Entanglement-Enhanced Quantum Key Distribution

Olli Ahonen, 1 Mikko Möttönen, 1, 2, 3 and Jeremy O’Brien 4

1 Department of Engineering Physics, Helsinki University of Technology, P.O. Box 5100, FI-02015 TKK, Finland
2 Australian Research Council Centre of Excellence for Quantum Computer Technology, The University of New South Wales, Sydney 2052, Australia
3 Low Temperature Laboratory, Helsinki University of Technology, P.O. Box 3500, FI-02015 TKK, Finland
4 Centre for Quantum Photonics, H. H. Wills Physics Laboratory & Department of Electrical and Electronic Engineering, University of Bristol, Merchant Venturers Building, Woodland Road, Bristol, BS8 1UB, UK

(Dated: October 7, 2008)

We present and analyze a quantum key distribution protocol based on sending entangled N-qubit states instead of single-qubit ones as in the trail-blazing scheme by Bennett and Brassard (BB84).

Since the qubits are sent and acknowledged individually, an eavesdropper is limited to accessing them one by one. In an intercept-resend attack, this fundamental restriction allows one to make the eavesdropper’s information on the transmitted key vanish if even one of the qubits is not intercepted.

The implied upper bound 1/(2N) for Eve’s information is further shown not to be the lowest, as the information can be reduced to less than 30% of that in BB84 in the case N = 2. In general, the protocol is at least as secure as BB84.

I. INTRODUCTION

Quantum information science 1 has emerged to answer the question: “What additional power and functionality can be gained by processing and transmitting information encoded in physical systems that exhibit uniquely quantum mechanical behavior?” Anticipated future quantum technologies include: quantum computing 2, which promises exponential speed-up for particular computational tasks; quantum metrology 3, which allows the fundamental precision limit to be reached; and quantum lithography 4, which could enable fabrication of devices with features much smaller than the wavelength of light. The most striking quantum technologies that have already reached commercial realization are in the area of quantum communication.

Quantum key distribution (QKD) offers secure communication based on the fundamental laws of physics—namely, that measurement of a quantum system being used to transmit information must necessarily disturb that system, and that this disturbance is detectable 5. The first QKD scheme was proposed by Bennett and Brassard in 1984 (BB84) and is based on generating a cryptographic secret key between two distant parties, Alice and Bob, by sending a random bit string encoded and measured in one of two randomly chosen mutually unbiased bases of a single qubit 6. Photons are the logical choice for transmitting quantum information and were used in the first experimental realization of BB84 7. Since then there have been several important theoretical improvements and experimental demonstrations of BB84 and other QKD protocols 8, 9, 10, 11, 12, 13, 14, 15, 16, which have culminated in commercial QKD systems. A major challenge facing future practical quantum networks is to increase the rate at which the secure key is generated. Most efforts in this direction are focused on improving the underpinning technology 9. Here we propose an alternative approach based on improving the underlying QKD protocol, which has been inspired by recent developments in optical quantum computing 17.

The ability to reliably entangle photons is a major goal of quantum information processing 12 and quantum communication. Recent demonstrations of strong coupling between semiconductor quantum dots and photonic crystal cavities has been reported 18, 19, 20. The generation and transfer of photons on a photonic crystal chip has been demonstrated 21, together with entangling photons in waveguides on silicon chips 22. The breakthrough proposal based on measurement induced nonlinearities 24, capable of entangling photons for optical quantum computing, was followed by important demonstrations of entangling logic gates 25, 26, 27. Recently, attention has focused on generating entangled states of many photons, and it was shown that atom-cavity systems can be used to generate an arbitrary entangled state of N photons 28. Thus the technology for performing an entangling transformation on several photons is now within sight.

Here we present a novel QKD protocol whose security is lower-bounded by BB84. The insight of the protocol relies on Alice entangling groups of qubits prior to their one-by-one transmission. Because successive qubits in each group are transmitted only after confirmation of reception by Bob, an eavesdropper only has access to the transmitted information one qubit at a time. The eavesdropper is thus unable to perfectly undo the entangling transformation even if aware of it. Qubits from different entangled groups can be sent interleaved to keep the quantum channel utilization high. We present the maximal mutual information on the established key
II. THE PROTOCOL

In our protocol, the initiator, Alice, generates a number of random bits, handled in groups of \( N \). Each group is an outcome of the random variable \( A = A_1 A_2 \cdots A_N \), composed of the binary random variables \( A_i \), for which the probabilities are \( p(A_i = 0) = p(A_i = 1) = \frac{1}{2} \), \( i = 1, \ldots, N \). Let the bit string \( a = a_1 a_2 \cdots a_N \) denote the outcome of \( A \). These bits form Alice’s raw key.

Alice uses a public quantum channel to transmit the raw key to the recipient, Bob. The basis of each qubit is random, being the eigenbasis of the Pauli matrix \( \sigma_z \), \( \{ 0, 1 \} \) or that of \( \sigma_x \), \( \{ + = (0) + (1) / \sqrt{2}, - = (0) - (1) / \sqrt{2} \} \) with equal probability. Let \( \alpha = \alpha_1 \alpha_2 \cdots \alpha_N \), with each \( \alpha_i \in \{ z, x \} \), denote Alice’s basis choices for an \( N \)-bit group.

Before transmission, Alice applies a fixed \( N \)-qubit gate \( U_N \), declared in public, to each group \( \{ |a_i; \alpha_i\rangle \}_{i=1}^N \). Thus the qubits are, in general, entangled. She then sends the qubits one by one to Bob, always waiting for Bob to acknowledge each qubit on a public authenticated classical channel before sending the next one. This waiting does not decrease the transmission rate: Individual qubits from different groups can be sent interleaved. Bob waits for \( N \) qubits to accumulate, and applies \( U_N^\dagger \) to the group. He projectively measures each qubit in the \( \sigma_z \) or \( \sigma_x \) eigenbasis, chosen at random, and obtains his raw key, consisting of the measurement results \( b_i \in \{ 0, 1 \} \).

Figure 1 shows the protocol as a quantum circuit for the \( N \) qubits. The quantum non-demolition (QND) measurements needed for Bob to detect the reception of each qubit are not shown. The QND measurements can be performed with high fidelity, as is demonstrated, for instance, in Ref. 29.

After the quantum transmission, Alice and Bob compare their basis choices over the classical channel, and discard the raw-key bits for which their bases did not coincide. Note that the entire \( N \)-bit group need not be discarded, only the individual incompatible results. The remaining bits form the participants’ sifted keys which may still contain differences due to noise or eavesdropping.

III. ANALYSIS

First, we point out that our protocol cannot be less secure than BB84, even if Eve is allowed any attack strategy. Giving Eve full control of the gates \( U_N \) and \( U_N^\dagger \) shown in Fig. 1 reduces the protocol to BB84 facing a coherent attack. Thus, the proofs of security for BB84 with coherent attacks allowed (Ref. 31 and references therein) also apply to our protocol, and Alice and Bob can ensure the secrecy of the generated key in our protocol, as well.

We continue our more refined analysis by studying the protocol under the IR attack. Potentially more efficient, e.g., cloning, attacks are to be studied in future work. In all attacks, the goal of the attacker is to obtain a copy of the sifted key for a minimal increase in the QBER, which is the only indicator of careful eavesdropping to Alice and Bob. In BB84, the IR attack is succinctly described as the eavesdropper, Eve, measuring the transmitted qubits in \( z \) or \( x \) basis and resending the obtained results to Bob. Independent of Eve’s choice of basis, she obtains on average at most 0.5 bits of information on each bit of the sifted key, and induces an average QBER of at least 25% 4. A slightly better strategy for Eve is to clone each
qubit imperfectly and measure the clone state \[32\]. The more information Eve extracts on the key, the larger the induced error rate is. Eve can also choose to interfere only with a fraction \(\xi \in [0, 1]\) of the transmitted qubits. Eve’s maximal information as a function of QBER is shown in Fig. 2 for these attacks.

![Image](image.png)

**FIG. 2:** Eve’s information per bit on Alice’s sifted key as a function of the observed QBER for BB84 with cloning and intercept-resend (IR) attacks (dashed lines), and for our protocol using \(U_2 = C(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})\) with the corresponding optimal IR attack (solid line). The arrow shows the effect of engaging \(U_2\) while keeping the fraction of intercepted qubits \(\xi = 0.8\) constant.

In our protocol, Eve’s choice of basis has a significant impact on her information and the induced QBER. Hence, we allow Eve to measure each qubit in any basis. This is equivalent to allowing Eve arbitrary single-qubit gates, and measurements in the \(z\) basis. For the group of \(N\) qubits, Eve’s measurement results are the outcomes \(e = e_1 e_2 \cdots e_N\) of the random variable \(E\), with each \(e_i \in \{0, 1\}\).

Once Eve has measured a qubit, the result \(e_i\) represents her best guess on Alice’s corresponding key bit. Therefore, to minimize the QBER, she constructs the state \(|e_i; z\rangle\) and then undoes the previously applied single-qubit gate before sending the qubit to Bob. Any single-qubit gate can be written as three successive rotations about the Bloch-sphere axes \(y\) and \(z\), \(R_z(\varphi) R_y(\beta) R_z(\gamma) e^{i\phi}\). Since Eve measures in the \(z\) basis, the final rotation \(R_z(\varphi)\) has no effect on the result. The global phase \(\phi\) is irrelevant as well. Eve’s attack is thus parametrized by the single-qubit gate rotation angles \((\beta_1, \gamma_1), \ldots, (\beta_N, \gamma_N)\).

The information Eve gains on the key is quantified by the mutual information of the random variables \(A\) and \(E\), defined as [1]

\[
I(A, E) = \frac{1}{N} [H(A) + H(E) - H(A, E)],
\]

where \(H(\cdot)\) denotes the Shannon entropy and \(H(\cdot, \cdot)\) the joint entropy. The factor \(\frac{1}{N}\) ensures that Eq. (1) yields the mutual information per bit, since \(A\) and \(E\) are both \(N\)-bit entities. The entropies must be averaged over Alice’s choice of bases \(\alpha\) which Eve eventually finds out. Thus, \(H(A, E) = -\frac{1}{N} \sum_{\alpha, e} p(\alpha, e|\alpha) \log_2 p(\alpha, e|\alpha)\), and \(H(E) = -\frac{1}{N} \sum_{\alpha} H_{\alpha}(E) = -\frac{1}{N} \sum_{\alpha, e} p(e|\alpha) \log_2 p(e|\alpha)\), where the probabilities are conditioned on \(\alpha\). The entropy \(H(A) = N\).

The QBER is defined as the average probability of a bit flip in the sifted key. For each individual qubit \(j = 1, \ldots, N\) it is

\[
\text{QBER}_j = \frac{1}{4} \sum_{a_j = z} \sum_{a_j = \bar{z}} p(B_j = \bar{a}_j|A_j = a_j; \alpha_j),
\]

where \(B_j\) is the random variable giving Bob’s measurement result \(b_j\) of \(j\)th qubit, and the bar denotes the logical \(\overline{\text{not}}\) operation. The QBER used in the following analysis is the average of the QBER’s of the \(N\) qubits.

For Alice and Bob to accept the sifted key for post-processing, the fraction of eavesdropped qubits \(\xi\) must be such that QBER \(\leq 0.25\). Typically, they set a suitable threshold value for acceptance in this regime, where the information gain of the eavesdropper is linear with respect to QBER in the IR attack. Therefore, Eve’s maximal information for a given QBER is determined by the maximum of the ratio \(I(A, E)/\text{QBER}\).

The final bit rate \(R_{\text{net}}\) is an important measure of efficiency for a QKD protocol. This is the rate at which Alice and Bob accumulate shared secret key bits, which contain no errors, and on which Eve’s information is negligible, i.e., below a known bound controlled by Alice and Bob. Since the transformations \(U_N\) and \(U_N^\dagger\) provide no new capabilities for Eve under the coherent attack model for BB84, the final bit rate of our protocol cannot be lower than in BB84, with an ideal quantum channel. However, innocent noise in the quantum channel may change this setting.

Let us present a recursive construction for the gate \(U_N\) which bounds the information of an IR attacker to at most \(1/(2N)\) for any QBER, a proof of which is given in the Appendix. We denote this gate by \(U_N^*\). The gate has two equivalent versions of different parity: \(U_N^{*, \text{even}}\) and \(U_N^{*, \text{odd}}\), either one can be used as \(U_N^*\). We define \(U_N^{*, \text{even}} = I_1\), the one-qubit identity operation, and \(U_N^{*, \text{odd}} = \sigma_y\). The unitary \((N+1)\)-qubit gate is obtained with the following rule:

\[
U_N^{*, N+1} = \frac{1}{\sqrt{2}} [I_1 \otimes U_N^{*, N} \pm i \sigma_y \otimes (P_N U_N^{*, N})],
\]

where \(P_N = \sigma_y \otimes I_N^{N-1}\) if \(N \geq 2\) and \(P_1 = \sigma_y\). At each step, either of the two signs can be chosen.

The fact that, with gate \(U_N^*\), Eve cannot miss even a single qubit unless she is content with zero information gain also protects the key distribution against photon-number splitting (PNS) attacks [32]. If the probability of an unwanted multi-photon pulse is \(\varepsilon\) and events are independent, the probability that Eve gains any information decreases at least as \(\varepsilon^N\).
In what follows, we study the case $N = 2$ in more detail. Arbitrary two-qubit gates have 16 degrees of freedom, several of which have no effect on Eve’s maximal information. First fixing the global phase of the gate and then following the treatment in Ref. [31], we obtain $U_2 = (k_{2,1} \otimes k_{2,2}) \times \exp \left[ \frac{i}{2} \left( c_1 \sigma_x \otimes \sigma_x + c_2 \sigma_y \otimes \sigma_y + c_3 \sigma_z \otimes \sigma_z \right) \right] \times (k_{1,1} \otimes k_{1,2})$, where $k_{j,1}$ are one-qubit gates and the middle gate, $C(c)$, has parameters $c = (c_1, c_2, c_3)$ with each $c_j \in [0, 2\pi]$. The local operation $k_{2,1} \otimes k_{2,2}$ can be directly undone by Eve, and is thus of no use to Alice and Bob. Hence, the interesting two-qubit gates are of the form $C(c)(k_{1,1} \otimes k_{1,2})$. To simplify the calculations, we set $k_{1,1} = k_{1,2}$. Removing this restriction can only improve the results presented in Sec. IV.

IV. RESULTS

Figure 3 shows Eve’s mutual information on Alice’s sifted key in the case $N = 2$, for an IR attack carried out using the $\sigma_x$ eigenbasis. The plot is obtained by a uniform sweep over the parameters $c \in [0, 2\pi]^3$, over which Alice can optimize the protocol. In the upper set of points, Eve measures both entangled qubits, and in the lower set only one of them. It makes no difference which qubit is measured, since here the gate $U_2$ is symmetric with respect to the two entangled qubits.

![Figure 3: Eve’s mutual information on Alice’s sifted key as a function of the induced QBER for different gates $U_2 = C(c)$. Eve uses the IR attack and measures in the $\sigma_x$ eigenbasis. The red dots (blue crosses) correspond to Eve measuring both (only one) of the two entangled qubits, in which case Eve’s maximal mutual information is between 0.5 and 0.125 (0.25 and 0).](image)

The topmost point in each set corresponds to $U_2$ being the two-qubit identity operation, with which our protocol reduces to BB84. At the undermost points of the two sets, $U_2 = C(0, \frac{\pi}{2}, 0) = U_2^\ast = (I_1 \otimes I_1 + \sigma_y \otimes \sigma_y) / \sqrt{2}$. As $c_2$ increases from 0 to $\frac{\pi}{2}$, the protocol continuously shifts from BB84 to the $U_2^\ast$-enhanced protocol. Eve achieves the maximal information $\frac{\Delta}{\pi} = 0.25$ by changing one of her measurement bases from $\sigma_z$ to $\sigma_y$.

![Figure 4: Eve’s mutual information on Alice’s sifted key as a function of the induced QBER sampled over all possible measurement bases for Eve. The entangling gate is fixed to $U_2 = C(c^\ast)$, where $c^\ast = (\frac{\pi}{32}, \frac{3\pi}{8}, \frac{\pi}{2})$. The red dots (blue crosses) correspond to Eve measuring both (only one) of the two entangled qubits, in which case Eve’s maximal mutual information is between 0 and 0.2237 (0 and 0.0284).](image)

Next, we show how to improve on the $1/(2N)$ bound in the case $N = 2$. We allow Eve to use any measurement bases. Thus, the task of finding the optimal $C(c)$ becomes a twofold optimization problem: Alice and Bob wish to minimize the maximal information Eve can obtain for a given QBER. We are thus interested in finding the value $\min_c \max_{(\beta_1, \gamma_1, \beta_2, \gamma_2)} \left[ I(A, E) / \text{QBER} \right]$ and the optimizing parameter values. We perform the optimization with the simplex search method [32]. One of the optimal choices of parameters for Alice and Bob is $c^\ast = (\frac{\pi}{32}, \frac{3\pi}{8}, \frac{\pi}{2})$, which leads to $I(A, E) \approx 0.2237$ and QBER $= 0.375$ for $\xi = 1$. Given $U_2 = C(c^\ast)$, an optimal choice for Eve is $(\beta_1, \gamma_1, \beta_2, \gamma_2) = (\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{4})$. Eve’s maximal information as a function of the QBER is shown as the solid line in Fig. 4. For a fixed QBER $\leq 25\%$, Eve’s information drops to less than $30\%$ of that in BB84.

Figure 4 elaborates on the consequences of Eve’s choices given $U_2 = C(c^\ast)$. In the upper (lower) set of points, Eve measures both (only one) of the qubits in different bases. The plot is generated by a uniform sweep over $(\beta_1, \gamma_1, \beta_2, \gamma_2) \in [0, 2\pi]^4$. Alice’s gate is fixed to $C(c^\ast)$ which, unlike $U_2^\ast$, is observed not to guarantee zero but still less than 0.03 bits of information leakage for one-qubit interceptions.

Let us present an approximate comparison between our protocol and BB84 in terms of the final bit rate. Following Ref. [33], we assume that during error correction Alice and Bob must exchange

$$nH_{\text{bin}}(q) = n[-q \log_2 q - (1 - q) \log_2(1 - q)]$$

bits, where $n$ is the length of the key material, and $q$ the QBER. We further make the safe assumption that this is the information, in bits, that is leaked to Eve. In BB84,
Eve’s information per bit after EC is

$$I_{EC}^{BB84}(q) = 2q + H_{bin}(q).$$

Let the optimal $N = 2$ setting represent our protocol, where Eve’s information after EC is

$$I_{EC}^{(2)}(q) = s\delta q + H_{bin}(\delta q),$$

where $s = 0.5965$ is the slope of the $I(A, E)$ curve shown in Fig. 2. The observed QBER is denoted by $\delta q$, so that $\delta$ is the factor by which the use of $U_2 = C(e^*)$ changes the QBER. The absolute key rate depends heavily on the practical implementation of the protocol, and we therefore use the relative key rate $r = R_{net}/R_{sift}$, where $R_{sift}$ is the rate at which sifted key bits are generated. We have

$$r(q) = I(A, B) - I(A, E) = 1 - H_{bin}(q) - I(A, E) = 1 - I_{EC}(q)$$

for both protocols.

In the following, we fix the QBER to $q = 6\%$, a typical value in a practical realization of BB84. Then, the relative key rate is $r_{BB84} = 0.553$ in BB84. The relative key rate for our two-qubit protocol is shown in Fig. 5 together with a protocol, for which $s = 0$. For example at $\delta = 1$ for both protocols, the gain of the two-qubit protocol over BB84 is $70\%$ of that of the protocol with $s = 0$. The relative key rate of BB84 is recovered at $\delta = 1.323$. Determining the exact value of $\delta$ and ways to decrease it is left for future research.

V. CONCLUSIONS

Our results show that entanglement can be employed to considerably improve the BB84-type key distribution, even in the case of two-qubit entanglement. The new protocol can be directly adapted to the several variants of BB84. We have demonstrated one promising scheme, where an IR eavesdropper must intercept every qubit in the entangled group to gain any information. Unfortunately, loss of qubits may pose a problem not only for Eve, but also for Bob. If one of the entangled qubits is completely lost, the QBER of the remaining qubits is likely to increase. Therefore, this protocol cannot be recommended for use at extreme distances where most transmitted qubits are lost. Making the protocol robust against qubit loss is a goal for future research.

Since the dimension of the total Hilbert space increases exponentially with the number of qubits, and the dimension of the subspace Eve can directly access increases only linearly, our scheme is expected to show even more pronounced benefits if applied to many-qubit entanglement. Further optimization for an arbitrary number of entangled qubits and assessment of more potential attacks is to be carried out in the future. Potential future research also includes methods for distinguishing between innocent noise in the quantum channel and that caused by eavesdropping, and determining the exact dependence of QBER on the innocent noise. The latter would enable definitive evaluation of the protocol final bit rate.

ACKNOWLEDGMENTS

We thank the Academy of Finland, Nokia Corporation, the Finnish Cultural Foundation, Väisälä Foundation, the EPSRC, the QIP IRC, and the Leverhulme Trust for financial support. In addition, Bob Clark is acknowledged for his warm hospitality at the Centre for Quantum Computer Technology.

APPENDIX

We show that the gate $U_2^N$ defined in Sec. III restricts the information provided by an intercept-resend attack to at most $1/(2N)$. First, note that $\sigma_y|a_j;\alpha_j\rangle = |\bar{a}_j;\alpha_j\rangle$ for $j = 1,...,N$. We claim that single-qubit measurements in any basis, applied to state $U_2^N (|a_1;\alpha_1\rangle|a_2;\alpha_2\rangle\cdots|a_N;\alpha_N\rangle)$, give uniformly random results until the last, $N$th, one. Thus it is not until the last measurement that Eve gets any information with the IR attack. Let us refer to this randomness of the first $N - 1$ measurements as property $R$.

To prove this first claim, we note that the transmitted states for gates $U_{2,even}^N$ and $U_{2,odd}^N$ are, respectively, $(|a_1;\alpha_1\rangle|a_2;\alpha_2\rangle \pm i|\bar{a}_1;\alpha_1\rangle|\bar{a}_2;\alpha_2\rangle)/\sqrt{2}$ and $(|a_1;\alpha_1\rangle|\bar{a}_2;\alpha_2\rangle \pm i|\bar{a}_1;\alpha_1\rangle|a_2;\alpha_2\rangle)/\sqrt{2}$, on which the
The remaining state to be that resulting from application of gate $U^*_N \in \mathcal{P}$, with an even or odd number of operators $\sigma_y$, according to the parity $\mathcal{P}$. The parity is invariant under the application of Eq. (10).

As the total number of different $N$-qubit tensor products of $I_1$ and $\sigma_y$ is $2^N$ and half of them have an even number of operators $\sigma_y$, the sum in Eq. (11) contains all possible $v^N_{\mathcal{P}}$ of the given parity $\mathcal{P}$. It follows that any permutation of qubits in the state $U^*_{N, \mathcal{P}}(\{a_1; \alpha_1\} \cdots |a_N; \alpha_N\rangle)$ results in essentially the same state, i.e., only the phases of the different terms change, which does not affect on the outcome of the following measurement. Hence, we can assume that the leftmost qubit is measured first, without restricting Eve’s actual order of measurements. Thus the application of the gate $U^*_N$ is not limited to IR attack, for which the measurement order of the eavesdropper is determined by Alice.

According to Eq. (12), the outcome of measuring the leftmost qubit in the state $U^*_{N, \mathcal{P}}(\{a_1; \alpha_1\} \cdots |a_N; \alpha_N\rangle)$ is uniformly random. Moreover, a correct result leads to the remaining state to be that resulting from application of gate $U^*_{N-1, \mathcal{P}}$, i.e., the gate of the same parity. An incorrect result leads to the state corresponding to $U^*_{N-1}$ of different parity. Thus, gate $U^*_N$ has property $\mathcal{R}$ for all $N > 1$. As Eve measures the qubits, she unwinds the recursion of Eq. (2) through even and odd states while learning nothing of the key until the remaining state has $N = 1$.

Let $E_1$ and $E_N$ denote the random variables of the outcomes of the first $N - 1$ measurements and the final, $N$th measurement Eve makes, respectively. Denote the conditional entropy of $E_N$ as $h_N = H(E_N | A, E_1)$. Note that $0 \leq h_N \leq 1$. If $U^*_N$ is used, $I(A, E_1) = 0$. The entropy $H(E_1) = N - 1$. Using the definition of conditional entropy $H(X|Y) = H(X, Y) - H(Y)$, we obtain:

$$I(A, E) = \frac{1}{N} \left[ 2N - H(A, E_1, E_N) \right]$$

$$= \frac{1}{N} \left[ 2N - h_N - H(E_1 | A) - H(A) \right]$$

$$= \frac{1}{N} \left[ N - h_N - H(E_1) \right]$$

$$= \frac{1}{N} (1 - h_N),$$

where we have recomposed random variables as $I(A, E) = I(A, E_1, E_N) = I(AE_1, E_N)$. Since the last measurement targets one qubit in a BB84 state, $h_N = \frac{1}{2}$ and $I(A, E) = \frac{1}{2N}$. This completes our proof that the gate $U^*_N$ limits the information provided by an intercept-resend attack to at most $1/(2N)$ per bit.

[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, 2000).
[2] D. Deutsch, Proc. R. Soc. Lond. A 400, 97 (1985).
[3] V. Giovannetti, S. Lloyd, and L. Maccone, Science 306, 1330 (2004).
[4] A. N. Boto, P. Kok, D. S. Abrams, S. L. Braunstein, C. P. Williams, and J. P. Dowling, Phys. Rev. Lett. 85, 2733 (2000).
[5] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. 74, 145 (2002).
[6] C. H. Bennett and G. Brassard, in Proceedings of IEEE International Conference on Computers, Systems and Signal Processing (IEEE, New York, 1984), pp. 175–179.
[7] C. H. Bennett, F. Bessette, G. Brassard, L. Salvail, and J. Smolin, J. Cryptology 5, 3 (1992).
[8] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[9] M. Lucamarini and S. Mancini, Phys. Rev. Lett. 94, 140501 (2005).
[10] L. Goldenberg and L. Vaidman, Phys. Rev. Lett. 75, 1239 (1995).
[11] D. Bruß, Phys. Rev. Lett. 81, 3018 (1998).
[12] X.-B. Wang, Phys. Rev. Lett. 94, 230503 (2005).
[13] B. Kraus, N. Gisin, and R. Renner, Phys. Rev. Lett. 95, 080501 (2005).
[14] H.-K. Lo, H. F. Chau, and M. Ardehali, J. Cryptology 18, 133 (2005).
[15] A. Acin, N. Gisin, and L. Masanes, Phys. Rev. Lett. 97, 120405 (2006).
[16] L.-P. Lamoureux, H. Bechmann-Pasquinucci, N. J. Cerf, N. Gisin, and C. Macchiavello, Phys. Rev. A 73, 032304 (2006).
[17] J. L. O’Brien, Science 318, 1567 (2007).
[18] T. Yoshie, A. Scherer, J. Hendrickson, G. Khitrova, H. M. Gibbs, G. Rupper, C. Ell, O. B. Shchekin, and D. G. Deppe, Nature 432, 200 (2004).
[19] K. Hennessey, A. Badolato, M. Winger, D. Gerace, M. Atatüre, S. Gulde, S. Fält, E. L. Hu, and A. Imamoglu, Nature 445, 896 (2007).
[20] D. Englund, A. Farao, I. Fushman, N. Stoltz, P. Petroff, and J. Vučković, Nature 450, 857 (2007).
[21] D. Englund, A. Farao, B. Zhang, Y. Yamamoto, and J. Vučković, Optics Express 15, 5550 (2007).
[22] A. S. Clark, J. Fulconis, J. G. Rarity, W. J. Wadsworth, and J. L. O’Brien (2008), arXiv:0802.1676.
[23] A. Politi, M. J. Cryan, J. G. Rarity, S. Yu, and J. L. O’Brien, Science 320, 646 (2008).
[24] E. Knill, R. Laflamme, and G. J. Milburn, Nature 409, 46 (2001).
[25] J. L. O’Brien, G. J. Pryde, A. G. White, T. C. Ralph, and D. Branning, Nature 426, 264 (2003).
[26] J. L. O’Brien, G. J. Pryde, A. Gilchrist, D. F. V. James,
N. K. Langford, T. C. Ralph, and A. G. White, Phys. Rev. Lett. 93, 080502 (2004).

[27] S. Gasparoni, J.-W. Pan, P. Walther, T. Rudolph, and A. Zeilinger, Phys. Rev. Lett. 93, 020504 (2004).

[28] S. J. Devitt, A. D. Greentree, R. Ionicioiu, J. L. O’Brien, W. J. Munro, and L. C. L. Hollenberg, Phys. Rev. A 76, 052312 (2007).

[29] G. J. Pryde, J. L. O’Brien, A. G. White, S. D. Bartlett, and T. C. Ralph, Phys. Rev. Lett. 92, 190402 (2004).

[30] G. Brassard and L. Salvail, in Advances in Cryptology—Eurocrypt ’93 Proceedings (Springer, Berlin, 1994), vol. 765 of Lecture Notes in Computer Science, pp. 410–423.

[31] H. Inamori, N. Lütkenhaus, and D. Mayers, Eur. Phys. J. D 41, 599 (2007).

[32] C. A. Fuchs, N. Gisin, R. B. Griffiths, C.-S. Niu, and A. Peres, Phys. Rev. A 56, 1163 (1997).

[33] S. Félix, N. Gisin, A. Stefanov, and H. Zbinden, J. Mod. Opt. 48, 2009 (2001).

[34] J. Zhang, J. Vala, S. Sastry, and K. B. Whaley, Phys. Rev. A 67, 042313 (2003).

[35] J. C. Lagarias, J. A. Reeds, M. H. Wright, and P. E. Wright, SIAM J. Optim. 9, 112 (1998).

[36] D. Bouwmeester, A. Ekert, and A. Zeilinger, eds., The Physics of Quantum Information (Springer-Verlag, Berlin, 2000), chap. 2, p. 37.

[37] A. Muller, H. Zbinden, and N. Gisin, Europhys. Lett. 33, 335 (1996).

[38] G. Ribordy, J.-D. Gautier, N. Gisin, O. Guinnard, and H. Zbinden, Electron. Lett. 34, 2116 (1998).

[39] T. Hirano, H. Yamanaka, M. Ashikaga, T. Konishi, and R. Namiki, Phys. Rev. A 68, 042331 (2003).

[40] R. Ursin, F. Tiefenbacher, T. Schmitt-Manderbach, H. Weier, T. Scheidl, M. Lindenthal, B. Blauensteiner, T. Jennewein, J. Perdigues, P. Trojek, et al., Nature Phys. 3, 481 (2007).