Attractor Solutions in $f(T)$ Cosmology

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Abstract: In this paper, we explore the cosmological implications of interacting dark energy model in a torsion based gravity namely $f(T)$. Assuming dark energy interacts with dark matter and radiation components, we examine the stability of this model by choosing different forms of interaction terms. We consider three different forms of dark energy: cosmological constant, quintessence and phantom energy. We then obtain several attractor solutions for each dark energy model interacting with other components. This model successfully explains the coincidence problem via the interacting dark energy scenario.

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I. INTRODUCTION

It is now well-accepted in astrophysics community that the observable universe is in a phase of rapid expansion whose rate of expansion is increasing, so called 'accelerated expansion’. This conclusion has been drawn by numerous recent cosmological and astrophysical data findings of supernovae SNe Ia [1], cosmic microwave background radiations via WMAP [2], galaxy redshift surveys via SDSS [3] and galactic X-ray [4]. This phenomenon is commonly termed 'dark energy' (DE) in the literature, and suggests that a cosmic dark fluid possessing negative pressure and positive energy density. Although the phenomenon of dark energy in cosmic history is very recent $z \sim 0.7$, it has opened new areas in cosmology research. Two important problems like ‘fine tuning’ and ‘cosmic coincidence’ are related to dark energy. It is thought that the most elegant solution to DE paradigm is the Einstein’s cosmological constant [5] but it cannot resolve the two problems mentioned above. Hence cosmologists looked for other theoretical models by considering the dynamic nature of dark energy like quintessence scalar field [6], a phantom energy field [7] and f-essence [8]. Another interesting set of proposals to DE puzzle is the 'modified gravity' which was proposed after the failure of general relativity (GR). This new set of gravity theories passes several solar system and astrophysical test successfully [9].

For the past few years, models based on DE interacting with dark matter or any other exotic component have gained great impetus. Such interacting DE models can successfully explain numerous cosmological puzzles including dynamic DE, phantom crossing, cosmic coincidence and cosmic age [10] and also in good compatibility with the astrophysical observations of cosmic microwave background, supernova type Ia, baryonic acoustic oscillations and galaxy redshift surveys [11]. There are some criticisms on interacting DE models for not being favored from observations and that the usual ΛCDM model is favorable [12]. However the thermal properties of this model in various gravities have been discussed in literature [13]. The model in which dark energy interacts with different fluids has been investigated in literature. In [14], the two fluids were dark matter and another was unspecified. However, in another investigation [15], the third component was taken as radiation to address the cosmic-triple-coincidence problem and study the generalized second law of thermodynamics. In a recent investigation, the authors investigated the coincidence problem in loop quantum gravity with triple interacting fluids including DE, dark matter and unparticle [16].

There is no need to construct a gravitational theory on a Riemannian manifold. A manifold can be divided into two separate but connected parts; one with Riemannian structure with a definite metric and another part, with a non-Riemannian structure and with torsion or non-metricity. That part which has the zero Riemannian tensor but has non zero torsion is based on a tetrad basis, and defines a Weitzenbock spacetime. $f(T)$ gravity is an alternative theory for GR, defined on the Weitzenbock non-Riemannian manifold, working only with torsion. This model firstly proposed by Einstein for unifying the electromagnetism and the gravity. If $f(T) = T$, this theory is called teleparallel gravity [17] [18]. It has been shown that with linear $T$, this model has many common features like GR and is in good agreement with some standard tests of the GR in solar system [17]. But introducing a general model $f(T)$ backs to few years [19]. This model has many features, for example Birkhoff’s theorem has been studied in this gravity [20]. Earlier the authors in [21] investigated perturbation in $f(T)$ and found that the perturbation in $f(T)$ gravity grows slower than that in Einstein general relativity. Bamba et al [22] studied the evolution of equation of state parameter and phantom crossing in $f(T)$ model. Emergent universes in chameleon $f(T)$ model is investigated in [23]. As a
thermodynamical view, there are many strange features. It has been shown in \( f(T) \) gravity, the famous formula of entropy-area of the black hole thermodynamics is not valid \[25\]. The reason goes back to the violation of the local lorentz invariance of this theory \[25, 26\]. It shows that it is not possible to use the Wald conjecture \[27\] for calculating the entropy as a Noether charge in \( f(T) \). The Hamiltonian formulation of \( f(T) \) gravity has been studied in \[24\] and shown that there are five degrees of freedom. In this paper we study the triple coincidence problem: why we happen to live during this special epoch when \( \rho_A \sim \rho_m \sim \rho_r \)? \[28\]. We investigate this problem in the framework of \( f(T) \) gravity.

We follow the plan: In section II we introduce the basic equations as an autonomous dynamical system. In section III we choose a model for \( f(T) \) gravity. In section IV we working on numerical analysis of the stability and the evolution of the functions of the model in details. We conclude and summarize in section V.

II. BASIC EQUATIONS

One suitable form of action for \( f(T) \) gravity in Weitzenbock spacetime is \[19\]

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{e} (T + f(T) + L_m)
\]

Here \( e = \text{det}(e^i_j), \kappa^2 = 8\pi G \) and \( e^i_j \) is the tetrad (vier-bein) basis. The dynamical quantity of the model is the scalar torsion \( T \) and \( L_m \) is the matter Lagrangian. We start with the Friedmann equation in this form of the \( f(T) \) model \[16\]

\[
H^2 = \frac{1}{1 + 2f_T} \left( \frac{\kappa^2}{3} \rho - \frac{1}{6} f_T \right), \quad (1)
\]

where \( \rho = \rho_m + \rho_d + \rho_r \), while \( \rho_m, \rho_d \) and \( \rho_r \) represent the energy densities of matter, dark energy and the radiation respectively.

Another FRW equation is

\[
\dot{H} = -\frac{\kappa^2}{2} \left( \frac{\rho + p}{1 + f_T + 2T f_{TT}} \right), \quad (2)
\]

For a spatially flat universe \( (k = 0) \), the total energy conservation equation is

\[
\dot{\rho} + 3H(\rho + p) = 0, \quad (3)
\]

where \( H \) is the Hubble parameter, \( \rho \) is the total energy density and \( p \) is the total pressure of the background fluid.

We assume a three component fluid containing matter, dark energy and radiation having an interaction. The corresponding continuity equations are \[16\]

\[
\begin{align*}
\dot{\rho}_d + 3H(\rho_d + p_d) &= \Gamma_1, \\
\dot{\rho}_m + 3H\rho_m &= \Gamma_2, \\
\dot{\rho}_r + 3H(\rho_r + p_r) &= \Gamma_3,
\end{align*}
\]

which satisfy collectively \[3\] such that \( \Gamma_1 + \Gamma_2 + \Gamma_3 = 0 \).

We define dimensionless density parameters via

\[
x \equiv \frac{\kappa^2 \rho_d}{3H^2}, \quad y \equiv \frac{\kappa^2 \rho_m}{3H^2}, \quad z \equiv \frac{\kappa^2 \rho_r}{3H^2}, \quad (5)
\]

The continuity equations \[11\] in dimensionless variables reduce to

\[
\begin{align*}
\frac{dx}{dN} &= 3x \left( \frac{x + y + z + \omega_d x + \omega_r z}{1 + f_T + 2T f_{TT}} \right) \quad (6) \\
-3x(1 + \omega_d) + \frac{\kappa^2}{3H^2} \Gamma_1,
\end{align*}
\]

\[
\begin{align*}
\frac{dy}{dN} &= 3y \left( \frac{y + z + \omega_d y + \omega_r y}{1 + f_T + 2T f_{TT}} \right) \quad (7) \\
-3y + \frac{\kappa^2}{3H^2} \Gamma_2,
\end{align*}
\]

\[
\begin{align*}
\frac{dz}{dN} &= 3z \left( \frac{z + \omega_d z + \omega_r z}{1 + f_T + 2T f_{TT}} \right) \quad (8) \\
-3z(1 + \omega_r) + \frac{\kappa^2}{3H^2} \Gamma_3,
\end{align*}
\]

where \( N \equiv \ln a \), is called the e-folding parameter. The coupling functions \( \Gamma_i, i = 1, 2, 3 \) are in general functions of the energy densities and the Hubble parameter i.e. \( \Gamma_i(H\rho_i) \). The system of equations in \( 6 \) is analyzed by first equating them to zero to obtain the critical points. Next we perturb equations up to first order about the critical points and check their stability. Below for computation, we shall assume \( \omega_m = 0, \omega_r = \frac{\alpha}{3} \) and \( \omega_d \) to be a general non-zero but negative parameter. We are interested in stable critical points (i.e. those points for which all eigenvalues of Jacobian matrix are negative) as these are attractor solutions of the dynamical system.

III. \( f(T) \) MODEL

To avoid analytic and computation problems, we choose a suitable \( f(T) \) expression which contains a constant, linear and a non-linear form of torsion, specifically

\[
f(T) = 2C_1 \sqrt{-T} + \alpha T + C_2, \quad (7)
\]

where \( \alpha, C_1 \) and \( C_2 \) are arbitrary constants. The first and the third terms (excluding the middle term) has correspondence with the cosmological constant EoS in \( f(T) \) gravity \[24\]. There are many kinds of such models, reconstructed from different kinds of the dark energy models. For example this form \( 7 \) may be inspired from a model for dark energy from proposed form of the Veneziano ghost \[30\]. But the linear term is needed to show the differences between our results in \( f(T) \) gravity from the Einstein gravity. Here we choose this model to simplify our numerical computations and for easier discussion on the difference of our results with the same results in GR. It is the minimum model, but our equations have been written for a general \( f(T) \) action. It
is possible by repeating the numerical steps as we done in this paper, discuss the stability of other models. Recently Capozziello et al.\cite{31} investigated the cosmography of $f(T)$ cosmology by using data of BAO, Supernovae Ia and WMAP. Following their interesting results, we notice that if we choose $C_2 = 0$, $\alpha = \Omega_{m0}$ and $C_1 = \sqrt{6}H_0(\Omega_{m0} - 1)$, then we can estimate the parameters of our proposed $f(T)$ model as a function of Hubble parameter $H_0$ and the cosmographic parameters and the value of matter density parameter.

IV. ANALYSIS OF STABILITY IN PHASE SPACE

In this section, we will construct four models by choosing different coupling forms $\Gamma_i$ and analyze the stability of the corresponding dynamical systems about the critical points. We shall plot the phase and evolutionary diagrams accordingly. For this reason, we must find the critical points of the $\frac{\partial}{\partial x}$, and then we linearize the system near the critical points up to first order.

A. Interacting model - I

We consider the model with the following interaction terms

$$\Gamma_1 = -6bH\rho_d, \quad \Gamma_2 = \Gamma_3 = 3bH\rho_d,$$

(8)

where $b$ is a coupling parameter and we assume it to be a positive real number of order unity. Thus (8) says that both matter and radiation have increase in energy density with time while dark energy loses its energy density. Therefore it is a decay of dark energy into matter and radiation.

Using (8), the system (6) takes the form

$$\frac{dx}{dN} = -3x(1 + w_d) + 3x\left(\frac{x + y + z + w_dx + wrz}{1 + \alpha}\right) - 6bx,$$

$$\frac{dy}{dN} = -3y + 3y\left(\frac{x + y + z + w_dx + wrz}{1 + \alpha}\right) + 3bx,$$

$$\frac{dz}{dN} = -3z(1 + w_r) + 3z\left(\frac{x + y + z + w_dx + wrz}{1 + \alpha}\right) + 3bx.$$

(9)

The critical points for this model are obtained by equating the left hand sides of (9) to zero. We obtain four critical points:

- Point $A_1: (\lambda_1 = 3(1 + w_d + 2b), \lambda_2 = 3(w_d + 3b), \lambda_3 = -1 + 3(w_d + 3b)),$
- Point $B_1: (\lambda_1 = 3(b - 1), \lambda_2 = -3(b - 1), \lambda_3 = -3(1 + 2b + w_d))$,  
- Point $C_1: (\lambda_1 = -1, \lambda_2 = 3(1 - b), \lambda_3 = -3(w_d + 3b))$,  
- Point $D_1: (\lambda_1 = 1, \lambda_2 = 4 - 3b, \lambda_3 = 1 - 3w_d - 9b)$

Point $A_1$ is stable when one of these conditions is satisfied:

$$w_d < -3, \quad b < 1/18. \quad \text{(10)}$$

$$w_d \geq -3, \quad w_d \leq -10/9, \quad b \leq 0. \quad \text{(11)}$$

$$w_d \geq -10/9, \quad w_d < 0, \quad b < -\frac{1}{2}(1 + w_d). \quad \text{(12)}$$

Fig. 1: Model I: Phase space for $w_d = -1.2, b = 0.5, \alpha = 0.5$. It shows an attractor behavior.

B. Interacting model - II

We study another model with the choice of the interaction terms

$$\Gamma_1 = -3bH\rho_d, \quad \Gamma_2 = 3bH(\rho_d - \rho_m), \quad \Gamma_3 = 3bH\rho_m.$$

(13)
FIG. 2: Model I: variation of $x, y, z$ as a function of the $N = \ln(a)$. The initial conditions chosen are $x(0) = 0.7, y(0) = 0.3, z(0) = 0.01, w_d = -1.2$ and $b = 0.5$.

FIG. 3: Model I: Phase space for $w_d = -0.5, b = 0.5, \alpha = 0.5$.

This model effectively describes the situation when dark energy loses energy density to matter while the radiation density increases due to interaction with the matter.

\[
\frac{dx}{dN} = 3x \left( \frac{x + y + z + w_dx + w_r z}{1 + \alpha} \right) - 3bx - 3x(1 + w_d),
\]

\[
\frac{dy}{dN} = 3y \left( \frac{x + y + z + w_dx + w_r z}{1 + \alpha} \right) + 3b(x - y) - 3y,
\]

\[
\frac{dz}{dN} = 3z \left( \frac{x + y + z + w_dx + w_r z}{1 + \alpha} \right) + 3by - 3z(1 + w_r),
\]

There are four critical points:

- Point $A_2 : (0, 0, 0)$,
- Point $B_2 : (0, 0, 1 + \alpha)$,
- Point $C_2 : (0, (1 - 3b)(1 + \alpha), 3b(1 + \alpha))$,
- Point $D_2 : \left( -\frac{3(2+w_d)(b-w_d-1)(b-w_d-7/3)(\alpha+1)}{b(\alpha+1)(b-w_d-7/3)(b-w_d-1)} \right.$

\[
\left. + \frac{3w_d+16-3b}{w_d^2+6b^2+3w_d^2+27w_d+8} b^2 \right)
\]

The eigenvalues of the Jacobian matrix for these critical points are:

- Point $A_2 : \lambda_1 = -4, \lambda_2 = -3(1 + b), \lambda_3 = 3(1 + w_d - b)$,
- Point $B_2 : \lambda_1 = 4, \lambda_2 = 1 - 3b, \lambda_3 = 7 + 3(w_d - b)$,
FIG. 6: Model I: variation of $x, y, z$ as a function of the $N = \ln(a)$. The initial conditions chosen are $x(0) = 0.7, y(0) = 0.3, z(0) = 0.01, w_d = -1$ and $b = 0.5$.

FIG. 7: Model II: Phase space for $w_d = -1.2, b = 0.5, \alpha = 0.5$.

- Point $C_2: \lambda_1 = 3(1+b), \lambda_2 = 3b-1, \lambda_3 = 3(w_d+2)$,
- Point $D_2: \lambda_1 = -3(w_d+2), \lambda_2 = 3(b-w_d-1), \lambda_3 = -7-3(b-w_d)$

$A_2, D_2$ are conditionally stable if $b > 1 + w_d$ (for $A_2$) and $w_d > -2$ and $b < 1 + w_d$ (for $D_2$). But $B_2$ and $C_2$ are unstable since $\lambda_1 > 0$.

In figures (7-12), we show the dynamics of Model-II. Attractor solutions are shown in figures 7, 9 and 11. In figure 8, we see that the energy density of dark energy decays like quintessence. Also the energy density of radiation first increases till $N \sim 1.6$ and then starts decreasing. The matter density always decreases and approaches zero nearly $N \sim 3$. In figure 10, the energy density of dark energy increases rapidly behaving like phantom energy, energy density of matter rises almost exponentially at later times, while radiation density increases slower compared to both matter and dark energy. These novel behaviors appear on account of interaction between three components. In figure 12, the dark energy density behaves like quintessence, while matter and radiation density falls with expansion.

C. Interacting Model - III

Let us take the interaction terms [16]:

$$\Gamma_1 = -6b\kappa^2 H^{-1} \rho_d \rho_r, \quad \Gamma_2 = \Gamma_3 = 3b\kappa^2 H^{-1} \rho_d \rho_r. \quad (17)$$
FIG. 10: Model II: variation of $x$, $y$, $z$ as a function of the $N = \ln(a)$. The initial conditions chosen are $x(0) = 0.7, y(0) = 0.3, z(0) = 0.01, w_d = -0.5$ and $b = 0.5$.

FIG. 11: Model II: Phase space for $w_d = -1, b = 0.5, \alpha = 0.5$.

The system in (6) takes the form

\[
\begin{align*}
\frac{dx}{dN} &= 3x\left(\frac{x+y+z+w_dx+w_ry}{1+\alpha}\right) - 3x - 3w_dx - 18bzx, \\
\frac{dy}{dN} &= 3y\left(\frac{x+y+z+w_dx+w_ry}{1+\alpha}\right) - 3y + 9bxz, \\
\frac{dz}{dN} &= 3z\left(\frac{x+y+z+w_dx+w_ry}{1+\alpha}\right) - 3z - 3w_rz + 9bxz.
\end{align*}
\]

(18)

There are six critical points:

- Point $A_3 : (0, 0, 0)$,
- Point $B_3 : (0, 1 + \alpha, 0)$,
- Point $C_3 : (0, 0, 1 + \alpha)$,
- Point $D_3 : (1 + \alpha, 0, 0)$,
- Point $E_3 : (\frac{4}{9\alpha}, \frac{2(1+w_d)}{9\alpha}, \frac{1+w_d}{6\alpha})$,
- Point $F_3 : \left(\frac{w_d+6b\alpha+6b-\frac{3}{2}+2w_d}{6b\alpha+6b-\frac{3}{2}+2w_d}\right)$

\[
\frac{-3b\alpha - 3b + 36b^2\alpha + 18b^2\alpha^2 + 9w_d\alpha + 18b^2 + w_d^2 - 2w_d + 9w_db + \frac{1}{2}}{36(6b\alpha + 6b - \frac{3}{2} + 2w_d)}
\]

The eigenvalues of the Jacobian matrix for these critical points are:

- Point $A_3 : \lambda_1 = -3, \lambda_2 = -4, \lambda_3 = -3(1 + w_d)$,
- Point $B_3 : \lambda_1 = 3, \lambda_2 = -1, \lambda_3 = -3w_d$,

FIG. 12: Model II: variation of $x$, $y$, $z$ as a function of the $N = \ln(a)$. The initial conditions chosen are $x(0) = 0.7, y(0) = 0.3, z(0) = 0.01, w_d = -1$ and $b = 0.5$.

FIG. 13: Model III: Phase space for $w_d = -1.2, b = 0.5, \alpha = 0.5$. 

- Point $D_3 : (1 + \alpha, 0, 0)$,
- Point $E_3 : (\frac{4}{9\alpha}, \frac{2(1+w_d)}{9\alpha}, \frac{1+w_d}{6\alpha})$,
FIG. 14: Model III: variation of $x, y, z$ as a function of the $N = \ln(a)$. The initial conditions chosen are $x(0) = 0.7, y(0) = 0.3, z(0) = 0.01, w_d = -1.2$ and $b = 0.5$.

FIG. 15: Model III: Phase space for $w_d = -0.5, b = 0.5, \alpha = 0.5$.

- Point $C_3$: $\lambda_1 = 3w_d, \lambda_2 = 3(1 + w_d), \lambda_3 = 9b\alpha + 9b - 1 + 3w_d$.
- Point $D_3$: $\lambda_1 = 4, \lambda_2 = 1 - 9b - 9b\alpha, \lambda_3 = -9b\alpha - 9b + 1 - 3w_d$.

$A_3$ is stable for $w_d > -1$. $B_3, D_3$ are unstable. $C_3$ is conditionally stable when $w_d < -1, b < \frac{1 - 3w_d}{9b + 9\alpha}$.

In figures (13-18), we have plotted the cosmological parameters of model-III. In figures 13, 15 and 17, we show the attractor solutions of the differential equations equation. In figure 14, we observe the oscillatory behavior of dark energy and other cosmic components. It shows that when DE energy density decays then corresponding densities of dark matter and radiation increases and vice versa. Figure 16 shows that all forms of energy densities vanish by $N \sim 4.5$. From figure 18, the radiation density stays zero while matter energy density decreases and vanish by $N \sim 2$. The dark energy density increases by $N \sim 2$ while it decreases and stays constant at later epochs.

**D. Interacting Model - IV**

Consider another model with the interaction terms

\[
\begin{align*}
\Gamma_1 &= -3bc^2 H^{-1} \rho_d \rho_r, \\
\Gamma_2 &= 3bc^2 H^{-1} (\rho_d \rho_r - \rho_m \rho_r), \\
\Gamma_3 &= 3bc^2 H^{-1} \rho_m \rho_r.
\end{align*}
\]


The system in (9) takes the form

\[
\begin{align*}
\frac{dx}{dN} &= 3x\left(\frac{x + y + z + w_d x + w_r z}{1 + \alpha}\right) - 3x - 3w_d x - 9b x z, \\
\frac{dy}{dN} &= 3y\left(\frac{x + y + z + w_d x + w_r z}{1 + \alpha}\right) - 3y + 9b(x z - y y) \\
\frac{dz}{dN} &= 3z\left(\frac{x + y + z + w_d x + w_r z}{1 + \alpha}\right) - 3z - 3w_r z + 9b y z.
\end{align*}
\]

There are seven critical points:

- Point \( A_4 : (0, 0, 0) \),
- Point \( B_4 : (1 + \alpha, 0, 0) \),
- Point \( C_4 : (0, 1 + \alpha, 0) \),
- Point \( D_4 : (0, 0, 1 + \alpha) \),
- Point \( E_4 : \left(\frac{4w_d}{3b(1 + w_d) - \frac{1}{2b}}, -\frac{1 + w_d}{3} \right)\),
- Point \( F_4 : \left(\frac{w_d + 3b + 3b\alpha - 1/3}{3b(3b + 3b\alpha - 1)}, -\frac{1/3}{3b(3b + 3b\alpha - 1)}\right)\),
- Point \( G_4 : (0, \frac{4b}{3b}, 0) \).

For points \( A_4, B_4, C_4, D_4, G_4 \), the eigenvalues of the Jacobian matrix are

- Point \( A_4 : \lambda_1 = -3, \lambda_2 = -4, \lambda_3 = -3(1 + w_d) \),
- Point \( B_4 : \lambda_1 = 3w_d, \lambda_2 = 3(1 + w_d), \lambda_3 = 3w_d - 1 \),
- Point \( C_4 : \lambda_1 = 3, \lambda_2 = -3w_d, \lambda_3 = 9b(1 + \alpha) - 1 \),
- Point \( D_4 : \lambda_1 = 3(1 + w_d), \lambda_2 = -9b - 9b\alpha + 1, \lambda_3 = -3w_d - 9b - 9b\alpha + 1 \),
- Point \( G_4 : \lambda_1 = -3w_d \),

\[
\lambda_2 = \frac{\sqrt{3}\sqrt{b(1 + \alpha)(3b + 3b\alpha - 4/9 + 3b + b)}}{b(1 + \alpha)}, \\
\lambda_3 = -\frac{\sqrt{3}\sqrt{b(1 + \alpha)(3b + 3b\alpha - 4/9 + 3b + b)}}{b(1 + \alpha)}.
\]

It is observed that in this model, \( A_4 \) is stable for \( w_d > -1 \), \( B_4 \) is stable for \( w_d < -1 \). \( C_4 \) is unstable. \( D_4 \) is stable for \( w_d < -1, b > \frac{1}{1 + \alpha} \). It’s not possible to determine the stability of the point \( E_4 \). But \( G_4 \) is unstable.

In figures (19-23), we give description about model IV. Figures 19,21,23 represent the phase space diagrams for different forms of dark energy. Figures 22 and 23 shows that energy density of dark energy decreases while in figure 20, the density of DE increases. It’s the typical
behavior of the phantom fields. Thus, our model predicts the correct evolutionary scheme for the DE density in regime of phantom.

V. CONCLUSION

$f(T)$ gravity is a powerful and novel theory for explanation of the acceleration expansion of the universe. In this paper we discussed the stability of the interactive models of the dark energy, matter and radiation in a FRW model, for a general $f(T)$ theory. We derived the equations and show that why we have some attractor solutions for some specific forms of the interactions. We numerically integrate the equations and show that the evolution of the dark energy density mimics three different behaviors phantom, quintessence and cosmological constant in some interactive forms. We like to comment that this interaction is purely phenomenological required to meet some observational consequences. Since the phase space of the system of the evolutionary equations has a definite end point, trackers are exist and depending on the initial conditions, there are different kinds of the trackers.

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