RKDG method solution for hyperbolic hyperelastic model

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Abstract. The natural for solid dynamics applications mathematical model has a form of the first-order hyperbolic system of conservation laws in the Eulerian reference frame. This model is proposed by Godunov and Romenski [1] and has a number of characteristic features from more traditional second-order formulations employed in elasticity. The most important is the presence of a complex structure of contact, shock and rarefaction waves and their interactions. Numerical solution of such problems is a difficult task due to a complex wave structure of the solution. Recently a number of efficient approaches were presented to handle a problem. Most of them are based on linearized HLL type solvers and high order WENO reconstruction techniques to obtain accurate fronts resolution in order to resolve complex solution structure. They also use Eulerian schemes based on solving Riemann problems for solid materials. In this paper we apply high order RKDG method as an alternative to WENO schemes to solve the problem. Up to 5-th order reconstruction is considered with 3-rd order timesteppng scheme. The Riemann-free-solver approach is used with simple flux reconstruction. As a trial problem one dimensional hyperelastic formulation is presented similar to [1]. The developed computer code utilizes automatic differentiation (AD) approach which allows for an easy switch of underlying EOS and strain measure. The presented results cover discussion of the problem statement, corresponding RKDG approximations and simulation results that demonstrate possibilities of RKDG approach applied for the hyperbolic hyperelastic formulations in the Eulerian reference frame.

1. Introduction
Recently it has been paid a substantial attention for simulation of high velocity and penetration problems using Eulerian formulations for solid dynamics with finite strain. The main advantage of such approach is possibility to handle large to extreme deformations when application of the more traditional arbitrary eulerian-lagrangian formulations are problematic. The common approach is to use a hyperelastic or Green elastic material model which is a type of constitutive model for ideally elastic material. In this model a strain-energy (or stored-energy) function $U$ is used to define a hyperelastic material by postulating that the derivative of $U$ can be used to obtain the stress-strain relationship. The most common example of this kind of material is rubber, whose stress-strain relationship can be defined as non-linearly elastic, isotropic, incompressible and generally independent of strain rate.

A significant number of literature has been published on modelling of hyperelastic material. A wide variety of works is associated with the need to introduce a proper strain energy function $U$, which can be specified in various ways, and accurate determination of needed material parameters [2].
In this paper the solid media is considered in the Eulerian reference frame. Such approach is potentially well suited for modelling large deformations and discontinuous waves, but it is rather complicated.

The purpose of this work is to apply high-order discontinuous Galerkin method with additional techniques to avoid oscillations near discontinuities or shocks which lead to unbounded computational solutions. This approach can be considered as an alternative to the most common approaches based on linearized HLL type solvers and high order WENO reconstruction. High-order schemes are necessary for modelling collision problems and interactions of continuous media.

2. Governing theory

To describe deformation of solid media in the Eulerian reference frame the model of Godunov and Romenski [1] is used.

The complete three-dimensional system of momentum, strain and energy conservation laws is given respectively by:

\[
\begin{align*}
\frac{\partial}{\partial t} (\rho u) + \frac{\partial (\rho u \otimes u - T)}{\partial x} &= 0, \quad (1a) \\
\frac{\partial}{\partial t} (\rho F) + \frac{\partial (\rho F \otimes u - \rho u \otimes F^T)}{\partial x} &= 0, \quad (1b) \\
\frac{\partial}{\partial t} (\rho E) + \frac{\partial (\rho u E - u \otimes T)}{\partial x} &= 0. \quad (1c)
\end{align*}
\]

Here \( \rho \) is density given by equation \( \rho = \rho_0 / \det (F) \), \( T \) is Cauchy stress tensor, \( \rho_0 \) is the initial density of unstressed medium, \( E = U + |u|^2 / 2 \) is the total energy. Primary variables characterized the solid media state are the elastic deformation gradient \( F = \nabla X x \), where \( x \) and \( X \) denote the eulerian and material coordinates of the initial state respectively, velocity \( u \) and entropy \( S \).

The combination of above equations recovers the continuity equation (mass conservation law) [3]: \( \partial \rho / \partial t + \nabla \cdot (\rho u) = 0 \).

To close the system a specific type of equation of state for specific internal energy is used. It is shown in [4] that the internal energy must be expressed in terms of some symmetric strain tensor to satisfy frame indifference:

\[
U = U (F, S). \quad (2)
\]

As mentioned above, there are a lot of forms of the potential (2). Formulations of a number of the common strain tensors in terms of \( F \) are given in [5]. In this paper Finger tensor is considered: \( G = F^{-T} F^{-1} \).

In terms of \( G \) the Cauchy stress tensor is given by Murnaghan formula [5]

\[
T = -2 \rho G \cdot \frac{\partial U}{\partial G}. \quad (3)
\]

According to (3), stress tensor is calculated as the derivative of specific thermodynamic potential. The main requirement imposed on the form of the potential is the symmetry of the resulting stress tensor.

It is shown in [5] that the specific internal energy for a hyperelastic isotropic medium depends on principle invariants of Finger tensor:

\[
U = U (I_1, I_2, I_3, S). \quad (4)
\]

Here \( I_1 = \text{tr} (G), I_2 = \left[ \text{tr} (G)^2 - \text{tr} (G^2) \right] / 2, I_3 = \det (G) \).
The specific internal energy can be divided into hydrostatic potential part $U^h$ and shear deformation term $U^{sh}$:

$$U = U(I_1, I_2, I_3, S) = U^h(I_3, S) + U^{sh}(I_1, I_2, I_3, S). \tag{5}$$

Further development of this model can be found in [8].

In present paper the isotropic hyperelastic equation of state from [8] is used. The corresponding terms are given by:

$$U^h(I_3, S) = \frac{K_0}{2\alpha^2} \left(I_{3}^{n/2} - 1\right)^2 + c_v T_0 I_{3}^{n/2} \left(\exp \{S/c_v\} - 1\right) \tag{6}$$

and

$$U^{sh}(I_1, I_2, I_3) = B_0 \frac{\beta^2}{\gamma^2} \left(\frac{I_1^2}{3} - I_2\right)/2. \tag{7}$$

Here $K_0 = c_0^2 - (4/3) b_1^2$, $B_0 = b_2^2$ are the squared bulk speed of sound and the squared speed of shear wave, respectively, $c_v$ is heat capacity at constant volume, $\alpha, \beta, \gamma$ are nonlinear coefficients of equation of state.

3. Numerical method
The system of equations (1a), (1b), (1c) is solved numerically. Two types of numerical schemes of different orders are implemented for the purpose of comparing the results.

3.1. First-order finite volume method
In one and three dimensional cases computational domain is discretized into cells. Each cell is denoted $I_i = [x_{i-1/2}, x_{i+1/2}]$ with the size $(\Delta x)_i = x_{i+1/2} - x_{i-1/2}$. The finite volume scheme is used to obtain the solution. The resulting system is given by:

$$\frac{dU_i}{dt} = F(U^n_i). \tag{8}$$

After time discretization the complete finite volume scheme is given by:

$$U^{n+1}_i = U^n_i - \frac{\Delta t}{\Delta x} \left(\frac{F^n_{i+1/2} - F^n_{i-1/2}}{2}\right), \tag{9}$$

where $U^{n+1}_i = U_i(t + \Delta t), U^n_i = U_i(t)$. Right-hand side is calculated using calculation of flux with Rusanov scheme:

$$F^{n+1}_{j+1/2} = F(U^n_j) + F(U^n_{j+1})/2 - c_{max} (U^n_{j+1} - U^n_j), \tag{10a}$$

$$F^{n}_{j-1/2} = F(U^n_j) + F(U^n_{j-1})/2 - c_{max} (U^n_j - U^n_{j-1}). \tag{10b}$$

Max wave propagation speed $c_{max}$ is estimated as maximal eigenvalue of acoustic tensor: $A = \rho^{-1} \{\partial T_{ki}/\partial F_{ij}\}$.

3.2. Runge-Kutta discontinuous Galerkin method
To increase the order of the method the discontinuous Galerkin approach (DG) is used [10].

The semi-discrete weak formulation employs local discontinuous Galerkin formulation in spatial variables within each cell $K$ is written as:

$$\int_{\partial D_k} \left(\frac{\partial h^{k}_{l_j} \phi_{l_j}}{\partial t} (x) - f^{l_j}_{k} \frac{d\phi_{l_j}}{dx} (x)\right) dx + \int_{D_k} n_f \phi_{l_j} (x) dx = 0, \tag{11}$$
where \( l^k_j(x) \) denote Lagrange polynomial basis. Solution representation in finite element is given by:

\[
u^k_h = \sum_{i=1}^p u^k_i l^k_i(x).
\]

(12)

The resulting system is given by:

\[
M \frac{du^k_h}{dt} = R(u^k_h, t),
\]

(13)

where \( M \) denotes the diagonal mass matrix, \( u^k_h \) is global vector of \( p \) degrees of freedom and \( R \) is the residual vector. The further discretization is obtained by using strong stability preserving TVD Runge-Kutta (TVDRK) method of the third order of accuracy [6]:

\[
U^{(1)} = U^n + \Delta t R(U^n), \quad U^{(2)} = \frac{3}{4} U^n + \frac{1}{4} U^{(1)} + \frac{\Delta t}{4} R(U^{(1)}), \quad U^{n+1} = \frac{1}{3} U^n + \frac{2}{3} U^{(1)} + \frac{2\Delta t}{3} R(U^{(2)}).
\]

3.3. Slope limiter

In order to reduce solution oscillations of high-order schemes the slope moment limiter introduced by Krivodonova [7] is used. It applies progressively limiting the high-order terms firstly if the solution starts to oscillate. Formally, it achieves the highest possible accuracy when some limiting is necessary. In this work it is conservative variables that are limited.

4. Implementation

The results are obtained using the automatic differentiation library STAN [9] to calculate quantities derived from EOS such as \( T \) in (3) and diffusive term of flux with its magnitude \( c_{\text{max}} \) in (10a), (10b). The equation of state is given analytically and is represented as an independent class in program implementation. This allows for the flexibility of specifying the equation of state. Furthermore, Riemann-solver-free discontinuous Galerkin method considered in this work could be implemented for systems without knowing the full eigenstructure. This avoids a number of difficulties and adds variability of approach. The numerical solution is obtained using C++ with template meta programming approach.

5. Results and discussion

The five wave example in [3] is taken as a basis for further validation of the numerical scheme presented in this paper. The initial conditions are

\[
X_L = \begin{cases} \boldsymbol{u}_l = \begin{pmatrix} 0 \\ 0.5 \\ 1 \end{pmatrix} \text{km s}^{-1}, & \boldsymbol{F} = \begin{pmatrix} 0.98 & 0 & 0 \\ 0.02 & 1 & 0.1 \\ 0 & 0 & 1 \end{pmatrix}, & S = 10^{-3} \text{kJ g}^{-1} \text{K}^{-1} \end{cases}, \]

(14)

\[
X_R = \begin{cases} \boldsymbol{u}_r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{km s}^{-1}, & \boldsymbol{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.1 \\ 0 & 0 & 1 \end{pmatrix}, & S = 0 \text{kJ g}^{-1} \text{K}^{-1} \end{cases}.
\]

(15)

An additional degree of shear deformation as described in [8] is added in order to obtain full seven wave structure to further comparison of different schemes approximation order. There are three left travelling rarefaction waves, one right travelling contact wave, right travelling shock wave and two right travelling rarefactions waves. Figures (1) and (2) show density profiles obtained using first-order scheme (fig. 1) and RKDG5 scheme (fig. 2) with 500 cells at time 0.6\( \mu \text{s} \). Figures (3) and (4) show velocity fields at the same conditions.

As a result of comparison with the first-order finite volume method it is noted that RKDG method allows to achieve high numerical accuracy. It can be implemented to hyperelastic model in order to obtain the solution of the collision and interaction problems.

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Figure 1. Density profile with first-order space and time scheme.

Figure 2. Density profile with RKDG5 scheme.

Figure 3. Velocity field with first-order space and time scheme.

Figure 4. Velocity field with RKDG5 scheme.

6. Conclusion
The purpose of the present study was to apply existing high-order RKDG method with Krivodonova limiter to the non-linear hyperelastic model in Eulerian reference frame. This method was implemented to resolve to a seven waves structures in the Riemann problem. The results of applying this method lead to the following conclusions:

• RKDG can be successfully applied to a given class of problems;
• In spite of the fact that the Riemann-solver-free approach is used with simple flux and the completion of simple or conservative variables, the RKDG method shows improvement in resolving waves and can be used;
• Results using high-order RKDG method are essentially non-oscillatory when Krivodonova limiter is implemented. Further research in this direction will be put to improve the current situation;
• In comparison with the currently known approaches, this approach shows similar results when comparing to WENO scheme using HLLC flux and characteristic based approximation of the Riemann problem [8].

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