Topological quantization of self-dual Chern-Simons vortices on Riemann Surfaces

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The self-duality equations of Chern-Simons Higgs theory in a background curved spacetime are studied by making use of the $U(1)$ gauge potential decomposition theory and $\phi$-mapping method. The special form of the gauge potential decomposition is obtained directly from the first of the self-duality equations. Using this decomposition, a rigorous proof of magnetic flux quantization in background curved spacetime is given. Furthermore, the precise self-dual vortex equation with topological term is obtained, in which the topological term has always been ignored.

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I. INTRODUCTION

In recent years, Chern-Simons gauge theories have attracted the attention in various subjects of both physics and mathematics. One of interdisciplinary topics attracted attention is so-called Bogomol’nyi-type vortices and self-dual solutions [1, 2, 3]. In Chern-Simons gauge theories coupled to scalar matter field, a important model is Chern-Simons Higgs model. The progress to this direction has also been achieved in Chern-Simons Higgs model coupled to background gravity [4, 5] and Einstein gravity [6].

In previous paper about Chern-Simons Higgs theory including gravity, none can indicate or prove quantization of magnetic flux, and moreover, the conventional self-dual vortex equation is everywhere away from the zeros of the scalar field and the equation is not meaningful at the zeros of the scalar field. In this paper we will discuss these two questions. Firstly, using Duan’s $\phi$-mapping theory [7, 8, 9, 10, 11], we study the topological inner structure of Bogomol’nyi self-duality equations and obtain directly one special form of the gauge potential decomposition. Using this decomposition, we firstly give a rigorous proof of magnetic flux quantization in background curved spacetime , and one sees that the inner structure of this vortex labelled only by the topological indices of the zero points of the

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complex scalar field. Secondly, we obtain the precise scalar field equation with a topological term, which differs from the conventional equation.

This paper is organized as follows. In the next section, we briefly review Bogomol’nyi bound of the Chern-Simons Higgs theory coupled to background gravity and discuss the conventional self-duality equations, Sec. III presents one special form of the general $U(1)$ gauge potential decomposition from the first self-duality equation. In Sec. IV we will give a rigorous proof of magnetic flux quantization. In Sec. V We then obtain the precise self-dual vortex equation with topological term. The conclusions of this paper are given in Sec. VI.

II. REVIEW OF BOGOMOL’NYI BOUND OF THE CHERN-SIMONS HIGGS IN BACKGROUND GRAVITY

In this section we review derivation of so-called Bogomol’nyi bound of the Chern-Simons Higgs theory coupled to background gravity. We consider a (2+1)-dimensional space-time $M^3$, the metric $g_{\mu\nu}$ of the (2+1)-dimensional manifold $M^3$ is determined by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = N^2(x^k)dt^2 - \gamma_{ij}(x^k)dx^i dx^j,$$

where $\gamma_{ij}$ is the metric of two-dimensional spatial hypersurface and $i, j, k = 1, 2$. The Chern-Simons Higgs Lagrangian is described by

$$S = \int d^3x \sqrt{|g|} \left[ \frac{\kappa}{4} \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} + \frac{1}{2} g^{\mu\nu} D_\mu \phi D_\nu \phi^* \right] - V(\|\phi\|),$$

where $\phi$ is the designated charged Higgs complex scalar field minimally coupled to an Abelian gauge field, $\frac{\kappa}{4} \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda}$ is the so-called Chern-Simons term in curved spacetime, and the covariant derivative is $D_\mu \phi = \partial_\mu \phi - ie A_\mu \phi$. Since the Bogomol’nyi limit is our interest, the form of the scalar potential $V(\|\phi\|)$ is taken to be

$$V(\|\phi\|) = \frac{e^4}{8R^2} \|\phi\|^2 (\|\phi\|^2 - v^2)^2.$$

The energy-momentum tensor is derived as usual

$$T_{\mu\nu} = \frac{1}{2} (D_\mu \phi^* D_\nu \phi + D_\nu \phi^* D_\mu \phi) - g_{\mu\nu} \left[ \frac{1}{2} g^{\rho\sigma} D_\rho \phi^* D_\sigma \phi \right] - V(\|\phi\|).$$
An appropriate rearrangement of stress components of the static energy-momentum tensor gives

\[ T^{ij} = \frac{1}{2} \gamma^{ij} \left[ \frac{\kappa^2}{2e^2 \| \phi \|^2} B^2 \right] - \frac{1}{2} \left( \gamma^{ij} \gamma^{kl} - \gamma^{ik} \gamma^{jl} - \gamma^{il} \gamma^{jk} \right) D_k \phi \ast D_l \phi - \frac{1}{2} \frac{\gamma^{ij}}{\| \phi \|^2} \left( B - \frac{e^3}{2\kappa^2} \| \phi \|^2 \right) \left( B + \frac{e^3}{2\kappa^2} \| \phi \|^2 \right) \times \left( \gamma^{ij} \gamma^{kl} - \gamma^{ik} \gamma^{jl} - \gamma^{il} \gamma^{jk} \right) \frac{D_k \phi \ast D_l \phi}{\| \phi \|^2} \]

\[ = \frac{\kappa^2}{2e^2 \| \phi \|^2} \left[ B - \frac{e^3}{2\kappa^2} \| \phi \|^2 (\| \phi \|^2 - v^2) \right] \left[ B + \frac{e^3}{2\kappa^2} \| \phi \|^2 (\| \phi \|^2 - v^2) \right] \times \left( \gamma^{ij} \gamma^{kl} - \gamma^{ik} \gamma^{jl} - \gamma^{il} \gamma^{jk} \right) \frac{D_k \phi \ast D_l \phi}{\| \phi \|^2} \]

\[ + \frac{1}{8} \left\{ \left( D^i \phi \pm \frac{i}{\sqrt{\gamma}} \epsilon_{ik} \gamma^{kl} D^l \phi \right) \left( D^j \phi \mp \frac{i}{\sqrt{\gamma}} \epsilon_{jm} \gamma^{mn} D^n \phi \right) \right\} \]

\[ + \left( D^i \phi \pm \frac{i}{\sqrt{\gamma}} \epsilon_{ik} \gamma^{kl} D^l \phi \right) \left( D^j \phi \mp \frac{i}{\sqrt{\gamma}} \epsilon_{jm} \gamma^{mn} D^n \phi \right) \]

\[ + \left( D^i \phi \pm \frac{i}{\sqrt{\gamma}} \epsilon_{ik} \gamma^{kl} D^l \phi \right) \left( D^j \phi \pm \frac{i}{\sqrt{\gamma}} \epsilon_{jm} \gamma^{mn} D^n \phi \right) \]

\[ = \frac{\kappa^2}{2e^2 \| \phi \|^2} \left[ B - \frac{e^3}{2\kappa^2} \| \phi \|^2 (\| \phi \|^2 - v^2) \right] \left[ B + \frac{e^3}{2\kappa^2} \| \phi \|^2 (\| \phi \|^2 - v^2) \right] \times \left( \gamma^{ij} \gamma^{kl} - \gamma^{ik} \gamma^{jl} - \gamma^{il} \gamma^{jk} \right) \frac{D_k \phi \ast D_l \phi}{\| \phi \|^2} \]

\[ + \frac{1}{8} \left\{ \left( D^i \phi \pm \frac{i}{\sqrt{\gamma}} \epsilon_{ik} \gamma^{kl} D^l \phi \right) \left( D^j \phi \mp \frac{i}{\sqrt{\gamma}} \epsilon_{jm} \gamma^{mn} D^n \phi \right) \right\} \]

\[ + \left( D^i \phi \pm \frac{i}{\sqrt{\gamma}} \epsilon_{ik} \gamma^{kl} D^l \phi \right) \left( D^j \phi \mp \frac{i}{\sqrt{\gamma}} \epsilon_{jm} \gamma^{mn} D^n \phi \right) \]

\[ + \left( D^i \phi \pm \frac{i}{\sqrt{\gamma}} \epsilon_{ik} \gamma^{kl} D^l \phi \right) \left( D^j \phi \pm \frac{i}{\sqrt{\gamma}} \epsilon_{jm} \gamma^{mn} D^n \phi \right) \}

\[ (5) \]

where \( \gamma^{ij} \) is inverse of the \( \gamma_{ij} \), \( \sqrt{\gamma} = \sqrt{\text{det} \gamma} \), and the magnetic field is defined by \( B = -\frac{e}{\sqrt{\gamma}} \frac{\epsilon_{ij}}{\sqrt{\gamma}} \partial_i A_j \).

We obtain the first-order Bogomol’nyi equations from Eq. (5)

\[ D_i \phi \pm \frac{i}{\sqrt{\gamma}} \epsilon_{ij} \gamma^{jk} D^k \phi = 0, \]

\[ B = \pm \frac{e^3}{2\kappa^2} \| \phi \|^2 (\| \phi \|^2 - v^2). \]

\[ (6) \]

When the scalar field \( \phi \) is decomposed into its phase and magnitude: \( \phi = \rho^2 e^{i\omega} \), the first of the self-duality equations \[ (6) \]

determines the gauge field:

\[ A_i = -\frac{1}{e} \partial_i w \pm \frac{1}{2e} \sqrt{\gamma} \epsilon_{ij} \gamma^{jk} \partial_j \ln \rho. \]

\[ (7) \]

We can see that the first equation expresses the spatial components of the gauge field \( A_i \) in terms of the scalar field. Substituting it into the second Bogomol’nyi equation \[ (6) \], the second equation reduces to a nonlinear elliptic equation for the scalar field density \( \rho \):

\[ \Box \ln \rho = \frac{e^4}{\kappa^2} \rho (\rho - v^2), \]

\[ (8) \]

where \( \Box = \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j \ln (\rho \phi^*)) \) is the Laplacian and this equation is not solvable, or even integrable.

### III. U(1) GAUGE POTENTIAL DECOMPOSITION OF SELF-DUALITY EQUATIONS

It is well known that the complex scalar field \( \phi \) can be looked upon as a section of a complex line bundle with base manifold \( M \). Denoting the charged Higgs complex scalar field \( \phi \) as

\[ \phi = \phi^1 + i\phi^2, \]

\[ (9) \]
where \( \phi^a (a = 1, 2) \) are two components of a two-dimensional vector field \( \vec{\phi} = (\phi^1, \phi^2) \) over the base space, one can introduce the two-dimensional unit vector

\[
n^a = \frac{\phi^a}{\|\phi\|}, \quad \|\phi\| = (\phi^* \phi)^{\frac{1}{2}}.
\] (10)

Let us consider the first of self-duality equations (11) (firstly, we choose the upper signs):

\[
D_1 \phi + i \sqrt{\gamma} \epsilon_{ijk} \gamma^{jk} D_k \phi = 0,
\] (11)

or,

\[
\begin{cases}
D_1 \phi + i \sqrt{\gamma} \gamma^{21} D_1 \phi + i \sqrt{\gamma} \gamma^{22} D_2 \phi = 0, \\
D_2 \phi - i \sqrt{\gamma} \gamma^{11} D_1 \phi - i \sqrt{\gamma} \gamma^{12} D_2 \phi = 0.
\end{cases}
\] (12)

Substituting Eq. (9) into the first of the above equations, we obtain two equations

\[
\partial_1 \phi^1 - \sqrt{\gamma} (\gamma^{21} \partial_1 \phi^2 + \gamma^{22} \partial_2 \phi^2) + e A_1 \phi^2 + \sqrt{\gamma} (\gamma^{21} e A_1 \phi^1 + \gamma^{22} e A_2 \phi^1) = 0,
\]

\[
\partial_1 \phi^2 + \sqrt{\gamma} (\gamma^{21} \partial_1 \phi^1 + \gamma^{22} \partial_2 \phi^1) - e A_1 \phi^1 + \sqrt{\gamma} (\gamma^{21} e A_1 \phi^2 + \gamma^{22} e A_2 \phi^2) = 0.
\] (13)

Making use of the above relations, we derive:

\[
\partial_1 \phi^* \phi - \partial_1 \phi \phi^* = -2 i e A_1 \|\phi\|^2 + i \sqrt{\gamma} \gamma^{22} (\partial_2 \phi^* \phi + \partial_2 \phi \phi^*) + i \sqrt{\gamma} \gamma^{21} (\partial_1 \phi^* \phi + \partial_1 \phi \phi^*),
\] (14)

\[
\sqrt{\gamma} \gamma^{22} (\partial_2 \phi^* \phi - \partial_2 \phi \phi^*) = -2 i e \sqrt{\gamma} \gamma^{22} A_2 \|\phi\|^2 - i (\partial_1 \phi^* \phi + \partial_1 \phi \phi^*) - i \gamma \gamma^{21} \gamma^{22} (\partial_1 \phi^* \phi + \partial_2 \phi \phi^*) - i \gamma \gamma^{21} (\partial_1 \phi^* \phi + \partial_1 \phi \phi^*).
\] (15)

To proceed, we need a fundamental identity-one that appear many times throughout our study in the gauge potential decomposition theory:

\[
\epsilon_{abc} n^a \partial_l n^b = \frac{1}{2 i \phi^* \phi} (\partial_l \phi^* \phi - \partial_l \phi \phi^*),
\] (16)

using this identity, Eq. (14) and Eq. (15) become

\[
e A_1 = -\epsilon_{abc} n^a \partial_1 n^b + \frac{1}{2} \sqrt{\gamma} \gamma^{22} \partial_2 \ln(\phi \phi^*) + \frac{1}{2} \sqrt{\gamma} \gamma^{21} \partial_1 \ln(\phi \phi^*),
\] (17)

\[
e A_2 = -\epsilon_{abc} n^a \partial_2 n^b - \frac{1}{2} \sqrt{\gamma} \gamma^{22} \partial_1 \ln(\phi \phi^*) - \frac{1}{2} \sqrt{\gamma} \gamma^{21} \partial_2 \ln(\phi \phi^*) - \frac{1}{2} \sqrt{\gamma} \gamma^{21} \gamma^{21} \partial_1 \ln(\phi \phi^*).\] (18)
Making use of the relation: $\det(\gamma^i)\det(\gamma_{jk}) = 1$, we can obtain a fundamental identity:

$$
-\frac{1}{2} \frac{1}{\sqrt{\gamma^{22}}} \partial_1 \ln(\phi \phi^*) - \frac{1}{2} \sqrt{\gamma^{21}} \gamma^{21} \partial_2 \ln(\phi \phi^*) - \frac{1}{2} \sqrt{\gamma^{12}} \partial_1 \ln(\phi \phi^*) - \frac{1}{2} \sqrt{\gamma^{11}} \partial_1 \ln(\phi \phi^*). \quad (19)
$$

Eqs. (11) and (12) can be rewritten as:

$$
e A_i = -\epsilon_{ab} n^a \partial_i n^b + \frac{1}{2} \epsilon_{ij} \sqrt{\gamma} \gamma^{jk} \partial_k \ln(\phi \phi^*). \quad (20)
$$

From the second of the equations (12), we also obtain the same conclusion.

Following the same discussion, we obtain the similar conclusion from $D_i \phi - i \sqrt{\gamma} \epsilon_{ij} \gamma^{jk} D_k \phi = 0$:

$$
e A_i = -\epsilon_{ab} n^a \partial_i n^b - \frac{1}{2} \epsilon_{ij} \sqrt{\gamma} \gamma^{jk} \partial_k \ln(\phi \phi^*). \quad (21)
$$

So, from the first self-duality equation $D_i \phi \pm i \sqrt{\gamma} \epsilon_{ij} \gamma^{jk} D_k \phi = 0$, we get

$$
A_i = -\frac{1}{e} \epsilon_{ab} n^a \partial_i n^b \pm \frac{1}{2e} \epsilon_{ij} \sqrt{\gamma} \gamma^{jk} \partial_k \ln(\phi \phi^*). \quad (22)
$$

It is obvious to see that we obtain one special form of the general $U(1)$ gauge potential decomposition in background curved spacetime.

As one has showed in [11], the $U(1)$ gauge potential in flat space can be decomposed by the Higgs complex scalar field $\phi$ as

$$
A_i = \beta \epsilon_{ab} \partial_i n^a n^b + \partial_i \lambda, \quad (23)
$$
in which $\beta$ is a constant and $\lambda$ is a phase factor. Comparing Eq. (22) with Eq. (23), we find that the term $\frac{1}{2e} \epsilon_{ij} \sqrt{\gamma} \gamma^{jk} \partial_k \ln(\phi \phi^*)$ on the RHS of Eq. (22) can not be expressed with the form of partial derivative, so, this term is not a phase factor denoting the $U(1)$ transformation in curved space.

### IV. TOPOLOGICAL QUANTIZATION OF SELF-DUAL CHERN-SIMONS VORTICES

Based on the decomposition of the gauge potential $A_i$ discussed in section III, in the following, we will immediately give a rigorous proof of magnetic flux quantization in background curved spacetime. Using the two-dimensional unit vector field (10), we can construct a topological current in curved spacetime:

$$
J^\mu = -\frac{1}{\sqrt{\gamma}} \epsilon^{\mu \nu \lambda} \partial_\nu A_\lambda = \frac{1}{e \sqrt{\gamma}} \epsilon^{\mu \nu \lambda} \epsilon_{ab} \partial_\nu n^a \partial_\lambda n^b, \quad (24)
$$
which is the special case of the general $\phi$-mapping topological current theory. Obviously, the current is conserved. Following the $\phi$-mapping theory, it can be rigorously proved that

$$J^\mu = \frac{2\pi}{e} \frac{1}{\sqrt{\gamma}} \delta^2(\vec{\phi}) D^\mu(\vec{\phi}) x, \quad (25)$$

where $D^\mu(\vec{\phi}) x = \frac{1}{2} \epsilon_{\mu\nu\lambda} c_{ab} \partial_\nu \phi^a \partial_\lambda \phi^b$ is the vector Jacobians. This expression provides an important conclusion: $J^\mu = 0$, if $\vec{\phi} \neq 0; J^\mu \neq 0$, if $\vec{\phi} = 0$. Suppose that the vector field $\vec{\phi}(\phi_1, \phi_2)$ possesses $l$ zeros, denoted as $z_i(i = 1, ..., l)$. According to the implicit function theorem, when the zero points $\vec{z}_i$ are the regular points of $\vec{\phi}$, that requires the Jacobians determinant

$$D(\vec{\phi}) x |_{z_i} \equiv D^0(\vec{\phi}) x |_{z_i} \neq 0. \quad (26)$$

The solutions of $\vec{\phi}(\phi_1, \phi_2) = 0$ can be generally obtained: $\vec{x} = \vec{z}_i(t), i = 1, 2, \cdots, l, x^0 = t$. It is easy to prove that

$$D^\mu(\vec{\phi}) x |_{z_i} = D(\vec{\phi}) x |_{z_i} \frac{dx^\mu}{dt}. \quad (27)$$

According to the $\delta$-function theory and the $\phi$-mapping theory, one can prove that

$$J^\mu = \frac{2\pi}{e} \frac{1}{\sqrt{\gamma}} \sum_{i=1}^{l} \beta_i \eta_i \delta^2(\vec{x} - \vec{z}_i) \frac{dx^\mu}{dt} |_{z_i}, \quad (28)$$

in which the positive integer $\beta_i$ is the Hopf index and $\eta_i = \text{sgn}(D(\vec{\phi}/x) z_i) = \pm 1$ is the Brouwer degree. Then the density of topological charge can be expressed as

$$J^0 = \frac{2\pi}{e} \frac{1}{\sqrt{\gamma}} \sum_{i=1}^{l} \beta_i \eta_i \delta^2(\vec{x} - \vec{z}_i). \quad (29)$$

From Eq. (24), it is easy to see that

$$J^0 = -\epsilon^{ij} \sqrt{\gamma} \partial_i A_j = B. \quad (30)$$

So, the total charge of the system can be written as

$$\Phi = \int B \frac{1}{\sqrt{\gamma}} dx^2 = \int J^0 \frac{1}{\sqrt{\gamma}} dx^2 = \Phi_0 \sum_{i=1}^{l} \beta_i \eta_i, \quad (31)$$

where $\Phi_0 = \frac{2\pi}{e}$ is the unit magnetic flux in curved spacetime, and the topological index $n$ in Eq. (31) has the following expression

$$n = \sum_{i=1}^{l} \beta_i \eta_i. \quad (32)$$

It is obvious to see that there exist $l$ isolated vortices in which the $i$th vortex possesses charge $\frac{2\pi}{e} \beta_i \eta_i$. The vortex corresponds to $\eta_i = +1$, while the antivortex corresponds to $\eta_i = -1$. One can conclude that vortex configuration given in Eq. (31) is a multivortex solution which possesses the inner structure described by expression (32).
In Sec. II the second Bogomol’nyi self-duality equation (8) is meaningless, when the field \( \phi = 0 \). Moreover, no exact solutions are known for this equation. In this section, based on the decomposition of \( U(1) \), we obtain the precise self-dual equation with topological term.

Firstly, based on the special form of the general \( U(1) \) decomposition of gauge potential (22), we get:

\[
B = \frac{1}{e} \frac{1}{\sqrt{\gamma}} \epsilon^{ij} \epsilon_{ab} \partial_i n^a \partial_j n^b \mp \frac{1}{2e} \epsilon^{ij} \epsilon_{jk} \frac{1}{\sqrt{\gamma}} \partial_l (\sqrt{\gamma} n^l \ln(\phi \phi^*)).
\] (33)

Substituting above equation into the second Bogomol’nyi self-duality equation (6), we obtain:

\[
\frac{1}{2} \Box \ln(\phi \phi^*) = \frac{e^4}{2 \kappa^2} ||\phi||^2 (||\phi||^2 - \nu^2) \mp \frac{1}{\sqrt{\gamma}} \epsilon^{ij} \epsilon_{ab} \partial_i n^a \partial_j n^b.
\] (34)

From Eqs. (24) and (25), we obtain the \( \delta \)-function form of topological term

\[
\frac{1}{\sqrt{\gamma}} \epsilon^{ij} \epsilon_{ab} \partial_i n^a \partial_j n^b = eJ^0 = \frac{2 \pi}{\sqrt{\gamma}} \delta^2(\vec{\phi}) D(\frac{\phi}{x}).
\] (35)

The second self-dual equation in Eq. (6) then can reduce to a nonlinear elliptic equation for the scalar field density \( \rho = \phi \phi^* \)

\[
\Box \ln \rho = \frac{e^4}{\kappa^2} \rho (\rho - \nu^2) \mp \frac{4 \pi}{\sqrt{\gamma}} \delta^2(\vec{\phi}) D(\frac{\phi}{x}).
\] (36)

Comparing Eq. (36) with Eq. (8), one can find that the conventional self-dual equation (8), in which the topological term has been ignored, is meaningless when the field \( \phi = 0 \); we get the self-dual equation with topological term, which is meaning when the field \( \phi = 0 \). Obviously, topological term is very important to the inner topological structure of the self-duality equations.

VI. CONCLUSION

Our investigation is based on the connection between the self-dual equation of Chern-Simons Higgs model coupled to background gravity and the \( U(1) \) gauge potential decomposition theory and \( \phi \)-mapping theory. First, we directly obtain one special form of \( U(1) \) gauge potential decomposition from self-duality equations. Making use of the decomposition, we give a rigorous proof of magnetic flux quantization in background curved spacetime. Moreover, we obtain the inner topological structure of the Chern-Simons vortex, the multicharged vortex has been found at the
Jacobian determinate $D(\phi/x) \neq 0$. It is also showed that the charge of the vortices is determined by Hopf indices and Brouwer degrees. Second, we establish the rigorous self-duality equations with topological term for the first time, in which the topological term is the density of topological charge of vortex: \( \frac{1}{2\pi} \delta^2(\phi) D(\phi) = 2eJ^0 \). In contrast with the conventional self-duality equation \( \mathbb{S} \), one can see that the self-duality equation with topological term is valid when the field $\phi = 0$; topological term vanishes and the self-duality equation becomes Eq. \( \mathbb{S} \) when the field $\phi \neq 0$.

VII. ACKNOWLEDGEMENTS

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[1] J. Hong, Y. Kim, and P. Y. Pac, Phys. Rev. Lett. 64, 2230 (1990); R. Jackiw and E. J. Weinberg, Phys. Rev. Lett. 64, 2234 (1990).
[2] R. Jackiw, K. Lee, and E. J. Weinberg, Phys. Rev. D 42, 3488 (1998).
[3] G. Dunne, Self-dual Chern-Simons Theories (Springer, Berlin, 1995).
[4] J. Schiff, J. Math. Phys. 32 (1991), 753; A. Comtet and A. Khare, Phys. Lett. B 278 (1992), 236;
[5] S. Kim and Y. Kim, J. Math. Phys. 43 2355 (2002).
[6] P. Valtancoli, Int. J. Mod. Phys. A 18 (1992), 4335; D. Cangemi and C. Lee, Phys. Rev. D 46 (1992), 4768; G. Clement, Phys. Rev. D 54 (1996) 1844.
[7] Y. S. Duan, SLAC-PUB-3301(1984); Y. S. Duan, L. B. Fu and G. Jia, J. Math. Phys. 41, 4379 (2000).
[8] Y. Duan, H. Zhang and S. Li, Phys. Rev. B58, 125 (1998).
[9] Y. S. Duan and M. L. Ge, Sci. Sin. 11, 1072 (1979); C. Gu, Phys. Rep. 80, 251 (1981).
[10] Y. S. Duan and S. L. Zhang, Int. J. Eng. Sci. 28, 689 (1990); 29, 1593 (1991); 30, 153 (1992).
[11] Y. S. Duan, in Proceedings of the Symposium on Yang-Mills Gauge Theories, Beijing, 1984; Y. S. Duan, G. H. Yang, and Y. Jiang, Gen. Relativ. Gravit. 29, 715 (1997).
[12] B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, Modern Geometry and Applications, Part II (Springer-Verlag, New York, 1984).
[13] E. Goursat, A course in Mathethematical Analysis, translated by E. R. Hedrick (Dover, New York, 1904), Vol. 1.
[14] J. A. Schouton, Tensor Analysis for Physicists (Clarendon, Oxford, 1951).
[15] H. Hopf, Math. Ann. 96, 209 (1929).