ON EXISTENCE OF PI-EXPONENTS OF CODIMENSION GROWTH

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Abstract. We construct a family of examples of non-associative algebras \( \{ R_\alpha \mid 1 < \alpha \in \mathbb{R} \} \) such that \( \exp(R_\alpha) = 1, \exp(R_\alpha) = \alpha \). In particular, it follows that for any \( R_\alpha \), an ordinary PI-exponent of codimension growth does not exist.

1. Introduction

We consider algebras over a field \( F \) of characteristic zero. Given an algebra \( A \) over \( F \), one can associate the sequence of non-negative integers \( \{ c_n(A) \} \), \( n = 1, 2, ... \), which is an important numerical characteristic of polynomial identities of \( A \). Study of asymptotic behavior of \( \{ c_n(A) \} \) for associative algebras was started in the beginning of 70’s (see, for example, [23], [15], [16]) and was continued during the subsequent decades (see, for example, [14], [6], [10], [3], [4], [5], [24] and also the bibliography in [11]). Later, similar numerical characteristics were considered for Lie algebras [17], [25], [18] and other non-associative algebras: Lie superalgebras [28] and their generalizations [22], Leibniz algebras [21], Jordan and alternative algebras [12], [9], Poisson and Novikov algebras [20], [8], etc.

For a wide class of algebras, the sequence \( \{ c_n(A) \} \) is bounded by exponential functions \( a^n \). This class contains all associative PI-algebras [23], all finite dimensional algebras [2], Kac-Moody algebras [26], infinite dimensional simple Lie algebras of Cartan type [18], and many others. Clearly, the inequality \( c_n(A) \leq a^n \) implies an existence of upper and lower limits

\[
\limsup_{n \to \infty} \sqrt[n]{c_n(A)} = \exp(A), \quad \liminf_{n \to \infty} \sqrt[n]{c_n(A)} = \underline{\exp(A)}
\]

called upper and lower PI-exponents of \( A \), respectively. If an ordinary limit of \( \sqrt[n]{c_n(A)} \) exists, that is, if \( \exp(A) = \underline{\exp(A)} \), it is called (an ordinary) PI-exponent of \( A \).

One of the main problems of the theory of numerical invariants of polynomial identities is the problem of existence of PI-exponent. At the end of 80’s Amitsur
conjectured that for any associative algebra with a non-trivial polynomial identity PI-exponent exists and it is a non-negative integer. Amitsur’s conjecture was confirmed in [10].

In Lie case existence and integrality of PI-exponent were proved for all finite dimensional algebras [27] and for some classes of infinite dimensional algebras (see, for example, [19]).

Up to now there was no example of an algebra $A$ with $\exp(A) \neq \exp(A)$. The main result of our paper is the following theorem.

**Theorem 1.1.** For any real number $\alpha > 1$ there exists an algebra $A$ such that $\exp(A) = 1$ while $\exp(A) = \alpha$.

All details about polynomial identities and their numerical characteristics one can find in [1], [7], [11].

2. Main definitions and constructions

Let $F$ be a field of characteristic zero and let $A$ be an algebra over $F$. Denote by $F\{X\}$ the absolutely free algebra over $F$ with the countable set of generators $X = \{x_1, x_2, \ldots\}$. The set $Id(A)$ of all identities of $A$ forms an ideal of $F\{X\}$. Consider the subspace $P_n \subset F\{X\}$ of all multilinear polynomials on $x_1, \ldots, x_n$. Then $P_n \cap Id(A)$ consists of all multilinear identities of $A$ of degree $n$. It is well-known that the family of subspaces $P_n \cap Id(A)$, $n = 1, 2, \ldots$, completely define all ideal $Id(A)$ in the case $\text{char } F = 0$.

Denote by

$$c_n(A) = \dim \frac{P_n}{P_n \cap Id(A)}.$$ 

The non-negative integer $c_n(A)$ is called $n^{th}$ codimension of $A$. The sequence $\{c_n(A)\}$ is one of the most important numerical characteristics of polynomial identities of $A$.

For proving our main result, we need some intermediate constructions and results. First, given an integer $T \geq 2$, we define an algebra $B_T$ by its basis

$$\{a, b_i, z_i^1, \ldots, z_i^T | i = 1, 2, \ldots\} \quad (2.1)$$

and by the multiplication table

$$z_i^1 a = z_i^2, z_i^2 a = z_i^3, \ldots, z_i^{T-1} a = z_i^T, z_i^T b_i = z_i^{T+1}, \quad i = 1, 2, \ldots.$$ 

We suppose that all other products of basis element are equal to zero. It is easy to see that $B_T$ is left nilpotent of step 2 algebra, that is, $x_1(x_2x_3) \equiv 0$ (2.2) is an identity of $B_T$. From (2.2), it follows that only left-normed products of basis elements may be non-zero. Therefore we will omit brackets in left-normed products of elements of $B_T$. That is we will write $y_1y_2y_3 = (y_1y_2)y_3$ and $y_1 \cdots y_ky_{k+1} = (y_1 \cdots y_k)y_{k+1}$ if $k \geq 3$.

First, we estimate codimension growth of $B_T$.

**Lemma 2.1.** Let $n \leq T$. Then $c_n(B_T) \leq 2n^3$.

**Proof.** Let $f = f(x_1, \ldots, x_n)$ be a multilinear polynomial on $x_1, \ldots, x_n$. Consider the following evaluations of the set of indeterminates $X$ in $B_T$: 

Since the total number of elements (2.5) is equal to $k$ modulo the ideal $\mathcal{I}_d$ where $S$ and, for example, $\alpha$ prove of lemma is completed.

Consider monomials

Proof. 

Any multilinear identity Lemma 2.3. $B$ algebra

Hence (2.3) and (2.4) imply the inequality

Note that $\varphi^k_{ij}(f) = 0$ if and only if $\varphi^k_{ij}(f) = 0$, that is, the kernels of all $\varphi^k_{ij}$, $i = 1, 2, \ldots$, coincide.

Since $\varphi^k_{ij}(x_{i_1} \cdots x_{i_n}) = z_{i_1+n-1}^1$ as soon as $i_1 = k, j + n - 1 \leq T$ while $\varphi^k_{ij}(x_{i_1} \cdots x_{i_n}) = 0$ in all other cases, then

$$\dim \text{Im} \varphi^k_{ij} = \text{codim}_n \ker \varphi^k_{ij} \leq 1. \quad (2.3)$$

Similarly, $\ker \varphi^k_{ij} = \ker \varphi^k_{ij}$ for all $j, k, r \geq 1$ and

$$\text{Im} \varphi^k_{ij} = \text{span} \left\{ \frac{z_1^1 a \cdots a b_1 a \cdots a}{p} \mid p + q = n - 2, j = p = T \right\}$$

$$\text{Im} = \text{span}(z_{q+1}^2).$$

Hence

$$\dim \text{Im} \varphi^k_{ij} = \text{codim}_n \ker \varphi^k_{ij} \leq 1. \quad (2.4)$$

Note that $f \in \text{Id}(B_T)$ if and only if $f \in K_1 \cap K_2$, where

$$K_1 = \bigcap_{j, k} \ker \varphi^k_{ij}, \quad K_2 = \bigcap_{j, k, r} \ker \varphi^{kr}_{ij}.$$ 

Note also that

$$\ker \varphi^k_{1,1} = \ker \varphi^k_{1,2} = \cdots = \ker \varphi^k_{1,T-n+1}.$$ 

Hence (2.3) and (2.4) imply the inequality

$$c_n(B_T) = \text{codim}_n K_1 \cap K_2 \leq n^3 + n^2 \leq 2n^3$$

provided that $n \leq T$, and we complete the proof. \qed

Lemma 2.2. Let $n = kT + 1$. Then $c_n(B_T) \geq k! = \frac{n!}{T!}!$.

Proof. Consider monomials

$$f_\sigma = x_0 x_1 \cdots x_{T-1} y_{\sigma(1)} x_T \cdots x_{2T-2} y_{\sigma(2)} \cdots y_{\sigma(k)}, \quad \sigma \in S_k,$$

where $S_k$ is the permutation group, and prove that they are linearly independent modulo the ideal $\text{Id}(B_T)$ of identities of $B_T$. Suppose that

$$h = \sum_{\sigma \in S_k} \alpha_{\sigma} f_\sigma \in \text{Id}(B_T) \quad (2.6)$$

and, for example, $\alpha_e \neq 0$, where $e$ is the unit of $S_k$. Then the evaluation

$$\varphi(x_0) = z_1^1, \varphi(y_1) = b_1, \ldots, \varphi(y_k) = b_k, \varphi(x_q) = a, \quad q = 1, \ldots, k(T - 1),$$

maps $f_\sigma$ to $z_{k+1}^1$ while $\varphi(f_\sigma) = 0$ for all $\sigma \neq e$. Hence $\varphi(h) = 0$, a contradiction. This means that $\alpha_e = 0$ in (2.6). Similarly, all other $\alpha_{\sigma}$ in (2.6) are equal to zero. Since the total number of elements (2.5) is equal to $k!$ and all $f_\sigma$ lie in $P_{kT+1}$, the proof of lemma is completed. \qed

Now we compare identities of small degree of algebras $B_T$ with distinct $T$.

Lemma 2.3. Any multilinear identity $f = f(x_1, \ldots, x_n)$ of degree $n \leq T$ of the algebra $B_T$ is an identity of $B_{T+1}$.
Proof. Let \( n \leq T \). It is sufficient to prove that if \( f = f(x_1, \ldots, x_n) \in P_n \) is not an identity of \( B_{T+1} \) then \( f \) is not an identity of \( B_T \). Let

\[
h = \sum_{\sigma \in S_n} \alpha_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(n)}
\]

and let \( \varphi : X \to B_{T+1} \) be an evaluation such that \( \varphi(x_1), \ldots, \varphi(x_n) \) are basis elements of \( B_{T+1} \) and \( \varphi(f) \neq 0 \). Then exactly one of \( x_1, \ldots, x_n \) should be replaced by some \( z_j^i \) while all remaining \( x_k \) should be replaced by \( a \) or \( b_m, m \geq 1 \). We may assume that \( \varphi(x_1) = z_j^i \).

First let \( \varphi(x_2) = \cdots = \varphi(x_n) = a \). Then

\[
\varphi(f) = f(z_j^i, a, \ldots, a) = \lambda z_{j+n-1}^i,
\]

where

\[
\lambda = \sum_{\sigma \in S_n \atop \sigma(1) = 1} \alpha_{\sigma}.
\]

Since \( \varphi(f) \neq 0 \) then \( \lambda \neq 0 \) and \( j + n - 1 \leq T + 1 \). If \( j + n - 1 \leq T \) then \( \varphi(f) = \lambda z_{j+n-1}^i \neq 0 \) in \( B_T \) for similar evaluation \( \varphi : X \to B_T \). If \( j + n - 1 = T + 1 \) then \( j \geq 2 \) since \( n \leq T \). Then

\[
f(z_{j-1}^i, a, \ldots, a) = \lambda z_{j+n-2}^i \neq 0
\]
in \( B_T \).

Now let \( f(z_j^i, a, \ldots, a) = 0 \) in \( B_{T+1} \). Then there exists an evaluation \( \varphi : X \to B_{T+1} \) such that \( \varphi(x_1) = z_j^i, \varphi(x_k) = b_i \) for some \( 2 \leq k \leq n \), \( \varphi(x_r) = a \) if \( r \neq 1, k \) and \( \varphi(f) \neq 0 \). As before, we can assume that \( k = 2 \). Then

\[
\varphi(f) = f(z_j^i, b_i, a, \ldots, a) = \left( \sum_{\sigma \in S_n \atop \sigma(1) = 1 \atop \sigma(j+p+1) = 2} \alpha_{\sigma} \right) z_j^i a \cdots b_i a \cdots a = \lambda z_{q+1}^{i+1}
\]
in \( B_{T+1} \) where

\[
\lambda = \left( \sum_{\sigma \in S_n \atop \sigma(1) = 1 \atop \sigma(j+p+1) = 2} \alpha_{\sigma} \right) \neq 0.
\]

Moreover,

\[
p + q = n - 2 \quad \text{and} \quad j + p = T + 1. \tag{2.7}
\]

From (2.7), it follows that \( j \geq 2 \) and \( q + 1 \leq T \). Hence

\[
f(z_{j-1}^i, b_i, a, \ldots, a) = \lambda z_{q+1}^{i+1}
\]
in \( B_T \) and \( f \) is not an identity of \( B_T \).

\[\square\]

3. Main result

Now we are ready to prove Theorem 1.1.

Proof. Fix a real number \( \alpha > 1 \). Denote by \( R_N \) the quotient algebra

\[
R_N = F[Y]_0/(Y^{N+1}),
\]

where \( Y \) is a non-commuting variable.
where $F[Y]_0$ is the ring of polynomials without free term and $(Y^{N+1})$ is its ideal generated by $Y^{N+1}$. Then $R_N^N \neq 0, R_N^{N+1} = 0$. Denote also $B(T, N) = B_T \otimes R_N$. We will construct an algebra $A$ with $\exp(A) = 1, \exp(A) = \alpha$ as a direct sum

$$A = B(T_1, N_1) \oplus B(T_2, N_2) \oplus \cdots. \tag{3.1}$$

The sequence $T_1 < N_1 < T_2 < N_2 < \ldots$ we will choose during the proof.

First note that if $T_1 < n \leq N_1$ then multilinear identities of $A$ of degree $n$ coincide with identities of

$$B(T_1, N_1) \oplus B(T_{i+1}, N_{i+1})$$

by Lemma 2.3. In particular,

$$c_n(B_{T_1}) + c_n(B_{T_{i+1}}) \geq c_n(B_{T_1}) \tag{3.2}$$

since $n \leq N_i$. In the case $N_i < n \leq T_{i+1}$, we have

$$P_n \cap \text{Id}(A) = P_n \cap \text{Id}(B(T_{i+1}, N_{i+1})) = P_n \cap \text{Id}(B_{T_{i+1}})$$

and

$$c_n(A) \leq 2n^3 \tag{3.3}$$

by Lemma 2.1.

First, we choose $T_1$ such that the inequality

$$2m^3 < \alpha^m \tag{3.4}$$

holds for all $m \geq T_1$. By Lemma 2.2, codimension growth of $B_{T_1}$ is overexponential. Hence, one can find $N_1 > T_1$ such that

$$c_n(B_{T_1}) < \alpha^n \text{ for all } n \leq N_1 - 1 \text{ and } c_{N_1}(B_{T_1}) \geq \alpha^{N_1}. \tag{3.5}$$

Now we take $T_2 = 2N_1$. Then by (3.3) and (3.4) we have

$$c_{N_1+1}(A) \leq 2(N_1 + 1)^3 < \alpha^{N_1+1} \tag{3.6}$$

and

$$c_n(A) \leq c_n(B_{T_1}) + c_n(B_{T_2}) < \alpha^n + 2n^3 < 2\alpha^n \tag{3.7}$$

for all $T_1 < n \leq N_1 - 1$ as follows from (3.2), (3.4), (3.5), and from Lemma 2.1.

On the next step we choose $N_2 > T_2$ satisfying the relations similar to (3.5), (3.6), and (3.7). Continuing this procedure we obtain an infinite sequence $T_1 < N_1 < T_2 < N_2 < \ldots$ such that

$$c_n(A) < 2\alpha^n \text{ if } N_i-1 < n \leq N_i - 1, \quad c_n(A) \geq \alpha^{N_i}, \quad c_{N_i+1}(A) \leq 2(N_i + 1)^3 \tag{3.8}$$

for all $i = 2, 3, \ldots$.

From (2.2), it follows that $c_n(A) \leq nc_{n-1}(A)$ for all $n \geq 2$. Hence (3.8) implies

$$\alpha^{N_i} \leq c_{N_i}(A) \leq 2N_i \alpha^{N_i-1}.$$

Therefore

$$\lim_{i \to \infty} (c_{N_i}(A))^{\frac{1}{N_i}} = \alpha.$$

On the other hand, by the choice of $N_1, N_2, \ldots$, we have

$$c_n(A) < 2\alpha^n$$

for all $n \neq N_1, N_2, \ldots$. It follows that

$$\exp(A) = \limsup_{n \to \infty} \sqrt[n]{c_n(A)} = \alpha.$$
Finally, the last inequality in (3.8) shows that
\[
\lim_{i \to \infty} \sqrt[i]{c_{m_i}(A)} \leq 1,
\]
where \( m_i = N_i + 1 \). Since \( c_n(A) \neq 0 \) for all \( n \geq 1 \), we have
\[
\exp(A) = \lim_{n \to \infty} \sqrt[n]{c_n(A)} = 1
\]
and the proof is completed. □

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