Estimating with kernel smoothers the mean of functional data in a finite population setting. A note on variance estimation in presence of partially observed trajectories

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Abstract

In the near future, millions of load curves measuring the electricity consumption of French households in small time grids (probably half hours) will be available. All these collected load curves represent a huge amount of information which could be exploited using survey sampling techniques. In particular, the total consumption of a specific customer group (for example all the customers of an electricity supplier) could be estimated using unequal probability random sampling methods. Unfortunately, data collection may undergo technical problems resulting in missing values. In this paper we study a new estimation method for the mean curve in the presence of missing values which consists in extending kernel estimation techniques developed for longitudinal data analysis to sampled curves. Three nonparametric estimators that take account of the missing pieces of trajectories are suggested. We also study pointwise variance estimators which are based on linearization techniques. The particular but very important case of stratified sampling is then specifically studied. Finally, we discuss some more practical aspects such as choosing the bandwidth values for the kernel and estimating the probabilities of observation of the trajectories.

Keywords. Functional data, Hájek estimator, Horvitz-Thompson estimator, Linearization, Missing values, Nonparametric estimation, Ratio estimator, Survey sampling.
1 Introduction

In the next few years in France, tens of millions of smart meters will be deployed and will collect the individual load curves, i.e. electricity consumption time series, of residential and business customers at short time steps (probably half hours). This deployment will result in a huge increase in the amount of available data for energy suppliers such as EDF (Electricité de France) and power grid managers. However, it may be complex and costly to stock and exploit such a large quantity of information, therefore it will be relevant to use sampling techniques to estimate load curves of specific customer groups (e.g. market segments, owners of a specific equipment or clients of an energy supplier). See Cardot and Josserand (2011) or Cardot et al. (2013b) for preliminary studies.

Unfortunately, data collection, like every mass process, may undergo technical problems at every point of the metering and collecting chain resulting in missing values. This problem is very similar to nonresponse in survey sampling: it deteriorates the accuracy of the estimators and may generate bias if the clients affected by missing values are different from the clients with complete curves.

There is a large literature dealing with the inference in presence of missing values (see e.g. Särndal & Lundström (2005) and Haziza (2009) for reviews) but as far as we know the case in which collected data are curves has not been addressed yet.

In this paper we will use functional data analysis methods, adapted to the sampling framework and to the presence of missing values, in order to take advantage of the specificities of our problematic that is to say the strong correlations between the consumptions at the various instants and the smoothness of the curves. More precisely, we suggest to adapt, in an unequal probability sampling context, kernel estimation techniques that have initially been developed to deal with longitudinal data (Staniswalis and Lee, 1998) and sparse functional data (Hall et al. 2006). The asymptotic behavior of the Mean Square Error of the estimators (a very close estimator in fact) is given by Hart & Wehrly (1986) and Hall et al. (2006) under the assumption that the number of measurements and the number of observations tends to infinity. In a finite population setting with unequal probability sampling designs, the properties of local polynomial smoothers with noisy measurements at at finite number of instants of time but without non response have been studied in Cardot et al. (2013a).

The context we consider in this work is different and new. We suppose that some curves of the finite population under study are partially observed during periods that are random. The second section fixes notations and presents the three proposed kernel estimators of the mean load curve when pieces of trajectories are missing. In Section 3, the approximate variance of these estimators are derived. Note that our derivations are quite general and remain true
for linear smoothers (local polynomials, series expansion, smoothing splines). The important particular case of stratified sampling is studied more precisely in Section 4 and we get that Hájek type estimators seem to be preferable, in this context, to weighted Horvitz-Thompson estimators. Finally some comments about estimation and choice of the tuning parameters are given in Section 5. Some technical details about kernels estimators are postponed in an Appendix.

2 The functional observations, the non response mechanism and the estimators

We consider a population \( U \), with known size \( N \), of (load) curves defined over a time interval \([0, 1]\): for each unit \( k \) in \( U \), we have a function of time \( Y_k(t), t \in [0, T] \), where the continuous index \( t \) represents time.

The aim is to estimate the mean load curve \( \mu \) (or the total trajectory = \( N\mu \)) over the population

\[
\mu(t) = \frac{1}{N} \sum_{k \in U} Y_k(t), \quad t \in [0, T],
\]

when only a sample (drawn randomly from the population \( U \)) of the units is available and some parts of the sampled trajectories are missing.

2.1 Kernel smoothing of the mean trajectory

With real data, the trajectories \( Y_k \) are not observed at all instants \( t \in [0, T] \) but at some discrete time instants, \( 0 \leq t_1 < \ldots < t_j < \ldots < t_d \leq T \), which are supposed to be the same for all data and equispaced, so that \( t_j = T(j-1)/(d-1) \). For example in Cardot and Josserand (2011), the measurements are made every half an hour over a period of two weeks.

In this ideal discretized framework a simple continuous approximation to the function \( \mu \), denoted by \( \tilde{\mu}(t) \), can be obtained at all instants \( t \in [0, T] \) by applying a kernel smoother (see Staniswalis and Lee, 1998). For that, let us introduce a kernel \( K(.,) \), i.e a continuous and positive function, symmetric around zero (see e.g. Hart, 1997 for a more precise definition as well as examples). Classical kernels are the Gaussian kernel defined by \( K(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \) and the Epanechnikov kernel which is defined by \( K(x) = \frac{3}{4}(1-x^2)1_{|x| \leq 1} \). For all instants \( t \in [0, T] \), employing kernel smoothing leads to the following smooth approximation to \( \mu(t) \),

\[
\tilde{\mu}(t) = \frac{1}{N} \sum_{k=1}^{N} \sum_{j=1}^{d} K\left(\frac{t-t_j}{h}\right) Y_k(t_j)
\]

\[
\frac{1}{N} \sum_{k=1}^{N} \sum_{j=1}^{d} K\left(\frac{t-t_j}{h}\right)
\]

(2)
with a bandwidth $h$ whose aim is to control the smoothness of the approximation. Larger values of $h$ lead to smoother estimates, with a larger bias and a smaller variance whereas small values of $h$ lead to estimates that may have many oscillations, with a small bias but a larger variability. When the design points $t_1, \ldots, t_p$ are the same for all the curves, expression (2) can be simplified as follows,

$$
\tilde{\mu}(t) = \sum_{j=1}^{d} w(t, t_j, h) \mu(t_j),
$$

with smoothing weights

$$
w(t, t_j, h) = \frac{K \left( \frac{t - t_j}{h} \right)}{\sum_{j=1}^{d} K \left( \frac{t - t_j}{h} \right)},
$$

so that $\tilde{\mu}(t)$ is just obtained by smoothing the discretized population mean trajectory, $(\mu(t_1), \ldots, \mu(t_d))$.

### 2.2 Sampling designs and kernel estimators

It is assumed now that only a part the population $U$ is observed and we denote by $s \subset U$ a sample drawn randomly from $U$, with fixed size $n$. For $k$ and $\ell \in \{1, \ldots, N\}$, we denote by $\pi_k$ and $\pi_{k\ell}$ the first and second order inclusion probabilities: $\pi_k = \mathbb{P}[k \in s]$ and $\pi_{k\ell} = \mathbb{P}[k \in s \& \ell \in s]$. These inclusion probabilities are supposed to be strictly positive.

A first simple Horvitz-Thompson estimator of the smooth approximation $\tilde{\mu}(t)$ is simply obtained by replacing the $\mu(t_j)$’s in (3) by their Horvitz-Thompson estimations $\hat{\mu}(t_j)$, so that we get, for $t \in [0, T]$,

$$
\hat{\mu}_{HT}(t) = \sum_{j=1}^{d} w(t, t_j, h) \hat{\mu}(t_j),
$$

where

$$
\hat{\mu}(t_j) = \frac{1}{N} \sum_{k \in s} \frac{Y_k(t_j)}{\pi_k}.
$$

Another estimator of $\hat{\mu}(t)$ can be defined by considering a ratio or Hájek point of view. The estimator is obtained by replacing the $\mu(t_j)$’s in by their Hájek estimator,

$$
\hat{\mu}_{Ha}(t) = \frac{\sum_{j=1}^{d} K \left( \frac{t - t_j}{h} \right) \frac{Y_k(t_j)}{\pi_k}}{\sum_{j=1}^{d} K \left( \frac{t - t_j}{h} \right) \frac{1}{\pi_k}} \quad \text{and} \quad
$$

$$
= \sum_{j=1}^{d} w(t, t_j, h) \left( \frac{\sum_{k \in s} Y_k(t_j) \frac{1}{\pi_k}}{\sum_{k \in s} \frac{1}{\pi_k}} \right).
$$
2.3 Nonparametric estimators with non response

The individual trajectories of the sample $s$ are not always observed at all the discretization points and some parts may be missing. To take account of the non response mechanism we introduce a response random variable defined as follows. We define the continuous time process $r_k(t)$ which takes value 1 if $Y_k$ can be observed at instant $t$ and 0 else. This binary continuous time stochastic process is supposed to be independent of the values of the trajectories as well as the sampling design. Nonetheless, the response probability is allowed to depend on $k$ and on time. For each unit $k$ in the population, we denote by

$$\theta_k(t_j) = \mathbb{P}[r_k(t_j) = 1],$$

the probability of response at instant $t_j$ and by

$$\theta_k(t_j, t_{j'}) = \mathbb{P}[r_k(t_j) = 1 \& r_k(t_{j'}) = 1],$$

the probability of response at both instants $t_j$ and $t_{j'}$. Note that, for simpler calculus, we could also assume that there is a small number of response homogeneity groups (see for example Särndal & Lundström, 2005), that is to say that all elements within one and the same group respond with the same probability, and in an independent manner.

Taking now the non response mechanism into account, we can consider three different estimators of $\tilde{\mu}(t)$ based on reweighting and smoothing. A first one, derived from (4), is a smoothed Horvitz-Thompson estimator that takes account of non response, 

$$\hat{\mu}_{r,HT}(t) = \frac{1}{N} \sum_{j=1}^{d} w(t, t_j, h) \left( \sum_{k \in s} r_k(t_j) Y_k(t_j) \theta_k(t_j) \pi_k \right). \tag{8}$$

The second one derived from (6), can be seen as a Hájek type estimator of two smoothed ratios,

$$\hat{\mu}_{r,Ha}^{(1)}(t) = \frac{\sum_{j=1}^{d} K(h^{-1}(t - t_j)) \left( \sum_{k \in s} r_k(t_j) Y_k(t_j) \theta_k(t_j) \pi_k \right)}{\sum_{j=1}^{d} K(h^{-1}(t - t_j)) \left( \sum_{k \in s} r_k(t_j) 1 \theta_k(t_j) \pi_k \right)}. \tag{9}$$

The last estimator can also be seen as a smoothed Hájek estimator and is derived from (7). It is defined, for $t \in [0, T]$, by

$$\hat{\mu}_{r,Ha}^{(2)}(t) = \sum_{j=1}^{d} w(t, t_j, h) \frac{\hat{Y}(t_j)}{\hat{N}(t_j)}, \tag{10}$$

where $\hat{Y}(t_j) = \sum_{k \in s} r_k(t_j) Y_k(t_j) \theta_k(t_j) \pi_k$ and $\hat{N}(t_j) = \sum_{k \in s} r_k(t_j) 1 \theta_k(t_j) \pi_k$. Note that if there is no nonresponse, the estimators $\hat{\mu}_{r,Ha}^{(1)}(t)$ and $\hat{\mu}_{r,Ha}^{(2)}(t)$ are the same.
3 Variance of the estimators

We first show, under general conditions, that the approximation error and the bias are negligible compared to the variance. This explains why we focus on variance estimation of the three proposed estimators in the presence of non response.

We denote in the following by $E_p$ the expectation with respect to the sampling design and by $E_R$ the expectation with respect to the non response mechanism. When there is no subscript, the expectation $E$ is considered both with respect to the sampling design and the non response random mechanism.

3.1 The approximation error and the bias are negligible

We first consider the Horvitz-Thompson estimator $\hat{\mu}_{r,HT}(t)$ defined in (8) and we clearly have that

$$E(\hat{\mu}_{r,HT}(t)) = \sum_{j=1}^{d} w(t,t_j,h) E \left[ \sum_{k \in s} \frac{r_k(t_j)Y_k(t_j)}{\theta_k(t_j)\pi_k} \right]$$

$$= \sum_{j=1}^{d} w(t,t_j,h) \mu(t_j)$$

$$= \bar{\mu}(t)$$

so that it is unbiased for $\bar{\mu}(t)$. Thus, the mean square error satisfies

$$E [\hat{\mu}_{r,HT}(t) - \mu(t)]^2 = |\bar{\mu}(t) - \mu(t)|^2 + \mathbb{V}(\hat{\mu}_{r,HT}(t)).$$

Furthermore, we can show under general regularity conditions on the mean trajectory given in the Appendix and if, as the population size $N$ tends to infinity, the bandwidth $h$ tends to zero and the number of design points tends to infinity, satisfying $2h > (d - 1)^{-1}$, that the approximation error is bounded, for some constant $C_t$, as follows

$$|\bar{\mu}(t) - \mu(t)| \leq C_t h^\beta.$$ (13)

Combining (12) and (13), this means that, provided that $\sqrt{n}h^\beta \to 0$ as the sample size $n \to \infty$, the approximation error, $\bar{\mu}(t) - \mu(t)$, is negligible compared to the sampling error, which tends to zero in probability at most at rate $1/\sqrt{n}$. Note that this condition on the bandwidth $h$, which must be small, and the sample size also implies that the grid of discretization points must be dense enough so that $\sqrt{n} \max_j |t_{j+1} - t_j|^\beta \to 0$. In that case, the mean square error of the Horvitz-Thompson estimator can be approximated by its variance,

$$E [\hat{\mu}_{r,HT}(t) - \mu(t)]^2 \approx \mathbb{V}(\hat{\mu}_{r,HT}(t)).$$
The estimators \( \hat{\mu}_{r,H_a}(t) \) and \( \tilde{\mu}_{r,H_a}(t) \) are not unbiased estimators of \( \mu(t) \) but they are ratio estimators with unbiased estimators of each component (numerator and denominator) of the ratio. Nevertheless, they are asymptotically unbiased for \( \mu(t) \) and under previous conditions, their squared bias plus their squared approximation error are negligible compared to their variance. This means that their mean square errors can also be approximated by their variances.

### 3.2 Variance approximation for the Horvitz-Thompson estimator

The derivations made below are greatly inspired by the Chapter 15 of Särndal et al. (1992). Let us first decompose the variance of \( \hat{\mu}_{r,HT}(t) \) by using the classical formula:

\[
\mathbb{V}(\hat{\mu}_{r,HT}(t)) = \mathbb{V}_R \mathbb{E}_p(\hat{\mu}_{r,HT}(t) | s_r) + \mathbb{E}_R \mathbb{V}_p(\hat{\mu}_{r,HT}(t) | s_r),
\]

where \( s_r \) is the set of samples of respondents at each instant \( \{t_1, \ldots, t_p\} \). For simpler notations and shorter equations, we denote \( w(t, t_j, h) \) by \( w_j(t) \). We have

\[
\mathbb{E}_p(\hat{\mu}_{r,HT}(t) - \mu(t)|s_r) = \frac{1}{N} \sum_{j=1}^{d} w_j(t) \left[ \sum_{k \in U} \frac{Y_k(t_j) \left( \frac{r_k(t_j)}{\theta_k(t_j)} - 1 \right)}{\theta_k(t_j)} \right]
\]

and, by independence between units in the non response mechanism,

\[
\begin{align*}
\mathbb{V}_R \mathbb{E}_p(\hat{\mu}_{r,HT}(t) - \mu(t)|s_r) &= \frac{1}{N^2} \sum_{k,l \in U} \frac{\Delta_{kl}}{\pi_k \pi_l} \left( \sum_{j=1}^{d} w_j(t) \frac{r_l(t_j)}{\theta_l(t_j)} Y_l(t_j) \right) \left( \sum_{j'=1}^{d} w_{j'}(t) \frac{r_k(t_{j'})}{\theta_k(t_{j'})} Y_k(t_{j'}) \right) \\
+ \frac{1}{N^2} \sum_{k \in U} \sum_{j=1}^{d} \sum_{j' \neq j} w_j(t) w_{j'}(t) Y_k(t_j) Y_k(t_{j'}) \frac{\theta_k(t_j, t_{j'}) - \theta_k(t_j) \theta_k(t_{j'})}{\theta_k(t_j) \theta_k(t_{j'})}
\end{align*}
\]

Define \( \Delta_{kl} = \pi_{kl} - \pi_k \pi_l \) for \( k \neq l \) and \( \Delta_{kk} = \pi_k(1 - \pi_k) \). We have

\[
\mathbb{V}_p(\hat{\mu}_{r,HT}(t) - \mu(t)|s_r) = \frac{1}{N^2} \sum_{k,l \in U} \frac{\Delta_{kl}}{\pi_k \pi_l} \left( \sum_{j=1}^{d} w_j(t) \frac{r_l(t_j)}{\theta_l(t_j)} Y_l(t_j) \right) \left( \sum_{j'=1}^{d} w_{j'}(t) \frac{r_k(t_{j'})}{\theta_k(t_{j'})} Y_k(t_{j'}) \right)
\]

and taking the expectation with respect to the non response mechanism, we get

\[
\begin{align*}
\mathbb{E}_R \mathbb{V}_p(\hat{\mu}_{r,HT}(t) - \mu(t)|s_r) &= \frac{1}{N^2} \sum_{k \in U} \frac{1 - \pi_k}{\pi_k} \sum_{j,j' \neq j} w_j(t) w_{j'}(t) Y_k(t_j) Y_k(t_{j'}) \frac{\theta_k(t_j, t_{j'}) - \theta_k(t_j) \theta_k(t_{j'})}{\theta_k(t_j) \theta_k(t_{j'})} \\
+ \frac{1}{N^2} \sum_{k \in U} \frac{1 - \pi_k}{\pi_k} \sum_{j=1}^{d} w_j^2(t) Y_k^2(t_j) \frac{1}{\theta_k(t_j)} \\
+ \frac{1}{N^2} \sum_{k \in U} \sum_{l \neq k} \frac{\Delta_{kl}}{\pi_k \pi_l} \left( \sum_{j=1}^{d} w_j(t) Y_k(t_j) \right) \left( \sum_{j'=1}^{d} w_{j'}(t) Y_l(t_{j'}) \right)
\end{align*}
\]
Combining (15) and (16) in (14), we get, after some algebra, the following expression for the variance of $\hat{\mu}_{r,HT}(t)$, at each instant $t$ in $[0, T]$,

$$
\mathbb{V}(\hat{\mu}_{r,HT}(t)) = \frac{1}{N^2} \sum_{k,l \in U} \frac{\Delta_{kl}}{\pi_k \pi_l} \left( \sum_{j=1}^{d} \bar{w}_j(t)Y_k(t_j) \right) \left( \sum_{j'=1}^{d} \bar{w}_{j'}(t)Y_l(t_{j'}) \right) + \frac{1}{N^2} \sum_{k \in U} \frac{1}{\pi_k} \sum_{j,j'} \bar{w}_j(t) \bar{w}_{j'}(t) Y_k(t_j) \frac{\theta_k(t_j,t_{j'}) - \theta_k(t_j) \theta_k(t_{j'})}{\theta_k(t_j) \theta_k(t_{j'})}
$$

(17)

with the convention that $\theta_k(t_j,t_j) = \theta_k(t_j)$. The part of the variance given in (17) corresponds to the sampling variance whereas the additional variance term in (18) is due to the non response.

### 3.3 Variance approximation for the Hájek estimators

The variance of the estimator $\hat{\mu}^{(1)}_{r,Ha}(t)$ defined in (9) can be approximated thanks to a linearization technique (see Deville, 1999) with respect to the sampling distribution and the non response mechanism. Indeed, it is a ratio of two linear estimators whose expressions are similar to the expression of $\hat{\mu}_{r,HT}(t)$. We have

$$
\mathbb{V}(\hat{\mu}^{(1)}_{r,Ha}(t)) \approx \mathbb{V}\left( \sum_{k \in s} \sum_{j=1}^{d} \frac{\bar{r}_k(t_j) \bar{u}_{kj}^{(1)}(t)}{\bar{\theta}_k(t_j)} \right)
$$

(19)

where the "linearized" variable $\bar{u}_{kj}(t)$ is defined as follows

$$
\bar{u}^{(1)}_{kj}(t) = \frac{1}{N} \bar{w}_j(t) (Y_k(t_j) - \bar{\mu}(t)).
$$

(20)

After some algebra, we get the following expression for the variance,

$$
\mathbb{V}(\hat{\mu}^{(1)}_{r,Ha}(t)) \approx \frac{1}{N^2} \sum_{k \in U} \sum_{i \in U} \frac{\Delta_{kl}}{\pi_k \pi_l} \bar{u}^{(1)}_{ki}(t) \bar{u}^{(1)}_{lj}(t) + \frac{1}{N^2} \sum_{k \in U} \frac{1}{\pi_k} \sum_{j,j'=1}^{d} \frac{\bar{u}^{(1)}_{kj}(t) \bar{u}^{(1)}_{kj'}(t)}{\bar{\theta}_k(t_j) \bar{\theta}_k(t_{j'})} \left( \bar{\theta}_k(t_j,t_{j'}) - \bar{\theta}_k(t_j) \bar{\theta}_k(t_{j'}) \right)
$$

(21)

where $\bar{u}^{(1)}_k(t)$ is, up to a multiplicative factor $N$, the smoothed linearized variable trajectory,

$$
\bar{u}^{(1)}_k(t) = \sum_{j=1}^{d} \bar{w}_j(t) (Y_k(t_j) - \bar{\mu}(t)).
$$

For the third estimator, $\hat{\mu}^{(2)}_{r,Ha}(t)$ defined in (10), we have

$$
\mathbb{V}(\hat{\mu}^{(2)}_{r,Ha}(t)) = \sum_{jj'=1}^{d} \bar{w}_j(t) \bar{w}_{j'}(t) \text{Cov} \left( \frac{\bar{Y}(t_j)}{\bar{N}(t_j)}, \frac{\bar{Y}(t_{j'})}{\bar{N}(t_{j'})} \right)
$$

(22)

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Employing again a linearization technique, we have
\[
\text{Cov} \left( \frac{\hat{Y}(t_j)}{N(t_j)}; \frac{\hat{Y}(t_j')}{N(t_j')} \right) \approx \text{Cov} \left( \frac{1}{N} \sum_{k \in s} \frac{r_k(t_j) (Y_k(t_j) - \mu(t_j))}{\pi_k}, \frac{1}{N} \sum_{k \in s} \frac{r_k(t_{j'}) (Y_k(t_{j'}) - \mu(t_{j'}))}{\pi_k} \right),
\]
so that
\[
\mathbb{V} \left( \hat{\mu}^{(2)}_{r,Ha}(t) \right) \approx \mathbb{V} \left( \sum_{j=1}^d w_j(t) \frac{1}{N} \sum_{k \in s} \frac{r_k(t_j) (Y_k(t_j) - \mu(t_j))}{\pi_k} \right) = \frac{1}{N^2} \mathbb{V} \left( \sum_{k \in s} \sum_{j=1}^d \frac{r_k(t_j) w_j(t) (Y_k(t_j) - \mu(t_j))}{\pi_k} \right). \tag{23}
\]
A direct comparison of (23) with (19) gives us that the approximated variance of \( \hat{\mu}^{(2)}_{r,Ha}(t) \) which is based on linearization is very similar to the approximated variance of \( \hat{\mu}^{(1)}_{r,Ha}(t) \). We have
\[
\mathbb{V} \left( \hat{\mu}^{(2)}_{r,Ha}(t) \right) \approx \frac{1}{N^2} \sum_{k \in U} \sum_{l \in U} \frac{\Delta_{kl}}{\pi_k \pi_l} \tilde{u}^{(2)}_k(t) \tilde{u}^{(2)}_l(t) + \sum_{k \in U} \frac{1}{\pi_k} \sum_{j,j'=1}^d u^{(2)}_{k,j}(t) \frac{u^{(2)}_{k,j'}(t)}{\theta_k(t_j) \theta_k(t_{j'})} (\theta_k(t_j, t_{j'}) - \theta_k(t_j) \theta_k(t_{j'})) \tag{24}
\]
where \( u^{(2)}_{k,j} = \frac{1}{N} w_j(t) (Y_k(t_j) - \mu(t_j)) \) is the "linearized" variable of \( \mu(t_j), j = 1, \ldots, p \) with \( \sum_{j=1}^p \sum_{k \in U} w_j(t) u^{(2)}_{k,j} = 0 \) and \( \tilde{u}^{(2)}_k(t) = \sum_{j=1}^d w_j(t) (Y_k(t_j) - \mu(t_j)) \). Since \( \sum_{j=1}^d w_j(t) = 1 \) we also have \( \tilde{u}^{(2)}_k(t) = \sum_{j=1}^d w_j(t) (Y_k(t_j) - \bar{\mu}(t)) \).

4 The particular case of stratified sampling with homogeneous response groups

We consider now a simpler but important particular case, stratified sampling with homogeneous response group within each stratum. The population \( U \) is divided into \( \Lambda \) strata, \( U_{\lambda}, \lambda = 1, \ldots, \Lambda, \) with size \( N_{\lambda}, \) so that \( U = \bigcup_{\lambda=1}^\Lambda U_{\lambda}, U_{\lambda} \cap U_{\ell} = \emptyset \) if \( \lambda \neq \ell \) and \( N = \sum_{\lambda=1}^\Lambda N_{\lambda}. \) The \( \Lambda \) strata are built thanks to auxiliary information that is relevant to model the shape of the individual trajectories. The mean trajectory can be written
\[
\mu = \sum_{\lambda=1}^\Lambda \frac{N_{\lambda}}{N} \mu_{\lambda}, \tag{25}
\]
where, for each \( \lambda \in \{1, \ldots, \Lambda\}, \mu_{\lambda} \) is the mean trajectory in subpopulation \( U_{\lambda}, \)
\[
\mu_{\lambda} = \frac{1}{N_{\lambda}} \sum_{k \in U_{\lambda}} Y_k. \tag{26}
\]

Different kernel smoothers may be considered in each stratum and the overall kernel approximation to \( \mu \) is obtained by linear combination
\[
\tilde{\mu} = \sum_{\lambda=1}^\Lambda \frac{N_{\lambda}}{N} \tilde{\mu}_{\lambda}. \tag{27}
\]
where, for each \( \lambda \) and each \( t \in [0,T] \),
\[
\tilde{\mu}_\lambda(t) = \sum_{j=1}^d w(t, t_j, h) \mu_\lambda(t_j),
\]
(28)
where the smoothing weights \( w(t, t_j, h) \) are defined as in (3) and are allowed to be different from one stratum to another.

4.1 The estimators of the mean trajectory

In each strata \( \lambda \), a sample \( s_\lambda \) of size \( n_\lambda \) is drawn with simple random sampling without replacement. The non response mechanism is supposed to be homogeneous within each stratum, so that the response probabilities are the same for all units within the stratum: if \( k \in U_\lambda \), then \( \mathbb{P}(r_k(t_j) = 1) = \theta_\lambda(t_j) \). For each stratum \( \lambda \), we can write, as in previous section, three different estimators of \( \tilde{\mu}_\lambda \). Using expressions (9) and (10) and the fact that \( \pi_k = n_\lambda/N_\lambda \) for \( k \in U_\lambda \), we get for \( t \in [0,T] \),
\[
\hat{\mu}_{\lambda,Ha}^{(1)}(t) = \frac{\sum_{j=1}^d K(h^{-1}(t - t_j)) \left( \frac{1}{n_\lambda} \sum_{k \in s_\lambda} Y_k(t_j) \frac{r_k(t_j)}{\theta_\lambda(t_j)} \right)}{\sum_{j=1}^d K(h^{-1}(t - t_j)) \left( \frac{1}{n_\lambda} \sum_{k \in s_\lambda} \frac{r_k(t_j)}{\theta_\lambda(t_j)} \right)},
\]
(29)
and
\[
\hat{\mu}_{\lambda,Ha}^{(2)}(t) = \sum_{j=1}^d w(t, t_j, h) \left( \frac{\sum_{k \in s_\lambda} Y_k(t_j) \frac{r_k(t_j)}{\theta_\lambda(t_j)}}{\sum_{k \in s_\lambda} \frac{r_k(t_j)}{\theta_\lambda(t_j)}} \right),
\]
(30)
whereas
\[
\hat{\mu}_{\lambda,HT}(t) = \sum_{j=1}^d w(t, t_j, h) \left( \frac{1}{n_\lambda} \sum_{k \in s_\lambda} Y_k(t_j) \frac{r_k(t_j)}{\theta_\lambda(t_j)} \right)
\]
(31)
is obtained by considering the estimator defined in (8). The estimator of the mean for the whole population is then obtained by combining the previous estimators,
\[
\hat{\mu}(t) = \sum_{\lambda=1}^\Lambda \frac{N_\lambda}{N} \hat{\mu}_\lambda(t)
\]
where \( \hat{\mu}_\lambda(t) \) is either the estimator defined in (29) or in (31).

4.2 Variance formula for stratified sampling

Note that by independence of the samples \( s_1, \ldots, s_\Lambda \), we have that
\[
\var(\hat{\mu}(t)) = \frac{1}{N^2} \sum_{\lambda=1}^\Lambda N_\lambda^2 \var(\hat{\mu}_\lambda(t)).
\]
(32)
When considering estimator $\hat{\mathbf{\mu}}_{\lambda,HT}(t)$ for $\mathbf{\mu}$, we get

$$
\mathbb{V}(\hat{\mathbf{\mu}}_{\lambda,HT}(t)) = \left(1 - \frac{\lambda}{N_{\lambda}}\right) \frac{1}{n_{\lambda}} \frac{1}{N_{\lambda} - 1} \sum_{k \in U_{\lambda}} \left(\bar{Y}_k(t) - \hat{\mathbf{\mu}}_{\lambda}(t)\right)^2
+ \frac{N_{\lambda}}{n_{\lambda}} \sum_{k \in U_{\lambda}} \sum_{j, j'}^d w_j(t)w_{j'}(t)Y_k(t_j)Y_k(t_{j'}) \frac{\theta_{\lambda}(t_j, t_{j'}) - \theta_{\lambda}(t_j)\theta_{\lambda}(t_{j'})}{\theta_{\lambda}(t_j)\theta_{\lambda}(t_{j'})}
$$

(33)

where $\bar{Y}_k(t) = \sum_{j=1}^d w_j(t)Y_k(t_j)$ is the smoothed trajectory for unit $k$, when there is no non-response. If we consider instead the ratio point of view, we have

$$
\mathbb{V}(\hat{\mathbf{\mu}}_{\lambda,Ha}(t)) \approx \left(1 - \frac{\lambda}{N_{\lambda}}\right) \frac{1}{n_{\lambda}} \frac{1}{N_{\lambda} - 1} \sum_{k \in U_{\lambda}} \left(\bar{Y}_k(t) - \hat{\mathbf{\mu}}_{\lambda}(t)\right)^2
+ \frac{N_{\lambda}}{n_{\lambda}} \sum_{k \in U_{\lambda}} \sum_{j, j'=1}^d w_j(t)w_{j'}(t)(Y_k(t_j) - \hat{\mathbf{\mu}}_{\lambda}(t))(Y_k(t_{j'}) - \hat{\mathbf{\mu}}_{\lambda}(t)) \frac{\theta_{\lambda}(t_j, t_{j'}) - \theta_{\lambda}(t_j)\theta_{\lambda}(t_{j'})}{\theta_{\lambda}(t_j)\theta_{\lambda}(t_{j'})}
$$

(34)

and

$$
\mathbb{V}(\hat{\mathbf{\mu}}_{\lambda,Ha}(t)) \approx \left(1 - \frac{\lambda}{N_{\lambda}}\right) \frac{1}{n_{\lambda}} \frac{1}{N_{\lambda} - 1} \sum_{k \in U_{\lambda}} \left(\bar{Y}_k(t) - \hat{\mathbf{\mu}}_{\lambda}(t)\right)^2
+ \frac{N_{\lambda}}{n_{\lambda}} \sum_{k \in U_{\lambda}} \sum_{j, j'=1}^d w_j(t)w_{j'}(t)(\hat{\mathbf{\mu}}_{\lambda}(t) - \mu_{\lambda}(t))(\hat{\mathbf{\mu}}_{\lambda}(t) - \mu_{\lambda}(t)) \frac{\theta_{\lambda}(t_j, t_{j'}) - \theta_{\lambda}(t_j)\theta_{\lambda}(t_{j'})}{\theta_{\lambda}(t_j)\theta_{\lambda}(t_{j'})}
$$

(35)

Note that, as expected, the part of the variance due to the sampling error is the same for both estimators, since they coincide when no data are missing.

4.3 A comparison of the variances

Defining

$$
\Delta_{\lambda}(j, j') = \frac{\theta_{\lambda}(t_j, t_{j'}) - \theta_{\lambda}(t_j)\theta_{\lambda}(t_{j'})}{\theta_{\lambda}(t_j)\theta_{\lambda}(t_{j'})},
$$

the difference between the variances of the estimators is approximated as follows:

$$
\mathbb{V}(\hat{\mathbf{\mu}}_{\lambda,Ha}(t)) - \mathbb{V}(\hat{\mathbf{\mu}}_{\lambda,HT}(t)) \approx \frac{N_{\lambda}}{n_{\lambda}} \sum_{j, j'=1}^d \Delta_{\lambda}(j, j')w_j(t)w_{j'}(t) \sum_{k \in U_{\lambda}} (\hat{\mathbf{\mu}}_{\lambda}(t) - Y_k(t_j))
$$

$$
= \frac{N_{\lambda}^2}{n_{\lambda}} \bar{\mu}_{\lambda}(t) \sum_{j, j'=1}^d \Delta_{\lambda}(j, j')w_j(t)w_{j'}(t) \left(\hat{\mathbf{\mu}}_{\lambda}(t) - \mu_{\lambda}(t_j) - \mu_{\lambda}(t_{j'})\right)
$$

Considering now the $d \times d$ matrix $\Delta_{\lambda}$, with generic elements $\Delta_{\lambda}(j, j')$, previous difference can be expressed as follows

$$
\mathbb{V}(\hat{\mathbf{\mu}}_{\lambda,Ha}(t)) - \mathbb{V}(\hat{\mathbf{\mu}}_{\lambda,HT}(t)) \approx \frac{N_{\lambda}^2}{n_{\lambda}} (\mathbf{w}(t)\bar{\mu}_{\lambda}(t))^T \Delta_{\lambda} \left(\mathbf{w}(t)\bar{\mu}_{\lambda}(t) - 2\hat{\mathbf{\mu}}_{\lambda}(t)\right)
$$
where \( \mathbf{w}(t) = (w_1(t), \ldots, w_d(t)) \) and \( \hat{\mathbf{\mu}}(t) = (\mu_{\lambda_1}(t_1), \ldots, \mu_{\lambda_d}(t_d)) \). Since the bandwidth \( h \) is small, \( w_j(t) \) is very small (and supposed to be negligible) if \( t \) is not very close to \( t_j \) and \( w_j(t) \approx 1 \) if \( t \approx t_j \). Thus, we can make the following approximation:

\[
(w(t)\hat{\mu}_\lambda(t) - 2\hat{\mu}_\lambda(t)) \approx -w(t)\hat{\mu}_\lambda(t).
\]

Remark that matrix \( \Delta_\lambda \) is non negative (it is a covariance matrix), we finally obtain that

\[
\mathbb{V}(\hat{\mu}^{(1)}_{\lambda,Ha}(t)) - \mathbb{V}(\hat{\mu}_{\lambda,HT}(t)) \approx -(w(t)\hat{\mu}_\lambda(t))^T \Delta_\lambda (w(t)\hat{\mu}_\lambda(t)) \leq 0.
\]

The Hájek estimator \( \hat{\mu}^{(1)}_{\lambda,Ha}(t) \) seems to be preferable to the Horvitz-Thompson estimator \( \hat{\mu}_{\lambda,HT}(t) \) since it has a smaller variance when the bandwidth value \( h \) is small, which has been supposed to have a negligible bias.

5 Some comments about estimation

We discuss in this section some strategies that can be employed to estimate the mean trajectories and the variance of the estimators in practice.

5.1 Variance estimation

For the Hájek type of estimators, we need to estimate the values of the linearized variables in order to build an estimator of the variance. We can consider for example the following variance estimator (see Ardilly & Tillé, 2006, Chapter 9):

\[
\hat{\mathbb{V}}(\hat{\mu}^{(1)}_{r,Ha}(t)) = \frac{1}{N^2} \sum_{k \in s} \sum_{l \in s} \frac{\Delta_{kl}}{\pi_k \pi_l} \hat{u}_k(t)\hat{u}_l(t) + \sum_{k \in s} \frac{1}{\pi_k} \sum_{j,j'=1}^p \hat{u}_{kj}(t)\hat{u}_{kj'}(t) \frac{\left( \theta_k(t_j, t_{j'}) - \theta_k(t_j)\theta_k(t_{j'}) \right)}{\theta_k(t_j)\theta_k(t_{j'})} r_k(t_j)r_k(t_{j'}). \tag{36}
\]

where

\[
\hat{u}_{kj}(t) = \frac{1}{N} w_j(t) \left( Y_k(t_j) - \hat{\mu}^{(1)}_{r,Ha}(t) \right)
\]

and

\[
\hat{w}_k(t) = \sum_{j=1}^d w_j(t) \left( \frac{r_k(t_j)}{\theta_k(t_j)} Y_k(t_j) - \hat{\mu}^{(1)}_{r,Ha}(t) \right).
\]

5.2 Suggestions on how to select the bandwidth values

As it is well known in nonparametric kernel regression, when having to analyze real data, the quality of a nonparametric estimator strongly depends on how the value of the bandwidth is
chosen. For example, it is shown in a similar context with a small simulation study in Cardot et al. (2013) that linear interpolation can outperform kernel smoothing, even if the noise level is rather high, if the value of the bandwidth is chosen by a classical cross-validation performed curve by curve. This individual procedure leads to oversmoothing (see also Hart & Werhly, 1993), so that the bias of the resulting mean estimator is much larger than its variance. As in Cardot, Degras & Josserand (2013), we suggest to use a modified cross-validation in order to choose the value of the bandwidth. This modified criterion takes account of the sampling design as well as the non response process, the bandwidth value is chosen to minimize

\[ CV(h) = \sum_{\lambda=1}^{A} \sum_{k \in s_{\lambda}} \frac{N_{\lambda}}{n_{\lambda}} \sum_{j=1}^{d} \frac{r_{k}(t_{j})}{\theta_{\lambda}(t_{j})} \left( Y_{k}(t_{j}) - \hat{\mu}^{(-k)}(t_{j}) \right)^2 \] (37)

where \( \hat{\mu}^{(-k)} \) is the estimator of the mean trajectory \( \tilde{\mu} \) built without considering trajectory \( Y_{k} \) in the sample \( s \). Note that considering different smoothing parameters in each stratum may not be more effective since it can lead, as noted before, to oversmoothing. Indeed, the best approximations, in terms of mean squared errors, of the mean of each subpopulation, may not lead to the best estimator of the overall mean function.

5.3 Estimation of the probabilities of response \( \theta_{\lambda}(t_{j}) \) and \( \theta_{\lambda}(t_{j}, t'_{j}) \)

In the general situation in which each unit \( k \) in the population is driven by a specific non trivial non response mechanism, it will be almost impossible to estimate the probability of response. If we suppose that, in the stratified sampling context, the units within each stratum obey the same response mechanism, we can consider Horvitz-Thompson estimators for the probabilities of responding. In sub-population \( U_{\lambda} \),

\[ \hat{\theta}_{\lambda}(t_{j}) = \frac{1}{n_{\lambda}} \sum_{k \in s_{\lambda}} r_{k}(t_{j}) \]

and

\[ \hat{\theta}_{\lambda}(t_{j}, t'_{j}) = \frac{1}{n_{\lambda}} \sum_{k \in s_{\lambda}} r_{k}(t_{j}) r_{k}(t'_{j}). \]

We may also suppose that the response process is second order stationary, that is to say \( \hat{\theta}_{\lambda}(t_{j}) \) does not depend on \( t_{j} \) and \( \hat{\theta}_{\lambda}(t_{j}, t'_{j}) \) only depends on \( |t_{j} - t'_{j}| \), so that we would get the following estimators

\[ \hat{\theta}_{\lambda} = \frac{1}{n_{\lambda}} \sum_{k \in s_{\lambda}} \sum_{j=1}^{d} r_{k}(t_{j}) \]

and, for each pair \( (t_{j}, t'_{j}) \) such that \( |t_{j} - t'_{j}| = \Delta_{t} \),

\[ \hat{\theta}_{\lambda}(\Delta_{t}) = \frac{1}{n_{\lambda}} \sum_{k \in s_{\lambda}} \sum_{j, j' \mid |t_{j} - t'_{j}| = \Delta_{t}} r_{k}(t_{j}) r_{k}(t'_{j}). \]
These estimations can be performed either directly on the dataset used for the mean load curve estimation or on a previous and larger one provided that it was collected by meters with the same technical characteristics.

Appendix : technical details

A1. We suppose that function \( \mu \) is \( \beta \)-Hölder. There is \( \beta \in [0, 1] \) and a constant \( C \) such that
\[
\forall (t, u) \in [0, T], \quad |\mu(t) - \mu(u)| \leq C|t - u|^\beta.
\]

A2. We suppose that kernel \( K \) is a continuous positive function with bounded support \([-1, 1]\).

A3. We assume that the instants \( 0 = t_1 < t_2 < \cdots < t_d = T \) are equidistant, \( t_j = (j - 1)/(d - 1) \), \( j = 1, \ldots, d \) and the bandwidth satisfies \( 2h > T(d - 1)^{-1} \).

Conditions A1 and A2 are classical hypotheses in non parametric regression. Assumption A2 is satisfied for example if \( K \) is the Epanechnikov kernel. Condition A3 ensures that the grid of discretization points is fine enough and that the bandwidth \( h \) is not too small so that the estimator is well defined.

Let us write \( \tilde{\mu}(t) - \mu(t) \) as follows
\[
\tilde{\mu}(t) - \mu(t) = \frac{1}{hd} \sum_{j=1}^{d} K \left( \frac{t - t_j}{h} \right) (\mu(t_j) - \mu(t))
\]
(38)

Since kernel \( K \) has a bounded support, we have that \( K \left( \frac{t_j - t}{h} \right) > 0 \) only if \( t_j \in [t - h, t + h] \). Since by assumption the instants of observation are equidistant in \([0, T]\), i.e. \( t_j = T(j - 1)/(p - 1) \), \( j = 1, \ldots, p \), there are at most \( 2h(d - 1)/T \) values of \( K \left( \frac{t_j - t}{h} \right) \) that are strictly positive and hypothesis A3 prevents all the terms in (38) from being equal to zero. Since function \( \mu \) is \( \beta \)-Hölder, we have
\[
\frac{1}{d} \sum_{j=1}^{d} K \left( \frac{t_j - t}{h} \right) (\mu(t_j) - \mu(t)) \leq \frac{1}{d} \sum_{t_j \in [t-h, t+h]} \frac{1}{h} K \left( \frac{t_j - t}{h} \right) |\mu(t_j) - \mu(t)|
\]
\[
\leq C|2h|^\beta \frac{1}{d} \sum_{t_j \in [t-h, t+h]} \frac{1}{h} K \left( \frac{t_j - t}{h} \right)
\]

By Riemann sum approximation and the fact that kernel \( K \) is continuous, with compact support, we get that, as \( d \to \infty \),
\[
\left| \frac{1}{d} \sum_{j=1}^{d} \frac{1}{h} K \left( \frac{t_j - t}{h} \right) - \frac{1}{h} \int K \left( \frac{u - t}{h} \right) du \right| \to 0
\]
and by the change of variable $x = (u - t)/h$ we have 
\[
\frac{1}{h} \int_{\mathbb{R}} K \left( \frac{u-t}{h} \right) \, du = \int_{\mathbb{R}} K(x) \, dx < +\infty.
\]
We have proved that the bound given in [13] is true.

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