Local Minimizers with Unbounded Vorticity for the 2D Ginzburg-Landau Functional

ANDRES CONTRERAS
New Mexico State University

ROBERT L. JERRARD
University of Toronto

Abstract
A central focus of Ginzburg-Landau theory is the understanding and characterization of vortex configurations. On a bounded domain $\Omega \subseteq \mathbb{R}^2$, global minimizers, and critical states in general, of the corresponding energy functional have been studied thoroughly in the limit $\epsilon \to 0$, where $\epsilon > 0$ is the inverse of the Ginzburg-Landau parameter. A notable open problem is whether there are solutions of the Ginzburg-Landau equation for any number of vortices below $h_{ex}^{-1/4}$, for external fields of up to superheating field strength.

In this paper, we prove that there are constants $K_1, \alpha > 0$ such that given natural numbers satisfying
\[ 1 \leq N \leq \frac{h_{ex}}{2\pi}(|\Omega| - h_{ex}^{-1/4}), \]
local minimizers of the Ginzburg-Landau functional with this many vortices exist, for fields such that $K_1 \leq h_{ex} \leq 1/\epsilon^\alpha$. Our strategy consists of combining: the minimization over a subset of configurations for which we can obtain a very precise localization of vortices; expansion of the energy in terms of a modified vortex interaction energy that allows for a reduction to a potential theory problem; and a quantitative vortex separation result for admissible configurations. Our results provide detailed information about the vorticity and refined asymptotics of the local minimizers that we construct. © 2021 Wiley Periodicals LLC.

1 Introduction
Let $\Omega$ be a bounded, open, simply connected subset of $\mathbb{R}^2$ with smooth boundary. Given $(u, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$, we define the Ginzburg-Landau functional
\[ GL_\epsilon(u, A) := \frac{1}{2} \int_\Omega \left( |(\nabla - iA)u|^2 + |\nabla \times A - h_{ex}|^2 + \frac{(1 - |u|^2)^2}{2\epsilon^2} \right). \]
Quantized vortices, described below in detail, are a prominent qualitative feature of a large class of critical points of $GL_\epsilon$, relevant to both physical phenomena and certain problems of a geometric flavour. The influential work [1] characterizes minimizers of a simplified version without magnetic field, where vortices emerge...
as a result of imposed topologically nontrivial boundary conditions. Later, this work was extended to a problem contemplating magnetic influences in [2]. In a series of works, starting with [21, 22]; continuing with [17, 18], the monograph [19] (and references therein); and culminating in [20], the vortex structure of global minimizers of the full model has been described in great detail for a wide range of values of the external field $h_{ex}$. It is known that minimizers transition from a vortexless state to one where a specific number of vortices is preferred, as the external field increases over a threshold called the first critical field. On the high end of strengths of applied fields considered in [20], the optimal number of vortices, which diverges as $\epsilon \to 0$, and their asymptotic distribution are obtained at main order.

A satisfactory picture of the moduli space of solutions to the Ginzburg-Landau equations should not only characterize global minimizers but also other stable equilibria. In 2D Ginzburg-Landau, the existence of branches of solutions with a prescribed number of vortices (different from those present in a global minimizer) in a range determined by the capacity of the applied field to contain them, is a known conjecture. This phenomenon is a mathematical manifestation of the expected hysteretic properties of vortex (and vortexless) configurations as noted in [14, 16]. Stable vortex states were obtained in [12] below the first critical field. On the other hand, it was noted as early as [21, 22] that local minimizers with a fixed number of vortices exist for applied fields near the first critical field; these results were extended in [23], in particular considering fields in a much larger interval $1 \ll h_{ex} \ll 1/\epsilon^s$, $0 < s < 1/2$. In [19], the authors obtain for the first time local minimizers with a possibly divergent number of vortices, although $N \ll C \abs{\log \epsilon}^{1/2}$ and close to the highest allowed numbers, these solutions exist for a limited range of external fields (smaller than any power of $1/\epsilon$). In the list of open questions in [19], it is asked to extend the results about branches of stable solutions in chapter 11, to a larger set of choices of numbers of vortices and applied fields. The work [4] partially addresses this question and proves, in particular, the existence of solutions with vortices up to $N \sim \abs{\log \epsilon}$ for fields sufficiently larger than the first critical field. The ranges obtained in [4] improve on previous constructions considerably; however, they are still far from establishing the folklore problem about local branches of minimizers with prescribed vorticity, and in fact they do not cover a noticeable portion of the expected range,

$$K_1 \ll h_{ex} \ll \frac{1}{\epsilon} \quad \text{and} \quad 1 \leq N \leq N^*(h_{ex}),$$

where $K_1$ is some, possibly large number and $N^*(h_{ex})$ is the maximum expected number of vortices that can be contained by a field of strength $h_{ex}$. The maximum allowed vorticity is believed to be $N^*(h_{ex}) = \frac{h_{ex}[2]}{2\pi}$ based on a free boundary problem associated to the corresponding mean field model [3, 4, 20] for $N \to \infty$ vortices. The condition $h_{ex} \ll \frac{1}{\epsilon}$ comes from the knowledge that the Meissner
(vortexless) solution is stable for fields of these strengths; the strength of the field for which the Meissner solution loses its stability is known as a superheating field.

Most of the above-mentioned works also give information about the location of vortices in terms of a renormalized energy or averaged versions of it. The work \[20\] introduces the Coulombian renormalized energy and global minimizers studied there, and the local minimizers found in \[4\] assort their vortices so as to asymptotically minimize this energy. Understanding reduced models for a divergent number of vortices in this and other related equations is of great interest \[3, 8, 11, 24\]; in the case of 2D Ginzburg-Landau, this interest is partially motivated by connections to problems of crystallization \[13\] \[20\]. In all instances of problems where one has to deal with very large vorticities, the analysis becomes very technically difficult, and these challenges are partly responsible for the lack of progress in the problem of obtaining stable vortex configurations with a very large number of vortices.

The constructions of local minimizers in the earlier works \[19, 21–23\] rely on two elements:

- Roughly speaking, the energy contribution of a vortex for \(E_\varepsilon\) (defined in (1.6) below) is \(\pi |\log \varepsilon|\), while the energy associated with interaction between vortices scales like \(O(1) \times (# \text{ of pairs of vortices}) = O(N^2)\). The admissible class of functions is chosen so that the energy contribution to \(E_\varepsilon\) due to interactions between vortices is known up to \(C |\log \varepsilon|\), for some \(C > 0\) sufficiently small.
- There are lower bounds for the energy of an \(N\)-vortex configuration of the form

\[
GL_\varepsilon(u, A) \geq a_0 h_{ex}^2 + \pi N |\log \varepsilon| + a_1 N^2 + a_2 N + error(N),
\]

for certain explicit constants \(a_0, a_1, a_2\) depending on \(\Omega, h_{ex}\), and (in a mild way) on \(N\) itself. These bounds can be used to show that minimizers are not on the boundary of the admissible class, as long as \(N\) is not too large.

The idea is that the total vorticity of minimizers in the class can be prescribed because their energy is compatible only with the desired vorticity; this is why accurate knowledge of the error is essential. In \[19\] \(error(N) = o(N^2)\), whence the restriction \(N^2 \ll |\log \varepsilon|\). In \[4\], the authors exploit lower bounds involving the Coulombian renormalized energy that follow from results and techniques in \[20\]. The improved lower bounds yield an expansion (1.1) (for different constants \(a_1, a_2\)) where \(error(N) = o(N)\). Given that the range of \(N\)’s was to be extended to values much higher than \(|\log \varepsilon|^{1/2}\), the first element cannot be combined with the lower bound to prescribe the vorticity as in \[19, 21–23\], although a lower bound for the vorticity is available. Instead, the authors of \[4\] devise a new approach whereby a new admissible class allows to bound the total vorticity from above indirectly. Even then the result can only cover a range of \(N\)’s where the error does not exceed the cost of a vortex.

In this paper we develop a new strategy that allows us to find local minimizers of \(GL_\varepsilon\) for much larger numbers \(N = N_\varepsilon\) of vortices and applied magnetic field
\( h_{\text{ex}} = h_{\text{ex},\epsilon} \). We will always assume that
\[
0 < \epsilon < \epsilon_0, \quad K_1 \leq h_{\text{ex}} \leq k_1 \epsilon^{-1/4},
\]
and
\[
1 \leq N \leq \min \left\{ \frac{h_{\text{ex}}}{2\pi(|\Omega| - h_{\text{ex}}^{-1/4})}, k_2 \epsilon^{-1/10} h_{\text{ex}}^{-1/5} \right\}
\]
where the constants, fixed below, depend only on \( \Omega \). (In general, we write \( k_j, K_j \) to denote small and large constants, and we always assume that \( k_j \leq 1 \leq K_j \).) In particular, for \( K_1 \leq h_{\text{ex}} \leq k_2 \epsilon^{-1/12} \), the entire range \( 1 \leq N \leq \frac{h_{\text{ex}}}{2\pi(|\Omega| - h_{\text{ex}}^{-1/4})} \) is included. No technical adaptation of earlier arguments seems likely to be of use in this whole range. The key new elements in our approach are:

1. The set over which we minimize prescribes the number of vortices directly: we work with functions \( u \) whose vorticity (see (1.14) below) is close to \( \pi \sum_{i=1}^{N} \delta_{a_i} \) where \( a = (a_1, \ldots, a_n) \) is an approximate constrained minimizer of a renormalized energy \( H_N^{\epsilon} \), defined in (1.9).

2. We derive lower bounds in terms of the renormalized energy. A similar renormalized energy has appeared before in [19]. Here, drawing on [7], we rigorously justify the renormalized energy for very large values of \( N \) and \( h_{\text{ex}} \). A drawback of these expressions is that the renormalized energy \( H_N^{\epsilon} \) tends to \(-\infty\) as vortices approach the boundary. (In particular, \( H_N^{\epsilon} \) does not attain its infimum.) Moreover, it loses accuracy as vortices approach the boundary or as any pair of vortices approach each other. These considerations give rise to the constraints to which we have alluded above on the configurations \( a_1, \ldots, a_n \) that we consider.

3. A major advantage of our approach is that, unlike earlier works, we do not require (although such an expansion could presumably be derived a posteriori from our results) an energy expansion of the form (1.1). But to handle the difficulties mentioned above, we need a priori lower bounds for \( \min \text{ dist}(a_i, \partial \Omega) \) and \( \min_{j \neq i} |a_i - a_j| \), when \( a = (a_1, \ldots, a_n) \) is an approximate constrained minimizer of \( H_N^{\epsilon} \). We also need to show that the “vorticity close to \( \pi \sum_{i=1}^{N} \delta_{a_i} \)” condition in point (1) above can be improved for \((u, A)\) minimizing \( GL_{\epsilon} \) in the admissible class. To do these, we introduce a modification of the renormalized energy that let us obtain minimizers of this energy in terms of an obstacle problem. From here we can study deviations in almost optimal configurations via a “screened” problem. We do this by means of a quantitative version of an argument used in [15] for a similar problem.

1.1 Main result

To formulate our results, we need some definitions. First, let \( G = G(x, y) \) be the Green’s function defined by
\[
-\Delta_x G + G = \delta_y \quad \text{for} \ x \in \Omega, \quad G(x, y) = 0 \quad \text{for} \ x \in \partial \Omega.
\]
We let $S(\cdot, \cdot)$ denote the regular part of $G$, defined by
\begin{equation}
S(x, y) = 2\pi G(x, y) + \log |x - y|.
\end{equation}
We define
\begin{equation}
E(\epsilon)(u) := \int_\Omega \frac{|\nabla u|^2}{2} + \frac{(|u|^2 - 1)^2}{4\epsilon^2},
\end{equation}
(1.7)
\begin{equation}
F(\xi) := \frac{1}{2} \int_\Omega |\nabla \xi|^2 + (\xi + 1)^2 \, dx.
\end{equation}
We will always write $\xi_0$ to denote the (unique) minimizer of $F$ in $H_0^1(\Omega)$. Thus $\xi_0$ satisfies
\begin{equation}
(-\Delta + 1)\xi_0 = -1 \quad \text{in } \Omega, \quad \xi_0 = 0 \quad \text{on } \partial\Omega.
\end{equation}
For $N \in \mathbb{N}$ and $a = (a_1, \ldots, a_N) \in \Omega^N$, we define the renormalized energy
\begin{equation}
H(\epsilon)(a) = \sum_{i=1}^N \left[ 2\pi h_{\text{ex}} \xi_0(a_i) + \pi S(a_i, a_i) \right] + 2\pi^2 \sum_{i \neq j} G(a_i, a_j),
\end{equation}
(1.19)
where we set $G(a, a) = +\infty$ for $a \in \Omega$. We will see that this approximately characterizes the least possible energy of a pair $(u, A)$ with vortices near $(a_1, \ldots, a_N)$, up to a constant that depends on $\epsilon, N$ and $h_{\text{ex}}$ but not on vortex locations.

We will always restrict our attention to pairs $(u, A)$ such that
\begin{equation}
\nabla \cdot A = 0 \quad \text{in } \Omega, \quad A \cdot v = 0 \quad \text{in } \partial\Omega.
\end{equation}
holds; in view of the basic gauge invariance property of the Ginzburg-Landau functional (see, for example, [19, sec. 2.1.3]), this does not entail any loss of generality. Recalling that $\Omega$ is simply connected, we can then write
\begin{equation}
A = \nabla^\perp B \quad \text{for some (unique) } B \in H^2 \cap H^1_0(\Omega).
\end{equation}
We will write $B = (\nabla^\perp)^{-1} A$ when \ref{1.11} holds, without indicating the role of the boundary conditions.

Define
\begin{equation}
M_{\epsilon, N} := \{ a = (a_1, \ldots, a_N) \in \Omega^N : \text{dist}(a_i, \partial\Omega) \geq h_{\text{ex}}^{-1/3} \},
\end{equation}
(1.12)
\begin{equation}
M^{*}_{\epsilon, N} := \{ a = (a_1, \ldots, a_N) \in M_{\epsilon, N} : H(\epsilon)(a) \leq \min_{M_{\epsilon, N}} H(\epsilon) + t_0 \}.
\end{equation}

for some positive $t_0$, to be chosen below (in the proof of Lemma \ref{3.4}). Thus, configurations in $M^{*}_{\epsilon, N}$ nearly minimize the renormalized energy $H(\epsilon)$, subject to the constraint that no $a_i$ is too close to $\partial\Omega$.

Given $(u, A)$ satisfying \ref{1.10}, writing $u = u^1 + i u^2 \cong (u^1, u^2)$, we define the associated vorticity, denoted $J u$, by
\begin{equation}
J u := \det \nabla u = \partial_1 u^1 \partial_2 u^2 - \partial_2 u^1 \partial_1 u^2.
\end{equation}
A pair \((u, A)\) is thus interpreted as having vortices near \(a \in \Omega^N\) if

\[
(1.15) \quad A \text{ satisfies } (1.10) \quad \text{and} \quad \left\| J u - \pi \sum_{i=1}^{N} \delta_{a_i} \right\|_{W^{-1,1}} \leq \sigma_e
\]

for some small \(\sigma_e\). This means roughly that the vorticity \(J u\) is strongly concentrated in a union \(\bigcup_{i=1}^{N} B_{r_i}(a_i)\) with \(\sum r_i \leq \sigma_e\), and with \(\int_{B_{r_i}(a_i)} J u \approx \pi\) for every \(i\).

In particular, we are interested in the set of pairs \((u, A)\) with vortices very close to a configuration \(a = (a_1, \ldots, a_N)\) that is a near-minimizer of \(H^N_\epsilon\). Thus, we define

\[
(1.16) \quad A^N_\epsilon := \{ (u, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) : \exists a \in M^*_{\epsilon, N} \text{ such that } (1.15) \text{ holds} \}
\]

for a choice of \(\sigma_e\) to be specified later; see (5.6). It is a standard fact that \(GL_\epsilon\) attains its minimum in \(A^N_\epsilon\). By using exactly the argument to prove existence of unconstrained minimizers for 2D Ginzburg-Landau (see, for example, proposition 3.5 in section 3.1.5 in [19]), one can extract a subsequence that converges weakly in \(H^1 \times H^1\) to a limit, with the energy of the limit bounded by \(\inf A^N_\epsilon GL_\epsilon\). We thus only need to prove that the limit belongs to \(A^N_\epsilon\), and this follows from weak continuity properties of the Jacobian, together with the fact that \(M^*_{\epsilon, N}\) is a closed set.

Our main result is the following:

**Theorem 1.1.** Assume that (1.2) and (1.3) hold, and let \((u_\epsilon, A_\epsilon)\) minimize \(GL_\epsilon\) in \(A^N_\epsilon\) for \(\sigma_\epsilon\) defined in (5.6), which in particular implies that \(\sigma_\epsilon \leq C \epsilon^{49/100}\).

Then \((u_\epsilon, A_\epsilon)\) belongs to the interior of \(A^N_\epsilon\). As a result, \((u_\epsilon, A_\epsilon)\) is a local minimizer of \(GL_\epsilon\), and hence a solution of the Ginzburg-Landau equations.

This shows both that there exists a local minimizer with \(N\) vortices, and that the vortices are located near points found by minimizing the renormalized energy.

Theorem 1.1 is the first result showing the existence of local minimizers of \(GL_\epsilon\) with a number of vortices much larger than \(\lfloor \log \epsilon \rfloor\), going all the way up to \(\epsilon^{-\alpha}\) for some positive \(\alpha\). For fields smaller than \(\epsilon^{-1/12}\), our result settles the conjecture about local minimizers covering the full range of \(N\)’s. Above this range, that is for \(\epsilon^{-1/12} < h_{ex} < \epsilon^{-1/4}\), Theorem 1.1 also greatly extends the previous best-known partial result [4] in two directions (see Figures 1.1, 1.2, and 1.3 for a comparison between Theorem 1.1 and previous works): the strength of the field is allowed to be as large as \(\epsilon^{-1/4}\), which is much larger than the \(\epsilon^{-1/7}\) in [4], and also the number of vortices is still allowed to get as big as \(\epsilon^{-1/12}\) in this range of fields.

Our proof also yields a great deal of information about the local minimizers that we construct. We show that their vortices are approximated with extreme precision by sums of point masses at points that asymptotically minimize \(H^N_\epsilon\). We also
describe their energy up to errors of order $o(1)$. For our local minimizers, our results would in principle make it possible to derive explicit estimates in terms of the Coulombian renormalized energy by directly studying the simple discrete energy $H_N^e$; this would allow to bypass the delicate mass displacement results used in [20]. We believe that our results may also have some implications for global minimizers, at least when $h_{ex}$ is not too large, but we do not explore that here.

The organization of the paper is as follows. First, in Section 2 we present the proof of Theorem 1.1, assuming various facts that are proved in the remainder of the paper. In Section 3 we introduce a modification of $H_N^e$ and study properties of near minimizers of this modification via an auxiliary screened problem. The localization results and corresponding lower bounds are proved in Section 4. Finally, in Section 5 the upper and lower bounds for minimizers in $A_N^e$ together with the
improved localization of vortices are collected, concluding the proof of Theorem 1.1.

2 Proof of Theorem 1.1

In this section we first describe the ingredients in our analysis, and we then show how these elements combine naturally to yield the proof of our main result. In doing so, we give a more detailed account of the overall strategy.

2.1 Ingredients in the proof

Interior near-minimizers of the renormalized energy

The following result provides information about points in $M^*_e$ (i.e., near-minimizers $a = (a_1, \ldots, a_N)$ of $H^N_e$, subject to the constraint that every $a_j$ stays a certain distance from $\partial \Omega$).

**Proposition 2.1.** Assume that (1.2) holds and that

$$1 \leq N \leq \frac{h_{ex}}{2\pi} (|\Omega| - h_{ex}^{-1/4})$$

(which is implied by (1.3)). Then there exists $c_0, c_1, t_0 > 0$, depending on $\Omega$, such that every $a = (a_1, \ldots, a_N) \in M^*_e$ satisfies

$$\text{dist}(a_i, \partial \Omega) \geq c_0 h_{ex}^{-1/4} \quad \text{for all } i,$$

$$|a_i - a_j| \geq c_1 h_{ex}^{-1/2} \quad \text{for all } i \neq j.$$

The proof, which we present in Section 3, uses ideas from [15, 19].

As mentioned earlier the full conjecture about stable vortex states with prescribed vorticity asks to show that the conclusions of Theorem 1.1 hold if assumption (1.3) is replaced by assumption (2.1), as long as $K_1 \leq h_{ex} \ll \frac{1}{e}$. For use toward a possible proof of this conjecture, Proposition 2.1 is sharp. The stronger requirement (1.3) arises from other parts of the proof, described below, involving upper and lower energy bounds in terms of the renormalized energy.

Lower energy bounds

In Proposition 4.1 we prove some results relating the Ginzburg-Landau energy and the renormalized energy. We show that if $(u, A)$ satisfies (1.15), then

$$GL_e(u, A) \geq H^N_e(a) + \kappa^G_e - \text{error terms}$$

where $\kappa^G_e$ is a constant defined in (4.5). The error terms are quite complicated and depend on $e, E_e(u), h_{ex}, \sigma_e, N, \rho_a$, where

$$\rho_a = \frac{1}{4} \min \left\{ \min_{i \neq j} |a_i - a_j|, \min_i \text{dist}(a_i, \partial \Omega) \right\}.$$

but will end up being small under assumptions (1.2) and (1.3), and our eventual choice of $\sigma_e$. The proof of this smallness uses the lower bound for $\rho_a$ that follows
from Proposition [2.1]. Estimate (2.4) is reasonably sharp in the sense that for every \( a \in \Omega^N \) such that \( \rho_a \) is not too small, there exist \((u, A)\) such that (1.15) holds and in addition

\[
(2.6) \quad GL_\epsilon(u, A) \leq H_\epsilon^N(a) + \kappa_\epsilon^{GL} + \text{error terms}
\]

for error terms of a similar character, and that are similarly small under our assumptions. This follows from arguments in the proof of Proposition 5.1.

These results are adaptations to our setting of estimates proved in [7], which dealt with the simplified functional \( E_\epsilon \), without magnetic field, rather than the full Ginzburg-Landau functional \( GL_\epsilon \). Similar results are proved in [10]. Bounds related to (2.4) and (2.6) can also be found in [19], in the formal discussion leading up to equation (9.3), or the rigorous derivation (10.2), which applies for a bounded number of vortices in the limit as \( \epsilon \to 0 \).

**Localization**

The next input needed for Theorem 1.1 is given in Proposition 4.2, also adapted from [7]. It involves the quantity

\[
\Sigma_\epsilon^{GL}(u, A, a) := GL_\epsilon(u, A) - (H^N_\epsilon(a) + \kappa_\epsilon^{GL}),
\]

which measures the excess energy of \((u, A)\) relative to the lower bound (2.4). The proposition shows that if \((u, A)\) satisfies (1.15) and \( \Sigma_\epsilon^{GL}(u, A, a) \) is small, then one can find \( \xi = (\xi_1, \ldots, \xi_N) \) near \( a \) such that

\[
(2.7) \quad \left\| J u - \pi \sum_{i=1}^N \delta_{\xi_i} \right\|_{W^{-1,1}} \leq \text{error terms}
\]

where the (complicated) error terms depend on the same parameters as (2.4), together with \( \Sigma_\epsilon^{GL}(u, A, a) \). This is a good estimate when the right-hand side is smaller than \( \sigma_\epsilon \), appearing in hypotheses (1.15); otherwise it is obvious.

**2.2 Proof of Theorem 1.1**

We now describe the proof of our main result. The inequalities appearing in the argument are all established in Proposition 5.1.

Let \((u_\epsilon, A_\epsilon)\) minimize \( GL_\epsilon \) in \( \mathcal{A}_\epsilon^N \), where the parameter \( \sigma_\epsilon \) in the definition of \( \mathcal{A}_\epsilon^N \) is in the range \( \epsilon^{49/100} \lesssim \sigma_\epsilon \lesssim \epsilon^{49/100} \). The precise choice will depend on \( h_\text{ex}, N, \) and \( \epsilon \); see (5.6).

We first verify, by construction of a competitor, that

\[
(2.8) \quad GL_\epsilon(u_\epsilon, A_\epsilon) \leq \min_{M_{\epsilon,N}} H^N_\epsilon + \kappa_\epsilon^{GL} + \frac{t_0}{3}
\]

whenever \( \epsilon \) is small enough. This is an instance of (2.6), and its proof essentially contains that of the general case (which we omit). We also show that \( E_\epsilon(u_\epsilon) \leq Ch_\text{ex}^2 \), which is needed to make effective use of the lower bound and localization results.
The definition of $A^N_\varepsilon$ implies that there exists some $a_\varepsilon \in M^{\ast}_{\varepsilon,N}$ such that $(u_\varepsilon, A_\varepsilon)$ and $a_\varepsilon$ satisfy (1.15). Then (2.8) immediately yields
\[
\sum_{\xi}^{GL}(u_\varepsilon, A_\varepsilon, a_\varepsilon) = GL_\varepsilon(u_\varepsilon, A_\varepsilon) - H^N_\varepsilon(a_\varepsilon) - \kappa^{GL}_\varepsilon \leq \frac{t_0}{3}.
\]

Note also that Proposition 2.1 provides a lower bound $\rho a_\varepsilon \geq c_1 h^{-1/2}_c$. These estimates and the scaling assumptions (1.2) and (1.3) allow us to control the error terms in (2.7) and finalize the choice of $\sigma_\varepsilon$ in such a way that
\[
\|Ju_\varepsilon - \pi \sum_{i} \delta_{\xi_i}\|_{\dot{W}^{-1,1}} \leq \frac{1}{2}\sigma_\varepsilon
\]
for some $\xi \in M_{\varepsilon,N}$. Once this is known, we can apply (2.4) to relate $GL_\varepsilon(u_\varepsilon, A_\varepsilon)$ to $H^N_\varepsilon(\xi)$. After controlling error terms as above, this yields
\[
GL_\varepsilon(u_\varepsilon, A_\varepsilon) \geq H^N_\varepsilon(\xi) + \kappa^{GL}_\varepsilon + \frac{t_0}{3}.
\]

Recalling (2.8), we deduce that
\[
H^N_\varepsilon(\xi) \leq \min_{M_{\varepsilon,N}} H^N_\varepsilon + \frac{2}{3}t_0.
\]

Thus $\xi \in M^{\ast}_{\varepsilon,N}$, and in fact Proposition 2.1 guarantees that $\xi \in (M^{\ast}_{\varepsilon,N})^{\text{int}}$. Then (2.2), (2.3), (2.9), and (2.10) imply that $(u_\varepsilon, A_\varepsilon) \in (A^N_\varepsilon)^{\text{int}}$, and is thus a local minimizer. This completes the proof.

\section{Near-Minimizers of $H^N_\varepsilon$ in $M_{\varepsilon,N}$}

In this section we prove Proposition 2.1. The crucial idea is to transform this problem into a local argument via a screening process. This screening is made possible by first identifying the leading-order distribution of vortices through an obstacle problem. In attempting to carry this out, we encounter a nontrivial technical challenge: the renormalized energy $H^N_\varepsilon$ is not bounded from below in $\Omega^N$, and this makes impossible a dual formulation. To overcome this difficulty, we modify the renormalized energy near the boundary so as to have the desired dual formulation, and to do this we need to estimate how fast the divergent parts of $H^N_\varepsilon$ go to $-\infty$ as some of the vortices approach $\partial \Omega$.

We remark that $H^N_\varepsilon$ depends on $\varepsilon$ only through $h_c$. Similarly, all quantities to be introduced in this section (such as auxiliary functions $\chi_\varepsilon, v_\varepsilon, w_\varepsilon = (-\Delta + 1)v_\varepsilon$, ...) that appear to depend on $\varepsilon$—in fact, depend only on $h_c$ (which, however, may depend on $\varepsilon$) Thus the right hypothesis in these results is not that $\varepsilon$ be sufficiently small, but rather that $h_c$ be sufficiently large (which however forces $\varepsilon$ to be rather small, in view of (1.2)).
3.1 Modification of $H^N_e$

Proposition 2.1 deals with near-minimizers $a = (a_1, \ldots, a_N)$ of $H^N_e$, which is unbounded below, subject to a constraint that $\text{dist}(a_j, \partial \Omega)$ is not too small. Our first lemma will allow us instead to analyze unconstrained near-minimizers of a function $H^N_e$ that is continuous on $\overline{\Omega}^N$ and in particular bounded below.

LEMMA 3.1. There exists a function $v \in C^\infty(\Omega)$ such that

\begin{align}
\|(-\Delta + 1)v\|_{L^\infty(\Omega)} & \leq Ch^{-1/3}\log h \tag{3.1} \\
v(x) & = \frac{1}{2h}S(x, x) \quad \text{if} \quad \text{dist}(a, \partial \Omega) \geq h^{-1/3} \tag{3.2}
\end{align}

The lemma will allow us to prove Proposition 2.1 by studying near-minimizers of

\begin{equation}
H^N_e(a) := \sum_{i=1}^N 2\pi h \xi_0(a_i) + v_k(a_i)] + 2\pi^2 \sum_{i \neq j} G(a_i, a_j), \tag{3.3}
\end{equation}

which coincides with $H^N_e$ in $M_{e, N}$ as a result of (3.2).

PROOF. Let $\chi_\epsilon : \Omega \to [0, 1]$ denote a smooth function such that
\begin{align}
\chi_\epsilon(x) & = 1 \quad \text{if} \quad \text{dist}(x, \partial \Omega) \geq h^{-1/3}, \tag{3.4} \\
\chi_\epsilon(x) & = 0 \quad \text{if} \quad \text{dist}(x, \partial \Omega) \leq \frac{1}{2}h^{-1/3},
\end{align}

and

\begin{align}
\|\nabla \chi_\epsilon\|_{\infty} & \leq C h^{1/3} \quad \|\nabla^2 \chi_\epsilon\|_{\infty} \leq Ch^{2/3} \tag{3.5}
\end{align}

We define

\[v_\epsilon(x) := \frac{\chi_\epsilon(x)}{2h}S(x, x)\]

Then (3.2) is immediate. For the proof of (3.1) we will write $s(x) = S(x, x)$. In view of (3.4) and (3.5), it suffices to show that

\begin{align}
|s(x)| & \leq C(|\log d(x)| + 1), \quad |\nabla s(x)| \leq Cd(x)^{-1}, \tag{3.6} \\
|\Delta s(x)| & \leq Cd(x)^{-2}
\end{align}

for $d(x) = \text{dist}(x, \partial \Omega)$.

Estimate of $|s(x)|$. Let

\begin{equation}
\tilde{S}(x, y) = 2\pi \left(G(y, x) - J_2(y - x)\right) \tag{3.7}
\end{equation}

where $J_2 \in L^2(\mathbb{R}^2)$ is the Bessel potential of order 2, that is, the unique square-integrable function on $\mathbb{R}^2$ solving $(-\Delta + 1)J_2 = \delta_0$. Well-known properties of $J_2$ include the fact that it is smooth away from the origin, radial, with exponential
decay as $|x| \to \infty$; see, for example, [25], where rather explicit formulas may be found. This formula, or the maximum principle, implies that $\mathcal{J}_2 \geq 0$. Also, 

$$(-\Delta + 1)(2\pi \mathcal{J}_2(\cdot) + \log |\cdot|) = \log |\cdot| \in W^{1,p}_{\text{loc}}(\mathbb{R}^2) \text{ for all } p \in [1, 2).$$

Elliptic regularity thus implies that

$$2\pi \mathcal{J}_2(\cdot) + \log |\cdot| \in W^{3,p}_{\text{loc}}(\mathbb{R}^2) \subset C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2)$$

for all $p \in [1, 2)$, with $\alpha = \frac{2p-2}{p} \in [0, 1)$. In particular, $L := \lim_{\varepsilon \to 0}(2\pi \mathcal{J}_2(\varepsilon) + \log |\varepsilon|)$ exists, and as a result,

$$s(x) = S(x, x) = \lim_{y \to x^+} [\widetilde{S}(x, y) + 2\pi \mathcal{J}_2(x - y) + \log |x - y|]$$

(3.9)

$$= \widetilde{S}(x, x) + L.$$ 

It follows from the definitions of $G$ (see (1.4)) and $\widetilde{S}$ that for every $y \in \Omega$, 

$$(3.10) \begin{cases} (-\Delta_x + 1)\tilde{S} = 0 & \text{for } x \in \Omega, \\
\tilde{S}(x, y) = -2\pi \mathcal{J}_2(y - x) & \text{for } x \in \partial \Omega. \end{cases}$$

Then, since $\mathcal{J}_2 \geq 0$, the maximum principle and (3.8) easily imply that for every $y$,

$$0 \geq \max_{x \in \Omega} \tilde{S}(x, y) \geq \min_{x \in \Omega} \tilde{S}(x, y)$$

(3.11)

$$= \frac{1}{\min_{x \in \partial \Omega} -2\pi \mathcal{J}_2(x - y) \geq \log(d(y)) - C(\Omega)}$$

for all $y \in \Omega$. This and (3.9) imply the estimate of $|s(x)|$ stated in (3.6).

**Estimate of $|\nabla s|$.** Next, we use the chain rule, (3.9), and (3.10) to compute

$$\nabla s(x) = (\nabla_x \tilde{S}(x, y) + \nabla_y \tilde{S}(x, y))|_{y = x} = 2 \nabla_y \tilde{S}(x, y)|_{y = x},$$

where the second equality follows from the standard fact that $\tilde{S}(x, y) = \tilde{S}(y, x)$ for all $x$ and $y$. By differentiating (3.10), we find that

$$\begin{cases} (-\Delta_x + 1)\partial_{x_j} \tilde{S} = 0 & \text{for } x \in \Omega, \\
\partial_{x_j} \tilde{S}(x, y) = -2\pi \partial_{y_j} \mathcal{J}_2(y - x) & \text{for } x \in \partial \Omega, \quad \text{for } j = 1, 2. \end{cases}$$

From [3.8], we see that $|\nabla \mathcal{J}_2(y - x)| \leq C d(y)^{-1}$ for $y \in \Omega$ and $x \in \partial \Omega$, so we again use the maximum principle to deduce that

$$\sup_x |\nabla_y \tilde{S}(x, y)| \leq C d(y)^{-1}, \quad \text{and thus } |\nabla s(x)| \leq C d(x)^{-1}. \tag{3.12}$$

**Estimate of $|\nabla^2 s|$.** Finally, we compute

$$\Delta s(x) = (\Delta_x \tilde{S}(x, y) + 2 \nabla_x \cdot \nabla_y \tilde{S}(x, y) + \Delta_y \tilde{S}(x, y))|_{y = x}$$

$$= -2\tilde{S}(x, x) + 2 \nabla_x \cdot \nabla_y \tilde{S}(x, y)|_{y = x}. \tag{3.13}$$

In general, if $w(x)$ satisfies $(-\Delta + 1)w = 0$ in a ball $B(r, a)$, then standard elliptic theory (after rescaling, this is equivalent to the claim that if $(-\Delta + r^2)v = 0$
on $B(1)$, then $|Dv(0)| \leq C\|v\|_{\infty}$. This is proved by noting that in this context, standard interior estimates $\|v\|_{H^k(B(2^{-k}))} \leq C(k)\|v\|_{L^2(B(1))}$ hold with constants independent of $r$ implies that

$$|\nabla w(a)| \leq C \sup_{B(r,a)} |w|.$$ 

Fixing $y \in \Omega$, we apply this to $w(x) = \partial_{y_j} S(x, y)$ in $B(d(y), y)$ and use (3.12) to conclude that

$$|\nabla_x \partial_{y_j} \tilde{S}(y, y)| \leq C d(y)^{-2}$$

for every $y \in \Omega$. Then (3.13) and our earlier estimate of $|\tilde{s}|$ imply that $|\Delta s(x)| \leq C d(x)^{-2}$. □

3.2 An obstacle problem

Having modified $H_{e}^N$, we aim to study near minimizers in $M_{e, N}^*$ by characterizing an associated coincidence set.

The main result of this section is the following lemma, which yields some auxiliary functions that will play a key role in the proof of Proposition 2.1. We introduce a family of obstacle problems indexed by a parameter $\lambda > 0$. For each $\lambda$ in a certain range, this obstacle problem yields, among other things, a coincidence set $\Sigma_\lambda$; see (3.14). We will see in Lemma 3.4 below that, given suitable $h_{ex}$ and $N$, most vortices are found in $\Sigma_\lambda$ for the particular choice $\lambda = h_{ex} / 2\pi N$.

This is very convenient, since the obstacle problem formulation allows for the use of barriers, not only to estimate how far the coincidence set is from $\partial \Omega$ but also, as we will see later, the minimum cost of a vortex lying outside the coincidence set in terms of its distance to it.

**Lemma 3.2.** Let $\xi_e = \xi_0 + v_e$, where $v_e$ is the function found in Lemma 3.1 Then for $h_{ex} \geq K_1$ and $\lambda > (|\Omega| - h_{ex}^{-1/4})^{-1}$, there exist $m(\lambda) > 0$, a function $\varphi_\lambda \in C_0^{1,1}(\Omega)$, and a set $\Sigma_\lambda \subset \Omega$ such that

(3.14) $\begin{align*}
\xi_e := \lambda \xi_0 + \varphi_\lambda &\geq -m(\lambda), \\
\Sigma_\lambda = \text{supp}((-\Delta + 1)\varphi_\lambda) &\supset \{x \in \Omega : \xi_\lambda(x) = -m(\lambda)\}.
\end{align*}$

and

(3.15) $\int_{\Omega} (-\Delta + 1)\varphi_\lambda \, dx = 1.$

Moreover, there exist positive constants $\lambda_0 > \frac{1}{|\Omega|}$ and $c_2, \ldots , c_5$ such that for all $\epsilon < \epsilon_0$, 

...
if \( j = h \leq 4 \) ex, then \[
\min_{\Omega} \xi - m(\lambda) \leq -c_2 (|\Omega| - \frac{1}{\lambda})^2.
\] (3.16)
\[
\xi(x) \geq -2 \sqrt{c_3 \lambda (|\Omega| - \frac{1}{\lambda}) d(x) + \lambda d^2(x)}
\] if \( d(x) < \sqrt{\frac{c_3}{\lambda} (|\Omega| - \frac{1}{\lambda})} \).
\end{equation}

where we continue to use the notation \( d(x) = \text{dist}(x, \partial \Omega) \).

The new part of the lemma, apart from the dependence on \( \epsilon \) (which is very mild and hence mostly suppressed in our notation), consists of conclusions (3.16) and (3.17), which of course imply lower bounds on the distance between \( \Sigma_\lambda \) and \( \partial \Omega \).

Following ideas from [20], in Appendix A we obtain \( \varphi_\lambda \) from an obstacle problem, stated as follows. For \( \lambda, \mu \geq 0 \), define
\[
O_{\lambda, \mu} := \{ \varphi \in H_0^1(\Omega) : \varphi \geq -\lambda \xi - m \},
\]
\[
I(\varphi) := \frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 + \varphi^2 \, dx.
\]
Since \( I(\cdot) \) is strictly convex and \( O_{\lambda, \mu} \) is convex and nonempty, we may define
\[
\varphi_{\lambda, \mu} := \text{the unique minimizer of } I(\cdot) \text{ in } O_{\lambda, \mu},
\]
\[
\xi_{\lambda, \mu} := \lambda \xi + \varphi_{\lambda, \mu},
\]
\[
\Sigma_{\lambda, \mu} := \{ x \in \Omega : \xi_{\lambda, \mu}(x) = -m \}.
\]
Below we will define \( \varphi_\lambda = \varphi_{\lambda, \mu} \) for a suitable choice \( m = m(\lambda) \).

Well-known results about the obstacle problem (see, for example, [9]), guarantee that \( \varphi_{\lambda, \mu} \) is \( C^{1,1} \) and that
\begin{equation}
(-\Delta + 1) \varphi_{\lambda, \mu} = 1_{\Sigma_{\lambda, \mu}} \cdot (\lambda w_e - m) \geq 0,
\end{equation}
\begin{equation}
\text{for } w_e := (\Delta - 1) \xi := 1 + o(h_{\text{ex}}^{-1/4}).
\end{equation}

It follows that \( \xi_{\lambda, \mu} \in C^{1,1}(\Omega) \), and
\begin{equation}
(-\Delta + 1) \xi_{\lambda, \mu} = -\lambda w_e + 1_{\Sigma_{\lambda, \mu}} \cdot (\lambda w_e - m) \geq -\lambda w_e.
\end{equation}

This equation allows us to control certain aspects of \( \xi_{\lambda, \mu} \) by constructing sub- and supersolutions.

**Lemma 3.3.** Assume that \( \eta \in C^{1,1}(\Omega) \) and that \( \eta \geq -m \) everywhere in \( \Omega \).

Upper barrier: If \( \eta \geq 0 \) on \( \partial \Omega \), and \( (-\Delta + 1) \eta \geq -\lambda w_e \) a.e. in \( \Omega \), then \( \eta \geq \xi_{\lambda, \mu} \) in \( \Omega \), and thus \( \{ \eta = -m \} \subset \Sigma_{\lambda, \mu} \).
LOWER BARRIER: If \( \eta \leq 0 \) on \( \partial \Omega \), and \((-\Delta + 1) \eta \leq -\lambda w_e + 1 \mathbb{1}_{\{\eta = -m\}} \cdot (\lambda w_e - m)\) a.e. in \( \Omega \), then \( \eta \leq \zeta_{\lambda, m} \) in \( \Omega \), and thus \( \Sigma_{\lambda, m} \subset \{ \eta = -m \} \).

PROOF. For an upper barrier \( \eta \), it is clear that \( \zeta_{\lambda, m} - \eta = -m - \eta \leq 0 \) on \( \Sigma_{\lambda, m} \), so the claim follows by applying the strong maximum principle (which in two dimensions requires only \( W^{2,p} \) regularity for \( p \geq 2 \); see, for example, [6, theorem 9.1]) to \( \zeta_{\lambda, m} - \eta \leq 0 \) in \( \Omega \setminus \Sigma_{\lambda, m} \). The other case similarly follows by applying the strong maximum principle to to \( \eta - \zeta_{\lambda, m} \) on \( \{ x \in \Omega : \eta(x) > -m \} \). \( \square \)

PROOF OF LEMMA 3.2

Step 1. Let

\[
\varphi_{\lambda, m} := \int_{\Omega} (\Delta + 1) \psi_{\lambda, m} \, dx = \int_{\Sigma_{\lambda, m}} (\lambda w_e - m) \, dx.
\]

Lemma 3.3 and (3.20) immediately imply that

if \( 0 < m_1 < m_2 \), then \( \frac{m_1}{m_2} \zeta_{\lambda, m_2} \geq \zeta_{\lambda, m_1} \) and hence \( \Sigma_{\lambda, m_2} \subset \Sigma_{\lambda, m_1} \).

For \( m_1 < m_2 \), since \((\lambda w_e - m) \geq 0\) on \( \Sigma_{\lambda, m} \), it follows that

\[
f(\lambda, m_2) = \int_{\Sigma_{\lambda, m_2}} (\lambda w_e - m_2) \, dx \leq \int_{\Sigma_{\lambda, m_1}} (\lambda w_e - m_1) \, dx
\]

\[
\leq \int_{\Sigma_{\lambda, m_1}} (\lambda w_e - m_1) \, dx = f(\lambda, m),
\]

with strict inequality if \( f(\lambda, m_2) > 0 \).

When \( m = 0 \), it is clear that \( \eta = 0 \) is both an upper and lower barrier, and hence that \( \zeta_{\lambda, 0} = 0 \) and \( \Sigma_{\lambda, 0} = \Omega \). Thus

\[
f(\lambda, 0) = \lambda \int_{\Omega} w_e \, dx = 1 + o(h_{\text{er}}^{-1/4})
\]

In particular, \( f(\lambda, 0) > 1 \) if \( \frac{1}{\lambda} > |\Omega| - h_{\text{er}}^{-1/4} \) and \( h_{\text{er}} \) is large enough.

Similarly, if \( m > \lambda |\xi_e|_{\infty} \), then \( \eta = \lambda \xi_e \) is both an upper and lower barrier. Hence \( \zeta_{\lambda, m} = \lambda \xi_e \) (that is, \( \varphi_{\lambda, m} = 0 \)), so \( \Sigma_{\lambda, m} = \emptyset \) and

\[
f(\lambda, m) = 0.
\]

Since \( f(\lambda, \cdot) \) is strictly decreasing on its support, and compactly supported, there must then exist a unique \( m(\lambda) \) such that \( f(\lambda, m(\lambda)) = 1 \).

We now define \( \zeta_{\lambda} = \varphi_{\lambda, m(\lambda)} \), and similarly we set \( \zeta_{\lambda, m} = \zeta_{\lambda, m(\lambda)} \) and so on. We have just shown that (3.15) holds. The regularity of \( \varphi_{\lambda} \) and properties (3.14) follow directly from the construction of \( \zeta_{\lambda, m} \) and facts such as (3.20) about the obstacle problem.

Step 2. It remains to prove (3.16) and (3.17). In doing so, we will first assume that

\[
(\Omega| - h_{\text{er}}^{-1/4})^{-1} \leq \lambda \leq \lambda_0
\]

EXTREME Hysteresis for 2D Ginzburg-Landau 2011
for $\lambda_0 \in \left( \frac{1}{|\mathbb{T}|}, \frac{2}{|\mathbb{T}|} \right)$ to be chosen below, independent of $\epsilon \in (0, \epsilon_0]$. For $r > 0$ we will write

$$N_r := \{ x \in \Omega : d(x) < r \}.$$

Fix $d_0$, depending on $\Omega$, such that the boundary distance function $d$ is smooth on $N_{2d_0}$, and for $0 < \delta \leq d_0$ define

$$\eta_{\delta, m} := f_{\delta, m} \circ d \quad \text{for} \quad f_{\delta, m}(s) = \begin{cases} \frac{-2ms}{\delta} + \frac{ms^2}{\delta^2} & \text{if } s \leq \delta, \\ -m & \text{if } s > \delta. \end{cases}$$

Then $\eta_{\delta, m}$ is $C^{1,1}$, and

$$(-\Delta + 1) \eta_{\delta, m} = \begin{cases} -\frac{2m}{\delta^2} + f'_{\delta, m}(d(x)) \Delta d(x) + f_{\delta, m}(d(x)) & \text{in } N_{\delta}, \\ -m & \text{in } \Omega \setminus N_{\delta}. \end{cases}$$

We claim that there exist $m_0$ and $K_1$, depending only on $\Omega$, such that if $0 < m < m_0$ and $h_{\text{ex}} \geq K_1$, then

$$\eta_{\delta, m} \text{ is an upper barrier if } \delta = \frac{2m}{\sqrt{m/\lambda}} \leq d_0,$$

lower barrier if $\delta = \frac{2m}{\sqrt{m/\lambda}} \leq d_0$.

For example, if $\delta = \frac{2m}{\sqrt{m/\lambda}}$, clearly $|f'_{\delta, m}| \leq \frac{-2m}{\delta} = 2\sqrt{m/\lambda}$ and $|f_{\delta, m}| \leq m$, so (as long as $\delta \leq d_0$)

$$(-\Delta + 1) \eta_{\delta, m} \leq -2\lambda + 2\sqrt{m\lambda\kappa} + m \quad \text{in } N_{\delta} \text{ for } \kappa := \| \Delta d \|_{L^\infty(N_{2d})}. \quad (3.22)$$

We require $K_1$ be large enough that $w_{\epsilon} < 3/2$ whenever $h_{\text{ex}} \geq K_1$. Then (recalling that $\lambda > |\Omega|^{-1/2}$) one sees that $\eta_{\delta, m}$ is a lower barrier if $m_0$ satisfies

$$m_0 \leq \frac{1}{4}|\Omega|^{-1}, \quad 2\sqrt{m_0}\kappa \leq \frac{1}{4}|\Omega|^{-1/2}, \quad m_0 \leq |\Omega|^{-1}d_0^2,$$

where the first two conditions guarantee that $2\sqrt{m\lambda\kappa} + m \leq \frac{1}{2}\lambda$, and the last condition guarantees that $\delta < d_0$. The case of an upper barrier is essentially identical. Thus we have proved (3.22).

Next, we assume that $h_{\text{ex}} \geq K_1$ and $0 < m < m_0$, and we estimate $f(\lambda, m)$. It follows from (3.22) that

$$\Omega \setminus N_{2\sqrt{m/\lambda}} \subset \Sigma_{\lambda, m} \subset \Omega \setminus N_{\sqrt{m/\lambda}}.$$

Recalling (3.21), we infer that there exist $c < C$, depending only on $\Omega$, such that

$$|\Omega| - C\sqrt{m} \leq |\Sigma_{\lambda, m}| \leq |\Omega| - c\sqrt{m}. \quad \text{Since } \lambda w_{\epsilon} - m = \lambda - m + o(h_{\text{ex}}^{-1/4}), \text{ it follows that}$$

$$f(\lambda, m) = \int_{\Sigma_{\lambda, m}} (\lambda w_{\epsilon} - m) \, dx \leq (\lambda - m) (|\Omega| - c\sqrt{m}) + o(h_{\text{ex}}^{-1/4}).$$
Thus if $c \sqrt{m} = 2(|\Omega| - \lambda^{-1}) \geq 2h_{\text{ex}}^{-1/4}$, then
\[
f(\lambda, m) \leq (\lambda - m)(\lambda^{-1} - h_{\text{ex}}^{-1/4}) + o(h_{\text{ex}}^{-1/4}) < 1
\]
after increasing $K_1$ if necessary. Similarly, if $C \sqrt{m} = \frac{1}{2}(|\Omega| - \lambda^{-1}) \geq \frac{1}{2}h_{\text{ex}}^{-1/4}$, then
\[
f(\lambda, m) \geq (\lambda - m)(\lambda^{-1} - C \sqrt{m}) + o(h_{\text{ex}}^{-1/4}) > 1
\]
after adjusting $K_1$. Since $f$ is a decreasing function of $m$, we conclude that when $h_{\text{ex}} \geq K_1$ and $0 < m < m_0$, if $f(\lambda, m) = 1$, then there exist constants $C > c > 0$ such that
\[
c(|\Omega| - \lambda^{-1})^2 \leq m \leq C(|\Omega| - \lambda^{-1})^2.
\]
In other words,
\[
(3.23) \quad c(|\Omega| - \lambda^{-1})^2 \leq m(\lambda) \leq C(|\Omega| - \lambda^{-1})^2.
\]
In particular (3.16) holds if (3.21) is satisfied. (The condition $m < m_0(\Omega)$ translates to the upper bound $\lambda \leq \lambda_0$ in (3.21).

It now follows from (3.22) that $\eta_{\delta, m}$, with $m = m(\lambda)$ and $\delta = \sqrt{m/\lambda}$, is a lower barrier for $\xi_\lambda = \xi_{\lambda, m(\lambda)}$. Since $m \mapsto \eta_{\delta, m}$ is a decreasing function, we conclude from this and (3.23) that $\xi_\lambda \geq \eta_{\delta, m}$ when $m = C(|\Omega| - \lambda^{-1})^2$ and $\delta = \sqrt{m/\lambda}$, as long as $d(x) < \frac{c}{\sqrt{\lambda}}(|\Omega| - \lambda^{-1})$. This is exactly conclusion (3.17).

**Step 3.** Finally, we prove (3.16) and (3.17) for $\lambda > \lambda_0$.

First, for any $\delta \in (0, \delta_0)$ and $m > 0$, we compute as above that
\[
(-\Delta + 1)\eta_{\delta, m} \geq \frac{2m}{\delta^2} - \frac{2m}{\delta} - m = -\left(\frac{2}{\delta^2} + \frac{2}{\delta} + 1\right)m.
\]
It follows that for any $\delta$ as above, there exists $\theta(\delta) > 0$ such that
\[
(-\Delta + 1)\eta_{\delta, m} \geq -\frac{1}{2} \lambda \quad \text{whenever } 0 < \theta < \theta(\delta),
\]
and hence that $\eta_{\delta, \theta(\delta)}$ is an upper barrier for $\xi_{\lambda, \theta(\delta)}$. Here we are using the assumption that $w_e \geq \frac{1}{2}$ everywhere, which we have already imposed as a condition on $K_1$.

Then arguing as above, we estimate
\[
f(\lambda, \theta(\delta)) \geq (|\Omega| - |N_{\delta}|)(1 - \theta + o(h_{\text{ex}}^{-1/4}))\lambda.
\]
Since $\lambda_0 > |\Omega|^{-1}$, we may fix $\delta_0$ and $\theta_0 < \theta(\delta_0)$ so small that
\[
(|\Omega| - |N_{\delta_0}|)(1 - \theta_0)\lambda_0 > 1
\]
and hence also $f(\lambda, \theta_0 \lambda) > 1$ when $\lambda \geq \lambda_0$.

The monotonicity of $f(\lambda, \cdot)$ then implies that $m(\lambda) \geq \theta_0 \lambda$ whenever $\lambda \geq \lambda_0$.

It is also clear that $\varphi_{\lambda, m} \geq 0$ for all choices of parameters, and hence that $\xi_\lambda \geq \lambda \xi_e$. Hopf’s lemma implies that there exists a positive constant $c$ such that $\xi_0(x) \leq -c d(x)$ for all $x$, and it follows that $\xi_e(x) \leq -(c/2)d(x)$ for all sufficiently large $h_{\text{ex}}$. This proves (3.18). \qed
3.3 Proof of Proposition 2.1

We now complete the proof of the proposition by studying near-minimizers of $H^N$, defined in (3.3). Here we turn the delicate problem of estimating deviations of the energy due to small variations in the position of a single vortex. We reduce this problem, which is clearly nonlocal, to an almost local one by a screening procedure. This is a quantitative version of an argument (attributed by the authors of [15] to unpublished work of Lieb) for a discrete energy similar to ours but simpler in some respects, to bound from below minimum neighbor distances in minimizers.

**Lemma 3.4.** There exists $c_0, c_1, t_0 > 0$ such that if $N, h_{ex}$ satisfy (1.2), (1.3), and $a \in M^*_{e,N} := \{ a = (a_1, \ldots, a_N) \in \Omega^N : H^N_e(a) \leq \inf_{\Omega^N} H^N_e + t_0 \}$, then

\begin{align*}
(3.24) & \quad \text{dist}(a_i, \partial \Omega) \geq c_0 h_{ex}^{-1/4} \quad \text{for all } i, \\
(3.25) & \quad |a_i - a_j| \geq c_1 h_{ex}^{-1/2} \quad \text{for all } i \neq j.
\end{align*}

Proposition 2.1 is an immediate corollary. Indeed, it follows from (3.2) that $H^N_e = H^N_e$ in $M_{e,N}$ (defined in (1.12)). Thus for $a \in M^*_{e,N} \subset M_{e,N},$

\begin{equation*}
H^N_e(a) = \inf_{M_{e,N}} H^N_e + t_0 = \inf_{M_{e,N}} H^N_e + t_0 \leq \inf_{\Omega^N} H^N_e + t_0.
\end{equation*}

Hence $a \in M^*_{e,N}$, so Lemma 3.4 implies that $a$ satisfies (3.24) and (3.25), proving Proposition 2.1.

**Proof of Lemma 3.4** Assume that $a = (a_1, \ldots, a_N) \in M^*_{e,N}.$ For the proof, we will write

\[ \mu_\lambda := (-\Delta + 1)\varphi_\lambda \, dx = 1_{\Sigma_\lambda}(\lambda u_e - m(\lambda)) \, dx, \]

that is, the measure on $\Omega$ whose density with respect to the Lebesgue measure is $(-\Delta + 1)\varphi_\lambda$. Recall also that $\mu_\lambda$ is a probability measure, by (3.15).

Let

\[ \lambda = \frac{h_{ex}}{2\pi N} \geq \left( |\Omega| - h_{ex}^{-1/4} \right)^{-1}. \]

Then for any $\tilde{a}_1 \in \Omega$, we use the functions $\varphi_\lambda$ and $\xi_\lambda = \lambda \xi_e + \varphi_\lambda$ from Lemma 3.2 to write

\begin{align*}
\frac{1}{4\pi^2 N} [H^N_e(a_1, a_2, \ldots, a_n) - H^N_e(\tilde{a}_1, a_2, \ldots, a_n)] \\
= \lambda [\xi_e(a_1) - \xi_e(\tilde{a}_1)] + \frac{1}{N} \sum_{j=2}^{N} (G(a_1, a_j) - G(\tilde{a}_1, a_j)) \\
= [\xi_\lambda(a_1) + U(a_1)] - [\xi_\lambda(\tilde{a}_1) + U(\tilde{a}_1)]
\end{align*}
for

\[ U(x) = -\varphi_\lambda(x) + \frac{1}{N} \sum_{j=2}^{N} G(x, a_j), \]

so that

\[ \begin{cases} (-\Delta + 1)U = -\mu_\lambda + \frac{1}{N} \sum_{j=2}^{N} \delta_{a_j} & \text{in } \Omega, \\ U = 0 & \text{on } \partial \Omega. \end{cases} \]

Since \( a \in M_{x, N}^* \), it follows that

\[ (3.26) \quad [\xi_\lambda + U](a_1) \leq \min_{\Omega} [\xi_\lambda + U] + \frac{t_0}{4\pi^2 N}. \]

We claim that

\[ (3.27) \quad \inf_{\Omega} U < 0. \]

To prove this, assume toward a contradiction that \( U \geq 0 \) in \( \Omega \). Then

\[ \int_{\partial \Omega} v \cdot \nabla U = -\int_{\Omega} \Delta U \leq \int_{\Omega} (-\Delta + 1)U = -1 + \frac{N-1}{N} < 0, \]

using the equation for \((-\Delta + 1)U\) and the fact that \( \mu_\lambda \) is a probability measure. It follows that \( v \cdot \nabla U(x) > 0 \) at some \( x \in \partial \Omega \). But since \( U = 0 \) on \( \partial \Omega \), this contradicts our assumption that \( U \geq 0 \) in \( \Omega \), proving (3.27).

Now the boundary condition for \( U \) implies that \( \min_{\Omega} U < 0 \) is attained. This cannot occur at any \( a_j, j = 2, \ldots, N \) (where \( U = +\infty \)) or any other point of \( \Omega \setminus \operatorname{supp}(\mu_\lambda) \), where \((-\Delta + 1)U = 0\). Thus all minima of \( U \) are contained in \( \Sigma_\lambda = \operatorname{supp}(\mu_\lambda) \), which is exactly the set where \( \xi_\lambda \) attains its minimum. It follows that

\[ \min_{\Omega} [\xi_\lambda + U] = \min_{\Omega} \xi_\lambda + \min_{\Omega} U \leq \min_{\Omega} \xi_\lambda + U(a_1). \]

Thus we infer from (3.26) that

\[ (3.28) \quad \xi_\lambda(a_1) \leq \min_{\Omega} \xi_\lambda + \frac{t_0}{4\pi^2 N}. \]

Now (3.24) follows from Lemma 3.2. To prove this we consider two cases:

Case 1. \( \frac{1}{\lambda_0} \leq \frac{1}{\lambda} = \frac{2\pi}{h_{ex}} \leq |\Omega| - h_{ex}^{-1/4} \).

In this case, (3.16) implies that

\[ m(\lambda) \geq c_2 \left( |\Omega| - \frac{1}{\lambda} \right)^2 \geq c_2 h_{ex}^{-1/2} \geq c_2 (2\pi \lambda_0 N)^{-1/2} \geq \frac{2t_0}{4\pi^2 N} \]

for any \( N \geq 1 \) if \( t_0 \) is small enough. As a result,

\[ \xi_\lambda(a_1) \leq \min_{\Omega} \xi_\lambda + \frac{t_0}{4\pi^2 N} = -m(\lambda) + \frac{t_0}{4\pi^2 N} \leq -\frac{1}{2} m(\lambda) \leq -\frac{1}{2} c_2 (|\Omega| - \lambda^{-1})^2. \]
On the other hand, clearly from (3.17),
\[ d(a_i) \geq -C \zeta_\lambda(x) (|\Omega| - \lambda^{-1})^{-1} \geq c(|\Omega| - \lambda^{-1}) \geq c \chi_{ex}^{-1/4}, \]
proving (3.24).

**Case 2.** \( \frac{1}{\lambda^2} \geq \frac{1}{\lambda} = \frac{2\pi N}{\lambda^2}. \)

Then (3.24) follows by similar arguments, using (3.18) in place of (3.16) and (3.17).

**Proof of (3.25).** We now wish to prove a lower bound for the distance from \( a_1 \) to another point, say \( a_2 \). First, note that
\[ U(a_1) \leq \min_{\Omega} U + \frac{t_0}{4\pi^2 N} < \frac{t_0}{4\pi^2 N}, \]
exactly as with (3.28). Next, we let
\[ r = c_6 h_{ex}^{-1/2} \]
for \( c_6 \) to be fixed below, and we decompose
\[ U = \left( -\varphi_{\text{near}} + \frac{1}{N} G(\cdot, a_2) \right) + \left( -\varphi_{\text{far}} + \frac{1}{N} \sum_{j=3}^{N} G(x, a_j) \right) =: U_{\text{near}} + U_{\text{far}} \]
where \( \varphi_{\text{near}} \) and \( \varphi_{\text{far}} \) vanish on \( \partial \Omega \) and satisfy
\[ (-\Delta + 1) \varphi_{\text{near}} = \mathbf{1}_{B(r, a_2)} \cdot (\lambda w_e - m), \]
\[ (-\Delta + 1) \varphi_{\text{far}} = -\mathbf{1}_{\Sigma \setminus B(r, a_2)} \cdot (\lambda w_e - m), \]
where \( m = m(\lambda) \). We will show below that if \( c_6 \) is sufficiently small, then
\[ 0 \leq \max_{\partial B_r(a_2)} U_{\text{near}} < \min_{\partial B_{r/2}(a_2)} U_{\text{near}} = \frac{t_0}{4\pi^2 N}. \]
For the moment, we assume toward a contradiction that \( |a_1 - a_2| \leq \frac{\pi}{2} \). Then
\[ U_{\text{far}}(a_1) = U(a_1) - U_{\text{near}}(a_1) \]
\[ \leq \left( \frac{t_0}{4\pi^2 N} \right) - U_{\text{near}}(a_1) \]
\[ \leq \min_{\partial B_r(a_2)} U_{\text{far}} + \max_{\partial B_r(a_2)} U_{\text{near}} + \frac{t_0}{4\pi^2 N} - U_{\text{near}}(a_1) \]
\[ < \min_{\partial B_r(a_2)} U_{\text{far}} \]
and
\[ U_{\text{far}}(a_1) = U(a_1) - U_{\text{near}}(a_1) < 0 \quad \text{by (3.29) and (3.30).} \]
It follows that $U_{\text{far}}$ attains a negative local minimum in $B_r(a_2)$. But this cannot happen, by the maximum principle, since

$$(-\Delta + 1)U_{\text{far}} = 1_{\Sigma} \cdot (\lambda w_e - m) + \sum_{j \geq 3, \ |a_j - a_2| < r} \delta_{a_j} \geq 0 \quad \text{in } B_r(a_2).$$

Thus $|a_1 - a_2| > \frac{r}{2}$, proving (3.25). \qed

**Proof of (3.30).** Let $r = c_6 h_{\text{ex}}^{-1/2} \leq \frac{1}{2}$. Recall from (1.3) that $G(x, y) = \frac{1}{2\pi} (-\log |x - y| + S(x, y))$. In addition, it follows from (3.7) and (3.8) that $S$ can be written as the sum of a radial $C^1,\alpha$ function and a function $S(x, y)$ that satisfies certain estimates recorded in (3.11) and (3.12). These imply that

$$(3.31) \quad |S(x, y)| \leq C - \log d(y), \quad |\nabla_x S(x, y)| \leq C d(x)^{-1}.$$ 

Since

$$\varphi_{\text{near}}(x) = \int_{\Omega} G(x, y)(-\Delta + 1)\varphi(y) dy = \int_{B(r, a_2)} G(x, y)(\lambda w_e(y) - m) dy,$$

we can write $U_{\text{near}}$ in the form

$$(3.32) \quad U_{\text{near}}(x) = -\frac{1}{2\pi N} \log |x - a_2| + \psi_1(x) + \psi_2(x)$$

where

$$\psi_1(x) := \frac{1}{2\pi} \int_{B(r, a_2)} \log |x - y|(\lambda w_e(y) - m) dy,$$

$$\psi_2(x) := \frac{1}{2\pi N} S(x, a_2) - \left( \frac{1}{2\pi} \int_{B(r, a_2)} S(x, y)(\lambda w_e(y) - m) dy \right).$$

We have arranged that $h_{\text{ex}}$ is large enough that $w_e \leq 2$. Noting that $\log |x - y| < 0$ for $x, y \in B_r(a_2)$, we therefore have

$$(3.33) \quad |\psi_1(x)| \leq \frac{\lambda}{\pi} \int_{B(r, a_2)} \log |x - y| dy \leq \lambda r^2 \left( |\log r| + \frac{1}{2} \right) \leq \frac{c_6^2}{\pi N} (\log h_{\text{ex}} + C)$$

for $x \in B(r, a_2)$, where we have used the classical fact that the integral is maximized when $x = a_2$, together with the choices of $\lambda$ and $r$. Similarly,

$$|\nabla \psi_1(x)| \leq 2\lambda r$$

and thus

$$(3.34) \quad |\psi_1(x) - \psi_1(y)| \leq 4\lambda r^2 = \frac{2c_6^2}{N \pi} \quad \text{for } x, y \in B_r(a_2).$$
Also, it follows easily from (3.31) and (3.24) that for \( x \in B_r(a_2) \),
\[
|\psi_2(x)| \leq \left( \frac{1}{2\pi N} + \frac{c_6^2}{2N} \right) \left( 1 + \log h_{\text{ex}}^{1/4} \right)
\]
and
\[
|\nabla \psi_2(x)| \leq \frac{C}{N}(1 + c_6^2)h_{\text{ex}}^{1/4}
\]
and thus
\[
|\psi_2(x) - \psi_2(y)| \leq \frac{C}{N} h_{\text{ex}}^{-1/4} \quad \text{for} \ x, y \in B_r(a_2).
\]

It follows from (3.32), (3.33), and (3.35) that if \( |x - a_2| = r = c_6 h_{\text{ex}}^{-1/2} \), then
\[
U_{\text{near}}(x) \geq \frac{1}{2\pi N} (\log h_{\text{ex}}^{1/2} - \log h_{\text{ex}}^{1/4}) - \frac{c_6^2}{N} (1 + \log h_{\text{ex}})
\]
\[
= \frac{1}{8\pi N} \log h_{\text{ex}} - \frac{c_6^2}{N} (1 + \log h_{\text{ex}}).
\]
This is clearly positive if \( c_6 \) is small enough, whence the first inequality in (3.30).
Similarly, if \( |x - a_2| = r \) and \( |y - a_2| \leq r/2 \), then we deduce from (3.32), (3.34),
and (3.36) that
\[
U_{\text{near}}(y) - U_{\text{near}}(x) \geq \frac{1}{2\pi N} \log 2 - \frac{C}{N} (c_6^2 + o(h_{\text{ex}}^{-1/4})).
\]
By choosing \( K_1 \) large, \( c_6 \) small and \( t_0 \) small, we can therefore clearly arrange that
\[
U_{\text{near}}(y) - U_{\text{near}}(x) > \frac{t_0}{4\pi^2 N},
\]
completing the proof of (3.30). \( \square \)

4 Lower Bounds and Localization

In this section, we prove some results about pairs \((u, A)\) for which there exists \( N \in \mathbb{N} \) and \( a = (a_1, \ldots, a_N) \in \Omega^N \) such that

\[
\left( J(u) - \sum_{j=1}^{N} \pi \delta_{a_j} \right)_{\dot{W}^{-1,1}(\Omega)} \leq \sigma_\varepsilon
\]
and \( \sigma_\varepsilon \ll \rho_\varepsilon = \frac{1}{\pi} \min \{ \min_{i\neq j} |a_i - a_j|, \min_{i} \text{dist}(a_i, \partial \Omega) \} \). These are all adapted from results in [7] about the simplified Ginzburg-Landau functional without magnetic field.

Our first result provides very precise lower bounds for \( GL_\varepsilon(u, A) \) when (4.1) holds and \( \sigma_\varepsilon \) is small enough, for suitable values of other parameters such as \( N \) and \( \rho_\varepsilon \).
PROPOSITION 4.1. Let \( \Omega \) be a bounded, open, simply connected subset of \( \mathbb{R}^2 \) with \( C^1 \) boundary. Then there exists absolute constants \( C \) and \( C_1 \) with the following property:

Assume that \( (u, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \), and that the Coulomb gauge condition \((1.10)\) holds. If there exists \( a = (a_1, \ldots, a_N) \in \Omega^N \) for some \( N \geq 1 \) such that \((4.1)\) holds and

\[
4\epsilon \sqrt{\ln(\rho_a/\epsilon)} \leq 4a_\epsilon \leq a^* := \left[ \frac{\rho_a}{N^3} (\sigma_a + \epsilon E_\epsilon(u)) \right]^{1/2} \leq \frac{\rho_a}{NC_1},
\]

then

\[
GL_\epsilon(u, A) \geq H_\epsilon^N(a) + \kappa_\epsilon^{GL} - C \left[ \frac{N^5}{\rho_a} (\sigma_a + \epsilon E_\epsilon(u)) \right]^{1/2} - \varpi,
\]

where

\[
\varpi = \varpi(u) = C (\epsilon E_\epsilon(u))^2 + \epsilon h_{\epsilon x}^4 + \sigma_\epsilon^{7/8} (E_\epsilon(u) + h_{\epsilon x}^2)^{5/8}
\]

and

\[
\kappa_\epsilon^{GL} = h_{\epsilon x}^2 F(\xi_0) + N \left( \pi \ln \frac{1}{\epsilon} + \gamma \right).
\]

Here \( F(\cdot) \) was defined in \((1.7)\), and the definition of \( \gamma \) is discussed following \((4.11)\) below.

The next proposition shows that if \((u, A)\) satisfies \((4.1)\) and nearly attains the energy lower bound in \((4.3)\), then in fact \((4.1)\) can be strengthened significantly, after possibly adjusting the points \( a \in \Omega^N \) slightly.

PROPOSITION 4.2. Let \( \Omega \) be a bounded, open, simply connected subset of \( \mathbb{R}^2 \) with \( C^1 \) boundary. Then there exist constants \( C \) and \( C_2 \), depending on \( \text{diam}(\Omega) \), with the following property:

Assume that \( (u, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \) and that the Coulomb gauge condition \((1.10)\) holds. If there exists \( a = (a_1, \ldots, a_N) \in \Omega^N \) for some \( N \geq 1 \) such that \((4.1)\) holds with

\[
(4.6) \quad \sigma_\epsilon \leq \frac{\rho_a}{8C_2 N^3},
\]

and if in addition

\[
(4.7) \quad E_\epsilon(u) \geq 1 \quad \text{and} \quad \left[ \frac{N^5}{\rho_a} E_\epsilon(u) + \frac{N^{10}}{\rho_a^2} \sqrt{E_\epsilon(u)} \right] \leq \frac{1}{\epsilon},
\]

then there exist \( (\xi_1, \ldots, \xi_N) \in \Omega^N \) such that \( |\xi_i - a_i| \leq \frac{\rho_a}{2C_2 N^4} \) for all \( i \), and

\[
(4.8) \quad \left\| J(u) - \pi \sum_{i=1}^N \delta_{\xi_i} \right\|_{\mathcal{W}^{-1,1}} \leq C \epsilon \left[ N(C + \Sigma_\epsilon^{GL})^2 e^{\frac{1}{8} \Sigma_\epsilon^{GL}} + (C + \Sigma_\epsilon^{GL}) \frac{N^5}{\rho_a} + E_\epsilon(u) \right]
\]
where
\[
\Sigma_{GL}^e := \Sigma_e^{GL}(u, A, a) = GL_e(u, A) - \kappa_e^{GL} - H_e^N(a),
\]
(4.9)
\[
\Sigma_{e}^{GL} := \Sigma_e^{GL}(u, A, a) + \sigma(u).
\]

Results very much like Proposition 4.1 are proved in [10, theorem 4.1], but with
the leading terms on the right-hand side of (4.3) appearing in a different form that
is not well-suited to our purposes. There does not seem to be a counterpart of
Proposition 4.2 in [10].

4.1 A reduction

Both propositions are proved by reducing them to results in [7]. These relate the
simplified Ginzburg-Landau energy \(E_e\) to the renormalized energy
\(W^N : \Omega^N \rightarrow \mathbb{R}\),
introduced in the pioneering book of Bethuel, Brezis, and Hélein [1], and defined
in our context by
\[
W^N(a) = -\pi \sum_{i \neq j} \log |a_i - a_j| + \pi \sum_{i, j} R(a_i, a_j).
\]
(4.10)
Here \(R(\cdot, \cdot)\) is the regular part of the Green’s function for the Laplace operator on
\(\Omega\), i.e.,
\[
R(x, y) = 2\pi \Gamma(x, y) + \log|x - y|,
\]
where \(\Gamma = \Gamma(x, y)\) is characterized by
\[
-\Delta \Gamma = 4\pi \delta_y \text{ for } x \in \Omega, \quad \Gamma(x, y) = 0 \text{ for } x \in \partial\Omega.
\]
The results we will use from [7] involve the quantity
\[
\Sigma_{e}^{BBH} = \Sigma_e^{BBH}(u, a) = E_e(u) - \kappa_e^{BBH} - W^N(a) \quad \text{for } \kappa_e^{BBH}
\]
(4.11)
\[
:= N\left(\pi \log \frac{1}{\epsilon} + \gamma\right).
\]
where \(\gamma\) is the same constant appearing in (4.5), whose definition (which we will
not actually need) can be found in [1], lemma IX.1, where the constant was first
introduced.

The reduction of Propositions 4.1 and 4.2 to results from [7] will be carried out
by proving, roughly speaking, that \(\Sigma_{e}^{GL}(u, A, a) \approx \Sigma_{e}^{BBH}(u, a)\) in the regimes
we are interested in. More precisely, we will prove the following:

**Lemma 4.3.** Assume that \(u \in H^1(\Omega; \mathbb{C})\) satisfies (4.1) and that \(N \leq Ch_{ex}\).
\[
(4.12) \quad \min_{A \text{ satisfying (1.10)}} \left|\Sigma_{e}^{GL}(u, A, a) - \Sigma_{e}^{BBH}(u, a)\right| \leq \sigma(u).
\]

If this is known, both propositions follow essentially by transcribing results
from [7]:
PROOF OF PROPOSITION 4.1. This is a direct consequence of Lemma 4.3 and [7, theorem 2]. Indeed, the hypotheses of Proposition 4.1 imply those of [7, theorem 2], and one of the conclusions of that result is that the lower bound
\[ E_e(u) \geq W^N(a) + \kappa_e^{BBH} + \text{other positive terms} - C \left( \frac{N^5}{\rho_a} (\sigma_e + \epsilon E_e(u)) \right)^{1/2}. \]
This and (4.12) immediately imply (4.3). In fact, they imply a stronger result, one that we have not recorded here, as we do not need the extra positive terms. See [7] or [10] for more. □

PROOF OF PROPOSITION 4.2. This follows immediately from Lemma 4.3 and [7, theorem 3]. Indeed, Proposition 4.2 and the cited result from [7] have exactly the same hypotheses. The conclusion of the result in [7] is (4.8), but with \( \Sigma_e^{BBH}(u, A, a) \) in place of \( \Sigma_e^{GL}(u, A, a) \). Since Lemma 4.3 implies that
\[ \Sigma_e^{BBH}(u, A, a) \leq \Sigma_e^{GL}(u, A, a), \]
the result follows. □

4.2 Proof of Lemma 4.3
The proof has two main steps. The first is to show that \( \Sigma_e^{BBH}(u, A, a) \) is close to
\[
\Sigma_e^{GL}(u, A, a) := GL_e^{\min}(u) - \min_{B \in H^2 \cap H^1_0} \left( \Phi(B) - 2\pi \sum_{i=1}^N B(a_i) \right) - W^N(a) - \kappa_e^{BBH},
\]
where
\[
GL_e^{\min}(u) := \min_{A \text{ satisfying (1.10)}} GL_e(u, A) \quad \text{and}
\]
\[
\Phi(B) := \frac{1}{2} \int_{\Omega} |\nabla \cdot B|^2 + (\Delta B + h_{ex})^2.
\]
We will then check that
\[
\min_{A \text{ satisfying (1.10)}} \Sigma_e^{GL}(u, A, a) = \Sigma_e^{BBH}(u, A, a).
\]

Lemma 4.4. Assume that \( u \in H^1(\Omega; \mathbb{C}) \) satisfies (4.1) and that \( N \leq h_{ex}|\Omega|/2\pi \). Then for any \( \theta \in (0, 1) \)
\[
\left| \Sigma_e^{GL}(u, A, a) - \Sigma_e^{BBH}(u, A, a) \right| \leq \omega.
\]
The implicit constants in (4.15) depend on \( \Omega \) and \( \theta \) and the assumed bound on \( N/h_{ex} \).
Step 1. Preliminaries. We start from the algebraic identity

\[ GL_e(u, A) = \int_\Omega j(u) \cdot A + \frac{1}{2} \int_\Omega |A|^2 + |\nabla \times A - h_{ex}|^2 + R(u, A) \]

where

\[ R(u, A) := \frac{1}{2} \int_\Omega (|u|^2 - 1)|A|^2. \]

We rewrite in terms of \( B = (\nabla^\perp)^{-1} A \) (see (1.11)), then integrate by parts to find that

\[ GL_e(u, \nabla^\perp B) = E_e(u) - 2 \int_\Omega B J u + \Phi(B) + R(u, \nabla^\perp B). \]

Thus

\[ GL_e(u, \nabla^\perp B) = -R(u, \nabla^\perp B) - 2 \int_\Omega B \left( J u - \pi \sum_{i=1}^N \delta a_i \right). \]

Step 2. Lower bound for \( \Sigma_e^{GL}(u, a) \). We now prove that

\[ \Sigma_e^{GL}(u, a) - \Sigma_e^{BBH}(u, a) \geq -e (E_e(u)^2 + h_{ex}^4) - \sigma_e \frac{1}{m} (E_e(u) + h_{ex})^{\frac{1}{2} + \frac{1}{m}}. \]

For this, let \( A_* = \nabla^\perp B_* \) minimize \( B \mapsto GL_e(u, \nabla^\perp B) \), so that \( GL_e^\min(u) = GL_e(u, \nabla^\perp B_*) \). Then

\[ E_e(u) + \frac{1}{2} |\Omega| h_{ex}^2 = GL_e(u, 0) \geq GL_e(u, \nabla^\perp B_*) \geq \int_\Omega |\Delta B_* + h_{ex}|^2. \]

Since \( B_* = 0 \) on \( \partial \Omega \), basic elliptic estimates (see, for example, [5, sec. 6.3.2, theorem 4] and the remark that follows) imply that

\[ \|B_*\|_{H^2} \leq C \|\Delta B_*\|_{L^2} \leq C (\|\Delta B_* + h_{ex}\|_{L^2} + h_{ex}) \leq C (E_e(u) + h_{ex}). \]

Since \( W^{1,m} \) embeds into \( H^2 \) for every \( m < \infty \) (and since we always assume \( h_{ex} \geq 1 \)), it follows that

\[ |R(u, \nabla^\perp B_*)| \leq \int_\Omega \frac{|u|^2 - 1)^2}{\epsilon} + e |\nabla^\perp B_*|^4 \leq C \epsilon (E_e(u)^2 + h_{ex}^4). \]

For the remaining term, for any \( m > 2 \) we estimate

\[ \left| \int B_* \left( J u - \pi \sum_{i=1}^N \delta a_i \right) \right| \leq \left\| J u - \pi \sum_{i=1}^N \delta a_i \right\|_{W^{-1,m}} \|B_*\|_{W^{0,1,m}}. \]
For any \( m > 2 \), an interpolation inequality and (4.1) yield

\[
\begin{align*}
\left\| J u - \pi \sum_{i=1}^{N} \delta a_i \right\| & \leq \left\| J u - \pi \sum_{i=1}^{N} \delta a_i \right\|_{W^{1-\frac{2}{m}}_{\text{rad}}} \\
& \leq \left\| J u - \pi \sum_{i=1}^{N} \delta a_i \right\|_{L^1} \\
& \leq \sigma e^{1-\frac{2}{m}} \left\| J u - \pi \sum_{i=1}^{N} \delta a_i \right\|_{L^1} \leq C \sigma e^{1-\frac{2}{m}} (E_e(u) + \pi N)^{\frac{2}{m}},
\end{align*}
\]

since

\[
\left\| J u - \pi \sum_{i=1}^{N} \delta a_i \right\|_{L^1} \leq \| J u \|_{L^1} + \left\| \pi \sum_{i=1}^{N} \delta a_i \right\|_{L^1} \leq C E_e(u) + \pi N.
\]

Since \( N \leq h_{\text{ex}}|\Omega|/2\pi \), it follows that

\[
GL_e^{\min}(u) = GL(u, \nabla B_B)
\]

\[
\geq E_e(u) + \left( \Phi(B_B) - 2\pi \sum_{i=1}^{N} B_B(a_i) \right) - C e (E_e(u)^2 + h_{\text{ex}}^2) - C \sigma e^{1-\frac{2}{m}} (E_e(u) + h_{\text{ex}}^2)^{\frac{1}{2} + \frac{2}{m}}.
\]

Subtracting \( W^N(a) + \kappa e_B B_B \) from both sides and rearranging, we obtain (4.19).

**Step 3.** Upper bound for \( \Sigma^G_L(u, a) \). The opposite inequality is proved in a very similar way. First, for any \( B \),

\[
\| B \|_{L^2} \leq C \| \Delta B \|_{L^2} \leq C(\| \Delta B + h_{\text{ex}} \|_{L^2} + h_{\text{ex}}) \leq C(\sqrt{\Phi(B)} + h_{\text{ex}}).
\]

Thus, using the bound \( N \leq h_{\text{ex}}|\Omega|/2\pi \),

\[
\Phi(B) - 2\pi \sum_{i=1}^{N} B_B(a_i) \geq \Phi(B) - NC(\sqrt{\Phi(B)} + h_{\text{ex}}).
\]

It follows from this and elementary inequalities, together with our standing assumption \( N \leq C|\Omega| \), that

\[
\Phi(B) - 2\pi \sum_{i=1}^{N} B_B(a_i) \geq -C h_{\text{ex}}^2.
\]

Thus standard lower semicontinuity arguments imply that the minimum of the left-hand side is attained. Let \( \beta \) denote a minimizer. It is clear that

\[
\Phi(\beta) - 2\pi \sum_{i=1}^{N} \beta(a_i) \leq 0,
\]

since otherwise we could decrease the value of the functional by taking \( \beta = 0 \). Then it follows from (4.26) that

\[
\Phi(\beta) \leq C h_{\text{ex}}^2 \quad \text{and hence} \quad \| \beta \|_{H^2}^2 \leq C h_{\text{ex}}^2.
\]
Thus $\beta$ satisfies the same estimate as $B_\ast$ in (4.21)—a slightly stronger estimate actually, although we will not use the improvement. We can thus estimate the error terms exactly as above to conclude that
\[
GL_{\min}^{\epsilon}(u) \leq GL(u, \nabla^2 \beta)
\]
\[
\leq E_{\epsilon}(u) + \Phi(\beta) - 2\pi \sum_{i=1}^{N} \beta(a_i)
+ Ce(E_{\epsilon}(u)^2 + h_{ex}^4) + C\alpha e^{1-\frac{2}{m}}(E_{\epsilon}(u) + h_{ex}^2)^{\frac{1}{2} + \frac{2}{m}}.
\]
Recalling the choice of $\beta$ and rewriting as above, this implies that
\[
\tilde{\Sigma}_{\epsilon}^{GL}(u, a) - \Sigma_{\epsilon}^{BBH}(u, A) \leq -\epsilon(E_{\epsilon}(u)^2 + h_{ex}^4) - \sigma e^{1-\frac{2}{m}}(E_{\epsilon}(u) + h_{ex}^2)^{\frac{1}{2} + \frac{2}{m}}.
\]
Choosing $m = \frac{1}{16}$ completes the proof. \qed

To finish the proof of Lemma 4.3, we will rewrite $\tilde{\Sigma}_{\epsilon}^{GL}(u, a)$, which was defined in (4.13) and (4.14). Toward this end, we fix $a = (a_1, \ldots, a_N) \in \Omega^N$, and we let $\beta \in H^2 \cap H_0^1(\Omega)$ denote the unique minimizer of
\[
B \mapsto \Phi(B) - 2\pi \sum_{i=1}^{N} B(a_i).
\]
We want to find a simple expression for $\Phi(\beta) - 2\pi \sum_{i=1}^{N} \beta(a_i)$.

**Lemma 4.5.** $\beta$ belongs to $W^{3,p}(\Omega)$ for every $p < 2$, and satisfies
\[
\Delta^2 \beta - \Delta \beta = 2\pi \sum_{i=1}^{N} \delta_{a_i} \quad \text{in } \Omega, \quad \beta = 0 \text{ on } \partial \Omega, \quad -\Delta \beta = h_{ex} \quad \text{on } \partial \Omega.
\]

We omit the proof, which is standard.

We now define $B_1 : \Omega \to \mathbb{R}$ as the solution of the boundary value problem
\[
\Delta^2 B_1 - \Delta B_1 = 2\pi \sum_{i=1}^{N} \delta_{a_i} \quad \text{in } \Omega, \quad B_1 = 0 \text{ on } \partial \Omega, \quad -\Delta B_1 = 0 \quad \text{on } \partial \Omega.
\]
With this notation we can state the following:

**Lemma 4.6.**
\[
\min_{B \in H_0^1 \cap H^2} \left[ \Phi(B) - 2\pi \sum_{i} B(a_i) \right]
= h_{ex}^2 F(\xi_0) + 2\pi h_{ex} \sum_{i} \xi_0(a_i) - \pi \sum_{i} B_1(a_i).
\]
where $\xi_0$ is defined in (1.8).
PROOF. By differentiating the equation satisfied by $\xi_0$, one finds that
\[ \Delta^2 \xi_0 - \Delta \xi_0 = 0 \text{ in } \Omega, \quad \xi_0 = 0 \text{ on } \partial \Omega, \quad -\Delta \xi_0 = -1 \text{ on } \partial \Omega. \]
It follows that
\[ \beta = -h_{\text{ex}} \xi_0 + B_1. \tag{4.29} \]
Defining $w_1 = -\Delta B_1$, it follows that $\Delta \beta = -h_{\text{ex}} (\xi_0 + 1) - w_1$, and hence that
\[ (h_{\text{ex}} + \Delta \beta)^2 = (h_{\text{ex}} \xi_0 + w_1)^2. \tag{4.30} \]
Furthermore,
\[ -\Delta (w_1 + B_1) = (-\Delta + 1) w_1 = (\Delta^2 - \Delta) B_1 = 2\pi \sum_i \delta_{a_i}. \tag{4.31} \]
Using (4.29) and (4.30), we rewrite
\[ \Phi(\beta) = \frac{1}{2} \int |\nabla \beta|^2 + (\Delta \beta + h_{\text{ex}})^2 \]
\[ = \frac{1}{2} \int |\nabla (h_{\text{ex}} \xi_0 - B_1)|^2 + (h_{\text{ex}} \xi_0 + w_1)^2 \]
\[ = h_{\text{ex}}^2 \left( \frac{1}{2} \int |\nabla \xi_0|^2 + \xi_0^2 \right) + h_{\text{ex}} \left( \int -\nabla \xi_0 \cdot \nabla B_1 + \xi_0 w_1 \right) \]
\[ + \left( \frac{1}{2} \int |\nabla B_1|^2 + w_1^2 \right). \]
For the second term on the right-hand side, note that
\[ \int \nabla \xi_0 \cdot \nabla B_1 - \xi_0 w_1 = 0 \quad \text{since } \xi_0 \in H_0^1 \text{ and } -\Delta B_1 = w_1. \]
For the final term on the right-hand side, since $B_1 = w_1 = 0$ on $\partial \Omega$ and $w_1 = -\Delta B_1$, we integrate by parts and use the equations (in particular (4.31)) to find that
\[ \frac{1}{2} \int |\nabla B_1|^2 + w_1^2 = \frac{1}{2} \int -B_1 \Delta B_1 - w_1 \Delta B_1 = -\frac{1}{2} \int (B_1 + w_1) \Delta B_1 \]
\[ = \pi \sum_i B_1(a_i). \]
The conclusion now follows by using these facts and (4.29) to rewrite $\Phi(\beta) = 2\pi \sum_i \beta(a_i)$. \hfill \Box

Finally, we complete the following:

PROOF OF LEMMA 4.3 In view of Lemma 4.4, we must show that
\[ \min_A \Sigma^{GL}_\varepsilon (u, A, a) = \Sigma^{GL}_\varepsilon (u, a). \]
By comparing the definitions, we see that this is the same as
\[ H^N_\varepsilon (a) + h_{\text{ex}}^2 F(\xi_0) = \min_{B \in H_0^1 \cap H^2} \left[ \Phi(B) - 2\pi \sum_i B(a_i) \right] + W^N(a). \tag{4.32} \]
Using the previous lemma and various definitions (see (1.5), (1.9), and (4.10)), this reduces to checking that
\[ \pi \sum_{i,j=1}^{N} S(a_i, a_j) = \pi \sum_{i,j=1}^{N} R(a_i, a_j) - \pi \sum_{i=1}^{N} B(a_i). \]
So we only need to prove that
\[ B_1(x) = \sum_{j} R(x, a_j) - S(x, a_j) \]
for all \( x \). To do this, we use the equations for \( G \) and \( \Gamma \) to compute
\[ -\Delta \left( \sum_{j} R(x, a_j) - S(x, a_j) \right) = -\Delta_x \left( 2\pi \sum_{j} \Gamma(x, a_j) - G(x, a_j) \right) = 2\pi \sum_{j} G(x, a_j). \]
Thus, applying \((-\Delta + 1)\) to both sides, we find that \( \sum_{j} R(., a_j) - S(., a_j) \) satisfies the boundary value problem that characterizes \( B_1 \). Thus we have completed the proof. \( \square \)

5 Energy-Minimizers in \( \mathcal{A}_{e}^N \)

The proposition below completes our argument; once it is known, the proof of Theorem 1.1 follows by exactly the argument given in Section 2.2.

Recall that \( t_0 \) is a constant that appears in the definition of \( \mathcal{A}_{e}^N \) and was fixed in the proof of Lemma 3.4.

**Proposition 5.1.** Assume that \((u_e, A_e)\) minimizes \( GL_e \) in \( \mathcal{A}_{e}^N \) and that \( A_e \) satisfies the Coulomb gauge condition (1.10). Then
\[ GL_e(u_e, A_e) \leq \min_{M_e,N} H_e^N + \kappa_e^{GL} + \frac{t_0}{3}, \]
and in addition, there exists \( \xi \in M_e,N \) such that
\[ \left\| J u_e - \pi \sum_{i=1}^{N} \delta_{\xi_i} \right\|_{W^{-1,1}} \leq \frac{1}{2} \sigma_e \]
and
\[ GL_e(u_e, A_e) \geq H_e^N(\xi) + \kappa_e^{GL} - \frac{t_0}{3}. \]

Before we can use results from the previous section effectively, we need to control \( E_e(u_e) \), which appears in many error terms. This is the point of our first lemma.
**Lemma 5.2.** Assume that \((u, A)\) satisfies (4.1) for some \(\sigma_0 > 0\) and \(a \in \Omega^N\), with \(N \leq h_{\text{ex}} |\Omega| / 2\pi\). Then
\[
E_\varepsilon(u) \leq GL_\varepsilon(u, A) + Ch_{\text{ex}}^2 + C\varepsilon h_{\text{ex}}^4 + C\varepsilon GL_\varepsilon(u, A)^2 \\
+ C\sigma_\varepsilon(GL_\varepsilon(u, A) + h_{\text{ex}}^2)^{3/2}.
\]
for constants depending only on \(\Omega\).

**Proof.** Recall from (4.18) the identity
\[
E_\varepsilon(u) = GL_\varepsilon(u, \nabla B) - \left(\Phi(B) - 2\pi \sum_{i=1}^N B(a_i)\right) \\
+ R(u, \nabla B) + 2\int_{\Omega} B \left(Ju - \pi \sum \delta_{ai}\right),
\]
where \(R(u, A) := \frac{1}{2} \int_{\Omega} (|u|^2 - 1)|A|^2\). Arguing as in (4.20) and (4.21), we see that \(B = (\nabla B)^{-1} A\) satisfies
\[
\|B\|_{H^2}^2 \leq C \left(GL_\varepsilon(u, A) + h_{\text{ex}}^2\right).
\]
As in (4.22), we deduce that
\[
|R(u, A)| \leq C\varepsilon(GL_\varepsilon(u, A)^2 + h_{\text{ex}}^4).
\]
Next, by combining (4.23) and (4.24) with (5.5) and a Sobolev embedding, we find that for any \(m > 2\),
\[
\left|\int_{\Omega} B \left(Ju - \pi \sum \delta_{ai}\right)\right| \leq C\sigma_\varepsilon \left(GL_\varepsilon(u, A) + h_{\text{ex}}^2\right) \left(E_\varepsilon(u) + \pi N\right)^{\frac{2}{m}}.
\]
Taking \(m = 6\) and using elementary inequalities, we deduce that the right-hand side is bounded by
\[
\frac{1}{3} E_\varepsilon(u) + C h_{\text{ex}} + C\sigma_\varepsilon(GL_\varepsilon(u, A) + h_{\text{ex}}^2)^{3/2}.
\]
The lemma follows by combining these estimates with (4.27). \(\square\)

**Proof of Proposition 5.1.**

Step 1. Our eventual aim is to use Propositions 4.1 and 4.2 from the previous section, adjusting the parameters in our scaling assumptions both to arrange that the hypotheses are satisfied and to control the error terms. We will go through the choice of parameters rather carefully to make it clear that there is nothing circular in our argument. We recall the assumptions:
\[
0 < \varepsilon < \varepsilon_0, \quad K_1 \leq h_{\text{ex}} \leq k_1 \varepsilon^{-1/4},
\]
\[
1 \leq N \leq \min \left\{\frac{h_{\text{ex}}}{2\pi} \left(|\Omega| - h_{\text{ex}}^{-1/4}\right), k_2 \varepsilon^{-1/10} h_{\text{ex}}^{-1/5}\right\}.
\]
For definition (1.16) of \(A_\varepsilon^N\), we will choose
\[
\sigma_\varepsilon = \varepsilon^{99/100} \max \left\{N^5 h_{\text{ex}}^{1/2}, h_{\text{ex}}^2\right\}.
\]
If $a \in M_{e,N}$, then it follows from Proposition 2.1 that
$$\rho_a \geq c_1 h_{ex}^{-1/2}.$$ 
Finally, we will only apply Propositions 4.1 and 4.2 to $(u, A)$ such that

$$E_\varepsilon(u) \leq C_3 h_{ex}^2$$

for $C_3(\Omega)$ to be determined below. (In fact, we will take $C_3 = \max\{C_5, C_6\}$, where these constants are identified below.) We have already imposed conditions on $K_1$ for example. We now adjust $\epsilon_0, k_1, k_2$ as follows.

First, by decreasing $\epsilon_0$ and $k_1$ as necessary,

$$\frac{1}{2} \sum_{\varepsilon} G_{\varepsilon} \leq t_0 \text{ and } u \text{ satisfies (5.7)},$$

$$\sigma(u) \leq \frac{t_0}{6} \text{ if } u \text{ satisfies (5.7)}.$$  

Similarly, by decreasing $k_2$ we may assume that

$$\text{hypotheses (4.6) and (4.7) of Proposition 4.2 hold if } u \text{ satisfies (5.7)}. $$

Hypothesis (4.2) of Proposition 4.1 involves both an upper and lower bound on $\sigma_\varepsilon$. The lower bound $\sigma_\varepsilon \geq \epsilon \sqrt{\log(\rho_0/\varepsilon)}$ follows from the form of $\sigma_\varepsilon$, after possibly adjusting $\epsilon_0$, and the upper bound is less stringent than (4.6), already satisfied. Finally, we claim that by further decreasing $\epsilon_0$, we can arrange that

$$N^{-5} \rho_a^{-1}(\sigma_\varepsilon + \epsilon E_\varepsilon(u))$$

is as small as we like, and hence, in view of (5.8), that

$$\text{right-hand side of (4.3)} \geq H^N_N(a) + \kappa^G_L - \frac{t_0}{3} \text{ if } u \text{ satisfies (5.7)}. $$

To see this, note that

$$N^{-5} \rho_a^{-1} \sigma_\varepsilon \leq CN^{-5} h_{ex}^{1/2} \sigma \leq C \epsilon^{99/100} \max\{N^{10} h_{ex}, N^{5} h_{ex}^{5/2}\}.$$  

Using $N \leq k_2(\epsilon h_{ex}^2)^{-1/10}$ for large $h_{ex}$ and $N \leq Ch_{ex}$ for small $h_{ex}$, we deduce that

$$N^{-5} \rho_a^{-1} \sigma_\varepsilon \begin{cases} C(k_2) \max\{\epsilon^{-1/10} h_{ex}^{1}, \epsilon^{49/100} h_{ex}^{3/2}\} & \text{if } \epsilon^{-1/12} \leq h_{ex} \leq \epsilon^{-1/4}, \\
C \epsilon^{99/100} h_{ex}^{11} & \text{if } h_{ex} \leq \epsilon^{-1/12}, \end{cases}$$

which can be made as small as we like by a suitable choice of $\epsilon_0$. Similar considerations show that the same holds for $N^{-5} \rho_a^{-1} \epsilon E_\varepsilon(u)$, subject to (5.7). Thus we may achieve (5.10).

Step 2. We now prove (5.1). We start by noting that for every $N \geq 1$ satisfying (1.3),

$$\min_{a \in M_{e,N}} H^N_N(a) = \min_{a \in \Omega^N} H^N_N \leq 0.$$ 

The equality on the left is clear from Lemma 3.4, which, for this range of $N$, implies that $\min_{\Omega^N} H^N_N$ is attained in $M_{e,N}$, in which $H^N_N = H^N_\varepsilon$. The inequality
follows by an easy induction argument, which relies on the fact that $\xi_0$ and $v_\epsilon$ vanish on $\partial \Omega$, as does $G(\cdot,a)$, for every $a \in \Omega$.

Let $a$ minimize $H^N_\epsilon$ in $M^*_\epsilon,N$. Then we deduce from (4.32), (4.27), and (5.11) that

$$W^N_\epsilon(a) \leq C_4 h^2_{\text{ex}}, \quad C_4 = C_4(\Omega).$$

According to lemma 14 in [7], there exists $u_\epsilon \in H^1(\Omega)$ such that

$$\|Ju_a - \pi \sum_{i=1}^N \delta_{a_i}\|_{W^{-1,1}} \leq C N\epsilon (1 + \epsilon N^2 \rho_a^{-2})$$

and

$$\Sigma^{BBH}_\epsilon (u_\epsilon,a, a) = E_\epsilon (u_\epsilon) - \kappa^G_{\epsilon} - W^N_\epsilon(a) \leq C N\epsilon^2 \rho_a^{-2}.$$ 

In particular, it follows from the above and (5.12) that

$$E_\epsilon (u_\epsilon) \leq C_5 h^2_{\text{ex}}$$

for $C_5$ depending only on $\Omega$. It then follows from (5.8) that

$$\overline{\omega}(u_\epsilon) \leq \frac{t_0}{6}.$$ 

Now let $A_\epsilon$ minimize $A \mapsto GL_e(u_\epsilon, A)$ among all competitors satisfying the Coulomb gauge condition (1.10). It follows that for $\epsilon < \epsilon_0$ with $\epsilon_0$ is small enough, then

$$(u_\epsilon, A_\epsilon) \in \mathcal{A}^N_{\epsilon}, \quad \Sigma^{BBH}_\epsilon (u_\epsilon, a) \leq \frac{t_0}{6}.$$ 

Then we infer from the above and Lemma 4.3 that the minimizer $(u_\epsilon, A_\epsilon)$ satisfies

$$GL_e(u_\epsilon, A_\epsilon) - \min_{M_{\epsilon,N}} H^N_\epsilon - \kappa^G_{\epsilon} \leq GL_e(u_\epsilon, A_\epsilon) - \min_{M_{\epsilon,N}} H^N_\epsilon - \kappa^G_{\epsilon}$$

$$= GL_e(u_\epsilon, A_\epsilon) - H^N_\epsilon(a) - \kappa^G_{\epsilon}$$

$$= \sum^G_{\epsilon}(u_\epsilon, A_\epsilon, a)$$

$$\leq \Sigma^{BBH}_\epsilon (u_\epsilon, a) + \overline{\omega}(u_\epsilon) \leq \frac{t_0}{3},$$

proving (5.1).

Step 3. From (5.1), (5.11), and (5.4), we see that

$$E_\epsilon (u_\epsilon) \leq C_6 h^2_{\text{ex}}.$$ 

Since $u_\epsilon \in \mathcal{A}^N_{\epsilon}$, there exists $a_\epsilon \in M^*_\epsilon,N$, such that (4.1) holds. We have arranged above that the other hypotheses (4.6) and (4.7) of Proposition 4.2 hold. Also, it follows from Step 1 above that

$$\sum^G_{\epsilon}(u_\epsilon, A_\epsilon, a_\epsilon) = \sum^G_{\epsilon}(u_\epsilon, A_\epsilon, a_\epsilon) + \overline{\omega}(u_\epsilon) \leq \frac{t_0}{2}.$$
Thus Proposition \(4.2\) (see in particular \((4.8)\)) and \((5.8)\) yield \(\xi \in \Omega^N\) such that \((5.2)\) is satisfied. It follows from this and \((4.1)\) that
\[
\left\| \pi \sum_{i=1}^{N} (\delta_{a_i} - \delta_{\xi_i}) \right\|_{\dot{W}^{-1,1}} \leq \frac{3}{2} \sigma_\varepsilon \leq \frac{\pi}{2} \h_{\text{ex}}^{-1/3} \quad \text{for } 0 < \varepsilon < \varepsilon_0,
\]
and this and \((3.24)\) imply that
\[
\xi \in M_{\varepsilon,N} \quad \text{or in other words} \quad d(\xi_i) = \text{dist}(\xi_i, \partial \Omega) \geq \h_{\text{ex}}^{-1/3} \text{ for every } i.
\]
To see this, assume toward a contradiction that
\[
d(\xi_i) < \h_{\text{ex}}^{-1/3} \quad \text{for some } i,
\]
and define
\[
f(x) := \begin{cases} 0 & \text{if } d(x) \leq \h_{\text{ex}}^{-1/3}, \\ d(x) - \h_{\text{ex}}^{-1/3} & \text{if } \h_{\text{ex}}^{-1/3} \leq d(x) \leq 2\h_{\text{ex}}^{-1/3}, \\ \h_{\text{ex}}^{-1/3} & \text{if } d(x) \geq 2\h_{\text{ex}}^{-1/3}.
\end{cases}
\]
It follows from \((3.24)\) and the assumption that \(d(\xi_i) < \h_{\text{ex}}^{-1/3} \quad \text{for some } i\) that
\[
\pi \int_{\Omega} f(x) \sum_i \delta_{a_i} = N \pi \h_{\text{ex}}^{-1/3}, \quad \pi \int_{\Omega} f(x) \sum_i \delta_{\xi_i} \leq (N - 1) \pi \h_{\text{ex}}^{-1/3}.
\]
On the other hand, from \((5.16)\) and the construction of \(f\), it is evident that
\[
\left\| \pi \int_{\Omega} f(x) \sum_i (\delta_{a_i} - \delta_{\xi_i}) \right\|_{\dot{W}^{-1,1}} \leq \left\| \pi \sum_{i=1}^{N} (\delta_{a_i} - \delta_{\xi_i}) \right\|_{\dot{W}^{-1,1}} \| f \|_{\dot{W}^{-1,1}} \leq \frac{\pi}{2} \h_{\text{ex}}^{-1/3}.
\]
(Here we are using the same convention as in \([7]\), which is that \(\| f \|_{\dot{W}^{-1,1}} = \| \nabla f \|_\infty \) for \(f\) vanishing on \(\partial \Omega\).) This is a contradiction, proving \((5.17)\), and completing the proof of \((5.2)\).

Step 4. Finally, \((5.3)\) follows immediately from Proposition \(4.1\) in view of \((5.10)\). (Note that the hypotheses of the proposition are satisfied due to \((5.2)\) and the choice of \(\varepsilon_0, k_1, \text{etc.}\)) \(\square\)

Acknowledgment.

The work of A.C. was partially supported by a grant from the Simons Foundation \# 426318. The work of R.J. was partially supported by the Natural Sciences and Engineering Research Council of Canada under operating Grant 261955. A.C. wishes to thank S. Serfaty for useful discussions.

Bibliography

[1] Bethuel, F.; Brezis, H.; Helein, F. *Ginzburg-Landau Vortices*. Progress in Nonlinear Differential Equations and their Applications, 13. Birkhauser, Boston, 1994. doi:10.1007/978-1-4612-0287-5

[2] Bethuel, F.; Rivière, T. Vortices for a variational problem related to superconductivity. *Annales de l'I.H.P. Analyse non linéaire* 12 (1995), no. 3, 243–303. doi:10.1016/S0294-1449(16)30157-3
[3] Chapman, J.S.; Rubinstein, J.; Schatzman, M. A mean-field model for superconducting vortices. *Eur. J. Appl. Math.* 7 (1996), no. 2, 97–111.

[4] Contreras, A.; Serfaty, S. Large vorticity stable solutions to the Ginzburg-Landau equations. *Indiana Univ. Math. J.* 61 (2012), no. 5, 1737–1763. doi:10.1512/iumj.2012.61.4818

[5] Evans, L.C. *Partial differential equations*. Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2010. doi:10.1090/gsm/019

[6] Gilbarg, D.; Trudinger, N. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer, Berlin, 2001.

[7] Jerrard, R. L.; Spirn, D. Refined Jacobian estimates and Gross-Pitaevsky vortex dynamics. *Arch. Ration. Mech. Anal.* 190 (2008), 425–475. doi:10.1007/s00205-008-0167-8

[8] Jerrard, R. L.; Spirn, D. Hydrodynamic limit of the Gross-Pitaevskii equation. *Comm. Partial Differential Equations* 40 (2015), no. 2, 135–190. doi:10.1080/03605302.2014.963604

[9] Kinderlehrer, D.; Stampacchia, G. *An introduction to variational inequalities and their applications*. Classics in Applied Mathematics, 31. SIAM, Philadelphia, 2000. doi:10.1137/1.9780898719451

[10] Kurzke, M.; Spirn, D. $\Gamma$-stability and vortex motion in type II superconductors. *Comm. Partial Differential Equations* 36 (2011), no. 2, 256–292.

[11] Lin, F.; Du, Q. Ginzburg–Landau vortices, dynamics, pinning and hysteresis. *SIAM J. Math. Anal.* 28 (1999), 1265–1293. doi:10.1137/S0036141096298060

[12] Petrache, M.; Serfaty, S. Crystallization for Coulomb and Riesz interactions via the Cohn-Kumar conjecture. Preprint.

[13] Rota Nodari, S.; Serfaty, S. Renormalized energy equidistribution and local charge balance in 2D Coulomb systems. *Int. Math. Res. Not. IMRN* (2015), no. 11, 3035–3093. doi:10.1093/imrn/rnu031

[14] Stein, E. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, 30. Princeton University Press, Princeton, N.J., 1970.
Received January 2020.