Perturbation Theory at Finite Extent of Fifth Dimension
for Vacuum Overlap Formula of Chiral Determinant

— Continuum Limit Case —

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Abstract

Taking into account of the boundary condition in the fifth direction which
is derived from the lattice Wilson fermion, we develop a theory of five-
dimensional fermion with kink-like and homogeneous masses in finite extent
of the fifth dimension. The boundary state wave functions are constructed
explicitly and the would-be vacuum overlap is expanded by using the propa-
gator of the theory. The subtraction is performed unambiguously at the finite
extent with the help of the dimensional regularization. Then the limit of
the infinite extent is evaluated. The consistent anomaly in four dimensional
theory is finitely obtained. Each contribution to the vacuum polarization is
vector-like. It is the lack of the massless mode in the fermion with negative
homogeneous mass that leads to the correct chiral normalization. Gauge non-
invariant piece remains due to the breaking of the boundary condition by the
dimensional regularization.

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I. INTRODUCTION

A nonperturbative regularization of chiral gauge theory, if it would exist, could offer a consistent framework for studying the dynamics of the standard model, especially the dynamics of spontaneous gauge symmetry breaking. Lattice regularization has succeeded to play such a role for understanding the QCD dynamics. For chiral theories, however, it suffers from the species doubling problem [1–4].

Recently new approaches by means of an infinite number of Fermi fields have been proposed for such a regularization [5–9]. A five-dimensional fermion has a chiral zero mode when coupled to a domain wall [10]. Kaplan formulated such a system on a lattice with Wilson fermion and discussed the possibility to simulate chiral fermions [5,11]. On the other hand, Frolov and Slavnov considered the possibility of regulating chiral fermion loops gauge-invariantly in the \( SO(10) \) chiral gauge theory with an infinite number of the Pauli-Villars-Gupta fields [6,12,13].

Unified point of view on these two approaches was given by Neuberger and Narayanan [7] and they have put forward the approach to derive a lattice vacuum overlap formula for the determinant of chiral fermion [8]. They first discarded the five dimensional nature of gauge boson in the Kaplan’s lattice setup. Then the five-dimensional fermion can be seen as a collection of infinitely many four-dimensional fermions labeled by the extra coordinate. They regarded the massive Dirac modes as regulator fields for the chiral mode (these correspond to the fermionic Pauli-Villars-Gupta fields in the method of Frolov and Slavnov) and gave a prescription to subtract irrelevant bulk effects of the massive modes by ordinary fermions with homogeneous masses (these correspond to the bosonic Pauli-Villars-Gupta fields). They emphasized the importance of infinite extent of the extra space to make sure a chiral content of fermions, which space is usually compactified on a lattice with periodic boundary condition. As a results of the infiniteness of the extra space, they obtained a vacuum overlap formula for the determinant of lattice chiral fermion, using transfer matrices in the direction of the extra space.

By means of the infinite number of the Pauli-Villars-Gupta fields, however, the odd-parity part can never be regularized because the regulator fields are the Dirac spinors [14]. Even for the even-parity part, there exist ambiguity in the summation over the infinite number of contributions. In order to make the summation well-defined, at the same time to make the number of regulator fields finite at the first stage, each contribution of the original or the regulator field should be made finite by a certain subsidiary regularization. Such subsidiary regularization necessarily breaks the gauge invariance in the contribution of the original chiral fermion. This may well lead to the gauge noninvariant result even in the limit of the infinite number of the regulator fields. The dimensional regularization is an example for such a subsidiary regularization [15].

In the formulation of the vacuum overlap, the problem of the odd-parity part is reflected in that we must fix the phase of the overlap by a certain guiding principle to reproduce the properties of the chiral determinant by this formula, especially anomaly. Fixing the phase...
of the overlap is equivalent to the choice of the wave functions of the boundary states. By this choice, we must carefully place a source of gauge noninvariance at the boundaries to reproduce the consistent anomaly.

Neuberger and Narayanan [8] proposed to fix the phase following the Wigner-Brillouin perturbation theory, referring to the ground state of the free Hamiltonian. By this prescription, the continuum two-dimensional overlap was examined and it was shown to reproduce the correct consistent anomaly. The lattice two- and four-dimensional overlap were also examined numerically and correct anomaly coefficients were observed for the Abelian background gauge field. Its behavior under the topologically nontrivial background gauge fields was also examined and promising result was obtained.

They also showed the gauge invariance of the even-parity part at nonperturbative level. Their discussions are based on the case of the periodic boundary condition in the fifth direction and also on the gauge transformation property of the ground states of the four-dimensional Hamiltonians.

On the other hand, in the viewpoint of the perturbation theory with the five(or three)-dimensional Wilson fermion, several authors discussed this problem. Shamir [16] considered the fermion in the infinite extent of the fifth dimension but restricted first the interaction with the gauge field into the finite region of the fifth space. By this restriction, the model becomes gauge noninvariant at the boundaries of the finite region. Then he examined how to take the limit of the infinite extent. He claimed that the limit should be taken uniformly at every interaction vertex with gauge boson and then showed that this way to introduce the gauge noninvariance leads to the correct consistent anomaly in two-dimensional model. In the similar approach, S. Aoki and R.B. Levien [17] performed a detailed study about the infinite extent of the extra dimension and the subtraction procedure in the lattice two-dimensional chiral Schwinger model. They showed that the scheme reproduces the desired form of the effective action: the gauge invariant real part and the consistent anomaly from the imaginary part.

Four-dimensional perturbative study of the vacuum overlap formula has been recently in progress. S. Randjbar-Daemi and J. Strathdee obtained the consistent anomaly by the four-dimensional Hamiltonian perturbation theory in the continuum limit [18]. Quite recently, they performed the similar analysis in the lattice regularization [19]. In the viewpoint of the theory with infinitely many regulator fields, it is also desirable to understand how the gauge-noninvariance put at the boundaries leads to the consistent anomaly and how the gauge-invariance of the even-parity part of the determinant is established by the subtraction. In this article, toward this goal, we further examine the four dimensional aspect of the vacuum overlap formula in the continuum limit.

Our approach is as follows. We consider the nonabelian background gauge field in general. We start from finite extent of the fifth direction in order to make the summation unambiguous. We first develop the theory of the free five-dimensional fermion with kink-like mass and positive(negative) homogeneous mass in the finite extent of fifth direction. As to the boundary condition in the fifth direction, we adopt the one derived from the Wilson
fermion, by which the Dirichlet and Neumann components are determined by the chiral projection.

From this free theory, the boundary state wave functions which correspond to the Wigner-Brillouin phase choice are explicitly constructed. Then we formulate the perturbation expansion of the vacuum overlap formula in terms of the propagator at the finite extent of the fifth dimension satisfying the boundary condition. After performing the subtraction at the finite extent of the fifth dimension, we examine the limit of the infinite extent.

As to a subsidiary regularization, we adopt the dimensional regularization. It turns out that the dimensional regularization cannot respect the boundary condition determined by the chiral projection. Although this fact reduces the ability of our analysis in the continuum limit, we believe that we can make clear in what way the vacuum overlap formula could give the perturbative properties of the chiral determinant in four dimensions. We also make another technical assumption that the dimensional regularization preserves the cluster property.

By this perturbation theory, we calculate the variation of the vacuum overlap under the gauge transformation which is induced by the boundary state wave function. We also calculate the two-point function of the external gauge boson. The variation is found to be finite and does not suffer from the subtlety of the dimensional regularization. It reproduces the consistent anomaly in four dimensions correctly. We also observe how the chiral normalization of the vacuum polarization is realized in the finite extent of the fifth direction. We, however, fail to establish its gauge invariance due to the dimensional regularization.

This article is organized as follows. In Sec. II, we discuss the lattice Schrödinger functional [20,21] to describe the evolution of the boundary state during a finite “time” interval in the fifth direction. It is naturally formulated from the transfer matrix given by Neuberger and Narayanan. The boundary condition of the fermion field is read off from the Wilson fermion action and the boundary term is derived. In Sec. III, we formulate the free theory of the five-dimensional fermion in the finite fifth space volume. We solve the field equation and obtain the complete set of solutions. The field operator is defined by the mode expansion and the propagator is derived. The Sommerfeld-Watson transformation is introduced, by which we rearrange the normal modes of the fifth momentum to be common among the fermions with the kink-like mass and the positive(negative) homogeneous mass. This makes it possible to do the subtraction at the finite extent of the fifth dimension. In Sec. IV, the perturbation theory for the vacuum overlap is developed. We first derive boundary state wave functions. Then we discuss the cluster property of the contribution induced by the boundary state wave function. By this cluster property, the boundary contribution turns out to be odd-parity in the limit of the infinite extent. In Sec. V, we perform the calculation of the anomaly induced by the boundary state wave function. In Sec. VI, we also perform the calculation of the vacuum polarization. Section VII is devoted to summary and discussion.
II. WOULD-BE VACUUM OVERLAP

In this section, we consider the five-dimensional Wilson fermion with kink-like mass and its finite “time” evolution in the fifth direction. In order to describe it, we introduce the Schrödinger functional, which is naturally formulated by the transfer matrix given by Neuberger and Narayanan. Through its path-integral representation, we read off the boundary condition imposed on the fermionic field and also derive the boundary terms. We also rewrite the functional in the form factorized into the determinant of the Dirac operator over the five-dimensional volume under the derived boundary condition and the contribution from the boundary terms. Next we will discuss how to prepare the boundary state wave functions which implement the Wigner-Brillouin phase choice. With these wave functions, we give the expression of the “would-be” vacuum overlap at finite extent of the fifth dimension. We also discuss its variation under the gauge transformation. Finally, we derive the counterpart in the continuum limit and in the Minkowski space. We also specify the regularization in the continuum limit analysis.

A. Boundary condition in the fifth direction and boundary terms

The action of the five-dimensional Wilson fermion with kink-like mass is given by

\[
A = \sum_{n,s} \left\{ \sum_\mu \frac{1}{2} \left[ \bar{\psi}(n,s)(1 + \gamma_\mu)U_\mu(n)\psi(n + \hat{\mu},s) + \bar{\psi}(n + \hat{\mu},s)(1 - \gamma_\mu)U^\dagger_\mu(n)\psi(n,s) \right] \\
+ \frac{1}{2} \left[ \bar{\psi}(n,s)(1 + \gamma_5)\psi(n,s + 1) + \bar{\psi}(n,s + 1)(1 - \gamma_5)\psi(n,s) \right] \\
+ \left( m_0 \text{sgn}(s + \frac{1}{2}) - 5 \right) \bar{\psi}(n,s)\psi(n,s) \right\} .
\]

(2.1)

Here we are considering SU(N) background gauge field in general.

The transfer matrix formulation for it was first given by Neuberger and Narayanan [8]. Let us assume that the Fock space is spaned by the operators \( \hat{c}_{\alpha i}(n) \) and \( \hat{d}^\dagger_{\alpha i}(n) \) satisfying the following commutation relations,

\[
\{ \hat{c}_{\alpha i}(n), \hat{c}^\dagger_{\beta j}(m) \} = \delta_{nm}\delta_{\alpha\beta}\delta_i^j , \quad \{ \hat{d}_{\alpha i}(n), \hat{d}^\dagger_{\beta j}(m) \} = \delta_{nm}\delta_{\alpha\beta}\delta_i^j .
\]

(2.2)

\[
\hat{c}_{\alpha i}(n) |0\rangle = 0 , \quad \hat{d}_{\alpha j}(n) |0\rangle = 0 .
\]

(2.3)

Note that \( \alpha, \beta \) denote the spinor index and \( i, j \) denote the index of the representation of SU(N) gauge group. Then the transfer matrix is given in terms of \( \hat{a} = (\hat{c}, \hat{d}^\dagger)^t \) and \( \hat{a}^\dagger = (\hat{c}^\dagger, \hat{d}) \) as,

\[
\hat{T}_{\pm} = \exp \left( \hat{a}^\dagger H_{\pm} \hat{a} \right) ,
\]

(2.4)

with the matrix
\[
\exp \left( H_{\pm} \right) \equiv \begin{pmatrix}
\frac{1}{B^z} & \frac{1}{B^z} C \\
\frac{1}{B^z} C^\dagger & \frac{1}{B^z} C + B^\pm
\end{pmatrix},
\]  
(2.5)

where

\[
B(n, m, s) = \left( 5 - m_0 \text{sgn}(s + \frac{1}{2}) \right) \delta_{n,m} \delta_{i}^j - \frac{1}{2} \sum_{\mu} \left( U_\mu(n)^j_i \delta_{n+\mu,m} + U_\mu^\dagger(m)^j_i \delta_{n,m+\mu} \right),
\]  
(2.6)

\[
C(n, m) = \frac{1}{2} \sum_{\mu} \sigma_{\mu\alpha\beta} \left( U_\mu(n)^j_i \delta_{n+\mu,m} - U_\mu^\dagger(m)^j_i \delta_{n,m+\mu} \right) \equiv \sum_{\mu} \sigma_{\mu\alpha\beta} \nabla_\mu (n, m),
\]  
(2.7)

and \( \sigma_\mu \equiv (1, i\sigma_i) \). \( B(n, m) \) can be shown to be positive definite for \( 0 < m_0 < 1 \). We also introduce the operator

\[
\hat{D}_\pm = \exp \left( \hat{a}^\dagger Q_\pm \hat{a} \right),
\]  
(2.8)

with

\[
\exp \left( Q_\pm \right) = \begin{pmatrix}
\frac{1}{\sqrt{B^z}} & \frac{1}{\sqrt{B^z}} C \\
\frac{1}{\sqrt{B^z}} C^\dagger & -\frac{1}{\sqrt{B^z}}
\end{pmatrix},
\]  
(2.9)

and we can show

\[
\exp \left( H_{\pm} \right) = \exp \left( Q_\pm \right)^\dagger \exp \left( Q_\pm \right), \quad \hat{T}_\pm = \hat{D}_\pm^\dagger \hat{D}_\pm.
\]  
(2.10)

We start from a finite “time” evolution in the fifth direction. We take the symmetric region \( s \in [-L-1, L] \). The evolution can be described by the Schrödinger kernel \([20, 21]\)

\[
\langle c_{-L-1}^*, d_{-L-1}^* \mid D_- (T_-)^L (T_+)^L D_+^\dagger \mid c_L, d_L \rangle,
\]  
(2.11)

in the coherent state basis,

\[
|c, d\rangle = \exp[-(c, \hat{c}^\dagger) - (d, \hat{d}^\dagger)] |0\rangle,
\]  
(2.12)

\[
\langle c^*, d^* \rangle = \langle 0 \mid \exp[-(\hat{c}, c^*) - (\hat{d}, d^*)].
\]  
(2.13)

Here we are following the notation in Ref. \( [8] \): \( (\bar{a}_s, b_s) \equiv \sum_{n, \alpha, \beta} \bar{a}_\alpha (n, s) b_\alpha(n, s) \).

In terms of the path integral, it reads

\[
\langle c_{-L-1}^*, d_{-L-1}^* \mid D_- (T_-)^L (T_+)^L D_+^\dagger \mid c_L, d_L \rangle \prod_{0 \leq s \leq L-1} (\det B_+)^2 \prod_{-L \leq s \leq -1} (\det B_-)^2
\]

\[
= \int \prod_{-L \leq s \leq -L-1} [D \psi_s D \tilde{\psi}_s] \exp \{-A[-L-1, L]\}
\]

\[
\equiv Z[\psi_L(-L-1), \tilde{\psi}_L(-L-1), \psi_R(L), \tilde{\psi}_R(L)].
\]  
(2.14)

The boundary variables are given by
\[ \psi_R(n, L) = \frac{1}{\sqrt{B_+}} \begin{pmatrix} c(n, L) \\ 0 \end{pmatrix}, \quad \bar{\psi}_R(n, L) = \begin{pmatrix} 0 & -b(n, L) \end{pmatrix} \frac{1}{\sqrt{B_+}}, \]  
(2.15)

\[ \psi_L(n, -L-1) = \frac{1}{\sqrt{B_-}} \begin{pmatrix} 0 \\ b^*(n, -L-1) \end{pmatrix}, \quad \bar{\psi}_L(n, -L-1) = \begin{pmatrix} c^*(n, -L-1) & 0 \end{pmatrix} \frac{1}{\sqrt{B_-}}. \]  
(2.16)

The action and the boundary terms are given by

\[ A[-L-1, L] = A + A_L + A_R, \]  
(2.17)

\[ A_L = \sum_n \sum_{s=-L}^{L-1} \left\{ \sum_{\mu} \bar{\psi}(n, s) \gamma_{\mu} \nabla_{\mu} \psi(n, s) - \bar{\psi}(n, s)B(n, m)\psi(n, s) + \frac{1}{2} \left[ \bar{\psi}(n, s)(1 + \gamma_5)\psi(n, s + 1) + \bar{\psi}(n, s + 1)(1 - \gamma_5)\psi(n, s) \right] \right\} \bigg|_{[-L, L]}, \]  
(2.18)

\[ A_R^L = \sum_n \left\{ \bar{\psi}(n, -L)P_L\psi(n, -L-1) + \bar{\psi}(n, -L-1)P_R\psi(n, -L) \right\} + \sum_n \sum_{\mu} \bar{\psi}(n, -L-1)\gamma_{\mu} \nabla_{\mu} P_L\psi(n, -L-1), \]  
(2.19)

\[ A_R^B = \sum_n \left\{ \bar{\psi}(n, L)P_L\psi(n, L-1) + \bar{\psi}(n, L-1)P_R\psi(n, L) \right\} + \sum_n \sum_{\mu} \bar{\psi}(n, L)\gamma_{\mu} \nabla_{\mu} P_R\psi(n, L), \]  
(2.20)

where \( \bigg|_{[-L, L]} \) stands for the homogeneous boundary condition:

\[ P_R\psi(n, L) = P_L\psi(n, -L-1) = 0, \quad \bar{\psi}(n, L)P_L = \bar{\psi}(n, -L-1)P_R = 0. \]  
(2.21)

We refer to this boundary condition as *chiral boundary condition* hereafter.

Since the boundary terms depending on \( \psi_R(L), \bar{\psi}_R(L) \) and \( \psi_L(-L-1), \bar{\psi}_L(-L-1) \) can be regarded to be source terms for the system with the homogeneous boundary condition, we obtain the following factorized form:

\[ Z[\psi_L(-L-1), \bar{\psi}_L(-L-1); \psi_R(L), \bar{\psi}_R(L)] = \det (K) \bigg|_{[-L, L]} \exp \left\{ -\left( \bar{\Psi}, M\Psi \right) \right\}, \]  
(2.22)

where \( \Psi(n) = (\psi(n, -L-1), \psi(n, L))^t \) and \( \bar{\Psi}(n) = (\bar{\psi}(n, -L-1), \bar{\psi}(n, L)) \), and

\[ K(n, s; m, t) = -\delta_{s,t} \left\{ \sum_{\mu} \gamma_{\mu} \nabla_{\mu} (n, m) - B(n, m; s) \right\} - \frac{1}{2} \left[ (1 + \gamma_5)\delta_{s+1,t} + (1 - \gamma_5)\delta_{s,t+1} \right], \]  
(2.23)

\[ M(n, m; [-L-1, L]) = \begin{pmatrix} P_R S(n, -L-1; m, -L-1)P_L + P_R \sum_{\mu} \gamma_{\mu} \nabla_{\mu} (n, m) & P_R S(n, -L-1; m, L)P_R \\ P_L S(n, L; m, -L-1)P_L & P_L S(n, L; m, L)P_R + P_L \sum_{\mu} \gamma_{\mu} \nabla_{\mu} (n, m) \end{pmatrix}, \]  
(2.24)
and

\[ S(n, s; m, t) = K^{-1}(n, s; m, t)|_{[-L, L]} . \] (2.25)

Note that the determinant and the inverse of \( K(n, s; m, t) \) should be evaluated taking into account of the chiral boundary condition (2.21).

**B. Boundary state wave functions and Gauge non-invariance**

Next we consider the “time” evolution of a boundary state at \( s = L \)

\[ |b_+ \rangle (at \ s = L), \] (2.26)

and its transition to another boundary state at \( s = -L-1 \),

\[ |b_- \rangle (at \ s = -L-1), \] (2.27)

The wave functions of these states, in general, can be given in the coherent state representation as

\[ \langle c^*, d^* | b_+ \rangle = \langle \psi_L, \bar{\psi}_L | b_+ \rangle, \quad \langle b_- | c^*, d^* \rangle = \langle b_- | \psi_R, \bar{\psi}_R \rangle . \] (2.28)

To reproduce the properties of the chiral determinant by the vacuum overlap formula, especially anomaly, we must carefully place the source of gauge noninvariance at the boundaries. This is equivalent to the choice of the wave functions of the boundary states. In order to implement a Wigner-Brillouin phase choice, we regard the boundary states as the states which evolved(would evolve) a certain period of “time” by the Hamiltonian without the gauge interaction. We take \( 2L' \) period of “time”. Then the boundary state \( |b_+ \rangle \) can be written as

\[ \langle \psi_L(l), \bar{\psi}_L(l) | b_\pm \rangle = \langle \psi_L(l), \bar{\psi}_L(l) | (T_+^{0})^{2L'} | b'_\pm \rangle . \] (2.29)

In the limit \( L' \to \infty \), we can expect that the ground state of \( \hat{H}_+^{0} \), which we denote as \( |0_+ \rangle \), is projected out for any choice of the boundary state \( |b'_+ \rangle \). Note that this is nothing but the way to introduce the breaking of gauge symmetry adopted in [16,17], by which the extent of the fifth dimension is infinite but the gauge field is introduced only in a finite range of the fifth dimension.

In terms of the path integral, Eq. (2.23) reads

\[
\begin{align*}
\langle \psi_L(l), \bar{\psi}_L(l) | b_\pm \rangle \\
= \int [D\psi(l + 2L')D\bar{\psi}(l + 2L')] \exp \left\{ \left( \bar{\psi}(l + 2L'), B_0^0 \psi(l + 2L') \right) \right\} \\
\times Z_+^0 [\psi_L(l), \bar{\psi}_L(l); \psi_R(l + 2L'), \bar{\psi}_R(l + 2L')] \langle \psi_L(l + 2L'), \bar{\psi}_L(l + 2L') | b'_\pm \rangle \\
= \int [D\psi(l + 2L')D\bar{\psi}(l + 2L')] \exp \left\{ \left( \bar{\psi}(l + 2L'), B_0^0 \psi(l + 2L') \right) \right\} \\
\times \det \left( K_+^0 \right)_{[L, L+2L']} \exp \left\{ - \left( \bar{\Psi}', M_+ \Psi' \right) \right\} \langle \psi_L(l + 2L'), \bar{\psi}_L(l + 2L') | b'_\pm \rangle ,
\end{align*}
\] (2.30)
where $\Psi'(n) = (\psi(n, l), \psi(n, l + 2l'))^t$ and $\bar{\Psi}'(n) = (\bar{\psi}(n, l), \bar{\psi}(n, l + 2l'))$, $|_{[L, L+2L']} \bar{\psi}$ stands for the boundary condition

$$P_R \psi(n, l + 2l') = P_L \psi(n, l) = 0, \quad \bar{\psi}(n, l + 2l') P_L = \bar{\psi}(n, l) P_R = 0,$$

and

$$M'_+(n, m; l, l + 2l') = \begin{pmatrix}
P_R S^0_+(n, L; m, L) P_L + P_R \sum_{\mu} \gamma_\mu \nabla^0_\mu (n, m) & \left. P_R S^0_+(n, L; m, L + 2L') P_R \right|_{[L, L+2L']}
\left. P_L S^0_+(n, L + 2L'; m, L) P_L \right|_{[L, L+2L']} & P_L S^0_+(n, L + 2L'; m, L + 2L') P_R + P_L \sum_{\mu} \gamma_\mu \nabla^0_\mu (n, m)
\end{pmatrix}.$$ \hspace{1cm} (2.32)

$$S^0_+(n, s; m, t) = K^{-1}(n, s; m, t)|_{[L, L+2L']} = K_+^{-1}(n, s; m, t)|_{[L, L+2L']}.$$ \hspace{1cm} (2.33)

The superscript 0 stands for the quantities in which the gauge link variable is set to unity.

At finite $L'$, we can find the following choice convenient:

$$|b'_+ \rangle = |0 \rangle.$$ \hspace{1cm} (2.34)

In this choice, the dependence of the wave function in the variables $\psi_L(l), \bar{\psi}_L(l)$ can be given explicitly as

$$\langle \psi_L(l), \bar{\psi}_L(l) | b_+ \rangle = c_+ \exp \left\{ -\left( \bar{\psi}(l), X'_+ \psi(l) \right) \right\}.$$ \hspace{1cm} (2.35)

where

$$X'_+(n, m) = P_R S^0_+(n, L; m, L) P_L + P_R \sum_{\mu} \gamma_\mu \nabla^0_\mu (n, m) + P_R S^0_+(L; L + 2L') P_R \left( B'_+ - P_L S^0_+ (L + 2L'; L + 2L') P_R - P_R \sum_{\mu} \gamma_\mu \nabla^0_\mu \right)^{-1} P_L S^0_+(L + 2L'; L) P_L (n, m),$$ \hspace{1cm} (2.36)

and $c_+$ is a certain constant depending on $l$ and $l + 2l'$.

We make a similar choice for the boundary state at $s = -L - 1$.

$$\langle b'_- | = |0 \rangle,$$ \hspace{1cm} (2.37)

and we obtain

$$\langle b_- | \psi_R(-L-1), \bar{\psi}_R(-L-1) \rangle = c'_+ \exp \left\{ -\left( \bar{\psi}(-L-1), X'_- \psi(-L-1) \right) \right\},$$ \hspace{1cm} (2.38)

where

$$X'_-(n, m) = P_L S^0_-(n, -L-1; m, -L-1) P_R + P_L \sum_{\mu} \gamma_\mu \nabla^0_\mu (n, m) + P_L S^0_-(L-1; L-1) P_R \left( B'_- - P_R S^0_- (L-1-2L'; L-1-2L') P_L - P_R \sum_{\mu} \gamma_\mu \nabla^0_\mu \right)^{-1} P_R S^0_- (L-1-2L'; L-1) P_R (n, m).$$ \hspace{1cm} (2.39)
C. Would-be vacuum overlap formula at finite extent of fifth dimension

Given the explicit form of the boundary state wave functions, the transition amplitude can be written as follows.

\[ \langle b_- | D_- (T_-)^L (T_+)^L D_+^T | b_+ \rangle \prod_{0 \leq s \leq L} (\det B_+)^2 \prod_{-L-1 \leq s \leq -1} (\det B_-)^2 \]

\[ = \int \prod_{s=-L-1}^{-1} [\mathcal{D} \psi(s) \mathcal{D} \bar{\psi}(s)] \exp \left\{ \sum_{s=-L-1}^{-1} \left( \bar{\psi}_s, B_s \psi_s \right) \right\} \times \langle b_+ | \psi_R(-L-1), \bar{\psi}_R(-L-1) \rangle Z[\psi_L, \bar{\psi}_L(-L-1); \psi_R, \bar{\psi}_R(L)] \langle \psi_L(L), \bar{\psi}_L(L) | b_- \rangle \]

\[ = \det (K) \left| \mathcal{D} \psi \right|_{[-L,L]} \exp \Phi(b_-, b_+), \tag{2.40} \]

where the boundary contribution \( \Phi(b_-, b_+) \) is given

\[ \exp \Phi(b_-, b_+) = \prod_{s=-L-1}^{-1} [\mathcal{D} \psi(s) \mathcal{D} \bar{\psi}(s)] \exp \left\{ \left( \bar{\Psi}, [B_{-+} - M - X_{-+}^{L'}] \Psi \right) \right\} c_+ c_+^* \]

\[ \equiv \det' \left( B_{-+} - M - X_{-+}^{L'} \right) c_+ c_+^*, \tag{2.41} \]

with

\[ B_{-+}(n, m) = \begin{pmatrix} B_-(n, m) & 0 \\ 0 & B_+(n, m) \end{pmatrix}, \tag{2.42} \]

and

\[ X_{-+}^{L'}(n, m) = \begin{pmatrix} X_{-+}^{L'}(n, m) & 0 \\ 0 & X_{++}^{L'}(n, m) \end{pmatrix}. \tag{2.43} \]

Note that \( \det' \) denotes the determinant over the four-dimensional surfaces at \( s = L \) and \(-L-1\) besides over the indices of spinor and the representation of gauge group.

It is also possible to write in a similar manner the transition amplitudes for the five-dimensional fermions with the positive and negative homogeneous masses, which are needed for the subtraction scheme proposed by Neuberger and Narayanan. Then the subtracted transition amplitude at finite \( L \) and \( L' \) can be written in a factorized form as

\[ \frac{\langle b_- | D_- (T_-)^L (T_+)^L D_+^T | b_+ \rangle}{\sqrt{\langle b_- | D_- (T_-)^{2L} D_+^T | b_+ \rangle \sqrt{\langle b_+ | D_+ (T_+)^{2L} D_+^T | b_+ \rangle}}} \]

\[ = \frac{\det (K)}{\sqrt{\det (K_-) \det (K_+)} |[-L,L]|} \exp \left\{ \Phi(b_-, b_+) - \frac{1}{2} \Phi_+(b_+, b_+) - \frac{1}{2} \Phi_-(b_-, b_-) \right\} \]

\[ = \frac{\det (K)}{\sqrt{\det (K_-) \det (K_+)} |[-L,L]|} \frac{\det' \left( B - M - X_{-+}^{L'} \right)}{\sqrt{\det' \left( B_+ - M_+ - X_{++}^{L'} \right) \det' \left( B_+ - M_+ - X_{++}^{L'} \right)}}. \tag{2.44} \]
Since the phase factor which comes from the constant $c_+$ and $c_-$, $e^{c_+}e^{c_-}$, does not depend on the gauge field and is irrelevant, we have omitted it. This is the expression from which we start our analysis in the continuum limit. Taking the limit $L' \to \infty$ first in the above formula, we obtain the would-be vacuum overlap formula at finite extent of the fifth dimension as

$$\exp \left\{ -S_i[U_\mu; L] \right\} = \frac{\det (K)}{\sqrt{\det (K_-) \det (K_+)} \left|_{-L,L} \right|} \exp \left\{ \Phi(0_-, 0_+) - \frac{1}{2} \Phi(0_+, 0+) - \frac{1}{2} \Phi(0_-, 0_-) \right\}$$

$$= \frac{\det (K)}{\sqrt{\det (K_-) \det (K_+)} \left|_{-L,L} \right|} \frac{\det' (B_{-+} - M - X_{++}^-)}{\sqrt{\det' (B_{--} - M - X_{--}^+)} \det' (B_{++} - M_+ - X_{++}^+)}.$$

(2.45)

Finally we give the expression for the variation of the effective action under the gauge transformation. Under the gauge transformation of the link variables:

$$U_\mu(n) \longrightarrow g(n)U_\mu(n)g^\dagger(n + \hat{\mu}), \quad g(n) \in SU(N),$$

we can easily see that the matrices $K, B$ and $M$ transform covariantly. For example,

$$M(n, m) \longrightarrow g(n)M(n, m)g^\dagger(m).$$

(2.47)

On the contrary, $X^\infty$ does not transform covariantly because it consists of $B^0$ and $M^0$ without gauge link variables in them. Therefore the variation of the effective action under the infinitesimal gauge transformation with $g(n) = 1 + i\omega(n)$ is given by

$$i\delta S_i[U_\mu; L] = \text{Tr}' \left\{ \left( B_{-+} - M - X_{++}^- \right)^{-1} \left( \omega X_{-+}^+ - X_{-+}^- \omega \right) \right\}$$

$$- \frac{1}{2} \text{Tr}' \left\{ \left( B_{++} - M_+ - X_{++}^+ \right)^{-1} \left( \omega X_{++}^- - X_{++}^+ \omega \right) \right\}$$

$$- \frac{1}{2} \text{Tr}' \left\{ \left( B_{--} - M_- - X_{--}^- \right)^{-1} \left( \omega X_{--}^+ - X_{--}^- \omega \right) \right\},$$

(2.48)

where Tr' denotes the trace over the four-dimensional surfaces at $s = L$ and $-L - 1$ besides over the indices of spinor and the representation of gauge group.

**D. Continuum limit counterpart**

We start to investigate the would-be vacuum overlap formula at finite extent of the fifth dimension in the continuum limit and in the Minkowski space. We first take the naive continuum limit of the action with the boundary terms, Eq. (2.17), (2.18), (2.19) and (2.20).

$$S[-L-1, L] = S + S^R_L + S^R_R,$$

(2.49)
\[ S = \int_{-\infty}^{\infty} d^4 x \int_{-L}^{L} ds \, \bar{\psi}(x, s) \{ i \gamma^\mu \left( \partial_\mu - igT^a A_\mu(x) \right) - [\gamma_5 \partial_s + M(s)] \} \psi(x, s) \bigg|_{[-L,L]}, \quad (2.50) \]

\[ S^B_L = - \int_{-\infty}^{\infty} d^4 x \left\{ \bar{\psi}(x, -L+0) P_L \psi(x, -L) + \bar{\psi}(x, -L) P_R \psi(x, -L+0) \right\}, \quad (2.51) \]

\[ S^B_R = - \int_{-\infty}^{\infty} d^4 x \left\{ \bar{\psi}(x, L) P_L \psi(x, L-0) + \bar{\psi}(x, L-0) P_R \psi(x, L) \right\}. \quad (2.52) \]

where \( \bigg| \bigg|_{[-L,L]} \) stands for the chiral boundary condition in the continuum limit:

\[ P_R \psi(x, L) = P_L \psi(x, -L) = 0, \quad \bar{\psi}(x, L) P_L = \bar{\psi}(x, -L) P_R = 0. \quad (2.53) \]

Note that the kinetic parts in the boundary terms, Eqs. (2.19) and (2.20), vanish in the continuum limit because they correspond to operators of dimension five. The kink-like mass is defined by

\[
M(s) \equiv \begin{cases} +M & s > 0 \\
-M & s < 0 
\end{cases} = M \epsilon(s) = M \int \frac{d\omega}{2\pi i} \left[ \frac{1}{\omega - i0} + \frac{1}{\omega + i0} \right] e^{iws}.
\]

We assume that the Latin indices run from 0 to 4 and the Greek ones from 0 to 3. The gamma matrices are defined as \( \{ \gamma^a, \gamma^b \} = 2\eta^{ab} \) with Minkowskian metric, \( \eta^{ab} = \text{diag.}(+1, -1, -1, -1, -1) \). We adopt the following chiral representation:

\[
\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\
\sigma^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = (1, \sigma^1), \quad \bar{\sigma}^\mu = (1, -\sigma^1), \quad (2.54)
\]

and

\[
\gamma^{a=4} = i\gamma_5 = i^2\gamma^0\gamma^1\gamma^2\gamma^3 = i \begin{pmatrix} 1 & 0 \\
0 & -1 \end{pmatrix}. \quad (2.55)
\]

\( T^a \) are the hermitian generators of \( SU(N) \) gauge group in a certain representation. We also denote the gauge potential in the matrix form as \( A_\mu(x) = -igT^a A^a_\mu(x) \).

In the continuum limit, \( B, K, \) its inverse, \( M, \) and \( X \) are given formally by

\[
B(x, y) = \delta^4(x - y), \quad (2.56)
\]

\[
K(x, s; y, t) = \{ i \gamma^\mu \left( \partial_\mu + A_\mu(x) \right) - [\gamma_5 \partial_s + M(s)] \} \delta^4(x - y) \delta(s - t), \quad (2.57)
\]

\[
S_F[A](n, s; m, t) = iK^{-1}(n, s; m, t) \bigg|_{[-L,L]}, \quad (2.58)
\]

\[
iM(x, y; -L, L) = \begin{pmatrix} P_R S_F[A](x, -L, y; -L) P_L & P_R S_F[A](x, -L, y; L) P_R \\
P_L S_F[A](x, L, y; -L) P_L & P_L S_F[A](x, L, y; L) P_R \end{pmatrix}. \quad (2.59)
\]
and

\[ iX_{x}^{L'}(x,y) = P_{L}S_{F+}^{0}(x,-L',y,-L)P_{R} \]
\[ + P_{L}S_{F-}^{0}(-L-2L';-L)L_{L}(1-P_{R}S_{F+}^{0}(-L-2L';-L)L_{L})^{-1}P_{R}S_{F-}^{0}(-L-2L';-L)L_{L}(x,y), \]
\[ iX_{x}^{L'}(x,y) = P_{R}S_{F+}^{0}(x,L,y,L)P_{L} \]
\[ + P_{R}S_{F-}^{0}(L+2L';L)P_{R}(1-P_{L}S_{F+}^{0}(L+2L';L)P_{R})^{-1}P_{L}S_{F-}^{0}(L+2L';L)P_{L}(x,y), \]
\[ X_{x}^{L'}(x,y) = \begin{pmatrix}
X_{x}^{L'}(x,y) & 0 \\
0 & X_{x}^{L'}(x,y)
\end{pmatrix}. \] (2.62)

\( S_{F+} \) and \( S_{F-} \) are the inverse of the Dirac operators of the five dimensional fermions with positive and negative homogeneous masses, taking into account of the chiral boundary conditions, \([-L+L']\) and \([-L-2L',-L]\), respectively. The superscript 0 stands for the quantities in which gauge interaction is switched off.

Therefore we arrive at the formal expression of the vacuum overlap formula at finite extent of the fifth dimension (\( L' \) is also kept finite) in the continuum limit.

\[ \frac{\det (K)}{\sqrt{\det (K_-) \det (K_+)} [-L,L]} \frac{\det' \left( 1 - M - X_{x}^{L'} \right)}{\sqrt{\det' \left( 1 - M_- - X_{x}^{L'} \right) \det' \left( 1 - M_+ - X_{x}^{L'} \right)}}. \] (2.63)

This expression is formal because we do not yet specify the regularization. In our continuum limit analysis, we adopt the dimensional regularization of 't Hooft and Veltman scheme. That is, we consider an extended space such that the four dimensional space \((\mu = 0, 1, 2, 3)\) is extended to the D-dimensional one, but keep the fifth space. The gamma matrices follow the convention such that

\[ \{ \gamma^{\mu}, \gamma^{\nu} \} = 2\eta^{\mu\nu} \quad (\mu = 0, 1, \ldots, D), \] (2.64)
\[ \gamma^{a=4} = i\gamma_{5} = i^{2}\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}, \quad [\gamma^{\mu}, \gamma^{a=4}] = 0 \quad (\mu = 4 \ldots, D). \] (2.65)

Actually, in the following section, we will find in the evaluation of the contribution of the five-dimensional determinant,

\[ \frac{\det (K)}{\sqrt{\det (K_-) \det (K_+)} [-L,L]} \] (2.66)

that the dimensional regularization cannot maintain the chiral boundary condition. Furthermore, we also find that the subtraction by the determinants of the fermions with the positive and negative homogeneous masses just correspond to the subtraction by only one bosonic Pauli-Villars-Gupta field. Because of these facts, we obtain a gauge noninvariant result for the vacuum polarization in four dimensions. Since the volume contribution is
expected to be gauge invariant in the lattice regularization, our choice of the dimensional regularization is not adequate in this case.

On the other hand, for the boundary contribution,
\[
\frac{\det^\prime(1 - M - X_{L}^L)}{\sqrt{\det^\prime(1 - M - X_{L}^L) \det^\prime(1 - M - X_{L}^L)}} ,
\]
(2.67)

once we assume that the dimensional regularization preserves the cluster property, we can obtain the following results: The boundary contribution is purely odd-parity. Its variation under the gauge transformation gives the consistent form of the gauge anomaly in four dimensions, which can be evaluated without any divergence and is actually originated from the source of gauge noninvariance at the boundary, but not due to the breaking of the chiral boundary condition by the dimensional regularization.

In this sense, the continuum limit analysis with the dimensional regularization cannot give the whole structure of the vacuum overlap defined in the lattice regularization. But we think that we can see rather clearly in what way the vacuum overlap formula could give the perturbative properties of the chiral determinant in four dimension.

III. FIVE-DIMENSIONAL FERMION WITH KINK-LIKE MASS

IN A FINITE FIFTH SPACE VOLUME

In this section, we develop the theory of free five-dimensional fermion in the finite fifth space volume. We solve the field equation and obtain the complete set of solutions. The field operator is defined by the mode expansion and the propagator is derived. The Sommerfeld-Watson transformation is introduced, by which we rearrange the normal modes of the fifth momentum to be common among the fermion with the kink-like mass and the fermion with the positive(negative) homogeneous mass.

A. Complete set of solutions

We solve the free field equations of the five dimensional fermion with kink-like mass and positive(negative) homogeneous mass under the chiral boundary condition, which are derived from the actions,
\[
S_0 = \int_{-\infty}^{\infty} d^4x \int_{-L}^{L} ds \bar{\psi}(x, s) \left\{ i \gamma^\mu \partial_\mu - [\gamma_5 \partial_s + M(s)] \right\} \psi(x, s) \bigg|_{[-L,L]} ,
\]
(3.1)
\[
S_{\pm} = \int_{-\infty}^{\infty} d^4x \int_{-L}^{L} ds \bar{\psi}(x, s) \left\{ i \gamma^\mu \partial_\mu - [\gamma_5 \partial_s \pm M \right\} \psi(x, s) \bigg|_{[-L,L]} ,
\]
(3.2)
and the chiral boundary condition reads
\[ P_R \psi(x, L) = P_L \psi(x, -L) = 0, \quad \bar{\psi}(x, L) P_L = \bar{\psi}(x, -L) P_R = 0. \]  

(3.3)

Note that, if the parity transformation here is defined by

\[ \psi(x_0, x_i, s) \rightarrow \psi'(x_0', x_i', s') \equiv \gamma^0 \psi(x_0, -x_i, -s), \]  

(3.4)

both the action and the chiral boundary condition are parity invariant for the fermion with the homogeneous mass. For the fermion with the kink-like mass, the parity transformation has the effect to change the signature of the mass parameter \( M \). This is also true when the gauge field is introduced provided that the gauge field transforms as

\[ (A_0(x_0, x_i), A_i(x_0, x_i)) \rightarrow (A_0'(x_0', x_i'), A_i'(x_0', x_i')) \equiv (A_0(x_0, -x_i), -A_i(x_0, -x_i)) , \]  

(3.5)

under the parity transformation.

We treat both \( S_0 \) and \( S_+ \) in a unified manner. The suffix \( \pm \) in the following denotes the solutions for \( S_0 \) and \( S_+ \), respectively. The solution for \( S_- \) can be obtained by setting all the mass \( M \) to \( -M \) in that for \( S_+ \). We work in the momentum space for all dimensions. \( \omega \) denotes the fifth component of five momentum.

i) Solution for \( s > 0 \)

General solution in the region \( s > 0 \) is given as follows:

\[ \begin{pmatrix} \sigma^\mu p^\mu \\ -i\omega + M \end{pmatrix} e^{-i\omega s}, \]  

(3.6)

where

\[ p^\mu p_\mu = \omega^2 + M^2. \]  

(3.7)

We have denoted the two independent solutions in the form of four-by-two matrix. Then the solution satisfying the chiral boundary condition at \( s = L \) is given by

\[ \begin{pmatrix} \sigma^\mu p^\mu \\ -i\omega + M \end{pmatrix} e^{-i\omega(s-L)} - \begin{pmatrix} \sigma^\mu p^\mu \\ i\omega + M \end{pmatrix} e^{i\omega(s-L)} = 2i \begin{pmatrix} \sigma^\mu p^\mu \sin \omega(L-s) \\ -\omega \cos \omega(L-s) + M \sin \omega(L-s) \end{pmatrix}, \]  

(3.8)

where \( \omega > 0 \).

ii) Solution for \( s < 0 \)

Similarly, general solution in the region \( s < 0 \) is given as follows:

\[ \begin{pmatrix} i\omega \mp M \\ \sigma^\mu p^\mu \end{pmatrix} e^{-i\omega s}. \]  

(3.9)
Then the solution satisfying the chiral boundary condition at \(s = -L\) is given by

\[
\begin{pmatrix}
  i\omega \mp M \\
  \sigma^\mu p_\mu
\end{pmatrix}
\begin{pmatrix}
  -i\omega \mp M \\
  \sigma^\mu p_\mu
\end{pmatrix}
\begin{pmatrix}
  e^{-i\omega(s+L)} \\
  e^{i\omega(s+L)}
\end{pmatrix} = 2
\begin{pmatrix}
  -\omega \cos \omega(L+s) \mp M \sin \omega(L+s) \\
  \sigma^\mu p_\mu \sin \omega(L+s)
\end{pmatrix},
\]

where \(\omega > 0\).

iii) Matching at \(s = 0\)

The solution should be continuous at \(s = 0\) and this condition determines the normal modes of \(\omega\). The general solution satisfying the chiral boundary condition can be written as follows:

\[
\phi(p, \omega; s) \equiv C_{>0} \begin{pmatrix}
  \bar{\sigma}^\mu p_\mu \sin \omega(L-s) \\
  -\omega \cos \omega(L-s) + M \sin \omega(L-s)
\end{pmatrix} \theta(s) + C_{<0} \begin{pmatrix}
  -\omega \cos \omega(L+s) \mp M \sin \omega(L+s) \\
  \sigma^\mu p_\mu \sin \omega(L+s)
\end{pmatrix} \theta(-s).
\]

(3.11)

For both two components in the above \(\phi(p, s)\) to match at \(s = 0\), we should have

\[
C_{>0} = C \sigma^\mu p_\mu \sin \omega L, \quad \omega > 0, \quad (3.12)
\]

\[
C_{<0} = C (-\omega \cos \omega L + M \sin \omega L), \quad \lambda > 0; \omega = i\lambda. \quad (3.13)
\]

and

\[
p^2 \sin^2 \omega L = (\omega \cos \omega L - M \sin \omega L)(\omega \cos \omega L \pm M \sin \omega L). \quad (3.14)
\]

iv) Spectrum of the normal modes of \(\omega\)

From the dispersion relation Eq. (3.7) and the matching condition Eq. (3.14), we can obtain the spectrum of the normal modes of \(\omega\).

\[
(\omega^2 + M^2) \sin^2 \omega L = (\omega \cos \omega L - M \sin \omega L)(\omega \cos \omega L \pm M \sin \omega L), \quad (3.15a)
\]

For the fermion with kink-like mass term we have

\[
(\omega^2 + M^2) = M^2 \frac{1}{\cos 2\omega L} \quad (\omega > 0), \quad (3.16)
\]

\[
(-\lambda^2 + M^2) = M^2 \frac{1}{\cosh 2\lambda L} \quad (\lambda > 0; \omega = i\lambda). \quad (3.17)
\]

For the fermion with ordinary positive mass we have
\[
\omega = M \tan 2\omega L \quad (\omega > 0),
\]
\[
\lambda = M \tanh 2\lambda L \quad (\lambda > 0, \omega = i\lambda; \text{only for } + M).
\]

Note that both sets of solutions have the bounded modes with the wave function which behave exponentially.

iv) \textit{Solutions over } \([-L, +L]\)

Taking into account of the mode equation Eq. (3.15a), the solution over the entire region can be rewritten as follows:

\[
\phi(p, \omega; s) = \begin{pmatrix}
\sigma^\mu p_\mu \sin \omega (L - s) \\
-\omega \cos \omega (L - s) + M \sin \omega (L - s)
\end{pmatrix} \theta(s) + \frac{(-\omega \cos \omega L + M \sin \omega L)}{\sigma^\mu p_\mu \sin \omega L} \begin{pmatrix}
-\omega \cos \omega (L + s) \mp M \sin \omega (L + s) \\
\sigma^\mu p_\mu \sin \omega (L + s)
\end{pmatrix} \theta(-s) \quad (3.20)
\]

\[
= \begin{pmatrix}
\sigma^\mu p_\mu \sin \omega (L - s) \\
-\omega \cot \omega L + M \sin \omega (L - s)
\end{pmatrix} \theta(s) + \begin{pmatrix}
\sigma^\mu p_\mu \sin \omega (L + s) \\
-\omega \cot \omega L - M \sin \omega (L + s)
\end{pmatrix} \theta(-s) \quad (3.21)
\]

\[
= \begin{pmatrix}
\sigma^\mu p_\mu \sin \omega (L - s) \\
(-\omega \cot \omega L + M) \sin \omega (L + s)
\end{pmatrix} \theta(s) + \begin{pmatrix}
\sigma^\mu p_\mu \sin \omega (L - s) \\
(-\omega \cot \omega L + M) \sin \omega (L + s)
\end{pmatrix} \theta(-s) \quad (3.22)
\]

\[
= \begin{pmatrix}
\sigma^\mu p_\mu [\sin \omega (L - s)]_+ \\
(-\omega \cot \omega L + M) [\sin \omega (L + s)]_-
\end{pmatrix}, \quad (3.23)
\]

where we have defined

\[
\sin_\pm \omega (L - s) \equiv \frac{\omega \cos \omega (L + s) \pm M \sin \omega (L + s)}{\omega \cot \omega L \pm M} \tag{3.24}
\]

\[
= \begin{cases}
\frac{\omega \cos \omega (L + s) + M \sin \omega (L + s)}{\omega \cot \omega L + M} & (\omega^2 + M^2 = M^2 / \cos 2\omega L, \omega > 0) \\
\sin \omega (L - s) & (\omega = M \tan 2\omega L, \omega > 0)
\end{cases}
\tag{3.25}
\]

and we use abbreviations for the generalized “sin” functions in \([-L, +L]\) as follows,

\[
[\sin \omega (L - s)]_\pm \equiv \sin \omega (L - s) \theta(s) \mp \sin \omega (L - s) \theta(-s). \tag{3.26}
\]
This generalized “sin” function satisfies the orthogonality:

\[
\int_{-L}^{L} ds [\sin \omega(L-s)]_{\pm} [\sin \omega'(L-s)]_{\pm} = N_{\pm}(\omega) \delta_{\omega \omega'}. \tag{3.27}
\]

where the normalization factor \(N_{\pm}(\omega)\) is given by

\[
N_{\pm}(\omega) = \left[ \frac{(\omega \cot \omega L \pm M) + (\omega \cot \omega L - M)}{(\omega \cot \omega L \pm M)} \right] \frac{1}{2} \left( L - \frac{\sin 2\omega L}{2\omega} \right) + \frac{\sin^2 \omega L}{(\omega \cot \omega L \pm M)} \equiv n_{\pm}(\omega) \frac{1}{(\omega \cot \omega L \pm M)}. \tag{3.28}
\]

For the fermion with kink-like mass term, it turns out to be

\[
N_{\pm}(\omega) = \frac{\omega \cot \omega L}{(\omega \cot \omega L + M)} \left[ L - \frac{\sin 4\omega L}{4\omega \cos^2 \omega L} \right]. \tag{3.29}
\]

For the fermion with ordinary positive mass term, it reads

\[
N_{-}(\omega) = \left[ L - \frac{\sin 4\omega L}{4\omega} \right]. \tag{3.30}
\]

Orthogonality of the general solutions over the entire region, \(\phi(p, \omega, s)\), is given as follows.

\[
\int_{-L}^{L} ds \phi^\dagger(p, \omega, s)\phi(p, \omega', s) = 2p_0 \bar{\sigma}^\mu p_\mu N(\omega) \delta_{\omega \omega'}. \tag{3.31}
\]

It is shown as

\[
\int_{-L}^{L} ds \phi^\dagger(p, \omega, s)\phi(p, \omega', s) = \bar{\sigma}^\mu p_\mu \sigma^\mu p_\mu \int_{-L}^{L} ds [\sin \omega(L-s)]_{\pm} [\sin \omega'(L-s)]_{\pm}
\]

\[
+(-\omega \cot \omega L + M)(-\omega' \cot \omega' L + M) \int_{-L}^{L} ds [\sin \omega(L-s)]_{-} [\sin \omega'(L-s)]_{-}
\]

\[
= \left( (\sigma^\mu p_\mu)^2 N_{\pm}(\omega) + (-\omega \cot \omega L + M)(-\omega \cot \omega L + M)N_{-}(\omega) \right) \delta_{\omega \omega'}
\]

\[
= \left( (\sigma^\mu p_\mu)^2 + (\omega \cot \omega L - M)(\omega \cot \omega L \pm M) \right) N_{\pm}(\omega) \delta_{\omega \omega'}
\]

\[
= 2p_0 \bar{\sigma}^\mu p_\mu N_{\pm}(\omega) \delta_{\omega \omega'}. \tag{3.32}
\]

Then the orthonormal positive- and negative-energy wave functions can be obtained as

\[
u(p, \omega; s) \equiv u(-p, \omega; -s)
\]

\[
v(p, \omega; s) = \left( 1 - \frac{\sigma^\mu p_\mu}{\omega L + M} \right) \left[ \bar{\sigma}^\mu p_\mu [\sin \omega(L-s)]_{\pm}
\]

\[
\sqrt{2(|p_0| + M)} \left( (-\omega \cot \omega L + M)[\sin \omega(L+s)]_{-} \right).
\]

\[
\sqrt{2(|p_0| + M)} \left( (\sigma^\mu p_\mu)^2 \right)
\]

\[
\left( \bar{\sigma}^\mu p_\mu [\sin \omega(L-s)]_{\pm}
\right), \tag{3.33}
\]

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Note, for later use, that

\[ v(p_0, -\vec{p}, \omega; s) = \frac{1 - \frac{\partial^\mu p_\mu}{-i\omega + M}}{\sqrt{2|p_0| + M}} \left( -\sigma^\mu p_\mu [\sin \omega(L - s)]_\pm \right. \]

\[ \left. (-\omega \cot \omega L + M) [\sin \omega(L + s)]_- \right) \]

\[ = \frac{1 - \frac{\partial^\mu p_\mu}{-i\omega + M}}{\sqrt{2|p_0| + M}} \omega \cot \omega L \pm M \left( (\omega \cot \omega L \pm M) [\sin \omega(L - s)]_\pm \right. \]

\[ \left. \sigma^\mu p_\mu [\sin \omega(L + s)]_- \right) \]

\[ = \frac{i\omega + M}{\omega \cot \omega L \pm M} \left( 1 - \frac{\partial^\mu p_\mu}{-i\omega + M} \right) \left( (\omega \cot \omega L \pm M) [\sin \omega(L - s)]_\pm \right. \]

\[ \left. \sigma^\mu p_\mu [\sin \omega(L + s)]_- \right). \quad (3.34) \]

B. Mode expansion and Equal-time commutation relation

We define the field operator by the mode expansion as follows.

\[ \psi(x, s) \equiv \int \frac{d^3 p}{(2\pi)^3} \sum_\omega \frac{1}{N_\pm(\omega)} \frac{1}{\sqrt{2p_0}} \left\{ b(\vec{p}, \omega) u(p, \omega; s) e^{-ipx} + d^+(\vec{p}, \omega) v(p, \omega; s) e^{+ipx} \right\}, \]

\[ (3.35) \]

where the canonical commutation relations are assumed as

\[ \left\{ b(\vec{p}, \omega), b^\dagger(\vec{q}, \omega') \right\} = \delta^3(\vec{p} - \vec{q}) \delta_{\omega\omega'}, \]

\[ \left\{ d(\vec{p}, \omega), d^\dagger(\vec{q}, \omega') \right\} = \delta^3(\vec{p} - \vec{q}) \delta_{\omega\omega'}, \]

\[ (3.36) \]

and other commutators vanish. Then the equal-time commutation relation follows.

\[ \left\{ \psi(x, s), \psi^\dagger(y, t) \right\} \bigg|_{x^0 = y^0} = \delta^3(\vec{x} - \vec{y}) \Delta_\pm(s, t), \]

\[ (3.37) \]

where \( \Delta_\pm(s, t) \) is defined by

\[ \Delta_\pm(s, t) \equiv \sum_\omega \frac{1}{N_\pm(\omega)} \frac{1}{2p_0} \left\{ u(p_0, \vec{p}, \omega; s) u^\dagger(p_0, \vec{p}, \omega; t) + v(p_0, -\vec{p}, \omega; s) v^\dagger(p_0, -\vec{p}, \omega; t) \right\} \]

\[ = \sum_\omega \frac{1}{n_\pm(\omega)} \begin{pmatrix} (\omega \cot \omega L \pm M) [\sin \omega(L - s)]_\pm [\sin \omega(L - t)]_\pm \hfill 0 \\ 0 \hfill (\omega \cot \omega L - M) [\sin \omega(L + s)]_- [\sin \omega(L + t)]_- \end{pmatrix}. \quad (3.38) \]

We have used the following relation to obtain the above result.

\[ \frac{1}{2p_0} \left\{ u(p_0, \vec{p}, \omega; s) u^\dagger(p_0, \vec{p}, \omega; t) + v(p_0, -\vec{p}, \omega; s) v^\dagger(p_0, -\vec{p}, \omega; t) \right\} \]

\[ = \frac{1}{2p_0} \left( \sigma^\mu p_\mu [\sin \omega(L - s)]_\pm \right. \]

\[ \left. (-\omega \cot \omega L + M) [\sin \omega(L + s)]_- \right) \left( \frac{\sigma^\mu p_\mu}{\omega^2 + M^2} \right) \]
Then we have as follows. We define the Green function by the time-ordered product as usual:

\[ \times \left( \tilde{\sigma}^\mu p_\mu [\sin \omega(L-t)]_\pm (-\omega \cot \omega L + M) [\sin \omega(L+t)]_- \right) \]

\[ + \frac{\omega \cot \omega L - M}{\omega \cot \omega L \pm M} \frac{1}{2p_0} \left( \frac{\omega \cot \omega L \pm M}{} [\sin \omega(L-s)]_\pm \left( \frac{\sigma^\mu p_\mu}{\omega^2 + M^2} \right) \right) \times \left( \omega \cot \omega L \pm M) [\sin \omega(L-t)]_\pm \tilde{\sigma}^\mu p_\mu [\sin \omega(L+t)]_- \right) \]

\[ = \left( \begin{array}{c} [\sin \omega(L-s)]_\pm [\sin \omega(L-t)]_\pm 0 \\ 0 \end{array} \right) \]

\[ = \frac{\omega \cot \omega L - M}{\omega \cot \omega L \pm M} [\sin \omega(L+s)]_- [\sin \omega(L+t)]_- . \] (3.39)

C. Propagator

Once we have defined the field operator, the two-point Green function can be obtained as follows. We define the Green function by the time-ordered product as usual:

\[ S_{F\pm}(x - y; s, t) \equiv \langle 0| T\psi(x, s)\bar{\psi}(y, t) |0 \rangle . \] (3.40)

Then we have

\[ S_{F\pm}(x - y; s, t) = \int \frac{d^4p}{i(2\pi)^4} \sum_\omega \frac{1}{N^\pm(\omega)} \frac{e^{-ip(x-y)}}{M^2 + \omega^2 - p^2 - i\epsilon} s_\pm(p, \omega; s, t) \] (3.41)

\[ = \int \frac{d^4p}{i(2\pi)^4} \sum_\omega \frac{1}{n^\pm(\omega)} \frac{e^{-ip(x-y)}}{M^2 + \omega^2 - p^2 - i\epsilon} (\omega \cot \omega L \pm M)s_\pm(p, \omega; s, t) \]

\[ = \int \frac{d^4p}{i(2\pi)^4} e^{-ip(x-y)} \left\{ P_R \gamma^0 \Delta_R \pm(p, s, t) + P_L \gamma^0 \Delta_L \mp(p, s, t) \right\} \]

\[ + P_R B_{RL\pm}(p, s, t) + P_L B_{LR\pm}(p, s, t) \} , \] (3.42)

where

\[ s_\pm(p, \omega; s, t) = u(p, \omega; s) \bar{u}(p, \omega; t) \]

\[ = \left( \tilde{\sigma}^\mu p_\mu [\sin \omega(L-s)]_\pm (-\omega \cot \omega L + M) [\sin \omega(L+s)]_- \left( \frac{\sigma^\mu p_\mu}{\omega^2 + M^2} \right) \right) \times \left( \tilde{\sigma}^\mu p_\mu [\sin \omega(L-t)]_\pm (-\omega \cot \omega L + M) [\sin \omega(L+t)]_- \right) \gamma^0 \]

\[ = P_R \gamma^0 [\sin \omega(L-s)]_\pm [\sin \omega(L-t)]_\pm \]

\[ + P_L \gamma^0 [\sin \omega(L+s)]_- [\sin \omega(L+t)]_- \left( \frac{\omega \cot \omega L - M}{\omega \cot \omega L \pm M} \right) \]

\[ + P_R (-\omega \cot \omega L + M) [\sin \omega(L-s)]_\pm [\sin \omega(L+t)]_- \]

\[ + P_L (-\omega \cot \omega L + M) [\sin \omega(L+s)]_- [\sin \omega(L-t)]_\pm , \] (3.43)
\[
\Delta_{R\pm}(p; s, t) = \sum_\omega \frac{(\omega \cot \omega L \pm M)}{n_\pm(\omega)} \frac{1}{M^2 + \omega^2 - p^2 - i\varepsilon} \sin \omega(L - s) \pm \sin \omega(L - t) \pm, \quad (3.44)
\]
\[
\Delta_{L-}(p; s, t) = \sum_\omega \frac{(\omega \cot \omega L - M)}{n_\pm(\omega)} \frac{1}{M^2 + \omega^2 - p^2 - i\varepsilon} \sin \omega(L + s) - \sin \omega(L + t), \quad (3.45)
\]
\[
B_{RL\pm}(p; s, t) = \sum_\omega \frac{(\omega \cot \omega L \pm M)}{n_\pm(\omega)} \frac{(-\omega \cot \omega L + M)}{M^2 + \omega^2 - p^2 - i\varepsilon} \sin \omega(L - s) \pm \sin \omega(L + t), \quad (3.46)
\]
\[
B_{LR\pm}(p; s, t) = B_{RL\pm}(p; t, s)
= \sum_\omega \frac{(\omega \cot \omega L \pm M)}{n_\pm(\omega)} \frac{(-\omega \cot \omega L + M)}{M^2 + \omega^2 - p^2 - i\varepsilon} \sin \omega(L + s) - \sin \omega(L - t) \pm. \quad (3.47)
\]

**D. Sommerfeld-Watson Transformation**

As we have shown in the previous subsection, the fermion with kink-like mass has different normal modes of \(\omega\) from those of the fermion with positive (negative) homogeneous mass. For the fermion with kink-like mass term, we have Eq. (3.16),

\[
(\omega^2 + M^2) = M^2 \frac{1}{\cos 2\omega L} \quad (\omega > 0),
\]
\[
(-\lambda^2 + M^2) = M^2 \frac{1}{\cosh 2\lambda L} \quad (\lambda > 0; \omega = i\lambda).
\]

On the other hand, for the fermion with positive homogeneous mass, we have Eq. (3.18),

\[
\omega = M \tan 2\omega L \quad (\omega > 0);
\]
\[
\lambda = M \tanh 2\lambda L \quad (\lambda > 0, \omega = i\lambda; \text{only for } + M).
\]

Therefore we encounter the summations over the modes of \(\omega\),

\[
\sum_\omega \frac{1}{n_+(\omega)} F_+(\omega) = \sum_{\omega>0} \frac{1}{n_+(\omega)} F_+(\omega) \bigg|_{\omega^2 + M^2 = M^2 / \cos 2\omega L} + \frac{1}{n_+(i\lambda)} F_+(i\lambda) \bigg|_{-\lambda^2 + M^2 = M^2 / \cosh 2\lambda L},
\]
\[
\sum_\omega \frac{1}{n_-(\omega)} F_-(\omega) = \sum_{\omega>0} \frac{1}{n_-(\omega)} F_-(\omega) \bigg|_{\omega = M \tan 2\omega L} + \frac{1}{n_+(i\lambda)} F_-(i\lambda) \bigg|_{\lambda = M \tan 2\lambda L},
\]

where \(F_\pm(\omega)\) are certain functions. In order to perform the subtraction at finite fifth space volume \(L\), we need to rearrange the modes such that the massive modes would be common. To achieve it, we consider the Sommerfeld-Watson transformation.

**1. General Case**

Let us consider the function

\[
\frac{1}{\sin^2 \omega L \Delta_\pm(\omega)} = \frac{1}{\sin^2 \omega L \left[(\omega^2 + M^2) - (\omega \cot \omega L - M)(\omega \cot \omega L \pm M) \right]}. \quad (3.48)
\]
This function has poles at the value of $\omega$ given by the mode equation Eq. (3.15a),

$$(\omega^2 + M^2) \sin^2 \omega L = (\omega \cos \omega L - M \sin \omega L)(\omega \cos \omega L \pm M \sin \omega L). \quad (3.49)$$

Since we can show

$$\frac{\partial}{\partial \omega} \{\sin^2 \omega L \Delta_{\pm}(\omega)\} = 2 \cot \omega L \{\sin^2 \omega L \Delta_{\pm}(\omega)\} + 2 \omega \left[ \left(\frac{\omega \cot \omega L \pm M}{2}\right)(L - \frac{\sin 2\omega L}{2\omega}) + \sin^2 \omega L \right], \quad (3.50)$$

the residue at the pole is given as

$$\frac{1}{2\omega n_{\pm}(\omega)}. \quad (3.51)$$

For the case $\omega = i\lambda$, this expression also holds true.

Accordingly we are declined to consider the following integral.

$$I_{\pm} \equiv \int_{C} \frac{d\omega}{2\pi i \sin^2 \omega L \Delta_{\pm}(\omega)} F_{\pm}(\omega), \quad (3.52)$$

with a contour $C$ shown bellow.

We assume that $F_{\pm}$ is an even function of $\omega$, it vanishes at the origin so that the whole integrand does not possess any singularity at the origin, and it also vanishes at infinity so that the contour integral at infinity vanishes. We allow $F_{\pm}(\omega)$ to have poles, for example, on the real axis. We showed them by white-circles.
\[
\frac{2\omega}{\sin^2 \omega L \Delta_\pm(\omega)} \text{ has poles on the real axis and two additional poles near the imaginary axis. For sufficiently large } L, \text{ we have}
\]
\[
\omega = i\lambda \simeq iM + \varepsilon.
\] (3.53)

Here we take into account of the Feynman boundary condition, that is, the infinitesimal imaginary part of the mass M. We showed these poles by black-circles. It also has a pole at the origin, but it is assumed not to contribute the integral due to the zero of \(F_\pm\).

Then the above integral leads to the identity
\[
\sum_\omega \frac{1}{n_\pm(\omega)} F_\pm(\omega) = -\sum_\omega' \frac{2\omega'}{\sin^2 \omega' L \Delta_\pm(\omega') \text{Res} F_\pm(\omega') \text{poles of } F_\pm}.
\] (3.54)

If \(F_\pm(\omega)\) possess common series of poles, we can rearrange the different series of modes into common series of modes. In some cases, it turns out to need several steps of the transformations to obtain the common series of modes.

2. Transformation of the functions \(\Delta_{R\pm}(p; s, t)\) and \(\Delta_{L-}(p; s, t)\)

a. Integrand in \(\Delta_{R\pm}\) and \(\Delta_{L-}\) given by Eq. (3.44) has the following Integrand.
\[
F_\pm(\omega) = \frac{1}{M^2 + \omega^2 - p^2 - i\varepsilon} \left[ \sin \omega(L - s) \sin \omega(L - t) (\omega \cot \omega L \pm M) \right.
\]
\[
+ \theta(s) \theta(t) \sin \omega(L + s) \sin \omega(L - t) (\omega \cos \omega(L + s)) (\omega \cos \omega(L + t)) \frac{\omega \cot \omega L \pm M}{\omega \cot \omega L \pm M}
\]
\[
+ \theta(s) \theta(-t) \sin \omega(L + s) \sin \omega(L - t) (\omega \cos \omega(L + t)) (\omega \cos \omega(L + s)) \sin \omega(L - t) \left. \right].
\] (3.55)

b. Poles There are three types of poles in the above \(F_\pm(\omega)\). We summarize the poles and their residues in Table I.

| Table I Poles and Residues in \(I_\pm\) for \(\Delta_{R\pm}\) |
|---|---|---|---|
| singular part in \(I_\pm\) | pole | residue | \(\frac{2\omega}{\sin^2 \omega L \Delta_\pm(\omega)}\) |
| \(\frac{2\omega}{\sin^2 \omega L \Delta_\pm(\omega)}\) | \(\sin^2 \omega L \Delta_\pm(\omega) = 0\) | \(\frac{1}{n_\pm(\omega)}\) | 1 |
| \(\omega \cot \omega L = \frac{\omega \cos \omega L}{\sin \omega L}\) | \(\sin \omega L = 0\) | \(\frac{\omega}{\omega L}\) | \(-\frac{2}{\omega}\) |
| \(\frac{1}{(\omega \cot \omega L \pm M)}\) | \(\omega \cot \omega L \pm M = 0\) | \(-\frac{\sin \omega L}{\omega L \frac{\sin \omega L}{2\omega L}}\) | \(\frac{2}{\omega}\) |
| \(\frac{1}{M^2 + \omega^2 - p^2 - i\varepsilon}\) | \(i\lambda \equiv i\sqrt{M^2 - p^2 - i\varepsilon}\) | \(\frac{1}{2i\lambda}\) | \(\frac{2i\lambda}{-\sin \omega L \Delta_\pm(i\lambda)}\) |
c. First Stage Transformation  By the Sommerfeld-Watson transformation given by

Eq. (3.54), $\Delta_{R\pm}$ can be written as

\[
\Delta_{R\pm}(p; s, t) = \sum_\omega \left( \frac{\omega \cot \omega L \pm M}{n_\omega(\omega)} \right) \frac{1}{M^2 + \omega^2 - p^2 - \imath \varepsilon} [\sin \omega(L - s)] \pm [\sin \omega(L - t)] \pm \\
\theta(s) \theta(t) \sum_{\sin \omega L} \frac{2}{L} \frac{1}{M^2 + \omega^2 - p^2 - \imath \varepsilon} \sin \omega s \sin \omega t \\
+ \theta(-s) \theta(-t) \sum_{\omega \cot \omega L \pm M} \left( \frac{2 \sin^2 \omega L}{L - \sin \frac{2\omega L}{2\omega}} \right) \frac{1}{M^2 + \omega^2 - p^2 - \imath \varepsilon} \times (\omega \cos \omega(L + s) \pm M \sin \omega(L + s)) (\omega \cos \omega(L + t) \pm M \sin \omega(L + t)) \\
- \frac{(P \coth PL \pm M)}{\sinh^2 PL} \frac{[\sinh P(L - s)] \pm [\sinh P(L - t)] \pm}{\sinh^2 PL},
\]

(3.56)

where we have used the relation,

\[
\frac{\sin \omega L}{\omega} (\omega \cos \omega(L + s) \pm M \sin \omega(L + s)) = -\sin \omega s,
\]

(3.57)

for $\omega$ satisfying $\omega \cot \omega L \pm M = 0$

d. Second Stage Transformation  We need further Sommerfeld-Watson transformation for the part in which the summation should be taken over the different normal modes of $\omega$ given by $\omega \cot \omega L \pm M = 0$,

\[
\sum_\omega \left( \frac{2}{L - \sin \frac{2\omega L}{2\omega}} \right) \frac{1}{M^2 + \omega^2 - p^2 - \imath \varepsilon} \sin \omega s \sin \omega t \bigg|_{\omega \cos \omega L \pm M \sin \omega L = 0}.
\]

(3.58)

For this purpose, we consider the following integral.

\[
J_\pm = \int_{C/2\pi i} \frac{d\omega}{\sin \omega L} \frac{-2\omega}{\omega \cos \omega L \pm M} \frac{1}{\omega \cos \omega L \pm M} \frac{1}{M^2 + \omega^2 - p^2 - \imath \varepsilon} \sin \omega s \sin \omega t.
\]

(3.59)

There are two types of poles in this case, which we summarized in Table II.

| singular part in $J_\pm$ | pole | residue | $\frac{-2\omega}{\sin^2 \omega L} \frac{1}{\omega \cos \omega L \pm M}$ |
|-------------------------|------|---------|-------------------------------------------------|
| $\frac{-2\omega}{\sin \omega L} \frac{1}{\omega \cos \omega L \pm M} \sin \omega L$ | $\omega \cos \omega L \pm M \sin \omega L = 0$ | $\frac{1}{L - \sin \frac{2\omega L}{2\omega}}$ | 1 |
| $\frac{-2\omega}{\sin \omega L} \frac{1}{\omega \cos \omega L \pm M} \sin \omega L$ | $\sin \omega L = 0$ | $\frac{2}{L}$ | 1 |
| $\frac{1}{M^2 + \omega^2 - p^2 - \imath \varepsilon}$ | $iP \equiv i\sqrt{M^2 - p^2 - \imath \varepsilon}$ | $\frac{1}{2iP}$ | $\frac{2}{\sinh^2 PL P \coth PL \pm M}$ |
By the Sommerfeld-Watson transformation at the second stage, we obtain
\[
\sum_{\omega} \left( \frac{2}{L - \sin 2\omega L} \right) \frac{1}{M^2 + \omega^2 - p^2 - i\varepsilon} \sin \omega s \sin \omega t \bigg|_{\omega \cos \omega L \pm M \sin \omega L = 0} \sin \omega s \sin \omega t + \frac{1}{P \coth PL \pm M} \sinh Ps \sinh Pt \cdot \left( \frac{1}{\sinh^2 PL} \right). \quad (3.60)
\]

e. Final result
The final form of \( \Delta_{R\pm} \) is given by
\[
\Delta_{R\pm}(p; s, t) = \sum_{\omega} \left( \frac{\omega \cot \omega L \pm M}{n_\pm(\omega)} \right) \frac{1}{M^2 + \omega^2 - p^2 - i\varepsilon} \sin \omega (L - s) \pm \sin \omega (L - t) \pm \\
= [\theta(s)\theta(t) + \theta(-s)\theta(-t)] \sum_{\sin \omega L} \left( \frac{2}{L} \frac{1}{M^2 + \omega^2 - p^2 - i\varepsilon} \sin \omega s \sin \omega t \right) + \frac{\omega \cot \omega L \pm M}{P \coth PL \pm M} \frac{1}{\sinh^2 PL} \sinh Ps \sinh Pt \cdot \left( \frac{1}{\sinh^2 PL} \right). \quad (3.61)
\]

Similarly we obtain,
\[
\Delta_{L-}(p; s, t) = \sum_{\omega} \left( \frac{\omega \cot \omega L - M}{n_\pm(\omega)} \right) \frac{1}{M^2 + \omega^2 - p^2 - i\varepsilon} \sin \omega (L + s) \pm \sin \omega (L + t) \pm \\
= [\theta(s)\theta(t) + \theta(-s)\theta(-t)] \sum_{\sin \omega L} \left( \frac{2}{L} \frac{1}{M^2 + \omega^2 - p^2 - i\varepsilon} \sin \omega s \sin \omega t \right) + \frac{\omega \cot \omega L - M}{P \coth PL - M} \frac{1}{\sinh^2 PL} \sinh Ps \sinh Pt \cdot \left( \frac{1}{\sinh^2 PL} \right). \quad (3.62)
\]

We can perform the similar transformation for \( B_{\pm}(p; s, t) \). In this case, however, we need to improve the convergence of the integration of \( \omega \), using the relation
\[
(\omega \cot \omega L \pm M) (\omega \cot \omega L - M) = \omega^2 + M^2 = \frac{M^2}{\cos(3+1)/2} \frac{1}{2\omega L}. \quad (3.63)
\]

IV. PERTURBATION THEORY AT FINITE EXTENT OF FIFTH DIMENSION

In this section, we formulate the perturbative expansion of the would-be vacuum overlap based on the theory of free five-dimensional fermion at finite extent of the fifth dimension, which takes into account of the chiral boundary condition. The expansion can be performed independently for the five-dimensional volume contribution and the four-dimensional boundary contribution. As a subsidiary regularization, we adopt the dimensional regularization.

As to the volume contribution, the subtraction can be performed at finite \( L \), thanks to the Sommerfeld-Watson transformation, in each order of the expansion. And then the limit of the infinite \( L \) can be evaluated.
As to the boundary contribution, we first derive the boundary state wave functions taking the limit $L' \to \infty$. Since the boundary contribution is given by the correlation between the boundaries, it is expected to be finite in the limit $L \to \infty$ and the subtraction to be irrelevant.

As far as we do not take into account of the breaking of the chiral boundary condition due to the dimensional regularization, we can see that it is actually the case and the cluster property holds: the boundary contribution consist of the sum of the contributions from the two boundaries and that of the fermion with kink-like mass can be replaced by that of the fermion with homogeneous positive(negative) mass. Then we make an assumption that this cluster property holds even under the dimensional regularization. From the cluster property and the parity invariance of the fermion with the homogeneous mass, we can show that the boundary contribution is odd-parity in the limit $L \to \infty$.

A. Perturbation expansion of the determinant of $K$

The perturbative expansion of the volume contribution, Eq. (2.66),

$$\frac{\det(K)}{\sqrt{\det(K_-) \det(K_+)}|_{[-L,L]}}$$

(4.1)

can be performed as follows.

$$\ln \det(K)|_{[-L,L]} = \ln \left( \left\{ 1 + i \mathcal{A} (K^0)^{-1}|_{[-L,L]} \right\} K^0 \right)|_{[-L,L]}$$

(4.2)

$$= \sum_{n=1}^{\infty} \frac{(-i)^n}{n} \left( \mathcal{A} (K^0)^{-1}|_{[-L,L]} \right)^n + \ln \det(K^0)|_{[-L,L]}$$

(4.3)

$$= \sum_{n=1}^{\infty} \frac{(-)^n}{n} \left( \mathcal{A} S_{F+}|_{[-L,L]} \right)^n + \ln \det(K^0)|_{[-L,L]}.$$  

(4.4)

This expansion should be evaluated with the propagator $S_{F+}$ supplemented by the dimensional regularization. Similar expansion can be performed for the determinant of $K_\pm$. The subtraction of them can be performed at each order of the expansion. Thus we have

$$\ln \left[ \frac{\det(K)}{\sqrt{\det(K_-) \det(K_+)}|_{[-L,L]}} \right] - \ln \left[ \frac{\det(K^0)}{\sqrt{\det(K_-^0) \det(K_+^0)}|_{[-L,L]}} \right]$$

$$= \sum_{n=1}^{\infty} \frac{(-)^n}{n} \left[ \mathcal{A} S_{F+}|_{[-L,L]} \right]^n - \frac{1}{2} \mathcal{A} S_{F-}(+M)|_{[-L,L]} \right|^n - \frac{1}{2} \mathcal{A} S_{F-}(-M)|_{[-L,L]} \right|^n$$

$$\equiv i \Gamma_K[A].$$

(4.5)

After the subtraction, the limit of the infinite $L$ can be evaluated. Explicit calculation of the vacuum polarization is given in the later section.


B. Perturbation expansion of the boundary term

Next we consider the perturbative expansion of the boundary contribution, Eq. (2.67),
\[
\frac{\det' \left( 1 - M - X_{L}^{L'} \right)}{\sqrt{\det' \left( 1 - M - X_{L}^{L'} \right) \det' \left( 1 - M_{-} - X_{L}^{L'} \right)}}. 
\]
(4.6)

1. Boundary state wave function

We first derive boundary state wave functions taking the limit \( L' \to \infty \). We have derived the propagator of the fermion with positive and negative homogeneous masses satisfying the chiral boundary condition \( \big| \left[ -L, L \right] \big| \) in the previous section. Since the translational invariance hold for the fermion with homogeneous mass, we can show
\[
S_{F}[-M](p; s + 2L', t + 2L') \big|_{L, L+2L'} = S_{F}[-M](p; s, t) \big|_{-L', L'}, 
\]
(4.7)
\[
S_{F}[-M](p; s - 2L', t - 2L') \big|_{-L-2L', -L} = S_{F}[-M](p; s, t) \big|_{-L', L'}. 
\]
(4.8)

Then, using the previous results, we can give the explicit form of the boundary state wave function Eqs.(2.35) and (2.38) and the explicit form of \( X_{L}^{L'} \), Eqs.(2.61), (2.62) and (2.62). Actually we have
\[
iS_{F}[-M](p; L', L') = P_{L} \sum_{\omega} \frac{1}{n_{-}(\omega)} \frac{1}{M^{2} + \omega^{2} - p^{2} - i\varepsilon (\omega \cot \omega L' - M)} = P_{L} \frac{1}{\Delta_{-}(iP)} \left( P \coth \frac{pL'}{M} \right) \frac{1}{\sinh^{2} \frac{pL'}{M}} + P_{L} \frac{1}{P \coth \frac{pL'}{M}}. 
\]
(4.9)

In the last equality, we have used the result of the Sommerfeld-Watson transformation, Eq. (3.62). Similarly, we obtain
\[
iS_{F}[-M](p; -L', -L') = P_{L} \frac{1}{\Delta_{-}(iP)} \left( P \coth \frac{pL'}{M} \right) \frac{1}{\sinh^{2} \frac{pL'}{M}} + P_{L} \frac{1}{P \coth \frac{pL'}{M}}. 
\]
(4.10)
\[
iS_{F}[-M](p; L', -L') = P_{L} B_{LR}-\left( p; L', -L' \right) 
\]
\[
= P_{L} \frac{1}{\Delta_{-}(iP \coth \frac{pL'}{M})} \frac{1}{\sinh^{2} \frac{pL'}{M} \Delta_{-}(iP)} \left( M^{2} - \frac{M^{2}}{\cos^{2} 2\omega L'} \right) 
\]
(4.11)
\[ iS_{F-}[+M](p; -\nu', \nu') = P_R \frac{1}{-\sinh^2 P'L' \Delta_- (iP)} \left( M^2 - \frac{M^2}{\cosh^2 2PL'} \right). \] (4.12)

The calculation of Eqs. (4.11) and (4.12) is given in the later subsection. Assuming that \( p^2 \neq 0 \), we take the limit \( L' \to \infty \) and have

\[ \lim_{L' \to \infty} iS_{F-}[+M](p; -\nu', \nu') = P_R \frac{1}{P - M}, \] (4.13)
\[ \lim_{L' \to \infty} iS_{F-}[+M](p; -\nu', -\nu') = P_L \frac{1}{P - M}, \] (4.14)
\[ \lim_{L' \to \infty} iS_{F-}[+M](p; \nu, -\nu') = 0, \] (4.15)
\[ \lim_{L' \to \infty} iS_{F-}[+M](p; -\nu, \nu') = 0. \] (4.16)

Therefore we obtain

\[ X_\infty^\pm (p) = \begin{pmatrix} X_\infty^\pm (p) & 0 \\ 0 & X_\infty^\mp (p) \end{pmatrix} = \begin{pmatrix} P_L \frac{1}{P + M} & 0 \\ 0 & P_R \frac{1}{P - M} \end{pmatrix}. \] (4.17)

In terms of the wave function, the explicit gauge symmetry breaking term can be written as follows.

\[ \langle b_- | \psi_R(-L), \psi_R^*(-L) \rangle \]
\[ = c_- \exp \left\{ - \int d^4x d^4y \psi_R(x, -L) \frac{d^4p}{i(2\pi)^4} e^{-ip(x-y)} \frac{\slashed{p}}{P + M} \psi_R(y, -L) \right\}, \] (4.18)
\[ \langle \psi_L(L), \psi_L^*(L) | b_+ \rangle \]
\[ = c_+ \exp \left\{ - \int d^4x d^4y \psi_L(x, L) \frac{d^4p}{i(2\pi)^4} e^{-ip(x-y)} \frac{\slashed{p}}{P - M} \psi_L(y, L) \right\}. \] (4.19)

2. Perturbation expansion of the boundary term

Given the explicit form of the boundary state wave function, we next consider the perturbative expansion of \( 1 - M - X_\infty^\pm \). \( S_{F\pm}[A](x - y; s, t) \) can be expanded as

\[ S_{F\pm}[A] = S_{F\pm} + \sum_{n=1}^{\infty} \{ S_{F\pm} \cdot (-) \mathcal{A} \}^n S_{F\pm}, \] (4.20)

where the following abbreviation is used,

\[ S_{F\pm} \cdot (-) \mathcal{A} \cdot S_{F\pm} \equiv \int d^4z \int_{-L}^{L} du S_{F\pm}(x - z; s, u)(-\mathcal{A}(z)S_{F\pm}(z - y; u, t). \] (4.21)

Then we obtain the expansion,
Here we have also introduced the abbreviation for the boundary condition:

\[
..P_R \psi(x, s) = P_R \psi(x, -L), \quad \tilde{\psi}(x, s) P_{L\ldots} = \tilde{\psi}(x, -L) P_L.
\]  

(4.23)

Then we obtain the perturbative expansion of the boundary contribution.

\[
\ln \text{det}' \left(1 - M - X_{-+}^\infty\right) - \ln \text{det}' \left(1 - M^0 - X_{-+}^\infty\right)
= \sum_{m=1}^\infty \frac{(-)^m}{m} \text{Tr}' \left\{ D^L \left(..P_R\right) \left[ \sum_{n=1}^\infty \left\{ S_{F+} \cdot (-.) A^+ \right\}^n S_{F+} \right] \left( P_{L\ldots} P_{R\ldots} \right) \right\}^m,
\]

(4.25)

where we denote the inverse of \(1 - M^0 - X_{-+}^\infty\) as \(D^L\),

\[
D^{-1}_L(p) \equiv \left(1 - M^0 - X_{-+}^\infty\right)(p)
= 1 + \left(..P_R\right) i S_{F+}(p; s, t) \left( P_{L\ldots} P_{R\ldots} \right) + \left( P_{L\ldots} \frac{1}{p^+M} \right) \left( P_{R\ldots} \frac{1}{p^-M} \right).
\]

(4.26)

The contribution of the fermion with positive and negative homogeneous mass can be expanded in a similar manner.

\[
\ln \text{det}' \left(1 - M_{\pm} - X_{\pm\pm}^\infty\right) - \ln \text{det}' \left(1 - M^0_{\pm} - X_{\pm\pm}^\infty\right)
= \sum_{m=1}^\infty \frac{(-)^m}{m} \text{Tr}' \left\{ D^L \left(..P_R\right) \left[ \sum_{n=1}^\infty \left\{ S_{F-}[\pm M] \cdot (-.) A^+ \right\}^n S_{F-}[\pm M] \right] \left( P_{L\ldots} P_{R\ldots} \right) \right\}^m,
\]

(4.27)

and

\[
D^{-1}_\pm(p) \equiv \left(1 - M^0_{\pm} - X_{\pm\pm}^\infty\right)
= 1 + \left(..P_R\right) i S_{F-}[\pm M](p; s, t) \left( P_{L\ldots} P_{R\ldots} \right) + \left( P_{L\ldots} \frac{1}{p^+M} \right) \left( P_{R\ldots} \frac{1}{p^-M} \right).
\]

(4.28)

These expression can be also regularized by the dimensional regularization.
3. Cluster property and parity

Next we consider taking the limit \( L \to \infty \). Since the boundary contribution is given by the correlations between the boundaries, it is expected to be finite in the limit \( L \to \infty \) and the subtraction to be irrelevant. As far as we do not take into account of the breaking of the chiral boundary condition due to the dimensional regularization, we will see that it is actually the case and the cluster property holds: the correlation between the two boundaries vanishes in the limit \( L \to \infty \) and the remaining diagonal contribution from each boundary is equal to that of the fermion with homogeneous mass (positive or negative according to the signature of mass at that boundary). We will make an assumption that this cluster property holds even under the dimensional regularization. Then, as a result of the cluster property and the parity invariance of the fermion with the homogeneous mass, we will show that the boundary contribution is odd-parity in the limit \( L \to \infty \).

We first consider the leading term in the perturbative expansion,

\[
\begin{pmatrix}
P_R \\
P_L 
\end{pmatrix} S_{F\pm}(p; s, t) \begin{pmatrix} P_L & P_R \end{pmatrix}.
\]

(4.29)

The diagonal components can be read off from Eqs. (3.61), (3.62).

\[
..P_R S_{F\pm}(p; s, t) P_L = P_R \delta \left\{ \frac{(P \coth PL - M)}{-\Delta_-(iP)} \frac{P^2}{\sinh^2 PL} + \frac{1}{P \coth PL \pm M} \right\}
\]

\[
P_L S_{F\pm}(p; s, t) P_R = P_L \delta \left\{ \frac{(P \coth PL \pm M)}{-\Delta_+(iP)} \frac{P^2}{\sinh^2 PL} + \frac{1}{P \coth PL \pm M} \right\}.
\]

(4.30)

Off diagonal components, which give the correlation between the boundaries at \( s = -L \) and \( s = L \), are given by the following summations.

\[
..P_R S_{F\pm}(p; s, t) P_R = -P_R \sum_{\omega} \frac{1}{n_\pm(\omega)} \frac{\omega^2}{M^2 + \omega^2 - p^2 - i\varepsilon}
\]

\[
P_L S_{F\pm}(p; s, t) P_L = -P_L \sum_{\omega} \frac{-1}{n_\pm(\omega)} \frac{\omega^2}{M^2 + \omega^2 - p^2 - i\varepsilon}
\]

(4.31)

These summations can be performed by the use of the technique of the Sommerfeld-Watson transformation. At first we need to improve the convergence of the integration of \( \omega \), using Eq. (3.63). Then referring to Table III,

| singular part in \( I_\pm \) | pole | residue | \( \frac{2\omega}{\sin^2 \omega \Delta_\pm(\omega)} \) |
|---------------------------|------|--------|-------------------|
| \( \frac{2\omega}{\sin^2 \omega \Delta_\pm(\omega)} \) | \( \sin^2 \omega L \Delta_\pm(\omega) = 0 \) | \( \frac{1}{n_\pm(\omega)} \) | 1 |
| \( \frac{M^2}{\cos(3\pi/4) \sqrt{2} \omega L} \) | \( \cos 2\omega L = 0 \) | \( \frac{M^2}{-2L \sin 2\omega L} \frac{1+1}{2} \) | \( \frac{2\omega}{M^2} \) |
| \( \frac{1}{M^2 + \omega^2 - p^2 - i\varepsilon} \) | \( iP \equiv i\sqrt{M^2 - p^2 - i\varepsilon} \) | \( \frac{1}{2iP} \) | \( \frac{2iP}{-\sinh^2 PL \Delta_\pm(iP)} \) |

Table III  Poles and Residues in \( I_\pm \) for \( ..P_R S_{F\pm} P_L \)..
we obtain
\[ \sum_\omega \frac{1}{n_\pm(\omega)} \frac{\omega^2}{M^2 + \omega^2 - p^2 - i\varepsilon} = \sum_\omega \frac{1}{n_\pm(\omega)} \frac{1}{M^2 + \omega^2 - p^2 - i\varepsilon} \left( \frac{M^2}{\cos(3\pi/2)2\omega L} - M^2 \right) \]
\[ = \frac{1}{\sinh^2 PL\Delta_\pm(iP)} \left( \frac{M^2}{\cosh(3\pi/2)2PL} - M^2 \right) + \frac{1}{2} \sum_{\omega L} \frac{1}{\omega} \left( \frac{1}{L \sin 2\omega L M^2 + \omega^2 - p^2 - i\varepsilon} \right) \]
\[ = \frac{1}{\sinh^2 PL\Delta_\pm(iP)} \left( \frac{M^2}{\cosh(3\pi/2)2PL} - M^2 \right) + \frac{1}{2} \sum_{n=0}^\infty \left( \frac{(-)^n (\pi + n\pi)}{2} \right)^2 + L^2 (M^2 - p^2 - i\varepsilon) \]
\[ = \frac{1}{\sinh^2 PL\Delta_\pm(iP)} \left( \frac{M^2}{\cosh(3\pi/2)2PL} - M^2 \right) + \frac{1}{2} \frac{1}{\cosh 2PL}. \] (4.32)

From these results, we can see that the correlations between the boundaries have definite limit when \( L \to \infty \) and we obtain
\[ \lim_{L \to \infty} \left( \begin{array}{c} P_R \\ P_L \end{array} \right)^i S_{F \pm}(p; s, t) \left( P_L.. P_R. \right) = \left( \begin{array}{cc} P_R \gamma^\dagger_{F \pm M} & 0 \\ 0 & P_L \gamma^\dagger_{F - M} \end{array} \right). \] (4.33)

Thus the correlation between the two boundaries vanishes and the remaining diagonal contribution from each boundary is equal to that of the fermion with homogeneous mass (positive or negative according to the signature of mass at that boundary).

Next we consider higher order terms,
\[ \left( \begin{array}{c} P_R \\ P_L \end{array} \right) \left[ i \sum_{n=1}^\infty \{ S_{F \pm} \cdot (-) A \cdot \}^n S_{F \pm} \right] \left( P_L.. P_R. \right) \] (4.34)

When the dimensional regularization is not taken into account, we can see by the explicit calculation that the cluster property holds (see Appendix [3]):
\[ \lim_{L \to \infty} \left( \begin{array}{c} P_R \\ P_L \end{array} \right) \left[ i \sum_{n=1}^\infty \{ S_{F \pm} \cdot (-) A \cdot \}^n S_{F \pm} \right] \left( P_L.. P_R. \right) \]
\[ = \lim_{L \to \infty} \left( \begin{array}{c} P_R \left[ i \sum_{n=1}^\infty \{ S_{F - [+M]} \cdot (-) A \cdot \}^n S_{F - [+M]} \right] P_L.. \right. \]
\[ \left. \begin{array}{c} 0 \\ 0 \end{array} \right) \]
\[ \left. \frac{1}{2} \sum_{n=1}^\infty \{ S_{F - [+M]} \cdot (-) A \cdot \}^n S_{F - [+M]} \right] P_R. \] (4.35)

For \( n = 1 \), we have
correlation between the two boundaries vanish in the limit of $L \to \infty$ 
the leading case by the Sommerfeld-Watson transformation. We obtain the result that the 
functions between the boundaries at the order $n$ functions of 

Then we can write 

$$
\Delta_{R_\pm} \cdot \Delta_{R_\pm} = \sum_{\omega} \frac{(\omega \cot \omega L \pm M)}{n_{\pm}(\omega)} \frac{[\sin \omega(L - s)]_{\pm}[\sin \omega(L - t)]_{\pm}}{[M^2 + \omega^2 - (p + k)^2 - i\varepsilon][M^2 + \omega^2 - p^2 - i\varepsilon]} \quad (4.38)
$$

Same is true for $B_{RL\pm} \cdot B_{LR\pm}$ and it satisfies the same boundary condition as $\Delta_{R\pm}$. Let us denote this similarity as follows.

$$
\Delta_{R_\pm} \cdot \Delta_{R_\pm} \sim B_{RL\pm} \cdot B_{LR\pm} \sim \Delta_{R_\pm}. \quad (4.39)
$$

Then we can write 

$$
\Delta_{L_-} \cdot \Delta_{L_-} \sim B_{LR\pm} \cdot B_{RL\pm} \sim \Delta_{L_-}. \quad (4.40)
$$

$$
\Delta_{R_\pm} \cdot B_{RL\pm} \sim B_{RL\pm} \cdot \Delta_{L_-} \sim B_{RL\pm}. \quad (4.41)
$$

$$
\Delta_{L_-} \cdot B_{LR\pm} \sim B_{LR\pm} \cdot \Delta_{R_\pm} \sim B_{LR\pm}. \quad (4.42)
$$

As expected from the similarity of the structure of the Green functions, the correlation functions between the boundaries at the order $n = 1$ can be evaluated in the same way as the leading case by the Sommerfeld-Watson transformation. We obtain the result that the correlation between the two boundaries vanish in the limit of $L \to \infty$, 

$$
\lim_{L \to \infty} \left( 
\begin{array}{c}
P_R \\
_P_L 
\end{array}
\right) \left[ i \{S_{F_\pm} \cdot (-)\gamma^\mu \cdot \} S_{F_\pm} \right] 
\left( 
\begin{array}{c} 
P_L \\
P_R 
\end{array}
\right) (k + p, k) 
= (i) \begin{pmatrix} P_R V^\mu_{\pm} (k + p, k) P_L & 0 \\ 0 & P_L V^\mu_{\pm} (k + p, k) P_R \end{pmatrix}, \quad (4.43)
$$

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where
\[
V_\pm^{\mu}(k + p, k) (P + K) = \frac{1}{[P \pm M][K \pm M]} \gamma^{\mu}.
\] (4.44)

\( p \) is assumed to be the momentum incoming from the external gauge boson attached to the vertex \( \gamma^{\mu} \). \( P \) and \( K \) are defined as \( P = \sqrt{M^2 - (k + p)^2 - i\varepsilon} \) and \( K = \sqrt{M^2 - k^2 - i\varepsilon} \), respectively.

For \( n > 1 \), the correlation functions between the boundaries can be evaluated in a similar manner. We obtain
\[
\lim_{L \to \infty} (..P_R \ldots P_L) \left[ i \prod_{i=1}^{n} \{ S_{F\pm}(k_i) \cdot (\ldots) \gamma^{\nu_i} \cdot S_{F\pm}(k_{n+1}) \} \left( P_L .. P_R. \right) \right] = (i)^n \sum_{n=1} \left( P_R V_\pm^{\nu_1\nu_2 \ldots \nu_{n+1}}(k_1, k_2, \ldots, k_{n+1}) \right) P_L \left( \begin{array}{cc} 0 & 1 \end{array} \right)
\] (4.45)

where
\[
V_\pm^{\nu_1\nu_2 \ldots \nu_{n+1}}(k_1, k_2, \ldots, k_{n+1}) = \sum_{0 \leq l \leq n+1} C_{2l}^{n+1} \left( \frac{1}{(M - K^2_i)} \prod_{i \neq j} \frac{1}{K_j^2 - K_i^2} \right).
\] (4.46)

(See Appendix [3] for the definition of \( C_{2l}^{n+1} \).) This result shows that the cluster property holds in each order of the expansion.

We will show the explicit results for \( n = 2 \) and \( n = 3 \) for later use. For \( n = 2 \),
\[
V_\pm^{\mu\nu}(k + p, k, k - q) (P + K)(K + Q)(Q + P)
= \frac{P + K + Q \pm M}{[P \pm M][K \pm M][Q \pm M]} \left[ (k' + q') \gamma^\mu \gamma^\nu (k' - q') \right] + \gamma^\mu \gamma^\nu (k' - q') \left[ (k' + q') \gamma^\mu \gamma^\nu (k' - q') \right].
\] (4.47)

\( q \) is assumed to be the momentum incoming from the external gauge boson attached to the vertex \( \gamma^\nu \) and \( Q \) is defined by \( Q = \sqrt{M^2 - (k - q)^2 - i\varepsilon} \).

For \( n = 3 \),
\[
V_\pm^{\mu\nu\lambda}(k + p, k, k - q, k - q - r) (P + K)(P + Q)(P + R)(K + Q)(K + R)(Q + R)
= \frac{A_\pm(P, K, Q, R)}{[P \pm M][K \pm M][Q \pm M][R \pm M]} \left[ (k' + q') \gamma^\mu \gamma^\nu \gamma^\lambda (k' - q') \right] \left[ (k' + q') \gamma^\mu \gamma^\nu \gamma^\lambda (k' - q') \right] + \gamma^\mu \gamma^\nu \gamma^\lambda (k' - q') B(P, K, Q, R),
\] (4.48)

and
\[ A_{\pm}(P, K, Q, R) = P^2(Q + R + K) + Q^2(P + R + K) + R^2(P + Q + K) + K^2(P + Q + R) \\
+ 2(QRK + PRK + PQK + PQR) \\
\pm M(P + K + Q + R)^2 + M^2(P + K + Q + R), \quad (4.49) \]

\[ B(P, K, Q, R) = (QRK + PRK + PQK + PQR) + M^2(P + K + Q + R). \quad (4.50) \]

\( r \) is assumed to be the momentum incoming from the external gauge boson attached to the vertex \( \gamma^\lambda \) and \( R \) is defined by
\[ R = \sqrt{M^2 - (k - q - r)^2 - i\varepsilon}. \]

If we take into account of the dimensional regularization, since \( \gamma^{a=5} \) commutes with the extended part of the gamma matrices \( \gamma^\mu(\mu = 5, \ldots, D) \), the boundary condition is not respected by the chiral structure of the gamma matrices. There occurs the mismatch of the components on which different boundary conditions are imposed. For example, let us consider the component,
\[ ..P_R \left[ i \sum_{n=1}^{\infty} \{ S_{F+} \cdot \gamma^{\mu_i} \}^n S_{F+} \right] P_R. \quad (4.51) \]

Beside the regular part, we encounter the following term at \( n = 1 \).
\[ P_{Rf} \gamma^\mu P_{Lg} ..\Delta_R \cdot \Delta_L. \quad (4.52) \]

Since the boundary condition mismatches, we cannot use the orthogonality of the generalized “sin” function and cannot evaluate the correlation straightforwardly. At \( n = 2 \), we find the following mismatched correlation functions.
\[ ..\Delta_R \cdot \Delta_L \cdot B_{RL}, \quad ..B_{RL} \cdot \Delta_L \cdot B_{RL}. \sim ..B_{RL} \cdot B_{RL}. \quad (4.53) \]

At \( n = 3 \),
\[ ..\Delta_R \cdot \Delta_L \cdot \Delta_R \cdot \Delta_L. \quad (4.54) \]
\[ ..B_{RL} \cdot \Delta_L \cdot \Delta_R \cdot \Delta_L. \sim ..B_{RL} \cdot \Delta_R \cdot \Delta_L. \quad (4.55) \]
\[ ..\Delta_R \cdot B_{LR} \cdot \Delta_R \cdot \Delta_L. \sim ..\Delta_R \cdot B_{LR} \cdot \Delta_L. \quad (4.56) \]

At \( n = 4 \),
\[ ..\Delta_R \cdot \Delta_L \cdot \Delta_R \cdot \Delta_L \cdot B_{RL}. \quad (4.57) \]
\[ ..B_{RL} \cdot \Delta_L \cdot \Delta_R \cdot \Delta_L \cdot B_{RL}. \sim ..B_{RL} \cdot \Delta_R \cdot \Delta_L \cdot B_{RL}. \quad (4.58) \]
\[ ..\Delta_R \cdot \Delta_L \cdot B_{RL} \cdot \Delta_L \cdot B_{RL}. \sim ..\Delta_R \cdot \Delta_L \cdot B_{RL} \cdot B_{RL}. \quad (4.59) \]
\[ ..B_{RL} \cdot \Delta_L \cdot B_{RL} \cdot \Delta_L \cdot B_{RL}. \sim ..B_{RL} \cdot B_{RL} \cdot B_{RL}. \quad (4.60) \]

Since the divergence appears in the order \( n \leq 4 \) by the power counting, it is enough to consider only such orders. Any mismatched correlation function up to \( n = 4 \) is “similar” to one of the above examples.

As to the simplest case, Eq. (4.53), we can see that it actually vanishes in the limit \( L \to \infty \) as follows. From Eq. (3.61) and Eq. (3.62), the correlation between the boundary at \( s = -L \) and that at \( s = L \) could emerge from the parts,
\[
\frac{(P \coth PL \pm M)}{-\Delta_\pm(iP)} \frac{[\sinh P(L - s)]_\pm[\sinh P(L - t)]_\pm}{\sinh^2 PL}, \quad (4.61)
\]

in \( \Delta_{R\pm} \) or
\[
\frac{(P \coth PL - M)}{-\Delta_- (iP)} \frac{[\sinh P(L + s)]_-[\sinh P(L + t)]_-}{\sinh^2 PL}, \quad (4.62)
\]

in \( \Delta_{L\pm} \). At the boundaries, on the other hand, the above correlationmediate-parts cannot contribute because
\[
\frac{[\sinh P(L - s)]_\pm}{\sinh PL} = \frac{1}{P} \frac{P}{\sinh PL \coth PL \pm M} \xrightarrow{L \to \infty} 0
\]
\[
\frac{[\sinh P(L + s)]_-}{\sinh PL} = \frac{1}{P} \frac{P}{\sinh PL \coth PL - M} \xrightarrow{L \to \infty} 0, \quad (4.63)
\]

As a result,
\[
\Delta_R \cdot \Delta_L \xrightarrow{L \to \infty} 0. \quad (4.64)
\]

We can expect that other mismatched correlation functions which appear in Eq. (4.34) also vanish in the limit \( L \to \infty \) in the similar reason. We do not enter the detail of the proof here. We rather assume that the dimensional regularization preserves the cluster property Eq. (4.33).

Using this cluster property, we obtain the formula of the perturbative expansion of the boundary contribution Eq. (2.67) as follows.

\[
\lim_{L \to \infty} \ln \left[ \sqrt[det'(1-M-X^-_{\infty})_{-\infty}^{\pm} det'(1-M^0_{+\infty} \mp X^+_{\infty})_{+\infty}^{\infty}} \right] - \lim_{L \to \infty} \left[ \sqrt[det'(1-M^0_{-\infty} \pm X^-_{\infty})_{-\infty}^{\pm} det'(1-M^0_{+\infty} \mp X^+_{\infty})_{+\infty}^{\infty}} \right] \\
= \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-)^m}{m} \text{Tr}' \left\{ d^\infty_\infty \left( \lim_{L \to \infty} .. P_R \left[ \int_1^n \{ S_{F-}[M] \cdot (-A_i) \}^n S_{F-}[M] P_L \right] \right) m \right\} \\
- \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-)^m}{m} \text{Tr}' \left\{ d^\infty_\infty \left( \lim_{L \to \infty} .. P_R \left[ \int_1^n \{ S_{F-}[M] \cdot (-A_i) \}^n S_{F-}[M] P_L \right] \right) m \right\} \\
+ \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-)^m}{m} \text{Tr}' \left\{ d^\infty_\infty \left( \lim_{L \to \infty} .. P_L \left[ \int_1^n \{ S_{F-}[M] \cdot (-A_i) \}^n S_{F-}[M] P_L \right] \right) m \right\} \\
- \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-)^m}{m} \text{Tr}' \left\{ d^\infty_\infty \left( \lim_{L \to \infty} .. P_L \left[ \int_1^n \{ S_{F-}[M] \cdot (-A_i) \}^n S_{F-}[M] P_L \right] \right) m \right\} \\
\equiv i \Gamma_X[A], \quad (4.65)
\]

where
\[
d^\infty_{\pm}(p) = \frac{P \mp M - \eta}{2P}. \quad (4.66)
\]

This is derived by using the result Eq. (4.33) as follows.
\[ D^\infty(p)^{-1} \equiv \lim_{L \to \infty} (1 - M^0 - X^\infty) \]
\[ = 1 + \begin{pmatrix} P_R \gamma^1 & 0 \\ 0 & P_L \gamma^1 \end{pmatrix} + \begin{pmatrix} P_L \gamma^1 & 0 \\ 0 & P_R \gamma^1 \end{pmatrix} \]
\[ = \left( 1 + \frac{\hat{\rho}}{P+M} \right) \equiv \left( d^\infty(p)^{-1} \right). \quad (4.67) \]

As an important consequence of the cluster property, we can show that the boundary contribution Eq. (2.67) is purely odd-parity. Since the fermion system with the homogeneous mass possesses the parity invariance, the propagator satisfies the relation,
\[ S_F^{-}[\pm M](x_0 - y_0, x_i - y_i, s, t) = \gamma^0 S_F^{-}[\pm M](x_0 - y_0, -x_i + y_i, -s, -t) \gamma^0. \quad (4.68) \]

We can also check explicitly that \( d^\infty(x_0 - y_0, x_i - y_i) \) satisfies the similar relation. From these properties, we can easily see that the boundary contribution given by the above trace formula is a parity-odd functional of the external gauge field potential.

Note also that it changes the sign when we change the sign of \( M \) to \( -M \). This means that it has the functional form as
\[ \Gamma_X[A] = MF_X[A; M^2]. \quad (4.69) \]

This property is useful to reduce the superficial degree of divergence of the loop integral in the perturbative evaluation.

**V. ANOMALY FROM BOUNDARY TERM**

In this section, as an application of the perturbation theory given in the previous section IV, we calculate the variation of the boundary term under the gauge transformation. We find that the consistent anomaly is actually reproduced by the gauge noninvariant boundary state wave functions. This shows that it is the correct choice to fix the phase of the overlap following the Wigner-Brillouin perturbation theory.

**A. variation under gauge transformation**

In order to examine the gauge symmetry breaking induced by the boundary state wave function Eq. (4.18) and Eq. (4.19), we consider the variation of the boundary contribution \( \Gamma_X[A] \) under the gauge transformation:
\[ A_\mu(x) \to A_\mu(x) + \partial_\mu \omega(x) - [\omega(x), A_\mu(x)], \quad \omega \in su(N). \quad (5.1) \]
First we note the Ward-Takahashi identity in which we take into account of the breaking of the chiral boundary condition due to the dimensional regularization,

\[ S_{F\pm}(k + p) \cdot \gamma \cdot S_{F\pm}(k) = S_{F\pm}(k + p) - S_{F\pm}(k) + \Delta_1(k + p) \cdot \gamma \cdot \Delta_2(k), \]  

(5.2)

where we have defined

\[ \Delta_1(k) \equiv -k^\mu \{ P_R \Delta_{R-}(k) - P_L \Delta_{L-}(k) \}, \]

(5.3)

\[ \Delta_2(k) \equiv [\gamma \cdot (\Delta_{R-}(k) - \Delta_{L-}(k)) + (B_{RL-}(k) - B_{LR-}(k))] \],

(5.4)

where \( k^\mu = \sum_{\mu=1}^D \gamma^\mu k_\mu \). From this identity, we obtain

\[ \delta_\omega \left[ i \sum_{n=0}^\infty \{ S_{F\pm} \cdot (-\gamma \cdot \gamma) \}^n S_{F\pm} \right] (x, y) = -\omega(x) \left[ i \sum_{n=0}^\infty \{ S_{F\pm} \cdot (-\gamma \cdot \gamma) \}^n S_{F\pm} \right] (x, y) + \left[ i \sum_{n=0}^\infty \{ S_{F\pm} \cdot (-\gamma \cdot \gamma) \}^n S_{F\pm} \right] (x, y) \omega(y) - \Delta_\omega(x, y), \]

(5.5)

where \( \Delta_\omega(x, y) \) stands for the gauge non-invariant correction due to the dimensional regularization,

\[ \Delta_\omega(x, y) = i \Delta_1 \cdot \partial \omega \cdot \Delta_2(x, y) - i [S_{F\pm} \cdot \gamma \cdot \Delta_1 \cdot \partial \omega \cdot \Delta_2] (x, y) - i [\Delta_1 \cdot \partial \omega \cdot \Delta_2 \cdot \gamma \cdot S_{F\pm}] (x, y) + \ldots \]

(5.6)

Then the variation of \( 1 - M - X_{X+}^\infty \) is given by

\[ \delta_\omega (1 - M - X_{X+}^\infty) = -\omega X_{X+}^\infty - X_{X+}^\infty \omega) - \left( \begin{array}{c} \cdot P_R \\ \cdot P_L \end{array} \right) [\Delta_\omega] \left( \begin{array}{c} P_L \cdot P_R \end{array} \right) - \left[ \omega, (1 - M - X_{X+}^\infty) \right]. \]

(5.7)

Therefore the variation of the determinant of \( 1 - M - X_{X+}^\infty \) can be written as

\[ \delta_\omega \ln \text{det}'(1 - M - X_{X+}^\infty) = \text{Tr}' \left\{ \left( \begin{array}{c} \cdot P_R \\ \cdot P_L \end{array} \right) [\Delta_\omega] \left( \begin{array}{c} P_L \cdot P_R \end{array} \right) \right\} \]

\[ \times \sum_{m=0}^\infty \left( -1 \right)^m \left\{ D^L_+ \left( \begin{array}{c} \cdot P_R \\ \cdot P_L \end{array} \right) \left[ i \sum_{n=1}^\infty \{ S_{F+} \cdot (-\gamma \cdot \gamma) \}^n S_{F+} \right] \left( \begin{array}{c} P_L \cdot P_R \end{array} \right) \right\}^m D^L_+ \}

(5.8)
The variation of the determinant of $1 - M_{\pm} - X_{\pm \pm}^\infty$ is obtained in a similar manner.

Taking into account of the cluster property in the limit $L \to \infty$, the variation of $\Gamma_X$ is given by

$$i \delta_\omega \Gamma_X[A] = \lim_{L \to \infty} \delta_\omega \ln \left[ \frac{\det(1 - M_{\pm} - X_{\pm \pm}^\infty)}{\sqrt{\det(1 - M_{-} - X_{- \pm}^\infty) \det(1 - M_{+} - X_{+ \pm}^\infty)}} \right]$$

$$= \frac{1}{2} \text{Tr}' \left\{ (-) \left( \omega X_{\pm} - X_{\pm}^\omega \right) + p_L \Delta_\omega P_{L.} \right\} \times \sum_{m=0}^\infty \left( - \right)^m \left\{ d_+^\infty \left( \lim_{L \to \infty} p_L \left[ i \sum_{n=1}^\infty \{ S_{F^{-}[+M]} \cdot (-) A \cdot \}^n S_{F^{-}[-M]} \right] P_{L.} \right) \right\} m d_+$$

$$+ \frac{1}{2} \text{Tr}' \left\{ (-) \left( \omega X_{\pm} - X_{\pm}^\omega \right) + p_L \Delta_\omega P_{R.} \right\} \times \sum_{m=0}^\infty \left( - \right)^m \left\{ d_+^\infty \left( \lim_{L \to \infty} p_L \left[ i \sum_{n=1}^\infty \{ S_{F^{-}[+M]} \cdot (-) A \cdot \}^n S_{F^{-}[-M]} \right] P_{R.} \right) \right\} m d_+$$

$$- \frac{1}{2} \text{Tr}' \left\{ (-) \left( \omega X_{\pm} - X_{\pm}^\omega \right) + p_L \Delta_\omega P_{L.} \right\} \times \sum_{m=0}^\infty \left( - \right)^m \left\{ d_+^\infty \left( \lim_{L \to \infty} p_L \left[ i \sum_{n=1}^\infty \{ S_{F^{-}[-M]} \cdot (-) A \cdot \}^n S_{F^{-}[+M]} \right] P_{L.} \right) \right\} m d_+$$

$$- \frac{1}{2} \text{Tr}' \left\{ (-) \left( \omega X_{\pm} - X_{\pm}^\omega \right) + p_L \Delta_\omega P_{R.} \right\} \times \sum_{m=0}^\infty \left( - \right)^m \left\{ d_+^\infty \left( \lim_{L \to \infty} p_L \left[ i \sum_{n=1}^\infty \{ S_{F^{-}[-M]} \cdot (-) A \cdot \}^n S_{F^{-}[+M]} \right] P_{R.} \right) \right\} m d_+ \right\}. \quad (5.9)$$

**B. Consistent anomaly**

Now we perform the calculation of Eq. (5.9). We express it in the momentum space as follows.

$$\delta_\omega \Gamma_X[A] = \sum_{n=1}^\infty (i)^{n+1} \int \frac{d^4l}{(2\pi)^4} \prod_{i=1}^n \frac{d^4p_i}{(2\pi)^4} (2\pi)^4 \delta \left( l + \sum_{i=1}^n p_i \right) \Gamma_X^{\nu_1 \nu_2 \cdots \nu_n} (p_1, \ldots, p_n) \text{tr} \{ \omega(l) \prod_{i=1}^n A_{\nu_i}(p_i) \}.$$

$$= \sum_{n=1}^\infty (i)^{n+1} \int \frac{d^4l}{(2\pi)^4} \prod_{i=1}^n \frac{d^4p_i}{(2\pi)^4} (2\pi)^4 \delta \left( l + \sum_{i=1}^n p_i \right) \Gamma_X^{\nu_1 \nu_2 \cdots \nu_n} (p_1, \ldots, p_n) \text{tr} \{ \omega(l) \prod_{i=1}^n A_{\nu_i}(p_i) \}.$$

(5.10)

1. **Finiteness of $\Gamma_X$**

Since $\Gamma_X[A]$ is a parity-odd functional of gauge field potential, it must involve the $\epsilon$-tenser in four-dimensions $\epsilon_{\nu_1 \nu_2 \nu_3 \nu_4}$ ($\nu_i = 0, 1, 2, 3$). It is also proportional to $M$. The Lorentz indices $\nu_1, \nu_2, \nu_3, \nu_4$ should be contracted with those of the gauge field potential $A^\nu$ and
momentum $p^\mu$'s. For $n = 1$, we can easily see that such structure cannot appear. For $n = 2$, we can have the form

$$M \epsilon_{\mu\nu\rho\sigma} p_1^\nu p_2^\rho F(p_1, p_2, M^2). \quad (5.11)$$

In this case, the superficial degree of divergence of $\Gamma_{\mu\nu}^X$ is two by the power counting rule of the five-dimensional theory. Since the above structure reduces it by three and we have minus one. This means that there does not appear ultraviolet divergence in this term. This also means that the additional term including $\Delta_\omega$ due to the dimensional regularization does not contribute. For $n = 3$, we can have the term

$$M \epsilon_{\mu\nu\rho\sigma} p_3^\nu F(p_1, p_2, p_3, M^2). \quad (5.12)$$

In this case, the superficial degree of divergence is also reduced to minus one and the additional term due to the dimensional regularization does not contribute. For $n = 4$, we can have the term

$$M \epsilon_{\mu\nu\rho\sigma}. \quad (5.13)$$

The superficial degree of divergence is zero and it is reduced by one because of the coefficient $M$. It is also finite. For $n \geq 5$, they are finite by the power counting rule. Therefore, in every orders of the expansion, the boundary contribution is finite and the additional term due to the dimensional regularization does not contribute. Therefore, in the following calculation, we can omit the terms due to the dimensional regularization.

2. First order

The first order term ($n = 1$) is given by the expression

$$\Gamma_{\mu\nu}^X(p) = \frac{1}{2} \int \frac{d^5 k}{(2\pi)^5} \text{tr} \left\{ P_L \left( \frac{k^\nu + p^\nu}{P + M} - \frac{k^\nu}{K + M} \right) P_R \right.$$ \nonumber

$$\cdot \left( P + M - \frac{(k^\nu + p^\nu)}{2P} \right) P_R V_+^\nu(k + p, k) P_L \frac{K + M - k^\nu}{2K} \right\}$$ \nonumber

$$+ \frac{1}{2} \int \frac{d^5 k}{(2\pi)^5} \text{tr} \left\{ i \Delta_1(k + p) \frac{k^\nu}{2P} \right.$$ \nonumber

$$\cdot \left( P + M - \frac{(k^\nu + p^\nu)}{2P} \right) P_R V_+^\nu(k + p, k) P_L \frac{K + M - k^\nu}{2K} \right\}$$ \nonumber

$$+ \ldots$$ \nonumber

$$= -\frac{1}{2} \int \frac{d^5 k}{(2\pi)^5} \text{tr} \left\{ \gamma_5 \left( \frac{k^\nu + p^\nu}{P + M} - \frac{k^\nu}{K + M} \right) \frac{P + M}{2P} V_+^\nu(k + p, k) \frac{K + M}{2K} \right\}$$ \nonumber

$$+ \frac{1}{2} \int \frac{d^5 k}{(2\pi)^5} \text{tr} \left\{ \gamma_5 \left( \frac{k^\nu + p^\nu}{P - M} - \frac{k^\nu}{K - M} \right) \frac{P - M}{2P} V_-^\nu(k + p, k) \frac{K - M}{2K} \right\}, \quad (5.14)$$
where \( ... \) stands for the contributions from the second, third and fourth terms in Eq. (5.9). \( \nu^\mu(k + p, k) \) is defined by Eq. (4.44). This contribution actually vanishes because of the trace over gamma matrices.

3. Second order

The second order term \((n = 2)\) is given by the following expression.

\[
\Gamma_\mu^\nu(p, q) = \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left\{ P_L \left( \frac{k + p}{P + M} - \frac{k - q}{Q + M} \right) P_R \\
+ \frac{P + M - (k + p)}{2P} \frac{V^\mu_+(k + p, k - q) P_L (Q + M - k - q)}{2Q} \right\}
\]

\[
-1 \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left\{ P_L \left( \frac{k + p}{P + M} - \frac{k - q}{Q + M} \right) P_R \\
+ \frac{P + M - (k + p)}{2P} \frac{V^\mu_+(k + p, k - q) P_L (Q + M - k - q)}{2Q} \right\}
\]

\[
+ \ldots
\]

\[
\Gamma_\mu^\nu(p, q) = -\frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left\{ \gamma_5 \left( \frac{k + p}{P + M} - \frac{k - q}{Q + M} \right) P^\mu_+(k + p, k - q) \frac{Q + M}{2Q} \right\}
\]

\[
+ \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left\{ \gamma_5 \left( \frac{k + p}{P - M} - \frac{k - q}{Q - M} \right) P^\mu_+(k + p, k - q) \frac{Q - M}{2Q} \right\}
\]

\[
+ \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left\{ \gamma_5 \left( \frac{k + p}{P + M} - \frac{k - q}{Q + M} \right) K + M \frac{V^\mu_+(k + p, k)}{2K} \right\}
\]

\[
- \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left\{ \gamma_5 \left( \frac{k + p}{P - M} - \frac{k - q}{Q - M} \right) K - M \frac{V^\mu_+(k + p, k)}{2K} \right\}
\]

where \( V^\mu_+(k + q + p, k + q, k) \) is defined by Eq. (4.47).

The first line (5.16) in the second equality can be evaluated as

\[
-\frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{4PQ(P + K)(K + Q)(Q + P)} \text{tr} \gamma_5 \left\{ (k + p)^2 \gamma^\mu \gamma^\nu (l - q) \frac{P + K + Q + M}{(P + M)(K + M)} \right\}
\]
of the property of the trace of gamma matrix, we can calculate it as follows.

\[ + [(k + p)^2 \gamma^\mu \gamma^\nu + (\not{q} + \not{p}) \gamma^\mu \gamma^\nu + (\not{q} + \not{p}) \gamma^\mu \gamma^\nu (\not{q} - \not{q})] (Q + M) \]
\[ + (k - q)^2 (\not{q} + \not{p}) \gamma^\mu \gamma^\nu \frac{P + K + Q + M}{(K + M)(Q + M)} \]
\[ + [(k - q)^2 \gamma^\mu \gamma^\nu + (\not{q} + \not{p}) \gamma^\mu \gamma^\nu (\not{q} - \not{q}) + \gamma^\mu \gamma^\nu (\not{q} - \not{q})] (P + M) \].

(5.20)

Subtracting the contribution with the mass of opposite signature (5.17), we obtain

\[ - \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \frac{2M}{4PQ(P + K)(K + Q)(Q + P)} \]
\[ \times \{ \gamma^\mu \gamma^\nu (\not{q} - \not{q}) (P + K + Q)(P + K) - (PK + M^2) \}
\[ + \frac{1}{8K^2(K^2 - M^2)} \]
\[ + [k + p)^2 \gamma^\mu \gamma^\nu + (\not{q} + \not{p}) \gamma^\mu \gamma^\nu + (\not{q} + \not{p}) \gamma^\mu \gamma^\nu (\not{q} - \not{q})] \]
\[ + (\not{q} + \not{p}) \gamma^\mu \gamma^\nu \frac{P + K + Q)(K + Q) - (KQ + M^2)}{(K^2 - M^2)} \]
\[ + [(k - q)^2 \gamma^\mu \gamma^\nu + (\not{q} + \not{p}) \gamma^\mu \gamma^\nu (\not{q} - \not{q}) + \gamma^\mu \gamma^\nu (\not{q} - \not{q})] \}

(5.21)

Assuming that \( p, q \ll M \), we make expansion with respect to \( p, q \). Taking into account
of the property of the trace of gamma matrix, we can calculate it as follows.

\[ -M \int \frac{d^Dk}{(2\pi)^D} \left\{ - \text{tr}(\gamma^\mu \gamma^\nu \not{q}) \frac{1}{8K^2(K^2 - M^2)} \frac{(P + 2K)(P + K) - (PK + M^2)}{(P + K)^2} \right\} \]
\[ + \text{tr}(\gamma^\mu \gamma^\nu \not{q}) \frac{1}{8K^2(K^2 - M^2)} \frac{(Q + 2K)(Q + K) - (KQ + M^2)}{Q(Q + K)^2} \]
\[ - 2 \text{tr}(\gamma^\mu \gamma^\nu \not{q}) \frac{1}{32K^5} \]
\[ - \text{tr}(\gamma^\mu \gamma^\nu \not{q}) \frac{1}{8K^2P(P + K)^2} + \text{tr}(\gamma^\mu \gamma^\nu \not{q}) \frac{1}{8K^2Q(Q + K)^2} \} + \mathcal{O} \left( \frac{1}{M} \right) \]
\[ = -M \int \frac{d^Dk}{(2\pi)^D} \left\{ - \text{tr}(\gamma^\mu \gamma^\nu \not{q}) \frac{1}{8K^2(K^2 - M^2)} \left( \frac{1}{4K^3} + 2 \frac{3}{4K^3} - K \frac{1}{4K^4} - M^2 \frac{1}{2K^5} \right) \right\} \]
\[ - \text{tr}(\gamma^\mu \gamma^\nu \not{q}) \frac{1}{8K^2(K^2 - M^2)} \left( \frac{1}{4K^3} + 2 \frac{3}{4K^3} - K \frac{1}{4K^4} - M^2 \frac{1}{2K^5} \right) \]
\[ - \text{tr}(\gamma^\mu \gamma^\nu \not{q}) \frac{1}{16K^5} \]
\[ - \text{tr}(\gamma^\mu \gamma^\nu \not{q}) \frac{1}{8K^2} \frac{1}{2K^5} - \text{tr}(\gamma^\mu \gamma^\nu \not{q}) \frac{1}{8K^2} \frac{1}{2K^5} \} + \mathcal{O} \left( \frac{1}{M} \right) \]
\[ = M \text{tr}(\gamma^\mu \gamma^\nu \not{q}) \right\} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{8K^5} + \mathcal{O} \left( \frac{1}{M} \right) \]
\[ = M \text{tr}(\gamma^\mu \gamma^\nu \not{q}) \right\} - \frac{i}{16\pi^2} \frac{1}{3} \frac{1}{M} + \mathcal{O} \left( \frac{1}{M} \right) \]
\[ = - \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} p^\rho q^\sigma + \mathcal{O} \left( \frac{1}{M} \right) \] (5.22)

On the other hand, the third line (5.18) can be evaluated as
Subtracting the contribution with the mass of opposite signature \(5.19\), we obtain

\[
- \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{8PKQ(P + K)(K + Q)} \text{tr}_{\gamma_5} \left\{ \left( \gamma^\mu \frac{(k + p)^2 k^2}{(P + M)(K + M)} + (\not k + \not p) \gamma^\mu \not k \right) \left( \frac{1}{k^2(k - q)^2} \frac{k^2}{(K + M)(Q + M)} + \gamma^\nu (Q + M) \right) \\
+ \left( (\not k + \not p) \gamma^\mu \not k \right) \left( \frac{1}{K + M} + \gamma^\nu (P + M) \right) \right\}.
\]

The expansion with respect to \( p, q \) leads to the result,

\[
-M \int \frac{d^D k}{(2\pi)^D} \left\{ \text{tr}_{\gamma_5} \left( (\not k + \not p) \gamma^\mu \not k \right) \frac{1}{16K^3(K^2 - M^2)} \frac{2PK + (K^2 + M^2)}{P(P + K)} \\
- \text{tr}_{\gamma_5} \left( (\not k + \not p) \gamma^\mu \gamma^\nu \not k \right) \frac{1}{16K^3(K^2 - M^2)} \frac{2QK + (K^2 + M^2)}{Q(Q + K)} \\
- \text{tr}_{\gamma_5} \left( (\not k + \not p) \gamma^\mu \gamma^\nu \not k \right) \frac{1}{16K^5} \\
- \text{tr}_{\gamma_5} \left( (\not k + \not p) \gamma^\mu \gamma^\nu \not k \right) \frac{1}{16K^3} \frac{1}{P(P + K)} + \text{tr}_{\gamma_5} \left( (\not k + \not p) \gamma^\mu \gamma^\nu \not k \right) \frac{1}{16K^3} \frac{1}{Q(Q + K)} \right\} + O \left( \frac{1}{M} \right)
\]

\[
= -M \int \frac{d^D k}{(2\pi)^D} \left\{ \text{tr}_{\gamma_5} \left( (\not k + \not p) \gamma^\mu \not k \right) \frac{(k \cdot p)}{16K^3(K^2 - M^2)} \left( 2K \frac{1}{4K^3} + (K^2 + M^2) \frac{3}{4K^4} \right) \\
+ \text{tr}_{\gamma_5} \left( (\not k + \not p) \gamma^\mu \gamma^\nu \not k \right) \frac{1}{16K^3(K^2 - M^2)} \left( 2K \frac{1}{4K^3} + (K^2 + M^2) \frac{3}{4K^4} \right) \\
- \text{tr}_{\gamma_5} \left( (\not k + \not p) \gamma^\mu \not k \right) \frac{1}{16K^5} \right\}.
\]
Therefore, we obtain

\[
\Gamma_{\chi}^{\mu\nu}(p, q) = -\frac{1}{24\pi^2}\varepsilon_{\mu\nu\rho\sigma}p_{\rho}q_{\sigma} + \mathcal{O}\left(\frac{1}{M}\right).
\]  

(5.26)

4. Third order

The third order term is given by the following expression.

\[
\Gamma_{\chi}^{\mu\nu\lambda}(p, q, r) = \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left\{ P_L \left( \frac{k^i + p^i}{P + M} - \frac{k^i - q^i - \psi - r^i}{R + M} \right) P_R 
\right.
\]

\[
\left. \cdot \frac{P + M - (k^i + p^i)}{2P} P_R V_{+}^{\mu\nu\lambda}(k + p, k, k - q, k - q - r) P_L \frac{R + M - (k^i - q^i - \psi - r^i)}{2R} \right\}
\]

\[
-\frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left\{ P_L \left( \frac{k^i + p^i}{P + M} - \frac{k^i - q^i - \psi - r^i}{R + M} \right) P_R 
\right.
\]

\[
\left. \cdot \frac{P + M - (k^i + p^i)}{2P} P_R V_{+}^{\mu\nu}(k + p, k, k - q) P_L \right\}
\]

\[
-\frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left\{ P_L \left( \frac{k^i + p^i}{P + M} - \frac{k^i - q^i - \psi - r^i}{R + M} \right) P_R 
\right.
\]

\[
\left. \cdot \frac{P + M - (k^i + p^i)}{2P} P_R V_{+}^{\mu\nu\lambda}(k, k - q, k - q - r) P_L \frac{R + M - (k^i - q^i - \psi - r^i)}{2R} \right\}
\]

\[
+\frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} \text{tr} \left\{ P_L \left( \frac{k^i + p^i}{P + M} - \frac{k^i - q^i - \psi - r^i}{R + M} \right) P_R 
\right.
\]

\[
\left. \cdot \frac{P + M - (k^i + p^i)}{2P} P_R V_{+}^{\mu\nu}(k + p, k) P_L \frac{R + M - (k^i - q^i - \psi - r^i)}{2R} \right\}
\]

\[
\frac{Q + M - (k^i - q^i)}{2K} P_R V_{+}^{\lambda}(k - q, k - q - r) P_L \frac{R + M - (k^i - q^i - \psi - r^i)}{2R} \}
\]
Subtracting the contribution with the mass of opposite signature (5.32), we finally obtain
\[
\Gamma_{\lambda}(p + q + r) = \frac{1}{2} \text{tr} \gamma_{\mu} \gamma_{\nu} \gamma_{\lambda} (p + q + r) + \Gamma_{\mu \nu \rho \sigma} (p + q + r) + \mathcal{O} \left( \frac{1}{M} \right).
\]

The expansion with respect to \( p, q \) and \( r \) in (5.28), (5.29), (5.30) and (5.31) leads to the result,
\[
\Gamma_{\mu \nu \rho \sigma} (p, q, r) = -\frac{1}{2} \text{tr} \gamma_5 \frac{1}{(2\pi)^D 2^D K^7} \left[ -10 Mk^2 - 4M k^2 + 20 M^3 + 2 Mk^2 - 10 M k^2 \right] + \mathcal{O} \left( \frac{1}{M} \right).
\]

Subtracting the contribution with the mass of opposite signature (5.32), we finally obtain
\[
\Gamma_{\lambda}(p + q + r) = \frac{1}{2} \text{tr} \gamma_{\mu} \gamma_{\nu} \gamma_{\lambda} (p + q + r) + \Gamma_{\mu \nu \rho \sigma} (p + q + r) + \mathcal{O} \left( \frac{1}{M} \right)
\]

where
\[
\Gamma_{\mu \nu \rho \sigma} (p, q, r) = \frac{1}{2} 24 \pi^2 \epsilon^{\mu \nu \rho \sigma} (p + q + r)_{\sigma} + \mathcal{O} \left( \frac{1}{M} \right).
\]
5. Forth order and Higher orders

As to the forth order term \((n = 4)\), we find that the leading term vanishes in the expansion with respect to the external momentum. Then we obtain

\[
\Gamma_\mu^\nu_{\rho\lambda}(p, q, r, s) = 0 + \mathcal{O}\left(\frac{1}{M}\right). \tag{5.35}
\]

The higher order terms \((n \geq 5)\) have negative mass dimensions and they are expected to be suppressed by the factor \(\frac{1}{M^{n-4}}\).

6. Final result

From these results, we obtain the variation of the boundary term under the gauge transformation as follows.

\[
\delta_\omega \Gamma_\mu_\nu[A] = \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \left\{ -\frac{i^3}{24\pi^2} \varepsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma \right\} \text{tr}\{\omega(-p - q)A_\mu(p)A_\nu(q)\}
\]

\[
+ \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{d^4 r}{(2\pi)^4} \left\{ -\frac{i^4}{24\pi^2} \varepsilon^{\mu\nu\lambda\sigma} (p + q + r)_\sigma \right\} \text{tr}\{\omega(-p - q)A_\mu(p)A_\nu(q)A_\lambda(r)\} + \mathcal{O}\left(\frac{1}{M}\right)
\]

\[
= \frac{i}{24\pi^2} \int dx^4 \varepsilon^{\mu\nu\rho\sigma} \text{tr}\left\{\omega(x) \left[ \partial_\mu A_\nu(x) \partial_\rho A_\sigma(x) + \frac{1}{2} \partial_\mu (A_\nu(x)A_\rho(x)) A_\sigma(x) \right] \right\} + \mathcal{O}\left(\frac{1}{M}\right). \tag{5.36}
\]

We can see that the consistent anomaly is correctly reproduced by the Wigner-Brillouin phase fixing procedure.

VI. VACUUM POLARIZATION

In this section, as another application of the perturbation theory, we perform the calculation of the two-point function (vacuum polarization function) in the expansion of the five-dimensional determinant contribution, Eq. \(\text{(4.3)}\). In the momentum space, it is written as follows.

\[
i\Gamma_K[A] = \sum_{n=1}^{\infty} (i)^n \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} (2\pi)^4 \delta \left( \sum_{i=1}^{n} p_i \right) \Gamma_+^{\nu_1, \nu_2, ..., \nu_n}(p_1, \ldots, p_n) \text{tr}\left\{ \prod_{i=1}^{n} A_{\nu_i}(p_i) \right\}, \tag{6.1}
\]

where

\[
\Gamma_+^{\nu_1, \nu_2, ..., \nu_n}(p_1, \ldots, p_n; L) = \Pi_+^{\nu_1, \nu_2, ..., \nu_n}(p_1, \ldots, p_n; L)
\]

\[
- \frac{1}{2} \Pi_-^{\nu_1, \nu_2, ..., \nu_n}(p_1, \ldots, p_n; L)[+M] - \frac{1}{2} \Pi_-^{\nu_1, \nu_2, ..., \nu_n}(p_1, \ldots, p_n; L)[-M], \tag{6.2}
\]

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\[ \Pi_{\pm}^{\mu_1 \mu_2 \cdots \mu_n}(p_1, \ldots, p_n) = \int \frac{d^Dk}{i(2\pi)^D} \int_{-L}^{+L} ds_i \text{Tr} \left\{ \prod_{i=1}^{n} \left[ \gamma^\mu i S_F^\pm(k + \sum_{j>i} p_j; s_i, s_{i+1}) \right] \right\}. \quad (6.3) \]

Note that \( s_{n+1} = s_1 \).

We will find that each contribution from the fermion with the kink-like mass or the fermion with the homogeneous mass is never chiral. The fermion with the homogeneous mass contains the light mode (massless mode in the limit \( L \to \infty \)) just as well as the fermion with the kink-like mass. On the contrary, the fermion with the negative homogeneous mass does not contain such light mode. Therefore, by the subtraction, the normalization of the vacuum polarization becomes correctly a half of that of the massless Dirac fermion.

Unfortunately, it turns out that the dimensional regularization is not adequate for the calculation of this volume contribution. It leads to the gauge noninvariant term proportional to \( M^2 \). Since in the lattice regularization the volume contribution is expected to be gauge invariant, this fact means our bad choice of the subsidiary regularization.

### A. Expression of Vacuum Polarization

The two point function is given explicitly as follows.

\[
\Pi_{\pm}^{\mu \nu}(p; L) = \int \frac{d^Dk}{i(2\pi)^D} \int_{-L}^{+L} ds dt \text{Tr} \left\{ \gamma^\mu i S_{\pm}(k + p; s, t) \gamma^\nu i S_{\pm}(k; t, s) \right\}
\]

\[
= \int \frac{d^Dk}{i(2\pi)^D} \left\{ \text{Tr} \left( \gamma^\mu P_R(k' + \gamma^\nu P_R k') \right) \int_{-L}^{+L} ds dt \Delta_{R^\pm}(k + p; s, t) \Delta_{R^\pm}(k; t, s) \right.
\]

\[
\left. + \text{Tr} \left( \gamma^\mu P_L(k' + \gamma^\nu P_L k') \right) \int_{-L}^{+L} ds dt \Delta_{L^\pm}(k + p; s, t) \Delta_{L^\pm}(k; t, s) \right. 
\]

\[
\left. + \text{Tr} \left( \gamma^\mu P_R(k' + \gamma^\nu P_L k') \right) \int_{-L}^{+L} ds dt B_{R^\pm}(k + p; s, t) B_{L^\pm}(k; t, s) \right. 
\]

\[
\left. + \text{Tr} \left( \gamma^\mu P_L(k' + \gamma^\nu P_R k') \right) \int_{-L}^{+L} ds dt B_{L^\pm}(k + p; s, t) B_{R^\pm}(k; t, s) \right. 
\]

\[
\left. + \text{Tr} \left( \gamma^\mu P_R(k' + \gamma^\nu P_R k') \right) \int_{-L}^{+L} ds dt \Delta_{R^\pm}(k + p; s, t) \Delta_{L^\pm}(k; t, s) \right. 
\]

\[
\left. + \text{Tr} \left( \gamma^\mu P_L(k' + \gamma^\nu P_L k') \right) \int_{-L}^{+L} ds dt \Delta_{L^\pm}(k + p; s, t) \Delta_{R^\pm}(k; t, s) \right\}. \quad (6.5)
\]

Using the orthogonality of the generalized “sin” function Eq. (3.27), we perform the integration over \( s, t \) and obtain,

\[
= \int \frac{d^Dk}{i(2\pi)^D} \sum_{\omega(\pm)} \frac{\text{Tr} \left( \gamma^\mu P_R(k' + \gamma^\nu P_R k') \right) + \text{Tr} \left( \gamma^\mu P_L(k' + \gamma^\nu P_R k') \right)}{[M^2 + \omega^2 - (k + p)^2 - i\varepsilon][M^2 + \omega^2 - k^2 - i\varepsilon]}
\]

\[
+ \int \frac{d^Dk}{i(2\pi)^D} \sum_{\omega(\pm)} \frac{(\omega^2 + M^2) \text{Tr} \left( \gamma^\mu \gamma^\nu \right)}{[M^2 + \omega^2 - (k + p)^2 - i\varepsilon][M^2 + \omega^2 - k^2 - i\varepsilon]}
\]

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\[
+ \int \frac{d^{d}k}{i(2\pi)^{d}} \left\{ \text{Tr} (\gamma_{\mu} P_{R}(k') + \gamma_{\nu} P_{L} k') \right\} \int_{-L}^{+L} dsdt \Delta_{R\pm}(k + p; s, t) \Delta_{L\mp}(k; t, s) \\
+ \text{Tr} (\gamma_{\mu} P_{L}(k') + \gamma_{\nu} P_{R} k') \int_{-L}^{+L} dsdt \Delta_{R\pm}(k + p; s, t) \Delta_{R\mp}(k; t, s) \right\} \\
= \int \frac{d^{d}k}{i(2\pi)^{d}} \sum_{\omega(\pm)} \frac{\text{Tr} (\gamma_{\mu}(k') + \gamma_{\nu}) \gamma_{\nu} k')}{[M^2 + \omega^2 - (k + p)^2 - i\varepsilon] [M^2 + \omega^2 - k^2 - i\varepsilon]} \\
+ \int \frac{d^{d}k}{i(2\pi)^{d}} \sum_{\omega(\pm)} \frac{(\omega^2 + M^2) \text{Tr} (\gamma_{\mu} \gamma_{\nu})}{[M^2 + \omega^2 - (k + p)^2 - i\varepsilon] [M^2 + \omega^2 - k^2 - i\varepsilon]} \\
+ \frac{1}{2} \int \frac{d^{d}k}{i(2\pi)^{d}} \left\{ \text{Tr} (\gamma_{\mu} k' \gamma_{\nu} k') \right\} \Delta_{\pm}(k + p, k),
\]

Therefore we can write the two-point function as follows.

\[
\Pi_{\pm}(p, L) = \int \frac{d^{d}k}{i(2\pi)^{d}} \left\{ \text{Tr} (\gamma_{\mu}(k') + \gamma_{\nu}) \gamma_{\nu} k') \right\} \mathcal{K}_{\pm}(k + p, k) \\
+ \int \frac{d^{d}k}{i(2\pi)^{d}} \left\{ \text{Tr} (\gamma_{\mu} \gamma_{\nu}) \right\} \mathcal{B}_{\pm}(k + p, k) \\
- \frac{1}{2} \int \frac{d^{d}k}{i(2\pi)^{d}} \left\{ \text{Tr} (\gamma_{\mu} k' \gamma_{\nu} k') \right\} \mathcal{M}_{\pm}(k + p, k),
\]

where

\[
\mathcal{K}_{\pm}(k + p, k) = \int_{-L}^{+L} dsdt \Delta_{R\pm(L\mp)}(k + p; s, t) \Delta_{R\pm(L\mp)}(k; t, s),
\]

\[
= \sum_{\omega(\pm)} \frac{1}{[M^2 + \omega^2 - (k + p)^2 - i\varepsilon] [M^2 + \omega^2 - k^2 - i\varepsilon]} \\
\mathcal{B}_{\pm}(k + p, k) = \int_{-L}^{+L} dsdt B_{\pm}(k + p; s, t) B_{\pm}(k; s, t)
\]

\[
= \sum_{\omega(\pm)} \frac{(\omega^2 + M^2)}{[M^2 + \omega^2 - (k + p)^2 - i\varepsilon] [M^2 + \omega^2 - k^2 - i\varepsilon]},
\]

\[
\mathcal{M}_{\pm}(k + p, k) = \int_{-L}^{+L} dsdt \left( \Delta_{R\pm}(k + p; s, t) - \Delta_{L\mp}(k + p; s, t) \right) \left( \Delta_{R\pm}(k; t, s) - \Delta_{L\mp}(k; t, s) \right).
\]

From this expression, we find that each term \(\Pi_{+}^{\mu\nu}\) is vector-like, is even-parity and does not have any chiral structure. (Even for the extra term due to the dimensional regularization.) However both \(\Pi_{-}^{\mu\nu}\) and \(\Pi_{+}^{\mu\nu}[+M]\) contain the contribution of the light mode (massless mode in the limit \(L \to \infty\)) given by Eq. (3.17) and Eq. (3.19). On the contrary, \(\Pi_{+}^{\mu\nu}[-M]\) does
not contain such a contribution of the light mode. By this fact, the subtracted two-point function shows the correct chiral normalization: one half of the vacuum polarization of the massless Dirac fermion. We can see it explicitly in the following calculation.

**B. Evaluation of Vacuum Polarization**

By the Sommerfeld-Watson transformation, it is possible to express $\mathcal{K}_\pm(k + p, k)$ and $\mathcal{B}_\pm(k + p, k)$ by the common normal modes. Then we can perform the subtraction explicitly at the finite extent of the fifth dimension.

$\mathcal{K}_\pm(k + p, k; s, t)$ has the following Integrand.

$$F_\pm(\omega) = \frac{n_\pm(\omega)}{[M^2 + \omega^2 - k^2 - i\varepsilon][M^2 + \omega^2 - (k + p)^2 - i\varepsilon]}$$  \hspace{1cm} (6.14)

where

$$n_\pm(\omega) = \left[ \left( \frac{\omega \cot \omega L \pm M}{2} \right) + \left( \frac{\omega \cot \omega L - M}{2} \right) \right] (L - \frac{\sin 2\omega L}{2\omega} + \sin^2 \omega L)$$  \hspace{1cm} (6.15)

The poles and its residues are summarized in Table IV.

| singular part in $\mathcal{K}_\pm$ | pole | residue | $\frac{2\omega}{\sin^2 \omega L \Delta_\pm(\omega)}$ |
|-----------------------------------|------|---------|----------------------------------|
| $\frac{\sin^2 \omega L \Delta_\pm(\omega)}{\sin^2 \omega L \Delta_\pm(\omega)}$ | $\sin^2 \omega L \Delta_\pm(\omega) = 0$ | $\frac{1}{n_\pm(\omega)}$ | 1 |
| $\frac{1}{M^2 + \omega^2 - p^2 - i\varepsilon}$ | $iP \equiv i\sqrt{M^2 - p^2 - i\varepsilon}$ | $\frac{1}{2iP}$ | $\frac{2P}{-\sinh^2 PL \Delta_\pm(iP)}$ |

By the Sommerfeld-Watson transformation, $\mathcal{K}_\pm(k + p, k)$ can be rewritten as

$$\mathcal{K}_\pm(k + p, k) = \sum_{\sin \omega L = 0} \frac{2}{[M^2 + \omega^2 - (k + p)^2 - i\varepsilon][M^2 + \omega^2 - k^2 - i\varepsilon]}$$

$$+ \frac{n_\pm(iP)}{\sinh^2 PL \Delta_\pm(iP)} \frac{1}{K^2 - P^2} + \frac{n_\pm(iK)}{\sinh^2 KL \Delta_\pm(iK)} \frac{1}{P^2 - K^2}.$$  \hspace{1cm} (6.16)

Similarly we obtain,

$$\mathcal{B}_\pm(k + p, k) = \sum_{\sin \omega L = 0} \frac{2(\omega^2 + M^2)}{[M^2 + \omega^2 - (k + p)^2 - i\varepsilon][M^2 + \omega^2 - k^2 - i\varepsilon]}$$

$$+ \frac{n_\pm(iP)}{\sinh^2 PL \Delta_\pm(iP)} \frac{M^2 - P^2}{K^2 - P^2} + \frac{n_\pm(iK)}{\sinh^2 KL \Delta_\pm(iK)} \frac{M^2 - K^2}{P^2 - K^2}.$$  \hspace{1cm} (6.17)

From these results, we see that the subtraction can be performed rather simply.
In order to evaluate the remaining terms in the limit $L \to \infty$, we need to know the limit of $\Delta_{\pm}(iP)$ and $\frac{n_{\pm}(iP)}{\sinh^2 PL}$. It is given by

$$\Delta_{\pm}(iP) = M^2 - P^2 - (P \coth PL \pm M)(P \coth PL - M)$$  \hspace{1cm} (6.18)

$$= \lim_{L \to \infty} \begin{cases} 2(M - P)(M + P) \\
2(M - P)P \end{cases}$$ \hspace{1cm} (6.19)

$$\frac{n_{\pm}(iP)}{\sinh^2 PL} = \left[ \frac{(P \coth PL \pm M) + (P \coth PL - M)}{2} \right] \left( L - \frac{\sinh^2 PL}{2P} \right) - \frac{\sinh^2 PL}{2P}$$  \hspace{1cm} (6.20)

$$= \lim_{L \to \infty} \begin{cases} \frac{-2}{-2P + M} \\
\frac{-2}{P} \end{cases}$$ \hspace{1cm} (6.21)

Then we have

$$\bar{K}(k + p, k) \equiv K_{\pm}(k + p, k) - \frac{1}{2} \{ K_{-}(k + p, k) + K_{-}(k + p, k)[M \to -M] \}$$ \hspace{1cm} (6.22)

$$= \lim_{L \to \infty} \frac{1}{2} \left\{ \frac{1}{P^2 - M^2} \left[ K^2 - M^2 \right] - \frac{1}{P^2K^2} \right\}.$$ \hspace{1cm} (6.23)

$$\bar{B}(k + p, k) \equiv B_{\pm}(k + p, k) - \frac{1}{2} \{ B_{-}(k + p, k) + B_{-}(k + p, k)[M \to -M] \}$$ \hspace{1cm} (6.24)

$$= \lim_{L \to \infty} \frac{1}{2} \left\{ -\frac{M^2}{P^2K^2} \right\}.$$ \hspace{1cm} (6.25)

Using these results, the two-point function is written as

$$\lim_{L \to \infty} \Pi_{\mu\nu}^\pm (p, L) = \frac{1}{2} \int \frac{d^Dk}{i(2\pi)^D} \left\{ \frac{\Tr (\gamma_\mu (k^\nu + p) \gamma_\nu k^\nu)}{[-(k + p)^2][-k^2]} - \frac{\Tr (\gamma_\mu (k^\nu + p) \gamma_\nu k^\nu) + \Tr (\gamma_\mu \gamma_\nu) M^2}{[M^2 - (k + p)^2][M^2 - k^2]} \right\}$$

$$- \lim_{L \to \infty} \frac{1}{2} \int \frac{d^Dk}{i(2\pi)^D} \left\{ \Tr \left( \gamma_\mu k^\nu \gamma_\nu k^\nu \right) \right\} \mathcal{M}_{\pm}(k + p, k).$$ \hspace{1cm} (6.26)

The first term in the r.h.s. is nothing but the contribution of massless Dirac fermion subtracted by one Pauli-Villars-Gupta bosonic spinor field with mass $M$, except for the factor one half before it. It is gauge invariant (even under the dimensional regularization). This factor gives the correct normalization of the vacuum polarization derived from the chiral determinant.

The remaining term is due to the dimensional regularization. In four dimensions, we can show that it gives a finite term proportional to $M^2$, which means quadratic divergence in the limit $M \to \infty$. It breaks gauge invariance.

The determinant of $K$, however, is expected to be gauge invariant in the lattice regularization as we can see from Eq. (2.48). Of course, this fact is not yet established at the
perturbative level. We need careful investigation of the two-point function \textit{in the continuum limit of the lattice theory}.

As far as the continuum limit theory is concerned, the above result tells that our choice of the dimensional regularization is not suitable for the calculation of the part of the determinant of $K$. Besides this failure due to the dimensional regularization, we think that our continuum limit analysis so far have clarified the structure of the vacuum overlap formula by taking the limit from at finite extent of the fifth dimension.

\section*{VII. SUMMARY AND DISCUSSION}

We have formulated the perturbation theory of the vacuum overlap formula, based on the theory of the fermion (with kink-like and homogeneous masses) in the finite extent of the fifth dimension. The chiral projection entered the boundary condition of the fermion field in the fifth direction. Different series of discrete normal modes of the fifth momentum occurred and they were rearranged by the Sommerfeld-Watson transformation. We have assumed that the dimensional regularization preserves the cluster property.

The gauge non-invariance introduced by the boundary state wave function actually led to the consistent anomaly. The normalization of the vacuum polarization is a half of the massless Dirac fermion. This is because both the fermion with kink-like mass and the fermion with positive homogeneous mass involve the light (massless) modes but the fermion with negative homogeneous mass does not.

We find that the dimensional regularization is not suitable as a subsidiary regularization. It cannot respect the chiral boundary condition and it induced a gauge non-invariant piece in the vacuum polarization of the four dimensional theory. The determinant of $K$, however, is expected to be gauge invariant in the lattice regularization as we can see from Eq. (2.48). Of course, this important point is not yet established at the perturbative level. We need careful investigation of the two-point function \textit{in the continuum limit of the lattice theory}.

Finally we make a comment about the case of the two-dimensional theory. In this case, the subtle point due to the dimensional regularization does not cause any difficulty. The \textit{once-subtraction by the Pauli-Villars-Gupta bosonic spinor field} is enough to make the two-point function finite. We find that the two-point function from the volume contribution is gauge invariant and has the correct chiral normalization. We also find that the boundary term reproduces the consistent anomaly.

Therefore the next desired step is to examine the perturbative aspect of the vacuum overlap in the lattice regularization in four dimensions. We hope that the technique developed in the continuum limit analysis given here may be also useful in the lattice case.
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APPENDIX A: ORTHOGONALITY OF GENERALIZED “SIN” FUNCTION

The orthogonality of the generalized “sin” function

\[
\int_{-L}^{L} ds \left[ \sin(\omega(L-s)) \right]_{\pm} \left[ \sin(\omega'(L-s)) \right]_{\pm} = N_{\pm}(\omega) \delta_{\omega'\omega}, \tag{A1}
\]

can be shown as follows.

\[
\int_{-L}^{L} ds \left[ \sin(\omega(L-s)) \right]_{\pm} \left[ \sin(\omega'(L-s)) \right]_{\pm} = \int_{0}^{L} ds \sin(\omega(L-s)) \sin(\omega'(L-s)) + \int_{0}^{L} ds \left[ \sin(\omega L - s) \sin(\omega'(L-s)) \right]_{\pm}
\]

\[
= \frac{1}{(\omega \cot \omega L \pm M)(\omega' \cot \omega' L \pm M)} \left( (\omega' + M^2) \left[ \frac{\sin(\omega - \omega')L}{2(\omega - \omega')} - \frac{\sin(\omega + \omega')L}{2(\omega + \omega')} \right] + \frac{\sin(\omega + \omega')L}{(\omega + \omega')} \right)
\]

\[
= \frac{1}{(\omega^2 - \omega'^2)(\omega \cot \omega L \pm M)(\omega' \cot \omega' L \pm M)} \times \left[ \sin \omega L (\omega' \cos \omega' L - M \sin \omega' L) - (\omega \cos \omega L - M \sin \omega L) \sin \omega' L \right]
\]

\[
+ (\omega' + M^2) \times \left[ \sin \omega L (\omega' \cos \omega' L \pm M \sin \omega' L) - (\omega \cos \omega L \pm M \sin \omega L) \sin \omega' L \right] + \omega '\left[ \sin \omega L \cos \omega' L + \cos \omega L \sin \omega' L \right] \pm M(\omega^2 - \omega'^2) \sin \omega L \sin \omega' L \tag{A2}
\]

using Eq. (3.15a)
\[ \begin{align*}
\times & \left( (\omega^2 + M^2) [(\omega \cos \omega L \pm M \sin \omega L) \sin \omega\prime L] \\
- & (\omega^2 + M^2) [\sin \omega L (\omega' \cos \omega' L \pm M \sin \omega' L)] \\
+ & (\omega \omega' + M^2) \\
\times [\sin \omega L (\omega' \cos \omega' L \pm M \sin \omega' L) - (\omega \cos \omega L \pm M \sin \omega L) \sin \omega' L] \\
+ & \omega \omega' (\omega - \omega') [\sin \omega L \cos \omega L + \cos \omega \sin \omega' L] \\
\pm & M (\omega^2 - \omega'^2) \sin \omega L \sin \omega' L \\
= & N_{\pm}(\omega) \delta_{\omega \omega'}.
\end{align*} \]

The normalization factor \( N_{\pm}(\omega) \) is evaluated by also using Eq. (3.19a) as

\[
N_{\pm}(\omega) = \left\{ 1 + \frac{(\omega^2 + M^2)}{(\omega \cot \omega L \pm M)^2} \right\} \frac{1}{2} \left( L - \frac{\sin 2\omega L}{2\omega} \right)
\]

\[
+ \frac{1}{(\omega \cot \omega L \pm M)^2} \left\{ \frac{1}{2} \sin 2\omega L \pm M \frac{M}{2} (1 - \cos 2\omega L) \right\}
\]

\[
= \left[ \frac{(\omega \cot \omega L \pm M) + (\omega \cot \omega L - M)}{(\omega \cot \omega L \pm M)} \right] \frac{1}{2} \left( L - \frac{\sin 2\omega L}{2\omega} \right) + \frac{\sin^2 \omega L}{(\omega \cot \omega L \pm M)}
\]

\[
\equiv n_{\pm}(\omega) \frac{1}{(\omega \cot \omega L \pm M)}.
\]

**APPENDIX B: CORRELATION FUNCTIONS BETWEEN BOUNDARIES**

In this appendix, we perform the calculation of the correlation functions between boundaries of the order \( n \geq 1 \) in the perturbative expansion given in Sec. IV. We do not take into account of the breaking of the chiral boundary condition due to the dimensional regularization. In the momentum space, it reads

\[
\left( \begin{array}{c}
..P_R \\
..P_L
\end{array} \right) \left[ i \sum_{n=1}^{\infty} \{ S_{F_{\pm}} \cdot (\cdot A_{\gamma}) \}^n S_{F_{\pm}} \right] \left( \begin{array}{c}
P_L.. \\
P_R.
\end{array} \right)
\]

\[
= \sum_{n=1}^{\infty} \int \frac{d^4k}{(2\pi)^4} \prod_{i=1}^{n} \frac{d^4p_i}{(2\pi)^4} \exp \{-ik_1x + ik_{n+1}y\}
\]

\[
\left( \begin{array}{c}
..P_R \\
..P_L
\end{array} \right) \left[ i \prod_{i=1}^{n} \{ S_{F_{\pm}}(k_i) \cdot (\cdot A_{\gamma^i}) \} S_{F_{\pm}}(k) \right] \left( \begin{array}{c}
P_L.. \\
P_R.
\end{array} \right) \prod_{i=1}^{n} A_{\gamma_i}(p_i),
\]

where \( p_i \) is assumed to be the momentum incoming from the external gauge boson attached to the vertex \( \gamma^i \) and \( k_i = k + \sum_{j-i} p_j \).

We first perform the integration over the extra coordinates at every vertices of external gauge field, with the help of the orthogonality of the generalized “sin”,

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\[
\int_{-L}^{L} ds \left[ \sin \omega(L - s) \right] \pm \left[ \sin \omega'(L - s) \right] \pm = N_\pm(\omega) \delta_{\omega \omega'} .
\]

Then we obtain
\[
.._P \left[ \prod_{i=1}^{n} \left\{ S_{F_\pm}(k_i) \cdot (-\gamma^\nu_i) \cdot S_{F_\pm}(k_{i+n+1}) \right\} \right] P_L.. = (i)^n \sum_{0 \leq 2l \leq n+1} P_R C_{2l}^{m} \Delta_{R_\pm}^{(n+1,2l)}, \quad (B3)
\]
\[
.._P \left[ \prod_{i=1}^{n} \left\{ S_{F_\pm}(k_i) \cdot (-\gamma^\nu_i) \cdot S_{F_\pm}(k_{i+n+1}) \right\} \right] P_R.. = (i)^n \sum_{0 \leq 2l \leq n+1} P_L C_{2l}^{m} \Delta_{L-}^{(n+1,2l)}, \quad (B4)
\]
\[
.._P \left[ \prod_{i=1}^{n} \left\{ S_{F_\pm}(k_i) \cdot (-\gamma^\nu_i) \cdot S_{F_\pm}(k_{i+n+1}) \right\} \right] P_R.. = (i)^n \sum_{1 \leq 2l+1 \leq n+1} P_R C_{2l+1}^{m} B_{\pm}^{(n+1,2l)}, \quad (B5)
\]
\[
.._P \left[ \prod_{i=1}^{n} \left\{ S_{F_\pm}(k_i) \cdot (-\gamma^\nu_i) \cdot S_{F_\pm}(k_{i+n+1}) \right\} \right] P_L.. = (i)^n \sum_{1 \leq 2l+1 \leq n+1} P_L C_{2l+1}^{m} B_{\pm}^{(n+1,2l)}, \quad (B6)
\]
and
\[
\Delta_{R_\pm}^{(n+1,2l)} = \sum_{\omega} \frac{1}{n_\pm(\omega) (\omega \cot \omega L \pm M)} (\omega^2 + M^2)^l \prod_{i=1}^{n+1} \frac{1}{[M^2 + \omega^2 - k_i^2 - i\varepsilon]}, \quad (B7)
\]
\[
\Delta_{L-}^{(n+1,2l)} = \sum_{\omega} \frac{1}{n_\pm(\omega) (\omega \cot \omega L - M)} (\omega^2 + M^2)^l \prod_{i=1}^{n+1} \frac{1}{[M^2 + \omega^2 - k_i^2 - i\varepsilon]}, \quad (B8)
\]
\[
B_{\pm}^{(n+1,2l)} = \sum_{\omega} \frac{1}{n_\pm(\omega)} (-\omega^2) (\omega^2 + M^2)^l \prod_{i=1}^{n+1} \frac{1}{[M^2 + \omega^2 - k_i^2 - i\varepsilon]} . \quad (B9)
\]

\( C_0^m \) is the product of the gamma matrices defined by
\[
C_0^m = \gamma_1^\nu_1 \gamma_2^\nu_2 \ldots \gamma_{l-2}^\nu_{l-2} \gamma_{l-1}^\nu_{l-1} \gamma_{l}^\nu_{l} \gamma_{l+1}^\nu_{l+1} ,
\]
and \( C_0^m \) is defined as the summation over the possible products of the gamma matrices which can be obtained from \( C_0^m \) by replacing \( m \)-number of \( i \) by the unit matrix.

The summations over the normal modes of \( \omega \) in \( \Delta_{R_\pm}^{(n+1,2l)} \), \( \Delta_{L-}^{(n+1,2l)} \) and \( B_{\pm}^{(n+1,2l)} \) can be performed by the use of the technique of the Sommerfeld-Watson transformation. Referring to the Tables I and II, we obtain
\[
\Delta_{R_\pm}^{(n+1,2l)} = \sum_{\omega} \frac{1}{n_\pm(\omega) (\omega \cot \omega L \pm M)} (\omega^2 + M^2)^l \prod_{i=1}^{n+1} \frac{1}{[M^2 + \omega^2 - k_i^2 - i\varepsilon]}
\]
\[
= - \sum_{i=1}^{n+1} \frac{1}{\sinh^2 K_i L \Delta_\pm (iK_i)} K_i^2 (M^2 - K_i^2)^l \prod_{j \neq i} \frac{1}{K_j^2 - K_i^2}
\]
\[
+ \sum_{\omega \cot \omega L \pm M} \frac{2 \sin^2 \omega L}{\omega^2 (L - \sin 2\omega L)} \omega^2 (\omega^2 + M^2)^l \prod_{i=1}^{n+1} \frac{1}{[M^2 + \omega^2 - k_i^2 - i\varepsilon]}
\]
\[
= - \sum_{i=1}^{n+1} \frac{1}{\sinh^2 K_i L \Delta_\pm (iK_i)} K_i^2 (M^2 - K_i^2)^l \prod_{j \neq i} \frac{1}{K_j^2 - K_i^2}
\]
\[
+ \sum_{i=1}^{n+1} \frac{(M^2 - K_i^2)^l}{(K_i \cot K_i L \pm M)} \prod_{j \neq i} \frac{1}{K_j^2 - K_i^2} . \quad (B11)
\]

53
\[ B^{(n+1,2l)}_\pm = \sum_\omega \frac{1}{n_\pm(\omega)} (-\omega^2) (\omega^2 + M^2)^l \prod_{i=1}^{n+1} \left[ \frac{1}{\omega^2 + M^2 - k_i^2 - i\varepsilon} \right] \]
\[ = \sum_{i=1}^{n+1} \frac{1}{\sinh^2 K_i L \Delta_\pm(iK_i)} K_i^2 (M^2 - K_i^2)^l \prod_{j \neq i} \frac{1}{K_j^2 - K_i^2} ; \]  
(B12)

In the limit \( L \to \infty \), we have
\[ \lim_{L \to \infty} \Delta^{(n+1,2l)}_{R\pm} = \sum_{i=1}^{n+1} \frac{(M^2 - K_i^2)^l}{(K_i \pm M)} \prod_{j \neq i} \frac{1}{K_j^2 - K_i^2} ; \]  
(B13)
\[ \lim_{L \to \infty} \Delta^{(n+1,2l)}_{L\pm} = \sum_{i=1}^{n+1} \frac{(M^2 - K_i^2)^l}{(K_i - M)} \prod_{j \neq i} \frac{1}{K_j^2 - K_i^2} ; \]  
(B14)
\[ \lim_{L \to \infty} B^{(n+1,2l)}_\pm = 0 , \]  
(B15)

Therefore we obtain
\[ \lim_{L \to \infty} \begin{pmatrix} \ldots P_R \\ P_L \end{pmatrix} \left[ i \prod_{i=1}^{n} \{ S_{F\pm}(k_i) \cdot (-\gamma^\nu) \cdot \} S_{F\pm}(k_{n+1}) \right] \begin{pmatrix} P_L \ldots P_R. \end{pmatrix} \]
\[ = (i)^n \sum_{n=1}^{\infty} \begin{pmatrix} P_R V_{\pm^n} (k_1, k_2, \ldots, k_{n+1}) P_L \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ P_L V_{-\pm^n} (k_1, k_2, \ldots, k_{n+1}) P_R \end{pmatrix} , \]  
(B16)

where
\[ V_{\pm^n} (k_1, k_2, \ldots, k_{n+1}) = \sum_{0 \leq 2l \leq n+1} C_{2l}^n \sum_{i=1}^{n+1} \frac{(M^2 - K_i^2)^l}{(K_i \pm M)} \prod_{j \neq i} \frac{1}{K_j^2 - K_i^2} . \]  
(B17)

This result shows that the cluster property holds:
\[ \lim_{L \to \infty} \begin{pmatrix} \ldots P_R \\ P_L \end{pmatrix} \left[ i \sum_{n=1}^{\infty} \{ S_{F\pm} \cdot (-\gamma^\nu) \}^n S_{F\pm} \right] \begin{pmatrix} P_L \ldots P_R. \end{pmatrix} \]
\[ = \lim_{L \to \infty} \begin{pmatrix} \ldots P_R \left[ i \sum_{n=1}^{\infty} \{ S_{F\pm} \cdot (+M) \cdot (-\gamma^\nu) \}^n S_{F\pm} \cdot (+M) \right] P_L \ldots \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ P_L \left[ i \sum_{n=1}^{\infty} \{ S_{F\pm} \cdot (+M) \cdot (-\gamma^\nu) \}^n S_{F\pm} \cdot (+M) \right] P_R. \end{pmatrix} , \]  
(B18)

For \( n = 1, 2, 3 \), the explicit form of \( V_{\pm^n}^{\mu} \) is given as follows. For \( n = 1 \),
\[ V_{\pm}^{\mu} (k + p, k) (P + K) = \left[ (\gamma^\nu \gamma^\mu) \gamma^\nu \gamma^\mu \right] \frac{1}{[P \pm M][K \pm M]} + \gamma^\mu . \]  
(B19)
\( p \) is assumed to be the momentum incoming from the external gauge boson attached to the vertex \( \gamma^\mu \). \( P \) and \( K \) is defined as \( P = \sqrt{M^2 - (k + p)^2 - i\varepsilon} \) and \( K = \sqrt{M^2 - k^2 - i\varepsilon} \).

For \( n = 2 \),
\[
V^\mu_\nu (k + p, k, k - q) (P + K)(K + Q)(Q + P) = \frac{[\gamma^\mu \gamma^\nu (k - q)]}{[P + K + Q \pm M]} \left[ \begin{array}{c} P + K + Q \pm M \\ P \pm M, K \pm M, Q \pm M \end{array} \right]
+ \left[ \gamma^\mu \gamma^\nu (k - q) \right] (P + K + Q + R)
\]
\[
= \frac{[\gamma^\mu \gamma^\nu (k - q)]}{[P + K + Q \pm M]} \left[ \begin{array}{c} P + K + Q \pm M \\ P \pm M, K \pm M, Q \pm M \end{array} \right]
+ \left[ \gamma^\mu \gamma^\nu (k - q) \right] (P + K + Q + R)
\]  \quad (B20)

\( q \) is assumed to be the momentum incoming from the external gauge boson attached to the vertex \( \gamma^\nu \) and \( Q \) is defined by \( Q = \sqrt{M^2 - (k - q)^2 - i\varepsilon} \).

For \( n = 3 \),
\[
V^\mu_\nu_\lambda (k + p, k, k - q, k - q - r) (P + K)(P + Q)(P + R)(K + Q)(K + R)(Q + R) = \frac{[\gamma^\mu \gamma^\nu \gamma^\lambda (k - q)]}{[P + K + Q \pm M]} \left[ \begin{array}{c} P + K + Q \pm M \\ P \pm M, K \pm M, Q \pm M, R \pm M \end{array} \right]
+ \left[ \gamma^\mu \gamma^\nu \gamma^\lambda (k - q) \right] (P + K + Q + R)
\]
\[
+ \left[ \gamma^\mu \gamma^\nu \gamma^\lambda \right] B(P, K, Q, R)
\]  \quad (B21)

and
\[
A_\pm (P, K, Q, R) = P^2 (Q + R + K) + Q^2 (P + R + K) + R^2 (P + Q + K) + K^2 (P + Q + R)
+ 2 (QRK + PRK + PQK + PQR)
\]
\[
\pm M (P + K + Q + R)^2 + M^2 (P + K + Q + R),
\]  \quad (B22)

\[
B(P, K, Q, R) = (QRK + PRK + PQK + PQR) + M^2 (P + K + Q + R).
\]  \quad (B23)

\( r \) is assumed to be the momentum incoming from the external gauge boson attached to the vertex \( \gamma^\lambda \) and \( R \) is defined by \( R = \sqrt{M^2 - (k - q - r)^2 - i\varepsilon} \).
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