0. Introduction.

The main purpose of this paper is to prove an important generalization of the construction of the Incidence Divisor given in [BMg1] in the case of an ambient manifold. Let us first recall briefly the setting: let $Z$ be a complex manifold and $(X_s)_{s \in S}$ an analytic family of (closed) $n$–cycles in $Z$ parametrized by a reduced complex space $S$.

To a $(n+1)$–codimensional subspace $Y$ in $Z$, which is assumed to be a locally complete intersection and to satisfy the following condition:

\[ (C1) \text{ the analytic set } (S \times |Y|) \cap |X| \text{ in } S \times Z \]  

\[ \text{(**) is } S \text{-proper and finite on its image } |\Sigma_Y| \text{ which is nowhere dense in } S, \]

an effective Cartier divisor $\Sigma_Y$ in $S$, called the "incidence divisor of $Y$ in $S"$, is defined with support $|\Sigma_Y|$, and nice functorial properties of this construction are proven. Of course, no assumption is made on the singularities of $S$.

A relative version is also given: when $Y$ varies in a flat family over $T$ in such a way that $(C1)$ remains true, $\Sigma_Y$ moves in a flat family over $T$. As a consequence, $\Sigma_Y$ depends only on the underlying cycle of $Y$ for a connected flat deformation of a locally complete intersection (nilpotent) structure inducing a fixed cycle.

But the general invariance question was not solved in [BMg1].

[Q1] Does the Cartier divisor $\Sigma_Y$ only depend on the cycle underlying the locally complete intersection ideal of $\mathcal{O}_Z$ defining $Y$?

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(**) $X$ denotes the graph of the family $(X_s)_{s \in S}$, which is a cycle in $S \times Z$ and $|X|$ the support of this cycle.
Of course another natural (but stronger) question arises if \([Q1]\) admit a positive answer:

\[ Q2 \] Is it possible to construct the incidence divisor \( \Sigma_Y \) (Cartier and effective) for any cycle \( Y \) satisfying (C1) (with nice functorial properties)?

Of course the construction of a line bundle given in \([K]\) suggested that the cohomological method could help to solve these questions. Also the hypothesis used in \([K]\) was weaker than (C1). In fact the condition (C1) has a nice geometrical meaning and is very common \textbf{generically} on \( S \). But in concrete examples some degenerations appear very often.

The following weaker condition seems much more attractive for applications:

\(\text{(C2) the analytic set } (S \times |Y|) \cap |X| \text{ in } S \times Z \text{ is } S-\text{proper and \textbf{generically} finite on its image } |\Sigma_Y| \text{ which is nowhere dense in } S.\)

But extending a Cartier divisor through an analytic set is, in general hopeless without strong assumptions on the singularities of \( S \). So the geometric method developed in \([BMg1]\) cannot treat this (C2) situation in general.

Our generalization gives the following positive answer:

\( \text{[Q2] has a positive answer with the assumption (C2) (see the Theorem in \S 2).} \)

In fact, using the cohomological tools built in \([K]\), that is to say the existence of a relative fundamental class in Deligne-Beilinson cohomology for the analytic family \((X_s)_{s \in S}\), we are able to prove also a relative version for \([Q2]\), which solves also \([Q1]\), assuming only (C2).

To conclude this article, we prove a formula computing the intersection number of the incidence divisor \( \Sigma_Y \) with a curve in \( S \) in terms of intersection numbers of \( Y \) in \( Z \). This is useful, for instance, to compute the multiplicities of the incidence divisor, but can be used also in a more global setting (\( Z \) projective) to obtain a simple criterion of nefness for the incidence divisor. Combined with results in \([BMg2]\) this may lead to vanishing theorems on \( S \).

Finally, let us remark that this article is a modest contribution, in the local analytic context and at the geometric level, to P. Deligne’s program (see \([D]\)), which has been developed in a global projective algebraic setting (with rather strong hypothesis) by R. Elkik in \([E]\).

In \( \S 1 \) we explain how to build up a Cartier divisor from ”special” cohomology classes in \( \mathcal{H}_I^2(S, \mathbb{Z}(1)_D) \) following in our local analytic setting some simple ideas from algebraic geometry (more precisely pseudo-divisors as described in \([F]\)). Of course, essential singularities have to be avoided.
In §2 we use the cohomological tools from [K] to build a cohomology class in $IH^2_T(S,\mathbb{Z}(1)_D)$ and we prove that it satisfies the criterion proved in §1.

In §3 we prove various functorial properties of our construction. This leads to the comparaison with [BMg 1] and give effectivity of our incidence divisor. The relative case and the positive answer to [Q1] conclude §3.

In §4 we compute the intersection number of the incidence divisor with a curve in $S$.

Finally, we have postponed till the Appendix some basic properties of (relative) fundamental classes in Deligne cohomology such as compatibility with intersection, projection formula and traces, for which it seems that no reference is avaible in our local analytic context.

1. A criterion.

Let $S$ be a reduced complex analytic space and $T \subset S$ a closed analytic subset of $S$ with empty interior. Denote by $j : S - T \hookrightarrow S$ the inclusion and by $\mathcal{O}_S^*$ the sheaf of abelian (multiplicative) groups of non vanishing germs of holomorphic functions on $S$. We shall consider $\mathcal{O}_S^*$ as a subsheaf of the sheaf $m_T^*$ of germs of meromorphic functions (not vanishing identically on any open set in $S$) with zeros and poles contained in $T$. So $m_T^*|_{S-T} = \mathcal{O}_S^*|_{S-T}$.

There is a canonical isomorphism of sheaves

$$j_!j^*\mathcal{O}_S^*/\mathcal{O}_S^* \xrightarrow{\sim} H^1_T(\mathcal{O}_S^*)$$

and a natural inclusion

$$m_T^* \hookrightarrow j_! j^* \mathcal{O}_S^*.$$ 

So the quotient sheaf $m_T^*/\mathcal{O}_S^*$ is naturally embedded as a subsheaf of $H^1_T(\mathcal{O}_S^*)$. We shall denote it by $H^1_T(\mathcal{O}_S^*)$.

Now let us consider the sheaf $C_T$ of Cartier divisors in $S$ with support in $T$.

This is a sheaf of abelian groups on $S$ supported in $T$ and we have a natural homomorphism of sheaves of abelian groups

$$\text{div} : m_T^* \to C_T$$

defined by $\text{div}\left(\frac{f}{g}\right) = \{f = 0\} - \{g = 0\}$ for $f, g \in \mathcal{O}(U)$ such that $\{f = 0\}$ and $\{g = 0\}$ are contained in $U \cap T$. 

The kernel of div is clearly \( \mathcal{O}_S^* \subset m_T^* \) and so we obtain an isomorphism of sheaves of abelian groups

\[
\text{Div} : \quad H^1_{[T]}(\mathcal{O}_S^*) \xrightarrow{\sim} \mathbb{C}.
\]

The reader may compare with [F] p. 31 for the algebraic version of these facts. We want to give a simple criterion in order that a section \( \sigma \) of \( H^1_{[T]}(\mathcal{O}_S^*) \) be a section of the subsheaf \( H^1_{[T]}(\mathcal{O}_S^*) \), and so, produces a Cartier divisor via Div.

Consider the normalisation map \( \nu : \tilde{S} \to S \) for \( S \) and the following sheaf homomorphisms

\[
d \log : \mathcal{O}_S^* \to \Omega^1_S \quad (g \mapsto dg/g)
\]

\[
\nu^* : \Omega^1_S \to \nu_*\Omega^1_S
\]

\[
q : \nu_*\Omega^1_S \to \nu_*\omega^1_S
\]

where \( \omega^1_S \) is the sheaf defined in [B.3].

The normality of \( \tilde{S} \) gives the isomorphism

\[
\omega^1_S \simeq i_*i^*\Omega^1_S
\]

where \( i : \tilde{S} - \text{Sing}(\tilde{S}) \hookrightarrow \tilde{S} \) is the inclusion of the smooth points in \( \tilde{S} \).

Let \( \tilde{T} := \nu^{-1}(T) \). Consider now the sheaf morphism

\[
\varphi : H^1_{[T]}(\mathcal{O}_S^*) \to H^1_{[\tilde{T}]}(\nu_*\omega^1_S) \simeq \nu_*H^1_{[\tilde{T}]}(\omega^1_S)
\]

deduced from \( d\log, \nu^* \) and \( q \).

Denote by \( H^1_{[1,\tilde{T}]}(\omega^1_S) \) the subsheaf of the sheaf \( H^1_{[\tilde{T}]}(\omega^1_S) \) of sections annihilated by the reduced ideal \( \mathcal{J}_{\tilde{T}} \) of \( \tilde{T} \) in \( \tilde{S} \). Note that it is equivalent for a section \( \sigma \) of \( H^1_{[\tilde{T}]}(\omega^1_S) \) to be annihilated by \( \mathcal{J}_{\tilde{T}} \) or to be annihilated by \( \mathcal{J}_{\tilde{T}} \) at generic points of codimension 1 of \( \tilde{T} \) because the sheaf \( \omega^1_S \) has no torsion and satisfies the analytic continuation property in codimension 2 (see [B.3]). But this latter condition is very simple to check because at generic points of codimension one of \( \tilde{T} \), \( \tilde{S} \) is smooth and \( \tilde{T} \) is a smooth hypersurface. So we are checking a pole of order one for a singularity of a holomorphic 1-form along a smooth hypersurface.

Our criterion is given by the following

**Proposition 1.**

In the above notation, we have

\[
\varphi^{-1}[\nu_*H^1_{[1,\tilde{T}]}(\omega^1_S)] = H^1_{[T]}(\mathcal{O}_S^*)
\]
Proof.
First, if \( g \in m_T^* \) we have \( \varphi[g] = \nu_*(\nu^*(\frac{dg}{g})) \) in \( \nu_*H^1_S(\omega_S) \) and we want to see that \( \nu^*(\frac{dg}{g}) \) is annihilated at generic codimension 1 points of \( \tilde{T} \) in \( \tilde{S} \) by \( J_{\tilde{T}} \).
This is clear. Conversely if \( g \in j_*j^*O_S^* \) we want to show that if \( \nu^*(\frac{dg}{g}) \) has at most simple poles at generic codimension 1 points in \( \tilde{T} \) then \( g \in m_T^* \). This is a consequence of the following classical Lemma

Lemma 1.
Let \( D^* := \{ z \in \mathbb{C} / 0 < |z| < 1 \} \) and let \( g \in \mathcal{O}(D^*) \). Assume that \( z\frac{g'}{g} \in \mathcal{O}(D) \). Then \( g \) is meromorphic at 0.

Proof.
Denote by \( h \) the holomorphic function \( z\frac{g'}{g} \) on \( D \). Then, on \( D^* \), \( g \) satisfies the differential equation \( g' = \frac{h}{z} \cdot g \). This equation has a simple pole at \( z = 0 \). So \( g \) has moderate growth at 0. The conclusion follows.

Using this Lemma we obtain that \( \nu^*g \) is meromorphic on \( \tilde{S} \) minus an analytic set of codimension \( \geq 2 \). So \( \nu^*g \) is meromorphic on \( \tilde{S} \) and also on \( S \).

2. The main construction.

Now we shall consider the following situation:
Let \( Z \) be a complex manifold of dimension \( n + p \) at let \( \{X_s\}_{s \in S} \) be an analytic family of closed \( n \)-cycles in \( Z \) parametrized by a reduced complex space \( S \). Let \( Y \) be a cycle of pure dimension \( (p - 1) \) in \( Z \) and assume the following condition:

\[
[AP](*) \quad \text{Let } X \subset S \times Z \text{ be the graph of the family } \{X_s\}_{s \in S}, \ p_X : X \rightarrow Z \text{ the projection induced by the canonical projection of } S \times Z \text{ on } Z \text{ and let } p_X^{-1}(|Y|) := (S \times |Y|) \cap |X|. \text{ We require that:}
\]

1) \( \text{codim } p_X^{-1}(|Y|) \) in \( S \times Z \) is \( n + p + 1 \) (the expected codimension)
2) the restriction of \( \pi : |X| \rightarrow S \) to \( p_X^{-1}(|Y|) \) is proper and generically finite on its image \( T := \pi(p_X^{-1}(|Y|)) \)

Remark Under \( [AP] \) one can define the intersection in \( S \times Z \) of the cycles \( p_X^*Y := X \underset{p}{\times} Y \) where \( p : S \times Z \rightarrow Z \) is the second projection(**) and also the

\((*) \) \( AP \) for admissible pole

\((** \) See [B.1] ch 6 §3
direct image \( \pi_*(p_X^*Y) \) as a cycle. We obtain in this way a codimension 1 Weil cycle in \( S \) with support in \( T \). Everything comes down to showing that this "rough" geometric construction underlies the construction of a natural Cartier divisor in \( S \) generalizing the results of [BMg1].

We shall prove the following result:

**Theorem.**

In the situation described above to each cycle \( Y \) satisfying [AP] we associate a Cartier divisor \( \Sigma_Y \) in \( S \) with support in \( T := \{ s \in S/|Y| \cap |X_s| \neq \emptyset \} \). This construction satisfies

1) For \( Z = \mathbb{C}, S = \text{Sym}^k(\mathbb{C}) \simeq \mathbb{C}^k/\sigma_k \), \( (X_s)_{s \in S} \) the universal family of 0-cycles of degree \( k \) in \( \mathbb{C} \) parametrized by \( S \) and \( Y = 1, \{0\} \), we have \( \Sigma_Y = \{ s_k = 0 \} \) where \( s_k : \text{Sym}^k(\mathbb{C}) \to \mathbb{C} \) is the \( k \)-th elementary symmetric function.

The map \( Y \to \Sigma_Y \) is compatible with

2) allowed base change (so is local in \( S \))

3) localization in \( Z \)

4) direct image of cycles

5) inverse image of cycles

6) analytic deformation of \( Y \) as a cycle.

Moreover these properties characterize our construction.

Precise formulations of conditions 2) to 6) will be given at the beginning of §3.

We shall now give the definition of \( \Sigma_Y \) using tools from [K] and the criterion of the previous paragraph. The functorial properties 2) to 6) will be proved in §3.

Let \( C_{X/S}^{S \times Z} \) be the relative fundamental class in Deligne cohomology for the family \((X_s)_{s \in S}\). Recall that \( C_{X/S}^{S \times Z} \) is a (global) section of the sheaf

\[
H^2_{\Delta}(\mathcal{H}^p(p_{D/S})
\]

where we denote by \( \mathcal{H}^p(p_{D/S}) \) the complex of sheaves on \( S \times Z \)

\[
\mathcal{H}^p(p_{D/S}) := (2i\pi)^p(\mathcal{Z}) \to \mathcal{O}_{S \times Z/S} \to \Omega_{S \times Z/S}^1 \to \cdots \to \Omega_{S \times Z/S}^{p-1}
\]

with the differential \( S \)-relative (for the construction see [K] Th. II).

Now let us consider \((Y_s)_{s \in S}\) the constant family of cycles in \( Z \) given by \( Y_s = Y \) for all \( s \in S \). We have also a relative fundamental class \( C_{S \times Y/S}^{S \times Z} \) which is the section of the
sheaf $H^{2(n+1)}_S(Z(n+1)_{D/S})$ obtained from $C^Z_Y$ (a section of $H^{2(n+1)}_Y(Z(n+1)_D)$ on $Z$) by pull back and the natural map

$$Z(n+1)_D \to Z(n+1)_{D/S}$$

deduced from the quotients $\Omega^\bullet_{S \times Z} \to \Omega^\bullet_{S \times Z/S}$. Now the cup-product $C^S_{S \times Z} \cup C^S_{S \times Y/S}$ is a section of the sheaf

$$H^{2(n+p+1)}_{p^{-1}(Y)}(Z(n+p+1)_{D/S})$$

( for the definition of the cup product in the absolute case in Deligne cohomology see [E.V]; the relative case is similar). Using the trace map (see [K] §V)

$$Tr_{/S} : \pi_* H^{2(n+p+1)}_{p^{-1}(Y)}(Z(n+p+1)_{D/S}) \to H^2_{T}(Z(1)_D)$$

we obtain a section $\alpha$ of the sheaf $H^2_{T}(Z(1)_D)$ on $S$. But we have

$$Z(1)_D := 2i\pi Z \to O_S$$

and so $Z(1)_D$ is quasi-isomorphic to the sheaf $O_S[-1]$ via the exponential. We shall then consider $\alpha$ as a section of the sheaf $H^1_{T}(O_S^*)$ and our goal is now to prove that we have in fact a section of the subsheaf $H^1_{T}(O_S^*)$ which is isomorphic via Div (see §1) to the sheaf $C_T$ of Cartier divisors supported in $T$.

So the key to our construction is to show that, near the generic points in $\tilde{T}$ of codimension 1 in $\tilde{S}$ (they are all of codimension one by our previous remark !), the image of $\alpha$ by $\varphi$ has at most a simple pole along $\tilde{T}$.

Then using the compatibility of our construction with base change as long as $Y$ satisfies [AP], we are reduced to show that, for $S$ smooth, $\alpha$ is given by our ”rough” geometric construction which could be seen a follows:

Let $C^S_{S \times Z}$ be the absolute fundamental class of the cycle $X$ in $S \times Z$, seen as a section of the sheaf $H^{2p}_X(Z(p)_D)$.

Let $C^S_{S \times Y}$ be the absolute fundamental class of the cycle $S \times Y$ in $S \times Z$.

Now the intersection $p^*_X Y := X.(S \times Y)$ is well defined as a cycle in $S \times Z$ ($S$ is smooth) and the absolute fundamental class of $p^*_X Y$ is the cup product

$$C^S_{S \times Z} \cup C^S_{S \times Y}$$

as a section of the sheaf $H^{2(n+p+1)}_{p^{-1}(Y)}(Z(n+p+1)_{D})$ (see Appendix II.A).
Of course via the obvious map
\[ H^{2(n+p+1)}_{p_X^{-1}(|Y|)}(\mathbb{Z}(n+p+1)_D) \rightarrow H^{2(n+p+1)}_{p_X^{-1}(|Y|)}(\mathbb{Z}(n+p+1)_{D/S}) \]
this cup product is sent to \( C^{S \times Z}_{X/S} \cup C^{S \times Z}_{S \times Y/S} \).

We have a commutative diagram of sheaves on \( S \):
\[ \begin{array}{ccc}
\pi_* H^{2(n+p+1)}_{p_X^{-1}(|Y|)}(\mathbb{Z}(n+p+1)_D) & \xrightarrow{\lambda} & \pi_* H^{n+p+1}_{p_X^{-1}(|Y|)}(\Omega^{n+p+1}_{S \times Z}) \\
\downarrow Tr & & \downarrow \pi_* \\
H^2_T(\mathbb{Z}(1)_D) & \xrightarrow{\pi_* dlog} & H^1_T(\Omega^1_S) \\
\end{array} \]

where the vertical arrows are absolute traces induced by integration maps which give, for \( \pi_* \), values in the sheaf of the holomorphic forms on \( S \) assumed to be smooth (see [K]) and where \( \lambda \) is deduced from the boundary map
\[ \partial : H^{2k}_{A^{-1}}(\mathbb{Z}(k)_D) \rightarrow H^{2k+1}_{A^{-1}}(\Omega^k[-k-1]) \]
of the exact sequence of complexes
\[ 0 \rightarrow \Omega^k[-k-1] \rightarrow \mathbb{Z}(k+1)_D \rightarrow \mathbb{Z}(k)_D \rightarrow 0. \]

For \( k = 1 \) and \( A = T \) in \( S \), it is a simple exercise to see that \( \partial \) gives \( d \log \left( \text{up to } \frac{1}{2i\pi} \right) \) using the quasi-isomorphism
\[ \mathbb{Z}(1)_D \simeq \mathcal{O}_S^*[-1] \]
and the exact sequence
\[ 0 \rightarrow \Omega^1[-2] \rightarrow \mathbb{Z}(2)_D \rightarrow \mathbb{Z}(1)_D \rightarrow 0. \]

In fact the commutativity of \((D)\) reflects the compatibility between direct images of cycles and the trace of their fundamental classes (in Deligne cohomology and in holomorphic cohomology).

Denote by \( \hat{T} \) the cycle \( \pi_*(p_X^*Y) \). We have
\[ \frac{1}{2i\pi} d\log Tr(C_{p_X^*Y}^{S \times Z}) = C_{\hat{T}}^S \in H^1_T(\Omega^1_S) \]
and the fundamental class $C^S_T$ is annihilated by the reduced ideal of $|\hat{T}|$ in $S$ (still assumed smooth).

So, to conclude that $\alpha$ is a section of $H^1_{|T|}(\mathcal{O}^*_S)$ it is enough to show that the following triangle commutes

$$
\begin{align*}
\pi_*H^{2(n+p+1)}_{p_X^1(|Y|)}(\mathbb{Z}(n+p+1)_{\mathcal{D}}) & \xrightarrow{\text{Tr}} \pi_*H^{2(n+p+1)}_{p_X^1(|Y|)}(\mathbb{Z}(n+p+1)_{\mathcal{D}/S}) \\
& \xrightarrow{\text{Tr}/S} H^2_{|T|}(\mathbb{Z}(1)_{\mathcal{D}})
\end{align*}
$$

for $S$ smooth. But, by definition of the trace $\text{Tr}$, it factorises through $\text{Tr}/S$ (see [K] p.320).

So $\alpha$ is a section of $H^1_{|T|}(\mathcal{O}^*_S)$ and define via $\text{Div} : H^1_{|T|}(\mathcal{O}^*_S) \rightarrow C_T$ a Cartier divisor in $S$ with support in $T$, denoted by $\Sigma_Y$.

It is clear from our construction that the Cartier divisor $\Sigma_Y$ in $S$ satisfies, as a codimension one cycle in $S$ (i.e as Weil divisor)

$$
\Sigma_Y = \pi_*(p^*X) Y
$$

where $p^*X := X_p Y$ in $S \times Z$.

But, of course, the proof that the Weil divisor $\pi_*(p^*X)$ is Cartier does not seem to be clear (for $S$ general) without using Deligne cohomology.

**Remark.**

With condition (C2), it is no hard to see that, for all $i \in \mathbb{N}$, [K] give us the following commutative diagram

$$
\begin{align*}
\mathcal{H}^{2n+i}_{|Y|}(Z, \Omega^*_Z/F^{n+1}\Omega^*_Z) & \xrightarrow{\Gamma(S, \mathcal{H}^i_T(\mathcal{O}_S))} \Gamma(S, \mathcal{H}^{2n+i+1}_T(\mathcal{O}^*_S)) \\
& \xrightarrow{\Gamma(S, \mathcal{H}^{2n+i+1}_T(\mathbb{Z}(1)))}
\end{align*}
$$

where $F^\bullet$ is the shifted filtration bête on the De Rham complex on $Z$.

(*) see [B.1] ch.6 §3 for the definition. Here $p : S \times Z \rightarrow Z$ is the projection and $Z$ is smooth
This gives, in particular, the map

\[ \Psi^X_{Z,S} : \mathcal{H}^0(Z, \mathcal{H}^{2n+2}_{|Y|}(Z, \mathbb{Z}(n+1)_\mathcal{D})) \to \Gamma(S, \mathcal{H}^1_T(O^*_S)) \]

### 3. Functorial properties.

We shall begin this section with precise statements and proofs of Properties 2) to 6) of the theorem.

2) **Base change.**

We consider a holomorphic map \( \lambda : S' \to S \) of reduced complex space and we assume that \( Y \) satisfies \([AP]\) for the analytic family \((X_s)_{s \in S}\) and also for the family \((X_{\lambda(s')})_{s' \in S'}\). This last condition reduces to the following hypothesis:

Let \( R \subset T \) be the analytic subset in \( T = |\pi^*(p_X(Y))| \) where \( \pi : |p_X Y| \to T \) has positive dimensional fibers. Then \( R \) is a closed analytic subset with empty interior in \( T \) from \([AP]\) (relative to the family \((X_s)_{s \in S}\)). Then \( Y \) will satisfy \([AP]\) for the family \((X_{\lambda(s')})_{s' \in S'}\) iff

\[
(H) \begin{cases} 
\lambda^{-1}(R) \text{ has no interior points in } \lambda^{-1}(T) \\
\text{which has no interior point in } S'.
\end{cases}
\]

Under these hypothesis, we want to show that

\[ \lambda^*(\sum^S_Y) = \sum^{S'}_Y \]

Observe that the pull back of the Cartier divisor is well defined under \((H)\)

**Proof.**

Let \( p' : S' \times Z \to Z \) be the projection. We want to prove the commutativity of the square

\[
\begin{array}{ccc}
\pi_* \mathcal{H}^{2(n+p+1)}_{p_X^{-1}(|Y|)}(\mathbb{Z}(n+p+1)_{\mathcal{D}/S}) & \xrightarrow{\lambda^*} & \pi'_* \mathcal{H}^{2(n+p+1)}_{p_X^{-1}(|Y|)}(\mathbb{Z}(n+p+1)_{\mathcal{D}/S}) \\
\downarrow T_{/S} & & \downarrow T'_{/S'} \\
\mathcal{H}^2_T(Z(1)_\mathcal{D}) & \xrightarrow{\lambda^*} & \mathcal{H}^2_{T'}(Z(1)_\mathcal{D})
\end{array}
\]

where \( T' := \pi'[p_X^{-1}(|Y|)] \).

This is consequence of the stability by base change of the relative trace in Deligne cohomology (see Appendix I and \([K]\)).
3) The construction is local on $\mathbb{Z}$.
Let $U$ an open set in $\mathbb{Z}$ and let
$$S_U := \{s \in S/|X_s| \cap |Y| \subset U\}.$$ Then $S_U$ is open in $S$ and we want to show that
$$\sum_Y |s_U = \sum_Y |U.$$ Notice that when $Y$ satisfies [AP], $Y \cap U$ satisfies [AP] for the family $(X_s \cap U)_{s \in S_U}$. The proof is a consequence of the compatibility of our constructions with restrictions to open sets. We have already put some emphasis on that by arguing at the level of sheaves!

4) Direct image of cycles.
Let $f : Z \to W$ be a holomorphic map between complex manifolds. Assume that $(X_s)_{s \in S}$ is an analytic family of $n$–cycles in $Z$ and $Y$ a codimension $(n + 1)$–cycle in $W$ such that
1) $(f_*(X_s))_{s \in S}$ is an analytic family of $n$–cycles in $W$ (*)
2) $f^*(Y)$ is a codimension $(n + 1)$ cycle in $Z$ (**) and satisfies [AP] for the family $(X_s)_{s \in S}$.
Then $Y$ satisfies [AP] for the family $(f_*(X_s))_{s \in S}$ and we have
$$\sum_{f^*(Y)} = \sum_Y$$

**Proof.**
Let $F := \text{id}_S \times f : S \times Z \to S \times W$. Condition 1) implies that $F : |X| \to |\hat{X}|$, where $\hat{X}$ is the graph of the family $(f_*(X_s))_{s \in S}$, is proper and generically finite on each fiber over $S$. So Condition [AP] for $f^*(Y)$ and the family $(X_s)_{s \in S}$ gives [AP] for $Y$ and the family $(f_*(X_s))_{s \in S}$ (recall that $F : |X| \to |\hat{X}|$ is surjective from 1)).

The relation $C^W_{X/S} = F_* (C^Z_{\hat{X}/S})$ is true in relative holomorphic cohomology (see [B.1] ch.IV and [B.2]) and then still true in relative Deligne cohomology (see [K], the semipurity properties); for this, see the Appendix [C].

(*) This is so iff $\forall s \in S$, $f|_{X_s}$ is proper and generically finite on its images ; see [B.1] ch.IV.
(**) This is satisfied iff $f^{-1}(|Y|)$ has pure codim $n + 1$ in $Z$ ; see [B.1] ch.VI §3.
Thanks to the projection formula (see Appendix 3), the section of the sheaf $\mathcal{H}^2_I(Z(1)_D)$ associated to $\Sigma_Y$ is given by:

$$Tr_{/S}(F_*(C_S^{X/S} \cup C_S^{S \times W})) = Tr_{/S}(F_*(C_S^{X/S} \cup F^*C_S^{S \times Y})) = \hat{Tr}_{/S}(C_S^{X/S} \cup F^*C_S^{S \times Y})$$

because $Tr_{/S}F_* = \hat{Tr}_{/S}$ (see Appendix [C]).

So we conclude that we have the same section of $\mathcal{H}^2_I(Z(1)_D)$ and $\Sigma_{f*Y} = \Sigma_Y$.

**Remark.**

Using Property 2) to normalize $S$ and Property 3) to localize at generic points of $T = |\Sigma_Y|$ we can reduce the proof of the formula $\Sigma_{f*Y} = \Sigma_Y$ to the equality of corresponding Weil divisors. This can be deduced from the projection formula (then $X$ and $\hat{X} = F_*X$ are seen as absolute cycles in $S \times Z$ and $S \times W$, with $S$ smooth).

This avoids to use compatibility for the relative trace in relative Deligne cohomology with direct images (proved in the Appendix [C]).

5) **Inverse image for cycles.**

Consider a holomorphic map $f : Z \to W$ between complex manifolds, $(X_s)_{s \in S}$ an analytic family of cycles in $W$ such that the family $(f^*X_s)_{s \in S}$ is an analytic family of $n$–cycles in $Z$ (see [B.1] ch.VI §3 for a definition) and $Y$ an $(n + 1)$–codimensional cycle in $Z$ such that $f_*Y$ is well defined in $W$.

Assume that $Y$ satisfies [AP] for the family $(f^*(X_s))_{s \in S}$, then we want to show that $f_*Y$ satisfies [AP] and that

$$\sum Y = \sum_{f_*Y}$$

**Proof.**

Again Condition [AP] for the cycle $f_*Y$ is easy to verify because $f|_Y$ is proper and surjective on $|f_*Y|$. Then using as before the projection formula (Appendix [B]) and the compatibility of the (relative) trace with direct image we have

$$Tr_{/S}(F_*(C_S^{X/S} \cup C_S^{S \times Z})) = Tr_{/S}(C_S^{X/S} \cup F_*(C_S^{S \times Y})) = \hat{Tr}_{/S}(F_*(C_S^{X/S} \cup C_S^{S \times W})).$$

Thus the corresponding sections of the sheaf $\mathcal{H}^2_I(Z(1)_D)$ coincide.

6) **Parameters on $Y$.**

Now we consider an analytic family $(Y_v)_{v \in V}$ of $(p - 1)$–cycles in $Z$ parametrized by a reduced complex space $V$ and such that, for any $v \in V$, the cycle $Y_v$ satisfies [AP] for the family $(X_s)_{s \in S}$ in $Z$. 
Then the family \((\Sigma_{Y_v})_{v \in V}\) is a flat family of Cartier divisors in \(S\) or an analytic family of cycles; for Cartier divisors it is equivalent (see [B.1] ch. V).

The proof here is a repetition of the absolute case where \(V\) is a point; the absolute fundamental class of \(Y\) in \(Z\) being replaced by the relative fundamental class of the family \((Y_v)_{v \in V}\).

Then, using \(S \times V\) as a parameter space, we conclude that there exists a Cartier divisor \(\Sigma_Y\) in \(S \times V\) such that (using base change) for any \(v \in V, \Sigma_Y|_{S \times \{v\}} = \Sigma_{Y_v}\). so \(\Sigma_Y\) is \(V\)-flat in \(S \times V\) and the proof is complete. ■

Now to complete the proof of Theorem 1 we shall show that Conditions 1) to 6) characterize our construction:

The first remark is that, using 2) and 3), our construction is uniquely determined by the case \(S\) smooth and \(|p^*_X Y|\) finite on \(|\Sigma_Y|\). Then using 6) we can move locally \(Y\) in order to have \(|Y|\) smooth around \(|Y| \cap |X_{s_0}|\) (see Appendix II.A for details on this argument). In this case we can assume that the cycle \(Y\) underlies a complete intersection ideal (locally in \(Z\)) and using 4) we can reduce the situation to \(Z = \mathbb{C}^{n+1}\) and \(Y = k\{0\}(*), (X_s)_{s \in S}\) being an analytic family of hypersurfaces near 0 in \(\mathbb{C}^{n+1}\).

Finally, using 5) to cut with a smooth curve through \(\{0\}\), we are reduced to the case \(Z = \mathbb{C}\) and \(Y = k\{0\}\) which is determined by 1). ■

**Remark 1.**

As we have seen in the end of the proof above, the condition 5) allows to cut the cycles \((X_s)_{s \in S}\) with a submanifold containing \(Y\) (when the intersections have the expected dimension).

**Remark 2.**

Assume that \(Y\) and \(Y'\) are admissible poles which are, as \((n+1)\)-codimensional cycles, rationally equivalent (i.e. \(\exists \mathcal{Y} \subset \mathbb{P}_1 \times Z\) codimension \(n+1\) cycle in \(\mathbb{P}_1 \times Z\), equidimensional on \(\mathbb{P}_1\), such that \(\mathcal{Y}_0 = Y\) and \(\mathcal{Y}_\infty = Y'\)). Then for \(S\) compact the corresponding Cartier divisors \(\Sigma_Y\) and \(\Sigma_{Y'}\) in \(S\) are linearly equivalent. This is an easy consequence of [K] because the holomorphic map \(\mathbb{P}_1 \rightarrow H^1(S, \mathcal{O}_S^*)\) obtained from the analytic family \((\mathcal{Y}_t)_{t \in \mathbb{P}_1}\) has to be constant.

Another easy consequence of [K] is the result concerning the Abel’s theorem (or problem b) ) obtained by Griffiths in [Gr].

(*) For this argument see [BMg1] p. 831.
4. Intersection number of the incidence divisor with a curve.

We consider the situation of the Theorem. Let $C$ be a curve in $S$ and assume

1) $|C| \cap |\Sigma_Y|$ is finite
2) for any $\sigma \in |C| \cap |\Sigma_Y|$, we have $|X_\sigma| \cap |Y|$ is finite.

Then the following proposition gives a very simple formula to compute $\#(C.\Sigma_Y)$.

Note that this intersection product is well defined because $C$ induces a homology class and $\Sigma_Y$ a cohomology class in $S$.

**Proposition 1.**

Let

- $(X_s)_{s \in S}$ be an analytic family of $n-$cycles in a complex manifold $Z$
- $Y$ be a $n+1-$codimensional cycle in $Z$ which is an admissible pole for the family $(X_s)_{s \in S}$ (see the condition $[AP] \ 2$)
- $C \hookrightarrow S$ be a curve satisfying conditions 1) and 2) above
- $X_C$ be the graph of the family obtained by pull back by $j$ from $(X_s)_{s \in S}$
- $p : S \times Z \to Z$ the canonical projection.

Then there exist a neighbourhood $W$ of $\bigcup_{\sigma \in |C| \cap |\Sigma_Y|} X_\sigma \cap Y$ for which the direct image $p'_*(X_C)$ (where $p'$ is the restriction of $p$ to $S \times W$) of the cycle $X_C$ is well defined and we have:

$$\#(p_*(X_C).Y) = \#(C.\Sigma_Y)$$

Remark that in $W$ the intersection $p'_*(X_C).Y$ is well defined because these cycles have dimension and codimension $n+1$ in $W$ and finite intersection by Assumptions 1) and 2).

**Proof.** The assumptions give us such a $W$. We denote $p$ the projection $p'$. Let $\nu : C' \to C$ be the normalization of $C$. The map $j \circ \nu : C' \to S$ is then an admissible pull back for the pole $Y$ and the family $(X_s)_{s \in S}$. So we are reduced to prove our formula when $S = C'$ is a smooth curve and all $X_\sigma \cap Y$ are finite. But in this case, as $S$ is smooth, we have

$$\Sigma_Y = \pi_*(X \cdot Y)$$

where $\pi$ is the projection of $S \times Z$ on $S$. Using the projection formula and the fact that, for finite cycles, the intersection number is preserved by direct image, we deduce that $\#(C.\Sigma_Y) = \#(X_C.p^*Y) = \#(p_*(X_C.Y))$. ■
Example. (very classical)
Let $Y$ be a codimension $n + 1$ cycle in $\mathbb{P}_N$ and let $(X_s)_{s \in S}$ be the universal family of $n$–planes in $\mathbb{P}_N$ (so $S = \text{Gr}(n + 1, N + 1)$ is the Grassmann manifold of $n$–plane in $\mathbb{P}_N$). Let $C$ be the line of $n$ planes in a $(n + 1)$–planes $\Pi$ containing a given $(n - 1)$–plane $P$ in $\Pi$.
Assume $P \cap Y = \emptyset$ and $Y \cap \Pi$ finite. Then $\#(\Sigma_Y \cap C) = \#(Y \cap \Pi) = \deg Y$.
So the intersection number of the Chow divisor $\Sigma_Y$ of $Y$ in $\text{Gr}(n + 1, N + 1)$ with such a generic ”line” is the degree of $Y$ in $\mathbb{P}_N$.

Remark.
If we allow that, for some $\sigma$ in $|C| \cap |\Sigma_Y|$,
$|X_{\sigma}| \cap |Y|$ is not finite (but necessarily compact) the right handside of the previous formula is not defined. One can try to replace it by $\text{Tr}_W(C^W_{p_*(X_C)} \cup C^W_Y)$ which is well defined and give the same number in the finite intersection case.
It is possible to extend the proposition with this formulation as soon as it is possible to move the curve $C$ in $S$ in an analytic family $(C_t)_{t \in T}$ such that $C_{t_0} = C$ and that for generic $t$ we have $|X_\sigma| \cap |Y|$ is finite for all $\sigma \in |C_t| \cap |\Sigma_Y|$ (*): then, using the stability of analytic family by direct image and the existence of relative fundamental class for analytic family of cycles in complex manifold, it is enough to prove, after weak normalization of $T$, that the family $(X_{C_t})_{t \in T}$ is analytic.
In this situation we shall set
$$\#(C_{t_0}, \Sigma_Y) = \#(C_t, \Sigma_Y) \quad \text{for generic } t \text{ (near } t_0)$$
and
$$\#(X_C, Y) = \text{Tr}_W(C^W_{p_*, X_{C_t}} \cup C^W_Y)$$
$$= \text{Tr}_W(C^W_{p_*, X_{C_{t_0}}} \cup C^W_Y)$$
because $t \rightarrow \text{Tr}_W(C^W_{p_*, X_{C_t}} \cup C^W_Y)$ will be weakly holomorphic (existence of a relative fundamental class for the analytic family $p_*(X_{C_t})_{t \in \hat{T}}$ in $W$ where $\hat{T}$ is the weak normalization of $T$) and with value in $\mathbb{N}$ for generic $t \in T$.
To complete this extention of the Proposition it is enough to prove the following

**Proposition 2.**
Let $\tilde{Z} \xrightarrow{\pi} S$ be a geometrically flat map relative to the geometric weight $\tilde{X} \subset S \times \tilde{Z}$ (see [BMg1] p.12). Let $(C_t)_{t \in T}$ be an analytic family of cycles in $S$. Define for each $t \in T$, $\tilde{X}_{C_t}$ as the cycle in $\tilde{Z} \simeq \{t\} \times \tilde{Z}$ defined as the graph of the family of cycles in $\tilde{Z}$ given by the graph of the base change $C_t \hookrightarrow S$ (see below for a more precise definition).

(*): for instance if $|C| \cap |\Sigma_Y|$ is in $\text{Reg}(S)$ this will be possible.
Let \( \hat{T} \rightarrow T \) be the weak normalization of \( T \). Then \((\tilde{X}_{C_t})_{t \in \hat{T}}\) is an analytic family of cycles in \( \tilde{Z} \).

**Proof.**

Let us, first, be more precise on the definition of the cycles \( \tilde{X}_{C_t} \): let \( Y = \sum_{\alpha \in A} m_{\alpha} Y_{\alpha} \) be a cycle in \( S \) with \( Y_{\alpha} \) irreducible and locally finite, \( m_{\alpha} \in \mathbb{N}^* \). We define \( \pi^*(Y) \) in \( \tilde{Z} \) as follows: let \( Y_{\alpha} \) be the cycle in \( \tilde{Z} \) which is the graph of the analytic family defined by the base change \( Y_{\alpha} \hookrightarrow S \) from the family \((\tilde{X}_s)_{s \in S}\) of fibers of \( \pi : (\tilde{Z}, \tilde{X}) \rightarrow S \). Then let \( \pi^*(Y) := \sum m_{\alpha} Y_{\alpha} \).

So in the Proposition \( \tilde{X}_{C_t} := \pi^*(C_t) \).

Now, using the fact that the change of projection (in a scale) is always meromorphic and continuous ([B.1] Th. 2 p.42; we can also use Proposition 3 of ch. III §3) it is enough to prove that for any \( t_0 \in T \) and any \( z_0 \in \pi^{-1}(|C_{t_0}|) \) we can find a scale in a neighbourhood of \( z_0 \) adapted to \( \pi^*(C_{t_0}) \) such that the family of branched covering associated to \( \pi^*(C_t) \) \( t \in T \) is analytic near \( t_0 \).

For that purpose, choose a scale adapted to \( C_{t_0} \) near \( \pi(z_0) \) given by a local embedding

\[ S \hookrightarrow V \times B' \]

and a scale adapted to \( \tilde{X}_{\pi(z_0)} \) near \( z_0 \) given by a local embedding:

\[ \tilde{Z} \hookrightarrow S \times U \times B \hookrightarrow V \times U \times B \times B'. \]

So we have holomorphic maps:

\[
\begin{align*}
f : S' \times U & \longrightarrow Sym^l(B) \\
g : T' \times V & \longrightarrow Sym^k(B')
\end{align*}
\]

defining respectively the fibers of \( \pi \) near \( \pi(z_0) \in S' \) and the family \((C_t)_{t \in T} \) near \( t_0 \in T' \) (here \( S' \) and \( T' \) are "small" open sets in \( S \) and \( T \)).

Remark that, by our assumptions, \( f \) and \( g \) are isotropic (see [B.1] ch. 2).

Then the family \( \pi^*(C_t)_{t \in T'} \) is defined in the scale given by \( \tilde{Z} \hookrightarrow (V \times U) \times (B \times B') \) (near \( z_0 \)) by the map

\[ T' \times V \times U \longrightarrow Sym^{kl}(B \times B') \]

obtained as follows: from \( g \) we have an holomorphic map \( G : T' \times V \longrightarrow Sym^k(V \times B') \) where \( G(t, v) = ((v, b'_1), \ldots, (v, b'_k)) \), if \( g(t, v) \) is the \( k- \)uple \( (b'_1, \ldots, b'_k) \). But, by assumption, \( (v, b'_j) \in S \) for \( j \in [1, k] \). So \( G \) factors through the holomorphic map

\[ G_0 : T' \times V \longrightarrow Sym^k(S'). \]
Now, using $G_0$ and $f$ we obtain an analytic map

$$T' \times V \times U \longrightarrow \text{Sym}^k(S' \times U) \xrightarrow{\text{Sym}^kF} \text{Sym}^{kl}(B \times B')$$

where $F : S' \times U \longrightarrow \text{Sym}^l(B \times B')$ is given by

$$F(s, u) = ((b_1, p(s)), \ldots, (b_l, p(s)))$$

if $f(s, u) = (b_1, \ldots, b_l)$ and $p : S' \longrightarrow B'$ is the composition of the embedding $S' \hookrightarrow V \times B'$ and the projection on $B'$.

So the map $T' \times V \times U \longrightarrow \text{Sym}^{kl}(B \times B')$ is holomorphic and this proves Proposition 2.

**Remark.**

It is not clear (an may be not true ) that the map $T' \times V \times U \longrightarrow \text{Sym}^{kl}(B \times B')$ is $T'$–isotropic when $f$ and $g$ are isotropic. That is the reason why we have to normalize weakly $T$ in our conclusion.

Of course, because the weak normalization is an homeomorphism, this is irrelevant in our generalisation of the proposition 1.

To conclude, let us state the simple case where this generalization works :

**Corollary.**

*In the situation of Proposition 1, assume

2') for any $\sigma \in |\mathcal{C}| \cap |\Sigma_Y|$, either $|X_\sigma| \cap |Y|$ is finite or $\sigma$ is a smooth point in $S$.

Then Hypothesis 1) and 2') imply

$$\#(\mathcal{C}.\Sigma_Y) = \text{Trace}_W(C^W_{p_*(X_c)} \cup C^W_Y)$$*
Appendix

I.

The formulas we want to prove in this Appendix (intersection, projection and compatibility between direct image and trace in Deligne cohomology) are consequences of

1) the existence in Deligne cohomology of a push-forward and a pull-back for an holomorphic map between complex manifolds.

2) the corresponding formulas in holomorphic cohomology

and we conclude using the injectivity result of [K] corollary 2 p.295.

So our first purpose is to prove 1).

Let \( f : Z \to W \) be a holomorphic map between connected complex manifolds and let \( d = \dim_{\mathbb{C}} W - \dim_{\mathbb{C}} Z \quad (d \in \mathbb{Z}) \).

To build a push-forward for \( k \in \mathbb{N} \)

\[
\begin{align*}
 f^D_* : Rf_! Z(k)_{\mathcal{D}, Z} & \to Z(k + d)_{\mathcal{D}, W} \,[2d] \\
 f^h_* : Rf_! \Omega^\bullet_Z & \to \Omega^{\bullet + d}_W \,[d] \\
 f^t_* : Rf_! Z_Z & \to Z_W \,[2d]
\end{align*}
\]

we shall use the holomorphic and topological push-forwards

\[
\begin{align*}
 f^h_* : Rf_! \Omega^\bullet_Z & \to \Omega^{\bullet + d}_W \,[d] \\
 f^t_* : Rf_! Z_Z & \to Z_W \,[2d]
\end{align*}
\]

which are defined as follows :

Using the factorization of \( f \) by its graph, it is enough to handle the case where \( f \) is the embedding of a closed submanifold and when \( f \) is the projection of a product (so \( Z = X \times W \) with \( \dim_{\mathbb{C}} X = d \)).

In the first case \( d = \text{codim}_Z W \), we have

\[
\begin{align*}
 f^h_* : \Omega^\bullet_Z & \to H^d_Z(\Omega^{\bullet + d}_W) \\
 \text{and} \quad f^t_* : Z_Z & \to H^{2d}_Z(Z_W)
\end{align*}
\]

which are quasi isomorphisms.

In the case of the projection \( Z = X \times W \to W \) we have the usual ”integration” maps.

\[
\begin{align*}
 f^h_* : R^d f_! \Omega^{\bullet+ d}_Z & \to \Omega^\bullet_W \\
 f^t_* : R^{2d} f_! Z_Z & \to Z_W
\end{align*}
\]

The push-forward (1) is now built by a decreasing induction on $k$.
For $k$ large enough we have, by the exactness of the holomorphic de Rham complex, a quasi-isomorphism
$$Z(k)_D \simeq (\mathbb{C}/\mathbb{Z})[-1]$$
on any complex manifold. So the topological push-forward $f^*_t$ with coefficient $\mathbb{C}/\mathbb{Z}$ is enough to define (1) for large $k$.
To define (1) for $k$ assuming that it is already built for $k + 1$, we consider the exact sequence of complexes on $Z$ and $W$
$$0 \longrightarrow \Omega^k[-k - 1] \longrightarrow Z(k + 1)_D \longrightarrow Z(k)_D \longrightarrow 0$$
and the diagram
$$\begin{array}{cccc}
Rf^!\Omega^k_Z[-k - 1] & \longrightarrow & Rf^!Z(k + 1)_{D,Z} & \longrightarrow & Rf^!Zk_{D,Z} \\
\downarrow f^*_h & & \downarrow f^*_p & & \downarrow \\
\Omega^k_W(d - k - 1)[2d] & \longrightarrow & Z(k + d + 1)_{D,W}[2d] & \longrightarrow & Z(k + d)_{D,W}[2d]
\end{array}$$
from which we deduce the definition of $f^*_p$ for $k$ (we assume inductively on $k$ the commutativity of this diagram).
We can claim that
1) The functor $Rf^!$ transforms distinguished triangles into distinguished triangles.
2) If $(A, B, C)$ and $(A', B', C')$ are two distinguished triangles with two morphisms $u : A \rightarrow A'$ and $u' : B \rightarrow B'$ such that the square $(A, B, A', B')$ is commutative then we have a morphism between these triangles.
The construction for the pull-back $(\forall k \in \mathbb{N})$
$$f^*_D : f^!Z(k)_{D,W} \longrightarrow Z(k - d)_{D,Z}$$
works in the same way :
We use the holomorphic and topological pull-back
$$(4) \quad f^*_h : Rf^!\Omega^*_W \longrightarrow \Omega^*_W[-d]$$
$$f^*_t : f^!Z_W \longrightarrow Z_Z[-2d]$$
defined, as before, by factorizing $f$ through its graph. Then for $f$ the inclusion of $Z$ as a codimension $d$ closed submanifold of $W$ we have
$$f^*_h : H^d_Z(\Omega^*_W) \longrightarrow Z_Z$$
and $$f^*_t : H^{2d}_Z(Z_W) \longrightarrow Z_Z$$
which are the holomorphic and topological residue maps (they are inverse of $f^*_h$ and $f^*_t$ respectively).

In the projection case we have

\[ f^*_h : f^!\Omega^*_W = f^*\Omega^*_W \otimes \Omega^d_{W \times Z/W}[d] \longrightarrow \Omega^*_Z[d] \]

and

\[ f^*_t : f^!Z_W = f^*Z_W \otimes Z_{Z \times W}[2d] \longrightarrow Z_Z[2d]. \]

Then we proceed, as before, by a decreasing induction on $k$ to construct the pull-back (3).

**Remark.**

By construction, it is obvious that (1) and (3) are compatible with (2) and (4).

\section{II.}

Our aim now is to describe more precisely the nice behaviour of fundamental classes in Deligne cohomology (absolute and relative case) for some simple operations. These results are not ”new” but we were unable to give references for these results in our local analytic context.

\subsection{A. Intersection.}

We shall prove the following

**Proposition 1.**

Let

\( Z \) be a complex manifold

\( S \) and \( T \) complex analytic reduced spaces

\( (X_s)_{s \in S} \) (resp. \( (Y_t)_{t \in T} \)) analytic family of pure \( p \) (resp. \( q \))-codimensional cycles in \( Z \) parametrized by the reduced analytic space \( S \) (resp. \( T \)).

\( C^2_{X/S} \) and \( C^2_{Y/T} \) the relative fundamental classes in Deligne cohomology for these families

\( p_1 : S \times T \times Z \rightarrow S \times Z \) (resp. \( q_1 : S \times T \times Z \rightarrow T \times Z \)) the canonical projection.

Assume that for any \((s, t) \in S \times T\) the closed analytic set \(|X_s| \cap |Y_t|\) has the expected codimension.

Then, \((X_s, Y_t)_{(s, t) \in S \times T}\) is an analytic family of cycles in \( Z \) and its relative fundamental class in Deligne cohomology is given by the cup product \( p_1^*C^2_{X/S} \cup q_1^*C^2_{Y/T} \) in

\[ \text{IH}_{p_1^{-1}(|X|) \cap q_1^{-1}(|Y|)}^{2(p+q)}(\mathbb{Z}(p+q)\mathcal{D}/S \times T) \]
Proof.
We shall proceed by successive reductions

First reduction
First, let \( X_1 := p_1^* (X) \) and \( Y_1 := q_1^* (Y) \).

Theorem. 10 (local) of [B.1] ch.VI says us that the family \((X_s, Y_t)_{(s,t) \in S \times T}\) is analytic.

For our purpose, it suffices to work in holomorphic cohomology because we have injective morphisms of sheaves:

\[
\begin{align*}
\mathbb{H}^2_{|X_1|}(\mathbb{Z}(p) \mathcal{D}/S) & \rightarrow \mathbb{H}^p_{|X_1|}(\Omega^p_{S \times Z/S}) \\
\mathbb{H}^2_{|Y_1|}(\mathbb{Z}(q) \mathcal{D}/S) & \rightarrow \mathbb{H}^q_{|Y_1|}(\Omega^q_{T \times Z/T}) \\
\mathbb{H}^{2(p+q)}_{|X_1| \cap |Y_1|}(\mathbb{Z}(p+q) \mathcal{D}/S \times T) & \rightarrow \mathbb{H}^{p+q}_{|X_1| \cap |Y_1|}(\Omega^{p+q}_{S \times T \times Z/S \times T})
\end{align*}
\]

(see [K] p. 320)

Second reduction
We can assume \( S \) and \( T \) are points, because the sheaf \( \mathbb{H}^{p+q}_{|X_1| \cap |Y_1|}(\Omega^{p+q}_{S \times T \times Z/S \times T}) \) has no torsion over \( \mathcal{O}_{S \times T} \)

Third reduction
Using diagonal trick, we can assume that \( Y \) is a smooth (closed) submanifold in \( Z \).

Now the case where \( X \) and \( Y \) are smooth and transversal along \(|X| \cap |Y|\), is trivial. We denote this intersection \( X \bullet Y \).

We are reduced to consider the case where \( X = |X| \) is irreducible of dimension \( n \) and where \( Y \) is smooth of codimension \( n - q \) in \( Z \) an open ball in \( \mathbb{C}^N \). Of course we assume that \( X \cap Y \) is of pure dimension \( q \). Then we consider the family of \( Y_s = Y + s \) (translated of \( Y \)) where \( s \in \mathbb{C}^N \) is small enough, say in \( S \). Then for any given open set \((*)\) in \( Z \), there exists an open dense set in \( S \) such that the intersection \( X.Y_s \) is smooth and transversal. So \( C^Z_X \cup C^Z_{Y/S} \) induces, for an open dense set in \( S \), the fundamental class of \( X \cap Y_s \). So \( C^Z_X \cup C^Z_{Y/S} \) has to be the relative fundamental class of the family \((X \cap Y_s)_{s \in S}\), which is analytic by [B.1] Th. 10 (local) and so has a relative fundamental class by [B.2]. So we conclude that for \( s = 0 \) we have

\[
C^Z_X \cup C^Z_Y = C^Z_{X \cap Y}.
\]

B. Projection formula.
We want now to prove the following

\((*)\) see [B.1] proposition 2 p. 138 for a more precise proof of this.
Proposition 2.
Let \( f : Z \rightarrow W \) be a holomorphic map between complex manifolds of dimension \( d \) and \( \delta \) respectively. Let \( X \) be a \( n \)-cycle in \( Z \) such that \( f|_{|X|} \) is proper and generically finite on its image; let \( Y \) be a \((n+q)\)-codimensional cycle in \( W \). Assume that \( X \cdot f(Y) \) is well defined(**). Then we can define \( f_*X \) in \( W \) and also \((f_*X) \cdot Y\). The direct image \( f_* (X \cdot f(Y)) \) is also well defined and the projection formula says

\[
(P_0) \quad f_*(X \cdot f(Y)) = (f_*X) \cdot Y.
\]

The corresponding formula in Deligne cohomology is the equality

\[
(P) \quad f_* (C^Z_X \cup f^*(C^W_Y)) = f_* (C^Z_X) \cup C^W_Y
\]

where the map \( f^*: \mathcal{H}^{2(n+q)}(W, \mathbb{Z}(n+q)_\mathcal{D}) \rightarrow \mathcal{H}^{2(n+q)}(Z, \mathbb{Z}(n+q)_\mathcal{D}) \) is the pullback and the maps

\[
\begin{align*}
  f_* : \mathcal{H}^{2(d-q)}_{|X| \cap f^{-1}(|Y|)}(Z, \mathbb{Z}(d-q)_\mathcal{D}) & \rightarrow \mathcal{H}^{2(d-q)}_{f(|X| \cap |Y|)}(W, \mathbb{Z}(d-q)_\mathcal{D}) \\
  f_* : \mathcal{H}^{2(d-n)}_{|X|}(Z, \mathbb{Z}(d-n)_\mathcal{D}) & \rightarrow \mathcal{H}^{2(d-n)}_{f(|X|)}(W, \mathbb{Z}(d-n)_\mathcal{D})
\end{align*}
\]

are the trace maps (described below via the holomorphic cohomology).

Proof.
As before it is enough to prove Formula \((P_0)\) in holomorphic cohomology. Recall that the trace map, for a closed set \( F \) such that \( f|_F \) is proper,

\[
f_* : H^{d-\alpha}_F(Z, \Omega^d_Z) \rightarrow H^{\delta-\alpha}_{f(F)}(W, \Omega^\delta_W)
\]

is given by direct image of hyperfunctions of type \((d-\alpha, d-\alpha), \overline{\partial}\)-closed with support in \( F \).

Because the part \( A \) and the previous remark we have only to prove that the formula

\[
C^W_{f_* (X)} = f_* (C^Z_X)
\]

is valid in holomorphic cohomology. We can use for that, the fact that the integration current \([X]\) on \( X \) represents the class \( C^Z_X \) in \( H^{d-n}_{|X|}(Z, \Omega^d_Z) \). Now the formula

\[
f_* [X] = [f_* X]
\]

(**) see [B.1] chap.6 §3. This hypothesis means that in \( Z \times W, Z \times Y \) meets the graph of \( f|_{|X|} \) in the expected dimension.
between currents in \( W \) is essentially a trivial change of variable formula. So the proof is complete. ■

Corollary.
The Proposition extends in a straightforward manner in the relative case \( ((X_s)_{s \in S} \text{ in } Z \text{ and } (Y_t)_{t \in T} \text{ in } W \) are now analytic families of cycles).

C. Compatibility between direct image and trace

We shall prove the following

**Proposition 3.**
Let \( f : Z \to W \) be a holomorphic map between complex manifolds of dimension \( d \) and \( \delta \) respectively. Then we have the following commutative triangle

\[
\begin{array}{ccc}
\mathcal{H}^2_{c}(\mathcal{Z}(2d),Z) & \xrightarrow{f^*} & \mathcal{H}^2_{c}(\mathcal{Z}(2\delta),W) \\
\downarrow \tilde{\text{Tr}} & & \downarrow \text{Tr} \\
\mathbb{C} & & \mathbb{C}
\end{array}
\]

where \( \tilde{\text{Tr}} \) and \( \text{Tr} \) are the trace maps.

**Proof.**
In fact if \( \mathcal{H}^d_c(Z,\Omega^d_Z) \xrightarrow{\int_Z} \mathbb{C} \) (resp. \( \mathcal{H}^\delta_c(W,\Omega^\delta_W) \xrightarrow{\int_W} \mathbb{C} \) is the usual integration map, \( \tilde{\text{Tr}} \) (resp. \( \text{Tr} \)) is the composition of \( \int_Z \) (resp. \( \int_W \)) with the connector of the long exact hypercohomology sequence of the short exact sequence of complexes

\[
0 \to \Omega^d[-d-1] \to \mathcal{Z}(d+1) \to \mathcal{Z}(d) \to 0.
\]

The commutativity of the triangle above is then a consequence of the compatibility of the integration maps with the direct image: if \( \varphi \) is a \( C^\infty(d,d) \) form with compact support on \( Z \), \( f_*\varphi \) is a \( (\delta,\delta) \) current with compact support in \( W \) and we have

\[
\int_W f_*\varphi := \langle f_*\varphi, 1 \rangle = \langle \varphi, f^*(1) \rangle = \int_Z \varphi.
\]

Corollary.

**Proposition 3** holds in the relative setting.
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