Robust Discrete-Time Pontryagin Maximum Principle on Matrix Lie Groups

Anant A. Joshi, Debasish Chatterjee, Ravi N. Banavar

Abstract—This article considers a discrete-time robust optimal control problem on matrix Lie groups. The underlying system is assumed to be perturbed by exogenous unmeasured bounded disturbances, and the control problem is posed as a min-max optimal control wherein the disturbance is the adversary and tries to maximise a cost that the control tries to minimise. Assuming the existence of a saddle point in the problem, we present a version of the Pontryagin maximum principle (PMP) that encapsulates first-order necessary conditions that the optimal control and disturbance trajectories must satisfy. This PMP features a saddle point condition on the Hamiltonian and a set of backward difference equations for the adjoint dynamics. We also present a special case of our result on Euclidean spaces.

I. INTRODUCTION

Optimal control is a cornerstone in the theory of modern control [1], [19]. The Pontryagin maximum principle [26] (henceforth referred to as the PMP) constitutes an integral part of optimal control theory, and it provides first-order necessary conditions for optimality. These necessary conditions make the process of finding an optimal control significantly simpler by narrowing the set of admissible controls into a subset of candidate optimisers, and they are used by numerical algorithms to search for an optimal control. It is, in fact, one of the most widely used tools in optimal control apart from the Hamilton-Jacobi-Bellman (HJB) approach. The PMP has traditionally been studied in a continuous-time setting, and while the discrete-time version [7] received less attention in the initial stages, there have recently been several investigations into the discrete time PMP, for instance, in the presence of constraints and in the geometric setting [17], [22], [23], [25].

The mathematical model of any engineering system is almost always an approximation of its actual behaviour, and consequently, it is essential to address the robustness aspect of control schemes. By this we mean the ability of the controller to give satisfactory performance under system modelling uncertainties or the presence of exogenous unmeasured disturbances. Our emphasis here will be on deterministic controllers formulated as min-max optimisation problems (also studied as games), in which the disturbance is the adversary and tries to maximise the cost which the control tries to minimise [2], [3], [9], [27].

PMPs for robust optimal control problems of the aforementioned type have been derived in [20] for continuous time systems with uncertainties, and in [6] for systems which have parametric uncertainties. For continuous time systems, there has been considerable work in the game theoretic framework on the min-max control for systems with disturbances, we pick the representative articles [4], [8] from that literature, since they demonstrate results which closely resemble the PMP. When [8] is specialized to our setting of a min-max problem, we arrive at a zero sum differential game which has a Nash equilibrium (a game theoretic analog of a saddle point). This gives rise to two distinct sub-problems, one of minimising the cost over allowable controls (and this problem is parametrised by the optimal disturbance) and the other of maximising the cost over allowable disturbances (which is parametrised by the optimal control). They proceed to apply the PMP on both sub-problems individually to arrive at two sets of two-point boundary value problems, each with their own Hamiltonians and covectors, each parametrized by the solution of the other. [4] takes this further by deriving a single maximum principle with a single Hamiltonian and single covector by employing the Isaacs equation [10] in an interesting fashion. Our approach here is derived from the ideas in these two articles, but we operate under a different context and regime that we now explain.

A broad class of aerospace and mechanical systems evolve on the class of smooth manifolds known as Lie groups. These state spaces lack a vector space structure which warrants new techniques to be developed in order to study these systems, and such techniques fall under the broad umbrella of geometric mechanics and control [1], [20]. Optimal control of such systems has received significant attention [1], [5]; in particular, PMPs for systems evolving on smooth manifolds has been studied both in continuous and discrete time [17], [22], [25] in considerable detail. More specifically, geometric discrete-time optimal control, designed specifically to respect the non-flat nature of the underlying state spaces, has been of great interest in the engineering community since the associated techniques eliminate problems that occur by using local parametrisation [14], [15], [24].

In [25], a discrete-time PMP for optimal control problems for systems evolving on matrix Lie groups was presented. That work however did not consider the performance of the controller under the effect of exogenous unmeasured disturbances. Our work here takes the problem a step further by incorporating the effect of bounded disturbances acting on the class of systems considered in [25] and by posing the optimal control problem as a min-max problem. Assuming the existence of a saddle point of the cost function, we provide first-order necessary conditions that the control and
disturbance satisfy for optimality are presented in the form of a modification of the PMP. We arrive at a saddle point condition on the Hamiltonian and backward difference equations for the covector (also termed adjoint) dynamics. We also present a specialised version of our result to Euclidean spaces for those interested in directly applying it to such systems. Our results are similar in spirit to [4] but different in three very significant aspects:

○ We consider a problem that evolves in discrete-time, for which it is not possible to derive the maximum principle from a min-max version of the Bellman’s equation. This is relevant, since in continuous-time the PMP can be derived using ideas from the HJB partial differential equation [19, Chapter 5], and similarly the modification of PMP in [4] can be derived using ideas from the Isaacs equation.

○ [4] hypothesizes that the Hamiltonian satisfies a saddle point condition before establishing their result. We do not do any such assumption, but instead the saddle point condition appears naturally in our development.

○ [4] assumes that the abnormal multiplier used in the Hamiltonian is non-zero at the outset, but we prove that it must always be non-zero in our setting.

The results closest to ours are in [6], but as noted earlier, [6] considers parametric uncertainty in the system. This paper is organized as follows. Section II contains preliminaries and the statement of the maximum principle for our case. Section III contains an aerospace example where our theory is applied. The proofs of the results stated are in [16].

II. PRELIMINARIES AND STATEMENT OF MAIN RESULT

A. Mathematical Preliminaries

We present some mathematical preliminaries in this subsection. \( \mathcal{W} \) will denote the set of non-negative integers. Given a vector space \( V \), let \( V^\ast \) denote its dual space, which is the set of all linear functionals (also termed covectors) on \( V \). Denote by \( \langle \cdot, \cdot \rangle : V^\ast \times V \to \mathbb{R} \) the duality pairing. Given a linear map between two vector spaces \( F : V_1 \to V_2 \), let \( F^\ast : (V_2)^\ast \to (V_1)^\ast \) denote the dual of \( F \) defined as \( \langle F^\ast(\eta), v \rangle = \langle \eta, F(v) \rangle \) for all \( \eta \in (V_2)^\ast, v \in V_1 \). The standard inner product on \( \mathbb{R}^n \) will also be denoted by \( \langle \cdot, \cdot \rangle \) since it is equivalent to the duality pairing on \( \mathbb{R}^n \). For a smooth map between two vector spaces \( F : V_1 \to V_2 \), for any \( x \in V_1 \), \( DF(x) \) will denote the derivative of \( F \) at \( x \) and \( D^2F(x) \) will denote the second derivative of \( F \) at \( x \). Given a third vector space \( V_3 \), for a smooth map \( V_1 \times V_2 \ni (x_1, x_2) \mapsto F(x_1, x_2) \in V_3 \), for any \( (\tilde{x}_1, \tilde{x}_2) \in V_1 \times V_2 \), \( D_{x_1}F(\tilde{x}_1, x_2) \) denotes the derivative of \( F(\cdot, x_2) \), evaluated at \( \tilde{x}_1 \) and \( D_{x_2}F(\tilde{x}_1, x_2) \) denotes the second derivative of \( F(\cdot, \cdot) \), evaluated at \( \tilde{x}_1 \). If two vector spaces \( V_1 \) and \( V_2 \) are isomorphic, it will be denoted by \( V_1 \cong V_2 \). The preceding material was from [12], [28].

Consider a smooth function \( F : \mathbb{R}^n \to \mathbb{R} \). Suppose that we desire to find argmin \( F(x) \) with \( x \in \Sigma \subset \mathbb{R}^n \). Then, as per [6, Theorem 6.1] \( x^* \in \Sigma \) is a minimum of \( F \) over \( \Sigma \) if and only if \( \Sigma \cap \Omega = \{x^*\} \), where \( \Omega := \{x \in \mathbb{R}^n \mid F(x) < F(x^*)\} \cup \{x^*\} \).

Definition 1 ([11, Definition 11.4]): Consider two arbitrary sets \( \mathcal{U} \) and \( \mathcal{D} \) and an arbitrary function \( F : \mathcal{U} \times \mathcal{D} \to \mathbb{R} \). \((u^*, d^*) \in \mathcal{U} \times \mathcal{D} \) is a saddle point of \( F \) if

\[
F(u^*, d) \leq F(u^*, d^*) \leq F(u, d^*) \quad \text{for all } u \in \mathcal{U}, d \in \mathcal{D}
\]

We consider a problem that evolves in discrete-time, for which it is not possible to derive the maximum principle from a min-max version of the Bellman’s equation. This is relevant, since in continuous-time the PMP can be derived using ideas from the HJB partial differential equation [19, Chapter 5], and similarly the modification of PMP in [4] can be derived using ideas from the Isaacs equation.

\[
\arg\min_{x} F(x)
\]

Given a third vector space \( \mathbb{R}^n \) and a vector \( d \), denote the set of non-negative integers. Given a smooth map \( F : M \to \mathbb{R} \) on these topics, we refer the reader to standard texts on differential geometry, for instance, [18, Chapter 3, 11]. For a smooth manifold \( M \), \( T_xM \) represents its tangent space at \( x \in M \) [18, Page 54]. Given two smooth manifolds \( M_1 \) and \( M_2 \), and a smooth map \( F : M_1 \to M_2 \), \( TF(x) : T_xM_1 \to T_{F(x)}M_2 \) will denote the tangent map of \( F \) at \( x \) (recall that the tangent map is the generalisation of the derivative in the context of smooth manifolds [18, Page 68]). The cotangent map of \( F \) at \( x \) (which is the dual of the tangent map [18, Page 284]) will be denoted as \( T^*F(x) : (T_xM_2)^\ast \to (T_{F(x)}M_2)^\ast \), and for any \( \eta \in (T_{F(x)}M_2)^\ast \), \( T^*F(x) \eta \) will denote its action on \( \eta \). Given a third smooth manifold \( M_3 \), let \( M_1 \times M_2 \ni (x_1, x_2) \mapsto F(x_1, x_2) \in M_3 \). Then \( T_{F(x)}F(x_1, x_2) \) denotes the tangent map of \( F(\cdot, \cdot) \), evaluated at \( x_1 \). The following notion will be used while defining covectors as tangent maps of real-valued functions. Given a smooth map \( F_1 : M_1 \to \mathbb{R} \), for any \( x \in M_1 \), \( \langle F_1(x), v \rangle \) for all \( v \in T_xM_1 \) and \( T_F(x) \) is a linear map. We can associate to \( T_F(x) \) a covector \( \eta \in (T_xM_1)^\ast \) such that \( \langle TF_1(x), v \rangle := \langle \eta, v \rangle = T F_1(x) \cdot v \) for all \( v \in T_x M_1 \). Therefore, given a smooth map
B. Problem Definition

Since our setting is that of a Lie group, let us quickly recall some preliminaries. Refer [13, Chapter 5.6] for more details on Lie groups. For our problem we will consider a $n$ dimensional matrix Lie group $G \subset \mathbb{R}^{M \times M}$ with identity element $I$ and Lie algebra $g \subset \mathbb{R}^{M \times M}$ [13, Chapter 5.1].

Let $\sigma : \mathbb{R}^n \to g$ be an isomorphism. The dual of $g$ is $g^* \subset \mathbb{R}^{M \times M}$ and $\langle \eta, v \rangle = \text{tr}(\eta^T v)$ for all $\eta \in g^*$, $v \in g$, where $\text{tr}(\cdot)$ denotes the trace, and $(\cdot)^T$ denotes the transpose.

Let $\Phi : G \times G \to G$ denote the group multiplication, which by definition, is matrix multiplication for matrix Lie groups, that is, $\Phi(g_1 g_2) = g_1 g_2$. Then $\Phi_g : G \to G$ for all $g \in G$ is a diffeomorphism, and $\Phi_{g_1}(g_2) = g_1 g_2$ [13, Chapter 5.2]. For any $g \in G$, the tangent space is parameterised as $T_g G = T \Phi_g(g)$. The tangent map of $\Phi$ simplifies to matrix multiplication as well, that is, given $g_1, g_2 \in G$ and $v \in g$, $T \Phi_{g_1} g_2 (T \Phi_{g_1} g_2 v) = g_2 v$ and therefore, $T \Phi_{g_1} g_2 (T \Phi_{g_1} g_2 v) = g_2 v$.

The exponential map is denoted as $\exp : g \to G$, which for matrix Lie groups is simply the matrix exponential [13, Chapter 5.4]. Given $g \in G$, the exp map can be used to define smooth curve on $G$ passing through $g$ with tangent vector $T_g \Phi g : v \to g \exp(vs) \in G$. This can be conveniently used to find tangent maps of real valued functions, say $f : G \to \mathbb{R}$ as $T \Phi f : g \to \frac{d}{ds} \big|_{s=0} f(g \exp(sv))$. The Adjoint action of $G$ on $g$ is [13, Definition 6.40]

$$G \times g \ni (g, v) \mapsto \text{Ad}_g v := \frac{d}{ds} \big|_{s=0} g \exp(sv) g^{-1} \in g$$

For any $g \in G$, let $\text{Ad}_g^*$ be the dual of $\text{Ad}_g$.

Remark 1: Although we restrict attention to matrix Lie groups here, our derivations proceed in full generality first and then are made specific to matrix Lie groups, thus making it applicable to any finite dimensional Lie group.

For any $n \in \mathbb{N}$, $[n] := \{0, 1, \ldots, n\}$. Let $N \in \mathbb{N}$ be fixed, and will be referred to as the control horizon throughout. Consider a discrete time control system evolving on $G \times \mathbb{R}^n$

$$g_{k+1} = g_k f_1(\omega_k, g_k) \quad \text{for all } k \in [N-1],$$

$$\omega_{k+1} = f_2(g_k, \omega_k, u_k, d_k) \quad \text{for all } k \in [N-1],$$

where

(Sys-i) $g_k \in G$ and $\omega_k \in \mathbb{R}^n$ (equivalently, $\sigma(\omega_k) \in g$) for all $k \in [N]$ are the states,

(Sys-ii) $f_1 : G \times \mathbb{R}^n \to G$ and $f_2 : G \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ are smooth and respectively represent the kinematics and dynamics of the system,

(Sys-iii) $u_k \in U_k \subset \mathbb{R}^m$ and $d_k \in D_k \subset \mathbb{R}^p$ respectively are the control and unmeasured external disturbance for $k \in [N-1]$.

Remark 2: Physical systems (whether on Euclidean spaces or manifolds) are typically modelled as evolving in continuous time as differential equations. However, for implementation purposes one finds it suitable to represent them as discrete-time systems using. Discretising systems that evolve on manifolds is a non-trivial task since the discretisation should respect the manifold structure of the configuration space of the system. Discrete mechanics finds applications here, see, e.g., [21]. We will suppose that $f_1$ has been obtained using such an appropriate discretisation technique.

Our aim is to obtain necessary conditions satisfied by the optimiser of the following optimal control problem:

$$\min_{u, d} \max_{g, \omega, d} J(g, \omega, u, d) := \sum_{k=0}^{N-1} c_k(g_k, \omega_k, u_k, d_k)$$

$$+ c_N(g_N, \omega_N) \quad (2a)$$

s.t. system (1),

$$u_k \in U_k, d_k \in D_k \quad \text{for all } k \in [N-1], \quad (2c)$$

$$g_0 = \bar{g}_0, \omega_0 = \bar{\omega}_0, \quad (2d)$$

where the notation goes as,

(N-i) $G \times \mathbb{R}^n \times \mathbb{R}^m \ni (g, \omega, u) \to c_k(g, \omega, u) \in \mathbb{R}$, and $G \times \mathbb{R}^n \ni (g, \omega) \to c_N(g, \omega) \in \mathbb{R}$ are suitable smooth functions for all $k \in [N-1]$.

(N-ii) $g := (g_0, g_1, \ldots, g_N) \in G \times G \times \cdots \times G$ and $\omega := (\omega_0, \omega_1, \ldots, \omega_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ are the state sequences,

(N-iii) $u := (u_0, u_1, \ldots, u_{N-1}) \in \mathbb{R}^m$ is the control sequence,

(N-iv) $d := (d_0, d_1, \ldots, d_{N-1}) \in \mathbb{R}^p$ is the disturbance sequence.

Assumption 1: The following assumptions (taken from [25, Assumption 1]) are required to transfer the optimisation problem into a Euclidean setting and to obtain necessary conditions that the optimiser satisfies.

- There exists an open set $O \subset g$ such that exp map restricted to $O$, i.e. $\exp : O \to \exp(O)$ is a diffeomorphism and the discretisation is such that $f_1 \in \exp(O)$.

- $U_k$ and $D_k$ are compact and convex sets for all $k \in [N]$.

We assume that the cost in (2a) admits a saddle point.

Assumption 2: $(g^*, \omega^*, u^*, d^*)$ is a saddle point for the optimization problem (2).

Then defining open sets $\mathcal{U} \subset \mathcal{U}_0 \times \mathcal{U}_1 \times \cdots \times \mathcal{U}_{N-1}$ and $\mathcal{D} \subset \mathcal{D}_0 \times \mathcal{D}_1 \times \cdots \times \mathcal{D}_{N-1}$, with $u^* \in \mathcal{U}$ and $d^* \in \mathcal{D}$,

$$J(g^*, \omega^*, u^*, d^*) \leq J(g, \omega, u, d^*) \quad \text{for all } u \in \mathcal{U},$$

$$J(g, \omega, u^*, d) \leq J(g^*, \omega^*, u^*, d^*) \quad \text{for all } d \in \mathcal{D}.$$
C. Main Result

Theorem 1: For the optimization problem in (2) let the optimal control and disturbance sequence be \( u^\star \) and \( d^\star \) respectively corresponding to which the state trajectory is \( g^\star \) and \( \omega^\star \). Define the Hamiltonian as

\[
[N-1] \times g^\star \times (\mathbb{R}^n)^* \times \mathbb{G} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \ni (k, \xi, g, \omega, u, d) \mapsto \mathcal{H}(k, \xi, g, \omega, u, d) := -c_k(\xi, g, \omega, u, d) + \langle \xi, f_1(g, \omega) \rangle + \langle \xi, f_2(g, \omega, u, d) \rangle.
\]

For all \( k \in [N-1] \), there exist covectors \( \xi^k \in \mathbb{g}^* \), \( \xi^k \in \mathbb{R}^n \), \( \gamma^k \in \mathbb{R}^n \), \( \gamma^k \in \mathbb{R}^n \), \( \gamma^k \in \mathbb{R}^n \), and \( \gamma^k \in \mathbb{R}^n \), and define \( \gamma^*_k := (\xi^k, \xi^k, \omega^k, u^k, d^k), \)

\[
\rho^k := T^\star \left( \text{exp}^{-1} \circ \Phi((g^*_k)^{-1}g_k) \right)(I)(\zeta^k),
\]

which satisfy the following necessary conditions

(H-i) Optimal state dynamics (for all \( k \in [N] \)):

\[
g^*_{k+1} = g_k^* \exp(D\zeta \mathcal{H}(\gamma^*_k)), \\
\omega^*_{k+1} = D\zeta \mathcal{H}(\gamma^*_k);
\]

(H-ii) Adjoint equations (for all \( k \in [N-1] \)):

\[
\xi^{k-1} = D\zeta \mathcal{H}(\gamma^*_k), \\
\rho^{k-1} = \text{Ad}_{\phi_k^*}^{-1}(D\zeta \mathcal{H}(\gamma^*_k)) \rho^k + T^* \phi_k^*(I)(T\theta \mathcal{H}(\gamma^*_k));
\]

(H-iii) Transversality relations:

\[
\xi^N = -D\omega c_N(g^*_N, \omega^*_N), \\
\rho^N = -T^* \phi_k^*(I)(T\theta c_N(g^*_N, \omega^*_N));
\]

(H-iv) Hamiltonian “saddle point” condition (for all \( k \in [N-1] \)):

\[
\langle D_a \mathcal{H}(\gamma^*_{k-1}), u_{k-1} \rangle \leq 0 \text{ for all } u_{k-1} \in U_{k-1}, \\
\langle D_d \mathcal{H}(\gamma^*_{k-1}), d_{k-1} \rangle \geq 0 \text{ for all } d_{k-1} \in D_{k-1}.
\]

Proof: The Proof is in [16, Section 3].

Remark 3: A comment on the notation used here, which is heavy in differential geometric jargon:

- Condition (H-iv) does not strictly denote a saddle point in the most general case. However, under suitable assumptions on \( \mathcal{H} \) it does, for instance, if the Hamiltonian is convex in \( u \) and concave in \( d \).
- The derivative of \( \mathcal{H} \) with respect to \( \omega, \xi, \zeta \) are well understood since these entities lie in vector spaces.
- Note that \( \rho, \zeta \) can be written as elements of \( \mathbb{R}^{M \times M} \).
- Since a co-vector can be easily visualised by its action on a vector, we will perform some calculations to make this apparent. To this end, let \( v \in \mathbb{g} \) be arbitrary and let \( A := \text{exp}(-D\zeta \mathcal{H}(\gamma^*_k)) \). To understand the relation between \( \rho^{k-1} \) and \( \rho^k \):

\[
\langle \rho^{k-1}, v \rangle = \langle \text{Ad}_x^* \rho^k + T^* \phi_k^*(I)(T\theta \mathcal{H}(\gamma^*_k)), v \rangle = \langle \rho^k, \text{Ad}_x A \rangle + \langle T\theta \mathcal{H}(\gamma^*_k), T\phi_k^*(I) \cdot v \rangle = \text{tr} \left( (\rho^k)^T (A^*A) \right) + \langle T\theta \mathcal{H}(\gamma^*_k), g_k^* \rangle v = \text{tr} \left( (\rho^k)^T (A^*A) \right) + \left. \frac{d}{ds} \right|_{s=0} \mathcal{H}(g_k^* \text{exp}(uv))
\]

To understand the relation between \( \rho^k \) and \( \zeta^k \):

\[
\langle \rho^k, v \rangle = \left( T^* \left( \text{exp}^{-1} \circ \Phi((g^*_k)^{-1}g_k) \right)(I)(\zeta^k), v \right) = \langle \zeta^k, T^* \left( \text{exp}^{-1} \circ \Phi((g^*_k)^{-1}g_k) \right)(I) \cdot v \rangle = \langle \zeta^k, \left. \frac{d}{ds} \right|_{s=0} \text{exp}^{-1} \left( ((g^*_k)^{-1}g_k) \text{exp}(vs) \right) \rangle = \langle \zeta^k, \left. \frac{d}{ds} \right|_{s=0} \text{exp}^{-1} \left( ((g^*_k)^{-1}g_k) \text{exp}(vs) \right) \rangle
\]

D. Specialisation to Euclidean Spaces

Let the control system (1a),(1b) evolve on \( \mathbb{R}^n \), i.e. \( \mathbb{G} = \mathbb{R}^n \) (although we require \( \mathbb{G} \) to be a matrix Lie group, as noted in Remark 1, the part of our presentation made in full generality holds for any finite dimensional Lie group). We will introduce new notation for the cost and dynamics to avoid clashing with previous notation. Since the system evolves on a Euclidean space in entirety, we club all state variables into \( \omega \in \mathbb{R}^n \) and kinematics and dynamics into \( f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n \). For complete clarity, we shall restate the specific form of the control system

\[
\omega_{k+1} = f(\omega_k, u_k, d_k) \quad \text{for all } k \in [N-1], \quad (3)
\]

and optimal control problem

\[
\min_u \max_d \quad J(\omega, u, d) := \sum_{k=0}^{N-1} c_k(\omega_k, u_k, d_k) + c_N(\omega_N) \quad (4a)
\]

s.t. \( \text{system (3)}, \)

\[
u_k \in U_k, d_k \in D_k \text{ for all } k \in [N-1], \quad (4c)
\]

\[
\omega_0 = \bar{\omega}_0, \quad (4d)
\]

where \( \mathbb{R}^n \times \mathbb{R}^m \ni (\omega, u) \mapsto c_k(\omega, u) \in \mathbb{R} \), and \( \mathbb{R}^n \ni \omega \mapsto c_N(\omega) \in \mathbb{R} \) are suitable smooth functions for all \( k \in [N-1] \), and the remaining notation is same as List (Sys-i),(Sys-iii) and List (N-ii),(N-iii),(N-iv) (accommodating for the change of Lie group \( \mathbb{G} \) to \( \mathbb{R}^n \)).

Corollary 1: For the optimisation problem in (4) let the optimal control and disturbance sequence be \( u^* \) and \( d^* \) respectively corresponding to which the state trajectory is \( g^* \) and \( \omega^* \). Define the Hamiltonian as

\[
[N-1] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \ni (k, \xi, \omega, u, d) \mapsto \mathcal{H}(k, \xi, \omega, u, d) := -c_k(\omega, u) + \langle \xi, f(\omega, u, d) \rangle.
\]

For \( k \in [N-1] \), there exist covectors \( \xi^k \in \mathbb{R}^n \), and define \( \gamma^k := (\xi^k, \omega^k, u^k, d^k), \) which satisfy the following necessary conditions

- Optimal state dynamics (for all \( k \in [N] \)):

\[
\omega^*_{k+1} = D\zeta \mathcal{H}(\gamma^*_k);
\]
• Adjoint equations (for all $k \in [N - 1]$):
  \[ \xi^{k-1} = D_\omega H(\gamma^*_k); \]

• Transversality relations:
  \[ \xi^{N-1} = -D_\omega c_N(\omega^*_N); \]

• Hamiltonian “saddle point condition” (for all $k \in [N - 1]$):
  \[ \langle D_u H(\gamma^*_{k-1}), u_{k-1} \rangle \leq 0 \text{ for all } u^*_{k-1} + u_{k-1} \in U_{k-1}, \]
  \[ \langle D_d H(\gamma^*_{k-1}), d_{k-1} \rangle \geq 0 \text{ for all } d^*_{k-1} + d_{k-1} \in D_{k-1}. \]

Proof: Direct application of Theorem 1 on optimisation problem (4) (noting en route the change of $G$ to $\mathbb{R}^n$).

Remark 4: We make some observations that simplify computations when $G = \mathbb{R}^n$. The group multiplication is then vector space addition on $\mathbb{R}^n$ i.e. $\Phi(\omega_1, \omega_2) = \omega_1 + \omega_2$ for all $\omega_1, \omega_2 \in \mathbb{R}^n$, the exp map is the identity map [20, Chapter 9, Page 274], and it is clear from the definition of the adjoint action that it simplifies to the identity transformation as well. Tangent maps simplify to derivatives, and for any $\omega \in \mathbb{R}^n$, $T\mathbb{R}^n(\omega) \cong \mathbb{R}^n$ (in particular, $g \cong \mathbb{R}^n$) and $(T\mathbb{R}^n(\omega))^* \cong \mathbb{R}^n$ (in particular, $g^* \cong \mathbb{R}^n$).

III. EXAMPLE

To demonstrate the application of Theorem 1, we use a simple example of single axis rotation of a spacecraft under bounded disturbance. It is essentially a continuation of the example presented in [25], to keep the spirit of extension of that work intact. Single axis rotation of a spacecraft evolves on $SO(2)$. This is one of the easiest and most intuitive examples of a Lie group, which still preserves the structure intrinsic to the problem. Consider the Lie group $SO(2)$ and its Lie algebra $so(2)$,

\[ SO(2) := \{ g \in \mathbb{R}^{2 \times 2} \mid g^T g = g g^T = I, \det(g) = 1 \} \]

Define an isomorphism $\sigma$, its inverse vex, and the exp map

\[ SO(2) \ni x \mapsto \sigma(x) := \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \in so(2), \text{ vex}(g) := \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} = x \]

By virtue of the Riesz representation theorem, we identify $so(2)^*$ with $so(2)$ and $^*$ with vex. For this Section, $\langle \eta, v \rangle = \frac{1}{2} \text{tr}(\eta^T v)$ for all $\eta \in (so(2))^*$, $v \in so(2)$.

We model single axis rotation as

\[
\begin{align*}
  g_{k+1} &= g_k q_1(g_k, \omega_k) \quad \text{for all } k \in [N - 1], \quad (5a) \\
  \omega_{k+1} &= q_2(g_k, \omega_k, u_k, d_k) \quad \text{for all } k \in [N - 1], \quad (5b)
\end{align*}
\]

where

\[ g_k \in SO(2), \quad \omega_k \in \mathbb{R} \quad \text{for all } k \in [N], \]

\[ \mathcal{U} := [-u_c, u_c], \quad \mathcal{D} := [-d_c, d_c] \quad u_c, d_c, h \in \mathbb{R}_{\geq 0} \text{ are fixed, and initial conditions } g_0 = \bar{g}_0, \omega_0 = \bar{\omega}_0 \text{ are known}, \]

\[ SO(2) \times \mathbb{R} \ni (g, \omega) \mapsto q_1(g, \omega) := \left( \begin{array}{c} \sqrt{1 - h^2\omega^2} \\
\frac{-h\omega}{\sqrt{1 - h^2\omega^2}} \end{array} \right) \in SO(2), \]

\[ SO(2) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ni (g, \omega, u, d) \mapsto q_2(g, \omega, u, d) := \omega + h(u + d). \]

The optimal control problem is

\[
\begin{align*}
\min_{u,d} \max_{g,\omega} & \quad \frac{1}{2} \left( \lambda^2 u_k^2 + \sum_{k=0}^{N-1} \lambda^2 u_k^2 + \lambda^2 \omega_k^2 - \mu^2 d_k^2 \right) \quad (6a) \\
\text{s.t.} & \quad u_k \in U_k, \quad d_k \in D_k \quad \text{for all } k \in [N - 1], \quad (6c) \\
& \quad g_0 = \bar{g}_0, \omega_0 = \bar{\omega}_0. \quad (6d)
\end{align*}
\]

Remark 5: The modelling interpretation of cost function is as follows. $\lambda^2 u_k^2$ minimises control effort and $\lambda^2 \omega_k^2$ maintain low angular velocity. $-\mu^2 d_k^2$ penalises the effort exerted by the external disturbance, the justification for which is that Nature also tries to exert minimum effort.

Remark 6: In (6), the problem is essentially a linear quadratic dynamic game (which is a robust linear system with quadratic cost). This is because $(g_k)_{k=1}^N$ doesn’t appear in (6a), hence the constraint imposed by (5a) can be trivially satisfied, and $u^*$ and $d^*$ respectively, corresponding to which the state trajectory is $g^*$ and $\omega^*$. Applying Theorem 1, we define the Hamiltonian

\[ (so(2))^* \times \mathbb{R}^* \times SO(2) \times \mathbb{R} \times \mathbb{R} \ni (\zeta_\sigma, \xi, g, \omega, u, d) \mapsto H(\zeta_\sigma, \xi, g, \omega, u, d) := -\left( \frac{\lambda^2 u^2 + \lambda^2 \omega^2 - \mu^2 d^2}{2} \right) + \langle \zeta_\sigma, \exp(-q_1(g, \omega)) \rangle + \xi q_2(g, \omega, u, d) \]

\[ = -\left( \frac{\lambda^2 u^2 + \lambda^2 \omega^2 - \mu^2 d^2}{2} \right) + \zeta \sin^{-1}(h\omega) + \xi (\omega + h(u + d)) \]

where $\zeta := \sigma^*(\zeta_\sigma) = \text{vex}(\zeta_\sigma)$. The optimal state dynamics and adjoint dynamics are obtained as

\[
\begin{align*}
  g_{k+1} &= g_k q_1(g_k, \omega_k); \quad \omega_{k+1} = q_2(g_k, \omega_k, u_k, d_k); \quad (5a) \\
  \zeta^k &= Ad_{\exp(-D_\omega H(\gamma^*_k))} \zeta^k; \quad (5b) \\
  = q_1(g^*_k, \omega^*_k) \zeta^k \langle q_1(g_k, \omega_k) \rangle^T = \zeta^k; \quad (5c) \\
  \xi^k &= D_\omega H(\gamma^*_k) = \frac{h^2 g^k}{\sqrt{1 - (h\omega^*_k)^2}} + \xi^k = -\lambda^2 \omega^*_k; \quad (5d) \\
  \zeta^N &= 0; \quad \xi^{N-1} = -D_\omega c_N(g_N, \omega^*_N) = -\lambda^2 \omega^*_N.
\end{align*}
\]
The adjoint dynamics simplify to the following
\[
\zeta^k = 0, \quad \xi^k = -\sum_{i=k+1}^{N} \Lambda^2 \omega^\star_i \text{ for all } k \in [N - 1].
\]
Since the Hamiltonian is convex in \( u \) and concave in \( d \), the optimal control and disturbance are easily derived as
\[
\begin{align*}
&u^*_k = \arg\max_{w \in [-u_c, u_c]} \mathcal{H}(\zeta^k, \xi^k, g^\star_k, \omega^\star_k, u_k, d_k, w), \\
&d^*_k = \arg\min_{w \in [-d_c, d_c]} \mathcal{H}(\zeta^k, \xi^k, g^\star_k, \omega^\star_k, u_k, d_k, w),
\end{align*}
\]
If we let the control and disturbance be unconstrained, that is, \( \mathcal{U} = \mathbb{R} \) and \( \mathcal{D} = \mathbb{R} \), then the solution of the resulting problem is well known [2, Chapter 3.2.1], and it satisfies the necessary conditions given by Theorem 1. For a proof of this claim, see [16, Appendix D]. For more involved example, see [16, Section 4].

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