Information-Theoretic Comparison of Quantum Many-Body Systems

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Abstract

An information measure inspired by Onicescu’s information energy and Uffink’s information measure (recently discussed by Brukner and Zeilinger) are calculated as functions of the number of particles $N$ for fermionic systems (nuclei and atomic clusters) and correlated bosonic systems (atoms in a trap). Our results are compared with previous ones obtained for Shannon’s information entropy, where a universal property was derived for atoms, nuclei, atomic clusters and correlated bosons. It is indicated that Onicescu’s and Uffink’s definitions are finer measures of information entropy than Shannon’s.

Onicescu [1] introduced the concept of information energy $E$ as a finer measure of dispersion distributions than that of Shannon’s information entropy [2, 3]. So far, only the mathematical aspects of this concept have been developed, while the physical aspects have been neglected [4].

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The information energy for a single statistical variable $x$ with the normalized density $\rho(x)$ is defined by

$$E(\rho) = \int \rho^2(x) \, dx$$  \hspace{1cm} (1)

For a Gaussian distribution of mean value $\mu$, standard deviation $\sigma$ and normalized density

$$\rho(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$  \hspace{1cm} (2)

relation (1) gives

$$E = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{\sigma^2}} \, dx$$

Thus

$$E = \frac{1}{2\sigma\sqrt{\pi}}$$  \hspace{1cm} (3)

Therefore, the greater the information energy $E$, the narrower the Gaussian distribution. $E$ does not have the dimension of energy, but it has been connected with Planck’s constant appearing in Heisenberg’s uncertainty relation [4, 5].

For a 3-dimensional spherically symmetric density distribution $\rho(r)$ the obvious generalization of (1) is

$$E_r = \int \rho^2(r)4\pi r^2 \, dr$$  \hspace{1cm} (4)

and

$$E_k = \int n^2(k)4\pi k^2 \, dk$$  \hspace{1cm} (5)

in position- and momentum-space respectively, where $n(k)$ is the density distribution in momentum-space.

$E_r$ has the dimension of inverse volume, while $E_k$ of volume. Thus the product $E_r E_k$ is dimensionless and is a measure of the concentration (or the information content) of the density distribution of a quantum system. As seen from (3) $E$ increases as $\sigma$ decreases (or the concentration increases) and Shannon’s information entropy (or uncertainty) $S$ decreases. Clearly, Shannon’s information $S$ and information energy $E$ are reciprocal. In order to be able to compare them, we define the quantity

$$S_E = \frac{1}{E_r E_k}$$  \hspace{1cm} (6)
as a measure of the information content of a quantum system in both position
and momentum spaces.

In place of Shannon information, Brukner and Zeilinger [6] propose the
quantity

\[ I = N \sum_{i=1}^{n} \left( p_i - \frac{1}{n} \right)^2 \]  

(7)

from which they derive their notion of information content of a discrete prob-
ability distribution \( p_1, p_2, \ldots, p_n \). The quantity \( \sum_{i=1}^{n} (p_i - \frac{1}{n})^2 \) is one of the
class of measures of the concentration of a probability distribution given by
Uffink [7, 8]. For a continuous 3-dimensional density distribution \( \rho(r) \), relation
(7) is extended as \( (N = 1) \)

\[ I_r = \int \left( \rho(r) - \tilde{\rho}(r) \right)^2 4\pi r^2 \, dr \]  

(8)

and

\[ I_k = \int \left( n(k) - \tilde{n}(k) \right)^2 4\pi k^2 \, dk \]  

(9)

in position- and momentum space respectively, \( \tilde{\rho}(r) \) is the equivalent uniform
distribution defined according to the relation

\[ \tilde{\rho}(r) = \begin{cases} \rho_0 & 0 < r < R_U \\ 0 & r > R_U \end{cases} \]  

(10)

where \( \rho_0 \) = constant and \( R_U = R_{\text{uniform}} \) are fixed by the relation

\[ \langle r^2 \rangle_U = \langle r^2 \rangle_{\rho(r)} \]  

(11)

where

\[ \langle r^2 \rangle_U = \int_0^{R_U} \rho_0 r^2 4\pi r^2 \, dr \]  

(12)

and

\[ \langle r^2 \rangle_{\rho(r)} = \int_0^{\infty} \rho(r) r^2 4\pi r^2 \, dr \]  

(13)

while

\[ R_U = \sqrt{\frac{5}{3} \langle r^2 \rangle_{\rho(r)}} \]  

(14)

and

\[ \rho_0 = \frac{3}{4\pi R_U^3} \]  

(15)
$\hat{n}(k)$ is the equivalent uniform distribution in momentum-space, defined in a similar way. Thus we define a measure of information content by the relation

$$S_I = \frac{1}{I_r I_k}$$

which gives (16) putting $\hat{\rho}(r) = \hat{n}(k) = 0$.

We calculate $S_E$ and $S_I$ as functions of the number of particles $N$ for three quantum many-body systems, where $\rho(r)$ and $\eta(k)$ are calculated numerically:

1. Nuclei, using the Skyrme III parametrization of the nuclear field [9]. Here $N$ is the number of nucleons in nuclei.

2. Atomic clusters, employing a Woods-Saxon potential parametrized by Ekardt [10]. Here $N$ is the number of valence electrons.

3. A correlated bosonic system (atoms in a trap) [11, 12]. Here $N$ is the number of atoms in the trap.

In Fig.1 we plot $S_E$ as a function of $N$ for nuclei and clusters and in Fig.2 $S_I(N)$ for the same systems. In Fig.3 we plot $S_E(N)$ and in Fig.4 $S_I(N)$ for a correlated bosonic system. It is seen that $S_E$ depends linearly on $N$ for both nuclei and atomic clusters. Also $S_I$ shows a similar trend (a power of $N$) for nuclei and clusters. However the dependence $S_E(N)$ and $S_I(N)$ is different for correlated bosons compared with nuclei and clusters.

Our fitted expressions are:

$$S_E(\text{clusters}) = 143.420 N, \quad S_E(\text{nuclei}) = 73.883 N \quad \text{(Fig.1)}$$

$$S_I(\text{clusters}) = 431.576 N^{1.719}, \quad S_I(\text{nuclei}) = 260.275 N^{1.554} \quad \text{(Fig.2)}$$

We can compare with the universal relation $S(N) = a + b \ln N$ ($a, b$ are constants depending on the system) obtained recently [13] for Shannon’s information entropy for fermionic systems (atoms, nuclei and atomic clusters) and correlated bosonic systems [11] (atoms in a trap). It was seen [13] that $S(N)$ shows the same dependence on $N$ for all the systems considered i.e. nuclei, clusters, atoms and correlated bosons.
It is conjectured that nuclei and atomic clusters are equivalent from an information-theoretic point of view in the following sense: under any definition of information content (e.g. Shannon, Onicescu or Uffink), the dependence of information shows a similar trend (linear on \( \ln N \) for Shannon, linear on \( N \) for Onicescu and a power of \( N \) for Uffink). However, the similarity breaks down for bosons. This indicates that \( S_E \) and \( S_I \) distinguish between fermions and correlated bosons i.e. they are finer measures of information than Shannon’s \( S \). Our results may contribute to the recent debate between Brukner-Zeilinger and Timpson for a possible inadequacy of the Shannon information \[6, 14\]

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Figure 1: Onisescu’s information entropy $S_E$ as function of $N$ & for nuclei (circles) and atomic clusters (squares)
Figure 2: The same as in Fig. 1 but for Uffink’s information & entropy $S_i$
Figure 3: Dependence of $S_E$ on $N$ for correlated bosons
Figure 4: The same as in Fig.3 but for $S_I$