SPINORIALITY OF ORTHOGONAL REPRESENTATIONS OF $\text{GL}(n,q)$

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ABSTRACT. We determine which orthogonal representations $V$ of $\text{GL}_n(\mathbb{F}_q)$ lift to the double cover $\text{Pin}(V)$ of the orthogonal group $O(V)$. We cover all $n \geq 1$ and prime powers $q$, except for $(n, q) = (3, 4)$.

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1. INTRODUCTION

Let $G$ be a finite group, and $\pi : G \to O(V)$ a complex orthogonal representation. We say that $\pi$ is spinorial, provided it lifts to the double cover $\text{Pin}(V)$ of $O(V)$. This is one of a series of papers investigating the spinoriality question for well-known groups. Please see [12] for criteria when $G$ is a connected reductive Lie group, [8] for orthogonal groups, and [9] for criteria when $G$ is a symmetric or alternating group. This paper covers the group $G = \text{GL}_n(\mathbb{F}_q)$, our foray into finite groups of Lie type.

In the case that $\det \pi = 1$, the representation $\pi$ is spinorial iff the 2nd Stiefel-Whitney class of its real form vanishes. (See [11].) Thus
the spinoriality of $\pi$ is equivalent to the existence of a spin structure for the vector bundle associated to $\pi$ over the classifying space $BG$. (See [2, Section 2.6], [13, Theorem II.1.7].) Determining spinoriality of Galois representations also has application in number theory: see [18], [5], and [15].

The group $G$ is a semidirect product of $H = \text{SL}_n(\mathbb{F}_q)$ by the cyclic group $\text{GL}_1(\mathbb{F}_q) \cong \mathbb{F}_q^*$. For $(n,q) \notin S = \{(2,2), (2,3), (2,4), (3,2), (3,4), (4,2)\}$, $H$ is perfect and has no central extensions of even degree. Therefore every orthogonal representation of $H$ is spinorial, and as we show, $\pi$ is spinorial iff its restriction to $\text{GL}_1(\mathbb{F}_q)$ is spinorial. The lifting problem for cyclic groups is not difficult.

To state our main theorem, let $a_1$ be the diagonal matrix in $\text{GL}_n(\mathbb{F}_q)$ whose first diagonal entry is $-1$ and the rest are $1$'s. Let $\Theta_\pi$ be the character of $\pi$, and put

$$m_\pi = \frac{\Theta_\pi(1) - \Theta_\pi(a_1)}{2}.$$ 

**Theorem 1.** Suppose $(n,q) \notin S$. An orthogonal representation $\pi$ of $\text{GL}_n(\mathbb{F}_q)$ is spinorial iff:

1. $m_\pi \equiv 0 \mod 4$, when $q \equiv 1 \mod 4$
2. $m_\pi \equiv 0$ or $3 \mod 4$, when $q \equiv 3 \mod 4$.

Moreover when $q$ is even, all orthogonal representations of $\text{GL}_n(\mathbb{F}_q)$ are spinorial.

The method adapts to similar families of finite groups of Lie type: to illustrate we produce a version of Theorem 1 for $\text{GSp}_{2n}(\mathbb{F}_q)$.

Using the well-known character table of $\text{GL}_2(\mathbb{F}_q)$, we catalogue which irreducible orthogonal representations are spinorial. Also for $n \geq 3$, we deduce the following:

**Theorem 2.** Let $q$ be odd, $n \geq 3$, and $G = \text{GL}_n(\mathbb{F}_q)$. Then all irreducible orthogonal cuspidal representations, all orthogonal principal series representations, and the Steinberg representation of $G$ are spinorial. However there exist aspinorial orthogonal representations of $G$.

We also give criterion for the “exceptional cases” when $(n, q) \in S$ for spinoriality in terms of character values, except when $(n, q) = (3,4)$.

The layout of this paper is as follows. In Section 2 we set up notation, and review the theory of orthogonal representations and group cohomology. The spinoriality problem for cyclic groups is settled in
Section 3. Theorem 1 is proved in Section 4, resulting from generalities about semidirect products. We also give the analogous result for $\text{GSp}_{2n}(\mathbb{F}_q)$. Section 5 is our enumeration of spinorial irreducible orthogonal representations of $\text{GL}_2(\mathbb{F}_q)$ with $q$ odd. In Section 6 we demonstrate Theorem 2 by using known character formulas. Finally in Section 7 we treat the cases of $(n, q) \in S$ but $(n, q) \neq (3, 4)$.

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## 2. Notation and Preliminaries

Let $G$ be a finite group. All representations we consider are on finite-dimensional complex vector spaces. A “linear character” is a one-dimensional representation. We denote the trivial linear character by ‘1’.

A linear character $\chi$ on $G$ is *quadratic* when it takes values in $\{\pm 1\}$. If $G$ is cyclic of even order, or if $G = \text{GL}_n(\mathbb{F}_q)$, with $q$ odd, then there is a unique nontrivial quadratic linear character of $G$, denoted ‘$\text{sgn}$’ or ‘$\text{sgn}_G$’. In the cyclic case, $\text{sgn}(g) = -1$ when $g$ is a generator of $G$. In the general linear case, $\text{sgn}_G = \text{sgn}_{\mathbb{F}_q} \circ \text{det}$, where ‘$\text{det}$’ is the determinant.

If $H < G$ is a subgroup, and $\pi$ is a representation of $G$, we write ‘$\text{Res}_H^G \pi$’ or ‘$\pi|_H$’ for the restriction of $\pi$ to $H$. Write ‘$\Theta_\pi$’ for the character of $\pi$. If $\sigma$ is a representation of $H$, write ‘$\text{Ind}_H^G \sigma$’ for the representation of $G$ induced from $\sigma$.

Let $(\pi_1, V_1), (\pi_2, V_2)$ be representations of finite groups $G_1, G_2$, respectively. We write $(\pi_1 \boxtimes \pi_2, V_1 \otimes V_2)$ for the external tensor product representation of $G_1 \times G_2$.

### 2.1. Orthogonal Representations

A representation $(\pi, V)$ of $G$ is *orthogonal*, provided it preserves a quadratic form on $V$, or equivalently a symmetric nondegenerate bilinear form on $V$. In this case, $\pi$ can be viewed as a homomorphism from $G$ to the orthogonal group $\text{O}(V)$.

Given a complex representation $(\pi, V)$, write $S(\pi)$ for the representation $\pi \oplus \pi^\vee$ on $V \oplus V^\vee$. Then $S(\pi)$ preserves the quadratic form

$$Q((v, v^*)) = (v^*, v),$$

and is thus orthogonal.
An orthogonal representation $\Pi$ of $G$ may be decomposed as

$$\Pi \cong S(\pi) \oplus \bigoplus_j \varphi_j,$$

where each $\varphi_j$ is irreducible orthogonal and $\pi$ does not have any irreducible orthogonal constituents.

Let us say that a representation $\pi$ is orthogonally irreducible, provided $\pi$ is orthogonal, and $\pi$ does not decompose into a direct sum of two orthogonal representations. Thus, an orthogonal representation $\pi$ is orthogonally irreducible iff $\pi$ is irreducible, or of the form $S(\varphi)$ where $\varphi$ is irreducible but not orthogonal. We’ll write ‘OIR’ for “orthogonally irreducible representation”.

Note that a linear character is orthogonal iff it is quadratic.

2.2. The Pin Group

Let $V$ be a finite-dimensional (complex) vector space, with a nondegenerate symmetric bilinear form $\Phi$. The Clifford algebra $C(V)$ is the quotient of the tensor algebra $T(V)$ by the two-sided ideal generated by the set

$$\{v \otimes v + \Phi(v,v) : v \in V\}.$$

It contains $V$ as a subspace. Write $C(V)^\times$ for its group of units. Then $\text{Pin}(V)$ is the subgroup of $C(V)^\times$ generated by the unit vectors in $V \subset C(V)$. There is a unique homomorphism $\rho : \text{Pin}(V) \to O(V)$ taking a unit vector $u$ to the reflection of $V$ determined by $u$. Note that $-1 = u^2 \in \text{Pin}(V)$ and $\rho(-1) = 1$. Since $O(V)$ is generated by reflections, $\rho$ is surjective; it is a nontrivial double cover of $O(V)$. See [7, Chapter 20] for details.

Suppose $\pi : G \to O(V)$ is an orthogonal representation, where $O(V)$ is determined by $\Phi$. We say that $\pi$ is spinorial, provided there is a homomorphism $\hat{\pi} : G \to \text{Pin}(V)$ so that $\rho \circ \hat{\pi} = \pi$. In this situation, $\hat{\pi}$ is called a lift of $\pi$.

**Lemma 1.** Let $A \in O(V)$ with $A^2 = 1$. Let $m$ be the multiplicity of $-1$ as an eigenvalue of $A$. Suppose $B \in \text{Pin}(V)$ with $\rho(B) = A$. Then $B^2 = 1$ iff $m \equiv 0, 3 \mod 4$.

**Proof.** Let $e_1, \ldots, e_{m_x}$ be an orthonormal basis of the $-1$-eigenspace of $A$, so that $A$ is the product of the reflections in each $e_j$. Therefore $B = \pm e_1 \cdots e_{m_x} \in \text{Pin}(V)$. One computes

$$B^2 = (-1)^{\frac{1}{2} m(m+1)},$$

and this exponent is even iff $m$ is congruent to 0 or 3 modulo 4. $\square$
Note that here

\[ m = \frac{\dim V - \text{trace}(A)}{2}. \]

2.3. Extensions and Cohomology

Recall [19, Section 6.6] that to an extension

\[ 1 \to A \to E \to G \to 1 \]

of a group \( G \) by an abelian group \( A \) corresponds a cohomology class \( c_E \in H^2(G, A) \). Moreover, \( E \) is a split extension iff \( c_E = 0 \). The extension \( \rho : \text{Pin}(V) \to \text{O}(V) \) does not split, and has fibre \( A = \mathbb{Z}/2\mathbb{Z} \).

A homomorphism \( \varphi : G' \to G \) induces a pullback extension

\[ 1 \to A \to E' \to G' \to 1, \]

where \( E' = E \times_G G' \) and \( E' \to G' \) is projection to the second component. Moreover, \( c_{E'} \) is the image of \( c_E \) under the induced map \( \varphi^* : H^2(G, A) \to H^2(G', A) \).

If \( (\pi, V) \) is an orthogonal representation of a group \( G \), then the pullback extension is split iff \( \pi \) is spinorial.

3. Abelian Groups

3.1. Cyclic Groups

Suppose first that \( n \) is odd. If \( \pi \) is an orthogonal representation of \( C_n \), then the pullback extension of \( C_n \) is split by the Schur-Zassenhaus Theorem. Therefore \( \pi \) is spinorial.

Now let \( n \) be even. Write \( C_n \) for the cyclic group of order \( n \). Fix a generator \( g \) of \( C_n \). We say a linear character \( \chi \) is \textit{even} provided \( \chi(g^{n/2}) = 1 \), and that it is \textit{odd} when \( \chi(g^{n/2}) = -1 \).

It is well-known [19, Section 6.2] that \( H^2(C_n, \mathbb{Z}/2\mathbb{Z}) \) has only one nonzero element; it corresponds to the nonsplit extension

\[ 1 \to C_2 \to C_{2^n} \to C_n \to 1. \tag{2} \]

**Proposition 1.** Suppose \( d \) is even, and \( n \) is a multiple of \( d \). Then the restriction map \( H^2(C_n, \mathbb{Z}/2\mathbb{Z}) \to H^2(C_d, \mathbb{Z}/2\mathbb{Z}) \) is an isomorphism. An orthogonal representation \( \pi \) of \( C_n \) is spinorial iff its restriction to \( C_d \) is spinorial.

**Proof.** It is enough to show that the pullback of (2) to \( C_d < C_n \) does not split. But this pullback is

\[ 1 \to C_2 \to C_{2d} \to C_d \to 1. \]

This gives the first statement, and the last statement follows. \( \square \)
Let \( \pi \) be an orthogonal representation of \( C_n \) with \( n \) even. Let \( m_\pi \) be the multiplicity of \(-1\) as an eigenvalue of \( \pi(g^{n/2}) \). By (1), we have
\[
m_\pi = \frac{\Theta_\pi(1) - \Theta_\pi(g^{n/2})}{2}.
\]

**Proposition 2.** With notation as above, the representation \( \pi \) is spinorial iff \( m_\pi \equiv 0, 3 \mod 4 \).

**Proof.** By Proposition 1, we may assume \( n = 2 \). The proposition then follows from applying Lemma 1 to \( A = \pi(g) \). \(\square\)

When \( n \) is a multiple of 4, the integer \( m_\pi \) is twice the multiplicity of \( i \) (or of \(-i\)) as an eigenvalue of \( \pi(g^n) \). In particular, \( m_\pi \) is even. Therefore in this case, \( \pi \) is spinorial iff \( m_\pi \) is a multiple of 4.

To summarize:

**Proposition 3.** Let \( \pi \) be an orthogonal representation of \( C_n \). For \( n \) even, let \( m_\pi \) be the multiplicity of \(-1\) as an eigenvalue of \( \pi(g^{n/2}) \).

1. If \( n \) is odd, then \( \pi \) is spinorial.
2. If \( n \equiv 0 \mod 4 \), then \( \pi \) is spinorial iff \( m_\pi \equiv 0 \mod 4 \).
3. If \( n \equiv 2 \mod 4 \), then \( \pi \) is spinorial iff \( m_\pi \equiv 0 \) or \( 3 \mod 4 \).

### 3.2. Elementary Abelian 2-groups

Let \( E \) be a rank \( d \) elementary 2-group, i.e., \( E \cong C_2^d \). Then \( E \) has the following presentation (via generators and relations):
\[
E = \langle e_1, \ldots, e_d \mid e_i^2, (e_ie_j)^2, 1 \leq i, j \leq d \rangle.
\]

**Proposition 4.** Let \( \pi \) be an orthogonal representation of \( E \). Then \( \pi \) is spinorial iff \( \forall e \in E \), the integer
\[
m_e = \frac{\Theta_\pi(1) - \Theta_\pi(e)}{2}
\]
is congruent to 0 or 3 mod 4.

**Proof.** From the presentation, \( \pi \) is spinorial iff the lift of each \( \pi(e_i) \) and \( \pi(e_ie_j) \) in \( \text{Pin}(V) \) squares to 1. If \( \pi \) is spinorial, then the lift of each \( \pi(e) \) squares to 1. The result then follows from Lemma 1. \(\square\)

### 4. Main Theorem

The group \( G = \text{GL}_n(F_q) \) is the semidirect product of \( H = \text{SL}_n(F_q) \) with the cyclic group \( \text{GL}_1(F_q) \). In this section we show that, except for finitely many \((n, q)\), an orthogonal representation of \( G \) is spinorial iff its restriction to this cyclic group is spinorial. Our main theorem will then follow from the previous section.
4.1. Semidirect Products

Let \( G \) be a group, and \( \pi \) a spinorial (orthogonal) representation of \( G \). The group \( \text{Hom}(G, \{\pm 1\}) \) of quadratic linear characters \( \chi \) acts on the set of lifts \( \hat{\pi} \) of \( \pi \) via
\[
(\chi \odot \hat{\pi})(g) = \chi(g)\hat{\pi}(g).
\]
This defines a simply transitive action of the group of orthogonal linear characters on the set of lifts of \( \pi \). In particular:

**Lemma 2.** If \( G \) has no subgroups of index 2, then the lift of \( \pi \) is unique.

**Proposition 5.** Let \( G \) be a finite group, and \( H, K \) subgroups of \( G \) so that \( H \) is normal in \( G \), \( H \cap K = \{1\} \), and \( G = HK \). Suppose further that \( H \) has no subgroups of index 2. If \( \pi \) is an orthogonal representation of \( G \), then \( \pi \) is spinorial iff its restrictions to both \( H \) and \( K \) are spinorial.

**Proof.** Suppose \( \pi|_H \) and \( \pi|_K \) are spinorial, let \( \hat{\pi}_H \) and \( \hat{\pi}_K \) be the respective lifts.

**Claim:** Given \( k_0 \in K \), we have
\[
(\hat{\pi}_K(k_0)) \circ \hat{\pi}_H \circ \text{Int}(k_0)^{-1} = \hat{\pi}_H.
\]
To see this, note that the LHS evaluated at \( h \in H \) equals
\[
\hat{\pi}_K(k_0)\hat{\pi}_H(k_0^{-1}hk_0)\hat{\pi}_K(k_0^{-1}),
\]
and applying \( \rho \) gives \( \pi(h) \). The claim then follows from Lemma 2.

Next, define \( \hat{\pi}(g) = \hat{\pi}_H(h)\hat{\pi}_K(k) \) when \( g = hk \) with \( h \in H \) and \( k \in K \). Using (3), it is straightforward to check that \( \hat{\pi} \) is a homomorphism, and indeed a lift of \( \pi \). \( \square \)

4.2. Case of Odd Schur Multiplier

Recall from [19, Section 6.9] that a perfect group \( G \) has a universal central extension \( \beta : \tilde{G} \to G \), in the sense that it factors through every central extension of \( G \). Let \( M(G) = \ker \beta \); this abelian group is the Schur multiplier of \( G \). It is isomorphic to the abelian group \( H^2(G, \mathbb{C}^\times) \).

**Proposition 6.** Let \( G \) be a perfect group with \( |M(G)| \) odd. Then every orthogonal representation \( \pi \) of \( G \) is spinorial.

**Proof.** Let \( \beta : \tilde{G} \to G \) be the universal central extension of \( G \).

The extension of \( G \) associated to \( \pi \) is a degree 2 central extension \( \rho' : G' \to G \). By the universality of \( \tilde{G} \), there exists a morphism \( \alpha : \tilde{G} \to G' \) of covers, i.e., so that \( \rho' \circ \alpha = \beta \). But \( \alpha(\ker \beta) \) is in the kernel of \( \rho' \), which has order 2. But since \( |\ker \beta| \) is odd, it must be
that $\alpha(\ker \beta) = 1$, so $\alpha$ factors to $\alpha' : G \to G'$, a splitting of $\rho'$. The composition of $\alpha'$ with the projection of $G'$ to $\text{Pin}(V)$ is a lift of $\pi$. □

4.3. Application to $\text{GL}_n(\mathbb{F}_q)$

Let $S = \{(2, 2), (2, 3), (2, 4), (3, 2), (3, 4), (4, 2)\}$. It is well-known that if $(n, q) \notin S$, then $\text{SL}_n(\mathbb{F}_q)$ is perfect, and $\text{M(\text{SL}_n(\mathbb{F}_q))}$ has odd order. In what follows we regard $\text{GL}_1(\mathbb{F}_q)$ as a subgroup of $\text{GL}_n(\mathbb{F}_q)$ in the usual way, via the top left entry.

**Corollary 1.** Let $G = \text{GL}_n(\mathbb{F}_q)$, and suppose $(n, q) \notin S$. Then an orthogonal representation $\pi$ of $G$ is spinorial iff its restriction to $\text{GL}_1(\mathbb{F}_q)$ is spinorial.

**Proof.** Take $H = \text{SL}_n(\mathbb{F}_q)$ and $K = \text{GL}_1(\mathbb{F}_q)$. Then $H, K$ satisfy the conditions of Proposition 5. By Proposition 6, the restriction of $\pi$ to $H$ is spinorial. □

**Remark 1.** This method was suggested by D. Prasad.

**Proposition 7.** Let $G = \text{GL}_n(\mathbb{F}_q)$, with $(n, q) \notin S$. Then

$$H^2(G, \mathbb{Z}/2\mathbb{Z}) \cong H^2(\text{GL}_1(\mathbb{F}_q), \mathbb{Z}/2\mathbb{Z}).$$

Thus when $q$ is odd, there is only one nontrivial extension of $G$ by $\mathbb{Z}/2\mathbb{Z}$.

**Proof.** Again let $H = \text{SL}_n(\mathbb{F}_q)$. Since $H$ is perfect, we have $H^1(H, \mathbb{C}^\times) = H^1(H, \mathbb{Z}/2\mathbb{Z}) = 0$. From the exact sequence of the squaring map,

$$1 \to \mu_2 \to \mathbb{C}^\times \to \mathbb{C}^\times \to 1,$$

we deduce an injection $H^2(H, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^2(H, \mathbb{C}^\times)$. But since $H^2(H, \mathbb{C}^\times) \cong \text{M}(H)$, which has odd order, it must be that $H^2(H, \mathbb{Z}/2\mathbb{Z}) = 0$. According to [17, Chapter 7, Section 6, Corollary], inflation gives an isomorphism

$$H^2(G/H, \mathbb{Z}/2\mathbb{Z}) \cong H^2(G, \mathbb{Z}/2\mathbb{Z}).$$

The proposition follows since $G/H \cong \text{GL}_1(\mathbb{F}_q)$. □

Let $a_1 = \text{diag}(-1, 1, 1, \ldots) \in G$.

**Theorem 3.** Let $G = \text{GL}_n(\mathbb{F}_q)$, with $(n, q) \notin S$, and $\pi$ an orthogonal representation of $G$. Put

$$m_\pi = \frac{\Theta_\pi(1) - \Theta_\pi(a_1)}{2}.$$

(1) If $q$ is even, then $\pi$ is spinorial.
(2) If $q \equiv 1 \mod 4$, then $\pi$ is spinorial iff $m_\pi \equiv 0 \mod 4$.
(3) If $q \equiv 3 \mod 4$, then $\pi$ is spinorial iff $m_\pi \equiv 0$ or $3 \mod 4$. 

Proof. Note that if \( g \) is a generator of \( \mathbb{F}_q^\times \), with \( q \) odd, then \( g^{(q-1)/2} = -1 \), as it is the unique element of order 2. The Theorem then follows from Corollary 11 and Proposition 3.

4.4. Permutation Matrices Detect Spinoriality

Consider the symmetric group \( S_n \) as a subgroup of \( G \) via permutation matrices.

**Proposition 8.** An orthogonal representation \( \pi \) of \( G \) as above is spinorial iff its restriction to \( S_n \) is spinorial.

**Proof.** This is clear if \( q \) is even, as every representation of \( G \) is spinorial. So suppose \( q \) is odd. Clearly if \( \pi \) is spinorial, then \( \pi|_{S_n} \) is spinorial. Suppose \( \pi|_{S_n} \) is spinorial. Any transposition in \( S_n \) is conjugate in \( G \) to \( a_1 \). According to Theorem 1.1 of [9], we have \( m_\pi \equiv 0 \text{ or } 3 \mod 4 \). Therefore \( \pi \) is spinorial.

4.5. Other Finite Groups of Lie Type

The proof of Theorem 8 adapts to other finite groups of Lie type. For instance, let \( G = \text{GSp}_{2n}(\mathbb{F}_q) \). To fix ideas say

\[
J = \begin{pmatrix}
1 \\
& 1 \\
& & \ddots \\
& & & 1 \\
-1 & -1 & \cdots & -1
\end{pmatrix},
\]

and

\[
G = \{ g \in \text{GL}_{2n}(\mathbb{F}_q) \mid \exists \lambda \in \mathbb{F}_q^\times \text{ so that } '{gJg = \lambda J}' \}.
\]

Then \( G \) is the semidirect product of \( H = \text{Sp}_{2n}(\mathbb{F}_q) \) with \( \text{GL}_1(\mathbb{F}_q) \). Barring finitely many \((n,q)\), \( H \) is perfect with Schur multiplier 1. Thus Theorem 8 holds for such \((n,q)\) with

\[
a_1 = \begin{pmatrix}
-1 \\
-1 & -1 \\
& & \ddots \\
& & & 1 \\
& & & & 1
\end{pmatrix},
\]

with the entry ‘−1’ repeated \( n \) times followed by \( n \) ‘1’s.
5. GL$_2$(\mathbb{F}_q)

Let $G = \text{GL}_2(\mathbb{F}_q)$, with $q$ odd. We write $A$ for the subgroup of diagonal matrices, $U$ for the upper unitriangular subgroup, and $B = AU$. Write $Z$ for the center of $G$. Choosing an $\mathbb{F}_q$-basis of $\mathbb{F}_q^2$ gives an elliptic torus $T < G$, which we identify with $\mathbb{F}_q^\times$.

5.1. Catalogue of Irreducible Representations of $G$

We follow the notation of [3, Chapter 2] to enumerate the irreducible representations of $G = \text{GL}_2(\mathbb{F}_q)$.

Let $\text{St}_G$ be the Steinberg representation, so that $\text{Ind}_G^B 1 = 1 \oplus \text{St}_G$. For $\chi, \chi'$ linear characters of $\mathbb{F}_q^\times$, let $\pi(\chi, \chi')$ be the parabolic induction $\text{Ind}_B^G(\chi \boxtimes \chi')$.

If $\theta$ is a character of $\mathbb{F}_q^\times$, write $\theta^\tau$ for its composition with the non-trivial element $\tau$ of the Galois group of $\mathbb{F}_q^2$ over $\mathbb{F}_q$. We say that $\theta$ is regular, provided $\theta \neq \theta^\tau$. Fix a nontrivial linear character $\psi$ of $U$, and define a linear character of $ZU$ by $\theta_{\psi}(zu) = \theta(z)\psi(u)$. For $\theta$ regular, put

$$\pi_\theta = \text{Ind}_{ZU}^G \theta_{\psi} - \text{Ind}_T^G \theta.$$ 

These are the cuspidal representations.

The irreducible representations of $G$ are as follows:

1. The linear characters,
2. The principal series representations $\pi(\chi, \chi')$, with $\chi \neq \chi'$ linear characters of $\mathbb{F}_q^\times$,
3. Twists $\text{St}_G \otimes \chi$ of the Steinberg for a linear character $\chi$ of $G$,
4. The cuspidal representations $\pi_\theta$, with $\theta$ a regular character of $T$.

The irreducible orthogonal representations of $G$ are:

1. $1$ and $\text{sgn}_G$,
2. $\pi(1, \text{sgn})$,
3. $\pi(\chi, \chi^{-1})$ with $\chi$ not quadratic,
4. $\text{St}_G$ and $\text{St}_G \otimes \text{sgn}_G$,
5. $\pi_\theta$, where $\theta^\tau = \theta^{-1}$.

Recall that the OIRs of $G$ are either irreducible orthogonal $\pi$, or $S(\pi)$ for $\pi$ irreducible but not orthogonal.

5.2. List of spinorial OIRS for $G$

We may immediately enumerate the spinorial OIRs of $G = \text{GL}_2(\mathbb{F}_q)$ by invoking Theorem 3, since the character table of $G$ is well-known.

**Theorem 4.** The following is the complete list of spinorial OIRs of $G$:
(1) Case $q \equiv 1 \pmod{8}$
   - $1, \text{sgn}_G$
   - $\pi(\chi, \chi^{-1})$ with $\chi$ even
   - $\text{St}_G$ and $\text{St}_G \otimes \text{sgn}_G$
   - $\pi(1, \text{sgn})$
   - All cuspidal OIRs
   - $S(\chi)$ with $\chi$ even and $\chi^2 \neq 1$
   - $S(\text{St}_G \otimes \chi)$ with $\chi$ even and $\chi^2 \neq 1$
   - $S(\pi(\chi_1, \chi_2))$ with $\chi_1 \cdot \chi_2$ even and $\chi_1^2 \neq 1$
   - $S(\pi_\theta)$, with $\theta^r \neq \theta^{-1}$

(2) Case $q \equiv 3 \pmod{8}$
   - $1$
   - $\pi(\chi, \chi^{-1})$ with $\chi$ odd
   - $S(\chi)$ with $\chi$ even and $\chi^2 \neq 1$
   - $S(\text{St}_G \otimes \chi)$ with $\chi$ odd and $\chi^2 \neq 1$
   - $S(\pi(\chi_1, \chi_2))$ with $\chi_1 \cdot \chi_2$ odd and $\chi_1^2 \neq 1$

(3) Case $q \equiv 5 \pmod{8}$
   - $1, \text{sgn}_G$
   - $\pi(\chi, \chi^{-1})$ with $\chi$ odd
   - $S(\chi)$ with $\chi$ even and $\chi^2 \neq 1$
   - $S(\text{St}_G \otimes \chi)$ with $\chi$ even and $\chi^2 \neq 1$
   - $S(\pi(\chi_1, \chi_2))$ with $\chi_1 \cdot \chi_2$ even and $\chi_1^2 \neq 1$
   - $S(\pi_\theta)$, with $\theta^r \neq \theta^{-1}$

(4) Case $q \equiv 7 \pmod{8}$
   - $1$
   - $\pi(\chi, \chi^{-1})$ with $\chi$ even
   - $\text{St}_G$ and $\text{St}_G \otimes \text{sgn}_G$
   - $\pi(1, \text{sgn})$
   - All cuspidal irreducible orthogonal representations
   - $S(\chi)$ with $\chi$ even and $\chi^2 \neq 1$
   - $S(\text{St}_G \otimes \chi)$ with $\chi$ odd and $\chi^2 \neq 1$
   - $S(\pi(\chi_1, \chi_2))$ with $\chi_1 \cdot \chi_2$ odd and $\chi_1^2 \neq 1$

6. Case of $n \geq 3$

In this section, we apply Theorem 3 to list all spinorial OIRs in familiar situations for $n \geq 3$. We assume that $q = p^r$ for some odd prime $p$.

6.1. Existence of Aspinorial Representations

First, note that aspinorial orthogonal representations always exist:
Proposition 9. For each \( n, q \) with \( q \) odd, there exist aspinorial orthogonal representations of \( \text{GL}_n(\mathbb{F}_q) \).

Proof. Take an odd character \( \chi_0 : \mathbb{F}_q^\times \to \mathbb{C}^\times \), and put \( \pi = S(\chi_0 \circ \det_G) \). Then \( \Theta_\pi(1) = 2 \) and \( \Theta_\pi(a_1) = -2 \), so \( m_\pi = 2 \), hence \( \pi \) is aspinorial by Theorem 3. \( \square \)

Remark 2. Also, \( \text{sgn}_G \) is spinorial iff \( q \equiv 1 \mod 4 \).

6.2. Cuspidal Representations

Fixing an \( \mathbb{F}_q \)-basis of \( \mathbb{F}_q^n \) allows us to identify \( \mathbb{F}_q^\times \) with a subgroup \( T < G \). In particular, \( T \) gets an action of the Galois group of \( \mathbb{F}_q^n \) over \( \mathbb{F}_q \). Let \( \theta \) be a regular linear character of \( T \), meaning it is unequal to any linear character in its Galois orbit. Associated to \( (T, \theta) \) is an irreducible representation \( \pi_\theta \) whose character is denoted \( R_{T,\theta} \) in [4]. Such representations are called cuspidal. The character of \( \pi_\theta \) is supported on elements of \( G \) conjugate to elements of \( T \). Thus if \( n \geq 3 \), then \( \Theta_{\pi_\theta} \) vanishes on \( a_1 \). Its degree is \( \psi_{n-1}(q) \), where generally

\[
\psi_m(q) = \prod_{i=1}^m (q^i - 1).
\]

It is clear that \( \psi_m(q) \equiv 0 \mod 8 \) for \( m \geq 2 \). Therefore for \( n \geq 3 \), we have \( m_\pi \equiv 0 \mod 8 \). We deduce that every orthogonal cuspidal representation of \( G \) is spinorial.

6.3. Steinberg Representation

Let \( \pi \) be the Steinberg representation, and \( a \in A \). Let \( \langle Z_G(a) \rangle \) denote the centralizer of \( a \) in \( G \). According to [4, Thm 6.4.7], we have

\[
\Theta_\pi(a) = p^k,
\]

where \( p^k \) is the highest power of \( p \) dividing \( |Z_G(a)| \). Thus

\[
\Theta_\pi(1) = q^{\frac{1}{2}n(n-1)},
\]

and

\[
\Theta_\pi(a_1) = q^{\frac{1}{2}(n-1)(n-2)}.
\]

From this we deduce:

Proposition 10. The Steinberg representation of \( \text{GL}_n(\mathbb{F}_q) \) is spinorial whenever \( n \geq 3 \).
6.4. Principal Series

Let $\chi_1, \ldots, \chi_n$ be linear characters of $\mathbb{F}_q^\times$, and let $B$ be the subgroup of upper triangular members of $G$. Put

$$\pi = \text{Ind}_B^G (\chi_1 \boxtimes \cdots \boxtimes \chi_n).$$

Such representations are called principal series representations of $G$. Then $\pi$ is orthogonal iff

$$\{\chi_1, \ldots, \chi_n\} = \{\chi_1^{-1}, \ldots, \chi_n^{-1}\}$$

as multisets.

Recall the $q$-factorial:

$$[n]_q! = \prod_{i=1}^{n} \frac{q^i - 1}{q - 1} = (q + 1)(q^2 + q + 1) \cdots (q^{n-1} + \cdots + 1).$$

**Theorem 5.** If $n \geq 3$, then all orthogonal principal series of $\text{GL}_n(\mathbb{F}_q)$ are spinorial.

*Proof.* Let $\pi$ be as in (4). The relevant character values of $\pi$ are

$$\Theta_{\pi}(1) = [n]_q!$$

and

$$\Theta_{\pi}(a_1) = [n - 1]_q! \cdot \sum_{i=1}^{n} \chi_i(-1).$$

These can be inferred from [10 Section 2], [14 Chapter IV, Section 3], or [4 Proposition 7.5.3].

Note that $[n]_q!$ is divisible by 8 whenever $n \geq 4$. Therefore both $\Theta_{\pi}(1)$ and $\Theta_{\pi}(a_1)$ are divisible by 8 unless $n \leq 4$. Thus we need only consider the cases of $n = 3, 4$.

**Case** $n = 3$: By (5), there is a linear character $\chi$ of $\mathbb{F}_q^\times$ so that

$$\{\chi_1, \chi_2, \chi_3\} = \{1, \chi, \chi^{-1}\} \text{ or } \{\text{sgn}, \chi, \chi^{-1}\}.$$

For $q \equiv 1 \pmod{4}$, we have $\text{sgn}(-1) = 1$, so

$$\Theta_{\pi}(1) - \Theta_{\pi}(a_1) = (q^2 + q + 1)(q + 1) - (q + 1)(1 \pm 2)$$

$$= (q + 1)(q^2 + q + 2),$$

which is evidently a multiple of 8. For $q \equiv 3 \pmod{4}$, it’s clear that $\Theta_{\pi}(1) - \Theta_{\pi}(a_1)$ is an even multiple of $q + 1$, and hence is also a multiple of 8. Thus $m_{\pi}$ is divisible by 4, and we conclude for $n = 3$ that $\pi$ is spinorial.
Case $n = 4$: Here $\Theta_\pi(1) = [4]_q$ is a multiple of 8, so it is enough to show the same for $\Theta_\pi(a_1)$. By (5), either there is a linear character $\chi$ so that

\[
\{\chi_1, \chi_2, \chi_3, \chi_4\} = \{1, \text{sgn} \chi, \chi^{-1}\},
\]

or there are linear characters $\chi_A, \chi_B$ so that

\[
\{\chi_1, \chi_2, \chi_3, \chi_4\} = \{\chi_A, \chi_A^{-1}, \chi_B, \chi_B^{-1}\}.
\]

Under the first possibility,

\[
\Theta_\pi(a_1) = (q^2 + q + 1)(q + 1)(2 + 2\chi(-1)),
\]

which is evidently divisible by 8. The second possibility is similar, and this completes the proof.

\[\square\]

7. Exceptional Cases

In conclusion we treat spinoriality for the groups $G = \text{GL}_n(\mathbb{F}_q)$ with $(n, q) \in S$, except $(n, q) = (3, 4)$. In each of the cases below, let $\pi$ be an orthogonal representation of $G$. We give a criterion in terms of character values of $\pi$ to determine its spinoriality. Unfortunately for $G = \text{GL}_3(\mathbb{F}_4)$ we do not currently have the tools to settle this question.

7.1. $G = \text{GL}_2(\mathbb{F}_2)$

Let $K$ be the subgroup generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $H$ the subgroup generated by $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then $|H| = 3$, $|K| = 2$, and the hypotheses of Proposition 5 hold. It follows that $\pi$ is spinorial iff $m_\pi \equiv 0, 3 \mod 4$, where

\[
m_\pi = \frac{\Theta_\pi(1) - \Theta_\pi \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)}{2}.
\]

7.2. $G = \text{GL}_2(\mathbb{F}_3)$

Let $A < G$ be the subgroup of diagonal members; it is a Klein 4-group. According to [16, Section 8], the restriction map

\[H^2(G, \mathbb{Z}/2\mathbb{Z}) \to H^2(A, \mathbb{Z}/2\mathbb{Z})\]

is injective. Therefore $\pi$ is spinorial iff its restriction to $A$ is spinorial.

Put

\[
m_\pi = \frac{\Theta_\pi(1) - \Theta_\pi \left( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right)}{2}.
\]
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and

\[ m'_\pi = \frac{\Theta_\pi(1) - \Theta_\pi\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)}{2}. \]

It follows from Proposition 4 that \( \pi \) is spinorial iff \( m_\pi, m'_\pi \equiv 0 \) or 3 mod 4.

7.3. \( G = \text{GL}_2(\mathbb{F}_4) \)

In this case \( G \) is the direct product of \( H = \text{SL}_2(\mathbb{F}_4) \) and the center \( K \), which is cyclic of order 3. From [20, 3.10], we may identify \( H \) with the alternating group \( A_5 \). By Proposition 5, \( \pi \) is spinorial iff \( \pi|_{A_5} \) is spinorial. We conclude from [9, Theorem 1.1] that \( \pi \) is spinorial iff \( \Theta_\pi(1) \equiv \Theta_\pi((12)(34)) \) mod 8. Here we are using the usual cycle notation for permutations.

7.4. \( G = \text{GL}_3(\mathbb{F}_2) \)

This is the simple group of order 168. The upper triangular subgroup \( U \) of \( G \) is a 2-Sylow subgroup. According to [11, Corollary 5.2, Chapter II], the restriction map

\[ H^2(G, \mathbb{Z}/2\mathbb{Z}) \to H^2(U, \mathbb{Z}/2\mathbb{Z}) \]

is an injection, and therefore \( \pi \) is spinorial iff its restriction to \( U \) is spinorial. The group \( U \) is dihedral of order 8. According to [6, Proposition 3.3, page 323], there are subgroups \( E_1, E_2 \), both Klein 4-groups, with the property that the sum of the restriction maps,

\[ H^2(U, \mathbb{Z}/2\mathbb{Z}) \to H^2(E_1, \mathbb{Z}/2\mathbb{Z}) \oplus H^2(E_2, \mathbb{Z}/2\mathbb{Z}), \]

is injective. It follows that \( \pi \) is spinorial iff its restrictions to \( E_1 \) and \( E_2 \) are both spinorial. Therefore \( \pi \) is spinorial iff \( \pi(u) \) has lift squaring to 1 for each \( u \in U \) of order 2. There are two \( G \)-conjugacy classes of order 2 elements in \( U \), represented by

\[ u_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 \\ 1 \end{pmatrix}, \]

and

\[ u_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 \\ 1 \end{pmatrix}. \]

For \( i = 1, 2 \), let

\[ m_{\pi,i} = \frac{\Theta_\pi(1) - \Theta_\pi(u_i)}{2}. \]

From the above and Lemma 1, we deduce:
Proposition 11. The representation $\pi$ is spinorial iff $m_{\pi,1}, m_{\pi,2} \equiv 0, 3 \mod 4$.

7.5. $G = \text{GL}_4(\mathbb{F}_2)$

From [20, 3.10], $G$ is isomorphic to $A_8$. Again by [9, Theorem 1.1], $\pi$ is spinorial iff $\Theta_\pi(1) \equiv \Theta_\pi((12)(34)) \mod 8$.

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