ABOUT THE LENSE-THIRRING
AND THIRRING EFFECTS

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Abstract. In Part I we prove that in the linearized version of GR the “dust-like” model of a spinning sphere cannot give the Lense-Thirring and Thirring effects. In Part II we give a proper and model-independent deduction of both effects within the linearized version of GR.

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Introduction – As it is known, in recent years various experiments with the object to verify the real existence of the Lense-Thirring effect [1] have been performed. And the researchers have concluded that this effect exists [2], [3]. The present paper regards exclusively the theoretical aspect of the question. In Part I we prove that the computations of Lense and Thirring [1] have a fundamental defect, which makes problematic their result. Actually, the linearized version of GR does not give any frame-dragging effect à la Lense-Thirring, if we employ the “dust-like” model for a spinning sphere. An analogous negative conclusion holds for Thirring’s articles quoted in [4]. In Part II we deduce both effects within the linear approximation of GR. Our results are model-independent and physically adequate – they have no components of centrifugal-like accelerations along the rotation axis of the spheres.

PART I

1. – The linear approximation of GR allows to start from eqs. (1) of Lense-Thirring [1]; $\gamma_{\mu\nu}$ is the Minkowskian Kronecker tensor:

\[
\begin{align*}
\delta_{\mu\nu} & = -\delta_{\mu\nu} + \gamma_{\mu\nu} ; \\
\gamma_{\mu\nu} & = \gamma'_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \gamma'_{\alpha\alpha} ; \\
\gamma'_{\mu\nu} & = -\frac{\kappa}{2\pi} \int \frac{T_{\mu\nu}(x', y', z', t - R)}{R} dV ; \ (\alpha, \mu, \nu = 1, 2, 3, 4 \ ; \ x_4 = i t) : \\
\kappa & \text{ is the Einsteinian gravitational constant}; \ dV = r^2 dr' \sin \theta' d\theta' d\phi' ; \ R \text{ is the distance between the field point and the integration element. For } T_{\mu\nu} \text{ the Authors choose (for simplicity) the energy tensor of a “cloud of dust”, i.e.:}
\end{align*}
\]
(2) \[ T_{\mu\nu} = T^{\mu\nu} = \varrho_0 \frac{dx_{\mu}}{ds} \frac{dx_{\nu}}{ds} \; ; \]

the scalar \( \varrho_0 \) is the invariant mass density. The *postulated* velocity components are:

\[
\begin{align*}
\frac{dx_1}{dx_4} &= -i \frac{dz'}{dt} = i r' \omega \sin \vartheta' \sin \varphi' \\
\frac{dx_2}{dx_4} &= -i \frac{dy'}{dt} = -i r' \omega \sin \vartheta' \cos \varphi' \\
\frac{dx_3}{dx_4} &= 0 ;
\end{align*}
\]

\( r', \vartheta', \varphi' \) are the polar coordinates of a point of the rotating sphere of “dust”; the rotation happens round the \( Z \)-axis with angular velocity \( \omega \).

Formulæ (3) play an essential role in the computations of the papers \[1\], \[4\]. Unfortunately, these formulæ are not consistent with the equations of motion of the “dust” particles that are prescribed by the linear theory, as we shall prove.

2. – It is well known that in the linear approximation of GR the ordinary divergence of \( T_{\mu\nu} \) is equal to zero. Now, in 1918 people did not know that the equations of matter motion are an analytical consequence of the gravitational field equations (both in the exact GR and in its linear approximation \[5\]). In our case, the equations \( \partial T_{\mu\nu}/\partial x_\nu = 0 \) imply the equations of motion of the “dust” particles. But these equations tell us that the particles describe rectilinear and uniform motions; any acceleration is excluded, in particular any rotation. The proof is quite trivial; let us consider the equations

\[
\frac{\partial}{\partial x_\nu} (\varrho_0 u^{\mu} u_\nu) = 0 
\]

where \( u_\mu = dx_\mu/ds \); from which

\[
\begin{align*}
\varrho_0 u_\mu \frac{\partial}{\partial x_\nu} (\varrho_0 u^{\mu}) + \varrho_0 u_\nu \frac{\partial u^{\mu}}{\partial x_\nu} = 0 ;
\end{align*}
\]

multiply these equations by \( u_\mu \); the second term gives zero (because \( u_\mu \partial u^{\mu}/\partial x_\nu = 0 \); we are left with \( \varrho_0 \partial (\varrho_0 u^{\mu})/\partial x_\nu = 0 \), the conservation equation of matter – and eqs. (5) reduce to

\[
\begin{align*}
u \frac{\partial u^{\mu}}{\partial x_\nu} = 0 \\
i.e. ;
\end{align*}
\]

\[
\frac{du^{\mu}}{ds} = 0 , \; q.e.d.
\]
This means that Lense-Thirring and Thirring effects could be possibly obtained only with computations that go beyond the linear stage of approximation, if we adopt the “dust-like” model for the spinning spheres. Such conclusion holds also for the frame-dragging effect induced by a generic non-zero four-acceleration of the “dust” particles [6].

3. – In their papers [1], [4] Thirring and Lense employed, more antiquo, an imaginary time coordinate \( x_4 = it \). This makes particularly intuitive at any computational step that the considered spacetime \( \mathfrak{P} \) is pseudo-Euclidean, i.e. that we are dealing with the special relativity. The \( \gamma_{\mu\nu} \)'s are the components of a symmetrical tensor-field of the second order in the Minkowskian spacetime \( \mathfrak{P} \).

The field equations of the linear approximation of GR remain unchanged if \( \gamma_{\mu\nu} \) is replaced by

\[
\gamma'_{\mu\nu} = \gamma_{\mu\nu} + \frac{\partial \xi_\mu}{\partial x_\nu} + \frac{\partial \xi_\nu}{\partial x_\mu} ;
\]

the four functions \( \xi_\mu \)'s can be chosen in such a way that \( \gamma'_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \gamma_{\alpha\alpha} \) satisfies the equations

\[
\frac{\partial \gamma'_{\mu\nu}}{\partial x_\nu} = 0 ,
\]

in correspondence with the differential conservation equations \( \partial T_{\mu\nu} / \partial x_\nu = 0 \).

Eqs. (8) have a twofold interpretation: i) from the Minkowskian standpoint they represent a gauge transformation which is quite analogous to a gauge transformation of Maxwell electrodynamics [5]; ii) from a general standpoint they are the result of an infinitesimal coordinate transformation \( x'_\mu = x_\mu + \xi_\mu(x) \) on \( g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \).

A last remark. Of course, the physically legitimate \( \gamma_{\mu\nu} \)'s allow to compute the geodesic of a test-particle in the gravitational potential \( g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \).

PART II

4. – Frame-dragging by a spinning full sphere – We use the previous notations, but employing the CGS system of units. The fundamental equations of the linear approximation of GR are:

\[
\Box \gamma'_{\mu\nu} = \frac{\kappa}{2\pi} T_{\mu\nu} ; \quad (x_4 = ict) .
\]

It follows from eqs. (10) that any material homogeneous sphere \( \Sigma \) (mass \( M \), radius \( l \)), which is at rest in our Cartesian orthogonal system \( S(X_1, X_2, X_3) \), with its centre at \( X_1 = X_2 = X_3 = 0 \), generates an external gravitational potential given by:
\[ \gamma'_{44}(0) = \frac{4GM/c^2}{R} \equiv \frac{4m_i}{R} ; \quad \gamma'_{\mu\nu}(0) = 0 , \quad \text{if } (\mu, \nu) \neq (4, 4) , \]

where \( R = (X_1^2 + X_2^2 + X_3^2)^{1/2} \geq l \).

Let us now assume that \( \Sigma \) rotates round the \( X_3 \)-axis, with a small angular velocity \( \omega \), with respect to \( S \). We compute the new \( \gamma'_{\mu\nu} \)'s, neglecting with Lense and Thirring (and for the identical reasons) the \( \gamma'_{\mu\nu} \)'s which contain \( \omega^2 \). Then, we compute the corresponding \( \gamma_{\mu\nu} \)'s. We get, if \( \chi \equiv \omega/c \):

\[
\begin{align*}
\gamma_{11} &= -\frac{2m_i}{R} ; \quad \gamma_{12} \approx 0 ; \quad \gamma_{13} = 0 ; \quad \gamma_{14} = \frac{4mi\chi}{R} X_2 ; \\
\gamma_{22} &= -\frac{2m_i}{R} ; \quad \gamma_{23} = 0 ; \quad \gamma_{24} = -\frac{4mi\chi}{R} X_1 ; \\
\gamma_{33} &= -\frac{2m_i}{R} ; \quad \gamma_{34} = 0 ; \quad \gamma_{44} = -\frac{2m_i}{R} .
\end{align*}
\]

Denoting with the small letters \( x_1, x_2, x_3, x_4, r \) the time-dependent dynamical variables which concern the geodesic motion of a test-particle through the gravitational field of eqs. (12), we have \( (j = 1, 2, 3) \):

\[
\ddot{x}_j = \frac{1}{2} \left( \gamma_{j\mu,\nu} + \gamma_{j\nu,\mu} - \gamma_{\mu\nu,j} \right) \dot{x}_\mu \dot{x}_\nu ,
\]

where an overdot means a derivative with respect to time \( t \), and the comma the derivative with respect to a coordinate \( x_\lambda \). Eqs. (13) are the equations of a geodesic with the approximation \( icdt \approx ds \) – the squares of all three-velocity components divided by \( c \) are neglected (as in \( \Pi \)). And in the three-acceleration \( \dddot{x}_j \) we retain only the terms with indices \( (\mu, \nu) = (14), (24), (34), (44) \). We have finally, if \( (x_1, x_2, x_3) \equiv (x, y, z) \):

\[
\begin{align*}
\dddot{x} &= 4m\omega \left( \frac{x^2 + y^2}{r^3} - \frac{3}{r} \right) \ddot{y} + 4m\omega y \left( \frac{x\ddot{x} + y\dddot{y} + 2z\dddot{z}}{r^3} \right) - \frac{GM}{r^2} x ; \\
\dddot{y} &= -4m\omega \left( \frac{x^2 + y^2}{r^3} - \frac{3}{r} \right) \ddot{x} - 4m\omega x \left( \frac{x\dddot{x} + y\ddot{y} + 2z\dddot{z}}{r^3} \right) - \frac{GM}{r^2} y ; \\
\dddot{z} &= -\frac{GM}{r^2} z .
\end{align*}
\]

In Appendix A we have transcribed formulae (15) of Lense and Thirring \( \Pi \) in the CGS system of units. The \( z \)-component of their acceleration contains the Coriolis-like term \((m/r^2)(\omega l^2/r)[(12z/5r)(x\dot{y} - y\dot{x})/r] \). This would be sufficient to conclude that their results are problematic.

5. – Frame-dragging by a spinning hollow sphere – We consider with Thirring an infinitely thin spherical shell \( \Sigma \) of radius \( a \) and mass \( M \). If
this shell is at rest in $S$, the *internal* gravitational potential can be suitably written as follows:

\[ \gamma_{44}'(0) = \frac{4m}{a} \quad \text{if } (\mu, \nu) \neq (4, 4) ; \]

this expression of the (constant) internal potential coincides with the value for $R = a$ of the external potential $4m/R$. Assume now that $\Sigma$ rotates round the $X_3$-axis. With Thirring [4], in the computation of the $\gamma_{\mu\nu}'s$ we take into consideration also the terms with $\omega^2(\equiv c^2\chi^2)$, that will give a kind of centrifugal force.

Derive the $\gamma_{\mu\nu}$'s from the $\gamma_{\mu\nu}'s$; putting for brevity $K \equiv \frac{4m}{a}$, we have:

\[
\begin{align*}
\gamma_{11} &= \frac{1}{2} K - \frac{1}{2} K\chi^2 (X_2^2 - X_1^2) \quad \gamma_{12} = K\chi^2 X_1 X_2 ; \\
\gamma_{13} &= 0 ; \quad \gamma_{14} = Ki\chi X_2 ; \\
\gamma_{22} &= -\frac{1}{2} K + \frac{1}{2} K\chi^2 (X_1^2 - X_2^2) \quad \gamma_{23} = 0 ; \\
\gamma_{24} &= -Ki\chi X_1 ; \\
\gamma_{33} &= \frac{1}{2} K + \frac{1}{2} K\chi^2 (X_1^2 + X_2^2) \quad \gamma_{34} = 0 ; \\
\gamma_{44} &= \frac{1}{2} K + \frac{1}{2} K\chi^2 (X_1^2 + X_2^2) .
\end{align*}
\]

The equations of the geodesic of a test-particle in the gravitational field of these $\gamma_{\mu\nu}$'s are identical to eqs. (13). And, as in sect. 4, we consider only the terms with indices $(\mu, \nu) = (14), (24), (34), (44)$. We get:

\[
\begin{align*}
\ddot{x} &= -\frac{12}{a} m \omega \dot{y} + 2 \frac{m}{a} \omega^2 x ; \\
\ddot{y} &= -\frac{12}{a} m \omega \dot{x} + 2 \frac{m}{a} \omega^2 y ; \\
\ddot{z} &= 0 .
\end{align*}
\]

In Appendix B we have transcribed formulae (22) of Thirring paper [4] in the CGS system of units. *Thirring’s formula for $\ddot{z}$ is a clear absurdity from the physical standpoint.* He tried to justify it with the following sentences: “Die dritte Gleichung [(22)] liefert das im ersten Augenblick überraschende Ergebnis, daß diese “Zentrifugalkraft” noch eine axiale Komponente besitzt [i.e., $-\frac{8m}{15a}\omega^2 z$]. Ihr Auftreten im Felde der rotierenden Kugel läßt sich folgendermaßen aufklären: Vom ruhenden Beobachter aus betrachtet haben jene Flächenelmente der Hohlkugel, welche sich in der Nähe des Äquators befinden, größere Geschwindigkeit, und infolgedessen auch größere scheinbare (träge und gravitierende) Masse als jene, die sich in der Umgebung der Pole befinden. Das Feld einer mit konstanter Flächendichte belegten rotierenden Hohlkugel entspricht also dem einer ruhenden Kugelschale, bei
welcher die Flächendichte mit wachsendem Polabstand \( \vartheta \) zunimmt. Daß im letzteren Falle Punkte [i.e. test-particles], die außerhalb der Äquatorebene liegen, in sie hineingezogen werden, ist ohne weiteres verständlich.” – This is a clever, but illogical (and \textit{a posteriori}) justification. Indeed, in the deduction of Thirring’s formulae (22) there is no use of a possible difference (by virtue of the difference of the respective velocities) between the masses of the equatorial zone and the masses of the polar zones.

We hope that our deduction of the Lense-Thirring effect will be appreciated by the teams of experimentalists, who, after years of refined efforts, have concluded in favour of the existence of this effect – notwithstanding the subtle difficulties to take properly into account the systematic errors of measurement. –

**APPENDIX A**

Formulae (15) of Lense and Thirring \cite{1} in the CGS system of units \((m \equiv GM/c^2; l \) is the radius of the sphere):

\[
\begin{align*}
\ddot{x} &= \frac{m \omega l^2}{r^2} \left[ \frac{4}{5} \frac{x^2 + y^2 - 2z^2}{r^2} \dot{y} + \frac{12}{5} \frac{yz}{r^2} \dot{z} \right] - \frac{GM}{r^2} \frac{x}{r} ; \\
\ddot{y} &= -\frac{m \omega l^2}{r^2} \left[ \frac{4}{5} \frac{x^2 + y^2 - 2z^2}{r^2} \dot{x} + \frac{12}{5} \frac{xz}{r^2} \dot{z} \right] - \frac{GM}{r^2} \frac{y}{r} ; \\
\ddot{z} &= \frac{m \omega l^2}{r^2} \left[ \frac{12}{5} \frac{z}{r} \frac{y}{r} \dot{x} + \frac{12}{5} \frac{xz}{r^2} \dot{y} \right] - \frac{GM}{r^2} \frac{z}{r} .
\end{align*}
\]

**APPENDIX B**

Formulae (22) of Thirring \cite{4} in the CGS system of units \((m \equiv GM/c^2; a \) is the radius of the spherical shell):

\[
\begin{align*}
\ddot{x} &= -\frac{8m}{3a} \omega \dot{y} + 4 \frac{m}{15a} \omega^2 x ; \\
\ddot{y} &= -\frac{8m}{3a} \omega \dot{x} + 4 \frac{m}{15a} \omega^2 y ; \\
\ddot{z} &= -\frac{8m}{15a} \omega^2 z .
\end{align*}
\]

(We have taken into account the \textit{Berichtigung} of 1921, see \cite{4} ii). –
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[4] i) H. Thirring, Physik. Zeitsch., 19 (1918) 33; ii) Idem, ibid., 22 (1921) 29.
[5] Cf. A. Loinger, arXiv:physics/0407134 (July 27th, 2004); in Appendix I have reproduced sects. 1 and 2 of a beautiful memoir by H. Weyl in Amer. J. Math., 66 (1944) 591.

(In the above Appendix the expressions that follow eq. (3) contain some trivial misprints; the right formulae are: $\Box \varphi_i$ and $\partial \varphi'/\partial x_i$ ($\varphi' = \partial \varphi^p \partial x_p$).)
[6] Cf. A. Einstein, The Meaning of Relativity, (Princeton University Press, Princeton, N.J.) 1955, pp.100÷103.

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