Q-ENTROPY FOR GENERAL TOPOLOGICAL DYNAMICAL SYSTEMS

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Abstract. The aim of this paper is to extend the q-entropy from symbolic systems to a general topological dynamical system. Using a (weak) Gibbs measure as the reference measure, this paper defines q-topological entropy and q-metric entropy, then studies basic properties of these entropies. In particular, this paper describes the relations between q-topological entropy and topological pressure for almost additive potentials, and the relations between q-metric entropy and local metric entropy. Although these relations are quite similar to that described in [19], the methods used here need more techniques from the theory of thermodynamic formalism with almost additive potentials.

1. Introduction. A basic issue in the theory of dynamical systems is the study of the complexity of orbits. This has led to the development of many different subjects in mathematics. Since the introduction by Kolmogorov and Sinai, who applied the notion of entropy from information theory to ergodic theory, the concept of entropy has played an important role in understanding the complexity of various dynamical systems. The two main types of entropy are metric entropy and topological entropy. The former measures the maximal loss of information of the iteration of finite partitions in a measure-preserving transformation. The latter measures the maximal exponential growth rate of orbits for any topological dynamical system.

In [19], utilizing a reference measure μ, which gives positive weight to any non-empty open set (but does not have to be invariant under the dynamics), the authors introduced the notions of q-topological and q-metric entropies for symbolic systems, where q ≥ 0 is a parameter. Particularly, in [19, Theorem 2.6], the authors described the relations of q-topological entropy with topological pressure provided that the reference measure is Gibbsian. The purpose of this paper is to investigate some

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properties about \( q \)-topological and \( q \)-metric entropies for general dynamical systems. Motivated by the recent developments of the theory of thermodynamic formalism of almost additive potentials, see [4] and references therein for details, we choose a (weak) Gibbs measure \( \mu \) for almost additive potentials as the reference measure in defining \( q \)-topological and \( q \)-metric entropies. Therefore, we use it to build a particular weight function in the Carathéodory construction as described in [14].

Along with the study in [19], this paper studies some basic properties of the \( q \)-topological and \( q \)-metric entropies, including the relations between \( q \)-topological entropy and almost additive topological pressure. Moreover, we find that the \( q \)-topological entropy is Lipschitz continuous with respect to the parameter \( q \). Furthermore, the dependence can be differentiable if the almost additive potential \( q \Phi \) has a unique equilibrium state for any \( q > 0 \). We also use the notion of \( q \)-topological entropy to introduce the Hentschel-Proccacia entropy spectrum, which is an anal-

The outline of this study is as follows. Section 2 introduces the notions of almost additive topological pressure and the weak Gibbs measure. Section 3 defines the \( q \)-topological entropy, the lower and upper \( q \)-topological entropies and describes some properties of those quantities. In the final part of the paper, Section 4 illustrates different forms of \( q \)-metric entropy and the relationships among those quantities.

2. Notation and preliminaries. Let \( f : X \to X \) be a continuous transformation on a compact metric space \( X \) equipped with metric \( d \). We first recall these definitions of almost additive topological pressure and (weak) Gibbs measure with respect to an almost additive sequence of continuous functions. Using the Carathéodory construction described in [14], we introduce \( q \)-topological entropy and \( q \)-metric entropy with a Gibbsian reference measure, which extends the corresponding notions in [19] to the general topological dynamical system.

2.1. Almost additive thermodynamic formalism. This subsection briefly recalls the notion of almost additive topological pressure which was introduced by Barreira [2] and Mummert [12] independently.

2.1.1. Almost additive potentials and bounded distortion property. Let \( \mathcal{C}(X) \) denote the Banach space of all continuous functions from \( X \) to \( \mathbb{R} \) equipped with the supremum norm \( \| \cdot \| \). A sequence of continuous functions \( \Phi = \{ \phi_n \}_{n \geq 1} \subset \mathcal{C}(X) \) is almost additive, if there exists a constant \( C > 0 \) such that

\[
\phi_n + \phi_m \circ f^m - C \leq \phi_m \circ f^n \leq \phi_n + \phi_m \circ f^n + C, \quad \forall m, n \geq 1.
\]

Denoted by \( \mathcal{M}_f \) and \( \mathcal{E}_f \) the set of all \( f \)-invariant and ergodic \( f \)-invariant Borel probability measures, respectively, on \( X \). For \( x, y \in X \), define a dynamical metric as \( d_n(x, y) = \max \{ d(f^ix,f^iy) : 0 \leq i < n \} \), and let \( B_n(x, \epsilon) = \{ y \in X : d_n(x, y) < \epsilon \} \) denote the dynamical ball centered at \( x \) of radius \( \epsilon \) and length \( n \).

An almost additive sequence of continuous functions \( \Phi = \{ \phi_n \}_{n \geq 1} \) always satisfies the weak bounded distortion property (see [20]), i.e.,

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \frac{\gamma_n(\Phi, \delta)}{n} = 0
\]
where $\gamma_n(\Phi, \delta) = \sup_{x \in X} \{|\phi_n(y) - \phi_n(z)| : y, z \in B_n(x, \delta)|$. Furthermore, if there exists $\delta > 0$ such that $\sup_{n \in \mathbb{N}} \gamma_n(\Phi, \delta) < \infty$, then we say that the sequence of functions $\Phi = \{\phi_n\}_{n \geq 1}$ satisfies the bounded distortion property.

2.1.2. Almost additive topological pressure. Let $Z \subseteq X$ be an arbitrary subset of $X$. Let $I$ be an index set. Fix $\epsilon > 0$, we call $\Gamma = \{B_n(x_i, \epsilon) : i \in I\}$ a cover of $Z$ if $Z \subseteq \bigcup_i B_n(x_i, \epsilon)$. Let $\Phi = \{\phi_n\}_{n \geq 1} \subseteq C(X)$ be an almost additive potential and $s \in \mathbb{R}$, set

$$M(Z, \Phi, s, N, \epsilon) = \inf_{\Gamma} \sum_i \exp(-sn_i + \phi_n(x_i)), \quad (2)$$

where the infimum is taken over all covers $\Gamma = \{B_n(x_i, \epsilon)\}$ of $Z$ with $n_i \geq N$ for all $i$. Since $M(Z, \Phi, s, N, \epsilon)$ is monotonically increasing with $N$, the following limit exists

$$m(Z, \Phi, s, \epsilon) = \lim_{N \to \infty} M(Z, \Phi, s, N, \epsilon).$$

For a fixed set $Z$, it is easy to see that $m(Z, \Phi, s, \epsilon)$ as a function of $s$ has a critical point so that

$$P_Z(f, \Phi, \epsilon) := \inf \{s : m(Z, \Phi, s, \epsilon) = 0\} = \sup \{s : m(Z, \Phi, s, \epsilon) = +\infty\}.$$

**Definition 2.1.** The quantity $P_Z(f, \Phi, \epsilon) := \lim_{\epsilon \to 0} P_Z(f, \Phi, \epsilon)$ is called the almost additive topological pressure of $f$ on the set $Z$ (w.r.t. $\Phi$).

**Remark 2.1.** The above definition is slightly different from the original one given by Barreira [2] and Mummert [12]. However, since an almost additive sequence of continuous functions always satisfies the weak bounded distortion property (1), these two definitions are equivalent (see [9, Proposition 5.2] or [1]).

In Definition 2.1, if $Z = X$ then the following variational principle holds (see [2] or [12]):

$$P_X(f, \Phi) = P_{\text{top}}(f, \Phi) = \sup \left\{h_{\mu}(f) + F_{\mu}(\mu) : \mu \in \mathcal{M}_f\right\}$$

where $F_{\mu}(\mu) := \lim_{n \to \infty} \frac{1}{n} \int \phi_n d\mu$, $P_{\text{top}}(f, \Phi)$ is the classical topological pressure defined via open covers and $h_{\mu}(f)$ is the metric entropy of $f$ with respect to $\mu$, see [17] for the detailed definitions. See [7] and [11] for the variational principle of topological pressure for a larger class of potentials. An $f$-invariant probability measure $\mu$ that attains the supremum is called an equilibrium state for $f$ with respect to $\Phi$.

Given a real number $s \in \mathbb{R}$ and a subset $Z \subset X$, define

$$R(Z, \Phi, s, N, \epsilon) = \inf_{\Gamma} \sum_i \exp(-sn + \phi_N(x_i))$$

where the infimum is taken over all covers $\Gamma$ of $Z$ with $n_i = N$ for all $i$. We set

$$g(Z, \Phi, s, \epsilon) = \lim_{N \to \infty} \inf_{N \to \infty} R(Z, \Phi, s, N, \epsilon)$$

$$\tau(Z, \Phi, s, \epsilon) = \lim_{N \to \infty} \sup_{N \to \infty} R(Z, \Phi, s, N, \epsilon)$$
and define the jump-up points of \( r(Z, \Phi, s, \epsilon) \) and \( \tau(Z, \Phi, s, \epsilon) \) respectively as

\[
CP_Z(f, \Phi, \epsilon) := \inf \{ s : r(Z, \Phi, s, \epsilon) = 0 \} = \sup \{ s : r(Z, \Phi, s, \epsilon) = +\infty \},
\]

\[
CP_Z(f, \Phi) := \lim_{\epsilon \to 0} CP_Z(f, \Phi, \epsilon).
\]

**Definition 2.2.** We call the quantities

\[
CP_Z(f, \Phi) := \lim_{\epsilon \to 0} CP_Z(f, \Phi, \epsilon),
\]

the lower and upper capacity topological pressures of \( f \) on the set \( Z \) (w.r.t. \( \Phi \)).

**Remark 2.2.** If \( Z \subset X \) is \( f \)-invariant, then \( CP_Z(f, \Phi) = CP_Z(f, \Phi) \). Furthermore, if \( Z \subset X \) is compact and \( f \)-invariant, then \( P_Z(f, \Phi) = CP_Z(f, \Phi) \) (see [14] or [1]).

**Remark 2.3.** The limits of \( \epsilon \to 0 \) in Definitions 2.1 and 2.2 exist (see [2] or [12] for a proof).

### 2.2. Gibbs measure

In this subsection, we recall the definition of (weak) Gibbs measure which will be used as a reference measure in defining \( q \)-entropy (see [19]).

**Definition 2.3.** Given an almost additive sequence of continuous functions \( \Phi = \{ \phi_n \}_{n \geq 1} \), we say that a probability measure \( \mu_\Phi \) (not necessary invariant) is a weak Gibbs measure with respect to \( \Phi \) on \( \Lambda \subset X \), if the set \( \Lambda \) has full \( \mu_\Phi \)-measure and there exists \( \epsilon_0 > 0 \) such that for every \( x \in \Lambda \) and \( 0 < \epsilon < \epsilon_0 \) there exists a sequence of positive constants \( \{ K_n \}_{n \geq 1} \) (depending only on \( \epsilon \)) satisfying

\[
\lim_{n \to \infty} \frac{1}{n} \log K_n = 0
\]

and for every \( n \geq 1 \)

\[
K_n^{-1} \leq \frac{\mu_\Phi(B_n(x, \epsilon))}{e^{-nP(\Phi)+\phi_n(x)}} \leq K_n
\]

where \( P(\Phi) = P_{top}(f, \Phi) \) is the topological pressure of \( f \). We say that \( \mu_\Phi \) is a Gibbs measure with respect to \( \Phi \), if there exists \( K > 0 \) such that the same property holds with \( K_n = K \) independent of \( n \).

The previous notion of Gibbs measure is a generalization of the usual one obtained in [2, 12] under the uniformly hyperbolic setting. In fact, in the uniformly hyperbolic setting or mixing subshift of finite type, an almost additive potential always has a weak Gibbs measure whose support is the whole space. Furthermore, if the almost additive potential satisfies the bounded distortion property, then there exists an ergodic Gibbs measure with full support. In the case of additive potentials, these weak Gibbs measures appear in dynamics with some non-uniform hyperbolicity, e.g. [16, 18].

### 3. \( q \)-topological entropy

Throughout this section, we always assume that the topological entropy of the system \( (X, f) \) is finite. This indicates that the almost additive topological pressure is also finite. Consider a reference measure to be a fully supported Gibbsian measure \( \mu_\Phi \) with respect to an almost additive potential \( \Phi = \{ \phi_n \}_{n \geq 1} \), following those methods in [19], we define the \( q \)-topological entropy and the lower and upper \( q \)-topological entropies and describe some properties of these quantities.
3.1. **Definition of q-topological entropy.** Given an almost additive potential \( \Phi = \{ \phi_n \}_{n \geq 1} \), we choose a weak Gibbs measure \( \mu_\Phi \) (with respect to \( \Phi \)) to be the reference measure. Furthermore, we assume that \( \text{supp}\mu_\Phi = X \), otherwise we could take \( X \) to be \( X \cap \text{supp}\mu_\Phi \). By Definition 2.3 we see that \( \mu_\Phi \) gives positive weight to any open ball \( B(x, \epsilon) := \{ y \in X : d(x, y) < \epsilon \} \).

Given \( s \in \mathbb{R}, q \geq 0 \) and \( Z \subset X \), let
\[
W_q(Z, s, N, \epsilon) = \inf \left\{ \sum_i \mu_\Phi(B_n(x_i, \epsilon))^q \exp(-sn_i) \right\},
\]
where the infimum is taken over all covers \( \Gamma = \{ B_n(x_i, \epsilon) \} \) of \( Z \) with \( n_i \geq N \) for all \( i \). Since \( W_q(Z, s, N, \epsilon) \) is monotonically increasing with \( N \), the limit exists and is denoted by
\[
w_q(Z, s, \epsilon) := \lim_{N \to \infty} W_q(Z, s, N, \epsilon).
\]
It is easy to see that \( w_q(Z, s, \epsilon) \) as a function of \( s \) has a critical point. Thus, we set
\[
h_q(f, Z, \epsilon) := \inf \{ s : w_q(Z, s, \epsilon) = 0 \} = \sup \{ s : w_q(Z, s, \epsilon) = \infty \}.
\]

**Definition 3.1.** The quantity \( h_q(f, Z) := \lim_{\epsilon \to 0} h_q(f, Z, \epsilon) \) is called the \( q \)-topological entropy of \( f \) on the set \( Z \).

Given \( s \in \mathbb{R}, N > 0, q \geq 0 \) and \( Z \subset X \), define
\[
S_q(Z, s, N, \epsilon) = \inf \left\{ \sum_i \mu_\Phi(B_N(x_i, \epsilon))^q \exp(-sN) \right\},
\]
where the infimum is taken over all covers \( \Gamma = \{ B_N(x_i, \epsilon) \} \) of \( Z \). We set
\[
\underline{S}_q(Z, s, \epsilon) = \liminf_{N \to \infty} S_q(Z, s, N, \epsilon),
\]
\[
\overline{S}_q(Z, s, \epsilon) = \limsup_{N \to \infty} S_q(Z, s, N, \epsilon)
\]
and define the “jump-up” points of \( \underline{S}_q(Z, s, \epsilon) \) and \( \overline{S}_q(Z, s, \epsilon) \) as
\[
\underline{h}_q(f, Z, \epsilon) = \inf \{ s : \underline{S}_q(Z, s, \epsilon) = 0 \} = \sup \{ s : \underline{S}_q(Z, s, \epsilon) = \infty \},
\]
\[
\overline{h}_q(f, Z, \epsilon) = \inf \{ s : \overline{S}_q(Z, s, \epsilon) = 0 \} = \sup \{ s : \overline{S}_q(Z, s, \epsilon) = \infty \}
\]
respectively.

**Definition 3.2.** The quantities
\[
\underline{h}_q(f, Z) := \lim_{\epsilon \to 0} \underline{h}_q(f, Z, \epsilon) \quad \text{and} \quad \overline{h}_q(f, Z) := \lim_{\epsilon \to 0} \overline{h}_q(f, Z, \epsilon)
\]
are called the lower and upper \( q \)-topological entropies of \( f \) on the set \( Z \), respectively.

**Remark 3.1.** Since \( h_q(f, Z, \epsilon), \underline{h}_q(f, Z, \epsilon) \) and \( \overline{h}_q(f, Z, \epsilon) \) are monotone with respect to \( \epsilon \), the limits in Definitions 3.1 and 3.2 do exist, which follow the generalized Carathéodory construction described in [14]. We refer the reader to [14] for detailed discussions of Carathéodory construction.

**Remark 3.2.** If \( q = 0 \), then \( h_q(f, Z), \underline{h}_q(f, Z) \) and \( \overline{h}_q(f, Z) \) are the standard Bowen’s topological entropy and Bowen’s lower and upper topological entropies of \( f \) on the set \( Z \) (see [5] or [14] for details). To avoid any confusion regarding the definition of metric entropies, although they depend on the reference Gibbsian measure \( \mu_\Phi \), we do not emphasize the dependence on \( \mu_\Phi \) in the notations.
3.2. Basic properties of $q$-topological entropy. The following properties follow immediately from the definitions and Theorems 1.1, 2.1 and 2.4 in [14].

**Proposition 3.1.** For any $q \geq 0$, the following statements hold:

1. if $Z_1 \subset Z_2$, then $\mathcal{H}_q(f, Z_1, \epsilon) \leq \mathcal{H}_q(f, Z_2, \epsilon)$ for each $\epsilon > 0$, and $\mathcal{H}_q(f, Z_1) \leq \mathcal{H}_q(f, Z_2)$ where $\mathcal{H}_q$ denotes either $h_q$ or $\overline{h}_q$ or $\underline{h}_q$;
2. if $Z_i \subset X$, $i \geq 1$ and $Z = \bigcup_{i \geq 1} Z_i$, then $h_q(f, Z) = \sup_{i \geq 1} h_q(f, Z_i)$, $\overline{h}_q(f, Z) \geq \sup_{i \geq 1} \overline{h}_q(f, Z_i)$;
3. $h_q(f, Z) \leq \overline{h}_q(f, Z) \leq \overline{h}_q(f, Z)$ for any $Z \subset X$.

Given a subset $Z \subset X$ and a real number $q \geq 0$, set

$$
\Lambda_q(Z, n, \epsilon) = \inf \left\{ \sum_i \mu(\text{B}(x_i, \epsilon))^q \right\},
$$

where the infimum is taken over all covers $\Gamma = \{\text{B}_n(x_i, \epsilon)\}$ of $Z$. The following equivalent description of the lower and upper $q$-topological entropies follows immediately from definitions and Theorem 2.2 in [14].

**Proposition 3.2.** For any subset $Z \subset X$ and any $q \geq 0$, the following properties hold:

$$
\underline{h}_q(f, Z, \epsilon) = \liminf_{n \to \infty} \frac{1}{n} \log \Lambda_q(Z, n, \epsilon),
$$

$$
\overline{h}_q(f, Z, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \Lambda_q(Z, n, \epsilon).
$$

Consequently, we have that

$$
\underline{h}_q(f, Z) = \liminf_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \Lambda_q(Z, n, \epsilon),
$$

$$
\overline{h}_q(f, Z) = \limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \Lambda_q(Z, n, \epsilon).
$$

3.3. Relations between $q$-topological entropies and topological pressures. The following property gives the relations between $q$-topological entropies and topological pressures, which extends Theorem 2.6 in [19] to general topological dynamical systems.

**Theorem 3.1.** Assume that $\mu_\Phi$ is a fully supported Gibbs measure with respect to an almost additive potential $\Phi = \{\phi_n\}_{n \geq 1}$. For any $Z \subset X$ and any $q \geq 0$, we have

$$
\mathcal{H}_q(f, Z) = -qP_X(f, \Phi) + \mathcal{P}_Z(f, q\Phi)
$$

where $\mathcal{H}_q$ denotes either $h_q$ or $\overline{h}_q$ or $\underline{h}_q$, and $\mathcal{P}_Z$ denotes either $P_Z$ or $\overline{P}_Z$ or $\underline{P}_Z$.

**Proof.** It yields from the Gibbs property of $\mu_\Phi$ that

$$
K(\epsilon)^{-1} \leq \frac{\mu_\Phi(\text{B}(x, \epsilon))}{e^{-n\text{P}_X(f, \Phi) + \phi_n(x)}} \leq K(\epsilon)
$$

for all $x \in X$ and all $n \geq 1$. Hence, for a fixed cover $\Gamma = \{\text{B}_n(x_i, \epsilon)\}$ of $Z$ with $n_i \geq N$ for all $i$, we have

$$
\sum_i \mu_\Phi(\text{B}(x_i, \epsilon))^q \exp(-sn_i) \leq K(\epsilon)^q \sum_i \exp(-n_i(qP_X(f, \Phi) + s) + q\phi_n(x_i)).
$$
It follows that
\[ W_q(Z, s, N, \epsilon) \leq K(\epsilon)^q M(Z, q\Phi, s + q P_X(f, \Phi), N, \epsilon) \]
where \( q\Phi = \{q\phi_n\}_{n \geq 1} \). Letting \( N \to \infty \), we have
\[ w_q(Z, s, \epsilon) \leq K(\epsilon)^q m(Z, q\Phi, s + q P_X(f, \Phi), \epsilon). \]
Hence,
\[ h_q(f, Z, \epsilon) \leq P_Z(f, q\Phi, \epsilon) - q P_X(f, \Phi). \]
This implies that
\[ h_q(f, Z) \leq P_Z(f, q\Phi) - q P_X(f, \Phi). \]
Analogously, using the left hand side of inequality (6) we obtain the following lower bound of the \( q \)-topological entropy
\[ h_q(f, Z) \geq P_Z(f, q\Phi) - q P_X(f, \Phi). \]
Hence,
\[ h_q(f, Z) = P_Z(f, q\Phi) - q P_X(f, \Phi). \]
The other two equalities for \( h_q \) and \( \overline{h}_q \) can be shown in a similar fashion. \( \Box \)

**Remark 3.3.** If \( \mu_{\Phi} \) is a fully supported weak Gibbs measure with respect to an almost additive potential \( \Phi = \{\phi_n\}_{n \geq 1} \), then using Proposition 3.2, we can still determine that \( h_q(f, Z) = -q P_X(f, \Phi) + CP_Z(f, q\Phi) \) and \( \overline{h}_q(f, Z) = -q P_X(f, \Phi) + \overline{CP}_Z(f, q\Phi) \). However, we do not know whether \( h_q(f, Z) = -q P_X(f, \Phi) + P_Z(f, q\Phi) \) holds in this case.

**Corollary 3.1.** For any \( q \geq 0 \) the following statements hold:
1. if \( Z \subset X \) is \( f \)-invariant, then \( h_q(f, Z) = \overline{h}_q(f, Z); \)
2. if \( Z \subset X \) is \( f \)-invariant and compact, then \( h_q(f, Z) = h_q(f, Z) = \overline{h}_q(f, Z). \)

**Proof.** These results follow from Theorem 3.1 and Remark 2.2. \( \Box \)

3.4. **Computation formula of \( q \)-topological entropies.** In this subsection, we describe a formula that allows one to compute the lower and upper \( q \)-topological entropies for any \( q \geq 1 \). Note that the function \( x \mapsto \mu_{\Phi}(B_n(x, \epsilon))^q - 1 \) is measurable. Due to boundness, the function is also integrable. For any measurable set \( Z \subset X \) and \( q \geq 1 \), set
\[ \varphi_q(Z, n, \epsilon) = \int_Z \mu_{\Phi}(B_n(x, \epsilon))^q - 1 \, d\mu_{\Phi}(x). \]
The same computation formula hold in symbolic dynamics (see Theorem 2.4 in [19]). However, the proof of the following theorem depends on the Gibbs property of the reference measure.

**Theorem 3.2.** The following statements hold:
1. for any \( q \geq 1 \) and any measurable set \( Z \subset X \),
\[ h_q(f, Z) \geq \liminf_{n \to \infty} \frac{1}{n} \log \varphi_q(Z_n, n, \epsilon), \quad \overline{h}_q(f, Z) \geq \limsup_{n \to \infty} \frac{1}{n} \log \varphi_q(Z_n, n, \epsilon), \]
and
\[ h_q(f, Z) \leq \liminf_{n \to \infty} \frac{1}{n} \log \varphi_q(Z_n, n, \epsilon), \quad \overline{h}_q(f, Z) \leq \limsup_{n \to \infty} \frac{1}{n} \log \varphi_q(Z_n, n, \epsilon), \]
where \( (Z)_n = \bigcup_{x \in Z} B_n(x, \epsilon); \)
2. for any set $Z$ of full $\mu_\Phi$-measure and any $q \geq 1$,

$$h_q(f, Z) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \varphi_q(Z, n, \epsilon), \quad \overline{h}_q(f, Z) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \varphi_q(Z, n, \epsilon).$$

Proof. Given a dynamical ball $B_n(x, \epsilon)$, for any $y \in B_n(x, \epsilon)$, it follows from the Gibbs property of $\mu_\Phi$ that

$$K(\epsilon)^{-2} \exp[-\gamma_n(\Phi, \epsilon)] \leq \frac{\mu_\Phi(B_n(x, \epsilon))}{\mu_\Phi(B(y, \epsilon))} \leq K(\epsilon)^2 \exp[\gamma_n(\Phi, \epsilon)], \quad (7)$$

see (1) for the definition of $\gamma_n(\Phi, \epsilon)$.

Fixing a small number $\beta > 0$, for any $n \in \mathbb{N}$ there is a cover $\Gamma = \{B_n(x_i, \epsilon)\}_{i \geq 1}$ of $Z$ such that

$$\sum_i \mu_\Phi(B_n(x_i, \epsilon))^q \leq \Lambda_q(Z, n, \epsilon) + \beta.$$

Using the estimation in (7), it can be seen that

$$\sum_i \mu_\Phi(B_n(x_i, \epsilon))^q = \sum_i \int_{B_n(x_i, \epsilon)} \mu_\Phi(B_n(x_i, \epsilon))^{q-1} \, d\mu_\Phi(x)$$

$$\geq K(\epsilon)^{-2(q-1)} \exp[-(q-1)\gamma_n(\Phi, \epsilon)] \sum_i \int_{B_n(x_i, \epsilon)} \mu_\Phi(B_n(x_i, \epsilon))^{q-1} \, d\mu_\Phi(x)$$

$$\geq K(\epsilon)^{-2(q-1)} \exp[-(q-1)\gamma_n(\Phi, \epsilon)] \varphi_q(Z, n, \epsilon).$$

Since $\beta$ can be chosen any arbitrarily small value, it follows that

$$\Lambda_q(Z, n, \epsilon) \geq K(\epsilon)^{-2(q-1)} \exp[-(q-1)\gamma_n(\Phi, \epsilon)] \varphi_q(Z, n, \epsilon).$$

By Proposition 3.2 and the weak bounded distortion property (1) of almost additive potentials, it can be seen that

$$h_q(f, Z) \geq \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \varphi_q(Z, n, \epsilon), \quad \text{and} \quad \overline{h}_q(f, Z) \geq \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \varphi_q(Z, n, \epsilon).$$

To prove the reverse inequality, given $n \in \mathbb{N}$, choose a maximal $(n, 2\epsilon)$-separated subset $E \subset Z$, i.e., $d_n(x, y) \geq 2\epsilon$ for all distinct $x, y \in E$. Moreover, we have that $Z \subset \bigcup_{x \in E} B_n(x, 2\epsilon)$ and $\{B_n(x, \epsilon) : x \in E\}$ are pairwise disjoint. Using (7), we have

$$\Lambda_q(Z, n, 2\epsilon) \leq \sum_{x_i \in E} \mu_\Phi(B_n(x_i, 2\epsilon))^q$$

$$\leq [K(2\epsilon)K(\epsilon)]^q \sum_{x_i \in E} \mu_\Phi(B_n(x_i, \epsilon))^q$$

$$= [K(2\epsilon)K(\epsilon)]^q \sum_{x_i \in E} \int_{B_n(x_i, \epsilon)} \mu_\Phi(B_n(x_i, \epsilon))^{q-1} \, d\mu_\Phi(x)$$

$$\leq C(\epsilon) \exp[(q-1)\gamma_n(\Phi, \epsilon)] \sum_{x_i \in E} \int_{B_n(x_i, \epsilon)} \mu_\Phi(B_n(x, \epsilon))^{q-1} \, d\mu_\Phi(x)$$

$$\leq C(\epsilon) \exp[(q-1)\gamma_n(\Phi, \epsilon)] \int_{\bigcup_{x \in E} B_n(x, \epsilon)} \mu_\Phi(B_n(x, \epsilon))^{q-1} \, d\mu_\Phi(x)$$

$$= C(\epsilon) \exp[(q-1)\gamma_n(\Phi, \epsilon)] \varphi_q(Z, n, \epsilon).$$

where we use the Gibbs property of $\mu_\Phi$ in the second inequality and $C(\epsilon) = [K(2\epsilon)K(\epsilon)]^q K(\epsilon)^2(q-1)$. This yields the desired results. The second statement now is a direct consequence of the first one.
Remark 3.4. The above theorem holds even if $\mu_\Phi$ is a fully supported weak Gibbs measure with respect to an almost additive potential.

3.5. **Dependence on the parameter $q$.** If one considers $q$-topological entropy and the lower and upper $q$-topological entropies as functions over $q \geq 0$, then we have the following proposition.

**Theorem 3.3.** Assume that $\mu_\Phi$ is a fully supported Gibbs measure with respect to an almost additive potential $\Phi = \{\phi_n\}_{n \geq 1}$. The following statements hold:

1. $\mathcal{H}_{q_1}(f, Z) \geq \mathcal{H}_{q_2}(f, Z)$ for any $Z \subset X$ and any $0 \leq q_1 \leq q_2$, where $\mathcal{H}_q$ denotes either $h_q$ or $\underline{h}_q$ or $\overline{h}_q$;
2. if $\mu_\Phi(Z) = 0$, then $h_1(f, Z) \leq h_1(f, Z) \leq \overline{h}_1(f, Z) \leq 0$ and hence, $h_q(f, Z) \leq \underline{h}_q(f, Z) \leq \overline{h}_q(f, Z) \leq 0$ for any $q \geq 1$;
3. if $\mu_\Phi(Z) > 0$, then $h_1(f, Z) = h_1(f, Z) = \overline{h}_1(f, Z) = 0$ and hence,
   
   $0 \leq h_q(f, Z) \leq \underline{h}_q(f, Z) \leq \overline{h}_q(f, Z) \quad \text{if } 0 \leq q \leq 1$

   and
   
   $h_q(f, Z) \leq \underline{h}_q(f, Z) \leq \overline{h}_q(f, Z) \leq 0 \quad \text{if } q \geq 1$;
4. for any fixed subset $Z \subset X$, the function $q \mapsto \mathcal{H}_q(f, Z)$ is Lipschitz continuous, where $\mathcal{H}_q$ denotes either $h_q$ or $\underline{h}_q$ or $\overline{h}_q$;
5. if the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous and for any $q > 0$, the almost additive potentials $q\Phi = \{q\phi_n\}_{n \geq 1}$ have a unique equilibrium Gibbs measure $\mu_q$, then the map $q \mapsto \mathcal{H}_q(f, X)$ is differentiable, where $\mathcal{H}_q$ denotes either $h_q$ or $\underline{h}_q$ or $\overline{h}_q$.

**Proof.** The first statement is obvious. For $q = 1$, $n \in \mathbb{N}$ and every small $\epsilon > 0$, choose a maximal $(n, \epsilon)$-separated subset $E \subset Z$ such that $Z \subset \bigcup_{x \in E} B_n(x, \epsilon)$ and $\{B_n(x, \epsilon/2) : x \in E\}$ are pairwise disjoint. Then, by (5) and the Gibbs property of $\mu_\Phi$, it can be seen that

$$
\mu_\Phi(Z) \leq K(\epsilon)K(\frac{\epsilon}{2}) \sum_{x_i \in E} \mu_\Phi(B_n(x_i, \epsilon) \leq K(\epsilon)K(\frac{\epsilon}{2}).
$$

(8)

The above fact together with Proposition 3.1 and the first statement yield the second statement.

To prove the third statement, first note that the inequality (8) implies that $\overline{h}_1(f, Z) = \overline{h}_1(f, Z) = 0$ in the case of $\mu_\Phi(Z) > 0$. On the other hand, for any $s < 0$ and any $\epsilon > 0$ by a direct computation we have

$$w_1(Z, s, \epsilon) = +\infty.$$

Since $s$ and $\epsilon$ can be chosen arbitrarily, this implies that $h_1(f, Z) \geq 0$. Hence, $h_1(f, Z) = \underline{h}_1(f, Z) = \overline{h}_1(f, Z) = 0$. The rest of the results follow from Proposition 3.1 and the first statement.

To prove the fourth statement, by Theorem 3.1 we have $h_q(f, Z) = -qP_X(f, \Phi) + P_Z(f, q\Phi)$. Hence, for any $q_1, q_2 > 0$, it can be seen that

$$|h_{q_1}(f, Z) - h_{q_2}(f, Z)| \leq |P_X(f, \Phi)| \cdot |q_1 - q_2| + |P_Z(f, q_1\Phi) - P_Z(f, q_2\Phi)|.$$

On the other hand, by the almost additivity of $\Phi$ we have that

$$-n(\|\phi_1\| + C) \leq \phi_n(x) \leq n(\|\phi_1\| + C), \quad \forall x \in X.$$
Hence, for \( q_1 > q_2 > 0 \), we find that
\[
M(Z, q_2 \Phi, s - (q_1 - q_2)(\|\phi_1\| + C), N, \epsilon) \\
\geq M(Z, q_1 \Phi, s, N, \epsilon) \\
\geq M(Z, q_2 \Phi, s + (q_1 - q_2)(\|\phi_1\| + C), N, \epsilon),
\]
which implies that
\[
-(q_1 - q_2)(\|\phi_1\| + C) \leq P_Z(q_2 \Phi) - P_Z(q_1 \Phi) \leq (q_1 - q_2)(\|\phi_1\| + C).
\]
Therefore, for any \( q_1, q_2 > 0 \) we have
\[
|h_{q_1}(f, Z) - h_{q_2}(f, Z)| \leq |q_2 - q_1|(P_X(f, \Phi) + \|\phi_1\| + C).
\]
This proves the Lipschitz continuity of the map \( q \mapsto h_q(f, Z) \). The Lipschitz continuity of the maps \( q \mapsto h_q(f, Z) \) and \( q \mapsto \overline{h}_q(f, Z) \) can be proven in a similar fashion.

To complete the proof of the theorem, we first note that \( h_q(f, X) = h_q(f, X) = \overline{h}_q(f, X) = -qP_X(f, \Phi) + P_X(f, q\Phi) \) by Theorem 3.1 and Remark 2.2. Since the entropy map \( \mu \mapsto h_q(\mu) \) is upper semi-continuous and the map \( \mu \mapsto F_* (\mu, \Phi) \) is continuous (see [3, Proposition 3(3)]) and since \( q\Phi \) has a unique equilibrium state \( \mu_q \) for every \( q > 0 \), by Theorem 2 in [3], it can been seen that
\[
\frac{d}{dq} P_X(f, q\Phi) \mid_{q=q_0} = F_* (\mu_{q_0}, \Phi).
\]
This yields that
\[
\frac{d}{dq} h_q(f, X) \mid_{q=q_0} = -P_X(f, \Phi) + F_* (\mu_{q_0}, \Phi). \tag{9}
\]
Therefore, this completes the proof of the theorem. \( \square \)

Some comments on the above theorem are in order: the first three statements are similar to those in [19, Proposition 2.5]; the last two statements are new phenomena, the fourth is a direct consequence of Theorem 3.1 and the last one follows from the theory of almost additive thermodynamic formalism. In particular, if \( q_0 = 1 \) it yields from (9) that
\[
\frac{d}{dq} h_q(f, X) \mid_{q=1} = -P_X(f, \Phi) + F_* (\mu_1, \Phi) = -h_{\mu_1}(f),
\]
since \( \mu_1 = \mu_\Phi \) is the unique equilibrium state for \( f \) with respect to \( \Phi \); if \( \Phi \) satisfies the bounded distortion property, then \( q\Phi \) also satisfies this property for every \( q > 0 \). Precisely, if \( \sup_{n \in \mathbb{N}} \gamma_n(\Phi, \delta) < \infty \) for some \( \delta > 0 \), then for each \( q > 0 \) we have that
\[
\sup_{n \in \mathbb{N}} \gamma_n(q\Phi, \delta) = q \sup_{n \in \mathbb{N}} \gamma_n(\Phi, \delta) < \infty.
\]

3.6. The Hentschel-Procaccia entropy spectrum. We follow the approach in [19] to introduce the notion of the Hentschel-Procaccia entropy spectrum, which is originally due to the Hentschel-Procaccia dimension spectrum in [14].

Let \( \mu_\Phi \) be a fully supported Gibbs measure of an almost additive potential \( \Phi = \{\phi_n\}_{n \geq 1} \), and for every \( q > 1 \), define
\[
\overline{h}_P_q(\mu_\Phi) = \frac{1}{q-1} \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \int_X \mu_\Phi(B_n(x, \epsilon))^{q-1} d\mu_\Phi(x),
\]
\[
\underline{h}_P_q(\mu_\Phi) = \frac{1}{q-1} \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \int_X \mu_\Phi(B_n(x, \epsilon))^{q-1} d\mu_\Phi(x).
\]
Definition 3.3. The one-parameter family of pairs of quantities
\[(\mathcal{H}_q^{\mathcal{P}}(\mu_\Phi), \mathcal{H}_q^{\mathcal{P}}(\mu_\Phi))\]
is called the \textit{HP-spectrum for entropies}.

It follows from Theorems 3.1 and 3.2 that
\begin{align*}
\mathcal{H}_q^{\mathcal{P}}(\mu_\Phi) &= \frac{1}{q-1} \mathcal{L}_q(f, X) = \frac{1}{q-1} (-qP_X(f, \Phi) + C_P X(f, q\Phi)), \\
\mathcal{H}_q^{\mathcal{P}}(\mu_\Phi) &= \frac{1}{q-1} \mathcal{L}_q(f, X) = \frac{1}{q-1} (-qP_X(f, \Phi) + C_P X(f, q\Phi))
\end{align*}
(10)

for any \(q > 1\). As a manifestation of formula (10) we obtain the following:

1. By Remark 2.2, we have \(\mathcal{H}_q^{\mathcal{P}}(\mu_\Phi) = \mathcal{H}_q^{\mathcal{P}}(\mu_\Phi)\). We denote the common value by \(\mathcal{H}_q^{\mathcal{P}}(\mu_\Phi)\). Moreover, by (10) and the Gibbs property of \(\mu_\Phi\), it can been seen that
\[
\lim_{n \to \infty} \frac{\log \int_X e^{(q-1)\phi_n(x)} d\mu_\Phi}{n} = (q-1)(P_X(f, \Phi) + \mathcal{H}_q^{\mathcal{P}}(\mu_\Phi)) = -P_X(f, \Phi) + P_X(f, q\Phi).
\]
The left hand side is very useful in the physics and large deviations literature since its Legendre transform is an upper bound for the measure of deviation sets. See [10, Lemma 2.1] for the case of topological mixing subshift of finite type with a H"older continuous potential.

2. It is a direct consequence of Theorem 3.3 and (10) that \(\mathcal{H}_q^{\mathcal{P}}(\mu_\Phi)\) is continuous over \(q > 1\).

3. Moreover, if the entropy map \(\mu \mapsto h_\mu(f)\) is upper semi-continuous and the almost additive potential \(\Phi\) satisfies those assumptions in Theorem 3.3, then the map \(q \mapsto \mathcal{H}_q^{\mathcal{P}}(\mu_\Phi)\) is differentiable over \(q > 1\).

4. \textbf{q-metric entropy.} In this section, following the approach described in [14], we introduce different types of \(q\)-metric entropy and study their properties and the relations between them. Throughout this section, \(\mu_\Phi\) is the fully supported reference Gibbsian measure with respect to an almost additive potential \(\Phi = \{\phi_n\}_{n \geq 1}\).

4.1. \textbf{Definition of the \(q\)-metric entropy.} Given \(q \geq 0\) and a Borel probability measure \(\nu\), let
\[
h_q(f, \nu) = \liminf_{\epsilon \to 0} \{h_q(f, Z, \epsilon) : \nu(Z) = 1\} = \lim_{\epsilon \to 0} \liminf_{\delta \to 0} \{h_q(f, Z, \epsilon) : \nu(Z) \geq 1 - \delta\}.
\]
We call the quantity \(h_q(f, \nu)\) the \textit{\(q\)-metric entropy} of \(f\) with respect to \(\nu\). Further, we set
\[
\underline{h}_q(f, \nu) = \liminf_{\epsilon \to 0} \liminf_{\delta \to 0} \{h_q(f, Z, \epsilon) : \nu(Z) \geq 1 - \delta\},
\]
\[
\overline{h}_q(f, \nu) = \liminf_{\epsilon \to 0} \liminf_{\delta \to 0} \{\overline{h}_q(f, Z, \epsilon) : \nu(Z) \geq 1 - \delta\}.
\]
We call the quantities \(\underline{h}_q(f, \nu)\) and \(\overline{h}_q(f, \nu)\), respectively the \textit{lower} and \textit{upper \(q\)-metric entropy} of \(f\) with respect to \(\nu\). We mention that the reference Gibbsian measure \(\mu_\Phi\) can be different from \(\nu\).
4.2. Relations between different \( q \)-metric entropies. This subsection studies the relations between various versions of \( q \)-metric entropies.

**Proposition 4.1.** The following statements hold:

1. the \( q \)-metric entropies with respect to the reference Gibbsian measure \( \mu_\Phi \) satisfies that \( h_1(f, \mu_\Phi) = h_q(f, \mu_\Phi) = \overline{h}_1(f, \mu_\Phi) = 0 \);
2. for any \( q \geq 0 \) and any Borel probability measure \( \nu \) we have
   \[
   h_q(f, \nu) \leq h_q(f, \nu) \leq \overline{h}_q(f, \nu);
   \]
3. for any \( q \geq 0 \) and any \( f \)-invariant ergodic measure \( \nu \) we have that
   \[
   h_q(f, \nu) = \overline{h}_q(f, \nu) = h_q(f) + qF_\ast(\nu, \Phi) - qP_X(f, \Phi).
   \]
   Consequently, we have that \( \overline{h}_q(f, \nu) \leq h_\nu(f) \) for any \( q \geq 0 \). Here \( h_\nu(f) \) is the standard metric entropy of \( f \) with respect to \( \nu \), see [17] for details;
4. furthermore, we have \( 0 \leq h_q(f, \mu_\Phi) \leq \overline{h}_q(f, \mu_\Phi) \leq \overline{h}_q(f, \mu_\Phi) \) if \( 0 \leq q \leq 1 \) and, \( h_q(f, \mu_\Phi) \leq \overline{h}_q(f, \mu_\Phi) \leq 0 \) if \( q \geq 1 \).

**Proof.** The first statement is a direct consequence of statement (3) of Theorem 3.3 and the definitions. The second statement follows immediately from statement (3) of Proposition 3.1.

To prove the third statement, first note that by the proof of Theorem 3.1, it can be seen that
\[
h_q(f, Z, \epsilon) = P_Z(f, q\Phi, \epsilon) - qP_X(f, \Phi), \quad \forall Z \subset X.
\]
By Theorem A in [8] we can obtain that
\[
h_\nu(f) + qF_\ast(\nu, \Phi) = \liminf_{\epsilon \to 0} \{ P_Z(f, q\Phi, \epsilon) : \nu(Z) = 1 \}.
\]
Hence,
\[
h_q(f, \nu) = h_\nu(f) + qF_\ast(\nu, \Phi) - qP_X(f, \Phi).
\]
The formula of the lower and upper \( q \)-metric entropies of \( f \) with respect to \( \nu \) can be proven in a similar fashion. Consequently, for any \( q \geq 0 \) by the variational principle of almost additive topological pressure we have
\[
\overline{h}_q(f, \nu) \leq h_\nu(f) + qF_\ast(\nu, \Phi) - q(h_\nu(f) + F_\ast(\nu, \Phi)) = (1 - q)h_\nu(f) \leq h_\nu(f).
\]
The last statement follows directly from Theorem 3.3 and the above statement (1). This completes the proof of the proposition. \( \square \)

**Remark 4.1.** If \( \mu_\Phi \) is the unique Gibbsian equilibrium state for \( f \) with respect to \( \Phi \), then it is ergodic and by statement (3) of the above proposition we have
\[
h_q(f, \mu_\Phi) = h_q(f, \mu_\Phi) = \overline{h}_q(f, \mu_\Phi) = h_{\mu_\Phi}(f) + qF_\ast(\mu_\Phi, \Phi) - q(h_{\mu_\Phi}(f) + F_\ast(\mu_\Phi, \Phi)) = (1 - q)h_{\mu_\Phi}(f).
\]

Given a Borel probability measure \( \nu \), set
\[
h_\nu(x) := \lim_{\epsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \nu(B_n(x, \epsilon)) \quad \text{and} \quad \overline{h}_\nu(x) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \nu(B_n(x, \epsilon)).
\]
These two quantities are called the local lower and upper metric entropy at point \( x \) with respect to \( \nu \), respectively. Particularly, if \( \nu \) is an \( f \)-invariant and ergodic measure, then \( h_\nu(x) = \overline{h}_\nu(x) = h_\nu(f) \) for \( \nu \)-almost every point \( x \in X \) (see [6]).
By Remark 4.1, if the reference Gibbsian measure $\mu_\Phi$ satisfies some additional conditions, then the $q$–metric entropy $h_q(f, \mu_\Phi)$ is linear over $q \geq 0$. However, in general we have the following results.

**Theorem 4.1.** The following statements hold:

1. For any $q > 0$, 
   \[(1 - q)\text{ess inf}_{x \in X} \overline{h}_{q_\Phi}(x) \leq h_q(f, \mu_\Phi) \leq \overline{h}_q(f, \mu_\Phi) \leq (1 - q)\text{ess inf}_{x \in X} \underline{h}_{q_\Phi}(x);\]
2. For any $0 \leq q < 1$, 
   \[(1 - q)\text{ess sup}_{x \in X} \underline{h}_{q_\Phi}(x) \leq h_q(f, \mu_\Phi) \leq \overline{h}_q(f, \mu_\Phi) \leq (1 - q)\text{ess sup}_{x \in X} \overline{h}_{q_\Phi}(x).\]

**Proof.** We will prove the first statement; the second one can be proven in a similar fashion. For each $x \in X$ and $\epsilon > 0$, set 
\[\underline{h}_{q_\Phi}(x, \epsilon) := \liminf_{n \to \infty} -\frac{1}{n} \log \mu_\Phi(B_n(x, \epsilon))\]
and 
\[\overline{h}_{q_\Phi}(x, \epsilon) := \limsup_{n \to \infty} -\frac{1}{n} \log \mu_\Phi(B_n(x, \epsilon)).\]

Given a small number $\beta > 0$ and a positive integer $M$, we define the set 
\[\mathcal{L}_M = \{x \in X : \underline{h}_{q_\Phi}(x) - \frac{\beta}{2} \leq \overline{h}_{q_\Phi}(x, \epsilon) \leq \overline{h}_{q_\Phi}(x) \leq \overline{h}_{q_\Phi}(x) + \frac{\beta}{2}, \forall 0 < \epsilon \leq \frac{1}{M}\}.\]
Clearly, $\mathcal{L}_{M_2} \subset \mathcal{L}_{M_1}$ if $M_1 > M_2$ and $X = \bigcup_{M=1}^{\infty} \mathcal{L}_M$. For any positive integer $M$, set 
\[\mathcal{L}_{M,N} = \{x \in \mathcal{L}_M : \underline{h}_{q_\Phi}(x) - \beta \leq -\frac{1}{n} \log \mu_\Phi(B_n(x, \epsilon)) \leq \overline{h}_{q_\Phi}(x) + \beta, \forall n \geq N\}\]
It is obvious that $\mathcal{L}_{M,N_2} \subset \mathcal{L}_{M,N_1}$ if $N_1 > N_2$ and $\bigcup_{N=1}^{\infty} \mathcal{L}_{M,N} = \mathcal{L}_M.

For any $\delta > 0$ there exist positive integers $M_1$ and $N_1$ such that $\mu_\Phi(\mathcal{L}_{M,N}) > 1 - \delta$ for any $M > M_1$ and any $N \geq N_1$. Take such positive integers $M$ and $N$. It is easy to see that for any $x \in \mathcal{L}_{M,N}$, $n \geq N$ and $0 < \epsilon \leq \frac{1}{M}$, 
\[\exp(-n(\overline{h}_{q_\Phi}(x) + \beta)) \leq \mu_\Phi(B_n(x, \epsilon)) \leq \exp(-n(\underline{h}_{q_\Phi}(x) - \beta)).\] (11)

To simplify our notations, let 
\[\underline{h} = \text{ess inf}_{x \in X} \underline{h}_{q_\Phi}(x)\] and 
\[\overline{h} = \text{ess inf}_{x \in X} \overline{h}_{q_\Phi}(x).\]

For $\mu_\Phi$–almost every $x$ we have $\underline{h}_{q_\Phi}(x) \geq \underline{h} - \beta$. Furthermore, there exists a subset $\mathcal{X}$ of positive $\mu_\Phi$–measure such that $\overline{h}_{q_\Phi}(x) \leq \overline{h} + \beta$ for every $x \in \mathcal{X}$. Take $Z \subset \mathcal{L}_{M,N} \cap \{x : \underline{h}_{q_\Phi}(x) \geq \underline{h} - \beta\}$ such that $\mu_\Phi(Z) > 1 - \delta$. For any $n \geq N$, choose a maximal $(n, \epsilon)$–separated subset $E \subset Z$ such that $Z \subset \bigcup_{n \in E} B_n(x, \epsilon)$ and $\{B_n(x, \epsilon/2) : x \in E\}$ are pairwise disjoint. By (11) and the Gibbs property of $\mu_\Phi$, for $q \geq 1$ we have that 
\[
\sum_{x \in E} \mu_\Phi(B_n(x, \epsilon))^q = \sum_{x \in E} \mu_\Phi(B_n(x, \epsilon))^{q-1} \mu_\Phi(B_n(x, \epsilon)) \\
\leq \exp\left(-n(\underline{h} - 2\beta)(q - 1)\right) \sum_{x \in E} \mu_\Phi(B_n(x, \epsilon)) \\
\leq \exp\left(-n(\underline{h} - 2\beta)(q - 1)\right) K(\epsilon) K\left(\frac{\epsilon}{2}\right) \sum_{x \in E} \mu_\Phi(B_n(x, \epsilon/2)) \\
\leq \exp\left(-n(\underline{h} - 2\beta)(q - 1)\right) K(\epsilon) K\left(\frac{\epsilon}{2}\right).\] (12)
Hence,
\[ \Lambda_q(Z, n, \epsilon) \leq \exp(-n(h - 2\beta)(q - 1))K(\epsilon)K(\frac{\epsilon}{2}). \]

In view of Proposition 3.2, this implies that \( \overline{h}_q(f, Z) \leq (1 - q)(h - 2\beta) \) and thus,
\[ \overline{h}_q(f, \mu_\Phi) \leq \liminf_{\delta \to 0} \{ \overline{h}_q(f, Z) : \mu_\Phi(Z) \geq 1 - \delta \} \leq (1 - q)(h - 2\beta). \]

Since \( \beta \) can be chosen arbitrarily small, it can be seen that \( \overline{h}_q(f, \mu_\Phi) \leq (1 - q)h. \)

To prove the reverse inequality, for sufficiently small \( \delta > 0 \) and \( \epsilon > 0 \) we choose a set \( Z \) with \( \mu(Z) > 1 - \delta \) for which \( h_{q, \delta}(f, \mu_\Phi) \geq h_{q, \delta}(f, Z, \epsilon) - \beta. \) Since \( \delta \) is sufficiently small the set
\[ Y = Z \cap \tilde{X} \cap \mathcal{L}_{M,N} \]
has positive measure. Let \( \{B_n(x_i, \epsilon)\}_{i \geq 1} \) be a cover of \( Y \) with \( n \geq N. \) It can be seen that
\[ \sum_i \mu_\Phi(B_n(x_i, \epsilon))q = \sum_i \mu_\Phi(B_n(x_i, \epsilon))q^{i-1} \mu_\Phi(B_n(x_i, \epsilon)) \geq \exp(-n(\overline{h} + 2\beta))q^{i-1} \mu_\Phi(Y). \]

Hence, by Propositions 3.1 and 3.2 we have that
\[ h_q(f, Z, \epsilon) \geq h_q(f, Y, \epsilon) \geq (1 - q)(\overline{h} + 2\beta). \]

This implies that
\[ h_q(f, \mu_\Phi) \geq h_q(f, Z, \epsilon) - \beta \geq h_q(f, Y, \epsilon) - \beta \geq (1 - q)(\overline{h} + 2\beta) - \beta. \]

Since \( \beta \) can be chosen arbitrarily small, this implies that \( h_q(f, \mu_\Phi) \geq (1 - q)\overline{h} \)
completing the proof of the theorem. \( \square \)

**Remark 4.2.** In fact, since \( \mu_\Phi \) is a Gibbs measure with respect to an almost additive potential \( \Phi = \{\phi_n\}_{n \geq 1} \) we can obtain that
\[ h_{\mu_\Phi}(x) = P_X(f, \Phi) - \limsup_{n \to \infty} \frac{\phi_n(x)}{n} \quad \text{and} \quad \overline{h}_{\mu_\Phi}(x) = P_X(f, \Phi) - \liminf_{n \to \infty} \frac{\phi_n(x)}{n} \]

If the measure \( \mu_\Phi \) is \( f \)-invariant, then \( \lim_{n \to \infty} \frac{\phi_n(x)}{n} \) exists almost everywhere, and is denoted the limit by \( \phi^*(x) \). By Theorem 4.1 and Remark 4.2, we have:
\[ h_q(f, \mu_\Phi) = \overline{h}_q(f, \mu_\Phi) = (1 - q)(P_X(f, \Phi) - \text{ess sup}_{x \in X} \phi^*(x)) \]
for \( q \geq 1 \), and
\[ h_q(f, \mu_\Phi) = \overline{h}_q(f, \mu_\Phi) = (1 - q)(P_X(f, \Phi) - \text{ess inf}_{x \in X} \phi^*(x)) \]
for \( 0 \leq q < 1 \).

**Theorem 4.2.** Assume that there exists a real number \( h \in \mathbb{R} \) such that
\[ h_{\mu_\Phi}(x) = \overline{h}_{\mu_\Phi}(x) := h \]
for \( \mu_\Phi \)-almost every point \( x \in X \). Then for any \( q \geq 0 \), we have
1. \( h_q(f, \mu_\Phi) = h_q(f, \mu_\Phi) = h_q(f, \mu_\Phi) = h(1 - q); \)
2. there exists a subset \( \mathcal{L}_h \subset X \) of positive \( \mu_\Phi \)-measure such that \( h_q(\sigma, \mathcal{L}_h) = h_q(f, \mathcal{L}_h) = h_q(f, \mathcal{L}_h) = h(1 - q). \)
Proof. If \( q = 1 \), the first statement is a direct consequence of Proposition 4.1 or Theorem 4.1 and the second statement follows from Theorem 3.3.

We now consider the case \( q \neq 1 \). By Theorem 4.1 we can obtain that
\[
h_q(f, \mu_\Phi) = \overline h_q(f, \mu_\Phi) = h(1 - q).
\]
By Proposition 4.1, to prove the first statement it suffices to prove that \( h_q(f, \mu_\Phi) \geq h(1 - q) \).

We shall now prove that \( h_q(f, Z, \epsilon/2) \geq h(1 - q) \) for any subset \( Z \subset X \) of full \( \mu_\Phi \)-measure and any sufficiently small \( \epsilon > 0 \). Choose a small number \( \eta > 0 \) and \( \delta \in (0, 1/2) \) and denote \( \lambda = (h - \eta)(1 - q) - \eta \) if \( 0 \leq q < 1 \) or \( \lambda = (h + \eta)(1 - q) - \eta \) if \( q > 1 \). As in the proof of Theorem 4.1, for any positive integers \( M \) and \( N \), define
\[
\mathcal{L}_M = \left\{ x \in X : h - \frac{\eta}{2} \leq h_{\mu_\Phi}(x, \epsilon) \leq \overline h_{\mu_\Phi}(x, \epsilon) \leq h + \frac{\eta}{2}, \forall 0 < \epsilon \leq \frac{1}{M} \right\}.
\]
and
\[
\mathcal{L}_{M, N} = \left\{ x \in \mathcal{L}_M : h - \eta \leq -\frac{1}{n} \log \mu_\Phi(B_n(x, \epsilon)) \leq h + \eta, \forall n \geq N \right\}.
\]
We have that the sets \( \mathcal{L}_M \) are nested and exhaust \( X \) up to a set of \( \mu_\Phi \)-measure zero. Furthermore, for any fixed positive integer \( M \) we have that \( \mathcal{L}_{M, N} \subset \mathcal{L}_{M, N-1} \) and \( \bigcup_{n=1}^{\infty} \mathcal{L}_{M, N} = \mathcal{L}_M \). One can find \( N_0 > 0 \) and \( M_0 > 0 \) such that \( \mu_\Phi(\mathcal{L}_{M_0, N_0}) > 1 - \delta \).

Let \( \mathcal{L}' = \mathcal{L}_{M_0, N_0} \cap Z \). Clearly, \( \mu(\mathcal{L}') \geq 1 - \delta \). By the choice of \( N_0 \) and \( M_0 \), one can have that for any \( x \in \mathcal{L}' \), \( n \geq N_0 \) and any sufficiently small \( \epsilon > 0 \)
\[
\exp(\-n(h + \eta)) \leq \mu_\Phi(B_n(x, \epsilon)) \leq \exp(\-n(h - \eta)).
\]
We may further assume that \( \mathcal{L}' \) is compact, since otherwise we can approximate it from within by a compact subset. Given any \( N > N_0 \), let \( \{B_n(x_i, \epsilon/2)\}_{i \geq 1} \) be a cover of \( \mathcal{L}' \) with \( n_i \geq N \) for all \( i \). Since \( \mathcal{L}' \) is compact, we may assume that the cover is finite and consists of dynamical balls \( B_n(x_1, \epsilon/2), \ldots, B_n(x_l, \epsilon/2) \).

Take \( y_i \in B_n(x_i, \epsilon/2) \cap \mathcal{L}' \) for all \( 1 \leq i \leq l \). Consider the cover \( \{B_n(y_i, \epsilon)\}_{i=1}^l \) of \( \mathcal{L}' \). Now for \( 0 \leq q < 1 \) we have
\[
\sum_{i=1}^l \mu_\Phi(B_n(y_i, \epsilon)) = \sum_{i=1}^l \mu_\Phi(B_n(x_i, \epsilon)) \geq [K(\frac{\epsilon}{2})K(\epsilon)]^{-q} \sum_{i=1}^l \mu_\Phi(B_n(y_i, \epsilon)) \geq [K(\frac{\epsilon}{2})K(\epsilon)]^{-q} \sum_{i=1}^l \mu_\Phi(B_n(y_i, \epsilon)) \geq [K(\frac{\epsilon}{2})K(\epsilon)]^{-q} \sum_{i=1}^l \mu_\Phi(B_n(y_i, \epsilon)) \]
where we use the Gibbs property of \( \mu_\Phi \) in the second inequality, use (1) in the third inequality and use (13) in the fourth inequality. Since the inequality (14) holds for any cover of \( \mathcal{L}' \), we conclude that \( W_q(\mathcal{L}', \lambda, N, \epsilon/2) \geq [K(\frac{\epsilon}{2})K(\epsilon)]^{-q}(1 - \delta) \). Hence,
\[
W_q(\mathcal{L}', \lambda, \epsilon/2) \geq [K(\frac{\epsilon}{2})K(\epsilon)]^{-q}(1 - \delta) > 0.
\]
This implies that
\[
h_q(f, \mathcal{L}', \epsilon/2) \geq (h - \eta)(1 - q) - \eta
\]
for $0 \le q < 1$. For $q > 1$, using the same arguments we obtain that

$$h_q(f, \mathcal{L}', \epsilon/2) \ge (h + \eta)(1 - q) - \eta. \quad (16)$$

Using Proposition 3.1, we find that

$$h_q(f, Z, \epsilon/2) \ge h_q(f, \mathcal{L}', \epsilon/2) \ge (h - \eta)(1 - q) - \eta, \quad \forall 0 \le q < 1$$

and

$$h_q(f, Z, \epsilon/2) \ge h_q(f, \mathcal{L}', \epsilon/2) \ge (h + \eta)(1 - q) - \eta, \quad \forall q > 1.$$

Therefore, by the definition of $q$–metric entropy, $h_q(f, \mu) \ge h(1 - q)$.

To prove the second statement, let $\mathcal{L}_h := \mathcal{L}_{M_0, N_0}$. First note that the inequality $h_q(f, \mathcal{L}_h) \ge h(1 - q)$ is contained in equalities (15) and (16), since $\mathcal{L}' \subset \mathcal{L}_h$.

Now, fix a small number $\eta > 0$ and a positive integer $N \ge N_0$. Choose a maximal $(n, 2\epsilon)$–separated subset $E \subset \mathcal{L}_h$ such that $\mathcal{L}_h \subset \bigcup_{x \in E} B_n(x, 2\epsilon)$ and $\{B_n(x, \epsilon) : x \in E\}$ are pairwise disjoint. By the first inequality of (13), the cardinality of the subset $E$ is less than or equal to $\exp[n(h + \eta)]$ for any $n \ge N$. For all sufficiently large $n$, by the Gibbs property of $\mu_{\Phi}$ and the above observation we have

$$\Lambda_q(\mathcal{L}_h, n, 2\epsilon) \le \sum_{x_i \in E} \mu_{\Phi}(B_n(x_i, 2\epsilon))^q \le [K(2\epsilon)K(\epsilon)]^q \sum_{x_i \in E} \mu_{\Phi}(B_n(x_i, \epsilon))^q \le [K(2\epsilon)K(\epsilon)]^q \exp[-nq(h - \eta)] \exp[n(h + \eta)] = [K(2\epsilon)K(\epsilon)]^q \exp[n((1 - q)h + (1 + q)\eta)].$$

It follows from Proposition 3.2 that

$$\overline{h}_q(f, \mathcal{L}_h, 2\epsilon) \le (1 - q)h + (1 + q)\eta.$$ 

Hence,

$$\overline{h}_q(f, \mathcal{L}_h) \le (1 - q)h.$$ 

This completes the proof of the theorem.

4.3. **The modified Hentschel-Procaccia entropy spectrum.** Following the approach in [15], we introduce the modified HP-entropy spectrum. Given the reference Gibbsian measure $\mu_{\Phi}$ and $q > 1$, define

$$\mathcal{HPM}_q(\mu_{\Phi}) = \frac{1}{q - 1} \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log \int Z \mu_{\Phi}(B_n(x, \epsilon))^q d\mu_{\Phi}(x),$$

$$\overline{\mathcal{HPM}}_q(\mu_{\Phi}) = \frac{1}{q - 1} \lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log \int Z \mu_{\Phi}(B_n(x, \epsilon))^q d\mu_{\Phi}(x),$$

where the infimum is taken over all sets $Z \subset X$ with $\mu_{\Phi}(Z) > 1 - \delta$.

**Definition 4.1.** The one-parameter family of pairs of quantities $(\mathcal{HPM}_q(\mu_{\Phi}), \overline{\mathcal{HPM}}_q(\mu_{\Phi}))$ is called the modified HP-spectrum for entropies.

By Theorem 3.2 and the definitions of the modified HP-spectrum for entropies, and the lower and upper $q$-metric entropies, one can immediately show that $\mathcal{HPM}_q(\mu_{\Phi}) \le \frac{1}{q - 1} \overline{h}_q(f, \mu_{\Phi})$ and $\overline{\mathcal{HPM}}_q(\mu_{\Phi}) \le \frac{1}{q - 1} \overline{h}_q(f, \mu_{\Phi})$ for any $q > 1$. 
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