Abstract

Microlocal defect functionals (H-measures, H-distributions, semiclassical measures, etc.) are objects which determine, in some sense, the lack of strong compactness for weakly convergent $L^p$ sequences. Recently, Luc Tartar introduced one-scale H-measures, a generalisation of H-measures with a characteristic length, which also comprehend the notion of semiclassical measures.

We present a self-contained introduction to one-scale H-measures, carrying out some alternative proofs, and strengthening some results, comparing these objects to known microlocal defect functionals. Furthermore, we improve and generalise Tartar’s localisation principle for these objects from which we are able to derive the known localisation principles for both H-measures and semiclassical measures. Moreover, we develop a variant of compactness by compensation suitable for equations with a characteristic length.

Keywords: semiclassical measure, H-measure, one-scale H-measure, localisation principle, compactness by compensation

Mathematics subject classification: 35B27, 35Q40, 35S05, 46G10
1. Introduction

Overview

In various situations concerning partial differential equations one often encounters $L^2$ weakly converging sequences, which do not converge strongly. For such a sequence $(u_n)$ it is natural to consider a $L^1$ bounded sequence $|u_n|^2$ which, in general, does not converge weakly in $L^1$, but only weakly $*$ in the space of bounded Radon measures $(\mathcal{M}_b = C^*_0)$, to a defect measure $\nu$.

Essentially, there are two distinctive types of non-compact sequences, with typical defect measures $\nu$ (for these two examples we fix $\varphi \in C^\infty_c(\mathbb{R}^d)$ such that $\|\varphi\|_{L^2(\mathbb{R}^d)} = 1$, $\varepsilon_n \to 0^+$, and note that in both cases below one has $u_n \rightharpoonup 0$):

a) concentration: $u_n(x) := \varepsilon_n^{-d/2} \varphi(\frac{x-x_0}{\varepsilon_n})$, where $\nu = \delta_{x_0}$, and

b) oscillation: $u_n(x) := \varphi(x)e^{\frac{2\pi i x \varepsilon_n}{\varepsilon_n}}$, where $\nu$ is actually equal to $|\varphi|^2 \lambda$ (i.e. to the measure having density $|\varphi|^2$ with respect to the Lebesgue measure $\lambda$ on $\mathbb{R}^d$).

However, defect measures are not enough to completely determine the difference between various non-compact sequences (e.g. the direction and frequency of oscillations). This lack of information could be (partially) overcome by introducing objects in full phase space such as H-measures and semiclassical measures.

H-measures, as originally introduced a quarter of century ago by Luc Tartar [26] and (independently) Patrick Gérard [14], are Radon measures on the cospherical bundle $\Omega \times S^{d-1}$ over a domain $\Omega \subseteq \mathbb{R}^d$. Since their introduction they saw quite a number of successful applications, many of which depend on the so called localisation principle, which is closely related to the generalisation of compactness by compensation method to variable coefficients. Let us mention the applications to small-amplitude homogenisation [26, 3], to the control theory [10, 20], to explicit formulæ and bounds in homogenisation [26, 4], to the existence of entropy solutions in the theory of conservation laws [23], and the velocity averaging results [14, 19]. This property is also crucial in applications of the propagation principle, as it allows certain control over the support of H-measure ([26, 1]).

Related to H-measures, but certainly different objects, are semiclassical measures first introduced by Gérard [15], and later renamed by Pierre-Luis Lions and Thierry Paul [21] as Wigner measures. They are Radon measures on the cotangential bundle $\Omega \times \mathbb{R}^d$ over a domain $\Omega \subseteq \mathbb{R}^d$.

In contrast to the H-measures, semiclassical measures depend upon characteristic length $(\omega_n)$, $\omega_n \to 0^+$, which makes them more suitable objects for the problems where such a characteristic length naturally appears, often related to highly oscillating problems for partial differential equations (such as the homogenisation limit [17, 21], to microlocal energy density for (semi)linear wave equation [16, 13], the semiclassical limit of Schrödinger equations [11, 29], or some other problems related to the quantum theory).

While the localisation principle for semiclassical measures was mentioned already in the first papers [15, 21] (sometimes called also the elliptic regularity [10]), it has been playing a less prominent role in applications.

In the next section we recall the definitions of both H-measures and semiclassical measures, state the corresponding localisation principles, and discuss their differences in the light of some examples. The third section is devoted to the objects recently introduced by Luc Tartar [27], which we call one-scale H-measures, in line with the term multi-scale H-measures used recently in [28]. As opposed to the historical approach in [27], we have chosen some variant proofs, and for the sake of completeness provided all the necessary details. The following section is dedicated to the localisation principle for one-scale H-measures, where we significantly extend Tartar’s result from [27]. In the last section we show how the localisation principles, both for H-measures and semiclassical measures, can be derived from the obtained corresponding principle for the one-scale H-measures. We also apply the localisation principle for one-scale H-measures in developing a variant of compactness by compensation suitable for equations containing a characteristic length.
Notation
Throughout the paper we denote by $\otimes$ the tensor product of vectors on $\mathbb{C}^r$, defined by $(a \otimes b) \nu = (\nu \cdot b) a$, where $\cdot$ stands for the (complex) scalar product $a \cdot b := \sum_{i=1}^{r} a_i b_i$, resulting in $[a \otimes b]_{ij} = a_i \delta_{ij}$, while $\otimes$ denotes the tensor product of functions in different variables. By $\langle \cdot, \cdot \rangle$ we denote any sesquilinear dual product, which we take to be antilinear in the first variable, and linear in the second. When matrix functions appear as both arguments in a dual product, we interpret it as $\langle A, B \rangle := \int B \cdot A = \int \text{tr}(BA^*)$. In order to have all formulæ written in matrix notation, we define dual product also for the case when matrix function appear in the first and scalar function in the second argument as $\langle A, \varphi \rangle = |\langle A^D, \varphi \rangle|_{ij}$.

By $\pi(\xi) := \frac{\xi}{|\xi|}$ we denote the projection on $\mathbb{R}_s^d := \mathbb{R}^d \setminus \{0\}$ along rays to the unit sphere $S^{d-1}$. The open (closed) ball in $\mathbb{R}^d$ or $\mathbb{C}^r$ centred at point $x$ with radius $r > 0$ we will denote by $K(x, r) \ (K[x, r])$.

The Fourier transform, defined as $\hat{u}(\xi) := \mathcal{F}u(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} u(x) \, dx$, and its inverse as $(u)_{\mathcal{F}}(\xi) := \mathcal{F}^{-1}u(\xi) := \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} u(x) \, dx$, allows for an elegant definition of Sobolev spaces on the whole space $\mathbb{R}^d$, in particular those modelled on $L^2(\mathbb{R}^d)$, which is sufficient for our present purpose. Indeed, for $s \in \mathbb{R}$ we take

$$H^s(\mathbb{R}^d) := \left\{ u \in S' : \left(1 + |\xi|^s\right)^{\text{sign } s} \hat{u} \in L^2(\mathbb{R}^d) \right\},$$

with the norm $||u||_{H^s(\mathbb{R}^d)} := ||(1 + |\xi|^s)^{\text{sign } s} \hat{u}||_{L^2(\mathbb{R}^d)}$.

However, in the sequel we shall also deal with functions defined on an open $\Omega \subseteq \mathbb{R}^d$. In order to apply the Fourier transform, as well as to simplify other arguments, we shall identify such a function by its extension by zero to the whole $\mathbb{R}^d$.

The local Sobolev spaces we are going to need are

$$H^s_{\text{loc}}(\Omega) := \left\{ u \in D'(\Omega) : (\forall \varphi \in C^\infty_c(\Omega)) \varphi u \in H^s(\mathbb{R}^d) \right\},$$

with the above mentioned identification of a distribution and its extension by zero assumed. Naturally, $H^s_{\text{loc}}(\Omega)$ will be endowed with the weakest topology in which every mapping $u \mapsto \varphi u$ is continuous (cf. [5, p. 1207] and the references mentioned there). In particular, the space $L^2_{\text{loc}}(\Omega; \mathbb{C}^r)$ is equipped with the standard Fréchet locally convex topology (cf. [2] and the references therein). Hence, a subset of $L^2_{\text{loc}}(\Omega; \mathbb{C}^r)$ is bounded if and only if it is bounded in the sense of seminorms which generate the corresponding locally convex topology (which is a stronger notion then metric boundedness).

2. Overview of H-measures and semiclassical measures

H-measures
For the sake of generality we shall deal with local spaces. In this context, the following theorem can be stated (cf. [14, Theorem 1] or [26, Theorem 1.1]):

**Theorem 1. (existence of H-measures)** For a weakly converging sequence $u_n \longrightarrow 0$ in $L^2_{\text{loc}}(\Omega; \mathbb{C}^r)$, there exists a subsequence $(u_{n'})$ and an $r \times r$ hermitian non-negative matrix Radon measure $\mu_H$ on $\Omega \times S^{d-1}$ such that for any $\varphi_1, \varphi_2 \in C^\infty_c(\Omega)$ and $\psi \in C(S^{d-1})$ one has:

$$\lim_{n' \to \infty} \int_{\mathbb{R}^d} \left( \varphi_1 \overline{u_{n'}}(\xi) \otimes \varphi_2 \overline{u_{n'}}(\xi) \right) \psi \left( \frac{\xi}{|\xi|} \right) \, d\xi = \langle \mu_H, (\varphi_1 \varphi_2) \otimes \psi \rangle$$

$$= \int_{\Omega \times S^{d-1}} \varphi_1(x) \varphi_2(x) \psi(\xi) \, d\mu_H(x, \xi).$$

The above measure $\mu_H$ is called the H-measure associated to the (sub)sequence $(u_{n'})$. 

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We shall often abuse the notation and terminology, assuming that we have already passed to a subsequence determining an H-measure. When the whole sequence admits the H-measure (i.e. the definition is valid without passing to a subsequence), we say that the sequence is pure.

The corresponding defect measure of the weakly converging (towards 0) sequence \((u_n)\) can be obtained simply by integrating the H-measure with respect to the Fourier space variable \(\xi\). As an immediate consequence we have that any H-measure associated to a strongly convergent sequence in \(L^2_{\text{loc}}(\Omega; \mathbb{C}^r)\) is necessarily zero, and vice versa, if the H-measure is trivial, then the corresponding (sub)sequence converges strongly in \(L^2_{\text{loc}}(\Omega; \mathbb{C}^r)\).

The majority of successful applications of H-measures use the localisation principle (cf. [26], [1]).

**Theorem 2. (localisation principle for H-measures)** Let \(u_n \rightharpoonup 0\) in \(L^2_{\text{loc}}(\Omega; \mathbb{C}^r)\), and let

\[
\sum_{|\alpha| \leq m} \partial_\alpha (A^\alpha u_n) \rightharpoonup 0 \quad \text{strongly in} \quad H^{-m}_{\text{loc}}(\Omega; \mathbb{C}^q),
\]

where \(A^\alpha \in C(\Omega; M_{q \times r}(\mathbb{C}))\) and \(\partial_\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}\) denotes partial derivatives in variable \(x\) in the physical space.

Then for the associated H-measure \(\mu_H\) we have

\[
p_{pr}(\mu_H) = 0,
\]

where

\[
p_{pr}(x, \xi) := \sum_{|\alpha| = m} (2\pi i)^m \left( \frac{\xi}{|\xi|} \right)^\alpha A^\alpha(x)
\]

is the principal symbol of the differential operator in (2.1).

This result implies that the support of \(\mu_H\) is contained in the set

\[
\Sigma_{p_{pr}} := \left\{(x, \xi) \in \Omega \times S^{d-1} : \text{rank} p_{pr}(x, \xi) < r \right\}
\]

of points where \(p_{pr}(x, \xi)\) is not left invertible.

**Remark 1.** The main difference between Tartar’s and Gérard’s approach in the construction of H-measures is in the choice of the space of test functions. Tartar based the proof on a commutation lemma (known under the name *First commutation lemma*) for which only the continuity of test functions is sufficient. On the other hand, Gérard used the theory of pseudodifferential operators in which the commutation lemma is trivially obtained (the difference of pseudodifferential operators of the same order and with the same principal symbol is compact), but requiring smooth test functions.

Recently, H-measures were extended in two directions: to *parabolic H-measures* [3, 5], adapted to a different scaling and projection \(\pi\), and to *H-distributions* [6], allowing the treatment of \(L^p\) weakly converging sequences for \(p \in (1, \infty)\) (see also [25]).

**Semiclassical measures**

We present the existence result for semiclassical measures in a simpler, but equivalent form to the original Gérard’s definition [15], without introducing the notion of (semiclassical) pseudodifferential operators (cf. [27, Chapter 32]).
Theorem 3.  (existence of semiclassical measures)  If \((u_n)\) is a bounded sequence in \(L^2_{\text{loc}}(\Omega; \mathbb{C}^r)\), and \((\omega_n)\) a sequence of positive numbers such that \(\omega_n \to 0^+\), then there exists a subsequence \((u_{n'}^r)\) and an \(r \times r\) hermitian non-negative matrix Radon measure \(\mu_{\omega_{n'}}^{(\omega_{n'})}\) on \(\Omega \times \mathbb{R}^d\) such that for any \(\varphi_1, \varphi_2 \in C_c^\infty(\Omega)\) and \(\psi \in \mathcal{S}(\mathbb{R}^d)\) one has:

\[
\lim_{n' \to \infty} \int_{\mathbb{R}^d} \left( \varphi_1 u_{n'}(\xi) \otimes \varphi_2 u_{n'}(\xi) \right) \psi(\omega_{n'} \xi) \, d\xi = \langle \mu_{\omega_{n'}}^{(\omega_{n'})}, (\varphi_1 \varphi_2) \otimes \psi \rangle \\
= \int_{\Omega \times \mathbb{R}^d} \varphi_1(x) \varphi_2(x) \psi(\xi) \, d\mu_{\omega_{n'}}^{(\omega_{n'})}(x, \xi).
\]

The above measure \(\mu_{\omega_{n'}}^{(\omega_{n'})}\) is called the semiclassical measure (with characteristic length \((\omega_{n'})\)) associated to the (sub)sequence \((u_{n'})\).

Remark 2.  For a bounded sequence in \(L^2(\Omega; \mathbb{C}^r)\), both the associated H-measure and the semiclassical measure are bounded Radon measures and formulae in Theorem 1 and Theorem 3 make sense for \(\varphi_1, \varphi_2 \in C_0(\Omega)\).

As it is done with H-measures, we shall often abuse the notation and terminology, assuming that we have already passed to a subsequence determining a semiclassical measure, and reduce the notation to \(\mu_{\text{sc}} = \mu_{\omega_{n}}^{(\omega_{n})}\), except in situations where it will not be clear which characteristic length do we use. When the whole sequence admits a semiclassical measure with characteristic length \((\omega_n)\) (i.e. the definition is valid without passing to a subsequence), we say that the sequence is \((\omega_n)\)-pure (in terms of semiclassical measures).

In the definition of semiclassical measures it is not necessary that the observed sequence tends (weakly) to zero, contrary to the definition of H-measures. This is the case because here, due to the characteristic length \((\omega_n)\), we have a stronger variant of the commutation lemma (see Lemma 2 in the next section). Consequently, in some situations semiclassical measures can be used not only to prove strong compactness of a sequence, but also to establish that the limit is zero. Moreover, we can decompose a semiclassical measure into two parts, one depending on the value of the limit, and the other associated to the sequence converging to zero. Namely, if \(u_n \rightharpoonup u\) in \(L^2_{\text{loc}}(\Omega; \mathbb{C}^r)\), then

\[
\mu_{\text{sc}} = (u \otimes u) \lambda \otimes \delta_0 + \nu_{\text{sc}},
\]

where \(\nu_{\text{sc}}\) is the semiclassical measure, with the same characteristic length as \(\mu_{\text{sc}},\) associated to the sequence \((u_n - u),\) while \(\lambda\) is the Lebesgue measure in \(x,\) and \(\delta_0\) the Dirac mass in \(\xi = 0.\)

In order to obtain the defect measure from a semiclassical measure, an additional assumption is required. First, recall that a sequence \((u_n)\) in \(L^2_{\text{loc}}(\Omega; \mathbb{C}^r)\) is called \((\omega_n)\)-oscillatory, where \(\omega_n \to 0^+\), if for any \(\varphi \in C_c^\infty(\Omega)\)

\[
\lim_{r \to \infty} \limsup_n \int_{|\xi| > \frac{1}{\omega_n}} |\varphi u_n|^2 \, d\xi = 0.
\]

An interpretation of this property is that information of the observed sequence is on the scale not greater than \(\frac{1}{\omega_n}\) in the Fourier space, so the semiclassical measure with characteristic length \((\omega_n)\) is able of capturing all the information and nothing is lost at infinity (cf. [13, Def. 3.3] and [17, Def. 1.6]). For example, sequences of concentration and oscillations from the introductory section are both \((\omega_n)\)-oscillatory if and only if \(\lim_n \frac{\omega_n}{\omega_{n'}} \in (0, \infty]\). One can notice here certain ambiguity in terminology, suggesting that a concentration effect provides oscillations in some sense.

If a sequence \((u_n)\) is bounded and \((\omega_n)\)-oscillatory, then the corresponding defect measure can be obtained by integrating \(\mu_{\omega_{n}}^{(\omega_{n})}\) with respect to the Fourier space variable \(\xi\). As an immediate consequence we have that, if for a bounded sequence \((u_n)\) in \(L^2_{\text{loc}}(\Omega; \mathbb{C}^r)\) there exists a sequence...
\((\omega_n), \omega_n \to 0^+,\) such that \((u_n)\) is \((\omega_n)-\)oscillatory and the semiclassical measure with characteristic length \((\omega_n)\) associated to \((u_n)\) is trivial, then \((u_n)\) converges strongly in \(L^2_{\text{loc}}(\Omega; C^q)\).

As it was the case with H-measures, it is of great interest to localise the support of semiclassical measures. The statement of the following theorem is slightly different from those in [10, 15, 21], as we have a system of differential relations, so the coefficients are matrix valued. Since the proof follows by minor modifications of the usual proof, while in the sequel a more general result will be presented, we shall omit it.

**Theorem 4. (localisation principle for semiclassical measures)** Let \((u_n)\) be bounded in \(L^2_{\text{loc}}(\Omega; C^r)\), \(\omega_n \to 0^+,\) and let for \(m \in \mathbb{N}\)

\[
\sum_{|\alpha| \leq m} \omega_n|\alpha| \partial_{\alpha}(A^\alpha u_n) \to 0 \quad \text{strongly in } L^2_{\text{loc}}(\Omega; C^q),
\]

where \(A^\alpha \in C^\infty(\Omega; M_{q \times r}(C))\).

Then for the associated semiclassical measure \(\mu_{sc} = \mu_{sc}(\omega_n')\) we have

\[
p_{sc}\mu_{sc}^\top = 0,
\]

where

\[
p_{sc}(x, \xi) := \sum_{|\alpha| \leq m} (2\pi i \xi)^\alpha A^\alpha(x).
\]

Analogously to the result for H-measures, this theorem constrains the support of \(\mu_{sc}\) within the set of points where the symbol \(p_{sc}(x, \xi)\) has no left inverse.

**Relation between H-measures and semiclassical measures**

In order to compare the two types of objects, let us consider an elementary example where both the H-measure and the semiclassical measure can be explicitly calculated (cf. [27, Lemma 32.2]).

**Example 1. (oscillation: one characteristic length)** Let \(\alpha > 0, k \in \mathbb{Z}^d \setminus \{0\},\) and consider

\[
u_n(x) := e^{2\pi i \alpha^\top k x} \to 0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^d).
\]

The sequence \((\nu_n)\) is pure and the associated H-measure is

\[
\mu_H = \lambda \boxtimes \delta_{\frac{k}{\alpha}}.
\]

Thus the H-measure contains important information on the direction of oscillation \(k\).

Furthermore, the sequence \((\nu_n)\) is \((\omega_n)-\)pure for any characteristic length \((\omega_n),\) and the corresponding semiclassical measure depends on the characteristic length:

\[
\mu_{sc}(\omega_n) = \lambda \boxtimes \begin{cases}
\delta_0, & \lim_n n^{\alpha} \omega_n = 0 \\
\delta_{ck}, & \lim_n n^{\alpha} \omega_n = c \in (0, \infty) \\
0, & \lim_n n^{\alpha} \omega_n = \infty
\end{cases}.
\]

Only in the case \(\lim_n n^{\alpha} \omega_n \in (0, \infty),\) where the characteristic length of semiclassical measure is of the same order as the wavelength of the observed sequence, the corresponding semiclassical measure contains information on the direction of oscillation. However, contrary to the H-measure, which cannot comprehend the frequency, for every \((\omega_n)\) the semiclassical measure contains information on the frequency of oscillation.

In the previous example one can notice that, in the case \(\lim_n n^{\alpha} \omega_n \in (0, \infty),\) the H-measure can be derived from the corresponding semiclassical measure by taking the projection to the unit sphere in the Fourier space. This can be justified by the well known result on the relation of the above measures [10, Proposition 4], [13, Lemma 3.4].
Theorem 5. Let \( u_n \rightharpoonup u \) in \( L^2_{\text{loc}}(\Omega; C^r) \), \( \omega_n \to 0^+ \), and let \( (u_n) \) be \((\omega_n)\)-pure with the corresponding semiclassical measure \( \mu_{sc} = \mu_{sc}(\omega_n) \).

If \( \text{tr} \mu_{sc}(\Omega \times \{0\}) = 0 \), then \( u = 0 \). Furthermore, if we additionally assume that \( (u_n) \) is \((\omega_n)\)-oscillatory, then the sequence \( (u_n) \) is pure (in the sense of H-measures), and for any \( \varphi \in C_c^\infty(\Omega) \) and \( \psi \in C^\infty(S^{d-1}) \) one has:

\[
\langle \mu_H, \varphi \otimes \psi \rangle = \langle \mu_{sc}, \varphi \otimes (\psi \circ \pi) \rangle,
\]

where \( \mu_H \) is the H-measure associated to \( (u_n) \).

Going back to Example 1, it can be checked (either by straightforward calculation, or by using the interpretation of the \((\omega_n)\)-oscillatory property) that the sequence \( (u_n) \) is \((\omega_n)\)-oscillatory in the case \( \lim_n n^\alpha \omega_n = 0 \). Furthermore, with the exception of the case when this limit is zero, the set \( \mathbb{R}^d \times \{0\} \) is of zero measure with respect to the corresponding semiclassical measure as well. Hence, by the previous theorem, the comment after Example 1 is justified. Moreover, we can see that the previous theorem is sharp, since in other two cases the H-measure cannot be derived from the semiclassical measure.

Although one can think, after the above arguments, that semiclassical measures are more general objects than H-measures (taking into account all possible characteristic lengths), this is not the case. Indeed, it is not hard to construct an example in which (for any characteristic length) one of the assumptions of the previous theorem is not satisfied.

Example 2. (oscillation: two characteristic lengths) Let \( 0 < \alpha < \beta \), while \( k, s \in \mathbb{Z}^d \setminus \{0\} \), and define two weakly convergent sequences in \( L^2_{\text{loc}}(\mathbb{R}^d) \):

\[
\begin{align*}
  u_n(x) &:= e^{2\pi in^\alpha k \cdot x} \rightharpoonup 0, \\
  v_n(x) &:= e^{2\pi in^\beta s \cdot x} \rightharpoonup 0.
\end{align*}
\]

We are interested in the measures associated to the sequence \((u_n + v_n)\), which is pure and \((\omega_n)\)-pure for any characteristic length \((\omega_n)\).

The corresponding measures \( \mu_H \) and \( \mu_{sc}(\omega_n) \) are given by:

\[
\begin{align*}
\mu_H &= \lambda \otimes \left( \frac{\delta_0}{|n|} + \frac{\delta_\infty}{|n|} \right), \\
\mu_{sc}(\omega_n) &= \lambda \otimes \begin{cases} 
2\delta_0, & \lim_n n^\beta \omega_n = 0 \\
(\delta_s + \delta_0), & \lim_n n^\beta \omega_n = c \in (0, \infty) \\
\delta_0, & \lim_n n^\beta \omega_n = \infty \& \lim_n n^\alpha \omega_n = 0 \\
\delta_{ck}, & \lim_n n^\alpha \omega_n = c \in (0, \infty) \\
0, & \lim_n n^\alpha \omega_n = \infty
\end{cases}
\end{align*}
\]

In neither of these cases the assumptions of the last theorem are satisfied, and one can not derive the H-measure from the semiclassical measure.

To conclude, we can say that semiclassical measures lose a part of information for some (or even all) characteristic lengths, while H-measures cannot catch frequency of an observed sequence. This was the motivation for Tartar to introduce a new object, the one-scale H-measure, that overcomes the shortcomings of the preceding tools, and which we study in the next sections.

3. Tartar’s one-scale H-measures

In the recent book [27, Chapter 32] Luc Tartar introduced improved objects, having the advantages of both H-measures and semiclassical measures, which he called H-measure variants with a characteristic length there. However, in line with [28], we call them one-scale H-measures in this paper.
Test functions

While the theorems on existence of both H-measures and semiclassical measures have the same form, the crucial difference is in the choice of test functions in the Fourier space. In the former case functions from $C(S^{d-1})$ are chosen (in fact, the continuous functions on $\mathbb{R}^d := \mathbb{R}^d \setminus \{0\}$ being homogeneous of order zero), while in the latter functions from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ are being used.

For the new variant, test functions are taken to be continuous functions on a suitable compactification of $\mathbb{R}^d_{\ast}$ in the Fourier space. More precisely, following Tartar, we denote by $K_{0,\infty}(\mathbb{R}^d)$ the compactification of $\mathbb{R}^d_{\ast}$ obtained by placing two unit spheres, $\Sigma_0$ at the origin, and $\Sigma_{\infty}$ at infinity, equipped with the topology making it homeomorphic to the $d$-dimensional spherical shell, as depicted on Figure 1. The points on these two spheres we denote by $0_{e}$ and $\infty_{e}$, respectively, where $e \in S^{d-1}$ is the direction of the point (i.e. $0_{e}$ and $\infty_{e}$ are the endpoints of a ray passing through $e$). In the plane, $K_{0,\infty}(\mathbb{R}^2)$ is homeomorphic to the closed annulus.

![Figure 1. $K_{0,\infty}(\mathbb{R}^d)$, the compactification of $\mathbb{R}^d_{\ast}$.](image)

As we have already seen in two previous examples, the semiclassical measures have a disadvantage of losing information at infinity and mixing various pieces of information at the origin, which motivated Tartar to introduce this compactification. The idea is to capture lost data on $\Sigma_0$ and $\Sigma_{\infty}$ (cf. Example 4 below).

For the sake of simplicity, we shall use the same notation for functions $\psi$ on $K_{0,\infty}(\mathbb{R}^d)$ and their corresponding pullbacks $\sigma^\ast \psi = \psi \circ \sigma$, where $\sigma$ is the embedding of $\mathbb{R}^d_{\ast}$ into $K_{0,\infty}(\mathbb{R}^d)$, while from the context it will be clear which object is used.

According to the previous identification, the space $C(K_{0,\infty}(\mathbb{R}^d))$ can be interpreted as consisting of all continuous functions $\psi$ on $\mathbb{R}^d_{\ast}$ such that there are functions $\psi_0, \psi_{\infty} \in C(S^{d-1})$ satisfying

$$\psi(\xi) - \psi_0\left(\frac{\xi}{|\xi|}\right) \to 0, \quad |\xi| \to 0,$$

$$\psi(\xi) - \psi_{\infty}\left(\frac{\xi}{|\xi|}\right) \to 0, \quad |\xi| \to \infty.$$  (3.1)

Furthermore, for an arbitrary $e \in S^{d-1}$ we have $\psi_0(e) = \psi(0_{e})$ and $\psi_{\infty}(e) = \psi(\infty_{e})$. This notation will be kept throughout the paper: for an arbitrary function $\psi$ from $C(K_{0,\infty}(\mathbb{R}^d))$, with indices $0$ ($\psi_0$) and $\infty$ ($\psi_{\infty}$), we shall denote functions with the above properties. Equipped by the $L^\infty$ norm, $C(K_{0,\infty}(\mathbb{R}^d))$ becomes a separable Banach space, even a Banach algebra with usual multiplication of functions.

For $\psi \in C(K_{0,\infty}(\mathbb{R}^d))$, $\psi - \psi_0 \circ \pi$ is constant (with zero value) on $\Sigma_0$, so it can be extended by continuity to the origin. Hence, we can say that $\psi - \psi_0 \circ \pi$ is contained in $C_{ab}(\mathbb{R}^d)$ (the space
of bounded uniformly continuous functions on $\mathbb{R}^d$), as both $\psi$ and $\psi_0 \circ \pi$ are bounded, while outside a sufficiently large compact set the difference can be approximated by $(\psi_\infty - \psi_0) \circ \pi$. Moreover, if $\psi_0$ is constant than we immediately have that $\psi \in C_{\text{ub}}(\mathbb{R}^d)$.

The space $C_0(\mathbb{R}^d)$ is continuously embedded in the space $C(K_{0,\infty}(\mathbb{R}^d))$. Indeed, a function $\psi \in C_0(\mathbb{R}^d)$ is continuous at the origin and tends to zero at infinity, so for constants $\psi_0 \equiv \psi(0)$ and $\psi_\infty \equiv 0$, $\psi$ satisfies the above conditions. It is even simpler to prove that $\pi^*(C(S^{d-1})) := \{\psi \circ \pi : \psi \in C(S^{d-1})\} \hookrightarrow C(K_{0,\infty}(\mathbb{R}^d))$, by taking $\psi_0 := \psi$ and $\psi_\infty := \psi$. This discussion we summarise in the form of a lemma.

Lemma 1.

\begin{enumerate}
  \item $C_0(\mathbb{R}^d) \hookrightarrow C(K_{0,\infty}(\mathbb{R}^d))$, and
  \item $\{\psi \circ \pi : \psi \in C(S^{d-1})\} \hookrightarrow C(K_{0,\infty}(\mathbb{R}^d))$.
\end{enumerate}

As $S(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d)$, it follows that the space $C(K_{0,\infty}(\mathbb{R}^d))$ subsumes the spaces of test functions for both $H$-measures and semiclassical measures.

An example of a function from $C(K_{0,\infty}(\mathbb{R}^d))$ which is not entirely contained in Lemma 1 and which will be important in the next section when discussing the localisation principle for one-scale $H$-measures, is given in the next example.

Example 3. Let $l, m \in \mathbb{N}$ and define $\psi^\alpha(\xi) := \frac{\xi^\alpha}{|\xi|}$ for $\alpha \in \mathbb{N}^d$, $l \leq |\alpha| \leq m$. The cases $|\alpha| = l$ and $|\alpha| = m$ are of particular interest, as they are not necessarily contained in the above lemma. Let us show that $\psi^\alpha$ is (a pullback of a function) in $C(K_{0,\infty}(\mathbb{R}^d))$. It is obvious that $\psi^\alpha$ is continuous on $\mathbb{R}^d$. For $l = m$, $\psi^\alpha$ is homogeneous of order zero, so Lemma 1 gives the statement. On the other hand, for $l \leq m - 1$ we need to make an analysis in dependence on $\alpha$. For $|\alpha| = l$ we have $\psi^\alpha_0(\xi) = \frac{\xi^\alpha}{|\xi|}$, while in the case $|\alpha| \geq l + 1$, $\psi^\alpha$ tends to zero at the origin, which implies $\psi^\alpha_0 \equiv 0$. At $\Sigma_\infty$ we have an opposite situation. For $|\alpha| \leq m - 1$, $\psi^\alpha$ tends to zero at infinity, which implies $\psi^\alpha_\infty \equiv 0$, but $\psi^\alpha_\infty(\xi) = \frac{\xi^\alpha}{|\xi|^\alpha}$ for $|\alpha| = m$.

Preliminary lemmata

The existence result for one-scale $H$-measures, resembling the one for $H$-measures, relies on a variant of the First commutation lemma for which we introduce two basic types of operators. For $\psi \in C_{\text{ub}}(\mathbb{R}^d)$, as well as for $\psi \in C(K_{0,\infty}(\mathbb{R}^d))$, one can define the Fourier multiplier operator:

$$A_\psi : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d), \quad A_\psi u := \hat{(\psi u)}^\vee.$$  

The definition is justified, as in both cases $\psi$ is in $L^\infty(\mathbb{R}^d)$ (in the latter case, we assume that $\psi$ is first restricted to $\mathbb{R}^d$, and then interpreted as an $L^\infty$ function). Also, for $\varphi \in L^\infty(\mathbb{R}^d)$ by

$$B_\varphi : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d), \quad B_\varphi u := \varphi u,$$

we denote the operator of multiplication by $\varphi$. In both cases, the above operators are bounded on $L^2(\mathbb{R}^d)$, with the norm equal to the $L^\infty$ norm of $\psi$, respectively $\varphi$.

In [27, Lemma 32.4] the following result is given:

Lemma 2. Let $\psi \in C_{\text{ub}}(\mathbb{R}^d)$, $\varphi \in C_0(\mathbb{R}^d)$, $\omega_n \rightarrow 0^+$, and denote $\psi_n(\xi) := \psi(\omega_n \xi)$. Then the commutator $C_n := [B_\varphi, A_\psi] = B_\varphi A_\psi - A_\psi B_\varphi$ tends to zero in the operator norm on $\mathcal{L}(L^2(\mathbb{R}^d))$.

Remark 3. Lemma 2 can be used to prove the known result for semiclassical measures given in Theorem 3 not only for (infinitely) smooth, but merely for continuous test functions both in the physical and Fourier space [cf. 28].

In comparison to the classical First commutation lemma, where the commutator is only a compact operator, here we have the convergence in the uniform (operator) norm to zero. This would allow us to weaken the requirement that $(u_n)$ weakly converges to zero, allowing any limit if test functions were taken from $C_{\text{ub}}(\mathbb{R}^d)$ in the Fourier space. Unfortunately, functions from $C(K_{0,\infty}(\mathbb{R}^d))$ are not continuous at the origin, therefore not in $C_{\text{ub}}(\mathbb{R}^d)$, so we cannot avoid the additional assumption that the weak limit of the observed sequence is zero. For that reason the following variant of the previous lemma will be used in the definition of one-scale $H$-measures.
Lemma 3. Let \( \psi \in C(K_{0,\infty}(\mathbb{R}^d)) \), \( \varphi \in C_0(\mathbb{R}^d) \), \( \omega_n \to 0^+ \), and denote \( \psi_n(\xi) := \psi(\omega_n \xi) \). Then the commutator can be expressed as a sum

\[
C_n := [B_\varphi, A_{\psi_n}] = \tilde{C}_n + K,
\]

where \( K \) is a compact operator on \( L^2(\mathbb{R}^d) \), while \( \tilde{C}_n \to 0 \) in the operator norm on \( L(L^2(\mathbb{R}^d)) \).

Dem. Since

\[
A_{\psi_n} = A_{\psi_n - \psi_0 \circ \pi} + A_{\psi_0 \circ \pi},
\]

we have that \( C_n = [B_\varphi, A_{\psi_n - \psi_0 \circ \pi}] + [B_\varphi, A_{\psi_0 \circ \pi}] =: \tilde{C}_n + K \). As \( \psi - \psi_0 \circ \pi \in C_{ub}(\mathbb{R}^d) \), we can apply Lemma 2 to this term in order to show \( \tilde{C}_n \to 0 \), while \( K = [B_\varphi, A_{\psi_0 \circ \pi}] \) is compact by the classical First commutation lemma [26, Lemma 1.7].

Q.E.D.

In [27, Chapter 32] and [28] one can find the proof of existence of one-scale H-measures (albeit only for the weakly converging sequences in \( L^2 \), not in the local space), starting from the corresponding H-measure in one dimension more. In the same references one can find the idea of an alternative proof based on Tartar’s original proof [26], which uses the construction with Hilbert-Schmidt operators [26, Lemma 1.10]. As we need several modifications, we shall refrain from stating that the construction can be done along the same lines, and try to outline all the necessary steps.

Another venue for the proof could be by using the Schwartz kernel theorem, but this would require introduction of a differential structure on \( K_{0,\infty}(\mathbb{R}^d) \), and the space of distributions on it, which we choose to avoid here. However, it could be said that we are providing a version of the kernel theorem in this special case.

Lemma 4. Let \( X \) and \( Y \) be open unit balls in Euclidean spaces (of dimension \( d \) and \( d' \)), and let \( B \) be a non-negative continuous bilinear form on \( C_c(X) \times C_c(Y) \). Then there exists a Radon measure \( \mu \in \mathcal{M}(X \times Y) \) such that

\[
(\forall f \in C_c(X))(\forall g \in C_c(Y)) \quad B(f, g) = \langle \mu, f \overset{\circ}{\otimes} g \rangle.
\]

Furthermore, the above remains valid if we replace \( C_c \) by \( C_0 \), and \( \mathcal{M} \) by \( \mathcal{M}_b \) (the space of bounded Radon measures, i.e. the dual of Banach space \( C_0 \)).

Dem. I. Friedrichs mollifiers and nested balls

Following Tartar’s ideas, we take standard Friedrichs mollifiers

\[
(M_n f)(x) := \int_{\mathbb{R}^d} m_n(x, x') f(x') \, dx',
\]

where the kernel \( m_n(x, x') := \rho_n(x - x') \), while \( \rho_n(x) := n^d \rho(nx) \) and

\[
\rho(x) := C_{\chi_{K(0,1)}}(x) e^{-\frac{1}{1-|x|^2}},
\]

with \( C \) chosen such that \( \int \rho = \int \rho_n = 1 \) (\( C \) depends on dimension \( d \)).

These kernels \( m_n \) are non-negative, continuous, supported in \( \{(x, x') \in X \times X : |x - x'| \leq 1/n \} \), and smoothing; in particular \( M_n \) maps Lebesgue functions into smooth ones (recall that we identify functions defined on subsets of \( \mathbb{R}^d \) to their continuation by zero to \( \mathbb{R}^d \)).

For an open unit ball \( X = K(0,1) \) in \( \mathbb{R}^d \) we have an increasing sequence of compacts \( K_m \) contained in it, such that their union is the whole ball, while \( K_m \subseteq \text{int} \, K_{m+1} \). For definiteness, let us take closed balls \( K_m := K[0,1-1/m] \). Recall that \( C_c(X) \) is a strict inductive limit of Banach spaces \( C_{K_m}(X) := \{ \varphi \in C_c(X) : \text{supp} \varphi \subseteq K_m \} \).

Of course, we can repeat the same construction on \( Y \), obtaining operators \( N_n \) and kernels \( n_n \), while the corresponding sequence of compacts we denote by \( L_m \) (we also write \( \rho_n', C' \) instead of \( \rho_n, C \)).
The continuity of $B$, as spaces $C_c$ are strict inductive limits, can be expressed by the continuity of restrictions, i.e. by
\begin{equation}
(3.2) \quad (\forall m \in \mathbb{N})(\exists C_m > 0)(\forall f \in C_{K_m}(X))(\forall g \in C_{L_m}(Y)) \quad |B(f, g)| \leq C_m \|f\|_{L^\infty(K_m)} \|g\|_{L^\infty(L_m)}.
\end{equation}

II. Construction of approximative Hilbert-Schmidt kernels

Let us define a bilinear functional $B^m_n : L^2(K_m) \times L^2(L_m) \rightarrow \mathbb{C}$ by the relation $B^m_n(f, g) := B(M_n f, N_n g)$. Note that for $n > m(m+1)$ the function $M_n f$ belongs to $C_{K_m}(X)$, and similarly for $N_n g$. Furthermore, we have the following estimate (by $\omega_d$ we denote the volume of $d$-dimensional unit ball):
\[
|B^m_n(f, g)| = |B(M_n f, N_n g)| \\
\leq C_{m+1} \|M_n f\|_{L^\infty(K_{m+1})} \|N_n g\|_{L^\infty(L_{m+1})} \\
\leq C_{m+1} \|\rho_n\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(K_m)} \|\rho_n\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(L_m)} \\
\leq C_{m+1} C C' e^{-2 \sqrt{\omega_d d^d n^{-\frac{d+2}{2}}} \|f\|_{L^2(K_m)} \|g\|_{L^2(L_m)}},
\]
so for any (large enough) $n$ we have that $B^m_n$ is a continuous bilinear form on $L^2(K_m) \times L^2(L_m)$. Therefore, it can be represented by an operator $T^m_n \in L(L^2(K_m); L^2(L_m))$ in the sense that
\[
\langle T^m_n f, g \rangle = B^m_n(f, g) = B(M_n f, N_n g).
\]

Of course, $T^m_n$ is not uniformly bounded with respect to $n$.

The operators $M_n$ and $N_n$ are Hilbert-Schmidt operators, with corresponding kernels $m_n$ and $n_n$ (for standard material on Hilbert-Schmidt operators the reader might consult [7, Chapter 12]). If we restrict operator $M_n$ to $L^2(K_m)$, in the sense that we also restrict the resulting function to $K_m$, we again obtain a Hilbert-Schmidt operator $M^m_n$, with kernel $k^m_n$ equal to $m_n$ restricted to $K_m \times K_m$. As $T^m_n$ is a bounded operator, the composition $T^m_n M^m_n$ is again a Hilbert-Schmidt operator, but from $L^2(K_m)$ to $L^2(L_m)$, with a kernel which we denote by $k^m_n \in L^2(K_m \times L_m)$.

III. Positivity and boundedness of approximative Hilbert-Schmidt kernels

Since $M_n$ and $N_n$ are positive operators, for non-negative functions $f$ and $g$ we have that
\[
\langle T^m_n f, g \rangle = B(M_n f, N_n g) \geq 0,
\]
so $T^m_n$ is a positive operator, thus $T^m_n M^m_n$ as well, which implies that its kernel $k^m_n \geq 0$.

On the other hand, for constants 1 on compacts $K_m$ and $L_m$ we have:
\[
0 \leq \langle T^m_n M^m_n \chi_{K_m}, \chi_{L_m} \rangle = B(M_n M^m_n \chi_{K_m}, N_n \chi_{L_m}) \\
= |B(M_n M^m_n \chi_{K_m}, N_n \chi_{L_m})| \\
\leq C_{m+1} \|M_n M^m_n \chi_{K_m}\|_{L^\infty(X)} \|N_n \chi_{L_m}\|_{L^\infty(Y)} \leq C_{m+1}.
\]

Note that $M^m_n \chi_{K_m}$ is identical to 1 on $K_{m-1}$, while supported in $K_m$. Further, $M_n M^m_n \chi_{K_m}$ is identical to 1 on $K_{m-2}$ (here we need to take $m \geq 4$), and supported in $K_{m+1}$. For $N_n \chi_{L_m}$ we similarly have that it is equal to 1 on $L_{m-1}$, while it is supported in $L_{m+1}$. Therefore, the $L^\infty$ norms in the above inequality are both equal to 1.

In particular, from the above we can conclude that $\|k^m_n\|_{L^1(K_m \times L_m)} \leq C_{m+1}$. It might be of interest to notice that positivity was crucial in obtaining the boundedness.

IV. Passage to the limit

By the continuity of bilinear form, for $m$ large enough we get
\begin{equation}
(3.3) \quad B(f, g) = \lim_n B(M_n M^m_n f, N_n g) = \lim_n \langle T^m_n M^m_n f, g \rangle = \lim_n \langle k^m_n, f \otimes g \rangle.
\end{equation}

Indeed, take $f \in C_c(X)$ and $g \in C_c(Y)$. Naturally, for some large enough $m$, supp $f \subseteq K_{m-1}$ and supp $g \subseteq L_{m-1}$, so we have $f \in C_{K_{m-1}}(X)$ and $g \in C_{L_{m-1}}(Y)$. By standard properties of
mollifiers, the latter immediately implies that \( N_n g \rightarrow g \) in the space \( C_{L_m}(Y) \). For the former, we first notice that \( M_n^m f \rightarrow f \) in \( C_{K_m}(X) \), thus \( h_n := M_n^m f - f \rightarrow 0 \) in the same space. Therefore
\[
|(M_n h_n)(x)| \leq \int_{K_m} \rho_n(x - y)|h_n(y)|dy < \varepsilon \int_{K_m} \rho_n(x - y)dy \leq \varepsilon,
\]
for \( n \geq n_0 \) such that \( \sup_{x \in X}|h_n(x)| < \varepsilon \), which furnishes the proof of (3.3).

As the sequence \((k_n^m)_{n \in \mathbb{N}}\) is bounded in \( L^1(K_{m_1} \times L_{m_1}) \) by \( C_{m+1} \) (actually, here we consider the restrictions of \( k_n^m \) to \( K_{m_1} \times L_{m_1} \)) without unnecessarily reflecting that in the notation; this restriction is needed in order to use (3.3)), it is weakly \(* \) precompact in the space of bounded Radon measures. Since any accumulation point satisfies (3.3), they all coincide on tensor products, so by density also on \( C(K_{m_1} \times L_{m_1}) \). Therefore the accumulation point is unique, and the whole sequence \( k_n^m \) converges to a non-negative Radon measure; let us denote it by \( \mu^m \).

Clearly, by (3.3) again, we have that \( \mu^{m+1} \) and \( \mu^m \) coincide on \( C(K_{m_1} \times L_{m_1}) \), so for any \( \Phi \in C_c(X \times Y) \) we can define \( \mu \) by
\[
\langle \mu, \Phi \rangle := \langle \mu^m, \Phi \rangle,
\]
for any \( m \) such that \( \text{supp } \Phi \subseteq K_{m_1} \times L_{m_1} \), and the definition is good, thus obtaining a non-negative unbounded Radon measure on \( X \times Y \). This measure \( \mu \) is indeed a representation of bilinear functional \( B \).

Furthermore, if \( B \) is a non-negative continuous bilinear form on \( C_0(X) \times C_0(Y) \), then the sequence of kernels \((k_n^m)\) is uniformly bounded in the space \( L^1(K_m \times L_m) \) both in \( n \) and \( m \) (as in this case one constant \( C_0 \) is good for any \( m \) in (3.2)), thus \( (\mu^m) \) is also uniformly bounded in \( m \) which shows that \( \mu \) is a bounded Radon measure.

Q.E.D.

In fact, we want to apply the above lemma to a bilinear form on \( C_c(\Omega) \times C(K_{0,\infty}(\mathbb{R}^d)) \). To this end, we can consider both \( \Omega \) and \( K_{0,\infty}(\mathbb{R}^d) \) as (nice) topological manifolds (for standard material on manifolds the reader might consult [18, Chapter XXII]), which can be covered by a countable atlas (even finite for the compact \( K_{0,\infty}(\mathbb{R}^d) \)), with the image of charts being the unit balls in the Euclidean space (for the \( K_{0,\infty}(\mathbb{R}^d) \), which is a manifold with boundary, some charts have unit semiballs \( K^+(0,1) := \{ x \in K(0,1) : x^d \geq 0 \} \) as images). The coordinate patches on both \( \Omega \) and \( K_{0,\infty}(\mathbb{R}^d) \) can be chosen in such a way that any compact set contained in the manifold intersects only a finite number of patches. In such an open cover we can inscribe a partition of unity, thus reducing the problem to the local one, considered in the lemma.

However, for the manifold with boundary \( K_{0,\infty}(\mathbb{R}^d) \), some coordinate images might be semiballs \( K^+(0,1) \), for which we first introduce an extended bilinear form \( \bar{B} \) on \( C_c(K(0,1)) \times C_c(K(0,1)) \) by \( \bar{B}(f,g) := B(f,g|_{K^+(0,1)}) \), and then apply Lemma 4 obtaining measure \( \bar{\mu} \) on \( K(0,1) \times K(0,1) \). Then we define
\[
\langle \mu, \Phi \rangle := \langle \bar{\mu}, \bar{\Phi} \rangle,
\]
where \( \bar{\Phi} \) is an extension of \( \Phi \) by reflection to the lower semiball in the second argument.

The above discussion, which can easily be expressed in terms of general manifolds \( X \) and \( Y \) instead of \( \Omega \) and \( K_{0,\infty}(\mathbb{R}^d) \), we summarise as a lemma.

**Lemma 5.** Let \( X \) and \( Y \) be two Hausdorff second countable topological manifolds (with or without a boundary), and let \( B \) be a non-negative continuous bilinear form on \( C_c(X) \times C_c(Y) \). Then the conclusions of Lemma 4 hold.

The existence

Similarly as we have done in the proof of Lemma 4 (for open unit ball \( X \)), let us denote by \( (K_m) \) a sequence of compacts in \( \Omega \) which exhaust \( \Omega \); more precisely, such that \( K_m \subseteq \text{Int } K_{m+1} \) and \( \bigcup_m K_m = \Omega \), and recall that \( C_c(\Omega) \) is a strict inductive limit of spaces \( C_{K_m}(\Omega) \). Moreover, note that each \( C_{K_m}(\Omega) \) (with appropriate restriction to \( K_m \)) is a closed subspace of separable Banach space \( C(K_m) \), therefore separable itself.
Theorem 6. (existence of one-scale H-measures) If \( u_n \to 0 \) in \( L^2_{\text{loc}}(\Omega; C^r) \) and \( \omega_n \to 0^+ \), then there exists a subsequence \((u_{n'})\) and an \( r \times r \) hermitian non-negative matrix Radon measure \( \mu_{K_{0,\infty}}^{(\omega_{n'})} \) on \( \Omega \times K_{0,\infty}(\mathbb{R}^d) \) such that for any \( \varphi_1, \varphi_2 \in C_c(\Omega) \) and \( \psi \in C(K_{0,\infty}(\mathbb{R}^d)) \) one has:

\[
\lim_{n'} \int_{\mathbb{R}^d} \left( \varphi_1 u_{n'}(\xi) \otimes \varphi_2 u_{n'}(\xi) \right) \psi(\omega_{n'} \xi) \, d\xi = \left( \mu_{K_{0,\infty}}^{(\omega_{n'})}, (\varphi_1 \varphi_2) \otimes \psi \right)_{\Omega \times K_{0,\infty}(\mathbb{R}^d)}
\]

The above measure \( \mu_{K_{0,\infty}}^{(\omega_{n'})} \) is called the one-scale H-measure (with characteristic length \( (\omega_{n'}) \)) associated to the (sub)sequence \((u_{n'})\).

Dem. For simplicity, we shall omit writing explicitly the characteristic length of the one-scale H-measure in the proof.

For any test functions \( \varphi_1 \) and \( \varphi_2 \) supported in a \( K_m \), and \( \psi \in C(K_{0,\infty}(\mathbb{R}^d)) \), by the Plancherel formula the following sequence of integrals is bounded:

\[
\left| \int_{\mathbb{R}^d} \left( \varphi_1 u_{n}(\xi) \otimes \varphi_2 u_{n}(\xi) \right) \psi(\omega_{n} \xi) \, d\xi \right| \leq \left( \limsup_{n} \| u_n \|_{L^2(K_m; C^r)} \right) \| \varphi_1 \|_{L^\infty(K_m)} \| \varphi_2 \|_{L^\infty(K_m)} \| \psi \|_{L^\infty(\mathbb{R}^d)},
\]

hence we can pass to a subsequence converging to a limit satisfying the same bound.

This passage should be done in such a way that we obtain a subsequence \((u_{n'})\) good for any choice of test functions. The space \( C(K_{0,\infty}(\mathbb{R}^d)) \) is a separable Banach space, so we can consider only a countable dense set of \( \psi \in C(K_{0,\infty}(\mathbb{R}^d)) \), and use the Cantor diagonal procedure. However, for \( C_c(\Omega) \) we cannot apply the same idea, even though it is separable. We have to consider separable Banach spaces of continuous functions with \( L^\infty \) norm, and then use the fact that the original space is a strict inductive limit of such spaces.

Thus for each \( m \in \mathbb{N} \) we consider countable dense subsets of \( C_{K_m}(\Omega) \) and \( C(K_{0,\infty}(\mathbb{R}^d)) \), and choose a subsequence of integrals in (3.4) that converge to a limit denoted by \( L(\varphi_1, \varphi_2, \psi) \in M_{r \times r} \), for any test functions from the chosen countable sets, and then by continuity for any \( \varphi_1, \varphi_2 \in C_{K_m}(\Omega) \) and \( \psi \in C(K_{0,\infty}(\mathbb{R}^d)) \). The details in a similar setting could be found in the proof of [3, Theorem 6], so we omit them here.

After choosing a good subsequence for \( m \), we pass to a subsequence again for \( m + 1 \). Applying the Cantor diagonal procedure once more, finally we arrive at a subsequence, denoted by \( n' \), which is good for any choice of test functions \( \varphi_1, \varphi_2 \in C_c(\Omega) \) as well.

Furthermore, \( L(\varphi_1, \varphi_2, \psi) \) depends only on the product \( \varphi_1 \varphi_2 \) (not on both \( \varphi_1 \) and \( \varphi_2 \) independently). Indeed, according to Lemma 3 we have: \( (A_{\psi_n} B_{\varphi_2} - B_{\varphi_2} A_{\psi_n}) B_{\varphi_1} u_n \to 0 \) in \( L^2(\mathbb{R}^d; C^r) \), hence by the Plancherel formula we have

\[
\lim_{n'} \int_{\mathbb{R}^d} \left( \varphi_1 u_{n'}(\xi) \otimes \varphi_2 u_{n'}(\xi) \right) \psi(\omega_{n'} \xi) \, d\xi = \lim_{n'} \int_{\mathbb{R}^d} B_{\varphi_2} A_{\psi_{n'}} B_{\varphi_1} u_{n'} \otimes u_{n'} \, dx
\]

\[
= \lim_{n'} \int_{\mathbb{R}^d} A_{\psi_{n'}} B_{\varphi_2} B_{\varphi_1} u_{n'} \otimes u_{n'} \, dx
\]

\[
= \lim_{n'} \int_{\mathbb{R}^d} \left( (\varphi_1 \varphi_2 u_{n'}) (\xi) \otimes u_{n'}(\xi) \right) \psi(\omega_{n} \xi) \, d\xi,
\]

so we can define \( L(\varphi_1 \varphi_2, \psi) := L(\varphi_1, \varphi_2, \psi) \). In particular, if we take \( \varphi_2 \) to be equal to 1 on \( \text{supp} \varphi_1 \), by

\[
L(\varphi_1, \psi) = L(\varphi_1 \varphi_2, \psi)
\]
we define an \( r \times r \) matrix, its entries being continuous bilinear forms on \( C_c(\Omega) \times C(K_{0,\infty}(\mathbb{R}^d)) \).

For real functions \( \varphi \) and \( \psi \) we have that \( \mathcal{L}(\varphi, \psi) \) is a hermitian matrix. Indeed,

\[
\mathcal{L}(\varphi, \psi) = \mathcal{L}(\varphi_2, \psi)
\]

\[
= \lim_{n'} \int_{\mathbb{R}^d} (\varphi_2 u_{n'}(\xi) \otimes \varphi_2 u_{n'}(\xi)) \psi(\omega_{n'}\xi) d\xi
\]

\[
= \lim_{n'} \int_{\mathbb{R}^d} (\varphi_2 u_{n'}(\xi) \otimes \varphi_2 u_{n'}(\xi))^* \psi(\omega_{n'}\xi) d\xi
\]

\[
= \left( \lim_{n'} \int_{\mathbb{R}^d} (\varphi_2 u_{n'}(\xi) \otimes \varphi_2 u_{n'}(\xi)) \psi(\omega_{n'}\xi) d\xi \right)^* = \mathcal{L}(\varphi, \psi)^*,
\]

where \( \varphi_2 \) is again taken to be equal to 1 on the support of \( \varphi \). By the just proved hermitian property we have for \( \lambda_1, \lambda_2 \in \mathbb{C}^r \) and real \( \varphi \in C_c(\Omega) \) and \( \psi \in C(K_{0,\infty}(\mathbb{R}^d)) \) that

\[
2\text{Re}\left( \mathcal{L}(\varphi, \psi) \lambda_1 \cdot \lambda_2 \right) = \mathcal{L}(\varphi, \psi)(\lambda_1 + \lambda_2) \cdot (\lambda_1 + \lambda_2) - \mathcal{L}(\varphi, \psi) \lambda_1 \cdot \lambda_1 - \mathcal{L}(\varphi, \psi) \lambda_2 \cdot \lambda_2,
\]

\[
2\text{Im}\left( \mathcal{L}(\varphi, \psi) \lambda_1 \cdot \lambda_2 \right) = \mathcal{L}(\varphi, \psi)(\lambda_1 + i\lambda_2) \cdot (\lambda_1 + i\lambda_2) - \mathcal{L}(\varphi, \psi) \lambda_1 \cdot \lambda_1 - \mathcal{L}(\varphi, \psi) \lambda_2 \cdot \lambda_2.
\]

Moreover, for \( \varphi, \psi \geq 0 \) and \( \lambda \in \mathbb{C}^r \) we have

\[
\mathcal{L}(\varphi, \psi) \lambda \cdot \lambda = \mathcal{L}(\sqrt{\varphi} \sqrt{\varphi}, \psi) \lambda \cdot \lambda = \lim_{n'} \int_{\mathbb{R}^d} |\sqrt{\varphi} u_{n'}(\xi) \cdot \lambda|^2 \psi(\omega_{n'}\xi) d\xi \geq 0.
\]

Therefore, for any \( \lambda \in \mathbb{C}^r \) \( \mathcal{L}(\cdot, \cdot) \lambda \cdot \lambda \) is a non-negative continuous bilinear form on \( C_c(\Omega) \times C(K_{0,\infty}(\mathbb{R}^d)) \). Hence, by Lemma 5 for any \( \lambda \in \mathbb{C}^r \) there exists \( \nu_\lambda \in \mathcal{M}(\Omega \times K_{0,\infty}(\mathbb{R}^d)) \) such that

\[
(\forall \varphi \in C_c(\Omega)) (\forall \psi \in C(K_{0,\infty}(\mathbb{R}^d))) \quad \mathcal{L}(\varphi, \psi) \lambda \cdot \lambda = \langle \nu_\lambda, \varphi \otimes \psi \rangle.
\]

Finally, let us define a matrix Radon measure \( \mu_{K_{0,\infty}} = [\mu_{K_{0,\infty}}^{ij}] \) by

\[
\text{Re} \mu_{K_{0,\infty}}^{ij} := \frac{1}{2}(\nu_{e_j + e_i} - \nu_{e_j} - \nu_{e_i});
\]

\[
\text{Im} \mu_{K_{0,\infty}}^{ij} := \frac{1}{2}(\nu_{e_j + ie_i} - \nu_{e_j} - \nu_{e_i});
\]

where vectors \( e_i, i \in 1..r \), form a canonical basis of \( \mathbb{C}^r \). By (3.5) we have that \( \mathcal{L} \) and \( \mu_{K_{0,\infty}} \) coincide on real functions, and by linearity we have the same for arbitrary \( \varphi \in C_c(\Omega) \) and \( \psi \in C(K_{0,\infty}(\mathbb{R}^d)) \). In addition, \( \mu_{K_{0,\infty}} \) inherits the hermitian and positivity property of \( \mathcal{L} \).

Q.E.D.

As with H-measures before, when there is no fear of ambiguity, we assume that we have already passed to a subsequence determining an one-scale H-measure, and reduce the notation to \( \mu_{K_{0,\infty}} = \mu_{K_{0,\infty}}^{(\omega_n)} \). When the whole sequence admits an one-scale H-measure with characteristic length \( (\omega_n) \) (i.e. the definition is valid without passing to a subsequence), we say that the sequence is \( (\omega_n) \)-pure (in terms of one-scale H-measures). For the sake of simplicity, here we use the same notion as in the case with semiclassical measures, while in the situations where there will not be clear which definition is used, additional explanation will be given. Let us just emphasise that from Corollary 6 it will be clear that the property of a sequence being \( (\omega_n) \)-pure in terms of one-scale H-measures implies both that the sequence is pure and \( (\omega_n) \)-pure in terms of semiclassical measures.
First properties

An immediate consequence of the definition is the following simple localisation property.

**Corollary 1.** Let $\mu_{K_{0,\infty}}$ be a one-scale H-measure with an arbitrary characteristic length, determined by the sequence $(u_n)$. If all components $u'_n = u_n \cdot e_i$ have their supports in closed sets $K_i \subseteq \Omega$ (respectively), then the support of $\mu_{K_{0,\infty}}^{ij} = \mu_{K_{0,\infty}} e_j \cdot e_i$ is necessary contained in $(K_i \cap K_j) \times K_{0,\infty}(\mathbb{R}^d)$.

**Dem.** For $\varphi_1 \in C_c(\Omega)$ such that $\text{supp} \varphi_1 \cap (K_i) \neq \emptyset$, we take $\varphi_2 \in C_c(\Omega)$ being equal to 1 on $K_i$ and vanishing on $K_i \cap K_j$. Since both $\varphi_1 u'_n$ and $\varphi_2 u'_n$ are zero, we have

$$\langle \mu_{K_{0,\infty}}^{ij}, \varphi_1 \boxtimes \varphi \rangle = \lim_{n} \int_{\mathbb{R}^d} \varphi_1 u'_n(\xi) \varphi_2 u'_n(\xi) \psi(\omega_n \xi) \, d\xi = 0,$$

where $\psi \in C(K_{0,\infty}(\mathbb{R}^d))$ is an arbitrary test function and $(\omega_n)$ is the characteristic length of the measure. Thus we get the claim.

Q.E.D.

The more precise relation between supports of diagonal and non-diagonal elements is a consequence of the Cauchy-Schwarz-Bunjakovskij inequality.

**Corollary 2.** Let $\mu_{K_{0,\infty}}$ be a one-scale H-measure with an arbitrary characteristic length, determined by the sequence $(u_n)$. The support of $\mu_{K_{0,\infty}}^{ij}$ is contained in the intersection of supports of the corresponding diagonal elements $\mu_{K_{0,\infty}}^{ii}$ and $\mu_{K_{0,\infty}}^{jj}$, i.e.

$$\text{supp } \mu_{K_{0,\infty}}^{ij} \subseteq \text{supp } \mu_{K_{0,\infty}}^{ii} \cap \text{supp } \mu_{K_{0,\infty}}^{jj}.$$

**Dem.** Let $\varphi_1, \varphi_2 \in C_c(\Omega)$ and $\psi \in C(K_{0,\infty}(\mathbb{R}^d))$. By the definition of one-scale H-measures and the Cauchy-Schwarz-Bunjakovskij inequality we have

$$|\langle \mu_{K_{0,\infty}}^{ij}, \varphi_1 \boxtimes \varphi \rangle| \leq \lim_{n} \sup_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \left| \varphi_1 u'_n(\xi) \right| \sqrt{\psi(\omega_n \xi)} \right) \left( \left| \varphi_2 u'_n(\xi) \right| \sqrt{\psi(\omega_n \xi)} \right) \, d\xi \leq \left( \lim_{n} \int_{\mathbb{R}^d} \left| \varphi_1 u'_n(\xi) \right|^2 \psi(\omega_n \xi) \, d\xi \right)^{\frac{1}{2}} \left( \lim_{n} \int_{\mathbb{R}^d} \left| \varphi_2 u'_n(\xi) \right|^2 \psi(\omega_n \xi) \, d\xi \right)^{\frac{1}{2}}$$

$$= \sqrt{\langle \mu_{K_{0,\infty}}^{ii}, |\varphi_1|^2 \boxtimes |\psi| \rangle} \sqrt{\langle \mu_{K_{0,\infty}}^{jj}, |\varphi_2|^2 \boxtimes |\psi| \rangle},$$

so $\langle \mu_{K_{0,\infty}}^{ii}, |\varphi_1|^2 \boxtimes |\psi| \rangle = 0$ or $\langle \mu_{K_{0,\infty}}^{jj}, |\varphi_2|^2 \boxtimes |\psi| \rangle = 0$ implies $\langle \mu_{K_{0,\infty}}^{ij}, \varphi_1 \boxtimes \varphi \rangle = 0$.

Q.E.D.

One-scale H-measures are in general associated to complex vector valued functions. However, if the observed sequence $(u_n)$ is real, the associated one-scale H-measure has an additional feature.

**Corollary 3.** Let $(u_n)$ be an $(\omega_n)$-pure sequence in $L^2_{\text{loc}}(\Omega; \mathbb{C}^r)$, and $\mu_{K_{0,\infty}}$ the corresponding one-scale H-measure. Then the sequence $(\bar{u}_n)$ is $(\omega_n)$-pure with associated one-scale H-measure $\nu_{K_{0,\infty}}$, such that $\nu_{K_{0,\infty}}(x, \xi) = \mu_{K_{0,\infty}}^{\top}(x, -\xi)$.

In particular, a one-scale H-measure $\mu_{K_{0,\infty}}$ associated to a real scalar sequence is antipodally symmetric, i.e. $\mu_{K_{0,\infty}}(x, \xi) = \mu_{K_{0,\infty}}(x, -\xi)$.

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Dem. Let us take test functions \( \varphi_1, \varphi_2 \in C_c(\Omega) \) and \( \psi \in C(K_{0,\infty}(\mathbb{R}^d)) \), and calculate the one-scale H-measure associated to the sequence \((\bar{u}_n)\).

\[
\int_{\mathbb{R}^d} \left( \varphi_1 \bar{u}_n(\xi) \otimes \varphi_2 \bar{u}_n(\xi) \right) \psi(\omega_n \xi) \, d\xi = \int_{\mathbb{R}^d} \left( (\varphi_1 u_n)^\dag(\xi) \otimes (\varphi_2 u_n)^\dag(\xi) \right) \psi(\omega_n \xi) \, d\xi \\
= \int_{\mathbb{R}^d} \left( (\varphi_1 u_n)^\dag(-\eta) \otimes (\varphi_2 u_n)^\dag(-\eta) \right) \tilde{\psi}(-\omega_n \eta) \, d\eta \\
= \int_{\mathbb{R}^d} (\varphi_1 \bar{u}_n(\eta) \otimes \varphi_2 \bar{u}_n(\eta)) \tilde{\psi}(\omega_n \eta) \, d\eta ,
\]

where we made use of a change of variable \((\eta = -\xi)\) in the second equality, while \(\tilde{\psi}\) denotes the sign change in argument \(\xi (\psi(\xi) = \psi(-\xi))\). Hence, the limit of the term on the left hand side exists and we have

\[
\langle \nu_{K_{0,\infty}}, \varphi_1 \varphi_2 \boxtimes \psi \rangle = \langle \mu_{K_{0,\infty}}, \varphi_1 \varphi_2 \boxtimes \psi \rangle \\
= \langle \hat{\mu}_{K_{0,\infty}}^\mathbb{R}, \varphi_1 \varphi_2 \boxtimes \psi \rangle = \langle \hat{\mu}_{K_{0,\infty}}^\mathbb{R}, \varphi_1 \varphi_2 \boxtimes \psi \rangle .
\]

The last equality is a consequence of the hermitian property of one-scale H-measures.

Q.E.D.

If \((u_n)\) is a weakly convergent sequence in \(L^2_{\text{loc}}(\Omega; \mathbb{C}^r)\), then \(u_n \otimes u_n\) is bounded in the space \(L^1_{\text{loc}}(\Omega; M_{r \times r})\), thus on a subsequence converges weakly to a hermitian non-negative Radon measure \(\nu\) (the defect measure). The relationship between one-scale H-measure and the defect measure is given by the following corollary.

**Corollary 4.** If \(u_n \to 0\) in \(L^2_{\text{loc}}(\Omega; \mathbb{C}^r)\) is such that \(u_n \otimes u_n \rightharpoonup_{\text{w}} \nu\), the limit being in the space of Radon measures, then for any \(\varphi \in C_c(\Omega)\)

\[
\langle \nu, \varphi \rangle = \langle \mu_{K_{0,\infty}}, \varphi \otimes 1 \rangle ,
\]

where \(\mu_{K_{0,\infty}}\) is a one-scale H-measure with an arbitrary characteristic length associated to a (sub)sequence \((u_n)\).

Dem. By inserting \(\psi \equiv 1\) in the definition of one-scale H-measures and using the Plancherel formula we get the claim.

Q.E.D.

From the previous corollary it is straightforward to conclude that a trivial one-scale H-measure of an arbitrary characteristic length implies strong convergence in \(L^2_{\text{loc}}(\Omega; \mathbb{C}^r)\) of the observed sequence. On the other hand, for a strongly convergent sequence the associated one-scale H-measure (of an arbitrary characteristic length) is trivial. Let us remember that for this property of semiclassical measures we need an additional assumption that the corresponding sequence is \((\omega_n)\)-oscillatory. As semiclassical measures associated to such sequences do not lose information at infinity (in the dual space), a simple characterisation of that assumption in terms of one-scale H-measures easily follows.

**Corollary 5.** If \(u_n \to 0\) in \(L^2_{\text{loc}}(\Omega; \mathbb{C}^r)\) is \((\omega_n)\)-pure (in the sense of one-scale H-measures), then \(\text{tr} \mu_{K_{0,\infty}}^{(\omega_n)}(\Omega \times \Sigma_{\infty}) = 0\) if and only if \((u_n)\) is \((\omega_n)\)-oscillatory.

Dem. By definition we have the following equivalence

\[
\text{tr} \mu_{K_{0,\infty}}(\Omega \times \Sigma_{\infty}) = 0 \iff \int_{\Omega \times K_{0,\infty}(\mathbb{R}^d)} |\varphi(x)|^2 \chi_{\Sigma_{\infty}}(\xi) \, d\mu_{K_{0,\infty}}(x, \xi) = 0,
\]

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where $\chi_{\Sigma_\infty}$ denotes the characteristic function of the sphere at $\infty$ in $K_{0,\infty}(\mathbb{R}^d)$.

By $\zeta \in C_c^\infty(\mathbb{R}^d)$ let us denote a smooth cutoff function such that $0 \leq \zeta \leq 1$ and such that it is identically equal to 1 on $K[0,1]$, while $\text{supp} \zeta \subseteq K(0,2)$. Further, we define $\zeta_m := \zeta(\frac{\cdot}{m})$.

By the Lebesgue dominated convergence theorem, non-negativity of diagonal elements and the definition of one-scale H-measures for an arbitrary $i \in 1..d$ we have

$$0 = \int_{\Omega \times K_{0,\infty}(\mathbb{R}^d)} |\varphi(x)|^2 \chi_{\Sigma_\infty}(\xi) \, d\mu_{K_{0,\infty}}^{ii}(x, \xi) = \lim_{m} \int_{\Omega \times K_{0,\infty}(\mathbb{R}^d)} |\varphi(x)|^2 (1 - \zeta_m(\xi)) \, d\mu_{K_{0,\infty}}^{ii}(x, \xi)$$

$$= \lim_{m} \lim_{n} \int_{\mathbb{R}^d} |\widehat{\varphi u_n}(\xi)|^2 (1 - \zeta_m(\omega_n \xi)) \, d\xi$$

$$\geq \lim_{n} \limsup \int_{|\omega_n \xi| \geq 2m} |\widehat{\varphi u_n^i}(\xi)|^2 \, d\xi \geq 0,$$

where in the last step we have used the fact that $\text{supp} \zeta_m \subseteq K(0,2m)$. Hence

$$(\forall i \in 1..d) \quad \limsup_{m} \int_{|\xi| \geq \frac{2m}{\omega_n}} |\widehat{\varphi u_n^i}(\xi)|^2 \, d\xi = 0,$$

and therefore $(u_n)$ is $(\omega_n)$-oscillatory.

The opposite implication follows from the estimate

$$(\forall i \in 1..d) \quad \int_{\mathbb{R}^d} |\widehat{\varphi u_n^i}(\xi)|^2 (1 - \zeta_m(\omega_n \xi)) \, d\xi \leq \int_{|\xi| \geq \frac{m}{\omega_n}} |\widehat{\varphi u_n^i}(\xi)|^2 \, d\xi.$$

Q.E.D.

We shall illustrate further advantages of one-scale H-measures by reconsidering sequences from examples 1 and 2.

**Example 4.** The one-scale H-measure with characteristic length $(\omega_n)$ associated to the sequence $(u_n)$ from Example 1 is given by

$$\mu_{K_{0,\infty}}^{(\omega_n)} = \lambda \bigotimes \begin{cases} \delta^\eta_{0,\infty}, & \text{lim}_{n} n^\alpha \omega_n = 0 \\ \delta^\eta_{\delta_{ck}}, & \text{lim}_{n} n^\alpha \omega_n = c \in (0, \infty) \\ \delta^\eta_{\delta_{\infty}}, & \text{lim}_{n} n^\alpha \omega_n = \infty \end{cases}.$$

Contrary to the semiclassical measure, one-scale H-measure contains information on the direction of oscillation for every characteristic length $(\omega_n)$.

For the sequence $(u_n + v_n)$ observed in Example 2, the associated one-scale H-measure with characteristic length $(\omega_n)$ is given by

$$\mu_{K_{0,\infty}}^{(\omega_n)} = \lambda \bigotimes \begin{cases} (\delta^\eta_{0,\infty} + \delta^\eta_{\delta_{ck}}), & \text{lim}_{n} n^\beta \omega_n = 0 \\ (\delta^\eta_{\delta_{\infty}}, \text{lim}_{n} n^\beta \omega_n = c \in (0, \infty) \\ (\delta^\eta_{\delta_{ck}}, \text{lim}_{n} n^\alpha \omega_n = 0. \\ (\delta^\eta_{\delta_{\infty}}, \text{lim}_{n} n^\alpha \omega_n = c \in (0, \infty) \\ (\delta^\eta_{\delta_{\infty}}, \text{lim}_{n} n^\alpha \omega_n = \infty \end{cases}.$$

It is important to notice that in both examples the respective H-measure can be derived from the one-scale H-measure with arbitrary characteristic length, which was not the case with the semiclassical measures.
By Lemma 1 and the known results for semiclassical and one-scale H-measures, we have that the semiclassical measure $\mu_{sc}$ and the one-scale H-measure $\mu_{K_{0,\infty}}$ associated to the same (sub)sequence and with the same characteristic length, coincide on $C^\infty_c(\Omega) \times S(\mathbb{R}^d)$ under necessary assumptions that the weak limit of the observed (sub)sequence is zero. Of course, by continuity we can extend this conclusion to $C_c(\Omega) \times C_0(\mathbb{R}^d)$.

Analogously, by using Lemma 1 and the definitions of H-measures and one-scale H-measures, we have:

$$
\langle \mu_{K_{0,\infty}}, (\varphi_1 \varphi_2) \boxtimes (\psi \circ \pi) \rangle = \lim_{n \to \infty} \int_{\mathbb{R}^d} \left( \varphi_1 \tilde{u}_n(\xi) \otimes \varphi_2 \tilde{u}_n(\xi) \right) \psi\left( \pi(\omega_n(\xi)) \right) \, d\xi
$$

$$
= \lim_{n \to \infty} \int_{\mathbb{R}^d} \left( \varphi_1 \tilde{u}_n(\xi) \otimes \varphi_2 \tilde{u}_n(\xi) \right) \psi\left( \frac{\xi}{|\xi|} \right) \, d\xi = \langle \mu_H, (\varphi_1 \varphi_2) \boxtimes \psi \rangle,
$$

where $\varphi_1, \varphi_2 \in C_c(\Omega)$ and $\psi \in C(S^{d-1})$ are arbitrary test functions, while the H-measure $\mu_H$ and the one-scale H-measure $\mu_{K_{0,\infty}}$ are associated to the same (sub)sequence. In particular, this relationship justifies the name one-scale H-measure, as an H-measure with one characteristic length. Let us summarise the previous observations.

**Corollary 6.** Let $\omega_n \to 0^+$ and let $\mu_H, \mu_{sc}^{(\omega_n)}$ and $\mu_{K_{0,\infty}}^{(\omega_n)}$ be the H-measure, the semiclassical and the one-scale H-measure (respectively), associated to the same (sub)sequence, weakly converging to zero. Then for any choice of test functions $\varphi_1, \varphi_2 \in C_c(\mathbb{R}^d)$ one has

1. $\langle \mu_{K_{0,\infty}}^{(\omega_n)}, (\varphi_1 \varphi_2) \otimes (\psi \circ \pi) \rangle = \langle \mu_H, (\varphi_1 \varphi_2) \boxtimes \psi \rangle$,
2. $\langle \mu_{sc}^{(\omega_n)}, (\varphi_1 \varphi_2) \boxtimes \psi \rangle = \langle \mu_{K_{0,\infty}}^{(\omega_n)}, (\varphi_1 \varphi_2) \otimes \psi \rangle$.

**Remark 4.** Although in the case when $u_n \to u \neq 0$ in $L^2_{\text{loc}}(\Omega; C^d)$ we cannot associate one-scale H-measure to $(u_n)$, the limit of (sub)sequence of integrals in Theorem 6 can be explicitly calculated. Indeed, inserting $u_n = (u_n - u) + u$, by Theorem 6 and the fact that the product of a weakly and a strongly converging sequence converges in $C$, we get

$$
\lim_{n' \to n} \int_{\mathbb{R}^d} \left( \varphi_1 \tilde{u}_{n'}(\xi) \otimes \varphi_2 \tilde{u}_{n'}(\xi) \right) \psi(\omega_{n'}(\xi)) \, d\xi = \langle \nu_{K_{0,\infty}}, (\varphi_1 \varphi_2) \otimes \psi \rangle + \int_{\mathbb{R}^d} \left( \varphi_1 \tilde{u}(\xi) \otimes \varphi_2 \tilde{u}(\xi) \right) (\psi \circ \pi)(\xi) \, d\xi,
$$

where $\nu_{K_{0,\infty}}$ is the one-scale H-measure associated to $(u_n - u)$, and $\psi_0 \in C(S^{d-1})$ is defined by (3.1). Moreover, if we take $\psi \in C_0(\mathbb{R}^d)$ (hence $\psi_0 \equiv \psi(0)$) and apply the Plancherel formula, by Corollary 6 the above resembles (2.3). Therefore, in the case when the limit of the observed sequence is known, we can reconstruct the semiclassical measure starting from the one-scale H-measure.

### 4. Localisation principle

Our next goal is to investigate the properties of one-scale H-measures determined by solutions to nonlinear differential relations with variable coefficients. More precisely, we consider a sequence of systems of partial differential equations of order $m \in \mathbb{N}$ with a characteristic length

$$
(4.1) \quad \sum_{l \leq |\alpha| \leq m} \varepsilon_n^{|\alpha| - l} \partial_\alpha (A^\alpha u_n) = f_n,
$$

where $(\varepsilon_n)$ is a bounded sequence of positive numbers, and $l \in 0..m$. Our aim is to develop a localisation principle for the corresponding one-scale H-measure, and eventually to deduce the conditions implying strong convergence of $(u_n)$.
A notable condition

To this effect we have to impose appropriate conditions on the terms entering (4.1). The characteristic length of the problem, expressed by the sequence $(\varepsilon_n)$, naturally appears in the assumption on the convergence of the right hand side. Namely, for a sequence $(f_n)$ in $H^{-m}(\mathbb{R}^d; C^q)$, the following convergence turns out to be important

$$\frac{\tilde{f}_n}{1 + k_n} \to 0 \quad \text{in} \quad L^2(\mathbb{R}^d; C^q),$$

where $k_n(\xi) := \sum_{s=1}^{m} \varepsilon_n^{-s-1} |\xi|^s$.

Actually, the local version will turn out to be more natural: if the sequence $(f_n)$ belongs only to $H^{-m}_{loc}(\Omega; C^q)$, we assume the local version of the above condition

$$\frac{\varphi f_n}{1 + k_n} \to 0 \quad \text{in} \quad L^2(\mathbb{R}^d; C^q).$$

The characterisation of the above convergence is given by the following lemma.

**Lemma 6.** Let $(\varepsilon_n)$ be a sequence of positive numbers. Then (4.2) is equivalent to

$$\frac{\tilde{f}_n}{1 + h_n} \to 0 \quad \text{in} \quad L^2(\mathbb{R}^d; C^q),$$

where $h_n(\xi) := |\xi|^l + \varepsilon_n^{-m-1} |\xi|^m$.

Furthermore, if we additionally assume that there are $\varepsilon_0, \varepsilon_\infty > 0$ such that $\varepsilon_0 \leq \varepsilon_n \leq \varepsilon_\infty$, then (4.2) is equivalent to the convergence in $H^{-m}(\mathbb{R}^d; C^q)$.

**Dem.** In order to prove the first statement, it is enough to show that $\frac{1 + h_n}{1 + k_n}$ is bounded on $\mathbb{R}^d$ by positive constants both from below and above. Clearly, it is bounded above by 1 for $l \neq m$, while for $l = m$ it is bounded by 2. From

$$\max\{1, \varepsilon_n |\xi|, \ldots, (\varepsilon_n |\xi|)^{m-l}\} = \max\{1, (\varepsilon_n |\xi|)^{m-l}\},$$

it follows that $1 + k_n(\xi) = 1 + |\xi|^l \sum_{s=1}^{m} (\varepsilon_n |\xi|)^{s-l} \leq 1 + (m - l + 1)h_n(\xi)$, which implies the lower bound.

In order to deduce the second statement, it is enough to observe the following elementary estimate

$$\frac{\min\{1, \varepsilon_\infty^{l-m}\}}{2} \frac{1}{1 + |\xi|^m} \leq \frac{1}{1 + |\xi|^l + \varepsilon_n^{-m-1} |\xi|^m} \leq \max\{1, \varepsilon_0^{l-m}\} \frac{1}{1 + |\xi|^m},$$

and to apply the sandwich theorem.

Q.E.D.

From the proof of the last lemma we conclude that if the sequence $(\varepsilon_n)$ is bounded from above, the condition (4.2) implies strong convergence to 0 in $H^{-m}(\mathbb{R}^d; C^q)$. On the other hand, the strong convergence in $H^{-l}(\mathbb{R}^d; C^q)$ is obviously a stronger notion than (4.2), implying that the latter is an intermediate convergence between those in $H^{-l}$ and $H^{-m}$ in this case.

Further relations among the above convergences are provided under additional assumption that $(f_n)$ is an $(\varepsilon_n)$-oscillating sequence. More precisely, the following result holds.

**Theorem 7.** Let $(\varepsilon_n)$ be a sequence in $\mathbb{R}^+$, and $(f_n)$ a sequence of vector functions in $H^{-l}(\mathbb{R}^d; C^q)$ satisfying (4.2) (with $m \geq 1$) and

$$\lim_{r \to \infty} \limsup_{n} \int_{|\xi| \geq \frac{r}{\varepsilon_n}} \left| \frac{\tilde{f}_n}{1 + |\xi|^l} \right|^2 d\xi = 0.$$

Then $f_n \to 0$ in $H^{-l}(\mathbb{R}^d; C^q)$. 

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Dem. Let us split \( \|f_n\|_{H^{-l}(\mathbb{R}^d; C^q)}^2 \) into two integrals:

\[
\int_{\mathbb{R}^d} \left| \frac{\hat{f}_n}{1 + |\xi|^l} \right|^2 d\xi = \int_{|\xi| < \frac{\varepsilon_n}{c}} \left| \frac{\hat{f}_n}{1 + |\xi|^l} \right|^2 d\xi + \int_{|\xi| > \frac{\varepsilon_n}{c}} \left| \frac{\hat{f}_n}{1 + |\xi|^l} \right|^2 d\xi.
\]

By (4.4) the first term on the right hand side is arbitrary small for \( r \) large enough (uniformly with respect to large enough \( n \)). For the second term we have

\[
\int_{|\xi| < \frac{\varepsilon_n}{c}} \left| \frac{\hat{f}_n}{1 + |\xi|^l} \right|^2 d\xi \leq \int_{|\xi| < \frac{\varepsilon_n}{c}} \left| \frac{\hat{f}_n}{1 + |\xi|^l + \varepsilon_n^m |\xi|^m} \right|^2 \left( 1 + \frac{\varepsilon_n^m}{\varepsilon_n^l + \varepsilon_n^m |\xi|^l} \right)^2 d\xi
\]

\[
\leq (1 + r^{m-l})^2 \int_{\mathbb{R}^d} \left| \frac{\hat{f}_n}{1 + |\xi|^l + \varepsilon_n^m |\xi|^m} \right|^2 d\xi,
\]

which by the assumptions goes to zero for every \( r \) as \( n \to \infty \).

Q.E.D.

Remark 5. Let us note that for \( l = 0 \) relation (4.4) resembles the assumption of the Riesz-Kolmogorov compactness theorem [8, Section 4.5], whereas the latter one is stronger with no \( (\varepsilon_n) \) scale included. The localised variant of condition (4.4) (with \( f_n \) replaced by \( \varphi f_n \) for an arbitrary \( \varphi \in C^\infty_c(\Omega) \)) is just a weaker version of the \( (\varepsilon_n) \)-oscillatory property adapted to \( H^{-l} \) setting.

In general, we shall deal with \( H^{-k} \) precompact sequences, where \( k \) is an arbitrary integer between \( l \) and \( m \), for which the following result holds.

Lemma 7. Let \( (\varepsilon_n) \) be a sequence in \( \mathbb{R}^+ \) bounded from above, and \( (f_n) \) a sequence of vector functions for some \( k \in l, m \) strongly converging to \( f \) in the space \( H^{-k}(\mathbb{R}^d; C^q) \). Then the sequence \( (\varepsilon_n^{-l}(f_n - f)) \) satisfies condition (4.2).

If we additionally assume that \( \varepsilon_n \) converges to zero, then

\[
(\varepsilon_n^{-k} f_n) \to \begin{cases} f & \text{if } k = l, \\ 0 & \text{if } k \geq l + 1, \end{cases}
\]

Dem. Let \( \varepsilon_\infty > 0 \) be an upper bound for \( \varepsilon_n \); from the estimate

\[
\frac{\varepsilon_n^{-k-l} |(\hat{f}_n - \hat{f})(\xi)|}{1 + k_n(\xi)} \leq \frac{\varepsilon_n^{-k-l} |(\hat{f}_n - \hat{f})(\xi)|}{1 + \varepsilon_n^{-k-l} |\xi|^k} \leq C \frac{|(\hat{f}_n - \hat{f})(\xi)|}{1 + |\xi|^l},
\]

where \( C = \max \{1, \varepsilon_\infty^{-l} \} \), we get the claim.

In order to prove the second statement, one rewrites the left hand side term of (4.5) by adding and subtracting \( f \) from \( f_n \). The term containing the difference \( f_n - f \) converges strongly to zero by the first part of the lemma, while the limit of the remaining one is obtained by the Lebesgue dominated convergence theorem.

Q.E.D.

If \( (f_n) \) satisfies (4.3), an application of the above results to \( \varphi f_n \), for an arbitrary \( \varphi \in C^\infty_c(\Omega) \), gives us that all the conclusions obtained for convergence (4.2) are also valid for its local counterpart (4.3), the only difference being the spaces \( H^{-k}(\mathbb{R}^d; C^q) \) replaced by their local versions \( H^{-k}_{\text{loc}}(\Omega; C^q) \).

The case \( \omega_n = \varepsilon_n \)

In the special case \( l = 1 \), Luc Tartar [27, Lemma 32.7] proved the localisation principle for one-scale H-measures with characteristic length equal to the characteristic length of the observed system (4.1). However, this result does not provide any information on the structure of one-scale H-measures on \( \Sigma_0 \). In the following theorems we extend that result to more general systems, simultaneously overcoming the mentioned shortcoming.

First we present results for one-scale H-measures with characteristic length \( (\omega_n) \) equal to the one of the system under consideration (like in Tartar’s approach), and later we generalise them to the case of an arbitrary characteristic length.
Theorem 8. Let $u_n \to 0$ in $L^2_{\text{loc}}(\Omega; C^r)$ and satisfy (4.1), where $A^{\alpha} \in C(\Omega; M_{q \times r}(C))$, $\varepsilon_n \to 0^+$, while $(f_n)$ is a sequence in $H^{-m}_{\text{loc}}(\Omega; C^q)$ satisfying (4.3).

Then the associated one-scale H-measure $\mu_{K_0,\infty}$ with characteristic length $(\varepsilon_n)$ satisfies:

$$(4.6) \quad p_1^{\top} \mu_{K_0,\infty} = 0,$$

where

$$(4.7) \quad p_1(x, \xi) := \sum_{\ell \leq |\alpha| \leq m} (2\pi i)^{\alpha} \frac{\xi^\alpha}{|\xi|^l + |\xi|^{m-\ell}} A^\alpha(x).$$

Dem. I. In the first step we localise (4.1) by multiplying it by a test function $\phi \in C^\infty_c(\Omega)$. With the aid of Lemma 8 below we get

$$\sum_{\ell \leq |\alpha| \leq m} \sum_{0 \leq \beta \leq \alpha} \frac{(-1)^{\beta}}{\beta} \varepsilon_n^{\alpha-l} \partial^{\alpha-\beta} (\partial^\beta \phi) A^{\alpha} u_n = \phi f_n.$$

Our goal is to show that the terms with $\beta \neq 0$ on the left hand side of this equality satisfy the convergence condition (4.2), i.e. that we can subsume them under the right hand side.

To this end, note that for given $\alpha$ and $\beta$ the sequence $\partial^{\alpha-\beta} (\partial^\beta \phi) A^{\alpha} u_n$ is supported in a fixed compact (supp $\varphi$), hence by the Rellich compactness theorem for $\beta \neq 0$ it is strongly precompact in $H^{-|\alpha|}(R^d; C^q)$. Thus by Lemma 7 we get that $\varepsilon_n^{\alpha-l} \partial^{\alpha-\beta} (\partial^\beta \phi) A^{\alpha} u_n$ satisfies (4.2).

Therefore the above equality can be simplified into the form

$$(4.8) \quad \sum_{\ell \leq |\alpha| \leq m} \varepsilon_n^{\alpha-l} \partial^{\alpha} (A^{\alpha} \phi u_n) = \tilde{f}_n,$$

where sequence $(\tilde{f}_n)$ satisfies (4.2).

II. Next we express the convergence in the terms of convergence in $L^2(R^d; C^q)$: by applying the Fourier transform to (4.8), and multiplying the whole equality by $\frac{1}{1 + |\xi|^l + \varepsilon_n^{m-\ell}|\xi|^m}$ we get

$$\sum_{\ell \leq |\alpha| \leq m} \varepsilon_n^{\alpha-l} (2\pi i)^{\alpha} \frac{\xi^\alpha A^{\alpha} \phi u_n}{1 + |\xi|^l + \varepsilon_n^{m-\ell}|\xi|^m} = \frac{\tilde{f}_n}{1 + |\xi|^l + \varepsilon_n^{m-\ell}|\xi|^m}.$$ 

By the assumption, after applying Lemma 6, the right hand side tends to 0 strongly in $L^2(R^d; C^q)$, so the same holds for the left hand side. Moreover, by Lemma 9 below we also obtain that

$$\sum_{\ell \leq |\alpha| \leq m} (2\pi i)^{\alpha} \frac{\varepsilon_n^{\alpha-l} \xi^\alpha}{|\xi|^l + \varepsilon_n^{m-\ell}|\xi|^m} A^{\alpha} \phi u_n \to 0 \quad \text{in} \quad L^2(R^d; C^q).$$

Indeed, for $|\xi| \geq r$ we have $|\xi|^l + \varepsilon_n^{m-\ell}|\xi|^m \geq r'$, while

$$\left| \frac{\varepsilon_n^{\alpha-l} \xi^\alpha}{|\xi|^l + \varepsilon_n^{m-\ell}|\xi|^m} \right| \leq \frac{|\varepsilon_n \xi|^{|\alpha|-l}}{1 + |\varepsilon_n \xi|^{m-l}} \leq 1,$$

which justifies the application of Lemma 9.

III. In order to introduce one-scale H-measures, we rewrite the last sum as

$$\sum_{\ell \leq |\alpha| \leq m} (2\pi i)^{\alpha} \frac{\varepsilon_n \xi^\alpha}{|\varepsilon_n \xi|^l + |\varepsilon_n \xi|^m} A^{\alpha} \phi u_n \to 0 \quad \text{in} \quad L^2(R^d; C^q).$$
where the function \( \xi \mapsto \frac{\xi^\alpha}{|\xi|^{1+|\xi|}} \) belongs to \( C(K_{0,\infty}(\mathbb{R}^d)) \) (see Example 3).

After multiplying the above sum by \( \psi(\varepsilon_n \cdot) \), where \( \psi \in C(K_{0,\infty}(\mathbb{R}^d)) \), and forming a tensor product with \( \check{\varphi}_1 u_n \), for \( \varphi_1 \in C_0^\infty(\Omega) \), the definition of one-scale H-measures implies

\[
0 = \lim_{n} \int_{\mathbb{R}^d} \psi(\varepsilon_n \xi) \left( \sum_{l \leq |\alpha| \leq m} (2\pi) |\alpha| \frac{\varepsilon_n \xi}{|\varepsilon_n \xi|^l + |\varepsilon_n \xi|^m} \Lambda^\alpha \varphi u_n \right) \otimes (\varphi_1 u_n) d\xi
\]

In particular, by choosing \( \varphi_1 \) to be 1 on the support of \( \varphi \) and using the hermitian property of \( \mu_{K_{0,\infty}} \), we get the claim.

Q.E.D.

Lemma 8. For any \( \alpha \in \mathbb{N}_0^d \), \( \varphi \in C_0^\infty(\Omega) \) and a distribution \( S \in \mathcal{D}'(\Omega) \) one has:

\[
\varphi \partial_\alpha S = \sum_{\beta \leq \alpha} (-1)^{|\beta|} (\varphi \partial_\beta S) \partial_{\alpha-\beta}.
\]

The proof of this lemma follows directly from the generalised Leibniz rule.

Next lemma is a generalisation of [4, Lemma 3].

Lemma 9. Let \( (f_n) \) be a sequence of measurable vector functions on \( \mathbb{R}^d \) and \( (h_n) \) a sequence of positive scalar functions which are uniformly bounded from below outside an arbitrary neighbourhood of the origin, i.e. which satisfy

\[
(\forall r > 0)(\exists \tilde{C} > 0)(\forall n \in \mathbb{N})(\forall \xi \in \mathbb{R}^d \setminus K(0,r)) \quad h_n(\xi) \geq \tilde{C}.
\]

Furthermore, let \( (u_n) \) be a bounded sequence in \( L^2(\mathbb{R}^d; C') \cap L^1(\mathbb{R}^d; C') \), such that

\[
f_n + h_n \cdot u_n \rightarrow 0 \quad \text{in} \quad L^2(\mathbb{R}^d).
\]

If \( h_n^{-2}|f_n|^2 \) is equiintegrable, i.e.

\[
(\forall \varepsilon > 0)(\exists \delta > 0)(\forall n \in \mathbb{N})(\forall A \in \mathcal{B}) \quad \lambda(A) < \delta \implies \int_A \left( \frac{|f_n|}{h_n} \right)^2 d\xi < \varepsilon,
\]

where \( \mathcal{B} \) denotes the Borel \( \sigma \)-algebra and \( \lambda \) the Lebesgue measure on \( \mathbb{R}^d \), then we also have

\[
\frac{f_n}{h_n} \cdot u_n \rightarrow 0 \quad \text{in} \quad L^2(\mathbb{R}^d).
\]

Dem. By the continuity of Fourier transform from \( L^1 \) to \( L^\infty \), \( S := \sup_n \|u_n\|_{L^\infty} \) is finite. From the equiintegrability of \( h_n^{-2}|f_n|^2 \) we have that for given \( \varepsilon > 0 \) we can take a ball \( K = K(0,\tilde{C}) \) such that \( \|h_n^{-1}f_n\|_{L^2(K;C')} < \frac{\varepsilon}{2C} \).

Thus we get

\[
\left\| \frac{f_n}{h_n} \cdot u_n \right\|_{L^2(K)} \leq \left\| \frac{f_n}{h_n} \right\|_{L^2(K;C')} \|u_n\|_{L^\infty(\mathbb{R}^d;C')} < \frac{\varepsilon}{2C},
\]

and in order to prove the lemma it remains to estimate the same term over \( \mathbb{R}^d \setminus K \).

By uniform boundedness from below, for \( \tilde{C} = 1/\tilde{C} + 1 \), outside \( K \) we have

\[
\left\| \frac{f_n \cdot u_n}{h_n} \right\|_{L^2(\mathbb{R}^d \setminus K)} \leq C \left\| \frac{f_n}{1 + h_n} \cdot u_n \right\|_{L^2(\mathbb{R}^d \setminus K)} \left\| \frac{f_n}{1 + h_n} \cdot u_n \right\|_{L^2(\mathbb{R}^d)}.
\]

As the last term above tends to 0 by assumption, there is an \( n_0 \in \mathbb{N} \) such that for any \( n \geq n_0 \) we have \( \left\| \frac{f_n}{1 + h_n} \cdot u_n \right\|_{L^2(\mathbb{R}^d)} < \frac{\varepsilon}{2C} \).

Q.E.D.
The localisation principle has been obtained under the assumption on the right hand side (4.3), which is adapted to the problem under consideration. However, we shall prove that the assumed convergence is also a necessary condition, similarly as it was the case with the localisation principle for H-measures [27, Lemma 28.18].

**Theorem 9.** Let \( u_n \to 0 \) in \( L^2_\text{loc}(\Omega; C^r) \) define one-scale H-measure \( \mu_{K_0,\infty} \) with the characteristic length \((\varepsilon_n)\), and \( A^\alpha \in C(\Omega; M_{q\times 1}(C)) \) for \( l \leq |\alpha| \leq m \).

If the conclusion of previous theorem holds, then (4.3) must also hold, with \( f_n \) being defined by (4.1).

**Dem.** By part 1 of the proof of Theorem 8 and Lemma 6 it is equivalent to prove that for an arbitrary \( \varphi \in C^\infty(\Omega) \)

\[
F\left(\sum_{l \leq |\alpha| \leq m} \varepsilon_n^{|\alpha|-l} \partial_\alpha (\varphi A^\alpha u_n)\right) \to 0 \quad \text{in} \quad L^2(R^d; C^r).
\]

Moreover, after applying the Fourier transform and taking into account simple inequality \( 1 + |\xi|^l + \varepsilon_n^{-l} |\xi|^m \geq |\xi|^l + \varepsilon_n^{-l} |\xi|^m \), it is sufficient to show that

\[
w_n := \sum_{l \leq |\alpha| \leq m} (2\pi)^{|\alpha|} \frac{(\varepsilon_n \xi)^\alpha}{|\varepsilon_n \xi|^l + |\varepsilon_n \xi|^m} A^\alpha \varphi u_n \to 0 \quad \text{in} \quad L^2(R^d; C^r).
\]

However, by the definition of one-scale H-measure we have

\[
\lim_{n} \int_{R^d} w_n \otimes w_n d\xi = \left< \overline{p_1 \mu_{K_0,\infty} p_1^*} |\varphi|^2 \otimes 1 \right> = 0,
\]

where in the last equality we have used the assumption (4.6), with \( p_1 \) given by (4.7), which completes the proof.

Q.E.D.

Let us now go back to our main result, Theorem 8, which easily extends to the case where the coefficients suitably depend on \( n \).

**Corollary 7.** Under the same assumptions on \((f_n) \) and \((\varepsilon_n)\) as in Theorem 8, for \( u_n \to 0 \) in \( L^2_\text{loc}(\Omega; C^r) \) additionally assume that

\[
(4.9) \quad \sum_{l \leq |\alpha| \leq m} \varepsilon_n^{|\alpha|-l} \partial_\alpha (A^\alpha_n u_n) = f_n,
\]

where \( A^\alpha_n \in C(\Omega; M_{q\times 1}(C)) \), such that for any \( \alpha \) the sequence \( A^\alpha_n \to A^\alpha \) in the space \( C(\Omega; M_{q\times 1}(C)) \) (in other words, \( A^\alpha_n \) converges locally uniformly to \( A^\alpha \)). Then we obtain the same conclusion as in Theorem 8, with the same symbol given by (4.7).

**Dem.** It is enough to prove that

\[
\sum_{l \leq |\alpha| \leq m} \varepsilon_n^{|\alpha|-l} \partial_\alpha \left((A^\alpha_n - A^\alpha)u_n\right)
\]

satisfies the convergence condition for the right hand side (4.3). As \((A^\alpha_n - A^\alpha)u_n\) strongly converges to 0 in \( L^2_\text{loc}(\Omega; C^r) \), so \( \partial_\alpha \left((A^\alpha_n - A^\alpha)u_n\right) \) strongly converges to 0 in \( H^\alpha_\text{loc}(\Omega; C^r) \), hence by Lemma 7 we have the claim.

Q.E.D.
Remark 6. The powers of scale \((\varepsilon_n)\) in (4.9) strictly follow the order of derivatives (up to constant \(l\)). However, by a slight modification of the convergence assumption on the right side, the above result easily extends to equations allowing more general relations between these terms.

Namely, let us suppose that we start with the equalities

\[
\sum_{l \leq |\alpha| \leq m} \varepsilon_n^{l|\alpha|-l} \partial_{\alpha} (\mathbf{A}_n^\alpha \mathbf{u}_n) = f_n ,
\]

with \(\delta_\alpha \in \mathbb{R}\). Multiplication of the above relation by \(\varepsilon_n^{-\delta}\), where \(\delta := \min \delta_\alpha\), implies

\[
\sum_{l \leq |\alpha| \leq m} \varepsilon_n^{l|\alpha|-l} \partial_{\alpha} \left( (\varepsilon_n^{-\delta} \mathbf{A}_n^\alpha) \mathbf{u}_n \right) = \varepsilon_n^{-\delta} f_n .
\]

For sequence \((f_n)\) we assume that \((\varepsilon_n^{-\delta} f_n)\) satisfies (4.3), which is a stronger assumption for \(\delta > 0\), and weaker for \(\delta < 0\), compared to the assumptions of the last corollary. Its application finally provides the localisation result (with symbol \(p_1\) not containing terms for which \(\delta_\alpha > \delta\)).

The case \(c := \lim_n \frac{\varepsilon_n}{\omega_n} \in [0, \infty]\)

Theorem 10. Let \(\mathbf{u}_n \to 0\) in \(L^2_{\text{loc}}(\Omega; \mathbb{C}^r)\) and satisfy (4.9), where \(\mathbf{A}_n^\alpha \to \mathbf{A}^\alpha\) in the space \(C(\Omega; M_{q \times r}(\mathbb{C}))\), while \((f_n)\) is a sequence in \(H^{-m}_{\text{loc}}(\Omega; \mathbb{C}^q)\) satisfying (4.3).

If \((\varepsilon_n)\) and \((\omega_n)\) are sequences of positive numbers such that \(c = \lim_n \frac{\varepsilon_n}{\omega_n}\) exists (in \([0, \infty]\)), then the associated one-scale \(H\)-measure \(\mu^{(\omega_n)}_{K_{0, \infty}}\) satisfies:

\[
p_c \left( \mu^{(\omega_n)}_{K_{0, \infty}} \right) ^\top = 0 ,
\]

where with respect to the value of \(c\) we have

a) \(c = 0\):

\[
p_0(x, \xi) := \sum_{|\alpha| = l} (2\pi i)^{|\alpha|} \frac{\xi^\alpha}{|\xi|^l + |\xi|^m} \mathbf{A}^\alpha(x) ,
\]

b) \(c \in (0, \infty)\):

\[
p_c(x, \xi) := \sum_{l \leq |\alpha| \leq m} (2\pi i c)^{|\alpha|} \frac{\xi^\alpha}{|\xi|^l + |\xi|^m} \mathbf{A}^\alpha(x) ,
\]

c) \(c = \infty\):

\[
p_\infty(x, \xi) := \sum_{|\alpha| = m} (2\pi i)^m \frac{\xi^\alpha}{|\xi|^l + |\xi|^m} \mathbf{A}^\alpha(x) .
\]

Dem. In both (a) and (b) cases, by rewriting (4.9) we infer

\[
\sum_{l \leq |\alpha| \leq m} \omega_n^{|\alpha| - l} \partial_{\alpha} (\mathbf{B}_n^\alpha \mathbf{u}_n) = f_n ,
\]

where \(\mathbf{B}_n^\alpha := (\frac{\varepsilon_n}{\omega_n})^{l|\alpha|-l} \mathbf{A}_n^\alpha\). As for \(n\) large enough there is \(\kappa > 0\) such that \(\omega_n > \kappa \varepsilon_n\), it can be easily seen that the right hand side also satisfies

\[
(\forall \varphi \in C_c^\infty(\Omega)) \quad \frac{\varphi f_n}{1 + |\xi|^l + \omega_n^{m-l} |\xi|^m} \to 0 \quad \text{in} \quad L^2(\mathbb{R}^d; \mathbb{C}^q) .
\]
Taking into account Corollary 7, and the fact that $B_n^\alpha$ converges to $c^{\prod |\alpha|-l}A^{\alpha}$ (in particular to $A^{\alpha}$ for $c = 0$ and $|\alpha| = l$), we get the claim.

c) After multiplying (4.9) by \( \left(\frac{\omega_n}{\alpha_n}\right)^{m-l} \), we obtain

\[
\sum_{l \leq |\alpha| \leq m} \omega_n^{\prod |\alpha|-l} \partial_\alpha (\tilde{B}_n^\alpha u_n) = g_n,
\]

where $\tilde{B}_n^\alpha := \left(\frac{\omega_n}{\alpha_n}\right)^{m-|\alpha|}A_n^\alpha$, while $g_n := \left(\frac{\omega_n}{\alpha_n}\right)^{m-l}$. In order to apply Corollary 7 it remains to check the condition satisfied by the right-hand side $g_n$.

For $\varphi \in C_\infty^\infty(\Omega)$ we have

\[
\frac{|\varphi g_n|}{1 + |\xi|^l + \omega_n^{m-l}|\xi|^m} = \frac{|\varphi f_n|}{\left(\frac{\omega_n}{\alpha_n}\right)^{m-l} (1 + |\xi|^l) + \epsilon_n^{m-l}|\xi|^m} \leq \frac{|\varphi f_n|}{1 + |\xi|^l + \epsilon_n^{m-l}|\xi|^m} \rightarrow 0 \quad \text{in} \quad L^2(\mathbb{R}^d),
\]

where the last inequality is valid for sufficiently large $n$. Thus Corollary 7, together with the fact that $\tilde{B}_n^\alpha$ converges to zero for $|\alpha| \leq m - 1$, provides the claim.

Q.E.D.

Remark 7. Note that for $c = 0$ and $c = \infty$ (with $l < m$) the previous theorem does not provide any information about the structure of one-scale H-measure on $\Sigma_\infty$ and $\Sigma_0$, respectively. This is caused by the fact that (4.11) is zero on $\Sigma_\infty$, while (4.13) vanishes on $\Sigma_0$.

Part (c) of the previous theorem, in the special case when $(\epsilon_n)$ is bounded by positive constants both from below and above, can be improved as a consequence of Lemma 6.

Theorem 11. Under the assumptions of Theorem 10 for $c = \infty$, additionally assume that $0 < \epsilon_0 \leq \epsilon_n \leq \epsilon_\infty$. Then the associated one-scale H-measure $\mu_{K_0,\infty}$ satisfies:

\[
P_{pr} \left( \mu_{K_0,\infty}^{(\omega_n)} \right)^\top = 0,
\]

where

\[
P_{pr}(x, \xi) = \sum_{|\alpha| = m} \frac{(2\pi i)^m \left(\frac{\xi}{\alpha}\right)^\alpha}{|\xi|^m} A^\alpha(x).
\]

Dem. Without loosing generality, we may assume that the coefficients do not depend on $n$, i.e. $A_n^\alpha = A^\alpha$. The general case can be treated as in Corollary 7.

By repeating part I of the proof of Theorem 8, we get (4.8). We aim to show that the lower order terms converge to zero in the sense of (the right hand side) convergence (4.2). In fact, Lemma 6 additionally gives us that $f_n$ converges to 0 in $H^{-m}(\mathbb{R}^d; \mathbb{C}^q)$, same as terms $\epsilon_n^{\prod |\alpha|-l} \partial_\alpha (\varphi A^\alpha u_n)$ for $|\alpha| < m$. This implies

\[
\sum_{|\alpha| = m} \epsilon_n^{m-l} \partial_\alpha (A^\alpha \varphi u_n) \rightarrow 0 \quad \text{in} \quad H^{-m}(\mathbb{R}^d; \mathbb{C}^q),
\]

and further

\[
\sum_{|\alpha| = m} (2\pi i)^m \frac{\xi^\alpha A^\alpha \varphi u_n}{1 + |\xi|^m} \rightarrow 0 \quad \text{in} \quad L^2(\mathbb{R}^d; \mathbb{C}^q),
\]
as $\varepsilon_n > \varepsilon_0$. Furthermore, Lemma 9 gives us
\[
\sum_{|\alpha|=m} (2\pi i)^m \frac{\xi^\alpha}{|\xi|^m} A_n^\alpha \varphi u_n \to 0 \text{ in } L^2(\mathbb{R}^d; \mathbb{C}^q).
\]
After forming the tensor product with $\psi(\omega_n)\varphi_1 u_n$, where $\psi \in C(\mathbb{K}_{0,\infty}(\mathbb{R}^d))$, $\varphi_1 \in C_c(\Omega)$, and integrating, we obtain
\[
0 = \lim_{n} \int_{\mathbb{R}^d} \psi(\omega_n) \xi \left( \sum_{|\alpha|=m} (2\pi i)^m \frac{\omega_n \xi^\alpha}{|\omega_n \xi|^m} A_n^\alpha \varphi u_n \right) \otimes \left( \varphi_1 u_n \right) d\xi
\]
\[
= \left( \sum_{|\alpha|=m} (2\pi i)^m \frac{\xi^\alpha}{|\xi|^m} A_n^\alpha \mu_{\mathbb{K}_{0,\infty}} \varphi \varphi_1 \psi \right).
\]
As $\varphi_1$ is arbitrary, we can choose it equal to 1 on supp $\varphi$, which gives the claim.

\[\text{Q.E.D.}\]

Let us note that in the previous proof we have not assumed that $(\varepsilon_n)$ has a unique accumulation point.

Theorem 11 is an improvement over Theorem 10(c) indeed, as the symbol (2.2) does not vanish on the sphere $\Sigma_0$ in the compactified domain $\mathbb{K}_{0,\infty}(\mathbb{R}^d)$, as it was the case with symbol (4.13). In such a way it provides information on the structure of the considered one-scale H-measure on its entire domain.

**Remark 8.** If $u_n \to u \neq 0$ in $L^2_{\text{loc}}(\Omega; \mathbb{C}^q)$, we cannot associate a one-scale H-measure to $(u_n)$, but, like in [27, Theorem 28.7], we can still derive the localisation principle for corresponding null sequence $(u_n - u)$. Namely, let $u_n$ satisfy (4.9), where $A_n^\alpha$ converges to $A^\alpha$ in $C(\Omega; M_{q\times d}(\mathbb{C}))$, let $\varepsilon_n \to 0^+$, and let $f_n \in H^{-m}_{\text{loc}}(\Omega; \mathbb{C}^q)$ be such that for any $\varphi \in C_c(\Omega)$, $\frac{\varepsilon_n f_n}{1+k_n}$ is precompact in the strong topology of $L^2(\mathbb{R}^d; \mathbb{C}^q)$.

As in the first step of the proof of Theorem 8 we localise the equation by multiplying it by $\varphi \in C_c(\Omega)$ in order to get
\[
\sum_{l=0}^{|\alpha| \leq m} \varepsilon_n^{|\alpha|-l} \partial_\alpha (A_n^\alpha \varphi u_n) = f_{1,n},
\]
where
\[
f_{1,n} := \varphi f_n - \sum_{l=0}^{|\alpha| \leq m} \sum_{0 \neq \beta \leq \alpha} (-1)^{|\beta|} \binom{\alpha}{\beta} \varepsilon_n^{|\alpha|-l} \partial_\alpha - \beta (\partial_\beta \varphi) A_n^\alpha u_n.
\]
Since $\partial_\alpha - \beta (\partial_\beta \varphi) A_n^\alpha u_n \to \partial_\alpha - \beta (\partial_\beta \varphi) A^\alpha u$ in $H^{-|\alpha|}(\mathbb{R}^d; \mathbb{C}^q)$, by Lemma 7 and the assumption on $(f_n)$, we have that $\frac{f_{1,n}}{1+k_n}$ is precompact in the strong topology of $L^2(\mathbb{R}^d; \mathbb{C}^q)$.

Next, after subtracting $\sum_{l=0}^{|\alpha| \leq m} \varepsilon_n^{|\alpha|-l} \partial_\alpha (A_n^\alpha \varphi u)$ in the localised equation, by the linearity of differential operator on the left hand side, $u_n$ is replaced by $u_n - u$, so the whole left hand side converges to zero in the sense of distributions. On the other hand, on the right hand side we obtained a sequence $(f_{2,n})$ which still satisfies that $\frac{f_{2,n}}{1+k_n}$ is precompact, while $f_{2,n} \to 0$ in the sense of distributions. By Lemma 10 below we get that $(f_{2,n})$ satisfies (4.2). Hence all the assumptions of Theorem 10 are satisfied and the one-scale H-measure associated to $(u_n - u)$ satisfies (4.10).

**Lemma 10.** Let $\varepsilon_n \to 0^+$ and let $(f_n)$ be a sequence of vector functions from $H^{-m}(\mathbb{R}^d; \mathbb{C}^q)$ such that $\frac{f_n}{1+k_n}$ is precompact in the strong topology of $L^2(\mathbb{R}^d; \mathbb{C}^q)$. Then there exists a subsequence $(f_{n'})$ and an $f \in H^{-1}(\mathbb{R}^d; \mathbb{C}^q)$ such that
\[
\frac{f_{n'} - f}{1+k_{n'}} \to 0.
\]
in $L^2(\mathbb{R}^d; \mathbb{C}^q)$. In particular, if $f_n \rightarrow 0$ in the sense of distributions, then $f \equiv 0$ and the whole sequence $(f_n)$ satisfies (4.2).

Dem. By the assumptions, there exists a subsequence $(f_{n'})$ and $F \in L^2(\mathbb{R}^d; \mathbb{C}^r)$ such that \( \hat{f}_{n'} \rightarrow F \) in $L^2(\mathbb{R}^d; \mathbb{C}^q)$. By using the Lebesgue dominated convergence theorem it follows that 
\[
\frac{\hat{f}_{n'} - \hat{f}}{1 + k_{n'}} \rightarrow 0
\]
in $L^2(\mathbb{R}^d; \mathbb{C}^q)$, with $f := \mathcal{F} \left( (1 + |\xi|^l)F \right) \in H^{-l}(\mathbb{R}^d; \mathbb{C}^q)$. By the remark after Lemma 6, the above implies $f_{n'} \rightarrow f$ in $H^{-m}(\mathbb{R}^d; \mathbb{C}^q)$.

If in addition we have that $f_n \rightarrow 0$ in the sense of distribution, then by the uniqueness of the limit, $f \equiv 0$ and hence $(f_{n'})$ satisfies (4.2). Since this holds for every subsequence the claim follows.

Q.E.D.

5. Some applications

Localisation principles revisited

As we have already noticed, from the one-scale H-measure one can get both the corresponding semiclassical measure, as well as the corresponding H-measure. In the following corollaries we show that the localisation principles for these two objects are also consequences of the localisation principle for one-scale H-measures.

Corollary 8. (localisation principle for H-measures) Under the assumptions of Theorem 11 one has the conclusion of Theorem 2.

Dem. Since the symbol (2.2) is homogeneous of order zero in $\xi$, by Theorem 11 and Corollary 6 we get the claim.

Q.E.D.

The statement of the previous corollary is more general then the one of Theorem 2 since it allows for coefficients depending on $n$ in the system under consideration. However, such a generalisation can also be obtained directly, by using the standard techniques for H-measures and the approach presented in Corollary 7.

Corollary 9. (localisation principle for semiclassical measures) Under the assumptions of Theorem 10 we have

\[
 p \mu_{sc} = 0 ,
\]

where $\mu_{sc}$ is a semiclassical measure with characteristic length $(\omega_n)$ associated to the sequence $(u_n)$, while

\[
p(x, \xi) := \begin{cases} 
\sum_{|\alpha| = l} (2\pi i)^l |\xi|^\alpha A^\alpha(x) & , \quad \lim_n \frac{\omega_n}{\omega_{n'}} = 0 \\
\sum_{l \leq |\alpha| \leq m} (2\pi i)^l |\xi|^\alpha A^\alpha(x) & , \quad \lim_n \frac{\omega_n}{\omega_{n'}} = c \in (0, \infty) \\
\sum_{|\alpha| = m} (2\pi i)^m |\xi|^\alpha A^\alpha(x) & , \quad \lim_n \frac{\omega_n}{\omega_{n'}} = \infty
\end{cases}
\]

Dem. Let $\lim_n \frac{\omega_n}{\omega_{n'}} = c \in (0, \infty)$. For $\psi \in S(\mathbb{R}^d)$ we have that $\xi \mapsto (|\xi|^l + |\xi|^m)\psi(\xi)$ is in $C(K_{0,\infty}(\mathbb{R}^d))$, so after applying the localisation principle, i.e. Theorem 10(b), to $\varphi(\xi)|\xi|^l + |\xi|^m)\psi$, 

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for $\varphi \in C_c(\Omega)$ one obtains

$$0 = \left\langle \sum_{l \leq |\alpha| \leq m} (2\pi ic)^{\alpha l} \frac{\xi^{\alpha l}}{|\xi|^l + |\xi|^m} A^\alpha \mu_{K_0,\infty}, \varphi \otimes (|\xi|^l + |\xi|^m) \psi \right\rangle = \sum_{l \leq |\alpha| \leq m} \left\langle A^\alpha \mu_{K_0,\infty} (2\pi ic)^{\alpha l} \varphi \otimes \xi^{\alpha l} \psi \right\rangle = \sum_{l \leq |\alpha| \leq m} \left\langle A^\alpha \mu_{sc} (2\pi ic)^{\alpha l} \varphi \otimes \xi^{\alpha l} \psi \right\rangle = \left\langle \sum_{l \leq |\alpha| \leq m} (2\pi ic)^{\alpha l} \xi^{\alpha l} A^\alpha \mu_{sc}, \varphi \otimes \psi \right\rangle,$$

where in the third equality the fact that $\xi^{\alpha l} \psi \in S(\mathbb{R}^d)$ was used, as well as Corollary 6.

The cases $\lim_{n} \frac{c_n}{\omega_n} = 0$ and $\lim_{n} \frac{c_n}{\omega_n} = \infty$ follow in the same manner.

Q.E.D.

The theory of semiclassical measures was developed by means of pseudodifferential calculus which requires smooth test functions. That was the main obstruction to weaker assumptions on the coefficients of the observed system. Following Tartar’s approach, the theory of one scale H-measures is constructed without the smoothness assumption, thus enabling us to generalise existing results for semiclassical measures to merely continuous functions without any technical difficulties. Furthermore, another improvement is based on the convergence of the right hand side (4.3), which is weaker than $L^2_{locc}$ convergence required in Theorem 4.

In addition, Corollary 9 provides a complete analysis of dependence on the relation between characteristic lengths of semiclassical measures and systems. This is useful in situations where the characteristic length of the observed problem cannot be easily identified, or where the same measure has to be applied to two (or more) systems with different characteristic scales.

Let us stress once more that in Theorem 4 the statement was valid for an arbitrary bounded sequence in $L^2_{locc}$, which has not been the case in the previous theorem. However, according to Remark 8 at the end of the previous section, we can still obtain information about the semiclassical measure associated to $(u_n - u)$, where $u_n \to u$. Furthermore, this result is not weaker then the result of Theorem 4 since $(u \otimes u)\lambda \otimes \delta_0$ cancels all the terms of symbol $p_{sc}$ from Theorem 4. Indeed, nontrivial is only the fact that $A^0(u \otimes u)\lambda \otimes \delta_0 = 0$, which is a consequence of $A^0u = 0$, following from the assumptions of Theorem 4.

**Compactness by compensation with a characteristic length**

One of the main purposes of one-scale H-measures, like of allmicrolocal defect tools, is to investigate conditions resulting in strong convergence of a bounded $L^2$ sequence under consideration. Hereby, localisation principles presented above play a significant rôle as they, possibly accompanied by some additional information, enable one to deduce that a (component of) measure $\mu_{K_0,\infty}$ associated to $u_n$ is trivial. This implies then either stronger $L^2$ convergence of $(u_n')$ (for a diagonal component) or vague convergence of $(u_n' u_n')$ (for a vanishing corresponding off-diagonal element). In the following example we shall present such a result obtained by the localisation principle derived in the previous section.

**Example 5.** Let $\Omega \subseteq \mathbb{R}^2$ be open, and let $u_n := (u_n^{1}, u_n^{2}) \to 0$ in $L^2_{locc}(\Omega; C^2)$ satisfies

$$
\begin{cases}
  u_n^1 + \varepsilon_n \partial_{x_1} (a_1 u_n^1) = f_n^1 \\
  u_n^2 + \varepsilon_n \partial_{x_2} (a_2 u_n^2) = f_n^2 ,
\end{cases}
$$

where $\varepsilon_n \to 0^+$, $f_n := (f_n^1, f_n^2) \in H^{-1}_{locc}(\Omega; C^2)$ satisfies (4.3) (with $l = 0, m = 1$), while $a_1, a_2 \in C(\Omega; \mathbb{R})$, $a_1, a_2 \neq 0$ everywhere.

By the localisation principle for one-scale H-measure $\mu_{K_0,\infty}$ with characteristic length $(\varepsilon_n)$ (i.e. $c = 1$) associated to $(u_n)$ we get the relation

$$\left( \begin{array}{cc}
  1 & 2\pi i \xi_1 \\
  1 & 1+|\xi|
\end{array} \right) \left( \begin{array}{cc}
  a_1(x) & 0 \\
  0 & a_2(x)
\end{array} \right) \mu_{K_0,\infty}^\top = 0 ,$$
whose \((1,1)\) component reads

\[
\left(\frac{1}{1 + |\xi|} + i \frac{2\pi \gamma_1}{1 + |\xi|} a_1(x)\right) \mu_{K_0,\infty}^{11} = 0.
\]

Since measure \(\mu_{K_0,\infty}^{11}\) is non-negative, we can separate its real and imaginary part in order to obtain

\[
\frac{1}{1 + |\xi|} \mu_{K_0,\infty}^{11} = 0, \quad \frac{\gamma_1}{1 + |\xi|} \mu_{K_0,\infty}^{11} = 0,
\]

where we have used the assumption that \(a_1 \neq 0\). Since \(\frac{1}{1 + |\xi|}\) differs from zero on \(K_{0,\infty}(\mathbb{R}^d) \setminus \Sigma_{\infty}\), from the first equation we get \(\text{supp} \mu_{K_0,\infty}^{11} \subseteq \Omega \times \Sigma_{\infty}\). Similarly, from the second one we deduce \(\text{supp} \mu_{K_0,\infty}^{11} \subseteq \Omega \times (\Sigma_0 \cup \{\xi_1 = 0\})\), so at the end we have

\[
\text{supp} \mu_{K_0,\infty}^{11} \subseteq \Omega \times \{\infty^{(0,-1)}, \infty^{(0,1)}\}.
\]

Analogously, from the \((2,2)\) component we get

\[
\text{supp} \mu_{K_0,\infty}^{22} \subseteq \Omega \times \{\infty^{(-1,0)}, \infty^{(1,0)}\},
\]

which by Corollary 2 implies that \(\mu_{K_0,\infty}^{22} = \mu_{K_0,\infty}^{21} = 0\). The very definition of one-scale H-measures gives \(u_n^1 u_n^2 \xrightarrow{\text{loc}} 0\).

**Remark 9.** If in the previous example we additionally have that \((u_n^1)\) is \((\varepsilon_n)\)-oscillatory, then Corollary 5 implies \(\mu_{K_0,\infty}^{11} (\Omega \times \Sigma_{\infty}) = 0\), hence by (5.2) \(\mu_{K_0,\infty}^{11}\) vanishes everywhere, which gives \(u_n^1 \xrightarrow{L^2_{\text{loc}}(\Omega; \mathbb{C})} 0\). As the product of a strongly and a weakly converging sequence, the convergence \(u_n^1 u_n^2 \xrightarrow{\text{loc}} 0\) is now a trivial consequence. An example of such a situation is the case when \(f_n^1\) is a sequence of oscillations, resulting in a solution \(u_n^1\) oscillating at the same frequency as \(f_n^1\). Therefore, with an appropriate choice of the frequency, one obtains solutions satisfying the \((\varepsilon_n)\)-oscillatory property.

Moreover, having an \((\varepsilon_n)\)-oscillating sequence of solutions, we can remove the assumption \(a_1 \neq 0\) in Example 5, since from the first equation in (5.1) we already have \(\text{supp} \mu_{K_0,\infty}^{11} \subseteq \Omega \times \Sigma_{\infty}\), hence still \(\mu_{K_0,\infty}^{11} = 0\). Thus in the special case when \(a_1\) is trivial, this provides an alternative proof of Theorem 7 for the case \(l = 0\).

The previous result is an example of a compactness by compensation theory, investigating conditions under which a product of weakly convergent sequences converges (in some suitable sense) to the product of corresponding limits. The theory was introduced by François Murat and Luc Tartar (see [9] and references therein), first for constant coefficients and then generalised to continuous coefficients in [26] by means of H-measures (cf. [27, Cor. 28.11] and [14, Theorem 2]; some recent extensions can be found in [22, 24]). Of course, the obtained results do not treat equations containing a characteristic length, which is now possible by the obtained localisation principle for one-scale H-measures.

**Theorem 12. (a variant of compactness by compensation)** Let \(u_n \xrightarrow{L^2_{\text{loc}}(\Omega; \mathbb{C}^d)} u\) satisfy (4.9), where \(A_n^\alpha \xrightarrow{\text{loc}} A^\alpha \in C(\Omega; M_{q\times 1}(\mathbb{C}))\), let \(\varepsilon_n \rightarrow 0^+\), and \(f_n \in H^{-m}_{\text{loc}}(\Omega; \mathbb{C}^d)\) be such that for any \(\varphi \in C_c^\infty(\Omega)\)

\[
\varphi f_n \xrightarrow{1 + k_n} \nu \text{ in } \mathcal{M}(\Omega).
\]

is precompact in \(L^2(\mathbb{R}^d; \mathbb{C}^d)\). Furthermore, let \(Q(x; \lambda) := Q(x)\lambda \cdot \lambda\), where \(Q \in C(\Omega; M_f(\mathbb{C}))\), \(Q^* = Q\), is such that \(Q(\cdot; u_n) \xrightarrow{\text{loc}} \nu\) in \(\mathcal{M}(\Omega)\).

Then we have
Let $\mu_{K_{0,\infty}}$ be the one-scale H-measure associated to $(u_n-u)$, with a characteristic length $(\omega_n)$ such that $\lim_{n} \frac{\omega_n}{\omega_n} = c$.

The limit $\nu$ we can express as a sum of $Q(\cdot;u)$ and the corresponding linear combinations of components of matrix defect measure $\nu_D$ associated to $(u_n-u)$. Namely, for an arbitrary $\varphi \in C_c(\Omega)$ we have

$$\langle \nu, \varphi \rangle = \langle Q(\cdot;u), \varphi \rangle + \left( Q \cdot \nu_D^\top, \varphi \right)$$

$$= \langle Q(\cdot;u), \varphi \rangle + \left( Q \cdot \mu_{K_{0,\infty}}^\top, \varphi \otimes 1 \right),$$

where in the second equality we have used Corollary 4.

We shall demonstrate that $Q \cdot \mu_{K_{0,\infty}}^\top$ is a non-negative measure (or trivial measure in part (b)), which implies the claim.

Let us introduce a non-negative Radon measure $\rho := \text{tr}\mu_{K_{0,\infty}} = \sum_{i=1}^r \mu_{ij}^i$. According to Corollary 2, for each $i, j \in 1..r$ we have that $\mu_{ij}^i < \rho$, so by the Lebesgue-Radon-Nikodým theorem there exists a measurable function $M^{ij}(x, \xi)$ such that $\mu_{ij}^i = M^{ij} \rho$. Since $\mu_{K_{0,\infty}}$ is hermitian and non-negative, $M := [M^{ij}]$ is also hermitian and non-negative. In terms of the matrix function $M$ it remains to prove that for $\rho$-a.e. $(x, \xi)$ we have $Q \cdot M^\top \geq 0$ (or equal to 0 in part (b)).

By the localisation principle given in Theorem 10 (which remains valid for $u \neq 0$ based on Remark 8) we have $p_{c} \mu_{K_{0,\infty}}^\top = 0$, hence $p_{c} M^\top = 0$ ($\rho$-a.e.), which implies that for $\rho$-a.e. $(x, \xi)$ the columns of $M^\top$ are in the characteristic space $\Lambda_{c;\xi}$. Let us take $(x, \xi) \in \Omega \times K_{0,\infty}(R^d)$ arbitrary, but fixed. As $M^\top(x, \xi)$ is hermitian, there is an orthonormal basis $\{\xi_1, \ldots, \xi_r\}$ of $C^r$ composed of eigenvectors of $M^\top(x, \xi)$, which are of course in $\Lambda_{c;\xi}$. Moreover, the non-negativity of $M^\top(x, \xi)$ implies that all eigenvalues are non-negative, therefore equal to $\|M^\top(x, \xi)\xi_i\|$. Let us define

$$\lambda_i := \sqrt{\|M^\top(x, \xi)\xi_i\|}, \quad i \in 1..r.$$ 

We have $\lambda_i \in \Lambda_{c;\xi}$, $i \in 1..r$, and it is not hard to check that $M^\top(x, \xi) = \sum_{i=1}^r \lambda_i \otimes \lambda_i$, so by the assumption it follows that $Q(x) \cdot M^\top(x, \xi) = \sum_i Q(x; \lambda_i) \geq 0$ (or equal to 0 in part (b)). The arbitrariness of $(x, \xi)$ gives us the claim.

Q.E.D.

**Remark 10.** Although the statement of the previous theorem is valid for all $c \in [0, \infty]$, the interesting case is only $c \in (0, \infty)$, as for the remaining ones the statement is trivial, at least for $l < m$. Indeed, in the case $c = 0$ we have $p_0(x, \xi) = 0$ for $\xi \in \Sigma_0$, implying $\Lambda_{0;\xi} = C^r$ for every $x$. Similarly, $p_\infty$ vanishes for $\xi \in \Sigma_0$, again resulting in $\Omega = 0$ (or equal to 0 in part (b)).

**Remark 11.** By splitting the matrix into hermitian and antihermitian part, we can generalise part (b) of the previous theorem to an arbitrary $Q \in C(\Omega, M_r(C))$ (cf. [14, Theorem 2(ii)])

Going back to Example 5, we can obtain the same conclusion by applying the last theorem. Indeed, a straightforward computation leads to

$$\bigcup_{c;\xi} \Lambda_{c;\xi} = \left\{ (\lambda_1, 0) : \lambda_1 \in C \right\} \cup \left\{ (0, \lambda_2) : \lambda_2 \in C \right\},$$

and on that set quadratic form $Q(\lambda) = \lambda_1 \lambda_2$ is zero. One needs to be aware that $Q$ is not a hermitian form, but Theorem 12 remains applicable according to the last remark.
Concluding remarks

We have shown that one-scale H-measures encompass features of both H-measures and semiclassical measures (Corollary 6), at the same time providing new properties (Corollary 5) and results. Namely, by improving and generalising Tartar’s localisation principle (theorems 10 and 11, Remark 8), we have obtained a compactness by compensation result suitable for problems with a characteristic length (Theorem 12), opening a new approach for investigating partial differential equations with a characteristic length. Moreover, a more detailed insight into the condition satisfied by the right hand side (4.3) leads to various useful properties (lemmas 6, 7 and 10) and, more importantly, nice behaviour with respect to $(\varepsilon_n)$-oscillatory sequences (Theorem 7) and the corresponding partial differential equations with a characteristic length (Theorem 9), so it seems to be the right choice for further studies. Of course, in applications a need for different scaling $(\varepsilon_n)$ in different variables on the left hand side of (4.9) might arise, which can be partially overcome by the procedure presented in Remark 6. However, a more systematic approach can be devised by introducing suitable variants [28] (see also [5] for problems without a characteristic length and the scaling 1:2).

Further improvement might consist in deriving the propagation principle, as it was the case with classical [26, 1] and parabolic [5] H-measures or, on the other hand, with semiclassical measures [11, 12, 17]. According to that principle, the measure propagates along bicharacteristics of a differential operator under consideration. By means of Corollary 5, its direct consequence in the case of a finite speed of propagation (i.e. for hyperbolic operators) will be $(\varepsilon_n)$-oscillating property of a corresponding sequence of solutions, under the assumption of $(\varepsilon_n)$-oscillatory initial conditions. Further results will rely on generalisations of existing applications of the propagation principle to problems with a characteristic length, as it was done here with the localisation principle.

Conflict of Interest: The authors declare that they have no conflict of interest.

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