Neutrino masses and mixings

She-Sheng Xue

INFN, Section of Milan, Via Celoria 16, Milan, Italy
Physics Department, University of Milan, Italy

Abstract

We propose a novel theoretical understanding of neutrino masses and mixings, which is attributed to the intrinsic vector-like feature of the regularized Standard Model at short distances. We try to explain the smallness of Dirac neutrino masses and the decoupling of the right-handed neutrino as a free particle. Neutrino masses and mixing angles are completely related to each other in the Schwinger-Dyson equations for their self-energy functions. The solutions to these equations and a possible pattern of masses and mixings are discussed.

May, 1997
PACS 11.15Ha, 11.30.Rd, 11.30.Qc

* E-mail address: xue@milano.infn.it
Since their appearance, neutrinos have always been extremely peculiar. Their charge neutrality, near masslessness, flavour mixing, and parity-violating coupling have been at the centre of a conceptual elaboration and an intensive experimental analysis that have played a major role in donating to mankind the beauty of the Standard Model (SM). In the present letter, we propose a novel theoretical understanding of neutrino masses and mixings in a left-right symmetric extension of the SM, inspired by the intrinsic vector-like feature of chiral gauge theories at high-energies.

The notion of the vector-like feature stands for: given any conserved quantum numbers of chiral gauge symmetries of a regularized quantum field theory, there must be the exactly equal numbers of left-handed and right-handed fermions, and parity-conserving gauge couplings at short distances. The feature is clearly phenomenologically unacceptable. However, it really implies right-handed neutrinos and parity-conserving gauge theories in short distances. This calls for the left-right symmetric model \((SU_L(2) \otimes SU_R(2) \otimes U_{B-L}(1))\), where parity is unbroken at high energies and its nonconservation at low-energies occurs through a spontaneous symmetry breakdown mechanism.

The left-right symmetric extension that we suggest still possesses \(SU_L(2) \otimes U_Y(1)\) gauge symmetries. The right-handed doublets are assigned to bound three-fermion states \((i = e, \mu, \tau)\):

\[
\begin{pmatrix}
\nu_3^i \\
\bar{i}_3^R
\end{pmatrix}_R; \quad \nu_3^i R \sim (\bar{\nu}_R \cdot \nu_L^i) \nu_R; \quad \bar{i}_3^R \sim (\bar{i}_R \cdot i_L) i_R,
\]

where \(\nu_R\) is a gauge singlet and a unique right-handed neutrino for three families. These three-fermion states carry the appropriate quantum numbers of the \(SU_L(2) \otimes U_Y(1)\) gauge symmetries so that the SM gauge symmetries are preserved. We do not need new elementary gauge and fermion fields for the \(SU_R(2)\) sector. The neutrino fields (both elementary and composite) considered in this letter are purely Weyl neutrinos.

These three-fermion states can be formed by effective high-dimension operators of fermionic fields, \(L_{h.d.o.}\), which are due to the underlying physics

1 This feature is generic, not only for the lattice regularization, since the “no-go” theorem that proves this feature actually is based on the argument of chiral gauge anomalies.

2 Although the solution to this “no-go” theorem has not been completely found.
at the cutoff Λ,
\[ L_{\text{effective}} = L_{\text{SM}} + L_{\text{h.d.o.}}. \] (2)

Before knowing what the dynamics of underlying physics is, \textit{a priori}, we conceive that the possibilities for \( L_{\text{h.d.o.}} \) are those allowed by the gauge symmetries of the SM or other unification models, e.g., \( SO(10) \)[7, 8].

In order to consistently achieve parity-violation at low-energies, we postulate that at an intermediate energy-threshold \( \epsilon \):
\[ 250 \text{GeV} \ll \epsilon < \Lambda, \] (3)

the three-fermion states \( \| \) dissolve into their constituents, i.e. turn into the virtual states of their constituents (three-fermion cuts). This is due to the vanishing of the binding energy of three-fermion states \( \| \) at the threshold \( \epsilon \)[6]. This phenomenon could be realized on the basis that the effective high-dimension operators for binding the three-fermion states are momentum-dependent. However, it is difficult to demonstrate this dynamics, since it relates to a non-perturbative issue of finding the inferred fix point and determining the spectrum and relevant high-dimension operators that realize the symmetries. It should be pointed out that this phenomenon is not spontaneous symmetry breaking and no Goldstone bosons occur.

In this letter, we consider the above extension of the SM as a model instead of a demonstrated theory. For the purpose of studying neutrino masses and mixings, we only discuss the three-fermion state \( \| \) and disregard the \( U_{B-L}(1) \) symmetry. Presumably, the spectrum of right-handed three-fermion states should be much richer than \( \| \), so as to preserve chiral gauge symmetries of a unified theory.

2. \( \nu_R \) couples to other fermions via \( L_{\text{h.d.o.}} \) in (2). The 1PI (one-particle irreducible) vertices between \( \nu_R \) and \( \nu_L^i \) give Dirac neutrino masses. Rather than the Sea-Saw mechanism[8], we discuss another possible dynamics for decoupling \( \nu_R \) and small Dirac neutrino masses.

We assume, the couplings of \( \nu_R \) low-frequency modes \( (p \ll \Lambda) \) to other fermions are suppressed by \( \left( \frac{p}{\Lambda} \right)^2 \). This is to mean that every external \( \nu_R \) line (with external momentum \( p \)) of any 1PI-vertices is associated with \( \left( \frac{p}{\Lambda} \right)^2 \). Thus, at low-energies, \( \nu_R \) decouples from other fermions, as a result, Dirac neutrino masses are very small and \( \nu_R \) is a free particle. On the other hand,
ν_R high-frequency modes (250GeV ≪ ϵ < p < Λ) couple strongly enough to other fermions to form three-fermion states (1).

This is equivalent to assuming that \( L_{h.d.o.} \) (2) possess the following shift-symmetry\[9\],
\[
\nu_R(x) \rightarrow \nu_R(x) + \delta,
\]
where \( \delta \) is a constant. The decoupling of \( \nu_R \) can be shown by the Ward identities of this shift-symmetry\[6\],
\[
\gamma_\mu \partial^\mu \nu'_R(x) + \left( \frac{\partial_\mu}{\Lambda} \right)^2 \langle \hat{O}(x) \rangle - \frac{\delta\Gamma}{\delta\nu'_R(x)} = 0,
\]
where “\( \Gamma \)” is the effective potential with non-vanishing external sources (\( J, \eta \)); the external field \( \nu'_R \equiv \langle \nu_R \rangle \), and \( \langle \hat{O}(x) \rangle \) is the expectation value of the operator \( \hat{O} \) w.r.t. the generating functional \( Z[J, \eta] \). Based on this Ward identity, we can get all 1PI vertices containing at least one external \( \nu'_R \).

The first are the self-energy functions \( \Sigma_{\nu_i}(p) \) (\( i = e, \mu, \tau \)) of neutrinos. Performing a functional derivative of eq. (5) w.r.t. the primed field \( \nu'_L(0) \) and then putting external sources \( \eta = J = 0 \), we obtain,
\[
\left( \frac{\partial_\mu}{\Lambda} \right)^2 \langle \hat{O}^i(x) \rangle_0 - \frac{\delta^2\Gamma}{\delta\nu'_L(0)\delta\nu'_R(x)} = 0,
\]
where \( \langle \hat{O}^i(x) \rangle_0 \equiv \langle \hat{O}^i(x) \rangle|_{J=\eta=0} \), and as a result,
\[
\int_x e^{-ipx} \frac{\delta^2\Gamma}{\delta\nu'_L(0)\delta\nu'_R(x)} = \frac{1}{2} \Sigma^i(p) = \left( \frac{p^2}{\Lambda^2} \right) \langle \hat{O}^i(0) \rangle_0.
\]
For low energies \( p \ll \Lambda \), Dirac neutrino masses \( \Sigma_{\nu_i}(p) \ll 1 \).

The second is the wave renormalization function. The functional derivative of eq. (6) w.r.t. \( \nu'_L(0) \) and then \( \eta = J = 0 \) lead to,
\[
(\gamma_\mu P_R)^{\beta\alpha} \partial^{\beta} \delta(x) - \frac{\delta^2\Gamma}{\delta\nu'_L(0)\delta\nu'_R(x)} = 0.
\]
The two-point function is then given by,
\[
\int_x e^{-ipx} \frac{\delta^{(2)}\Gamma}{\delta\nu'_R(x)\delta\nu'_L(0)} = i\gamma_\mu P^\mu;
\]
indicating that $\nu_R$ does not receive wave-function renormalization $Z_3$.

The third are the $n$-point ($n > 2$) 1PI interacting vertices. Analogously, we can obtain,

$$\frac{\delta^{(n)}\Gamma}{\delta^{(n-1)}(\cdots)\partial^p_R(x)} \sim O\left(\frac{p^2}{\Lambda^2}\right), \quad n > 2. \quad (10)$$

where $\delta^{(n-1)}$ indicates $(n - 1)$ derivatives w.r.t. other external fields.

These three identities, eqs.(7,9) and (10), show us two conclusions owing to the $\nu_R$ shift-symmetry (or only $\nu_R$ high-frequency modes coupling to other fermions): (i) the Dirac neutrino masses due to high-dimension operators ($L_{h.d.o.}$) are extremely small; (ii) the right-handed neutrino $\nu_R$ at low-energies is a free particle and decouples from other physical particles.

3. Once the soft spontaneous symmetry breaking occurs at the weak scale, the right-handed fermion states are the mixed states comprising the elementary state $\nu_R(i_R)$ and the composite state $\nu^3_i R(i^3_R)$ ($i = e, \mu, \tau$):

$$\Psi_{\nu_R}^i = (\nu_R, \nu^3_i R); \quad \Psi_{i_R}^i = (i_R, i^3_R). \quad (11)$$

The self-energy functions $\Sigma_{\nu_i}(p)$ ($\Sigma_i(p)$) of neutrinos (charged leptons) are couplings between $\nu^i_L(i_L)$ and mixing right-handed fermion states $\Psi_{\nu_R}^i(\Psi_{i_R}^i)$, rather than the couplings between only elementary states $\nu^i_L(i_L)$ and $\nu_R(i_R)$ in the SM. This will become very clear in the Schwinger-Dyson (SD) equations for the self-energy functions.

For the reason that the three-fermion states (1) carry the $SU_L(2)$ quantum number, there must be an interacting 1PI vertex between $W^\pm, Z^0$ bosons and composite right-handed fermions (1) in the high-energy region. We may write this effective 1PI coupling for $W^\pm$ as,

$$\Gamma_{\mu}^{ij}(q) = i \frac{g_2}{2\sqrt{2}} V_{ij}\gamma_{\mu}(P_L + f(q)) \quad (12)$$

$$f(q) \neq 0, \quad q \geq \epsilon, \quad (13)$$

where $g_2$ is the $SU_L(2)$ coupling and $V_{ij}$ is the CKM matrix, and that for the $Z^0$ is similar. At the energy threshold (3), where the three-fermion states dissolve into their constituents, the effective 1PI vertex function $f(q)$ must vanish,

$$f(q)|_{q \rightarrow \epsilon + 0^+} \rightarrow 0, \quad (14)$$
which results in the parity-violating gauge-couplings in low-energies.

In $L_{h.o.d.}$ of the effective lagrangian (2), we may have following gauge-invariant operators interacting between up quarks ($q^u_i = u, c, t$) and neutrinos, and between down quarks ($q^d_i = d, s, b$) and charged leptons ($i = e, \mu, \tau$),

\[
G \bar{\psi}^\beta_i \mathcal{L} \cdot \left[ \frac{\partial^2}{\Lambda^2} \nu_R(x) \right] \bar{q}^u_i R(x) \cdot Q^\beta_i \mathcal{L}(x), \tag{15}
\]

\[
G \bar{\psi}^\beta_i \mathcal{L} \cdot i_R(x) \bar{q}^d_i R(x) \cdot Q^\beta_i \mathcal{L}(x), \tag{16}
\]

where $\psi^\beta_i$ and $Q^\beta_i$ ($\beta = 1, 2$) are the $SU_L(2)$ doublet of leptons and quarks respectively. Once quarks are massive, these operators are the sources providing explicit chiral symmetry breaking to generate lepton masses.

Turning off all gauge interactions and putting the four-fermion coupling $G$ to its critical value $G \rightarrow 4 + 0^+$ given by the $\bar{t}t$-condensate model[11], we obtain the simplest SD gap-equations at the cutoff ($p = \Lambda$),

\[
\Sigma^\nu_i(p) = \Sigma^\nu_i(\Lambda) + Z^\nu_i(p) + W^\nu_i(p), \tag{17}
\]

\[
\Sigma^\nu_i(\Lambda) = \Sigma^q_i(\Lambda), \tag{18}
\]

where we are henceforth in the basis of mass eigenstates. Eq.(17) shows that the neutrino $\Sigma^\nu_i(p)$ decouples from the quark $\Sigma^q_i(\Lambda)$ for $p \ll \Lambda$, which agrees with (7). Eq.(18) is reminiscent of the predictions in the $SU(5)$ unification model[12].

It should be pointed out that there are other possible gauge-invariant four-fermion interactions and their corresponding tadpole diagrams contribute to eqs.(17,18) as well. We consider all such contributions to eqs.(17,18) as lepton’s bare masses at the cutoff, which are actually explicit symmetry breaking terms in the full SD equations (19, 20).

4. Turning on all gauge interactions, we study the full SD equations for the lepton self-energy functions. In the rainbow approximation and the Landau gauge, these equations can be written as,

\[
\Sigma^\nu_i(p) = \Sigma^\nu_i(\Lambda) + Z^\nu_i(p) + W^\nu_i(p), \tag{19}
\]

\[
\Sigma_i(p) = \Sigma_i(\Lambda) + Z^1_i(p) + Z^3_i(p) + W_i(p) + \gamma_i(p), \tag{20}
\]

\^The same strength $G$ of all four-fermion couplings is assigned for a potential unification.
where

\[
\begin{align*}
\gamma_{i}(p) &= 3e^2 \int_{p'}^{\Lambda} \frac{1}{(p - p')^2 p'^2 + \Sigma_{i}(p')} \Sigma_{i}(p') \\
Z_{i}^{1}(p) &= 3\lambda^2 \int_{p'}^{\Lambda} \frac{1}{(p - p')^2 + M_{i}^2 p'^2 + \Sigma_{i}^{2}(p'^2)} \Sigma_{i}(p')
\end{align*}
\]  

\( \lambda = g_{2}\tan\theta_{w}(\sin^{2}\theta_{w} - \frac{1}{2}) \) and \( \theta_{w} \) is the Weinberg angle.

Because of the effective 1PI coupling (12) and three-fermion states (1) above the threshold \( \epsilon \) (4), \( W^{\pm} \) and \( Z^{3} \) bosons contribute to eqs.\( (19,20) \). These most peculiar contributions are \( W_{\nu_{i}}(p), Z_{\nu_{i}}(p) \) and \( W_{i}(p), Z_{i}^{3}(p) \):

\[
\begin{align*}
W_{\nu_{i}}(p) &= \left( \frac{g_{2}}{2\sqrt{2}} \right)^{2} |V_{ij}|^2 \int_{|p'| \geq \epsilon}^{\Lambda} \frac{f(p' - p)}{(p - p')^2 + M_{i}^2 p'^2 + \Sigma_{j}(p'^2)} \Sigma_{j}(p'^2) \\
Z_{\nu_{i}}(p) &= \left( \frac{g_{2}}{2\cos\theta_{w}} \right)^{2} |V_{ij}|^2 \int_{|p'| \geq \epsilon}^{\Lambda} \frac{f(p' - p)}{(p - p')^2 + M_{i}^2 p'^2 + \Sigma_{j}(p'^2)} \Sigma_{j}(p'^2) \\
W_{i}(p) &= \left( \frac{g_{2}}{2\sqrt{2}} \right)^{2} |V_{ij}|^2 \int_{|p'| \geq \epsilon}^{\Lambda} \frac{f(p' - p)}{(p - p')^2 + M_{i}^2 p'^2 + \Sigma_{j}(p'^2)} \Sigma_{j}(p'^2) \\
Z_{i}^{3}(p) &= \left( \frac{g_{2}\cos2\theta_{w}}{2\cos\theta_{w}} \right)^{2} |V_{ij}|^2 \int_{|p'| \geq \epsilon}^{\Lambda} \frac{f(p' - p)}{(p - p')^2 + M_{i}^2 p'^2 + \Sigma_{j}(p'^2)} \Sigma_{j}(p'^2)
\end{align*}
\]

where the integration of the internal momentum \( p' \) starts from the intermediate threshold \( \epsilon \) to the cut-off \( \Lambda \).

We note that eqs.\( (17,18,21,22) \) are the 1PI couplings between elementary left-handed and right-handed fields, whereas eqs.\( (23)-(26) \) are the 1PI couplings between elementary left-handed fields and right-handed three-fermion fields (1). Thus, we clarify that the full self-energy functions \( \Sigma_{\nu_{i}}(p) \) (19) and \( \Sigma_{i}(p) \) (20) are the 1PI couplings between elementary left-handed fields and mixed right-handed fields (11). At low-energies, external momenta \( p \ll \epsilon \), \( \Sigma_{\nu_{i}}(p) \) (13) and \( \Sigma_{i}(p) \) (20) are very small, because eqs.\( (17), \text{ and } (23) \) turn to zero, these latter are due to disappearance of three-fermion states and the 1PI-vertex (14).

5. We are in the position to solve the SD eqs.\( (19,20) \). Assuming the scale \( \epsilon \) is large enough and \( p' > \epsilon \gg 1 \), we approximate the inhomogeneous terms \( (23,25) \) to be,

\[
W_{\nu_{i}}(p) \simeq \alpha_{w}(p)|V_{ij}|^2 \Sigma_{j}(\Lambda), \quad W_{i}(p) \simeq \alpha_{w}(p)|V_{ij}|^2 \Sigma_{j}(\Lambda),
\]  

\( \alpha_{w} \) is the Weinberg angle.
where

\[ \alpha_w(p) \simeq \left( \frac{g_2}{2\sqrt{2}} \right)^2 \int_{|p'| \geq \epsilon} f(p' - p) \frac{1}{(p - p')^2 + M_w^2 p'^2}. \]  

In the high-energy region \( (x = p^2 > \epsilon \gg 1) \), where \( M_w^2, M_z^2 \) and non-linearity are negligible, the SD integral equations (19,20) can be converted to the following boundary value problems:

\[
\frac{d}{dx} \left( x^2 \Sigma_{\nu i}(x) \right) + \frac{f_n}{4} \Sigma_{\nu i}(x) = 0, \tag{29}
\]

\[
\Lambda^2 \Sigma_{\nu i}(\Lambda^2) + \Sigma_{\nu i}(\Lambda^2) = \Sigma_{\nu i}^0(\Lambda) + \alpha_w(\Lambda) |V_{ij}|^2 \Sigma_j(\Lambda); \tag{30}
\]

\[
\frac{d}{dx} \left( x^2 \Sigma_i(x) \right) + \frac{f_c}{4} \Sigma_i(x) = 0, \tag{31}
\]

\[
\Lambda^2 \Sigma_i(\Lambda^2) + \Sigma_i(\Lambda^2) = \Sigma_i^0(\Lambda) + \alpha_w(\Lambda) |V_{ij}|^2 \Sigma_{\nu j}(\Lambda), \tag{32}
\]

where \( f_n, f_c \) are perturbative functions of electroweak couplings. These are differential equations with the inhomogeneous boundary conditions at the cutoff. The generic solutions to eqs. (29,31) for \( (x \gg 1) \) are given by,

\[
\Sigma_{\nu i}(x) \simeq \frac{A_{\nu i} \mu^2}{\sqrt{x}} \text{sinh} \left( \frac{1}{2} \sqrt{1 - f_n \ell n \left( \frac{x}{\mu^2} \right)} \right), \tag{33}
\]

\[
\Sigma_i(x) \simeq \frac{A_i \mu^2}{\sqrt{x}} \text{sinh} \left( \frac{1}{2} \sqrt{1 - f_c \ell n \left( \frac{x}{\mu^2} \right)} \right), \tag{34}
\]

where \( A_{\nu i}, A_i \) are arbitrary constants and \( \mu \) is an inferred scale. Substituting (33) into (30) and (34) into (32) in the low-energy limit \( (\mu \ll \Lambda) \), we obtain the gap-equations:

\[
\alpha_w(\Lambda) |V_{ij}|^2 \Sigma_j(\Lambda) = \frac{1}{2} \Sigma_{\nu i}(\Lambda) + \frac{1}{2} \sqrt{1 - f_n \ell n \left( \frac{x}{\mu^2} \right)} \Sigma_{\nu i}(\Lambda) - \Sigma_{\nu i}^0(\Lambda); \tag{35}
\]

\[
\alpha_w(\Lambda) |V_{ij}|^2 \Sigma_{\nu j}(\Lambda) = \frac{1}{2} \Sigma_i(\Lambda) + \frac{1}{2} \sqrt{1 - f_c \ell n \left( \frac{x}{\mu^2} \right)} \Sigma_i(\Lambda) - \Sigma_i^0(\Lambda). \tag{36}
\]

Since the lepton bare masses (17,18) are defined when all gauge interactions are turned off, we can rewrite the RHS of the gap-equations (35,36),

\[
\alpha_w(\Lambda) |V_{ij}|^2 \Sigma_j(\Lambda) = -\frac{f_n}{4} \Sigma_{\nu i}(\Lambda), \tag{37}
\]

\[
\alpha_w(\Lambda) |V_{ij}|^2 \Sigma_{\nu j}(\Lambda) = -\frac{f_c}{4} \Sigma_i(\Lambda). \tag{38}
\]

4 Analogous technique can be found in ref. [13]
As a consequence of the gap-equations (35,36), the lepton self-energy functions must be non-trivial

\[ \Sigma_{\nu_i}(\Lambda) \neq 0; \quad \text{and} \quad \Sigma_j(\Lambda) \neq 0, \quad (39) \]

if quarks are massive. The gap-equations (18) strongly imply that the hierarchical pattern of charged lepton masses is mainly due to the hierarchical pattern of down quark masses. Eqs.(37,38) show that the pattern of neutrino masses is determined by the CKM-mixing angles and charged lepton masses.

6. The six gap-equations (37,38) relate neutrino and charged lepton masses at the cutoff. Noticing the fact that \( \Sigma(\vec{p}) \) must be continuous functions of \( \vec{p} \) from \( p = \Lambda \) to \( p \ll \epsilon \) for the locality of quantum field theories, and the ratios of \( \Sigma(p) \)'s in the same charge sector (but different generations) should be scale invariant (renormalization group invariant), we take ratios between the two equations of eqs.(37), the two equations of eqs.(38) respectively and scale them down to the low-energy scale. We end up with four independent equations:

\[
\begin{align*}
    m_{\nu_e} &= \frac{|V_{\nu_{ee}}|^2 m_e - |V_{\nu_{e\mu}}|^2 m_\mu + |V_{\nu_{e\tau}}|^2 m_\tau}{|V_{\nu_{ee}}|^2 m_e - |V_{\nu_{e\mu}}|^2 m_\mu + |V_{\nu_{e\tau}}|^2 m_\tau}, \\
    m_{\nu_\mu} &= \frac{|V_{\nu_{\mu e}}|^2 m_e - |V_{\nu_{\mu\mu}}|^2 m_\mu + |V_{\nu_{\mu\tau}}|^2 m_\tau}{|V_{\nu_{\mu e}}|^2 m_e - |V_{\nu_{\mu\mu}}|^2 m_\mu + |V_{\nu_{\mu\tau}}|^2 m_\tau}, \\
    m_{\nu_\tau} &= \frac{|V_{\nu_{\tau e}}|^2 m_e - |V_{\nu_{\tau\mu}}|^2 m_\mu + |V_{\nu_{\tau\tau}}|^2 m_\tau}{|V_{\nu_{\tau e}}|^2 m_e - |V_{\nu_{\tau\mu}}|^2 m_\mu + |V_{\nu_{\tau\tau}}|^2 m_\tau},
\end{align*}
\]

and

\[
\begin{align*}
    m_e &= \frac{|V_{e\nu_e}|^2 m_{\nu_e} - |V_{e\nu_\mu}|^2 m_{\nu_\mu} + |V_{e\nu_\tau}|^2 m_{\nu_\tau}}{|V_{e\nu_e}|^2 m_{\nu_e} - |V_{e\nu_\mu}|^2 m_{\nu_\mu} + |V_{e\nu_\tau}|^2 m_{\nu_\tau}}, \\
    m_\mu &= \frac{|V_{\mu\nu_e}|^2 m_{\nu_e} - |V_{\mu\nu_\mu}|^2 m_{\nu_\mu} + |V_{\mu\nu_\tau}|^2 m_{\nu_\tau}}{|V_{\mu\nu_e}|^2 m_{\nu_e} - |V_{\mu\nu_\mu}|^2 m_{\nu_\mu} + |V_{\mu\nu_\tau}|^2 m_{\nu_\tau}}, \\
    m_\tau &= \frac{|V_{\tau\nu_e}|^2 m_{\nu_e} - |V_{\tau\nu_\mu}|^2 m_{\nu_\mu} + |V_{\tau\nu_\tau}|^2 m_{\nu_\tau}}{|V_{\tau\nu_e}|^2 m_{\nu_e} - |V_{\tau\nu_\mu}|^2 m_{\nu_\mu} + |V_{\tau\nu_\tau}|^2 m_{\nu_\tau}}.
\end{align*}
\]

In these equations, all fermion masses are defined at the same low-energy scale. We make an appropriate chiral rotation in the second family and all fermion masses are positive. Given charged lepton masses, four relationships satisfied by four mixing angles and three neutrino masses, which are no longer seven free parameters. These equations give a class of solutions for the possible patterns of neutrino masses and mixing angles.
Setting $\theta_{13} = 0$ so that $\theta_{13}$ and $\delta_{13}$ decouple from eqs. (40–43), and assuming $m_{\nu_e} \ll m_{\nu_\mu} \ll m_{\nu_\tau}$, we get a possible pattern:

$$\tan^2 \theta_{12} = \frac{m_e}{m_\mu} + O \left( \frac{m_{\nu_e}}{m_{\nu_\tau}} \right), \quad m_{\nu_e} \sim \sin^2 \theta_{13} = 0,$$  \hspace{1cm} (44)$$

$$\sin^2 \theta_{23} = \frac{m_\mu + m_e}{m_e + m_\mu + m_\tau} + O \left( \frac{m_{\nu_\mu}}{m_{\nu_\tau}} \right), \quad \frac{m_{\nu_\mu}}{m_{\nu_\tau}} \simeq 5.8 \cdot 10^{-3}. \hspace{1cm} (45)$$

This coincides with the “standard” scenario of neutrino masses and mixings that can also be realized by the sea-saw mechanism in GUT models [1, 2]. Given this hierarchical pattern and $m_{\nu_e} \sim O(eV)$ for the desired HDM of the Universe, the results (44)–(45) are consistent with the small angle MSW solution to the solar neutrino problem, and however we may need an addition sterile neutrino $\nu_s$ [1] to explain the atmospheric neutrino deficit.

Many other possible patterns of neutrino masses and mixings are discussed in the literature [1], phenomenologically based on cosmological constraints and three neutrino experiments (the solar neutrinos, atmospheric neutrino, LSND). While, on the other hand, these neutrino experiments need to be substantiated [3].

At this point, we should emphasize that in this extension of the SM, the four CKM mixing angles ($\theta_{ij}$) are totally extrinsic elements, qualifying the true pattern of neutrino masses as real as compared (and opposed) to any other possible pattern, where the mixing angles can be anything one wishes. A priori, no theoretical reason can determine which patterns is real. In the quark sector, that the observed CKM mixing angles are almost trivial and $\theta_{12} \gg \theta_{23} \gg \theta_{13}$ completely qualify the observed hierarchical pattern of quark masses [14]. In the lepton sector, it seems unlikely to have exactly the same pattern for the following observation: the hierarchical pattern of charged lepton masses is originated dominantly from the hierarchical pattern of down quark masses [14], instead of a possible hierarchical pattern of the CKM-mixing angles, as that in the quark sector. We have no theoretical reasons to preclude the possibilities of degenerate neutrino masses and large mixing angles.

We expect that in eqs. (40)–(43), there exists a particular solution giving the real pattern of neutrino masses and mixings in Nature, even though we still have no power of making predictions. Nevertheless, these relationships, originated from the purely theoretical stipulation, are certainly facing all
ongoing and future neutrino experiments.

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