Zykov sums of digraphs with diachromatic number equal to their harmonious chromatic number

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Abstract

The dichromatic number and the diachromatic number are generalizations of the chromatic number and the achromatic number for digraphs considering acyclic colorings. In this paper, we determine the diachromatic number of digraphs arising from the Zykov sum of digraphs that admit a complete $k$-coloring with $k = \frac{1 + \sqrt{1 + 4m^2}}{2}$ for a suitable $m$. Consequently, the diachromatic number equals the harmonious number for every digraph in this family. In particular, we study the chromatic number, the diachromatic number, and the harmonious chromatic number of the Zykov sum of cycles.

Keywords. Dichromatic number, factorization, detachments, composition of digraphs, lexicographic product.

1 Introduction

A coloring of a digraph can be understood as a function that maps elements of a graph, usually vertices, into some set, usually numbers, which are called colors and such that some property is satisfied, usually related to the arcs. More precisely, a $k$-coloring of a digraph $D$ is a proper vertex-coloring, that is, each chromatic class induces a subdigraph with no arcs. The chromatic number $\chi(D)$ of $D$ is the smallest $k$ for which there exists a $k$-coloring of $D$ [6]. The original concept of the chromatic number comes from graphs and then extended to digraphs in this natural way, as well as the following colorings and parameters.

A coloring of $D$ is called harmonious if for every ordered pair $(i,j)$ of different colors there is at most one arc $uv$ such that $u$ is colored $i$ and $v$ is colored $j$. The harmonious chromatic number $h(D)$ of $D$ is the smallest $k$ for which there exists a harmonious $k$-coloring of $D$ [15] [17]. A coloring of $D$ is complete if for every ordered pair $(i,j)$ of different colors there is at least one arc $uv$ such that $u$ is colored $i$ and $v$ is colored $j$ [12]. The achromatic number $\psi(D)$ of $D$ is the largest $k$ for which there is a complete $k$-coloring of $D$ [12]. Therefore, the size $m$ of a digraph $D$ is bounded below by $2^{\frac{\lambda(D)}{2}}$, hence, $\psi(D)$ is bounded above by $\frac{1 + \sqrt{1 + 4m^2}}{2}$ and both coincide if and only if there are exactly two arcs between every two chromatic classes.

For graphs (which can be seen as symmetric digraphs), such parameters are called the chromatic number $\chi$, the harmonious chromatic number $h$ and the achromatic number $\psi$, respectively. Edwards [12] proved that determining the exact value of the harmonious chromatic number is NP-hard for digraphs of bounded degree and that for a given digraph the existence of a complete coloring is NP-complete. For this reason, we use the following generalization for the chromatic and the achromatic numbers of a digraph.

A vertex-coloring of a digraph $D$ is acyclic if each chromatic class induces a subdigraph with no directed cycles. The dichromatic number $dc(D)$ of $D$ is the smallest $k$ for which there exists an acyclic coloring of

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D using \( k \) colors \([24]\). Any \( \text{dc}(D) \)-coloring of \( D \) is also complete. The **diachromatic number** \( \text{dac}(D) \) of \( D \) is the largest \( k \) for which there is an acyclic and complete coloring of \( D \) using \( k \) colors \([1]\). Therefore, for any digraph \( D \) of size \( m \), we have that

\[
\text{dc}(D) \leq \text{dac}(D) \leq \frac{1+\sqrt{1+4m}}{2} \leq h(D).
\]

On the other hand, the study of parameters arising of complete colorings into graph products can be found in \([5, 9, 18, 21, 29]\) with results for the cartesian product or join of graphs. Let \( D \) be a digraph and \( X = \{ H_u : u \in V(D) \} \) a family of nonempty mutually vertex-disjoint digraphs. The **Zykov sum** \( D[X] \) of \( X \) over \( D \) is a digraph with vertex set \( \bigcup_{u \in V(D)} V(H_u) \) and arc set

\[
\bigcup_{u \in V(D)} A(H_u) \cup \{ ab : a \in V(H_u), b \in V(H_v), uv \in A(D) \}.
\]

The corresponding operation for graphs is called **generalized composition**. If \( H_u \cong H \) for every \( u \in V(D) \), then \( D[X] \) is called **lexicographic product** (also called **digraph composition**) and is denoted by \( D[H] \). It is known that the complexity of testing whether an arbitrary graph can be written nontrivially as the composition of two smaller graphs is the same as the complexity of testing whether two graphs are isomorphic \([13]\) which can be solved in quasipolynomial time according to \([4]\). The dichromatic number of Zykov sums and composition of digraphs were studied in \([19, 25]\) and the chromatic number of the lexicographic product of graphs were studied in \([10, 13]\).

In this paper, we determine digraphs, arising from the Zykov sum of digraphs, which accept a complete \( k \)-coloring with \( k \) equals their harmonious number. As a consequence, we obtain results about graphs, arising from the lexicographic product of graphs, with achromatic number equals their harmonious number. Then, we analyze conditions to apply the results with particular attention to factorizations of the complete graphs into Hamiltonian cycles. Also, we study a recursive application of the results.

## 2 Main result

In this paper, we consider only finite simple digraphs. Let \([n]\) denote the set \( \{1, 2, \ldots, n\} \) and let \( m \geq 2 \). For the case of digraphs, \( K_m \) denotes the complete symmetric digraph and for graphs, \( K_m \) denotes the complete graph. A **factor** \( H_j \) of the complete digraph (resp. graph) \( K_m \) is a spanning subdigraph (resp. subgraph). A **factorization** \( Y \) of \( K_m \) is a set of \( q \) pairwise arc-disjoint (resp. edge-disjoint) factors \( H_j \), for \( j \in [q] \), such that these factors induce a partition of the arcs (resp. edge) of \( K_m \). If \( H_j \cong H \) (for all \( j \in [q] \)) then it is called an **H-factorization**. Given a factorization \( Y \), a **relabel factorization** \( X \) of \( Y \) is a relabeling of the vertices of \( Y \) in the following way: the vertices \( \{v^1, v^2, \ldots, v^m\} \) of the factor \( H_j \) is relabeled into \( \{v_j^1, v_j^2, \ldots, v_j^m\} \).

Let \( D \) be a \( k \)-diachromatic digraph (resp. \( k \)-achromatic graph) with a \( k \)-coloring \( \varphi \). Let \( \{V_1, V_2, \ldots, V_k\} \) be the set of chromatic classes for \( \varphi \) with \( |V_i| = q_i \). For each \( i \in [k] \), denote the vertices of the chromatic class \( V_i \) by \( \{u_{i,1}, u_{i,2}, \ldots, u_{i,q_i}\} \). In this case \( V(D) = \bigcup_{i=1}^{k} V_i \). For each \( i \in [k] \), let \( X_i = \{H_{u_{i,1}}, H_{u_{i,2}}, \ldots, H_{u_{i,q_i}}\} \) be a relabel factorization of \( K_{q_i} \) into \( q_i \) factors. We consider the Zykov sum \( D[X] \), where \( X = \bigcup_{i=1}^{k} X_i = \{H_{u_{i,j}} : u_{i,j} \in V(D)\} \).

Before proving our first theorem, we require the following result and definitions. For two nonempty vertex sets \( V_1, V_2 \) of a digraph \( D \), we define \( [V_1, V_2] = \{(x, y) \in A(D) \mid x \in V_1, y \in V_2\} \). A digraph \( D \) is \( k \)-**minimal** if \( \text{dac}(D) = k \) and \( \text{dac}(D - f) < k \) for all \( f \in A(D) \).

**Theorem 1.** \([1]\) Let \( D \) be a digraph with diachromatic number \( k \). Then, \( D \) is \( k \)-minimal if and only if \( D \) has size \( k(k-1) \).

**Theorem 2.** Let \( D \) be a \( k \)-minimal digraph of order \( n \) with a \( k \)-coloring \( \varphi \). Let \( \{V_1, V_2, \ldots, V_k\} \) be the set of chromatic classes for \( \varphi \) with \( |V_i| = q_i \). For each \( i \in [k] \) let \( V_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,q_i}\} \) and let \( X_i = \)
\{H_{u_{1,1}}, H_{u_{1,2}}, \ldots, H_{u_{i,q_i}}\} be relabel factorizations of $K_{m_i}$ into $q_i$ factors. Then $D[X]$ is $t$-minimal, where $X = \bigcup_{i=1}^{k} X_i$ and $t = \sum_{i=1}^{k} m_i$.

Proof. We take a partition of $K_m$ into $q_i$ factors. In order to have a set of colored and sorted vertices arising from $V(K_{m_i}) = \{v_{l}^1, v_{l}^2, \ldots, v_{l}^{m_i}\}$, we define the following coloring. Let $f_i: V(K_{m_i}) \to [m_i]$ be the complete $m_i$-colorings of $K_{m_i}$, such that $f_i(v_{l}^i) = l$ for each $l \in [m_i]$. Let $f_{i,j}: V(H_{u_{i,j}}) \to [m_i]$ the natural restriction of $f_i$ into each factor $H_{u_{i,j}}$, that is, $f_{i,j}(v_{u}^i) = f_i(v_{u}^i) = l$ for any vertex $v_{u}^i \in V(H_{u_{i,j}})$, with $i \in [k]$, $j \in [q_i]$ and $l \in [m_i]$, see Figure 1 for an example.

Let $\varsigma: V(D[X]) \to [t]$ be a $t$-coloring such that for each $l \in [m_i]$

$$\varsigma(v_{u}^{i,j}) = c(i,l) := \sum_{a=0}^{i-1} m_a + l,$$

with $m_0 = 0$.

That is, if $i$ and $l$ are fixed, for each $j \in [q_i]$ the vertex $v_{u}^{i,j}$ in the factor $H_{u_{i,j}}$ has color $c(i,l)$. Thus, the set of vertices colored $c(i,l)$ of $\varsigma$ is

\{v_{u_{1,1}}^{i,j}, v_{u_{1,2}}^{i,j}, \ldots, v_{u_{i,q_i}}^{i,j}\}.

Since the Zykov sums of empty graphs is empty, the coloring is proper and then acyclic due to the fact that the induced subgraph by \{v_{u_{1,1}}^{i,j}, v_{u_{1,2}}^{i,j}, \ldots, v_{u_{i,q_i}}^{i,j}\} of $D[X]$ is empty.

Next, we claim the $\varsigma$ coloring is minimal and complete. Let $c(i,l)$ and $c(i',l')$ be two colors of $\varsigma$ with $i, i' \in [k]$, $l \in [m_i]$ and $l' \in [m_{i'}]$. If $i = i'$, since each $H_{u_{i,j}}$ has the $m_i$ colors of $f_i$, then $v_{u}^{i,j}$ is the unique arc of $H_{u_{i,j}}$ for some $j$ and then there exists a unique arc between $c(i,l)$ and $c(i',l')$. On the other hand, since $\varphi$ is minimal and complete, if $i \neq i'$ there exists a unique arc $u^{i,j}u^{i',j'}$ such that $\varphi(u^{i,j}) = i$ and $\varphi(u^{i',j'}) = l'$ with $j \in [q_i]$ and $j' \in [q_i]$. Therefore, $[\varphi(H_{u_{i,j}}), \varphi(H_{u_{i',j'}})]$ is a bipartition of a directed complete bipartite subdigraph of $D[X]$. In consequence, for a fixed $l$ and $l'$ the arc $v_{u}^{i,j}v_{u}^{i',j'}$ is the unique arc from a vertex of color $c(i,l)$ to a vertex with color $c(i',l')$. \hfill \Box

Figure 1 shows the example of the Zykov sum $\overrightarrow{C}_k[X]$ for some set of digraphs $X$, where $X = \{X_1, X_2, X_3\}$ and $X_1 = \{H_{u_{1,1}}, H_{u_{1,2}}\}$, $X_2 = \{H_{u_{2,1}}, H_{u_{2,2}}\}$ and $X_3 = \{H_{u_{3,1}}, H_{u_{3,2}}\}$ are relabel factorizations of $K_2$, $K_3$ and $K_4$, respectively. In order to avoid drawing all the arc between subdigraphs we use the symbol $\Rightarrow$. An edge represents a couple of symmetric arcs.

Now, we have the following corollaries due to Theorem 2. For this, we recall that a balanced coloring is a coloring in such a way that any two chromatic classes have the same cardinality.

**Corollary 3.** Let $D$ be a $k$-minimal digraph of order $n$ with a balanced $k$-coloring such that $qk = n$. Let $X_i$ be relabel factorizations of $K_m$ into $q$ factors, that is, $X_i = \{H_{u_{i,1}}, H_{u_{i,2}}, \ldots, H_{u_{i,q}}\}$ for $i \in [k]$. Then $D[X]$ is $km$-minimal with a balanced $km$-coloring.

**Corollary 4.** Let $D$ be a $k$-minimal digraph of order $n$ with a balanced $k$-coloring such that $qk = n$. If $K_m$ has a relabel $H$-factorization into $q$ factors, then $D[H]$ is $km$-minimal with a balanced $km$-coloring.

### 3 The lexicographic product of cycles

Now, we proceed to construct families of digraphs obtained by Zykov sums $D$ and $H$ that satisfy the hypothesis of Theorem 2 and then we give results for $\overrightarrow{C}_m[\overrightarrow{C}_n]$ for some values of $m$. We recall some definitions given in [1].

Let $u$ and $v$ be two vertices of a digraph $D$ such that $uv$ is an arc of $D$. We say that $u$ is incident to $v$ and $v$ is incident from $u$. The out-neighborhood $N^+(u)$ of a vertex $u$ is the set of vertices that are incident from $u$. Similarly, the in-neighborhood $N^-(v)$ of a vertex $v$ is the set of vertices incident to $v$. 

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Figure 1: The Zykov sum \( \vec{C}_6[X] \) where \( X \) is a relabel factorization of \( K_2 \), \( K_3 \) and \( K_4 \).
Two vertices are adjacent if they are in a 2-cycle. To obtain an elementary dihomomorphism of a digraph $D$, identify two nonadjacent vertices $u$ and $v$ of $D$. The resulting vertex, when identifying $u$ and $v$, may be denoted by either $u$ or $v$. A sequence of elementary dihomomorphisms is a dihomomorphism. For graphs, a dihomomorphism image corresponds to a usual homomorphism image also called amalgamation, see [23]. A graph $G$ is an amalgamation of a graph $H$ if and only if $H$ is a detachment of $G$, see [11].

An elementary dihomomorphism preserving the cardinality of arcs is called elementary identification $\epsilon$, that is, let $D$ be a digraph and $u, v \in V(D)$ two independent vertices such that $N^+(u) \cap N^+(v) = \emptyset$ and $N^-(u) \cap N^-(v) = \emptyset$, then $\epsilon$ is the elementary dihomomorphism obtained by identifying $u$ and $v$. A digraph $D'$ is an identification image of a digraph $D$ if and only if $D'$ can be obtained by a sequence of elementary identifications beginning with $D$.

An elementary unfold is the inverse of an elementary identification and an unfold is the inverse of an identification. For graphs, an exact graph has exactly $\binom{k}{2}$ edges for some integer $k$, see [11]. Then, we say a digraph is exact if it has exactly $k(k - 1)$ arcs for some integer $k$, and we call an identification image as an exact amalgamation and an unfold image as an exact detachment. For example, an exact detachment of $K_5$ is $\overrightarrow{C}_{20}$ if we follow an Eulerian circuit of $K_5$, and vice versa, an exact amalgamation of $\overrightarrow{C}_{20}$ is $K_5$. Also, it is clear that a factorization of a digraph can be understood as an exact detachament.

Remark 5. A digraph $D$ is $k$-minimal if and only if there exists an identification $\Gamma$ from the digraph $D$ to the complete digraph $K_k$.

As a consequence of Remark 5, we have that $K_k$ can be unfolded in the cycle $\overrightarrow{C}_{k(k-1)}$. The induced $k$-coloring of $\overrightarrow{C}_{k(k-1)}$ is balanced where each chromatic class has $k - 1$ vertices. On the other hand, $K_k$ accepts a $\overrightarrow{C}_k$-factorization into $k - 1$ factors for $k \neq 4, 6$, see [7, 25].

Corollary 6. Let $n$ be a natural number such that $n \neq 4, 6$. The digraph $D = \overrightarrow{C}_{n^2-n}[\overrightarrow{C}_n]$ is $n^2$-minimal with a balanced $n^2$-coloring. Hence $\text{dac}(D) = h(D) = n^2$.

Proof. Since, $K_n$ can be unfold into $n - 1$ digraphs isomorphic to $\overrightarrow{C}_n$, and $K_n$ can be unfold into $\overrightarrow{C}_{n^2-n}$ with a balance $n$-coloring where each chromatic class has $n - 1$ vertices. Therefore, $\overrightarrow{C}_{n^2-n}[\overrightarrow{C}_n]$ is $n^2$-minimal and the result follows.

3.1 On the diachromatic number

In this subsection, we bound the diachromatic number of $D = \overrightarrow{C}_m[\overrightarrow{C}_n]$ for $m$ close to $n^2 - n$.

To begin with, we improve the bound in Equation 1 for $m = n^2 - n + t$ following the idea of comparing two functions one of which determines the maximum possible number of chromatic classes of order $x$ and the other one determines how many chromatic classes can be incident to one chromatic class of order $x$, where $x$ is the order of the smallest chromatic class.

Theorem 7. Let $m, n$ be natural numbers such that $m, n \geq 3$. For any complete coloring of $D = \overrightarrow{C}_m[\overrightarrow{C}_n]$ using $k$ colors

$$k \leq \max \{\min\{f_n(x), g_n(x)\} \mid x \in \mathbb{N}\}$$

where $f_n(x) = \lceil mn/x \rceil$ and $g_n(x) = x(n + 1) + 1$.

Proof. Let $\varsigma: V(D) \to [k]$ be a complete $k$-coloring of $D$. Let $x = \min\{|\varsigma^{-1}(i)| : i \in [k]\}$ be the cardinality of the smallest chromatic class of $\varsigma$. Without loss of generality, suppose that $x = |\varsigma^{-1}(k)|$. Since $\varsigma$ defines a partition of $V(D)$, it follows that $k \leq mn/x$ and then $k \leq f_n(x)$.

On the other hand, there are $x(n + 1)$ vertices from any smallest chromatic class, then there are at most $x(n + 1) + 1$ chromatic classes, and it follows that $k \leq g_n(x)$. Thus, we have that $k \leq \min\{f_n(x), g_n(x)\}$. Finally, we obtain that

$$k \leq \max \{\min\{f_n(x), g_n(x)\} \mid x \in \mathbb{N}\}.$$
Since \( f_n(x) \) is a hyperbola and \( g_n(x) \) is a concave parabola, we are interested in the positive solution \( x_0 \) for which \( f_n(x_0) = g_n(x_0) \) is the largest possible, it is not hard to see that it happens when \( x \approx \sqrt{m} \).

**Corollary 8.** For the digraph \( \overrightarrow{C}_{n^2-n+t}^{\ell}[\overrightarrow{C}_n] \) with \( 1 \leq t \leq n \), we have that
\[
\text{dac}(\overrightarrow{C}_{n^2-n+t}^{\ell}[\overrightarrow{C}_n]) \leq n^2.
\]
**Proof.** Consider \( x = n \) and \( x = n-1 \), then \( f_n(n) = n^2 - n + t, g_n(n) = n^2 + n + 1 \) and \( \min\{f_n(n), g_n(n)\} = f_n(n) = n^2 - n + t \). Next, \( f_n(n - 1) = \left\lceil \frac{n^2 + t + \frac{f}{n-1} \right\rceil \), \( g_n(n - 1) = n^2 \) and \( \min\{f_n(n - 1), g_n(n - 1)\} = g_n(n - 1) = n^2 \). It follows that \( \text{dac}(\overrightarrow{C}_{n^2-n+t}^{\ell}[\overrightarrow{C}_n]) \leq n^2 \). \( \square \)

In order to give a lower bound for \( \text{dac}(\overrightarrow{C}_{n^2-n+t}^{\ell}[\overrightarrow{C}_n]) \), we use a similar coloring of \( \overrightarrow{C}_{n^2-n}[\overrightarrow{C}_n] \) given in Corollary 8 and the fact of a direct cycle has dichromatic number 2.

**Theorem 9.** Let \( n \) be a natural number such that \( n \neq 4, 6 \). For the digraph \( \overrightarrow{C}_{n^2-n+t}^{\ell}[\overrightarrow{C}_n] \) with \( 1 \leq t \leq n \),
\[
\text{dac}(\overrightarrow{C}_{n^2-n+t}^{\ell}[\overrightarrow{C}_n]) = n^2.
\]
**Proof.** We can extend a diachromatic coloring \( \varsigma \) of \( \overrightarrow{C}_{n^2-n} \) with the colors \([n]\) and chromatic classes \( \varsigma^{-1}(i) = \{x_1, x_2^1, \ldots, x_{n-1}^i\} \) to a coloring \( \varsigma' \) of \( \overrightarrow{C}_{n^2-n+t} \) as follows. Without loss of generality, we suppose that \( x_1^1 x_2^2 \) is the arc with the colors 1 and 2 of \( \overrightarrow{C}_{n^2-n+t} \). Make \( t \) subdivisions to \( x_1^1 x_2^2 \) obtaining a directed path, i.e. \( (x_1^1, x_0, x_1, \ldots, x_{t-1}, x_t^2) \). Color the vertices \( \{x_0, x_1, x_2, \ldots, x_{t-1}\} \) with the alternating colors 2, 1, 2, 1, … then the coloring is also acyclic and complete using \( n \) colors.

Now, consider the complete graph \( K_n \) with the vertex-set \( \{v_1, v_2, \ldots, v_n\} \). Take a factorization of \( K_n \) into \( n-1 \) Hamiltonian cycles \( H_j \cong \overrightarrow{C}_n \), for \( j \in [n-1] \). We define the coloring \( \varsigma_j : V(H_j) \to [n] \) such as \( \varsigma_j(v_k) = k \). The digraph \( \overrightarrow{C}_{n^2-n+t}^{\ell}[\overrightarrow{C}_n] \) has vertices \( (x_j^i, v_k) \) where \( v_k \in V(H_j) \) and \( (x_l, v_k) \), where \( v_k \in V(H_l) \) if \( l \) is odd and \( v_k \in V(H_2) \) if \( l \) is even. Color the vertices of \( \overrightarrow{C}_{n^2-n+t}^{\ell}[\overrightarrow{C}_n] \) with \( n^2 \) colors such that \( (x_j^i, v_k) \to (\varsigma(x_j^i), \varsigma_j(v_k)) \), \( (x_l, v_k) \to (\varsigma'(x_l), \varsigma_1(v_k)) \) if \( l \) is odd and \( (x_l, v_k) \to (\varsigma'(x_l), \varsigma_2(v_k)) \) if \( l \) is even. The result follows due to Corollary 8. \( \square \)

### 3.2 On the harmonious chromatic number

In this subsection, we bound the harmonious chromatic number of \( D = \overrightarrow{C}_{m}[\overrightarrow{C}_n] \) for \( m \) close to \( n^2 - n \). A close relationship between this subsection and the previous one can be observed.

First, we improve the upper bound of Equation 1 for \( m = n^2 - n - t \) following the idea of comparing two functions.

**Theorem 10.** Let \( m, n \) be natural numbers such that \( m, n \geq 3 \). For any harmonious coloring of \( D = \overrightarrow{C}_{m}[\overrightarrow{C}_n] \) using \( k \) colors
\[
k \geq \min \{\max\{f_n(x), g_n(x)\} \text{ with } x \in \mathbb{N}\}
\]
where \( f_n(x) = \lceil mn/x \rceil \) and \( g_n(x) = x(n+1) + 1 \).

**Proof.** Let \( \varsigma : V(D) \to [k] \) be an harmonious \( k \)-coloring of \( D \). Let \( x = \max\{|\varsigma^{-1}(i)| : i \in [k]\} \), that is, let \( x \) be the cardinality of the largest chromatic class of \( \varsigma \). Without loss of generality, suppose that \( x = |\varsigma^{-1}(k)| \). Since \( \varsigma \) defines a partition of \( V(D) \) it follows that \( k \geq mn/x \) and then \( k \geq f_n(x) \).

On the other hand, there are \( x(n+1) \) vertices from any largest chromatic class, and then there are at least \( x(n+1) + 1 \) chromatic classes, it follows that \( k \geq g_n(x) \). Thus, we have that \( k \geq \max\{f_n(x), g_n(x)\} \).

Finally, we obtain that
\[
k \geq \min \{\max\{f_n(x), g_n(x)\} \text{ with } x \in \mathbb{N}\}.
\]
\( \square \)
Corollary 11. For the digraph $\overrightarrow{C}_{n^2-n-t}[\overrightarrow{C}_n]$ with $1 \leq t < n$, we have that
\[ h(\overrightarrow{C}_{n^2-n-t}[\overrightarrow{C}_n]) \geq n^2. \]

Proof. Consider $x = n - 1$ and $x = n - 2$, then $f_n(n - 1) = \left\lceil \frac{n^2 - x}{n-1} \right\rceil$, $g_n(n - 1) = n^2$ and $\max\{f_n(n - 1), g_n(n - 1)\} = g_n(n - 1) = n^2$. Next, $f_n(n - 2) = \left\lceil \frac{n^2 + n + 1}{n-2} - \frac{3}{n-2} \right\rceil$, $g_n(n - 2) = n^2 - n - 1$ and $\max\{f_n(n - 2), g_n(n - 2)\} = f_n(n - 2) = \left\lceil n^2 + n + 1 - \frac{3}{n-2} \right\rceil$. It follows that $h(\overrightarrow{C}_{n^2-n-t}[\overrightarrow{C}_n]) \geq n^2$. \[\Box\]

In order to give an upper bound for $h(\overrightarrow{C}_{n^2-n-t}[\overrightarrow{C}_n])$, we color $\overrightarrow{C}_{n^2-n}[\overrightarrow{C}_n]$ using the technique given in Corollary 6 and a particular unfold of $K_n - A$ where $A$ is a particular set of edges.

Theorem 12. Let $n$ be a natural number such that $n \neq 4, 6$. For the digraph $\overrightarrow{C}_{n^2-n-t}[\overrightarrow{C}_n]$ with $1 \leq t < n$, we have that
\[ h(\overrightarrow{C}_{n^2-n-t}[\overrightarrow{C}_n]) = n^2. \]

Proof. Consider the graph $K_n - A$ where $A$ is a set of edges incident to a vertex $u \in V(K_n)$, with $|A| = t$. Now, we obtain the graph $G$ which is an unfold $K_n - A$ such that each edge of $A$ is a leaf where its vertex of degree 1 is the corresponding vertex $u$ in $K_n - A$.

Next, we obtain the directed cycle $\overrightarrow{C}_{n^2-n-t}$ to unfold $G$ since $G$ is an Eulerian digraph. We can color $\overrightarrow{C}_{n^2-n-t}[\overrightarrow{C}_n]$ following the same technique of Theorem 2 and obtaining our coloring.

The result follows due to Corollary 11 and this upper bound. \[\Box\]

4 Recursive results

The lexicographic product of digraphs $D$ and $H$ is an operation that produces a digraph $D[H]$, and then we can obtain the digraph $D[H][H]$ and so on. We define $D[H]^i := (D[H]^{i-1})[H]$ with $D[H]^1 := D[H]$. If $D \cong H$, we write $[H]^{i+1}$.

Note that Corollary 11 produces a $k$-minimal digraph for which, their chromatic classes $\{v_{u,1}, v_{u,2}, \ldots, v_{u,q}\}$ have cardinality equal to $q$, therefore this digraph and the relabel $H$-factorizations fulfills the hypothesis, hence, a recursive construction can be done given an initial digraph $D$ and an $H$-factorization.

Corollary 13. Let $D$ be a $k$-minimal digraph of order $n$ with a balanced $k$-coloring, such that $qk = n$. If $K_m$ has a relabel $H$-factorization into $q$ factors, then $D[H]^i$ is $k^i m$-minimal with a balanced $k^i m$-coloring, for all $i \in \mathbb{Z}^+$.

Hence, we can extend Corollary 6 as follows.

Corollary 14. Let $n$ be a natural number such that $n \neq 4, 6$. The digraph $D = \overrightarrow{C}_{n^2-n}[\overrightarrow{C}_n]$ is $n^{i+1}$-minimal with a balanced $n^{i+1}$-coloring. Hence, for all $i \in \mathbb{Z}^+$,
\[ \text{dac}(D) = h(D) = n^{i+1}. \]

Now, we determine the dichromatic number of $D = \overrightarrow{C}_{n^2-n}[\overrightarrow{C}_n]$.

First, we determine a result of [25] in the following lemma, namely, Proposition 32 (iii) and Proposition 34 together with Corollary 43. We omit the proof because it is analogous to the original one generalizing from transitive tournaments to acyclic digraphs.

Theorem 15. Let $D, H$ be digraphs such that $D$ has order $m$ and $\text{dc}(H) = k$. If $r$ is the maximum order of an acyclic set of vertices of $D$, then

- $\text{dc}(D[H]) \geq \left\lceil \frac{km}{r} \right\rceil$. 

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• Moreover, if \( D \) contains a spanning subdigraph isomorphic to a circulant digraph \( \overrightarrow{C}_m(J) \) such that the induced subdigraph of the vertices \( \{0, 1, \ldots, r\} \) is acyclic, then
\[
dc(D[H]) = \left\lceil \frac{k \cdot m}{r} \right\rceil.
\]

Next, we use a result concerning the recurrence relation that appears in the solution of the well-known Josephus Problem, see [15, 26] and see [19] for an application in digraphs.

**Theorem 16.** [26] Consider the recurrence relation
\[
T_n(i) = \left\lceil \frac{n}{n-1} T_n(i-1) \right\rceil \quad \text{with} \quad i \geq 1 \text{ and } T_n(0) = 1.
\]

1. For each integer \( n \geq 2 \) there is a real number \( c_n \) such that
   \[
   T_n(i) = c_n \left( \frac{n}{n-1} \right)^i + e_{i,n}
   \]
   where \( n \geq 4 \) and \(-n + 2 < e_{i,n} \leq 0\).
2. \( T_3(i) = \left\lfloor c_3 \left( \frac{3}{2} \right)^i \right\rfloor \) where \( c_3 \approx 1.62227 \ldots \) is an irrational number.
3. \( T_2(i) = 2^i \).

Now, we can determine \( dc(\overrightarrow{C}_n^i) \) and then \( dc(\overrightarrow{C}_n^{n^2-n}\overrightarrow{C}_n^i) \).

**Lemma 17.** \( dc(\overrightarrow{C}_n^i) = T_n(i) \).

**Proof.** Clearly, the maximal order of an acyclic set of vertices of \( \overrightarrow{C}_n \) is \( n-1 \), and \( \overrightarrow{C}_n \) contains an isomorphic copy of a circulant digraph over \( \mathbb{Z}_n \) as a spanning subdigraph. By Theorem 15, it follows that
\[
dc(\overrightarrow{C}_n^i) = \left\lceil \frac{n}{n-1} \left\lceil \frac{n}{n-1} \right\rceil \right\rceil.
\]
Repeating this argument \( i-1 \) times, it follows that Theorem 16 completes the proof.

As a consequence of Theorem 15 and Lemma 17 it follows that

**Theorem 18.** \( dc(\overrightarrow{C}_n^{n^2-n}\overrightarrow{C}_n^i) = \left\lceil \frac{n^2-n}{n^2-n-1} T_n(i) \right\rceil \).

5 Final remarks

Different factorizations can be considered, for instance, in [20] answered the question about a factorization of \( K_n \) into Hamiltonian directed paths, that is, if the elements of some group of order \( n \) can be arranged in a sequence \( c_1, c_2, \ldots, c_n \) such that \( c_1 c_2 c_3 \ldots c_i = c_1 c_2 c_3 \ldots c_j \) whenever \( i \neq j \). This is shown to be possible for any Abelian group with exactly one element of order 2 and for the non-Abelian group of order 21. Then, we have the following corollary.

**Corollary 19.** If \( K_n \) accepts a \( \overrightarrow{P}_n \)-factorization, then the digraph \( D = \overrightarrow{C}_n^{n(n+1)}[\overrightarrow{P}_n]^i \) is \( n(n+1)^i \)-minimal with a balanced \( n(n+1)^i \)-coloring. Hence
\[
dac(D) = h(D) = n(n+1)^i,
\]
for all \( i \in \mathbb{Z}^+ \).
In [2] was given $H$-factorizations of the complete graphs via quadratic residues where $H$ is a circulant graph. There are several types of factorizations of the complete graphs, for instance, see [22, 27].

Another possible interesting problem is due to Theorem 18 and Corollary 14. For any pair of positive integers $i, n$, we have the dichromatic numbers of $\overrightarrow{C}_{n^2-n}[\overrightarrow{C}_n]^i$. Although these results provide an infinite number of pairs of integers $a \leq b$ such that there exists an oriented graph $D$ satisfying that $dc(D) = a$ and $ dac(D) = b$. For a given $a \leq b$, there exists a oriented graph $D$ such that $dc(D) = a$ and $ dac(D) = b$? It is known that the question for graphs is answered by Bhave, see [8].

Finally, we remark that for graphs, an elementary identification is called as a harmonious homomorphism, see [3]. For digraphs, we need to extend proper coloring to digraphs. In this paper, we use the extension given by Hedge and Castelino [16, 17] and Edwards [12]. The other possibility is via acyclic coloring. We define the harmonious dichromatic number $dh(D)$ of $D$ as the smallest $k$ for which there exists an acyclic and harmonious coloring of $D$ using $k$ colors. Clearly, $dh(D) \leq h(D)$ but the lower bound of Equation 1 of $h(D)$ is not necessarily a lower bound of $dh(D)$.

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