\) be an exact filling with a non-degenerate contact boundary, we say the completion \(\widehat{W}\) is \(k\)-uniruled if there exists a \(\Lambda > 0\), such that for every \(p \in W^\circ\) and every admissible almost complex structure \(J\) that is integrable near \(p\), there is a rational holomorphic curve passing through \(p\) with at most \(k\) positive punctures and contact energy of the curve is at most \(\Lambda\).

**Proposition 5.12.** An exact filling \((W, \lambda)\) is \(k\)-uniruled iff \(\widehat{W}_e\) is \(k\)-uniruled, where \(W_e\) is Liouville homotopic to \(W\) with a non-degenerate contact boundary.

**Proof.** We first show that \((W, \lambda)\) is \(k\)-uniruled implies \(\widehat{W}_e\) is \(k\)-uniruled. WLOG, we can take \(W_e \subset W\), since we can rescale \(W_e\). By assumption there is a \(\Lambda > 0\) such that for any \(p \in W^\circ\) and any \(J\) integrable near \(p\) and convex near \(\partial W\), there is a \(J\)-rational curve \(u : S \to W\) with \(\int_S u^*d\lambda < \Lambda\) and \(H_1(S; \mathbb{Q}) \leq k - 1\). In particular, we can choose \(J\) to be cylindrical convex near \(\partial W_e\). Then by applying neck-stretching along \(\partial W_e\), we must have a rational holomorphic curve \(u : S \to \widehat{W}_e\) passing through \(p\) with contact energy smaller than \(\Lambda\). We know that \(S_i\) is a punctured sphere, as \(\partial W_e\) is non-degenerate. It sufficient to prove rank \(H_1(S; \mathbb{Z}) = \rank H_1(S; \mathbb{Q}) \leq k - 1\). Assume otherwise, we know that \(H_1(S; \mathbb{Z}) \to H_1(S; \mathbb{Q})\) is not injective, for otherwise, we have rank \(H_1(S; \mathbb{Q}) \geq \rank H_1(S; \mathbb{Z}) \geq k\). Therefore we find a class \([\gamma] \in H_1(S; \mathbb{Z})\), such that \([\gamma]\) is represented by a disjoint union \(\gamma\) of possibly multiply covered loops around punctures of \(S_i\) and there is an immersed surface \(A\) in \(S\setminus S_e\). Then in the fully stretched case, \(u|_A\) corresponds to a holomorphic building with only negative punctures, which is impossible for energy reasons.

Now we assume \(\widehat{W}_e\) is \(k\)-uniruled. WLOG, we can assume \(W \subset W_e\). By [83, Proposition 5.3], any convex almost complex structure on \(W\) can be extended to an admissible almost complex structure on \(\widehat{W}_e\). By assumption, there is a rational curve \(u : S \to \widehat{W}_e\) passing through the chosen point \(p \in \widehat{W}_e\) with \(S\) an at most \(k\) punctured sphere and the contact energy of \(u\) is at most \(\Lambda\). Let \(S'\) be the connected component of \(u^{-1}(W^\circ)\) containing the point mapped to \(p\). It clear that the area of \(u|_{S'}\) is bounded by \(\Lambda\). We claim that \(H_1(S'; \mathbb{Z}) \to H_1(S; \mathbb{Z})\) is injective. For otherwise, there is a class \(A \in H_2(S, S'; \mathbb{Z})\) mapped to a nontrivial element by \(H_2(S, S'; \mathbb{Z}) \to H_1(S'; \mathbb{Z})\). Then we can find a \(S'' \subset S'\) such that \(\lambda \circ J = d\phi\) on \(u|_{S'' \setminus S'}\), where
\(\phi\) is the function in Definition 5.1. Then by excision, we have \(A\) represented by an immersed surface in \(S\backslash S''\) not contained completely in \(S'\backslash S''\) with boundary in \(S'\backslash S''\). Let \(\hat{\phi}\) be the extension of \(\phi\) on \(\hat{W}_\epsilon\) by \([83, \text{Proposition 5.3}]\). In particular, the maximum principle holds for \(\hat{\phi}\). Then we reach at a contradiction, since \(\hat{\phi}(u)|_{\partial A} < \hat{\phi}(u)|_{A}\). Since \(Q\) is flat, we know that \(H_1(S'; \mathbb{Q}) \to H_1(S; \mathbb{Q})\) is also injective, hence \(\operatorname{rank} H_1(S'; \mathbb{Q}) \leq \operatorname{rank} H_1(S; \mathbb{Q}) \leq k - 1\).

\(\square\)

5.2. Uniruledness and planarity. The main theorem of this section is following.

**Theorem 5.13.** If \(P(Y) = k\), then any exact filling of \(Y\) is \(k\)-uniruled.

Since an exact filling \(W\) gives rise to a \(BL_\infty\) augmentation \(\epsilon_W\) over \(\mathbb{Q}\). As a consequence we have a chain morphism \(\hat{\epsilon}_{\bullet, \epsilon_W} : BV \to Q\) after fixing a point \(o\) in \(Y\) and an auxiliary data. We can define a different map \(\eta_W : BV \to Q\) by

\[
\eta_W(q^{\Gamma^+}) = \sum_A \frac{1}{\mu_{\Gamma^+}} \# \mathcal{M}_{W, A, \gamma}(\Gamma^+, 0)
\]

for a fixed point \(p \in W\) such that \(p, o\) are in the same connected component of \(W\), and \(|\Gamma^+| = k\).

**Proposition 5.14.** \(\eta_W\) is a chain morphism and is homotopic to \(\hat{\epsilon}_{\bullet, \epsilon_W}\) with appropriate choices of auxiliary data, where \(\epsilon_W\) is the augmentation from \(W\). Moreover \(\eta_W\) is compatible with the word length filtration, i.e. \(\eta_W\) is a chain map from \(B^k V\) for any \(k \geq 1\), and is homotopic to \(\hat{\epsilon}_{\bullet, \epsilon_W}\) on \(B^k V\).

**Proof.** Let \(\gamma\) be a path in \(W\) connecting \(p\) to \(o\) and we use \(\hat{\gamma}\) to denote the completion of \(\gamma\) in \(\hat{W}\). Then the homotopy \(H : BV \to Q\) is defined by

\[
H(q^{\Gamma^+}) = \sum_A \frac{1}{\mu_{\Gamma^+}} \# \mathcal{M}_{W, A, \gamma}(\Gamma^+, 0),
\]

The realization of those operators using virtual techniques is similar to \(\phi_\bullet\) in (4) of Theorem 3.9 in §3.6. The homotopy relation comes from the boundary of 1-dimensional \(\mathcal{M}_{W, A, \gamma}(\Gamma^+, 0)\). It is clear that both \(\eta_W\) and \(H\) are compatible with the word length.

**Proof of Theorem 5.13.** The theory of \(BL_\infty\) algebra considered for contact manifold is equipped with a filtration by the contact action, where the action \(A(q_\gamma)\) of a generator is \(\int \gamma^* \alpha\). Then the action can be extended to \(EV\) and \(SV\) uniquely by the following two property \(A(x \cdot \gamma \circ y) = A(x) + A(y)\) and \(A(x + y) \leq \max\{A(x), A(y)\}\). Then all of the operators for contact manifolds and exact cobordisms will decrease the action as explained in §3.7. Although it may not be true that the spectral invariant for \(P(Y) = k\) is bounded for all \(BL_\infty\) augmentations. But for the augmentation \(\epsilon\) from an exact filling \(W\), we have the spectral invariant is bounded, i.e. there is a \(\Lambda > 0\) such that there is \(x \in B^k V\) with \(A(x) \leq \Lambda\), \(\hat{\gamma}(x) = 0\), and \(\hat{\bullet}_\epsilon(x) = 1\). Then by Proposition 5.14, we have \(\eta_W(x) = 1\). By Axiom 3.25, we must have the geometric \(\mathcal{M}_{W, A, \gamma}(\Gamma^+, 0)\) is not empty for some \(\Gamma^+\) with \(|\Gamma^+| \leq k\) and \(\sum_{\gamma \in \Gamma^+} \int \gamma^* \alpha \leq \Lambda\). This shows that \(\hat{W}\) is \((k, \Lambda)\) uniruled, by Proposition 5.12, \(W\) is \(k\)-uniruled.

**Theorem 5.13** provides a lower bound for \(P\).

**Corollary 5.15.** Let \(W\) be an exact filling of \(Y\), then \(P(Y) \geq U(W)\). If \(W\) is an affine variety, then \(P(Y) \geq AU(W)\).
6. Upper bounds for planarity

The strategy for obtaining an upper bounds $P(Y) \leq k$ on the planarity of a contact manifold $(Y, \xi)$ is via the following algebraic-geometric condition:

**Lemma 6.1.** Let $(Y, \xi)$ be a contact manifold. Assume the following holds:

$(\ast)_k$ There exists a point $o \in \mathbb{R} \times Y$, a contact form $\alpha$ for $\xi$, a choice of $\alpha$-compatible cylindrical almost complex structure $J$ on $\mathbb{R} \times Y$, and some collection $\Gamma = (\gamma_1, \ldots, \gamma_k)$ of precisely $k$ distinct, non-degenerate and simply-covered $\alpha$-Reeb orbits, for which the following holds:

1. If $\Gamma^+ \subseteq \Gamma$ and $\Gamma^- \neq \emptyset$, then $\mathcal{M}_{Y, A}(\Gamma^+, \Gamma^-) = \emptyset$ for every homology class $A$.
2. The moduli space $\mathcal{M}_{Y, A, o}(\Gamma, \emptyset)$ is transversely cut out for every $A$ with expected dimension $0$.
3. For some choice of coherent orientations, the algebraic count of the $k$-punctured spheres in $\bigcup_{A, \dim = 0} \mathcal{M}_{Y, A, o}(\Gamma, \emptyset)$ is nonzero.

Then $P(Y) \leq k$.

**Proof.** By the first property, we have $\mathcal{M}_{Y, A, o}(\Gamma^+, \Gamma^-) = \emptyset$ for any $\Gamma^+ \subseteq \Gamma$ and $\Gamma^- \neq \emptyset$. Therefore by Axiom 3.25, the second and third condition implies that $\hat{l}_\epsilon(q^\Gamma) \neq 0$ for any augmentation $\epsilon$ (if there is no augmentation, then $P(Y) = 0$ by definition). Moreover by the first property, we have $q^\Gamma$ is closed in $(\mathcal{S}Y, \hat{\ell}_\epsilon^\Gamma)$ for any augmentation $\epsilon$. Then the claim follows.

**Remark 6.2.** In some situations, we will need to relax (1) to the following:

1. If $\Gamma^+ \subseteq \Gamma$ and $\Gamma^- \neq \emptyset$, then $\mathcal{M}_{Y, A}(\Gamma^+, \Gamma^-) = \emptyset$ for every homology class $A$, unless $|\Gamma^+| = |\Gamma^-| = 1$. In addition, we have $\mathcal{M}_{Y, A, o}(\{\gamma^+\}, \{\gamma^-\}) = \emptyset$ for any $\gamma^+ \in \Gamma^+$ and any $\gamma^-$. Moreover the $\mathcal{M}_{Y, A}(\{\gamma^+\}, \{\gamma^-\})$ is cut out transversely for any $\gamma^+ \in \Gamma^+$ and any $\gamma^-$ and the compactification $\overline{\mathcal{M}}_{Y, A}(\{\gamma^+\}, \{\gamma^-\})$ only involves cylinders. Finally $\#\mathcal{M}_{Y, A}(\{\gamma^+\}, \{\gamma^-\}) = 0$ when the expected dimension is 0.

We also need to modify (2) accordingly.

2. The moduli space $\mathcal{M}_{Y, A, o}(\Gamma, \emptyset)$ is transversely cut out for every $A$ with expected dimension 0. Assume $\{\gamma^+_1, \ldots, \gamma^+_j\} \subset \Gamma$, such that there are $\gamma^-_1, \ldots, \gamma^-_j$ and $A_i, \ldots, A_j$ with $\mathcal{M}_{Y, A_i}(\gamma^+_i, \gamma^-_i) \neq \emptyset$ for $1 \leq i \leq j$, then we have $\mathcal{M}_{Y, A', o}(\Gamma', \emptyset) = \emptyset$, where $\Gamma' = (\Gamma \setminus \{\gamma^+_1, \ldots, \gamma^+_j\}) \cup \{\gamma^-_1, \ldots, \gamma^-_j\}$ and the expected dimension of $\mathcal{M}_{Y, \#A, \#A', o}(\Gamma, \emptyset)$ is zero.

Along with (3) and (2), we have $P(Y) \leq k$.

**Proof.** By (1), we have $\mathcal{M}_{Y, A}(\Gamma^+, \Gamma^-) = \emptyset$ unless $|\Gamma^+| = |\Gamma^-| = 1$. And when $|\Gamma^+| = |\Gamma^-| = 1$, we have $\overline{\mathcal{M}}_{Y, A}(\Gamma^+, \Gamma^-)$ is cut out transversely with algebraic count 0 when the expected dimension is 0. Therefore by Axiom 3.25, $q^\Gamma$ is a closed class for any augmentation. Since we know that $\mathcal{M}_{Y, A, o}(\Gamma^+, \Gamma^-) = \emptyset$ as long as $\Gamma^- \neq \emptyset$ by (1). We know that $\hat{l}_\epsilon(q^\Gamma)$ is solely contributed by $\overline{\mathcal{M}}_{Y, A, o}(\Gamma^+, \emptyset)$. Moreover, in the compactification $\overline{\mathcal{M}}_{Y, A, o}(\Gamma^+, \emptyset)$, we only need to worry about buildings containing upper levels without the point constraint. Then by (2), either the upper level is empty or the lower level is empty. In particular, we have $\overline{\mathcal{M}}_{Y, A, o}(\Gamma^+, \emptyset) = \mathcal{M}_{Y, A, o}(\Gamma^+, \emptyset)$ and it is cut out transversely with total algebraic count $\neq 0$. As a consequence, $\hat{l}_\epsilon(q^\Gamma) \neq 0$ for any augmentation by Axiom 3.25. Hence $P(Y) \leq k$.

In practice, we usually construct a homological foliation on the symplectization $\mathbb{R} \times Y$ (i.e. a moduli space of curves for which the above algebraic-geometric condition holds for generic points $o$), with strong
uniqueness properties, although this is stronger than needed. We shall do this for a sufficiently large class of examples, as follows.

6.1. Iterated planar open books.

Definition 6.3 ([2]). An iterated planar Lefschetz fibration \( f : (W^{2n}, \omega) \to \mathbb{D}^2 \) on a 2n-dimensional Weinstein domain \((W^{2n}, \omega)\) is an exact symplectic Lefschetz fibration satisfying the following properties:

1. There exists a sequence of exact symplectic Lefschetz fibrations \( f_i : (W^{2i}, \omega_i) \to \mathbb{D}^2 \) for \( i = 2, \ldots, n \) with \( f = f_n \).
2. The total space \((W^{2i}, \omega_i)\) of \( f_i \) is a regular fiber of \( f_{i+1} \), for \( i = 2, \ldots, n - 1 \).
3. \( f_2 : (W^4, \omega_2) \to \mathbb{D}^2 \) is a planar Lefschetz fibration, i.e. the regular fiber of \( f_2 \) is a genus zero surface with nonempty boundary, which we denote by \( W^2 \).

Definition 6.4 ([2]). An iterated planar open book decomposition of a contact manifold \((Y^{2n+1}, \xi)\) is an open book decomposition for \( Y \) whose page \( W \) admits an iterated planar Lefschetz fibration, which supports the contact structure \( \xi \) in the sense of Giroux. We say that \((Y, \xi)\) is iterated planar (IP).

If the number of boundary components of \( W^2 \) in the above definition is \( k \), we say that \((Y, \xi)\) is \( k \)-iterated planar or \( k \)-IP. We remark that the collection of IP contact manifolds is already a large class of examples, as e.g. the fundamental group is not an obstruction in any fixed dimension at least 5 [4, Theorem 1.4].

Theorem 6.5. Let \((Y, \xi)\) be a \( k \)-IP contact manifold. Then \( \text{P}(Y) \leq k \).

Proof. We proceed by induction on dimension.

If \( \dim Y = 3 \), then an IP contact 3-manifold is simply a planar contact 3-manifold. Fix a choice of planar open book supporting the contact structure, with page a sphere with \( k \)-disks removed, and so with binding consisting of \( k \) circles. One then constructs an adapted Giroux form, so that each component of the binding is a non-degenerate and simply-covered orbit, and a holomorphic open book as e.g. in [71]. This provides a Fredholm-regular foliation of \( \mathbb{R} \times Y \) whose leaves are either trivial cylinders over the binding, or holomorphic Fredholm-regular \( k \)-punctured spheres projecting to pages and asymptotic to the binding. One can prove via standard 4-dimensional arguments coming from Siefring intersection theory (the same as in higher-dimensions, as used below), that any curve whose positive asymptotics are a subset of the binding, is a leaf of this foliation. While this a priori holds for an almost complex structure which is compatible with a SHS deforming the contact form, one may perturb this SHS to nearby contact data without changing the isotopy class of the contact form. After perturbing the almost complex structure to make it compatible with this nearby contact data and generic, the curves in the foliation survive by Fredholm regularity, and the uniqueness statement still holds if the perturbation is small enough (as follows easily from a SFT compactness argument). In particular, (1)\(_k \) and (2)\(_k \) in Lemma 6.1 hold, for the perturbed \( J \). In this case the geometric (and hence the algebraic) count of these curves with a point constrain is 1 for any generic point \( o \), and so (3)\(_k \) is also satisfied. We fix such a \( o \) for which we have this uniqueness property.

If \( \dim Y \geq 5 \), we fix an IP open book \( \pi : Y \setminus B \to S^1 \) supporting \( \xi \), with binding \( B \subset Y \), a codimension-2 contact submanifold. Since \( B \) is also \( k \)-IP if \( Y \) is, we may assume by induction that (\( * \)\(_k \) holds for \( B \). We may then extend the Giroux contact form on \( B \) for which (\( * \)\(_k \) holds to a Giroux contact form on \( Y \), in such a way that all \( k \) Reeb orbits \( \Gamma = (\gamma_1, \ldots, \gamma_k) \) from the induction step are still non-degenerate orbits in \( Y \). On the other hand, the holomorphic open book construction can also be done in arbitrary dimensions (again, after deforming the Giroux form away from \( B \) to a stable Hamiltonian structure, cf. [19, Appendix A], [58, 59]). The choice of almost complex structure can be taken to agree with the one from the inductive step along
we have a decomposition
We first note that
Proof. Theorem 6.7. We have
where \( Y \) lies in making the virtual machinery compatible with the geometry for general fibrations. The geometric intuition behind the conjecture is clear and was used in Theorem 6.5, the difficulty

An appeal to Lemma 6.1 finishes the proof. □

It is clear from the definition that the Weinstein conjecture holds for contact manifolds with finite planarity. In the case of iterated planar open books, the Weinstein conjectures was proven for dimension 3 in [1] and higher dimensions in [2, 5]. Theorem 6.5 proves the Weinstein conjecture for a slightly larger class of contact manifolds than iterated planar. In view of the proof of Weinstein conjecture, Theorem 6.5 is of the same spirit as the proofs in [1, 2, 5]. However, more importantly, Theorem 6.5 endows the holomorphic curve with SFT meaning. We further remark that the proof of the above theorem in the 5-dimensional case actually provides a foliation, as opposed to a homological one, as shown in [59].

Theorem 6.5 can be viewed as a special case of the following conjecture.

**Conjecture 6.6.** Let \( Y \) be an open book whose page is \( W \), then \( P(Y) \leq P(W) \) and \( SD(Y) \leq SD(W) \).

In the context of semi-dilations in symplectic cohomology, such claim was proven in [78] for Lefschetz fibrations. The geometric intuition behind the conjecture is clear and was used in Theorem 6.5, the difficulty lies in making the virtual machinery compatible with the geometry for general \( Y \) and \( W \).

**6.2. Trivial planar SOBDs.** We now consider a related example as to the ones considered above. Fix \((S_k, d\lambda)\) a sphere with \( k \)-disks removed together with a Liouville form \( \lambda \), and let \((M, d\alpha)\) be any Liouville domain. Define \((V := S_k \times M, \omega = d(\lambda + \alpha))\), endowed with the product Liouville domain structure. Let \((Y = \partial V, \xi = \text{ker}(\alpha + \lambda))\), the contact manifold filled by \( V \).

**Theorem 6.7.** We have \( P(Y) \leq k \).

**Proof.** We first note that \( Y \) admits a supporting (trivial) SOBD, as considered in [52, 58]. In other words, we have a decomposition
\[
Y = Y_S \cup Y_P,
\]
where \( Y_S = \partial S_k \times M \) is the spine and \( Y_P = S_k \times \partial M \) is the paper, and we have trivial fibrations
\[
\pi_S : Y_S \to M,
\]
\[
\pi_P : Y_P \to \partial M.
\]
We view the first one as a contact fibration over a Liouville domain, and the second, as a Liouville fibration over a contact manifold (its fibers are called the pages).

We then construct a holomorphic foliation of $\mathbb{R} \times Y$, i.e. we make our SOBD holomorphic (this is the analogous construction for trivial SOBDs, to the one considered in the proof of Theorem 6.5 for the case of open books; see [58] for full details), as follows. By choosing a Morse function $H$ on the vertebrae $M$ (the base of the spine) which vanishes near $Y_P$ and has a unique maximum and no minimum, we may perturb the contact form along $Y_S$ to $e^{\epsilon H}(\alpha + d\theta)$, where $\theta \in S^1$ parametrizes each connected component of $\partial S_k$. As explained in [58], each critical point $p \in M$ of $H$ corresponds to a Reeb orbit of the form $\gamma_p = S^1 \times \{p\}$ (one for each of the $k$ components of $Y_S$; they are non-degenerate by non-degeneracy of $H$). One then deforms the contact form to a stable Hamiltonian structure which coincides with $H_{\partial S}$ of $(S, \alpha, \omega)$ (the base of the spine) which vanishes near $Y_P$ and has a unique maximum and no minimum, we may perturb the contact form along $Y_S$ to $e^{\epsilon H}(\alpha + d\theta)$, where $\theta \in S^1$ parametrizes each connected component of $\partial S_k$. As explained in [58], each critical point $p \in M$ of $H$ corresponds to a Reeb orbit of the form $\gamma_p = S^1 \times \{p\}$ (one for each of the $k$ components of $Y_S$; they are non-degenerate by non-degeneracy of $H$). One then deforms the contact form to a stable Hamiltonian structure which coincides with $H = (\alpha, d(\alpha + \lambda))$ on $Y_P$, and so its kernel there is $TS_k \oplus \ker \alpha$, tangent to $S_k$. After this, one can construct a compatible cylindrical almost complex structure, for which there exist a foliation of $\mathbb{R} \times Y$ by Fredholm regular curves whose leaves come in three types: trivial cylinders $\mathbb{R} \times \gamma_p$ over critical points, holomorphic pages (which are Liouville completions of $S_k$, project to the pages, and are asymptotic to the trivial cylinders at infinity), and holomorphic flow-line cylinders (which project to $M$ as Morse flow lines of $H$, and are also asymptotic to trivial cylinders). A generic point $p$ on $Y_S$ lies in a flow-line which enters $Y_S$ from $Y_P$ and reaches the maximum. The corresponding flow-line cylinder is an asymptotic of a generic page, which has Fredholm index $\dim M$ (and so has index zero after including a point constraint). By the corresponding version of the uniqueness theorem [58, Theorem 3.9], which proves that any holomorphic curve with positive asymptotics a subset of the $\gamma_p$ must be a curve in the foliation, we see that $(\star)_k^o$ is satisfied for generic choice of $o \in Y$. This uniqueness statement also survives sufficiently small perturbations to nearby contact data and adapted generic $J$. The result follows.

Remark 6.8. The same exact proof works for the SOBDs considered in [58]. The difference for those examples is that the paper has two connected components $Y_P^+, Y_P^-$ having genus zero ones (i.e. the SOBD is not symmetric). While $(\star)_k^o$ is satisfied for generic points $o \in Y_P^-$, this is not true for $o \in Y_P^+$. This is explained as follows: these examples have finite algebraic planar torsion by the proof of [58, Theorem 1.4], in particular, there is no $BL_\infty$ augmentation, hence the planarity is 0. Note that the proof of Lemma 6.1 and Remark 6.2 shows that the ruling curve has homological meaning only if there are augmentations.

Corollary 6.9. Let $V$ be an affine variety, such that $\text{AU}(V) \geq k$. Then $P(\partial(S_k \times V)) = k$.

Proof. By Theorem 6.7, we have $P(\partial(S_k \times V)) \leq k$. One the other hand it is easy to see that $\text{AU}(S_k \times V) = k$, as every algebraic curve in $V \times S_k$ projects two algebraic curve in both $V$ and $S_k$. Then the claim follows from Corollary 5.15. □

6.3. IP Bourgeois examples. In [8], given a contact manifold $(Y, \xi)$ together with a supporting open book decomposition, Bourgeois constructs an associated contact structure $\xi_{BO}$ on $Y \times \mathbb{T}^2$. These contact manifolds were studied more systematically in [51, 19]. Amongst these examples, the natural candidates for which we may estimate the planarity is precisely those for which the initial open book is iterated planar. We will focus on the 5-dimensional case, i.e. when $Y$ is a 3-manifold and the open book is planar, since controlling holomorphic curves becomes much more approachable, due to dimensional reasons having to do with intersection theory. We say that the 5-dimensional Bourgeois manifold $(Y \times \mathbb{T}^2, \xi_{BO})$ is $k$-planar if the starting open book has genus zero pages with $k$-boundary components, $k \geq 1$. 
Theorem 6.10. If \((Y \times \mathbb{T}^2, \xi_{BO})\) is \(k\)-planar, then \(P(Y \times \mathbb{T}^2) \leq k\).

Proof. If \((Y, \xi) = \text{OBD}(\Sigma, \phi)\) is supported by an open book with page \(\Sigma\) and monodromy \(\phi\), we consider the associated SOBD \((Y \times \mathbb{T}^2, \xi_{BO}) = Y_S \cup Y_P\) supporting the Bourgeois contact structure, where \(Y_S = B \times \mathbb{D}^* \mathbb{T}^2\), with \(B = \partial \Sigma\) the binding in \(Y\), and \(Y_P = \Sigma \times \mathbb{T}^2\), with \(\Sigma\) the mapping torus of \(\phi\) (see [19, §2]). This SOBD can be made holomorphic, i.e. we may consider the codimension-4 holomorphic foliation \(\mathcal{F}\) of [19, Appendix A, §7.2], whose leaves are either trivial cylinders over the SOBD binding \(\mathcal{B} = B \times \mathbb{T}^2 \times \{0\}\) (a \(\mathbb{T}^2\) Morse-Bott family of Reeb orbits), or Liouville completions of \(\Sigma\) which are asymptotic to orbits in \(\mathcal{B}\), one in each of the \(k\) components of \(\mathcal{B}\). If we introduce a Morse perturbation by choosing a Morse function on \(\mathbb{T}^2\), critical points of this function corresponds to non-degenerate orbits lying in \(\mathcal{B}\), and we further obtain flow-line holomorphic cylinders.

We claim that, before or after a Morse perturbation, any holomorphic curve \(u\) in the symplectization \(\mathbb{R} \times Y \times \mathbb{T}^2\), whose positive asymptotics are simply covered, and each one lie in a different connected component of \(\mathcal{B}\), is necessarily a leaf in \(\mathcal{F}\). This claim implies that \((\ast)_k\) in Lemma 6.1 is satisfied (and this still holds after perturbing to nearby contact data) and so the theorem follows. This is again proved by an adaptation of [58, Theorem 3.9], as in [19, Lemma 7.4] (note, however, that [19, Lemma 7.4] assumes the stronger assumption that the curve has precisely \(k\) asymptotics, which is not enough for the conditions of Lemma 6.1; this is not the case for [58, Theorem 3.9], which is more general).

We give a guide to the argument for convenience of the reader. First, one can arrange that the negative ends of \(u\) also asymptote orbits in \(\mathcal{B}\), and their number is bounded above by the number of positive asymptotics of \(u\) [19, Lemma 7.2]. Then one separates two cases: either \(u\) lies completely in \(Y_S\) (case A), or it does not (case B). In case A, \(u\) has only one positive end by assumption, and since orbits are non-contractible in \(Y_S\), \(u\) has precisely one negative end. The Morse-Bott case is then easily dealt with using energy (\(u\) is necessarily a trivial cylinder); the Morse case is obtained from gluing analysis for holomorphic cascades as in [7] (see [58, Theorem 3.9, case A]). The proof for Case B is then almost word by word as that in [19, Lemma 7.4]; see also [58, Theorem 3.9, case B].

Remark 6.11. For the higher-dimensional case, it suffices to show that a holomorphic curve \(u\) as in the proof above lies in a leaf of \(\mathcal{F}\), and appeal to what we proved above for the case of IP-contact manifolds. While we expect this to hold, its proof needs relies on a different argument, since the foliation is now not 2-dimensional (nor 2-codimensional), and so the intersection theory is not so useful a priori.

7. Examples and applications

In this section, we will discuss two more classes of examples, where we can compute the hierarchy functors. The first case is smooth affine varieties with a \(\mathbb{CP}^n\) compactification, or more generally a Fano hypersurface compactification. The second case is links of singularities, including links of Brieskorn singularities and quotient singularities by the diagonal action of cyclic groups. In particular, we will finish the proof of Theorem B.

7.1. Affine varieties. Let \(V\) be a smooth affine variety, then \(V\) is naturally a Weinstein manifold by viewing \(V \subset \mathbb{C}^N\) and the function \(|x - x_0|^2\) on \(\mathbb{C}^N\) restricted to \(V\) is a Morse function with finitely many critical points for a generic \(x_0 \in \mathbb{C}^N\) [57, §6]. In particular, we obtain a contact manifold by taking the intersection of \(V\) with a large enough ball. We will use \(\partial V\) to denote the contact boundary.

An alternative way of associating a Weinstein structure to \(V\) is by using a smooth projective compactification \(\overline{V}\) with an ample line bundle \(\mathcal{L}\) with a holomorphic section \(s\) such that \(s^{-1}(0)\) is normal crossing and \(V = \overline{V \setminus s^{-1}(0)}\). We choose a metric on \(\mathcal{L}\), such that the curvature is a Kähler form \(\omega\) on \(\overline{V}\). Then by [68,
Lemma 4.3], $h = -\log |s|$ and $-d^c h$ defines a Weinstein structure (possibly after perturbation) on $V$. The equivalence of these two definitions can be found in [55].

We first give a description on the embedding relations of affine varieties with the same projective compactification.

**Lemma 7.1.** Let $X$ be a smooth projective variety with a very ample line $L$. For $s \in H^0(L)$, we use $V_s$ to denote the Liouville domain associated to the affine variety $X \setminus s^{-1}(0)$. Then for $s \neq 0 \in H^0(L)$, there exists $\epsilon > 0$, such that for all $t \in H^0(L)$ with $|s - t| < \epsilon$, we have $V_s$ embeds exactly into $V_t$.

Proof. With the very ample line bundle $L$, $X$ can be embedded in $\mathbb{P}H^0(L)$ such that every $s \in H^0(L)$ corresponds to a hyperplane $H_s \subset \mathbb{P}H^0(L)$ and $s^{-1}(0) = X \cap H_s$. Since we can view the Liouville domain $V_s$ as the intersection of $X$ with a large ball in the identification of $\mathbb{C}^N$ with $\mathbb{P}H^0(L) \setminus H_s$. Then for $t$ sufficiently close to $s$, i.e. $H_t$ sufficiently closed to $H_s$, the Liouville form of $V_t$ restricts $V_t \cap S_R$ is a contact form, where $S_R$ the radius $R \gg 0$ sphere in $\mathbb{C}^N$. The Gray stability theorem implies that all of them induced the same contact structure on $\partial V_s$ for $t$ sufficiently close to $s$, hence $V_s$ embeds exactly into $V_t$. \qed

**Remark 7.2.** In principle, the exact embedding from $V_s$ to $V_t$ should be built from a Weinstein cobordism. Hence one expects a more precise description of the Weinstein cobordism, which depends the deformation from $s$ to $t$. Some results in this direction can be found in [3, 61].

Roughly speaking, we should have a stratification on $\mathbb{P}H^0(L)$ indexed by the singularity type of $s^{-1}(0)$. The index set forms a category by declaring a morphism from stratum $A$ to stratum $B$ if the closure of $B$ contains $A$. Then Lemma 7.1 implies that we have a functor from the index set (which should be a poset) to $\mathcal{C}n_*$. Making such description precise is not easy, as we do not have a classification of singularities of $s^{-1}(0)$. However, we can describe some subcategory of the index set. The following lemma is also very useful in understanding the embedding relations of affine varieties arose from different line bundles.

**Lemma 7.3** ([68, Lemma 4.4]). Assume the smooth affine variety $V$ has a smooth projective compactification $\overline{V}$. Assume there are two ample line bundles $L_i$ with sections $s_i$, such that $s_1^{-1}(0) = s_2^{-1}(0) = \overline{V} \setminus V$ is normal crossing, but possibly with different multiplicities. Then the Liouville structures on $V$ defined by $s_i$ are homotopic.

**Example 7.4.** $\mathbb{C}P^n$ minus $k$ generic hyperplanes can be viewed as the complement of $(s_1 \otimes \ldots \otimes s_k)^{-1}(0)$ for generic sections $s_i$ of $O(1)$. On the other hand, $\mathbb{C}P^n$ minus $k - 1$ generic hyperplanes can be viewed as the complement of $(s_1 \otimes s_2 \otimes s_3 \otimes \ldots \otimes s_k)^{-1}(0)$ by Lemma 7.3. As a consequence of Lemma 7.1, we have an exact embedding of $\mathbb{C}P^n$ minus $k - 1$ generic hyperplanes to $\mathbb{C}P^n$ minus $k$ generic hyperplanes. As a simple example, $\mathbb{C}P^2$ minus a line is $\mathbb{C}^2$, $\mathbb{C}P^2$ minus two generic lines is $\mathbb{C} \times T^*S^1$ and $\mathbb{C}P^2$ minus three generic lines is $T^*T^2$. It is clear that we have the embedding relations. Moreover some of the relations can not be reversed, e.g. $T^*T^2$ can not be embedded exactly into $\mathbb{C}^2$ or $\mathbb{C} \times T^*S^1$. But $\mathbb{C} \times T^*S^1$ can be embedded back into $\mathbb{C}^2$ by adding an 2 handle corresponding to the positive Dehn twist in the trivial open book for $\partial(\mathbb{C} \times T^*S^1)$. More generally, $\mathbb{C}P^n$ minus $k$ generic hyperplanes is $\mathbb{C}^{n+1-k} \times T^*T^{k-1}$ for $k \leq n$, and they can be embedded into each other exactly.

**Example 7.5.** $\mathbb{C}P^2$ minus 3 hyperplanes passing through the same point is $\mathbb{C} \times S_3$, where $S_3$ is the thrice punctured sphere. Since $\mathbb{C}P^2$ minus 2 hyperplanes can still be viewed as a further degeneration, we have $\mathbb{C} \times T^*S^1$ embeds to $\mathbb{C} \times S_3$, which is obviously true. On the other hand, $\mathbb{C}P^2$ minus 3 generic hyperplanes, i.e. $T^*T^2$ contains $\mathbb{C} \times S_3$ as an exact domain. Moreover, $\mathbb{C}P^2$ minus a smooth degree 2 curve is $T^*\mathbb{R}P^2$, which is obtained from attaching a 2-handle to $\mathbb{C} \times T^*S^1$, i.e. $\mathbb{C}P^2$ minus 2 generic lines. $\mathbb{C}P^2$ minus a
smooth degree 3 curve can be described as attaching three 2-handles to $T^*T^2$, see [3] for details. It is not clear if the complement of a smooth degree 2 curve embeds exactly into the complement of a smooth degree 3 curve. However, the former embeds exactly into the complement of a smooth degree 4 curve by Lemma 7.1 and Lemma 7.3.

Let $D$ be a divisor, we use $D^c$ to denote the complement affine variety. Our main theorem in this section is the following.

**Theorem 7.6.** Let $D$ be $k$ generic hyperplanes in $\mathbb{C}P^n$ for $n \geq 2$, then we have the following.

1. $P(\partial D^c) \geq k + 1 - n$ for $k > n + 1$.
2. $P(\partial D^c) = k + 1 - n$ for $n + 1 < k < \frac{3n-1}{2}$ and $n$ odd.
3. $P(\partial D^c) = 2$ for $k = n + 1$.
4. $\mathbb{H}_k(\partial D^c) = 0^\text{SD}$ for $k \leq n$.

The strategy to obtain Theorem 7.6 is first prove $P(\partial D^c) \geq \max\{1, k + 1 - n\}$ by index computation, then we obtain that the planarity of the affine variety $D^c$ is at most $\max\{1, k + 1 - n\}$ by looking at the affine variety $D^\gamma_c$, where $D^\gamma$ is the smoothing of $D$, i.e. a smooth degree $k$ hypersurface. Finally, we use index computation to show that the relevant portion of computation is independent of the $BL_\infty$ augmentation for RSFT(\partial D^c) when $n + 1 < k < \frac{3n-1}{2}$. The $n$ being odd condition is to obtain automatic closedness for a chain in $\mathcal{SV}_{D^\gamma}$ for any augmentations and is expected to be irrelevant. However, to drop this constraint, we need to use stronger transversality properties supplied by [82], see Remark 7.20 for more discussion. The $n + 1 < k < \frac{3n-1}{2}$ condition is likely not optimal and it is not clear whether it is necessary. It is a difficult task to compute planarity for all augmentations. There are many affine varieties with a $\mathbb{C}P^n$ compactification such that the contact boundary has infinite planarity, while the planarity of the affine domain, i.e. using the augmentation for the affine variety is finite, see Theorem 7.12. In particular, different augmentations do make a difference. Therefore it is a subtle question to determine which affine variety has finite planarity. In general, we need to develop a computation method of RSFT from (log/relative) Gromov-Witten invariants like the symplectic (co)homology computation in [27].

**7.1.1. Reeb dynamics on the divisor complement.** In this part, we describe the Reeb dynamics on the boundary of a tubular neighborhood of a simple normal crossing divisor. The general description was obtained in [54], see also [35, 36, 56]. For our purpose, we are only interested in the following two special cases.

A smooth degree $k$ hypersurfaces in $\mathbb{C}P^n$ for $n \geq 2$. Let $D$ denote a smooth degree $k$ hypersurface in $\mathbb{C}P^n$, then the contact boundary of the concave boundary of the $O(k)$ line bundle over $D$ carries a natural Morse-Bott contact form. We can pick a $C^2$-small Morse function $f$ on $D$, such that Reeb orbits (up to an arbitrarily high action threshold) have the following properties.

1. There is a simple Reeb orbit $\gamma_p$ over every critical point $p$ of $f$ and these are all of the simple Reeb orbits. We use $\gamma_p^m$ to denote the $m$-th cover of $\gamma_p$. All of the Reeb orbits are good and non-degenerate.
2. $[\gamma_p] \in H_1(\partial D^c) \subset H_1(D^c)$ is a generator of order $k$.
3. Using the obvious disk bounded by $\gamma_p^m$ in $O(k)|_D$, which induces an trivialization of $\det_\mathbb{C} \xi$, we have that the Conley-Zehnder index has the following property,

$$n - 3 - \mu_{CZ}(\gamma_p^m) = 2m - 2 + \text{ind}(p),$$

where $\text{ind}(p)$ is the Morse index of $p$.

$k$ generic hyperplanes in $\mathbb{C}P^n$ for $k \geq n + 1$. Let $D_1, \ldots, D_k$ denote the $k$ hyperplanes. Let $I \subset \{1, \ldots, k\}$ be a set of cardinality at most $n$. We define $D_I$ to be the intersection $\cap_{i \in I} D_i$ which is a copy of $\mathbb{C}P^{n-|I|}$. 

Let Proposition 7.7. We define \( \hat{\mathcal{D}}_I \) by \( D_I \setminus \cup_{i \in I} D_i \). We pick exhausting a Morse function \( f_I \) on each \( \hat{D}_I \). The Reeb dynamics has the following properties.

1. For each critical point \( p \) of \( f_I \) and a function \( s : \{1, \ldots, k\} \to \mathbb{N}^k \) with \( \text{supp} \; s = I \), we have a \( T^{|I|-1} \)-Morse-Bott family of Reeb orbits \( \gamma^s_p \).
2. \( H_1(D^c) = H_1(\partial D^c) \) is generated by the simple circles \( \{[\beta_i]\} \) wrapping around \( D_i \) once subject to the relation \( \sum_{i=1}^k [\beta_i] = 0 \). The homology class \( [\gamma^s_p] \) over \( \hat{D}_I \) is \( \sum_{i \in I} s(i)[\beta_i] \).
3. The generalized Conley-Zehnder index using the obvious disk whose the intersection number with \( D_i \) is \( s(i) \) for \( i \in I \), is given by

\[
    n - 3 - \mu_{\text{CZ}}(\gamma^s_p) = 2 \sum s(i) - 2 + \text{ind}(p) + \frac{|I| - 1}{2}.
\]

After the perturbation, the \( T^{|I|-1} \) family of Reeb orbits degenerate to \( 2^{|I|-1} \) many non-degenerate orbits, the Conley-Zehnder indices span the following region,

\[
    n - 3 - \mu_{\text{CZ}} \in [2 \sum (s(i) - 2 + \text{ind}(p), \; 2 \sum s(i) - 2 + \text{ind}(p) + |I| - 1].
\]

We use \( \hat{\gamma}^s_p \) to denote the orbit with \( n - 3 - \mu_{\text{CZ}}(\hat{\gamma}^s_p) = 2 \sum s(i) - 2 + \text{ind}(p) \) and \( \hat{\gamma}^s_p \) to denote the orbit with \( n - 3 - \mu_{\text{CZ}}(\hat{\gamma}^s_p) = 2 \sum s(i) - 2 + \text{ind}(p) + |I| - 1 \).

**Proposition 7.7.** Let \( D = D_1 \cup \ldots \cup D_k \) denote the \( k > n + 1 \) generic hyperplanes in \( \mathbb{CP}^n \) for \( n \geq 2 \).

1. For any Reeb orbits set \( \Gamma := \{\gamma_1, \ldots, \gamma_r\} \) for \( r < k + 1 - n \) with \( \sum [\gamma_i] = 0 \in H_1(D^c) \), the virtual dimension of the moduli space \( \mathcal{M}_{D^c,A,0}(\Gamma, \emptyset) \) is less than 0 for any \( A \).
2. For any Reeb orbits set \( \Gamma := \{\gamma_1, \ldots, \gamma_{k+1-n}\} \) with \( \sum [\gamma_i] = 0 \in H_1(D^c) \) and the virtual dimension of the moduli space \( \mathcal{M}_{D^c,A,0}(\Gamma, \emptyset) \geq 0 \), then there is a partition of \( \{1, \ldots, k\} \) into \( I_1, \ldots, I_{k+1-n} \), such that \( \Gamma = \{\gamma_{p_i, \text{min}}\} \), where \( p_i, \text{min} \) is the minimum on \( D_{I_i} \) and \( \sigma_{I_i} \) is the indication function supported on \( I_i \).

**Proof.** Note that \( c_1(D^c) = 0 \), we have the virtual dimension does not depend on \( A \), hence we will abbreviate it in the following discussion (the same applies to everywhere in this subsection). Given a curve \( u \) in the same homotopy class of a curve in \( \mathcal{M}_{D^c,A,0}(\Gamma, \emptyset) \), we use \( u_i \) to denote the natural disk cap of \( \gamma_i \), then we have

\[
    \text{ind}(u) + \sum_{i=1}^r (n - 3 - \mu_{\text{CZ}}(\gamma_i)) = 2c_1(u_{i=1}^r u_i) - 4.
\]

We assume \( \gamma_i \) is from \( \gamma_{p_i}^s \) after perturbation, then they are subject to the condition \( \sum_{i=1}^r s_i = (N, \ldots, N) \) and \( c_1(u_{i=1}^r u_i) = N(n+1) \). Then we have

\[
    \text{ind}(u) \leq 2N(n+1) - 4 - \sum_{i=1}^r (2 \sum s_i - 2 + \text{ind}(p_i)) \tag{7.1}
\]

\[
    \leq 2N(n+1) - 4 - 2 \sum_{i=1}^r \sum s_i + 2r \tag{7.2}
    = 2N(n+1-k) - 4 + 2r
    = 2(N-1)(n+1-k) + 2(r+n-k-1) < 0
\]
when \( r < k + 1 - n \). If \( r = k + 1 - n \), to have \( \text{ind}(u) \geq 0 \), we must have \( N = 1 \). In this case, both inequalities (7.1) and (7.2) must be equality. In particular, \( \text{ind}(p_i) = 0 \) and \( \gamma_i \) must be a check orbit, i.e. the claim holds.

Then the by Proposition 5.14, we have the following.

**Corollary 7.8.** If \( s \) is a perturbation of the \( k \) generic hyperplanes for \( k > n + 1 \) and \( n \geq 2 \), then \( P(\partial V_s) \geq k + 1 - n \).

**Proof.** Let \( D \) be \( k \) generic hyperplanes, then \( \eta D^c \) on \( B^k V_{\partial D^c} \) in Propitiation 5.14 is zero for \( r < k + 1 - n \) by dimension reasons. Therefore \( P(\partial D^c) \geq k + 1 - n \). The remaining of the claim follows from Lemma 7.1. \( \square \)

7.2. A neck-stretching argument.

**Proposition 7.9.** Let \( D_s \) be a smooth degree \( k \) hypersurfaces in \( \mathbb{C}P^n \), then the following holds.

1. If \( k \leq n \), for a point \( o \in D^c \), there is a Reeb orbit \( \gamma^k_o \) with \( \text{ind}(p) = 2(n - k) \) and an admissible complex structure, such that \( \mathcal{M}_{D_s,o}(\{ \gamma^k_o \}, \emptyset) \) is cut out transversely and \( \# \mathcal{M}_{D_s,o}(\{ \gamma^k_o \}, \emptyset) \neq 0 \).
2. If \( k \geq n + 1 \), for a point \( o \in D^c_s \), there are two Reeb orbits \( \gamma^n_{p_{\text{min}}}, \gamma^n_{p_{\text{min}}} \) with \( p_{\text{min}} \) is the minimum on \( D_s \) and an admissible almost complex structure, such that \( \mathcal{M}_{D_s,o}(\{ \gamma^n_{p_{\text{min}}}, \ldots, \gamma^n_{p_{\text{min}}} \}, \emptyset) \) is cut out transversely with nontrivial algebraic count.

**Proof.** This follows from applying neck-stretching to \( \mathbb{C}P^n \) along \( \partial D^c_s \). We denote the relative Gromov-Witten invariant that counts genus 0 holomorphic curves in class \( A \) with \( k \) marked point going through \( C_1, \ldots, C_k \in H_s(\mathbb{C}P^n) \) and \( l \) marked point going through \( E_1, \ldots, E_l \in H_s(D_s) \) and intersect \( D_s \) with multiplicity at least \( s_1, \ldots, s_l \) respectively by \( \text{GW}_{\mathbb{C}P^n,D_s}^{0,k,(s_1, \ldots, s_l),A}(C_1, \ldots, C_k, E_1, \ldots, E_l) \) [46]. The source of holomorphic curves is from the non-vanishing relative Gromov-Witten invariants \( \text{GW}_{\mathbb{C}P^n,D_s}^{0,1,(k),A}(\{pt\}, \{D_s\} \cap n-k[H]) \) and \( \text{GW}_{\mathbb{C}P^n,D_s}^{0,1,(n,1,\ldots,1),A}(\{pt\}, \{D_s\}, \ldots, \{D_s\}) \) respectively from [38], where \( H \in H_{2n-2}(\mathbb{C}P^n) \) is the hyperplane class and \( A \) is the generator of \( H^2(\mathbb{C}P^n) \). Since the curve is necessarily somewhere injective and not contained in \( D_s \) because we can choose the \( \{pt\} \) class in \( D_s^c \), one can assume transversality in the process of neck-stretching.

In the fully stretched picture, the bottom curve has at most \( \max\{1, k + 1 - n\} \) positive punctures for otherwise genus has to be created. If the bottom curve has \( 0 < r \leq \max\{1, k + 1 - n\} \) positive punctures, in particular, \( k \geq n + 1 \), we assume the positive asymptotics are \( \Gamma^+ = \{\gamma^k_{p_i}\} \). Then by homology reasons we have \( \sum d_i = km \). Then the expected dimension of \( \mathcal{M}_{D_s,o}(\Gamma^+, \emptyset) \) is given by

\[
\text{ind}(u) = 2m(n + 1) - 4 - \sum_{i=1}^{r} (n + 3 - \mu CZ(\gamma^k_{p_i}))
\]

\[
\leq 2m(n + 1) - 4 - 2 \sum_{i=1}^{r} d_i + 2r
\]

\[
= 2m(n + 1 - k) + 2r - 4 \leq 2(m - 1)(n + 1 - k) + 2(r + n - k - 1) < 0
\]

Therefore the bottom curve must have \( \max\{1, k + 1 - n\} \) positive punctures. Moreover, from the above computation, we separate the proof into three cases.
(1) $k > n + 1$. To have $\text{ind}(u) \geq 0$, we must have that $p_i$ is the minimum $p_{\text{min}}$ and $m = 1$. That is the positive asymptotics of the bottom curve are $\{\gamma_{p_{\text{min}}}^{d_i}\}_{1 \leq i \leq k+1-n}$ with $\sum d_i = k$. Then other levels must be either a cylinder in the symplectization or a disk in the symplectic cap. Note that the disk $v$ in the symplectic cap that intersect $D_s$ with order $n$ must be asymptotic to $\gamma_q^n$ for a critical point $q$. Assume otherwise that the multiplicity is $n + km$ for $m \geq 1$, then the relative homology class of $v$ is the same as the sum of the natural disk of $\gamma_q^{n+km}$ and $-mA$ for the positive generator $A \in H_2(D_s) \to H_2(\mathbb{CP}^n)$ when $n \geq 3$. Then the symplectic action of such disk is negative for $m \geq 1$ for an appropriate symplectic form on the cap, i.e. a sufficiently small neighborhood of the divisor. When $n = 2$, since $\pi_2(D_s) = 0$, it is necessarily to have $m = 0$ as $\gamma_q^{n+km}$ and $\gamma_q^n$ are not homotopic in the neighborhood of $D$. Therefore the negative asymptotics of the disks in the symplectic cap must be $\gamma_q^n$ and $\gamma_q^r$. Then by computing the expected dimension of the cylinders in the symplectization, the only possibility (i.e. those with non-negative dimension) is $q,q'$ are the minimum $p_{\text{min}}$ and there is no nontrivial curve in the symplectization. The bottom curve moduli space $M_{D_s,o}(\{\gamma_{p_{\text{min}}}^{n}, \gamma_{p_{\text{min}}}^{n}, \ldots, \gamma_{p_{\text{min}}}^{n}\}, \emptyset)$ consists of somewhere injective curves, for otherwise, assume $u \in M_{D_s,o}(\{\gamma_{p_{\text{min}}}^{n}, \gamma_{p_{\text{min}}}^{n}, \ldots, \gamma_{p_{\text{min}}}^{n}\}, \emptyset)$ is a branched cover over $u'$, then we can cap off $u'$ with natural disks to obtain a homology class $A$ in $H_2(\mathbb{CP}^2)$ with $A \cap D_s < k$, which is a contradiction. It is direct to check that the holomorphic disks in the symplectic cap (i.e. disk fibers) are cut out transversely, hence transversality holds for the fully stretched situation. Therefore we have $\#M_{D_s,o}(\{\gamma_{p_{\text{min}}}^{n}, \gamma_{p_{\text{min}}}^{n}, \ldots, \gamma_{p_{\text{min}}}^{n}\}, \emptyset) \neq 0$.

(2) $k = n + 1$. Then to have $\text{ind}(u) \geq 0$, we must have that $p_i$ is the minimum of $p_{\text{min}}$ but $m \geq 1$. By the same area argument for the cap, we have the negative asymptotics of the disks must be $\gamma_q^n$ and $\gamma_q^r$. Then we must have $m = 1$, for otherwise the total contact action of the symplectization levels is negative. Then the remaining of the argument is the same as before.

(3) $k \leq n$. Since the bottom level has one positive puncture, that is asymptotic to $\gamma_q^{km}$. By the same area and action argument, we have $m = 1$. Assume the negative asymptotic of the disk is $\gamma_p^k$, then we must have $\text{ind}(p) \geq 2(n-k)$ to have non-negative expected dimension for the disk. On the other hand, for the bottom curve, we must have $\text{ind}(q) \leq 2(n-k)$ to have non-negative expected dimension. Therefore if $p \neq q$, the expected dimension of the cylinders in the symplectization is negative. Hence we have $p = q$ with $\text{ind}(p) = 2(n-k)$. Then we know that there is at least one critical point $p$ with $\text{ind}(p) = 2(n-k)$ such that $\#M_{D_s,o}(\{\gamma_{p_{\text{min}}}^{k}\}, \emptyset) \neq 0$ and the unstable manifold of $p$ represents multiples of $[D_s] \cap \gamma_{p_{\text{min}}}^{n-k} [H]$.

Corollary 7.10. Let $D_s$ be the smooth degree $k$ hypersurface in $\mathbb{CP}^n$ for $k > n + 1$ and $n \geq 2$. Then $\eta_{D_s} q_{p_{\text{min}}}^{-m} q_{p_{\text{min}}}^{k-n} \neq 0$ and $q_{p_{\text{min}}}^{-m} q_{p_{\text{min}}}^{k-n}$ is closed in $(\overline{B^{n+1-k}}, \overline{\ell_{D_s}})$. 

Proof. We may assume the Morse function on $D_s$ is perfect, this follows from direct check for $n = 2$, [43] for $n = 3$, and the Lefschetz hyperplane theorem and the $h$-cobordism theorem for $n \geq 4$. Then Proposition 7.9 implies that $\eta_{D_s} q_{p_{\text{min}}}^{-m} q_{p_{\text{min}}}^{k-n} \neq 0$. It suffices to prove that $q_{p_{\text{min}}}^{-m} q_{p_{\text{min}}}^{k-n}$ is closed. Since we have the parity of the SFT grading is the same as the Morse index, and $q_{p_{\text{min}}}^{-m} q_{p_{\text{min}}}^{k-n}$ has even grading, we only need to consider $\langle \ell_{D_s} q_{p_{\text{min}}}^{-m} q_{p_{\text{min}}}^{k-n}, q_{p_{\text{min}}} \rangle$ with $\text{ind}(p) = n-1$ for $n$ even. As a consequence, we need consider $\overline{M_{D_s}}(\Gamma^+, \Gamma^-)$ for $\Gamma^+ \subset \{\gamma_{p_{\text{min}}}^{m}, \gamma_{p_{\text{min}}}^{d_1}, \ldots, \gamma_{p_{\text{min}}}^{d_k}\}$ and $\Gamma^- = \{\gamma_{p_{\text{min}}}^{m}, \gamma_{p_{\text{min}}}^{d_1}, \ldots, \gamma_{p_{\text{min}}}^{d_k}\}$, then we close off $\{\gamma_{p_{\text{min}}}^{d_i}\}$ by the augmentation from $D_s$. On the other hand, by homology reason, we know the sum of multiplicities of $\Gamma^-$ equals to the sum of multiplicities of $\Gamma^+$ which is no larger than $k$. As a consequence, there is no subset of $\{\gamma_{p_{\text{min}}}^{d_i}\}$ whose sum
represents a null-homologous class in $D^c_s$. In particular, to consider $⟨\ell_{ε_{D^c_{s}}} (q_{p_{\min}}^{n}, d_{p_{\min}}^{k-n}), q_{p}^{m}⟩$, we only need to consider $\mathcal{M}_{\partial D^c_{s}}(\Gamma^{+}, \{γ_{p}^{n}\})$ where $m$ is the sum of the multiplicities of $\Gamma^{+}$. It is direct to check the expected dimension of this moduli space is $(n - 2) + 2|\Gamma^{+}| - 2$, which is strictly positive whenever $n \geq 3$. When $n = 2$, it is direct check that only case with expected dimension 0 is $\mathcal{M}_{\partial D^c_{s}}(\{γ_{p_{\min}}^{2}\}, \{γ_{p}^{1}\})$ and $\mathcal{M}_{\partial D^c_{s}}(\{γ_{p_{\min}}^{2}\}, \{γ_{p}\})$, each of them corresponds to moduli space of gradient trajectories from minimum $p_{\min}$ to the index 1 critical point $p$, whose algebraic count is zero, as our Morse function is perfect [27, 35]. Therefore $q_{p_{\min}}^{n} d_{p_{\min}}^{k-n}$ is closed in $(B^{k+1-n} V_{D^c_{s}}, \ell_{ε_{D^c_{s}}})$.

**Remark 7.11.** In the proof of Corollary 7.10, we use the topology of the filling $D^c$ to get some restrictions on the augmentation, in particular the augmentation respects the homology classes of orbits. However, we can not run such argument for general augmentations to obtain Theorem F and we use $n$ being odd to get automatic closedness.

Roughly speaking, we proved that $P(D^c_s) \leq k + 1 - n$ for a smooth degree $k > n + 1$ divisor $D_s$. If the functoriality of $P$ for exact domains was proven in Claim 4.9, then we can conclude that $P(D^c) \leq k + 1 - n$ for $D$ the $k$ generic hyperplanes. We still need to argue that the computation is independent of augmentation. In the following, we first show that the computation of $P(\partial D^c_s)$ does depend on augmentations in some cases

**Theorem 7.12.** Let $D_s$ be a smooth degree $k > 2n - 3$ hypersurface in $\mathbb{C}P^n$ for $n \geq 3$ odd, then $P(\partial D^c_s) = \infty$.

**Proof.** As assume as before the Morse function on $D_s$ is perfect. Since $n \geq 3$ odd, all generators have SFT grading even, as a consequence, we have any $ε^k : SV_{\partial D^c_s} \to \mathbb{Q}$ is a $BL_{\infty}$ augmentation. We claim if we pick $ε^k = 0$ for $k \geq 1$, then the order of planarity is $\infty$. To obtain this, we need to prove that $\# \mathcal{M}_{\partial D^c_{s}, o}(\Gamma^{+}, \emptyset) = 0$. For this we use a cascades model (but only the compactness), i.e. we consider the Boothby-Wang contact form on $\partial D^c_{s}$.

Following the compactness argument in [13], if we degenerate the contact form on $D_s$ (as perturbed by the Morse function) to the Boothby-Wang contact form, the $\mathcal{M}_{\partial D^c_{s}, A, o}(\Gamma^{+}, \emptyset)$ degenerates to cascades. But since $|\Gamma^{+}| \neq \emptyset$, there is one level containing nontrivial holomorphic curves in the symplectization of the Boothby-Wang contact form, which projects to a holomorphic sphere in $D_s$. However since $k > 2n - 3$, there is no holomorphic sphere in $D_s$. As a consequence $\mathcal{M}_{\partial D^c_{s}, A, o}(\Gamma^{+}, \emptyset) = \emptyset$ and the claims follows. □

**Proposition 7.13.** Let $D_s$ be a smooth degree $k \geq n + 1$ hypersurface in $\mathbb{C}P^n$, assume $\Gamma^{+}$ is a proper subset of $\{γ_{p_{\min}}^{n}, γ_{p_{\min}}, \ldots, γ_{p_{\min}}^{n}\}$. Then for $\Gamma^{-} \neq \emptyset$, $\# \mathcal{M}_{\partial D^c_{s}, o}(\Gamma^{+}, \Gamma^{-}) = 0$ unless $\Gamma^{+} = \{γ_{p_{\min}}^{n}\}$, $\Gamma^{-} = \{γ_{p_{\max}}^{n}\}$ or $\Gamma^{+} = \{γ_{p_{\min}}^{n}\}$, $\Gamma^{-} = \{γ_{p_{\max}}^{n}\}$.

**Proof.** We can assume $\Gamma^{-} = \{γ_{p_1}^{d_1}, \ldots, γ_{p_r}^{d_r}\}$ with $\sum d_i = d$ by homology and action reasons. Then we can run the Morse-Bott compactness argument as in Theorem 7.12 for $\mathcal{M}_{\partial D^c_{s}, A, o}(\Gamma^{+}, \Gamma^{-})$, in the limit cascades moduli space, we necessarily have the holomorphic curve part have zero energy and hence a constant. Therefore due to the generic point constraint $o$, we must have $p_1 = \ldots = p_r = p_{\max}$. Then the expected dimension of such moduli space is

$$2d - 2|\Gamma^{+}| + \text{vdim} \mathcal{M}_{\partial D^c_{s}, A, o}(\Gamma^{+}, \Gamma^{-}) - 2|\Gamma^{-}| - 2d = -4.$$ 

Hence $\text{vdim} \mathcal{M}_{\partial D^c_{s}, A, o}(\Gamma^{+}, \Gamma^{-}) = 2|\Gamma^{+}| + 2|\Gamma^{-}| - 4$, which is zero iff $|\Gamma^{+}| = |\Gamma^{-}| = 1$. Hence the claim follows. □

**Remark 7.14.** In the case considered in Proposition 7.13, the only non-empty moduli spaces contributing to the pointed map are $\mathcal{M}_{\partial D^c_{s}, o}(\{γ_{p_{\min}}^{n}\}, \{γ_{p_{\max}}^{n}\})$ and $\mathcal{M}_{\partial D^c_{s}, o}(\{γ_{p_{\min}}^{n}\}, \{γ_{p_{\max}}^{n}\})$. Moreover, the algebraic count is not
zero as the gradient trajectories from \( p_{\text{min}} \) to \( p_{\text{max}} \) traverse the whole manifold. This follows from a cascades construction with gluing as in [13].

Let \( D \) be \( k \) generic hyperplanes and \( D_\epsilon \) the degree \( k \) smooth hypersurface. We use \( X \) to denote the exact cobordism from \( \partial D^c \) to \( \partial D^s_\epsilon \). Our strategy for Theorem 7.6 is showing \( P(\partial D^c) \) is independent of \( \epsilon \) for any augmentation \( \epsilon \) of RSFT(\( \partial D^c \)) and \( \phi \) is the \( BL_{\infty} \) morphism induced from \( X \). Then we use the functoriality in Proposition 2.22 and argue that the computation we did with the filling \( D^c_\epsilon \) in Corollary 7.10 is in the form \( \epsilon_D \circ \phi \), where \( \epsilon_D \) is the augmentation of RSFT(\( \partial D^c \)) from \( D^c \). In principle, this involves a homotopy argument by neck-stretching. To avoid the overhead of introducing homotopies of \( BL_{\infty} \) morphisms, we show that the formula can be identified on the nose, due to the fact that when transversality in neck-streaking holds, we can identify a fully-stretched moduli space with a sufficiently stretched moduli space by classical gluing. In the following, we first prove a property explaining the role of \( k < \frac{3n-1}{2} \).

**Proposition 7.15.** Let \( X \) be the cobordism from \( \partial D^c \) to \( \partial D^s_\epsilon \) as above. Assume \( \Gamma^+ = \{ k_1^{\min}, k_2^{\min}, \ldots, k_s^{\min} \} \) for \( \sum_{i=1}^s k_i \leq k \), we have \( \text{vdim} M_X(\Gamma^+, \Gamma^-) < 0 \) if \( \Gamma^- \neq \emptyset \) and \( s < \frac{n+1}{2} \).

**Proof.** Assume \( \Gamma^- = \{ \gamma_r \}_{r \in R} \), which are perturbations from \( \{ \gamma_r^{s+} \}_{r \in R} \), with \( \sum \gamma_r = \sum_{i=1}^s k_i \) by homology and action reasons. Then the expected dimension of \( M_{X,A}(\Gamma^+, \Gamma^-) \), i.e. \( \text{ind}(u) \) for \( u \in M_{X,A}(\Gamma^+, \Gamma^-) \), satisfies

\[
2n - 2 + \sum_{i=1}^s (2(k_i - 1) + \text{ind}(u)) + \sum_{r=1}^R (\mu_{CZ}(\gamma_r) + n - 3) \geq 2n - 6.
\]

Since \( (\mu_{CZ}(\gamma_s) + n - 3) \geq 2n - 3 - 2 \sum s_r - \text{ind}(p) - |\text{supp} s_r| \) and \( D_{\text{supp} s_r} \) is Weinstein by \( k \geq n + 1 \), we have \( \text{ind}(p) \leq n - |\text{supp} s_r| \) and \( (\mu_{CZ}(\gamma_s) + n - 3) \geq (n - 3) - 2 \sum s_r \). As a consequence, we have

\[
\text{ind}(u) \leq n - 4 - |R|(n - 3) + 2s.
\]

In particular, \( \text{ind}(u) < 0 \) if \( |R| \neq 0 \) and \( s < \frac{n+1}{2} \). \( \square \)

**Proposition 7.16.** Let \( \phi \) denote the \( BL_{\infty} \) morphism from the cobordisms \( X \) and \( \epsilon_{D^c_\epsilon}, \epsilon_{D^c} \) augmentations from \( D^c_\epsilon, D^c \) respectively, then we have

\[
\hat{\Phi}_{\epsilon_{D^c}, \epsilon_{D^c}}(q^{k-n}_{\gamma^{\min}}, q^{k-n}_{\gamma^{\min}}) = \hat{\Phi}_{\epsilon_{D^c}, \epsilon_{D^c}}(q^{k-n}_{\gamma^{\min}}, q^{k-n}_{\gamma^{\min}}) \neq 0,
\]

where \( \hat{\Phi}_{\epsilon_{D^c}, \epsilon_{D^c}} \) is defined in Proposition 2.22, i.e. the map on the bar complex for the linearized \( L_{\infty} \) morphism from \( (V_{\partial D^c}, \{ e_k^{\epsilon_{D^c}} \}_{k \geq 1}) \) to \( (V_{\partial D^c}, \{ e_k^{\epsilon_{D^c}} \}_{k \geq 1}) \).

**Proof.** We will apply a neck-stretching for \( \mathcal{M}_{D^c_\epsilon \circ \{ \gamma^n_{\gamma^{\min}}, \gamma^{\min}, \ldots, \gamma^{\min} }, \emptyset \) in (2) of Proposition 7.9 along \( \partial D^c \) for \( o \in D^c \). Every curve in \( \mathcal{M}_{D^c_\epsilon \circ \{ \gamma^n_{\gamma^{\min}}, \gamma^{\min}, \ldots, \gamma^{\min} }, \emptyset \) is somewhere injective. For otherwise, assume \( u \in \mathcal{M}_{D^c_\epsilon \circ \{ \gamma^n_{\gamma^{\min}}, \gamma^{\min}, \ldots, \gamma^{\min} }, \emptyset \) is a branched cover over \( u' \), then we can cap off \( u' \) with natural disks to obtain a homology class \( A \in H_2(\mathbb{CP}^2) \) with \( A \cap D_\epsilon < k \), which is a contradiction. Therefore it is safe to assume \( \mathcal{M}_{D^c_\epsilon \circ \{ \gamma^n_{\gamma^{\min}}, \gamma^{\min}, \ldots, \gamma^{\min} }, \emptyset \) is cut out transversely for the stretching \( J_\epsilon \). In the fully stretched picture, the bottom level containing the marked point \( o \) must have \( k + 1 - n \) positive punctures. This is because we must have the number of positive punctures no larger than \( k + 1 - n \) for otherwise genus has to be created. If there are fewer punctures, then by Proposition 7.7, the curve cannot exist. By the same capping argument, we know that the bottom curve is necessarily somewhere injective. Then by the dimension computation in Proposition 7.7, the only possible bottom level is described in Proposition 7.7. As a consequence, all the levels above the bottom level must be unions of cylinders because of the number.
of positive punctures. Then by considering homology of the cobordism \( X \), we must have the positive asymptotics of the bottom level is the form of \( \bar{\gamma}^{\sigma} \cup \{ \gamma_{p_{\min}} \}_{i \in I} \), where \( I \subset \{ 1, \ldots, k \} \) is a subset of size \( n \), \( p_{\text{I}_{\min}}, I_{\min} \) are minimums. Then by a dimension argument, it is easy to obtain every level above the bottom is also cut out transversely. In fact, we only one more level consists of \( M_{X}(\{ \gamma_{n_{\min}} \}, \{ \bar{\gamma}_{\sigma_{I}} \}_{I_{\min}}) \) and \( k - n \) copies of \( M_{X}(\{ \gamma_{p_{\min}} \}, \{ \gamma_{p_{\min}} \}) \). The transversality of neck-stretching, implies that this 2-level breaking can be identified with \( M_{D_{c}^{s}, o}(\{ \gamma_{n_{\min}}^{p_{\min}} \}, \{ \gamma_{p_{\min}}^{p_{\min}} \}) \) for sufficiently stretched \( J_{t} \). By Axiom 3.25, we can count them to obtain that

\[
\eta_{D_{c}^{s}}(q_{p_{\min}}^{n_{\min}}, q_{p_{\min}}^{k_{\min}}) = \eta_{D_{c}^{s}} \circ \phi_{D_{c}^{s}}(q_{p_{\min}}^{n_{\min}}, q_{p_{\min}}^{k_{\min}}).
\]

Then we can use Proposition 5.14 to relate \( \eta \) back to \( \hat{\ell}_{\epsilon} \), since \( q_{p_{\min}}^{n_{\min}} \) is closed in \((B^{k+1-n}V_{\partial D_{c}^{s}}, \hat{\ell}_{\epsilon D_{c}^{s}})\) by Corollary 7.10. The non-vanishing follows from Corollary 7.10.

**Proposition 7.17.** If \( k < \frac{3n-1}{2} \), then \( \hat{\ell}_{\epsilon} \circ \phi_{D_{c}^{s}}(q_{p_{\min}}^{n_{\min}}, q_{p_{\min}}^{k_{\min}}) \neq 0 \) is independent of the augmentation \( \epsilon \) of RSFT(\( \partial D_{c}^{c} \)).

**Proof.** When \( k < \frac{3n-1}{2} \), we have \( 1 + k - n < \frac{n+1}{2} \). Then the independence follows from Proposition 7.13 and Proposition 7.15, as a component to \( \hat{\ell}_{\epsilon} \circ \phi_{D_{c}^{s}}(q_{p_{\min}}^{n_{\min}}, q_{p_{\min}}^{k_{\min}}) \) with influence from \( \epsilon \) is described in the graph below, which does not exist by dimension reasons by Proposition 7.15.

![Figure 12](image.png)

The non-vanishing then follows from Proposition 7.16.

**Proof of Theorem 7.6.** If \( k \leq n \), then \( D_{c}^{c} = T^{*}T^{k-1} \times \mathbb{C}^{n-k+1} \), then \( H_{cx}(\partial D_{c}^{c}) = 0_{SD}^{SFT} \) by Theorem 7.31. If \( k = n + 1 \), then \( P(\partial D_{c}^{c}) = 2 \) by Corollary 6.9. For \( k > n + 1 \), the lower bound follows from Corollary 7.8. When \( k < \frac{3n-1}{2} \) and \( n \) odd, we have for any augmentation \( \epsilon \) of RSFT(\( \partial D_{c}^{c} \)), \( q_{p_{\min}}^{n_{\min}}q_{p_{\min}}^{k_{\min}} \) represents a closed class in \((B^{k+1-n}V_{\partial D_{c}^{s}}, \hat{\ell}_{\epsilon D_{c}^{s}})\), as the SFT grading of RSFT(\( \partial D_{c}^{c} \)) is even for all generators. In particular,
\( \widehat{\phi}_\epsilon(q_{\gamma_{\min} \ q_{\gamma_{\min}}}^k \ q_{\gamma_{\min}}^{k-n}) \) is closed in \( (\overline{B}_k^{k+1-n} V_{\partial D^2}, \widehat{\ell}_\epsilon) \) for any \( \epsilon \). Then by Proposition 7.17, \( \widehat{\ell}_{\bullet, \epsilon} \circ \widehat{\phi}_\epsilon (q_{\gamma_{\min}}^k \ q_{\gamma_{\min}}^{k-n}) \neq 0 \) for any \( \epsilon \), and we conclude that \( P(\partial D^c) = k + 1 - n \) if \( n + 1 < k < \frac{3n-1}{2} \).

\[ \square \]

**Remark 7.18.** Our computation method above can be summarized as finding a curve contributing to the planarity by relative Gromov-Witten invariants and then argue the independence of augmentation by dimension computation. The trick we use is arguing closedness in the smooth divisor, where generators are simpler, and prove the upper bounds using the functoriality and arguing everything interesting about the functoriality happens purely in \( X \) (i.e. not dependent on augmentation for RSFT(\( \partial D^c \))). A more scientific way of computing planarity is deriving a formula for the BL\( \infty \) algebra as well as the augmentation from the affine variety using log/relative Gromov-Witten invariants. In the context of symplectic (co)homology, such formula was obtained in [27].

**Theorem 7.19.** Assume \( D_s \) is a smooth degree \( 2 \leq k < \frac{n+1}{2} \) hypersurface in \( \mathbb{C}P^n \) for \( n \geq 3 \), then \( P(\partial D^c_s) = 1 \) and \( H_{\text{cx}}(\partial D^c_s) \leq (k^2 - 1)^{\text{SD}} \). When \( n \) is odd, then same holds for \( 2 \leq k < n \), and moreover we have \( H_{\text{cx}}(\partial D^c_s) \geq (k - 1)^{\text{SD}} \).

**Proof.** Let \( p \) be the critical point in (1) of Proposition 7.9, Then we have \( \eta_{D^c_s}(q_{\gamma_p}) \neq 0 \) by the same argument of Corollary 7.10. We can pick the Morse function on \( D_s \) similar to [79, Proposition 3.1], such that the perturbed contact form has the following property,

\( \langle 7.3 \rangle \)

\[ \int \alpha^* \gamma_p^d - \sum_{i=1}^{j} \int \alpha^* \gamma_{p_i}^{d_i} < 0, \]

for \( d \leq k \), \( \sum d_i = d \) and one of \( p_i \) has the property that \( \text{ind}(p_i) < \text{ind}(p) \)\(^{14} \). This energy constraint will help us exclude certain configurations.

**Claim.** \( q_{\gamma_p} \) is closed in \( (\overline{B}_1 V_{\partial D_s^c}, \ell_{\epsilon}' \ell) \) for any augmentation \( \epsilon \) for \( 2 \leq k < \frac{n+1}{2} \) or \( n \) odd with \( 2 \leq k < n \).

**Proof.** Since the parity of the SFT grading of \( \gamma_p^d \) is the same as the parity of \( \text{ind}(q) \). As a consequence, we only need to consider \( \ell_{\epsilon}'(q_{\gamma_p}^d, q_{\gamma_q}^d) \) for \( \text{ind}(q) = n - 1 \) when \( n \) is even. By \( k < \frac{n+1}{2} \), we have \( \text{ind}(p) = 2n - 2k > \text{ind}(q) = n - 1 \). Then by (7.3), all relevant moduli spaces are empty by action reasons. Hence the claim follows.

**Claim.** \( \ell_{\bullet, \epsilon}'(q_{\gamma_p}^d) \) is independent of \( \epsilon \).

**Proof.** It is sufficient to show that \( \overline{\mathcal{M}}_{\partial D_s^c, \epsilon}(\{q_{\gamma_p}^d\}, \Gamma^-) \) is empty for \( \Gamma^- \neq \emptyset \). If \( \Gamma^- \neq \emptyset \), then \( \Gamma^- = \{\gamma_{p_i}^d\}_{1 \leq i \leq r} \) with \( \sum d_i = k \). Then claim follows from the same argument of Proposition 7.13 and \( \text{ind}(p) > 0 \).

**Claim.** We have \( H_{\text{cx}}(\partial D^c_s) \leq (k^2 - 1)^{\text{SD}} \).

**Proof.** We need to show that \( u^k(q_{\gamma_p}^d) = 0 \) for any augmentation. By homology reason and (7.3), for \( d \leq k \), \( u(q_{\gamma_p}^d) \) can only have nontrivial coefficient for \( q_{\gamma_q}^{d'} \) for \( d' < d \) and \( \text{ind}(q') \geq \text{ind}(q) \) and \( \text{ind}(q') = \text{ind}(q) \mod 2 \), and for \( q_{\gamma_q}^{d'} \) for \( \text{ind}(q') > \text{ind}(q) \) and \( \text{ind}(q') = \text{ind}(q) \mod 2 \). Therefore, we have \( u^k(q_{\gamma_p}^d) = 0 \) for any augmentation.

\(^{14}\)Note that in our setup here higher \( \text{ind}(p) \) means smaller contact action, since we apply the perturbation in the cap of the positive prequantization bundle instead of the filling of the negative prequantization bundle, see [56]. In particular, the order is reversed compared to [79] and the proof of Theorem 7.28 below.
Claim. When \( n \) is odd, we have \( H_{cx}(\partial D^c_s) \geq (k-1)^{SD} \).

Proof. The linearized contact homology has an action filtration, such the filtered theory around period \( \int (\gamma^k_p)^* \alpha = k \int \gamma^k_p \alpha \), is generated by the \( k \)th covered orbits. By the argument in [12], as transversality holds in this case, the \( u \) map on this filtered theory is the same as the \( u \) map on the filtered \( S^1 \)-equivariant symplectic cohomology represented by multiplying \( c_1(\mathcal{O}(k)|_D) \) to the cochain represented by the critical point, i.e. the Poincaré dual of the unstable manifold. We have \( u^{k-1}(q_{\| p}) = k^{-1}q_{\| p}^k \) plus terms lower multiplicities for the maximum \( p \) with \( \text{ind}(p) = 2n - 2 \). When \( n \) is odd, all generators have even SFT degree, hence \( u^{k-1}(q_{\| p}) \neq 0 \) in homology.

\( \square \)

Remark 7.20. The \( n \) being odd condition in Theorem 7.6, 7.12, 7.19, as well as 7.22 below is not necessary, as one can show \( q_n^{k-n} p_{\min} \) is always closed. This is because a differential from \( q_n^{k-n} p_{\min} \) involves counting \( M_Y(\Gamma^+,\Gamma^-) \) with \( \Gamma^+ \) is a subset of \( \{\gamma^n_{\text{Poin}}, \gamma_{\text{min}}, \ldots, \gamma_{\text{min}}\} \) and \( \Gamma^- = \{\gamma_{\text{Poin}}, 1 \leq i \leq r \} \) for \( d_i = k \). Therefore if we use the Morse-Bott contact form and cascades construction, the relevant holomorphic curve must be covers of trivial cylinders. Then the moduli space \( M_Y(\Gamma^+,\Gamma^-) \) is the fiber product of \( D \times M \) with unstable/stable manifolds of \( p_{\min}, p_i \), where \( M \) is the space of meromorphic functions on \( \mathbb{C}^p \) with a pole of order \( n \) and \( k-n \) simple poles and a zero with order \( p_i \) for all \( 1 \leq i \leq r \) modulo the \( \mathbb{R} \) rescaling on meromorphic functions and the automorphism of the punctured Riemann surface. Then by the nontrivial \( S^1 \) action on meromorphic functions, we have \# \( M_Y(\Gamma^+,\Gamma^-) \) = 0 unless \( |\Gamma^+| = |\Gamma^-| = 1 \). If \( |\Gamma^+| = |\Gamma^-| = 1 \), \( M_Y(\Gamma^+,\Gamma^-) \) is identified with Morse trajectories (here the \( S^1 \) action on meromorphic function is identical with the \( S^1 \) action in the automorphism group of surface, hence is trivial on the quotient), whose algebraic count is zero, as we assume the Morse function is perfect. To make this precise, one can follow a Morse perturbation of the contact form as before. And we use a \( J \) that is \( S^1 \)-invariant under the rotation in the fiber direction, then apply the \( S^1 \)-equivariant transversality for quotients from [82], we can argue that \# \( M_Y(\Gamma^+,\Gamma^-) \) = 0 unless \( |\Gamma^+| = |\Gamma^-| = 1 \) similar to Floer’s proof of the isomorphism between Hamiltonian Floer cohomology and Morse cohomology. This argument requires building our functors using polyfolds as in [33].

Nevertheless we have the following result that is independent of the parity of \( n \) or size of \( k \).

Theorem 7.21. Let \( D \) be \( k \) generic hyperplanes in \( \mathbb{C}^p \) for \( n \geq 1 \), then \( U(D^c) = \max\{1, k + 1 - n\} \).

Proof. The \( n = 1 \) case is obvious. For \( n \geq 2 \), we can use Proposition 7.7 to claim that \( \overline{M}_{D^c,\emptyset}(\Gamma^+,\emptyset) = \emptyset \) for generic \( J \) as long as \( |\Gamma^-| < \max\{1, k + 1 - n\} \). This is because we can obtain the classical transversality of \( \overline{M}_{D^c,\emptyset}(\Gamma^+,\emptyset) = \emptyset \), as every curve is a branched cover of a somewhere injective curve with negative expected dimension. Therefore, we have \( U(D^c) \geq \max\{1, k + 1 - n\} \) by Proposition 5.12. On the other hand, the nontrivial relative Gromov-Witten invariant used in Proposition 7.9 implies that \( U(D^c) \geq \max\{1, k + 1 - n\} \) by neck-stretching.

We also have the following generalization of Theorem 7.6 by the same argument.

Theorem 7.22. Let \( X \) be a smooth degree \( m \) hypersurface in \( \mathbb{C}^{p+1} \) for \( 2 \leq m \leq n \) and \( D \) be \( k \geq n \) generic hyperplanes, i.e. \( D = (H_1 \cup \ldots \cup H_k) \cap X \) for \( H_i \) is a hyperplane in \( \mathbb{C}^{p+1} \) in generic position with each other and \( X \), then \( P({\partial D}^c) = k + m - n \) for \( n \) odd and \( k + m < \frac{2n + 1}{2} \).

Proof. We separate the proof into several steps. The Reeb dynamics on \( \partial D^c \) has the same property with the \( \mathbb{C}^p \) case, with the only difference that the minimal Chern number of \( X \) is \( n + 2 - m \), which will enter into the computation of virtual dimensions.
Claim. For any Reeb orbits set \( \Gamma := \{ \gamma_1, \ldots, \gamma_r \} \) for \( r < k + m - n \) with \( \sum [\gamma_i] = 0 \in H_1(D^c) \), the virtual dimension of the moduli space \( \overline{M}_{D^c,A,o}(\Gamma, \emptyset) \) is negative for any \( A \).

Proof. This follows from the same argument in Proposition 7.7, with the difference that \( c_1(u_{\#_{i=1}^r u_i}) = 2N(n + 2 - m) \). Therefore we have

\[
\text{ind}(u) \leq 2N(n + 2 - m) - 4 - \sum_{i=1}^r (2 \sum s_i - 2 + \text{ind}(p_i)) \\
\leq 2N(n + 2 - m) - 4 - 2kN - 4 + 2r \\
= 2(N - 1)(n + 2 - m - k) + 2(r + n - m - k) < 0,
\]

since \( r < m + k - n \) and \( k \geq n, m \geq 2 \). This computation also implies the following claim. This also shows the lower bound.

\[\square\]

Claim. Assume \( D^c \) in the generic intersection of a degree \( k \) hypersurface in \( \mathbb{CP}^{n+1} \) with \( X \). Then we have

\[\eta_{D^c}(q_{\gamma_{\min}}^{k+m-n-1}) \neq 0,\]

and \( q_{\gamma_{\min}}^{k+m-n-1} \) is closed in \( (B^{k+n-m} \cap \partial D^c, \hat{\ell}_{D^c}) \).

Proof. That \( \eta_{D^c}(q_{\gamma_{\min}}^{k+m-n-1} \cdot q_{\gamma_{\min}}^{n}) \neq 0 \) follows from the non-vanishing of \( \text{GW}^{X,D}_{0,1,(n+1-m,1),A}([pt],[D_{k-m-n}],\ldots,[D_{k-m-n}]) \)

from [38] for \( A \) is the positive generator of \( H_2(X) \) and the same argument in Proposition 7.9. The remaining of the argument is exactly same as Corollary 7.10.

Then by the same neck-stretching argument in Proposition 7.16, we have

\[\hat{\ell}_{\cdot,D^c}(q_{\gamma_{\min}}^{k+m-n-1} \cdot q_{\gamma_{\min}}^{n}) = \hat{\ell}_{\cdot,D^c} \circ \hat{\phi}_{D^c}^{-1}(q_{\gamma_{\min}}^{k+m-n-1} \cdot q_{\gamma_{\min}}^{n}) \neq 0.\]

Next Proposition 7.13 and Proposition 7.15 also holds, as the dimension computation there is essentially for trivial homology class, which does not depend on \( m \). It is important to note that in the proof of Proposition 7.15, we use that \( \partial D^c \) is Weinstein is obtain an upper bound of Morse indices. Such property also holds here as we assume \( k \geq n \). Then the remaining of the proof is the same as Theorem 7.6.

From the proof above, the source of holomorphic curves is supplied by the degree 1 holomorphic curves in \( X \) for \( m \leq n \). For \( m = n + 1 \), the degree 1 curve does not unirule \( X \) anymore, but a degree 2 curve unirules \( X \). In the proof Theorem 7.6 and Theorem 7.22, being degree 1 is used in several places to obtain somewhere injectivity (the capping argument). Indeed, for \( m = n + 1 \), the situation is different, we will prove \( \text{P}(\partial D^c) \geq 2 \) for \( D^c \) is a generic intersection of \( X \) with a hyperplane in \( \mathbb{CP}^{n+1} \). For \( m \geq n + 2 \), then \( X \) is not uniruled, which implies \( D^c \) is not k uniruled for any \( k \) by [55], therefore \( \text{P}(\partial D^c) = \infty \) by Corollary 5.15.

In view of Theorem 5.13, Theorem 7.6, Theorem 7.21, and Theorem 7.22, we make the following conjecture.

**Conjecture 7.23.** \( V \) is a \( k \)-uniruled affine variety then \( \text{P}(V) < \infty \) and \( \text{P}(V) = \text{U}(V) = \text{AU}(V) \).

On the other hand, by Theorem 7.12, it is not true that any uniruled affine variety has a contact boundary with finite planararity. It is subtle question to determine which affine variety with a \( \mathbb{CP}^n \) compactification has a finite planararity boundary.

**Question 7.24.** Let \( D \) be \( k \) generic hyperplanes in \( \mathbb{CP}^n \), is \( \text{P}(\partial D^c) \) always finite?
Theorem 7.6 and Theorem 7.22 along with Lemma 7.1 and Lemma 7.3 imply that there are many sequences of contact manifolds where exact cobordisms only exist in one direction. On the other hand, exact embedding problems in the flavor of Theorem 7.21 are studied in [37]. It is an interesting question to determine whether those embedding obstructions can lift to cobordism obstructions.

7.3. Links of singularities. Another natural source of contact manifolds is links of isolated singularities. In the following, we will consider the Brieskorn singularities and quotient singularities from cyclic actions on \( \mathbb{C}^n \).

7.3.1. Brieskorn singularities. A Brieskorn singularity is of the following form

\[
x_0^{a_0} + \ldots + x_n^{a_n} = 0,
\]

for \( 2 \leq a_0 \leq \ldots \leq a_n \). We use \( \bar{a} \) to denote the sequence, the link \( \text{LB}(\bar{a}) \) is defined to be the intersection \( \text{LB}(\bar{a}) := \{(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} | x_0^{a_0} + \ldots + x_n^{a_n} = 0\} \cap S^{2n+1} \), which is \((2n-1)\)-dimensional contact manifold. Moreover, \( \text{LB}(\bar{a}) \) is exactly fillable by the smooth affine variety \( x_0^{a_0} + \ldots + x_n^{a_n} = 1 \), which is called the Brieskorn variety. Moreover, we have the following fact about embedding relations for Brieskorn varieties.

**Proposition 7.25.** We say \( \bar{a} \leq \bar{b} \) iff \( a_i \leq b_i \) for all \( i \). Then if \( \bar{a} \leq \bar{b} \), the Brieskorn variety of \( \bar{a} \) embeds exactly into the Brieskorn variety of \( \bar{b} \). In particular \( \text{LB}(\bar{a}) \leq \text{LB}(\bar{b}) \) in \( \text{Con}_{\leq} \).

Brieskorn varieties were shown in [78, Theorem A] to support \( k \)-dilations. In the following, we will compute the hierarchy functor \( H_{cx} \) for \( \text{LB}(\bar{a}) \). In particular, we will have either a computation or an estimate of \( H_{cx}(\text{LB}(\bar{a})) \) for any \( \bar{a} \) from the theorem below and Proposition 7.25.

**Theorem 7.26.** We use \( \text{LB}(k,n) \) to denote the contact link of the Brieskorn singularity \( x_0^k + \ldots + x_n^k = 0 \), then \( H_{cx}(\text{LB}(k,n)) \) is

1. \((k-1)^{\text{SD}}\) if \( k < n \);
2. \((k-1)^{\text{SD}}\) if \( k = n \) and \( > 1^P \) if \( k = n + 1 \);
3. \( \infty^P \), if \( k > n + 1 \).

**Proof.** The associated Brieskorn variety \( V(k,n) \) carries a \( k-1 \) dilation by [78, Theorem A] when \( k \leq n \). Moreover, the dilation is provided by a simple Reeb orbit \( \gamma_p \) with \( \text{ind}(p) = (2n-2k) \). Since we can consider the filtered theory generated by all simple Reeb orbits, transversality conditions in [12] hold, and the order of semi-dilation of \( \text{LB}(k,n) \) for augmentation from \( V(k,n) \) is \( k-1 \). In particular, \( H_{cx}(\text{LB}(k,n)) \) is \((k-1)^{\text{SD}}\) for \( k \leq n \). Next we need to argue that this semi-dilation is independent of the augmentation when \( k < n \). To see that, we first claim that \( q_{\gamma_p} \) contribute to \( P = 1 \) is independent of augmentation. If not, we have a non-empty moduli space \( \mathcal{M}_{\text{LB}(k,n),o}(\{\gamma_p\}, \{\gamma_q\}) \), whose expected dimension is \( \text{ind}(q) - \text{ind}(p) + 2 - 2n = \text{ind}(q) + 2k + 2 - 4n < 0 \) when \( k < n \) since \( \text{ind}(q) \leq 2n - 2 \). Therefore planarity of is always 1 if \( k < n \). Moreover, \( u'(q_{\gamma_p}) \) is independent of augmentation as we are at the minimal period, there is no room for \( u \) to depend on augmentations. To see the case \( k = n + 1 \), since the log-Kodaira dimension of the corresponding Brieskorn variety \( V \) is 0, we know that \( V \) is not algebraically 1-uniruled. Hence the planarity is greater than 1 by Corollary 5.15. When \( k > n + 1 \), the Brieskorn variety admits a compactification that is not uniruled, hence the planarity is infinity by Corollary 5.15.

**Remark 7.27.** If Conjecture 6.6 was proven, one can get better estimate for \( H_{cx}(\text{LB}(\bar{a})) \) by writing \( \text{LB}(\bar{a}) \) as an open book with a Brieskorn variety page. In the context of symplectic cohomology, computation in such spirit can be found in [78, Proposition 3.27].
Proof of Theorem B. This theorem is a combination of Theorem 3.16, Theorem 3.17, Theorem 3.21, Corollary 5.15, Theorem 6.5, Corollary 6.9 and Theorem 7.26.

7.3.2. Quotient singularities by cyclic groups. Let \( \mathbb{Z}_k \) acts on \( \mathbb{C}^n \) on the diagonal action by multiplying \( e^{2\pi i k} \), then the link of the quotient singularity \( \mathbb{C}^n/\mathbb{Z}_k \) is the quotient contact manifold \((S^{2n-1}/\mathbb{Z}_k, \xi_{std})\). Such contact manifolds provide many examples of strongly fillable but not exactly fillable contact manifolds [79]. In fact, the symplectic part of [79] is a computation of the hierarchy functor \( H_{\text{cx}} \) in the context of symplectic cohomology, which will be rephrased as follows.

**Theorem 7.28.** Let \( Y \) be the quotient \((S^{2n-1}/\mathbb{Z}_k, \xi_{std})\) by the diagonal action by \( e^{2\pi i k} \) for \( n \geq 2 \).

1. If \( n > k \), we have \( H_{\text{cx}}(Y) = 0^{SD} \).
2. If \( n \leq k \), we have \( H_{\text{cx}}(Y) \leq (n-1)^{SD} \).

**Proof.** We follow the same setup as in [79, Proposition 3.1]. We have a non-degenerate contact form on \( \xi_{std} \) by perturbing with a \( C^2 \)-small perfect Morse function \( f \) on \( \mathbb{CP}^{n-1} \), such that Reeb orbits are the following.

1. Reeb orbits of period smaller than \( k+1 \) are \( \gamma_i^j \) for \( 0 \leq i \leq n-1, 1 \leq j \leq k \), where \( \gamma_i^j \) is the \( j \)-multiple cover of \( \gamma_i \) and \( \gamma_i \) projects to the \( i \)th critical point \( q_i \) of \( f \) with \( \text{ind}(q_i) = 2i \).
2. The period of \( \gamma_j \) is \( 1 + \epsilon_j \).
3. \( \epsilon_j < \frac{\epsilon_{j+1}}{k} \).
4. The Conley-Zehnder index of \( \gamma_i^j \) with the natural disk in \( \mathcal{O}(-k) \) satisfies \( \mu_{CZ}(\gamma_i^j) + n - 3 = 2i + 2j - 2 \).

**Claim.** We have \( P(Y) = 1 \) for \( n \geq 2, \forall k \).

**Proof.** By the same argument as [79, Step 3 of Proposition 3.1], we have \( \# \mathcal{M}_{Y,o}(\gamma_i^j, \emptyset) = k \) for \( n \geq 2 \), which is induced from the holomorphic curve in the symplectization of the standard sphere. When \( \Gamma^{-} \neq \emptyset \), we have \( \# \mathcal{M}_{Y,o}(\gamma_i^j, \Gamma^{-}) = 0 \) by action reasons, unless \( \Gamma^{-} = \{ \gamma_0^d \}_{1 \leq i \leq r} \) for \( \sum d_i = k \). In this case, a curve in \( \mathcal{M}_{Y,o}(\gamma_0^k, \Gamma^{-}) \) is necessarily a branched cover over a trivial cylinder. In particular, \( \mathcal{M}_{Y,o}(\gamma_0^k, \Gamma^{-}) = \emptyset \) for generic \( o \). Since all Reeb orbits have even SFT degree, we have \( q_{\gamma_0^k} \) is closed in any linearized contact homology, and the planarity is 1 for any augmentation (which exists) by \( q_{\gamma_0^k} \).

**Claim.** If \( k < n \), then we have \( H_{\text{cx}}(Y) = 0^{SD} \).

**Proof.** By action reasons, \( u(q_{\gamma_0^k}) \) can only have nontrivial coefficients in \( q_{\gamma_0^d} \) for \( d < k \). Note that the filtered linearized contact homology with action supported around \( d \) is generated by \( a_{q_{\gamma_0^d}} \). Since the transversality for all moduli spaces for this filtered homology holds, the argument in [12] implies that it is isomorphic to the filtered \( S^1 \) symplectic cohomology with action centered around \( d \), i.e., the homology is \( H^*(\mathbb{CP}^{n-1}) \) with the \( u \) map is the multiplication by \( c_1(\mathcal{O}(k)) \). As a consequence, we have \( u(q_{\gamma_0^k}) = kq_{\gamma_0^{d_1}} + \sum_{i=1}^{d-1} \sum_{j=0}^{r} a_{ij} q_{\gamma_i^j} \) by action reasons. Therefore for any augmentation, there exist \( c_{ij} \) such that \( u(q_{\gamma_0^k} + \sum_{i=1}^{k-1} \sum_{j=1}^{k-i} c_{ij} q_{\gamma_i^j}) \) by the same argument as [79, (3.2)]. In order to finish the proof, it is sufficient to prove \( \mathcal{M}_{Y,o}(\gamma_i^j, \Gamma^{-}) = \emptyset \) for \( i + j \leq k \) and \( j > 0 \). This follows from the same dimension computation in [79, Step 7 of Proposition 3.1] and is the place where \( n > k \) is essential. Then \( q_{\gamma_0^k} + \sum_{i=1}^{k-1} \sum_{j=1}^{k-i} c_{ij} q_{\gamma_i^j} \) also contributes \( P = 1 \) and is killed by \( u \). In particular, \( H_{\text{cx}}(Y) = 0^{SD} \).

\[ ^{15} \text{It is important to note that now we perturb } f \text{ is the prequantization filling in [79], therefore higher Morse index means lager period, which is reverse to Theorem 7.6.} \]
Claim. If \( k \geq n \), then we have \( 0^{\text{SD}} \leq H_{\text{cx}}(Y) \leq (n-1)^{\text{SD}} \).

**Proof.** First note that all generators have even degree, hence any maps \( \{e^k\}_{k \geq 1} \) form an augmentation. By the argument in Proposition 3.13 and Remark 7.14, we have that \( \#M_{Y,o}(\{\gamma_{0}^{d}\}, \{\gamma_{0}^{d}\}) = 1 \) for \( 1 \leq d \leq k-1 \). On the other hand, following the argument to obtain \([27, \langle \partial(\bar{p}_k^+), \bar{p}_k^- \rangle \) in Theorem 9.1, Lemma 9.4], we know that \( \langle u(q_{\gamma_{k+}}), q_{\gamma_{k-}} \rangle = (k_+ - k_-)e^1(q_{\gamma_{k+}}) \) for augmentation \( \{e^k\}_{k \geq 1} \). If for every \( 1 \leq i \leq k-n \), we have \( e^1(q_{\gamma_{0}}) = 0 \). Then we have \( u^n(q_{\gamma_{0}}) = 0 \), since we have \( u(q_{\gamma_{0}}) = \sum_{j=1}^{k-1}(k-j)e^1(q_{\gamma_{0}})q_{\gamma_{j}} \). Otherwise, we assume \( i \) is the minimum among \( \{1, \ldots, k-n\} \) such that \( e^1(q_{\gamma_{0}}) \neq 0 \). As a consequence, we have planarity 1 contributed by \( q_{\gamma_{2n-2}} \) by \( \#M_{Y,o}(\{\gamma_{2n-2}^{i}\}, \{\gamma_{i}\}) = 1 \). Since \( i \) is the minimal one with nontrivial augmentation, we know that \( u^n(q_{\gamma_{i}}) = 0 \). Hence we have \( H_{\text{cx}}(Y) \leq (n-1)^{\text{SD}} \). \( \square \)

Claim. If \( k = n \), then we have \( H_{\text{cx}}(Y) \geq 1^{\text{SD}} \)

**Proof.** It suffices to find one augmentation such that the order of semi-dilation is 1. We choose our augmentation to be \( e^1(q_{\gamma_{0}}) = -n \) and \( e^k = 0 \) in all other cases. We first list the following expected dimensions of various moduli spaces.

1. \( \text{vdim} \ M_{Y,o}(\{\gamma_{0}^{m}\}, \emptyset) = 2i + 2n(m - 1) \geq 0 \), which is positive, unless \( i = 1, m = 1 \), where we know \( \#M_{Y,o}(\{\gamma_{0}^{0}\}, \emptyset) = 0 \).
2. For \( l > 0 \), \( \text{vdim} \ M_{Y,o}(\{\gamma_{0}^{m+l}\}, \{\gamma_{p_0}, \ldots, \gamma_{p_0}\}) = 2i + 2l + 2n(m - 1) \). Then it is zero iff \( i = n - 1, l = 1, m = 0 \).
3. For \( l > 0 \), \( \text{vdim} \ M_{Y}^{1}(\{\gamma_{0}^{m+l}, \gamma_{j}^{i}\}, \emptyset) = \text{vdim} \ M_{Y}^{2}(\{\gamma_{0}^{m+l}, \gamma_{j}^{i}, \emptyset, \emptyset\} = 2mn + 2i - 2j - 2 \). Then it is zero iff \( m = 0, i = j + 1 \), which corresponds to \( \langle u(q_{\gamma_{i+1}}), q_{\gamma_{j}} \rangle = n \).
4. For \( l, s > 0 \), \( \text{vdim} \ M_{Y}^{1}(\{\gamma_{0}^{m+l+s}, \gamma_{j}^{i}, \{\gamma_{p_0}, \ldots, \gamma_{p_0}\}) = 2mn + 2s + 2i - 2j - 2 \). Then it is zero iff \( m = 0, s = 1, j = i \), which corresponds to \( \langle u(q_{\gamma_{i+1}}), q_{\gamma_{j}} \rangle = e^1(q_{\gamma_{0}}) = -n \).

By (1) and (2), to supply for planarity 1, we must have \( aq_{\gamma_{0}} + bq_{\gamma_{n-1}} \) with \( a - b \neq 0 \). Note that \( u(aq_{\gamma_{0}} + bq_{\gamma_{n-1}}) = -anq_{\gamma_{n-1}} + bnq_{\gamma_{n-2}} \neq 0 \). Therefore if the order of semi-dilation of this augmentation is smaller than 1, then there exists \( A \) generated by generators other than \( q_{\gamma_{n-1}} \), \( q_{\gamma_{n-1}} \), such that \( u(A) = anq_{\gamma_{n-1}} - bnq_{\gamma_{n-2}} \). The only way to eliminate \( -bnq_{\gamma_{n-2}} \) is to have \( bq_{\gamma_{n-2}} \) in \( A \), which adds \( bnq_{\gamma_{n-3}} \) to \( u(A) \). The only way to compensate such term is add a \( bq_{\gamma_{n-3}} \) to \( A \). We can keep the argument going, and claim that \( A \) has \( b \sum_{i=2}^{n-1} q_{\gamma_{n-i}} \), then \( u(A) = bnq_{\gamma_{n-1}} - bnq_{\gamma_{n-2}} \neq anq_{\gamma_{n-1}} - bnq_{\gamma_{n-2}}, \) since \( a - b \neq 0 \). The claim follows. \( \square \)

\footnote{Although such structure originally appears as part of differential in symplectic cochain complex, it contributes to the \( u \)-map in the \( S^3 \)-equivariant symplectic cohomology, see [81, §5] for discussion.}
When \( n \leq k \), there are augmentations with zero order of semi-dilation. For example, one can use the augmentation from natural prequantization bundle filling, then the order of semi-dilation is 0 since the symplectic cohomology vanishes [67]. However, we conjecture that \( H_{cx}(Y) \geq 1^{SD} \) whenever \( n \leq k \). It is possible that there are \( BL_{\infty} \) augmentations that are not from (even singular) fillings. Note that \( n > k \) is the region where the quotient singularity is terminal. Hence we ask whether there is relation between this algebro-geometric property with contact property of the link via the hierarchy functor \( H_{cx} \).

**Conjecture 7.29.** For discrete \( G \subset U(n) \), if \( \mathbb{C}^n/G \) is an isolated singularity, then \( H_{cx}(S^{2n-1}/G, \xi_{std}) = 0^{SD} \) if the singularity is terminal.

Combining with Theorem 7.26, we can also ask the following question.

**Question 7.30.** Is the planarity of an isolated terminal singularity always 1? Is it true for hypersurface singularities?

By a similar argument to Theorem 7.28, we have the following.

**Theorem 7.31.** Let \( V \) be an exact domain, then \( H_{cx}(\partial(V \times \mathbb{D})) = 0^{SD} \).

**Proof.** By Corollary 6.9, we have \( P(\partial(V \times \mathbb{D})) = 1 \). Note the planarity is provided the simple Reeb orbit wrapping around \( V \). Then the transversality in [12] holds, we can use the symplectic cohomology description to compute the order of semi-dilation, which is zero for augmentation from \( V \times \mathbb{D} \) by [63]. The relevant \( u \)-map is again independent of augmentations by action reasons similar to Theorem 7.26. The claim follows. \( \square \)

The computation in Theorem 7.31 carried out in the form of symplectic cohomology is the symplectic input used to prove uniqueness results of exact fillings for \( \partial(V \times \mathbb{D}) \) in [80].

### 7.4. An obstruction to IP. 
In dimension 3, obstructions to planar open book decomposition were studied from many different perspectives in [31, 64]. In higher dimensions, obstructions to supporting an iterated planar structure were found in [5]. By Corollary 5.15 and Theorem 6.5, we the following easy to check obstruction to iterated planar structure.

**Corollary 7.32.** If contact manifold \( Y \) admits an exact filling that is not \( k \)-uniruled for any \( k \), then \( Y \) is not iterated planar.

As an application of this corollary, we have the following.

**Corollary 7.33.** Let \( Q \) be a hyperbolic manifold of dimension \( \geq 3 \), then \( S^*Q \) is not iterated planar.

**Proof.** The claim follows from a result of Viterbo [28, Theorem 1.7.5] that \( T^*Q \) is not \( k \)-uniruled for any \( k \). \( \square \)

For other classes of cosphere bundles, by Theorem 7.26, \( H_{cx}(S^*S^n) = 1^{SD} \) for \( n \geq 2 \). By Corollary 6.9, \( H_{cx}(S^*T^n) = 2^P \) for \( n \geq 2 \). Assuming Claim 4.9 holds, since \( SH^*(T^*Q) \neq 0 \) for any \( Q \), we know \( H_{cx}(S^*Q) > 0^{SD} \). As a consequence there is no exact cobordism from \( S^*Q \) to \( \partial(V \times \mathbb{D}) \) for any Liouville domain \( V \), which is a generalization of a result of Gromov [42]. By [78, Proposition 5.1], \( T^*Q \) admits a \( k \)-dilation for some \( k \geq 1 \) for rationally-inessential \( n \)-manifold \( Q \), i.e. if \( H_n(Q; \mathbb{Q}) \to H_n(B\pi_1(Q); \mathbb{Q}) \) vanishes, then we can update the estimate \( H_{cx}(S^*Q) \) by figuring out \( k \). For Lagrangian \( Q \) that is a \( K(\pi, 1) \) space, we have \( H_{cx}(S^*Q) \geq 2^P \), since \( T^*Q \) carries no \( k \)-semi-dilation for any \( k \).
Corollary 7.34. For every \( n \geq 3 \), there exists a tight \( S^{2n-1} \) with the standard almost contact structure that is not iterated planar.

Proof. Note that the contact boundary of the Brieskorn variety \( x_0^{n+2} + \ldots + x_n^{n+2} = 1 \) has planarity order \( \infty \) by Theorem 7.26. Then we can increases the indexes to get another Brieskorn manifold \( Y \) which is an exotic sphere. By the monoidal structure and functoriality of \( H_{\text{cx}} \), we know \( H_{\text{cx}}(\#^k Y) = \infty^P \), where \( \# \) is the contact connected sum. Since there exists \( k \) such that \( \#^k Y \) is the standard smooth sphere with the standard almost contact structure. The claim follows.

Corollary 7.35. In all dimension \( \geq 5 \), if \((Y,J) \) is an almost contact manifold which has an exactly fillable contact representation \((Y,\xi)\). Then there is a contact structure \( \xi' \) in the homotopy class of \( J \), such that \((Y,\xi')\) is not iterated planar. In particular, any almost contact simply connected 5-manifolds admits a contact representation which is not iterated planar.

Proof. Let \( Y' \) be the tight sphere from corollary 7.34, since \( P(Y) > 0 \) as \( Y \) has an exact filling, then \( H_{\text{cx}}(Y\# Y') = \infty^P \). By Corollary 7.32, \( Y\# Y' \) is not iterated planar. The last claim follows from any almost contact simply connected 5-manifold is almost Weinstein fillable [39], in particular, there is a contact representation that is Weinstein fillable by [20].

7.5. The order on \( \text{Con} \). It is a natural question to ask whether the poset \( \text{Con}_{\leq} \) is a totally ordered set. For this, we need to throw out \( \emptyset \), since an overtwisted contact manifold is obviously not comparable with \( \emptyset \). We use \( \text{Con}_{\neq \emptyset} \) denote the full subcategory without \( \emptyset \). We first give a simple argument for Weinstein cobordisms.

Proposition 7.36. In any odd dimension \( \geq 15 \), There exists two contact manifolds, such that there are no Weinstein cobordisms in either direction.

Proof. Take \( Y = \partial(V \times T^*S^3) \) such that \( V \) is the 2n-dimensional Liouville domain that is not Weinstein constructed in [53] for \( n \geq 5 \). Then \( Y \) is asymptotically dynamically convex by [83, Theorem K] and \( \text{im} \Delta_g \) contains an element of degree \( 2n-1 \), hence \( Y \) is not Weinstein fillable by [83, Corollary 4.19]. Therefore there is no Weinstein cobordism from \((S^{2n+5},\xi_{\text{std}})\) to \( Y \). On the other hand, since any exact filling of \((S^{2n+5},\xi_{\text{std}})\) has vanishing symplectic cohomology [68, Corollary 6.5] but \( SH^*(V \times T^*S^3) \neq 0 \), we know that there is no exact cobordism from \( Y \) to \((S^{2n+5},\xi_{\text{std}})\).

Remark 7.37. One can replace \((S^{2n+5},\xi_{\text{std}})\) in the proof above by a flexibly fillable representative of the maximal element in the almost Weinstein cobordism category with vanishing first Chern class [18, Theorem 1.2]. Then at least one side of the non-existence does not follow from a topological obstruction.

On the other hand, when we consider strong cobordisms, there are a lot more morphisms. In particular, if we include \( \emptyset \) into the discussion, the existence of symplectic cap [25, 49] implies that anything with a strong filling is equivalent to \( \emptyset \) in \( \text{Con}_{\leq} \). Even if we throw out \( \emptyset \) and even restrict to the case of connected strong cobordisms, the existence of strong cobordism is much less rigid compared to the the counterparts for exact or Weinstein cobordisms by [74]. Nevertheless, the following question seems to be open.

Question 7.38. Is there a pair of contact manifolds without connected strong cobordisms in either direction?

We can also consider the analogous question in \( \text{Con} \).

Question 7.39. Is there a pair of contact manifolds without connected exact cobordisms in either direction?
When dimension is 3, an affirmative answer to the above question is explained to us by Chris Wendl based on the not exactly fillable contact manifold \((Y, \eta_0)\) found by Ghiggini [40]. As observed in [16], \(Y\) admits a Liouville pair \((\eta_0, \eta)\) [53, Definition 1], so in particular there is a connected exact filling for \((Y, \eta_0) \sqcup (-Y, \eta)\). If there is an exact cobordism from \((Y, \eta_0)\) to \((S^3, \xi_{std})\), then there is a connected exact filling of \((S^3, \xi_{std}) \sqcup (-Y, \eta)\), contradicting that \((S^3, \xi_{std})\) is not co-fillable [31]. On the other hand, by [40], there is no exact cobordism from \((S^3, \xi_{std})\) to \((Y, \eta_0)\). In the following, we will explain a strategy for higher dimensional cases.

We adopt the notation of [19], and consider the Bourgeois manifold \(\text{BO}(D^*S^n, \tau^k)\) associated to the open book \(\text{OBD}(D^*S^n, \tau^k)\), where \(n \geq 1, k \geq 1\), and \(\tau\) is the Dehn-Seidel twist. It was shown in [19, Theorem G and Remark 1.2] that the set

\[
\text{Fill}(n) = \{ k \in \mathbb{Z} \mid \text{BO}(D^*S^n, \tau^k) \text{ is strongly fillable} \}
\]

is a subgroup, and satisfies

\[
\text{Fill}(n) = k_0(n)\mathbb{Z},
\]

for some \(k_0(n) > 1 \in \mathbb{N}\). In particular, for every \(n\), there exists arbitrarily large \(k\) for which \(\text{BO}(D^*S^n, \tau^k)\) is not strongly fillable.

On the other hand, those contact structures are weakly fillable by [51, Theorem A]. By [53, Proposition 6], a weak filling can always be deformed into a stable Hamiltonian filling near the positive end\(^{17}\), and a stable Hamiltonian filling for a contact structure has all the essential structures to define SFT, c.f. [24, 48]. One place we need to pay attention to is potential holomorphic caps possibly with multiple negative puncture, which will break the \(BL_\infty\) algebra structure (without deformation) in general.

**Definition 7.40.** Let \(M\) be a \(2n - 1\) oriented dimensional manifold. A SHS is a pair \((\lambda, \omega)\) of a one form \(\lambda\) and a two form \(\omega\) such that the following holds.

1. \(\lambda \wedge \omega^{n-1} > 0\).
2. \(\ker \omega \subset \ker d\lambda\).

The Reeb vector field \(R\) associated to \((\lambda, \omega)\) is determined by \(\lambda(R) = 1\) and \(\iota_R \omega = 0\). To relate SHS with contact structure, we will also consider the following cobordisms, which is a combination of [48, Definition 1.9] and [76, §6.2].

**Definition 7.41.** A symplectic manifold \((X, \omega)\) is a stable cobordism from a SHS \((M_-, (\lambda_-, \omega_-))\) to a (co-oriented) contact \((M_+, \xi)\) if the following holds.

1. \(\partial W = M_- \sqcup M_+\).
2. \(\omega_- = |\omega|_{M_-}, \omega|_{\xi}\) is non-degenerate and the induced orientation is the same as the orientation on \(\xi\).
3. The stabilizing vector field \(V_-\) determined \(\lambda_- = \iota_{V_-} \omega\) near \(M_-\) points inward along \(M_-\).
4. There exists a non-degenerate contact form \(\alpha\) for \(\xi\), such that the Reeb vector field \(R_\alpha\) generates the characteristic line field on \(M_+\). The stabilizing vector field \(V_+\) determined \(\alpha = \iota_{V_+} \omega\) near \(M_+\) points outward along \(M_+\).
5. \(\xi\) admits a complex structure \(J\) which is tamed by both \(d\lambda|_{\xi}\) and \(\omega|_{\xi}\).

We refer readers to [76, §6] for almost complex structure \(J\) on the completion \(\hat{X}\). One key condition is that \(J\) is tamed by both \(d\lambda\) and \(\omega\) on \(\xi\) on the positive cylindrical end. We shall call such almost complex structure admissible. The importance of the last two conditions of the Definition 7.41 is that the compactification of

\(^{17}\)This is not true for negative end.
Proposition 7.42. There is stable cobordism \( X := ([0,1] \times \text{OBD}(D^*S^n, \tau^k)) \times \mathbb{T}^2, \omega \) from the SHS \( \mathcal{H}_- := (\text{OBD}(D^*S^n, \tau^k) \times \mathbb{T}^2, (\alpha, \omega T_2 + \omega_{\tau^2})) \) to the contact manifold \( Y_+ := \text{BO}(D^*S^n, \tau^k) \), where \( \alpha \) is any contact form on \( \text{OBD}(D^*S^n, \tau^k) \). Moreover, we have \( \omega = \pi^* \omega_{\tau^2} \in H^2(X) \) for \( \pi : X \to \mathbb{T}^2 \) with \( \omega_{\tau^2} \) is a rational volume form on \( \mathbb{T}^2 \).

We note the following.

Lemma 7.43. For any admissible \( J \) on the cobordism in Proposition 7.42, there exists no non-constant punctured rational \( J \)-holomorphic curve with no positive ends.

Proof. Assume that \( u \) is a non-constant rational holomorphic curve having only negative ends, if any. Since the Reeb vector field of \( \mathcal{H}_- \) is \( R_{\alpha} \) on \( \text{OBD}(D^*S^n, \tau^k) \), the map \( v = \pi \circ u \), where \( \pi \) is the projection to \( \mathbb{T}^2 \) is the natural projection, extends to a continuous map on the closed sphere. Since \( \pi_2(\mathbb{T}^2) = 0 \), \( v \) is necessarily contractible, and so \( \int_u \omega_{\tau^2} = \int_u \omega_{T^2} = 0 \). If \( A \geq 0 \) denotes the total \( \alpha \)-action of the negative asymptotics of \( u \), we then see that

\[
0 < \int_u \omega = -A \leq 0,
\]

a contradiction.

Due to the absence of caps, we can define a \( BL_\infty \) algebra for \( \mathcal{H}_- \) and the stable cobordism in Proposition 7.42 defines a \( BL_\infty \) morphism from RSFT\((Y_+)\) to RSFT\((\mathcal{H}_-)\), with one caveat that we need to use a group ring coefficient as in [48, Proposition 2.5] to obtain compactness. More precisely, RSFT\((\mathcal{H}_-)\) is defined using \( \text{Q}[H_2(\mathcal{H}_-)/\text{ker} \omega_{\tau^2}] \) (the completion of the group ring with respect to the evaluation by \( \omega_{\tau^2} \)). It is natural to expect that the hierarchy of the split \( \mathcal{H}_- \) using the group ring coefficient is at least the hierarchy of \( \text{OBD}(D^*S^n, \tau^k) \) as we can use a split \( J \) for the SHS. The functorial property of the stable cobordism holds for RSFT\((Y_+)\) with coefficient in \( \text{Q}[H_2(Y_+)/\text{ker} \omega|_{Y_+}] \). Since \( H^\text{cx}(\text{BO}(D^*S^n, \tau^k)) \geq 1^{SD} \) by Theorem 7.26, whose argument works for any coefficient for \( k \geq 2 \), then we have the hierarchy of \( Y_+ \) using \( \text{Q}[H_2(Y_+)/\text{ker} \omega|_{Y_+}] \) coefficient is at least \( 1^{SD} \) for \( k \geq 2 \). Hence, we ask the following question.

Question 7.44. Is \( H^\text{cx}(\text{BO}(D^*S^n, \tau^k)) \geq 1^{SD} \) (in \( \text{Q} \) coefficient) for \( k \geq 2 \)? In general, is \( H^\text{cx}(\text{BO}(V, \phi)) \geq H^\text{cx}(\text{OBD}(V, \phi))? \)

If the answer to the above question is affirmative, then \( \text{BO}(D^*S^n, \tau^k), (S^{2n+3}, \xi_{\text{std}}) \) are pair of contact manifolds without exact cobordisms in either directions for \( k \geq 2 \not\in \text{Fill}(n) \) by Theorem A, as the hierarchy of \( (S^{2n+1}, \xi_{\text{std}}) \) is \( 0^{SD} \) for any coefficient by the argument of Theorem 7.31. In the group ring context, since the group ring is generated by \( \mathbb{T}^2 \), the functorial property of \( BL_\infty \) morphism for the hypothetical exact cobordism from \( \text{BO}(D^*S^n, \tau^k) \) to \( (S^{2n+3}, \xi_{\text{std}}) \) holds for the group ring coefficient if \( H_2(\text{BO}(D^*S^n, \tau^k)) \to H_2(X) \) is injective on \( \mathbb{T}^2 \), see [48, §2.1] for more discussion on group ring coefficients.

Proposition 7.45. There is no exact cobordism \( X \) from \( Y := \text{BO}(D^*S^n, \tau^k) \) for \( k \geq 2 \) to \( (S^{2n+3}, \xi_{\text{std}}) \) such that \( H_2(Y) \to H_2(X) \) is injective on \( \mathbb{T}^2 \).

\(^{18}\)The rationality of \( \omega_{\tau^2} \) is required in [24, Prop 2.18] to obtain [53, Proposition 6].
The hierarchy functor $\mathcal{H}_{cx}$ can only obstruct exact cobordisms in one direction. Another natural way to answer Question 7.39 is building a functor from $\mathcal{C}on$ to a poset, then the preimage of two incomparable elements from the poset will be incomparable in $\mathcal{C}on$. A natural candidate theory is the grid of torsions (or the analogous grid of planarity) introduced in §3.8. However, it is a much harder task to construct examples that we can apply the machinery effectively to answer Question 7.39. One potential example with interesting torsions was discussed in [58].

It is known that a overtwisted contact manifold is the least element in $\overline{\mathcal{C}on}_{<W}$ in dimension 3 [32], i.e. there is an exact cobordism from an overtwisted manifold to any contact 3-manifold. It is natural to question about the other direction. An element in a poset is called maximal, if there is no element strictly greater than it. An element is greatest if every element is smaller than it.

**Question 7.46.** Is there a maximal element in $\overline{\mathcal{C}on}_{<}$? Is there is a greatest element in $\overline{\mathcal{C}on}_{<}$?

In the context of Weinstein cobordism, there are geometric constructions of a contact manifold $Y$ that is Weinstein cobordant both from $Y_1$ and $Y_2$ given that there are Weinstein cobordism from $Y'$ to $Y_1, Y_2$ in dimension $\geq 5$ [49]. However it is not clear if such construction would yield something larger in $\overline{\mathcal{C}on}_{<}$ or $\overline{\mathcal{C}on}_{<,W}$. On the other hand, in $\overline{\mathcal{C}on}_{<,S}$, $\emptyset$ is the greatest element. Even in $\overline{\mathcal{C}on}_{<,S}^{\neq \emptyset}$, there are still greatest element by those co-fillable manifolds [75], which exists in all dimensions [53].

In many cases, algebraic torsions and algebraic planar torsions can not be used to differentiate between strongly fillable and exact fillable. One of the reasons is that we can derive a $BL_{\infty}$ augmentation over $\mathbb{Q}$ from a $BL_{\infty}$ augmentation over $\Lambda$ or more generally the group ring coefficient, if some finiteness assumption is satisfied, e.g. Theorem 7.28, we can still have $BL_{\infty}$ augmentations over $\mathbb{Q}$, while the existence of exact fillings are obstructed [79]. Therefore a natural question is the following.

**Question 7.47.** Is there a SFT obstruction sensitive to strong/exact fillings?

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