Generalized forward scattering amplitudes in QCD at high

temperature

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We extend to a general class of covariant gauges an approach which relates the thermal Green functions to forward scattering amplitudes of thermal particles. A brief discussion of the non-transversality of the thermal gluon polarization tensor is given in this context. This method is then applied to the calculation of the \( \ln(T) \) contributions associated with general configurations of 2 and 3-point gluon functions. The results are Lorentz covariant and have the same structure as the ultraviolet divergent contributions which occur at zero temperature.

There have been many studies of the high-temperature behavior of Green functions in thermal QCD [1–5]. These investigations have been mainly concerned with hard thermal loops, which are important elements for the re-summation of the QCD thermal perturbation theory [6]. The work in reference [7] describes a method for calculating thermal loops, which are expressed as a momentum integral of forward scattering amplitudes (summed over spins and internal quantum numbers) of thermal particles. This approach has been further elaborated in the Feynman gauge and shown to be very useful for determining the partition function in QCD at high temperature [8].

The main purpose of this brief report is to extend the analysis in [8] to a general class of covariant gauges, characterized by a gauge parameter \( \xi \) (the Feynman gauge corresponds to the special case when \( \xi = 1 \)). It has been shown that the thermal gluon polarization tensor is in general not transverse, except in the Feynman gauge [9,10]. We point out
that this property may be simply understood when one extends, at finite temperature, a
procedure which implements the transversality condition at zero temperature. The distinct
behavior in the Feynman gauge arises in consequence of cancellations between ghost and
gluon contributions, which occur in the longitudinal part of the gluon self-energy. Another
application of the above approach is the evaluation of the sub-leading \( \ln(T) \) contributions
associated with general configurations of 2 and 3-point gluon functions. After a rather
involved calculation, we obtain a simple Lorentz covariant result. This provides a nontrivial
verification of a previous argument \[1\], which shows that these contributions should appear
with the same coefficient as the ultraviolet pole part of the zero temperature amplitudes.

In order to derive the relation between thermal loops and forward scattering amplitudes
we treat, for definiteness, the case of the gluon self-energy, but the result generalizes in a
obvious way to higher order Green functions (see Eq. \[19\]). Since the quark loops are
independent of the gauge parameter and have been already considered in \[8\], we shall focus
here on the thermal gluon loops.

Consider the contributions associated with the diagram in Fig. 1a, where we suppress
for simplicity the color indices. We shall employ an analytic continuation of the imaginary
time formalism \[12\], where these contributions may be written as

\[
\Pi_{\mu_1,\mu_2}^{1}(k, u) = \frac{1}{2\pi i} \int \frac{d^3Q}{(2\pi)^3} \int_{C} dQ_0 N(Q_0) i^{\alpha\beta\rho\sigma}_{\mu_1,\mu_2} (Q_0, Q, Q_0 + k_0, Q + k) D_{\alpha\beta}(Q) D_{\rho\sigma}(Q + k),
\]

(1)

where \( u \equiv (1, 0, 0, 0) \) specifies the rest-frame of the plasma. The tensor \( i^{\alpha\beta\rho\sigma}_{\mu_1,\mu_2} \) is the numerator
which is of degree two in its arguments and \( D_{\alpha\beta}(Q) \) is the bare gluon propagator given by

\[
D_{\alpha\beta}(Q) = \frac{1}{Q^2} \left[ \eta_{\alpha\beta} - (1 - \xi) \frac{Q_\alpha Q_\beta}{Q^2} \right].
\]

(2)

\( N(Q_0) = [\exp(Q_0/T) - 1]^{-1} \) is the Bose statistical distribution, which has poles along the
imaginary \( Q_0 \) axis at \( Q_0 = 2\pi n iT \ (n = 0, \pm 1, \pm 2 \ldots) \). The contour \( C \) surrounds all the
poles of \( N(Q_0) \) in a anti-clockwise sense.
\[ Q + k \]  
\[ k, \mu_j \]  
\[ -k, \mu_2 \]  

(a) The gluon self-energy thermal loop. Contributions from internal ghost particles are to be understood. (b) An example of the forward scattering graph connected with diagram (a).

In the contour on the left-hand side of the imaginary axis, we let \( Q_0 \to -Q_0 \) and make also the change of variable \( Q \to -Q \). Using the relation \( N(Q_0) + N(-Q_0) = -1 \) and ignoring the temperature independent part, we can write the thermal contribution of (1) as

\[
\Pi_{\mu_1 \mu_2}^1 (k, u) = \frac{1}{(2\pi)^3} \int \frac{d^3 Q}{(2\pi)^3} \int_{-i\infty + \epsilon}^{i\infty + \epsilon} dQ_0 N(Q_0) \times \{ t_{\alpha \beta \rho \sigma}^{\mu_1 \mu_2} (Q_0, Q, Q_0 + k, Q + k) D_{\alpha \beta} (Q) D_{\rho \sigma} (Q + k) + (Q \to -Q) \}.
\]

This can be evaluated with the help of Cauchy’s theorem, by encircling the poles in the Feynman denominators in the right-half plane. Consider first the poles at \( Q_0 = |Q| \) present in \( D_{\alpha \beta} (Q) \). Evaluating the contributions from simple and double poles, we get

\[
\Pi_{\mu_1 \mu_2}^1 (k, u) = -\int \frac{d^3 Q}{(2\pi)^3} \left\{ \eta_{\alpha \beta} N(|Q|) - (1 - \xi) \frac{\bar{d}}{dQ_0} \frac{Q_\alpha Q_\beta}{Q_0 + |Q|} N(Q_0) \right\} \times \left\{ t_{\alpha \beta \rho \sigma}^{\mu_1 \mu_2} (Q_0, Q, Q_0 + k, Q + k) D_{\rho \sigma} (Q + k) + (Q \to -Q) \right\} \bigg|_{Q_0 = |Q|},
\]

where the derivative \( \bar{d}/dQ_0 \) acts on all terms on its right. At this point, it is convenient to introduce the operator

\[
P_{\alpha \beta} (Q) \equiv \eta_{\alpha \beta} - \frac{1 - \xi}{2} \frac{\bar{d}}{dQ_0} \frac{Q_\alpha Q_\beta}{Q_0}
\]

which is related to the sum over the polarization states of the thermal gluon. In terms of this quantity, equation [4] may be rewritten as

\[
P_{\alpha \beta} (Q) N(Q_0) \times \left\{ t_{\alpha \beta \rho \sigma}^{\mu_1 \mu_2} (Q_0, Q, Q_0 + k, Q + k) D_{\rho \sigma} (Q + k) + (Q \to -Q) \right\} \bigg|_{Q_0 = |Q|}.
\]
Now consider the contributions to $\Pi_{\mu_1\mu_2}^1(k, u)$ from the poles on the right-hand side of the imaginary axis present in $D_{\rho\sigma}(Q + k)$. Since in the imaginary time formalism, $k_0 = 2\pi i m T$, such a pole occurs only at $Q_0 = -k_0 + |Q + k|$. Here we make the change of variable $Q \to Q - k$ and use the relation

$$N(Q_0 - k_0) = N(Q_0).$$  \hspace{1cm} (7)

We then get a contribution similar to that in (6), but with $k \to -k$. Thus, the total result for the thermal part of (1) is

$$\Pi_{\mu_1\mu_2}^1(k, u) = \frac{-1}{(2\pi)^3} \int \frac{d^3Q}{2|Q|} \left\{ P_{\alpha\beta}(Q) N(Q_0) \right. \times \left[ t_{\mu_1\mu_2}^{\alpha\beta\rho\sigma}(Q_0, Q, Q_0 + k_0, Q + k) D_{\rho\sigma}(Q + k) + (k \rightarrow -k) + (Q \rightarrow -Q) \right]\} \bigg|_{Q_0 = |Q|}. \hspace{1cm} (8)

At this stage, after using (7), the integrand in Eq. (8) is a rational function of $k_0$ which may be analytically continued to all continuous values of the external energy. The first term in the square bracket can be represented by the forward scattering amplitude $A_{\mu_1\mu_2}^{\alpha\beta}(Q, k, -k)$ shown in Fig. 1b. The term containing $(k \to -k)$ corresponds to a diagram obtained from Fig. 1b by a permutation of the external momenta.

We must also consider the contributions associated with the ghost loop. Here it is useful to define

$$\tilde{A}_{\mu_1\mu_2}^{\alpha\beta}(Q, k, -k) \equiv \frac{P_{\alpha\beta}}{3 + \xi} A_{\mu_1\mu_2}^{\text{ghost}}(Q, k, -k),$$  \hspace{1cm} (9)

where $A_{\mu_1\mu_2}^{\text{ghost}}(Q, k, -k)$ represents the forward-scattering amplitude of a ghost particle by the external gluon fields. The above definition is convenient because of the identity $P_{\alpha\beta} \tilde{A}_{\mu_1\mu_2}^{\alpha\beta} = A_{\mu_1\mu_2}^{\text{ghost}}$, which holds due to the fact that the ghost particle is on-shell.

In order to obtain the full thermal amplitude for the gluon self-energy, we must include the contributions from the tadpole graph shown in Fig. 2. Denoting by $A_{\mu_1\mu_2}^{\alpha\beta}(Q, k, -k)$ the total forward scattering amplitude, we can express the complete thermal contribution in terms of a momentum integral of $A_{\mu_1\mu_2}^{\alpha\beta}$, as

$$\Pi_{\mu_1\mu_2}(k, u) = \frac{-1}{(2\pi)^3} \int \frac{d^3Q}{2|Q|} \left\{ P_{\alpha\beta}(Q) N(Q_0) \left[ A_{\mu_1\mu_2}^{\alpha\beta}(Q, k, -k) + (Q \rightarrow -Q) \right] \right\} \bigg|_{Q_0 = |Q|}. \hspace{1cm} (10)$$
Using this expression, it can be verified that for the exact gluon self-energy, $k^{\mu_1}\Pi_{\mu_1\mu_2} \neq 0$, a property that was previously investigated \[3,9,10\]. Multiplying the forward scattering amplitude in Eq. (10) by $k^{\mu_1}$, we find that the longitudinal part of the polarization tensor can be expressed as

$$k^{\mu_1}\Pi_{\mu_1\mu_2} = (1 - \xi) \frac{g^2 C_G}{(2\pi)^3} \int \frac{d^3 Q}{|Q|} \left\{ \left( \frac{k^2}{k^2 + 2k \cdot Q} + \frac{1}{2dQ_0} \frac{k \cdot Q}{Q_0} \right) \right.$$  

$$\times \frac{k \cdot Qk_{\mu_2} - k^2Q_{\mu_2}}{k^2 + 2k \cdot Q} N(|Q_0|) + (Q \rightarrow -Q) \right\}_{Q_0=|Q|}, \quad (11)$$  

where $C_G$ is the gluon Casimir invariant. This relation shows that the thermal part of $\Pi_{\mu_1\mu_2}$ is transverse only in the Feynman gauge (but $k^{\mu_1}k^{\mu_2}\Pi_{\mu_1\mu_2} = 0$ for all values of $\xi$).

FIG. 2. (a) The tadpole loop diagram and the corresponding forward-scattering graph(b).

There is an interesting connection between this behavior and the transversality condition which must hold at zero temperature. To see this, we recall that the integrand appearing in the self-energy function at finite temperature is similar, apart from the Bose factor, to the one which occurs at zero temperature \[12\]. Thus, we are lead to consider the way that the zero temperature property $k^{\mu_1}\Pi_{\mu_1\mu_2} = 0$ is actually implemented at the one-loop order. In the Feynman gauge, due to cancellations between gluon and ghost loops, this condition is verified just by making a shift $Q \rightarrow Q - k$ in some terms in the integrand. Such a shift can be performed at zero temperature using the dimensional regularization scheme and is also allowed at finite temperature in the imaginary time formalism, in view of property (7). In a general covariant gauge, the ghost contribution remains the same since it is independent of the gauge parameter $\xi$. But the longitudinal part of the gluon loop contribution differs
at the integrand level from that obtained in the Feynman gauge by terms proportional to \((1 - \xi)\). In this case, the verification of the transversality condition at zero temperature requires, in addition, the use of dimensionally regularized integrals like

\[
\int \frac{d^nQ}{(Q^2)\alpha} f(Q) = 0. \tag{12}
\]

However, such a result does not hold at finite temperature when \(f(Q) = N(Q_0)\), in which case (12) becomes instead proportional to \(T^{n-2\alpha}\). Consequently, the above longitudinal part of the gluon loop contribution will not vanish at finite temperature, in agreement with the result given by equation (11).

Let us next consider the contributions of (10) coming from the hard thermal region where \(|Q| \gg |k|, k_0\). This region is relevant for the determination of \(T^2\) and \(\ln(T)\) contributions (unlike the terms linear in \(T\) which come also from soft momenta). We will argue that in this case we can effectively commute the differential operator \(P_{\alpha\beta}\) and the Bose factor \(N(Q_0)\). To this end, let us study the contribution obtained when the derivative from \(P_{\alpha\beta}\) acts on \(N(Q_0)\).

This leads to an integrand proportional to a factor \(N'(|Q|)/|Q|^2\) times \(Q_\alpha Q_\beta A^{\alpha\beta}_{\mu_1\mu_2}\), where the thermal particle four-vector \(Q\) is on shell. As a consequence of the gauge invariance of the forward scattering amplitude, \(Q_\alpha Q_\beta A^{\alpha\beta}_{\mu_1\mu_2}\) vanishes when the external gauge fields are on shell and transverse. It is not difficult to show that even for general values of \(k\) there will be strong cancellations, so that \(Q_\alpha Q_\beta A^{\alpha\beta}_{\mu_1\mu_2}\) actually becomes a function of zero degree in \(Q\). Together with the above factor, such a function will produce in (10) an integral of the form

\[
\int_{\tau}^{\infty} d|Q|N'(|Q|) = \frac{1}{1 - \exp(\tau/T)}, \tag{13}
\]

where the lower limit \(\tau \gg |k|, k_0\) delimitates the hard thermal region. Since (13) does not yield at high temperature \(T^2\) or \(\ln(T)\) contributions, it may be neglected for our purpose so that (10) becomes equivalent to

\[
\Pi_{\mu_1\mu_2}^{ht}(k, u) = -\frac{1}{(2\pi)^3} \int \frac{d^3Q}{2|Q|} N(|Q|) \left\{ P_{\alpha\beta}(Q)A^{\alpha\beta}_{\mu_1\mu_2}(Q, k; -k) + (Q \rightarrow -Q) \right\}_{Q_0=|Q|}. \tag{14}
\]

We may now extract from (14) a series of high temperature terms which come from the
region of large $Q$. Since the four momentum of the thermal particle is ultimately set on shell, the denominators in $A_{\mu_1 \mu_2}^{\alpha \beta}$ can be expanded as follows

$$\frac{1}{(k + Q)^2} \bigg|_{Q_0 = |Q|} = \frac{1}{2k \cdot Q} - \frac{k^2}{(2k \cdot Q)^2} + \cdots.$$  \hspace{1cm} (15)

We may also expand the numerator in powers of $k_\mu/|Q|$. The first term has a denominator of the form $(k \cdot Q)^{-1}$ and a numerator quadratic in $Q$ which is independent of $k$. But such terms cancel out in (14) by symmetry under $Q \rightarrow -Q$. (For the same reason, any terms odd in $Q$ will generally cancel out). The next contributions are down by a power of $k_\mu/|Q|$ and arise from those terms in $A_{\mu_1 \mu_2}^{\alpha \beta}$ which are of zero degree in $Q$. Such terms give the well known leading $T^2$ contributions, which are gauge independent. The following non-vanishing contributions arise from those terms in $A_{\mu_1 \mu_2}^{\alpha \beta}$ which are of degree $-2$ in $Q$. By power counting, such terms will produce $\ln(n)$ contributions. The corresponding term in (14) coming from the trace $A_{\mu_1 \mu_2}^{\alpha \alpha}$ yields, after performing the $Q$-integration, a Lorentz covariant expression. The other part of the integrand involves the operator $\bar{d}/dQ_0 Q_0^{-1}$ applied to a covariant function of zero degree in $Q$. Such a function can be generated by differentiating $\ln(k \cdot Q)$ with respect to $k_\mu$. We are thus lead to consider the integral

$$\int \frac{d^3Q}{|Q|} N(|Q|) \left\{ \frac{d}{dQ_0} \left( \frac{1}{Q_0} \ln(k \cdot Q) \right) \right\}_{Q_0 = |Q|} = \pi \ln(T) \ln(k^2) + \cdots,$$

where we have written explicitly only the $k$-dependent $\ln(T)$ term. Then the $\ln(k^2)$ term will generate, after differentiations with respect to $k_\mu$, a Lorentz covariant expression. Adding all such contributions, we obtain for the logarithmic dependence on the temperature the simple result

$$\Pi^{\ln}_{\mu_1 \mu_2} = - \left( \frac{13}{3} - \xi \right) \frac{g^2 C_G}{(4\pi)^2} \ln(T) \left( \eta_{\mu_1 \mu_2} k^2 - k_{\mu_1} k_{\mu_2} \right).$$  \hspace{1cm} (17)

The important point about (17) is the fact that it has the same structure as the ultraviolet divergent contribution of the gluon self-energy at zero temperature (18)

$$\Pi^{\ln}_{\mu_1 \mu_2} = \left( \frac{13}{3} - \xi \right) \frac{g^2 C_G}{(4\pi)^2} \left( \frac{1}{2\epsilon} + \ln(M) \right) \left( \eta_{\mu_1 \mu_2} k^2 - k_{\mu_1} k_{\mu_2} \right),$$  \hspace{1cm} (18)
where the space-time dimension is $4 - 2\epsilon$ and $M$ is the renormalization scale.

The result (10) for the thermal gluon self-energy can be generalized in a straightforward way to higher order $n$-point gluon functions. In this case, let us denote by $A_{\mu_1 \mu_2 \ldots \mu_n}^{\alpha \beta} (Q, k_1, k_2, \ldots, k_n)$ the forward scattering amplitude of a thermal particle with momentum $Q$ by external fields with momenta $k_1, k_2, \ldots, k_n$ such that $k_1 + k_2 + \cdots + k_n = 0$. (We have omitted, for simplicity of notation, the color indices). The thermal part of the one-loop $n$-point gluon function can then be expressed in terms of a momentum integral of the forward scattering amplitude as

$$
\Gamma_{\mu_1 \ldots \mu_n} (k_1, \ldots, k_n, u) = -\frac{1}{(2\pi)^3} \int \frac{d^3Q}{2|Q|} \left\{ P_{\alpha \beta}(Q) N(Q_0) \left[ A_{\mu_1 \ldots \mu_n}^{\alpha \beta} (Q, k_1, \ldots, k_n) + (Q \rightarrow -Q) \right] \right\}_{Q_0=|Q|},
$$

(19)

where $u$ is the four-velocity of the heat bath, $P_{\alpha \beta}(Q)$ indicates the operator defined in (5) and $N(Q_0)$ is the Bose distribution function.

Let us apply this result to the evaluation of the hard thermal contributions associated with the 3-point gluon function. The corresponding forward scattering amplitude is represented in Fig. 3, where graphs obtained by all permutations of external momenta and indices are to be understood. Proceeding in a similar way to that indicated following Eq. (14), one finds that the terms of zero degree in $Q$ from $A_{\mu_1 \mu_2 \mu_3}^{\alpha \beta} (Q, k_1, k_2, k_3)$ yield the well known leading $T^2$ contributions, which are gauge invariant. The terms from $A_{\mu_1 \mu_2 \mu_3}^{\alpha \beta} (Q, k_1, k_2, k_3)$ which are of degree -2 in $Q$ give the high temperature $\ln(T)$ contributions. After a very involved computation which uses an extension of the approach described in (14), we obtain for these contributions the simple Lorentz covariant result

$$
\Gamma_{\mu_1 \mu_2 \mu_3}^{\text{in}} = -\left( \frac{17}{6} - \frac{3\xi}{2} \right) \frac{g^2C_G}{(4\pi)^2} \ln(T) i \left( \eta_{\mu_1 \mu_2} (k_1 - k_2)_{\mu_3} + \eta_{\mu_2 \mu_3} (k_2 - k_3)_{\mu_1} + \eta_{\mu_3 \mu_1} (k_3 - k_1)_{\mu_2} \right).
$$

(20)

This is proportional to the basic 3-gluon vertex and has the same structure as the ultraviolet divergent part of the 3-point gluon function at zero temperature, which is given by (13).
\[ \Gamma_{\mu_1\mu_2\mu_3} = \left( \frac{17}{6} - \frac{3\xi}{2} \right) g^2 C_G \left( \frac{1}{2\epsilon} + \ln (M) \right) i \left( \eta_{\mu_1\mu_2} (k_1 - k_2)_{\mu_3} + \eta_{\mu_2\mu_3} (k_2 - k_3)_{\mu_1} + \eta_{\mu_3\mu_1} (k_3 - k_1)_{\mu_2} \right). \]

(21)

FIG. 3. Examples of forward scattering amplitudes associated with the thermal 3-gluon loop diagrams. Graphs involving the forward scattering of ghost particles should be included.

We note that the \( \ln(T) \) in Eqs. (17) and (20) may be combined with the \( \ln(M) \) from Eqs. (18) and (21) respectively, to yield \( \ln(T/M) \) contributions. Although these terms have gauge-dependent coefficients, the quantity

\[ Z_g^{-1}(T/M) \equiv \left[ 1 - \left( \frac{13}{3} - \xi \right) \frac{g^2 C_G}{4\pi^2} \ln \left( \frac{T}{M} \right) \right]^{3/2} \left[ 1 - \left( \frac{17}{6} - \frac{3\xi}{2} \right) \frac{g^2 C_G}{4\pi^2} \ln \left( \frac{T}{M} \right) \right]^{-1} \]

\[ \simeq \left[ 1 + \frac{11g^2 C_G}{48\pi^2} \ln \left( \frac{T}{M} \right) \right]^{-1} \]

(22)

is gauge invariant. This factor, which sums up the one-loop \( \ln(T) \) contributions to the running coupling constant \( \bar{g}(T) \) at high temperature, is relevant, for example, in the calculation of the pressure in thermal QCD [14,15].

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REFERENCES

[1] D. I. Gross, R. D. Pisarski and L. G. Yaffe, Rev. Mod. Phys. 53, 43 (1981).

[2] A. H. Weldon, Phys. Rev. D 26, 1394 (1982).

[3] K. Kajantie and J. Kapusta, Ann Phys, 160, 477 (1985).

[4] N. P. Landsman and Ch. G. Van Weert, Phys. Rep. 145, 141 (1987).

[5] E. Braaten and R. D. Pisarski, Phys. Rev. D 45, 1827 (1992).

[6] E. Braaten and R. D. Pisarski, Nucl. Phys. B337, 569 (1990); B339, 310 (1990).

[7] G. Barton, Ann. Phys. 200, 271 (1990).

[8] J. Frenkel and J. C. Taylor, Nucl. Phys. B334, 199 (1990); B374, 156 (1992).

[9] K. Kobes, G. Kunstatter and K. W. Mak, Z. Phys. C 45, 129 (1989)

[10] H. Weldon, Structure of the gluon propagator at finite temperature, e-Print Archive: hep-ph/9701279

[11] F. T. Brandt and J. Frenkel, High temperature ln(T) contributions in thermal field theory, e-Print Archive: hep-th/9701021, to be published in Phys. Rev. D.

[12] J. Kapusta, Finite Temperature Field Theory (Cambridge University Press, Cambridge, England, 1989).

[13] T. Muta, Foundations of Quantum Chromodynamics (World Scientific, Singapore, 1987)

[14] P. Arnold and C. Zhai, Phys. Rev. D 51, 1906 (1995).

[15] E. Braaten and A. Nieto, Phys. Rev. D 53, 1996 (1996).