Catenaries and Singular Minimal Surfaces in the Simply Isotropic Space

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Abstract. This paper investigates the hanging chain problem in the simply isotropic plane and its 2-dimensional analog in the simply isotropic space. The simply isotropic plane and space are two- and three-dimensional geometries equipped with a degenerate metric whose kernel has dimension 1. Although the metric is degenerate, the hanging chain and surface problems are well-posed if we employ the relative arc length and relative area to measure the weight. Here, the concepts of relative arc length and relative area emerge by seeing the simply isotropic geometry as a relative geometry. In addition to characterizing the simply isotropic catenary, i.e., the solutions to the hanging chain problem, we also prove that it is the generating curve of a minimal surface of revolution in the simply isotropic space. Finally, we obtain the 2-dimensional analog of the catenaries, the so-called singular minimal surfaces, and determine the shape of a hanging surface of revolution in the simply isotropic space.

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1. Introduction

The problem of finding the shape of a hanging inextensible chain has received much attention from scientists since the times of Galileo. The problem was solved at the end of the XVII century by Hooke, Leibniz, Huygens, and Bernoulli, among others. The solution is the curve known as the catenary. Later, Euler proved in 1744 that the catenoid is the only non-planar minimal surface of revolution and that the generating curve of this surface is just the
catenary. Thus, the catenary appears in two different scenarios, first as a solution to the physical problem of a hanging chain and second as a solution to the problem of finding the surfaces of revolution with minimum surface area. Recently, the study of catenaries has also been extended to ambient spaces of constant curvature \([11,12]\), where, among other things, it is shown that non-zero curvature implies that solving the hanging chain problem is no longer equivalent to finding minimal surfaces of revolution.

In this paper, we propose the hanging chain problem in the simply isotropic plane \(\mathbb{I}^2\). Similarly to the Euclidean setting, we will investigate whether the solution curve is the generating curve of a minimal surface of revolution of the 3-dimensional simply isotropic space \(\mathbb{I}^3\). For this last question, the plane \(\mathbb{I}^2\) will be immersed in \(\mathbb{I}^3\), and the reference line that serves to define the weight for curves of \(\mathbb{I}^2\) will be the rotation axis of surfaces of revolution of \(\mathbb{I}^3\).

The simply isotropic plane \(\mathbb{I}^2\) is the plane \(\mathbb{R}^2\) endowed with the degenerate metric \(d_{2} = dx^2\) where \(x\) and \(z\) are the canonical Cartesian coordinates of \(\mathbb{R}^2\). Analogously, the simply isotropic space \(\mathbb{I}^3 = \{(x, y, z) \in \mathbb{R}^3\}\) is the space equipped with the degenerate metric \(d_{3} = dx^2 + dy^2\). The fact that the metric is degenerate is an obstacle regarding minimizing the weight of a curve in \(\mathbb{I}^2\). Indeed, let \(\gamma(x) = (x, z(x)), x \in [a, b]\), be a curve in \(\mathbb{I}^2\). If we followed the same steps as in the Euclidean version of the hanging chain problem, first, we would fix a reference line to measure the weight of a curve. Supposing that the reference line is the \(z\)-axis, \(L_z\), which is an isotropic line of \(\mathbb{I}^2\), i.e., a line of length zero, the weight of a curve would be calculated by means of the distance between the points of \(\gamma\) and \(L_z\). The hanging chain would then minimize the gravitational potential, which corresponds to the functional

\[
\mathcal{F}[\gamma] = \int_{\gamma} \text{dist}(\gamma(x), L_z) \, ds - \lambda \int_{\gamma} ds,
\]

where \(\mathcal{F}\) is defined among all curves with the same length and endpoints \(\gamma(a)\) and \(\gamma(b)\). The constant \(\lambda\) is a Lagrange multiplier due to the length constraint. However, since \(\text{dist}(\gamma(x), L_z)\) in \(\mathbb{I}^2\) is just \(|x|\), it follows that \(ds = dx\), from which we conclude that the functional \(\mathcal{F}\) can be calculated explicitly, obtaining

\[
\mathcal{F} = \frac{1}{2}(b^2 - a^2) - \lambda(b - a).
\]

In other words, the functional would be constant for all curves with the same endpoints, and the hanging chain problem would be trivial. A similar situation occurs if the reference line is the \(x\)-axis.

We can obtain a well-posed and non-trivial hanging chain problem by resorting to concepts of relative geometry. In \(\mathbb{I}^2\), the points at a constant distance from a center form a pair of lines parallel to the \(z\)-axis, which does not lead to a manageable notion of a unit normal. The idea is to replace the unit circle with a curve of constant curvature 1, a parabola in \(\mathbb{I}^2\) whose axis is parallel to the \(z\)-axis. By viewing \(\mathbb{I}^2\) as a relative geometry, the role of the unit normal is then played by the vector connecting the focus of a unit parabola \(\Sigma^1\) to the point of \(\Sigma^1\) whose tangent line is parallel to the tangent of the curve, called the relative normal. As a byproduct of using a relative
normal, one may introduce a new notion of arc length known as the relative arc length denoted by \( ds^* \). In this work, we propose to replace the simply isotropic arc length element \( ds \) with the relative arc length element \( ds^* \) in the definition of the gravitational potential of a curve in \( I^2 \). We will show that the modified potential functional leads to non-trivial solutions, where the graph of the natural logarithm then plays the role of a catenary in the simply isotropic space. Similarly to the Euclidean setting, the revolution of the logarithm graph around an isotropic axis also leads to a minimal surface in the simply isotropic space.

We may extend the hanging chain problem to surfaces whose solutions are called isotropic singular minimal surfaces. The surface is suspended under its weight measured with respect to a reference plane that can be isotropic or non-isotropic. As in the one-dimensional case, we replace the area element with the relative area. In contrast to the study of isotropic catenaries, we could not obtain an explicit parametrization of the isotropic singular minimal surfaces because the elliptic equation describing the surface cannot be integrated. However, we will focus on the classification of the isotropic singular minimal surfaces of revolution, obtaining a complete classification.

We divide this paper as follows. In Sect. 2, we revise the basic notions of the differential geometry of curves and surfaces in the simply isotropic spaces \( I^2 \) and \( I^3 \), with a particular emphasis on the notions of normal vector fields of curves and surfaces. In Sect. 3, we solve the hanging chain problem in the isotropic plane and obtain the concept of the isotropic catenary. In addition, we provide several characterizations of the isotropic catenary in terms of the curvature of the curve and certain vector fields of \( I^2 \). In Sect. 4, by rotating isotropic catenaries around an isotropic axis, we obtain minimal surfaces of revolution in \( I^3 \). In Sect. 5, we extend the hanging chain problem to dimension two, where the concept of isotropic singular minimal surface arises as a generalization of the isotropic catenary. As in the one-dimensional problem, it is necessary to distinguish between the cases where the reference plane is isotropic and non-isotropic. Finally, Sect. 6 fully classifies invariant singular minimal surfaces, i.e., the surfaces generated by a 1-parameter subgroup of isotropic rigid motions, described in Sect. 2.4, which then generalizes the study of revolution and helicoidal surfaces in Euclidean space. Indeed, first, we show no helicoidal singular minimal surfaces exist. Then, for surfaces of revolution in \( I^3 \), we characterize surfaces of Euclidean revolution in Sect. 6.1 and surfaces of parabolic revolution in Sect. 6.2.

2. Preliminaries

In this section, we review some basic notions of the differential geometry of curves and surfaces in the simply isotropic spaces \( I^2 \) and \( I^3 \). For more details, we refer the reader to [3, 4], or to [15, 16] for textbook sources (in German),
though part of the discussion on curves appears here for the first time, to the best of our knowledge.

The simply isotropic space \( \mathbb{I}^3 \) corresponds to the canonical real vector space \( \mathbb{R}^3 \) with Cartesian coordinates \((x, y, z)\) equipped with the degenerate metric

\[
\langle u, v \rangle = u^1 v^1 + u^2 v^2,
\]

where \( u = (u^1, u^2, u^3) \) and \( v = (v^1, v^2, v^3) \). A non-zero vector \( v \) is said to be isotropic if \( \langle v, v \rangle = 0 \). In addition, on the set of isotropic vectors, i.e., \( \{ v \in \mathbb{I}^3 : v = (0, 0, u^3) \} \), we shall use the secondary metric

\[
\llangle u, v \rrangle = u^3 v^3.
\]

Therefore, the space \( \mathbb{I}^3 \) is an example of a Cayley-Klein vector space [18].

A plane is said to be isotropic if it contains an isotropic vector. Thus, in \( \mathbb{I}^3 \) an isotropic vector and an isotropic plane are vertical, that is, parallel to the \( z \)-axis. The inner product induces a semi-norm \( \| u \| = \sqrt{\langle u, u \rangle} \). The top-view projection of a vector \( u \) is the projection \( \tilde{u} \) over the \( xy \)-plane,

\[
u = (u^1, u^2, u^3) \mapsto \tilde{u} \equiv (u^1, u^2, 0).
\]

In the following, it will be useful to resort to the Euclidean inner and vector products respectively written as

\[
u \cdot v = u^1 v^1 + u^2 v^2 + u^3 v^3
\]

and

\[
u \times v = (u^2 v^3 - u^3 v^2, u^3 v^1 - u^1 v^3, u^1 v^2 - u^2 v^1).
\]

Analogously, the simply isotropic plane \( \mathbb{I}^2 \) is defined as the canonical real vector space \( \mathbb{R}^2 = \{(x, z) : x, z \in \mathbb{R}\} \) equipped with the degenerate metric

\[
\langle u, v \rangle = u^1 v^1,
\]

where \( u = (u^1, u^2) \) and \( v = (v^1, v^3) \). The secondary metric \( \mathbb{I}^2 \) is also defined as in Eq. (2). Note we can alternatively see \( \mathbb{I}^2 \) isometrically embedded in \( \mathbb{I}^3 \) as the surface implicitly defined by the equation \( y = 0 \).

2.1. Geometry of Curves in the Simply Isotropic Plane

Let \( I \subset \mathbb{R} \) denote an interval and \( \gamma(t) = (x(t), z(t)) \), \( t \in I \), a smooth curve in \( \mathbb{I}^2 \). In what follows, we shall restrict our discussion to admissible curves, i.e., curves whose tangent lines are not isotropic, i.e., \( dx/dt \neq 0 \) for all \( t \in I \). Note that a curve \( \gamma : I \to \mathbb{I}^2 \) is parametrized by arc length \( s \) if

\[
\gamma(s) = (\pm s, z(s)).
\]

The normal to \( \gamma \) with respect to the simply isotropic metric is the isotropic vector \( \mathcal{N} = (0, 1) \), which is normalized by the secondary metric: \( \llangle \mathcal{N}, \mathcal{N} \rrangle = 1 \). The unit tangent is \( T(s) = (\pm 1, z'(s)) \). Thus

\[
T' = (0, z'') = \kappa \mathcal{N}, \quad \kappa = z''.
\]

The function \( \kappa \) is the (signed) simply isotropic curvature of \( \gamma \).
For a generic parameter \( t \), let us write \( \gamma(t) = (x(t), z(t)) \). We shall distinguish between derivatives with respect to the arc length parameter \( s \) and a generic parameter \( t \) by respectively using a prime and a dot: e.g., \( x' = dx/ds \) and \( \dot{x} = dx/dt \). For a generic parametrization, the unit tangent becomes
\[
\mathbf{T} = \frac{\dot{\gamma}}{\|\dot{\gamma}\|} = \left( \frac{\dot{x}, \dot{z}}{\sqrt{\dot{x}^2}} \right) = \left( \pm 1, \frac{\dot{z}}{\dot{x}} \right).
\]
On the other hand, to find the curvature, we use the chain rule
\[
\kappa \mathbf{N} = \frac{d}{ds} \mathbf{T} = \frac{1}{\dot{x}} \left( 0, \frac{\ddot{z}\dot{x} - \dot{z}\ddot{x}}{\dot{x}^2} \right) \Rightarrow \kappa = \frac{\ddot{x}\dot{z} - \ddot{z}\dot{x}}{\dot{x}^3}, \quad \dot{x} > 0. \tag{4}
\]
From now on, we may assume for simplicity that \( \dot{x} > 0 \), after applying a simply isotropic rigid motion \( (x, z) \mapsto (\pm x + a, z) \) if necessary.

**Example 1.** The parabola \( \gamma(t) = (t, \frac{1}{2}ct^2 + bt + a) \) has constant curvature \( c \). Conversely, any curve with constant curvature is a parabola. Indeed, if \( \kappa = c \), and if we write \( \gamma(t) = (t, z(t)) \), then \( \kappa = c = z''(t) \), so \( z(t) = \frac{1}{2}c t^2 + bt + a \), \( a, b, c \in \mathbb{R} \). We include the case \( c = 0 \), corresponding to \( \gamma \) being a straight line.

For a curve \( \gamma : I \rightarrow \mathbb{E}^2 \) in the Euclidean plane \( \mathbb{E}^2 \) parametrized by arc length \( s \), the (signed) curvature is computed as \( \kappa = \gamma'' \cdot \mathbf{N} \), where \( \mathbf{N} \) is a unit vector field normal to the curve. For a generic parameter \( t \), we define the first and second fundamental forms of \( \gamma \) as \( g_{11} = \dot{\gamma} \cdot \dot{\gamma} \) and \( h_{11} = \ddot{\gamma} \cdot \mathbf{N} \), respectively. Then, using the chain rule,
\[
\kappa = \gamma'' \cdot \mathbf{N} = \left[ \left( \frac{dt}{ds} \right)^2 \dot{\gamma} + \frac{dt}{ds} \frac{d}{dt} \left( \frac{dt}{ds} \right) \dot{\gamma} \right] \cdot \mathbf{N} = \left( \frac{dt}{ds} \right)^2 \ddot{\gamma} \cdot \mathbf{N} = \frac{h_{11}}{g_{11}}. \tag{5}
\]

Analogously, we define the first fundamental form for curves \( \gamma(t) = (x(t), z(t)) \) in \( \mathbb{I}^2 \) as \( g_{11} = \langle \dot{\gamma}, \dot{\gamma} \rangle = \dot{x}^2 \). To introduce the second fundamental form, we may enforce the condition that \( \kappa = h_{11}/g_{11} \):
\[
\kappa = \frac{\ddot{z}\dot{x} - \ddot{x}\dot{z}}{\dot{x}^3} = \frac{1}{\dot{x}^2} \frac{\ddot{z}\dot{x} - \ddot{x}\dot{z}}{\dot{x}} \Rightarrow h_{11} = \frac{\ddot{z}\dot{x} - \ddot{x}\dot{z}}{\dot{x}}.
\]
We cannot obtain the coefficient \( h_{11} \) by taking an inner product of the acceleration vector \( \ddot{\gamma} \) with some normal vector. To achieve that, we may resort to the metric of the Euclidean plane. Indeed, if we define the *minimal normal* vector field (see Fig. 1)
\[
\mathbf{N}_{\text{min}} = \left( -\frac{\dot{z}}{\dot{x}}, 1 \right) = \frac{1}{\dot{x}} J(\dot{\gamma}), \tag{6}
\]
where \( J \) is the counter-clockwise (Euclidean) \( \frac{\pi}{2} \)-rotation on the \( xz \)-plane, we can finally write the second fundamental form of a curve \( \gamma : I \rightarrow \mathbb{I}^2 \) as
\[
h_{11} = \ddot{\gamma} \cdot \mathbf{N}_{\text{min}} = \det(\ddot{\gamma}, \dot{\gamma}). \tag{7}
\]
Remark 1. The terminology “minimal normal” comes from the similar construction for surfaces in the space $\mathbb{I}^3$ [9]. More precisely, if one tries to see $-d\mathbf{N}_{\text{min}}$ as a shape operator, its trace vanishes identically for any surface in $\mathbb{I}^3$. (See the next subsection for further details on the simply isotropic geometry of surfaces.)

2.2. Simply Isotropic Relative Geometry

To further study the geometry of curves and surfaces in the simply isotropic space, we shall resort to some ideas from the so-called relative geometry [17], a topic whose origins can be traced back to the contributions of E. Müller in the 1920s [13]. The reader may consult [17], or Sect. 2 of [19] for further information.

The basic idea of relative geometry comes from the following reasoning. In Euclidean space, many properties associated with the unit normal $\mathbf{N}$ to a hypersurface $\mathbf{x}: U \subset \mathbb{R}^n \rightarrow M^n \subset \mathbb{E}^{n+1}$ do not depend on the orthogonality. For example, the definition of the Christoffel symbols of an affine connection relies on the property that $\mathbf{N}$ is transversal to the tangent space: $\mathbf{x}_{ij} = \Gamma_{ij}^k \mathbf{x}_k + h_{ij} \mathbf{N}$. As another example, the definition of the shape operator, and consequently of the Gaussian and mean curvatures, relies on the property that $T_p M$ and $T_{\mathbf{x}(p)} S^n$ are parallel and that $-d\mathbf{N}$ takes values on $T_p M^2$.

Definition 1. Let $\mathbf{y}$ be a vector field along $M^n \subset \mathbb{R}^{n+1}$ and taking values in $\mathbb{R}^{n+1}$. The vector field $\mathbf{y}$ is a relative normal of $M$ if it is (i) transversal to $T_p M$, i.e., $\mathbf{y} \not\in T_p M$ and (ii) equiaffine, i.e., $\forall v \in T_p M, \, dy(v) \in T_p M$.

Let $\bar{\Sigma}^n \subset \mathbb{R}^{n+1}$ be a hypersurface with relative normal $\bar{\mathbf{y}}$. The Peterson mapping of a hypersurface $M^n$ is a map $\mathcal{P}$ that sends a point $p \in M$ to a point $q \in \bar{\Sigma}^n$ such that $T_p M$ and $T_q \bar{\Sigma}$ are parallel. We may introduce a relative normal $\mathbf{y}$ for $M$ with respect to $\bar{\Sigma}$ by defining

$$\mathbf{y}(p) = \bar{\mathbf{y}}(\mathcal{P}(p)).$$

The relative shape operator $A$ of $M$ is then given by $A = -dy$. In this work, we shall concentrate on the use of the centro-affine normal, i.e., $\bar{\mathbf{y}}$ is the position vector of $\bar{\Sigma}$. We shall refer to $\bar{\Sigma}$ as the relative sphere.

In the relative approach to the simply isotropic plane, we may take as the relative sphere the circle of parabolic type

$$\Sigma^1 = \left\{ (x, z) \in \mathbb{I}^2 : z = \frac{1}{2} - \frac{x^2}{2} \right\}. \quad (8)$$

We then obtain a relative normal $\mathbf{N}_{\text{par}}$, named the parabolic normal, as an alternative to the minimal normal. More precisely, the parabolic normal vector field $\mathbf{N}_{\text{par}}$ is defined as (see Fig. 1)

$$\mathbf{N}_{\text{par}} = \left( -\frac{\dot{z}}{\dot{x}} \frac{1}{2} - \frac{\dot{z}^2}{2\dot{x}^2} \right). \quad (9)$$
Figure 1. Minimal ($N_{\text{min}}$) and parabolic ($N_{\text{par}}$) normal vector fields of an admissible curve in $\mathbb{L}^2$. Note that $N_{\text{min}}$, Eq. (6), is a multiple of the Euclidean normal, while $N_{\text{par}}$, Eq. (9), does not come from a notion of orthogonality. Figure generated with Mathematica

**Proposition 1.** The parabolic normal $N_{\text{par}}$ of an admissible curve $\gamma : I \to \mathbb{L}^2$ is a relative normal.

**Proof.** We must prove that $N_{\text{par}}$ is equiaffine and transversal. The parabolic normal $N_{\text{par}}$ is equiaffine:

$$-\frac{dN_{\text{par}}}{dt} = \left( \frac{\ddot{z}\dot{x} - \dot{z}\ddot{x}}{\dot{x}^2}, \frac{\ddot{x}}{\dot{x}} - \frac{\dot{z}\ddot{x} - \dot{z}\ddot{x}}{\dot{x}^2} \right) = \kappa \dot{\gamma}.$$ 

Finally, the parabolic normal $N_{\text{par}}$ is transversal. Indeed, since $\dot{x} \neq 0$, we obtain

$$\det \left( \begin{array}{cc} \dot{x} & \frac{\dot{z}}{\dot{x}} \\ \frac{\ddot{x}}{\dot{x}} - \frac{\dot{z}^2}{2\dot{x}^2} & \frac{\ddot{z}}{\dot{x}} \end{array} \right) = \frac{\dot{x}^2 + \dot{z}^2}{2\dot{x}} \neq 0.$$

As a corollary from the fact that $N_{\text{par}}$ is a relative normal, it follows we may alternatively compute the second fundamental form of $\gamma$ as

$$h_{11} = \kappa \langle \dot{\gamma}, \dot{\gamma} \rangle = \langle -dN_{\text{par}}(\dot{\gamma}), \dot{\gamma} \rangle.$$ 

(10)

An important concept in the context of relative geometry of surfaces is played by the *relative area* $A^*$. More precisely, given a surface $M^2 : (u, v) \mapsto r(u, v)$ with relative normal $y$, we define

$$A^* = \int_M |\det(r_u, r_v, y)| \, du \, dv.$$ 

(11)
Analogously, we define the relative arc length $s^*$ for a curve $\gamma : [a, b] \rightarrow \mathbb{I}^2$ with parabolic normal $\mathbf{N}_{\text{par}}$ as

$$s^* = \int_a^b |\det(\mathbf{N}_{\text{par}}, \dot{\gamma})| \, dt = \int_a^b |\mathbf{N}_{\text{par}} \cdot J(\dot{\gamma})| \, dt$$

$$= \int_a^b |\mathbf{N}_{\text{par}} \cdot \mathbf{N}_{\text{min}}| \, \dot{x} \, dt.$$  \hspace{1cm} (12)

Thus, from Eqs. (6) and (9), the relative arc length element of a curve $\gamma(t) = (x(t), z(t))$ satisfies

$$ds^* = (\mathbf{N}_{\text{par}} \cdot \mathbf{N}_{\text{min}}) \, ds = \left(\frac{1}{2} + \frac{\dot{z}^2}{2x^2}\right) \, ds = \left(\frac{\dot{x}}{2} + \frac{\dot{z}^2}{2x}\right) \, dt.$$  \hspace{1cm} (13)

Remark 2. In the Euclidean plane, the relative arc length with respect to the relative normalization given by the Euclidean normal coincides with the usual arc length: $s^* = \int \det(\mathbf{N}, \dot{\gamma}) \, dt = \int \mathbf{N} \cdot \mathbf{N} \sqrt{\dot{x}^2 + \dot{z}^2} \, dt = \int ds = s$, where we used that $\mathbf{N} = J(\dot{\gamma})/\sqrt{\gamma \cdot \dot{\gamma}}$ and $J(\mathbf{u}) \cdot \mathbf{v} = \det(\mathbf{u}, \mathbf{v})$.

2.3. Geometry of Surfaces in the Simply Isotropic Space

Let $U \subset \mathbb{R}^2$ be an open set and $\mathbf{r} : U \rightarrow M^2 \subset \mathbb{I}^3$ a regular parametrized surface. In what follows, we shall focus on admissible surfaces, i.e., surfaces whose tangent planes are not isotropic. Equivalently, admissibility implies $x^1_i x^2_j - x^1_j x^2_i \neq 0$, where $\mathbf{r}(u^1, u^2) = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2))$ and $x^j_i = \partial x^j / \partial u^i$. Note that every admissible surface in $\mathbb{I}^3$ is locally parametrized as the graph of a real function $\mathbf{r} = (u^1, u^2, f(u^1, u^2))$, called the normal form of $M^2$.

The first fundamental form is defined as usual $g_{ij} = \langle \mathbf{r}_i, \mathbf{r}_j \rangle$, where $\mathbf{r}_i = \partial \mathbf{r} / \partial u^i$. The normal to a surface $M^2$ with respect to the simply isotropic metric is given by the isotropic vector $\mathbf{N} = (0, 0, 1)$ normalized by the secondary metric. The second fundamental form $h_{ij}$ is defined by the expression

$$\mathbf{r}_{ij} = R^k_{ij} \mathbf{r}_k + h_{ij} \mathbf{N},$$  \hspace{1cm} (14)

where we shall adopt the convention of summing over repeated indices. If we parametrize $M$ in its normal form, the first and second fundamental forms are

$$I = (du^1)^2 + (du^2)^2 \quad \text{and} \quad II = f_{ij} du^i du^j.$$  \hspace{1cm} (15)

We can compute the second fundamental form $h_{ij}$ by forcing an analogy with Euclidean space by using the minimal normal $\mathbf{N}_{\text{min}}$:

$$h_{ij} = \frac{\partial^2 \mathbf{r}}{\partial u^i \partial u^j} \cdot \mathbf{N}_{\text{min}} = \frac{\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_{ij})}{\sqrt{\det g_{ij}}} \quad \text{and} \quad \mathbf{N}_{\text{min}} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\sqrt{g_{11}g_{22} - g_{12}^2}}.$$  \hspace{1cm} (16)

The map $\mathbf{N}_{\text{min}}$ is not a relative normal. Indeed, the minimal normal $\mathbf{N}_{\text{min}}$ is not equiaffine since $d\mathbf{N}_{\text{min}}$ is always a horizontal vector and, as a
consequence, it generally fails to be tangent to $M^2$:

$$
N_{\text{min}} = \left( \frac{X_{23}}{X_{12}}, \frac{X_{31}}{X_{12}}, 1 \right), \quad X_{ij} = \det \begin{pmatrix}
x_1^i & x_1^j \\
x_2^i & x_2^j
\end{pmatrix},
$$

where we may assume $X_{12} > 0$ by exchanging $u^1 \leftrightarrow u^2$ if necessary.

The mean, $H$, and the Gaussian, $K$, curvatures are given as usual by

$$
H = \frac{1}{2} \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{g_{11}g_{22} - g_{12}^2}, \quad K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}. \quad (17)
$$

In the simply isotropic space, we may take as the relative sphere the sphere of parabolic type

$$
\Sigma^2 = \{ (x, y, z) \in \mathbb{I}^3 : z = \frac{1}{2} - \frac{x^2}{2} - \frac{y^2}{2} \}. \quad (18)
$$

We then obtain a relative normal $N_{\text{par}}$, named parabolic normal, as an alternative to the minimal normal. The parabolic normal vector field $N_{\text{par}}$ is defined as

$$
N_{\text{par}} = \left( \frac{X_{23}}{X_{12}}, \frac{X_{31}}{X_{12}}, \frac{1}{2} - \frac{X_{23}^2 + X_{31}^2}{2X_{12}^2} \right). \quad (19)
$$

Proposition 2. The parabolic normal $N_{\text{par}}$ of an admissible surface $r : U \to M^2 \subset \mathbb{I}^3$ is a relative normal.

Proof. From the definition of $N_{\text{par}}$ in (19), we have

$$
\det(r_1, r_2, N_{\text{par}}) = \frac{(X_{23})^2 + (X_{31})^2 + (X_{12})^2}{2X_{12}} > 0,
$$

which implies $N_{\text{par}}$ is transversal to $M^2$. Let us parametrize $M^2$ in its normal form $r = (u^1, u^2, f(u^1, u^2))$. Then, the parabolic normal is given by $N_{\text{par}} = (-f_1, -f_2, \frac{1}{2} - \frac{f_1^2 + f_2^2}{2})$, from which follows that $\partial N_{\text{par}}/\partial u^i = -f_i; r_1 - f_2; r_2$. Therefore, $N_{\text{par}}$ is equiaffine. \qed

As a byproduct of using the parabolic normal $N_{\text{par}}$, we may alternatively compute the second fundamental form, mean, and Gaussian curvatures as

$$
h_{ij} = \langle -dN_{\text{par}}(r_j), r_i \rangle, \quad H = \text{tr}(-dN_{\text{par}}), \quad \text{and} \quad K = \det(-dN_{\text{par}}). \quad (20)
$$

Now, let us compute the relative area of $M^2$. From (11), we have

$$
A^* = \int_M N_{\text{min}} \cdot N_{\text{par}} \sqrt{g_{11}g_{22} - g_{12}^2} \, du^1 \, du^2 = \int_M (N_{\text{min}} \cdot N_{\text{par}}) \, dA. \quad (21)
$$

If we parametrize $M$ in its normal form, the relative area takes the form

$$
A^* = \int \frac{1 + f_1^2 + f_2^2}{2} \, du^1 \, du^2. \quad (22)
$$
2.4. Invariant Surfaces in the Simply Isotropic Space

We could construct the simply isotropic geometry as a Cayley–Klein geometry. First, one fixes a set of objects in projective space $\mathbb{P}^3$ called the absolute figure; in our case, a plane $\omega : x_0 = 0$ and a degenerate quadric $Q : x_0^2 + x_1^2 + x_2^2 = 0$, where we use homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$ in $\mathbb{P}^3$. Then, we consider the subgroup $G$ of projectivities of $\mathbb{P}^3$ that leave the absolute figure invariant. Finally, the simply isotropic geometry corresponds to the study of the properties of $I^3 = \mathbb{P}^3/\omega$ invariant by the action of $G = \text{ISO}(I^3)$ [16]. (See also Eq. (2.1) of [4].) Alternatively, the group $\text{ISO}(I^3)$ of simply isotropic rigid motions can be built as the group of affine transformations that preserve the primary and secondary metrics. The group $\text{ISO}(I^3)$ comprises translations and isotropic special orthogonal transformations, i.e., linear transformations of determinant 1 preserving the primary and secondary metrics [4].

An invariant surface is a surface $\Sigma^2 \subset I^3$ invariant by the action of a 1-parameter subgroup $\mathbb{R}$ of the group of simply isotropic isometries $\text{ISO}(I^3)$: $\Sigma^2 = G_t(\Sigma^2)$. In $I^3$, there are seven types of 1-parameter subgroups of simply isotropic rigid motions, which can be divided into two main families. Namely, helicoidal motions and parabolic rotations [4,16].

Let $L_z$ be the isotropic $z$-axis. A helicoidal surface with screw axis $L_z$ is the surface invariant under the action of the one-parameter group $G_p = \{H_\theta : \theta \in \mathbb{R}\}$ of helicoidal motions [4], where

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} \mapsto H_\theta
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} =
\begin{pmatrix}
  \cos \theta & -\sin \theta & 0 \\
  \sin \theta & \cos \theta & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} +
\begin{pmatrix}
  0 \\
  0 \\
  c \theta
\end{pmatrix},
\]

(23)

When $c = 0$, the subgroup $H_\theta$ gives rise to (Euclidean) rotations, denoted by $R_\theta$, which leave $L_z$ point-wise fixed. On the other hand, a surface of parabolic revolution is invariant under the action of the one-parameter group $G_p = \{P_\theta : \theta \in \mathbb{R}\}$ of parabolic revolutions

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} \mapsto P_\theta
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  c_1 \theta & c_2 \theta & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} +
\begin{pmatrix}
  a \theta \\
  b \theta \\
  c \theta + \frac{ac_1 + bc_2}{2} \theta^2
\end{pmatrix},
\]

(24)

where $a, b, c, c_1, c_2 \in \mathbb{R}$.

3. The Solution of the Simply Isotropic Hanging Chain Problem

In the hanging chain problem in $I^2$, the gravitational potential of the curve is calculated using the distance to a straight-line $L$ of $I^2$. Since there are isotropic and non-isotropic straight lines, it will be necessary to distinguish between both cases. Without loss of generality, if $L$ is isotropic, it will be assumed to be the

---

1A 1-parameter subgroup $G$ is a group homomorphism between $\mathbb{R}$ equipped with the addition and $\text{ISO}(I^3)$, i.e., $G_{t+s} = G_t \circ G_s$ and $G_0 = \text{Id}$, where $G_t = \mathcal{G}(t)$. 

---
\[ z \text{-axis } L_z = \{ (0, z) : z \in \mathbb{R} \} \text{ and if } L \text{ is non-isotropic, then the line will be the} \]
\[ x \text{-axis } L_x = \{ (x, 0) : x \in \mathbb{R} \}. \]
In each case, the functional
\[ F[\gamma] = \int_\gamma \text{dist}(\gamma(x), L) \, ds^* - \lambda \int_\gamma ds^* \] (25)
will be denoted by \( F_z \) and \( F_x \). In addition, all curves will be contained in one of the two half-planes determined by \( L_z \) and \( L_x \), namely, \( \{(x, z) \in \mathbb{R}^2 : x > 0\} \) and \( \{(x, z) \in \mathbb{R}^2 : z > 0\} \), respectively.

The first case to consider is when the reference line is the isotropic line \( L_z \).

**Theorem 1.** The critical points of the functional \( F_z \) are the curves
\[ \gamma(t) = (t, c \ln(t - \lambda) + d), \quad c, d \in \mathbb{R}. \] (26)

From now on, the curves (26) when \( c \neq 0 \) will be called isotropic catenaries with respect to \( L_z \).

**Proof.** Let us consider the parametrization \( \gamma(t) = (t, z(t)) \). Using (13), the relative arc length element becomes \( ds^* = \frac{1 + \dot{z}^2}{2} \, dt \). The functional \( F_z \) writes as
\[ F_z[\gamma] = \int_a^b F(t, z, \dot{z}) \, dt, \quad F(t, z, \dot{z}) = (t - \lambda) \left( \frac{1}{2} + \frac{\dot{z}^2}{2} \right). \]
The Euler–Lagrange equation is
\[ \frac{\partial F}{\partial z} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{z}} \right) = 0. \] (27)
Notice that \( F \) does not depend on \( z \). Thus, the Euler–Lagrange equation gives
\[ 0 = \frac{d}{dt} ((t - \lambda) \dot{z}) = \dot{z} + (t - \lambda) \ddot{z}. \] (28)

Then, there exists \( c \in \mathbb{R} \) such that \( (t - \lambda) \dot{z} = c \). If \( c = 0 \), then \( z \) is a constant function, \( z(t) = d, \ d \in \mathbb{R} \). This is a particular case of (26). If \( c \neq 0 \), then a direct integration gives Eq. (26). \( \square \)

Isotropic catenaries with respect to \( L_z \) can also be characterized as solutions of a coordinate-free prescribed curvature problem involving the curvature \( \kappa \) of \( \gamma \), the parabolic normal vector \( N_{\text{par}} \), and the unit vector field of \( \mathbb{I}^2 \) which is orthogonal to \( L_z \), i.e., \( X = \partial_x \).

**Theorem 2.** Let \( \gamma \) be a curve in \( \mathbb{I}^2 \) and \( X = \partial_x \in \mathfrak{X}(\mathbb{I}^2) \). Then, \( \gamma \) is an isotropic catenary with respect to \( L_z \) if, and only if, its curvature \( \kappa \) satisfies
\[ \kappa = \frac{\langle N_{\text{par}}, X \rangle}{\langle \gamma, X \rangle - \lambda}. \] (29)
Proof. From Eq. (4), the curvature of \( \gamma(t) = (t, z(t)) \) is \( \kappa = \ddot{z} \). This implies that Eq. (28) writes simply as \( \kappa = -\dot{z}/(t - \lambda) \). Finally, from Eq. (6), we have \( N_{par} = (-\dot{z}, \frac{1}{2} - \frac{\dot{z}^2}{2}) \), hence \( \langle N_{par}, X \rangle = -\dot{z} \). These expressions prove Eq. (29). \( \square \)

The notion of catenary can be generalized if a power \( \alpha \in \mathbb{R} \) is introduced in the functional \( F_z \). More precisely, define the functional

\[
F_z^\alpha[\gamma] = \int_{\gamma} (z^\alpha - \lambda) \, ds^*.
\]

Theorem 3. Let \( \gamma \) be a curve in \( \mathbb{I}^2 \) and \( X = \partial_z \in \mathcal{X}(\mathbb{I}^2) \). Then, \( \gamma \) is a critical point of \( F_z^\alpha \) if, and only if, its curvature \( \kappa \) satisfies

\[
\kappa = \alpha \frac{\langle \gamma, X \rangle}{\langle \gamma, X \rangle^\alpha - \lambda} \cdot \langle N_{par}, X \rangle.
\] (30)

Proof. The computations are similar to those in the proof of Theorem 2. If \( \gamma(t) = (t, z(t)) \), then the functional \( F_z^\alpha \) becomes

\[
F_z^\alpha[\gamma] = \int_a^b (t^\alpha - \lambda) \left( \frac{1}{2} + \frac{\dot{z}^2}{2} \right) dt.
\]

The computation of the Euler–Lagrange equation (27) gives

\[
\alpha t^{\alpha-1} \dot{z} + (t^\alpha - \lambda) \ddot{z} = 0,
\] (31)

proving the theorem. \( \square \)

By analogy with the Euclidean space \([6,10]\), and taking \( \lambda = 0 \) in Eq. (30), we give the following definition:

**Definition 2.** A curve \( \gamma \) in \( \mathbb{I}^2 \) is said to be an isotropic \( \alpha \)-catenary with respect to \( L_z \) if its curvature satisfies

\[
\kappa = \alpha \frac{\langle N_{par}, X \rangle}{\langle \gamma, X \rangle}.
\]

The case \( \alpha = 1 \) corresponds to the isotropic catenary (26). The expression characterizing \( \alpha \)-catenaries, Eq. (31), can be solved, and the solution provides an explicit parametrization. Indeed,

**Corollary 1.** Let \( \gamma \) be an \( \alpha \)-catenary. Then, it is parametrized as

\[
\gamma(t) = \begin{cases} (t, c \ln t + d), & \text{if } \alpha = 1, \\ (t, c t^{1-\alpha} + d), & \text{if } \alpha \neq 0, \end{cases} \quad c, d \in \mathbb{R}.
\] (32)

We conclude this section by investigating the hanging chain problem when the reference line is the non-isotropic line \( L_x \). In this case, we must resort to the distance as computed using the secondary metric: \( d(x, y) = \sqrt{\langle x - y, x - y \rangle} \). Thus, the functional to minimize is

\[
F_x^\alpha = \int_{\gamma} (z^\alpha - \lambda) \, ds^*.
\] (33)
Now, the computations follow the same steps as in Theorems 1 and 2.

**Theorem 4.** Let $\gamma$ be a curve in $\mathbb{I}^2$ parametrized by $\gamma(t) = (t, z(t))$ and $Z = \partial_z \in \mathcal{X}(\mathbb{I}^2)$. The following statements are equivalent:

1. The curve $\gamma$ is a critical point of the functional $\mathcal{F}_x^\alpha$;
2. The function $z$ satisfies
   
   $$(z^\alpha - \lambda)\dddot{z} = \alpha z^{\alpha-1} \left(1 - \frac{\dot{z}^2}{2}\right),$$
   
   (34)
3. The curvature $\kappa$ satisfies
   
   $$\kappa = \alpha \langle \gamma, Z \rangle^{\alpha-1} \left\langle \mathbb{N}_{par}, Z \right\rangle \langle \gamma, Z \rangle^{\alpha} - \lambda.$$

These curves will be called *isotropic $\alpha$-catenaries* with respect to $L_x$.

**Proof.** The functional $\mathcal{F}_x^\alpha$ in (33) is

$$\mathcal{F}_x^\alpha[\gamma] = \int_\gamma (z^\alpha - \lambda) ds^* = \int_a^b (z^\alpha - \lambda) \left(\frac{1}{2} + \frac{\dot{z}^2}{2}\right) dt.$$

The corresponding Euler–Lagrange equation is

$$\alpha z^{\alpha-1} \frac{1 + \frac{\dot{z}^2}{2}}{2} = \frac{d}{dt} \left[(z^\alpha - \lambda)\dddot{z}\right] = \alpha z^{\alpha-1} \dddot{z} + (z^\alpha - \lambda)\dddot{z} \Rightarrow \dddot{z} = \alpha z^{\alpha-1} \frac{1 + \frac{\dot{z}^2}{2}}{2}.$$

(35)

Since $\langle \mathbb{N}_{par}, Z \rangle = \langle (-\dot{z}, \frac{1}{2} - \frac{\dot{z}^2}{2}, 0, 1) \rangle = \frac{1}{2}(1 - \dot{z}^2)$, $\langle \gamma, Z \rangle = z$, and the curvature is $\kappa = \dddot{z}$, we finally obtain the desired chain of equivalences. □

**Remark 3.** It is worth mentioning that there is an isometry between the simply isotropic plane $\mathbb{I}^2$ and $\mathbb{R}^2$ equipped with $\langle \cdot, \cdot \rangle$ and with $\langle \cdot, \cdot \rangle$ as the secondary metric. (The latter has the $x$-axis as an isotropic direction, while in the former, this role is played by the $z$-axis.) Therefore, it comes as no surprise the similarity between Theorems 2 and 3, where distances are measured using $\langle \cdot, \cdot \rangle$, and Theorem 4, where distances are measured using $\langle \cdot, \cdot \rangle$.

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### 4. The Catenary as the Generating Curve of Minimal Surfaces of Revolution

In this section, it will be derived that Euclidean rotational minimal surfaces in $\mathbb{I}^3$, i.e., surfaces in $\mathbb{I}^3$ rotated around an isotropic axis, are generated by isotropic catenaries. This will extend Euler’s result [1,7] to the ambient of simply isotropic spaces.

Let $I = [a, b] \subset \mathbb{R}$, $a > 0$, and let $\gamma : I \to \Pi_{xz} \subset \mathbb{R}^3$ be a smooth curve parametrized by $\gamma(t) = (t, 0, z(t))$, $t > 0$, where $\Pi_{xz}$ is the isotropic plane of equation $y = 0$. Let $R_{\theta}$ be the (Euclidean) rotation around the isotropic line
$L_z$, Eq. (23) with $c = 0$, and $S_\gamma = \{ R_\theta \cdot \gamma(t) : t \in I, \theta \in \mathbb{R} \} \subset \mathbb{H}^3$ be the corresponding invariant surface. A parametrization of $S_\gamma$ is

$$r(t, \theta) = (t \cos \theta, t \sin \theta, z(t)), \quad t \in I, \theta \in \mathbb{R}. \quad (36)$$

If $N_{\text{min}}$ and $N_{\text{par}}$ are the minimal and parabolic normal of $S_\gamma$, the relative area (21) is given by $dA^* = (N_{\text{par}} \cdot N_{\text{min}}) dA$. Thus, the problem of minimum area for surfaces of revolution consists in minimizing the relative area

$$A[\gamma] = \int_a^b \int_0^{2\pi} dA^*$$

among all curves $\gamma(t) = (t, 0, z(t))$ for which $z(a) = r_1$ and $z(b) = r_2$ are fixed. Here, $r_1$ and $r_2$ are the radii of the circles forming the boundary of $S_\gamma$. As for catenaries, the area $\int dA$ must be replaced by the relative area to obtain surfaces with $H = 0$ [5].

**Theorem 5.** Suppose $S_\gamma$ is a surface of minimum (relative) area. In that case, $\gamma$ is a horizontal line (and $S_\gamma$ is a horizontal plane) or $\gamma$ is an isotropic catenary with respect to $L_z$ given in Eq. (26) for $\lambda = 0$.

**Proof.** If $S_\gamma$ has minimum area, its generating curve $\gamma$ is a critical point of the functional $A[\gamma]$. Let us compute $dA^*$. The first fundamental form and area element of $S_\gamma$ with respect to the parametrization $r = r(t, \theta)$ are given by $I = dt^2 + t^2 d\theta^2$ and $dA = t dt d\theta$. The minimal and parabolic normal are

$$N_{\text{min}} = (-z' \cos \theta, -z' \sin \theta, 1) \quad \text{and} \quad N_{\text{par}} = \left(-z' \cos \theta, -z' \sin \theta, \frac{1}{2} - \frac{z'^2}{2}\right).$$

Therefore, the relative area is computed as

$$dA^* = (N_{\text{par}} \cdot N_{\text{min}}) dA = t \left(\frac{1}{2} + \frac{z'^2}{2}\right) dt d\theta \Rightarrow A[\gamma] = \pi \int_a^b t(1 + z'^2) dt.$$

Notice that the Lagrangian $J(t, z, z') = t(1 + z'^2)$ does not depend on $z$. Thus, the Euler–Lagrange equation (27) reduces to

$$2tz' = \frac{\partial J}{\partial z'} = 2c, \quad c \in \mathbb{R}. \quad (37)$$

On the one hand, if $c = 0$, then $z = z(t)$ is a constant function, which implies that $\gamma$ is a horizontal line and $S_\gamma$ is a horizontal plane. On the other hand, if $c \neq 0$, integration of Eq. (37) gives $z(t) = c \ln(t) + d$, where $c, d \in \mathbb{R}$.

The Theorem 5 is analogous to Euler’s result in the simply isotropic space. Proceeding with the motivation provided by the catenoid, it is natural to ask whether an isotropic catenoid connects any two coaxial circles of $\mathbb{H}^3$. It is known that the existence of a Euclidean catenoid joining two coaxial circles depends on the distance between both circles [2,8,14]. If the circles are sufficiently close (depending on the radii), then two catenoids connect both circles. However, if
the distance between the circle is large, no catenoid exists connecting them. The following result shows that this problem in $\mathbb{I}^3$ has a different solution.

**Theorem 6.** Let $\Gamma_1$ and $\Gamma_2$ be two coaxial circles in $\mathbb{I}^3$ with respect to $L_z$ and with distinct radii. Then, a unique isotropic catenoid exists with axis $L_z$ and connecting $\Gamma_1$ and $\Gamma_2$. On the other hand, if $\Gamma_1$ and $\Gamma_2$ have the same radii, no catenoid is joining them.

**Proof.** The assumption that the circles have different radii is necessary since the profile curve of the isotropic catenoid, namely the isotropic catenary, is monotonic as a function of the distance to $L_z$. Let $(r_i, z_i) \in \mathbb{R}^2_+$ be the intersection of $\Gamma_i$ with the $xz$-plane, $i = 1, 2$. By hypothesis, $r_1 \neq r_2$. Since the isotropic catenary is $\gamma(t) = (t, 0, c \ln(t) + d)$, the proof is completed if it is established the existence of $c, d \in \mathbb{R}$ such that

$$c \ln(r_1) + d = z_1 \quad \text{and} \quad c \ln(r_2) + d = z_2.$$  \hspace{1cm} (38)

Without loss of generality, we may assume that $z_1 < z_2$. (A similar reasoning holds if $z_1 > z_2$.) In particular, $c > 0$ in order to ensure that the function $t \mapsto c \ln(t) + d$ is increasing. From the first equation of (38), it is deduced that $d = z_1 - c \ln(r_1)$. By using the second equation of (38), the problem reduces to finding $c > 0$ such that

$$c \ln(r_2) + z_1 = z_2.$$  \hspace{1cm} (38)

Using that $r_2/r_1 > 1$, we have $\lim_{c \to 0^+} f(c) = z_1$ and $\lim_{c \to \infty} f(c) = \infty$. Then, the Intermediate Value Theorem assures the existence of $c > 0$ such that $f(c) = z_2$. On the other hand, since the function $f = f(c)$ is strictly increasing, the value $c$ such that $f(c) = z_2$ is unique. \hfill $\Box$

**Remark 4.** In isotropic space $\mathbb{I}^3$, there are two types of revolution surfaces: Euclidean rotational surfaces and parabolic rotational surfaces. The minimal surfaces of revolution were classified in [4], obtaining the logarithmoid of revolution and the hyperbolic paraboloid, respectively. (See our Proposition 3 for the characterization of minimal surfaces of parabolic revolution.) While the hyperbolic paraboloid is self-conjugate, the logarithmoid of revolution is conjugate to a helicoid [5]. This property of the logarithmoid of revolution and the fact that its generating curve is an isotropic catenary show that this surface is the simply isotropic analog of the catenoid.

### 5. Simply Isotropic Singular Minimal Surfaces

In this section, we extend the notion of the catenary of $\mathbb{I}^2$ to the problem of the hanging surface in $\mathbb{I}^3$. Consider a surface $M^2$ of constant mass density suspended from a given closed curve $\Gamma$. Let $A_0 > 0$ be the relative area of $M^2$. The hanging surface problem consists in finding the shape of $M^2$ when $M^2$ is suspended under its weight, where the gravitational potential is measured using the distance with respect to a plane of $\mathbb{I}^3$. As in the case of the catenary, the computation of the potential depends on whether the reference
plane is isotropic or not. In both situations, the problem is equivalent to finding the surfaces which are critical points of the gravitational potential among all surfaces with the same boundary curve $\Gamma$ and the same relative area $A_0$.

Similarly to Sect. 1, for the hanging chain problem of $I^2$, we shall calculate the gravitational potential of a surface in $I^3$ using the relative area. Otherwise, the use of the area element would lead to trivial conclusions. Indeed, if we were using the regular area element, then the potential of $M^2 : r(y,z) = (u(y,z), y, z)$ computed with respect to an isotropic plane $\Pi_{yz} = \{x = 0\}$ would be

$$\mathcal{F} = \int_{M^2} d(r(y,z), \Pi_{yz}) dA = \int_\Omega u u_z dydz = \int_{\partial \Omega} \left(0, \frac{u^2}{2}\right) \cdot \nu ds,$$

where $\nu$ is the outer unit normal vector along the boundary $\partial \Omega$, and we used the divergence theorem on the third equality. (Here, $u : \Omega \subset \Pi_{yz} \to \mathbb{R}$ is a smooth function over the open set $\Omega$.) This integral depends only on $\Omega$ and $\partial \Omega$; consequently, it is constant for all surfaces with the same boundary.

Similar arguments can be employed for the potential measured with respect to a non-isotropic plane. In conclusion, we must replace the area element with the relative area one.

In the following, we shall distinguish between isotropic and non-isotropic reference planes. Let $M^2$ be an admissible surface of $\mathbb{R}^3$ parametrized in its normal form: $r(x, y) = (x, y, u(x, y))$. From $r_x = (1, 0, u_x)$ and $r_y = (0, 1, u_y)$, the first and second fundamental forms and the area element are

$$I = dx^2 + dy^2, \quad II = u_{xx} dx^2 + 2u_{xy} dxdy + u_{yy} dy^2, \quad \text{and} \quad dA = dxdy. \quad (39)$$

The formula (17) of the mean curvature $H$ of $M^2$ is

$$H = \frac{u_{xx} + u_{yy}}{2}. \quad (40)$$

In addition, the minimal and parabolic normal vector fields are

$$N_{\text{min}} = (-u_x, -u_y, 1) \quad \text{and} \quad N_{\text{par}} = \left(-u_x, -u_y, \frac{1}{2} - \frac{u_x^2 + u_y^2}{2}\right). \quad (41)$$

Hence, the relative area element of $M^2$ is

$$dA^* = (N_{\text{min}} \cdot N_{\text{par}}) dA = \frac{1}{2} \left(1 + u_x^2 + u_y^2\right) dxdy. \quad (42)$$

We begin considering the gravitational potential of $M^2$ calculated by measuring the distance between the points of $M^2$ and the plane $\Pi_{yz}$ of equation $x = 0$. This distance is $d((x, y, z), \Pi_{yz}) = |x|$. From now on, and without loss of generality, it will be assumed that all surfaces are included in the half-space
\[ \Pi_{yz}^+ = \{(x, y, z) \in \mathbb{I}^3 : x > 0\}. \]
The potential of \( M^2 \) is \( \int_{M^2} x \, dA^* \), where \( dA^* \) is the relative area element of \( M^2 \). Thus, the functional to minimize is
\[
E_{yz}[M^2] = \int_{M^2} x \, dA^* - \lambda \int_{M^2} dA^* = \int_{M^2} (x - \lambda) \, dA^*,
\] (43)
where \( \lambda \) is a Lagrange multiplier due to the constraint on the relative area. Let \( \Gamma \) be the boundary curve of \( M^2 \) and \( A_0 \) its relative area. Let \( C_{\Gamma,A_0}^{yz}(\Omega) \) be the class of all graphs \( G \subset \Pi_{yz}^+ \) over \( \Omega \) with boundary \( \Gamma \) and relative area \( A_0 \).

**Theorem 7.** Let \( M^2 \subset \Pi_{yz}^+ \) be the graph of a function \( u : \Omega \subset \Pi_{yz} \to \mathbb{R} \), where \( \Omega \) is a bounded domain, and \( X = \partial_x \in \mathfrak{X}(\mathbb{I}^3) \). Then, \( M^2 \) is a critical point of \( E_{yz} \) in \( C_{\Gamma,A_0}^{yz}(\Omega) \) if, and only if, its mean curvature \( H \) satisfies
\[
H(r) = \frac{\langle N_{par}, X \rangle}{2(x - \lambda)} = \frac{\langle N_{par}, X \rangle}{2(d(r, \Pi_{yz}^+ \setminus \Omega) - \lambda)},
\] (44)

**Proof.** Using Eq. (42), the functional \( E_{yz} \) is
\[
E_{yz}[M^2] = \int_{M^2} (x - \lambda) \, dA^* = \int_{\Omega} \frac{(x - \lambda)}{2} \left(1 + p^2 + q^2\right) \, dx \, dy,
\]
where \( p = u_x \) and \( q = u_y \). To find the critical points of \( E_{yz} \) in \( C_{\Gamma,A_0}^{yz} \), we use the Euler–Lagrange equation
\[
\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial p} \right) - \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial q} \right) = 0, \quad F(x, y, u, p, q) = \frac{(x - \lambda)}{2} \left(1 + p^2 + q^2\right).
\] (45)

Notice that \( F \) does not depend on \( u \). Thus, Eq. (45) leads to
\[
((x - \lambda)p)_x + ((x - \lambda)q)_y = 0 \implies (x - \lambda)(p_x + q_y) + p = 0.
\]
Using Eq. (40), we have \( 2H(x - \lambda) = -p \). Finally, from Eq. (41) and the definition of \( X \), we conclude \( \langle N_{par}, X \rangle = -p \), proving the validity of Eq. (44). \( \square \)

Equation (44) is the two-dimensional version of Eq. (29) after replacing the curvature \( \kappa \) with the mean curvature \( H \) of \( M^2 \). In (44), the vector field \( X = \partial_x \) is unitary and orthogonal to the reference plane \( \Pi_{yz} \), and it is the velocity vector field of the lines used to measure the distance in the computation of the gravitational potential.

Now, consider the gravitational potential of \( M^2 \) calculated by measuring the distance between the points of \( M^2 \) with respect to \( \Pi_{xy} \). Since the distance to the \( xy \)-plane vanishes if we consider the simply isotropic metric, we shall employ the secondary metric to compute this distance. Therefore, we have \( d((x, y, z), \Pi_{xy}) = |z| \). Again, we will assume that \( M^2 \) is contained in one of the two half-spaces determined by \( \Pi_{xy} \), which will be \( \Pi_{xy}^+ = \{(x, y, z) \in \mathbb{I}^3 : z > 0\} \). The potential of \( M^2 \) is \( \int_{M^2} z \, dA^* \). Thus, the functional to minimize is
\[
E_{xy}[M^2] = \int_{M^2} z \, dA^* - \lambda \int_{M^2} dA^* = \int_{M^2} (z - \lambda) \, dA^*.
\] (46)
Here, $\lambda$ is again a Lagrange multiplier due to the constraint on the relative area. Let $\Gamma$ and $A_0$ be the boundary curve and the relative area of $M^2$, respectively. Let $C_{\Gamma, A_0}^{xy}(\Omega)$ be the class of all graphs $G$ over $\Omega$, $G \subset \Pi_{xy}^+$, with boundary $\Gamma$ and relative area $A_0$. Let $Z = \partial_z \in \mathfrak{X}(\Pi^3)$, i.e., $Z$ is the velocity vector field of the lines used to measure distances in the computation of the potential.

**Theorem 8.** Let $M^2 \subset \Pi_{xy}^+$ be the graph of a function $u : \Omega \subset \Pi_{xy} \to \mathbb{R}$, where $\Omega$ is a bounded domain. Then, $M^2$ is a critical point of $E_{xy}$ in $C_{\Gamma, A_0}^{xy}$ if, and only if, its mean curvature $H$ satisfies

$$H(r) = \langle N_{\text{par}}, Z \rangle = \langle N_{\text{par}}, Z \rangle = \frac{d(r, \Pi_{xy}^+)}{2} - \lambda). \quad (47)$$

**Proof.** The functional $E_{xy}$ to minimize in $C_{\Gamma, A_0}^{xy}$ writes as

$$E_{xy}[M^2] = \int_{M^2} (z - \lambda) \, dA^* = \int_{\Omega} \frac{u - \lambda}{2} \left(1 + p^2 + q^2\right) \, dx \, dy.$$

The corresponding Euler–Lagrange equation is calculated with $F(x, y, u, p, q) = \frac{u - \lambda}{2} (1 + p^2 + q^2)$. Computing (45), we obtain

$$1 + p^2 + q^2 - (2(u - \lambda)p)_x - (2(u - \lambda)q)_y = 0 \Rightarrow 1 - p^2 - q^2$$

$$= 2(u - \lambda)(p_x + q_y).$$

This identity, together with Eq. (40), gives $H = (1 - u_x^2 - u_y^2)/4(u - \lambda)$. Then, Eq. (47) follows from the expression of $N_{\text{par}}$ in Eq. (41): $\langle N_{\text{par}}, \partial_z \rangle = (1 - p^2 - q^2)/2$. \hfill \square

Note that for the surfaces described in Theorems 8 and 9, we may set $\lambda = 0$ after a convenient translation of $I^3$. We have the following definition.

**Definition 3.** Let $\Pi \subset I^3$ be a plane and $V \in \mathfrak{X}(I^3)$ be a unit vector field orthogonal to $\Pi$. A surface $M^2$ in $I^3$ is an isotropic singular minimal surface with respect to $\Pi$ if its mean curvature satisfies

$$H(p) = \begin{cases} 
\langle N_{\text{par}}, V \rangle & p \in M^2, \quad \Pi \text{ isotropic} \\
\langle N_{\text{par}}, V \rangle & 2 \, d(p, \Pi) \end{cases}.$$

Note that this definition extends the notion of isotropic catenary to dimension 2. It is also possible to extend this definition if we introduce a power $\alpha$ in the functional $E_{yz}$ and $E_{xy}$, replacing $x - \lambda$ with $x^\alpha - \lambda$ and $z - \lambda$ with $z^\alpha - \lambda$, respectively. The critical points of these functionals generalize $\alpha$-catenaries and Eqs. (44) and (47) characterize them after multiplying the right-hand sides by the factor $\alpha$. The case $\alpha = 0$ corresponds to minimal surfaces.
6. Simply Isotropic Invariant Singular Minimal Surfaces

In this section, we study the isotropic singular minimal surfaces that are invariant and whose generating curves lie in the $yz$-plane. We show that there exist no singular minimal helicoidal surfaces. For surfaces of revolution, either Euclidean or parabolic, we completely classify all singular minimal surfaces when the gravitational potential is measured with respect to an isotropic plane. On the other hand, when the potential is measured with respect to a non-isotropic plane, the generating curve is associated with the solution of a non-linear ordinary differential equation of second order.

Invariant surfaces will be important in studying singular minimal surfaces. In addition to the fact that we must study an ordinary differential equation over the generating curve instead of a partial differential one, using invariant surfaces allows for establishing an important connection between simply isotropic catenaries and singular minimal surfaces.

6.1. Singular Minimal Surfaces of Euclidean Revolution

Applying helicoidal motions to $\gamma(t) = (t, 0, z(t))$, Eq. (23), gives the invariant surface $S_\gamma = \mathcal{H}_\theta(\gamma)$ parametrized as

$$\mathbf{r}(t, \theta) = (t \cos \theta, t \sin \theta, c \theta + z(t)).$$

(48)

For $c = 0$, $\mathcal{H}_\theta = \mathcal{R}_\theta$ and we obtain surfaces of (Euclidean) revolution as in Eq. (36). The mean curvature $H$ and parabolic normal $\mathbf{N}_{par}$ of $S_\gamma$ are

$$H = \frac{z' + tz''}{2t} \quad \text{and}$$

$$\mathbf{N}_{par} = \left( \frac{c \sin \theta}{t} - z' \cos \theta, - \frac{c \cos \theta}{t} - z' \sin \theta, 1 - \frac{c^2}{t^2} - z'^2 \right).$$

(49)

**Theorem 9.** If $S_\gamma$ is a helicoidal singular minimal surface, then $S_\gamma$ must be a surface of Euclidean revolution, i.e., $c = 0$. In addition,

1. If the reference plane is $\Pi_{yz}$ (isotropic), then $z(t) = \frac{z_2}{t} + z_1$, $z_1, z_2 \in \mathbb{R}$. In particular, this includes horizontal planes.

2. If the reference plane is $\Pi_{xz}$ (non-isotropic), then $z = z(t) > 0$ satisfies

$$z''(t) + \frac{z'(t)}{t} = \frac{1 - z'(t)^2}{2z(t)}.$$

(50)

**Proof.** We distinguish between two cases depending on the type of plane.

1. The reference plane is $\Pi_{yz}$. Then, $\langle \mathbf{N}_{par}, X \rangle = c \sin \theta / t - z' \cos \theta$. Since the distance to $\Pi_{yz}$ in Eq. (44) is the $x$-coordinate, we have $d(p, \Pi_{yz}) = t \cos \theta$ and the Eq. (44) for the mean curvature of a helicoidal singular minimal surface reduces to

$$\frac{z' + tz''}{t} = \frac{z \sin \theta - z' \cos \theta}{t \cos \theta} \iff t(2z' + tz'') \cos \theta - c \sin \theta = 0.$$

(51)
Since \( \cos \theta \) and \( \sin \theta \) are linearly independent, we deduce that \( c = 0 \) and, consequently, \( S_\gamma \) is a surface of revolution. Moreover, \( z = z(t) \) satisfies

\[
2z' + tz'' = 0,
\]

whose solution is \( z(t) = z_2 t^{-1} + z_1 \), where \( z_1, z_2 \in \mathbb{R} \).

2. The reference plane is \( \Pi_{xy} \). For the computation of Eq. (47), we have

\[
\langle N_{par}, Z \rangle = \frac{1}{2} (1 - c^2/t^2 - z'^2) \quad \text{and} \quad d(p, \Pi_{xy}) = z + c\theta.
\]

Using Eq. (49), it follows that Eq. (47) is

\[
\frac{z' + tz''}{2t} = \frac{1}{2} (1 - c^2/t^2 - z'^2) \Leftrightarrow c(z' + tz'')\theta + z(z' + tz'')
\]

\[
- \frac{t}{2} \left(1 - \frac{c^2}{t^2} - z'^2\right) = 0.
\]

Since \( \{1, \theta\} \) is linearly independent, we deduce that

\[
c(z' + tz'') = 0 \quad \text{and} \quad z(z' + tz'') - \frac{t}{2} \left(1 - \frac{c^2}{t^2} - z'^2\right) = 0.
\]   (52)

From the first equation of (52), we distinguish between two cases according to whether \( c \) is 0 or not. If \( c \neq 0 \), then \( z' + tz'' = 0 \). If, in addition, \( z' = 0 \), then the second equation of (52) is simply \( \frac{t}{2} (1 - \frac{c^2}{t^2}) = 0 \), which leads to a contradiction, namely, \( t \) cannot be constant. On the other hand, if \( c \neq 0 \) but \( z' \neq 0 \), then the equation \( z' + tz'' = 0 \) yields \( z' = m/t \) for some constant \( m > 0 \). Using this, the second equation of (52) is \( \frac{1 - m^2}{2t} - \frac{c^2}{2t} = 0 \), which also leads to a contradiction. Therefore, we must have \( c = 0 \) and, in particular, \( S_\gamma \) is a surface of revolution. Coming back to (52), we conclude that

\[
z(z' + tz'') - \frac{t}{2} (1 - \frac{c^2}{t^2} - z'^2) = 0,
\]

which is nothing but Eq. (50).

\[\square\]

The solution \( \gamma \) in item 1 of Theorem 9 coincides with the 2-catenary with respect to \( L_z \) that appeared in Eq. (32). This observation is a particular instance of a more general result.

**Corollary 2.** Let \( S_\gamma \) be a surface of revolution in \( \mathbb{E}^3 \) with respect to \( L_z \). Then, \( S_\gamma \) is an isotropic \( \alpha \)-singular minimal surface with respect to \( \Pi_{yz} \) if, and only if, \( \gamma \) is an isotropic \( (\alpha + 1) \)-catenary with respect to \( L_z \).

**Proof.** Suppose that \( \gamma(t) = (t, 0, z(t)) \). Using the parametrization in Eq. (36), it follows from Eqs. (49) that Eq. (44) implies

\[
\frac{z' + tz''}{2t} = -\alpha \frac{z' \cos \theta}{2t \cos \theta} = -\alpha \frac{z'}{2t} \Rightarrow (\alpha + 1)z' + tz'' = 0.
\]

This equation coincides with Eq. (31) for \( \alpha + 1 \).

\[\square\]

We conclude this section by studying singular minimal surfaces of revolution with respect to non-isotropic planes that intersect the rotation axis. The generating curve \( \gamma = \gamma(t) \) is defined over an interval \( I = (t_0, t_1) \subset (0, \infty) \).
and, therefore, if \( t_0 = 0 \), then \( \gamma \) meets the rotation axis \( L_z \). We are interested in the configuration where the intersection is orthogonal, which implies that the surface \( S_\gamma \) that generates \( \gamma \) is smooth at \( S_\gamma \cap L_z \).

The height function \( z = z(t) \) of \( \gamma(t) = (t, 0, z(t)) \) satisfies Eq. (50). For \( a, b \in \mathbb{R} \), \( a > 0 \), consider the following Initial Value Problem (IVP)

\[
z''(t) + \frac{z'(t)}{t} = \frac{1 - z'(t)^2}{2z(t)}, \quad z(t_0) = a, \quad z'(t_0) = b. \quad (53)
\]

For any \( t_0 > 0 \), the existence and uniqueness of the IVP (53) is assured by the standard theory of Ordinary Differential Equations. However, if \( t_0 = 0 \), Eq. (53) is degenerate at \( t = 0 \), and the existence may be lost. To establish the existence of a surface of revolution intersecting the axis \( L_z \) orthogonally, it is necessary to solve the IVP (53) at \( t = 0 \) under the condition \( z'(0) = 0 \).

**Theorem 10.** For any \( a > 0 \), the initial value problem (53) with initial conditions \( z(0) = a \) and \( z'(0) = 0 \) has a solution \( z \in C^2([0, R]) \) for some \( R > 0 \). In addition, the solution depends continuously on the parameter \( a \).

**Proof.** Multiplying Eq. (50) by \( t \), the equation becomes \( (tz')' = t(1 - z'^2)/(2z) \).

Define the operator

\[
(Tz)(t) = a + \int_0^t \frac{1}{r} \left( \int_0^r \frac{\tau(1 - z'(\tau)^2)}{2z(\tau)} \, d\tau \right) \, dr, \quad z(t) \in C^1([0, R]).
\]

It is immediate that \( z \) is a solution of the problem (53) with \( z'(0) = 0 \) if \( u \) is a fixed point of the operator \( T \). Let \( C^1([0, R]) \) be considered as a Banach space endowed with the usual norm \( \| z \| = \| z \|_\infty + \| z' \|_\infty \). It will be proved the existence of \( R > 0 \) such that \( T \) is a contraction in some closed ball \( \overline{B}(a, \epsilon) \). First, we prove that \( T \) is a self-map in a closed ball \( \overline{B}(a, \epsilon) \) for some \( \epsilon > 0 \) and next, that \( T \) is a contraction.

1. **Claim:** there exists \( \epsilon > 0 \) such that \( T(\overline{B}(a, \epsilon)) \subset \overline{B}(a, \epsilon) \).

Indeed, let \( \epsilon > 0 \) such that \( \epsilon < a \), which is fixed. Consider \( R > 0 \) such that

\[
R \leq \min \left\{ \sqrt{\frac{4e(a - \epsilon)}{1 + e^2}}, \frac{2e(a - \epsilon)}{1 + e^2} \right\}.
\]

If \( z \in \overline{B}(a, \epsilon) \), we have \( |z(t) - a| \leq \epsilon \) and \( |z'(t)| \leq \epsilon \) for \( t \in [0, R] \). Then

\[
| (Tz)(t) - a | \leq \int_0^t \frac{1}{r} \left| \int_0^r \frac{\tau(1 - z'(\tau)^2)}{2z(\tau)} \, d\tau \right| \, dr \leq \int_0^t \frac{1}{r} \int_0^r \frac{\tau(1 + e^2)}{2(a - \epsilon)} \, d\tau \, dr
\]

\[
= \frac{t^2(1 + e^2)}{8(a - \epsilon)} \leq \frac{R^2(1 + e^2)}{8(a - \epsilon)} \leq \frac{\epsilon}{2}.
\]

On the other hand,

\[
| (Tz - a)'(t) | \leq \frac{1}{t} \int_0^t \frac{\tau(1 + e^2)}{2(a - \epsilon)} \, d\tau = \frac{t(1 + e^2)}{4(a - \epsilon)} \leq \frac{R(1 + e^2)}{4(a - \epsilon)} \leq \frac{\epsilon}{2}.
\]
This proves that \( \|Tz - a\| \leq \epsilon \). Hence, \( Tz \in \overline{B(a, \epsilon)} \).

As a consequence of the claim, \( T \) is a self-map \( \overline{B(a, \epsilon)} \rightarrow \overline{B(a, \epsilon)} \).

2. **Claim:** the operator \( \overline{T} : \overline{B(a, \epsilon)} \rightarrow \overline{B(a, \epsilon)} \) is a contraction.

Indeed, let \( L_1 \) and \( L_2 \) respectively denote the Lipschitz constants of the functions \( x \mapsto 1/(2x) \) and \( x \mapsto 1 - x^2 \) in \([a - \epsilon, a + \epsilon]\), provided that \( \epsilon < a \).

Let \( L = L_1L_2 \). For all \( z_1, z_2 \in C^1([0, R]) \), we have

\[
\|Tz_1 - Tz_2\| = \|Tz_1 - Tz_2\|_\infty + \|(Tz_1)' - (Tz_2)'\|_\infty.
\]

We study the term \( \|Tz_1 - Tz_2\|_\infty \). Let \( z_1, z_2 \) be two functions in the ball \( \overline{B(a, \epsilon)} \) of \( C^1([0, R]), \| \cdot \| \). For all \( t \in [0, R] \), where \( R \) will be determined later, we have

\[
|(Tz_1)(t) - (Tz_2)(t)| \leq \int_0^t \frac{1}{r} \left( \int_0^r L \|z_1 - z_2\|_\infty \tau \, d\tau \right) \, dr = \frac{Lt^2}{4} \|z_1 - z_2\|_\infty,
\]

Similarly, for \( \|(Tz_1)' - (Tz_2)'\|_\infty \), we have

\[
|(Tz_1)'(t) - (Tz_2)'(t)| \leq \frac{L}{t} \int_0^t \|z_1 - z_2\|_\infty \, d\tau = \frac{Lt}{2} \|z_1 - z_2\|_\infty \leq \frac{Lt}{2} \|z_1 - z_2\|.
\]

If \( R \leq \{ \sqrt{2/L}, 1/L \} \), then (54) and (55) imply \( \|Tz_1 - Tz_2\| < \|z_1 - z_2\| \), proving that the operator \( T \) is a contraction in \( C^1([0, R]) \).

Once the two claims have been proved, the Fixed Point Theorem asserts the existence of a fixed point \( z \in C^1([0, R]) \cap C^2((0, R]) \). This function \( z \) is then a solution of (53) with \( z'(0) = 0 \). Finally, we prove that the solution \( u \) extends with \( C^2 \)-regularity at \( t = 0 \). By taking limits in (53) as \( t \to 0 \), and by L'Hôpital rule on the quotient \( z'(t)/t \), we conclude

\[
\frac{1}{2a} = \lim_{t \to 0} \frac{1 - z'(t)^2}{2z(t)} = \lim_{t \to 0} z''(0) + \lim_{t \to 0} \frac{z'(t)}{t} = \lim_{t \to 0} z''(0) + \lim_{t \to 0} z''(t) = 2 \lim_{t \to 0} z''(0).
\]

This proves that \( z''(0) = \frac{1}{4a^3} \). The continuous dependence of local solutions on \( a \) follows from the continuous dependence of the fixed points of \( T \) on \( a \).

### 6.2. Singular Minimal Surfaces of Parabolic Revolution

Applying parabolic revolutions to \( \gamma(t) = (t, 0, z(t)) \), Eq. (24), gives the invariant surface \( S_\gamma = R_\theta(\gamma) \) parametrized as

\[
r(t, \theta) = \left( a\theta + t, b\theta, c\theta + \frac{ac_1 + bc_2}{2} \theta^2 + c_1 t\theta + z(t) \right), \quad b \neq 0.
\]
The $\alpha$-catenaries, Eq. (32), with $\alpha = -1$, $\alpha = 1$, and $\alpha = 2$. (Center) Singular minimal surface obtained as a warped translation surface, Eq. (56), and whose generating curve (dashed black line) is a catenary $z(t) = z_2 \ln t + z_1$ (See Theorem 11). Note the surface is ruled. (Right) Singular minimal surface obtained as a surface of parabolic revolution, Eq. (56), and whose generating curve (dashed black line) is a combination of a $(-1)$-catenary and a catenary: $z(t) = -\frac{c_2}{4b} t^2 + z_2 \ln t + z_1$ (See Theorem 11). Here, the surface is foliated by isotropic circles. (In the figures, the parameters take the values: (Left) $c = 1$ and $d = 0$; (Center) $a = 0$, $b = 1$, $c = 0$, $c_1 = 0$, and $c_2 = 0$; and (Right) $a = 0$, $b = 1$, $c = \frac{1}{2}$, $c_1 = 0$, and $c_2 = 1.0$.) Figures generated with Mathematica.

If $ac_1 + bc_2 = 0$, then $S_\gamma$ is called a surface of warped translation (see Fig. 2). The mean curvature $H$ of $S_\gamma$ is

$$H = \frac{a^2 + b^2}{2b^2} z'' + \frac{bc_2 - ac_1}{2b^2}. \quad (57)$$

The parabolic normal vector field of $S_\gamma$ is

$$N_{par} = \left(-c_1 \theta - z', \frac{az' - bc_2 \theta - c - c_1 t}{b} \frac{1}{2} - \frac{F(t)}{2}\right), \quad (58)$$

where

$$F(t) = \frac{(c + c_1 t)^2}{b^2} - \frac{2a(c + c_1 t)}{b^2} z' + \frac{a^2 + b^2}{b^2} z'^2 - \frac{2t}{b} [(ac - bc_1) z' - c_2 (c + c_1 t)] + t^2 (c_1^2 + c_2^2). \quad (59)$$

**Theorem 11.** Let $S_\gamma$ be a singular minimal surface of parabolic revolution.

1. If the reference plane is $\Pi_{yz}$ (isotropic), then we have two cases:
   
   (a) if $a = 0$, then $c_1 = 0$ and
   
   $$z(t) = -\frac{c_2}{4b} t^2 + z_2 \ln t + z_1, \quad z_1, z_2 \in \mathbb{R} \quad (60)$$
(b) if \( a \neq 0 \), then \( ac_2 + 2bc_1 = 0 \) and
\[
z(t) = \frac{c_1}{2a} t^2 + z_1, \quad z_1 \in \mathbb{R}. \tag{61}
\]

2. If the reference plane is \( \Pi_{xz} \) (non-isotropic), then \( c = c_1 = 0 \) and \( z(t) \) is a solution of
\[
(2z + bc_2t^2)z'' + z' - \frac{2abc_2}{a^2 + b^2} tz' + \frac{2bc_2}{a^2 + b^2}(z + bc_2t^2) - \frac{b^2}{a^2 + b^2} = 0.
\]
\[
(2z + bc_2t^2)z'' + z' - \frac{2abc_2}{a^2 + b^2} tz' + \frac{2bc_2}{a^2 + b^2}(z + bc_2t^2) - \frac{b^2}{a^2 + b^2} = 0. \tag{62}
\]

**Proof.** We distinguish between two cases depending on the type of the reference plane. In what follows, we employ the shorthand notation \( A = a_2 + b_2 \) and \( B = \frac{bc_2 - ac_1}{b_2} \), which implies \( H = \frac{(A z'' + B)}{2} \).

1. The reference plane is \( \Pi_{yz} \). Then, \( \langle N_{\text{par}}, X \rangle = -c_1 \theta - z' \). Since the distance to \( \Pi_{yz} \) in Eq. (44) is the \( x \)-coordinate, we have \( d(p, \Pi_{yz}) = t + a \theta \).

Equation (44) for the mean curvature reduces to
\[
A z'' + B = -\frac{c_1 \theta + z'}{2(t + a \theta)} \iff (aA z'' + aB + c_1) \theta + (A t z'' + z' + tB) = 0.
\]

Since \( \{1, \theta\} \) forms a set of linearly independent functions, we find
\[
A z'' + B = 0 \quad \text{and} \quad z'' + \frac{z'}{t} + \frac{B}{A} = 0. \tag{63}
\]

We have two sub-cases.

(a) Case \( a = 0 \). Thus, \( A = 1 \) and \( B = c_2/b \). Moreover, \( c_1 = 0 \) and the second equation of (63) is \( z'' + \frac{z'}{t} + \frac{c_2}{b} = 0 \), whose solution is
\[
z(t) = -\frac{c_2}{4b} t^2 + z_2 \ln(t) + z_1, \quad z_1, z_2 \in \mathbb{R}.
\]

(b) Case \( a \neq 0 \). The first equation of the system (63) gives
\[
z(t) = -\frac{aB + c_1}{2aA} t^2 + z_2 t + z_1, \quad z_1, z_2 \in \mathbb{R}.
\]

Substituting this solution for \( z(t) \) in the second equation of (63),
\[
-\frac{aB + c_1}{aA} + \frac{z_1 - \frac{aB + c_1}{aA} t}{tA} + \frac{B}{A} = 0 \iff \frac{z_1}{t} - \frac{aB + c_1}{a} - \frac{aB + c_1}{aA} + B = 0.
\]

Thus, we must have \( z_2 = 0 \) and \( 0 = aAB - A(aB + c_1) - aB - c_1 = -(c_1 + Ac_1 + aB) \). From the definition of \( A \) and \( B \), we conclude that the parameters characterizing a singular minimal surface of parabolic revolution are subjected to the constraint \( ac_2 + 2bc_1 = 0 \).

Finally, using that \( ac_2 + bc_1 = -bc_1 \) implies that the solution for \( z(t) \) takes the form
\[
z(t) = -\frac{aB + c_1}{2aA} t^2 + z_1 = \frac{c_1}{2a} t^2 + z_1, \quad z_1 \in \mathbb{R}.
\]
2. The reference plane is $\Pi_{xy}$. For the computation of Eq. (47), we have $\langle N_{\text{par}}, Z \rangle = \frac{1}{2}(1 - F(t))$ and $d(p, \Pi_{xy}) = c\theta + \frac{ac_1 + bc_2}{2}t^2 + c_1t\theta + z$. Using Eq. (57), it follows that Eq. (47) is

$$B + Az'' = \frac{1 - F(t; z, z')}{2 \left[ (c + c_1t)\theta + \frac{1}{2}(ac_1 + bc_2)t^2 + z \right]}.$$  

Using again that $\{1, \theta\}$ is linearly independent, we have

$$2(c + c_1t)(Az'' + B) = 0 \quad \text{and} \quad F - 1 + [2z + (ac_1 + bc_2)t^2](Az'' + B) = 0. \quad (64)$$

(a) Case $Az'' + B \neq 0$ at some point $t = t_0$. Around an interval of $t_0$, we have $c = c_1 = 0$. In particular, from (59), the function $F$ is $F = Az'^2 - \frac{2atc_2}{b}z' + t^2c_2^2$. With this value of $F$, the second equation of (64) is

$$(2z + bc_2t^2)z'' + z'^2 - \frac{2abc_2}{a^2 + b^2}tz' + \frac{b^2c_2^2}{a^2 + b^2}t^2 + \frac{bc_2}{a^2 + b^2}(2z + bc_2t^2) - \frac{b^2}{a^2 + b^2} = 0.$$  

(b) Case $Az'' + B = 0$. Then

$$z(t) = -\frac{B}{2A}t^2 + z_2t + z_1 = \frac{ac_1 - bc_2}{a^2 + b^2} \frac{t^2}{2} + z_2t + z_1, \quad z_1, z_2 \in \mathbb{R}.$$  

The second equation of (64) gives $F = 1$. From the value of $F$ in (59), we find a polynomial of degree 2 in $t$. The quadratic coefficient becomes

$$\frac{(c_1^2 + c_2^2)[b^2 + (1 + a)^2]}{a^2 + b^2} = 0.$$  

However, this constraint would imply $b = 0$, which is not allowed by hypothesis. This proves that the case $Az'' + B = 0$ is not possible.

Considering an isotropic reference plane, the family of generating curves of singular minimal surfaces of parabolic revolution contains isotropic catenaries when $c_2 = 0$ and isotropic circles when $z_2 = 0$. In the former class, we have surfaces of warped translation (see Fig. 2, Center). In the latter, the surfaces have constant mean curvature: $H = \frac{c_1^2}{2b}$ if $a = 0$ and $H = -\frac{c_1^2}{2a}$ if otherwise. Constant mean curvature surfaces of parabolic revolution are implicitly defined as parabolic quadrics. From the proposition to be proved below, the type of the parabolic quadric is determined by a parameter $\Lambda$, which is $\Lambda = -c_1^2$ if $a = 0$ and $\Lambda = -2c_1^2 - \frac{c^2_2}{2}$ if otherwise. In both cases, the corresponding surface of parabolic revolution is a hyperbolic paraboloid.

**Proposition 3.** An admissable surface of parabolic revolution $M^2 \subset \mathbb{R}^3$ has constant mean curvature $H_0$ if, and only if, it is a parabolic quadric, where the parabolas that foliate the surfaces are isotropic circles, i.e., parabolas whose
axes are an isotropic line. In addition, the type of the parabolic quadric is determined by the parameter \( \Lambda = 2(ac_1 + bc_2)H_0 - (c_1^2 + c_2^2) \): \( M^2 \) is an elliptic paraboloid if \( \Lambda > 0 \); \( M^2 \) is a parabolic cylinder if \( \Lambda = 0 \); and \( M^2 \) is a hyperbolic paraboloid if \( \Lambda < 0 \). Finally, the only minimal surfaces of parabolic revolution in \( \mathbb{R}^3 \) are the hyperbolic paraboloids implicitly defined by \( z = \lambda(x^2 - y^2) \).

Proof. Let \( M^2 \) be a surface of parabolic revolution with constant mean curvature (CMC) \( H_0 \). Then, \( M^2 \) is generated by an isotropic circle \( z(t) = z_0 + z_1 t + z_2 t^2 \), where the mean curvature \( H_0 \) depends on the parameters defining the surface by the relation \( z_2 = (ac_1 - bc_2 + 2b^2H_0)/2(a^2 + b^2) \) (see Ref. [4], Example 5.4). These surfaces are implicitly given by the equation \( z - z_0 = Ax^2 + 2Bxy + Cy^2 + Dx + Ey \), where the coefficients are (see Ref. [9], Fig. 3) \( A = z_2, B = \frac{c_1 - 2a_2 z_2}{2b}, C = \frac{2a_1 z_2 - ac_1 + bc_2}{2b^2}, D = z_1, \) and \( E = \frac{c - a_2 z_1}{b} \).

The type of the CMC surface is determined by the sign of the parameter \( \Lambda = 2(ac_1 + bc_2)H_0 - (c_1^2 + c_2^2) \): elliptic paraboloid if \( \Lambda > 0 \); parabolic cylinder if \( \Lambda = 0 \); and hyperbolic paraboloid if \( \Lambda < 0 \). For minimal surfaces, \( \Lambda < 0 \) and, therefore, the surface is a hyperbolic paraboloid. Finally, the trace of the quadratic part defining the surface vanishes, which implies that a minimal surface of parabolic revolution is implicitly given, up to rigid motions, by the equation \( z = \lambda(x^2 - y^2) \).

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Data Availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Declarations

Conflict of interest The authors declare they have no conflict of interest.
References

[1] Barbosa, J.L.M., Colares, A.G.: Minimal Surfaces in $\mathbb{R}^3$. Springer, Berlin (1986)
[2] Bliss, G.: Lectures on the Calculus of Variations. University of Chicago Press, Chicago (1946)
[3] da Silva, L.C.B.: The geometry of Gauss map and shape operator in simply isotropic and pseudo-isotropic spaces. J. Geom. 110, 31 (2019)
[4] da Silva, L.C.B.: Differential geometry of invariant surfaces in simply isotropic and pseudo-isotropic spaces. Math. J. Okayama Univ. 63, 15 (2021)
[5] da Silva, L.C.B.: Holomorphic representation of minimal surfaces in simply isotropic space. J. Geom. 112, 35 (2021)
[6] Dierkes, U.: Singular Minimal Surfaces. In: Hildebrandt, S., Karcher, H. (eds.) Geometric Analysis and Nonlinear Partial Differential Equations, pp. 177–193. Springer, Berlin (2003)
[7] Euler, L.: Methodus inveniendi curvas lineas maximi minimive proprietate gaudentes sive solution problematis isoperimetrici latissimo sensu accepti, Lausanne. Reprinted as Opera omnia Ser. 1, V. 24 (1952)
[8] Isenberg, C.: The Science of Soap Films and Soap Bubbles. Dover (1992)
[9] Kelleci, A., da Silva, L.C.B.: Invariant surfaces with coordinate finite-type Gauss map in simply isotropic space. J. Math. Anal. Appl. 495, 124673 (2021)
[10] López, R.: Invariant singular minimal surfaces. Ann. Glob. Anal. Geom. 53, 521–541 (2018)
[11] López, R.: The hanging chain problem in the sphere and in the hyperbolic plane. arXiv:2208.13694
[12] López, R.: A characterization of minimal rotational surfaces in the de Sitter space. Mediterr. J. Math. 20, 68 (2023)
[13] Müller, E.: Relative Minimalflächen. Monatsh. Math. Phys. 31, 3–19 (1921)
[14] Nitsche, J.C.C.: Lectures on Minimal Surfaces. Cambridge University Press, Cambridge (1989)
[15] Sachs, H.: Ebene Isotrope Geometrie. Friedr. Vieweg & Sohn, Braunschweig (1987)
[16] Sachs, H.: Isotrope Geometrie des Raumes. Friedr. Vieweg & Sohn, Braunschweig (1990)
[17] Simon, U., Schwenk-Schellschmidt, A., Viesel, H.: Introduction to the Affine Differential Geometry of Hypersurfaces. Science University of Tokyo, Tokyo (1991)
[18] Struve, R.: Orthogonal Cayley–Klein groups. Results Math. 48, 168 (2005)
[19] Verpoort, S.: A characterisation of Manhart’s relative normal vector fields. Adv. Geom. 12, 29–42 (2012)

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