GLOBAL WEAK SOLUTIONS TO INVISCID BURGERS-VLASOV EQUATIONS

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Abstract. In this paper, we consider the existence of global weak solutions to a one dimensional fluid-particles interaction model: inviscid Burgers-Vlasov equations with fluid velocity in $L^\infty$ and particles' probability density in $L^1$. Our weak solution is also an entropy solution to inviscid Burgers' equation. The approach is adding ingeniously artificial viscosity to construct approximate solutions satisfying $L^\infty$ compensated compactness framework and weak $L^1$ compactness framework. It is worthy to be pointed out that the bounds of fluid velocity and the kinetic energy of particles' probability density are both independent of time.

Keywords. weak solution; fluid-particles interaction; $L^\infty$ velocity; $L^1$ density; compensated compactness; Dunford-Pettis therom.

AMS subject classifications. 76T10, 35F20, 35Q35, 35Q72, 45K05, 82D05.

1. Introduction

We consider the following inviscid Burgers-Vlasov equations:

\[
\begin{aligned}
&u_t + uu_x = \int_{\mathbb{R}} f vdv - u \int_{\mathbb{R}} fdv, \\
&f_t + vf_x + (f(u-v))_v = 0,
\end{aligned}
\]  

(1.1)

with the initial data

\[
u(x,0) = u_0(x), \quad f(x,v,0) = f_0(x,v) \geq 0.
\]

(1.2)

The system (1.1) is one kind of simple model about inviscid fluid-particles interaction. The motion of the fluid with bulk velocity $u(x,t)$ is modeled by the inviscid Burgers equation, while the dispersed particles with probability density function $f(x,v,t)$ is described by Vlasov like equation. The interaction between the fluid and the dispersed particles is achieved by a friction term between the bulk velocity of fluid and velocity of the dispersed particles, namely the drag force term $\int_{\mathbb{R}} f(v-u)dv$.

For related fluid-structure models, we shall first mention the following diffusive system, Burgers-Vlasov equations:

\[
\begin{aligned}
&\rho_g(u_t + uu_x - \nu_g u_{xx}) = E_d, \\
&f_t + vf_x + (F_d f)_v = 0,
\end{aligned}
\]

(1.3)

in which a dispersed phase interacts with a viscous gas. Here $\rho_g$ is the density of gas. The force term $E_d$ describes the exchange of impulse between the gas and particles, and the drag force $F_d$ is used to describe the friction of viscous gas on the droplets. The force terms are related by the following formulas:

\[
\begin{aligned}
&E_d = C(r)\rho_p(u_p-u), \quad F_d = C(r)(u(x,t)-v), \\
&\rho_p = \frac{4\pi}{3}\rho_p r^3 \int_{\mathbb{R}} f(x,v,t)dv, \quad \rho_p u_p = \frac{4\pi}{3}\rho_p r^3 \int_{\mathbb{R}} f(x,v,t)v dv.
\end{aligned}
\]

(1.4)
In (1.4), $\rho_t$ is the density of the liquid and $C(r)$ is a constant depending on the radius $r$ of droplets in the spray consisting of dispersed particles. After assuming that the gas is of constant mass density $\rho_g$ and simplified the momentum equation of the gas, the Burgers’ equation, i.e. the first equation in (1.3), is utilized to model the evolution of the viscous gas. Further assuming the spray is of enough dilution and neglecting gravity effect, a Vlasov like equation, i.e. the second equation in (1.3), is then applied to govern the evolution of the particles. Other detailed information about the derivations and assumptions on (1.3) can also be found in [10,12,28]. For the mathematical analysis of (1.3), as far as we know, the first global existence and uniqueness of classical solutions to the Cauchy problem has been considered in [10], in which the Burgers-Vlasov equations are equipped with regular and compact supported initial data. Meanwhile, the stability of travelling wave are also considered. When the initial data is less regular than which in [10], the global existence and uniqueness of finite energy solutions are proved in [12]. The second related simpler 1-D model on fluid-structure interaction

\[
\begin{cases}
  u_t + uu_x = \lambda(h'(t) - u(t,h(t)))\delta_{h(t)}, \\
  mh''(t) = -\lambda(h'(t) - u(t,h(t))),
\end{cases}
\]  

(1.5)

is also considered in [19], where $u$ is the velocity of the inviscid fluid and $h(t)$ is the location of the particles. $\lambda$ is the positive friction constant and $\delta_{h(t)}$ is the Dirac measure at $h(t)$. Global entropy weak solution involving shock waves to the system (1.5) are obtained in [19]. There are also some other fluid-kinetic models: compressible/incompressible Euler/Navier-Stokes equations coupled with Vlasov/-Fokker-Plank equations. Weak solutions or classical solution close to the equilibrium are studied in [20,22,25,26,29]. Some asymptotic problems such as hydrodynamic limit/stratified limit of viscous Burgers-Vlasov equation and Euler/Navier-Stokes equations coupling with Vlasov equation are also considered in [3,12–16,21,24].

In this paper, we investigate the existence of global weak solutions to the Cauchy problem (1.1)-(1.2). Since the derivation of the model in [9] is in 1D, and as far as the authors know, references on 2D or 3D cases are not founded for such model, we consider the 1D inviscid Burgers-Vlasov equations. Comparing with the diffusive system (1.3), without viscosity term, shock wave may exist when the initial data are given arbitrarily large. Hence we consider the entropy weak solutions to inviscid Burgers’ equation. Because of the nonlocal source term in the Burgers’ equation, we require the solution of the Vlasov equation be of finite kinetic energy. Consequently, we define $L^\infty - L^1$ weak solution to (1.1).

**Definition 1.1.** For any fixed $T \in (0, \infty)$, a pair of functions $u : \mathbb{R} \times [0,T] \to \mathbb{R}$, $f : \mathbb{R}^2 \times [0,T] \to [0,\infty)$ is called a global $L^\infty - L^1$ weak solution of Cauchy problem (1.1)-(1.2) if the following statements hold:

1. $u(x,t) \in L^\infty([0,T])$ and $f(x,v,t) \in L^\infty([0,T], (1+\nu^2) L^1(\mathbb{R}^2))$.
2. $u(x,t)$ is an entropy solution to Burgers’ equation, i.e. for any $\phi \in C^1_c(\mathbb{R} \times [0,T])$,

\[
\int_{\mathbb{R}} \phi(x,0)u_0(x)dx + \int_0^T \int_{\mathbb{R}} \left( u\phi_t + \frac{1}{2} u^2 \phi_x + \phi \int_{\mathbb{R}} f(v-u)dv \right) dxdt = 0,
\]  

(1.6)

and for any convex entropy pair $(\eta,q)$ the following entropy inequality

\[
\eta(u)_t + q(u)_x + \eta'(u) \int_{\mathbb{R}} f(u-v)dv \leq 0
\]  

(1.7)
holds in the sense of distributions, where \((\eta, q)\) satisfies \(\eta'(u)u = q'(u)\) and \(\eta(u)\) is convex with respect to \(u\).

(3) \(f(x,v,t)\) is a weak solution to Vlasov equation, i.e. for any \(\psi \in C^1_c(\mathbb{R}^2 \times [0,T])\),

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \psi(x,v,0)f_0(x,v)dvdx + \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} (f\psi_t + v f\psi_x + \psi_u f(u-v))dvdxdt = 0.
\]

(1.8)

Now we are ready to state our main result.

**Theorem 1.1.** [Main theorem] Let initial data \((u_0,f_0)\) satisfy

\[
\|u_0(x)\|_{L^\infty(\mathbb{R})} + \|(1+v^2)f_0(x,v)\|_{L^1(\mathbb{R}^2)} \leq M_0
\]

for some positive constant \(M_0\). Then there exists a global \(L^\infty - L^1\) weak solution to (1.1)-(1.2) in the sense of Definition 1.1 and there is a constant \(M\) depending solely on \(M_0\) such that

\[
\|u(x,t)\|_{L^\infty(\mathbb{R} \times [0,T])} + \|(1+v^2)f(x,v,t)\|_{L^\infty([0,T],L^1(\mathbb{R}^2))} \leq M.
\]

(1.10)

Hereafter, \(M\) denotes the constant depending only on \(M_0\) and it may vary from line to line.

**Remark 1.1.** Our uniform bounds of velocity \(\|u\|_{L^\infty}\) and kinetic energy of Vlasov equation \(\int_{\mathbb{R}} \int_{\mathbb{R}} (1+v^2)dvdx\) are both independent of time \(T\).

Our strategy of proving Theorem 1.1 is to construct approximate solutions by adding artificial viscosity to Burgers equation technically and regard the nonlocal term \(\int_{\mathbb{R}} f(v-u)dv\) as a dissipative source term in some sense. Maximum principles of parabolic equation and transport equation are applied to establish the uniform bound. Besides, in order to get the uniform estimates (also be independent of time \(T\)) of approximate viscosity solution \(u^\varepsilon\), we add some novel viscosity term and choose a special control function. In proving almost everywhere convergence of \(u^\varepsilon\), we employ \(L^\infty\) compensated compactness framework. More information about compensated compactness framework of \(L^p\) or \(L^\infty\) space can be found in [6-8,11,23] and the reference therein. On the other hand, to show the weak \(L^1\) convergence of \(f^\varepsilon\), we apply Dunford-Pettis theorem and analyze kinetic energy of \(f^\varepsilon\) and evolution of sets. In the proof, we also came across the difficulty on the weak convergence of \(\int vf^\varepsilon dv\), which is overcame again by applying the uniform bound of kinetic energy.

To be concise, in the present paper, we use \(\int\) instead of \(\int_{\mathbb{R}}\). \(C(\cdot)\) denotes constant depending on the parameters in the bracket. The rest of the paper is organised as follows. Section 2 is devoted to construct approximate solutions and prove their global existence. The proof of Theorem 1.1 is given in Section 3.

**2. Approximate solutions** In this section, we construct the globally existing approximate solutions to problem (1.1)-(1.2) by adding artificial viscosity and choosing the initial data technically. That is, we consider the approximate problem

\[
\begin{cases}
    u^\varepsilon_t + u^\varepsilon u^\varepsilon_x = \varepsilon u^\varepsilon_{xx} + \varepsilon \left( \int f^\varepsilon dv \right)_x + \int f^\varepsilon vdv - u^\varepsilon \int f^\varepsilon dv, \\
    f^\varepsilon_t + vf^\varepsilon_x + (f^\varepsilon(u^\varepsilon - v))_v = 0,
\end{cases}
\]

(2.1)

and the carefully selected initial data

\[u^\varepsilon(x,0) = u_0^\varepsilon(x) = u_0(x) * j_\varepsilon(x),\]
\[ f^\varepsilon(x,v,0) = f_0^\varepsilon(x,v) = \left[ \min\{\varepsilon^{-1/6}, f_0(x,v)\mathbf{1}_{\{|x|+|v| \leq \varepsilon^{-1/6}\}} \} \right] * j_\varepsilon(x) * j_\varepsilon(v) \geq 0, \quad (2.2) \]

where \( j_\varepsilon \) is standard one dimensional mollifier with parameter \( \varepsilon \). The initial data here is chosen to make \( f^\varepsilon \) be of explicit \( \varepsilon \)-depending compact support and \( L^\infty \) bound for later use. Our idea of adding the above viscosity is based on the following reasons. As is known, linear transport equation preserves the regularity of initial data and inviscid Burgers’ equation may formulate shock wave. Thus we add parabolic viscosity term to Burgers’ equation only. Moreover, to gain the uniform bound of \( u^\varepsilon \), we not only make full use of Vlasov equation and flux term in Burgers equation but also carefully choose control function. The viscosity term \( \varepsilon \int_R f^\varepsilon dv \) is introduced due to derivatives of our control function.

For any \( \sigma \in (0,1) \), we use \( C^{2+\sigma}(\mathbb{R}^2 \times [0,T]) \) and \( C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R} \times [0,T]) \) to denote usual and parabolic Hölder space respectively. We now would like to consider the global existence and uniqueness of smooth solutions \((u^\varepsilon, f^\varepsilon)\) to the Cauchy problem \((2.1)-(2.2)\) with \( u^\varepsilon(x,t) \in C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R} \times [0,T]), 0 \leq f^\varepsilon(x,v,t) \in C^{2+\sigma}(\mathbb{R}^2 \times [0,T]) \). For simplicity, in this section, the up index \( \varepsilon \) in \( u^\varepsilon \) and \( f^\varepsilon \) will be dropped.

### 2.1. Local existence.

We first consider the local existence of smooth solution of the Cauchy problem \((2.1)-(2.2)\). Let

\[
G(x,t) = \begin{cases} 
\delta(x), & t = 0, \\
\frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}, & t > 0,
\end{cases}
\]

denote the kernel of the homogeneous heat equation \( u_t = \varepsilon u_{xx} \). Then from the Burgers’ equation, i.e. the first equation in \((2.1)\), Duhamel principle tells us

\[
u(x,t) = \int G(x-y,t)u_0(y)dy + \int_0^t \int G(x-y,t-s) \left( \int fvdv - u \int fdv \right)(y,s)dyds \\
+ \int_0^t \int G(x-y,t-s) \left( \varepsilon \int fvdv - \frac{1}{2} u^2 \right) y(y,s)dyds.
\]

Integrating by parts gives

\[
u(x,t) = \int G(x-y,t)u_0(y)dy + \int_0^t \int G(x-y,t-s) \left( \int fvdv - u \int fdv \right)(y,s)dyds \\
+ \int_0^t \int G_y(x-y,t-s) \left( \varepsilon \int fvdv - \frac{1}{2} u^2 \right)(y,s)dyds,
\]

which implicitly gives the solution to the Burgers’ equation. For the Vlasov equation, i.e. the second equation in \((2.1)\), we rewrite it as

\[
f_t + vf_x + (u-v)f_v = f,
\]

which is a transport equation. One can integrate along the backward characteristic curves

\[
\frac{d}{ds} X(s;x,v,t) = V(s;x,v,t), \quad X(t;x,v,t) = x, \quad (2.4)
\]

\[
\frac{d}{ds} V(s;x,v,t) = u(X(s;x,v,t),s) - V(s;x,v,t), \quad V(t;x,v,t) = v, \quad (2.5)
\]
to get
\[ f(x,v,t) = f_0(X(0;x,v,t), V(0;x,v,t))e^t. \]  
(2.6)

Note that for smooth \( u \), the system of (2.4) and (2.5) has a unique smooth solution \((X(s;x,v,t), V(s;x,v,t))\):
\[
X(s;x,v,t) = x + \int_t^s V(\tau;x,v,t)d\tau, \quad (2.7)
\]
\[
V(s;x,v,t) = ve^t - s + \int_t^s e^{\tau-s}u(X(\tau;x,v,t), \tau)d\tau. \quad (2.8)
\]

Hence, from (2.3) and (2.6) we construct the approximate solutions of (2.1) in the following way. Set \( u^{(0)} = u_0^0, f^{(0)} = f_0^0 \), then there exists a \( K = C(\varepsilon, M_0) \) such that for any \( t > 0 \)
\[
\| u^{(0)} \|_{C^{2+\sigma,1+\sigma}([0,t] \times \mathbb{R}^2)} \leq K, \quad \| f^{(0)} \|_{C^{2+\sigma}([0,t] \times \mathbb{R}^2)} \leq K, \quad |\text{supp} f^{(0)}| \leq K,
\]
where we used the definition
\[
|\text{supp} f| = \max\{|x|, |v| : f(x,v,\cdot) > 0\}.
\]

For \( k \geq 1 \), we define
\[
u^{(k)}(x,t) = \int G(x-y,t)u_0(y)dy + \int_0^t \int G(x-y,t-s) \left( \int f^{(k-1)}v dq - u^{(k-1)} \int f^{(k-1)} dv \right) (y,s)dyds + \int_0^t \int G_y(x-y,t-s) (\varepsilon \int f^{(k-1)} dv - \frac{1}{2}(u^{(k-1)})^2) (y,s)dyds,
\]
and
\[
f^{(k)}(x,v,t) = f_0(X^{(k)}(0;x,v,t), V^{(k)}(0;x,v,t))e^t,
\]
where \((X^{(k)}(s;x,v,t), V^{(k)}(s;x,v,t))\) is defined using (2.4) and (2.5) as
\[
\frac{d}{ds} X^{(k)}(s;x,v,t) = V^{(k)}(s;x,v,t),
\]
\[
X^{(k)}(t;x,v,t) = x,
\]
\[
\frac{d}{ds} V^{(k)}(s;x,v,t) = u^{(k-1)}(X^{(k)}(s;x,v,t), s) - V^{(k)}(s;x,v,t),
\]
\[
V^{(k)}(t;x,v,t) = v,
\]
and \((x,v) \in \text{supp} f^{(k)}\). It is easy to see that \((X^{(k)}, V^{(k)})\) is well-defined. Thus \( f^{(k)} \) and \( u^{(k)} \) make sense. Besides, one can also see that \((u^{(k)}, f^{(k)})\) solves the following approximate equations
\[
\begin{aligned}
\begin{cases}
  u^{(k)}_t + u^{(k-1)}u^{(k-1)} = \varepsilon u^{(k)} u^{k-1} + \varepsilon \left( \int f^{(k-1)} dv \right)_x + \int f^{(k-1)}v dv - u^{(k-1)} \int f^{(k-1)} dv,
  
  f^{(k)}_t + v f^{(k)}_x + (u^{(k-1)} - v) f^{(k)}_v = f^{(k)}.
\end{cases}
\end{aligned}
\]
(2.9)
Obviously, from (2.2), for any \((x,v) \in \text{supp} f^{(1)}\),
\[ |X^{(1)}(0; x, v, t)| + |V^{(1)}(0; x, v, t)| \leq \varepsilon^{-1/6}. \]

Using (2.7), (2.8) and the bound of \(u^{(0)}\), we have
\[
|v| \leq |V^{(1)}(0; x, v, t)e^{-t}| + \left| \int_0^t e^{\tau-t} u^{(0)}(X^{(1)}(\tau; x, v, t), \tau) d\tau \right| \leq \varepsilon^{-1/6} + K \leq 2K,
\]
\[
|x| \leq |X^{(1)}(0; x, v, t)| + \left| \int_0^t V^{(1)}(\tau; x, v, t) d\tau \right| \leq \varepsilon^{-1/6} + 3Kte^t \leq 2K,
\]
provided \(t\) is sufficiently small, where we have used the fact that
\[
|V^{(1)}(s; x, v, t)| \leq |ve^{t-server}\|u^{(0)}\|_L^2| \leq 3Ke^t, \quad \text{for} \ 0 \leq s \leq t.
\]

Thus \(|\text{supp} f^{(1)}| \leq 2K\). Moreover, we have the following conclusion.

**Lemma 2.1.** There exists a small \(t_0 > 0\) such that the sequence \(\{(u^{(k)}, f^{(k)})\}_{k \geq 0}\) constructed above is a contraction sequence in
\[
S(t) = \{(u, f) \|\|u\|_{C^{2+s,1+\beta}([\mathbb{R}^2 \times [0, t])} \leq 2K, \quad \|f\|_{C^{2+s,1+\beta}([\mathbb{R}^2 \times [0, t])} \leq 2K, \quad |\text{supp} f| \leq 2K\}
\]
for all \(t \in (0, t_0)\).

**Proof.** It is easy to see that \((u^{(0)}, f^{(0)}) \in S(t)\). Suppose that for \(k \geq 1\), \((u^{(k-1)}, f^{(k-1)})\) has been shown in \(S(t)\). We then estimate \((u^{(k)}, f^{(k)})\).

Taking derivatives with respect to \(x\), one has for \(\ell = 0, 1, 2\),
\[
\partial_x u^{(k)}(x, t) = \int \partial_x G(x - y, t)u_0(y) dy + \int_0^t \int \partial_x G(x - y, t - s) \left( \int f^{(k-1)}dv - u^{(k-1)} \int f^{(k-1)}dv \right)(y, s) dy ds
\]
\[
+ \int_0^t \int \partial_y G_y(x - y, t - s) \left( \varepsilon \int f^{(k-1)}dv - \frac{1}{2}(u^{(k-1)})^2 \right)(y, s) dy ds.
\]

Using the symmetry of \(G(x - y)\), one further has
\[
\partial_x u^{(k)}(x, t) = \int \partial_x G(x - y, t)u_0(y) dy + \int_0^t \int (-1)^\ell \partial_y G(x - y, t - s) \left( \int f^{(k-1)}dv - u^{(k-1)} \int f^{(k-1)}dv \right)(y, s) dy ds
\]
\[
+ \int_0^t \int (-1)^\ell \partial_y G_y(x - y, t - s) \left( \varepsilon \int f^{(k-1)}dv - \frac{1}{2}(u^{(k-1)})^2 \right)(y, s) dy ds.
\]

Integration by parts gives
\[
\partial_x u^{(k)}(x, t) = \int G(x - y, t)\partial_x u_0(y) dy + \int_0^t \int G(x - y, t - s) \left( \partial_y \int f^{(k-1)}dv \right)(y, s) dy ds
\]
For any \((x,v)\) \(\in \text{supp} f^{(k)}\), \(t < t_1 \ll 1\), one has
\[
|X^{(k)}(0;x,v,t)| + |V^{(k)}(0;x,v,t)| \leq \varepsilon^{-1/6},
\]
and
\[
|v| \leq |V^{(k)}(0;x,v,t)e^{-t}| + \int_0^t e^{\tau-t} u^{(k-1)}(X^{(k)}(\tau;x,v,t),\tau)d\tau \leq K + 2Kt \leq 2K,
\]
\[
|V^{(k)}(s;x,v,t)| \leq |ve^{-s}| + \int_s^t e^{\tau-s} u^{(k-1)}(X^{(k)}(\tau;x,v,t),\tau)d\tau \leq 4e^{t}K \leq 6K,
\]
\[
|x| \leq |X^{(k)}(0;x,v,t)| + \int_0^t V^{(k)}(\tau;x,v,t)d\tau \leq K + 4Kte^t \leq 2K
\]
\[
|X^{(k)}(s;x,v,t)| \leq |x| + t|V^{(k)}(s;x,v,t)| \leq 2K + 4te^tK \leq 6K
\]
Thus
\[
|\text{supp} f^{(k)}| \leq 2K. \tag{2.10}
\]
Noticing the estimates of heat kernel
\[
\int G(x,t)dx = 1, \quad \int |G_x(x,t)|dx \leq \frac{C}{\sqrt{\varepsilon t}},
\]
using the bound of \(\text{supp} f^{(k-1)}\), one has
\[
\|u^{(k)}(\cdot,t)\|_{C^2(\mathbb{R})} \leq K + C(\varepsilon,K)(t + \sqrt{t}),
\]
then further gets
\[
\|u^{(k)}\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R} \times [0,t])} \leq K + C(\varepsilon,K)(t + \sqrt{t}) \leq 2K, \tag{2.11}
\]
provided \(t < t_2 \ll 1\).

For the estimates of \(f^{(k)}\) and for any \(0 \leq |\alpha| \leq 2\), taking derivatives one has
\[
(\partial^\alpha f^{(k)})_t + v(\partial^\alpha f^{(k)})_x + (u^{(k-1)} - v)(\partial^\alpha f^{(k)})_v = B(\alpha)\partial^\alpha f^{(k)} + B(\alpha'), \tag{2.12}
\]
where \(\partial^\alpha\) denotes the mix derivatives of \(x, v\) and \(t\) with order \(|\alpha|\), \(B(\alpha)\) is a linear function of \(v\) and \(\partial^\beta_x u^{(k-1)}\), with \(|\beta| \leq |\alpha|\) and \(B(\alpha')\) is a linear combination of \(\partial^\alpha f^{(k)}\) with \(|\alpha'| < |\alpha|\), whose coefficients are linear functions of \(v\) and \(\partial^\beta_x u^{(k-1)}\), with \(|\beta| \leq |\alpha|\). Integrating (2.12) along the characteristic curves \((X^{(k)}, V^{(k)})\), one has
\[
\partial^\alpha f^{k} = \langle \partial^\alpha f_0 \rangle (X^{(k)}, V^{(k)}) + \int_0^t B(\alpha)(X^{(k)}, V^{(k)})d\tau
\]
\[
+ \int_0^t e^{\int_0^s B(\alpha)(X^{(k)}, V^{(k)})ds} B(\alpha')(X^{(k)}, V^{(k)})d\tau,
\]
which then implies
\[ \| f^{(k)} \|_{C^2([0,t])} \leq Ke^{tC(\varepsilon,K)} + C(\varepsilon,K)te^{tC(\varepsilon,K)} \leq 2K. \]

We further gain the following Hölder estimate
\[ \| f^{(k)} \|_{C^{2+\sigma}([0,t])} \leq 2K, \quad (2.13) \]
provided \( t < t_3 \ll 1 \). Hence from (2.10), (2.11) and (2.13), one has
\[ (u^{(k)}, f^{(k)}) \in \mathcal{S}(t). \]

Moreover, consider the equation for \( u^{(k)} - u^{(k-1)} \) and \( f^{(k)} - f^{(k-1)} \), similar to above calculation, we get
\[ \| u^{(k+1)} - u^{(k)} \|_{C^{2+\sigma,1+\delta}([\mathbb{R} \times [0,t])} \leq C(\varepsilon,K)\int_0^t \| u^{(k)} - u^{(k-1)} \|_{C^{2+\sigma,1+\delta}([\mathbb{R} \times [0,t])} \]
\[ \leq \frac{1}{2} \| u^{(k)} - u^{(k-1)} \|_{C^{2+\sigma,1+\delta}([\mathbb{R} \times [0,t])} \]
\[ \| f^{(k+1)} - f^{(k)} \|_{C^{2+\sigma}([0,t])} \leq C(\varepsilon,K)(e^t - 1) \| f^{(k)} - f^{(k-1)} \|_{C^{2+\sigma}([\mathbb{R} \times [0,t])} \]
\[ \leq \frac{1}{2} \| f^{(k)} - f^{(k-1)} \|_{C^{2+\sigma}([\mathbb{R} \times [0,t])}, \]
provided \( t < t_4 \ll 1 \). Let \( t_0 = \min\{t_i, i = 1, \cdots, 4\} \). Then we gain \( (u^{(k)}, f^{(k)}) \) is a contraction sequence in \( \mathcal{S}(t) \) with \( t \in (0, t_0) \) and end the proof.

Applying fixed point theorem to \((u^{(k)}, f^{(k)})\), combing with Lemma 2.1, one gains that there exists a pair of functions \((u, f)\) such that
\[ u^{(k)} \to u \text{ in } C^{2+\sigma,1+\delta}([\mathbb{R} \times [0, t_0]), \]
\[ f^{(k)} \to f \text{ in } C^{2+\sigma}([\mathbb{R} \times [0, t_0]), \]
and \((u, f)\) is the unique smooth solution of Cauchy problem (2.1)-(2.2) by taking limit in (2.9).

**2.2. Uniform estimates.** We will apply maximum principles of parabolic equation and transport equation to bound \( \| u \|_{L^\infty([\mathbb{R} \times [0,T])} \) and \( \| f \|_{L^\infty([0,T], 1+\mu^2 L^1([\mathbb{R}^2]))} \).

**Lemma 2.2.** For the approximate solutions constructed above, there exists a constant \( M \) depending only on \( M_0 \) such that (1.10) holds for \((u^\varepsilon, f^\varepsilon)\).

**Proof.** Obviously, from (2.2) and (2.6) one has \( f(x,v,t) \geq 0 \). For any \((x,v) \in \text{supp } f\), by (2.2) and (2.8), one has
\[ |v| \leq |V(0;x,v,t)e^{-t}| + \left| \int_0^t e^{-\tau-t} u(X(\tau;x,v,t), \tau) d\tau \right| \]
\[ \leq e^{-1/6} + \hat{C}(\varepsilon,M_0,T), \quad (2.14) \]
after priori assuming
\[ \| u \|_{L^\infty} \leq \hat{C}(\varepsilon,M_0,T) \text{ for some large enough } \hat{C}(\varepsilon,M_0,T). \]

Besides,
\[ |x| \leq |X(0;x,v,t)| + \left| \int_0^t V(\tau;x,v,t) d\tau \right| \]
\[ \leq \varepsilon^{-1/6} + (2\hat{C}(\varepsilon, M_0, T) + \varepsilon^{-1/6})Te^T. \] (2.15)

Hence by (2.14)-(2.15), \( f \) enjoys compact support (depending on \( \varepsilon, T \)) and
\[ \lim_{|x| \to \infty \text{ or } |v| \to \infty} f(x, v, t) = 0. \]

Thus integrating the Vlasov equation over \( \mathbb{R}^2 \times [0, t] \) with respect to \( (x, v, t) \) gives
\[ \int \int f(x, v, t) dx dv = \int \int f_0(x, v) dx dv \leq M_0, \] (2.16)
where we have used (1.9). Moreover, integration over \( \mathbb{R} \) with respect to \( v \) also gives
\[ \left( \int f dv \right)_t + \left( \int vf dv \right)_x = 0. \] (2.17)
Define control function
\[ \psi(x, t) = \int_\infty^{-\infty} \int f(y, v, t) dv dy, \]
then using (2.17) one has
\[ \psi_t = \int_x^{-\infty} \left( \int f(y, v, t) dv \right)_t dy = -\int_x^{-\infty} \left( \int vf(y, v, t) dv \right)_y dy = \int v dv, \]
\[ \psi_x = -\int f dv, \ \psi_{xx} = -\left( \int f dv \right)_x. \]
Thus one is also able to derive the equation for \( u - \psi \)
\[ (u - \psi)_t + u(u - \psi)_x = \varepsilon u_{xx} + \varepsilon \left( \int f dv \right)_x + \int vf dv - u \int f dv \]
\[ -\psi_t - u \psi_x \]
\[ = \varepsilon(u - \psi)_{xx}. \]
Applying maximum principle for the above parabolic equation with respect to \( u - \psi \), we obtain
\[ \|u - \psi\|_{L^\infty} \leq \|u_0 - \psi(x, 0)\|_{L^\infty} \leq M_0 + \int \int f_0 dv dx \leq 2M_0. \]

Thus, we have
\[ \|u(x, t)\|_{L^\infty} \leq \|\psi\|_{L^\infty} + \|u - \psi\|_{L^\infty} \leq 3M_0 \] (2.18)
by using (2.16), which also close our priori assumption
\[ \|u\|_{L^\infty} \leq 3M_0 < \hat{C}(\varepsilon, M_0, T). \]
Besides, similar to the calculations of (2.14) and (2.15), one has
\[ |v| \leq \varepsilon^{-1/6} + 3M_0, \ \ |x| \leq \varepsilon^{-1/6} + (\varepsilon^{-1/6} + 2M_0)Te^T. \] (2.19)
Furthermore, multiplying the Vlasov equation by $v^2$ and integrating over $\mathbb{R}^2$ with respect to $(v,x)$ we have
\[
\frac{d}{dt} \int \int f v^2 dv dx = \int \int (2 f u - 2 f v^2) dv dx \leq \int \int f (u^2 - v^2) dv dx \\
\leq \|u\|_{L^\infty}^2 \int \int f dv dx - \int \int f v^2 dv dx \\
\leq 9 M_0^3 - \int \int f v^2 dv dx,
\]
where we have used (2.16). Gronwall inequality yields
\[
\int \int f v^2 dv dx \leq e^{-t} \int \int f_0 v^2 dv dx + 9 M_0^3 (1 - e^{-t}) \leq M_0 + 9 M_0^3.
\]
(2.20)
thus (2.16), (2.18) and (2.20) conclude the present Lemma. \[\square\]

2.3. Conclusion. Standard theory of quasilinear parabolic equation (see [18]) can be applied to the equation for $u - \psi$
\[
(u - \psi)_x + u (u - \psi)_x = \varepsilon (u - \psi)_{xx}
\]
to get
\[
\| (u - \psi)_{x} \|_{C^0(\mathbb{R})} \leq C(\varepsilon, T),
\]
where we have used the uniform bound on $\|u\|_{C^0(\mathbb{R})} \leq C$ and $C(\varepsilon, T)$ is increasing function of $T$. So we have
\[
\| u_{x} \|_{C^0(\mathbb{R})} \leq C(\varepsilon, T).
\]
Then using the Vlasov equation, we can also get
\[
\| f_{x} \|_{C^0(\mathbb{R}^2)} + \| f_{v} \|_{C^0(\mathbb{R}^2)} \leq C(\varepsilon, T).
\]
By standard bootstrap argument we have the following estimate
\[
\| u(x,t) \|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R} \times [0,T])} + \| f(x,v,t) \|_{C^{2+\sigma}(\mathbb{R}^2 \times \mathbb{R} \times [0,T])} \leq C(\varepsilon, T),
\]
With the local existence in Subsection 2.1, the global existence of solution $u(x,t) \in C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R} \times [0,T])$ and $0 \leq f(x,v,t) \in C^{2+\sigma}(\mathbb{R}^2 \times [0,T])$ to the Cauchy problem (2.1)-(2.2) is obtained. Thus we got the following conclusion.

THEOREM 2.1. For any $T > 0$, any fixed $\varepsilon$, there exist a unique global solution $u^\varepsilon(x,t) \in C^{2+\sigma,1+\frac{\sigma}{2}}(\mathbb{R} \times [0,T])$, $0 \leq f^\varepsilon(x,v,t) \in C^{2+\sigma}(\mathbb{R}^2 \times [0,T])$ to Cauchy problem (2.1)-(2.2).

3. Proof of main theorem
In this section, we will establish the convergence of $(u^\varepsilon, f^\varepsilon)$, whose limit is just an $L^\infty - L^1$ weak solution to Cauchy problem (1.1)-(1.2).

3.1. Limit of functions. We first show some convergence results related to $f^\varepsilon$. Recall the well-known weak $L^1$ compactness framework, i.e. Dunford-Pettis Theorem (see [17] Theorem 8 or [2], page 167).

PROPOSITION 3.1. [Dunford-Pettis] A sequence $\{f^\varepsilon\}$ is weakly compact in $L^1(\mathbb{R}^2)$ if and only if $\{f^\varepsilon\}$ satisfies the following conditions:
(1) The sequence $f^\varepsilon$ is equibounded in $L^1(\mathbb{R}^2)$, i.e.
$$\sup_{\varepsilon} \| f^\varepsilon \|_{L^1(\mathbb{R}^2)} < \infty.$$ 

(2) The sequence $f^\varepsilon$ is equiintegrable, i.e.

(2a) For any $\delta > 0$, there exists measurable set $A \subset \mathbb{R}^2$ with $|A| < \infty$ such that
$$\int_{\mathbb{R}^2 \setminus A} f^\varepsilon dxdv < \delta.$$ 

(2b) For any $\delta > 0$, there exists $\kappa > 0$ such that for any measurable set $E \subset \mathbb{R}^2$, with $|E| \leq \kappa$, there holds
$$\int_{E} f^\varepsilon dxdv < \delta.$$ 

We shall verify (1) and (2) for $f^\varepsilon(x,v,t)$ with any fixed $t > 0$.

Verification of (1): Obviously, from Lemma 2.2, one finds that
$$\| f^\varepsilon \|_{L^\infty([0,T],(1+v^2)L^1(\mathbb{R}^2))} \leq M,$$ (3.1)
which means $f^\varepsilon(\cdot,\cdot,t)$ is uniformly equibounded with respect to $\varepsilon$ and $t$ in $L^1(\mathbb{R}^2)$.

Verification of (2a): For any $\delta > 0$, we can choose $A = \{(x,v)||x| \leq \Lambda,|v| \leq \Lambda\}$ with $\Lambda \geq \frac{M}{\delta}$ where $M$ comes from (3.1), so we have for any $t > 0$,
$$\int_{\mathbb{R}^2 \setminus A} f^\varepsilon(x,v,t)dxdv \leq \frac{1}{\Lambda^2+1} \int_{\mathbb{R}^2 \setminus A} f^\varepsilon(x,v,t)(1+v^2)dxdv \leq \delta,$$
which implies that (2a) is satisfied by $f^\varepsilon(x,v,t)$.

Verification of (2b): By the fact that $f_0^\varepsilon \rightharpoonup f_0$ in $L^1(\mathbb{R}^2)$, Dunford-Pettis theorem tells us that for any $\delta$, there exists $\kappa_0$ such that for any $E_0 \subset \mathbb{R}^2$ with $|E_0| \leq \kappa_0$ it holds that
$$\int_{E_0} f_0^\varepsilon(x,v)dxdv \leq \delta.$$ 

On the other hand, considering the variable transformation
$$\mathcal{J} : (x,v) \mapsto (X^\varepsilon,V^\varepsilon),$$
from (2.4) and (2.5), one is able to show that the Jacobian
$$J(\tau) = \det \nabla_{x,v}(X^\varepsilon,V^\varepsilon)$$
of map $\mathcal{J}$ is positive and satisfies the following ODE
$$\begin{cases}
\frac{dJ(\tau)}{d\tau} = - J(\tau), \\
J(t) = 1.
\end{cases}$$
Then we have $J(\tau) = e^{t-\tau}$. For any $T \in (0,\infty)$, one can take $\kappa = e^{-T}\kappa_0$. Then for any measurable set $E \subset \mathbb{R}^2$ with $|E| \leq \kappa$, one has $|\mathcal{J}(0)(E)| \leq e^{t-\tau}\kappa_0 \leq \kappa_0$ for any $t \in [0,T]$. Hence one further has
$$\int_{\mathcal{J}(0)(E)} f_0^\varepsilon(x,v)dxdv \leq \delta.$$
Finally, with (2.6), one gains
\[
\int_E f^\varepsilon(x,v,t) dx dv = \int_{\mathcal{J}(0)(E)} f_0^\varepsilon(X^\varepsilon(0;x,v,t), V^\varepsilon(0;x,v,t)) e^t J(0)^{-1} dX^\varepsilon dV^\varepsilon
\]
\[
= \int_{\mathcal{J}(0)(E)} f_0^\varepsilon(X^\varepsilon(0;x,v,t), V^\varepsilon(0;x,v,t)) dX^\varepsilon dV^\varepsilon \leq \delta,
\]
which then gives (2b).

Therefore, applying Proposition 3.1, we get some subsequence (for simplicity we still denote) \( f^\varepsilon \) and a nonnegative function \( f \in L^\infty([0,T],L^1(\mathbb{R}^2)) \) such that
\[
f^\varepsilon(x,v,t) \rightharpoonup f(x,v,t) \text{ weakly in } L^1(\mathbb{R}^2), \text{ for any } t > 0 \tag{3.2}
\]
and
\[
\iint f dx dv \leq M. \tag{3.3}
\]

Next we shall show the convergence of \( \int v f^\varepsilon dv \). In fact, using the convexity of kinetic energy and weak convergence of \( f^\varepsilon \), one is able to get from (2.20) that
\[
\iint f v^2 dv dx \leq \liminf_{\varepsilon \to 0} \iint f^\varepsilon v^2 dv dx \leq M. \tag{3.4}
\]

For the weak convergence of \( \int f^\varepsilon v dv \), we utilize the approach in [1]. Denote \( 1_{[-1,1]}(s) \) as \( \omega(s) \), for any \( \varphi \in C_c^\infty(\mathbb{R} \times [0,T]) \), for arbitrary \( L > 0 \), one can derive
\[
\int_0^T \int \left( \sqrt{\frac{v}{L}} v f^\varepsilon dv - \int \varphi dx \right) dt
\]
\[
= \int_0^T \int \left( \int \omega\left(\frac{v}{L}\right) v f^\varepsilon dv - \int \omega\left(\frac{v}{L}\right) v f dv \right) \varphi dx dt
\]
\[
+ \int_0^T \int \int f^\varepsilon (1 - \omega\left(\frac{v}{L}\right)) dv \varphi dx dt
\]
\[
- \int_0^T \int \int f (1 - \omega\left(\frac{v}{L}\right)) dv \varphi dx dt.
\]

By (3.2), i.e. the weak convergence of \( f^\varepsilon \) in \( L^1 \), one can get that the first term on the right hand side converges to 0 as \( \varepsilon \to 0 \). For the last two terms, using (2.20) and (3.4) we also has
\[
\left| \int_0^T \int \int v f^\varepsilon (1 - \omega\left(\frac{v}{L}\right)) dv \varphi dx dt \right| \leq \frac{\|\varphi\|_{L^\infty}}{L} \int_0^T \int \int f^\varepsilon v^2 dv dx dt \leq \frac{MT\|\varphi\|_{L^\infty}}{L},
\]
\[
\left| \int_0^T \int \int v f (1 - \omega\left(\frac{v}{L}\right)) dv \varphi dx dt \right| \leq \frac{\|\varphi\|_{L^\infty}}{L} \int_0^T \int \int f v^2 dv dx dt \leq \frac{MT\|\varphi\|_{L^\infty}}{L},
\]
both of which go to 0 when \( L \to \infty \). Therefore, we have
\[
\int f^\varepsilon v dv \rightharpoonup \int v f dv \text{ in the sense of distribution.} \tag{3.5}
\]
Similarly, we also have
\[ \int f^\varepsilon dv \to \int f dv \text{ in the sense of distribution.} \quad (3.6) \]

Now, we consider the convergence of \( u^\varepsilon \). We also recall the \( L^\infty \) compensated compactness framework and Murat’s Lemma:

**Proposition 3.2.** ([5]) Assume that a sequence \( u^\varepsilon(x,t) \) satisfies
\[ \|u^\varepsilon\|_{L^\infty} \leq C, \]
and
\[ \eta(u^\varepsilon)_t + q(u^\varepsilon)_x \text{ is compact in } H^{-1}_{loc}(\mathbb{R} \times [0,T]) \]
for any entropy pair \((\eta,q)\) with \( \eta'(u)u = q'(u) \) (or two special entropy pairs in Theorem 2.7 of [5]). Then there exist a subsequence \( \{u^\varepsilon_k\}_{k=1}^{\infty} \subset \{u^\varepsilon\}_{\varepsilon>0} \) and function \( u(x,t) \) such that
\[ u^\varepsilon_k \to u, \quad (u^\varepsilon_k)^2 \to u^2, \text{ a.e. as } k \to \infty. \]

**Lemma 3.1.** ([4, 27]) Let \( \Omega \subset \mathbb{R}^n \) be a open bounded subset, then
\( (\text{compact set of } W^{-1,a}_{loc}(\Omega)) \cap (\text{bounded set of } W^{-1,b}_{loc}(\Omega)) \subset (\text{compact set of } H^{-1}_{loc}(\Omega)), \)
where \( a \) and \( b \) are constants satisfying \( 1 < a \leq 2 < b \).

From uniform estimate (2.18), there exists a \( u(x,t) \in L^\infty(\mathbb{R} \times [0,T]) \) such that
\[ \|u\|_{L^\infty(\mathbb{R} \times [0,T])} \leq M \text{ and } u^\varepsilon(x,t) \rightharpoonup u(x,t), \text{ weak * in } L^\infty(\mathbb{R} \times [0,T]). \quad (3.7) \]

To prove the strong convergence of \( u^\varepsilon \), we need to get entropy dissipation estimate. For any compact set \( \Omega \subset \mathbb{R} \times [0,T], \) for any \( \varphi \in C^\infty_c(\mathbb{R} \times [0,T]) \) with \( \varphi|_\Omega = 1 \), multiplying the first equation in (2.1) by \( u^\varepsilon \varphi^2 \), integrating over \( \mathbb{R} \times [0,T], \) one has
\[ \int_0^T \int \left[ \frac{1}{2}(u^\varepsilon)_t^2 \varphi^2 + \frac{1}{3}(u^\varepsilon)_x^3 \varphi^2 \right] dxdt = \int_0^T \int \left[ \varepsilon u^\varepsilon u^\varepsilon \varphi^2 + \varepsilon \left( \int f^\varepsilon dv \right) u^\varepsilon \varphi^2 \right] dxdt + \int_0^T \int \left[ u^\varepsilon \varphi^2 \int f^\varepsilon dv - \varphi^2 (u^\varepsilon)^2 \int f^\varepsilon dv \right] dxdt. \]

Integrating by parts gives
\[ \int_0^T \int \varepsilon \varphi^2 (u^\varepsilon)_x^2 dxdt = \int_0^T \int \left[ (u^\varepsilon)^2 \varphi \varphi_x + \frac{2}{3}(u^\varepsilon)^3 \varphi \varphi_x - 2\varepsilon \varphi \varphi_x u^\varepsilon \int f^\varepsilon dv \right] dxdt + \int_0^T \int \left[ u^\varepsilon \varphi^2 \int f^\varepsilon dv - \varphi^2 (u^\varepsilon)^2 \int f^\varepsilon dv \right] dxdt - \int_0^T \int \left[ 2\varepsilon \varphi \varphi_x u^\varepsilon u^\varepsilon_x + \varepsilon \varphi^2 u^\varepsilon_x \int f^\varepsilon dv \right] dxdt. \]

From Lemma 2.2, one is easy to get the first two terms are bounded. Applying Hölder inequality and (3.1), the last term is bounded by
\[ \frac{1}{2} \int_0^T \int_\Omega \varepsilon (u^\varepsilon)_x^2 dxdt + M \varepsilon \int_0^T \int \varphi^2 \left( \int f^\varepsilon dv \right)^2 dxdt + C(M_0,T). \]
From (2.2) and (2.6), one has
\[ \|f\|_{L^\infty} \leq e^T \|f_0\|_{L^\infty} \leq e^{-1/6}e^T. \]

By (2.19), we further have
\[ \varepsilon \int_0^T \varphi^2 \left( \int f^\varepsilon dv \right)^2 dx dt \leq C(M_0, T)\varepsilon^{1/3}. \]

So we gain
\[ \int \int_\Omega \varepsilon(u_x^\varepsilon)^2 dx dt \leq C(M_0, T). \]

Now we are ready to show the entropy dissipation of (2.1). For any weak entropy-entropy flux (\( \eta, q \)) with \( \eta \in C^2 \), one has
\[ \eta(u_x^\varepsilon)_t + q(u_x^\varepsilon)_x = \varepsilon \eta(u_x^\varepsilon)_{xx} - \varepsilon \eta''(u_x^\varepsilon)(u_x^\varepsilon)^2 + \eta'(u_x^\varepsilon) \int f^\varepsilon (v - u_x^\varepsilon) dv \]
\[ + \varepsilon \eta'(u_x^\varepsilon) \left( \int f^\varepsilon dv \right)_x = \sum_{k=1}^4 I_k, \]
which is got by multiplying (2.1) by \( \eta'(u_x^\varepsilon) \). Obviously, from the uniform boundedness of \( u_x^\varepsilon \) (2.18), using the estimates (3.1) and (3.9), we obtain \( I_2 + I_3 \) is bounded in \( L_{loc}^1(\mathbb{R} \times [0, T]) \). Thus by embedding theorem and Schauder theorem \( I_2 + I_3 \) is compact in \( W_{loc}^{-1, \alpha}(\mathbb{R} \times [0, T]) \) with some \( 1 < \alpha < 2 \). For \( I_1 \), we also have
\[ \left| \int_0^T \int_\Omega \varepsilon(\eta(u_x^\varepsilon)_{xx} \varphi) dx dt \right| = \left| \int_0^T \int_\Omega \varepsilon \eta'(u_x^\varepsilon) u_x^\varepsilon \varphi dx dt \right| \]
\[ \leq M \sqrt{\varepsilon} \left( \int_0^T \int_\Omega (u_x^\varepsilon)^2 dx dt \right)^{1/2}, \]
for any compact set \( \Omega \subset \mathbb{R} \times [0, T] \), and any \( \varphi \in C^\infty_c(\Omega) \). So we have \( I_1 \) is compact in \( H_{loc}^{-1}(\mathbb{R} \times [0, T]) \). For \( I_4 \), using (3.8) and (3.9), one also has
\[ \left| \int_0^T \int_\Omega \varepsilon \eta'(u_x^\varepsilon) \left( \int f^\varepsilon dv \right)_x \varphi dx dt \right| \]
\[ \leq \left| \int_0^T \int_\Omega \varepsilon \eta'(u_x^\varepsilon) \int f^\varepsilon dv \varphi dx dt \right| + \left| \int_0^T \int_\Omega \varepsilon \varphi \eta''(u_x^\varepsilon) u_x^\varepsilon \int f^\varepsilon dv dx dt \right| \]
\[ \leq M \varepsilon + M \left( \int_0^T \int_\Omega (u_x^\varepsilon)^2 dx dt \right)^{1/2} \left( \varepsilon \int_0^T \int_\Omega (f^\varepsilon dv)^2 dx dt \right)^{1/2} \]
\[ \leq M \varepsilon^{1/6}, \]
which also implies \( I_4 \) is compact in \( H_{loc}^{-1}(\mathbb{R} \times [0, T]) \). Putting things together, one has
\[ \eta(u_x^\varepsilon)_t + q(u_x^\varepsilon)_x \text{ is compact in } W_{loc}^{-1, \alpha}(\mathbb{R} \times [0, T]) \text{ for some } 1 < \alpha < 2. \] (3.10)
From the $L^\infty$ bound of $u^\varepsilon$, one also has
\[ \eta(u^\varepsilon)_t + q(u^\varepsilon)_x \text{ is bounded in } W^{-1,\infty}_{\text{loc}}(\mathbb{R} \times [0,T]). \] (3.11)

Applying Murat’s lemma (see Lemma 3.1) to (3.10) and (3.11), we finally have
\[ \eta(u^\varepsilon)_t + q(u^\varepsilon)_x \text{ is compact in } H^{-1}_{\text{loc}}(\mathbb{R} \times [0,T]). \] (3.12)

Therefore, using $L^\infty$ compensated compactness framework (see Proposition 3.2 or Theorem 2.7 in [5]), (3.7) and (3.12), we have
\[ u^\varepsilon(x,t) \to u(x,t), \text{ a.e. in } \mathbb{R} \times [0,T], \]
\[ u^\varepsilon(x,t) \to u(x,t), \text{ in } L^p_{\text{loc}}(\mathbb{R} \times [0,T]), \quad \forall p \in [1,\infty). \] (3.13)

### 3.2. Limit of equations.

To show $(u,f)$ is weak solution to (1.1), we need to verify (1.6) and (1.8). For simplicity, here we only show (1.6), since (1.8) can be treated similarly. Multiplying the first equation in (2.1) by $\phi \in C_\infty(\mathbb{R} \times [0,T))$, integrating over $\mathbb{R} \times [0,T]$ and using integration by parts, we obtain
\[
\int_0^T \phi(x,0)u^\varepsilon_0(x)dx + \int_0^T \int \left( u^\varepsilon \phi_t + \frac{1}{2}(u^\varepsilon)^2 \phi_x + \phi \int f^\varepsilon(v-u^\varepsilon)dv \right) dxdt \\
+ \varepsilon \int_0^T \int \left( u_x^\varepsilon \phi_x + \int f^\varepsilon dv \phi_x \right) dxdt = 0.
\]

For the last two term in the left hand side, using (3.9), one can derive
\[
\left| \varepsilon \int_0^T \int u_x^\varepsilon \phi_x dxdt \right| \leq \sqrt{\varepsilon} \left( \int_0^T \int \varepsilon(u_x^\varepsilon)^2 dxdt \right)^{\frac{1}{2}} \left( \int_0^T \int \phi_x^2 dxdt \right)^{\frac{1}{2}} \\
\leq C(M_0,\|\phi\|_{H^1(\mathbb{R})},T) \sqrt{\varepsilon},
\]
and
\[
\left| \varepsilon \int_0^T \int \phi_x \int f dv dxdt \right| \leq C(M_0,T,\|\phi\|_{C^1(\mathbb{R}))}) \varepsilon,
\]
both of which go to 0 when $\varepsilon \to 0$. It only remains to show the convergence of $u^\varepsilon \int f^\varepsilon dv$. In fact, observing that
\[
u^\varepsilon \int f^\varepsilon dv - u \int f dv \\
= (u^\varepsilon - u) \int f^\varepsilon dv + \left( \int f^\varepsilon dv - \int f dv \right) u,
\]
using (3.6) and (3.13) we have
\[ u^\varepsilon \int f^\varepsilon dv \to u \int f dv \text{ in the sense of distribution.} \]

Thus we have (1.6).
3.3. Entropy inequality. We shall also show entropy inequality for Burgers equation, i.e. (1.7). Multiply the first equation in (2.1) by $\eta'(u^\varepsilon)\varphi$, where $\eta$ is convex and $\varphi \in C_c^\infty(\mathbb{R} \times (0,T))$ is nonnegative, and integrate the result over $\mathbb{R} \times (0,T)$ we get

$$
\int_0^T \int_{\mathbb{R}} \left( \eta(u^\varepsilon) \varphi_t + q(u^\varepsilon) \varphi_x + \varphi \eta'(u^\varepsilon) \int_{\mathbb{R}} f^\varepsilon(v-u^\varepsilon) dv \right) dx dt \\
= - \int_0^T \int_{\mathbb{R}} \left( \varepsilon u^\varepsilon_{xx} + \varepsilon \left( \int_{\mathbb{R}} f^\varepsilon dv \right) \right) \eta'(u^\varepsilon) \varphi dx dt \\
= \varepsilon \int_0^T \int_{\mathbb{R}} u^\varepsilon \eta'(u^\varepsilon) \varphi_x dx dt + \varepsilon \int_0^T \int_{\mathbb{R}} \eta''(u^\varepsilon) (u^\varepsilon_x)^2 \varphi dx dt \\
+ \int_0^T \int_{\mathbb{R}} \varepsilon \varphi_x \eta'(u^\varepsilon) \int_{\mathbb{R}} f^\varepsilon dv dx dt + \int_0^T \int_{\mathbb{R}} \varepsilon \eta''(u^\varepsilon) u^\varepsilon_x \varphi \int_{\mathbb{R}} f^\varepsilon dv dx dt \\
\geq - C(\varphi)(\varepsilon + \varepsilon^{1/2} + \varepsilon^{1/6}),
$$

where we have used (3.8) and (3.9). We also have

$$
\eta'(u^\varepsilon) \int_{\mathbb{R}} v f^\varepsilon dv - \eta'(u) \int_{\mathbb{R}} v f dv \\
= (\eta'(u^\varepsilon) - \eta'(u)) \int_{\mathbb{R}} v f^\varepsilon dv + \left( \int_{\mathbb{R}} v f^\varepsilon dv - \int_{\mathbb{R}} v f dv \right) \eta'(u),
$$

and

$$
u^\varepsilon \eta'(u^\varepsilon) \int_{\mathbb{R}} f^\varepsilon dv - u \eta'(u) \int_{\mathbb{R}} f dv \\
= (u^\varepsilon \eta'(u^\varepsilon) - u \eta'(u)) \int_{\mathbb{R}} f^\varepsilon dv + \left( \int_{\mathbb{R}} f^\varepsilon dv - \int_{\mathbb{R}} f dv \right) u \eta'(u).
$$

Using (3.5), (3.6) and (3.13) we get the entropy inequality (1.7) by letting $\varepsilon \to 0$.

Finally, combining with (3.3), (3.4), (3.7), we complete the proof of Theorem 1.1.

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