Supersymmetry of $\mathcal{PT}$- symmetric tridiagonal Hamiltonians

Mohammad Walid AlMasri$^1$

1Department of Physics, Koç University,
Rumelifeneri Yolu, 34450 Sarıyer, Istanbul, Turkey

We extend the study of supersymmetric tridiagonal Hamiltonians to the case of non-Hermitian Hamiltonians with real or complex conjugate eigenvalues. We find the relation between matrix elements of the non-Hermitian Hamiltonian $H$ and its supersymmetric partner $H^+$ in a given basis. Moreover, the orthogonal polynomials in the eigenstate expansion problem attached to $H^+$ can be recovered from those polynomials arising from the same problem for $H$ with the help of kernel polynomials. Besides its generality, the developed formalism in this work is a natural home for using the numerically powerful Gauss quadrature techniques in probing the nature of some physical quantities such as the energy spectrum of $\mathcal{PT}$-symmetric complex potentials. Finally, we solve the shifted $\mathcal{PT}$-symmetric Morse oscillator exactly in the tridiagonal representation.

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I. INTRODUCTION

Supersymmetric quantum mechanics (SUSY-QM) plays an important role in the study of exactly solvable potentials and its properties mainly due to its powerful factorization techniques which could be used in constructing new analytically solvable potential [1-7]. The relation between one-dimensional solvable potentials and SUSY was addressed by Gendenshtein through the concept of shape-invariant potentials [8]. By definition the potential is said to be shape invariant if its supersymmetric partner has the same spatial dependence as the original potential with possibly adjusted parameters.

It has been known for a long time that unperturbed partial-wave kinetic energy operator

*Electronic address: mwalmasri2003@gmail.com*
\( H^0 = -\frac{1}{2} \frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} \) has a tridiagonal representation in a complete Slater or oscillator basis set \( \{|\phi_n\rangle\} \) [9]. This was extended later to the case of Coulomb, Morse and Rosen-Morse potentials [10]. The Supersymmetry of tridiagonal Hamiltonians was studied in [11]. It was shown that if a positive semi-definite Hamiltonian has a tridiagonal representation in a given orthonormal basis \( \{|\phi_n\rangle\} \), its supersymmetric partner Hamiltonian \( H^+ \) will have the same matrix representation in the same basis \( \{|\phi_n\rangle\} \)[11]. With the help of kernel polynomials, the orthogonal polynomials in the eigenstate expansion problem attached to \( H^+ \) can be recovered from those polynomials arising from the same problem for \( H \). This method was elaborated to solve exactly the supersymmetry of Morse oscillator [12].

All previous studies were done with Hermitian tridiagonal Hamiltonians, however it is known that special class of non-Hermitian Hamiltonians may posses real eigenvalues [13–17]. One feature of these Hamiltonians is that they are all \( \mathcal{PT} \)-symmetric, note that the opposite of this statement is not true in general since not all \( \mathcal{PT} \)- symmetric Hamiltonians have real eigenvalues in their energy spectrum. These observations lead to enormous studies of \( \mathcal{PT} \)-symmetric Hamiltonians and their properties especially in optics [18–21]. In [22], a relation between non-Hermitian systems with the Hermitian scattering systems was found. More recently Zhang et al. proposed a non-Hermitian supersymmetric array with high-order exceptional point of arbitrary order and investigated the topological properties of exceptional point [23]. Possible extension to the ideas of \( \mathcal{PT} \)-symmetric Hamiltonian would be found in some magnetic systems such as quantum spin chains and \( \mathcal{PT} \)-symmetric magnonic waveguides [24, 25]. The supersymmetric quantum mechanical treatment of \( \mathcal{PT} \)-symmetric Hamiltonians was done by Znojil et al [26] and the supersymmetric method for complex potentials was investigated in [27] and can be read from the general chapter by Levai [28].

Throughout the present work, we study the supersymmetry of tridiagonal non-Hermitian Hamiltonians with real or complex conjugate pair eigenvalues. We formulate the problem using pseudo-Hermiticity notion where the Hilbert space of quantum states is endowed with a Hermitian indefinite inner product. Very recently, supersymmetry of tridiagonal non-Hermitian Hamiltonians was studied in [29]. However, our formalism is different from that presented in [29] since it depends on the Askey scheme of classical orthogonal polynomials.
Also we considered cases where energy spectrum can have discrete and continuous values simultaneously unlike [29] where the energy outcomes were discrete (shifted harmonic oscillator and squeezed Hamiltonian). The main motivation of this work is to develop the notion of \(J\)-matrix method for complex potentials and study the supersymmetry of these complex potentials in the tridiagonal representation. Finally we tested the formalism by solving the \(\mathcal{PT}\)-symmetric Morse potential exactly in the tridiagonal representation domain with the help of orthogonal polynomials.

The organization of paper goes as follows, in section (II) we give a formal definition of supersymmetric quantum mechanics and its pseudo-Hermitian generalization. We briefly discuss the hierarchy method of \(\mathcal{PT}\)-symmetric tridiagonal Hamiltonians in section (III). In section (IV), we express the non-Hermitian Hamiltonians in tridiagonal representation and calculate the matrix elements for \(H\) and its superpartner \(H^{(+)}\). The eigenstates expansion in terms of orthogonal polynomials are presented in (V). Finally we apply the developed formalism to the case of \(\mathcal{PT}\)-symmetric shifted Morse oscillator.

II. SUPERSYMMETRIC QUANTUM MECHANICS AND PSEUDO-HERMITICITY

Supersymmetry combines the bosonic and fermionic degrees of freedom in a unified algebra. It was first proposed in the context of unification of particle physics models [30-32]. However it was realized later that some ideas of supersymmetry could be used in solving nonrelativistic quantum mechanical problems [1]. In SUSY-QM, the Hilbert space consists of bosonic \(\mathcal{H}_+\) and fermionic \(\mathcal{H}_-\) sub-Hilbert spaces i.e. \(\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-\) for a given Hamiltonian \(H\).

We define the non-Hermitian supersymmetric charges or generators \(Q^I, I = 1, \ldots N\) as

\[
Q^I(\mathcal{H}_\pm) = \mathcal{H}_\mp, \quad (2.1)
\]

\[
Q^{I \dagger}(\mathcal{H}_\pm) = \mathcal{H}_\mp, \quad (2.2)
\]

where \(\{,\}\) is the anti-commutator. These generators satisfy the following super-algebra conditions

\[
\{Q^I, Q^{I \dagger}\} = 2\delta^{IJ} H, \quad (2.3)
\]
\{Q^I, Q^J\} = \{Q^{J\dagger}, Q^{I\dagger}\} = 0, \quad (2.4)

where \(I, J = 1, \ldots N\) and \(\delta^{IJ}\) is the Kronecker delta.

The equations (2.3) and (2.4) can also be rephrased in the form

\[
H = \frac{1}{2}(Q^I Q^{I\dagger} + Q^{I\dagger} Q^I),
\]

\[
(Q^I)^2 = (Q^{I\dagger})^2 = 0,
\]

(2.5), (2.6)

For all \(I = 1, \ldots N\).

To simplify the formalism we write the \(2N\)-Hermitian SUSY charges as

\[
Q^I_1 := \frac{1}{\sqrt{2}}(Q^I + Q^{I\dagger}),
\]

(2.7)

\[
Q^I_2 := \frac{1}{\sqrt{2}}(Q^I - Q^{I\dagger}).
\]

(2.8)

and the anti-commutators are

\[
\{Q^I_{\alpha}, Q^J_{\beta}\} = 2\delta_{\alpha\beta}\delta^{IJ}H,
\]

\[
\{Q^I_1, Q^I_2\} = 0,
\]

(2.9), (2.10)

where \(\alpha, \beta = 1, 2\) and \(I, J = 1, 2, \ldots N\). Alternatively, the Hamiltonian takes the form

\[
H = (Q^I_1)^2 = (Q^I_2)^2
\]

(2.11)

In this work we concentrate on \(\mathcal{PT}\)-symmetric SUSY-QM [26]. By definition, a \(\mathcal{PT}\)-symmetric Hamiltonian satisfies

\[
\mathcal{PT}H(\mathcal{PT})^{-1} = \mathcal{PT}H\mathcal{PT} = H,
\]

(2.12)

Where \(\mathcal{P}\) and \(\mathcal{T}\) are the parity and time-reversal operators respectively. These operators are defined in the following way

\[
\mathcal{P} \ x \ \mathcal{P} = -x, \quad \mathcal{P} \ p \ \mathcal{P} = -p, \quad \mathcal{T} \ i \ \mathcal{T} = -i\mathbb{I}.
\]

(2.13)

In the case of \(\mathcal{PT}\)-symmetric Hamiltonians, the inner product of two states \(\phi_1\) and \(\phi_2\) is defined in the Hilbert space \(\mathcal{H}\) as [14]

\[
\langle\langle \phi_1 | \phi_2 \rangle\rangle = \langle \phi_1 | \mathcal{P} | \phi_2 \rangle.
\]

(2.14)
which reduces to the ordinary Hermitian inner product when $\mathcal{P} = 1$. In is worthy to note that $\{\mathcal{P}, Q\} = 0$. The inner product defined in (2.14) is invariant under time translation generated by the Hamiltonian $H$ if and only if $H$ is $\mathcal{P}$ pseudo-Hermitian as shown in [33]. The pseudo-superalgebra is given by [26] (we omitted the index $I$ for the sake of simplicity)

$$\{Q, \tilde{Q}\} = 2H, \quad Q^2 = \tilde{Q}^2 = 0,$$

(2.15)

Where $\tilde{Q} = \mathcal{P}^{-1}Q^\dagger\mathcal{P}$.

The Hamiltonian $H_+ = BA$ and $H_- = AB$ can be constructed from the shifting operator $A : \mathcal{H}_+ \to \mathcal{H}_-$ which maps the eigenvectors of $H_+$ to those of $H_-$ while $B$ does the opposite action.

### III. FACTORIZATION OF $\mathcal{PT}$-SYMMETRIC TRIDIAGONAL HAMILTONIANS

One advantage of supersymmetric quantum mechanics is the factorization techniques that allow us to factorize Schrödinger equation by expressing the Hamiltonian as $H = A^\dagger A + \text{constant term(s)}$ [4, 28, 34]. The generic Hamiltonian with tridiagonal representation reads

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{2r^2} + V_1(r)$$

(3.1)

so that $V_1^{\text{eff}} = \frac{\hbar^2}{2m} \frac{\Psi_0''}{\Psi_0}$ where $'$ denotes the derivative with respect to the radial coordinate $r$ and $V_1^{\text{eff}} = \frac{\ell(\ell + 1)}{2r^2} + V_1$ is the centrifugal effective potential which appears usually in three-dimensional quantum problems with spherical symmetry. It can be defined using the superpotential $W = -\frac{\hbar}{\sqrt{2m}} \left( \frac{d\ln \Psi_0}{dr} \right)$ as [4]

$$V_1^{\text{eff}} = W^2 - \hbar \frac{dW}{\sqrt{2m}} \frac{dW}{dr}$$

(3.2)

The ground state wavefunction can be computed from the superpotential up to a normalization constant as

$$\Psi_0 \sim \exp\left[-\frac{\sqrt{2m}}{\hbar} \int^r W(r')dr'\right]$$

(3.3)

The Hamiltonian [3.1] factorizes as

$$H = A^\dagger A$$

(3.4)

$$H^\dagger = AA^\dagger$$

(3.5)
Where the bosonic operator $A$ and its Hermitian conjugate are

$$A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dr} + W(r), \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dr} + W(r). \quad (3.6)$$

For unbroken supersymmetry, the ground state energy is zero. Thus, we have

$$H_1 \Psi_0 = -\frac{\hbar^2}{2m} \frac{d^2 \Psi_0}{dr^2} + \frac{\ell(\ell + 1)}{2r^2} \Psi_0 + V_1 \Psi_0 = 0 \quad (3.7)$$

We can solve the differential equation $3.7$ and find an expression for $V_1^{\text{eff}}$ defined with respect to the ground state wave function and its second derivative with respect to the radial coordinate $r$,

$$V_1^{\text{eff}} = \frac{\hbar^2}{2m} \frac{\Psi''_0}{\Psi_0} \quad (3.8)$$

The Hamiltonian $3.7$ factorizes as

$$H_1 = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V_1^{\text{eff}}(r) \quad (3.9)$$

$$= A_1 A_1^\dagger + E_1^{(0)}. \quad (3.10)$$

After some calculations we get the Riccati equation for the superpotential $W_1$,

$$W_1^2 - W_1' = \frac{2m}{\hbar^2} (V_1^{\text{eff}}(r) - E_1^{(0)}). \quad (3.11)$$

The supersymmetric partner of Hamiltonian $H_1$ is

$$H_2 = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V_2^{\text{eff}}(r) \quad (3.12)$$

$$= A_2^\dagger A_2 + E_2^{(0)}. \quad (3.13)$$

and the corresponding Riccati equation takes the form

$$W_2^2 + W_2' = \frac{2m}{\hbar^2} (V_2^{\text{eff}}(r) - E_2^{(0)}). \quad (3.14)$$

By iteration we can express the general Hamiltonian and Riccati equations respectively as

$$H_n = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V_n^{\text{tot}}(r) \quad (3.15)$$

$$= A_n A_n^\dagger + E_n^{(0)}, \quad n = 1, 3, 5 \ldots$$

$$H_n = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V_n^{\text{eff}}(r) \quad (3.16)$$

$$= A_n^\dagger A_n + E_n^{(0)}, \quad n = 2, 4, 6 \ldots$$
\begin{align}
W_n^2 - W'_n &= \frac{2m}{\hbar^2} (V^n_{\text{tot}}(r) - E^n_0), n = 1, 3, 5 \ldots \tag{3.17}
\W_n^2 + W'_n &= \frac{2m}{\hbar^2} (V^n_{\text{eff}}(r) - E^n_0), n = 2, 4, 6 \ldots \tag{3.18}
\end{align}

For unbroken supersymmetry, the partner Hamiltonian obey the following expressions

\[ E^0_{n+1} = E^1_n \tag{3.19} \]

with \( E^0_0 = 0 \) and \( n = 0, 1, 2, \ldots \).

It is not difficult to notice that such procedure should be modified for complex potentials with \( \mathcal{PT} \)-symmetry since \( V^{\text{eff}} \) and \( W \) are complex and should be decomposed to its real and imaginary parts \[27\].

In order to factorize any \( \mathcal{PT} \)-symmetric Hamiltonians, we introduce the following four related operators \[26\]

\[ A^{(\pm)} = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W^{(\pm)}(x), \quad B^{(\pm)} = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W^{(\pm)}(x) \tag{3.20} \]

and the corresponding Riccati equation is

\[ H^{(\pm)} = B^{(\pm)} A^{(\pm)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + [W^{(\pm)}(x)]^2 - \frac{\hbar}{\sqrt{2m}} [W^{(\pm)}(x)]' \tag{3.21} \]

where \( t \) is the derivative with respect to the coordinate \( x \). In the incoming sections we will choose the \( + \) operators only since the second case follows the same procedures. The term \( \frac{\ell (\ell + 1)}{2 r^2} \) appears in the three-dimensional problems such as the radial Coulomb potential and isotropic harmonic oscillator \[3\].

**IV. J-MATRIX METHOD FOR COMPLEX TRIDIAGONAL HAMILTONIANS**

In ordinary quantum mechanics, it is customary to expand the Hamiltonian in a basis that makes it diagonal. This simplifies the process of computing the energy eigenvalues especially for large-sized Hamiltonian matrices. On the other hand, this approach reduces the class of possible solutions of the corresponding wave equation of the system. One possible extension of the space of solutions is known as J-matrix method which assumes the matrix representation of the Hamiltonian to be tridiagonal in a given orthogonal polynomial basis \[9, 10, 36\]. In the J-matrix method, we expand the wavefunctions over a complete set of square-integrable functions \( L^2(\mathbb{R}) \) in configuration space \( |\psi(E, x)\rangle = \sum_n f_n(E) |\phi_n(x)\rangle \). The
coefficients are some functions of energy. They encode all physical information about the system. The wave equation $H |\psi\rangle = E |\psi\rangle$ becomes

$$\sum_n f_n H |\psi\rangle = E \sum_n f_n |\phi_n\rangle$$  \hspace{1cm} (4.1)$$

By projecting $\langle \phi_m$ from left we find

$$\sum_n f_n \langle \phi_m | H |\phi_n\rangle = E \sum_n f_n \langle \phi_m | \phi_n\rangle$$  \hspace{1cm} (4.2)$$

which can written in matrix from as $\sum_n H_{mn} f_n = E \sum_n \Omega_{mn} f_n$ where $\Omega_{mn} = \langle \phi_m | \phi_n\rangle$ is the overlap identity matrix of the basis elements. Our main task in this approach is to calculate the matrix elements of the tridiagonal and symmetric operator $J_{mn}$

$$J_{mn} = H_{mn} - E \Omega_{mn} = (a_n - z) \delta_{n,m} + b_n \delta_{n,m-1} + b_{n-1} \delta_{n,m+1}$$  \hspace{1cm} (4.3)$$

In this work we extend J-matrix method to complex potentials with $\mathcal{PT}$ symmetry. For this reason we assume the matrix $J_{mn}$ to be non-self adjoint so that the relation (4.3) will be written generally as

$$J_{mn} = H_{mn} - \varepsilon \Omega_{mn} = (a_n - z) \delta_{n,m} + b_n^* \delta_{n,m-1} + b_{n-1} \delta_{n,m+1}$$  \hspace{1cm} (4.4)$$

where in this case the coefficients $a_n$ and $b_n$ are complex quantities in general. For a given non-self adjoint Hamiltonian with one-dimensional complex potential $V$, the reality of its energy bound states requires $[\mathcal{PT}, H] = 0$ and $(V(-x))^* = V(x)$ \[37\]

The radial Coulomb potential with imaginary charge $iz$ in the atomic units $\hbar = m = 1$

$$H = -\frac{1}{2} \frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{2r^2} + \frac{iz}{r}$$  \hspace{1cm} (4.5)$$

The Hamiltonian (4.5) has a tridiagonal matrix representation in the non-orthogonal Laguerre basis \[9\]

$$\phi_n(x) = B_n(\lambda r)^{\ell + 1} e^{-\frac{\lambda}{2} r^2} L_{2\ell+1}^{\ell} (\lambda r), \hspace{0.5cm} n = 0, 1, 2 \ldots$$  \hspace{1cm} (4.6)$$

$$B_n = \sqrt{\frac{\lambda n!}{\Gamma(n + 2\ell + 2)}}$$
Here \( \lambda \) is a scale parameter. If we define the \( J \)-matrix as \( J = (H - \varepsilon) \), then the only non-zero elements of \( J \) are

\[
(J)_{n,n} = \langle \langle \phi_n | H - \varepsilon | \phi_n \rangle \rangle = i\lambda z - (\varepsilon - \frac{\lambda^2}{8})(2n + 2\ell + 2) \quad (4.7)
\]

\[
(J)_{n,n+1} = \langle \langle \phi_n | H - \varepsilon | \phi_{n+1} \rangle \rangle = (\varepsilon + \frac{\lambda^2}{8})\sqrt{(n + 1)(2n + 2\ell + 2)} \quad (4.8)
\]

The discrete spectrum can be deduced by imposing the conditions given in [10] for specific cases, which apply here, and by the procedure explained in [12] in the general case. Here, we first find the result of demanding that \( J_{\mu,\mu+1} = 0 \) and then use the result in the requirement that \( J_{\mu,\mu} = 0 \). These conditions finally give

\[
(\lambda)_{\mu} = \frac{2iz}{\mu + \ell + 1} \quad (4.9)
\]

\[
(\varepsilon)_{\mu} = \frac{z^2}{2(\mu + \ell + 1)^2} \quad (4.10)
\]

From these results we notice that Coulomb potential with imaginary charge has positive energy spectrum. Furthermore we notice that there are an infinite number of positive energy bound-states in the interval \( 0 \leq (\varepsilon)_{\mu} \leq \frac{z^2}{2(\ell+1)^2} \). Also the wave-function associate with these bound states exhibit sinusoidal behavior as shown in figure [IV]. Remarkably, the bound-states exist for both the system with positive imaginary charge \(+iz\) as well as for negative imaginary charge \(−iz\). This stands in contrast to usual Coulomb Hamiltonian which support bound states only for the case that the charge \( z \) is negative. However it is not \( \mathcal{PT} \)-symmetric so we did not pursue this direction.

For one-dimensional complex potentials the term \( \frac{\ell(\ell+1)}{2r^2} \) is absent. We apply the J-matrix method for the case of \( \mathcal{PT} \)-symmetric Rosen-Morse II potential [38] which has applications in optics as shown in [39]. The \( \mathcal{PT} \)-symmetric Rosen-Morse II potential is

\[
V(x) = 2iB \tanh \delta x - A(A + \frac{\delta \hbar}{\sqrt{2m}}) \sech^2 \delta x \quad (4.11)
\]

where \( \alpha \) is a free real parameter, \( A, B \) are coefficients and \( x \) is the coordinate variable in one-dimension. The corresponding associated energy eigenvalues are

\[
E_n = -(A - \frac{n\hbar \delta}{\sqrt{2m}})^2 + \frac{B^2}{(A - \frac{n\hbar \delta}{\sqrt{2m}})^2} \quad (4.12)
\]
Figure 1: The real and imaginary part of the bound-state wavefunction for the Coulomb potential with complex charge and different configurations shown above each plot.

which are real. The basis function for this potential can be written in virtue of Jacobi polynomials $P_{n}^{\mu,\nu}$ with complex parameters since $\mu, \nu \in \mathbb{C}$ such that $\mu = \overline{\nu}$

$$\psi_n(y) = C_n(1 - y)^{\alpha}(1 + y)^{\beta}P_{n}^{\mu,\nu}(y)$$

(4.13)

where $y = \tanh \delta x$ and The parameters $\mu = s - n + a$, $\nu = s - n - a$ where
The generalized Rosen-Morse II

\[ V_g(x) = 2iB \tanh \alpha x - A(A + \frac{\alpha \hbar}{\sqrt{2m}}) \text{sech}^2 \alpha x + iC \tanh \alpha x \text{sech}^2 \alpha x \quad (4.14) \]

It has tridiagonal representation in Jacobi polynomial basis \( \psi_n(x) = (1 - \tanh(\alpha x))^\frac{\nu}{2}(1 + \tanh(\alpha x))^\frac{\nu}{2} P_n^{(\mu,\nu)}(\tanh(\alpha x)) \) in the case \((\alpha, \beta) = (\frac{s-n+a}{2}, \frac{s-n-a}{2})\) where \(2iB = (\frac{\delta \nu}{2})^2 \left( \frac{\delta \mu}{2} \right) \).

Another interesting example is the \( \mathcal{PT} \)-symmetric Scarf II potential \( V_{Scarf}(x) = -V_1 \text{sech}^2 x + iV_2 \text{sech} x \tanh x \) for \( V_1 > 0 \) and \( V_2 \neq 0 \) which was found to host an exceptional point in its energy spectrum (zero-width resonance or spectral singularity in other references) for some wavefunction \[10, 40-42\]. Similar to the complex Rosen-Morse II potential, the complex Scarf II potential can be solved exactly in a Jacobi polynomial basis \[42, 43\].

V. PSEUDO-HERMITIAN SUPERSYMMETRIC HAMILTONIANS IN TRIDIAGONAL REPRESENTATION

We assume the matrix representation of a given \( \mathcal{PT} \)-symmetric Hamiltonian \( H = BA \) in a complete orthonormal basis \( \{ |\phi_n\rangle \} \) where \( n = 0, 1, 2, \ldots \), to be tridiagonal in the following manner

\[ \langle \langle \phi_n | H | \phi_m \rangle \rangle = b_{n-1} \delta_{n,m+1} + a_n \delta_{n,m} + b_n^* \delta_{n,m-1}, \quad (5.1) \]

where \( \{a_n, b_n\} \) are complex coefficients, and \( b_{-1} = 0 \). We define the operator \( A \) by its action on each element of the basis \( \{ |\phi_n\rangle \} \) as

\[ A |\phi_n\rangle = c_n |\phi_n\rangle + d_n |\phi_{n-1}\rangle. \quad (5.2) \]

where the coefficients \( \{c_n, d_n\} \) are to be defined later except to \( d_0 = 0 \). The action of operator \( B \) on a given basis \( \{ |\phi_n\rangle \} \) is

\[ B |\phi_n\rangle = c_n^* |\phi_n\rangle + d_{n+1}^* |\phi_{n+1}\rangle. \quad (5.3) \]
\( n = 0, 1, \ldots, \) and \(*\) is the complex conjugation operation. Note that when \( c_n = 0 \) in equations \ref{eq:5.2} and \ref{eq:5.3} the operators \( A \) and \( B \) become the lowering and raising operators, respectively. The product operator \( BA \) has the following tridiagonal representation in the given basis \( \{ | \phi_n \rangle \} \) as follows

\[
\langle \langle \phi_n | BA | \phi_m \rangle \rangle = c_m d_{m+1}^* \delta_{n,m+1} + (|c_m|^2 + |d_m|^2) \delta_{m,n} + d_m c_{m-1}^* \delta_{n,m-1}.
\]  

(5.4)

where \( n, m = 1, 2, \ldots \). We prove this last result by acting with \ref{eq:5.2} and \ref{eq:5.3} on a given basis \( \{ | \phi_n \rangle \} \), we obtain

\[
BA | \phi_m \rangle = B(A | \phi_m \rangle) = B(c_m | \phi_m \rangle + d_m | \phi_{m-1} \rangle)
\]

(5.5)

\[
= c_m c_m^* | \phi_m \rangle + c_m d_{m+1}^* | \phi_{m+1} \rangle + d_m c_{m-1}^* | \phi_{m-1} \rangle + d_m d_{m-1}^* | \phi_m \rangle
\]

\[
= c_m d_{m+1}^* | \phi_{m+1} \rangle + (|c_m|^2 + |d_m|^2) | \phi_m \rangle + d_m c_{m-1}^* | \phi_{m-1} \rangle
\]

Next, the matrix elements \( \langle \langle \phi_m | BA | \phi_n \rangle \rangle \) can be calculated by taking the inner product of the previous relation alongside with the orthonormality condition \( \langle \langle \phi_m | \phi_n \rangle \rangle = \langle \phi_m | \mathcal{P} | \phi_n \rangle = \delta_{m,n} \). This gives the desired result

\[
\langle \langle \phi_n | BA | \phi_m \rangle \rangle = c_n d_{n+1}^* \delta_{m,n+1} + (|c_n|^2 + |d_n|^2) \delta_{m,n} + d_n c_{n-1}^* \delta_{m,n-1}.
\]

(5.6)

The coefficients \( \{ a_n, b_n \} \) defined in \ref{eq:5.1} are connected with \( \{ c_n, d_n \} \) via the relations

\[
a_n = |c_n|^2 + |d_n|^2; \quad a_n = c_n d_{n+1}^* \quad b_n = c_n d_{n+1}^*
\]

(5.7)

(5.8)

From \ref{eq:5.7} and \ref{eq:5.8}, we find the following condition between the coefficients

\[
a_0 d_1^* = |b_0|^2.
\]

(5.9)

Now, let us calculate the supersymmetric partner Hamiltonian \( H^+ \) in the same basis \( \{ | \phi_m \rangle \} \) of the original Hamiltonian. It is not difficult to notice that \( H^+ \) has a tridiagonal representation in the same basis of \( H \) as follows

\[
\langle \langle \phi_m | H^+ | \phi_n \rangle \rangle = b_n^+ \delta_{n,m+1} + a_n^+ \delta_{n,m} + (b_n^*)^+ \delta_{n,m-1},
\]

(5.10)

If the operators \( A \) and \( B \) act on the basis elements \( \{ | \phi_n \rangle \} \) according to \ref{eq:5.2} and \ref{eq:5.3} respectively. The product operator \( AB \) has the following tridiagonal representation in the
given basis \{\mid \phi_n \rangle \} as follows:

\[ \langle \langle \phi_n | AB | \phi_m \rangle \rangle = d_{m+1}^* c_{m+1} \delta_{m,n} + (|c_m|^2 + |d_{m+1}|^2) \delta_{m,n+1} + c_m^* d_m \delta_{m,n-1}. \]

where \( n, m = 1, 2, \ldots \). The coefficients in 5.10 are connected with \( c_n \) and \( d_n \) in the following way

\[ a_n^+ = |c_n|^2 + |d_{n+1}|^2, \]
\[ b_n^+ = d_{n+1}^* c_{n+1}. \]

If we consider \( A \) and \( B \) as lowering and raising operators respectively, we would then have vanishing \( c_n \) and \( c_n^* \) coefficients. In that case, both \( H \) and \( H^{(+)} \) are diagonal in this basis \{\mid \phi_n \rangle \} as follows

\[ H_{nm} = |d_n|^2 \delta_{m,n}, \]
\[ H_{nm}^{(+)} = |d_{n+1}|^2 \delta_{m,n}. \]

As a final comment, the energy eigenvalues \( E_n^{(+)} \) of the superpartner Hamiltonian \( H^{(+)} \) are connected with the energy eigenvalues \( E_n \) of the Hamiltonian \( H \) through the relation

\[ E_n^{(+)} = E_{n+1} \]

as expected for any SUSY quantum mechanical model.

VI. EIGENSTATES EXPANSION USING ORTHOGONAL POLYNOMIALS

The action of Hamiltonian \( H \) in the tridiagonal representation is

\[ H | \phi_n \rangle = b_{n-1} | \phi_{n-1} \rangle + a_n | \phi_n \rangle + b_n^* | \phi_{n+1} \rangle \]

\( n = 0, 1, \ldots \). One other hand, the Hamiltonian \( H \) obeys the following eigenvalue equation

\[ H | \psi_E \rangle = E | \psi_E \rangle, \]

where \( | \psi_E \rangle \) is the set of energy eigenstates. It is convenient in our case to expand \( | \psi_E \rangle \) in term of the basis set \{\mid \phi_n \rangle \} as

\[ | \psi_E \rangle = \sum_{n=0}^{\infty} C_n(E) \mid \phi_n \rangle. \]
Here $C_n(E)$ are defined as complex coefficients in general. Then, making use of equation 6.3 and the orthonormality of basis set $\{|\phi_n\rangle\}$ in 6.1 gives the following recurrence relation of the expansion coefficients $C(E)$

\[ E \ C_n(E) = b_{n-1} \ C_{n-1}(E) + a_n \ C_n(E) + b_n^* \ C_{n+1}(E). \]  

(6.4)

The coefficients $C_n(E)$ satisfy the following orthonormality condition in case of discrete energy spectrum

\[ \int_{\Omega(E)} C_n^*(E) C_m(E) \, dE = \int_{\Omega(E)} \langle \langle \phi_n|\psi_E \rangle \rangle \langle \langle \psi_E|\phi_m \rangle \rangle \, dE = \langle \langle \phi_n|\phi_m \rangle \rangle = \delta_{n,m}. \]  

(6.5)

where $\Omega(E)$ is the energy support of the Hamiltonian $H$. For general case, when we have both discrete and continuous energy spectrum, the orthonormality condition becomes

\[ \delta_{m,n} = \Sigma_\alpha C_\alpha^*(E_{\alpha}) C_m(E_{\alpha}) + \int_{\Omega(E)} C_n^*(E) C_m(E) \, dE. \]  

(6.6)

Let us define $P(E)$ as

\[ P_n(E) = \frac{C_n(E)}{C_0(E)} \]  

(6.7)

with $P_0(E) = 1$ and $P_1(E) = (E - a_0)(b_0^*)^{-1}$. By plugging 6.7 in 6.4 we obtain the following three-recursion formula

\[ E P_n(E) = b_{n-1} P_{n-1}(E) + a_n P_n(E) + b_n^* P_{n+1}(E). \]  

(6.8)

The coefficients $(c_n, d_n)$ are related to the values at zero of consecutive polynomial $P_n(0)$ as

\[ (d_{n+1}^*)^2 = -b_n \frac{P_n(0)}{P_{n+1}(0)}, \]  

\[ |c_n|^2 = -b_n^* \frac{P_{n+1}(0)}{P_n(0)}. \]  

(6.9)

(6.10)

Since the supersymmetric partner Hamiltonian has the same tridiagonal representation in the same basis, we write

\[ E P_n^{(+)}(E) = b_{n-1}^{(+)} P_{n-1}^{(+)} + a_n^{(+)} P_n^{(+)}(E) + (b_n^{(+)})^* P_{n+1}^{(+)}(E). \]  

(6.11)

With the initial conditions $P_0^{(+)}(E) = 1$ and $P_1^{(+)}(E) = \frac{(E - a_0^+)}{(b_0^+)^*}$. The polynomials $P_n(E)$ are related to their supersymmetric partner $P_n^{(+)}(E)$ via the following relation

\[ P_n^{(+)}(E) = \sqrt{\frac{b_0 P_1(0)}{b_n P_n(0) P_{n+1}(0)}} \, K_0(E, 0), \]  

(6.12)

where $K_n(E, 0) = \Sigma_j P_j(E)P_j(0)$ denotes the kernel polynomials [11].
VII. SUPERSYMMETRY OF NON-HERMITIAN $\mathcal{PT}$-SYMMETRIC MORSE OSCILLATOR

The generalized non-Hermitian $\mathcal{PT}$-symmetric Morse potential is [48–51]

$$V(x) = V_1 e^{-2i\alpha x} - V_2 e^{-i\alpha x}$$  \hspace{1cm} (7.1)

$$= V(e^{-2i\alpha x} - qe^{-i\alpha x}).$$  \hspace{1cm} (7.2)

where $V = V_1$ and $q = \frac{V_2}{V_1}$.

We choose $V = V_0$ and $q = 2$, then the potential reads

$$V(x) = V_0(e^{-2i\alpha x} - 2e^{-i\alpha x}),$$  \hspace{1cm} (7.3)

where $V_0$ and $\alpha$ are real parameters of the potential. One interesting feature of the Morse Hamiltonian is the possibility of having a discrete energy spectrum depending on the parameter $V_0$ besides its continuous ones. However, in contrast to real case, the potential $V_0$ is bounded and periodic function for fixed values of $V_0$. The total Hamiltonian $H$ is unbounded non-self adjoint operator. Moreover, it has imaginary and real parts. Another alternative way for obtaining real eigenvalues of energy was addressed in [49] for generalized Morse potential defined in [7.1]. We apply our developed formalism in earlier sections to the case of $\mathcal{PT}$-symmetric shifted Morse Hamiltonian written in atomic units $\hbar = m = 1$ as [12]

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + V_0 (e^{-2i\alpha x} - 2e^{-i\alpha x}) - \frac{1}{2} \alpha^2 D^2,$$  \hspace{1cm} (7.4)

where $D = \sqrt{-\frac{2V_0}{\alpha^2}} - \frac{1}{2}$.

Before we start our analysis it would be important to trace the energy spectrum of the $\mathcal{PT}$-symmetric Morse potential and verify it is real on some interval. To answer this exactly we numerically find the eigenvalues of $V_0$ in a Hermite orthogonal basis endowed with a scale parameter $\lambda$

$$\psi_n(x) = A_n e^{-\frac{\lambda x^2}{2}} H_n(\lambda x),$$  \hspace{1cm} (7.5)

where $A_n = \sqrt{\frac{\lambda}{2^n n! \sqrt{\pi}}}$. This basis does not make $7.3$ tridiagonal but it is nice on two counts. First it is complete in the $x$-space over the interval $(-\infty, \infty)$ which is the range of Morse potential. Second, for any desired value $N$, it generates two sets of numbers , the
Figure 2: The imaginary part of the $\mathcal{PT}$-symmetric Morse potential with respect to the real part with $\lambda = 1.5$ and $N = 70$. We notice that complex (also the real) eigenvalues creep usually in pairs.

discrete points $\{x_\mu\}_{\mu=0}^{N-1}$ and the associated matrix $\{\Gamma_{n,\mu}\}_{\mu=0}^{N-1}$ to enable us doing integrals approximately, but accurately, simply as a quadrature

$$S_{n,m} \equiv \int dx \, \psi_n(x) S(x) \psi_n(x) \simeq \sum_{\mu=0}^{N-1} \Gamma_{n,\mu} \, S(x_\mu) S_{m,\mu}. \quad (7.6)$$

For any function $S(x)$ such as $\mathcal{PT}$-symmetric Morse potential. For $N = 70$ and $\lambda = 12$ we were able to verify the reality of eigenvalues for the $\mathcal{PT}$-symmetric Morse potential. For lower scale parameter some complex eigenvalues creep usually in pairs as shown in Figure 1 and 2. If we go higher than $\lambda = 10$, we have a pure real spectrum. The Gauss quadrature treatment can be used for all complex potentials. It is convenient in this case to consider orthogonal polynomials with three-recurrence relation on the interval of study $[45, 47]$.

The complex Morse Hamiltonian has tridiagonal representation using the following orthogonal basis

$$\phi_n(x) = \sqrt{\frac{i\alpha n!}{\Gamma(n+2\gamma+1)}} \, (\xi)^{\gamma+\frac{1}{2}} \, e^{-\frac{1}{2}\xi} \, L_n^{2\gamma}(\xi), \quad (7.7)$$

where $\xi = \sqrt{-\frac{8V_0}{\alpha^2}} \, e^{-i\alpha x}$ and $L_n^{2\gamma}(\xi)$ is the associated Laguerre polynomial $[52]$. The parameter $\gamma$ can take any value except $-\frac{1}{2}$. In contrast to the real case, $L_n^{2\gamma}(z)$ is a function of complex variable $z$. The basis $[7.7]$ vanishes asymptotically when $x \to \pm \infty$ if and only if $V_0$ is negative $[37]$. We considered one set of basis $[7.7]$ for simplicity. However as emphasized in $[49]$ it
Figure 3: The imaginary part of the $\mathcal{PT}$-symmetric Morse potential with respect to the real part with $\lambda = 10$ and $N = 70$. We get a totally real spectrum except for one pair of complex energy eigenvalues. If we jack up the value of scale parameter higher than $\lambda = 10$ this last pair disappear, and we have a pure real spectrum.

is possible to write the total wavefunction for the one-dimensional $\mathcal{PT}$-symmetric Morse potential as a sum of two terms one of them is what I considered in 7.7 and other term which is the same but with $\gamma \rightarrow -\gamma$. The basis set 7.7 can be written in term of confluent hypergeometric function of the first kind $1F_1(a; b; x)$ since $L_{2\gamma}^n(\xi) = \frac{(2\gamma+1)_n}{n!} 1F_1(-n; 2\gamma+1; \xi)$ where $(2\gamma+1)_n$ is the Pochhammer symbol [53].

The matrix representation of the shifted Morse Hamiltonian is

$$\langle \phi_n | H | \phi_m \rangle = b_{n-1} \delta_{n,m+1} + a_n \delta_{n,m} + b_\ast \delta_{n,m-1}. \quad (7.8)$$

Since the Hamiltonian $H$ obeys the eigenvalue equation $H | \psi_E \rangle = E | \psi_E \rangle$ we can make the following expansion $| \psi_E \rangle = \sum_{n=0}^{\infty} C_n(E) | \phi_n \rangle$. We note that $C(E)$ satisfies a recursion relation similar to 6.4. More explicitly, we have the following

$$E \ C_n(E) = -\frac{\alpha^2}{2} \sqrt{n(n+2\gamma)} \left( n - \frac{1}{2} + \gamma - D \right) C_{n-1}(E)$$

$$+ \frac{\alpha^2}{2} \left( \left( n + \gamma + \frac{1}{2} - D \right)^2 + n(n+2\gamma) - D^2 \right) C_n(E)$$

$$- \frac{\alpha^2}{2} \sqrt{(n+1)(n+2\gamma+1)(n+\gamma+\frac{1}{2} - D)} \ C_{n+1}(E). \quad (7.9)$$

In contrast to the real case, $D$ is imaginary for positive $V_0$. The polynomials $C_n(E)$ can be
defined in terms of the continuous dual Hahn polynomials $S_n$ as

$$C_n(E) = \sqrt{\frac{\Gamma(n + 2\gamma + 1)}{n!}} S_n(\lambda^2; -D, \gamma + \frac{1}{2}, \gamma + \frac{1}{2})$$

where $\lambda = \sqrt{-2\alpha^2E - D^2}$ and $W_n$ is the Wilson polynomials with $d$ being an arbitrary parameter [53]. From the previous relation we obtain the following exact expression for $P_n$,

$$P_n(E) = \frac{C_n(E)}{C_0(E)}$$

Following the developed formalism in [V], we need to calculate the coefficients $(c_n, d_n)$ in order to find $a_n^+$ and $b_n^+$ that appear in the matrix representation of supersymmetric partner Hamiltonian $H^+ [5.10]$. Thus, we have the following relations

$$c_n = \frac{i\alpha}{\sqrt{2}} (n + \gamma + \frac{1}{2} - D),$$

$$d_{n+1} = -\frac{i\alpha}{\sqrt{2}} \sqrt{(n+1)(n+1+2\gamma)}.$$  

By plugging 7.12, 7.13 in 5.12 and 5.13, we find

$$a_n^{(+)} = \frac{\alpha^2}{2} ((n + 1)(n + 2\gamma + 1)) + (n + \gamma + \frac{1}{2} - D)^2$$

$$b_n^{(+)} = -\frac{\alpha^2}{2} \sqrt{(n+1)(n+2\gamma+1)} \left( n + \gamma + \frac{3}{2} - D \right)$$

for every $n = 0, 1, 2, \ldots$. From 6.12 we can find the relation for $P_n^{(+)}(E)$ provided that we have determined $P_n(E)$ in 7.11

$$P_n^{(+)}(E) = \frac{(-1)^n (\gamma + \frac{3}{2} - D)}{\sqrt{n!(2\gamma+2)}} \times$$

$$3F_2 \left( \begin{array}{c} -n \ 1 - D + i\lambda \ 1 - D - i\lambda \\ \gamma + \frac{1}{2} - D \ \gamma + \frac{1}{2} - D \end{array} | 1 \right)$$

comparing 7.16 with 7.11 we found that the supersymmetric partner of $P_n(E)$ can be obtained by the substitution $\gamma \rightarrow \gamma + 1$ as expected for supersymmetric theory.
VIII. CONCLUSION

In this work, we have extended the study of supersymmetric tridiagonal Hamiltonians to include $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians. Our developed formalism can be used for both Hermitian and non-Hermitian Hamiltonians so it provides a generalization to [11]. Moreover, we developed the notion of $J$-matrix method for complex potentials. We have calculated the matrix coefficients for the Hamiltonian $H$ and its supersymmetric partner $H^{(+)}$. In contrast to the Hermitian Hamiltonian case described in [11], we found the coefficients $a_n$ and $b_n$ in eq. 5.1 to be imaginary. Moreover we have expanded the eigenstates of $H$ and its supersymmetric partner in terms of orthogonal polynomials. We proved that orthogonal polynomials attached to $H^{(+)}$ can be determined from the orthogonal polynomials attached to $H$ with the help of kernel polynomials. We examined our formalism by solving the $\mathcal{PT}$-symmetric shifted Morse potential exactly. In this case, we found the coefficient polynomials $P_n(E)$ and its supersymmetric partner $P^+_n(E)$ to be given in term of the hypergeometric function $\mathbf{3F}_2$. These results make our task of classifying complex potentials with real energy eigenvalues easier when one consider the Askey scheme which is based mainly hypergeometric functions [53].

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