THE MATROID STRATIFICATION OF \((\mathbb{P}^1)^{[k]}\)

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Abstract. Given a homogeneous ideal \(I\) in a polynomial ring over a field, one may record, for each degree \(d\) and for each polynomial \(f \in I_d\), the set of monomials in \(f\) with nonzero coefficients. These sets form a sequence of matroids indexed by \(d\), collectively called the tropicalization of \(I\). Tropicalizing ideals induces a “matroid stratification” on any (multi-)graded Hilbert scheme. Very little is known about the structure of these stratifications; our motivation in studying them is to understand torus orbits in the Hilbert scheme of points \((\mathbb{A}^2)^{[n]}\).

In this paper, we explore many examples of matroid strata and give a convenient way of visualizing them. We focus on the special case of principal ideals in \(k[x, y]\), i.e. collections of points in \(\mathbb{P}^1\). In this case, we find that the matroid strata in \((\mathbb{P}^1)^{[k]}\) (of which there are infinitely many, so they are not Zariski-locally closed) are exactly cut out by the Schur polynomials in \(k\) variables. We find certain minimal strata (rational curves) in \((\mathbb{P}^1)^{[k]}\) that are in natural bijection with binary necklaces with \(k\) black beads and any number of white beads, if we are over a field containing appropriate roots of unity. We then give an application: by intersecting these special principal ideals with monomial ideals, we find corresponding strata in \((k^2)^{[n]}\); classifying this graph is a longstanding open problem studied by Altmann-Sturmfels, Herding-Maclagan, and others.

1. Introduction

Let \(k\) be a field. Given a homogeneous ideal \(I \subseteq k[x_1, \ldots, x_r]\), each graded piece \(I_d\) is a subspace of the free vector space on the set \(\text{Mon}_d\) degree-\(d\) monomials. The data of all supports of polynomials in \(I_d\) — the support of a vector in a free vector space is the set of coordinates whose coefficients are nonzero — comprise a combinatorial portrait called the matroid of \(I_d\), denoted \(M(I_d)\). In this paper, we study \(I\) via the infinite sequence of matroids \((M(I_d))_{d \geq 0}\). We may consider in this way ideals homogeneous with respect to any positive grading (or multigrading) on \(k[x_1, \ldots, x_r]\). One can recover various properties of \(I\) from its sequence of matroids, including (importantly) its initial ideal with respect to a term order, see Observation 3.7. The sequence of matroids \((M(I_d))_{d \geq 0}\) is equivalent data to the tropicalization of \(I\) [MR18], where \(k\) is considered as a valued field with trivial valuation.

In any moduli space of homogeneous ideals (a multigraded Hilbert scheme in the sense of [HS04]), the assignment \(I \mapsto (M(I_d))_{d \geq 0}\) defines a “matroid stratification,” possibly with countably many strata, analogous to the matroid stratification on \(Gr(m, n)\). The first natural question is which strata are nonempty, i.e. the “realizability question”: given a sequence of matroids \((M_d)_{d \geq 0}\), where \(M_d\) has groundset \(\text{Mon}_d\), is there a homogeneous ideal \(I\) such that \(M(I_d) = M_d\) for all \(d\)? There is a simple compatibility condition between the matroids that is necessary for such an \(I\) to exist — \((M_d)_{d \geq 0}\) must be a tropical ideal in the sense of [MR18] — and we restrict our attention to such sequences (see Definition 3.9). This generalizes the usual realizability problem for matroids, which asks whether a given matroid \(M\) with groundset \(S\) is the matroid of a subspace of \(\mathbb{K}S\). It also generalizes the “\(T\)-graph problem” for multigraded Hilbert schemes, see [AS05] [HM12]. (This was our motivation for this project — see the discussion in Section 2.) The \(T\)-graph problem for a multigraded Hilbert scheme — or more generally for a variety on which a torus acts with finitely many fixed points — asks which pairs of torus-fixed points are the endpoints of a 1-dimensional orbit.

We study in detail the tropicalizations of principal homogeneous ideals in \(k[x, y]\) (i.e. points of \(\text{Hilb}^{(k)}(\mathbb{P}^1)\) or \(\text{Hilb}^{(k)}(\mathbb{P}(a, b))\)). First we identify the matroid stratification in this case:

**Theorem 4.9** The matroid stratification on \(\text{Hilb}^{(k)}(\mathbb{P}^1)\) is the stratification generated by all Schur polynomials \(s_{\lambda}\) in \(k\) variables. (Schur polynomials define divisors on \(\text{Hilb}^{(k)}(\mathbb{P}^1)\) via the identification \(\text{Hilb}^{(k)}(\mathbb{P}^1) \cong \text{Sym}^k(\mathbb{P}^1)\).)

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By analyzing the common vanishing loci of sets of Schur polynomials, we find certain minimal strata with extra combinatorial structure:

**Corollary 4.22** Let $d > k$, and let $\mathcal{D}_{d,k} \subseteq \text{Hilb}^{(k)}(\mathbb{P}^1)$ be the subscheme consisting of those ideals containing a polynomial of the form $ax^d + by^d$. If $k$ contains $d$ distinct $d$th roots of unity, then $\mathcal{D}_{d,k} \subseteq \text{Hilb}^{(k)}(\mathbb{P}^1)$ is a finite set of rational curves, in natural bijection with the set of binary necklaces with $k$ black and $d - k$ white beads.

Lastly, in Section 5, we apply these results to study the $T$-graph of $\text{Hilb}^{(m)}(\mathbb{A}^2)$. We do this by intersecting ideals in $\text{Hilb}^{(k)}(\mathbb{P}^1)$ with monomial ideals to get finite-cotangent ideals. We refer to the resulting ideals as **monoprincipal**. We use them to draw conclusions about certain edges of the $T$-graph, e.g.

**Corollary 5.11** Let $k = \mathbb{C}$. Let $k \geq 1$ and $m > k$. Consider ideals in $\text{Hilb}^{(km)}(\mathbb{A}^2)$ whose initial ideals with respect to the two term orders $x \prec y$ and $y \prec x$ are, respectively, $(x^k, y^m)$ and $(x^m, y^k)$. This is a finite set of rational curves, in natural bijection with the set of binary necklaces with $k$ black and $m - k$ white beads. In particular, $(x^k, y^m)$ and $(x^m, y^k)$ are connected by an edge in the $T$-graph of $\text{Hilb}^{(km)}(\mathbb{A}^2)$.

Most of our constructions — dependence loci, the “edge moduli spaces” we use to study 1-dimensional torus orbits — are defined in Sections 2 and 3 for general multi-graded Hilbert schemes. In particular, we hope to generalize these combinatorial results to the case of hypersurfaces in $\mathbb{P}^n$. The forthcoming paper [FGG] of Fink-Giansiracusa-Giansiracusa is closely related to this one. Motivated by understanding “tropical Hilbert schemes,” which are moduli spaces of tropical ideals over arbitrary valued fields, they also investigate the tropicalizations of ideals of points in $\mathbb{P}^1$. Our results complement each other: this paper considers trivially valued fields, and Hilbert schemes of arbitrarily many points on $\mathbb{P}^1$, while they consider arbitrary valued fields, but have results mainly for $\leq 2$ points in $\mathbb{P}^1$. We hope that these perspectives can be merged to describe tropical Hilbert schemes of arbitrarily many points in $\mathbb{P}^1$.

Zajaczewska’s Ph.D thesis [Zaj18] studied the tropical Hilbert schemes of hypersurfaces of degrees 1 and 2 in $\mathbb{P}^1$ and $\mathbb{P}^2$. Among other things, the thesis contains a version of the special case $k = 2$ of Corollary 4.22. (More specifically, the case $k = 2$ of Proposition 4.19)

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## 2. Torus orbits on multigraded Hilbert schemes

### 2.1. Multigraded Hilbert schemes

Multigraded Hilbert schemes are the natural moduli spaces of homogeneous ideals in a polynomial ring. Let $k$ be a field, and consider the polynomial ring $R = k[x_1, \ldots, x_r]$. We use the following notion of multigrading, which is not the most general version:

**Definition 2.1.** For $b \in \mathbb{Z}_{>0}^r$, a $\mathbb{Z}^b$-multigrading $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_r)$ on $R$ is an assignment of a positive multidegree $\mathbf{a}_i \in \mathbb{Z}^b_{\geq 0} \setminus \{(0, \ldots, 0)\}$ to each variable $x_i$. A multigrading is **nondegenerate** if the rowspan of $\mathbf{a}$ is a rank-$b$ lattice in $\mathbb{Z}^r$.

All multigradings from now on are assumed to be nondegenerate. A $\mathbb{Z}^b$-multigrading defines a decomposition $R = \bigoplus_{d \in \mathbb{Z}^b_{\geq 0}} R_d$. For any $\mathbf{a}$-homogeneous ideal $I \subseteq R$, there is a **multigraded Hilbert function** $h : \mathbb{Z}^b_{\geq 0} \to \mathbb{N}$ that assigns to each element $d \in \mathbb{Z}^b_{\geq 0}$ the dimension $\dim_k(R_d/(I \cap R_d))$.}

Haiman and Sturmfels [HS04] define (in considerably more generality than we have presented) a **multigraded Hilbert scheme** $\text{Hilb}_{\mathbf{a}}(\mathbb{A}^r)$ that is a projective fine moduli space for $\mathbf{a}$-homogeneous ideals with multigraded Hilbert function $h$. For each $d \in \mathbb{Z}^b_{\geq 0}$, there is a short exact sequence of vector bundles on $\text{Hilb}_{\mathbf{a}}(\mathbb{A}^r)$:

$$
0 \to I_d \to R_d \to Q_d \to 0,
$$

(1)
where $\mathcal{I}_d$ is the universal ideal sheaf, $\mathcal{R}_d$ denotes the trivial sheaf with fiber $R_d$, and $\mathcal{Q}_d$ is the rank-$h(d)$ universal quotient sheaf. Instead of $a$-homogeneous ideals, one may consider points of $\text{Hilb}_d^h(\mathbb{A}^r)$ to be subspaces of $\mathbb{A}^r$ that are invariant under the action of a certain subtorus of $T := (k^*)^r$ — specifically, the image of the homomorphism $(k^*)^b \rightarrow (k^*)^r$ defined by the matrix of exponents $a$.

**Example 2.2.** The ideal $(x^2y + z^6) \subseteq k[x, y, z]$ is homogeneous with respect to the grading $((3, 0), (0, 6), (1, 1))$. This ideal is invariant under elements of $(k^*)^3$ of the form $(\lambda_1^3, \lambda_2^6, \lambda_1\lambda_2)$, which act on the polynomial $x^2y + z^6$ by multiplication by $\lambda_1^3\lambda_2^6$.

The torus $T$ acts on $\text{Hilb}_d^h(\mathbb{A}^r)$. The orbit-stabilizer theorem implies that the dimension of any $T$-orbit $T \cdot I$ is at most $r - m$ (by nondegeneracy). There is a stratification of $\text{Hilb}_d^h(\mathbb{A}^r)$ by $T$-orbit dimension; the monomial ideals with graded Hilbert function $h$, which are $T$-fixed, are the minimal strata.

**Example 2.3.** By $a$-homogeneity of $I$, $\dim_k(R/I) = \sum_{d \in \mathbb{Z}_{\geq 0}} h(d)$ when either side (hence both) is finite. That is, $V(I)$ has finite length if and only if $h(d) = 0$ for all but finitely many $d \in \mathbb{Z}_{\geq 0}$. In this case, there is a natural embedding $\text{Hilb}_d^h(\mathbb{A}^r) \rightarrow \text{Hilb}[\sum h(d) ](\mathbb{A}^r)$ into the Hilbert scheme of points in $\mathbb{A}^r$. When $r = 2$ and $\sum h(d) < \infty$, monomial ideals $M$ are in bijection with Young diagrams by $M \mapsto \{(a, b) : x^a y^b \in M\}$.

**Example 2.4.** If $b = 1$ and $\sum h(d)$ is not finite, then $\text{Hilb}^{(a_1, \ldots, a_r)}_d(\mathbb{A}^r)$ has a natural map to a Hilbert scheme of subschemes of the weighted projective space $\mathbb{P}(a_1, \ldots, a_r)$, cut out by the same ideal. This map may not be an embedding, essentially due to the fact that the primary decomposition of $I \in \text{Hilb}^{(a_1, \ldots, a_r)}_d(\mathbb{A}^r)$ could have $(x, y)$ as an embedded prime. We use this map in Section 3

**Remark 2.5.** In this paper we will be concerned mainly with the case $r = 2, b = 1$, and $k = \mathbb{C}$; however, everything we define makes sense over an arbitrary field. We restrict to the case $k = \mathbb{C}$ in Section 4.2 in order to do explicit computations involving roots of unity.

### 2.2. $T$-curves

Suppose $b = r - 1$. Then for any $h$, the points of $\text{Hilb}_d^h(\mathbb{A}^r)$ are ideals whose $T$-orbit is at most 1-dimensional. The case $\sum_{d} h(d) < \infty$ is of particular interest; $T$-curves in $\text{Hilb}^h(\mathbb{A}^r)$ have been studied before [Iar72, Evar02, Evar04, AS05, HM12]. By [HM12], every nonmonomial ideal $I$ in $\text{Hilb}^h(\mathbb{A}^r)$ has exactly two initial monomial ideals $M_1$ and $M_2$; these are the “endpoints” of the orbit $T \cdot I$. The subscheme of $\text{Hilb}_d^h(\mathbb{A}^r)$ consisting of ideals whose endpoints are two fixed monomial ideals $M_1$ and $M_2$ is called the edge scheme $E(M_1, M_2)$. An algorithm to compute $E(M_1, M_2)$ using Gröbner bases was developed by Altmann-Sturmfels [AS05], and was implemented in Macaulay2 by Maclagan [Mac09].

**Remark 2.6.** In the case $r = 2$ (and $b = 1$) these are simply the initial ideals with respect to the two term orders $x > y$ and $y < x$. Also, in this case $M_1$ and $M_2$ determine the grading $(a_1, a_2)$ (with gcd$(a_1, a_2) = 1$) and $h$. Finally, $M_1$ and $M_2$ can only be the endpoints of $T \cdot I$ if the Young diagram of $M_1$ can be obtained from that of $M_2$ (or vice versa) by moving boxes up and to the left, in direction $(-a_2, a_1)$. (See [HM12], Section 2.) This defines a partial order $<$ on Young diagrams.

**Example 2.7.** On the left in Figure 1 is a diagram of $\text{Hilb}_{(1,2,2,1,1,0,0,\ldots)}^1(\mathbb{A}^2)$, computed using TEdges. Note that in particular $E(\begin{array}{ll}1 & 1 \\ 1 & 1 \end{array}, \begin{array}{ll}2 & 0 \\ 0 & 0 \end{array})$ is empty, even though $\begin{array}{ll}1 & 1 \\ 1 & 1 \end{array} < \begin{array}{ll}2 & 0 \\ 0 & 0 \end{array}$ (because $\begin{array}{ll}2 & 0 \\ 0 & 0 \end{array}$ can be obtained from $\begin{array}{ll}1 & 1 \\ 1 & 1 \end{array}$ by moving boxes up and to the left in direction $(-1, 1)$.) This is the smallest example of an “unexpectedly” empty edge scheme. (Note, however, that this is not really unexpected, see Example 3.11.)

Also, we see here a kind of degeneration of orbits that may happen in a multigraded Hilbert scheme — namely, orbits in $E(\begin{array}{ll}1 & 1 \\ 1 & 1 \end{array}, \begin{array}{ll}2 & 0 \\ 0 & 0 \end{array})$ may degenerate into a union of two orbits, one in the edge scheme $E(\begin{array}{ll}1 & 1 \\ 1 & 1 \end{array}, \begin{array}{ll}1 & 1 \\ 1 & 1 \end{array})$ and the other in $E(\begin{array}{ll}1 & 1 \\ 1 & 1 \end{array}, \begin{array}{ll}0 & 0 \\ 0 & 0 \end{array})$. (Alternatively, one in $E(\begin{array}{ll}1 & 1 \\ 1 & 1 \end{array}, \begin{array}{ll}1 & 1 \\ 1 & 1 \end{array})$ and the other in $E(\begin{array}{ll}1 & 1 \\ 1 & 1 \end{array}, \begin{array}{ll}0 & 0 \\ 0 & 0 \end{array})$.)

**Example 2.8.** On the right in Figure 1 is a diagram of $\text{Hilb}_{(1,2,1,0,0,\ldots)}^1(\mathbb{A}^2)$. The closure of the relevant edge scheme is in fact isomorphic to $\mathbb{P}^2$, and the orbit-closures of the $T$-action are identified with the conics $sxy = t^2$ for $|s : t| \in \mathbb{P}^1$. We see here another kind of degeneration of orbits — as $s \rightarrow 0$, the conics degenerate into a “doubled” line.
Remark 2.9. By viewing $T$-orbit-closures as rational curves in $\text{Hilb}_h(A^r)$, and using machinery of unbroken stable maps [OP10], one may associated to each edge scheme a moduli space $M(M_1, M_2)$, whose points correspond to $T$-orbit-closures or their degenerations. (More specifically, the moduli space parametrizes $T$-invariant maps $f : C \to \text{Hilb}_h(A^r)$, possibly ramified, from nodal rational curves to $\text{Hilb}_h(A^r)$, such that $f$ is locally $T$-equivariantly smoothable at every node of $C$.) In Example 2.7, one can show that $M(\emptyset, \emptyset) \cong \mathbb{P}^1$, with two points corresponding to the two degenerations of orbits into nodal rational curves (unions of orbits). In Example 2.8, one can show that $M(\emptyset, \emptyset) \cong \mathbb{P}(2,1)$, a weighted projective stack. The orbifold point corresponds to (mapping to a double cover of) the “doubled” limit curve, and another point corresponds to the nodal degeneration of orbits.

We finish this section with two examples that show “bad” behaviors exhibited by edge schemes. We have already seen (Example 2.7) that $E(M_1, M_2)$ may be empty.

Example 2.10. Let $r = 2$, and $b = 1$. Let $M_1 = (x^k, y^\ell)$, and $M_2 = (x^\ell, y^k)$, where $k < \ell$. By Section 5, $E(M_1, M_2)$ is a finite collection of rational curves, in natural bijection with the set of binary necklaces with $k$ black and $\ell - k$ white beads. In particular, $E(M_1, M_2)$ need not be connected, and may have arbitrarily many components.

Example 2.11. Using TEdges, we compute that $E(\emptyset, \emptyset)$ is empty. However, $\text{M}(\emptyset, \emptyset)$ consists of a single point — a map from a nodal curve to the union of two orbits, with the node mapping to $\emptyset$.

Question 2.12. Very little is known about edge schemes or the associated moduli spaces; we have computed them explicitly through $n = 7$, and most of the moduli spaces $\text{M}(M_1, M_2)$ are orbifold points or weighted projective lines. For example, we do not know the answer to the following: do there exist monomial ideals $M_1, M_2$ such that $\text{M}(M_1, M_2)$ is singular (as an orbifold)? Irrational?

3. Homogeneous ideals and matroids

3.1. Matroids and vector subspaces. We briefly recall the basics of matroid theory.

Definition 3.1. A matroid $M = (E, r)$ is the data of a finite set $E$, called the groundset, together with a function $r : 2^E \to \mathbb{Z}_{\geq 0}$ (where $2^E$ is the power set of $E$) called the rank function, such that:

1. $r(\emptyset) = 0$,
2. For all subsets $S, S' \subseteq E$, $r(S \cup S') + r(S \cap S') \leq r(S) + r(S')$, and
3. For every subset $S \subseteq E$ and every element $x \in E \setminus S$, $r(S \cup \{x\}) \leq r(S) + 1$.

A subset $S \subseteq E$ is called dependent if $r(S) < |S|$, and independent otherwise. A maximal independent subset is called a basis, and a minimal dependent subset is called a circuit.
A subspace of $k^n$ gives rise to a matroid, as follows. (Note: This is dual to the definition found in some literature.)

**Definition 3.2.** Let $V \subseteq k^n$ be a subspace. The matroid of $V$, denoted $M(V)$, has groundset $[n] = \{1, \ldots, n\}$, and rank function $r(S) = \dim(k^S/(V \cap k^S))$, where $k^S = \text{Span}(e_i : i \in S)$.

**Example 3.3.** Let $E$ be a finite set, and let $0 \leq b \in \mathbb{Z}$. The uniform matroid $U_{k,E}$ of rank $k$ is the matroid with rank function

$$r(S) = \begin{cases} |S| & |S| \leq k \\ k & |S| \geq k. \end{cases}$$

We will use the following standard fact.

**Lemma 3.4.** Let $V \subseteq k^n$ be a subspace, and let $S \subseteq [n]$. Then $S$ is a union of circuits in $M(V)$ if and only if there exists $v \in V$ such that $\text{supp}(v) = S$.

### 3.2. The tropicalization of a homogeneous ideal.

Our main objects of study are tropical ideals, as defined in [MRIS]. However, we study them only “over the Boolean semifield,” corresponding to the fact that $k$ is assumed to have trivial valuation. For the rest of the paper, we suppress the phrase “over the Boolean semifield” and refer simply to “tropical ideals.”

**Definition 3.5.** Let $\mathbf{a} = (\vec{a}_1, \ldots, \vec{a}_r)$ be a positive multigrading on $k[x_1, \ldots, x_r]$. Let $\text{Mon}_d(\mathbf{a})$ denote the set of monomials of degree $d$ with respect to the grading $\mathbf{a}$. A tropical (homogeneous) ideal $\mathcal{M} = \{\mathcal{M}_d : d \geq 0\}$ with respect to the grading $\mathbf{a}$ (over the Boolean semifield) is the data of, for each $d \in \mathbb{Z}_{\geq 0}^+$, a matroid $\mathcal{M}_d = (\text{Mon}_d(\mathbf{a}), r_d)$, such that for any circuit $S$ of $\mathcal{M}_d$, and any monomial $m' \in \text{Mon}_{d'}(\mathbf{a})$, $m'S$ is a union of circuits in $\mathcal{M}_{d+d'}(\mathbf{a})$. The graded Hilbert function of a tropical homogeneous ideal $\mathcal{M}$ is the function $d \mapsto r_d(\text{Mon}_d(\mathbf{a}))$.

Just as a subspace of $k^n$ gives rise to a matroid (a “tropical linear space over the Boolean semifield”), a homogeneous ideal with respect to the grading $\mathbf{a}$ gives rise to a tropical homogeneous ideal with grading $\mathbf{a}$, as follows.

**Definition 3.6.** Let $I \subseteq \text{Hilb}_n(k^r)$. The tropicalization $\text{Trop}(I)$ of $I$ is $\{\text{Trop}(I)_d : d \in \mathbb{Z}_{\geq 0}^+\}$, where $\text{Trop}(I)_d = M(I_d)$. (Note that Lemma 3.4 implies $\text{Trop}(I)$ is a tropical ideal with grading $\mathbf{a}$ whose graded Hilbert function agrees with that of $I$.)

We have seen that our motivating questions about torus orbits can be stated using the language of initial ideals. Initial ideals of $I$ are determined from $\text{Trop}(I)$, as follows. If $M$ is a matroid and $\preceq$ is a total order on the groundset of $M$, the initial set of $M$ with respect to $\preceq$ is

$$\text{in}_\preceq(M) = \{\text{min}_\preceq(S) : S \text{ a circuit of } M\}.$$  

The following is then immediate from Lemma 3.4.

**Observation 3.7.** Let $I \subseteq R$ be a homogeneous ideal, and let $\preceq$ be a term order. The initial ideal of $I$ with respect to $\preceq$ is given by $\text{in}_\preceq(I)_d = \text{Span}(\text{in}_\preceq(\text{Trop}(I)_d))$.

The following gives a useful restriction on $\text{Trop}(I)$ in terms of the collection of initial ideals of $I$ with respect to various term orders.

**Lemma 3.8.** Suppose $M = (E, r_M)$ is a matroid, and $m \in E$. Suppose there exists a circuit of $M$ that properly contains $m$. Then for any total order $\preceq$ on $E$,

$$|\{m' \in \text{in}_\preceq(M) : m' \preceq m\}| - |\{m' \in \text{in}_\preceq(M) : m' \preceq m\} > |\{m' \in \text{in}_\preceq(M) : m' \succeq m\}| - |\{m' \in \text{in}_\preceq(M) : m' \succeq m\}|.$$ 

\footnote{That is, $\{m\}$ is neither a loop (1-element circuit) nor a coloop (set not contained in any circuit) of $M$.}
Proof. The following are easy to check using matroid contraction and deletion operations:

\[ |\{m' \in \text{in}_{\leq} (M) : m' \leq m\}| = r(\{m' \in E : m' \leq m\}) \]
\[ |\{m' \in \text{in}_{\geq} (M) : m' \geq m\}| = r(E) - r(\{m' \in E : m' \succ m\}) \]
\[ |\{m' \in \text{in}_{<} (M) : m' \geq m\}| = r(E) - r(\{m' \in E : m' \prec m\}) \]
\[ |\{m' \in \text{in}_{>} (M) : m' \geq m\}| = r(\{m' \in E : m' \geq m\}) \].

Note that \(r(\{m' \in E : m' \leq m\}) + r(\{m' \in E : m' \succ m\}) \geq r(E)\), with equality if and only if \(M\) is the direct sum of matroids on the groundsets \(\{m' \in E : m' \leq m\}\) and \(\{m' \in E : m' \succ m\}\). Similarly, \(r(\{m' \in E : m' \prec m\}) + r(\{m' \in E : m' \geq m\}) \geq r(E)\), with equality if and only if \(M\) is the direct sum of matroids on the groundsets \(\{m' \in E : m' \prec m\}\) and \(\{m' \in E : m' \geq m\}\). Thus the left side of (2) is nonnegative, the right side is nonpositive, and both are zero if and only if \(M\) is a direct sum of matroids on the groundsets \(\{m' \in E : m' \prec m\}\), \(\{m\}\), and \(\{m' \in E : m' \succ m\}\). If so, then \(m\) is either a 1-element circuit of \(M\), or is contained in no dependent sets.

\[\square\]

Remark 3.9. Since the torus action on \(R\) preserves the supports of polynomials, it follows that \(\text{Trop}(t \cdot I) = \text{Trop}(I)\) for \(t \in (k')^r\).

From this setup we get an easy necessary condition for an edge moduli space from Section 2.2 to be nonempty:

Observation 3.10. Suppose \(b = r-1\), and suppose \(M_1, M_2\) are monomial ideals. Then \(E(M_1, M_2) = \emptyset\) if there is no tropical ideal whose two initial ideals (see Section 2.2) are \(M_1\) and \(M_2\), respectively.

Example 3.11. Consider again \(E(\underline{2}, \underline{4})\) from Example 2.7. Suppose there existed an ideal \(I\) with these two initial ideals. We attempt to describe \(\text{Trop}(I)\). Since \(\text{Trop}(I)_0\) and \(\text{Trop}(I)_1\) have corank zero, they must be the matroids of the zero subspace in \(k\) and \(k^2\), respectively. The two initial sets of \(\text{Trop}(I)_2\) are equal: the only way this can happen is if \(\text{Trop}(I)_2 = \text{Trop}(\underline{2})_2\). (To see this, we apply Lemma 3.8 to each of the three monomials \(x^2, xy, y^2\).) Thus \(xy\) is a 1-element circuit of \(\text{Trop}(I)\), as are all of its monomial multiples. In degree 3, since \(x^3\) is in the initial set corresponding to \(\underline{4}\), but not \(\underline{2}\), we must have a circuit strictly containing \(x^3\), and the only choice is \(\{x^3, y^3\}\). On the other hand, this implies \(\{x^3y, y^4\}\) is a union of circuits. Since \(\{x^3y\}\) is a circuit, we conclude \(\{y^4\}\) is a circuit, contradicting the fact that \(y^4\) is missing from the initial set corresponding to \(\underline{4}\). This explains why we computationally found the edge scheme to be empty.

Remark 3.12. Observation 3.10 is stronger than the necessary condition given in Corollary 3.4 of [HMT12] — a tropical ideal has an associated “arrow map.” (This has been observed by Maclagan.) However, Observation 3.10 is much more difficult to check than the existence of an arrow map. (There is an intermediate necessary condition that is also easy to check; we do not include the procedure, as we do not know of an application.)

3.3. Tropically principal tropical ideals. Let \(\text{Mon}_d(a)\) be the set of degree-\(d\) monomials with respect to the multigrading \(a\). Let \(m_d(a) = |\text{Mon}_d(a)|\).

Definition 3.13. Let \(\mathcal{M} = \{\mathcal{M}_d : d \in \mathbb{Z}_{\geq 0}^b\}\) be a tropical ideal with grading \(a\). We say that \(\mathcal{M}\) is tropically principal if there exists \(c \in \mathbb{Z}_{\geq 0}^b\) such that

\[ \text{rk}(\mathcal{M}_d) = m_d(a) - m_{d-c}(a) \]

for all \(d \in \mathbb{Z}_{\geq 0}^b\). We say that \(\mathcal{M}\) is generated in degree \(c\).

Note that \(\mathcal{M}_c\) has corank 1, hence has a unique circuit. The following is standard in commutative algebra:

Proposition 3.14. Let \(I \subseteq R\) be a homogeneous ideal with grading \(a\). Then \(I\) is principal if and only if \(\text{Trop}(I)\) is tropically principal.

Remark 3.15. The colength of a principal ideal \(I \subseteq R\) is always infinite if \(r > 1\). In the case \(r = 2\), if \(I = (f)\) is \((a, b)\)-homogeneous of degree \(k\), then \(I\) is a point of \(\text{Hilb}_{(b)}(\mathbb{P}(a, b))\). We study these ideals in Section 4 and show in Section 5 how to relate them to finite-colength ideals.
3.4. Pictures of tropical ideals. When $r = 2$, we draw tropical ideals as follows. We draw a grid representing monomials in two variables, where the bottom-leftmost square represents the monomial 1. We draw each circuit of a tropical ideal as a line segment connecting a collection of dots in squares of the grid; these dots correspond to monomials in the circuit. We also color-code monomials $m$: red if $m \not\in \text{in}_q(\mathcal{M})$, blue if $m \not\in \text{in}_y(\mathcal{M})$, purple if both are true, white if neither is true, and green if the $m$ is itself a circuit. (Note that $m$ is colored green in the diagram of Trop$(I)$ if and only if $m \in I$.)

To avoid clutter, we may omit a circuit $S$ of $\mathcal{M}_d$ if we deem it “uninformative,” i.e. if $S$ is “forced” to be a dependent by the existence of a circuit in lower degree. Precisely, from now on we omit a circuit $S$ in degree $d$ if there exists a circuit $S'$ in degree $d' < d$ and a collection $T$ of degree-$(d - d')$ monomials such that $S \subseteq \bigcup_{m \in T} mS'$ and $|S| > r(\bigcup_{m \in T} mS')$.

**Example 3.16.** The ideal $I = (x^3 + x^2y + 2xy^3 + 3y^3, x^5, xy^4)$ has tropicalization pictured (fully) on the left in Figure 2. When we omit uninformative circuits, the tropicalization appears as the diagram on the right in Figure 2. Note that Trop$(I)_4$ has rank 3; therefore the five circuits are uninformative, as they each have four elements, and 4-element sets must be dependent. We omit the 2-element circuits in Trop$(I)_5$ for the same reason.

3.5. Dependence loci. The operation of tropicalization defines a stratification of Hilb$(\mathbb{A}^r)$ that refines the grading stratification, as follows. Fix a multigrading $\mathbf{a}$ and a collection $U \subseteq \text{Mon}_d(\mathbf{a})$, with $\ell := |U|$. We restate the condition that $U$ be dependent in Trop$(I)_d$. Consider the tautological sequence (1) on Hilb$(\mathbb{A}^r)$. The collection $U$ defines, up to sign, an element of $\prod^\ell R_d$. The wedge power of the map $R_d \to Q_d$ gives a global section $\sigma_U$ of $\bigwedge^\ell Q_d$. This section vanishes at $I$ if and only if the monomials in $U$ are not linearly independent modulo $I$, i.e. if $U$ is a dependent set in Trop$(I)$. Motivated by this, we define:

**Definition 3.17.** The dependence scheme $\mathcal{D}(U) \subseteq \text{Hilb}_a(\mathbb{A}^r)$ of $U$ is defined as $V(\sigma_U)$.

It is immediate that dependence schemes $\mathcal{D}(U)$ for various $U$ are closed subschemes. Since matroids are uniquely defined by their dependent sets, we define:

**Definition 3.18.** Let $\mathcal{M}$ be a tropical ideal. The matroid stratum $\mathcal{D}(\mathcal{M}) \subseteq \text{Hilb}_a(\mathbb{A}^r)$ is the locally closed subscheme

$$U \text{ dependent in } \mathcal{M} \quad \bigcap \quad U \text{ independent in } \mathcal{M} \quad \bigcap \quad \mathcal{D}(U) \cap \mathcal{D}(U)^C.$$ 

Note that the intersection on the right is a priori infinite. (So is the intersection on the left, but it is the intersection of Zariski-closed sets, so this is not an issue.) However, if we restrict to finite-length subschemes of a given length, there are no independent sets in large degree. This implies there are finitely many matroid strata in Hilb$(\mathbb{A}^r)$, and they are Zariski-locally closed. In the infinite-length case this is not true — in particular, in Section 4 we consider principal ideals $I \in \text{Hilb}^{(k)}(\mathbb{P}(a,b))$, where we see infinitely many strata, which are not Zariski-locally closed. (See Theorem 4.9 and Remark 4.12.)
It is a basic property of matroids that all bases have the same size. It immediately follows that a subset $U$ with $|U| \leq k$ is dependent in a rank-$k$ matroid if and only if every $k$-element subset $W$ containing $U$ is dependent. We have the following scheme-theoretic version:

**Proposition 3.19.** Let $U \subseteq \text{Mon}_d(a)$ with $|U| \leq P(d)$. Fix a graded Hilbert function $P$. The dependence scheme $\mathcal{D}(U) \subseteq \text{Hilb}^d_a(\mathbb{A}^r)$ satisfies

$$\mathcal{D}(U) = \bigcap_{W \supseteq U, |W| = P(d)} \mathcal{D}(W).$$

**Proof.** Consider the sequence of maps

$$\bigwedge^{|U|} \text{Span}(U) \xrightarrow{\tau} \bigwedge^{|U|} \mathcal{Q}_d \xrightarrow{w} \bigoplus_{U' \in \binom{\text{Mon}_d}{|U|-|U'|}} \bigwedge^{|U'|} \mathcal{Q}_d, \quad \bigoplus_{U' \in \binom{\text{Mon}_d}{|U|-|U'|}} \bigwedge^{|U'|} \mathcal{Q}_d,$$

where $\tau$ is the inclusion, $p$ is the projection, and $w(\alpha) = \alpha \wedge \bigwedge_{u \in U} u$. If $\alpha \in \ker(w)$, then $\alpha$ is in the kernel of the pairing $\bigwedge^{|U|} \mathcal{Q}_d \otimes \bigwedge^{k-|U|} \mathcal{Q}_d$, as it pairs to zero with the spanning set $\{\bigwedge_{u \in U} u : U' \in \binom{\text{Mon}_d}{|U|-|U'|}\}$ of $\bigwedge^{k-|U|} \mathcal{Q}_d$. This pairing is nondegenerate, so we conclude that $w$ is injective. Thus $V(w(\sigma_U)) = V(\sigma_U)$.

Also, $p \circ w \circ \tau$ is zero, so $w \circ \tau$ factors through $\ker(p) = \bigoplus_{U' \in \binom{\text{Mon}_d}{|U|-|U'|}} \bigwedge^{|U'|} \mathcal{Q}_d$. Thus

$$\mathcal{D}(U) = V(\sigma_U) = V(w(\sigma_U)) = \bigcap_{U' \in \binom{\text{Mon}_d}{|U|-|U'|}, U' \cap U = \emptyset} V(\sigma_{U \cup U'}) = \bigcap_{U' \in \binom{\text{Mon}_d}{|U|-|U'|}, U' \cap U = \emptyset} \mathcal{D}(U' \cup U). \quad \square$$

**Remark 3.20.** The tautological exact sequence is naturally $T$-equivariant, and $\sigma_U$ is an invariant section. That is, all dependence loci $\mathcal{D}(U)$, and thus all matroid strata $\mathcal{D}(\mathcal{M})$, are $T$-invariant. This is essentially a scheme-theoretic restatement of Remark 3.9.

3.6. **$T$-curves in \text{Hilb}(\mathbb{A}^r) and the tropical ideal realizability problem.** Recall the “$T$-graph problem” from Section 2

1. Let $a$ be a $\mathbb{Z}^{r-1}$-multigrading. Given two monomial ideals $M_1, M_2 \in \text{Hilb}_a(\mathbb{A}^r)$ whose multigraded Hilbert functions agree, does there exist an $a$-homogeneous ideal $I$ whose two initial ideals are $I_1$ and $I_2$?

From Observation 3.7, the following “realizability problem” for tropical ideals is a refinement:

2. Given a tropical ideal $\mathcal{M} = \{\mathcal{M}_d : d \geq 0\}$ with grading $a$, does there exist an $a$-homogeneous ideal such that $\text{Trop}(I) = \mathcal{M}$?

**Remark 3.21.** Question 2 is a generalization of the matroid realizability problem, in the following trivial way. Let $r = 2$, and $a = (1, 1)$. For a matroid $M$ whose groundset $E$ has size $c + 1$, we choose an identification of $E$ with $\{x^e, x^{e-1}y, \ldots, y^e\}$, and define a tropical ideal $\mathcal{M}$ by

$$\mathcal{M}_d = \begin{cases} U_{d+1, \text{Mon}_d(1, 1)} & d < c \\ M & d = c \\ U_{d, \text{Mon}_d(1, 1)} & d > c \end{cases}$$

where $U_{k, E}$ is the uniform matroid from Example 3.3. If $M$ is realizable by $V$, then the ideal generated by $V$ and all monomials of degree greater than $c$ provides a realization of $\mathcal{M}$. Conversely, if $\mathcal{M}$ is realizable by $I$, then $I_{k, c}$ is a realization of $M$.

**Remark 3.22.** Consider the two questions above in the principal case:

1. Given two principal monomial ideals $I_1, I_2 \subseteq k[x_1, \ldots, x_r]$ whose Hilbert functions with respect to $a$ agree, does there exist an $a$-homogeneous ideal $I$ whose initial ideals are $I_1$ and $I_2$?

2. Given a tropically principal tropical ideal $\mathcal{M} = \{\mathcal{M}_d : d \geq 0\}$ with grading $(a, b)$, does there exist a homogeneous ideal with grading $(a, b)$ such that $\text{Trop}(I) = \mathcal{M}$?
It is immediate that the answer to Question 3.24 is always “yes;” we may write \( I_1 = (x_1^{b_1} \cdots x_r^{b_r}) \) and \( I_2 = (x_1^{b_1} \cdots x_r^{b_r}) \), and let \( I = (x_1^{b_1} \cdots x_r^{b_r} + tx_1^{b_1} \cdots x_r^{b_r}) \). Question 3.25, while much less difficult than Question 2 above, is still very interesting — in the case \( r = 2 \), it is the subject of Section 4.

Remark 3.23. Fink-Giansiracusa-Giansiracusa [FGG] use the notion of matroid union to define an “expected” tropically principal tropical ideal of degree \( c \) with a fixed degree-\( c \) part. They then show by example that this tropical ideal may not be realizable if \( r \geq 3 \); that is, there is a sense in which every principal homogeneous ideal whose generator has a given support must have “unexpected” relations in higher degrees.

3.7. **Recursivity of the matroid stratification.** We next prove a recursivity result, which will allow us to focus on ideals \( I \) not of the form \( mI' \) for some monomial \( m \). For simplicity we consider only the case \( r = 2 \) and \( b = 1 \). Let \( I = (f) \in \text{Hilb}^{(k-1)}(\mathbb{P}^1) \). There are natural embeddings \( \iota_x, \iota_y : \text{Hilb}^{(k-1)}(\mathbb{P}^1) \to \text{Hilb}^{(k)}(\mathbb{P}^1) \) given by \( \iota_x(I) = xI = (xf) \) and \( \iota_y(I) = yI = (yf) \). The embeddings are compatible with the matroid stratification:

**Proposition 3.24.** Let \( U \subseteq \text{Mon}_d \) with \( |U| = k \). Then the pullback of the dependence scheme \( \mathcal{D}(U) \subseteq \text{Hilb}^{(k)}(\mathbb{P}^1) \) under \( \iota_x \) is

\[
\mathcal{D}(U) \times_{\text{Hilb}^{(k)}(\mathbb{P}^1)} \text{Hilb}^{(k-1)}(\mathbb{P}^1) = \mathcal{D}\left(\frac{1}{x}(U \setminus \{y^d\})\right) \subseteq \text{Hilb}^{(k-1)}(\mathbb{P}^1),
\]

and similarly for \( \iota_y \).

**Remark 3.25.** Proposition 3.24 shows that when studying the dependence loci in \( \text{Hilb}^{(k)}(\mathbb{P}^1) \), we may restrict our attention (by induction on \( k \)) to those \( U \subseteq \text{Mon}_d \) such that \( x^d, y^d \in U \). (C.f. Section 4.2.) A similar statement may be made for \( \mathbb{P}(a,b) \).

**Proof of Proposition 3.24.** We have maps on tautological bundles:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}_{d-1} & \longrightarrow & \mathcal{R}_{d-1} & \longrightarrow & \mathcal{Q}_{d-1} & \longrightarrow & 0 \\
\vee & & \downarrow x & & \downarrow x & & \downarrow x & & \\
0 & \longrightarrow & i^*_x \mathcal{I}_d & \longrightarrow & i^*_x \mathcal{R}_d & \longrightarrow & i^*_x \mathcal{Q}_d & \longrightarrow & 0 \\
\end{array}
\]

The middle map is injective, and the left map is an isomorphism. Thus (by the four lemma) the right map is injective.

If \( y^d \notin U \), consider the induced square of wedge powers

\[
\begin{array}{ccccccccc}
\wedge^{[U]} \mathcal{R}_{d-1} & \longrightarrow & \wedge^{[U]} \mathcal{Q}_{d-1} \\
\downarrow x & & \downarrow x \\
\wedge^{[U]} i^*_x \mathcal{R}_d & \longrightarrow & \wedge^{[U]} i^*_x \mathcal{Q}_d \\
\end{array}
\]

Then the special section \( \wedge_{m \in U} m \) of \( \wedge^{[U]} i^*_x \mathcal{R}_d \) is in the image of the left map; it is the image of \( \wedge_{m \in (1/x)U} m \). Thus \( \sigma_U \in H^0(\wedge^{[U]} i^*_x \mathcal{Q}_d) \) is the image under the right map of \( \sigma_{(1/x)U} \in H^0(\mathcal{Q}_{d-1}) \). This implies

\[
\mathcal{D}(U)_{\text{Hilb}^{(k)}(\mathbb{P}^1)} \text{Hilb}^{(k-1)}(\mathbb{P}^1) = \mathcal{D}((1/x)U).
\]

If \( y^d \in U \), consider instead the square

\[
\begin{array}{ccccccccc}
\wedge^{[U]-1} \mathcal{R}_{d-1} & \longrightarrow & \wedge^{[U]-1} \mathcal{Q}_{d-1} \\
\downarrow x & & \downarrow x \\
\wedge^{[U]-1} i^*_x \mathcal{R}_d & \longrightarrow & \wedge^{[U]-1} i^*_x \mathcal{Q}_d \\
\downarrow \wedge y^d & & \downarrow \wedge y^d \\
\wedge^{[U]} i^*_x \mathcal{R}_d & \longrightarrow & \wedge^{[U]} i^*_x \mathcal{Q}_d \\
\end{array}
\]

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4. **T-curves in \( \text{Hilb}^k(\mathbb{P}^1) \)**

For the rest of the paper, we assume \( r = 2 \) and \( b = 1 \). Our motivation was to study finite-colength homogeneous ideals in \( k[x,y] \) (i.e. finite-length subschemes of \( k^2 \)). In this section, however, we turn our attention to certain infinite-colength homogeneous ideals with respect to the grading \((1,1)\); namely, the ideals associated to finite collections of points in \( \mathbb{P}^1 \). We will characterize the tropicalization of such an ideal, which will turn out to depend in an interesting way upon the collection of points. This characterization will also provide information about the finite-colength case via monoprincipal ideals; see Section 5.

For any \( k \), we have

\[
\text{Hilb}^k(\mathbb{P}^1) \cong \text{Sym}^k(\mathbb{P}^1) \cong \mathbb{P}^k,
\]

where \([a_0 : a_1 : \cdots : a_k] \in \mathbb{P}^k\) corresponds to the principal ideal \( I = (a_0x^k + a_1x^{k-1}y + \cdots + a_ky^k) \in \text{Hilb}^k(\mathbb{P}^1)\), and to the roots counted with multiplicity, \( V(I) \in \text{Sym}^k(\mathbb{P}^1)\).

**Remark 4.1.** The line bundle \( \mathcal{O}(m) \) on \( \mathbb{P}^k \) is identified with the bundle on \( \text{Sym}^k(\mathbb{P}^1) \) whose sections are \( S_k \)-symmetric rational functions in \( k \) pairs of variables \( x_1, y_1, \ldots, x_k, y_k \), bihomogeneous in each pair of variables of degree \( m \). One may check that the tautological line bundle \( \mathcal{I}_k \) on \( \text{Hilb}^k(\mathbb{P}^1) \) (see \( \mathbb{P}^k \)) is identified with the bundle \( \mathcal{O}(-1) \) on \( \mathbb{P}^k \).

**4.1. Matroid strata are determined by vanishing of Schur polynomials.** In this section, we characterize the tropicalization of an ideal \( I \in \text{Hilb}^k(\mathbb{P}^1) \) in terms of which of the Schur polynomials in \( k \) variables vanish at \( V(I) \in \text{Sym}^k(\mathbb{P}^1) \).

**Remark 4.2.** Schur polynomials are indexed by Young diagrams, which also appear in this paper in relation to monomial ideals. To avoid confusion, we draw Young diagrams related to Schur polynomials with the longest row on top (English notation), as opposed to the way we have been drawing monomial ideals thus far (French notation).

We now give a correspondence between Young diagrams and sets of monomials.

**Definition 4.3.** Let \( \lambda = (\lambda_1, \ldots, \lambda_{m-1}, \lambda_m) \) be a partition, in nonincreasing order, with \( m \) parts. We visualize \( \lambda \) as a Young diagram in English notation. Let \( h \geq \lambda_1 \) and let \( k \geq m \). The width-\( h \), height-\( k \) rim path of \( \lambda \) is the lattice path in \( \mathbb{Z}^2 \) that starts at \((h,0)\), and follows the edge of the Young diagram down to the left until it reaches \((0,-k)\). The width-\( h \), height-\( k \) rim path sequence

\[
u_{\lambda}^{h,k} : \{0,1,\ldots,h+k-1\} \to \text{down, left}
\]

of \( \lambda \) is the sequence of directions of the rim path.

The width-\( h \), height-\( k \) rim path monomial set of \( \lambda \) is the set

\[
U_{\lambda}^{h,k} = \{x^{h+k-1-i}y^i \in \text{Mon}_{h+k-1} : \nu_{\lambda}^{h,k}(i) = \text{down}\}.
\]

(See Figure 3 left.) Note that \( |U_{\lambda}^{h,k}| = k \).

**Remark 4.4.** The operation of taking the width-\( h \), height-\( k \) rim path monomial set gives a bijection between partitions with at most \( k \) parts and \( \lambda_1 \leq h \), and \( k \)-element subsets of \( \text{Mon}_{h+k-1} \). Thus Proposition 3.19 implies:

**Proposition 4.5.** For any subset \( U \subseteq \text{Mon}_{h+k-1} \),

\[
\mathcal{D}(U) = \bigcap_{\lambda, U_{\lambda}^{h,k} \supseteq U} \mathcal{D}(U_{\lambda}^{h,k}).
\]

Proposition 4.5 can be made more visual; the condition \( U_{\lambda}^{h,k} \supseteq U \) says that certain entries of the rim path sequence of \( \lambda \) must be “down.” Entries of the rim path sequence correspond to diagonals in the Young diagram. We see what this restriction looks like in an example.
Example 4.6. Let $h = 7$ and $k = 5$. Let $U = \{x^{11}, x^6y^5, x^3y^2\}$. If $U_{\lambda}^{7,5}$ contains $U$, then the 0th, 5th, and 9th steps of the rim path are vertical. Since each step is either down by 1 unit or left by 1 unit, the sum of the horizontal and vertical coordinates decreases by 1 at each step along the rim path.

Since the 0th step of the path begins at $(7, 0)$, and $7 + 0 = 7$, the 5th step must begin at a point $(a, 2 - a)$ for some $2 \leq a \leq 7$. Since the 5th step is known to be vertical, it ends at $(a, 1 - a)$. In particular, the rim path may not contain a segment from $(a, 2 - a)$ to $(a - 1, 2 - a)$ for any $a$. Similarly, from the restriction on the 9th step, the rim path may not contain a segment from $(a - 2 - a)$ to $(a - 1, -2 - a)$ for any $a$. The disallowed segments are dashed in the Figure 3 and the allowed example $\lambda = (7, 5, 4, 1, 0)$ is drawn. (Segments are dotted if they are not explicitly disallowed, but nonetheless cannot appear in any allowed rim path.)

\[
\begin{array}{cccc}
 & x^7 & x^3y^4 & x^4y^3 \\
 & x^6y & x^5y^2 & x^2y^5 \\
y^7 &  &  &  \\
\end{array}
\]

**Figure 3.** Left: The width-5, height-3 rim path monomial set of $\lambda = (4, 1)$ is $U_{(4,1)}^{5,3} = \{x^6y, x^2y^5, y^7\}$. Right: Allowed rim path segments from Example 4.6.

By Proposition 4.5, the matroid stratification of Hilb$^{(k)}(\mathbb{P}^1)$ is “generated” (via taking intersections) by the loci $D(U_{\lambda}^{h,k})$. Our next goal is Theorem 4.9, which describes these loci as vanishing sets of certain symmetric polynomials.

**Definition 4.7.** Let $k \geq 1$. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition, in nonincreasing order, with at most $k$ parts. We write $\lambda_i = 0$ for $m < i \leq k$. The **bihomogeneous Schur polynomial** $s_{\lambda}$ in $k$ variables is defined by

\[
s_{\lambda}(x_1, y_1, x_2, y_2, \ldots, x_k, y_k) = \frac{a_{(\lambda_1+k-1, \lambda_2+k-2, \ldots, \lambda_{k}+0)}(x_1, y_1, \ldots, x_k, y_k)}{a_{(k-1-k-2, \ldots, 0)}(x_1, y_1, \ldots, x_k, y_k)},
\]

where

\[
a_{(i_1, i_2, \ldots, i_k)}(x_1, y_1, x_2, y_2, \ldots, x_k, y_k) = \det(x_j^{i_1} y_j^{1-i_1})
\]

is the Vandermonde determinant. This is a bihomogenization of the usual Schur polynomials $s_{\lambda}(x_1, \ldots, x_k)$ in each variable. Similarly, the **bihomogeneous elementary symmetric polynomials** $e_j$ are defined by

\[
e_j(x_1, y_1, \ldots, x_k, y_k) = \sum_{A \subseteq [k]} \prod_{i \in A} x_i \prod_{i \in A^c} y_i
\]

Note that $e_j$ is bihomogeneous of degree 1 in each pair of variables $x_i, y_i$, and $e_0 = y_1 \cdots y_k$.

**Remark 4.8.** Note that via the identification from Remark 4.1, $s_{\lambda}(x_1, \ldots, y_k)$ can be viewed as a global section of the bundle $O(\lambda_1)$ on $\mathbb{P}^k$, and $e_j(x_1, \ldots, y_k)$ can be viewed as a global section of $O(1)$ on $\mathbb{P}^k$.

**Theorem 4.9.** The dependence subscheme $D(U_{\lambda}^{h,k}) \subseteq \text{Hilb}^{(k)}(\mathbb{P}^1)$ is the vanishing locus of the polynomial $e_0(x_1, \ldots, y_k)^{h-\lambda_1} s_{\lambda}(x_1, \ldots, y_k)$.

**Remark 4.10.** We note that A. Fink independently observed that matroid strata in Hilb$^{(k)}(\mathbb{P}^1)$ are defined by Schur polynomials.

**Proof.** Recall the tautological sequences [1]. Let $f$ be a nonvanishing local section of the line bundle $\mathcal{I}_k$. Then expressed in terms of the roots $[x_1 : y_1], \ldots, [x_k : y_k]$, we have

\[
f = \prod_{i=1}^{k} (y_i x - x_i y) = e_0 x^k - e_1 x^{k-1} y + \cdots + (-1)^k e_k y^k,
\]
where \( e_i \) denotes the \( i \)-th bihomogeneous elementary symmetric polynomial in \( x_1, \ldots, y_k \).

**Step 0.** It follows from the definition that \( e_0(x_1, \ldots, y_k) x^{h-\lambda} s_\lambda(x_1, \ldots, y_k) \) is homogeneous in \( y_1, \ldots, y_k \). Therefore \( s_\lambda(x_1, y_1, \ldots, x_k, y_k) = \pm s_\lambda(x_1, -y_1, \ldots, x_k, -y_k) \), so we may instead work with the negative roots, and write

\[
f = \prod_{i=1}^{k} (y_i x + x_i y) = e_0 x^k + e_1 x^{k-1} y + \cdots + e_k y^k.
\]

The lack of signs will allow us to apply the Jacobi-Trudi formula more easily in Step 2.

**Step 1.** First, since \( I_{h+k-1} \) is a more concrete object that \( Q_{h+k-1} \), we reinterpret \( \sigma_{U_{h,k}} \) in terms of \( I_{h+k-1} \).

By definition, \( D(U_{\lambda}^{h,k}) \) is the vanishing locus of the section

\[
\sigma_{U_{\lambda}^{h,k}} \in \bigwedge^k Q_{h+k-1} = \text{Hom} \left( \bigwedge^k \text{Span}(U_{\lambda}^{h,k}), \bigwedge^k Q_{h+k-1} \right)
\]

defined as the \( k \)-th wedge of the chain of maps

\[
\text{Span}(U_{\lambda}^{h,k}) \hookrightarrow R_{h+k-1} \rightarrow Q_{h+k-1}.
\]

By duality, there is a natural isomorphism

\[
\text{Hom} \left( \bigwedge^k \text{Span}(U_{\lambda}^{h,k}), \bigwedge^k Q_{h+k-1} \right) \cong \text{Hom} \left( \left( \bigwedge^k Q_{h+k-1} \right)^\vee, \left( \bigwedge^k \text{Span}(U_{\lambda}^{h,k}) \right)^\vee \right).
\]

The two exact sequences

\[
0 \rightarrow I_{h+k-1} \rightarrow R_{h+k-1} \rightarrow Q_{h+k-1} \rightarrow 0
\]

and

\[
0 \rightarrow \text{Span}(U_{\lambda}^{h,k}) \rightarrow R_{h+k-1} \rightarrow Q_{h+k-1} \rightarrow 0
\]

give identifications

\[
\left( \bigwedge^k Q_{h+k-1} \right)^\vee \cong \bigwedge^h I_{h+k-1},
\]

\[
\left( \bigwedge^k \text{Span}(U_{\lambda}^{h,k}) \right)^\vee \cong \bigwedge^h R_{h+k-1} / \text{Span}(U_{\lambda}^{h,k}).
\]

Thus \( \sigma_{U_{\lambda}^{h,k}} \) is identified with the section of \( \text{Hom} \left( \bigwedge^h I_{h+k-1}, \bigwedge^h R_{h+k-1} / \text{Span}(U_{\lambda}^{h,k}) \right) \) defined as the \( h \)th (top) wedge of the chain of maps

\[
I_{h+k-1} \xrightarrow{A} R_{h+k-1} \xrightarrow{B} R_{h+k-1} / \text{Span}(U_{\lambda}^{h,k}),
\]

i.e. \( \det(B \circ A) \). Note that

\[
\text{Hom} \left( \bigwedge^h I_{h+k-1}, \bigwedge^h R_{h+k-1} / \text{Span}(U_{\lambda}^{h,k}) \right) \cong \text{Hom} \left( \bigwedge^h (R_{h-1} \otimes I_k), \bigwedge^h R_{h+k-1} / \text{Span}(U_{\lambda}^{h,k}) \right)
\]

\[
\cong \text{Hom} \left( \bigwedge^h R_{h-1} \otimes I_k^h, \bigwedge^h R_{h+k-1} / \text{Span}(U_{\lambda}^{h,k}) \right)
\]

\[
\cong (I_k)^h \otimes \mathcal{O}(h),
\]

**Step 2.** By principality, \( I_{h+k-1} \) is spanned by all monomial multiples \( x^i y^{h-1-i} f \) of \( f \), i.e. there is a natural isomorphism

\[
R_{h-1} \otimes I_k \rightarrow I_{h+k-1}
\]
that sends $p \otimes f \mapsto pf$. The inclusion $A$ from \[3\] has the following matrix $X_A$ with respect to the basis\{ $m \otimes f :$ monomials $m \in R_{h-1}$\} for $I_{h+k-1}$ and the monomial basis for $R_{h+k-1}$:

\[
X_A = (e_{h+j})_{0 \leq j \leq h-1} = \begin{pmatrix}
e_0 & 0 & \cdots & 0 \\
e_1 & e_0 & \cdots & 0 \\
e_2 & e_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & e_1 \\
e_k & e_{k-1} & \cdots & e_2 \\
0 & e_k & \cdots & e_2 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e_k 
\end{pmatrix}
\]

(4)

The matrix $X_{B o A}$ of the composition $B \circ A$ is obtained by deleting the rows corresponding to elements of $U_{\lambda}^{h,k}$. Let $X'_{B o A}$ be the matrix obtained by reversing the order of the rows and the order of the columns in $X_{B o A}$.

Note that rows of $X'_{B o A}$ correspond to rightward steps in the reverse width-$h$, height-$k$ rim path sequence of $\lambda$ — that is, to columns in the Young diagram of $\lambda$. Now, consider the entries of the $i$-th row of $X'_{B o A}$ (starting with $i = 0$), which is (say) the $b$-th row of $X_A$. These entries are the bihomogeneous elementary symmetric polynomials $e_i, e_{i+1}, \ldots, e_{i+h-1}$, where

\[
\ell_i = k - i - \# \{ b \leq b_i : x^by^{h+k-1-b} \in U_{\lambda}^{h,k} \}.
\]

Since elements of $\{ b \leq b_i : x^by^{h+k-1-b} \in U_{\lambda}^{h,k} \}$ correspond to upward steps in the reverse rim path sequence, we see that $\ell_i + i = k - \# \{ b \leq b_i : x^by^{h+k-1-b} \}$ is the negative $y$-coordinate of the $i$-th rightward step in the reverse rim path sequence; that is, $\ell_i + i$ is the $i$-th entry of the conjugate partition $\lambda'$. (Note that $\lambda'$ here has exactly $h$ entries, some of which may be zero.)

On the other hand, $e_{i+i}$ is the $i$-th diagonal entry of $X'_{B o A}$. Thus $X'_{B o A} = (e_{\lambda'_i+1,j})_{i,j=0}^{h-1}$. Note that $\ell_i + i = 0$ for $\lambda_1 < i \leq h$. Expanding the determinant along the last $h - \lambda_1$ rows gives

\[
\det(X_{B o A}) = e_{0}^{h-\lambda_1} \det((e_{\lambda'_i+1,j})_{i,j=0}^{h-1} = \det(X_{B o A}).
\]

The last matrix appears in the second Jacobi-Trudi formula; applying the formula, we get $e_{0}^{h-\lambda_1} s_{\lambda} = \det(X_{B o A})$.

**Step 3.** From Step 1, $s_{\lambda}^{h,k}$ is naturally identified with $\det(B \circ A)$. We have chosen an isomorphism $R_{h-1} \to R_{h+k-1}/\text{Span}(U_{\lambda}^{h,k})$, which identifies $\det(B \circ A)$ with $\det(X_{B o A})$. Thus

\[
\mathcal{D}(U_{\lambda}^{h,k}) = V(\det(X_{B o A})) = e_{0}^{h-\lambda_1} s_{\lambda}.
\]

**Remark 4.11.** The second Jacobi-Trudi formula states that $s_{\lambda} = \det(e_{\lambda'_i+1,j})_{i,j=0}^{h-1}$. If $\lambda_k > 0$, then the entries to the right of the diagonal vanish in the first $\lambda_k$ rows, and we may expand the determinant along these rows to find

\[
s_{\lambda} = e_{k}^{h-k} s_{\lambda_1-\lambda_k,\lambda_2-\lambda_k,\ldots,\lambda_{k-1}-\lambda_k}(x_1, \ldots, y_k).
\]

In particular, Theorem 4.9 now implies that

\[
\mathcal{D}(U_{\lambda}^{h,k}) = V(e_{0}^{h-\lambda_1} e_{k}^{h-k} s_{\lambda_1-\lambda_k,\lambda_2-\lambda_k,\ldots,\lambda_{k-1}-\lambda_k}(x_1, \ldots, y_k)).
\]

**Remark 4.12.** In the case $r = 2$, Question [2] from Remark 3.22 is reduced to the question: “What is the intersection complex of the (infinitely many) divisors $\mathcal{D}(U_{\lambda}^{h,k})$?”

**4.2. One-dimensional dependence loci and binary necklaces.** Let $U_d = \{ x^d, y^d \}$. In this section we consider the structure of the loci $\mathcal{D}_{d,k} := \mathcal{D}(U_d) \subseteq \text{Hilb}^{(k)}(\mathbb{P}^1)$.

**Proposition 4.13.** The partitions $\lambda$ such that $U_{\lambda}^{d-k+1,k} \supseteq U_d$ are exactly those with at most $k - 1$ parts, and $\lambda_1 = d - k + 1$. 

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Proof. This is immediate from the visualization in Example 4.14. The only dashed edges will be the one from \((d-k,0)\) to \((d-k+1,0)\) and the one from \((0,-k)\) to \((1,-k)\), as shown in Figure 4. This rules out the possibilities \(\lambda_1 < d-k+1\) and \(\lambda_k > 0\), but these are the only restrictions. \(\square\)

From Proposition 4.15 and Theorem 4.9, \(D_{d,k}\) is defined by the Schur polynomials \(s_{\lambda}(x_1,\ldots,y_k)\) for partitions \(\lambda\) such that \(\lambda_1 = d-k+1\) and \(\lambda_k = 0\).

Example 4.14. If \(k = 2\), then \(|U_d| = k = 2\), and \(D_{d,k} = D_{d,2}\) is the divisor \(U_{(d-k+1,k)} = U^{d-1,2}_{(d-1)} = V(s_{(d-1)})\).

In this case it is reasonably easy to describe \(D_{d,2}\). For example, \(D_{6,2}\) is defined by the Schur polynomial

\[
\begin{align*}
    s_{(5)} &= x_2^5 y_1^5 + x_1 x_2^4 y_2 y_1^4 + x_1^2 x_2^2 y_2^2 y_1^3 + x_1^3 x_2 y_2^3 y_1^2 + x_1^4 x_2 y_2^4 y_1 + x_1^5 y_2^5, \\
    &= (x_2 y_1 + x_1 y_2) \left(x_2^2 y_1^2 - x_1 x_2 y_1 y_2 + x_1^2 y_2^2\right) \left(x_2^2 y_1^2 + x_1 x_2 y_1 y_2 + x_1^2 y_2^2\right).
\end{align*}
\]

The second and third factors can be factored further, e.g.

\[
x_2^2 y_1^2 - x_1 x_2 y_1 y_2 + x_1^2 y_2^2 = (x_1 y_2 - \zeta_6 x_2 y_2)(x_1 y_2 - \zeta_6^{-1} x_2 y_2),
\]

but the irreducible factors are then no longer symmetric, but rather permuted by the \(S_2\) action. Thus \(D_{6,2}\) consists of 3 irreducible curves, one of degree 1 and two of degree 2. (Observe that all three curves pass through \((0:1],[0:1])\) and \((1:0],[1:0])\); these two points of \(\text{Sym}^2 \mathbb{P}^1\) correspond to the monomial ideals \((x^2)\) and \((y^2)\) in \(\text{Hilb}^2(\mathbb{P}^1)\). By Remark 3.20, these three curves must be \(T\)-curves. The tropical ideal corresponding to each curve is shown in Figure 5 on page 17. (Note that in Figure 5 as in Example 3.16 “uninformative” circuits are omitted.)

Note from the equations of the three curves (or from the tropical ideals pictured in Figure 5) that \(T\) acts on the degree-1 curve with weight 2 and on the degree-2 curves with weight 1. (From Section 2.2, the associated moduli space consists of two points, together with an orbifold point with isotropy group \(\mathbb{Z}/2\mathbb{Z}\).)

Example 4.15. If \(k > 2\), the equations very quickly become impossible to solve explicitly. For example, \(D_{5,3}\) is defined by the Schur polynomials

\[
\begin{align*}
    s_{(3,3)} &= x_1^3 x_2^3 y_3^3 + x_1^2 x_2^2 x_3^2 y_1 y_2 y_3 + x_1 x_2 x_3^2 y_2^3 + \cdots, \\
    s_{(3,2)} &= x_1^3 x_2^2 y_2^3 + x_1^3 x_2 x_3^2 y_3 + x_1^2 x_3^2 y_1 y_2 y_3^2 + \cdots, \\
    s_{(3,1)} &= x_1^3 x_2 y_2^3 y_3 + x_1^2 x_2 x_3 y_1 y_2 y_3^2 + x_1^2 x_3 y_1 y_2^3 y_3 + \cdots, \\
    s_{(3)} &= x_1^3 y_3^3 + x_1^2 x_2 y_3^3 y_3 + x_1 x_2 x_3 y_1 y_2 y_3^2 + x_1 x_2 x_3 y_2^2 y_3^2 + \cdots,
\end{align*}
\]

where “\(\cdots\)” denotes the missing \(S_3\)-translates of the terms listed. By inspection, the two triples \((0:1],[0:1])\) and \((1:0],[1:0])\) are solutions, but it is not clear from the equations if there are others. In the rest of this section, we identify \(D_{d,k}\) completely.

Proposition 4.16. \(D_{d,k}\) does not intersect \(V(x_1 x_2 \cdots x_k y_1 y_2 \cdots y_k)\) except at the two torus-fixed points \((x^k)\) and \((y^k)\).

Proof. If \(I = (f) \in D_{d,k}\), then \(\{x^d, y^d\}\) is dependent in \(\text{Trop}(I)\). Thus \(\{x^d, y^d\}\) contains a circuit; it is either \(\{x^d\}, \{y^d\}\), or \(\{x^d, y^d\}\). The first case implies \(f\) divides \(x^d\), so \(f = x^k\). Similarly the second case implies \(f = y^k\). In the third case, by Lemma 3.4 we have \(x^d + cy^d \in I\) for some \(c \neq 0\). On the other hand, if \(I \in V(x_1 x_2 \cdots x_k y_1 y_2 \cdots y_k)\), then \(c_0\) or \(c_k\) vanishes, i.e. \(f\) is divisible by \(x\) or \(y\). Thus \(x^d + cy^d\) cannot be a multiple of \(f\), a contradiction. \(\square\)
We now work in $(\mathbb{P}^1)^k$ instead of $\text{Hilb}^k(\mathbb{P}^1)$; we will translate the story back to $\text{Hilb}^k(\mathbb{P}^1)$ in Remark 4.21. Let $\mathcal{D}_{d,k} = \mathcal{D}_{d,k} \times_{\text{Hilb}^k(\mathbb{P}^1)} (\mathbb{P}^1)^k$. Note that $\mathcal{D}_{d,k}$ is cut out by the same Schur polynomials as $\mathcal{D}_{d,k}$. Note also that the torus action on $\text{Hilb}^k(\mathbb{P}^1)$ is descended from the diagonal action of $\mathbb{C}^*$ on $(\mathbb{P}^1)^k$.

We look locally at $P_0$. Note that $\mathcal{D}_{d,k}$ is a union of $T$-orbit-closures, which all contain $P_0$ by Proposition 4.16. Also, $T$ acts on the tangent space $T_{P_0}(\mathbb{P}^1)^k$ by scaling; thus any $T$-orbit-closure $C$ is determined by its tangent space at $P_0$. In other words, if we consider the blow-up $\mathcal{B}^*_P(\mathbb{P}^1)^k$, then $C$ is determined by the point where its strict transform intersects the exceptional divisor. The exceptional divisor is isomorphic to $\mathbb{P}^{k-1}$, with coordinates $[z_1 : \ldots : z_k]$, where $z_i = x_i/y_i$. We denote by $\mathcal{D}'_{d,k}$ the intersection of the strict transform of $\mathcal{D}_{d,k}$ with the exceptional divisor.

**Proposition 4.17.** $\mathcal{D}'_{d,k} = V(s(d-k+1), s(d-k+2), \ldots, s(d-1))$, where $s(j) = s(j)(z_1, \ldots, z_k)$ are (usual, not bihomogeneous) Schur polynomials. Note that $s(j)$ is also the $j$th complete symmetric polynomial in $z_1, \ldots, z_k$.

**Proof.** Let $\lambda$ be a partition with $\lambda_1 = d - k + 1$ and $\lambda_k = 0$. $\mathcal{D}'_{d,k}$ is cut out by the (usual, not bihomogeneous) Schur polynomials $s_\lambda(z_1, \ldots, z_k)$. By the first Jacobi-Trudi formula, $s_\lambda = \det(s(\lambda_i + j - 1))_{i,j=1}^{k-1}$. The first row of this matrix is $s(d-k+1), s(d-k+2), \ldots, s(d-1)$; this immediately implies that $s_\lambda \in (s(d-k+1), s(d-k+2), \ldots, s(d-1))$. We must show that all of $s(d-k+1), s(d-k+2), \ldots, s(d-i)$ are in the ideal generated by $\{s_\lambda : \lambda_1 = d - k + 1, \lambda_k = 0\}$.

Consider $\lambda = (d - k + 1, 1, \ldots, 1, 0, \ldots, 0)$, where $1 \leq i \leq k - 1$. The first Jacobi-Trudi formula gives

\[
\begin{vmatrix}
  s(d-k+1) & s(d-k+2) & \cdots & s(d-k+1) \\
  1 & s(1) & \cdots & s(1) \\
  0 & 1 & s(1) & \cdots & s(1) \\
  \vdots & \vdots & \ddots & \ddots & \ddots \\
  0 & 0 & \cdots & 0 & 1 \\
  \vdots & \vdots & \ddots & \ddots & \ddots \\
  0 & 0 & \cdots & 0 & 0 & 1
\end{vmatrix}_{(i-1) \text{ times}} = \begin{vmatrix}
  s(d-k+1) & s(d-k+2) & \cdots & s(d-k+1) \\
  1 & s(1) & \cdots & s(1) \\
  0 & 1 & s(1) & \cdots & s(1) \\
  \vdots & \vdots & \ddots & \ddots & \ddots \\
  0 & 0 & \cdots & 0 & 1 \\
  \vdots & \vdots & \ddots & \ddots & \ddots \\
  0 & 0 & \cdots & 0 & 0 & 1
\end{vmatrix}_{(i-1) \text{ times}}
\]

(Note that $i = 1$ gives the expression $s(d-k+1) = s(d-k+1)$, the base case.) Expanding this determinant along the first row gives $s_\lambda$ as a $k[x, y]$-linear combination of $s(d-k+1), \ldots, s(d-k+i)$, where the coefficient of $s(d-k+i)$ is $\pm 1$. By induction on $i$, $s(d-k+1), \ldots, s(d-k+i-1)$ are in the ideal generated by $\{s_\lambda : \lambda_1 = d - k + 1, \lambda_k = 0\}$. Thus so is $s(d-k+i)$.

**Remark 4.18.** Proposition 4.17 shows that $\mathcal{D}'_{d,k}$ is defined by $k - 1$ equations, whereas before it had been defined by $(d-1)/(k-2)$ equations. Therefore $\mathcal{D}'_{d,k}$ is expected to have dimension zero (and $\mathcal{D}_{d,k}$ is expected to have dimension 1). Indeed, $\mathcal{D}'_{d,k}$ has dimension at least zero, i.e. the associated moduli space is nonempty.

**Assumption.** We are now concerned with solving various combinations of Schur polynomials, and to do so, we restrict our attention to the case where $k = \mathbb{C}$.

**Proposition 4.19.** Let $z_1, \ldots, z_k \in \mathbb{C}^*$ be such that the ratios $\{z_j/z_1 : i = 1, \ldots, k\}$ are distinct $d$th roots of unity. Then $s(j)(z_1, z_2, \ldots, z_k) = 0$ for $j = d - k + 1, d - k + 2, \ldots, d - 1$.

**Proof.** We may assume $z_1 = 1$, and thus $z_i = \zeta_d^{a_i}$ for $\zeta_d$ a primitive $d$th root of unity. By the definition of Schur polynomials,

\[
(5) \quad s(j)(1, z_2, \ldots, z_k) = \frac{1}{a_{(n-1,\ldots,0)}}(1, z_2, \ldots, z_k) \det \begin{pmatrix}
  1 & \zeta_d^{a_1(j+1-k)} & \cdots & \zeta_d^{a_k(j+k-1)} \\
  \zeta_d^{a_2} & \cdots & \zeta_d^{a_k(d-k)} \\
  1 & \zeta_d^{a_2(j+1-k)} & \cdots & \zeta_d^{a_k(j+k-1)} \\
  \vdots & \ddots & \ddots & \ddots \\
  1 & \zeta_d^{a_2} & \cdots & \zeta_d^{a_k} \\
  1 & 1 & 1 & 1
\end{pmatrix},
\]
where \( a_{(n-1,0,\ldots,0)} \) is the Vandermonde determinant from Definition 4.7. The determinant on the right in (4) vanishes if two rows coincide, which must happen if \( j + k - 1 \) differs from any of \( 0, 1, \ldots, k - 2 \) by a multiple of \( d \). In particular, this is true if \( 0 \leq j + k - 1 - d \leq k - 2 \), i.e. if \( d - k + 1 \leq j \leq d - 1 \).

The vanishing of the right factor in (5) does not guarantee that \( s(j) \) vanishes, since the Vandermonde determinant could also vanish. On the other hand, we have the alternate expression

\[
a_{(n-1,0,\ldots,0)}(z_1, z_2, \ldots, z_k) = \prod_{1 \leq i \leq j \leq k} (z_i - z_j).
\]

By assumption \( z_1, \ldots, z_k \) are distinct, so \( a_{(n-1,0,\ldots,0)}(z_1, z_2, 1, \ldots, z_k, 1) \) does not vanish. We conclude that \( s(j)(z_1, z_2, \ldots, z_k) = 0 \) for \( d - k + 1 \leq j \leq d - 1 \).

Proposition 4.19 together with Proposition 4.17 produces a collection of points of \( \tilde{D}'_{d,k} \). In fact, these are everything:

**Proposition 4.20.** For distinct integers \( 1 \leq a_2, \ldots, a_k \leq d - 1 \), let \( P_{\alpha_2, \ldots, \alpha_k} = [1 : \zeta_{d}^{\alpha_2} : \cdots : \zeta_{d}^{\alpha_k}] \in (\mathbb{P}^1)^k \), for \( \zeta_{d} \) a primitive \( d \)-th root of unity. Then \( \tilde{D}'_{d,k} \) is the union of all such points \( P_{\alpha_2, \ldots, \alpha_k} \).

**Proof.** We count degrees. Remember, we are working in the exceptional divisor \( \mathbb{P}^{k-1} \subseteq \mathcal{B}(\mathcal{P}_0((\mathbb{P}^1)^k)) \), where \( \mathcal{P}_0 = \{[0 : 1], \ldots, [0 : 1]\} \). Since \( s(j) \) is homogeneous of degree \( j \) in \( z_1, \ldots, z_k \), we expect \( V(s(d-k+1), \ldots, s(d-1)) \) to be a collection of \((d-1)(d-2) \cdots (d-k+1)\) points. This is the same as the number of points \( P_{\alpha_2, \ldots, \alpha_k} \), since there are \( d - 1 \) choices of \( \alpha_2, d - 2 \) choices of \( \alpha_3 \), and so on. It is thus sufficient to show that \( V(s(d-k+1), \ldots, s(d-1)) \) is zero-dimensional, i.e. that \( s(d-k+1), \ldots, s(d-1) \) is a regular sequence. This follows from Proposition 2.9 of [CKW09]. □

**Remark 4.21.** Recall that \( D_{d,k} = \tilde{D}_{d,k}/S_k \). In order to take this quotient, first consider

\[
\tilde{D}'_{d,k} := \{(\zeta_{d}^{\alpha_1} : \cdots : \zeta_{d}^{\alpha_k}) : 0 \leq a_1, \ldots, a_k \leq d - 1, a_1, \ldots, a_k \ \text{distinct}\} \subseteq \mathbb{C}^k.
\]

Let \( \mu_d \subseteq \mathbb{C}^* \) be the group of \( d \)th roots of unity, acting on \( \mathbb{C}^k \) by multiplication. Note that \( S_k \) acts freely on \( \tilde{D}'_{d,k} \). From Proposition 4.20 \( \tilde{D}'_{d,k}/\mu_d = \tilde{D}'_{d,k} \). Thus \( \tilde{D}_{d,k} \) consists of a single \( T \)-orbit-closure for each \( \mu_d \)-orbit in \( \tilde{D}'_{d,k} \). This means that \( D_{d,k} \) consists of a single \( T \)-orbit-closure for each \( \mu_d \)-orbit in \( \tilde{D}'_{d,k}/S_k \cong \binom{\mu_d}{k} \), where \( \binom{\mu_d}{k} \) is the set of \( k \)-element subsets of the \( d \)th roots of unity. By placing a black bead at each of the \( k \) chosen \( d \)th roots of unity, and a white bead at each of the others, one obtains a binary necklace. This proves:

**Corollary 4.22.** \( D_{d,k} \) is the scheme theoretic union of distinct \( T \)-orbit-closures through \((x^k)\) and \((y^k)\), and these orbit-closures are in natural bijection with the set \( N_{d,k} \) of binary necklaces with \( k \) black beads and \( d - k \) white beads. (As usual, necklaces are considered up to rotation.)

**Remark 4.23.** Because the \( T \)-action on \( \text{Hilb}^{(k)}(\mathbb{P}^1) \) is descended from \((\mathbb{P}^1)^k \), the symmetries of necklaces are respected in the following sense. If a necklace has an order \( d' \) rotational symmetry, where \( d'|d \), then the element \( (\zeta_{d})^{d'/d} \in \mu_d \subseteq \mathbb{C}^* \) acts trivially on \( \binom{\mu_d}{k} \). This means that \( T \) acts with weight \( d' \) on the corresponding orbit in \( \tilde{D}'_{d,k} \). In particular, the associated moduli space is naturally isomorphic as an orbifold to the moduli space of necklaces \( N_{d,k} = \binom{\mu_d}{k}/\mu_d \).

### 4.3. The tropical ideal associated to a necklace

In Example 4.14 we saw that \( D_{6,2} \) is the union of three \( T \)-orbits. Each \( T \)-orbit had an associated tropicalization; on the other hand, we now know that the \( T \)-orbits are in natural bijection with the 3-element set \( N_{6,2} \) of necklaces with 2 black and 4 white beads; see Figure 5 (The necklace on the left has a nontrivial automorphism.)

**Definition 4.24.** For \( \gamma \in N_{d,k} \), let \( \text{Trop}(\gamma) = \text{Trop}(I_{\gamma}) \) for any \( I_{\gamma} \) in the \( T \)-orbit corresponding to \( \gamma \).

**Question 4.25.** Given \( \gamma \in N_{d,k} \), is there a combinatorial algorithm to compute \( \text{Trop}(\gamma) \)?

We do not have a full answer to this question, but we now discuss it further. First, we characterize for which integers \( d' \) the subset \( U_{d'} \subset \text{Trop}(\gamma) \) is dependent in \( \text{Trop}(\gamma) \).

Consider the inclusion \( \iota_m : N_{d,k} \to N_{md,k} \), where \( \iota_m(\gamma) \) is the necklace obtained by adding \( m - 1 \) white beads between each pair of consecutive beads in \( \gamma \). (See Figure 6.) Note that \( \gamma \in N_{d,k} \) and \( \iota_m(\gamma) \) correspond to the same \( T \)-orbit in \( \text{Hilb}^{(k)}(\mathbb{P}^1) \), and hence \( \text{Trop}(\iota_m(\gamma)) = \text{Trop}(\gamma) \). This shows:
**Proposition 4.26.** Let $\gamma \in N_{d,k}$, and let $d'$ be the gcd of the $k$ distances between consecutive beads in $\gamma$. (Since the sum of these distances is $d$, $d'|d$.) Then $U_{d'}$ is dependent in $\text{Trop}(\gamma)$ if and only if $d'$ is a multiple of $d/d'$.

This explains Figure 5:
- In the first column, $b = \gcd(3, 6) = 3$, and $U_{2m} = U_{(d/b)m}$ is dependent for all $m$.
- In the second column, $b = \gcd(1, 6) = 1$, and $U_{6m} = U_{(d/b)m}$ is dependent for all $m$.
- In the third column, $b = \gcd(2, 6) = 2$, and $U_{3m} = U_{(d/b)m}$ is dependent for all $m$.

We also note the following condition, which implies certain necklaces have the same tropicalization.

**Proposition 4.27.** Let $\gamma \in N_{d,k}$, and let $a \in (\mathbb{Z}/d\mathbb{Z})^\times$. We define $a\gamma$ to be the necklace obtained by traversing $\gamma$ by jumps of length $a$. For example, if $\gamma = \text{\textbullet}\text{\textbullet}\text{\textbullet}$, then $3\gamma = \text{\textbullet}\text{\textbullet}\text{\textbullet}$ = $\text{\textbullet}\text{\textbullet}\text{\textbullet}$. Then $\text{Trop}(\gamma) = \text{Trop}(a\gamma)$.

**Proof.** The independence of any $k$-element set $U_{h,k}^a$ in $\text{Trop}(\gamma)$ is determined by the nonvanishing of an element of $\mathbb{C}$ obtained by field operations applied to a primitive $d$th root of unity $\zeta$ (namely, the determinant of the associated Schur matrix). This nonvanishing is preserved by the field automorphism that sends $\zeta \mapsto \zeta^a$, which determines the independence of $U_{h,k}^a$ in $\text{Trop}(\gamma)$. \qed

**Question 4.28.** Observe that in Figure 6, necklaces $\gamma_1, \gamma_2 \in N_{8,4}$ satisfy $\text{Trop}(\gamma_1) = \text{Trop}(\gamma_2)$ if and only if $\gamma_2 = a\gamma_1$ for some $a \in (\mathbb{Z}/8\mathbb{Z})^\times$. It is therefore natural to ask if the converse of Proposition 4.27 holds.

In order to fully characterize $\text{Trop}(\gamma)$, we need to know not only which sets $U_{d'}$ are dependent, but which sets $U_{h,k}^a$ are dependent.

**Definition 4.29.** Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition with at most $k$ parts and maximum part at most $h$. We associate to $\lambda$ the collection $\gamma(\lambda) = \{\zeta_{d}^{\lambda_1+k-i-1} : i = 1, \ldots, k\}$ of roots of unity.
Note that the collection $\gamma(\lambda)$ is exactly the sequence of exponents appearing in the matrix defining $s_\lambda$. If $\lambda$ is such that two elements of $\gamma(\lambda)$ coincide, then $s_\lambda$ vanishes on any $k$-tuple of $d$th roots of unity, since two rows of the defining matrix are equal. (In particular, for such $\lambda$, $D_{d,k} \subseteq D(U_{\lambda}^{h,k})$.)

On the other hand, if $\gamma(\lambda)$ contains $k$ distinct elements, then $\gamma(\lambda)$ naturally corresponds to a necklace with $k$ black beads and $d-k$ white beads. In particular, reordering and scaling $\gamma(\lambda)$ corresponds to reordering and scaling the rows of the matrix in the definition of $s_\lambda$, which does not affect its rank – hence, the question of whether $\gamma \in D(U_{\lambda}^{h,k})$ for some $\gamma \in N_{d,k}$ depends only on the necklace $\gamma(\lambda)$, not $\lambda$ itself. This dependence is, interestingly, commutative in the following sense.

**Proposition 4.30.** Let $\gamma \in N_{d,k}$ such that $\gamma = \gamma(\lambda)$. Then $\gamma \in D(U_{\lambda}^{h,k})$ if and only if $\gamma(\lambda') \in D(U_{\lambda'}^{h,k})$.

**Proof.** This follows immediately from $\det(A) = \det(A^T)$. □

Answering Question 4.25 now boils down to:

**Question 4.31.** We define a (symmetric) subset $P_{d,k}$ of $N_{d,k}$ as follows. For $\gamma_1, \gamma_2 \in N_{d,k}$ we say $(\gamma_1, \gamma_2) \in P_{d,k}$ if the determinant of any corresponding matrix $(x_i^{\gamma_1(i)}, y_i^{\gamma_2(i)})_{i,j}$ vanishes. Here $\gamma_1, \gamma_2$ are identified, up to translation, with $k$-element subsets $\{\gamma_1(i)\}$ and $\{\gamma_2(j)\}$ of $\mathbb{Z}/d\mathbb{Z}$. Is there a combinatorial description of $P_{d,k}$?

**Remark 4.32.** Experimentally, one may find sufficient conditions for a pair $(\gamma_1, \gamma_2)$ to be in $P_{d,k}$. In particular, one may prove a statement of the following form: if $a$ divides $d$, and the $k$ black beads of $\gamma_1$ are distributed “sufficiently unequally” among the $\mu_a$-orbits of the $d$th roots of unity, and the $k$ black beads of $\gamma_2$ are distributed “sufficiently unequally” among the $\mu_d\mu_a$-orbits of the $d$th roots of unity, then $(\gamma_1, \gamma_2) \in P_{d,k}$. However, we do not know of any necessary conditions; an additional idea would be needed to prove that the matrices in question have nonzero determinant.

### 4.4. Other gradings: T-curves in $\text{Hilb}^{(k)}(\mathbb{P}(a_1, a_2))$.

There is a straightforward analog of Theorem 4.9 for gradings other than $(1,1)$. For simplicity, we scale the grading $(a_1, a_2)$ so that $x$ has degree $1/a_2$ and $y$ has degree $1/a_1$, where $\gcd(a_1, a_2) = 1$. Let $\text{Hilb}^{(k)}(\mathbb{P}(a_1, a_2))$ denote the Hilbert scheme of subschemes of $\mathbb{P}(a_1, a_2)$ cut out by degree-$k$ polynomials.

**Theorem 4.33.** Let $U$ be a $m_k(a_1, a_2)$-element subset of $\text{Mon}_d(a_1, a_2)$, where $d$ is such that $m_d-k(a_1, a_2) > 0$. As in Section 4 there is a corresponding partition $\lambda$, with $m_k(a_1, a_2) - 1$ parts, each at most $m_d(a_1, a_2) - m_k(a_1, a_2) + 1$. Then $D(U) \subseteq \text{Hilb}^{(k)}(\mathbb{P}(a_1, a_2))$ is the vanishing locus of the polynomial

$$e_0(x_{a_2}^{a_2}, y_{a_1}^{a_1}, \ldots, y_{a_k}^{a_k})^{m_d(a_1, a_2) - m_k(a_1, a_2) + 1} - \lambda_1 s_\lambda(x_{a_2}^{a_2}, \ldots, y_{a_k}^{a_k}).$$

**Proof.** Note that in $\mathbb{P}(a_1, a_2)$, a point $[x_0 : y_0]$ with $x_0, y_0 \neq 0$ has degree 1, and is defined by the degree-1 polynomial $y_0^{a_1} - x_0^{a_2} y_0^{a_1}$. Meanwhile, $[0 : 1]$ has degree $1/a_2$ (and is an orbifold point of degree $a_2$) and $[1 : 0]$ has degree $1/a_1$ (and is an orbifold point of degree $a_1$). If $k = k_0 + \ell_1/a_1 + \ell_2/a_2$ with $\ell_1, \ell_2$ minimal, then the coefficient of $x_2^{a_2} y_1^{a_1} x_2^{a_2} y_1^{a_1} + \ldots$ in a degree-$k$ polynomial is given in terms of its (non-fractional) roots by $e_j(x_{a_2}^{a_2}, y_{a_1}^{a_1}, \ldots, x_{a_k}^{a_k}, y_{a_k}^{a_k})$. Note $m_k(a_1, a_2) = k_0 + 1$.

From here, the proof of Theorem 4.9 goes through the the following modifications. The matrix (4) now has rows indexed by $\text{Mon}_{d-k}(a_1, a_2)$, and rows indexed by $\text{Mon}_{d-k}(a_1, a_2)$. The entries are $e_0, \ldots, e_{k_0}$ rather than $e_0, \ldots, e_k$. (The first and last rows may be all zeros.) The second Jacobi-Trudi formula applies as in the proof of Theorem 4.9. □

### 5. Applications to finite-length Hilbert schemes

We now apply the results of the last section to draw conclusions about $T$-curves in $\text{Hilb}^{(n)}(\mathbb{C}^2)$. The key observation is that one can obtain a finite-colength ideal from a principal ideal by adding an appropriate monomial ideal: given a homogeneous ideal $I$ with respect to a grading $a$, and a monomial ideal $N$, the ideal $I + N$ is homogeneous with respect to $a$, though has a different graded Hilbert polynomial. (Of course, not all finite-colength ideals can be obtained this way, e.g. $(x^2 - xy, xy - y^2, x^3)$ cannot.)

**Example 5.1.** Consider the ideal $I = (x^2 - y^2) \in \text{Hilb}^{(k)}(\mathbb{P}^1)$. Adding the monomial ideal $N = (x^3)$ yields $I + N = (x^2 - y^2, x^3)$, an ideal of colength 6, with tropicalization shown in Figure 7. Note that adding $N$ does not commute with taking initial ideals; for example, in $x(I + N) = (x^2) + (x^3) = (x^2)$, which has infinite colength, while $\text{in}_x(I + N) = (x^2, xy^2, y^4)$.
Let Proposition 5.4. Using Proposition 5.4, we have

\[
\text{Proof.}
\]

For any homogeneous ideal \( M \), it does behave well in matroid strata:

\[
\text{Corollary 5.5.}
\]

It immediately follows that while adding monomial ideals does not behave well in families (see Example 5.1), it does behave well in matroid strata:

\[
\text{Proposition 5.4.}
\]

\[
\text{Definition 5.2.}
\]

A homogeneous ideal \( I \subseteq R \) is called monoprincipal if \( I = (f) + N \) for some \( f \in R \) homogeneous, and some monomial ideal \( N \).

The analogous operation of matroids is the “looped contraction.” (We do not know of a standard term for this operation.)

\[
\text{Definition 5.3.}
\]

Let \( M = (E, r) \) be a matroid, and let \( S \subseteq E \). The contraction \( M/S \) of \( M \) at \( S \) is the matroid with groundset \( E \setminus S \) whose circuits are the minimal elements of \( \{S' \cap (E \setminus S) : S' \text{ a circuit of } M\} \). In other words, for \( T \subseteq E \setminus S \),

\[
\text{Proof.}
\]

\[
\text{Let } S \subseteq E \text{ such that } r(S) = 0. \text{ Note } M \div S \text{ has groundset } E \text{ and rank } r(M) - r(S). \text{ The rank function is given, for } T \subseteq E,
\]

\[
\text{or}
\]

\[
\text{Proposition 5.4.}
\]

Let \( V \subseteq k^n \) be a subspace, and \( S \subseteq [n] \). Then \( M(V) \div S = M(V + kS) \).

\[
\text{Proof.}
\]

The rank function of \( M(V + kS) \) is, by definition,

\[
\text{Corollary 5.5.}
\]

For any homogeneous ideal \( I \) and any monomial ideal \( N = \bigoplus_{d \geq 0} k\{m_{d,1}, \ldots, m_{d,a_d}\} \),

\[
\text{Trop}(I + N) = \text{Trop}(I) \div N.
\]

\[
\text{Proof.}
\]

Using Proposition 5.4, we have

\[
\text{It immediately follows that while adding monomial ideals does not behave well in families (see Example 5.1), it does behave well in matroid strata:}
\]

\[
\text{Figure 7. Tropicalization of } I + N = (x^2 - y^2, x^3)
\]
Corollary 5.6. Let $\mathcal{M}$ be a tropically principal tropical ideal of degree $k$, and $N$ a monomial ideal. Then $I \rightarrow I + N$ defines a natural morphism $\mathcal{D}(\mathcal{M}) \rightarrow \mathcal{D}(\mathcal{M} \div N)$. (Note that $\mathcal{D}(\mathcal{M} \div N)$ lies in a single multigraded Hilbert scheme.)

Corollary 5.5 immediately implies (cf. Proposition 3.14):

Proposition 5.7. If $(f) + N \subseteq R$ is monoprincipal, then $\text{Trop}((f) + N)$ is monoprincipal.

However, the converse does not hold, as in the following example.

Example 5.8. Let $k = \mathbb{C}$. The matroid $\mathcal{M}$ in Figure 8 is monoprincipal, since $\mathcal{M} = \text{Trop}((f) + N)$, where $f = x^3 + x^2y + 2xy^2 + y^3$ and $N = \{x^4, x^3y^2, x^2y^3, y^4\}$. (It is straightforward to check that the roots of $f$ do not differ by 4th roots of unity, hence $f \not\in \mathcal{D}(U_4)$. This implies that $\text{Trop}((f) + N)_4$ has rank 1, as shown.) On the other hand, we also have $\mathcal{M} = \text{Trop}(((x - y)(x - iy)(x + y), x^3y + 2xy^2) + N)$, as follows. Since $((x - y)(x - iy)(x + y)) \in \mathcal{D}_{4,3}$, Section 4.2 implies that $\text{Trop}(((x - y)(x - iy)(x + y))_4 + N)$ has rank 2, and adding in the polynomial $x^3y + 2xy^2$ reduces the rank to 1. (It is again easy to check that neither ideal contains any extra monomials in degree 4.)

Lastly, we observe that $((x - y)(x - iy)(x + y), x^3y + 2xy^2) + N$ is not monoprincipal. If it were, it would necessarily be generated in degree 4 by $\{x(x - y)(x - iy)(x + y), y(x - y)(x - iy)(x + y), x^4, y^4\}$; these span too small a subspace.

The following is the key observation for applying Section 4 to Hilbert schemes of finite-length subschemes.

Lemma 5.9. Let $I = (f) + N$ be a monoprincipal ideal. Let $U \subseteq N_d$ be a set of monomials such that $|U| > r_{\text{Trop}((f))}(\text{Mon}_d(a_1, a_2)) - r_{\text{Trop}((f))}(\text{Mon}_d(a_1, a_2))$. Then $(f) \in \mathcal{D}(U)$.

Proof. By Corollary 5.5, $\text{Trop}(I) = \text{Trop}((f)) \div N$, so

$$r_{\text{Trop}((f))}(\text{Mon}_d(a_1, a_2)) = r_{\text{Trop}((f))}(\text{Mon}_d(a_1, a_2)) - r_{\text{Trop}((f))}(N_d).$$

By assumption,

$$|U| > r_{\text{Trop}((f))}(\text{Mon}_d(a_1, a_2)) - r_{\text{Trop}((f))}(\text{Mon}_d(a_1, a_2)) = r_{\text{Trop}((f))}(N_d) \geq r_{\text{Trop}((f))}(U),$$

so $U$ is dependent. □

Remark 5.10. We will apply Lemma 5.9 as follows. Often, it can be argued that a given matroid stratum (or edge scheme) $B$ must consist only of monoprincipal ideals, say of degree $k$. In this case, forgetting all data but the generator defines a natural embedding $B \hookrightarrow \text{Hilb}^k(\mathbb{P}(a_1, a_2))$. Lemma 5.9 then says that the embedding factors through $\bigcap U \mathcal{D}(U) \subseteq \text{Hilb}^k(\mathbb{P}(a_1, a_2))$, where $U$ runs over sets satisfying the condition in the hypothesis.

We now illustrate how this works in our motivating example.

Corollary 5.11. Let $k \geq 1$ and $d_0 > k$. Let $M_1$ (resp. $M_2$) be the partition whose Young diagram is an $d_0 \times k$ (resp. $k \times d_0$) rectangle. Then the edge moduli space $\mathcal{M}(M_1, M_2)$ is isomorphic to $N_{d_0,k}$.
Remark 5.13

Further, it should be noted that Theorem 4.9 can also be used to study non-principal ideals, provided they “resemble a principal ideal locally near some box.” In particular, if \( f \in I \) has degree \( c \) and support \( S \), and \( I \cap (\text{Mon}_d(a_1, a_2)S) \) is generated by multiples of \( f \), then Theorem 4.9 can be applied to subsets\[ U \subseteq \text{Mon}_d(a_1, a_2)S \]
It is plausible that a sufficiently clever application of Theorem 4.9 might give an answer to Question 1 in the case $r = 2$; experimentally, all edge schemes we have computed are cut out by Schur polynomials. However, we do not see how to do this.

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