Uncountable locally free groups and their group rings

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Abstract. In this note, we show that an uncountable locally free group, and therefore every locally free group, has a free subgroup whose cardinality is the same as that of $G$. This result directly improve the main result in [4] and establish the primitivity of group rings of locally free groups.

1. INTRODUCTION

A group $G$ is called locally free if all of its finitely generated subgroups are free. As a consequence of Nielsen-Schreier theorem, a free group is always locally free. If the cardinality $|G|$ of $G$ is countable, then $G$ is locally free if and only if $G$ is an ascending union of free groups. In particular, $G$ is a locally free group which is not free provided that it is a properly ascending union of non-abelian free groups of bounded finite rank. In fact, in this case, $G$ is infinitely generated and Hopfian and so it is not free (also see [3] and [6]). If $|G|$ is uncountable, that is $|G| > \aleph_0$, then it was studied in the context of almost free groups, and it is also known that there exists an uncountable locally free group which is not free ([2]).

Now, clearly, if $G$ is a locally free group with $|G| = \aleph_0$, then $G$ has a free subgroup whose cardinality is the same as that of $G$. In the present note, we shall show that it is true for locally free groups of any cardinality. In fact, we shall prove the following theorem:

Theorem 1.1. If $G$ is a locally free group with $|G| > \aleph_0$, then for each finitely generated subgroup $A$ of $G$, there exists a subgroup $H$ of $G$ with $|H| = |G|$ such that $AH \simeq A \ast H$, the free product of $A$ and $H$.

In particular, $G$ has a free subgroup of the same cardinality as that of $G$.

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Theorem 1.1 means that there is no need to assume the existence of free subgroups in [4, Theorem 1]. That is, we can improve the theorem and establish the primitivity of group rings of locally free groups, where a ring \( R \) is (right) primitive provided it has a faithful irreducible (right) \( R \)-module.

2. Proof of the theorem

In order to prove Theorem 1.1, we prepare necessary notations and some lemmas which include a result due to Mal’cev [3]. Some of them might be trivial for experts but we include their proofs for completeness.

For a finitely generated subgroup \( H \) of a locally free group \( G \), \( \mu_G(H) \) is defined to be the least positive integer \( m \) such that \( H \subseteq F_m \) for some free subgroup \( F_m \) of rank \( m > 0 \) in \( G \). The rank \( r(G) \) of \( G \) is defined to be the maximum element in

\[
\{ \mu_G(H) \mid H \text{ is a finitely generated subgroup in } G \}
\]
or \( r(G) = \infty \). We should note that \( r(G) \) is finite if and only if for each finitely generated subgroup \( H \) of \( G \), there exists a free subgroup \( N \) of rank \( r(G) \) such that \( H \subseteq N \). We should also note that for subgroups \( H, M \) and \( N \) of \( G \) with \( H \subseteq M \subseteq N \), \( \mu_M(H) \geq \mu_N(H) \) holds.

Lemma 2.1. Let \( G \) be a locally free group, \( D \) a finitely generated subgroup of \( G \) with \( \mu_G(D) = m \) and \( M \) a subgroup of \( G \) such that \( D \subseteq M \) and \( r(M) = m \). For \( g \in G \setminus M \), set \( M_g = M \langle g \rangle \); the subgroup of \( G \) generated by \( g \) and the elements in \( M \), and let \( r(M_g) = n \).

Then, if \( n > m \), then \( n = m + 1 \) and there exists a free subgroup \( F_m \) of rank \( m \) in \( M \) such that \( D \subseteq F_m \) and \( F_m(g) \) is isomorphic to the free product \( F_m \ast \langle g \rangle \).

Proof. Since \( r(M_g) = n \), there exists a finitely generated subgroup \( C \) of \( M_g \) such that \( \mu_{M_g}(C) = n \). Since \( C \) is finitely generated, it can be easily seen that there exists finite number of elements \( a_1, \ldots, a_l \) in \( M \) such that \( C \subseteq \langle a_1, \ldots, a_l, g \rangle \). We have then that there exists a free group \( F_m \) of rank \( m \) in \( M \) such that \( \langle a_1, \ldots, a_l \rangle D \subseteq F_m \) because of \( r(M) = m \). Since \( C \subseteq F_m(g) \) and \( \mu_{M_g}(C) = n \), we see that \( r(F_m(g)) \geq n \). On the other hand, \( r(F_m(g)) \leq m + 1 \) because of \( g \notin F_m \). Combining these with the assumption \( n > m \), we get that \( n = m + 1 = r(F_m(g)) \), which implies \( F_m(g) \cong F_m \ast \langle g \rangle \).

For a locally free group of finite rank, the following result due to Mal’cev is well-known.

Lemma 2.2. (See [3]) If \( G \) is a locally free group of finite rank, then the cardinality of \( G \) is countable; namely \( |G| = \aleph_0 \).
On the other hand, if $G$ is not of finite rank, then we have the following property:

**Lemma 2.3.** If $G$ is a locally free group whose rank is not finite, then for each finitely generated subgroup $A$ of $G$, there exists an element $x \in G$ with $x \notin A$, such that $A\langle x \rangle \simeq A * \langle x \rangle$, the free product of $A$ and $\langle x \rangle$.

**Proof.** Let $A$ be a finitely generated subgroup of $G$. We have then that $A$ is a free group of finite rank, because $G$ is locally free. Since the rank of $G$ is not finite, there exists a free subgroup $F$ of $G$ such that $A \subsetneq F$ and $\rho(F) > \rho(A)$. Let $A = \langle y_1, \ldots, y_l \rangle$ and $F = \langle x_1, \ldots, x_n \rangle$, where $l = \rho(A)$ and $m = \rho(F)$. If for $A_1 = \langle y_1, \ldots, y_l, x_1 \rangle$, $\rho(A_1) = l + 1$, then $A_1 \simeq A * \langle x_1 \rangle$. If $\rho(A_1) \leq l$ then there exists $i \in \{2, \ldots, m\}$ such that $\rho(A_i) = \rho(A_{i-1}) + 1$, where $A_i = \langle y_1, \ldots, y_l, x_1, \ldots, x_i \rangle$. We have then that $A_i \simeq A_{i-1} * \langle x_i \rangle$. Since $A \subseteq A_i$, we have thus seen that $A\langle x \rangle \simeq A * \langle x \rangle$ for some $x \in G \setminus A$. \hfill \Box

Let $G = A \ast B$ be the free product of $A \neq 1$ and $B \neq 1$. Clearly, if $|G| = \aleph_0$, then $G$ has a free subgroup whose cardinality is the same as that of $G$. If $|G| > \aleph_0$, then either $|A| = |G|$ or $|B| = |G|$, say $|A| = |G|$. Let $I$ be a set with $|I| = |A|$, and for each $i \in I$, let $a_i$ be in $A$ such that $a_i \neq a_j$ for $i \neq j$. We have then that for $1 \neq b \in B$, the elements $(a_i b)^2$ over $i \in I$ freely generate the subgroup of $G$ whose cardinality is the same as that of $G$. Hence we have

**Lemma 2.4.** If $G = A \ast B$ is the free product of $A$ and $B$, then $G$ has a free subgroup whose cardinality is the same as that of $G$.

We are now read to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $A$ be a finitely generated subgroup of $G$. We set

$\mathcal{B} = \{ B \mid B$ is a non-trivial subgroup of $G$ such that $AB \simeq A \ast B \}$.

Since $G$ is locally free, $A$ is a free group of finite rank. By assumption, $|G| > \aleph_0$, and so the rank of $G$ is not finite by Lemma 2.2. Hence, by Lemma 2.3, there exists an element $g \in G \setminus A$ such that $A\langle g \rangle \simeq A \ast \langle g \rangle$, whence $\langle g \rangle \in \mathcal{B}$; thus $\mathcal{B} \neq \emptyset$. Let $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_i \subseteq \cdots$ be a chain of $B_i$’s in $\mathcal{B}$, and let $B^* = \bigcup_{i=1}^{\infty} B_i$. We can see that $B^*$ belongs to $\mathcal{B}$. In fact, if not so, then $AB^* \not\simeq A \ast B^*$, and so there exists a finitely generated subgroup $C$ of $B^*$ such that $AC \neq A \ast C$. However, because $C$ is finitely generated in $B^*$, we have $C \subseteq B_i$ for some $i$, which implies $AB_i \not\simeq A \ast B_i$, a contradiction. We have thus shown that $(\mathcal{B}, \subseteq)$ is an inductively ordered set. By Zorn’s lemma, there exists a maximal
element $H$ in $(B, \subseteq)$. We shall show $|H| = |G|$, which completes the proof of the theorem. In fact, $AH \simeq A \ast H$ and it has also a free subgroup whose cardinality is the same as that of $G$ by Lemma 2.4.

Suppose, to the contrary, that $|H| < |G|$. Set $N = AH(\simeq A \ast H)$, and for a finitely generated subgroup $D$ of $N$ with $\mu_G(D) = m$, let $\mathcal{M}(D)$ be the set of subgroups $M$ of $G$ such that $D \subseteq M$ and $r(M) = m$. We can see that $(\mathcal{M}(D), \subseteq)$ is an inductively ordered set as follows: Since $m = \mu_G(D)$, there exists a free subgroup $F_m$ of rank $m$ in $G$ such that $D \subseteq F_m$. Hence $F_m \in \mathcal{M}(D)$, whence $\mathcal{M}(D) \neq \emptyset$. Let $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq \cdots$ be a chain of $M_i$'s in $\mathcal{M}$, and let $M^* = \bigcup_{i=1}^{\infty} M_i$. Clearly, $D \subseteq M^*$. By the definition of the rank, $r(M^*) \geq \mu_{M^*}(D)$. Since $M^* \subseteq G$, we have that $\mu_{M^*}(D) \geq \mu_G(D) = m$; thus $r(M^*) \geq m$. On the other hand, by the definition of $r(M^*)$, there exists a finitely generated subgroup $C$ of $M^*$ such that $\mu_{M^*}(C) = r(M^*)$. Since $C$ is finitely generated, there exists $i > 0$ such that $C \subseteq M_i$, and then $\mu_{M^*}(C) \leq r(M_i)$, which implies that $r(M^*) = \mu_{M^*}(C) \leq \mu_{M^*}(C) \leq r(M_i) = m$; thus $r(M^*) \leq m$. Hence we have $r(M^*) = m$. We have thus proved that $M^* \in \mathcal{M}(D)$ and that $(\mathcal{M}(D), \subseteq)$ is an inductively ordered set.

Again by Zorn’s lemma, there exists a maximal element $M(D)$ in $(\mathcal{M}(D), \subseteq)$. Let $L = \bigcup_{D \in \mathcal{D}} M(D)$, where $\mathcal{D}$ is the set consisting of all finitely generated subgroups of $N$. Since $r(M(D))$ is finite for each $D \in \mathcal{D}$, it follows from Lemma 2.2 that $|M(D)| = \aleph_0$ for each $D \in \mathcal{D}$. Hence we have $|L| < |G|$ because $|\mathcal{D}| = |N| < |G|$. In particular, there exists $g \in G$ such that $g \not\in L$. Note that $g \not\in N$ because of $N \subseteq L$, and so $g \not\in H$. We shall show that $N \langle g \rangle \simeq N \ast \langle g \rangle$. In order to do this, it suffices to show that for each $D \in \mathcal{D}$, $D \langle g \rangle \simeq D \ast \langle g \rangle$ holds.

Let $r(M(D)) = m$ for $D \in \mathcal{D}$ and let $M_g = M(D) \langle g \rangle$. Since $D \subseteq M_g$ and $\mu_{M_g}(D) \geq \mu_G(D) = m$, it follows that $r(M_g) \geq m$. Moreover, since $g \not\in M(D)$, we have $M(D) \subsetneq M_g$. Hence the maximality of $M(D)$ implies $r(M_g) > m$. It follows from Lemma 2.1 that there exists a free subgroup $F_m$ of rank $m$ in $M(D)$ such that $D \subseteq F_m$ and $F_m \langle g \rangle \simeq F_m \ast \langle g \rangle$. Hence we have $D \langle g \rangle \simeq D \ast \langle g \rangle$.

We have thus shown that $N \langle g \rangle \simeq N \ast \langle g \rangle$, which contradicts the maximality of $H$ because $N \ast \langle g \rangle = A \ast (H \ast \langle g \rangle)$. This completes the proof of the theorem. □

Theorem 1.1 shows that the assumption on existence of free subgroups in [1] Theorem 1] can be dropped. That is, we have the following theorem:
Theorem 2.5. Let $G$ be a non-abelian locally free group. If $R$ is a domain with $|R| \leq |G|$, then $RG$ is primitive.
In particular, $KG$ is primitive for any field $K$.

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