The effective potential in the presence of several mass scales

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Abstract

We consider the problem of improving the effective potential in mass independent schemes, as e.g. the $\overline{\text{MS}}$ or $\text{DR}$ renormalization scheme, in the presence of an arbitrary number of fields with $\phi$-dependent masses $M_i(\phi_c)$. We use the decoupling theorem at the scales $\mu_i = M_i(\phi_c)$ such that the matching between the effective (low energy) and complete (high energy) one-loop theories contains no thresholds. We find that for any value of $\phi_c$, there is a convenient scale $\mu^* \equiv \min_i\{M_i(\phi_c)\}$, at which the loop expansion has the best behaviour and the effective potential has the least $\mu$-dependence. Furthermore, at this scale the effective potential coincides with the (improved) tree-level one in the effective field theory. The decoupling method is explicitly illustrated with a simple Higgs-Yukawa model, along with its relationship with other decoupling prescriptions and with proposed multi-scale renormalization approaches. The procedure leads to a nice suppression of potentially large logarithms and can be easily adapted to include higher-loop effects, which is explicitly shown at the two-loop level.

IEM-FT-181/98
September 1998

*Work supported in part by the European Union (contract CHRX/CT92-0004) and CICYT of Spain (contract AEN95-0195).
1 Introduction

Unlike the case of the standard model (SM), where one-loop radiative corrections to the effective potential are essentially dominated by the top quark, all extensions thereof involve new mass scales corresponding to new physics. A well-celebrated example is the supersymmetric standard model (MSSM) [1] where, beside the top quark scale $m_t$, there are mass scales corresponding to the stops $m_{\tilde{t}_{L,R}}$ which are strongly coupled to the Higgs sector and thus provide very important contributions to the effective potential: it is a typical multi-scale case. Moreover, if there is a large mixing in the stop sector, the two stop mass eigenvalues become very split, which represents by itself a multi-scale problem. Understanding the behaviour of the effective potential in multi-scale cases is then essential to handle a number of physical problems associated with them. One example is the supersymmetric standard model and, in particular, the calculation of the Higgs boson mass spectrum from the effective potential [2]. A good treatment of the effective potential would permit, in particular, to attack the above-mentioned case of large stop splitting, which still remains as an open question. Another application would be to establish the relationship between the scale of new physics and the possible instability scale of the standard model [3]. This precise relation can be of the highest interest specially in the case that the standard model Higgs boson is light.

Let us briefly review the state of the art concerning the computation of the effective potential, focusing in the case of multi-scale scenarios. The improved effective potential in any mass-independent renormalization scale can be expanded in a loop expansion as [4]:

\[ V_{\text{eff}}(\lambda_a(t), \phi(t), \mu(t)) = V^{(0)} + V^{(1)} + \cdots \]  

(1.1)

where $\lambda_a(t)$ are all dimensionless and dimensionful couplings of the theory, running with the renormalization group equation (RGE) scale $\mu(t) \equiv e^{t}\mu$, and $\phi(t) = \xi(t)\phi_c$ is (are) the running field(s) with $\xi(t) = \exp(-\int_0^t \gamma(t')dt')$, $\gamma(t)$ being the anomalous dimension of the $\phi$-field. In the loop expansion (1.1), $V^{(0)}$ is the improved tree level potential while the one-loop correction can be written as

\[ V^{(1)} = \kappa \sum_i V^{(1)}_i \]

\[ V^{(1)}_i = \frac{n_i}{4} \frac{M_i^4(\phi)}{\mu_i^2} \log \frac{M_i^2(\phi)}{\mu_i^2} \]  

(1.2)

where $\kappa = 1/16\pi^2$, $n_i$ is the number of degrees of freedom of the $i$-th field (negative for fermions), $M_i^2(\phi)$ are the (tree-level) mass eigenvalues, $\mu_i^2 \equiv e^{C_i}\tilde{\mu}^2$ ($\tilde{\mu}$ is the renormalization scale in the corresponding scheme, typically $\overline{\text{MS}}$ or $\overline{\text{DR}}$), and $C_i$ are scheme dependent constants [4].

\footnote{For instance, in the $\overline{\text{MS}}$ scheme, $C_i = 3/2$ (5/6) for scalar bosons and fermions (gauge bosons), respectively.}
It is evident from (1.2) that in the case of a single mass scale, say \( M_i(\phi) = m(\phi) \), a judicious choice of the renormalization scale \( \mu^2 \sim m^2(\phi) \) removes large logarithms from the one-loop corrections. Actually, this choice amounts to a resummation of all the leading-logarithms at all order in loop perturbation theory \( [1] \) (if the \( \lambda_a, \phi \) parameters are running with the two-loop RGE, the next-to-leading logarithms are also resummed \( [3] \)). However, if there are very different mass scales, e.g. \( m(\phi) \) and \( M(\phi) \), any choice of the renormalization scale will leave large logarithms in (1.2), as \( \log[M^2(\phi)/m^2(\phi)] \) which jeopardize the validity of perturbation theory.

There are two proposed solutions in the literature to the multi-scale improved effective potential. One, introduced by Einhorn and Jones \( [6] \), and recently modified by Ford and Wiesendanger \( [7] \), proposing the presence of independent renormalization scales \( \mu_i \) which can then be adjusted differently to take care of the different mass scales. This procedure is however highly heterodox, involves as many RGE for each parameter as mass scales in the problem and seems very difficult to be applied to realistic cases with possibly many mass scales. Another solution, more physical in our opinion, was studied by Bando, Kugo, Maekawa and Nakano \( [8] \), and consists in recognizing that the origin of the problem in mass-independent renormalization schemes is that the decoupling is not automatically included in the formalism and has to be incorporated \( [3] \). Then all masses smaller than a given scale behave as massless while larger masses decouple and never generate problems with perturbation theory. Below a decoupling scale, the theory is an effective field theory with new RGE and, eventually, with threshold effects taking care of the matching of both theories at the decoupling scale \( [10] \). These authors, however, have identified the decoupling scale of the \( i \)-th particle with the \( \phi \)-independent mass \( m_i \) eventually contributing to the mass eigenvalue \( M_i(\phi) \). This is not a very convenient and/or physical choice because: \( \text{a)} \) In the presence of the background field \( \phi \), particles propagate with the mass \( M_i(\phi) \), which is therefore the relevant scale for decoupling; \( \text{b)} \) For large values of the field \( \phi \), such that the main contribution to the mass \( M_i(\phi) \) does not come from \( m_i \), it cannot be a good approximation; \( \text{c)} \) It does not contemplate the case where the decoupled particle does not have an invariant mass; \( \text{d)} \) It does not fit some generic situations, as e.g. the case where there are large mixings between different states. This is the case of the stops \( \tilde{t}_{L,R} \) where besides the invariant masses \( m_{\tilde{t}_L} \) and \( m_{\tilde{t}_R} \) there is the mixing parameter \( A_t \). In that case decoupling at \( m_{\tilde{t}_L} \) and \( m_{\tilde{t}_R} \) would while \( C_i = \frac{3}{2} \) for all fields in the \( \overline{\text{DR}} \) scheme. In a theory where all fields have equal value of \( C_i \), as e.g. in a theory without gauge bosons, renormalized in the \( \overline{\text{MS}} \) scheme, or an arbitrary theory renormalized in the \( \overline{\text{DR}} \) scheme, we can reabsorb all finite constants in a redefinition of the renormalization scale: \( e^{C_i} \mu^2 \rightarrow \mu^2 \). This will be the case in the simple example considered in this paper.

\( \text{2A modified mass dependent renormalization scheme has been proposed in Ref.} \ [9] \) to handle with the multimass problem.
be clearly inappropriate for large mixing. In addition, the choice of the value of the renormalization scale proposed in Ref. [8] in order to evaluate the potential, namely $\mu$ equal to the fermion mass, is only well defined when there is a single fermion in the model and, even in that case, it turns out to be inappropriate in many instances, as we will see.

In this paper we will consider the effective potential in the multi-scale case, proposing a new and simple approach based upon the decoupling theorem [11]. The method is fully consistent with perturbation theory, even when there are very different scales in the game, which normally leads to potentially dangerous logarithms. We will illustrate the method with a simple Higgs-Yukawa model and present the rationale to be applied to more realistic and physical models. The general decoupling method is introduced in Section 2, and applied to a simple Higgs-Yukawa model in Section 3, where the numerical analysis has been explicitly performed (in the one-loop approximation) and illustrated with plots. In Section 4 we compare our decoupling approach with the multi-scale renormalization proposals. In Section 5 we generalize the analysis of Section 3 by including two-loop corrections. We explicitly show how the dangerous logarithms continue to be suppressed at this level, confirming the perturbative validity of the approach. Section 6 is devoted to our conclusions and outline.

2 Our method

Our proposal in this paper is to use the decoupling theorem [11] to handle with the problem of multi-scale in the effective potential, using as decoupling scale for the mass $M_i(\phi)$ the scale

$$\mu^d_i = M_i(\phi) \quad (2.1)$$

In other words we replace the improved potential (1.2) by

$$V^{(1)} = \kappa \sum_i V_i^{(1)} \theta_i \quad (2.2)$$

where $\theta_i \equiv \theta(\mu_i - M_i(\phi))$ is the Heaviside theta function. Using the above prescription we obtain, as a bonus, that there are no one-loop threshold corrections neither for the couplings nor for the effective potential. In this way the effective theory is the usual one, containing just the light degrees of freedom while all the effects of integrating out the heavy modes are encoded in the initial conditions of the low energy couplings at the decoupling scale $\mu^d_i$. Let us recall that in a mass-independent renormalization scheme, such as the $\overline{\text{MS}}$ or $\text{DR}$, the decoupling must be implemented by hand, being justified
by physical reasons\footnote{In section 3 we will discuss how to incorporate the decoupling in this kind of schemes in a natural way, showing that the previous procedure is favoured not just by physical considerations but also by the demand of highest calculational accuracy.}.

Since the effective potential satisfies the RGE

\[
\mathcal{D}V \equiv \left( \mu \frac{\partial}{\partial \mu} + \beta_a \frac{\partial}{\partial \lambda_a} - \gamma_\phi \frac{\partial}{\partial \phi} \right) V = 0
\] (2.3)

the decoupling prescription for the \( \beta \) functions is inherited from the corresponding one imposed on the effective potential. For instance, at one-loop:

\[
\left( \beta_a \frac{\partial}{\partial \lambda_a} - \gamma_\phi \frac{\partial}{\partial \phi} \right) V^{(0)} = \frac{1}{32\pi^2} \sum_i n_i M_i^4 \theta_i
\] (2.4)

and so the \( \beta \) and \( \gamma \)-functions decouple at the same points as the effective potential (2.2). Notice that, as for the effective potential, with this prescription there are no one-loop threshold corrections for the \( \lambda_a \) couplings\footnote{Observe that \( \theta_i \equiv \theta \left( -V_i^{(1)} \right) \) and, as a consequence, \( V_i^{(1)} \mathcal{D} \theta_i = -V_i^{(1)} \delta \left( V_i^{(1)} \right) \mathcal{D}V_i^{(1)} \equiv 0 \). Therefore the presence of Heaviside \( \delta \)-functions in the effective potential does not yield extra \( \delta \)-function terms to the RGE which would translate, upon integration, into threshold effects for the coupling constants.}. For a fixed value of the classical field, \( \phi_c \), the effective potential given by Eqs. (1.1), (2.2), with the \( \lambda_a \) parameters running in this way, represents the one–loop approximation to \( V(\phi_c) \), which is \( \mu \)-independent (see Eq. (2.3)) up to two loop corrections. These corrections, however, may be very large due to the presence of potentially large logarithms in \( V^{(n)} \). Thus, one has to be careful in choosing a value of \( \mu \) which gives the finest approximation to \( V(\phi_c) \). We will come back to this point shortly.

Let us now turn to a subtle point. It is usually assumed that the initial conditions for the \( \lambda_a(\mu) \) parameters are given at some large value of the scale (larger than all the involved masses), which can be even at some grand unified value. This is reasonable since the SM is supposed to be a low-energy effective theory coming from a more fundamental theory. The values of \( \lambda_a(\mu) \) at low scales are the result of the RGE running from the high-energy scale, i.e. a top–down procedure. In between both scales, the successive particles will be decoupled according to the values given by (2.1). Since the decoupling scales \( \mu_i^d \) depend on \( \phi_c \), it is clear that the low-energy values of \( \lambda_a(\mu) \) are (logarithmically) \( \phi_c \)-dependent, so we can denote them as \( \lambda_a(\mu, \phi_c) \). Of course, in many physical instances what is known is the “experimental” values of the parameters, \( \lambda_a^{\text{exp}} \). These correspond to the values of \( \lambda_a(\mu = \mu_0, \phi_c = v) \), where \( \mu_0 \) is the low-energy scale at which experiments are done and \( v = \langle \phi_c \rangle \) is the physical VEV of the scalar field\footnote{From a low-energy point of view, what actually makes sense are the coefficients of the \( n \)-point operators, which are physically defined at the energy scale \( \mu_0 \).}. In that case, the RGE running and thresholds (2.1) permit to evaluate in a...
bottom–up procedure the couplings $\lambda_a(\mu, v)$ at any large scale. Notice that for a large enough value of $\mu$, say $\mu$ greater than all masses $M_i(\phi_c)$, the values of $\lambda_a(\mu, \phi_c)$ do not depend on $\phi_c$. Then, one can come back downstairs (top–down procedure) with the RGE and any desired value of $\phi_c$, using the corresponding decouplings. In this way one can reconstruct the $\phi_c$ dependence of $\lambda_a$ for any scale $\mu$. This will be illustrated in the next section.

Let us now examine what is the optimum value of the renormalization scale, say $\mu = \mu^*$, at which the one-loop potential $V_{\text{eff}}(\phi_c, \mu)$ should be evaluated. There are two sensible criteria for this choice. The first one is to take $\mu^*$ as the scale at which the one-loop potential has the least $\mu$–dependence. This is reasonable since the complete potential satisfies $dV/d\mu \equiv 0$ and a substantial dependence of the one–loop potential on $\mu$ clearly induces large uncertainties in the evaluation itself. The second criterion is to take $\mu^*$ as the scale at which the loop expansion has the best apparent behavior, i.e. $V^{(1)} = 0$. Again, a substantial departure from this criterion amounts to large uncertainties concerning higher order corrections. Obviously, the most desirable situation occurs when the two criteria give (essentially) the same value of $\mu^*$.

It is clear that for scales below $\min_i\{M_i(\phi)\}$ the effective potential is contributed only by the tree level potential. Hence any $\mu^* \leq \min_i\{M_i(\phi)\}$ trivially satisfies the second criterion, $V^{(1)} = 0$, as well as the first criterion, $dV_{\text{eff}}/d\mu \equiv 0$, since all the particles get decoupled from the RGE. Consequently, this will be our criterion to choose $\mu^*$. Nevertheless, it is not obvious that, in particular models, there are not other values of $\mu$ satisfying the required conditions. This will be examined in the next section.

Including two-loop corrections to the previous approach is straightforward. It can be done in two possible ways: Either by modifying the prescription (2.1) for the decoupling scales $\mu^*_i$ in order to cancel one– plus two–loop corrections to the effective potential for every particle species, or just maintaining prescription (2.1) in which case there will appear two-loop threshold effects in the potential and in the couplings at every decoupling scale.

In the next section we will illustrate numerically the above ideas with a simple Higgs-Yukawa model and postpone more realistic theories, as extensions of the standard model, for a future analysis although it is clear from the present section that the generalization is straightforward.

operators $(\phi - v)^n$. These can be straightforwardly related to the $\lambda_a(\mu = \mu_0, \phi_c = v)$ couplings. Notice that, in a precise way, the (logarithmic) dependence of $\lambda_a$ on $\phi$ should be taken into account when establishing this connection.
3 The Higgs-Yukawa model

This model contains a Higgs boson $\phi$ and an arbitrary number $N$ of Dirac fermions $\psi_i$ coupled to the Higgs with a Yukawa coupling $g$. The lagrangian is given by

$$L = \frac{1}{2}(\partial \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \bar{\psi}(i\gamma^\mu + g\phi)\psi - \Lambda,$$  \hfill (3.1)

where $\psi \equiv (\psi_1, \ldots, \psi_N)$ and $\Lambda$ is a cosmological constant term. Using the decoupling prescription proposed in (2.2), the improved one-loop effective potential and $\beta$ and $\gamma$-functions for this model can be written as:

$$V^{(0)} = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + \Lambda$$

$$V^{(1)} = \kappa \left(V_B^{(1)}\theta_B + V_F^{(1)}\theta_F\right)$$

$$V_B^{(1)} = \frac{1}{4}M_B^4 \log \frac{M_B^2}{\mu^2}$$

$$V_F^{(1)} = -NM_F^4 \log \frac{M_F^2}{\mu^2}$$  \hfill (3.2)

where

$$M_B^2 = \frac{1}{2}\lambda\phi^2 + m^2, \quad M_F^2 = g^2\phi^2$$  \hfill (3.3)

are the Higgs boson and fermion masses, respectively, $\theta_{B,F} = \theta(\mu - M_{B,F})$, $\mu^2 \equiv \tilde{\mu}^2 e^{3/2}$, and

$$\beta_\lambda = \frac{1}{(4\pi)^2} \left[3\lambda^2\theta_B + 8g^2N\left(\lambda - 6g^2\right)\theta_F\right]$$

$$\beta_{m^2} = \frac{1}{(4\pi)^2} \left[\lambda\theta_B + 4Ng^2\theta_F\right] m^2$$

$$\beta_\lambda = \frac{1}{(4\pi)^2} \frac{1}{2}m^4\theta_B$$

$$\beta_{g^2} = \frac{1}{(4\pi)^2} \left[6\theta_B\theta_F + 4N\theta_F\right] g^4$$

$$= \frac{1}{(4\pi)^2} \left[6\theta(M_B - M_F)\theta_B + (6\theta(M_F - M_B) + 4N)\theta_F\right] g^4$$

$$\gamma_\phi = \frac{1}{(4\pi)^2} 2g^2N\theta_F$$  \hfill (3.4)

All the couplings, including the mass parameter, and the field $\phi$ in (3.2) are running with the renormalization scale $\mu$. As described in the previous section, we will start with initial boundary conditions for all the couplings at a high scale. In particular we will fix $\lambda = 0.1$, $g = 1$ and $m = 1$ TeV at the high scale $\mu_M = e^{3/4} 10^4$ GeV \(\simeq 2 \times 10^4\)
GeV. Then, running the RGE in a top–down procedure, passing through the various mass thresholds of the model, we can get the couplings $\lambda, m, etc.$ at any value of $\mu$ (we will run downwards until the low scale $\mu_m = 100$ GeV). As stressed in the previous section, since the decoupling scales associated with the thresholds (in this case $M_B$ and $M_F$) are $\phi_c$-dependent, so the low-energy parameters are. This is illustrated in Fig. 1a for the $m$ parameter and several values of $\phi_c$. There is a value of $\mu$, corresponding to $\max\{M_F, M_B\}$, at which the heaviest particle decouples and the top–down running of $m(\mu)$ departs from the complete RGE. Then, after passing through the lower threshold, at $\min\{M_F, M_B\}$, the value of $m(\mu)$ becomes frozen. Since both $M_B$ and $M_F$ are $\phi_c$-dependent, see (3.3), so the low-energy value of $m(\mu)$ is. An analogous behavior occurs for the remaining parameters. Figs. 1b, 2 show explicitly the variation of $m(\mu_m)$ and $\lambda(\mu_m)$ with $\phi_c$.

Figure 1: a) Plot of the $m$ parameter as a function of $\mu$ for several values of $\phi_c$. b) Plot of the $m(\mu^*)$ parameter as a function of $\phi$, where $\mu^* = \min\{M_F, M_B\}$. In both figures, $\lambda = 0.1$, $g = 1$ and $m = 1$ TeV at high scale $\mu_M \simeq 2 \times 10^4$ GeV.

Let us now consider the one-loop effective potential, $V^{(0)} + V^{(1)}$, for fixed $\phi_c$, as a function of $\mu$. The effective potential undergoes the same decouplings as the $m, \lambda, g$ parameters. As $\mu$ decreases, depending on the value of $\phi_c$, the fermion will be decoupled first (the case $M_F > M_B$) or last (the case $M_B > M_F$). Both cases are respectively illustrated in Fig. 3, where the solid lines denote the total one-loop potential $V^{(0)} + V^{(1)}$ and the dotted lines the tree level approximation $V^{(0)}$. The decoupling scales, corresponding to $M_B$ and $M_F$, are marked by arrows. Incidentally, the one-loop potential between $M_B$ and $M_F$ in Fig. 3 (left panel) is a very slightly increasing function of $\mu$, which is hardly appreciable from the plot. On the other hand, in Fig. 3 (right panel) the

\[ y \equiv \text{sign}(V - V(0)) \times \log_{10} \left[ \frac{|V - V(0)|}{T_{eV}^4} + 1 \right] 10^{-6} \]

that has been plotted in Fig. 3 has been chosen in order to give a continuous and faithful representation of $V$ in logarithmic units.
Figure 2: Plot of the $\lambda(\mu^*)$ parameter as a function of $\phi$, where again $\mu^* = \min\{M_F, M_B\}$ and the initial condition are the same as in Fig. 1.

one-loop potential in the analogous region is exactly $\mu$-independent. In either case, this is a peculiarity of the toy model at hand: in more general models the one-loop potential is only $\mu$-flat in the $\mu \leq M_B, M_F$ region, i.e. where all the particles get decoupled.

Figure 3: Plot of the total one-loop potential (solid line) and the tree level potential (dotted line) as a function of the scale $\mu$ for two different values of $\phi_c$. In the left panel $\phi_c = 3 \times 10^3$ GeV, so that $M_F > M_B$, while in the right panel $\phi_c = 200$ GeV, so that $M_B > M_F$. The decoupling scales, corresponding to $M_B$ and $M_F$, are marked by arrows. The vertical axis variable, $y$, represents the scalar potential in a convenient choice of units as described in the text.

Let us now turn to the key question: which value of $\mu$, say $\mu^*$, should be chosen to evaluate $V(\phi)$ in the most reliable way? As explained in the previous section, there are two sensible criteria to choose $\mu^*$:

\[ V^{(1)}(\mu = \mu^*) = 0 \]
\[
\frac{d}{d\mu} \left[ V^{(0)} + V^{(1)} \right] \bigg|_{\mu=\mu^*} = 0 \tag{3.5}
\]

They correspond respectively to the (apparently) best perturbative convergence and the least \(\mu\)-sensitivity. Consistency of a given choice of \(\mu^*\) requires that both \(i\) and \(ii\) are simultaneously satisfied, at least in an approximate way. From Fig. 3 it appears that there are two possible choices of \(\mu^*\) satisfying \(i\) and \(ii\) with various degrees of accuracy.

First, as explained in the previous section, any \(\mu^* \leq M_B, M_F\) automatically fulfills \(i\) and \(ii\) in an exact way since all the particles get decoupled for the radiative corrections. This is apparent from Fig. 3, since for \(\mu^* \leq M_B, M_F\) the solid and dotted lines coincide and are flat. Second, there is another region of \(\mu\), just above \(\text{max}\{M_B, M_F\}\), where conditions \(i\) and \(ii\) are approximately satisfied. This shows up in Fig. 3 as the region between the crossing of the solid line with the dotted line and stationary point. Let us denote these two regions as \((a)\) and \((b)\) respectively, i.e.

Which region, \((a)\) or \((b)\), should be selected to choose \(\mu^*\)? From Fig. 3, it could seem that this question is not crucial since both choices provide similar values of \(V(\phi)\). However, in less simplified models this is not true anymore. In fact, in models like the MSSM there may be not just two, but several possible choices of \(\mu^*\) approximately satisfying \(i\) and \(ii\), and leading to substantially different results. In order to discriminate between \((a)\) and \((b)\) it is convenient to plot the corresponding values of \(\mu^*\) for different values of \(\phi_c\), see Fig. 4. For region \((a)\), condition \(i\) (solid line) and condition \(ii\) (circles) lead exactly to the same value of \(\mu^*\), namely any \(\mu^* \leq M_B, M_F\). For the sake of definiteness we take \(\mu^* = \text{min}\{M_F, M_B\}\). On the contrary, for region \((b)\) [the region between solid line and circles] conditions \(i\) (solid line) and \(ii\) (circles) lead to different values of \(\mu^*\). These values get increasingly split as \(\phi_c\) grows, eventually becoming enormously far away from each other. In other words, in region \((b)\) criteria \(i\) and \(ii\) cannot be in general simultaneously satisfied, which makes the evaluation of \(V(\phi)\) unreliable. Significantly, as region \((b)\) becomes more and more unreliable it also gets more and more separate from region \((a)\). Since \(V(\phi)\) must be, of course, a continuous function of \(\phi\), region \((b)\) should be in general disregarded. It is worth-noticing that the region of maximum reliability of region \((b)\), see Fig. 4, corresponds to the values of \(\phi_c\) for which \(M_B \sim M_F\). This is satisfactory since in that case we do not have a multi-scale problem and the usual choice \(\mu^* \sim M_B, M_F\) is good enough.

In summary, we implement in the model the decoupling conditions at \(M_B, M_F\), both for the \(m, \lambda, g\) parameters and for the one-loop effective potential. Then, we evaluate the potential at a scale \(\mu^* = \text{min}\{M_F, M_B\}\), where all the particles get decoupled. Notice that at this scale the tree-level and the one-loop potentials coincide since \(V^{(1)} = 0\). So, at the end of the day we just need to evaluate the tree-level potential \(V^{(0)}\) at low
energy in order to get the best estimate of $V(\phi)$. In doing this it is important to keep in mind that the values of the parameters at low energy depend on the value of $\phi_c$ in the above-explained way, see Figs. 1–2.

4 Relation with the multi-scale-renormalization approach

The most sophisticated attempt up to now to deal with the effective potential in the presence of several mass scales was introduced by Einhorn and Jones (EJ) [6], and recently improved by Ford and Wiesendanger (FW) [7]. It is extremely interesting to study the relation of our method with theirs. In doing that we will appreciate the underlying equivalence of both approaches and the advantages of the decoupling approach proposed in this paper, both from the conceptual and the technical points of view. This will also shed light on the physical significance of the EJ and FW methods.

The key point of the EJ and FW approaches is to extend the usual $\overline{\text{MS}}$ renormalization scheme by using several independent renormalization scales $\kappa_1, \kappa_2, \ldots$, which can then be adjusted differently to take care of the different mass scales. Let us briefly review the EJ-FW method using the simple Higgs-Yukawa model with the lagrangian written in Eq. (3.1).

In the $\overline{\text{MS}}$ scheme the one-loop correction to the effective potential is given by

$$V^{(1)} = \frac{1}{64\pi^2} \left[ M_B^2 \log \frac{M_B^2}{\mu^2} - 4N M_F^2 \log \frac{M_F^2}{\mu^2} \right].$$

Here $\mu^2 \equiv \bar{\mu}^2 e^{3/2}$, where $\bar{\mu}$ is the $\overline{\text{MS}}$ renormalization scale, and $M_B^2, M_F^2$ are the two mass scales in the theory, Eqs. (3.3). Clearly, it is not possible to track both scales in
\( \text{(1.1)} \) by a single choice of \( \mu \). The goal of the EJ-FW method in this case is to introduce two renormalization scales, \( \kappa_1, \kappa_2 \), so that \( V^{(1)} \) becomes

\[
V^{(1)} = \frac{1}{64\pi^2} \left[ M_B^4 \log \frac{M_B^2}{\kappa_1^2} - 4NM_F^4 \log \frac{M_F^2}{\kappa_2^2} \right], \tag{4.2}
\]

This would allow to take \( \kappa_1 = M_B, \kappa_2 = M_F \), avoiding the undesirable presence of potentially large logarithms. More precisely, the EJ idea is to modify the standard \( \overline{\text{MS}} \) bare lagrangian

\[
\mathcal{L}_{\text{Bare}} = \frac{1}{2} Z_\phi (\partial \phi)^2 - \frac{1}{2} Z_\phi Z_{m\phi} m^2 \phi^2 - \frac{1}{4!} \mu^4 Z_\phi^2 \mu^4 \lambda \phi^4 + \kappa_1 \frac{1}{2} \psi_i \bar{\psi} \psi_j \psi_i \psi_j + Z_\phi \bar{Z}_\phi \phi^2 \phi - \Lambda + (Z - 1) \mu^{-1} \lambda^{-1} \quad \text{replacing the occurrences of } \mu \text{ with independent scales } \kappa_1, \kappa_2, \ldots, \text{in such a way that the one-loop effective potential gets the form (4.2). Actually, they go to something similar to Eq. (4.2) but with extra terms involving the potentially large log } \kappa_1^2 / \kappa_2^2. \text{ Hence, their attempt was not completely successful. FW modified the EJ scheme allowing the RG independent scales } \kappa_1, \kappa_2 \text{ to be attached to the kinetic terms in a clever way. In particular they proposed to re-write } \mathcal{L}_{\text{Bare}} \text{ as}
\]

\[
\mathcal{L}^{\text{FW}}_{\text{Bare}} = \frac{1}{2} \kappa_1^{\frac{1}{2} \epsilon} Z_\phi (\partial \phi)^2 - \frac{1}{2} Z_\phi Z_{m\phi} m^2 \phi^2 - \frac{1}{4!} Z_\phi^2 \mu^4 \lambda \phi^4 + \kappa_2 \frac{1}{2} \psi_i \bar{\psi} \psi_j \psi_i \psi_j + Z_\phi \bar{Z}_\phi \phi^2 \phi - \Lambda + (Z - 1) \mu^{-1} \lambda^{-1}. \tag{4.3}
\]

The relation between the bare and renormalized parameters is then given by

\[
\begin{align*}
\lambda_B &= \kappa_1^{\epsilon} \lambda, & g_B^2 &= \kappa_1^{\frac{1}{2} \epsilon} \kappa_2^{\frac{1}{2} \epsilon} \lambda g^2, \\
m_B^2 &= \kappa_1^{\frac{1}{2} \epsilon} Z_{m\phi} m^2, & \phi_B &= \kappa_1^{\frac{1}{2} \epsilon} \kappa_2^{\frac{1}{2} \epsilon} \phi, \\
\psi_B &= \kappa_2^{\frac{1}{2} \epsilon} Z_{\bar{\psi} \psi} \psi, & \Lambda_B &= \lambda + (Z - 1) \mu^{-1} \lambda^{-1}. \tag{4.5}
\end{align*}
\]

Then, the one-loop effective potential really gets the form (1.2). The use of two independent RG scales \( \kappa_1, \kappa_2 \) implies double number of RGEs. Namely, for each generic parameter \( h \) there are two \( \beta \)-functions, defined as

\[
i_{\beta h} = \kappa_i \frac{d}{d\kappa_i} h, \quad i = 1, 2 \tag{4.6}
\]

Consistency of the approach requires \( i_{\beta h} + 2 \beta h = \beta h \), i.e. the complete \( \overline{\text{MS}} \) \( \beta \)-function. For the particular choice \( \kappa_1 = \kappa_2 \) the standard \( \overline{\text{MS}} \) scheme is recovered. The \( i_{\beta h} \) functions are evaluated from the \( Z \) factors in a standard way. For example,

\[
\begin{align*}
i_{\beta \lambda}^{\text{FW}} &= -\epsilon \lambda - \kappa_1 \left[ \frac{\partial z_\lambda}{\partial \lambda} \frac{\partial \lambda}{\partial \kappa_1} + \frac{\partial z_\lambda}{\partial g^2} \frac{\partial g^2}{\partial \kappa_1} + \ldots \right] = -\epsilon \lambda + \lambda (\mu \partial \lambda + \frac{1}{2} g^2 \partial g^2) z_\lambda, \\
i_{\beta \lambda}^{\text{FW}} &= \kappa_2 \left[ \frac{\partial z_\lambda}{\partial \lambda} \frac{\partial \lambda}{\partial \kappa_2} + \frac{\partial z_\lambda}{\partial g^2} \frac{\partial g^2}{\partial \kappa_2} + \ldots \right] = \frac{1}{2} \lambda g^2 \partial g^2 z_\lambda. \tag{4.7}
\end{align*}
\]
where $z_\lambda$ denotes the $1/\epsilon$ coefficient in $Z_\lambda$. The one-loop contribution to the $z$'s is identical to the $\overline{\text{MS}}$ one. In particular

$$z_\lambda = \frac{1}{(4\pi)^2} (3\lambda + 8Ng^2 - 48Ng^4\lambda^{-1}) \quad (4.8)$$

Thus,

$$1^{\beta}_{\lambda}^{\text{FW}} = \frac{1}{(4\pi)^2} (3\lambda^2 + 4Ng^2\lambda), \quad 2^{\beta}_{\lambda}^{\text{FW}} = \frac{1}{(4\pi)^2} (4Ng^2\lambda - 48Ng^4) \quad (4.9)$$

For the other parameters the expressions are analogous (for more details, see Ref. [7]).

FW give an additional condition for the consistency of their approach, namely

$$[D_1, D_2] = 0 \quad (4.10)$$

where $D_i \equiv d/d\kappa_i$. This arises as an integrability condition for the RGE satisfied by the effective potential, $D_i V = 0$. As was shown by FW, the implementation of Eq. (4.10) requires in general that beyond one-loop-order the $i\beta$-functions contain logarithms of the $\kappa_i$ ratios. We will turn to this point in section 5.

The previous FW procedure is ingenious and successful, in the sense that it leads to the one-loop effective potential written in the form (4.2). However, it is clearly involved and seems hard to be extended to more complicated models, such as the SM and the MSSM. Moreover, there is a lack of intuition about the physical role of the several $\kappa_i$ scales. In this sense, attempts to associate the $\kappa_i$ running with the usual $\overline{\text{MS}}$ running when all the particles but one (the one whose mass is tracked by $\kappa_i$) are decoupled, were unfruitful [12]. For example, in the case at hand the $i\beta_\lambda$-functions given in Eq. (4.9) do not correspond to the presence of just the boson or the fermion respectively. If they did, the two beta functions would have been $(4\pi)^21\beta_\lambda \rightarrow 3\lambda^2$, $(4\pi)^22\beta_\lambda \rightarrow 8g^2N(\lambda - 6g^2)$ [see Eq. (3.4)]. This shortcoming prevents from giving a clear physical interpretation to the $\kappa_i$ scales. We will come back to this point later.

Let us now turn to the question: Is it possible to reproduce the results of the EJ-FW multi-scale renormalization with our decoupling approach? We will see that the answer is yes. First of all, let us see how our decoupling approach can be interpreted as a scenario with several independent scales. The key point is to allow the decoupling scales to be free parameters. Let us illustrate this by using the simple Higgs-Yukawa model. Then, instead of decoupling the boson (fermion) at $\mu = M_B$ ($\mu = M_F$), as described in section 3, we will do it at $\mu = \mu_B$ ($\mu = \mu_F$), where $\mu_B$, $\mu_F$ are in principle arbitrary. This procedure is acceptable since in a mass-independent renormalization scheme, such as the $\overline{\text{MS}}$, the decoupling is implemented by hand and there is no indication about the value of the decoupling scale. Besides, the complete effective potential is $\mu$-independent, which in principle allows to decouple at any scale.
Decoupling at scales $\mu_B, \mu_F$ for the one-loop effective potential amounts to replace the $\theta_{B,F} = \theta(\mu - M_{B,F})$ functions in (3.2) by

$$\theta_{B,F} = \theta(\mu - \mu_{B,F})$$

Then, there appear threshold corrections in $V^{(1)}$ [see Eq. (3.2)]. Keeping these threshold pieces explicitly, the one-loop effective potential, $V^{(0)} + V^{(1)}$, reads

$$V^{(0)} = \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 + \Lambda$$

$$V^{(1)} = \frac{1}{64 \pi^2} \left[ M_B^4 \log \frac{M_B^2}{\mu_B^2} + M_F^4 \log \frac{M_F^2}{\mu_F^2} \theta_B 
- 4NM_B^4 \log \frac{M_B^2}{\mu_B^2} - 4NM_F^4 \log \frac{M_F^2}{\mu_F^2} \right].$$

The one-loop $\beta$-functions are extracted from (4.12) in the usual way, applying the RG condition $dV/d\mu = 0$. They are exactly as in Eq. (3.4), but with $\theta_{B,F}$ defined as in (4.11). Notice that all of them have the same structure. Namely, for a generic coupling $h$, the $\beta$-function has the form

$$\beta_h = \mu \frac{dh}{d\mu} = B \beta_h \theta_B + F \beta_h \theta_F.$$  

At low scale, i.e. $\mu_m \leq \mu_B, \mu_F$, $V^{(1)}$ gets the form

$$V^{(1)} = \frac{1}{64 \pi^2} \left[ M_B^4 \log \frac{M_B^2}{\mu_B^2} - 4NM_B^4 \log \frac{M_F^2}{\mu_F^2} \right].$$

This is exactly as Eq. (4.2) from the FW approach, where the role of $\kappa_{1,2}$ is now played by $\mu_{B,F}$. Since $\mu_{B,F}$ are arbitrary scales we can choose $\mu_B = M_B, \mu_F = M_F$, getting rid of the potentially large logarithms and recovering the decoupling approach in the form described in sections 2, 3 [1]. Hence, at this stage we notice a perfect equivalence between the FW approach and our method. Concerning the $i\beta$-functions, defined in Eq. (4.6) in the FW approach, we must study here the dependence of the various parameters with respect to $\mu_B, \mu_F$. Notice that a generic parameter $h(\mu = \mu_m)$ is given by

$$h(\mu_m) = h(\mu_M) - \int_{\mu_m}^{\mu_M} d\mu \frac{dh}{d\mu},$$

where $\mu_M$ is the initial high scale. So, using $d\theta(\mu - \mu_{B,F})/d\mu_{B,F} = \delta(\mu - \mu_{B,F})$, it is clear that

$$\mu_B \frac{dh}{d\mu_B} = B \beta_h$$

$$\mu_F \frac{dh}{d\mu_F} = F \beta_h.$$  

Incidentally, this shows that the $M_B, M_F$ scales are the most convenient ones to implement the decoupling from the calculational point of view. This point was mentioned in section 2.
where \( B, F, \beta_h \) have been defined in Eqs. (4.13), (4.4). Obviously, \( B\beta_h + F\beta_h = \beta_h \), again as in the FW approach. However, there is an important difference: the values of \( B, F, \beta \) for the various parameters of the model do not coincide with those of \( 1, 2, \beta_{FW} \) in the FW approach [see Eq. (4.9)]. For example,

\[
B_\beta = \frac{1}{(4\pi)^2} 3\lambda^2, \quad F_\beta = \frac{1}{(4\pi)^2} 8g^2 N(\lambda - 6g^2)
\]

(4.17) [to be compared to (4.9)]. It is interesting to note that in our approach the \( B_\beta \)- and \( F_\beta \)-functions do correspond to the \( \beta \)-functions when all the particles except the boson or the fermion (respectively) are decoupled. This is precisely what was (unsuccessfully) required in Ref. [12] to interpret the EJ-FW method as an implementation of the decoupling. In our case this emerges automatically by construction.

Our final goal in this section is to prove that there is a (slight variation of the) FW approach that is exactly equivalent to our decoupling scheme. First of all, notice that the FW lagrangian of Eq. (4.4) is not the only consistent multiscale-renormalization proposal. For example, one may simply replace

\[
k_1^{-\frac{x}{4}} \rightarrow k_1^{-\frac{(1+x)e}{4}} k_2^{-x}, \quad k_2^{-\frac{y}{4}} \rightarrow k_1^{-\frac{(1+y)e}{4}} k_2^{-y}
\]

in Eq. (4.4), which leads to straightforward modifications in Eq. (4.5). In this case, it is easy to verify that \( x = 0, y = 0 \) are in fact the only constant values of \( x, y \) leading to the desired form (4.2) for \( V^{(1)} \). However, there is still a possible modification of the lagrangian of Eq. (4.4) that apparently was not considered by FW. Namely, let us replace (4.4) by

\[
\mathcal{L}_{\text{Bare}} = \frac{1}{2} k_1^{-\frac{x}{4}} f_1(k_1, k_2) Z_\phi (\partial \phi)^2 - \frac{1}{2} Z_\phi Z_m^2 m^2 \phi^2 - \frac{1}{4!} Z_\phi^2 Z_\lambda \phi^4 + \frac{1}{2} k_2^{-\frac{y}{4}} f_2(k_1, k_2) Z_\psi \bar{\psi} \psi \bar{\psi} \psi - \Lambda - (Z_\lambda - 1) m^4 \lambda^{-1}, \quad (4.18)
\]

where \( f_1, f_2 \) are dimensionless functions of \( k_1, k_2 \). They must satisfy \( f_{1,2}(k, k) = 1 \) in order to ensure that the MS scheme is recovered for \( k_1 = k_2 \). Moreover, the usual tree-level lagrangian must be recovered when the one-loop corrections are neglected. This means that the \( f \)-functions can only be non-trivial beyond tree-level; in other words, \( f_{1,2} = 1 + O(\hbar) \).

It is now straightforward to show that taking

\[
\begin{align*}
f_1(k_1, k_2) &= \left[ \frac{k_2}{k_1} \right]^{a/2} \\
f_2(k_1, k_2) &= \left[ \frac{k_1}{k_2} \right]^{a/4} \frac{k_1^{b/2} k_2^{c/2}}{[k_1 \theta_{12} + k_2 \theta_{21}][b+c]/2}
\end{align*}
\]

(4.19)

where

\[
a = \frac{1}{(4\pi)^2} 4N g^2
\]

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\[ b = \frac{1}{(4\pi)^2}(-2N - 3)g^2 \]
\[ c = \frac{1}{(4\pi)^2}(2N - 3)g^2 \]
\[ \theta_{ij} = \theta(\kappa_i - \kappa_j) \quad (4.20) \]

the \( 1\beta \)- and \( 2\beta \)-functions for all the parameters exactly coincide with the \( B\beta \)- and \( F\beta \)-functions readable from Eq. (3.4). To check that, notice that from (4.18), (4.19) the relation between the bare and renormalized parameters is now given by [compare to (4.21)]

\[
\lambda_B = \kappa_1^t \left[ \frac{\kappa_2}{\kappa_1} \right]^{-a} Z_\lambda \lambda, \quad g_B^2 = \kappa_1^t \kappa_2^t \frac{[\kappa_1 \theta_{12} + \kappa_2 \theta_{21}][b+c]}{\kappa_1^b \kappa_2^c} Z g^2 g^2;
\]
\[
m_B^2 = \kappa_1^t \left[ \frac{\kappa_2}{\kappa_1} \right]^{-a/2} Z m^2 m^2, \quad \phi_B = \kappa_1^t \left[ \frac{\kappa_2}{\kappa_1} \right]^{a/4} Z^{1/2} \phi,
\]
\[
\psi_B = \kappa_2^t \left[ \frac{\kappa_1}{\kappa_2} \right]^{a/8} \frac{\kappa_1^{b/4} \kappa_2^{c/4}}{[\kappa_1 \theta_{12} + \kappa_2 \theta_{21}][b+c]/4} Z^{1/2} \psi
\]
\[
\Lambda_B = \Lambda + (Z_\Lambda - 1) m^4 \lambda^{-1}. \quad (4.21)
\]

Then, the \( \beta \)-functions are modified in a straightforward way. For example,

\[
_{1\beta} = _{1\beta}^{FW} - \frac{1}{(4\pi)^2} 2g^2 N = \frac{1}{(4\pi)^2} 3\lambda^2 
\]
\[
_{2\beta} = _{2\beta}^{FW} + \frac{1}{(4\pi)^2} 2g^2 N = \frac{1}{(4\pi)^2} 8g^2 (\lambda - 6g^2) 
\]
\[
_{1\beta} = _{1\beta}^{FW} - \frac{1}{(4\pi)^2} 2g^2 N m^2 = \frac{1}{(4\pi)^2} \lambda m^2 
\]
\[
_{2\beta} = _{2\beta}^{FW} + \frac{1}{(4\pi)^2} 2g^2 N m^2 = \frac{1}{(4\pi)^2} 4Ng^2 m^2 
\]
\[
_{1\gamma} = _{1\gamma}^{FW} - \frac{1}{(4\pi)^2} g^2 N = 0 
\]
\[
_{2\gamma} = _{2\gamma}^{FW} + \frac{1}{(4\pi)^2} 2g^2 N = \frac{1}{(4\pi)^2} 2g^2 N \quad (4.22)
\]

Thus, lagrangian (4.18) with \( f_{1,2} \) defined as in Eq. (4.19) exactly implements the decoupling approach. Still, the one-loop potential \( V^{(1)} \) maintains the desired form (4.2). Of course the renormalization scheme defined by Eqs. (4.18), (4.19) may a priori seem absolutely ad hoc. Its usefulness is to prove the equivalence of our decoupling approach (which does emerge in a natural way) with a particular multi-scale renormalization scheme.
5 Two-loop effects

The procedure of section 3 can be straightforwardly generalized to include two-loop corrections to the effective potential. In particular, in the Higgs-Yukawa model, these corrections have been computed in Ref. [13] as

$$V^{(2)} = \kappa^2 \left( V^{(2)}_B + V^{(2)}_{BF} \right)$$

(5.1)

where $V^{(2)}_B$ includes the sunset and figure eight diagrams with boson internal lines, and $V^{(2)}_{BF}$ is the sunset diagram with a fermion loop line. The precise expressions for these terms can be found in Ref. [13]. Using the fact that the decoupling of $V^{(2)}_B$ is controlled by the scale $M_B$ while that of $V^{(2)}_{BF}$ is controlled by both $M_B$ and $M_F$, we can generalize the one-loop improved potential in the presence of decoupling, given by Eqs. (2.2), (3.2), as:

$$V^{(1)} + V^{(2)} = \kappa \left[ V^{(1)}_B + \kappa V^{(2)}_B + \kappa \theta(M_B - M_F)V^{(2)}_{BF} \right] \theta_B^{(2)}$$

$$+ \kappa \left[ V^{(1)}_F + \kappa \theta(M_F - M_B)V^{(2)}_{BF} \right] \theta_F^{(2)}$$

(5.2)

where the $\theta$-functions are now defined by,

$$\theta_B^{(2)} = \theta \left( -V^{(1)}_B - \kappa V^{(2)}_B - \kappa \theta(M_B - M_F)V^{(2)}_{BF} \right)$$

$$\theta_F^{(2)} = \theta \left( -V^{(1)}_F - \kappa \theta(M_F - M_B)V^{(2)}_{BF} \right).$$

(5.3)

From the property $\theta \left( f(\mu) \right) = \theta(\mu - \mu_0)$, where $f(\mu)$ is a monotonically increasing function with $f(\mu_0) = 0$, we can write (5.3) in a two-loop expansion as $\theta_B^{(2)} = \theta \left( \mu - \mu_B^{(2)} \right)$, $\theta_F^{(2)} = \theta \left( \mu - \mu_F^{(2)} \right)$, where now the decoupling scales $\mu_B^{(2)}, \mu_F^{(2)}$ are given by the two-loop expansion

$$\mu_B^{(2)} = M_B + \frac{2}{M_B^3} \kappa \left. \left[ V^{(2)}_B + \theta(M_B - M_F)V^{(2)}_{BF} \right] \right|_{\mu=M_B} + O(h^2)$$

$$\mu_F^{(2)} = M_F - \frac{1}{2NM_F^2} \kappa \theta(M_F - M_B)V^{(2)}_{BF} \right|_{\mu=M_F} + O(h^2)$$

(5.4)

Decoupling bosons and fermions at scales (5.4) there are no threshold corrections at two-loop, neither in the effective potential nor in the coupling constants, since the property $fD\theta(f) = f\delta(f)Df \equiv 0$ guarantees the absence of $\delta$-functions in the RGE. It might seem that by taking the decoupling conditions (5.3) we can get rid of the potentially large logarithms at all. This would contradict the statement of Ref. [7] concerning the unavoidable appearance of logarithms beyond one-loop order. However, there is no contradiction, since the logarithms are now hidden in the definitions of $\mu_B^{(2)}$ and $\mu_F^{(2)}$. Nevertheless, it is amazing to check the logarithms do never spoil the perturbative
validity of the approach, since they cancel each other in the potentially dangerous limits $M_F/M_B \to 0, \infty$. To see this, it is convenient to explicitly write the decoupling scales (5.4)

$$\mu_B^{(2)} = M_B \left[ 1 + \frac{4\kappa g^2 N\theta(M_B - M_F)}{M_B^4} \left( 3M_F^4 - M_B^2 M_F^2 \right) \log \frac{M_F^2}{M_B^2} \right. $$

$$- \left( M_F^4 - \frac{3}{2} M_B^2 M_F^2 + \frac{1}{4} M_B^4 \right) \log^2 \frac{M_F^2}{M_B^2} - \frac{1}{2} (2M_F^2 - \frac{1}{2} M_B^2) \xi(M_F, M_B) \right]$$

$$+ \text{non-logarithms}$$

$$\mu_F^{(2)} = M_F \left[ 1 - \frac{\kappa g^2 \theta(M_F - M_B)}{2M_F^4} \left( M_B^2 M_F^2 - \frac{1}{2} M_B^4 \right) \log \frac{M_F^2}{M_B^2} \right.$$

$$- \frac{1}{2} \left( 2M_F^2 - \frac{1}{2} M_B^2 \right) \xi(M_F, M_B) \right] + \text{non-logarithms} \right], \quad (5.5)$$

where $\xi(M_F, M_B)$ is defined as

$$\xi(M_F, M_B) = 4 \left| M_B^4 - 4M_F^2 M_B^2 \right|^{1/2} - M(-\vartheta_F) + M(\vartheta_F) + M(\vartheta_B) \quad \text{if} \quad M_B^2 > 4M_F^2$$

$$\left( 2L(\varphi_F) + L(\varphi_B) - \frac{\pi}{2} \log 2 \right) \quad \text{otherwise} \quad (5.6)$$

with

$$M(t) = - \int_0^t dx \log \sinh x, \quad L(t) = - \int_0^t dx \log \cos x,$$

$$\vartheta_F = \coth^{-1} \left( \frac{M_B^2}{(M_B^4 - 4M_F^2 M_B^2)^{1/2}} \right), \quad \vartheta_B = \coth^{-1} \left( \frac{2M_F^2 - M_B^2}{(M_B^4 - 4M_F^2 M_B^2)^{1/2}} \right),$$

$$\varphi_F = \tan^{-1} \left( \frac{M_B^2}{(4M_F^2 M_B^2 - M_B^4)^{1/2}} \right), \quad \varphi_B = \tan^{-1} \left( \frac{2M_B^2 - M_B^2}{(4M_F^2 M_B^2 - M_B^4)^{1/2}} \right). \quad (5.7)$$

It is now straightforward to check that in the limit $M_F/M_B \to 0$

$$\xi(M_F, M_B) \to M_B^2 \left( \frac{\pi^2}{3} + \log^2 \frac{M_F^2}{M_B^2} + \mathcal{O} \left( \frac{M_F^2}{M_B^2} \right) \right), \quad (5.8)$$

so that

$$\mu_B^{(2)} \to M_B \left[ 1 + \kappa \left\{ \text{constant} + \mathcal{O} \left( \frac{M_F^2}{M_B^2} \right) \right\} \right]$$

$$\mu_F^{(2)} \to M_F \quad (5.9)$$

Analogously, in the other potentially dangerous case, $M_F/M_B \to \infty$, one gets

$$\xi(M_F, M_B) \to 0 \quad (5.10)$$

and

$$\mu_B^{(2)} \to M_B \left[ 1 + \kappa \left\{ \text{constant} \right\} \right]$$

$$\mu_F^{(2)} \to M_B \left[ 1 + \kappa \left\{ \text{constant} \right\} \right] \quad (5.11)$$
Eqs. (5.9), (5.11) show that the potentially dangerous limits \( M_F/M_B \to 0, \infty \) are in fact well-behaved. Consequently, our decoupling approach remains consistent with perturbation theory, even when the relevant scales are very different. This is highly satisfactory.

It is interesting to mention that had we kept the one-loop choice of decoupling scales for the two-loop level, i.e. \( \mu^{(2)}_{B,F} = M_B, M_F \), we would have found non-vanishing threshold effects involving \( \log M_F^2/M_B^2 \) terms. These terms are in direct correspondence to the logarithms found in the two-loop \( \beta \) - functions of the FW approach. They are in fact needed to implement the FW condition (4.10). On the other hand, the \( \mu^{(2)}_{B,F} \) scales defined in (5.4) are not bound to satisfy condition (4.10), which is consistent with the absence of logarithmic threshold corrections.

6 Conclusions

In this paper we have dealt with the problem of multi-scale theories in mass-independent renormalization schemes. We have proposed a method based on the use of the decoupling theorem at the field-dependent scales \( M_i(\phi_c) \), which are the relevant mass-scales in effective potential calculations. The successive decouplings of the fields lead to a simple prescription for the scale at which the effective potential has the least scale-dependence and the loop expansion has the best behaviour, as \( \min_i \{ M_i(\phi_c) \} \). Our method does not yield any threshold corrections for any of the parameters appearing in the effective potential, a feature which can be implemented to any arbitrary order in perturbation theory.

We have compared our decoupling approach with previously proposed methods to deal with the multi-scale problem in effective potential calculations. In particular we have compared our approach with the multi-scale renormalization approach and found the particular multi-scale formalism equivalent to our decoupling approach at one-loop. However, unlike the multi-scale formalism, our decoupling method can avoid the presence of large logarithms corresponding to very different scales, as we have explicitly exhibited at two-loop in the Higgs-Yukawa model. On the other hand previous methods to deal with the multi-scale problem based on decoupling used as decoupling scales invariant masses in the lagrangian. Clearly these methods were insufficient to extend the effective potential to large values of the field, where field-dependent masses are larger than field independent ones, or to decouple any particle without an invariant mass.

We have chosen a particularly simple case, the Higgs-Yukawa model, to illustrate the main features of the method, but clearly it can be implemented in more realistic
models as the Standard Model and the MSSM. In both cases the method present in some cases advantages over previous studies. In the case of the SM, as we have stated above, our method allows to go to large values of the classical Higgs field while properly decoupling all particles at their corresponding mass scales. In particular the problem of the stability of the electroweak minimum and its relation to the scale of new physics can be safely considered. A result which can be of great interest if a light Higgs is found at present or future colliders. In the case of the MSSM our method is appropriate to deal with some extreme situations in the calculations of the radiatively corrected Higgs mass spectrum, in particular with those involving a large mixing in the left-right stop sector. A case which is difficult to deal with in present effective potential calculations. Those issues will be the object of future investigations.
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