Four-quark state in QCD

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Abstract

The spectra of some $0^{++}$ four-quark states, which are composed of $\bar{q}q$ pairs, are calculated in QCD. The light four-quark states are calculated using the traditional sum rules while four-quark states containing one heavy quark are computed in HQET. For constructing the interpolating currents, different couplings of the color and spin inside the $\bar{q}q$ pair are taken into account. It is found that the spin and color combination has little effect on the mass of the four-quark states.

1 Introduction

The introduction of color and the development of QCD explain the classification and many spectroscopic characteristics of hadrons, they predict the possibility of the existence of exotic hadronic structures such as glueball and multiquark states too. However, in contrast with the normal hadrons, the existence of multiquark states is neither forbidden nor required by color confinement, so the identification of multiquark states is an interesting topic. It is possible too that four-quark configurations lead to hadron-hadron potentials, which play an important role in final state interactions. Therefore, even though there exists no multiquark state, it is helpful to study multiquark configurations to understand final state interactions.

The search for exotics has gone on for a long time, especially in light hadrons energy region. As we know, there are several ambiguous resonances among the light resonances, such as the $\sigma(400 - 1200)$, $a_0(980)$, $f_0(980)$, $f_0(1500)$, $f_2(1710)$, etc. There are many analyses of these resonances. However, since we know little about glueball and multiquark states, especially because they are complicated by mixing among these hadron states, we could not identify them at present. These light hadron states need further investigation. Not until we have complete knowledge about the normal hadron, glueball and multiquark state, can we classify the hadron zoo unambiguously. Experimentally, there are several four-quark candidates, such as $a_0(980)$, $f_0(980)$, $f_1(1420)$, $f_2(1565)$, and $\Psi(4040)$ etc. The $a_0(980)$ and $f_0(980)$ lie below the threshold of $K\bar{K}$; the rest lie below the threshold of some other meson pairs ($K\bar{K}^*$, $\omega\omega$, and $D^*D^*$) too. In practice, their characteristics of decay imply intensely their multiquark bound state role.

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Theoretically, the four-quark system has previously been studied in the frameworks of the bag model, of nonrelativistic potential models, and from many other points of view, but the conclusions for the existence of four-quark state are quite model dependent, and few investigations are based on QCD. The $a_0(980)$ and $f_0(980)$ have been interpreted as the four-quark states in many papers, but none of them has been definitely confirmed as the four-quark state by experiments so far. It is widely believed that the QCD sum rule is a capable nonperturbative method to extract the properties of hadrons, but there exist few works on multiquark states, especially systematic calculations of four-quark states. So we hope to examine the properties of four-quark states with this method in this paper.

Since there are four quarks (including antiquarks) in this multibody system, the analysis is much more complicated than that for the normal hadron. Apart from the complication of dynamical interaction among the quarks, the couplings of color, spin, and flavor among the quarks are not unique either. It is obvious that the four-quark state can consist of diquark-antidiquark or $q\bar{q}$ pairs (meson-meson like). We will concentrate our analysis on the simple $0^{++}$ meson-meson-like four-quark states only.

The spin of the quarks in each $q\bar{q}$ pair can couple to both singlets and triplets, while color can couple to both singlets and octets, so there are different combinations for the spin and color between quarks. The QCD sum rule is based on the assumption of the existence of bound state and resonance, but it cannot distinguish the interaction between quarks inside the hadron directly. However, we try to construct a different sum rule with different current, in which couplings of the spin and color are different, to detect some information about the four-quark states. For light quark systems, it may be insufficient to predict the characteristics of the interaction inside the hadron. However, in the case of heavy quark systems, it is possible to study the interactions using the sum rule from the analysis of the wave function of this system. For the presence of interactions, different four-quark states with the same overall quantum numbers will mix with each other, which makes our analyses much more difficult. Accordingly, the mixing between these currents has not been taken into account either. To find out any difference between the light four-quark states and heavy four-quark states explicitly, we give the sum rules and conclusions for them, respectively. Surely, we have no intention of identifying the $a_0(980)$ and $f_0(980)$ only through analyses of the spectra of four-quark states, and no attention has been placed on them.

This paper is organized as follows. The features of four-quark state have been analyzed in Sec. 2. In Sec. 3, the sum rules for the $0^{++}$ light four-quark states have been constructed. The heavy four-quark states with one heavy quark have been analyzed in heavy quark effective theory (HQET) in Sec. 4. We give the numerical results of the spectra in Sec. 5. The last section is reserved for the conclusion and discussion.

2 Review of the features of four-quark state

In the bag model, once the confinement has been imposed on by the boundary condition and four quarks inside the bound state have been arranged symmetrically about their center of mass, the spectra and dominant decay couplings were calculated. It was found that $q^2\bar{q}^2$ resonances were generally too broad and heavy to show up as bumps in mass spectra except the $0^{++}$ state, which can be seen in phase shift analyses. The lighter $0^{++}$ four-quark states
were predicted to couple strongly to two pseudoscalars while the heavier coupled strongly
to two vectors. The observed $a_0(980)$ and $f_0(980)$ have been identified as the isospin-1 and
isospin-0 four-quark states, respectively.

In the nonrelativistic potential model\textsuperscript{[1],[3],} after the introduction of the color dependent
confinement force and hyperfine interactions, the existence of four-quark state is predicted
as a dynamical solution of the Schrödinger equation. In contrast with the prediction of bag
model, it was found that normally the ground state of this four-quark system consisted of
two free mesons except for the $K\bar{K}$ system(named for the $K\bar{K}$ molecule), which was in fact
a weakly bound $0^{++}$ state. There does not exist a rich discrete spectrum of four-quark states
either. The $a_0(980)$ and $f_0(980)$ have been interpreted as this kind of isospin-1 and isospin-0
$K\bar{K}$ molecule, respectively. There are some other points of view too\textsuperscript{[6]}.

For the combination of four color quarks into a color singlet hadron, there exist different
ways. We can combine two color triplet q’s into a color 6 or 3; similarly, we can combine
two antitriplet $\bar{q}$’s into 3 or 6. Therefore, there are two ways to combine the four quarks
$q\bar{q}q\bar{q}$ into the final color singlet: 33 and 66. The four-quark states can consist of two $q\bar{q}$ pairs
too; then we can combine the $q\bar{q}$ pair into color singlet 1 and color octet 8, and obtain the
final color singlet for the four-quark states from color combination: 11 and 88. It is more
complicated because the 33 66 couplings can mix to give the 11-88 color configurations.

Neither $\vec{L}$ nor $\vec{S}$ is conserved in a relativistic quark model. Nevertheless, if we consider
only the S-wave sector, the algebra generated by the states and their currents is an $SU(2)$.
Therefore, the spin couplings between two quarks can be symmetric triplet and asymmetric
singlet. For example, the four-quark scalar states can be considered either as bound states
of two ordinary pseudoscalar mesons or as bound states of two ordinary vector mesons. As
for the flavor combination, the literature\textsuperscript{[4]} has a detailed description. The special character
of the four-quark states may lie in the existence of flavor exotics, such as $Q = 2$ or $S = 2$,
which may be our best chance to find genuine multiquark states.

It was suggested too that, similar to the bound states below the $K\bar{K}$ threshold, bound
states $D\bar{K}$ and $DK$ should exist near and possibly below the $DK$ threshold\textsuperscript{[6]}. However,
except for the shortness of experiment, there existed few quantitative calculations of this
kind of heavy quark system either. The development of heavy quark effective theory provides
us a capable tool to deal with with these systems.

In the analyses of the literatures mentioned above, all the interactions were put into
theory by hand, so conclusions about the properties of interactions between quarks inside
four-quark states are not conclusive either. Taking into account the complication of couplings
of spin and color inside the possible physical four-quark states, we try to construct sum rules
with different currents to explore them. No matter how much we can extract from these
analyses, they are helpful to us from the point of view of the sum rule itself. For flavor, all
the following calculations are kept in the unbroken symmetric $SU(3)$.

3 Sum rules for the light four-quark state

First, let us consider the light $q\bar{q}$ pairs four-quark states, in which the color couples to a
singlet. For the scalar bound states, they can be regarded as either two bound pseudoscalars
or two two bound vectors. So we choose the current as
\[ j_1(x) = (q\Gamma^m q)(\bar{q}\Gamma^n q)(x), \]  
(1)
where for the pseudoscalar quark pairs, \( \Gamma = \gamma_5 \), while \( \Gamma = \gamma_\mu \) for the vector pairs. \( \Lambda^m \) is the generator of flavor \( SU(3) \).

To obtain the operator product expansion, two kinds of Feynman diagram should be taken into account: one is unbound while the other is bound. For the pseudoscalar current, the contribution of the bound one is only \( 1/12 \) of the unbound one; it is the same case for other currents (the suppressed factor may not be \( 1/12 \) too). Therefore, we will omit the contributions of bound diagram in the following calculations. All our calculations are in the \( x \) representation\(^{[9]} \), and only those terms contributing to the sum rules after Borel transformation are kept in the following formulas.

The operator product expansion for the correlation function with pseudoscalar pairs inside the currents can be expressed as
\[ \Pi(q^2) = i \int dx e^{ixq} \langle T\{j_1(x), j_1^\dagger(0)\} \rangle, \]  
(2)
\[ = -A(q^2)^4 \ln(-q^2) - B(q^2)^2 \ln(-q^2) - Cq^2 \ln(-q^2), \]
where \( A, B \) and \( C \) correspond to the perturbative contribution, two-gluon condensate, and four-quark condensate, respectively, while two-quark condensate vanishes.

For \( \Gamma = \gamma_5 \),
\[ A = \frac{1}{163840\pi^6}, B = \frac{3}{2^{11}\pi}, C = \frac{\langle \bar{q}q \rangle^2}{64\pi^2}; \]  
(3)
while
\[ A = \frac{1}{40960\pi^6}, B = \frac{\langle \bar{q}q \rangle^2}{32\pi^2}; \]  
(4)
in the case of \( \Gamma = \gamma_\mu \), where the two-gluon condensate vanishes also.

The imaginary part of the correlation functions can be represented as
\[ Im\Pi(s) = \pi f_0^2(m^4)^2\delta(s - m^2) + \pi(As^4 + Bs^2 + Cs)\theta(s - s_0), \]  
(5)
where the first term is from the lowest lying bound state or resonance and the second one is from higher resonances or continuum states.

Similarly, the \( \bar{q}q \) pair can couple to a color octet too, and the currents are chosen as
\[ j_2(x) = f^{abc_1} f^{abcd_2}(\bar{q}^{b_1}\Gamma^m q^{c_1})(\bar{q}^{b_2}\Gamma^m q^{c_2})(x), \]  
(6)
where \( \Gamma \) is the same as the definition below formula (1).

The coefficients of the correlation functions for \( \Gamma = \gamma_5 \) and \( \Gamma = \gamma_\mu \) are
\[ A = \frac{3}{163840\pi^6}, B = \frac{9}{2^{11}\pi}, C = \frac{3}{64\pi^2}. \]  
(7)
and
\[ A = \frac{3}{40960\pi^6}, \quad C = \frac{3\langle \bar{q}q \rangle^2}{8\pi^2}, \] (8)
respectively.

Then after equating the quark sides with the hadron sides with the dispersion relation, we obtain the mass of the four-quark state,
\[ m^2(s_0, \tau) = \frac{R_{k+1}(s_0, \tau)}{R_k(s_0, \tau)}, \] (9)
where \( s_0 \) is the continuum threshold, \( \tau \) is the Borel transformation variable and
\[ R_k(\tau, s_0) = \frac{1}{\tau^2}L[(q^2)^k \{\Pi(Q^2) - \sum_{k=0}^{n-1} a_k(q^2)^k\}] - \frac{1}{\pi} \int_{s_0}^{+\infty} s^k e^{-s\tau} Im\Pi^{pert}(s) ds \] (10)
\[ = \frac{1}{\pi} \int_{0}^{s_0} s^k e^{-s\tau} Im\Pi(s) ds. \]

4 Sum rules for the heavy four-quark state

In this section, we will discuss the scalar four-quark systems with one heavy quark. With the same consideration as the previous section, the interpolating current corresponding to the color singlet of a quark-antiquark pair is chosen as
\[ j_3(x) = (\bar{q}\Gamma h_v)(\bar{q}\Gamma \Lambda^m q) \] (11)
where \( q(x) \) is the light quark field, \( h_v(x) \) is the heavy quark effective field, and \( v \) is the velocity of the heavy quark.

Then, we construct the correlation function as
\[ \Pi(\omega) = i \int d^4x e^{i\omega x} \langle 0|T\{j_3(x), j_3^\dagger(0)\}|0\rangle, \] (12)
where
\[ \omega = 2q \cdot v. \] (13)

After twice suitable Borel transformations, the imaginary part of it is obtained
\[ Im\Pi(\tau) = Ar^8 + B\langle \bar{q}q \rangle \tau^5 + C\langle \alpha_s G^2 \rangle \tau^4 + D\langle \bar{q}q \rangle^2 \tau^2. \] (14)

When \( \Gamma = \gamma_5 \),
\[ A = \frac{9}{2^9 \cdot 8!\pi^5}, \quad B = -\frac{3}{2^7 \cdot 5!\pi^3}, \quad C = \frac{9}{2^{10} \cdot 4!\pi^4}, \quad D = \frac{1}{2^6 \pi}. \] (15)

For \( \Gamma = \gamma_\mu \),
\[ A = \frac{9}{2^7 \cdot 8!\pi^5}, \quad B = -\frac{3}{2^6 \cdot 5!\pi^3}, \quad D = \frac{1}{2^5 \pi}. \] (16)

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and C vanishes. To obtain the results above, we have taken use of the infinite heavy quark mass limit.

The current with the color octet quark-antiquark pair is chosen as
\[ j_4(x) = f^{abc1} f^{abc2} (q^b \Gamma l^c_1)(q^b \Gamma^m q^c_2). \]  

(17)

The coefficients of \( A, B, C \) and \( D \) for \( \Gamma = \gamma_5, \gamma_\mu \) are
\[ A = \frac{27}{8! \pi^5}, \quad B = \frac{9}{5! \pi^3}, \quad C = \frac{27}{4! \pi^1}, \quad D = \frac{3}{2^6 \pi}, \]  

(18)

and
\[ A = \frac{27}{8! \pi^5}, \quad B = \frac{-9}{5! \pi^3}, \quad D = \frac{3}{2^5 \pi}, \]  

(19)

respectively.

On the phenomenal side, the correlation function is expressed as
\[ \Pi(\omega) = \frac{F^2_{H^+}}{(2\Lambda - \omega)} + \int_{\omega_c}^{\infty} d\omega' \frac{Im\Pi_s(\omega')}{\omega' - \omega}, \]  

(20)

where the first term on the right side is the dominant pole term resulting from the lowest lying resonance contribution, the second term represents the contribution of the continuum state and higher resonances, and \( \omega_c \) is the continuum threshold.

Making use of the dispersion relations for the correlation functions to equate the quark sides with hadron sides, we obtain
\[ \frac{F^2_{H^+}}{(2\Lambda - \omega)} = \frac{1}{\pi} \int_{0}^{\infty} d\omega' \frac{Im\Pi(\omega')}{\omega' - \omega}; \]  

(21)

after Borel transformation, they are turned into
\[ F^2_{H^+} e^{-2\Lambda/T} = \frac{1}{\pi} \int_{0}^{\infty} d\omega' Im\Pi(\omega') e^{-\omega'/T}, \]  

(22)

where \( T \) is the Borel transformation variable. So the \( \Lambda \) can be determined as
\[ 2\Lambda = \frac{\int_{0}^{\infty} d\omega' \omega' Im\Pi(\omega') e^{-\omega'/T}}{\int_{0}^{\infty} d\omega' Im\Pi(\omega') e^{-\omega'/T}}. \]  

(23)

5 Numerical results of the spectra for the four-quark states

In this section, we will give the numerical results of the spectra for the four-quark states. To proceed with the calculation, the mass of the b and c quarks are chosen as 4.7 GeV and 1.3 GeV, respectively. The condensates are chosen as
\[ \langle 0 | \bar{q} q | 0 \rangle = -(0.24 GeV)^3, \langle 0 | \alpha_s G^2 | 0 \rangle = 0.06 GeV^4. \]  

(24)
A few words should be given about the technical details first. In the light quark case, we tried the calculation with \( s_0 = 1.0, 1.5 \) and 2.5 GeV, respectively. The mass square of them is displayed in Figs. 1 and 2 though the platform is not satisfactory. For the heavy four-quark states, the 2\( \Lambda \) obtained by us is shown in Figs. 3 and 4, where the \( \omega_c \) are chosen as 2.0, 3.0, and 4.0 GeV, respectively. To find which \( s_0 \) or \( \omega_c \) is the suitable one for our sum rules, the ordinary criteria\[10\] of the determination of the threshold and platform are taken into account. All the results are collected in Table I.

From the results obtained here, the light four-quark states are found to be light. When the mass of the \( s \) quark is taken into account, the conclusion will not change. So it is reasonable to search for four-quark states in the light hadron regions, while the heavy four-quark states with one \( c \) or \( b \) quark are found to lie below 2.0 GeV or 5.5 GeV, respectively. The mass prediction\[6\] of the \( D\bar{K} \) or \( DK \) four-quark states is higher than the result here.

As for the effect of interactions between quarks on the mass of four-quark states, in principle, it cannot be studied through sum rule methods directly. There is no correspondence between the relative interpolating currents and the bound states. Nevertheless, we can proceed with the sum rules process with different currents as we did previously. Especially, we believe that sum rules may be capable of finding some information about the interactions in heavy quark systems, where there exists a nonrelative limit. It is found that the couplings of spin and color inside the \( q\bar{q} \) pair have little effect on the mass determination of them. The four different combinations between the spin and color inside the currents result in similar masses in both light and heavy quark systems.

Through the lessons from analyses of mesons and glueballs, we have learned that perturbative contributions play a dominant role in the sum rules. However, in the present case, the four-quark condensate plays the dominant role, and all the other contributions(including the perturbative term) are negligible. The results obtained here infer that either the interactions inside the four-quark states are some what special or it is not suitable to deal with these states using QCD sum rules.

For systems with one heavy quark, there exists some symmetries and a nonrelativistic model may work well. Therefore, sum rule analyses with different combinations of spin and color in the interpolating currents in heavy quark effective theory may explore some physical information about the four-quark states. It can be seen that the platform for these systems is much better than that for light quark systems. Especially, in contrast with the light quark case, the perturbative contribution dominates the sum rule here, which means that application of the HQET sum rule to heavy four-quark states is suitable.

### 6 Conclusion and discussion

The spectra of \((q\bar{q})(q\bar{q})\) meson-meson like four-quark states have been calculated using

| (GeV) | \( \gamma_5 \otimes singlet(c) \) | \( \gamma_\mu \otimes singlet(c) \) | \( \gamma_5 \otimes octet(c) \) | \( \gamma_\mu \otimes octet(c) \) |
|-------|--------------------------|---------------------------|--------------------------|--------------------------|
| light\((m)\) | 0.74-0.87 | 0.71-0.89 | 0.74-0.90 | 0.71-0.87 |
| heavy\((2\Lambda)\) | 0.95-1.26 | 0.97-1.24 | 0.95-1.24 | 0.97-1.24 |
QCD sum rules. Four-quark states with light quarks are found below 1.0 GeV. The mass of heavy four-quark states with a c or b quark has been predicted too.

In other models, to extract the properties of hadrons, the characteristics of interactions were put into the theory by hand. Unlikely, both the hadron properties and the interactions inside are predicted on basis of QCD in the framework of sum rules. For four-quark states, whether they are compact states such as the ordinary mesons or weakly bound states such as molecules is of the key concern with us. Moreover, the complicated couplings of spin and color between the quarks inside the bound states need not be speculated about either. A possible choice is to construct suitable currents corresponding to the topic with which we are concerned, and the analysis of the reasonableness for choosing currents is important to us.

In this paper, we proceeded with sum rule analyses with different currents, where different combinations of spin and color inside the \(q\bar{q}\) pair have been tried. In fact, it may be a good way to detect the physical couplings of spin and color inside the bound states in heavy quark systems. Even though there is no correspondence between the currents and physical bound states at all, it is necessary to find which interpolating current is the right one for the special question. Previous results confirm that different combinations of spin and color inside the currents have little effect on the mass determination of four-quark states.

Experimentally, we cannot confirm the four-quark state yet; we have observed only some final strong coupling meson-meson channel. For this reason, we consider only the \((q\bar{q})(q\bar{q})\) four-quark currents in our sum rules. For a complete sum rule analysis, the \((q\bar{q})(q\bar{q})\) four-quark states should be taken into account too. Moreover, besides the mixing between ordinary hadrons and four-quark states, there exists mixing between the four-quark states with the same overall quantum number also. Accordingly, the mixing between different currents should be considered. Unfortunately, we know little about the mixing, both in QCD and in some models. This problem has remained beyond the scope of this paper.

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Figure caption

Fig. 1: Mass square of $0^{++}$ four-quark state from current $j_1(x)$ versus Borel variable $\tau$.

Fig. 2: Mass square of $0^{++}$ four-quark state from current $j_2(x)$ versus Borel variable $\tau$.
Fig. 3: $2\Lambda$ of $0^{++}$ heavy-light four-quark state from current $j_3(x)$ versus Borel variable $T$.

Fig. 4: $2\Lambda$ of $0^{++}$ heavy-light four-quark state from current $j_4(x)$ versus Borel variable $T$. 
\[ M^2 \text{(GeV}^2) \]

vs.

\[ \tau \text{(GeV}^2) \]

Fig. 1
Fig. 2

\(M^2 \text{ (GeV}^2)\)

\(\tau \text{ (GeV}^2)\)

\(\tau = \frac{1}{s_0} (\text{GeV}^2)^{-2}\)

For different values of \(s_0\):
- \(s_0 = 2.5\)
- \(s_0 = 1.5\)
- \(s_0 = 1.0\)
Fig. 3

\( \Lambda (\text{GeV}) \)

\( \tau (\text{GeV}^{-1}) \)

\( \omega_l = 4.0 \)
\( \omega_l = 3.0 \)
\( \omega_l = 2.0 \)
Fig. 4

\[ 2\Lambda (GeV) \]

\[ \tau (GeV^{-1}) \]