A FIVE-VARIABLE GENERALIZATION OF
RAMANUJAN’S RECIPROCITY THEOREM
AND ITS APPLICATIONS

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Abstract. By virtue of Bailey’s well-known bilateral \( \psi_6 \) summation formula and Watson’s transformation formula, we extend the four-variable generalization of Ramanujan’s reciprocity theorem due to Andrews to a five-variable one. Some relevant new \( q \)-series identities including a new proof of Ramanujan’s reciprocity theorem and of Watson’s quintuple product identity only based on Jackson’s transformation are presented.

Keywords: \( q \)-series; reciprocity theorem; Ramanujan’s \( \psi_1 \) summation formula; Jacobi’s triple product identity; Watson’s quintuple product identity; transformation formula.

1. Introduction

In his lost notebook \([11, p.40]\), Ramanujan offered without proof a beautiful \( q \)-series identity, which is now called Ramanujan’s reciprocity theorem.

Theorem 1.1. For \( a, b \neq q^{-n} \), it holds

\[
\rho(a, b) - \rho(b, a) = \left( \frac{1}{b} - \frac{1}{a} \right) \frac{(q, aq/b, bq/a; q)_{\infty}}{(-aq, -bq; q)_{\infty}},
\]

where

\[
\rho(a, b) = \left( 1 + \frac{1}{b} \right) \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{(-aq; q)_k} \left( \frac{a}{b} \right)^k.
\]

Ramanujan’s reciprocity theorem has been proved to be very useful to partial theta function identities. For further details on this subject, the reader can refer the forthcoming second volume of Ramanujan’s lost notebook \([4]\) by Andrews and Berndt. It has been an active topic, in the past years, to find possibly short and easy proofs for Theorem 1.1. Up to now, various approaches have been found by many mathematicians. For our purpose of this paper, we only mention a few remarkable results. As is known to us, Andrews \([3, Theorem 1]\) gave the first proof of Theorem 1.1 whereas it seems a bit complicated, by setting a four-variable generalization of Theorem 1.1 \([3, Theorem 6]\). In 2003, Liu \([10, Theorem 6]\) showed the four-variable generalization of Andrews by \( q \)-exponential operator identity. During the last two years, Adiga-Anitha...
[1], Berndt et al [5] found independently that this theorem can be verified by Heine’s transformation for $_2\phi_1$ series. Besides, Berndt et al presented another simpler analytic proof and an elegant combinatorial proof in [5]. An interesting phenomenon is that almost all known analytic proofs utilized Ramanujan’s $_1\psi_1$ summation formula and the Rogers-Fine identity. In a very recent paper [12], Kang rederived Andrews’ four-variable generalization from Sears’ three-term relation between $_3\phi_2$-series [8, III.33], which had been claimed as a long-waited problem proposed by Andrews and Agarwal.

We now restate the four-variable generalization of Theorem 1.1 in Kang’s form for further discussion.

**Theorem 1.2.** For four parameters $a, b, c, d$ satisfying $c, d \neq -aq^{-m}, -bq^{-n}$, $n, m \geq 0$, $0 < |d| < |b|$, it holds

\[
\rho(a, b; c, d) - \rho(b, a; c, d) = \left( \frac{1}{b} - \frac{1}{a} \right) \frac{(q, aq/b, bq/a, c, d, cd/ab; q)_{\infty}}{(-aq, -bq, -c/b, -d/b; q)_{\infty}}.
\]

where

\[
\rho(a, b; c, d) = \left( 1 + \frac{1}{b} \right) \sum_{k=0}^{\infty} \frac{(c, -aq/d; q)_{k}}{(-aq; q)_{k}(-c/b; q)_{k+1}} \left( -\frac{d}{b} \right)^{k}
\]

\[
= \left( 1 + \frac{1}{b} \right) \sum_{k=0}^{\infty} q^{\frac{k+1}{2}} \frac{(1 + cdq^{2k}/b)(c, d, cd/ab; q)_{k}}{(-aq; q)_{k}(-c/b; q)_{k+1}} \left( -\frac{a}{b} \right)^{k}.
\]

For the connections of this four-variable reciprocity theorem with some well known $q$-series identities such as Jacobi’s triple and Watson’s quintuple product identity, the reader can consult [12]. Here, we only point out that Identity (1.3) given by Kang is equivalent to the four-variable generalization of Theorem 1.1 of Andrews [3, Theorem 6]. This fact is stated clearly by (4.10) of [12]. One of the most important results of Kang, in the author’s point of view, is that she established the new expression (1.5) for the function $\rho(a, b; c, d)$.

In the present paper, motivated by the method of Kang, we will extend Theorem 1.2 to the following five-variable form.

**Theorem 1.3.** For five parameters $a, b, c, d, e$ satisfying

\[ 0 < |cde| < |abq|, c, d, e \neq -aq^{-m}, -bq^{-n}, n, m \geq 0, \]

it holds

\[
\rho(a, b; c, d, e) - \rho(b, a; c, d, e) = \left( \frac{1}{b} - \frac{1}{a} \right) \frac{(q, aq/b, bq/a, c, d, ce/(ab), de/(ab); q)_{\infty}}{(-aq, -bq, -c/a, -c/b, -d/a, -d/b, -e/a, -e/b, cde/(abq); q)_{\infty}}.
\]
where

$$\rho(a, b; c, d, e) = \sum_{k=0}^{\infty} \left( 1 - \frac{aq^{2k+1}}{b} \right) \frac{(-1/b; q)_{k+1}}{(-c/b, -d/b, -e/b; q)_{k+1}} \times \frac{(-aq/c, -aq/d, -aq/e; q)_k}{(-aq; q)_k} \frac{(cde)^k}{(abq)}.$$

(1.7)

As application of Theorem 1.3, several new \(q\)-series identities will also be derived.

To make our paper self-contained, we will repeat a few standard notation and terminology for basic hypergeometric series (or \(q\)-series) found in [8]. Given a (fixed) complex number \(q\) with \(|q| < 1\), a complex number \(a\) and an integer \(n\), define the \(q\)-shifted factorials \((a; q)_{\infty}\) and \((a; q)_n\) as

$$\begin{align*}
(a; q)_\infty &= \prod_{k=0}^{\infty} (1 - aq^k), \\
(a; q)_n &= (a; q)_\infty (aq^n; q)_\infty.
\end{align*}$$

(1.8)

We also employ the following compact multi-parameter notation:

$$\prod_{k=0}^{\infty} (1 - aq^k), \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}.$$

In addition, the compact notation \(r\phi_{r-1}\left[ a_1, \ldots, a_r; b_1, \ldots, b_{r-1}; q, z \right]\) denotes the special case of the above \(r\phi_{r-1}\) called a very-well-poised series, in which all parameters satisfy the relations

$$qa_1 = b_1a_2 = \cdots = b_{r-1}a_r; a_2 = q\sqrt{a_1}, a_3 = -q\sqrt{a_1}.$$

2. Proof of the main result

Our argument entirely relies on Bailey’s very-well-poised \(6\psi_6\) summation formula of bilateral \(q\)-series.

Lemma 2.1 (Bailey’s very-well-poised \(6\psi_6\) summation formula). (cf. [8 II.33])

$$\begin{align*}
6\psi_6 \left[ \frac{q\sqrt{a}, -q\sqrt{a}, b, c, d, e; q^{2q}}{\sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e; bcdde} \right] &= \left( q, aq, qa/q/a, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/dec; q \right)_{\infty} \\
&= \left( aq/b, aq/c, aq/d, aq/e, qa/b, qa/c, qa/d, qa/e, a^2q/(bcde); q \right)_{\infty}.
\end{align*}$$
Proof of Theorem 1.3. First, let’s define for \(0 < |cde| < |abq|\) that

(2.1) \[
\rho_0(a, b; c, d, e) = \sum_{k=0}^{\infty} \left(1 - \frac{aq^{2k+1}}{b} \right) (-q/b, -aq/c, -aq/d, -aq/e; q)_k \frac{(cde)^k}{(abq)}.
\]

Then according to (1.8)

(2.2) \[
(a; q)_n = \frac{1}{(q/a; q)_n} \left(-\frac{q}{a}\right)^n q^{\binom{n}{2}}
\]

we can proceed to the following computation

\[
\rho_0(b, a; c, d, e) = \sum_{k=0}^{\infty} \left(1 - \frac{bq^{2k+1}}{a} \right) (-q/a; q)_k (-bq/c, -bq/d, -bq/e; q)_k \frac{(cde)^k}{(abq)}
\]

\[
= c_1 \sum_{k=0}^{\infty} \left(1 - \frac{bq^{2k+1}}{a} \right) (-aq/a; q)_k (-bq/c, -bq/d, -bq/e; q)_k \frac{(cde)^k}{(abq)}
\]

\[
= \ast \sum_{k=-\infty}^{-1} \left(1 - \frac{bq^{-2k-1}}{a} \right) (-1/b; q)_{k+1} (-aq/c, -aq/d, -aq/e; q)_k \frac{(cdeq)^k}{ab}
\]

\[
= c_1 c_2 \sum_{k=-\infty}^{-1} \left(1 - \frac{aq^{2k+1}}{b} \right) (-aq/a; q)_k (-aq/c, -aq/d, -aq/e; q)_k \frac{(cde)^k}{abq}
\]

where the line marked with “\(\ast\)” has been justified by the replacement \(k \rightarrow -k - 1\) and the three constant \(c_i\) with \(i = 1, 2, 3\) are given respectively by

\[
c_1 = \frac{1}{(1+b/c)(1+b/d)(1+b/e)}, \quad c_2 = \frac{a^2 b^2 q}{cde};
\]

\[
c_3 = -\frac{b(1+1/b)(1+c/b)(1+d/b)(1+e/b)}{aq(1+a)}.
\]

It is easy to verify that

\[
c_1 c_2 c_3 = -\frac{a(1+b)}{b(1+a)}.
\]

Consequently, we find that

(2.3) \[
\sum_{k=-\infty}^{-1} \left(1 - \frac{aq^{2k+1}}{b} \right) (-aq/a; q)_k (-aq/c, -aq/d, -aq/e; q)_k \frac{(cde)^k}{abq}.
\]
Observing that the right members of (2.1) and (2.3) form a bilateral $q$-series, we have
\[
\rho_0(a, b; c, d, e) - \frac{b(1 + a)}{a(1 + b)} \rho_0(b, a; c, d, e) = \frac{(1 - aq/b)}{(1 + c/b)(1 + d/b)(1 + e/b)}
\]
\[
\rho_0(a, b; c, d, e) = (1 - aq/b) \times 6\psi_6 \left[ q \sqrt{aq/b}, -q \sqrt{aq/b}, -q/b, -aq/c, -aq/d, -aq/e, cde/abq \right].
\]

Recalling Bailey’s very-well-poised $6\psi_6$ summation formula displayed in Lemma 2.1, we get
\[
\rho_0(a, b; c, d, e) = (1 - aq/b) \times 6\psi_6 \left[ q, aq/b, bq/a, c, d, e, cd/(ab), ce/(ab), de/(ab); q \right]_\infty.
\]

Multiplying both sides of this identity by $a(1 + b)$ and simplifying the resulting identity, we finally get
\[
(2.4)
\]
\[
(a - b) \rho_0(a, b; c, d, e) = (a + b) \rho_0(a, b; c, d, e) - b(1 + a) \rho_0(b, a; c, d, e)
\]
\[
= (a - b) \frac{(q, aq/b, bq/a, c, d, e, cd/(ab), ce/(ab), de/(ab); q)_{\infty}}{(-aq, -bq, -e/a, -e/b, -c/b, -d/b, -e/a, -d/a, cde/(abq); q)_{\infty}}.
\]

Define
\[
\rho(a, b; c, d, e) = \left(1 + \frac{1}{b}\right) \rho_0(a, b; c, d, e).
\]

Dividing both sides of (2.4) by $ab$ and rewriting the left-hand side of the resulting expression in terms of $\rho(a, b; c, d, e)$, we have the formula stated in the theorem.

We remark that once letting $e \mapsto 0$ in Theorem 1.3 with the expressions of $\rho(a, b; c, d, e)$ given in the next section, we get Theorem 1.2 while letting $c, d, e \mapsto 0$ simultaneously, Theorem 1.1 follows. The limiting procession is guaranteed by the convergent condition.

3. Applications

By finding new representations for both $\rho(a, b; c, d)$ and $\rho(a, b; c, d, e)$, we shall establish, in this section, some new $q$-series identities. This will be realized by following Kang’s approach [12] and employing Watson’s $q$-analogue of Whipple’s transformation formula between $8W_7$ and $4\phi_3$ series (cf. [8, III.17]).

Lemma 3.1. Let $a, b, c, y, z, w$ be such complex numbers that the following $8W_7$ series is convergent and the $4\phi_3$ series is terminating. Then it holds
\[
8W_7 \left( a; b, c, y, z, w; q, \frac{a^2q^2}{bcyzw} \right) = \frac{(aq, aq/(yz), aq/(yw), aq/(zw); q)_{\infty}}{(aq/y, aq/z, aq/w, aq/(yzw); q)_{\infty}}
\]
\[
\times 4\phi_3 \left[ aq/(bc), y, z, w \right. \left. aq/b, aq/c, yzw/a ; q, q \right].
\]
Observe that the limiting case \( n \mapsto \infty \) of Watson’s transformation (3.1), under the specification that
\[
b \mapsto c/b, c \mapsto aq/c, w \mapsto q^{-n},
\]
turns out to be
\[
\sum_{k=0}^{\infty} \frac{(b, y, z; q)_{k}}{(q, c, abq/c; q)_{k}} \left(\frac{aq}{yz}\right)^{k} = \frac{(aq/y, aq/z; q)_{\infty}}{(aq, aq/(yz); q)_{\infty}}
\]
\[
\times \sum_{k=0}^{\infty} (-1)^{k} q^{\binom{k}{2}} \frac{1 - aq^{2k}}{(1 - a)q} \left(\frac{ab}{c/b, c, aq/c, aq/y, aq/z; q}_{k}\right) \left(\frac{abq}{yz}\right)^{k}.
\]
For \( z = q \), this becomes

**Lemma 3.2.** For \( \max\{|a/y|, |ab/y|\} < 1 \), there holds
\[
\sum_{k=0}^{\infty} \frac{(b, y; q)_{k}}{(c, abq/c; q)_{k}} \left(\frac{a}{y}\right)^{k} = \sum_{k=0}^{\infty} (-1)^{k} q^{\binom{k}{2}} \frac{1 - aq^{2k}}{(1 - a/yq)} \left(\frac{ab}{c/b, aq/c, aq/y, aq/z; q}_{k}\right) \left(\frac{abq}{yz}\right)^{k}.
\]

Now, we are in a position to show

**Theorem 3.1.** Let \( \rho(a, b; c, d) \) be the same as in Theorem 1.2. Then
\[
\rho(a, b; c, d)
\]
\[
= \sum_{k=0}^{\infty} q^{\binom{k}{2}} (1 - aq^{2k+1}/b) \left(-\frac{1/b}{q}\right)_{k+1} (1 - aq/c, -aq/d; q)_{k+1} \left(\frac{cd}{b}\right)^{k}
\]
\[
\times \sum_{k=0}^{\infty} (-1)^{k} q^{\binom{k+1}{2}} \left(1 + \frac{1}{b}\right) (c, d, cd/(ab); q)_{k} \left(-\frac{a/b}{q}\right)_{k+1} \left(\frac{abq}{c/b, -d/b; q}_{k}\right) \left(\frac{a/b}{cd}\right)^{k}.
\]

**Proof.** Define
\[
h(a, b; c, d) = \sum_{k=0}^{\infty} \frac{(c, -aq/d; q)_{k}}{(-aq; q)_{k}(-c/b; q)_{k+1}} \left(-\frac{d}{b}\right)^{k}.
\]
By making the substitutions
\[
\begin{align*}
a & \mapsto aq/b \\
b & \mapsto c \\
c & \mapsto -aq \\
y & \mapsto -aq/d
\end{align*}
\]
and
\[
\begin{align*}
a & \mapsto -cd/b \\
b & \mapsto -aq/d \\
c & \mapsto -aq \\
y & \mapsto c
\end{align*}
\]
in (3.3) and then dividing the resulting identities by \( 1 + c/b \), we obtain
\[
h(a, b; c, d) = \sum_{k=0}^{\infty} q^{\binom{k}{2}} (1 - aq^{2k+1}/b) \left(-\frac{a/b}{q}\right)_{k+1} (c, d, cd/(ab); q)_{k} \left(-\frac{a/b}{q}\right)_{k+1} \left(\frac{a/b}{cd}\right)^{k}
\]
\[
= \sum_{k=0}^{\infty} (-1)^{k} q^{\binom{k+1}{2}} (1 + cdq^{2k}/b) \left(-\frac{a/b}{q}\right)_{k+1} \left(\frac{a/b}{cd}\right)^{k}.
\]
Keeping in mind of the fact that
\[ \rho(a, b; c, d) = \left(1 + \frac{1}{b}\right) h(a, b; c, d), \]
we get the desired result.

Obviously, (3.4) was missed by Kang and not recorded in Theorem 1.2. On taking the convergent conditions of Watson’s transformation into account, we obtain an alternative representation for \( \rho(a, b; c, d, e) \).

**Theorem 3.2.** Let \( \rho(a, b; c, d, e) \) be the same as in Theorem 1.3, at least one of the parameters \( c, -aq/d, -aq/e \) be of the form \( q^{-m}, m \geq 0 \). Then
\[
\rho(a, b; c, d, e) = 1 + b \left(1 + \frac{1}{b}\right) \left(1 - \frac{aq}{b}\right) \sum_{k=0}^{\infty} \frac{(c, -aq/d, -aq/e; q)_k}{(-aq, -cq/b, abq^2/(de); q)_k} q^k.
\]

**Proof.** Note that in this case, we are able to apply Watson’s transformation to Theorem 1.3 in order to get
\[
\rho(a, b; c, d, e) = c_4 c_5 \sum_{k=0}^{\infty} \frac{(c, -aq/d, -aq/e; q)_k}{(-aq, -cq/b, abq^2/(de); q)_k} q^k,
\]
where the two constant \( c_4 \) and \( c_5 \) are defined by
\[
c_4 = \frac{(1 + 1/b)(1 - aq/b)}{(1 + c/b)(1 + d/b)(1 + e/b)}, \quad c_5 = \frac{(1 + d/b)(1 + e/b)}{(1 - aq/b)(1 - de/abq)}.
\]
with their product equal to
\[
c_4 c_5 = \frac{1 + b}{(b + c)(1 - de/(abq))}.
\]
Therefore Theorem 3.2 follows.

A few special cases of interest may be displayed as follows.

**Corollary 3.1.** For two integers \( r, s \geq 0 \), it holds
\[
\begin{align*}
1 + b & \sum_{k=0}^{r} \frac{r + s - k}{c + b} \left(1 + \frac{1}{b}\right) \left(1 - \frac{aq}{b}\right) \sum_{k=0}^{s} \frac{r + s - k}{c + a} \frac{(c, -aq/d; q)_k}{(-aq, -cq/b; q)_k} q^{(s+1)k} \\
&= \frac{1 + b}{a} \frac{1}{1 + r + s} \frac{(aq/b; q)_r(bq/a; q)_s(c; q)_1}{(-aq; q)_r(-aq; q)_s(-c/b; q)_{r+1}(-c/a; q)_{s+1}}.
\end{align*}
\]
where the \( q \)-binomial coefficient \( \left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}} \).
Proof. It follows from combination of Theorem 1.3 and Theorem 3.2 with \( d = -aq^{1+r} \) and \( e = -bq^{1+s} \). We are not going to produce the tedious simplification involved.

Putting \( b \rightarrow \infty \) (or \( b \rightarrow 0 \)) in (3.7), then a curious \( q \)-series identity follows.

**Corollary 3.2.** Let \( a \neq 0 \). Then for any two integers \( r, s \geq 0 \), there holds

\[
\sum_{k=0}^{r} \frac{r + s - k}{r - k} \binom{r}{k} \frac{(c; q)_k}{(aq; q)_k} q^{(s+1)k} = \frac{-1 + a}{c + a} \sum_{k=0}^{s} \frac{r + s - k}{s - k} \binom{r}{k} \frac{(c; q)_k}{(-cq/a; q)_k} \left( \frac{-1}{a} \right)^k.
\]

(3.8)

Further, if we let \( a \) tend to zero in (3.8), then we have

**Corollary 3.3.** Let \( c, q \neq 0 \). Then for any two integers \( r, s \geq 0 \), there holds

\[
\sum_{k=0}^{r} \frac{r + s - k}{r - k} \binom{r}{k} \frac{(c; q)_k}{(aq; q)_k} q^{(s+1)k} = (c^{-1}; q^{-1})_{1+s} (cq^{1+s}; q)_r c^{-1} \sum_{k=0}^{s} \frac{r + s - k}{s - k} \binom{r}{k} (c^{-1}; q^{-1})_k q^{-k}.
\]

(3.9)

Next, letting \( q \rightarrow 1 \) in (3.9) and replacing \( c \) by \( 1/(1-x) \), we obtain a finite series transformation of interest, which may be considered as a supplement to the classical Pfaff transformation for the Gauss hypergeometric function \( \text{$_2F_1(x)$} \) [2, p.68, Theorem 2.2.5].

**Corollary 3.4.** For any two integers \( r, s \geq 0, x \neq 1 \),

\[
\sum_{k=0}^{r} \binom{r + s - k}{s} \frac{x^k}{(x-1)^r} = \frac{x^{r+s+1}}{(x-1)^r} + (1-x) \sum_{k=0}^{s} \binom{r + s - k}{r} x^k,
\]

(3.10)

where \( \binom{n}{k} \) stands for the usual binomial coefficient.

We remark that the special case \( r = s = n \) and \( x = 2 \) of (3.10) is revealed to be (1.81) by Gould [9]:

\[
\sum_{k=0}^{n} \binom{2n - k}{n} 2^k = 2^{2n}.
\]

The case that \( c \) is of the form \( q^{-m} \) in Theorem 3.2 also deserves our consideration.
Corollary 3.5. Assume the conditions in Theorem 3.3. Let \(a, b \neq -1\) and \(m\) is a nonnegative integer. Then

\[
\frac{1 + b}{1 + bq^m} \sum_{k=0}^{m} \frac{(q^{-m}, -aq/d, -aq/e; q)_k}{(-aq, q^{-1-m}/b, abq^2/(de); q)_k} q^k = \frac{1 + a}{1 + aq^m} \sum_{k=0}^{m} \frac{(q^{-m}, -bq/d, -bq/e; q)_k}{(-bq, q^{-1-m}/a, abq^2/(de); q)_k} q^k.
\]

(3.11)

Proof. It suffices to insert \(\rho(a, b; c, d, e)\) given by Theorem 3.2 and then set \(c = q^{-m}\) in Theorem 1.3. After a bit of simplification, it yields the result as claimed.

Since

\[
\frac{(q^{-m}; q)_k}{(-q^{-1-m}/b; q)_k} = \frac{(q; q)_m(-b; q)_{m-k}}{(q; q)_{m-k}(-b; q)_m} \left(\frac{-b}{q}\right)^k,
\]

there exists the limiting case \(m \to \infty\) of (3.11), which may be stated as

Corollary 3.6. For \(|a|, |b| < 1\), it holds

\[
(1 + b) \sum_{k=0}^{\infty} \frac{(-aq/d, -aq/e; q)_k}{(-aq, abq^2/(de); q)_k} (-b)^k = (1 + a) \sum_{k=0}^{\infty} \frac{(-bq/d, -bq/e; q)_k}{(-bq, abq^2/(de); q)_k} (-a)^k.
\]

(3.12)

In fact, Corollary 3.6 is a generalization of the symmetric property of the Rogers-Fine function [7, Eq.(6.3)]:

\[
(1 - b) \sum_{k=0}^{\infty} \frac{(aq/d; q)_k b^k}{(aq; q)_k} = (1 - a) \sum_{k=0}^{\infty} \frac{(bq/d; q)_k a^k}{(bq; q)_k}.
\]

(3.13)

Two special cases of this corollary are of interest.

Corollary 3.7. For \(|d| < |q|, a \neq -q^m, m \leq 0\), the following hold:

\[
\sum_{k=0}^{\infty} \frac{(-aq/d; q)_k}{(-a; q)_{k+1}} \left(\frac{d}{q}\right)^k = \frac{q}{q - d};
\]

(3.14)

\[
\sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} a^k}{(-a; q)_{k+1}} = 1.
\]

Proof. It suffices to show (3.13) since (3.14) is the case \(d = 0\) of it. For this, write

\[
f(a, d) = \sum_{k=0}^{\infty} \frac{(-aq/d; q)_k}{(-a; q)_{k+1}} \left(\frac{d}{q}\right)^k.
\]
Actually it holds that
\[
\begin{align*}
    f(a, d) &= \frac{1}{1 + a} \lim_{e \to 0} \sum_{k=0}^{\infty} \frac{(-aq/d, -aq/e; q)_k}{(-aq, abq^2/(de); q)_k} (-b)^k \\
    &\overset{(3.12)}{=} \frac{1}{1 + b} \lim_{e \to 0} \sum_{k=0}^{\infty} \frac{(-bq/d, -bq/e; q)_k}{(-bq, abq^2/(de); q)_k} (-a)^k
    = f(b, d),
\end{align*}
\]
which means that \( f(a, d) \) is independent of the variable \( a \). Hence, we have
\[
f(a, d) = f(0, d) = \frac{q}{q - d}.
\]
The theorem is proved.

Identity (3.14) as well as its combinatorial interpretation was also discovered by Kang [12, Corollary 7.4].

4. Some remarks on Theorem [1.1]

We end this paper by offering a new form of Ramanujan’s reciprocity theorem, which in turn leads us to a new proof for itself and for the two-variable generalization of Watson’s quintuple product identity given by Berndt et al. A comprehensive survey on the history and various proofs for the latter can be found in [6].

**Theorem 4.1.** For \(|x| < 1, |q| < |a|\), it holds

\[
(4.1) \quad \xi(a, x) - \frac{q}{ax} \xi(q/x, q/a) = \frac{(q, ax, q/(ax); q)_{\infty}}{(x, q/a; q)_{\infty}},
\]
where

\[
(4.2) \quad \xi(a, x) = \sum_{k=0}^{\infty} (a; q)_k x^k = \sum_{k=0}^{\infty} \frac{q(k)}{q} (-ax)^k (x; q)_{k+1}.
\]

**Proof.** Observe that the case \( c = 0 \) of Jackson’s transformation [8 III.4]

\[
(4.3) \quad \sum_{k=0}^{\infty} \frac{(a, y; q)_k}{(q, q)_k} x^k = \frac{(xy; q)_{\infty}}{(x; q)_{\infty}} \sum_{k=0}^{\infty} q^{(k)} (-ax)^k (y; q)_k (xy; q)_k
\]

In particular, when \( y = q \), it follows that

\[
(4.4) \quad \xi(a, x) = \sum_{k=0}^{\infty} q^{(k)} (-ax)^k (x; q)_{k+1}.
\]

Hence, by (4.4) and (2.2), it is easily found that

\[
(4.5) \quad \xi(a, x) - \frac{q}{ax} \xi(q/x, q/a) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{(k)}}{(x; q)_{k+1}} (ax)^k.
\]
Evaluating the last sum by Ramanujan’s \(_1\psi_1\) summation formula
\[
\sum_{k=-\infty}^{\infty} \frac{(-1)^k q^k}{(x; q)_{k+1}} (ax)^k = \frac{(q, ax, q/(ax); q)_\infty}{(x, q/a; q)_\infty},
\]
we finally get
\[
\xi(a, x) - q\frac{a}{ax} \xi(q/x, q/a) = \frac{(q, ax, q/(ax); q)_\infty}{(x, q/a; q)_\infty},
\]
which proves the theorem.

It is also worth pointing out that Ramanujan’s original reciprocity theorem, i.e., Theorem 1.1 follows from (4.1) by setting \(x = -aq, a = -1/b\). Note that in this case
\[
\rho(a, b) = \frac{1}{b} \xi(-1/b, -aq).
\]
This fact was used by Kang. See [12, Eq.(3.5)]. From (4.5) it is clear why almost all known analytic proofs employed Ramanujan’s \(_1\psi_1\) summation formula.

The next is a new proof of Theorem 1.1 i.e., (4.1), without invoking Ramanujan’s \(_1\psi_1\) summation formula.

**Proof of Theorem 1.1** At first, replace \(a\) instead of \(b\) by \(q\) in (4.3) to get
\[
(4.6) \quad \xi(y, x) = \frac{(xy; q)_\infty}{(x; q)_\infty} \lim_{b \to 0} _2\phi_1 \left[ x/t, y \quad xy; q, tq \right].
\]
On the other hand, by making the substitution
\[
a \mapsto x/t, b \mapsto y, c \mapsto xy, z \mapsto tq
\]
in the three-term transformation formula [8 III.31], we obtain that
\[
_2\phi_1 \left[ x/t, y \quad xy; q, tq \right] - \frac{q}{xy} \frac{(x, q^2/xy, ty; q)_\infty}{(q/y, xy, tq/x; q)_\infty} _2\phi_1 \left[ q/ty, q/x \quad q^2/xy; q, tq \right] = \frac{q(q/(xy); q)_\infty}{(q/y, tq/x; q)_\infty},
\]
from which it follows that
\[
(4.7) \quad \lim_{b \to 0} _2\phi_1 \left[ x/t, y \quad xy; q, tq \right] = \frac{q}{xy} \frac{(x, q^2/xy; q)_\infty}{(q/y, xy; q)_\infty} \lim_{b \to 0} _2\phi_1 \left[ q/ty, q/x \quad q^2/xy; q, tq \right] = \frac{(q, q/(xy); q)_\infty}{(q/y; q)_\infty}.
\]
Taking (4.6) into account and multiplying both sides of (4.7) by \((xy; q)_\infty/(x; q)_\infty\) we have the formula stated in the theorem.

With Theorem 1.1 it will be easier to derive the two-variable generalization of Watson’s quintuple product identity due to Berndt et al. [3 Theorem 3.1].
Corollary 4.1.

\[
\sum_{k=0}^{\infty} \frac{(1 - xzq^{2k})(z; q)_k}{(x; q)_{k+1}} q^{k(3k-1)/2} (-1)^k \left( x^2 z \right)^k
\]

(4.8)

\[
- \frac{q}{zx} \sum_{k=0}^{\infty} \frac{1 - q^{2k+2}/(xz)}{(q/z; q)_{k+1}} \frac{(q/x; q)_k}{(q/z; q)_{k+1}} q^{k(3k-1)/2} (-1)^k \left( q^3/(z^2 x) \right)^k
\]

\[
= \frac{(q, zx, q/(zx); q)_\infty}{(x, q/z; q)_\infty}.
\]

Proof. Recall that the limiting case \( c \to 0 \) while \( b = c^2 \) of (3.2) yields

\[
\sum_{k=0}^{\infty} \frac{(y, z; q)_k}{(q; q)_k} \frac{aq}{yz} = \frac{(aq/y, aq/z; q)_\infty}{(aq, aq/(yz); q)_\infty}
\]

\[
\times \sum_{k=0}^{\infty} \frac{1 - aq^{2k}}{1 - a} \frac{(a, y, z; q)_k}{(q, aq/y, aq/z; q)_k} (-1)^k q^{3k(3)} \frac{a^2 q^2}{yz}.
\]

Next, make the substitution \( y \mapsto q, a \mapsto xz \) in this identity so that the both sides become respectively

\[
\xi(z, x) = \sum_{k=0}^{\infty} (z; q)_k x^k
\]

\[
= \sum_{k=0}^{\infty} \frac{(1 - xzq^{2k})(z; q)_k}{(x; q)_{k+1}} q^{k(3k-1)/2} (-1)^k \left( x^2 z \right)^k.
\]

Substituting the last relation in (4.1) leads us to the claimed result. \( \blacksquare \)

In particular when we set \( x = -a, z = -aq \) in (4.8), then Watson’s celebrated quintuple product identity [6] follows immediately

\[
\sum_{k=-\infty}^{\infty} (a^2 q^{2k+1} - 1)a^{3k+1} q^{k(3k+1)/2} = (q, a, q/a; q)_\infty ( qa^2, q/a^2; q^2)_\infty.
\]

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