Global Conserved Quantities and Unfree Gauge Symmetry

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Abstract—We consider a class of theories with unfree gauge symmetry, whose gauge parameters are restricted by differential equations. We demonstrate that such theories admit global conserved quantities, whose on-shell values are defined by asymptotics of the fields rather than Cauchy data. The global conserved quantities can be deduced proceeding from the equations restricting gauge parameters, and they are treated differently by two BRST complexes corresponding to a system with unfree gauge symmetry.

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1. GLOBAL CONSERVED QUANTITIES IN CONSTRAINED HAMILTONIAN FORMALISM

Let us consider the constrained Hamiltonian system

\[ S_H = \int dt(p, q) - H(q, p) - \lambda^a T_a(q, p), \]  

with involution relations

\[ \{ T_a, H \} = \tau_a \Gamma^a, \quad \{ \tau_a, H \} = T_a \nu^a, \]  

where \( \Gamma^a \) and \( V^a \) are supposed to be finite. The irreducible secondary constraints \( \tau_a \) contain modular parameters, which do not reduce to the linear combination of modular parameters. The simplest example of a modular parameter is the cosmological constant of unimodular gravity. Its analogues for higher spin field theories with transverse symmetry [2, 3] are found in [4, 5]. The completion functions (3) and their differential consequences can be resolved with respect to the modular parameters, so there exist global conserved quantities \( J^a \), whose specific values are defined by asymptotics of the fields and their derivatives.

Any action with unfree gauge symmetry admits the alternative reducible symmetry with unconstrained gauge parameters [6]. The global conserved quantities are treated differently depending on the type of symmetry. In the next section, we demonstrate it in constrained Hamiltonian formalism.
there exist null-vectors $Z^A_{i A}$ and $Z^A_{2,A}$, such that the
involution relations read
\begin{equation}
\langle T_\alpha, H \rangle = \tilde{\tau}_\alpha, \quad \langle \tilde{\tau}_\alpha, H \rangle = T_\beta \tilde{\tau}_\alpha,
\end{equation}
where $\tilde{\tau}_\alpha Z^A_{i A} = 0$, $Z^A_{i A} Z^A_{2,A} = 0$,
and as physical observables in a theory with reducible
gauge symmetry. In the next section, we demonstrate
this inequivalence in the model of Maxwell-like spin 2 model.

2. EXAMPLE:
MAXWELL-LIKE SPIN 2 MODEL
Let us consider a theory of symmetric traceful sec-
ond-rank tensor in 4d Minkowski space with the metric
$\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$, described by the Max-
well-like Lagrangian [3],
\begin{equation}
\mathcal{L} = -\frac{1}{2} \partial_\nu h^\mu_{\nu\sigma} \partial^\sigma h^\mu_{\nu\sigma} + \partial^\mu h^\nu_{\nu\sigma} \partial_\nu h^\sigma_{\mu\nu},
\end{equation}
where $\mu, \nu, \rho = 0,1,2,3$. The modified Noether identi-
ties (1) read
\begin{equation}
-2 \partial^\mu \frac{\delta S}{\delta h^\mu_{\nu\sigma}} + \partial_\nu \tau = 0, \quad \tau \equiv -2 \partial_\mu \partial_\nu h^\mu_{\nu\sigma}.
\end{equation}
A single completion function, $\bar{\mathcal{F}} = \tau - \Lambda$, can be resolved with respect to $\Lambda$, $\Lambda = \tau$, i.e. the global conserved quantity $J$ coincides with $\tau$.

The unfree gauge symmetry transformations (2) for the theory (11) read
\begin{equation}
\delta_\chi h^\mu_{\nu\sigma} = \partial^\mu \chi^\nu + \partial^\nu \chi^\mu, \quad \partial_\mu \chi^\mu = 0,
\end{equation}
while corresponding reducible transformations read
\begin{equation}
\delta_\chi h^\mu_{\nu\sigma} = \partial^\mu \delta_\chi \chi^\nu + \partial^\nu \delta_\chi \chi^\mu,
\end{equation}
\begin{equation}
\delta_\chi \chi^\mu = \partial_\mu \epsilon^\nu, \quad \delta_\chi \chi^\nu = -\epsilon^\nu \partial_\mu \delta_\chi \chi^\mu,
\end{equation}
where $\epsilon^\mu, \omega^{\nu\mu\rho}$ are totally antisymmetric tensors, $\epsilon^{\mu
u\rho}$ is the Levi–Civita symbol, $\epsilon^\mu = \partial_\nu \epsilon^\nu$.

Introducing the momenta
\begin{equation}
\Pi_\mu \equiv -h_\mu + \partial_\mu \chi^\nu + \partial_\nu \chi^\mu, \quad \Pi \equiv h + 2 \partial_\mu \chi^\mu,
\end{equation}
we arrive to the Hamiltonian action for the theory (11),
\begin{equation}
S_H = \int d^4x \langle \Pi_\mu \dot{h}^\mu + \Pi \dot{h} - H - \lambda^i \dot{T}_i \rangle,
\end{equation}
\begin{equation}
\bar{Q} = T_\alpha C^\alpha + \tilde{\tau}_\alpha C^\alpha + \bar{\bar{T}} \alpha Z^A_{i A} \epsilon^A + \bar{\bar{T}} \alpha Z^A_{2,A} \epsilon^A
+ \pi_\alpha \rho^\alpha + \pi_\alpha \rho^A + \pi_\alpha \rho^A + \pi_\alpha \rho^A,
\end{equation}
all the ghosts and Lagrange multipliers are introduced in Table 2. For more details see [5].

So, the same Hamiltonian system (4) admits two
different BRST complexes. They are related, as $Q$ (7)
and $\bar{Q}$ (10) are connected by some non-local canoni-
tcal transformation [5], but non-equivalent. Indeed,
the irreducible secondary constraints $\tau_\alpha$, being the
Hamiltonian counterparts of the completion functions
$\bar{\bar{T}}_\alpha$ (3), are BRST-exact with respect to $Q$ (7), and
BRST-closed, but not BRST-exact with respect to
\begin{equation}
\langle T_\alpha, H \rangle = \tilde{\tau}_\alpha, \quad \langle \tilde{\tau}_\alpha, H \rangle = T_\beta \tilde{\tau}_\alpha,
\end{equation}
In this case, we arrive at unfree symmetry transformations
\[ \delta_{\epsilon} h^{ij} = \delta' e^i + \delta' e^j, \quad \delta_{\epsilon} h = -2\partial \epsilon, \]
\[ \delta \lambda = \epsilon + \partial \epsilon, \quad \epsilon + \partial \epsilon = 0, \tag{18} \]
cf. (6). The corresponding BRST-charge reads
\[ Q = 2(\partial \Pi - \partial' \Pi), -2(\Delta h + \partial \partial' h) + \pi_i p^i. \tag{19} \]

If we consider \( \tilde{\tau} \equiv -\partial \tau \) as reducible secondary constraints, the corresponding gauge symmetry will be reducible, cf. (9),
\[ \delta_{\epsilon} h^{ij} = \delta' (e^i + \partial \xi^{ij}) + \partial' (e^i + \partial \xi^{ij}), \quad \delta_{\epsilon} h = -2\partial \epsilon, \]
\[ \delta \lambda = \epsilon + \partial \epsilon, \quad \delta \epsilon = -\partial \delta, \tag{20} \]
\[ \tilde{\tau} = -\partial \tau \]
where \( \epsilon^{ij}, \tilde{\epsilon}^{ij} \) are totally antisymmetric tensors, and \( \epsilon^{ijk} \equiv \epsilon^{jik} \) is the Levi–Civita symbol. The corresponding BRST charge (10) has the form
\[ \tilde{Q} = 2(\partial \Pi - \partial' \Pi) - 2(\Delta h + \partial \partial' h) - \pi_i p^i. \tag{21} \]

The theory (16) admits two different BRST complexes defined by (19) and (21). The irreducible secondary constraint \( \tau \) (17) is BRST-exact with respect to \( Q \) (19), \( \tau = \{\tilde{P}, \tilde{Q}\} \), so the modular parameter \( \Lambda \) is a trivial quantity that corresponds to the case of fixed field asymptotics. At the same time, \( \tau \) is BRST-closed, but not BRST-exact with respect to \( Q \) (21),
\[ \{\tau, \tilde{Q}\} = 0, \quad \tau = \{\psi, \tilde{Q}\}, \quad \psi = \tilde{P}, \psi', \tag{22} \]
as \( \tau = \tau' \psi' \), where \( \psi' \) is a non-local operator inverse to partial derivative. So, \( \Lambda \) is a physical quantity, and various field asymptotics are admissible.

CONCLUSIONS

We demonstrated that the theories with unfree gauge symmetry can be self-consistently described by two different BRST complexes. One of them corresponds to the unfree gauge symmetry as such, while another one is connected with the alternative description of the same symmetry with reducible gauge transformations with unrestricted gauge parameters. These two descriptions are related, but not equivalent. The choice between them depends on the physical problem setting. For the first BRST complex, the global conserved quantities are BRST-exact, and for the second one, they are BRST-closed, but non-trivial.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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