A simpler description of the $\kappa$-topologies on the spaces $\mathcal{D}_{L^p}$, $L^p$, $\mathcal{M}^1$

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Abstract
For the spaces $\mathcal{D}_{L^p}$, $L^p$ and $\mathcal{M}^1$, we consider the topology of uniform convergence on absolutely convex compact subsets of their (pre-)dual space. Following the notation of J. Horváth’s book we call these topologies $\kappa$-topologies. They are given by a neighbourhood basis consisting of polars of absolutely convex and compact subsets of their (pre-)dual spaces. In many cases it is more convenient to work with a description of the topology by means of a family of semi-norms defined by multiplication and/or convolution with functions and by classical norms. We give such families of semi-norms generating the $\kappa$-topologies on the above spaces of functions and measures defined by integrability properties. In addition, we present a sequence-space representation of the spaces $\mathcal{D}_{L^p}$ equipped with the $\kappa$-topology, which complements a result of J. Bonet and M. Maestre. As a byproduct, we give a characterisation of the compact subsets of the spaces $\mathcal{D}_{L^p}$, $L^p$ and $\mathcal{M}^1$.

KEYWORDS
compact sets, locally convex distribution spaces, $p$-integrable smooth functions, topology of uniform convergence on compact sets

MSC (2010)
46A13, 46A50, 46B50, 46E10, 46E35, 46F05

1 INTRODUCTION

In the context of the convolution of distributions, the notion of integrable distributions plays an important role. Using this notion, it is possible to define the “general” convolution of two distributions, see e.g. [24, Exp. n° 21] or [19]. Recall that the space $\mathcal{D}_{L^1}$ of integrable distributions is defined as the dual space of the space

$$
\mathcal{D} = \left\{ \varphi \in \mathcal{C}^\infty : \partial^\alpha \varphi \in \mathcal{C}_0 \text{ for all } \alpha \in \mathbb{N}_0^n \right\}
$$

of smooth functions all of whose derivatives are vanishing at infinity. The topology of the space $\mathcal{D}$ is generated by the sequence of seminorms

$$
\mathcal{D} \to \mathbb{R}, \quad \varphi \mapsto p_m(\varphi) := \sup_{|\alpha| \leq m} \| \partial^\alpha \varphi \|_\infty.
$$
Since for test functions $\varphi \in \mathcal{D}$, we have
\[
\int_{\mathbb{R}^n} \varphi(x) \, dx = \langle \varphi, 1 \rangle
\]
it is natural to define the integral of a distribution as the evaluation at the constant one function. Unfortunately, this function is not an element of the space $\hat{\mathcal{B}}$ but only of its bidual space $(\hat{\mathcal{B}})^{\prime\prime} = \mathcal{D}_{L^\infty}$ of bounded smooth functions with bounded derivatives; see below more on this space. On the other hand, the dual space of $\mathcal{D}_{L^\infty}$ is much bigger than $\mathcal{D}_{L^1}'$ and, maybe more importantly, not canonically embedded into the space of distributions $\mathcal{D}'$.

For this reason on $\mathcal{D}_{L^\infty}$ the topology $\chi(\mathcal{D}_{L^\infty}, \mathcal{D}_{L^1}')$ of uniform convergence on absolutely convex compact subsets of $\mathcal{D}_{L^1}'$ became of interest since $\mathcal{D}_{L^1}$ is precisely the dual space of $\mathcal{D}_{L^\infty}$ equipped with this topology; see e.g. [26, p. 128]. For more general domains than $\mathbb{R}^n$, this dual pair was investigated in detail by P. Dierolf and S. Dierolf in [11].

Also in the context of the convolution, L. Schwartz introduced the spaces $\mathcal{D}_{L^p}$ of $C^\infty$-functions whose derivatives of any order are contained in $L^p$ and the spaces $\mathcal{D}_{L^q}'$ of finite sums of derivatives of $L^q$-functions, $1 \leq p, q \leq \infty$. The topology of $\mathcal{D}_{L^p}$ is defined by the sequence of (semi-)norms

\[ p_{\mathcal{B}} : \mathcal{D}_{L^p} \to \mathbb{R}, \quad p_{\mathcal{B}}(\varphi) = \sup_{S \in \mathcal{B}} |\langle \varphi, S \rangle|, \quad \mathcal{B} \subseteq \mathcal{D}_{L^q}' \text{ bounded}, \]

whereas $\mathcal{D}_{L^q}' = (\mathcal{D}_{L^p})'$ for $\frac{1}{p} + \frac{1}{q} = 1$ if $p < \infty$ and $\mathcal{D}_{L^1}' = (\hat{\mathcal{B}})'$ carry the strong dual topology. Equivalently, for $1 \leq p < \infty$, since $\mathcal{D}_{L^1}'$ is barrelled, the topology of $\mathcal{D}_{L^p}$ is also the topology $\beta(\mathcal{D}_{L^\infty}, \mathcal{D}_{L^1})$ of uniform convergence on the bounded sets of $\mathcal{D}_{L^1}'$.

Following L. Schwartz in [27, p. 59], $\hat{\mathcal{B}}'$ denotes the closure of $\mathcal{E}'$ in $\mathcal{B}' = \mathcal{D}_{L^\infty}'$ which is not the dual space of $\hat{\mathcal{B}}$ but in some sense is its analogon for distributions instead of smooth functions. For $p = \infty$, $\mathcal{D}_{L^\infty}$ carries the topology $\beta(\mathcal{D}_{L^\infty}, \mathcal{D}_{L^1})$ due to $(\mathcal{D}_{L^1})' = \mathcal{D}_{L^\infty}$ [27, p. 200].

In the case of $p = 1$, the duality relation $(\hat{\mathcal{B}}')' = \mathcal{D}_{L^1}$, see [3, Prop. 7, p. 13], provides that $\mathcal{D}_{L^1}$ also has the topology of uniform convergence on bounded sets of $\hat{\mathcal{B}}$.

By definition, the topology of $\mathcal{D}_{L^p}'$ is the topology of uniform convergence on the bounded sets of $\mathcal{D}_{L^p}$ and $\hat{\mathcal{B}}$ for $q > 1$ and $q = 1$, respectively.

In [26, p. 59], the spaces $\mathcal{D}_{L^p,c}$ and $\mathcal{D}_{L^q,c}'$ are considered, where the index $c$ designates the topologies $\chi(\mathcal{D}_{L^p}, \mathcal{D}_{L^q})$ and $\chi(\mathcal{D}_{L^q}', \mathcal{D}_{L^p})$ of uniform convergence on absolutely convex compact sets of $\mathcal{D}_{L^q}'$ and $\mathcal{D}_{L^p}$, respectively.

Before giving three further reasons for our interest in the spaces $L^q_c$ and $\mathcal{D}_{L^q,c}'$, let us recall some notation.

We follow [27, p. 36] in denoting by $Y$ the Heaviside-function. The translate of a function $f$ by a vector $h$ is denoted by $(t_h f)(x) := f(x - h)$. Besides the spaces $L^p$, $\mathcal{D}_{L^p}$, $1 \leq p \leq \infty$, we use the space $\mathcal{M}^1$ of integrable measures which is the strong dual of the space $\mathcal{C}_0$ of continuous functions vanishing at infinity. In measure theory the measures in $\mathcal{M}^1$ usually are called bounded measures whereas J. Horváth, in analogy to the integrable distributions, calls them integrable measures. Here we follow J. Horváth’s naming convention.

The topology of $\mathcal{D}_{L^p}$, $1 \leq p \leq \infty$, can be described either by the seminorms $p_m$ or, equivalently, by

\[ p_{\mathcal{B}} : \mathcal{D}_{L^p} \to \mathbb{R}, \quad p_{\mathcal{B}}(\varphi) = \sup_{S \in \mathcal{B}} |\langle \varphi, S \rangle|, \quad B \subseteq \mathcal{D}_{L^q}' \text{ bounded}, \]

the topology of $\mathcal{D}_{L^p,c}$ only is described by

\[ p_{\mathcal{C}} : \mathcal{D}_{L^p} \to \mathbb{R}, \quad p_{\mathcal{C}}(\varphi) = \sup_{S \in \mathcal{C}} |\langle \varphi, S \rangle|, \quad C \subseteq \mathcal{D}_{L^q}' \text{ compact} \]

for $1 < p \leq \infty$ and

\[ p_{\mathcal{C}} : \mathcal{D}_{L^1} \to \mathbb{R}, \quad p_{\mathcal{C}}(\varphi) = \sup_{S \in \mathcal{C}} |\langle \varphi, S \rangle|, \quad C \subseteq \hat{\mathcal{B}}' \text{ compact}. \]

An analogue statement holds for $L^p_c$, $1 < p \leq \infty$ and $\mathcal{M}^1_c$. 
Thus, our task is the description of the topologies $\gamma(\mathcal{D}_{L^p}, \mathcal{D}'_{L^q})$, $\gamma(\mathcal{D}_{L^1}, \mathcal{B}')$, $\gamma(L^p, L^q)$ and $\gamma(\mathcal{M}^1, \mathcal{C}_0)$ (for $1 < p \leq \infty$) by seminorms involving functions and not sets (Propositions 3.2, 2.5, 6.1 and 6.3). As a byproduct, the compact subsets of $\mathcal{D}'_{L^q}$ and $L^q$ for $1 \leq q < \infty$ are characterised in Proposition 2.4 and in Proposition 6.2, respectively. In addition, we also give characterisations of the compact subsets of $\mathcal{B}'$ and $\mathcal{M}^1$.

The notation generally adopted is the one from [25–29] and [18]. However, we deviate from these references by defining the Fourier transform as

$$\mathcal{F}\varphi(y) = \int_{\mathbb{R}^n} e^{-iy\cdot x} \varphi(x) \, dx, \quad \varphi \in \mathcal{S}.$$ 

Given a dual pair $(E, F)$ of Hausdorff locally convex spaces $E, F$ we use the following topologies on $E$:

- $\beta(E, F)$, the topology (on $E$) of uniform convergence on bounded sets of $F$,
- $\gamma(E, F)$, the topology (on $E$) of uniform convergence on absolutely convex compact subsets of $F$, see [18, p. 235].

Thus, if $1 < p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\mathcal{M}^1_c = (\mathcal{M}^1, \gamma(\mathcal{M}^1, \mathcal{C}_0)),
\quad L^p_c = (L^p, \gamma(L^p, L^q)),
\quad \mathcal{D}_{L^1,c} = (\mathcal{D}_{L^1}, \gamma(\mathcal{D}_{L^1}, \mathcal{B}')),
\quad \mathcal{D}_{L^p,c} = (\mathcal{D}_{L^p}, \gamma(\mathcal{D}_{L^p}, \mathcal{D}'_{L^q})).$$

We use the potential spaces

$$H^{s,p} = \mathcal{F}^{-1}\left( (1 + |x|^2)^{-s/2} \mathcal{F}L^p \right), \quad 1 \leq p \leq \infty, s \in \mathbb{R},$$

see [23, Def. 3.6.1, p. 108], [1, 7.63, p. 252] or [37] and the weighted $L^p$-spaces

$$L^p_k = (1 + |x|^2)^{-k/2} L^p, \quad k \in \mathbb{Z}.$$ 

In addition, we consider the following sequence spaces. By $s$ we denote the space of rapidly decreasing sequences, by $s'$ its dual, the space of slowly increasing sequences. Moreover, we consider the space $c_0$ of null sequences and its weighted variant

$$(c_0)_{-k} = \left\{ x \in \mathbb{C}^\mathbb{N} : \lim_{j \to \infty} j^{-k} x(j) = 0 \right\}.$$ 

The space $s'$ is the non-strict inductive limit

$$s' = \lim_k (c_0)_{-k}$$

with compact embeddings, i.e. an (LS)-space, see [13, p. 132].

Now, let us mention three observations which sparked our interest in the spaces $L^q_c$ and $\mathcal{D}'_{L^q,c}$.

1. Let $E$ and $F$ be distribution spaces. If $E$ has the $\varepsilon$-property introduced by L. Schwartz in [26, p. 53], a kernel distribution $K(x, y) \in \mathcal{D}'_x (F_y)$ belongs to the space $E_x (F_y)$ if it does so “scalarly”, i.e.,

$$K(x, y) \in E_x (F_y) \iff \forall f \in F' : \langle K(x, y), f(y) \rangle \in E_x.$$
The spaces $L^q$, $1 < q \leq \infty$, and $\mathcal{D}'_{L^q,c}$, $1 \leq q \leq \infty$, have the $\varepsilon$-property [26, Prop. 16, p. 59] whereas for $1 \leq p \leq \infty$, $L^p$ and $\mathcal{D}'_{L^p}$ do not. For $L^p$, $1 \leq p < \infty$, this can be seen by checking that $K(x, k) = (k^{1/p} / (1 + k^2 x^2)) \in \mathcal{D}'(\mathbb{R}) \hat{\otimes} c_0$ satisfies $\langle K(x, k), a \rangle \in L^p$ for all $a \in c^1$ but is not contained in $L^p \hat{\otimes} c_0$; in the case of $p = \infty$ one takes $K(x, k) = Y(x - k)$ instead. For $\mathcal{D}'_{L^q}$ a similar argument with $K(x, k) = \delta(x - k)$ can be used.

(2) The kernel $\delta(x - y)$ of the identity mapping $E_x \to E_y$ of a distribution space $E$ belongs to $E'_{c,y} \in E_x$. Thus, e.g., the equation

$$\delta(x - y) = \sum_{k=0}^{\infty} H_k(x)H_k(y)$$

where the $H_k$ denote the Hermite functions, due to G. Arfken in [2, p. 529] and P. Hirvonen in [16] only formally, is valid in the spaces $L^2 \hat{\otimes} c_0 = \mathcal{S}' \hat{\otimes} \mathcal{S}$; see [27, p. 260–262].

(3) The classical Fourier transform

$$\mathcal{F} : L^p \to L^q, \quad 1 \leq p \leq 2, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

is well-defined and continuous by the Hausdorff–Young theorem. By means of the kernel $e^{-ixy}$ of $\mathcal{F}$ we can express the Hausdorff–Young theorem in a different way. In the case $p > 1$, if we identify the mapping $\mathcal{F}$ with its kernel, the Hausdorff–Young theorem can be interpreted as

$$e^{-ixy} \in \mathcal{D}'(L^p_x, L^q_y).$$

Note that the topology on the space of linear mappings does not play any role in the above statement. Now using [26, p. 11–12] and the fact that $L^q$ has the approximation property, we obtain

$$e^{-ixy} \in \mathcal{D}'(L^p_x, L^q_y) \cong \mathcal{L}'(L^q_x, L^p_{c,y}) = L^q_y \hat{\otimes} c_0 L^p_{c,x} = L^q_{c,y} \hat{\otimes} c_0 L^p_{c,x}.$$ 

In the case $p = 1$, we use the mapping properties of the Fourier transform together with the fact that by [26, p. 18] $L^1 = \left( L^\infty_{c,x} \right)'$, we obtain

$$e^{-ixy} \in \mathcal{D}'(L^1_x, c_0 y) \cong \mathcal{L}'(L^\infty_{c,x} c_0 y) \cong L^\infty_{c,x} \hat{\otimes} c_0 c_0 y.$$ 

Obvious generalizations are

$$e^{-ixy} \in \mathcal{D}'_{L^q,c,x} \left( \bigcup_{k=0}^{\infty} (L^q)_{-k,y} \right), \quad 2 < q < \infty, \quad \text{and}$$

$$e^{-ixy} \in \mathcal{D}'_{L^\infty,c,x} (\theta^0_{c,y});$$

see [19, p. 173] for the space $\theta^0_{c,y}$.

In order to obtain a convenient representation of the spaces $\mathcal{D}'_{L^p}$ and $\mathcal{B}'$ as inductive limits, we consider the uniquely determined temperate fundamental solution of the iterated metaharmonic operator $\left( 1 - \Delta_n \right)^k$, where $\Delta_n$ is the $n$-dimensional Laplacean. In [27, p. 47], up to using the finite part, this fundamental solution is given by

$$L^q_{2k} = \frac{1}{2^{k+n/2-1} \pi^{n/2} \Gamma(k)} |x|^{k-n/2} K_{n/2-k}(|x|) = \mathcal{F}^{-1} \left( (1 + |x|^2)^{-k} \right).$$
Note that defining the distribution

\[ L_s = \frac{1}{2^{s+n-1} \pi^{n/2} \Gamma(s/2)} |x|^{s-n} K_{s-1}(x) \]

by analytic continuation with respect to \( s \in \mathbb{C} \) allows us to drop the symbol \( \text{Pf} \) used in [27, (VI, 8; 5), p. 204] because that way \( L_s \in \mathcal{H}(\mathbb{C}_s) \otimes \mathscr{S}' \) can be considered as an entire holomorphic function with values in \( \mathscr{S}'(\mathbb{R}^n_x) \). In virtue of

\[ \mathcal{F} L_s = (1 + |x|^2)^{-s/2} \in \mathcal{O}_M \]

we, in fact, have \( L_s \in \mathcal{H}(\mathbb{C}_s) \otimes \mathcal{O}'_C \) and \( L_s \ast L_t = L_{s+t} \) for \( s, t \in \mathbb{C} \) [27, (VI, 8; 5), p. 204].

Particular cases of this formula are:

\[ L_0 = \delta , \quad L_{-2k} = (1 - \Delta_n)^k \delta , \quad (1 - \Delta_n)^k L_{2k} = \delta , \quad k \in \mathbb{N} . \]

For \( s > 0 \), \( L_s > 0 \) and \( L_s \) decreases exponentially at infinity. Moreover, \( L_s \in L^1 \) for \( \text{Re} s > 0 \). In contrast to [30, p. 131] and [37], we maintain the original notation \( L_s \) for the Bessel kernels and we write \( L_s \ast \) instead of \( \mathfrak{J}_s \). The spaces \( \mathcal{D}'_{L,q} \) can be described as the inductive limit

\[ \bigcup_{m=0}^{\infty} (1 - \Delta_n)^m \mathcal{L}_q = \lim_{m} H^{-2m,q} , \]

see, e.g., [27, p. 205]. The space \( \mathcal{B}' \) of distributions vanishing at infinity has the similar representation

\[ \mathcal{B}' = \bigcup_{m=0}^{\infty} (1 - \Delta_n)^m \mathcal{C}_0 = \lim_{m} (1 - \Delta_n)^m \mathcal{C}_0 , \]

by [27, p. 205]. If we equip \( (1 - \Delta_n)^m \mathcal{C}_0 \) with the final locally convex topology with respect to \( (1 - \Delta_n)^m \) an application of de Wilde’s closed graph theorem provides the topological equality since by [3, Prop. 7, p. 65] \( \mathcal{B}' \) is ultrabornological and \( \lim_{m} (1 - \Delta_n)^m \mathcal{C}_0 \) has a completing web since it is a Hausdorff inductive limit of Banach spaces.

## 2 "FUNCTION"-SEMINORMS IN \( \mathcal{D}'_{L,p,c} \), \( 1 < p \leq \infty \)

In order to describe the topology of \( \mathcal{D}'_{L,p,c} (1 < p < \infty) \) by “function”-seminorms it is necessary to characterise the compact sets of the dual space \( \mathcal{D}'_{L,q} (1 < q < \infty) \), defined in [27, p. 200] as

\[ \mathcal{D}'_{L,q} = \left( \mathcal{D}_{L,p} \right)' , \quad \frac{1}{p} + \frac{1}{q} = 1 , \]

and endowed with the strong topology \( \mathcal{B} \left( \mathcal{D}'_{L,q}, \mathcal{D}_{L,p} \right) \). The description of \( \mathcal{D}'_{L,\infty,c} \) is already well-known [11].

We recall the following known description of the space \( \mathcal{D}'_{L,p} \) as an inductive limit.

**Proposition 2.1.** The space \( \mathcal{D}'_{L,p} \) has the representation

\[ \mathcal{D}'_{L,p} = \lim_{m} H^{-2m,q} \]

as a regular inductive limit of potential spaces.
Due to the definition of the space $\mathcal{D}_L^p$, 1 $\leq$ $p$ $<$ $\infty$, as the countable projective limit $\bigcap_{m=0}^{\infty} H^{2m,p}$ of the Banach spaces $H^{2m,p}$, which are called “potential spaces” in [30, p. 135], we conclude that the strong dual $\mathcal{D}'_L^q$ coincides with the countable projective limit $\bigcap_{m=0}^{\infty} H^{2m,p}$. Note that the topological identity follows from the fact that $(\mathcal{D}'_L^q, \beta(\mathcal{D}'_L^q, \mathcal{D}_L^p))$ is ultrabornological, which follows for example by the sequence-space representation $\mathcal{D}_L^p \cong s^\prime \hat{\otimes} \ell^p$ given independently by D. Vogt in [32] and by M. Valdivia in [31], by means of Grothendieck's Théorème B [15, p. 17]. The completeness of $\mathcal{D}'_L^q$ implies the regularity of the inductive limit $\bigcup_{m=0}^{\infty} H^{−2m,q}$ [6, p. 77]. □

An alternative proof of the representation of $\mathcal{D}'_L^q$, as the inductive limit of the potential spaces above can be given using [4, Thm. 5] and the fact that $1 − \Delta_n$ is a densely defined and invertible, closed operator on $L^2$.

We first show that the $(LB)$-space $\mathcal{D}'_L^q$, 1 $\leq$ $q$ $<$ $\infty$, is compactly regular [6, 6. Def. (c), p. 100]. Recall that a Hausdorff locally convex inductive limit $E = \lim_{\alpha} E_\alpha$ is compactly regular if every compact subset $K$ of $E$ is already a compact subset of some $E_\alpha$.

**Proposition 2.2.** If 1 $\leq$ $q$ $<$ $\infty$, the $(LB)$-space $\mathcal{D}'_L^q$ is compactly regular.

**Proof.**

Compactly regular (LF)-spaces are characterised by condition (Q) ([35, Thm. 2.7, p. 252], [36, Thm. 6.4, p. 112]) which in our case reads as:

$$\forall m \in \mathbb{N}_0 \ \exists k > m \ \forall \varepsilon > 0 \ \forall \ell > k \ \exists C > 0 : \ ||S||_{2k,q} \leq \varepsilon ||S||_{2m,q} + C ||S||_{2\ell,q} \ \text{for all } S \in H^{−2m,q}.$$  

Note that $H^{−2m,q} \hookrightarrow H^{−2k,q} \hookrightarrow H^{−2\ell,q}$. For the condition (Q) see [33, Prop. 2.3, p. 62]. By definition,

$$||L_{2k} * S||_q \leq \varepsilon ||L_{2m} * S||_q + C ||L_{2\ell} * S||_q \ \text{for all } S \in H^{−2m,q}$$

is equivalent to

$$||L_{2(k−\ell)} * S||_q \leq \varepsilon ||L_{2(m−\ell)} * S||_q + C ||S||_q \ \text{for all } S \in H^{−2(m−\ell),q}.$$  

But this inequality follows from Ehrling’s inequality in [37, p. 101], which states that for 1 $\leq$ $q$ $<$ $\infty$ and 0 $<$ $s$ $<$ $t$,

$$\forall \varepsilon > 0 \ \exists C > 0 : \ ||L_s * \varphi||_q \leq \varepsilon ||\varphi||_q + C ||L_t * \varphi||_q, \ \varphi \in S.$$  

By density of $s$ in $H^{−2(m−\ell),q}$ this implies the validity of (Q). □

**Remarks 2.3.**

(a) By means of M. Valdivia’s and D. Vogt’s sequence space representation $\mathcal{D}_L^p \cong s^\prime \hat{\otimes} \ell^p$ given in [31, Thm. 1, p. 766], and [32, (3.2) Thm., p. 415] and $\mathcal{B} \cong c_0 \hat{\otimes} s$, we obtain by [15, Ch. II, Thm. XII, p. 76] that $\mathcal{D}'_L^q \cong \ell^q \hat{\otimes} s'$, 1 $\leq$ $q$ $\leq$ $\infty$. Using this representation, a further proof of the compact regularity of the $(LB)$-space $\mathcal{D}'_L^q$ is given in Section 4.

(b) A different proof of the compact regularity of the $(LB)$-space $\mathcal{D}'_L^1$ is given in [11, (3.6) Prop., p. 71].

(c) If $q = 2$, the space $\mathcal{D}'_L^2$ is isomorphic to the $(LB)$-space $\bigcup_{k=0}^{\infty} (L^2)_{−k}$. The compact regularity of the $(LB)$-spaces

$$\bigcup_{k=0}^{\infty} (L^p)_{−k}, \quad 1 \leq p \leq \infty,$$

immediately follows from the validity of condition (Q). For $p = 1$ the compact regularity of the space $\bigcup_{k=0}^{\infty} (L^1)_{−k}$ was shown in [10, (3.8), Satz (a), (b), p. 28; (3.9) Bem., (a), p. 29].

The next proposition characterises compact sets in $\mathcal{D}'_L^q$.

**Proposition 2.4.** Let 1 $\leq$ $q$ $<$ $\infty$. A set $C \subseteq \mathcal{D}'_L^q$ is compact if and only if for some $m \in \mathbb{N}_0$, $L_{2m} * C$ is compact in $L^q$. 


Proof. “⇐”: The compactness of \( L_{2m} \ast C \) in \( L^q \) implies its compactness in \( \mathcal{D}'_{L^q} \) and, hence, \( C = L_{-2m} \ast (L_{2m} \ast C) \) is compact in \( \mathcal{D}'_{L^q} \).

“⇒”: In virtue of Proposition 2.2 there is \( m \in \mathbb{N}_0 \) such that \( C \) is compact in \( H^{-2m,q} \). The continuity of the mapping \( \varphi \mapsto L_{2m} \ast \varphi, H^{-2m,q} \to L^q \) implies the compactness of \( L_{2m} \ast C \) in \( L^q \).

The following proposition generalizes the description of the topology of the space \( \mathcal{B}_c = \left( \mathcal{D}_L, \kappa \left( \mathcal{D}_L, \mathcal{D}'_{L^1} \right) \right) \) by the “function”-seminorms

\[
p_{g,m}(\varphi) = \sup_{|\alpha| \leq m} \| g^{\alpha} \varphi \|_\infty, \quad g \in \mathcal{C}_0, \; m \in \mathbb{N}_0,
\]

for \( \varphi \in \mathcal{B} = \mathcal{D}_L \) in [11, (3.5) Cor., p. 71].

**Proposition 2.5.** Let \( 1 < p \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). The topology \( \kappa \left( \mathcal{D}_{L^p}, \mathcal{D}'_{L^q} \right) \) of \( \mathcal{D}_{L^p,c} \) is generated by the seminorms

\[
\mathcal{D}_{L^p} \to \mathbb{R}_+, \quad \varphi \mapsto p_{g,m}(\varphi) := \| g(1 - \Delta_n)^m \varphi \|_p, \quad g \in \mathcal{C}_0, m \in \mathbb{N}_0,
\]

or equivalently by

\[
\varphi \mapsto \sup_{|\alpha| \leq m} \| g^{\alpha} \varphi \|_p, \quad g \in \mathcal{C}_0, m \in \mathbb{N}_0.
\]

**Proof.** Due to [11, (3.5) Cor., p. 71] it suffices to assume \( 1 < p < \infty \). We denote the topology on \( \mathcal{D}_{L^p} \) generated by \( \{ p_{g,m} : g \in \mathcal{C}_0, m \in \mathbb{N}_0 \} \) by \( t \). Moreover, \( B_{1,p} \) shall denote the unit ball in \( L^p \).

(a) \( t \subseteq \kappa \left( \mathcal{D}_{L^p}, \mathcal{D}'_{L^q} \right) \):

If \( \mathcal{U}_{g,m} := \{ \varphi \in \mathcal{D}_{L^p} : p_{g,m}(\varphi) \leq 1 \} \) is a neighborhood of 0 in \( t \) we have \( \mathcal{U}_{g,m} = \left( \left( \mathcal{U}_{g,m} \right)^{\circ} \right)^{\circ} \) by the theorem on bipolars. We show that \( \mathcal{U}_{g,m}^{\circ} \) is a compact set in \( \mathcal{D}'_{L^q} \). We have

\[
\varphi \in \mathcal{U}_{g,m}^{\circ} \iff g(L_{-2m} \ast \varphi) \in B_{1,p}
\]

\[
\iff \sup_{\psi \in B_{1,q}} \left| \left< \psi, g(L_{-2m} \ast \varphi) \right> \right| \leq 1
\]

\[
\iff \sup_{\psi \in B_{1,q}} \left| \left< L_{-2m} \ast (g\psi), \varphi \right> \right| \leq 1
\]

\[
\iff \varphi \in \left( L_{-2m} \ast \left( GB_{1,q} \right) \right)^{\circ}.
\]

Hence, \( \mathcal{U}_{g,m}^{\circ} = L_{-2m} \ast \left( GB_{1,q} \right) \subseteq \mathcal{D}'_{L^q} \). By Proposition 2.4, \( \mathcal{U}_{g,m}^{\circ} \) is compact in \( \mathcal{D}'_{L^q} \) if there exists \( \ell \in \mathbb{N}_0 \) such that

\[
L_{2\ell} \ast \mathcal{U}_{g,m}^{\circ} = L_{2(\ell - m)} \ast \left( GB_{1,q} \right)
\]

is compact in \( L^q \). Choosing any \( \ell > m \), it suffices to show that \( C := L_{2(\ell - m)} \ast \left( GB_{1,q} \right) \) satisfies the three criteria of the M. Fréchet–M. Riesz–A. Kolmogorov–H. Weyl Theorem [29, Thm. 6.4.12, p. 140] or [38, p. 275]:

(i) it is bounded;
(ii) it is uniformly small at infinity, i.e.

\[
\lim_{R \to \infty} \sup_{f \in \mathcal{C}} \| Y(|x| - R)f \|_p = 0
\]

where \( | \cdot | \) denotes the Euclidean norm on \( \mathbb{R}^n \).
(iii) it is uniformly \( p \)-equicontinuous, i.e.
\[
\lim_{s \to 0} \sup_{f \in C} \| \tau_s f - f \|_p = 0,
\]
where \( \tau_s f(x) = f(x - s) \).

Now, let us check that these conditions are indeed satisfied.

(i) Because \( \ell > m \), \( \mu := L_{2(\ell - m)} \in L^1 \) and hence, for \( \varphi \in B_{1,q} \),
\[
\| \mu \ast (g \varphi) \|_q \leq \| \mu \|_1 \| g \|_\infty,
\]
i.e., \( C \) is bounded in \( L^q \).

(ii) The set \( C \) has to be small at infinity: for \( \varphi \in B_{1,q} \) and \( R > 0 \),
\[
\begin{align*}
\| Y(|.| - R)(\mu \ast (g \varphi)) \|_q &\leq \| Y(|.| - R) \left( Y \left( \frac{R}{2} - |.| \right) \mu \right) \ast (g \varphi) \|_q + \| Y(|.| - R) \left( \frac{Y(|.| - R)}{2} \mu \right) \ast (g \varphi) \|_q \\
&\leq \left( \int_{|x| \geq R} \left( \int_{|\xi| \leq R/2} |\mu(\xi)(g \varphi)(x - \xi)|^q d\xi \right)^{1/q} dx \right)^{1/q} + \| Y(|.| - R/2) \mu \ast (g \varphi) \|_q \\
&\leq \left( \int_{|\xi| \leq R/2} \left( \int_{|z| \geq R/2} |(g \varphi)(z)|^q dz \right)^{1/q} d\xi + \| Y(|.| - R/2) \mu \|_1 \cdot g \|_{\infty} \right)^{1/q} \\
&\leq \| \mu \|_1 \| Y(|.| - R/2)g \|_{\infty} + \| Y(|.| - R) \mu \|_1 \cdot g \|_{\infty}.
\end{align*}
\]
Hence, \( \lim_{R \to \infty} \| Y(|.| - R)(\mu \ast (g \varphi)) \|_q = 0 \) uniformly for \( \varphi \in B_{1,q} \).

(iii) \( C \) is \( L^q \)-equicontinuous because
\[
\begin{align*}
\| \tau_h (\mu \ast (g \varphi)) - \mu \ast (g \varphi) \|_q &\leq \| \tau_h \mu - \mu \|_1 \cdot g \|_{\infty} \\
tends to 0 if \( h \to 0 \), uniformly for \( \varphi \in B_{1,q} \).
\]

(b) \( \kappa \left( \mathcal{D}_{L^p}, \mathcal{D}_{L^q}' \right) \subset t \):

If \( C^o \) is a \( \kappa \left( \mathcal{D}_{L^p}, \mathcal{D}_{L^q}' \right) \)-neighborhood of 0 with \( C \) a compact set in \( \mathcal{D}_{L^q}' \), then, by Proposition 2.4, there exists \( m \in \mathbb{N}_0 \) such that the set \( L_{2m} \ast C \) is compact in \( L^q \). By means of Lemma 2.6 below there is a function \( g \in C_0 \) such that \( L_{2m} \ast C \subseteq g B_{1,q} \), hence \( C \subseteq L_{-2m} \ast (gB_{1,q}) \subseteq \mathcal{U}_{g,m}^e \). Thus, \( C^o \supseteq \mathcal{U}_{g,m} \), i.e., \( C^o \) is a neighborhood in \( t \).

Lemma 2.6. Let \( 1 \leq q < \infty \). If \( K \subseteq L^q \) is compact, then there exists \( g \in C_0 \) such that \( K \subseteq g B_{1,q} \).

Proof. In order to apply the Cohen–Hewitt factorization theorem \([12, (17.1), p. 114]\) to the bounded subset \( K \) of the (left) Banach module \( L^q \) with respect to the multiplicative Banach algebra \( C_0 \) we have to check that
\[
\left\| e^{-k^2|x|^2} f - f \right\|_p \to 0 \quad \text{for} \quad k \to 0
\]
uniformly with respect to \( f \in K \) which follows from a direct computation. This shows that \( K \subset gB \) for some \( g \in C_0 \) and a bounded set \( B \). Without loss of generality, we may assume \( B = B_{1,q} \) which finishes the proof. 

**Remark 2.7.** Denoting by \( \tau(\mathcal{B},\mathcal{D}_{L,1}') \) the Mackey-topology on \( \mathcal{B} = \mathcal{D}_{L,\infty} \) we even have for \( p = \infty \), \( \mathcal{B} = \mathcal{D}_{L,\infty} \):

\[
\mathcal{B}_c = \left( \mathcal{B}, \kappa(\mathcal{B}, \mathcal{D}_{L,1}') \right) = \left( \mathcal{B}, \tau(\mathcal{B}, \mathcal{D}_{L,1}') \right),
\]

since \( \mathcal{D}_{L,1}' \) is a Schur space [11, p. 52].

**3  ** THE CASE \( p = 1 \)

Using the sequence-space representation \( \hat{\mathcal{B}}' \cong s' \otimes c_0 \) given in [3, Thm. 3, p. 13], the compact regularity of the \( (LB) \)-space \( \hat{\mathcal{B}}' \) can be shown similarly to the proof of Proposition 4.1. Moreover, one has the following characterisation of the compact sets of \( \hat{\mathcal{B}}' \).

**Proposition 3.1.** A set \( C \subseteq \hat{\mathcal{B}}' \) is compact if and only if for some \( m \in \mathbb{N}_0 \), \( L^2_m \ast C \) is compact in \( \mathcal{C}_0 \).

**Proof.** The proof is completely analogous to the one of Proposition 2.4. \( \square \)

**Proposition 3.2.** The topology \( \kappa(\mathcal{D}_{L,1}, \hat{\mathcal{B}}') \) of \( \mathcal{D}_{L,1,c} \) is generated by the seminorms

\[
\mathcal{D}_{L,1} \to \mathbb{R}_+ , \quad \varphi \mapsto p_{g,m}(\varphi) : = \| g(1 - \Delta_n)^m \varphi \|_1 , \quad g \in C_0 , m \in \mathbb{N}_0 ,
\]

or equivalently by

\[
\varphi \mapsto \sup_{|\alpha| \leq m} \| g \partial^\alpha \varphi \|_1 , \quad g \in C_0 , m \in \mathbb{N}_0 .
\]

**Proof.** We first show that the topology \( t \) generated by the above seminorms is finer than the topology of uniform convergence on compact subsets of \( \hat{\mathcal{B}}' \). Similarly to the proof of Proposition 2.5, we have to show that the set \( L^2(\ell - m) \ast \left( B_1, \mathcal{C}_0 \right) \) is a relatively compact subset of \( \hat{\mathcal{B}}' \), where \( B_1, \mathcal{C}_0 \) denotes the unit ball of \( \mathcal{C}_0 \). We pick \( \ell > m \) and, by the compact regularity of \( \hat{\mathcal{B}}' \) and the Arzela–Ascoli theorem, we have to show that \( L^2(\ell - m) \ast \left( B_1, \mathcal{C}_0 \right) \) is bounded as a subset of \( \mathcal{C}_0 \) and equicontinuous as a set of functions on the Alexandroff compactification of \( \mathbb{R}_n \). Since \( \ell > m \), we have that \( L^2(\ell - m) \in L^1 \).

For every \( \varphi \in B_1, \mathcal{C}_0 \), Young’s convolution inequality implies

\[
\| L^2(\ell - m) \ast (g \varphi) \|_\infty \leq \| L^2(\ell - m) \|_1 \| g \varphi \|_\infty \leq \| L^2(\ell - m) \|_1 \| g \|_\infty ,
\]

i.e. \( L^2(\ell - m) \ast \left( B_1, \mathcal{C}_0 \right) \subseteq \mathcal{C}_0 \) is a bounded set.

Since a translation of a convolution product can be computed by applying the translation to one of the factors, we can again use Young’s convolution inequality to obtain

\[
\left\| \tau_h(L^2(\ell - m) \ast (g \varphi)) - L^2(\ell - m) \ast (g \varphi) \right\|_\infty \leq \left\| \tau_h(L^2(\ell - m) - L^2(\ell - m)) \ast (g \varphi) \right\|_\infty \\
\leq \left\| \tau_h(L^2(\ell - m) - L^2(\ell - m)) \right\|_1 \| g \|_\infty
\]

for all \( \varphi \) in the unit ball of \( \mathcal{C}_0 \). From this inequality we may conclude that \( L^2(\ell - m) \ast \left( g B_1, \mathcal{C}_0 \right) \) is equicontinuous at all points in \( \mathbb{R}^n \). Therefore we are left to show that it is also equicontinuous at infinity, i.e., for every \( k \in \mathbb{N} \) there is an \( R_k \) such that \( |h(x)| \leq 1/k \) for all \( h \in L^2(\ell - m) \ast \left( g B_1, \mathcal{C}_0 \right) \) and all \( |x| > R_k \). In order to do this, first observe that \( \| g \| \in \mathcal{C}_0 \) and \( \| L^2(\ell - m) \| = L^2(\ell - m) \). Moreover, Lebesgue’s theorem on dominated convergence implies that the convolution of a function
in \( C_0 \) and an \( L^1 \)-function is contained in \( C_0 \). Finally, by the above reasoning the inequality
\[
\left| \left( L_2(\ell - m) \ast (g\varphi) \right)(x) \right| \leq \int_{\mathbb{R}^n} |g(x - \xi)| |L_2(\ell - m)(\xi)| \, d\xi = \left( L_2(\ell - m) \ast |g| \right)(x)
\]
shows that \( L_2(\ell - m) \ast (gB_{1,c_0}) \) is equicontinuous at infinity.

The proof that \( \kappa(D_{L^1}, B') \) is finer than \( t \) is completely analogous to the corresponding part of Proposition 2.5 if we can show that every compact subset of \( C_0 \) is contained in \( gB_{1,c_0} \) for some \( g \in C_0 \). Let \( C \subseteq C_0 \) be a compact set. By the Arzela–Ascoli theorem, it is equicontinuous at infinity, i.e., for every \( k \in \mathbb{N} \) there is an \( R_k \) such that \( |h(x)| \leq 1/k \) for \( h \in C \) and all \( |x| > R_k \). This condition implies the existence of the required function \( g \in C_0 \) with the above property. \( \square \)

4  |  SEQUENCE SPACE REPRESENTATIONS OF THE SPACES \( D_{L^p,c}, D'_{L^q,c} \) AND THE COMPACT REGULARITY OF \( D'_{L^q} \)

P. and S. Dierolf conjectured in [11, p. 74] the isomorphy
\[
D_{L^\infty,c} = B_c \cong \ell^\infty_c \hat{\otimes} s,
\]
where \( \ell^\infty_c = (\ell^\infty, \kappa(\ell^\infty, \ell^1)) = (\ell^\infty, \tau(\ell^\infty, \ell^1)) \). This conjecture is proven in [8, 1. Thm., p. 293]. More generally, we obtain:

**Proposition 4.1.** Let \( 1 \leq p, q \leq \infty \).

(a) \( D_{L^p,c} \cong \ell^p_c \hat{\otimes} s \);
(b) \( D'_{L^q,c} \cong \ell^q_c \hat{\otimes} s' \).

**Proof.** (1) By means of M. Valdivia’s isomorphy (see Remarks 2.3 (a)) \( D'_{L^q} \cong \ell^q \hat{\otimes} s' \) it follows, for \( q < \infty \), by [9, 4.1 Thm., p. 52 and 2.2 Prop., p. 46]:
\[
D_{L^p,c} \cong \ell^p_c \hat{\otimes} s, \quad \frac{1}{p} + \frac{1}{q} = 1.
\]
If \( p = 1 \), \( D_{L^1,c} \cong (B')_c \cong (c_0 \hat{\otimes} s')_c \cong \ell^1_c \hat{\otimes} s \) by [9, 4.1 Thm., p. 52 and 2.2 Prop., p. 46] and [3, Prop. 7, p. 13]. The case \( p = \infty \) is a special case of [8, 1. Thm., p. 293].

(2) The second Theorem on duality of H. Buchwalter [22, (5), p. 302] yields for two Fréchet spaces \( E, F \):
\[
(\mathcal{E} \hat{\otimes} \mathcal{F})'_c \cong E'_c \hat{\otimes} F'_c,
\]
and hence, for \( E = \ell^p \) and \( F = s, E \hat{\otimes} F = \ell^p \hat{\otimes} s \) which implies
\[
D'_{L^q,c} \cong \ell^q_c \hat{\otimes} s',
\]
if \( 1 < q \leq \infty, \frac{1}{p} + \frac{1}{q} = 1 \). If \( q = 1 \),
\[
D'_{L^1,c} \cong (B')_c \cong (c_0 \hat{\otimes} s)'_c \cong \ell^1_c \hat{\otimes} s'.
\]
An alternative proof for (2) can also be given using [9, 4.1 Thm., p. 52 and 2.2 Prop., p. 46] again. \( \square \)

**Remark 4.2.** The sequence space representations of \( D_{L^p,c} \) and \( D'_{L^q,c} \) yield a further proof of the quasinormability of these spaces. Indeed, \( \ell^p_c \) and \( \ell^q_c \) are quasinormable since their duals are Banach spaces and thus, they fulfill the strict Mackey
convergence condition [14, p. 106]. The claim follows from the fact that the completed tensor product with $s$ or $s'$ remains quasinormable by [15, Ch. II, Prop. 13.b, p. 76].

The compact regularity of $\mathcal{D}'_L$, as a countable inductive limit of Banach spaces is proven in Proposition 2.2. By means of the sequence space representation

$$\mathcal{D}'_L \cong \ell^q \hat{\otimes} s'$$

which has been presented in Remarks 2.3 (a) we can give a second proof:

**Proposition 4.3.** Let $E$ be a Banach space.

(a) The inductive limit representations

$$s' \hat{\otimes} E = \lim_k \left( (c_0)_{-k} \hat{\otimes} E \right) = \lim_k \left( c_0 \hat{\otimes} E \right)_{-k} = \lim_k \left( c_0(E) \right)_{-k}$$

are valid.

(b) The inductive limit $\lim_k (c_0(E))_{-k}$ is compactly regular.

**Proof.** (a) The assertion follows from [17, Thm. 4.1, p. 55].

(b) The compact regularity of $\lim_k (c_0(E))_{-k}$ is a consequence of [36, Thm. 6.4, p. 112] (or [35, Thm. 2.7, p. 252]) and the validity of condition $(Q)$ (see [33, Prop. 2.3, p. 62]). The condition $(Q)$ reads as:

$$\forall m \in \mathbb{N}_0 \ \exists k > m \ \forall \varepsilon > 0 \ \forall l > k \ \exists C > 0 \ \forall x = (x_j) \in (c_0(E))_{-m} :$$

$$\sup_j j^{-k} \|x_j\| \leq \varepsilon \sup_j j^{-m} \|x_j\| + C \sup_j j^{-l} \|x_j\|.$$

For the sequences $(x_j) \in (c_0(E))_{-m}$ the sequence $(\|x_j\|)_j$ is contained in $s' = \lim_k (c_0)_{-k}$. Thus, $(Q)$ is fulfilled because $s'$ is an (LS)-space. □

**Remark 4.4.** The above results can also be obtained from 5.10 Proposition in [7, p. 157] using that the inductive limit $\lim_k (c_0(E))_{-k}$ is compactly regular and by an application of Proposition 4.4 and Theorem 4.1 in [17, p. 55].

**Corollary 4.5.** The spaces $\hat{\mathcal{B}}' \cong \lim_k (c_0(E))_{-k}$ and $\mathcal{D}'_L \cong \lim_k (c_0(\ell^q))_{-k}$, for $1 \leq q \leq \infty$, are compactly regular countable inductive limits of Banach spaces.

### 5 | PROPERTIES OF THE SPACES $\mathcal{D}_{L,p,c}$

In [26, p. 127], L. Schwartz proves that the spaces $\mathcal{B}_c = \mathcal{D}_{L,\infty,c}$ and $\mathcal{B} = \mathcal{D}_{L,\infty}$ have the same bounded sets, and that on these sets the topology $\kappa(\mathcal{B}, \mathcal{D}'_{L,1})$ equals the topology induced by $\mathcal{C} = C^\infty$. Moreover, $\kappa(\mathcal{B}, \mathcal{D}'_{L,1})$ is the finest locally convex topology with this property. By an identical reasoning we obtain:

**Proposition 5.1.** Let $1 \leq p < \infty$.

(a) The spaces $\mathcal{D}_{L,p}$ and $\mathcal{D}_{L,p,c}$ have the same bounded sets. These sets are relatively $\kappa(\mathcal{D}_{L,p}, \mathcal{D}'_{L,p})$-compact and relatively $\kappa(\mathcal{D}_{L,1}, \mathcal{B}')$-compact for $1 < p < \infty$ and $p = 1$, respectively.
(b) The topology \( \kappa \left( \mathcal{D}_{L^p}, \mathcal{D}'_{L^q} \right) \) of \( \mathcal{D}_{L^p} \) is the finest locally convex topology on \( \mathcal{D}_{L^p} \) which induces on bounded sets the topologies of \( \mathcal{E} \) or \( \mathcal{D}' \) or \( \mathcal{D}_{L^p} \).

In the following proposition we collect some further properties of the spaces \( \mathcal{D}_{L^p} \). As a tool, we need the following lemma.

**Lemma 5.2.** The space \( \ell^p_c \), for \( 1 \leq p \leq \infty \) is not nuclear.

**Proof.** We first consider the case \( p > 1 \). The nuclearity of \( \ell^p_c \) would imply

\[
\ell^1 \{ \ell^p_c \} = \ell^1 \hat{\otimes} \ell^p_c = \ell^1 \hat{\otimes} \ell^p_c = \ell^1 \langle \ell^p_c \rangle
\]

where \( \ell^1 \{ \ell^p_c \} \) and \( \ell^1 \langle \ell^p_c \rangle \) is the space of absolutely summable and of unconditionally summable sequences in \( \ell^p_c \), respectively, see [20, pp. 341, 359]. We now proceed by giving an example of an element of the space at the very right which is not contained in the space at the very left. Fix \( \varepsilon > 0 \) small enough. Using Hölder’s inequality, we observe that

\[
\left\| \sum_{k=1}^{\infty} \frac{\delta_{jk}}{k^{(1+\varepsilon)/p}} f_k \right\|_1 = \left\| \sum_{j=1}^{\infty} \frac{1}{j^{(1+\varepsilon)/p}} f_j \right\|_1 \leq C \| f \|_q
\]

for every \( f = (f_k)_{k=1}^{\infty} \in \ell^q \) which together with the characterisation of unconditional convergence in [34, Thm. 1.15] and the condition that compact subsets of \( \ell^q \) are small at infinity implies that the sequence is unconditionally convergent. On the other hand taking \( g = (k^{-\alpha})_{k=1}^{\infty} \in c_0 \) with \( \alpha = 1 - \frac{1+\varepsilon}{p} \) yields

\[
\sum_{j=1}^{\infty} \left\| \delta_{jk} k^{-(1+\varepsilon)/p} g_k \right\|_p = \sum_{j=1}^{\infty} j^{-(1+\varepsilon)/p - \alpha} = \sum_{j=1}^{\infty} \frac{1}{j} = \infty
\]

from which we may conclude that \( \left( \delta_{jk} k^{-(1+\varepsilon)/p} \right)_{j,k} \) is not an absolutely summable sequence in \( \ell^p_c \) (see the seminorms in the beginning of Section 6).

For the case \( p = 1 \) we use the Grothendieck–Pietsch criterion, see [20, p. 497], and observe that \( \Lambda (c_{0,+}) = \ell^1_c \). Choosing \( \alpha = \left( \frac{1}{k} \right)_{k=1}^{\infty} \) provides the necessary sequence with \( (\alpha_k/\beta_k)_{k=1}^{\infty} \notin \ell^1 \) for every \( \beta \in c_{0,+} \) with \( \beta \geq \alpha \). □

**Proposition 5.3.** Let \( 1 \leq p \leq \infty \). The spaces \( \mathcal{D}_{L^p,c} \) are complete, quasinormable, semi-Montel and hence semireflexive. \( \mathcal{D}_{L^p,c} \) is not infrabarrelled and hence neither barrelled nor bornological. \( \mathcal{D}_{L^p,c} \) is a Schwartz space but not a nuclear space.

**Proof.**

1. From Corollary 4.5 we may conclude that \( \mathcal{D}'_{L^q} \) and \( \mathcal{B}' \) are bornological. Hence the completeness follows either from [21, (1), p. 385] or from [18, Ex. 7(b), p. 243] which show that the dual space of a bornological space, when equipped with the \( \kappa \)-topology, is complete.

2. \( \mathcal{D}_{L^p,c} \) is quasinormable since its dual \( \mathcal{D}'_{L^q} \) is boundedly retractive (see the argument in [11, p. 73] and use [14, Def. 4, p. 106]) which, by [6, 7, Prop., p. 101] is equivalent with its compact regularity (Proposition 2.2).

3. Since by Proposition 5.1 bounded and relatively compact sets coincide in \( \mathcal{D}_{L^p,c} \) it is a semi-Montel space.

4. Being infrabarrelled together with the Montel-property would imply that \( \mathcal{D}_{L^p,c} \) is Montel which in turn implies the coincidence of the topologies \( \kappa \left( \mathcal{D}_{L^p}, \mathcal{D}'_{L^q} \right) \) and \( \beta \left( \mathcal{D}_{L^p}, \mathcal{D}'_{L^q} \right) \). This is a contradiction, since together with the compact regularity of the inductive limit representation \( \mathcal{D}'_{L^q} = \bigcup H^{-2m,q} \) this would imply that for every \( m \) there is \( m' \) such that the unit ball of \( H^{-2m,q} \) is contained and relatively compact in \( H^{-2m,q} \). In case \( m' \leq m \), the continuous inclusion \( H^{-2m,q} \subseteq H^{-2m,q} \) would give that the unit ball of \( H^{-2m,q} \) is compact, i.e., that \( H^{-2m,q} \) is finite dimensional; in case \( m' \geq m \), by transposition the inclusion \( H^{2m,q} \rightarrow H^{2m,q} \) and a fortiori the inclusion \( H^{2m,q} \rightarrow L^q \) would be compact, which cannot be the case [1, Example 6.11, p. 173].
(5) \( \mathcal{D}_{L^p,c} \) is a Schwartz space [14, Def. 5, p. 117] because by (2) it is quasinormable and by (3) it is semi-Montel.

(6) Using the sequence-space representation \( \mathcal{D}_{L^p,c} \cong \ell^p_c \otimes s \), we can conclude by [15, Ch. II §3 n°2, Prop. 13, p. 76] that \( \mathcal{D}_{L^p,c} \) is nuclear if and only if \( \ell^p_c \) is nuclear. Since by Lemma 5.2, the space \( \ell^p_c \) is not nuclear the assertion follows. □

Remark 5.4. Similar to (6) in the above proof, we may also use Lemma 5.2 to conclude that also the space \( \mathcal{D}'_{L^p,c} \) for \( 1 \leq p \leq \infty \), is not nuclear.

Analogous to the table with properties of the spaces \( \mathcal{D}^F, \mathcal{D}'^F \) (defined in [18, p. 172,173]) in [5, p. 19], we list properties of \( \mathcal{D}_{L^p} \) and \( \mathcal{D}_{L^p,c} \) in the following table (\( 1 \leq p \leq \infty \)):

| property       | \( \mathcal{D}_{L^p} \) | \( \mathcal{D}_{L^p,c} \) |
|----------------|--------------------------|--------------------------|
| complete       | +                        | +                        |
| quasinormable  | +                        | +                        |
| metrizable     | +                        | -                        |
| bornological   | +                        | -                        |
| barrelled      | +                        | -                        |
| (semi-)reflexive | + (\( 1 < p < \infty \)) | +                        |
| semi-Montel    | -                        | +                        |
| Schwartz       | -                        | +                        |
| nuclear        | -                        | -                        |

### 6 “FUNCTION”-SEMINORMS IN \( L^p_c \) AND \( M^1_c \)

Motivated by the description of the topology \( \kappa(\ell^p, \ell^q) \) of the sequence space \( \ell^p_c \) by the seminorms

\[
\ell^p \rightarrow \mathbb{R}, \quad x = (x_j)_{j \in \mathbb{N}} \mapsto p_g(x) = \left( \sum_{j=1}^{\infty} g(j)|x(j)|^p \right)^{1/p} = \| g \cdot x \|_p,
\]

\( g \in c_0 \) and \( g \geq 0 \), see 4. Theorem (b) in [20, p. 47] with \( X = \mathbb{N} \) for the case \( p = \infty \), we also investigate the description of the topologies \( \kappa(L^p, L^q) \) and \( \kappa(M^1, c_0) \) by means of “function”-seminorms.

Denoting the space of bounded and continuous functions by \( \mathcal{B}^0 \) (see [25, p. 99]) we recall that \( \mathcal{B}^0_c = (\mathcal{B}^0, \kappa(\mathcal{B}^0, M^1)) \) has the topology of R. C. Buck; see Proposition 1.2.1 in [23, p. 6].

**Proposition 6.1.** Let \( 1 < p \leq \infty \). The topology \( \kappa(L^p, L^q) \) of \( L^p_c \) is generated by the family of seminorms

\[
L^p \rightarrow \mathbb{R}, \quad f \mapsto p_{g,h}(f) = \| h \ast (g f) \|_p, \quad g \in c_0, h \in L^1.
\]

The space \( L^p_c \) is a complete Schwartz space.

**Proof.** (1) \( t \subseteq \kappa(L^p, L^q) \) where \( t \) denotes the topology generated by the seminorms \( p_{g,h} \) above and let

\[
\mathcal{U}_{g,h} := \{ f \in L^p : p_{g,h}(f) \leq 1 \}
\]

be a neighborhood of 0 in \( t \). We show that

(i) \( \mathcal{U}_{g,h} = (g(\tilde{h} \ast B_{1,q}))^\circ \),

(ii) \( g(\tilde{h} \ast B_{1,q}) \) is compact in \( L^q \).
(i): If \( S \in g(\hat{h} * B_{1,q}) \subseteq L^q \) there exists \( \psi \in B_{1,q} \) with \( S = g(\hat{h} * \psi) \). For \( f \in L^p \) with \( p_{g,h}(f) \leq 1 \) we obtain

\[
|\langle f, S \rangle| = \left| \langle f, g(\hat{h} * \psi) \rangle \right| = |\langle h * (gf), \psi \rangle|
\]

and

\[
\sup_{S \in g(\hat{h} * B_{1,q})} |\langle f, S \rangle| = \|h * (gf)\|_p = p_{g,h}(f),
\]

so (i) follows.

(ii): The 3 criteria of the M. Fréchet–M. Riesz–A. Kolmogorov–H. Weyl Theorem are fulfilled (cf. the proof of Proposition 2.5):

(a) The set \( g(\hat{h} * B_{1,q}) \) is bounded in \( L^q \).

(b) The set \( g(\hat{h} * B_{1,q}) \) is small at infinity: \( \forall \varepsilon > 0 \exists R > 0 \) such that \( |g(x)| Y(|x| - R) < \varepsilon \) for all \( x \in \mathbb{R}^n \) and hence,

\[
\|Y(|.| - R)g(\hat{h} * B_{1,q})\|_q \leq \varepsilon \|h\|_1.
\]

(c) The set \( g(\hat{h} * B_{1,q}) \) is \( L^q \)-equicontinuous:

\[
\|\tau_s(g(\hat{h} * f)) - g(\hat{h} * f)\|_q \leq \|\tau_s g - g\|_q \|h\|_1 + \|g\|_\infty \|\tau_s \hat{h} - \hat{h}\|_1.
\]

If \( s \) tends to 0, \( \|\tau_s g - g\|_\infty \to 0 \) because \( g \) is uniformly continuous, and \( \|\tau_s \hat{h} - \hat{h}\|_1 \to 0 \) because the \( L^1 \)-modulus of continuity is continuous.

(2) \( t \supseteq \chi(L^p, L^q) \):
For a compact set \( C \subseteq L^q \) we have to show that there exist \( g \in \mathcal{C}_0, h \in L^1 \) such that

\[
p_{g,h}(f) = \|g(h * f)\|_p \geq \sup_{S \in C} |\langle f, S \rangle|
\]

for \( f \in L^p \). By applying the factorization theorem \([12, (17.1). p. 114]\) twice, there exist \( g \in \mathcal{C}_0, h \in L^1 \) such that \( C \subseteq g(h * B_{1,q}) \). More precisely, take \( L^1 \) as the Banach algebra with respect to convolution and \( \mathcal{C}_0 \) as the Banach algebra with respect to pointwise multiplication, \( L^p \) as the Banach module. As approximate units we use \( e^{-k^2|x|^2}, k > 0 \), in the case of \( \mathcal{C}_0 \) and \( (4\pi t)^{-n/2}e^{-|x|^2/4t}, t > 0 \), in the convolution algebra \( L^1 \). Then,

\[
\sup_{S \in C} |\langle f, S \rangle| \leq \sup_{\psi \in B_{1,q}} \left| \langle f, g(\hat{h} * \psi) \rangle \right| = \sup_{\psi \in B_{1,q}} |\langle h * (gf), \psi \rangle| = \|h * (gf)\|_p,
\]

which finishes the proof that the above family of seminorms generates the topology.

(3) The proofs for the completeness and for the Schwartz space property are completely analogous to the corresponding part of the proof of Proposition 5.3.

Proposition 6.2. Let \( 1 \leq q < \infty \) and \( C \subseteq L^q \). Then,

\[
C \text{ compact} \iff \exists g \in \mathcal{C}_0 \exists h \in L^1 : C \subseteq g(h * B_{1,q}).
\]

An exactly analogous reasoning as for Proposition 6.1 yields
Proposition 6.3. The topology \( \kappa(\mathcal{M}_1, \mathcal{C}_0) \) of the space \( \mathcal{M}_1 \) is generated by the seminorms

\[
\mathcal{M}_1 \to \mathbb{R}, \quad \mu \mapsto p_{g,h}(\mu) := \|h \ast (g \mu)\|_1, \quad g \in \mathcal{C}_0, h \in L^1.
\]

The space \( \mathcal{M}_1 \) is a complete Schwartz space.

Proof. For this, note that \( \|\mu\|_1 = \sup \{ |\langle \varphi, \mu \rangle| : \varphi \in \mathcal{C}_0, \|\varphi\|_{\infty} \leq 1 \} \) and that the multiplication

\[
\mathcal{C}_0 \times \mathcal{M}_1 \to \mathcal{M}_1, \quad (g, \mu) \mapsto g \cdot \mu
\]

and the convolution

\[
L^1 \times \mathcal{M}_1 \to L^1, \quad (h, \nu) \mapsto h \ast \nu
\]

are continuous [29, Thm. 6.4.20, p. 150]).

The proofs for the completeness and for the Schwartz space property are completely analogous to the corresponding part of the proof of Proposition 5.3. \( \square \)

Proposition 6.4. Let \( C \subseteq \mathcal{M}_1 \). Then,

\[
C \text{ compact} \iff \exists g \in \mathcal{C}_0 \exists h \in L^1 : C \subseteq g \ast \{ h \ast B_{1,1} \}.
\]

Acknowledgments

E. A. Nigsch was supported by the Austrian Science Fund (FWF) grants P26859 and P30233. The authors wish to thank two anonymous referees for their valuable comments which improved the presentation of this article.

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**How to cite this article:** Bargetz C, Nigsch EA, Ortner N. A simpler description of the \( \kappa \)-topologies on the spaces \( \mathcal{D}_{L^p}, L^p, M^1 \). *Mathematische Nachrichten*. 2020;293:1691–1706. [https://doi.org/10.1002/mana.201900109](https://doi.org/10.1002/mana.201900109)