$L^2$ harmonic forms on complete special holonomy manifolds

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Abstract
In this article, we consider $L^2$ harmonic forms on a complete non-compact Riemannian manifold $X$ with a nonzero parallel form $\omega$. The main result is that if $(X, \omega)$ is a complete $G_2$- (or $\text{Spin}(7)$-) manifold with a $d$-linear $G_2$- (or $\text{Spin}(7)$-) structure form $\omega$, then the $L^2$ harmonic 2-forms on $X$ vanish. As an application, we prove that the instanton equation with square-integrable curvature on $(X, \omega)$ only has trivial solution. We would also consider the Hodge theory on the principal $G$-bundle $E$ over $(X, \omega)$.

Keywords $L^2$ harmonic form · $G_2$- ($\text{Spin}(7)$-)manifold · $d$-linear-form · Gauge theory

1 Introduction

Let $X$ be a $C^\infty$-manifold equipped with a differential form $\omega$. This form is called parallel if $\omega$ is preserved by the Levi-Civita connection: $\nabla \omega = 0$. This identity gives a powerful restriction on the holonomy group $\text{Hol}(X)$. In Kähler geometry, the parallel forms are the Kähler form and its powers. The algebraic geometers obtained many results of topological and geometric on studying the corresponding algebraic structure. In $G_2$- or $\text{Spin}(7)$-manifold, the parallel form is the $G_2$- or $\text{Spin}(7)$-structure. In [36], the author had generalized some of these results on Kähler manifolds to other manifolds with a parallel form, especially the parallel $G_2$-manifolds. The results which obtained on [36] can be summarized as Kähler identities for $G_2$-manifolds.

The theory of $G_2$-manifolds is one of the places where mathematics and physics interact most strongly [29,31]. In string theory, $G_2$-manifolds are expected to play the same role as Calabi–Yau manifolds in the usual A and B models of type-II string theories. There are many results on the construction of $G_2$-manifolds [2,27,28,30]. Hitchin constructed a geometry flow [19] which physicists called Hichin’s flow; it turned out to be extremely important in string physics.

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A basic question, pertaining both the function theory and topology on $X$, is: when are there nontrivial harmonic forms on $X$? When $X$ is not compact, a growth condition on the harmonic forms at infinity must be imposed, in order that the answer to this question be useful. A natural growth condition is square integrable; if $\Lambda^p_{(2)}(X)$ denotes the $L^2$ $p$-forms on $X$ and $\mathcal{H}^p_{(2)}(X)$ the harmonic forms in $\Lambda^p_{(2)}(X)$. One version of this basic question is: what is the structure of $\mathcal{H}^p_{(2)}(X)$? The study of $L^2$ harmonic forms on a complete Riemannian manifold is a very interesting and important subject; it also has numerous applications in the field of mathematical physics (see for example [18]).

In [16], Gromov states that if the Kähler form $\sigma$ on a complete Kähler satisfies $\sigma = d\theta$, where $\theta$ is a bounded one form, the only $L^2$ harmonic forms lie in the middle dimension. There are many complete Kähler manifolds with a exact Kähler form $\omega$ [4,16,26]. In [4,26], they extended Gromov’s theorem to the case of the one form $\theta$ is linear growth.

A $G_7$- or a $\text{Spin}(7)$-structure of a 7-, 8-manifold is given by a parallel 3-form $\phi$ or 4-form $\Omega$ (see [28] Sect. 10). It is also very intriguing to construct some examples of the $G_2$- or $\text{Spin}(7)$-manifolds with a $d$(linear) structure form.

**Example 1.1** There are some trivial examples of $G_2$-manifolds and $\text{Spin}(7)$-manifolds satisfy the growth conditions required.

1. Let $X$ be a complete connected manifold with zero sectional curvature and $\omega$ be a parallel differential $k$-form on $X$, then the Theorem 1.1 [4] states that $\omega$ is $d$(linear). But the Killing–Hopf theorem states that $X$ is isometric to a quotient of a Euclidean space by a group acting freely and properly discontinuously.

2. $\mathcal{C}(X) = (\mathbb{R}^+ \times X, \bar{g})$ with $\bar{g} = dr^2 + r^2 g$ as the Riemannian or metric cone over $X$. It is well known that $X$ admits a real Killing spinor if and only if $\mathcal{C}(X)$ admits a parallel spinor. Then, $\mathcal{C}(X)$ has restricted holonomy, for any nearly Kähler 6-manifold $X$, $\mathcal{C}(X)$ has holonomy $G_2$, and for any nearly parallel $G_2$-manifold $X$, $\mathcal{C}(X)$ has holonomy $\text{Spin}(7)$. We can show that the cone $\mathcal{C}(X)$ is also the model for the growth conditions required (see Sect. 3). But following Hopf–Rinow theorem, it implies that the cone manifold $\mathcal{C}(X)$ is incomplete.

In this article, we could prove that if $X$ is a complete $G_2$- (or $\text{Spin}(7)$-) manifold with a $d$(linear) $G_2$- (or $\text{Spin}(7)$-) structure form $\phi$ (or $\Omega$), then $\mathcal{H}^k_{(2)}(X) = \{0\}$ for $k = 0, 1, 2$, see Theorems 3.4 and 3.10. We will also show that the structure form could not be $d$(bounded), see Proposition 2.13.

**Remark 1.2** It is well known that if $X$ is a connected, complete, non-compact manifold with nonnegative Ricci curvature, then $\mathcal{H}^i_{(2)}(X) = \{0\}$, $i = 0, 1$. One knows that $G_2$-manifold or $\text{Spin}(7)$-manifold is Ricci flat, then $\mathcal{H}^i_{(2)}(X) = \{0\}$, $i = 0, 1$.

Instantons on the higher dimension, proposed in [7] and studied in [5,9,10,37], are important in mathematics [10] and string theory [15]. Instantons are important objects in modern field theories. To construct nontrivial solutions of instanton equations over a non-compact manifold is very important for high-energy physics. It is well known that the structures of the cylinders and metric cones over the six-, seven- and eight-dimensional manifolds with structure group $SU(3)$, $G_2$ and $\text{Spin}(7)$ are inherited from the base manifolds [13]. Constructions of solutions of the instanton equations on cylinders over nearly Kähler 6-manifolds and nearly parallel $G_2$ manifold were considered in [1,17,24,25]. In [13], they were interested in cone structures constructed over nearly Kähler sixfold $X^6$. Its metric cone has $G_2$-holonomy if we normalize the nearly Kähler manifold such that its Einstein constant is 5. The cylinder over a
parallel $G_2$-manifold has $\text{Spin}(7)$-holonomy. They showed that there was a $G_2$-instanton on these $G_2$-manifolds which given rise to a $\text{Spin}(7)$-instanton in eight dimensions.

In this article, we observe that if $(X^n,\omega)$ is a complete Riemannian manifold with a $d$-(linear) $k$-form $\omega$, then $\int_X \alpha \wedge \omega = 0$, where $\alpha$ is a closed $(n-k)$-form in $L^1$, see Lemma 4.1. We can prove a vanishing theorem as follows: if $X$ is a complete $G_2$-(or $\text{Spin}(7)$-) manifold with a $d$-(linear) $G_2$-(or $\text{Spin}(7)$-) structure $\phi$ (or $\Omega$), the $L^2$ solutions of the instanton equation are trivial, see Theorem 4.3. In [25] Sect. 4, the authors confirmed that the standard Yang–Mills functional was infinite on their solutions. The author was inspired by those results; he proved that the solutions of instantons with square-integrable curvature on the cylinder over a compact Riemannian manifold with a real Killing spinor are trivial [20]. We observe that the cylinder $\text{Cyl}(X):=(\mathbb{R} \times X, dr^2+ g_X)$ over a closed Riemannian manifold $X$ is complete. Combining Corollary 4.2, we can give another way to prove the vanishing theorem in [20]. Furthermore, we also prove that if the curvature of the connection satisfies a mild condition, then the instanton is a flat connection, see Theorem 4.6.

**Remark 1.3** The vanishing theorem 4.3 only means that the nontrivial instantons on a complete Riemannian manifold with a $d$-(linear) parallel form must have infinite standard Yang–Mills action. However, we cannot catch any information of the topological numbers associated with the instanton solutions, they might even be finite. For example, $\mathbb{R}^8$ is a model for the growth conditions required. The well-known $\text{Spin}(7)$-instanton solution on $\mathbb{R}^8$ constructed in S. Fubini and H. Nicolai [12] has infinite Yang–Mills action but finite topological numbers.

We also consider the Hodge theory on a principal $G$-bundle $E$ over a complete manifold $X$. We denote $H_p^P(X, E)$ by the space of $L^2$ harmonic $p$-forms $\Lambda^p(X, E)$ with respect to the Laplace–Beltrami operator $\Delta := d^* d + d d^*$ (see Definition 4.7). The space $H_p^P(X, E)$ depends on the connection. In this article, we assume that $E$ possesses a flat connection $d_A$ which means that $F_A = 0$, or equivalently, that $E$ is given by a representation $\pi_X \to U(r)$. Then, we would prove that if $(X, \phi)$ is a complete $G_2$-manifold with a $d$-(linear) $G_2$-structure $\phi$, then $H_p^P(X, E) = 0$ unless $p \neq 3, 4$, see Theorem 4.10.

## 2 Riemannian manifolds with a parallel differential form

In this section, we recall some notations and definitions on differential geometry [36]. Let $X$ be a $C^\infty$-manifold. We denote by $\Lambda^*(X)$ the smooth forms on $X$. Given an odd or even from $\alpha \in \Lambda^*(X)$, we denote by $\overline{\alpha}$ its parity, which is equal to 0 for even forms and 1 for odd forms. An operator $f \in \text{End}(\Lambda^*(X))$ preserving parity is called even, and one exchanging odd and even forms is odd, $f$ is equal to 0 for even forms and 1 for odd ones.

Given a $C^\infty$-linear map $\Lambda^1(X) \xrightarrow{P} \Lambda^{\text{odd}}(X)$ or $\Lambda^1(X) \xrightarrow{P} \Lambda^{\text{even}}(X)$, $p$ can be uniquely extended to a $C^\infty$-linear derivation $\rho$ on $\Lambda^*(X)$, using the rule

$$\rho|_{\Lambda^0(X)} = 0,$$

$$\rho|_{\Lambda^1(X)} = p,$$

$$\rho(\alpha \wedge \beta) = \rho(\alpha) \wedge \beta + (-1)^{\bar{\beta} \bar{\alpha}} \alpha \wedge \rho(\beta).$$

Verbitsky gave a definition of the structure operator of $(X, \omega)$, see [36] Definition 2.1.
**Definition 2.1** Let $X$ be a Riemannian manifold equipped with a parallel differential $k$-form $\omega$. Consider an operator $C : \Lambda^1(X) \to \Lambda^{k-1}(X)$ mapping $\alpha \in \Lambda^1(X)$ to $\ast(\ast \omega \wedge \alpha)$. The corresponding differentiation

$$C : \Lambda^n(X) \to \Lambda^{n+k-2}(X)$$

is called the structure operator of $(X, \omega)$.

**Lemma 2.2** Let $X$ be a Riemannian manifold equipped with a parallel differential $k$-form $\omega$ and $L_\omega$ be the operator $\alpha \mapsto \alpha \wedge \omega$. Then,

$$dC = \{L_\omega, d\}$$

where $dC$ is the supercommutator $\{d, C\} := dC - (-1)^{\tilde{C}} Cd$.

We recall some Generalized Kähler identities which proved by Verbitsky (see [36] Proposition 2.5). Here, we give a proof in detail for the reader’s convenience.

**Proposition 2.3** Let $X$ be a Riemannian manifold equipped with a parallel differential $k$-form $\omega$, $dC$ the twisted de Rham operator constructed above and $d_C^*$ its Hermitian adjoint. Then,

(i) The following supercommutators vanish:

$$\{d, dC\} = 0, \{d, dC^*\} = 0, \{d^*, dC\} = 0, \{d^*, dC^*\} = 0.$$

(ii) The Laplacian $\Delta = \{d, d^*\}$ commutes with $L_\omega : \alpha \mapsto \alpha \wedge \omega$ and its adjoint operator, denoted as $\Lambda_\omega : \Lambda^i(X) \to \Lambda^{i-k}(X)$.

**Proof** Let $\delta$ be an odd element in a graded Lie superalgebra $A$ satisfying $\{\delta, \delta\} = 0$. Using the graded Jacobi identity, we obtain

$$\{\delta, \{\delta, \chi\}\} = -\{\delta, \{\delta, \chi\}\} + \{\{\delta, \delta\}, \chi\}.$$

This gives $2\{\delta, \{\delta, \chi\}\} = 0$.

Now, $\{d, dC\} = \{d, \{d, dC\}\} = 0$ and $\{d^*, dC\} = \{d^*, \{d^*, L_\omega\}\} = 0$ by Lemma 2.2. Taking Hermitian adjoints of these identities, we obtain the other two equations of Proposition 2.3 (i).

Now, the graded Jacobi identity implies

$$[L_\omega, \Delta] = [L_\omega, \{d, d^*\}] = (-1)^{\tilde{\omega}}[d, \{L_\omega, d^*\}]$$

we use $\{L_\omega, d\} = 0$ as $\omega$ is closed. This gives

$$[L_\omega, \Delta] = (-1)^{\tilde{\omega}}[d, dC] = 0$$

as Proposition 2.3 implies. Taking the Hermitian adjoint, we also obtain $[\Lambda_\omega, \Delta] = 0$. $\square$

**Corollary 2.4** ([36] Corollary 2.9) Let $(X, \omega)$ be a Riemannian manifold equipped with a parallel differential $k$-form $\omega$ and $\alpha$ a harmonic form on $X$. Then, $\alpha \wedge \omega$ is harmonic.

**Proof** It follows from Proposition 2.3 (ii). $\square$

**Remark 2.5** If $(X, \omega)$ is a $G_2$- or $\text{Spin}(7)$-manifold, Proposition 2.3 gives the Laplacian $\Delta$ commutes between the operators $L_\omega, \Lambda_\omega, L_{\ast \omega}, \Lambda_{\ast \omega}$. $\square$
We begin the proof of Theorem 2.9 by recalling some basic facts in Hodge theory. If $X$ is an oriented complete Riemannian manifold, let $d^*$ be the adjoint operator of $d$ acting on the space of $L^2$ $k$-forms. Denoted by $\Lambda^k_{(2)}(X)$ and $\mathcal{H}^k_{(2)}(X)$ the spaces of $L^2$ $k$-forms and $L^2$ harmonic $k$-forms, respectively. By elliptic regularity and completeness of the manifold, a $k$-form in $\mathcal{H}^k_{(2)}(X)$ is smooth, closed and co-closed.

**Definition 2.6** A differential form $\omega$ on a complete non-compact Riemannian manifold is called $d$ (linear) if there exists a differential form $\beta$ and a number $c > 0$ such that 

$$\omega = d\beta, \quad |\omega(x)| \leq c,$$

$$|\beta(x)| \leq c(1 + \rho(x_0, x)),$$

where $\rho(x_0, x)$ stands for the Riemannian distance between $x$ and a base point $x_0$.

Jost and Zuo’s theorem stated that if a complete Kähler manifold $X$ with a $d$ (linear) Kähler form $\omega$, then the only $L^2$-harmonic forms lie in the middle dimension. In [4], Cao–Xavier also obtained the same result of Jost–Zuo by another way.

**Theorem 2.7** Let $(X, \omega)$ be a complete Kähler $n$-manifold with a $d$ (linear) Kähler form. Then, all $L^2$-harmonic $p$-forms for $p \neq n$ vanish.

**Example 2.8** Let $(X, \eta, \omega)$ be a Sasakian $2n + 1$-fold, $\eta$ is a contact 1-form on $X$. Denoted by $C(X)$ the Riemannian cone of $(X, g)$. By definition, the Riemannian cone is a product, $\mathbb{R}^+ \times X$, equipped with a metric $dr^2 + r^2 g$, where $r$ is a unit parameter of $\mathbb{R}^+$. Then, the Riemannian cone $C(X)$ is a Kähler manifold with a Kähler form $\omega$ defined by

$$\omega = r^2 d\eta + 2r dr \wedge \eta,$$

Since $\Omega = d(r^2 \eta) = d\beta$ and $\rho(x_0, x) = O(r)$, then the Riemannian cone $C(X)$ is also the model for the growth conditions required.

We extend the idea of Cao–Xavier’s to the case of Riemannian manifold equipped with a parallel differential form. Then, we have

**Theorem 2.9** Let $(X, \omega)$ be a Riemannian manifold equipped with a parallel differential $k$-form $\omega$. If $\omega$ is also $d$ (linear), then for any $\alpha \in \mathcal{H}^p_{(2)}(X)$, we have 

$$\omega \wedge \alpha = 0.$$

**Proof** Let $\eta : \mathbb{R} \to \mathbb{R}$ be smooth, $0 \leq \eta \leq 1$,

$$\eta(t) = \begin{cases} 1, & t \leq 0 \\ 0, & t \geq 1 \end{cases}$$

and consider the compactly supported function

$$f_j(x) = \eta(\rho(x_0, x) - j),$$

where $j$ is a positive integer.

Let $\alpha$ be a harmonic $p$-form in $L^2$ and consider the form $v = \beta \wedge \alpha$. Observing that $d^*(\omega \wedge \alpha) = 0$ since $\omega \wedge \alpha \in \mathcal{H}^{p+k}_{(2)}(X)$ and noticing that $f_j v$ has compact support, one has

$$0 = \langle d^*(\omega \wedge \alpha), f_j v \rangle_{L^2(X)} = \langle \omega \wedge \alpha, d(f_j v) \rangle_{L^2(X)}.$$
We further note that, since $\omega = d\beta$ and $d\alpha = 0$,
\[
0 = \langle \omega \wedge \alpha, d(f_j \nu) \rangle_{L^2(X)}
\]
\[
= \langle \omega \wedge \alpha, f_j d\nu \rangle_{L^2(X)} + \langle \omega \wedge \alpha, d f_j \wedge \nu \rangle_{L^2(X)}
\]
\[
= \langle \omega \wedge \alpha, f_j f_j \wedge \alpha \rangle_{L^2(X)} + \langle \omega \wedge \alpha, d f_j \wedge \beta \wedge \alpha \rangle_{L^2(X)}.
\]
(2.1)

Since $0 \leq f_j \leq 1$ and $\lim_{j \to \infty} f_j(x)(\omega \wedge \alpha)(x) = (\omega \wedge \alpha)(x)$, it follows from the dominated convergence theorem that
\[
\lim_{j \to \infty} \langle \omega \wedge \alpha, f_j \omega \wedge \alpha \rangle_{L^2(X)} = \| \omega \wedge \alpha \|_{L^2(X)}^2.
\]
(2.2)

Since $\omega$ is bounded, $\text{supp}(d f_j) \subset B_{j+1} \setminus B_j$ and $|\beta(x)| = O(\rho(x_0, x))$, one obtains
\[
|\langle \omega \wedge \alpha, d f_j \wedge \beta \wedge \alpha \rangle_{L^2(X)}| \leq (j + 1) C \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 dx,
\]
(2.3)

where $C$ is a constant independent of $j$.

We claim that there exists a subsequence $\{j_i\}_{i \geq 1}$ such that
\[
\lim_{i \to \infty} (j_i + 1) \int_{B_{j_i+1} \setminus B_{j_i}} |\alpha(x)|^2 dx = 0.
\]
(2.4)

If not, there would exist a positive constant $a$ such that
\[
\lim_{i \to \infty} (j_i + 1) \int_{B_{j_i+1} \setminus B_{j_i}} |\alpha(x)|^2 dx \geq a > 0, \quad j \geq 1.
\]

This inequality implies
\[
\int_X |\alpha(x)|^2 dx = \sum_{j=0}^{\infty} \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 dx \geq a \sum_{j=0}^{\infty} \frac{1}{j + 1} = +\infty
\]
a contradiction to the assumption $\int_X |\alpha(x)|^2 dx < \infty$. Hence, there exists a subsequence $\{j_i\}_{i \geq 1}$ for which (2.4) holds. Using (2.3) and (2.4), one obtains
\[
\lim_{i \to \infty} \langle \omega \wedge \alpha, d f_j \wedge \beta \wedge \alpha \rangle_{L^2(X)} = 0
\]
(2.5)

It now follows from (2.1), (2.2) and (2.5) that $\omega \wedge \alpha = 0$. \hfill \qed

**Remark 2.10** There are many complete manifolds with a $d$-linear parallel differential form. If $X$ is a complete simply connected manifold of non-positive sectional curvature and $\omega$ is a parallel differential $k$-form on $X$, then the Theorem 1.1 on [4] states that $\omega$ is $d$-linear.

**Corollary 2.11** Let $(X, \omega)$ be a Riemannian manifold equipped with a nonzero parallel differential $k$-form $\omega$. If $\omega$ is also $d$-linear, then $\mathcal{H}^0_2(X) = 0$.

**Proof** We denote by $f$ a $L^2$-harmonic function on $X$. Then, following Theorem 2.9, $f \omega = 0$. Since $\omega$ is nonzero all over $X$, it follows that $f$ vanish. \hfill \qed
As we derive estimates in this section (and also following sections), there will be many constants which appear. Sometimes, we will take care to bound the size of these constants, but we will also use the following notation whenever the value of the constants are unimportant. We write $\alpha \lesssim \beta$ to mean that $\alpha \leq C \beta$ for some positive constant $C$ independent of certain parameters on which $\alpha$ and $\beta$ depend. The parameters on which $C$ is independent will be clear or specified at each occurrence. We also use $\beta \lesssim \alpha$ and $\alpha \approx \beta$ analogously.

If we suppose the parallel $k$-form $\omega$ is $d$-(bounded), following the idea of Gromov [16], we can give a lower bound on the spectrum of the Laplace operator $\Delta$ on $\Lambda^0(X)$.

**Proposition 2.12** Let $(X, \omega)$ be a Riemannian $n$-manifold equipped with a parallel, nonzero, differential $k$-form $\omega$. If $\omega$ is $d$-(bounded), i.e., there exists a bounded $k$-1-form $\theta$ such that $\omega = d\theta$, then any $\alpha \in \Lambda^0(X)$ satisfies the inequality

$$\|\alpha\|^2_{L^2(X)} \leq C \|\theta\|^2_{L^\infty(X)} \langle \Delta \alpha, \alpha \rangle_{L^2(X)},$$

where $C = C(X, n)$ is a positive constant.

**Proof** Since $\omega$ is a parallel differential form, then $\nabla|\omega|^2 = 0$, i.e., $|\omega| = \text{constant}$. Denoted by $u \in \Lambda^0(X)$, we observe that:

$$|u \wedge \omega|^2 = *(u \wedge \omega) = \text{constant}|u|^2,$$

and

$$\Delta(u \wedge \omega) \wedge *(u \wedge \omega) = \Delta(u \wedge \omega) \wedge *(u \wedge \omega) = \text{constant}(\Delta u \wedge *u).$$

These imply that

$$\|u\|_{L^2(X)} = \text{constant}\|u \wedge \omega\|_{L^2(X)}, \quad \langle \Delta(u \wedge \omega), u \wedge \omega \rangle_{L^2(X)} = \text{constant}\langle \Delta u, u \rangle_{L^2(X)}.$$ 

Now, we write $\beta = \alpha \wedge \omega = d\eta - \tilde{\alpha}$, for $\eta = \alpha \wedge \theta$ and $\tilde{\alpha} = d\alpha \wedge \theta$, and observe that

$$\|\eta\|_{L^2(X)} \lesssim \|\theta\|_{L^\infty(X)} \|\alpha\|_{L^2(X)}.$$ 

Next, since

$$\|\tilde{\alpha}\|_{L^2(X)} \lesssim \|d\alpha\|_{L^2(X)} \|\theta\|_{L^\infty(X)} \lesssim \langle \Delta \alpha, \alpha \rangle_{L^2(X)}^{1/2} \|\theta\|_{L^\infty(X)},$$

we have

$$\|\beta\|^2_{L^2(X)} \leq |\langle \beta, d\eta \rangle_{L^2(X)}| + |\langle \beta, \tilde{\alpha} \rangle_{L^2(X)}| \leq |\langle d^* \beta, \eta \rangle_{L^2(X)}| + |\langle \beta, \tilde{\alpha} \rangle_{L^2(X)}| \lesssim \langle \Delta \beta, \beta \rangle_{L^2(X)}^{1/2} \|\theta\|_{L^\infty(X)} \|\beta\|_{L^2(X)} + \|\beta\|_{L^2(X)} \|d\alpha\|_{L^2(X)} \|\theta\|_{L^\infty(X)} \lesssim \langle \Delta \alpha, \alpha \rangle_{L^2(X)}^{1/2} \|\theta\|_{L^\infty(X)} \|\beta\|_{L^2(X)}.$$ 

This yields the desired estimate

$$\|\alpha\|^2_{L^2(X)} \lesssim \|\beta\|^2_{L^2(X)} \lesssim \|\theta\|^2_{L^\infty(X)} \langle \Delta \alpha, \alpha \rangle_{L^2(X)}.$$ 

We complete this proof.
In [6], Cheng and Yau proved that the first eigenvalue of Laplace operator $\Delta$ is zero on a complete Ricci-flat manifold. Hence, one can easily see the $G_2$- or $\text{Spin}(7)$-structure could not be $d(\text{bounded})$ since the Proposition 2.12 states that the first eigenvalue is nonzero if the structure form is $d(\text{bounded})$.

**Proposition 2.13** If $\phi$ (or $\Omega$) is the $G_2$- (or $\text{Spin}(7)$-) structure from over a complete, non-compact $G_2$- (or $\text{Spin}(7)$-) manifold, then $\phi$ (or $\Omega$) could be not $d(\text{bounded})$.

### 3 Special holonomy manifolds

#### 3.1 $G_2$-manifolds

**Definition 3.1** A $G_2$-manifold is a 7-manifold $X$ equipped with a torsion-free $G_2$-structure $\phi$, that is

$$\nabla_{g_\phi} \phi = 0,$$

where $g_\phi$ is the metric induced by $\phi$.

Under the action of $G_2$, the space $\Lambda^2(X)$ splits into irreducible representations, as follows:

$$\Lambda^2(X) = \Lambda^2_7(X) \oplus \Lambda^2_{14}(X).$$

(3.1)

where $\Lambda^i_j$ is an irreducible $G_2$-representation of dimension $j$. These summands can be characterized as follows:

- $\Lambda^2_7(X) = \{\alpha \in \Lambda^2(X) \mid \ast(\alpha \wedge \phi) = 2\alpha\}$,
- $\Lambda^2_{14}(X) = \{\alpha \in \Lambda^2(X) \mid \ast(\alpha \wedge \phi) = -\alpha\} = \{\alpha \in \Lambda^2(X) \mid \alpha \wedge \ast \phi = 0\}$.

From the construction, it is clear that the splitting (3.1) can be obtained via the operator $L_\phi$, $\Lambda_\phi$, $L_{\ast \phi}$, $\Lambda_{\ast \phi}$. By Proposition 2.3 these operators commute with the Laplacian. Therefore, the harmonic forms also split:

$$\mathcal{H}^2_2(X) = \mathcal{H}^2_7(X) \oplus \mathcal{H}^2_{14}(X).$$

**Example 3.2** Let $(X, \omega, \Omega)$ be a nearly Kähler sixfold, see [34,35]. There is a $(3,0)$-form $\Omega$ with $|\Omega| = 1$, and

$$d\omega = 3\lambda Re\Omega, \quad dIm\Omega = -2\lambda \omega^2,$$

where $\lambda$ is a nonzero real constant. For simply, we choose $\lambda = 1$. Denoted by $C(X)$ the Riemannian cone of $(X, g)$. The Riemannian cone $(C(X), dr^2 + r^2 g)$ is a $G_2$-manifold with torsion-free $G_2$-structure $\phi$ defined by

$$\phi := r^2 \omega \wedge dr + r^3 Re\Omega.$$

Since $\phi = d(\frac{1}{2} r^3 \omega) = d\beta$ and $\rho(x_0, x) = O(r)$, then the Riemanniann cone $C(X)$ is also the model for the growth conditions required.

We will show that the map $L_\phi : \Lambda^p \rightarrow \Lambda^{p+3}$ on the complete $G_2$-manifold is injective for $p = 0, 1, 2$. 
Lemma 3.3  Let $(X, \phi)$ be a complete $G_2$-manifold, for any $\alpha \in \Lambda^k(X)$, $k = 0, 1, 2$, satisfies the inequalities

\[
\|\alpha\|_{L^2(X)} \approx \|\alpha \wedge \phi\|_{L^2(X)}, \\
(\Delta \alpha, \alpha)_{L^2(X)} \approx (\Delta(\alpha \wedge \phi), \alpha \wedge \phi)_{L^2(X)}.
\]

Proof  Let $\alpha, \beta \in \Lambda^0(X)$, we observe that:

\((\alpha \wedge \phi) \wedge *(\beta \wedge \phi) = 7\alpha\beta \ast 1.\)

We take $\beta = \alpha$, then

\[
\|\alpha\|_{L^2(X)}^2 = \frac{1}{7}\|\alpha \wedge \phi\|_{L^2(X)}^2, \\
(\Delta \alpha, \alpha)_{L^2(X)} = \frac{1}{7}(\Delta(\alpha \wedge \phi), \alpha \wedge \phi)_{L^2(X)}.
\]

Let $\alpha, \beta \in \Lambda^1(X)$, we also observe that:

\[*(\alpha \wedge \phi) \wedge (\beta \wedge \phi) = 4 \ast \alpha \wedge \beta;\]

here, we use the fact $*(\alpha \wedge \phi) \wedge \phi = -4 \ast \alpha$ (see [3]). We take $\beta = \alpha$, then

\[
\|\alpha\|_{L^2(X)}^2 = \frac{1}{4}\|\alpha \wedge \phi\|_{L^2(X)}^2, \\
(\Delta \alpha, \alpha)_{L^2(X)} = \frac{1}{4}(\Delta(\alpha \wedge \phi), \alpha \wedge \phi)_{L^2(X)}.
\]

Let $\alpha \in \Lambda^2(X)$, we can write $\alpha = \alpha^7 + \alpha^{14}$, then $\alpha \wedge \phi = 2 \ast \alpha^7 - \ast \alpha^{14}$. Hence,

\[
\|\alpha \wedge \phi\|_{L^2(X)}^2 = 4\|\alpha^7\|_{L^2(X)}^2 + \|\alpha^{14}\|_{L^2(X)}^2 \approx \|\alpha\|_{L^2(X)}^2.
\]

Since $[\Delta, L_{\phi}] = 0$, we have $\Delta(\alpha \wedge \phi) = \Delta \alpha \wedge \phi = \ast \Delta(2\alpha^7 - \alpha^{14})$. Then,

\[
(\Delta(\alpha \wedge \phi), \alpha \wedge \phi)_{L^2(X)} = (\ast \Delta(2\alpha^7 - \alpha^{14}), \ast(2\alpha^7 - \alpha^{14}))_{L^2(X)}
\]

\[
= 4\langle \Delta \alpha^7, \alpha^7 \rangle_{L^2(X)} + \langle \Delta \alpha^{14}, \alpha^{14} \rangle_{L^2(X)}
\]

\[
\approx (\Delta \alpha, \alpha)_{L^2(X)}.
\]

\[\square\]

Theorem 3.4  Let $(X, \phi)$ be a complete $G_2$-manifold with a $d$(linear) $G_2$-structure. Then, $\mathcal{H}^k_2(X) = \{0\}$ for $k = 0, 1, 2$.

Proof  We denote $\alpha$ by a harmonic $p$-form $\alpha$. Following the hypothesis of the structure form $\phi$, we have $\alpha \wedge \phi = 0$ (see Lemma 3.3). Since $L_{\phi} : \Lambda^p(X) \rightarrow \Lambda^{p+3}(X)$ is injective for $p = 0, 1, 2$ (see Lemma 3.3), we have $\alpha = 0$. \[\square\]

If we suppose that the $G_2$-structure 4-form $\ast \phi$ is $d$(linear), we would also prove another vanishing theorem.

Theorem 3.5  Let $(X, \phi)$ be a complete $G_2$-manifold. If $\ast \phi$ is a $d$(linear) form, then $\mathcal{H}^2_2(X) = \{0\}$.

Proof  We denote $\alpha$ by a harmonic $L^2$-form of degree 2. We also consider the form $\alpha \wedge \ast \phi$, following Theorem 2.9, $\alpha \wedge \ast \phi = 0$, i.e., $\alpha + \ast(\alpha \wedge \phi) = 0$. On this time, the map $L_{\ast \phi} : \Lambda^2(X) \rightarrow \Lambda^6(X)$ is not injective. But tr$(\alpha \wedge \alpha)$ is closed $L^1$ form on $X$, following Lemma 4.1, $\|\alpha\|_{L^2(X)}^2 = \int_X \operatorname{tr}(\alpha \wedge \alpha \wedge \phi) = 0$, i.e., $\alpha = 0$. \[\square\]
Example 3.6 Let \((X, \eta, \omega)\) be a Sasakian–Einstein fivefold; \(\eta\) is a contact 1-form on \(X\). The metric cone \(C(X)\) is a Calabi–Yau manifold. There are Kähler form \(\omega = d(\frac{1}{2}r^2 \eta)\) and volume form \(\Omega \in \Lambda^{3,0}(X)\) which satisfies \(\nabla \Omega = 0\). Denoted by \(Cyl(C(X))\) the cylinder over the Calabi–Yau manifold \(C(X)\). We can use the \(\omega, \Omega\) on the base \(C(X)\) to define a \(G_2\)-structure:

\[
\phi = dt \wedge \omega + Im \Omega
\]

and

\[
* \phi = \frac{1}{2} \omega^2 + dt \wedge Re \Omega.
\]

where the metric on \(Cyl(C(X))\) is \(dr^2 + dr^2 + r^2 g_X\). Since \(* \phi = d(\omega \wedge \frac{1}{2}r^2 \eta + t Re \Omega)\) and \(\rho(x_0, x) = O((r^2 + r^2)^{1/2})\), then the \(G_2\)-manifold \(Cyl(C(X))\) has a linear growth parallel form \(* \phi\).

3.2 \textit{Spin}(7)-manifolds

Definition 3.7 A \textit{Spin}(7)-manifold is a 8-manifold \(X\) equipped with a torsion-free \textit{Spin}(7)-structure \(\Omega\), that is

\[
\nabla g_\Omega \Omega = 0,
\]

where \(g_\Omega\) is the metric induced by \(\Omega\).

Under the action of \textit{Spin}(7), the space \(\Lambda^2(X)\) splits into irreducible representations, as follows:

\[
\Lambda^2(X) = \Lambda^2_7(X) \oplus \Lambda^2_{21}(X). \tag{3.2}
\]

These summands can be characterized as follows:

\[
\Lambda^2_7(X) = \{\alpha \in \Lambda^2(X) \mid * (\alpha \wedge \Omega) = 3\alpha\},
\]

\[
\Lambda^2_{21}(X) = \{\alpha \in \Lambda^2(X) \mid *(\alpha \wedge \Omega) = -\alpha\}.
\]

From the construction, it is clear that the splitting (3.2) can be obtained via the operator \(L_\Omega, \Lambda_\Omega\). By Proposition 2.3, these operators commute with the Laplacian. Therefore, the harmonic forms also split:

\[
\mathcal{H}^2_{(2)}(X) = \mathcal{H}^2_{7,(2)}(X) \oplus \mathcal{H}^2_{21,21}(X).
\]

Example 3.8 Let \((X, \phi)\) be a nearly parallel \(G_2\)-manifold (see [23]). There is a 3-form \(\phi\) with \(\|\phi\|^2 = 7\) such that

\[
d\phi = 4 * \phi.
\]

Then, the Riemannian cone \((C(X), dr^2 + r^2 g)\) is a \textit{Spin}(7)-manifold with \textit{Spin}(7)-structure \(\Omega\) defined by

\[
\Omega := r^3 dr \wedge \phi + r^4 * \phi.
\]

Since \(\phi = d(\frac{1}{4}r^4 \phi) = d\beta\) and \(\rho(x_0, x) = O(r)\), the Riemannian cone \(C(X)\) is also the model for the growth conditions required.

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We will also show that the map $L_\Omega : \Lambda^p \to \Lambda^{p+4}$ on the complete Spin(7)-manifold is injective for $p = 0, 1, 2$.

**Lemma 3.9** Let $(X, \Omega)$ be a complete Spin(7)-manifold, for any $\alpha \in \Lambda^k(X), k = 0, 1, 2$, satisfies the inequalities

\[
\|\alpha\|_{L^2(X)}^2 = \|\alpha \wedge \Omega\|_{L^2(X)}, \quad \langle \Delta \alpha, \alpha \rangle_{L^2(X)} = \frac{1}{12} \langle \Delta (\alpha \wedge \Omega), \alpha \wedge \Omega \rangle_{L^2(X)}.
\]

**Proof** Let $\alpha, \beta \in \Lambda^0(X)$, we observe that:

\[
(\alpha \wedge \Omega) \wedge * (\beta \wedge \Omega) = 14 \alpha \beta \wedge 1,
\]

then

\[
\|\alpha\|_{L^2(X)}^2 = \frac{1}{14} \|\alpha \wedge \Omega\|_{L^2(X)}^2, \quad \langle \Delta \alpha, \alpha \rangle_{L^2(X)} = \frac{1}{12} \langle \Delta (\alpha \wedge \Omega), \alpha \wedge \Omega \rangle_{L^2(X)}.
\]

Let $\alpha, \beta \in \Lambda^1(X)$, we also observe that:

\[
*(\alpha \wedge \Omega) \wedge (\beta \wedge \Omega) = 4 * \alpha \wedge \beta;
\]

here, we use the fact $*(\alpha \wedge \Omega) \wedge \Omega = 4 \alpha$. We take $\beta = \alpha$, then

\[
\|\alpha\|_{L^2(X)}^2 = \frac{1}{4} \|\alpha \wedge \Omega\|_{L^2(X)}^2, \quad \langle \Delta \alpha, \alpha \rangle_{L^2(X)} = \frac{1}{4} \langle \Delta (\alpha \wedge \Omega), \alpha \wedge \Omega \rangle_{L^2(X)}.
\]

Let $\alpha \in \Lambda^2(X)$, we write $\alpha = \alpha^7 + \alpha^{21}$, then $\alpha \wedge \Omega = 3 \alpha^7 - \alpha^{21}$, Hence,

\[
\|\alpha \wedge \Omega\|_{L^2(X)}^2 = 9 \|\alpha^7\|_{L^2(X)}^2 + \|\alpha^{21}\|_{L^2(X)}^2 \approx \|\alpha\|_{L^2(X)}^2.
\]

Since $[\Delta, L_\Omega] = 0$, we have $\Delta (\alpha \wedge \Omega) = \Delta \alpha \wedge \Omega = * \Delta (3\alpha^7 - \alpha^{21})$. Then,

\[
\langle \Delta (\alpha \wedge \Omega), \alpha \wedge \Omega \rangle_{L^2(X)} = \langle * \Delta (3\alpha^7 - \alpha^{21}), * (3\alpha^7 - \alpha^{21}) \rangle_{L^2(X)}
\]

\[
= 9 \langle \alpha^7, \alpha^{21} \rangle_{L^2(X)} + \langle \alpha^{21}, \alpha^{21} \rangle_{L^2(X)} \approx \langle \Delta \alpha, \alpha \rangle_{L^2(X)}.
\]

\[\Box\]

**Theorem 3.10** Let $(X, \Omega)$ be a complete Spin(7)-manifold with a d(linear) Spin(7)-structure. Then, $H^k_{(1)}(X) = \{0\}$ for $k = 0, 1, 2$.

**Proof** We denote $\alpha$ by a harmonic $p$-form $\alpha$. Following the hypothesis of the structure form $\Omega$, we have $\alpha \wedge \Omega = 0$ (see Lemma 3.3). Since $L_\Omega : \Lambda^p(X) \to \Lambda^{p+4}(X)$ is injective for $p = 0, 1, 2$ (see Lemma 3.9), we have $\alpha = 0$. \[\Box\]

### 4 Gauge theory

#### 4.1 Instantons

We consider the instanton equation on the geometries discussed in the previous section. Let $E$ be a principal $G$-bundle over a complete Riemannian manifold $X$, with dimension $n$ and $A$ be a connection on bundle $E$ over $X$. The instanton equation on $X$ can be introduced as follows. Assume there is a 4-form $Q$ on $X$. Then, a $(n - 4)$-form $\ast Q$ exists, where $\ast$ is the
Hodge operator on $X$. A connection $A$ is called an anti-self-dual instanton, when it satisfies the instanton equation

$$\ast F_A + \ast Q \wedge F_A = 0$$

(4.1)

When $n > 4$, these equations can be defined on the manifold $X$ with a special holonomy group, i.e., the holonomy group $\text{Hol}(X)$ of the Levi–Civita connection on the tangent bundle $TX$ is a subgroup of the group $SO(n)$. Each solution of Eq. (4.1) satisfies the Yang–Mills equation. The instanton equation (4.1) is also well defined on a manifold $X$ with non-integrable $G$-structures, but Eq. (4.1) implies the Yang–Mills equation will have torsion. For our purposes, $X$ is a $G_2$-manifold and $\ast Q$ is the $G_2$-structure 3-form or $X$ is a $\text{Spin}(7)$-manifold and $\ast Q$ is the $\text{Spin}(7)$-structure 4-form.

**Lemma 4.1** Let $(X^n, \omega)$ be a complete Riemannian $n$-manifold with a $d$-linear $k$-form $\omega$. Suppose that $\omega$ is bounded. If $\alpha$ is a closed $L^1$ form of degree $n - k$, then

$$\int_X \alpha \wedge \omega = 0.$$  

**Proof** Let $\alpha$ be a closed $(n - k)$-form in $L^1$, and noticing that $f_j$ is as the cutoff function in the proof of Theorem 2.9, one has

$$\langle f_j \alpha, \ast \omega \rangle_{L^2(X)} = \langle f_j \alpha, \ast \beta \rangle_{L^2(X)} = (\pm) \langle d(f_j \alpha), \ast \beta \rangle_{L^2(X)} = (\pm) \langle (d f_j \wedge \alpha, \ast \beta)_{L^2(X)} + (f_j \wedge \alpha, \ast \beta)_{L^2(X)} \rangle = (\pm) \langle d f_j \wedge \alpha, \ast \beta \rangle_{L^2(X)}. \quad (4.2)$$

Since $0 \leq f_j \leq 1$ and $\lim_{j \to \infty} f_j(x) \alpha(x) = \alpha(x)$, it follows from the dominated convergence theorem that

$$\lim_{j \to \infty} \langle f_j \alpha, \ast \omega \rangle_{L^2(X)} = \int_X \alpha \wedge \omega. \quad (4.3)$$

Since $\omega$ is bounded, $\text{supp}(d f_j) \subset B_{j+1} \setminus B_j$ and $|\beta(x)| = O(\rho(x_0, x))$, one obtains

$$|(d f_j \wedge \alpha, \ast \beta)_{L^2(X)}| \leq (j + 1) C \int_{B_{j+1} \setminus B_j} |\alpha(x)| dx, \quad (4.4)$$

where $C$ is a constant independent of $j$. Using the similar proof in Theorem 2.9, we can prove that there exists a subsequence $\{j_i\}_{i \geq 1}$ such that

$$\lim_{i \to \infty} (j_i + 1) C \int_{B_{j_i+1} \setminus B_{j_i}} |\alpha(x)| dx = 0. \quad (4.5)$$

It now follows from (4.2), (4.3) and (4.5) that $\int_X \alpha \wedge \omega = 0$. \hfill $\Box$

**Corollary 4.2** Let $(X^n, \omega)$ be a complete Riemannian manifold with a $d$-linear $(n - 4)$-form $\omega$, $E$ be a principal $G$-bundle on $X$ and $A$ be a smooth connection on $E$. Suppose that $\omega$ is bounded. If the curvature $F_A$ is in $L^2$, then

$$\int_X tr(F_A \wedge F_A) \wedge \omega = 0.$$
**Proof** From the Bianchi identity $d_A F_A = 0$, we have

$$dtr(F_A \wedge F_A) = \text{tr} (d_A (F_A \wedge F_A)) = 0.$$ 

Thus, $dtr(F_A \wedge F_A)$ is an $L^1$ closed form. Following Lemma 4.1, we can complete the proof of this Corollary. \hfill \Box

We then have a vanishing theorem on the $G_2$-(or $\text{Spin}(7)$-) instantons over a complete manifold with $d$(linear) structure form.

**Theorem 4.3** Let $X$ be a complete $G_2$-(or $\text{Spin}(7)$-) manifold with a $d$(linear) $G_2$-(or $\text{Spin}(7)$-) structure $\phi$ (or $\Omega$), $E$ be a $G$-bundle on $X$ and $A$ be a smooth connection on $E$. If the connection $A$ is a $G_2$-(or $\text{Spin}(7)$-) instanton with square-integrable curvature $F_A$, then $A$ is a flat connection.

**Proof** By the hypothesis of the connection $A$, the Yang–Mills energy functional on a complete $G_2$-manifold is

$$\text{YM}(A) = \int_X \text{tr}(F_A \wedge F_A) \wedge \phi.$$ 

Following Corollary 4.2, we obtain $\text{YM}(A) = 0$, i.e., $F_A \equiv 0$. \hfill \Box

Let $(X, g)$ be a real Killing spinor compact manifold of dimension $n$, i.e., there are 3-form $P$ and 4-form $Q$ which satisfy

$$dP = 4Q, \quad d\ast_X Q = (n - 3) \ast_X P,$$

where $\ast_X$ is the Hodge star operator on $X$. For $n > 3$, the Chern–Simons functional can then be written as

$$\text{CS}(A) = -\frac{1}{2(n - 3)} \int_X \text{tr}(F_A \wedge F_A) \wedge \ast_X Q,$$ 

which is gauge-invariant. We consider the cylinder $\text{Cyl}(X) := \mathbb{R} \times X$ over $X$. Then, we can define a 4-form $\Omega$ on $\text{Cyl}(X)$ as

$$\Omega = dt \wedge P + Q,$$

with $t$ the linear coordinate on $\mathbb{R}$. Let $A$ be a gauge field on $\text{Cyl}(X)$ with the property that $dt \wedge A$, which is simply a choice of gauge. The instanton equation on the cylinder splits into the two equations

$$\ast_X \frac{\partial A}{\partial t} = -\ast_X P \wedge F_A,$$

$$\ast_X F_A = \ast_X Q \wedge F_A - \frac{\partial A}{\partial t} \wedge \ast_X P.$$ 

(4.7)

The gradient flow of Chern–Simons functional (4.6) is equivalent to the first of equations (4.7). We denote $\ast$ by the Hodge star operator on $\text{Cyl}(X)$, $D$ by the exterior derivative on $T^*(\text{Cyl}(X))$. We also denote $\tilde{P} = dt \wedge P$, $\tilde{Q} = dt \wedge Q$. Then, the forms $\tilde{P}$, $\tilde{Q}$ satisfy

$$\ast \tilde{P} = \ast_X P, \quad \ast \tilde{Q} = \ast_X Q,$$

and

$$D \tilde{P} = 4 \tilde{Q}, \quad D \ast \tilde{Q} = (n - 3) \ast \tilde{P}.$$
The Yang–Mills energy function is
\[
YM(A) := \| F_A \|^2_{L^2(Cyl(X))} = - \int_{\mathbb{R} \times X} \text{tr}(F_A \wedge F_A) \wedge \ast \Omega
\]
\[
= - \int_{\mathbb{R} \times X} \text{tr}(F_A^2) \wedge \ast \bar{P} - \int_{\mathbb{R} \times X} \text{tr}(F_A^2) \wedge \ast X Q \wedge dt.
\]
We observe that
\[
- \int_{\mathbb{R} \times X} \text{tr}(F_A^2) \wedge \ast \bar{P} = - \frac{1}{n-3} \int_{\mathbb{R} \times X} \text{tr}(F_A^2) \wedge D \ast \bar{Q}.
\]
We also observe that
\[
F_A = \frac{\partial A}{\partial t} \wedge dt + F_A
\]
and
\[
-\text{tr}(F_A^2) \wedge \ast \bar{P} = -2\text{tr} \left( \frac{\partial A}{\partial t} \wedge dt \wedge F_A \right) \wedge \ast X P = 2 \left| \frac{\partial A}{\partial t} \right|^2 dt \wedge dvol;
\]
here, we use the first equation on (4.7). Thus
\[
- \int_{\mathbb{R} \times X} \text{tr}(F_A^2) \wedge \ast \bar{P} = 2 \int_{\mathbb{R} \times X} \left| \frac{\partial A}{\partial t} \right|^2 dt \wedge dvol. \tag{4.8}
\]
In [20], the author proved a vanishing theorem as follows:

**Theorem 4.4** ([20] Theorem 1.2) *If the connection \( A \) is a solution of \( \Omega \)-instanton equation with square-integrable curvature \( F_A \) over \( Cyl(X) \), where \( X \) is a compact real Killing spinor manifold, then \( A \) is flat.*

**Proof** If \( F_A \) is in \( L^2(Cyl(X)) \), following the Corollary 4.2, then it implies that
\[
\int_{\mathbb{R} \times X} \text{tr}(F_A^2) \wedge \ast \bar{P} = 0.
\]
Combining Equation (4.8) gives \( \frac{\partial A}{\partial t} = 0 \), i.e., the connection \( A \) is not dependence on parameter \( t \). Thus, the Yang–Mills functional \( YM(A) = \int_{\mathbb{R}} dt \int_X |F_A|^2 dvol \) is finite if only if \( F_A = 0 \). \( \square \)

In this article, we will show that if the standard Yang–Mills functional on \( Cyl(X) \) satisfies some mild conditions, then the solution of \( \Omega \)-instanton equation is trivial. One also can see Sect. 4 on [22]. We define the energy density \( \rho(A) \) by
\[
\rho(A) := \lim_{T \to \infty} \frac{1}{2T} \int_{(-T,T) \times X} |F_A|^2 dvol dt.
\]

**Lemma 4.5** ([22] Lemma 4.2) *Let \( X \) be a complete manifold of dimension \( n \) with a \( d \)-bounded \( k \)-form \( \omega \), i.e., there exist a \( (k-1) \)-form \( \theta \) such that \( \omega = d\theta \), \( \alpha \) be a closed form of degree \( n-k \). If \( \alpha \) satisfies
\[
\lim_{r \to \infty} \frac{1}{r} \int_{B_r(x_0)} |\alpha| dvol = 0, \tag{4.9}
\]
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where $x_0$ is a point on $X$, $B_r(x_0)$ is a geodesic ball, then there exists a sequence $\{j_i\}_{i \geq 1}$ such that
\[
\lim_{i \to \infty} \int_{B_{j_i}(x_0)} \alpha \wedge \omega = 0.
\]

**Proof** We denote $f_j$ by the cutoff function in the proof of Theorem 2.9. We consider the form $\beta := \alpha \wedge \omega = d(\alpha \wedge \theta)$. We have $f_j \beta = d(f_j \alpha \wedge \theta) - d f_j \wedge (\alpha \wedge \theta)$. By Stokes formula, we obtain
\[
\left| \int_X f_j \beta \right| \leq \left| \int_X d f_j \wedge (\alpha \wedge \theta) \right| \lesssim \int_{B_{j+1} \setminus B_j} |\alpha|,
\]
and
\[
\left| \int_{B_j} \beta \right| \leq \left| \int_X f_j \beta \right| + \int_{B_{j+1} \setminus B_j} |\beta| \lesssim \left| \int_X f_j \beta \right| + \int_{B_{j+1} \setminus B_j} |\alpha|.
\]
Thus,
\[
\left| \int_{B_j} \beta \right| \lesssim \int_{B_{j+1} \setminus B_j} |\alpha|. \tag{4.10}
\]
By the hypothesis (4.9), there exists a subsequence $\{j_i\}_{i \geq 1}$ such that
\[
\lim_{i \to \infty} \int_{B_{j_i+1} \setminus B_{j_i}} |\alpha| = 0. \tag{4.11}
\]
It now follows (4.10), (4.11) that $\lim_{i \to \infty} \int_{B_{j_i}(x_0)} \alpha \wedge \omega = 0$. \qed

We then have

**Theorem 4.6** ([22] Theorem 4.3) Let $\text{Cyl}(X)$ be the cylinder over a compact real Killing spinor manifold $X$, $A$ be a solution of $\Omega$-instanton equation. If $\rho(A) = 0$, then $A$ is a flat connection.

**Proof** Since $\rho(A) = 0$ and $|Tr(F_A^2)| \lesssim |F_A|^2$, we observe that
\[
\lim_{T \to \infty} \frac{1}{T} \int_{(-T, T) \times X} |Tr(F_A^2)| = 0. \tag{4.12}
\]
Since $Tr(F_A^2)$ is a closed 4-form on $\text{Cyl}(X)$, it also satisfies Eq. (4.12) and $* \tilde{P}$ is a $D$(bounded) $(n - 4)$-form, then following Lemma 4.5, there exist a sequence $\{j_i\}_{i \geq 1}$ such that
\[
\lim_{i \to \infty} \int_{(-j_i, j_i) \times X} tr(F_A^2) \wedge * \tilde{P} = 0. \tag{4.13}
\]
It now follows (4.13), (4.8) that
\[
\lim_{i \to \infty} \int_{(-j_i, j_i) \times X} |\frac{\partial A}{\partial t}|^2 dt \wedge dvol = 0,
\]
i.e., $\frac{\partial A}{\partial t} = 0$. The connection $A$ is not dependence on parameter $t$. Thus
\[
\rho(A) = \int_X |F_A|^2 dvol.
\]
By the hypothesis of energy density $\rho(A)$, we obtain that $F_A = 0$. We complete this proof. \hfill $\Box$

### 4.2 Hodge theory on bundle $E$

In this section, we consider the Hodge theory on principal bundle over the complete $G_2$-manifold equipped with a $d$-linear $G_2$-structure. At first, we recall some definitions on differential geometry. Let $E$ be a principal $G$-bundle over a complete Riemannian manifold $X$. Assume now that $d_A$ is a smooth connection on $E$. The formal adjoint operator of $d_A$ acting on $\Lambda^p(X, E) := \Lambda^p(X) \otimes E$ is $d^*_A := (\pm) \ast d_A \ast$.

**Definition 4.7**  The Laplace–Beltrami operator associated to $d_A$ is the second-order operator

$$\Delta_A := d_A d_A^* + d_A^* d_A.$$  

The space of $L^2$-harmonic forms of degree of $p$ with respect to the Laplace–Beltrami operator $\Delta_A$ is defined by

$$H^p_{(2)}(X, E) = \{ \alpha \in \Lambda^p_{(2)}(X, E) : \Delta_A \alpha = 0 \}.$$

**Proposition 4.8**  Let $(X, \omega)$ be a complete Riemannian manifold equipped with a nonzero parallel $k$-form $\omega$, $E$ be a principal $G$-bundle over $X$ and $A$ be a smooth connection on $E$. If $\omega$ is $d$-linear, then

$$H^0_{(2)}(X, E) = \{ 0 \}.$$

Furthermore, if the Ricci curvature is flat, $H^1_{(2)}(X, E) = \{ 0 \}$.

**Proof**  For any $\alpha \in H^0_{(2)}(X, E)$, the Weitzenböck formula gives:

$$0 = (d_A^* d_A \alpha, \alpha)_{L^2(X)} = (\nabla_A^* \nabla_A \alpha, \alpha)_{L^2(X)} = \| \nabla_A \alpha \|^2_{L^2(X)}.$$

Using the Kato inequality, $|\nabla \alpha| \leq |\nabla_A \alpha|$, we have $|\nabla \alpha| = 0$, i.e., $\alpha$ is a harmonic function over $X$. Then following Corollary 2.11, $|\alpha| \equiv 0$, i.e., $\alpha \equiv 0$.

Next, we will show that if Ricci curvature is flat, $H^1_{(2)}(X, E) = \{ 0 \}$. For any $\alpha \in H^1_{(2)}(X, E)$, the Weitzenböck formula gives:

$$0 = (\Delta_A \alpha, \alpha)_{L^2(X)} = (\nabla_A^* \nabla_A \alpha, \alpha)_{L^2(X)} = \| \nabla_A \alpha \|^2_{L^2(X)}.$$

here, we use the fact the connection $A$ is flat. By Kato inequality, $\nabla |\alpha| = 0$, i.e., $|\alpha|$ is also a harmonic function over $X$. Thus $\alpha \equiv 0$. \hfill $\Box$

The operator $\Delta_A$ always does not commute with $L_\omega$, where $\omega$ is parallel form on a complete manifold $X$. We cannot extend the idea of Theorem 2.9 to the principal bundle $E$. But on a complete $G_2$-manifold $X$, there exists a structure operator $C$ on $X$ (see Definition 2.1). Then $C$ induces isomorphisms $\Lambda^1(X, E) \to \Lambda^2(X, E)$. To be more specific, we can compose $\alpha = \alpha^7 + \alpha^{14}$ for any $\alpha \in \Lambda^2(X, E), \alpha^i \in \Lambda^2_i \otimes E$. There exists a one form $\beta$ such that

$$C(\beta) := \ast (\ast \phi \wedge \beta) = \alpha^7, \text{ i.e., } \beta = \frac{1}{3}(\ast (\alpha^7 \wedge \ast \phi)). \quad (4.14)$$

**Lemma 4.9**  Let $A$ be a connection on a complete $G_2$-manifold, $\alpha$ be a harmonic 2-form with respect to $\Delta_A$. If $X$ is non-compact, suppose also that $\alpha \in L^2$, then we have following identities:

$$d_A^* \beta = 0, \; \Pi^2_7(d_A \beta) = 0. \quad (4.15)$$
where $\beta$ is defined as (4.14) and $\Pi_7^2$ denote a projection map $\Lambda^2 \rightarrow \Lambda^2_7$. Further more, if $A$ is a flat connection on $X$, then $\beta$ is also closed with respect to $d_A$.

**Proof** Our proof uses the author’s argument in [21] for Yang–Mills connections. We compose $\alpha = \alpha^7 + \alpha^{14}$; thus, $\alpha^7 \wedge *\phi = \alpha \wedge *\phi$. From the identity $d_A \alpha = 0$ and the fact $d \ast \phi = 0$, we have

$$0 = d_A (\alpha^7 \wedge *\phi) = d_A (\alpha \wedge *\phi) = 3d_A \ast \beta.$$  

Further more, using the fact $d_A^* \alpha = d_A \alpha = 0$ and $\alpha^7 = \frac{1}{3} (\alpha + \ast (\alpha \wedge \phi))$, we have

$$d_A^* \alpha^7 = \frac{1}{3} \ast d_A (\alpha \wedge \phi) = 0. \quad (4.16)$$

Applying operator $d_A^*$ to $C(\beta) = \alpha^7$, following Eq. (4.16), we get

$$\ast (d_A \beta \wedge *\phi) = 0,$$

i.e., $\Pi_7^2(d_A \beta) = 0$. \quad (4.17)

If $\alpha$ is in $L^2(X)$, by the definition of $\beta$, we obtain that $|\beta| \lesssim |\alpha^7|$, i.e., $\beta$ is also in $L^2$. Furthermore, if $A$ is a flat connection, we have

$$0 = d_A^* \Pi_7^2 (d_A \beta) = d_A^* d_A \beta + \ast d_A (d_A \beta \wedge \phi) = d_A^* d_A \beta.$$

Then, $d_A \beta = 0$. We complete this proof. \hfill \Box

**Theorem 4.10** Let $(X, \phi)$ be a complete $G_2$-manifold with a $d$(linear) $G_2$-structure $\phi$, $E$ be a principal $G$-bundle over $X$ and $A$ be a smooth connection on $E$. If $A$ is a flat connection, then $H^p_{(2)}(X, E) = 0$ unless $p \neq 3, 4$.

**Proof** Following Proposition 4.8, we obtain that $H^k_{(2)}(X, E) = 0$, $k = 0, 1$. We only need to show $H^2_{(2)}(X, E) = \{0\}$. We denote $\alpha \in H^2_{(2)}(X, E)$; $\beta$ is defined as (4.14). If $A$ is a flat connection, following Lemma 4.9, $\beta$ is also harmonic with respect to $\Delta_A$. By Proposition 4.8, $\beta = 0$, i.e., $\alpha^7 = 0$. It implies that the $L^2$-harmonic 2-form $\alpha$ also on $\Lambda^2_{(2)}(X) \otimes E$, i.e., $\alpha + \ast (\alpha \wedge \phi) = 0$. Thus, we have an identity, $\|\alpha\|^2_{L^2(X)} = -\int_X tr(\alpha \wedge \alpha) \wedge \phi$. It is easy to see $tr(\alpha \wedge \alpha)$ is an closed $L^1$ form; following Lemma 4.1, we have $\|\alpha\|_{L^2(X)} = 0$, i.e., $\alpha = 0$. \hfill \Box

Let us recall that from Bishop–Gromov’s volume comparison theorem, we can define the asymptotic volume ratio

$$V_X := \lim_{r \rightarrow \infty} \frac{V(r)}{r^n}$$

where $V(r)$ is the volume of geodesic ball $B(r)$ centered at $p$ with radius $r$. And the above definition is independent of $p$, so we omit $p$ here. If $V_X > 0$, we say that $(X, g)$ has maximal volume growth. We suppose that the complete manifold $X$ is Ricci flat, then $X$ has maximal volume growth is equivalence to any $u \in C^\infty_c(X)$ satisfies the Sobolev inequality [33]:

$$\|u\|_{L^\infty \cap L^2(X)} \lesssim \|\nabla u\|_{L^2(X)}.$$

We then prove an useful
Lemma 4.11 Let \((X^n, \omega)\) be a complete Ricci-flat Riemannian manifold with maximal volume growth, \(E\) be a principal \(G\)-bundle over \(X\) and \(A\) be a smooth connection on \(E\). Then, there is a positive constant \(\delta\) with following significance. If the curvature \(F_A\) obeys

\[
\|F_A\|_{L^2(X)} \leq \delta, \tag{4.18}
\]

then any \(\alpha \in \Lambda^1(X, E)\) satisfies the inequality

\[
\|\alpha\|^2_{L^{2n/3}(X)} \leq c \langle \Delta A \alpha, \alpha \rangle_{L^2(X)}.
\]

In particular, \(H^1_{(2)}(X, E) = \{0\}\).

Proof We observe that

\[
|\langle F_A, [\alpha \wedge \alpha] \rangle_{L^2(X)}| \lesssim \|F_A\|_{L^2(X)} \|\alpha\|^2_{L^{2n/3}(X)};
\]

thus, we have

\[
\langle \Delta A \alpha, \alpha \rangle_{L^2(X)} \geq \|\nabla A \alpha\|^2_{L^2(X)} - C_1 \|F_A\|_{L^2(X)} \|\alpha\|^2_{L^{2n/3}(X)}
\]

\[
\geq \|\nabla |\alpha|^2_{L^2(X)} - C_1 \|F_A\|_{L^2(X)} \|\alpha\|^2_{L^{2n/3}(X)}
\]

\[
\geq (C_2 - C_1 \|F_A\|_{L^2(X)}) \|\alpha\|^2_{L^{2n/3}(X)}
\]

where \(C_1, C_2\) are positive constant only dependent on \(X\). We can choose \(\delta\) sufficiently small to ensure that \(\|F_A\|_{L^2(X)} \leq C_2^2 / 2C_1\), thus we complete the proof of this lemma.

\[\square\]

Theorem 4.12 Let \((X, \phi)\) be a complete \(G_2\)-manifold with maximal volume growth, \(E\) be a principal \(G\)-bundle over \(X\) and \(A\) be a smooth connection on \(E\). If the \(G_2\)-structure \(\phi\) is \(d(\text{linear})\), then there is a positive constant \(\delta\) with following significance. If the curvature \(F_A\) obeys

\[
\|F_A\|_{L^2(X)} \leq \delta,
\]

then

\[
H^2_{(2)}(X, E) = \{0\}.
\]

Proof We denote \(\alpha \in H^2_{(2)}(X, E)\), and \(\beta\) is defined as (4.14). Then following Lemma 4.17, \(\beta\) satisfies

\[
0 = d_A^* d_A \beta + *[F_A \wedge \beta \wedge \phi].
\]

Taking the inner product of this equation with \(\beta\) yields

\[
0 = \langle \Delta A \beta, \beta \rangle_{L^2(X)} + \int_X \text{tr}(F_A \wedge [\beta \wedge \beta]) \wedge \phi.
\]

For a smooth connection \(A\) with \(\|F_A\|_{L^2(X)} \leq \delta\), where \(\delta\) is a constant in the hypotheses of Lemma 4.11, we have

\[
\|\beta\|^2_{L^{14/3}(X)} \lesssim \langle \Delta A \beta, \beta \rangle_{L^2(X)}.
\]

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We also observe that

\[ \int_X \text{tr}(F_A \wedge [\beta \wedge \beta]) \wedge \phi \lesssim \| F_A \|_{L^2(X)} \| \beta \|_{L^{14/5}(X)}^2. \]

Combining the preceding inequalities gives

\[ 0 \geq \langle \Delta_A \beta, \beta \rangle_{L^2(X)} - C_3 \| F_A \|_{L^2(X)}^2 \| \beta \|_{L^{14/5}(X)}^2 \geq (C_4 - C_3 \| F_A \|_{L^2(X)}^2) \| \beta \|_{L^{14/5}(X)}^2. \]

where \( C_3, C_4 \) are positive constants dependent on \( X \). We can choose \( \delta \) sufficiently small to ensure that \( \| F_A \|_{L^2(X)} \leq C_4 \delta \); hence, \( \beta \equiv 0 \). It implies that \( \alpha \in \Lambda_{21}^2(X) \otimes E \). Hence, following Lemma 4.1, \( \| \alpha \|_{L^2(X)}^2 = -\int_X \text{tr}(\alpha \wedge \alpha) \wedge \phi = 0 \), i.e., \( \alpha = 0 \).

A connection is called a Yang–Mills connection if it is a critical point of the Yang–Mills functional \( \text{YM}(A) \), i.e., \( d^* A F_A = 0 \). In addition, all connections satisfy the Bianchi identity \( dA F_A = 0 \). It implies that the Yang–Mills connection is a harmonic 2-form with respect to \( \Delta_A \). There are very few gap results of Yang–Mills connection over non-compact, complete manifold, for example [8,11,14,32]. Their results all depend on some positive conditions of Riemannian curvature tensors. Following Theorem 4.12, we have a gap result for Yang–Mills connection on a complete \( G_2 \)-manifold.

**Corollary 4.13** Let \((X, \phi)\) be a complete \( G_2 \)-manifold with a d(linear) \( G_2 \)-structure \( \phi \), \( E \) be a principal \( G \)-bundle over \( X \) and \( A \) be a smooth Yang–Mills connection on \( E \). If \( X \) has maximal volume growth, then there exists a positive constant \( \delta \in (0, 1] \) with following significance. If the curvature \( F_A \in L^2(X) \) obeys

\[ \| F_A \|_{L^2(X)} \leq \delta, \]

then \( A \) is a flat connection.

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