Applications of Cohomology to Set Theory I: Hausdorff Gaps

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Abstract. We explore an application of homological algebra to set theoretic objects by developing a cohomology theory for Hausdorff gaps. This leads to a natural equivalence notion for gaps about which we answer questions by constructing many simultaneous gaps. The first result is proved in ZFC while new combinatorial hypotheses generalizing ♣ are introduced to prove the second result. The cohomology theory is introduced with enough generality to be applicable to other questions in set theory.

Additionally, the notion of an incollapsible gap is introduced and the existence of such a gap is shown to be independent of ZFC.

Section 1 Introduction

Hausdorff gaps appear in a wide scope of applications in the literature on set theory and a fitting and voluminous tribute to the importance of these objects has recently been published by M. Scheepers, [Sc]. Hence, I shall not attempt to provide further incentive for their study at this time, but anticipate an interest in understanding their structure.

In this article we explore a new approach to viewing gaps, adapting tools from cohomology to describe them. In fact, although new information about gaps is gained in this way, this framework for investigating set theoretic structures is of as much interest to the author as the information about gaps. The apparatus of cohomology will be shown to be an appropriate tool for understanding and directing questions about objects in set theory by exploring this particular example. We will see how cohomology provides insight into the pertinent issues underlying gaps and from the direction so obtained will formulate and answer questions about gaps.

In a future paper, we will examine the case of $\omega_1$-trees and develop a cohomology theory for a class of Aronszajn trees. From this cohomological formulation of trees, we will be able to describe a connection between gaps and trees. This will add evidence to the circumstantial similarities between gaps and trees that presently exist because of their constructions and similar behavior under the alchemy of forcing.

This cohomological framework is thus initially unifying. More enticing is the vast amount of technical apparatus available in the various guises of the categorical theory of homological algebra and derived functors.

The modus operandi for this investigation is as follows: 1. Describe the choices and interpretations of chain groups and connecting maps for the selected context; 2. Show that these interpretations capture the given objects; 3. Manipulate the cohomological apparatus to extract information about the associated groups; 4. Interpret this information and use it to direct further investigation of the given objects. One benefit of this approach is the appearance of other venues of generalization not visible from a traditionally set theoretic viewpoint. Further, topological cohomology offers some useful mnemonic and visual associations.

This method of exploration will be both economical and flexible. The set theorist with little familiarity with cohomology should find the objects familiar enough to make the terminology understandable. For the

*Results in this paper constitute a substantial portion of my dissertation completed at The University of Michigan in Ann Arbor under the supervision of Andreas Blass.
reader familiar with cohomology but unfamiliar with set theory, the approach should permit her or him to be introduced to the pertinent objects within a familiar context.

**Notation and Conventions.**

Notation used is essentially standard for set theory. We use [Ku] as a reference. Additionally, we use \( \Delta \) to denote the symmetric difference of sets. \( P(\omega)/\text{fin} \) is considered as boolean algebra, though it should be noted that it maintains a group structure (actually a \( \mathbb{Z}/2 \)-module structure) under the operation induced by \( \Delta \). The adverb “almost” will mean “modulo a finite quantity”.

The functions max and min take as arguments a list of numbers, while sup, inf and sup\(^+\) (the proper supremum of a set of ordinals) take a set as an argument.

\( \alpha, \beta, \xi, \eta, \) etc., will always denote ordinals, usually countable. Whenever possible, we will follow the convention that \( \beta \leq \alpha \) and \( \eta \leq \xi \). \( \lambda \) will always represent a countable limit ordinal and \( \Lambda \) will be the collection of all countable limit ordinals. \( \kappa \) will usually denote an infinite cardinal. \( n, m, i \) will usually denote elements in \( \omega \). If the context is clear, then quantification of these variables will implicitly be over objects of the proper type. Thus in a proof where \( \beta \) is used as an arbitrary countable ordinal, \( (\forall \beta) \) will mean \( (\forall \beta \in \omega_1) \).

**Introduction to Hausdorff Gaps.**

The usual formulation of a Hausdorff gap is as a pair of sequences of length \( \omega_1 \) of subsets of \( \omega \), one almost increasing (increasing modulo finite) and the other almost decreasing, with the decreasing sequence “above” the increasing sequence (i.e., every element of the decreasing sequence almost includes every element of the increasing sequence as a subset). The defining property of a gap is that these sequences have no space “above” the increasing sequence (i.e., every element of the decreasing sequence almost includes every element of the increasing sequence as a subset). This can

If each element of the top (decreasing) sequence of sets in a gap is complemented relative to \( \omega \), the result is a pair of increasing sequences such that any pair of elements, one from each sequence, is almost disjoint. The defining property then becomes: There is no subset of \( \omega \) containing one sequence and remaining almost disjoint from each element of the other. This is the formulation of gap used most frequently in this paper. Increasing sequences will be called towers.

**Definition 1** Let \( \subseteq^* \) be the pre-order on \( P(\omega) \) given by \( A \subseteq^* B \) if and only if \( A \setminus B \) is finite. If this relationship holds, we say \( A \) is almost included in \( B \) and \( B \) almost includes \( A \). If \( A \subseteq^* B \) and additionally \( B \setminus A \) is infinite, we say \( B \) is strictly above \( A \) and write \( A \nsubseteq^* B \). Denote by \( A =^* B \) the statement that \( A \Delta B \) is finite and say \( A \) and \( B \) are almost equal. We use this notation for sets in general.

**Definition 2** Let \( P(\omega)/\text{fin} \) denote the Boolean algebra obtained by taking the subsets of the natural numbers modulo the ideal of finite sets. The partial order in this algebra is induced by \( \subseteq \), or equivalently, by \( \subseteq^* \). Hence we use \( \subseteq^* \) as the order symbol on the Boolean algebra and may conflate elements of the Boolean algebra with representatives when convenient.

**Definition 3** Two elements of \( P(\omega) \) are said to be *almost disjoint* if their intersection is finite. This can also be denoted \( A \cap B =^* \emptyset \). Unless explicitly noted otherwise, we use the notion upwards hereditarily: That is, two collections of subsets of \( \omega \) are said to be almost disjoint if every pair of elements, one from either collection, is almost disjoint.

**Definition 4** A tower, \( T = \langle T_\alpha : \alpha \in \kappa \rangle \), in \( P(\omega) \) is an indexed sequence of elements of \( P(\omega) \) increasingly linearly ordered by \( \subseteq^* : \beta < \alpha \implies T_\beta \subseteq^* T_\alpha \). The ordinal \( \kappa \) is said to be the height of the tower. \( T_\alpha \) will be called the \( \alpha \)th level of \( T \). A subtower, \( S_\alpha \), of \( T \) is a tower satisfying for each \( \alpha \), \( S_\alpha \subseteq T_\alpha \) (levelwise inclusion) and additionally for each \( \beta < \alpha \), \( S_\alpha \cap (T_\beta \setminus S_\beta) \) is finite. (Restriction to lower levels of the subtower
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Theorem. This clearly implies the previous minimize requirement, but the proof, under this hypothesis, that the pair \( \langle A, B \rangle \) is a gap is slightly different. It is a variant of this second condition which is used in the \( \aleph_1 \) Gap Theorem.

Definition 5 A pregap (or an \( \omega_1 \)-pregap) is a pair of towers \( \langle A, B \rangle \), each of height \( \omega_1 \) such that for each \( \alpha < \omega_1 \), \( A_\alpha \cap B_\alpha =^* \emptyset \). Note that both \( A \) and \( B \) are subtowers of the tower \( A \cup B \), the level-wise union of \( A \) and \( B \).

Definition 6 A Hausdorff gap is a pregap, \( \langle A, B \rangle \), satisfying: There is no \( X \) in \( \mathcal{P}(\omega) \) such that for all \( \alpha \), \( X \supseteq^* A_\alpha \) and \( X \) is almost disjoint from \( B_\alpha \). We say such an \( X \) separates or fills a pregap. The definition is actually symmetric between \( A \) and \( B \) as can be seen by taking the complement of \( X \) relative to \( \omega \).

We will usually be interested in gaps that exist inside a given tower. Hence we give the following definition.

Definition 7 Given a tower, \( T \), in \( \mathcal{P}(\omega) \) of height \( \omega_1 \), and a subtower of \( T \), \( A \), let \( B \) be the levelwise complement of \( A \) in \( T \): \( B_\alpha := T_\alpha \setminus A_\alpha \). We say \( A \) is a gap in \( T \) if and only if the pair \( \langle A, B \rangle \) is a Hausdorff gap. Assuming only that \( A \) is a subtower of \( T \), it is easy to show the tower \( B \) is a subtower of \( T \) and that the pair \( \langle A, B \rangle \) is a pregap.

Notice that a subtower, \( A \), is a gap in a tower \( T \) just in case there is no \( X \subseteq \omega \) with \( X \cap T_\alpha =^* A_\alpha \) for all \( \alpha \). As above, we say such an \( X \) fills or separates \( A \). Restricting attention to gaps inside of a tower does not reduce the generality of the considerations since for any Hausdorff gap \( \langle A, B \rangle \), \( A \) is a gap in the tower \( A \cup B \).

The existence of a gap, proven from the axioms of ZFC, was demonstrated by Hausdorff in the first decade of this century. It is this theorem which we shall generalize in later sections. We produce many “different” gaps simultaneously, building them side by side. The number of simultaneous gaps produced is enormous, hence the \( \aleph_0 \) and \( \aleph_1 \) Gap Theorems. In fact, we do substantially better in each case, at least for our purposes, as indicated by the corollaries following each theorem. The motivation for these theorems and the notion of “different” are developed in the next section. We first state the basic theorem and sketch the methods of generalization.

Theorem 8 (The Basic Gap Theorem) [Ha]. There is a Hausdorff gap.

For a proof of this theorem, see [Fr, p. 36]. It is this proof from which we depart in the generalizations which follow. We take a moment to review the important elements of the proof and indicate the directions in which changes will be made.

Two towers are simultaneously constructed by recursion on the level. There are two conflicting requirements to fulfill during the construction. The first is a “minimize intersection” requirement to ensure that sets in the different towers are almost disjoint. The second is a “maximize intersection” requirement to ensure the result is in fact a gap. If the sets being built are \( A_\alpha \) and \( B_\alpha \) for \( \alpha \in \omega_1 \) then one possible maximization requirement is phrased:

\[
(\forall \alpha)(\forall r \in \omega) \{ \beta < \alpha : A_\alpha \cap B_\beta \subseteq r \} \text{ is finite.}
\]

This (with the minimization requirement) is sufficient to ensure the pair \( \langle A, B \rangle \) is a gap and is used in the \( \aleph_0 \) Gap Theorem.

There is an alternative to these requirements (attributed to Todorčević) which can be stated as follows:

\[
(\forall \alpha)(\forall \beta) \quad A_\alpha \cap B_\beta = \emptyset \iff \alpha = \beta.
\]

This clearly implies the previous minimize requirement, but the proof, under this hypothesis, that the pair \( \langle A, B \rangle \) is a gap is slightly different. It is a variant of this second condition which is used in the \( \aleph_1 \) Gap Theorem.
In the Basic Gap Theorem, most of the work occurs at limit stages. At those stages, a first approximation to one side of the gap is defined satisfying the minimization requirement. This set is carefully enlarged to satisfy the maximization requirement without ruining the previous work. This is the process which is recursively applied in the \( \aleph_0 \) Gap Theorem at limit stages. However, in the \( \aleph_1 \) Gap Theorem, we depart from this procedure adapting (the term is used loosely) the techniques of forcing to obtain the desired result.

**Section 2 The Gap Cohomology Group**

We turn now to the motivation for the theorems which are to follow. In fact, though substantial and technical, those theorems only begin to explore the potential generalizations and applications of homological algebra to problems in set theory and other areas in logic. The original observation that gaps are cohomological in nature is due to Blass.

The following is a list of homological ideas used but not defined here: short and long exact sequences; the properties of boundary maps and how they produce cocycles and coboundaries; the definition of a cohomology group and how it is a measure of failed exactness. The unfamiliar reader can find a reasonable introduction to this material in [Vi] or [H-Y]. More advanced uses of homological algebra are alluded to, but the results do not depend upon them.

Herein, fix a tower, \( T \), of height \( \omega_1 \) which may be referred to as the ambient tower. Let \( D \subseteq \mathcal{P}(\omega) \) be the family of subsets of \( \omega \) generated by the closure under “finite upward modification” of sets in \( T \); i.e.

\[
D \supseteq T_\alpha \land D \setminus T_\alpha \text{ finite } \implies D \in D.
\]

Note that if \( \alpha, \beta \in \omega_1 \) then \( T_\alpha \cup T_\beta \in D \). More generally, \( (D, \subseteq) \) as a partial order is directed upwards. To each element \( D \in D \), associate the following coefficient groups:

- \( G_D := D(\mathbb{Z}/2) \), the functions from \( D \) into \( \mathbb{Z}/2 \), which will be associated with subsets of \( D \),
- \( F_D := \bigoplus_D(\mathbb{Z}/2) \), the finitely supported functions from \( D \) into \( \mathbb{Z}/2 \), associated with finite subsets of \( D \), and
- \( (G/F)_D := G_D/F_D \), associated with the Boolean algebra \( \mathcal{P}(D)/\text{fin} \).

These function groups have their group operation induced componentwise from \( \mathbb{Z}/2 \). We interpret such functions as characteristic functions following the convention that \( f \) is the characteristic function of the set \( f^{-1}\{1\} \) which will be denoted by \( \tilde{f} \). (Also, if \( s(\xi) \) is a function into \( \{0, 1\} \) then \( \tilde{s}(\xi) \) is the pre-image of \( \{1\} \).) In this case, the sum of two functions is the characteristic function of the symmetric difference of the represented sets.

If \( D_1 \supseteq D_2 \) then there is a natural restriction map, \( \rho_2 \) from \( G_{D_1} \) to \( G_{D_2} \) namely \( f \mapsto f \upharpoonright D_2 \); similarly for \( F \) and \( G/F \). We may now define cochain groups for each collection of coefficients. A 0-cochain is just a choice function fixing for each \( D \in D \) an element of \( G_D \). An \( n \)-cochain associates to each linearly ordered \( n+1 \) element set, \( D_0 \supseteq D_1 \supseteq \cdots \supseteq D_n \), an element of the “smallest” coefficient group, \( G_{D_n} \). Hence we define

\[
C^n(D,G) := \prod_{D_0 \supseteq D_1 \supseteq \cdots \supseteq D_n} G_{D_n}.
\]

The coboundary operator connects the \( n \)- and \( n+1 \)-cochain groups. The elements it sends to 0 can be thought of as locally patching. We employ functional notation to denote the evaluation of a cochain on its arguments. For \( x \in C^n(D,G) \) we have \( \delta x \in C^{n+1}(D,G) \) given by

\[
(\delta x)(D_0 \supseteq \cdots \supseteq D_{n+1}) := \sum_{i=0}^{n} \left[ (-1)^i x(D_0 \ldots \hat{D}_i \ldots D_{n+1}) \right] + (-1)^{n+1} \rho_{n+1}(x(D_0 \ldots D_n))
\]
where $\hat{D}_i$ means $D_i$ is removed from the sequence. This is essentially cohomology on a presheaf. See [Jn]. The corresponding definitions for $F$ and $G/F$ are similar.

Having made these definitions, we introduce a useful cohomological tool that relates the three cohomologies. From the short exact sequence of coefficient groups,

$$0 \rightarrow F \rightarrow G \rightarrow G/F \rightarrow 0$$

we induce the following long exact sequence of cohomology groups:

$$0 \rightarrow H^0(\mathcal{D}, F) \xrightarrow{i^*} H^0(\mathcal{D}, G) \xrightarrow{\pi^*} H^0(\mathcal{D}, G/F) \xrightarrow{\delta^*} H^1(\mathcal{D}, F) \rightarrow H^1(\mathcal{D}, G) \rightarrow H^1(\mathcal{D}, G/F) \rightarrow \ldots$$

(1)

It is time to state a few properties of these groups and understand what is represented in the above sequence. Again, assume for the following that $T$ is an $\omega_1$ tower in $\mathcal{P}(\omega)$ and $\mathcal{D}$ is generated from $T$ by closing under finite upward modification. We additionally assume that $\bigcup_{\alpha \in \omega_1} T_\alpha = \omega$, though this is inconsequential.

**Proposition 9**

(a) $H^0(\mathcal{D}, G) \cong \mathcal{P}(\omega)$ with $\Delta$ (symmetric difference) as the group operation in $\mathcal{P}(\omega)$.

(b) $H^0(\mathcal{D}, F)$ depends on the structure of $T$. If $T$ is an inextendable tower then $H^0(\mathcal{D}, F) \cong fin$ (the finite subsets of $\omega$) with $\Delta$ as the group operation.

(c) $H^1(\mathcal{D}, G) = 0$.

This is where the real connection between the set theoretic property of being a tower, for example, and cohomological properties is made. This proposition deals with the “concrete” case where there is no concern about modulo finite.

**Proof of Proposition:** (a) If $x \in H^0(\mathcal{D}, G)$ and $D_0 \supseteq D_1 \in \mathcal{D}$, then the coboundary condition on $x$ indicates $x(D_0) \upharpoonright D_1 = x(D_1)$. Thus we can define without ambiguity a function $\bigcup_{D \in \mathcal{D}} x(D)$. This is the characteristic function of a subset of $\omega$. Conversely, a subset of $\omega$, $\omega$, induces an element of $H^0(\mathcal{D}, G)$ via $x(D) = X \cap D$ for each $D \in \mathcal{D}$ (confuting sets and their characteristic functions). Since distinct subsets of $\omega$ give rise to distinct cocycles and every cocycle arises in this way, we have a bijective correspondence. The group structure is preserved because the operation in $H^0(\mathcal{D}, G)$ is induced by the operation in $\mathbb{Z}/2$.

(b) If $x \in H^0(\mathcal{D}, F)$, the above applies as well. If $T$ is an inextendable tower, then for any infinite $X \subseteq \omega$, there is an $\alpha$ with $T_\alpha \cap X$ infinite. Thus the function $\bigcup_{D \in \mathcal{D}} x(D)$ must have finite support. Consequently, there is always a $T$, though not necessarily of height $\omega_1$, with $H^0(\mathcal{D}, F) \cong fin$. If $T$ is extendable, let $X \subseteq \omega$ be an infinite set almost disjoint from all levels of $T$. Then the cochain given by $x(D) := D \cap X$ is a cocycle (in $H^0(\mathcal{D}, F)$). However, distinct $X$’s may not give distinct cochains.

For one last example here, consider an extendible $\omega_1$ tower, $T$, still satisfying the weak condition $\bigcup_{\alpha} T_\alpha = \omega$. If $X$ is an infinite set almost disjoint from each $T_\alpha$ (as $T$ is extendible) and $X' \subseteq X$ is also infinite, then there is an $\alpha$ with $T_\alpha \cap X' \neq T_\alpha \cap X$ (by the assumption that $\bigcup_{\alpha} T_\alpha = \omega$). Thus the $F$ cohomology classes associated with $X$ and $X'$ are not equal. As a consequence, $|H^0(\mathcal{D}, F)| = 2^{\aleph_0}$: the previous sentence demonstrates $\geq$ while the fact that an element in $H^0(\mathcal{D}, F)$ defines a subset of $\omega$ gives the reverse inequality. It is unknown whether under $\neg CH$ other alternatives are possible.

(c) This is an example of the facility with which homological algebra can make statements about these structures. The presheaf of interest, $G$, is flasque (also called flabby) which implies the higher derived functors of $\lim$ (the cohomology groups of interest) are trivial. See [Jn]. However, the statement can also be proved directly and doing so reveals how topological visualization can guide our proofs. We wish to show that given a cocycle $x \in C^1(\mathcal{D}, G)$ there is an element $y \in C^0(\mathcal{D}, G)$ such that $\delta y = x$.

First, assume that $T_0 \subseteq T_\alpha$ for all $\alpha$. (Alternatively, we may assume $T_0 = \emptyset$.) We need to define $y(D)$ for each $D \in \mathcal{D}$ so that

$$\forall D_1 \supseteq D_2 \ x(D_1, D_2) = y(D_2) - y(D_1) \upharpoonright D_2.$$  

(2)
For each \( n \in \omega \) we will simultaneously define \( y(D)(n) \) for each \( D \) containing \( n \). We then verify the above equation holds for each \( n \in D_2 \).

For each \( n \in \omega \) we define \( y(D)(n) \) for those \( D \in \mathcal{D} \) with \( n \in D \) by:

\[
y(D)(n) := -x(D, T_0 \cup \{n\})(n).
\]

Applying this definition to the right hand side of Equation (2), we have

\[
y(D_2)(n) - y(D_1)(n) = x(D_1, T_0 \cup \{n\})(n) - x(D_2, T_0 \cup \{n\})(n).
\]

which, by the cochain condition on \( x \) applied to the triple \( D_1 \supseteq D_2 \supseteq T_0 \cup \{n\} \), is equal to \( x(D_1, D_2)(n) \) as desired.

If we do not assume that \( T_0 \subseteq T_\alpha \) for all \( \alpha \), then the proof can be completed by defining the operation \( D^* := D \cup T_0 \) and defining

\[
y(D)(n) := (x(D^*, D) - x(D^*, T_0 \cup \{n\}))(n).
\]

The motivation for these definitions is in fact geometrical as will be described below.

Suppose \( D_1 \) and \( D_2 \) are arbitrary elements of \( \mathcal{D} \) and that \( n \in D_2 \). Consider the following diagram

\[
\begin{array}{ccc}
D_1 & \xleftarrow{\subseteq} & D_1^* \\
\| & & \| \\
D_2 & \xleftarrow{\subseteq} & D_2^* \\
& \xrightarrow{\supseteq} & T_0 \cup \{n\}
\end{array}
\]

The arrows represent the restriction maps induced on the coefficient groups. The diagram resembles a simplicial complex. We can “push off” the information from \( x(D_1, D_2) \) to the other edges of the 2-simplex containing that edge because \( x \) satisfies the cochain condition. This was the motivation for the definition of \( y \). With this in mind, we present the formal argument, guided by this process of “pushing off” via the cochain condition.

\[
(y(D_2) - y(D_1))(n) = (x(D_2^*, D_2) - x(D_2^*, T_0 \cup \{n\}))(n) - (x(D_1^*, D_1) - x(D_1^*, T_0 \cup \{n\}))(n)
\]

\[
= x(D_2^*, D_2)(n) - x(D_1^*, D_1)(n) + (x(D_1^*, T_0 \cup \{n\}) - x(D_2^*, T_0 \cup \{n\}))(n)
\]

\[
= x(D_2^*, D_2)(n) - (x(D_1^*, D_2) - x(D_1, D_2))(n) + x(D_1^*, D_2^*)(n)
\]

\[
= x(D_1, D_2)(n)
\]

where the first equality is by definition of \( y \), the second by re-arrangement, the third by two applications of the cochain condition on \( x \) to the triples \((D_1^*, D_1, D_2)\) and \((D_2^*, D_2, T_0 \cup \{n\})\), and the last equality by rearrangement and another application of the cochain condition on \( x \) to the triple \((D_1^*, D_2^*, D_2)\). This gives the desired equality and shows \( y \) is sent to \( x \) by \( \delta \). \( \square \)

For the following proposition, we shift to the more natural context of \( \mathcal{P}(\omega)/\text{fin} \). Recall that the partial order in \( \mathcal{P}(\omega)/\text{fin} \) induced by \( \subseteq \) is in fact \( \subseteq^* \) and that towers and gaps transfer to this context. Notice that in examining elements of \( H^n(D, G/F) \), we need only consider the original tower as all other elements of \( \mathcal{D} \) are finite modifications of elements of the tower and such finite modification is “washed away” by our definitions.
Definition 10  If \( x \in \text{H}^0(D, G/F) \), then \( x \) is an equivalence class (modulo boundaries) of equivalence classes (modulo finite). By a representative of \( x(T_\alpha) \) I mean a subset of \( \omega \) whose characteristic function is a representative (under the modulo finite equivalence relation) of the evaluation at \( T_\alpha \) of a cochain representative of \( x \).

Proposition 11  There is a bijective correspondence between \( \text{H}^0(D, G/F) \) and subsets of \( \omega \), and a cochain (as a cocycle) for subsets of each element of the tower patch in \( \mathcal{P}(\omega)/\text{fin} \). Further, if \( x \) and \( A \) are so related, then \( A \) is a gap in \( T \) if and only if \( x \) does not lift under \( \pi^* \) in the long exact sequence, (1).

Through this proposition, we see that the cohomology structure completely captures the notion of Hausdorff gap within a given tower.

Proof of Proposition:  If \( x \in \text{H}^0(D, G/F) \), then \( x \) is a gap in \( T_\alpha \), \( \alpha \) is a gap in \( T \). Let \( \pi_x \) be the map \( \text{H}^0(D, G/F) \to \text{H}^0(D, G/F) \). Let \( Y \subseteq \omega \) be the set given for \( y \) by Proposition 9 and \( A \) the subtower of \( T \) induced by \( x \) given by the preceding paragraph. We wish to see that \( Y \) fills the pregap induced by \( A \), that is, \( Y \cap T_\alpha = * A_\alpha \). But this follows immediately from the fact that \( Y \cap T_\alpha = y(T_\alpha) \) and \( \pi^* y = x \).

Conversely, suppose \( A \) is a subtower of \( T \) and \( x \in \text{H}^0(D, G/F) \) is the cocycle associated with \( A \). Suppose \( Y \subseteq \omega \) fills \( A \). Then the cocycle \( y \in \text{H}^0(D, F) \) associated with \( Y \) has \( (\pi^* y)(T_\alpha) \) induced by \( Y \cap T_\alpha \) which is almost equal to \( A_\alpha \). Thus \( \pi^* y = x \) as desired. □

From this proposition, we also see a new equivalence relation arising for gaps within a tower. Hence, we give the following definition.

Definition 12  We say that two gaps \( A \) and \( A' \) in a tower \( T \) are cohomologous if the levelwise symmetric difference is not a gap in \( T \); i.e., if the subtower given by \( \langle A_\alpha \triangle A'_\alpha : \alpha \in \omega_1 \rangle \) is not a gap in \( T \).

By the previous proposition, this is equivalent to saying that the difference of the \( G/F \) cohomology classes associated with the gaps does not lift under \( \pi^* \). As one example of the reasonableness of such a relation, we have

Proposition 13  If \( A \) and \( A' \) are two gaps in \( T \) which are cohomologous and \( \mathbb{P} \) is a notion of forcing such that it is forced by \( \mathbb{P} \) that \( A \) is not a gap, then it is similarly forced by \( \mathbb{P} \) that \( A' \) is not a gap.

Proof of Proposition:  We argue in the generic extension under the stated assumptions. Let \( X \) be a set that fills \( A \). Thus \( X \cap T_\alpha = A_\alpha \) for each \( \alpha \). Let \( Y \) be a set that fills \( A \triangle A' \) by the assumption that \( A \) and \( A' \) are cohomologous. It is straightforward to check that \( X \triangle Y \) fills \( A' \).

As a corollary to this proposition, we have: If \( A \) is indestructible under notions of forcing that preserve \( \omega_1 \) and \( A \) is cohomologous to \( A' \) then \( A' \) is similarly indestructible.

Up to this point, we have gaps associated to the quotient of a cohomology group rather than simply to a cohomology group. We remedy this with the following

Theorem 14  The group \( \text{H}^1(D, F) \) is isomorphic to the set of gaps in \( T \) modulo the equivalence relation “cohomologous”, with group operation being levelwise symmetric difference.

Proof of Theorem.

Examining Sequence (1) and Proposition 9 (a) and (c) we see

\[
\text{H}^1(D, F) \cong \text{H}^0(D, G/F)/\ker(\delta^*) \cong \text{H}^0(D, G/F)/\text{im}(\pi^*)
\]

where the first \( \cong \) is true because \( \delta^* \) is a surjection (by Proposition 9 (c)) while the second is true as Sequence (1) is exact. But the previous proposition gives the desired correspondence between this last group and the gaps in \( T \). □
Thus, there is a cohomology group which represents gaps. Notice that the characteristic properties of gaps are captured by finite sets. As a result of this proposition, we give

**Definition 15**: \( H^1(D, F) \) will be called the \textit{gap cohomology group}.

Next, we state a few more properties about this equivalence relationship on gaps.

**Proposition 16**

(a) If \( A \) is a gap in \( T \) and \( A' \) is (almost) obtained from \( A \) by symmetric difference by a constant set (i.e., there is an \( X \subseteq \omega \) such that for each \( \alpha, A'_\alpha = A_\alpha \triangle (X \cap T_\alpha) \)) then \( A \) and \( A' \) are cohomologous.

(b) If \( A \) and \( A' \) are two gaps in \( T \) such that for all \( \alpha \) we have \( A_\alpha = A'_\alpha \) then \( A \) and \( A' \) are cohomologous.

(c) If \( A \) and \( A' \) are two gaps in \( T \) such that for cofinally many \( \alpha \) we have \( A_\alpha = A'_\alpha \) then \( A \) and \( A' \) are cohomologous.

**Proof of Proposition**: For (a), we have \( A'_\alpha \triangle A_\alpha = A_\alpha \cap T_\alpha \) for each \( \alpha \). Thus \( X \) fills the pregap \( A' \triangle A \).

Part (b) follows from (a) with \( X = \emptyset \). The supposition in (c) implies the stronger condition used in (b): Let \( \beta \) be given and let \( \alpha \geq \beta \) satisfy \( A_\alpha = A'_\alpha \). Now for each subtower \( A \) and \( A' \), we know \( A_\alpha \cap T_\beta = A_\beta \), etc. Thus \( A_\beta = A_\alpha \cap T_\beta = A'_\alpha \cap T_\beta = A_\beta \). Since \( \beta \) was arbitrary, the condition in (b) is satisfied.

\( \square \)

**Section 3** The \( \aleph_0 \) Gap Theorem

Having shown that cohomology induces an equivalence relation on the gaps within a tower, being cohomologous, we can ask what properties the equivalence classes have. We have seen in Proposition 16 that this equivalence relation smooths out some unimportant differences in gaps. However, it is conceivable that every pair of gaps is cohomologous. The main theorems of this paper, the \( \aleph_0 \) and \( \aleph_1 \) Gap Theorems indicate that this is not the case as is explained in the corollaries following each. The constructions, however, are of interest in their own right, and the additional hypotheses that have arisen in consideration of these questions seem important as well.

We now present the \( \aleph_0 \) Gap Theorem, so called because it is based on the construction of \( \aleph_0 \) simultaneous subtowers in a given tower. It implies that the size of the gap cohomology group is at least \( 2^{\aleph_0} \).

**Theorem 17** The \( \aleph_0 \) Gap Theorem. Let \( T = \langle T_\alpha : \alpha \in \omega_1 \rangle \) be a tower in \( \mathcal{P}(\omega) \). Then there is an \( \omega \) by \( \omega_1 \) matrix, \( (A(m, \alpha) : m \in \omega, \alpha \in \omega_1) \), with the following properties:

(\( \aleph_0 \) 1) For each \( m \in \omega, (A(m, \alpha) : \alpha \in \omega_1) \) is a subtower of \( T \).

(\( \aleph_0 \) 2) \( (\forall m_0)(\exists m_1)(\forall \alpha)(\forall r \in \omega) \{ \beta < \alpha : A(m_0, \alpha) \cap A(m_1, \beta) \subseteq r \} \) is finite.

(\( \aleph_0 \) 3) \( (\forall \alpha \in \omega_1)(\forall \beta < \alpha)(\exists m_0)(\forall m > m_0) A(m, \alpha) \cap T_\beta = A(m, \beta) \).

(\( \aleph_0 \) 4) \( (\forall \alpha \in \omega_1) T_\alpha = \bigcup \{ A(m, \alpha) : m \in \omega \} \), a disjoint union.

The intuition behind the construction is to see \( A(\text{column}, \text{row}) \) with \((0, 0)\) at the lower left. Then each column is a tower growing upwards and each row is a partition of the associated level of the tower \( T \). The first and second conditions ensure that each pair of columns is a Hausdorff gap. See [Fr, p.36] for a proof of this. Hence each column is a gap in \( T \).

As previously discussed, to be a gap requires “interaction” (non-empty intersection) between (sets in different) columns, a property which is assured by condition (\( \aleph_0 \) 2). But to obtain the following corollary where unions of collections of columns are gaps (in particular, are towers) requires the intersections to be controlled as formalized by condition (\( \aleph_0 \) 3) which can be seen as a more complicated “minimize intersection” requirement. This increases the delicateness and technicality required in the proof making it reminiscent of a priority argument.

**Corollary 18** The cardinality of the gap cohomology group is at least \( 2^{\aleph_0} \).

**Proof of Corollary**: The third condition ensures that

\[
X \subseteq \omega \implies \left( \bigcup \{ A(m, \alpha) : m \in X \} : \alpha \in \omega_1 \right)
\]
is a tower by the following argument. Let \( A(X, \alpha) \) be the \( \alpha \text{th} \) level of this sequence. We must check that

\[
\beta \leq \alpha \implies A(X, \beta) \subseteq^* A(X, \alpha).
\]

The problem is that the finite differences, \( A(m, \beta) \setminus A(m, \alpha) \), that exist between different levels of a given column may accumulate to an infinite quantity under the infinite union. Examining (N₀ 3) reveals that for a fixed \( \alpha \) and \( \beta \) only finitely many columns can contribute to \( A(X, \beta) \setminus A(X, \alpha) \) which hence is finite.

If the set \( X \neq \omega, \emptyset \) then it is easy to see that \( A(X) \) is a gap in \( T \). Recall that two gaps are not cohomologous iff their levelwise symmetric difference is still a gap. But it is clear that for \( X, Y \subseteq \omega \) we have \( A(X, \alpha) \triangle A(Y, \alpha) = A(X \triangle Y, \alpha) \) and hence if \( X \neq Y \) and \( X \neq \omega \setminus Y \) then \( A(X) \) is not cohomologous with \( A(Y) \). Thus the cardinality of the gap cohomology group is at least that of the continuum. ■

**Proof of Theorem.**

We construct \( A(m, \alpha) \) by induction on \( \alpha \), and then, for limit stages only, induction on \( m \). I shall refer to the restrictions of (N₀ 1–4) to an ordinal \( \gamma \) (replacing \( \omega_1 \) by \( \gamma \) or by \( \lambda \) during a limit stage) as the induction hypotheses. The stage \( \alpha = 0 \) is inconsequential so long as condition (N₀ 4) is fulfilled.

**Successor Stage:** Suppose \( A(m, \beta) \) have been constructed satisfying the induction hypotheses for \( \beta \leq \alpha \) and \( m \in \omega \). As \( T_{\alpha+1} \setminus T_\alpha \) is infinite, it can be partitioned into infinitely many disjoint sets, \( S(m) \), \( m \in \omega \). Define

\[
A(m, \alpha + 1) := (A(m, \alpha) \cap T_{\alpha+1}) \cup S(m).
\]

We check the induction hypotheses are maintained. (N₀ 1) and (N₀ 4) are immediate.

For (N₀ 2), fix \( m_0 \) and \( m_1 \). We need to show that the set \( \{ \beta < \alpha + 1 : A(m_0, \alpha + 1) \cap A(m, \beta) \subseteq r \} \) is finite. This set is

\[
\subseteq \{ \beta < \alpha : A(m_0, \alpha) \cap T_{\alpha+1} \cap A(m_1, \beta) \subseteq r \} \cup \{ \alpha \}
\]

\[
\subseteq \{ \beta < \alpha : A(m_0, \alpha) \cap A(m_1, \beta) \subseteq \sup (r \cup A(m_0, \alpha) \setminus T_{\alpha+1}) \} \cup \{ \alpha \},
\]

which is finite by induction hypothesis.

For (N₀ 3) we need to show

\[
(\forall \beta \leq \alpha)(\exists m_0)(\forall m > m_0) A(m, \alpha + 1) \cap T_\beta = A(m, \beta).
\]

For \( \beta = \alpha \), this is true by the definition of \( A(m, \alpha + 1) \) and:

(a) \((\forall m) S(m) \cap T_\alpha = \emptyset\).

(b) \(T_\alpha \subseteq^* T_{\alpha+1} \implies (\exists m_1)(\forall m > m_1) A(m, \alpha) \subseteq T_{\alpha+1} \).

Now fix \( \beta < \alpha \). Then we have:

(c) By induction hypothesis (N₀ 3)

\[
(\exists m_2)(\forall m > m_2) A(m, \alpha) \cap T_\beta = A(m, \beta).
\]

(d) As \( T_\beta \subseteq^* T_\alpha \), \( S(m) \) are disjoint, and for all \( m \), \( S(m) \cap T_\alpha = \emptyset \), so

\[
(\exists m_3)(\forall m > m_3) S(m) \cap T_\beta = \emptyset.
\]

Fix \( m_1, m_2 \) and \( m_3 \) as in (b), (c) and (d) respectively. Let \( m_0 = \max(m_1, m_2, m_3) \) and express \( A(m, \alpha + 1) \) as \( (A(m, \alpha) \cap T_{\alpha+1}) \cup S(m) \) to derive (5).

**End of Successor Stage.**

**Limit Stage:** Now suppose \( \lambda \) is a limit ordinal and for \( m \in \omega \) and \( \alpha < \lambda \), sets \( A(m, \alpha) \) have been constructed fulfilling the induction hypotheses. Fix a function \( f : \omega \to \lambda \) which is increasing and cofinal in \( \lambda \), with \( f(0) = 0 \).
We now construct the sets \( A(m, \lambda) \) by recursion on \( m \in \omega \). Assume that the sets \( A(s, \lambda) \) have been constructed for \( s < m \). For notation, let \( T(m, \lambda) \) denote the set \( T_\lambda \setminus \bigcup \{ A(s, \lambda) : s < m \} \), the space remaining in which to build \( A(m, \lambda) \). At stage \( m \), we have as induction hypotheses the restrictions of \((\aleph_0 \ 1)\) and \((\aleph_0 \ 2)\) and additionally:

1. \((\forall m', m)(\forall \beta < \lambda) T(m, \lambda) \supseteq A(m', \beta)\).
2. \((A(s, \lambda) : s < m)\) is a disjoint family.

Fix \( m \in \omega \) and assume \( A(s, \lambda) \) are defined for \( s < m \), satisfying the induction hypotheses.

For \( n \in \omega \), define \( K(m, n) \in \omega \) to be the minimum number satisfying the following three conditions:

1. \( K(m, n) \supseteq A(m, f(n)) \cap \bigcup \{ A(m', f(p)) : p, m' < n \land m' \neq m \} \);
2. \( K(m, n) \supseteq A(m, f(n)) \setminus T(m, \lambda) \);
3. \( K(m, n) \supseteq \bigcup \{ A(m, f(n)) \cap T_f(p) \setminus A(m, f(p)) : p < n \} \).

To see that \( K(m, n) \) is finite, we examine each item individually. For (K1), \( A(m, f(n)) \cap A(m', f(p)) \) is finite as \( m' \neq m \) and the argument of \( \bigcup \) is finite. For (K2), this is finite by (IH1) (with \( m' = m \)). For (K3), fix \( n \) and \( p < n \). Then \( A(m, f(n)) \cap T_f(p) = A(m, f(p)) \) by induction hypothesis.

\( K(m, n) \) is the amount of \( A(m, f(n)) \) to be “removed” in order to satisfy \( A(m, \lambda) \supseteq A(m, f(n)) \).

Further, item (K1) ensures almost disjointness between sets in different columns is maintained. This will ensure (IH1) is maintained.

Let \( B(m) := \bigcup \{ A(m, f(n)) \setminus K(m, n) : n \in \omega \} \). (Though including \( m \) in this notation may seem redundant, other values of \( B(m) \) will be referred to later in the proof.) \( B(m) \) is a first approximation to \( A(m, \lambda) \). Notice that induction hypotheses \((\aleph_0 \ 1)\) and (IH1) would be satisfied if we defined \( A(m, \lambda) \) to be \( B(m) \) since:

1. \( B(m) \supseteq A(m, \beta) \) for all \( \beta < \lambda \) because for some \( n \), \( A(m, \beta) \subseteq A(m, f(n)) \subseteq B(m) \), and
2. For \( m' \neq m \) and any \( i \), we have \( B(m) \cap A(m', f(i)) \) finite (because once \( n > i, m' \), then \( K(m, n) \supseteq A(m, f(n)) \cap A(m', f(i)) \) by (K3)). This immediately gives for all \( \beta < \lambda \), \( B(m) \cap A(m', \beta) \) is finite. Then by (IH1) for \( m' \), for any \( m' \geq m + 1 \) and any \( \beta < \lambda \), we would have \( A(m', \beta) \subseteq T(m + 1, \lambda) \) as needed for (IH1) for \( m + 1 \).

For \( \beta < \lambda \), let \( n_\beta \) denote the unique value of \( n \) satisfying \( f(n) \leq \beta < f(n + 1) \). Define

\[
J(m) := \{(m', \beta) : m < m' < n_\beta \land A(m', \beta) \cap B(m) \subseteq n_\beta \}.
\]

\( J(m) \) is the set of (indices of) sets in the matrix with which \( B(m) \) does not yet have “large” intersection (in the sense of \((\aleph_0 \ 2)\)).

**Lemma 19** \((\forall n) \{ \beta : n_\beta < n \land (\exists m')(m', \beta) \in J(m) \} \) is finite.

**Proof of Lemma:** Fix \( n \in \omega \). It is enough to show that for each \( m' \) with \( m < m' < n \) the set \( pr_{m'}J(m) := \{ \beta : n_\beta < n \land (m', \beta) \in J(m) \} \) is finite since for each \( \beta \) there are only finitely many \( m' \) with \( (m', \beta) \in J(m) \).

By the definition of \( B(m) \), \( A(m, f(n + 1)) \setminus K(m, n + 1) \subseteq B(m) \), and so

\[
pr_{m'}J(m) \subseteq \{ \beta < f(n + 1) : A(m', \beta) \cap A(m, f(n + 1)) \setminus K(m, n + 1) \subseteq n \}
\]

\[
\subseteq \{ \beta < f(n + 1) : A(m', \beta) \cap A(m, f(n + 1)) \subseteq \max(n, K(m, n + 1)) \}
\]

which is finite by induction hypothesis.

Note that for any \( \beta < \lambda \), if \( m' > n_\beta \) then \( (m', \beta) \notin J(m) \). This and the previous lemma imply

**Lemma 20** \((\forall \beta < \lambda) J(m) \cap (\omega \times \beta) \) is finite.

Define the function \( j^m : J(m) \to \omega \) as follows. Suppose that \( (m', \beta) \in J(m) \). Then let \( j^m(m', \beta) \) be

\[
\inf \left[ \left( A(m', \beta) \cap A(m', f(n_\beta)) \setminus \bigcup \{ T_f(l) : l < n_\beta \} \right) \cap T(m, \lambda) \setminus n_\beta \right]
\]
Note that, as \( m' > m \), \( A(m', \beta) \subseteq^* T(m, \lambda) \) by the induction hypotheses. (Also note that \( A(m', \beta) \cap A(m', f(n_\beta)) =^* A(m', f(n_\beta)) \) and is thus infinite.) It is not difficult to check in addition that the argument of \( \inf \) is an infinite set. Finally, note that \( j^{m}(m', \beta) \geq n_\beta \).

Let \( A(m, \lambda) := B(m) \cup \ran(j^{m}) \). We check that the induction hypotheses are maintained.

\((\text{IH}1)\) As \( A(m, \lambda) \supseteq B(m) \supseteq^* A(m', \beta) \) for \( \beta < \lambda \), so \((\forall \beta < \lambda) A(m, \lambda) \supseteq^* A(m, \beta) \) as required to continue the tower.

\((\text{IH}2)\) Let \( m' > m \) and \( r \in \omega \). We need to show that \( \{ \beta < \lambda : A(m, \lambda) \cap A(m', \beta) \subseteq r \} \) is finite. Defining

\[
S_n := \{ \beta : n_\beta = n \land A(m, \lambda) \cap A(m', \beta) \subseteq r \},
\]

we have the set of interest equal to \( \bigcup \{ S_n : n \in \omega \} \).

**Claim:** For all \( n, S_n \) is finite.

**Proof of Claim:**

\[
S_n \subseteq \{ \beta < f(n + 1) : A(m, \lambda) \cap A(m', \beta) \subseteq r \}
\subseteq \{ \beta < f(n + 1) : B(m) \cap A(m', \beta) \subseteq r \}
\subseteq \{ \beta < f(n + 1) : (A(m, f(n + 1)) \cap K(m, n + 1) \cap A(m', \beta) \subseteq r \}
\subseteq \{ \beta < f(n + 1) : A(m, f(n + 1)) \cap A(m', \beta) \subseteq \max(r, K(m, n + 1)) \}
\]

which is finite by induction hypothesis.

**Claim:** If \( n > \max(r, m') \) then \( S_n = \emptyset \).

**Proof of Claim:** Suppose \( \beta \in S_n \) where \( n > \max(r, m') \). Then \( n_\beta = n \) and \( A(m, \lambda) \cap A(m', \beta) \subseteq r \subseteq n \). Since \( B(m) \subseteq A(m, \lambda) \) this gives \( B(m) \cap A(m', \beta) \subseteq n \). But \( m < m' < n \), and so \( (m', \beta) \in J(m) \). Consequently, \( j^{m}(m', \beta) \in A(m, \lambda) \cap A(m', \beta) \cap n \), which contradicts the deduction that this set is empty.

Thus \((\text{IH}2)\) is verified.

\((\text{IH}1)\). Fix \( m' > m + 1 \) and \( \beta < \lambda \). We wish to show that \( T(m + 1, \lambda) \supseteq^* A(m', \beta) \). Since by induction hypothesis, \( T(m, \lambda) \supseteq^* A(m', \beta) \), it is sufficient to show that \( A(m, \lambda) \cap A(m', \beta) \) is finite. Further, we may assume \( \beta \) is of the form \( f(n) \) for some \( n \in \omega \). (See the proof of \((\text{IH}3)\) below for the proof of a similar statement.)

Assume \( \beta = f(n) \). It was noted that \( B(m) \cap A(m', \beta) \) is finite. To establish \( \ran(j^{m}) \cap A(m', \beta) \) is finite, we show

\[
\ran(j^{m}) \cap A(m', \beta) \subseteq \ran(j^{m} \upharpoonright [J(m) \cap (\omega \times f(n_\beta + 1))])
\]

and appeal to Lemma 20 above. Let \( (p, \gamma) \in J(m) \setminus (\omega \times f(n_\beta + 1)) \). Then \( \gamma \geq f(n_\beta + 1) \) and so \( n_\gamma > n_\beta \). By the definition of \( j^{m} \), we have

\[
j^{m}(p, \gamma) \notin T_{f(n_\beta)} = T_\beta \supseteq A(m', \beta)
\]

as desired. This completes the proof for \((\text{IH}1)\).

\((\text{IH}2)\) is immediate since \( A(m, \lambda) \subseteq T(m, \lambda) \) which is disjoint from \( A(s, \lambda) \) for \( s < m \).

**End of Limit Construction.**

It remains to check that \((\text{IH}3)\) and \((\text{IH}4)\) are satisfied after the completion of the construction of \( A(m, \lambda) \) for \( m \in \omega \).

\((\text{IH}3)\). We must show that

\[
(\forall \beta < \lambda)(\exists m_0)(\forall m > m_0) A(m, \lambda) \cap T_\beta = A(m, \beta).
\]
Claim: It is sufficient to check this for $\beta$ of the form $f(n)$. For suppose it holds of such ordinals and $\beta \in \omega_1$ is arbitrary. Fix $n$ such that $\beta < f(n)$. Then there are $m_0, m_1$ and $m_2$ such that:

1. $(\forall m > m_0) A(m, \lambda) \cap T_{f(n)} = A(m, f(n))$, i.e., $(B_0\,3)$ holds for $f(n)$,
2. $(\forall m > m_1) A(m, f(n)) \cap T_\beta = A(m, \beta)$, which holds by induction hypothesis,
3. $(\forall m > m_2) A(m, \lambda) \cap (T_\beta \setminus T_{f(n)}) = \emptyset$ which is possible as $T_\beta \setminus T_{f(n)}$ is finite and the $A(m, \lambda)$ are disjoint.

Then for $m > \max(m_0, m_1, m_2)$,

$$A(m, \lambda) \cap T_\beta = A(m, \lambda) \cap ([T_\beta \setminus T_{f(n)}] \cup (T_\beta \setminus T_{f(n)}))$$

$$= \emptyset \cup (A(m, \lambda) \cap T_{f(n)} \cap T_\beta)$$

$$= A(m, f(n)) \cap T_\beta$$

$$= A(m, \beta)$$

$$= A(m, \beta)$$

We use induction on $n$. Fix $n \in \omega$ and assume the claim holds for all $p < n$. We wish to show

$$\exists m_0 \forall m > m_0 \ A(m, \lambda) \cap T_{f(n)} = A(m, f(n)).$$

Since there are only finitely many $p < n$, we have by induction hypothesis that

$$\exists m_1 \forall p < n \forall m > m_1 \ A(m, f(n)) \cap T_{f(p)} = A(m, f(p)).$$

Fixing such an $m_1$, this gives

$$\forall p \leq n \forall m > m_1 \forall m' \neq m \ A(m, f(n)) \cap A(m', f(p)) = \emptyset$$

(where we use additionally that $A(m, f(p)) \cap A(m', f(p)) = \emptyset$). Next we have

$$\exists m_2 \forall m > m_2 A(m, f(n)) \subseteq T_\lambda.$$ 

Let $m_0 := \max(m_1, m_2, n)$.

Claim: $(\forall m > m_0) A(m, \lambda) \cap T_{f(n)} = A(m, f(n))$.

Proof of Claim: This is demonstrated by establishing the following three facts.

A. $(\forall m > m_0) \text{ran}(j^m) \cap T_{f(n)} = \emptyset$.

B. $(\forall m > m_0) B(m) \cap T_{f(n)} \subseteq A(m, f(n))$.

C. $(\exists m > m_0) K(m, n) = 0$.

C implies $(\forall m > m_0) B(m) \supseteq A(m, f(n))$ which combined with (B) gives $(\forall m > m_0) B(m) \cap T_{f(n)} = A(m, f(n))$. With (A) we can deduce the claim since $A(m, \lambda) = B(m) \cup \text{ran}(j^m)$.

Proof of (A). Show $(\forall m > m_0) \text{ran}(j^m) \cap T_{f(n)} = \emptyset$.

Let $(m', \beta) \in J(m)$. We may infer that $n \leq m_0 < m < m' < n_\beta$, where the first inequality holds by the definition of $m_0$ and so $f(n) < f(n_\beta) \leq \beta$. Thus $j^m(m', \beta) \notin T_{f(n)}$ by the definition of $j^m$. This establishes (A).

Proof of (B). Show $(\forall m > m_0) B(m) \cap T_{f(n)} \subseteq A(m, f(n))$ or, from the definition of $B(m)$, show that

$$\bigcup_{p' \in \omega} (A(m, f(p')) \cap T_{f(n)} \setminus K(m, p')) \subseteq A(m, f(n)).$$

We examine the two cases when $p' \leq n$ and when $p' > n$. In the first case we appeal to display (7) which implies $(\forall m > m_0) (\forall p \leq n) A(m, f(p)) \subseteq A(m, f(n))$. 
When \( p' > n \), \( K(m, p') \supseteq [A(m, f(p')) \cap T_{f(n)} \setminus A(m, f(n)) \] by the third part of the definition of \( K(m, p') \). Consequently \( A(m, f(p')) \cap T_{f(n)} \setminus K(m, p') \subseteq A(m, f(n)) \). Hence the displayed union over \( p' \) is contained in the desired set, \( A(m, f(n)) \). This establishes claim (B).

**Proof of (C).** Show \((\forall m > m_0) K(m, n) = \emptyset\).

There are three parts to the definition of \( K(m, n) \). It is sufficient to show that for \( m > m_0 \) the right hand side of each part is empty.

For (K1) we must show

\[
(\forall m > m_0) \left( A(m, f(n)) \cap \bigcup \{A(m', f(p)) : p, m' < n \text{ and } m' \neq m \} \right) = \emptyset.
\]

This is immediate from the definition of \( m_0 \) and display (8).

For (K2), show \((\forall m > m_0) A(m, f(n)) \setminus T(m, \lambda) = \emptyset\). Using the definition of \( T(m, \lambda) \), and since \( m > m_0 \) implies that \( A(m, f(n)) \subseteq T_\lambda \), this reduces to \((\forall m > m_0)(\forall s < m) A(m, f(n)) \cap A(s, \lambda) = \emptyset\). As \( A(s, \lambda) = B(s) \cup \text{ran}(j^s) \), we show

(i) \((\forall m > m_0)(\forall s < m) A(m, f(n)) \cap \text{ran}(j^s) = \emptyset\), and

(ii) \((\forall m > m_0)(\forall s < m) A(m, f(n)) \cap B(s) = \emptyset\).

For (i), suppose for arbitrary \( \beta, m' \), that \((m', \beta) \in J(s)\). We show that \( j^s(m', \beta) \notin A(m, f(n)) \) in two cases. First, if \( n_\beta \leq n \) then \( m \geq m_0 \geq n' \geq n_\beta > m' \) (where the last inequality follows from the definition of \( J(s) \)) gives \( m \neq m' \) which by display (8) gives \( A(m, f(n)) \cap A(m', f(n_\beta)) = \emptyset \). But \( j^s(m', \beta) \in A(m', f(n_\beta)) \). In the second case, \( n_\beta > n \). From the definition of \( j^s \), \( j^s(m', \beta) \notin T_{f(n)} \supseteq A(m, f(n)) \). This proves (i).

For (ii), note that

\[
A(m, f(n)) \cap B(s) = \bigcup_{p \in \omega} [A(m, f(n)) \cap A(s, f(p)) \setminus K(s, p)],
\]

and again we have two cases. If \( p \leq n \) then immediately \( A(m, f(n)) \cap A(s, f(p)) = \emptyset \) by display (8) and the fact that \( s \neq m \).

In the second case, where \( p > n \), we have by the third clause in the definition of \( K(s, p) \) that

\[
K(s, p) \supseteq A(s, f(p)) \cap T_{f(n)} \setminus A(s, f(n))
\supseteq A(s, f(p)) \cap A(m, f(n)) \setminus A(s, f(n))
\supseteq A(s, f(p)) \cap A(m, f(n)).
\]

where the last line is true because \( A(m, f(n)) \cap A(s, f(n)) = \emptyset \). Thus \( A(s, f(p)) \cap A(m, f(n)) \setminus K(s, p) = \emptyset \) as desired. This establishes (ii) which completes the proof for (K2).

For (K3), we show that

\[
\bigcup_{p < n} [A(m, f(n)) \cap T_{f(p)} \setminus A(m, f(p))] = \emptyset.
\]

This follows immediately from display (7) since

\[
A(m, f(p)) = A(m, f(n)) \cap T_{f(p)} \implies A(m, f(n)) \cap T_{f(p)} \setminus A(m, f(p)) = \emptyset.
\]

The premise of this display holds when \( m > m_0 \). This completes the proof for (K3) which finishes claim (C) that \( K(m, n) = \emptyset \) for all but finitely many \( m \).

By the reasoning presented after the statements of (A)—(C), we conclude the claim that for all but finitely many \( m \), \( A(m, \lambda) \cap T_{f(n)} = A(m, f(n)) \). This completes the proof that \((\aleph_0 \ 3)\) holds through the limit stage.
The sets $A(m, \lambda)$, $m \in \omega$ are disjoint by construction. Suppose that their union does not exhaust $T_{\lambda}$. Notice that the quantity of $T_{\lambda}$ remaining must be almost disjoint from $T_{\beta}$ for all $\beta < \lambda$ (and hence almost disjoint from each $A(m, \beta)$ for $m \in \omega$ and $\beta < \lambda$). This follows from ($\aleph_0$ 3). Consequently, $A(0, \lambda)$ can be expanded to contain this set without affecting the other hypotheses. Note in particular that $\langle A(0, \alpha) : \alpha \leq \lambda \rangle$ will be continue to be a subtower of $T$.

**End of Limit Stage.**

This completes the construction of the desired $\omega \times \omega_1$ matrix. Since the properties of this matrix are all stated with quantifiers over countable ordinals, the proofs of the persistence of the induction hypotheses through the recursion establishes that the matrix has the stated properties. ■

Immediate attempts to improve this theorem were resisted by apparently combinatorial complications. These difficulties had the flavor of independence results and indeed the only successful attacks on the problem have relied on combinatorial principles which are known to be consistent with and independent of the axioms of ZFC. At this point, it is necessary that these principles be introduced in their proper context.

**Section 4 New Combinatorial Hypotheses**

We turn now to the combinatorial hypothesis that will be used to prove the $\aleph_1$ Gap Theorem. This and related hypotheses seem to be of interest in their own right, and so I take the opportunity to prove some statements about their relative consistency.

An important type of object for these definitions is the following. The reader can find further information on such objects in [D-S].

**Definition 21** A ladder system, $\langle \varphi_\lambda : \lambda \in \Gamma \rangle$, on a set of limit ordinals of countable cofinality, $\Gamma$, is a $\Gamma$-indexed collection of increasing $\omega$-sequences, $\varphi_\lambda$, each cofinal in its respective $\lambda$.

Recall the definition of ♣ from [Os]:

**Definition 22** ♣ is the statement that there is a ladder system, $\langle \varphi_\lambda : \lambda \in \Lambda \rangle$, such that for every uncountable set $X \subseteq \omega_1$ there is a $\lambda \in \Lambda$ with $\varphi_\lambda \subseteq X$.

Most of the hypotheses involved follow the basic form of ♣. That is, they state the existence of a sequence of sets having some property with respect to other sets. In general, we will follow the convention that a sequence satisfying these properties is called a ♣-sequence ($\lozenge$-sequence, H2-sequence, etc.)

The reader is referred to [Ku, p. 80] for the statements of the hypotheses in the $\lozenge$ family.

Blass has pointed out that H2, defined below, can be phrased as a negative partition relation connecting these ideas to the work of Todorčević and others. This seems to reflect the implicit connection between the properties used to ensure a pair of towers is a gap—an event occurring between sets at different levels—and partition relations on pairs of ordinals. In addition, it has led to weakened forms of the hypotheses, also given below, which are more easily seen to be independent of ZFC.

**Definition 23** H0 is the statement that there is a ladder system $\langle \varphi_\lambda : \lambda \in \Lambda \rangle$ such that for each stationary subset $S$ of $\omega_1$ there is a $\lambda \in S$ such that $\varphi_\lambda \subseteq S$.

Compared to ♣, we have strengthened the statement in requiring the “self-reference” of $S$, while weakening the universal quantifier to stationary sets. In any case, this turns out to be inconsistent with ZFC.

**Claim:** H0 is not consistent with ZFC.

**Proof of Claim:** Suppose $\langle \varphi_\lambda : \lambda \in \Lambda \rangle$ were an H0-sequence. Inductively define a set $S$ such that

$$\lambda \in S \iff \varphi_\lambda \not\subseteq S$$

We show that $S$ is stationary, immediately contradicting H0. Let $C$ be a club and suppose $C \cap S = \emptyset$. By the assumption of H0, there is a $\lambda \in C$ such that $\varphi_\lambda \subseteq C$. But then $\lambda \in S$, a contradiction. Hence $S$ is stationary.

Fortunately, the same fate does not befall the following weakenings of H0.
Definition 24 H1 is the statement that there is a ladder system \( \varphi : \lambda \in \Lambda \) such that for each stationary subset \( S \) of \( \omega_1 \) there is a \( \lambda \in S \) such that \( |\varphi_\lambda \cap S| = \aleph_0 \).

Definition 25 H2 is the statement that there is a ladder system \( \varphi : \lambda \in \Lambda \) such that for each stationary subset \( S \) of \( \omega_1 \) there is a \( \lambda \in S \) such that \( \varphi_\lambda \cap S \neq \emptyset \).

Proposition 26 \( \diamond^* \implies H1 \implies H2 \). In particular, H1 and H2 are consistent with ZFC.

Proof of Proposition: The second implication is immediate. Let \( \langle D_\alpha : \alpha \in \Lambda \rangle \) be a \( \diamond^* \)-sequence. For each \( \lambda \in \Lambda \) define \( \varphi_\lambda \) a cofinal \( \omega \)-sequence in \( \lambda \) such that for each \( D \in D_\lambda \) which is cofinal in \( \lambda \), \( \varphi_\lambda \cap D \) is infinite. This is done by enumerating the \( D \in D_\lambda \) which are cofinal in \( \lambda \) and recursively defining \( \varphi_\lambda \). If there are no cofinal elements in \( D_\lambda \), then let \( \varphi_\lambda \) be arbitrary.

I claim that this \( \varphi_\lambda \) sequence is an H2-sequence. For let \( S \) be a stationary subset of \( \omega_1 \) and let \( C \) be a club as in the definition of the \( \diamond^* \)-sequence, that is, where \( S \) is predicted. Let \( \lambda \in C \cap S \) such that \( C \cap S \) is cofinal in \( \lambda \). This is possible because \( C \cap S \) is stationary. As \( \lambda \in C \), we know \( S \cap \lambda \in D_\alpha \) and \( S \cap \lambda \) is cofinal in \( \lambda \). By the definition of \( \varphi_\lambda \), we have the desired statement that \( \varphi_\lambda \cap S \) is infinite. This shows that H1 and H2 hold in L and are thus consistent with ZFC.

Of course, \( \diamond \implies CH \), and we are interested in statements about gaps when CH does not hold, too. Further, the use of \( \diamond^* \) in the above proof seems to be more than is necessary. It would be more satisfying to have a better understanding of the power of H2. Towards this end, we will show H2 is consistent with \( \neg CH \). In fact, the proof below shows H2 is consistent with the continuum being anything reasonable and can be easily adapted to show the same for H1.

This theorem is proved by showing that an H2-sequence is preserved under notions of forcing that satisfy an apparent strengthening of a previously known condition. We begin by stating this new condition and proving lemmas that will help show familiar notions of forcing satisfy the condition.

Definition 27 We say that a notion of forcing, \( P \), has property SK if and only if for any sequence of conditions \( \langle p_\alpha : \alpha \in S \rangle \) indexed by a stationary set \( S \subseteq \omega_1 \) there is a stationary \( T \subseteq S \) such that for all \( \alpha, \beta \in T \), \( p_\alpha \) and \( p_\beta \) are compatible.

SK can be read as strong Knaster or stationary Knaster as this is a strengthening of property K.

Lemma 28 If \( \langle A_\alpha : \alpha \in S \rangle \) is a collection of finite subsets of \( \omega_1 \) with \( S \subseteq \omega_1 \) stationary, then there is a stationary \( T \subseteq S \) such that \( \langle A_\alpha : \alpha \in T \rangle \) is a \( \Delta \)-system.

Proof of Lemma: (Blass) Thinning \( S \), we may assume all \( A_\alpha \) have the same cardinality, \( n \); for \( k < n \) let \( a_\alpha(k) \) be the \( k^{th} \) element of \( A_\alpha \). If there is a stationary set of \( \alpha \)'s for which \( a_\alpha(n-1) \) is bounded, then for these \( \alpha \)'s there are only countably many different \( A_\alpha \)'s and so stationarily many are the same.

Otherwise, let \( k \) be the least number such that \( a_\alpha(k) \) is unbounded on every stationary set of \( \alpha \)'s. Note that the same must hold for all \( i \) between \( k \) and \( n \). As above, we can thin the index set to a stationary set such that \( \{a_\alpha(0), \ldots, a_\alpha(k-1)\} \) is independent of \( \alpha \). As \( a_\alpha(k) \) cannot be a regressive function of \( \alpha \) on any stationary set, we can thin to arrange that \( \alpha \leq a_\alpha(k) \) for all \( \alpha \). Further, by intersecting the index set with a suitable club (namely, \( \{ \alpha : (\forall \beta < \alpha) a_\beta(n-1) < \alpha \} \) we have \( a_\beta(n-1) < \alpha \) whenever \( \beta < \alpha \). This collection of \( A_\alpha \)'s forms a \( \Delta \)-system with kernel \( \{a_\alpha(0), \ldots, a_\alpha(k-1)\} \).

Now we demonstrate that two of the most familiar notions of forcing have property SK.

Lemma 29 If \( \kappa \geq \omega_1 \) and \( P \) is the set of finite partial functions from \( \kappa \) into 2, then \( P \) has property SK.

(I.e., Cohen forcing has property SK.)

Proof of Lemma: Let \( p_\alpha \in P, \alpha \in S \), where \( S \subseteq \omega_1 \) is stationary be given. Notice that the cardinality of the set of all finite partial functions on the union of the domains of the \( p_\alpha \) is \( \aleph_1 \). As the previous lemma addresses only properties of extensionality and cardinality, we can apply it to \( \langle dom(p_\alpha) : \alpha \in S \rangle \) to get a \( \Delta \)-system on a stationary \( T' \subseteq S \). We may then reduce to a set \( T \subseteq T' \) such that the restriction of \( p_\alpha \) to the root of the \( \Delta \)-system is independent of \( \alpha \) in \( T \). Then the union of any two conditions indexed by \( T \) is a common extension of each.
Lemma 30  Random forcing has property SK.

The proof actually shows that any ∆-linked forcing has property SK.

Proof of Lemma: We use the following fact: If \( p \) is a Borel set of positive measure, then for almost all \( x \in p \), the density of \( p \) in intervals around \( x \) goes to 1 as the interval width goes to 0. (This result, known as the Lebesgue density theorem, can be established by showing sets without this property have measure 0.) So for \( q \in \mathbb{Q} \), the rationals, and \( n \in \omega \), consider the set

\[
S_n^q := \{ p : \text{The density of } p \text{ in } (q - \frac{1}{n}, q + \frac{1}{n}) \text{ is greater than } \frac{1}{2} \}.
\]

Notice that any pair of conditions in this set are compatible (have intersection with positive measure.) Secondly, for any \( p \), there is a \( q \in \mathbb{Q} \) and an \( n \in \omega \) with \( p \in S_n^q \). If \( S \) is stationary and indexes a set of conditions, then there must be a \( q \) and an \( n \) with \( \{ \alpha : p_\alpha \in S_n^q \} \) stationary. This completes the lemma.  □

Lemma 31  Having property SK is preserved under finite support iteration. I.e., if \( \langle Q_i, \pi_i : i \in \kappa \rangle \) is a finite support iteration and \( \vdash \gamma \) “\( Q_\gamma \) has property SK” for all \( \gamma < \kappa \), then the resulting partial order has property SK.

Proof of Lemma: Let \( \langle Q_i, \pi_i : i \in \kappa \rangle \) be a finite support iteration and \( p_\alpha \in \mathbb{P} \) for \( \alpha \in S \) where \( S \subseteq \omega_1 \) is stationary. Since \( spt(p_\alpha) \in [\kappa]^{<\aleph_0} \), we may apply Lemma 28 to get a stationary \( T_0 \subseteq S \). For each \( \gamma \) in the root of this ∆-system, successively get \( T_{n+1} \subseteq T_n \) stationary such that

\[
\vdash_{T_n} (\forall \alpha, \beta \in T_{n+1}) \text{ “} p_\alpha(\gamma) \text{ is compatible with } p_\beta(\gamma) \text{”,}
\]

by applying the fact that \( Q_\gamma, \pi_\gamma \) has property SK. As the original root was a finite set, there is a single \( T_N \) demonstrating the lemma.  □

We are finally ready to state and prove the theorem.

Theorem 32  \( \text{Con}(\text{ZFC} + \text{H2} + \neg \text{CH}) \). That is, if ZFC is consistent, then so is ZFC + H2 + ¬CH.

Proof of Theorem.

Start with a model of H2, for example a model of \( \Diamond^* \). Add \( \aleph_2 \) Cohen reals. I intend to show that the H2-sequence in the ground model continues to enjoy this property in the extension.

Let \( \dot{S} \) be the name of a stationary subset of \( \omega_1 \) and fix \( p \in \mathbb{P} \). Define \( p_\alpha := || \alpha \in \dot{S} \|^\mathbb{P} \wedge p \). As \( \vdash \text{“} \dot{S} \text{ is stationary”,} \) we have \( \{ \alpha : p_\alpha \text{ is not 0} \} \) is stationary in the generic extension (associated with any generic set containing \( p \)) as it contains \( \dot{S} \). Hence it is stationary in the ground model where it is definable. Since \( \mathbb{P} \) has property SK, there is a stationary \( T \subseteq \omega_1 \) such that \( \{ p_\alpha : \alpha \in T \} \) is a pairwise compatible set.

By H2 in the ground model, there is a \( \lambda \in T \) such that \( \varphi_\lambda \cap T \neq \emptyset \). So for some \( \alpha \in \varphi_\lambda \), we have \( p_\alpha \) and \( p_\lambda \) are compatible, \( p_\alpha \wedge p_\lambda \leq p \) and

\[
p_\alpha \wedge p_\lambda \vdash \lambda \in \dot{S} \wedge \dot{S} \cap \varphi_\lambda \neq \emptyset.
\]

Since we started with an arbitrary condition and an arbitrary name for a stationary set and we found an extension of the condition which forces that the set has non-empty intersection with some \( \varphi_\lambda \) and forces that \( \lambda \) is in the stationary set, we see that the H2 sequence in the ground model remains an H2 sequence in the extension. We have thus completed the proof of the theorem. □

Notice \( \aleph_2 \) Cohen reals in the above proof could be replaced with any number of Cohen or random reals (added in a finite support iteration), or any other notion of forcing known to have property SK.

Since the development of H2 and the above discourse, Blass has noted that a weakening of H2 is all that is really needed in the proof of the \( \aleph_1 \) Gap Theorem below. Additionally, this weakening follows from \( \Diamond \) rather than \( \Diamond^* \).

Definition 33  Weak-H1 is the statement that there is a ladder system \( \langle \varphi_\lambda : \lambda \in \Lambda \rangle \) such that for all functions \( f : \omega_1 \to \omega \) there is a limit \( \lambda \) such that \( \{ \alpha \in \varphi_\lambda : f(\lambda) = f(\alpha) \} \) is infinite.
**Definition 34**  Weak-H2 is the statement that there is a ladder system \( \langle \varphi_\lambda : \lambda \in \Lambda \rangle \) such that for all functions \( f : \omega_1 \rightarrow \omega \) there is a limit \( \lambda \) and an \( \alpha \in \varphi_\lambda \) such that \( f(\lambda) = f(\alpha) \).

Notice that H1 implies weak-H1 and H2 implies weak-H2. These can be phrased in terms of partitions as well. For example, for a weak-H2 sequence, given any partition of \( \omega_1 \) into countably many pieces, there is necessarily one piece which contains both some \( \lambda \) and an element of that \( \varphi_\lambda \).

**Proposition 35**  (Blass) \( \Diamond \implies \) weak-H1 \( \implies \) weak-H2.

*Proof of Proposition:*  Fix a \( \Diamond \)-sequence of functions \( f_\alpha : \alpha \rightarrow \omega \). Let \( \varphi_\lambda \) be an \( \omega \)-sequence increasing to \( \lambda \) additionally satisfying

\[
(\forall n \in \omega) \text{ If } f^{-1}_\lambda(n) \text{ is cofinal in } \lambda \text{ then it meets } \varphi_\lambda \text{ in an infinite set.}
\]

Such a \( \varphi_\lambda \) may be constructed recursively by infinitely often addressing each \( n \) for which \( f^{-1}_\lambda(n) \) is cofinal in \( \lambda \). We show this sequence satisfies weak-H1.

Let \( g : \omega_1 \rightarrow \omega \). Let \( A := \{ n \in \omega : g^{-1}(n) \text{ is uncountable} \} \). Define \( \beta := \sup^+ \bigcup \{ g^{-1}(n) : n \in \omega \setminus A \} \). Notice that \( \beta < \omega_1 \) and that if \( \xi > \beta \) then \( g(\xi) \in A \).

For each \( n \in A \), the set of limit points of \( g^{-1}(n) \) is a club; let \( C \) be the intersection of these countably many clubs. Because the \( f_\alpha \)'s form a \( \Diamond \)-sequence, there is a limit ordinal \( \lambda \in C \) with \( \lambda > \beta \) and \( g \upharpoonright \lambda = f_\lambda \).

Fix such a \( \lambda \) and let \( n = g(\lambda) \). By the above remark, \( n \in A \). As \( \lambda \in C \), it follows that \( \lambda \) is a limit point of \( g^{-1}(n) \cap \lambda = f^{-1}_\lambda(n) \). So, by construction, \( \varphi_\lambda \) meets \( f^{-1}_\lambda(n) \), say at \( \gamma \). Then \( g(\gamma) = f_\lambda(\gamma) = n \) and \( g(\lambda) = n \) and \( \gamma \in \varphi_\lambda \) as required.

The second implication is immediate from the definitions.

There is another reason for the introduction of the weak forms of these hypotheses. Given a ladder system, it is relatively easy to describe a c.c.c. notion of forcing which adjoins, via \( \mathcal{N}_1 \) many dense sets, a function from \( \omega_1 \) to \( \omega \) which demonstrates that the given ladder system is not a weak-H2 sequence. Consequently, we have

**Proposition 36**  Under MA, weak-H2 does not hold; hence neither do weak-H1, H2 nor H1 hold.

Before leaving these hypotheses, there is one more curiosity to point out. It is well known that in the conclusion of \( \Diamond \), it is equivalent to assume that the \( \Diamond \)-sequence guesses correctly just once or to assume that the set of correct guesses is stationary. Not surprisingly, this turns out to be true of H2 as well. In particular

**Proposition 37**  If \( \langle \varphi_\lambda : \lambda \in \Lambda \rangle \) is an H2-sequence, then for any stationary set \( S, \{ \lambda \in S : S \cap \varphi_\lambda \neq \emptyset \} \) is stationary.

There is a similar proposition for H1. What is more surprising than this proposition is the following:

**Proposition 38**  If \( \langle \varphi_\lambda : \lambda \in \Lambda \rangle \) is an H2-sequence, then for any set \( C \subseteq \omega_1 \) which contains a club, the set \( \{ \lambda \in C : C \cap \varphi_\lambda \neq \emptyset \} \) contains a club.

*Proof of Proposition:*  Let \( \langle \varphi_\lambda : \lambda \in \Lambda \rangle \) be an H2-sequence and suppose \( C \subseteq \omega_1 \) is a set containing a club for which the set in question does not contain a club. Then the complement of this set, namely \( \{ \lambda : \lambda \notin C \vee C \cap \varphi_\lambda = \emptyset \} \), is stationary. We may intersect this set with \( C \) to get a stationary set: \( A := \{ \lambda \in C : C \cap \varphi_\lambda = \emptyset \} \). Since the \( \varphi_\lambda \) form an H2-sequence, there is a \( \lambda \in A \) such that \( A \cap \varphi_\lambda \neq \emptyset \). But \( A \subseteq C \) implies \( \varphi_\lambda \cap C \neq \emptyset \) which by the definition of \( A \) implies \( \lambda \notin A \), a contradiction.

As an aside, Proposition 37 indicates there is an intermediary between H2 and weak-H2, namely:

**Definition 39**  Not-as-weak-H2 is the statement that there is a ladder system \( \langle \varphi_\lambda : \lambda \in \Lambda \rangle \) such that for all functions \( f : \omega_1 \rightarrow \omega \), the set \( \{ \lambda : f^{-1}(\lambda) \cap \varphi_\lambda \neq \emptyset \} \) is stationary.

Again, there is a similar statement for H1.


Section 5 The $\aleph_1$ Gap Theorem

In this section, we prove a strengthening of the $\aleph_0$ Gap Theorem. This theorem is not stated as a consistency result. It is a construction that occurs in ZFC. However, to show that the constructed object is in fact a large collection of gaps, and in particular to prove the corollary corresponding to that following the $\aleph_0$ Gap Theorem, we use hypothesis H2. Towards this corollary, we first prove the following lemma which shows that under H2 a weak condition on a pregap makes it a gap.

Lemma 40 (H2 and Gaps Lemma.) Assume H2 holds for the ladder system $⟨\varphi_λ : λ \in Λ⟩$. Let $⟨A, B⟩$ be a pregap satisfying $A_α \cap B_α = \emptyset$ for all $α$ and

$$(∃η)(∃\text{stationary } S \subseteq Λ)(∀λ \in S)(∀β \in \varphi_λ \setminus (η + 1)) A_λ \cap B_β \neq \emptyset.$$ 

Then $⟨A, B⟩$ is a gap.

The appearance of $η$ in this lemma is for technical reasons that will be clear in its application after the proof of the $\aleph_1$ Gap Theorem. The proof of the lemma is better understood ignoring $η$.

Proof of Lemma: Suppose $D \subseteq ω$ separated $⟨A, B⟩$. Then there is an $n \in ω$ and a stationary set $T \subseteq S \setminus (η + 1)$ such that $∀λ, λ' \in T$

$$A_λ \setminus n \subseteq D \quad (B_λ \setminus n) \cap D = \emptyset$$ 

$$A_λ \cap n = A_λ \cap n \quad B_λ \cap n = B_λ \cap n.$$ 

By H2, $∃λ \in T) \varphi_λ \cap T \neq \emptyset$. Let $β \in ϕ_λ \cap T$. Since $β \in ϕ_λ$, $A_λ \cap B_β \neq \emptyset$, while $β$ and $λ$ being in $T$ implies $A_λ \cap B_β = \emptyset$, by the previous displayed equations. This is a contradiction, and hence no such $D$ exists. □

We now state and prove the title theorem of this section.

Theorem 41 The $\aleph_1$ Gap Theorem. Let the following objects be given.

(G1) A tower $T = ⟨T_α : α \in ω_1⟩$ with $T_0 = \emptyset$.

(G2) A ladder system $⟨φ_λ : λ \in Λ⟩$.

(G3) A collection of disjoint stationary subsets of $Λ$: $⟨S(ξ, η) : η < ξ < ω_1⟩$. We shall also assume that each $λ \in Λ$ is in some $S(ξ, η)$ and that

$$(λ \in S(ξ, η)) \implies (η < ξ < λ).$$

Then there is a collection $⟨A(ξ, α) : ξ < α < ω_1⟩$ of subsets of $ω$ satisfying:

(N1) 1) $∀λ \in ω_1)(A(ξ, α) : ξ < α < ω_1)$ is a subtower of $T$.

(N1) 2) $∀α \in T_α) T_α = ∪{A(ξ, α) : ξ < α}$, a disjoint union.

(N1) 3) $∀η < ξ < ω_1)(∀λ \in S(ξ, η))(∀β ∈ φ_λ, β > η)A(ξ, λ) \cap A(η, β) \neq \emptyset$.

(N1) 4) $∀β < α < ω_1)$ the set $\{ξ : A(ξ, β) \not\subseteq A(ξ, α)\}$ is finite.

Thinking of the indexing of $A(ξ, α)$ as $A$ (column, row), we can visualize the result as an $ω_1 \times ω_1$ upper triangular matrix with $(0, 0)$ at the lower left corner. This follows the intention of building $ω_1$-towers (the columns) while keeping each row a countable disjoint collection of sets whose union is $T_α$.

These conditions on the matrix will be satisfied by a recursive construction of sets at level, $γ$ ($λ$ when working with limit stages). Consequently we will frequently refer to the “restrictions” of (N1 1–4) as the induction hypotheses, which are obtained by replacing $ω_1$ by $γ$ (or $λ$) and quantifying over the sets constructed to that point in the proof. When no confusion can result, we shall refer to these restrictions as (N1 1), etc.

There is a corollary to this theorem corresponding to the one after the $\aleph_0$ Gap Theorem. However, the corollary is stated in terms of an independence result and its proof is more involved, so we delay its consideration until after the construction.

Proof of Theorem.

Assume the objects in (G1)–(G3) have been fixed. We construct sets $A(ξ, α)$ with the convention that $A(ξ, α) = \emptyset$ for $ξ ≥ α$. We first establish the following lemma which provides a convenient equivalent formulation of (N1 4).
Lemma 42  Suppose that sets $A(\xi, \alpha)$ for $\xi < \alpha < \gamma$ satisfy the restrictions of $(\aleph_1)$ and $(\aleph_1)$ 2. Fix $\beta < \alpha < \gamma$. Then the following are equivalent:

(A) The set $U_1 := \{\xi : A(\xi, \beta) \not\subseteq A(\xi, \alpha)\}$ is finite.

(B) The set $U_2 := \{\xi : A(\xi, \alpha) \cap T_\beta \neq A(\xi, \beta)\}$ is finite.

Proof of Lemma:

(A) $\implies$ (B): Assume $U_1$ is finite. If $\xi \in U_2$ then at least one of the following three conditions holds of $\xi$:

(i) $A(\xi, \beta) \not\subseteq T_\alpha$, or

(ii) The set $(A(\xi, \beta) \setminus A(\xi, \alpha)) \cap T_\alpha$ is non-empty, or

(iii) The set $A(\xi, \alpha) \cap T_\beta \setminus A(\xi, \beta)$ is non-empty.

It is sufficient to show that there are only finitely many $\xi$ satisfying each of (i), (ii) and (iii). This is true for (i) by our assumption that $(\aleph_1)$ 1 and $(\aleph_1)$ 2 hold. Next, if $\xi$ satisfies (ii) then $A(\xi, \beta) \not\subseteq A(\xi, \alpha)$, and so $\xi \in U_1$ which is assumed to be finite.

If $\xi$ satisfies (iii) then by $(\aleph_1)$ 2 for level $\beta$, $(\exists \eta) A(\xi, \alpha) \cap A(\eta, \beta) \neq \emptyset$. Let

$$U_2^+(\eta) := \{\xi : A(\xi, \alpha) \cap A(\eta, \beta) \neq \emptyset \land \eta \neq \xi\}.$$ Then the set of $\xi$ satisfying (iii) is equal to $\bigcup_{\eta < \gamma} U_2^+(\eta)$. By $(\aleph_1)$ 2 for level $\alpha$, if $U_2^+(\eta) \neq \emptyset$ then $\eta \in U_1$. Since we are assuming that $U_1$ is finite, it must be the case that for only finitely many $\eta$ is the set $U_2^+(\eta)$ non-empty. It remains to show that each $U_2^+(\eta)$ is finite. By $(\aleph_1)$ 2, the $A(\xi, \alpha)$ are disjoint. So, if for some $\eta$ the set $U_2^+(\eta)$ were infinite, then $A(\eta, \beta) \not\subseteq A(\eta, \alpha)$, contradicting the restriction of $(\aleph_1)$ 1 for column $\eta$. Hence only finitely many $\xi$ satisfy (iii).

(B) $\implies$ (A): This is immediate since $A(\xi, \beta) \not\subseteq A(\xi, \alpha)$ implies that $A(\xi, \alpha) \cap T_\beta \neq A(\xi, \beta)$.

Consequently, we may assume (B) holds, but need only prove that (A) is maintained.

The Construction of the Matrix.

The stage $\gamma = 0$ is trivial.

Successor Stage: $\gamma = \alpha + 1$.

Assume that for $\xi < \beta < \gamma$, sets $A(\xi, \beta)$ have been constructed satisfying the induction hypotheses. Let $\langle B(\xi) : \xi < \gamma \rangle$ be a partition of $T_\gamma \setminus T_\alpha$ into infinite disjoint sets. Define

$$A(\xi, \gamma) := (A(\xi, \alpha) \cap T_\gamma) \cup B(\xi)$$

for each $\xi < \gamma$.

(Recall that by convention $A(\alpha, \alpha) = \emptyset$.)

Clearly $(\aleph_1)$ 1 and $(\aleph_1)$ 2 now hold. The condition in $(\aleph_1)$ 3 will be satisfied at limit stages. So it remains to check $(\aleph_1)$ 4, in particular that

$$\forall \beta < \gamma \{\xi : A(\xi, \beta) \not\subseteq A(\xi, \gamma)\}$$

is finite.

Case 1: $\beta = \alpha$. Since $T_\alpha \setminus T_\gamma$ is finite and since the rows are disjoint families, it follows from the definition of $A(\xi, \gamma)$ that the desired set is finite. In particular, it is contained in the set $\{\xi : A(\xi, \alpha) \cap (T_\alpha \setminus T_\gamma) \neq \emptyset\}$.

Case 2: $\beta < \alpha$. Taking the contrapositive of

$$[A(\xi, \beta) \subseteq A(\xi, \alpha) \land A(\xi, \alpha) \subseteq A(\xi, \gamma)] \implies A(\xi, \beta) \subseteq A(\xi, \gamma)$$

we have

$$\{\xi : A(\xi, \beta) \not\subseteq A(\xi, \gamma)\} \subseteq \{\xi : A(\xi, \beta) \not\subseteq A(\xi, \alpha)\} \cup \{\xi : A(\xi, \alpha) \not\subseteq A(\xi, \gamma)\}.$$ Both of the sets on the right are finite, the first by induction hypothesis $(\aleph_1)$ 4 and the second by Case 1.

This establishes the preservation of $(\aleph_1)$ 4 through the successor stage and so completes this part of the construction. We now address the arduous
Limit Stage: λ.

Assume that \( \lambda \in S(\xi_0, \eta_0) \) for some \( \eta_0 < \xi_0 < \lambda \), and that \( A(\xi, \alpha) \) are constructed for \( \xi < \alpha < \lambda \) satisfying the induction hypotheses. Fix bijections, \( f : \omega \to \lambda \), and \( g : \omega \to (\lambda \times 2) \cup T\lambda \).

The following is a brief description of the construction which is to follow. The method is expressed in the notation and ideas of forcing. However, all objects involved are countable so no new generic objects are needed to obtain the result. Nonetheless, future extensions of this theorem may take advantage of this methodology.

We describe six different properties obtainable by objects of the form \( \langle r, s, n \rangle \) where \( n \in \omega \), and \( r \) and \( s \) are finite partial functions on \( \lambda \) with codomains \( \omega \) and \( \omega \times 2 \) respectively. Those tuples satisfying these properties will be called conditions.

The function \( s(\xi) \) is a finite approximation of (the characteristic function of) the set \( A(\xi, \lambda) \). If \( \xi \in \text{dom}(s) \) and \( \beta \in \text{dom}(r) \), we consider this a promise to satisfy \( A(\xi, \beta) \setminus n \subseteq A(\xi, \lambda) \). The value of \( r(\beta) = m \in \omega \) will be a promise to satisfy \((\mathbb{N}_1 4)\) between rows \( \lambda \) and \( \beta \) “above” \( m \) (really, to satisfy the condition for \( \xi \) whose \( f \)-preimage is greater than \( m \)). Thus if \( f^{-1}(\xi) > r(\beta) \) then we promise to have \( A(\xi, \beta) \subseteq A(\xi, \lambda) \).

The idea of forcing is implicit in this description. Here is an informal list of what a condition forces.

\( \text{(F1)} \) A number, \( i \in T\lambda \), is explicitly forced into some \( A(\xi, \lambda) \) by \( \langle r, s, n \rangle \) if \( \xi \in \text{dom}(s) \) and \( s(\xi)(i) = 1 \).

\( \text{(F2)} \) A number, \( i \in T\lambda \) may be implicitly forced into some \( A(\xi, \lambda) \) by \( \langle r, s, n \rangle \) due to the “almost containment” for the tower: \( i \geq n, \xi \in \text{dom}(s) \) and there is a \( \beta \in \text{dom}(r) \) with \( i \in A(\xi, \beta) \), or

\( \text{(F3)} \) A number \( i \in T\lambda \) can also be implicitly forced into \( A(\xi, \lambda) \) due to \((\mathbb{N}_1 4)\): there is a \( \beta \in \text{dom}(r) \) with \( f^{-1}(\xi) > r(\beta) \) and \( i \in A(\xi, \beta) \).

\( \text{(F4)} \) A number \( i \in T\lambda \) is forced out of an \( A(\xi, \lambda) \) by \( \langle r, s, n \rangle \) explicitly if \( s(\xi)(i) = 0 \) or implicitly just in case it is forced into \( A(\eta, \lambda) \) for some \( \eta \neq \xi \). Of course, \( i \) is forced out of all sets if \( i \notin T\lambda \).

\( \text{(F5)} \) Hence, \( \langle r, s, n \rangle \) forces \( A(\xi, \beta) \setminus n \subseteq A(\xi, \lambda) \) whenever \( \xi \in \text{dom}(s) \) and \( \beta \in \text{dom}(r) \).

\( \text{(F6)} \) To ensure \((\mathbb{N}_1 3)\) is satisfied, \( \langle r, s, n \rangle \) will explicitly force that \( A(\eta_0, \delta) \cap A(\xi_0, \lambda) \) is non-empty for all \( \delta \) satisfying \( \delta \in \varphi_\lambda \cap \sup^+ \text{dom}(r) \setminus \delta > \eta_0 \).

Most of the definition of \( \mathbb{P} \) below can be seen as consistency requirements for this “forcing”.

A partial order called extension will be defined on conditions with the intuition being that an extension contains more information about the sets being constructed. A four part extension lemma is proved with the following implications: \((\mathbb{E}_1)\) permits \( n \) to be incremented and is essentially a service lemma for the following parts. \((\mathbb{E}_2)\) and \((\mathbb{E}_3)\) permit the extension of the domains of \( s \) and \( r \), respectively, by an element. \((\mathbb{E}_4)\) permits the addition of an arbitrary element of \( T\lambda \) into some \( A(\xi, \lambda) \).

A recursive definition is given starting with the initial condition \( \langle \emptyset, \{ (\xi_0, \emptyset) \}, 0 \rangle \), to which is applied the appropriate extension lemma which is dictated by the type of \( g(m) \) where \( m \) is the stage of the definition.

This generates a chain of objects from which the \( A(\xi, \lambda) \) are derived. \( g \) is a bookkeeping function that ensures all the desired properties are obtained. This ends the description of the proof mentioned above.

For a function \( s(\xi) \) mapping into \( 2 \), recall the notation that \( \bar{s}(\xi) := (s(\xi))^{-1}(1) \).

**Definition 43** Define the set

\[
\mathbb{P} \subseteq \{ \langle r, s, n \rangle : \text{dom}(r), \text{dom}(s) \in [\lambda]^{\aleph_0} \land n \in \omega \land \\
\text{ran}(r) \subseteq \omega \land \text{ran}(s) \subseteq \omega \times 2 \}
\]

as follows.

\( \langle r, s, n \rangle \in \mathbb{P} \) if and only if the following requirements are satisfied. Note that the parenthetical statements are meant as explanation, not as part of the definition.
\((P1)\) \(\langle s(\xi) : \xi \in \text{dom}(s) \rangle\) is a collection of pairwise disjoint subsets of \(T_\lambda\). (\(s\) builds the final sets. See (F1).)

\((P2)\) \(\forall \alpha \in \text{dom}(r) (\forall \xi \in \text{dom}(s) (\forall m > r(\alpha))

\((P2A)\) \(\xi = f(m) \implies A(\xi, \alpha) \cap n \subseteq s(\xi)\), \((N_1\ 4)\) satisfied with row \(\alpha\). See (F3).)

\((P2B)\) \(\xi \neq f(m) \implies A(f(m), \alpha) \cap s(\xi) = \emptyset\). (Allows \(f(m)\) to be added to the domain of \(s\).)

\((P2C)\) \(A(f(m), \alpha) \subseteq T_\lambda\).

\((P3)\) \(\forall \alpha, \beta \in \text{dom}(r) (\forall \xi \in \text{dom}(s) (\forall m > r(\alpha))

\((P4)\) \(\forall \alpha, \beta \in \text{dom}(r) (\forall m_0 > r(\alpha)) (\forall m_1 > r(\beta))\)

\((P5)\) \(\forall \xi \neq \xi' \in \text{dom}(s) (\forall \beta, \beta' \in \text{dom}(r)) A(\xi, \beta) \cap A(\xi', \beta') \setminus n = \emptyset\). (Needed to allow almost containment to be satisfied. See (F5).)

\((P6)\) \(\xi_0 \in \text{dom}(s)\) and \(\forall \delta \in \varphi_\lambda \cap \sup^+ \text{dom}(r). \delta > \eta_0\) \(A(\eta_0, \delta) \cap s(\xi_0) \neq \emptyset\). (This will ensure condition \((N_1\ 3)\) is satisfied. See (F6).)

**Definition 44** If \(p = \langle r, s, n \rangle\) and \(p' = \langle r', s', n' \rangle\) are elements of \(\mathbb{P}\), say \(p'\) extends \(p\) and write \(p' \leq p\) if and only if the following five conditions are satisfied:

\((e1)\) \(r' \supseteq r\);

\((e2)\) \(n' \geq n\);

\((e3)\) \(\text{dom}(s') \supseteq \text{dom}(s)\);

\((e4)\) \(\forall \xi \in \text{dom}(s) s'(\xi) \supseteq s(\xi)\);

\((e5)\) \(\forall \xi \in \text{dom}(s) (\forall \beta \in \text{dom}(r))\; \tilde{s}'(\xi) \supseteq (A(\xi, \beta) \cap (n' \setminus n)).\) (See (F5).)

We now state the Extension Lemmas. For all these statements, let \(p = \langle r, s, n \rangle \in \mathbb{P}\).

**Lemma 45** \((E1)\). (The Simple Extension Lemma.) There is an \(s'\) such that

\((a)\) \(\langle r, s', n + 1 \rangle \in \mathbb{P}\),

\((b)\) \(\langle r, s', n + 1 \rangle \leq p\),

\((c)\) \(\text{dom}(s') = \text{dom}(s)\).

**Lemma 46** \((E2)\). (Extending the domain of \(s\).) For any \(\xi \in \lambda\) there are an \(s'\) and an \(n'\) such that

\((a)\) \(\langle r, s', n' \rangle \in \mathbb{P}\),

\((b)\) \(\langle r, s', n' \rangle \leq p\),

\((c)\) \(\text{dom}(s') = \text{dom}(s) \cup \{\xi\}\).

**Lemma 47** \((E3)\). (Extending the domain of \(r\).) For any \(\beta \in \lambda\) there are an \(s'\), an \(n'\) and an \(m'\) such that

\((a)\) \(\langle r \cup \{\beta, m'\}, s', n' \rangle \in \mathbb{P}\),

\((b)\) \(\langle r \cup \{\beta, m'\}, s', n' \rangle \leq p\),

\((c)\) \(\text{dom}(s') = \text{dom}(s)\).

**Lemma 48** \((E4)\). (Adding an element of \(T_\lambda\).) For any \(i \in T_\lambda\) there are an \(s'\) and an \(n'\) such that

\((a)\) \(\langle r, s', n' \rangle \in \mathbb{P}\),

\((b)\) \(\langle r, s', n' \rangle \leq p\),

\((c)\) \((\exists \xi \in \text{dom}(s')) i \in \tilde{s}'(\xi)\),

\((d)\) \(|\text{dom}(s') \setminus \text{dom}(s)| \leq 1\).

**Proof of Lemma:** \((E1)\).

Let \(\langle r, s, n \rangle \in \mathbb{P}\) be given. To show the claim, we must extend each function \(s(\xi)\) by one place such that the resulting triple \(\langle r, s', n + 1 \rangle\) satisfies \((P1)\) through \((P6)\). Notice that since \(r\) and \(\text{dom}(s)\) do not change, \((P2C)\) and \((P3)\) through \((P6)\) will necessarily be satisfied. By \((P1)\), \(n\) should belong to at most one \(s(\xi)\). We perform a minimal extension to satisfy \((P2A)\) and the definition of extension, \((e5)\). That is, define
that as $\xi / \beta > \alpha$

Proof of Claim: We first show that only one such $\xi$ can satisfy this second requirement. That is if $\langle r, s, n \rangle \in \mathbb{P}$, then for at most one $\xi$ is there an $\alpha \in \text{dom}(r)$ with $n \in A(\xi, \alpha)$. But this is exactly what $(\mathbb{P}5)$ states for $\langle r, s, n \rangle$. Hence $\langle r, s', n + 1 \rangle$ satisfies $(\mathbb{P}1)$.

The condition defining $\langle r, s', n + 1 \rangle$ gives $(\mathbb{P}2A)$ immediately. If $(\mathbb{P}2B)$ failed, there would be an $\alpha \in \text{dom}(r)$, a $\xi \in \text{dom}(s)$, and an $m > r(\alpha)$ such that $\xi \neq f(m)$ and $n \in A(\xi, \alpha) \cap A(f(m), \alpha)$. But this contradicts the fact that $(\mathbb{P}3)$ held for $\langle r, s, n \rangle$. This completes the proof that $\langle r, s', n + 1 \rangle \in \mathbb{P}$. \hfill \qed

Before continuing, we state and prove a lemma necessary for the remaining proofs. While $(\mathbb{P}1)$ stated

**Lemma 49.** Given $\langle r, s, n \rangle \in \mathbb{P}$, and $\xi \notin \text{dom}(s)$ there is an $n' > n$ satisfying the following two conditions:

(a) $(\forall \alpha \neq \beta \in \text{dom}(r))(\forall m > r(\alpha)) A(\xi, \alpha) \cap A(f(m), \beta) \setminus n' = \emptyset$.

(b) $(\forall \eta \in \text{dom}(s))(\forall \alpha \neq \beta \in \text{dom}(r)) A(\xi, \alpha) \cap A(\eta, \beta) \setminus n' = \emptyset$.

Proof of Lemma: Since (a) and (b) are preserved as $n'$ grows and since there are only finitely many triples $\eta \in \text{dom}(s)$, $\alpha \neq \beta \in \text{dom}(r)$, it suffices to show such an $n'$ exists for an arbitrary such triple. For (b), notice that as $\xi \notin \text{dom}(s)$ we have $\xi \neq \eta$. Since $A(\xi, \alpha)$ and $A(\eta, \beta)$ are almost disjoint, (b) follows.

For (a) there are two cases depending on the order of $\alpha$ and $\beta$.

- $\beta > \alpha$: By induction hypothesis and Lemma 42, $\{ \mu : A(\mu, \beta) \cap T_{\alpha} \neq A(\mu, \alpha) \}$ is finite. So there is an $m_0$ such that for all $m$

  $$ m > m_0 \land \xi \neq f(m) \implies A(f(m), \beta) \cap T_{\alpha} = A(f(m), \alpha) $$

But there are only finitely many $m$ with $r(\beta) < m \leq m_0$ while $\xi \neq f(m)$ implies $A(f(m), \beta)$ and $A(\xi, \alpha)$ are almost disjoint.

- $\beta < \alpha$: Similarly to the previous case, the set $\{ \mu : A(\mu, \beta) \notin A(\mu, \alpha) \}$ is finite. So there is an $m_0$ such that for all $m$

  $$ m > m_0 \land \xi \neq f(m) \implies A(f(m), \beta) \subseteq A(f(m), \alpha) $$

The case is completed as above. \hfill \qed

Proof of Lemma: $(\mathbb{E}2)$.

Fix $\langle r, s, n \rangle \in \mathbb{P}$ and $\xi \in \lambda \setminus \text{dom}(s)$. We wish to add $\xi$ to the domain of $s$. This is done in two steps, first by extending the functions $s(\eta)$ for the $\eta \in \text{dom}(s)$ (i.e., increasing $n$) and then by defining $s'(\xi)$.

**Step 1:** Fix $n'$ as in Lemma 49 for $\langle r, s, n \rangle$. By iterated application of $(\mathbb{E}1)$, we may extend $\langle r, s, n \rangle$ to $\langle r, s', n' \rangle \in \mathbb{P}$. This ensures that $(\mathbb{P}3)$ and $(\mathbb{P}5)$ will be satisfied by the new condition.

**Step 2:** We now add $\xi$ to $\text{dom}(s)$ conforming to condition $(\mathbb{P}2A)$. For $\eta \in \text{dom}(s)$, let $s'(\eta) := s''(\eta)$. Define $s'(\xi)$ by

$$ s'(\xi) := \{ i < n' : (\exists \alpha \in \text{dom}(r)) i \in A(\xi, \alpha) \land f^{-1}(\xi) > r(\alpha) \}. $$

Claim: $\langle r, s', n' \rangle \in \mathbb{P}$.

Proof of Claim: Sketch.
The construction assures that \((P2A), (P3)\) and \((P5)\) are satisfied. \((P2C), (P4)\) and \((P6)\) persist from \(\langle r, s'', n' \rangle\). If \((P1)\) fails for \(\langle r, s', n' \rangle\), then \((P2B)\) would not hold for \(\langle r, s'', n' \rangle\). If \((P2B)\) fails for \(\langle r, s', n' \rangle\), then \((P4)\) would not hold for \(\langle r, s'', n' \rangle\).

Hence \(\langle r, s', n' \rangle \in P\). The other desired conditions for \((E2)\) follow from the construction. This completes the proof of \((E2)\).

\(^\Box\)

**Proof of Lemma:** \((E3)\).

**Part 1:** Let \(\beta \in \lambda \setminus \text{dom}(r)\). We wish to add \(\beta\) to the domain of \(r\). The initial part of the construction is more complicated if \(\beta > \max(\sup(\text{dom}(r)), \eta_0)\). If this is not the case, we may let \(\langle r, s'', n'' \rangle := \langle r, s, n \rangle\) and skip to **Part 2** of the construction.

But suppose \(\beta > \max(\sup(\text{dom}(r)), \eta_0)\). Recall that \(\lambda \in S(\xi, \alpha)\) and by \((P6), \xi_0 \in \text{dom}(s)\). We must ensure that \((P6)\) is satisfied by the extension, in particular for each \(\delta \in (\varphi_\lambda \cap (\beta + 1) \setminus (\eta_0 + 1))\), \(A(\eta_0, \delta) \cap s'(\xi_0) \neq \emptyset\).

Consider an arbitrary \(\delta \in (\varphi_\lambda \cap (\beta + 1) \setminus (\sup(\text{dom}(r)), \eta_0 + 1))\), a finite set. We wish to find an “unrestricted” number in \(A(\eta_0, \delta)\) to force into \(s'(\xi_0)\). Say that \(i \in A(\eta_0, \delta) \setminus n\) is restricted (i.e., is already forced into another set) if either of the following conditions are met:

\(\begin{align*}
1) & \quad (\exists \alpha \in \text{dom}(r))(\exists m > r(\alpha)) \ i \in A(f(m), \alpha), \\
2) & \quad (\exists \alpha \in \text{dom}(r))(\exists \xi \in \text{dom}(s)) \ i \in A(\xi, \alpha).
\end{align*}\)

Since \(\delta > \sup(\text{dom}(r))\), the set \(A(\eta_0, \delta) \setminus \bigcup \{T_\alpha : \alpha \in \text{dom}(r)\}\) is infinite. (Actually, we also use the following facts: \((R_4)\) is satisfied as an induction hypothesis, sets in different columns are almost disjoint and \(A(\eta_0, \delta) \setminus A(\eta_0, \alpha)\) is infinite for each \(\alpha \in \text{dom}(r)\).) This means there are arbitrarily large \(i\) which are not restricted.

Perform the following extension process for each relevant \(\delta\) in turn. Let \(i_\delta\) be the least \(i \geq n\) such that

\[i_\delta \in \left(A(\eta_0, \delta) \setminus \bigcup \{T_\alpha : \alpha \in \text{dom}(r)\}\right) \cap T_\lambda.\]

(That is, \(i_\delta\) is not restricted.) Repeatedly apply \((E1)\) to get \(\langle r, s'', i_\delta \rangle \leq \langle r, s, n \rangle\) with \(\text{dom}(s'') = \text{dom}(s)\). Extend this in the obvious minimal way to satisfy \(i_\delta \in s'(\xi_0)\). Notice that as \(i_\delta\) is not restricted, this is a condition extending \(\langle r, s, n \rangle\). Let the result of this finite iteration be called \(\langle r, s'', n'' \rangle\).

**Part 2:** Assume we have a condition \(\langle r, s', n'' \rangle\) satisfying \((P6)\) for \(\delta \in \varphi_\lambda \cap (\beta + 1) \setminus (\eta_0 + 1)\). We now extend \(s''\) and \(n''\) to satisfy \((P3)\) and \((P5)\). By a slight variation of the Lemma 49, there is an \(n' \geq n''\) such that

\(\begin{align*}
(a) & \quad \text{If } \alpha \in \text{dom}(r), m > r(\alpha) \text{ and } \xi \in \text{dom}(s) \text{ satisfy } \xi \neq f(m) \text{ then } A(\xi, \beta) \cap A(f(m), \alpha) \setminus n' = \emptyset. \\
(b) & \quad \text{If } \xi \neq \xi' \in \text{dom}(s'') \text{ and } \beta' \in \text{dom}(r) \text{ then } A(\xi, \beta) \cap A(\xi', \beta') \setminus n' = \emptyset.
\end{align*}\)

Iteratively applying \((E1)\), obtain \(\langle r, s', n' \rangle \leq \langle r, s'', n'' \rangle\).

**Part 3:** Defining \(r(\beta)\).

**Claim:** There is an \(m' \in \omega\) satisfying the following three conditions:

\(\begin{align*}
1) & \quad (\forall m > m') A(f(m), \beta) \cap n' = \emptyset, \\
2) & \quad (\forall \alpha \in \text{dom}(r))(\forall m > m')(\forall l > r(\alpha)) \\
& \quad l \neq m \implies A(f(l), \alpha) \cap A(f(m), \beta) = \emptyset, \\
3) & \quad (\forall m > m') A(f(m), \beta) \subseteq T_\lambda.
\end{align*}\)

**Proof of Claim:** Since there are only finitely many requirements on \(m'\), and since the properties of interest are preserved upward (for larger \(m'\)), we may consider each one separately. (1) follows from the fact that \(\{A(\eta, \beta) : \eta < \beta\}\) is a disjoint family while (3) follows because \(T_\beta \setminus T_\lambda\) is finite.

For (2) fix \(\alpha \in \text{dom}(r)\). If \(\alpha < \beta\), take \(m' > \max(f^{-1}\{\xi : A(\xi, \beta) \not\subseteq A(\xi, \alpha)\})\), a finite set by hypothesis. If \(\alpha > \beta\), let \(m' > \max(f^{-1}\{\xi : A(\xi, \beta) \not\subseteq A(\xi, \alpha)\})\), again finite by hypothesis. In either case, if \(\xi \neq f(m)\) and \(m > m'\) then \(A(f(m), \beta) \cap A(\xi, \alpha) = \emptyset\).

Fix \(m'\) as in the previous claim.
Claim: \( \langle r \cup \{ (\beta, m') \}, s', n' \rangle \in \mathcal{P} \).

Proof of Claim: All the work has been done in the previous lemmas and claims. For example, (P1) holds for \( \langle r, s', n' \rangle \) and hence holds for \( \langle r \cup \{ (m', \beta) \}, s', n' \rangle \). (P2) holds by choice of \( m' \), condition (3), as does (P3) when \( \beta \) is \( \alpha \). When \( \beta \) instantiates \( \beta \) in (P3), the choice of \( n' \), part (2), ensures the condition is met. (P4) holds by choice of \( m' \), part (3), (P5) by choice of \( n' \) part (2), and finally (P6) holds because of the construction of \( \langle r, s'', n'' \rangle \).

The other properties of this condition are clear from the construction and so this completes the proof of (E3).

Proof of Lemma: (E4). Fix \( i \in T_\lambda \). We wish to find an \( s' \) and an \( n' \) such that there is a \( \xi \in dom(s') \) with \( i \in s'(\xi) \) and with \( dom(s') \) having at most 1 new element. The conclusion is immediate if \( i \) is in \( s(\xi) \) for some \( \xi \) in \( dom(s) \), so assume otherwise. We examine three cases.

Case 1: \( \langle r, s, n \rangle \) implicitly “forces” \( i \in A(\xi, \lambda) \) for some \( \xi \). See (F2) and (F3). Either:

1) There is an \( \alpha \in dom(r) \) and an \( m > r(\alpha) \) such that \( i \in A(f(m), \alpha) \), or

2) \( i > n \) and there is an \( \alpha \in dom(r) \) and \( \xi \in dom(s) \) such that \( i \in A(\xi, \alpha) \).

Either way, apply (E1) and (E2) to ensure that \( \langle r, s', n' \rangle \leq \langle r, s, n \rangle \), \( n' > i \) and \( \xi \in dom(s') \). Then the conclusion holds by (P2) applied to \( \langle r, s', n' \rangle \) or by (e5) according to whether (1) or (2) holds, respectively.

Case 2: Case 1 fails and \( i < n \). Then we may fix \( \xi \notin dom(s) \) and define a new condition that satisfies \( i \in s'(\xi) \). This case uses (P4) on \( \langle r, s, n \rangle \) to ensure the result holds.

Case 3: Case 1 fails and \( i \geq n \). Extend by (E1) to \( \langle r, s', i \rangle \) and put \( i \) in any \( s'(\xi) \).

These exhaust the cases need to establish (E4) and complete the proof of the Extension Lemmas.

Completion of the Construction.

We recursively define a sequence of conditions \( p_m \) for \( m \in \omega \). Let \( p_0 := \langle \emptyset, \{ (\xi_0, 0) \} \rangle, 0 \rangle \) and note that \( p_0 \in \mathcal{P} \). Recall that \( g : \omega \to (\lambda \times 2) \cup T_\lambda \) is a bijection. Suppose that \( p_m = \langle r_m, s_m, n_m \rangle \) is defined. We define \( p_{m+1} \leq p_m \) according to the value of \( g(m) \) as follows:

Case 1: \( g(m) = \langle \xi, 0 \rangle \) for some \( \xi < \lambda \). Put \( \xi \) into the domain of \( s \). Apply Extension Lemma (E2) to \( p_m \) and \( \xi \) to get \( p_{m+1} = \langle r_{m+1}, s_{m+1}, n_{m+1} \rangle \leq p_m \) with \( r_{m+1} = r_m \) and \( \xi \in dom(s_{m+1}) \).

Case 2: \( g(m) = \langle \beta, 1 \rangle \) for some \( \beta < \lambda \). Put \( \beta \) into the domain of \( r \). Apply Extension Lemma (E3) to get \( p_{m+1} = \langle r_{m+1}, s_{m+1}, n_{m+1} \rangle \leq p_m \) with \( \beta \in dom(r_{m+1}) \).

Case 3: \( g(m) \in T_\lambda \). Ensure \( g(m) \) is in some \( s(\xi) \). Apply Extension Lemma (E4) to obtain \( p_{m+1} = \langle r_{m+1}, s_{m+1}, n_{m+1} \rangle \leq p_m \) with \( g(m) \in s_{m+1}(\xi) \) for some \( \xi \in dom(s_{m+1}) \).

This completes the definition of \( p_m \) and we are ready to define the \( A(\xi, \lambda) \). Fix \( \xi < \lambda \). Since \( g \) is a bijection, there is an \( l \in \omega \) with \( g(l) = \langle \xi, 0 \rangle \). By the definition of \( p_m \), if \( m > l \) then \( \xi \in dom(s_m) \). Let

\[
A(\xi, \lambda) := \bigcup \{ s_m(\xi) : m \geq 1 \}.
\]

This completes the construction of \( A(\xi, \lambda) \).

It remains to show that the induction hypotheses (N1 1) through (N1 4) hold for these \( A(\xi, \lambda) \). Unless otherwise noted, \( p_m = \langle r_m, s_m, n_m \rangle \).

Claim: (N1 1). For each \( \xi < \lambda \), \( A(\xi, \alpha) : \xi < \alpha \leq \lambda \) is a tower.

Proof of Claim: It suffices to show for \( \alpha \) with \( \xi < \alpha \leq \lambda \) that \( A(\xi, \alpha) \subseteq A(\xi, \lambda) \). So fix such an \( \alpha \). Since \( A(\xi, \alpha) \setminus T_\lambda \) is finite, there is an \( i_0 \in T_\lambda \) with \( i_0 > \text{sup}(A(\xi, \alpha) \setminus T_\lambda) \). Let \( m > \text{sup}(g^{-1}(i_0, \langle \xi, 0 \rangle, (\alpha, 1))) \). This choice ensures that \( \xi \in \text{dom}(s_m) \) and \( \alpha \in \text{dom}(r_m) \). We show that \( A(\xi, \alpha) \setminus n_m \subseteq A(\xi, \lambda) \). Suppose \( i \in A(\xi, \alpha) \setminus n_m \). By the definition of \( i_0 \) and since \( i \in A(\xi, \alpha) \setminus i_0 \) we know \( i \in T_\lambda \). Next, \( m' := g^{-1}(i) > m \), since if \( m' \leq m \) then \( i \) would be in some \( \tilde{s}_m(\xi) \) because of the construction at stage \( m' \) and this implies \( i \leq n_m \), contrary to the choice of \( i \). Hence \( p_{m'} \leq p_m \) and by the definition of extension, in particular by (e5), we may conclude \( i \in \tilde{s}_{m'}(\xi) \).

Claim: (N1 2). \( T_\lambda \) is the disjoint union of the \( A(\xi, \lambda) \) with \( \xi < \lambda \).
Proof of Claim: Only elements of \( T_\lambda \) can be put in any \( A(\xi, \lambda) \). That \( T_\lambda \) is contained in the union follows from the third case in the definition of \( p_m \) and the fact that \( T_\lambda \) is contained in the image of \( g \). The property of disjointness follows from condition (P1) in the definition of \( \mathbb{P} \).

Claim: (81 3). For each \( \beta \in \varphi _\lambda \setminus (\eta _0 + 1) \), \( A(\xi _0, \lambda) \cap A(\eta _0, \beta) \neq \emptyset \).

Proof of Claim: Fix \( \beta \in \varphi _\lambda \setminus (\eta _0 + 1) \) and let \( m > g^{-1}(\beta, 1) \). Then \( \beta \in \text{dom}(r_m) \) and by (P6) we conclude \( \bar{s}(\xi _0) \cap A(\eta _0, \beta) \neq \emptyset \). But \( \bar{s}(\xi _0) \subseteq A(\xi _0, \lambda) \) and so \( A(\xi _0, \lambda) \cap A(\eta _0, \beta) \neq \emptyset \) as desired. 

Claim: (81 4). If \( \alpha < \lambda \) then \( \{ \xi : A(\xi, \alpha) \not\subseteq A(\xi, \lambda) \} \) is finite.

Proof of Claim: Temporarily denote \( \{ \xi : A(\xi, \alpha) \not\subseteq A(\xi, \lambda) \} \) by \( X(\alpha) \). Fix \( \alpha < \lambda \) and let \( m := g^{-1}(\alpha, 1) + 1 \). We show \( X(\alpha) \subseteq f^m(\alpha) + 1 \).

Recall that \( f : \omega \to \lambda \) is a bijection. It is sufficient to show that \( \lambda \setminus f^m(\alpha) + 1 \cap X(\alpha) = \emptyset \). Fix \( \xi \) such that \( f^{-1}(\xi) > r_m(\alpha) \) and fix \( i \in A(\xi, \alpha) \). Note by (P2C) that \( i \in T_\xi \). Let \( l > \max(g^{-1}(i), g^{-1}(\xi, 0), m) \).

Since \( p_l \leq p_m \), we have \( r_l(\alpha) = r_m(\alpha) \). Further, \( \xi \in \text{dom}(s_i) \) and \( i < n_m \). By (P2A) we can conclude that \( i \in \bar{s}_m(\xi) \), hence \( i \in A(\xi, \lambda) \). Since \( i \) was an arbitrary element of \( A(\xi, \alpha) \), we have \( A(\xi, \alpha) \subseteq A(\xi, \lambda) \), so \( \xi \notin X(\alpha) \). This gives the desired result.

This completes the proof that the induction hypotheses are satisfied through the limit stage. Continuing the construction through the countable ordinals gives the desired \( \omega _1 \times \omega _1 \) matrix.

We are now in a position to apply the theorem to obtain information about the gap cohomology. According to Lemma 40, if H2 holds of the ladder system \( \langle \varphi _\lambda : \lambda \in \Lambda \rangle \) and we apply the \( 81 \) gap construction with this system, then for each pair \( \eta < \xi < \omega _1 \), the pair of towers \( \langle A(\eta, \alpha) \rangle \) is a Hausdorff gap. This is where the appearance of \( \eta \) in Lemma 40 is used and is necessary because of the form of the gaps constructed.

Suppose that \( X \subseteq \omega _1 \) is non-empty and not equal to \( \omega _1 \). Define

\[
A(X, \alpha) := \bigcup \{ A(\xi, \alpha) : \xi \in X \cap \alpha \}.
\]

Claim: The tower \( A(X) := \langle A(X, \alpha) : \alpha \in \omega _1 \rangle \) is a gap in \( T \).

Proof of Claim: Assume that \( \mu _0 \in X \) and \( \mu _1 \notin X \). In fact, if \( A(X) \) is truly a subtower, it is easy to see that it is a gap in \( T \). This is simply because the tower \( A(\mu _0) \) is a subtower of \( A(X) \) while \( A(\mu _1) \) is a subtower of its levelwise complement. Any separation of \( A(X) \) from its levelwise complement would provide a separation of \( A(\mu _0) \) from \( A(\mu _1) \).

So it remains to show that \( A(X) \) is a tower. Fix \( \beta < \alpha \). There are three things to check:

(a) \( A(X, \beta) \subseteq ^* A(X, \alpha) \),
(b) \( A(X, \beta) \neq ^* A(X, \alpha) \), (i.e. \( A(X, \alpha) \setminus A(X, \beta) \) is infinite) and
(c) The set \( A(X, \alpha) \cap T_{\beta} \setminus A(X, \beta) \) is finite.

(a) and (b) show \( A(X) \) is a tower while (c) ensures \( A(X) \) satisfies the “faithful restriction” clause in the definition of subtower.

For (a) we need to check that \( A(X, \beta) \setminus A(X, \alpha) \) is finite. This set is equal to

\[
\bigcup_{\eta \in X \cap \beta} \left( A(\eta, \beta) \setminus \bigcup_{\xi \in X \cap \alpha} A(\xi, \alpha) \right).
\]

Note that this displayed set is contained in \( \bigcup \{ A(\xi, \beta) \setminus A(\xi, \alpha) : \xi \in X \cap \beta \} \) since we have made the set being subtracted smaller. For each \( \xi \in X \), we have \( A(\xi, \beta) \subseteq ^* A(\xi, \alpha) \). However, if \( X \cap \beta \) is infinite, there is the possibility that the finite sets \( A(\xi, \beta) \setminus A(\xi, \alpha) \) might accumulate to an infinite set. This is the reason the construction insists that \( A(\xi, \beta) \subseteq A(\xi, \alpha) \) (not almost containment) “most of the time”. In particular, for this fixed \( \alpha \) and \( \beta \), for all but finitely many \( \xi \) we have \( A(\xi, \beta) \setminus A(\xi, \alpha) = \emptyset \), and hence the displayed set is finite. This gives (a).
For (b), we need to show $A(X, \alpha) \setminus A(X, \beta)$ is infinite. Since $A(\mu_0, \alpha) \subseteq A(X, \alpha)$, it suffices to show that $A(\mu_0, \alpha) \setminus T_\beta$ is infinite because $A(X, \beta)$ is contained in $T_\beta$. Recall that the set $\{ \eta : A(\eta, \beta) \notin A(\eta, \alpha) \}$ is finite and that the levels of the matrix are disjoint families. These, together with the fact that $T_\beta \subseteq T_\alpha$, imply that $A(\mu_0, \alpha) \setminus T_\beta =^* A(\mu_0, \beta)$. But $A(\mu_0, \alpha) \setminus A(\mu_0, \beta)$ is an infinite set.

For (c), the set of interest is contained in $\bigcup \{ A(\xi, \alpha) \setminus T_\beta \}$, $\forall \xi$, $A(\xi, \alpha) \setminus T_\beta =^* A(\xi, \beta)$. Hence only finitely many sets contribute to the noted union, and each only a finite amount. Thus $A(X)$ satisfies the “faithful restriction” requirement.

This claim leads to the following

**Corollary 50** It is consistent with the statement $2^{\aleph_1} > 2^{\aleph_0}$ that the gap cohomology group have cardinality $2^{\aleph_1}$.

**Proof of Corollary:** We have shown that H2 is consistent in the preceding section. Build the $\omega_1 \times \omega_1$ matrix in the $\aleph_1$ Gap Theorem with the ladder system in (G2) satisfying H2. As noted in the corollary to the $\aleph_0$ Gap Theorem, if $X$ and $Y$ are non-trivial subsets of $\omega_1$ then $A(X) \triangle A(Y) = A(X \triangle Y)$. In the discussion of the gap cohomology, recall that two gaps represent different cohomology classes just in case their levelwise symmetric difference is again a gap. By the immediately preceding claim, this is the case whenever $X \neq Y$ and $X \neq \omega_1 \setminus Y$.

This ends the discussion of cohomology and gaps for this article. In attempting to settle the issue of, for example, the possible size of the gap cohomology group, it may be useful to look at definable properties of gaps. One example of this is tight gaps which are described in [Sc] and [Ra]. The next section introduces another example, incollapsible gaps.

**Section 6 Incollapsible Gaps**

Let $\langle A, B \rangle$ be a Hausdorff gap. We ask: On what subsets of $\omega$ does the restriction of $\langle A, B \rangle$ remain a gap? Note that the question only makes sense for subsets on which $\langle A, B \rangle$ remains a pregap. For a tower, $A$, in $\mathcal{P}(\omega)$ let $A \upharpoonright Z$ for $Z \subseteq \omega$ be the tower $\langle A_\alpha \cap Z : \alpha \in \omega_1 \rangle$.

**Definition 51** Say that a gap, $\langle A, B \rangle$, collapses on $Z \subseteq \omega$ if and only if $\langle A \upharpoonright Z, B \upharpoonright Z \rangle$ is a pregap but not a gap. That is, there is some $Y \subseteq Z$ that fills $\langle A \upharpoonright Z, B \upharpoonright Z \rangle$.

**Definition 52** Say a gap, $\langle A, B \rangle$, is incollapsible if

$$\forall Z \subseteq \omega (\langle A, B \rangle \text{ does not collapse on } Z).$$

**Definition 53** Let IG be the statement: There is an incollapsible gap.

**Theorem 54** (Incollapsible Gaps.) IG is independent of ZFC.

**Proof of Theorem.**

We show that MA + $\neg CH \vdash \neg IG$ and CH $\vdash IG$. Similar investigations have been undertaken in [K-vD-vM] who show under MA + $\neg CH$ that for any gap there is an infinite proper subset of $\omega$ on which the gap remains a gap when restricted.

**Sketch of proof of** MA + $\neg CH \vdash \neg IG$.

The “obvious” partial order works. Fix $\langle A, B \rangle$, a gap. Let

$$\mathbb{P} := \{ \langle z, y, s, t \rangle : (\exists n \in \omega) z, y \in n \{0, 1\} \land s, t \in [\omega_1]^{<\omega} \}.$$ 

$z$ will build a set $Z$, while $y$ will build a set $Y \subseteq Z$. The set $s$ is a list of ordinals, $\alpha$, for which we promise to keep $A_\alpha \cap Z$ inside $Y$ “from now on” and $t$ is a list of ordinals keeping $B_\alpha \cap Z$ out of $Y$ “from now on”.

The goal is to have $\langle A \upharpoonright Z, B \upharpoonright Z \rangle$ a pregap filled by $Y$. Recall the notation from previous chapters that $\bar{z}$ is the set $z^{-1}(1)$, etc. Let $p = \langle z, y, s, t \rangle$ and $p' = \langle z', y', s', t' \rangle$ and define $\leq$ on $\mathbb{P}$ by $p \leq p'$ if and only if

1. $z \supseteq z'$, $y \supseteq y'$, $s \supseteq s'$ and $t \supseteq t'$.

2. $\mathbb{P} := \{ \langle z, y, s, t \rangle : (\exists n \in \omega) z, y \in n \{0, 1\} \land s, t \in [\omega_1]^{<\omega} \}.$
Proof of Claim: A pair of these is a common extension.

\(\aleph_0\) be fixed. Define

\[ \text{Claim: } P \text{ has the c.c.c. (In fact, it is } \sigma\text{-centered.)} \]

**Proof of Claim:** Fix an uncountable subcollection of \(P\), \(S\). By thinning, we may assume all the \(z\) and \(y\) components are the same, independent of the condition chosen from \(S\). The componentwise union of any pair of these is a common extension.

We now describe the dense sets that ensure \(\langle A, B \rangle\) will collapse as desired. Let \(\beta < \alpha < \omega_1\) and \(m \in \omega\) be fixed. Define

\[ D_{\alpha \beta}^m := \{ p = \langle z, y, s, t \rangle : |\bar{z} \cap A_\alpha \setminus A_\beta| \geq m \}. \]

**Claim:** \(D_{\alpha \beta}^m\) is dense for all \(\alpha, \beta\) and \(m\).

**Proof of Claim:** By induction on \(m\). The claim is clear for \(m = 0\). Suppose it is true for \(m\). Fix \(p_0 \in P\) and \(p_1 \leq p_0\) (with the obvious notational extensions) so that \(|\bar{z}_1 \cap A_\alpha \setminus A_\beta| \geq m\). Let \(r \in \omega\) be so large that

\[ \left( \bigcup_{\alpha' \in s_1 \cup t_1} A_{\alpha'} \setminus r \right) \cap \left( \bigcup_{\alpha' \in s_1 \cup t_1} B_{\alpha'} \setminus r \right) = \emptyset \]

which is possible since \(s_1 \cup t_1\) is a finite set. Fix \(n \in A_\alpha \setminus A_\beta\), an infinite set, such that \(n > r, n_1\). Let \(z \supseteq z_1\) and \(n \in \bar{z}\), and let \(y \supseteq y_1\) with \(n \in \bar{y}\). Then \(\langle z, y, s, t \rangle \leq p_0\) and \(|\bar{z} \cap A_\alpha \setminus A_\beta| \geq m + 1\).

The point of this claim is that \(A \upharpoonright Z\) will still be an \(\omega_1\) tower. A similar argument shows the same for \(B \upharpoonright Z\) and so \(\langle A \upharpoonright Z, B \upharpoonright Z \rangle\) remains a pregap. To show that the set \(Y\) separates the restricted pregap, note that if \(\alpha \in s\) and \(\langle z, y, s, t \rangle \in G\) for \(G\) generic then \(Y \supseteq (Z \cap A_\alpha)\). So it suffices to show the set \(E_\alpha := \{ \langle z, y, s, t \rangle \in P : \alpha \in s \}\) is dense which is left to the reader.

**Sketch of proof of CH ⊨ IG.**

This is a standard diagonalization argument on \([\omega]^{\aleph_0}\) (the infinite subsets of \(\omega\)) which under CH has cardinality \(\aleph_1\). Fix an enumeration of \([\omega]^{\aleph_0}, Y_\alpha\) for \(\alpha < \omega_1\). Recursively define \(A_\alpha\) and \(B_\alpha\) so that \(Y_\alpha\) will not fill \((A \upharpoonright Z, B \upharpoonright Z)\) for any \(Z \supseteq Y_\alpha\). Suppose \(A_\beta\) and \(B_\beta\) are defined for \(\beta < \alpha\) and form a pregap. If for some \(\beta < \alpha\) we have \(Y_\alpha \subset * A_\beta\) then there is nothing to worry about. Otherwise \(Y_\alpha \setminus A_\beta\) is infinite for each \(\beta < \alpha\). As the set of \(A_\beta\)'s is an increasing chain, the collection of \(Y_\alpha \setminus A_\beta\) has the strong finite intersection property. So there is an infinite set \(X \supseteq * Y_\alpha \setminus A_\beta\) and thus \(X\) is almost disjoint from each \(A_\beta\).

Since \(\alpha\) is countable, there is a set \(X'\) such that for all \(\beta < \alpha\), \(X' \supseteq * B_\beta\) and \(X' \cap A_\beta = * \emptyset\). Let \(B_\alpha := X \cup X'\) and notice now that \(Y_\alpha\) cannot separate \(A\) from \(B\) since \(B_\alpha \cap Y_\alpha\) is infinite. Let \(A_\alpha\) be any set such that \(A_\alpha \supseteq * A_\beta\) for \(\beta < \alpha\) and \(A_\alpha\) is almost disjoint from \(B_\alpha\).

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