Mathematical conditions for and physical meaning of a maximum of the determinant of $K_T$ in the prebuckling regime

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Abstract

It is shown that the determinant of the tangent stiffness matrix has a maximum in the prebuckling regime if and only if the determinant of a specific linear combination of the first and the third derivative of this matrix with respect to a dimensionless load factor vanishes. The mathematical tool for this proof is the so-called consistently linearized eigenproblem in the frame of the Finite Element Method. The physical meaning of the mentioned maximum is the one of a minimum of the percentage bending energy of the total strain energy. The paper provides mathematical and physical background knowledge on numerical results that were obtained 35 years ago.

1. Introduction

The vanishing of the determinant of the tangent stiffness matrix $\tilde{K}_T$, i.e.

$$\text{Det}\tilde{K}_T = 0,$$

(1)

is an indispensable ingredient of academic teaching of computational structural stability with emphasis on the Finite Element Method (FEM). Eq. (1) represents a necessary and sufficient condition for loss of static stability of structures subjected to conservative loading $\lambda P$, where $P$ denotes the vector of node forces, in the context of the FEM, that are work-equivalent to the actual loading, and $\lambda$ is a dimensionless load factor. In general, $\tilde{K}_T$ depends on the vector of node displacements $q(\lambda)$, i.e.

$$\tilde{K}_T := \tilde{K}_T(q(\lambda)).$$

(2)

The tilde in $\tilde{K}_T$ serves the purpose of distinguishing the tangent stiffness matrix from tensor functions $K_T$ which refer to internal forces that are not in equilibrium with the external forces [1].

In contrast to Eq. (1), the exact physical meaning of

$$\frac{d}{d\lambda} (\text{Det}(\tilde{K}_T)) = 0, \quad \frac{d^2}{d\lambda^2} (\text{Det}(\tilde{K}_T)) < 0,$$

(3)

in the prebuckling regime seems to be unknown. This would explain why mathematical conditions, alternative to (3), analogous to the alternative condition

$$\tilde{K}_T \cdot v = 0$$

(4)
to (1), where $v$ is the eigenvector of $\tilde{K}_T$, could not be found in the literature.

The mathematical definition of

$$\frac{d}{d\lambda} (\text{Det}(A)).$$

(5)

where each element of $\text{Det}(A)$ which is of order $n$ is a differentiable function of $\lambda$, reads as follows [2]: The derivative of $\text{Det}(A)$ with respect to $\lambda$ is equal to the sum of $n$ derivatives, the $i$th one of which is identical with $\text{Det}(A)$ except for the $i$th row, which consists of the derivatives of the elements of the $i$th row of $\text{Det}(A)$. This definition is not very helpful, because normally the elements of $\text{Det}(A)$ are numerical quantities that are obtained in the frame of incremental-iterative nonlinear Finite Element Analysis (FEA).

The objective of this work is to derive mathematical conditions for (3) and to explain the physical meaning of a maximum of $\text{Det}(\tilde{K}_T)$ in the prebuckling regime. The tool for this derivation is the so-called consistently linearized eigenproblem (CLE) [3].

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In Section 2, a brief summary of the CLE will be given. In Section 3, the aforementioned conditions will be derived, and in Section 4 the physical meaning of these conditions will be explained. Section 5 is devoted to a numerical example. Conclusions drawn from this work will be given in Section 6.

2. Consistently linearized eigenproblem

The formulation of the CLE for the first eigenpair \((\lambda_1', \mathbf{v}_1')\) reads as follows [3]:

\[
A_1(\lambda) \cdot \mathbf{v}_1(\lambda) = 0, \quad A_1 := \mathbf{K}_T + (\lambda_1' - \lambda) \mathbf{K}_T, \quad |\mathbf{v}_1'| = 1, \tag{6}
\]

where \(\mathbf{K}_T\) is the first derivative of \(\mathbf{K}_T\) with respect to \(\lambda\) along a direction parallel to the primary path, i.e.

\[
\mathbf{K}_T = \frac{d}{dz} \mathbf{K}_T(q(\lambda)) := \frac{d}{dz} \mathbf{K}_T(q(\lambda) + z\mathbf{q}(\lambda)). \tag{7}
\]

In contrast to the matrix \(\mathbf{K}_T\), which is positive definite in the prebuckling regime, \(\mathbf{K}_T\) is an indefinite matrix. Apart from one special case [4], \(\mathbf{v}_1'\) does not become an eigenvector of \(\mathbf{K}_T\) in the prebuckling regime. Hence, the eigenvalue \(\lambda_1' - \lambda\) remains finite. Fig. 1 illustrates a typical function \(\lambda_1'(\lambda)\) for bifurcation bucking from a nonlinear load–displacement path. At the starting point \(A\), \(\lambda = 0\). At the stability limit \(S:\)

\[
\lambda_1' - \lambda = 0 \Rightarrow \lambda_1' = \lambda = \lambda_S, \quad \lambda_1^* = 0. \tag{8}
\]

[3] and

\[
\mathbf{v}_1'(\lambda_S) := \mathbf{v}_1. \tag{9}
\]

(For snap-through, \(\lambda\) is a nonmonotonic function of the displacements. This suggests replacing \(\lambda\) by a suitable parameter \(\xi_0\).)

In order to derive the mathematical conditions for a maximum of \(\text{Det} \mathbf{K}_T\), the first three derivatives of (6.1) with respect to \(\lambda\) are needed. They are obtained as follows [4]:

\[
\begin{align*}
\mathbf{A}_1 & = \frac{d}{d\lambda} A_1 \cdot \mathbf{v}_1 + A_1 \cdot \frac{d}{d\lambda} \mathbf{v}_1 = 0, \tag{10} \\
\mathbf{A}_1 & = \frac{d}{d\lambda} \mathbf{v}_1 + 3\mathbf{A}_1 \cdot \mathbf{v}_1 + 3\frac{d}{d\lambda} \mathbf{A}_1 \cdot \mathbf{v}_1 = 0, \tag{11} \\
\mathbf{A}_1 & = \lambda_1' \mathbf{K}_T + (\lambda_1' - \lambda) \mathbf{K}_T, \tag{12} \\
\mathbf{A}_1 & = \lambda_1' \mathbf{K}_T + (2\lambda_1' - 1) \mathbf{K}_T + (\lambda_1' - \lambda) \mathbf{K}_T, \tag{13}
\end{align*}
\]

where

\[
\begin{align*}
\mathbf{A}_1 & = \lambda_1' \mathbf{K}_T + (\lambda_1' - \lambda) \mathbf{K}_T, \tag{14}
\end{align*}
\]

Fig. 1. \(\lambda_1'\) versus \(\lambda\), with \(\lambda_1'(\lambda) - \lambda\) as a typical eigenvalue function for bifurcation bucking [4].

\[
\mathbf{A}_1 = \lambda_1' \mathbf{K}_T + 3\lambda_1' \mathbf{K}_T + (3\lambda_1' - 2) \mathbf{K}_T + (\lambda_1' - \lambda) \mathbf{K}_T, \tag{15}
\]

and

\[
\begin{align*}
\mathbf{v}_1' & = \sum_{j=2}^{N} c_{ij} \mathbf{v}_j', \quad \text{with} \quad c_{ij} = \frac{\lambda_1' - \lambda}{\lambda_1' - \lambda_j}, \quad \mathbf{v}_j' = \mathbf{K}_T \cdot \mathbf{v}_j. \tag{16}
\end{align*}
\]

\[
\begin{align*}
\mathbf{v}_1' & = \sum_{j=2}^{N} \left( c_{ij} \mathbf{v}_j' + c_{ij} \mathbf{v}_j' \right), \tag{17} \\
\mathbf{v}_1' & = \sum_{j=2}^{N} \left( c_{ij} \mathbf{v}_j' + 2c_{ij} \mathbf{v}_j' + c_{ij} \mathbf{v}_j' \right). \tag{18}
\end{align*}
\]

where \(\lambda_1' - \lambda_j\) represents the difference of the first and the \(j\)th eigenvalue and \(\mathbf{v}_j'\) denotes the \(j\)th eigenvector. Following from normalization of \(\mathbf{v}_1'\) according to (6.3):

\begin{align*}
\mathbf{v}_1' \cdot \mathbf{v}_1' & = 0, \tag{19} \\
\mathbf{v}_1' \cdot \mathbf{v}_1' + \mathbf{v}_1' \cdot \mathbf{v}_1' & = 0, \tag{20} \\
3\mathbf{v}_1' \cdot \mathbf{v}_1' + \mathbf{v}_1' \cdot \mathbf{v}_1' & = 0. \tag{21}
\end{align*}

Substitution of (8.1) into (16.2) and insertion of the result into (16.1) gives

\[
\mathbf{v}_1'(\lambda_S) = \mathbf{0}. \tag{22}
\]

3. Derivation of mathematical conditions for a maximum of \(\text{Det} \mathbf{K}_T\)

Premultiplication of (11) by \(\mathbf{v}_1'\) and use of (6.1) gives

\[
\mathbf{v}_1' \cdot \mathbf{A}_1 \cdot \mathbf{v}_1' + 2\mathbf{v}_1' \cdot \mathbf{A}_1 \cdot \mathbf{v}_1' = 0. \tag{23}
\]

Substitution of (13) and (14) into (23) yields

\[
\begin{align*}
\mathbf{v}_1' \cdot \left( \lambda_1' \mathbf{K}_T + (2\lambda_1' - 1) \mathbf{K}_T + (\lambda_1' - \lambda) \mathbf{K}_T \right) & \cdot \mathbf{v}_1' + 2\mathbf{v}_1' \\
& \cdot \lambda_1' \mathbf{K}_T + (\lambda_1' - \lambda) \mathbf{K}_T \right) \cdot \mathbf{v}_1' = 0. \tag{24}
\end{align*}
\]

Substitution of (16) into the bilinear form \(\mathbf{v}_1' \cdot \mathbf{K}_T \cdot \mathbf{v}_1'\) in (24), consideration of the orthogonality relations

\[
\mathbf{v}_j' \cdot \mathbf{K}_T \cdot \mathbf{v}_1' = 0, \quad j = 2, 3, \ldots, N, \tag{25}
\]

following from (6.1), and use of \(\mathbf{v}_1' \cdot \mathbf{A}_1 \cdot \mathbf{v}_1' = 0\), following from premultiplication of (10) by \(\mathbf{v}_1'\) and consideration of (6.1), gives

\[
\begin{align*}
\lambda_1' (\lambda_1' - \lambda - 1) & = \frac{1}{\mathbf{v}_1' \cdot \mathbf{K}_T \cdot \mathbf{v}_1'} \cdot \left( 2\mathbf{v}_1' \cdot \mathbf{K}_T \cdot \mathbf{v}_1' + \frac{\mathbf{v}_1' \cdot \mathbf{K}_T \cdot \mathbf{v}_1'}{\mathbf{v}_1' \cdot \mathbf{K}_T \cdot \mathbf{v}_1'} \right)^2. \tag{26}
\end{align*}
\]

Substitution of (8) into

\[
\begin{align*}
d \frac{\lambda_1' \lambda_1' - \lambda - 1}{\mathbf{v}_1' \cdot \mathbf{K}_T \cdot \mathbf{v}_1'} \left( \lambda_1' - \lambda - 1 \right) = \frac{\lambda_1' (\lambda_1' - \lambda - 1)}{(\lambda_1' - \lambda)^2}, \tag{27}
\end{align*}
\]

and
\[
\frac{d^2}{d\lambda^2} \left( \frac{\lambda_1}{\lambda_1 - \lambda} \right) = \frac{\lambda_1 (\lambda_1 - \lambda) - \lambda_1 (\lambda_1 - \lambda)^2 - 2 (\lambda_1 (\lambda_1 - \lambda) - \lambda_1 (\lambda_1 - 1)) (\lambda_1 - \lambda) (\lambda_1 - 1)}{[(\lambda_1 - \lambda)^4]},
\]

(28)

and use of de l'Hôpital's rule yields
\[
\left. \frac{d}{d\lambda} \left( \frac{\lambda_1}{\lambda_1 - \lambda} \right) \right|_{\lambda = \lambda_0} = -\frac{1}{2} \left( \lambda_1^2 + \lambda \right),
\]

(29)

and
\[
\left. \frac{d^2}{d\lambda^2} \left( \frac{\lambda_1}{\lambda_1 - \lambda} \right) \right|_{\lambda = \lambda_0} = -\frac{1}{6} (15 \lambda_1^2 + 2 \lambda_1),
\]

(30)

respectively. For a general stress state, resulting from a combination of membrane and non-membrane action, in addition to (8):
\[
\frac{d^2}{d\lambda^2} \left( \frac{\lambda_1}{\lambda_1 - \lambda} \right) = 0,
\]

(31)

indicating a point of inflection of the function \( \lambda_1/(\lambda_1 - \lambda) \) at \( \lambda = \lambda_0 \) (see the situation at point \( S \) in Fig. 2b). Substitution of (31) into (30) gives
\[
15 \lambda_1^2 + 2 \lambda_1 = 0.
\]

(32)

For the limiting case of buckling from a state of pure bending (lateral torsional buckling):
\[
\lambda_1 = \lambda_0.
\]

(33)

For the limiting case of buckling from a membrane stress state, in addition to (31):
\[
\left. \frac{d}{d\lambda} \left( \frac{\lambda_1}{\lambda_1 - \lambda} \right) \right|_{\lambda = \lambda_0} = 0,
\]

(34)

indicating a saddle point of the function \( \lambda_1/(\lambda_1 - \lambda) \) at \( \lambda = \lambda_0 \). Substitution of (34) into (29) gives
\[
\lambda_1^2 + \lambda_1 = 0.
\]

(35)

Substitution of (35) into (32) yields
\[
-15 \lambda_1^2 + 2 \lambda_1 = 0.
\]

(36)

For a maximum of \( \text{Det} \mathbf{K}_T \) in the prebuckling regime:
\[
\frac{d}{d\lambda} \left( \frac{\lambda_1}{\lambda_1 - \lambda} \right) \bigg|_{\lambda = \lambda_0} = 0,
\]

\[
\frac{d^2}{d\lambda^2} \left( \frac{\lambda_1}{\lambda_1 - \lambda} \right) \bigg|_{\lambda = \lambda_0} < 0.
\]

(37)

indicating a maximum of the function \( \lambda_1/(\lambda_1 - \lambda) \) at \( \lambda = \lambda_0 \) (see the situation at point \( R \) in Fig. 2b). The rationale of this assertion becomes evident from substitution of (37.1) into (27), resulting in
\[
\lambda_0 (\lambda_0 - 1) - \lambda_0 (\lambda_0 - 1) = 0.
\]

(38)

This relation indicates a disintegration of (26), analogous to the disintegration of (26) at the stability limit, albeit one of a different kind. As will be shown later:
\[
\lambda_1 \big|_{\lambda = \lambda_0} = \frac{1}{2}.
\]

(39)

(see the situation at point \( R \) in Fig. 2a).

Representing part of the solution of the CLE, the eigenvalue \( \lambda_1 - \lambda \) is known for all values of \( \lambda \) in the prebuckling regime. In order to determine the two unknown quantities \( \lambda_1 \) and \( \lambda_1 \), at \( \lambda = \lambda_0 \), in (38), an additional condition is needed. As part of a disintegration of (12), this condition is obtained as
\[
\mathbf{A} \cdot \mathbf{i}_1 = 0.
\]

(40)

In this context, it is worthy of note that (12) also disintegrates at the stability limit, albeit differently from the disintegration at point \( R \). Substitution of (14) into (40) gives
\[
\left[ \lambda_1 \mathbf{K}_T + \left( 2 \lambda_1 - 1 \right) \mathbf{K}_T + (\lambda_1 - \lambda) \mathbf{K}_T \right] \cdot \mathbf{v}_1 = 0.
\]

(41)

Elimination of \( \lambda_1 \) in (41) by means of (38) yields
\[
\left[ \frac{\lambda_1 (\lambda_1 - 1)}{\lambda_1 - \lambda} \mathbf{K}_T + \left( 2 \lambda_1 - 1 \right) \mathbf{K}_T + (\lambda_1 - \lambda) \mathbf{K}_T \right] \cdot \mathbf{v}_1 = 0.
\]

(42)

Substitution of (16) into (42) and premultiplication by \( \mathbf{v}_1^* \) results in
\[
\mathbf{v}_1^* \cdot \left[ \frac{\lambda_1 (\lambda_1 - 1)}{\lambda_1 - \lambda} \mathbf{K}_T + \left( 2 \lambda_1 - 1 \right) \mathbf{K}_T + (\lambda_1 - \lambda) \mathbf{K}_T \right] \cdot \left( \sum_{j=2}^{N} c_{ij} \mathbf{v}_j^* \right) = 0.
\]

(43)

Consideration of the orthogonality condition (25) in (43) gives
\[
\mathbf{v}_1^* \cdot \left[ \left( 2 \lambda_1 - 1 \right) \mathbf{K}_T + (\lambda_1 - \lambda) \mathbf{K}_T \right] \cdot \left( \sum_{j=2}^{N} c_{ij} \mathbf{v}_j^* \right) = 0.
\]

(44)
Substitution of (38) into (26) yields
\[ 2v_i^* \cdot K_{TT} \cdot v_i^* + (v_i^* \cdot K_{TT} \cdot v_i^*) (v_i^* \cdot K_{TT} \cdot v_i^*) - (v_i^* \cdot K_{TT} \cdot v_i^*)^2 = 0. \]
which implies that
\[ K_{TT} = 0 \quad \forall \lambda, \]
which confirms (39), and that
\[ v_i^* \cdot K_{TT} \cdot v_j^* = 0, \quad j = 2, 3, \ldots, N. \]
This permits interpretation of \( \lambda_1^*/(\lambda_1^* - \lambda) \) (see Fig. 2b) as the increase of the aggregate nonlinearity of the underlying stability problem, normalized with respect to the eigenvalue. Consequently, point \( R \) in Fig. 2b indicates a maximum of such an increase.

Linear stability problems are a special case of buckling from a membrane stress state [4]. To obtain a mathematical condition for buckling from such a stress state, (11) is premultiplied by \( v_i^* \):
\[ v_i^* \cdot (A_1 \cdot v_i^* + 2A_1 \cdot v_i^* + A_1 \cdot v_i^*) = 0. \]
For buckling from a membrane stress state, (56) disintegrates into
\[ v_i^* \cdot A_1 \cdot v_i^* = 0 \quad \text{and} \quad v_i^* \cdot (2A_1 \cdot v_i^* + A_1 \cdot v_i^*) = 0. \]
Eq. (57.1) indicates that \( v_i^* \) is orthogonal to \( A_1 \cdot v_i^* \) for all values of \( \lambda \) in the prebuckling regime. Disintegration of (56) into (57) does not imply disintegration of (11), as was originally assumed [6]. Comparing (57) with (40) reveals that buckling from a membrane stress state precludes the possibility of a maximum of \( \text{Det} K_T \) in the prebuckling regime.

Buckling from a state of pure bending (lateral torsional buckling) is characterized by [4]
\[ v_i^* = v \quad \forall \lambda. \]
Hence:
\[ v_i^* = 0 \quad \forall \lambda. \]
Substitution of (59) into (10) and consideration of (58) yields
\[ A_1 \cdot v_i = 0. \]
Comparing (60) with (40) shows that also buckling from a state of pure bending precludes the possibility of a maximum of \( \text{Det} K_T \) in the prebuckling regime.

Consequently, such a maximum is only possible for buckling from a general stress state, characterized by
\[ 0 < \frac{U_b}{U_M + U_b} < 1 \quad \forall \lambda, \]
where \( U_M \) denotes the membrane energy and \( U_b \) stands for the complement of \( U_M \) to the total strain energy. For convenience’s sake, \( U_b \) is termed as the bending energy. Recalling that a maximum of \( \text{Det} K_T \) in the prebuckling regime correlates with a maximum of the function \( \lambda_1^*/(\lambda_1^* - \lambda) \), i.e. with a maximum increase of the aggregate normalized nonlinearity of the underlying stability problem, such a maximum must correspond to a minimum percentage bending energy of the total strain energy, characterized by
\[ \frac{\text{d}^2}{\text{d} \lambda^2} \left( \frac{U_b}{U_M + U_b} \right) > 0 \Rightarrow \frac{U_b}{U_M + U_b} \bigg( \frac{U_b}{U_M + U_b} \bigg)^2 \frac{U_M}{U_M + U_b} > 0. \]
Fig. 3 elucidates the physical meaning of a maximum of \( \text{Det} K_T \) in the prebuckling regime. According to (64), the tangent to the curve \( U_b/\text{Det} K_T \) at point \( R \) is equal to the chord. As follows from (63) and (67), the curvature of this curve at point \( R \) is positive. Analogous to the situation at \( R \), the tangent to the curve \( U_b/\text{Det} K_T \) at the stability limit \( S \) is also equal to the chord. However, at \( S \), in contrast to the situation at \( R \):
\[ \frac{\text{d}^2}{\text{d} U_M^2} \left( \frac{U_b}{U_M + U_b} \right) \bigg( \frac{U_M + U_b}{U_M} \bigg)^2 \frac{U_M}{U_M + U_b} < 0. \]
indicating a planar point of the curve $U_B(U_M)$ at the stability limit.

5. Numerical example

Following the failure of a large cylindrical fluid coker with a shallow spherical bottom head while undergoing its hydrostatic proof test at Avon, California, in 1956, interest in the problem of buckling of such pressure vessels under internal pressure was growing. Fig. 4 shows a buckled test specimen of a pressure vessel head that was subjected to internal pressure [7].

Fig. 5 illustrates geometrical details of a pressure vessel head that was investigated by means of the FEM 35 years ago by Kanoedia, Gallagher, and the senior writer [8].

Fig. 6 shows the distribution of the circumferential membrane force $n_a$ due to internal pressure $p = 703.8 \text{kN/m}^2$, in the meridional (axial) direction [8].

Fig. 7 shows a plot of the normalized determinant of a reduced form of the tangent stiffness matrix $\mathbf{K}_T$ versus the load factor $\lambda$ [8].

cylinder and $D$ stands for its diameter. Fig. 7 shows a plot of the normalized determinant of a reduced form of the tangent stiffness matrix $\mathbf{K}_T$, containing the small-displacement stiffness matrix $\mathbf{K}_0$ and the so-called first-order (linear) geometric stiffness matrix $\lambda \mathbf{N}_1$, versus the load factor $\lambda$. Normalization is performed with respect to $\det \mathbf{K}_0$. It is seen that the determinant increases up to a value of the internal pressure close to the buckling pressure, indicating the initial load-hardening nature of the behavior of the structure. The conjecture that the vanishing of the determinant of $\mathbf{K}_T$ is the condition for a maximum of the determinant of $\mathbf{K}_T$ [9] is falsified by (50), according to which such a maximum occurs if the determinant of a specific linear combination of $\mathbf{K}_T$ and $\mathbf{K}_T$ vanishes.

6. Conclusions

1. The determinant of the tangent stiffness matrix has a maximum in the prebuckling regime if and only if the determinant of a specific linear combination of the first and the third derivative of the tangent stiffness matrix vanishes.
2. A necessary condition for a maximum of the determinant in the prebuckling regime is a general stress state, resulting from a combination of membrane and bending action.
3. The physical meaning of a maximum of the determinant of the tangent stiffness matrix in the prebuckling regime is the one of a minimum of the percentage bending energy of the total strain energy.
4. The present work is a good example of the lasting usefulness of mathematical and physical background knowledge about numerical results that were obtained long ago.
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