IMPROVED MULTILINEAR ESTIMATES AND GLOBAL REGULARITY FOR GENERAL NONLINEAR WAVE EQUATIONS IN \((1 + 3)\) DIMENSIONS

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Abstract. This paper is devoted to the investigation of long-time behaviour of solutions to wave equations with quadratic nonlinearity and cubic Dirac equations with Hartree-type nonlinearity. We consider the nonlinearity here with enough simplicity so that we can treat it as a toy model and simultaneously with enough generality so that we can apply our result to wave and Dirac equations with various nonlinearities.

The challenging point is that nonlinearity possesses singularity near the origin. Our strategy is to relax such a singularity by exploiting fully an angular momentum operator. In this manner we establish scattering for the critical Sobolev data.

1. Introduction

For several decades wave equations have appealed to a lot of interest and have been extensively studied in many works of literature. It plays a role as a model to explain various physical phenomena and in the mathematical literature, the study of the wave equations becomes the very first step to shattering the light on the investigation of hyperbolic partial differential equations.

We are interested in time-evolution of solutions to wave equations with various nonlinearities for low regularity initial data. In the investigation it is important to control the nonlinearity in terms of the initial data. In other words, we have to prove that the presence of nonlinearity turns out to be nothing but a small perturbation. Such a perturbative method to wave equations and even more general dispersive equations is a typical approach to the study of Cauchy problems. The first well-known tool is so-called Stricharz estimates

\[ \| e^{-it|\nabla|} P_1 f \|_{L^q_t L^r_x (\mathbb{R}^{1+n})} \lesssim \| P_1 f \|_{L^2_x} \]

for any function \( f \). Here \( P_1 f \) is the projection onto the unit frequency and \((q,r)\) is a proper admissible pair. However, the linear estimate is not sufficient to control over the frequency-interactions between the products of homogeneous solutions especially when we are concerned with well-posedness problem for a low regularity data. This problem requires one to delicately consider the following bilinear estimates

\[ \| e^{-it|\nabla|} f e^{-it|\nabla|} g \|_{L^2_t L^2_x} \lesssim \| f \|_{L^2_x} \| g \|_{L^2_x}. \]

In fact, when nonlinearity is given by power-type, nonlinear estimates are reduced to bilinear estimates. Recently there has been a huge amount of progress on bilinear estimates of wave-type by many works of \cite{13, 19, 20, 25, 33, 34, 35, 36, 37} and long-time behaviour of solutions to nonlinear wave equations and even more complicated systems such as the Maxwell-Klein-Gordon or the Yang-Mills equations is well-known for \((1 + 4)\) dimensions and higher dimensional setting \cite{23, 24, 31, 27}.

However, global regularity is still open for the most of wave equations in a low dimensional setting such as \((1 + 3)\) or \((1 + 2)\) dimensions. At a first glimpse this is obviously because of the weaker time-decay of

\[ e^{-it|\nabla|} \]

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solutions in a low dimensional setting \( \|e^{-it|\nabla|}P_1f\|_{L^\infty_t(L^n)} \lesssim t^{-\frac{n}{2}}\|P_1f\|_{L^1_t} \). Moreover, at the nonlinear level, one can see that resonant interactions grows stronger as the spatial dimensions decrease. Even further, when nonlinearity possesses singularity near the origin, one may encounter more serious situation since the singularity grows harsher in a low dimension. From these several problems one may have a question whether it is possible to establish global well-posedness and scattering for the scale-invariant Sobolev data. To overcome this difficulty we equip the Sobolev spaces with an extra weighted smoothness assumptions with respect to the angular variables. Indeed, we invoke the infinitesimal rotation generators \( \Omega_{ij} = x_i \partial_j - x_j \partial_i \). In the spirit of [30], \( \Omega_{ij} \) plays a crucial role in the aspect of both linear and multilinear estimates. More precisely, one enjoy a significant improvement of linear estimates. At the nonlinear estimates, the rotation operator helps to overcome the resonant interactions. Even more, the rotation can relax the harshness of the singularity since the operator \( \Omega_{ij} \) works very favourably in the low-output interactions. In this manner, it is possible to improve the bilinear estimates.

Now we turn to an application of an improved multilinear estimate on \( \mathbb{R}^{1+3} \). We are concerned with somewhat a general class of quadratic nonlinear wave equations and the Hartree-type nonlinear Dirac equations which becomes a toy model for several nonlinear wave and Dirac equations. The following equations we shall present seem too primitive at a first glimpse, however, by the primitiveness and generality of a toy model we can attack efficiently even more complicated system such as gauge-field-theoretic wave equations which represent a genuinely nonphysical model.

1.1. Quadratic nonlinear wave equations. Firstly we aim to investigate global-in-time evolution of wave equations in \( \mathbb{R}^{1+3} \) with quite a general quadratic nonlinearity given by

\[
\begin{align*}
\Box u &= |\nabla|^{-1}Q(\nabla, u), \\
(u, \partial_t u)|_{t=0} &= (u_0, u_1),
\end{align*}
\]

where \( u \) is a complex-valued function on \( \mathbb{R}^{1+3} \) and \( Q : (u, v) \mapsto Q(u, v) \) is a bilinear form which is a finite linear combination of the standard \( Q \)-type null forms

\[Q_{ij}(u,v) = \partial_i u \partial_j v - \partial_j u \partial_i v, \quad Q_0(u,v) = \partial_t u \partial_t v - \nabla u \cdot \nabla v,\]

which give the cancellation by angle between input-frequency\(^1\). More precisely the Fourier transform of \( Q_{ij}(u,v) \) is

\[|\hat{Q}_{ij}(\xi, \eta)|(\xi) \lesssim \int_{\xi+\eta} \xi(\xi, \eta)\xi||\eta|\eta(\xi)|\bar{\eta}(\eta)\,d\xi d\eta.\]

The wave equation (1.1) has the scaling symmetry, i.e., if \( u = u(t,x), \quad (t,x) \in \mathbb{R}^{1+3} \) is a solution of (1.1) then the scaled function \( \lambda^{-1}u(\lambda^{-1}t, \lambda^{-1}x) \) will be also a solution to the equation (1.1) for any \( \lambda > 0 \) and hence the scale-invariant Sobolev space for the initial data \( (u_0, u_1) \) is \( \dot{H}^s_{\sigma} \times H^{-\frac{s}{2}} \), where \( \dot{H}^s \) is the usual homogeneous Sobolev space. Now we define the angularly regular space \( \dot{H}^s_{\sigma} \) to be \( \|f\|_{\dot{H}^s_{\sigma}} = \|\Omega\|^s f \|_{\dot{H}^s} \), where \( \langle \Omega \rangle^s = (1 - \Delta_{\mathbb{S}^2})^\frac{s}{2} \) and \( \Delta_{\mathbb{S}^2} \) is the Laplace-Beltrami operator on the unit sphere \( S^2 \subset \mathbb{R}^3 \). The inhomogeneous Sobolev space with angular regularity \( H^s_{\sigma} \) is defined in the obvious way. We state our first main result.

\(^1\)In fact, the null form \( Q_0 \) gives stronger cancellation, and we can overcome the singularity \( |\nabla|^{-1} \) more easily by exploiting the \( Q_0 \) null form. However, for the generality of our result, we focus on the \( Q_{ij} \) null form.
Theorem 1.1. Let \( \sigma = 1 \). Suppose that the initial datum \((u_0, u_1) \in H^\frac{2}{3}_\sigma \times \dot{H}^{-\frac{1}{6}}_\sigma\) satisfies
\[
\| (u_0, u_1) \|_{H^\frac{2}{3}_\sigma \times \dot{H}^{-\frac{1}{6}}_\sigma} = \| u_0 \|_{H^\frac{2}{3}_\sigma} + \| u_1 \|_{\dot{H}^{-\frac{1}{6}}_\sigma} \ll 1.
\]
The Cauchy problem for the equation \((1.1)\) is globally well-posed and scatters to free solutions as \( t \to \pm \infty \).

1.1. Application to the Maxwell-Klein-Gordon equations in the Coulomb gauge. We would like to mention here briefly an application of Theorem 1.1. The Maxwell-Klein-Gordon system is a physical model for the interaction of a spin 0 particle with electromagnetic fields. We define the real-valued gauge potentials \( A_\mu \), \( \mu = 0, 1, \cdots, 3 \) on the Minkowski space \((\mathbb{R}^{1+3}, m)\), where the metric \( m \) is given by \( m = \text{diag}(-1, 1, 1, 1) \). The covariant derivative is given by \( D_\mu = \partial_\mu + i A_\mu \). The electromagnetic field \( F \) associated to the potential \( A_\mu \) is defined by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). Then the covariant form of the Maxwell-Klein-Gordon system presents
\[
\partial_\mu F^\mu\nu = -\text{Im}(\phi D^\nu \phi),
\]
\[
D_\mu D^\mu \phi = 0,
\]
where \( \text{Im}(A) \) is the imaginary part of \( A \). Note that we adapt the usual summation convention with respect to repeated indices. The Maxwell-Klein-Gordon system has a gauge-invariance. Indeed, if \((A_\mu, \phi)\) is a solution to the system, then for any real-valued smooth function \( \Lambda \) on \( \mathbb{R}^{1+3} \), the set \((A_\mu + \partial_\mu \Lambda, e^{-i\Lambda} \phi)\) is also a solution to the system. This observation allows one to enjoy the gauge-freedom. Now we impose the Coulomb gauge condition: \( \text{div} A = \partial_j A^j = 0 \). Then after an application of the projection \( P = -\frac{\text{curl}^2}{4} \) we see that the spatial parts of the gauge potentials obey the following wave equation
\[
\Box A_j = -\text{Im}(P(\phi D_j \phi)).
\]
Then the quadratic nonlinearity in the wave equation \((1.3)\) presents a finite linear combination of the \( Q \)-type null forms as \( \Delta^{-1} \partial_k Q_{ij}(\psi, \bar{\psi}) \), which turns out to be the nonlinearity in our toy model \((1.1)\). We refer the readers to [29] for more details on the Maxwell-Klein-Gordon system.

The Maxwell-Klein-Gordon system is one of well-studied gauge-field-theoretic wave equations. In \((1 + 4)\) dimensional setting, the global dynamics of solutions to the system are shown by Oh and Tataru [27, 28, 29]. However, global solutions to the system in \((1 + 3)\) dimensions is still open. The main drawback of the system is the strong singularity in the quadratic nonlinearity \( |\nabla|^{-1} Q_{ij}(\psi, \bar{\psi}) \). Our first main result provides a partial answer on the question of the scattering property of solutions to the Maxwell-Klein-Gordon system for the scale-invariant Sobolev regularity.

1.2. Cubic Dirac equations. Secondly we would like to investigate long-time behaviour of solutions to cubic Dirac equations with the Hartree-type nonlinearity
\[
\begin{aligned}
&-i\gamma^\mu \partial_\mu \psi + m \psi = [V_b \ast (\psi^\dagger \psi)]^0 \psi, \\
&\psi|_{t=0} = \psi_0,
\end{aligned}
\]
where \( V_b = V_b(x) \) is the Yukawa-type potential given by
\[
V_b(x) = \frac{1}{4\pi} \frac{e^{-b|x|}}{|x|}, \quad b > 0,
\]
and \( m > 0 \) is a positive mass. Recall that we adapt the summation convention. Here \( \psi : \mathbb{R}^{1+3} \to \mathbb{C}^4 \) is the Dirac spinor field and \( \psi^\dagger \) is the complex conjugate transpose of \( \psi \), i.e., \( \psi^\dagger = (\psi^*)^T \). The Dirac gamma
matrices \( \gamma^\mu \) are the \( 4 \times 4 \) complex matrices given by

\[
\gamma^0 = \begin{bmatrix} I_{2 \times 2} & 0 \\ 0 & -I_{2 \times 2} \end{bmatrix}, \quad \gamma^j = \begin{bmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{bmatrix},
\]

with the Pauli matrices \( \sigma^j, j = 1, 2, 3 \) given by

\[
\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

As Theorem 1.1 we prove the global well-posedness and scattering for the scaling critical Sobolev data.

**Theorem 1.2.** Let \( \sigma = 1 \). Suppose that the initial data \( \psi_0 \in L^2_\sigma \) satisfies \( \| \psi_0 \|_{L^2_\sigma} \ll 1 \). The Cauchy problem for the equation (1.4) is globally well-posed and scatters to free solutions as \( t \to \pm \infty \).

### 1.2.1. Application to nonlinear Dirac equations.

Now we shall discuss an application of Theorem 1.2. First of all it is instructive to introduce the general form of the Dirac-Klein-Gordon system. Indeed, cubic Dirac equations of the form (1.4) can be obtained by uncoupling the Dirac-Klein-Gordon system

\[
\begin{cases}
(-i\gamma^\mu \partial_\mu + M)\psi = g\phi \Gamma \psi, \\
(\Box + m^2)\phi = -g\psi^\dagger \gamma^0 \Gamma \psi.
\end{cases}
\]

Here \( g \) is a coupling constant and we put \( g = 1 \) for simplicity. The \( 4 \times 4 \) matrix \( \Gamma \) can be chosen properly by researchers, for example, \( \Gamma = I_{4 \times 4}, \gamma^0, -\gamma^0 \gamma^1 \gamma^2 \gamma^3 \). From (1.5) one can obtain cubic Dirac equations of the form

\[
(-i\gamma^\mu \partial_\mu + M)\psi = V_b^\dagger (\psi^\dagger \gamma^0 \Gamma \psi) \Gamma \psi.
\]

We refer the readers to [38, 39, 40] for more detailed derivation from the system (1.5) to (1.6). Recently the nonlinear Dirac systems (1.5) and (1.6) with \( \Gamma = I_{4 \times 4} \) have been extensively studied. See [1, 3, 5, 6, 7, 9, 11] and reference therein. For the case \( \Gamma = \gamma^\mu \) and the Klein-Gordon field \( \phi \) replaced by the vector potential \( A_\mu \) with \( m = 0 \), the system (1.5) becomes the Maxwell-Dirac system [2, 12]. In the case \( \Gamma = I_{4 \times 4} \), it is crucial to exploit the null structure in the bilinear form \( \psi^\dagger \gamma^0 \psi \) to attain low regularity well-posedness. If \( \Gamma = \gamma^0 \), however, one cannot enjoy such an advantage and in consequence it is not easy to obtain global well-posedness for a low regularity data. Our second main result says that one can establish scattering property even when it is not possible to take an advantage of null structures.

We would like to mention the Cauchy problems for the boson star equation (or the semi-relativistic equation) with the Hartree-type nonlinearity on \( \mathbb{R}^{1+3} \):

\[
\begin{cases}
-i\partial_t u + \sqrt{m^2 - \Delta} u = (V_b * |u|^2)u, \\
u|_{t=0} = u_0
\end{cases}
\]

We refer to [11, 15, 16] for this well-studied equation. After the use of the Dirac projection operators (see Section 2.2) our Dirac equations (1.4) is of the form (1.7). Thus as a direct application of Theorem 1.2 we have

**Corollary 1.3.** Let \( \sigma = 1 \). Suppose that the initial data \( u_0 \in L^2_\sigma \) satisfies \( \| u_0 \|_{L^2_\sigma} \ll 1 \). The Cauchy problems for the equation (1.7) is globally well-posed and scatters to free solutions as \( t \to \pm \infty \).

By Corollary 1.3, we improve the previous results on the Cauchy problems for (1.7) and attain the scaling critical regularity.
The rest of this paper is organised as follows. In the next section, we give some preliminaries which include half-wave decompositions, Dirac operators, multipliers, definition and basic properties on $U^p - V^p$ spaces and auxiliary estimates. Section 3 and Section 4 are devoted to the proof of our main results, Theorem 1.1 and Theorem 1.2 respectively.

**Notations.**

1. As usual different positive constants, which are independent of dyadic numbers $\mu, \lambda$, and $h$ are denoted by the same letter $C$, if not specified. The inequalities $A \lesssim B$ and $A \gtrsim B$ means that $A \leq CB$ and $A \geq C^{-1}B$, respectively for some $C > 0$. By the notation $A \approx B$ we mean that $A \lesssim B$ and $A \gtrsim B$, i.e., $\frac{1}{C}B \leq A \leq CB$ for some absolute constant $C$. We also use the notation $A \ll B$ if $A \leq C^{-1}B$ for some large constant $C$. Thus for quantities $A$ and $B$, we can consider three cases: $A \approx B$, $A \ll B$ and $A \gg B$. In fact, $A \approx B$ means that $A \lesssim B$ or $A \ll B$.

The spatial and space-time Fourier transform are defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) \, dx, \quad \tilde{u}(\tau, \xi) = \int_{\mathbb{R}^{1+3}} e^{-i(t\tau + x \cdot \xi)} u(t, x) \, dtdx.$$  

We also write $F_{x}(f) = \hat{f}$ and $F_{t,x}(u) = \tilde{u}$. We denote the backward and forward wave propagation of a function $f$ on $\mathbb{R}^3$ by

$$e^{-\theta|\nabla|f} = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix \cdot \xi} e^{-\theta|\xi|} \hat{f}(\xi) \, d\xi;$$

where $\theta \in \{+, -\}$.

2. We fix a smooth function $\rho \in C_0^\infty(\mathbb{R})$ such that $\rho$ is supported in the set $\{\frac{1}{2} < r < 2\}$ and we let

$$\sum_{\lambda \in 2^\mathbb{N}} \rho(\lambda) = 1,$$

and write $\rho_1 = \sum_{\lambda \leq 1} \rho(\lambda)$ with $\rho_1(0) = 1$. Now we define the standard Littlewood-Paley multipliers for $\lambda \in 2^\mathbb{N}$ and $\lambda > 1$:

$$P_\lambda = \rho\left(\frac{|-i\nabla|}{\lambda}\right), \quad P_1 = \rho_1(|-i\nabla|).$$

2. Preliminaries

2.1. **Half-wave decomposition of the d’Alembertian.** We formulate nonlinear wave equations $\Box u = F$ as a first-order system, which clarifies the dispersive properties of a nonlinear wave. (See also [17].) We first write

$$\frac{\partial}{\partial t} \left[ \begin{array}{c} u \\ \partial_t u \end{array} \right] = \left[ \begin{array}{cc} 0 & 1 \\ \Delta & 0 \end{array} \right] \left[ \begin{array}{c} u \\ \partial_t u \end{array} \right] + \left[ \begin{array}{c} 0 \\ F \end{array} \right].$$

We make use of the transform

$$(u, \partial_t u) \rightarrow (u_+, u_-), \quad (0, F) \rightarrow (F_+, F_-),$$

where

$$u_\pm = \frac{1}{2} \left( u \mp \frac{1}{i|\nabla|} \partial_t u \right), \quad F_\pm = \mp \frac{1}{2i|\nabla|} F,$$

with $|\nabla| = \sqrt{-\Delta}$, which yields the following diagonal system

$$\frac{\partial}{\partial t} \left[ \begin{array}{c} u_+ \\ u_- \end{array} \right] = \left[ \begin{array}{cc} -i|\nabla| & 0 \\ 0 & +i|\nabla| \end{array} \right] \left[ \begin{array}{c} u_+ \\ u_- \end{array} \right] + \left[ \begin{array}{c} F_+ \\ F_- \end{array} \right].$$
This is equivalent to the following half-wave equations

\begin{equation}
(-i\partial_t + \theta|\nabla|)u_\theta = \theta\frac{1}{2|\nabla|} F,
\end{equation}

where \( \theta \in \{+,-\} \). Thus we conclude that the initial value problems for the equation (1.1) is reduced to the following first-order system of nonlinear Klein-Gordon equations

\begin{equation}
\begin{cases}
(-i\partial_t + \theta|\nabla|)u_\theta = \theta|\nabla|^{-2}Q(\vec{\sigma},u), \\
u_\theta|_{t=0} = u_{0,\theta}.
\end{cases}
\end{equation}

2.2. **Dirac projection operators.** Recall the Dirac equations (1.4)

\[-i\gamma^\mu \partial_\mu \psi + m\psi = [V * (\psi^\dagger\psi)]\gamma^0\psi.\]

We would like to decompose the Dirac equations and obtain a similar form of a first-order system of half-wave equations as we have done in the previous section. To do this, we first introduce the projections for \( \theta \in \{+,-\} \)

\begin{equation}
\Pi_\theta(\xi) = \frac{1}{2} \left( I_{4\times4} + \theta\xi^j\gamma^0\gamma^j + m\gamma^0 \right),
\end{equation}

where we used the summation convention and the gamma matrices \( \gamma^\mu \in \mathbb{C}^{4\times4}, \mu = 0, 1, 2, 3 \) are given by

\[
\gamma^0 = \begin{bmatrix} I_{2\times2} & 0 \\ 0 & -I_{2\times2} \end{bmatrix}, \quad \gamma^j = \begin{bmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{bmatrix},
\]

with the Pauli matrices \( \sigma^j \in \mathbb{C}^{2\times2}, j = 1, 2, 3 \) given by

\[
\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

Now we define the Fourier multiplier by the identity \( F_x[\Pi_\theta f](\xi) = \Pi_\theta(\xi)\hat{f}(\xi) \). By an easy computation one easily see the identity \( \Pi_+\Pi_- = \Pi_-\Pi_+ = 0 \) and \( \Pi_+\Pi_-\psi = 0 \). We also have \( \psi = \Pi_+\psi + \Pi_-\psi \). Then we see that

\[
(-i\gamma^\mu \partial_\mu + m)\Pi_\theta \psi = \gamma^0(-i\partial_t + \theta|\nabla|m)\psi
\]

and hence we conclude that the initial value problems for the equations (1.4) is reduced to the following first-order system of nonlinear Klein-Gordon equations

\begin{equation}
\begin{cases}
(-i\partial_t + \theta|\nabla|m)\psi_\theta = \Pi_\theta[|V\ast(\psi^\dagger\psi)|], \\
\psi_\theta|_{t=0} = \psi_{0,\theta},
\end{cases}
\end{equation}

where \( \psi_\theta = \Pi_\theta\psi \).

2.3. **Multipliers.** We define \( Q_\mu \) to be a finitely overlapping collection of cubes of diameter \( \frac{\alpha}{1000} \) covering \( \mathbb{R}^3 \), and let \( \{\rho_\delta\}_{\delta \in Q_\mu} \) be a corresponding subordinate partition of unity. For \( q \in Q_\mu, d \in 2^2 \) let

\[
P_q = \rho_q(-i\nabla), \quad C^d_\alpha = \rho \left( \frac{|-i\partial_t + \theta|\nabla|}{d} \right).
\]

We define \( C^\theta_{\leq d} = \sum_{\delta \leq d} C^\theta_\delta \). Given \( 0 < \alpha \lesssim 1 \), we define \( C_\alpha \) to be a collection of finitely overlapping caps of radius \( \alpha \) on the sphere \( S^2 \). If \( \kappa \in C_\alpha \), we let \( \omega_\kappa \) be the centre of the cap \( \kappa \). Then we define \( \{\rho_\kappa\}_{\kappa \in C_\alpha} \) to be a smooth partition of unity subordinate to the conic sectors \( \{\xi \neq 0, \frac{\xi}{|\xi|} \in \kappa\} \) and denote the angular localisation Fourier multipliers by \( R_\kappa = \rho_\kappa(-i\nabla) \).
2.4. Analysis on the sphere. We recall some basic facts from harmonic analysis on the unit sphere. We refer the readers to [30] for the most of ingredients in this section. We let \( Y_\ell \) be the set of homogeneous harmonic polynomial of degree \( \ell \) on \( \mathbb{R}^3 \). Then define \( \{ y_{\ell,n} \}_{n=0}^{2\ell} \) a set of orthonormal basis for \( Y_\ell \), with respect to the inner product:

\[
\langle y_{\ell,n}, y_{\ell',n'} \rangle_{L^2_\omega(S^2)} = \int_{S^2} y_{\ell,n}(\omega) y_{\ell',n'}(\omega) \, d\omega.
\]

Given \( f \in L^2_\omega(\mathbb{R}^3) \), we have the orthogonal decomposition as follow:

\[
f(x) = \sum_{\ell} \sum_{n=0}^{2\ell} \langle f(|x|\omega), y_{\ell,n}(\omega) \rangle_{L^2_\omega(S^2)} y_{\ell,n}(\frac{x}{|x|}).
\]

For a dyadic number \( N > 1 \), we define the spherical Littlewood-Paley decompositions by

\[
H_N(f)(x) = \sum_{\ell} \sum_{n=0}^{2\ell} \rho \left( \frac{\ell}{N} \right) \langle f(|x|\omega), y_{\ell,n}(\omega) \rangle_{L^2_\omega(S^2)} y_{\ell,n}(\frac{x}{|x|}),
\]

\[
H_1(f)(x) = \sum_{\ell} \sum_{n=0}^{2\ell} \rho_{\leq 1}(\ell) \langle f(|x|\omega), y_{\ell,n}(\omega) \rangle_{L^2_\omega(S^2)} y_{\ell,n}(\frac{x}{|x|}).
\]

Since \(-\Delta_{S^2} y_{\ell,n} = \ell(\ell + 1)y_{\ell,n}\), by orthogonality one can readily get

\[
\| \langle \Omega \rangle^\sigma f \|_{L^2_\omega(S^2)} \approx \left\| \sum_{N \in 2^{\mathbb{N}}} N^\sigma H_N f \right\|_{L^2_\omega(S^2)}.
\]

**Lemma 2.1** (Lemma 7.1 of [30]). Let \( N \geq 1 \). Then \( H_N \) is uniformly bounded on \( L^p(\mathbb{R}^3) \) in \( N \), and \( H_N \) commutes with all radial Fourier multipliers. Moreover, if \( N' \geq 1 \), then either \( N \approx N' \) or

\[
H_N \Pi_0 H_{N'} = 0.
\]

As an application of Lemma 2.1 one can say that the spherical harmonic projections \( H_N \) commutes with the Littlewood-Paley projections such as \( P_\alpha \) and \( C^\alpha_{\beta} \). Furthermore the orthogonality of the spherical harmonics still holds when one deals with the Dirac projections.

2.5. Adapted function spaces. We discuss the basic properties of function spaces of \( U^p \) and \( V^p \) type. We refer the readers to [31] for more details. Let \( \mathcal{I} \) be the set of finite partitions \(-\infty = t_0 < t_1 < \cdots < t_K = \infty \) and let \( 1 \leq p < \infty \).

**Definition 2.2.** A function \( a : \mathbb{R} \to L^2_x \) is called a \( U^p \)-atom if there exists a decomposition

\[
a = \sum_{j=1}^{K} \chi_{[t_{k-1},t_k)}(t) f_j - 1
\]

with

\[
\{ f_j \}_{j=0}^{K-1} \subset L^2_x, \quad \sum_{j=0}^{K-1} \| f_j \|^p_{L^2_x} = 1, \quad f_0 = 0.
\]

Furthermore, we define the atomic Banach space

\[
U^p := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j : a_j \text{ \( U^p \)-atom, } \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}
\]
with the induced norm
\[ \|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j, \lambda_j \in \mathbb{C}, a_j U^p\text{-atom} \right\}. \]

We list some basic properties of $U^p$ spaces.

**Proposition 2.3** (Proposition 2.2 of [14]). Let $1 \leq p < q < \infty$.

1. $U^p$ is a Banach space.
2. The embeddings $U^p \subset U^q \subset L^\infty(\mathbb{R}; L^2_\mathbb{R})$ are continuous.
3. For $u \in U^p$, $u$ is right-continuous.

We also define the space $U^p_\theta$ to be the set of all $u \in \mathbb{R} \to L^2_\mathbb{R}$ such that $e^{-\theta|\nabla|}u \in U^p$ with the obvious norm $\|u\|_{U^p_\theta} := \|e^{-\theta|\nabla|}u\|_{U^p}$. We define the 2-variation of $v$ to be
\[ |v|_{V^2} = \sup_{\{t_k\}_{k=0}^K} \left( \sum_{k=0}^{K} \|v(t_k) - v(t_{k-1})\|_{L^2_\mathbb{R}}^2 \right)^{\frac{1}{2}}. \]

Then the Banach space $V^2$ can be defined to be all right-continuous functions $v : \mathbb{R} \to L^2_\mathbb{R}$ such that the quantity
\[ \|v\|_{V^2} = \|v\|_{L^\infty_t L^2_x} + |v|_{V^2} \]
is finite. Set $\|u\|_{V^2_\theta} = \|e^{-\theta|\nabla|}u\|_{V^2}$. We recall basic properties of $V^2_\theta$ space from [5, 6, 14]. In particular, we use the following lemma to prove the scattering result.

**Lemma 2.4** (Lemma 7.4 of [5]). Let $u \in V^2_\theta$. Then there exists $f \in L^2_\mathbb{R}$ such that $\|u(t) - e^{-\theta|\nabla|}f\|_{L^2_\mathbb{R}} \to 0$ as $t \to \pm\infty$.

The following lemma is on a simple bound in the high-modulation region.

**Lemma 2.5** (Corollary 2.18 of [14]). Let $2 \leq q \leq \infty$. For $d \in 2\mathbb{Z}$ and $\theta \in \{+,-\}$, we have
\[ \|C_\theta^d u\|_{L^q_t L^2_x} \lesssim d^{-\frac{d}{4}} \|u\|_{V^2_\theta}, \]

**Lemma 2.6** (Lemma 2.2 of [7]). Let $u \in U^p$ be absolutely continuous with $1 < p < \infty$. Then
\[ \|u\|_{U^p} = \sup \left\{ \left| \int \langle u'(t), v(t) \rangle_{L^2_\mathbb{R}} dt \right| : v \in C_0^\infty, \|v\|_{V^2_t} = 1 \right\}. \]

We define the Banach space associated with the homogeneous Sobolev space to be the set
\[ \hat{F}^s_\theta = \left\{ u \in C(\mathbb{R} : \langle \Omega \rangle^{-\sigma} \dot{H}^s) : \|u\|_{\hat{F}^s_\theta} < \infty \right\}, \]
where the norm is defined by
\[ \|u\|_{\hat{F}^s_\theta} = \left( \sum_{\lambda \in \mathbb{Z}} \sum_{N \geq 1} \lambda^{2s} N^{2\sigma} \|P_\lambda H_N u\|_{U^0_\theta}^2 \right)^{\frac{1}{2}}. \]

Similarly we define the Banach space $F^s_\theta$ associated to the inhomogenous Sobolev space in the obvious way.

**Remark 2.7.** So far we have defined the adapted function spaces for the wave operator. However, with a slight modification we can also define the adapted function spaces for the Klein-Gordon-type operator and all the above lemma also holds for the Klein-Gordon operator. In consequence, for brevity we allow abuse of notation and simply use the notation $U^p_\theta$ and $V^p_\theta$ for both wave and Klein-Gordon operators. We also refer the readers to [5].
2.6. Auxiliary estimates. We begin with very basic Sobolev estimates which is also known as the Bernstein inequality.

Lemma 2.8. Let $0 < \alpha \lesssim 1$ and $\kappa \in C_\alpha$. Let $\lambda > 0$ be a dyadic number. For any test function $f$ on $\mathbb{R}^3$ we have
\[
\| R_\kappa P_\lambda f \|_{L^p_x} \lesssim (\lambda^3 \alpha^2)^{\frac{1}{2}} \| f \|_{L^p_x}.
\]

To study the dispersive property of solutions it is of great importance to exploit so-called the Strichartz estimates [18, 32]. In this paper we use an improved Strichartz estimate which is obtained by spending an extra regularity with respect to the angular variables. (See also [8, 30].)

Proposition 2.9. For $\frac{1}{\eta} \geq \eta > 0$, let $q_\eta = \frac{1}{1-\eta}$. We have the improved Strichartz estimates by imposing angular regularity as follow:
\[
\| e^{\theta t|\nabla|} P_\lambda H_N f \|_{L^p_t L^q_x} \lesssim \lambda^{1 - \frac{\eta}{\eta q_\eta}} N^{\frac{1}{2} + \eta} \| P_\lambda H_N f \|_{L^q_x}.
\]

Note that the estimates hold when we replace the propagator $e^{-\mu t|\nabla|}$ with $e^{-\mu t|\nabla|}$. The space-time estimates (2.10) say that one can obtain an improved bound when dealing with multilinear estimates. However, the singularity $|\nabla|^{-2}$ in the nonlinearity in (1.1) is too strong and we cannot obtain the desired well-posedness for the critical Sobolev data by simply using the estimate (2.10). This is why the low-output frequency interaction becomes the most serious case. To overcome this problem, we apply the almost orthogonal decompositions of conic sectors. The question is whether one can obtain the better estimates by exploiting the localisation into the conic sectors. The following lemma which is also known as angular concentration estimates answers this question.

Lemma 2.10 (Lemma 8.5 of [5]). Let $2 \leq p < \infty$, and $0 \leq s < \frac{2}{p}$. If $\lambda, N \geq 1$, $\alpha \gtrsim \lambda^{-1}$, and $\kappa \in C_\alpha$, then we have
\[
\| R_\kappa P_\lambda H_N f \|_{L^p_x(\mathbb{R}^3)} \lesssim (\alpha N)^{\frac{s}{2}} \| P_\lambda H_N f \|_{L^p_x(\mathbb{R}^3)}.
\]

We refer the readers to [31] for the proof. Note that in the Bernstein inequality, it is no harm to put $f \rightarrow R_\kappa P_{2\lambda} f$ with $\kappa' \in C_{2\alpha}$. With cube localisation $P_q$ of size $\mu \leq \lambda$, we use in order the Bernstein inequality, angular concentration estimates, and then the improved Strichartz estimates. Here one should note that $\frac{1}{2} - 2\eta < \frac{2}{\eta q_\eta} = \frac{1-\eta}{2}$. Then we see that
\[
\| P_q R_\kappa P_\lambda H_N u \|_{L^2_t L^\infty_x} \lesssim (\mu^3 \alpha^2)^{\frac{1}{4}} \| P_\lambda H_N u \|_{L^2_t L^\infty_x}
\]
\[
\lesssim (\mu^3 \alpha^2)^{\frac{1}{4}} (\alpha N)^{\frac{1}{2} - 2\eta} \| P_\lambda H_N u \|_{L^2_t L^\infty_x}
\]
\[
\lesssim \mu^{\frac{3}{4}} \alpha^{\frac{1}{4} + 2\eta} N^{\frac{1}{2} - 2\eta} \lambda^{1 - \frac{\eta q_\eta}{\eta q_\eta}} \| P_\lambda H_N u \|_{L^2_x}.
\]

The above argument will be often used in the proof of Theorem 1.1 and Theorem 1.2.

3. Bilinear estimates: Proof of Theorem 1.1

Now we arrive at the proof of Theorem 1.1. First we define the Duhamel integral
\[
\mathcal{J}^\theta[F] = \int_0^t e^{-\theta (t-t')|\nabla|} F(t') \, dt'.
\]
Then $\mathcal{J}^\theta[F]$ solves the equation
\[
(-i\partial_t + \theta|\nabla|) \mathcal{J}^\theta[F] = F.
\]
with vanishing data at \( t = 0 \). To prove Theorem 1.1, it is enough to show the following bilinear estimates
\[
\left\| \mathfrak{G}^{\mu} \mathfrak{G}^{\nu} \mathfrak{G}^{\lambda} \right\|_{F^{2,1}_2} \lesssim \left\| u \right\|_{F^{2,1}_2} \left\| v \right\|_{F^{2,1}_2}.
\]
Then an application of the standard contraction argument gives the desired global solutions to the equation \([1.1]\) when we have the smallness assumptions on the initial datum: \( \left\| (u_0, u_1) \right\|_{H^1_x \times H^1_x} \ll 1 \). Moreover, the continuous embedding \( U^2 \subset V^2 \) and Lemma 2.4 imply the scattering in \( U^2_2 \) space. By the duality in \( U^2 - V^2 \) Lemma 2.0 we obtain the trilinear expression as follows
\[
\left\| \mathfrak{G}^{\mu} \mathfrak{G}^{\nu} \mathfrak{G}^{\lambda} \mathfrak{G}^{\| | |^{-2} Q(\mathfrak{U}, \mathfrak{V})} \right\|_{F^{2,1}_2} \lesssim \sum_{\mu < 2^N} \sum_{N \geq 1} (\mu^{\frac{1}{2}} N)^2 \left\| \mathcal{P}_H \mathfrak{G}^{\mu} \mathfrak{G}^{\nu} \mathfrak{G}^{\lambda} \mathfrak{G}^{\| | |^{-2} Q(\mathfrak{U}, \mathfrak{V})} \right\|_{U^2_2}^2
\]
\[
\lesssim \sum_{\mu < 2^N} \sum_{N \geq 1} (\mu^{\frac{1}{2}} N)^2 \sup_{\| \mathcal{P}_H w \|_{U^2_2} \leq 1} \left| \int_{\mathbb{R}^{1+3}} P_{H} \mathfrak{G}^{\mu} \mathfrak{G}^{\nu} \mathfrak{G}^{\lambda} \mathfrak{G}^{\| | |^{-2} Q(\mathfrak{U}, \mathfrak{V})} dx dt \right|^2.
\]
Thus our main bilinear estimates can be obtained provided that the following frequency-localised trilinear estimates holds:

**Lemma 3.1.** Let \( 0 < \eta \ll 1 \) be a small positive number. For some \( \frac{1}{8} < \delta \leq \frac{1}{4} \), we have
\[
\left| \int_{\mathbb{R}^{1+3}} w_{\mu, N} |\nabla|^{-2} Q(u_{\lambda_1, N_1}, v_{\lambda_2, N_2}) dx dt \right|
\]
\[
\lesssim (\min\{\lambda_1, \lambda_2\})^{\frac{1}{2}} \left( \frac{\min\{\mu, \lambda_1, \lambda_2\}}{\max\{\mu, \lambda_1, \lambda_2\}} \right)^{\delta} (\min\{N_1, N_2\})^{1-\eta} \| w_{\mu, N} \|_{H^3} \| u_{\lambda_1, N_1} \|_{U^2_2} \| v_{\lambda_2, N_2} \|_{U^2_2},
\]
where we put \( w_{\mu, N} = \mathcal{P}_H w, u_{\lambda_1, N_1} = \mathcal{P}_{\lambda_1} \mathcal{P}_H u, \) and \( v_{\lambda_2, N_2} = \mathcal{P}_{\lambda_2} \mathcal{P}_H v \) for brevity. To obtain (3.2), we shall consider all possible frequency interactions. In view of the standard Littlewood-Paley trichotomy one can easily see that the integral in (3.2) vanishes unless the following interactions hold:
\[
\min\{\lambda_1, \lambda_2\} \lesssim \text{med}\{\mu, \lambda_1, \lambda_2\} \approx \max\{\mu, \lambda_1, \lambda_2\},
\]
\[
\min\{N_1, N_2\} \lesssim \text{med}\{N_1, N_2\} \approx \max\{N_1, N_2\}.
\]
We first decompose the integrand in (3.2) with respect to the modulation as follows
\[
\int_{\mathbb{R}^{1+3}} P_{H} \mathfrak{G}^{\mu} \mathfrak{G}^{\nu} \mathfrak{G}^{\lambda} \mathfrak{G}^{\| | |^{-2} Q(\mathfrak{U}, \mathfrak{V})} dx dt
\]
\[
= \sum_{d \in \mathbb{Z}^2} \int_{\mathbb{R}^{1+3}} C^d_{\mu} w_{\mu, N} |\nabla|^{-2} Q(C^d_{\lambda_1} u_{\lambda_1, N_1}, C^d_{\lambda_2} v_{\lambda_2, N_2}) dx dt
\]
\[
+ \sum_{d \in \mathbb{Z}^2} \int_{\mathbb{R}^{1+3}} C^d_{\mu} w_{\mu, N} |\nabla|^{-2} Q(C^d_{\lambda_1} u_{\lambda_1, N_1}, C^d_{\lambda_2} v_{\lambda_2, N_2}) dx dt
\]
\[
+ \sum_{d \in \mathbb{Z}^2} \int_{\mathbb{R}^{1+3}} C^d_{\mu} w_{\mu, N} |\nabla|^{-2} Q(C^d_{\lambda_1} u_{\lambda_1, N_1}, C^d_{\lambda_2} v_{\lambda_2, N_2}) dx dt
\]
\[
:= \sum_{d \in \mathbb{Z}^2} I_0 + I_1 + I_2.
\]

### 3.1. Low modulation.
Now we consider the low-modulation regime \( d \lesssim \min\{\mu, \lambda_1, \lambda_2\} \). In this regime we will pay special attention to the low-output interaction, i.e., \( \mu \ll \lambda_1 \approx \lambda_2 \). The main problem is that even when we can take an advantage of the presence of null forms very favourably in the low-output interactions, the Fourier multiplier \( |\nabla|^{-2} \) gives rise to the serious singularity and the cancellation property given by null structure is not sufficient to cover all such a bad interaction. To overcome this problem we adapt fully angular momentum operator and exploit angular concentration phenomena via bilinear decompositions by
conic sectors. The key point is that when one input-frequency is localised in a conic sector of a small angle, the other input-frequency should be also localised in another conic sector of a compatible size. On the other hand, in the high-output interaction the null structure no longer plays any crucial role compared to the low-output case since we only have bilinear decompositions by a rather wide-angle. This is not problematic however, the Fourier multiplier $|\nabla|^{-2}$ no longer is a serious singularity, instead, it plays a crucial role as strong decay. In consequence, the high-output interaction becomes the easiest case in the proof.

### 3.1.1. Case 1: $\mu \ll \lambda_1 \approx \lambda_2$.

It is no harm to put $\lambda_1 = \lambda_2 = \lambda$ in our argument. Put $\alpha = \left(\frac{2}{q}\right)^{\frac{1}{s}}$. We first use an almost orthogonal decomposition by smaller cubes and angular sectors and obtain

$$
\mathcal{I}_0 \lesssim \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{\kappa_1, \kappa_2 \in \mathbb{C}_\alpha} \left| \int_{\mathbb{R}^{d+1}} C^\theta_{d,w_{\mu,N}} |\nabla|^{-2} Q(P_{q_1}, R_{\kappa_1}, C^\theta_{d,w_{\mu,N},1}, P_{q_2}, R_{\kappa_2}, C^\theta_{d,w_{\mu,N},2}) \ dt dx \right|
$$

$$
\lesssim \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{\kappa_1, \kappa_2 \in \mathbb{C}_\alpha} \left\| C^\theta_{d,w_{\mu,N}} \right\|_{L^1_t L^\infty_x} \left\| |\nabla|^{-2} Q(P_{q_1}, R_{\kappa_1}, C^\theta_{d,w_{\mu,N},1}, P_{q_2}, R_{\kappa_2}, C^\theta_{d,w_{\mu,N},2}) \right\|_{L^2_t L^\infty_x}
$$

$$
\lesssim d^{-\frac{1}{2}} \left\| w_{\mu,N} \right\|_{V^2_q} \left( \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{\kappa_1, \kappa_2 \in \mathbb{C}_\alpha} \left\| |\nabla|^{-2} Q(P_{q_1}, R_{\kappa_1}, C^\theta_{d,w_{\mu,N},1}, P_{q_2}, R_{\kappa_2}, C^\theta_{d,w_{\mu,N},2}) \right\|_{L^2_t L^\infty_x} \right)^{\frac{1}{2}}
$$

where we used the simple bound for a high-modulation-regime \[2.8\] for $w_{\mu,N}$. Now we exploit the null structure in the bilinear form $Q$ and then use the Hölder inequality and Bernstein inequality for $u_{\lambda,N_1}$. In sequel we put $q = \frac{4}{1-\eta}$ for a small $\eta > 0$. Then we have

$$
\mathcal{I}_0 \lesssim d^{-\frac{1}{2}} \mu^{-2} \alpha^2 \left\| w_{\mu,N} \right\|_{V^2_q} \left( \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{\kappa_1, \kappa_2 \in \mathbb{C}_\alpha} \left\| R_{\kappa_1}, C^\theta_{d,w_{\mu,N},1} \right\|_{L^2_t L^\infty_x} \right)^{\frac{1}{2}} \left\| P_{q_1} R_{\kappa_2} C^\theta_{d,w_{\mu,N},2} \right\|_{L^2_t L^\infty_x}
$$

$$
\lesssim d^{-\frac{1}{2}} \mu^{-2} \alpha^2 \left\| \right\|_{V^2_q} \sup_{\kappa_1 \in \mathbb{C}_\alpha} \left( \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{\kappa_1, \kappa_2 \in \mathbb{C}_\alpha} \left\| R_{\kappa_1}, C^\theta_{d,w_{\mu,N},1} \right\|_{L^2_t L^\infty_x} \right)^{\frac{1}{2}}
$$

The final step is an application of the angular concentration estimates Lemma \[2.10\] with $s = \frac{1}{2} - 2\eta < \frac{2}{q}$ and then the improve Strichartz estimates \[2.11\], which gives

$$
\mathcal{I}_0 \lesssim d^{-\frac{1}{2}} \mu^{-2} \alpha^2 \left( \frac{1}{2} \right) \left\| w_{\mu,N} \right\|_{V^2_q} \left( \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{\kappa_1, \kappa_2 \in \mathbb{C}_\alpha} \left\| R_{\kappa_1}, C^\theta_{d,w_{\mu,N},1} \right\|_{L^2_t L^\infty_x} \right)^{\frac{1}{2}} \left\| P_{q_1} R_{\kappa_2} C^\theta_{d,w_{\mu,N},2} \right\|_{L^2_t L^\infty_x}
$$

$$
\lesssim d^{-\frac{1}{2}} \mu^{-2} \alpha^2 \left( \frac{1}{2} \right) \left\| w_{\mu,N} \right\|_{V^2_q} \left( \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{\kappa_1, \kappa_2 \in \mathbb{C}_\alpha} \left\| R_{\kappa_1}, C^\theta_{d,w_{\mu,N},1} \right\|_{L^2_t L^\infty_x} \right)^{\frac{1}{2}} \left\| P_{q_1} R_{\kappa_2} C^\theta_{d,w_{\mu,N},2} \right\|_{L^2_t L^\infty_x}
$$

The summation with respect to $d \lesssim \mu$ yields

$$
\sum_{d, d \leq \mu} \mathcal{I}_0 \lesssim \mu^{-1+\frac{s}{2}-2\eta} \left\| w_{\mu,N} \right\|_{V^2_q} \left( \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{\kappa_1, \kappa_2 \in \mathbb{C}_\alpha} \left\| R_{\kappa_1}, C^\theta_{d,w_{\mu,N},1} \right\|_{L^2_t L^\infty_x} \right)^{\frac{1}{2}} \left\| P_{q_1} R_{\kappa_2} C^\theta_{d,w_{\mu,N},2} \right\|_{L^2_t L^\infty_x}
$$

If $N_1 \gg N_2$, then we simply interchange the role of $u_{\lambda,N_1}$ and $v_{\lambda,N_2}$ and then obtain exactly the same bound. We now consider $\mathcal{I}_1$. As we have done in the previous estimate, we use an almost orthogonal decomposition of cubes and angular sectors to get

$$
\mathcal{I}_1 \lesssim \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{\kappa_1, \kappa_2 \in \mathbb{C}_\alpha} \left| \int_{\mathbb{R}^{d+1}} C^\theta_{d,w_{\mu,N}} |\nabla|^{-2} Q(P_{q_1}, R_{\kappa_1}, C^\theta_{d,w_{\mu,N},1}, P_{q_2}, R_{\kappa_2}, C^\theta_{d,w_{\mu,N},2}) \ dt dx \right|
$$
The next step is to exploit the null structure and use the Hölder inequality as the previous estimate

\[ I_1 \lesssim \mu^{-2} \lambda^2 \sum_{q, r \in \mathbb{Q}_\mu} \sum_{k_1, k_2 \in \mathbb{C}_\alpha} \int_{\mathbb{R}^{n+1}} C_{\leq d}^\theta w_{\mu, \xi} P_{q_1} R_{k_1} C_{d}^\theta u_{\lambda, \xi} P_{q_2} R_{k_2} C_{\leq d}^\theta v_{\lambda, \xi} \, dx dt \]

\[ \lesssim \mu^{-2} \lambda^2 \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{k_1, k_2 \in \mathbb{C}_\alpha} \| C_{\leq d}^\theta w_{\mu, \xi} \|_{L^2 \mathcal{L}_x^2} \| P_{q_1} R_{k_1} C_{d}^\theta u_{\lambda, \xi} \|_{L^2 \mathcal{L}_x^2} \| P_{q_2} R_{k_2} C_{\leq d}^\theta v_{\lambda, \xi} \|_{L^2 \mathcal{L}_x^\infty} \]

\[ \lesssim \mu^{-2} \lambda^2 \| C_{\leq d}^\theta w_{\mu, \xi} \|_{L^2 \mathcal{L}_x^2} \left( \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{k_1, k_2 \in \mathbb{C}_\alpha} \| P_{q_1} R_{k_1} C_{d}^\theta u_{\lambda, \xi} \|_{L^2 \mathcal{L}_x^2}^2 \right)^{1/2} \]

Then we use the Bernstein inequality for \( v_{\lambda, \xi} \) and then Lemma \( \ref{lemma Bernstein} \) and the Strichartz estimates \( \ref{Strichartz estimates} \).

\[ I_1 \lesssim \mu^{-2} \lambda^2 \alpha^2 \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{k_1, k_2 \in \mathbb{C}_\alpha} \| R_{k_1} C_{d}^\theta u_{\lambda, \xi} \|_{L^2 \mathcal{L}_x^2} \left( \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{k_1, k_2 \in \mathbb{C}_\alpha} \| P_{q_1} R_{k_1} C_{d}^\theta u_{\lambda, \xi} \|_{L^2 \mathcal{L}_x^2}^2 \right)^{1/2} \]

\[ \lesssim \mu^{-2} \lambda^2 \alpha^2 \| R_{k_1} C_{d}^\theta u_{\lambda, \xi} \|_{L^2 \mathcal{L}_x^2} \left( \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{k_1, k_2 \in \mathbb{C}_\alpha} \| P_{q_1} R_{k_1} C_{d}^\theta u_{\lambda, \xi} \|_{L^2 \mathcal{L}_x^2}^2 \right)^{1/2} \]

\[ \lesssim \mu^{-2} \lambda^2 \alpha^2 \| R_{k_1} C_{d}^\theta u_{\lambda, \xi} \|_{L^2 \mathcal{L}_x^2} \left( \sum_{q_1, q_2 \in \mathbb{Q}_\mu} \sum_{k_1, k_2 \in \mathbb{C}_\alpha} \| P_{q_1} R_{k_1} C_{d}^\theta u_{\lambda, \xi} \|_{L^2 \mathcal{L}_x^2}^2 \right)^{1/2} \]

where we used the bound \( \ref{C^0 bounded} \) for \( C_{d}^\theta \mu \). The summation with respect to the modulation \( d \lesssim \mu \) gives the desired bound. If \( N_1 \ll N_2 \), we can simply interchange the role of \( u_{\lambda, \xi} \) and \( v_{\lambda, \xi} \) and follow the above argument. The estimate of \( I_2 \) can be obtained in the identical manner as the estimate of \( I_1 \). We omit the details.

3.1.2. Case 2: \( \lambda_1 \lesssim \mu \approx \lambda_2 \). The case \( \lambda_2 \lesssim \mu \approx \lambda_1 \) would readily follow by symmetry and we focus on the case \( \lambda_1 \ll \lambda_2 \). The high-output case is much easier than the low-output case, i.e., \( \min \{ \mu, \lambda_1, \lambda_2 \} = \mu \), since the Fourier multiplier \( |\nabla|^{-2} \) in the integrand is not the serious singularity, even further it plays a role as a strong decay. We only treat the estimate of \( I_1 \) with \( N_1 \gg N_2 \) in this paper, since this case is the most serious interaction in the high-output interaction. We put \( \beta = (d_{-1})^{1/2} \) and use the orthogonal decompositions

\[ I_1 \lesssim \sum_{q, r \in \mathbb{Q}_{\lambda_1}} \sum_{k_1, k_2 \in \mathbb{C}_{\beta}} \int_{\mathbb{R}^{n+1}} P_{q_1} R_{k_1} C_{d}^\theta |\nabla|^{-2} Q(R_{k_1} C_{d}^\theta u_{\lambda_1, \xi}, P_{q_2} R_{k_2} C_{\leq d}^\theta v_{\lambda_2, \xi}) \, dx dt \]

\[ \lesssim \mu^{-2} \lambda_1 \lambda_2 \beta \sum_{q_1, q_2 \in \mathbb{Q}_{\lambda_1}} \sum_{k_1, k_2 \in \mathbb{C}_{\beta}} \int_{\mathbb{R}^{n+1}} P_{q} R_{k_1} C_{d}^\theta R_{k_2} C_{d}^\theta u_{\lambda_1, \xi} P_{q_2} R_{k_2} C_{\leq d}^\theta v_{\lambda_2, \xi} \, dx dt. \]

Then we use the Hölder inequality and then the Cauchy-Schwarz inequality in \( \kappa_1 \) to get

\[ I_1 \lesssim \mu^{-2} \lambda_1 \lambda_2 \beta \left( \sum_{\kappa_1} \| R_{\kappa_1} C_{d}^\theta u_{\lambda_1, \xi} \|_{L^2 \mathcal{L}_x^2}^2 \right)^{1/2} \]

\[ \times \left( \sum_{\kappa_1} \sum_{q_1, q_2} \| P_{q} R_{k_1} C_{d}^\theta w_{\mu, \xi} \|_{L^2 \mathcal{L}_x^2} \| P_{q_2} R_{k_2} C_{\leq d}^\theta v_{\lambda_2, \xi} \|_{L^2 \mathcal{L}_x^\infty} \right)^{1/2}. \]
We use the Bernstein inequality for $v_{\lambda_2, N_2}$ and obtain
\[
\mathcal{I}_1 \lesssim \mu^{-2} \lambda_1 \lambda_2 \beta \gamma \lambda_1^{\frac{1}{2}} \beta \gamma \lambda_2 \beta \gamma \frac{d}{\mu} \left\| C_d^\theta u_{\lambda_1, N_1} \mid_{L_t^2 L_x^2} \sup_{\kappa_2} \| R_{\kappa_2} C_{\leq d}^\theta v_{\lambda_2, N_2} \|_{L_t^2 L_x^2} \right\| \left( \sum \sum \| P_{\varphi} R_{\kappa} C_{\leq d}^\theta w_{\mu, N} \left\|_{L_t^\infty L_x^2}^2 \right\| \frac{1}{2} \right)^{\frac{1}{2}}.
\]

The remaining step is to apply the bound for the high-modulation-region (2.8) for $C_d^\theta u$ and then Lemma 2.10 followed by the Strichartz estimate (2.10) for $v_{\lambda_2, N_2}$
\[
\mathcal{I}_1 \lesssim \mu^{-2} \lambda_1 \lambda_2 \beta \gamma \lambda_1^{\frac{1}{2}} \beta \gamma \lambda_2 \beta \gamma \frac{d}{\mu} \left\| \left( \beta N_2 \right)^{\frac{1}{2}} - \eta \lambda_2^{\frac{1}{2} + \eta} d - \frac{1}{2} \right\|_{V_t^2}^2 \mu \| u_{\lambda_1, N_1} \|_{V_t^2_{\alpha}} \| v_{\lambda_2, N_2} \|_{V_t^2_{\alpha}}.
\]

The summation with respect to $d \lesssim \lambda_1$ yields
\[
\sum_{d \lesssim \lambda_1} \mathcal{I}_1 \lesssim \lambda_1^{\frac{1}{2}} \left( \frac{\lambda_1}{\lambda_2} \right)^{\frac{1}{2}} \left( \frac{\lambda_2}{\mu} \right)^{-2} \lambda_2^{1-\eta} \| w_{\mu, N} \|_{V_t^2}^2 \| u_{\lambda_1, N_1} \|_{V_t^2_{\alpha}} \| v_{\lambda_2, N_2} \|_{V_t^2_{\alpha}},
\]

where we used the continuous embedding $U^2 \subset V^2$ for $u_{\lambda_1, N_1}$. (See [14] Proposition 2.4.) This completes the proof of (3.2) in the low-modulation regime.

### 3.2. High modulation

From now on we shall consider the high-modulation region: $d \gg \min \{ \mu, \lambda_1, \lambda_2 \}$. In this regime we only consider low-output interaction, i.e., $\mu \ll \lambda_1 \approx \lambda_2$; the Fourier multiplier $|\nabla|^{-2}$ yields good decay in the high-output interaction and hence it is much easier than the low-output case. As Section 3.1, we put $\lambda_1 = \lambda_2 = \lambda$. Note that the angle between the Fourier support of $u_{\lambda}$ and $v_{\lambda}$ is less than $\frac{\mu}{\lambda}$.

We put $\alpha = \frac{\mu}{\lambda}$. By the orthogonal decompositions by smaller cubes of size $\mu$ and conic sectors of size $\alpha$ we follow the similar approach as we have done in Section 3.1. For $d \gtrsim \lambda$, we have
\[
\mathcal{I}_0 \lesssim \sum \sum \int_{R_{\kappa+\varphi} \subset \subset C_{\kappa}} C_d^\theta w_{\mu, N} \| \nabla \|^2 Q \left( P_{\varphi} \left( R_{\kappa_1} C_{\leq d}^\theta u_{\lambda_1, N_1} \right), P_{\varphi_2} R_{\kappa_2} C_{\leq d}^\theta v_{\lambda_2, N_2} \right) \| dt \| dx
\]
\[
\lesssim \mu^{-\frac{1}{2}} \mu^{-2} \alpha \lambda_2 \| w_{\mu, N} \|_{V_t^2} \left( \sum \sum \| P_{\varphi} R_{\kappa} C_{\leq d}^\theta u_{\lambda_1, N_1} \|_{L_t^2 L_x^2} \| P_{\varphi_2} R_{\kappa_2} C_{\leq d}^\theta v_{\lambda_2, N_2} \|_{L_t^\infty L_x^2} \right)^{\frac{1}{2}}
\]
\[
\lesssim \mu^{-\frac{1}{2}} \mu^{-2} \alpha \lambda_2^{\frac{1}{2}} \mu^{\frac{1}{2}} \lambda_1^{\frac{1}{2}} \mu^{\frac{1}{2}} (\alpha N_1)^{\frac{1}{2}} \eta \lambda_1^{\frac{1}{2} + \eta} \| w_{\mu, N} \|_{V_t^2} \| u_{\lambda_1, N_1} \|_{V_t^2_{\alpha}} \| v_{\lambda_2, N_2} \|_{V_t^2_{\alpha}}
\]
\[
\lesssim \lambda \left( \frac{\mu}{\lambda} \right)^{\frac{1}{2}} \mu^{\frac{1}{2} + \eta} \left( \frac{\lambda}{\mu} \right)^{\frac{1}{2}} \lambda_1^{1-\eta} \| w_{\mu, N} \|_{V_t^2} \| u_{\lambda_1, N_1} \|_{V_t^2_{\alpha}} \| v_{\lambda_2, N_2} \|_{V_t^2_{\alpha}},
\]
which gives the required bound after the summation with respect to the modulation $d; d \gtrsim \lambda$. On the other hand, if $\mu \ll d \ll \lambda$, we see that

$$I_0 \lesssim \sum_{q_1, q_2 \in \mathbb{Q}_\mu, \kappa_1, \kappa_2 \in \mathcal{C}_\alpha} \int_{\mathbb{R}^{3+1}} C^\theta_d w_{\mu, N} |\nabla|^{-2} Q(P_{q_1} R_{\kappa_1} C^{\theta_1}_{d; d} u_{\lambda, N_1}, P_{q_2} R_{\kappa_2} C^{\theta_2}_{d; d} v_{\lambda, N_2}) \, dt \, dx$$

$$\lesssim d^{-\frac{1}{4}} \mu^{-2} \alpha \lambda^2 \|w_{\mu, N}\|_{V^2_{\theta}} \left( \sum_{q_1, q_2 \in \mathbb{Q}_\mu, \kappa_1, \kappa_2 \in \mathcal{C}_\alpha} \|P_{q_1} R_{\kappa_1} C^{\theta_1}_{d; d} u_{\lambda, N_1}\|_{L^2_{t} L^\infty_x} \|P_{q_2} R_{\kappa_2} C^{\theta_2}_{d; d} v_{\lambda, N_2}\|_{L^\infty_{t} L^2_x} \right)^{\frac{1}{2}}$$

$$\lesssim d^{-\frac{1}{4}} \mu^{-2} \alpha \lambda^2 \|w_{\mu, N}\|_{V^2_{\theta}} \sup_{\kappa_1 \in \mathcal{C}_\alpha} \|R_{\kappa_1} C^{\theta_1}_{d; d} u_{\lambda, N_1}\|_{L^2_{t} L^2_x} \times \left( \sum_{q_1, q_2 \in \mathbb{Q}_\mu, \kappa_1, \kappa_2 \in \mathcal{C}_\alpha} \|P_{q_2} R_{\kappa_2} C^{\theta_2}_{d; d} v_{\lambda, N_2}\|_{L^\infty_{t} L^2_x} \right)^{\frac{1}{2}} \lesssim d^{-\frac{1}{4}} \mu^{-2} \alpha \lambda^2 \|w_{\mu, N}\|_{V^2_{\theta}} \|u_{\lambda, N_1}\|_{V^2_{\theta}} \|u_{\lambda, N_2}\|_{V^2_{\theta}}$$

and the summation with respect to the modulation $d; \mu \ll d \ll \lambda$ gives the desired estimate. The estimates of $I_1$ and $I_2$ follow by the similar way. We omit the details. This completes the proof of the main trilinear estimates (3.2).

4. TRILINEAR ESTIMATES: PROOF OF THEOREM 1.2

This section is devoted to the proof of Theorem 1.2. As the previous section, we define the Duhamel integral

$$\hat{\mathcal{G}}^\theta[F] = \int_0^t e^{-\theta(t-t') \langle \nabla \rangle} F(t') \, dt'.$$

Then $\hat{\mathcal{G}}^\theta[F]$ solves the equation

$$(-i \partial_t + \theta \langle \nabla \rangle) \hat{\mathcal{G}}^\theta[F] = F,$$

with vanishing data at $t = 0$. From now on we put $m = 1$ for simplicity. We are left to prove the following trilinear estimates

$$\|\hat{\mathcal{G}}^\theta[V \ast (\varphi^1 \phi)]\|_{F^0,1_{\theta}} \lesssim \|\varphi\|_{F^{0,1}_{\tilde{\theta}_1}} \|\phi\|_{F^{0,1}_{\tilde{\theta}_2}} \|\psi\|_{F^{0,1}_{\tilde{\theta}_3}}$$

which imply the global well-posedness and scattering in the $U^2$-space provided that the smallness condition for the initial data is given. The use of duality in $U^2 - V^2$ gives

$$\|\hat{\mathcal{G}}^\theta[V \ast (\varphi^1 \phi)]\|_{F^0,1_{\theta}}^2 \lesssim \sum_{\lambda_1, N_4 \geq 1} (N_4)^2 \|P_{\lambda_4} H_{N_4} \hat{\mathcal{G}}^\theta[V \ast (\varphi^1 \phi)]\|_{U^2_{\theta_4}}^2$$

$$\lesssim \sum_{\lambda_4, N_4 \geq 1} (N_4)^2 \sup_{\lambda_4, N_4 \geq 1} \|P_{\lambda_4} H_{N_4} \psi\|_{V^1_{\theta_4}} \leq 1 \left[ \int_{\mathbb{R}^{1+3}} V_{\theta} \ast (\varphi^1 \phi) P_{\lambda_4} H_{N_4} \psi \, dx \right]^2.$$
Then dyadic decompositions and the Hölder inequality yield

$$\|3^{\theta_4}[V_b \ast (\varphi^1 \phi)\psi]\|^2_{F_{\theta_4}} \leq \sum_{\lambda_j \geq 1, j=0,1,\ldots,4} \sum_{N_j \geq 1, j=0,1,\ldots,4} \sup_{\|P_{\lambda_j}H_{N_j}\|_{\lambda_j}^2 1} \left| \int_{\mathbb{R}^{1+3}} \langle \nabla \rangle_{b}^{-2} P_{\lambda_0} H_{N_0} (\varphi_{\lambda_1,N_1}^1 \phi_{\lambda_2,N_2}) P_{\lambda_0} H_{N_0} (\psi_{\lambda_4,N_4}^1 \psi_{\lambda_3,N_3}) \, dt dx \right|^2$$

Thus our main trilinear estimates follow from the following frequency-localised $L^2$-bilinear estimates:

**Lemma 4.1.** Let $0 < \eta \ll 1$ be a small positive number. For some $\frac{1}{8} \leq \delta \leq \frac{1}{4}$, we have

$$\|P_{\lambda_0}(\varphi_{\lambda_1,N_1}^1 \phi_{\lambda_2,N_2})\|_{L^2_t L^2_x} \lesssim \lambda_0 \left( \frac{\min\{\lambda_0, \lambda_1, \lambda_2\}}{\max\{\lambda_0, \lambda_1, \lambda_2\}} \right) \delta (\min\{N_1, N_2\})^{1-\eta} \|\varphi_{\lambda_1,N_1}\|_{U_{\delta_1}^2} \|\phi_{\lambda_2,N_2}\|_{U_{\delta_2}^2}.$$  

To prove [4.2] we need to deal with the frequency interactions:

$$\lambda_0 \ll \lambda_1 \approx \lambda_2, \quad \lambda_1 \ll \lambda_0 \approx \lambda_2, \quad \lambda_2 \ll \lambda_0 \approx \lambda_1.$$  

Then it suffices to consider the bilinear estimates

$$\|P_{\mu}(\varphi_{\lambda,N_1}^1 \phi_{\lambda,N_2})\|_{L^2_t L^2_x}, \quad \|P_{\lambda}(\varphi_{\mu,N_1}^1 \phi_{\lambda,N_2})\|_{L^2_t L^2_x}$$

for $\mu \ll \lambda$. We first consider the first bilinear form. As the proof of Theorem 1.1 we apply the orthogonal decomposition of cubes of size $\mu$ and conic sectors of size $\alpha$ with $\alpha = \frac{\eta}{4}$ and we use in order the Hölder inequality and the Bernstein inequality and then Lemma 2.10 and the Strichartz estimates (2.10) for $\varphi_{\lambda,N_1}$

$$\|P_{\mu}(\varphi_{\lambda,N_1}^1 \phi_{\lambda,N_2})\|_{L^2_t L^2_x} \lesssim \left( \sum_{q_1, q_2 \in \Omega_{\mu}} \sum_{\kappa_1, \kappa_2 \in C, \mu} \|P_{\mu}(P_{q_1} R_{\kappa_1} \varphi_{\lambda,N_1}^1 P_{q_2} R_{\kappa_2} \phi_{\lambda,N_2})\|_{L^2_t L^2_x}^{\frac{2}{\delta}} \right)^{\frac{1}{2}}$$

$$\lesssim \left( \sum_{q_1, q_2 \in \Omega_{\mu}} \sum_{\kappa_1, \kappa_2 \in C, \mu} \|P_{q_1} R_{\kappa_1} \varphi_{\lambda,N_1}\|_{L^2_t L^\infty_x}^{2} \|P_{q_2} R_{\kappa_2} \phi_{\lambda,N_2}\|_{L^\infty_t L^2_x}^{2} \right)^{\frac{1}{2}}$$

$$\lesssim \mu^{\frac{\delta}{2}} \left( \frac{\mu}{\lambda} \right)^{\frac{\delta}{2}} \sup_{\kappa_1} \|R_{\kappa_1} \varphi_{\lambda,N_1}\|_{L^2_t L^\infty_x} \times \left( \sum_{q_1, q_2 \in \Omega_{\mu}} \sum_{\kappa_1, \kappa_2 \in C, \mu} \|P_{q_2} R_{\kappa_2} \phi_{\lambda,N_2}\|_{L^\infty_t L^2_x}^{2} \right)^{\frac{1}{2}}$$

$$\lesssim \mu^{\frac{\delta}{2}} \left( \frac{\mu}{\lambda} \right)^{\frac{\delta}{2}} \left( \frac{\mu}{\lambda} \right)^{\frac{\delta}{2} - 2\eta} \lambda^{-\frac{\delta}{2}} N_1^{\frac{\delta}{2} - \eta} \|\varphi_{\lambda,N_1}\|_{U_{\delta_1}^2} \|\phi_{\lambda,N_2}\|_{U_{\delta_2}^2} \lesssim \mu \left( \frac{\mu}{\lambda} \right)^{\frac{\delta}{2} + \frac{\delta}{2} - \eta} N_1^{1-\eta} \|\varphi_{\lambda,N_1}\|_{U_{\delta_1}^2} \|\phi_{\lambda,N_2}\|_{U_{\delta_2}^2}.$$
If $N_2 \ll N_1$, then we interchange the role of $\varphi$ and $\phi$. For the second bilinear form, we are only concerned with the case $N_1 \gg N_2$. We make the use of $L^2$-duality and then orthogonal decompositions of cubes and angular sectors of size $c > 0$ where $c$ is a small constant and we have
\[
\| P_\lambda (\varphi^\dagger_{\mu,N_1} \phi_{\lambda,N_2}) \|_{L^2_t L^2_x} \lesssim \sup_{\| \psi \|_{L^2_t L^2_x} \lesssim 1} \left( \sum_{q,q_2 \in \mathbb{Q}_\mu} \sum_{\kappa_1, \kappa_2 \in \mathcal{C}_c} \int_{\mathbb{R}^{1+3}} P_q R_{\kappa_1} \varphi^\dagger_{\mu,N_1} P_{q_2} R_{\kappa_2} \phi_{\lambda,N_2} \, dt dx \right) \lesssim \sup_{\| \psi \|_{L^2_t L^2_x} \lesssim 1} \left( \sum_{q,q_2 \in \mathbb{Q}_\mu} \sum_{\kappa_1, \kappa_2 \in \mathcal{C}_c} \| P_q R_{\kappa_1} \varphi_{\mu,N_1} P_{q_2} R_{\kappa_2} \phi_{\lambda,N_2} \|_{L^2_t L^2_x}^2 \right)^{1/2} \lesssim \mu^{2/3} \sup_{\kappa_2} \| R_{\kappa_2} \phi_{\lambda,N_2} \|_{L^2_t L^2_x} \| \phi_{\mu,N_1} \| \| \varphi_{\mu,N_1} \|_{V^2_{q_1}} \lesssim \mu^{2/3} \left( cN_2 \right)^{1/2 - 2\eta} N_2^{1-\eta} \| \phi_{\lambda,N_2} \|_{U^2_{q_2}} \lesssim \lambda \left( \frac{\mu}{\lambda} \right)^{\frac{2}{3}} N_2^{1-\eta} \| \phi_{\lambda,N_2} \|_{U^2_{q_2}},
\]
where we used the continuous embedding $U^2 \subset V^2$. This completes the proof of the main bilinear estimates.

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