A Representation for Compound Quantum Systems as Individual Entities: Hard Acts of Creation and Hidden Correlations.

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Abstract
We introduce an explicit definition for 'hidden correlations' on individual entities in a compound system: when one individual entity is measured, this induces a well-defined transition of the 'proper state' of the other individual entities. We prove that every compound quantum system described in the tensor product of a finite number of Hilbert spaces can be uniquely represented as a collection of individual(ized) (pseudo-)entities between which there exist such hidden correlations. We investigate the significance of these hidden correlation representations within the so-called “creation-discovery-approach” and in particular their compatibility with the “hidden measurement formalism”. This leads us to the introduction of the notions of ‘soft’ and ‘hard’ ‘acts of creation’ and to the observation that our approach can be seen as a theory of (pseudo-)individuals when compared to the standard quantum theory. (For a presentation of some of the ideas proposed in this paper within a quantum logical setting, yielding a structural theorem for the representation of a compound quantum system in terms of the Hilbert space tensor product, we refer to [18].)

Key words: state, compound system, Hilbert space tensor product, act of creation, hidden measurement.

1 Introduction.

Already for some time the study of compound systems has been one of the main issues of the foundations of quantum physics, studied in great detail by many authors (see for example [3], [12], [20], [22], [2] and [33]). Nevertheless, to the present author’s knowledge, all literature before the 80’s only refers to compound quantum systems in so called ‘entangled states’, i.e., compound systems described in the tensor product of two Hilbert spaces. In 1981 Aerts showed that separated quantum entities cannot be described by the property lattice of a tensor product (see [1], [8] and [35]). As a consequence, alternative ways of describing compound systems should be considered1. In particular, one should try to find a scheme for the representation of compound systems in which as well the separated as the non-separated case fit in a conceptually consistent way. Such an attempt has been made in [3] and in [15]: the entities in the compound system are considered as individual(ized) (pseudo-)entities between which there exist a specific kind of correlations, called correlations of the second kind. These correlations of the second kind were defined by Aerts in [3] as correlations which were not present before the measurement but that are created during and by the measurement process. In fact, as we show in the next section, this specific approach towards compound systems can be considered as an aspect of a more general approach, namely the so-called creation-discovery approach, discussed in [3] and [15] (a brief outline of this approach can be found in the next section). As Aerts shows in [3], it is the presence of a non-empty act of creation, formalized as correlations of the second kind, which is responsible for the violation of Bell-like inequalities (what we precisely mean by this ‘act of creation’ gets clear in the next sections). In the case that the individual entities in the compound system are initially separated, we will consider this act of creation as empty. Thus, this concept of correlations of the second kind delivers a tool to consider ‘entangled’ and ‘separated’ entities within one approach. However, the approaches introduced in [3] and in [15] are essentially metaphorical. Moreover, the model systems that have been introduced only relate to a few compound quantum systems with a description in $\mathbb{C}^2 \otimes \mathbb{C}^2$ or $\mathbb{C}^3$, and although all these model systems have

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1See also [3] and [15] for recent perspectives.
from a structural point of view a similar kind of correlations of the second kind, an explicit mathematical definition has never been given.

In this paper we state a more explicit definition for correlations of the second kind, which is fulfilled by the model systems of 3 and 18, and which enables a generalization beyond the Hilbert space tensor product representation used in quantum theory, in the sense that we are not bound to the Hilbert space tensor product for the description of compound systems, nor are we bound to the Hilbert space itself for the representation of the states and the properties of the entities. However, in order to obtain the right results for the specific case of pure quantum theory, we prove that every compound quantum system described in the tensor product of a finite number of Hilbert spaces has a unique representation as a collection of individual entities on which we introduce our correlations of the second kind. As a consequence, the approach introduced in this paper generalizes the quantum description for compound systems.

The general conceptual nature of the representation introduced in this paper will be discussed in detail in the following section, but we already mention some of it in a brief way. Following Piron’s conception of state we consider the state of a quantum system as a complete representation of the entity’s elements of reality, i.e., the state represents all actual properties of it, where the collection of all properties is defined as the equivalence classes of the collection of all possible questions corresponding with an explicit experimental procedure, and where actuality means that we obtain a positive answer with certainty when the procedure is performed (see 1, 27 and 30). As a consequence, with ‘state’ we mean ‘pure state’: the mixed states that occur in standard quantum theory should be considered as due to a lack of knowledge on the state of the entity. Since this point of view works very well (read, is compatible) with the creation-discovery approach (see 8), we take it as a starting point. Since in this paper, we want to consider individual entities within the compound system, we will be forced to extend this conception of state. This extension of Piron’s concept of a state will be called the proper state of an individual entity within a compound system.

In our definition of correlations of the second kind we require a deterministic dependence of the created correlations on the transition of the proper state of the individual entity which is submitted to a measurement. As a consequence, this approach to compound systems can be considered as conceptually compatible with another ingredient of the creation-discovery approach, namely the hidden measurement formalism for quantum measurements[4] which will be discussed in the next section. Therefore we call the specific kind of correlations of the second kind that we define in this paper hidden correlations. In section 8 we formally combine hidden measurement representations and hidden correlations. From this will follow that our approach can be considered as a ‘theory of individuals’, compared to the standard treatment of compound systems in quantum theory.

The mathematical core of the construction that we make for the case of quantum theory is essentially based on Schmidt’s biorthogonal decomposition lemma (see 33), which has been used by von Neumann to explain how physical quantities of the subsystems of compound quantum systems are related (see 86 and 4). We want to remark that the use of the biorthogonal decomposition within our construction is not of the same fundamental nature as it is the case for the modal interpretations of quantum mechanics (see for example 32 and 25). In fact, in our case it could even be avoided in the main formulation of the representation, but this causes much more complicated expressions. Moreover, the uniqueness of the representation is due to the existence of the ‘almost’ uniqueness of the biorthogonal decomposition[5].

Finally we want to remark that our approach/representation for obtaining all possible tensor product states through ‘hidden correlations’ has also from a purely formal point of view some interesting extra

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2The present author is aware of the discussions, i.e., the criticisms and replies, which are still running on this approach to physics. For an overview on this debate we refer to 3, 21, 22, 23 and 24. However, since in the past most criticism seems to be due to misunderstandings (see 3, 21 and 24), we do choose to apply the conception of a state as it occurs within this approach.

3This idea of hidden measurements has been introduced by Aerts in 8, it has been developed into a mathematical framework in 32, 33 and 34, and constitutes a fundamental aspect of the creation-discovery approach (see 8 and 35). For the specific case of the tensor product of two Hilbert spaces, there is some similarity with Herbut and Vujčić’s representation for two spatially separated correlated systems by two density matrices and a correlation operator (introduced and studied in 29 and 31).

4For the present author, it is true that “the distinguishing feature of the modal interpretations of quantum mechanics is their abandonment of the orthodox eigenstate-eigenvalue rule” (see 33) in contrary to our representation where Pauli’s definition of a measurement of the first kind (see 32 and 33), a generalization of von Neumann’s eigenstate-eigenvalue rule, plays a crucial role as a formalization of the explicit state transitions which we consider as an essential ingredient of a measurement (a more detailed explanation of this matter is contained in the following sections of this paper).

5I would like to thank Dennis Dieks and Pieter Vermaas for pointing out the different nature of the biorthogonal decomposition in our construction and in the modal interpretation in an explicit way. The kernel of this fundamental difference is the fact that “the distinguishing feature of the modal interpretations of quantum mechanics is their abandonment of the orthodox eigenstate-eigenvalue rule” (see 33) in contrary to our representation where Pauli’s definition of a measurement of the first kind (see 32 and 33), a generalization of von Neumann’s eigenstate-eigenvalue rule, plays a crucial role as a formalization of the explicit state transitions which we consider as an essential ingredient of a measurement (a more detailed explanation of this matter is contained in the following sections of this paper).
features relative to for example Herbut and Vujčić’s representation, due to the generalization to more than two Hilbert spaces, and due to the very natural way in which we obtain uniqueness. Thus, we think that the results presented in this paper contain some new formal ingredients, even in absence of the extended creation-discovery approach. A first application of these ingredients can be found in [7], where we apply the formal results of this paper in order to construct a representation for a spin-$S$ entity as a compound system consisting of $2S$ individual spin-1/2 entities between which there exist hidden correlations.

2 The creation-discovery approach and compound systems.

In this section we introduce the general philosophical framework, called the creation-discovery approach, that will be considered as the general discourse and mode of thinking in this paper. As already remarked in the previous section, some formal aspects of the representation could be considered without the framework presented in this section. However, the embedment of the representation within a clear and global approach accentuates the motivation behind it. We also are aware of the fact that the philosophical framework presented in this section requires much more pages of explanation for really deserving the name of a ‘philosophical framework’. Nonetheless, we think that this more or less intuitive presentation preserves a sufficient explanation of our motivation.

2.1 The creation-discovery approach.

D. Aerts proposed in the 80’s a way to identify the physical aspects that are at the origin of the structural differences between quantum and classical theories (see [11] and [3]). It are mainly two aspects that determine these structural differences, in the sense that we obtain quantum-like probability structures if the measurements needed to test the properties of the system are such that:

1. The measurements are not just observations but provoke a real change of the state of the system. 2. There exists a lack of knowledge on what precisely happens during the measurement process.

The first aspect, the change of state, can be interpreted as an ‘act of creation’ on the entity under study. It is indeed the external device that provokes the change of state during the interaction with the entity. If there is not such a change of state, we call the measurement a discovery. The second aspect, the presence of the lack of knowledge on the precise act of creation which results from an interaction with the measurement context, lies at the origin of the so called indeterministic nature of quantum measurements and can be formalized as a lack of knowledge on the precise measurement that is actually performed:

1. With each real measurement $e$ corresponds a collection of deterministic measurements $e_\lambda$, called ‘hidden measurements’. 2. When a measurement $e$ is performed on an entity in a state $p$, then one of the hidden measurements $e_\lambda$ takes place. The probability finds its origin in the lack of knowledge about which one of the hidden measurements effectively takes place.

In [2] and [4] it is proved that such a representation exists for quantum measurements described in finite dimensional Hilbert spaces, and a proof for the infinite dimensional case is delivered in [14] and [16]. It is important to remark that in this hidden measurement formalism:

The state $p$ does not depend on the parameter $\lambda$ and the selection of $\lambda$ is also independent of the state $p$.

As is shown in [3] and [4], these hidden measurement representations are not in contradiction with the NoGo-theorems about hidden variables (all inspired by the von Neumann proof in [36]) since the variables in the hidden measurement approach are contextual by definition. For more details on this hidden measurement approach we refer to [4] and [5]. Another model system in which we encounter an introduction of parameters representative for this kind of lack of knowledge situation is the Gisin-Piron model in [14]. However, this aspect of the model was not the main topic of their paper. Gisin and Piron mainly wanted to show that it is possible to find a dynamical equation for a ‘state transition’ during a measurement, i.e., a collapse without going through a mixture.

In fact, in a somewhat modified version of the hidden measurement formalism elaborated in [4], this is not true anymore. As a consequence, we loose the clean cut between ‘measurement-dependence’ and ‘entity-dependence’ of relevant parameters in a formal description of a measurement. This altered version of the original hidden measurement approach has the great advantage that it takes in an a priori way the property structure of the entity into account, which leads to some remarkable uniqueness theorems. Unfortunately, one loses the extreme philosophical simplicity of the original hidden measurement approach. We also want to remark that for all results presented in this paper, and also for the presented conceptual framework in relation to the description of compound systems, it wouldn’t pose any additional problem if we would replace the hidden measurement ingredient by this recently introduced alternative for it. However, we would loose some of the transparency obtained in the resulting models, and therefore we choose for the purpose of this paper to stick to the original hidden measurement formalism.
The formalism has been generalized beyond the quantum framework in [4], [14] and [16]. We briefly present the hidden measurement formalism following the definitions of [16]. Let $\Sigma$ be a collection of states, $\Lambda$ the collection of parameters for the hidden measurements in a hidden measurement representation and let $\mu : B(\Lambda) \rightarrow [0, 1]$, with $B(\Lambda)$ a $\sigma$-field of subsets of $\Lambda$, determine the relative frequency of occurrence of $\lambda \in \Lambda$. For the purpose of this paper, we suppose that due to the state transition aspect of a measurement $e$, the outcomes of this measurement can be identified in a one to one way by a collection of outcome states $\Sigma_e$. For a fixed $\lambda \in \Lambda$, the measurement process is strictly classical: for every hidden measurement $e_\lambda$ there exists a strictly classical observable $\varphi_\lambda : \Sigma \rightarrow \Sigma_e$. Thus we can represent the unknown but relevant content of the measurement interaction for the measurement process as a couple consisting of a set of strictly classical observables and a probability measure:

$$\begin{align*}
\Phi &= \{\varphi_\lambda : \Sigma \rightarrow \Sigma_e | \lambda \in \Lambda\} \\
\mu : B(\Lambda) &\rightarrow [0, 1]
\end{align*}$$

(1)

For every possible initial state $p \in \Sigma$, a measurement $e$ is characterized by a probability measure $P_{p,e} : B(\Sigma_e) \rightarrow [0, 1]$ ($B(\Sigma_e)$ is a $\sigma$-field of subsets of $\Sigma_e$). Also every hidden measurement $e_\lambda$ corresponds with a probability measure, namely $P_{p,\lambda} : B(\Sigma_e) \rightarrow \{0, 1\}$ which is such that

$$P_{p,\lambda}(\{q\}) = 1 \iff \varphi_\lambda(p) = q$$

(2)

Thus we can relate the above introduced probability measures:

$$P_{p,e}(q) = \mu(\{\lambda | P_{p,\lambda}(\{q\}) = 1\}) = \mu(\{\lambda | \varphi_\lambda(p) = q\})$$

(3)

All this has led to a general axiomatics for context dependence and a classification of all possible hidden measurement representations (see [16]). From this classification it follows that in general, the hidden measurement approach delivers no a priori way to represent the 'true structure' of a physical entity (see [16] and [15]). In fact, it is one of the main aims of this paper to introduce a way to describe compoundness of physical systems within this very general approach. More precisely, the hidden measurement representations for the composing entities together with the hidden correlations deliver a new hidden measurement representation for the compound system which takes into account the true nature of the compound quantum entity, i.e., the compoundness itself. As already mentioned in the introduction, this specific 'compatibility', or even better 'complementarity', is formally elaborated in section 3 of this paper.

Although we are always able to formalize quantum-like probability structures as a lack of knowledge on the measurement that will take place, this does not exclude the occurrence of lack of knowledge on the initial state (for example: an entity in a statistical ensemble in which appear different states). In such a case, it is more natural to formalize this 'lack of knowledge on the initial state'-aspect by a probability measure defined on measurable subsets of the state space. For the case of quantum theory, the combination of the probabilistic behavior in a measurement for a well-defined initial state (i.e., a positive normalized measure defined on the subspaces of the Hilbert space which is additive on every countable subset of mutual orthogonal subspaces) and a lack of knowledge on the initial state (i.e., a probability measure defined on the set of states) corresponds due to Gleason’s theorem in a one to one way with a density matrix, i.e., with a positive operator on the Hilbert space of unit trace (see [24]).

However, although formally we have a situation of a probability measure on the set of states and a probability measure on the set of hidden measurements, there is a very important conceptual difference between the two kinds of uncertainty that correspond with them. Namely, at the initial stage of an experimental setup, the entity is in a state, we only don’t know which one. On the contrary, one cannot speak about an initial unknown hidden measurement, but only about a formalization of the interaction that will take place when we actually decide to perform the measurement, i.e., if there is no measurement, the parameters $\lambda$ that characterize the possible formalizations of the deterministic measurement process through hidden measurements are meaningless. As a consequence, the word 'hidden' represents in the creation-discovery approach a combination of an aspect of potentiality (i.e., what might become but what is not) combined with an aspect of uncertainty. We will see in the following sections, this same point of view applies to the hidden correlations.

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8To assure the existence of the following expressions all arguments of probability measures need to be measurable. A necessary and sufficient condition to achieve this can be found in [4].

9This new status that we attach to the word 'hidden' definitely differs from its historical origin, namely the hidden variable disputes. Therefore one might think that for our purpose, the word hidden might be badly chosen. However, according to Aerts, at the time he introduced his first hidden measurement model, the living paradigms and the not yet sufficiently refined ideas he had on the hidden measurements did justify that choice of the name.
2.2 Soft and hard acts of creation.

For the purpose of this paper, we need to refine the notion of act of creation. In the previous subsection we have identified the change of state that occurs during a measurement as an 'act of creation'. In the case of a pure quantum measurement, this change of state corresponds with a unitary (read 'structure preserving') transition. Thus, the act of measurement doesn’t imply a change of the set of properties, and thus, also not a change of the set of states, but only a change of the actual properties. However, one could imagine some more drastic acts of creation. Consider for example a uniform sphere. The states of such a sphere correspond with possible spatial coordinates of its center, and a state transition corresponds with a translation. Suppose now that one cuts up the sphere in two half-spheres. Every one of these half-spheres has now, apart from their translational properties, also angular properties. Thus, due to this cutting, new properties occur, and thus, we obtain a set of states that is different from the original one. Moreover, even if we keep the two half-spheres together, and consider them as one entity, the presence of the cut yields a different set of states than the one for the uniform sphere. Thus, by 'cutting' we perform an 'act of creation': we create new properties. In fact, since a lot of authors define an entity by a set of states, one might say that we create new entities. To distinguish between these two different acts of creation, respectively the one provoking a change of state and the one provoking a change of the set of states, we will call them respectively soft and hard acts of creation. Another example is an Aspect-like measurement, where one destroys the entanglement and creates new properties, which refer to the polarization angles, in order to obtain two separate photons. Nonetheless, as shown in [3], in this case we also need a soft act of creation in order to obtain the required quantum structure. Thus, in general we will have to consider both kinds of acts of creation together.

We will now analyze the change of the state space due to the hard act of creation for the case of two individual spin-1/2 entities described by the so called singlet state, an example which is conceptually almost identical to an aspect-like experiment and which was also considered by Aerts in [7]. Before we perform the measurement, the angular properties are traditionally described by the singlet state. In fact, from the experimental data that result from measurements on it, it follows that 'being in a singlet state' means nothing else than 'having no angular properties at all', since for every possible measurement referring to spin we have a uniform probability distribution for the possible outcomes (see [3]). After a measurement we obtain two separated spin-1/2 entities, in standard quantum mechanics described as products in the tensor product, but which might of course as well be described by a cartesian product of two spin-1/2 state spaces. Thus we have a change of the states space from a singleton to a cartesian product, due to the hard act of creation. The power of the standard quantum description lies in the fact that within one representation space, the 'large' space $\mathbb{C}^2 \otimes \mathbb{C}^2$, we are able to represent both state spaces relatively to each other. However, this requires the introduction of superselection rules to forbid the use of 'meaningless' elements in $\mathbb{C}^2 \otimes \mathbb{C}^2$. Unfortunately, when we start to consider measurements on compound quantum systems consisting of more than two individual entities things get more complicated. This is discussed in the next section.

2.3 Individual entities and proper states.

As already mentioned in the introduction, if one wants to talk about the elements of reality of an individual entity within a compound system, the concept of a state of [1], [11], [27] and [30] should be extended (read weakened or fuzzyfied) beyond its explicit operational definition. We first consider the example of an Aspect-like experiment or equivalently, the case of two individual spin-1/2 entities described by the so called singlet state: before a measurement on one of the individual entities has been performed, there exist no properties in Piron's sense referring to spin, and thus, strictly spoken, nothing can be said on it; in this case of two individual spin-$\frac{1}{2}$ entities described by the so called singlet state this causes not too much problems since before the actual act of creation took place there are no spin tendencies at all (in the same sense as the absence of angular tendencies for a uniform sphere) and after a measurement on one of the individual entities these entities are separated such that both have taken a spin-$\frac{1}{2}$ state; however, as it will follow from the results that we obtain in section 4 of this paper, when we consider three or more individual spin-$\frac{1}{2}$ states in a compound system, after a measurement on one of the individual entities, the two not yet measured entities do have tendencies towards certain states, although they do...

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10Since in this subsection we are discussing Piron’s concept of ‘state’, which is itself based on a very rigorous notion of ‘property’, we will from now on avoid the use of the word property in its more intuitive sense where it refers to a certain tendency of a system in a measurement, by using the word 'tendency' itself.
we refer to a set indexed over all \( \nu \) into a state that corresponds with the obtained outcome, and we call this state an first kind (see [29] and [30]), i.e., we suppose that after a measurement on an entity, its state has changed the state transition aspect of a soft act of creation, we follow Pauli’s definition of a measurement of the referring to the states the individual entity can take when it is separated from the other ones. Due to Definition 1

We say that there exist 'hidden correlations' between individual entities in a collection during the measurement only become relevant characterizations of the individual entity after the hard act of creation that happens a proper state it makes no sense anymore to consider a lack of knowledge on the state, since these states with one on the state transition that will take place when we perform the measurement): in the case of (i.e., a representation for a measurement that formalizes a lack of knowledge on the initial state combined the possible states for the entity after the measurement. In the case of individual quantum entities we will see that that this complex representation reduces into one formal object, namely a density matrix (see section 4). However, it is clear that within our creation-discovery approach there is a definite conceptual difference between this use of the density matrix as a representation for proper states and its usual use (i.e., a representation for a measurement that formalizes a lack of knowledge on the initial state combined with one on the state transition that will take place when we perform the measurement): in the case of a proper state it makes no sense anymore to consider a lack of knowledge on the state, since these states only become relevant characterizations of the individual entity after the hard act of creation that happens during the measurement.\(^\text{11}\)

2.4 Hidden correlations.

Let \( S \) be a compound system consisting of a finite collection of individual entities \( \{S_\nu\}_\nu \) (with this notation we refer to a set indexed over all \( \nu \); the symbol \( \nu \) will consequently be used as a parameter that takes \( \alpha, \beta, \gamma, \ldots, \kappa, \lambda, \mu, \ldots, \xi, \upsilon, \zeta \); i.e., the labels of individual entities in the compound system, as values). Every individual entity \( S_\nu \in \{S_\nu\}_\nu \) has a set of proper states represented by the set \( \Sigma_\alpha \) and a set \( \Sigma_\alpha \) referring to the states the individual entity can take when it is separated from the other ones. Due to the state transition aspect of a soft act of creation, we follow Pauli’s definition of a measurement of the first kind (see [29] and [30]), i.e., we suppose that after a measurement on an entity, its state has changed into a state that corresponds with the obtained outcome, and we call this state an outcome state. Thus, a measurement on \( S_\alpha \) which is such that after the measurement the individual entities in the compound system are separated, is characterized by a set of outcome states \( \{\phi_{\alpha,i}\}_i \) (with this notation we refer to a set indexed over all 'relevant' indices \( i \); what we mean by the relevant indices will follow from the context where we apply this notation). In fact, in this paper we will always consider such measurements.\(^\text{12}\) If we consider an obtained outcome state in a measurement on \( S_\alpha \) we denote it as \( \phi_{\alpha,i} \). A measurement on the compound system \( S \) corresponds with one measurement on every \( S_\alpha \), and an outcome of this measurement can be written as a cartesian product \( \times_\nu \phi_{\nu} \).

**Definition 1** We say that there exist 'hidden correlations' between individual entities in a collection

\(^{11}\)In fact, in accordance with the meaning that we have given to the word hidden in 'hidden measurement’, we might even call these states 'hidden states': they formalize what might become when we perform a measurement, i.e., when we perform an act of creation. However, since this expression 'hidden state' is explicitly used in the hidden variable context it is better to avoid the use of it in this paper.

\(^{12}\)In [29] we consider measurements that do not separate the individual entities. However, the 'proper outcome states' can be 'represented' in a one to one way by 'states', and thus the outcomes of a measurement on the compound systems can be written as a product \( \times_\nu \phi_{\nu} \). As a consequence, the obtained results in this paper can be applied in [30].
Following the above stated definitions we have: 1. a measurement on one individual entity \( S_\alpha \) induces a change of the proper state of the other individual entities 2. this change of proper state depends deterministically on the proper state transition of \( S_\alpha \). 3. after this measurement, the state of \( S_\alpha \) cannot be influenced anymore by measurements on other entities.

It is clear that the use of the word ‘hidden’ is justified by the fact that the specific correlation that occurs depends on the effect that the act of creation of the measurement has on the proper state of the individual entity \( S_\alpha \). In the next section we present a representation that fulfills the above stated definition for the case of the quantum formalism. For every measurement on an individual entity \( S_\alpha \), we have to define one map for every other individual entity \( S_\beta \) that characterizes the induced change of proper state. In the specific representation that we introduce in the following section, these maps only depend in an explicit way on the outcome state of the first measurement. Thus, we will have to define:

\[
f_{\beta \alpha} : \Sigma_\alpha \to \Sigma_\beta : \phi_\alpha \mapsto \omega_{\beta \alpha}
\]

which determines a state transition of \( S_\beta \) from a proper state \( \omega_\beta \) into a proper state \( \omega_{\beta \alpha} \), due to a transition of the proper state of \( S_\alpha \) from \( \omega_\alpha \) into an outcome state \( \phi_\alpha \). Analogously we define:

\[
f_{\mu \nu \lambda \kappa \ldots \alpha} : \Sigma_\lambda \to \Sigma_\mu : \phi_\lambda \mapsto \omega_{\mu \nu \lambda \kappa \ldots \alpha}
\]

such that if due to a measurement on \( S_\lambda \) after we have already performed measurements on \( S_\alpha, \ldots, S_\kappa \), the proper state of \( S_\lambda \) becomes \( \phi_\lambda \), the proper state of every not yet measured individual entity \( S_\mu \) changes to \( \omega_{\mu \nu \lambda \kappa \ldots \alpha} = f_{\mu \nu \lambda \kappa \ldots \alpha}(\phi_\lambda) \). Of course, in general one could consider more general kinds of \( f_{\mu \nu \lambda \kappa \ldots \alpha} \) in the sense that they do not only depend on \( \phi_\lambda \) but also on \( \omega_{\mu \nu \lambda \kappa \ldots \alpha} \), on \( \omega_{\mu \nu \lambda \kappa \ldots \alpha} \), or even on an additional variable. However, in theorem (8) in section (2) we will prove that this specific kind of dependence on the state transition covers all possible hidden correlation representations for compound quantum systems described in a tensor product. Also for the purpose of showing some peculiar features of our approach in contrary to the standard quantum approach, these kinds of hidden correlations suffice.

We remark that one can consider the change of state of the individual entities (caused by a measurement on one of them) as due to the presence of a ‘preserved quantity’ (such as momentum in classical physics), in the sense that the change of state of the measured individual entity (‘forced’ by the act of creation in the measurement on it) is compensated by the change of state of the other individual entities. The deterministic dependence represented by the map \( f_{\mu \nu \lambda \kappa \ldots \alpha} \) is an obvious assumption since the ‘preserved quantity’ is an intrinsic quantity of the collection of entities, and thus, these changes of state are not due to an act of creation during interaction with an external context.

### 3 Combining hidden measurements and hidden correlations.

Suppose that we have a given hidden measurement representation for the measurements on the individual entities in a compound system \( S \), i.e., for all \( \alpha \) there exists:

\[
\begin{align*}
\{ \Phi_\alpha &= \{ \varphi_{\alpha, \lambda_\alpha} : \Sigma_\alpha \to \{ \phi_{\alpha, i}\}_i | \lambda_\alpha \in \Lambda_\alpha \} \\
\mu_\alpha &= B(\Lambda_\alpha) \to [0, 1]
\end{align*}
\]

and suppose that we also have a hidden correlation representation which formalizes the specific compoundness of the system. Then we can define a new hidden measurement representation for the compound system \( S \). We represent the proper states of the individual systems in a cartesian product, i.e., \( \Sigma_S = \times_\alpha \Sigma_\alpha \), and we also define \( \Lambda_S = \times_\alpha \Lambda_\alpha \). For a measurement on the individual entities according to the ordering \( \alpha, \beta, \ldots, v, \zeta \) we define:

\[
\varphi_{\nu \lambda} : \Sigma_S \to \times_\alpha \{ \phi_{\alpha, i}\}_i : \times_\alpha \omega_\alpha \mapsto \left( \begin{array} {c} \varphi_{\alpha, \lambda_\alpha}(\omega_\alpha) \\ \varphi_{\beta, \lambda_\beta} \circ f_{\beta \alpha} \circ \varphi_{\alpha, \lambda_\alpha}(\omega_\alpha) \\ \vdots \\ \varphi_{\zeta, \lambda_\zeta} \circ f_{\zeta \nu \lambda_\nu \ldots \lambda_\alpha} \circ \varphi_{\nu, \lambda_\nu} \circ \ldots \circ \varphi_{\beta, \lambda_\beta} \circ f_{\beta \alpha} \circ \varphi_{\alpha, \lambda_\alpha}(\omega_\alpha) \end{array} \right)
\]

Following the above stated definitions we have:

\[
\phi_\alpha = \varphi_{\alpha, \lambda_\alpha}(\omega_\alpha)
\]
Since we have $\omega = f_{\beta \alpha}(\phi) = f_{\beta \alpha} \circ \varphi_{\alpha, \lambda}(\omega)$ and since $\phi = \varphi_{\beta, \lambda}(\omega)$ we also have:

$$\phi = \varphi_{\beta, \lambda} \circ f_{\beta \alpha} \circ \varphi_{\alpha, \lambda}(\omega)$$ \hspace{1cm} (9)

By an analogous induction on the number of consecutively performed measurements on the individual entities we obtain:

$$\phi = \varphi_{\beta, \lambda} \circ f_{\beta \alpha} \circ \varphi_{\alpha, \lambda}(\omega) = \varphi_{\beta, \lambda} \circ f_{\beta \alpha} \circ \varphi_{\alpha, \lambda}(\omega)$$ \hspace{1cm} (10)

As a consequence, eq.(7) displays the outcome of a measurement on $S$ up to a value in $\Lambda_S$, by taking into account the transitions of proper states of the individual entities that happen 'between' the measurements on the individual entities. Thus, we obtain a new hidden measurement representation:

$$\left\{ \Phi_S = \{\varphi_{S, \lambda} \mid \lambda \in \Lambda_S \}\right\}$$

which is such that $\times_\alpha \mu_\alpha$ is equal to $\mu_S$ for these subsets of $\Lambda_S$ where both are defined.

We remark that in this new hidden measurement representation, the formal ingredient that represents the specific kind of compoundness of the system is included in $\Phi_S$ and not in $\Sigma_S$. Thus, in our representation for measurements on compound systems, we deal in the same way with the interaction of an individual entity with the other individual entities as with the interaction with the measurement context.

In the case of quantum mechanics, the formal ingredient that represents the kind of compoundness is included in the so called 'state' of the compound system, i.e., in the representative vector of the tensor product. This observation allows us to say that our approach can be considered as a theory of individuals, compared to the standard treatment of compound systems in quantum theory. In general, the representation of eq.(7) and eq.(11) depends on the order of the measurements. Since this order in which the measurements on the individual entities are performed is definitely an ingredient of the 'context' of the individual entities and not of the individual entities or even the compound system itself, this fact seems to be very obvious. Thus, when we consider standard quantum theory, the description of the compound system seems to include an ingredient that represents the order of the measurements on the individual entities, and this can at least be called strange. However, in this quantum case we have the exceptional situation that the obtained probability structure does not depend on the order of the measurements on the individual entities, such that we do not encounter equivalently strange phenomena on the formal level.

To conclude: although this reasoning might be called somewhat intuitive, it definitely indicates that in the standard quantum theory there are serious problems when we want to consider individual entities within a compound system.

4 A hidden correlation representation for compound quantum systems described in the tensor product of Hilbert spaces.

According to quantum mechanics, the states in $\Sigma_\alpha$ are described in a Hilbert space $H_\alpha$ and the compound system is described by $\Psi_S \in \otimes_\nu H_\nu$. In this section we will show that it requires and suffices to consider the density matrices as representations for the proper states, consequently denoted by $\bar{H}_\alpha$. For reasons of simplicity we will refer to these density matrices as proper states. Also for reasons of simplicity we suppose that all measurements have a non-degenerate discrete spectrum (this does not cause a loss of generality). The outcome states for a measurement on individual entities correspond with the eigenvectors of the self-adjoint operator that represents the measurement, and an outcome of a measurement on the compound system $S$, which we have denoted by $\times_\nu \phi_\nu$, can now equivalently be represented as a vector in $\otimes_\nu H_\nu$, namely $\otimes_\nu \phi_\nu$.

\(^{13}\)We remark that from a geometric point of view, the density matrices can be seen as the convex closure of the states.
4.1 Explicitation of the representation.

Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be two Hilbert spaces. For every $\Psi \in \mathcal{H} \otimes \tilde{\mathcal{H}}$ there always exists a biorthogonal decomposition\footnote{In most of the cases, this biorthogonal decomposition is unique. It is possible to show that the representation that we are going to construct will not depend on the choice of the decomposition for the cases that it is not unique. The proof of this can be found in the appendix at the end of this paper.} such that we can write:

$$\Psi = \sum_i \langle \psi_i \otimes \tilde{\psi}_i | \Psi \rangle \psi_i \otimes \tilde{\psi}_i$$

(12)

where $\{\psi_i\}_i$ and $\{\tilde{\psi}_i\}_i$ are two orthonormal sets of vectors respectively in $\mathcal{H}$ and $\tilde{\mathcal{H}}$. For a proof we refer to [35] or [36]. Clearly, $\{\psi_i\}_i$ (resp. $\{\tilde{\psi}_i\}_i$) can always be extended to a base of $\mathcal{H}$ (resp. $\tilde{\mathcal{H}}$). If no confusion is possible, we will apply the same notation $\{\psi_i\}_i$ (resp. $\{\tilde{\psi}_i\}_i$) for such a base of $\mathcal{H}$ (resp. $\tilde{\mathcal{H}}$).

One easily verifies that for all ‘extra’ vectors that we introduce in order to obtain a base we have that $\langle \psi_i \otimes \tilde{\psi}_i | \Psi \rangle$ is equal to zero or it does not exist (this might happen in the case that $\mathcal{H}$ or $\tilde{\mathcal{H}}$ are not equal dimensional). We’ll assume that in eq.(12) the sum runs over all $i$ which are such that $\langle \psi_i \otimes \tilde{\psi}_i | \Psi \rangle$ exists.

Let $\Psi_S \in \otimes_{\nu} \mathcal{H}_{\nu}$. For every $S_\alpha$, there exists such a decomposition for $\Psi_S$ if we consider $\otimes_{\nu} \mathcal{H}_{\nu}$ as a tensor product of two Hilbert spaces $\mathcal{H}_\alpha$ and $\otimes_{\nu \neq \alpha} \mathcal{H}_{\nu}$, i.e., we consider $\Psi_S \in \mathcal{H}_\alpha \otimes (\otimes_{\nu \neq \alpha} \mathcal{H}_{\nu})$. In this case, we denote the two orthonormal bases in the biorthogonal decomposition of $\Psi_S$ as $\{\psi_{\alpha,i}\}_i$ (an orthonormal base for $\mathcal{H}_\alpha$) and $\{\tilde{\psi}_{\alpha,i}\}_i$ (an orthonormal base for $\otimes_{\nu \neq \alpha} \mathcal{H}_{\nu}$). Given this decomposition, we define for every $S_\alpha$ a map:

$$R_\alpha : \otimes_{\nu} \mathcal{H}_{\nu} \to \mathcal{H}_\alpha : \Psi \mapsto \omega_\alpha$$

(13)

where $\omega_\alpha$ is the diagonal density matrix in the base $\{\psi_{\alpha,i}\}_i$ which is such that the $i$th diagonal element is given by $|\langle \psi_{\alpha,i} \otimes \tilde{\psi}_{\alpha,i} | \Psi \rangle|^2$. We define a map:

$$T_\alpha : \mathcal{H}_\alpha \to \otimes_{\nu \neq \alpha} \mathcal{H}_{\nu} : \phi \mapsto \frac{1}{N(\phi)} \sum_i \langle \psi_{\alpha,i} \otimes \tilde{\psi}_{\alpha,i} | \Psi_S \rangle \langle \phi | \psi_{\alpha,i} \rangle \tilde{\psi}_{\alpha,i}$$

(14)

where:

$$N(\phi) = \sqrt{\sum_i |\langle \psi_{\alpha,i} \otimes \tilde{\psi}_{\alpha,i} | \Psi_S \rangle|^2 |\langle \phi | \psi_{\alpha,i} \rangle|^2}$$

(15)

We also define a map for every two individual entities $S_\alpha$ and $S_\beta$:

$$R_{\beta \alpha} : \otimes_{\nu} \mathcal{H}_{\nu} \to \mathcal{H}_\beta : \Psi \mapsto \omega_{\beta \alpha}$$

(16)

in an analogous way as $R_\alpha$, but now by considering a biorthogonal decomposition of $\Psi \in \otimes_{\nu \neq \alpha} \mathcal{H}_{\nu}$ in the tensor product $\mathcal{H}_\beta \otimes (\otimes_{\nu \neq \alpha, \beta} \mathcal{H}_{\nu})$ in stead of in $\mathcal{H}_\alpha \otimes (\otimes_{\nu \neq \alpha} \mathcal{H}_{\nu})$, i.e., if $\Psi = \sum_i \langle \psi_{\beta \alpha, i} \otimes \tilde{\psi}_{\beta \alpha, i} | \Psi_S \rangle \psi_{\beta \alpha, i} \otimes \tilde{\psi}_{\beta \alpha, i}$ is again a biorthogonal decomposition with $\{\psi_{\beta \alpha, i}\}_i$ as a base for $\mathcal{H}_\beta$ and $\{\tilde{\psi}_{\beta \alpha, i}\}_i$ as a base for $\otimes_{\nu \neq \alpha, \beta} \mathcal{H}_{\nu}$, then $R_{\beta \alpha}(\Psi) = \omega_{\beta \alpha}$ is the diagonal density matrix with $|\langle \psi_{\beta \alpha, i} \otimes \tilde{\psi}_{\beta \alpha, i} | \Psi \rangle|^2$ as respective elements in the base $\{\tilde{\psi}_{\beta \alpha, i}\}_i$. Finally, we define:

$$f_{\beta \alpha} : \mathcal{H}_\alpha \to \mathcal{H}_\beta$$

(17)

such that:

$$f_{\beta \alpha} = R_{\beta \alpha} \circ T_\alpha$$

(18)

If we perform a measurement on $S_\alpha$ and we obtain an outcome state $\phi_\alpha$, then the proper state of $S_\beta$ (for all $\beta \neq \alpha$) changes to $f_{\beta \alpha}(\phi_\alpha)$. We still have to explain what happens in consecutive measurements. Suppose that after the measurement on $S_\alpha$, we perform a measurement on $S_\beta$. We define a map:

$$T_{\beta \alpha} : \mathcal{H}_\beta \to \otimes_{\nu \neq \alpha, \beta} \mathcal{H}_{\nu}$$

(19)

in analogy with $T_\alpha$, but now by using a biorthogonal decomposition of the vector $T_\alpha(\phi_\alpha)$ in $\mathcal{H}_\beta \otimes (\otimes_{\nu \neq \alpha, \beta} \mathcal{H}_{\nu})$ in stead of the vector $\Psi_S$ in $\mathcal{H}_\alpha \otimes (\otimes_{\nu \neq \alpha} \mathcal{H}_{\nu})$. For every third individual entity $S_\gamma$ we define a map:

$$R_{\gamma \alpha \beta} : \otimes_{\nu \neq \alpha, \beta} \mathcal{H}_{\nu} \to \mathcal{H}_\gamma : \Psi \mapsto \omega_{\gamma \alpha \beta}$$

(20)
in an analogous way as $R_{\beta\alpha}$ and $R_\alpha$, but now by considering biorthogonal decompositions in $H_\gamma \otimes (\otimes_{\nu \neq \alpha, \beta, \gamma} H_\nu)$. Thus, we can define
\[
f_{\gamma\beta\alpha} : H_\beta \to H_\gamma
\]
such that:
\[
f_{\gamma\beta\alpha} = R_{\gamma\beta\alpha} \circ T_{\beta\alpha}
\tag{22}
\]
If we perform this measurement on $S_\beta$ and we obtain an outcome state $\phi_\beta$, then the proper state of $S_\gamma$ (for all $\gamma \neq \alpha, \beta$) changes to $f_{\gamma\beta\alpha}(\phi_\beta)$. Clearly, we can proceed with this procedure for some more consecutive measurements. After we have performed consecutive measurements on $S_\alpha, \ldots, S_\kappa$, they are in the states respectively denoted as $\phi_\alpha, \ldots, \phi_\kappa$, and the proper state of every not yet measured individual entity $S_\lambda$ is given by $\omega_{\lambda\kappa,\ldots,\alpha} = f_{\lambda\kappa,\ldots,\alpha}(\phi_\kappa)$. If we perform a next measurement on $S_\lambda$, the proper state of a not yet measured individual entity $S_\mu$ changes to $f_{\mu\lambda\kappa,\ldots,\alpha}(\phi_\lambda)$, where $\phi_\lambda$ is the outcome state of $S_\lambda$

\[
f_{\mu\lambda\kappa,\ldots,\alpha} : H_\lambda \to \tilde{H}_\mu
\tag{23}
\]
is defined in an analogous way as $f_{\gamma\beta\alpha}$, but now by considering a biorthogonal decomposition of $T_{\kappa,\ldots,\alpha}(\phi_\kappa)$ in stead of $T_\alpha(\phi_\alpha)$. We can summarize all this in the following definition:

**Definition 2** If $S$ is a compound quantum system consisting of a finite collection of individual entities $\{S_\nu\}_\nu$, and such that $S$ is described by $\Psi_S \in \otimes_{\nu} H_\nu$, then we can introduce a hidden correlation representation for $S$: 1. Initially, every $S_\alpha$ is in a proper state $\omega_\alpha = R_\alpha(\Psi_S)$. 2. If $S_\lambda$ takes the state $\phi_\lambda$ due to a measurement on it after we have already performed measurements on $S_\alpha, \ldots, S_\kappa$, then the proper state of every not yet measured individual entity $S_\mu$ changes to $f_{\mu\lambda\kappa,\ldots,\alpha}(\phi_\lambda) = R_{\mu\lambda\kappa,\ldots,\alpha} \circ T_{\lambda\kappa,\ldots,\alpha}(\phi_\lambda)$.

### 4.2 Correctness of the representation.

Now we will prove that our representation respects the specific probabilistic nature of the quantum description of a compound system.

**Theorem 1** The probabilities that we obtain in the representation introduced in definition 4 are the quantum probabilities.

**Proof:** Suppose that the quantum system $S$ consists of the individual entities $\{S_\nu\}_\nu$, that $S$ is described by $\Psi_S \in \otimes_{\nu} H_\nu$ and that we measure these individual entities one by one according to the ordering $\alpha, \beta, \gamma, \ldots, \xi, \nu, \zeta$. First we calculate, according to the rules of quantum mechanics, the probability to obtain an outcome state $\otimes_{\nu} \phi_\nu$ in a measurement that has $\otimes_{\nu} \phi_\nu$ as an eigenvector, taking into account the existence of representations as a biorthogonal decomposition for vectors described in the tensor product of two Hilbert spaces (to distinguish between the indices of the different bases corresponding with the different Hilbert spaces in $\{H_\nu\}_\nu$ we introduce a subscript in the indices):

\[
P_{QM}(\Psi_S \mid \phi_\alpha \otimes \ldots \otimes \phi_\zeta) = |\langle \Psi_S | \phi_\alpha \otimes \ldots \otimes \phi_\zeta \rangle|^2
\]

\[
= |(\sum_{i_\alpha} \langle \psi_{\alpha,i_\alpha} | \tilde{\psi}_{\alpha,i_\alpha} \rangle |\Psi_S \rangle \psi_{\alpha,i_\alpha} \otimes \tilde{\psi}_{\alpha,i_\alpha} | \phi_\alpha \otimes \ldots \otimes \phi_\zeta \rangle|^2
\]

\[
= |(\sum_{i_\alpha} \langle \psi_{\alpha,i_\alpha} | \tilde{\psi}_{\alpha,i_\alpha} \rangle |\Psi_S \rangle \langle \psi_{\alpha,i_\alpha} \otimes \tilde{\psi}_{\alpha,i_\alpha} | \phi_\alpha \otimes \ldots \otimes \phi_\zeta \rangle|^2
\]

\[
= |(\sum_{i_\alpha} \langle \psi_{\alpha,i_\alpha} | \tilde{\psi}_{\alpha,i_\alpha} \rangle |\Psi_S \rangle \langle \psi_{\alpha,i_\alpha} | \phi_\alpha \rangle \langle \tilde{\psi}_{\alpha,i_\alpha} | \phi_\beta \otimes \ldots \otimes \phi_\zeta \rangle|^2
\]

\[
= |(\sum_{i_\alpha} \langle \psi_{\alpha,i_\alpha} | \tilde{\psi}_{\alpha,i_\alpha} \rangle |\Psi_S \rangle \langle \psi_{\alpha,i_\alpha} | \phi_\alpha \rangle \tilde{\psi}_{\alpha,i_\alpha} | \phi_\beta \otimes \ldots \otimes \phi_\zeta \rangle|^2
\]

For $\sum_{i_\alpha} \langle \psi_{\alpha,i_\alpha} | \tilde{\psi}_{\alpha,i_\alpha} |\Psi_S \rangle \langle \phi_\alpha | \psi_{\alpha,i_\alpha} \rangle \tilde{\psi}_{\alpha,i_\alpha}$ we have the following biorthogonal decomposition in $H_\beta \otimes (H_\gamma \otimes \ldots \otimes H_\zeta)$:

\[
\sum_{i_\beta} \langle \psi_{\beta,i_\beta} | \tilde{\psi}_{\beta,i_\beta} |\sum_{i_\alpha} \langle \psi_{\alpha,i_\alpha} | \tilde{\psi}_{\alpha,i_\alpha} |\Psi_S \rangle \langle \phi_\alpha | \psi_{\alpha,i_\alpha} \rangle \tilde{\psi}_{\alpha,i_\alpha} \rangle | \psi_{\beta,i_\beta} \otimes \tilde{\psi}_{\beta,i_\beta} \rangle
\]
thus one finds that $P_{QM}(\Psi_S \mid \phi_\alpha \otimes \ldots \otimes \phi_\zeta)$ is given by:

$$
\left| \sum_{i_\beta} \langle \psi_{\beta,i_\beta} \otimes \tilde{\psi}_{\beta,i_\beta} | \sum_{i_\alpha} \langle \psi_{\alpha,i_\alpha} \otimes \tilde{\psi}_{\alpha,i_\alpha} | \Psi_S \rangle \langle \phi_\alpha | \psi_{\alpha,i_\alpha} \rangle \tilde{\psi}_{\alpha,i_\alpha} \psi_{\beta,i_\beta} \otimes \tilde{\psi}_{\beta,i_\beta} | \phi_\beta \otimes \ldots \otimes \phi_\zeta \rangle \right|^2
$$

$$
= \left| \sum_{i_\alpha} \sum_{i_\beta} \langle \psi_{\alpha,i_\alpha} \otimes \tilde{\psi}_{\alpha,i_\alpha} | \Psi_S \rangle \langle \phi_\alpha | \psi_{\alpha,i_\alpha} \rangle \langle \psi_{\beta,i_\beta} \otimes \tilde{\psi}_{\beta,i_\beta} | \psi_{\alpha,i_\alpha} \rangle \langle \psi_{\beta,i_\beta} \otimes \tilde{\psi}_{\beta,i_\beta} | \phi_\beta \otimes \ldots \otimes \phi_\zeta \rangle \right|^2
$$

$$
= \left| \sum_{i_\alpha} \sum_{i_\beta} \langle \psi_{\alpha,i_\alpha} \otimes \tilde{\psi}_{\alpha,i_\alpha} | \Psi_S \rangle \langle \phi_\alpha | \psi_{\alpha,i_\alpha} \rangle \langle \psi_{\beta,i_\beta} \otimes \tilde{\psi}_{\beta,i_\beta} | \tilde{\psi}_{\alpha,i_\alpha} \rangle \langle \psi_{\beta,i_\beta} | \phi_\beta \rangle \langle \tilde{\psi}_{\beta,i_\beta} | \phi_\gamma \otimes \ldots \otimes \phi_\zeta \rangle \right|^2
$$

$$
= \left| \sum_{i_\alpha} \langle \psi_{\alpha,i_\alpha} \otimes \tilde{\psi}_{\alpha,i_\alpha} | \Psi_S \rangle \langle \phi_\alpha | \psi_{\alpha,i_\alpha} \rangle \langle \psi_{\beta,i_\beta} \otimes \tilde{\psi}_{\beta,i_\beta} | \tilde{\psi}_{\alpha,i_\alpha} \rangle \langle \psi_{\beta,i_\beta} | \phi_\beta \rangle \langle \tilde{\psi}_{\beta,i_\beta} | \phi_\gamma \otimes \ldots \otimes \phi_\zeta \rangle \right|^2
$$

Also for $\sum_{i_\beta} \langle \psi_{\beta,i_\beta} \otimes \tilde{\psi}_{\beta,i_\beta} | \tilde{\psi}_{\alpha,i_\alpha} \rangle \langle \phi_\beta | \psi_{\beta,i_\beta} \rangle \tilde{\psi}_{\beta,i_\beta}$ we can consider a biorthogonal decomposition in the base $\{ \psi_{\gamma,i_\gamma} \otimes \psi_{\chi,i_\chi} \}_{i_\gamma}$. If we proceed along the same lines we obtain (we can omit the complex conjugation since it applies to all factors of which we take the norm):

$$
\left| \sum_{i_\alpha} \ldots \sum_{i_\beta} \langle \psi_{\alpha,i_\alpha} \otimes \tilde{\psi}_{\alpha,i_\alpha} | \Psi_S \rangle \langle \psi_{\beta,i_\beta} \otimes \tilde{\psi}_{\beta,i_\beta} | \tilde{\psi}_{\alpha,i_\alpha} \rangle \ldots \langle \psi_{\chi,i_\chi} \otimes \tilde{\psi}_{\chi,i_\chi} | \tilde{\psi}_{\alpha,i_\alpha} \rangle \right|^2
$$

Now we calculate the probabilities in our representation. Let us use $P(\psi \to \phi)$ as a notation for the probability of the transition of $\psi$ into $\phi$. The probability to obtain an outcome $\phi_\alpha \otimes \ldots \otimes \phi_\zeta$ for the measurement on $S$ is given by the product:

$$
P(\omega_\alpha \to \phi_\alpha) P(f_{\beta_\alpha} (\phi_\alpha) \to \phi_\beta) \ldots P(f_{\zeta_\alpha} (\phi_\zeta) \to \phi_\zeta)
$$

(although the initial proper state for every next measurement depends on the outcome state of the previous one, the measurements themselves are independent). For the first factor we have:

$$
P(\omega_\alpha \to \phi_\alpha) = \sum_{i_\alpha} \left| \langle \psi_{\alpha,i_\alpha} \otimes \tilde{\psi}_{\alpha,i_\alpha} | \Psi_S \rangle \right|^2 P(\psi_{\alpha,i_\alpha} \to \phi_\alpha)
$$

$$
= \sum_{i_\alpha} \left| \langle \psi_{\alpha,i_\alpha} \otimes \tilde{\psi}_{\alpha,i_\alpha} | \Psi_S \rangle \right|^2 |\langle \psi_{\alpha,i_\alpha} | \phi_\alpha \rangle|^2
$$

$$
= N(\phi_\alpha)^2
$$

Thus, $N(\phi_\alpha)^2$ is the probability to obtain an outcome state $\phi_\alpha$ when we are in proper state $\omega_\alpha$, which is itself is determined by $\Psi_S$. Since:

$$
f_{\beta_\alpha}(\phi_\alpha) = R_{\beta_\alpha}(T_\alpha(\phi_\alpha))
$$

we have (according to the definition of $R_{\beta_\alpha}$):

$$
P(f_{\beta_\alpha} (\phi_\alpha) \to \phi_\beta) = \sum_{i_\beta} \left| \langle \psi_{\beta,i_\beta} \otimes \tilde{\psi}_{\beta,i_\beta} | T_\alpha(\phi_\alpha) \rangle \right|^2 P(\psi_{\beta,i_\beta} \to \phi_\beta)
$$

$$
= \sum_{i_\beta} \left| \langle \psi_{\beta,i_\beta} \otimes \tilde{\psi}_{\beta,i_\beta} | T_\alpha(\phi_\alpha) \rangle \right|^2 |\langle \psi_{\beta,i_\beta} | \phi_\beta \rangle|^2
$$

$$
= N(\phi_\beta)^2
$$

where we applied the same biorthogonal decomposition as in the calculation of the quantum case, since $T_\alpha(\phi_\alpha)$ equals $\sum_{i_\alpha} \langle \psi_{\alpha,i_\alpha} \otimes \psi_{\alpha,i_\alpha} | \Psi_S \rangle \langle \phi | \psi_{\alpha,i_\alpha} \rangle \psi_{\alpha,i_\alpha}$ up to a the constant $N(\phi_\alpha)$, and where $N(\phi_\beta)$ is defined by eq. (13). This gives:

$$
P((\omega_\alpha, \ldots, \omega_\zeta) \to (\phi_\alpha, \ldots, \phi_\zeta)) = N(\phi_\zeta)^2 \ldots N(\phi_\beta)^2 P(f_{\beta_\alpha \ldots \alpha_\zeta} (\phi_\zeta) \to \phi_\zeta)
$$
For the last factor we have that $T_v(\phi_v) \in \mathcal{H}_\zeta$ and thus:

$$
P(f_\zeta \circ \omega_\alpha(\phi_v) \to \phi_\zeta) = P(T_v(\phi_v) \to \phi_\zeta)
$$

$$
= |\langle \phi_\zeta | T_v(\phi_v) \rangle|^2
$$

$$
= |\langle \phi_\zeta | T_v(\phi_v) \rangle|^2
= \frac{1}{N(\phi_v)^2} \sum_{i_v} \langle \psi_{v,i_v} \otimes \bar{\psi}_{v,i_v} | T_v(\phi_\zeta) \rangle \langle \phi_v | \psi_{v,i_v} \rangle |\bar{\psi}_{v,i_v} \rangle|^2
$$

$$
= \frac{1}{N(\phi_v)^2} \sum_{i_v} \langle \psi_{v,i_v} \otimes \bar{\psi}_{v,i_v} | T_v(\phi_\zeta) \rangle \langle \phi_v | \psi_{v,i_v} \rangle |\bar{\psi}_{v,i_v} \rangle|^2
$$

A consecutive substitution of $T_v(\phi_\zeta), \ldots, T_v(\phi_{\alpha})$ in $P(f_\zeta \circ \omega_\alpha(\phi_v) \to \phi_\zeta)$ gives:

$$
\frac{1}{N(\phi_v)^2} \sum_{i_v} \sum_{i_v} \cdots \sum_{i_v} \langle \psi_{\omega_{i_v}} \otimes \bar{\psi}_{\omega_{i_v}} | \Psi_{S} \rangle \langle \psi_{\beta,i_\beta} \otimes \bar{\psi}_{\beta,i_\beta} | \bar{\psi}_{\alpha,i_\alpha} \rangle \cdots \langle \psi_{\alpha,i_\alpha} \otimes \bar{\psi}_{\alpha,i_\alpha} | \bar{\psi}_{\zeta,i_\zeta} \rangle
$$

$$
|\langle \psi_{\alpha,i_\alpha} | \psi_{\beta,i_\beta} \rangle |\psi_{\alpha,i_\alpha} \rangle |\bar{\psi}_{\beta,i_\beta} \rangle |\bar{\psi}_{\alpha,i_\alpha} \rangle |\bar{\psi}_{\zeta,i_\zeta} \rangle|^2
$$

and thus, we find for $P((\omega_\alpha, \ldots, \omega_\zeta) \to (\phi_\alpha, \ldots, \phi_\zeta))$ the same expression as we have found in the quantum calculation. •

4.3 Uniqueness of the representation.

Now we will prove that the representation of definition 2 is the only possible one that fulfills definition 1.

**Theorem 2** The representation introduced in section 2 is unique, i.e., there exist no other hidden correlation representations for compound quantum systems described in the tensor product of a finite number of Hilbert spaces.

**Proof:** We have to prove that the density matrix representations $\{\omega_\eta\}_\eta$ for the proper states, as well as the transitions of them induced by measurements on individual entities are uniquely determined by $\Psi_S$. Suppose that there exist other $\{\omega'_\eta\}_\eta$ which fulfill definition 1 (initially we allow the state transitions to depend on other variables than the final state of the performed measurement). For all $\nu \neq \alpha$, let $\{\phi_{v,i_v}\}_{i_v}$ be an orthonormal base. We have for every $\phi_\alpha \in \mathcal{H}_\alpha$:

$$
\sum_{i_\beta} \cdots \sum_{i_\zeta} P_{QM} \left( \Psi_S \mid \phi_\alpha \otimes \phi_{\beta,i_\beta} \otimes \cdots \otimes \phi_{\zeta,i_\zeta} \right)
$$

$$
= \sum_{i_\beta} \cdots \sum_{i_\zeta} P(\omega'_\alpha \to \phi_\alpha) P(f'_{\beta,0\alpha}(\phi_\alpha, \ldots) \to \phi_{\beta,i_\beta}) \cdots P(f'_{\zeta,0\alpha}(\phi_{v,i_v}, \ldots) \to \phi_{\zeta,i_\zeta})
$$

$$
= P(\omega'_\alpha \to \phi_\alpha) \left( \sum_{i_\beta} P(f'_{\beta,0\alpha}(\phi_\alpha, \ldots) \to \phi_{\beta,i_\beta}) \cdots \sum_{i_\zeta} P(f'_{\zeta,0\alpha}(\phi_{v,i_v}, \ldots) \to \phi_{\zeta,i_\zeta}) \right)
$$

$$
= P(\omega'_\alpha \to \phi_\alpha) P_{QM} (\omega'_\alpha | \phi_\alpha)
$$

since all sums between brackets are equal to one (as a consequence of the normalization of the Hilbert in-product). Due to the independence of $\sum_{i_\beta} \cdots \sum_{i_\zeta} P_{QM} \left( \Psi_S \mid \phi_\alpha \otimes \phi_{\beta,i_\beta} \otimes \cdots \otimes \phi_{\zeta,i_\zeta} \right)$ on $\omega'_\alpha$, we have for every $\phi_\alpha \in \mathcal{H}_\alpha$ that $P_{QM}(\omega'_\alpha | \phi_\alpha) = P_{QM}(\omega'_\alpha | \phi_\alpha)$, and thus, $\omega'_\alpha = \omega'_\alpha$, since this expression implicitly defines a density matrix as a positive normalized measure, $\sigma$-additive on mutual orthogonal subspaces, on the subspaces of the Hilbert space. In an analogous way we prove the same for the other proper states in $\{\omega'_\eta\}_\eta$, by considering the first measurement performed on them. In a rather similar way we also prove the uniqueness of the imposed transitions of these proper states. We illustrate this for the proper state transition of $S_{\beta}$ induced by a measurement on $S_{\alpha}$, i.e., $\omega_{\beta}$ changes into $f'_{\beta,0\alpha}(\phi_\alpha, \ldots)$ due to a transition of $\omega'_\alpha$ into $\phi_\alpha$:

$$
\sum_{i_\gamma} \cdots \sum_{i_\zeta} P_{QM} \left( \Psi_S \mid \phi_\alpha \otimes \phi_{\beta,i_\beta} \otimes \phi_{\gamma,i_\gamma} \otimes \cdots \otimes \phi_{\zeta,i_\zeta} \right)
$$

$$
= P(\omega'_\alpha \to \phi_\alpha) P(f'_{\beta,0\alpha}(\phi_\alpha, \ldots) \to \phi_{\beta}) \left( \sum_{i_\gamma} P(f'_{\gamma,0\alpha}(\phi_{\gamma, \ldots}) \to \phi_{\gamma,i_\gamma}) \cdots \sum_{i_\zeta} P(f'_{\zeta,0\alpha}(\phi_{v,i_v, \ldots}) \to \phi_{\zeta,i_\zeta}) \right)
$$

$$
= P(\omega'_\alpha \to \phi_\alpha) P_{QM}(f'_{\beta,0\alpha}(\phi_\alpha, \ldots) | \phi_{\beta})
$$

12
The state transition of $S_\beta$ from $\omega_\beta$ into $f_{\beta\alpha}(\phi_\alpha,\ldots)$ happens after a transition of the proper state $\omega_\alpha$ into $\phi_\alpha$ and thus $P(\omega_\alpha \rightarrow \phi_\alpha) \neq 0$. Thus we have:

$$P_{QM}(f_{\beta\alpha}(\phi_\alpha,\ldots)|\phi_\beta) = \frac{\sum_{i_1} \cdots \sum_{i_c} P_{QM}(\Psi_S | \phi_\alpha \otimes \phi_\beta \otimes \phi_{i_1} \otimes \cdots \otimes \phi_{i_c})}{P(\omega_\alpha \rightarrow \phi_\alpha)}$$

(24)

for every $\phi_\beta \in \mathcal{H}_\beta$. As a consequence, $f_{\beta\alpha}(\phi_\alpha,\ldots)$ is equal to $f_{\beta\alpha}(\phi_\alpha)$. $ullet$

5 Summary and conclusion.

To conclude, for all states described in the tensor product of Hilbert spaces there exists a unique representation of the kind we have defined in this paper, i.e., a representation as a collection of individual entities each in a proper state, and such that a measurement on one of the entities induces a transition of the proper states of the other ones. Due to the uniqueness theorem, the representation introduced in this paper gives a definite characterization of the new kind of compoundness that can be described in a tensor product and not in a cartesian product. The approach can be embedded in a very natural way within Aerts’ creation-discovery approach and Piron’s property-approach if we introduce the notions of proper state, individual entity and hard and soft acts of creation. In particular does this approach deliver the appropriate tools to express compoundness of physical systems within the hidden measurement formalism. From the explicit construction of such a hidden measurement representation for a compound system it follows that our approach can be considered as a theory of individuals relative to the quantum description in a tensor product of Hilbert spaces. This is due to the fact that in our approach, the interaction of an individual entity with the other individual entities is threaded in the same way as the interaction with the measurement context. We also delivered an argument why our theory of individuals might be considered as preferable: the choice of the order in which we perform the measurements on the individual entities in the compound system should be an ingredient of the description of the context and not of the entity itself.

In fact, in this paper we only studied the situation in which after the measurements on the individual entities, these entities are separated due to a hard act of creation. In [16] we introduce a representation for a spin-$S$ entity as a compound system consisting of $2S$ individual spin-$\frac{1}{2}$ entities. Although in that paper we focus more on the formal ingredients of the representation, and in particular on the connection with Majorana’s representation, the representation itself delivers an example within quantum theory where a hidden correlation representation is applied in order to describe a situation where there are no hard acts of creation: the individual entities in the compound systems are not separated by the measurements since the spin-$S$ entity is still a spin-$S$ entity after the measurement, i.e., there is no change of the state space.

Of course, a lot of questions, as well on the formal as on the philosophical level remain unanswered. However, we think that the explicit nature of the representations for compound systems introduced in this paper allows it to be explicitly confronted with some other points of view, and to contribute in a constructive way to some of the remaining problems in the understanding of physics. For example: a somewhat metaphorical characterization of symmetric and anti-symmetric superpositions has been given in [13], and from this might result an answer to the question why we only encounter them in nature; a more general and explicit answer might result from the representation introduced in this paper, by an explicitation in it of the ideas launched in [15].

6 Appendix: independence of the choice of the biorthogonal decomposition.

In this appendix we show the well-definedness of our representation, i.e., it does not depend on the choice of the biorthogonal decomposition if there exist more than one. Let $\sum_i a_i^l \psi_i \otimes \tilde{\psi}_i$ be a biorthogonal decomposition of $\Psi$ and suppose that $\sum_i a_i^l \psi_i^t \otimes \tilde{\psi}_i^t$ is a second one. Since the vectors in $\{\psi_i\}_i$, $\{\tilde{\psi}_i\}_i$, $\{\psi_i^t\}_i$ and $\{\tilde{\psi}_i^t\}_i$, that appear explicitly in the biorthogonal decompositions can always be extended to orthonormal bases of equal dimensional Hilbert spaces, there exist two unitary matrices $\{U_{i,j}\}_{i,j}$ and $\{V_{i,j}\}_{i,j}$ such that for all $i$ we have $\psi_i^t = \sum_j U_{i,j} \psi_j$ and $\tilde{\psi}_i^t = \sum_j V_{i,j} \tilde{\psi}_j$. Since both biorthogonal
decomposition are representations of the same state we have for all $i$ and $j$:

$$\langle \psi_i \otimes \tilde{\psi}_j | \sum_k a'_{k} \psi'_k \otimes \tilde{\psi}'_k \rangle = \langle \psi_i \otimes \psi_j | \sum_k a_k \psi_k \otimes \tilde{\psi}_k \rangle = \delta_{i,j} a_i$$

(25)

Since,

$$\langle \psi_i \otimes \tilde{\psi}_i | \sum_k a'_{k} \psi'_k \otimes \tilde{\psi}'_k \rangle = \sum_k a'_k U_{k,i} V_{k,j}$$

(26)

we find:

$$\sum_k a'_k U_{k,i} V_{k,j} = \delta_{i,j} a_i$$

(27)

As a consequence, we also have:

$$\sum_j \sum_k a'_k U_{k,i} V_{k,j} \sum_l a'_l U_{l,m} V_{l,j} = |a_i|^2 \delta_{i,m}$$

(28)

Since,

$$\sum_j \sum_k a'_k U_{k,i} V_{k,j} \sum_l a'_l U_{l,m} V_{l,j} = \sum_k |a'_k|^2 U_{k,i} \bar{U}_{k,m}$$

(29)

we obtain:

$$\sum_k |a'_k|^2 U_{k,i} \bar{U}_{k,j} = |a_i|^2 \delta_{i,j}$$

(30)

For the second biorthogonal decomposition we find:

$$\frac{1}{N' (\phi)} \sum_i a_i \langle \phi | \psi'_i \rangle \tilde{\psi}'_i = \frac{1}{N' (\phi)} \sum_{i,j,k} a_{i,j} U_{i,k} V_{j,k} \langle \phi | \psi'_i \rangle \tilde{\psi}'_k = \frac{1}{N' (\phi)} \sum_j a_j \langle \phi | \psi_j \rangle \tilde{\psi}_j$$

by applying eq. (27), and where:

$$N' (\phi) = \sqrt{\sum_i |a'_i|^2 |\langle \sum_j U_{i,j} \psi_j | \phi \rangle|^2} = \sqrt{\sum_j |a_j|^2 |\langle \psi_j | \phi \rangle|^2} = N (\phi)$$

by applying eq. (30). Along the same lines we can also prove that the choice of the biorthogonal decomposition doesn’t influence $T_{\xi_0 \ldots \alpha}$. Since $N (\phi)$ is equal to the probability of a transition to a state $\phi$ when we are in a proper state $\omega_{\xi_0 \ldots \alpha}$, we have also proved that both biorthogonal decompositions always determine the same map $R_{\xi_0 \ldots \alpha}$.

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References

[1] D. Aerts, *The One and the Many*, Doctoral Dissertation, Free University of Brussels (1981); Found. Phys. 12, 1131 (1982).

[2] D. Aerts, *J. Math. Phys.* 27, 202 (1986).

[3] D. Aerts, *Helv. Phys. Acta* 64, 1 (1991).

[4] D. Aerts, *Found. Phys.* 24, 1227 (1994).

[5] D. Aerts, 'The Entity and Modern Physics: The Creation-Discovery View of Reality', in *Interpreting Bodies: Classical and Quantum Objects in Modern Physics*, pp. 223-257, ed. E. Castellani, Princeton University Press, New Jersey (1998).
[6] D. Aerts and B. Coecke, 'The Creation-Discovery-View : Towards a Possible Explanation of Quantum Reality', in Language, Quantum, Music, pp. 105–116, eds. M.L. Dalla-Ciara et al., Kluwer Academic Publishers, Dordrecht (1999).

[7] J. Bell, Physics 1, 195 (1964).

[8] E.G. Beltrametti and G. Cassinelli, The Logic of Quantum Mechanics, Addison-Wesley Publishing Company, London (1981).

[9] E. Bitsakis, Scientia 117, 561 (1982).

[10] G. Cattaneo and G. Nistico, Int. J. Theor. Phys. 30, 1293 (1991).

[11] G. Cattaneo and G. Nistico, Int. J. Theor. Phys. 32, 407 (1993).

[12] J.F. Clauser and A. Shimony, Rep. Prog. Phys. 41, 1881 (1978).

[13] R. Clifton, British J. Phil. Sc. 46, 33 (1995).

[14] B. Coecke, Found. Phys. Lett. 8, 437 (1995).

[15] B. Coecke, Int. J. Theor. Phys. 35, 1217 (1996).

[16] B. Coecke, Hidden Measurement Systems, Doctoral Dissertation, Free University of Brussels (1996); Helv. Phys. Acta 70, 442 (1997); Helv. Phys. Acta 70, 462 (1997); arXiv: quant-ph/0008061 & 0008062.

[17] B. Coecke, Found. Phys. 28, 1347 (1998).

[18] B. Coecke, Int. J. Theor. Phys. 39, 581 (2000); arXiv: quant-ph/0008054.

[19] B. Coecke and F. Valckenborgh, Int. J. Theor. Phys. 37, 311 (1998).

[20] B. d’Espagnat, Conceptual Foundations of Quantum Mechanics, W.A. Benjamin, London (1976).

[21] D.J. Foulis and C.H. Randall, Found Phys. 14, 65 (1984).

[22] S.J. Freedman and J.F. Clauser, Phys. Rev. Lett. 28, 938 (1972).

[23] N. Gisin and C. Piron, Lett. Math. Phys. 5, 379 (1981).

[24] A.M. Gleason, J. Math. Mech. 6, 885 (1957).

[25] R. Healey, The Philosophy of Quantum Mechanics, Cambridge Univ. Press, Cambridge (1989).

[26] F. Herbut and M. Vujčić, Ann. Phys. 96, 382 (1976).

[27] D.J. Moore, Helv. Phys. Acta 68, 658 (1995).

[28] D.J. Moore, Stud. Hist. Phil. Mod. Phys. 30, 61 (1999).

[29] W. Pauli, Die Allgemeinen Prinziopen der Wellenmechanic, Handbuch der Physik Vol. V, Part I, Springer-Verlag, Berlin (1958).

[30] C. Piron, Foundations of Quantum Physics, W.A. Benjamin, London (1976).

[31] E. Schmidt, Math. Ann. 63, 433 (1907).

[32] E. Schrödinger, Proc. Cambridge Philos. Soc. 31, 555 (1935).

[33] E. Schrödinger, Proc. Cambridge Philos. Soc. 32, 446 (1936).

[34] F. Valckenborgh, 'Operational Axiomatics and Compound Systems', in Current Research in Operational Quantum Logic: Algebras Categories, Languages, eds. B. Coecke, D.J. Moore and A. Wilce, Kluwer Academic Publishers, Dordrecht (2000).

[35] B.C. van Fraassen, Quantum Mechanics, Clarendon Press, Oxford (1991).

[36] J. von Neumann, The Mathematical Foundations of Quantum Mechanics, Princeton University Press, Princeton (1955).

[37] M. Vujčić and F. Herbut, J. Math. Phys. 25, 2253 (1984).