Stability of large scale chromomagnetic fields in the early universe

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It is well known that Yang-Mills theory in vacuum has a perturbative instability to spontaneously form a large scale magnetic field (the Savvidy mechanism) and that a constant field is unstable so that a possible ground state has to be inhomogeneous over the non-perturbative scale $\Lambda$ (the Copenhagen vacuum). We argue that this spontaneous instability does not occur at high temperature when the induced field strength $gB \sim \Lambda^2$ is much weaker than the magnetic mass squared $(g^2T)^2$. At high temperature oscillations of gauge fields acquire a thermal mass $M \sim gT$ and we show that this mass stabilizes a magnetic field which is constant over length scales shorter than the magnetic screening length $(g^2T)^{-1}$. We therefore conclude that there is no indication for any spontaneous generation of weak non-abelian magnetic fields in the early universe.

I. INTRODUCTION

The perturbative ground state in non-abelian gauge theory is unstable towards a generation of a background field. This was first discovered by Savvidy \cite{1} who calculated the effective potential in a uniform background magnetic field and found a non-trivial minimum. Shortly after Nielsen and Olesen \cite{2} noticed that this minimum is unstable and this discovery stimulated a whole series of papers about how a non-uniform background field develops and stabilizes the system. \cite{3}

One question that has been asked many times is whether this instability occurs also at high temperature. The gauge bosons in the electroweak model have large masses generated by the Higgs mechanism so they screen any $SU(2)$ magnetic field on very short scales below the critical temperature. At high temperature QCD can to some extent be described by perturbation theory around the trivial vacuum. To find observable effects from a magnetic instability it is therefore motivated to go to the very early universe. In particular it was argued in \cite{4} that a ferromagnetic phase of Yang-Mills theory could possibly be responsible for generating the seed field that is needed in many models to explain the observed intergalactic magnetic field. \cite{5}

In this article we are going to argue that no such instability occurred in the early universe. The simple argument is that the typical long wavelength modes that are responsible for the instability in the first place have typically a much longer correlation length than the magnetic screening length. Therefore, these modes do not see a uniform background field but only statistical fluctuations and the whole mechanism for the instability is not present. We furthermore derive the dispersion relation for excitations in the presence of a constant magnetic field (typically much stronger than the one generated by the Savvidy mechanism) and we find that there are stable modes for fields much weaker than the plasma mass, while for strong fields the situation is much less clear. These background fields are however not generated spontaneously. The reason for the stability is simply that the plasma generates a mass for all propagating modes at long wavelengths.

II. THE INSTABILITY AND HOW IT IS SCREENED

A constant field strength is not a gauge invariant concept for non-abelian field theories but one can define classes of gauge equivalent gauge potentials that can be transformed into a given constant field strength by a gauge transformation. It has been shown \cite{6} that there are actually two gauge-inequivalent classes. Let us for simplicity consider $SU(2)$ Yang-Mills theory throughout the paper and give examples of the two classes. For a constant magnetic field $F_{xy}^3 = -B$, a representative of the first class is

\begin{equation}
A_x^1 = \sqrt{B/g}, \quad A_y^1 = -\sqrt{B/g}, \quad (II.1)
\end{equation}

where the field strength is generated by the non-trivial commutation relations of the different components in Eq. (II.1). This field strength is not covariantly conserved and requires therefore a uniform source to fulfill the equations of motion. Since we do not know of any reason to expect such a coherent source over large scales in the early universe we shall instead concentrate on the other class of gauge fields which are the close analogue of the usual abelian magnetic field

\begin{equation}
A_\mu^a = \delta^{a3}(0,0,-Bx,0), \quad (II.2)
\end{equation}
where the field points in the 3-direction in group space. Although this field is gauge dependent we only calculate quantities that are gauge independent so we can safely choose a convenient gauge as above.

The energy spectrum of charged vector fluctuations in the background field in Eq. (II.3) is

\[ E^2(k_z, l, \sigma) = k_z^2 + (2l + 1)gB - 2\sigma gB \,, \quad (II.3) \]

where the term \((2l + 1)gB, l = 0, 1, 2, \ldots\), comes from the orbital angular momentum and the term \(2\sigma gB, \sigma = \pm 1\), is the spin energy. For spin one particles the spin coupling to the magnetic field overcompensates the cost of confining the excitation in the plane orthogonal to the magnetic field and this is why there is an unstable mode for \(k_z^2 < gB\). The real part of the effective potential at zero temperature for a \(SU(N)\) theory \([1]\)

\[ \text{Re} V(B) = \frac{1}{2} B^2 + \frac{11N}{96\pi^2} g^2 B^2 \left( \ln \frac{g B}{\mu^2} - \frac{1}{2} \right) \,, \quad (II.4) \]

has a minimum at the renormalization group invariant scale

\[ gB_{\text{min}} = \Lambda^2 = \mu^2 \exp \left( -\frac{48\pi^2}{11N g^2(\mu)} \right) \,. \quad (II.5) \]

There is also an imaginary part of the effective potential with the interpretation that a constant background field is unstable and decays to a non-uniform configuration \([3]\).

The length scale over which the magnetic field varies is given by the only dimensionful constant in the problem, namely \(\Lambda\).

A. Scales

Like in all other problems in physics we need first to sort out the different scales that enter into the problem and see if there is any hierarchy between them.

\(T\): The highest energy scale is given by the temperature for the situations we have in mind. This is the typical energy of particles in the plasma and the inter-particle distance is \(\sim 1/T\).

\(gT\): The interaction of soft particles \((\rho \sim gT)\) with hard particles \((\rho \sim T)\) generates a thermal mass of order \(gT\). Though static magnetic correlation functions are unscreened by this mechanism all propagating modes pick up a plasma mass of this order.

\(g^2T\): On this momentum scale Yang-Mills theory becomes non-perturbative. All diagrams contribute to the same order. We can therefore not analytically calculate correlation functions on too large length scale. On the other hand, lattice simulations and arguments from dimensional reduction show that non-abelian magnetic fields are screened over the length scale \(1/g^2T\). Thus, no magnetic fields can be constant on this scale.

A: The strong coupling scale below which the vacuum theory becomes non-perturbative depends on the particular theory we have in mind. For QCD we have \(\Lambda_{\text{QCD}} \approx 200\text{ MeV}\). For an \(SU(2)\) theory with a coupling constant \(g(\mu) = 0.65\) at \(\mu = M_Z = 91\text{ GeV}\) the scale is \(\Lambda_{SU(2)} = 8 \times 10^{-10}\text{ GeV}\). A hypothetical \(SU(5)\) GUT with \(g(\mu) = 0.7\) at \(\mu = 10^{15}\text{ GeV}\) would have \(\Lambda_{SU(5)} = 2 \times 10^{-4}\mu = 2 \times 10^{11}\text{ GeV}\).

In order to possibly have an instability the symmetry has to be unbroken and so the temperature has to be much higher than \(\Lambda\) for the examples we give above. In this paper we only consider \(SU(N)\) explicitly (and mostly only \(SU(2)\)) and in that case one can derive the bound \(g(T)^2 T N > 58\Lambda\) from Eq. (II.3). Therefore, we shall assume that we have a clear scale separation with the hierarchy

\[ \Lambda \ll g^2 T \ll gT \ll T \,. \quad (II.6) \]

Now we have to put the in the scale \(gB\) and we are going to discuss two possibilities.

B. Screening of the Savvidy Instability

In the Savvidy mechanism for generating a magnetic field the magnitude is given in Eq. (II.3) and is thus much weaker than any thermal scale. One should then remember that in order to derive Eq. (II.4) a constant field was assumed and it was the modes with momentum \(k_z^2 < gB, \ (k_z^2 + k_y^2) = gB\) that gave rise to both the instability and the imaginary part at the non-trivial minimum. These modes have a spatial extension that is of the order \(1/\sqrt{gB}\) and thus much larger than the magnetic screening length \(1/g^2 T\). Even though it has not been possible to rigorously calculate the magnetic screening length analytically there is overwhelming numerical evidence \([4]\), as well as theoretical arguments \([5]\), that non-abelian magnetic fields are screened over the distance \(1/g^2 T\). The modes that are responsible for the instability do therefore not experience a constant field but only random thermal fluctuations. Consequently, there are no indications that any large scale magnetic field is present in a high temperature Yang-Mills gas. It should however be kept in mind that the physics of such a gas on large length scales is very poorly known and it is difficult to make quantitative statements.
C. External magnetic fields

Even if there is no spontaneous generation of magnetic fields it is still interesting to study the properties of a field on scales shorter than the screening length. Such a field could be generated in first order phase transitions and in cosmic strings. The smoothness of such fields depends on the exact situation but we shall concentrate on fields which are constant over the size of the lowest Landau level, i.e. \( L \sim 1/\sqrt{gB} \). For very strong fields we expect that thermal effects do not matter and we would end up in the unstable situation just like in the vacuum. To start from such a state and try to add thermal corrections is bound to be difficult since the starting point is unstable.

We shall therefore start from the stable high temperature phase and turn on a weak external field. The relevant scale to compare with is the thermal mass \( m_{\text{Th}} \) which is only marginally valid in many theories of a stable dispersion relations of propagating modes while as perturbation theory. We now expect that for field in the unstable situation just like in the vacuum. To start

\[
\Lambda^2 \ll (g^2T)^2 \ll gB \ll (gT)^2 \ll T^2 .
\]  

This separation of scales assumes a small coupling constant which is only marginally valid in many theories of interest but this is the only way of making sense of perturbation theory. We now expect that for field in the range of Eq. (II.7) there are only small corrections to the stable dispersion relations of propagating modes while as \( gB \to (gT)^2 \) an instability may develop.

III. SELF ENERGY IN A MAGNETIC FIELD

We shall now calculate the self-energy in a background magnetic field at high temperature and see how the dispersion relations are affected. As a concrete example we study a \( SU(2) \) gauge theory with a background field given by Eq. (II.2). The Lagrangian in a background field is given by

\[
\mathcal{L} = -\frac{1}{4} F^2 - \frac{1}{2} \left( (\partial_{\mu} Z_{\nu})^2 - (1 - \frac{1}{\xi}) (\partial_{\mu} Z^{\nu})^2 \right)
- \left( |D_{\mu} W_{\nu}|^2 - (1 - \frac{1}{\xi}) |D_{\mu} W^{\nu}|^2 \right)
- igF_{\mu\nu} W^\dagger_{\mu} W_{\nu} - ig\partial_{\mu} Z_{\nu}(W^\dagger_{\mu} W_{\nu} - W^\dagger_{\mu} W_{\nu})
- ig(Z_{\mu} W^\dagger_{\nu} - Z_{\nu} W^\dagger_{\mu}) D_{\mu} W_{\nu}
- ig(D_{\mu} W_{\nu})^\dagger (Z_{\mu} W_{\nu} - Z_{\nu} W_{\mu})
\]

\[
-\frac{g^2}{2} \left( (W^\dagger \cdot W)^2 - |W \cdot W|^2 
+ 2Z^2 W^\dagger \cdot W - 2|Z \cdot W|^2 \right)
+ \text{ghost terms}.
\]  

The background field points in the 3-direction in group space and we use the notation \( Z_{\mu} = W^e_{\mu} \) and \( W_{\mu} = (W^e_{\mu} + iW^o_{\mu})/\sqrt{2} \). The diagrams that contribute to the polarization tensor at one loop are the same as usual but with propagators in a background field and the 3-point vertex with derivatives has covariant derivatives with respect to the charged particle lines. We shall use the Feynman gauge \( \xi = 1 \). The \( W \)-boson propagator in the Schwinger representation is then

\[
G_{\mu\nu}^{W}(x, x') = \phi(x, x') \int \frac{d^4P}{(2\pi)^4} e^{-iP(x-x')} G_{\mu\nu}^{W}(P)
\]

\[
G_{\mu\nu}^{W}(P) = -\int_0^\infty \frac{ds}{\cos(gBs)} (\exp[-2sgF])_{\mu\nu}
\times \exp[i(P_\parallel^2 + \frac{\tan gBs}{gBs} P_\|^2 )],
\]

where the gauge dependence is contained in the phase factor

\[
\phi(x, x') = \exp\{ig \int_{x'}^x d{x'}^\mu [A_\mu + \frac{1}{2} F_{\mu\nu}(\xi - x')^\nu] \}
\]

the rest being both translational invariant and invariant under abelian gauge transformations generated by \( U = e^{i\alpha(x)a^3} \). The exponential of the field strength can be expanded as

\[
(\exp[-2sgF])_{\mu\nu} = g_{\mu\nu} - gF_{\mu\nu} \frac{\sin(2gBs)}{gB}
- \frac{(F^2)_{\mu\nu}}{(gB)^2} (1 - \cos(2gBs)) .
\]

In the bubble diagram it is convenient to integrate the 3-point vertex derivatives by parts so they only act on the propagators inside the loop. Then one can use the rule

\[
(i\partial_{\mu} + gA_\mu)G_{\alpha\beta}^{W}(x, x')
= \phi(x, x') \int \frac{d^4P}{(2\pi)^4} e^{-iP(x-x')}
\times \left( -\frac{i}{2} gF_{\mu\nu} \frac{\partial}{\partial P_{\nu}} + P_{\mu} \right) G_{\alpha\beta}^{W}(P) \]

It follows from the form of the propagators that the polarization tensor can be written as
\[ \Pi_{\mu\nu}(x, x') = \phi(x, x') \int \frac{d^4 P}{(2\pi)^4} e^{-iP(x-x')} \Pi_{\mu\nu}(P) , \]

(III.6)

where again the gauge dependence is only in phase factor.

One problem we encounter immediately if we try to do a strict perturbative expansion in the background field is that the Landau level solutions in a background field cannot be approximated by small perturbations from a plane wave basis. We shall therefore use Landau levels \( W_\mu(\kappa, x) \) as external states and calculate the polarization tensor for small field strengths. Another concern is gauge invariance of the external fields and the phase factor in Eq. (III.4). We shall see in Section IV that when calculating expectation values of the form

\[
\int d^4 x d^4 x' \, W_\mu^\dagger(x, x') \times [ -\delta(x - x')(g^{\mu\nu} D^2 - D^\mu D^\nu - 2igF^{\mu\nu}) \\
-\Pi^{\mu\nu}(x, x')] \, W_\nu(\kappa', x')
\]

(III.7)

the phase factor in \( \Pi_{\mu\nu} \) combine with the external Landau levels to give a factor \( \delta(\kappa - \kappa') \), where \( \kappa = (k_0, k_z, k_y, n) \) are the quantum numbers of the Landau levels. We shall therefore now calculate \( \Pi_{\mu\nu}(P) \) for weak fields and evaluate the expectation values in Section IV.

As pointed out above, the diagrams to one-loop are the standard ones except that the full \( W \)-propagators and covariant derivatives should be used. It is then trivial to see that at lowest order the standard hard thermal loop result is recovered (assuming \( T \gg K \))

\[
\Pi(K)_{\mu\nu} = \frac{3M^2}{2} \int \frac{d^4 K}{4\pi} \left[ g_{\mu\nu} - \frac{K_\mu u_\nu + u_\mu K_\nu}{u \cdot K} \right],
\]

(III.8)

where \( M^2 = \frac{e^2 N_C T^2}{6} \), \( u_\mu = (1, \vec{u}) \) and the integral is over the angles of \( \vec{u} \). The first order correction in \( gB \) is, not surprisingly, IR divergent. The full calculation is difficult to perform, but for small loop momenta, and assuming that there is regularizing mass \( m \), one can show that correction to \( \Pi(K) \) goes like

\[
g^2 T \left[ \sqrt{m^2 + 2gB} - \sqrt{m^2 - 2gB} \right]. \tag{III.9}
\]

If the thermal gluon mass on the electric scale \( m \sim gT \) is the regularizing scale, as suggested in [10], this gives

\[
\Delta \Pi \sim g^2 T gB/m \sim g gB \ll gB , \tag{III.10}
\]

which is much smaller than the tree level \( gB \). However, static magnetic modes are not screened in the HTL resummation scheme. If the mass is instead generated on the magnetic scale \( m \sim g^2 T \), with our assumption of a clear scale separation with \( g^2 T \ll \sqrt{gB} \), the instability is instead regularized by the field itself

\[
\Delta \Pi \sim g^2 T \sqrt{gB(1 + i)} \ll gB , \tag{III.11}
\]

and again smaller than the tree level \( gB \). An imaginary part appears here due to the unstable mode on tree level, but it is of the same order as the real part, and thus negligible. Furthermore, we find that the contributions linear in \( B \) from hard internal momenta are suppressed compared to the tree level \( gB \). We can therefore conclude that it is enough to consider the leading HTL term for \( \Pi(K) \).

It may now seem that we have approximated away all non-trivial effects from having a heat-bath and a magnetic field. This is however not so because when we calculate the expectation value of \( \Pi_{\mu\nu}(K) \) to find the mass gap the orthogonal momentum is \( (k_0^2 + k_z^2) = (2l + 1)gB \), where \( l = 0, 1, 2, \ldots \) is the orbital angular momentum quantum number. Therefore the thermal mass is shifted by an amount \( g^2 T^2 (k_0^2/k_y^2) \sim g^2 T^2 gB/(g^2 T^2) \sim gB \).

The fact that even the lowest Landau level is localized makes a shift through the self-energy that is of the same order as the tree level term and is therefore the dominant correction.

**IV. SCREENING IN THE LOWEST LANDAU LEVEL**

The equation of motion for charged YM fields in a background of a constant chromo-magnetic field is given by

\[
D^2 W_\mu - D_\mu D \cdot W - 2igF_\mu^\nu W_\nu = 0 , \tag{IV.1}
\]

where \( F_\mu^\nu \) is the background field strength and \( D_\mu = \partial_\mu - igA_\mu \) is the background covariant derivative. \( W_0 \) is not a physical degree of freedom but it is algebraically related to the currents and the gauge is fixed using the condition \( D \cdot W = 0 \) as usual. For the general solution of Eq. (IV.1) in the gauge \( A_\mu = (0, 0, -Bx, 0) \) we use the Ansatz

\[
W_\mu(k_0, k_z, k_y, n; x) \equiv \sum_a c_a W^{(a)}_\mu(k; x)
\]

(IV.2)

where \( I_{n, k_y}(x) \) can be expressed in terms of Hermite polynomials as

\[
I_{n, k_y}(x) = \left( \frac{gB}{\pi} \right)^{1/4} \exp \left[ -\frac{gB}{2} (x + \frac{k_y}{gB})^2 \right] \times \frac{1}{\sqrt{m!}} H_m \left[ \sqrt{2gB(x + \frac{k_y}{gB})} \right] . \tag{IV.3}
\]
For the vector structure of $W_{\mu}$ we choose the basis
\[ w^{(0)\mu} = (1, 0, 0, 0) \]
\[ w^{(z)\mu} = (0, 0, 0, 1) \]
\[ w^{(+\mu} = \frac{1}{\sqrt{2}} (0, 1, i, 0) \]
\[ w^{(-\mu} = \frac{1}{\sqrt{2}} (0, 1, -i, 0) \]
and they are normalized to $w^{(a)\dagger}_\mu w^{(b)\mu} = g^{ab} = \text{diag}(1, -1, -1, -1)$. They have the convenient properties
\[ F^\mu_{\nu} w^{(0, z\mu)} = 0 \]
\[ F^\mu_{\nu} w^{(\pm\mu)} = \pm i B w^{(\pm\mu)} \]
Energy eigenfunctions are then given by Eq. (IV.2) with $\nu = 0$ and $\nu = 1$ equivalently that
\[ W(I) = (2 \pi)^2 \int \exp\left[ -i \sum_j (2 \pi)^2 \nu_j \right] \phi(x, x') \]
\[ = (-k_0^2 + k_z^2 + (2n - 1)gB)I \phi(x, x') \]
The gauge condition $D \cdot W = 0$ should now be imposed on this solution and that gives us the constraint
\[ c_0 k_0 - c_z k_z - ic_+ \sqrt{gB(n - 1) + ic_- \sqrt{gB}} = 0 \]
The factor $w^{(a)\dagger}_\mu \Pi_{\mu\nu}(k_\parallel, k_z, p_\parallel) w^{(b)\mu}$ depends only on rotational invariant quantities apart from factors of $p_\parallel \pm i p_y$. The angular integral over $p_\parallel$ then constrains the total power of $p_\parallel \pm i p_y$ to zero. This gives a $\delta$-function between $n_a$ and $n'_a$ such that, depending on $a$ and $b$, different energy levels have vanishing overlap.

In particular we can study the polarization tensor in the lowest Landau level where we only have the polarization state $w^{(-)}_\mu$ which automatically fulfills the gauge condition. The dispersion relation takes the form
\[ k_0^2 + gB - k_z^2 + \int_0^\infty dp_\perp 2p_\perp e^{-\frac{p_\perp^2}{gB}} \times w^{(\dagger)}_\mu \Pi_{\mu\nu}(k_0, k_z, p_\perp) w^{(-)}_\nu = 0 \]
It is straightforward to expand $w^{(\dagger)}_\mu \Pi_{\mu\nu}(k_0, k_z, p_\perp) w^{(-)}_\nu$ using $\Pi_{\mu\nu} = \mathcal{P}_{\mu\nu} \Pi_T + \mathcal{Q}_{\mu\nu} \Pi_L$ where
\[ \mathcal{P}_{ij}(K) = -\delta_{ij} + \frac{k_i k_j}{k^2} \]
\[ \mathcal{Q}_{\mu\nu}(K) = -\frac{1}{k^2} (k^2 - k_0 k) \]
From Eq. (III.8) we have that
\[ \Pi_T(K) = \frac{3M^2}{2} \left[ \frac{k_0^2}{k^2} + 1 - \frac{k_0^2}{k^2} \frac{k_0}{2k} \ln \left( \frac{k_0 + k}{k_0 - k} \right) \right] \]
\[ \Pi_L(K) = 3M^2 \left[ 1 - \frac{k_0}{2k} \ln \left( \frac{k_0 + k}{k_0 - k} \right) \right] \]
In the lowest Landau level, we thus get
\[ w^{(-)}_\mu \Pi_{\mu\nu}(k_0, k_z, p_\perp) w^{(-)}_\nu = -\frac{2k^2}{2(k_z^2 + p_\perp^2 - (k_0^2 - k_0^2)(k_0^2 + p_\perp^2))} \Pi_L \]
It is interesting to notice that the longitudinal polarization contributes to this lowest mode. It is so because $w^{(-)}_\mu$ points in the plane perpendicular to the magnetic field and the momentum cannot be zero in the plane. For the question of stability of this mode we are interested low momenta so we put $k_z = 0$ which in this context means that $k_z^2 \ll gB$ while being larger than $g^2T$ since we cannot probe the magnetic scale. The mass shell condition $k_0^2 = M^2$ including $O(gB)$ corrections then becomes
\[ k_0^2 + gB - M^2 \left( 1 + \frac{2gB}{5M^2} \right) = 0 \]
that is $k_0 \simeq \mathcal{M}(1 - 3gB/10M^2)$. A weak field results in a small reduction of the plasma mass but plasmons
remains stable (up to higher order in $g$). We notice that Eq. (IV.12) vanishes for $k_0 = 0$ and it is natural to wonder whether there is any new low energy mode that potentially becomes unstable as the field is switched on. Expanding Eq. (IV.13) in powers of $k_0$ we find the dispersion relation

$$k_0^2 + gB + i \frac{3\pi^{3/2}}{8} k^2_0 B^2 = 0 .$$

There is no solution to this equation with $k_0^2 \sim gB$ and the dominating imaginary part comes from Landau damping. i.e. scattering with particles in the heat bath.

As the field is increased the tree level $gB$ becomes comparable to the plasma mass $\mathcal{M}^2$ and one could expect the instability to reappear. The analysis in Section II showed that Eq. (IV.9) contains the dominant correction as long as $g$ is small and $gB \gg (g^2T)^2$. It can therefore be used all the way up to $gB \lesssim \mathcal{M}^2$, but the analysis in terms of a dispersion relation breaks down when the imaginary part of the self-energy becomes large. The integration in Eq. (IV.9) goes below the light cone ($k_0^2 < k_0^2 + p^2_\perp$) where the polarization tensor has an imaginary part due to Landau damping and synchrotron absorption (emission is not possible in the lowest Landau level). It increases with the field and eventually the width of the excitation is so large that it is not possible to talk about a quasi-particle.

In order to estimate what happens as the field is increased we have solved the dispersion relation in Eq. (IV.9) numerically using only the real part. The result is shown in Fig. 1. In the same figure we show the imaginary part for the same value of $gB$ and $k_0$. Our interpretation is that for weak fields ($gB \lesssim 0.2\mathcal{M}^2$) there is only one stable mode. For $gB \gtrsim 0.2\mathcal{M}^2$ the imaginary part becomes so large that the spectral function looks more like a broad resonance. Long before $gB$ reaches $\mathcal{M}^2$ the imaginary part is so large that the quasi-particle language is meaningless. It should be emphasized that the imaginary part we are discussing is not the one that indicates a spontaneous generation of a magnetic field, but simply the Landau damping that attenuates all excitations. In particular, the sign of the imaginary part corresponds to an exponentially damped mode.

To get further insight in what modes are present for weak fields we have calculated the spectral weight of the $k_0 \simeq \mathcal{M}$ mode using

$$Z^{-1}(k_0, gB) = \frac{1}{2k_0} \frac{d}{dk_0} \left( k_0^2 + gB - \int dp_\perp \frac{2p^2_\perp}{gB} e^{-\frac{p^2_\perp}{2gB}} \times w_{\mu}^{(-)}(k_0, z, p_\perp) w_{\mu}^{(-)}(k_0, z, p_\perp) \right) ,$$

and evaluate it on the solution to Eq. (IV.9). This formula makes sense only if the branch is a narrow resonance. We have normalized $Z$ so that it equals to one for free a particle. In Fig. 1 $Z$ is plotted for weak fields and it is obvious that it is very close to one, leaving no spectral weight to other modes. To conclude: for weak fields there is only one propagating mode and it has a large massgap $\simeq \mathcal{M}$ while for increasing fields Landau damping and synchrotron absorption rapidly damps all modes.

**Figure 1**: The solid lines (fat and thin) show the solutions to Eq. (IV.3) for $k_0 = 0$ using only the real part of the self-energy. There are two branches of which the fat line corresponds to the standard plasmon. These two lines are only meaningful if $-\text{Im} \Pi/k_0$, is much smaller than $k_0$. Therefore, we plot $-\text{Im} \Pi/k_0$ for the two branches using dashed lines. Only the fat line can satisfy the condition $-\text{Im} \Pi \ll k_0^2$ for small fields while the thin mode is simply an artefact of neglecting the imaginary part of the self-energy. To further emphasize that there is only one propagating mode we plot the spectral weight of the fat mode (dotted line), as calculated in Eq. (IV.15), and it shows that the upper mode saturates the spectral weight for weak fields.
V. CONCLUSIONS

As discussed in the introduction, rather soon after the non-trivial minimum of the effective potential in vacuum was found, it was also realized that this minimum is unstable. Moreover, it was realized that the starting point of a constant field is inconsistent and one is forced to consider non-uniform background fields. The characteristic length scale on which the background field has to vary is given by the renormalization scale $\Lambda$. For QCD this would be $\Lambda_{\text{QCD}} \approx 200$ MeV while for a $SU(2)$ theory $\Lambda_{SU(2)} \approx 8 \times 10^{-10}$ GeV if we take the values of $g(\mu)$ from the $SU(2)$ sector of the electroweak theory. In a GUT we made an estimate of $\Lambda_{SU(5)} \approx 10^{11}$ GeV. As long as we consider theories at temperature far above non-perturbative scale $\Lambda$ we would typically have that $\Lambda \ll g^2 T$ and we do not expect the perturbative instability to have any consequence for the high temperature behaviour. In particular we do not expect that the free energy (or effective potential) should have any non-trivial minimum. The non-trivial minimum in vacuum comes entirely from the unstable mode. We have now found that this mode is stabilized by the thermal heat bath and therefore we expect only a trivial minimum. It is difficult to say anything rigorous about what happens with the background field on the scale $(g^2 T)^{-1}$ and larger, but since $\Lambda \ll g^2 T N$ and magnetic fields are expected to be screened by the mass $g^2 T N$ we do not find any support for the idea that a nontrivial field configuration should be generated spontaneously in the early universe.

The situation is very different if we consider an external magnetic field that can be tuned at will. For weak fields the lowest Landau level has a weakly damped massive plasma mode but it acquires a large imaginary part from Landau damping for $gB \gtrsim 0.2 M^2$. The equilibrium configuration for stronger external fields is not known but one can expect that for strong enough fields the situation is similar to the vacuum case.

Though the propagating mode is screened we find that in the static limit $k_0 = 0$ the expectation value of the polarization tensor vanishes (see Eq. (IV.12)) which would at first indicate that the instability persists even at finite temperature. It should however be noticed that an unstable mode is not static, and as soon as $k_0 \neq 0$ the polarization tensor has both a real part of order $(gT)^2 k_0^2 / k^2$, and for $k_0 < k$ and imaginary part or order $(gT)^2 k_0 / k$ which are typically much larger than $gB$. Also, the spectral weight for the lowest Landau level around $k_0 = 0$ is very small. We can therefore conclude that any potentially increasing uniform mode would rapidly be Landau damped and disappear. In more realistic models with Higgs field and fermions the situation is more complicated. In the low temperature phase of the electroweak theory field strength larger than $m_W^2 / c$ is needed to have an instability [1], but the situation has not been studied carefully including thermal damping effects. We expect the results in this paper to give a good picture of what happens in the high temperature phase.

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