Matrix product states and the decay of quantum conditional mutual information

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A uniform matrix product state defined on a tripartite system of spins, denoted by ABC, is shown to be an approximate quantum Markov chain when the size of subsystem B, denoted |B|, is large enough. The quantum conditional mutual information (QCMI) is investigated and proved to be bounded by a function proportional to exp(-q(|B| - K) + 2K ln |B|), with q and K computable constants. The properties of the bounding function are derived by a new approach, with a corresponding improved value given for its asymptotic decay rate q. We show the improved value of q to be optimal. Numerical investigations of the decay of QCMI are reported for a collection of matrix product states generated by selecting the defining isometry with respect to Haar measure.

I. INTRODUCTION

In this article, we consider a one dimensional, homogeneous spin chain that is subdivided into a tripartite system with regions A and C separated by “buffer region” B. The strong subadditivity of quantum entropy theorem states that for any density operator ρABC on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, the quantum conditional mutual information (QcMI) of subsystems A and C given B, denoted I(A : C|B), is non-negative.

$I(A : C|B) := S(\rho_{AB}) + S(\rho_{CB}) - S(\rho_{ABC}) - S(\rho_B) \geq 0,$

where $S(\rho_B) = -\text{tr}(\rho_B \ln \rho_B)$ denotes the quantum entropy of the region B. The states $\rho_{ABC}$ for which equality holds are the quantum Markov chains and these have the property that they may be recovered exactly from a reduced density operator; that is, there exists a completely positive map which induces the transformation $\rho_{AB} \mapsto \rho_{ABC}$ by acting on the spins in the separating region B only. While exact recovery is associated with the quantum states for which $I(A : C|B) = 0$, approximate recovery requires investigation of states for which QCMI may be small $I(A : C|B) < g(\varepsilon)$, where $g$ is a known positive function and $\varepsilon := \left\| \rho_{ABC} - \rho_{ABC}^{\text{approx}} \right\|_1$ determines the recovery error for the recovery map $\rho_{ABC}^{\text{approx}}$ with domain the support of the reduced density operator $\rho_{BC}$. Conversely, there exists a so-called universal recovery map $\rho_{BC}^{\text{approx}}$, which guarantees that when $I(A : C|B)$ is small, the recovery error is
developed.

The recovery of quantum states from a quantum memory that has been locally corrupted in region C, say, is thus bounded by the QCMI $I(A : C|B)$ when the process of encoding implementing the universal recovery map $\rho_{ABC}^{\text{approx}}$. It follows, in the case that $I(A : C|B)$ is a rapidly decaying function of the size of the buffer region, the number of spins |B|, that the quantum state may be globally recovered by experiments on the well-localized interface region B and |C|.

The many-body ground states of spin lattices described by local gapped Hamiltonians are well approximated by matrix product states (MPS) in one dimension and their generalization, the projected entangled pair states (PEPS) in higher dimensions. The MPS first effectively appeared as finitely correlated states (FCS) and were defined on infinite one-dimensional spin chains. (More precisely, MPS correspond to the purely generated FCS (pgFCS), which comprise a subclass of FCS.) The MPS provide an alternative description of the previously developed density matrix renormalization group algorithm and are also useful in time dependent studies of weakly entangled systems. In one dimension, MPS may be generalized to matrix product density operators (MPDO) which, in the translationally invariant infinite case correspond to the FCS. The MPDO may be used to approximate thermal states of local gapped Hamiltonians. Controlled experimental preparation techniques for MPS have been proposed in the literature.

Given the significance of ground and thermal states of local gapped Hamiltonians, their quantum-information theoretic properties are of great interest. As are the properties of their approximants – the MPS and MPDO, which are often easier to study, given their more explicit structure.

In a recent paper Ref. 33 we proved that the pgFCS (infinite uniform MPS (uiMPS), with reduced density operators $\rho_{ABC}$, are approximate quantum Markov chains for the one-dimensional configuration space $ABC$ discussed above. Related to this result, the QCMI $I(A : C|B)$ decays and is bounded by a (sub)exponentially decreasing function in the size of the separating region $B$. The goal of the current paper is to reexamine this problem and report an improved prediction for the exponential decay rate, differing by a factor of four from our earlier prediction. We also present numerical experiments in support of this result.

Our method of estimating the QCMI builds on earlier works Refs. 18, 20, 21 and 29 and our work Ref. 33 where we construct a quantum Markov chain $\tilde{\rho}_{ABC}$ approximating $\rho_{ABC}$. Specifically, the method is based on (i) bounding the QCMI by the trace distance between the exact density operator $\rho_{ABC}$ and its approximant $\tilde{\rho}_{ABC}$, and (ii) bounding this trace distance by the distance in diamond norm between the induced quantum channels $\rho_{BC}$ and $\tilde{\rho}_{BC}$ (our notation is explained below). In our earlier work Ref. 33 the combination of Alićki-Fannes inequality and Strinseoppen Continuity theorem provided (i) a QCMI bound linear in trace distance, and (ii) the trace distance bounded by the square root of the channel distance. In the present work the QCMI bound is improved to a quadratic dependence on trace distance, while the lat-
ter is bounded by a linear function of the channel distance. To achieve these improvements, use of the Alicki-Fannes inequality has been replaced by a bound based on the Taylor expansion of a logarithmic function representation of the density operator, with estimate of the remainder. The continuity of Stinespring dilation is no longer employed and instead a construction used in the proof of Choi’s theorem takes advantage of the nearness of the quantum channels, and provides a sharper tool in this context. Overall we go from a bound on QCMI which had square root dependence in the quantum channel distance, to one with quadratic dependence. This leads to a factor of four change in our predicted decay rate, a result which appears to be corroborated by our numerical experiments, and present numerical results for comparison with Theorem III.1 in Section V. Section VI contains concluding remarks. A series of appendices contains technical information.

II. THEORETICAL FORMULATION

A pure state vector for a system of \( n \) spins, each associated Hilbert space \( \mathcal{H}_s \) and orthonormal basis \( \{ s \} : s = 1, 2, \ldots, d_s \}, may be expressed in the form

\[
|\Psi\rangle = \sum_{s_1, s_2, \ldots, s_n=1}^{d_s} \langle L|M^{s_n} \ldots M^{s_2} M^{s_1}|R\rangle |s_n\rangle \otimes \cdots \otimes |s_2\rangle \otimes |s_1\rangle.
\]

(1)

Without loss of generality, we may assume that \(|\Psi\rangle\) is in canonical form, so that \(M^i\) satisfies the condition diagrammatically depicted in Fig 1(a), and by means of the equation

\[
\sum_{s=1}^{d_s} M^{s_i} M^i = \mathbb{1}_{M},
\]

(2)

which may be achieved by a suitable definition of \(|L\rangle\) and \(|R\rangle\). Another, more compact, way to represent a MPS is by introducing the isometry \(V : \mathcal{H}_M \rightarrow \mathcal{H}_R \otimes \mathcal{H}_M\),

\[
V = \sum_{s=1}^{d_s} |s\rangle \otimes M^s,
\]

satisfying \(V^\dagger V = \mathbb{1}_{d_M}\). With the help of \(V\), we may express

\[
\rho_n = \sum_{s_1, s_2, \ldots, s_n=1}^{d_s} \text{tr}_M(M^{s_n} \ldots M^{s_2} M^{s_1} \sigma^{s_1} \ldots \sigma^{s_2} \ldots \sigma^{s_n}) |s_n\rangle \langle s_n| \otimes \cdots \otimes |s_2\rangle \langle s_2| \otimes |s_1\rangle \langle s_1|,
\]

(3)

where \(\text{tr} \rho_n = 1\), and \(\sigma\) is a fixed point of the quantum channel (completely positive trace-preserving (CPTP) map) \(\mathcal{E}^\sigma\).
\[ E(j'|i') = \begin{pmatrix} \sigma_j \sigma_{j'} \end{pmatrix} \begin{pmatrix} M \end{pmatrix} = \begin{pmatrix} \sigma_j \sigma_{j'} \end{pmatrix} \begin{pmatrix} M \end{pmatrix} \]

Figure 1. Diagrammatic depictions of: (a) the iuMPS in canonical form, as expressed in (2); (b) the density operator \( \sigma \), the fixed point of the quantum channel \( \sigma' \), defined in (3); (c) the transfer matrix \( E \) defined in (4); (d) the (right) eigenvalue equation (5) for \( E \).

\[ \mathcal{B}(\mathcal{H}_M) \to \mathcal{B}(\mathcal{H}_M), \quad \mathcal{E}(X) = \sum_{s=1}^{d} M^s X M^s. \]  

The latter is conveniently represented as a linear operator and defines the transfer matrix \( E \in \mathbb{C}^{d_M \times d_M} \),

\[ E = \sum_{s=1}^{d} M^s \otimes \bar{M}^s, \]  

where \( \bar{M}^s \) is the complex conjugate of \( M^s \). Without loss of generality, the density matrix \( \sigma \) may be considered to be full-rank, which we will use in our derivations. The transfer matrix is depicted in Fig. 1(c), with the matrix elements \( E_{jj',ii'} = \langle j| \otimes \langle i'| E |i \rangle \otimes |i'\rangle \). To uniquely define an iuMPS we must specify a family of \( d_M \times d_M \) matrices \( \{M^s\}_{s=1}^{d} \) (or, alternatively, the isometry \( V \)), and the fixed point density operator \( \sigma \) that satisfies \( \mathcal{E}(\sigma) = \sigma \) (refer to Fig. 1(b)).

The eigenvalue problem, depicted in Fig. 1(d), and given by

\[ E|X_k\rangle = \nu_k|X_k\rangle, \quad k = 1, \ldots, k_{\text{max}}, \]  

Figure 3. The tensor \( \bar{M}_B \) of the dimension \( d_M \times d_M \times d_M \), and its representation as the matrix elements of the collection of \( d_M \times d_M \) matrices \( \bar{M}_B^{s_1,\ldots,s_B} \), where \( s_1, s_2, \ldots, s_B = 1, 2, \ldots, d \).

...
where $|X_k\rangle \in \mathcal{H}_M \otimes \mathcal{H}_M = C^{d_M} \otimes C^{d_M}$, determines the spectrum of the transfer matrix. Note that since $E$ is generally non-Hermitian, then $\lambda_{\text{max}} \leq d_M^2$. Nonetheless, the equation (9) provides all eigenvalues of $E$, which is sufficient at this step. Alternatively, we may formulate the eigenvalue problem for the quantum channel $\mathcal{E}$,

$$ \mathcal{E}(X_k) = v_k X_k, \quad k = 1, \ldots, k_{\text{max}}, $$

where $X_k \in \mathcal{B}$. For our purposes the most significant spectral parameter characterizing the transfer matrix is the largest absolute value of its non-peripheral eigenvalues,

$$ v_{\text{gap}} = \max_{|v_k| < 1} |v_k|. $$

Note that $v_{\text{gap}} < 1$ for any $E$, even if several eigenvalues of $E$ have absolute value one. For an injective iuMPS, the quantity $1 - v_{\text{gap}}$ is the usually defined spectral gap, which motivates our choice of the subscript in $v_{\text{gap}}$

Using the notation introduced in (5) we may express $\rho_n$ as

$$ \rho_n = \sum_{j_1, j_2, \ldots, j_n = 1}^{d_M} (|M_{j_1} \otimes \bar{M}_{j_1}| \cdots (M_{j_n} \otimes \bar{M}_{j_n}) (M_{j_1} \otimes \bar{M}_{j_1}) |\sigma\rangle |s_n\rangle \cdots |s_2\rangle |s_1\rangle |s_1\rangle |s_1\rangle, \quad (7) $$

where $|+\rangle = \sum_{i = 1}^{d_M} |i\rangle \otimes |i\rangle$ and $|\sigma\rangle = \sum_{i = 1}^{d_M} |i\rangle \otimes |i\rangle$.

In this paper we divide $n$ spins into three continuous subregions $A$, $B$, and C, and denote the reduced density operator $\rho_n$ as $\rho_{ABC}$. A diagrammatic representation of $\rho_{ABC}$ is given in Fig. 2(c). For $\rho_{ABC}$ we investigate the behavior of $I(A : C|B)$ as a function of the size of the buffer region $B$, denoted by $|B|$.

### III. MAIN THEOREM AND ITS PROOF

Consider the configuration $ABC$ shown in Fig 2(a) and (b), and prepared in an infinite uniform matrix product state, the quantum conditional mutual information (QCMI) obeys the bound

$$ I(A : C|B) \leq Q e^{-q|\langle B|K\rangle|+2K \ln|B|}, $$

for $|B|$ large enough. Here $q = -2 \ln v_{\text{gap}}, K$ is a non-negative integer, and $Q$ does not depend on $|B|$. The rate $q = -2 \ln v_{\text{gap}}$ cannot be further improved.

Compared to the similar bound in Ref. 33 with $q = -\frac{1}{2} \ln v_{\text{gap}}$, the value predicted here is $q = -2 \ln v_{\text{gap}}$.

**Remarks:**

1. For a given iuMPS $\rho_{ABC}$, the non-negative integer $K$ may be extracted from the Jordan decomposition of the corresponding transfer matrix [3]: $K + 1$ is the largest dimension of the Jordan blocks corresponding to eigenvalues of magnitude $v_{\text{gap}}$. We provide the detailed computation of $K$ in Appendix III.

2. We prove that the decay rate $q = -2 \ln v_{\text{gap}}$ cannot be further improved in Appendix III. There we provide an example of an injective iuMPS (with $K = 0$) for which QCMI may be explicitly calculated to be $I(A : C|B) = O(e^{-q|B|})$.

We prove Theorem III.1 building on the derivation of Theorem II.1 in Ref. 33. For simplicity, in subsections III A and III C we prove Theorem III.1 for the case of injective iuMPS, i.e. the case for which the transfer matrix [5] has a unique eigenvector $|\sigma\rangle$ with eigenvalue of magnitude 1. The generalization to the case of an arbitrary iuMPS is straightforward, but more technical, and will be discussed in subsection III D.

The key idea of the proof is identical to the one used in Theorem II.1 of Ref. 33 and involves the construction of a density operator $\tilde{\rho}_{ABC}$, that is simultaneously a quantum Markov chain and a good approximation to $\rho_{ABC}$ in trace norm. By comparison with Ref. 33 however, we use different methods to estimate $I(A : C|B)$ and $\|\rho_{ABC} - \tilde{\rho}_{ABC}\|_1$, which allow for a tighter bound on $I(A : C|B)$. The density operator $\tilde{\rho}_{ABC}$ is illustrated diagrammatically in Fig 2(d) with the related complex tensor $M_{\rho}$ of dimensions $d_M \times d_M \times d_M$, depicted in Fig 3.

Throughout the Subsections III A, III C, and III D we exploit the properties of the quantum channel $\mathcal{E} : \mathcal{B}(\mathcal{H}_M) \rightarrow \mathcal{B}(\mathcal{H}_M)$ (and its matrix representation $\hat{E} : \mathcal{H}_M \otimes \mathcal{H}_M \rightarrow \mathcal{H}_M \otimes \mathcal{H}_M$), which corresponds to the peripheral spectrum of $\mathcal{E}$. It is straightforward to define $\hat{E}$, by first defining $\hat{E}$ through

$$ \hat{E} = \text{PEP,} $$

where $P$ is the projector onto the subspace span$\{|u_i\rangle | E|u_i\rangle = v_i|u_i\rangle, \{v_i| = 1\}$. The map $\hat{E}$ is then defined by

$$ \langle \hat{E}(i)\rangle |(i')\rangle = \langle (i)\rangle \otimes \langle (i')\rangle |(i)\rangle \otimes \langle (i')\rangle. $$

It may be shown that the map $\hat{E}$ is indeed a quantum channel (see Lemma III.2 in Ref. 33).

The key properties that make $\hat{E}$ (and $\hat{E}$) useful, are the bounds on $\|E^n - E^n\|$ derived in Theorem III.2 of Ref. 39 and satisfied for large enough $n$ (specified in Appendix III).

$$ c_1 n^K v_{\text{gap}} - K \leq \|E^n - E^n\| $$

$$ \leq c_2 n^K v_{\text{gap}} - K, $$

where $E^n$ is the $n$-fold convolution of $E$. The quantity $\|E^n - E^n\|$ is useful because it is equal to $\|\rho_{ABC} - \rho_{ABC}\|_1$ for $\rho_{ABC}$ and $\rho_{ABC}$ the density operators of $\rho_{ABC}$ and $\rho_{ABC}$, respectively.
where \( n \in \mathbb{N}, K \) is a non-negative integer, and \( c_1 \) and \( c_2 \) are some constants. In Appendix F we obtain the values of \( c_1, c_2 \), and \( K \) from the Jordan decomposition of \( E \).

A. A trace distance bound on QCMI.

In this subsection we bound \( I(A : C | B) \) in terms of \( \| \rho_{ABC} - \tilde{\rho}_{ABC} \|_1 \), with the assumption that \( \tilde{\rho}_{ABC} \) is a quantum Markov chain. We will present the construction of \( \tilde{\rho}_{ABC} \) that is indeed a quantum Markov chain in Subsection III B. Here we will derive only the leading contribution to the bound of QCMI. We delegate the careful treatment of subleading contributions that arise from the expansion (11) below to Appendix B.

In QCMI

\[
I(A : C | B) = \text{tr}[\rho_{ABC} (\ln\rho_{ABC} + \ln\rho_B - \ln\rho_A - \ln\rho_{BC})],
\]

we express

\[
\rho_{ABC} = \rho_{ABC} + \Delta \rho_{ABC},
\]

with similar expressions for the marginals of \( \rho_{ABC} \) obtained by suitable partial traces. Then, using the integral representation for the logarithm of an operator

\[
\ln\rho = \int_0^\infty ds \left( \frac{1}{1 + s} \right),
\]

we expand (10) to second order in \( \Delta \rho_{ABC} \) and its marginals. In particular for a typical density operator, and suppressing indices, we have

\[
\ln(\rho + \Delta \rho) = \ln\rho + \int_0^{\infty} ds \left( \frac{1}{\rho + s\|} \Delta \rho \frac{1}{\rho + s\|} \right)
\]

we obtain the asymptotic bound

The first order contribution has a term that vanishes for the same reason,

\[
\text{tr}[\Delta \rho_{ABC} (\ln\rho_{ABC} + \ln\rho_B - \ln\rho_A - \ln\rho_{BC}) = 0.
\]

The other terms of first order in \( \Delta \rho \) are all proportional to an expression of the following kind,

\[
\text{tr} \left( \rho \int_0^{+\infty} ds \frac{1}{\rho + s\|} \Delta \rho \frac{1}{\rho + s\|} \right) = \text{tr} \left( \rho \int_0^{+\infty} ds \frac{1}{\rho + s\|} \Delta \rho \frac{1}{\rho + s\|} \Delta \rho \right) = \text{tr}(\rho^{-1} \Delta \rho) = \text{tr}(\Delta \rho) = 0.
\]

since \( \text{tr}\Delta \rho_{ABC} = \text{tr}\rho_{ABC} - \text{tr}\tilde{\rho}_{ABC} = 0 \) (hence, \( \text{tr}\Delta \rho_{AB} = 0 \), etc.).

Thus, the leading non-vanishing terms in the expansion of \( I(A : C | B) \) are of the second order in \( \Delta \rho \), coming from the terms of the type

\[
\text{tr} \left( \rho \int_0^{+\infty} ds \frac{s}{\rho + s\|} \Delta \rho \frac{1}{\rho + s\|} \Delta \rho \frac{1}{\rho + s\|} \right),
\]

which can be estimated as

\[
\text{tr} \left( \int_0^{+\infty} ds \frac{s}{\rho + s\|} \Delta \rho \frac{1}{\rho + s\|} \Delta \rho \frac{1}{\rho + s\|} \right) \leq \int_0^{+\infty} ds \left\| \frac{s}{\rho + s\|} \Delta \rho \right\|_1 \leq \frac{\|\Delta \rho\|_1^2}{2\lambda_{\text{min}}},
\]

where \( \lambda_{\text{min}} \) is the smallest non-zero eigenvalue of \( \rho \).

Expansion of each of the four terms in (10) contributes a term of the type (13) to the bound of \( I(A : C | B) \). Moreover, for each of the four terms in (10), we need to account for the contribution emerging from the remainder term in the expansion (11). This contribution, for large enough \( |B| \), may also be bounded by the contribution resulting from terms of the second order (13), as shown on Appendix B. Thus, combining all contributions together, for \( |B| \) large enough, we can guarantee the asymptotic bound

\[
I(A : C | B) \leq \frac{4}{\lambda_{\text{min}}} \|\Delta \rho_{ABC}\|_1^2.
\]
We provide the detailed derivation of the bound \(^{14}\) in Appendix \(^{3}\).

**Remark:** The inequality \(^{25}\) is an upper bound on QCMI quadratic in \(\|\Delta\rho_{ABC}\|_1\), in contrast to the upper bound linear in \(\|\Delta\rho_{ABC}\|_1\) that we used in Ref. \(^{33}\). In Subsection IIIC we will see that this improved bound is responsible for changing the rate of exponential decay of the QCMI bounding function by a factor of 2, compared to that in Theorem II.1 in Ref. \(^{33}\).

### B. Construction of \(\tilde{\rho}_{ABC}\) and evaluation of \(\lambda_{\text{min}}\).

In this subsection we implicitly construct the tensor \(\tilde{M}_B\), such that \(\tilde{\rho}_{ABC}\) defined in Fig.2 becomes a quantum Markov chain, and we show that its smallest non-zero eigenvalue satisfies (for injective iuMPS)

\[
\lambda_{\text{min}} = \sigma_{\text{min}}^2. \tag{15}
\]

We represent the tensor \(\tilde{M}_B\) as a collection of \(d_s^{|B|}\) complex \(d_M \times d_M\) matrices \(\tilde{M}_B^{s_1 \ldots s_{|B|}}\), where \(s_k = 1, 2, \ldots, d_s\) for \(k = 1, 2, \ldots, |B|\). The matrices \(\tilde{M}_B^{s_1 \ldots s_{|B|}}\) are defined by their matrix elements in Fig. 3.

For an injective iuMPS, the quantum channel \(\tilde{E}\) has the form \(^{20}\)

\[
\tilde{E}(X) = \text{tr}(X)\sigma.
\]

Its matrix representation \(\tilde{E}\) is then obtained with the use of \(\mathbf{S}\).

\[
\tilde{E} \equiv \tilde{E}^n = |\sigma\rangle\langle +|.
\]

In Section IV.A of Ref. \(^{33}\) it was proved, that if we impose on \(\tilde{M}_B\) the condition

\[
\sum_{s_1, \ldots, s_{|B|}} d_s \tilde{M}_B^{s_1 \ldots s_{|B|}} \sigma_{s_1} \cdots \sigma_{s_{|B|}} = |\sigma\rangle\langle +|, \tag{16}
\]

or, equivalently, the condition

\[
\tilde{E}^{(B)}(X) = \sum_{s_1, \ldots, s_{|B|}} d_s \tilde{M}_B^{s_1 \ldots s_{|B|}} X \tilde{M}_B^{s_1 \ldots s_{|B|}} = \text{tr}(X)\sigma, \tag{17}
\]

then the induced \(\tilde{\rho}_{ABC}\) is a quantum Markov chain. The \(d_M \times d_M\) complex matrices \(\tilde{M}_B^{s_1 \ldots s_{|B|}}\) and \(\tilde{M}_B^{s_1 \ldots s_{|B|}}\) are the element-wise complex conjugate and the complex conjugate transpose of \(\tilde{M}_B^{s_1 \ldots s_{|B|}}\), respectively. We will now show that the conditions \(\text{(16)}\) and \(\text{(17)}\) result in the desired bound \(^{26}\).

Notice that the isometries \(\tilde{M}_B\) represent Stinespring dilations \(^{56}\) for the quantum channels \(\tilde{E}^{(B)}\) and \(\tilde{E}^{(B)}\), respectively, with \(\mathcal{H}_B\) serving as the dilating Hilbert space, i.e.,

\[
\tilde{E}^{(B)}(X) = \text{tr}(V_B X \tilde{V}_B^\dagger), \tag{18}
\]

\[
\tilde{E}^{(B)}(X) = \text{tr}(\tilde{V}_B X \tilde{V}_B^\dagger) = \text{tr}(X)\sigma. \tag{19}
\]

Moreover, we can conveniently reexpress \(\tilde{M}_B\) in terms of the isometry \(\tilde{V}_B\) as

\[
\tilde{M}_B^{s_1 \ldots s_{|B|}} = (\langle s_1 | \otimes \langle s_2 | \otimes \cdots \otimes \langle s_{|B|} |) \tilde{V}_B, \tag{20}
\]

which allows us to work with \(\tilde{V}_B\) satisfying \(\text{(18)}\) instead of \(\tilde{M}_B\) satisfying \(\text{(17)}\).

We notice that we can express \(\rho_{ABC}\) and \(\tilde{\rho}_{ABC}\) in terms of \(V_B\) and \(\tilde{V}_B\) as

\[
\rho_{ABC} = \text{tr}_M(V_C V_B A \sigma A V_B^\dagger V_C^\dagger),
\]

\[
\tilde{\rho}_{ABC} = \text{tr}_M(V_C \tilde{V}_B A \sigma A \tilde{V}_B^\dagger V_C^\dagger).
\]

Now we estimate \(\lambda_{\text{min}}\) with the help of a purification argument. The density operator \(\tilde{\rho}_{ABC}\) can be purified with the use of the auxiliary space \(\mathcal{H}_{M'}\) (note \(\mathcal{H}_{M'} = \mathcal{H}_M\)).

\[
\tilde{\rho}_{ABC} = \text{tr}_{MM'}((V_C \tilde{V}_B A \otimes 1_{M'}) |\sqrt{\sigma}\rangle \langle \sqrt{\sigma}| \\
\times (V_A^\dagger V_B^\dagger V_C^\dagger \otimes 1_{M'})),
\]

where \(|\sqrt{\sigma}\rangle = \sum_i |\sqrt{\sigma}|i\rangle \otimes |i\rangle \in \mathcal{H}_M \otimes \mathcal{H}_{M'}\), and \(V_A, V_B,\) and \(V_C\) act on the space \(\mathcal{H}_{M'}\). This implies that \(\tilde{\rho}_{ABC}\) has the same spectrum as

\[
\text{tr}_{ABC}((V_C \tilde{V}_B A \otimes 1_{M'}) |\sqrt{\sigma}\rangle \langle \sqrt{\sigma}| \\
\times (V_A^\dagger V_B^\dagger V_C^\dagger \otimes 1_{M'})) = (\tilde{E}^{(C)} |\sqrt{\sigma}\rangle \langle \sqrt{\sigma}| \\
\times (\tilde{E}^{(B)} |\sqrt{\sigma}\rangle \langle \sqrt{\sigma}|),
\]

and hence, assuming \(\dim \mathcal{H}_{ABC} \geq \dim (\mathcal{H}_M \otimes \mathcal{H}_{M'}) = d_M^2\).
that
\[ \tilde{\rho}_{ABC} = J(\rho^{[C]} \odot \rho^{[B]} \odot \rho^{[A]} \otimes \text{id}_{M'}) \left( (\sqrt{\sigma})(\sqrt{\sigma}) \right) J^\dagger, \]  
(21)
where \( J : \mathcal{H}_M \otimes \mathcal{H}_M' \rightarrow \mathcal{H}_{ABC} \) is an isometry.

Now notice that \( \tilde{\rho} \circ \rho = \tilde{\rho} \circ \rho = \rho^2 \), hence
\[ \tilde{\rho}_{ABC} = J(\rho^{[ABC]} \otimes \text{id}_{M'}) \left( (\sqrt{\sigma})(\sqrt{\sigma}) \right) J^\dagger, \]  
(22)
Moreover, since any of the regions \( ABC, AB, BC, \) or \( B \) contains the subregion \( B \), then any \( \tilde{\rho}_R \) may be expressed as
\[ \tilde{\rho}_R = J(\rho^{[R]} \otimes \text{id}_{M'}) \left( (\sqrt{\sigma})(\sqrt{\sigma}) \right) J^\dagger, \]  
(23)
where \( R \) may be any of \( ABC, AB, BC \) or \( B \), and \( J : \mathcal{H}_M \otimes \mathcal{H}_M' \rightarrow \mathcal{H}_R \) is an isometry appropriate to region \( R \).

For an injective iuMPS, \( \tilde{\rho} : \mathcal{B}(\mathcal{H}_M) \rightarrow \mathcal{B}(\mathcal{H}_M) \) acts as \( \tilde{\rho}(X) = \text{tr}(X)\sigma \), and hence
\[ \tilde{\rho}_R = J\sigma \otimes \sigma J^\dagger, \]  
(24)
which implies that the smallest eigenvalue of \( \rho_R \) is \( \lambda_{\text{min}} = \sigma^{2}_{\text{min}} \), where \( \sigma_{\text{min}} := \min(\sigma_1, \ldots, \sigma_{d_M}) \). Recall that \( \sigma \) is full-rank, hence \( \sigma_{\text{min}} > 0 \). With this information, we can update the bound (14) to
\[ I(A : C|B) \leq \frac{4}{\sigma_{\text{min}}^2} \| \Delta \rho_{ABC} \|^2. \]  
(25)

C. A (sub)exponential bound on trace distance.

We now show that \( \| \Delta \rho_{ABC} \|_1 \) in (25) may be bounded as
\[ \| \Delta \rho_{ABC} \|_1 \leq O\left( \| E^{[B]} - \tilde{E}^{[B]} \| \right) \]  
(26)
\[ \leq O\left( B^{K} \| V_{\text{gap}}^{[B]} \|^{-K} \right). \]

We first recall from Ref. [33] that
\[ \| \Delta \rho_{ABC} \|_1 \leq 2\| V_B - \tilde{V}_B \|. \]  
(27)
Since the dimension of the dilating Hilbert space \( d_B^{[B]} \) is exponentially large, an explicit construction of \( \tilde{V}_B \) is not useful. Instead we notice that the quantum channels \( \rho^{[B]} \) and \( \tilde{\rho}^{[B]} \) may be dilated with a Hilbert space of dimension \( d_M^{[B]} \), i.e. a space isomorphic to \( \mathcal{H}_M \otimes \mathcal{H}_M \). We will thus explicitly construct the isometries \( W, \tilde{W} : \mathcal{H}_M \rightarrow \mathcal{H}_M \otimes \mathcal{H}_M \otimes \mathcal{H}_M \), representing Stinespring dilations of \( \rho^{[B]} \) and \( \tilde{\rho}^{[B]} \), respectively. We may then express \( V_B \) and \( \tilde{V}_B \) in terms of these isometries, as we now show. Since \( W \) and \( V_B \) represent dilations of the same quantum channel \( \rho^{[B]} \), then, by Stinespring’s theorem[26] and assuming \( \mathcal{H}_B \geq d_M^{[B]} \), there exists an isometry \( U : \mathcal{H}_M \otimes \mathcal{H}_M \rightarrow \mathcal{H}_B \), such that
\[ V_B = (U \otimes \text{1}_M)W. \]  
(28)
Using this isometry \( U \), we may re-express \( \tilde{V}_B \) as
\[ \tilde{V}_B := (U \otimes \text{1}_M)\tilde{W}. \]  
(29)
We will be able to construct \( \tilde{W} \) explicitly, and thus \( \tilde{V}_B \) follows from (29) and \( \tilde{M}_B \) follows from (20).

We may now express the bound (27) in terms of the norm distance between \( W \) and \( \tilde{W} \),
\[ \| \Delta \rho_{ABC} \|_1 \leq 2\| V_B - \tilde{V}_B \| \]  
(30)
\[ = 2\|(U \otimes \text{1}_M)(W - \tilde{W})\| \]  
\[ = 2\| W - \tilde{W} \|, \]
eliminating the need to explicitly specify \( U \), which we will not do.

We now construct \( W \) and \( \tilde{W} \) using Choi’s methods[35] which will enable us to show that \( \| \Delta \rho_{ABC} \|_1 \leq O\left( \| E^{[B]} - \tilde{E}^{[B]} \| \right) \).

Let \( C \) and \( \tilde{C} \) be the Choi matrices of \( E^{[B]} \) and \( \tilde{E}^{[B]} \), respectively,
\[ C = \left( \text{id} \otimes E^{[B]} \right) (|+ \rangle \langle +|), \]  
(31)
\[ \tilde{C} = \left( \text{id} \otimes \tilde{E}^{[B]} \right) (|+ \rangle \langle +|) = \sigma \otimes 1_M, \]
where we have used (18). We further define
\[ \Delta C := C - \tilde{C} = \left( \text{id} \otimes (E^{[B]} - \tilde{E}^{[B]}) \right) (|+ \rangle \langle +|), \]  
(32)
from which it follows
\[ \| \Delta C \|_1 \leq d_M^{[B]} \| E^{[B]} - \tilde{E}^{[B]} \|_{2 \rightarrow 2} \]  
(33)
\[ \leq c_2 d_M^{[B]} \| V_{\text{gap}}^{[B]} \|^{-K}, \]
\[ \| \Delta C \| \leq d_M \| E^{[B]} - \tilde{E}^{[B]} \|_{2 \rightarrow 2} \]  
\[ \leq c_2 d_M \| V_{\text{gap}}^{[B]} \|^{-K}, \]
where we have used (9). We aim to express \( \| W - \tilde{W} \| \) in terms of \( \| \Delta C \|_1 \) and \( \| \Delta C \| \), obtaining the desired bound (26).

We now employ Choi’s construction[35] Recall that \( |+ \rangle := \sum_{i=1}^{d_B} |i \rangle \otimes |i \rangle \) in (31) is defined in our chosen orthonormal product basis, which satisfies \( (1 \otimes \sigma)|i \rangle \otimes |j \rangle = (\sigma|i \rangle \otimes |j \rangle \) and \( (\sigma \otimes 1)|i \rangle \otimes |j \rangle = (\sigma|i \rangle \otimes |j \rangle \) for all \( i, j = 1, 2, \ldots, d_M \). With respect to these bases, we decompose
\[ \tilde{C} = \sum_{i,j=1}^{d_M} |\tilde{\psi}_i \rangle \langle \tilde{\psi}_j|, \]  
(34)
\[ |\tilde{\psi}_j \rangle := \sqrt{\sigma} |i \rangle \otimes |j \rangle. \]
Notice that the vectors \( |\tilde{\psi}_j \rangle \) are pairwise orthogonal, \( \langle \tilde{\psi}_{j'}| \tilde{\psi}_j \rangle = \sqrt{\sigma} \delta_{jj'} \delta_{j'j} \), and form a basis in \( \mathcal{H}_M \otimes \mathcal{H}_M \). We define a set of projections \( P_k : \mathcal{H}_M \otimes \mathcal{H}_M \rightarrow \mathcal{H}_M \) with \( k = 1, 2, \ldots, d_M \) by
\[ P_k := 1_M \otimes |k \rangle \]  
(35)
Now Choi’s construction[35] provides the isometry \( \tilde{W} \) dilating \( \tilde{E}^{[B]} \) explicitly in the form
\[ \tilde{W} := \sum_{k,i,j=1}^{d_M} (P_k |\tilde{\psi}_j \rangle \otimes |i \rangle \otimes |j \rangle) |k \rangle. \]  
(36)
Now we assume a decomposition of $C$ of the form,
\[
\Gamma := \sum_{i,j=1}^{d_M} \langle \gamma_{ij} \rangle_{\psi_{ij}},
\]
(37)
where the vectors $|\psi_{ij}\rangle$ are not necessarily orthogonal. Recall that Choi’s construction does not require the vectors $|\psi_{ij}\rangle$ in the decomposition (37) to be orthogonal. Using the projector $P_i$ defined in (35), we employ Choi’s construction again, obtaining the isometry
\[
W := \sum_{i,j=1}^{d_M} (P_i^* |\psi_{ij}\rangle \otimes |i\rangle \otimes |j\rangle \langle k|),
\]
(38)
which dilates $\delta^{[B]}$. Since $|\tilde{\psi}_{ij}\rangle$ form a basis, we can express,
\[
|\psi_{ij}\rangle = |\tilde{\psi}_{ij}\rangle + \sum_{m,n=1}^{d_M} \gamma_{mnij} |\psi_{nm}\rangle
:= (1_M \otimes (\Gamma + \Gamma)) |\tilde{\psi}_{ij}\rangle,
\]
(39)
where $\gamma_{mnij} \in \mathbb{C}$ are some coefficients, and where we have defined
\[
\Gamma := \sum_{i,j,m,n=1}^{d_M} \gamma_{mnij} \sigma_i^{-1} |\tilde{\psi}_{mn}\rangle \langle \tilde{\psi}_{ij}|.
\]
(40)
We express the distance in operator norm between the constructed isometries $W$ and $\Gamma$ in terms of $\Gamma$,
\[
\|W - \Gamma\|^2 = \|W - \Gamma\|^2 = \|W - \Gamma\|^2
\]
(41)
transforming the equation (40) into
\[
\Gamma' + \Gamma^\dagger + \Gamma' \Gamma^\dagger = \Delta C'.
\]
(42)
Notice that from (41) it follows that
\[
\|\Delta C'\|_1 \leq \frac{1}{\sigma_{\min}} \|\Delta C\|_1 \leq \frac{c_2 d_M^2}{\sigma_{\min}} |B|^{K} |\Delta - K|, \tag{43}
\]
where we have used (33). This implies that $\Delta C' \to 0$ as $|B| \to \infty$. We wish to find $\Gamma'$ satisfying (42), such that $\|\Gamma'\|_1 = O(\|\Delta C\|_1)$, and via (43), $\|\Gamma'\|_1 = O(\|\Delta C\|_1)$. Since any solution of (42) satisfying this condition will suffice, and observing that $\Delta C'$ is Hermitian, we shall seek a Hermitian solution, $\Gamma' = \Gamma^{\dagger}$.

For $\Gamma$ Hermitian, equation (42) reduces to a quadratic equation,
\[
\Gamma' + 2\Gamma' - \Delta C' = 0,
\]
which indeed has a Hermitian solution of small norm,
\[
\Gamma' = \sqrt{1_M \otimes M + \Delta C^2 - 1_M \otimes M}, \tag{44}
\]
provided that $1_M \otimes M + \Delta C' \geq 0$, which is satisfied for large enough $|B|$. Notice that through (41), (39), and (38), $\Gamma'$ explicitly defines $W$.

From (44) follows,
\[
\Gamma' \Gamma^{\dagger} = \Gamma' \Gamma = 21_M \otimes M + \Delta C - 2 \sqrt{1_M \otimes M + \Delta C}. \tag{45}
\]
Assuming $|B|$ large enough that $\|\Delta C'\| \leq 2^{-\frac{3}{2}}$, by Taylor expansion, we estimate
\[
0 \leq \Gamma' \leq \frac{\Delta C'^2}{2},
\]
(46)
Expressing $\|W - \Gamma\|^2$ in terms of $\Gamma'$, we find
\[
\|W - \tilde{\Gamma}\|^2 \leq \text{tr} (\Gamma' \tilde{\Gamma}^{\dagger}) = \text{tr} (\tilde{\Gamma} C' \Gamma^{\dagger}),
\]
leading to
\[
\|W - \tilde{\Gamma}\|^2 \leq \frac{1}{2} \text{tr} (\tilde{\Gamma} C' \Gamma^{\dagger}), \tag{47}
\]
where we have used (31), (33), (41), and $\|\tilde{\Gamma}^{-1}\| = \sigma_{\min}^{-1}$. Combining (27), (30), (46), and (47) together, we have
\[
\|\Delta \rho_{ABC}\| \leq c_2 d_M \sqrt{2 d_M |B|^{K} |\Delta - K|} \tag{48},
\]
(48)
proving that (an injective) iuMPS is an approximate quantum Markov chain.

Finally, using [25], we obtain the desired bound on QCMI,

$$I(A : C|B) \leq \frac{8d^3M^2}{\lambda^3} e^{-2\ln v_{\text{gap}}^1(|B| - K) + 2 K \ln |B|},$$

(48)

which corresponds to the statement of Theorem III.1 with $Q = \frac{8d^3M^2}{\lambda^3} q = 2 \ln v_{\text{gap}}^1$. This proves Theorem (III.1) for the case of injective iuMPS.

**Remarks:**

1. Notice that the inequality [47] has the form $\|\Delta \rho_{ABC}\|_1 \leq O(\|\rho_{AB} - \rho_{AB}\|_{2 \rightarrow 2})$, in contrast to the bound $\|\Delta \rho_{ABC}\|_1 \leq O(\sqrt{\|\rho_{AB} - \rho_{AB}\|_{2 \rightarrow 2}})$ that we have obtained in Theorem II.1 of Ref. [33]. This improvement modifies the rate of exponential decay in (48) by a factor of 2, in addition to the factor of 2 discussed in Remark 1 in Subsection III.A. The rate of the exponential decay in the final bound (48) is thus four times larger than the one in Theorem II.1 of Ref. [33].

2. The numerical results presented in Section IV are much closer to the new bound on the decay rate than the earlier value given in Ref. [33].

3. Another improvement over Theorem II.1 of Ref. [33] is that the prefactor in (48) is independent of the size of any region of the spin chain. As a drawback, we gained the dependence on $\sigma_{\text{min}}^{-3}$, which may be a large number.

**D. General case: infinite uniform matrix product states.**

Now we outline the changes required to extend the proof to the general class of iuMPS. While the main steps of the proof remain the same, we have to account for the cases in which iuMPS is not injective. These entail changes in the peripheral part $E$ [8] of the transfer matrix $E$ [5]. Since the modifications we make involve the equations [23] and [31], formulated in terms of the quantum channel $E$ (represented by $E$), we focus attention on $E$, rather than $E$.

In the case of a general iuMPS, the quantum channel $E : B(\mathcal{H}_M) \rightarrow B(\mathcal{H}_M)$ has the following properties [20] (see also Propositions 1 and 2 in Ref. [33]):

- The quantum channel $E$ decomposes into a sum of components, $E = \sum_{\alpha=1}^{N} E_\alpha$, $E_\alpha : \Pi^{\alpha} B(\mathcal{H}_M) \Pi^{\alpha} \rightarrow \Pi^{\alpha} B(\mathcal{H}_M) \Pi^{\alpha}$, defined by the set of projectors $\{\Pi^{\alpha}\}_{\alpha=1}^{N}, \sum_{\alpha=1}^{N} \Pi^{\alpha} = 1_{M}$. This implies that the fixed point $\sigma_{\text{min}}$ of $E$ and $E$ satisfies $\sigma_{\text{min}} = \sum_{\alpha=1}^{N} \Pi^{\alpha} \sigma_{\text{min}}$.

- For each $\alpha$, the component $E_\alpha$ admits a further decomposition,

$$E_\alpha(X) = \sum_{k=0}^{p - 1} \frac{p - k}{k!} \Pi^{\alpha}_{k} X \Pi^{\alpha}_{k+1} \Pi^{\alpha}_{k+1},$$

(50)

defined by the set of projectors $\Pi^{\alpha}_{k}$ with $k = 0, \ldots, p - 1$, satisfying $\sum_{k=0}^{p - 1} \Pi^{\alpha}_{k} = \Pi^{\alpha}$. The index $k$ in $\Pi^{\alpha}_{k}$ is taken modulo $p$. This implies that $\sigma$ satisfies $\sigma = \sum_{\alpha=1}^{N} \Pi^{\alpha}_{k} \sigma^{\alpha}_{\Pi^{\alpha}_{k} \sigma^{\alpha}_{\Pi^{\alpha}_{k}}}$. While for $p > 1$, the matrix $\tilde{C}$ is not full-rank, it still satisfies $\text{supp}(\Delta C) \subseteq \text{supp}(\tilde{C})$ and $\text{supp}(C) \subseteq \text{supp}(\tilde{C})$, which we prove in Appendix C. Hence, the derivations of Section III.C may still be applied, if we consider expressions to be restricted to the support of $\tilde{C}$ where necessary. The inverse of $\tilde{C}$ should be considered to be defined on $\text{supp}(\tilde{C})$. With this convention we may estimate,

$$\|\tilde{C}^{-1}\| = \frac{1}{\rho \sigma_{\text{min}}}.$$

The bound (46) is still valid if we update the value of $\|\tilde{C}^{-1}\|$, $\|W - \tilde{W}\| \leq \frac{d^3}{2 \rho \sigma_{\text{min}}} |\rho_{AB} - \rho_{AB}|^{2 \rightarrow 2}$. The density operator $\tilde{\rho}_{R}$, defined in [25], changes from (24) to

$$\tilde{\rho}_{R} = \rho J \sum_{k=0}^{p - 1} \Pi_{k} \sigma_{\Pi_{k}} \otimes \sqrt{\Pi_{k+|R|} \Pi_{k+|R|}} + (\sigma \otimes \sigma) \pi_{R} \Pi_{k+|R|} J^{\dagger},$$

where $J : \mathcal{H}_{R} \otimes \mathcal{H}_{R} \rightarrow \mathcal{H}_{R}$ is an isometry, and where we have used $\sigma = \sum_{k=0}^{p - 1} \Pi_{k} \sigma_{\Pi_{k}}$. This implies that for any $\tilde{\rho}_{R}$ its smallest eigenvalue $\lambda_{\text{min}}$ is bounded from below by

$$\lambda_{\text{min}} \geq \rho \sigma_{\text{min}}^{-1}.$$ 

This modifies the pre-factor in the bound [25] from $\frac{4}{\rho \sigma_{\text{min}}}$ to $\frac{4}{\rho \sigma_{\text{min}}}$. Thus, the overall bound (48) gains a factor $p^{-2} \leq 1$,

$$I(A : C|B) \leq \frac{8d^3M^2}{\lambda^3} e^{-2\ln v_{\text{gap}}^1(|B| - K) + 2 K \ln |B|},$$

(49)

proving Theorem (III.1) for the case $N = 1$ and $p \geq 1$.

Generalizing to the case of $N \geq 1$ we notice that $\tilde{C}$ obtains a direct sum structure,

$$\tilde{C} = \bigoplus_{\alpha=1}^{N} p_{\alpha} \sum_{k=0}^{p - 1} \Pi_{k+|B|} \sigma_{\Pi_{k+|B|}} \otimes \Pi^{\alpha}_{k}.$$
The density operator $\tilde{\rho}_R$ also gets a direct sum structure,

$$\tilde{\rho}_R = \bigoplus_{\alpha=1}^N \rho_\alpha \sum_{k=0}^{p_\alpha-1} \left( \Pi^\alpha_k \otimes \Pi^\alpha_{k+|R|} \right)$$

(51)

Proceeding as in the case of $N = 1$, the equations (50) and (51) imply that the bound (47) is modified to

$$\|\Delta_{ABC}\|_1 \leq \max_{\alpha} \left( \frac{d_{M_{\alpha}}^3}{p_\alpha} \right) \epsilon \sum_{\sigma} \frac{2}{\sigma_{\min}} |B|^K v^{[|B|]}_{\text{gap}},$$

(52)

and the bound (49) to

$$I(A : C|B) \leq \max_{\alpha} \left( \frac{d_{M_{\alpha}}^3}{p_\alpha} \right) \epsilon^2 \sum_{\sigma} \frac{2}{\sigma_{\min}} |B|^K v^{[|B|]}_{\text{gap}},$$

(53)

where $d_{M_{\alpha}} = \dim(\Pi^\alpha, \mathcal{H}_M)$. The bound (52) implies that any iuMPS is an approximate quantum Markov chain for sufficiently large $|B|$. We note that since $p_\alpha \geq 1$ and $d_{M_{\alpha}} \leq d_M$, the bounds (47) and (48) hold for the general class of iuMPS.

### IV. NUMERICAL CALCULATION OF QCMI

#### A. Constructing iuMPS

Recall that an iuMPS is completely characterized by reduced density operators $\rho_n$ on finite continuous regions of $n$ sites (refer to (7)). In turn, $\rho_n$ is characterized by the fixed point of the quantum channel $E$ and the choice of the set $\mathcal{S}$. The quantum channel $E$ is represented by a transfer matrix $M$ that may have a single or multiple fixed point(s) (eigenvectors corresponding to eigenvalue 1 of $E$). There are three cases that account for all iuMPS:

**Case 1.** When $E$ has a single fixed point, and,

$$\mathcal{S} := \left\{ M^s \in \mathbb{C}^{d_M \times d_M}, \forall s \in \{1,2,\ldots,d_s\} \right\},$$

**Case 2.** When $E$ has two fixed points and $\rho_n$ is of period 1, and,

$$\mathcal{S} := \left\{ M^s := \begin{bmatrix} M_1^s & 0 \\ 0 & M_2^s \end{bmatrix} \bigg| M_1^s, M_2^s \in \mathbb{C}^{d_M \times d_M/2}, \right.$$  

$$M^s \in \mathbb{C}^{d_M \times d_M}, \forall s \in \{1,2,\ldots,d_s\} \right\},$$

**Case 3.** When $E$ has two fixed points and $\rho_n$ is of period 2,

$$\mathcal{S} := \left\{ M^s := \begin{bmatrix} 0 & M_1^s \\ M_2^s & 0 \end{bmatrix} \bigg| M_1^s, M_2^s \in \mathbb{C}^{d_M \times d_M/2}, \right.$$  

$$M^s \in \mathbb{C}^{d_M \times d_M}, \forall s \in \{1,2,\ldots,d_s\} \right\},$$

The matrices $M^s$ in the set $\mathcal{S}$ in Case 1 are constructed as follows,

$$\langle M^s \rangle_{ij} = \sum_{d_s=1}^{d_s} \sum_{d_s=1}^{d_s} U_{ij}^{s',s} \Psi_{s'}^s, \quad (54)$$

where the unitary $U \in \mathbb{C}^{d_M \times d_M}$ is selected randomly with respect to Haar measure and $\Psi \in \mathbb{C}^{d_M \times d_M}$ is given by,

$$\Psi := \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \otimes \mathbb{I}_{d_M}, \quad (55)$$

where we fix $d_s = 3$ and $d_M = 4$ in our numerical calculations. In Case 2 and Case 3, each of the two matrices $M_1^s$ and $M_2^s$ is constructed using an equation of the form (54) and (55), but with $d_M$ replaced by $d_M/2$, and different and independent choices of the unitary $U$.

After constructing $\mathcal{S}$, the left and right fixed points of the transfer matrix $E$ denoted by $|+\rangle$ and $|\sigma\rangle$ in (7), respectively, remain to be found to completely specify the iuMPS. Recall, fixed points of $E$ are its left and right eigenvectors corresponding to a left and right eigenvalue of 1, respectively. If there are
multiple fixed points of $E$, in other words, if (left and/or right) eigenvalue of 1 of $E$ is degenerate, then the fixed point used to define iuMPS is the uniform linear combination of the multiple fixed points. That is, the left and/or right eigenvectors are defined as the normalized sum of all left and/or right eigenvectors, respectively.

**B. Calculating entropies**

QCMI is computed from von Neumann entropies of the states on the continuous subregions $AB$, $BC$, $B$ and $ABC$ that are in turn computed from the eigenvalues of the reduced density operators for the respective subregions. A numerical check of Theorem III.1 requires QCMI for increasing size, $|B|$, of the separating region $B$. Direct diagonalization is computationally inefficient, because even linear growth of $|B|$ would imply exponential growth of the Hilbert space dimension of any subregion that includes $B$. To overcome this fundamental issue of dimensionality, we use network contraction techniques to project reduced density operators on to their respective supports.\[11\]

Consider an iuMPS, $\rho_n$ given by (7). The support of $\rho_n$ is constructed as follows. Define $G \in \mathbb{C}^{d_M \times d_M}$ by,

$$G := E_n = \sum_{s_1, s_2, \ldots, s_n=1}^{d_M} (M^{s_n} \otimes \bar{M}^{s_n}) \ldots \ldots (M^{s_2} \otimes \bar{M}^{s_2}) (M^{s_1} \otimes \bar{M}^{s_1}),$$

(56)

which can be numerically computed efficiently. Then, express the matrix elements of the permutation of $G$, defined by $G_{i,j;i',j'} := G_{j,i;i',j'}$, in terms of its spectral decomposition,

$$G_{i,j;i',j'} := \sum_{l,p=1}^{d_M^2} W_{i,l} \Sigma_{l,p} W_{j',l'}^*,$$

(57)

which can also be computed efficiently, and where $i, j, i', j' \in \{1, 2, \ldots, d_M\}$, unitary $W \in \mathbb{C}^{d_M^2 \times d_M^2}$, and $\Sigma$ is the diagonal matrix of eigenvalues of $\bar{G}$. Equations (56) and (57) are diagrammatically depicted in Fig. 4. The equation (57) implies that the support of $\rho_n$, denoted as supp($\rho_n$), is isomorphic to $\mathcal{H}_M \otimes \bar{\mathcal{H}}_M$. We define isometry $P: \mathcal{H}_M \otimes \bar{\mathcal{H}}_M \rightarrow \mathcal{H}_s \otimes \mathcal{H}_s$, with range($P$) = supp($\rho_n$), defined by its matrix elements,

$$P_{i_1,i_2,\ldots,i_n;j_1,j_2,\ldots,j_n} := \sum_{i,j=1}^{d_M} \sum_{k=1}^{d_M^2} (M^{s_n} \ldots M^{s_2} \bar{M}^{s_2} \bar{M}^{s_1})_{ij} \times W_{s_1,k} \Sigma_{k,l}^{-\frac{1}{2}}.$$

(58)

The isometry $P$ satisfies

$$P^\dagger 1_{d_M^2} P = 1_{d_M^2},$$

(59)

as diagrammatically verified in Fig. 5. The projection of $\rho_n$ onto its support is given by

$$P^\dagger \rho_n P = \Sigma_1^\dagger W_{1} (\sigma \otimes 1_{d_M}) W_{1}^\dagger;$$

(60)

a $d_M^2 \times d_M^2$ matrix, with non-zero eigenvalues identical to those of $\rho_n$. Figure 6 depicts $P^\dagger \rho_n P$. From (60) we observe that the $d_M^2 \times d_M^2$ matrix $P^\dagger \rho_n P$ can be efficiently computed from $W$ and $\Sigma$, without explicitly computing $P$, which would be prohibitively expensive. The von Neumann entropy, denoted by $S(\rho_n)$, is then found by computing the eigenvalues $\lambda_i$ of $P^\dagger \rho_n P$, and is given by

$$S(\rho_n) := \sum_{i=1}^{d_M^2} \lambda_i \ln \lambda_i.$$  

(61)

Finally, we repeat the above projection procedure, finding the non-zero eigenvalues and entropies for each of the four reduced density operators appearing in the definition of QCMI 10).

**Remark:**

Alternatively, to calculate the entropy $S(\rho_R)$ for the reduced density operator on the continuous region $R$, we could have used the relation analogous to (23), realising that $\rho_R$ and

$$J^\dagger \rho_R J = (\sigma |R| \otimes 1_{d_M}) (|\sqrt{\sigma} \rangle \langle \sqrt{\sigma}|) \in \mathcal{B}(\mathcal{H}_M \otimes \bar{\mathcal{H}}_M)$$

(62)

have the same eigenvalues and, thus, entropies. The numerical calculation would be of comparable complexity to the one we have chosen to employ.
V. NUMERICAL VERIFICATION OF THEOREM III.1

In this section, we present numerical calculations of QCMI for a randomly generated collection of iuMPS. The main result is given in Fig. 8 which (1) shows that the QCMI decays approximately exponentially as a function of $|B|$, and (2) compares its numerical decay rate with the decay rate $q = 2\ln(v_{\text{gap}})$ of the bounding function

$$I(A : C|B) \leq Q e^{-q(|B|-K) + 2K\ln|B|},$$
derived in Theorem III.1. Before presenting Fig. 8 we discuss the choice of parameters for the numerical calculations and the results of benchmarking our tensor network algorithm against the analytical example in Appendix A.

A. Choice of parameters

There are 4 parameters in the numerical calculations: sizes of subregions $A$ and $C$ and the dimensions of the physical and virtual spaces $d_s$ and $d_M$, respectively. We fix $d_s = 3$ and $d_M = 4$, and the sizes of subregions $A$ and $C$ are held fixed at 1 lattice site each. $|B|$ is varied from 2 to 40 in multiples of 2 (to decrease the complexity of the tensor network algorithm for projecting the reduced density operators onto their supports).

B. Benchmarking the numerical calculations

In the example discussed in Appendix A, quantum mutual information (QMI), defined by

$$I(A : C|B) = S(\rho_A) + S(\rho_B) - S(\rho_{AC}),$$
is analytically shown to converge to the non-zero finite value, $I_{\text{th}} := \lim_{|B| \to \infty} I(A : C|B)$, given by,

$$I_{\text{th}} = 17 \ln(2)/16 - 9 \ln(3)/8 + 5 \ln(5)/16,$$

$$\approx 3.5 \times 10^{-3}.$$
right (black) hand side vertical axes represent the natural logarithms to a finite value and exponentially to 0, respectively. Left (red) and right (black) hand side vertical axes represent the natural logarithms of QMI and QCMI, respectively. The horizontal axes represents the size of the separating region, $|B|$. The solid light red and gray lines connecting the red circles and black diamonds guide the readers’ eyes towards the trends in QMI and QCMI, respectively. The theoretically computed QMI value for $|B| \to \infty$ is denoted by $I_{th}$ (dotted black line) and is given in the text. The choice of parameter values is given in Sec. IV.

$|B| = 26$ the QMI is identical to $I_{th}$ down to machine precision ($O(10^{-15})$). This agreement between the analytical and numerical QMI values benchmarks our tensor network algorithm. With the same algorithm, we compute the QCMI for the considered example and show the result in black in Fig. 7. We observe that the QCMI decays exponentially, which is consistent with Theorem III.1.

C. Main Numerical Result

Let $S = \{1, 2, \ldots, N\}$ denote a subset of integers, and take $i \in S$ to label each instance of $N$ randomly generated iuMPS constructed using the methods described in Sec. IV here $N = 6 \times 10^4$. Define the function $f_i : B_i \to \mathbb{R}$, where $i \in S$ and $B_i := \{0, 2, 4, \ldots, |B_i|\}$, such that,

$$f_i(|B|) := \ln I(A : C|B|)/2\ln(v_{\text{gap}}^{-1}).$$

(63)

Then the inset panel in Fig. 8 displays the collection of graphs $\Gamma = \{G_i : i \in S\}$ where,

$$G_i := \{(x-x_i,f_i(x)-f_i(x_i)) : x = |B|, x_i = |B_i|\}.$$  

(64)

Note that for graph $G_i$ the values of $|B| \leq |B_i|$, which is the maximum size of region $B$. Several graphs could have the same value of $|B_i|$, i.e. the function $i \mapsto |B_i|$ is many to one. In practice $|B_i|$ is found by the first instance that the condition $\exp(2\ln(v_{\text{gap}}^{-1})f_i(|B_i|)) > 10^{-k}$ does not hold, where $k = 12$ is determined independently by errors in numerical eigenvalue computations as discussed in Appendix D. In Fig. 8, the blue graphs display the slopes $-1$ and help compare graph data with the theorem. The origin $(0, 0)$ of both the inset and the main panels in Fig. 8 corresponds to the maximum dimension $|B| = |B_i|$. By comparing the graphs (black lines) against the blue line of slope $-1$ in the inset panel, we see that the slopes of most graphs in $\Gamma$ are eventually less than or equal to $-1$. This indicates, according to Theorem III.1, that for most graphs in $\Gamma$, $|B| = |B_i|$ explores the asymptotic regime in the numerical calculations. In the main panel of Fig. 8, we display the same graph data as a 2D histogram, where the bins of the histogram $H_{i,j} \subset \mathbb{R}^2$ are given by,

$$H_{i,j} = \left(-2i+1\right) \times \left(2j+2\right),$$

(65)

for $i, j \in \{0, 1, 2, \ldots, 19\}$. We observe that for a large fraction of graphs in $\Gamma$, the slopes are less than or equal to $-1$ and the occupancy of the bins just below the diagonal of slope $-1$ increases as $|B| \to |B_i|$. In Fig. 8 we further probe the decay of QCMI by plotting the cumulative distribution of the QCMI decay rates. The decay rates are extracted by linear regression from the graphs $G_i \in \Gamma$. Note that for all the iuMPS in our generated sample, the constant $K$ is zero, and the logarithmic correction is absent, which we verify during the computation. This may be observed from the fact, that in the bound given in Theorem III.1 the constant $K$ is non-zero, only if the Jordan decomposition of the transfer matrix contains non-trivial Jordan blocks (i.e., contains nilpotent operators) for some eigenvalues with magnitude less than 1. (For the eigenvalues of mag-
Normalized decay rate

Figure 9. This figure depicts the cumulative distribution of the QCMI decay rates normalized by $2\ln v^{-1}_{\nu - 1}\text{gap}$ computed (i) for all $G_i \in \Gamma$, illustrated by the dashed black line, and (ii) for subset of $\Gamma$ corresponding to $|B_i| = 40$, illustrated by the solid black line. The vertical solid blue line corresponds to the slope of the solid blue line in Fig. 8. Note that the cumulative distribution extends further than 1.22 on the horizontal axis, but is not significant to our discussion and is omitted.

VI. CONCLUSION

In this paper we considered a system ABC as shown in Fig. 2(a) and (b), prepared in an infinite uniform matrix product state with density operator $\rho_{ABC}$, illustrated in Fig. 2(c). We proved that the QCMI, denoted $I(A:C|B)$, is bounded by an exponentially decaying function in the asymptotic limit of large size $|B|$ of the separating region. By taking advantage of the more direct methods of estimation than in Ref. 33, we have improved the exponent of the asymptotic bounding function for QCMI by a factor of 4. We showed that the newly obtained exponent cannot be further improved. We have also shown that $\rho_{ABC}$ is close to an exact quantum Markov chain $\tilde{\rho}_{ABC}$ in the trace distance, which is also bounded by an exponentially decaying function. Compared to Ref. 33, the exponent of the latter function is improved by a factor of 2. The theory was supported by numerical experiments in which the QCMI is computed for a collection of matrix product states generated by sampling their generating isometries with respect to Haar measure.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.
Appendix A: An example: convergence of QMI to a non-zero value

In this appendix we specify the parameters of the iuMPS used in Fig.7 and Fig.8, and the asymptotic limit of its QMI

$$I(A : C) = S(\rho_A) + S(\rho_C) - S(\rho_{AC}),$$

which we use to benchmark our numerical computations.

The iuMPS is generated by

$$M' = \begin{bmatrix} M'_1 & 0 \\ 0 & M'_2 \end{bmatrix},$$

i.e., belongs to Case 2 defined in Sec. (V A) with submatrices specified by,

$$M'_1 = M'_2 = \begin{bmatrix} 0 & -1 \sqrt{2} \\ 0 & 0 \end{bmatrix},$$

$$M'_1 = M'_2 = \begin{bmatrix} -1 \sqrt{2} & 0 \\ 0 & 1 \sqrt{2} \end{bmatrix}, \quad (A1)$$

$$M'_1 = M'_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and by the fixed point $|\sigma\rangle = (1/2, 1/2, 1/2)^T$.

As was shown in Appendix D of Ref. [33], if we set $|A| = |C| = 1$, then, when $|B| \to \infty$, the reduced density operators $\rho_{AC}$, $\rho_A$, and $\rho_C$ converge to

$$\rho_{AC} = \frac{1}{32} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho_A = \frac{1}{8} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \rho_C = \frac{1}{8} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

and by the fixed point $|\sigma\rangle = (1/2, 1/2, 1/2)^T$.

Appendix B: Taylor expansion with remainder

In this appendix we provide details of the derivation of the bound (25), that we have omitted in Subsection III A

Recall that we are aiming to bound the QCMI

$$I(A : C|B) = \text{tr}(\rho_{ABC} \ln \rho_{ABC} + \ln \rho_B - \ln \rho_{AB} - \ln \rho_{BC})$$

(B1)

in terms of $||\Delta \rho_{ABC}||_1$, where $\Delta \rho_{ABC} = \rho_{ABC} - \tilde{\rho}_{ABC}$. We achieve this by making the Taylor expansion of all the logarithms, which, with an explicit remainder term, takes the form

$$\ln(\tilde{\rho} + \Delta \rho) = \ln \tilde{\rho} + \int_0^{\infty} ds \left( \frac{1}{\tilde{\rho} + s \Delta \rho} \frac{1}{\tilde{\rho} + s \Delta \rho} - \frac{1}{\tilde{\rho} + s \Delta \rho} \frac{1}{\tilde{\rho} + s \Delta \rho} \frac{1}{\tilde{\rho} + s \Delta \rho} \right)$$

and where we dropped index $R$ that can be any of $ABC$, $AB$, $BC$, or $B$.

We have already shown in Subsection III A that the contributions to $I(A : C|B)$ of the zeroth and the first order in

By direct calculation, the QMI $I(A : C)$, converges to the value $I_B = 17 \ln(2)/16 - 9 \ln(3)/8 + 5 \ln(5)/16 \approx 3.5 \times 10^{-3}$. In contrast, as is proven by Theorem III 1 or by Theorem II.1 of Ref. [33] the QCMI converges to zero.

We have also estimated the contribution of any of the four terms of the second order in $\Delta \rho_{ABC}$.
\[
\text{tr} \left( \Delta \rho \int_0^{\infty} ds \frac{1}{\hat{\rho} + s} \Delta \rho \frac{1}{\hat{\rho} + s} \Delta \rho \frac{1}{\hat{\rho} + s} \right) + \text{tr} \left( \left( \hat{\rho} + \Delta \rho \right) \int_0^{\infty} ds \int_0^1 dt (1-t)^3 \frac{1}{\hat{\rho} + t\Delta \rho + s} \Delta \rho \frac{1}{\hat{\rho} + t\Delta \rho + s} \Delta \rho \frac{1}{\hat{\rho} + t\Delta \rho + s} \right). \]

The first of these may be estimated as

\[
\left| \text{tr} \left( \Delta \rho \int_0^{\infty} ds \frac{1}{\hat{\rho} + s} \Delta \rho \frac{1}{\hat{\rho} + s} \Delta \rho \frac{1}{\hat{\rho} + s} \right) \right| \leq \int_0^{\infty} ds \left\| \frac{1}{\hat{\rho} + s} \right\|_1^3 \| \Delta \rho \|_3^3 \\
\leq \int_0^{\infty} ds \frac{1}{\left( \lambda_{\text{min}} + s \right)^3} \| \Delta \rho \|_3^3 \\
\leq \frac{\| \Delta \rho \|_3^3}{2\lambda_{\text{min}}^2}.
\]

The second term may be estimated as

\[
\left| \text{tr} \left( \left( \hat{\rho} + \Delta \rho \right) \int_0^{\infty} ds \int_0^1 dt (1-t)^3 \frac{1}{\hat{\rho} + t\Delta \rho + s} \Delta \rho \frac{1}{\hat{\rho} + t\Delta \rho + s} \Delta \rho \frac{1}{\hat{\rho} + t\Delta \rho + s} \right) \right| \leq \left\| \hat{\rho} + \Delta \rho \right\|_1 \left( \int_0^{\infty} ds \int_0^1 dt (1-t)^3 \right) \left\| \frac{1}{\hat{\rho} + t\Delta \rho + s} \right\|_4 \| \Delta \rho \|_3^3 \\
\leq \int_0^{\infty} ds \frac{1}{\left( \lambda_{\text{min}} + s \right)^3} \| \Delta \rho \|_3^3 \\
= \frac{\| \Delta \rho \|_3^3}{3\lambda_{\text{min}}^3} \\
\leq \frac{\| \Delta \rho \|_3^3}{3\lambda_{\text{min}}^3},
\]

where \( 0 \leq t' \leq 1 \) and \( \lambda_{\text{min}}' \) is the smallest eigenvalue of \( \hat{\rho} + t'\Delta \rho \).

We may estimate \( \lambda_{\text{min}}' \) using the inequality

\[
|\lambda_{\text{min}}' - \lambda_{\text{min}}| \leq \| \rho_{ABC} + t'\Delta \rho_{ABC} - \hat{\rho}_{ABC} \|_1 \leq \| \Delta \rho_{ABC} \|_1 \\
\leq c_2 d_M \sqrt{\frac{2d_M}{\sigma_{\text{min}}} |B|^{K_{\text{gap}} - K}}.
\]

Combining (13), (B3), and (B4), and recalling that there are four logarithms contributing to \( I(A : C|B) \), (B1), we estimate,
\[ I(A : C|B) \leq 4 \left( \frac{\|\Delta \rho_{ABC}\|_1^2 \lambda_{\min}}{2\lambda_{\min}} + \frac{\|\Delta \rho_{ABC}\|_1^2}{2\lambda_{\min}} + \frac{\|\Delta \rho_{ABC}\|_1^2}{3\lambda_{\min}^3} \right) = \frac{2\|\Delta \rho_{ABC}\|_1^2}{\lambda_{\min}} \left( 1 + \frac{1 + 2\lambda_{\min}}{3\lambda_{\min}^3} \right) \]

where we have redefined \( \lambda_{\min} \) to be the smallest eigenvalue of all \( \tilde{\rho}_R \), where \( R \) can be any of \( ABC, AB, BC \), or \( B \). We have similarly redefined \( \lambda_{\min}^{\prime} \) with \( \rho_R + t\Delta \rho_R \) replacing \( \tilde{\rho}_R \). We have also used the inequality \( \|\Delta \rho_R\|_1 \leq \|\Delta \rho_{ABC}\|_1 \).

From (B5) we observe that \( \lambda_{\min}^{\prime} \) converges to \( \lambda_{\min} \), while (from (27)) the trace distance \( \|\Delta \rho_{ABC}\|_1 \) vanishes as \( |B| \to \infty \). Then, for large enough \( |B| \),

\[ \left( \frac{1}{\lambda_{\min} + \frac{2\lambda_{\min}}{3\lambda_{\min}^3}} \right) \|\Delta \rho_{ABC}\|_1 \leq 1. \tag{B6} \]

This leads to the bound

\[ I(A : C|B) \leq \frac{4\|\Delta \rho_{ABC}\|_1^2}{\lambda_{\min}}. \]

Recalling that from (24) \( \lambda_{\min} = \sigma_{\min}^2 \),

\[ I(A : C|B) \leq \frac{4\|\Delta \rho_{ABC}\|_1^2}{\sigma_{\min}^2}. \]

Appendix C: Sufficient conditions on \( |B| \)

In this appendix we combine together all the requirements imposed on the size \( |B| \) of the separating region in order to get a unified sufficient condition.

To satisfy the requirement (B6) we impose

\[ \|\Delta \rho_{ABC}\|_1 \leq \frac{3}{19} \lambda_{\min}^2, \]

which supersedes the condition (C1). Using (27) we arrive at the condition

\[ c_2 d_M \left( \frac{2d_M}{\sigma_{\min}} |B|^K v_{\text{gap}}^{K-K} \right) \leq \frac{3}{19} \lambda_{\min}^2 \]

Recalling that \( \lambda_{\min} = \sigma_{\min}^2 \), we re-express the above inequality in the form

\[ |B|^K v_{\text{gap}}^{K-K} \leq \frac{3}{19} \frac{\sigma_{\min}^2}{\sqrt{2} c_2 d_M} \tag{C2} \]

First, we require \( |B| \) to be large enough, that the bound (B6) is satisfied. We derive the following conditions on \( |B| \) in Appendix F:

\[ |B| > 2K, \]

\[ \left( \frac{|B|}{K} \right)^K v_{\text{gap}}^{K-K} \geq (K_{v_i} + 1) \left( \frac{|B|}{K_{v_i}} \right)^{\frac{K_{v_i}}{K_{v_i}}} |B|_{v_i}^{K-K_{v_i}}, \]

for all \( v_i \in Y \setminus Y_+ \).

The constants \( K \) and \( K_{v_i} \) may be extracted from the Jordan decomposition of \( E \), as described in Appendix F.

Second, we consider the requirement imposed on \( |B| \) in Appendix B in (B6). Using (B5) we may estimate

\[ \lambda_{\min}^{\prime} \geq 1 - \frac{\|\Delta \rho_{ABC}\|_1}{\lambda_{\min}}. \]

Now we impose the condition

\[ \frac{\|\Delta \rho_{ABC}\|_1}{\lambda_{\min}} \leq \frac{1}{2}, \tag{C1} \]

which leads to

\[ \frac{\lambda_{\min}}{\lambda_{\min}^{\prime}} \leq 2. \]

This allows to estimate the left-hand side of (B6) as

\[ \left( 1 + \frac{2\lambda_{\min}^2}{3\lambda_{\min}^3} \right) \frac{\|\Delta \rho_{ABC}\|_1}{\lambda_{\min}} \leq \left( 1 + \frac{16\lambda_{\min}^2}{3\lambda_{\min}} \right) \frac{\|\Delta \rho_{ABC}\|_1}{\lambda_{\min}} \leq \frac{19}{3} \frac{\|\Delta \rho_{ABC}\|_1}{\lambda_{\min}^2}. \]

We imposed a further condition on \( |B| \), namely \( \|\Delta C\| \leq 1 \), in order to derive (44), with \( \Delta C \) defined by (32) and (41). It was superseded by the stronger condition \( \|\Delta C\| \leq 2\sqrt{2} \), which we used to guarantee the inequality (45). By using (33), this condition leads to the inequality

\[ \frac{1}{\sigma_{\min} c_2 d_M |B|^K v_{\text{gap}}^{K-K}} \leq \frac{1}{2\sqrt{2}}. \]

Expressing this inequality in the same form as (C2), we arrive at the condition

\[ |B|^K v_{\text{gap}}^{K-K} \leq \frac{1}{2\sqrt{2} c_2 d_M}. \]

This condition is, however, redundant, since it is strictly weaker than (C2).
We have imposed one more condition on $|B|$, in Subsection III A, by requiring $\dim \mathcal{H}_B \geq d_M^2$.

$$|B| \geq 2 \ln d_M \ln d_e.$$  

Thus, the size $|B|$ should be large enough to satisfy the conditions

$$|B| > 2K,$$

$$(|B|/K)^{\nu_1} \geq (K_{\nu_1} + 1) \left( |B|/K_{\nu_1} \right)^{\nu_1},$$

for all $\nu_1 \in \mathcal{Y} \backslash \mathcal{Y}_c$, $|B|^2 \geq 19 \sigma_{\min}^2 / 2 c_2 d_M^2$,

$$|B| \geq 2 \ln d_M \ln d_e.$$  

The constant $c_2$ is specified in Appendix D.

We remind the reader that the constraints (C3) should not be taken as necessary to prove Theorem III.1 and only sufficient (and convenient) to guarantee validity of the steps in the proof.

We performed the derivation above for the case of an injective inuMPS. If we repeat the same derivation for a general case of inuMPS, we will have to replace (with respect to the definitions made in Section III D) the factors $d_M^{-3/2} \rightarrow \max_{\alpha} \{ d_M^{-3/2} p^2_{\alpha} \}$ and $\ln d_M \rightarrow \max_{\alpha} \ln d_{M_{\alpha}}$ in the third and in the fourth lines of (C3), respectively. Alternatively, we can use (C3) without any changes, since the described alternations lead to weaker conditions.

Appendix D: An example of errors in eigenvalue calculations

Errors resulting from the numerical computation of eigenvalues of reduced density operators dictate the smallest value of QCMI that we can reliably compute. In turn, this restricts the maximum size of the separating region $|B|$ we can explore. Since $I(A : C|B) = S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_B)$, the error incurred in calculating QCMI is estimated as

$$\delta I(A : C|B) \sim \delta S,$$

where

$$\delta S := \max \{ \delta S(\rho_{ABC}), \delta S(\rho_B), \delta S(\rho_{AB}), \delta S(\rho_{BC}) \},$$

is the maximum error in a calculation of the entropies. We cannot reliably compute values of QCMI smaller than $\delta I$, and thus are forced to stop calculations when the size of the separating region $|B|$ becomes large enough that $I(A : C|B)$ decays to values of the order $O(\delta I)$.

As discussed in Section IVB, each of the density operators $\rho_{ABC}, \rho_{AB}, \rho_{BC}$, and $\rho_B$ is supported on a subspace of dimension $d_M^2$ so

$$S = - \sum_{i=1}^{d_M^2} \lambda_i \ln \lambda_i.$$  

Hence, we may estimate the error in calculating entropies as

$$\delta S \leq \sum_{i=1}^{d_M^2} \delta \ln \lambda_i^{-1} - \delta \lambda \leq \sum_{i=1}^{d_M^2} \delta \lambda \ln \lambda_i^{-1} \leq d_M^2 \delta \lambda \ln \lambda^{-1},$$

where $\lambda_{\min} := \min \lambda$, and $\delta \lambda > 0$ is the maximum error in computing the eigenvalues. In practice, we use the MATLAB’s built in eigenvalue calculation function “eig” to compute eigenvalues, with an error of order $O(10^{-14})$, as explained in the following paragraph. Hence we find

$$\delta I \sim \delta S \sim 2^2 \cdot 10^{-14} \cdot \ln 10^{14} \sim 10^{-12}.$$  

In our numerical calculations, referring to (63) and below (64), for each $i \in \mathcal{P}$, we denote $|B_i|$ to be the maximum size of the separating region such that $\exp(2 \ln (\nu^{-1}_{\text{gap}}) f_i(|B_i|)) > 10^{-k}$. From the above discussion, we can reliably compute QCMI as small as $O(10^{-12})$, and, thus, we take $k = 12$ for our numerical calculations.

To gauge the size of errors incurred in eigenvalue calculations we compute numerically computed values against an exact result. We consider the transfer matrix eigenvalue problem of Case I in Sec. IV A and recall that one of the eigenvalues $\nu_1 = 1$. The numerically computed value incurs an error of $O(10^{-14})$. This estimate of the error is derived empirically from Fig. 10(a), which depicts the absolute difference between $\nu_1$, and its numerically computed value for $2 \times 10^4$ different instances of transfer matrices.

Appendix E: Spectrum of the transfer matrix

In this appendix we discuss the spectrum for the transfer matrix of an inuMPS, which is sampled with respect to the Haar measure (as defined in Section IV). For the purpose of comparing theoretically expected eigenvalues of the transfer matrix to numerically computed eigenvalues, both denoted by $\nu_i$ and delineated by context in what follows, we assume the ordering $|\nu_1| \geq |\nu_2| \geq \cdots \geq |\nu_{d_M^2}|$. Recall that we define

$$\nu_{\text{gap}} = \max_{|\nu_i| < 1} |\nu_i|.$$  

In Fig. 10(a) we depict the magnitude of difference between 1, the known largest eigenvalue, and $|\nu_1|$, the largest numerically computed value. As mentioned in Appendix D we infer from the figure that eigenvalue computations incur a numerical error of the order $O(10^{-14})$.

In Fig. 10(b), the magnitude of the difference between $|\nu_1|$ and $|\nu_2|$ is shown. We observe a gap between the computed eigenvalue magnitudes that clearly exceeds the numerical error. The gap is, however, an artifact of the finite sampling of Haar random isometries used to generate the inuMPS’s, as a simple example illustrates. Consider the family of quantum channels $\mathcal{E} : \mathcal{C}^4 \rightarrow \mathcal{C}^4$, generated by the dilating isometry $V : \mathcal{C}^4 \rightarrow \mathcal{C}^3 \otimes \mathcal{C}^4$, defined by

$$V = \sqrt{\beta} |1 \rangle \otimes |+ \rangle + \sqrt{1 - \beta} |0 \rangle \otimes |+ \rangle \otimes |1 \rangle \otimes |1 \rangle + \sqrt{1 - \beta} |0 \rangle \otimes |+ \rangle \otimes |1 \rangle \otimes |1 \rangle,$$

where $\beta$ is a parameter.
where we have decomposed $\mathbb{C}^4 \cong \mathbb{C}^2 \otimes \mathbb{C}^2$, the bases of $\mathbb{C}^2$ and $\mathbb{C}^3$ are denoted by $\{|+,−\rangle\}$ and $\{|1\rangle,|0\rangle,|−1\rangle\}$, respectively, and where $0 \leq \beta \leq 1$. The isometry induces the quantum channel

$$\mathcal{E} = \text{tr}_3(\mathcal{V}\mathcal{X}\mathcal{V}^\dagger) = (1 - \beta)\text{id}_2 \otimes \text{id}_2 + \beta \mathcal{E}_1 \otimes \text{id}_2,$$

where $\text{id}_2 : \mathbb{C}^2\times2 \rightarrow \mathbb{C}^2\times2$ is the identity channel, and the quantum channel $\mathcal{E}_1 : \mathbb{C}^2\times2 \rightarrow \mathbb{C}^2\times2$ is defined by

$$\mathcal{E}_1(|+\rangle\langle+|) = |−\rangle\langle−|,$$

$$\mathcal{E}_1(|−\rangle\langle−|) = |+\rangle\langle+|,$$

$$\mathcal{E}_1(|+\rangle\langle−|) = \mathcal{E}_1(|−\rangle\langle+|) = 0.$$

For this quantum channel, we may readily compute

$$|v_1| - |v_2| = 2\beta,$$

which may be arbitrarily close to 0, thus closing the gap in Fig. 10(b). These channels belong to the family represented by Case 1 in Section IV A. Our numerical results, however, suggest that the probability of sampling channels of this kind is small. The spectrum of random quantum channels (Haar-distributed or otherwise) has been studied in previous works Ref. [43–47], including the gap in the spectrum of quantum channels. The presence of a gap in a typical - with a particular sense of this word depending on the theoretical study - quantum channel was proven, for example, for Gaussian distributed quantum channels[43] and Haar-distributed quantum channels[43] in an asymptotic limit of large dimensions. These settings are closely related to this paper, but our numerical computations involve iuMPS with the virtual dimension $d_M = 4$ and physical dimension $d_s = 3$, both of which are too small for the results of Ref. [43] to be applicable. Nevertheless, our numerical experiments indicate that the gap in the spectrum of Haar-random quantum channels is present even for such small dimensions.

In Fig. 10(c), we show the magnitude of the numerically computed difference between $|v_2|$ and $|v_3|$. We observe that, in contrast to the difference $||v_1| - |v_2||$ in Fig. 10(b), for a non-zero fraction of the sampled transfer matrices the second and the third largest eigenvalue magnitudes are equal (or differ by less than numerical error). We speculate that these occurrences might be explained as follows. Recall that eigenvalues of a transfer matrix come in complex conjugate pairs. Hence, if we sample a transfer matrix with complex eigenvalues, we will necessarily find complex conjugate pairs of equal modulus. Thus, together, Figs. 10(b) and (c) indicate that, using the Haar measure, it is possible to sample with non-vanishing probability transfer matrices that have complex eigenvalues of magnitude strictly less than 1 (Fig. 10(c)); the probability of sampling transfer matrices with more than one eigenvalue of magnitude one (including cases of non-real eigenvalues) is, however, vanishingly small (Fig. 10(b)). (See Ref. [45] for a relevant study of distributions of eigenvalues of random quantum channels.)

The gap in the sampled values of $||v_2| - |v_3||$ is, as in the case of $||v_1| - |v_2||$, a numerical artifact. For the collection of transfer matrices generated by the one-parameter family of isometries

$$\mathcal{V} = (1 - \beta)\left(\frac{2}{3}\sqrt{\beta} - \frac{1}{3}\beta\frac{\sqrt{3}}{2}\right)|1\rangle\otimes|−\rangle\langle+|\otimes\mathbb{1}_2$$

$$- \left(1 - \beta\right)\left(\frac{1}{\sqrt{3}} - \frac{1}{2}\beta\frac{1}{2}\right)|0\rangle\otimes|+\rangle\langle+|\otimes\mathbb{1}_2$$

$$+ \left(1 - \beta\right)\left(\frac{1}{\sqrt{3}} - \frac{1}{2}\beta\frac{1}{2}\right)|0\rangle\otimes|−\rangle\langle−|\otimes\mathbb{1}_2$$

$$- \left(1 - \beta\right)\left(\frac{2}{3} + \frac{\sqrt{3}}{2}\right)|1\rangle\otimes|−\rangle\langle+|\otimes\mathbb{1}_2,$$

with $0 \leq \beta \leq 1$, the analytically calculated value,

$$|v_2| - |v_3| = \frac{\sqrt{6} - 2}{\sqrt{3}}\beta + O(\beta^2),$$

may be arbitrarily close to zero.

**Appendix F: Values of constants**

In this appendix we specify the constants $K$, $c_1$, and $c_2$ in the bound (9), as well as the alternative value for $c_2$, denoted as $c_3$.

First, we determine the values of $c_1$ and $K$, and the value of $c_2$, that follow from Theorem III.2 of Ref. [39]. We perform the Jordan decomposition of the transfer matrix $E$,

$$E = S\bigoplus_{v_i}(v_i\bar{P}_{v_i} + \bar{N}_{v_i})S^{-1}, \quad (F1)$$

where $S$ is an invertible matrix, $v_i$ is an eigenvalue, $\bar{P}_{v_i}$ is the projector onto the subspace corresponding to the eigenvalue $v_i$, and $\bar{N}_{v_i}$ is a matrix with the elements immediately above the main diagonal equal to 1, and all other elements equal to 0, possessing the index of nilpotency equal to $(K_{v_i} + 1)$ (i.e., $\bar{N}_{v_i}^{K_{v_i}+1} = 0$). Let us set

$$K := \max_{|v_i|=V_{\text{gap}}} \bar{K}_{v_i}.$$

Let us introduce the set $\Upsilon$ containing all distinct values of $v_i$. The Jordan decomposition (F1) can contain terms corresponding to equal eigenvalues. For each distinct eigenvalue, let us combine such terms together, arriving to the decomposition

$$E = S\bigoplus_{v_{i}\in\Upsilon}(v_iP_{v_i} + N_{v_i})S^{-1}, \quad (F2)$$

where $P_{v_i}$ is still a projector, and $N_{v_i}$ is still a nilpotent matrix with the index of nilpotency $K_{v_i}$ being the largest one among the nilpotency indices $\bar{K}_{v_i}$ of the combined operators $\bar{N}_{v_i}$. We define $\Upsilon_{|K} := \{v_i \in \Upsilon | K_{v_i} = K\}$, the subset of $\Upsilon$.

Then, by Theorem III.2 in Ref. [39] the bound (9) is satisfied for $n$ large enough that

$$\|\bigoplus_{v_i\in\Upsilon}(v_iP_{v_i} + N_{v_i})^n\| = \max_{v_i\in\Upsilon_{|K}}(v_iP_{v_i} + N_{v_i})^n\|,$$
The precise condition on how large the numerically computed first, second and third largest absolute eigenvalues, respectively, of the transfer matrices of Case I in Sec. IV A. The black circles in (b) represents the minimum of the graph. The black squares in (c) represent the bounds on the region around $10^{-9}$ where no numerically sampled differences are observed.

which may be guaranteed by satisfying the conditions,

$$n > 2K,$$

$$\left(\frac{n}{K}\right)^K v_{\text{gap}}^{n-K} \geq (K+1) \left(\frac{e}{K}\right)^K |v_i|^n - K_{v_i},$$

for all $v_i \in \mathcal{T}\backslash \mathcal{Y},$

with the constants

$$c_1 := ||S||^{-1} ||S^{-1}||^{-1},$$

$$c_2 := (K + 1) \left(\frac{e}{K}\right)^K ||S|| ||S^{-1}||.$$
of Theorem IV.3 of Ref. 39 leads to the qualitatively same conclusion as using the bound of Theorem III.2 of Ref. 39.

With (34), we may bound the $g(n)$ from above,

$$ g(n) \leq \frac{16e^2\sqrt{D(D+1)}}{\sqrt{2}(1-v_{\text{gap}}^2)^2} \left(1 - v_{\text{gap}}^2\right)^{K+1} \left(\frac{2}{\Delta}\right)^{D-K-1}, $n

where

$$ \Delta = \min_{v_i \in \mathcal{Y}, v_i \neq v_j} |v_i - v_j| \leq 2. $$

Thus, for large enough $n$, we may bound $g(n) \leq c_3$, which leads to $\|E^n - E^n\| \leq c_3 n^K v_{\text{gap}}^{n-K}$, with

$$ c_3 := \frac{16e^2\sqrt{D(D+1)}}{\sqrt{2}(1-v_{\text{gap}}^2)^2} \left(1 - v_{\text{gap}}^2\right)^{K+1} \left(\frac{2}{\Delta}\right)^{D-K-1} \leq \frac{16e^2d_M(D+1)}{\sqrt{2}(1-v_{\text{gap}}^2)^2} \left(1 - v_{\text{gap}}^2\right)^{K+1} \left(\frac{2}{\Delta}\right)^{d_M-K-1}. $$

### Appendix G: Containment of supports

In this appendix we show that for $R$ any of the regions $ABC$, $AB$, $BC$, or $B$,

$$ \text{supp}(\rho_R) \subseteq \text{supp}(\tilde{\rho}_R), $$

$$ \text{supp}(C) \subseteq \text{supp}(\tilde{C}), $$

with $C$ and $\tilde{C}$ defined in (31), for a general iuMPS (both injective and non-injective cases). We separate the latter statement and its proof into Lemma G.1, which might be of independent interest for the reader, since the lemma is valid for any quantum channel with the same input and output spaces, and not only for the quantum channels to which it is applied in this paper.

In the case when an iuMPS has a component with period $p_a > 1$, the Choi matrix $\tilde{C}$ is not of full-rank. It follows, as a consequence of Lemma G.1 proved below, that the Choi matrix $C$ is also not of full-rank. To accommodate for such a general case of iuMPS, we must modify the decomposition (34) in which the index pair $(i, j)$ of the vector $|\psi_{ij}\rangle$, lies in the Cartesian product $\hat{\mathcal{O}} := \{1, 2, \ldots, d_M^2\} \times \{1, 2, \ldots, d_M\}$. We now need, however, to limit the choice of pairs $(i, j)$ to the set

$$ \mathcal{O} := \{(i, j) \in \hat{\mathcal{O}} | \exists \alpha \in \{1, \ldots, N\}, \exists k \in \{1, \ldots, p_a\} : |i\rangle \in \Pi_{k_a}^{\alpha_a} \mathcal{H}_M, |j\rangle \in \Pi_{k_a}^{\alpha_a} \mathcal{H}_M\}. $$

(The particular definition of the set $\mathcal{O}$ does not play a role in any further arguments and is included for the sake of completeness.) Thus, the decomposition (34) becomes

$$ \tilde{C} = \sum_{(i, j) \in \mathcal{O}} |\psi_{ij}\rangle\langle\psi_{ij}|, \quad \text{(G1)} $$

where $|\psi_{ij}\rangle$ are mutually orthogonal. Lemma G.1 permits us to decompose $C$ similarly to (37) in the form,

$$ C = \sum_{(i, j) \in \mathcal{O}} |\psi_{ij}\rangle\langle\psi_{ij}|. \quad \text{(G2)} $$

Now we are ready to show that $\text{supp}(\rho_R) \subseteq \text{supp}(\tilde{\rho}_R)$ follows from $\text{supp}(C) \subseteq \text{supp}(\tilde{C})$. Consider, for example,

$$ \rho_{ABC} = \text{tr}_M(V_C V_B \sigma V_A \rho_{V_C V_B V_A}) \rho_{V_C V_B V_A}^T, $$

$$ \tilde{\rho}_{ABC} = \text{tr}_M(V_C V_B \sigma V_A \tilde{\rho}_{V_C V_B V_A}) \tilde{\rho}_{V_C V_B V_A}^T. $$

Using (28) and (29), we may express $\rho_{ABC}$ and $\tilde{\rho}_{ABC}$ as

$$ \rho_{ABC} = \text{tr}_M(V_C (U \otimes 1_M) W V_B \sigma V_A \rho_{V_C V_B V_A}^T (U^T \otimes 1_M) V_A^T), \quad \text{(G3)} $$

$$ \tilde{\rho}_{ABC} = \text{tr}_M(V_C (U \otimes 1_M) \tilde{W} V_B \sigma V_A \tilde{\rho}_{V_C V_B V_A}^T (U^T \otimes 1_M) V_A^T), $$

where the isometries $W, \tilde{W} : \mathcal{H}_M \rightarrow \mathcal{H}_M \otimes \mathcal{H}_N \otimes \mathcal{H}_N$ are defined similarly to (36) and (38), but with a modification needed to account for Choi matrices $C$ and $\tilde{C}$ not being full-rank.

$$ W = \sum_{k=1}^{d_M} \sum_{(i, j) \in \mathcal{O}} (P_k |\psi_{ij}\rangle \otimes |i\rangle \otimes |j\rangle) \langle k|, $$

$$ \tilde{W} = \sum_{k=1}^{d_M} \sum_{(i, j) \in \mathcal{O}} (P_k |\psi_{ij}\rangle \otimes |i\rangle \otimes |j\rangle) \langle k|. $$

The projectors $P_k$ and the isometry $U$ were defined in Section III.C. From (G3) and (G4) it is clear that if span$\{|\psi_{ij}\rangle\} \subseteq$ span$\{|\tilde{\psi}_{ij}\rangle\}$, then sup$\{(\rho_{ABC})\} \subseteq$ sup$\{(\tilde{\rho}_{ABC})\}$. Moreover, since span$\{|\psi_{ij}\rangle\} \subseteq$ sup$\{(\rho_{ABC})\}$ and span$\{|\psi_{ij}\rangle\} =$ sup$\{(\tilde{\rho}_{ABC})\}$, it suffices to prove that sup$\{(C)\} \subseteq$ sup$\{(\tilde{C})\}$, which we do in Lemma G.1 below. Besides $R = ABC$, we may also perform a similar argument for $R = AB, BC$, and $B$.

We will be considering a quantum channel $\mathcal{J}$, which is an automorphism of the linear space $\mathcal{B}(\mathcal{H})$, i.e., $\mathcal{J} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a finite-dimensional Hilbert space of dimension $d := \dim \mathcal{H}$. The Lemma G.1 is valid for any quantum channel of this type. For a quantum channel $\mathcal{J}$ one can define its peripheral part $\tilde{\mathcal{J}}$ as follows. The linear space $\mathcal{B}(\mathcal{H})$ may be considered a Hilbert space with respect to a Hilbert-Schmidt inner product. We may choose some basis $\{|i\rangle \}$ in $\mathcal{H}$ and vectorize $\mathcal{B}(\mathcal{H})$ through the usual process of identifying $|i\rangle \langle j| \leftrightarrow |i\rangle \otimes |j\rangle$. In the vectorized space $\mathcal{H} \otimes \mathcal{H}$, the quantum channel $\mathcal{J}$ is represented by a (generally non-Hermitian) matrix $T : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$. For the matrix $T$ we may perform the Jordan decomposition

$$ T = \sum_{k=1}^{k_{\text{max}}} (V_k D_k + N_k), $$

where $D_k$ are projections and $\Pi_k$ are nilpotent matrices, satisfying $D_k D_{\ell} = \delta_{k\ell} D_k$ and $D_k N_{\ell} = N_{\ell} D_k = 0$. Moreover, since $\mathcal{J}$ is a quantum channel, then $|v_k| \leq 1$ for $k = 1, \ldots, d^2$ and $N_k$ is 0 when $|v_k| = 1$. Then we may separate the part of
Consider a quantum channel $\tilde{T}$: $\mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$, which can be shown to be a quantum channel (see, for example, Lemma III.2 in Ref. [13]).

**Lemma G.1.** Consider a quantum channel $\mathcal{T}: \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$ and a quantum channel $\tilde{T}: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$, which is a peripheral part of $\mathcal{T}$ (as defined above). Let $C$ be Choi matrices for $\mathcal{T}$ and $\tilde{T}$, respectively. Then,

$$\ker(\tilde{C}) \subseteq \ker(C),$$

or equivalently,

$$\text{supp}(C) \subseteq \text{supp}(\tilde{C}).$$

**Proof.** To start the proof we notice that, since $C = \tilde{C} + \Delta C$, it suffices to show that $\text{supp}(\Delta C) \subseteq \text{supp}(\tilde{C})$, or $\ker(\tilde{C}) \subseteq \ker(\Delta C)$. We will prove the latter statement by obtaining a matrix inequality of the form

$$a_1 \tilde{C} \leq \Delta C \leq a_2 \tilde{C},$$

where $a_1, a_2 \in \mathbb{R}$ are some constants.

Since $C$ is a Choi matrix, it is positive, and we have immediately

$$\Delta C \geq -\tilde{C}. \quad \text{(G6)}$$

Now recall that

$$C = (\text{id} \otimes \mathcal{T})(\{+\} \langle + |), \quad \text{(G7)}$$

where $|+\rangle = \sum_{n=1}^{d} |n\rangle \otimes |n\rangle$ is a maximally entangled vector on $\mathcal{H}$, with $d := \dim \mathcal{K}$ and $\{|n\rangle\}_{n=1}^{d}$ being some orthonormal basis on $\mathcal{K}$. The state $|+\rangle \langle +|$ is defined on the space $\mathcal{B}(\mathcal{K}) \otimes \mathcal{B}(\mathcal{H})$; each factor $\mathcal{B}(\mathcal{K})$ may be considered a Hilbert space with respect to Hilbert-Schmidt inner product. On each factor $\mathcal{B}(\mathcal{K})$ we introduce a so-called Schwinger basis of unitary matrices $Y_k$, with $k = 1, 2, \ldots, d^2$, which satisfy

$$\text{tr} \left( Y^\dagger_k Y_k \right) = d \delta_{kk},$$

$$Y_1 = \mathbb{1}_d.$$  \quad \text{(G8)}

One may think of the Schwinger basis as a generalization of the Pauli matrix basis for $\mathbb{C}^{2 \times 2}$. It immediately follows that

$$\text{tr} (Y_k) = 0, \text{ for } k = 2, 3, \ldots, d^2.$$  \quad \text{(G9)}

The state $|+\rangle \langle +|$ may then be expressed as

$$|+\rangle \langle +| = \frac{d^2}{d} \sum_{k=1}^{d^2} Y_k \otimes \tilde{Y}_k. \quad \text{(G10)}$$

Then,

$$C = \frac{1}{d} \mathbb{1}_d \otimes \mathcal{T}(\mathbb{1}_d) + \frac{d}{d} \sum_{k=2}^{d^2} Y_k \otimes \mathcal{T}(\tilde{Y}_k), \quad \text{(G11)}$$

where we have explicitly separated the term corresponding to $k = 1$.

From Propositions 3.2 and 3.3 of Ref. [20] and Appendix I of Ref. [33] it follows that $Y_1 = \mathbb{1}_d$ may be decomposed as $\mathbb{1}_d = \sum_{a=1}^{N_N} \sum_{k_a=1}^{\Pi_{k_a}^\dagger} \Pi_{k_a}^\dagger$ (in the notation introduced in Section III D), so this family of projectors $\Pi_{k_a}^\dagger$ spans the subspace of $\mathcal{B}(\mathcal{K})$ corresponding to the peripheral spectrum of $\mathcal{T}$. This implies that $\mathcal{T}(\Pi_{k_a}^\dagger) = \mathcal{T}(\Pi_{k_a}^\dagger)$, hence $\mathcal{T}(\mathbb{1}_d) = \mathcal{T}(\mathbb{1}_d)$, allowing us to express $\tilde{C}$ and $\Delta C$ in terms of the Schwincker basis,

$$\tilde{C} = \frac{1}{d} \mathbb{1}_d \otimes \mathcal{T}(\mathbb{1}_d) = \frac{1}{d} \mathbb{1}_d \otimes \mathcal{T}(\mathbb{1}_d), \quad \text{(G12)}$$

$$\Delta C = \frac{1}{d} \sum_{k=2}^{d^2} Y_k \otimes \mathcal{T}(\tilde{Y}_k).$$

We define a completely positive map $\mathcal{N}: \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{K})$ by

$$\mathcal{N}(X) := \sum_{k=2}^{d^2} Y_k X Y_k^\dagger. \quad \text{(G13)}$$

Using the relation $\text{tr}(X) \mathbb{1}_d = \frac{1}{d} \sum_{k=1}^{d^2} Y_k X Y_k^\dagger$,  \quad \text{(G14)}

which is (up to normalization) a Kraus representation of the completely mixing quantum channel, we infer that

$$\mathcal{N}(X) = d \text{tr}(X) \mathbb{1}_d - X. \quad \text{(G15)}$$

Now consider action of $\mathcal{N}$ on each of the Schwinger matrices $Y_k$, recalling that $Y_1 = \mathbb{1}_d$ and $\text{tr}(Y_n) = 0$ for $n = 2, \ldots, d^2$,

$$\mathcal{N}(Y_1) = \sum_{k=2}^{d^2} Y_k Y_1 Y_k^\dagger = (d^2 - 1) Y_1, \quad \text{(G16)}$$

$$\mathcal{N}(Y_n) = \sum_{k=2}^{d^2} Y_k Y_n Y_k^\dagger = -Y_n, \text{ for } n = 2, \ldots, d^2. \quad \text{(G17)}$$

Since the map $\mathcal{N}$ is completely positive, the map $\mathcal{N} \otimes \text{id}: \mathcal{B}(\mathcal{K}) \otimes \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{K}) \otimes \mathcal{B}(\mathcal{K})$ is positive, and its action on the Choi matrix $C$ preserves its positivity, leading to the inequality

$$0 \leq (\mathcal{N} \otimes \text{id})(C) \quad \text{(G18)}$$

$$= \frac{1}{d} \sum_{k=1}^{d^2} \mathcal{N}(Y_k) \otimes \mathcal{T}(\tilde{Y}_k)$

$$= \left( d^2 - 1 \right) \frac{1}{d} \mathbb{1}_d \otimes \mathcal{T}(\mathbb{1}_d) - \frac{1}{d} \sum_{k=2}^{d^2} Y_k \otimes \mathcal{T}(\tilde{Y}_k)$

$$= \left( d^2 - 1 \right) \tilde{C} - \Delta C.$$
Combining (G6) and (G17), we obtain
\[-C \leq \Delta C \leq (d^2 - 1)C.\] (G18)

For any \(|\phi\rangle \in \ker(\hat{C})\) we observe
\[0 = -\langle \phi | \hat{C} | \phi \rangle \leq \langle \phi | \Delta C | \phi \rangle \leq (d^2 - 1)\langle \phi | \hat{C} | \phi \rangle = 0.\] (G19)

The result \(\langle \phi | \Delta C | \phi \rangle = 0\), and the polarization identity applied
to the complex subspace \(\ker(\hat{C})\), implies that \(\Delta C\) is the zero
operator on \(\ker(\Delta C)\). Hence,
\[\ker(\hat{C}) \subseteq \ker(\Delta C),\] (G20)
completing the proof. \(\square\)

**Remark:** For the quantum channel \(\mathcal{J} = \mathcal{E}^{[B]}\) a stronger result
holds,
\[\text{supp}(C) = \text{supp}(\hat{C}).\] (G21)

Indeed, Lemma [G.1] validates the decomposition [39],
\[|\psi_{ij}\rangle = \left(\mathbb{1}_{M \otimes M} + \Gamma \right) |\psi_{ij}\rangle,\]
for any \((i, j) \in \mathcal{E}\), where \(|\Gamma| \to 0\) as \(|B| \to 0\), and vectors \(|\psi_{ij}\rangle\)
are mutually orthogonal. The equations (G1), (G2), and (39)
together imply that for large enough \(|B|\) the matrix \(\mathbb{1}_{M \otimes M} + 1\)
is full-rank and invertible, hence for the quantum channel \(\mathcal{E}^{[B]}\),
\[\text{supp}(\hat{C}) = \text{span}\{|\psi_{ij}\rangle\} = \text{span}\{|\psi_{ij}\rangle\} \subseteq \text{supp}(C).\] (G22)

Combined with (G5), this implies (G21).

**Appendix H: Optimality of the decay rate**

In this appendix we show that the decay rate \(q = -2 \ln v_{\text{gap}}\)
cannot be improved (at least without additional assumptions
imposed on iuMPS). We prove this statement by providing an
example of an injective iuMPS for which QCMI may be ex-
plicitly evaluated to be \(I(A : C|B) = O\left(e^{-q|B|}\right)\). (Recall that
for an injective iuMPS \(K = 0\) in Theorem [111], and the poly-
nomial factor in the bound for QCMI is absent.

For our example we set \(\mathcal{H}_c = \mathbb{C}^8\) and \(\mathcal{H}_M = \mathbb{C}^2\). We
will treat \(\mathcal{H}_c = \mathbb{C}^8\) as \(\mathbb{C}^4 \oplus \mathbb{C}^4\), and set orthonormal bases
\(\{|\xi_{ij}\rangle\}_{i,j=1}^2\) and \(\{|\zeta_{ij}\rangle\}_{i,j=1}^2\) in the first and the second
copies of \(\mathbb{C}^4\), respectively. We denote Pauli matrices as \(\sigma_x, \sigma_y,\)
and \(\sigma_z\), and the eigenvalues of \(\sigma_x\) as \(\sigma_{x,1} := 1\) and \(\sigma_{x,2} := -1\). We
set \(\sigma = \frac{\sqrt{2}}{2}\). We also set some constant \(0 < \kappa < 1\).

We construct the operator
\[V = \sum_{i,j=1}^2 \left(\sqrt{\sigma_i} |\xi_{ij}\rangle + \sqrt{\frac{\kappa}{2}} \sigma_{x,i} |\zeta_{ij}\rangle |\xi_{ij}\rangle \right) \otimes |i\rangle \langle j|,\] (H1)

which is an isometry,
\[V^\dagger V = \sum_{i,j,j'=1}^2 \left(\sqrt{\sigma_i} |\zeta_{ij'}\rangle + \sqrt{\frac{\kappa}{2}} \sigma_{x,i} |\zeta_{ij'}\rangle |\xi_{ij}\rangle \otimes |j\rangle \otimes |j'\rangle\right),\]
\[= \sum_{i,j=1}^2 \left(\sigma_i + \frac{\kappa}{2} \sigma_{x,i} \sigma_{x,j} \right) |j\rangle \langle j| + \frac{2}{\sqrt{2}}\delta_{j,j'} \mathbb{1},\]
\[= \mathbb{1}.\]

We choose the constructed isometry (H1) to define the iuMPS
in our example.

The isometry (H1) induces the quantum channel \(\mathcal{E} : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^2)\), defined by
\[\mathcal{E}(X) = \text{tr}_r(V XV^\dagger),\]
\[= \sum_{i,j=1}^2 \sigma_i |j\rangle \langle j| |i\rangle + \sum_{i,j=1}^2 \frac{\kappa}{2} \sigma_{x,i} |j\rangle \langle j| \sigma_{x,j} |i\rangle \langle i|.\]
\[= \text{tr}(X) \sigma + \frac{\kappa}{2} \text{tr}(\sigma X) \sigma_x,\]

with eigenvectors and eigenvalues
\[\mathcal{E}(\sigma) = \sigma,\]
\[\mathcal{E}(\sigma_x) = 0,\]
\[\mathcal{E}(\sigma_y) = 0,\]
\[\mathcal{E}(\sigma_z) = \kappa \sigma_z.\]

It follows that for \(\mathcal{E}\), the density operator \(\sigma = \frac{\sqrt{2}}{2}\) is the fixed
point and \(v_{\text{gap}} = \kappa\).

Further we will need the expression for the \(n\)-fold composition
of \(\mathcal{E}\),
\[\mathcal{E}^n(X) = \text{tr}(X) \sigma + \frac{\kappa^n}{2} \text{tr}(\sigma X) \sigma_x.\] (H2)

To calculate the entropies in the expression for \(I(A : C|B)\), we
will use the relation (62), from which it follows that
\[S(\rho_K) = S\left(\mathcal{E}^{[|B|]} \otimes \text{id}_M\right)(|\sqrt{\sigma}| \langle \sqrt{\sigma}|\),\]
where \(|\sqrt{\sigma}| = \sum_{i=1}^2 \sqrt{\sigma_i} |i\rangle \otimes |i| \in \mathcal{H}_M \otimes \mathcal{H}_M\). For the sake of convenience, we denote
\[\rho'_K := (\mathcal{E}^{[|B|]} \otimes \text{id}_M)(|\sqrt{\sigma}| \langle \sqrt{\sigma}|).\]

Using (H2), we may calculate
\[
\rho'_R = (\sigma^{[R]} \otimes \text{id}_{M'}) (|\sqrt{\sigma}\rangle \langle \sqrt{\sigma}|)
\]
\[
= \left(\sigma^{[R]} \otimes \text{id}_{M'}\right) \left(\sum_{i,j=1}^{2} \sqrt{\sigma_i \sigma_j} |i\rangle \langle j| \otimes |j\rangle \langle i| \right)
\]
\[
= \sigma \otimes \sum_{i,j=1}^{2} \sqrt{\sigma_i \sigma_j} \delta_{ij} |i\rangle \langle j| + \frac{\kappa^{[R]}_{z}}{2} \sigma_z \otimes \sum_{i,j=1}^{2} \sqrt{\sigma_i \sigma_j} |i\rangle \langle j| \otimes |j\rangle \langle i| \\
= \frac{1}{4} \left(1 \otimes 1 + \kappa^{[R]} \sigma_z \otimes \sigma_z\right),
\]

where we have used \(\sigma_1 = \sigma_2 = 1/2\) to get to the last line. Since both matrices \(1 \otimes 1\) and \(\sigma_z \otimes \sigma_z\) are diagonal and

\[
\ln \rho'_R = \ln \left(\frac{1}{4} \left(1 \otimes 1 + \kappa^{[R]} \sigma_z \otimes \sigma_z\right)\right)
\]
\[
= \ln \left(\frac{1}{4} \otimes 1\right) + \ln \left(1 \otimes 1 + \kappa^{[R]} \sigma_z \otimes \sigma_z\right)
\]
\[
= -2 \ln 2 \otimes 1 + \frac{1}{2} \left(\ln (1 + \kappa^{[R]}) + \ln (1 - \kappa^{[R]})\right) \otimes 1 + \frac{1}{2} \left(\ln (1 + \kappa^{[R]}) - \ln (1 - \kappa^{[R]})\right) \sigma_z \otimes \sigma_z.
\]

This allows us to obtain an explicit expression for \(S(\rho_R)\),

\[
S(\rho_R) = S(\rho'_R)
\]
\[
= -\text{tr}(\rho'_R \ln \rho'_R)
\]
\[
= \text{tr} \left(\frac{1}{4} \left(1 \otimes 1 + \kappa^{[R]} \sigma_z \otimes \sigma_z\right)\right)
\]
\[
\times \left(2 \ln 2 \otimes 1 - \frac{1}{2} \left(\ln (1 + \kappa^{[R]}) + \ln (1 - \kappa^{[R]})\right) \otimes 1 - \frac{1}{2} \left(\ln (1 + \kappa^{[R]}) - \ln (1 - \kappa^{[R]})\right) \sigma_z \otimes \sigma_z\right)
\]
\[
= 2 \ln 2 - \frac{1}{2} \left(\ln (1 + \kappa^{[R]}) + \ln (1 - \kappa^{[R]})\right) - \frac{\kappa^{[R]}_{z}}{2} \left(\ln (1 + \kappa^{[R]}) - \ln (1 - \kappa^{[R]})\right).
\]

Technically, we could use the above expression to calculate QCMI, but it is more convenient to make the Taylor expansion of \(S(\rho_R)\) in powers of \(\kappa\),

\[
S(\rho_R) = 2 \ln 2 - \frac{\kappa^{[R]}_{2z}}{2} + O(\kappa^{[R]}_{4z}).
\]

We may then calculate QCMI,

\[
I(A:C|B) = -\frac{\kappa^{2(|A|+|B|)}}{2} + O(\kappa^{4(|A|+|B|)}) - \frac{\kappa^{2(|B|+|C|)}}{2} + O(\kappa^{4(|A|+|B|)+|C|}) + \frac{\kappa^{2(|A|+|B|)+|C|}}{2} + O(\kappa^{4(|A|+|B|)+|C|}) + \frac{\kappa^{2|B|}}{2} + O(\kappa^{4|B|})
\]
\[
= \frac{(1 - \kappa^{2|A|})(1 - \kappa^{2|B|})}{2} e^{2 \ln \kappa^{[R]}} + O(\kappa^{4|B|}).
\]
Thus, for the chosen example of iuMPS, QCMI cannot be bounded by any function $Q e^{-\nu|B|}$ from the statement of Theorem III.1 (recall that $K = 0$ for the considered iuMPS) with $q > -2\ln \nu_{\text{gap}}$.

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