COMPLETENESS OF COMPACT LORENTZIAN MANIFOLDS
WITH SPECIAL HOLONYMOY

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Abstract. We address the problem of finding conditions under which a compact
Lorentzian manifold is geodesically complete, a property, which always holds for
compact Riemannian manifolds. It is known that a compact Lorentzian manifold is
geodesially complete if it is homogeneous, has constant curvature, or admits a time-
like conformal vector field. We consider certain Lorentzian manifolds with special
holonomy, the so called pp-waves, which, in general, do not satisfy any of the above
conditions. They are defined by the existence of a parallel null vector field and
an additional curvature condition. We show that compact pp-waves are universally
covered by a vector space, determine the metric on the universal cover, and prove that
they are geodesically complete. Using this, we show that every Ricci-flat compact
pp-wave is a plane wave, and determine the metric on its universal cover.

1. Introduction and statement of results

An important class of Lorentzian manifolds are those with special holonomy. Following the terminology in Riemannian geometry, these are Lorentzian manifolds for
which the connected holonomy group acts indecomposably, i.e., the manifold does not
locally decompose into a product, but still is reduced from the full orthogonal group.
In contrast to the Riemannian situation, the latter prevents the holonomy group from
acting irreducibly and equips the manifold with a null line bundle in the tangent bundle
which is invariant under parallel transport. We will study the geodesic completeness of
Lorentzian manifolds with special holonomy. A semi-Riemannian manifold is geodesi-
cally complete, or for short complete, if all maximal geodesics are defined on \( \mathbb{R} \).

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After Berger’s classification of connected irreducibly acting Riemannian holonomy groups [4], the quest for complete or compact Riemannian manifolds with holonomy groups from Berger’s list produced some of the highlights of modern differential geometry: for example, the construction of complete metrics by Calabi [9], Bryant and Salamon [6, 7], and LeBrun [31], Yau’s proof of Calabi’s conjecture [51], the construction of compact hyper-Kähler manifolds by Beauville [2], or of compact manifolds with exceptional holonomy by Joyce [27, 26].

Based on results in [3], the classification of the connected components of indecomposable Lorentzian holonomy groups was obtained in [34]. Moreover, in [22] a construction method for Lorentzian metrics was developed, which showed that indeed all possible groups can be realised as holonomy groups. We should mention that possible holonomy groups of four-dimensional space times where classified much earlier [45, 48]. A survey about the general classification is given in [23]. Furthermore, a first attempt to investigate the full holonomy group and global properties of the manifolds, such as global hyperbolicity, was made in [1]. Compact Lorentzian manifolds with special holonomy have also been studied in [15, 17], in [30], and in [47].

Since there are no proper connected irreducible subgroups of the Lorentz group [18], and since the construction of Lorentzian manifolds with special holonomy in some parts relies on the existence results in Riemannian geometry, finding Lorentzian manifolds with prescribed holonomy is easier than in the Riemannian context. However, the relation between compact and complete examples is more subtle, since — in sharp contrast to the Riemannian world — compact Lorentzian manifolds do not have to be complete. The standard example of this phenomenon is the Clifton-Pohl torus, which is compact, but geodesically incomplete [37, Example 7.16]. Hence, finding compact Lorentzian manifolds with special holonomy does not automatically provide geodesically complete examples.

Under some strong assumptions, a compact Lorentzian manifold is complete, for example, if it is flat [13], has constant curvature [29], or if it is homogeneous. In fact, Marsden proved in [35] that any compact homogeneous semi-Riemannian manifold is complete. Moreover, compact, locally homogeneous 3-dimensional Lorentzian manifolds are complete [19]. Finally, in [42] it was shown that compact Lorentzian manifolds with a time-like conformal Killing vector field are complete (see also [28] or [40, 41]).

We will consider manifolds with a very degenerate curvature tensor, but which, in general, are not locally homogeneous, not of constant curvature and do not admit a time-like conformal Killing vector field.

Definition 1. A Lorentzian manifold \((\mathcal{M}, g)\) is called **pp-wave**\(^2\) if it admits a global parallel null vector field \(V \in \Gamma (T \mathcal{M})\), i.e., \(V \neq 0\), \(g(V, V) = 0\) and \(\nabla V = 0\), and if its curvature tensor \(R\) satisfies

\[
R(U, W) = 0, \text{ for all } U, W \in V^\perp.
\]

\(^1\)In fact, in [13] Carrièrè proved a much more general result for affine manifolds. A direct proof for the flat case was given in [52]. However, this proof has gaps as it was pointed out in [59].

\(^2\)In the following we will consider compact manifolds of this type. We are aware that for compact manifolds the term wave might not be appropriate, but we use this term since it is established in the literature for manifolds with the given curvature properties. Later we will see that an appropriate name would be screen flat, but this term has other obvious problems.
Here $\nabla$ denotes the Levi-Civita connection of $g$ and $R$ the curvature tensor of $\nabla$, $R \in \Lambda^2 T^* \mathcal{M} \otimes \text{End}(T \mathcal{M})$ of $(\mathcal{M}, g)$. Note that the global null vector field $V$ forces pp-waves to be time-orientable. Similar to spaces of constant curvature, a pp-wave metric locally depends only on one function, i.e., for a pp-wave $(\mathcal{M}, g)$ there are local coordinates $(\mathcal{U}, (u, v, x^1, \ldots, x^n))$ such that

$$g|_{\mathcal{U}} = 2du(dv + Hdu) + \delta_{ij}dx^i dx^j,$$

where $\dim \mathcal{M} = n + 2$, $H = H(u, x^1, \ldots, x^n)$ is a smooth function on the coordinate patch $\mathcal{U}$ not depending on $v$. If $\mathcal{M} = \mathbb{R}^{n+2}$ and $g$ is globally of the form (2) we call $(\mathcal{M}, g)$ a pp-wave in standard form or a standard pp-wave. Four-dimensional standard pp-waves were discovered by Brinkmann in the context of conformal geometry [5], and then played an important role in general relativity (e.g., see [20], where also the name pp-wave for plane fronted with parallel rays was introduced). More recently, higher dimensional pp-waves appeared in supergravity theories, e.g. in [25], and there is now a vast physics literature on them.

Generically, pp-waves are manifolds with special holonomy. Indeed, the existence of a parallel null vector field implies that pp-waves have their holonomy contained in the Abelian ideal $\mathbb{R}^n$ of the stabiliser $O(n) \ltimes \mathbb{R}^n$ of a null vector. Moreover, under a mild genericity condition on the function $H$, their holonomy is equal to $\mathbb{R}^n$ and hence indecomposable.

Our results about compact pp-waves can be summarised in two theorems:

**Theorem A.** The universal cover of an $(n+2)$-dimensional compact pp-wave is globally isometric to a standard pp-wave

$$(\mathbb{R}^{n+2}, g^H = 2dudv + 2H(u, x^1, \ldots, x^n)du^2 + \delta_{ij}dx^i dx^j).$$

Under this isometry, the lift of the parallel null vector field is mapped to $\frac{\partial}{\partial v}$.

The proof in Section 4.1 uses the so-called screen bundle $\Sigma = V^\perp/V \to \mathcal{M}$ and the induced screen distributions (as described in Section 2). They can be used to define Riemannian metrics on the leaves of $V^\perp$, which are flat in case of pp-waves. This yields a detailed description of the universal cover of pp-waves in Section 3. Then, results by Candela et al. [10] about the completeness of certain non-compact Lorentzian manifolds, which apply to pp-waves in standard form, enable us in Section 4.2 to prove

**Theorem B.** Every compact pp-wave $(\mathcal{M}, g)$ is geodesically complete.

As a consequence we obtain that the examples of compact pp-waves we give below, are also geodesically complete examples of Lorentzian manifolds of special holonomy.

Theorem B is somewhat surprising when recalling that Ehlers and Kundt posed the following problem [20, Section 2-5.7]:

“Prove the plane waves to be the only $g$-complete pp-waves, no matter which topology one chooses.”

The plane waves mentioned in the problem are a special class of pp-waves:

**Definition 2.** A pp-wave $(\mathcal{M}, g)$ with parallel null vector field $V$ is a plane wave if

$$\nabla R = V^\flat \otimes Q,$$

where $R$ is the curvature tensor of $(\mathcal{M}, g)$, $Q$ is a $(0,4)$-tensor field, and $V^\flat := g(V, \cdot)$.

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3We thank Wolfgang Globke for pointing us to this reference.
For a plane wave, the function $H$ in the local form of the metric is quadratic in the $x^i$-coordinates

$$H(u, x^1, \ldots, x^n) = a_{ij}(u)x^ix^j, \quad \text{with } a_{ij} = a_{ji} \in C^\infty(\mathbb{R}).$$

This can be used to show that plane waves (in standard form, i.e., with $\mathcal{M} = \mathbb{R}^{n+2}$ and $g = g^H$ with $H$ as in (4)) are always geodesically complete ([10, Proposition 3.5], we review this result in Section 3.1). Our Theorem B shows that any compact pp-wave is complete, even if it is not a plane wave.

**Example 1.** Let $\eta$ be the flat metric on the $n$-torus $\mathbb{T}^n$ and $H \in C^\infty(\mathbb{T}^n)$ a smooth function on $\mathbb{T}^n$. On $\mathcal{M} := \mathbb{T}^2 \times \mathbb{T}^n$ we consider the Lorentzian metric

$$g^H = 2d\theta d\phi + 2Hd\theta^2 + \eta,$$

where $d\theta$ and $d\phi$ is the standard coframe on $\mathbb{T}^2$. This metric is a complete pp-wave metric on the torus $\mathbb{T}^{n+2}$, and one can choose $H$ in a way that it is not a plane wave. Indeed, computing $\nabla R$ shows that for any function $H$ with non-vanishing third partial derivatives with respect to the $x^i$-coordinates, the equality (3) is violated. More examples are given in [30] and [1], and in our Example 2.

However, this example is not in contradiction to the claim in the Ehlers-Kundt problem because there, pp-waves are understood to be solutions of the Einstein vacuum field equations and hence, in addition to Definition 1, are assumed to be Ricci flat. But the metric (5) is Ricci flat if and only if $H$ is harmonic with respect to the flat metric on the torus, which forces $H$ to be constant and $g^H$ to be flat. In fact, Theorem A and results in Section 4.2 allow us to generalise this observation.

**Corollary 1.** Every compact Ricci-flat pp-wave is a plane wave.

This solves the Ehlers-Kundt problem in case of compact manifolds. We should mention that, using their results in [10], another partial answer to the Ehlers-Kundt problem is given in [21, Theorem 4]. We describe this in our Section 3.1 and provide more examples of (non-compact) complete pp-waves that are not plane waves with our Lemma 10 and the corresponding Remark 5 in Section 4.

Finally, motivated by Corollary 1 in Section 4.2 we apply Theorems A and B to compact plane waves and conclude not only that they are complete but also covered by a plane wave in standard form. This and results about compact plane waves with parallel Weyl tensor in [15, 17], suggest the problem of classifying compact quotients of plane waves, however, such a classification is beyond the scope of this paper.

Of course, it would be interesting to study geodesic completeness for the larger class of Lorentzian manifolds with special holonomy, i.e., with parallel null vector field, or more generally, with parallel null line bundle. We believe that some of our methods can be generalised to this setting, although such a generalisation is not straightforward as our Example 3 on page 28 shows. For instance, an interesting question is, whether the manifolds constructed in [22] to realise all possible (connected) holonomy groups are complete. These problems will be subject to further research.

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2. Screen distributions on special Lorentzian manifolds

2.1. Lorentzian manifolds with special holonomy. Let \((M, g)\) be a Lorentzian manifold of dimension \((n + 2)\), \(\nabla\) the Levi-Civita connection of \(g\), and denote by \(H\) its holonomy group at \(p \in M\) and by \(H^0\) its connected component, the reduced holonomy group. We say that \((M, g)\) has special holonomy if \(H^0\) acts indecomposably, i.e., the metric degenerates on every holonomy invariant subspace, but is not equal to the connected component of the special orthogonal group in Lorentzian signature, \(SO^0(T_p M, g_p)\). Since \(SO^0(T_p M, g_p)\) has no proper irreducible subgroups [18], this means that \(H^0\) leaves invariant a degenerate subspace \(W \subset T_p M\) of the tangent space. This subspace defines an \(H^0\)-invariant line \(L := W \cap W^\perp\) which is null, i.e. \(g|_{L \times L} = 0\), and \(H^0\) is contained in the stabiliser in \(SO^0(T_p M, g_p)\) of this line \(L\). Then also the full holonomy \(H\) is contained in this stabiliser, see [1].

Geometrically this means that, in \(TM\) there is a bundle of null lines \(L\) that is parallel, i.e., which is invariant under parallel transport, or equivalently, if \(V\) is a local section of \(L\), then \(\nabla_X V\) is again a local section of \(L\) for all \(X \in \Gamma(TM)\). This implies that the distribution of hyperplanes

\[ L^\perp = \{X \in \Gamma(TM) \mid g(X, L) = 0 \text{ for } L \in L\} \]

is parallel in the same sense, and we have a filtration of \(TM\) into parallel distributions

\[ L \subset L^\perp \subset TM. \]

In particular, both distributions are involutive and define foliations \(\mathcal{L}\) and \(\mathcal{L}^\perp\) of the manifold \(M\). The line bundle \(L\) admits a global section if and only if \((M, g)\) is time-orientable [1]. Such a global section is then given by a recurrent vector field \(V \in \Gamma(TM)\), i.e., a vector field satisfying

\[ \nabla V = \varphi \otimes V, \]

for a one-form \(\varphi\). Such a vector field can be rescaled to a parallel vector field if and only if \(d\varphi = 0\).

If \((M, g)\) not only admits a parallel line bundle, but a global parallel vector field \(V \in \Gamma(TM)\), i.e., \(\nabla V = 0\), it is sometimes called a Brinkmann wave after [5]. In this case the full holonomy group of \((M, g)\) is contained in the stabiliser in \(O(1, n)\) of a null vector. Since \(V\) is a global null vector field, \((M, g)\) is time-orientable. We denote by \(V^\perp\) the hyperplane distribution \(L^\perp\) that is orthogonal to \(L = \mathbb{R} \cdot V\). Obviously, pp-waves are a special class of Lorentzian manifolds with parallel null vector field and we will return to them later.

Locally, a Lorentzian manifold \((M, g)\) of dimension \(n + 2\) with parallel null line bundle \(L\) admits coordinates \((U, \varphi = (u, v, x^1, \ldots, x^n))\) such that the metric is given as

\[ g|_U = 2du(dv + Hdu + \mu_i dx^i) + \hat{g}_{ij}dx^idx^j \]

where \(H\) is a smooth function on \(U\), \(\hat{g}_{ij} = \hat{g}_{ij}(u, x^1, \ldots, x^n)\) and \(\mu_i = \mu_i(u, x^1, \ldots, x^n)\) are smooth functions on \(U\), not depending on \(v\), which can be viewed as coefficients of \(u\)-dependent families of Riemannian metrics and one-forms, respectively, on the manifold \(u, v \equiv \text{constant}\). In these coordinates, the recurrent vector field \(V|_U\) is given as \(\partial_v\) and the leaves of \(V^\perp|_U\) are given by \(u \equiv \text{constant}\).

If \((M, g)\) admits in addition a parallel null vector field, the function \(H\) can be chosen to be independent of the coordinate \(v\), \(H = H(u, x^1, \ldots, x^n)\). It is known, see [10] or [24], that in this case these coordinates can be chosen in a way that \(\mu_i \equiv 0\).
2.2. The screen bundle and screen distributions. Most of what will be said in this subsection also holds for a time-orientable Lorentzian manifold with parallel null line bundle $L$, but for our purposes we will restrict ourselves from now on to the case when $L$ is spanned by a global parallel null vector field. Let $(\mathcal{M},g)$ be a Lorentzian manifold of dimension $n = m + 2 > 2$ with parallel null vector field $V \in \Gamma(TM)$. The filtration $\mathbb{R} \cdot V \subset V^\perp \subset TM$ defines a vector bundle, the screen bundle

$$\Sigma := V^\perp / V \longrightarrow \mathcal{M},$$

which is equipped with a positive definite metric induced by $g$,

$$g^\Sigma([X],[Y]) := g(X,Y),$$

and a covariant derivative $\nabla^\Sigma$ induced by the Levi-Civita connection $\nabla$ of $g$,

$$\nabla^\Sigma_X [Y] = [\nabla_X Y].$$

**Definition 3.** Let $(\mathcal{M},g)$ be a Lorentzian manifold of dimension $n + 2 > 2$ with parallel null vector field $V$. A screen distribution $S$ is a subbundle of $V^\perp$ of rank $n$ on which the metric $g$ is non-degenerate. A null vector field $Z$ such that $g(V,Z) \equiv 1$ is called a screen vector field.

Every screen vector field defines a screen distribution via $S := V^\perp \cap Z^\perp$, where by $\perp$ we always denote the orthogonal space with respect to the metric $g$. In fact, we have a one-to-one correspondence between screen distributions $S$, screen vector fields $Z$, and null one-forms $\zeta$ such that $\zeta(V) = 1$. The correspondence is given by

$$Z \quad \mapsto \quad S = V^\perp \cap Z^\perp$$

$$S = \text{Ker}(V^\flat,\zeta) \quad \mapsto \quad \zeta$$

$$\zeta \quad \mapsto \quad \zeta^\sharp = Z$$

where $\sharp : T^*\mathcal{M} \rightarrow TM$ denotes the isomorphism which is the inverse of $\flat : TM \rightarrow T^*\mathcal{M}$ defined by $Z^\flat = g(Z,\cdot)$ and $\text{Ker}(\cdot)$ is the annihilator of the one-forms in the argument.

By partition of unity, $\mathcal{M}$ always admits a screen distribution which is a non-canonical splitting of the exact sequence $0 \rightarrow L \rightarrow L^\perp \rightarrow \Sigma \rightarrow 0$. Then the bundle of null lines transversal to $V^\perp$ has a global nowhere-vanishing section $Z$, which defines a screen vector field, [1, Proposition 2].

Note that, for a screen distribution $S$ defined by a screen vector field $Z$, the image of $\nabla Z \in T^*\mathcal{M} \otimes TM$ always lies in $S$, i.e.,

$$\nabla U Z \in \Gamma(S) \quad \text{for all} \quad U \in \Gamma(TM).$$

Indeed, $g(\nabla Z, V) = -g(Z, \nabla V) = 0$ and $2g(\nabla Z, Z) = d(g(Z, Z)) = 0$.

In the following we will make use of the Riemannian metric $h = h^S$ defined by a screen distribution $S$ or, equivalently, by a screen vector field $Z$ via

$$h(V,\cdot) := g(Z,\cdot), \quad h(Z,\cdot) := g(V,\cdot), \quad h(X,\cdot) := g(X,\cdot) \quad \text{for} \quad X \in S.$$

and extension by linearity.
2.3. **Horizontal and involutive screen distributions.** In the following we will call a screen distribution horizontal if for every $p \in M$ exists a neighborhood $U$ and local frame fields $S_1, \ldots, S_n \in \Gamma(S|U)$ of $S$ such that
\begin{equation}
[V, S_i] \in \Gamma(S|U),
\end{equation}
and involutive if
\begin{equation}
[S_j, S_i] \in \Gamma(S|U).
\end{equation}
Note that both conditions are independent of the chosen frame fields for $S$. Note also that the coordinates in [8] provide us with a local horizontal screen distribution spanned by $\partial_1 + \mu_1 \partial_v, \ldots, \partial_n + \mu_n \partial_v$ with corresponding screen vector field
\[
Z = \partial_u - H \partial_v - 2 g^{ij} \mu_i \partial_j,
\]
where $g^{ij}$ is the inverse matrix of $g(\partial_i, \partial_j)$. Moreover, if the $\mu_i$’s in [9] are the coefficients of a $u$-dependent family of closed one-forms, this screen is also involutive.

The term horizontal for a given screen distribution comes from the following observation about the Riemannian metric defined in [8].

**Proposition 1.** Let $(M, g)$ be a Lorentzian manifold with parallel null vector field $V$. By $N$ we denote a leaf of the integrable distribution $V^\perp$. Furthermore, let $S$ be a screen distribution defining a Riemannian metric $h$ on $M$ and induced Riemannian metrics $h^N$ on each leaf $N$ by restriction. Then we have
\begin{enumerate}
\item $S$ is horizontal if and only if on each leaf $N$ of $V^\perp$, $V$ defines an isometric Riemannian flow, i.e., $V$ is a Killing vector field of constant length for the metric $h^N$ on $N$.
\item Fix a leaf $N$ and assume that, along $N$, there is a screen distribution $S$ which is involutive and horizontal. Then $V \in \Gamma(TN)$ is parallel on $(N, h)$, i.e., with respect to the Levi-Civita connection $\nabla^h$ of $h$. In particular, the leaves of $S$ in $N$ are totally geodesic for $h$.
\end{enumerate}

**Proof.** (1) For the only non-vanishing term of $\mathcal{L}_V h|_{V^\perp \times V^\perp}$ we compute for $X \in S$,
\[
\mathcal{L}_V h(V, X) = h([X, V], V) = g([X, V], Z).
\]
This is zero if and only if $S$ is horizontal. By definition, $V$ has also constant length.

(2) As $h(V, V) \equiv 1$ we clearly have $h(\nabla^h V, V) = 0$. Furthermore, for $X, Y \in \Gamma(S|N)$ the Koszul formula for $\nabla^h$ gives
\[
h(\nabla^h V, X) = -h(\nabla^h V, X) = -h([V, X], V) = 0,
\]
since $S$ is horizontal, i.e., $[V, X] \in \Gamma(S)$, as well as for $X, Y \in S$,
\[
2h(\nabla^h_X V, Y) = V(h(X, Y)) + h([X, V], Y) + h([Y, V], X) + h([Y, X], V)
= V(g(X, Y)) + g([X, V], Y) + g([Y, V], X) + g([Y, X], V)
= 0,
\]
since $\nabla V = 0$ and $S$ horizontal. Here, for the second equality we use that $S$ is horizontal and involutive, i.e., that $h([Y, X], V) = 0$ and $h([X, V], Y) = g([X, V], Y)$. \hfill $\square$

Now we derive some criteria for a screen distribution to be horizontal and involutive in terms of the one-form $Z^\perp := g(Z, \cdot)$. An immediate consequence of $S = V^\perp \cap Z^\perp$ is:
Lemma 1. Let $\mathcal{S}$ be a screen distribution with screen vector field $Z$. Then $\mathcal{S}$ is involutive and horizontal if and only if $dZ^b|_{V^\perp \wedge V^\perp} = 0$, which is equivalent to $V^\perp \wedge dZ^b = 0$.

Assuming that the screen bundle is globally trivialisable, which will be relevant in the next section, we obtain:

Lemma 2. Let $\mathcal{S}$ be a screen distribution defined by a screen vector field $Z$. Assume that the screen bundle is globally trivial (i.e., admits $n$ linearly independent global sections). Then there exists a global orthonormal frame field $S_1, \ldots, S_n$ of $\mathcal{S}$. Furthermore, $\mathcal{S}$ is horizontal if and only if the global one-forms $\alpha^i \in \Gamma(T^*M)$ defined by

$$\alpha^i := g(\nabla S_i, Z),$$

satisfy

$$\alpha^i(V) = 0,$$

and $\mathcal{S}$ is involutive if and only if

$$\alpha^i(S_j) - \alpha^j(S_i) = 0.$$

Proof. Let $\mathcal{S}$ be a screen distribution. Then the bundle projection

$$\mathcal{S} \ni X \mapsto [X] \in \Sigma$$

can be used to define a global frame field $S_1, \ldots, S_n$ for $\mathcal{S}$ from a trivialization of $\Sigma$. Then

$$\alpha^i(V) = g(\nabla V S_i, Z) = g([V, S_i], Z)$$

shows this first equivalence and

$$\alpha^i(S_j) - \alpha^j(S_i) = g([S_i, S_j], Z)$$

the second one. □

Now we compute the difference between two screen vector fields and their screen distributions, still under the assumption that the screen bundle $\Sigma$ is globally trivial, so that we have a global orthonormal frame field $\sigma_1, \ldots, \sigma_n$ of $\Sigma$. Then, if $\mathcal{S}$ and $\hat{\mathcal{S}}$ are two screen distributions, the sections $\sigma_i$ define sections $S_i \in \Gamma(\mathcal{S})$ and $\hat{S}_i \in \Gamma(\hat{\mathcal{S}})$, both orthonormal with respect to $g$, which are related by

$$\hat{S}_i = S_i - b^i V \mapsto \sigma_i = [S_i] \in \Gamma(\Sigma),$$

for smooth functions $b^i$ on $M$. The corresponding screen vector fields $Z$ and $\hat{Z}$ are then related by

$$\hat{Z} = Z + \sum_{k=1}^n b^k S_k - \frac{1}{2} \sum_{k=1}^n (b^k)^2 V,$$

and for the differentials of the duals we get

$$d\hat{Z}^b = dZ^b + \sum_{k=1}^n \left( d\hat{S}_k^b + b^k dS_k^b - b^k d\hat{b}^k \wedge V^b \right),$$

Then, computing the differentials of $Z^b$ and $d\hat{S}_i^b$ we get

$$dZ^b = S_k^b \wedge \alpha^k$$

with $\alpha^k$ defined in (11), and

$$dS_i^b = \omega_i^k \wedge S_k^b + \alpha^i \wedge V^b,$$
where $\omega_i^j$ is the part of the connection one-form defined by
\begin{equation}
\omega_i^j := g(\nabla S_i, S_j).
\end{equation}
This allows us to express the differential of $\hat{Z}^b$ in terms of a basis of the old screen, its connection coefficients and the functions $b^i$ as
\begin{equation}
d\hat{Z}^b = (db^k - \alpha^k + b^i \omega_i^k) \wedge S_k^b + b^k(\alpha^k - db^k) \wedge V^b,
\end{equation}
in which we omit the sum symbol and use the summation convention. This, together with Lemma 1 gives us

**Proposition 2.** Let $(\mathcal{M}, g)$ be a Lorentzian manifold with parallel null vector field $V$ and a screen bundle that is defined by global sections $S_1, \ldots, S_n$, and let $\alpha^i$ and $\omega_i^j$ be the corresponding connection forms defined in (11) and (14). Then there is an involutive and horizontal screen distribution if and only if there are smooth functions $b^1, \ldots, b^n$ on $\mathcal{M}$ which are solutions to the differential system
\begin{equation}
0 = (db^k - \alpha^k + b^i \omega_i^k) \wedge S_k^b |_{V^\perp \wedge V^\perp}.
\end{equation}
In particular, if there exist functions $b^i$ such that
\begin{equation}
(db^i - \alpha^i + b^k \omega_i^k)|_{V^\perp} = 0,
\end{equation}
then there is a horizontal and involutive screen distribution spanned by $S_i - b^i V$.

### 2.4. Manifolds with trivial screen holonomy and pp-waves.
We say that a Lorentzian manifold $(\mathcal{M}, g)$ has trivial screen holonomy if the full holonomy group of $\nabla^\Sigma$ is trivial, i.e., consists only of the identity transformation. This is related to the notion of pp-waves as in Definition 1.

**Proposition 3.** Let $(\mathcal{M}, g)$ be a Lorentzian manifold with parallel null vector field $V$. The following statements are equivalent:

(a) $(\mathcal{M}, g)$ is a pp-wave.
(b) For all $W \in V^\perp$ and $X, Y \in \Gamma(T \mathcal{M})$ it holds $R(X, Y)W \in \mathbb{R} V$.
(c) The screen bundle $(\Sigma, \nabla^\Sigma)$ is flat, i.e., the curvature of $\nabla^\Sigma$ vanishes.
(d) The connected component of the holonomy group of $(\mathcal{M}, g)$ is contained in $\mathbb{R}^n \subset SO(1, n + 1)$.
(e) There exist local sections $S_1, \ldots, S_n$ of $V^\perp$ with $g(S_i, S_j) = \delta_{ij}$ and local one-forms $\alpha^i$ such that
\begin{equation}
\nabla S_i = \alpha^i \otimes V.
\end{equation}
In this case, the one-forms satisfy $d\alpha^i |_{V^\perp \wedge V^\perp} = 0$.

The proof is a straightforward computation carried out in [3, 4]. The property for the differentials of the $\alpha^i$’s follows from the following computation: Let $Z$ be a screen vector field, $X \in V^\perp$ and $S_i$ frame fields as in (1d). Then
\begin{equation}
\begin{aligned}
d\alpha^i(S_j, X) &= g(R(S_j, X)S_i, Z) = g(R(S_i, Z)S_j, X) = d\alpha^i(S_i, Z)g(V, X) = 0, \\
\text{for all } i, j = 1, \ldots, n, \text{ i.e., } d\alpha^i |_{V^\perp \wedge V^\perp} = 0.
\end{aligned}
\end{equation}
Clearly, manifolds with trivial screen holonomy are pp-waves, but for non simply connected manifolds the converse is not true (see [1] for examples).
Locally, for a pp-wave the coordinates in (6) can be chosen in a way such that $\mu_i \equiv 0$ and $\hat{g}_{ij} \equiv \delta_{ij}$, i.e., with $\hat{g}$ being the standard flat metric for all $u$, i.e.,

$$g = 2du(dv + Hdu) + \delta_{ij}dx^i dx^j$$

where $H = H(u, x^1, \ldots, x^n)$ is a smooth function. In these coordinates, $\nabla \partial_v = 0$ and

$$\nabla \partial_i \partial_j = 0, \quad \nabla \partial_i \partial_u = \partial_i (H) \partial_v, \quad \nabla \partial_u \partial_u = \partial_u (H) \partial_v - \sum_{i=1}^n \partial_i (H) \partial_i$$

which implies that the only non-vanishing curvature terms of $g$, up to symmetries, are (17)

$$R(\partial_i, \partial_u, \partial_u, \partial_j) = -\partial_i \partial_j H,$$

where our sign convention is $R(X, Y)U = [\nabla_X, \nabla_Y]U - \nabla_{[X,Y]}U$, and for the Ricci curvature, $\text{Ric} = \text{trace}_{[2,3]} R$,

$$\text{Ric} (\partial_u, \partial_u) = \Delta (H)$$

where $\Delta = -\sum_{i=1}^n \partial_i^2$ is the flat Laplacian. Formula [17] shows that the connected holonomy of a pp-waves is equal to $\mathbb{R}^n$, and hence indecomposable, if there is a point in $\mathcal{M}$ with local coordinates such that the Hessian of $H$ is non-degenerate at this point.

Since the distribution $V^\perp$ is parallel and thus defines a foliation of $\mathcal{M}$ into totally geodesic leaves of codimension one, the flatness of the screen bundle can be stated as

**Lemma 3.** Let $(\mathcal{M}, g)$ be a Lorentzian manifold with parallel null vector field $V$ and foliation $V^\perp$. Then $(\mathcal{M}, g)$ is a pp-wave if and only if, for each leaf $\mathcal{N}$ of $V^\perp$, the linear connection which is induced on $\mathcal{N}$ by the Levi-Civita connection of $g$ is flat.

**Proof.** Let $\nabla^\mathcal{N}$ be the linear connection defined by $\nabla$ on a leaf $\mathcal{N}$ of $V^\perp$, i.e., $\nabla^\mathcal{N}_U W := \nabla_U W \in V^\perp|_{\mathcal{N}}$ for $U, W \in \Gamma (TN)$, where $TN = V^\perp|_{\mathcal{N}}$. Hence, for the curvature $R$ of $\nabla$ and $R^\mathcal{N}$ of $\nabla^\mathcal{N}$ we have

$$R^\mathcal{N} (U, W) S = R(U, W) S \quad \forall U, W, S \in V^\perp|_{\mathcal{N}}.$$

This term vanishes if and only if $g(R(U, W) S, X) = 0$ for all $X \in \Gamma (TM)$, which is equivalent to [11] in the definition of pp-waves on page 2.

**Proposition 4.** Let $(\mathcal{M}, g)$ be a Lorentzian manifold with parallel null vector field $V$ and trivial screen holonomy, or equivalently, with $\text{Hol} (\mathcal{M}, g) = \mathbb{R}^n$. Then, for each screen distribution $\mathcal{S} = V^\perp \cap Z^\perp$ with screen vector field $Z$, there is a global frame field $S_1, \ldots, S_n$ of $\mathcal{S}$ and the $\alpha^i$ defined in (11) satisfy

$$d\alpha^i (X, Y) = R(X, Y, S_i, Z),$$

and hence

$$d\alpha^i|_{V^\perp \cap V^\perp} = 0.$$

Furthermore, a given screen distribution $\mathcal{S}$ can be changed to an involutive and horizontal one, if there exist functions $b^1, \ldots, b^n$ on $\mathcal{M}$ such that

$$(db^i - \alpha^i)|_{V^\perp} = 0.$$

**Proof.** Since $\Sigma$ is assumed to have trivial holonomy, we find global basis sections $\sigma_i$ of $\Sigma$ such that $\nabla^\Sigma \sigma_i = 0$. Hence, for a given screen distribution, the induced frame fields $S_i$ satisfy $[\nabla_X S_i] = \nabla^\Sigma_X \sigma_i = 0$ and thus

$$\nabla S_i = g(\nabla S_i, Z) = \alpha^i \otimes V,$$
or equivalently, $\omega_i^j = 0$. As above, we have

$$R(X,Y)S_i = d\alpha_i(X,Y) \cdot V,$$

since $V$ is parallel. Given functions $b^i$ with $(db^i - \alpha^i)|_{V^\perp} = 0$, from $\omega_i^j = 0$ and equation (16) in Proposition 2 we see that $\hat{S}_k = S_k - b^kV$ defines a horizontal and integrable screen distribution.

For Lorentzian manifolds with trivial screen holonomy admitting a horizontal and integrable screen distribution, we can strengthen Proposition 1 in the following way:

**Proposition 5.** Let $(\mathcal{M}, g)$ be a Lorentzian manifold with parallel null vector field $V$ and with trivial screen holonomy. Let $\mathcal{N}$ be a leaf of the integrable distribution $V^\perp$. Assume that, along $\mathcal{N}$, there is a screen distribution $\mathcal{S}$ which is involutive and horizontal. Then the Riemannian metric $h$ on $\mathcal{N}$ that is defined by $\mathcal{S}$ through the relations [S] is flat and the frame $S_i$ in Proposition 4 together with $V$ constitute a $\nabla^h$-parallel frame field for $(\mathcal{N}, h)$.

**Proof.** Let $\mathcal{S}$ be an involutive and horizontal screen distribution along a $V^\perp$-leaf $\mathcal{N}$. By Proposition 4, we have sections $S_1, \ldots, S_n$ of $\mathcal{S}$ with [S], i.e.,

$$g(\nabla_X S_i, Y) = 0$$

for all $X, Y \in \Gamma(T\mathcal{N})$. Writing out the Koszul formula for this term we get

$$0 = X(g(S_i, Y)) + S_i(g(X, Y)) - Y(g(S_i, X)) + g([X, S_i], Y) + g([Y, S_i], X) + g([Y, X], S_i)$$

This equation holds for all $X, Y \in \Gamma(T\mathcal{N})$, but, since $\mathcal{S}$ was assumed to be horizontal and involutive, we have that the brackets $[X, S_i]$, $[Y, S_i]$ and $[X, Y]$ are in $\mathcal{S}$. Hence, when recalling the definition of $h$ in [S], in the above expression we can replace the metric $g$ by the Riemannian metric $h$ on $\mathcal{N}$, which shows that

$$h(\nabla^h_X S_i, Y) = 0,$$

where $\nabla^h$ is the Levi-Civita connection of $(\mathcal{N}, h)$. Hence, the $S_i$ are parallel vector fields on $(\mathcal{N}, h)$. But we have already seen in Proposition 4 that $V$ is also parallel for $h$. Hence, we have a $h$-orthonormal frame of $\mathcal{N}$ which is parallel for $\nabla^h$ yielding the flatness of $(\mathcal{N}, h)$. □

**Remark 1.** Let $(\mathcal{M}, g)$ be a pp-wave and $\mathcal{S}$ an involutive and horizontal screen distribution on $\mathcal{M}$ and denote by $h$ the Riemannian metric on $\mathcal{M}$ defined by $\mathcal{S}$. In this situation, we have seen that, on each leaf $\mathcal{N}$ of $V^\perp$, the connection induced by $\nabla$ coincides with the Levi-Civita connection of the Riemannian metric $h^V$ which is induced by $h$ on $\mathcal{N}$, and in fact, both are flat. However, on $\mathcal{M}$ the Levi-Civita connections of $(\mathcal{M}, g)$ and $(\mathcal{M}, h)$ do not coincide, not even along $\mathcal{N}$. For example, if $S_1, \ldots, S_n$ is a frame of $\mathcal{S}$ with $\nabla S_i|_{V^\perp} = 0$ we have that $g(\nabla_V S_i, V) = 0$ but $\nabla^h_V S_i$ has a transversal component,

$$g(\nabla^h_V S_i, V) = h(\nabla^h_V S_i, Z) = \frac{1}{2} h([Z, S_i], V) = \frac{1}{2} g([Z, S_i], Z) = -\frac{1}{2} g(\nabla Z S_i, Z)$$

which in general is not zero.
3. The Universal Cover of a pp-Wave

3.1. Review of completeness results for Lorentzian manifolds with parallel null vector field. Before we turn to the universal cover of pp-waves and to the proof of the main theorems for compact manifolds, we want to recall results about the completeness of Lorentzian manifolds with parallel null vector fields in a more general setting. To our knowledge, the strongest of such results can be found in [10] and they hold for a special class of these manifolds described in the next theorem.

**Theorem 1** ([10, Theorem 3.2 and Corollary 3.4]). Let \((S, h)\) be a connected Riemannian manifold of dimension \(n\) and let \(H \in C^\infty(\mathbb{R} \times S)\) be a smooth function. On the manifold \(M := \mathbb{R}^2 \times S\) define the Lorentzian metric \(g\) by

\[
g|_{(u,v,x)} = 2\, du\, dv + 2H(u,x)\, du^2 + h|_x,
\]

where \(x \in S\) and \((u,v)\) are the global coordinates on \(\mathbb{R}^2\).

1. The Lorentzian manifold \((M, g)\) is geodesically complete if and only if the Riemannian manifold \((S, h)\) is complete and the solutions \(s \mapsto \gamma(s)\) of the ODE

\[
\nabla^h \dot{\gamma}(s) = \text{grad}^h H(s, \gamma(s))
\]

are defined on the whole real line. Here \(\nabla^h\) is the Levi-Civita connection of \(h\).

2. If \((S, h)\) is geodesically complete and the function \(H\) does not depend on \(u\) and is at most quadratic at spatial infinity, i.e., there exist \(x_0 \in S\) and real constants \(r, c > 0\) such that

\[
H(x) \leq c \cdot d_S(x_0, x)^2 \quad \text{for all } x \in S \text{ with } d_S(x_0, x) \geq r,
\]

then \((M, g)\) is geodesically complete. Here \(d_S\) is the distance function of \((S, h)\).

This theorem applies to pp-waves in standard form, and to the more general class of pp-waves that are globally of the form \([18]\) with \((S, h)\) a flat Riemannian manifold, not necessarily the \(\mathbb{R}^n\). For plane waves (as defined in the introduction) it holds

**Proposition 6** ([10, Proposition 3.5]). Let \((M, g)\) be a Lorentzian metric of the form \([18]\) and assume that its universal cover is globally isometric to a standard plane-wave

\[
(\mathbb{R}^{n+2}, \bar{g} = 2\, du\, dv + (a_{ij}(u)\, x^i x^j)\, du + \delta_{ij}\, dx^i\, dx^j),
\]

with \(a_{ij} = a_{ji} \in C^\infty(\mathbb{R})\). Then \((M, g)\) is geodesically complete.

Regarding the Ehlers-Kundt problem, these results imply:

**Proposition 7.** [21, Theorem 4] Any gravitational (Ricci-flat and four-dimensional) pp-wave (in standard form) such that \(H\) behaves at most quadratically at spatial infinity (in the sense of Theorem 1) is a (necessarily complete) plane wave.

We should mention the symmetric spaces amongst the pp-waves, the Cahen-Wallach spaces \([8]\). For these, the function \(H\) is just a quadratic polynomial with constant coefficients. As symmetric spaces they are automatically complete.

Not assuming the existence of a parallel vector field but a timelike conformal Killing field, more completeness results were obtained in \([10, 11, 12]\). Since pp-waves admit a parallel null vector field, one might be tempted to adapt these proofs of completeness to the existence of a conformal Killing vector field that is causal instead of time-like.
However, the Clifton-Pohl torus and the example in [42] of a compact but incomplete Lorentzian manifold with a causal Killing vector field show that this might be more difficult than expected, and that these results in general do not apply to our setting.

3.2. Horizontal and involutive screen distributions on the universal cover.

In this section we will deal with compact pp-waves but first present some results which hold in a more general setting. A vector field is complete if its maximal integral curves are defined on \( \mathbb{R} \).

**Proposition 8.** Let \( M \) be a manifold with a closed nowhere-vanishing one-form \( \eta \). Assume that there is a complete vector field \( Z \) with \( \eta(Z) = 1 \). Then the leaves of the distribution \( \text{Ker}(\eta) \) are all diffeomorphic to each other under the flow \( \phi_t \) of \( Z \), and the universal cover \( \tilde{M} \) of \( M \) is diffeomorphic to \( \mathbb{R} \times \tilde{N} \) with the diffeomorphism given as

\[
\Phi : \mathbb{R} \times \tilde{N} \ni (u, p) \mapsto \tilde{\phi}_u(p) \in \tilde{M},
\]

where \( \tilde{N} \) is the universal cover of a leaf of the distribution \( \text{Ker}(\eta) \) and \( \tilde{\phi} \) is the flow of the lift of \( Z \). If \( M \) is compact with closed \( \eta \), all of the above is satisfied.

**Proof.** The idea of the proof can be found in [36, Thm. 3.1]. Since \( \eta \) is closed, the distribution \( \text{Ker}(\eta) \) is involutive. For each \( t \in \mathbb{R} \), the flow \( \phi_t \) of the complete vector field \( Z \) is a diffeomorphism of \( M \). Since \( \eta \) is closed and \( \eta(Z) = 1 \) we get for the Lie derivative of \( g \) that

\[
\mathcal{L}_Z \eta(X) = d\eta(Z, X) + X(\eta(Z)) \equiv 0,
\]

which shows that \( \phi_t \) maps the leaves of the distribution \( \text{Ker}(\eta) \) diffeomorphically onto each other.

Let \( \tilde{\eta} \) and \( \tilde{Z} \) be the lifts of \( \eta \) and \( Z \) to the universal cover \( \tilde{M} \). Then \( \tilde{Z} \) is still a complete vector field with \( \tilde{\eta}(\tilde{Z}) = 1 \) and \( d\tilde{\eta} = 0 \). Hence, there is a real function \( f \in C^\infty(\tilde{M}) \) such that \( \tilde{\eta} = df \). Let \( \tilde{\phi}_t, t \in \mathbb{R} \), denote the flow of \( \tilde{Z} \). Then, for each \( p \in \tilde{M} \), the function \( \tau_p : \mathbb{R} \to \mathbb{R} \) defined by \( \tau_p(t) := f(\tilde{\phi}_t(p)) \in \mathbb{R} \) satisifies

\[
\tau_p'(t) = df_{\tilde{\phi}_t(p)}(\tilde{Z}) = \eta_{\tilde{\phi}_t(p)}(\tilde{Z}) \equiv 1.
\]

Hence, \( \tau(t) = t + f(p) \). This shows that \( f : \tilde{M} \to \mathbb{R} \) is surjective and that two level sets \( \tilde{N}_a = f^{-1}(a), a \in \mathbb{R} \), are diffeomorphic under the flow,

\[
\tilde{\phi}_{b-a} : \tilde{N}_a \simeq \tilde{N}_b.
\]

This defines a diffeomorphism

\[
\Phi : \mathbb{R} \times \tilde{N}_0 \to \tilde{M} \quad (u, p) \mapsto \tilde{\phi}_u(p)
\]

the inverse of which is given by

\[
\Phi^{-1}(p) = (f(p), \tilde{\phi}_{-f(p)}(p)) \in \mathbb{R} \times \tilde{N}_0.
\]

Being simply connected, \( \tilde{N} := \tilde{N}_0 \) is the universal cover of the leaves of \( \text{Ker}(\eta) \).

The proposition applies in particular to a compact Lorentzian manifold with parallel vector field \( V \) defining a closed one form \( \eta = g(V, \cdot) \). Here \( \tilde{N} \) is given as an integral
manifold of the distribution $\tilde{V}^\perp$. Because of $d\Phi(u,p)(\partial_u) = \frac{\partial}{\partial r}(\tilde{\phi}(p)) = \tilde{Z}|_{\phi(p)}$, for the pull-back of the metric $\tilde{g}$ on the universal cover $\tilde{M}$ we have

$$\Phi^*\tilde{g}(\partial_u, \partial_u) = g(Z, Z).$$

The proposition implies

**Theorem 2.** Let $(\mathcal{M}, g)$ be an $(n + 2)$-dimensional pp-wave with parallel null vector field $V \in \Gamma(T\mathcal{M})$ and complete screen vector field. Then there exists a horizontal and involutive screen distribution $S$ on the universal cover $(\tilde{\mathcal{M}}, \tilde{g})$ of $(\mathcal{M}, g)$. In particular, there are linear independent vector fields $S_i \in \Gamma(S)$ on $\tilde{\mathcal{M}}$, $i = 1, \ldots, n$, with $\tilde{\nabla}_X S_i = 0$, for all $X \in \tilde{V}^\perp$ and $\tilde{\nabla}$ the Levi-Civita connection of $\tilde{g}$. Moreover, $(V, S_1, \ldots, S_n)$ is a $\nabla^h$-parallel orthonormal frame on $(\mathcal{N}, h)$, where $h$ is the Riemannian metric defined by $S$ on the leaves of $\tilde{V}^\perp$.

**Proof.** By a tilde we denote the lift of any object to the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$. Let $S = V^\perp \cap Z^\perp$ be a screen distribution defined by a complete screen vector field $Z$. By assumption, the bundle $\tilde{\Sigma} \to \tilde{\mathcal{M}}$ is flat, and, since $\tilde{\mathcal{M}}$ is simply connected, has trivial holonomy. Thus we obtain linearly independent global parallel sections $\sigma_1, \ldots, \sigma_n \in \Gamma(\tilde{\Sigma})$ which give rise to $n$ sections $S_1, \ldots, S_n \in \Gamma(S)$ with $\tilde{\nabla} S_i = \alpha_i \otimes \tilde{V}$ with $\alpha_i := \tilde{g}(\tilde{\nabla} S_i, \tilde{Z})$. Since $\mathcal{M}$ is a pp-wave, according to Proposition 4 they satisfy

$$\left.\frac{d\alpha_i}{\alpha_i}\right|_{\tilde{V}^\perp, \tilde{\Sigma}^\perp} = 0.$$  

By Proposition 3 the universal cover $\tilde{\mathcal{M}}$ is diffeomorphic to $\mathbb{R} \times \mathcal{N}$, where $\mathcal{N}$ is the universal cover of the leaves of the distribution $V^\perp$, and the map $\mathbb{R} \times \mathcal{N} \to \tilde{\mathcal{M}}$ is given by the flow of $\tilde{Z}$. Now, for each $r \in \mathbb{R}$, let

$$\iota(r) : \mathcal{N} \to \mathbb{R} \times \mathcal{N}, \quad \iota(r) = \begin{array}{c} \mathbb{R} \times \mathcal{N} \\ x \mapsto (r, x) \end{array}$$

denote the inclusion of $\mathcal{N}$ into $\mathbb{R} \times \mathcal{N}$. We use these to pull back the $\alpha_i$'s to $\mathcal{N}$,

$$\alpha^i_r := (\iota^*_r) \alpha^i,$$

which is now a one-parameter family of one-forms on $\mathcal{N}$, depending smoothly on the parameter $r \in \mathbb{R}$. Because of equation (20), all $\alpha^i_r$ are closed,

$$d\alpha^i_r = d(\iota^*_r \alpha^i) = \iota^*_r d\alpha^i = 0.$$  

Fixing $x_0 \in \mathcal{N}$, since $\mathcal{N}$ is simply connected, for each $i = 1, \ldots, n$ and each $r \in \mathbb{R}$ we find a unique function $b^i_r \in C^\infty(\mathcal{N})$ such that

$$db^i_r = \alpha^i_r, \quad \text{and} \quad b^i_r(x_0) = 0,$$

where the differential is the differential on $\mathcal{N}$. Hence we obtain smooth functions $b^i \in C^\infty(\mathbb{R} \times \mathcal{N})$ defined by

$$b^i(r, x) = b^i_r(x).$$

We have to verify that these functions are smooth on $\mathbb{R} \times \mathcal{N}$: Take an arbitrary $\hat{x} \in \mathcal{N}$ and fix coordinates $(U, \varphi = (x^1, \ldots, x^n))$ around $\hat{x}$ such that $\varphi(\hat{x}) = 0$. Over $U$ we
write $\alpha^i_{(r)}$ as

$$\alpha^i_{(r)}|_{(r, \varphi^{-1}(x^1, \ldots, x^n))} = \sum_{k=1}^{n} \alpha^i_k(r, x^1, \ldots, x^n)dx^k$$

with $\alpha^i_k$ smooth functions on $\mathbb{R} \times \varphi(\mathcal{U})$ and the solutions $b^i_{(r)}$ are given by

$$b^i_{(r)}(r, \varphi^{-1}(x^1, \ldots, x^n)) = b^i_{(r)} \circ \varphi^{-1}(x^1, \ldots, x^n)$$

$$= \sum_{k=1}^{n} x^k \int_{0}^{1} \alpha^i_k(r, t(x^1, \ldots, x^n))dt + b^i(r, \hat{x}),$$

for all $(x^1, \ldots, x^n) \in \varphi(\mathcal{U})$. Since $\alpha^i_{(r)}$ and hence $\alpha^i_k$ depend smoothly on $r$, this is clearly smooth in $r$ and $x^i$.

Using these $b^i \in C^\infty(\mathbb{R} \times \mathcal{N})$ we define the new screen distribution as

$$\tilde{\mathcal{S}} := \text{span}\{\tilde{S}_1, \ldots, \tilde{S}_n\} \quad \text{with} \quad \tilde{S}_i := S_i - b^i\tilde{V}.$$  

For every $\tilde{X} = dt_{(r)}x(X) \in V^\perp_{(r, x)} \subset T_{(r, x)}(\mathbb{R} \times \mathcal{N})$ with $X \in T_x\mathcal{N}$ the $\tilde{S}_i$ satisfy

$$\nabla_{\tilde{X}}\tilde{S}_i = (\alpha^i(\tilde{X}) - db^i(\tilde{X}))\tilde{V} = (\alpha^i_{(r)}(X) - db^i_{(r)}(X))\tilde{V} = 0,$$

which shows that $\tilde{\mathcal{S}}$ is involutive and horizontal. Finally, the $\nabla^h$-parallelity of the frame $(V, S_1, \ldots, S_n)$ on $(\mathcal{N}, h)$ follows from Proposition 5.

Remark 2. The horizontal and involutive screen distribution on the universal cover obtained by this result does not necessarily descend to a horizontal and involutive one on the base manifold. In fact, the next example shows a compact pp-wave for which no horizontal and involutive realization of the screen bundle $\Sigma$ exists. The construction can also be found in [30, Proposition 2.42].

Example 2. We will give an example of a compact pp-wave that does not admit an involutive screen distribution. Let $\mathcal{N} := T^{n+1}$ be the $(n + 1)$-torus and let $c \in H^2(T^{n+1}, \mathbb{Z})$ a non-zero cohomology class, $\omega \in c$ a closed two-form representing $c$ in the de Rham cohomology. Now let $\pi : \mathcal{M} \to T^{n+1}$ be the circle-bundle over $T^{n+1}$ with first Chern class being equal to $c$. Furthermore, let $A \in T^*\mathcal{M} \otimes \mathbb{R}$ be the corresponding $S^1$-connection with curvature $F := dA = -2\pi i\eta^*\omega$. Now let $\partial_0, \ldots, \partial_n$ be the canonical frame and $\xi^0, \ldots, \xi^n$ be the canonical coframe on $T^{n+1} = S^1 \times \ldots \times S^1$. Denote by

$$\eta := \pi^*\xi^0, \quad \sigma^i := \pi^*\xi^i, \quad i = 1, \ldots n$$

their pull-backs to $\tilde{\mathcal{M}}$ and by $U, S_i, i = 1, \ldots, n$ the $A$-horizontal lifts of $\partial_0$ and $\partial_i$, $i = 1, \ldots, n$ respectively. Hence, we have

$$d\pi_p(U) = \partial_0|_{\pi(p)}, \quad d\pi_p(S_i) = \partial_i|_{\pi(p)}$$

and

$$A(U) = A(S_i) = 0, \quad \sigma^i(S_j) = \delta^i_j, \quad \sigma^i(U) = 0, \quad \eta(S_i) = 0, \quad \eta(U) = 1.$$  

Let $V$ be the fundamental vector field of the $S^1$-action on $\mathcal{M}$, i.e., with $A(V) = i$. Since $U$ and the $S_i$’s are defined as horizontal lifts of the $\partial_i$’s we have that $[U, S_i]$ and $[S_i, S_j]$ are vertical vector fields and moreover that

$$[V, U] = [V, S_i] = 0.$$
Having \((n + 2)\)-linearly independent nowhere-vanishing one forms, and choosing a smooth function \(H \in C^\infty(T^{n+1})\), enables us to define a Lorentzian metric \(g\) on \(\mathcal{M}\) by

\[
g = 2(\eta H - iA) \cdot \eta + \sum_{i=1}^{n} (\sigma^i)^2.
\]

The Koszul formula, the verticality of \([U, S_i]\) and \([S_i, S_j]\) together with equation (21) show that \(V\) is a parallel vector field for \(g\). An obvious screen vector field is given by

\[Z := U - HV,
\]

with the metric dual given by

\[Z^\flat = g(Z, \cdot) = H\eta - iA.
\]

Moreover, we have \(\nabla_V S_i = 0\), and, again using the Koszul formula

\[
\begin{align*}
\nabla S_i S_j &= \frac{i}{2} F(S_i, S_j) V \\
\n\nabla_U S_i &= (S_i(H) - iF(S_i, U)) V + \frac{i}{2} \sum_{k=1}^{n} F(S_i, S_k) S_k,
\end{align*}
\]

where \(F = dA\) is the curvature of \(A\). For the curvature \(R\) of \(g\) this implies

\[
R(S_k, U, S_i, S_j) = \frac{i}{2} S_k(F(S_i, S_j)) = \frac{i}{2} \langle \nabla S_k F \rangle(S_i, S_j).
\]

Hence, in order to make \(g\) a pp-wave metric, we have to assume that

\[
\nabla F|_{V^\perp \otimes V^\perp \wedge V^\perp} = 0.
\]

Then, for the curvature of \(g\) we obtain

\[
R(S_i, U, S_j, U) = \partial_i \partial_j (H) - \frac{i}{2} \left(S_i(F(S_j, U)) + \sum_{k=1}^{n} F(S_i, S_k) F(S_k, S_j)\right).
\]

Turning to the properties of the screen, \(\nabla_V S_i = 0\) implies that the screen distribution \(S\) defined by \(Z\) is horizontal. In order to show that \(S\) is not involutive, we obtain from equation (22) that

\[Z^\flat([S_i, S_j]) = iF(S_i, S_j).
\]

Hence, the screen defined by \(Z\) is involutive, if and only if

\[F|_{V^\perp \wedge V^\perp} = 0,
\]

or equivalently \(0 = \eta \wedge F\), since \(V^\flat = g(V, \cdot) = \eta\). As an example with condition (24), but with \(\eta \wedge F \neq 0\), in \(F = -2\pi i \pi^* \omega\) we can choose \(\omega \in \Omega^2(T^{n+1})\) as

\[
\omega = \frac{i}{2} \sum_{i,j=1}^{n} a_{ij} \xi^i \wedge \xi^j \neq 0
\]

with constants \(a_{ij} = -a_{ji}\). Moreover, since the \(\xi^i\)'s are not exact, we can choose these constants such that \([\omega] \neq 0 \in H^2(T^{n+2}, \mathbb{Z}).\) For such an example, the screen defined by \(Z\) is not involutive, since

\[iF = \sum_{i,j=1}^{n} a_{ij} \sigma^i \wedge \sigma^j.
\]
does not vanish on $V^\perp \wedge V^\perp$. Then, $\mathcal{M} = S^1 \times \mathcal{P}$, where $\pi: \mathcal{P} \to T^n$ is the circle bundle with first Chern class $c = [\omega] \in H^2(T^n, \mathbb{Z})$.

Now assume that $\hat{\mathcal{S}} = \text{span}(\hat{S}_1, \ldots, \hat{S}_n)$ is any other screen distribution defined by smooth functions $b_i$ on $\mathcal{M}$ via $\hat{S}_i = S_i - b_i V$. This screen is horizontal since

$$0 = \nabla_V \hat{S}_i = -V(b_i)V,$$

and the functions $b_i$ descend to smooth functions on $T^{n+1}$. Furthermore, if $\hat{\mathcal{S}}$ was involutive, we would have that

$$0 = iF(S_i, S_j) - (db_j(S_i) - db_i(S_j)),$$

on $\mathcal{M}$, or equivalently, for the one-form $\beta = \sum_{i=1}^n \varphi_i \xi^i$ on $T^n$,

$$\omega(\partial_i, \partial_j) = d\beta(\partial_i, \partial_j),$$

where $\varphi_i(x) := b_i(1, y)$ for $y \in \pi^{-1}(x)$. Hence, $\omega = d\beta$ which contradicts $[\omega] \neq 0$.

### 3.3. The universal cover of pp-waves with completeness assumptions

Now we will use the results in the previous section to show that, under some completeness conditions, the universal cover of a pp-waves is globally of standard form.

**Theorem 3.** Let $(\mathcal{M}, g)$ be a pp-wave with parallel null vector field $V$ satisfying the following completeness assumptions:

1. The maximal geodesics along the leaves of the parallel distribution $V^\perp$ are defined on $\mathbb{R}$, and
2. there exists a complete screen vector field $Z$.

Then the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^{n+2}$. Moreover, the universal cover $(\tilde{\mathcal{M}}, \tilde{g})$ is globally isometric to a standard pp-wave

$$g^H = 2dudv + 2H(u, x^1, \ldots, x^n)du^2 + \delta_{ij}dx^ix^j).$$

Under this isometry, the lift $\tilde{V}$ of the parallel vector field $V$ is mapped to $\partial_u$.

**Proof.** By a tilde we shall denote the lift of an object to the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$. First, let $Z$ be the complete screen vector field and $\mathcal{S}$ the corresponding screen distribution on $\mathcal{M}$. Since $Z$ is complete, we can apply Proposition 2 to $Z$ and $\eta := g(V, .)$ and obtain that the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \tilde{N}$, where $\tilde{N}$ is the universal cover of a leaf $\mathcal{N}$ of the distribution $V^\perp$ of $\mathcal{M}$. Clearly, $\tilde{N}$ is also a leaf of the horizontal $\tilde{V}^\perp$ on $\tilde{\mathcal{M}}$. Again, as $Z$ is complete we can apply Theorem 2 and obtain a horizontal and involutive realization $\hat{\mathcal{S}}$ of the screen bundle $\mathcal{S}$ on the universal cover $\tilde{\mathcal{M}}$ and a corresponding screen vector field $\hat{Z} \in \Gamma(T\tilde{\mathcal{M}})$, together with a frame $\hat{S}_i$ of $\hat{\mathcal{S}}$ with

$$\nabla_X \hat{S}_i = 0, \text{ for all } X \in \tilde{V}^\perp.$$

Furthermore, consider the Riemannian metric $\hat{h}$ on $\tilde{\mathcal{N}}$ defined by $\hat{\mathcal{S}}$,

$$\hat{h}(\tilde{V}, \tilde{V}) = 1, \quad \hat{h}|_{\hat{\mathcal{S}} \times \hat{\mathcal{S}}} = g|_{\mathcal{S} \times \mathcal{S}}, \quad \hat{h}(\tilde{V}, .)|_{\mathcal{S}} = 0.$$

Hence, by Proposition 3 the simply connected Riemannian manifold $(\tilde{\mathcal{N}}, \hat{h})$ is flat and admits a frame $(\tilde{V}, \tilde{S}_1, \ldots, \tilde{S}_n)$ of $\nabla^h$-parallel vector fields which are orthonormal with respect to $\hat{h}$. Moreover, by the first completeness assumption and relation (27), they are complete. Using a result by Palais [38, Theorem VIII, Chapter IV] (see also [50].
Proposition 1.9] for a proof), we conclude that the manifold \( \tilde{N} \) has a unique structure of an Abelian Lie group for which the frame and its (in our case vanishing) Lie brackets define the Lie algebra, and therefore, being simply connected, \( \tilde{N} \) is diffeomorphic to \( \mathbb{R}^{n+1} \). This proves the first part of the statement.

The proof of the second part, that the lifted metric \( \tilde{g} \) is isometric to a standard pp-wave, is more involved and requires some auxiliary statements. It follows ideas in [16]. We start with

**Lemma 4.** Let \( (\mathcal{M}, \nabla) \) be smooth manifold with a torsion free connection \( \nabla \) and let \( \delta : I \rightarrow \mathcal{M} \) be a curve in \( \mathcal{M} \) with \( 0 \in I \subset \mathbb{R} \). Then a vector field \( X \in \Gamma(\delta^* T\mathcal{M}) \) along the curve \( \delta \) is parallel along \( \delta \) if and only if the vector field \( Y \in \Gamma(\delta^* T\mathcal{M}) \) with \( Y(t) := t \cdot X(t) \) satisfies \( \frac{\nabla^2}{dt^2} Y(t) = 0 \).

**Proof.** One direction of the proof is trivial, so let us assume that

\[
\frac{\nabla^2}{dt^2} Y(t) = 0.
\]

By the Leibniz rule this implies that

\[
2 \cdot \frac{\nabla}{dt} X(t) + t \cdot \frac{\nabla^2}{dt^2} X(t) = 0
\]

for \( t \in I \). Now let \( E_i(t) := \mathcal{P}_{\delta(t)}^\nabla(e_i) \) with fixed \( x^0 = \delta(0) \in \mathcal{M} \) and a basis \( e_1, \ldots, e_n \) in \( T_{x^0} \mathcal{M} \). Consequently, we can write

\[
X(t) = \sum_{i=1}^n \xi_i(t) \cdot E_i(t),
\]

which implies that \( X \) is parallel along \( \delta \) if \( \xi_i' \equiv 0 \) on \( I \) for all \( i = 1, \ldots, n \). If we write \( X \) in the form \( (30) \), formula \( (29) \) implies that the coefficient functions \( \xi_i \in C^\infty(I) \) of \( X \) must satisfy the ordinary differential equation \( 2\xi_i'(t) + t \cdot \xi_i''(t) = 0 \) with the initial values given by \( X(0) \in T_{x^0} \mathcal{M} \). Each such equation only has the constant solution defined on \( I \): A discussion of the solutions \( y \) on \( (0, \infty) \cap I \) or \( (-\infty, 0) \cap I \) of \( 2y(t) + ty'(t) = 0 \) yields \( |y(t)| = \frac{1}{t^2} \). Therefore, \( y \equiv 0 \) is the only solution, defined on \( 0 \) since otherwise it must be equal to \( \pm \frac{1}{t} \) on \( (0, \infty) \cap I \) or \( (-\infty, 0) \cap I \) - a contradiction. Hence, if \( \xi_i \in C^\infty(I) \) with \( 0 \in I \) solves \( 2\xi_i'(t) + t \cdot \xi_i''(t) = 0 \), then \( y := \xi_i' \) is defined on \( 0 \in I \) and solves the second differential equation of order one. Consequently, \( y = \xi_i' \) is identically zero. \( \square \)

To prove the second part of Theorem \( \mathbb{K} \) let \( \tilde{Z} \) and \( \tilde{S} \) denote the lifts to \( \tilde{\mathcal{M}} \). Let \( \gamma : \mathbb{R} \rightarrow \tilde{\mathcal{M}} \) be the integral curve of the complete vector field \( \tilde{Z} \) through \( x^0 \in \tilde{\mathcal{M}} \) and \( S_1, \ldots, S_n \in \Gamma(\tilde{S}) \) such that \( \tilde{g}(S_i, S_j) = \delta_{ij} \) and \( \nabla S_i = \alpha_i \otimes \tilde{V} \), see Proposition \( \mathbb{A} \). We define the smooth map \( \Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \rightarrow \tilde{\mathcal{M}} \) by

\[
\Phi(u, v, x^1, \ldots, x^n) := \exp_{\gamma(u)}(v \cdot \tilde{V}(\gamma(u)) + \langle x, \tilde{S}(u) \rangle),
\]

where \( \exp \) is the exponential map of \( \tilde{g} \) and where we define \( \langle x, \tilde{S}(u) \rangle := \sum_{k=1}^n x^k S_k(\gamma(u)) \). Then we show

**Lemma 5.** The smooth map \( \Phi \) in \( (31) \) is well defined and a diffeomorphism.
Proof. By the first completeness assumption, the exponential \( \exp_p : \tilde{V}_p^\perp = T_p \tilde{N} \rightarrow \tilde{N} \) is defined on the whole tangent space for each leaf \( \tilde{N} \) through \( p \in \tilde{M} \) and moreover, it is a diffeomorphism, since \( (\tilde{N}, \tilde{\nabla}|_p) \) is a complete, flat and simply connected manifold. Hence, in order to prove that \( \Phi \) is injective, it suffices to show that \( \Phi(u_1, v_1, x) \neq \Phi(u_2, v_2, y) \) for all \( v_1, v_2 \in \mathbb{R} \) and \( x, y \in \mathbb{R}^n \) whenever \( u_1 \neq u_2 \). But for \( u_1 \neq u_2 \) we have

\[
\gamma(u_1) \text{ and } \gamma(u_2) \text{ are contained in two disjoint leaves } \tilde{N}_1 \text{ and } \tilde{N}_2, 
\]

respectively. To see this, recall that — as we have seen in the proof of Proposition 8 — it holds \( \tilde{N} = df \) for some \( f \in C^\infty(\tilde{M}) \) and \( \tilde{N} := \tilde{g}(\tilde{V}, \cdot) \) such that \( f(\gamma(u)) = u + f(x^0) \). In this situation, each leaf is given as a level set of \( f \) and \( \tilde{N}_1 = f^{-1}(u_1 + f(x^0)) \), \( \tilde{N}_2 = f^{-1}(u_2 + f(x^0)) \) which implies that \( \gamma(u_1) \) and \( \gamma(u_2) \) cannot lie within the same leaf. But then

\[
\exp_{\gamma(u_1)}(\tilde{V}_{\gamma(u_1)}) = \tilde{N}_1 \quad \text{and} \quad \exp_{\gamma(u_2)}(\tilde{V}_{\gamma(u_2)}) = \tilde{N}_2
\]

and since \( \tilde{N}_1 \cap \tilde{N}_2 = \emptyset \), this yields \( \Phi(u_1, v_1, x) \neq \Phi(u_2, v_2, y) \) for all \( v_1, v_2 \in \mathbb{R} \) and \( x, y \in \mathbb{R}^n \).

For proving the surjectivity of \( \Phi \), let \( p \in \tilde{M} \) be arbitrary and \( \tilde{N}_p \) be the leaf through \( p \). Then \( f|_{\tilde{N}_p} \equiv c \) for some \( c \in \mathbb{R} \). The point \( (c_0, v, x) \) with \( c_0 := c - f(x^0) \) and \( \exp_{\gamma(c_0)}(v\tilde{V}(\gamma(c_0))) + \langle x, \tilde{S}(c_0) \rangle \) is then a preimage of \( p \).

Now we will show that the pull-back of \( \tilde{g} \) by \( \Phi \) is of the form \( g^H \) as in (26). For \( k = 1, \ldots, n \), let

\[
\mathcal{V}(u, v, x) := d\Phi(u, v, x)(\partial_v),
\]

\[
\mathcal{X}_k(u, v, x) := d\Phi(u, v, x)(\partial_k),
\]

\[
\mathcal{Z}(u, v, x) := d\Phi(u, v, x)(\partial_u)
\]

denote the push-forward vector fields. Then, since the leaves \( \tilde{N} \) of \( \tilde{V}^\perp \) are totally geodesic, we have that \( \mathcal{V}(u, v, x) \in \tilde{V}^\perp_{\Phi(u, v, x)} \) and \( \mathcal{X}_k(u, v, x) \in \tilde{V}^\perp_{\Phi(u, v, x)} \) for \( k = 1, \ldots, n \). Furthermore, along the integral curve \( \gamma \) of \( \tilde{Z} \), we have \( \mathcal{V}(u, 0, 0) = \tilde{V}(\gamma(u)) \) and \( \mathcal{X}_k(u, 0, 0) = \tilde{S}(\gamma(u)) \). Moreover we can show

**Lemma 6.** For each \( (u, v, x) \in \mathbb{R}^{n+2} \), consider the geodesic

\[
\mathbb{R} \ni t \mapsto \delta(t) := \Phi(u, tv, tx) = \exp_{\gamma(u)}(t(v\tilde{V}(\gamma(u)) + \langle x, \tilde{S}(u) \rangle)).
\]

The vector fields \( t \mapsto \mathcal{V}(u, tv, tx) \) and \( t \mapsto \mathcal{X}_k(u, tv, tx) \) are parallel transported along \( \delta \).

**Proof.** For each \( (u, v, x) \in \mathbb{R}^{n+2} \) consider the geodesic variation \( F : \mathbb{R} \times (-\varepsilon, \varepsilon) \rightarrow \tilde{M} \),

\[
F(t, s) = \Phi(u, t(v + s), tx) = \exp_{\gamma(u)} \left( t((v + s)\tilde{V}(\gamma(u)) + \langle x, \tilde{S}(u) \rangle) \right)
\]

of the geodesic \( \delta(t) := F(t, 0) \). The variation vector field along \( \delta \) is given as

\[
t \mapsto \frac{\partial F}{\partial s}(t, 0) = dF|_{(t, 0)} \left( \frac{\partial}{\partial s} \right) = d\Phi(u, tv, tx)(t\partial_v) = t \mathcal{V}(u, tv, tx).
\]

Thus, as the variation vector field of a variation of \( \delta(t) \) by geodesics, \( Y(t) := t\mathcal{V}(u, tv, tx) \) is a Jacobi vector field along \( \delta \). Hence, since \( \delta'(t) \in \tilde{V}^\perp_{\delta(t)} \) as well as \( Y(t) \in \tilde{V}^\perp_{\delta(t)} \), we
have
\[
\nabla^2 dt^2 Y(t) = \tilde{R}(\delta'(t), Y(t))\delta(t) = 0,
\]
by the curvature properties of a pp-wave. We can apply Lemma 4 and obtain that
\[
t \mapsto V(u, tv, tx)
\]
is parallel transported along the geodesic \( \delta \). The same argument, using the geodesic variation
\[
F_k(t, s) := \Phi(u, tv, tx + se_k),
\]
shows that the \( \mathcal{X}_k \) are parallel transported along \( \delta \). \( \square \)

Recall that for \( t = 0 \) we know that \( V(u, 0, 0) = \tilde{V}(\gamma(u)) \) and \( \mathcal{X}_k(u, 0, 0) = S_k(\gamma(u)) \).
On the one hand, since \( \tilde{V} \) is parallel, in particular along \( \delta \), this implies that \( \forall (u, tv, tx) = \tilde{V}(\delta(t)) \) and hence \( \tilde{V} = V \) everywhere on \( \mathcal{M} \). On the other hand, it implies that
\[
\begin{align*}
\tilde{g}_{\phi(u,v,x)}(V(u, v, x), V(u, v, x)) &= \tilde{g}_{\gamma(u)}(\tilde{V}(\gamma(u)), \tilde{V}(\gamma(u))) = 0, \\
\tilde{g}_{\phi(u,v,x)}(V(u, v, x), \mathcal{X}_k(u, v, x)) &= \tilde{g}_{\gamma(u)}(\tilde{V}(\gamma(u)), S_k(\gamma(u))) = 0, \\
\tilde{g}_{\phi(u,v,x)}(\mathcal{X}_i(u, v, x), \mathcal{X}_j(u, v, x)) &= \tilde{g}_{\gamma(u)}(S_i(\gamma(u)), S_j(\gamma(u))) = \delta_{ij},
\end{align*}
\]
and thus
\[
\Phi^*\tilde{g}(\partial_\nu, \partial_\nu) = 0, \Phi^*\tilde{g}(\partial_\nu, \partial_\kappa) = 0 \text{ and } \Phi^*\tilde{g}(\partial_\nu, \partial_\kappa) = \delta_{ij}.
\]
It remains to show that \( \Phi^*\tilde{g}(\partial_\kappa, \partial_\kappa) = 0 \) and \( \Phi^*\tilde{g}(\partial_\kappa, \partial_\nu) = 1 \). For the second equation consider for fixed \( v \in \mathbb{R} \) and \( x \in \mathbb{R}^n \) the variation
\[
\nu(u, s) := \Phi(u, sv, sx)
\]
and let \( \nu_u(u, s) := \partial_{\nu u}(u, s) \) and \( \nu_s(u, s) := \partial_{\nu s}(u, s) \). Observe that \( \nu_s \in \tilde{V}^\perp \) and \( Z(u, sv, sx) = \nu_u(u, s) \). Consequently, by Schwarz’ lemma and the parallelity of \( \tilde{V}^\perp \),
\[
\nabla ds \nu_u(u, s) = \nabla du \nu_s(u, s) \in \tilde{V}^\perp.
\]
This implies
\[
\frac{d}{ds} \tilde{g}_{\nu(u, s)}(\nu_u(u, s), \tilde{V}(\nu(u, s))) \equiv 0,
\]
i.e., \( s \mapsto \tilde{g}(\nu_u(u, s), \tilde{V}(\nu(u, s))) \) is constant and thus equals its value in \( s = 0 \), which is
\[
\tilde{g}_{\nu(u, 0)}(\tilde{\gamma}(u), \tilde{V}(\gamma(u))) = \tilde{g}_{\gamma(u)}(\tilde{Z}(\gamma(u)), \tilde{V}(\gamma(u))) \equiv 1,
\]
since \( \nu(u, 0) = \gamma(u) \), \( \nu_u(u, 0) = \tilde{\gamma}(u) \) and since \( \gamma \) is an integral curve of \( \tilde{Z} \). This proves \( \Phi^*\tilde{g}(\partial_\nu, \partial_\nu) = 1 \).

To see \( \Phi^*\tilde{g}(\partial_\kappa, \partial_\nu) = 0 \), consider the identity
\[
R\tilde{g}(\nu_u, \nu_s) = \nabla ds \nabla du \nu_s - \nabla du \nabla ds \nu_s.
\]
Since \( s \mapsto \nu(u, s) \) is a geodesic for every \( u \in \mathbb{R} \), it holds \( \nabla ds \nu_s = 0 \). Taking into account that \( \nu_s \in \tilde{V}^\perp \) we have by the definition of a pp-wave, see also Proposition 3[11], that \( R\tilde{g}(\nu_u, \nu_s) \nu_s \in \mathbb{R} \cdot \tilde{V} \) and hence \( 32 \) and \( 33 \) yield
\[
\nabla ds \nabla du \nu_u(u, s) = \varphi(\nu(u, s)) \cdot \tilde{V}(\nu(u, s))
\]
Thus, the only non-constant term in the metric $\Phi^*C$ is obtained at Remark 3. Note that, at this stage we do not make a claim about the geodesic completeness of $\gamma(u)$, hence

This finishes the proof of the second statement of Theorem 3.

Finally, this and Lemma 7 imply that

$$\frac{d}{ds}g_{\nu(u,s)}(\sum \frac{d}{ds}v_{u}(u,s), X_k(u,sv,sx)) = \frac{d}{ds}(\sum \frac{d}{ds}v_{u}(u,s), X_k(u,sv,sx)) = 0,$$

because of Lemma 6. Hence,

$$s \mapsto g_{\nu(u,s)}(\sum \frac{d}{ds}v_{u}(u,s), X_k(u,sv,sx))$$

is constant and equals its value in $s = 0$. But for $s = 0$ we have

$$g_{\nu(u,0)}(\sum \frac{d}{ds}v_{u}(u,0), X_k(u,0,0)) = 0,$$

since $X_k(u,0,0) = S_k(\gamma(u))$, and

$$\frac{d}{ds}v_{u}(u,0) = \frac{d}{d\nu}v_{u}(u,0) = \frac{d}{du} \left( \frac{d}{ds} \left( \exp_{\gamma(u)}(s(\nu V(\gamma(u)) + \langle x, \nu S(\gamma(u)) \rangle) \right) |_{s=0} \right)$$

$$= \frac{d}{du} \left( v \cdot \nu V(\gamma(u)) + \sum_{k=1}^{n} x^k S_k(\gamma(u)) \right)$$

$$= \sum_{k=1}^{n} x^k \alpha_k(\gamma(u)) \nu V(\gamma(u)),$$

by the Schwarz lemma, the parallelity of $\nu V$ and the property of $S_k$. Note that we use here that $\sum_{x=1}^{2}(x^k S_k) = x^k \sum_{x=1}^{2} S_k$ since the $x^k$ are constant along the curve $\gamma(u)$. Hence,

$$g_{\nu(u,s)}(\sum \frac{d}{ds}v_{u}(u,s), X_k(u,sv,sx)) \equiv 0.$$

Finally, this and Lemma 6 imply that

$$\frac{d}{ds}g_{\nu(u,s)}(v_{u}(u,s), X_k(u,sv,sx)) = \frac{d}{ds}(\sum \frac{d}{ds}v_{u}(u,s), X_k(u,sv,sx)) = 0.$$

Hence, also $s \mapsto g_{\nu(u,s)}(v_{u}(u,s), X_k(u,sv,sx))$ is constant and as $v_{u}(0,0) = \dot{\gamma}(u)$ we obtain at $s = 0$:

$$g_{\nu(0,0)}(\dot{\gamma}(u), S_k(\gamma(u))) = g_{\nu(t)}(\dot{\gamma}(t), S_k(\gamma(u))) = 0.$$

Thus, the only non-constant term in the metric $\Phi^*\nu V$ on $\mathbb{R}^{n+2}$ is the function $H \in C^\gamma(\mathbb{R} \times \mathbb{R})$ defined by

$$2H := (\Phi^*\nu V)(\partial_u, \partial_u).$$

This finishes the proof of the second statement of Theorem 3. □

Remark 3. Note that, at this stage we do not make a claim about the geodesic completeness of pp-waves satisfying the assumptions of Theorem 3. This will depend on the function $H$. We will give a sufficient condition in Lemma 10 in the next section, which we will then use to establish completeness for compact pp-waves.

Remark 4. In the proof of the first part of Theorem 3 we used the result by Palais that a simply connected manifold which admits a parallelism consisting of complete vector fields with constant Lie brackets, i.e., constant linear combinations of these vector fields, admits a unique Lie group structure, for which the vector fields of the parallelism are left-invariant. For the sake of being self contained, we will prove directly the weaker statement that we need for our proof.

Lemma 7. Let $\mathcal{N}$ be a manifold of dimension $n$ and $X_1, \ldots, X_n$ complete vector fields.
(1) If the $X_i$ commute with each other, i.e., $[X_i, X_j] = 0$, then every vector field $X$ that is a constant linear combination of the $X_i$’s, i.e., $X = \sum_{i=1}^{n} a^i X_i$ with $a^i \in \mathbb{R}$, is complete.

(2) If the $X_i$ are linearly independent at each point in $\mathcal{N}$ and if $h$ is a semi-Riemannian metric on $\mathcal{N}$ such that the $X_i$ are parallel with respect to the Levi-Civita connection $\nabla^h$, then $(\mathcal{N}, h)$ is geodesically complete. In particular, if $\mathcal{N}$ is simply connected, then $(\mathcal{N}, h)$ is isometric to the standard semi-Euclidean metric on $\mathbb{R}^n$.

Proof. (1) Let $X = \sum_{i=1}^{n} a^i X_i$ with $a^i \in \mathbb{R}$ and $\phi^i : \mathbb{R} \times \mathcal{N} \ni (t, p) \mapsto \phi^i_t(p) \in \mathcal{N}$ be the flow of $X_i$. Let $p \in \mathcal{N}$ fixed. Then we claim that the map

$$
\phi : \mathbb{R} \times \mathcal{N} \ni (t, p) \mapsto \phi^1_{a^1 t} \circ \ldots \circ \phi^n_{a^n t}(p) \in \mathcal{N}.
$$

is the flow of $X$ through $p$. Indeed, if we define

$$
g : \mathbb{R} \ni t \longmapsto t \sum_{i=1}^{n} a^i e_i \in \mathbb{R}^n
$$

where $e_i$ is the standard basis of $\mathbb{R}^n$ and

$$
f : \mathbb{R}^n \ni \sum_{i=1}^{n} a^i e_i \longmapsto \phi^1_{a^1 t} \circ \ldots \circ \phi^n_{a^n t}(p) \in \mathcal{N}
$$

then we have that $\phi = f \circ g$. For the differential of $g$ we clearly have

$$
\frac{dg}{dt}(\partial_t) = \sum_{i=1}^{n} a^i e_i \in \mathbb{R}^n.
$$

Since the $X_i$’s commute with each other, their flows commute, i.e., $\phi^i_t \circ \phi^j_t = \phi^j_t \circ \phi^i_t$, which implies that the differential of $f$ is given as

$$
\frac{df}{dt}(\sum_{i=1}^{n} a^i e_i(e_j) = \frac{d}{dt} \left( \phi^i_j(\circ \ldots \circ \phi^n_{a^n t}(p)) \right) \bigg|_{t=0} = X_j(\circ \ldots \circ \phi^n_{a^n t}(p)).
$$

Hence, by the chain rule we obtain

$$
\frac{d}{dt} \phi_t(p) = d(f \circ g)_t(\partial_t) = df_{g(t)} dg_t(\partial_t) = \sum_{i=1}^{n} a^i X_i(\phi_t(p)) = X(\phi_t(p))
$$

as claimed.

(2) Clearly, if the $X_i$ are parallel with respect to $\nabla^h$, they commute with each other and we can apply (1). Now, let $w := \sum_{i=1}^{n} a^i X_i(p) \in T_p \mathcal{N}$ be an arbitrary tangent vector at an arbitrary point $p \in \mathcal{N}$. Then the vector field $X = \sum_{i=1}^{n} a^i X_i$ is parallel and, by (1), complete. Hence, the geodesic starting at $p$ with initial speed $w$ is given by the flow of $X$ through $p$, and thus defined on all of $\mathbb{R}$.

Finally, if $(\mathcal{N}, h)$ admits $n$ parallel complete vector fields, it is flat and geodesically complete, and thus, with the assumption that $\mathcal{N}$ is simply connected, the Killing-Hopf theorem gives us that $(\mathcal{N}, h)$ is isometric to the standard Euclidean vector space of dimension $n$. □
Theorem 4. Let $\mathcal{M}$ be assumed to be compact, every screen vector field is complete. Hence, we have to verify that a compact pp-wave satisfies the other assumption of Theorem 3.

**Theorem 4.** Let $(\mathcal{M}, g)$ be a compact pp-wave with parallel null vector field $V$. Then the maximal geodesics along the leaves of the parallel distribution $V^\perp$ are defined on $\mathbb{R}$.

**Proof.** Again, by a tilde we shall denote the lift of an object to the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$. As $\mathcal{M}$ is compact, $Z$ is complete and we can apply Proposition 8 to $Z$ and $\eta := g(V, \cdot)$ and obtain that the universal cover $\tilde{\mathcal{M}}$ of $\mathcal{M}$ is diffeomorphic to $\mathbb{R} \times \tilde{\mathcal{N}},$ where $\tilde{\mathcal{N}}$ is the universal cover of a leaf $\mathcal{N}$ of the distribution $V^\perp$ of $\mathcal{M}$. Clearly, $\tilde{\mathcal{N}}$ is also a leaf of the distribution $\tilde{V}^\perp$ on $\tilde{\mathcal{M}}$. Since $(\mathcal{M}, g)$ is a pp-wave, the lift $\tilde{\mathcal{S}}$ of the screen distribution $\mathcal{S}$ comes with a global frame field $S_i \in \Gamma(\tilde{\mathcal{S}})$, $i = 1, \ldots, n$, on $\mathcal{M}$ satisfying the relations (see Proposition 8)

$$\tilde{\nabla}_X S_i = \alpha_i(X) \tilde{V}. \quad (34)$$

Note that the $S_i$ are not necessarily lifts of global vector fields on the compact $\mathcal{M}$, however we will show that they are complete.

To this end, consider the Riemannian metric $h$ on $\mathcal{M}$ defined by the original screen distribution $\mathcal{S}$ on $\mathcal{M}$ via

$$h(V, V) = h(Z, Z) = 1, \quad h(V, Z) = h(V, X) = h(Z, X) = 0, \quad h(X, Y) = g(X, Y),$$

for all $X, Y \in \Gamma(\mathcal{S})$. As a Riemannian metric on a compact manifold $\mathcal{M}$ it is geodesically complete, and so is its restriction to the leaves $\mathcal{N}$ of $V^\perp$, see for example [14, Exercise 10.4.28]. Therefore, the lifted Riemannian metric $\tilde{h}$ on $\tilde{\mathcal{N}}$ is geodesically complete. Now one computes that the vector fields $S_1, \ldots, S_n$ on $\tilde{\mathcal{N}}$, which are $\tilde{h}$-orthonormal, span the lifted screen $\tilde{\mathcal{S}}$ and satisfy equation (34), are in fact geodesic vector fields for $(\tilde{\mathcal{N}}, \tilde{h})$. Indeed, from the Koszul formula we get

$$0 = \tilde{g}(\tilde{\nabla}_{S_i} S_i, X) = S_i(\tilde{g}(S_i, X)) + \tilde{g}([X, S_i], S_i) = S_i(\tilde{h}(S_i, X)) + \tilde{h}([X, S_i], S_i) = \tilde{h}(\nabla^{\tilde{h}}_{S_i} S_i, X),$$

for all $X \in \Gamma(T\tilde{\mathcal{N}})$. Here the replacement of $\tilde{g}$ by $\tilde{h}$ is justified since

$$\tilde{g}(S_i, \cdot)|_{T\tilde{\mathcal{N}}} = \tilde{h}(S_i, \cdot)|_{T\tilde{\mathcal{N}}}.$$ 

With $(\tilde{\mathcal{N}}, \tilde{h})$ being geodesically complete and $S_i$ being geodesic vector fields, this yields the conclusion that the $S_i$ are complete vector fields. Now we need

**Lemma 8.** Let $(\mathcal{M}, g)$ be a pp-wave with a complete parallel null vector field $V$ and assume that there is a complete screen vector field $Z$. Then there is a horizontal and involutive realization $\hat{\mathcal{S}}$ of the screen bundle $\Sigma$ on the universal cover $\hat{\mathcal{M}}$, and the leaves $\hat{\mathcal{N}}$ of $\hat{V}^\perp$ are diffeomorphic to $\mathbb{R} \times \hat{\mathcal{S}}$, where $\hat{\mathcal{S}}$ is a leaf of the distribution $\hat{\mathcal{S}}$. In particular, $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^2 \times \hat{\mathcal{S}}$.

**Proof.** Since $Z$ is complete we can apply Theorem 3 and obtain a horizontal and involutive realization $\hat{\mathcal{S}}$ of the screen bundle $\Sigma$ on the universal cover $\hat{\mathcal{M}}$ and a corresponding
screen vector field \( \hat{Z} \in \Gamma(T\hat{M}) \). Furthermore, consider the Riemannian metric \( \hat{h} \) on \( \hat{N} \) defined by
\[
\hat{h}(\hat{V}, \hat{V}) = 1, \quad \hat{h}|_{\hat{S} \times \hat{S}} = g|_{\hat{S} \times \hat{S}}, \quad \hat{h}(\hat{V}, \cdot)|_{\hat{S}} = 0,
\]
and \( \hat{\eta} \in \Gamma(T^*\hat{N}) \) defined by \( \hat{\eta}(X) = \hat{h}(\hat{V}, X) \). Then \( \hat{\eta} \) is closed, since \( \hat{S} \) is integrable and hence
\[
d\hat{\eta}(X,Y) = \hat{h}([X,Y], \hat{V}) = 0.
\]
By assumption, \( \hat{V} \) and hence its lift \( \hat{\hat{V}} \) are complete vector fields. Thus we can again apply Proposition\(^3\) this time to \( \hat{\hat{N}} \), \( \hat{\eta} \) and \( \hat{\hat{V}} \), to conclude the proof. \( \square \)

Let \( \hat{\hat{S}} \) be a horizontal and involutive screen distribution, obtained from Theorem\(^2\) with corresponding Riemannian metric \( \hat{h} \) on \( \hat{N} \). Consider the \( \hat{h} \)-orthonormal vector fields \( \hat{S}_1, \ldots, \hat{S}_n \in \Gamma(\hat{S}) \) with \( \nabla \hat{S}_i|_{\hat{N}} = 0 \) and given by
\[
\hat{S}_i = S_i - b_i \hat{\hat{V}}
\]
for some real functions \( b_i \in C^\infty(\hat{M}) \). According to Proposition\(^5\) \( \hat{\hat{V}} \) together with the \( \hat{S}_i \)'s form a frame of \( T\hat{N} \) consisting of \( \hat{h} \)-parallel vector fields. Using Lemma\(^8\) for these we prove

**Lemma 9.** The vector fields \( \hat{S}_i \) are complete.

**Proof.** We saw that the vector fields \( S_i \) are complete, i.e., we obtain their flows as
\[
\phi^i : \mathbb{R} \times \hat{N} \rightarrow \hat{N}.
\]
Recall that, by Proposition\(^3\) the leaf \( \hat{N} \) is diffeomorphic to \( \mathbb{R} \times \hat{S} \) via
\[
\Psi : p \in \hat{N} \mapsto (\phi(p), \psi_-(p))(p) \in \mathbb{R} \times \hat{S},
\]
were \( \{\psi_t\} \) is the flow of \( \hat{\hat{V}} \) and \( \phi \in C^\infty(\hat{N}) \), such that \( \hat{h}(\hat{\hat{V}}, \cdot)|_{\hat{\hat{V}}} = d\phi \) (see the proof of Proposition\(^3\)). Under this diffeomorphism, the flows \( \{\phi^i\} \) are given as
\[
\phi^i_t(p) := \Psi(\phi^i_t(p)) = (\nu^i_t(p), \phi^i_t(p)),
\]
with
\[
\nu^i_t(p) := \phi^i_t(\phi(p)) = \phi^i_t(\phi(p)), \quad \phi^i_t(p) := \psi_{-\nu^i_t(p)}(\phi^i_t(p)),
\]
both defined for all \( t \in \mathbb{R} \). We do now claim that \( \{\phi^i_t\} \) is the flow of \( \hat{S}_i \). Indeed, on the one hand we have that
\[
\frac{d}{dt}\phi^i_t(p) = (\frac{d}{dt}\nu^i_t(p), \frac{d}{dt}\phi^i_t(p)) = (\frac{d}{dt}\nu^i_t(p), 0) + (0, \frac{d}{dt}\phi^i_t(p)).
\]
On the other hand we compute, using Lemma\(^3\) the chain rule and the linearity of the differential
\[
\frac{d}{dt}\phi^i_t(p) = d\Psi_{\phi^i_t(p)}(S_i(\phi^i_t(p))) = d\Psi_{\phi^i_t(p)}(\hat{S}_i(\phi^i_t(p))) + b_i(\phi^i_t(p))d\Psi_{\phi^i_t(p)}(\hat{\hat{V}}(\phi^i_t(p))).
\]

\(^4\)One can also argue in the following way: From Proposition\(^5\) we know that \( \hat{\hat{V}} \) is a parallel vector field on the Riemannian manifold \( (\hat{N}, \hat{h}) \) but also that \( \hat{\hat{V}} \) as a lift of the complete vector field \( V \) is complete. Hence, as \( \hat{N} \) is simply connected, the flow of \( \hat{\hat{V}} \) separates a line \( \mathbb{R} \) from \( \hat{N} \) with orthogonal complement being the leaves \( \hat{S} \) of the integrable distribution \( \hat{S} \), again proving the lemma.
Temporarily denoting by $\{\xi^i_t\}$ the flow of $\hat{S}_i$, for the first term we get

$$d\Psi_{\phi^i_t(p)}(\hat{S}_i(\phi^i_t(p))) \equiv \frac{d}{dt}\Psi(\xi^i_t(\phi^i_t(p)))|_{t=0} = \frac{d}{dt}(\nu_t(p), \psi_t \circ \xi^i_t(\phi^i_t(p)))|_{t=0}$$

then

$$= \frac{d}{dt}(\nu_t(p), \xi^i_t \circ \psi_t(\phi^i_t(p)))|_{t=0}$$

and

$$= \frac{d}{dt}(\nu_t(p), \xi^i_t(\phi^i_t(p)))|_{t=0}$$

$$= (0, \hat{S}_i(\phi^i_t)) \in \mathbb{R} \oplus T\hat{S}$$

in which we were allowed to commute the flows $\xi^i_t$ and $\psi^i_t$ because of $[\tilde{V}, \hat{S}_i] = 0$. For the second term in (36) we recall that

$$\frac{d}{dt}\tilde{V}(\phi^i_t(p)))|_{t=0} = d\tilde{V}(\phi^i_t(p)) = \hat{h}(\tilde{V}, \tilde{V})|_{\phi^i_t(p)} \equiv 1,$$

which implies $\varphi(\psi(\phi^i_t(p))) = \tau + c$ for a constant $c$. Hence, we get

$$d\Psi_{\phi^i_t(p)}(\tilde{V}(\phi^i_t(p))) = \frac{d}{dt}\Psi(\phi^i_t(p)))|_{t=0}$$

then

$$= \frac{d}{dt}(\varphi(\psi(\phi^i_t(p)))))|_{t=0}$$

and

$$= \frac{d}{dt}(\tau + c, \psi \circ \psi(\phi^i_t(p))))|_{t=0}$$

$$= (1, 0) \in \mathbb{R} \oplus T\hat{S}$$

Both computations show that (36) becomes

$$\frac{d}{dt}\hat{\phi}^i_t(p) = (b_i(\phi^i_t(p)), \hat{S}_i(\phi^i_t)) \in \mathbb{R} \oplus T\hat{S},$$

which, together with (35), shows that $\frac{d}{dt}\hat{\phi}^i_t(p) = \hat{S}_i(\phi^i_t(p))$. Hence, $\hat{\phi}^i_t$ is the flow of $\hat{S}_i$ which is defined on $\mathbb{R}$. This proves the lemma. \qed

Thus, applying Lemma 9 we can proceed as in the proof of Theorem 3 and obtain that, on the simply connected Riemannian manifold $(N, \hat{h})$, we have a frame $(\tilde{V}, \hat{S}_1, \ldots, \hat{S}_n)$ of complete vector fields which are $\hat{V}$-parallel and $\hat{h}$-orthonormal, implying again that $(N, \hat{h})$ is geodesically complete. On $N$ the frame $(\tilde{V}, \hat{S}_1, \ldots, \hat{S}_n)$ is parallel for both, the Levi-Civita connections $\nabla g$ of $g$ and $\hat{\nabla}$ of $\hat{h}$, the connections are equal, and whence, the leaf $\tilde{N}$ of $\tilde{V}$ is geodesically complete for the metric $g$. Hence, the leaves $N$ of $V^{\perp}$ on $\mathcal{M}$ are geodesically complete for $g$. \qed

4.2. Proof of Theorem 3 and Corollary 11. Finally, the proof of Theorem 3 is based on Theorem A and a version of results by Candela et al. [10] adapted to our situation.\footnote{In fact, during the preparation of the paper we learned that Lemma 10 follows from stronger results by Candela et al. [11, Theorems 1 and 2]. However, for the sake of being self-contained we include a proof of the lemma. For further results and comments see [12, 13].}

**Lemma 10.** The pp-wave metric on $\mathbb{R}^{n+2}$ in standard form

$$g^H = 2du(du + H(u,x^1,\ldots,x^n)du) + \delta_{ij}dx^i dx^j$$

is geodesically complete if all second $x^i$-derivatives of $H$ are bounded, $\left|\frac{d^2 H}{dx^i dx^j}\right| \leq c$ for a positive constant $c$ and $1 \leq i, j \leq n$.\footnote{In fact, during the preparation of the paper we learned that Lemma 10 follows from stronger results by Candela et al. [11, Theorems 1 and 2]. However, for the sake of being self-contained we include a proof of the lemma. For further results and comments see [12, 13].}
Lemma 11. Let bounded second derivatives unless they are quadratic and thus a pp-wave. Again, these examples cannot be Ricci-flat, since harmonic functions do not have Lemma 10 provides us with many examples of pp-waves that are not plane waves.

be the metric on the universal cover $R$ in regard to the Ehlers-Kundt problem mentioned in the introduction, Remark 5. so we have in fact that $0 \leq \mathcal{H}(s, \gamma) := \text{grad}_{\mathbb{R}^n} H(s, \gamma)$ is defined on the whole real line. Now, recall the following fact, see for example \cite[Theorem A.1.7]{49}:[10] Let $F : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^n$ be globally Lipschitz on every set of the form $I \times \mathbb{R}^{2n}$, where $I$ is a closed interval, then, for every initial value $(t_0, x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{2n}$ there is a solution $x : \mathbb{R} \to \mathbb{R}^n$ of the initial value problem $\dot{x} = F(t, x, \dot{x})$ with $x(t_0) = x_0$ and $\dot{x}(t_0) = x_1$. We thus have to show, that the function $F : [a, b] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ with $F(s, x, y) = F(s, x)$ defined in \eqref{37} is Lipschitz for arbitrary $a, b \in \mathbb{R}$. Clearly, by the mean value theorem for functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, if every partial derivative of $F$ is bounded, then $F$ is Lipschitz. But every partial derivative in the second argument of $F = (F_1, \ldots, F_n)$ is given by

$$\frac{\partial F_i}{\partial x_j}(t, x) = \frac{\partial}{\partial x_j} \left( \frac{\partial H}{\partial x_i}(t, x) \right),$$

and thus bounded by assumption. We conclude that $F$ must be Lipschitz on every set $[a, b] \times \mathbb{R}^n$ which guarantees that the maximal solutions $\gamma$ of \eqref{37} are defined on $\mathbb{R}$. \hfill $\square$

Remark 5. In regard to the Ehlers-Kundt problem mentioned in the introduction, Lemma \[10] provides us with many examples of pp-waves that are not plane waves. Again, these examples cannot be Ricci-flat, since harmonic functions do not have bounded second derivatives unless they are quadratic and thus a pp-wave.

The proof of Theorem \[11] will follow from

Lemma 11. Let $(\mathcal{M}, g)$ be a compact pp-wave and let $g^H = 2du(dv + H \, du) + \delta_{ij} \, dx^i \, dx^j$ be the metric on the universal cover $\mathbb{R}^{n+2}$ of $\mathcal{M}$ that is globally isometric to the lift of $g$. Then all second covariant derivatives of $H$ in $x^i$-directions are bounded,

$$0 \leq \partial_i \partial_j H \leq c, \quad \text{for all } i, j = 1, \ldots, n.$$

Proof. Let $\phi : (\mathbb{R}^{n+2}, g^H) \to (\mathcal{M}, g)$ denote the isometric universal covering map from Theorem \[11]. Let $Z \in \Gamma(T\mathcal{M})$ be an arbitrarily chosen screen vector field and $\bar{Z} \in \Gamma(T\bar{\mathcal{M}})$ its pullback to $\bar{\mathcal{M}}$. Note that we have particularly shown in Theorem \[11] that $g(d\phi(\partial_u), V) = 1$, and hence we have that

$$d\phi(\partial_u) = Z + \sum_{i=1}^n b_i S_i + cV$$

for smooth functions $b_i, c \in C^\infty(\bar{\mathcal{M}})$ and $S_i$ a basis of the screen distribution corresponding to $Z$. Now we define a symmetric $(0, 2)$-tensor field on $\mathcal{M}$ as

$$Q(X, Y) := R(X, Z, Z, Y).$$

Since $\mathcal{M}$ is compact, the function $Q(Q, Q)$, where $Q$ denotes the metric induced by $g$ on $(0, 2)$-tensor fields, is bounded, i.e., $-C^2 < Q(Q, Q) < C^2$ for some constant $C \in \mathbb{R}^+$. Computing $Q(Q, Q)$ in a frame $V, Z, E_1, \ldots, E_n$ with $E_i$ an orthonormal frame of the screen defined by $Z$, the obvious equation $Q(V, \cdot) = 0$ gives us

$$Q(Q, Q) = \sum_{i,j=1}^n Q(E_i, E_j)^2 = \sum_{i,j=1}^n R(E_i, Z, Z, E_j)^2,$$

so we have in fact that $0 \leq Q(Q, Q) < C^2$. 

Pulling back $Q$ to the universal cover $(\mathbb{R}^{n+2}, g^H)$ by the isometric covering map $\phi$, using \cite{35, 17} and \cite{11}, we get that $\phi^* Q(\partial_v, \partial_i) = 0$ and

$$\phi^* Q(\partial_i, \partial_j)_x = R_{\phi(x)}(d\phi_x(\partial_i), Z, Z, d\phi_x(\partial_j)) = R_{\phi(x)}(d\phi_x(\partial_i), d\phi_x(\partial_u), d\phi_x(\partial_u), d\phi_x(\partial_j)) = \phi^* R_x(\partial_i, \partial_u, \partial_u, \partial_j) = R^H_x(\partial_i, \partial_u, \partial_u, \partial_j) = -\partial_i \partial_j H(x).$$

Hence, by using a frame $(\partial_v, \partial_u - H \partial_v, \partial_i)$ on $(\mathbb{R}^{n+2}, g^H)$ to compute $\overline{g^H} (\phi^* Q, \phi^* Q)$, at each point in $\mathbb{R}^{n+2}$ we have

$$C^2 > \overline{g^H}(\phi^* Q, \phi^* Q) = \sum_{i,j=1}^n \phi^* Q(\partial_i, \partial_j)^2 = \sum_{i,j=1}^n (\partial_i \partial_j H)^2,$$

which shows that all $\partial_i \partial_j H$ are bounded. \hfill $\Box$

**Proof of Theorem \cite{A}** Let $(\mathcal{M}, g)$ be a compact pp-wave. Because of Theorem \cite{A} the universal cover is isometric to a standard pp-wave $(\mathbb{R}^{n+2}, g^H)$, and by Lemma \cite{11} all $\partial_i \partial_j H$ are bounded. Then, by Lemma \cite{11} $(\mathbb{R}^{n+2}, g^H)$ is complete, and thus $(\mathcal{M}, g)$ itself is complete. \hfill $\Box$

Lemma \cite{11} also provides us with a proof of Corollary \cite{11}

**Proof of Corollary \cite{11}** Let $(\mathcal{M}, g)$ be a compact pp-wave and let $(\mathbb{R}^{n+2}, g^H)$ be the standard pp-wave that is globally isometric to the universal cover of $(\mathcal{M}, g)$. Lemma \cite{11} tells us that the $\partial_i \partial_j H$ are bounded. If $g$ is Ricci-flat, so is $g^H$, and thus $H$ is harmonic with respect to the $x^i$-directions, i.e., $\sum_{i=1}^n \partial_i^2 (H) = 0$. But this implies that also $\partial_i \partial_j H$ is harmonic in the same sense, and thus, by the maximum principle for harmonic functions, independent of the $x^i$ components. Hence,

$$H = \sum_{i,j=1}^n a_{ij}(u)x^ix^j + b_i x^i + c$$

with $a_{ij}$, $b_i$ and $c$ functions of $u$ only, which implies that $(\mathcal{M}, g)$ is a plane wave. \hfill $\Box$

**4.3. Plane waves.** Finally, we apply Theorems \cite{A} and \cite{B} to plane waves as defined in Definition \cite{2}.

**Corollary 2.** An $(n+2)$-dimensional compact plane-wave is geodesically complete and its universal cover is isometric to $\mathbb{R}^{n+2}$ with the metric $g^H$ defined in Theorem \cite{A} where $H(u, x) = \sum_{k,l=1}^n a_{kl}(u)x^k x^l$ for some $a_{kl} = a_{lk} \in C^\infty(\mathbb{R})$.

**Proof.** Since plane waves are pp-waves, Theorem \cite{B} implies that compact plane waves are complete. Furthermore, by Theorem \cite{A} we have for the universal covering that

$$R^{g^H}(\partial_i, \partial_u, \partial_u, \partial_j) = \text{Hess } H(\partial_i, \partial_j) = \partial_i(\partial_j(H)).$$

The additional plane wave condition $\nabla R = V^b \otimes Q$ implies for the universal cover that

$$0 = (\nabla_{\partial_k} R^{g^H})(\partial_i, \partial_u, \partial_u, \partial_j) = -\partial_k \partial_i \partial_j (H),$$
since $\nabla_{\partial_k} \partial_u \in \partial_v^\perp$. This implies that
\[
H(u, x) = \sum_{k,l=1}^n a_{kl}(u)x^kx^l + \sum_{k=1}^n b_k(u)x^k + c(u).
\]
Getting rid of the linear and constant terms in this expression is achieved by a coordinate transformation of the form
\[
\tilde{v} = v - \dot{\beta}_i(u)x^i + \gamma(u), \quad \tilde{x}^i = x^i + \beta_i(u), \quad \tilde{u} = u
\]
where $\beta$ and $\gamma$ are obtained by integrating
\[
\ddot{\beta}_i(u) = -b_i(u), \quad \dot{\gamma}(u) = c(u) - \frac{1}{2} \sum_{i=1}^n \dot{\beta}_i(u)^2,
\]
with initial conditions $\beta_i(0) = 0$ and $\gamma(0) = 0$. \hfill $\square$

In view of Corollary 1, note that plane waves in standard form are Ricci flat if and only if the matrix $a_{ij}$ is trace-free.

**Remark 6.** If we weaken the assumption made within this paper that the null vector field $V$ is parallel, i.e. with $\nabla V = \varphi \otimes V$, then a compact Lorentzian manifold with the curvature condition of a pp-wave but with such a recurrent vector field\footnote{In \cite{33} we called these Lorentzian manifolds \textit{pr-waves for plane fronted with recurrent rays}.} is not necessarily complete. This means Theorem B cannot be generalized to compact Lorentzian manifolds with the curvature conditions of a pp-wave but with recurrent null vector field. Even if $\varphi(X) = 0$ for all $X \in V^\perp$ such that the 1-form $g(V, \cdot)$ is still closed, the result is false in general, as the following example shows.

**Example 3.** Consider $\tilde{M} := \mathbb{R}^{n+2}$ endowed with the metric
\[
\tilde{g}(u,v,x_1,\ldots,x_n) := 2dudv - 2H(v,x_1,\ldots,x_n)du^2 + \sum_{i=1}^n dx_i^2
\]
for
\[
H(v,x_1,\ldots,x_n) := \sin(v) - \sum_{i=1}^n a_i(\cos(x_i) - 1)
\]
for constants $a_i$. Being $2\pi$-periodic, the metric $\tilde{g}$ descends to a metric $g$ on the torus $\mathbb{T}^{n+2} := \mathbb{R}^{n+2}/2\pi\mathbb{Z}^{n+2}$. The inextensible (transversal) geodesic
\[
\tilde{\gamma}(t) := (\ln(t), 0, \ldots, 0)
\]
then defines an inextensible geodesic $\gamma : (0, \infty) \rightarrow \mathbb{T}^{n+2}$ on the compact Lorentz manifold $(\mathbb{T}^{n+2}, g)$ by $\gamma(t) := \pi(\tilde{\gamma}(t))$, with $\pi : \mathbb{R}^{n+2} \rightarrow \mathbb{T}^{n+2}$ denoting the canonical projection. For $a_i = 0$ this is a version of the Clifton-Pohl torus. See also results by Sánchez \cite{43} on (incomplete) Lorentzian 2-tori.

However, we do not know, if at least the geodesics along the leaves of $V^\perp$ are all complete. For the case $\varphi(X) = 0$ for all $X \in V^\perp$ our proofs seem to be adaptable to this situation since, in this case, $V^\flat$ is still closed.
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