QED corrections to leptonic decay rates

P. A. Boyle\textsuperscript{2}, V. Gülbers\textsuperscript{1}, A. Jüttner\textsuperscript{1}, C. Lehner\textsuperscript{3}, F. Ó hógáin\textsuperscript{2}, A. Portelli\textsuperscript{2}, J. P. Richings\textsuperscript{1}, C. T. Sachrajda\textsuperscript{1},

\textsuperscript{1}School of Physics and Astronomy, University of Southampton, Southampton SO17 1BJ, UK
\textsuperscript{2}School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, UK
\textsuperscript{3}Physics Department, Brookhaven National Laboratory, Upton, NY 11973, USA

E-mail: j.p.richings@soton.ac.uk

RBC/UKQCD is preparing a calculation of leptonic decay rates including isospin breaking corrections. We use a perturbative approach to include NLO contributions from QED effects, following N. Carassco et al. We present preliminary results for a contribution to the leptonic decay rate of a pion. We also report on techniques developed to treat the leptonic decays in an all-to-all approach following Foley et al.
1. Introduction

In this contribution we outline RBC/UKQCD’s work towards a calculation of the isospin breaking (IB) corrections to leptonic decay rate for pions and kaons. This study is motivated by the sub percent level precision achieved in Lattice QCD for the calculation of $f_\pi$ and $f_K$, by some collaborations using various lattice actions in the isospin symmetric limit where up and down quarks are treated as identical particles [3]. The approach used to extract elements of the CKM matrix from leptonic decays (Figure 1) is to calculate decay constants and use experimental results for the decay rates to yield the CKM matrix elements. For example, the pion decay rate at tree-level is,

$$\Gamma(\pi^+ \rightarrow l^+ \nu) = \frac{m_\pi}{8\pi} G_F^2 |f_{\pi^+}|^2 |V_{ud}|^2 \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2. \quad (1.1)$$

The pion decay constant $f_{\pi^+}$ is defined in terms of the QCD matrix element, $\langle 0 | \bar{d} \gamma_\mu \gamma_5 u | \pi^+ (p) \rangle = ip_\mu f_{\pi^+}$, which is computed from two point correlation functions. To improve precision isospin breaking (IB) effects must be taken into account. These effects are due to the different masses of the light quarks and the difference in the QED coupling between up and down type quarks. Through power counting one expects these effects to enter at the percent level.

We focus our discussion to the QED isospin breaking corrections to leptonic decay of pions. Following the approach developed in [1], where the QCD+QED path integral is expanded in $\alpha$ and IR divergences are dealt with consistently. We work in the framework of QEDL [4, 5, 6, 7] and use QED quenched ensembles. Preliminary results are shown for an implementation using a perturbative expansion in the photon coupling as in [1].

We are also developing a strategy to implement QED corrections in an all-to-all approach. As detailed later this is based on work by Foley et al. [2] and where we construct matrices from all-to-all propagators, low mode eigen-vectors and stochastic noise. These matrices are summed over the spacial dimensions of a lattice making them suitable for storage on disk. Once generated, these objects can be multiplied to construct correlators. This approach should allow for a variety of physical quantities to be calculated from the set of stored matrices. We discuss the set of such matrices required to investigate IB corrections to leptonic decays of a pion as an example and how one can construct the correlators for the QED correction from these matrices.

![Figure 1: Pion decay into leptons via a weak current without QED contributions.](image)

2. QED Isospin Breaking corrections to leptonic decay rates

In order to calculate an infra-red (IR) finite order $\alpha$ decay rate, we must consider contributions from graphs with and without final state photons to cancel IR divergences [9]. In this work we will follow the strategy outlined in [1] to carefully deal with IR divergences, where the contributions with final state photons are treated analytically using the point-like approximation. Therefore we will focus our discussion to the lattice calculation of diagrams without final state photons. In order to calculate these corrections a perturbative approach [10] is used, where we expand the QCD
+ QED path integral in powers of the electromagnetic coupling $\alpha$. The set of connected graphs generated by this perturbative expansion are given in Figures 2. In our calculation QED$_L$ [4] is used and we choose the Feynman gauge. Therefore the photon propagator takes the form,

$$\Delta_{\mu\nu}(x-y) = \delta_{\mu\nu} \frac{1}{N} \sum_{k,k\neq 0} e^{ik(x-y)} \frac{1}{k^2} = \langle A_\mu(x)A_\nu(y) \rangle,$$

where $\hat{k}$ is the lattice momentum of the photon. In practice the photon propagator is generated by inserting stochastic photons [11]. Figure 2 are contributions where the photon only couples to quarks. Figure 2a is a contribution where the photon couples to quarks and leptons. We implement the lepton propagator on the lattice as a free domain wall fermion. As a test the calculation of the diagram in Figure 2a using sequential propagators [12] on a $24^3 \times 64$ lattice was performed.

The ensemble used has an isospin symmetric pion mass of 340 MeV and inverse lattice spacing of $a^{-1} = 1.78$ GeV [13].

The results for diagram 2a illustrated in Figure 3 shows an encouraging signal but we also investigate other methods which might offer better signal-to-noise ratio and more flexibility.

3. The all-to-all approach and meson fields

We follow the all-to-all approach in [2] where the propagator is decomposed into a number of exact low mode eigenvectors and a component that is solved stochastically. This offers a relatively convenient way of structuring the calculation of a correlator in terms of meson fields.
3.1 All-to-all propagator

The all-to-all propagator can be constructed from two sets of vectors, \( v_i(x) \) and \( w_j(x) \), such that

\[
D_{A2A}^{-1}(x, y) = \sum_{i=1}^{N_{\text{modes}}} v_i(x) w_j^\dagger(y). \quad (3.1)
\]

These vectors can be decomposed into \( (N_{\text{modes}} = N_{\text{high}} + N_{\text{low}}) \) high and low modes,

\[
D_{A2A}^{-1}(x, y) = \sum_{i=1}^{N_{\text{modes}}} v_i(x) w_j^\dagger(y) = \sum_{i=1}^{N_{\text{high}}} v_i(x) w_j^\dagger(y) + \sum_{h=N_{\text{high}}+1}^{N_{\text{modes}}} v_h(x) w_j^\dagger(y). \quad (3.2)
\]

The low modes,

\[
v_l(x) = \phi_l(x) \quad w_l(y) = \phi_l(y)/\lambda_l,
\]

where \( \phi_l(x) \) is the \( l \)th eigenvector with eigenvalue \( \lambda_l \) of the Dirac operator are exact, they are generated in advance, e.g., using a Lanczos algorithm \([14]\). To calculate the high modes \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) stochastic noise sources \( w_h(y) = \eta_h(y) \) are used and the low mode contribution to the propagator is projected out,

\[
v_h(x) = D_{\text{def}}^{-1} \eta_h(x) = (D^{-1} - \sum_{l=1}^{N_{\text{high}}} \phi_l(x) \phi_l^\dagger(x)/\lambda_l) \eta_h(x). \quad (3.4)
\]

A number of inversions are required to gain the benefits of full spin-colour and time dilution. In practice we don’t use eigenvectors of \( D(x, y) \) but eigenvectors of the preconditioned matrix.

3.2 Two point correlation function

We can consider a two point correlation function and rewrite it in terms of all-to-all propagators using (3.1),

\[
C_{2pt}(t_y - t_x) = \sum_{x,y} Tr[\Gamma_1 S(x, y) \Gamma_2 S(y, x)]
\]

\[
= \sum_{x,y} \sum_{i,j}^{N_{\text{modes}}} v_i(x) w_j^\dagger(y) \Gamma_1 \sum_{j=1}^{N_{\text{modes}}} v_j(y) w_i^\dagger(x) \quad (3.6)
\]

\[
= \sum_{i,j}^{N_{\text{modes}}} \sum_{x,y} w_j^\dagger(x) \Gamma_1 v_i(x) \left[ \sum_{y} w_i^\dagger(y) \Gamma_2 v_j(y) \right]. \quad (3.7)
\]

In equation (3.7) the cyclic property of the trace has been used forming a product of meson fields,

\[
\Pi_{ij}(t_x; \Gamma) = \sum_{x} w_i^\dagger(x) \Gamma v_j(x),
\]

where \( \Gamma \) is any gamma matrix. Meson fields \( \Pi_{ij} \) are of size \( N_T \times N_{\text{modes}} \), where \( N_T \) is the time extent of the lattice. The spatially summed meson fields can be stored to disk and retrieved to form correlators. This allows for more complicated correlators to be formed without the need to perform additional inversions.
4. Meson fields and Isospin breaking corrections to leptonic decays

In order to use the all-to-all method we have to form the meson fields required to construct the IB corrections of leptonic decays. We have already discussed the pion two point function. The other correlation function required to determine the pion decay constant without QED can be written in terms of meson fields as,

$$C(t_s - t_s) = \sum_{l,j} tr \left[ \Pi_{lj}(t_j; \gamma_l) \Pi_{ji}(t_i; \gamma_i \gamma_j) \right]. \quad (4.1)$$

Where the appropriate choice of gamma structure has been applied to (3.7) at source and sink. To construct the QED IB correction to the decay rate we need to construct a meson field for the conserved current, tadpole insertion and the decay operator.

4.1 Point split operator meson fields

We require meson fields for operators with point split structure such as conserved vector current. The conserved current for Wilson fermions is,

$$V^i_{\mu}(x) = \frac{1}{2} \left[ \bar{\psi}(x + \mu)(1 + \gamma_5)U^\dagger_\mu(x)\psi(x) - \bar{\psi}(x)(1 - \gamma_5)U^\dagger_\mu(x)\psi(x + \mu) \right]. \quad (4.2)$$

The exchange diagram 2b, with photons inserted using conserved vector currents, has the form,

$$C = \sum_{x,y} \langle \bar{\psi}(y)\Gamma(x)\psi(y)\psi(x)\Gamma_2(y)\psi_2(x)\Delta_{\mu\nu}(x) \rangle \quad (4.3)$$

We need to rewrite the conserved current as a meson field. Consider one of the two currents in this correlator. The two terms in (4.2) are converted to a meson field in the same way so we pick the second term and rewrite it as \(\bar{\psi}(x)\Gamma(x)\psi(x + \mu)\) where \(\Gamma(x) = (1 - \gamma_5)U^\dagger_\mu(x)\). The section of the correlator we will focus on is,

$$\sum_y \langle ... \Gamma_1\psi(x)|\psi(y)\Gamma(y)\psi(y + \mu)|\psi(z)\Gamma_2 ... \rangle = \sum_y \langle ... \Gamma_1S(x,y)\Gamma(y)S(y + \mu,z)\Gamma_2 ... \rangle, \quad (4.4)$$

replacing the propagators with the all-to-all decomposition (3.1),

$$\sum_y \langle ... \sum_f \bar{v}_f(x)w^+_f(y)\Gamma(y)\sum_f v_f(y + \mu)w^+_f(z) ... \rangle = \sum_{ij} \langle ... v_i(x)\left[ \sum_y w^+_i(y)\Gamma(y)v_j(y + \mu) \right] w^+_j(z) ... \rangle \quad (4.5)$$

The square brackets on the right of (4.5) contain an object of the form of a meson field. If the first term in (4.2) is treated in the same way then we have a meson field for the conserved vector current. For our purposes we must also include a stochastic photon field, \(A_\mu\). The mesons field is then,

$$\Pi_{ij}\left[t_x, V^i_{\mu}A_\mu \right] = \sum_{x,\mu} \left[ w^+_i(x + \mu)(1 + \gamma_5)U^\dagger_\mu(x)A_\mu(x)v_j(x) - w^+_i(x)(1 - \gamma_5)U^\dagger_\mu(x)A_\mu(x)v_j(x + \mu) \right]. \quad (4.6)$$

This method for dealing with conserved currents in the all-to-all set-up has also been understood for the DWF and overlap cases.

Further more, we require an additional meson field for the lepton coupling diagram in Figure 2a, where we place the leptonic part \(L = \Gamma^W_D^{-1}V^L_{\mu}A_\mu\) with the left-handed V-A current \(\Gamma^W_\mu = \gamma_\mu(1 - \gamma_5)\) in the meson field for the decay operator. This completes the set of meson fields required to construct the QED IB corrections to the leptonic decay.
4.2 Leptonic decay corrections from meson fields

Using the meson fields discussed it is possible to construct all the diagrams required for a calculation of corrections to the decay rate. This is illustrated in Figures 5-8 where the colours correspond to the different meson fields required to construct each graph. In particular, blue for $\gamma_5$, red for $\gamma_0\gamma_5$, green for the conserved vector current with a photon insertion, light blue for the weak Hamiltonian and lepton insertion and pink for the tadpole insertion. In total five meson fields are required to determine the QED IB corrections to the decay rate.

The quark-disconnected diagrams (Figure 9) can be formed from the same set of meson fields. If the meson fields are stored then construction of the disconnected diagrams is possible without any further inversions.

5. Conclusion

Progress is being made towards a determination of the isospin breaking corrections to leptonic decays of pions and kaons. It is possible to construct meson fields for the operators required to study QED corrections using the all-to-all methodology. This offers a convenient approach for computing n-point functions once the meson fields are generated. We are in the process of testing our all-to-all approach and implementation of meson field generation. Once we verify the all-to-all method for the calculation of QED effects we aim to calculate IB correction to leptonic decays for both the pion and kaon in this way. If this approach is successful a number of physics processes can be calculated from a set of stored meson fields, increasing the physics output from consumed computer time.
Acknowledgements

A.P, V.G and F.O received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme under grant agreement No 757646. A.J received funding from STFC consolidated grant ST/P000711/1 and from the European Research Council under the European Union’s Seventh Framework Program (FP7/2007-2013) / ERC Grant agreement 279757. C.T.S is partially supported by an Emeritus Fellowship from the Leverhulme Trust. J.R acknowledges support from STFC for his studentship. This work used the DiRAC Extreme Scaling service at the University of Edinburgh, operated by the Edinburgh Parallel Computing Centre on behalf of the STFC DiRAC HPC Facility (www.dirac.ac.uk). This equipment was funded by BEIS capital funding via STFC capital grant ST/R00238X/1 and STFC DiRAC Operations grant ST/R001006/1. DiRAC is part of the National e-Infrastructure. The authors acknowledge the use of the IRIDIS High Performance Computing Facility in the completion of this work.

References

[1] N. Carrasco, V. Lubicz, G. Martinelli, C. T. Sachrajda, N. Tantalo, C. Tarantino, and M. Testa. QED corrections to hadronic processes in lattice QCD. Phys. Rev. D, 91:074506, Apr 2015.

[2] J. Foley, K. Jimmy Juge, A. O’Cais, M. Peardon, S. M. Ryan and J. I. Skullerud, Comput. Phys. Commun. 172 (2005) 145 doi:10.1016/j.cpc.2005.06.008 [hep-lat/0505023].

[3] S. Aoki et al Review of lattice results concerning low-energy particle physics. The European Physical Journal C, 77(2):112, Feb 2017.

[4] M. Hayakawa and S. Uno, Prog. Theor. Phys. 120 (2008) 413 doi:10.1143/PTP.120.413 [arXiv:0804.2044 [hep-ph]].

[5] S. Borsányi et al., Science 347 (2015) 1452 doi:10.1126/science.1257050 [arXiv:1406.4088 [hep-lat]].

[6] T. Blum, R. Zhou, T. Doi, M. Hayakawa, T. Izubuchi, S. Uno and N. Yamada, Phys. Rev. D 82 (2010) 094508 doi:10.1103/PhysRevD.82.094508 [arXiv:1006.1311 [hep-lat]].

[7] Z. Davoudi, J. Harrison, A. Jijttner, A. Portelli and M. J. Savage, arXiv:1810.05923 [hep-lat].

[8] P. Boyle, V. Gulpers, J. Harrison, A. Jüttner, C. Lehner, A. Portelli, and C.T. Sachrajda. Isospin breaking corrections to meson masses and the hadronic vacuum polarization: a comparative study. Journal of High Energy Physics, 2017(9):153, Sep 2017.

[9] F. Bloch and A. Nordsieck, Phys. Rev. 52 (1937) 54. doi:10.1103/PhysRev.52.54

[10] G. M. de Divitiis et al. [RM123 Collaboration], Phys. Rev. D 87 (2013) no.11, 114505 doi:10.1103/PhysRevD.87.114505 [arXiv:1303.4896 [hep-lat]].

[11] D. Giusti, V. Lubicz, C. Tarantino, G. Martinelli, S. Sanfilippo, S. Simula and N. Tantalo, Phys. Rev. D 95 (2017) no.11, 114504 doi:10.1103/PhysRevD.95.114504 [arXiv:1704.06561 [hep-lat]].

[12] P. A. Boyle, A. Jüttner, C. Kelly and R. D. Kenway, JHEP 0808 (2008) 086 doi:10.1088/1126-6708/2008/08/086 [arXiv:0804.1501 [hep-lat]].

[13] C. Allton et al. [RBC-UKQCD Collaboration], Phys. Rev. D 78 (2008) 114509 doi:10.1103/PhysRevD.78.114509 [arXiv:0804.0473 [hep-lat]].

[14] C. Lanczos, Journal of Research of the National Bureau of Standards Vol. 45, No.4, October 1950, Research Paper 2133