INVERSE PROBLEMS FOR NON-LINEAR SCHRÖDINGER EQUATIONS WITH TIME-DEPENDENT COEFFICIENTS

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Abstract. The paper studies the inverse problem of reconstructuring the coefficient $\beta(t, x)$ of the non-linear term and the potential $V(t, x)$ of a non-linear Schrödinger equation in time-domain, $(i \frac{\partial}{\partial t} + \Delta + V) u + \beta u^2 = f$ in $(0, T) \times M$, where $M \subset \mathbb{R}^n$ is a convex and compact set with smooth boundary. We consider measurements in a neighborhood $\Omega \subset M$ that is a neighborhood of the boundary of $M$ and the source-to-solution map $L_{\beta, V}$ that maps a source $f$ supported in $\Omega \times [0, T]$ to the restriction of the solution $u$ in $\Omega \times [0, T]$. We show that the map $L_{\beta, V}$ uniquely determines the time-dependent potential and the coefficient of the non-linearity, for a second-order non-linear Schrödinger equation and for the Gross-Pitaevskii equation, with a cubic non-linear term $\beta |u|^2 u$, that is encountered in quantum physics.

1. Introduction

1.1. Statement of results. Let $T > 0$ and $M \subset \mathbb{R}^n$ be a convex and compact set with a smooth boundary $\partial M$. Let $\Omega \subset M$ be a neighbourhood of $\partial M$, let $V \in C_0^\infty((0, T) \times (M \setminus \Omega))$, and let $\beta$ be either a non-zero constant or a function $\beta \in C_0^\infty((0, T) \times (M \setminus \Omega))$ which is non-zero everywhere in $\text{supp}(V)$.

Let $u$ be a solution of a second order non-linear Schrödinger equation given a source term $f$

$$
\begin{cases}
\left( i \frac{\partial}{\partial t} + \Delta + V(t, x) \right) u(t, x) + \beta(t, x) u(t, x)^2 = f(t, x) & \text{on } (0, T) \times M, \\
u(t, x)|_{x \in \partial M} = 0, \\
u(t, x)|_{t=0} = 0,
\end{cases}
$$

(1)

where $\Delta$ denotes the spatial Laplacian, $\Delta u(t, x) = \sum_{i=1}^n \left( \frac{\partial}{\partial x_i} \right)^2 u(t, x)$.

Let $\kappa \in \mathbb{N}$ be sufficiently large, and let

$$
H_0^{2\kappa} := \{ f \in H^{2\kappa}((0, T) \times M) : \partial_t^m f|_{t=0} = 0 \ \forall \ m \leq 2\kappa - 1 \}.
$$

We consider the source-to-solution map $L_{\beta, V}$ for the equation problem (1) via

$$
L_{\beta, V} f = u|_{(0,T)\times \Omega}.
$$
that is defined for the sources in \( \{ f \in H : \text{supp}(f) \subset (0, T) \times \Omega \} \), where \( H \) is a sufficiently small neighbourhood of the zero function in the space \( H^2_{0\kappa} \), and \( u \) is the unique solution of the non-linear Schrödinger equation (1) corresponding to the source term \( f \). In particular, it can be shown that the map \( L_{\beta,V} \) is well-defined (see section 2 for details).

The inverse problem of coefficient determination for the equation (1) consists of determining the potential \( V \) and coefficient \( \beta \) of the non-linearity from knowledge of the source-to-solution map \( L_{\beta,V} \). In particular, we establish the following result:

**Theorem 1.** Let \( n \geq 2, T > 0, M \subset \mathbb{R}^n \) be a convex and compact set with a smooth boundary, and \( \Omega \) be a neighbourhood of \( \partial M \). Moreover, for \( j = 1, 2 \), let \( \beta_j \) be either a function in \( C_0^\infty((0,T) \times (M \setminus \Omega)) \) or non-zero constant function and let \( V_j \in C_0^\infty((0,T) \times (M \setminus \Omega)) \). Suppose that \( \beta_j \) are non-zero everywhere in \( \text{supp}(V_j) \).

If the source-to-solution maps satisfy \( L_{\beta_1,V_1} = L_{\beta_2,V_2} \) then \( \beta_1 = \beta_2 \) and \( V_1 = V_2 \).

The proof of the above result relies on the use of boundary sources that give rise to geometric optics solutions of the linearised problem. The product of these solutions can be chosen to focus at a given point, and we can then exploit the non-linear interactions of the solutions to determine the coefficients at that point.

In fact, this method can be applied, with modifications, to a variety of non-linear Schrödinger equations. We consider, as an example, the Gross-Pitaevskii equation in the case where \( M \subset \mathbb{R}^n \) is a convex and compact set with a smooth boundary:

\[
\begin{align*}
\left( i \partial_t + \Delta + V(t,x) \right) u + \beta(t,x)|u|^2u &= f(t,x) \quad \text{on } (0, T) \times M, \\
\{ u(t,x) \}_{x \in \partial M} &= 0, \\
\{ u(t,x) \}_{t=0} &= 0.
\end{align*}
\]

(2)

Again, we denote the source-to-solution map for the above problem by \( L_{\beta,V} f = u|_{(0,T)\times \Omega} \), defined for sources in \( \{ f \in H : \text{supp}(f) \subset (0, T) \times \Omega \} \), where \( u \) is the unique solution that solve (2). We have the following result:

**Theorem 2.** Let \( T > 0, M \subset \mathbb{R}^n, \Omega \subset M \) and functions \( \beta_j \) and \( V_j \) be as in Theorem 1. Let \( L_{\beta_j,V_j}, j = 1, 2 \) be the source-to-solution maps for the Gross-Pitaevskii equation with coefficients \( V_j \) and \( \beta_j \). If \( L_{\beta_1,V_1} = L_{\beta_2,V_2} \), it follows that we have \( \beta_1 = \beta_2 \) and \( V_1 = V_2 \).

The remainder of the present work is organised as follows. In section 2 we show that the source-to-solution map for (1) is well-defined and smooth in a neighbourhood...
of zero. In section 3, we give the proof of Theorem 1, and in section 4, we show how this proof can be modified for the Gross-Pitaevskii equation.

1.2. Earlier studies and related problems. The non-linear Schrödinger equations arise in the study of Bose-Einstein condensates [50] and the propagation of light in nonlinear optical fibers [46]. They also appear in the study of gravity waves on water and the models in waves in plasma [46].

Literature dealing with the linearized problem of recovering the time-dependent potentials of the dynamic Schrödinger equation is reasonably plentiful. It was initially shown by Eskin [18] that the time-dependent electromagnetic potentials are uniquely determined by the Dirichlet-to-Neumann map. Logarithmic stability estimates for this recovery were established in [8, 14], and further stability estimates of Hölder-type were established by Kian and Soccorsi [35]. Let us also mention the work of Bellassoued and ben Fraj [5], which establishes logarithmic and double-logarithmic stability estimates for the same problem with partial data. In the Riemannian setting, Hölder-stable recovery of the potentials from the Dirichlet-to-Neumann was first established for time-independent potentials [4, 6, 7], and then for time-dependent potentials by [36]. Lastly, there is the work [61], which uniquely recovers the time-dependent Hermitian coefficients appearing in the dynamic Schrödinger equation on a trivial vector bundle.

The inverse problem studied here is a generalization of the inverse problem introduced by Calderón [11], where the objective is to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary. It is closely related to the problem of determining an unknown potential \( q(x) \) in a fixed-energy Schrödinger operator \( \Delta + q(x) \) from boundary measurements, first solved by Sylvester and Uhlmann [60] in dimensions \( n \geq 3 \) and by Bukhgeim [10] in the 2-dimensional space. For the inverse conductivity problem, the first global solution in two dimensions is due to Nachman [45] for conductivities with two derivatives and by Astala and Päivärinta [3] the uniqueness of the inverse problem was proven general isotropic conductivities in \( L^\infty \).

The inverse problems for nonlinear elliptic equations have also been widely studied. A standard method is to show that the first linearization of the nonlinear Dirichlet-to-Neumann map is actually the Dirichlet-to-Neumann map of a linear equation, and to use the theory of inverse problems for linear equations. For the semilinear stationary Schrödinger equation \( \Delta u + a(x, u) = 0 \), the problem of recovering the potential \( a(x, u) \) was studied in [28, 57] in dimensions \( n \geq 3 \), and in [27, 57, 26] when
In addition, inverse problems have been studied for quasilinear elliptic equations \([59, 58, 31]\). Certain Calderón type inverse problems for quasilinear equations on Riemannian manifolds were recently considered in \([43]\).

This paper uses extensively the non-linear interaction of solutions to solve the inverse problems. In this approach, non-linearity is used as a tool that helps in solving inverse problems and the reconstruction applies the higher order linearizations of the source-to-solution map. An Inverse problem for a non-linear scalar wave equation with a quadratic non-linearity was studied in \([40]\) using the multiple-fold linearization and non-linear interaction of linearized solutions. For the direct problem, the analysis of non-linear interaction for hyperbolic equations started in the studies of Bony \([9]\), Melrose and Ritter \([48]\), and Rauch and Reed \([52]\), see also \([53, 54]\). These studies used microlocal analysis and conormal singularities, see \([24, 25, 47]\).

The inverse problem for a semi-linear wave equation in \((1+3)\)-dimensional Lorentzian space with quadratic non-linearities was studied in \([40]\) using interaction of four waves. This approach was extended for a general semi-linear term in \([29, 44]\) and with quadratic derivative in \([63]\). In \([39]\), the coupled Einstein and scalar field equations were studied. The result has been more recently strengthened in \([62]\) for the Einstein scalar field equations with general sources. The inverse for semi-linear and quasi-linear wave equations in \((1+n)\)-dimensional space are studied in \([21]\) using the three wave interactions. In \([22, 41, 42]\) similar multiple-fold linearization methods have been introduced to study inverse problems for elliptic non-linear equations, see also \([37]\).

In recent works \([12, 13, 23]\), the authors have also studied problems of recovering zeroth and first order terms for semi-linear wave equations with Minkowski metric. The three wave interactions were used in \([12, 13]\) to determine the lower order terms in the equations and in modelling non-linear elastic scattering from discontinuities \([15, 16]\).

For the linear wave equation, the determination of general time-dependent coefficients have been studied using the propagation of singularities. In the studies of recovery of sub-principal coefficients for the linear wave equation, we refer the reader to the recent works \([19, 20, 55]\) for recovery of zeroth and first order coefficients and to \([56]\) for a reduction from the boundary data for the inverse problem associated to linear wave equation to the study of geometrical transforms of the domain. This latter approach has been recently extended to general real principal type differential operators \([49]\). Let us also mention here the recent works \([1, 2]\) which recover zeroth
order coefficients of the wave equation on Lorentzian manifolds from the Dirichletto-Neumann map under suitable geometric assumptions.

2. The Source-to-Solution Map

The aim of this section is to establish that the source-to-solution map for the problem (1) is smooth in a neighbourhood of zero. We have the following result:

**Proposition 3.** Let \( \kappa > \frac{n+1}{2} \) be an integer and \( \mathcal{H} \) be a sufficiently small neighbourhood of the zero function in the space \( H^2_0 \). Then for any \( f \in \mathcal{H} \) there is a unique \( u \in H^2_0 \) that satisfies the equation (1). Moreover, the map \( L_{\beta,V} : \mathcal{H} \to H^2_0 \) is smooth.

In order to prove this result, we need some higher order energy estimates for the linearized problem. Thus, we begin by recalling the inhomogeneous linear Schrödinger equation

\[
\begin{cases}
(i \partial_t + \Delta + V) u = f \quad \text{on } (0, T) \times M, \\
u |_{x \in \partial M} = 0, \\
u |_{t=0} = 0,
\end{cases}
\]

and let \( S \) denote the solution operator for the above equation, defined by \( S(f) = u \).

Let us also define the energy space

\[ H^{r,s}((0, T) \times M) = H^r(0, T; L^2(M)) \cap L^2(0, T; H^s(M)), \]

together with the associated norm \( \| \cdot \|_{H^{r,s}((0, T) \times M)} = \| \cdot \|_{H^r(0, T; L^2(M))} + \| \cdot \|_{L^2(0, T; H^s(M))} \).

We now establish the desired higher order energy estimates for the linearized problem (3).

**Lemma 4.** The problem (3) satisfies the estimate

\[
\| u \|_{H^2(0, T) \times M} \leq C \| f \|_{H^2(0, T) \times M}
\]

for any source term \( f \in H^2_0 \).
We can then proceed to apply this estimate to \((\partial_t^m f|_{t=0} = 0\) for \(m \leq 2\kappa - 1\), together with the fact that \(u|_{t=0} = 0\), immediately implies that
\[(8) \quad \partial_t^m u|_{t=0} = 0, \text{ when } m \leq 2\kappa.\]

The proof of the estimate (7) is by induction. The case \(\kappa = 0\) is implied by (4). Therefore, suppose that we have shown the estimate (7) holds for \(\kappa \leq K - 1\). Then it suffices to show that, for \(\rho, \sigma \in \mathbb{N}\) such that \(\rho + 2\sigma = 2K\), we have
\[(9) \quad \|\partial_t^\rho \Delta^\sigma u\|_{L^2((0,T)\times M)} \leq \|f\|_{H^{2K}((0,T)\times M)}.

We begin by applying \(\partial_t\) to (3), and observe that
\[(10) \quad (i\partial_t + \Delta + V)\partial_t u = \partial_t f - (\partial_t V)u.

Then, since \(\partial_t u\) satisfies the zero initial condition \(\partial_t u|_{t=0} = 0\), we can apply (5) to equation (10) to observe that
\[\|\partial_t^2 u\|_{L^\infty(0,T;L^2(M))} \leq C\|f\|_{H^{2,0}(0,T)\times M} + C\|\partial_t V u\|_{H^{1,0}(0,T)\times M},\]
and using the fact that \(V \in C_0^\infty((0,T) \times (M \setminus \Omega))\), together with the estimates (4) and (5), we conclude that
\[\|\partial_t^2 u\|_{L^\infty(0,T;L^2(M))} \leq C\|f\|_{H^{2,0}(0,T)\times M}.

We can then proceed to apply this estimate to (10) and use the previously derived estimates to obtain, in turn, the bound
\[\|\partial_t^\rho \Delta^\sigma u\|_{L^\infty(0,T;L^2(M))} \leq C\|f\|_{H^{3,0}(0,T)\times M},\]
and bootstrapping in this manner we can conclude that
\[(11) \quad \|\partial_t^m u\|_{L^\infty(0,T;L^2(M))} \leq C\|f\|_{H^m(0,T)\times M}.

In particular, we note that estimate (11) holds for all \(m \leq 2K\), not just when \(m\) is even, and this estimate establishes (9) in the case where \(\sigma = 0\). Let us now consider the case where \(\sigma = 1\). Since \(u\) satisfies the Schrödinger equation, it follows that
\[(12) \quad \|\partial_t^{2K-2} \Delta u\|_{L^2((0,T)\times M)} \leq C\|\partial_t^{2K-2}(f - i\partial_t u + Vu)\|_{L^2((0,T)\times M)}
\[\leq C\|f\|_{H^{2K-2}(0,T)\times M} + C\|\partial_t^{2K-1} u\|_{L^2((0,T)\times M)} + C\|\partial_t^{2K-2} (Vu)\|_{L^2((0,T)\times M)}.

From (11), we deduce that the second term on the right-hand side of (12) is bounded by \(\|f\|_{H^{2K-1}(0,T)\times M}\). Further, since \(V \in C_0^\infty((0,T) \times (M \setminus \Omega))\), the induction hypothesis implies that the third term is bounded by \(\|f\|_{H^{2K-2}(0,T)\times M}\). Therefore, it follows that
\[\|\partial_t^{K-2} \Delta u\|_{L^2((0,T)\times M)} \leq C\|f\|_{H^{2K-1}(0,T)\times M},\]
which establishes (9) in the case where $\sigma = 1$. It remains only to deal with the case where $\sigma \geq 2$. In this case, note that

$$
\partial_t^\rho \Delta^\sigma u = \partial_t^\rho \Delta^{\sigma-1}(f - i\partial_t u - V u)
= \partial_t^\rho \Delta^{\sigma-1}(f - V u) - i\partial_t^{\rho+1} \Delta^{\sigma-2} u
= \partial_t^\rho \Delta^{\sigma-1}(f - V u) - i\partial_t^{\rho+1} \Delta^{\sigma-2}(f - i\partial_t u - V u).
$$

In particular, since the derivatives of $u$ and $f$ in the last expression are all of order $2K - 2$ or lower, the induction hypothesis then implies that

$$
\|\partial_t^\rho \Delta^\sigma u\|_{L^2((0,T) \times M)} \leq C \|f\|_{H^{2K-2}((0,T) \times M)},
$$

and this finishes the proof of (9) for $\sigma \geq 2$. \qed

In light of the above, we can now proceed to the proof of proposition 3.

**Proof of Proposition 3.** Let us first address the issue of uniqueness. Suppose that for some $f$, we have two solutions of (1), which we denote by $u$ and $v$. Then it follows that 

$$
(i\partial_t + \Delta + V)(u - v) + \beta(u + v)(u - v) = 0,
$$

and applying the energy estimate (4) for the linear problem, we conclude that $u - v = 0$, whence $L_{\beta,V}$ is indeed unique.

As $\kappa \in \mathbb{N}$ satisfies $\kappa > \frac{n+1}{2}$, the space $H^{2\kappa}_0$ is a Banach algebra. It follows from the trace theorem that $H^{2\kappa}_0$ is likewise a Banach algebra, and we can therefore define the map

$$
\mathcal{K} : H^{2\kappa}_0 \times H^{2\kappa}_0 \to H^{2\kappa}_0
$$

via the expression

$$
\mathcal{K}(u, f) = f - \beta u^2.
$$

We now consider the map $\Phi(u, f) = u - S\mathcal{K}(u, f)$, and observe that $\Phi(u, f) = 0$ implies that $u$ is a solution of the non-linear Schrödinger equation (1). Observe also that

$$
S : H^{2\kappa}_0 \to H^{2\kappa}_0
$$

by the result of Lemma 4 together with (8). Therefore it follows that

$$
\Phi(u, f) : H^{2\kappa}_0 \times H^{2\kappa}_0 \to H^{2\kappa}_0.
$$

We note that the map $\Phi$ is smooth in $u$ and $f$, since $\mathcal{K}(u, f)$ is a polynomial and $S$ is linear. We can use the chain rule to compute $\partial_u \Phi(0, 0) = \text{Id}$, and the implicit function theorem gives a smooth map $f \mapsto u$ from a neighbourhood $\mathcal{H}$ of the zero function in $H^{2\kappa}_0$ to $H^{2\kappa}_0$, such that we have $\Phi(u(f), f) = 0$ for all $f \in \mathcal{H}$. This map must coincide with $L_{\beta,V}$ in $\mathcal{H}$, by the uniqueness already established. \qed
3. Proof of the Main Result with the 2nd Order Non-Linear Term

Let \( T > 0 \). Next we consider the solutions of the non-linear Schrödinger equation (1)

\[
\begin{aligned}
&i \partial_t u + \Delta u + Vu + \beta u^2 = f \quad \text{on} \ (0, T) \times M, \\
&u|_{x \in \partial M} = 0, \\
&u|_{t=0} = 0
\end{aligned}
\]

when the source \( f \) varies.

Before entering into the technical details of the proof of Theorem 1, we will first give a brief explanation of how the non-linearity is used. Let \( f_1, f_2 \in H^k_0 \) be supported in \((0, T) \times \Omega\), and consider the two-parameter family of source terms

\[
f_\varepsilon := \varepsilon_1 f_1 + \varepsilon_2 f_2, \quad \forall (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2.
\]

For small enough \( \varepsilon_1, \varepsilon_2 \) it follows that \( f_\varepsilon \in \mathcal{H} \), and we let \( w_\varepsilon \) denote the unique solution of (1) with this choice of source term. Then

\[
w := \partial_{\varepsilon_1} \partial_{\varepsilon_2} w_{\varepsilon}|_{\varepsilon=0}
\]
solves the linear Schrödinger equation (3) with

\[
f = -2\beta U_1 U_2, \quad U_j = \partial_{\varepsilon_j} w_{\varepsilon}|_{\varepsilon=0},
\]

and \( U_j \) satisfies the same equation with \( f = f_j \). In the proof of Theorem 1, we choose \( f_j \) which generate geometric optics solutions \( U_j \) supported near lines which intersect at some \( p \in M \). As a result of this, we will recover information about the coefficients \((\beta, V)\) at the point \( p \in M \).

The remainder of this section is divided as follows. We briefly recall the construction of approximate geometric optics solutions for the linear Schrödinger equation in section 3.1. In section 3.2, we use the convexity of \( M \) to show that the source-to-solution map determines the amplitudes of these solutions in \( \Omega \), and hence also the sources \( f_j \) up to a small error. Finally, in section 3.3, we show how the solutions generated by these sources can be used to recover the coefficients.

3.1. Geometric optics solutions. In this section, we recall the construction of approximate geometric optics solutions to the linear Schrödinger equation. The details are largely the same as those elsewhere in the literature (e.g., [35] and [32, 34, 51]), but we give them below for the reader’s convenience. Let us begin by considering the homogeneous linear Schrödinger equation

\[
\begin{aligned}
i \partial_t u + \Delta u + Vu &= 0 \quad \text{in} \ (0, T) \times M, \\
&u|_{t=0} = 0 \quad \text{in} \ M.
\end{aligned}
\]
The construction is based on the use of the ansatz

\[ U(t, x) = e^{i\tau(\xi \cdot x - ct)} a(\tau; t, x) = e^{i\tau(\xi \cdot x - ct)} \left( \sum_{k=0}^{N} \frac{a_k(t, x)}{\tau^k} \right), \]

where \( \tau > 0 \) is a large parameter, \( c > 0 \) is some constant, \( \xi \in \mathbb{R}^n \), and the \( a_k \) are functions to be determined.

Computing the Schrödinger operator applied to \( U \) immediately yields

\[
(i\partial_t + \Delta + V) U = e^{i\tau(\xi \cdot x - ct)}(i\partial_t + \Delta + V) a + \tau^2 e^{i\tau(\xi \cdot x - ct)}(c - |\xi|^2) a \\
+ 2i\tau e^{i\tau(\xi \cdot x - ct)}(T_\xi a),
\]

where \( T_\xi = \sum_{l=1}^{n} \xi^l \partial_{x^l} \) is the transport operator in the \( \xi \) direction. We require the right-hand side of the above to vanish in powers of \( \tau \). In particular, this imposes the condition that

\[
|\xi|^2 = c,
\]

and that the amplitude functions \( a_k \) satisfy the transport equations

\[
T_\xi a_0 = 0 \\
T_\xi a_1 = \frac{i}{2} (i\partial_t + \Delta + V) a_0 \\
: \\
T_\xi a_N = \frac{i}{2} (i\partial_t + \Delta + V) a_{N-1}.
\]

Let us now fix some \( y \in M \), and denote by \( \gamma_{y, \xi} \) the line through the point \( y \) with direction \( \xi \), as parametrised by \( \gamma_{y, \xi}(s) = s\xi + y \). We can choose vectors \( \omega_l \in \mathbb{R}^n \) such that

\[
\frac{\xi}{|\xi|}, \omega_1, \ldots, \omega_{n-1}
\]
forms an orthonormal basis of \( \mathbb{R}^n \) with respect to the Euclidean metric. Then, for some small \( \delta > 0 \), we choose the zeroth amplitude to be

\[
a_0(t, x) = \phi(t) \prod_{l=1}^{n-1} \chi_\delta(\omega_l \cdot (x - y)),
\]

where \( \chi_\delta \in C_0^\infty(-\delta, \delta), \) and \( \phi \in C_0^\infty((0, T)) \) is a smooth cutoff. Therefore it follows that for all \( t \in (0, T) \) the amplitude \( a_0(t, \cdot) \) is supported in a \( \delta \)-neighbourhood of the line \( \gamma_{y, \xi}(\mathbb{R}) \).
We can then use the transport equations (17), with vanishing initial conditions imposed upon the subset
\[ \Sigma_{y,\xi} = \{ x \in \mathbb{R}^n : \xi \cdot (x - y) = 0 \}, \]
to iteratively compute the remaining amplitudes \( a_k \) for \( k \geq 1 \). Thus we have that
\[ a_k(s\xi + y) = \frac{i}{2} \int_0^s [(i\partial_t + \Delta + V)a_{k-1}](s\xi + y) ds. \]

It follows then from the above that \( U(t, \cdot) \) is compactly supported in a \( \delta \)-neighbourhood of \( \gamma_{y,\xi}(\mathbb{R}) \) for all \( t \in (0, T) \). Then, by using (16) and (17) in the expression (15), we can deduce that
\[ (i\partial_t + \Delta + V)U = \tau^{-N} e^{i\tau(x - ct)}(i\partial_t + \Delta + V)a_N, \]
and then a direct computation shows that
\[ \| (i\partial_t + \Delta + V)U \|_{H^\sigma((0,T) \times M)} \lesssim \tau^{-N+2\sigma}. \]

We can turn \( U(t, x) \) into a corresponding exact solution of (14) via
\[ u = U + R_\tau, \]
where the remainder term \( R_\tau \) solves
\[ \begin{cases} 
(i\partial_t + \Delta + V)R_\tau = -(i\partial_t + \Delta + V)U & \text{in } (0, T) \times M, \\
R_\tau|_{x \in \partial M} = 0, \\
R_\tau|_{t=0} = 0.
\end{cases} \]

Similarly, we can turn \( U(t, x) \) into a corresponding exact solution \( u = U + R_\tau \) of the adjoint problem, by choosing instead the initial condition \( R_\tau|_{t=T} = 0 \) in (22). In either case, the energy estimate (7) immediately implies that
\[ \| u - U \|_{H^\sigma((0,T) \times M)} \lesssim \tau^{-N+2\sigma} \]
for even \( \sigma \in \mathbb{N} \), and we have verified that the ansatz \( U(t, x) \) is indeed an approximate solution of the linear Schrödinger equation.

3.2. Determination of the boundary sources. In the present section, we show that the amplitudes of the geometric optics solutions are determined in \( (0, T) \times \Omega \) by the source-to-solution map \( L_{\beta,V} \). We begin by defining the map \( L_V \) via
\[ L_V f = u|_{(0,T) \times \Omega}, \]
where \( f \) is supported in \( (0, T) \times \Omega \), and \( u \) solves the linear Schrödinger equation
\[ \begin{cases} 
(i\partial_t + \Delta + V)u = f & \text{on } (0, T) \times M, \\
u|_{x \in \partial M} = 0, \\
u|_{t=0} = 0.
\end{cases} \]
Note that \( L_{\beta,V} \) determines \( L_V \) for any \( \beta,V \) via the expression
\[
L_V f = \partial_\varepsilon L_{\beta,V}(\varepsilon f)|_{\varepsilon=0}.
\]

We consider also the adjoint (backwards-in-time) equation
\[
\begin{cases}
(i\partial_t + \Delta + V)w = h & \text{on } (0,T) \times M, \\
w|_{x \in \partial M} = 0, \\
w|_{t=T} = 0.
\end{cases}
\]

(25)

Then, it holds that \( L_V^* h = w|_{(0,T) \times \Omega} \) with \( h \) supported in \((0,T) \times \Omega\). In fact, we can compute that for source terms \( f,h \) supported in \((0,T) \times \Omega\) we have
\[
\langle L_V f, h \rangle_{L^2((0,T) \times \Omega)} = \langle f, w \rangle_{L^2((0,T) \times \Omega)}.
\]

(26)

For \( j = 1,2 \), let us consider the potentials \( V_j \in C^\infty_\partial((0,T) \times (M \setminus \Omega)) \). Given any line \( \gamma_{q,\xi} \) with initial point \( q \in \partial M \) and initial direction \( \xi \in \mathbb{R}^n \), we can define a sequence of functions \( a_k^{(j)} \) corresponding to \( V_j \) as follows. First, let us choose vectors \( \omega_1, \ldots, \omega_{n-1} \in \mathbb{R}^n \) such that \( \xi, \omega_1, \ldots, \omega_{n-1} \) is an orthonormal basis of \( \mathbb{R}^n \) with respect to the Euclidean metric. Then, for some small \( \delta > 0 \), we once again choose the zeroth amplitude
\[
a_0^{(j)}(t,x) = \phi(t) \prod_{l=1}^{n-1} \chi_\delta(\omega_l \cdot (x-q)),
\]
where \( \chi_\delta \in C^\infty_\partial(-\delta,\delta) \), and \( \phi \in C^\infty(0,T) \) is a smooth cutoff. We can define the subsequent functions by solving the following transport equations
\[
2i\tau \xi a_{k+1}^{(j)} + (i\partial_t + \Delta + V_j) a_k^{(j)} = 0, \quad a_{k+1}^{(j)}|_{\Sigma_{q,\xi}} = 0.
\]

(27)

Observe that the coefficients \( a_k^{(j)} \) can be explicitly computed along \( \gamma_{q,\xi} \) up to the point where the line leaves \( \Omega \), since \( V_1 = V_2 = 0 \) in this region. We can define, for each \( \tau > 0 \) and \( N \in \mathbb{N} \), an approximate geometric optics solution \( U_j \) corresponding to the choice of \( a_0^{(j)} \) and \( \gamma_{q,\xi} \) through the expression
\[
U_j(t,x) = e^{i(\tau \xi \cdot x - |\xi|^2 t)} \sum_{k=0}^N \tau^{-k} a_k^{(j)}(t,x).
\]

We denote by \( u_j \) the corresponding exact solution of the Schrödinger equation
\[
\begin{cases}
(i\partial_t + \Delta + V_j)u_j = 0 & \text{on } (0,T) \times M, \\
u_j|_{t=0} = 0.
\end{cases}
\]

(28)
as constructed in \((21)\), and note \(U_j\) coincides with \(u_j\) up to a small error \(O(\tau^{-N})\) in \(L^2\).

Let \(\eta \in C_0^\infty(M)\) satisfy \(\eta = 1\) in \(M \setminus \Omega\), and choose \(f_j = (i\partial_t + \Delta + V_j)(\eta u_j)\). Then, we note that the function \(\eta u_j\) solves the Schrödinger equation

\[
\begin{cases}
(i\partial_t + \Delta + V_j)U = f_j & \text{on } (0, T) \times M, \\
U|_{x=\partial M} = 0, \\
U|_{t=0} = 0,
\end{cases}
\]  
(29)

with the source \(f_j\) supported in \((0, T) \times \Omega\). In a similar manner, we can consider the geometric optics solution of the adjoint (backwards-in-time) problem,

\[
\begin{cases}
(i\partial_t + \Delta + V_2)w = 0 & \text{on } (0, T) \times M, \\
w|_{t=T} = 0,
\end{cases}
\]  
(30)

and observe that \(w\) can be similarly approximated to \(O(\tau^{-N})\) by the expression

\[
e^{i(\tau \xi \cdot x - |\xi|^2 \tau^2 t)} \sum_{k=0}^N \tau^{-k} w_k(\tilde{t}, x),
\]

where \(w_k\) is a sequence of the form \((27)\) for \(V = V_2\). Further, letting \(\tilde{\eta} \in C_0^\infty(M)\) also satisfy \(\tilde{\eta} = 1\) in \(M \setminus \Omega\), we observe further that, for \(h = (i\partial_t + \Delta + V_2)(\tilde{\eta} w)\), the function \(\tilde{\eta} w\) solves

\[
\begin{cases}
(i\partial_t + \Delta + V_2)W = h & \text{on } (0, T) \times M, \\
W|_{x=\partial M} = 0, \\
W|_{t=T} = 0,
\end{cases}
\]  
(31)

**Lemma 5.** Suppose that \(\mathcal{L}_{V_1} = \mathcal{L}_{V_2}\). Then \(a_k^{(1)} = a_k^{(2)}\) in \((0, T) \times \Omega\) for all \(k \in \mathbb{N}\).

**Proof.** The proof is by induction in \(k\). For \(k = 0\), the result is trivial. Suppose the result holds for \(k \leq K - 1\). Then the following holds when \(N > K\) is a sufficiently large integer. We note that \(a_k^{(1)} = a_k^{(2)}\) along \(\gamma_{q,\xi}\) until the line leaves \(\Omega\), since \(V_1 = V_2 = 0\) in this region. For the inductive step, let us first observe that \([(i\partial_t + \Delta + V_j)\eta] = 2\nabla \eta \cdot \nabla + \Delta \eta\), whence we deduce that

\[
(i\partial_t + \Delta + V_2)\left(\tilde{\eta} e^{i(\tau \xi \cdot x - |\xi|^2 \tau^2 t)} \sum_{k=0}^N \tau^{-k} w_k\right) = e^{i(\tau \xi \cdot x - |\xi|^2 \tau^2 t)} (2i\tau (\mathcal{T}_\xi \tilde{\eta}) w_0 + O(1))
\]  
(32)
and further, modulo a small error of \( O(\tau^{-N}) \), that

\[
(i\partial_t + \Delta + V_j)\left(\eta e^{i(\tau \xi \cdot x - |\xi|^2 \tau^2 t)} \sum_{k=0}^{N} \tau^{-k} a_k^{(j)}\right) = \]

\[
= e^{i(\tau \xi \cdot x - |\xi|^2 \tau^2 t)} \left(2i\tau (T_\xi \eta) + 2\nabla \eta \cdot \nabla + \Delta \eta\right) \sum_{k=0}^{N} \tau^{-k} a_k^{(j)}.
\]

Then, using the fact that \( L_{V_1} = L_{V_2} \), the identity (26) implies that

\[
0 = \langle (L_{V_1} - L_{V_2}) f_j, h \rangle_{L^2((0,T) \times \Omega)} = \langle \eta u_j, h \rangle_{L^2((0,T) \times \Omega)} - \langle f_j, \tilde{\eta} w_0 \rangle_{L^2((0,T) \times \Omega)}
\]

Taking the difference of the expression (34) for \( j = 1 \) and \( j = 2 \), we deduce that

\[
0 = \langle \eta \tau^{-K}(a_K^{(1)} - a_K^{(2)}) + O(\tau^{-K-1}), 2i\tau (T_\xi \tilde{\eta}) w_0 + O(1) \rangle - \langle 2i\tau (T_\xi \eta) + 2\nabla \eta \cdot \nabla + \Delta \eta\rangle \tau^{-K}(a_K^{(1)} - a_K^{(2)}) + O(\tau^{-K-1}), \tilde{\eta} w_0 + O(\tau^{-1}) \rangle.
\]

By considering the leading order term, we can deduce that

\[
0 = \langle \eta T_\xi \tilde{\eta} + \tilde{\eta} T_\xi \eta)(a_K^{(1)} - a_K^{(2)}), w_0 \rangle_{L^2((0,T) \times \Omega)};
\]

and if we choose \( \eta \) so that \( \eta = 1 \) in the support of \( \tilde{\eta} \), we have

\[
\langle T_\xi \tilde{\eta}(a_K^{(1)} - a_K^{(2)}), w_0 \rangle_{L^2((0,T) \times \Omega)} = 0.
\]

We then fix a set \( Q \) such that \( \overline{M \setminus \Omega} \subseteq Q \subseteq M^{int} \), with smooth boundary \( \partial Q \) such that the line \( \gamma_{q,\xi} \) intersects \( \partial Q \) exactly twice, and such that, near the point \( x \) where \( \gamma_{q,\xi} \) exits \( Q \), the hyperplane \( \Sigma_{x,\xi} \) coincides with \( \partial Q \). It follows from the convexity of \( M \) that we can always construct a set with these properties, e.g. see Figure 1 below.
We can then choose \( \tilde{\eta} \) which converges to the indicator function of \( Q \), so that, in a neighborhood of \( x \), \( T_{x} \tilde{\eta} \) converges to the Dirac delta distribution on \( \Sigma_{x,\xi} \). Observe that we can then choose \( w_{0} = \phi_{1} \phi_{2} \), where \( \phi_{1} \) converges to the delta distribution at \( t \in (0, T) \) and \( \phi_{2} \) converges to the delta distribution on \( \gamma_{q,\xi} \). Thus, we can conclude from (35) that \( a_{K}^{(1)} = a_{K}^{(2)} \), as required. \( \square \)

3.3. Recovery of the coefficients. We consider once again the problem (1) for \( V \in C_{0}^{\infty}((0, T) \times (M \setminus \Omega)) \) and \( \beta \in C_{0}^{\infty}((0, T) \times (M \setminus \Omega)) \) which is non-zero almost everywhere in \( \text{supp}(V) \).

Let us consider a point \( p \in M \setminus \Omega \), a unit vector \( \xi_{0} \in \mathbb{R}^{n} \), and a small parameter \( \lambda \in (0, 1) \). We choose vectors \( \xi_{1}, \xi_{2} \in \mathbb{R}^{n} \) such that they satisfy

\[
\xi_{0} = \xi_{1} + \xi_{2},
\]

the vectors \( \xi_{0}, \xi_{1} \) and \( \xi_{2} \) pairwise non-collinear, and we have

\[
|\xi_{0}|^2 = 1, \quad |\xi_{1}|^2 = 1 - \lambda^2, \quad |\xi_{2}|^2 = \lambda^2.
\]
Figure 2. The vectors $\xi_0$, $\xi_1$ and $\xi_2$.

Let us define $q_j \in \partial M$ to be the point at which the line $\gamma_{p,\xi_j}$ intersects $\partial M$, defined such that the vector $p - q_j$ is a positive multiple of $\xi_j$. Then, fixing $N \in \mathbb{N}$ sufficiently large, we define the amplitude functions $a_k^{(j)}$ as follows. We first construct $a_0^{(j)}$ by letting $\xi = \xi_j$ in expression (18), and define the remaining amplitudes $a_1^{(j)}, \ldots, a_N^{(j)}$ by letting $\xi = \xi_j$ in (20) for all $y$ in the hyperplane $\Sigma_{q_j,\xi_j}$. Then, for $\tau > 0$, we can define the approximate geometric optics solutions

$$U_j(t, x) = e^{i(\tau \xi_j \cdot x - \tau^2 |\xi_j|^2 t)} \left( \sum_{k=0}^{N} \tau^{-k} a_k^{(j)}(t, x) \right).$$

Further, for $U_1, U_2$, we can define the corresponding exact solution $u_j = U_j + R_j$ as in (21). We can also define for $U_0$ the exact solution $u_0 = U_0 + R_0$ of the adjoint problem, where $R_0$ is obtained by choosing the initial condition $R_0|_{t=0} = 0$ in (22). Letting $\eta \in C_0^\infty(M)$ satisfy $\eta = 1$ in $M \setminus \Omega$, we define the source terms $f_j = (i\partial_t + \Delta + V)(\eta u_j)$. Then the function $\eta u_j$ solves

$$\begin{cases}
  i\partial_t U_j + \Delta U_j + VU_j = f_j & \text{on } (0, T) \times M, \\
  U_j|_{x \in \partial M} = 0, \\
  U_1|_{t=0} = U_2|_{t=0} = U_0|_{t=T} = 0.
\end{cases}$$

We note that the sources $f_j$ are supported in $(0, T) \times \Omega$. Further, it can be shown that $f_j \in H_0^{2k}$ is determined by the source-to-solution map $L_{\beta, V}$, up to any error $O(\tau^{-K})$ in the $H^{2k}$-norm. To see this, let us first recall that

$$[i\partial_t + \Delta + V, \eta] = 2\nabla \eta \cdot \nabla + \Delta \eta,$$

whence it follows that $f_j$ is given by the expression

$$f_j = 2\nabla \eta \cdot \nabla U_j + \Delta \eta U_j + 2\nabla \eta \cdot \nabla R_j + \Delta \eta R_j.$$
We note that the first two terms on the right-hand side are uniquely determined by $L_{\beta,V}$ as a result of Lemma 5. On the other hand, we can apply the estimate (23) to the last two terms to conclude that they are $O(\tau^{-N+4\kappa+2})$ in the $H^{2\kappa}$-norm.

Then, letting $\varepsilon_1, \varepsilon_2 > 0$ be small, we set $\varepsilon = (\varepsilon_1, \varepsilon_2)$ and define the source term $f = \varepsilon_1 f_1 + \varepsilon_2 f_2$. For small enough $\varepsilon$, it holds that $f \in \mathcal{H}$, and we observe that

$$
-\frac{1}{2} \partial_{\varepsilon_1} \partial_{\varepsilon_2} L_{\beta,V} f|_{\varepsilon=0} = w|_{(0,T) \times \Omega},
$$

with $w$ the solution of the linear Schrödinger equation

$$
\begin{aligned}
&i \partial_t w + \Delta w + V w = \beta U_1 U_2 \\
&w|_{x \in \partial M} = 0 \\
&w|_{t=0} = 0.
\end{aligned}
$$

(37)

Then it follows that

$$
\int_{(0,T) \times \Omega} \partial_{\varepsilon_1} \partial_{\varepsilon_2} L_{\beta,V} f|_{\varepsilon=0} \S_{0} \, dx \, dt = \int_{(0,T) \times M} w (i \partial_t + \Delta + V) U_0 \, dx \, dt.
$$

(38)

Recall that $L_{\beta,V}$ is a continuous map from $\mathcal{H}$ into $H^{2\kappa}_0$, and that $f \in \mathcal{H}$ is determined by $L_{\beta,V}$ up to $O(\tau^{-N+4\kappa+2})$. Thus, the map $L_{\beta,V}$ determines the left-hand side of (38), up to this error. We now integrate the right-hand side of (38) by parts, and observe that it is given by

$$
\int_{(0,T) \times M} \beta U_0 U_1 U_2 \, dx \, dt.
$$

(39)

We would like to approximate $U_j$ in this integral by $\eta U_j$. Therefore, let $\kappa$ be large enough that $H^{2\kappa}_0$ is a Banach algebra, and let $N \geq 4\kappa + 4$. We see that, up to an error $O(\tau^{-2})$, the integral (39) coincides with the integral

$$
\int_{(0,T) \times M} \beta \eta^3 U_0 U_1 U_2 \, dx \, dt.
$$

Then since $\phi(t)$ appearing in the definition of $a_0^{(i)}$ is arbitrary, it follows that $L_{\beta,V}$ determines for all $t \in (0,T)$ the integral

$$
I = \int_{M} \beta(t, \cdot) \eta^3 \overline{U_0(t, \cdot)} U_1(t, \cdot) U_2(t, \cdot) \, dx,
$$

(40)

up to a small error $O(\tau^{-2})$. Henceforth, we shall suppress this $t$-dependence in our notation for the approximate geometric optics solutions. We now expand the above integral (40) in powers of $\tau$ as

$$
I = I_0 + I_1 \tau^{-1} + O(\tau^{-2}).
$$
Observe that the phases of the approximate geometric optics solutions cancel each other in the product $U_0 U_1 U_2$, and further that $\eta = 1$ in supp $(U_0 U_1 U_2)$.

Therefore, it follows that the source to solution map $L_{\beta,V}$ determines the integrals

$$I_0 = \int_M \beta(t,\cdot) a_0^{(0)} a_0^{(1)} a_0^{(2)} dx$$

and

$$I_1 = \sum_{|e|=1} \int_M \beta(t,\cdot) a_e^{(0)} a_e^{(1)} a_e^{(2)} dx,$$

where $e$ is a multi-index, $e = (e_0, e_1, e_2) \in \mathbb{N}^3$.

Then, letting $\chi_\delta$ in the definition (19) of $a_0^{(j)}$ converge to the indicator function of the interval $(-\delta, \delta)$, we obtain

$$I_0 = \int_{P_\delta} \beta(t,\cdot) dx,$$

where $P_\delta$ is a small neighbourhood of $p$ contained within a ball of radius $\delta$. More precisely, $P_\delta$ is the intersection of $\delta$-neighbourhoods of the lines $\gamma_{p,\xi_j}$ for $j = 0, 1, 2$. Thus, by letting $\delta \to 0$, we can recover the quantity

$$\lim_{\delta \to 0} \frac{1}{|P_\delta|} I_0 = \beta(t, p).$$

Since the choice of $p \in M \setminus \Omega$ was arbitrary, we recover the function $\beta(t, x)$ in its entirety. To recover the potential $V$, we now consider the integral $I_1$.

We recall from (20) that $a_1^{(j)}$ is of the form $a_1^{(j)} = b_1^{(j)} + c_1^{(j)}$, where it holds that

$$b_1^{(j)}(s\xi_j + y) = \frac{i}{2} \int_0^s \left[ (i\partial_t + \Delta) a_0^{(j)} \right](\tilde{s}\xi_j + y) d\tilde{s}$$

$$c_1^{(j)}(s\xi_j + y) = \frac{i}{2} \int_0^s \left[ V a_0^{(j)} \right](\tilde{s}\xi_j + y) d\tilde{s},$$

for all $y$ in the hyperplane $\Sigma_{q_j,\xi_j} = \{ x \in \mathbb{R}^n : \xi_j \cdot (x - q_j) \}$.

In particular, since $a_0^{(j)}$ is independent of $V$, so is $b_1^{(j)}$. Thus $L_{\beta,V}$ determines the quantity

$$J = \int_M c_1^{(0)} a_0^{(1)} a_0^{(2)} \beta dx + \int_M a_0^{(0)} c_1^{(1)} a_0^{(2)} \beta dx + \int_M a_0^{(0)} a_0^{(1)} c_1^{(2)} \beta dx.$$
Then, by letting $\chi_\delta$ in the definition of $a_0^{(j)}$ converge to the indicator function of the interval $(-\delta, \delta)$, we deduce that $L_{\beta,V}$ determines the quantity

$$J_\delta = 2 \sum_{j=0}^2 \int_{P_\delta} \beta c_j \, dx, \quad c_j(s\xi_j + y) = \frac{i}{2} \int_0^s V(\tilde{s}\xi_j + y) \, d\tilde{s}$$

for a small neighbourhood $P_\delta$ of $p$. Then, since $\beta(t, p)$ is known and non-zero, we can let $\delta \to 0$ to recover the quantity

$$\frac{1}{\beta(t, p)} \lim_{\delta \to 0} \frac{1}{|P_\delta|} J_\delta = \sum_{j=0}^2 c_j(p). \tag{42}$$

It remains only to show that $V$ can be recovered from (42). To this end, let $\hat{\xi}$ denote the vector of unit length in the direction of $\xi_2$, so that $\xi_2 = \lambda \hat{\xi}$, and let $s_0 \in \mathbb{R}$ be such that $\gamma_{q_2}(s_0) = p$. Then, it holds that

$$-2ic_2(p) = \int_0^{s_0/\lambda} V(s\hat{\xi}_2 + q_2) \, ds = \lambda^{-1} \int_0^{s_0} V(s\hat{\xi} + q_2) \, ds,$$

and we can similarly check that $c_j(p) = O(1)$ as $\lambda \to 0$ for $j \neq 2$. Therefore, we have shown that we can recover from $L_{\beta,V}$ the quantity

$$-2i \lim_{\lambda \to 0} \left( \lambda \sum_{j=0}^2 c_j(p) \right) = \int_0^{s_0} V(s\hat{\xi} + q_2) \, ds.$$

But this is precisely the truncated ray transform of $V$, which we can differentiate with respect to $s_0$ to recover $V$.

4. The Gross-Pitaevskii Equation

We again fix $T > 0$ and consider the case where $M$ is a Euclidean domain of $\mathbb{R}^n$ and $\Omega$ is a neighbourhood of $\partial M$. For a potential $V \in C_0^\infty((0,T) \times (M \setminus \Omega))$, and a coupling coefficient $\beta \in C_0^\infty((0,T) \times (M \setminus \Omega))$ such that $\beta$ is non-zero almost everywhere in $\text{supp}(V)$, we consider the problem of finding $u$ which, for a given source term $f$, solves the Gross-Pitaevskii equation

$$\begin{cases}
(i\partial_t + \Delta + V + \beta|u|^2)u = f & \text{on } (0,T) \times M, \\
u|_{x \in \partial M} = 0, \\
u|_{t=0} = 0.
\end{cases} \tag{43}$$

We now let $L_{\beta,V}$ denote the source-to-solution map, defined by

$$L_{\beta,V} f = u|_{(0,T) \times \Omega}.$$
where \( f \) is supported in \((0,T) \times \Omega\), and \( u \) solves the Gross-Pitaevskii equation (43) for the chosen source term \( f \). We note that this \( L_{\beta,V} \) is also a smooth map from some \( H \) to \( H^{2\kappa}_0 \) for large enough \( \kappa \), by the same argument used for the non-linear Schrödinger equation in section 2.

Let us now fix some \( p \in M \setminus \Omega \), and some small \( \lambda > 0 \). Let \( \xi_0, \ldots, \xi_3 \) be vectors in \( \mathbb{R}^n \) which depend on \( \lambda \), where the direction of \( \xi_3 \) is the same for all \( \lambda > 0 \). Observe that it is possible to choose such vectors so that they satisfy

\[
\xi_0 + \xi_1 = \xi_2 + \xi_3,
\]

with the \( \xi_j \) pairwise non-colinear, and such that they satisfy the conditions

\[
|\xi_0|^2 = 1/2, \quad |\xi_1|^2 = 1/2, \quad |\xi_2| = 1 - \lambda^2, \quad |\xi_3|^2 = \lambda^2.
\]

Let us define \( q_j \in \partial M \) to be the initial point of the line \( \gamma_{p,\xi_j} \), as we did previously. Then for \( \tau > 0 \) set

\[
U_j(t,x) = e^{i(\tau \xi_j \cdot x - \tau^2 |\xi_j|^2 t)} \left( \sum_{k=0}^{N} \tau^{-k} a_k^{(j)}(t,x) \right)
\]

where, for each \( j \), we construct \( a_0^{(j)} \) by letting \( \xi = \xi_j \) in (19), and define the remaining amplitudes \( a_k^{(j)} \) by taking \( \xi = \xi_j \) in (20). For \( U_1, U_2, U_3 \), we once again define the corresponding exact solution \( u_j = U_j + R_j \) as in (22), and for \( U_0 \) we define the corresponding exact solution of the adjoint problem \( u_0 = U_0 + R_0 \) by choosing \( R_0|_{t=T} = 0 \) in (22). Letting \( \eta \in C_0^\infty(M) \) satisfy \( \eta = 1 \) in \( M \setminus \Omega \), we choose \( f_j = (i\partial_t + \Delta + V)(\eta u_j) \),

\[\text{FIGURE 3. The vectors } \xi_0, \xi_1, \xi_2 \text{ and } \xi_3.\]
and observe that $\eta u_j$ solves

\begin{equation}
\begin{cases}
i\partial_t U_j + \Delta U_j + V U_j = f_j & \text{on } (0, T) \times M, \\
U_j|_{x \in \partial M} = 0, \\
U_j|_{t=0} = U_j|_{t=T} = 0.
\end{cases}
\end{equation}

(44)

Then, letting $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ be small, we set $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and define the source term $f = \varepsilon_1 f_1 + \varepsilon_2 f_2 + \varepsilon_3 f_3$. For small enough $\varepsilon$, it holds that $f \in \mathcal{H}$, and by linearizing in equation (1) we deduce that

\[-\frac{1}{6} \partial_{\varepsilon_1} \partial_{\varepsilon_2} \partial_{\varepsilon_3} L_{\beta,V} f|_{\varepsilon=0} = w|_{(0,T) \times \Omega},\]

where $w$ solves the linear Schrödinger equation

\begin{equation}
\begin{cases}
i\partial_t w + \Delta w + V w = \beta \left( U_1 U_2 U_3 + U_1 \overline{U}_2 U_3 + U_1 U_2 \overline{U}_3 \right) \\
w|_{x \in \partial M} = 0 \\
w|_{t=0} = 0.
\end{cases}
\end{equation}

(45)

Therefore, it holds that

\[-\frac{1}{2} \int_{(0,T) \times \Omega} \partial_{\varepsilon_1} \partial_{\varepsilon_2} L_{\beta,V} f|_{\varepsilon=0} \, dx \, dt = \int_{(0,T) \times M} w \left( i\partial_t + \Delta + V \right) U_0 \, dx \, dt\]

We can integrate by parts to see that the right-hand side of the above is given by

\begin{align*}
I &= \int_{(0,T) \times M} \beta U_0 U_1 U_2 U_3 dxdt + \int_{(0,T) \times M} \beta U_0 \overline{U}_1 U_2 U_3 dxdt + \int_{(0,T) \times M} \beta U_0 U_1 \overline{U}_2 U_3 dxdt \\
&\quad + \frac{1}{2} \int_{(0,T) \times \Omega} \partial_{\varepsilon_1} \partial_{\varepsilon_2} L_{\beta,V} f|_{\varepsilon=0} \, dx \, dt \\
&= \int_{(0,T) \times M} \beta U_0 U_1 U_2 U_3 dxdt + \frac{1}{2} \int_{(0,T) \times \Omega} \partial_{\varepsilon_1} \partial_{\varepsilon_2} L_{\beta,V} f|_{\varepsilon=0} \, dx \, dt + O(\tau^{-K}).
\end{align*}

Note that the phases of the geometric optics solutions cancel only in the first integral appearing in $I$. Therefore, since the integrands above are supported away from $\partial M$, an integration by parts tells us that, provided we choose large enough $N$ in (44), we have for any $K \in \mathbb{N}

\begin{align*}
I &= \int_{(0,T) \times M} \beta U_0 U_1 U_2 U_3 dxdt + O(\tau^{-K}).
\end{align*}

Then, choosing $K \geq 2$, we can expand the integral $I$ as

\begin{align*}
I &= I_0 + I_1 \tau^{-1} + O(\tau^{-2}),
\end{align*}

and arguing as we did for the non-linear Schrödinger equation in section 3, we deduce that the source-to-solution map $L_{\beta,V}$ determines for all $t \in (0, T)$ the integrals

\begin{align*}
I_0 &= \int_{M} \beta(t,x) a_0^{(0)} a_0^{(1)} a_0^{(2)} a_0^{(3)} dx \quad \text{and} \quad I_1 = \sum_{|\varepsilon| = 1} \int_{M} \beta(t,x) a_0^{(0)} a_0^{(1)} a_0^{(2)} a_0^{(3)} dx,
\end{align*}
where \( e \) is a multi-index, \( e = (e_0, e_1, e_2, e_3) \in \mathbb{N}^4 \).

Then, letting \( \chi_\delta \) in the definition of \( a_0^{(j)} \) converge to the indicator function of the interval \((-\delta, \delta)\), we obtain
\[
I_0 = \int_{P_\delta} \beta(t, x)dx,
\]
where \( P_\delta \) is a small neighbourhood of \( p \) contained in a ball of radius \( \delta \). Taking the limit as \( \delta \to 0 \), we can recover the quantity
\[
\lim_{\delta \to 0} \frac{1}{|P_\delta|} I_0 = \beta(t, p).
\]
Since the choice of \( p \in M \setminus \Omega \) was arbitrary, we thus recover the function \( \beta(t, x) \) in its entirety.

To recover the potential \( V \), we turn to the integral \( I_1 \). Arguing as we did for the non-linear Schrödinger equation in section 3, we conclude that \( L_{\beta,V} \) determines the quantity
\[
\sum_{j=0}^{3} c_j(p)
\]
where, once again, \( c_j \) is given by
\[
c_j(s\xi + y) = \frac{i}{2} \int_0^s V(s\xi + y)ds
\]
for all \( y \in \Sigma_{q_j,\xi_j} \) and \( \xi_j \) as in Figure 3. We now let \( \tilde{\xi} \) denote the vector of unit length in the direction \( \xi_3 \), so that we have \( \xi_3 = \lambda \tilde{\xi} \), and let \( s_0 \in \mathbb{R} \) be such that \( \gamma_{q_3,\xi}(s_0) = p \). Then, it follows from the definition of \( c_j \) that
\[
-2ic_3(p) = \lambda^{-1} \int_0^{s_0} V(s\tilde{\xi} + q_3)ds
\]
and similarly that \( c_j(p) = O(1) \) as \( \lambda \to 0 \) for \( j \neq 3 \). Thus, we can recover from \( L_{\beta,V} \) the quantity
\[
-2i \lim_{\lambda \to 0} \left( \lambda \sum_{j=0}^{3} c_j(p) \right) = \int_0^{s_0} V(s\tilde{\xi} + q_3)ds.
\]
But this is just the truncated ray-transform of \( V \), and we can differentiate with respect to \( s_0 \) in order to recover \( V \).

Acknowledgements: The authors are thankful for Ali Feizmohammadi on discussions that were crucial for the ideas used in Section 3.2. M.L., L.O., M.S were
supported by the Finnish Centre of Excellence of Inverse Modelling and Imaging. AT was supported by EPSRC DTP studentship EP/N509577/1. M.L was also partially supported by Academy of Finland, grants 284715, 312110 and M.S. was supported by ERC.

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