Are scattering amplitudes dual to super Wilson loops?

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Abstract

The MHV scattering amplitudes in planar $\mathcal{N} = 4$ SYM are dual to bosonic light-like Wilson loops. We explore various proposals for extending this duality to generic non-MHV amplitudes. The corresponding dual object should have the same symmetries as the scattering amplitudes and be invariant to all loops under the chiral half of the $\mathcal{N} = 4$ superconformal symmetry. We analyze the recently introduced supersymmetric extensions of the light-like Wilson loop (formulated in Minkowski space-time) and demonstrate that they have the required symmetry properties at the classical level only, up to terms proportional to field equations of motion. At the quantum level, due to the specific light-cone singularities of the Wilson loop, the equations of motion produce a nontrivial finite contribution which breaks some of the classical symmetries. As a result, the quantum corrections violate the chiral supersymmetry already at one loop, thus invalidating the conjectured duality between Wilson loops and non-MHV scattering amplitudes. We compute the corresponding anomaly to one loop and solve the supersymmetric Ward identity to find the complete expression for the rectangular Wilson loop at leading order in the coupling constant. We also demonstrate that this result is consistent with conformal Ward identities by independently evaluating corresponding one-loop conformal anomaly.
1 Introduction

Recent studies revealed that scattering amplitudes in planar \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory (SYM) possess a new hidden symmetry \[1, 2\]. The latter appears in addition to the conventional superconformal symmetry of the Lagrangian of the theory and leads to powerful constraints on the form of the all-loop scattering amplitudes.

A distinguishing feature of \( \mathcal{N} = 4 \) SYM is that all on-shell states (gluons with helicity \( \pm 1 \), gluinos with helicity \( \pm 1/2 \) and scalars) can be encoded in a single superstate \( \Phi(p_i, \eta_i) \) \[3, 4\]. States with different helicities appear as coefficients in the expansion of the superstate in powers of the odd variables \( \eta_i^A \) (with \( A = 1, 2, 3, 4 \)). As a consequence, all \( n \)-particle scattering amplitudes can be combined into a single superamplitude \( \mathcal{A}_n \). Supersymmetry restricts the form of the superamplitude to be

\[
\mathcal{A}_n = \mathcal{A}_n^{\text{MHV}} + \mathcal{A}_n^{\text{NMHV}} + \ldots + \mathcal{A}_n^{N^{0-4}\text{MHV}},
\]

where the \( k \)th term in the sum, \( \mathcal{A}_n^{N^k\text{MHV}} \), describes the scattering amplitudes with total helicity of the particles \((-n + 4 + 2k)\). It has the following general form \[4\]

\[
\mathcal{A}_n^{N^k\text{MHV}} = i(2\pi)^4 \delta^{(4)}(p^{\dot{\alpha}}) \delta^{(8)}(q^A) \langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle \widehat{\mathcal{A}}_{n;k}(\lambda, \tilde{\lambda}, \eta; a),
\]

where \( p^{\dot{\alpha}} = \sum_i p_i^{\dot{\alpha}} \) and \( q^A = \sum_i \lambda^A_i \eta^A_i \) is the total momentum and chiral supercharge of the \( n \) particles. Here we used the spinor-helicity formalism to parameterize the light-like momenta in terms of two-component commuting spinors \( p_i^{\dot{\alpha}} = \bar{\lambda}_i^\dot{\alpha} \lambda^\alpha_i \) and the angular brackets \( \langle jk \rangle \) are defined in Appendix A. The dependence of the scattering amplitude on the ’t Hooft coupling constant \( a = g^2 N_c/(4\pi^2) \) is carried by the nontrivial function \( \widehat{\mathcal{A}}_{n;k}(\lambda, \tilde{\lambda}, \eta; a) \) which is given by a homogenous polynomial in the \( \eta \)'s of degree \((4k)\). At tree level, the function \( \widehat{\mathcal{A}}_{n;k}(\lambda, \tilde{\lambda}, \eta) \) enjoys the full \( \text{PSU}(2, 2|4) \) superconformal symmetry of the \( \mathcal{N} = 4 \) SYM, while at loop level the dilatations, the special conformal boosts and their supersymmetric extension are broken by infrared divergences \[5\].

As we already mentioned, the superamplitude in planar \( \mathcal{N} = 4 \) SYM has another, hidden symmetry. This symmetry, called “dual superconformal symmetry” in Ref. \[1\], acts naturally on the so-called region supermomenta \((x_i, \theta_i)\) defined as

\[
p_i^{\dot{\alpha}} = \bar{\lambda}_i^\dot{\alpha} \lambda^\alpha_i = (x_i - x_{i+1})^{\dot{\alpha}}, \quad \lambda^\alpha_i \eta^A_i = (\theta_i - \theta_{i+1})^\alpha A.
\]

In spite of the fact that \((x_i, \theta_i)\) have the meaning of momenta, the dual symmetry acts on them as if they were coordinates in configuration (rather than momentum) space \[6\] \[7\] \[1\]. At tree level, the superamplitude \( \mathcal{A}_n^{(0)} \) enjoys exact dual superconformal symmetry, while at loop level some of the dual symmetries become anomalous (see below).

Dual superconformal symmetry emerges as a hidden property of the scattering amplitudes in planar \( \mathcal{N} = 4 \) SYM. For the simplest, maximally helicity violating superamplitude \( \mathcal{A}_n^{\text{MHV}} \), the dual conformal symmetry becomes manifest through the conjectured duality between the functions \( \widehat{\mathcal{A}}_{n,0} \) defining the perturbative loop corrections to the MHV superamplitude, Eq. \(2\),
\[ \ln \hat{A}_{n;0} = \ln W_n + O(\varepsilon), \quad W_n = \frac{1}{N_c} \left\langle \text{tr} \exp \left( ig \int_{C_n} dx \cdot A(x) \right) \right\rangle, \tag{4} \]

where \( O(\varepsilon) \) stands for terms vanishing in dimensional regularization and the Wilson loop is evaluated along the polygon light-like contour \( C_n = [x_1, x_2] \cup [x_2, x_3] \ldots [x_n, x_1] \) formed by the particle momenta \( p_i = x_i - x_{i+1} \). The power of the duality relation (4) exhibits itself through the fact that the dual conformal symmetry of the MHV superamplitude is mapped into the conventional conformal symmetry of the Wilson loop in \( \mathcal{N} = 4 \) SYM. First hinted at in Ref. [10] by the strong coupling analysis for \( n = 4, 5 \) via the AdS/CFT correspondence, the duality between the two objects was understood as the invariance of the string sigma model on an AdS5 × S5 background under \( T \)-duality transformations for the bosonic [11] and fermionic variables [2]. At weak coupling the proposal (4) was put forward in Ref. [9] and confirmed at one loop for an arbitrary number of cusps \( n \) in Ref. [12] and at two loops for \( n = 4, 5 \)–point Wilson loops [13, 14] by confronting these predictions with the available results for gluon scattering amplitudes [15, 16]. At the moment, the most thorough test of the duality (4) comes from the comparison of the two-loop results for the \( n = 6 \) (hexagon) Wilson loop [17, 18, 19, 20] with the six-gluon MHV scattering amplitude computed to the same order in the coupling [21, 22].

The MHV superamplitude is the simplest among all superamplitudes in (1). In particular, the corresponding function \( \hat{A}_{n;0} \) defined in Eq. (2) only depends on the bosonic variables and is independent of the odd \( \eta \)-variables. At tree level, it is given by \( \hat{A}_{n;0} = 1 + O(a) \) and, according to the duality relation (4), its loop corrections coincide (upon appropriate identification of the regularization parameters) with those of the bosonic light-like Wilson loop \( W_n \). A natural question arises whether the duality relation (4) can be extended to the full superamplitude \( A_n \). If such a duality relation exists, the bosonic Wilson loop in the right-hand side of (4) should appear as the first term in the expansion of the dual object \( \mathcal{W}_n \) in powers of the \( \eta \)-variables analogous to (1). Viewed as a function of the dual coordinates \((x_i, \theta_i)\), it should possess the same symmetry as the superamplitude. Namely, the duality between \( A_n \) and \( \mathcal{W}_n \) would imply that the dual superconformal symmetry of the former follows from the conventional superconformal symmetry of the latter in \( \mathcal{N} = 4 \) SYM. Moreover, since some of the symmetries of the superamplitudes are broken at the quantum level, \( \mathcal{W}_n \) should have the same anomalies. Regarding the \( \mathcal{N} = 4 \) supersymmetry, the dual object \( \mathcal{W}_n \) should have the following unusual property – it depends only on the chiral \( \theta \)-variables but not on \( \bar{\theta} \). The reason for this can be traced back to the chiral formulation of scattering amplitudes in the on-shell superspace formalism. In addition, a serious complication is that \( \mathcal{N} = 4 \) SYM only exists on shell, i.e. the algebra of the supersymmetry transformations closes modulo field equations. This makes the construction of \( \mathcal{W}_n \) somewhat tricky and, as we will see later in the paper, leads to various subtleties.

Recently, two different proposals for \( \mathcal{W}_n \) were put forward. In Ref. [23] Mason and Skinner put forward a formulation of \( \mathcal{W}_n \) in twistor space. Yet another form of \( \mathcal{W}_n \), this time in Minkowski space, was suggested by Caron-Huot in Ref. [24]. It was claimed in [23] that all amplitudes in \( \mathcal{N} = 4 \) SYM are described by a supersymmetric Wilson loop in twistor space and the same statement was apparently implied for its Minkowski transform. The Minkowski version takes the

\footnote{The duality between planar amplitudes and light-like Wilson loops also holds in gauge theories with less or no supersymmetry, including QCD. However, in distinction with \( \mathcal{N} = 4 \) SYM, there the relation (4) holds in the high-energy (Regge) limit only [8, 9].}
following form \[25\]

\[W_n = \frac{1}{N_c} \left\langle \text{tr} \exp \left( ig \int_{C_n} dx^\mu A_\mu(x, \theta) + ig \int_{C_n} d\theta A_{\alpha A}(x, \theta) \right) \right\rangle, \quad (5)\]

where the integration goes over a contour \(C_n\) in superspace formed by \(n\) straight segments connecting the points \((x_i, \theta_i)\) and defined by the supermomenta of the scattered particles \(3\). Here \(A_\mu\) and \(F_{\alpha A}\) are the bosonic and fermionic superspace gauge connections, respectively, given by a \(\theta\)-expansion whose coefficients are the various component fields (gluons, gauginos and scalars) of \(\mathcal{N} = 4\) SYM. Suppressing the dependence on the \(\theta\)-variables, the supersymmetrized Wilson loop \(5\) reduces to the bosonic Wilson loop \(W_n\) in \(4\). Another proposal for \(W_n\) was presented in \(24\) where certain perturbative data and BCFW-like recursion relations \(26, 27\) were used to advocate a particular supersymmetric generalization of the bosonic light-like Wilson loop. The resulting supersymmetric Wilson loop was argued to be equivalent to \(5\). Thus, both proposals suggest the duality \(A_n \sim W_n\).

By construction, the supersymmetric Wilson loop \(W_n\) automatically reproduces the MHV amplitudes through the duality relation \(4\). In addition, it yields a definite prediction for non-MHV amplitudes to all loops. In this paper, we verify the duality between the superamplitudes \(A_n\) and the supersymmetric Wilson loop \(W_n\) by performing one-loop calculations of \(5\) in \(N = 4\) SYM. We show by an explicit one-loop computation that, contrary to the above expectations, the perturbative corrections to \(W_n\) do not match those of the non-MHV amplitudes, thus invalidating the duality relation \(A_n \sim W_n\) beyond the MHV amplitudes. We show that the reason for this has to do with the fact that \(W_n\) is invariant only under on-shell chiral \(\mathcal{N} = 4\) supersymmetric transformations, that is up to contributions proportional to the equations of motion. At loop level, due to the specific light-cone singularities of the Wilson loops \(28\), the equations of motion generate a nontrivial finite effect. As a result, the supersymmetry of \(W_n\) gets broken by quantum corrections. We compute the corresponding anomaly to one-loop order and demonstrate that the result obtained for \(W_n\) is in a perfect agreement with the anomalous supersymmetric Ward identities.

An alternative proposal for the dual description of scattering amplitudes in planar \(\mathcal{N} = 4\) SYM via the light-cone limit of correlation functions of certain half-BPS operators was put forward in Refs. \(29, 30\). It proves to be immune to the problems that one encounters in the construction of the super Wilson loops, as we discuss in the Conclusions.

The paper is organized as follows. After a discussion of the basic symmetries that any proposal for the dual description of superamplitudes should fulfill, in Sect. 2 we introduce the supersymmetric generalizations of the bosonic Wilson loop proposed in Refs. \(23, 24\) and explicitly demonstrate that the two differ by the presence of equations of motion when considered off shell. Then, in Sect. 3 we derive the Ward identities resulting from the the transformation properties of the Wilson loops under Poincaré supersymmetry and calculate the corresponding anomalies. In Sect. 4 we solve the anomalous Ward identities and reconstruct the explicit form of the one-loop correction to the rectangular Wilson loop \(W_4\). Sect. 5 contains concluding remarks. Details on our conventions and normalizations are given in the Appendix A. Finally, the Ward identities corresponding to the conformal transformations of supersymmetric Wilson loop are discussed in Appendix B, where we demonstrate their complete consistency with our findings in the main text.
2 Supersymmetric Wilson loop

As already reviewed in the Introduction, the symmetries of the scattering amplitudes extend far beyond the conventional superconformal symmetry of $\mathcal{N} = 4$ SYM. If a generalization of the Wilson loop dual to superamplitudes exists, both objects will possess the same symmetries. This implies that the dual superconformal symmetry of the amplitudes should match the conventional superconformal $\mathcal{N} = 4$ symmetry of the supersymmetric Wilson loop.

2.1 Matching the symmetries of the $\mathcal{N} = 4$ amplitudes

Let us assume for a moment that the $\mathcal{N} = 4$ superamplitudes are indeed dual to $\mathcal{W}_n$ and let us summarize the constraints imposed by the dual superconformal symmetry of the amplitude $A_n$ on the properties of $\mathcal{W}_n$.

By construction, $\mathcal{W}_n$ depends on the Grassmann variables $\theta_i^A$ carrying $SU(4)$ charges. Then by virtue of $SU(4)$ invariance, it admits an expansion in powers $(\theta_i^A)^k$:

$$\mathcal{W}_n = \mathcal{W}_{n;0}(x; a) + \mathcal{W}_{n;1}(x, \theta_i^A; a) + \ldots + \mathcal{W}_{n;n}(x, \theta_i^A; a).$$

Here $\mathcal{W}_{n;k}$ is a homogeneous $SU(4)$ invariant polynomial of degree $4k$ in the Grassmann variables $\chi_i^A = \langle i \theta_i^A \rangle$ (see Eq. (26) below) and the expansion runs up to $k = n$. The lowest term of the expansion does not depend on the odd variables and, therefore, it coincides with the bosonic Wilson loop $\mathcal{W}_{n;0}(x; a) = W_n$, (see Eq. (4)).

The expansion (6) is similar to that of the scattering amplitude (1). In fact, the conjectured duality between superamplitudes and supersymmetric Wilson loops establishes the correspondence between the two expansions \cite{23, 24}

$$\mathcal{W}_{n;k}(x, \theta; a) = a^k \hat{A}_{n;k}(\lambda, \tilde{\lambda}, \eta; a),$$

where the functions $\hat{A}_{n;k}$ define the perturbative corrections to the $N^k$MHV superamplitude, Eq. (2). For $k = 0$ the relation (7) reproduces the duality between the MHV amplitude and the bosonic Wilson loop, Eq. (3). Notice the appearance of a power of the coupling constant in the right-hand side of (7). The reason for it is that the perturbative expansion of $\mathcal{W}_{n;k}$ starts at order $O(a^k)$. Then, the duality relation (7) leads to the following iterative structure of loop corrections: Having computed $\mathcal{W}_n$ to, say, $k$—loops and expanding the result in powers of $\theta$’s as in (6), we will be able to determine the $k$—loop corrections to the MHV amplitudes, the $(k - 1)$—loop corrections to the NMHV amplitudes and so on until we reach the tree-level expression for $N^k$MHV amplitudes.

We observe that, for a given number of external particles $n$, the expansion in (6) terminates at $k = n - 4$ and, therefore, $\hat{A}_{n;k} = 0$ for $k \geq n - 3$. This property is an immediate consequence of the conventional on-shell Poincaré supersymmetry of the all-loop amplitudes in $\mathcal{N} = 4$ SYM. Then, the duality relation (7) implies that the top four components of the expansion (6) should vanish to all loops

$$\mathcal{W}_{n;n-3}(x, \theta; a) = \mathcal{W}_{n;n-2}(x, \theta; a) = \mathcal{W}_{n;n-1}(x, \theta; a) = \mathcal{W}_{n;n}(x, \theta; a) = 0.$$

\footnote{Following \cite{24}, the factor $a^k$ in the right-hand side of (7) can be eliminated through a redefinition of the odd variables $\eta \rightarrow a^{-1/4} \eta$.}
We recall that the perturbative expansion of $W_{n,n−3}(x, θ; a)$ only starts at $(n − 3)$–loops and accordingly for the remaining functions in this relation. Here we inserted the question mark (see also Eqs. (9), (12) and (13) below) to indicate that these relations have the status of a conjecture. Below we shall check them by explicit one-loop calculations.

Tree-level amplitudes are free from infrared divergences and, as a result, they enjoy both conventional and dual superconformal symmetries. Then, the duality relation (7) predicts that the lowest $O(α̅)$ correction to $W_{n,k}(x, θ; a)$ should respect the dual superconformal symmetry. At loop level, the scattering amplitudes suffer from infrared divergences and some of the symmetries become anomalous. It is paramount for our purposes that the dual $Q$– and $S$–supersymmetries remain unbroken to all loops. The reason for this is as follows. As it is evident from its definition, the $Q$–supersymmetry generates shifts of the odd variables $δQθ_iA = ε^A_i$. Since the scattering amplitudes depend only through the differences (3), they stay invariant under the action of $Q$–supersymmetry. In a similar manner, the dual $S$–supersymmetry coincides with the conventional $q$–supersymmetry of the $N = 4$ Lagrangian. As long as we employ a regularization that preserves Poincaré supersymmetry, $q$–supersymmetry remains unbroken and the same should be true for the dual $S$–supersymmetry, $δS(θ_i)^A = (x_i)_{αα}ξ^A$. Thus, the duality relation (7) implies that the supersymmetric Wilson loop should preserve $Q$– and $S$–supersymmetries to all loops

$$Q^A_n W_n(x, θ; a) = S_{αA} W_n(x, θ; a) = 0.$$  (9)

To discuss the consequence of these relations, it is convenient to introduce new variables [33], the so-called momentum supertwistors $Z^A = (λ_i, μ_i, χ_i)$ with

$$χ_i^A = ⟨iθ_i^A⟩, \quad μ_α = λ^α_i(x_i)_{αα}.$$  (10)

The dual superconformal symmetry acts on $Z^A$ as linear $GL(4|4)$ transformations. In particular, the action of the dual $Q$– and $S$–supersymmetries corresponds to

$$χ^A_i → χ^A_i + ⟨iε^A⟩ + [μ_α, ξ^A],$$  (11)

with $ε^A_α$ and $ξ^A_{αα}$ being the parameters of the corresponding transformations. Using the invariance of the supersymmetric Wilson loop [33] under the transformations (11), we can always choose the parameters $ε^A_α$ and $ξ^A_{αα}$ in such a manner as to set 16 components of the Grassmann variables to zero, e.g., $χ^A_1 = χ^A_3 = χ^A_3 = χ^A_4 = 0$. This immediately implies that the supersymmetric Wilson loop depends on $(n−4)$ odd variables $χ_i$ and, therefore, its expansion (8) terminates at the term $W_{n,n−4}$, in agreement with [33]. In particular, in the special case of $n = 4$, the supersymmetric Wilson loop $W_4$ cannot depend on the odd variables and, therefore, it should coincide with the bosonic Wilson loop

$$W_4(x, θ; a) = W_4(3; a).$$  (12)

This is in accord with the well-known fact that for $n = 4$ particles the only nonzero amplitude in $N = 4$ SYM is the MHV one.

Infrared divergences break the dual conformal symmetry of the amplitudes making special conformal $K$–symmetry anomalous, $K^{αα}A_{n,k} ≠ 0$. Since the infrared divergences have a universal form independent of the helicity configuration of the scattered particles, they cancel in the ratio of amplitudes $A_n/A_{n,0}$. As a consequence, dual conformal symmetry gets restored in
the ratio, \( K^{\hat{\alpha}\hat{\alpha}}(\hat{A}_n/\hat{A}_{n,0}) = 0 \). Together with the duality relation (7) this leads to the following constraint on the supersymmetric Wilson loop

\[
K^{\hat{\alpha}\hat{\alpha}}(\mathcal{W}_n/\mathcal{W}_{n,0}) \equiv 0.
\]

(13)

In other words, the ratio of the supersymmetric Wilson loop and its lowest bosonic component should be invariant under the special conformal transformations in \( \mathcal{N} = 4 \) SYM.

2.2 Supersymmetric connections

Let us start with reviewing the construction of the supersymmetric Wilson loop (5) introduced in Refs. [23, 24]. The super Wilson loop in (5) is given by the path-ordered product of supersymmetric Wilson lines calculated along the segments of the polygon contour \( C_n \)

\[
\mathcal{W}_n = \frac{1}{N_c} \langle \text{tr} (\mathcal{W}_{[1,2]} \mathcal{W}_{[2,3]} \cdots \mathcal{W}_{[n,1]}) \rangle.
\]

(14)

Here the Wilson line \( \mathcal{W}_{[i,i+1]} \) is evaluated along the straight line connecting the points \((x_i, \theta_i)\) and \((x_{i+1}, \theta_{i+1})\) in the chiral superspace

\[
x(t) = x_i - t_i x_{i+1}, \quad \theta(t) = \theta_i - t_i \theta_{i+1}, \quad (0 \leq t_i \leq 1),
\]

(15)

with implied periodicity conditions \( x_{i+n} \equiv x_i \) and \( \theta_{i+n} \equiv \theta_i \). Then, the supersymmetric Wilson line takes the form

\[
\mathcal{W}_{[i,i+1]} = T \exp \left( ig \int_0^1 dt_i B(t_i) \right),
\]

(16)

where the (M)inkowski (S)uperspace connection \( B(t_i) \) is the sum of the bosonic and fermionic connections projected on the tangent direction to the trajectory,

\[
B^{\text{MS}}(t_i) = \frac{1}{2} \dot{x}^{\hat{\alpha}\hat{\alpha}}(t_i) A_{\hat{\alpha} \hat{\alpha}} + \dot{\theta}^{\alpha A}(t_i) F_{\alpha A}.
\]

(17)

The supersymmetric Wilson loop (14) is invariant under non-Abelian gauge transformations

\[
B(t_i) \rightarrow U(t_i) B(t_i) U(t_i)^\dagger + \frac{i}{g} U(t_i)^\dagger \partial_t U, \quad \mathcal{W}_{[i,i+1]} \rightarrow U(t_i)^\dagger \mathcal{W}_{[i,i+1]} U(0),
\]

(18)

with \( U = U(t_i) \). For the bosonic and fermionic connections the gauge transformations look as

\[
A_{\alpha \hat{\alpha}} \rightarrow U(t_i)^\dagger A_{\alpha \hat{\alpha}} U + \frac{i}{g} U(t_i)^\dagger \partial_{\alpha \hat{\alpha}} U,
\]

(19)

\[
F_{\alpha A} \rightarrow U(t_i)^\dagger F_{\alpha A} U + \frac{i}{g} U(t_i)^\dagger \partial_{\alpha A} U,
\]

where \( \partial_{\alpha \hat{\alpha}} \equiv \partial / \partial x^{\hat{\alpha}\hat{\alpha}} \) and \( \partial_{\alpha A} \equiv \partial / \partial \theta^{\alpha A} \).

Let us now present the explicit expressions for the bosonic, \( A(x, \theta) \), and fermionic, \( F(x, \theta) \), connections. Both of them are given by expansions in the Grassmann variable \( \theta^A_\alpha \) with the

\footnote{This property has been conjectured in Ref. [1] and was recently verified in Ref. [35] by an explicit two-loop calculation of the \( n = 6 \) NMHV amplitude in \( \mathcal{N} = 4 \) SYM.}
coefficients of increasing scaling dimension built from the various field components (gluon $A^{\dot{\alpha}\alpha}$, gauginos $\psi^{A}_{\dot{\alpha}}$, $\bar{\psi}^{A}_{\dot{\alpha}}$ and scalars $\phi^{AB}$, $\bar{\phi}_{AB}$) and their gauge covariant derivatives $D_{a\dot{a}} = \partial_a \phi^{\dot{a}} - ig[A_{a\dot{a}}]$.

\begin{equation}
\mathcal{A} = A + i[\theta^A] [\bar{\psi}_A] + \frac{i}{2!} [\theta^A] [\theta^B] D \bar{\phi}_{AB} - \frac{1}{3!} \epsilon_{ABCD} [\theta^A] [\theta^B] D \langle \theta^C \psi^D \rangle \tag{20}
\end{equation}

\begin{equation}
\mathcal{F}_A = \frac{i}{2} \bar{\psi}_{AB} [\theta^B] - \frac{1}{3!} \epsilon_{ABCD} [\theta^B] D \langle \theta^C | F | \theta^D \rangle + \ldots,
\end{equation}

where we used a compact notation for the quantities carrying spinor indices, e.g. $[\theta^A] \equiv \theta^A_{\dot{a}}$ and $[\bar{\psi}_A] \equiv \bar{\psi}_{a\dot{a}}$ (see Appendix A for explanations). Notice that in the expression for $\mathcal{A}$, Eq. (20), the covariant derivatives act on the quantum fields only. The expansion of $\mathcal{A}$ starts with the gauge field and the additional factor $1/2$ in front of the bosonic connection in (17) is introduced to get the correct normalization for the lowest component, i.e., $\frac{1}{2} dx^{a\dot{a}} A_{a\dot{a}} = dx^\mu A_\mu$. The ellipses in the right-hand sides of (20) and (21) denote higher-order terms in the $\theta$’s up to order $O(\theta^g)$. A characteristic feature of such terms is that, in the interaction-free limit (for $g = 0$), they are proportional to the free equations of motion. Thus, the free-theory expansion of the bosonic and fermionic connections terminates at orders $O(\theta^4)$ and $O(\theta^3)$, respectively.

The form of the connections and the values of the rational coefficients in (20) and (21) are fixed from the requirement for the supersymmetric Wilson loop (14) to be invariant under the shift of the odd variables, $\delta \theta^A_{\dot{a}} = \epsilon^A_{\dot{a}}$, and the simultaneous chiral supersymmetric transformation of fields (see Eq. (21) in Appendix A). It is a well-known feature of all supersymmetric gauge theories, considered in a non-supersymmetric gauge (Wess-Zumino gauge), that the supersymmetry algebra closes modulo compensating gauge transformations with a field-dependent parameter. In addition, in the absence of auxiliary fields, as is the case of $\mathcal{N} = 4$ SYM, the algebra closes on the shell of the field equations. An explicit calculation yields the following result for the $Q$–variation of both connections

\begin{equation}
\delta_Q A_{a\dot{a}} = \partial_a \omega + ig[\omega, A_{a\dot{a}}] + \Omega_{a\dot{a}}, \tag{22}
\end{equation}

\begin{equation}
\delta_Q F_A = \partial_A \omega + ig[\omega, F_A],
\end{equation}

with $\omega$ being the field-dependent gauge transformation parameter

\begin{equation}
\omega = \langle \epsilon^A \theta^B \rangle \left[ -\frac{i}{2!} \bar{\phi}_{AB} + \frac{1}{3!} \epsilon_{ABCD} [\theta^C \psi^D] - \frac{i}{4!} \epsilon_{ABCD} [\theta^C | F | \theta^D] + \ldots \right], \tag{23}
\end{equation}

and $\epsilon^A_{\dot{a}}$ being the parameter of the $Q$–transformations. Also, $\Omega_{a\dot{a}}$ in the first relation in (22) is given by

\begin{equation}
\Omega_{a\dot{a}} = -2 \epsilon_{ABCD} (\epsilon^B_{\dot{b}} \theta^B_{\dot{b}}) \theta^C_{\dot{a}} \left[ \frac{1}{3!} (\Omega_{\dot{a}})_{\alpha}^D - \frac{i}{4!} \theta^D_{\dot{b}} (\Omega_{\dot{a}})_{\gamma}^\dot{a} + \ldots \right], \tag{24}
\end{equation}

This is obvious from the supersymmetric transformations of the holomorphic gluon field strength $\delta_Q F^{a\beta} = \left[ \epsilon^A_{\dot{a}} (\Omega_{\dot{a}})_{\beta}^A + \epsilon^A_{\dot{a}} (\Omega_{\dot{a}})_{\dot{\alpha}}^A \right] / 4 + O(g)$ with $(\Omega_{\dot{a}})_{\alpha}^a = \partial^a \alpha \bar{\psi}^\dot{a}._A.$
and it is proportional to the fermion and gluon equations of motion
\[
(\Omega_t)_a^{\dot{\alpha} \dot{\beta}} = D_{\dot{\alpha}} \bar{\psi}_A^\gamma \psi_A^\gamma, \quad (\Omega_y)_a^{\alpha \beta} = D_\alpha F_{\gamma}^\beta.
\] 

Let us now examine the variation of the supersymmetric Wilson loop under the supersymmetric transformations (22). Comparing the relations (22) and (19), we observe that the $\omega$–dependent terms in (22) can be eliminated in $W_n$ by a compensating (infinitesimal) gauge transformation (19) with the parameter $U = 1 + ig\omega$. Thus, we are left over with the variation of the bosonic connection proportional to the equations of motion, $\delta Q A_{a \dot{a}} = \Omega_{a \dot{a}}$. Although such terms vanish on shell, that is for quantum fields satisfying their equations of motion, they do provide a nonvanishing contribution to the variation of $W_n$ under off-shell supersymmetry transformations. As we will see in a moment, this subtlety plays a very important rôle in testing the duality between supersymmetric Wilson loops and scattering amplitudes.

### 2.3 Equivalent form of the supersymmetric Wilson loop

As was explained in Sect. 2.1 to discuss the duality between supersymmetric Wilson loops and superamplitudes it is convenient to switch from the $(x, \theta)$–coordinates to the momentum superwistor $(\lambda, \mu, \chi)$–variables defined in (10). Inverting the relations (10) we find with the help of (3)
\[
x^{\dot{\alpha} \alpha} = \frac{\mu_{i-1}^{\dot{\alpha} \alpha} \chi^{\alpha} - \mu_{i}^{\dot{\alpha} \alpha} \chi^{\alpha}}{\langle i - 1 \rangle}, \quad \theta_i^{\alpha A} = \frac{\chi_i^{\alpha A} - \chi_i^{\alpha A}}{\langle i - 1 \rangle}.
\]

Rewriting the supersymmetric Wilson loop (14) and (16) in terms of these variables, we obtain the equivalent form of $W_n$ introduced in Ref. [24].

To begin with, we examine the expression for the Wilson line (16) and replace the bosonic and fermionic connections in (17) by their explicit expressions (20) and (21). Then, we use the definition (26) to get, after a rather lengthy calculation,
\[
B^{\text{MS}}(t_j) = E_j(t_j) + \left( \frac{d}{dt_j} V_j(t_j) + ig[V_j(t_j), E_j(t_j)] \right) + \Delta \Omega_j(t_j),
\]

where the three terms in the right-hand side have the following meaning. The first term $E_j(t_j)$ depends on the single odd variable $\chi^3_j$ and its expansion runs to order $O(\chi^3_j)$ only
\[
2E_j(t) = -\langle j|A[j] - i\chi^3_j[\bar{\psi}_A[j] + \frac{i}{2!} \chi^A_j \chi^B_j \frac{\langle j-1|D[j]}{\langle j-1 \rangle} \delta^{AB} - \frac{1}{3!} \epsilon ABCD \chi^A_j \chi^B_j \chi^C_j \frac{\langle j-1|D[j]}{\langle j-1 \rangle^2} \langle j-1 \rangle + \frac{1}{4!} \epsilon ABCD \chi^A_j \chi^B_j \chi^C_j \chi^D_j \frac{\langle j-1|D[j]}{\langle j-1 \rangle^3} \langle j-1 \rangle,
\]

where we used the compact spinor notations explained in Appendix A.

The second term in the right-hand side of (27) involves the covariant derivative of the function
\[
V_j = \langle j|\bar{\chi}^{\dot{\alpha}}_j \chi^{\alpha}_j \frac{\langle j-1, \theta^B \rangle}{\langle j-1 \rangle} \delta^{AB} - \frac{1}{3!} \epsilon ABCD \chi^A_j \langle j-1, \theta^B \rangle \langle j-1 \rangle^2 \langle L^C_j | \psi^D \rangle + \frac{1}{4!} \epsilon ABCD \chi^A_j \langle j-1, \theta^B \rangle \langle j-1 \rangle^3 \langle L^C_j | \psi^D \rangle + \chi^C_j \chi^D_j \langle j-1 \rangle \rangle + \ldots,
\]

8
where $\theta^A = \theta^A(t_j)$ was defined in (15) and a shorthand notation was introduced for the combination
\[ |\Lambda^C_j\rangle = |j\rangle\langle j - 1|\theta^C(t_j)\rangle - 2|j - 1\rangle\chi^C_j. \] (30)

The contribution of $V(t_j)$ to the Wilson line (16) can be factored out into boundary terms depending on $V(t_j = 1)$ and $V(t_j = 0)$. In the formulation of Ref. [24], such terms get absorbed into the so-called vertex operators localized at the vertices of the super-polygon $C_n$.

Finally, the last term in the right-hand side of (27) is proportional to the equations of motion
\[ 2\Delta \Omega_j = -\frac{1}{3!} \epsilon_{ABCD} \chi^A_j \chi^B_j \langle j - 1 \theta^C \rangle - \frac{i}{4!} \epsilon_{ABCD} \chi^A_j \chi^B_j \langle j - 1 \theta^C \rangle \langle \Lambda^C_j |\Omega^f_j| \rangle, \] (31)

where $\Omega^f$ and $\Omega^g$ were defined in (25). Neglecting the contribution of the latter to (27), we can define another connection
\[ B^{CH}(t_j) = E_j(t_j) + \left( \frac{d}{dt_j} V_j(t_j) + ig[V_j(t_j), E_j(t_j)] \right). \] (32)

Then, we can use this function to construct the Wilson line (16) and define the corresponding supersymmetric Wilson loop (14). The resulting expression for $W^{CH}_n$ coincides with the supersymmetric Wilson loop introduced in Ref. [24]. However, since by construction
\[ B^{MS}(t_j) - B^{CH}(t_j) = \Delta \Omega_j(t_j), \] (33)

the two Wilson loops proposed in Refs. [23, 24] are identical only on shell. Thus, the difference between the all-loop expressions for the two Wilson loops can be attributed to the contribution of the field-equation operators.

As a consequence of our analysis, the natural question arises whether the equations of motion can produce a nonvanishing contribution to the expectation value of the supersymmetric Wilson loop (5). If they do not, then the two Wilson loops $W^{MS}_n$ and $W^{CH}_n$ respect the $Q$—supersymmetry and they are identical at the quantum level. However, if the contribution of the equations of motion is different from zero, then $W^{MS}_n \neq W^{CH}_n$ and both Wilson loops are not $Q$—supersymmetry invariants. In the latter case, neither of the Wilson loops can be dual to the superamplitudes in $\mathcal{N} = 4$ SYM since the latter are invariant under the $Q$—supersymmetry.

## 3 Simple Super Ward identities

To understand better what happens to supersymmetry at the quantum level, let us derive the Ward identities for the two supersymmetric Wilson loops introduced in the previous section.

We start with the path integral representation for the vacuum expectation value of the Wilson loop
\[ \langle W_n(x_i, \theta_i) \rangle = \int [DX] W_n(x_i, \theta_i) e^{iS_N=4[X]}, \] (34)

where the integration goes over all fields in $\mathcal{N} = 4$ SYM collectively called $X$. Then, we make the shift $\theta_i \rightarrow \theta_i + \epsilon$ in both sides of (34) and perform a compensating supersymmetry transformation of the fields (91) inside the path integral, $X \rightarrow X + \delta_Q X$. Making use of the invariance of the
\( \mathcal{N} = 4 \) action, \( S_{\mathcal{N}=4}[X] = S_{\mathcal{N}=4}[X + \delta X] \) we arrive at\(^5\)

\[
(\epsilon \cdot Q) \langle W_n \rangle \equiv \sum_{i=1}^{n} \epsilon^A_i \partial_{\theta^A_i} \langle W_n \rangle = \langle \delta_Q W_n \rangle .
\] (35)

To make use of this relation we have to analyze the variation of the supersymmetric Wilson loops \( \delta_Q W_n \), which we come to do next.

It follows from the definitions (14) and (16) that the supersymmetry variation of the Wilson loop \( W_n \) amounts to inserting a local operator \( \delta \mathcal{B}(t_j) \) on the super-polygon \( \mathcal{C}_n \)

\[
\langle \delta_Q W_n \rangle = \frac{1}{N_c} \langle \text{tr} \left[ W_{[1,2]} \cdots \delta_Q W_{[j,j+1]} \cdots W_{[n,1]} \right] \rangle
\]

\[
= \frac{1}{N_c} \langle \text{tr} \left( ig \int_0^1 dt_j \delta_Q \mathcal{B}(t_j) \right) W_n(t_i) \rangle .
\] (36)

Here \( W_n(t_i) \) stands for the supersymmetric Wilson line evaluated along an open contour in superspace that starts at the point \( (x(t_i), \theta(t_i)) \), goes along the polygon \( \mathcal{C}_n \) and returns to the starting point.

### 3.1 Anomalies

Let us first compute \( \delta_Q \mathcal{B}(t_j) \) for the supersymmetric Wilson loop \( \mathcal{W}_n^{\text{MS}} \) defined in (17), (20) and (21). We apply the relation \( (22) \) to get

\[
\delta_Q \mathcal{B}^{\text{MS}}(t_j) = \frac{1}{2} \dot{x}^\alpha (t_j) \delta_Q A_{\alpha \dot{\alpha}} + \dot{\theta}^\alpha (t_j) \delta_Q F_{\alpha A}
\]

\[
= \frac{d\omega}{dt_j} + ig[\omega, \mathcal{B}^{\text{MS}}(t_j)] - \frac{1}{2} \langle j | \Omega(t_j) | j \rangle ,
\] (37)

with \( \omega \) and \( \Omega \) defined in (23) and (24), respectively. As was explained in Sect. 2.2 the terms involving \( \omega \) do not contribute to the right-hand side of (36) by virtue of gauge invariance. Therefore, computing (36) we can retain only the last term in (37)

\[
\delta_Q \mathcal{B}^{\text{MS}}(t_j) = \Omega^{\text{MS}} = \epsilon^{ABCD} \langle A^B \theta^C \rangle \left\{ \frac{1}{3!} [\Omega^D | j] + \frac{i}{4!} (\theta^D | \Omega_g | j] + \ldots \right\} ,
\] (38)

where \( \theta^A = \theta^A(t_j) \) was defined in (15) and the ellipses denote terms vanishing in the free theory limit (i.e., for \( g = 0 \)).

The analysis of the Wilson loop \( \mathcal{W}_n^{\text{CH}} \) goes along the same lines. We use the relations (28) and (32) to verify that, modulo gauge transformations,

\[
\delta_Q \mathcal{B}^{\text{CH}}(t_j) = \Omega^{\text{CH}} = \frac{1}{4} \epsilon^{ABCD} \frac{\chi^A_j \chi^B_j (j - 1 \epsilon^C) (j - 1 \epsilon^D)}{[j - 1]} \langle \Omega^D | j \rangle
\]

\[
- \frac{i}{12} \epsilon^{ABCD} \frac{\chi^A_j \chi^B_j \chi^C_j (j - 1 \epsilon^D)}{[j - 1] \epsilon^C} \langle j - 1 | \Omega_g | j \rangle + \ldots ,
\] (39)

\(^5\)More precisely, the \( Q \)-variation of the gauge-fixed action is different from zero, but for the supersymmetry preserving regularization, i.e., dimensional reduction, \( \delta_Q S_{\mathcal{N}=4} \) contains BRST-exact operators only and thus does not produce nontrivial contributions when inserted into the correlation function with the gauge-invariant Wilson loop, \( \langle W_n | i \delta Q S_{\mathcal{N}=4} \rangle = 0 \) (see, e.g., [32]).
where $\chi^A_j = \langle j \theta^A_j \rangle$. The obtained expressions for the anomalies have to satisfy a consistency condition that follows from (33)

$$
\delta_Q B^{\text{MS}} - \delta_Q B^{\text{CH}} = \Omega_j^{\text{MS}} - \Omega_j^{\text{CH}} = \delta_Q \Delta \Omega_j(t_j).
$$

(40)

Replacing $\Omega_j(t_j)$ by its explicit expression (31) we find after lengthy calculations that this relation is indeed satisfied.

Substituting the obtained expressions for $\delta_Q B$ into (36) and (35) we deduce the Ward identity for the supersymmetric Wilson loop

$$
\leqno(41)
(\epsilon \cdot Q) \langle W_n \rangle = \sum_{j=1}^n ig \int_0^1 dt_j \frac{1}{N_c} \text{tr} \left[ \Omega_j(t_j) W_n(t_j) \right],
$$

where $\Omega_j(t_j)$ is given by (38) and (39) for the two supersymmetric Wilson loops defined above and the operator $(\epsilon \cdot Q)$ admits two equivalent forms

$$
\leqno(42)
(\epsilon \cdot Q) = \sum_{j=1}^n \epsilon^A_{\alpha} \frac{\partial}{\partial \theta^A_{j\alpha}} = \sum_{j=1}^n (i \epsilon^A) \frac{\partial}{\partial \chi^A_j}.
$$

We would like to emphasize that the right-hand side of (41) involves the correlation function of the equations of motion with the Wilson loop. For the Wilson loop to be invariant under the $Q$—supersymmetry, the right-hand side of (41) should vanish. Otherwise, the $Q$—supersymmetry will be broken.

### 3.2 One-loop calculation

Let us now compute the one-loop corrections to the right-hand sides of the Ward identities (41) for both Wilson loops. We recall that the anomalies (38) and (39) are given by a linear combination of the fermion and gluon equations of motion, $\Omega_f$ and $\Omega_g$, defined in (25).

We start with (38) and retain for the moment only the terms involving $\Omega_f^A = \partial_{\alpha} \gamma \psi^A_\gamma + O(g)$. Their contribution to the first term in the sum, i.e., for $j = 1$, in the right-hand side of (41) is given by (the remaining terms can be obtained through a cyclic shift of the indices)

$$
\leqno(43)
\frac{ig}{3!} \epsilon_{ABCD} \int_0^1 dt_1 \langle \epsilon^A \theta^B(t_1) \rangle \langle 1 \theta^C(t_1) \rangle \left( \frac{1}{N_c} \text{tr} \left[ \Omega_1^D(x(t_1)), 1 \right] \right) \left( ig \int_{c_n} dt \mathcal{B}(t) \right) + O(g^4),
$$

where we replaced the Wilson line $W_n(t_1)$ by its lowest order expansion. Here the connection $\mathcal{B}(t)$ is integrated over the contour $c_n$ and the fermionic operator is inserted at the point $y = x(t_1)$ on the segment $[x_1, x_2]$. Computing the correlation function in (43) to the lowest order in the coupling, we have to Wick contract $\psi^A_\beta(x(t_1))$ from $\Omega_1^D$ with the fermion field $\bar{\psi}_{\beta B}(x(t))$ inside $\mathcal{B}(t)$. It is easy to see from (27), (28) and (32) that the corresponding terms look alike for both connections and for the $k$th segment they are given by

$$
\leqno(44)
\mathcal{B}(t_k) = -\frac{i}{2} \chi^A_k \left[ \bar{\psi}_A(x(t_k)) k \right] + \ldots
$$

In Eq. (43), the $\bar{\psi}$—field can be located at any of the segments of the contour $c_n$ including $[x_1, x_2]$. In that case, the two fields $\psi^A_\beta(x(t_1))$ and $\bar{\psi}_{\beta B}(x(t_k))$ belong to the same segment and
they become light-like separated (recall that $x^2_{12} = 0$). Due to light-cone singularities, the product of two quantum fields separated by a light-like interval is not well defined. Therefore, in order to define the corresponding contribution to (43), we have to introduce a regularization.

In what follows we shall use the Four-Dimensional Helicity (FDH) regularization of $\mathcal{N} = 4$ SYM. The main advantage of this scheme is that it preserves the Poincaré supersymmetry of $\mathcal{N} = 4$ SYM (at least to the lowest order in coupling as we do in our analysis) and allows us to use the spinor decomposition for super-momenta [3] without it interfering with the change of dimensionality of Minkowski space-time. Then, the regularized correlator of the gaugino field with its equation of motion operator (25) reads, in

$$\frac{1}{N_c} \text{tr}(\partial_\alpha \psi^A_\beta(y) \bar{\psi}_\beta B(x)) = i C_F \delta^A_B \frac{\epsilon \Gamma(2 - \epsilon)}{\pi^{2-\epsilon}} \frac{\epsilon \delta_{\alpha\beta}}{[-(x - y)^2 + i0]^{2-\epsilon}}, \quad (45)$$

where $C_F = (N_c^2 - 1)/(2N_c)$ is the quadratic Casimir of the $SU(N_c)$ gauge group. According to the standard prescription for tadpoles [34], the correlation function (45) vanishes for $(x - y)^2 = 0$ within the framework of dimensional regularization. This implies that (43) receives zero contribution when $B(t)$ is integrated along the segment $[x_1, x_2]$.

Notice that the two-point function (43) is proportional to the parameter of dimensional regularization $\epsilon$. Therefore, for the correlation function in (43) to be different from zero as $\epsilon \to 0$ the integration over the position of the fields in (43) should produce a pole $1/\epsilon$. As follows from (45), this could only happen if the two-point function (45) is integrated through a region where $(x - y)^2 \to 0$. As an example, let us consider the contribution to (43) when the connection $B(t)$ is integrated along the segment $[x_2, x_3]$, that is for $k = 2$ in (44).

Eq. (43) = \frac{-1}{3!} \frac{g^2 C_F}{2\pi^2} \epsilon_{ABCD} \chi_1^A \chi_2^B [12] \int_0^1 dt_1 \int_0^1 dt_2 \frac{\epsilon \Gamma(2 - \epsilon)}{[-x_{13}^2(1 - t_1)t_2 + i0]^{2-\epsilon}}, \quad (46)

where $x(t_i) = x_i - t_i x_{i+1}, \theta(t_i) = \theta_i - t_i \theta_{i+1}$. We used the identities $\langle i \theta^A(t_i) \rangle = \chi_i^A$ and

$$\langle x(t_1) - x(t_2) \rangle^2 = (x_{12}(1 - t_1) + x_{23} t_2)^2 = x_{13}^2 (1 - t_1) t_2. \quad (47)$$

Here it is crucial that the two segments are light-like, $x_{13}^2 = 0$. We observe that the integration in the right-hand side of (46) around $t_2 = 0$, $t_1 = 1$ produces a pole $1/\epsilon$. It compensates the factor of $\epsilon$ coming from (45) and produces a finite contribution

Eq. (43) = \frac{-1}{3!} \frac{g^2 C_F}{2\pi^2} \frac{[12]}{x_{13}^2} \epsilon_{ABCD} \chi_1^A \chi_2^B + O(\epsilon). \quad (48)

This example illustrates the general mechanism which is at work for the fermion equations of motion and which produces a nonvanishing contribution to the right-hand side of (41). Repeating the analysis for $k = 3, \ldots, n$ in (43) it is straightforward to show that the right-hand side of (43) only receives a nonzero contribution from four segments of the contour $C_n$ adjacent to $[x_1, x_2]$, that is from those with $k = 2, 3, n - 1, n$.

The analysis of the contribution from the gluon equations of motion to the right-hand side of (41) goes along the same lines. To lowest order in the coupling, the gluon equation-of-motion operator (25) entering $\Omega_j(t_j)$ can only interact with the bosonic part of $\mathcal{W}_n$ given by \( i \frac{1}{2} \int_{C_n} dx A_\alpha^\alpha A_\alpha^\alpha \). The correlation function of the gauge field with its equation-of-motion operator (25) reads, in
the Feynman gauge and to lowest order in the coupling,
\[
\frac{1}{N_c} \text{tr} \langle A_{\alpha\dot{\alpha}}(x_1) \Omega_{\dot{\gamma}}^\beta(x_0) \rangle = i \frac{C_F}{2\pi^{2-\varepsilon}} \Gamma(2 - \varepsilon) \left[ (1 + \varepsilon) \frac{(\varepsilon \beta \delta^\beta_\alpha)}{(-x_{\alpha 01}^2 + i0)^{2-\varepsilon}} + (1 - \varepsilon) \frac{(x_{01})_\beta\alpha(x_{01})_\dot{\alpha}^\beta}{(-x_{\dot{\alpha} 01}^2 + i0)^{2-\varepsilon}} \right].
\]
(49)

Making use of this relation we find the correlation function of the gluon equation of motion with the bosonic Wilson loop as
\[
\frac{1}{N_c} \text{tr} \oint_{C_n} dx^{\dot{\alpha}} A_{\alpha\dot{\alpha}}(x) \Omega_{\dot{\gamma}}^\beta(x_0) = i \frac{C_F}{\pi^{2-\varepsilon}} \varepsilon \Gamma(1 - \varepsilon) \sum_{j=1}^n (x_{j,j+1}^2 + x_{0j}^2) \frac{(x_{j,j+1}^2 + i0)_{\beta\dot{\alpha}}}{x_{j+1,0}^2 - x_{j0}^2}.
\]
(50)

where the notation was introduced for
\[
D_\varepsilon(x_{j+1,0}, x_{j0}) = \frac{(-x_{j+1,0}^2 + i0)^{\varepsilon-1} - (-x_{j0}^2 + i0)^{\varepsilon-1}}{x_{j+1,0}^2 - x_{j0}^2}.
\]
(51)

In distinction with (49), the relation (50) is gauge invariant.

Comparing (50) with (45) we observe the same pattern. Though both correlation functions vanish for \(\varepsilon \rightarrow 0\), they produce a nonvanishing contribution to the right-hand side of (41) upon integration over the polygon \(C_n\). We would like to emphasize that one may miss this contribution if one performs the calculation in \(D = 4\) dimensions without properly regularizing the light-cone singularities of the correlation functions.

### 3.3 Supersymmetry anomalies

It becomes straightforward to compute the one-loop correction to the right-hand side of (41) for both Wilson loops, \(W_n^{MS}\) and \(W_n^{CH}\), by making use of the relations (45) and (50). For the sake of simplicity we present here the explicit expressions for the simplest case of \(n = 4\), that is for the Wilson loop defined over the rectangular contour \(C_4\) in the superspace. The generalization of our analysis to arbitrary \(n\) is straightforward.

For the Wilson loop \(W_4^{CH}\) involving the superconnection (32) the supersymmetric Ward identity reads
\[
(\epsilon \cdot Q) W_4^{CH} = -\frac{g^2 C_F}{4\pi^2} \epsilon_{ABCD} \chi_i^A \chi_i^B \frac{\langle 4^{C} \rangle}{\langle 41 \rangle} \frac{\langle 13 \rangle}{x_{13}^2 x_{24}^2} \left( \chi_i^D + \frac{1}{3} \langle 3^{4} \rangle \chi_i^1 \right) + (\text{cyclic}),
\]
(52)

where ‘(cyclic)’ stands for terms obtained by the cyclic shift of indices \(i \rightarrow i + 1\) subject to the periodicity condition \(i + 4 \equiv i\). Here the first and second terms inside the braces arise from the fermion and gauge equations of motion, respectively.

For the Wilson loop \(W_4^{MS}\) involving the superconnection (17) the analogous supersymmetric
Ward identity looks as

\[
(\epsilon \cdot Q) W_{4}^{\text{MS}} = \frac{g^{2}C_{F}}{2\pi^{2}} \frac{1}{3!} \epsilon_{ABCD} \left\{ \langle \epsilon^{A} \theta_{12}^{B} \rangle \langle 1 \theta_{1}^{C} \rangle \left( \frac{[12]}{(x_{13}^{2})^{2}} \langle 2 \theta_{2}^{D} \rangle + \frac{[41]}{(x_{24}^{2})^{2}} \langle 4 \theta_{4}^{D} \rangle \right) \\
- \frac{[13]}{x_{13} x_{24}} \left( \langle \epsilon^{A} \theta_{1}^{B} \rangle + \langle \epsilon^{A} \theta_{2}^{B} \rangle \right) \langle 1 \theta_{1}^{C} \rangle \langle 3 \theta_{3}^{D} \rangle \right\} \\
- \frac{g^{2}C_{F}}{2\pi^{2}} \frac{1}{4!} \epsilon_{ABCD} \left\{ \frac{[12]}{(x_{13}^{2})^{2}} \left( \langle \epsilon^{A} \theta_{12}^{B} \rangle \langle 1 \theta_{1}^{C} \rangle \langle 2 \theta_{2}^{D} \rangle + \langle \epsilon^{A} \theta_{2}^{B} \rangle \langle 1 \theta_{2}^{C} \rangle \langle 2 \theta_{12}^{D} \rangle \right) \\
- \frac{[13]}{x_{13} x_{24}} \left( \langle \epsilon^{A} \theta_{1}^{B} \rangle \langle 1 \theta_{1}^{C} \rangle \langle 3 \theta_{4}^{D} \rangle + \langle \epsilon^{A} \theta_{2}^{B} \rangle \langle 1 \theta_{2}^{C} \rangle \langle 3 \theta_{12}^{D} \rangle \right) \\
+ \frac{[41]}{(x_{24}^{2})^{2}} \left[ \langle \epsilon^{A} \theta_{12}^{B} \rangle \langle 1 \theta_{1}^{C} \rangle \langle 4 \theta_{4}^{D} \rangle + \langle \epsilon^{A} \theta_{2}^{B} \rangle \langle 1 \theta_{2}^{C} \rangle \langle 4 \theta_{12}^{D} \rangle \right] \right\} + \text{(cyclic)}. \tag{53}
\]

Here the two terms in the right-hand side again describe the contribution of the fermion and gluon equations of motion, respectively.

The following comments are in order.

The very fact that the right-hand sides of the relations \((52)\) and \((53)\) are different from zero immediately implies that the chiral \(Q\)–supersymmetry of both Wilson loops is broken already at one loop. Moreover, it can be verified that the two anomalies do not respect the conformal symmetry (see Sect. \(4.2\) below)\(^6\).

We recall that, if the supersymmetric Wilson loop \(W_{4}\) respected the supersymmetry and conformal symmetry, it would be independent of the odd variables, Eq. \((12)\). In the next section, we will reconstruct the complete one-loop expression for the Wilson loops \(W_{4}\) by solving the Ward identities \((52)\) and \((53)\) and we will demonstrate that relation \((12)\) is invalidated by the anomalies.

We established in this section, that the correlation functions of the fermion and gluon equation of motion with the Wilson loop induce a nontrivial contribution to the supersymmetric Ward identity. Let us now show that the same correlation functions allow us to compute the difference between the two Wilson loops under consideration, \(W_{4}^{\text{MS}} - W_{4}^{\text{CH}}\). Indeed, it follows from \((53)\) that to lowest order in the coupling

\[
W_{4}^{\text{MS}} - W_{4}^{\text{CH}} = ig \int_{0}^{1} dt_{1} \left\langle \frac{1}{N_{c}} \text{tr} \left[ \Delta \Omega_{1}(t_{1}) \left( ig \int_{C_{1}} dt B(t) \right) \right] \right\rangle + \text{(cyclic)}, \tag{54}
\]

where \(\Delta \Omega_{1}(t_{1})\) is defined in \((31)\). The latter is given by a linear combination of fermion and gluon equations of motion. As a result, the calculation of the correlation function in the right-hand side of \((54)\) can be easily performed along the same lines as in \((41)\) and \((43)\). Going through the

\(^6\)In general, for some quantity depending on \((ij)\) and \([ij]\) to be invariant under conformal transformations, the indices should satisfy the conditions \(|i - j| = 1\). This is not the case for the one-loop expressions for the anomalies, Eqs. \((52)\) and \((53)\), which involve the square brackets \([13]\).
derivation we find
\[ W_4^{\text{CH}} - W_4^{\text{MS}} \]
\[ = \frac{g^2 C_F}{4 \pi^2} \frac{1}{3!} \epsilon_{ABCD} \chi_1^A \chi_1^B \left\{ \left[ \frac{13}{12} \eta_1^C \chi_1^D \right] + \left[ \frac{12}{41} \eta_1^C \chi_2^D \right] + \left[ \frac{41}{12} \eta_1^C \chi_4^D \right] - \frac{2}{3} \eta_1^C \chi_3^D \right\} \]
\[ - \frac{g^2 C_F}{4 \pi^2} \frac{1}{4!} \epsilon_{ABCD} \chi_1^A \chi_1^B \left\{ \frac{12}{x_{13}^2 x_{24}^2} \eta_1^C \chi_2^D + \frac{41}{x_{13}^2 x_{24}^2} \eta_1^C \chi_4^D + \frac{13}{41} \frac{1}{x_{13}^2 x_{24}^2} \left[ \chi_4^C \left( \langle 13 \rangle \chi_4^D + 2 \langle 34 \rangle \chi_1^D \right) \right] \right\} \right\} \}
\]
+ \left( \chi_4^C - \langle 41 \rangle \eta_1^C \right) \left[ \left( \langle 13 \rangle \chi_4^D - \langle 34 \rangle \eta_1^D \right) + 2 \langle 34 \rangle \chi_1^D \right] \}
\] +(cyclic),
\]
where \( \eta_1 \) was introduced in \( \text{(3)} \). It can be re-expressed in terms of \( \chi \)'s using Eq. \( \text{(26)} \) as follows
\[ \eta_1^A = \frac{\chi_4^A}{\langle 41 \rangle} + \frac{\chi_2^A}{\langle 12 \rangle} + \frac{\langle 24 \rangle \chi_1^A}{\langle 41 \rangle \langle 12 \rangle}. \]
\]
Applying the differential operator \( \text{(42)} \) to both sides of \( \text{(55)} \) we verified that \((\epsilon \cdot Q)(W_4^{\text{CH}} - W_4^{\text{MS}})\) coincides with the difference of the expressions in the right-hand sides of Eqs. \( \text{(52)} \) and \( \text{(53)} \).

4 Supersymmetric Wilson loops at one loop

In this section, we will compute one-loop correction to the supersymmetric Wilson loop \( W_4 \). As follows from its definition \( \text{(14)} \) and \( \text{(16)} \), the Wilson loop is given to this order by the following expression
\[ W_4 = 1 + \sum_{1 \leq j \leq k \leq 4} (ig)^2 \int_0^1 dt_j \int_0^1 dt_k \frac{1}{N_c} \text{tr} \left[ B_j(t_j) B_k(t_k) \right] + O(g^4), \]
where the superconnection \( B \) is defined in \( \text{(27)} \) and \( \text{(32)} \). The explicit expression for the superconnection involves the sum over all quantum fields in \( N = 4 \) SYM. As a consequence, the direct calculation of \( \text{(57)} \) yields a large number of contributing terms and makes the analysis very cumbersome. There is however another method that allows one to efficiently fix the form of the one-loop Wilson loop with little effort. Namely, we will solve the supersymmetric Ward identities, Eqs. \( \text{(52)} \) and \( \text{(53)} \), and reconstruct the explicit one-loop expression for the Wilson loop \( W_4 \) using some input requiring only a very small number of diagrams being computed explicitly.

We recall that the expansion of \( W_4 \) in powers of the odd variables has the general form \( \text{(3)} \). To one-loop order, this expansion terminates at
\[ W_4 = W_{4;0} + W_{4;1} + W_{4;2} + O(g^4), \]
where \( W_{4;0} \) is the bosonic light-like Wilson loop and \( W_{4;k=1,2} \) are given by homogenous polynomials of degree \( 4k \) in the \( \chi \)'s. According to \( \text{(52)} \) and \( \text{(53)} \), the anomaly \( (\epsilon \cdot Q) W_4 \) is given to one-loop order by a homogenous polynomial in the odd variables of degree 4. Then, replacing \( W_4 \) by its general expression \( \text{(58)} \) and matching the degree of the odd variables we find that \( W_{4;0} \) and \( W_{4;2} \) should be annihilated by \((\epsilon \cdot Q)\). For the bosonic component \( W_{4;0} \) this is obvious, while for \( W_{4;2} \) it

\[ \text{To see this we notice that the one-loop correction to } W_4 \text{ in } \text{(57)} \text{ is bilinear in the superconnection } B(t) \text{ which in turn is a polynomial of degree 4 in the Grassmann variables.} \]
leads to a nontrivial constraint \((\epsilon \cdot Q) \mathcal{W}_{4,2} = O(g^4)\). As we will see in a moment, \(\mathcal{W}_{4,2}\) takes zero value at one loop. Thus, to this order, the supersymmetric Ward identities become anomalous for the component \(\mathcal{W}_{4,1}\) only. By definition, \(\mathcal{W}_{4,1}\) is a homogenous polynomial of degree 4 in \(\chi_i\) (with \(i = 1, \ldots, 4\)). To one-loop order, we shall use the following ansatz for \(\mathcal{W}_{4,1}\):

\[
\mathcal{W}_{4,1} = \frac{g^2 C_F}{4 \pi^2} \sum_{0 \leq k_i \leq 4 \atop k_1 + k_2 + k_3 + k_4 = 4} c_{k_1 k_2 k_3 k_4} (\chi_1)^{k_1} (\chi_2)^{k_2} (\chi_3)^{k_3} (\chi_4)^{k_4},
\]

where \(c_{k_1 k_2 k_3 k_4}\) are bosonic coefficient functions. As follows from Eq. (14), the Wilson loop \(\mathcal{W}_4\) is invariant under the cyclic shift of indices, thus we have to require that the right-hand side of (59) should be cyclically symmetric as well.

The general solution to the Ward identities (52) and (53) is defined up to an arbitrary function depending on the invariants of the \(Q\)–supersymmetry transformations, \(\chi_i^A \to \chi_i^A + \langle i\epsilon^A \rangle\). Such invariants depend on three points and have the following form:

\[
\Theta_{ijk}^A = \chi_i^A \langle jk \rangle + \chi_j^A \langle ki \rangle + \chi_k^A \langle ik \rangle.
\]

For 4 points there are \((n - 2)\) linear independent invariants. For \(n = 4\) we can choose them to be \(\Theta_{412}^A\) and \(\Theta_{234}^A\). Then, the solution to the Ward identities are defined modulo the substitution

\[
\mathcal{W}_4 \to \mathcal{W}_4 + f_0(x) + f_1(x; \Theta_{412}, \Theta_{234}) + f_2(x)(\Theta_{412})^4(\Theta_{234})^4,
\]

where \(f_0(x)\) and \(f_2(x)\) are arbitrary functions of the bosonic variables and \(f_1(x; \Theta_{412}, \Theta_{234})\) is an arbitrary homogenous polynomial in the odd variables of degree 4.

### 4.1 Boundary conditions

To define a unique solution to the Ward identities, we have to fix the ambiguity in (61), or equivalently determine the functions \(f_0, f_1, f_2\). The function \(f_0(x)\) affects only the bosonic component and, therefore, its form is fixed by the one-loop correction to the bosonic Wilson loop \(\mathcal{W}_4\). To determine the remaining functions \(f_1\) and \(f_2\), we shall compute one-loop corrections to \(\mathcal{W}_4\) in the gauge \(\chi_2^A = \chi_4^A = 0\). In this gauge, two major simplifications occur. Firstly, the expression for the superconnections significantly reduce (see, e.g., Eqs. (28) and (29)), thus minimizing the number of one-loop Feynman diagrams contributing to the loop. Secondly, the \(Q\)–invariants now depend on a single odd variable, \(\Theta_{412} = \langle 24 \rangle \chi_1\) and \(\Theta_{234} = -\langle 24 \rangle \chi_3\), and their contribution to (61) takes a particularly simple form.

Let us start with the last term in the right-hand side of (61). In the gauge \(\chi_2 = \chi_4 = 0\), it is proportional to \(\eta_1^4 \eta_3^4\). Using the expression for one-loop corrections to the Wilson loop (57), we find that \(O(\eta_1^4 \eta_3^4)\) term could only appear from the correlation between \(O(\eta_1^4)\) and \(O(\eta_3^4)\) terms inside \(B_1(t_1)\) and \(B_3(t_3)\). The corresponding correlation function involves chiral components of the gauge field strength tensor located at two different segments of the polygon.

---

8Here, to simplify the notation, we stripped the \(SU(4)\) indices off the Grassmann variables and the accompanying Levi-Civita tensor these are contracted with. As explained in Appendix A in this form the \(\chi\)'s can be treated as commuting variables.

9To construct these invariants we use the transformations \(\chi_i^A \to \chi_i^A + \langle i\epsilon^A \rangle\) and choose \(\epsilon^A\) to put to zero two of the \(\chi\)'s, e.g. \(\chi_j = \chi_k = 0\). Then, the remaining \(\chi\)–variables will be automatically invariant under \(Q\)–supersymmetry.
\[ \langle F_{\alpha\beta}(x(t_1))F_{\alpha'\beta'}(x(t_3)) \rangle \]. In \( D = 4 \) dimension it vanishes, while for \( D = 4 - 2\varepsilon \) it is proportional to \( \varepsilon \). Therefore, for the result to differ from zero, the integral over \( t_1 \) and \( t_3 \) should produce a pole \( 1/\varepsilon \). A simple calculation shows that this does not happen and, therefore, the coefficient in front of \( O(\eta_i^\dagger \eta_i^0) \) term in \( \mathcal{W}_4 \) equals zero. This immediately implies that the one-loop correction to \( \mathcal{W}_4 \) does not involve terms of degree 8 in odd variables,

\[ \mathcal{W}_{4;2} = 0 + O(g^4). \]  

We now turn to computing corrections to (57) of the Grassmann degree 4. They have the general form (59). For \( \chi_2 = \chi_4 = 0 \) we are left with the terms of the form \( \chi_1^{j_1} \chi_3^{4-j_1} \) with \( j_1 = 0, \ldots, 4 \). As we will show in the next subsection, to construct a unique solution to the Ward identity it is sufficient to identify the contribution of two terms only, \( O(\chi_1^2 \chi_3^2) \) and \( O(\chi_1 \chi_3^3) \).

Using explicit expressions for the superconnections, Eqs. (20) and (21), it is easy to see that such terms arise from the correlation between scalar and fermion fields, respectively, entering into the expansion of \( B(t_1) \) and \( B(t_3) \).

For the Wilson loop \( W \), we now turn to computing corrections to (57) of the Grassmann degree 4. They have the same result (67) can be deduced from (55) by examining the coefficient in front of \( \chi_1 \chi_3^3 \) in \( \mathcal{W}_4^{MS} - \mathcal{W}_4^{CH} \).
4.2 Solving the Ward identities

We are now in a position to construct the solution to the Ward identities \((52)\) and \((53)\). We start with the former and substitute the ansatz \((59)\) into the left-hand side of \((52)\). In this way we obtain a relation both sides of which depend on the parameters of the supersymmetric transformation \((i\epsilon^A)\) (with \(i = 1, \ldots , 4\)). Notice that among the four parameters only two are linearly independent

\[
\langle 2\epsilon \rangle = \langle 1\epsilon \rangle \frac{23}{13} + \langle 3\epsilon \rangle \frac{12}{13} , \quad \langle 4\epsilon \rangle = -\langle 1\epsilon \rangle \frac{34}{13} - \langle 3\epsilon \rangle \frac{41}{13} . \tag{69}\]

Then, comparing the coefficients in front of various terms involving \(\langle 1\epsilon \rangle\), \(\langle 3\epsilon \rangle\) and different powers of \(\chi_i\), we obtain a system of linear inhomogeneous equations for the coefficients \(c_{k_1k_2k_3k_4}\) entering \((59)\). In addition, we impose the boundary conditions \((64)\) and \((66)\). The resulting system of equations is overdetermined but it has a unique solution leading to the following expression for the one-loop Wilson loop

\[
\mathcal{W}_{4;1}^{\text{MS}} = \frac{g^2 C_F}{4\pi^2} \left( \frac{1}{x_{13}^2} + \frac{1}{x_{24}^2} \right) \left\{ \frac{24}{12} \frac{\langle 24 \rangle}{\langle 24 \rangle} - \frac{\langle 24 \rangle}{\langle 12 \rangle} \frac{\chi_1^4}{\langle 12 \rangle} + \frac{\langle 24 \rangle}{\langle 12 \rangle} \frac{\chi_2^4}{\langle 12 \rangle} \right\} \frac{1}{6} + \left( \frac{\chi_2 \chi_3}{\langle 12 \rangle \langle 13 \rangle} - \frac{\chi_2 \chi_4}{\langle 41 \rangle \langle 13 \rangle} - \frac{\chi_3 \chi_4}{\langle 41 \rangle \langle 13 \rangle} - \frac{1}{4} \frac{3\langle 24 \rangle}{\langle 12 \rangle \langle 13 \rangle} + \frac{2\langle 34 \rangle}{\langle 12 \rangle \langle 13 \rangle} \right\} } + \text{(cyclic)} \tag{70}\]

In a similar fashion, solving the Ward identity \((52)\) subject to the boundary condition \((64)\) and \((67)\) we find

\[
\mathcal{W}_{4;1}^{\text{CH}} = \frac{g^2 C_F}{4\pi^2} \left( \frac{1}{x_{13}^2} + \frac{1}{x_{24}^2} \right) \left\{ \frac{23}{12} \frac{\langle 23 \rangle}{\langle 13 \rangle} \frac{\chi_1^4}{\langle 12 \rangle} + \frac{\langle 23 \rangle}{\langle 12 \rangle} \frac{\chi_2^4}{\langle 12 \rangle} + \frac{\langle 23 \rangle}{\langle 12 \rangle} \frac{\chi_3^4}{\langle 12 \rangle} + \frac{\langle 23 \rangle}{\langle 12 \rangle} \frac{\chi_4^4}{\langle 12 \rangle} \right\} \frac{1}{6} + \left( \frac{\chi_2 \chi_3}{\langle 12 \rangle \langle 13 \rangle} - \frac{\chi_2 \chi_4}{\langle 12 \rangle \langle 13 \rangle} - \frac{\chi_3 \chi_4}{\langle 12 \rangle \langle 13 \rangle} - \frac{\chi_2 \chi_4}{\langle 12 \rangle \langle 13 \rangle} \right\} + \text{(cyclic)} . \tag{71}\]

We verified that these relations obey several consistency conditions. By construction, they satisfy the Ward identities \((52)\) and \((53)\), respectively, and, therefore, they do not respect \(Q\)–supersymmetry

\[
Q_\alpha^A \mathcal{W}_{4;1} \neq 0 . \tag{72}\]

Also, the difference between the two Wilson loops is in agreement with \((55)\).

Next, we computed \(O(\chi_1 \chi_2 \chi_3 \chi_4)\) and \(O(\chi_1^4)\) terms in the right-hand side of \((57)\) and checked that they are correctly reproduced in \((70)\) and \((71)\). In the first case, the terms involving all four \(\chi\)–variables cannot be produced by the correlation function of two superconnections (see \((57)\)) and, therefore, they should be absent in the one-loop approximation. In the second case, to identify the terms \(O(\chi_1^4)\) we can set \(\chi_2 = \chi_3 = \chi_4 = 0\) thus reducing significantly the number of contributing diagrams.

We recall that the relations \((70)\) and \((71)\) define (together with \((58)\)) the part of the one-loop correction to the Wilson loop \((58)\) depending on the Grassmann variables. The bosonic component of the Wilson loop, \(\mathcal{W}_{4;0}\), has been previously computed in Ref. \([9]\). To one-loop order,
it develops a double pole $1/\varepsilon^2$ due to the presence of specific ultraviolet (UV) cusp singularities \cite{28}. In distinction with $W_{4;0}$, the expressions in the right-hand side of (71) and (70) are free from any divergences and are well-defined for $\varepsilon \to 0$. This implies that UV divergences cancel in the ratio of the supersymmetric Wilson loop and its bosonic component,

$$W_4/W_{4;0} = 1 + W_{4;1} + O(g^4),$$

and, therefore, the scaling (dilatation) invariance is restored in their ratio, $D(W_4/W_{4;0}) = 0$.

But what about special conformal transformation? The special conformal boosts are given by the superposition of translations and inversions, $K^\alpha = IP^\alpha I$. Translations shift $x_i$ but they do not affect $\lambda_i$ and $\chi_i$. At the same time, the inversion acts as

$$I[(x_i)_{\alpha\dot{\alpha}}] = \frac{x_i^{\dot{\alpha}}}{x_i^2}, \quad I[\lambda_i^\alpha] = (\mu_i)_{\dot{\alpha}}, \quad I[\chi_i] = \chi_i,$$

with $\mu_i$ defined in (10). A close examination of (71) and (70) shows that both expressions depend on the angle brackets of the form $\langle ii+1 \rangle$ and $\langle ii+2 \rangle$. Using (74) we find that the former brackets are transformed covariantly under the inversion, $I[\langle ii+1 \rangle] = \langle i|x_i|x_{i+1}|i+1 \rangle = x_i^2 \langle ii+1 \rangle$, but this is not the case for the brackets of the type $\langle ii+2 \rangle$. It is then straightforward to verify that both expressions are transformed nontrivially under inversion and, therefore, do not respect the special conformal invariance

$$K^\alpha \lambda_i \neq 0.$$ (75)

This conformal anomaly is explicitly calculated in Appendix B where we show complete consistency with the solution of the supersymmetric Ward identity that we constructed in Sect. 3.

Combining together the relations (72) and (75) we conclude that the ratio of the Wilson loops in (73) does possess neither $Q-$supersymmetry, nor conformal (super)symmetry already at one loop. As was already explained in Introduction, this fact is in contradiction with the relations (9) and (13) which follow in their turn from the conjectured duality between supersymmetric Wilson loops and scattering amplitudes in $\mathcal{N} = 4$ SYM.

5 Conclusions

The MHV scattering amplitudes in planar $\mathcal{N} = 4$ SYM are dual to bosonic Wilson loops. In this paper, we explored two recent proposals for extending this duality to generic non-MHV amplitudes. The corresponding dual object should have the same symmetries as the scattering amplitudes and be invariant to all loops under the chiral half ($Q-$ and $S-$symmetries) of the $\mathcal{N} = 4$ superconformal symmetry. The supersymmetric extensions of the bosonic Wilson loop proposed in Ref. \cite{24} comply with this condition at the classical level but only up to terms proportional to field equations. We would like to point out that while our conclusions apply to the supersymmetric Wilson loop in Minkowski space, the twistor space version from Ref. \cite{23} appears to be formulated off-shell and thus the question of whether any pathologies permeate this formalism as well requires extra studies.

Examining the properties of the supersymmetric Wilson loops at the quantum level, one encounters the following complication. According to its definition, the perturbative expansion of
the Wilson loop involves products of various fields (scalars, gauginos and gluon) integrated along a closed contour. Since the integration contour is uniquely determined by the light-like momenta of the scattered particles, these fields inevitably become light-like separated. In the classical theory such an object is well defined in $D = 4$ dimensions, while at the quantum level it suffers from light-cone singularities and requires regularization. This subtlety is not a specific feature of the supersymmetric Wilson loop and it is already present in the bosonic light-like Wilson loop. In the latter case, due to the light-cone singularities of the two-point functions of the gauge fields, the quantum corrections to the bosonic Wilson loop generate ultraviolet divergences which appear as double poles in $\varepsilon$ in the dimensional regularization scheme with $D = 4 - 2\varepsilon$. By the same token, the calculation of loop corrections to the supersymmetric Wilson loop demands the use of a regularization which we chose to be the supersymmetry preserving dimensional reduction. We would like to emphasize that even though the final expressions for the one-loop correction to $W_{4,1}$ (see Eqs. (71) and (70)) are finite as $\varepsilon \to 0$, they arise by adding together the contributions from several Feynman diagrams, each of which develops light-cone divergences and, therefore, requires a regularization.

Since the calculation of the quantum corrections to the supersymmetric Wilson loop involves going away from the critical dimension $D = 4$, one might suspect that some of the classical symmetries will be broken at the quantum level. Indeed, we demonstrated in this paper that, for the supersymmetric Wilson loops under consideration, both the chiral supersymmetry and conformal invariance are already broken at one loop. The underlying mechanism looks as follows. The variation of the Wilson loop under supersymmetry $Q$-transformations is proportional to a contribution from the equation of motion operators. In dimensional regularization with $D = 4 - 2\varepsilon$, the latter scales as $O(\varepsilon)$ and vanishes when the number of space-time dimensions is set to four. However, the correlation function of the equations of motion with the light-like Wilson loop produces a divergent $O(1/\varepsilon)$ contribution. Combining together the two effects, we obtain a nontrivial finite $O(\varepsilon^0)$ contribution to the corresponding supersymmetric Ward identity. The latter takes the form of a differential equation in the Grassmann variables and completely fixes the form of the super-Wilson loops with a minimal perturbative input. The unique solutions that we found were tested against explicit one-loop calculations of the Wilson loops.

Our analysis explicitly demonstrates that the quantum anomalies break the chiral supersymmetry of the Wilson loops, thus invalidating the conjectured duality with the scattering amplitudes in $\mathcal{N} = 4$ SYM. We recall that for the amplitudes the same symmetry (the dual chiral $Q$-supersymmetry) remains exact at the quantum level. In fact, it is a trivial consequence of the way the dual variables are introduced in Eq. (3). Notice that the general solution to the anomalous supersymmetry Ward identity is defined up to an arbitrary supersymmetric invariant function which could be compared with the scattering amplitude. In application to the Wilson loop this would mean that while the super Wilson loop is affected by nonvanishing contributions of the equations of motion, it is its supersymmetric invariant part with the properly subtracted anomaly which is matched with the scattering amplitudes. This would require, however, a detailed knowledge of the all-loop anomaly as well as a consistent formulation of a scheme for

\footnote{It is important to point out that there is a difference in the use of dimensional reduction and dimensional regularization already at one loop. Had we used dimensional regularization instead, the supersymmetry Ward identities would not be fulfilled indicating inconsistencies in the treatment at the quantum level.}

\footnote{This effect is reminiscent to the so-called $\mu^2$-terms in the scattering amplitudes. The latter appear as $O(\varepsilon)$ terms in the expression for the integrand of the dimensionally regularized amplitudes, but they produce a nontrivial contribution after integration over loop momenta starting from two loops.}
separation between anomalous and invariant terms inside the Wilson loop.

The fact that the anomaly in the $Q$–supersymmetry is due to the contributions of the field equations to the Wilson loop correlation function is closely related to the necessity to work with an on-shell realization of the $\mathcal{N} = 4$ supersymmetry algebra. The problem we encounter can be presented as follows. The bosonic superconnection $A_{a\dot{a}}(x, \theta)$ introduced in (20), can be viewed as the supersymmetric extension of the gauge field $A_{a\dot{a}}(x)$ obtained by a finite chiral supersymmetry transformation with generators $Q^a_\alpha$ and with the Grassmann variables $\theta^a_\alpha$ in the role of the parameter:

$$A_{a\dot{a}}(x, \theta) = e^{(\theta \cdot Q)} A_{a\dot{a}}(x), \quad (76)$$

with $(\theta \cdot Q) = \theta^A_\alpha Q^a_\alpha$. In expanding the exponential we make use of the relation $\{Q^a_\alpha, Q^B_\beta\} = 0$ which follows from the chiral supersymmetry algebra. However, in the $\mathcal{N} = 4$ case even this chiral subalgebra of the full Poincaré supersymmetry closes only on shell. Then it is clear that the bosonic connection constructed from (76) is invariant under $Q$–transformations (up to a compensating gauge transformation) only modulo field equations.

In supersymmetric theories the well-known remedy consists in adding sets of so-called auxiliary fields, whose equations of motion are algebraic and whose role is to maintain the closure of the supersymmetry algebra off shell. If we had the relevant auxiliary fields at hand, our super Wilson loop would presumably not suffer from the anomaly we observed. An example, to be presented elsewhere, is provided by $\mathcal{N} = 1$ SYM. In this simplest supersymmetric gauge theory one needs to just add a single scalar auxiliary field in order to close the algebra off shell. The $U(1)$ R symmetry of $\mathcal{N} = 1$ SYM does not allow us to construct a purely chiral extension of the Wilson loop. Instead, we may consider the fully supersymmetric $\mathcal{N} = 1$ super Wilson loop, with and without the auxiliary field in it. This will give us an alternative view on the anomaly mechanism. However, coming back to the $\mathcal{N} = 4$ case, we have to recall the very old result of [37] on the absence of auxiliary fields for $\mathcal{N} = 4$ SYM. In view of this, the $\mathcal{N} = 4$ super Wilson loop seems condemned to suffer from supersymmetry anomalies. The only escape might be that the no-go argument of [37] applies to the full $\mathcal{N} = 4$ supersymmetry, while here we need just its chiral half. So, this old issue needs to be revisited.

Finally, we would like to comment on another approach to the dual description of scattering amplitudes proposed in Refs. [29]. The starting point of this proposal is the correlation function of 1/2 BPS bilinear scalar operators $\mathcal{O}(x) = \text{tr} (\phi^2)$. It was found that, in the multiple light-cone limit $x_{i,i+1}^2 \to 0$, the leading asymptotic behavior of the correlation function $\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle$ is given by a product of free propagators multiplied by a light-like bosonic Wilson loop squared. This result was used to demonstrate the duality between the correlation function of bosonic operators in the light-cone limit $x_{i,i+1}^2 \to 0$ and the MHV amplitudes at loop level. Recently, the duality between the correlation functions and amplitudes was further extended to non-MHV amplitudes in Refs. [30]. In $\mathcal{N} = 4$ SYM, the scalar operator $\mathcal{O}(x)$ is the lowest component of the so-called stress-tensor supermultiplet described by the 1/2 BPS short superfields $\mathcal{T}(x, \theta, \bar{\theta})$. Then, a natural supersymmetric generalization of the correlation function is

$$G_n = \langle \mathcal{T}(x_1, \theta_1, \bar{\theta}_1) \ldots \mathcal{T}(x_n, \theta_n, \bar{\theta}_n) \rangle . \quad (77)$$

This correlation function depends on both Grassmann variables, $\theta_i$ and $\bar{\theta}_i$, and it enjoys the full $\mathcal{N} = 4$ superconformal symmetry to all loops. Then, the duality relation between the correlation function and the superamplitudes [2] is established by setting all $\bar{\theta}_i = 0$ and in the limit where
the points \((x_i, \theta_i)\) coincide with the vertices of the light-like polygon \(C_n\)

\[
\lim_{x_{i,i+1}^2 \to 0} G_n(x_i, \theta_i, \tilde{\theta}_i = 0) \sim \left( \sum_{k=0}^{n-4} a_k \tilde{A}_{n,k}(\lambda_i, \tilde{\lambda}_i, \eta_i; a) \right)^2,
\]

(78)

where the two sets of variables \((x_i, \theta_i)\) and \((\lambda_i, \tilde{\lambda}_i, \eta_i)\) are related to each other through (3) and the proportionality factor is given by the product of \(n\) consecutive scalar propagators \(1/(x_{12}^2 \ldots x_{n1}^2)\). This relation has been checked in Ref. [30] for \(n = 4, 5, 6\) amplitudes at tree- and one-loop level, as well as for the NMHV tree-level amplitudes general \(n\). Unlike the supersymmetric Wilson loop, the \(Q\)–supersymmetry of the correlation function in the left-hand side of (78) is not broken. The reason for this is that the correlation function, viewed as a function of \(x_{i,i+1}^2\), is a less singular object as compared with the Wilson loop. For \(x_{i,i+1}^2 \neq 0\) the former is well-defined in \(D = 4\) dimensions, while the latter suffers from UV divergences due to the presence of (non light-like) cusps on the integration contour and, therefore, requires a regularization [28].

The situation here is very much reminiscent of the one with the operator product expansion (OPE). Let us consider a product of two protected operators in \(\mathcal{N} = 4\) SYM like \(T(x, \theta, \tilde{\theta})\). It is well-defined in \(D = 4\) and does not require any regularization as long as the operators are not null separated. However, expanding the product of operators into the sum of local (Wilson) operators we find that the latter develop anomalous dimensions whose calculation requires introducing a UV regularization and whose explicit expressions depend (starting from two loops) on the choice of the regularization scheme. We recall however that each Wilson operator is accompanied by corresponding coefficient function. It is this coefficient function that insures independence of the product of operators on the renormalization scale as well as its scheme independence. Coming back to the correlation function (77), we expect that in the light-cone limit (78), it reduces, in complete analogy with conventional OPE, to the product of the supersymmetric Wilson loop and a coefficient function. Along these lines, all anomalies of the supersymmetric Wilson loops that we identified in this paper should be compensated by the coefficient function in such a manner that their product is anomaly free.

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A Notations and conventions

We adopt spinor notations from Ref. [36]. We use the following conventions for raising/lowering indicies

\[
\varepsilon^{\alpha\beta} \lambda_\beta = \lambda_\alpha, \quad \lambda^\beta \epsilon_{\beta\alpha} = \lambda_\alpha, \quad \tilde{\lambda}_{\dot{\alpha}} \varepsilon^{\dot{\alpha}\dot{\beta}} = \tilde{\lambda}_{\dot{\beta}}, \quad \epsilon_{\dot{\alpha}\dot{\beta}} \lambda^{\dot{\beta}} = \tilde{\lambda}_{\dot{\alpha}},
\]

(79)

and notations for angle and square brackets

\[
\langle j k \rangle = \lambda_j^\alpha \lambda_{k\alpha}, \quad [j k] = \lambda_{j\dot{\alpha}} \lambda^\dot{\alpha}_k.
\]

(80)
Then
\[ x^{\dot{\alpha}} = \sigma_\mu {^\mu} x^\mu, \quad \partial^{\dot{\alpha}} = \sigma_\mu \dot{\partial} \sigma^\mu. \] (81)

Such that, for instance,
\[ \partial^{\dot{\alpha}} x_{\dot{\beta}} = 2\varepsilon^{\dot{\beta}} \varepsilon_{\dot{\alpha}} \], \quad \partial^{\dot{\alpha}} x^2 = 2x^{\dot{\alpha}}, \] (82)

where we used
\[ g^{\mu\nu} \sigma_\mu \dot{\sigma}_\nu = 2\varepsilon^{\dot{\beta}} \varepsilon_{\dot{\alpha}}. \] (83)

Using these definitions, one can find that
\[ x_{\dot{j}j+1}^{\dot{\alpha}} \partial_{\dot{a}\dot{a}} f(x) = -2\partial f(x), \] (84)

for \( x = x_j - t x_{j+1} \).

In notations of Ref. [36], the \( \mathcal{N} = 4 \) SYM Lagrangian has the following form
\[ \mathcal{L}_{\mathcal{N}=4} = \text{tr} \left\{ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi^{AB} D^\mu \bar{\phi}_{AB} + \frac{1}{8} g^2 [\phi^{AB}, \phi^{CD}] [\bar{\phi}_{AB}, \bar{\phi}_{CD}] + i\bar{\psi}_\dot{a} A^{\dot{a}} D^\mu \psi^A - i(D^\mu \bar{\psi}_\dot{a}) A^{\dot{a}} \psi^A - \sqrt{2} g \bar{\psi}^A \phi^{AB} [\bar{\phi}_{AB}, \psi^A] + \sqrt{2} g \bar{\psi}_\dot{a} A^{\dot{a}} [\phi^{AB}, \bar{\psi}_\dot{b}] \right\} \] (85)

where all fields are in the adjoint representation of the gauge group \( SU(N_c) \), and the generators are normalized as relations
\[ \text{tr}(t^a t^b) = \frac{1}{2} \delta^{ab}, \quad t^a t^a = C_F = \frac{N_c^2 - 1}{2N_c}. \] (86)

It is invariant under (chiral) \( Q \)-supersymmetry transformations of fields
\[ \delta_Q A^\mu = -i \xi^A \sigma^\mu_{\dot{\alpha} \dot{\beta}} \bar{\psi}_A^\dot{\beta}, \]
\[ \delta_Q \phi^{AB} = -i \sqrt{2} \left\{ \xi^A \phi_B^A - \xi^B \phi_A^A \right\}, \]
\[ \delta_Q \psi^A = i \frac{1}{2} F_{\mu\nu} \sigma^\mu_{\dot{\alpha} \dot{\beta}} \xi^A + i g [\phi^{AB}, \bar{\phi}_{BC}] \xi^C, \]
\[ \delta_Q \bar{\psi}_A = \sqrt{2} (D_\mu \bar{\phi}_{AB}) \sigma^\mu_{\dot{\alpha} \dot{\beta}} \xi^B. \] (87)

However, we found it more convenient to redefine these elementary fields and transformation parameter as follows
\[ A^{\dot{\alpha}} = A^\mu \sigma^\alpha_\mu, \quad \bar{\psi}_A^\dot{\alpha} = \sqrt{2} \bar{\psi}_A^\dot{\alpha}, \quad \psi^A = \frac{1}{\sqrt{2}} \psi^A, \quad F^{\alpha\beta} = \frac{1}{4} \sigma^\mu_{\alpha\beta} F^{\mu\nu}, \quad \xi^A = \sqrt{8} \xi^A. \] (88)

Notice that this transformation does not change the kinetic term of the fermions. Here we defined the derivative as \( \partial^{\dot{\alpha}} = \sigma_\mu \dot{\partial} \sigma^\mu \).

In order to simplify notations we use a uniform way of contracting the \( SL(2) \) indices: undotted indices from upper left to lower right and dotted one lower left to upper right, that is \( A^\alpha B_{\alpha\dot{\alpha}} \) and \( A_{\dot{\alpha}} B^{\alpha\dot{\alpha}} \) (instead of \( A^\alpha B_{\alpha\dot{\alpha}} \) and \( A^{\dot{\alpha}} B^{\alpha\dot{\alpha}} \)), and use ket and bra notations
\[ A_{\alpha} \equiv |A\rangle, \quad A^\alpha \equiv \langle A|, \quad A^{\dot{\alpha}} \equiv |A\rangle, \quad A_{\dot{\alpha}} = [A| \] (89)

In these notations, contractions of spinors take the conventional form
\[ A^\alpha B_{\alpha} = \langle AB|, \quad A_{\dot{\alpha}} B^{\dot{\alpha}} = [AB]. \] (90)
Then the $Q$–supersymmetric transformations take the following concise form

$$
\delta_Q A = -i |\epsilon^A| [\bar{\psi}_A],
$$
$$
\delta_Q \phi^{AB} = -i \left\{ (\epsilon^A \psi^B) - (\epsilon^B \psi^A) \right\},
$$
$$
\delta_Q |\psi^A| = i F |\epsilon^A| + \frac{i}{2} \theta [\phi^{AB}, \bar{\phi}_{BC}] |\epsilon^C|,
$$
$$
\delta_Q |\bar{\psi}_A| = D \bar{\phi}_{AB} |\epsilon^B|.
$$

(91)

The Grassmann variables have the following properties

$$
\langle \epsilon^A \theta^B \rangle = \langle \theta^B \epsilon^A \rangle, \quad |\epsilon^A| \langle \theta^B \psi^D \rangle + |\theta^B| \langle \psi^D \epsilon^A \rangle + |\psi^D| \langle \epsilon^A \theta^B \rangle = 0,
$$
$$
\epsilon_{ABCD} \chi_1^A \chi_2^B \chi_3^C \chi_4^D = \epsilon_{ABCD} \chi_2^A \chi_1^B \chi_3^C \chi_4^D = \cdots = \epsilon_{ABCD} \chi_1^A \chi_2^B \chi_3^C \chi_4^D.
$$

(92)

Denoting $\chi_1 \chi_2 \chi_3 \chi_4 \equiv \epsilon_{ABCD} \chi_1^A \chi_2^B \chi_3^C \chi_4^D$ we observe that in the last relation we can deal with $\chi$–variables as if they were commuting variables stripped from their $SU(4)$ indices

$$
\chi_1 \chi_2 \chi_3 \chi_4 = \chi_2 \chi_1 \chi_3 \chi_4 = \cdots = \chi_4 \chi_1 \chi_2 \chi_3.
$$

(93)

## B Conformal anomaly

In this appendix, we elucidate the origin of the conformal anomaly of the supersymmetric Wilson loop (73) and present its explicit one-loop calculation.

To simplify the analysis we shall examine the component of the supersymmetric Wilson loop $\mathcal{W}_{4,1}^{\text{MS}}$ defined in (38) and (59) in the special case where $\chi_2 = \chi_3 = \chi_4 = 0$. According to (70), the corresponding one-loop expression for $\mathcal{W}_{4,1}^{\text{MS}}$ takes the form

$$
\mathcal{W}_{4,1} = -\frac{g^2 C_F}{4\pi^2} \left( \frac{1}{x_{13}^2} + \frac{1}{x_{24}^2} \right) \frac{(24)^2}{\langle 12 \rangle^2 \langle 41 \rangle^2} 24 \equiv W_1 + W_2.
$$

(94)

Here, in the right-hand side, we split $\mathcal{W}_{4,1}^{\text{MS}}$ into the sum of $W_1$ and $W_2$ corresponding to the two terms inside the parentheses. Let us examine the action of the conformal boost $K^{\hat{\alpha}\alpha}$ on the one-loop expression (94). We use $K = IPI$ with the inversion acting on the spinors according to (74), to get from (94)

$$
\delta_K W_1 = W_1 \left\{ 2 \frac{\langle 2 | x_2 \kappa | 4 \rangle}{\langle 24 \rangle} + 2 \frac{\langle 2 | \kappa x_4 | 4 \rangle}{\langle 4 \rangle} - 4 \kappa \cdot (x_1 + x_2) + 2 \kappa \cdot (x_1 + x_3) \right\}
$$
$$
\delta_K W_2 = W_2 \left\{ 2 \frac{\langle 2 | x_2 \kappa | 4 \rangle}{\langle 24 \rangle} + 2 \frac{\langle 2 | \kappa x_4 | 4 \rangle}{\langle 4 \rangle} - 4 \kappa \cdot (x_1 + x_2) + 2 \kappa \cdot (x_2 + x_4) \right\}
$$

(95)

with $\kappa^{\hat{\alpha}\alpha}$ being the transformation parameter. Combining together (95) and (94) we obtain the one-loop expression for the conformal anomaly $\delta_K \mathcal{W}_{4,1} = \delta_K W_1 + \delta_K W_2$.

Let us reproduce the same result by making use of the conformal Ward identity for the supersymmetric Wilson loop. The analysis goes along the same lines as in Sect. 3. Namely, we start with the path integral representation for the supersymmetric Wilson loop (54) and perform a conformal transformation of the (super) coordinates combined with a compensating transformation of the fields inside the path integral. The important difference from the supersymmetric Ward identity discussed in Sect. 3 is that the supersymmetric Wilson loop $\mathcal{W}_4$ stays invariant
under conformal transformations whereas the dimensionally regularized $N = 4$ action changes by an amount proportional to $(4 - D) = 2\varepsilon$ leading to

$$K^{\dot{\alpha}\alpha}\langle W_4 \rangle = -4i\varepsilon \int d^{4-2\varepsilon} x \ x^{\dot{\alpha}\alpha} \langle \mathcal{L}_{N=4}(x) W_4 \rangle .$$

Here the expression in the right-hand side involves the insertion of the $N = 4$ Lagrangian into the supersymmetric Wilson loop. It proves convenient to introduce into consideration the following quantity

$$\tilde{W}_4(k) = -i \int d^{4-2\varepsilon} x \ e^{ikx} \langle \mathcal{L}_{N=4}(x) W_4 \rangle .$$

Then, comparing the last two relations we find that the conformal anomaly is related to the behavior of $\tilde{W}_4(k)$ around $k = 0$

$$K^{\dot{\alpha}\alpha}\langle W_4 \rangle = -2i\varepsilon \frac{\partial \tilde{W}_4(k)}{\partial k_{\dot{\alpha}\alpha}} \bigg|_{k=0} .$$

Moreover, the value of the same function at the origin $\tilde{W}_4(0)$ is given by the insertion of the $N = 4$ action into the supersymmetric Wilson loop and is related to the derivative with respect to the coupling constant

$$\tilde{W}_4(0) = g^2 \frac{\partial \langle W_4 \rangle}{\partial g^2} = \langle W_4 \rangle_{1\text{-loop}} + O(g^4) .$$

As follows from (98), in order for the conformal anomaly $K^{\dot{\alpha}\alpha}\langle W_4 \rangle$ to be different from zero the derivative in the right-hand side of (98) should develop a pole at $\varepsilon = 0$.

For our purposes, it suffices to compute the one-loop correction to $\tilde{W}_4(k)$ proportional to $\chi^4$, which we denote by $\tilde{W}_{4;1}(k)$. To this order in the coupling, the following simplifications occur. Firstly, in the expression for the bosonic and fermion connections, Eqs. (20) and (21), entering into the definition (14) – (17) of the supersymmetric Wilson, we are allowed to retain only the terms involving the gauge field $A$ and the field-strength tensor $F$. Moreover, the Lagrangian $\mathcal{L}_{N=4}(x)$ can be replaced in (97) by the kinetic term for the gauge field. Its net effect is to modify the free gluon propagator. Going through a lengthy calculation we obtain at small $k$

$$\tilde{W}_{4;1}(k) = W_1 \left[ 1 - \frac{i}{\varepsilon} \left( 3\langle 2|k|2 \rangle + \frac{\langle 24 \rangle \langle 1|k|2 \rangle}{\langle 41 \rangle} + O(\varepsilon) \right) + O(k^2) \right]$$

$$+ W_2 \left[ 1 + \frac{i}{\varepsilon} \left( 3\langle 4|k|4 \rangle + \frac{\langle 42 \rangle \langle 1|k|4 \rangle}{\langle 21 \rangle} + O(\varepsilon) \right) + O(k^2) \right] ,$$

with $W_1$ and $W_2$ defined in (94). Notice that the terms linear in $k$ develop a pole in $\varepsilon$ thus indicating that the correlation function (97) is singular for $\varepsilon \to 0$. We can easily verify using (99) that the expression (100) is in agreement with the one-loop result (94). Then, substitution of (100) into (98) yields the one-loop result for the conformal anomaly

$$(\kappa \cdot K) W_{4;1} = -W_1 \left( 3\langle 2|\kappa|2 \rangle + \frac{\langle 24 \rangle \langle 1|\kappa|2 \rangle}{\langle 41 \rangle} \right) + W_2 \left( 3\langle 4|\kappa|4 \rangle + \frac{\langle 42 \rangle \langle 1|\kappa|4 \rangle}{\langle 21 \rangle} \right) .$$

It is now straightforward to confirm that this expression coincides with the expected result for the one-loop conformal anomaly $\delta_K W_{4;1} = \delta_K W_1 + \delta_K W_2$ given by (95).
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