UNIQUENESS IN LAW FOR THE ALLEN-CAHN SPDE VIA CHANGE OF MEASURE

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Abstract. We start by first using change of measure to prove the transfer of uniqueness in law among pairs of parabolic SPDEs differing only by a drift function, under an almost sure $L^2$ condition on the drift/diffusion ratio. This is a considerably weaker condition than the usual Novikov one, and it allows us to prove uniqueness in law for the Allen-Cahn SPDE driven by space-time white noise with diffusion function $\sigma(t,x,u) = Cu^\gamma$, $1/2 \leq \gamma \leq 1$ and $C \neq 0$. The same transfer result is also valid for ordinary SDEs and hyperbolic SPDEs.

1. Introduction.

We start by considering the pair of parabolic SPDEs

\begin{equation}
\begin{aligned}
\frac{\partial U}{\partial t} & = \Delta_x U + b(t,x,U) + a(t,x,U) \frac{\partial^2 W}{\partial t \partial x}; \\
U_x(t,0) & = U_x(t,L) = 0; \\
U(0,x) & = h(x);
\end{aligned}
\end{equation}

(1.1)

and

\begin{equation}
\begin{aligned}
\frac{\partial V}{\partial t} & = \Delta_x V + (b+d)(t,x,V) + a(t,x,V) \frac{\partial^2 W}{\partial t \partial x}; \\
V_x(t,0) & = V_x(t,L) = 0; \\
V(0,x) & = h(x);
\end{aligned}
\end{equation}

(1.2)

on the space-time rectangle $\mathcal{R}_{T,L} \triangleq [0,T] \times [0,L]$, where $\tilde{\mathcal{R}}_{T,L} \triangleq (0,T] \times (0,L)$. $W(t,x)$ is the Brownian sheet corresponding to the driving space-time white noise, written formally as $\partial^2 W/\partial t \partial x$. As in Walsh [1], white noise is regarded as a continuous orthogonal martingale measure, which we denote by $\mathcal{W}$, with the corresponding Brownian sheet as the random field induced by $\mathcal{W}$ in the usual way: $W(t,x) = \mathcal{W}([0,t] \times [0,x]) \triangleq \mathcal{W}(0,x)$. The diffusion $a(t,x,u)$ and the drifts $b(t,x,u)$ and $d(t,x,u)$ are Borel-measurable $\mathbb{R}$-valued functions on $\mathcal{R}_{T,L} \times \mathbb{R}$; and $h : \mathcal{R}_{T,L} \to \mathbb{R}$ is a Borel-measurable function. Henceforth, we will denote (1.1) and (1.2) by $e^{\text{heat}}_{\text{Neu}}(a,b,h)$ and $e^{\text{heat}}_{\text{Neu}}(a,b+d,h)$, respectively. When $b \equiv 0$, we denote (1.1) by $e^{\text{heat}}_{\text{Neu}}(a,0,h)$. In the interest of getting quickly to our main results, we relegate to the Appendix the rigorous interpretation of all SPDEs considered in this paper.

Proceeding toward a precise statement of the main result, we adopt some convenient notation. Let $R_u(t,x) \triangleq d(t,x,u)/a(t,x,u)$, for any $(t,x,u) \in [0,T] \times [0,L] \times \mathbb{R}$.
Theorem 1.1. Assume that $R_U$ and $R_V$ are in $L^2(\mathbb{R}_{T,L}, \lambda)$, almost surely, whenever the random fields $U$ and $V$ solve (weakly or strongly) $e_{\text{heat}}^N(a, b, h)$ and $e_{\text{heat}}^N(a, b+d, h)$, respectively (see Remark 4.2). Then, uniqueness in law holds for $e_{\text{heat}}^N(a, b, h)$ iff uniqueness in law holds for $e_{\text{heat}}^N(a, b+d, h)$.

By transferring uniqueness in law under weaker conditions (almost surely $L^2$ vs. Novikov’s), Theorem 1.1 makes more applicable the notion of Girsanov equivalence in our earlier work (Theorem 3.3.2 in [3] or Theorem 4.2 in [2]). The Neumann conditions in $e_{\text{heat}}^N(a, b, h)$ and $e_{\text{heat}}^N(a, b+d, h)$ may be changed to Dirichlet conditions without affecting the conclusions of Theorem 1.1.

An interesting application of Theorem 1.1 is provided in Theorem 1.2 below for the stochastic Allen-Cahn equation driven by space-time white noise

\begin{eqnarray}
\frac{\partial V}{\partial t} = \Delta_x V + 2V(1-V^2) + CV\gamma \frac{\partial^2 W}{\partial t \partial x}; \quad (t, x) \in \mathbb{R}_{T,L}, \\
V_x(t,0) = V_x(t,L) = 0; \quad 0 < t \leq T, \\
V(0, x) = h(x); \quad 0 < x < L,
\end{eqnarray}

in the case $C \neq 0$ and $\frac{1}{2} \leq \gamma \leq 1$. The deterministic Allen-Cahn PDE was introduced by Allen and Cahn [1] as a model for grain boundary motion. It has since become an important PDE for many mathematicians (see e.g. Katsoulakis et al. [7], Sowers et al. [13], and the references therein); and we intend to investigate, in a future paper, further properties of its solutions in the presence of a driving space-time white noise. We are thankful to Markos Katsoulakis for interesting Allen-Cahn conversations.

Theorem 1.2. Consider the stochastic Allen-Cahn equation (1.3) $(C \neq 0)$. If $\gamma \in \left[\frac{1}{2}, 1\right]$ then uniqueness in law holds for (1.3).

Theorem 1.2 follows since

- The Allen-Cahn SPDE (1.3) satisfies our transfer condition, and
- A result of Mytnik [9] gives us uniqueness in law for the SPDE (1.1) with $b \equiv 0$ and $a(t, x, u) = Cu^\gamma$, $\gamma \in (1/2, 1)$, the case $a(t, x, u) = Cu^{1/2}$ admits uniqueness in law as discussed in [8] p. 326 and in [12], and a classic result of Walsh gives us uniqueness for (1.1) when $b \equiv 0$ and $a(t, x, u) = Cu$.

With minor adaptations, the same uniqueness transfer result in Theorem 1.1 holds for ordinary SDEs and hyperbolic (wave) SPDEs (see [2] Theorem 3.6, Theorem 5.2, and their proofs for our uniqueness and existence transfer result for space-time SDEs and their rotationally-equivalent wave SPDEs, using change of measure under Novikov’s condition). We note that the existence of solutions to heat SPDEs with continuous diffusion coefficients $a$ satisfying a linear growth condition was established in [8] [10] [11]. In [3], we used an approximating system of stochastic differential-difference equations (SDDEs) to give a new proof of Reimers’ existence result, then we used our Girsanov theorem from [2] to extend the result to measurable drifts, under Novikov’s condition.
2. Proof of the Main Result.

We begin by adapting the well known Novikov condition to our setting: we say that a predictable random field \( X \) on the probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) (see Walsh [14]) satisfies Novikov’s condition on \( \mathcal{F} \),

\[
\mathbb{E}_\mathbb{P} \left[ \exp \left( \frac{1}{2} \int_{\mathcal{F}} X^2(t, x) dt dx \right) \right] < \infty.
\]

Remark 2.1. It is clear that if \( R_t \) is uniformly bounded for \((t, x, u) \in \mathcal{R}_{T,L} \times \mathbb{R} \), then \( R_t \) satisfies Novikov’s condition for every predictable random field \( Z \).

Proof of Theorem 1.1. Assume that uniqueness in law holds for \( e^{\text{New}}_{\text{heat}}(a, b, h) \), and suppose that

\[
(V^{(i)}, \tilde{W}^{(i)}), \ (\Omega^{(i)}, \mathcal{F}^{(i)}, \{\mathcal{F}_t^{(i)}\}, \tilde{\mathbb{P}}^{(i)}); \ i = 1, 2,
\]

are solutions to \( e^{\text{New}}_{\text{heat}}(a, b + d, h) \). By assumption

\[
\tilde{\mathbb{P}}^{(i)} \left[ \int_{\mathcal{R}_{T,L}} R_{V^{(i)}}^2(t, x) dt dx < \infty \right] = 1, \ i = 1, 2.
\]

Now take \( \{\tau_n^{(i)}\} \) to be the sequence of stopping times

\[
\tau_n^{(i)} \overset{\triangle}{=} T \wedge \inf \left\{ 0 \leq t \leq T; \int_{\mathcal{R}_{T,L}} R_{V^{(i)}}^2(s, x) ds dx = n \right\}; \ n \in \mathbb{N}, \ i = 1, 2.
\]

Let \( W^{(i)} = \{W^{(i)}_t(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathbb{B}([0, L])\} \) be given by

\[
W^{(i)}_t(B) \overset{\triangle}{=} \tilde{W}^{(i)}_t(B) + \int_{[0,t] \times B} R_{V^{(i)}}(s, x) ds dx; \ i = 1, 2.
\]

Novikov’s condition [20] and Girsanov’s theorem for white noise (see Corollary 3.1.3 in [3]) imply that \( W^{(i)}_t = \{W^{(i)}_t(t, x, u); 0 \leq t \leq T, B \in \mathbb{B}([0, L])\} \) is a white noise stopped at time \( \tau_n^{(i)} \), under the probability measure \( \mathbb{P}_n^{(i)} \) defined on \( \mathcal{F}_T^{(i)} \) by the recipe

\[
\frac{d\mathbb{P}_n^{(i)}}{d\mathbb{P}^{(i)}} = \mathbb{I}_{\mathbb{T}_n^{(i)}(B)} \mathbb{I}_{\mathbb{L}_n^{(i)}(B)} (0, L); \ n \in \mathbb{N}, \ i = 1, 2,
\]

where

\[
\mathbb{I}_{\mathbb{T}_n^{(i)}(B)} \mathbb{I}_{\mathbb{L}_n^{(i)}(B)}
\]

\[
\overset{\triangle}{=} \exp \left[ - \int_{[0,t \wedge \tau_n^{(i)}] \times B} R_{V^{(i)}}(s, x) \tilde{\mathbb{W}}^{(i)}(ds, dx) - \frac{1}{2} \int_{[0,t \wedge \tau_n^{(i)}] \times B} R_{V^{(i)}}^2(s, x) ds dx \right];
\]

\( 0 \leq t \leq T, B \in \mathbb{B}([0, L]) \). It follows that \( (V^{(i)}, W^{(i)}_n), (\Omega^{(i)}, \mathcal{F}^{(i)}_T, \{\mathcal{F}_t^{(i)}\}, \mathbb{P}_n^{(i)}) \) is a solution to \( e^{\text{New}}_{\text{heat}}(a, b, h) \) on \( \mathcal{R}_{T \wedge \tau_n^{(i)}, L} = [0, T \wedge \tau_n^{(i)}] \times [0, L] \) for each \( i = 1, 2 \) and
n \in \mathbb{N}. \text{ Of course, for } i = 1, 2,

(2.4) \quad \frac{d\bar{P}_n^{(i)}}{dP_n} = \Xi_{T \wedge \tau_n^{(i)}}^{R_{V^{(i)}}(1), \mathcal{W}^{(i)}}([0, L])

\Delta \exp \left[ \int_{[0, T \wedge \tau_n^{(i)}] \times [0, L]} R_{V^{(i)}}(s, x) \mathcal{W}^{(i)}(ds, dx) - \frac{1}{2} \int_{[0, T \wedge \tau_n^{(i)}] \times [0, L]} R_{V^{(i)}}^2(s, x) ds dx \right];

n \in \mathbb{N}. \text{ Consequently, for any set } \Lambda \in \mathcal{B}(C(\mathbb{R}_{T,L}, \mathbb{R}))

\tilde{P}^{(1)}(V^{(1)} \in \Lambda, \tau_n^{(1)} = T) = \mathbb{E}_{P_n^{(1)}} \left[ 1_{\{V^{(1)} \in \Lambda, \tau_n^{(1)} = T\}} \Xi_{T \wedge \tau_n^{(i)}}^{R_{V^{(1)}}(1), \mathcal{W}^{(i)}}([0, L]) \right]

= \mathbb{E}_{P_n^{(2)}} \left[ 1_{\{V^{(2)} \in \Lambda, \tau_n^{(2)} = T\}} \Xi_{T \wedge \tau_n^{(2)}}^{R_{V^{(2)}}(2), \mathcal{W}^{(2)}}([0, L]) \right]

= \tilde{P}^{(2)}(V^{(2)} \in \Lambda, \tau_n^{(2)} = T); \forall n \in \mathbb{N},

where we have used the uniqueness in law assumption on \( \exp_{\text{heat}}^{Neu}(a, b, h) \) (comparing the \( V^{(i)} \)'s only on \( \Omega_n^{(i)} = \{ \tau_n^{(i)} = T \} \) for each \( n \)), (2.3), and (2.4) to get the second equality in (2.5). By (2.2) and (2.3), we get that \( \lim_{n \to \infty} \tilde{P}^{(i)}(\tau_n^{(i)} = T) = 1 \) for \( i = 1, 2 \). We then see that passing to the limit as \( n \to \infty \) in (2.5) gives us that the law of \( V^{(1)} \) under \( \tilde{P}^{(1)} \) is the same as that of \( V^{(2)} \) under \( \tilde{P}^{(2)} \). I.e., we have uniqueness in law for \( \exp_{\text{heat}}^{Neu}(a, b + d, h) \). The proof of the other direction is similar and is omitted.

\( \square \)

Our Uniqueness result for the Allen-Cahn SPDE (1.3) can now be proved.

**Proof of Theorem 1.2** By Theorem 1.1 the proof essentially reduces to checking whether the random fields \( R_U \) and \( R_V \) are in \( L^2(\mathbb{R}_{T,L}, \lambda) \), almost surely, whenever \( U \) solves (weakly or strongly) \( \exp_{\text{heat}}^{Neu}(a, 0, h) \) (with \( a(t, x, u) \equiv Cu^\gamma \) and \( \frac{1}{2} \leq \gamma \leq 1 \)) and \( V \) solves (weakly or strongly) the Allen-Cahn SPDE (1.3). That is true can easily be seen since, in this case,

(2.6) \quad R_U^2(t, x) = 4C^2U^{2(1-\gamma)}(U^2 - 2U^2 + 1), \quad R_V^2(t, x) = 4C^2V^{2(1-\gamma)}(V^4 - 2V^2 + 1).

The continuity of \( U \) and \( V \) implies that \( R_U^2 \) and \( R_V^2 \) are continuous, for any \( 0 \leq \gamma \leq 1 \). Therefore, if \( U \) and \( V \) are defined on the usual probability spaces \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) and \((\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{\mathbb{P}})\), respectively, then

\[ |R_U^2(t, x, \omega)| \leq K(\omega, \gamma) < \infty; \text{ for all } (t, x) \in \mathbb{R}_{T,L}, \gamma \in [0, 1] \text{ a.s. } \mathbb{P}, \]

\[ |R_V^2(t, x, \tilde{\omega})| \leq \tilde{K}(\tilde{\omega}, \gamma) < \infty; \text{ for all } (t, x) \in \mathbb{R}_{T,L}, \gamma \in [0, 1] \text{ a.s. } \tilde{\mathbb{P}}, \]

where \( K \) and \( \tilde{K} \) depend only on \((\omega, \gamma) \in \Omega \times [0, 1]\) and \((\tilde{\omega}, \gamma) \in \tilde{\Omega} \times [0, 1]\), respectively. It follows that, for any fixed but arbitrary \( \gamma \in [0, 1] \), \( R_U \) and \( R_V \) are in \( L^2(\mathbb{R}_{T,L}, \lambda) \), almost surely. The assertion of Theorem 1.2 then follows from Theorem 1.1 and the fact that uniqueness in law holds for \( \exp_{\text{heat}}^{Neu}(a, 0, h) \) when \( a(t, x, u) \equiv Cu^\gamma \) and \( \gamma \in [\frac{1}{2}, 1] \) (see [8, 12, 9] and [14]). \( \square \)
Appendix

We collect here definitions and conventions that are used throughout this article. Filtrations are assumed to satisfy the usual conditions (completeness and right continuity), and any probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) with such a filtration is termed a usual probability space. The space of continuous functions on \(\mathcal{R}_{T,L}\) is denoted by \(C(\mathcal{R}_{T,L})\).

**Definition A.1** (Strong and Weak Solutions to \(e^{\text{Heat}}_{\text{heat}}(a,b,h)\)). We say that the pair \((U, W)\) defined on the usual probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) is a solution to the stochastic heat equation \(e^{\text{Heat}}_{\text{heat}}(a,b,h)\) if \(W\) is a space-time white noise on \(\mathcal{R}_{T,L}\); the random field \(U(t,x)\) is predictable (as in [14]), with continuous paths on \(\mathcal{R}_{T,L}\); and the pair \((U, W)\) satisfies the test function formulation:

\[
\int_0^L (U(t,x) - h(x))\varphi(x)dx - \int_0^L \int_0^t U(s,x)\varphi''(x)dsdx
\]

\[
= \int_0^L \int_0^t [a(s,x, U(s,x))\varphi(x)W(ds,dx) + b(s,x, U(s,x))\varphi(x)dsdx] \; ; \; 0 \leq t \leq T,
\]

as \(\mathbb{P}\), for every \(\varphi \in \Theta^L_0 \triangleq \{ \varphi \in C^\infty_c(\mathbb{R}; \mathbb{R}) : \varphi(t) = \varphi'(L) = 0 \} \) \((C^\infty_c(\mathbb{R}; \mathbb{R})\) being the collection of smooth \(\mathbb{R}\)-valued function on \(\mathbb{R}\) with compact support). A solution is said to be strong if the white noise \(W\) and the usual probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) are fixed a priori and \(\mathcal{F}_t\) is the augmentation of the natural filtration for \(W\) under \(\mathbb{P}\). It is termed a weak solution if we are allowed to choose the usual probability space and the white noise \(W\) on it, without requiring that the filtration be the augmented natural filtration of \(W\).

**Remark A.2.** We often simply say that \(U\) solves \(e^{\text{Heat}}_{\text{heat}}(a,b,h)\) (weakly or strongly) to mean the same thing as above.

**Definition A.2** (Uniqueness for SPDEs). We say that uniqueness in law holds for \(e^{\text{Heat}}_{\text{heat}}(a,b,h)\) if the laws \(\mathbb{L}^{U^{(i)}}_{\mathbb{P}^{(i)}}\) of \(U^{(i)}\) under \(\mathbb{P}^{(i)}\); \(i = 1,2\), are the same on \((C(\mathcal{R}_{T,L}), \mathbb{B}(C(\mathcal{R}_{T,L})))\) whenever \((U^{(i)}, W^{(i)}), (\Omega^{(i)}, \mathcal{F}^{(i)}, \{\mathcal{F}^{(i)}_t\}, \mathbb{P}^{(i)}); i = 1,2\), are solutions to \(e^{\text{Heat}}_{\text{heat}}(a,b,h)\).

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