SECONDARY BROWN-KERVAIRE QUADRATIC FORMS AND \( \pi \)-MANIFOLDS

FUQUAN FANG AND JIANZHONG PAN

Abstract. In this paper we assert that for each \( \Phi \)-oriented \( 2n \)-manifold (c.f : Definition 1.1) \( M \) where \( n \geq 4 \) and \( n \neq 3(\text{mod} 4) \), there is a well-defined quadratic function \( \phi_M : H^{n-1}(M, \mathbb{Z}_4) \to \mathbb{Q}/\mathbb{Z} \), we call the secondary Brown-Kervaire quadratic forms, so that

- \( \phi_M(x + y) = \phi_M(x) + \phi_M(y) + j(x \cup Sq^2 y)[M] \),
- the Witt class of \( \phi_M \) is a homotopy invariant, if the Wu class \( v_{n+2-2i}(\nu_M) = 0 \) for all \( i \).

where \( j : \mathbb{Z}_2 \to \mathbb{Q}/\mathbb{Z} \) is the inclusion homomorphism and \( \nu_M \) the stable normal bundle of \( M \).

Among the applications we obtain a complete classification of \((n - 2)\)-connected \( 2n \)-dimensional \( \pi \)-manifolds up to homeomorphism and homotopy equivalence, where \( n \geq 4 \) and \( n + 2 \neq 2^i \) for any \( i \). In particular, we prove that the homotopy type of such manifolds determine their homeomorphism type.

1. Introduction

Let \( M \) be a \( 2n \)-dimenisonal framed manifold (i.e. a \( \pi \)-manifold with a framing) where \( n = 1(\text{mod} 2) \). The Kervaire invariant of \( M \) is the Arf invariant of a \( \mathbb{Z}_2 \)-valued Kervaire quadratic form of \( M \)

\[ q_M : H^n(M, \mathbb{Z}_2) \to \mathbb{Z}_2 \]

satisfying

\[ q_M(x + y) = q_M(x) + q_M(y) + (x \cup y)[M]_2 \] (1.1)

It was invented by Kervaire to find the first example of non-smoothable PL-manifold. Kervaire invariants and its various generalizations, e.g.
the Brown-Kervaire invariants[4], play very important roles in geometric topology. Formally, $q_M$ is a “quadratic form” subject to the symmetric bilinear form

$$H^n(M, \mathbb{Z}_2) \times H^n(M, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$$

$$(x, y) \rightarrow x \cup y \mod 2$$

For a Spin manifold of even dimension, there is another symmetric bilinear form $\mu_M$ studied by Landweber and Stong [16]:

$$\mu_M : H^{n-1}(M, \mathbb{Z}_2) \times H^{n-1}(M, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$$

$$(x, y) \rightarrow Sq^2(x) \cup y \mod 2$$

A natural algebraic question to ask is whether there is an intrinsic “quadratic form” of $M$ subject to $\mu_M$. To answer this turns out to be the main novelty of this paper. For a large family of Spin manifolds including all $\pi$-manifolds, the so called $\Phi$-oriented manifolds, we will define a $\mathbb{Q}/\mathbb{Z}$-form subject to $\mu_M$, which resembles to the Brown-Kervaire quadratic forms in the formulation. It has the most similar properties of the Brown-Kervaire quadratic forms, e.g., the isomorphism class of the form is a homotopy invariant if the manifold has vanishing Wu classes. A bit surprising to us, this invariant applies to give a classification of $(n-2)$-connected $2n$-dimensional $\pi$-manifolds up to homotopy equivalence and homeomorphism ($n \geq 4$).

To state our main results, let us start with some notations.

Let $\{Y_k\}_{k \in \mathbb{N}}$ be a connected spectrum with $U \in H^0(Y) \cong \mathbb{Z}$ a generator so that $i^*U \in H^0(S^0)$ a generator, where $i : S^0 \rightarrow Y$ is the inclusion map of the spectrum.

**Definition 1.1.** (i) $\{Y_k\}_{k \in \mathbb{N}}$ is called $\Phi$-orientable if $Sq^2U = 0$, $\chi(Sq^{n+2})(U) = 0$ and $0 \in \Phi(U)$, where $\Phi$ is a secondary cohomology operator associated with the Adem relation (see Section 3 for the definition):

$$\chi(Sq^n)Sq^3 + \chi(Sq^{n+2})Sq^1 + Sq^1\chi(Sq^{n+2}) = 0 \quad n = 2(mod 4)$$

$$\chi(Sq^n)Sq^3 + Sq^1\chi(Sq^{n+2}) = 0 \quad n = 0(mod 4)$$

$$\chi(Sq^{n+1})Sq^2 + Sq^1\chi(Sq^{n+2}) = 0 \quad n = 1(mod 4)$$

where $\chi : \mathcal{A}_2 \rightarrow \mathcal{A}_2$ is the anti-automorphism of the Steenrod algebra $\mathcal{A}_2$.

A spherical fibration $\xi$ (a manifold) is called $\Phi$-orientable if its Thom spectrum $T\xi$ (stable normal bundle $\nu_M$) is. We define the universal $\Phi$-orientable $\Omega$-spectrum $\tilde{W}(n)$ by setting $\tilde{W}_k(n)$ to be the total space of the following Postnikov tower:
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\[
\begin{array}{ccc}
\tilde{W}_k(n) & \downarrow \Pi_2 & K_{k+n+2} \\
W_k(n) & \xrightarrow{k_2} & K_{k+n+2} \\
\downarrow \Pi_1 & & \\
K(Z, k) & \overset{Sq^2 \times \chi(Sq^{n+2})}{\longrightarrow} & K_{k+2} \times K_{k+n+2}
\end{array}
\]

where \( K_i = K(Z_2, i) \), \( K(Z, i) \) are the Eilenberg-Maclane spaces, \( k_2 \in \Phi(\Pi_1) \) and \( l_k \) is the basic class.

Note that a spectrum \( Y \) is \( \Phi \)-orientable if and only if \( U \in H^0(Y) \) can be lifted to a map \( w : Y \to \tilde{W}(n) \). We call such a lifting a \( \Phi \)-orientation of \( Y \). A \( \Phi \)-orientation of a manifold is understood as a \( \Phi \)-orientation of its Thom spectrum.

**Remark 1.2.** The sphere spectrum \( S^0 \) is \( \Phi \)-orientable. Thus stably parallelizable manifolds are \( \Phi \)-orientable.

Our main results are:

**Theorem 1.3.** Let \( M \) be a \( \Phi \)-oriented manifold of dimension \( 2n \), where \( n \neq 3(\text{mod} \ 4) \). Then there is a function \( \phi_M : H^{n-1}(M, Z_4) \to \mathbb{Q}/\mathbb{Z} \) such that, for all \( x, y \in H^{n-1}(M, Z_4) \),

\[
\phi_M(x + y) = \phi_M(x) + \phi_M(y) + j(x \cup Sq^2y)[M],
\]

where \( j : Z_2 \to \mathbb{Q}/\mathbb{Z} \) is the inclusion.

**Remark 1.4.** In general, \( \phi_M \) depends on the \( \Phi \)-orientation, just like the Kervaire quadratic form depends on the framing of the manifold. We will prove that \( \phi_M(x) \) depends only on the \( \Phi \)-oriented bordism class \( [M, x] \).

**Remark 1.5.** If \( n = 3(\text{mod} \ 4) \), the analogous definition gives only a linear function.

Let \( BSpin_G \) be the classifying space for spherical Spin fibrations. By Brown[4], a Wu orientation of a Spin spherical fibration \( \xi \searrow M \) is a lifting of the classifying map \( \xi : M \to BSpin_G \) to \( BSpin_G(v_{n+2}) \). A Wu orientation of \( \nu_M \), the stable normal bundle of \( M \), is understood as a Wu orientation of \( M \), where \( BSpin_G(v_{n+2}) \to BSpin_G \) is a principal fibration with \( v_{n+2} \in H^{n+2}(BSpin_G, Z_2) \) as the \( k \)-invariant.

We call quadratic forms \( \phi_{M_i} : H^{n-1}(M_i, Z_4) \to \mathbb{Q}/\mathbb{Z}, i = 1, 2 \) Witt equivalent if there exists an isomorphism \( \tau : H^{n-1}(M_1, Z_4) \to H^{n-1}(M_2, Z_4) \) so that \( \phi_{M_2}(\tau x) = \phi_{M_1}(x) \) for all \( x \in H^{n-1}(M_1, Z_4) \).
Theorem 1.6. Let $M_1$ and $M_2$ be $\Phi$-oriented $2n$-manifolds. Suppose that the Wu classes $v_{n+2-2^j}(v_{M_1}) = 0$ for all $2^j \leq n + 2$. If $f : M_1 \to M_2$ is a homotopy equivalence preserving the spin structure (resp. Wu orientation) if $n = 0, 1(\text{mod} 4)$ (resp. $n = 2(\text{mod} 4)$). Then

$$\phi_{M_1}(f^*x) = \phi_{M_2}(x)$$

for all $x \in H^{n-1}(M_2, \mathbb{Z}_4)$.

Since the Wu class $v_0 = 1$, the assumption in the above theorem implies that $n + 2 \neq 2^i$ for any integer $i$.

For framed manifolds, the Brown-Kervaire secondary quadratic forms have the following property:

Proposition 1.7. If $M$ is a framed manifold of dimension $2n$, where $n \neq 3(\text{mod} 4)$. Then $\phi_M$ factors through $\mathbb{Z}_4 \subset \mathbb{Q}/\mathbb{Z}$ (resp. $\mathbb{Z}_2 \subset \mathbb{Q}/\mathbb{Z}$), provided $n = 2(\text{mod} 4)$ (resp. $n = 0, 1(\text{mod} 4)$).

To state the next results, we need some preliminaries. Let $H$ be a finitely generated abelian group, and

$$\mu : \text{Hom}(H, \mathbb{Z}_2) \otimes \text{Hom}(H, \mathbb{Z}_2) \to \mathbb{Z}_2$$

be a symmetric bilinear form. We say that $\mu$ is of diagonal zero if $\mu(x, x) = 0$ for each $x \in \text{Hom}(H, \mathbb{Z}_2)$. A function $\phi : \text{Hom}(H, \mathbb{Z}_4) \to \mathbb{Q}/\mathbb{Z}$ is called \underline{quadratic} with respect to $\mu$ if

$$\phi(x + y) = \phi(x) + \phi(y) + j(\mu(x, y))$$

where $j : \mathbb{Z}_2 \to \mathbb{Q}/\mathbb{Z}$ is the inclusion. This gives a triple $(H, \mu, \phi)$. We say triples $(H_1, \mu_1, \phi_1)$, and $(H_2, \mu_2, \phi_2)$ are isometric if there exists an isomorphism $\tau : H_1 \to H_2$ such that $\mu_1(x, y) = \mu_2(\tau x, \tau y)$ and $\phi_1(x) = \phi_2(\tau x)$ for all $x, y$. We denote by $[H, \mu, \phi]$ the isometry class of a triple.

Remark 1.8. Since the natural map $\text{Hom}(H_{n-1}(M), \mathbb{Z}_2) \to H^{n-1}(M, \mathbb{Z}_2)$ is not an isomorphism in general, the notions of isometry associated with $\mu$ and $\mu_M$ as above are different. They do agree however for $(n-2)$-connected manifolds which we will assume in the later application. We will use both of them when necessary.

Let $i$ denote the maximal exponent of the 2-torsion subgroup of $H_{n-1}(M)$ and let $Sq^1_i \in H^n(K(\mathbb{Z}_2, n-1), \mathbb{Z}_2) \cong \mathbb{Z}_2$ be the unique generator. Considering $Sq^1_i$ as a cohomology operation we get a function

$$q_M(Sq^1_i) : H^{n-1}(M, \mathbb{Z}_2) \to \mathbb{Z}_2.$$

This gives a homomorphism since $Sq^1_i x \cup Sq^1_i y = Sq^1_i(x \cup Sq^1_i y) = 0$ for $x, y \in H^{n-1}(M, \mathbb{Z}_2)$. We denote by $[H_{n-1}(M), \mu_M, q_M(Sq^1_i)]$ for the
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isometry class of the triple. By [3], the Kervaire invariant of a smooth framed manifold of dimension $2n$, where $n \neq 2^i - 1$, is zero. For $i \leq 5$, there are smooth manifolds of dimension $2^{i+1} - 2$ of Kervaire invariant 1. It is still an open problem whether there is such a manifold for $i \geq 6$.

The Kervaire invariant does not depend on the framings of the underlying $2n$-manifold if $n \neq 1, 3, 7$ and the manifold is highly connected, e.g. $(n - 2)$-connected. Moreover, by [4] the Kervaire form is a homotopy invariant if $n \neq 1, 3, 7$ and $(n - 2)$-connected.

Let $M$ be a $(n - 2)$-connected $2n$-dimensional $\pi$-manifold. Observe that if $n \geq 3$, there exists a $(n - 2)$-connected $\pi$-manifold, $N$, so that $M = N \# X$ and $H_n(N, \mathbb{Q}) = 0$, where $X$ is a $(n - 1)$-connected $2n$-manifold. Since the classification of $(n - 1)$-connected $2n$-manifolds is well understood [25], for convenience in the following theorem we assume that $H_n(M, \mathbb{Q}) = 0$. For such a manifold, consider the correspondence

$$
\pi : M \mapsto [H_{n-1}(M), \mu_M, \phi_M](\text{ resp. } [H_{n-1}(M), \mu_M, \phi_M, q_M(Sq^1_i)])
$$

if $n = 0(\text{mod} 2)$ (resp. $n = 1(\text{mod} 2)$).

In the following theorem let $\alpha(n + 2)$ be the number of 1’s in the binary expansion of $n + 2$.

**Theorem 1.9.** Suppose $n \geq 4$ and $\alpha(n + 2) \geq 2$. Then $\pi$ gives a 1-1 correspondence between the homeomorphism types (resp. homotopy types) of $(n - 2)$-connected $2n$-dimensional $\pi$-manifolds $M$ so that $H_n(M, \mathbb{Q}) = 0$ with the following algebraic data

(a) $\wp_n = \{[H, \mu, \phi] : \text{ diag } \mu = 0 \text{ and } \phi \text{ factors through } j : \mathbb{Z}_2 \to \mathbb{Q}/\mathbb{Z}\}$ if $n = 2(\text{mod} 4)$,

(b) $\wp_n = \{[H, \mu, \phi] : \phi \text{ factors through } j : \mathbb{Z}_2 \to \mathbb{Q}/\mathbb{Z}\}$ if $n = 0(\text{mod} 4)$,

(c) $\wp_n = \{[H, \mu, \phi, \omega] : \omega \in \text{Hom}(\text{tor}H) \otimes \mathbb{Z}_2, \mathbb{Z}_2) \text{, } \phi \text{ factors through } j : \mathbb{Z}_2 \to \mathbb{Q}/\mathbb{Z}\}$ if $n = 1(\text{mod} 4)$,

(d) $\wp_n = \{[H, \mu, \omega] : \omega \in \text{Hom}(\text{tor}H) \otimes \mathbb{Z}_2, \mathbb{Z}_2)\}$ if $n = 3(\text{mod} 4)$.

where $i$ is the highest exponent of the 2-cyclic subgroup of $H$ and if $n = 1(\text{mod} 2)$, the pairing $\mu(x, x) = 0$ (resp. $\delta \omega(x)$) if $x$ can be lifted to a $\mathbb{Z}_4$ class with order 4 (resp. $x$ is of order 2), $\delta \in \{0, 1\}$ is ambiguous.

**Remark 1.10.** The classification of $(n-2)$-connected $2n$-manifolds with torsion free homology groups has been given by Ishimoto[9][10]. But his method does not work if the homology group has torsion.

The organization of this paper is as follows.

In §3 we define the secondary Brown-Kervaire form and state its basic properties.

In §4 we set up the necessary foundations on the stable homotopy theory of the Eilenberg-MacLane spaces.
In §4, we are addressed to show Theorems 1.3 and 1.6.
In §5, we prove Theorem 1.9.

2. A $\mathbb{Q}/\mathbb{Z}$-quadratic form of $\Phi$-oriented manifolds

Let us begin with some conventions. All homology/cohomology groups will be with integral coefficients unless otherwise stated. All spaces will have base points. Let
(i) $[X,Y]$ denote the set of homotopy classes of pointed maps from $X$ to $Y$.
(ii) $\{X,Y\} = \lim [S^kX, S^kY]$.
(iii) $\pi^*_s(X)$ be its 2-localization to simplify the notation.

Let $\kappa: K(\mathbb{Z}_4, n - 1) \times K(\mathbb{Z}_4, n - 1) \to K(\mathbb{Z}_4, n - 1)$ be the multiplication of $K(\mathbb{Z}_4, n - 1)$ and let $H(\kappa)$ be the Hopf construction of $\kappa$.

**Proposition 2.1.** The homomorphism

$$H(\kappa)_*: \pi^s_{2n}(K(\mathbb{Z}_4, n - 1) \wedge K(\mathbb{Z}_4, n - 1)) \to \pi^s_{2n}(K(\mathbb{Z}_4, n - 1))$$

is injective if $n \neq 3(\text{mod}4)$, and zero if $n = 3(\text{mod}4)$.

**Remark 2.2.** If $\mathbb{Z}_4$ is replaced by $\mathbb{Z}_2$, then $H(\kappa)_*$ is trivial.

By Theorem 3.1 and the proof of it, we obtain

$$\pi^s_{2n}(K(\mathbb{Z}_4, n - 1) \wedge K(\mathbb{Z}_4, n - 1)) \cong \mathbb{Z}_2 \text{ if } n \geq 4,$$

$$\pi^s_{2n}(K(\mathbb{Z}_4, n - 1)) \cong \mathbb{Z}_4 \text{ if } n = 2(\text{mod}4).$$

Let $\lambda_0$ be a generator of $\text{Im } H(\kappa)_*$ if $n \neq 2(\text{mod}4)$, and a specified generator of $\pi^s_{2n}(K(\mathbb{Z}_4, n - 1)) \cong \mathbb{Z}_4$ otherwise. For a given spectrum $Y$, let

$$H_*(K(\mathbb{Z}_4, n - 1); Y) = \lim \pi^*_{s+k}(K(\mathbb{Z}_4, n - 1) \wedge Y_k).$$

**Theorem 2.3.** Suppose that $\{Y_k\}_{k \in \mathbb{N}}$ is a $\Phi$-orientable spectrum. Then there exists a homomorphism

$$h : H_{2n}(K(\mathbb{Z}_4, n - 1); Y) \to \mathbb{Q}/\mathbb{Z}$$

such that $h(\lambda) = \frac{1}{4}$ (resp. $\frac{1}{2}$) if $n = 2(\text{mod}4)$ (resp. $n = 0, 1(\text{mod}4)$), where $\lambda = i_*(\lambda_0)$ and $i_* : H_{2n}(K(\mathbb{Z}_4, n - 1); S^0) \to H_{2n}(K(\mathbb{Z}_4, n - 1); Y)$ is induced by the inclusion.
Definition 2.4. A Poincaré triple $(M, \xi, \alpha)$ of dimension $2n$ consists of
(i) A CW complex $M$ with finitely generated homology.
(ii) A fibration $\xi$ over $M$ with fiber homotopy equivalent to $S^{k-1}$, $k$ large.
(iii) $\alpha \in \pi_{2n+k}(T\xi)$ such that an $(2n+k)$ Spanier-Whitehead S-duality is given by
\[ S^{2n+k} \xrightarrow{\alpha} T\xi \xrightarrow{\Delta} T\xi \wedge M^+ \]
where $T\xi$ is the Thom complex of $\xi$ and $\Delta$ is the diagonal map.

Let $A_\alpha : \{M_+, K(\mathbb{Z}_4, n-1)\} \to \{S^{2n+k}, T\xi \wedge K(\mathbb{Z}_4, n-1)\}$ be the $S$-duality map.

Definition 2.5. Let $(M, \xi, \alpha)$ be a Poincaré triple and $w$ is a $\Phi$-orientation of the Thom spectrum $T\xi$. For a homomorphism $h$ in Theorem 2.3, let
\[ \phi_{w,h} : H^{n-1}(M, \mathbb{Z}_4) \to \mathbb{Q}/\mathbb{Z} \]
be defined by setting
\[ \phi_{w,h}(x) = h([(w \wedge id)A_\alpha(x)]). \]

Theorem 2.6. Let $\phi_{w,h}$ be defined as above. Then for all $x, y \in H^{n-1}(M, \mathbb{Z}_4)$,
(i) If $n \neq 3(\text{mod } 4)$, the function is quadratic, i.e.
\[ \phi_{w,h}(x + y) = \phi_{w,h}(x) + \phi_{w,h}(y) + j(x \cup Sq^2 y)[M] \]
where $j : \mathbb{Z}_2 \to \mathbb{Q}/\mathbb{Z}$ is the inclusion;
(ii) If $n = 3(\text{mod } 4)$, $\phi_{w,h}$ is linear, i.e.
\[ \phi_{w,h}(x + y) = \phi_{w,h}(x) + \phi_{w,h}(y). \]

Now we want to study how the function $\phi_{w,h}$ depends on the choice of the orientation of the Thom spectrum $T\xi$.

Let $w_i, i = 1, 2$ are orientations of the Thom spectrum $T\xi$. Let
\[ d_1(w_1, w_2) \in H^1(T\xi) \oplus H^{n+1}(T\xi) \]
denote the difference of the composition maps $\Pi_2w_1$ and $\Pi_2w_2$, where $\Pi_2$ is as in the definition of the universal $\Omega$-spectrum $\widetilde{W}(n)$. Clearly, $w_1$ and $w_2$ are homotopy if and only if $d_1(w_1, w_2) = 0$ and a secondary obstruction vanishes. The following theorem shows that the secondary obstruction does not affect our quadratic function $\phi_{w,h}$.

Theorem 2.7. Let $\phi_{w_i,h}$ be the quadratic forms associated with $(w_i, h)$, $i = 1, 2$. If $d_1(w_1, w_2) = 0$, then $\phi_{w_1,h}(x) = \phi_{w_2,h}(x)$ for all $x \in H^{n-1}(M, \mathbb{Z}_4)$. 

In general, the quadratic form $\phi_{w,h}$ does depend on the choice of $w$ and $h$. In order to obtain a well-defined invariant of the $\Phi$-oriented manifold, we now choose certain type of $\Phi$-orientations of the Thom spectrum $T\xi$ in an universal way and then define the Brown-Kervaire secondary quadratic forms to be the quadratic functions associated to those $\Phi$-orientations.

Let $\gamma \rightarrow BSpin_G$ be the universal Spin spherical fibration and $U \in H^0(MSpin_G, \mathbb{Z}_2)$ the universal Thom class. Note that

\[
\chi(Sq^{n+2})U = \chi(Sq^{n+1})Sq^1U = 0 \text{ if } n \text{ is odd},
\]

\[
\chi(Sq^{n+2})U = \chi(Sq^n)Sq^2U = 0 \text{ if } n = 0(\text{mod}4),
\]

Thus $U$ lifts to a map $f : MSpin_G \rightarrow W(n)$. By the Thom isomorphism, $f^*k_2$ gives an element of $\tilde{k}_2 \in H^{n+2}(BSpin_G, \mathbb{Z}_2)$. Let $\pi : BSpin_G(\tilde{k}_2) \rightarrow BSpin_G$ be the principal fibration with $k$-invariant $\tilde{k}_2$.

If $n = 2(\text{mod}4)$, we get a similar principal fibration $\pi : BSpin_G(\tilde{k}_2) \rightarrow BSpin_G(v_{n+2})$, where $BSpin_G(v_{n+2}) \rightarrow BSpin_G$ is the fibration with fibre $K_{n+1}$ and $k$-invariant $v_{n+2}$.

It is easy to see that the fibration $\pi^*\gamma$ is $\Phi$-orientable. Clearly the classifying map of every $\Phi$-orientable stable spherical fibration lifts to $BSpin_G(\tilde{k}_2)$.

**Definition 2.8.** The fibration $\pi^*\gamma$ is called the universal $\Phi$-orientable spherical Spin fibration. Its Thom spectrum, $MSpin_G(\tilde{k}_2)$, is called the universal $\Phi$-orientable Thom spectrum.

For a closed $\Phi$-orientable manifold $M^{2n}$, there is a Poincaré triple $(M, \nu_M, \alpha)$ where $\nu_M$ is the stable normal bundle and $\alpha \in \pi_{2n+k}(T\nu_M)$ is the normal invariant of $M$ (obtained by the Thom-Pontryagin construction.)

**Definition 2.9.** Fix a connected spectral map $u : MSpin_G(\tilde{k}_2) \rightarrow \tilde{W}(n)$ and a homomorphism $h$ in Theorem 2.3. For a $\Phi$-orientable manifold $M$, let

$$\phi_M = \phi_{w,h}$$

where $w = u \circ T(v)$ and $T(v)$ is the Thom map of a classifying bundle map of the stable bundle $\nu_M$.

Now we prove Theorem 1.6 assuming Theorem 2.7.

**Proof of Theorem 1.6.** Let $\xi_i = \nu_{M_i}$ be the stable normal bundle of $M_i$ and $\alpha_i \in \pi_{2n+k}(T\xi_i)$ be the normal invariant, $i = 1, 2$. By the definition, $\phi_{M_i} = \phi_{w_i,h}$ where $w_i = u \circ T(v_i)$ and $T(v_i) : T(\xi_i) \rightarrow MSpin_G(\tilde{k}_2)$ the Thom map.
Let $\tilde{f} : f^*\xi_2 \to \xi_2$ be a bundle map over the homotopy equivalence $f$. Let $\alpha_3 = T(\tilde{f})^{-1} \alpha_2$, where $T(\tilde{f})$ is the Thom map of $\tilde{f}$. The Poincaré triple $(M_1, f^*\xi_2, \alpha_3)$ together with the $\Phi$-orientation $w_2 \circ T(\tilde{f})$ gives a quadratic form $\phi_3$, where $w_2 = u \circ T(v_2)$ is a $\Phi$-orientation of $M_2$. By 2.5 we get that

$$\phi_3(f^* x) = \phi_{M_2}(x)$$

for all $x \in H^{n-1}(M_2, \mathbb{Z})$.

To prove the desired result, it suffices to prove $\phi_3 = \phi_{M_1}$.

Note that $f^*\xi_2$ and $\xi_1$ are stably equivalent as spherical fibration since $f$ is a homotopy equivalence. Thus we can regard $f^*\xi_2$ and $\xi_1$ as the the same and so get two orientations for $\xi_1$, $(u \circ T(v_1), h)$ and $(u \circ T(v_2) \circ T(\tilde{f}), h)$. Since $f$ preserves the Spin structures/Wu orientations, $\pi \circ v_2 \circ f \simeq \pi \circ v_1$, where $\pi : \text{BSpin}_G(k_2) \to \text{BSpin}_G/\text{BSpin}_G(v_{n+2})$ is the principal fibration as above. This clearly implies that there exists a fibre automorphism $g \in \text{Aut}(\xi_1)$ over the identity such that

$$T(\pi \circ v_2 \circ \tilde{f}) \simeq T(\pi \circ v_1) \circ T(g).$$

Notice that $g$ gives a unique element $g_0 \in [M_1, G_k]$, where $G_k$ is the space of self homotopy equivalences of $S^k$. By a formula in Brown \cite{brown}, the $(n + 1)$-dimensional component of $d_1(u \circ T(v_1) \circ T(g), u \circ T(v_1))$ is $\sum v_{n+2i} \cup g_0^* u_{2i-1}$, where $u_{2i-1}$ is the transgression of $w_{2i}$ in $H^{2i}(BG_k, \mathbb{Z})$. By assumption, it must vanish since the Wu classes vanish. On the other hand, the 1-dimensional component of $d_1(u \circ T(v_1) \circ T(g), u \circ T(v_1))$ is determined by the Spin structures and so it vanishes since $f$ preserves the Spin structures. By Theorem 2.7 it follows that

$$\phi_{M_1} = \phi_4,$$

the quadratic form associated with the Poincaré triple $(M_1, \xi_1, \alpha_1)$ and the $\Phi$-orientation $w_2 \circ T(\tilde{f})$.

Note that in the definitions of $\phi_3$ and $\phi_4$ the only different ingredients are the normal invariants, after identifying $\xi_1$ with $f^*\xi_2$. By Theorem 2.7 once again $\phi_3 = \phi_4$. This implies the desired result. \qed

Now we prove Proposition 1.7.

Proof of Proposition 1.7. Since $M$ is a framed manifold, the stable normal bundle is trivial, i.e. the classifying map of $\nu_M$ factors through a point. Choose a $\Phi$-orientation $w = u \circ T(v) : \nu_M$ with $v$ the bundle map of $\nu_M$ to the trivial $k$-bundle on a point, then $\phi_M(x)$ factors through the stable homotopy group $\pi_{2n}^s(K(\mathbb{Z}_2, n-1))$. By Theorem 3.1 $\pi_{2n}^s(K(\mathbb{Z}_2, n-1)) \cong \mathbb{Z}_4$ if $n = 2(mod 4)$ and the order of elements in $\pi_{2n}^s(K(\mathbb{Z}_4, n-1))$ is at most 2 if $n = 0, 1(mod 4)$. On the other hand, by Theorem 1.6 the definition of $\phi_M$ does not depend on the choice of
the Φ-orientations since $M$ is a framed manifold. This completes the proof.

3. SOME PRELIMINARIES ON STABLE HOMOTOPY THEORY

In this section we calculate the stable homotopy groups $\pi_{2n}^s(K(\pi, n - 1))$ (see Theorem 3.1). We will also introduce some 2-stage Postnikov tower which will give the secondary cohomology operation $\Phi$ used in Section §1.

**Theorem 3.1.** The 2n-th stable homotopy group of $K(\pi, n - 1)$ for $n \geq 4$ is as follows:

\[
\begin{array}{|c|c|c|c|c|}
\hline
n \geq 4 & 0(\text{mod}4) & 1(\text{mod}4) & 2(\text{mod}4) & 3(\text{mod}4) \\
\pi_{2n}^s(K(\pi, n - 1)) & (\mathbb{Z}_2)^{2(t+k)+s+p} & (\mathbb{Z}_2)^{t+2k+s+p} & (\mathbb{Z}_4)^{t+k+\oplus(\mathbb{Z}_2)^s+p} & (\mathbb{Z}_2)^{k+s+p} \\
\hline
\end{array}
\]

where $p = \frac{t+k+s}{2}$ and $\pi = G_0 \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \times \mathbb{Z}_s$, $ij \geq 2$ if $1 \leq j \leq k$ and $G_0 \otimes \mathbb{Z} = 0$.

When $\pi = \mathbb{Z}$, Theorem 3.1 follows from [17].

**Proof.** It is easy to know that (since we are computing the 2-localization)

$\pi_{2n}^s(K(\pi, n - 1)) = \pi_{2n}^s(K(\pi/G_0, n - 1))$

Assume $G_0 = 0$ from now on. If $\pi = \pi_1 \oplus \pi_2$ with $\pi_1$ nontrivial and $\pi_2$ a nontrivial cyclic group, then

$K(\pi, n - 1) = K(\pi_1, n - 1) \times K(\pi_2, n - 1)$

and we have by a result in [3] that

$\pi_{2n}^s(K(\pi, n - 1)) = \bigoplus_{i=1,2} \pi_{2n}^s(K(\pi_i, n - 1)) \bigoplus \pi_{2n}^s(K(\pi_1, n - 1) \wedge K(\pi_2, n - 1))$

An easy calculation shows that $H_{n+1}(K(G_1, n - 1), G_2) = \mathbb{Z}_2$ if $G_1, G_2$ are nontrivial cyclic groups and thus $H_{n+1}(K(\pi_1, n - 1), \pi_2) = \mathbb{Z}_2^{t+k+s-1}$.

On the other hand we know groups $\pi_{2n}^s(K(\mathbb{Z}, n - 1))$ and $\pi_{2n}^s(K(\mathbb{Z}_2, n - 1))$ by results in [20, 17]. To complete the proof it remains to calculate $\pi_{2n}^s(K(\mathbb{Z}_2, n - 1))$ for $2 \leq i < \infty$ which will be given in the following results.

Recollect that for each locally finite connected CW complex $X$ we can define a space

$\Gamma_q X = S^{q-1} \times^T X \wedge X = S^{q-1} \wedge (X \wedge X) / \{(x, y, z) \sim (-x, z, y); (x, *) \sim *\}$
for every $q \in \mathbb{Z}_+$. By [20] Theorem 1.11, for a $(n-2)$-connected space $X$, $\Gamma_qX$ is $(2n-3)$-connected. Moreover, if $X = K(\pi, n - 1)$, we have a fibration
\[ G_q \to \Sigma^q K(\pi, n - 1) \to K(\pi, q + n - 1) \]
where $G_q \simeq \Sigma^q \Gamma_q K(\pi, n - 1)$ through dimension $(3n + q - 3)$. Thus $\pi_i^*(K(\pi, n - 1)) \cong \pi_i^*(\Gamma_q K(\pi, n - 1))$ for $n < i < 3n - 3$.

When $q = 1$, $\Gamma_qX = X \wedge X$. The corresponding sequence is:
\[ \Sigma F_{n-1}(\pi) \xrightarrow{H(\psi)} \Sigma K(\pi, n - 1) \to K(\pi, n) \]
where $F_{n-1}(\pi) = K(\pi, n - 1) \wedge K(\pi, n - 1)$. After $q - 1$ time suspensions we get a fibration sequence at least in dimensions less than $3n + q - 4$
\[ \Sigma^q F_{n-1}(\pi) \xrightarrow{\Sigma^q H(\psi)} \Sigma^q K(\pi, n - 1) \to \Sigma^q K(\pi, n) \]
Let $q$ be large enough so that we are always in the stable range and let $r = q + 2n$, then we have an exact sequence
\[ \cdots \to \pi^a_{2n+2}(K(\pi, n)) \xrightarrow{\partial} \pi^a_r(\Sigma^q F_{n-1}(\pi)) \to \]
\[ \to \pi^a_{2n}(K(\pi, n - 1)) \to \pi^a_{2n+1}(K(\pi, n)) \xrightarrow{\partial} \cdots \]
Since we know that $\pi_r(\Sigma^q F_{n-1}(\pi)) = \mathbb{Z}_2$ for $\pi = \mathbb{Z}$ and $\mathbb{Z}_2$, we can determine inductively $\pi^a_{2n}(K(\pi, n - 1))$ up to extension if we know the map $\partial$.

**Lemma 3.2.** For $\pi = \mathbb{Z}$ or $\mathbb{Z}_2$, a homotopy class $[g] \in \pi^a_{2n+2}(K(\pi, n))$ has $\partial g \neq 0$ if and only if $g^*(\Sigma^q H(\psi)) \neq 0$ where $g : S^{r+1} \to \Sigma^q K(\pi, n), \nu \in H^n(K(\pi, n), \mathbb{Z}_2)$ is the generator and $g^* : H^*(\Sigma^q K(\pi, n), \mathbb{Z}_2) \to H^*(S^{r+1}, \mathbb{Z}_2)$.

The proof is similar to that of Lemma 1.3 in [17]. The key points are the followings:
- $g^*$ can be nonzero only on element $\Sigma^q H(\psi)$
- the Hurewicz homomorphism $H : \mathbb{Z}_2 \cong \pi_r(\Sigma^q F_{n-1}(\pi)) \to H_r(\Sigma^q F_{n-1}(\pi))$
  is nonzero

The first statement is clear while the second is an easy consequence of the Whitehead exact sequence (c.f. [20], page 555).

With the lemma above we can now prove the Proposition 2.1.

**Proof of Proposition 2.1.** It suffices to prove that $\partial$ is trivial if $n = 0, 1, 2(\text{mod} 4)$ and nontrivial if $n = 3(\text{mod} 4)$.

For $n = 0, 1, 2(\text{mod} 4)$ there is no $g : S^{r+1} \to \Sigma^q K(\mathbb{Z}, n)$ such that $g^*(\Sigma^q H(\psi)) \neq 0$ since $\Sigma^q H(\psi)$ is detected by the secondary cohomology operation $\varphi_n$ in [18]. Thus there is no $g : S^{r+1} \to \Sigma^q K(\mathbb{Z}_2, n)$ such that $g^*(\Sigma^q H(\psi)) \neq 0$ by the naturality of
secondary cohomology operation and the fact that \( \rho_{2^i} : K(\mathbb{Z}, n) \to K(\mathbb{Z}_{2^i}, n) \) corresponding to mod \( 2^i \) reduction induces a homomorphism sending \( \Sigma^{q-1}(\iota \cup Sq^2 \iota) \) to the corresponding element. It follows that \( \partial = 0 \)

When \( n = 3(\text{mod} 4) \), there is a map \( g : S^{r+1} \to \Sigma^{q-1}K(\mathbb{Z}, n) \) such that \( g^*(\Sigma^{q-1}(\iota \cup Sq^2 \iota)) \neq 0 \) since otherwise \( \pi_{2n}^s(K(\mathbb{Z}, n-1)) \neq 0 \).

By the above fact on map \( \rho_{2^i} \), it is easy to see that there is a map \( h : S^{r+1} \to \Sigma^{q-1}K(\mathbb{Z}_{2^i}, n) \) such that \( h^*(\Sigma^{q-1}(\iota \cup Sq^2 \iota)) \neq 0 \). It follows from the lemma above that \( \partial h \neq 0 \).

With the help of Proposition 2.1 and the known results about \( \pi_{2n+j}^s(K(\pi, n)) \) for \( j = 0, 1 \), we can now determine the group \( \pi_{2n}^s(K(\mathbb{Z}_{2^i}, n-1)) \).

Assume \( i \geq 2 \) in the following unless otherwise stated.

**Proposition 3.3.** If \( n = 0(\text{mod} 2) \), then

\[
\rho_{2^i} : \pi_{2n}^s(K(\mathbb{Z}, n-1)) \to \pi_{2n}^s(K(\mathbb{Z}_{2^i}, n-1))
\]

is an isomorphism.

Before the proof of the Proposition, let’s give two remarks which are clear from the proof of the Proposition.

**Remark 3.4.** If \( i = 1 \), \( \rho_{2^i} \) is onto.

**Remark 3.5.** If \( n = 0(\text{mod} 4) \), then the spherical cohomology class in \( \pi_{2n}^s(K(\mathbb{Z}, n-1)) \) does not belong to the image of the natural map:

\[
\pi_r(Sq^{n-1}_F \mathbb{Z}) \to \pi_{2n}^s(K(\mathbb{Z}, n-1)).
\]

**Proof.** Note that we have a commutative diagram

\[
\begin{array}{cccc}
\pi_r(Sq^{n-1}_F \mathbb{Z}) & \longrightarrow & \pi_{2n}^s(K(\mathbb{Z}, n-1)) & \longrightarrow & \pi_{2n+1}^s(K(\mathbb{Z}, n)) \\
\rho_{2^i} \downarrow & & \rho_{2^i} \downarrow & & \rho_{2^i} \downarrow \\
\pi_r(Sq^{n-1}_F \mathbb{Z}_{2^i}) & \longrightarrow & \pi_{2n}^s(K(\mathbb{Z}_{2^i}, n-1)) & \longrightarrow & \pi_{2n+1}^s(K(\mathbb{Z}_{2^i}, n))
\end{array}
\]

In the above diagram, the two left horizontal maps are injective by Lemma 3.2, the left vertical map is obviously an isomorphism while the fact that the right vertical one is also an isomorphism follows by comparing the Whitehead exact sequences of \( \Gamma_{q-1}(K(\mathbb{Z}, n)) \) and \( \Gamma_{q-1}(K(\mathbb{Z}_{2^i}, n)) \). On the other hand, the fact that the right horizontal map on the bottom line is onto follows from the long exact sequence and the known results about \( \pi_{2n+j}^s(K(\mathbb{Z}_{2^i}, n)) \) for \( j = 0, 1 \).

**Proposition 3.6.** For \( n = 1(\text{mod} 2) \), \( \pi_{2n}^s(K(\mathbb{Z}_{2^i}, n-1)) = \pi_{2n}^s(K(\mathbb{Z}, n-1)) \oplus \mathbb{Z}_{2^i} \).
Proof. The relevant commutative diagram in this case is
\[ \begin{array}{ccc}
\pi_r(\Sigma^q F_{n-1}(\mathbb{Z})) & \longrightarrow & \pi^s_{2n}(K(\mathbb{Z}, n-1)) \\
\rho_{2i*} & & \rho_{2i*} \\
\pi_r(\Sigma^q F_{n-1}(\mathbb{Z}_2)) & \longrightarrow & \pi^s_{2n}(K(\mathbb{Z}_2, n-1)) \longrightarrow \pi^s_{2n+1}(K(\mathbb{Z}_2, n)) \longrightarrow \partial,
\end{array} \]

By the same argument as in the last Proposition, we know the map \( \partial \) is onto. If \( n = 3 \mod 4 \), the two left horizontal maps are trivial by Lemma 3.2, thus \( \pi^s_{2n}(K(\mathbb{Z}_2, n-1)) \cong \text{coker} \partial \cong \mathbb{Z}_2 \).

If \( n = 1 \mod 4 \), what we can get is an exact sequence
\[ 0 \rightarrow \pi^s_{2n}(K! \mathbb{Z}, n-1)) \rightarrow \pi^s_{2n}(K(\mathbb{Z}_2, n-1)) \rightarrow \mathbb{Z}_2 \rightarrow 0. \]

To complete the proof, it suffices to prove the last map in the above sequence has a section.

To do this we need another diagram
\[ \begin{array}{ccc}
\pi^s_{2n}(K_{n-1}) & \longrightarrow & \pi^s_{2n+1}(K_n) \\
\rho_1 & & \rho_2 \\
\pi^s_{2n}(K(\mathbb{Z}_2, n-1)) & \longrightarrow & \pi^s_{2n+1}(K(\mathbb{Z}_2, n)) \longrightarrow \pi^s_{2n+2}(K(\mathbb{Z}_2, n)) \longrightarrow \partial,
\end{array} \]

where \( j : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \) is the natural inclusion.

The same argument as above combined with the proof of Theorem 10.9 in [20] shows that the two \( \partial \)'s are onto and \( j_* \) induces an isomorphism between kernels of two \( \partial \)'s. Finally we get the following diagram which gives the desired section.

\[ \begin{array}{ccc}
\pi^s_{2n}(K_{n-1}) & \longrightarrow & \mathbb{Z}_2 \\
\rho_1 & & \rho_2 \\
\pi^s_{2n}(K(\mathbb{Z}_2, n-1)) & \longrightarrow & \mathbb{Z}_2
\end{array} \]

Lemma 3.7. If \( n \) is odd and \( Sq^1 \in H^n(K(\mathbb{Z}_2, n-1), \mathbb{Z}_2) \) is a generator. Then
(\( Sq^1 \)) : \( \pi^s_{2n}(K(\mathbb{Z}_2, n-1)) \rightarrow \pi^s_{2n}(K(\mathbb{Z}_2, n)) \cong \mathbb{Z}_2 \)

is an epimorphism.

Proof. It suffices to prove that the following map \( Sq^1 : K(\mathbb{Z}_2, n-1) \rightarrow K(\mathbb{Z}_2, n) \) induces an isomorphism on \( 2n \)-th stable homotopy group. By the calculation in Milgram’s book [20], the first group is generated by the class corresponding to \( Sq^1(t) \otimes Sq^1(t) \) and the second by \( s \otimes s \) where \( s, t \) are the fundamental classes of the corresponding groups.
Now what we want follows from the fact that $Sq^1$ induces a homomorphism mapping $s \otimes s$ to $Sq^1(t) \otimes Sq^1(t)$.

**Proposition 3.8.** Let $\tilde{E}_{n+q}$ ($q$ large) be the following 2-stage Postnikov tower. Then there is a map $f : \Sigma^q K(\mathbb{Z}_4, n - 1) \to \tilde{E}_{n+q}$ such that the composite $(\Sigma^q F_{n-1}(\mathbb{Z}) \to) \Sigma^q K(\mathbb{Z}, n - 1) \xrightarrow{\Sigma^q f} \Sigma^q K(\mathbb{Z}_4, n - 1) \to \tilde{E}_{n+q}$ if $n = 1, 2(\text{mod} 4)$ (or, $n = 0(\text{mod} 4)$) induces an isomorphism on $\pi_r$ where $r = q + 2n$ as above.

(1). $n = 2(\text{mod} 4)$

$$
\begin{array}{c}
K_r \\
K_{r-2} \times K_r
\end{array} \xrightarrow{i_2} \xrightarrow{i_1} \begin{array}{c}
\tilde{E}_{n+q} \\
K_{r+1}
\end{array}$$

$\Sigma^q K(\mathbb{Z}_4, n - 1) \xrightarrow{\Sigma^q i_{n-1}} K(\mathbb{Z}_4, q + n - 1) \xrightarrow{Sq^n \times Sq^{n+2}} K_{r-1} \times K_{r+1}$

where $i_1^*(\omega_2) = Sq^2 Sq^1 l_{r-2} + Sq^1 l_r$.

(2). $n = 0(\text{mod} 4)$

$$
\begin{array}{c}
K_r \\
K_{r-2} \times K_r
\end{array} \xrightarrow{i_2} \xrightarrow{i_1} \begin{array}{c}
\tilde{E}_{n+q} \\
K_{r+1}
\end{array}$$

$\Sigma^q K(\mathbb{Z}_4, n - 1) \xrightarrow{\Sigma^q i_{n-1}} K(\mathbb{Z}_4, q + n - 1) \xrightarrow{Sq^n} K_{r-1}$

where $i_1^*(\omega_2) = Sq^2 Sq^1 l_{r-2}$.

(3). $n = 1(\text{mod} 4)$

$$
\begin{array}{c}
K_r \\
K_r
\end{array} \xrightarrow{i_2} \xrightarrow{i_1} \begin{array}{c}
\tilde{E}_{n+q} \\
K_r
\end{array}$$

$\Sigma^q K(\mathbb{Z}_4, n - 1) \xrightarrow{\Sigma^q i_{n-1}} K(\mathbb{Z}_4, q + n - 1) \xrightarrow{Sq^{n+1}} K_r$

where $i_1^*(\omega_2) = Sq^2 l_{r-1}$.

**Proof.** Denote the tower in the Proposition by $\tilde{E}_{n+q}(\mathbb{Z}_4)$. Denote by $\tilde{E}_{n+q}(\mathbb{Z})$ a similar tower in which $K(\mathbb{Z}_4, n + q - 1)$ is replaced by $K(\mathbb{Z}, n + q - 1)$. By Remark 3.3, it is easy to see that there is a map from $\Sigma^q F_{n-1}(\mathbb{Z})$ to $\tilde{E}_{n+q}(\mathbb{Z})$ which induces an isomorphism on $\pi_r$ when $n = 0(\text{mod} 4)$. On the other hand it is not difficult to see that there is a map from the tower $\tilde{E}_{n+q}(\mathbb{Z})$ to the tower $\tilde{E}_{n+q}(\mathbb{Z}_4)$ which induces an
isomorphism on $\pi_r$. It remains to prove that the natural map $\Sigma^q\iota_{n-1} : \Sigma^qK(Z_4, n-1) \to K(Z_4, n+q-1)$ can be lifted to $\tilde{E}_{n+q}(Z_4)$ and the lifting is compatible to that of the map $\Sigma^q\iota_{n-1} : \Sigma^qK(Z, n-1) \to K(Z, n+q-1)$ to $\tilde{E}_{n+q}(Z)$.

We will give a proof only for $n = 2(\text{mod}4)$, the other cases are similar. Consider the fiber inclusion map $h : \Sigma^q\Gamma_q \to \Sigma^qK(Z_4, n-1)$, we have the following Peterson-Stein formula

$$h^*\Psi(\Sigma^q\iota_{n-1}) \in H^{r+1}(\Sigma^q\Gamma_q, Z)/Q$$

where $Q = Sq^2S^1(Qm\theta) + Sq^1(I\theta) = Sq^1(I\theta)$.

By Theorem 4.6 [20] and a familiar diagram chase argument as in the proof of Proposition 2 in Chap.16 [19] (see also [23]), we have $\Sigma^q(\theta\otimes\theta) \in S^q_0(\Sigma^q\iota_{n-1})$ and $\Sigma^q(\theta\otimes\theta) \in S^q_0(\Sigma^q\iota_{n-1})$. It follows easily that $h^*\Psi(\Sigma^q\iota_{n-1}) = 0 \in H^{r+1}(\Sigma^q\Gamma_q, Z)/Q$. It is not difficult to see from this and a simple computation that $\Psi(\Sigma^q\iota_{n-1}) = 0$ and a lifting can be chosen such that $\omega_2$ lies in its kernel.

To complete the proof, note that, as mentioned before, there is a commutative diagram up to homotopy

$$\begin{array}{ccc}
\tilde{E}_{n+q}(Z) & \xrightarrow{\rho} & \tilde{E}_{n+q}(Z_4) \\
\downarrow & & \downarrow \\
E_{n+q}(Z) & \xrightarrow{\rho} & E_{n+q}(Z_4) \\
\downarrow & & \downarrow \\
K(Z, n+q-1) & \xrightarrow{\rho} & K(Z_4, n+q-1) \\
\Sigma^q\iota_{n-1} & \uparrow & \Sigma^q\iota_{n-1} \\
\Sigma^qK(Z, n-1) & \xrightarrow{\rho} & \Sigma^qK(Z_4, n-1)
\end{array}$$

The lifting from $\Sigma^qK(Z, n-1)$ of $\Sigma^q\iota_{n-1}$ and the lifting from $\Sigma^qK(Z_4, n-1)$ of $\Sigma^q\iota_{n-1}$ can be made compatible by a modification of the lifting from $\Sigma^qK(Z_4, n-1)$ of $\Sigma^q\iota_{n-1}$. The same way the liftings to $\tilde{E}_{n+q}$ can also be made compatible. Thus we have the following commutative diagram up to homotopy which completes the proof.

$$\begin{array}{ccc}
\tilde{E}_{n+q}(Z) & \xrightarrow{\rho} & \tilde{E}_{n+q}(Z_4) \\
\uparrow & & \uparrow \\
\Sigma^qK(Z, n-1) & \xrightarrow{\rho} & \Sigma^qK(Z_4, n-1)
\end{array}$$

$\square$
Remark 3.9. The 2nd k-invariant $\omega_2$ in the Postnikov tower above gives an unique secondary cohomology operator $\Psi$ (with $\mathbb{Z}_4$-coefficients) associated with the Adem relation

\begin{align*}
S_q^2S_q^1S_q^n + S_q^1S_q^{n+2} &= 0 \quad n = 2(mod 4) \\
S_q^2S_q^1S_q^n &= 0 \quad n = 0(mod 4) \\
S_q^2S_q^{n+1} &= 0 \quad n = 1(mod 4)
\end{align*}

Note that $E_{n+q}$ is the universal example of the operator $\Psi$.

By Peterson-Stein\[21\], there are operators $\Phi$ which are $S$-dual to $\Psi$(which is uniquely determined by $\Psi$) so it is a secondary operator associated with the Adem relations:

\begin{align*}
\chi(S_q^n)S_q^3 + \chi(S_q^{n+2})S_q^1 + S_q^1\chi(S_q^{n+2}) &= 0 \quad n = 2(mod 4) \\
\chi(S_q^n)S_q^3 + S_q^1\chi(S_q^{n+2}) &= 0 \quad n = 0(mod 4) \\
\chi(S_q^{n+1})S_q^2 + S_q^1\chi(S_q^{n+2}) &= 0 \quad n = 1(mod 4)
\end{align*}

as we stated in [3].

4. Proofs of Theorems 2.3, 2.6 and 2.7

Proof of Theorem 2.3. First note that it suffices to show this for the universal spectrum $\widetilde{W}(n)$ since the map $i : S^0 \to \widetilde{W}(n)$ factors through $i : S^0 \to Y$. Notice that $H_i(\tilde{W}_k(n)/S^k) = 0$ for $i \leq k + 2$. Thus in the following proof, we may assume that $Y_k/S^k$ satisfies the same for $k$ large. Assuming $k$ large, without loss of generality we can assume that $Y_k$ is a finite complex. Write $Y^*_k$ for the $m$-S-dual of $Y_k$ and $g : Y^*_k \to S^{m-k}$ for the $S$-dual of the inclusion $i : S^k \to Y_k$. Note that $g^*(\varsigma_{S}^{m-k}) \neq 0$, where $\varsigma_{S}^{m-k}$ is the cohomology fundamental class of the sphere. By the $S$-duality we get a commutative diagram

\[
\begin{array}{ccc}
\{S^{2n+k}, S^k \wedge K(\mathbb{Z}_4, n-1)\} & \xrightarrow{i^*} & \{S^{2n+k}, Y_k \wedge K(\mathbb{Z}_4, n-1)\} \\
\downarrow \cong & & \downarrow \cong \\
\{S^{2m+n}, S^m \wedge K(\mathbb{Z}_4, n-1)\} & \xrightarrow{g^*} & \{S^{2n+k} \wedge Y^*_k, S^m \wedge K(\mathbb{Z}_4, n-1)\} \\
\downarrow & & \downarrow q_2^* \\
[S^{2n+m}, \widetilde{E}_{n+m}] & \xrightarrow{g^*} & [S^{2n+k} \wedge Y^*_k, \widetilde{E}_{n+m}] \\
\end{array}
\]

where $\widetilde{E}_{n+m}$ is the tower in Proposition [5.8] and $q_2 : S^m \wedge K(\mathbb{Z}_4, n-1) \to \widetilde{E}_{n+m}$ is a lifting of $\Sigma^n l_{n-1}$. From the diagram above and Proposition [5.8] it suffices to show that the homomorphism $g^*$ at the bottom line is injective. From now on we will restrict to the case $n \equiv 2(mod 4)$. The other cases are similar. Let $i_0 : F \to \widetilde{E}_{n+m}$ be the fibre of the composite $\Pi_1 \circ \Pi_2$. Note that $F$ can be viewed as a fibration over $K_{2n+m-2}$ with fibre $\tilde{K}(\mathbb{Z}_4, 2n + m)$ and $k$-invariant $j_*(Sq^2Sq^1)$; where

\[j_* : H^{m+2n+1}(-, \mathbb{Z}_2) \to H^{m+2n+1}(-, \mathbb{Z}_4)\]
is the homomorphism induced by the inclusion $\mathbb{Z}_2 \subset \mathbb{Z}_4$ and $l$ is the basic class of $K_{m+2n-2}$.

Consider the following commutative diagrams

\[
\begin{array}{ccc}
[S^{2n+m}, F] & \xrightarrow{\sim} & [S^{2n+m}, \tilde{E}_{n+m}] \\
\downarrow J := g^* & & \downarrow g^* \\
[S^{2n+k} \wedge Y_k^*, K(\mathbb{Z}_4, n + m - 2)] & \xrightarrow{i_1} & [S^{2n+k} \wedge Y_k^*, F] \\
& i_0 & \xrightarrow{\sim} \ [S^{2n+k} \wedge Y_k^*, \tilde{E}_{n+m}] \\
\end{array}
\]

and

\[
[S^{2n+m}, K(\mathbb{Z}_4, 2n + m)] \xrightarrow{\sim} [S^{2n+m}, F] \\
\downarrow g^* \\
[S^{2n+k} \wedge Y_k^*, K_{2n+m-3}] \xrightarrow{j_*(Sq^2 Sq^1)} [S^{2n+k} \wedge Y_k^*, K(\mathbb{Z}_4, 2n + m)] \xrightarrow{\sim} [S^{2n+k} \wedge Y_k^*, F]
\]

where $i_1 : K(\mathbb{Z}_4, n + m - 2) \to F$ is the homotopy fibre of $i_0$. $j_*(Sq^2 Sq^1)$ in the second diagram above is zero since $Sq^3 U_k = 0$ and thus by duality $\chi(Sq^3 H^{m-k-3}(Y_k^*)) = Sq^2 Sq^1 H^{m-k-3}(Y_k^*) = 0$. Thus the second diagram implies that $J$ is a monomorphism. To complete the proof, it suffices to show $\text{Ker}(i_0)_* = I(m(i_1))_* = 0$ in the first diagram above.

Let $q = m-n-k-1$, if $x \in H^{q-1}(Y_k^*, \mathbb{Z}_4)$, then $Sq^n(x) \in H^{n+q-1}(Y_k^*, \mathbb{Z}_4) \cong (H^{k+2}(Y_k, \mathbb{Z}_2))^* = 0$. On the other hand, by duality $\chi(Sq^{n+2}) U_k = 0$ implies that $Sq^{n+2} H^{q-1}(Y_k^*, \mathbb{Z}_2) = 0$. Thus

\[
x \in \text{Ker} Sq^n \cap \text{Ker} Sq^{n+2}
\]

Since $Y_k$ is $\Phi$-orientable, i.e., $0 \in \Phi(U_k)$. By [21] that $0 \in \Psi(x)$. Thus $x$ can be lifted to $\tilde{E}_{q-1}$ and so $(i_1)_*(x) = 0$. This completes the proof. \[\square\]

For simplicity, denote by $F_{n-1}(\mathbb{Z}_4)$ the space $K(\mathbb{Z}_4, n - 1) \wedge K(\mathbb{Z}_4, n - 1)$ as before in the following proof.

**Proof of Theorem 2.6.** For $x \in H^{n-1}(M, \mathbb{Z}_4)$, let $f(x) = (w \wedge i_1(n)) A_n(x) \in H_{2n}(K(\mathbb{Z}_4, n - 1) ; \tilde{W}(n))$. For $k$ large, $f(x + y)$ is the following composition of maps

\[
S^1 \wedge S^{2n+k} \xrightarrow{id \wedge \Delta_k} S^1 \wedge T \xi \wedge M_+ \xrightarrow{id \wedge w \wedge (x y)}
\]

$\to S^1 \wedge \tilde{W}(n)_k \wedge (K(\mathbb{Z}_4, n - 1) \times K(\mathbb{Z}_4, n - 1)) = \tilde{W}(n)_k \wedge S^1 \wedge (K(\mathbb{Z}_4, n - 1) \times K(\mathbb{Z}_4, n - 1)) \xrightarrow{id \wedge \kappa} \tilde{W}(n)_k \wedge S^1 \wedge K(\mathbb{Z}_4, n - 1),$

where $\kappa^* (l) = l \otimes 1 + 1 \otimes l$ for the basic class $l \in H^{n-1}(K(\mathbb{Z}_4, n - 1), \mathbb{Z}_4)$.

Identifying $\tilde{W}(n)_k \wedge S^1 \wedge (K(\mathbb{Z}_4, n - 1) \times K(\mathbb{Z}_4, n - 1))$ with

\[
\{\tilde{W}(n)_k \wedge S^1 \wedge K(\mathbb{Z}_4, n - 1)\} \vee \{\tilde{W}(n)_k \wedge S^1 \wedge K(\mathbb{Z}_4, n - 1)\} \\vee \{\tilde{W}(n)_k \wedge S^1 \wedge F_{n-1}(\mathbb{Z}_4)\}.
\]

It is readily to see that $f(x + y) = f(x) + f(y) + g$, here $g$ is the composition

\[
\{\tilde{W}(n)_k \wedge S^1 \wedge K(\mathbb{Z}_4, n - 1)\} \vee \{\tilde{W}(n)_k \wedge S^1 \wedge K(\mathbb{Z}_4, n - 1)\} \vee \{\tilde{W}(n)_k \wedge S^1 \wedge F_{n-1}(\mathbb{Z}_4)\}.
\]
Proof of Theorem 2.7. Let \( \mu : K_{n+k+1} \times \widetilde{W}_k(n) \to \widetilde{W}_k(n) \) denote the fiber multiplication. Since \( d_1(w_1, w_2) = 0 \), \( w_2 \) is the composition

\[
T \xi \xrightarrow{\Delta} T \xi \times T \xi \xrightarrow{w_1 \times vU_k} \widetilde{W}_k(n) \times K_{n+k+1} \xrightarrow{\mu} \widetilde{W}_k(n),
\]

where \( vU_k \in H^{k+n+1}(T \xi, Z_2) \) is the second difference of \( w_1 \) and \( w_2 \), i.e., the secondary obstruction to deform \( w_1 \) to \( w_2 \). Consider the commutative diagram:

\[
\begin{array}{ccc}
S^{2n+k} & \xrightarrow{\alpha'} & (T \xi \wedge M_+) \vee (T \xi \wedge M_+) & \xrightarrow{\alpha} & \widetilde{W}_k(n) \wedge K(Z_4, n-1) \\
\| & & \| & & \\
S^{2n+k} & \xrightarrow{\Delta \alpha} & (T \xi \times T \xi) \wedge M_+ & \xrightarrow{b} & \widetilde{W}_k(n) \wedge K(Z_4, n-1)
\end{array}
\]

where \( \alpha' \) is a lifting of \( \Delta \alpha \), \( b = \mu(w_1 \times vU_k) \wedge x \), \( a = (w_1 \wedge x) \vee c \), and \( c = i(vU_k) \wedge x \), \( i : K_{n+k+1} \to \widetilde{W}_k(n) \) the inclusion of the fibre. Write \( \alpha' = \alpha_1 + \alpha_2 \), here \( \alpha_1 \) and \( \alpha_2 \) are the factors of the wedge. Note that \( \phi_2(x) = h(b \circ \Delta \alpha) = h(a_1) + h(a_2) = \phi_1(x) + h(a \alpha_2) \).

We are going to show \( h(a \alpha_2) = 0 \).
As $a_2$ factors through the map $i \wedge id : K_{n+k+1} \wedge K(\mathbb{Z}_4, n-1) \to \widetilde{W}_k(n) \wedge K(\mathbb{Z}_4, n-1)$, it suffices to prove that

$$(i \wedge id)_* : \pi_{2n+k}(K_{n+k+1} \wedge K(\mathbb{Z}_4, n-1)) \to \pi_{2n+k}(\widetilde{W}_k(n) \wedge K(\mathbb{Z}_4, n-1))$$

is zero. Note the homomorphism

$$(Sq^1 \wedge id)_* : \pi_{2n+k}(K_{n+k} \wedge K(\mathbb{Z}_4, n-1)) \to \pi_{2n+k}(K_{n+k+1} \wedge K(\mathbb{Z}_4, n-1)) \cong \mathbb{Z}_2$$

is an isomorphism as it induces an isomorphism on the $(2k+n)$-th homology groups. The composition $K_{n+k} \xrightarrow{Sq^1} K_{n+k+1} \xrightarrow{i} \widetilde{W}_k(n)$ is null homotopy. Thus $(i \wedge id)_* = 0$. This completes the proof.

5. Proof of Theorem 1.9

In this section we prove Theorem 1.9. We first study the properties of the invariants $\mu_M$ and $q_M(Sq^1_i)$ defined in §1.

**Lemma 5.1.** Let $M$ be a framed manifold of dimension $2n$ with $n$ odd. Let $q_M : H^n(M, \mathbb{Z}_2) \to \mathbb{Z}_2$ be the Kervaire quadratic form. For $x \in H^{n-1}(M, \mathbb{Z}_2)$,

(i) $n \equiv 3(\text{mod} 4)$, $[M, x]$ is reduced bordant to zero iff $q_M(Sq^1_i)x = 0$.

(ii) $n \equiv 1(\text{mod} 4)$, $[M, x]$ is reduced bordant to $[M', x']$ where $x' \in H^{n-1}(M')$ iff $q_M(Sq^1_i)x = 0$.

**Proof.** Identify the reduced framed bordism group $\tilde{\Omega}^{fr}_{2n}(-)$ with the stable homotopy group $\pi_{2n}^s(-)$. Recall that $\pi_{2n}^s(K(\mathbb{Z}_2, n)) = \mathbb{Z}_2$. By [4] it is easy to see that the homomorphism

$$(Sq^1_i)_* : \pi_{2n}^s(K(\mathbb{Z}_2, n-1)) \to \pi_{2n}^s(K(\mathbb{Z}_2, n))$$

is identified with the following geometrically defined homomorphism

$$\tilde{\Omega}^{fr}_{2n}(K(\mathbb{Z}_2, n-1)) \xrightarrow{} \mathbb{Z}_2$$

$$[M, x] \xrightarrow{} q_M(Sq^1_i)x$$

By Theorem 3.1 and Lemma 3.7 (i) follows since $(Sq^1_i)_*$ is an isomorphism. To prove (ii), note that there is an exact sequence by Proposition 3.6 and Lemma 3.7

$$\pi_{2n}^s(K(\mathbb{Z}, n-1)) \to \pi_{2n}^s(K(\mathbb{Z}_2, n-1)) \xrightarrow{(Sq^1_i)_*} \pi_{2n}^s(K_n).$$

This completes the proof.
Now we want to study which bilinear forms $\mu$ can be realized by $(n-2)$-connected $2n$-dimensional $\pi$-manifolds. Note that a sphere bundle over $S^{n+1}$ with fiber $S^{n-1}$ is a $\pi$-manifold if the characteristic map of the bundle, $\theta \in \pi_n(SO(n))$, belongs to the kernel of the stabilization homomorphism $S_*: \pi_n(SO(n)) \to \pi_n(SO)$. Recall that the homotopy groups of $\pi_n(SO(n))$ are as follows (c.f. [11]):

$$\pi_n(SO(n)), n \geq 3, \neq 6$$

| $n \geq 3, \neq 6$ | 8s | 8s + 1 | 8s + 2 | 8s + 3 | 8s + 4 | 8s + 5 | 8s + 6 | 8s + 7 |
|---------------------|-----|-------|-------|-------|-------|-------|-------|-------|
| \[ \pi_n(SO(n)) \] | $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_4$ | $\mathbb{Z}$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_4$ | $\mathbb{Z}$ |

and $\pi_6(SO(6)) = 0$.

Let $\pi: SO(n) \to S^{n-1}$ be the canonical $SO(n-1)$-fibration. For a $S^{n-1}$-bundle over $S^{n+1}$ with characteristic map $\theta \in \pi_n(SO(n))$, say $M_\theta$, it is easy to see that $Sq^2 : H^{n-1}(M_\theta, \mathbb{Z}_2) \to H^{n+1}(M_\theta, \mathbb{Z}_2)$ is an isomorphism if and only if $\pi_*\theta \in \pi_n(S^{n-1}) = \mathbb{Z}_2$ is nonzero. By duality this implies that $\mathcal{Z} \cup Sq^2z = 0$ for all $z \in H^{n-1}(M_\theta, \mathbb{Z}_2)$ if and only if $\pi_*\theta = 0$. The latter is equivalent to the fact that the bundle has a section.

**Lemma 5.2.** Let $M$ be a $\pi$-manifold of dimension $2n$. Then

1. $\mu_M(x, x) = 0, \forall x \in H^{n-1}(M, \mathbb{Z}_2)$ if $n = 2(\text{mod} 4)$.
2. $\mu_M(x, x) = 0, \forall x \in \text{Im}(\rho_2 : T \subset H^{n-1}(M, \mathbb{Z}_4) \to H^{n+1}(M, \mathbb{Z}_2))$, if $n$ is odd where $T$ is the set of elements of order $4$.
3. If $n = 0(\text{mod} 4)$, then there is a $S^{n-1}$-bundle over $S^{n+1}$, $M$, so that $\mu_M(x, x) \neq 0$, where $x \in H^{n-1}(M, \mathbb{Z}_2)$ is a generator.

**Proof.** For each $x \in H^{n-1}(M, \mathbb{Z}_2)$, consider the reduced bordism class $[M, x] \in \Omega_2^{fr}(K_{n-1}) \cong \mathbb{Z}_2$. It is easy to see that $x \cup Sq^2x[M]$ is a bordism invariant. One verifies the following map defines a homomorphism

$$\Omega_2^{fr}(K_{n-1}) \to \mathbb{Z}_2$$

$$[M, x] \to x \cup Sq^2x[M]$$

By Remark 3.4 the reduction homomorphism

$$\Omega_2^{fr}(K, n - 1)) \to \Omega_2^{fr}(K_{n-1})$$

is surjective if $n$ is even.

If $n = 2(\text{mod} 4)$, let $\theta \in \pi_n(SO(n))$ be a generator. By the tables (I)(II) of [11] it follows that $\theta$ lies in the image of the inclusion map $\pi_n(SO(n-1)) \to \pi_n(SO(n))$. By the remark above this implies that the sphere bundle $M_\theta$ has a section. Therefore $z \cup Sq^2z = 0$ for all $z \in H^{n-1}(M_\theta, \mathbb{Z}_2)$. On the other hand, one can verify that $[M_\theta, z] \in \Omega_2^{fr}(K_{n-1})$ is a generator if $z \in H^{n-1}(M_\theta, \mathbb{Z}_2)$ is nonzero. This proves (i).

If $n = 0(\text{mod} 4)$, by [11] there is an element $\beta \in \ker S_* : \pi_n(SO(n)) \to \pi_n(SO)$ so that $\pi_*\beta$ is nonzero. This proves (iii).
If \( n \) is odd, by Lemma 5.1 the homomorphism
\[
q(Sq^1): \tilde{\Omega}^{fr}_{2n}(K_{n-1}) \rightarrow \mathbb{Z}_2
\]
for all \([M, x]\). In particular, if \( x \) can be lifted to the \( \mathbb{Z}_4 \)-coefficient class with order 4, \( Sq^1 x = 0 \) and so \( x \cup Sq^2 x = 0 \). This completes the proof.

Now we are ready to prove Theorem 1.9.

**Proof of Theorem 1.9.** By Theorem 1.6 the data of invariants are homotopy invariants of the manifolds. Thus the homotopy and homeomorphism classification of such manifolds are the same.

There is an isomorphism
\[
\tilde{\Omega}^{fr}_{2n}(K(H, n-1)) \cong \pi_{2n}^s(K(H, n-1)).
\]
Therefore from Theorem 3.1 there is a reduced framed bordism class \([M, f]\) corresponding to the given algebraic data \([H, \mu, \phi]\) (resp. \([H, \mu, \phi, \omega]\)) if \( n \) is even (resp. odd). This together with Lemmas 5.1 and 5.2 implies this is an 1-1 correspondence.

Add some \( S^{n-1} \times S^{n+1} \) to \( M \) if necessary so that \( f^*: H_{n-1}(M) \rightarrow H \) is surjective. By surgery on \( M \) we may assume further that \( f^*: H_{n-1}(M) \rightarrow H \) is an isomorphism and \( H_n(M, \mathbb{Q}) = 0 \). Therefore the data can be realized by a \((n-2)\)-connected \( 2n \)-dimensional \( \pi \)-manifold, \( M \), so that \( H_n(M, \mathbb{Q}) = 0 \) and \( \pi(M) = [H, \mu, \phi] \) (resp. \([H, \mu, \phi, \omega]\)).

Now it suffices to prove that the map \( \pi \) is injective.

Suppose that \( M_i, i = 1, 2 \), are two framed smooth manifolds with the same data (for TOP manifold, the similar argument works identically). Note that the Kervaire invariants of \( M_i \) must vanish since \( H_n(M_i, \mathbb{Q}) = 0 \). By the assumption there are maps \( f_i: M_i \rightarrow K(H, n-1) \), so that \((M_1, f_1)\) and \((M_2, f_2)\) are reduced framed bordant, where \( f_i \) induces an isomorphism on the \((n-1)\)-th homology groups. Since both \( M_i \) framed cobordant to some homotopy spheres, there is a framed homotopy sphere, \( \Sigma \), so that \((M_1, f_1)\) and \((M_2\# \Sigma, f_2)\) are framed bordant. By Freedman [8] or Kreck [14] it follows that \( M_1 \) and \( M_2\# \Sigma \) are diffeomorphic since \( H_n(M_i, \mathbb{Q}) = 0 \). Therefore \( M_1 \) and \( M_2 \) are almost diffeomorphic. The same argument as above applies to show that \( M_1 \) and \( M_2 \) are homeomorphic to each other. This completes the proof.

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Nankai Institute of Mathematics, Nankai University, Tianjin 300071, P.R.C

E-mail address: ffang@sun.nankai.edu.cn

Institute of Math., Academia Sinica, Beijing 100080, China and Department of Mathematics Education, Korea University, Seoul, Korea

E-mail address: pjz62@hotmail.com