Characterizing Sets of Lower Bounds: a Hidden Convexity Result

Emil Ernst¹ · Alberto Zaffaroni²

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Abstract  This study addresses sets of lower bounds in a vector space ordered by a convex cone. It is easy to see that every set of lower bounds must be simultaneously downward and bounded from above, and must possess the further property that it contains the supremum of any of its subsets which admits one. Our main result proves that these conditions are also sufficient, provided that the ordering cone is polyhedral. Simple counter-examples prove that the sufficiency fails when the polyhedrality assumption is dropped.

Keywords  Partially ordered vector spaces · Polyhedral cone · Set of lower bounds

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1 Introduction

Given \((X, \leq)\) a real vector space which is partially ordered by means of a pointed convex cone \(K \subseteq X\), we shall call a non empty and proper subset \(A\) of \(X\) a set of lower bounds, if

\[ A = \{ x \in X : x \leq y \quad \forall y \in S \} \]

for some set \(S \subset X\).

Dedicated to Michel Théra in honor of his 70th birthday.

¹ Aix Marseille University, CNRS, Centrale Marseille, I2M, Marseille, France
² Università di Modena e Reggio Emilia, Modena, Italy
The aim of this paper is to study how to characterize sets of lower bounds. This is a general question in the theory of ordered vector spaces which did not receive an appropriate answer. The relevance of this notion is made clear, for instance, by MacNeille’s completion theorem ([7, Theorem 25]), which states that, when endowed with its natural order, the class of all the sets of lower bounds of \((X, \leq)\) becomes a Dedekind complete lattice, precisely the smallest Dedekind order complete lattice with \((X, \leq)\) embedded in it.

Although the statement of this problem is fully general, and despite our intention to broaden the analysis in a future research, the main results of the present paper only deal with the particular situation of a finite dimensional space ordered by a polyhedral cone.

When \((X, \leq)\) is a Dedekind complete vector lattice, that is when any non-empty subset of \(X\) which is bounded from above possesses a supremum, it is easy to see that the only sets of lower bounds are the translates of the cone \(-K\), that is the sets of the form \(a - K\), with \(a \in X\).

In this respect, it is relevant to recall the celebrated Choquet-Meyer theorem ([4]), saying that \((X, \leq)\) is a Dedekind complete vector lattice if and only if the ordering cone \(K\) has a base which is a Choquet simplex (see [3]). When the dimension of \(X\) is finite, additional information is provided by the Rogers-Shephard theorem (see [9]), which proves that a set is a Choquet simplex if and only if it is the convex hull of finitely many affinely independent points, as for instance when the vector space \(X = \mathbb{R}^n\) is ordered by the cone \(K = \mathbb{R}^n_+\).

In the general case of a partially ordered space \((X, \leq)\) which is not necessarily a Dedekind complete lattice, the class of sets of lower bounds lacks a clear description. Indeed, again as a consequence of the Choquet-Meyer theorem, it follows that, when the ordering cone is not simplicial, there are lower bounded sets which do not admit an infimum. Hence, it is possible that there are sets of lower bounds which are not mere translates of \(-K\).

Let us notice that, regardless of specific assumption on the ordered vector space \((X, \leq)\), any set \(A\) of lower bounds must satisfy the following properties:

- **P1:** \(a_1 \in A\) and \(a_2 \leq a_1\) imply \(a_2 \in A\) (\(A\) is downward);
- **P2:** there exists \(z \in X\) such that \(a \leq z\), for all \(a \in A\) (\(A\) is bounded from above);
- **P3:** if \(B \subseteq A\) admits a supremum \(y = \sup B\), then \(y \in A\) (\(A\) is sup-containing).

Indeed, any set of the form \(a - K\), with \(a \in X\), obviously satisfies all the above properties, and it is an easy task to prove that the intersection of a family of sets fulfills properties P1-3, provided that the same holds true for every set in the family. A non-empty set \(A\) is said to be *regularly sup-containing* if the three properties above are satisfied. It is then tempting to conclude that properties P1-3 completely characterize the sets of lower bounds, and thus state the following conjecture:

**C:** A non empty proper set \(A \subset X\) is a set of lower bounds if and only if it is regularly sup-containing.

Surprisingly enough, the validity of our conjecture strongly depends on the ordering cone. Thus, the main result of the present study (Theorem 1, Section 3), proves the conjecture for the particular case of a finite dimensional space \(X\) ordered by a polyhedral cone \(K\), while Proposition 7 (Section 4.3) describes the example of a pointed convex cone \(K \subset \mathbb{R}^2\) for which the conjecture is false. Hence, properties P1-3 either give an accurate description of sets of lower bounds, or completely fail in doing so, depending on finely tuned properties of the ordering cone.

Let us also remark that a set of lower bounds is given by the intersection of all the translates of \(-K\) containing it, being hence a convex set. We may accordingly see Theorem 1 as
being a hidden convexity result, as none of the properties P1-3 explicitly requires convexity, and since, as proved by Proposition 7, for a different choice of the ordering cone, some non-convex sets also satisfy properties P1-3.

Exploiting this convexity feature, we prove two different characterizations in dual terms. One, Proposition 3, is based on separation by means of the linear functionals which define the ordering cone. The other, Proposition 5, is based on the properties of the support function.

While the validity of Conjecture C in the case of a closed, convex, solid ordering cone forms an open question, as we are not able to either prove it, or disprove it, the two above mentioned dual characterizations are specific for polyhedral orderings, as shown by a counterexample in \( \mathbb{R}^3 \).

The concluding section of our study gives us the opportunity to discuss this question in some detail.

2 Properties of Polyhedral Cones in Finite Dimensional Spaces

We consider a finite dimensional partially ordered vector space \((X, \preceq)\), in which the relation \( \preceq \) is induced by a convex cone \( K \) which is nontrivial (\( K \neq X \) and \( K \neq \{0\} \)) and solid (its interior is nonempty). Given \( \langle \cdot, \cdot \rangle \) some dot product on \( X \), we denote by \( K^+ \) its dual cone,

\[
K^+ := \{ x \in X : \langle x, y \rangle \geq 0 \ \forall y \in K \},
\]

and by \( \text{ext} K^+ \) the subset of \( K^+ \) gathering all the extreme rays of \( K^+ \).

Let \( S \subseteq X \) be a subset of \( X \); we say that \( z \in X \) is a lower (resp. upper) bound of \( S \) if \( z \preceq s \) (resp. \( s \preceq z \)) for all \( s \in S \). Thus the set \( L(S) \) of lower bounds of \( S \) is given by

\[
L(S) := \{ z \in X : z \preceq s, \ \forall s \in S \} = \bigcap_{s \in S} s - K,
\]

while

\[
U(S) := \{ z \in X : s \preceq z, \ \forall s \in S \} = \bigcap_{s \in S} s + K
\]

stands for the set of upper bounds of \( S \). The set \( L(S) \) is thus defined as the intersection of all the translates of \(-K\) of the form \( s - K \) with \( s \in S \). If \( L(S) \) admits a greatest element \( y \), then \( y \) is called an infimum of \( S \) (\( y = \inf S \)); in this case \( L(S) = y - K \). Analogously, if \( U(S) \) admits a least element \( v \), then \( v \) is called supremum of \( S \), indicated \( v = \sup S \).

In this section, we gather several, more or less elementary, facts about solid polyhedral cones in a finite dimensional space \( X \), that is solid cones which may be expressed as the intersection a finite number of closed half-spaces of \( X \).

For each such cone \( K \), there is a non empty and finite set \( I \) of vectors from \( X \) such that

\[
K = \{ x \in X : \langle x, y \rangle \geq 0 \ \forall y \in I \}. \tag{1}
\]

Of course, for a given polyhedral cone \( K \), the set \( \mathcal{J}(K) \) of all the finite sets \( I \) such that relation (1) holds true is infinite; let us denote by \( F = \{ f_i : i = \overline{1, m} \} \) an element of \( \mathcal{J}(K) \) which is minimal with respect to set inclusion. Thus

\[
K = \{ x \in X : \langle x, f_i \rangle \geq 0 \ \forall i = \overline{1, m} \}. \tag{2}
\]
As a consequence of the previous relation, we are in a position to provide a first characterization of the set $L(S)$.

**Proposition 1** Let $S$ be a nonempty subset of $X$ possessing at least one lower bound, and set $\alpha_k := \inf_{s \in S} \langle s, f_k \rangle$, for $k = 1, m$. Then

$$L(S) = \{z \in X : \langle z, f_k \rangle \leq \alpha_k \ \forall k = 1, m\}.$$ 

**Proof of Proposition 1:** Let $x \in L(S) = \bigcap_{s \in S} s - K$. Then $x \in s - K$ for all $s \in S$, which yields $\langle s, f_k \rangle \geq \langle x, f_k \rangle$ for all $s \in S$ and all $k = 1, m$. Hence

$$\langle x, f_k \rangle \leq \inf_{s \in S} \langle s, f_k \rangle = \alpha_k.$$ 

Conversely, let us assume that $\langle x, f_k \rangle \leq \alpha_k$ for $k = 1, m$, and take some $s \in S$. Then we have $\langle s, f_k \rangle \geq \alpha_k$, and

$$\langle s - x, f_k \rangle \geq 0 \quad \forall k = 1, m.$$ 

(3)

Relation (2) obviously implies the following equivalence:

$$[x \leq y] \iff [\langle x, f_i \rangle \leq \langle y, f_i \rangle \ \forall i = 1, m].$$ 

(4)

By combining relations (3) and (4), it follows that $s - x \in K$ for all $s \in S$, and therefore $x \in \bigcap_{s \in S} s - K$.

Without extra cost, one can give a similar result in a general, not necessarily polyhedral, setting. Indeed, if the ordering cone $K$ is such that

$$K = \{x \in X : \langle x, f \rangle \geq 0 \ \forall f \in F\}$$ 

for some non-empty subset $F$ of $X$, then the proof of Proposition 1 may be used with virtually no modification to deduce that

$$L(S) = \{z \in X : \langle z, f \rangle \leq \inf_{s \in S} \langle s, f \rangle \ \forall f \in F\}.$$ 

(5)

Relation (5) immediately yields

$$\sup_{z \in L(S)} \langle z, f \rangle \leq \inf_{s \in S} \langle s, f \rangle \quad \forall f \in F.$$ 

However, since in our study the set $F$ is minimal and finite, it is possible to prove (see Proposition 2) a much more accurate result.

The next statement (a well-known part of the folklore on the theory of faces of polyhedral cones, proved here for the reader’s convenience) shows that it is possible to pick an element from $K$ such that all but one among the $m$ inequalities in (2) are strict.

**Lemma 1** For every $k = 1, m$, there exists $w_k$ in the boundary of $K$ such that

$$\langle w_k, f_k \rangle = 0 \quad \text{and} \quad \langle w_k, f_i \rangle > 0 \quad \forall i = 1, m \quad \text{s.t.} \quad i \neq k.$$ 

(6)

**Proof of Lemma 1:** Let us pick $k = 1, m$; as $F$ is minimal in $\mathcal{J}(K)$ and $F \setminus \{f_k\} \subset F$, it follows that the closed convex cone

$$K_k := \{x \in X : \langle x, f_i \rangle \geq 0 \ \forall i = 1, m \quad \text{s.t.} \quad i \neq k\}$$

is strictly larger than $K$. But $K$ is solid, so the larger cone $K_k$ must also be solid. Moreover, like any closed convex set with non empty interior, $K_k$ is the closure of its interior.
the interior of \( K_k \) cannot be completely contained in \( K \), as this would imply that \( K_k \) itself lies within \( K \), a contradiction. One can thus find a vector, say \( v_k \), laying in the interior of \( K_k \) but outside \( K \), so that
\[
\langle v_k, f_k \rangle < 0, \quad \langle v_k, f_i \rangle > 0 \quad \forall i = 1, m \text{ s.t. } i \neq k.
\] (7)

Finally, let us pick \( v \), a (non null) vector from the interior of \( K \); hence
\[
\langle v, f_i \rangle > 0 \quad \forall i = 1, m.
\] (8)

From relations (7) and (8) we deduce that there is some \( \lambda_k \in (0, 1) \) such that
\[
\langle w_k, f_k \rangle = 0 \quad \text{and} \quad \langle w_k, f_i \rangle > 0 \quad \forall i = 1, m \text{ s.t. } i \neq k,
\] where \( w_k = \lambda_k v_k + (1 - \lambda_k) v \). Obviously, such a vector \( w_k \) must lie in the boundary of the cone \( K \).

3 The Main Results

3.1 Separating \( L(S) \) and \( S \) by Means of Linear Functionals

Let \( S \) be a nonempty set possessing at least a lower bound, and \( f \in K^+ \). It is obvious that
\[
\sup_{v \in L(S)} \langle v, f \rangle \leq \inf_{w \in S} \langle w, f \rangle ;
\] (10)
this subsection addresses the study of the gap which may exists between the two sides of the above inequality.

Let us first consider the case of a set \( S \) composed by two points \( x, y \in X \) (the more general case of a finite set \( S \) may be studied in a very similar way). It is a well known property (see [6] and also [1]) that, if \( f \in \text{ext} K^+ \), then for every \( \varepsilon > 0 \), there exists an element \( z_\varepsilon \in X \) such that \( z_\varepsilon \leq x, z_\varepsilon \leq y \), and
\[
\langle z_\varepsilon, f \rangle \geq \min (\langle x, f \rangle , \langle y, f \rangle ) - \varepsilon; \] (11)
in other words, the inequality (10) becomes an equality.

The following technical lemma exploits polyhedrality in order to extend relation (11) to the case \( \varepsilon = 0 \).

**Lemma 2** Given \( x, y \), two vectors of \( X \), and \( k = 1, m \) a fixed index, there is an element \( z_k \in X \) such that \( z_k \leq x, z_k \leq y \), and
\[
\langle z_k, f_k \rangle = \min (\langle x, f_k \rangle , \langle y, f_k \rangle ).
\] (12)

**Proof of Lemma 2:** Without restricting the generality, we may assume that \( \langle x, f_k \rangle \leq \langle y, f_k \rangle \), so
\[
\langle x, f_k \rangle = \min (\langle x, f_k \rangle , \langle y, f_k \rangle ).
\]

Set the real number \( \sigma \) by the following relation
\[
\sigma := \frac{\langle y - x, f_k \rangle}{\langle v, f_k \rangle},
\]
where \( v \) is a vector from the interior of \( K \), and define

\[ y_\sigma := y - \sigma v. \]

Obviously, \( \sigma \geq 0 \), so \( y_\sigma \leq y \), and

\[ \langle y_\sigma, f_k \rangle = \langle y, f_k \rangle - \sigma \langle v, f_k \rangle = \langle y, f_k \rangle - \frac{\langle y - x, f_k \rangle}{\langle v, f_k \rangle} \langle v, f_k \rangle = \langle x, f_k \rangle. \]

(13)

Let us now define

\[ t_i := \max \left( 0, \frac{\langle y_\sigma - x, f_i \rangle}{\langle w_k, f_i \rangle} \right) \quad \forall i = 1, m \text{ s.t. } i \neq k, \]

where the vector \( w_k \) is given by relation (6), and set

\[ \tau := \max \{ t_i : i = 1, m \text{ s.t. } i \neq k \}. \]

Clearly, \( \tau \geq 0 \), so \( y_\sigma - \tau w_k \leq y_\sigma \leq y \). Moreover,

\[ \langle y_\sigma - \tau w_k, f_k \rangle = \langle y_\sigma, f_k \rangle - \tau \langle w_k, f_k \rangle = \langle y_\sigma, f_k \rangle, \]

so from relation (13) it follows that

\[ \langle y_\sigma - \tau w_k, f_k \rangle = \langle x, f_k \rangle, \]

(14)

while

\[ \langle y_\sigma - \tau w_k, f_i \rangle = \langle y_\sigma, f_i \rangle - \tau \langle w_k, f_i \rangle \]

\[ \leq \langle y_\sigma, f_i \rangle - \frac{\langle y_\sigma - x, f_i \rangle}{\langle w_k, f_i \rangle} \langle w_k, f_i \rangle = \langle x, f_i \rangle \quad \forall i = 1, m \text{ s.t. } i \neq k. \]

(15)

Relations (14), (15) and (4) prove that \( y_\sigma - \tau w_k \leq x \). Accordingly, the vector \( y_\sigma - \tau w_k \) is smaller then both \( x \) and \( y \), and (in virtue of relation (14)) it also satisfies relation (12); Lemma 2 is thus fulfilled for \( z_k := y_\sigma - \tau w_k \). \( \square \)

The case of a not necessarily finite set \( S \) is much more complicated. Indeed, standard examples (see for instance the one illustrated in Section 4.2) prove that the gap between \( \sup_{v \in L(S)} \langle v, f \rangle \) and \( \inf_{w \in S} \langle w, f \rangle \) may be nonzero even for vectors \( f \) lying in \( \operatorname{ext} K^+ \).

When \( K \) is polyhedral, we can prove that the inequality in (10) becomes an equality for all \( f \in \operatorname{ext}(K^+) \). Thus, for any given \( f_k \in F \), we can deduce the largest value of \( f_k \) on \( L(S) \) only by looking at the values of \( f_k \) on \( S \).

**Proposition 2** Let \( X \) be a finite dimensional space ordered by a solid polyhedral cone \( K \). Given a nonempty and lower bounded set \( S \subseteq X \), and \( f \in \operatorname{ext} K^+ \), then it holds that

\[ \sup_{v \in L(S)} \langle v, f \rangle = \inf_{w \in S} \langle w, f \rangle. \]

(16)

**Proof of Proposition 2:** Reasoning by contradiction, assume that, for some \( k = 1, m \), it holds that

\[ \alpha_k := \inf_{w \in S} \langle w, f_k \rangle > \sup_{v \in L(S)} \langle v, f_k \rangle =: \beta_k, \]

and choose \( y \in X \) such that \( \langle y, f_k \rangle = \gamma \in (\beta_k, \alpha_k) \).
Now take $w_k$ as in Lemma 1 and define

$$t_i := \frac{\langle y, f_i \rangle - \alpha_i}{\langle w_k, f_i \rangle} \quad \forall i = 1, m \text{ s.t. } i \neq k,$$

$t = \max_{i \neq k} t_i$, and $y_k = y - tw_k$. Then we have, for all $i \neq k$, that

$$\langle y_k, f_i \rangle = \langle y, f_i \rangle - t \langle w_k, f_i \rangle \leq \langle y, f_i \rangle - t_i \langle w_k, f_i \rangle = \alpha_i.$$

Moreover we have

$$\langle y_k, f_k \rangle = \langle y, f_k \rangle - t \langle w_k, f_k \rangle = \langle y, f_k \rangle = \gamma < \alpha_k.$$

Thus it holds

$$\langle y_k, f_i \rangle \leq \alpha_i \quad \forall i = 1, m,$$

while the inequality

$$\langle y_k, f_k \rangle = \gamma > \beta_k = \sup_{z \in L(S)} \langle z, f_k \rangle$$

shows that $y_k \notin L(S)$, contradicting in this way Proposition 1.

By a geometric point of view, we can visualize the previous result by saying that $S$ and $A := L(S)$ are separated with no gap by every functional in $\text{ext } K^+$, provided that the ordering cone $K$ is polyhedral.

### 3.2 Direct and Dual Characterizations of a Set of Lower Bounds

The results presented above become relevant in the description of sets of lower bounds in the general setting in which the vectors $f_1, ..., f_m$ are not linearly independent, and thus Choquet-Meyer’s Theorem does not apply.

Obviously, $A$ is a set of lower bounds if and only if it satisfies the following separation property:

$$\forall y \notin A, \exists z \in X : A \subseteq z - K \text{ and } y \notin z - K. \quad (17)$$

Using relation (17), we obtain a description of sets $A$ of lower bounds by means of translates of $-K$: a description “from outside”. This description is somehow analogous to the well-known characterization “from outside” of closed convex sets as intersection of closed half-spaces. Of course, the family of closed, convex sets can also be described “from inside”; conjecture $C$ attempts to provide an analogous “from inside” description for the case of sets of lower bounds.

Despite being a simple rewording of the definition, the separation property (17) is worth being emphasized. While there are many instances in which separation is used to characterize classes of sets more general than convex ones, here the elementary separating sets are convex sets of a special type, and thus we obtain a kind of separation which is less general than the one in which half-spaces are used, and we can use relation (17) to characterize a particular subclass of convex sets in $X$.

A very similar situation is encountered for instance when closed balls of some normed space $X$ are used as separating sets in the “from outside” description: in this framework, it can be seen (see [5]) that every Banach space whose norm is Frechét differentiable enjoys the so-called Mazur Intersection Property, that is every closed, bounded, convex set is the intersection of closed balls containing it.
The following result offers the first characterization of sets of lower bounds. It can be seen as a dual description, in terms of extreme functionals in $F$, of the separation property (17) by means of translates of $-K$, which holds for sets of lower bounds.

**Proposition 3** Let $X$ be a finite dimensional space ordered by a solid polyhedral cone $K$. The set $A \subset X$ is a set of lower bounds if and only if it is upper bounded and for every $y \notin A$ there exists $f \in F$ such that

$$\langle y, f \rangle > \sup_{a \in A} \langle a, f \rangle.$$ 

**Proof of Proposition 3:** If $A = L(S)$ for some set $S \subset X$ and $y \notin A$, then there exists $s \in S$ such that $y \notin s - K$ and $A \subseteq s - K$. The latter implies $\sup_{a \in A} \langle a, f_i \rangle \leq \langle s, f_i \rangle$ for all $i = 1, \ldots, m$. Moreover there exists $k = 1, \ldots, m$ such that $\langle y, f_k \rangle > \sup_{x \in s - K} \langle x, f_k \rangle$. Observe that $\sup_{x \in s - K} \langle x, f_i \rangle = \langle s, f_i \rangle$, for all $i = 1, \ldots, m$, so that $\langle y, f_k \rangle > \sup_{a \in A} \langle a, f_k \rangle$.

To prove the converse we will show that (17) holds. To this aim, consider $y \notin A$ and apply the assumptions to find $f \in F$ such that $\langle y, f \rangle > \sup_{a \in A} \langle a, f \rangle$. Now use Proposition 2 with $S = U(A)$, so that $\langle y, f \rangle > \inf_{x \in S} \langle s, f \rangle$ and there exists $z \in U(A)$ such that $\langle y, f \rangle > \langle z, f \rangle = \sup_{x \in z - K} \langle x, f \rangle$, and $y \notin z - K$. Moreover $z \in U(A)$ means that $A \subseteq z - K$, and the result is proved. □

**Remark 1** The type of linear separation indicated in Proposition 3 was named $H$-convexity by Boltyanski [2]. Given a normed space $X$ and a family $H \subseteq S^*$, the dual unit sphere, it refers to the possibility of representing a set $A$ as the intersection of closed halfspaces defined by means of functionals in $H$. The original definition also asks that $H$ is not one-sided, that is $H$ is not included in a closed halfsphere of $S^*$. This does not hold true for $F$. But for instance [8] drops the latter requirement.

As a consequence of Proposition 2, it is possible to provide a description of the sets $B \subset X$ possessing a supremum.

**Proposition 4** Let $X$ be a finite dimensional space ordered by a solid polyhedral cone $K$. Given $B \subset X$, then it holds that $y = \sup B$ if and only if

$$\sup_{b \in B} \langle b, f_i \rangle = \langle y, f_i \rangle \quad \forall i = 1, m. \tag{18}$$

**Proof of Proposition 4:** If relation (18) holds, then we have

$$\langle b - y, f_i \rangle \leq 0 \quad \forall b \in B \quad \forall i = 1, m,$$

which yields $B \subseteq y - K$. This implies $y \in U(B)$ and $y + K \subseteq U(B)$. To prove that $U(B) = y + K$ suppose by contradiction that there exists $w \in U(B)$ such that $w - y \notin K$. Then there exists $k = 1, m$ such that $\langle w - y, f_k \rangle < 0$.

Since $B \subseteq w - K$, then

$$\sup_{b \in B} \langle b, f_k \rangle \leq \sup_{z \in w - K} \langle z, f_k \rangle = \langle w, f_k \rangle < \langle y, f_k \rangle. \tag{19}$$

But the strict inequality in (19) contradicts (18).
Conversely, let us consider $B$ a set admitting a supremum, and set $y = \sup B$. Clearly, $U(B) = y + K$ and relation
\[ \sup_{b \in B} \langle b, f_i \rangle = \inf_{z \in y + K} \langle z, f_i \rangle = \langle y, f_i \rangle \quad \forall i = 1, m. \]
stems from Proposition 2.

We are now in a position to prove the conjecture $C$ for the particular case of a finite dimensional vector space ordered by a cone which is solid and polyhedral.

Our result says that a $A \subset X$ is a set of lower bounds if and only if it is regularly sup-containing. Thus, we obtain the desired ”from inside” description of sets of lower bounds. Moreover, we prove that regularly sup-containing sets are automatically closed and convex, provided that the ordering cone is polyhedral.

**Theorem 1** Let $X$ be a finite dimensional space ordered by a solid polyhedral cone $K$. Let us also consider a nonempty proper subset $A \subset X$ satisfying relations P1-3, and let us set $\alpha_i := \sup_A \langle \cdot, f_i \rangle$, and define $S := U(A) = \{ x \in X : \langle x, f_i \rangle \geq \alpha_i \quad \forall i = 1, m \}$. Then
\[ A = \{ x \in X : \langle x, f_i \rangle \leq \alpha_i \quad \forall i = 1, m \} = L(S). \quad (20) \]

**Proof of Theorem 1:** We deduce from Proposition 1 that
\[ L := \{ x \in X : \langle x, f_i \rangle \leq \alpha_i \quad \forall i = 1, m \} \]
is the set of lower bounds of $S$. Then we need to show that $A = L$. As the relation $A \subseteq L$ is obvious, we have to prove that
\[ A \supset \{ x \in X : \langle x, f_i \rangle \leq \alpha_i \quad \forall i = 1, m \}. \quad (21) \]

To this aim, let us consider an element $x \in X$ such that
\[ \langle x, f_i \rangle \leq \alpha_i \quad \forall i = 1, m, \]
and let us pick $k = 1, m$. Since $\alpha_k$ is the supremum of the function $\langle \cdot, f_k \rangle$ over $A$, we deduce that there is a sequence $(y_{k,n})_{n \in \mathbb{N}} \in A$ such that
\[ \lim_{n \to +\infty} \langle y_{k,n}, f_k \rangle = \alpha_k. \]

By applying the conclusion of Lemma 2 to $x$ and $y_{k,n}$, we obtain a vector $z_{k,n}$ such that $z_{k,n} \leq y_{k,n}$, $z_{k,n} \leq x$, and
\[ \langle z_{k,n}, f_k \rangle = \min \left( \langle x, f_k \rangle, \langle y_{k,n}, f_k \rangle \right). \quad (22) \]

We have thus defined a sequence $(z_{k,n})_n$ of elements from $A$ $(z_{k,n}) \in A$ since $A$ is downward, $y_{k,n} \in A$ and $z_{k,n} \leq y_{k,n}$), which is also bounded above by $x$ (indeed, $z_{k,n} \leq x$). If moreover we let $n$ go to $+\infty$ in (22), we deduce that
\[ \lim_{n \to +\infty} \langle z_{k,n}, f_k \rangle = \min \left( \lim_{n \to +\infty} \langle y_{k,n}, f_k \rangle, \langle x, f_k \rangle \right) = \min (\alpha_k, \langle x, f_k \rangle) = \langle x, f_k \rangle. \quad (23) \]

Accordingly, the set
\[ B := \{ z_{i,n} : i = 1, m \text{ and } n \in \mathbb{N} \} \]
is a subset of $A$, it is bounded from above by $x$, and, by virtue of relation (23), it holds that

$$\sup_B \langle \cdot, f_i \rangle = \langle x, f_i \rangle \quad \forall i = 1, m.$$  \hspace{1cm} (24)

Obviously, $x$ is an upper bound for $B$, and, if $w \in X$ is another upper bound for $B$, it follows from relation (4) that

$$\langle w, f_i \rangle \geq \sup_B \langle \cdot, f_i \rangle \quad \forall i = 1, m;$$ \hspace{1cm} (25)

combine relations (24), (25) and relation (4) to conclude that $x$ is smaller than any upper bound of $B$, that is $x$ is a supremum of the subset $B$ of $A$.

Since $A$ satisfies property P3, $x$ is an element of $A$, and relation (21) is true. Thus the proof of Theorem 1 is completed.

Once we are able to identify the sets of lower bounds as the ones which are regularly sup-containing, then we can exploit properties P1-3 to obtain a dual characterization of a set $A$ of lower bounds in terms of its support function $\sigma_A : X^* \rightarrow \mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$, given by

$$\sigma_A(f) = \sup_{a \in A} \langle a, f \rangle.$$ 

We can prove that, provided $\sigma_A$ majorizes a linear function on $F$, then it majorizes the same function on $K^+$. 

**Proposition 5** Let $X$ be a finite dimensional space ordered by a solid polyhedral cone $K$. The set $A \subseteq X$ is a set of lower bounds if and only if its support function $\sigma_A$ satisfies:

i) the effective domain of $\sigma_A$ is $K^+$;

ii) if $\langle y, f \rangle \leq \sigma_A(f) \quad \forall f \in F$, then $\langle y, \ell \rangle \leq \sigma_A(\ell) \quad \forall \ell \in K^+.$

**Proof of Proposition 5:** Property P2 yields $\sigma_A(\ell) \leq \sigma_{-K}(\ell) = \langle z, \ell \rangle$, for all $\ell \in K^+$ and $z \in U(A)$, while P1 says that $\sigma_A(\ell) = +\infty$ if $\ell \notin K^+$, so that the validity of P1 and P2 is equivalent to (i). Moreover the inclusion $B \subseteq A$ is equivalent to $\sigma_B \leq \sigma_A$ on $X^*$, while the relation $y = \sup B$ amounts to $\sigma_B(f) = \langle y, f \rangle$ for all $f \in F$. Hence P3 becomes (ii).

### 4 Three Examples

In this section we show by means of three examples that our assumptions cannot be completely dispensed with.

#### 4.1 Partially Ordered Sets

The notions of lower and upper bounds, infimum and supremum, and (regularly) sup-containing set, can be easily extended to partially ordered sets which are not linear spaces.

It is not very complicated to prove that if $X$ is a finite poset with four or less elements, the conjecture C is automatically verified. The following example address the case of a poset $X$ containing five elements, and constructs a regular sup-containing set $A$ which is not a set of lower bounds.

Let indeed $X = \{a, b, c, d, e\} \subseteq \mathbb{R}^2$ with $a = (2, 2), b = (1, 1), c = (0, 3), d = (5, 4)$ and $e = (4, 5)$ with the component-wise order relation $\leq$, and let $A = \{b, c\}$. It is easy to see that $A$ is regularly sup-containing but it is not a set of lower bounds.
4.2 An Example in $\mathbb{R}^3$

The assumption that the ordering cone is polyhedral in Proposition 2 cannot be extended to include any closed, convex, pointed, solid cone $K$. Indeed consider $X = \mathbb{R}^3$ with $K = \{(x, y, z) \in \mathbb{R}^3 : z \geq \sqrt{x^2 + y^2}\}$ the so-called Lorentz cone. Let $A = -K \cap H$, with $H = \{(x, y, z) : x + z \leq -1\}$. The set $A$ is not a set of lower bounds, as it holds $K = U(A)$, so that $A$ admits the origin as its supremum, so the intersection of all translates of $-K$ containing $A$ is $-K$, which is different from $A$. For all linear functionals $f$ which are extreme directions of $K^+ = K$, it holds $\sup_{a \in A} \langle a, f \rangle = 0$, except for (the ones proportional to) $\bar{f} = (1, 0, 1)$, for which $\sup_{a \in A} \langle a, \bar{f} \rangle = -1$ holds.

This example contradicts Proposition 2 as the valuation of $\bar{f}$ on $A$ and $U(A)$ shows a gap, as well as one of the implications in both Propositions 3 and 4.

4.3 An Example in $\mathbb{R}^2$

As for the validity of Theorem 1, although there is no evidence that it cannot be extended to closed, convex, solid cones, we can show that a convex, solid ordering cone is not enough. Indeed, let us consider $X = \mathbb{R}^2$ and $K = \{(0, 0)\} \cup \mathbb{R}^2_{++}$. In this particular setting, the following very strong result is in order.

Proposition 6 If the supremum of a set exists, then it belongs to the set itself.

Proof of Proposition 6: Let $x = (x_1, x_2) \in \mathbb{R}^2$ be the supremum of the set $A \subset \mathbb{R}^2$. Since $x$ is the smallest upper bound for $A$, and since $x$ and $y := (x_1 + 1, x_2)$ are incomparable (indeed, the vector $y - x = (1, 0)$ belongs neither to $K$, nor to $-K$), we can conclude that $y$ is not an upper bound of $A$.

Consequently, $A$ does not entirely lie in $y - K$; as on the other hand, $A$ is contained in $x - K$, it follows that $A$ has at least an element in $(x - K) \setminus (y - K)$:

$$A \cap ((x - K) \setminus (y - K)) \neq \emptyset.$$ 

Since $(x - K) \setminus (y - K) = \{x\}$, then $A \cap \{x\} \neq \emptyset$, which is another way of saying that $x$ belongs to $A$. \hfill $\square$

In this case, any subset of $X$ satisfies property P3, and the following result is not a surprise.

Proposition 7 Given $X = \mathbb{R}^2$ and $K = \{(0, 0)\} \cup \mathbb{R}^2_{++}$, the set $A := ((-1, 0) - K) \cup ((0, -1) - K)$ fulfills properties P1-3, but it is not a set of lower bounds.

Proof of Proposition 7: It is obvious that any union of translates of $-K$ satisfies property P1; in particular, the set $A$, as being the union of two translates of $-K$, fulfills this requirement.

It is again easy to see that $(1, 1)$ is an upper bound for $A$, so property P2 is equally fulfilled.

Since any subset of $\mathbb{R}^2$ satisfies property P3, it means that we have proved that all conditions P1-3 are fulfilled by the set $A$. 

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On the other hand, although the vectors \((-1, 0)\) and \((0, -1)\) are members of \(A\), their mid-point, \((-\frac{1}{2}, -\frac{1}{2})\) does not belong to \(A\) (indeed, \(A\) can alternatively be described as the union of all the vectors lower than at least one between the two vectors \((-1, 0)\) and \((0, -1)\); and \((-\frac{1}{2}, -\frac{1}{2})\) is incomparable to both the vectors \((-1, 0)\) and \((0, -1)\)).

Accordingly, the set \(A\) is not convex, so it cannot be a set of lower bounds.

5 Conclusions

In the case of a finite dimensional vector space ordered by a solid and polyhedral cone, Theorem 1 indicates a simple manner to verify that a given set is a set of lower bounds. Indeed, in order to decide if a set \(A\) is the intersection of a family of translates of the cone \(-K\), it is enough to verify that \(A\) is a bounded from above downward set, which also contains the supremum of each of its subsets possessing one.

When the polyhedrality of the ordering cone is dropped, Proposition 7 informs us about the possibility that a non-convex set verifies conditions P1-3. It is hence natural to ask under what assumptions on the ordering cone \(K\), fulfilling conditions P1-3 is sufficient for a set to be a set of lower bounds. At our best knowledge, this question is open, and no simple solution seems in reach.

An important step in answering this question is to completely understand the role of the polyhedrality of the ordering cone in the validity of the conjecture \(C\): an example of a non-polyhedral cone granting that the conjecture \(C\) holds true will be pivotal for a detailed analysis of this problem.

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