Hybrid-NLIE for the AdS/CFT spectral problem

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Abstract: Hybrid-NLIE equations, an alternative finite NLIE description for the spectral problem of the super sigma model of AdS/CFT and its $\gamma$-deformations are derived by replacing the semi-infinite SU(2) and SU(4) parts of the AdS/CFT TBA equations by a few appropriately chosen complex NLIE variables, which are coupled among themselves and to the Y-functions associated to the remaining central nodes of the TBA diagram. The integral equations are written explicitly for the ground state of the $\gamma$-deformed system. We linearize these NLIE equations, analytically calculate the first correction to the asymptotic solution and find agreement with analogous results coming from the original TBA formalism. Our equations differ substantially from the recently published finite FiNLIE formulation of the spectral problem.
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1. Introduction

One of the most important problems in testing the AdS/CFT correspondence [1] in the planar limit is to determine the finite size spectrum of the $AdS_5 \times S^5$ superstring sigma model [2]. After integrability was discovered in the string worldsheet theory, the mirror Thermodynamic Bethe Ansatz (TBA) technique was proposed [3, 4] to determine non-perturbatively the spectrum of the string theory. The TBA equations for AdS/CFT were first derived for the ground state [5, 6, 8, 7, 9] and then using an analytic continuation trick [10] TBA equations were conjectured for magnon excited states [7, 1, 12, 13] and bound states [14] in the $\mathfrak{s}(2)$ and $\mathfrak{su}(2)$ sectors of the theory. The Y-system (and T-system) associated to the AdS/CFT problem were proposed first in [15] and were later derived from the mirror TBA equations.

Analogous progress has been made for the deformed, but still integrable cousins of the superstring sigma model [16, 17, 18, 19, 21, 20]. An important integrability preserving deformation is the so-called $\gamma$-deformed theory [18] which has three deformation parameters. The implementation of the deformations in the integrable context [23, 24, 25, 26, 22] led to the TBA formulation of the finite size problem in the deformed theory [27]. An important property of the $\gamma$-deformed theory is that since supersymmetry is not preserved, unlike in the undeformed case, the ground state energy is non-vanishing. Recently [28] the most general integrable deformations (obtained by orbifolding and TsT-transforming the original string sigma model) were found and the corresponding TBA system was proposed.

In the undeformed theory the correctness of the (conjectured) TBA equations has been nicely demonstrated by the convincing agreement with gauge theory results [29, 30, 31, 32] in the weak coupling limit [15, 33, 34, 13] through the generalized Lüscher approach [35, 36, 37, 38, 39], and with string theory results in the strong coupling limit [1, 10, 11, 12, 43].

In the $\gamma$-deformed theory the TBA and Y-system has been checked [24, 27] in the large volume and small coupling regimes against 1st order Lüscher formulae and 2nd order [27] order Lüscher formulae and direct field theory computations [44].

In spite of the success of the TBA technique in AdS/CFT, it has some obvious disadvantages as well. First of all, like all the TBA equations of known sigma-models it contains infinitely many unknown functions, which makes the study of their properties, both analytically and numerically, difficult.

A possible way to give a simpler and finite formulation of the spectral problem of an integrable model is the so-called NLIE formulation, where only a few unknown functions appear in the resulting set of nonlinear integral equations.

The NLIE approach for finite size physics of integrable models originates from the paper [45], where it was discovered that the basic objects of the integrable structure like T- and Q-functions, their functional relations and analytic properties in the complex rapidity plane allow one to set up, though in a non-unique and non-trivial way, a compact set of non-linear integral equations that governs the finite size spectrum of the theory. Later this method has been developed further and extended to describe lattice models [46, 47, 48, 49, 50, 51] and quantum field theories [52, 53, 54, 55, 56, 57, 58, 59] as well.
In the case of quantum field theories the basic starting point for the NLIE approach is the reformulation of the TBA equations through functional relations \[60\]. It is known that the infinite Y-system can be rephrased as a T-system, a set of discrete Hirota equations, which can be solved in terms of a few Q-functions. This offers the possibility to replace the infinite set of TBA variables by a finite set of only a few variables. In order to transform the functional relations into integral equations the analyticity properties of the Y, T, and Q-functions must be known. In AdS/CFT the analyticity part of the problem is more complicated than it is in the previously elaborated relativistically invariant examples \[52, 53, 54, 55, 56, 57, 59, 58\] since Y-functions are defined on an infinite genus Riemann surface, such that the physical and mirror sheets are not equivalent and Y-functions satisfy non-trivial discontinuity relations on the mirror sheet \[63\]. This is responsible for the non-local properties of the mirror TBA. However it is possible to bring the TBA equations into a quasi-local form \[64\] which contains only next to nearest neighbor interactions among Y-functions opening the way to apply techniques worked out for relativistic models \[52, 53, 54, 55, 56, 57, 59, 58\].

Recently it has been shown that there exists a “magic” sheet with short cuts where the T-functions have very simple discontinuity structure and they admit a $\mathbb{Z}_4$ symmetry which supplemented by analyticity requirements dictated by the asymptotic solution and group-theoretical constraints allows one to derive both the mirror TBA and its simplified finite reformulation FiNLIE \[61\] as well. This was the first finite reformulation of the AdS/CFT spectral problem. In \[61\] left-right symmetric states (sl(2) or su(2) states) were considered and the 3 unknowns of the FiNLIE are discontinuities of two T-functions along short cuts in the “magic” sheet and the discontinuity of a certain gauge transformation along the real axis and they encode all information on the T- and Y-systems of the AdS/CFT.

In this paper we will give another finite reformulation of the spectral problem, but remaining on the mirror sheet. We follow the approach of the very first NLIE paper \[15\], where the basic objects of the integrable structure like T- and Q-functions are used to build the NLIE unknowns and where the functional relations they satisfy and their analyticity properties determine the actual form of the NLIE. Our approach may be called hybrid-NLIE, since by appropriate NLIEs we resum the semi-infinite SU(2) and SU(4) parts of the mirror TBA equations. This approach was initiated in \[66\] by resumming the two SU(2) wings similarly as it had been done in the case of the SU(2) related relativistic models \[53, 54, 55, 56, 57\]. The basic idea of the construction as follows.

The set of Y-functions appearing in the AdS/CFT spectral problem can be naturally grouped into four subsets (see Figure 1):

- right-wing nodes: $Y_{m|w}^{(+)}$ for $m \geq 2$.
- left-wing nodes: $Y_{m|w}^{(-)}$ for $m \geq 2$.
- upper nodes: $Y_{m|vw}^{(\pm)}$ for $m \geq 2$ and $Y_Q$ for $Q \geq 3$.
- central nodes: $Y_1, Y_2, Y_{1|w}^{(\pm)}, Y_{1|vw}^{(\pm)}$, and $Y_{\pm}^{(\pm)}$. 

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The first step [66] of the construction of the hybrid-NLIE is to replace the right-wing SU(2) type nodes by a complex NLIE variable coupled to itself and to the central nodes. The coupling of the central nodes to \( Y^{(+)}_{2|w} \) is replaced by coupling to the NLIE variable. Similar considerations apply to the left-wing part of the problem.

The second step is to replace the Y-functions corresponding to the upper SU(4) type nodes by 12 NLIE functions. These are coupled to themselves as well as to the 1st row of upper nodes \( (Y_3, Y^{(\pm)}_{2|vw}) \) while the 1st row of the upper nodes are coupled to the central nodes and the upper NLIE functions.

Thus we can replace the semi-infinite SU(2) and SU(4) parts of the TBA diagram by two SU(2) and an SU(4) type relativistic NLIEs which are sewn together by the quasi-local TBA equations for the central nodes. We call the final equations hybrid-NLIE for AdS/CFT. Though we have more unknown functions than in [61], our NLIE equations are based on the Bäcklund transformations of the corresponding T-system and can be straightforwardly generalized to a wide range of relativistically invariant integrable models. In this paper we will complete the derivation of our equations only for the ground state, but in order to get non-trivial results, in the \( \gamma \)-deformed AdS/CFT model, because in this case the ground state solution of the Y-system is non-trivial and the ground state equations can be checked by analytical computations using the 2nd order Lüschel formula [27].

Though at the level of equations we concentrate on the ground state, all our considerations concerning the analyticity properties and the construction of NLIE unknowns given in sections 4-6 are general and valid for excited states of the model as well.

The paper is organized as follows: in section 2 we give our starting point by recalling the quasi-local form of the TBA equations. In section 3 we review the Bäcklund transformations and analyticity strips which form the basis of the construction of our variables. In section 4

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\(^1\)Note that a resummation of the horizontal nodes, similarly to what is used for the FiNLIE, was already proposed in [57].
we construct the NLIE functions and the functional relations they satisfy. In section 5 the asymptotic solution is given at all levels of the nesting for the upper nodes and in section 6 the NLIE construction for the horizontal part is given. Section 7 contains the derivation of the SU(2) and SU(4) type NLIEs. In section 8 we linearize our NLIE equations for the \( \gamma \)-deformed ground state, analytically calculate the first correction to the asymptotic solution and show the agreement with analogous results coming from the original TBA formalism. In section 9 we summarize our results by collecting all NLIE equations for the description of the \( \gamma \)-deformed ground state. Appendix A contains our basic notations and the kernels of the TBA. We collected the basic building blocks of the asymptotic solution in appendix B. The paper is closed by appendix C where the kernels of the SU(4) type NLIE are listed.

2. Quasi-local twisted TBA

In this section we review the quasi-local form of the mirror TBA for the three parameter \( \gamma \)-deformed AdS/CFT theories \([16,17,18,19,62]\). Contrary to the undeformed or \( \beta \)-deformed theories, in the most general case no supersymmetry is preserved and the ground state energy is non-zero. This allows us to test our ideas on the ground state equations directly. For this reason and for the sake of simplicity at the level of equations we will restrict our attention to the ground state of the model though as it has already been mentioned in the introduction all of our equations can be extended to excited states without difficulties.

At the level of the worldsheet scattering theory the \( \gamma \)-deformation can be implemented in two ways: either imposing operatorial (particle number dependent) twisted boundary conditions \([25]\), or by imposing a (c-number) twisted boundary condition and considering a twisted scattering matrix for the excitations \([23,26]\).

The ground state TBA equations of the \( \gamma \)-deformed theory have been derived in \([27]\) and it was found that the twist parameters enter the so-called canonical version of the equations as if they were Y-system preserving chemical potentials in the undeformed theory \([63,8]\). Thus the Y-system for the \( \gamma \)-deformed theory is identical to that of the undeformed theory. On the other hand the twist parameters cancel from the simplified version of the twisted TBA and they re-appear as parameters of the boundary conditions imposed on the Y-functions at large \( u \). These canonical and simplified twisted TBA equations can be reformulated in a quasi-local form \([64]\) which we review now for the ground state.

The quasi-local formulation of the mirror TBA is possible since all kernels \( K_Q \equiv K_Q(u,v) \) entering the TBA equations satisfy the important identity:

\[
K_Q - s \ast (K_{Q-1} + K_{Q+1}) = \delta K_Q, \quad K_0 \equiv 0, \quad Q = 1, 2, ... \quad (2.1)
\]

with \( \delta K_Q \) vanishing with the exception of a few values of the index \( Q \). Explicit formulas are given at the end of appendix A. In order to compactly present the quasi-local TBA equations we introduce the notations:

\[
L_Q = \ln(1 + Y_Q), \quad R_Q = \ln \left( 1 + \frac{1}{Y_Q} \right) \quad Q = 1, 2, ...
\]
We use the kernels equations and they take the form:

\[ r_m^{(\alpha)} = \ln(1 + Y_{m|vw}^{(\alpha)}), \quad L_{\pm}^{(\alpha)} = \ln \left( -\left( 1 - \frac{1}{Y_{\pm}^{(\alpha)}} \right) \right), \quad m = 1, 2, \ldots, \quad \alpha = \pm \]  \hspace{1cm} (2.2)

\[ H^{(\alpha)} = \ln \left( \frac{1 - Y_{-}^{(\alpha)}}{1 - Y_{+}^{(\alpha)}} \right), \quad \alpha = \pm. \]

Next we introduce the linear functional\(^2\) of the vector kernel \(K_Q\) with the definition:

\[ \Omega(K_Q) = R_2 * (\delta K_2 + 2s_{1/2} * \delta K_2) + R_2 * \sigma_{1/2} * K_1 - R_1 * \sigma_{1/2} * K_2 + \sum_{\alpha=\pm} \left( r_1^{(\alpha)} * \sigma_{1/2} * \delta K_2 + r_1^{(\alpha)} * s_{1/2} * K_1 - H^{(\alpha)} * s_{1/2} * K_2 \right). \]  \hspace{1cm} (2.3)

We use the kernels

\[ s(u) = \frac{g}{4 \cosh \frac{\pi gu}{2}}, \quad s_{1/2}(u) = \frac{1}{2} s \left( \frac{u}{2} \right), \quad \sigma_{1/2}(u) = \frac{g}{2 \sqrt{2} \cosh \frac{\pi gu}{2}}, \]  \hspace{1cm} (2.4)

and \(*\) and \(\hat{\star}\) denote convolutions running on \(\mathbb{R}\) and \([-2, 2] \subset \mathbb{R}\) respectively. See appendix A.

The quasi-local TBA equations for the ground state are composed of two groups of equations. Equations in the first group follow from the Y-system relations, they are local and their form is the same as in the simplified version of the equations:

\[ Y_{m|vw}^{(\alpha)} = \exp \left\{ \ln \left[ \frac{(1 + Y_{m|vw}^{(\alpha)})(1 + Y_{m-1|vw}^{(\alpha)})}{(1 + Y_{m+1})} \right] * s \right\}, \quad m \geq 2, \]  \hspace{1cm} (2.5)

\[ Y_{1|vw}^{(\alpha)} = \exp \left\{ \ln \left[ \frac{(1 + Y_{2|vw}^{(\alpha)})}{(1 + Y_{2})} \right] * s + \ln \left[ \frac{1 - Y_{-}^{(\alpha)}}{1 - Y_{+}^{(\alpha)}} \right] * \hat{s} \right\}, \]  \hspace{1cm} (2.6)

\[ Y_{m}^{(\alpha)} = \exp \left\{ \ln \left[ (1 + Y_{m|vw}^{(\alpha)})(1 + Y_{m-1|vw}^{(\alpha)}) \right] * s \right\}, \quad m \geq 2, \]  \hspace{1cm} (2.7)

\[ Y_{1|w}^{(\alpha)} = \exp \left\{ \ln \left[ 1 + Y_{2|w}^{(\alpha)} \right] * s + \ln \left[ \frac{1 - Y_{-}^{(\alpha)}}{1 - Y_{+}^{(\alpha)}} \right] * \hat{s} \right\}, \]  \hspace{1cm} (2.8)

\[ Y_Q = \exp \left\{ \ln \left[ \frac{Y_{Q+1} Y_{Q-1}(1 + Y_{Q+1|vw}^{(\alpha)})(1 + Y_{Q-1|vw}^{(\alpha)})}{Y_{Q-1|vw}^{(-)} Y_{Q-1|vw}^{(-)}(1 + Y_{Q+1})(1 + Y_{Q-1})} \right] * s \right\}, \quad Q \geq 2. \]  \hspace{1cm} (2.9)

The second group consists of quasi-local (next to nearest neighbor interacting) central node equations and they take the form:

\[ \frac{Y_{-}^{(\alpha)}}{Y_{+}^{(\alpha)}} = \exp \left\{ -L_1 * K_{1y} - \Omega(K_{Qy}) \right\}, \]  \hspace{1cm} (2.10)

\[ Y_{+}^{(\alpha)} Y_{-}^{(\alpha)} = \exp \left\{ 2 \ln \left[ \frac{1 + Y_{1|vw}^{(\alpha)}}{1 + Y_{1|vw}^{(\alpha)}} \right] * s + L_1 * \left[ -K_1 + 2K_{xv}^{11} * s \right] \right. \]

\[ - \Omega(K_Q) + 2 \Omega(K_{Q1}^{Q1} * s) \right\}, \]  \hspace{1cm} (2.11)

\(^2\)The expression \(\Omega(K_Q) = \sum_{Q=1}^{\infty} L_Q * K_Q\) in the non-local versions of the mirror TBA.
\[ \ln Y_1 = - L \tilde{E}_1 + \sum_{a=\pm} r_1^{(a)} \ast s \ast K_{y1} \]

\[ = - \sum_{a=\pm} \left( \ln \left[ \frac{1 - Y_{-}^{(a)}}{1 - Y_{+}^{(a)}} \right] \ast s \ast K_{\text{vix}}^{11} + \mathcal{L}_{-}^{(a)} \ast K_{y1}^{1} + \mathcal{L}_{+}^{(a)} \ast K_{y1}^{1} \right) \]

\[ + L_1 \ast K_{\text{vix}}^{11} + \Omega(K_{\text{vix}}^{Q1}) + 2 \Omega(s \ast K_{\text{vix}}^{Q1}) \]

These equations are parameter-free, but the \( \gamma \)-deformation parameters enter the conditions determining the asymptotic large \( u \) behavior of \( Y \)-functions:

\[ Y_{m|w}^{(a)} \rightarrow m (m + 2), \quad Y_{m|w}^{(+)} \rightarrow [m]_q [m + 2]_q, \quad Y_{m|w}^{(-)} \rightarrow [m]_q [m + 2]_q, \quad m = 1, 2, \ldots \]

and

\[ Y_{\pm}^{(+)} \rightarrow \frac{2}{[2]_q}, \quad Y_{\pm}^{(-)} \rightarrow \frac{2}{[2]_q} \]

where \([m]_q\) denotes the \( q \)-number \([m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}\). They correspond to the (c-number) boundary conditions in the description of [23, 26], and they are expressed by the deformation parameters of the \( \gamma \)-deformed theory as follows: \( q = e^{i \frac{2\pi}{L}} \), \( \hat{q} = e^{i \frac{2\pi}{L}} \). The parameter \( \gamma_1 \) does not enter the ground state equations since it corresponds to the twist parameter of the S-matrix.

To complete the quasi-local TBA description of the \( \gamma \)-deformed AdS/CFT we should supplement the integral equations (2.5-2.12) with the energy formula. In the quasi-local description the energy expression is a function of the central nodes only:

\[ \mathcal{E} = L_1 \ast \tilde{J}_1 + (R_2 - R_2^0 + L_2^0) \ast \sigma_{1/2} \ast \tilde{J}_1 - (R_1 - R_1^0 + L_1^0) \ast \sigma_{1/2} \ast \tilde{J}_2 \]

\[ + \sum_{a=\pm} \left( (r_1^{(a)} - r_1^{(a)^0}) \ast s_{1/2} \ast \tilde{J}_1 - (H^{(a)} - H^{(a)^0}) \ast s_{1/2} \ast \tilde{J}_2 \right), \]

where we introduced the notation \( \tilde{J}_Q(u) = -\frac{1}{2\pi} \frac{d\tilde{g}_Q}{du} \) and the upper index \( ^0 \) means that the corresponding expression should be taken at the asymptotic solution. This representation is necessary for all integrals in (2.15) to converge. In the \( \gamma \)-deformed model the asymptotic solutions are identical with the large \( u \) limits for \( Y \)-functions appearing in (2.13), (2.14) while the asymptotic solution for the momentum carrying nodes takes the form

\[ Y_Q^{\alpha} \sim (2 - [2]_q)(2 - [2]_q) Q^2 e^{-L \tilde{E}_Q} \]

The quasi-local TBA equations presented in this section are the starting point for our NLIE description. We will transform the semi-infinite set of TBA equations (2.7) and (2.5), (2.9) to NLIE equations of SU(2) and SU(4) type, respectively by (the nested hierarchy of) Bäcklund transformations. This is described in the next sections.

### 3. Hierarchy of Bäcklund transformations and analyticity strips

In this paper we will denote the AdS/CFT \( Y \)-functions in the index conventions\(^3\) of [13] by \( y_{a,s} \) and the corresponding T-system elements by \( t_{a,s} \). They satisfy the usual Y-T relations

\[ y_{a,s} = \frac{t_{a,s+1} t_{a,s-1}}{t_{a+1,s} t_{a-1,s}} \]

\( ^3\)The precise relation between our \( Y \)-functions and those of ref. [13] is \( Y_{a,s}(u) = y_{a,s}(-2u/g) \).
and the T-system equations
\[ t_{a,s}^{+} t_{a,s}^{-} = t_{a+1,s} t_{a-1,s} + t_{a,s+1} t_{a,s-1}. \]  (3.2)

The T-functions \( t_{a,s} \) for \( s = 0, 1, 2 \) have been constructed explicitly in [65] in a particular gauge called the BA (Bethe Ansatz) gauge. It was found that their analytic properties can be summarized as:
- \( t_{a,0} \) is of type \((-1 - a, a + 1), \quad a \geq 1\),
- \( t_{a,1} \) is of type \((-a, a), \quad a \geq 1\),
- \( t_{a,2} \) is of type \((1 - a, a - 1), \quad a \geq 2\).

A function \( f(u) \) is called of type \((c, d)\) if it is meromorphic in the strip \(c/g < \text{Im}u < d/g\).

We can extend the above solution of T-functions for \( s = -1, -2 \) by the Y-T relations (3.1). Using the fact that \( y_{a,0} = Y_{a} \) is of type \((-a, a)\) and the relation
\[ t_{a,-1} = y_{a,0} \frac{t_{a+1,0} t_{a-1,0}}{t_{a,1}} \]  (3.3)
we see that
- \( t_{a,-1} \) is of type \((-a, a), \quad a \geq 1\),

and similarly from
\[ t_{a,-2} = y_{a,-1} \frac{t_{a+1,-1} t_{a-1,-1}}{t_{a,0}} \]  (3.4)
and using the fact that \( y_{a,-1} = Y_{a-1}^{(-)} \) is of type \((1 - a, a - 1)\) we find that
- \( t_{a,-2} \) is of type \((1 - a, a - 1), \quad a \geq 2\).

The AdS/CFT T-system satisfies the boundary condition \( t_{a,\pm 3} = 0, \quad a \geq 3 \) and hence the boundary T-functions \( t_{a,\pm 2}, \quad a \geq 2 \) are solutions of the discrete Laplace equation. These are parametrized by the four functions \( A, B, C \) and \( D \):
\[ t_{a,2} = A^{[a]} B^{[-a]}, \quad t_{a,-2} = C^{[a]} \]  (3.5)

This implies
\[ \frac{t_{a+1,2}}{t_{a,2}} = B^{[-a]}, \quad B = \frac{B^{-}}{B^{+}}, \]  (3.6)
\[ \frac{t_{a+1,2}}{t_{a,2}} = A^{[a]}, \quad A = \frac{A^{+}}{A^{-}}, \]  (3.7)
\[ \frac{t_{a+1,-2}}{t_{a,-2}} = D^{[-a]}, \quad D = \frac{D^{-}}{D^{+}}, \]  (3.8)
\[ \frac{t_{a+1,-2}}{t_{a,-2}} = C^{[a]}, \quad C = \frac{C^{+}}{C^{-}}. \]  (3.9)

From (3.6) we obtain that \( B \) is of type \((-2a, -2)\) and since this is true for any \( a \geq 2 \) finally we can conclude that \( B \) is of type \((-\infty, -2)\). Similarly we find that \( D \) is also of type \((-\infty, -2)\) and \( A \) and \( C \) are of type \((2, \infty)\).
3.1 Principal chiral model conventions

In this paper we will also use an alternative notation for the Y and T-functions corresponding to the upper nodes \(a \geq 2\) of the AdS/CFT TBA diagram. Using this new notation this part becomes identical to the corresponding functional relations of an SU(4) principal chiral model. In this latter model, for general SU\((k)\), there are T-functions \(T_{a,s} a = 0,1,\ldots,k\) satisfying the T-system equations

\[
T^+_{a,s} T^-_{a,s} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}
\]

(3.10)

and the boundary conditions

\[
T_{-1,s} = T_{k+1,s} = 0.
\]

(3.11)

The mapping of the AdS/CFT T-functions to the SU(4) principal model variables is given by

\[
t_{a,2} = T_{0,a}, \quad t_{a,1} = T_{1,a}, \quad t_{a,0} = T_{2,a}, \quad t_{a,-1} = T_{3,a}, \quad t_{a,-2} = T_{4,a}
\]

(3.12)

and we can summarize the analytic properties of the T-functions in this notation as

\[
T_{-1,s} \text{ and } T_{0,s} \text{ are of type } (1-s,s-1)\]

\[
T_{3,s} \text{ and } T_{1,s} \text{ are of type } (-s,s)\]

and \(T_{2,s}\) is of type \((-1-s,s+1)\). The principal chiral model Y-T relations are of the same form as (3.1):

\[
y_{a,s} = T^+_{a,s} T^-_{a,s}, \quad Y_{a,s} = 1 + y_{a,s} = \frac{T^+_{a,s} T^-_{a,s}}{T^+_{a+1,s} T^-_{a-1,s}}, \quad a = 1,\ldots,k-1.
\]

(3.13)

Note that the identification (3.12) implies the exchange of the indices \(a \leftrightarrow s\) and consequently the relation among the Y-functions is given by

\[
Y_{1,s} = 1 + \frac{1}{Y_{s,1}} = 1 + Y^{(+)}_{s-1|vw}, \quad Y_{2,s} = 1 + \frac{1}{Y_{s,0}} = 1 + \frac{1}{Y_s}, \quad Y_{3,s} = 1 + \frac{1}{Y_{s,-1}} = 1 + Y^{(-)}_{s-1|vw}.
\]

(3.14)

3.2 Bäcklund transformations

The advantage of using the principal model conventions is that in this language it is easy to formulate the hierarchy of Bäcklund transformations that will play an important role in our considerations. In addition, while the SU(4) case is relevant for the upper nodes, similar considerations, but for the SU(2) case, are relevant for the right-wing nodes and similarly for the left-wing nodes.

Given a set of T-functions satisfying the T-system equations (3.10) and boundary conditions (3.11) we can find the set of F-functions, \(F_{a,s}\), \(a = 0,\ldots,k-1\) with boundary condition

\[
F_{-1,s} = F_{k,s} = 0
\]

(3.15)

by solving the equations [69] (Bäcklund transformation) for \(a = 0,1,\ldots,k-1\)

\[
T_{a+1,s+1} F_{a,s} = T^+_{a+1,s} F_{a,s+1} + T^-_{a,s} F_{a+1,s+1};
\]

(3.16)

\[
T^+_{a,s+1} F_{a,s} = T_{a,s} F^+_{a,s+1} + T^+_{a+1,s} F_{a-1,s+1}.
\]

(3.17)
It can be shown that the F-functions also satisfy the T-system relations \((3.10)\) (with \(k - 1\)) thus the mapping \(T \rightarrow F\) is from a solution of the SU\((k)\) T-problem to a solution of the SU\((k - 1)\) T-problem. Hence it is natural to define the analogs of the Y-functions corresponding to the F-functions:

\[
\begin{align*}
  w_{a,s} &= \frac{F_{a,s+1} F_{a,s-1}}{F_{a+1,s} F_{a-1,s}}, \quad W_{a,s} = 1 + w_{a,s} = \frac{F_{a,s}^+ F_{a,s}^-}{F_{a+1,s} F_{a-1,s}}, \quad a = 1, \ldots, k - 2. \\

\end{align*}
\]

Let us recall that the boundary components satisfy the discrete Laplace equation and can be factorized as

\[
T_{0,s} = A[s] B[−s], \quad T_{k,s} = C[s] D[−s] \tag{3.19}
\]

and define the ratios

\[
A = \frac{A^+}{A^-}, \quad B = \frac{B^-}{B^+}, \quad C = \frac{C^+}{C^-}, \quad D = \frac{D^-}{D^+}. \tag{3.20}
\]

Similarly for the F-functions we have

\[
F_{0,s} = \alpha[s] \beta[−s], \quad F_{k-1,s} = \gamma[s] \delta[−s] \tag{3.21}
\]

and

\[
a = \frac{\alpha^+}{\alpha^-}, \quad b = \frac{\beta^-}{\beta^+}, \quad c = \frac{\gamma^+}{\gamma^-}, \quad d = \frac{\delta^-}{\delta^+}. \tag{3.22}
\]

It is easy to see from the Bäcklund transformations \((3.16)\) and \((3.17)\) that one of the boundary ratios is preserved on both boundary lines:

\[
a = A, \quad d = D^+. \tag{3.23}
\]

By studying the analytic properties of the T-functions appearing in the Bäcklund transformations and assuming maximal possible analyticity (meromorphicity) strips for the resulting F-functions we find that \((s \geq 2)\):

- \(F_{0,s}\) is of type \((1-s,s-1)\),
- \(F_{1,s}\) is of type \((-s,s)\),
- \(F_{2,s}\) is of type \((-1-s,s-1)\),
- \(F_{3,s}\) is of type \((-s,s-2)\).

The T-system equations are invariant under the gauge transformations

\[
T_{a,s} \rightarrow \tilde{T}_{a,s} = \lambda_{a,s} T_{a,s}, \tag{3.24}
\]

where the gauge function is of the form

\[
\lambda_{a,s} = f_1^{[−s−a]} f_2^{[s−a]} f_3^{[a−s]} f_4^{[s+a]}. \tag{3.25}
\]

The Y-functions defined by \((3.13)\) are gauge invariant and if we define the gauge transformation of the F-functions by

\[
F_{a,s} \rightarrow \tilde{F}_{a,s} = \omega_{a,s} F_{a,s}, \tag{3.26}
\]

with

\[
\omega_{a,s} = f_1^{[−s−a]} f_2^{[s−a]} f_3^{[2+a−s]} f_4^{[s+a]}. \tag{3.27}
\]
then also the Bäcklund transformations remain invariant. Similarly the W-functions defined by (3.18) are also gauge invariant.

We have seen that the properties of the F-functions are very similar to those of the T-functions apart from the reduction $k \to k - 1$. We can now formulate a Bäcklund transformation starting from the F-functions playing the role of the T-functions and by repeating the steps build the chain of Bäcklund transformations corresponding to

$$SU(k) \to SU(k - 1) \to SU(k - 2)\ldots$$  \hspace{1cm} (3.28)

To unify and simplify the notation we now introduce the family of T-functions $T^{(r)}_{a,s}$ where $r = 1, \ldots, k$ indicates the Bäcklund level, the range of the index $a$ is $a = 0, \ldots, r$ and the boundary conditions are

$$T^{(r)}_{a,s} = T^{(r)}_{a,s}, \quad T^{(r)}_{r,s} = F^{(r)}_{a,s}$$  \hspace{1cm} (3.29)

With this notation

$$A^{(r)} = A^{(r)}_+, \quad B^{(r)} = B^{(r)}_+, \quad C^{(r)} = C^{(r)}_+, \quad D^{(r)} = D^{(r)}_+$$  \hspace{1cm} (3.30)

and we can identify

$$A^{(k)} = A, \quad B^{(k)} = B, \quad C^{(k)} = C, \quad D^{(k)} = D$$  \hspace{1cm} (3.31)

and

$$A^{(k-1)} = a, \quad B^{(k-1)} = b, \quad C^{(k-1)} = c, \quad D^{(k-1)} = d$$  \hspace{1cm} (3.32)

Further we define

$$A^{(r)} = A^{(r)}_+, \quad B^{(r)} = B^{(r)}_+, \quad C^{(r)} = C^{(r)}_+, \quad D^{(r)} = D^{(r)}_+$$  \hspace{1cm} (3.33)

Here

$$A^{(k)} = A, \quad B^{(k)} = B, \quad C^{(k)} = C, \quad D^{(k)} = D$$  \hspace{1cm} (3.34)

and so on. Continuing (3.23) we have

$$A^{(k)} = A^{(k-p)}, \quad D^{(k)} = D^{(k-p)[-p]}, \quad p = 1, \ldots, k - 1.$$  \hspace{1cm} (3.35)

Let us also introduce the family of Y-functions using the notation

$$y^{(r)}_{a,s}, \quad Y^{(r)}_{a,s} = 1 + y^{(r)}_{a,s}, \quad a = 1, \ldots, r,$$  \hspace{1cm} (3.36)

where $r = 1, \ldots, k - 1$ and

$$y^{(k-1)}_{a,s} = y_{a,s}, \quad y^{(k-2)}_{a,s} = w_{a,s},$$  \hspace{1cm} (3.37)
and so on.

Returning to the SU(4) special case the analyticity strips of \( T_{a,s}^{(4)} = T_{a,s} \) and \( T_{a,s}^{(3)} = \frac{F_{a,s}}{T_{-1,s}} \) are already determined above and using the lower level Bäcklund equations can be established also for \( T_{a,s}^{(2)} \) and \( T_{a,s}^{(1)} \). We find that

\[
\begin{align*}
T_{0,s}^{(2)} & \text{ is of type } (1 - s, s - 1), \\
T_{1,s}^{(2)} & \text{ is of type } (-s, s - 2), \\
T_{2,s}^{(2)} & \text{ is of type } (-1 - s, s - 1), \\
T_{0,s}^{(1)} & \text{ is of type } (1 - s, s - 1), \\
T_{1,s}^{(1)} & \text{ is of type } (-2 - s, s - 4).
\end{align*}
\]

For the boundary ratios we find that

\[
\begin{align*}
A^{(4)}(a,s) & \text{ are of type } (2, \infty), \\
B^{(4)}(a,s) & \text{ are of type } (-\infty, -2), \\
C^{(4)}(a,s) & \text{ are of type } (2, \infty), \\
D^{(4)}(a,s) & \text{ are of type } (-\infty, -2).
\end{align*}
\]

For completeness we also list the analyticity strips for the Y-functions:

\[
\begin{align*}
y_{1,s}^{(3)} & \text{ is of type } (1 - s, s - 1), \\
y_{2,s}^{(3)} & \text{ is of type } (-s, s), \\
y_{3,s}^{(3)} & \text{ is of type } (1 - s, s - 1), \\
y_{1,s}^{(2)} & \text{ is of type } (1 - s, s - 1), \\
y_{2,s}^{(2)} & \text{ is of type } (-s, s - 2), \\
y_{1,s}^{(1)} & \text{ is of type } (1 - s, s - 3).
\end{align*}
\]

4. NLIE variables and functional equations

The gauge invariant Y-functions can be obtained from the T-system equations by dividing the equations by one of the terms. Similarly we can form gauge invariant ratios by dividing the Bäcklund equations by one of the three terms. These gauge invariant ratios will be the variables in our NLIE equations.

Using the first Bäcklund equation (3.16) we can form the ratios

\[
\begin{align*}
b_{a,s} &= \frac{T_{a,s}^+ F_{a-1,s+1}}{T_{a-1,s} F_{a,s+1}}, \\
B_{a,s} &= \frac{T_{a,s+1} F_{a-1,s}}{T_{a-1,s} F_{a,s+1}}, \\
B_{a,s} &= 1 + b_{a,s}, \\
D_{a,s} &= 1 + d_{a,s},
\end{align*}
\]

In terms of these variables the (3.16) equations simply become

\[
B_{a,s} = 1 + b_{a,s}, \quad a = 1, \ldots, k - 1.
\]

Similarly we obtain from the second Bäcklund equation (3.17)

\[
\begin{align*}
d_{a,s} &= \frac{T_{a,s}^- F_{a,s+1}}{T_{a+1,s} F_{a-1,s+1}}, \\
D_{a,s} &= 1 + d_{a,s},
\end{align*}
\]

In terms of these variables the (3.17) equations simplify to

\[
D_{a,s} = 1 + d_{a,s}, \quad a = 1, \ldots, k - 1.
\]
Since we know the analyticity strips of the T and F-functions we can see that for our SU(4) case the NLIE functions are of type:

\[ b_{1,s} : (1 - s, s - 1), \quad d_{1,s} : (1 - s, s + 1), \]
\[ b_{2,s} : (-s, s), \quad d_{2,s} : (-s, s), \]
\[ b_{3,s} : (-1 - s, s - 1), \quad d_{3,s} : (1 - s, s - 1). \]

Generalizing the definitions (4.1) and (4.3) to all Bäcklund levels we can introduce the corresponding NLIE functions:

\[ b^{(r)}_{a,s}, \quad d^{(r)}_{a,s}, \quad B^{(r)}_{a,s} = 1 + b^{(r)}_{a,s}, \quad D^{(r)}_{a,s} = 1 + d^{(r)}_{a,s}, \quad a = 1, \ldots, r, \quad r = 1, \ldots, k - 1, \quad (4.5) \]

where

\[ b^{(k-1)}_{a,s} = b_{a,s}, \quad d^{(k-1)}_{a,s} = d_{a,s} \quad (4.6) \]

and the analyticity strips are

\[ b^{(2)}_{1,s} : (1 - s, s - 1), \quad d^{(2)}_{1,s} : (1 - s, s - 1), \]
\[ b^{(2)}_{2,s} : (-s, s - 2), \quad d^{(2)}_{2,s} : (-s, s - 2), \]
\[ b^{(2)}_{3,s} : (-1 - s, s - 3), \quad d^{(2)}_{3,s} : (1 - s, s - 3). \]

It is easy to verify using the definitions (4.1) and (4.3) that the NLIE functions satisfy the functional equations

\[ b_{a,s} d_{a+1,s} = Y_{a,s}, \quad a = 1, \ldots, k - 1, \quad (4.7) \]
\[ d_{a,s} b_{a+1,s} = W_{a,s+1}, \quad a = 1, \ldots, k - 2, \quad (4.8) \]
\[ \beta_{a,s} D_{a,s} b_{a+1,s} B_{a,s+1}^{+} = W_{a,s}, \quad a = 1, \ldots, k - 2, \quad (4.9) \]
\[ B_{a,s} D_{a,s}^{+} \beta_{a+1,s} D_{a,s}^{-} = Y_{a,s+1}, \quad a = 1, \ldots, k - 1, \quad (4.10) \]

where

\[ \beta_{a,s} = \frac{b_{a,s}}{B_{a,s}}, \quad a = 1, \ldots, k - 1, \quad \beta_{k,s} = 1 \quad (4.11) \]

and

\[ \delta_{a,s} = \frac{d_{a,s}}{D_{a,s}}, \quad a = 1, \ldots, k - 1, \quad \delta_{0,s} = 1. \quad (4.12) \]

Again, we can write analogous relations for all Bäcklund levels:

\[ b_{a,s}^{(r)} d_{a,s}^{(r)} = Y_{a,s}^{(r)}, \quad a = 1, \ldots, r, \quad (4.13) \]
\[ d_{a,s}^{(r)} b_{a+1,s}^{(r)} = Y_{a,s+1}^{(r-1)}, \quad a = 1, \ldots, r - 1, \quad (4.14) \]
\[ \beta_{a,s}^{(r)} D_{a,s}^{(r)} \delta_{a+1,s}^{(r)} B_{a,s+1}^{(r)+} = Y_{a,s}^{(r-1)}, \quad a = 1, \ldots, r - 1, \quad (4.15) \]
\[ B_{a,s}^{(r)} D_{a,s}^{(r)+} \beta_{a+1,s}^{(r)} \delta_{a-1,s}^{(r)} = Y_{a,s+1}^{(r)}, \quad a = 1, \ldots, r \quad (4.16) \]

with

\[ \beta_{a,s}^{(r)} = \frac{b_{a,s}^{(r)}}{B_{a,s}^{(r)}}, \quad a = 1, \ldots, r, \quad \beta_{r+1,s}^{(r)} = 1 \quad (4.17) \]
and

\[ \delta_{a,s}^{(r)} = \frac{d_{a,s}^{(r)}}{D_{a,s}^{(r)}}, \quad a = 1, \ldots, r, \quad \delta_{0,s}^{(r)} = 1. \quad (4.18) \]

Using the functional equations (4.13-4.16) we can extend the analyticity strips for some NLIE functions. For example, the relation

\[ B_{1,s}^{(3)} - D_{1,s}^{(3)} + \beta_{2,s}^{(3)} = Y_{1,s+1}^{(3)} \quad (4.19) \]

should be valid in the strip \((-s, s)\) and allows us to extend the analyticity strip of \(b_{1,s}^{(3)}\) to \((-1 - s, s - 1)\). Similarly we can make the extensions to:

\[ d_{3,s}^{(2)}: (1 - s, s + 1), \quad b_{1,s}^{(2)}: (-1 - s, s - 1), \quad d_{2,s}^{(2)}: (-s, s), \quad b_{1,s}^{(1)}: (-1 - s, s - 3), \quad d_{1,s}^{(1)}: (1 - s, s - 1). \]

Actually, the NLIE functions \(b_{1,2}^{(1)}\) and \(d_{1,2}^{(1)}\) had no range at all before this extension and even more seriously the \(s = 2\) functional equations (4.15) for \(r = 2\) and \(a = 1\) and (4.13) for \(r = 1\) and \(a = 1\) have no range (even after the extensions). We conclude that our system of functional equations is meaningful for \(s \geq 3\) only.

For completeness, we here summarize the extended analyticity strips of our NLIE functions:

\[ b_{1,s}^{(3)}: (-1 - s, s - 1), \quad d_{1,s}^{(3)}: (1 - s, s + 1), \]
\[ b_{2,s}^{(3)}: (-s, s), \quad d_{2,s}^{(3)}: (-s, s), \]
\[ b_{3,s}^{(3)}: (-1 - s, s - 1), \quad d_{3,s}^{(3)}: (1 - s, s + 1), \]
\[ b_{1,s}^{(2)}: (-1 - s, s - 1), \quad d_{1,s}^{(2)}: (1 - s, s - 1), \]
\[ b_{2,s}^{(2)}: (-s, s - 2), \quad d_{2,s}^{(2)}: (-s, s), \]
\[ b_{1,s}^{(1)}: (-1 - s, s - 3), \quad d_{1,s}^{(1)}: (1 - s, s - 1). \]

We will see that the system of functional equations (4.13-4.16) cannot be translated to a closed set of NLIE integral equations. They have to be completed with further relations, which we will call the “half-plane” functional relations (because their building blocks have good analyticity properties either in the upper or the lower half plane). They can also be obtained from the definitions (4.1) and (4.3) and are of the form

\[ \beta_{1,s} = \frac{T_{1,s}^+}{T_{1,s+1}^-} b_{[1-s]}, \quad \delta_{k-1,s} = \frac{T_{k-1,s}^-}{T_{k-1,s+1}^+} c^{[s]}. \quad (4.20) \]

While the NLIE functional equations (4.13-4.16) are written in terms of gauge invariant variables this is apparently not the case for the half-plane functional equations (4.20). We can however reformulate them such that they contain explicitly gauge invariant combinations only. To find such a form, we first have to express the T-functions \(T_{a,s}, a = 1, \ldots, k-1\) in terms of the gauge invariant Y-functions and the boundary (factorized) variables \(T_{0,s}\)
and $T_{k,s}$. We start by writing the logarithmic derivative of the Y-T relations (3.13) as

$$dl T_{a,s}^+ + dl T_{a,s}^- - \sum_{b=1}^{k-1} \Delta_{ab} dl T_{b,s} = dl Y_{a,s} + \delta_{a1} dl T_{0,s} + \delta_{ak-1} dl T_{k,s},$$

(4.21)

where

$$\Delta_{ab} = \delta_{a,b+1} + \delta_{a,b-1}, \quad a,b = 1, \ldots, k-1$$

(4.22)

and we introduce the notation

$$dl F(u) = \frac{d}{du} \ln F(u) = \frac{F'(u)}{F(u)}$$

(4.23)

for any function $F(u)$.

Using Fourier transformation techniques, we can calculate the inverse of the linear operator appearing on the LHS of the equation (4.21). We will denote this inverse operator by $M_{ab}$ and write the solution as

$$dl T_{a,s} = \sum_{b=1}^{k-1} M_{ab} \star dl Y_{b,s} + M_{a1} \star dl T_{0,s} + M_{a,k-1} \star dl T_{k,s}, \quad a = 1, \ldots, k-1.$$  

(4.24)

(4.24) in this form is valid only up to source terms, i.e. the contribution of pointlike singularities within the analyticity (meromorphicity) strips. Using this result, we can write (again up to source terms)

$$dl \beta_{1,s} = \sum_{b=1}^{k-1} M_{1b} \star dl \frac{Y_{b,s}^+}{Y_{b,s+1}} - M_{11} \star dl B^{-s} - M_{1,k-1} \star dl D^{-s} + dl b^{-1-s},$$

$$dl \delta_{k-1,s} = \sum_{b=1}^{k-1} M_{k-1,b} \star dl \frac{Y_{b,s}^-}{Y_{b,s+1}} - M_{k-1,1} \star dl A^s - M_{k-1,k-1} \star dl C^s + dl c^s.$$  

(4.25)

Using this result, we now write the “half-plane” equations for all ($r = 1, 2, 3$) levels of our SU(4) problem:

$$dl \beta_{1,s}^{(r)} = \sum_{b=1}^r M_{1b}^{(r)} \star dl \frac{Y_{b,s}^{(r)+}}{Y_{b,s+1}^{(r)}} + \ldots,$$

$$dl \delta_{r,s}^{(r)} = \sum_{b=1}^r M_{rb}^{(r)} \star dl \frac{Y_{b,s}^{(r)-}}{Y_{b,s+1}^{(r)}} + \ldots,$$

(4.26)

where $M_{ab}^{(r)}$, $a,b = 1, \ldots, r$ is the $r \times r$ matrix kernel at Bäcklund level $r$ and the dots indicate that the equations are valid up to source terms and also terms vanishing (after Fourier transformation), similarly to what is explained later before the equations (6.21) and (6.22), for negative and positive frequencies, for the upper and lower equations, respectively.

---

4It will be explicitly given in section 7.
5. Asymptotic solution of the Bäcklund hierarchy

In this section we calculate the asymptotic solution of the whole Bäcklund hierarchy and of the corresponding NLIE variables and lower level Y-functions. All asymptotic functions are indicated by an upper index \(^0\). The asymptotic hierarchy of Bäcklund T-functions is given by the Bethe Ansatz and we start with recalling the Bethe Ansatz solution for the SU(2|2) fat-hook [68].

5.1 Bethe Ansatz solution of the SU(2|2) fat-hook

To construct the hierarchy of T-functions relevant for the upper part of our AdS/CFT T-system we will use the solution of the analogous problem for the SU(2|2) fat-hook. We will denote this set of T-functions by \(T^{(k,m)}(a,s,u)\), \(k = 0,1,2\), \(m = 0,1,2\). (5.1)

(For a general SU\((K|\)M\) fat-hook one has \(k = 0,\ldots K\), \(m = 0,\ldots ,M\).) The solution will be given [68] in terms of 9 Q-functions

\[Q^{(k,m)}(u), \quad k,m = 0,1,2\] (5.2)

listed in appendix [B]. These are not all independent, they satisfy a number of quadratic QQ-relations [68]. The meaning of these Q-functions is that in terms of these the boundary values of the T-functions are given as

\[T^{(k,m)}(0,s,u) = Q^{(k,m)[-s]}(u), \quad -\infty < s < \infty,\] (5.3)

\[T^{(k,m)}(a,0,u) = Q^{(k,m)[a]}(u), \quad 0 \leq a < \infty,\] (5.4)

\[T^{(k,m)}(k,s,u) = Q^{(k,0)[s+k]}(u)Q^{(0,m)[-s-k]}(u), \quad m \leq s < \infty,\] (5.5)

\[T^{(k,m)}(a,m,u) = (-1)^{m(a-k)}Q^{(k,0)[a+m]}(u)Q^{(0,m)[-a-m]}(u), \quad k \leq a < \infty.\] (5.6)

We also introduce the simplified notation

\[T^{(2,2)}(a,s,u) = T(a,s,u), \quad T^{(2,1)}(a,s,u) = F(a,s,u)\] (5.7)

for the most important members of the hierarchy.

The Bäcklund transformation we need is similar to (but not identical with) the transformations of section 3:

\[F(a,s+1,u)T(a-1,s,u) - F(a-1,s,u)T(a,s+1,u) + F^{+}(a,s,u)T^{-}(a-1,s+1,u) = 0,\] (5.8)

\[-F^{-}(a,s,u)T(a-1,s,u) + F(a-1,s,u)T^{-}(a,s,u) + F(a,s-1,u)T^{-}(a-1,s+1,u) = 0.\] (5.9)

Before this Bethe Ansatz solution can be used for our purposes, we have to go to the (1,1) gauge, where the T-functions are equal to unity along the left and lower boundaries of the fat-hook. Denoting the (1,1) gauge T-functions by \(\hat{T}\), the relation between this solution
and the asymptotic limit of our T-functions (both in the original conventions and in the
principal chiral model conventions) is given as

$$\hat{T}(a, s, u) = \frac{T(a, s, u)}{Q^{(2,2)|a-s|(u)}} = t_{a,s}^o(u).$$  \hfill (5.10)

In particular

$$\hat{T}(0, s, u) = t_{0,s}^o(u) = 1,$$  \hfill (5.11)
$$\hat{T}(a, 0, u) = t_{a,0}^o(u) = T_{2,a}^o(u) = 1,$$  \hfill (5.12)
$$\hat{T}(a, 1, u) = t_{a,1}^o(u) = T_{1,a}^o(u) = t_a(u),$$  \hfill (5.13)
$$\hat{T}(a, 2, u) = t_{a,2}^o(u) = T_{0,a}^o(u) = A^{[a]}(u)B^{[a]}(u).$$  \hfill (5.14)

Here

$$A^o = \frac{Q^{(2,0)++}}{Q^{(2,2)--}}, \quad B^o = \frac{Q^{(0,2)--}}{Q^{(2,2)--}}.$$  \hfill (5.15)

Introducing the function

$$E(u) = e^{\frac{2u\pi}{2}}, \quad E^+ = iE, \quad E^{[2\sigma]} = (-1)\sigma E$$  \hfill (5.16)

and defining

$$\beta^o = E^- Q^{(0,1)}, \quad \gamma^o = E \frac{Q^{(2,1)+}}{Q^{(2,2)--}}$$  \hfill (5.17)

the Bäcklund transformation (5.8) with $s = 0$ can be rewritten as

$$\gamma^{[a-1]} t_a^+ - \gamma^{[a+1]} t_{a-1} = A^{[a]} \beta^{[a]}$$  \hfill (5.18)

and the Bäcklund transformation (5.9) with $s = 1$ as

$$\beta^{[1-a]} t_a - \beta^{[1-a]} t_{a-1} = \gamma^{[a]} B^{[1-a]}.$$  \hfill (5.19)

### 5.2 Asymptotic solution of the AdS/CFT T-system

In this subsection we recall the asymptotic solution of the AdS/CFT Y-system and T-
system, which is given by two (left and right) copies of the SU(2|2) fat-hook. We will use
the notations of [65]. Let us denote these two copies by

$$t^X_{a,s}, \quad s = 0, 1, 2, \quad X = L, R.$$  \hfill (5.20)

Further definitions and relations are:

$$t^X_{0,s} = t^X_{a,0} = 1, \quad t^X_{a,1} = r_a, \quad t^X_{a,1} = \ell_a,$$  \hfill (5.21)
$$t^X_{a,2} = A^X a B^X {[a]}$$  \hfill (5.22)

and we also define $\beta^X, \gamma^X$ for $X = L, R$ using (5.17).

The asymptotic solution for the massive nodes on the AdS/CFT Y-system is given by

$$y^o_{a,0} = \eta_a t^R_{a,1} t^L_{a,1} = \eta_a r_a \ell_a.$$  \hfill (5.23)
where the prefactor $\eta_a$ is given in (B.25), (B.26). The prefactor satisfies the discrete Laplace equation
\[
\eta^+_a \eta^-_a = \eta_{a+1} \eta_{a-1} \quad (\eta_0 = 1)
\]  
and actually it can be written in the form
\[
\eta_a = \frac{\psi[a]}{\psi[-a]},
\]  
where $\psi$ has cuts along the real axis and is exponentially small, $O(\varepsilon)$, in the upper half plane and is exponentially large, $O(1/\varepsilon)$, in the lower half plane. Thus the ratio (5.25) is $O(\varepsilon^2)$ for $\frac{-a}{g} < \text{Im}u < \frac{a}{g}$.

The asymptotic solution of the complete AdS/CFT T-system is now constructed as follows. The right half of the upper part is
\[
t^o_{a,2} = t^R_{a,2} = A^a_R B^{-a}_R,
\]
\[
t^o_{a,1} = t^R_{a,1} = r_a,
\]
\[
t^o_{a,0} = 1
\]
and using the relation
\[
\frac{t^o_{a,-1} t^o_{a,1}}{t^o_{a+1,0} t^o_{a-1,0}} = y^o_{a,0} = \eta_a t^R_{a,1} t^L_{a,1}
\]
we obtain
\[
t^o_{a,-1} = \eta_a t^L_{a,1} = \eta_a \ell_a
\]
and similarly we get
\[
t^o_{a,-2} = \eta^+_a \eta^-_a t^L_{a,2} = \eta^+_a \eta^-_a A^a_L B^{-a}_L.
\]
Comparing this to (3.5) we find
\[
A^o = A_R, \quad B^o = B_R, \quad C^o = \psi^+ \psi^- A_L, \quad D^o = \frac{B_L}{\psi^+ \psi^-}
\]
and the asymptotic limit of the ratios (3.6-3.9) are given by
\[
A^o = \frac{A^{o+}}{A^{o-}} = \frac{A^+_R}{A^-_R} = A_R, \quad (5.33)
\]
\[
B^o = \frac{B^{o+}}{B^{o-}} = \frac{B^+_R}{B^-_R} = B_R, \quad (5.34)
\]
\[
C^o = \frac{C^{o+}}{C^{o-}} = \frac{\psi^{++} A^+_L}{\psi^{--} A^-_L} = \eta_2 A_L, \quad (5.35)
\]
\[
D^o = \frac{D^{o+}}{D^{o-}} = \frac{\psi^{++} B^+_L}{\psi^{--} B^-_L} = \eta_2 B_L. \quad (5.36)
\]

All four functions $A^o, B^o, C^o, D^o$ have cuts at $\pm 2i/g$. For completeness we record that in the principal chiral model conventions
\[
T^o_{0,s} = t^R_{s,2}, \quad T^o_{1,s} = t^R_{s,1}, \quad T^o_{2,s} = 1, \quad T^o_{3,s} = \eta_s t^L_{s,1}, \quad T^o_{4,s} = \eta^+_s \eta^-_s t^L_{s,2}. \quad (5.37)
\]
We now discuss a special case of the gauge transformations defined by (3.24-3.25). Let us choose
\[ f_1 = \frac{1}{f_2}, \quad f_3 = f_2^{-1}, \quad f_4 = \frac{1}{f_3}, \quad (5.38) \]
where \( f_2 \) is the solution of
\[ \frac{f_2}{f_2^+} = \psi^{++}. \quad (5.39) \]
This means that
\[ \lambda_{a,s} = \frac{f_2^{a-s-4} f_2^{s-a}}{f_2^{s-a} f_2^{s+a-4}} \quad (5.40) \]
and in particular
\[ \lambda_{0,s} = \frac{\eta_s^+ \eta_s^-}, \quad \lambda_{1,s} = \eta_s, \quad \lambda_{2,s} = 1, \quad \lambda_{3,s} = \frac{1}{\eta_s}, \quad \lambda_{4,s} = \frac{1}{\eta_s^+ \eta_s^-}. \quad (5.41) \]

After this gauge transformation we have
\[ \tilde{T}_{0,s} = \eta_s^+ \eta_s^- t_{s,2}, \quad \tilde{T}_{1,s} = \eta_s t_{s,1}, \quad \tilde{T}_{2,s} = 1, \quad \tilde{T}_{3,s} = t_{s,1}, \quad \tilde{T}_{4,s} = t_{s,2}. \quad (5.42) \]

This means that before the gauge transformation the components of \( T_{a,s} \) behave as
\[ T_{a,s}: \mathcal{O}(1,1,1,\varepsilon^2,\varepsilon^4) \quad \text{for } a = 0, 1, 2, 3, 4, \text{ respectively}, \]
and after the gauge transformation we have
\[ \tilde{T}_{a,s}: \mathcal{O}(\varepsilon^4,\varepsilon^2,1,1,1) \quad \text{for } a = 0, 1, 2, 3, 4, \text{ respectively}. \]

Note that we have chosen the gauge transformation \((5.41)\) such that in the new gauge the asymptotic solution is the mirror image (under the exchange of left and right) of the original. This implies (among other things) the above mirror symmetric \(\mathcal{O}(\varepsilon)\) behaviour of the asymptotic solution in the two gauges.

Here are the gauge functions corresponding to the lower Bäcklund levels:
\[ \omega_{0,s} = \frac{\psi^{[s+1]} \psi^{[s-1]}}{\psi^{[1-s]}}, \quad \omega_{1,s} = \psi^{[s]}, \quad \omega_{2,s} = \psi^{[1-s]}, \quad \omega_{3,s} = \frac{\psi^{[2-s]} \psi^{[-s]}}{\psi^{[s]}}, \quad (5.43) \]
\[ \nu_{0,s} = \psi^{[s+1]} \psi^{[s-1]}, \quad \nu_{1,s} = \psi^{[2-s]} \psi^{[s]}, \quad \nu_{2,s} = \psi^{[3-s]} \psi^{[1-s]}, \quad (5.44) \]
\[ \rho_{0,s} = \psi^{[3-s]} \psi^{[s+1]} \psi^{[s-1]}, \quad \rho_{1,s} = \psi^{[s]} \psi^{[4-s]} \psi^{[2-s]} . \quad (5.45) \]

The gauge functions \(\omega_{a,s}, \nu_{a,s}, \rho_{a,s}\) correspond to the gauge transformation of the lower level T-functions \(T_{a,s}^{(r)}\) for \(r = 3, 2, 1\), respectively.

So far we have constructed the asymptotic solution of the T-functions \(T_{a,s} = T_{a,s}^{(4)}\) at the highest level of the Bäcklund hierarchy. We now proceed to the T-functions at lower levels. We start with \(F_{a,s} = T_{a,s}^{(3)}\). We assume the pattern
\[ F_{a,s}: \mathcal{O}(1,1,\varepsilon,\varepsilon^3) \quad \text{for } a = 0, 1, 2, 3, \text{ respectively}, \]
which implies, using the formulae \((5.43)\)
\[ \tilde{F}_{a,s}: \mathcal{O}(\varepsilon^3,\varepsilon,1,1) \quad \text{for } a = 0, 1, 2, 3, \text{ respectively}. \]
The explanation of this behaviour is as follows. It is easy to see that the first two \(a = 0, 1\) Bäcklund equations are naturally solved by \(\mathcal{O}(1)\) F-functions since the T-functions...
occurring in them are also $O(1)$. Similarly after gauge transformation the last two ($a = 2, 3$) $F$-components are naturally $O(1)$. Together with (5.43) they already fix the above pattern. According to this pattern the Bäcklund equations take the following asymptotic form:

$$
T^o_{1,s+1}F^o_{0,s} = T^o_{1,s}F^o_{0,s+1} + T^o_{0,s}F^o_{1,s+1};
$$

(5.46)

$$
T^o_{2,s+1}F^o_{1,s} = T^o_{2,s}F^o_{1,s+1};
$$

(5.47)

$$
\tilde{T}^o_{3,s+1}F^o_{2,s} = \tilde{T}^o_{3,s}F^o_{2,s+1} + \tilde{T}^o_{2,s}F^o_{3,s+1};
$$

(5.48)

$$
\tilde{T}^o_{4,s+1}F^o_{3,s} = \tilde{T}^o_{4,s}F^o_{3,s+1}
$$

(5.49)

and

$$
T^{o+}_{0,s+1}F^o_{0,s} = T^{o+}_{0,s}F^o_{0,s+1};
$$

(5.50)

$$
T^{o+}_{1,s+1}F^o_{1,s} = T^{o+}_{1,s}F^o_{1,s+1} + T^{o+}_{2,s}F^o_{0,s+1};
$$

(5.51)

$$
\tilde{T}^{o+}_{2,s+1}F^o_{2,s} = \tilde{T}^{o+}_{2,s}F^o_{2,s+1};
$$

(5.52)

$$
\tilde{T}^{o+}_{3,s+1}F^o_{3,s} = \tilde{T}^{o+}_{3,s}F^o_{3,s+1} + \tilde{T}^{o+}_{4,s}F^o_{2,s+1};
$$

(5.53)

Here some $O(\varepsilon)$ terms were omitted from (5.47) and (5.52) and then all equations are written in terms of $O(1)$ variables. Using the asymptotic solution (5.47) and (5.42) and also the identities (5.18) and (5.19) we find that the solution of (5.46-5.49) and (5.50-5.53) is given by

$$
F^o_{0,s} = A^R_{[s]} \beta_R^{-[s]}, \quad F^o_{1,s} = \gamma_R^{[s]}, \quad \tilde{F}^o_{2,s} = \beta_L^{-[s]}, \quad \tilde{F}^o_{3,s} = B^L_{[1-s]}\gamma_L^{[s]}.
$$

(5.54)

Next we solve the asymptotic equations for $G^o_{a,s} = T^{(2\alpha)}_{a,s}$. We assume that

$G^o_{a,s} : O(1, \varepsilon, \varepsilon^2)$ and $\tilde{G}^o_{a,s} : O(\varepsilon^2, \varepsilon, 1)$ for $a = 0, 1, 2$, respectively.

Using this pattern (and the previous one for $F^o_{a,s}$) we have

$$
F^o_{1,s+1}G^o_{0,s} = F^{o+}_{1,s}G^o_{0,s+1};
$$

(5.55)

$$
\tilde{F}^o_{2,s+1}\tilde{G}^o_{1,s} = \tilde{F}^{o+}_{2,s}\tilde{G}^o_{1,s+1} + \tilde{F}^o_{1,s}G^o_{2,s+1};
$$

(5.56)

$$
\tilde{F}^o_{3,s+1}\tilde{G}^o_{2,s} = \tilde{F}^{o+}_{3,s}\tilde{G}^o_{2,s+1}
$$

(5.57)

and

$$
F^{o+}_{0,s+1}G^o_{0,s} = F^o_{0,s}G^o_{0,s+1};
$$

(5.58)

$$
F^{o+}_{1,s+1}G^o_{1,s} = F^o_{1,s}G^o_{1,s+1} + F^{o+}_{2,s}G^o_{0,s+1};
$$

(5.59)

$$
\tilde{F}^{o+}_{2,s+1}\tilde{G}^o_{2,s} = \tilde{F}^o_{2,s}\tilde{G}^o_{2,s+1}.
$$

(5.60)

The solution of (5.55-5.57) and (5.58-5.60) is

$$
G^o_{0,s} = A^R_{[s]}, \quad G^o_{1,s} = \frac{1}{\gamma_1^{[1-s]}\beta_1^L} (w^{[s]} + y^{[2-s]}), \quad \tilde{G}^o_{2,s} = B^L_{[2-s]},
$$

(5.61)

where the functions $w$ and $y$ are the solutions of

$$
w^- - w^+ = \frac{A_R}{\gamma_R + \gamma_R}, \quad y^+ - y^- = \frac{B_L}{\beta_L \beta_L}.
$$

(5.62)
We note that (5.61) is actually an exact solution of the SU(2) T-system for $T^{(2)_o}_{a,s} = C^{a}_{o,s}$. Completed with

$$
T^{(1)_o}_{0,s} = \frac{A^{[s]}_{R} \beta^{[2-s]}_{L}}{\psi^{[3-s]}_{o}}, \quad T^{(1)_o}_{1,s} = -\frac{B^{[3-s]}_{L} \gamma_{R}^{[s]}}{\psi^{[2-s]}_{o} \psi^{[4-s]}_{R}} \quad (5.63)
$$

they form an exact solution of the SU(2) Bäcklund system. The lowest level solution shows the following pattern.

$T^{(1)_o}_{a,s}: O(\varepsilon,\varepsilon^2)$ and $\tilde{T}^{(1)_o}_{a,s}: O(\varepsilon^2, \varepsilon)$ for $a = 0, 1$, respectively.

### 5.3 Asymptotic solution for Y-functions and NLIE variables

For completeness we here summarize the asymptotic solution for the gauge invariant objects: Y-functions and NLIE variables for all Bäcklund levels.

$$
y_{1,s}^{(3)_o} = \frac{r^{s+1} r^{s-1}}{A^{[s]}_{R} B^{[2-s]}_{R}}, \quad y_{1,s}^{(3)_o} = \frac{r^{s+1} r^{s-1}}{A^{[s]}_{R} B^{[2-s]}_{R}}, \quad (5.64)
$$

$$
y_{2,s}^{(3)_o} = Y_{2,s}^{(3)_o} = \frac{1}{\eta s r^{s} \ell^{s}}, \quad (5.65)
$$

$$
y_{3,s}^{(3)_o} = \frac{\ell_{s+1} \ell_{s-1}}{A^{[s]}_{L} B^{[2-s]}_{L}}, \quad Y_{3,s}^{(3)_o} = \frac{\ell_{s+1} \ell_{s-1}}{A^{[s]}_{L} B^{[2-s]}_{L}}, \quad (5.66)
$$

$$
y_{1,s}^{(2)_o} = Y_{1,s}^{(2)_o} = \left( \frac{\psi^{s+1}}{\beta^{[s]}_{R} \ell^{[s]}_{s}} \right) \left( \frac{1}{w^{[s-1]} - w^{[s+1]}} \right), \quad (5.67)
$$

$$
y_{2,s}^{(2)_o} = Y_{2,s}^{(2)_o} = \frac{1}{(\psi \gamma^{[s]}_{R} \gamma^{[s]}_{R} \gamma^{[s]}_{R})^{(s) - y^{[2-s]} - y^{[-s]}}}, \quad (5.68)
$$

$$
y_{1,s}^{(1)_o} = \frac{w^{[s+1]} + y^{[1-s]}}{w^{[s+1]} - w^{[s+1]}} \left( \frac{w^{[s-1]} + y^{[3-s]}}{y^{[3-s]} - y^{[1-s]}} \right), \quad (5.69)
$$

$$
y_{1,s}^{(1)_o} = \frac{w^{[s+1]} + y^{[3-s]}}{w^{[s+1]} - w^{[s+1]}} \left( \frac{w^{[s-1]} + y^{[1-s]}}{y^{[3-s]} - y^{[1-s]}} \right), \quad (5.70)
$$

$$
y_{1,s}^{(3)_o} = r^{s+1} \frac{\beta^{[-2-s]}_{R}}{B^{[-2-s]}_{R} \gamma^{[s]}_{R}}, \quad B_{1,s}^{(3)_o} = r^{s+1} \frac{\beta^{[-2-s]}_{R}}{B^{[-2-s]}_{R} \gamma^{[s]}_{R}}, \quad (5.71)
$$

$$
b_{2,s}^{(3)_o} = B_{2,s}^{(3)_o} = \psi \frac{\gamma^{[s]}_{R}}{\ell^{[s]}_{s+1}}, \quad (5.72)
$$

$$
y_{3,s}^{(3)_o} = \frac{\ell_{s+1}}{A^{[s]}_{L} B^{[2-s]}_{L}}, \quad D_{3,s}^{(3)_o} = \frac{\ell_{s+1}}{A^{[s]}_{L} B^{[2-s]}_{L}}, \quad (5.73)
$$

$$
d_{1,s}^{(3)_o} = r^{s+1} \frac{\gamma^{[s+1]}_{s}}{A^{[s]}_{R} \beta^{[2-s]}_{R}}, \quad D_{1,s}^{(3)_o} = r^{s+1} \frac{\gamma^{[s+1]}_{s}}{A^{[s]}_{R} \beta^{[2-s]}_{R}}, \quad (5.74)
$$

$$
d_{2,s}^{(3)_o} = D_{2,s}^{(3)_o} = \frac{1}{\psi^{[s]}_{s} \ell^{[s]}_{s+1}}, \quad (5.75)
$$

$$
d_{3,s}^{(3)_o} = \frac{\ell^{s+1}}{A^{[s]}_{L} \beta^{[2-s]}_{L}}, \quad D_{3,s}^{(3)_o} = \frac{\ell^{s+1}}{A^{[s]}_{L} \beta^{[2-s]}_{L}}. \quad (5.76)
$$
A sub-subsection for the ground state of the $\gamma$ case of a general (excited) state in the original, non-deformed model and later in a separate (either $R$ or $Q$) we will specify our building blocks in terms of the Bethe Ansatz solution of [15] (see appendix B and [65]):

determined from the relations

$$t R Q$$

The building blocks, in terms of which the asymptotic solutions are expressed are $w$, $w^o$, $A^o$, $B^o$, $\beta^o$ and $\gamma^o$. $w^o$ and $y^o$ can then be determined from the relations

$$w^o- - w^o+ = \frac{A^o}{\gamma^o+\gamma^o-}, \quad y^o+ - y^o- = \frac{B^o}{\beta^o\beta^o-}. \quad (5.83)$$

In terms of the 9 Q-functions $Q^{(k,m)}$, $k, m = 0, 1, 2$, the building blocks are expressed as

$$A^o = \frac{Q^{(2,0)++}}{Q^{(2,2)}-}, \quad B^o = \frac{Q^{(0,2)-}}{Q^{(0,1)}}, \quad \beta^o = E^{-}Q^{(0,1)}, \quad \gamma^o = E \frac{Q^{(2,1)+}}{Q^{(2,2)}-}, \quad (5.84)$$

where $E$ was defined in (5.16). It is also useful to know the building blocks as expressed in terms of the Bethe Ansatz solution of $[\overline{13}]$ (see appendix B and [33]):

$$A^o = -F^{(0)} \frac{R_m}{R_\ell} \frac{1}{Q_1^{-}Q_3^{-}}, \quad \beta^o = E^{-}Q_3^{+}, \quad (5.85)$$

$$B^o = -G^{(0)} \frac{R_m}{R_\ell} Q^{++} \frac{Q_3^{+}}{Q_1^{-}}, \quad \gamma^o = E \frac{Q_1}{Q^{-}} R_m. \quad (5.86)$$

The expression of the T-system elements $t_s$ in terms of the same Bethe Ansatz functions is also given in appendix B.

The combination necessary to calculate the function $w$ becomes

$$\frac{A^o}{\gamma^o+\gamma^o-} = \frac{Q}{E^2} \left\{ \frac{Q_2^{2+}}{Q_2(Q_1Q_3)^+} + \frac{Q_2^{-}}{Q_2(Q_1Q_3)^-} - \frac{R_m}{R_\ell(Q_1Q_3)^-} - \frac{R_\ell^+}{R_m(Q_1Q_3)^+} \right\}, \quad (5.87)$$

5.4 Identification of $Q^{(k,m)}$

We will specify our building blocks $Q^{(k,m)}$ completely in two cases. We first discuss the case of a general (excited) state in the original, non-deformed model and later in a separate sub-subsection for the ground state of the $\gamma$-deformed model.

The building blocks, in terms of which the asymptotic solutions are expressed are $r_s$, $A_R$, $B_R$, $\beta_R$, $\gamma_R$, $w$, $t_s$, $A_L$, $B_L$, $\beta_L$, $\gamma_L$ and $y$. For each of the Bethe Ansatz solutions (either $R$ or $L$) we thus need to know $t_s$, $A^o$, $B^o$, $\beta^o$ and $\gamma^o$. $w^o$ and $y^o$ can then be determined from the relations

$$w^o- - w^o+ = \frac{A^o}{\gamma^o+\gamma^o-}, \quad y^o+ - y^o- = \frac{B^o}{\beta^o\beta^o-}. \quad (5.83)$$

In terms of the 9 Q-functions $Q^{(k,m)}$, $k, m = 0, 1, 2$, the building blocks are expressed as

$$A^o = \frac{Q^{(2,0)++}}{Q^{(2,2)}-}, \quad B^o = \frac{Q^{(0,2)-}}{Q^{(0,1)}}, \quad \beta^o = E^{-}Q^{(0,1)}, \quad \gamma^o = E \frac{Q^{(2,1)+}}{Q^{(2,2)}-}, \quad (5.84)$$

where $E$ was defined in (5.16). It is also useful to know the building blocks as expressed in terms of the Bethe Ansatz solution of $[\overline{13}]$ (see appendix B and [33]):

$$A^o = -F^{(0)} \frac{R_m}{R_\ell} \frac{1}{Q_1^{-}Q_3^{-}}, \quad \beta^o = E^{-}Q_3^{+}, \quad (5.85)$$

$$B^o = -G^{(0)} \frac{R_m}{R_\ell} Q^{++} \frac{Q_3^{+}}{Q_1^{-}}, \quad \gamma^o = E \frac{Q_1}{Q^{-}} R_m. \quad (5.86)$$

The expression of the T-system elements $t_s$ in terms of the same Bethe Ansatz functions is also given in appendix B.

The combination necessary to calculate the function $w$ becomes

$$\frac{A^o}{\gamma^o+\gamma^o-} = \frac{Q}{E^2} \left\{ \frac{Q_2^{2+}}{Q_2(Q_1Q_3)^+} + \frac{Q_2^{-}}{Q_2(Q_1Q_3)^-} - \frac{R_m}{R_\ell(Q_1Q_3)^-} - \frac{R_\ell^+}{R_m(Q_1Q_3)^+} \right\}, \quad (5.87)$$
which, for the case of an even number of particles in the \( \mathfrak{sl}(2) \) sector, simplifies to

\[
\frac{A^o}{\gamma^o+\gamma^o} = -\frac{1}{E^2} \left\{ 2Q - R_m B^- - R_p ^+ B_m^+ \right\}. \tag{5.88}
\]

Similarly we have

\[
\frac{B^o}{\beta^o \beta^o} = -\frac{Q}{E^2} \left\{ \frac{Q_{2+}^+}{Q_{2}(Q_{1}Q_{3})^{+}} + \frac{Q_{2}^-}{Q_{2}(Q_{1}Q_{3})^{-}} - \frac{B_m}{B_p ^-(Q_{1}Q_{3})^-} - \frac{B_p ^+}{B_m ^+(Q_{1}Q_{3})^+} \right\}, \tag{5.89}
\]

and for the case of an even number of particles in the \( \mathfrak{sl}(2) \) sector

\[
\frac{B^o}{\beta^o \beta^o} = -\frac{1}{E^2} \left\{ 2Q - R_p ^- B^- - R_m ^+ B_m ^+ \right\}. \tag{5.90}
\]

Using (5.88) and (5.90) for the case of the Konishi operator corresponding to two particles with rapidities \( u_1 = -u_2 = \omega \) and

\[
x_1^+ = x_s \left( \omega + \frac{i}{g} \right) = \xi, \quad x_2^+ = -\xi^*, \quad x_1^- = \xi^*, \quad x_2^- = -\xi \tag{5.91}
\]

we can explicitly solve (5.88):

\[
w^o(u) = \frac{1}{E^2} \left\{ (h - h^*) \left( x - \frac{1}{x} \right) + iguH + w_c \right\} = w(u), \tag{5.92}
\]

\[
y^o(u) = \frac{1}{E^2} \left\{ (h - h^*) \left( x - \frac{1}{x} \right) - iguH + y_c \right\} = y(u), \tag{5.93}
\]

where

\[
h = \xi - \frac{1}{\xi}, \quad H = 4 + hh^* + \frac{1}{g^2} - \omega^2 \tag{5.94}
\]

and \( w_c, y_c \) are “integration” constants. (Only their sum enters the asymptotic expressions.)

### 5.4.1 \( \gamma \)-deformed constant solution

We also give here the identification of the Q-functions in the case of the constant (vacuum) solution for the \( \gamma \)-deformed model. Following ref. [27], the right-wing part of the problem is characterized by the constant \( q \) and the analogous left-wing part by the constant \( \dot{q} \). To distinguish it from the asymptotic solution discussed above (valid for arbitrary excited state in the undeformed model) the Q-functions for the deformed vacuum will be denoted by lower case \( q^{(k,m)} \). For the right-wing case we have

\[
q^{(2,2)} = 1, \quad q^{(2,1)} = 1 - q, \quad q^{(2,0)} = (1 - q)^2,
\]

\[
q^{(1,2)} = S_o, \quad q^{(1,1)} = S_o, \quad q^{(1,0)} = S_o,
\]

\[
q^{(0,2)} = \left( 1 - \frac{1}{q} \right)^2, \quad q^{(0,1)} = \left( 1 - \frac{1}{q} \right), \quad q^{(0,0)} = 1,
\]

where \( S_o \) is the solution of

\[
S_o^+ = qS_o^- \tag{5.96}
\]
The solution (5.95) can be obtained from the results of ref. [24]. In this case the building blocks are

\[ t_s = (-1)^s + 1 \frac{(1 - q)^2}{q} \]  \hspace{1cm} (5.97)

and

\[ A^o = (1 - q)^2, \quad \beta^o = E^o \left(1 - \frac{1}{q}\right), \quad \gamma^o = E(1 - q). \]  \hspace{1cm} (5.98)

It is easy to solve (5.83):

\[ w^o(u) = -\frac{iq}{2E^2}(u + w_c), \quad y^o(u) = \frac{iq}{2E^2}(u - y_c). \]  \hspace{1cm} (5.99)

The asymptotic solution of the gauge invariant functions is as follows.

\[ y_{1,s}^{(3)o} = y_{3,s}^{(3)o} = s^2 - 1, \quad Y_{1,s}^{(3)o} = Y_{3,s}^{(3)o} = s^2, \]  \hspace{1cm} (5.100)

\[ y_{2,s}^{(3)o} = Y_{2,s}^{(3)o} = \frac{1}{\eta_s} \frac{q\dot{\eta}}{s^2 (1 - q)^2 (1 - \dot{q})^2}, \]  \hspace{1cm} (5.101)

\[ y_{1,s}^{(2)o} = Y_{1,s}^{(2)o} = -\psi^{[1-s]} \frac{q\dot{\psi}}{(1 - q)(1 - \dot{q})}, \]  \hspace{1cm} (5.102)

\[ y_{2,s}^{(2)o} = Y_{2,s}^{(2)o} = -\frac{1}{\psi^{[s]}} \frac{1}{(1 - q)(1 - \dot{q})}, \]  \hspace{1cm} (5.103)

\[ y_{1,s}^{(1)o} = (s - \Delta)(s - 2 - \Delta), \quad Y_{1,s}^{(1)o} = (s - 1 - \Delta)^2, \]  \hspace{1cm} (5.104)

where the constant \( \Delta \) is a multiple of the sum of the arbitrary constants \( w_c \) and \( y_c \).

\[ b_{1,s}^{(3)o} = b_{3,s}^{(3)o} = d_{1,s}^{(3)o} = d_{3,s}^{(3)o} = s, \]  \hspace{1cm} (5.105)

\[ B_{1,s}^{(3)o} = B_{3,s}^{(3)o} = D_{1,s}^{(3)o} = D_{3,s}^{(3)o} = s + 1, \]  \hspace{1cm} (5.106)

\[ b_{2,s}^{(3)o} = B_{2,s}^{(3)o} = -\psi^{[-s]} \frac{q\dot{\psi}}{s(1 - q)(1 - \dot{q})}, \]  \hspace{1cm} (5.107)

\[ d_{2,s}^{(3)o} = D_{2,s}^{(3)o} = -\frac{1}{\psi^{[s]}} \frac{1}{s(1 - q)(1 - \dot{q})}, \]  \hspace{1cm} (5.108)

\[ b_{1,s}^{(2)o} = B_{1,s}^{(2)o} = \psi^{[1-s]} \frac{q\dot{\psi}}{s - \Delta \left(1 - q\right)(1 - \dot{q})}, \]  \hspace{1cm} (5.109)

\[ b_{2,s}^{(2)o} = d_{1,s}^{(2)o} = \Delta - s, \quad B_{2,s}^{(2)o} = D_{1,s}^{(2)o} = \Delta + 1 - s, \]  \hspace{1cm} (5.110)

\[ d_{2,s}^{(2)o} = D_{2,s}^{(2)o} = \frac{1}{\psi^{[s]}} \frac{1}{s - \Delta \left(1 - q\right)(1 - \dot{q})}, \]  \hspace{1cm} (5.111)

\[ b_{1,s}^{(1)o} = d_{1,s}^{(1)o} = s - 1 - \Delta, \quad B_{1,s}^{(1)o} = D_{1,s}^{(1)o} = s - \Delta. \]  \hspace{1cm} (5.112)

6. The horizontal SU(2) problem

In this section we study the problem of finding the NLIE description of the TBA system corresponding to the right-wing nodes. (The problem of the left-wing nodes is completely analogous.) Our discussion of the Bäcklund transformation, the construction of the NLIE
variables and the NLIE functional equations and the asymptotic solution of these variables will be very similar to, but considerably simpler than, what has been discussed in the preceding three sections. The construction here is based on [47], [57] and [66].

We introduce the notation

\[ Y_{m|w}^{(+)} = y_{1,m+1} = x_m, \quad X_m = 1 + x_m. \] (6.1)

These functions are of type \((-m,m)\) and satisfy the SU(2) Y-system equations

\[ x_m^+ x_m^- = X_{m+1} X_{m-1}, \quad m = 2, 3, \ldots. \] (6.2)

It is easy to construct the corresponding T-system as follows. It will turn out to be convenient to work in a gauge different from what has been used so far. In this gauge we start with the construction of the first two T-functions \(\tau_1, \tau_2\) from

\[ \tau_m^+ \tau_m^- = X_m, \quad m = 1, 2. \] (6.3)

The solution of this type of functional equations is the basic problem in the theory of TBA integral equations [73]. (See also the TBA lemmas of [65].) Next we define

\[ \tau_3 = \frac{x_2}{\tau_1}, \] (6.4)

satisfying

\[ \tau_3^+ \tau_3^- = \frac{x_3^+ x_3^-}{\tau_1^+ \tau_1^-} = \frac{X_1 X_3}{X_1} = X_3. \] (6.5)

Proceeding similarly we can construct \(\tau_m\) for all \(m = 1, 2, \ldots\) satisfying

\[ \tau_{m+1} \tau_{m-1} = x_m, \quad m = 2, 3, \ldots, \quad \tau_m^+ \tau_m^- = X_m, \quad m = 1, 2, \ldots \] (6.6)

and also the T-system equations of the form

\[ \tau_m^+ \tau_m^- = 1 + \tau_{m+1} \tau_{m-1}, \quad m = 2, 3, \ldots \] (6.7)

The \(\tau_m\) functions constructed this way are of type \((-1-m,m+1)\). The SU(2) T-system \(\text{(6.7)}\) is in the gauge

\[ t_{2,m} = t_{0,m} = 1, \quad t_{1,m} = \tau_m. \] (6.8)

Next we consider the SU(2) Bäcklund transformations. Specifying \(\text{(3.16)}\) for \(a = 0, 1\) we get the two equations

\[ \tau_{m+1} f_{0,m} = \tau_m^+ f_{0,m+1} + f_{1,m+1}, \quad f_{1,m} = f_{1,m+1}. \] (6.9)

The second one implies that there exists some function \(Q\) such that

\[ f_{1,m} = Q^{[m+1]}. \] (6.10)

Similarly from

\[ f_{0,m} = f_{0,m+1}, \quad \tau_{m+1} f_{1,m} = \tau_m f_{1,m+1}^+ + f_{0,m+1}, \] (6.11)
which are the $a = 0, 1$ components of (3.17), it follows that with some function $\tilde{Q}$

$$f_{0,m} = \tilde{Q}^{-m}.$$  \hspace{1cm} (6.12)

Rewriting (6.9) and (6.11) in terms of $Q$ and $\tilde{Q}$, we have

$$\tau_{m+1}Q^{-m} = \tau_{m}Q^{-m-2} + Q^{m+2},$$  \hspace{1cm} (6.13)

$$\tau_{m+1}^+Q^{m+1} = \tau_{m}Q^{m+3} + \tilde{Q}^{-m-1},$$  \hspace{1cm} (6.14)

which form a variant of Baxter’s famous TQ-relations. From these equations we find that $Q$ is of type $(0, 2m)$ and $\tilde{Q}$ is of type $(-2m - 2, 0)$. Since $m$ is arbitrary, $Q$ is free of cuts in the entire upper half plane and $\tilde{Q}$ is free of cuts in the lower half plane.

Having found the solution of the Bäcklund system we can construct the NLIE variables in analogy to (4.1) and (4.3):

$$b_m = \frac{\tau_m^- \tilde{Q}^{-m-2}}{Q^{m+2}}, \quad d_m = \frac{\tau_m^- Q^{m+2}}{Q^{-m-2}},$$

$$B_m = \frac{\tau_{m+1}^- \tilde{Q}^{-m}}{Q^{m+2}}, \quad D_m = \frac{\tau_{m+1}^+ Q^m}{Q^{-m-2}}.$$  \hspace{1cm} (6.15)

Baxter’s equations are equivalent to the relations

$$B_m = 1 + b_m, \quad D_m = 1 + d_m$$  \hspace{1cm} (6.16)

and the analogues of the NLIE functional equations (4.7), (4.10) are

$$b_m d_m = X_m, \quad B_m^{-1}D_m^+ = X_{m+1}.$$  \hspace{1cm} (6.17)

The NLIE functions $b_m$ and $d_m$ are of type $(-m - 2, m)$ and $(m, m + 2)$, respectively.

We have “half-plane” relations in this case as well. We write the ratios

$$\beta_m = \frac{b_m}{B_m} = \frac{\tau_m^-}{\tau_{m+1}^-} b_o^{-m-1}$$  \hspace{1cm} (6.18)

and

$$\delta_m = \frac{d_m}{D_m} = \frac{\tau_m^-}{\tau_{m+1}^-} c_o^m$$  \hspace{1cm} (6.19)

and since

$$b_o = \frac{\tilde{Q}^-}{Q^+} \quad \text{and} \quad c_o = \frac{Q^+}{Q}$$  \hspace{1cm} (6.20)

are of type $(-\infty, -1)$ and $(0, \infty)$, respectively, the second terms on the right hand side of the first and second equations

$$dl \frac{b_m}{B_m} = M^{(1)}_{11} \ast dl \frac{X_m^+}{X_{m+1}} + dl b_o^{-m-1},$$  \hspace{1cm} (6.21)

$$dl \frac{d_m}{D_m} = M^{(1)}_{11} \ast dl \frac{X_m^-}{X_{m+1}} + dl c_o^m$$  \hspace{1cm} (6.22)
do not contribute (after Fourier transformation) for negative and positive frequencies, respectively. \( M^{(1)}_{11} \) was defined in section 4 and its Fourier transform is given by \((7,8)\).

Next we discuss the asymptotic solution for the NLIE functions introduced above. We go back to the Bethe Ansatz solution discussed in the previous section and introduce the notation
\[
T^{(2,2)}(a, s, u) = t(a, s, u), \quad T^{(1,2)}(a, s, u) = f(a, s, u).
\]
In this section we will make use the set of Bäcklund transformations \([68]\) (similar to \((3.16-3.17)\))
\[
t(a + 1, s, u)f^+(a, s, u) - t^+(a, s, u)f(a + 1, s, u) = 0,
\]
\[
t^+(a + 1, s - 1, u)f(a, s + 1, u) = 0,
\]
\[
t^+(a + 1, s + 1, u)f(a, s, u) - t(a, s, u)f^+(a + 1, s + 1, u) = 0.
\]
Using \((6.24)\) for \( a = 0 \) and \((6.25)\) for \( a = 1 \) and the boundary relations
\[
t(0, m, u) = Q^{(2,2)[−m]}(u),
\]
\[
f(0, m, u) = Q^{(1,2)[−m]}(u),
\]
\[
t(2, m, u) = Q^{(2,0)[m+2]}(u)Q^{(0,2)[−m−2]}(u),
\]
\[
f(1, m, u) = Q^{(1,0)[m+1]}(u)Q^{(2,0)[−m−1]}(u).
\]
we can identify the asymptotic solution of the building blocks \( τ^o_m, Q^o \) and \( \tilde{Q}^o \):
\[
Q^o = \frac{Q^{(1,0)+}_{k_2}}{k_2}, \quad \tilde{Q}^o = k_1^− \frac{Q^{(1,2)−}_{Q^{(0,2)−}}}{Q^{(2,2)[−m]}},
\]
\[
τ^o_m = \frac{t(1, m + 1, u)}{k_2^{[m+1]}k_1^{[−m−1]}Q^{(2,2)[−m]}}.
\]
Here and below in this section the argument of all functions is \( u \). The factors \( k_1 \) and \( k_2 \) are the solutions of the relations
\[
k_1^+k_1^− = Q^{(0,2)−}_{Q^{(2,2)++}}, \quad k_2^+k_2^− = Q^{(2,0)++}.
\]

We now write down the asymptotic form of the NLIE functions. In these formulas we use the asymptotic T-functions in the \((1,1)\) gauge used in the previous section.

\[
b^o_m = \frac{\hat{T}^+(1, m + 1, u)Q^{(2,2)[1−m]}Q^{(1,2)[−3−m]}}{Q^{(1,0)[m+3]}Q^{(0,2)[−3−m]}Q^{(2,2)[−1−m]}},
\]
\[
B^o_m = \frac{\hat{T}^+(1, m + 2, u)Q^{(1,2)−[−1−m]}}{Q^{(1,0)[m+3]}Q^{(0,2)[−3−m]}},
\]
\[
d^o_m = \frac{\hat{T}^−(1, m + 1, u)Q^{(2,2)[−1−m]}Q^{(1,0)[m+3]}}{Q^{(2,0)[m+3]}Q^{(1,2)[−3−m]}},
\]
\[
D^o_m = \frac{\hat{T}^−(1, m + 2, u)Q^{(2,2)[−1−m]}Q^{(1,0)[m+1]}}{Q^{(2,0)[m+3]}Q^{(1,2)[−3−m]}},
\]
We know that the right-wing part of the asymptotic solution, in particular the T-functions \( \tilde{T}(1, s, u) = \tilde{t}_1^s(u) \) in the (1,1) gauge we are using are stained with several cuts close to the real axis. Most of these cuts are spurious and are not present in the gauge invariant Y-functions. This is also the case for the gauge invariant NLIE functions (6.15).

We can see this by considering the “cut-free” representation of the building blocks. Using the definition

\[
\frac{R_p}{R_m} = \frac{\Omega^+}{\Omega^-}
\]

we can write

\[
\tilde{T}(2, s, u) = \tilde{t}_s^2 \left( \frac{\Omega^{[2-s]}}{\Omega^{[s]}} \right)^2,
\]

where \( \tilde{t}_s^2 \) has cuts only at \( \pm \frac{i}{g} (s \pm 1) \) and

\[
\tilde{T}(1, s, u) = \tilde{t}_s^1 \frac{\Omega^{[1-s]}}{\Omega^{[s-1]}},
\]

where \( \tilde{t}_s^1 \) has cuts only at \( \pm \frac{2i}{g} \). The Y-functions are given by

\[
x_m^o = \frac{\tilde{T}(1, m + 2, u) \tilde{T}(1, m, u)}{\tilde{T}(2, m + 1, u)} = \frac{\tilde{t}_m^{1+} + \tilde{t}_m^{-1}}{\tilde{t}_m^2} \left( \frac{R_p}{R_m} \right)^{[m]} \left( \frac{R_m}{R_p} \right)^{[-m]},
\]

which has cuts only at \( \pm \frac{im}{g} \) and \( \pm (m + 2) \frac{i}{g} \). Similarly we have

\[
x_m^o = \frac{\tilde{T}^+(1, m + 1, u) \tilde{T}^-(1, m + 1, u)}{\tilde{T}(2, m + 1, u)} = \frac{\tilde{t}_m^{1+} + \tilde{t}_m^{-1}}{\tilde{t}_m^2} \left( \frac{R_p}{R_m} \right)^{[m]} \left( \frac{R_m}{R_p} \right)^{[-m]}.
\]

We further define the functions \( \tilde{Q}^{(k,m)} \), which have better analytic behaviour than the corresponding \( Q^{(k,m)} \). We write

\[
Q^{(2,2)} = \frac{\tilde{Q}^{(2,2)}}{\Omega^2}; \quad \tilde{Q}^{(2,2)} \text{ has no cuts},
\]

\[
Q^{(1,2)} = \frac{\tilde{Q}^{(1,2)}}{\Omega^{++}}; \quad \tilde{Q}^{(1,2)} \text{ has cuts only at } \frac{-i}{g},
\]

\[
Q^{(1,0)} = \frac{\tilde{Q}^{(1,0)}}{\Omega^{--}}; \quad \tilde{Q}^{(1,0)} \text{ has cuts only at } \frac{i}{g},
\]

\[
Q^{(0,2)} = \frac{\tilde{Q}^{(0,2)}}{\Omega^2}; \quad \tilde{Q}^{(0,2)} \text{ has cuts only at } \frac{i}{g}, \frac{-3i}{g},
\]

\[
Q^{(2,0)} = \frac{\tilde{Q}^{(2,0)}}{\Omega^2}; \quad \tilde{Q}^{(2,0)} \text{ has cuts only at } \frac{-i}{g}, \frac{-3i}{g}.
\]

Finally the “cut-free” representation of the NLIE functions is as follows.

\[
b_m^o = \left( \frac{R_m}{R_p} \right)^{[-m]} \frac{\tilde{t}_m^{1+} \tilde{Q}^{(2,2)[1-m]} \tilde{Q}^{(1,2)[-3-m]}}{\tilde{Q}^{(1,0)[m+3]} Q^{(0,2)[-3-m]} \tilde{Q}^{(2,2)[-1-m]}},
\]

\[
B_m^o = \left( \frac{R_m}{R_p} \right)^{[-m]} \frac{\tilde{t}_m^{1} \tilde{Q}^{(1,2)[-1-m]}}{\tilde{Q}^{(1,0)[m+3]} Q^{(0,2)[-3-m]}}.
\]
and
\[
d^0_m = \left\{ \left( \frac{R_p}{R_m} \right)^{[m+2]} \right\}^2 \left( \frac{R_p}{R_m} \right)^m \frac{\tilde{Q}^{(1,0)[m+3]}}{Q^{(2,0)[m+3]}(1,2)[m-3-m]}.
\]
\[
D^0_m = \left\{ \left( \frac{R_p}{R_m} \right)^{[m+2]} \right\}^2 \left( \frac{R_p}{R_m} \right)^m \frac{\tilde{Q}^{(1,0)[m+1]}}{Q^{(2,0)[m+3]}(1,2)[m-3-m]}.
\]

We end this section by giving the asymptotic solution corresponding to the deformed ground state using the building blocks (5.95). The T-functions in the (1,1) gauge are
\[
\hat{T}(1, s, u) = \frac{1 - q}{1 + q} q^{-s} (1 - q^{2s}), \quad \hat{T}(2, s, u) = \frac{(1 - q)^4}{q^2}.
\]

Further we have
\[
x^0_m = \frac{q^{-m}}{1 - q^2} (1 - q^{2m+2}), \quad Q^o = \tilde{Q}^o = \frac{S^+}{1 - q},
\]
and
\[
b^o_m = \frac{q^{-2m-2} - 1}{1 - q^2}, \quad B^o_m = \frac{q^{-2m-2} - q^2}{1 - q^2},
\]
\[
d^o_m = \frac{q^2 - q^{2m+4}}{1 - q^2}, \quad D^o_m = \frac{1 - q^{2m+4}}{1 - q^2}.
\]

7. NLIE for the ground state

In this section we will obtain the NLIE integral equations and thus reduce the system of integral equations to a finite set. We will transform the functional equations (4.13-4.16) into integral equations which together with (4.20) form a complete set of NLIE integral equations equivalent to the TBA integral equations for the \((a > 3)\) part of the upper nodes. But we start by considering the analogous but much simpler (and already solved) problem corresponding to the right-wing (and left-wing) nodes. We will see that the two constructions proceed along very similar lines.

In this paper we will consider the NLIE integral equations only up to source terms, i.e. the NLIE for the ground state problem. Although the addition of source terms is in principle straightforward, we leave the elaboration of the excited state problem to future work.

Before we start the construction we fix some notations and conventions. We will denote the Fourier transform of the function \(f(u)\) by \(\hat{f}(\omega)\) and use the definition
\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} du e^{i\omega u} f(u).
\]
In particular, the Fourier transform of a logarithmic derivative will be denoted by
\[
\tilde{dl}f(\omega) = \int_{-\infty}^{\infty} du e^{i\omega u} \frac{f'(u)}{f(u)}.
\]
We note that if we introduce the shorthand notation
\[ p = e^{\frac{\mathbf{i}}{\mathbf{g}}} \]  
we can write
\[ \tilde{f}^{[\gamma]}(\omega) = p^\gamma \tilde{f}(\omega) \]  
and in particular
\[ \tilde{f}^+(\omega) = p\tilde{f}(\omega), \quad \tilde{f}^-(\omega) = \frac{1}{p}\tilde{f}(\omega). \]  
We also note that if \( f(u) \) is analytic in the upper complex \( u \) plane then \( \tilde{f}(\omega) = 0 \) for \( \omega > 0 \) and likewise if \( f(u) \) is analytic in the lower complex \( u \) plane then \( \tilde{f}(\omega) = 0 \) for \( \omega < 0 \).

**7.1 NLIE integral equations for the right-wing nodes**

The NLIE integral equations for the one-dimensional SU(2) TBA chain are well known \[45, 52, 47\]. Here we will follow the construction in \[57\] and \[66\]. We start by rewriting (6.17) in Fourier space as
\[ \tilde{d}l b_m + \tilde{d}l d_m - \tilde{d}l X_m, \quad \frac{1}{p}\tilde{d}l B_m + p\tilde{d}l D_m - \tilde{d}l X_{m+1}. \]  
This has to be supplemented by the Fourier space version of (6.21) and (6.22). According to what we noted above, there is no contribution coming from the second term on the right hand side of (6.21) and (6.22) for negative and positive frequencies, respectively. For the first one we have
\[ \tilde{d}l b_m - \tilde{d}l B_m = \tilde{s}(p\tilde{d}l X_m - \tilde{d}l X_{m+1}), \quad p < 1, \]  
where
\[ \tilde{s}(\omega) = \frac{1}{p + \frac{1}{p}} = \tilde{M}_1^{(1)}(\omega) \]  
and for the second
\[ \tilde{d}l d_m - \tilde{d}l D_m = \tilde{s}(\frac{1}{p}\tilde{d}l X_m - \tilde{d}l X_{m+1}), \quad p > 1. \]  
Eliminating \( \tilde{d}l X_{m+1} \) from the equations we get
\[ \tilde{d}l b_m = \frac{\tilde{d}l B_m - \tilde{d}l D_m + p^2\tilde{d}l X_m}{1 + p^2}, \quad p > 1, \]  
\[ \tilde{d}l d_m = \frac{\tilde{d}l D_m - \tilde{d}l B_m + \tilde{d}l X_m}{1 + p^2}, \]  
\[ \tilde{d}l b_m = \frac{p^2\tilde{d}l B_m - p^2\tilde{d}l D_m + p^2\tilde{d}l X_m}{1 + p^2}, \quad p < 1, \]  
\[ \tilde{d}l d_m = \frac{p^2\tilde{d}l D_m - p^2\tilde{d}l B_m + \tilde{d}l X_m}{1 + p^2}, \]  
\[ \tilde{d}l b_m = \frac{p^2\tilde{d}l B_m - p^2\tilde{d}l D_m + p^2\tilde{d}l X_m}{1 + p^2}, \quad p < 1. \]  
\[ \tilde{d}l d_m = \frac{p^2\tilde{d}l D_m - p^2\tilde{d}l B_m + \tilde{d}l X_m}{1 + p^2}, \]  
\[ \tilde{d}l b_m = \frac{p^2\tilde{d}l B_m - p^2\tilde{d}l D_m + p^2\tilde{d}l X_m}{1 + p^2}, \quad p < 1. \]  
\[ \tilde{d}l d_m = \frac{p^2\tilde{d}l D_m - p^2\tilde{d}l B_m + \tilde{d}l X_m}{1 + p^2}, \]  
\[ \tilde{d}l b_m = \frac{p^2\tilde{d}l B_m - p^2\tilde{d}l D_m + p^2\tilde{d}l X_m}{1 + p^2}, \quad p < 1. \]  
\[ \tilde{d}l d_m = \frac{p^2\tilde{d}l D_m - p^2\tilde{d}l B_m + \tilde{d}l X_m}{1 + p^2}, \]  
\[ \tilde{d}l b_m = \frac{p^2\tilde{d}l B_m - p^2\tilde{d}l D_m + p^2\tilde{d}l X_m}{1 + p^2}, \quad p < 1. \]  
\[ \tilde{d}l d_m = \frac{p^2\tilde{d}l D_m - p^2\tilde{d}l B_m + \tilde{d}l X_m}{1 + p^2}, \]  
\[ \tilde{d}l b_m = \frac{p^2\tilde{d}l B_m - p^2\tilde{d}l D_m + p^2\tilde{d}l X_m}{1 + p^2}, \quad p < 1. \]  
\[ \tilde{d}l d_m = \frac{p^2\tilde{d}l D_m - p^2\tilde{d}l B_m + \tilde{d}l X_m}{1 + p^2}, \]  
\[ \tilde{d}l b_m = \frac{p^2\tilde{d}l B_m - p^2\tilde{d}l D_m + p^2\tilde{d}l X_m}{1 + p^2}, \quad p < 1. \]  
\[ \tilde{d}l d_m = \frac{p^2\tilde{d}l D_m - p^2\tilde{d}l B_m + \tilde{d}l X_m}{1 + p^2}, \]  
\[ \tilde{d}l b_m = \frac{p^2\tilde{d}l B_m - p^2\tilde{d}l D_m + p^2\tilde{d}l X_m}{1 + p^2}, \quad p < 1. \]  
\[ \tilde{d}l d_m = \frac{p^2\tilde{d}l D_m - p^2\tilde{d}l B_m + \tilde{d}l X_m}{1 + p^2}, \]  
\[ \tilde{d}l b_m = \frac{p^2\tilde{d}l B_m - p^2\tilde{d}l D_m + p^2\tilde{d}l X_m}{1 + p^2}, \quad p < 1. \]  
\[ \tilde{d}l d_m = \frac{p^2\tilde{d}l D_m - p^2\tilde{d}l B_m + \tilde{d}l X_m}{1 + p^2}, \]  
\[ \tilde{d}l b_m = \frac{p^2\tilde{d}l B_m - p^2\tilde{d}l D_m + p^2\tilde{d}l X_m}{1 + p^2}, \quad p < 1. \]  
\[ \tilde{d}l d_m = \frac{p^2\tilde{d}l D_m - p^2\tilde{d}l B_m + \tilde{d}l X_m}{1 + p^2}, \]  
\[ \tilde{d}l b_m = \frac{p^2\tilde{d}l B_m - p^2\tilde{d}l D_m + p^2\tilde{d}l X_m}{1 + p^2}, \quad p < 1. \]  
\[ \tilde{d}l d_m = \frac{p^2\tilde{d}l D_m - p^2\tilde{d}l B_m + \tilde{d}l X_m}{1 + p^2}, \]
\( \tilde{d} X_{m+1} \) is determined from
\[
\tilde{d} X_{m+1} = \frac{1}{p} \tilde{d} B_m + p \tilde{d} D_m. \tag{7.12}
\]

Having solved the problem in Fourier space we now return to rapidity space. In order that the kernels appearing in (7.10) and (7.11) can be transformed (back) to rapidity space, we have to shift the argument of the unknowns \( b_m(u) \) and \( d_m(u) \) in the imaginary direction. To emphasize that our unknowns in the NLIE integral equations are functions after these shifts, we introduce the notation
\[
\begin{align*}
\mathfrak{b}_m(u) &= b_m^\eta(u) = b_m(u - \frac{i\eta}{g}), & \mathfrak{B}_m(u) &= B_m^\eta(u) = B_m(u - \frac{i\eta}{g}), \\
\mathfrak{d}_m(u) &= d_m^\eta(u) = d_m(u + \frac{i\eta}{g}), & \mathfrak{D}_m(u) &= D_m^\eta(u) = D_m(u + \frac{i\eta}{g}),
\end{align*} \tag{7.13}
\]
which corresponds to the Fourier space relations
\[
\begin{align*}
\tilde{d} \mathfrak{b}_m &= p^{-\eta} \tilde{d} b_m, & \tilde{d} \mathfrak{B}_m &= p^{-\eta} \tilde{d} B_m, \\
\tilde{d} \mathfrak{d}_m &= p^{\eta} \tilde{d} d_m, & \tilde{d} \mathfrak{D}_m &= p^{\eta} \tilde{d} D_m. \tag{7.14}
\end{align*}
\]
Besides the basic TBA kernel function \( s(u) \), whose Fourier transform is given by (7.8) the kernel \( H(u) \) with Fourier transform
\[
\tilde{H}(\omega) = \tilde{s}(\omega) e^{-\frac{|\omega|}{\eta}}, \tag{7.15}
\]
enters the rapidity space version of the NLIE equations:
\[
\begin{align*}
\begin{align*}
\tilde{d} \mathfrak{b}_m &= H \ast \tilde{d} \mathfrak{B}_m - H^{[-2\eta]} \ast \tilde{d} \mathfrak{D}_m + s^{[1-\eta]} \ast \tilde{d} X_m, & \tilde{d} \mathfrak{d}_m &= H \ast \tilde{d} \mathfrak{D}_m - H^{[2\eta]} \ast \tilde{d} \mathfrak{B}_m + s^{[\eta-1]} \ast \tilde{d} X_m. \tag{7.16}
\end{align*}
\end{align*}
\]
Since \( s(u) \) is analytic in the strip \((-1,1)\) and \( H(u) \) in the strip \((-2,2)\), the above NLIE equations are well defined if the parameter \( \eta \) is chosen in the range \( 0 < \eta < 1 \). Integrating the equations once, we obtain the final form of the ground state NLIE equations for the right-wing nodes:
\[
\begin{align*}
\begin{align*}
\ln \mathfrak{b}_m &= H \ast \ln \mathfrak{B}_m - H^{[-2\eta]} \ast \ln \mathfrak{D}_m + s^{[1-\eta]} \ast \ln X_m + Cb_m, & \ln \mathfrak{d}_m &= H \ast \ln \mathfrak{D}_m - H^{[2\eta]} \ast \ln \mathfrak{B}_m + s^{[\eta-1]} \ast \ln X_m + Cd_m, \tag{7.17}
\end{align*}
\end{align*}
\]
where the integration constants \( Cb_m, Cd_m \) can be calculated using the large \( u \) asymptotics of the functions \( \mathfrak{b}_m, \mathfrak{d}_m \) and \( X_m \). Finally we note that we are going to apply this construction with a fixed value of \( m \). \( X_{m+1} \) only appears in the equation for the node \( x_m \) in the form \( s \ast \ln X_{m+1} \). We can write this combination using (7.12) as
\[
\begin{align*}
\begin{align*}
s \ast \ln X_{m+1} &= s^{[\eta-1]} \ast \ln \mathfrak{B}_m + s^{[1-\eta]} \ast \ln \mathfrak{D}_m. \tag{7.20}
\end{align*}
\end{align*}
\]
Thus all Y-functions with index larger than \( m \) are replaced by the two NLIE variables \( \mathfrak{b}_m, \mathfrak{d}_m \) and the set of integral equations for this truncated set is closed.
7.2 NLIE equations for the upper nodes

In this subsection we derive the NLIE equations for the upper nodes. This will be done in the same spirit as in ref. [51]. The setting up of the equations is based on the same principles, but the resulting set of equations (even the number of NLIE variables) is completely different. We start by rewriting the functional relations in terms of logarithmic derivatives in Fourier space. From (4.13) we obtain

\[ \tilde{d} b^{(3)}_{1,s} + \tilde{d} d^{(3)}_{1,s} = \tilde{d} Y_{1,s}^{(3)}, \]  

(7.21)

\[ \tilde{d} b^{(3)}_{2,s} + \tilde{d} d^{(3)}_{2,s} = \tilde{d} Y_{2,s}^{(3)}, \]  

(7.22)

\[ \tilde{d} b^{(3)}_{3,s} + \tilde{d} d^{(3)}_{3,s} = \tilde{d} Y_{3,s}^{(3)}, \]  

(7.23)

\[ \tilde{d} b^{(1)}_{1,s} + \tilde{d} d^{(1)}_{1,s} = \tilde{d} Y_{1,s}^{(1)}. \]  

(7.26)

From (4.14) we get

\[ \tilde{d} b^{(3)}_{2,s} + p \tilde{d} d^{(3)}_{1,s} = \tilde{d} Y_{1,s+1}^{(2)}, \]  

(7.27)

(7.28)

\[ \tilde{d} b^{(2)}_{2,s} + p \tilde{d} d^{(2)}_{1,s} = \tilde{d} Y_{1,s+1}^{(1)}. \]  

(7.29)

From (4.15) we get

\[ \tilde{d} b^{(3)}_{1,s} - \tilde{d} b^{(3)}_{1,s} + \tilde{d} D^{(3)}_{1,s} + p \tilde{d} d^{(3)}_{2,s} - p \tilde{d} D^{(3)}_{2,s} + p \tilde{d} B^{(3)}_{2,s} = \tilde{d} Y_{1,s}^{(2)}, \]  

(7.30)

\[ \tilde{d} b^{(3)}_{2,s} - \tilde{d} b^{(3)}_{2,s} + \tilde{d} D^{(3)}_{2,s} + p \tilde{d} d^{(3)}_{3,s} - p \tilde{d} D^{(3)}_{3,s} + p \tilde{d} B^{(3)}_{3,s} = \tilde{d} Y_{2,s}^{(2)}, \]  

(7.31)

\[ \tilde{d} b^{(2)}_{1,s} - \tilde{d} b^{(2)}_{1,s} + \tilde{d} D^{(2)}_{1,s} + p \tilde{d} d^{(2)}_{2,s} - p \tilde{d} D^{(2)}_{2,s} + p \tilde{d} B^{(2)}_{2,s} = \tilde{d} Y_{1,s}^{(1)}. \]  

(7.32)

Finally we have from (4.16)

\[ \frac{1}{p} \tilde{d} b^{(3)}_{2,s} + p \tilde{d} d^{(3)}_{2,s} + \tilde{d} b^{(3)}_{3,s} - \tilde{d} b^{(3)}_{2,s} + \tilde{d} d^{(3)}_{1,s} - \tilde{d} D^{(3)}_{1,s} = \tilde{d} Y_{2,s+1}^{(3)}, \]  

(7.33)

\[ \frac{1}{p} \tilde{d} b^{(3)}_{1,s} + p \tilde{d} D^{(3)}_{1,s} + \tilde{d} b^{(3)}_{2,s} - \tilde{d} b^{(3)}_{2,s} = \tilde{d} Y_{1,s+1}^{(3)}, \]  

(7.34)

\[ \frac{1}{p} \tilde{d} b^{(3)}_{2,s} + p \tilde{d} D^{(3)}_{2,s} + \tilde{d} d^{(3)}_{2,s} - \tilde{d} D^{(3)}_{2,s} = \tilde{d} Y_{3,s+1}^{(3)}, \]  

(7.35)

\[ \frac{1}{p} \tilde{d} b^{(2)}_{1,s} + p \tilde{d} D^{(2)}_{1,s} + \tilde{d} b^{(2)}_{2,s} - \tilde{d} b^{(2)}_{2,s} = \tilde{d} Y_{1,s+1}^{(2)}, \]  

(7.36)

\[ \frac{1}{p} \tilde{d} b^{(2)}_{2,s} + p \tilde{d} D^{(2)}_{2,s} + \tilde{d} d^{(2)}_{1,s} - \tilde{d} D^{(2)}_{1,s} = \tilde{d} Y_{2,s+1}^{(2)}, \]  

(7.37)

\[ \frac{1}{p} \tilde{d} b^{(1)}_{1,s} + p \tilde{d} D^{(1)}_{1,s} = \tilde{d} Y_{1,s+1}^{(1)}. \]  

(7.38)

The above set of equations has to be completed by the ones following from the “halfplane” relations (4.26). Here we have to use the “halfplane” properties of the boundary
ratios discussed in section 3 to obtain for \( p < 1 \) (\( \omega < 0 \))

\[
\tilde{d}l_{1,s}^{(3)} - \tilde{d}l_{1,s}^{(3)} = \frac{1}{\Sigma^2 - 2\Sigma} \left\{ (\Sigma^2 - 1) \left( \frac{1}{p} \tilde{d}l_{1,s}^{(3)} - \tilde{d}l_{1,s+1}^{(3)} \right) \right\},
\]

\( \Sigma = p + \frac{1}{p} = \frac{1}{s(\omega)} \)

is used to express various components of the Fourier space kernels \( \tilde{M}_{ab}^{(r)} \) using the general formula

\[
\tilde{M}_{ab}(\omega) = \frac{\cosh(k - |a - b|)\mu - \cosh(k - a - b)\mu}{2\sinh k\mu\sinh \mu}, \quad \mu = \frac{\omega}{g}
\]

with \( k = r + 1 \).

Next we solve the set of equations (7.21-7.44) in the following sense. We want to write down equations, which (after going back to rapidity space) allow us to determine the unknown functions \( b_{a,s}^{(r)}, d_{a,s}^{(r)} \) for \( r = 1, 2, 3, a = 1, \ldots, r \) in terms of \( Y_{a,s} = Y_{a,s}^{(3)} \), \( a = 1, 2, 3 \), which serve as “input” from the TBA equations of the central nodes. We have \( 6 + 3 + 3 + 6 + 3 = 21 \) equations (both for the \( p > 1 \) and for the \( p < 1 \) cases) for the \( 6 + 6 \) unknowns and the \( 6 + 3 \) \( Y \)-functions \( (Y_{a,s+1}^{(r)}, r = 1, 2, 3 \) and \( Y_{a,s}^{(r)}, r = 1, 2 \), which have to be eliminated from the equations. We see that we have just the right number of equations that allow us to obtain the NLIE integral equations first in Fourier space and then in rapidity space.

We note that the counting works similarly in the case of a general SU\((k)\) TBA system. From the functional equations (1.13-1.16) we get

\[
\binom{k}{2} + \binom{k-1}{2} + \binom{k-1}{2} + \binom{k}{2} = 2(k-1)^2
\]

equations, which, together with \( k - 1 \) “halfplane” relations form a total of \( (k - 1)(2k - 1) \) equations. This number exactly matches the sum of the number of unknowns \( (2 \times \binom{k}{2} = k(k - 1)) \) and the number of \( Y \)-functions to be eliminated \( (\binom{k}{2} + \binom{k-1}{2} = (k - 1)^2) \).
Solving for the unknowns in our case in Fourier space we obtain
\[
\tilde{d} b^{(r)}_{a,s} = \sum_{p=1}^{3} \sum_{a'=1}^{p} (\tilde{K}_{bB})_{aa'}^{(rp)} \tilde{d} l B^{(p)}_{a',s} + \sum_{p=1}^{3} \sum_{a'=1}^{p} (\tilde{K}_{bD})_{aa'}^{(rp)} \tilde{d} l D^{(p)}_{a',s} + \sum_{a'=1}^{3} (\tilde{K}_{bY})_{aa'}^{(r)} \tilde{d} l Y^{(r)}_{a',s},
\]
\[
\tilde{d} a^{(r)}_{a,s} = \sum_{p=1}^{3} \sum_{a'=1}^{p} (\tilde{K}_{bD})_{aa'}^{(rp)} \tilde{d} l B^{(p)}_{a',s} + \sum_{p=1}^{3} \sum_{a'=1}^{p} (\tilde{K}_{bD})_{aa'}^{(rp)} \tilde{d} l D^{(p)}_{a',s} + \sum_{a'=1}^{3} (\tilde{K}_{bY})_{aa'}^{(r)} \tilde{d} l Y^{(r)}_{a',s}.
\]

(7.47)

The matrices of Fourier space kernels \( \tilde{K}_{bB}, \tilde{K}_{bD} \) etc. are listed in appendix \( \mathbb{C} \).

We still have to express \( Y_{a,s+1}^{(3)} = Y_{a,s+1}^{(3)} \) in terms of the NLIE variables in order to close the set of equations for the central nodes. We achieve this goal by combining the Y-system equations for \( y_{a,s} = y_{a,s}^{(3)} \) with (7.33, 7.35):
\[
\tilde{d} l y_{a,s} = \tilde{s} \tilde{d} l \left( B_{a,s}^{(3)} - D_{a,s}^{(3)} + \frac{b_{a+1,s}^{(3)} - b_{a,s-1,s}^{(3)}}{B_{a+1,s}^{(3)} - D_{a-1,s}^{(3)}} \right) + \tilde{s} \tilde{d} l Y_{a,s-1}^{(3)} + \tilde{s} \tilde{d} l \left( \frac{y_{a+1,s} - y_{a-1,s}}{Y_{a+1,s} - Y_{a-1,s}} \right).
\]

(7.48)

We understand that terms appearing on the RHS of (7.48) with \( a \)-type indices “out of range” (\( \neq 1, 2, 3 \)) must be omitted.

In order to be able to transform the equations (7.47) back into rapidity space (where the kernel multiplications become convolutions), the Fourier form of the kernels should satisfy the following requirements:

- they must be continuous at \( \omega = 0 \),
- they must tend to zero exponentially as \( \omega \to \pm \infty \).

The \( \omega = 0 \) condition is to ensure that kernels decay as \( 1/u^2 \) at infinity and we can check (by comparing the \( \omega \to \pm \infty \) limits of the representations valid for positive and negative frequencies) that it is satisfied by our kernels.

The second requirement (which is necessary to ensure that the inverse Fourier transformation exists) is not automatically satisfied for all matrix elements of the kernel matrices listed in appendix \( \mathbb{C} \). Fortunately the problem can be solved by a simple redefinition of the NLIE variables. It can be shown that if we redefine our NLIE functions by shifting their arguments appropriately:

\[
\begin{align*}
&b^{(r)}_{a,s}(u) \to \tilde{b}^{(r)}_{a,s}(u) = b^{(r)}_{a,s}(u + (i/g)(r - 3 + \gamma_a^{(r)})), \quad r = 1, 2, 3 \quad a = 1, \ldots, r \\
d^{(r)}_{a,s}(u) \to \tilde{d}^{(r)}_{a,s}(u) = d^{(r)}_{a,s}(u + (i/g)\eta_a^{(r)}), \quad r = 1, 2, 3 \quad a = 1, \ldots, r \\
y_{a,s}(u) \to \tilde{y}_{a,s}(u) = y_{a,s}(u + (i/g)\epsilon_a), \quad a = 1, 2, 3
\end{align*}
\]

then the kernels entering the NLIE of these redefined variables do satisfy the requirements imposed above provided the shift parameters satisfy the inequalities:

\[
\begin{align*}
-\frac{1}{2} &< \gamma_1^{(2)} < \gamma_2^{(3)} < \epsilon_2 < \eta_1^{(3)} < \eta_2^{(2)} \frac{1}{2}, \\
-\frac{1}{2} &< \gamma_1^{(3)} < \epsilon_1 < \eta_1^{(3)} < \eta_1^{(2)} < \gamma_1^{(1)} < \gamma_2^{(2)} < \gamma_3^{(3)} < \epsilon_3 < \eta_3^{(3)} \frac{1}{2}.
\end{align*}
\]

(7.49, 7.50)
The NLIE for the new variables in rapidity space takes the form:

\[
\ln b_{a,s}^{(r)} = \sum_{p=1}^{3} \sum_{a' = 1}^{p} (G_{bB})_{aa'}^{(rp)} \ln B_{a',s}^{(p)} + \sum_{p=1}^{3} \sum_{a' = 1}^{p} (G_{bD})_{aa'}^{(rp)} \ln D_{a',s}^{(p)} + \sum_{a' = 1}^{3} (G_{bY})_{aa'}^{(r)} \ln Y_{a',s},
\]

\[
\ln d_{a,s}^{(r)} = \sum_{p=1}^{3} \sum_{a' = 1}^{p} (G_{dB})_{aa'}^{(rp)} \ln B_{a',s}^{(p)} + \sum_{p=1}^{3} \sum_{a' = 1}^{p} (G_{dD})_{aa'}^{(rp)} \ln D_{a',s}^{(p)} + \sum_{a' = 1}^{3} (G_{dY})_{aa'}^{(r)} \ln Y_{a',s},
\]

(7.51)

where we have used the notation \(B_{a,s}^{(p)} = 1 + b_{a,s}^{(p)}\), \(D_{a,s}^{(p)} = 1 + d_{a,s}^{(p)}\), \(Y_{a,s} = 1 + \eta_{a,s}\) and the kernels are rapidity space representations of the appropriately modified Fourier kernels \([C.2],[C.13]):\n
\[
(\hat{G}_{bB})_{aa'}^{(rr')}(\omega) = (\hat{K}_{bB})_{aa'}^{(rr')}(\omega) p^{r'-r+\gamma_a^{(r)} - \gamma_{a'}^{(r')}} ,
\]

\[
(\hat{G}_{bD})_{aa'}^{(rr')}(\omega) = (\hat{K}_{bD})_{aa'}^{(rr')} (\omega) p^{r-3+\gamma_a^{(r)} - \gamma_{a'}^{(r')}},
\]

etc. The rapidity space version of (7.48) in the new variables becomes

\[
\ln \eta_{a,s} = s^{-1+\epsilon_a-\gamma_a^{(3)}} \ln \Omega_{a,s}^{(3)} + s^{[1+\epsilon_a-\eta_a^{(3)}} \ln \Omega_{a,s}^{(3)} + s^{[\epsilon_a-\gamma_a^{(3)}]} \ln \left( \frac{b_{a+1,s}}{b_{a+1,s}} \right) + s^{[\epsilon_a-\eta_a-1]} \ln \left( \frac{\eta_{a-1,s}}{\eta_{a-1,s}} \right) + s^{[\epsilon_a-\eta_a+1]} \ln \left( \frac{\eta_{a+1,s}}{\eta_{a+1,s}} \right).
\]

(7.53)

We note that (7.53) is well-defined since (7.49) and (7.51) imply

\[
\epsilon_a - \gamma_a^{(3)} > 0, \quad \epsilon_a - \eta_a^{(3)} < 0, \quad a = 1, 2, 3.
\]

(7.54)

8. Linearized equations

In this section we compute the leading \(O(\varepsilon^2)\) corrections to the asymptotic solution corresponding to the ground state of the \(\gamma\)-deformed model. These corrections have been calculated recently \([27]\) using the TBA equations. In this paper their contribution to the ground state energy was also calculated and it was found to be in agreement with the NLO Lüscher formula. Here we calculate these corrections from the ground state NLIE and show that they are identical to the ones obtained from the TBA integral equations directly. This agreement is a useful analytical evidence of the equivalence of the two approaches.

8.1 Linearization of the right-wing SU(2) problem

We start with linearizing the NLIE equations corresponding to the right-wing nodes. Of course, the equivalence of the SU(2) type TBA equations with the corresponding NLIE equations is well known. Here we consider this problem not only for completeness but also
because the logic of the calculation in this case is similar to what we will follow in the technically much more complicated case of the upper nodes.

We will write for any function $f$

$$f = f^o(1 + f^{(1)} + \ldots),$$

where $f^o$ is the asymptotic limit of the function and $f^{(1)}$ is the corresponding $O(\varepsilon^2)$ correction. With this notation

$$\tilde{d} l \frac{f}{f^o} = \tilde{f}^{(1)} + \ldots$$

and for the corresponding $F = 1 + f$ we have

$$F^o = 1 + f^o, \quad \tilde{d} l \frac{F}{F^o} = \frac{f^o}{F^o} \tilde{f}^{(1)} + \ldots$$

The NLIE integral equations of subsection 7.1 are based on the NLIE functional relations (6.17) and (6.21-6.22). The same relations are also satisfied by the asymptotic limit of the NLIE functions and therefore the NLIE integral equations are also valid if written for the ratios of type $f/f^o$ (or equivalently in Fourier space for the differences of the logarithmic derivatives). Introducing the notations

$$\tilde{a}_m = \tilde{b}_m^{(1)}; \quad \tilde{f}_m = \tilde{d}_m^{(1)}; \quad \tilde{\xi}_m = \tilde{X}_m^{(1)}$$

(8.4)

and

$$\frac{b^o_m}{B^o_m} = \beta^o_m; \quad \frac{d^o_m}{D^o_m} = \delta^o_m; \quad p_o = e^{-\frac{1}{\tilde{s}}}$$

(8.5)

we can write the linearization of (7.10-7.11) in Fourier space as

$$\tilde{a}_m = p_o \tilde{s} (\beta^o_m \tilde{a}_m - \delta^o_m \tilde{f}_m) + p \tilde{s} \tilde{\xi}_m, \quad \tilde{f}_m = p_o \tilde{s} (\delta^o_m \tilde{f}_m - \beta^o_m \tilde{a}_m) + \frac{1}{p} \tilde{s} \tilde{\xi}_m$$

(8.6)

and from (7.12) we have

$$\tilde{\xi}_{m+1} = \frac{1}{p} \beta^o_m \tilde{a}_m + p \delta^o_m \tilde{f}_m.$$

(8.7)

We first solve (8.6):

$$\tilde{a}_m = \frac{(p - p_o \delta^o_m) \tilde{\xi}_m}{1 + p_o (1 - \beta^o_m - \delta^o_m)}, \quad \tilde{f}_m = \frac{\frac{1}{p} - p_o \beta^o_m) \tilde{\xi}_m}{1 + p_o (1 - \beta^o_m - \delta^o_m)},$$

(8.8)

then use the result in (8.7) and find

$$\tilde{\xi}_{m+1} = \frac{\beta^o_m \delta^o_m + \frac{1}{p_o^2} (\beta^o_m \delta^o_m - \beta^o_m - \delta^o_m)}{(\beta^o_m + \delta^o_m - 1) - \frac{1}{p_o^2}} \tilde{p}_o \tilde{\xi}_m.$$

(8.9)

Recalling the definition of $q$-numbers

$$[N]_q = \frac{q^N - q^{-N}}{q - 1/q}$$

(8.10)

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we rewrite the asymptotic limits (3.53)
\[ \beta_m^o = \frac{1}{q} \frac{[m+1]_q}{[m+2]_q}, \quad \delta_m^o = q \frac{[m+1]_q}{[m+2]_q}, \] (8.11)
and analogously
\[ x_m^o = [m]_q [m+2]_q, \quad X_m^o = ([m+1]_q)^2. \] (8.12)
The recursion relation (8.9) becomes
\[ \tilde{\xi}_{m+1} = \frac{[m+1]_q}{[m+2]_q} p_o \frac{[m+1]_q - \frac{1}{p_o} [m+3]_q}{[m]_q - \frac{1}{p_o} [m+2]_q} \tilde{\xi}_m. \] (8.13)
We note that
\[ \tilde{\xi}_m = \tilde{x}_m^{(1)r} = \frac{x_m^{(1)r}}{X_m^o} = \frac{[m]_q [m+2]_q x_m^{(1)r}}{([m+1]_q)^2}, \] (8.14)
and using this relation it is easy to see that our result follows from eq. (4.68) of ref. [27],
which in our notation reads
\[ \tilde{x}_m^{(1)} = \text{const.} p_o^{m+2} \left( \frac{[m+1]_q - \frac{1}{p_o} [m+3]_q}{[m]_q} \right), \] (8.15)
where the complicated constant is \( m \)-independent.

8.2 Linearization of the upper Y-system

We saw above that the final result of the NLIE linearization procedure is a recursion relation among \( O(\varepsilon^2) \) coefficients of TBA Y-functions. In the next subsection we will go through this procedure for the upper nodes and will show that they agree with the relations obtained directly from TBA. Although the latter problem has been solved in ref. [27], in this subsection we reproduce the derivation and present the relations in a form easily comparable to the NLIE results.

We start here by recalling the upper node (SU(4) problem) Y-system equations:
\[ y_{1,s}^o y_{1,s}^o = Y_{1,s+1} Y_{1,s-1} \frac{y_{2,s}}{Y_{2,s}}, \] (8.16)
\[ y_{2,s}^o y_{2,s}^o = Y_{2,s+1} Y_{2,s-1} \frac{y_{1,s}}{Y_{1,s}} \frac{y_{3,s}}{Y_{3,s}}, \] (8.17)
\[ y_{3,s}^o y_{3,s}^o = Y_{3,s+1} Y_{3,s-1} \frac{y_{2,s}}{Y_{2,s}}. \] (8.18)
The asymptotic solution\(^5\) can be recalled from subsection 5.4.1
\[ y_{1,s}^o = y_{3,s}^o = s^2 - 1, \quad Y_{1,s}^o = Y_{3,s}^o = s^2, \quad y_{2,s}^o = Y_{2,s}^o = \frac{1}{\Psi_s}, \] (8.19)
where
\[ \Psi_s = s^2 \left( \frac{1-q}{q} \right)^2 \left( \frac{1-\dot{q}}{\dot{q}} \right)^2 \frac{\psi[s]}{\psi[-s]}. \] (8.20)

\(^5\) (8.19) is an exact solution of the Y-system relations, it is only the relations \( Y_{a,s}^o = 1 + y_{a,s}^o \) which are not exactly satisfied for \( a = 2 \).
The new feature of the asymptotic solution here is that for $a = 2$ it is $O(1/\varepsilon^2)$ and for this reason if we introduce the $O(\varepsilon^2)$ coefficients $z_{a,s}, Z_{a,s}$ by

$$ y_{a,s} = y_{a,s}^0(1 + z_{a,s} + \ldots), \quad Y_{a,s} = Y_{a,s}^0(1 + Z_{a,s} + \ldots) \quad (8.21) $$

we find

$$ a = 1, 3 : \quad Z_{a,s} = r_s z_{a,s}, \quad r_s = \frac{s^2 - 1}{s^2} \quad (8.22) $$

and

$$ a = 2 : \quad Z_{2,s} = z_{2,s} + \Psi_s. \quad (8.23) $$

As in the previous subsection, the ratios

$$ x_{a,s} = \frac{y_{a,s}}{y_{a,s}^0}, \quad X_{a,s} = \frac{Y_{a,s}}{Y_{a,s}^0} \quad (8.24) $$

are exactly solving the Y-system equations, which can be rewritten as TBA integral equations as follows.

$$ \ln x_{1,s} = s \ast \{ \ln X_{1,s+1} + \ln X_{1,s-1} + \ln x_{2,s} - \ln X_{2,s} \}, \quad (8.25) $$

$$ \ln x_{2,s} = s \ast \{ \ln X_{2,s+1} + \ln X_{2,s-1} + \ln x_{1,s} + \ln x_{3,s} - \ln X_{1,s} - \ln X_{3,s} \}, \quad (8.26) $$

$$ \ln x_{3,s} = s \ast \{ \ln X_{3,s+1} + \ln X_{3,s-1} + \ln x_{2,s} - \ln X_{2,s} \}. \quad (8.27) $$

Going to Fourier space after linearizing the above system we get

$$ \Sigma \tilde{z}_{1,s} = r_{s+1} \tilde{z}_{1,s+1} + r_{s-1} \tilde{z}_{1,s-1} - \tilde{\Psi}_s, \quad (8.28) $$

$$ \Sigma \tilde{z}_{2,s} = \tilde{z}_{2,s+1} + \tilde{z}_{2,s-1} + \tilde{\Psi}_{s+1} + \tilde{\Psi}_{s-1} + (1 - r_s)(\tilde{z}_{1,s} + \tilde{z}_{3,s}), \quad (8.29) $$

$$ \Sigma \tilde{z}_{3,s} = r_{s+1} \tilde{z}_{3,s+1} + r_{s-1} \tilde{z}_{3,s-1} - \tilde{\Psi}_s. \quad (8.30) $$

### 8.2.1 $\tilde{z}_{1,s}, \tilde{z}_{3,s}$ problem

Since the linearized equations are separated, we first study the $a = 1, 3$ cases, which are of the form

$$ \Sigma \tilde{z}_s = r_{s+1} \tilde{z}_{s+1} + r_{s-1} \tilde{z}_{s-1} - \tilde{\Psi}_s. \quad (8.31) $$

Let us introduce a few building blocks in terms of which we will write the solution.

$$ N_s = s p_o^2 - s - 2, \quad (8.32) $$

$$ \alpha_s = p_o^s \frac{s + 1}{s(s + 2)} N_s = p_o^s \frac{s + 1}{s(s + 2)} [s(p_o^2 - 1) - 2], $$

$$ \beta_s = \alpha_{-s} = p_o^{-s} \frac{s - 1}{s(s - 2)} [s(p_o^2 - 1) + 2] $$

and further

$$ f_k = \frac{p_o^4}{(p_o^2 - 1)^2} \beta_k r_{k+1} \tilde{\Psi}_{k+1}, \quad g_k = \frac{p_o^2}{(p_o^2 - 1)^2} \alpha_k r_{k+1} \tilde{\Psi}_{k+1}. \quad (8.33) $$
We note that the quantities introduced above satisfy the following identities.

\[
\begin{align*}
\Sigma \alpha_{s-1} &= r_{s+1} \alpha_s + r_{s-1} \alpha_{s-2}, \\
\beta_s \alpha_{s-1} - \beta_{s+1} \alpha_{s-2} &= \frac{s(s-1)}{(s-2)(s+1)} \frac{(p_o^2 - 1)^3}{p_o^3}, \\
\Sigma \beta_{s+1} &= r_{s-1} \beta_s + r_{s+1} \beta_{s+2}.
\end{align*}
\] (8.34)

The main observation that allows one to present the recursion relations in a simple form is that the second order difference equations (8.31) are equivalent to the first order difference equations

\[
b_s = \frac{\tilde{z}_{s+1}}{\alpha_s} - \frac{\tilde{z}_s}{\alpha_{s-1}}
\] (8.35)

provided the new function \(b_s\) satisfies

\[
r_{s+1} \alpha_s b_s - r_{s-1} \alpha_{s-2} b_{s-1} = \tilde{\Psi}_s.
\] (8.36)

This can be easily shown using the identities (8.34), which are also useful to verify that the general solution of (8.36) is of the form

\[
b_s = p_o \left( \frac{\beta_{s+2}}{\alpha_s} - \frac{\beta_{s+1}}{\alpha_{s-1}} \right) \left( A_2 - \sum_{r=1}^{s-1} g_r \right),
\] (8.37)

where \(A_2\) is an arbitrary “integration” constant. It is also easy to write down the general solution of (8.35):

\[
\tilde{z}_s = \frac{\alpha_{s-1}}{p_o} \left( \sum_{r=1}^{s-1} f_r - A_1 \right) + p_o \beta_{s+1} \left( A_2 - \sum_{r=1}^{s-1} g_r \right)
\] (8.38)

containing an other arbitrary constant \(A_1\). This is the general solution of (8.31), but we are interested in particular solutions satisfying the requirement \(\lim_{s \to \infty} \tilde{z}_s = 0\). This is a boundary condition (at infinity) and requires

\[
A_2 = \sum_{r=1}^{\infty} g_r.
\] (8.39)

(The other integration constant \(A_1\) remains arbitrary. It is a complicated expression that can be determined from the coupling of the upper nodes to the rest of the AdS/CFT Y-functions.)

For our purposes (comparison to the recursion relations coming from the linearized NLIE) the relations (8.33) are sufficient. After imposing the boundary condition they are uniquely determined and can be rewritten in the form

\[
\tilde{Z}_{1,s+1} = A_s \tilde{Z}_{1,s} + B_s, \quad \tilde{Z}_{3,s+1} = A_s \tilde{Z}_{3,s} + B_s,
\] (8.40)

where

\[
A_s = \frac{r_{s+1} \alpha_s}{r_s \alpha_{s-1}} = \frac{sp_o}{s + 1} \frac{N_s}{N_{s-1}}, \quad B_s = r_{s+1} \alpha_s b_s = -\frac{sp_o}{N_{s-1}} \sum_{r=s}^{\infty} \frac{p_o^{r-s}}{s + 1} N_r \tilde{\Psi}_{r+1}.
\] (8.41)
8.2.2 \( \tilde{z}_{2,s} \) problem

Having found the solution for \( \tilde{z}_{a,s} \) for \( a = 1, 3 \) we now turn to (8.29), which we rewrite as

\[
\Sigma \tilde{Z}_{2,s} = \tilde{Z}_{2,s+1} + \tilde{Z}_{2,s-1} + \Sigma \hat{\Psi}_s + (1 - r_s)(\tilde{z}_{1,s} + \tilde{z}_{3,s}). \tag{8.42}
\]

Again, it is easy to verify that this is equivalent to the first order equation

\[
\tilde{Z}_{2,s+1} = p_o \tilde{Z}_{2,s} + C_s(\tilde{Z}_{1,s} + \tilde{Z}_{3,s}) + D_s, \quad C_s = -\frac{p_o^2}{(s + 1)N_{s-1}}, \tag{8.43}
\]

provided \( D_s \) satisfies

\[
D_s - \frac{1}{p_o} D_{s-1} + \left( p_o + \frac{1}{p_o} \right) \hat{\Psi}_s - \frac{2B_{s-1}}{(s - 1)N_{s-1}} = 0. \tag{8.44}
\]

Similarly to what we saw above for the \( a = 1, 3 \) cases, if we find a “good” solution \( D_s^{(g)} \) (satisfying the boundary condition at infinity), the general solution for (8.44) will be

\[
D_s = D_s^{(g)} + d_o p_o^{-s}, \tag{8.45}
\]

where \( d_o \) is arbitrary, however, imposing the boundary condition requires \( d_o = 0 \) again, making the “good” solution unique. The general solution of (8.43) for \( \tilde{Z}_{2,s} \) contains a term \( \zeta_o p_o^s \) with arbitrary constant \( \zeta_o \), but in the next section we will only need the relation (8.43) itself, which is unique.

8.3 Linearization of the NLIE equations for the upper nodes

Here again it is useful to start from the NLIE equations written for the ratios of functions divided by their asymptotic values. Let us introduce the shorthand notations\(^6\)

\[
\begin{align*}
\hat{b}_{i,r} & = \tilde{b}_{i,s} - \bar{b}_{i,s}, & \hat{B}_{i,r} & = \tilde{B}_{i,s} - \bar{B}_{i,s}, \\
\hat{d}_{i,r} & = \tilde{d}_{i,s} - \bar{d}_{i,s}, & \hat{D}_{i,r} & = \tilde{D}_{i,s} - \bar{D}_{i,s}, \\
\hat{\bar{p}}_i & = \tilde{Y}_i - \bar{Y}_i, & \hat{\bar{Y}}_i & = \tilde{Y}_{i+1} - \bar{Y}_{i+1}.
\end{align*}
\]

The NLIE equations for these variables are of the same form as (7.47).

\[
\begin{align*}
\hat{b}_{i,r} & = \sum_j (\tilde{K}_{bB})_{ij}^{(rw)} \hat{B}_{j,w} + \sum_j (\tilde{K}_{bD})_{ij}^{(rw)} \hat{D}_{j,w} + \sum_j (\tilde{K}_{bY})_{ij}^{(rw)} \hat{\bar{p}}_j, \tag{8.47} \\
\hat{d}_{i,r} & = \sum_j (\tilde{K}_{dB})_{ij}^{(rw)} \hat{B}_{j,w} + \sum_j (\tilde{K}_{dD})_{ij}^{(rw)} \hat{D}_{j,w} + \sum_j (\tilde{K}_{dY})_{ij}^{(rw)} \hat{\bar{p}}_j, \tag{8.48} \\
\hat{\bar{y}}_i & = \sum_j (\tilde{K}_{Yb})_{ij}^{(w)} \hat{B}_{j,w} + \sum_j (\tilde{K}_{YD})_{ij}^{(w)} \hat{D}_{j,w} + \sum_j (\tilde{K}_{YY})_{ij} \hat{\bar{p}}_j. \tag{8.49}
\end{align*}
\]

The kernels occurring in (8.47) and (8.48) are listed in appendix C. New kernels appear in (8.49). These can be calculated by substituting (8.47) and (8.48) into (7.33–7.35).

\(^6\)Note that the index \( s \) is arbitrary, but fixed in our present considerations. To simplify the notation, \( s \) is omitted from most of our formulas.
Let us recall the asymptotic solutions found in subsection 5.4.1:

\[ b_{1,s}^{(3)o} = b_{3,s}^{(3)o} = d_{1,s}^{(3)o} = d_{3,s}^{(3)o} = s, \]
\[ B_{1,s}^{(3)o} = B_{3,s}^{(3)o} = D_{1,s}^{(3)o} = D_{3,s}^{(3)o} = s + 1, \]
\[ b_{2,s}^{(3)o} = B_{2,s}^{(3)o} = -\frac{1}{sC_{2,s}}, \]
\[ d_{2,s}^{(3)o} = D_{2,s}^{(3)o} = -\frac{1}{sC_{1,s}}, \]
\[ b_{1,s}^{(2)o} = B_{1,s}^{(2)o} = \frac{1}{(s - \Delta)C_{2,s-1}}, \]
\[ b_{2,s}^{(2)o} = d_{1,s}^{(2)o} = \Delta - s, \quad B_{2,s}^{(2)o} = D_{1,s}^{(2)o} = \Delta + 1 - s, \]
\[ d_{2,s}^{(2)o} = D_{2,s}^{(2)o} = \frac{1}{(s - \Delta)C_{1,s}}, \]
\[ b_{1,s}^{(1)o} = d_{1,s}^{(1)o} = s - 1 - \Delta, \quad B_{1,s}^{(1)o} = D_{1,s}^{(1)o} = s - \Delta. \]

Here

\[ C_{1,s} = (1 - q)(1 - q^{\prime}) \psi^{[s]}, \quad C_{2,s} = \frac{(1 - q)(1 - q^{\prime})}{q^{\prime}} \frac{1}{\psi^{[s]}}. \]

Both functions are \( \mathcal{O}(\varepsilon) \) and

\[ \Psi = s^2 C_{1,s} C_{2,s} \sim \mathcal{O}(\varepsilon^2). \]

A serious complication as compared to the cases discussed so far is that some of the asymptotic solutions above are \( \mathcal{O}(1/\varepsilon) \). This implies that the leading corrections to the asymptotic solution are (relatively) \( \mathcal{O}(\varepsilon) \). We will call this order 1/2. We are interested in the corrections \( \mathcal{O}(\varepsilon^2) \) (order 1), which are unfortunately NLO corrections in this expansion.

Let us denote our variables generically by \( x \) and write \( X = 1 + x \). (\( x = b_{i,s}^{(r)}, X = B_{i,s}^{(r)}, \text{ etc.} \)) Define the expansion coefficients by

\[ x = x^o(1 + x^{(1/2)} + x^{(1)} + \ldots), \quad x^{(1/2)} \sim \mathcal{O}(\varepsilon), \quad x^{(1)} \sim \mathcal{O}(\varepsilon^2), \]
\[ X = X^o(1 + X^{(1/2)} + X^{(1)} + \ldots), \quad X^{(1/2)} \sim \mathcal{O}(\varepsilon), \quad X^{(1)} \sim \mathcal{O}(\varepsilon^2). \]

The ratio \( x_c = x^o/X^o \) is a constant for those variables for which \( x^o \) is \( \mathcal{O}(1) \) (constant) and \( x_c = 1 \) for the cases where \( x^o \) is \( \mathcal{O}(1/\varepsilon) \). Accordingly,

\[ x^o \sim \mathcal{O}(1) : X^{(1/2)} = x_c x^{(1/2)}, \quad X^{(1)} = x_c x^{(1)}, \]
\[ x^o \sim \mathcal{O}(1/\varepsilon) : X^{(1/2)} = \frac{1}{x^o} + x^{(1/2)}, \quad X^{(1)} = x^{(1)}. \]

Up to \( \mathcal{O}(\varepsilon^2) \),

\[ \dd x - \dd x^o = \dd x^{(1/2)} + \dd x^{(1)} - \dd x^{(2/2)} + \ldots, \quad x^{(2/2)} = x^{(1/2)} x^{(1/2)}. \]

Here we have used the Fourier transforms

\[ x^{(1/2)} \rightarrow \dd x^{(1/2)}, \quad x^{(1)} \rightarrow \dd x^{(1)}, \quad x^{(1/2)} x^{(1/2)} \rightarrow \dd x^{(2/2)} \]
and we will use the analogously defined objects for $X$.

After substituting the expansion (8.60) into the NLIE equations the problem can be solved order by order. We find it convenient to treat $x^{(1/2)}$ as our independent variable at order 1/2 and write

$$x^0 \sim O(1) : X^{(1/2)} = x_c x^{(1/2)}, \quad x^0 \sim O(1/\varepsilon) : X^{(1/2)} = \frac{1}{x^0} + x^{(1/2)},$$  \hspace{1cm} (8.64)

while for order 1 we introduce the independent variable $x^{(1m)}$ and write

$$X^{(1)} - X^{(2/2)} = x_c x^{(1m)}, \quad x^{(1)} - x^{(2/2)} = x^{(1m)} + x^{(1d)},$$  \hspace{1cm} (8.65)

where

$$x^0 \sim O(1) : x^{(1d)} = (x_c - 1) x^{(1/2)} x^{(1/2)},$$  

$$x^0 \sim O(1/\varepsilon) : x^{(1d)} = \frac{1}{x^0} \left( \frac{1}{x^0} \right)' + \left( \frac{1}{x^0} x^{(1/2)} \right)'.$$  \hspace{1cm} (8.66)

Let us start at order 1/2 and introduce the shorthand notation

$$\tilde{a}_{i,r} = \hat{b}_{i,s}^{(1/2)r}, \quad \tilde{A}_{i,r} = \hat{B}_{i,s}^{(1/2)r},$$  

$$\tilde{f}_{i,r} = \hat{d}_{i,s}^{(1/2)r}, \quad \tilde{F}_{i,r} = \hat{D}_{i,s}^{(1/2)r},$$  \hspace{1cm} (8.67)

and

$$c_{1,s} = \overline{C}_{1,s}, \quad c_{2,s} = \overline{C}_{2,s}.$$  \hspace{1cm} (8.68)

At order 1/2 we have to make the substitutions

$$\hat{b}_{i,r} \rightarrow \tilde{a}_{i,r}, \quad \hat{d}_{i,r} \rightarrow \tilde{f}_{i,r}, \quad \hat{B}_{i,r} \rightarrow \tilde{A}_{i,r}, \quad \hat{D}_{i,r} \rightarrow \tilde{F}_{i,r}, \quad \hat{p}_i \rightarrow 0$$  \hspace{1cm} (8.69)

in (8.47)-(8.49) and use the relations

$$\tilde{A}_{1,3} = \frac{s}{s + 1} \tilde{a}_{1,3}, \quad \tilde{F}_{1,3} = \frac{s}{s + 1} \tilde{f}_{1,3},$$

$$\tilde{A}_{2,3} = \tilde{a}_{2,3} - s c_{2,s}, \quad \tilde{F}_{2,3} = \tilde{f}_{2,3} - s c_{1,s},$$

$$\tilde{A}_{3,3} = \frac{s}{s + 1} \tilde{a}_{3,3}, \quad \tilde{F}_{3,3} = \frac{s}{s + 1} \tilde{f}_{3,3},$$

$$\tilde{A}_{1,2} = \tilde{a}_{1,2} + (s - \Delta) p c_{2,s}, \quad \tilde{F}_{1,2} = \frac{s - \Delta}{s - \Delta - 1} \tilde{f}_{1,2},$$

$$\tilde{A}_{2,2} = \frac{s - \Delta}{s - \Delta - 1} \tilde{a}_{2,2}, \quad \tilde{F}_{2,2} = \tilde{f}_{2,2} + (s - \Delta) c_{1,s},$$

$$\tilde{A}_{1,1} = \frac{s - \Delta - 1}{s - \Delta} \tilde{a}_{1,1}, \quad \tilde{F}_{1,1} = \frac{s - \Delta - 1}{s - \Delta} \tilde{f}_{1,1}.$$  \hspace{1cm} (8.70)

We have to solve the set of linear equations obtained by these substitutions from (8.47) and (8.48) for the unknowns $\tilde{a}_{i,r}, \tilde{f}_{i,r}$. The source terms of the equations are proportional to $c_{1,s}$ or $c_{2,s}$. These Fourier space source functions can be characterized by the following properties.

$$c_{1,s}(\omega) = 0, \quad \omega > 0, \quad c_{2,s}(\omega) = 0, \quad \omega < 0$$  \hspace{1cm} (8.71)
The resulting solution is also proportional to $c_{1,s}(\omega)$, $c_{2,s}(\omega)$. We have

$$a_{i,r}(\omega), \ f_{i,r}(\omega) \sim \begin{cases} c_{2,s}(\omega) & \omega > 0, \\ c_{1,s}(\omega) & \omega < 0. \end{cases}$$

Substituting the solution for $\tilde{a}_{i,r}$ and $\tilde{f}_{i,r}$ into (8.43) we find $\hat{Y}_i = 0$, as expected (to this order), since the expansion of the $Y$-functions starts at $\mathcal{O}(\varepsilon^2)$, order 1.

We now turn to order 1. We make the substitutions

$$\hat{B}_{j,w} \rightarrow (b_{j,s}^{(w)})_c \left(\frac{b_{j,s}^{(w)}}{(1m)}\right), \quad \hat{b}_{i,r} \rightarrow \left(\frac{b_{i,s}^{(r)}}{(1m)}\right) + \left(\frac{b_{i,s}^{(1d)}}{(1d)}\right),$$

$$\hat{D}_{j,w} \rightarrow (d_{j,s}^{(w)})_c \left(\frac{d_{j,s}^{(w)}}{(1m)}\right), \quad \hat{a}_{i,r} \rightarrow (d_{i,s}^{(r)}) + (d_{i,s}^{(1d)}),$$

and

$$\hat{p}_j \rightarrow \tilde{Z}_{j,s}, \quad \hat{Y}_i \rightarrow \tilde{Z}_{i,s+1}. \quad (8.75)$$

Recall that we treat the variables of type $x_{(1m)}$ as our independent variables and solve (8.47) and (8.48) for them in terms of the sources: the variables $\tilde{Z}_{j,s}$ and the variables of type $x_{(1d)}$. This solution is then substituted into (8.43) and the result is that $\tilde{Z}_{i,s+1}$ is expressed in terms of the sources. Due to the linearity of the problem we can consider the two sources separately.

First we solve the problem corresponding to the sources $\tilde{Z}_{j,s}$. Substituting this part of the solution to (8.43) we find that the homogeneous parts (terms proportional to $A_s$ and $p_o$, $C_s$) of (8.40) and (8.43) are reproduced.

The case of the inhomogeneous terms is much more complicated. Here $x_{(1d)}$ terms act as sources for the solution. These are known from the order 1/2 solution. More precisely, the functions

$$a_{i,r} = b_{i,s}^{(r)(1/2)}, \quad f_{i,r} = d_{i,s}^{(r)(1/2)} \quad (8.76)$$

can be considered as known from the order 1/2 solution, since the order 1/2 solution is given in terms of the Fourier transform of their derivatives:

$$\tilde{a}_{i,r} = \tilde{b}_{i,r}, \quad \tilde{f}_{i,r} = \tilde{f}_{i,r}' \quad (8.77)$$

The inhomogeneous case sources are given in terms of the functions (8.76):

$$b_{1,s}^{(3)(1d)} = -\frac{1}{s+1}a_{1,3}a_{1,3}' \quad (8.78)$$

$$b_{1,s}^{(2)(1d)} = (a_{2,3} - sC_{2,s})(a_{2,3} - sC_{2,s})' - a_{2,3}a_{2,3}' \quad (8.79)$$

$$b_{1,s}^{(3)(1d)} = \frac{-1}{s+1}a_{3,3}a_{3,3}' \quad (8.80)$$

$$b_{1,s}^{(2)(1d)} = (a_{1,2} + (s - \Delta)C_{2,s-1})(a_{1,2} + (s - \Delta)C_{2,s-1})' - a_{1,2}a_{1,2}' \quad (8.81)$$

$$b_{2,s}^{(2)(1d)} = \frac{1}{s-\Delta}a_{2,2}a_{2,2}' \quad (8.82)$$

$$b_{1,s}^{(1)(1d)} = -\frac{1}{s-\Delta}a_{1,1}a_{1,1}' \quad (8.83)$$
We can introduce the notation for the contributions:

\[ d_{1,s}^{(3)(1d)} = -\frac{1}{s + 1} f_{1,3} f'_{1,3}, \]  
\[ d_{2,s}^{(3)(1d)} = (f_{2,3} - s C_{1,s})(f_{2,3} - s C_{1,s})' - f_{2,3} f'_{2,3}, \]  
\[ d_{3,s}^{(3)(1d)} = -\frac{1}{s + 1} f_{3,3} f'_{3,3}, \]  
\[ d_{1,s}^{(2)(1d)} = \frac{1}{s - \Delta - 1} f_{1,2} f'_{1,2}, \]  
\[ d_{2,s}^{(2)(1d)} = (f_{2,2} + (s - \Delta) C_{1,s})(f_{2,2} + (s - \Delta) C_{1,s})' - f_{2,2} f'_{2,2}, \]  
\[ d_{1,s}^{(1)(1d)} = -\frac{1}{s - \Delta} f_{1,1} f'_{1,1}. \]  

We have to compute the Fourier transform of the functions listed above. The generic structure of this calculation is as follows. Given the Fourier transform of the derivative of some function \( \mathcal{F} \),

\[ \tilde{\mathcal{F}}_1(\omega) = \int_{-\infty}^{\infty} du e^{i\omega u} \mathcal{F}'(u) = f_p(\omega) + f_m(\omega), \]  

where

\[ f_p(\omega) = 0, \quad \omega < 0, \quad f_m(\omega) = 0, \quad \omega > 0, \]  

find the Fourier transform

\[ \tilde{\mathcal{F}}_2(\omega) = \int_{-\infty}^{\infty} du e^{i\omega u} \mathcal{F}(u) \mathcal{F}'(u). \]  

In all cases

\[ f_p(\omega) \sim c_{2,s}(\omega) \sim a_2(\omega), \quad f_m(\omega) \sim c_{1,s}(\omega) \sim a_1(\omega). \]  

\( \tilde{\mathcal{F}}_2(\omega) \) is given by the integrals

\[ \left\{ \begin{array}{l} \frac{i \omega}{4 \pi} \int_{0}^{\omega} \frac{dv}{\nu(\omega - \nu)} f_p(\nu) f_p(\omega - \nu) + \frac{i \omega}{2 \pi} \int_{-\infty}^{0} \frac{dv}{\nu(\omega - \nu)} f_m(\nu) f_p(\omega - \nu), \quad \omega > 0, \\ \frac{i \omega}{4 \pi} \int_{-\omega}^{0} \frac{dv}{\nu(\omega - \nu)} f_m(\nu) f_m(\omega - \nu) + \frac{i \omega}{2 \pi} \int_{0}^{\infty} \frac{dv}{\nu(\omega - \nu)} f_p(\nu) f_m(\omega - \nu), \quad \omega < 0. \end{array} \right. \]  

Thanks to the linearity of the problem, it is sufficient to solve the problem for the integrands as sources, moreover for each type of integrand separately. We have four types of contributions:

\[ f_p f_p \text{ type terms } \sim c_{2,s}(\nu)c_{2,s}(\omega - \nu) \]  
\[ f_m f_p \text{ type terms } \sim c_{1,s}(\nu)c_{2,s}(\omega - \nu) \]  
\[ f_m f_m \text{ type terms } \sim c_{1,s}(\nu)c_{1,s}(\omega - \nu) \]  
\[ f_p f_m \text{ type terms } \sim c_{2,s}(\nu)c_{1,s}(\omega - \nu) \]

We can introduce the notation for the \( f_m f_p \) part of the function \( G(\omega) \)

\[ G_{(mp)}(\omega) = \frac{i \omega}{2 \pi} \int_{-\infty}^{0} \frac{dv}{\nu(\omega - \nu)} G^{(1)}_{(mp)}(\omega, \nu)c_{1,s}(\nu)c_{2,s}(\omega - \nu) \]  

\[ G^{(1)}_{(mp)}(\omega, \nu) = \tilde{\mathcal{F}}_{1}(\nu) \tilde{\mathcal{F}}_{2}(\omega - \nu) \]  

\[ f_p f_p \text{ type terms } \sim c_{1,s}(\nu)c_{1,s}(\omega - \nu) \]  
\[ f_m f_p \text{ type terms } \sim c_{2,s}(\nu)c_{1,s}(\omega - \nu) \]  
\[ f_m f_m \text{ type terms } \sim c_{1,s}(\nu)c_{1,s}(\omega - \nu) \]  
\[ f_p f_m \text{ type terms } \sim c_{2,s}(\nu)c_{1,s}(\omega - \nu) \]
and similarly for \( G_{(pm)}(\omega) \sim G^{(I)}_{(pm)}(\omega, \nu) \), etc.

The formula for the Fourier transform of the derivative of \( \Psi_s \) has a similar structure:

\[
-i\omega \tilde{\Psi}_s(\omega) = \begin{cases}
\frac{is^2\omega}{2\pi} \int_0^\infty \frac{d\nu}{\nu(\omega - \nu)} c_{1,s}(\nu)c_{2,s}(\omega - \nu), & \omega > 0, \\
\frac{is^2\omega}{2\pi} \int_0^\infty \frac{d\nu}{\nu(\omega - \nu)} c_{2,s}(\nu)c_{1,s}(\omega - \nu), & \omega < 0.
\end{cases}
\] (8.98)

Let us concentrate on the \( s \) dependence of the integrand in the above formula:

\[
\text{integrand of } \tilde{\Psi}_s(\omega) \sim \begin{cases}
s^2 \left( \frac{k^2}{p} \right)^s a_1(\nu)a_2(\omega - \nu), & \omega > 0, \\
s^2 \left( \frac{p}{k^2} \right)^s a_2(\nu)a_1(\omega - \nu), & \omega < 0.
\end{cases}
\] (8.99)

Here we introduced the notation

\[ k = e^{\frac{s}{s}}. \] (8.100)

Using the simple \( s \)-dependence of the integrand, the summation in (8.41) can be performed and we find for the corresponding integrands

\[ B_{s(mp)}^{(I)}(\omega, \nu) = B_{s(o)}(\omega, \nu), \] (8.101)

where

\[
B_{s(o)} = -\frac{s^2 p^2 k_o^2}{[(s-1)p_o^2 - s - 1](1 - p_o^2 k_o^2)^3} \left\{ p_o^2 s(s - 1) p_o^4 k_o^4 - 2(s^2 - 1) p_o^2 k_o^2 + s(s + 1) - [s(s + 1) p_o^4 k_o^4 - 2s(s + 2) p_o^2 k_o^2 + (s + 1)(s + 2)] \right\}
\] (8.102)

and \( k_o = k \). \( B_{s(mp)}^{(I)} \) is given by the same formula, but with \( k_o = 1/k \).

If we now solve the inhomogeneous part of the order 1 problem at the level of integrands and substitute the solution to (8.49) we find that all \( f_p f_p \) and \( f_m f_m \) type contributions vanish and the \( f_m f_p \) (\( \omega > 0 \)) and \( f_p f_m \) (\( \omega < 0 \)) contributions to \( \tilde{Z}_{1,s+1} \) and \( \tilde{Z}_{3,s+1} \) can be given by integrands that are precisely the same as (8.102). Analogously, the \( f_m f_p \) and \( f_p f_m \) type contributions to the integrand of \( \tilde{Z}_{2,s+1} \) are of the form

\[
D_{s(mp)}^{(I)}(\omega, \nu) = D_s(o) \quad \text{with} \quad k_o = k,
\]

\[
D_{s(mp)}^{(I)}(\omega, \nu) = D_s(o) \quad \text{with} \quad k_o = 1/k,
\] (8.103)

where

\[
D_s(o) = \frac{p_o k_o^2}{(s-1)p_o^2 - s - 1}(1 - p_o^2 k_o^2)^3 \left\{ k_o^4 p_o^8 s^3 - k_o^4 p_o^8 s^2 - 2k_o^4 p_o^6 s^2 - k_o^4 p_o^4 s^3 \right.
\]

\[
- k_o^2 p_o^4 s^2 - 2k_o^2 p_o^2 s^3 + 3k_o^2 p_o^4 s^2 - k_o^2 p_o^6 + 4k_o^2 p_o^4 s^2 + 2k_o^2 p_o^6 + 2k_o^2 p_o^2 s^3 + 4k_o^2 p_o^2 s^2
\]

\[
+ k_o^2 p_o^2 s - k_o^2 p_o^4 + p_o^4 s^3 + p_o^4 s^2 - p_o^4 s - p_o^2 - 2p_o^2 s - s^3 - 3s^2 - 3s - 1 \}
\] (8.104)

and the coefficient functions satisfy the recursion relation

\[
D_s(o) - \frac{D_{s-1}(o)}{p_o^2 k_o^2} + \left( p_o + \frac{1}{p_o} \right) s^2 - \frac{2B_{s-1}(o)}{(s-1)p_o k_o^2[(s-1)p_o^2 - s - 1]} = 0,
\] (8.105)
which is equivalent to (8.44), taking into account the \((p_0 k^2_o)^s\) factor in \(c_{1,s} c_{2,s}\). Note that the arbitrary constant \(\Delta\) is absent from the final formulas (8.102) and (8.104), which is an important check on the overall consistency of our results.

9. Summary

In this paper we derived an alternative finite NLIE description for the AdS/CFT spectral problem, which we call hybrid-NLIE. The term hybrid-NLIE was first used in [54] referring to the property of the equations that semi-infinite parts of the infinite Y-system are resummed by appropriate NLIE functions which are coupled to the rest (unsummed part) of the Y-functions. Our equations differ in various aspects from the recently published finite FiNLIE [61] formulation of the spectral problem.

The main differences (apart from the obvious differences in the derivation as well as the construction of NLIE unknowns) are as follows: our NLIE is defined on the mirror sheet, while in ref. [61] also the “magic sheet” is important. Our equations have a more conventional form than those of ref. [61] but at the price of having more variables than in the FiNLIE. The unknowns of [61] are discontinuities along (short) cuts of the “magic sheet”, while our unknowns are complex functions on the whole real line. In [61] the derivation is an appropriate generalization of the methods of [58], [59] and it is based on the Wronskian solutions of the T-system [41], [74] on the “magic sheet”. In our derivation we remain on the mirror sheet and work in the spirit of [45], the very first NLIE paper in the literature, and its generalizations [47], [53], [54], where the NLIE originates from the TQ-relations of the integrable model under consideration. Here because of the involved nesting structure of the problem the derivation is based on a set of hierarchical Bäcklund equations of the corresponding T-system.

The starting point of the derivation of the hybrid-NLIE was the quasi-local formulation of the mirror TBA [64]. In this reformulation of the simplified mirror TBA equations [3] two SU(2) and an SU(4) type semi-infinite sub Y-systems are coupled by the quasi-local TBA equations to the central Y-functions. Our approach to get the hybrid-NLIE description was to transform the SU(\(N\)) type semi-infinite sub Y-systems into hybrid-NLIEs. This required to derive two SU(2) type and an SU(4) type hybrid-NLIEs. The SU(2) type hybrid-NLIEs have already been derived in [66], and in this paper we completed the derivation by constructing the missing SU(4) type hybrid-NLIE. The derivation of these hybrid-NLIEs proceeds in three steps: finding the proper set of unknown variables, the functional relations they satisfy and their analytic properties.

To each semi-infinite SU(\(N\)) type sub Y-system of the whole AdS/CFT Y-system, there corresponds an infinite SU(\(N\)) type sub T-system. The rank (w.r.t. SU(\(N\))) of these sub T-systems can be reduced via subsequent Bäcklund transformations, connecting the T-systems corresponding to neighboring levels of the nesting procedure. The unknown functions are constructed from the Bäcklund equations corresponding to this nested hierarchy. Every unknown is a simple multiplicative expression composed of T-functions of neighboring levels and their inverses. This multiplicative structure allowed us to derive the functional relations connecting the NLIE unknowns and the Y-functions at each level.
of the nesting. Another advantage of this construction is that analyticity information is available for the NLIE unknowns if the analytic properties of the T-functions are known at all levels of the nesting.

The analyticity information on the nested T-functions consists of 2 pieces. First of all, in order to be able to derive the NLIE for a given state, we have to know the analyticity domains where the T-functions are free of discontinuities and are close to the asymptotic solution. Furthermore we need to know (qualitatively) the positions of their point-like singularities (poles and zeroes). The information on the analyticity domains were extracted from the known analyticity properties of Y-functions at the highest level of the nesting and the explicit construction of Bäcklund transformations allowed us to determine the analyticity domains at lower levels of the nesting as well. The qualitative information on point-like singularities is encoded in the asymptotic solution that was calculated in section 5 for an arbitrary state in the undeformed model and for the ground state of the $\gamma$-deformed theory.

The knowledge of the functional relations satisfied by the NLIE unknowns, of their analyticity domains together with their qualitative singularity structure given by the asymptotic solution makes it possible to determine the form of the hybrid-NLIE of AdS/CFT for any excited state of the theory in a quite straightforward manner\textsuperscript{7}. However, for our present purposes in this paper it was sufficient to give the explicit form of the hybrid-NLIE only for the simplest nontrivial state, namely the ground state of the $\gamma$-deformed theory. This state has non-zero energy, has the simplest singularity structure and therefore it is ideal for describing the structure of the hybrid-NLIE and to test it analytically in the large volume (small coupling) limit.

The hybrid-NLIE equations can be grouped into four sets of equations. Three of them correspond to the left-wing and right-wing SU(2) type hybrid-sub-NLIEs and the SU(4) type hybrid-sub-NLIE. These three hybrid-sub-NLIEs are joined together to a closed set of equations by the fourth group of equations: the so-called central node quasi-local TBA equations. We list below all four groups of equations specified for the the ground state of the $\gamma$-deformed theory.

The first two groups of equations are formed by the left and right wing SU(2) hybrid-sub-NLIEs. These equations tell us how the SU(2) NLIE functions couple to each other and the rest of the Y-functions. They are obtained from (7.18,7.19) with $m = 1$ to get the truncation with a minimal number of unknowns:

\begin{align}
\ln b_1^{(\alpha)} &= H \ln \mathcal{B}_1^{(\alpha)} - H^{-2\alpha} \ln \mathcal{D}_1^{(\alpha)} + s^{[1-\eta]} \ln (1 + Y_{1|w}^{(\alpha)}) + C b_1^{(\alpha)}, \\
\ln d_1^{(\alpha)} &= H \ln \mathcal{D}_1^{(\alpha)} - H^{2\eta} \ln \mathcal{B}_1^{(\alpha)} + s^{[\alpha-1]} \ln (1 + Y_{1|w}^{(\alpha)}) + C d_1^{(\alpha)},
\end{align}

\begin{equation}
Y_{1|w}^{(\alpha)} = \exp \left\{ s^{[\eta-1]} \ln \mathcal{B}_1^{(\alpha)} + s^{[1-\eta]} \ln \mathcal{D}_1^{(\alpha)} + \ln \left[ \frac{1 - Y_{1|w}^{(\alpha)}}{1 + Y_{1|w}^{(\alpha)}} \right] \ast s \right\}, \quad \alpha = \pm,
\end{equation}

\textsuperscript{7}For excited states the hybrid-NLIE has to be supplemented by quantization conditions. The most important of these are the exact Bethe equations, but they pose no extra problem since only the central Y-functions appear in the formulas.
where the Fourier form of the kernel $H(u)$ is given by (7.13), $0 < \eta < 1$ is a small shift parameter, and the $+$ and $-$ values of the index $(a)$ refer to the right and left wing SU(2) NLIEs, respectively.

$b_1^{(a)}, \delta_1^{(a)}$ together with $Y_{1w}^{(a)}$ constitute the set of unknowns of the SU(2) type hybrid-sub-NLIEs. We recall that $\mathfrak{C}^{(a)}_1 = 1 + b_1^{(a)}, \mathfrak{D}^{(a)}_1 = 1 + b_1^{(a)}$. The integration constants given in terms of the deformation parameters of the $\gamma$-deformed theory are as follows: $Cd_1^{(+)} = -Cb_1^{(+)}, Cd_1^{(-)} = -Cb_1^{(-)}.$

The next group of equations form the upper SU(4) hybrid-NLIE. They are given by (7.51) and (7.53) taken at $s = 3$:

\[
\ln b_{\alpha,3}^{(r)} = \sum_{p=1}^{3} \sum_{a'=1}^{p} (G_{bb})_{aa'}^{(rp)} \star \ln b_{a',3}^{(p)} + \sum_{p=1}^{3} \sum_{a'=1}^{p} (G_{bb})_{aa'}^{(rp)} \star \ln b_{a',3}^{(p)} + \sum_{p=1}^{3} \sum_{a'=1}^{p} (G_{dd})_{aa'}^{(rp)} \star \ln \mathfrak{D}_{a',3}^{(p)} + \sum_{p=1}^{3} \sum_{a'=1}^{p} (G_{dd})_{aa'}^{(rp)} \star \ln \mathfrak{D}_{a',3}^{(p)} + \sum_{p=1}^{3} \sum_{a'=1}^{p} (G_{dd})_{aa'}^{(rp)} \star \ln \mathfrak{D}_{a',3}^{(p)}
\]

\[
\ln \mathfrak{D}_{a,3}^{(r)} = \sum_{p=1}^{3} \sum_{a'=1}^{p} (G_{dd})_{aa'}^{(rp)} \star \ln \mathfrak{D}_{a',3}^{(p)} + \sum_{p=1}^{3} \sum_{a'=1}^{p} (G_{dd})_{aa'}^{(rp)} \star \ln \mathfrak{D}_{a',3}^{(p)} + \sum_{p=1}^{3} \sum_{a'=1}^{p} (G_{dd})_{aa'}^{(rp)} \star \ln \mathfrak{D}_{a',3}^{(p)} + \sum_{p=1}^{3} \sum_{a'=1}^{p} (G_{dd})_{aa'}^{(rp)} \star \ln \mathfrak{D}_{a',3}^{(p)} + \sum_{p=1}^{3} \sum_{a'=1}^{p} (G_{dd})_{aa'}^{(rp)} \star \ln \mathfrak{D}_{a',3}^{(p)}
\]

\[
\ln \eta_{a,3} = s^{[1+\epsilon_a-\gamma_{a}]} \star \ln \mathfrak{D}_{a,3}^{(3)} + s^{[1+\epsilon_a-\gamma_{a}]} \star \ln \mathfrak{D}_{a,3}^{(3)} + s^{[1+\epsilon_a-\gamma_{a}]} \star \ln \mathfrak{D}_{a,3}^{(3)} + s^{[1+\epsilon_a-\gamma_{a}]} \star \ln \mathfrak{D}_{a,3}^{(3)} + s^{[1+\epsilon_a-\gamma_{a}]} \star \ln \mathfrak{D}_{a,3}^{(3)}
\]

Here the first two equations are for the 6 + 6 NLIE variables $b_{\alpha,3}^{(r)}$ and $\mathfrak{D}_{a,3}^{(r)}, r = 1, 2, 3, a = 1, \ldots, r$ and the kernels appearing here are given in appendix C. In the third equation $\mathfrak{D}_{a,2}$ are given by (7.14) taken at $s = 2$ and the additional variables are $(\eta_{1,3}, \eta_{2,3}, \eta_{3,3}) = (\gamma_{2}^{(r)}[\epsilon_1], 1/3, \gamma_{2}^{(r)}[\epsilon_1])$. Here it is understood that terms with $a$-type indices “out of range” are omitted. Recall that $\mathfrak{B}_{a,3}^{(p)} = 1 + \mathfrak{b}_{a,3}^{(p)}$ etc.

The last group of equations is given by the central part of the quasi-local TBA equations:

\[
\ln Y_2 = -s \star \ln \left(1 + \frac{1}{Y_1}\right) - s^{[1+\epsilon_2]} \star \ln \mathfrak{D}_{2,3} + \sum_{a=\pm} \ln \left(1 + \frac{1}{Y_{1w}^{(a)}}\right) \star s,
\]

\[
\ln Y_{1w}^{(+)} = s^{[1+\epsilon_1]} \star \ln \mathfrak{D}_{1,3} + s \star \ln (1 + Y_2) + \ln \frac{1 - Y_{1w}^{(+)}}{1 - Y_{1w}^{(+)}},
\]

\[
\ln Y_{1w}^{(-)} = s^{[1+\epsilon_3]} \star \ln \mathfrak{D}_{3,3} - s \star \ln (1 + Y_2) + \ln \frac{1 - Y_{1w}^{(-)}}{1 - Y_{1w}^{(-)}},
\]

\[
\frac{Y_{1w}^{(a)}}{Y_{1w}^{(a)}} = \exp \{-L_1 \star K_1y - \Omega(K_2y)\}
\]
\[ Y_+^{(a)} Y_-^{(a)} = \exp \left\{ 2 \ln \left[ \frac{1 + Y_1^{(a)}}{1 + Y_1^{(a)}} \right] * s + L_1 * \left[ -K_1 + 2K_{xv}^{11} * s \right] \right\} \]
\[ - \Omega(K_Q) + 2 \Omega(K_{xv}^{Q1} * s) \], \quad (9.9) \]

\[ \ln Y_1 = - L_1 \tilde{E}_1 + \sum_{\alpha=\pm} r_1^{(a)} * s * K_{y1} \]
\[ - \sum_{\alpha=\pm} \left( \ln \left[ \frac{1 - Y_1^{(a)}}{1 - Y_1^{(a)}} \right] * s * K_{xv}^{11} + \mathcal{L}_-^{(a)} * K_{y1}^{01} + \mathcal{L}_+^{(a)} * K_{y1}^{11} \right) \]
\[ + L_1 * K_{s(2)}^{11} + \Omega(K_{s(2)}^{Q1}) + 2 \Omega(s * K_{xv}^{Q-1}) \], \quad (9.10) \]

where \( \Omega(K_Q) \) is a linear functional of a vector kernel \( K_Q \) given by (2.3) and depends also linearly on the logarithmic expressions (2.2) of the central Y-functions: \( Y_1, Y_2, Y_{1|vw}^{(a)}, Y_{1|w}^{(a)}, Y_{\pm}^{(a)} \).

The explicit forms of the kernels of this group of equations are listed in appendix A.

The ground state problem consists of these four sets of equations supplemented by requirements on the large \( u \) behavior, the boundary conditions (2.13) and (2.14). This problem turned out to be an ideal analytical testing ground for the correctness of the NLIE equations. In section 8 we have shown that the hybrid-NLIE results for the wrapping corrections up to 2nd order are in agreement with those of earlier TBA computations [27].

The equations for excited states and their numerical solution will be discussed in a future publication.

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A. Notations and TBA kernels

In this paper we adopted the definitions and conventions of ref. [12]. For completeness, in this appendix we collect these definitions and give a list of all kernel functions used in the paper.

We use the notation \( f^{\pm}(u) = f(u \pm \frac{i}{g}) \) for any function \( f \) and in general \( f^{[a]}(u) = f(u + \frac{i}{g}a) \). We will also use \( w^{\pm} = w \pm \frac{i}{g} \) for \( w \) some parameter.

Most of the kernels and also the asymptotic solution of the Y-system is expressed in terms of the function \( x(u) \):

\[ x(u) = \frac{1}{2} (u - i \sqrt{4 - u^2}), \quad \text{Im} \ x(u) < 0, \quad (A.1) \]

which maps the \( u \)-plane with cuts \([-\infty, -2] \cup [2, \infty]\) onto the physical region of the mirror theory, and the function \( x_s(u) \)

\[ x_s(u) = \frac{u}{2} \left( 1 + \sqrt{1 - \frac{4}{u^2}} \right), \quad |x_s(u)| \geq 1, \quad (A.2) \]
which maps the $u$-plane with the short cut $[-2,2]$ onto the physical region of the string theory. Both functions satisfy the identity $x(u) + \frac{1}{x(u)} = u$ and they are related by $x(u) = x_s(u)$, and $x(u) = 1/x_s(u)$ in the lower and upper halves of the complex plane respectively.

The momentum $\tilde{p}_Q$ and the energy $\tilde{E}_Q$ of a mirror $Q$-particle are expressed in terms of $x(u)$ as follows

$$\tilde{p}_Q = gx(u - \frac{i}{g}Q) - gx(u + \frac{i}{g}Q) + iQ, \quad \tilde{E}_Q = \log \frac{x(u - \frac{i}{g}Q)}{x(u + \frac{i}{g}Q)}.$$  \hspace{1cm} (A.3)$$

Two different types of convolutions appear in the quasi-local TBA equations. These are:

$$f * K(v) \equiv \int_{-\infty}^{\infty} du f(u) K(u,v), \quad f * \hat{K}(v) \equiv \int_{-2}^{2} du f(u) K(u,v).$$

In addition, we also use the standard definition for convolutions with kernel functions depending on rapidity differences only:

$$k * f(v) = \int_{-\infty}^{\infty} du k(v-u)f(u).$$ \hspace{1cm} (A.4)$$

This definition is convenient because it is equivalent to ordinary multiplication in Fourier space. It was used for example in (4.24) and also for the NLIE equations in section 7.

The kernels and kernel vectors entering the mirror TBA equations can be grouped into two sets. The kernels from the first group are functions of only the difference of the rapidities, while kernels form the other group are not of difference type.

We start with listing kernels depending on a single variable:

$$s(u) = \frac{1}{2\pi i} \frac{d}{du} \log t^-(u) = \frac{g}{4 \cosh \frac{\pi u}{2}}, \quad t(u) = \tanh\left[\frac{\pi g}{4} u\right],$$

$$K_Q(u) = \frac{1}{2\pi i} \frac{d}{du} \log S_Q(u) = \frac{1}{\pi} \frac{gQ}{Q^2 + g^2u^2}, \quad S_Q(u) = \frac{u - \frac{iQ}{g}}{u + \frac{iQ}{g}},$$

$$K_{MN}(u) = \frac{1}{2\pi i} \frac{d}{du} \log S_{MN}(u) = K_{M+N}(u) + K_{N-M}(u) + 2 \sum_{j=1}^{M-1} K_{N-M+2j}(u),$$

$$S_{MN}(u) = S_{M+N}(u)S_{N-M}(u) \prod_{j=1}^{M-1} S_{N-M+2j}(u)^2 = S_{NM}(u).$$ \hspace{1cm} (A.5)$$

The fundamental building block of kernels which are not of difference type is:

$$K(u,v) = \frac{1}{2\pi i} \frac{d}{du} \log S(u,v) = \frac{1}{2\pi i} \frac{\sqrt{4 - u^2}}{\sqrt{4 - v^2}} \frac{1}{u-v}, \quad S(u,v) = \frac{x(u) - x(v)}{x(u)x(v) - 1}. \hspace{1cm} (A.6)$$

Using the kernels $K(u,v)$ and $K_Q(u,v)$ it is possible to define a series of kernels which are connected to the fermionic $Y^{(a)}_{\pm}$-functions. They are:

$$K_{Qy}(u,v) = K(u - \frac{i}{g}Q,v) - K(u + \frac{i}{g}Q,v),$$ \hspace{1cm} (A.7)$$

$$K^{Qy}_{\pm}(u,v) = \frac{1}{2} \left( K_Q(u-v) \pm K_Q(u,v) \right).$$ \hspace{1cm} (A.8)$$
and
\[ K_{yQ}(u, v) = K(u, v + \frac{i}{g}Q) - K(u, v - \frac{i}{g}Q), \tag{A.9} \]
\[ K_{xQ}^{\pm}(u, v) = \frac{1}{2} \left( K_{yQ}(u, v) \mp K_Q(u - v) \right). \tag{A.10} \]

The kernels entering the right hand sides of (2.11),(2.12) are
\[ K_{xv}^{QM}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{xv}^{QM}(u, v), \]
\[ S_{xv}^{QM}(u, v) = \frac{k(u - i\frac{Q}{g}) - k(v + i\frac{M}{g})}{k(u + i\frac{Q}{g}) - k(v - i\frac{M}{g})} \frac{k(u - i\frac{Q}{g}) - k(v + i\frac{M}{g})}{k(u + i\frac{Q}{g}) - k(v - i\frac{M}{g})} \]
\[ \times \prod_{j=1}^{M-1} \frac{u - v - i\frac{j}{g}(Q - M + 2j)}{u - v + i\frac{j}{g}(Q - M + 2j)} \tag{A.11} \]
and
\[ K_{vwx}^{QM}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{vwx}^{QM}(u, v), \]
\[ S_{vwx}^{QM}(u, v) = \frac{k(u - i\frac{Q}{g}) - k(v + i\frac{M}{g})}{k(u + i\frac{Q}{g}) - k(v - i\frac{M}{g})} \frac{k(u - i\frac{Q}{g}) - k(v + i\frac{M}{g})}{k(u + i\frac{Q}{g}) - k(v - i\frac{M}{g})} \]
\[ \times \prod_{j=1}^{Q-1} \frac{u - v - i\frac{j}{g}(M - Q + 2j)}{u - v + i\frac{j}{g}(M - Q + 2j)}. \tag{A.12} \]

The equations for the momentum carrying node (2.12) contain the dressing phase, an important building block of the \(\mathfrak{sl}(2)\) S-matrix of the model [2]. It is of the form
\[ S_{\mathfrak{sl}(2)}^{QM}(u, v) = S_{QM}(u - v)^{-1} \Sigma_{QM}(u, v)^{-2}, \tag{A.13} \]
where \(\Sigma_{QM}\) is the improved dressing factor [71]. The corresponding \(\mathfrak{sl}(2)\) and dressing kernels are defined in the usual way
\[ K_{\mathfrak{sl}(2)}^{QM}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log S_{\mathfrak{sl}(2)}^{QM}(u, v), \quad K_{\Sigma_{QM}}^{QM}(u, v) = \frac{1}{2\pi i} \frac{d}{du} \log \Sigma_{QM}(u, v). \tag{A.14} \]

The asymptotic solution and the source terms in the excited state generalization of the TBA equations involve the \(\mathfrak{sl}(2)\) S-matrix analytically continued to the physical region in the first argument.
\[ S_{\mathfrak{sl}(2)}^{1,M}(u, v) = \frac{1}{S_{1M}(u - v)^{-}\Sigma_{1,M}(u, v)^{-2}}. \]

Explicit expressions for the improved dressing factors \(\Sigma_{QM}(u, v)\) and \(\Sigma_{1,M}(u, v)\) can be found in section 6 of ref. [71].

In the quasi-local TBA formulation the vector \(\delta K_Q\) defined by (2.1) must be known. Since in the expression (2.3) and thus in the quasi-local TBA formulation \(\delta K_Q\) with \(Q \geq 2\) appear only, we list them for this set of the indexes. They are given by:
\[ \delta K_Q = 0, \quad \delta K_{Qy} = 0, \quad \delta K_{xQ}^{Q1} = 0, \quad Q \geq 2, \tag{A.15} \]
\[ \delta(s * K_{vwx}^{Q-1,1}) = \delta Q, s * s * K_{y1}, \quad \delta K_{\mathfrak{sl}(2)}^{Q1} = -\delta Q, s, \quad Q \geq 2. \tag{A.16} \]
Using \((A.17, A.16)\) and \((2.3)\) all the \(\Omega(K_Q)\) terms can be explicitly evaluated. The substitution of these expressions into the equations \((2.5, 2.12)\) completes the quasi-local form of the mirror TBA equations.

### B. Asymptotic solution

The asymptotic solutions of the vertical SU(4) and the horizontal SU(2) parts of the T-hook were given at all levels of the nesting in section 5 and 6. The solution is built from the asymptotic Q-functions of the left (L) and right (R) SU(2|2) T-systems to which the T-hook T-system splits in the asymptotic limit. In this appendix we collect the T-functions and the basic building elements such as the Q-functions of the relevant SU(2|2) fat-hook solution \([15]\). We consider \(N\) magnon states with magnon rapidities \(u_j\) and introduce the following functions:

\[
R_m(u) = \prod_{j=1}^{N} \frac{x(u) - x_j^+}{(x_j^+)^\frac{1}{2}}, \quad B_m(u) = \prod_{j=1}^{N} \frac{1}{x(u) - x_j^+} \quad (B.1)
\]

\[
R_p(u) = \prod_{j=1}^{N} \frac{x(u) - x_j^-}{(x_j^-)^\frac{1}{2}}, \quad B_p(u) = \prod_{j=1}^{N} \frac{1}{x(u) - x_j^-} \quad (B.2)
\]

where \(x_j^\pm = x_s(u_j \pm \frac{i}{g})\). These functions satisfy the relation

\[
R_m^+(u) B_m^+(u) = R_p^-(u) B_p^-(u) = (-1)^N Q(u), \quad (B.3)
\]

with \(Q(u) = \prod_{j=1}^{N} (u - u_j)\). For general states auxiliary Bethe roots will also appear in the formulae. To take into account their contribution as well, we need to introduce the following functions:

\[
R_l(u) = \prod_{j=1}^{K_l} \frac{x(u) - y_{l,j}}{(y_{l,j})^\frac{1}{2}}, \quad B_l(u) = \prod_{j=1}^{K_l} \frac{1}{x(u) - y_{l,j}}, \quad Q_l(u) = \prod_{j=1}^{K_l} (u - u_{l,j}), \quad l = 1, 2, 3, \quad (B.4)
\]

where \(y_{l,j} = x(u_{l,j})\) and they satisfy the relation

\[
R_l(u) B_l(u) = (-1)^{K_l} Q_l(u), \quad l = 1, 2, 3. \quad (B.5)
\]

The sets \(\{u_{l,j}\}_{l=1,2,3}\) form the 3 family of Bethe roots corresponding to the 3 levels of the SU(2|2) nested Bethe Ansatz. Choosing the \(\mathfrak{sl}(2)\) grading for the reference state they satisfy the asymptotic SU(2|2) Bethe equations \([15]\):

\[
\frac{Q_2^-(u_{1,j})}{Q_2^+(u_{1,j})} B_p(u_{1,j}) B_m(u_{1,j}) = 1, \quad j = 1, \ldots, K_1 \quad (B.6)
\]

\[
\frac{Q_2^+(u_{2,j})}{Q_2^-(u_{2,j})} Q_1^-(u_{2,j}) Q_3^+(u_{2,j}) = -1, \quad j = 1, \ldots, K_2 \quad (B.7)
\]

\[
\frac{Q_2^-(u_{3,j})}{Q_2^+(u_{3,j})} Q_2^+(u_{3,j}) R_p(u_{3,j}) = 1, \quad j = 1, \ldots, K_3. \quad (B.8)
\]
Roughly speaking each type of Bethe root corresponds to a zero of a Q-function of the system. There are nine $Q^{(k,m)}$ functions in an SU(2|2) problem. They are not all independent, but connected by the so-called QQ-relations [68]:

$$Q^{(k,m)} Q^{(k+1,m+1)} = Q^{(k+1,m)} Q^{(k,m+1)} = Q^{(k+1,m)} Q^{(k,m)}$$

(B.9)

which express the fact that starting from different reference states in the Bethe ansatz description leads to the same final result for the eigenvalues of T-functions.

The $W$ quantum characteristic function [70] and its inverse, are generators of the T-functions in the symmetric and antisymmetric representations respectively,

$$W = \sum_{s=0}^{\infty} t_{1,s}^{[s-1]} e^{2s \partial_u}, \quad W^{-1} = \sum_{a=0}^{\infty} (-1)^a t_{a,1}^{[a-1]} e^{2a \partial_u}. \quad (B.10)$$

They were explicitly given in the $sl(2)$ grading for the SU(2|2) asymptotic solution in [15] and can be expressed in terms of the $Q^{(k,m)}$ functions as follows:

$$W = \left( 1 - \frac{Q^{(2,2)} - Q^{(2,1)+} Q^{(2,1)-} e^{2\partial_u}}{Q^{(2,2)}} \right) \left( 1 - \frac{Q^{(1,1)+} - Q^{(1,1)-} e^{2\partial_u}}{Q^{(1,1)}} \right)$$

$$\left( 1 - \frac{Q^{(0,1)+} - Q^{(0,1)-} e^{2\partial_u}}{Q^{(0,1)}} \right)^{-1} \left( 1 - \frac{Q^{(0,0)+} - Q^{(0,0)-} e^{2\partial_u}}{Q^{(0,0)}} \right). \quad (B.11)$$

Comparing the expression for $W$ given in [15], to (B.11) and using the QQ-relations (B.9) the 9 Q-functions of the SU(2|2) fat-hook can be obtained in the $sl(2)$ grading, namely as functions of the Bethe roots of equations (B.6-B.8). Using the function $\Omega$ defined by

$$\Omega^+ \Omega^- = \frac{R_p}{R_m}, \quad (B.12)$$

the nine Q-functions take the form:

$$Q^{(2,2)} = \frac{(-1)^N Q}{\Omega^2}, \quad Q^{(2,1)} = \frac{Q_1}{\Omega^2}, \quad Q^{(2,0)} = \frac{Q^{(0,0)} - Q^{(0,1)} Q^{(0,2)} e^{2\partial_u}}{\Omega^2} \left( \frac{R_p}{Q_3} \right)^2 \left( \frac{R_p}{R_m} \right) \left( \frac{R_m}{R_p} \right)^{[3]}, \quad (B.13)$$

$$Q^{(1,0)} = \frac{-1}{\Omega} \frac{Q_2^+}{Q_3} \left( 1 - \frac{Q_2}{Q_3} \frac{R_m}{R_p} \right), \quad Q^{(1,1)} = \frac{Q_2}{\Omega}, \quad (B.14)$$

$$Q^{(1,2)} = \frac{(-1)^N Q^{++} Q_2}{\Omega^{++}} \left( 1 - \frac{Q^{++}}{Q_3^+} \frac{R_p}{R_m} \right), \quad (B.14)$$

$$Q^{(0,0)} = 1, \quad Q^{(0,1)} = Q_3^+, \quad Q^{(0,2)} = (-1)^N \frac{Q^{(0,2)} + Q^{(0,1)} Q^{(0,2)} e^{2\partial_u}}{\Omega^2} \left( \frac{R_p}{Q_3^+} \right) \left( \frac{R_m}{R_p} \right)^{[3]}, \quad (B.15)$$

8Fitting the conventions appropriately
where are $\mathcal{F}^{(0)}$ and $\mathcal{G}^{(0)}$ are given by [55]:

$$
\mathcal{F}^{(0)} = -\frac{Q_3^-}{Q_3^+} + \frac{R_x^+}{R_p^+} \left( \frac{Q_2^+}{Q_2^-} \frac{Q_3^-}{Q_3^+} + \frac{Q_2^-}{Q_2^+} \frac{Q_3^+}{Q_3^-} \right), \quad (B.16)
$$

$$
\mathcal{G}^{(0)} = -\frac{Q_3^-}{Q_3^+} + \frac{B_m^+}{B_p^+} \left( \frac{Q_2^+}{Q_2^-} \frac{Q_3^-}{Q_3^+} + \frac{Q_2^-}{Q_2^+} \frac{Q_3^+}{Q_3^-} \right), \quad (B.17)
$$

Expanding $W^{-1}$ the explicit form of the T-functions corresponding to the antisymmetric representations can be obtained [55]:

$$
t_{a,1}^o = (-1)^a \left\{ \frac{Q_3^-[-a]}{Q_3^+[a]} Q_1^+[a] B_p^+[a] \left\{ \frac{Q_1^+[a]}{Q_3^-[a+1]} B_p^+[a] + \frac{Q_3^-[a]}{Q_1^+[a-1]} B_m^-[a] \right\} + \Theta(a-2) \sum_{n=0}^{a-2} \frac{Q_1^-[a-1-2n]}{Q_3^-[a-2-2n]} \frac{Q_2^-[a-2-2n]}{Q_2^+[a-2-2n]} \left( \frac{B_m^-[a-2-2n]}{B_p^+[a-2-2n]} + \frac{R_m^-[a-2-2n]}{R_p^+[a-2-2n]} \right) \right. \\
\left. - \Theta(a-1) \sum_{n=0}^{a-1} \frac{Q_1^-[a-1-2n]}{Q_3^-[a-2-2n]} \frac{Q_2^-[a-2-2n]}{Q_2^+[a-2-2n]} \frac{Q_3^-[a-3-2n]}{Q_3^+[a-3-2n]} + \frac{Q_3^-[a-2-2n]}{Q_3^+[a-2-2n]} \right\}, \quad (B.18)
$$

where $\Theta(x)$ is the unit step function such that $\Theta(0) = 1$. The functions defined in (B.1), (B.2), (B.3) have their only discontinuities along the real line, and the B-type functions are analytical continuations of the R-type functions through the real cut line, this is why in spite of the seemingly complicated discontinuity structure of (B.18), it can be shown that $t_{a,1}^o$ is a $(-a, a)$ function.

The T-functions in the symmetric representations can be obtained by expanding $W$:

$$
t_{1,s}^o = \frac{1}{Q_1^{-[s]} Q_3^{-[s]}} \prod_{j=0}^{s-1} \frac{R_m^{[2j-s]}}{R_p^{[2j-s]}} \left\{ \frac{Q_2^{-[s-1]} Q_2^{[s+1]} R_m^{[s]}}{R_p^{[s]}} \sum_{k=0}^{s} F_{s,k} \right. \\
\left. - \Theta(s-1) \left( \frac{Q_2^{-[s-1]} Q_2^{[-s-1]} R_m^{[s]} R_m^{[s]} \sum_{k=1}^{s} F_{s,k}}{Q_2^{-[s]} Q_2^{[-s]} R_p^{[s]} R_p^{[s]} \sum_{k=1}^{s} F_{s,k}} \right) + \Theta(s-2) \frac{Q_2^{-[s]} Q_2^{[-s]} R_m^{[s]} R_m^{[s]} \sum_{k=1}^{s} F_{s,k}}{Q_2^{-[s]} Q_2^{[-s]} R_p^{[s]} R_p^{[s]} \sum_{k=1}^{s} F_{s,k}} \right\}, \quad (B.19)
$$

where

$$
F_{s,k} = \frac{Q_1^{[2k-s]} Q_3^{[2k-s]} Q_2^{[2k-1-s]} Q_2^{[2k+1-1-s]}}{Q_2^{[2k-1-s]} Q_2^{[2k+1-1-s]}}, \quad (B.20)
$$

Finally the form of the T-functions in the $(1, 1)$ gauge on the interior boundaries of the fat-hook can be read off from the boundary conditions (5.3-5.6), (B.11-B.15) and using (B.13),(B.15) their form can be given explicitly:

$$
t_{a,2}^o = \left(P^{(0)} R_p^+ R_p^- \frac{Q_1^-}{Q_3^-} \right)^{[a]} \left( G^{(0)} R_m^+ R_m^- \frac{Q_3^+}{Q_1^+} \right)^{[-a]}_2, \quad a \geq 2 \quad (B.21)
$$

$$
t_{2,s}^o = \left( P^{(0)} R_p^+ R_p^- \frac{Q_1^-}{Q_3^-} \right)^{[s]} \left( G^{(0)} R_m^+ R_m^- \frac{Q_3^+}{Q_1^+} \right)^{[-s]} \left( \prod_{k=1}^{s-1} \frac{R_m^{[2k+1-1-s]}}{R_p^{[2k+1-1-s]}} \right)^2, \quad s \geq 2. \quad (B.22)
$$
In the treatment of the horizontal SU(2) wing of the problem the T-functions were used in a "cut-free" gauge. Their relation to the (1, 1) gauge expressions (B.19) and (B.22) is given by:

\[
\tilde{t}_1^s = t_{1,s}^o \prod_{j=1}^{s-1} \frac{R_{p}^{[2j-s]}}{R_{m}^{[2j-s]}}, \quad \tilde{t}_2^s = t_{2,s}^o \left( \prod_{j=1}^{s-1} \frac{R_{p}^{[2j+1-s]}}{R_{m}^{[2j+1-s]}} \right)^2.
\]  

(B.23)

It can be seen from the above explicit formulas that \(\tilde{t}_1^s\) are \((-s, s)\) functions and \(\tilde{t}_2^s\) are of type \((1-s, s-1)\).

In order to complete the asymptotic solution of AdS/CFT we should discuss the asymptotic behavior of massive Y-functions. The asymptotic solution for the massive nodes on the AdS/CFT Y-system is given by

\[
y_{a,0}^o = \eta_a t_{a,1}^R t_{a,1}^L.
\]  

(B.24)

where the prefactor is a solution of the discrete Laplace equation and takes the form:

\[
\eta_a = \left( \frac{x[a]}{x[-a]} \right)^{J_{eff}} D_a \frac{\phi[-a]}{\phi[a]},
\]  

(B.25)

with

\[
\phi = \frac{B_{1,L} B_{1,R}}{B_{3,L} B_{3,R}}, \quad D_a = \prod_{k=0}^{a-1} D_1^{[a-1-2k]} \equiv \prod_{j=1}^{N} S_{sl(2)}^{1+}(u, u_j).
\]  

(B.26)

The parameter \(J_{eff}\) is an effective length composed of the \(J\)-charge and the numbers of the auxiliary Bethe roots by the formula \(J_{eff} = J + \frac{K_{3,L} - K_{1,L} + K_{3,R} - K_{1,R}}{2}\). The factor \(D_1\) is the product of the dressing phases of fundamental magnons in the mirror-physical channel

\[
D_1(u) \equiv \prod_{j=1}^{N} S_{sl(2)}^{1+}(u, u_j),
\]  

for the dressing factor in the physical-physical region.

The analytical properties of the dressing phase and the asymptotic T-functions imply that the \(y_{a,0}^o\) functions are of type \((-a, a)\) and they decay with a high power of \(u\) at infinity.

The requirement that on the physical sheet \(1 + y_{a,0}^o\) is zero at the positions of magnon rapidities leads to the Beisert-Staudacher asymptotic Bethe equations \(\textbf{[72]}\) :

\[
\left( \frac{x^+}{x^-} \right)^{J_{eff}} S_{sl(2)} \left| \begin{array}{c}
B_{1,L}^- & R_{3,L}^- & B_{1,R}^- & R_{3,R}^- \\
R_{1,L}^- & B_{3,L}^+ & R_{1,R}^- & B_{3,R}^+
\end{array} \right|_{u_k} = -1, \quad k = 1, \ldots, N,
\]  

(B.27)

where for short we introduced the notation

\[
S_{sl(2)}(u) = \prod_{j=1}^{N} S_{sl(2)}^{1+}(u, u_j).
\]  

(B.28)

for the dressing factor in the physical-physical region.
C. NLIE kernels

All the kernels appearing in the equations (7.47) for \(b_{a,s}^{(r)}\) and \(d_{a,s}^{(r)}\) are composed as sums of their negative and positive frequency parts. For instance in Fourier space:

\[
(\tilde{K}_{bB}^{(r)}(\omega))_{aa'} = \Theta(\omega) (\tilde{K}_{bB}^{(+)}(\omega))_{aa'} + \Theta(-\omega) (\tilde{K}_{bB}^{(-)}(\omega))_{aa'},
\]

\((\tilde{K}_{bB}^{(\pm)}(\omega))_{aa'}\) are the positive and negative frequency parts respectively and \(\Theta(\omega)\) is the Heaviside function.

Here we list below the negative and positive frequency parts of the kernels. Recall that \(p = e^{\frac{\omega}{2}}\) and introduce two important functions that eliminate the denominator parts of the kernels.

\[
\tilde{N}_+(p) = \frac{1}{1+2p^2+p^8} \quad \omega \geq 0, \quad \tilde{N}_-(p) = \frac{1}{1+2p^{-2}+p^{-8}} \quad \omega \leq 0. \tag{C.1}
\]

Then the kernels \(\tilde{K}_{bB}, \tilde{K}_{bD}, \tilde{K}_{dB}, \tilde{K}_{dD}\) in (7.47) are arranged into \(6 \times 6\) matrices by identifying the 6 index pairs \((a,r) = (a,s) = 1 \ldots 6\) to the indexes of the \(6 \times 6\) matrices as follows: \((1,3) \rightarrow 1\), \((2,3) \rightarrow 2\), \((3,3) \rightarrow 3\), \((1,2) \rightarrow 4\), \((2,2) \rightarrow 5\), \((1,1) \rightarrow 6\).

Based on the same identification the kernels \(\tilde{K}_{bY}, \tilde{K}_{dY}\) are represented as \(6 \times 3\) matrices.

Then the positive and negative parts of the kernels in (7.47) take the form in Fourier space as follows:

\[
\begin{align*}
\frac{K_{bB}^{(+)}(\omega)}{\tilde{N}_+(p)} &= \begin{pmatrix}
 1 - p^2 & p^2 - p^4 - 2p^2 + 2 & -p^6 - p^4 - 2p^2 - 2 & p^7 + 2p & -p^7 - p - \frac{1}{2} & p^3 + p \\
p^3 - p^4 & -p^6 - p^4 & p^7 + p & p^5 + 2p & \frac{1}{2} - p^6 - p^4 & -p^4 - 2p^2 - 1 \\
-p^8 - p^4 & p^4 - 1 & 2p^3 & 2p^3 + 2p & -p^4 - p^3 + 1 & -p^4 - p^3 - 1 \\
-p^8 + p^4 & p^8 - p^4 & p^6 + p^2 & 2p^3 + 2p & -p^4 - p^3 + 1 & -p^4 - p^3 - 1 \\
2p^3 & -p^8 - p^4 & p^6 + p^2 & 2p^3 + 2p & -p^4 - p^3 + 1 & -p^4 - p^3 - 1 \\
-p^8 - p^4 & -p^8 + p^4 & p^6 + p^2 & 2p^3 + 2p & -p^4 - p^3 + 1 & -p^4 - p^3 - 1
\end{pmatrix},
\end{align*}
\]

\[
\frac{K_{bB}^{(-)}(\omega)}{\tilde{N}_-(p)} = \begin{pmatrix}
 1 - p^2 & -p^3 - p & p^4 + p^2 - 2 & -p^6 + p^4 + 2p^2 - 2 & -p^7 - p^8 + 2p - 1 & p^7 - p^3 - \frac{1}{2} \\
-p^2 - p^4 & -p^7 + p & p^4 + p^2 & -p^7 + p^5 + 3p + p & -p^7 - p^6 - 2p - 1 & -p^7 - p^3 \\
p^3 + p^4 & -p^6 - p^2 & 1 - p^7 & -p^7 + p^5 + 3p - p & -p^7 - p^6 - 1 & -p^7 - p^3 - 1 \\
p^4 - p^2 & -p^8 + p^4 & p^6 + p^2 & -p^7 + p^5 + 3p - 1 & -p^7 - p^6 - 1 & -p^7 - p^3 - 1 \\
-p^6 + p^2 & -p^8 + p^4 & p^6 + p^2 & -p^7 + p^5 + 3p - 1 & -p^7 - p^6 - 1 & -p^7 - p^3 - 1 \\
p^6 + p^2 & -p^8 + p^4 & p^6 + p^2 & -p^7 + p^5 + 3p - 1 & -p^7 - p^6 - 1 & -p^7 - p^3 - 1
\end{pmatrix},
\]

\[
\frac{K_{bD}^{(+)}(\omega)}{\tilde{N}_+(p)} = \begin{pmatrix}
 -p^2 - 1 & -p^7 - p & p^8 - p^6 + 2p^2 - 2 & -p^8 - 2p^2 + 1 & p^7 - p^3 + 1 & -p^7 - 1 \\
-p^3 - p & -p^8 + p^4 & -p^7 - p & p^7 + p^5 + p + p & -p^7 - p^6 - 2p - 1 & p^7 - p^3 \\
-p^4 - p^2 & p^8 + p^2 & -p^7 - p & p^7 + p^5 + p + p & -p^7 - p^6 - 1 & p^7 - p^3 - 1 \\
-p^6 + p^2 & -p^8 + p^4 & p^7 - p^3 & -p^7 - p^5 + p^3 + 2p^2 - 1 & -2p^8 - 2p^2 & -p^8 + p^4 + 1 \\
-p^8 + p^2 & -p^8 + p^4 & p^7 - p^3 & -p^7 - p^5 + p^3 + 2p^2 - 1 & -2p^8 - 2p^2 & -p^8 + p^4 + 1 \\
-p^8 + p^2 & -p^8 + p^4 & p^7 - p^3 & -p^7 - p^5 + p^3 + 2p^2 - 1 & -2p^8 - 2p^2 & -p^8 + p^4 + 1
\end{pmatrix},
\]

\[
\frac{K_{bD}^{(-)}(\omega)}{\tilde{N}_-(p)} = \begin{pmatrix}
 -p^2 + 1 & -p^2 + 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 \\
-p^2 + 1 & -p^2 + 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 \\
-p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 \\
-p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 \\
-p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 \\
-p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1 & -p^2 - 1
\end{pmatrix}.
\]
\[
\frac{\mathcal{K}_{\omega}^{(\pm)}(\omega)}{N_{\pm}(\omega)} = \begin{pmatrix}
\begin{array}{cccc}
p^2 & p^3 - p & p^4 & p^5 - p \\
p^2 & p^3 - p & p^4 & p^5 - p \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
p^2 & p^3 + 2p^2 + p^4 & p^5 - p & -2p^3 \\
\end{array}
\end{pmatrix}
\end{equation}
\]
Then it can be seen from the Fourier representations (C.3.13), (7.52) that the rapidity space representations of all matrix elements of the kernels in (7.51) can be expressed as linear combinations of the two denominator functions \( N_\pm(u) \) with appropriately shifted arguments. For example let us consider the upper left corner matrix elements in (C.2-C.3). They correspond to \((G_{bb})_{11}^{(33)}(u)\) and yield the expressions in rapidity space as follows:

\[
(G_{bb})_{11}^{(33)}(u) = (K_{bb})_{11}^{(33)}(u) = N_+(u) - N_+(u + \frac{2i}{g}) + N_-(u) - N_-(u - \frac{2i}{g}),
\]

where \( N_\pm(u) \) are given as inverse Fourier transforms of \( \tilde{N}_\pm(e^{\omega/g}) \):

\[
N_+(u) = \int_0^\infty \frac{d\omega}{2\pi} e^{-i\omega u} \tilde{N}_+(e^{\omega/g}), \quad N_-(u) = \int_{-\infty}^0 \frac{d\omega}{2\pi} e^{-i\omega u} \tilde{N}_-(e^{\omega/g}), \quad N_+(u) = N_-(u)
\]

and can be expressed in terms of the incomplete beta function\(^9\) \( B_\pm(a, b) \) as follows. Denote by \( a_1, a_2, a_3, a_4 \) the four zeroes of the denominator polynomial \( 1 + 2x + x^4 \). They take the form:

\[
\begin{align*}
a_1 &= -1 \\
a_2 &= \frac{1}{3} \left( 1 - \frac{2}{\sqrt[3]{3\sqrt{33} - 17}} + \frac{3}{\sqrt[3]{3\sqrt{33} - 17}} \right) \approx -0.543689 \\
a_3 &= \frac{1}{3} + \frac{1 + i\sqrt{3}}{3\sqrt[3]{3\sqrt{33} - 17}} - \frac{1}{6} \left( 1 - i\sqrt{3} \right) \sqrt[3]{3\sqrt{33} - 17} \approx 0.771845 + 1.11514 i \\
a_4 &= \frac{1}{3} + \frac{1 - i\sqrt{3}}{3\sqrt[3]{3\sqrt{33} - 17}} - \frac{1}{6} \left( 1 + i\sqrt{3} \right) \sqrt[3]{3\sqrt{33} - 17} \approx 0.771845 - 1.11514 i \quad (C.16)
\end{align*}
\]

Then \( N_+(u) = N_-(u) \) is given by:

\[
N_+(u) = g \left( a_1^{1-\frac{i}{2}} B_{a_1} \left( \frac{igu}{2} + 1, 0 \right) - a_2^{1-\frac{i}{2}} B_{a_2} \left( \frac{igu}{2} + 1, 0 \right) \right) + g \left( a_3^{1-\frac{i}{2}} B_{a_3} \left( \frac{igu}{2} + 1, 0 \right) - a_4^{1-\frac{i}{2}} B_{a_4} \left( \frac{igu}{2} + 1, 0 \right) \right) + g \frac{4\pi}{(a_1-a_3)(a_2-a_4)(a_3-a_4)} - g \frac{4\pi}{(a_4-a_1)(a_2-a_3)(a_3-a_4)} \quad (C.17)
\]

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\(^9\)The incomplete beta function is defined by its integral representation: \( B_\pm(a, b) = \int_0^1 dt t^{a-1} (1 - t)^{b-1} \)
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