M-indeterminate distributions in quantum mechanics and the non-overlapping wave function paradox

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Abstract: We consider the non-overlapping wave function paradox of Aharanov et al., wherein the relative phase between two wave functions cannot be measured by the moments of position or momentum. We show that there is an unlimited number of other expectation values that depend on the phase. We further show that the Wigner distribution is M-indeterminate, that is, a distribution whose moments do not uniquely determine the distribution. We generalize to more than two non-overlapping functions. We consider arbitrary representations and show there is an unlimited number of M-indeterminate distributions. The dual case of non-overlapping momentum functions is also considered.

Keywords: M-indeterminate quantum distributions; non-overlapping wave functions; Wigner distribution; characteristic function; momentum distribution.

1 Introduction

In a series of papers Aharanov et al. \textsuperscript{1-5}, and others \textsuperscript{6-8}, discussed a paradox arising when one considers a wave function that consists of the sum of two non-overlapping functions. In particular they considered a wave function of the form

$$\psi(x) = \frac{1}{\sqrt{2}} (\psi_1(x) + e^{i\alpha} \psi_2(x))$$

(1)

where $\alpha$ is real and is the relative phase between the two functions $\psi_1(x)$ and $\psi_2(x)$, each of which is normalized to one. The issue is how to determine the relative phase for the case when $\psi_1(x)$ and $\psi_2(x)$ are of finite extent and do not overlap,

$$\psi_1^*(x)\psi_2(x) = 0$$

(2)

Accordingly, the position distribution,

$$P(x) = |\psi(x)|^2 = \frac{1}{2} |\psi_1(x) + e^{i\alpha} \psi_2(x)|^2 = \frac{1}{2} \left(|\psi_1(x)|^2 + |\psi_2(x)|^2\right)$$

(3)

is independent of $\alpha$ and hence so, too, are the position moments. Indeed, the expected value of any function of position will be independent of $\alpha$. Aharanov et al. then showed that the momentum moments

$$\langle p^n \rangle = \frac{1}{2} \int \left(\psi_1^*(x) + e^{-i\alpha} \psi_2^*(x)\right) \left(\frac{\hbar}{i} \frac{d}{dx}\right)^n \left(\psi_1(x) + e^{i\alpha} \psi_2(x)\right) dx$$

(4)

$$= \frac{1}{2} \int \psi_1^*(x) \left(\frac{\hbar}{i} \frac{d}{dx}\right)^n \psi_1(x) dx + \frac{1}{2} \int \psi_2^*(x) \left(\frac{\hbar}{i} \frac{d}{dx}\right)^n \psi_2(x) dx$$

(5)

likewise are independent of $\alpha$. This results in an apparent paradox. We consider various aspects of the problem in the context of what are known as M-indeterminate distributions, that is, a distribution whose moments do not uniquely determine the distribution \textsuperscript{9}.
2 Momentum probability distribution

We begin with the case of the sum of two non-overlapping wave functions, Eq. (1), that are translated versions of each other,

\[
\psi_1(x) = \begin{cases} f(x) & 0 \leq x \leq a \\ 0, & \text{otherwise} \end{cases}, \quad \psi_2(x) = \begin{cases} f(x - L) & a < L \leq x \leq L + a \\ 0, & \text{otherwise} \end{cases}
\]

The momentum wave function is

\[
\varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x) e^{-ipx/\hbar} dx = \frac{1}{\sqrt{2}} \left( \varphi_1(p) + e^{i\alpha} \varphi_2(p) \right)
\]

where \(\varphi_1(p)\) and \(\varphi_2(p)\) are the momentum wave function of \(\psi_1(x)\) and \(\psi_2(x)\) respectively.

Letting

\[
F(p) = \varphi_1(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^a f(x) e^{-ipx/\hbar} dx
\]

we have that

\[
\varphi(p) = \frac{1}{\sqrt{2}} F(p)(1 + e^{i(\alpha - pL/\hbar)})
\]

Hence, the probability distribution of momentum is given by

\[
P(p) = |\varphi(p)|^2 = |F(p)|^2 [1 + \cos (pL/\hbar - \alpha)]
\]

Observe that the momentum distribution depends on \(\alpha\). However, as mentioned above the moments \(\langle p^n \rangle\) do not. Before considering this apparent paradox further, we give an alternate derivation of the moment independence on \(\alpha\) directly from the distribution, as follows.

Note that

\[
|F(p)|^2 = \frac{1}{2\pi\hbar} \left| \int_0^a f(x) e^{-ipx/\hbar} dx \right|^2
\]

is a proper normalized distribution, meaning \(\int |F(p)|^2 dp = 1\). Thus, we must have

\[
\int |F(p)|^2 \cos (pL/\hbar - \alpha) \ dp = 0
\]

Differentiating Eq. (11) with respect to \(\alpha\) yields

\[
\int |F(p)|^2 \sin (pL/\hbar - \alpha) \ dp = 0
\]

If we further differentiate Eq. (11) and Eq. (12) with respect to \(L\), it follows that

\[
\int p^n |F(p)|^2 \cos (pL/\hbar - \alpha) \ dp = 0
\]

and therefore, as previously shown by way of Eq. (5), the moments are independent of \(\alpha\) and are given by

\[
\langle p^n \rangle = \int p^n |F(p)|^2 \ dp
\]

Thus the moments of \(|F(p)|^2\) and \(P(p)\) are identical, even though \(P(p) \neq |F(p)|^2\). Distributions of the form of Eq. (9) are moment-indeterminate (or “M-indeterminate”), an observation first made by Semon and Taylor in the context of the Aharonov-Bohm Effect \[7\]. We revisit this distribution and the non-overlapping wave function paradox from a number of perspectives. First, we explain the M-indeterminate nature of the distribution in terms of the characteristic function. This understanding of the root of the paradox allows us to give a general condition for an unlimited number of expectation values that do depend on the phase factor. We generalize to more than two non-overlapping functions and obtain a general expression for a family of M-indeterminate
momentum distributions. We also generalize to arbitrary representations, which allows the generation of an infinite number of M-indeterminate distributions from the wave function. Furthermore, we show that while the quantum mechanical current does not depend on the phase factor, its dual, which we call the quantum mechanical group delay, does. We show that the Wigner distribution, among other phase space distributions, is M-indeterminate for non-overlapping wave functions. We also consider the dual problem, where the momentum wave function consists of two non-overlapping functions.

Also, we point out that while the position wave function has no interference term, the momentum wave function does since it extends over all momentum space. This must always be the case since the position and momentum wave functions are Fourier transform pairs, and Fourier transform pairs can not both be of finite extent.

3 Momentum characteristic function

The problem of M-indeterminate distributions has a long history, with a majority of the focus being on devising such distributions as well as determining if a given distribution is M-indeterminate [9, 10, 11, 12, 13, 14, 15]. Our interest is in M-indeterminate distributions in quantum mechanics, which provides unique challenges and opportunities because of the way probabilities are obtained.

One way to study aspects of the M-indeterminacy problem, which does not seem to have been as extensively explored, is by way of the characteristic function. The distribution \( P(p) \) and characteristic function \( M(\theta) \) are Fourier transform pairs,

\[
M(\theta) = \int e^{i\theta p} P(p) dp = \langle e^{i\theta p} \rangle; \quad P(p) = \frac{1}{2\pi} \int e^{-i\theta p} M(\theta) d\theta
\]  

For our situation we have

\[
M(\theta) = \langle e^{i\theta p} \rangle = \int e^{i\theta p} |F(p)|^2 [1 + \cos(pL/\hbar - \alpha)] dp
\]

Since \( |F(p)|^2 \) is a probability distribution, we define its characteristic function by

\[
M_F(\theta) = \int |F(p)|^2 e^{i\theta p} dp
\]

by which we obtain

\[
M_F(\theta) = \left\{ \begin{array}{ll}
\int_a^0 f(x)f^*(x - \theta \hbar) dx, & -a/\hbar \leq \theta \leq a/\hbar \\
0, & \text{otherwise}
\end{array} \right.
\]

Notice that as a direct consequence of the finite extent of \( f(x) \), the characteristic function \( M_F(\theta) \) is also of finite extent (it is zero for \( |\theta| > a/\hbar \)). Accordingly, by Fourier properties, the momentum distribution \( P(p) \) extends over all \( p \).

The characteristic function \( M(\theta) \) as given by Eq. (16) may be expressed in terms of \( M_F(\theta) \) as

\[
M(\theta) = M_F(\theta) + \frac{1}{2} e^{-i\alpha} M_F(\theta + L/\hbar) + \frac{1}{2} e^{i\alpha} M_F(\theta - L/\hbar)
\]

Note that, like the momentum distribution, the characteristic function depends on \( \alpha \). Since in general the moments of a distribution can be obtained from the characteristic function by

\[
\langle p^n \rangle = \frac{1}{i^n} \frac{\partial^n}{\partial \theta^n} M(\theta) \bigg|_{\theta = 0}
\]

we see again that the moments are independent of \( \alpha \) since, with \( L > a \), we have that

\[
M_F(\theta \pm L/\hbar) \big|_{\theta = 0} = 0
\]

Note that this result highlights the root of the indeterminacy: Eq. (19) shows that the terms that depend on \( \alpha \) are shifted such that they are not centered about \( \theta = 0 \). Because these terms are finite extent and the shift is
greater than the half-width of the “unshifted” characteristic function $M_F(\theta)$, the dependence on $\alpha$ is lost when we evaluate the characteristic function at $\theta = 0$ to obtain the moments. Specifically, we have that
\[
\frac{1}{i^n} \frac{\partial^n}{\partial \theta^n} \left\{ e^{-i \alpha M_F(\theta + L/\hbar)} + e^{i \alpha M_F(\theta - L/\hbar)} \right\} \bigg|_{\theta = 0} = 0
\] (22)

Therefore, although $P(p) \neq |F(p)|^2$, they have identical moments
\[
\langle p^n \rangle = \frac{1}{i^n} \frac{\partial^n}{\partial \theta^n} M(\theta) \bigg|_{\theta = 0} = \frac{1}{i^n} \frac{\partial^n}{\partial \theta^n} M_F(\theta) \bigg|_{\theta = 0}
\] (23)

Hence, the family of distributions $P(p)$ (parameterized by $\alpha$) are M-indeterminate.

Historically, the first example of an M-indeterminate distribution was devised by Stieltjes but perhaps the best known one is the log-normal distribution,
\[
P_{\text{LN}}(x) = \frac{1}{x\sqrt{2\pi}} \exp \left[ -\frac{(\ln x)^2}{2} \right] \quad 0 < x < \infty
\] (24)

for which the moments are $\langle x^n \rangle = e^{n^2/2}$, but they do not uniquely determine the distribution, since
\[
P(x) = P_{\text{LN}}(x) \left[ 1 + \beta \sin(2\pi \ln x) \right], \quad -1 \leq \beta \leq 1
\] (25)
is a proper probability distribution that has the same moments as $P_{\text{LN}}(x)$. Thus, the log-normal is M-indeterminate.

There are two well known criteria for the moment indeterminacy problem, one that deals with the moments directly and the other deals with the distribution. We consider distributions that range from $-\infty$ to $\infty$ which is called the Hamburg case otherwise it is called the Stieltjes case. The Carleman condition is that if all the moments, $\langle x^n \rangle$, of a distribution are finite and if
\[
\sum_{n=1}^{\infty} \frac{1}{\langle x^{2n} \rangle^{1/2n}} = \infty
\] (26)

then the distribution having these moments is unique, that is, it is M-determinate. This is a sufficient but not necessary condition. The other criterion is the Krein condition: If
\[
- \int_{-\infty}^{\infty} \log P(x) \frac{dx}{1 + x^2} < \infty
\] (27)

then the moments do not determine a unique distribution, that is, it is an M-indeterminate distribution. Again, this is a sufficient but not necessary condition.

In our case we have the distribution $P_F(p)$ given by
\[
P_F(p) = |F(p)|^2 = \frac{1}{2\pi \hbar} \left| \int_0^a f(x) e^{-ipx/\hbar} dx \right|^2
\] (28)

which we have shown is M-indeterminate since
\[
\int p^n |F(p)|^2 dp = \int p^n |F(p)|^2 \left[ 1 + \cos (pL/\hbar - \alpha) \right] dp
\] (29)

However even though we have shown that by construction the distribution $P_F(p)$ is M-indeterminate, it would be interesting to apply the above criteria. A challenge is that because $f(x)$ is of finite extent, and hence so is the characteristic function, the moments $\langle p^n \rangle$ may not exist for all $n$. In particular, if a function $g(x)$ is infinitely differentiable, such that all of its Fourier (i.e., $p$-) moments exist, it does not necessarily follow that the function $f(x) = g(x)u(x)u(a-x)$ is infinitely differentiable because of the singularities arising from derivatives of the step function $u(x)$ (1 for $x \geq 0$ and zero otherwise). Accordingly, it follows from the differentiation theorem of the Fourier transform that not all of the moments $\langle p^n \rangle$ of the momentum distribution corresponding to $f(x)$
will exist. In particular, if the $n = N^{th}$ derivative of $f(x)$ contains a singularity, i.e., a Dirac delta function, then $|F(p)|^2 \sim 1/p^{2N}$ for $p \gg 1$.

For all of the moments to exist, we require finite extent functions that are infinitely differentiable, such as “bump” functions. However, there are many finite extent functions that are not infinitely differentiable. Hence, there clearly are M-indeterminate distributions that do not have all finite moments, as given by Eq. (29) (equivalently Eq. (23)) when $f(x)$ is not infinitely differentiable, yet in such cases, the Carleman condition can not be used. We consider it an interesting problem to find conditions of M-indeterminacy when the moments are not all finite but we are not aware of any results in that regard.

4 Expectation values that depend on the phase factor

We now show that, although the momentum moments do not depend on the phase factor, there is an unlimited number of other expectation values that do. That this should be so is of course not surprising, given that the momentum distribution depends on the phase factor, as does the characteristic function, Eq. (16), which we consider it an interesting problem to find conditions of M-indeterminacy when the moments are not all finite but we are not aware of any results in that regard.

Therefore at least one of the expectation values $\langle \cos(\theta p) \rangle$ or $\langle \sin(\theta p) \rangle$ must depend on $\alpha$ for some value(s) of $\theta$. This is analogous to the expectations of the shift operator proposed by Aharanov et al. which in the momentum representation is given by $e^{ipL/\hbar}$ (they further noted that operators that are “exponentials of the position and momentum” will also depend on $\alpha$). Here, we give general conditions on functions $g(p)$ so that the expectation value $\langle g(p) \rangle$ depends on $\alpha$ for the distributions given by Eq. (19). To achieve this, we require that

$$\int g(p) |F(p)|^2 \cos (pL/\hbar - \alpha) \, dp \neq 0$$

Equivalently, in terms of the characteristic function we have, by Eq. (19) and the multiplication/convolution property of the Fourier transform,

$$\left\{ e^{-i\alpha} \int_{-(a+L)/\hbar}^{(a-L)/\hbar} G(\theta' - \theta) \, M_F(\theta' + L/\hbar) \, d\theta' + e^{i\alpha} \int_{-(a-L)/\hbar}^{(a+L)/\hbar} G(\theta' - \theta) \, M_F(\theta - L/\hbar) \, d\theta' \right\}_{\theta=0} \neq 0$$

where the limits of integration follow from the fact that $M_F(\theta)$ is zero for $|\theta| > a/\hbar$, and

$$G(\theta) = \int g(p) e^{-i\theta p} \, dp$$

Thus, our aim is to find functions $G(\theta)$ such that

$$\int_{-(a-L)/\hbar}^{(a-L)/\hbar} G(\theta') \, M_F(\theta' + L/\hbar) \, d\theta' \neq 0 \quad \text{and/or} \quad \int_{-(a+L)/\hbar}^{(a+L)/\hbar} G(\theta') \, M_F(\theta - L/\hbar) \, d\theta' \neq 0$$

This condition can be satisfied by many functions $G(\theta)$ that are non-zero over the region of integration. For example taking $G(\theta) = 1$ results in $g(p) = \delta(p)$ by which it follows that

$$\langle g(p) \rangle = \int \delta(p) |F(p)|^2 \cos (pL/\hbar - \alpha) \, dp = |F(0)|^2 \cos (-\alpha)$$

More generally, the function

$$G(\theta) = \sqrt{\frac{\pi}{\beta}} e^{-\theta^2/(4\beta)}$$

gives

$$g(p) = e^{-\beta p^2}$$

which will generally satisfy Eq. (34) and hence $\langle e^{-\beta p^2} \rangle$ will depend on $\alpha$. 

5
5 Quantum mechanical current and group delay

The quantum mechanical current is also independent of $\alpha$ [4]. However, as we show here, its dual, which we define analogous to the current but in terms of the momentum representation, depends on $\alpha$.

To obtain the current, we express $\psi(x), \psi_1(x), \text{and } \psi_2(x)$ of Eq. (11) in terms of their respective amplitude and phase,

$$\psi(x) = R(x)e^{iS(x)/\hbar} = \sqrt{\frac{1}{2}} \left( R_1(x)e^{iS_1(x)/\hbar} + e^{i\alpha}R_2(x)e^{iS_2(x)/\hbar} \right)$$

(38)

Then, the current is

$$j(x) = \frac{\hbar}{2i} \left( \psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx} \right) = R^2(x)S'(x)$$

(39)

where we have taken the mass to be equal to one. Following derivations analogous to [16, 17], we have two equivalent expressions for the current

$$j(x) = R^2(x)S'(x) = R_x^2 S_1' + R^2_2 S_2' + \frac{1}{2} R_1 R_2 (S_1' + S_2') \cos(S_1/\hbar - S_2/\hbar - \alpha)$$

$$- \frac{\hbar}{2}(R_1 R_2' - R_1' R_2) \sin(S_1/\hbar - S_2/\hbar - \alpha)$$

(40)

and

$$j(x) = \frac{1}{2} (S_1' + S_2') R^2 + \frac{1}{2} (S_1' - S_2') (R_1^2 - R_2^2) - \hbar (R_1 R_2' - R_1' R_2) \sin(S_1/\hbar - S_2/\hbar - \alpha)$$

(41)

where

$$R^2 = R^2_1 + R^2_2 + 2R_1 R_2 \cos(S_1/\hbar - S_2/\hbar - \alpha)$$

(42)

Because the functions do not overlap, we have that $R_1 R_2 = 0$ and hence the current is independent of $\alpha$ and is given by

$$j(x) = R_x^2 S_1' + R^2_2 S_2'$$

(43)

This result generalizes as well for $N > 2$ non-overlapping functions [17]: that is, the current is a weighted sum of the individual currents of each wave function, and is independent of any constant phase terms.

We also note that one can think of quantum mechanical current as the local expectation value of momentum, namely

$$\langle p \rangle_x = \int p W_\psi(x,p)dp$$

(44)

where $W_\psi(x,p)$ is the Wigner distribution of the wave function $\psi(x)$,

$$W_\psi(x,p) = \frac{1}{2\pi} \int \psi^*(x - \frac{\hbar}{2}\tau) \psi(x + \frac{\hbar}{2}\tau) e^{-i\tau p} d\tau$$

(45)

In this case, Eq. (44) equals the current,

$$\langle p \rangle_x = j(x) = R_x^2 S_1' + R^2_2 S_2'$$

(46)

We point out that many other quasi-distributions [18] also yield this result.

**Quantum group delay.** Analogous to group delay in pulse propagation, we define the quantum group delay in terms of the momentum representation as the dual to quantum mechanical current. Writing the momentum wave function in terms of its amplitude and phase,

$$\varphi(p) = B(p)e^{i\eta(p)/\hbar}$$

(47)

we define the quantum group delay, $\tau(p)$, as

$$\tau(p) = \frac{\hbar}{2i} \left( \varphi^* \frac{d\varphi}{dp} - \varphi \frac{d\varphi^*}{dp} \right) = B^2(p)\eta'(p)$$

(48)
For the case where we have two arbitrary wave functions with a relative constant phase shift (Eq. (38)), the momentum wave function in terms of amplitude and phase is

\[ \varphi(p) = B(p) e^{i\eta_1(p)/\hbar} = \sqrt{\frac{1}{2}} \left( B_1(p) e^{i\eta_1(p)/\hbar} + e^{i\alpha} B_2(p) e^{i\eta_2(p)/\hbar} \right) \]  

(49)

Accordingly, the quantum group delay is

\[ \tau(p) = B^2 \eta' = \frac{1}{2} B^2 (\eta'_1 + \eta'_2) + \frac{1}{2} (\eta'_1 - \eta'_2) (B_1^2 - B_2^2) - \hbar (B_1 B'_2 - B'_1 B_2) \sin (\eta_1/\hbar - \eta_2/\hbar - \alpha) \]  

(50)

where

\[ B^2 = B_1^2 + B_2^2 + 2B_1 B_2 \cos (\eta_1/\hbar - \eta_2/\hbar - \alpha) \]  

(51)

Note that, unlike with the expression for the current, here we have that \( B_1(p)B_2(p) \neq 0 \) in general, even if \( \psi_1(x)\psi_2(x) = 0 \). Hence, in general, the quantum mechanical group delay depends on the relative phase, \( \alpha \).

For the case of two non-overlapping wave functions where one is a translated version of the other as considered in Sctn. 4 we have

\[ B_2(p) = B_1(p); \quad \eta_2(p) = \eta_1(p) - pL/\hbar \]  

(52)

by which it follows that

\[ \tau(p) = \frac{1}{2} (2\eta'_1 - L/\hbar) |F(p)|^2 \left( 1 + \cos \left( \frac{pL/\hbar}{\hbar} \right) \right) = (2\eta'_1 - L/\hbar) |F(p)|^2 \cos^2 \left( \frac{pL/\hbar - \alpha}{2} \right) \]  

(53)

Analogous to the quantum mechanical current, the group delay can be obtained as the local expectation value of position from the Wigner distribution,

\[ \langle x \rangle_p = \int x W_\psi(x,p) \, dx = \tau(p) \]  

(54)

6 M-indeterminate quantum phase space distributions

The above considerations lead us to determine whether or not the Wigner distribution of two non-overlapping wave functions is M-indeterminate. In particular, are the mixed moments \( \langle x^n p^m \rangle \) of the Wigner distribution, independent of \( \alpha \)? For the wave function given by Eq. (1), the Wigner distribution is

\[ W_\psi(x,p) = \frac{1}{2} W_{\psi_1}(x,p) + \frac{1}{2} W_{\psi_2}(x,p) + e^{i\alpha} W_{12}(x,p) + e^{i\alpha} W_{21}(x,p) \]  

(56)

where \( W_{12} \) is the cross Wigner distribution of the functions \( \psi_1 \) and \( \psi_2 \),

\[ W_{12}(x,p) = \frac{1}{2\pi} \int \psi_1^*(x - \frac{\hbar \tau}{2}) \psi_2(x + \frac{\hbar \tau}{2}) e^{-ip\tau} \, d\tau \]  

(57)

and similarity for \( W_{21} \). Thus, the Wigner distribution depends on the phase factor \( \alpha \).

To examine whether the moments as given by Eq. (56) are dependent on \( \alpha \) we note that the first two terms in Eq. (56) do not depend on \( \alpha \) and hence we have to examine whether

\[ \langle x^n p^m \rangle_{12} = \int \int x^n p^m W_{12}(x,p) \, dx \, dp \]  

(58)

is zero or not. Substituting Eq. (57) into Eq. (58) gives

\[ \langle x^n p^m \rangle_{12} = \frac{i^m}{2\pi} \int \int \int x^n \left( \int \psi_1^*(x - \frac{\hbar \tau}{2}) \psi_2(x + \frac{\hbar \tau}{2}) \frac{\partial^m}{\partial \tau^m} e^{-ip\tau} \, d\tau \right) \, dx \, dp \]  

(59)
and straightforward integration by parts yields

\[ \langle x^n p^m \rangle_{12} = i \sum_{k=0}^{m} \binom{n}{k} (-1)^k \int x^n \left( \frac{\partial^k}{\partial x^k} \psi_1^*(x) \right) \left( \frac{\partial^{m-k}}{\partial x^{m-k}} \psi_2(x) \right) \, dx \] (60)

which could have also been obtained using the Weyl-McCoy correspondence for \( x^n p^m \). Since the wave functions do not overlap we have that

\[ \langle x^n p^m \rangle_{12} = 0 \] (61)

This shows that the mixed moments are independent of \( \alpha \) and hence the Wigner distribution is M-indeterminate. Indeed, there is an unlimited number of M-indeterminate phase space distributions; in particular, if their mixed moments can be written in terms of a (finite) sum of the form

\[ \int \psi^*(x) x^n p^m \psi(x) \, dx \] (62)

then the distribution will be M-indeterminate.

7 Multiple non-overlapping wave functions

For the case of more than two non-overlapping wave functions we define

\[ \psi(x) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \psi_n(x) \] (63)

and take the individual wave functions to be non-overlapping,

\[ \psi_n^*(x) \psi_m(x) = 0, \quad n \neq m \] (64)

Let

\[ \psi_n(x) = e^{i\alpha_n f (x - nL)} \] (65)

As with the \( N = 2 \) case, the position moments are independent of the phases \( \alpha_n \), since the probability distribution of position is given by

\[ |\psi(x)|^2 = \left| \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \psi_n(x) \right|^2 = \frac{1}{N} \sum_{n=1}^{N} |\psi_n(x)|^2 \] (66)

The momentum wave function is

\[ \varphi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x) e^{-ixp/\hbar} \, dx = \frac{F(p)}{\sqrt{N}} \sum_{n=1}^{N} e^{i(\alpha_n - npL/\hbar)} \] (67)

The summation depends on the specific \( \alpha_n \) and can not in general be further simplified, but clearly the momentum distribution \( |\varphi(p)|^2 \) depends on the phases \( \alpha_n \). If we make the simplifying assumption that \( \alpha_n = n\alpha \), which imposes a constant relative phase difference between the adjacent \( \psi_n(x) \), then we have

\[ \varphi(p) = \frac{F(p)}{\sqrt{N}} \sum_{n=1}^{N} e^{i(n\alpha - npL/\hbar)} = F(p) e^{i(n\alpha - npL/\hbar)} \frac{\sin((N\alpha - npL/\hbar)/2)}{\sin((\alpha - pL/\hbar)/2)} \] (68)

Therefore, the momentum distribution is

\[ |\varphi(p)|^2 = \frac{|F(p)|^2}{N} \sum_{n=1}^{N} \sum_{k=1}^{N} e^{i(n-k)\alpha - \alpha(n-k)-k} e^{-i(n-k)pL/\hbar} = \frac{|F(p)|^2}{N} \frac{\sin((N\alpha - npL/\hbar)/2)}{\sin((\alpha - pL/\hbar)/2)} \] (69)
Analogous to the situation in Sctn. 3, we have that

\[ M \]  

where \( M \) is the characteristic function of \( |F(p)|^2 \) appearing frequently in sonar, radar, optics and digital image processing, such as for example a grating (or line array) of \( N \) equi-spaced apertures. It goes by different names, including the “periodic sinc,” “aliased sinc,” “circular sinc” and “Dirichlet function,” although this latter term is used to refer to other functions as well.

We now show that the momentum distribution for \( N \) non-overlapping shifted functions, Eq. (69), is \( M \)-indeterminate by showing that its moments

\[ \langle p^n \rangle = \int p^n |\varphi(p)|^2 \, dp \]  

are independent of \( \alpha \). We approach the issue by way of the characteristic function, which is given by

\[ M(\theta) = \langle e^{i\theta p} \rangle = \int e^{i\theta p} \left| \frac{F(p)}{N} \right|^2 \sum_{n=1}^{N} \sum_{k=1}^{N} e^{i\alpha(n-k)} e^{-i\alpha L(n-k)/\hbar} dp \]

\[ = \frac{1}{N} \sum_{n=1}^{N} \sum_{k=1}^{N} e^{i\alpha(n-k)} M_F(\theta - L(n-k)/\hbar) \]

where \( M_F \) is the characteristic function of \( |F(p)|^2 \) as given by Eq. (17). Extracting the \( n = k \) term we have

\[ M(\theta) = M_F(\theta) + \frac{1}{N} \sum_{n,k=1; n \neq k}^{N} e^{i\alpha(n-k)} M_F(\theta - L(n-k)/\hbar) \]

\[ = M_F(\theta) + \frac{1}{N} \sum_{n > k}^{N} e^{i\alpha(n-k)} M_F(\theta - L(n-k)/\hbar) + \frac{1}{N} \sum_{n < k}^{N} e^{-i\alpha(n-k)} M_F(\theta + L(n-k)/\hbar) \]

Analogous to the situation in Sctn. 3, we have that \( M_F(\theta \pm L(n-k)/\hbar) \big|_{\theta=0} = 0 \) and hence the moments are given by

\[ \langle p^n \rangle = \frac{1}{i^n} \left. \frac{\partial^n}{\partial \theta^n} M(\theta) \right|_{\theta=0} = \frac{1}{i^n} \left. \frac{\partial^n}{\partial \theta^n} M_F(\theta) \right|_{\theta=0} \]

and therefore the momentum distribution for \( N \) non-overlapping shifted functions is \( M \)-indeterminate.

### 8 Probability of other observables

We consider here the transformation of the wave function given by Eq. (1) to a general representation, and whether or not the corresponding distribution is \( M \)-indeterminate.

For an observable represented by the Hermitian operator \( A \) the eigenvalue problem (we consider the discrete case),

\[ A \, u_n(x) = a_n \, u_n(x) \]

results in real eigenvalues, \( a_n \), which are the measurable quantities. The eigenfunctions, \( u_n(x) \), are complete and orthogonal

\[ \int u_k^*(x) \, u_n(x) \, dx = \delta_{kn} ; \quad \sum_n u_n^*(x') \, u_n(x) = \delta(x-x') \]

Expanding the wave function, \( \psi(x) \), as

\[ \psi(x) = \sum_n c_n \, u_n(x) \]

gives the probability, \( P(a_n) \), of measuring \( a_n \),

\[ P(a_n) = |c_n|^2 = \left| \int \psi(x) u_n^*(x) \, dx \right|^2 \]
with
\[ c_n = \int \psi(x) u_n^*(x) dx \] (80)

For the wave function given by Eq. (1), we have
\[ c_n = \frac{1}{\sqrt{2}} (c_n^{(1)} + e^{i\alpha} c_n^{(2)}) \] (81)

where
\[ c_n^{(1)} = \int \psi_1(x) u_n^*(x) dx \quad c_n^{(2)} = \int \psi_2(x) u_n^*(x) dx \] (82)

Therefore
\[ P(a_n) = \left| \int \psi(x) u_n^*(x) dx \right|^2 = \frac{1}{2} \int \int (\psi_1^*(x') + e^{-i\alpha} \psi_2^*(x')) u_n(x') (\psi_1(x) + e^{i\alpha} \psi_2(x)) u_n^*(x) dx dx' \] (83)

which gives
\[ P(a_n) = \frac{1}{2} (P_1(a_n) + P_2(a_n)) + \frac{1}{2} \left( e^{i\alpha} c_n^{(1)} c_n^{(2)} + e^{-i\alpha} c_n^{(2)} c_n^{(1)} \right) \] (84)

where
\[ P_1(a_n) = \left| c_n^{(1)} \right|^2 \quad P_2(a_n) = \left| c_n^{(2)} \right|^2 \] (85)

Now generally speaking, while \( \psi_1^*(x) \psi_2(x) = 0 \), we have
\[ c_n^{(1)} c_n^{(2)} \neq 0 \] (86)

and hence the probability distribution will depend on the phase factor.

Now consider the moments,
\[ \langle A^n \rangle = \sum_{n=0}^{\infty} a^n |c_n|^2 = \frac{1}{2} \int (\psi_1^*(x) + e^{-i\alpha} \psi_2^*(x)) \ A^n (\psi_1(x) + e^{i\alpha} \psi_2(x)) \ dx \] (87)

\[ = \frac{1}{2} \langle A^n \rangle_1 + \frac{1}{2} \langle A^n \rangle_2 + 2 \text{Re} \left( e^{-i\alpha} \int \psi_2^*(x) A^n \psi_1(x) dx \right) \] (88)

where \( \langle A^n \rangle_1 \) and \( \langle A^n \rangle_2 \) are the expectation values taken with the wave functions \( \psi_1(x) \) and \( \psi_2(x) \), respectively, and they do not depend on \( \alpha \). The question then is, are there Hermitian operators such that \( \int \psi_2^*(x) A^n \psi_1(x) dx \neq 0 \)? If \( A^n \psi_1(x) \) has the same support as \( \psi_1(x) \), that is, it is zero over the same interval as \( \psi_1(x) \), then the integral will be zero. So, for example if \( A \) is the sum of finite polynomials in position and momentum then the last term in Eq. (88) will be zero and hence the moments will be independent of the phase factor. Therefore, for non-overlapping wave functions, there are many observables that result in M-indeterminate distributions. Generalization to continuous representations and distributions is straightforward.

We note that in general there will be interference terms in these representations even though \( \psi_1^*(x) \psi_2(x) = 0 \).

An interesting aspect of the above consideration is that it generates an infinite number of M-indeterminate distributions, both continuous and discrete. That is achieved by choosing representations that are generated by a Hermitian operator. This will be developed in a future paper.

9 Dual Case

Finally, we briefly consider the dual case, namely a momentum wave function that consists of the sum of two non-overlapping momentum wave functions,
\[ \varphi(p) = \frac{1}{\sqrt{2}} (\varphi_1(p) + e^{i\alpha} \varphi_2(p)) \] (89)
where \(\alpha\) is the relative phase between the two momentum wave functions \(\varphi_1(p)\) and \(\varphi_2(p)\), each of which is normalized to one and where \(\varphi_1(p)\) and \(\varphi_2(p)\) are of finite extent

\[
\varphi_1(p) = \begin{cases} h(p) & 0 \leq p \leq b \\ 0, & \text{otherwise} \end{cases} \quad \varphi_2(p) = \begin{cases} h(p - L) & b < p \leq L + b \\ 0, & \text{otherwise} \end{cases}
\]  

(90)

such that

\[\varphi_1^*(p)\varphi_2(p) = 0\]  

(91)

For this case, the momentum distribution, \(P(p)\), is independent of \(\alpha\),

\[P(p) = |\varphi(p)|^2 = \frac{1}{2} |\varphi_1(p) + e^{i\alpha}\varphi_2(p)|^2 = \frac{1}{2} \left( |\varphi_1(p)|^2 + |\varphi_2(p)|^2 \right)\]  

(92)

whereas the position distribution is not,

\[P(x) = |\psi(x)|^2 = |H(x)|^2 [1 + \cos (xL/h - \alpha)]\]  

(93)

where

\[H(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^a h(p) e^{ipx/\hbar} \, dp\]  

(94)

Hence, analogous to the previous case, the position and momentum moments are again independent of \(\alpha\), but now it is the position distribution that is M-indeterminate. An example of this case is considered in [19].

It follows that all of the previous results apply here, after transcribing \(x\) for \(p\) and vice versa. In other words, where in the previous case the quantum mechanical current was independent of \(\alpha\) but the quantum group delay was not, here the situation is reversed: the quantum mechanical current depends on \(\alpha\) while the quantum group delay does not. Explicitly, for the calculation of the current we write

\[\psi(x) = R(x) e^{iS(x)/\hbar} = \frac{1}{\sqrt{2}} \left( R_1(x)e^{iS_1(x)/\hbar} + e^{i\alpha} R_2(x)e^{iS_2(x)/\hbar} \right)\]  

(95)

for which the current is given by Eq. (40). However, unlike the previous case, here \(R_1R_2 \neq 0\), and hence the current depends on \(\alpha\). As a special case, for

\[\varphi_2(x) = \varphi_1(x - L)\]  

(96)

the current is

\[j(x) = S_1(x) - R^2 L/2\]  

(97)

where \(R^2\) depends on \(\alpha\) and is given by Eq. (42).

### 10 Conclusion

We considered the non-overlapping wave function paradox in quantum mechanics, wherein expectation values of position and momentum are independent of the relative phase between the wave functions, from the perspective of M-indeterminate distributions. We showed that, not only do non-overlapping wave functions with a relative phase difference give rise to M-indeterminate momentum distributions, but also there is an infinite number of M-indeterminate distributions, each associated with a different physical observable.

What is particularly interesting about the non-overlapping wave function paradox is the way in which moments and probability distributions are calculated in quantum mechanics. We showed that the characteristic function approach is particularly powerful for addressing the non-overlapping wave function paradox and highlights precisely why the moments are independent of the phase difference.

With this insight afforded by the characteristic function approach, we obtained general conditions for other expectation values that depend on the relative phase. We also defined the quantum mechanical group delay as
the dual to the quantum mechanical current, and showed that the group delay is a function of the phase difference even though the current is not. This result is a direct consequence of the fact that, while the non-overlapping wave functions do not interfere in position space, they do interfere in momentum space.

Considerations of the current and group delay lead naturally to phase space quasi-distributions, such as the Wigner distribution. In particular, the group delay and current are local expectation values of the Wigner (and many other) distributions. We showed that non-overlapping wave functions have M-indeterminate phase space distributions. A particularly interesting aspect of the problem here is that phase space distributions may be negative, or even complex, which is why they are often called “quasi-distributions.” As such, the usual tests for M-indeterminate distributions may not apply. We consider it an interesting future avenue of exploration to study these situations.

We also considered the dual problem, namely non-overlapping wave functions in momentum space, and showed that all of our results pertain to this case with a simple transcription of variables and quantities. For non-overlapping momentum functions, the quantum mechanical current will depend on the relative phase difference, as will the position distribution but not its moments. For this case it is the position distribution, not the momentum distribution, that is M-indeterminate.

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