MAXIMUM PRINCIPLES FOR A FULLY NONLINEAR NONLOCAL EQUATION ON UNBOUNDED DOMAINS

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Abstract. In this paper, we study equations involving fully nonlinear nonlocal operators
\[
F_\alpha(u(x)) = C_{n,\alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{G(u(x) - u(z))}{|x - z|^{n+\alpha}} \, dz = f(u(x)), \quad x \in \mathbb{R}^n.
\]
We shall establish a maximum principle for anti-symmetric functions on any half space, and obtain key ingredients for proving the symmetry and monotonicity for positive solutions to the fully nonlinear nonlocal equations. Especially, a Liouville theorem is derived, which will be useful in carrying out the method of moving planes on unbounded domains for a variety of problems with fully nonlinear nonlocal operators.

1. Introduction and main results. In this paper, we consider the nonlinear equations involving fully nonlinear non-local operators
\[
F_\alpha(u(x)) = f(u(x)), \quad x \in \mathbb{R}^n,
\]
where the operator \(F_\alpha\) with \(0 < \alpha < 2\), is given by
\[
F_\alpha(u(x)) = C_{n,\alpha} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{G(u(x) - u(y))}{|x - y|^{n+\alpha}} \, dy
\]
\[
= C_{n,\alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{G(u(x) - u(y))}{|x - y|^{n+\alpha}} \, dy.
\]
In this formulation, P.V. stands for the Cauchy principal value of the integral, \(G\) is a nonlinear operator and is at least local Lipschitz continuous function defined on \(\mathbb{R}\), and satisfying the following conditions.
\(G_1\): \(G(-t) = -G(t), G \in C^1(\mathbb{R})\) and \(G(t)\) is strictly increasing for \(t \in \mathbb{R}\).
\(G_2\): \(G(t) - G(s) \geq c_1 |t - s|^{\tau}\) for some \(c_1 > 0, \tau > 0\) and all \(t, s \in R\) with \(t \geq s\).
\(G_3\): \(|G'(t)| \leq c_2 |t|^{\beta}\) for some constant \(c_2 > 0, \beta \geq 0\).

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In order to make sense for the integral, we define

\[ \mathcal{L}_\alpha = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{G(1 + |u(x)|)}{1 + |x|^{n+\alpha}} \, dx < \infty \right\}, \]

then it is easy to see that for \( u \in C^{1,1}_{\text{loc}} \cap \mathcal{L}_\alpha \), \( \mathcal{F}_\alpha(u) \) is well defined. This kind of operator was introduced by Caffarelli and Silvestre [6].

In the special case when \( G(\cdot) \) is an identity map, \( \mathcal{F}_\alpha \) reduces to the fractional Laplacian \( (-\Delta)^{\alpha/2} \), \( 0 < \alpha < 2 \). When \( G(t) = |t|^{p-2}t \), \( \alpha = sp \), \( 0 < s < 1 \) and \( 1 < p < \infty \), \( \mathcal{F}_\alpha \) becomes fractional \( p \)-Laplacian \( (-\Delta)^{s}_p \). The nonlocal nature of these operators make them difficult to study. To circumvent this, Caffarelli and Silvestre [5] introduced the extension method which turns the nonlocal problem involving the fractional Laplacian into a local one in higher dimensions. This method has been applied successfully to study the fractional Laplacian equations and a series of fruitful results have been obtained, we refer to [14] and the references therein. Another way is using the integral equations method, such as the moving planes in integral forms and regularity lifting to investigate equations involving fractional Laplacian by first showing that they are equivalent to the corresponding integral equations (see [11, 12, 24]). For more articles concerning the method of moving planes for monlocal equations and for integral equations, see [16, 21] and references therein.

In 2017, Chen, Li and Li [10] developed a systematic approach to carry out the method of moving planes for problems with fractional Laplace operator. Subsequently, by using this direct method, many authors investigated different equations involving fractional Laplacian, see for instance, Chen and Li [8]; Chen and Wu [13]; Cheng [15] and the references therein.

However, these two effective tools fail to work for the fractional fully nonlinear equations due to the full nonlinearity of the operator, while this kind of operators have been recently used in many applications, including continuum mechanics, population dynamics, and many different non-local diffusion problems, see [1, 2] for more applied backgrounds. It is also applied in studying the non-local "Tug-of-War" games [3]. In [9], Chen, Li and Li developed a new method that one can treat with the fully nonlinear nonlocal equations directly. Recently, Wang and Niu [23] studied a fully nonlinear nonlocal system with special nonhomogeneous terms which have \( u(x) \) and \( v(x) \) simultaneously while \( u(x) \) and \( v(x) \) have positive coefficients.

Back to equation (1.1), when \( G(t) = t, \alpha = 2 \), it is reduced to the well known classical elliptic equation

\[ -\Delta u = f(u) \text{ in } \mathbb{R}^n. \]  

(1.3)

The well known classical Liouville’s theorem for harmonic functions states that: If \( u \) is bounded from below or from above and

\[ \Delta u = 0, \forall x \in \mathbb{R}^n, \]

then it must be constant.

One of its important applications is in the proof of the Fundamental Theorem of Algebra. It is also a key ingredient in deriving point-wise a priori estimates for solutions to a family of elliptic equations in bounded domains with prescribed boundary values, see [18]. Liouville’s theorem can also be used to study geometrical and reaction diffusion problems [20], and also to derive singularity and decay estimates, see [22].

A similar result has been obtained for \( s \)-harmonic functions in [4, 7]:
Assume that $0 < s < 1$. If $u$ is bounded from below or from above and

$$(-\Delta)^s u = C_{n,\alpha} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy = 0, \ x \in \mathbb{R}^n,$$

then it must be constant.

In order for the integral convergence, the function $u \in C^{1,1}_{loc} \cap L^{2s}$ with

$$L^{2s} = \left\{ u \in L^1_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |u(x)|^{n+2s}} dx < \infty \right\},$$

is required.

Besides the above mentioned applications, this Liouville theorem has also been used to prove the equivalences between fractional nonlinear equations and the integral equations, thus one can employ integral equations methods, such as method of moving planes in integral forms to study qualitative properties of the solutions for the original fractional equations.

To study fractional harmonic functions, one powerful tool is the Poisson representation [19]:

$$u(x) := \int_{|z| > r} P_r(z, x) u(z) dz, \ \text{for} \ |x| < r,$$

where $P_r(z, x)$ is the Poisson kernel:

$$P_r(z, x) = \begin{cases} \frac{\Gamma(n/2)}{\pi^{n/2+1}} \frac{\pi \alpha}{2} \left[ \frac{r^2 - |x|^2}{|z|^2 - t^2} \right]^{\frac{n}{2}} \frac{1}{|x - z|^n}, & |z| > r, \\ 0, & |z| < r. \end{cases}$$

If $u$ is bounded from one side, differentiating under the integral sign and letting $r \to \infty$, we may show that

$$\frac{\partial u(x)}{\partial x_i} = 0, \ i = 1, 2, \cdots, n, \ x \in \mathbb{R}^n,$$

and this implies that $u$ is constant.

The other effective way in studying fractional harmonic functions in $\mathbb{R}^n$ is the Fourier transform $\mathcal{F}$ by

$$0 = (-\Delta)^s u(x) = \mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi),$$

from which one has that

$$\mathcal{F}(u)(\xi) = 0 \ \text{for} \ \xi \neq 0.$$

So, $\mathcal{F}(u)$ consists of a finite combination of the Dirac’s delta measure and its derivatives. Hence $u$ is a polynomial. Under further restrictions that $u \in L^{2s}$ and bounded from one side, it must be constant.

As usual, one of the fundamental problems in studying the fractional fully nonlinear equation is the uniqueness of fractional harmonic functions. Since the conventional methods do not work for Eq. (1.1), so far as we know, there has not been any results in this respect.

Motivated by the direct methods introduced in [8, 10, 13], in this paper we consider the fully nonlinear nonlocal equation (1.1) and prove a maximum principle for anti-symmetric functions on any half space, which contains key ingredients in proving the symmetry and monotonicity for positive solutions. Especially, we prove that if $u$ is a bounded fractional harmonic function, then it is symmetric about any hyper-plane in $\mathbb{R}^n$, and therefore, it must be constant. We summarize it as the following Liouville type theorem:
Theorem 1.1 (Liouville theorem). Assume that \( u \in C_{loc}^{1,1} \cap L_{\alpha}(\mathbb{R}^n) \) is bounded, if
\[
\mathcal{F}_{\alpha}(u(x)) = C_{n,\alpha} \, P.V. \int_{\mathbb{R}^n} \frac{G(u(x) - u(z))}{|x - z|^{n+\alpha}} \, dz = 0, \quad \forall x \in \mathbb{R}^n,
\]
then
\[
u(x) \equiv C, \quad \forall x \in \mathbb{R}^n.
\]

Remark 1.2. For unbounded functions, there are obvious counter examples. For instance, if \( u(x) = x_i, i = 1, 2, \ldots, n, \) then one can verify that
\[
\mathcal{F}_{\alpha}(u(x)) = 0, \quad x \in \mathbb{R}^n.
\]

To study the symmetry of \( u \) with respect to a given hyper-plane \( T \), we denote \( w(x) = u(\tilde{x}) - u(x) \), with \( \tilde{x} \) being the reflection of \( x \) with respect to plane \( T \). We intend to prove that
\[
w(x) \leq 0 \quad \text{for all } x \text{ in the half space on one side of the plane.}
\]
This is actually a maximum principle for anti-asymmetric functions on a half space, in an unbounded region, without assuming that the function vanishes near infinity.

Theorem 1.3 (Maximum principle). Let \( T \) be any given hyper-plane in \( \mathbb{R}^n \) and \( \Sigma \) be the half space on one side of the plane. Let
\[
w(x) = u(\tilde{x}) - u(x)
\]
with \( \tilde{x} \) being the reflection of \( x \) with respect to \( T \). Assume that \( w(x) \) is bounded in \( \Sigma \) and
\[
\mathcal{F}_{\alpha}(u(\tilde{x})) - \mathcal{F}_{\alpha}(u(x)) \leq 0
\]
at the points \( x \in \Sigma \) where \( u(\tilde{x}) > u(x) \). Then
\[
w(x) \leq 0, \quad \forall x \in \Sigma.
\]

This maximum principle will be powerful in carrying out the method of moving plane on unbounded domains. To illustrate this, we use the well known De Giorgi Conjecture as a simple example, which is stated as

**De Giorgi Conjecture** [17]. If \( u \) is a solution of equation
\[
- \Delta u(x) = u(x) - u^3(x), \quad x \in \mathbb{R}^n
\]
such that
\[
|u(x)| \leq 1, \quad \lim_{x_n \to \pm \infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1}, \quad \text{and} \quad \frac{\partial u}{\partial x_n} > 0.
\]
Then there exists a vector \( \mathbf{a} \in \mathbb{R}^{n-1} \) and a function \( u_1 : \mathbb{R} \to \mathbb{R} \) such that
\[
u(x', x_n) = u_1(\mathbf{a} \cdot x' + x_n), \quad \forall x \in \mathbb{R}^n.
\]

Now we consider the fractional fully nonlinear version of (1.7):
\[
\mathcal{F}_{\alpha}(u(x)) = f(u(x)), \quad x \in \mathbb{R}^n.
\]

Then applying Theorem 1.3 we will be able to derive

**Theorem 1.4.** Suppose that \( u \in C_{loc}^{1,1} \cap L_{\alpha}(\mathbb{R}^n) \) is a solution of the equation (1.8) and verifies that
\[
|u(x)| \leq 1, \quad \forall x \in \mathbb{R}^n,
\]
\[
u(x', x_n) \to \pm 1 \quad \text{uniformly in } x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, \quad \text{as } x_n \to \pm \infty,
\]
and
\[
f(t) \text{ is nonincreasing for } |t| \text{ sufficiently close to } 1,
\]
then there exists $M > 0$ such that
\[
\frac{\partial u}{\partial x_n} \geq 0 \quad \text{for all } x \text{ with } |x_n| \geq M. 
\] (1.11)

**Remark 1.5.** (i) Notice that the condition on $f$ in Theorem 1.4 is satisfied for $f(u) = u - u^3$, as given in the De Giorgi Conjecture.

(ii) The inequality (1.11) obtained in Theorem 1.4 helps build De Giorgi Conjecture for the fully nonlinear nonlocal problems.

To see how Theorem 1.3 implies Theorem 1.4, let
\[
T_\lambda = \{ x \in \mathbb{R}^n : x_n = \lambda \} \quad \text{and} \quad \Sigma_\lambda = \{ x \in \mathbb{R}^n : x_n > \lambda \}.
\]
For $x \in \Sigma_\lambda$, let $x^\lambda$ be the reflection of the point $x$ with respect to plane $T_\lambda$. Set
\[
w_\lambda(x) = u(x^\lambda) - u(x).
\]
Then, for sufficiently large $\lambda$, (1.9) and (1.10) imply that
\[
F_\alpha(u(x^\lambda)) - F_\alpha(u(x)) \leq 0
\]
at the points $x \in \Sigma_\lambda$ where
\[
u(x^\lambda) > u(x).
\]
Now it follows from Theorem 1.3 that
\[
w_\lambda(x) \leq 0, \quad \forall x \in \Sigma_\lambda \text{ for all sufficiently large } \lambda. \quad (1.12)
\]
Consequently, a standard arguments will lead to (1.11) for all $|x_n| \geq M$.

We remark that inequality (1.12) actually provides a starting point to move the plane $T_\lambda$ in studying the symmetry and monotonicity of solutions for the fully nonlinear fractional equation (1.1). If we can move the plane all the way down, then we prove that
\[
\frac{\partial u(x)}{\partial x_n} > 0, \quad \forall x \in \mathbb{R}^n. \quad (1.13)
\]
For this purpose, we need to establish a narrow region principle on unbounded domains, without assuming that the function vanishes near infinity, which will be presented in our next paper. Note that precisely in carrying out the method of moving planes on unbounded domains, people usually require that the function $u$ vanishes near infinity, or consider $u(x)/g(x)$ for a proper choice of $g(x)$, so that $u(x)/g(x)$ vanishes near infinity, see for example, [9, 23]. Unfortunately, the latter two approaches fail to work on the fully nonlinear equation (1.1). Hence there is a pressing need to develop a method of moving planes for such operators that applies to unbounded domains while only assuming the function be bounded. As illustrated above, our Theorem 1.3 provides a starting point to move the planes in such a situation.

The paper is organized as follows. In Section 2 we shall establish the maximum principle and hence prove Theorem 1.3. In Section 3, we will use the maximum principle to derive Theorem 1.1 (Liouville Theorem) and Theorem 1.4.

2. The maximum principle and its proof. In this section we shall prove the maximum principle for fully nonlinear fractional equations on unbounded domains. We first present a preliminary lemma.
Lemma 2.1. Assume that \( u \in C^{1,1}_{loc} \cap L_\alpha(\mathbb{R}^n) \) and \( \psi \in C_0^\infty(\mathbb{R}^n) \), then for all small \( \delta > 0 \), there holds

\[
|F_\alpha(u + \varepsilon\psi) - F_\alpha(u)| \leq \varepsilon C_\delta + C\delta^{\beta+2-\alpha},
\]

where \( C \) is independent of \( \varepsilon \) while \( C_\delta \) may dependent on \( \delta \).

Proof. For simplicity of notation, denote \( v_\epsilon(x) = u(x) + \varepsilon\psi(x), C_{n,\alpha} = 1 \), then

\[
F_\alpha(u + \varepsilon\psi) - F_\alpha(u) = P.V. \int_{\mathbb{R}^n} \frac{G(v_\epsilon(x) - v_\epsilon(y))}{|x - y|^{n+\alpha}} dy - P.V. \int_{\mathbb{R}^n} \frac{G(u(x) - u(y))}{|x - y|^{n+\alpha}} dy,
\]

we shall divide \( \mathbb{R}^n \) into two regions: \( B_\delta(x) \) and \( B_\delta^c(x) \).

(i) In \( B_\delta^c(x) \). Note that

\[
G(v_\epsilon(x) - v_\epsilon(y)) - G(u(x) - u(y)) = \varepsilon(\psi(x) - \psi(y)) \int_0^1 G'(u(x) - u(y) + t(\psi(x) - \psi(y))) dt := \varepsilon H(x, y),
\]

and \( u \in C^{1,1}_{loc} \cap L_\alpha(\mathbb{R}^n) \) and \( \psi \in C_0^\infty(\mathbb{R}^n) \), we have

\[
\left| P.V. \int_{B_\delta^c(x)} \frac{\varepsilon H(x, y)}{|x - y|^{n+\alpha}} dy \right| \leq \varepsilon C_\delta. \tag{2.1}
\]

(ii) In the ball \( B_\delta(x) \). In this case, by Taylor expansion, we have

\[
v_\epsilon(x) - v_\epsilon(y) = \nabla v_\epsilon(x) \cdot (x - y) + O(|x - y|^2).
\]

The anti-symmetry of \( \nabla v_\epsilon(x) \cdot (x - y) \) for \( y \in B_\delta(x) \), and \( (G_1) \) imply that

\[
P.V. \int_{B_\delta(x)} \frac{G(\nabla v_\epsilon(x) \cdot (x - y))}{|x - y|^{n+\alpha}} dy = 0.
\]

By \( (G_3) \), we obtain

\[
\left| P.V. \int_{B_\delta(x)} \frac{G(v_\epsilon(x) - v_\epsilon(y))}{|x - y|^{n+\alpha}} dy \right| = \left| P.V. \int_{B_\delta(x)} \frac{G(v_\epsilon(x) - v_\epsilon(y))}{|x - y|^{n+\alpha}} dy - P.V. \int_{B_\delta(x)} \frac{G(\nabla v_\epsilon(x) \cdot (x - y))}{|x - y|^{n+\alpha}} dy \right| \\
\leq C_2 \int_{B_\delta(x)} \frac{(||\nabla v_\epsilon(x) \cdot (x - y)|| + O(|x - y|^2))\beta O(|x - y|^2)}{|x - y|^{n+\alpha}} dy \\
\leq C_1 \int_{B_\delta(x)} \frac{O(|x - y|^{\beta+2})}{|x - y|^{n+\alpha}} dy \leq C_2 \delta^{\beta+2-\alpha}, \tag{2.2}
\]

where \( C_2 \) is independent of \( \varepsilon \), since for any fixed \( x \), we have

\[
|\nabla v_\epsilon(x)| \leq |\nabla u(x)| + \varepsilon|\nabla \psi(x)| \leq |\nabla u(x)| + |\nabla \psi(x)|, \text{ if } \varepsilon \leq 1.
\]

Similarly, we have

\[
\left| P.V. \int_{B_\delta(x)} \frac{G(u(x) - u(y))}{|x - y|^{n+\alpha}} dy \right| \leq C_3 \delta^{\beta+2-\alpha}. \tag{2.3}
\]

Combining (2.1), (2.2) and (2.3), we have

\[
|F_\alpha(u + \varepsilon\psi) - F_\alpha(u)| \leq \varepsilon C_\delta + C\delta^{\beta+2-\alpha},
\]

which completes the proof. \( \square \)
Proof of Theorem 1.3. We use the contradiction arguments. Suppose that (1.6) is not valid. Then we have
\[ \sup_{x \in \Sigma} w(x) := A > 0. \] (2.4)
If the supremum can be attained, say at some point \( x \), then we have
\[ F_\alpha(u(x)) = F_\alpha(u(\bar{x})) \]
\[ = C_{n,\alpha} P.V. \int_{x^n} \frac{G(u(\bar{x}) - u(y)) - G(u(x) - u(y))}{|x - y|^{n+\alpha}} dy \]
\[ = C_{n,\alpha} P.V. \int_{x^n} \frac{G(u(x) - u(y)) - G(u(x) - u(y))}{|x - y|^{n+\alpha}} dy \]
\[ + C_{n,\alpha} P.V. \int_{x^n} \frac{G(u(x) - u(y)) - G(u(x) - u(y))}{|x - y|^{n+\alpha}} dy \]
\[ = C_{n,\alpha}(L_1 + L_2). \] (2.5)
To estimate \( L_1 \), we first note that
\[ \frac{1}{|x - y|} > \frac{1}{|x - \bar{y}|}, \forall x, y \in \Sigma. \]
While for the second part in the integral, we have
\[ G(u(\bar{x}) - u(y)) - G(u(x) - u(y)) \geq 0 \text{ but } \neq 0; \]
due to the strictly monotonicity of \( G \) and the fact that
\[ [u(\bar{x}) - u(y)] - ([u(x) - u(y)] = w(x) - w(y) \geq 0 \text{ but } \neq 0. \]
It follows that
\[ L_1 > 0. \] (2.6)
To estimate \( L_2 \), we regroup the terms and apply the mean value theorem to derive that
\[ L_2 = \int_{x^n} \left[ \frac{G(u(x) - u(y)) - G(u(x) - u(y))}{|x - y|^{n+\alpha}} \right] dy \]
\[ = w(x) \int_{x^n} \frac{G'(\xi(y)) + G'(\eta(y))}{|x - y|^{n+\alpha}} dy \geq 0 \] (2.7)
where we have used the condition \((G_1)\), and
\[ t_1(y) < \xi(y) < t_2(y) \text{ and } t_3(y) < \eta(y) < t_4(y), \]
with
\[ t_1(y) = u(\bar{x}) - u(y), \quad t_2(y) = u(x) - u(y), \]
\[ t_3(y) = u(\bar{x}) - u(y), \quad t_4(y) = u(x) - u(y). \]
Combining (2.5)-(2.7), we deduce
\[ F_\alpha(u(\bar{x})) - F_\alpha(u(x)) > 0. \]
This contradicts our assumption.
Now the main problem is that $\Sigma$ is an unbounded domain, and hence $\sup_{\Sigma} w(x)$ may not be attained. To circumvent this difficulty, we choose a point $x_0$ at which $w$ is very close to $A$. That is to say, for any $\sigma \in (0,1)$, there exists $x_0 \in \Sigma$, such that $w(x_0) \geq \sigma A$.

By translation, rotation and re-scaling arguments, we may assume that
\[ T = \{x : x_1 = 0\}, \quad \Sigma = \{x : x_1 < 0\}, \text{ and } x_0 = (-2, 0, \cdots, 0). \]

Then the reflection of the point $x$ about the plane $T$ is
\[ \bar{x} = (-x_1, x_2, \cdots, x_n). \]

We denote by
\[ u_0(x) = u(\bar{x}), \quad \text{and} \quad w(x) = u_0(x) - u(x). \]

Define the function
\[ \gamma(x) = \begin{cases} a e^{\frac{|x|^2}{|x_1|^2 - 1}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \]
with $a = e$ such that $\gamma(0) = \max_{\mathbb{R}^n} \gamma(x) = 1$. It is well-known that $\gamma(x) \in C_0^\infty(\mathbb{R}^n)$, and hence
\[ \mathcal{F}_\alpha(\gamma(x)) \leq C, \quad \forall \, x \in \mathbb{R}^n. \]

Also it is monotone decreasing with respect to $|x|$.

We denote
\[ \phi(x) = \gamma(x + x_0) \quad \text{and} \quad \phi_0(x) = \phi(\bar{x}) = \gamma(x - x_0). \]

Then $\phi_0(x) - \phi(x)$ is an anti-symmetric function with respect to the plane $T$.

Now take $\varepsilon > 0$ such that
\[ w(x_0) + \varepsilon(\phi_0(x_0) - \phi(x_0)) \geq A, \]

since $\phi(x_0) = 0$ and $\phi_0(x_0) = 1$, and we only need to set
\[ \varepsilon = (1 - \sigma)A. \quad (2.8) \]

Notice that, for any $y \in \Sigma \setminus B_1(x_0)$, $w(y) \leq A$ and $\phi_0(y) = \phi(y) = 0$, we conclude
\[ w(x_0) + \varepsilon(\phi_0(x_0) - \phi(x_0)) \geq w(x) + \varepsilon(\phi_0(x) - \phi(x)), \quad \forall \, x \in \Sigma \setminus B_1(x_0), \]

which implies that the supremum of the function $w(x) + \varepsilon(\phi_0(x) - \phi(x))$ in $\Sigma$ is attained on $B_1(x_0)$, so there exists a point $\bar{x} \in B_1(x_0)$ such that
\[ w(\bar{x}) + \varepsilon(\phi_0(\bar{x}) - \phi(\bar{x})) = \max_{\Sigma} \{w(x) + \varepsilon(\phi_0(x) - \phi(x))\} \geq A. \quad (2.9) \]

Now we shall estimate the lower bound and the upper bound of
\[ Q := \mathcal{F}_\alpha(u_0 + \varepsilon \phi_0)(\bar{x}) - \mathcal{F}_\alpha(u + \varepsilon \phi)(\bar{x}) \quad (2.10) \]
respectively, to derive a contradiction.

We first estimate the lower bound of (2.10) by direct calculations. For simplicity, we set $C_{n,\alpha} = 1$, then
\[
\mathcal{F}_\alpha(u_0 + \varepsilon \phi_0)(\bar{x}) - \mathcal{F}_\alpha(u + \varepsilon \phi)(\bar{x}) = P.V. \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n+\alpha}} \left[ G(u_0(\bar{x}) + \varepsilon \phi_0(\bar{x}) - u_0(y) - \varepsilon \phi_0(y)) - G(u(\bar{x}) + \varepsilon \phi(\bar{x}) - u(y) - \varepsilon \phi(y)) \right] dy
\]
\[ = P.V. \int_\Sigma \frac{1}{|\bar{x} - y|^{n+\alpha}} \left[ G(u_0(\bar{x}) + \varepsilon \phi_0(\bar{x}) - u_0(y) - \varepsilon \phi_0(y)) \right] dy \]
−G(u(\bar{x}) + \varepsilon \phi(\bar{x}) - u(y) - \varepsilon \phi(y)] \, dy
+ \text{P.V.} \int_\Sigma \frac{1}{|x - y|^{n+\alpha}} \left[ G(u_0(\bar{x}) + \varepsilon \phi_0(\bar{x}) - u(y) - \varepsilon \phi(y))
- G(u(\bar{x}) + \varepsilon \phi(\bar{x}) - u_0(y) - \varepsilon \phi_0(y))] \, dy
= \text{P.V.} \int_\Sigma \left( \frac{1}{|x - y|^{n+\alpha}} - \frac{1}{|x - \bar{y}|^{n+\alpha}} \right) \left[ G(u_0(\bar{x}) + \varepsilon \phi_0(\bar{x}) - u_0(y) - \varepsilon \phi_0(y))
- G(u(\bar{x}) + \varepsilon \phi(\bar{x}) - u(y) - \varepsilon \phi(y))] \, dy
+ \text{P.V.} \int_\Sigma \frac{1}{|x - y|^{n+\alpha}} \left\{ \left[ G(u_0(\bar{x}) + \varepsilon \phi_0(\bar{x}) - u(y) - \varepsilon \phi(y))
- G(u(\bar{x}) + \varepsilon \phi(\bar{x}) - u(y) - \varepsilon \phi(y))] + [G(u_0(\bar{x}) + \varepsilon \phi_0(\bar{x}) - u_0(y) - \varepsilon \phi_0(y))
- G(u(\bar{x}) + \varepsilon \phi(\bar{x}) - u(y) - \varepsilon \phi(y))] \right\} \, dy
:= I_1 + I_2. \quad (2.11)

To estimate $I_1$, we first notice that
\[ \frac{1}{|x - y|^{n+\alpha}} - \frac{1}{|x - \bar{y}|^{n+\alpha}} > 0 \text{ for all } y \in \Sigma, \]
while for the second part in the integral, for any $y \in \Sigma$, we have
\[ G(u_0(\bar{x}) + \varepsilon \phi_0(\bar{x}) - u_0(y) - \varepsilon \phi_0(y)) - G(u(\bar{x}) + \varepsilon \phi(\bar{x}) - u(y) - \varepsilon \phi(y)) \geq 0 \]
due to the strict monotonicity of $G$ and the fact that
\[ (u_0(\bar{x}) + \varepsilon \phi_0(\bar{x}) - u_0(y) - \varepsilon \phi_0(y)) - (u(\bar{x}) + \varepsilon \phi(\bar{x}) - u(y) - \varepsilon \phi(y)) = (u(\bar{x}) + \varepsilon \phi(\bar{x}) - u(y) - \varepsilon \phi(y)) \geq 0. \]

Consequently,
\[ I_1 = \text{P.V.} \int_\Sigma \left( \frac{1}{|x - y|^{n+\alpha}} - \frac{1}{|x - \bar{y}|^{n+\alpha}} \right) \cdot \left[ G(u_0(\bar{x}) + \varepsilon \phi_0(\bar{x}) - u_0(y) - \varepsilon \phi_0(y)) - G(u(\bar{x}) + \varepsilon \phi(\bar{x}) - u(y) - \varepsilon \phi(y))] \, dy \geq 0. \quad (2.12) \]

Now we estimate the term $I_2$ in (2.11). By (2.9), for any $y \in \Sigma$ we have
\[ (u_0(\bar{x}) + \varepsilon \phi_0(\bar{x}) - u(y) - \varepsilon \phi(y)) - (u(\bar{x}) + \varepsilon \phi(\bar{x}) - u(y) - \varepsilon \phi(y)) = w(\bar{x}) + \varepsilon \phi_0(\bar{x}) - \varepsilon \phi(\bar{x}) \geq A, \]
where $A$ is a constant.
and
\[
(u_0(\overline{x}) + \varepsilon \phi_0(\overline{x}) - u_0(y) - \varepsilon \phi_0(y)) - (u(\overline{x}) + \varepsilon \phi(\overline{x}) - u_0(y) - \varepsilon \phi_0(y)) = w(\overline{x}) + \varepsilon \phi_0(\overline{x}) - \varepsilon \phi(\overline{x}) \geq A.
\]

By (G_2), we infer that
\[
I_2 = P.V. \int_\Omega \frac{1}{|x-y|^{n+\alpha}} \left\{ [G(u_0(\overline{x}) + \varepsilon \phi_0(\overline{x}) - u(\overline{x}) - \varepsilon \phi(\overline{x}))
- G(u(\overline{x}) + \varepsilon \phi(\overline{x}) - u_0(y) - \varepsilon \phi_0(y))] + [G(u_0(\overline{x}) + \varepsilon \phi_0(\overline{x}) - u_0(y) - \varepsilon \phi_0(y))
- G(u(\overline{x}) + \varepsilon \phi(\overline{x}) - u_0(y) - \varepsilon \phi_0(y))] \right\} dy
\geq 2c_1 \int_\Omega \frac{1}{|x-y|^{n+\alpha}} (w(\overline{x}) + \varepsilon \phi_0(\overline{x}) - \varepsilon \phi(\overline{x})) dy
\geq 2c_1 \int_\Omega \frac{1}{|x-y|^{n+\alpha}} \int_{A^r} A dy \geq C_3 A^r. \tag{2.13}
\]

Let \(D = \{y : 2 < y_1 = x_1^* < 3, |y' - (x_0)'| < 1\}, s = y_1 - x_1^*, \rho = |y' - (x_0)'|\) and \(\omega_{n-2}\) denotes the area of unit sphere in \(\mathbb{R}^{n-1}\). Now we evaluate the last integral in (2.13) as
\[
\int_\Omega \frac{1}{|x-y|^{n+\alpha}} dy \geq \int_D \frac{1}{|x-y|^{n+\alpha}} dy = \int_0^1 \int_0^{s_1} \omega_{n-2}^{n-2} d\rho \frac{n+\alpha}{n} ds
= \int_0^1 \int_0^{s_1} \frac{\omega_{n-2}(st)^{n-2} ds dt}{s^{n+\alpha}(1 + t^2)^{n+\alpha}} \geq \int_0^1 \frac{1}{s^{n+\alpha}} \int_0^{\frac{s}{1+t^2}} \omega_{n-2}^{n-2} dt ds
\geq C_1 \int_0^{s_1} \frac{1}{s^{n+\alpha}} ds = C_2 > 0. \tag{2.14}
\]

Thus, by (2.13), (2.14) we get
\[
I_2 \geq C_3 A^r. \tag{2.15}
\]

Combining (2.11)-(2.15), we deduce
\[
Q := \mathcal{F}_\alpha(u_0 + \varepsilon \phi_0)(\overline{x}) - \mathcal{F}_\alpha(u + \varepsilon \phi)(\overline{x}) \geq C_3 A^r. \tag{2.16}
\]

In order to derive a contradiction, we also estimate the upper bound of (2.10). For this aim, using Lemma 2.1, we derive
\[
\mathcal{F}_\alpha(u_0 + \varepsilon \phi_0)(\overline{x}) - \mathcal{F}_\alpha(u + \varepsilon \phi)(\overline{x})
\leq \mathcal{F}_\alpha(u_0(\overline{x})) - \mathcal{F}_\alpha(u(\overline{x})) + 2\varepsilon C_3 + C_4 \delta^{2-\alpha} \leq 2\varepsilon C_3 + C_4 \delta^{2-\alpha}. \tag{2.17}
\]

Combining (2.16) and (2.17), we have
\[
C_3 A^r \leq 2\varepsilon C_3 + C_4 \delta^{2-\alpha}. \]

We first choose \(\delta\) small such that
\[
C_4 \delta^{2-\alpha} \leq \frac{C_3}{3} A^r,
\]
then for such \(\delta\), let \(\sigma\) be sufficiently close to 1, hence \(\varepsilon = (1 - \sigma)A\) is small such that
\[
2\varepsilon C_3 \leq \frac{C_3}{3} A^r,
\]
which contradicts with \(A > 0\). This implies that (1.6) holds true, and completes the proof of the theorem.
3. The proofs of Theorems 1.1 and 1.4. In this section we shall use the maximum principle established in the previous section to prove Theorems 1.1 and 1.4.

Proof of Theorem 1.1. We show that \( u \) is symmetric with respect to any hyperplane. To this aim, let \( x_n \) be any given direction in \( \mathbb{R}^n \) and set

\[
T_\lambda = \{ x \in \mathbb{R}^n : x_n = \lambda \text{ for } \lambda \in \mathbb{R} \}
\]

be a plane perpendicular to \( x_n \)-axis. Let \( \Sigma_\lambda = \{ x \in \mathbb{R}^n : x_n > \lambda \} \) be the right region about the plane \( T_\lambda \). For \( x \in \Sigma_\lambda \), let

\[
x^\lambda = (x_1, x_2, \ldots, 2\lambda - x_n)
\]

be its reflection about the plane \( T_\lambda \). Denote

\[
w_\lambda(x) = u(x^\lambda) - u(x).
\]

Under the assumptions of Theorem 1.1, we see that \( w_\lambda(x) \) is bounded, and verifies

\[
\mathcal{F}_\alpha(u(x^\lambda)) - \mathcal{F}_\alpha(u(x)) = 0.
\]

Applying Theorem 1.3, we arrive immediately \( w_\lambda(x) \leq 0 \) in \( \Sigma_\lambda \).

Similarly, we can prove that \( w_\lambda(x) \geq 0 \) in \( \Sigma_\lambda \), hence,

\[
w_\lambda(x) \equiv 0 \text{ in } \Sigma_\lambda. \quad (3.1)
\]

These imply that \( u(x) \) is symmetric with respect to plane \( T_\lambda \) for any \( \lambda \in \mathbb{R} \).

Since the \( x_n \)-direction can be chosen arbitrarily, (3.1) implies \( u(x) \equiv C \), which completes the proof of Theorem 1.1.

We next prove Theorem 1.4.

Proof of Theorem 1.4. It is sufficient to prove that

\[
w_\lambda(x) = u(x^\lambda) - u(x) \leq 0 \text{ for sufficiently large } \lambda.
\]

Suppose, by contradiction that

\[
\sup_{\Sigma_\lambda} w_\lambda(x) := A > 0. \quad (3.2)
\]

Then for any \( \sigma \in (0, 1) \), there exists \( x_0 \in \Sigma_\lambda \) such that

\[
w_\lambda(x_0) \geq \sigma A.
\]

By re-scaling, we may suppose that \( \text{dist}(x_0, T_\lambda) = 2 \). Define

\[
\eta(x) = \begin{cases} 
ae^{-\frac{|x|^2}{\alpha^2}}, & |x| < 1, 
0, & |x| \geq 1,
\end{cases}
\]

with \( a = e \) such that \( \eta(0) = \max_{\mathbb{R}^n} \eta(x) = 1 \). Set

\[
\phi(x) = \eta(x - x_0^\lambda) \text{ and } \phi_\lambda(x) = \eta(x - x_0).
\]

Then \( \phi_\lambda(x) - \phi(x) \) is an anti-symmetric function with respect to the plane \( T_\lambda \).

Now choose \( \varepsilon = (1 - \sigma)A > 0 \) such that

\[
w_\lambda(x_0) + \varepsilon(\phi_\lambda(x_0) - \phi(x_0)) \geq A.
\]
Similar to the proof in Theorem 1.3, it follows that there exists a point $x \in B_1(x_0)$ such that
\[
    w_\lambda(x) + \varepsilon(\phi_\lambda(x) - \phi(x)) = \max_{\Sigma_\lambda}\{w_\lambda(x) + \varepsilon(\phi_\lambda(x) - \phi(x))\} \geq A. \tag{3.3}
\]
Then we shall be able to estimate
\[
    Q_\lambda := F_\alpha(u_\lambda + \varepsilon\phi_\lambda(x)) - F_\alpha(u + \varepsilon\phi)(x) \tag{3.4}
\]
at the maximum point $x$.

As in the proof of Theorem 1.3, we can obtain the lower bound of $Q_\lambda$ as
\[
    Q_\lambda \geq CA^\tau. \tag{3.5}
\]
On the other hand, since
\[
    w_\lambda(x) + \varepsilon(\phi_\lambda(x) - \phi(x)) \geq w_\lambda(x_0) + \varepsilon(\phi_\lambda(x_0) - \phi(x_0)),
\]
$\phi(x) = \phi(x_0) = 0$ and $1 = \phi_\lambda(x_0) \geq \phi_\lambda(x)$, we derive
\[
    w_\lambda(x) \geq w_\lambda(x_0) > 0.
\]
Consequently,
\[
    u_\lambda(x) > u(x).
\]
By the monotonicity of $f$ and Lemma 2.1, we deduce that
\[
    Q_\lambda := F_\alpha(u_\lambda + \varepsilon\phi_\lambda(x)) - F_\alpha(u + \varepsilon\phi)(x)
    \leq F_\alpha(u_\lambda(x)) - F_\alpha(u(x)) + 2\varepsilon C_\delta + C_1 \delta^{2+2-\alpha}
    = f(u_\lambda(x)) - f(u(x)) + 2\varepsilon C_\delta + C_1 \delta^{2+2-\alpha}
    \leq 2\varepsilon C_\delta + C_1 \delta^{2+2-\alpha}.
\]
Therefore, we derive a contradiction when $\delta$ is small and $\sigma$ is sufficiently close to 1. This completes the proof of Theorem 1.4.

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REFERENCES

[1] F. Andreu, J. Mazon, J. Rossi and J. Toledo, Nonlocal Diffusion Problems, Vol. 165, American Mathematical Society, Providence, RI, 2010.
[2] D. Applebaum, Lévy processes-from probability to finance and quantum groups, Notices Amer. Math. Soc., 51 (2004), 1336–1347.
[3] C. Bjorland, L. Caffarelli and A. Figalli, Nonlocal tug-of-war and the infinity fractional Laplacian, Commun. Pure Appl. Math., 65 (2012), 337–380.
[4] K. Bogdan, T. Kuleczycki and A. Nowak, gradient estimates for harmonic and $q$-harmonic functions of symmetric stable processes, Illinois J. Math., 46 (2002), 541–556.
[5] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Commun. Pure Appl. Equ., 32 (2007), 1245–1260.
[6] L. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Commun. Pure Appl. Math., 62 (2009), 597–638.
[7] W. Chen, L. D’Ambrosio and Y. Li, Some Liouville theorems for the fractional Laplacian, Nonlinear Anal. Theory Meth. Appl., 121 (2015), 370–381.
[8] W. Chen and C. Li, Maximum principles for the fractional $p$-Laplacian and symmetry of solutions, Adv. Math., 335 (2018), 735–758.
[9] W. Chen, C. Li and G. Li, Maximum principle for a fully nonlinear fractional order equation and symmetry of solutions, Calc. Var. Partial Differ. Equ., 56 (2017), 29 pp.
[10] W. Chen, C. Li and Y. Li, A direct method of moving planes for the fractional Laplacian, Adv. Math., 308 (2017), 404–437.
[11] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, *Commun. Pure Appl. Math.*, **59** (2006), 330–343.

[12] W. Chen, C. Li and B. Ou, Qualitative properties of solutions for an integral equation, *Discrete Contin. Dyn. Syst.*, **12** (2005), 347–354.

[13] W. Chen and L. Wu, A maximum principle on unbounded domains and a Liouville theorem for fractional p-harmonic functions, preprint, arXiv:1905.09986.

[14] W. Chen and J. Zhu, Indefinite fractional elliptic problem and Liouville theorems, *J. Differ. Equ.*, **260** (2016), 4758–4785.

[15] T. Cheng, Monotonicity and symmetry of solutions to fractional Laplacian equation, *Discrete Contin. Dyn. Syst.*, **37** (2017), 3587–3599.

[16] T. Cheng, G. Huang and C. Li, The maximum principles for fractional Laplacian equations and their applications, *Commun. Contemp. Math.*, **19** (2017), Art. 1750018.

[17] E. De Giorgi, Convergence problems for functionals and operators, in *Proc. Int. Meeting on Recent Methods in Nonlinear Analysis*, Rome, 1978, Pitagora, (1979), 131–188.

[18] B. Gidas and J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, *Commun. Partial Differ. Equ.*, **6** (1981), 883–901.

[19] X. Han, G. Lu and J. Zhu, Characterization of balls in terms of Bessel-potential integral equation, *J. Differ. Equ.*, **252** (2012), 1589–1602.

[20] C. Kenig and W. Ni, An exterior Dirichlet problem with applications to some nonlinear equations arising in geometry, *Amer. J. Math.*, **106** (1984), 689–702.

[21] L. Ma and D. Chen, A Liouville type theorem for an integral system, *Commun. Pure Appl. Anal.*, **5** (2006), 855–859.

[22] P. Polacik, P. Quittner and P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems, Part I: Elliptic equations and systems, *Duke Math. J.*, **139** (2007), 555–579.

[23] P. Wang and P. Niu, A direct method of moving planes for a fully nonlinear nonlocal system, *Commun. Pure Appl. Anal.*, **16** (2017), 1707–1718.

[24] R. Zhuo, W. Chen, X. Cui and Z. Yuan, Symmetry and nonexistence of solutions for a nonlinear system involving the fractional Laplacian, *Discrete Contin. Dyn. Syst.*, **36** (2016), 1125–1141.

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