MIRKOVIĆ-VILONEN CYCLES AND POLYTOPES IN TYPE A

JARED ANDERSON AND MIKHAIL KOGAN

Abstract. We study, in type A, the algebraic cycles (MV-cycles) discovered by I. Mirković and K. Vilonen [MV]. In particular, we partition the loop Grassmannian into smooth pieces such that the MV-cycles are their closures. We explicitly describe the points in each piece using the lattice model of the loop Grassmannian in type A. The partition is invariant under the action of the coweights and, up to this action, the pieces are parametrized by the Kostant parameter set. This description of MV-cycles allows us to prove the main result of the paper: the computation of the moment map images of MV-cycles (MV-polytopes) by identifying the vertices of each polytope.

(Mathematics subject classification number: 14L35)

1. Introduction

Let $G$ be a connected complex algebraic reductive group. The loop Grassmannian $\mathcal{G}$ for $G$ is the quotient $G(\mathbb{K})/G(\mathbb{O})$ where $\mathcal{O} = \mathbb{C}[t]$ is the ring of formal power series and $\mathcal{K} = \mathbb{C}((t))$ is its field of fractions, the ring of formal Laurent series. The same set is obtained using polynomials instead of power series: $G = G(\mathbb{C}[t]/G(\mathbb{C}[t]))$. $G$ may be realized as an increasing union of finite-dimensional complex algebraic varieties [L]; more formally, it has the structure of an ind-scheme whose set of geometric points is $G(\mathbb{K})/G(\mathcal{O})$. In type A, when $G = SL_n$ or $G = GL_n$, there is a simple, well-known lattice model for the loop Grassmannian, described in Section 2.1. It allows one to study $\mathcal{G}$ using finite-dimensional linear algebra, as we will do throughout.

Our main results are concerned with certain finite-dimensional algebraic subvarieties of $G$, which we call MV-cycles. These were discovered by Mirković and Vilonen [MV] and provide a canonical basis of algebraic cycles for the intersection homology of the closures of the strata of $\mathcal{G}$, for the natural stratification by $G(\mathcal{O})$-orbits.

To define MV-cycles, we first need some more notation. Choose a maximal torus $T$ of $G$ and let $Λ = \text{Hom}(\mathbb{C}^*, T)$ be the coweight lattice. A coweight $λ$, viewed as a map from $\mathbb{C}^*$ to $T$, is given by a trigonometric series, which, after substituting $t$ for $e^{iθ}$, produces an element in $G(\mathbb{K})$, and hence in $G$, which we denote by $λ$. These are the $T$-fixed points in $G$. A coweight $λ$, treated as an element of $G(\mathbb{K})$, defines a shift $σ_λ$ of $G$ by left multiplication. This is an isomorphism of $G$, taking any fixed point $α$ to the fixed point $α + λ$.

Denote by $N$ and $N^-$ the unipotent radicals of opposite Borel subgroups of $G$ intersecting along $T$. Each $N(\mathbb{K})$-orbit on $G$ contains a unique $T$-fixed point $α$; similarly for $N^-(\mathbb{K})$. For $α, β ∈ Λ$, set $S_{α,β} = N(\mathbb{K})α ∩ N^-(\mathbb{K})β$; this is nonempty if and only if $α − β$ is a linear combination of positive coroots with nonnegative integer coefficients. We define the MV-cycles with highest coweight $α$ and lowest coweight $β$ to be the irreducible components of the closure of $S_{α,β}$. As shown in [AP], the results of [MV] imply that this definition of MV-cycles is equivalent (up to shifts) to the one in [MV]. Except for this equivalence of definitions, this paper is independent of the results of [MV]. In particular, we provide, in type A, an independent proof of the important purity property of MV-cycles established in [MV], by showing that $S_{α,β}$ is a complex algebraic variety of pure complex dimension equal to the height of $α − β$.

As shown in [AP], the left action of the maximal compact torus $T_K ⊂ T$ on $\mathcal{G}$ can be viewed as a Hamiltonian action whose moment map values lie in Lie($T_K$). Given an MV-cycle $M$, we denote

MK was supported by the NSF Postdoctoral Fellowship.
its image under this moment map by $P_M$ and call it an \textit{MV-polytope}. See \cite{A2} for many examples of MV-polytopes in low rank groups.

Our aim is to provide a detailed, explicit, and useful description and parametrization of MV-cycles and MV-polytopes in type A. This is not straightforward because of the use of irreducible components in the definition. I. Mirković and M. Finkelberg, using work of G. Lusztig, have a different way of finding a dense open subset of each cycle, which works for any type $\mathbf{A}$. The results of Gaussent and Littelmann \cite{GL}, which we learned of when this paper was almost completed, also provide a different explicit description of MV-cycles, which works for any type. It would be very interesting to compare these approaches, particularly with a view to calculating MV-polytopes in all types.

There are many reasons for wanting to understand MV-cycles as explicitly as possible; let us mention three of them. First, these varieties are intrinsically interesting since they provide a canonical basis for representations of the (Langlands dual) group. Second, as shown in \cite{A2}, MV-polytopes allow for combinatorial calculations in representation theory: weight multiplicities and tensor product multiplicities equal the number of MV-polytopes fitting inside a certain region. Third, as conjectured in \cite{A1}, there is a natural Hopf algebra structure on the vector space spanned by (equivalence classes under shifts of) MV-cycles, isomorphic to the algebra of functions on the group $N$. Moreover there seems to be a canonical set of generators and relations, which were calculated in the cases of $Sp_4$ and $SL_n$, $n \leq 5$, by looking at MV-polytopes. As observed by B. Sturmfels and A. Zelevinsky, these provide examples of cluster algebras in the sense of Fomin and Zelevinsky \cite{FZ2}. We hope that the explicit understanding of MV-cycles here will eventually allow for the construction of this cluster algebra structure in type A.

Using the lattice model for the loop Grassmannian in type A, namely when $G = SL_n$ or $G = GL_n$, we provide, in Theorem \textbf{A} a decomposition of $G$ into smooth pieces whose closures are the MV-cycles. The partition is invariant under the action of the coweights and, up to this action, the pieces are parametrized by the Kostant parameter set. The description of the pieces is very explicit and in the spirit of the definition of Bruhat cells in the Grassmannian: just as every Bruhat cell contains those vector spaces for which the dimensions of intersections with a fixed flag is determined by a partition, every piece of our decomposition contains those lattices for which dimensions (understood in the proper sense) of intersections with a fixed collection of infinite-dimensional vector spaces is determined by a Kostant partition.

Using this decomposition, we prove our main result: the computation of MV-polytopes by explicitly identifying their vertices in Theorem \textbf{B}. The key is a combinatorial algorithm presented in Section 2.4 which, given an MV-polytope, constructs certain lower-dimensional MV-polytopes that are faces of it. Repeated application of the algorithm yields the vertices. While the algorithm is combinatorial in nature, we don’t know how to prove some of its properties without using the geometry of the loop Grassmannian, our decomposition in particular.

The paper is largely self-contained, using only linear algebra, combinatorics, and the most basic complex algebraic geometry. One exception is that we take as given, in Section 5 the existence and basic properties of the moment map for the torus action.

We begin, in Section 2. by describing the lattice model for $G$ in type A and use it to state our main results: Theorem \textbf{A} and Theorem \textbf{B}. In Section 3 we discuss certain combinatorial properties of the abovementioned algorithm. Section 4 is the heart of the paper and uses the lattice model to construct open dense sets of points in each cycle. Included here is an explicit identification (Proposition 1.2) of each piece of our decomposition with an open Zariski subset of a product of projective spaces. In Section 5 which is the only one containing results for all types, we discuss the moment map and the moment images of three well-known coarse decompositions of the loop Grassmannian by certain group actions. Theorem \textbf{D} describes the general shape of moment images of subvarieties of $G$. In Section 6 we discuss the relation of our decomposition of the loop Grassmannian to the finer decomposition into torus orbit types. We calculate moment images of torus orbits and, in Theorem \textbf{E} identify all torus orbits inside each MV-cycle for which the moment map image of the orbit’s closure is the whole MV-polytope. We end this section by proving our main results.
Acknowledgments. Both authors thank R. MacPherson, I. Mirković, M. Vybornov and A. Zelevinsky for helpful discussions and suggestions. MK is especially indebted to A. Zelevinsky for introducing him to the subject of cluster algebras and motivating this work by pointing to the connection between MV-cycles and cluster algebras.

2. Main results

In this section we give precise statements of our main results, which are in type A. We partition the loop Grassmannian into smooth pieces such that the MV-cycles are the closures of these pieces. This decomposition is invariant under shifts by coweights, and, up to shifts, the pieces are parametrized by the Kostant parameter set. We give a combinatorial algorithm for constructing the vertices of the MV-polytope corresponding to an element of the Kostant parameter set.

2.1. Lattice model. Following Lusztig [L], we will use the lattice model of the loop Grassmannian $G$ in type A. Points in $G$ are subspaces of a certain infinite-dimensional complex vector space, satisfying some extra conditions. The vector space $X$ is defined by specifying a basis: if $e_1, e_2, \ldots, e_n$ denotes the standard basis for $C^n$, then $X$ is the span of $te_i$ for $1 \leq i \leq n$, $-\infty < j < \infty$, where we regard this as a symbol with two indices. We usually picture these basis vectors in an array consisting of $n$ columns, infinite in both directions. Let $t$ be the invertible linear operator on $X$ that sends each $te_i$ to $t^{i+1}e_i$. Then $G_{\text{GL}}$ consists of those subspaces $Y$ of $X$ such that

1. $Y$ is closed under the action of $t$.
2. $t^N X_0 \subseteq Y \subseteq t^{-N} X_0$ for some $N$, where $X_0$ is the span of those $te_i$ with $j \geq 0$.

We call such $Y$ *lattices*. Three examples of lattices are illustrated by the pictures in Figure 1. Each dot, or set of dots connected by line segments, represents a vector; these, together with all the dots $te_i$, $j \geq 3$ below the pictures are a $C$-basis for the lattice. The first picture represents the torus fixed point associated, by our convention, to the coweight $(2,0,-2,0,-1,-2)$. The second picture represents the lattice that is generated as a $C[t]$-module by the vectors $t^{-2}e_1 + 3e_2 + 4e_3, 3te_2 + 4te_3, -te_3 + 6e_5, t^2e_4 + 2te_5 - te_6, t^2e_5$ and $t^2e_6$. (Of course, this set of generators is not unique; for example $t^2e_5$ can be replaced by $t^2e_3$.) The third picture represents the lattice that is generated as a $C[t]$-module by the vectors $e_2 + 2e_3 + 3e_4 + 2e_5 + t^{-2}e_6, t^2e_1 + t^2e_2 + te_2 + 2te_3 + 3te_4, te_5 + t^{-1}e_6, e_6, t^2e_2$ and $t^2e_3$.

![Figure 1. Examples of lattices.](image)

The space $G_{\text{GL}}$ is the loop Grassmannian for $GL_n$: one checks that $GL_n(C[t, t^{-1}])$ acts transitively on lattices and that the stabilizer of $X_0$ is $GL_n(C[t])$. The action is the obvious one suggested by the notation: $GL_n(C)$ acts on $e_1, \ldots, e_n$ by the standard representation, and letting this commute with $t$ provides an action of $GL_n(C[t, t^{-1}])$ on $X$, which induces an action on lattices.

The relative dimension of a lattice $Y$ with respect to $X_0$ is defined to be $\dim(Y/Y \cap X_0) - \dim(X_0/Y \cap X_0)$. The lattices in Figure 1 have relative dimensions $-3, -6$ and $-9$. The connected components of $G_{\text{GL}}$ are parametrized by relative dimension and are isomorphic to each other by appropriate shifts $\sigma_\lambda$. The component consisting of the lattices with relative dimension 0 is the loop Grassmannian $G_{\text{GL}} \subset G_{\text{GL}}$ for $SL_n$: one checks that $SL_n(C[t, t^{-1}])$ acts transitively on such lattices and that the stabilizer of $X_0$ is $SL_n(C[t])$.

Let $\Lambda_{\text{GL}} = \mathbb{Z}^n$ and $\Lambda_{\text{GL}} = \{ (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\text{GL}} | \sum_{i=1}^n \lambda_i = 0 \}$ denote the coweight lattices of $GL_n$ and $SL_n$. For any lattice $Y$, define $\delta(Y)$ to be the $n$-tuple of integers $(\delta_1(Y), \ldots, \delta_n(Y))$ where $\delta_i(Y)$

The space $G_{\text{GL}}$ is the loop Grassmannian for $GL_n$: one checks that $GL_n(C[t, t^{-1}])$ acts transitively on lattices and that the stabilizer of $X_0$ is $GL_n(C[t])$. The action is the obvious one suggested by the notation: $GL_n(C)$ acts on $e_1, \ldots, e_n$ by the standard representation, and letting this commute with $t$ provides an action of $GL_n(C[t, t^{-1}])$ on $X$, which induces an action on lattices.

The relative dimension of a lattice $Y$ with respect to $X_0$ is defined to be $\dim(Y/Y \cap X_0) - \dim(X_0/Y \cap X_0)$. The lattices in Figure 1 have relative dimensions $-3, -6$ and $-9$. The connected components of $G_{\text{GL}}$ are parametrized by relative dimension and are isomorphic to each other by appropriate shifts $\sigma_\lambda$. The component consisting of the lattices with relative dimension 0 is the loop Grassmannian $G_{\text{GL}} \subset G_{\text{GL}}$ for $SL_n$: one checks that $SL_n(C[t, t^{-1}])$ acts transitively on such lattices and that the stabilizer of $X_0$ is $SL_n(C[t])$.

Let $\Lambda_{\text{GL}} = \mathbb{Z}^n$ and $\Lambda_{\text{GL}} = \{ (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\text{GL}} | \sum_{i=1}^n \lambda_i = 0 \}$ denote the coweight lattices of $GL_n$ and $SL_n$. For any lattice $Y$, define $\delta(Y)$ to be the $n$-tuple of integers $(\delta_1(Y), \ldots, \delta_n(Y))$ where $\delta_i(Y)$
is the largest $j$ such that $t^{-j} e_i \in Y$. Let $Y_0$ denote the vector space spanned by all $t^{-j} e_i \in Y$ and let $\dim_0 Y = \dim(Y/Y_0)$. Notice that for $Y \in G_{GL}$ we have $\delta(Y) \in \Lambda_{GL}$; for $Y \in G_{SL}$ we have $\delta(Y) \in \Lambda_{SL}$ if and only if $Y = Y_0$. Conversely, for each coweight $\lambda$, the lattice $\Lambda$ is spanned by all $t^{-j} e_i$ with $j \leq \lambda_i$. We choose the maximal torus consisting of diagonal matrices, and opposite Borels consisting of upper and lower triangular matrices. Then the dominant coweights are those $\lambda$ for which $i_1 < i_2$ implies $\lambda_{i_1} \geq \lambda_{i_2}$; the coweight is called strictly dominant if $i_1 < i_2$ implies $\lambda_{i_1} > \lambda_{i_2}$.

2.2. Kostant parameter set. Now we define the parameter set. Recall that the Dynkin diagram for $SL_n$ is $n-1$ dots in a row, one for each simple root (connected by line segments, which we will not draw). We denote a positive root by a loop around a sequence of consecutive dots in the Dynkin diagram, and we call the number of dots it encloses the length of the loop. The simple roots are the loops of length 1. The other positive roots are loops of length $\geq 2$; each is the sum of the simple roots corresponding to the dots it encloses. (Of course the word loop here has nothing to do with its use in loop Grassmannian.) A Kostant picture is a picture of the Dynkin diagram together with a finite number of such loops. We draw the loops so that if the dots contained in one loop are a proper subset of the dots contained in another, then the one is encircled by the other; if two loops contain precisely the same dots, we still draw one encircled by the other. In this way, the loops in a Kostant picture are partially ordered by encirclement. We write $L \subset L'$ if loop $L'$ encircles loop $L$, and $L \subset L'$ if either $L = L'$ or $L'$ encircles $L$.

Define the length $\text{len}(p)$ of a Kostant picture $p$ to be the sum of the lengths of the loops in $p$, and $|p|$ to be the number of loops in $p$. The Kostant parameter set $K$ is the collection of all Kostant pictures. Examples of Kostant pictures for $n = 6$ are in Figure 2. If $\alpha_{ij}$ is the root that is the sum of simple roots $\alpha_i + \cdots + \alpha_{j-1}$, then these three pictures represent the following Kostant partitions: $\alpha_3 + \alpha_2 + \alpha_3 + \alpha_5 + 3\alpha_{46}, \alpha_2 + \alpha_3 + \alpha_{35} + \alpha_{46}$ and $\alpha_5 + \alpha_2 + \alpha_4 + \alpha_{25} + \alpha_{26}$.

![Figure 2. Examples of Kostant pictures.](image)

It will be useful to imagine each of the $n-1$ dots of the Dynkin diagram as lying on the boundary between two of the $n$ columns of basis vectors for $X$. Then, associated to each loop $L$, will be the vector subspace $V_L$ spanned by the columns the loop passes through. To be precise, for each $i = 1, 2, \ldots, n$, let $V_i$ be the span of the basis vectors $t^j e_i$ in the $i$th column. If we number the dots of the Dynkin diagram 1, 2, $\ldots, n-1$ and if a loop $L$ contains dots $\ell, \ell + 1, \ldots, r - 1$ then $V_L = V_\ell \oplus V_{\ell+1} \oplus \cdots \oplus V_r$. We say that the loop $L$ has its left end at column $\ell$, its right end at column $r$, and passes through columns $\ell, \ell + 1, \ldots, r$.

2.3. Parametrization of MV-cycles. We partition $G_{GL}$ into pieces parametrized by $K \times \Lambda_{GL}$ by defining a map which sends $Y \in G_{GL}$ to $(p(Y), \lambda(Y)) \in K \times \Lambda_{GL}$ as follows. Let $p(Y)$ be the Kostant picture in which the number of loops encircling the same dots as a loop $L$ is

$$n_L = \dim_0(Y \cap V_L) - \dim_0(Y \cap V_{L+}) - \dim_0(Y \cap V_{L-}) + \dim_0(Y \cap V_{L \pm})$$

$$= \dim \left( \frac{(Y \cap V_L)}{(Y \cap V_{L+}) + (Y \cap V_{L-})} \right).$$

In the above expressions, $L_+, L_-$ and $L_{\pm}$ are the loops obtained from $L$ by removing the leftmost, rightmost, or both of these dots respectively. (If $L$ has length 1 then $V_{L+}, V_{L-}$ should be taken as the zero vector space; if $L$ has length 1 or 2, then $V_{L_{\pm}}$ should be taken as the zero vector space.) Also, we are viewing $Y \cap V_L$ as a lattice in $V_L$, as we shall throughout the paper. We let $\lambda(Y) = (\lambda_1, \ldots, \lambda_n)$ be the coweight such that $\lambda_i = \delta_i(Y) + l_i$ where $l_i$ is the number of loops in $p(Y)$ whose left end is at column $i$. Then the loop Grassmannian $G_{GL}$ decomposes into pieces $M(p, \lambda)$, each of which contains those lattices $Y$ with $(p(Y), \lambda(Y)) = (p, \lambda)$. Notice that both $\lambda(Y)$ and $Y$ have the same relative
dimension $|p|$ with respect to $Y_0$; therefore if $Y \in G_{SL}$ then $\lambda(Y) \in \Lambda_{SL}$. So the parametrization restricts from $\Lambda_{GL}$ to $\Lambda_{SL}$: we have $Y \in G_{SL} \mapsto (p(Y), \lambda(Y)) \in \mathbb{K} \times \Lambda_{SL}$, and if $\lambda \in \Lambda_{SL}$ then $M(p, \lambda) \subset G_{SL}$.

A related decomposition of $G_{GL}$ is the one into torus orbit types: let $G^T$ be the set of torus fixed points, and, for a subset $S$ of $G^T$, let $M_S$ be the set of lattices $Y$ such that $S$ is the set of fixed points contained in the closure of the torus orbit through $Y$. The loop Grassmannian is the disjoint union of all nonempty $M_S$'s. Our first theorem identifies MV-cycles as the closures of the pieces $M(p, \lambda)$ and states that the decomposition of the loop Grassmannian into the $M_S$'s is a refinement of the decomposition into the $M(p, \lambda)$'s.

**Theorem A.** Every $M(p, \lambda)$ is a smooth algebraic variety of dimension $\text{len}(p)$ and is a union of a finite number of $M_S$'s. The MV-cycles with highest coweight $\lambda$ are $\{M(p, \lambda) \mid p \in \mathbb{K}\}$.

Remarks.

(1) In Proposition 4.2, we will canonically identify each $M(p, \lambda)$ with an open Zariski subset of a product of $|p|$ projective spaces.

(2) The usual Grassmannian of $k$-planes in $\mathbb{C}^n$ also has a decomposition into torus orbit types, defined just as above; this is discussed in [GGMS]. It is still unknown, even for the Grassmannian, which $M_S$ are nonempty. In the case of the loop Grassmannian, using Theorems 4 and 5 it will be easy to identify certain sets $S$ for which $M_S$ is nonempty.

The $M_S$ decomposition is invariant under the action of the affine Weyl group, but the $M(p, \lambda)$ decomposition is of course not; it is invariant only under the affine part, namely translations. But it is easy to construct from it a finer, Weyl-invariant decomposition: for every element $w$ of the (ordinary) Weyl group $W$ there exists a decomposition of the loop Grassmannian into pieces $M^w(p, \lambda)$, each of which is $M(p, \lambda)$ acted upon by $w$. Intersecting these $|W|$ decompositions—so that each piece has the form $\bigcap_{w \in W} M^w(p, \lambda)$—is a Weyl-invariant decomposition of the loop Grassmannian. In the remark after Lemma 6.4, we will show that this decomposition is identical with the $M_S$ decomposition.

(3) Using Theorem A together with the definition of MV-cycles, it is easy to identify all lattices in an orbit $N(K)\Lambda$. The resulting description of the orbit was originally obtained in [Ngo]. □

2.4. Vertices of MV-polytopes. Fix $(p, \lambda) \in \mathbb{K} \times \Lambda_{GL}$; by Theorem A there corresponds an MV-cycle $M = M(p, \lambda)$ and, applying the moment map, an MV-polytope $P_M = P(p, \lambda)$. We will describe $P_M$ by constructing a map from the Weyl group $W$ onto its vertices; $W$ is the group of permutations of $\{1, 2, \ldots, n\}$ and the vertices are coweights.

Fix $w \in W$. We will give an $n$-step construction of the corresponding vertex $\nu = \nu(w)$. In terms of the lattice $\nu$, the idea is essentially simple: starting with column $w(1)$ we put in it as many basis vectors as possible, according to the length of the longest chain of nested loops in $p$ passing through that column; then, after removing these loops from $p$ in a certain sense, we put as many basis vectors as possible in column $w(2)$; and so on up to column $w(n)$. But it will take a little work to state this precisely.

An example is drawn in Figure 5; the Kostant picture $p$ is drawn to the right of the corresponding MV-polytope; the directions of the two simple roots are shown on the left; the vertices $\nu(w)$ are labelled by the pictures of the lattices $\nu(w)$, with Weyl group element $w$ written below (an $i$ written below column $j$ means $w(i) = j$); the top right vertex is the highest coweight $\lambda$.

To define $\nu(w)$, we first construct, for any Kostant picture $p$ in type $A_{r-1}$ and for any column $i$ ($1 \leq i \leq r$), a Kostant picture $p_i$ in type $A_{r-2}$, called the collapse of $p$ along column $i$. Let $q$ be the Kostant picture consisting of all loops in $p$ that pass through column $i$, and rank these loops by level of encirclement: level $1$ consists of all those loops in $q$ that don’t encircle any loops in $q$; level $2$ consists of all those loops in $q$ that don’t encircle any loops in $q$ except those in level $1$; level $3$ consists of all those loops in $q$ that don’t encircle any loops in $q$ except those in levels $1$ and $2$; etc.

Fix an arbitrary level. Notice that the loops in this level do not encircle each other and thus are naturally ordered from left to right, say $L_1, L_2, \ldots, L_k$. Form a new sequence with one fewer loop, $M_1, M_2, \ldots, M_{k-1}$ as follows: each $M_i$ is the join of loops $L_i$ and $L_{i+1}$, meaning that the left end
of $M_i$ is the left end of $L_i$ and the right end of $M_i$ is the right end of $L_{i+1}$. (Of course if $k = 1$ then this new sequence is empty.)

Do this for each level, and let $q'$ be the Kostant picture consisting of all the new loops formed for all levels. Let $p'$ be the Kostant picture obtained by replacing $q$ by $q'$; that is, $p'$ consists of all loops that are in $p$ but not in $q$ together with all loops in $q'$. It is easy to see that no loop of $p'$ has its left end or its right end at column $i$. So if we simply cross out column $i$ and identify the two dots of the Dynkin diagram on either side, we see that $p'$ may be viewed as a Kostant picture in one rank lower, namely type $A_{r-2}$. (In fact it is the Kostant picture corresponding to a particular face of the MV-polytope.) This is the Kostant picture $\hat{p}_i$. Notice that $|\hat{p}_i|$ equals $|p|$ minus the number of levels.

This process is illustrated in Figure 4 for the Kostant picture $p$ at the left of the figure and $i = 3$ as indicated by the arrow; $\hat{p}_3$ is drawn on the right; $q$ contains eight of the twelve loops, which have been separated into four levels in the second column of the figure.

Now we successively collapse the columns in the order determined by $w$. For an ordered subset $I = (i_1, \ldots, i_k)$ of $\{1, \ldots, n\}$ let $\hat{p}_I$ denote the Kostant picture that results from collapsing columns $i_1, \ldots, i_k$ in this order. (Actually, since after each collapse there is one fewer column, and the remaining columns are renumbered, we should, to be precise, say it like this: Let $h(m)$ be the number of $j < m$ with $i_j < i_m$. Then $\hat{p}_I$ is produced by collapsing column $i_1$, then column $i_2 - h(2)$, then column $i_3 - h(3)$ and so on.)

Now we can define $\nu = \nu(w)$. As before, let $l_i$ denote the number of loops in $p$ whose left end is at column $i$. For $1 \leq k \leq n$, let $I_k = (w(1), \ldots, w(k))$, and let $N_{w(k)} = |\hat{p}_{I_k}| - |\hat{p}_{I_{k-1}}|$ denote the number of loops removed at the $k^{th}$ step during the collapse along column $w(k)$. Set

\[ (2.1) \quad \nu_i = \lambda_i - l_i + N_i. \]

Note that if $\lambda \in A_{SL}$ then $\nu \in A_{SL}$ since $\sum \nu_i - \lambda_i = |p| - \sum l_i = 0$.

Notice that if $w$ is the identity permutation then $\nu(w) = \lambda$; this is the highest coweight vertex of the polytope.
To illustrate how $\nu(w)$ is defined, let $p$ be the leftmost Kostant picture in Figure 5, $\lambda$ the coweight $(2, 0, 1, 0, -1, -2)$, and $w$ the permutation $(345612)$—that is $w(1) = 3, w(2) = 4$ and so on. To define $\nu(w)$ we have to perform the collapses shown in Figure 5. So $N_{w(1)} = N_3 = 4$ is the number of loops removed during the first collapse, $N_{w(2)} = N_4 = 4$ is the number of loops removed during the second collapse and so on: $N_{w(3)} = N_6 = 3$, $N_{w(4)} = N_5 = 0$, $N_{w(5)} = N_1 = 1$, and $N_{w(6)} = N_2 = 0$. Hence, by (2.1), $\nu_1 = 2 - 4 + 1, \nu_2 = 0 - 2 + 0$, and so on, so that $\nu$ is the coweight $(-1, -2, 2, 2, -2, 1)$. Note that the first collapse is shown in detail in Figure 4.

**Figure 4.** Construction of $\hat{p}_i$ out of $p$.

**Theorem B.** The vertices of the MV-polytope $P(p, \lambda)$ are $\{\nu(w) \mid w \in W\}$.

**Remark.** I. Mirković and M. Vybornov [MV] have found a decomposition of the loop Grassmannian in type A into the quiver varieties of Nakajima [Na1, Na2]. Like MV-cycles, these varieties are torus invariant and it would be very interesting to understand their moment map images.

3. Combinatorics of collapses

This section explains certain combinatorial properties of the algorithm defined in Section 2.4. Most proofs are deferred until Section 4 since they rely on geometry. We end, however, with a purely combinatorial proof of a key claim used in the proof of Proposition 4.5.

3.1. Properties of the algorithm. Recall that $\hat{p}_i$ denotes the Kostant picture produced by collapsing $p$ along column $i$ and that if $I$ is the ordered set $(i_1, \ldots, i_k)$ then $\hat{p}_I$ is produced by collapsing along columns $i_1, \ldots, i_k$ in this order.

An unexpected property of this algorithm is that it is commutative:

**Theorem C.** For any Kostant picture $p$ and two columns $i_1, i_2$ we have $\hat{p}_{(i_1, i_2)} = \hat{p}_{(i_2, i_1)}$. 

**Figure 5.** Construction of $\nu(w)$. 

The proof of Theorem C will follow from certain geometric properties of the decomposition $M(p, \lambda)$, as explained in the remark preceding Proposition 4.3. We would be interested in knowing a combinatorial proof of it.

The next proposition gives some insight into how the algorithm behaves with multiple collapses and into why Theorem C is true. For a Kostant picture $p$ and an ordered set $I$, let $M$ be a loop of $p$. Define the *ancestry* of $M$ to be the set of all the loops of $p$ used to produce $M$. So if during the collapse of the last column of $I$, loop $M$ is produced by joining two loops $M'$ and $M''$, then the ancestry of $M$ is the union of the ancestries of $M'$ and $M''$. The ancestry depends in general on the order of collapse, as can be seen, for example, by collapsing the first Kostant picture in Figure 2 along $i = 3, 4$ in both orders; in only one of these does the ancestry of the single resulting loop contain the loop of length 1 around the third dot.

Let $L_1, \ldots, L_s$ be the loops from the ancestry of $M$ that do not encircle any other loop from the ancestry. We say that $M$ is the *join* of $L_1, \ldots, L_s$. Since the loops $L_1, \ldots, L_s$ do not encircle each other, we can assume they are ordered from left to right: the left and right ends of $L_m$ are to the left of the corresponding ends of $L_{m+1}$.

**Proposition 3.1.** Suppose a loop $M$ is the join of loops $L_1, \ldots, L_s$. Then the number of columns in the set $I$ that are passed through by at least one of these loops is exactly $s - 1$.

**Remark.** Theorem C and Proposition 3.1 suggest that there should be a non-recursive definition of $p$, that describes the simultaneous collapse $p_I$ along the columns $I$; here $I$ is the underlying unordered set of $I$. Each loop $M$ of $p_I$ should be a join of loops $L_1, \ldots, L_s$ from $p$ not encircling each other, passing through exactly $s - 1$ columns of $I$, and satisfying additional conditions that we only know how to state in the case $|I| = 1$.

Proposition 3.1 follows from Claim 3.2 proved combinatorially below, and the fact, proved geometrically during the proof of Proposition 4.3, that we necessarily have $|J| + 1 = s$ in this claim.

**Claim 3.2.** Suppose a loop $M$ is the join of loops $L_1, \ldots, L_s$. Let $J = (j_1 < \cdots < j_{|J|}) \subseteq I$ be the set of all columns passed through by at least one of these loops. Then $|J| + 1 \leq s$. Moreover, if $|J| + 1 = s$, then loop $L_m$ passes through columns $j_m - 1$ and $j_m$.

**3.2. Proof of Claim 3.2** We say that a loop $L$ is a *top* loop of a Kostant picture if it is not encircled by any other loop of this picture. Without loss of generality we can assume that the top loops of $p$ are the loops $L_1, \ldots, L_s$. Indeed, removing any loop of $p$ that is not encircled by one of the $L_j$ does not change the ancestry of $M$. After this, it is clear that we can also assume that $I = J$; so $J = (j_1 < \cdots < j_k)$ is a reordering of the set $I = (i_1, \ldots, i_k)$.

Set $J_m = (i_1, \ldots, i_m)$ for $1 \leq m \leq k$; so $I = J_k$. We will prove that the number of top loops of $p_{J_m}$ is strictly greater than the number of top loops of $p_{J_{m+1}}$. Since the Kostant picture $p$ has only one top loop, $M$, this will imply the first part of the claim.

By induction, it is enough to prove that $p_{J_1} = p_{J_i}$ has fewer top loops than $p$. First consider those loops of $p$ that do not pass through column $i_1$. Whether or not any particular one of them is a top loop clearly does not change after the collapse along column $i_1$; indeed, if it is a top loop, it cannot be encircled by the join of two loops in the collapse since it does not pass through column $i_1$; and if it is not a top loop it cannot be uncovered by the vanishing of a loop in the collapse since loops in the ancestry—top loops in particular—do not vanish.

Now consider those loops of $p$ passing through column $i_1$. Recall that the algorithm defining $p_{i_1}$ decomposes them into levels and then joins loops in each level. Every loop on the highest level is obviously a top loop of $p$. The algorithm joins loops of the highest level producing top loops of $p_{i_1}$, and the number of loops decreases by one.

Now consider an arbitrary level consisting of loops $R_1, \ldots, R_t$, so that loops $R_m$ and $R_{m+1}$ join to produce loop $S_m$ of $p_{i_1}$. To finish the proof of the first part of the claim, we will show that the number of top loops among $S_1, \ldots, S_{t-1}$ is not greater than the number of top loops among loops $R_1, \ldots, R_t$. This follows immediately from the following two statements:

1. If $R_m$ and $R_{m+1}$ are not top loops of $p$ then $S_m$ is not a top loop of $p_{i_1}$.
(2) If for some \( \ell < r - 1 \), loops \( R_\ell \) and \( R_r \) are not top loops of \( p \), but all the loops \( R_{\ell + 1}, \ldots, R_{r - 1} \) are top loops of \( p \), then loops \( S_r, \ldots, S_{r - 1} \) are not top loops of \( p_i \).

To prove (1) consider rightmost top loop \( L_j \) that encircles \( R_m \) and the leftmost top loop \( L_{j'} \) that encircles \( R_{m + 1} \). During the collapse of column \( i_1 \) every top loop passing through column \( i_1 \) (in particular loops \( L_j \) and \( L_{j'} \)) has to be joined with another loop, since otherwise this top loop will not be in the ancestry of \( M \). If \( L_j \) encircles \( R_{m + 1} \), then any join of \( L_j \) with another loop will encircle \( S_m \). In the case \( L_j \) does not encircle \( R_{m + 1} \), loop \( L_j \) must be to the left of loop \( L_{j'} \). Without loss of generality, we may assume that the level of loop \( L_j \) is not bigger than the level of \( L_{j'} \). Then there are loops on the level of loop \( L_j \) to the right of \( L_j \). In particular, \( L_j \) gets joined with another loop \( L' \) which is to the right of \( L_j \). Loop \( L' \) must encircle at least one loop in the level \( R_1, \ldots, R_t \). Moreover, since we assumed that \( L_j \) is the rightmost top loop that encircles \( R_m \), loop \( L' \) must encircle at least one \( R_{m'} \) with \( m' \geq m + 1 \). So the join of \( L_j \) and \( L' \) encircles \( S_m \).

To prove (2) repeat the argument for (1) with all indices \( m \) and \( m + 1 \) replaced by \( \ell \) and \( r \) and loop \( S_m \) replaced by the loop \( S' \) whose left end coincides with the left end of \( S_t \) and right end coincides with the right end of \( S_{r - 1} \). The whole argument goes through and since loop \( S' \) encircles all the loops \( S_t, \ldots, S_{r - 1} \), the first part of the claim is proved.

For the second part of the claim notice that if \( |J| + 1 = s \) then at each step of the algorithm the number of top loops has to decrease by exactly one. So the situation described in (2) can never happen; if it did, the number of top loops would decrease by at least \( r - \ell \), since \( S_t, \ldots, S_{r - 1} \) are not top loops.

For \( 1 \leq a \leq s - 1 \), let \( L_{m_a} \) be the rightmost loop among \( L_1, \ldots, L_s \) passing through column \( j_a \); in other words, \( m_a \) is the number of top loops of \( p \) that pass through one of the columns \( j_1, \ldots, j_a \). We claim that \( m_a > a \).

It is clear that if we first collapse along one of the columns \( j_1, \ldots, j_a \), then the number of top loops passing through one of these columns decreases by one (since the total number of top loops decreases by one at each collapse and all loops which do not pass through these columns remain unchanged).

Let us show that the number of top loops passing through columns \( j_1, \ldots, j_a \) does not increase if our first collapse is along column \( j_b \) with \( b > a \). The only way this number might increase is if one of the loops \( L_m \) with \( m > m_a \) is joined with another loop \( L \) which passes through \( j_a \). If this happens, then \( L \subseteq L_{m - 1} \). In particular, the top loop \( L_{m - 1} \) passes through column \( j_a \) and the top loop \( L_m \) does not. Hence \( m - 1 = m_a \). Moreover, the level of loop \( L_{m_a} \) is not smaller than the level of loop \( L_{m_a + 1} \), so that at least two of the loops \( L_1, \ldots, L_{m_a + 1} \) are in the highest level. Thus, after collapsing, these loops produce at most \( m_a \) top loops passing through one of the columns \( j_1, \ldots, j_a \).

Since at the end \( p_1 \) contains exactly one top loop passing through columns \( j_1, \ldots, j_a \), we conclude, by induction, that \( m_a > a \). By the symmetric argument we conclude that if \( L_{m_a} \) is the leftmost loop passing through column \( j_a \), then \( n_a \leq a \). This proves that each \( L_a \) passes through columns \( j_a - 1 \) and \( j_a \).

\[ \square \]

4. Lattices and their compatibility to Kostant pictures

In this section \( G = GL_n \), so that \( \mathcal{G} = \mathcal{G}_{GL} \); we use the lattice model and the terminology of Sections 2.1, 2.2 and 2.3. We explain the basic relationship between Kostant pictures and lattices, whereby loops correspond to basis vectors. Proofs of the main results will depend on understanding the collection of lattices “compatible” to a Kostant picture, with respect to three possible degrees of compatibility.

4.1. Bases of lattices. We say that a set of vectors \( y_1, \ldots, y_k \in X \) is a basis of a lattice \( Y \), if \( k = \dim_0(Y) \) and

\[ Y = Y_0 \oplus \langle y_1, \ldots, y_k \rangle, \]

where \( \langle y_1, \ldots, y_k \rangle \) is the \( \mathbb{C} \)-span of \( y_1, \ldots, y_k \). Given a basis of \( Y \), every vector \( y \in Y \) can be uniquely written as a linear combination of \( y_1, \ldots, y_k \) plus a vector \( y_0 \in Y_0 \).
4.2. \( p \)-flags. Suppose \( p \) is a Kostant picture and \( Y \) is a lattice with basis \( \{y_L\}_{L \in p} \) indexed by the loops of \( p \). Moreover, assume that for each \( L \in p \), we have \( y_L \in V_L \) and \( t \cdot y_L \in Y_0 \oplus \langle y_L \rangle' \subseteq L \). The collection \( \{Y_L\}_{L \in p} \) of \( t \)-invariant subspaces of \( Y \) defined by

\[
Y_L = (Y_0 \cap V_L) \oplus \langle y_L \rangle' \subseteq L
\]

is called the \( p \)-flag generated by the basis. Note that if \( L \subseteq L' \) then \( Y_L \subseteq Y_{L'} \) and that \( \dim_0(Y_L) \) equals the number of \( L' \in p \) with \( L' \subseteq L \).

For example, the four vectors in the second picture of Figure 1 form a basis that generates a \( p \)-flag for the lattice, where \( p \) is the second Kostant picture of Figure 2 similarly for the third pictures.

By a subpicture \( p' \) of \( p \) we mean a subset of the loops of \( p \); we say it is an inner subpicture if for every loop \( L \) of \( p' \), all the loops of \( p \) encircled by \( L \) are in \( p' \). For a subpicture \( p' \) let \( V_{p'} \) be the sum of all \( V_L \) for loops \( L \) in \( p' \). A \( p \)-flag in \( Y \) naturally extends to give a lattice \( Y_{p'} \) in \( V_{p'} \) for every inner subpicture \( p' \) of \( p \); let \( Y_{p'} \) be the sum of the subspaces \( Y_L \) for \( L \) in \( p' \). It is easy to see that \( \dim_0(Y_{p'}) \) equals the number of loops in \( p' \). In particular, we will use this construction for the inner subpicture \( p^L \) consisting of all loops encircled by a fixed loop \( L \) in \( p \) and we let \( Y^L_0 = Y_{p^L} + (Y_0 \cap V_L) \).

Note that \( Y^L_0 \subseteq Y_L \) and \( \dim(Y_L/Y^L_0) = 1 \).

4.3. Lattices weakly compatible to Kostant pictures. We say that a lattice \( Y \) is weakly compatible to \( p \) if there is a \( p \)-flag \( \{Y_L\}_{L \in p} \) in \( Y \) such that for every loop \( L \) of \( p \)

1. \( \text{Proj}_{Y_L}(Y') \neq \text{Proj}_{Y_L}(Y^L_0) \) for the left and right ends \( i \) of \( L \),

where \( \text{Proj}_{Y_L}(W) \) stands for the orthogonal projection of \( W \) onto \( V_L \), treating the vectors \( t \cdot e_i \) as an orthogonal basis of \( X \). Notice that by \( t \)-invariance, \( \text{Proj}_{Y_L}(Y_L) = \langle t^j e_i \rangle \) for some integers \( j \) and \( k \); condition 1 just says that \( k \) is unique rather than \( k = j \).

For example, the second lattice in Figure 1 is weakly compatible to the second Kostant picture in Figure 2. This would not be the case, however, if, for instance, the vector \( t^2 e_1 + 3 e_2 + 4 e_3 \) were changed to \( 3 e_2 + 4 e_3 \).

Similarity, the third lattice is weakly compatible to the third Kostant picture.

\textbf{Proposition 4.1.} A lattice \( Y \) is weakly compatible to a Kostant picture \( p \) if and only if \( p = \mathbf{p}(Y) \), the Kostant picture associated to \( Y \). Moreover, if \( p = \mathbf{p}(Y) \) then the \( p \)-flag \( \{Y_L\}_{L \in p} \) in \( Y \) is unique.

\textbf{Proof.} Assume we are given a \( p \)-flag \( \{Y_L\}_{L \in p} \) satisfying 1. Let \( L_1 \subseteq \cdots \subseteq L_k \) be all the loops of \( p \) with left end at column \( \ell \) and right end at column \( r \). Then, by definition, the number of such loops \( L \) in \( p(Y) \) is equal to

\[
n_L = \dim \left( \left( Y_0 \cap V_L \right) / \left( \left( Y_0 \cap V_{L_+} \right) + \left( Y_0 \cap V_{L_-} \right) \right) \right) = \dim(Y_L/Y^L_0) = k;
\]

where we used condition 1 in the last two equalities. Hence \( p(Y) = p \).

Conversely, assume that \( Y \) is a lattice with \( p(Y) = p \). We inductively define a \( p \)-flag in \( Y \). For a loop \( L \) of \( p \) that encircles no loops, let

\[
Y_L = Y_0 \cap t^{-1}(Y_0 \cap V_L).
\]

By induction assume the spaces \( Y_L' \) are defined for all loops \( L' \) encircled by loop \( L \) and define a \( p^L \)-flag in \( Y^L_0 \), where the loops in \( p^L \) satisfy condition 1. So, by hypothesis, we must have

\[
Y_0 \cap V_{L_+} \subseteq Y^L_0 \quad \text{and} \quad Y_0 \cap V_{L_-} \subseteq Y^L_0.
\]

Set

\[
Y_L = Y_0 \cap t^{-1}Y^L_0.
\]

Because of 1 no vector in \( Y_L - Y^L_0 \) can restrict trivially to \( V_i \) for the left or right end \( i \) of \( L \); since \( \dim \left( \frac{\text{Proj}_{Y_L}(Y_L)}{\text{Proj}_{Y_L}(Y^L_0)} \right) \leq 1 \) for the left and right ends, we conclude that \( \dim(Y_L/Y^L_0) \leq 1 \).

To show \( \{Y_L\}_{L \in p} \) is a \( p \)-flag it remains to show \( Y_L \neq Y^L_0 \). Assume \( L \) passes through columns \( \ell, \ldots, r \). Let \( L_1, \ldots, L_k \) be all the loops of \( p \) with left end \( \ell \) and right end \( r \). So \( L = L_m \) for some \( m \).
If $Y_L = Y_L^p$, then $Y_L^\circ = Y \cap t^{-1}Y_L^p = Y \cap V_L$. Hence $u_L = \dim \left( (Y \cap V_L)/((Y \cap V_{L+}) + (Y \cap V_{L-})) \right) = m - 1 < k$, a contradiction.

This finishes the construction of the collection $\{Y_L\}_{L \in \mathbb{P}}$. Its uniqueness follows immediately from the fact that for any $p$-flag we have $Y_L \subset Y \cap t^{-1}Y_L^p$ and hence definition 133 is the only one possible.

We have just shown that all lattices in $M(p, \lambda)$ are weakly compatible to $p$. We now identify $M(p, \lambda)$ with an open Zariski subset of a product of projective spaces. For a loop $L$ with left end $\ell$ and right end $r$, let $\mathbb{P}^L$ be the projective space of complex lines inside $\langle \ell, \ldots, r \rangle$. Set $\mathbb{P}^p = \bigoplus_{L \in \mathbb{P}} \mathbb{P}^L$.

If loop $L$ encircles no loop with right end $r$ then let $m = r$; otherwise let $M$ be the largest loop with right end $r$ encircled by $L$, and let $m$ be its left end; note that $m = \ell$ is possible. Let $\mathbb{P}_L$ be the set of lines in $\langle \ell, \ldots, r \rangle$ that project nontrivially onto $\langle \ell \rangle$ and $\langle m \rangle$. Define $\mathbb{P}_p = \bigoplus_{L \in \mathbb{P}} \mathbb{P}_L$.

**Proposition 4.2.** $M(p, \lambda)$ is isomorphic to $\mathbb{P}_p$.

*Proof.* Fix $p$ and $\lambda$. Without loss of generality, we may assume, using a shift by an appropriate coweight, that $\lambda$ is such that $Y_0 = X_0$ for any lattice $Y \in M(p, \lambda)$. We will construct an algebraic map $\pi : \mathbb{P}_p \to \mathcal{G}$ which will provide the required isomorphism with the image $M(p, \lambda)$. To do so, given a point $p \in \mathbb{P}_p$, we will construct a basis $\{Y_L\}_{L \in \mathbb{P}}$ that generates a $p$-flag $\{Y_L\}_{L \in \mathbb{P}}$ satisfying (i).

Moreover, each $y_L$ will be of the form $t^j e_{j'} + z$ for some $j$ and $z \in V_{r+1} \oplus \cdots \oplus V_r$, where $\ell$ and $r$ are the left and right ends of $L$.

The construction proceeds by induction on the number of loops of $p$. If $p$ contains only one loop $L$, say with left end $\ell$ and right end $r$, then $p$ defines a line in $\langle \ell, \ldots, r \rangle$ with nontrivial projections onto $\langle \ell \rangle$ and $\langle r \rangle$. Let $x$ be the unique point in the line $p$ with $\Proj_{\langle r \rangle} x = e_r$. Let $Y_L = Y_0 \oplus t^{-1}p$ and $y_L = t^{-1}x$. Clearly, the constructed lattice $Y = Y_L$ is in $M(p, \lambda)$.

Now assume that a loop $L$ is not encircled by any other loop of $p$. Then $p' = p - \{L\}$ is an inner subpicture of $p$. By inductive assumption, given $p' \in \mathbb{P}_p$, we can construct a $p'$-flag $\{Y_L\}_{L \in \mathbb{P}}'$ for $Y_{p'}$ together with a basis $\{y_L\}_{L \in \mathbb{P}}'$ for which each $y_{L'}$ is of the form $t^j e_{j'} + z$ for some $j$ and $z \in V_{r+1} \oplus \cdots \oplus V_r$, with $\Proj_{V_{r'}} y_{L'}$ nonzero; here $\ell'$ and $r'$ are the left and right ends of $L'$. We need to show how to construct $Y_L$ and $y_L$, given the line $p_L$ in $\langle \ell, \ldots, r \rangle$, the $L$th component of $p$.

Recall that $Y_L$ must lie inside $t^{-1}Y_L^\circ$ and $\dim(Y_L/Y_L^\circ) = 1$. So, if we can identify $t^{-1}Y_L^\circ/Y_L^\circ$ with $\langle \ell, \ldots, r \rangle$ then the line $p_L$ defines a line inside $t^{-1}Y_L^\circ/Y_L^\circ$ and hence defines $Y_L$.

For $\ell \leq j \leq r$, let $L_k$ be the largest loop encircled by $L$ whose left end is at column $k$ and set $y_k = y_k + z_k$ if no loop encircled by $L$ has column $k$ as its left end, set $y_k = 0$. We claim that $y_\ell, \ldots, y_r$ is a $\mathbb{C}[t]$-basis of $Y_L^\circ$; that is, $Y_L^\circ$ is the direct sum of $\langle y_k \rangle$, the $\mathbb{C}[t]$-span of the vector $y_k$.

To prove it, let us show that any vector $y \in Y_L^\circ$ can be uniquely written as a $\mathbb{C}[t]$-linear combination of the vectors $y_k$. Let $k$ be the smallest number such that $\Proj_{V_k} y \neq 0$. Then, since $y_k$ has the form $t^j e_k + z$, there is a unique linear combination $\sum c_j t^j y_k$ such that its difference with $y$ projects trivially onto $V_k$. If we can show that $c_j = 0$ for $j < 0$, then induction on $k$ proves that $y_\ell, \ldots, y_r$ is a $\mathbb{C}[t]$-basis of $Y_L^\circ$. Since $y$ can be written uniquely as a linear combination of basis vectors plus a vector from $Y_0$, it is enough to assume $y$ is a basis vector $y_{L'}$ for a loop $L'$ whose left end is at column $k$. But since $L' \subseteq L_k$, every $c_j$ must be 0 for negative $j$.

This implies that we can identify $t^{-1}Y_L^\circ/Y_L^\circ$ with $(t^{-1}y_\ell, \ldots, t^{-1}y_r)$. Mapping $t^{-1}y_k$ to $e_k$ provides the isomorphism between $t^{-1}Y_L^\circ/Y_L^\circ$ and $\langle \ell, \ldots, r \rangle$. So, as explained before, the line $p_L$ in $\langle \ell, \ldots, r \rangle$ defines $Y_L$. To define the vector $y_L$, let $x_L = e_\ell + \sum_{k=\ell+1}^r a_k e_k$ be the unique point in $p_L$ with $\Proj_{\langle \ell \rangle} x_L = e_\ell$. Then set $y_L = t^{-1}y_\ell + \sum_{k=\ell+1}^r a_k t^{-1}y_k$.

To finish the construction of $\pi$, we have to show that (i) holds for loop $L$. Since $\Proj_{Y_L} t^{-1}y_k \notin \Proj_{Y_L} Y_L^\circ$ we conclude $\Proj_{Y_L} y_L \notin \Proj_{Y_L} Y_L^\circ$; hence (i) holds for the left end of $L$. If $L$ encircles no loop with right end $r$ then $\Proj_{Y_L} y_L = 0$ for $k < r$ so that $\Proj_{Y_L} y_L = a_r t^{-1}y_r = a_r t^{-1}e_r \notin \Proj_{Y_L} Y_L^\circ$. Otherwise, recall that $M$ is the largest loop encircled by $L$ with right end $r$; it has left end $m$, and $p_L$ projects nontrivially onto $\langle m \rangle$. Then $M = L_m$ and $a_m \neq 0$. Since $\Proj_{Y_L} t^{-1}y_m \notin \Proj_{Y_L} Y_L^\circ$ we conclude $\Proj_{Y_L} y_L \notin \Proj_{Y_L} Y_L^\circ$; hence (i) holds for the right end of $L$ in either case.
It remains to show that for every $Y \in M(p, \lambda)$ there is a unique $p \in \hat{\mathbb{P}}^p$ with $\pi(p) = Y$. Let $\{Y_L\}_{L \in \mathcal{P}}$ be the $p$-flag for $Y$. Then from the above discussion it is clear that we can always construct a basis $\{y_L\}_{L \in \mathcal{P}}$ such that for each $L$, and basis vectors $y_i, \ldots, y_r$ as before, we have $y_L = t^{-1}y_r + \sum_{k=\ell+1} a_k t^{-1}y_k$. Given such a basis of $Y$, define $p \in \hat{\mathbb{P}}^p$ by setting $p_L$ to be the line passing through $e_t + \sum_{k=\ell+1} a_k e_k$. Clearly, $\pi(p) = Y$ and such $p$ is unique.

4.4. **Lattices compatible to Kostant pictures.** We say that a lattice $Y$ is compatible to $p$ if it has a $p$-flag $\{Y_L\}_{L \in \mathcal{P}}$ such that for every loop $L$ of $p$,

(i') $\text{Proj}_{Y_i}(Y_L) \neq \text{Proj}_{Y_i}(Y_L')$ for every column $i$ that $L$ passes through.

Since (i) is stronger than (i'), every lattice compatible to $p$ is weakly compatible to $p$. We denote by $M^\dagger(p, \lambda)$ the set of all lattices in $M(p, \lambda)$ that are compatible to $p$.

For example, the third lattice in Figure 1 is compatible to the third Kostant picture in Figure 2. But the second lattice in Figure 1 is not compatible to the second Kostant picture in Figure 2 since (i) fails for column 4 and the loop encircling dots 3 and 4; they are, however, weakly compatible.

**Proposition 4.3.** $M^\dagger(p, \lambda)$ is a dense Zariski open subset of $M(p, \lambda)$.

**Proof.** For an inner subpicture $p'$ of $p$, denote by $M_{p'}^\dagger(p, \lambda)$ the set of all lattices $Y \in M(p, \lambda)$ whose $p'$-flag satisfies (i) for all loops in $p'$. We will show that if $L \in p$ is a loop for which $p' \cup \{L\}$ is also an inner subpicture then $M_{p' \cup L}^\dagger(p, \lambda)$ is a dense Zariski open dense of $M_{p'}^\dagger(p, \lambda)$. The proof then follows by induction on the number of loops of $p$.

Given $Y \in M_{p'}^\dagger(p, \lambda)$, consider the basis and $p$-flag constructed for it in the proof of proposition 4.2. For loop $L$, there was a $\mathbb{C}[t]$-basis $y_i, \ldots, y_r$ of $Y_L^\circ$, and $y_L$ was defined as a linear combination $\sum a_i t^{-1}y_i$ with $a_i = 1$. For $\ell \leq k \leq r$, let $j_k$ be defined by $\text{Proj}_{Y_i}(Y_L^\circ) = \langle t^{j_k} e_k \rangle = \langle t^{j_k} e_k, t^{j_k+1} e_k, \ldots \rangle$. Then condition (i) for loop $L$ and column $k$ is equivalent to $\sum a_i \text{Proj}_{Y_i}(Y_L^\circ) y_i \neq 0$. Combining such conditions for all columns $L$ through $r$ defines a dense Zariski open subset of $M_{p'}^\dagger(p, \lambda)$; this subset is $M_{p' \cup L}^\dagger(p, \lambda)$.

4.5. **Lattices strongly compatible to Kostant pictures.** Recall that in Section 2.4 we defined the collapse $\hat{p}_i$ of a Kostant picture $p$ along column $i$. We also define the collapse $\hat{Y}_i$ of a lattice $Y$ along column $i$ as the intersection of $Y$ with the direct sum of all columns other than $i$. Just as for $\hat{p}_i$, we may view $\hat{Y}_i$ as a lattice one rank lower by removing the collapsed column and renumbering the remaining columns.

**Lemma 4.4.** If $Y$ is compatible to $p$ then $\hat{Y}_i$ is weakly compatible to $\hat{p}_i$.

**Proof.** Let $\{Y_L\}_{L \in \mathcal{P}}$ be the $p$-flag for $Y$ satisfying (i). We must produce a $p_i$-flag $\{\hat{Y}_M\}_{M \in \mathcal{P}_i}$ for the lattice $\hat{Y}_i$ satisfying (i).

As explained in Section 2.4, every loop $M$ of $\hat{p}_i$ is either a loop of $p$ or a join of two loops of $p$. If $M$ is a loop of $p$, set $\hat{Y}_M = Y_M$. If $M$ is the join of loops $L'$ and $L''$, set $\hat{Y}_M = \hat{V}_i \cap (Y_{L'} + Y_{L''})$, where $\hat{V}_i = \bigoplus_{j \neq i} V_j$.

Choose a basis $\{y_L\}_{L \in \mathcal{P}}$ of $Y$ that generates the $p$-flag $\{Y_L\}_{L \in \mathcal{P}}$. For a loop $L$ passing through column $i$, denote by $j_L$ the largest power of $t$ for which $y_L$ projects nontrivially onto $\langle t^{-j_L} e_i \rangle$. Without loss of generality we can assume that $\text{Proj}_{Y_i}(Y_L) = t^{-j_L} e_i$. Indeed, this can be achieved by adding to each $y_L$ a linear combination of the vectors $t^j y_L$ for $j \geq 1$, multiplying $y_L$ by a scalar, and adding a vector from $Y_0$.

Recall that in the definition of $\hat{p}_i$, the loops of $p$ passing through column $i$ were separated into levels, such that no loop encircles a loop from a higher level. Since $\{Y_L\}_{L \in \mathcal{P}}$ satisfies (i), it is easy to see that $j_L$ is equal to $\delta(Y) + \text{level of loop } L$. In particular, if loop $M$ of $\hat{p}_i$ is the join of two loops $L'$ and $L''$, necessarily from the same level, then $j_L = j_{L'} = j_{L''}$. Hence we can define $\hat{y}_M = y_{L'} - y_{L''} \in \hat{Y}_i$. For a loop $M$ of $p_i$ that is a loop of $p$, set $\hat{y}_M = y_M$.

Clearly the basis $\{\hat{Y}_M\}_{M \in \mathcal{P}_i}$ generates the $p_i$-flag $\{\hat{Y}_M\}_{M \in \mathcal{P}_i}$ for $\hat{Y}_i$. It remains to show that for every loop $M$ of $p_i$, property (i) holds. This is obvious if $M$ is a loop of $p$. If $M$ is the join of $L'$
and $L''$, then the left end $\ell$ of $M$ is the left end of $L'$, and $L''$ does not pass through this column. Hence
\[
\Proj_{Y_M} y_M = \Proj_{Y_L} (y_{LL'} - y_{L''}) = \Proj_{Y_L}(y_{LL'}) \notin \Proj_{Y_L}(Y_M^0) = \Proj_{Y_L}(\hat{Y}_M^0).
\]
This proves (i) for the left end, and an analogous argument proves it for the right end. □

The definition of strongly compatible is inductive on the number of columns $n$: For $n = 2$, a lattice $Y$ is said to be strongly compatible to a Kostant picture $p$ if it is compatible to $p$. For $n > 2$ a lattice $Y$ is said to be strongly compatible to $p$ if it is compatible to $p$ and if for every column $i$ the lattice $\hat{Y}_i$ is strongly compatible to $\hat{p}_i$. We denote by $M^1(p, \lambda)$ the set of all lattices in $M^1(p, \lambda)$ that are strongly compatible to $p$.

For example, the third lattice in Figure 1 is compatible but not strongly compatible to the third Kostant picture in Figure 2. Indeed, if we collapse the Kostant picture along column 4, the resulting Kostant picture contains only two loops: the rightmost loop of length 1, which is unchanged, and a new loop $M$. The lattice $\hat{Y}_3$ is generated by two corresponding basis vectors: $te_5 + t^{-1}e_6$ and $t^2e_1 + t^2e_2 - te_5$ (where we have not renumbered the columns after the collapse). Since the latter vector contains no $e_3$ component, we see that (i) fails for loop $M$ and column 3. If, however, we alter the original vector $te_2 + 2te_3 + 3te_4 + te_5$ by adding to it any nonzero multiple of $t^2e_3$, then the lattice becomes strongly compatible to the Kostant picture.

Remark. We will see later, in Theorem 13, that $M^1(p, \lambda)$ is the largest piece of the decomposition into torus orbit types contained in $M(p, \lambda)$: that is, the strongly compatible lattices are precisely those for which the moment map image of the closure of the torus orbit through the lattice is the entire MV-polytope.

We need notation for collapsing lattices along two or more columns. For a set $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ define
\[
V_I = V_{i_1} \oplus \cdots \oplus V_{i_k}.
\]
In particular, if $I = [\ell..r] = \{\ell, \ldots, r\}$ and $L$ is the loop with left end $\ell$ and right end $r$, then $V_I = V_L$.

For a lattice $Y$, set $Y_I = Y \cap V_I$. Denote by $\hat{I}$ the complement of $I$ in $[1..n]$. Let $\hat{Y}_I = Y_I$, the collapse of the lattice $Y$ along columns $I$; in particular when $I = \{i\}$ we get $\hat{Y}_i$.

For collapsing a Kostant picture along several columns $I$, the order in which the columns collapse matters, a priori. Let $I = (i_1, \ldots, i_k)$ be an ordering of $I$. As in Section 2.4 denote by $\hat{p}_I$ the Kostant picture produced out of $p$ by collapsing columns $I$ in the given order.

Remark. Once we have proved, in Proposition 15, that there always exists a lattice $Y$ strongly compatible to $p$, we will know that the order of collapse doesn’t matter; indeed, $\hat{p}_I = p(\hat{Y}_I)$ for any ordering of the set $I$. Thus, we will have proved Theorem 13 from Section 4.

Notice that a lattice $Y$ is strongly compatible to $p$ if and only if $\hat{Y}_I$ is compatible to $\hat{p}_I$ for every ordered set $I$. Indeed, if $Y$ is strongly compatible to $p$, then every $\hat{Y}_I$ must be strongly compatible (hence compatible) to $\hat{p}_I$. Conversely, if the lattice $\hat{Y}_I$ is compatible to $\hat{p}_I$ for every $I$, then, by induction on $n$, we can assume every $\hat{Y}_i$ is strongly compatible to $\hat{p}_i$; therefore $Y$ is strongly compatible to $p$.

Proposition 4.5. $M^1(p, \lambda)$ is a dense Zariski open subset of $M^1(p, \lambda)$.

Proof. We fix $p$ and $\lambda$ throughout the proof. For an ordered set $I = (i_1, \ldots, i_k)$, let $I_m = (i_1, \ldots, i_m)$ for $1 \leq m \leq k$. Let $I$ and $I_m$ denote the corresponding unordered sets. Let $S_I$ be the subset of $M^1(p, \lambda)$ of lattices $Y$ for which $Y_{I_m}$ is compatible to $\hat{p}_{I_m}$ for every $1 \leq m \leq k$. Then, as mentioned before, $M^1(p, \lambda)$ is the intersection of all $S_I$. Since every $S_I$ is clearly an open Zariski subset of $M^1(p, \lambda)$, it is enough to show that every $S_I$ is not empty.

Fix $I$. Let $M^m$ be the loop of $\hat{p}_{I_m}$ whose left end is furthest left; if there is more than one such loop, let it be the largest. Similarly, let $M$ be the largest leftmost loop of $p$. Then, recalling terminology from Section 4, if a loop of $\hat{p}_{I_m}$ is a join of loops, one of which is $M$, this loop must be $M^m$, and we say that the loop $M$ survives until step $m$; that is, $M^m$ is the only loop of $\hat{p}_{I_m}$ whose ancestry contains $M$. 

Set \( q = p - \{ M \} \) and let \( Q = Y_q + Y_0 \) denote the lattice associated to this inner subpicture. It is easy to see that if \( M \) does not survive until step \( m \) then \( q_{\ell m} = p_{\ell m} - \{ M'' \} \)

We will use induction on the number of loops of \( p \), and, within that, induction on the length \( k \) of \( I \) to prove:

(\*) There exists a lattice \( Y \in S_l \) for which \( Q \) is strongly compatible to \( q \).

The base case for the outer induction is for a Kostant picture for no loops, which is trivial. The base case for the inner induction is that there exists a lattice \( Y \) for which \( Q \) is strongly compatible to \( q \). Since \( q \) contains one fewer loop, we inductively know the existence of a lattice \( Q \) strongly compatible to \( q \), which may be extended by a single vector \( y_M \) to give \( Y \).

For the induction on \( k \), pick a lattice \( Y \) that satisfies (\*) for \( I_{k-1} \). First consider the case that \( M \) does not survive until step \( k \). We claim that the same lattice \( Y \) satisfies (\*) for \( I_k \). It is enough to show that \( Y_{I_k} = \tilde{Q}_{I_k} \) since by inductive assumption \( \tilde{Q}_{I_k} \) is compatible to \( \tilde{p}_I = \tilde{q}_I \). Obviously \( \tilde{Q}_{I_k} \subseteq Y_{I_k} \). Since \( Y_{I_{k-1}} \) is compatible to \( \tilde{p}_{I_{k-1}} \), we know that \( Y_{I_k} \) is weakly compatible to \( \tilde{p}_{I_k} \) by Lemma 4.4. Hence \( \dim_0(Y_{I_k}) = |\tilde{p}_{I_k}| = |\tilde{q}_{I_k}| = \dim_0(\tilde{Q}_{I_k}) \), and since \( Y_0 = Q_0 \), we must have \( Y_{I_k} = \tilde{Q}_{I_k} \).

If, however, \( M \) survives until step \( k \), we will show that there exists a lattice arbitrarily close to \( Y \) that satisfies (\*) for \( I_k \). Let \( \{ y_L \}_{L \in \tilde{p}} \) be a basis of \( Y \) that generates its \( \tilde{p} \)-flag. To shorten some notations, let us introduce the extended Kostant picture \( \tilde{p} = p \cup \{ 1..n \} \). So \( L \in \tilde{p} \) means either \( L \) is a loop of \( p \) or an integer between 1 and \( n \). (It might help to think about the integer \( i \) as a loop of length zero passing through the single column \( i \).) Of course we say that \( L \) encircles integer \( i \) if and only if \( L \) passes through column \( i \). Set \( y_i = t^{-v_i}(Y)_L \).

Since for a loop \( L \) of \( p \) we have \( t \cdot y_L \in Y_L \), we can uniquely write the corresponding basis vector as

\[
y_L = y_L(t) + \sum_{L' \in \tilde{p}} a_{L'} y_{L'},
\]

where \( v \equiv 0 \) means \( v - w \in Y_0 \), and \( a_{L'} = 0 \) unless \( L' \subset L \). Conversely, any set of numbers \( a_{L'} \) with this property uniquely defines a lattice. We will perturb \( Y \) by specifying how to change these numbers. As long as the changes we make are arbitrarily small, we may assume that the conditions in (\*) still hold since they are open conditions. So to show that (\*) holds for \( I \), we need only find an arbitrarily small perturbation of numbers \( a_{L'} \) such that \( Y_I \) is compatible to \( \tilde{p}_I \).

Let \( \{ Y_L \}_{L \in \tilde{p}} \) be a \( \tilde{p} \)-flag in \( Y_I \). The columns of \( Y_I \) are naturally identified with those \( i \) not in \( I \), and we will keep this numbering. We need only show that \( Y_{M^k} \) satisfies (\#), since the other \( Y_L \) satisfy it by the second part of the induction hypothesis. By Lemma 4.4, \( Y_I \) is weakly compatible to \( \tilde{p}_I \). So there exists a vector \( y \in Y_I \) such that \( Y_{M^k} = Y_{M^k} \oplus (y) \). If \( y \) satisfies \( \text{Proj}_V \neq \text{Proj}_V Y_{M^k} \) for every \( i \notin I \) that \( M^k \) passes through, then we are done. If not there exists a vector \( v \in Y_{M^k} \), as small as we would like it to be, such that \( y + t^{-v} \) satisfies this. Suppose \( M^k \) is the join of loops \( M = L_1, \ldots, L_s \). We will show how to change the numbers \( a_{L'}^{M_k} \) by small amounts so that \( y \) is replaced by \( y + t^{-v} \).

Let \( m \) denote the Kostant subpicture of \( \tilde{p} \) consisting of \( L_1, \ldots, L_s \) together with all the loops they encircle, and let \( m^* = m - \{ L_1, \ldots, L_s \} \). So \( Y_m = Y_{L_1} + \cdots + Y_{L_s} \) and \( Y_{m^*} = Y_{L_1}^\circ + \cdots + Y_{L_s}^\circ \). By the proof of Lemma 4.4 we know that \( Y_{M^k} = V_I \cap Y_m \) and \( Y_{M^k}^\circ = V_I \cap Y_{m^*} \). So we can uniquely write

\[
y \equiv \sum_{L \in m} b_L y_L,
\]

\[
v \equiv \sum_{L \in m^*} c_L y_L.
\]
If \( b_{L_j} \neq 0 \) for all \( j = 1, \ldots, s \), the perturbation is easy: for every nonzero \( c_L \), pick a loop \( L_j \) that encircles \( L \) and change \( a_{L_j}^L \) to \( a_{L_j}^L + \frac{c_L}{b_{L_j}} \). This changes \( y_{L_1}, \ldots, y_{L_s} \) so that the same linear combination \( \sum_{j \in J} y_j \) now equals \( y + t^{-1}v \); so \( \breve{Y}_j \) is now compatible to \( \breve{p}_j \).

It remains to show that we can perturb \( Y \) slightly so that every \( b_{L_j} \) is nonzero. Let \( J = (j_1, \ldots, j_{|J|}) \subseteq I \) denote those columns of \( I \) passed through by at least one of the loops \( L_1, \ldots, L_s \). Let us show that \( J \) contains exactly \( s - 1 \) elements, which will prove Proposition \ref{prop:small_change}.

Let \( \ell \) be the left end of \( L_1 \) and \( r \) the right end of \( L_s \), and let \( J \) be the complement of \( J \) in \( [\ell, r] \). Consider these finite-dimensional vector spaces:

\[
U = t^{-1}Y_{m^\circ}/Y_{m^\circ},
V = Y_m/Y_{m^\circ},
Z = (Y_{M^\circ} + Y_{m^\circ})/Y_{m^\circ},
W = ((t^{-1}Y_{m^\circ} \cap V_J) + Y_{m^\circ})/Y_{m^\circ}.
\]

Then \( V, Z \) and \( W \) are subspaces of \( U \). Moreover, \( \dim V = s \), \( \dim Z = 1 \), \( \dim W = |\breve{J}| \), and \( \dim U = r - \ell + 1 = |J| + |\breve{J}| \), since for any lattice \( Y^\circ \), the dimension of \( t^{-1}Y^\circ/Y^\circ \) equals the number of columns. Since \( V \cap W = Z \), we have \( \dim V + \dim W \leq \dim U + 1 \), so that \( s - 1 = \dim V - 1 \leq \dim U - \dim W = |J| \). Together with Claim \ref{claim:dim_relations} we conclude \(|J| = s - 1 \). So \( J = (j_1, j_2, \ldots, j_{s - 1}) \) and, by the second part of the claim, each \( L_m \) passes through columns \( j_{m-1} \) and \( j_m \).

Now we will work in the \( s - 1 \) dimensional vector space \( U/W \). Let \( \breve{y}_{L_j} \in U/W \) be the vectors corresponding to \( y_{L_j} \). Then, since \( y \in W \), we clearly have

\[
0 = \sum_{j = 1}^s b_{L_j} \breve{y}_{L_j},
\]

Also let \( \breve{y}_j \in U/W \) correspond to \( t^{-1}y_j \); note that if \( j \in J \) then \( \breve{y}_j \) is nonzero.

Consider perturbations just of the numbers \( a_{L_i}^i \) and \( a_{L_i}^{i-1} \), which are allowed by the second part of Claim \ref{claim:dim_relations}. We claim that there exists an arbitrarily small change of these numbers such that every \( s - 1 \) vectors among \( \breve{y}_{L_1}, \ldots, \breve{y}_{L_s} \) are linearly independent and, in particular, all \( b_{L_j} \) are nonzero.

First, let’s perturb these numbers to make \( \breve{y}_{L_1}, \ldots, \breve{y}_{L_{s-1}} \) independent. We will prove by induction on \( m \) that the vectors \( \breve{y}_{L_1}, \ldots, \breve{y}_{L_m} \) can be made independent in \( U/W_m \), where

\[
W_m = ((t^{-1}Y_{m^\circ} \cap V_{j_m}) + Y_{m^\circ})/Y_{m^\circ},
\]

and \( J_m = (j_1 < \cdots < j_m) \). Indeed, if this is true, then the vectors \( \breve{y}_{L_1}, \ldots, \breve{y}_{L_m}, \breve{y}_{j_{m+1}} \) are linearly independent inside \( U/W_{m+1} \). Moreover, if \( \breve{y}_{L_1}, \ldots, \breve{y}_{L_{m+1}} \) are linearly dependent inside \( W_{m+1} \), the dependency relation must have a nonzero coefficient in front of \( \breve{y}_{j_{m+1}} \); therefore, an alternation of \( a_{L_{m+1}}^{j_{m+1}} \) by any amount, which can be taken arbitrarily small, will make them independent.

For the \( s - 1 \) vectors \( \breve{y}_{L_1}, \ldots, \breve{y}_{L_{s-1}}, \breve{y}_{L_{s+1}}, \ldots, \breve{y}_{L_s} \), the same argument works, except that we have to alter numbers \( a_{L_1}^1, \ldots, a_{L_{s-1}}^{j_{s-1}}, a_{L_{s+1}}^{j_{s+1}}, \ldots, a_{L_s}^{j_s-1} \). Obviously we can successively choose \( a_{L_1}^1, \ldots, a_{L_s}^{j_s-1} \) arbitrarily small so that linear independence simultaneously holds for each of these \( s \) subsets of \( s - 1 \) vectors.

\[\square\]

### 5. Decompositions of \( \mathcal{G} \) and the moment map.

Throughout this section, \( G \) is any connected simply-connected semisimple complex algebraic group, and \( \mathcal{G} \) its loop Grassmannian. This section contains the facts about \( \mathcal{G} \) that we can prove for all types. We review three known decompositions of \( \mathcal{G} \) and discuss the moment map images of the pieces. The main theorem states that all moment map images of compact irreducible algebraic subvarieties of \( \mathcal{G} \) have the same shape: an intersection of certain cones spanned by coroots.
5.1. **Decompositions of \( G \).** First, the left action of \( G(\mathcal{O}) \) on \( G \) decomposes it into orbits. We let \( G_\lambda \) denote the \( G(\mathcal{O}) \)-orbit through \( \lambda \). The coweight \( \lambda \) is determined up to the action of the Weyl group, and we have

\[
G = \bigcup_{\lambda \in \Lambda^+} G_\lambda
\]

where \( \Lambda^+ \) is the set of all dominant coweights. Each closure \( \overline{G_\lambda} \) is a finite-dimensional projective variety and is the union of the \( G_\mu \) with \( \mu \leq \lambda \) and \( \mu \in \Lambda^+ \). (Here \( \leq \) is the usual partial order on coweights; \( \mu \leq \lambda \) means that \( \lambda - \mu \) is a nonnegative integral linear combination of simple coroots.)

Second, we have the finer \textit{Bruhat decomposition} of \( G \) into \( B(\mathcal{O}) \)-orbits, where \( B \) is a Borel subgroup of \( G \). Each orbit contains exactly one torus fixed point \( \lambda \) and is an affine cell \[ \text{AP} \].

\[
G = \bigcup_{\lambda \in \Lambda} B_\lambda
\]

Third, we need the decomposition of \( G \) into \( N(K) \)-orbits, where \( N \) is the unipotent radical of \( B \). Each orbit is infinite-dimensional and again contains exactly one \( \lambda \); we denote it by \( S_\lambda \).

\[
G = \bigcup_{\lambda \in \Lambda} S_\lambda
\]

The closure \( \overline{S_\lambda} \) is the union \( \bigcup_{\mu \leq \lambda} S_\mu \). Note that we get such a decomposition of \( G \) for each element \( w \) of the Weyl group: we set \( S_\lambda^w = wN(K)w^{-1} \). The following lemma uses the torus action to identify in which piece of this decomposition a given \( x \in \mathcal{O} \) lies.

Every coweight \( \beta \) defines a one-parameter subgroup of the maximal torus \( T \) whose elements are given by \( \exp(s\beta) \) for \( s \in \mathbb{C} \). If \( \tau = e^\alpha \in \mathbb{C} - \{0\} \) we write \( \tau^\beta = \exp(s\beta) \).

**Lemma 5.1.** A point \( x \in \mathcal{O} \) is in \( S_\lambda^w \) if and only if

\[
\lim_{\tau \to 0} w\tau^\beta w^{-1}x = \lambda
\]

for all strictly dominant coweights \( \beta \), that is, for dominant coweights in the interior of the positive Weyl chamber.

**Proof.** Let us recall some results from [FZ]. Let \( e_i, h_i, f_i \) for \( i = 1, \ldots, r \) be the standard generators of the Lie algebra \( \mathfrak{g} \) of \( G \). Denote by \( \alpha_i, \ldots, \alpha_r \in \mathfrak{h}^* \) the simple roots of \( \mathfrak{g} \). For \( p \in K \) define

\[
\exp(p\alpha_i) = x_i(p) = \exp(pe_i).
\]

Then the following commutation relation holds

\[
\tau^\beta x_i(p) = x_i(\tau^{\alpha_i(\beta)}p)\tau^\beta
\]

Given a reduced word \( i = (i_1, \ldots, i_k) \) of the longest element \( w_0 \) of the Weyl group, it is shown by Fomin and Zelevinsky in [FZ] that every element \( n \) of \( N \) can be written as a product

\[
n = x_{i_1}(p_{i_1}) \cdots x_{i_k}(p_k)
\]

with \( p_i \in \mathbb{C} \). Moreover, they express the \( p_i \)'s in terms of generalized minors of \( n \). Since generalized minors can be defined for the group \( G(K) \) the same way they were defined for \( G \) in [FZ], it is clear that \( \text{(5.1)} \) holds for \( n \in N(K) \) and \( p_i \in K \).

Assume \( x = wnw^{-1} \cdot \Delta \) and \( n \in N(K) \) decomposes as in \( \text{(5.1)} \); then

\[
\lim_{\tau \to 0} w\tau^\beta w^{-1}x = \lim_{\tau \to 0} w\tau^\beta x_{i_1}(p_{i_1}) \cdots x_{i_k}(p_k) w^{-1}\Delta = \lim_{\tau \to 0} w\tau^{\alpha_i(\beta)p_1} \cdots x_{i_k}(\tau^{\alpha_k(\beta)p_k}) w^{-1}\Delta.
\]

If \( \beta \) is a strictly dominant coweight, then all \( \alpha_i \) are positive integers, so that all \( x_{i_1}(\tau^{\alpha_i(\beta)p_1}) \cdots x_{i_k}(\tau^{\alpha_k(\beta)p_k}) \) approach the identity as \( \tau \) goes to zero. This shows that the above limit is equal to \( \Delta \).

This finishes the proof, since for a fixed \( w \), \( G \) is the union of all \( S_\lambda^w \). \( \square \)
5.2. Moment map images of strata. As explained in [AP], the loop Grassmannian can be thought of as an infinite-dimensional symplectic manifold with a Hamiltonian action of the maximal compact torus $T_K \subset T$. In particular, it is possible to define a moment map $\Phi$ on $G$ with range in the Lie algebra, $\text{Lie}(T_K)$. (Although the usual codomain for moment maps is the dual $\text{Lie}(T_K)^*$, here it is most naturally $\text{Lie}(T_K)$, the vector space in which the coweights of $G$ lie. This is because moment map images are closely related to the representation theory of the Langlands dual group [MV, C], whose weights are identified with the coweights of $G$.) We do not reproduce the definition of the moment map, but state only the two properties of this map from [AP] which we are going to use:

1. The moment map image of the fixed point $\Lambda$ of the $T$ action is the corresponding coweight $\lambda$.
2. For a dominant coweight $\lambda$, the moment map image of both $G_\lambda$ and $B_\lambda$ is the convex hull of the Weyl group orbit containing $\lambda$.

Note that every piece of each of the decompositions of $G$ is $T$-invariant; indeed, this will be true of every variety we consider.

One of our main interests is in the MV-polytopes, which are moment map images of MV-cycles. To compute these, we will need the moment map images of the MV-cycles. Let $C_\lambda$ be the cone inside $\text{Lie}(T_K)$ with vertex $\lambda$ and spanned by the negatives of the simple roots; note that $\alpha \leq \lambda$ if and only if $\alpha \in C_\lambda$. Let $C_\lambda^w$ be the cone produced by acting on $C_\lambda$ by the Weyl group element $w$.

**Lemma 5.2.** The moment map image of $S^w_x$ is $C^w_\lambda$.

**Proof.** It suffices to prove the statement for $S_\lambda$. Using property (2) of the moment map and the fact that $\sigma_\alpha B_\lambda B_\alpha \subseteq S_\lambda$ for any dominant coweight $\alpha$, we see that $\Phi(S_\lambda)$ contains $C_\lambda$. Conversely, suppose $x \in S_\lambda$. Then the closure $\overline{T \cdot x}$ of the torus orbit through $x$ is contained in $S_\lambda$. By [Ati82], $\Phi(\overline{T \cdot x})$ is a convex polytope. Every vertex $\beta$ of this polytope is a coweight contained in $\Phi(S_\lambda)$; hence $\beta \leq \lambda$ and so $\beta \in C_\lambda$. So $C_\lambda$ contains the polytope, which contains $\Phi(x)$. $\square$

5.3. Moment map images of algebraic subvarieties. By an algebraic subvariety of $G$, we mean any algebraic subvariety of one of the $G_\lambda$. Our interest in the following theorem is that it applies to MV-cycles.

**Theorem D.** The moment map image of a compact irreducible torus-invariant algebraic subvariety $Z$ of $G$ is a polytope given by intersecting cones $C^w_\lambda$, one cone for every element of the Weyl group.

**Proof.** The moment map image of every such variety is a convex polytope [BG].

Fix $w$ in the Weyl group. The intersection $Z \cap S^w_x$, if nonempty, contains some torus orbit; since $Z$ is closed, it follows by Lemma 5.1 that $\Lambda \in Z$. So there are only finitely many coweights $\lambda_1, \ldots, \lambda_k$ for which this intersection is nonempty, and all are contained in $\Phi(Z)$; hence $\Phi(Z) \subseteq \bigcup \overline{C^w_\lambda}$ by Lemma 5.2. Among these coweights, there must exist one, $\lambda_w$, that is greater than all the others; for otherwise there would be two of them such that the line segment joining them was not contained in $\Phi(Z)$, contradicting its convexity.

We have just shown that $\Phi(Z)$ lies inside the convex polytope $P = \bigcap_w S^w_x$ and contains the $\lambda_w$’s. To prove equality of the polytopes, it remains only to show that $P$ has no vertices besides the $\lambda_w$’s. Let $\beta$ be any vertex of $P$ and fix a hyperplane through $\beta$ intersecting $P$ only in $\nu$ and generic in the sense that it contains no root direction. This hyperplane determines a set of positive roots and a positive root cone $C$. Near $\beta$, $P$ is the intersection of some set of half-spaces, each on one side of a hyperplane spanned by roots; therefore $P$ must be contained in $C$. Since $C$ is a translation of $C^w_x$ for some $w$, we must have $\beta = \lambda_w$. $\square$

**Remark.** As explained in Section 2.1, every connected component of $G_{\text{GL}}$ is isomorphic to $G_{\text{SL}}$. Hence Theorem D also holds for $G_{\text{SL}}$, even though it is not simply-connected or semisimple. □
6. Moment map images of torus orbits

In this section we study moment map images of torus orbits—in particular of orbits through lattices strongly compatible to a given Kostant picture. This allows us to compute the moment map images of the cycles $\bar{M}(p, \lambda)$ and prove Theorems A and B.

6.1. Moment maps of general torus orbits. For a lattice $Y$, denote by $P(Y)$ the moment map image of the closure of the torus orbit through $Y$. We now describe the vertices of $P(Y)$.

Recall that for a subset $I$ of $\{1, \ldots, n\}$, we defined $Y_I = Y \cap V_I$ as the intersection of $Y$ with the columns $I$. Further define $d_I(Y) = \dim_0(Y_I)$. Given any permutation $w$ of $\{1, \ldots, n\}$, define the coweight $\mu^w(Y) = (\mu_1, \ldots, \mu_n)$ by

$$\mu_{w(j)} = \delta_{w(i)}(Y) + d_{w([i..n])}(Y) - d_{w([i+1..n])}(Y),$$

where $[i..n]$ is short for $\{i, i+1, \ldots, n\}$. Equivalently, $\mu_{w(i)}$ is the largest $j$ such that $t^{-j}c_{w(i)}$ is in $\Proj_{V_{w(i)}}(Y \cap V_{w([i..n])})$.

**Lemma 6.1.** $P(Y)$ is a convex polytope whose vertices are $\mu^w(Y)$.

**Proof.** By Theorem D and Lemmas 5.1 and 5.2, it is enough to show that

$$\lim_{\tau \to 0} \tau^\beta \mu^{-1}(Y) = \mu^w(Y)$$

for any strictly dominant coweight $\beta = (\beta_1 > \cdots > \beta_n)$, where the action of $\tau^\beta$ on a lattice is given by multiplying each $V_i$ by $\tau^{\beta_i}$. Using Weyl group symmetry, it suffices to prove it for the identity permutation id only.

By definition of $\mu^{id}(Y)$ there exists a basis of $Y$ such that exactly $k_i = d_{[i..n]}(Y) - d_{[i+1..n]}(Y)$ vectors $y_1^i, \ldots, y_{k_i}^i$ of this basis lie in $V_{[i..n]}$ and have linearly independent projections onto $V_i$. Moreover, since $\beta = (\beta_1 > \cdots > \beta_n)$, we have

$$\lim_{\tau \to 0} \tau^\beta \langle y_m^i \rangle = \lim_{\tau \to 0} \tau^\beta \langle \tau^{-\beta_i} y_m^i \rangle = \langle \Proj_{V_i} y_m^i \rangle.$$

Therefore

$$\lim_{\tau \to 0} \tau^\beta Y = Y_0 \oplus \lim_{\tau \to 0} \tau^\beta \langle y_m^i \rangle_{1 \leq i \leq n, 1 \leq m \leq k_i} = Y_0 \oplus \langle \Proj_{V_i} y_m^i \rangle_{1 \leq i \leq n, 1 \leq m \leq k_i} = \mu^{id}(Y).$$

Remarks.

(1) Specifying the numbers $\delta_i(Y)$ and $d_I(Y)$ for all $I$ is equivalent to specifying the polytope $P(Y)$. Indeed, Lemma 6.1 gives the vertices in terms of them. Conversely, these numbers are determined by the vertices: $\delta_I(Y)$ equals $\mu_{w(n)}$ for any choice of $w$ with $w(n) = j$; then $d_I(Y) = \sum_{i=k}^n (\mu_{w(i)} - \delta_{w(i)}(Y))$ for any choice of $w$ with $w([k..n]) = I$. The very interesting question of which polytopes arise is equivalent to the question of which combinations of numbers $d_I(Y)$ are possible.

(2) Specifying the numbers $\delta_i(Y)$ and $d_I(Y)$ for any interval $I = [\ell..r]$ is by definition equivalent to specifying the pair $(\lambda(Y), p(Y))$.

(3) If $P$ is a polytope, let $M_P$ denote the set of lattices $Y$ for which $P = P(Y)$. This gives the decomposition of $G$ into torus orbit types, and it is now easy to see that it is a refinement of the decomposition into the $M(p, \lambda)$'s. Indeed, if $Y$ is a lattice and we set $P = P(Y)$, $p = p(Y)$, $\lambda = \lambda(Y)$ then we conclude by remarks (1) and (2) that $M_P \subseteq M(p, \lambda)$.

(4) For each element $w$ of the Weyl group $W$ there is a decomposition $M^w(p, \lambda)$ analogous to $M(p, \lambda)$, where $M(p, \lambda) = M^{id}(p, \lambda)$: for a lattice $Y$ let $w \cdot Y$ be the lattice obtained by reordering the columns according to $w$ and set $M^w(p, \lambda) = \{ Y \mid p = p(w \cdot Y) \}$ and $\lambda = \mu^w(Y)$. Then the intersection of these $n!$ decompositions is the decomposition into torus orbit types. Indeed, since every subset $I$ of $\{1, \ldots, n\}$ is mapped onto an interval $[\ell..r]$ by some Weyl group element, we see by remarks (1) and (2) that $M_P(Y) = \bigcap_{w \in W} M^w(p(w \cdot Y), \mu^w(Y))$. □
6.2. Moment images of torus orbits through weakly compatible lattices. Given a lattice \( Y \), let us call \( \mu(Y) \) and \( \nu(Y) \) the highest and lowest vertices of \( P(Y) \).

**Lemma 6.2.** All polytopes \( P(Y) \) for lattices \( Y \in M(p, \lambda) \) have the same highest and lowest vertices. Moreover the highest vertex is \( \lambda \).

**Proof.** This is an immediate consequence of Lemma 6.1 and the definitions of \( p(Y) \) and \( \lambda(Y) \). \( \square \)

**Remark.** The lowest vertex is the coweight with \( i^{th} \) component \( \lambda_i - l_i + r_i \), where \( l_i \) is the number of loops of \( p \) with left end \( i \) and \( r_i \) is the number of loops of \( p \) with right end \( i \). \( \square \)

6.3. Moment images of torus orbits through strongly compatible lattices.

**Theorem E.** A lattice \( Y \) is in \( M^I(p, \lambda) \) if and only if the polytope \( P(Y) \) is equal to \( P(p, \lambda) \).

**Proof.** Suppose \( Y \in M^I(p, \lambda) \). To show \( P(Y) = P(p, \lambda) \), we show that these polytopes have the same vertices; that is, for each permutation \( w \), the coweight \( \mu^w(Y) = (\mu_1, \ldots, \mu_n) \) equals the coweight \( \nu(w) = (\nu_1, \ldots, \nu_n) \) defined in Section 2.3. Since the numbers \( \mu_w(1), \ldots, \mu_w(n) \) depend only on the lattice \( Y \) strongly compatible to \( p \), it suffices by induction to show that \( \mu_w(1) = \nu_w(1) \). Recall that \( \mu_w(1) = \delta_w(1)(Y) + \dim_0(Y) - \dim(Y) \) and \( \nu_w(1) = \lambda_w(1) - l_w(1) + N_w(1) \); we will check that the right sides are equal to each other. By definition of \( \lambda = \lambda(Y) \) we have \( \delta_w(1) = \lambda_w(1) - l_w(1) \).

Since \( Y \) is compatible to \( p \), we know that \( \hat{Y} \) is weakly compatible to \( \hat{p} \), therefore \( \dim_0(Y) - \dim(Y) = |p| - |\hat{p}| = N_w(1) \).

To show the converse, recall that the polytope \( P(Y) \) uniquely determines the numbers \( d_l(Y) \). It follows, by what we just proved, that if \( Y \in M^I(p, \lambda) \) then \( d_l(Y) = |\hat{p}| \) for any ordered set \( \bar{I} \) and the corresponding unordered complement \( \bar{I} \). So we will show that if a lattice \( Y \in M(p, \lambda) \) is not strongly compatible to \( p \) then

\( (*) \) there exists an ordered set \( I \) with \( d_l(Y) \neq |\hat{p}| \).

Since \( Y \) is not strongly compatible to \( p \), there exists an ordered set \( I \) such that \( \hat{Y} \) is weakly compatible to \( \hat{p} \) but not compatible to it. Obviously, if we can prove \( (*) \) for \( \hat{Y} \) instead of for \( Y \) it will also follow for \( Y \), so without loss of generality we assume that \( Y \) is not compatible to \( p \). In particular, there exists a loop \( L \) passing through columns \( \ell, \ldots, r \) for which condition \( (\textbf{I}) \) on page 12 fails. Consider the ordered set

\[ J = (1, \ldots, \ell - 1, n, \ldots, r + 1). \]

If \( L' \) is the largest loop in \( p \) encircling the same dots as \( L \), then it is easy to see that \( \hat{p}_I = p^{L'} \) and \( \hat{Y} = Y_{L'} \). So, if we can prove \( (*) \) for \( Y_{L'} \) and \( p^{L'} \) instead of for \( Y \) and \( p \), it will also follow for \( Y \).

So, without loss of generality, we can assume that \( Y \) is in \( M(p, \lambda) \) but is not compatible to \( p \) and that \( (\textbf{I}) \) fails for a column \( i \) and a loop \( L \) encircling all dots. We claim that \( \dim_0(Y_i) \neq |\hat{p}_i| \), which implies \( (*) \) for \( \bar{I} = (i) \). Indeed, \( \dim_0(Y_i) = |p| - \dim(\text{Proj}_{\bar{I}}(Y)/\text{Proj}_{\bar{I}}(Y_0)) \) and \( |\hat{p}_i| = |p| - N_i \) where \( N_i \) is the number of levels into which the picture \( p \) is broken during the collapse along column \( i \). Notice that loop \( L \) is removed during this collapse, since it is the only loop in its level. Because \( (\textbf{I}) \) fails for loop \( L \) at column \( i \) we conclude that \( \dim(\text{Proj}_{\bar{I}}(Y)/\text{Proj}_{\bar{I}}(Y_0)) < N_i \); hence \( \dim_0(Y_i) > |\hat{p}_i| \) and we have proved \( (*) \). \( \square \)

6.4. Proofs of the main theorems. Given two coweights \( \alpha \) and \( \beta \), recall that MV-cycles are defined to be the irreducible components of the closure of the set \( S_{\alpha, \beta} = S_\alpha \cap S_\beta \). By Lemmas 5.1 and 5.2 \( S_{\alpha, \beta} \) contains all lattices \( Y \) for which the highest and lowest vertices of \( P(Y) \) are \( \alpha \) and \( \beta \) respectively. Therefore, by Lemma 6.2 \( S_{\alpha, \beta} \) is a union of several pieces of the decomposition \( M(p, \alpha) \). By Proposition 4.2 all these pieces are Zariski open sets of the same dimension, \( \text{len}(p) \); hence each closure \( \overline{M(p, \alpha)} \) is an MV-cycle. On the other hand, for any \( M(p, \lambda) \), there exists a \( \beta \) for which \( M(p, \lambda) \subseteq S_{\lambda, \beta} \), by Lemma 6.2. This proves Theorem A. By Propositions 5.3 and 5.4 the closure of \( M^I(p, \lambda) \) is the same as the closure of \( M(p, \lambda) \). By Theorem B, the moment map image of \( \overline{M^I(p, \lambda)} \) and hence \( \overline{M(p, \lambda)} \) is \( P(p, \lambda) \). This proves Theorem B.
References

[A1] J. Anderson, *On Mirković and Vilonen’s Intersection Homology Cycles for the Loop Grassmannian*, PhD Thesis, Princeton University, 2000.

[A2] J. Anderson, *A polytope calculus for semisimple groups*, Duke Math. J. **107** (2003) no. 3, 567–588.

[Ati82] M.F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), 1-15.

[AP] M. F. Atiyah, A. N. Pressley, *Convexity and Loop Groups*, Arithmetic and Geometry, Vol II (Progr. Math 36) Birhäuser, Boston, 1983, 33-66.

[Bri] M. Brion, *Sur l’image de l’application moment*, Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin (Paris, 1986), 177–192, Lecture Notes in Math., **1296**, Springer, Berlin, 1987.

[FZ] S. Fomin, A. Zelevinsky, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. **12** (1999), no. 2, 335–380.

[FZ2] S. Fomin, A. Zelevinsky, *Cluster algebras I. Foundations*, J. Amer. Math. Soc. **15** (2002), no.2 497–529.

[GL] S Gaussent, P Littelmann, *LS-Galleries, the path model and MV-cycles* arXiv:math.RT/0307122

[GGMS] Gel’fand, I. M.; Goresky, R. M.; MacPherson, R. D.; Serganova, V. V. *Combinatorial geometries, convex polyhedra, and Schubert cells*, Adv. in Math. **63** (1987), no. 3, 301–316.

[G] V. Ginzburg, *Perverse sheaves on a loop group and Langlands’ duality*, preprint, 1995.

[L] G. Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities*, Astérisque **101-102** (1983), 208–229.

[M] I. Mirković, *Parameterization of canonical bases*, personal communication (2003).

[MV] I. Mirković and K. Vilonen, *Perverse Sheaves on affine Grassmannians and Langlands Duality*. Math. Res. Lett. **7** (2000) no. 1, 13–24.

[MVy] I. Mirković and M. Vybornov *On quiver varieties and affine Grassmannians of type A*. arXiv:math.AG/0206084

[Na1] H. Nakajima *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*. Duke Math. J. **76** (1994) no. 2, 365-416.

[Na2] H. Nakajima *Quiver varieties and Kac-Moody algebras*. Duke Math. J. **91** (1998) no. 3, 515-560.

[Ngo] B. C. Ngô, *Preuve d’une conjecture de Frenkel-Gaitsgory-Kazhdan-Vilonen pour les groupes lineaires généraux*, Israel J. Math. **120** (2000), part A, 259-270.

University of Pittsburgh, Pittsburgh, PA, USA

E-mail address: jared@math.pitt.edu

Northeastern University, Boston, MA, USA

E-mail address: misha@research.neu.edu