Reality problems for the Algebro-Geometric Solutions of Fokas-Lenell hierarchy

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Abstract

In a previous study, we obtained the algebro-geometric solutions and \( n \)-dark solitons of Fokas-Lenells (FL) hierarchy using algebro-geometric method. In this paper, we construct physically relevant classes of solutions for FL hierarchy by studying the reality conditions for \( q = \pm \bar{r} \) based on the idea of Vinikov’s homological basis.

Introduction

The Fokas-Lenells (FL) system [2, 3, 4, 5]

\[
\begin{align*}
q_{xt} - q_{xx} + iq q_x r - 2i q_x + q &= 0, \\
r_{xt} - r_{xx} - iq r r_x + 2i r_x + r &= 0,
\end{align*}
\]  

(0.0.1)

where \( q, r \) is a complex-valued function of \( x \) and \( t \), has been studied extensively in relation with various aspects such as Lax integrability, bi-Hamiltonian structure, various kinds of exact solutions, etc. [6, 7, 8, 9, 10, 15]. In ref. [15], we have constructed the FL hierarchy using polynomial recursion formalism, then derived the algebro-geometric solutions of the whole hierarchy and degenerated them into \( n \)-dark solitons. To find solutions we go through an intermediate step, a complexified version of FL hierarchy. However, in physical applications, two natural reductions of (0.0.1), the defocusing case, with \( r = -\bar{q} \),

\[
\text{FL}_- : \quad q_{xt} - q_{xx} - i|q|^2 q_x - 2i q_x + q = 0, 
\]  

(0.0.2)

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and focusing case, with $r = 7$,

$$FL_+ : \quad q_{xt} - q_{xx} + i|q|^2q_x - 2iq_x + q = 0,$$  \hspace{1cm} (0.0.3)

are of interest. Therefore, a natural question arising here is how to derive the solutions of FL from the solutions of \textbf{(0.0.1)}. From the modern mathematical point of view, the algebro-geometric solutions of FL hierarchy can be more or less solved if the formulas (1.2.26) and (1.2.27) are derived. However, this correspondence is only formal to some extent and needs further analysis about the parameters in the solutions (1.2.26) and (1.2.27). As remark in [1], the one-soliton of AKNS system, depends on four complex parameters $a, b, \phi, \psi$. Setting

$$\ell(x, t) = \exp (ax + f_0(a)t + \phi), \quad m(x, t) = \exp (bx - f_0(-b)t + \psi),$$  \hspace{1cm} (0.0.4)

the one-soliton is given by

$$q = (a + b)m/(1 - \ell m),$$  \hspace{1cm} (0.0.5)

$$r = (a + b)\ell/(1 - \ell m).$$  \hspace{1cm} (0.0.6)

It can easily be arranged at the denominator does not vanish, say, for $t = 0$ and all $x \in \mathbb{R}$, but that there are singularities after a finite time $t_0$. Hence the AKNS system is not specific enough to prevent solutions from exploding. However, after the reduction $r = -7$, we arrive at the NLS equation and the one-soliton of NLS is

$$q(x, t) = -Re(a)e^{-Im(a)x + [Im(a)^2 - Re(a)^2]t \cosh^{-1}(Re(a)[x + 2Im(a)t])}.$$  \hspace{1cm} (0.0.7)

Solution \textbf{(0.0.7)} is smooth without any singularity. The reason why the solution of AKNS system shows different properties with that of NLS equation is obvious: the solutions of AKNS system contains more parameters and hence can be more complicated than those of NLS equation. Therefore, finding useful formulas about parameters in solutions of FL hierarchy is an efficient and important way to obtain the solution of FL$_\pm$ hierarchy.

The application of this idea to the algebro-geometric method can be found in the wonderful works [17]. In [15] we have constructed the algebro-geometric solutions of FL hierarchy using the algebro-geometric method [12]. The basic strategy we shall take is to apply the idea of [17] to [15] and the technique developed in this text seems be available to solve the reality problems of the algebro-geometric solutions of other integrable equations via
the algebro-geometric method [12]. Here we should emphasize the work of
the authors in [12]. They have discussed the reduction conditions for the
algebro-geometric solutions of AKNS hierarchy using Vinikov’s basis (for
the definition, see Proposition 1.1.1). However, it may be very complicated
to depict the Vinikov’s basis for a general real curve, although this problem
become simple in some special cases (e.g. all the branch points of the spectral
curve are real). Moreover, various of path integration in explicit algebro-
geometric solutions and the action of holomorphic involution on spectral
curve very often involves the specific path, which inevitably connects with
the choice of homology basis. Thus, we shall solve the reality problem
by using very specific curve and homology basis and every parameter in
the finally expressions of solutions is very clear. This is also the biggest
difference from ‘abstract’ Vinikov’s basis.

This paper is organized as follows. To be self-contained, we give a funda-
mental discription of real Riemann surface theory and general results about
the algebro-geometric solutions of FL hierarchy in section 1. In section 2 we
obtain two complementary results, which indicates the information of the
algebro-geometric solutions is totally included in the spectral data. Section
3 is the main part of this text and we shall discuss the reality conditions for
q = ±r.

1 Preliminaries

In this section we first recall some basic facts from the theory of symmetric
Riemann surfaces [11, 16] and main results in our previous work [15].

1.1 Basic facts on Riemann surface

A Riemann surface $\mathcal{K}_n$ of genus $n$ is called real if it admits an antiholomor-
phic involution $\tau : \mathcal{K}_n \to \mathcal{K}_n$, $\tau^2 = \text{id}|_{\mathcal{K}_n}$. Let $\mathcal{R} = \{P \in \mathcal{K}_n|\tau(P) = P\}$
be the set of fixed points of $\tau$. The connected components of $\mathcal{R}$ are called
the real ovals of $\tau$. The number $r$ of the real ovals satisfies the relations
$0 \leq r \leq n + 1$. If $r = n + 1$, then $\mathcal{K}_n$ is called an M-curve. Moreover, the set
$\mathcal{K}_n \setminus \mathcal{R}$ has either one or two connected components. We call $\mathcal{K}_n$ a dividing
curve if $\mathcal{K}_n \setminus \mathcal{R}$ has two components and nondividing if $\mathcal{K}_n \setminus \mathcal{R}$ is connected.
Obviously, an M-curve is always a dividing curve.

We fix a canonical homology basis $\{a_j, b_j\}_{j=1}^n$ on $\mathcal{K}_n$ in such a way that
the intersection matrix of the cycles satisfies

\[ a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j, k = 1, \ldots, n. \]  (1.1.1)
Let \( \omega_1, \ldots, \omega_j \) be the basis for holomorphic differentials on \( K_n \), normalized with respect to the homology basis

\[
\int_{a_k} \omega_j = \delta_{j,k}, \quad j, k = 1, \ldots, n
\]  

and the period matrix

\[
\Gamma = (\Gamma_{j,k}) = \left( \int_{b_k} \omega_j \right), \quad j, k = 1, \ldots, n.
\]

Associated with \( \Gamma \) one defines the period lattice \( L_n \) in \( \mathbb{C}^n \) by

\[
L_n = \{ z \in \mathbb{C}^n \mid z = N + \Gamma M, \; N, M \in \mathbb{Z}^n \}.
\]

The Riemann theta function associated with Riemann surface \( K_n \) and the homology basis \( \{a_j, b_j\}_{j=1,\ldots,n} \) is given by

\[
\theta(z) = \sum_{\mathbf{n} \in \mathbb{Z}^n} \exp \left( 2\pi i (\mathbf{n}, z) + \pi i (\mathbf{n}, \Gamma \mathbf{n}) \right), \quad z \in \mathbb{C}^n,
\]

where \( (\mathbf{A},\mathbf{B}) = \sum_{j=1}^n A_j B_j \) denotes the inner product in \( \mathbb{C}^n \). Then the Jacobi variety \( J(K_n) \) of \( K_n \) is defined by

\[
J(K_n) = \mathbb{C}^n / L_n
\]

and the Abel maps are defined by

\[
\mathbf{A}_{Q_0} : K_n \to J(K_n), \quad P \mapsto \mathbf{A}_{Q_0}(P) = \left( \int_{Q_0} \omega_1, \ldots, \int_{Q_0} \omega_n \right) = (A_{Q_0,1}(P), \ldots, A_{Q_0,n}(P))
\]

and

\[
\alpha_{Q_0} : \text{Div}(K_n) \to J(K_n), \quad \mathcal{D} \mapsto \alpha_{Q_0}(\mathcal{D}) = \sum_{P \in K_n} \mathcal{D}(P) \mathbf{A}_{Q_0}(P) = (\alpha_{Q_0,1}(\mathcal{D}), \ldots, \alpha_{Q_0,n}(\mathcal{D}))
\]

where \( Q_0 \) is a fixed base point and the same path is chosen from \( Q_0 \) to \( P \) in (1.1.4) and (1.1.5).
Proposition 1.1.1. (Vinnikov, [11]) There exists a canonical homology basis \( \{a_j, b_j\}_{j=1}^n \) ("Vinnikov basis") on \( K_n \) such that the action of \( \tau \) on the homology group \( H_1(K_n, \mathbb{Z}) \) is

\[
\begin{pmatrix}
\tau(a) \\
\tau(h)
\end{pmatrix}
= \begin{pmatrix}
I_n & 0 \\
H & -I_n
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix},
\] (1.1.6)

where

\[
a = (a_1, \ldots, a_n), \quad b = (b_1, \ldots, b_n),
\]

\[
\tau(a) = (\tau(a_1), \ldots, \tau(a_n)),
\]

\[
\tau(h) = (\tau(b_1), \ldots, \tau(b_n)),
\]

and \( H \) is a block diagonal \( n \times n \) matrix defined as follows
1) If \( R \neq \emptyset, \)

\[
H = \begin{pmatrix}
\sigma_1 & \cdot & \cdot & \cdot \\
\cdot & \sigma_1 & 0 & \cdot \\
\cdot & \cdot & \sigma_1 & 0 \\
\cdot & \cdot & \cdot & \cdot
\end{pmatrix}
\] if \( R \) is dividing,

\[
H = \begin{pmatrix}
1 & \cdot & \cdot & \cdot \\
\cdot & 1 & 0 & \cdot \\
\cdot & \cdot & 1 & 0 \\
\cdot & \cdot & \cdot & 0
\end{pmatrix}
\] if \( R \) is nondividing,

\((\text{rank}H = n + 1 - r).\)

2) If \( R = \emptyset, \)

\[
H = \begin{pmatrix}
\sigma_1 & \cdot & \cdot \\
\cdot & \sigma_1 & \cdot \\
\cdot & \cdot & \sigma_1
\end{pmatrix}
\] or \( H = \begin{pmatrix}
\sigma_1 & \cdot & \cdot \\
\cdot & \sigma_1 & \cdot \\
\cdot & \cdot & 0
\end{pmatrix},
\]

\((\text{rank} H = n \text{ if } n \text{ is even}, \text{rank} H = g - 1 \text{ if } g \text{ is odd}),\)

\[
\sigma_1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]
Proposition 1.1.2. ([11], Proposition 2.3) The Riemann theta function \( \theta(\bar{z}) = c\theta(z + \frac{1}{2}\text{diag}(H)) \), \( c \in \mathbb{C} \), \( |c| = 1 \), \( z \in \mathbb{C}^n \), (1.1.7) (the constant \( c \) is independent of \( \Gamma \), but may dependent on \( H \)).

For a Riemann matrix \( \Gamma \) associated with a general homology basis \( \{a_j, b_j\}_{j=1}^n \), we have the following result.

Proposition 1.1.3. If \( \Gamma = \Lambda - \Gamma \), where \( \Lambda = (\Lambda_{ij}) \), \( \Lambda_{ij} \in \mathbb{Z} \), \( \Lambda_{ij} \) is independent of \( \Gamma \), but may dependent on \( H \), then \( \theta(\bar{z}) = \theta(z + \frac{1}{2}\text{diag}(\Lambda)) \).

Proof. The proof follows from the definition of the function \( \theta(\bar{z}) \). \( \Box \)

1.2 Algebro-geometric solutions of FL hierarchy

In this text, the Forkas-Lenells type nonsingular curve \( K_n \) [15] reads

\[
K_n : \mathcal{F}(\xi, y) = y^2 - \prod_{m=0}^{2n+1} (\xi - E_m) = 0,
\]

where \( E_m \neq E_{m'} \) for \( m \neq m', m, m' = 0, 1, \cdots, 2n + 1 \), \( \{E_m\}_{m=0}^{2n+1} \subset \mathbb{C} \setminus \{0\} \), and \( n = n_+ + n_- - 1 \in \mathbb{N} \), \( n_+, n_- \in \mathbb{N}_0 \). Based on the fundamental Riemann surface theory, \( K_n \) is compactified by joining two points at infinity \( P_{\infty_+}, P_{\infty_-} \neq P_{\infty_-} \), but for notational simplicity the compactification is also denoted by \( K_n \). Points \( P \) on

\[
K_n \setminus \{P_{\infty_+}, P_{\infty_-}\}
\]

are represented as pairs \( P = (\xi, y(P)) \), where \( y(\cdot) \) is the meromorphic function on \( K_n \) satisfying

\[
\mathcal{F}_n(\xi, y(P)) = 0.
\]

For convenience we use the notation \( \xi^\pm = (\xi, \pm \sqrt{\prod_{m=0}^{2n+1} (\xi - E_m)}) \) by introducing an appropriate choice of the square root branch. Especially, a branch point \( P = (\xi, 0) \in K_n \) is then denoted by \( \xi \). The complex structure on \( K_n \) is defined in the usual way by introducing local coordinates

\[
\zeta_{Q_0} : P \to (\xi - \xi_0)
\]
near points \( Q_0 = (\xi_0, y(Q_0)) \in \mathcal{K}_n \), which are neither branch nor singular points of \( \mathcal{K}_n \); near the points \( P_{\infty \pm} \in \mathcal{K}_n \), the local coordinates are

\[ \zeta_{P_{\infty \pm}} : P \rightarrow \xi^{-1}, \]

and similarly at branch and singular points of \( \mathcal{K}_n \). Hence \( \mathcal{K}_n \) becomes a two-sheeted Riemann surface of topological genus \( n \) in a standard manner.

The holomorphic differentials \( \eta_\ell(P) \) on \( \mathcal{K}_n \) are defined by

\[ \eta_\ell(P) = \frac{\xi_\ell^{-1}}{y(P)} d\xi, \quad \ell = 1, \ldots, n, \tag{1.2.2} \]

Associated with \( \mathcal{K}_n \) one introduces an invertible matrix \( E \in \text{GL}(n, \mathbb{C}) \)

\[ E = (E_{j,k})_{n \times n}, \quad E_{j,k} = \int_{a_k} \eta_j, \quad \zeta(k) = (c_1(k), \ldots, c_n(k)), \quad c_j(k) = (E^{-1})_{j,k}, \tag{1.2.3} \]

and the normalized holomorphic differentials

\[ \omega_j = \sum_{\ell=1}^n c_j(\ell) \eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad \int_{b_k} \omega_j = \Gamma_{j,k}, \quad j, k = 1, \ldots, n. \tag{1.2.4} \]

Apparently, the Riemann matrix \( \Gamma = (\Gamma_{i,j}) \) is symmetric and has a positive-definite imaginary part. Let \( \eta \in \mathbb{C}, \ |\eta| < \min\{|E_0|^{-1}, |E_1|^{-1}, |E_2|^{-1}, \ldots, |E_{2n+1}|^{-1}\} \) and abbreviate \( E = (E_0, \ldots, E_{2n+1}), \ E^{-1} = (E_0^{-1}, \ldots, E_{2n+1}^{-1}) \). It turns out that [12]

\[ \left( \prod_{m=0}^{2n+1} (1 - E_m \eta) \right)^{-1/2} = \sum_{k=0}^{\infty} \hat{c}_k(E) \eta^k, \tag{1.2.5} \]

where \( \hat{c}_0(E) = 1, \quad \hat{c}_1(E) = \frac{1}{2} \sum_{m=0}^{2n+1} E_m, \) and

\[ \hat{c}_k(E) = \sum_{j_0, \ldots, j_{2n+1}=0}^{k} \frac{(2j_0)! \cdots (2j_{2n+1})!}{2^k (j_0)!^2 (j_{2n+1})!^2} E_0^{j_0} \cdots E_{2n+1}^{j_{2n+1}}, \quad k \in \mathbb{N}. \]

Similarly, one derives

\[ \left( \prod_{m=0}^{2n+1} (1 - E_m \eta) \right)^{1/2} = \sum_{k=0}^{\infty} c_k(E) \eta^k, \]
where \( c_0(E) = 1 \), \( c_1(E) = -\frac{1}{2} \sum_{m=0}^{2n+1} E_m \), and
\[
c_k(E) = \sum_{j_0, \ldots, j_{2n+1}=0}^{k} \frac{(2j_0)! \cdots (2j_{2n+1})! E_0^{j_0} \cdots E_{2n+1}^{j_{2n+1}}}{2^{2k}(j_0)!^2 \cdots (j_{2n+1})!^2 (2j_0-1) \cdots (2j_{2n+1}-1)}, \quad k \in \mathbb{N}.
\]

The Abel differentials \( \Omega_{P_{\pm},s-1}^{(2)} \) of second kind are defined by

\[
\Omega_{P_{\pm},s-1}^{(2)} = \pm \frac{\sum_{j=0}^{s} c_j(E) \xi^{n+s-j}}{\sqrt{\prod_{m=0}^{2n+1} (\xi - E_m)}} d\xi + \sum_{i=1}^{n} c_{j_i,\pm}^{(s-1)} \omega_j,
\]

(1.2.6)

\[
\Omega_{P_{b,\pm},s-1}^{(2)} = \pm \frac{(-1)^{n+1} \prod_{m=0}^{2n+1} E_m^{1/2} \sum_{j=0}^{s} c_j(E) \xi^{-(s+1)+j}}{\sqrt{\prod_{m=0}^{2n+1} (\xi - E_m)}} d\xi
\]

\[
+ \sum_{i=1}^{n} c_{j_i,\pm}^{(s-1)} \omega_j,
\]

(1.2.7)

where the constants \( c_{j_i,\pm}^{(s-1)}, c_{j_i,\pm}^{(s-1)}, m_j^{\pm}, \tilde{m}_j^{\pm} \) are determined by the normalization conditions

\[
\int_{a_j}^{b_j} \Omega_{P_{\pm},s-1}^{(2)} = 0, \quad j = 1, \ldots, n,
\]

(1.2.8)

\[
\int_{a_j}^{b_j} \Omega_{P_{b,\pm},s-1}^{(2)} = 0, \quad j = 1, \ldots, n.
\]

(1.2.9)
By introducing

$$\tilde{\Omega}_{L}^{(2)} = \left( \sum_{s=1}^{r_-} \int_{\Omega_{0}}^{P} q_{r_- - s} \left( \Omega_{P_{0,+},s-1}^{(2)} - \Omega_{P_{0,-},s-1}^{(2)} \right) ight)\Omega_{r_+ - s}^{(2)} + \sum_{s=1}^{r_+} \int_{\Omega_{0}}^{P} \tilde{c}_{r_+ - s} \left( \Omega_{P_{0,+},s-1}^{(2)} - \Omega_{P_{0,-},s-1}^{(2)} \right) \right)$$

(1.2.10)

$$\tilde{\Omega}_{L}^{0,+} = \lim_{P \to P_{0,+}} \left( \int_{\Omega_{0}}^{P} \tilde{\Omega}_{L}^{(2)} + \sum_{s=1}^{r_-} \tilde{c}_{r_- - s} \zeta^{-s} \right), \quad \tilde{c}_s \in \mathbb{R}, \quad (1.2.11)$$

$$\Omega_{0}^{(2)} = \Omega_{P_{0,+},0}^{(2)} - \Omega_{P_{0,-},0}^{(2)}$$

(1.2.12)

$$\int_{\Omega_{0}}^{P} \Omega_{0}^{(2)} = e_{0,+} + e_{1,+} \zeta + O(\zeta^2), \quad P \to P_{0,+}, \quad (1.2.13)$$

$$\mathcal{U}_{j}^{(2)} = (\mathcal{U}_{j,1}, \ldots, \mathcal{U}_{j,n}), \quad \mathcal{U}_{0,j}^{(2)} = \frac{1}{2\pi i} \int_{b_{j}} \Omega_{0}^{(2)}, \quad j = 1, \ldots, n, \quad (1.2.14)$$

$$\mathcal{U}_{j}^{(2)} = (\mathcal{U}_{j,1}, \ldots, \mathcal{U}_{j,n}), \quad \mathcal{U}_{0,j}^{(2)} = \frac{1}{2\pi i} \int_{b_{j}} \tilde{\Omega}_{L}^{(2)}, \quad j = 1, \ldots, n, \quad (1.2.15)$$

the solution of the initial problem \[15\]

$$\mathcal{F}_{L}^{(q,r)}(q,r) = 0, \quad (q,r)|_{t_{L}=t_{0,L}} = (q^{(0)}, r^{(0)}) \quad (1.2.16)$$

has the form

$$q(x, t_{L}) = q(x, t_{0,L}) \frac{\theta(\xi(P_{0,-}, \hat{\mu}(x, t_{0,L})))}{\theta(\xi(P_{0,-}, \hat{\mu}(x, t_{L})))} \frac{\theta(\xi(P_{0,+}, \hat{\mu}(x, t_{0,L})))}{\theta(\xi(P_{0,+}, \hat{\mu}(x, t_{L})))} \times \exp \left( i(x - x_{0})(e_{0,-} - e_{0,+}) + i(t_{L} - t_{0,L})(\tilde{\Omega}_{L}^{0,-} - \tilde{\Omega}_{L}^{0,+}) \right),$$

(1.2.17)

$$r(x, t_{L}) = r(x, t_{0,L}) \frac{\theta(\xi(P_{0,-}, \hat{\mu}(x, t_{0,L})))}{\theta(\xi(P_{0,-}, \hat{\mu}(x, t_{L})))} \frac{\theta(\xi(P_{0,+}, \hat{\mu}(x, t_{0,L})))}{\theta(\xi(P_{0,+}, \hat{\mu}(x, t_{L})))} \times \exp \left( -i(x - x_{0})(e_{0,-} - e_{0,+}) - i(t_{L} - t_{0,L})(\tilde{\Omega}_{L}^{0,+} - \tilde{\Omega}_{L}^{0,-}) \right),$$

(1.2.18)

$$q(x, t_{0,L})r(x, t_{0,L}) = \frac{\theta(\xi(P_{0,-}, \hat{\mu}(x, t_{0,L})))}{\theta(\xi(P_{0,-}, \hat{\mu}(x, t_{L})))} \frac{\theta(\xi(P_{0,+}, \hat{\mu}(x, t_{0,L})))}{\theta(\xi(P_{0,+}, \hat{\mu}(x, t_{L})))} \times \exp \left( \omega_{0,-} - \omega_{0,+} \right),$$

(1.2.19)
and $\Xi_{Q_0}$ is the vector of Riemann constants (cf. (A.45) [12]). Moreover, The Abel map linearizes the auxiliary divisors $D_{\tilde{\mu}(x,t_2)}$, $D_{\tilde{\mu}(x,t_2)}$ in the sense that

$$
\alpha_{Q_0}(D_{\tilde{\mu}(x,t_2)}) = \alpha_{Q_0}(D_{\tilde{\mu}(x_0,t_{0,x})}) - i\tilde{\nu}_0^{(2)}(x - x_0) - i\tilde{\nu}_r^{(2)}(t - t_{0,x}),
$$

(1.2.21)

$$
\alpha_{Q_0}(D_{\tilde{\mu}(x,t_2)}) = \alpha_{Q_0}(D_{\tilde{\mu}(x_0,t_{0,x})}) - i\tilde{\nu}_0^{(2)}(x - x_0) - i\tilde{\nu}_r^{(2)}(t - t_{0,x}).
$$

(1.2.22)

For convenience we denote by

$$
Z = \Xi_{Q_0} - \Delta_{Q_0}(P_{0,-}) + \alpha_{Q_0}(D_{\tilde{\mu}(0,0)}),
$$

$$
Y = \Xi_{Q_0} - \Delta_{Q_0}(P_{0,-}) + \alpha_{Q_0}(D_{\tilde{\mu}(0,0)}),
$$

$$
T = \Delta_{P_{0,-}}(P_{0,+}), 
$$

$$
V = -i\tilde{\nu}_0^{(2)}, 
$$

$$
W = -i\tilde{\nu}_r^{(2)}
$$

and then (1.2.17), (1.2.18) can be rewritten as (taking $(x_0, t_{0,x}) = 0$)

$$
q(x, t_x) = q(0,0)\frac{\theta(Z)}{\theta(Z - T)} \frac{\theta(Z - T + Vx + Wt_x)}{\theta(Z + Vx + Wt_x)} \times \exp \left( i(e_{0,-} - e_{0,+})x + i(\tilde{\Omega}_L^{0,-} - \tilde{\Omega}_L^{0,+})t_x \right),
$$

(1.2.23)

$$
r(x, t_x) = r(0,0)\frac{\theta(Y)}{\theta(Y - T)} \frac{\theta(Y - T + Vx + Wt_x)}{\theta(Y + Vx + Wt_x)} \times \exp \left( -i(e_{0,-} - e_{0,+})x - i(\tilde{\Omega}_L^{0,-} - \tilde{\Omega}_L^{0,+})t_x \right),
$$

(1.2.24)

$$
q(0,0)r(0,0) = \frac{\theta(Y)}{\theta(Z)} \frac{\theta(Z - T)}{\theta(Y - T)},
$$

(1.2.25)

that is,

$$
q(x, t_x) = q(0,0)\frac{\theta(Z)}{\theta(Z - T)} \frac{\theta(Z - T + Vx + Wt_x)}{\theta(Z + Vx + Wt_x)} \times \exp \left( i(e_{0,-} - e_{0,+})x + i(\tilde{\Omega}_L^{0,-} - \tilde{\Omega}_L^{0,+})t_x \right),
$$

(1.2.26)

$$
r(x, t_x) = \frac{1}{q(0,0)} \frac{\theta(Y)}{\theta(Z)} \frac{\theta(Y + Vx + Wt_x)}{\theta(Y - T + Vx + Wt_x)} \times \exp \left( -i(e_{0,-} - e_{0,+})x - i(\tilde{\Omega}_L^{0,-} - \tilde{\Omega}_L^{0,+})t_x \right).
$$

(1.2.27)
It should be noticed that the following relation
\[ Z = Y - T - Q, \quad Q = A_{P_0}(P_{\infty}) \]  
holds by the equivalence of the divisors \( D_{P_0} - \hat{\nu}(x,t_\xi) \sim D_{P_\infty} + \hat{\mu}(x,t_\xi). \)

2 Two results about the spectral data

In this section, we give two results about the spectral data

\[ \{ \hat{\mu}_j(0,0), \hat{\nu}_j(0,0), E_m; q(0,0), r(0,0) \}_{j=1,...,n,m=0,...,2n+1}. \]

The first theorem (cf. Theorem 2.0.1) shows any smooth solution in \( M^{\mu,\nu}_0 \) can be locally and uniquely determined by the spectral data. And the inverse is also true. Therefore, the reality conditions for \( q = \pm \bar{r} \) are totally reflected on the spectral parameters. The Theorem 2.0.4 can be regarded as a complementary illustration that all the theta functions appeared in this paper is not identical to zero.

In this text, a function \( p = p(x,t_\xi) \) in two real variables \( x,t_\xi \) is called proper at \((0,0)\) if there exists an open interval \( \Omega \subset \mathbb{R}^2 \) with \((0,0) \in \Omega \) such that \( p \) is smooth at all points \( x \in \Omega \). Keeping \( n, t_\xi \) fixed, we denote by \( M_{n,t_\xi} \) all the smooth proper solutions of the Fokas-Lenells initial value problem (1.2.16) at \((0,0)\). The subset \( M_{n,t_\xi,0} \subset M_{n,t_\xi} \) is a collection of smooth solutions in \( M_{n,t_\xi} \) such that the polynomials \( F_n(x,t_\xi; \xi), H_n(x,t_\xi; \xi) \) in \( \xi \) only have simple roots at \((x,t_\xi) = (0,0)\), respectively.

**Theorem 2.0.1.** Assume \( q, r \in C^\infty(\Omega) \) on some open set \( \Omega \subset \mathbb{R}^2 \) with \((0,0) \in \Omega \). Moreover, assume \( \mu_j(0,0) \neq \mu_k(0,0), \nu_j(0,0) \neq \nu_k(0,0), \) and \( \mu_j(0,0) \neq 0, \nu_j(0,0) \neq 0 \) for \( j \neq k, j,k = 1, \ldots, n \). Then there exists a one-to-one correspondence between the spectral data (2.0.29) and \( M_{n,t_\xi}^{\mu,\nu} \)

\[ \{ \hat{\mu}_j(0,0), \hat{\nu}_j(0,0), E_m; q(0,0), r(0,0) \}_{j=1,...,n,m=0,...,2n+1} \equiv M_{n,t_\xi}^{\mu,\nu}(\Omega). \]

**Proof.** We first prove (2.0.30) holds for stationary case, that is, there exists an open interval \( \mathcal{I}_1 \subset \mathbb{C}, 0 \in \mathcal{I}_1 \) such that the correspondence

\[ \{ \mu_j(0), \nu_j(0), E_m; q(0), r(0) \}_{j=1,...,n,m=0,...,2n+1} \equiv \{ q(x), r(x) \} \]

is one to one on \( \mathcal{I}_1 \). It suffices to build the relation from the algebro-geometric data \( \{ \mu_j(x_0), \nu_j(x_0), E_m; q(x_0), r(x_0) \}_{j=1,...,n,m=0,...,2n+1} \) to stationary solu-
tions \(q(x), r(x)\) since another direction is obvious. The Dubrovin-type equations (13, Lemma 3.2)

\[
\begin{align*}
\mu_{j,x}(x) &= \frac{-2iy(\hat{\mu}_j(x))}{\prod_{k=1, k \neq j}^{n} (\mu_j(x) - \mu_k(x))}, \quad j = 1, \ldots, n, \\
\nu_{j,x}(x) &= \frac{-2iy(\hat{\nu}_j(x))}{\prod_{k=1, k \neq j}^{n} (\nu_j(x) - \nu_k(x))}, \quad j = 1, \ldots, n,
\end{align*}
\]

has a unique solution

\[
\hat{\mu}_j \in C^\infty(\Omega'_\mu, \mathcal{K}_n), \quad \hat{\nu}_j \in C^\infty(\Omega'_\nu, \mathcal{K}_n), \quad j = 0, \ldots, n
\]
on some neighborhood \(\Omega'_\mu, \Omega'_\nu \subset \mathbb{R}\) with \(0 \in \Omega'_\mu, \Omega'_\nu\). Using the trace formulas (13, Theorem 3.3)

\[
\begin{align*}
\frac{iq}{2q_x} &= (-1)^n \prod_{j=1}^{n} \mu_j, \\
\frac{ir}{2r_x} &= (-1)^{n-1} \prod_{j=1}^{n} \nu_j,
\end{align*}
\]

one gets \(q, r\) has a unique solution

\[
\begin{align*}
q(x) &= q(0) \exp \left( \int_{0}^{x} \frac{i}{2(-1)^n \prod_{j=1}^{n} \mu_j(x')} dx' \right) \in C^\infty(\Omega_\mu, \mathcal{K}_n), \\
r(x) &= r(0) \exp \left( \int_{0}^{x} \frac{i}{2(-1)^{n-1} \prod_{j=1}^{n} \nu_j(x')} dx' \right) \in C^\infty(\Omega_\nu, \mathcal{K}_n).
\end{align*}
\]

Taking \(I_1 = \Omega'_\mu \cap \Omega'_\nu\), then we prove this Theorem for stationary case. Analogous to time-dependent case, (2.0.34) and (2.0.35) change to

\[
\begin{align*}
q(x, t_\pm) &= q(0, t_\pm) \exp \left( \int_{0}^{x} \frac{i}{2(-1)^n \prod_{j=1}^{n} \mu_j(x', t_\pm)} dx' \right), \\
r(x, t_\pm) &= r(0, t_\pm) \exp \left( \int_{0}^{x} \frac{i}{2(-1)^{n-1} \prod_{j=1}^{n} \nu_j(x', t_\pm)} dx' \right),
\end{align*}
\]

and

\[
\mu_j(x, t_\pm) \in C^\infty(\Omega_\mu), \quad \nu_j(x, t_\pm) \in C^\infty(\Omega_\nu), \quad (0, 0) \in \Omega_\mu, \Omega_\nu \subseteq \mathbb{R}^2.
\]

In other way, one can prove \(f_{\ell, \pm}(0, t_\pm), g_{\ell, \pm}(0, t_\pm), h_{\ell, \pm}(0, t_\pm)\) satisfying the following first order autonomous system

\[
\begin{align*}
f_{\ell, \pm}(0, t_\pm) &= \mathcal{F}_{\ell, \pm}(f_{\ell, \pm}(0, t_\pm), g_{\ell, \pm}(0, t_\pm), h_{\ell, \pm}(0, t_\pm)), \quad \ell = 0, \ldots, n_\pm, \\
g_{\ell, \pm}(0, t_\pm) &= \mathcal{G}_{\ell, \pm}(f_{\ell, \pm}(0, t_\pm), g_{\ell, \pm}(0, t_\pm), h_{\ell, \pm}(0, t_\pm)), \quad \ell = 0, \ldots, n_\pm - \delta_\pm, \\
h_{\ell, \pm}(0, t_\pm) &= \mathcal{H}_{\ell, \pm}(f_{\ell, \pm}(0, t_\pm), g_{\ell, \pm}(0, t_\pm), h_{\ell, \pm}(0, t_\pm)), \quad \ell = 0, \ldots, n_\pm,
\end{align*}
\]

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where $F_{\ell, \pm}, G_{\ell, \pm}, H_{\ell, \pm}$ are polynomials and $\delta_+ = 1, \delta_- = 0$. This yields the local smooth solution $f_{\ell, \pm}(0, t_\nu), g_{\ell, \pm}(0, t_\nu), f_{\ell, \pm}(0, t_\nu)$ on some open interval $I_2$ with $0 \in I_2$. Noticing that the Fokas-Lenells hierarchy ([15], (2.37)) is equivalent to the following first-order system (taking $x = 0$ here)

$$q(x, t_\nu) = f_{n, -, +}(x, t_\nu) - f_{n, -1, +}(x, t_\nu) + C_1, \quad (2.0.39)$$

$$r(x, t_\nu) = h_{n, -1, +}(x, t_\nu) - f_{n, -, +}(x, t_\nu) + C_2, \quad C_1, C_2 \in \mathbb{C}, \quad (2.0.40)$$

one derives

$$q(0, t_\nu), r(0, t_\nu) \in C^\infty(I_2, \mathbb{R}). \quad (2.0.41)$$

Finally, taking $I = (I_1 \times I_2) \cap \Omega \cap \Omega$, and using (2.0.36), (2.0.38) and (2.0.41), we complete the proof. \hfill \Box

**Remark 2.0.2.** The choice of the point $(0,0)$ is not essential. In fact, every point in $(x_0, t_0, r) \in \Omega$, there exists such a kind of structure.

**Remark 2.0.3.** Due to the equivalence of the divisors $D_{\mathbb{P}_0, \hat{\mathbb{P}}(x, t_\nu)} \sim D_{\mathbb{P}_m, \hat{\mathbb{P}}(x, t_\nu)}$ and (1.2.25), the spectral data (2.0.29) can be more precisely to be written as

$$\{\mu_j(0, 0), E_m : q(0, 0)\} \quad (j = 1, \ldots, n, m = 0, \ldots, 2n+1) \quad (2.0.42)$$

in the algbro-geometric background [12].

Following the methodology of [13] handling $(1 + 1)$-dimensional discrete model, we shall prove the following results.

**Theorem 2.0.4.** Assume $q, r \in C^\infty(\Omega)$ on some open set $\Omega \subset \mathbb{R}^2$ with $(0,0) \in \Omega$. Let $D_{\hat{\mu}, D_{\hat{\nu}, \hat{\mu}}} = \{\hat{\mu}_1, \ldots, \hat{\mu}_n\}, \hat{\nu} = \{\hat{\nu}_1, \ldots, \hat{\nu}_n\}$, be the pole and zero divisor of degree $n$, respectively, associated with $q, r$ and $\phi(x, t_\nu) \in K_n$ according to ([15], (4.16) and (4.17)),

$$\hat{\mu}_j(x, t_\nu) = (\mu_j(x, t_\nu), -\mu_j(x, t_\nu)^{2n} - G_{\beta}(\mu_j(x, t_\nu), x, t_\nu)), \quad j = 1, \ldots, n, \quad (2.0.43)$$

$$\hat{\nu}_j(x, t_\nu) = (\nu_j(x, t_\nu), \nu_j(x, t_\nu)^{2n} - G_{\beta}(\nu_j(x, t_\nu), x, t_\nu)), \quad j = 1, \ldots, n, \quad (2.0.44)$$

Then $\hat{\mu}_j(x, t_\nu)$ and $\hat{\nu}_j(x, t_\nu)$ are nonspecial for all $(x, t_\nu) \in \Omega$.

**Proof.** The divisor $D_{\hat{\mu}(x, t_\nu)}$ is nonspecial if and only if $\{\hat{\mu}_1(x, t_\nu), \ldots, \hat{\mu}_n(x, t_\nu)\}$ contains one pair of $\{\hat{\mu}_j(x, t_\nu), \hat{\nu}_j(x, t_\nu)\}$. Hence, $D_{\hat{\mu}(x, t_\nu)}$ is nonspecial as long as the projection $\mu_j$ of $\hat{\mu}_j$ are mutually distinct, $\mu_j(x, t_\nu) \neq \mu_k(x, t_\nu)$.
for \( j \neq k \). If two or more projection coincide for some \((x_0, t_{0,\mathcal{L}}) \in \mathbb{R}^2\), for instance,
\[
\mu_{j_1}(x_0, t_{0,\mathcal{L}}) = \ldots = \mu_{j_k}(x_0, t_{0,\mathcal{L}}) = \mu_0, \quad k > 1,
\]
then there are two cases in the following associated with \( \mu_0 \).

(i) \( \mu_0 \notin \{E_0, E_1, \ldots, E_{2p+1}\} \); we have \( G_{\mathfrak{n}}(\mu_0, x_0, t_{0,\mathcal{L}}) \neq 0 \) and \( \hat{\mu}_{j_1}(n_0), \ldots, \hat{\mu}_{j_k}(n_0) \) all meet in the same sheet. Hence no special divisor can arise in this manner.

(ii) \( \mu_0 \in \{E_0, E_1, \ldots, E_{2p+1}\} \); we assume \( \mu_0 = E_0 \) without loss of generality.

One concludes \( F_n(\mu_0, x_0, t_{0,\mathcal{L}}) = O((\xi - E_0)^2) \) and \( G_n(\mu_0, x_0, t_{0,\mathcal{L}}) = 0 \).

Hence
\[
R_n(\xi, x_0, t_{0,\mathcal{L}}) = G^2_n(\xi, x_0, t_{0,\mathcal{L}}) + F_n(\xi, x_0, t_{0,\mathcal{L}}) H_n(\xi, x_0, t_{0,\mathcal{L}}) = O((\xi - E_0)^2).
\]
This conclusion contradicts the hypothesis \((1.2.1)\) that the curve is non-singular. As a result, we have \( k = 1 \) and \( \hat{\mu}_j, j = 1, \ldots, n \) are pairwise distinct. Then we have completed the proof.

\[ \square \]

3 Reality conditions for \( q = \pm \mathcal{F} \)

The solution \((1.2.23), (1.2.24)\), which does not involve a specific basis of cycles, did not lead to a expression for the reality condition. As we know now, a formula for the reality condition can only be written in a very specific basis. In this section, we impose some constraints on \( \mathcal{K}_n \) and then study the parameters in the algebro-geometric solutions of FL hierarchy.

First we consider the simplest case that all the the branch points \( \{E_m\}_{m=0}^{2n+1} \) of \( \mathcal{K}_n \) are real, that is,
\[
E_m \in \mathbb{R}, \quad m = 0, \ldots, 2n + 1 \tag{3.0.45}
\]
and \( \{E_m\}_{m=0}^{2n+1} \) are ordered such that
\[
E_0 < \ldots < E_{2n+1}. \tag{3.0.46}
\]

For convenience we denote by \( \mathcal{K}_n^1 \) the curve \( \mathcal{K}_n \) \((1.2.1)\) under the constraint \((3.0.45)\) and \((3.0.46)\). One defines the antiholomorphic involution on \( \mathcal{K}_n^1 \)
\[
\tau_1(\xi, y) = (\bar{\xi}, \bar{y}), \tag{3.0.47}
\]
where \( \bar{\xi}, \bar{y} \) is the complex conjugate of \( \xi, y \in \mathbb{C} \), respectively. Here we choose the square root in \( \sqrt{\prod_{m=0}^{2n+1}(\xi - E_m)} \) such that \( \sqrt{\prod_{m=0}^{2n+1}(\xi - E_m)} = \)
\[ \sqrt{\prod_{m=0}^{2n+1} (\xi - E_m)} \text{ for } \xi \in \mathbb{C} \setminus \bigcup_{j=0}^{n}[E_{2j},E_{2j+1}] . \] Moreover, the homology basis is explicitly shown in Figure 1.

Figure 1: Homology basis on the real curves \( \mathcal{K}_n^1 \), contours on sheet 1 are solid, contours on sheet 2 are dashed.

Figure 2: The path from \( P_{0,+} \) to \( P_{0,-} \), \( \tilde{Q}_0 \) is the nearest branch point whose projection to \( \xi \)-sphere has the minimal distance to the point 0

Obviously, the involution \( \tau_1 \) transforms the homology \( \{a_j,b_j\}_{j=1}^n \) as follows:

\[ \tau_1(a_j) = a_j, \quad \tau_1(b_j) = -b_j. \] (3.0.48)

In other way, the real ovals of \( \mathcal{K}_n^1 \) w.r.t \( \tau_1 \) has \( n + 1 \) components whose projection to \( \mathbb{C}\mathbb{P}^1 \) are

\[ [E_1,E_2], [E_3,E_4], \ldots, [E_{2n+1},E_0] \]

and hence \( \mathcal{K}_n^1 \) is an \( M \)-curve. This turns out that \( H = 0 \) in Proposition 1.1.1. Therefore, the homology depicted in Figure 1 satisfies the condition
of Proposition 1.1.1. Furthermore, the properties of Riemann theta function in Proposition 1.1.2 holds w.r.t the homology in Figure 1.

Next we consider the action $\tau_1$ on the Abel differentials. Let $\tau_1^*\omega_j$ be the action of $\tau_1$ lifted to the holomorphic differentials,

$$\tau_1^*\omega_j(P) = \omega_j(\tau_1P),$$

and the definitions for the Abel differentials of the second kind and third kind are similar. The normalization condition (1.2.4) indicates

$$\int_{a_k} \tau_1^*\omega_j = \int_{\tau_1(a_k)} \omega_j = \int_{a_k} \omega_j = \delta_{j,k} = \int_{a_k} \omega_j = \int_{a_k} \omega_j,$$

and therefore

$$\int_{a_k} (\tau_1^*\omega_j - \omega_j) = 0, \quad j, k = 1, \ldots, n.$$  (3.0.50)

Therefore, by the uniqueness theorem for holomorphic Abel differentials of first kind, $\tau_1^*\omega_j = \omega_j, j = 1, \ldots, n$ and $\Gamma = -\Gamma$. To be more exactly, the $k$-th column vector of the matrix $\Gamma$

$$\Gamma^{(k)} = \begin{pmatrix} \int_{b_k} \omega_1 \\ \vdots \\ \int_{b_k} \omega_n \end{pmatrix} = \begin{pmatrix} \int_{\beta_k} \omega_1 - \int_{\tau(\beta_k)} \omega_1 \\ \vdots \\ \int_{\beta_k} \omega_n - \int_{\tau(\beta_k)} \omega_n \end{pmatrix} = 2 \begin{pmatrix} \sum_{j=1}^{k} E_{2j-2} \omega_1 \\ \vdots \\ \sum_{j=1}^{k} E_{2j-2} \omega_n \end{pmatrix} \in i\mathbb{R}^n,$$  (3.0.52)

and

$$\int_{b_{n+1}} \omega_j = 2 \int_{\beta_{n+1}} \omega_j = -\sum_{k=1}^{n} \int_{a_k} \omega_j = -2 \sum_{k=1}^{n} \int_{E_{2k-1}} \omega_j \in \mathbb{R},$$  (3.0.53)

where $\beta_k$ is an oriented curve from $\tilde{E}_{2k-1}(k = 1, \ldots, n + 1)$ to $\tilde{E}_0$. One easily proves $\tau_1^*\eta_j = \eta_j, j = 1, \ldots, n$, from which it follows $c_j(\ell) \in \mathbb{R}, j, \ell = 1, \ldots, n$. Due to the normalization conditions (1.2.8) and (1.2.9), one easily
finds $c_{j;\pm}^{(2s-1)} \in \mathbb{R}$. Therefore, the action of $\tau$ on $\Omega_{\infty,\pm,2s-1}^{(2)}$, $\Omega_{P_{b,\pm,2s-1}}^{(2)}$ is

$$
\tau_1^* \Omega_{P_{\infty,\pm,2s-1}}^{(2)} = \Omega_{P_{\infty,\pm,2s-1}}^{(2)},
$$

(3.0.54)

$$
\tau_1^* \Omega_{P_{b,\pm,2s-1}}^{(2)} = \Omega_{P_{b,\pm,2s-1}}^{(2)}.
$$

(3.0.55)

Choose $Q_0 = \tilde{Q}_0$ (cf. Figure 2) as the base point of the Abel map and then the following lemma describes the properties of the parameters with respect to the constraint $(K_{n,1})$.

**Theorem 3.0.5.** Assume the algebro-geometric solution $\{q,r\} \in \mathcal{M}_{0,0}^{\mathbb{C}}$ (see (1.2.26), (1.2.27)) and let $\{\hat{\mu}_j(0,0), E_m; q(0,0)\}$ be the spectral data associated with $q,r$. In particular, the divisor $D_{\hat{\mu}(0,0)},D_{\hat{E}(0,0)}$ are nonspecial. Moreover, suppose the symmetry constraints (3.0.45)-(3.0.47) hold. Then the solution $q$ given in (1.2.26) is the algebro-geometric solution of $FL_+$ hierarchy if and only if all the branch points $E_m < 0$, $m = 0, \ldots, 2n+1$, and

$$
\text{Re}(Z) = -\frac{1}{2} \text{Re} Q, \ (\text{mod } Z^n).
$$

(3.0.56)

Under the constraint (3.0.56), the initial value $q(0,0)$ is not arbitrary and should be taken as the form

$$
q(0,0) = \frac{\theta(Z-T)}{\theta(Z)} e^{i\vartheta},
$$

(3.0.57)

where $\vartheta \in \mathbb{R}$ is an arbitrary but fixed constant. Finally, the algebro-geometric solutions of $FL_+$ hierarchy is given by

$$
q(x,t_r) = \frac{\theta(Z-T+Vx+Wt_r)}{\theta(Z+Vx+Wt_r)} \times \exp \left( i(e_{0,-} - e_{0,+})x + i(\tilde{\Omega}_{1,-}^{0,-} - \tilde{\Omega}_{1,+}^{0,+})t_r + i\vartheta \right).
$$

(3.0.58)

**Proof.** In ref. [15] (cf. Theorem 5.1), we have proved the $j$-th components $V_j, W_j$ of vectors $V, W$ are explicitly written as

$$
V_j = -2i c_j(n),
$$

(3.0.59)

$$
W_j = 2i (-1)^{n+1} \prod_{m=0}^{2n+1} E_m^{-1/2} \sum_{s=1}^{r_{+}} \tilde{c}_{r-s,-} \sum_{\ell=1}^{s} c_j(\ell) \tilde{c}_{s-\ell}(E^{-1})
$$

$$
- 2i \sum_{s=1}^{r_{+}} \tilde{c}_{r+s,+} \sum_{\ell=1}^{n} c_j(\ell) \tilde{c}_{\ell+s-n-1}(E).
$$

(3.0.60)
and hence one gets $V = -\nabla, W = -\bar{\nabla}$. Choose the integration paths $\gamma_{\pm}$ from $\tilde{Q}_0$ to the point $P$ near near $P_{0, \pm}$, which do not intersect the cycles $a_j$ (see Figure 2) and one finds

$$
\tilde{\Omega}_{x \pm}^{0, \pm} = \lim_{P \to P_{0, \pm}} \left( \int_{Q_0}^{P} \Omega_x^{(2)} + \sum_{s=1}^{r_-} \tilde{c}_{r_- - s} \zeta^{-2s} \right)
$$

$$
= \lim_{P \to P_{0, \pm}} \left( \int_{\gamma_{\pm}} \tau^* \Omega_x^{(2)} + \sum_{s=1}^{r_-} \tilde{c}_{r_- - s} \zeta^{-2s} \right)
$$

$$
= \lim_{P \to P_{0, \pm}} \left( \int_{\gamma_{\pm}} \tilde{\Omega}_x^{(2)} + \sum_{s=1}^{r_-} \tilde{c}_{r_- - s} \zeta^{-2s} \right)
$$

$$
= \tilde{\Omega}_x^{0, \pm}.
$$

(3.0.61)

Here we use $\gamma_{\pm}$ is invariant with respect to the action $\tau_1$, i.e. $\tau(\gamma_{\pm}) = \gamma_{\pm}$. The statement $\epsilon_{0, \pm} = \epsilon_{0, \pm}$ is also true by using the similar procedure. It should be also noticed, although we won’t use it right now, that $\Xi_{Q_0}$ is half-period, that is,

$$
2\Xi_{Q_0} = 0, \quad \text{(mod } L_n). \quad (3.0.62)
$$

Combining (1.1.7) with (3.0.62), then yields

$$
\Xi_{Q_0} = \Xi_{Q_0} \in (\mathbb{Z}/2)^n.
$$

(3.0.63)

According to the integration path between $P_{0, \pm}$ described in Figure 2, the parameter $T$ satisfies $T = \bar{T}$. Using above analysis about $V, W, T, \tilde{\Omega}_x^{0, \pm}, \epsilon_{0, \pm}$ and Proposition 1.1.2, one finds the condition $q(x, t_x) = \bar{r}(x, t_x)$ is equivalent to

$$
q(0, 0) \frac{\theta(Z - T + Vx + Wt_x)}{\theta(Z - T)} \frac{\theta(Z - T + Vx + Wt_x)}{\theta(Z + Vx + Wt_x)}
$$

$$
\times \exp \left( i(e_{0, -} - e_{0, +})x + i(\tilde{\Omega}_x^{0, -} - \tilde{\Omega}_x^{0, +})t_x \right)
$$

$$
= \frac{1}{q(0, 0)} \frac{\theta(Z - T)}{\theta(Z)} \frac{\theta(Z - T + Vx + Wt_x)}{\theta(Z - T + Vx + Wt_x)}
$$

$$
\times \exp \left( i(e_{0, -} - e_{0, +})x + i(\tilde{\Omega}_x^{0, -} - \tilde{\Omega}_x^{0, +})t_x \right)
$$

$$
= \frac{1}{q(0, 0)} \frac{\theta(Z - T)}{\theta(Z)} \frac{\theta(-V + Vx + Wt_x)}{\theta(-V + T + Vx + Wt_x)}
$$

$$
\times \exp \left( i(e_{0, -} - e_{0, +})x + i(\tilde{\Omega}_x^{0, -} - \tilde{\Omega}_x^{0, +})t_x \right).
$$

(3.0.64)
Then we get
\[
\left| q(0, 0) \frac{\theta(Z)}{\theta(Z - T)} \right|^2 = \frac{\theta(Z + Vx + Wt_x)}{\theta(-\bar{Y} + Vx + Wt_x)} \frac{\theta(-\bar{Y} + Vx + Wt_x)}{\theta(Z - T + Vx + Wt_x)}. 
\]  
\( (3.0.65) \)

The equality \( (3.0.65) \) is self-contained only under the constraint
\[
Z = -\bar{Y} + T + m + n \Gamma, 
\]  
\( (3.0.66) \)
for some \( m, n \in \mathbb{Z}^n \) and hence \( (3.0.65) \) changes to
\[
\left| q(0, 0) \frac{\theta(Z)}{\theta(Z - T)} \right|^2 = e^{-2\pi i < n, T>}, 
\]  
\( (3.0.67) \)
where we used the quasi-periodic property of Riemann theta function:
\[
\theta(z + m + n \Gamma) = \theta(z) \exp (-2\pi i < n, z> - \pi i < n, n \Gamma>). 
\]

The equality \( (3.0.67) \) makes sense if and only if \( < n, T> = 0 \) (mod \( \mathbb{Z} \)), which gives rise to \( (3.0.57) \). Moreover, combing \( (1.2.28) \) with \( (3.0.66) \) yields
\[
Z = -\bar{Z} - \bar{Q} + m + n \Gamma. 
\]  
\( (3.0.68) \)
Thus,
\[
2ReZ = -ReQ + m, 
\]  
\( (3.0.69) \)
\[
0 = -ImQ + nIm \Gamma. 
\]  
\( (3.0.70) \)
by taking real and imaginary parts in \( (3.0.68) \). We claim that
\[
\bar{n} = 0. 
\]  
\( (3.0.71) \)
To prove this conclusion, we first look at the distribution of the branch points \( E_m, m = 0, \ldots, 2n + 1 \) on the line. By the condition \( P_0, \pm = (0, \pm \frac{1}{2}) \in \mathcal{K}_n^1, \) one infers the number of branch points in positive real axis is even. Then using \( (3.0.52) \) and \( (3.0.53) \) we find the imaginary part of \( Q \) is of the type \( \frac{k}{2} \Gamma, k \in \mathbb{Z}^n \). However, the solvability of \( (3.0.70) \) requires \( k=0 \), i.e. \( E_m < 0, m = 0, \ldots, 2n + 1 \), which indicates \( (3.0.71) \). Therefore, \( (3.0.56) \) holds by relations \( (3.0.69) \), \( (3.0.71) \). Finally, expression \( (3.0.58) \) is the direct result of \( (1.2.26) \), \( (3.0.57) \).
Remark 3.0.6. The strategy in Theorem 3.0.5 is invalid when applied to analyze the reality condition for \( q = -\bar{r} \). In fact, in this case (3.0.67) changes to

\[
- \left| q(0, 0) \frac{\theta(Z)}{\theta(Z - T)} \right|^2 = e^{-2\pi i < n, T>} = 1,
\]

which is meaningless.

Next we consider the smoothness condition for the algebro-geometric solutions (3.0.58) of \( FL_+ \) hierarchy constructed from (3.0.56) and (3.0.57).

**Theorem 3.0.7.** Assume the conditions of Theorem 3.0.5 and the conclusions (3.0.56)-(3.0.58) hold. Then the solution \( q \) given in (3.0.58) is smooth if and only if \( \text{Re}(Q) \in (2\mathbb{Z})^n \).

**Proof.** The solution (3.0.58) can have the singularities only on account of the zeros of the Riemann theta function \( \theta(Z + Vx + Wt_r) \) and its argument \( Z + Vx + Wt_r \) belongs to the set \( \{ \chi \in J(K_1^n) | \tau_1^j \chi + \chi = \frac{1}{2} \text{diag}(H) \} \) (in this case \( H=0 \)). Here \( \tau_1^j \) is the anti-holomorphic involution on Jacobian \( J(K_1^n) \) induced by \( \tau_1 \), given by (in this case \( \tau_1(Q_0) = Q_0 \))

\[
\tau_1^j(\chi) = \chi - n_\chi \alpha_{Q_0}(\tau_1(Q_0)),
\]

where \( n_\chi \in \mathbb{Z}, n_\chi \leq n \), is the degree of the divisor \( D \) such that \( \alpha_{Q_0}(D) = \chi \). Then by [11] (Corollary 4.3), the solution (3.0.58) is smooth if and only if \( Z + Vx + Wt_r \in i\mathbb{R}^n \) and hence \( Z \in i\mathbb{R}^n \). We complete the proof. \( \square \)

Next we shall consider the reality condition for \( q = -\bar{r} \). In this case, we assume all the branch points \( E_m, m = 0, \ldots, 2n + 1 \) of \( K_n \) are pairwise conjugate. The antiholomorphic involution is still defined by (3.0.47) on \( K_n \) and the homology basis is explicitly shown in Figure 3.

To avoid confusion we denote by \( K_n^2 \) the curve \( K_n \) with pairwise distinct branch points. It is easy to see that the holomorphic involution \( \tau_1 \) transforms the homology \( \{ a_j, b_j \}^n_{j=1} \) on \( K_n^2 \) as follows:

\[
\tau_1(a_j) = -a_j, \quad \tau_1(b_j) = b_j - \sum_{k \neq j} a_k.
\]

From

\[
\int_{a_j} \tau_1^* \eta_k = \int_{\tau_1(a_j)} \eta_k = \int_{a_j} \tau_1^* \eta_k, \quad j, k = 1, \ldots, n
\]
Figure 3: Homology basis on the real curves $\mathcal{K}_n^2$, contours on sheet 1 are solid, contours on sheet 2 are dashed.

Figure 4: $\gamma_0$ is a path from $P_{0,-}$ to $P_{0,+}$ and $a_k$ is a cycle around the branch cut whose projection to $\xi$-sphere has negative real part and the minimal distance to 0.
one gets \( \tau_1^* \eta_k = \eta_k, k = 1, \ldots, n \). In analogy to the process (3.0.50), the conditions (1.1.2) and (3.0.72) lead to

\[
\tau^* \omega_j = -\omega_j, \quad j = 1, \ldots, n.
\] (3.0.73)

Then it follows \( c_j(\ell) \in i \mathbb{R}, j, \ell = 1, \ldots, n \) by (1.2.4). Using (3.0.72), (3.0.73), one finds

\[
\int_{b_j} \omega_k = \int_{b_j} \omega_k = -\int_{\tau_1(b_j)} \omega_k = -\int_{b_j} \omega_k + \sum_{\ell \neq j} \omega_k = -\int_{b_j} \omega_k + \sum_{\ell \neq j} \delta_{\ell k}, \quad (3.0.74)
\]

or equivalently,

\[
\Gamma = -\Gamma + \mathcal{P}, \quad \mathcal{P} = (P_{ij})_{n \times n}, \quad P_{ij} = 1 - \delta_{ij}. \quad (3.0.75)
\]

By (1.1.2), (1.2.6), (1.2.7), (1.2.8), (1.2.9) and (3.0.72), one derives that the constants \( c_j^{(2s-1)} \), \( c_j^{(2s-1)} \) \( j, \pm \) \( i \mathbb{R} \) and hence the action of \( \tau_1 \) on \( \Omega^{(2)}_{\infty, 2s-1} \), \( \Omega^{(2)}_{0, 2s-1} \) reads

\[
\tau_1^* \Omega^{(2)}_{\infty, 2s-1} = \Omega^{(2)}_{\infty, 2s-1}, \quad (3.0.76)
\]

\[
\tau_1^* \Omega^{(2)}_{0, 2s-1} = \Omega^{(2)}_{0, 2s-1}. \quad (3.0.77)
\]

and hence

\[
\tau_1^* \Omega^{(2)}_0 = \overline{\Omega^{(2)}_0}, \quad \tau_1^* \Omega^{0, \pm}_2 = \overline{\Omega^{0, \pm}_2}. \quad (3.0.78)
\]

**Theorem 3.0.8.** Assume the algebro-geometric solution \( \{q, r\} \in \mathcal{M}_{0}^{\pm} \) (see (1.2.26), (1.2.27)) and let \( \{\hat{\mu}_j(0, 0), E_m; q(0, 0)\} \) be the spectral data associated with \( q, r \). In particular, the divisor \( D_{\hat{\mu}_j(0, 0)}, D_{E_m} \) are nonspecial. Moreover, suppose the symmetry constraints (3.0.45)-(3.0.47) hold. Then the expression \( \Omega \) given in (1.2.26) is the algebro-geometric solution of \( FL_- \) hierarchy if and only if

\[
\text{Im} Z = -\frac{1}{2} \text{Im} Q + m \text{Im} \Gamma, \quad m \in \mathbb{Z}^n \setminus (2\mathbb{Z})^n. \quad (3.0.79)
\]

and in this constraint, the initial value \( q(0, 0) \) is not arbitrary and should be taken as the form

\[
q(0, 0) = \frac{\theta(Z - T)}{\theta(Z)} \exp \left( -\pi <m, \text{Im} T > +i\varphi \right), \quad (3.0.80)
\]
where $\varphi \in \mathbb{R}$ is an arbitrary but fixed constant. Finally, the algebro-geometric solutions of $FL_-$ hierarchy is given by

$$q(x, t_\mathbf{z}) = \exp \left( -\pi <m, \text{Im} T > \right) \frac{\theta(Z - T + \bar{V}x + \bar{W}t_\mathbf{z})}{\theta(Z + \bar{V}x + \bar{W}t_\mathbf{z})} \times \exp \left( i(e_{0,-} - e_{0, +})x + i(\bar{\Omega}_0^0 - \bar{\Omega}_0^0 + )t_\mathbf{z} + i\varphi \right).$$  (3.0.81)

**Proof.** For the components of $V, W$ we get

$$V_j = -\frac{1}{2\pi} \int_{b_j} \Omega_0^{(2)} = -\frac{1}{2\pi} \int_{b_j} \tau^*_1 \Omega_0^{(2)}$$

$$= -\frac{1}{2\pi} \int_{\tau^*_1(b_j)} \Omega_0^{(2)} = -\frac{1}{2\pi} \int_{b_j} \Omega_0^{(2)} = V_j,$$  (3.0.82)

and similarly $W_j = W_j$, $j = 1, \ldots, n$. To determine value of the constant $T$ let us choose a specific path $\gamma_0$ from $P_{0,-}$ to $P_{0,+}$ which is shown in Figure 4. It is easy to see that the holomorphic involution $\tau_1$ acts on the contour $\gamma_0$ as follows,

$$\tau_1(\gamma_0) = \gamma_0 + a_k + \gamma_1,$$  (3.0.83)

where $\gamma_1$ denotes a positively oriented circle around the point $P_{0,+}$ (see Figure 4). It is clear that

$$\Omega_0^{0, \pm} = \lim_{P \to P_{0, \pm}} \left( \int_{Q_0}^{P} \Omega_0^{(2)} + \sum_{s=1}^{r_-} \bar{c}_{r_- - s} \zeta^{-s} \right)$$

$$= \lim_{P \to P_{0, \pm}} \left( \pm \frac{1}{2} \int_{\gamma_0 + a_k + \gamma_1} \Omega_0^{(2)} + \sum_{s=1}^{r_-} \bar{c}_{r_- - s} \zeta^{-s} \right)$$

$$= \lim_{P \to P_{0, \pm}} \left( \pm \frac{1}{2} \int_{\gamma_0} \Omega_0^{(2)} + \sum_{s=1}^{r_-} \bar{c}_{r_- - s} \zeta^{-s} \right)$$

$$= \Omega_0^{0, \pm}.$$  (3.0.84)

and similarly we obtain $e_{0, \pm} = e_{0, \pm}$. From (3.0.73) and (3.0.83) one infers
that

\[
T = A_{\rho_0-}(P_{0+}) = - \left( \int_{\gamma_0}^{\tau_0^+} \omega_1, \ldots, \int_{\gamma_0}^{\tau_0^+} \omega_n \right) = - \left( \int_{\gamma_0}^{\tau_0^+} \omega_1, \ldots, \int_{\gamma_0}^{\tau_0^+} \omega_n \right) = - T + I_k,
\]

where \( I_k = (0, \ldots, 1 \ldots, 0) \) is the \( k \)-th row of unit matrix \( I \). Next we try to search for the reality condition for \( q(x, t_\xi) = -r(x, t_\xi) \). Using Proposition 1.1.3 one derives

\[
q(0, 0) \frac{\theta(Z)}{\theta(Z - T)} \frac{\theta(Z + Vx + Wt_\xi)}{\theta(Z + Q + Vx + Wt_\xi)} \times \exp \left( i(e_{0, -} - e_{0, +})x + i(\tilde{\Omega}^0_{\xi} - \tilde{\Omega}^{0, +}_{\xi})t_\xi \right) = - \frac{1}{q(0, 0)} \frac{\theta(Z - T)}{\theta(Z)} \frac{\theta(Z + T + Q + Vx + Wt_\xi)}{\theta(Z + Q + Vx + Wt_\xi)} \times \exp \left( i(e_{0, -} - e_{0, +})x + i(\tilde{\Omega}^0_{\xi} - \tilde{\Omega}^{0, +}_{\xi})t_\xi \right),
\]

i.e.

\[
\left| q(0, 0) \frac{\theta(Z)}{\theta(Z - T)} \right|^2 = - \frac{\theta(Z + Vx + Wt_\xi)}{\theta(Z + Q + Vx + Wt_\xi)} \frac{\theta(Z + T + Q + Vx + Wt_\xi)}{\theta(Z - T + Vx + Wt_\xi)}.
\]

Taking into account (3.0.85), we conclude that

\[
\left| q(0, 0) \frac{\theta(Z)}{\theta(Z - T)} \right|^2 = - \frac{\theta(Z + Vx + Wt_\xi)}{\theta(Z + Q + Vx + Wt_\xi)} \frac{\theta(Z - T + Q + Vx + Wt_\xi)}{\theta(Z - T + Vx + Wt_\xi)}.
\]

The equality (3.0.88) is self-contained only under the condition

\[
Z + Q = Z + \bar{n} + m\Gamma, \quad \bar{n}, m \in \mathbb{Z}^n.
\]
By comparing the real part and imaginary part of (3.0.89) we arrive to the relations

\[-2ImZ = ImQ + mIm\Gamma,\]  \hspace{1cm} (3.0.90)

\[\frac{n}{2} + \frac{1}{2}mP = 0.\]  \hspace{1cm} (3.0.91)

On the other hand, from (3.0.85), (3.0.88) and (3.0.89) one obtains

\[|q(0,0)\frac{\theta(Z)}{\theta(Z-T)}|^2 = -\exp(2\pi i <m,T>)\]
\[= -\exp(2\pi i <m,ReT + iImT>)\]
\[= -\exp(\pi i <m, I_k>) \exp(-2\pi <m, ImT>).\]  \hspace{1cm} (3.0.92)

It is easy to see that \(<m, I_k> = m_k\) and hence we know the k-th component of \(m\) must be odd by comparing both sides of (3.0.92). Taking into account (3.0.91) one infers

\[m \in \mathbb{Z}^n \backslash (2\mathbb{Z})^n.\]  \hspace{1cm} (3.0.93)

Moreover, by (3.0.92) and (3.0.93) we get

\[|q(0,0)\frac{\theta(Z)}{\theta(Z-T)}|^2 = \exp(-2\pi <m, ImT>).\]  \hspace{1cm} (3.0.94)

which indicates (3.0.80). (3.0.81) is the direct result of (1.2.26), (3.0.80).

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