Projection-free Distributed Online Learning with Strongly Convex Losses

Yuanyu Wan  
wanyy@lamda.nju.edu.cn
Guanghui Wang  
wanggh@lamda.nju.edu.cn
Lijun Zhang  
zhanglj@lamda.nju.edu.cn
National Key Laboratory for Novel Software Technology
Nanjing University, Nanjing 210023, China

Abstract

To efficiently solve distributed online learning problems with complicated constraints, previous studies have proposed several distributed projection-free algorithms. The state-of-the-art one achieves the $O(T^{3/4})$ regret bound with $O(\sqrt{T})$ communication complexity. In this paper, we further exploit the strong convexity of loss functions to improve the regret bound and communication complexity. Specifically, we first propose a distributed projection-free algorithm for strongly convex loss functions, which enjoys a better regret bound of $O(T^{2/3}/\log T)$ with smaller communication complexity of $O(T^{1/3})$. Furthermore, we demonstrate that the regret of distributed online algorithms with $C$ communication rounds has a lower bound of $\Omega(T/C)$, even when the loss functions are strongly convex. This lower bound implies that the $O(T^{1/3})$ communication complexity of our algorithm is nearly optimal for obtaining the $O(T^{2/3}/\log T)$ regret bound up to polylogarithmic factors. Finally, we extend our algorithm into the bandit setting and obtain similar theoretical guarantees.

Keywords: Projection-free, Distributed Online Learning, Strongly Convex, Communication Complexity, Regret

1. Introduction

Online convex optimization (OCO) is a powerful paradigm for sequential decision making over multiple rounds, and has become of great interest to both researchers and practitioners in the machine learning community (Shalev-Shwartz [2011] [2016]. In each round $t$ of OCO, a learner is required to select a decision $x(t)$ from a convex set $K \subseteq \mathbb{R}^d$, and then suffers a loss $f_t(x(t))$, where $f_t(x) : K \mapsto \mathbb{R}$ is a convex function chosen by an adversary. The gap between the total loss of the learner and an optimal fixed decision

$$R_T = \sum_{t=1}^{T} f_t(x(t)) - \min_{x \in K} \sum_{t=1}^{T} f_t(x)$$

is referred to as regret and commonly used to measure its performance, where $T$ is the number of total rounds. To yield optimal regret bounds for functions with different properties, many algorithms have been proposed (Zinkevich [2003] [2007], Hazan et al. [2007], Shalev-Shwartz [2007], Shalev-Shwartz and Singer [2007]).

A common and fundamental step in these algorithms is to compute projections onto the decision set $K$, which is required to make all decisions feasible. However, in many high-dimensional problems such as semidefinite programs (Hazan [2008] and multiclass...
classification (Hazan and Luo, 2016), the decision set $K$ is complicated or structural, which makes the projection step become a computational bottleneck. To address this computational issue, Hazan and Kale (2012) proposed online conditional gradient (OCG) by replacing the time-consuming projection step with one iteration of the classical conditional gradient (CG) method (Frank and Wolfe, 1956; Jaggi, 2013), and attained a regret bound of $O(T^{3/4})$ for convex functions. Although the $O(T^{3/4})$ regret of OCG is worse than the $O(\sqrt{T})$ regret achieved by projection-based algorithms, OCG is a projection-free algorithm, which only needs to perform a more efficient linear optimization step per round.

To solve distributed problems with large-scale streaming data and complicated constraints, Zhang et al. (2017) further proposed distributed online conditional gradient (D-OCG) by extending OCG into a distributed variant of OCO. Different from OCO only with 1 learner, the distributed OCO includes $n$ local learners connected by a network (Hosseini et al., 2013; Yan et al., 2013; Shahrampour and Jadbabaie, 2018). In each round $t$, each local learner $i$ selects a decision $x_i(t) \in K$, and only receives a local loss function $f_{t,i}(x) : K \rightarrow \mathbb{R}$ chosen by the adversary. The goal of each local learner $i$ is to minimize the regret defined as

$$R_{T,i} = \sum_{t=1}^{T} \sum_{j=1}^{n} f_{t,j}(x_i(t)) - \min_{x \in K} \sum_{t=1}^{T} \sum_{j=1}^{n} f_{t,j}(x).$$

Similar to OCG for the standard OCO, Zhang et al. (2017) proved that D-OCG attains a regret bound of $O(T^{3/4})$ for the distributed OCO. However, all local learners in D-OCG need to communicate with their neighbors in each round, which results in $O(T)$ communication complexity. For this reason, Wan et al. (2020) proposed distributed block online conditional gradient (D-BOCG), which achieves the same regret bound of $O(T^{3/4})$ but only requires $O(\sqrt{T})$ communication complexity. Moreover, by combining with the classical one-point gradient estimator (Flaxman et al., 2005), they proposed a bandit variant of D-BOCG and achieved similar theoretical guarantees.

Recently, in the standard OCO, two similar variants of OCG for strongly convex functions have been introduced, which are able to utilize the strong convexity to attain a better regret bound of $O(T^{2/3})$ (Garber and Kretzu, 2020b; Wan and Zhang, 2020). It is therefore natural to ask whether the strong convexity can be utilized to improve the regret of projection-free algorithms in the distributed OCO. In this paper, we give an affirmative answer by first proposing a variant of D-BOCG for strongly convex functions, namely D-BOCG$_{sc}$, which reduces the regret bound from $O(T^{3/4})$ to $O(T^{2/3} \log T)$ regret bound. Surprisingly, our D-BOCG$_{sc}$ also reduces the communication complexity from $O(T^{1/2})$ to $O(T^{1/3})$. Inspired by previous studies (Garber and Kretzu, 2020b; Wan and Zhang, 2020), the main idea is to design new surrogate loss functions for strongly convex functions. Furthermore, we provide a lower bound of $\Omega(T/C)$ for the regret of distributed online algorithms with $C$ communication rounds, even when the loss functions are strongly convex. This lower bound implies that the $O(T^{1/3})$ communication complexity of our D-BOCG$_{sc}$ is nearly optimal for obtaining the $O(T^{2/3} \log T)$ regret bound up to polylogarithmic factors. Finally, inspired by the bandit variant of D-BOCG (Wan et al., 2020), we also propose a bandit variant of our D-BOCG$_{sc}$ by combining with the one-point gradient estimator (Flaxman et al., 2005). Our

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1. The conditional gradient method is also known as the Frank-Wolfe method. Similarly, online conditional gradient is also known as online Frank-Wolfe.
theoretical analysis reveals that the bandit variant of D-BOCG \(_{sc}\) obtains an expected regret bound of \(O(T^{2/3}\log T)\) with \(O(T^{1/3})\) communication complexity, which is better than the bandit algorithm in [Wan et al. (2020)].

2. Related Work

The starting point for studies on projection-free algorithms is the seminal work of Frank and Wolfe (1956), which originally proposed CG for smooth convex optimization (SCO) over polyhedral sets. Over the past decades, there are many studies that extended the application and theory of CG to SCO over other sets [Hazan (2008), Clarkson (2010), Jaggi (2013), Garber and Hazan (2015, 2016), Garber (2016, 2020)]. In these studies, the closest one related to our work is that of Jaggi (2013), which applied CG to SCO over general convex sets and established the convergence rate of \(O(1/L)\), where \(L\) is the number iterations.

Due to the emergence of large-scale problems, Hazan and Kale (2012) proposed OCG by extending CG to OCO. Specifically, after choosing an arbitrary \(x(1) \in \mathcal{K}\), in each round \(t\), OCG updates the decision by utilizing one iteration of CG, as follows

\[
\begin{align*}
v &= \arg\min_{x \in \mathcal{K}} \nabla F_t(x(t))^\top x \\
x(t + 1) &= x(t) + s_t(v - x(t))
\end{align*}
\]

where \(F_{t+1}(x) = \eta \sum_{k=1}^{t} \nabla f_k(x(k))^\top x + \|x - x(1)\|_2^2\) is a surrogate loss function, \(s_t\) and \(\eta\) are two parameters. It is the first projection-free algorithm for OCO, and has a regret bound of \(O(T^{3/4})\). Since the \(O(T^{3/4})\) regret of OCG is worse than the \(O(\sqrt{T})\) regret achieved by projection-based algorithms, many recent studies have proposed to improve the regret of OCG by introducing additional assumptions on loss functions and decision sets.

For smooth functions, Hazan and Minasyan (2020) proposed the online smooth projection free algorithm, and obtained an expected regret bound of \(O(T^{2/3})\) as well as a high-probability regret bound of \(O(T^{2/3}\log T)\). If the functions are \(\alpha\)-strongly convex, Wan and Zhang (2020) proposed a strongly convex variant of OCG by redefining the surrogate loss function as

\[
F_{t+1}(x) = \sum_{k=1}^{t} \left( \nabla f_k(x(k))^\top x + \frac{\alpha}{2} \|x - x(t)\|_2^2 \right)
\]

and using a line search rule to select the parameter \(s_t\). Their algorithm can enjoy a regret bound of \(O(T^{2/3})\) for strongly convex functions, and a very similar algorithm was also proposed by Garber and Kretzu (2020b). Moreover, if the decision sets are polyhedral, a variant of OCG in Garber and Hazan (2016) enjoys an \(O(\sqrt{T})\) regret bound for convex functions and an \(O(\log T)\) regret bound for strongly convex functions, respectively. For OCO over smooth sets, Levy and Krause (2019) attained similar regret bounds by a variant of online gradient descent (Zinkevich, 2003), instead of OCG.

Furthermore, OCG has also been extended to two more practical variants of OCO including the bandit setting [Flaxman et al. (2005), Agarwal et al. (2010), Saha and Tewari, 2011, Hazan and Levy (2014)] and the distributed OCO (Hosseini et al. (2013), Yan et al. (2013), Shahriarpour and Jadababaid (2018). In the bandit setting, where only the loss value is available to the learner, Chen et al. (2019) proposed the first projection-free algorithm.
by combining OCG with the one-point gradient estimator (Flaxman et al., 2005), and
established an expected regret bound of $O(T^{3/5})$. Later, it was improved to attain $O(T^{3/4})$
regret for convex functions (Garber and Kretzu, 2020a) and $O(T^{2/3} \log T)$ regret for strongly
convex functions (Garber and Kretzu, 2020b), respectively.

To handle the distributed OCO, Zhang et al. (2017) proposed D-OCG by maintaining
OCG for each local learner. Moreover, to achieve the $O(T^{3/4})$ regret measured by the global
loss $\sum_{t=1}^{n} f_t(x)$, in each round, local learners in D-OCG needs to communicate with their
neighbors to share the local gradients. Therefore, the communication complexity of D-OCG
is on the order of $O(T)$. Recently, D-BOCG was proposed to achieve the same regret bound
of $O(T^{3/4})$, while only requires the communication complexity of $O(\sqrt{T})$ (Wan et al., 2020).

When the distributed OCO meets the bandit setting, they further proposed distributed block
bandit conditional gradient (D-BBCG), a bandit variant of D-BOCG, which can achieve
a high-probability regret bound of $O(T^{3/4} \log T)$ with the communication complexity of $O(\sqrt{T})$.

3. Main Results

In this section, we first introduce necessary preliminaries including standard definitions,
common assumptions and two basic algorithmic ingredients. Then, we present our D-BOCG_sc
for strongly convex functions and the corresponding theoretical guarantee. Furthermore,
a nearly matching lower bound is proposed. Finally, we extend D-BOCG_sc to the bandit
setting.

3.1 Preliminaries

We first recall the standard definitions for smooth and strongly convex functions (Boyd and
Vandenberghe, 2004).

**Definition 1** Let $f(x) : \mathcal{K} \rightarrow \mathbb{R}$ be a function over $\mathcal{K}$. It is called $\beta$-smooth over $\mathcal{K}$ if for
all $x, y \in \mathcal{K}$

$$f(y) \leq f(x) + \nabla f(x)^\top (y - x) + \frac{\beta}{2} \|y - x\|^2.$$  

**Definition 2** Let $f(x) : \mathcal{K} \rightarrow \mathbb{R}$ be a function over $\mathcal{K}$. It is called $\alpha$-strongly convex over $\mathcal{K}$
if for all $x, y \in \mathcal{K}$

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x) + \frac{\alpha}{2} \|y - x\|^2.$$  

According to Hazan and Kale (2012), for any $\alpha$-strongly convex function $f(x)$ and any
$x \in \mathcal{K}$, it holds that

$$\frac{\alpha}{2} \|x - x^*\|^2 \leq f(x) - f(x^*)$$  \hspace{1cm} (2)

where $x^* = \arg\min_{x \in \mathcal{K}} f(x)$.

Then, following previous studies on the distributed OCO (Yan et al., 2013; Zhang et al.,
2017; Wan et al., 2020), the local learners are connected by a network $G = (V, E)$, where
$V = [n]$ is the node set and $E \subseteq V \times V$ is the edge set. Each local learner $i \in V$ is only
allowed to communicate with its immediate neighbors

$$N_i = \{j \in V | (i, j) \in E\}.$$  

4
A non-negative weight matrix $P \in \mathbb{R}^{n \times n}$ is utilized to model the communication between the local learners, which satisfies the following assumption.

**Assumption 1** The non-negative weight matrix $P \in \mathbb{R}^{n \times n}$ is symmetric and doubly stochastic, which satisfies

- $P_{ij} > 0$ only if $(i, j) \in E$;
- $\sum_{j=1}^{n} P_{ij} = \sum_{j \in N_i} P_{ij} = 1, \forall i \in V$; $\sum_{i=1}^{n} P_{ij} = \sum_{i \in N_j} P_{ij} = 1, \forall j \in V$.

In the following, $\sigma_2(P)$ will be used to denote the second largest eigenvalue of $P$.

Moreover, there are also other common assumptions in the distributed OCO (Yan et al., 2013; Zhang et al., 2017; Wan et al., 2020).

**Assumption 2** At each round $t$, each local loss function $f_{t,i}(x)$ is $G$-Lipschitz over $\mathcal{K}$, i.e., $|f_{t,i}(x) - f_{t,i}(y)| \leq G \|x - y\|_2$ for any $x \in \mathcal{K}, y \in \mathcal{K}$.

**Assumption 3** At each round $t$, each local loss function $f_{t,i}(x)$ is $\alpha$-strongly convex over $\mathcal{K}$.

**Assumption 4** The convex decision set $\mathcal{K}$ is full dimensional and contains the origin. Moreover, there exist two constants $r, R > 0$ such that $rB^d \subseteq \mathcal{K} \subseteq RB^d$ where $B^d$ denotes the unit Euclidean ball centered at the origin in $\mathbb{R}^d$.

**Assumption 5** At each round $t$, each local loss function $f_{t,i}(x)$ is bounded over $\mathcal{K}$, i.e., $|f_{t,i}(x)| \leq M$, for any $x \in \mathcal{K}$.

Finally, we introduce two basic algorithmic ingredients. The first is the CG algorithm (Frank and Wolfe, 1956; Jaggi, 2013) outlined in Algorithm 1, which can iteratively minimize a smooth function $F(x) : \mathcal{K} \to \mathbb{R}$, where $x_0$ is the initial solution and $L$ is the total number of iterations. Inspired by Wan et al. (2020), we will utilize CG as a subroutine of our projection-free algorithms.

The second is the one-point gradient estimator (Flaxman et al., 2005). For a function $f(x)$, we define its $\delta$-smoothed version as

$$\hat{f}_\delta(x) = \mathbb{E}_{u \sim B^d}[f(x + \delta u)].$$

Flaxman et al. (2005) presented the following lemma, which provides an unbiased estimator of the gradient $\nabla \hat{f}_\delta(x)$ by only observing the single value $f(x + \delta u)$.

**Lemma 1** (Lemma 1 in Flaxman et al. (2005)) Given $\delta > 0$, we have

$$\nabla \hat{f}_\delta(x) = \mathbb{E}_{u \sim S^d} \left[ \frac{d}{\delta} f(x + \delta u)u \right]$$

where $S^d$ denotes the unit sphere in $\mathbb{R}^d$.

This technique is commonly used in the bandit setting (Garber and Kretzu, 2020a,b; Wan et al., 2020).
Algorithm 1 CG

1: Input: feasible set $\mathcal{K}$, $L$, $F(x)$, $x_{in}$
2: $c_0 = x_{in}$
3: for $\tau = 0, \cdots, L - 1$ do
4: \quad $v_\tau \in \text{argmin}_{x \in \mathcal{K}} \nabla F(c_\tau)^T x$
5: \quad $s_\tau = \text{argmin}_{s \in [0,1]} F(c_\tau + s(v_\tau - c_\tau))$
6: \quad $c_{\tau + 1} = c_\tau + s_\tau(v_\tau - c_\tau)$
7: end for
8: return $x_{out} = c_L$

3.2 An Algorithm for Full Information Setting

To reduce the communication complexity for the distributed OCO, the original D-BOCG (Wan et al., 2020) divides the total $T$ rounds into $B = T/K$ blocks, where $K$ is the block size and $B$ is assumed to be an integer without loss of generality. In each block $m$, D-BOCG maintains two variables $x_i(m)$ and $z_i(m)$ for each local learner $i$, where $x_i(1) = 0$ and $z_i(1) = 0$. The cumulative gradient of local learner $i$ in block $m$ can be computed by

$$\hat{g}_i(m) = \sum_{t \in \mathcal{T}_m} \nabla f_{t,i}(x_i(m))$$

where $\mathcal{T}_m = \{(m-1)K + 1, \cdots, mK\}$. At the beginning of each block $m > 1$, each local learner $i$ communicates with its neighbors to share $z_i(m - 1)$ and updates it as

$$z_i(m) = \sum_{j \in N_i} P_{ij}z_j(m - 1) + \hat{g}_i(m - 1)$$  \hfill (3)

which is utilized to define a surrogate loss function as

$$F_{m,i}(x) = \eta z_i(m)^T x + \|x\|_2^2.$$  \hfill (4)

Then, the decision $x_i(m)$ for block $m$ is updated by minimizing the surrogate loss function with CG as follows

$$x_i(m) = \text{CG}(\mathcal{K}, L, F_{m,i}(x), x_i(m - 1)).$$

Since the local learners in D-BOCG only communicate with their neighbors and update their local variables once per block, the communication complexity is only $O(T/K)$. Although Wan et al. (2020) showed that D-BOCG with $K = L = \sqrt{T}$ obtains a regret bound of $O(T^{3/4})$ with $O(\sqrt{T})$ communication complexity, D-BOCG does not utilize the strong convexity of the loss functions.

To address this limitation, we first introduce our algorithm for the simple case with $n = 1$ and then extend it to the general case for any $n \geq 1$.

The Simple Case  If $n = 1$, the distributed OCO reduces to the standard OCO. For this case, inspired by [1], one can simply define a surrogate loss function as

$$F_{m,1}(x) = \sum_{\tau=1}^{m-1} \left( \sum_{t \in \mathcal{T}_\tau} \nabla f_{t,1}(x_1(\tau))^T x + \sum_{t \in \mathcal{T}_\tau} \frac{\alpha}{2}\|x - x_1(\tau)\|_2^2 \right).$$
Algorithm 2 D-BOCGsc

1: Input: feasible set $K$, $\eta$, $L$, $\alpha$ and $K$
2: Initialization: choose $\{x_i(1) = 0 \in K| i \in V\}$ and set $\{z_i(1) = 0| i \in V\}$
3: for $t = 1, \cdots, T$ do
4: $m_t = \lceil t/K \rceil$
5: for each local learner $i \in V$ do
6: if $t > 1$ and mod($t, K$) = 1 then
7: $\hat{g}_i(m_t - 1) = \sum_{k=m_t-1}^{t-1} g_i(k)$
8: $z_i(m_t) = \sum_{j \in N_i} P_{ij} z_j(m_t - 1) + \hat{g}_i(m_t - 1) - \alpha K x_i(m_t - 1)$
9: define $F_{m_t,i}(x) = z_i(m_t) \top x + \frac{(m_t - 1)\alpha K}{2} \|x\|^2$
10: $x_i(m_t) = CG(K, L, F_{m_t,i}(x), x_i(m_t - 1))$
11: end if
12: play $x_i(m_t)$ and observe $g_i(t) = \nabla f_{t,i}(x_i(m_t))$
13: end for
14: end for

for $\alpha$-strongly convex loss functions.

Since $\|x_1(\tau)\|^2$ does not affect the minimizer of the function $F_{m,1}(x)$, it can be simplified as

$$F_{m,1}(x) = \sum_{\tau=1}^{m-1} \left( \left( \sum_{t \in T_{\tau}} \nabla f_{t,1}(x_1(\tau)) - \alpha K x_1(\tau) \right) \top x + \frac{\alpha K}{2} \|x\|^2 \right).$$

By updating $z_1(m)$ as

$$z_1(m) = z_1(m - 1) + \hat{g}_1(m - 1) - \alpha K x_1(m - 1),$$

we can rewrite $F_{m,1}(x)$ as

$$F_{m,1}(x) = z_1(m) \top x + \frac{(m - 1)\alpha K}{2} \|x\|^2.$$

Finally, we only need to update the decision $x_1(m)$ as the same way of D-BOCG.

The General Case To handle the general case, one may directly apply the above procedures for each local learner $i$. However, in this way, each local learner can only use its local information, which cannot achieve a regret bound measured by the global loss $\sum_{j=1}^{n} f_{t,j}(x)$.

Therefore, inspired by [Wan et al. 2020], we need to update $z_i(m)$ as

$$z_i(m) = \sum_{j \in N_i} P_{ij} z_j(m - 1) + \hat{g}_i(m - 1) - \alpha K x_i(m - 1)$$

for each local learner $i$. Compared with (5), the above update further incorporates the information from the neighbors of local learner $i$. Moreover, the surrogate loss function for
each local learner is defined as

$$F_{m,i}(x) = z_i(m) \mathbf{x} + \frac{(m-1)\alpha K}{2} \|\mathbf{x}\|^2_2.$$  \hspace{1cm} (7)

Replacing (3), (4) in the original D-BOCG with (6), (7) respectively, we develop distributed block online conditional gradient for strongly convex functions (D-BOCG$_{sc}$), and the detailed procedures are presented in Algorithm 2.

We provide the theoretical guarantee of Algorithm 2 in the following theorem.

**Theorem 1** Under Assumptions \textbf{1, 2, 3 and 4} for any $i \in [n]$, Algorithm 2 ensures

$$R_{T,i} \leq \left( \frac{2nK(G + 2\alpha R)^2}{\alpha} + \frac{3GK(G + \alpha R)n^{3/2}}{\alpha(1 - \sigma_2(P))} \right) \left( 1 + \ln \frac{T}{K} \right) + 3nG \frac{4RT}{\sqrt{L} + 2}.$$

**Remark** From Theorem 1 by setting $K = L = T^{2/3}$, the regret bound of Algorithm 2 is on the order of $O(T^{2/3} \log T)$ for strongly convex functions, and the number of linear optimization steps is equal to $TL/K = T$. In contrast, D-BOCG (Wan et al., 2020) only achieves a regret bound of $O(T^{3/4})$ with the same number of linear optimization steps. Moreover, since $T/K = T^{1/3}$, the communication complexity of Algorithm 2 is only on the order of $O(T^{1/3})$, which is smaller than the $O(\sqrt{T})$ communication complexity of D-BOCG.

### 3.3 A Lower Bound

Furthermore, we provide a lower bound for the regret of distributed OCO with strongly convex losses when the number of communication rounds is limited. For simplicity, let the number of local learners be $n = 2$ and the number of communication rounds be $C$.

**Theorem 2** Suppose $\mathcal{K} = \left[-R/\sqrt{d}, R/\sqrt{d}\right]^d$, which satisfies Assumption \textbf{4}. For distributed OCO with $n = 2$ local learners over $\mathcal{K}$ and any distributed online algorithm communicating $C$ rounds before the round $T$, there exists a sequence of function pairs

$$\{f_{1,1}(\mathbf{x}), f_{1,2}(\mathbf{x})\}, \cdots, \{f_{T,1}(\mathbf{x}), f_{T,2}(\mathbf{x})\}$$

satisfying Assumption \textbf{2} with $G = 2\alpha R$ and Assumption \textbf{3} respectively such that

$$R_{T,1} \geq \frac{\alpha R^2 T}{4(C + 1)}.$$

**Remark** The above theorem essentially establishes an $\Omega \left(T^{1/3}/\log T\right)$ lower bound on the communication rounds required by any online algorithm achieving the $O(T^{2/3} \log T)$ regret bound, which almost matches the $O(T^{1/3})$ communication complexity of our D-BOCG$_{sc}$ up to polylogarithmic factors.

### 3.4 An Extension to Bandit Setting

To handle the bandit setting, we need to combine our D-BOCG$_{sc}$ with the one-point gradient estimator (Flaxman et al., 2005). To this end, we first define a smaller decision set $\mathcal{K}_\delta$ as

$$\mathcal{K}_\delta = (1 - \delta/r)\mathcal{K} = \{(1 - \delta/r)\mathbf{x} | \mathbf{x} \in \mathcal{K}\}$$
Algorithm 3 D-BBCGsc

1: **Input:** feasible set $\mathcal{K}$, $\delta, \eta, L, \alpha$ and $K$
2: **Initialization:** choose $\{x_i(1) = 0 \in \mathcal{K}_\delta | i \in V\}$ and set $\{z_i(1) = 0 | i \in V\}$
3: for $t = 1, \ldots, T$
   4: $m_t = \lceil t/K \rceil$
   5: for each local learner $i \in V$
      6: if $t > 1$ and mod$(t, K) = 1$ then
         7: $\hat{g}_i(m_t - 1) = \sum_{k=1}^{t-1} \mathcal{K}_\delta g_i(k)$
         8: $z_i(m_t) = \sum_{j \in \mathcal{N}_i} P_{ij} z_j(m_t - 1) + \hat{g}_i(m_t - 1) - \alpha K x_i(m_t - 1)$
         9: define $F_{m_t,i}(x) = z_i(m_t)^\top x + \frac{(m_t - 1) - \alpha K}{2} \|x\|^2$
      10: $x_i(m_t) = \text{CG}(\mathcal{K}_\delta, L, F_{m_t,i}(x), x_i(m_t - 1))$
   end if
   11: $u_i(t) \sim S^d$
   12: $y_i(t) = x_i(m_t) + \delta u_i(t)$ and observe $f_{t,i}(y_i(t))$
   13: $g_i(t) = \frac{d}{3} f_{t,i}(y_i(t)) u_i(t)$
   14: end for
   15: end for

where $0 < \delta < r$. Then, according to the one-point gradient estimator, we replace the gradient $g_i(t) = \nabla f_{t,i}(x_i(m_t))$ used in D-BOCGsc with $g_i(t) = \frac{d}{3} f_{t,i}(y_i(t)) u_i(t)$, where $y_i(t) = x_i(m_t) + \delta u_i(t)$ is the actual decision and $u_i(t) \sim S^d$. To ensure $y_i(t) \in \mathcal{K}$, we replace $\mathcal{K}$ in step 10 of D-BOCGsc with $\mathcal{K}_\delta$.

The detailed procedures are presented in Algorithm 3, which is named as distributed block bandit conditional gradient for strongly convex functions (D-BBCGsc). Following previous studies (Garber and Kretzu 2020a, b; Wan et al. 2020), we assume that the adversary is oblivious, and present the following theorem for D-BBCGsc.

**Theorem 3** Under Assumptions 1, 2, 3, 4, 5, 6, 7, 8 and 9 for any $i \in [n]$ and $c > 0$ such that $\frac{cT^{-1/3}}{r} < 1$, Algorithm 3 with $B = T/K, K = L = T^{2/3}$ and $\delta = cT^{-1/3}$ ensures

$$
\mathbb{E}[R_{T,i}] \leq \frac{1 + \ln T^{1/3}}{\alpha} \left( \frac{8d^2M^2}{c^2} + 8G^2 + 12\alpha^2 R^2 \right) T^{2/3} + 3cnGT^{2/3} + \frac{cnGT^{2/3}}{r}
+ 3nGT^{2/3} \left( 4R + \sqrt{\frac{d^2M^2}{c^2} + G^2 + \alpha^2 R^2 \frac{\ln(1 + \ln T^{1/3})}{\alpha(1 - \sigma(P))}} \right).
$$

**Remark** According to Theorem 3 for strongly convex functions, Algorithm 3 with $K = L = T^{2/3}$ achieves an expected regret bound of $O(T^{2/3} \log T)$ with $T$ linear optimization steps and $T^{1/3}$ communication rounds, which is very similar to D-BOCGsc. Moreover, compared with the $O(T^{3/4} \log T)$ regret and $O(\sqrt{T})$ communication complexity of D-BBCG (Wan et al. 2020), our Algorithm 3 achieves a significant improvement for strongly convex functions.

### 4. Theoretical Analysis

In this section, we provide proofs for our theoretical guarantees.
Algorithm 2 with \( L_i \) is defined in Algorithm 2. Under Assumption 4, for any according to Algorithm 2, Assumptions 2 and 4, we have
\[
\|d(m)\|_2 = \frac{1}{n} \sum_{i=1}^{n} \|d_i(m)\|_2 = \frac{1}{n} \sum_{i=1}^{n} \hat{g}_i(m) - \alpha Kx_i(m)
\]
According to Algorithm 2, it is easy to verify that
\[
\bar{z}(m+1) = \frac{1}{n} \sum_{i=1}^{n} z_i(m+1)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j \in N_i} P_{ij} z_j(m) + \hat{g}_i(m) - \alpha Kx_i(m) \right)
\]
= \bar{z}(m) + d(m).
Then, we define
\[
\bar{F}_{m+1}(x) = \bar{z}(m+1)^T x + \frac{m\alpha K}{2} \|x\|_2^2
\]
and \( \bar{x}(1) = 0, \bar{x}(m+1) = \arg\min_{x \in \mathcal{K}} \bar{F}_{m+1}(x) \) for \( m \geq 1 \). Moreover, let \( \bar{x}_i(m) = \arg\min_{x \in \mathcal{K}} z_i(m)^T x + \frac{(m-1)\alpha K}{2} \|x\|_2^2 \) for \( m \geq 2 \) and \( x^* = \arg\min_{x \in \mathcal{K}} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{i,j}(x)_i \).

Now, we derive an upper bound of \( \|x_i(m) - \bar{x}(m)\|_2 \) with the following three lemmas.

**Lemma 2** (Lemma 6 in Zhang et al. (2017)) Let \( z_i(1) = 0, z_i(m+1) = \sum_{j \in N_i} P_{ij} z_j(m) + d_i(m) \) and \( \bar{z}(m) = \frac{1}{n} \sum_{i=1}^{n} \bar{z}_i(m) \) for \( m \in [B] \), where \( P \) satisfies Assumption 1. For any \( i \in V \) and \( m \in [B] \), assume \( \|d_i(m)\|_2 \leq \beta \), we have
\[
\|z_i(m) - \bar{z}(m)\|_2 \leq \frac{\beta \sqrt{n}}{1 - \sigma_2(P)}.
\]

**Lemma 3** (Lemma 5 in Duchi et al. (2011)) Let \( \Pi_\mathcal{K}(u, \eta) = \arg\min_{x \in \mathcal{K}} \eta x^T x + \|x\|_2^2 \). We have
\[
\|\Pi_\mathcal{K}(u, \eta) - \Pi_\mathcal{K}(v, \eta)\|_2 \leq \frac{\eta}{2} \|u - v\|_2.
\]

**Lemma 4** Let \( \bar{x}_i(m) = \arg\min_{x \in \mathcal{K}} F_{m,i}(x) \), for any \( i \in V \) and \( m \in \{2, \cdots, B\} \), where \( F_{m,i}(x) \) is defined in Algorithm 2. Under Assumption 4, for any \( i \in V \) and \( m \in \{2, \cdots, B\} \), Algorithm 2 with \( L \geq 1 \) has
\[
F_{m,i}(x_i(m)) - F_{m,i}(\bar{x}_i(m)) \leq \frac{8(m-1)\alpha K R^2}{L + 2}.
\]

According to Algorithm 2, Assumptions 2 and 4, we have
\[
\|d_i(m)\|_2 = \|\hat{g}_i(m) - \alpha Kx_i(m)\|_2 = \left\| \sum_{t \in T_m} g_i(t) - \alpha Kx_i(m) \right\|_2
\]
\[
\leq \sum_{t \in T_m} \left\| g_i(t) \right\|_2 + \alpha K \|x_i(m)\|_2 \leq K(G + \alpha R).
\]
where $\mathcal{T}_m = \{(m-1)K + 1, \cdots, mK\}$. 
Applying Lemma 2 with $\|d_i(m)\|_2 \leq K(G + \alpha R)$, we have
\begin{equation}
\|z_i(m) - \bar{z}(m)\|_2 \leq \frac{K(G + \alpha R)^{\sqrt{n}}}{1 - \sigma_2(P)}. \tag{8}
\end{equation}

For any $m \in \{2, \cdots, B\}$, we notice that
\begin{align*}
\hat{x}_i(m) &= \min_{x \in \mathcal{K}} z_i(m)^\top x + \frac{(m-1)\alpha K}{2} \|x\|_2^2 \\
&= \min_{x \in \mathcal{K}} \frac{2}{(m-1)\alpha K} z_i(m)^\top x + \|x\|_2^2
\end{align*}
and
\begin{align*}
x(m) &= \min_{x \in \mathcal{K}} z(m)^\top x + \frac{(m-1)\alpha K}{2} \|x\|_2^2 \\
&= \min_{x \in \mathcal{K}} \frac{2}{(m-1)\alpha K} z(m)^\top x + \|x\|_2^2.
\end{align*}

Therefore, applying Lemma 3 with (8), for any $m \in \{2, \cdots, B\}$, we have
\begin{align*}
\|\hat{x}_i(m) - \bar{x}(m)\|_2 &\leq \frac{1}{(m-1)\alpha K} \|z_i(m) - \bar{z}(m)\|_2 \\
&\leq \frac{(G + \alpha R)^{\sqrt{n}}}{(m-1)\alpha(1 - \sigma_2(P))}
\end{align*}
which further implies that
\begin{align*}
\|x_i(m) - \bar{x}(m)\|_2 &\leq \|x_i(m) - \hat{x}_i(m)\|_2 + \|\hat{x}_i(m) - \bar{x}(m)\|_2 \\
&\leq \sqrt{\frac{2F_{m,i}(x_i(m)) - 2F_{m,i}(\hat{x}_i(m))}{(m-1)\alpha K}} + \frac{(G + \alpha R)^{\sqrt{n}}}{(m-1)\alpha(1 - \sigma_2(P))} \tag{9}
\end{align*}
where the second inequality is due to the fact that $F_{m,i}(x)$ is $(m-1)\alpha K$-strongly convex and (2), and the last inequality is due to Lemma 4.

Let $\epsilon_m = \frac{4R}{\sqrt{L+2}} + \frac{(G+\alpha R)^{\sqrt{n}}}{(m-1)\alpha(1 - \sigma_2(P))}$ for any $m \in \{2, \cdots, B\}$ and $\epsilon_1 = 0$. For any $i, j \in V$ and $t \geq K + 1$, according to Assumptions 2 and 3, we have
\begin{align*}
f_{t,j}(x_i(m_t)) - f_{t,j}(x^*) &\leq f_{t,j}(\bar{x}(m_t)) + G\|\bar{x}(m_t) - x_i(m_t)\|_2 - f_{t,j}(x^*) \\
&\leq f_{t,j}(x_j(m_t)) + G\|\bar{x}(m_t) - x_j(m_t)\|_2 - f_{t,j}(x^*) + G\epsilon_{m_t} \\
&\leq \nabla f_{t,j}(x_j(m_t))^\top (x_j(m_t) - x^*) - \frac{\alpha}{2} \|x_j(m_t) - x^*\|_2^2 + 2G\epsilon_{m_t}
\end{align*}
where the last inequality is due to the strong convexity of $f_{t,j}(x)$. 

11
Moreover, if \( t \in \{1, \ldots, K\} \), for any \( i, j \in V \), we have \( x_i(m_t) = x_j(m_t) = 0 \), which implies that

\[
  f_{t,j}(x_i(m_t)) - f_{t,j}(x^*) = f_{t,j}(x_j(m_t)) - f_{t,j}(x^*) \\
  \leq \nabla f_{t,j}(x_j(m_t))^\top (x_j(m_t) - x^*) - \frac{\alpha}{2} \|x_j(m_t) - x^*\|_2^2 + 2G\epsilon_{m_t}.
\]

Summing over \( t = 1, \ldots, T \), for any \( i, j \in V \), we have

\[
  \sum_{t=1}^T (f_{t,j}(x_i(m_t)) - f_{t,j}(x^*)) \\
  \leq \sum_{t=1}^T \nabla f_{t,j}(x_j(m_t))^\top (x_j(m_t) - x^*) - \frac{\alpha}{2} \sum_{t=1}^T \|x_j(m_t) - x^*\|_2^2 + 2G \sum_{t=1}^T \epsilon_{m_t} \\
  = \sum_{t=1}^T \nabla f_{t,j}(x_j(m_t))^\top (x_j(m_t) - \bar{x}(m_t)) + \sum_{t=1}^T \nabla f_{t,j}(x_j(m_t))^\top (\bar{x}(m_t) - x^*) \\
  - \frac{\alpha}{2} \sum_{t=1}^T \|x_j(m_t) - x^*\|_2^2 + 2G \sum_{t=1}^T \epsilon_{m_t} \\
  \leq \sum_{t=1}^T \|\nabla f_{t,j}(x_j(m_t))\|_2 \|x_j(m_t) - \bar{x}(m_t)\|_2 + \sum_{t=1}^T \nabla f_{t,j}(x_j(m_t))^\top (\bar{x}(m_t) - x^*) \\
  - \frac{\alpha}{2} \sum_{t=1}^T \|x_j(m_t) - x^*\|_2^2 + 2G \sum_{t=1}^T \epsilon_{m_t} \\
  \leq \sum_{t=1}^T \nabla f_{t,j}(x_j(m_t))^\top (\bar{x}(m_t) - x^*) - \frac{\alpha}{2} \sum_{t=1}^T \|x_j(m_t) - x^*\|_2^2 + 3G \sum_{t=1}^T \epsilon_{m_t}
\]

where the last inequality is due to Assumption 2 and (9).

We note that

\[
  \|x_j(m_t) - x^*\|_2^2 \\
  = \|x_j(m_t)\|_2^2 - 2x_j(m_t)^\top x^* + \|x^*\|_2^2 \\
  = \|x_j(m_t) - \bar{x}(m_t)\|_2^2 - 2x_j(m_t)^\top (x^* - \bar{x}(m_t)) + \|x^*\|_2^2 - \|\bar{x}(m_t)\|_2^2.
\]

Therefore, for any \( i, j \in V \), it holds that

\[
  \sum_{t=1}^T (f_{t,j}(x_i(m_t)) - f_{t,j}(x^*)) \leq \sum_{t=1}^T (\nabla f_{t,j}(x_j(m_t)) - \alpha x_j(m_t))^\top (\bar{x}(m_t) - x^*) \\
  - \frac{\alpha}{2} \sum_{t=1}^T (\|x^*\|_2^2 - \|\bar{x}(m_t)\|_2^2) + 3G \sum_{t=1}^T \epsilon_{m_t}.
\]
Furthermore, summing over $j = 1, \cdots, n$, for any $i \in V$, we have

$$
\sum_{t=1}^{T} \sum_{j=1}^{n} f_{t,j}(x_i(m_t)) - \sum_{t=1}^{T} \sum_{j=1}^{n} f_{t,j}(x^*) \\
\leq \sum_{t=1}^{T} \sum_{j=1}^{n} (\nabla f_{t,j}(x_j(m_t)) - \alpha x_j(m_t))^{\top}(\bar{x}(m_t) - x^*) \\
- \frac{\alpha}{2} \sum_{t=1}^{T} \sum_{j=1}^{n} (\|x^*\|_2^2 - \|x(m_t)\|_2^2) \leq 3nG \sum_{t=1}^{T} \epsilon_{m_t} \\
= \sum_{m=1}^{B} \sum_{t \in T_m} \sum_{j=1}^{n} \left((\nabla f_{t,j}(x_j(m)) - \alpha x_j(m))^{\top}(\bar{x}(m) - x^*) - \frac{\alpha}{2} (\|x^*\|_2^2 - \|\bar{x}(m)\|_2^2)\right) \\
+ 3nG \sum_{m=1}^{B} \sum_{t \in T_m} \epsilon_{m} \\
= \sum_{m=1}^{B} \sum_{j=1}^{n} \left(\tilde{g}_j(m) - \alpha K x_j(m))^{\top}(\bar{x}(m) - x^*) - \frac{\alpha K}{2} (\|x^*\|_2^2 - \|\bar{x}(m)\|_2^2)\right) \\
+ 3nGK \sum_{m=1}^{B} \epsilon_{m} \\
= \sum_{m=1}^{B} \left(\tilde{d}(m)^{\top}(\bar{x}(m) - x^*) - \frac{\alpha K}{2} (\|x^*\|_2^2 - \|\bar{x}(m)\|_2^2)\right) + 3nGK \sum_{m=1}^{B} \epsilon_{m}.
$$

To bound $\sum_{m=1}^{B} (\tilde{d}(m)^{\top}(\bar{x}(m) - x^*) - \frac{\alpha K}{2} (\|x^*\|_2^2 - \|\bar{x}(m)\|_2^2))$, we introduce the following lemma.

Lemma 5 (Lemma 6.6 in Garber and Hazan (2016)) Let $\{f_t(x)\}_{t=1}^{T}$ be a sequence of loss functions and let $x^* \in \arg\min_{x \in K} \sum_{t=1}^{T} f_t(x)$ for any $t \in [T]$. Then, it holds that

$$
\sum_{t=1}^{T} f_t(x^*_t) - \min_{x \in K} \sum_{t=1}^{T} f_t(x) \leq 0.
$$

Before applying Lemma 5, we define $\tilde{f}_m(x) = \tilde{d}(m)^{\top} x + \frac{\alpha K}{2} \|x\|_2^2$. According to the definition, we have

$$
\bar{x}(m + 1) = \arg\min_{x \in K} \tilde{z}(m + 1)^{\top} x + \frac{m \alpha K}{2} \|x\|_2^2 \\
= \arg\min_{x \in K} \sum_{\tau=1}^{m} \tilde{f}_{\tau}(x).
$$
So, applying Lemma 5 with the loss functions \( \{ \tilde{f}_m(x) \} \) and the decision set \( \mathcal{K} \), we have

\[
\begin{align*}
\sum_{m=1}^{B} & \left( \mathbf{d}(m)^\top (\bar{x}(m) - x^*) - \frac{\alpha K}{2} (\|x^*\|_2^2 - \|\bar{x}(m)\|_2^2) \right) \\
= & \sum_{m=1}^{B} \left( \tilde{f}_m(\bar{x}(m)) - \tilde{f}_m(x^*) \right) \\
= & \sum_{m=1}^{B} \left( \tilde{f}_m(\bar{x}(m+1)) - \tilde{f}_m(x^*) \right) + \sum_{m=1}^{B} \left( \tilde{f}_m(\bar{x}(m)) - \tilde{f}_m(\bar{x}(m+1)) \right) \\
\leq & \sum_{m=1}^{B} \left( \tilde{f}_m(\bar{x}(m)) - \tilde{f}_m(\bar{x}(m+1)) \right) \\
\leq & \sum_{m=1}^{B} \nabla \tilde{f}_m(\bar{x}(m))^\top (\bar{x}(m) - \bar{x}(m+1)) \\
\leq & \sum_{m=1}^{B} \|\mathbf{d}(m) + \alpha K \bar{x}(m)\|_2 \|\bar{x}(m) - \bar{x}(m+1)\|_2.
\end{align*}
\]

(11)

It is easy to verify that \( \bar{F}_{m+1}(x) \) is \( m\alpha K \)-strongly convex, which implies that

\[
\begin{align*}
\frac{m\alpha K}{2} \|\bar{x}(m) - \bar{x}(m+1)\|_2^2 \leq & \bar{F}_{m+1}(\bar{x}(m)) - \bar{F}_{m+1}(\bar{x}(m+1)) \\
= & \bar{F}_m(\bar{x}(m)) + \bar{f}_m(\bar{x}(m)) - \bar{F}_m(\bar{x}(m+1)) - \bar{f}_m(\bar{x}(m+1)) \\
\leq & \|\mathbf{d}(m) + \alpha K \bar{x}(m)\|_2 \|\bar{x}(m) - \bar{x}(m+1)\|_2
\end{align*}
\]

where the first inequality is due to (2).

The above inequality can be simplified as

\[
\|\bar{x}(m) - \bar{x}(m+1)\|_2 \leq \frac{2\|\mathbf{d}(m) + \alpha K \bar{x}(m)\|_2}{m\alpha K}.
\]

(12)

Since \( \|\mathbf{d}_i(m)\|_2 \leq K(G + \alpha R) \), we have

\[
\begin{align*}
\|\mathbf{d}(m) + \alpha K \bar{x}(m)\|_2 \leq & \|\mathbf{d}(m)\|_2 + \|\alpha K \bar{x}(m)\|_2 \\
\leq & \left\| \frac{1}{n} \sum_{i=1}^{n} \mathbf{d}_i(m) \right\|_2 + \alpha KR \\
\leq & K(G + 2\alpha R).
\end{align*}
\]

(13)

Substituting (12) and (13) into (11), we have

\[
\begin{align*}
\sum_{m=1}^{B} \left( \mathbf{d}(m)^\top (\bar{x}(m) - x^*) - \frac{\alpha K}{2} (\|x^*\|_2^2 - \|\bar{x}(m)\|_2^2) \right) \leq & \sum_{m=1}^{B} \frac{2\|\mathbf{d}(m) + \alpha K \bar{x}(m)\|_2^2}{m\alpha K} \\
\leq & \sum_{m=1}^{B} \frac{2K(G + 2\alpha R)^2}{m\alpha}.
\end{align*}
\]

(14)
Finally, substituting (14) into (10), we have

$$
\sum_{t=1}^{T} \sum_{j=1}^{n} f_{t,j}(x_i(m_t)) - \sum_{t=1}^{T} \sum_{j=1}^{n} f_{t,j}(x^*)
\leq n \sum_{m=1}^{B} \frac{2K(G + 2\alpha R)^2}{m\alpha} + 3nGK \sum_{m=1}^{B} \epsilon_m
\leq \left( \frac{2nK(G + 2\alpha R)^2}{\alpha} + \frac{3GK(G + \alpha R)n^{3/2}}{\alpha(1 - \sigma_2(P))} \right) \sum_{m=1}^{B} \frac{1}{m} + 3nGK \frac{4RB}{\sqrt{L + 2}}
\leq \left( \frac{2nK(G + 2\alpha R)^2}{\alpha} + \frac{3GK(G + \alpha R)n^{3/2}}{\alpha(1 - \sigma_2(P))} \right) (1 + \ln B) + 3nGK \frac{4RB}{\sqrt{L + 2}}.
$$

We complete this proof with $B = T/K$.

4.2 Proof of Theorem 2

In each round $t$, for the local learner 1, we simply set $f_{t,1}(x) = \frac{g}{2}\|x\|_2^2$ that satisfies Assumption 2 with $G = 2\alpha R$ and Assumption 3. Moreover, we select $f_{t,2}(x)$ for the local learner 2 with a more carefully strategy. In this way, the global loss function is $f_t(x) = f_{t,1}(x) + f_{t,2}(x) = f_{t,2}(x) + \frac{g}{2}\|x\|_2^2$. Since the local loss function $f_{t,2}(x)$ is only revealed to the local learner 2, the local learner 1 cannot access to the time-varying part of the global loss unless it communicates with the local learner 2. Therefore, we can maximize the impact of communication on the regret of the local learner 1.

Without loss of generality, we denote the set of communication rounds by $\mathcal{C} = \{c_1, \cdots, c_C\}$, where $1 \leq c_1 < \cdots < c_C < T$. Let $c_0 = 0, c_{C+1} = T$. Then, we can divide the total $T$ rounds into the following $C + 1$ intervals

$$[c_0 + 1, c_1], [c_1 + 1, c_2], \cdots, [c_C + 1, c_{C+1}].$$

For any $i \in \{0, \cdots, C\}$ and $t \in [c_i + 1, c_{i+1}]$, we will set $f_{t,2}(x) = h_i(x)$, which is revealed to the local learner 1 after the decision $x_1(c_{i+1})$ is made.

For any distributed online algorithm with communication rounds $\mathcal{C} = \{c_1, \cdots, c_C\}$, we denote the sequence of decisions made by the local learner 1 by

$$x_1(1), \cdots, x_1(T).$$

For any $i \in \{0, \cdots, C\}$, we note that the decisions $x_1(c_i + 1), \cdots, x_1(c_{i+1})$ are made before the loss function $h_i(x)$ is revealed.

We first utilize a randomized strategy to select $h_i(x)$ for any $i \in \{0, \cdots, C\}$, and derive an expected lower bound for $R_{T,1}$. Specifically, we independently select $h_i(x) = w_i^T x + \frac{g}{2}\|x\|_2^2$ for any $i \in \{0, \cdots, C\}$, where the coordinates of $w_i$ are $\pm\alpha R/\sqrt{d}$ with probability 1/2. So,
$h_i(x)$ satisfies Assumption 2 with $G = 2\alpha R$ and Assumption 3 respectively. Moreover, let $w = \frac{1}{\alpha T} \sum_{i=0}^{C} (c_{i+1} - c_i) w_i$. In this way, the total loss for any $x \in K$ is equal to

\[
\sum_{t=1}^{T} f_t(x) = \sum_{t=1}^{T} \left( f_{t,2}(x) + \frac{\alpha}{2} \|x\|_2^2 \right) \\
= \sum_{i=0}^{C} (c_{i+1} - c_i) \left( w_i^\top x + \alpha \|x\|_2^2 \right) \\
= \alpha T w^\top x + \alpha T \|x\|_2^2 \\
= \alpha T \left\| x + \frac{w}{2} \right\|_2^2 - \frac{\alpha T}{4} \|\bar{w}\|_2^2.
\]

Since the absolute value of each element in $w_i$ is equal to $\alpha R / \sqrt{d}$, we note that the absolute value of each element in $\bar{w}/2$ is bounded by

\[
\frac{1}{2\alpha T} \sum_{i=0}^{C} \frac{(c_{i+1} - c_i) \alpha R}{\sqrt{d}} = \frac{R}{2\sqrt{d}} \leq \frac{R}{\sqrt{d}}
\]

which implies that $-\bar{w}/2$ belongs to $K = \left[-R/\sqrt{d}, R/\sqrt{d}\right]^d$.

Combining with (15), we have

\[
\arg\min_{x \in K} \sum_{t=1}^{T} f_t(x) = -\frac{\bar{w}}{2} \quad \text{and} \quad \min_{x \in K} \sum_{t=1}^{T} f_t(x) = -\frac{\alpha T}{4} \|\bar{w}\|_2^2.
\]

Then, we have

\[
E_{w_0, \ldots, w_C}[R_{T,1}] = E_{w_0, \ldots, w_C} \left[ \sum_{t=1}^{T} f_t(x_1(t)) - \min_{x \in K} \sum_{t=1}^{T} f_t(x) \right] \\
= E_{w_0, \ldots, w_C} \left[ \sum_{i=0}^{C} \sum_{t=c_i+1}^{c_{i+1}} h_i(x_1(t)) + \frac{\alpha T}{4} \|\bar{w}\|_2^2 \right] \\
\geq \min_{w_0, \ldots, w_C} \left[ \sum_{i=0}^{C} \sum_{t=c_i+1}^{c_{i+1}} w_i^\top x_1(t) + \frac{\alpha T}{4} \|\bar{w}\|_2^2 \right] \\
= E_{w_0, \ldots, w_C} \left[ \frac{\alpha T}{4} \|\bar{w}\|_2^2 \right]
\]

where the inequality is due to $\alpha \|x\|_2^2 \geq 0$ for any $x$ and the last equality is due to $E_{w_0, \ldots, w_C}[w_i^\top x_1(t)] = 0$ for any $t \in [c_i + 1, c_{i+1}]$. 


Let $\epsilon_{01}, \ldots, \epsilon_{0d}, \ldots, \epsilon_{C1}, \ldots, \epsilon_{Cd}$ be independent and identically distributed variables with $\Pr(\epsilon_{ij} = \pm 1) = 1/2$ for $i \in \{0, \ldots, C\}$ and $j = 1, \ldots, d$. Then, we have

\[
\mathbb{E}_{w_0, \ldots, w_C} \left[ \alpha^T \frac{\|\bar{w}\|_2^2}{4} \right] = \frac{1}{4\alpha^T} \mathbb{E}_{w_0, \ldots, w_C} \left[ \left\| \sum_{i=0}^{C} (c_{i+1} - c_i) w_i \right\|_2^2 \right] = \frac{1}{4\alpha^T} \mathbb{E}_{\epsilon_{01}, \ldots, \epsilon_{Cd}} \left[ \sum_{j=1}^{d} \left( \sum_{i=0}^{C} (c_{i+1} - c_i) \frac{\epsilon_{ij} \alpha R}{\sqrt{d}} \right)^2 \right] = \frac{\alpha R^2}{4T} \mathbb{E}_{\epsilon_{01}, \ldots, \epsilon_{C1}} \left[ \left( \sum_{i=0}^{C} (c_{i+1} - c_i) \epsilon_i \right)^2 \right] \geq \frac{\alpha R^2}{4T} \sum_{i=0}^{C} (c_{i+1} - c_i)^2 \geq \frac{\alpha R^2}{4T} \frac{(C_{C+1} - c_0)^2}{C + 1} = \frac{\alpha R^2 T}{4(C + 1)}
\]

where the first inequality is due to the Khintchine inequality and the second inequality is due to the Cauchy-Schwarz inequality.

Combining (17) with (16), we derive an expected lower bound as

\[
\mathbb{E}_{w_0, \ldots, w_C}[R_{T,1}] \geq \frac{\alpha R^2 T}{4(C + 1)}
\]

which implies that for any distributed online algorithm with communication rounds $C = \{c_1, \ldots, c_C\}$, there exists a particular choice of $w_0, \ldots, w_C$ such that

\[
R_{T,1} \geq \frac{\alpha R^2 T}{4(C + 1)}.
\]

### 4.3 Proof of Theorem 3

Similar to the proof of Theorem 1, we first define several auxiliary variables.

Let $\bar{z}(m) = \frac{1}{n} \sum_{i=1}^{n} z_i(m)$, $d_i(m) = \bar{g}_i(m) - \alpha K x_i(m)$ and $d(m) = \frac{1}{n} \sum_{i=1}^{n} d_i(m)$. Then, we define

\[
\bar{F}_{m+1}(x) = \bar{z}(m + 1)^\top x + \frac{m\alpha K}{2} \|x\|_2^2
\]

and $\bar{x}(1) = 0, \bar{x}(m+1) = \argmin_{x \in \mathcal{K}_\delta} \bar{F}_{m+1}(x)$ for $m \geq 1$. Similarly, let $\hat{x}_i(m) = \argmin_{x \in \mathcal{K}_\delta} z_i(m)^\top x + \frac{(m-1)\alpha K}{2} \|x\|_2^2$ for $m \geq 2$.

Moreover, we introduce the following lemmas.

**Lemma 6** Let $f(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ be $\alpha$-strongly convex and $G$-Lipschitz over a convex and compact set $\mathcal{K} \subset \mathbb{R}^d$. Then, $\hat{f}_\delta(x)$ has the following properties.
• \( \hat{f}_\delta(x) \) is \( \alpha \)-strongly convex over \( \mathcal{K}_\delta \);
• \( |\hat{f}_\delta(x) - f(x)| \leq \delta G \) for any \( x \in \mathcal{K}_\delta \);
• \( \hat{f}_\delta(x) \) is \( G \)-Lipschitz over \( \mathcal{K}_\delta \).

Lemma 7 For any \( i \in V \) and \( m \in [B] \), Algorithm 3 ensures that
\[
\mathbb{E}[\|d_i(m)\|_2^2] \leq \mathbb{E}[\|d_i(m)\|_2^2] \leq 2K \left( \frac{dM}{\delta} \right)^2 + 2K^2G^2 + 2(\alpha KR)^2.
\]

Lemma 8 Let \( \tilde{x}_i(m) = \arg\min_{x \in \mathcal{K}_\delta} F_{m,i}(x) \), for any \( i \in V \) and \( m \in \{2, \ldots, B\} \), where \( F_{m,i}(x) \) is defined in Algorithm 3. Under Assumption 4, for any \( i \in V \) and \( m \in \{2, \ldots, B\} \), Algorithm 3 with \( L \geq 1 \) has
\[
F_{m,i}(x_i(m)) - F_{m,i}(\tilde{x}_i(m)) \leq \frac{8(m-1)\alpha KR^2}{L + 2}.
\]

Let \( x^* = \arg\min_{x \in \mathcal{K}} \sum_{t=1}^T f_t(x) \) and \( \bar{x}^* = (1 - \delta/r)x^* \). For any \( i, j \in V \), we have
\[
\begin{aligned}
&\sum_{t=1}^T f_{t,i}(y_i(t)) - \sum_{t=1}^T f_{t,i}(x^*) \\
&= \sum_{t=1}^T f_{t,i}(x_i(m_i)) + \delta u_i(t) - \sum_{t=1}^T f_{t,i}(x^*) \\
&\leq \sum_{t=1}^T (f_{t,i}(x_i(m_i)) + G\|\delta u_i(t)\|_2 - \sum_{t=1}^T (f_{t,i}(\bar{x}^*) - G\|\bar{x}^* - x^*\|_2) \\
&\leq \sum_{t=1}^T f_{t,i}(x_i(m_i)) - \sum_{t=1}^T f_{t,i}(\bar{x}^*) + \delta G + \frac{\delta GRT}{r} \\
&\leq \sum_{t=1}^T (\hat{f}_{t,i,\delta}(x_i(m_i)) + \delta G) - \sum_{t=1}^T (\hat{f}_{t,i,\delta}(\bar{x}^*) - \delta G) + \delta G + \frac{\delta GRT}{r} \\
&\leq \sum_{t=1}^T (\hat{f}_{t,i,\delta}(x_i(m_i)) - \hat{f}_{t,i,\delta}(\bar{x}^*)) + 3\delta G + \frac{\delta GRT}{r}
\end{aligned}
\]  
where the first inequality is due to Assumption 2 and the third inequality is due to Lemma 6.

Then, similar to the proof of Theorem 1, we derive an upper bound of \( \|x_i(m) - \bar{x}(m)\|_2 \) by further introducing the following lemma.

Lemma 9 (Variant of Lemma 6 in Zhang et al. (2017)) Let \( z_i(1) = 0 \), \( z_i(m + 1) = \sum_{j \in N} P_{ij}z_j(m) + d_i(m) \) and \( \bar{z}(m) = \frac{1}{n} \sum_{i=1}^n z_i(m) \) for \( m \in [B] \), where \( P \) satisfies Assumption 4. For any \( i \in V \) and \( m \in [B] \), assume \( \mathbb{E}[\|d_i(m)\|_2] \leq \beta \), \( \beta \), we have
\[
\mathbb{E}[\|z_i(m) - \bar{z}(m)\|_2] \leq \frac{\beta \sqrt{n}}{1 - \sigma_2(P)}.
\]
Combining Lemmas 7 with 9, we have
\[
\mathbb{E}[\|z_i(m) - \tilde{z}(m)\|_2] \leq \sqrt{2K \left( \frac{dM}{\delta} \right)^2 + 2K^2G^2 + 2(\alpha KR)^2} \frac{\sqrt{n}}{1 - \sigma_2(P)}.
\] (19)

Applying Lemma 3 with (19), for any \( m \in \{2, \ldots, B\} \), we have
\[
\mathbb{E}[\|\hat{x}_i(m) - \tilde{x}(m)\|_2] \leq \frac{1}{(m-1)\alpha K} \mathbb{E}[\|z_i(m) - \tilde{z}(m)\|_2]
\]
\[
\leq \sqrt{2K \left( \frac{dM}{\delta} \right)^2 + 2K^2G^2 + 2(\alpha KR)^2} \frac{\sqrt{n}}{(m-1)\alpha K (1 - \sigma_2(P))}.
\]

Due to the fact that \( F_{m,i}(x) \) is \((m-1)\alpha K\)-strongly convex and \(2\), for any \( m \in \{2, \ldots, B\} \), it holds that
\[
\|x_i(m) - \hat{x}_i(m)\|_2 \leq \sqrt{2F_{m,i}(x_i(m)) - 2F_{m,i}(\hat{x}_i(m))} \leq \frac{4R}{\sqrt{L+2}}
\]
where the last inequality is due to Lemma 8.

Combining the above two inequalities, for any \( m \in \{2, \ldots, B\} \), we have
\[
\mathbb{E}[\|x_i(m) - \hat{x}(m)\|_2] \leq \mathbb{E}[\|x_i(m) - \tilde{x}_i(m)\|_2] + \mathbb{E}[\|\tilde{x}_i(m) - \tilde{x}(m)\|_2]
\]
\[
\leq \frac{4R}{\sqrt{L+2}} + \sqrt{2K \left( \frac{dM}{\delta} \right)^2 + 2K^2G^2 + 2(\alpha KR)^2} \frac{\sqrt{n}}{(m-1)\alpha K (1 - \sigma_2(P))}.
\] (20)

Applying Lemma 6, we have
\[
\tilde{f}_{i,j,\delta}(x_i(m_t)) - \tilde{f}_{i,j,\delta}(\tilde{x}) \leq \tilde{f}_{i,j,\delta}(x_j(m_t)) - \tilde{f}_{i,j,\delta}(\tilde{x}) + G\|\tilde{x}(m_t) - x_i(m_t)\|_2
\]
\[
\leq \tilde{f}_{i,j,\delta}(x_j(m_t)) - \tilde{f}_{i,j,\delta}(\tilde{x}) + G\|\tilde{x}(m_t) - x_j(m_t)\|_2 + G\|x_j(m_t) - x_i(m_t)\|_2
\]
\[
\leq \nabla \tilde{f}_{i,j,\delta}(x_j(m_t))^\top (x_j(m_t) - \tilde{x}) - \frac{\alpha}{2}\|x_j(m_t) - \tilde{x}\|^2_2
\]
\[
+ G\|\tilde{x}(m_t) - x_j(m_t)\|_2 + G\|\tilde{x}(m_t) - x_i(m_t)\|_2
\]
where the first two inequalities are due to the fact that \( \tilde{f}_{i,j,\delta}(x) \) is \(G\)-Lipschitz over \( K_\delta \), and the last inequality is due to the strong convexity of \( \tilde{f}_{i,j,\delta}(x) \).

Let \( \epsilon_m = \frac{4R}{\sqrt{L+2}} + \sqrt{2K \left( \frac{dM}{\delta} \right)^2 + 2K^2G^2 + 2(\alpha KR)^2} \frac{\sqrt{n}}{(m-1)\alpha K (1 - \sigma_2(P))} \) for any \( m = 2, \ldots, B \) and \( \epsilon_1 = 0 \). Combining the above inequality with (20) and \( \tilde{x}(1) = x_i(1) = 0, \forall i \in V \), for any \( i, j \in V \) and \( t \in [T] \), we have
\[
\mathbb{E}[\tilde{f}_{i,j,\delta}(x_i(m_t)) - \tilde{f}_{i,j,\delta}(\tilde{x})] \leq \mathbb{E} \left[ \nabla \tilde{f}_{i,j,\delta}(x_j(m_t))^\top (x_j(m_t) - \tilde{x}) - \frac{\alpha}{2}\|x_j(m_t) - \tilde{x}\|^2_2 \right] + 2G\epsilon_m. \]
Summing over \( t = 1, \ldots, T \), the first term in the right side of (18) can be bounded as

\[
\sum_{t=1}^{T} \mathbb{E}[\tilde{f}_{t,i,\delta}(x_i(m_t)) - \tilde{f}_{t,i,\delta}(\bar{x}^*)] \\
\leq \sum_{t=1}^{T} \mathbb{E} \left[ \nabla \tilde{f}_{t,i,\delta}(x_j(m_t))^\top (x_j(m_t) - \bar{x}^*) - \frac{\alpha}{2} \|x_j(m_t) - \bar{x}^*\|_2^2 \right] + 2G \sum_{t=1}^{T} \epsilon_{m_t} \\
= \sum_{t=1}^{T} \mathbb{E} \left[ \nabla \tilde{f}_{t,i,\delta}(x_j(m_t))^\top (x(m_t) - \bar{x}^*) - \frac{\alpha}{2} \|x_j(m_t) - \bar{x}^*\|_2^2 \right] \\
+ \sum_{t=1}^{T} \mathbb{E} [\nabla \tilde{f}_{t,i,\delta}(x_j(m_t))^\top (x_j(m_t) - \bar{x}(m_t))] + 2G \sum_{t=1}^{T} \epsilon_{m_t} \\
\leq \sum_{t=1}^{T} \mathbb{E} \left[ \nabla \tilde{f}_{t,i,\delta}(x_j(m_t))^\top (x(m_t) - \bar{x}^*) - \frac{\alpha}{2} \|x_j(m_t) - \bar{x}^*\|_2^2 \right] \\
+ \sum_{t=1}^{T} \mathbb{E} [G \|\bar{x}(m_t) - x_j(m_t)\|_2] + 2G \sum_{t=1}^{T} \epsilon_{m_t} \\
\leq \sum_{t=1}^{T} \mathbb{E} \left[ \nabla \tilde{f}_{t,i,\delta}(x_j(m_t))^\top (x(m_t) - \bar{x}^*) - \frac{\alpha}{2} \|x_j(m_t) - \bar{x}^*\|_2^2 \right] + 3G \sum_{t=1}^{T} \epsilon_{m_t}
\]

where the third inequality is due to the fact that \( \tilde{f}_{t,i,\delta}(x) \) is \( G \)-Lipschitz over \( \mathcal{K}_\delta \), and the last inequality is due to (20) and \( \bar{x}(1) = x_i(1) = 0, \forall i \in V \).

We note that

\[
\|x_j(m_t) - \bar{x}^*\|_2^2 = \|x_j(m_t) - \bar{x}(m_t)\|_2^2 - 2x_j(m_t)^\top (\bar{x}^* - \bar{x}(m_t)) + \|\bar{x}^*\|_2^2 - \|\bar{x}(m_t)\|_2^2.
\]

Therefore, for any \( i, j \in V \), it holds that

\[
\sum_{t=1}^{T} \mathbb{E}[\tilde{f}_{t,i,\delta}(x_i(m_t)) - \tilde{f}_{t,i,\delta}(\bar{x}^*)] \\
\leq \sum_{t=1}^{T} \mathbb{E} \left[ (\nabla \tilde{f}_{t,i,\delta}(x_j(m_t)) - \alpha x_j(m_t))^\top (\bar{x}(m_t) - \bar{x}^*) - \frac{\alpha}{2} (\|\bar{x}^*\|_2^2 - \|\bar{x}(m_t)\|_2^2) \right] + 3G \sum_{t=1}^{T} \epsilon_{m_t}.
\]

Combining (18) and the above inequality, for any \( i \in V \), we have

\[
\sum_{t=1}^{T} \sum_{j=1}^{n} \mathbb{E}[f_{t,j}(y_i(t)) - f_{t,j}(x^*)] \\
\leq \sum_{t=1}^{T} \sum_{j=1}^{n} \mathbb{E} \left[ (\nabla \tilde{f}_{t,i,\delta}(x_j(m_t)) - \alpha x_j(m_t))^\top (\bar{x}(m_t) - \bar{x}^*) - \frac{\alpha}{2} (\|\bar{x}^*\|_2^2 - \|\bar{x}(m_t)\|_2^2) \right] \\
+ 3\delta n G T + \frac{\delta n G T}{\rho} + 3n G \sum_{t=1}^{T} \epsilon_{m_t}.
\]
Due to Lemma 1, we have

\[
\sum_{t=1}^{T} \sum_{j=1}^{n} E \left[ (\nabla \hat{f}_{t,j}(x_{j}(m_{t})) - \alpha x_{j}(m_{t}))^\top (\bar{x}(m_{t}) - \bar{x}^*) - \frac{\alpha}{2} (\|\bar{x}^*\|_2^2 - \|\bar{x}(m_{t})\|_2^2) \right]
\]

\[
= \sum_{m=1}^{B} \sum_{t \in T_{m}} \sum_{j=1}^{n} E \left[ (\nabla \hat{f}_{t,j}(x_{j}(m)) - \alpha x_{j}(m))^\top (\bar{x}(m) - \bar{x}^*) - \frac{\alpha}{2} (\|\bar{x}^*\|_2^2 - \|\bar{x}(m)\|_2^2) \right]
\]

\[
= \sum_{m=1}^{B} \sum_{j=1}^{n} E \left[ (g_{j}(m) - \alpha K x_{j}(m))^\top (\bar{x}(m) - \bar{x}^*) - \frac{\alpha K}{2} (\|\bar{x}^*\|_2^2 - \|\bar{x}(m)\|_2^2) \right]
\]

\[
= n \sum_{m=1}^{B} E \left[ \tilde{d}(m)^\top (\bar{x}(m) - \bar{x}^*) - \frac{\alpha K}{2} (\|\bar{x}^*\|_2^2 - \|\bar{x}(m)\|_2^2) \right].
\]

Let \( \bar{f}_{m}(x) = \tilde{d}(m)^\top x + \frac{\alpha K}{2} \|x\|_2^2 \). According to the definition, we have

\[
\bar{x}(m + 1) = \arg\min_{x \in \mathcal{K}_{\delta}} \bar{z}(m + 1)^\top x + \frac{m \alpha K}{2} \|x\|_2^2 = \arg\min_{x \in \mathcal{K}_{\delta}} \sum_{\tau=1}^{m} \bar{f}_{\tau}(x).
\]

So, applying Lemma 5 with the loss functions \( \{ \bar{f}_{m}(x) \}_{m=1}^{B} \) and the decision set \( \mathcal{K}_{\delta} \), we have

\[
\sum_{m=1}^{B} \left( \tilde{d}(m)^\top (\bar{x}(m) - \bar{x}^*) - \frac{\alpha K}{2} (\|\bar{x}^*\|_2^2 - \|\bar{x}(m)\|_2^2) \right)
\]

\[
= \sum_{m=1}^{B} \left( \bar{f}_{m}(\bar{x}(m)) - \bar{f}_{m}(\bar{x}^*) \right)
\]

\[
= \sum_{m=1}^{B} \left( \bar{f}_{m}(\bar{x}(m + 1)) - \bar{f}_{m}(\bar{x}^*) \right) + \sum_{m=1}^{B} \left( \bar{f}_{m}(\bar{x}(m)) - \bar{f}_{m}(\bar{x}(m + 1)) \right)
\]

\[
\leq \sum_{m=1}^{B} \left( \bar{f}_{m}(\bar{x}(m)) - \bar{f}_{m}(\bar{x}(m + 1)) \right)
\]

\[
\leq \sum_{m=1}^{B} \nabla \bar{f}_{m}(\bar{x}(m))^\top (\bar{x}(m) - \bar{x}(m + 1))
\]

\[
\leq \sum_{m=1}^{B} \|\tilde{d}(m) + \alpha K \bar{x}(m)\|_2 \|\bar{x}(m) - \bar{x}(m + 1)\|_2.
\]

It is easy to verify that \( \bar{F}_{m+1}(x) \) is \( m \alpha K \)-strongly convex, which implies that

\[
\frac{m \alpha K}{2} \|\bar{x}(m) - \bar{x}(m + 1)\|_2^2 \leq \bar{F}_{m+1}(\bar{x}(m)) - \bar{F}_{m+1}(\bar{x}(m + 1))
\]

\[
= \bar{F}_{m}(\bar{x}(m)) + \bar{f}_{m}(\bar{x}(m)) - \bar{F}_{m}(\bar{x}(m + 1)) - \bar{f}_{m}(\bar{x}(m + 1))
\]

\[
\leq \|\tilde{d}(m) + \alpha K \bar{x}(m)\|_2 \|\bar{x}(m) - \bar{x}(m + 1)\|_2.
\]
where the first inequality is due to (2).

The above inequality can be simplified as

\[
\|\bar{x}(m) - \bar{x}(m+1)\|_2 \leq \frac{2\|d(m) + \alpha K \bar{x}(m)\|_2}{m\alpha K}.
\] (22)

Combining (21) and (22), we have

\[
\sum_{m=1}^{B} E \left[ d(m) \top (\bar{x}(m) - \bar{x}^*) - \frac{\alpha K}{2} (\|\bar{x}^*\|_2^2 - \|\bar{x}(m)\|_2^2) \right] \leq \sum_{m=1}^{B} E \left[ \frac{2\|d(m) + \alpha K \bar{x}(m)\|_2}{m\alpha K} \right].
\]

Due to Lemma 7, we have

\[
E\|d(m) + \alpha K \bar{x}(m)\|_2^2 \leq E[2\|d(m)\|_2^2] + E[2\|\alpha K \bar{x}(m)\|_2^2] \\
\leq 4K \left( \frac{dM}{\delta} \right)^2 + 4K^2 G^2 + 6(\alpha K R)^2.
\]

Therefore, we have

\[
\sum_{m=1}^{B} E \left[ d(m) \top (\bar{x}(m) - \bar{x}^*) - \frac{\alpha K}{2} (\|\bar{x}^*\|_2^2 - \|\bar{x}(m)\|_2^2) \right] \\
\leq \frac{1 + \ln B}{\alpha} \left( 8 \left( \frac{dM}{\delta} \right)^2 + 8K^2 G^2 + 12K \alpha^2 R^2 \right).
\]

Then, it is easy to verify that

\[
\sum_{t=1}^{T} \sum_{j=1}^{n} E[f_{t,j}(y_i(t)) - f_{t,j}(\bar{x}^*)] \\
\leq \frac{(1 + \ln B)}{\alpha} \left( 8 \left( \frac{dM}{\delta} \right)^2 + 8K^2 G^2 + 12K \alpha^2 R^2 \right) + 3\delta n G T + \frac{\delta n G R T}{r} + 3n G \sum_{t=1}^{T} \epsilon_{m_t} \\
= \frac{(1 + \ln B)}{\alpha} \left( 8 \left( \frac{dM}{\delta} \right)^2 + 8K^2 G^2 + 12K \alpha^2 R^2 \right) + 3\delta n G T + \frac{\delta n G R T}{r} + 3n G \sum_{m=1}^{B} \epsilon_m \\
\leq \frac{(1 + \ln B)}{\alpha} \left( 8 \left( \frac{dM}{\delta} \right)^2 + 8K^2 G^2 + 12K \alpha^2 R^2 \right) + 3\delta n G T + \frac{\delta n G R T}{r} \\
+ 3n G K \sum_{m=2}^{B} \left( \frac{4R}{\sqrt{L + 2}} + \sqrt{2K \left( \frac{dM}{\delta} \right)^2 + 2K^2 G^2 + 2(\alpha K R)^2 \sqrt{n} \over (m-1)\alpha K (1 - \sigma_2(P))} \right) \\
\leq \frac{(1 + \ln B)}{\alpha} \left( 8 \left( \frac{dM}{\delta} \right)^2 + 8K^2 G^2 + 12K \alpha^2 R^2 \right) + 3\delta n G T + \frac{\delta n G R T}{r} \\
+ 3n G K \left( \frac{4RB}{\sqrt{L + 2}} + \sqrt{2K \left( \frac{dM}{\delta} \right)^2 + 2K^2 G^2 + 2(\alpha K R)^2 \sqrt{n}(1 + \ln B) \over \alpha K (1 - \sigma_2(P))} \right)
\]
Finally, substituting $B = T/K$, $K = L = T^{2/3}$ and $\delta = cT^{-1/3}$, we have

$$
\sum_{t=1}^{T} \sum_{j=1}^{n} \mathbb{E}[f_{t,j}(y_i(t)) - f_{t,j}(x^*)] 
\leq \frac{1 + \ln T^{1/3}}{\alpha} \left( \frac{8d^2M^2}{c^2} + 8G^2 + 12\alpha^2R^2 \right) T^{2/3} + 3cnGT^{2/3} + \frac{cnGRT^{2/3}}{r} + \frac{3cnGRT^2}{r^2} \sqrt{2n(1 + \ln T^{1/3})}
+ \frac{3cnGT^2}{r^2} \sqrt{2n(1 - \sigma_2(P))}.
$$

4.4 Proof of Lemmas 4 and 8

This proof is based on the convergence rate of CG, which is presented in the following lemma.

Lemma 10 (Derived from Theorem 1 of Jaggi (2013)) If $F(x) : K \to \mathbb{R}$ is a convex and $\beta$-smooth function and $\|x\|_2 \leq R$ for any $x \in K$, Algorithm 4 with $L \geq 1$ ensures

$$
F(x_{out}) - F(x^*) \leq \frac{8\beta R^2}{L + 2}.
$$

where $x^* \in \arg\min_{x \in K} F(x)$.

We first prove Lemma 4. For any $m = 2, \cdots B$, according to Algorithm 2, we have

$$
x_i(m) = CG(K, L, F_{m,i}(x), x_i(m - 1)).
$$

Since $F_{m,i}(x)$ is $(m - 1)\alpha K$-smooth, according to Lemma 10 and Assumption 4, it is easy to verify that

$$
F_{m,i}(x_i(m)) - F_{m,i}(x_i(m)) \leq \frac{8(m - 1)\alpha KR^2}{L + 2}.
$$

Furthermore, we can prove Lemma 8 in the same way by noticing $K_\delta \subseteq K$.

4.5 Proof of Lemma 6

The first and second properties have been presented in Lemma 2.6 of Hazan (2016). The last property is proved by

$$
|f_{\delta}(x) - f_{\delta}(y)| = |E_{u \sim B^d}[f(x + \delta u)] - E_{u \sim B^d}[f(y + \delta u)]| 
= |E_{u \sim B^d}[f(x + \delta u) - f(y + \delta u)]| 
\leq E_{u \sim B^d}||f(x + \delta u) - f(y + \delta u)|| 
\leq E_{u \sim B^d}[G||x - y||_2] 
= G||x - y||_2
$$

where the first inequality is due to the Jensen’s inequality, and the second inequality is due to the fact that $f(x)$ is $G$-Lipschitz over $K$. 

23
4.6 Proof of Lemma 7

We first notice that

\[ \|\mathbf{d}_i(m)\|_2^2 = \|\hat{\mathbf{g}}_i(m) - \alpha K \mathbf{x}_i(m)\|_2^2 \leq 2 \|\hat{\mathbf{g}}_i(m)\|_2^2 + 2 \|\alpha K \mathbf{x}_i(m)\|_2^2 \leq 2 \|\hat{\mathbf{g}}_i(m)\|_2^2 + 2(\alpha KR)^2 \]

where the last inequality is due to Assumption 4.

Moreover, at the proof of Lemma 10 in Wan et al. (2020), they have proved that

\[ \mathbb{E}[\|\hat{\mathbf{g}}_i(m)\|_2^2] \leq K \left( \frac{dM}{\delta} \right)^2 + K^2 G^2. \]

Therefore, we have

\[ \mathbb{E}[\|\mathbf{d}_i(m)\|_2^2] \leq 2K \left( \frac{dM}{\delta} \right)^2 + 2K^2 G^2 + 2(\alpha KR)^2. \]

Moreover, according to Jensen’s inequality, we have

\[ \mathbb{E}[\|\mathbf{d}_i(m)\|_2^2] \leq \mathbb{E}[\|\mathbf{d}_i(m)\|_2^2]. \]

5. Conclusion

In this paper, we consider distributed OCO with strongly convex functions, and first propose a strongly convex variant of an existing D-BOCG algorithm, namely D-BOCG_{sc}. Compared with D-BOCG, our D-BOCG_{sc} can reduce the regret bound from \( O(T^{3/4}) \) to \( O(T^{2/3} \log T) \), and reduce the communication complexity from \( O(\sqrt{T}) \) to \( O(T^{1/3}) \) by utilizing the strong convexity. Furthermore, we provide a lower bound to show that the \( O(T^{1/3}) \) communication complexity of our D-BOCG_{sc} is nearly optimal for obtaining the \( O(T^{2/3} \log T) \) regret bound up to polylogarithmic factors. Finally, we propose a bandit variant of our D-BOCG_{sc}, namely D-BBCG_{sc}, which obtains an expected regret bound of \( O(T^{2/3} \log T) \) with \( O(T^{1/3}) \) communication complexity, and is better than an existing D-BBCG algorithm.

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