The ARMA($k$) Gaussian Feedback Capacity

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Abstract
Using Kim’s variational formulation [1] (with a slight yet important modification), we derive the ARMA($k$) Gaussian feedback capacity, i.e., the feedback capacity of an additive channel where the noise is a $k$-th order autoregressive moving average Gaussian process. More specifically, the ARMA($k$) Gaussian feedback capacity is expressed as a simple function evaluated at a solution to a system of polynomial equations, which has only finitely many solutions for the cases $k = 1, 2$ and possibly beyond.

1 Introduction

We consider the following additive Gaussian channel with feedback

$$Y_i = X_i(M, Y_{i-1}^i) + Z_i, \quad i = 1, 2, \ldots (1)$$

where $M$ denotes the message to be communicated through the channel, the noise $\{Z_i\}$, which is independent of $M$, is a zero mean stationary Gaussian process, and $X_i$, the channel input at time $i$, may depend on $M$ and previous channel outputs $Y_{i-1}^i$. And we assume the channel input $\{X_i\}$ satisfies the following average power constraint: there is $P > 0$ such that for all $n$,

$$\frac{1}{n} \sum_{i=1}^{n} E[(X_i(M, Y_{i-1}^i))^2] \leq P.$$ 

Let $C_{FB}$ denote the capacity of the channel [1], which is often referred to as Gaussian feedback capacity in the literature.

It is well known that the non-feedback capacity of [1] can be obtained through the power spectral density water-filling method [2]. As a matter of fact, when the channel noise is white (i.e., $\{Z_i\}$ is i.i.d.), Shannon [3] showed that feedback does not increase capacity, which means, like its non-feedback counterpart, the feedback capacity features an explicit and simple formula. Here we note that in [4], [5], Kadota, Zakai and Ziv also proved this statement for continuous-time white Gaussian channels. On the other hand though, if the channel is not white, feedback may increase capacity (see [6], [7]), and little has been known about its feedback capacity despite a number of papers [8], [9], [10], [11] relating the two
capacities. Computing $C_{FB}$ has been a long-standing open problem that is of fundamental importance in information theory.

An prominent approach to tackle Gaussian feedback capacity can be found in a pioneering work [11], where Cover and Pombra characterized the capacity through the sequence of the so-called “$n$-block feedback capacity”:

$$C_{n,FB} = \max_{\text{tr}(K_{X,n}) \leq nP} \frac{1}{2n} \log \frac{\det(K_{Y,n})}{\det(K_{Z,n})},$$

(2)

where $K_{X,n}$, $K_{Y,n}$, $K_{Z,n}$ stand for the covariance matrices of $X^n$, $Y^n$ and $Z^n$, respectively. It is also shown that the maximization can be taken over $X^n$ of the special form $X^n = B_n Z^n + V^n$, where $B_n$ is a strictly lower-triangular $n \times n$ matrix and the Gaussian vector $V^n$ is independent of $Z^n$. So, (2) can be rewritten as

$$C_{n,FB} = \max_{B_n, K_{V,n}} \frac{1}{2n} \log \frac{\det((B_n + I)K_{Z,n}(B_n + I)^T + K_{V,n})}{\det(K_{Z,n})},$$

(3)

subject to the constraint

$$\text{tr}(B_n K_{Z,n}B_n^T + K_{V,n}) \leq nP,$$

where $K_{V,n}$ is a non-negative definite $n \times n$ matrix. Then, using the asymptotic equipartition property for arbitrary (non-stationary non-ergodic) Gaussian processes, a coding theorem can then be proved to characterize the Gaussian feedback capacity as the limiting expression below:

$$C_{FB} = \lim_{n \to \infty} C_{n,FB}.$$  

(4)

Though considerable efforts have been devoted to follow up the Cover-Pombra formulation, a “computable” formula for the Gaussian feedback capacity does not seem to be within sight: it is already difficult to find the sequence of the optimal $\{B_n, K_{V,n}\}$ achieving $\{C_{n,FB}\}$, and its limiting behavior seems to be as evasive.

Another prominent approach came along in a recent work of Kim [1], which led to a number of breakthroughs deepening our understanding of Gaussian feedback capacity. Roughly speaking, instead of examining the channel (1) over a finite time window, Kim justifies certain interchanges between limits and integrals when evaluating (3) and (4) and recast the problem of computing $C_{FB}$ as an infinite-dimensional optimization problem. Below, we state one of the theorems in [1] that is relevant to our results.

**Theorem 1.1** (Theorem 4.1 of [1]). Suppose that the power spectral density (PSD) $S_Z(e^{i\theta})$ of the Gaussian noise process $\{Z_i\}_{i=1}^\infty$ is bounded away from 0, and has a canonical spectral factorization $S_Z(e^{i\theta}) = |H_Z(e^{i\theta})|^2$, where $H_Z(e^{i\theta}) \subset H_2$. Then the feedback capacity $C_{FB}$ is given by

$$C_{FB} = \max_{B} \frac{1}{2} \int_{-\pi}^{\pi} \log |1 + B(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi},$$

(5)

where the maximum is taken over all strictly causal $B(e^{i\theta})$ satisfying the power constraint

$$\int_{-\pi}^{\pi} |B(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi} \leq P.$$

Furthermore, a filter $B^*(e^{i\theta})$ attains the maximum in (5) if and only if
i) Power:
\[
\int_{-\pi}^{\pi} |B^*(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi} = P;
\]

ii) Output spectrum:
\[
\eta := \text{essinf}_{\theta \in [-\pi, \pi)} |1 + B^*(e^{i\theta})|^2 S_Z(e^{i\theta}) > 0;
\]

iii) Strong orthogonality: For some \( 0 < \lambda \leq \eta \)
\[
\frac{\lambda}{(1 + B^*(e^{i\theta})) H_Z(e^{i\theta})} - B^*(e^{-i\theta}) H_Z(e^{-i\theta})
\]
is causal.

Using Theorem 1.1 and relevant tools from the theory of Hardy spaces \([12]\), Kim further characterized the capacity achieving \(B(e^{i\theta})\) for the special case that \(\{Z_i\}\) is a \(k\)-th order autoregressive moving average (ARMA(\(k\))) Gaussian process. Roughly speaking, the following theorem says that the optimal \(B\) must be rational satisfying three conditions corresponding to those in Theorem 1.1. Here we note that following the notational convention in \([1]\), we use \(B(e^{i\theta})\) and \(B(z)\) (and the like) interchangeably, and when there is no risk of confusion, we may even suppress the notational dependence on \(e^{i\theta}\) or \(z\) for simplicity.

**Theorem 1.2** (Proposition 5.1 of \([1]\)). Suppose the noise \(\{Z_i\}\) is not white and is an ARMA(\(k\)) Gaussian process with parameters \(\alpha_i, \beta_i, |\alpha_i| < 1, |\beta_i| < 1\) for all \(i = 1, 2, \ldots, k\), namely, it has the power spectral density
\[
S_Z(e^{i\theta}) = |H_Z(e^{i\theta})|^2 = \left| \frac{P(e^{i\theta})}{Q(e^{i\theta})} \right|^2 = \left| \frac{\prod_{i=1}^{k} (1 + \alpha_i e^{i\theta})}{\prod_{i=1}^{k} (1 + \beta_i e^{i\theta})} \right|^2.
\]
(6)

Then the feedback capacity \(C_{FB}\) in (5) is necessarily achieved by a filter \(B\) of the form
\[
B(e^{i\theta}) = b(e^{i\theta}) R(e^{i\theta}) P(e^{i\theta})^{-1},
\]
(7)
where \(R(z)\) is a stable polynomial whose degree is at most \(k\), and
\[
b(z) = \frac{A(z)}{A^\#(z)} = \frac{\prod_{n}(1 - \gamma_n^{-1} z)}{\prod_{n}(1 - \gamma_n z)}
\]
is a normalized Blaschke product of at most \(k\) zeros. Furthermore, a filter \(B^*(e^{i\theta})\) of the form \([7]\) is optimal if and only if the following hold:

i) Power:
\[
\int_{-\pi}^{\pi} |B^*(e^{i\theta})|^2 S_Z(e^{i\theta}) \frac{d\theta}{2\pi} = P;
\]

ii) Output spectrum: For all zeros \(\gamma_n\) of \(b(z)\)
\[
0 < S_{Y}^*(\gamma_n) = \min_{\theta \in [-\pi, \pi)} S_Y^*(e^{i\theta});
\]
iii) Factorization:

\[ P(z)A^\#(z) - R(z)A(z) \]

has a factor \( Q(z) \).

When applied to the case \( k = 1 \), Theorem 1.2 readily yields a rather tractable expression for the capacity achieving \( B \) and gives a simple and explicit formula for \( C_{FB} \), as detailed in the following theorem.

**Theorem 1.3** (Theorem 5.3 in [1]). Suppose the noise process \( \{Z_i\} \) is an ARMA(1) Gaussian process with parameters \( \alpha \) and \( \beta \), \( |\alpha| < 1, |\beta| < 1 \). Then, the Gaussian feedback capacity is given by

\[ C_{FB} = -\frac{1}{2} \log x^2, \]  

where \( x \) is the unique root of the following fourth-order polynomial

\[ P_x^2 = \frac{(1 - x^2)(1 + \alpha x)^2}{(1 + \beta x)^2}, \]  

satisfying

\[ x \in \begin{cases} 
(-1, 0) & \text{if } \alpha \geq \beta, \\
(0, 1) & \text{if } \alpha < \beta.
\end{cases} \]  

We now digress a bit to briefly mention related results on the ARMA(1) Gaussian feedback capacity in the literature: Generalizing the celebrated Schalkwijk-Kailath scheme [13], [14], Butman [15] obtained a lower bound of the feedback capacity of AR(1) channel (a special ARMA(1) channel with \( \alpha = 0 \)). Butman’s bound was shown to be optimal under some cases of linear feedback schemes by Wolfowitz [16] and Tiernan [17]. Tiernan and Schalkwijk [18] also found an upper bound of AR(1) Gaussian channel capacity, which is equal to Butman’s lower bound at very low and very high signal-to-noise ratio. It was shown [19] that Butman’s lower bound is indeed the capacity, and the capacity of MA(1) channel (a special ARMA(1) channel with \( \beta = 0 \)) was also derived in the same paper. More recently, Yang, Kavčić and Tatikonda [20] studied the ARMA(1) Gaussian channel by analyzing the structure of the optimal input distribution and reformulating the problem as a stochastic control optimization problem. And based on a speculation of the limiting behavior of the optimal input distribution, they derived the formula (8) and conjectured that it gives the ARMA(1) Gaussian feedback capacity.

As mentioned above, the power of the variational formulation as in Theorem 1.1 has been showcased in Theorem 1.3, where the conjecture of [20] has been confirmed and the ARMA(1) Gaussian feedback capacity is given as an explicit and simple formula. To the best of our knowledge, the ARMA(1) Gaussian feedback channel is the only non-trivial scenario whose Gaussian feedback capacity is “computable”. The success by the variational formulation approach, contrasted by all the above-mentioned other approaches that have been struggling dealing with special cases of an ARMA(1) channel, naturally posed the question of whether it can be extended to deal with more general channels, for instance, ARMA(1) Gaussian feedback channels. Attempts in this direction, however, have somehow encountered certain technical barriers, due to the fact that the form in (7) is “less manageable” (see Page 78
in [1]). As a matter of fact, instead of following the variational formulation framework, an alternative state-space representation approach has been proposed in [1] to deal with the ARMA($k$) Gaussian feedback capacity, only to yield an intractable optimization problem (see Theorem 6.1 in [1]). Here we remark that prior to [1], a result of similar nature has also been derived in Theorem 6 of [20], which however appears to be at least as uncanny.

In this paper, we will naturally extend Theorem 1.3 and derive a computable formula for the ARMA($k$) Gaussian feedback capacity as a simple function evaluated at a solution to a system of equations which proves to have only finitely many solutions for at least some small $k$. Our starting point is precisely Theorem 1.1, but we choose a slightly different direction afterwards to obtain Theorem 2.3, a “more manageable” version of Theorem 1.2 and a natural extension to Theorem 1.3 combined. Instead of considering the filter $B(e^{i\theta})$, we use the “dull” method of “change of variables” and consider instead $C(e^{i\theta}) \triangleq B(e^{i\theta})HZ(e^{i\theta});$ (11) here we note that since $B(e^{i\theta})$ is strictly causal and $HZ(e^{i\theta}) \subset H_2$, it is obvious that $C(e^{i\theta})$ is also strictly causal, and thereby can be written as $C(e^{i\theta}) = \sum_{k=1}^{\infty} c_ke^{ik\theta}$ for some $c_1, c_2, \ldots \in \mathbb{R}$. Apparently, (11) can be used to reformulate other quantities, such as the output PSD $S_Y(e^{i\theta}) = |C(e^{i\theta}) + H(e^{i\theta})|^2$,

and eventually reformulate Theorem 1.1 as follows:

**Theorem 1.4** (Theorem 4.1 of [1] reformulated). Suppose that the power spectral density (PSD) $S_Z(e^{i\theta})$ of the Gaussian noise process $\{Z_i\}_{i=1}^\infty$ is bounded away from 0, and has a canonical spectral factorization $S_Z(e^{i\theta}) = |HZ(e^{i\theta})|^2$, where $HZ(e^{i\theta}) \subset H_2$. Then the feedback capacity $C_{FB}$ is given by

$$C_{FB} = \max_C \frac{1}{2} \int_{-\pi}^{\pi} \log |C(e^{i\theta}) + H(e^{i\theta})|^2 \frac{d\theta}{2\pi};$$ (12)

where the maximum is taken over all strictly causal $C(e^{i\theta})$ satisfying the power constraint

$$\int_{-\pi}^{\pi} |C(e^{i\theta})|^2 \frac{d\theta}{2\pi} \leq P.$$ (13)

Furthermore, a $C^*(e^{i\theta})$ attains the maximum in (12) if and only if

i) Power:

$$\int_{-\pi}^{\pi} |C^*(e^{i\theta})|^2 \frac{d\theta}{2\pi} = P;$$ (14)

ii) Output spectrum:

$$\eta := \inf_{\theta \in [-\pi, \pi]} \max_{\theta} |C^*(e^{i\theta}) + H(e^{i\theta})| > 0;$$ (15)

iii) Strong orthogonality: For some $0 < \lambda \leq \eta$

$$\frac{\lambda}{(C^*(e^{i\theta}) + H(e^{i\theta})) - C^*(e^{-i\theta})}$$ (16)

is causal.
Therefore, for the optimal solution $C$, we have

\[ S_Y(e^{i\theta}) = S_Y^{**}(e^{i\theta}). \]

**Proof.** We first prove that for any $C(e^{i\theta})$ satisfying (13),

\[ \int_{-\pi}^{\pi} \frac{|C(e^{i\theta}) + H_Z(e^{i\theta})|^2}{|C^*(e^{i\theta}) + H_Z(e^{i\theta})|^2} d\theta \leq 1. \]

To see this, note that

\[
\int_{-\pi}^{\pi} \frac{|C + H_Z|^2}{|C^* + H_Z|^2} \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} \frac{|C + H_Z + C - C^*|^2}{|C^* + H_Z|^2} \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} \frac{|C^* + H_Z|^2 + |C - C^*|^2 + 2(C^* + H_Z)(C - C^*)}{|C^* + H_Z|^2} \frac{d\theta}{2\pi} = 1 + \int_{-\pi}^{\pi} \frac{|C - C^*|^2}{|C^* + H_Z|^2} d\theta + 2 \int_{-\pi}^{\pi} \frac{C}{C^* + H_Z} d\theta - 2 \int_{-\pi}^{\pi} \frac{C^*}{C^* + H_Z} d\theta.
\]

Note that by (16), we have for almost all $\theta$,

\[ |C^* + H_Z|^2 \geq \lambda, \]

and

\[ \int_{-\pi}^{\pi} \left( \frac{1}{C^* + H_Z} - \frac{C^*}{\lambda} \right) C^* d\theta = 0, \quad \int_{-\pi}^{\pi} \left( \frac{1}{C^* + H_Z} - \frac{C^*}{\lambda} \right) C d\theta = 0. \]

It then follows that for any $C(e^{i\theta})$ satisfying (13), we have

\[
\int_{-\pi}^{\pi} \frac{|C + H_Z|^2}{|C^* + H_Z|^2} \frac{d\theta}{2\pi} \leq 1 + \frac{1}{\lambda} \int_{-\pi}^{\pi} |C - C^*|^2 \frac{d\theta}{2\pi} - \frac{2P}{\lambda} + \frac{2}{\lambda} \int_{-\pi}^{\pi} C C^* \frac{d\theta}{2\pi} = 1 + \frac{1}{\lambda} \int_{-\pi}^{\pi} |C|^2 \frac{d\theta}{2\pi} + \frac{1}{\lambda} \int_{-\pi}^{\pi} |C^*|^2 \frac{d\theta}{2\pi} - \frac{2}{\lambda} \int_{-\pi}^{\pi} C C^* \frac{d\theta}{2\pi} + \frac{2P}{\lambda} + \frac{2}{\lambda} \int_{-\pi}^{\pi} C C^* \frac{d\theta}{2\pi} \leq 1.
\]

Therefore, for the optimal solution $C^{**}$, we have

\[
\int_{-\pi}^{\pi} \log S_Y^{**} \frac{d\theta}{2\pi} \leq \int_{-\pi}^{\pi} \frac{S_Y^{**}}{S_Y} - 1 + \log S_Y \frac{d\theta}{2\pi} \leq \int_{-\pi}^{\pi} \log S_Y \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} \log S_Y^{**} \frac{d\theta}{2\pi},
\]
which in turns implies that
\[
\int_{-\pi}^{\pi} \log \frac{S^{**}_Y}{S^*_Y} \frac{d\theta}{2\pi} = \int_{-\pi}^{\pi} \left( \frac{S^{**}_Y}{S^*_Y} - 1 \right) \frac{d\theta}{2\pi}.
\]
Noting that for all \(\theta\),
\[
\log \frac{S^{**}_Y}{S^*_Y} \leq \frac{S^{**}_Y}{S^*_Y} - 1,
\]
we conclude that for almost all \(\theta\),
\[
\frac{S^{**}_Y}{S^*_Y} = 1.
\]
The proof of the theorem is then complete.

The following lemma will be used in the proof of Theorem 2.3, whose proof closely follows that of Proposition 4.2 in [1] and is included for completeness.

**Lemma 2.2.** If \(C^*\) is optimal, then \(\overline{C^*}(C^* + H_Z)\) is causal.

**Proof.** Suppose, by way of contradiction, that \(\overline{C^*}(C^* + H_Z)\) is not causal, then for some \(n \geq 1\), we have
\[
\int_{-\pi}^{\pi} \overline{C^*}(C^* + H_Z)e^{in\theta} \frac{d\theta}{2\pi} = \gamma \neq 0.
\]
Let \(A(e^{i\theta}) = xe^{in\theta}\) with \(|x| < 1\). Then, for \(C^{**} \triangleq (1 + A)(C^* + H_Z) - H_Z\), one verifies that it is also strictly causal, and furthermore,
\[
\log S^{**}_Y = \log |C^{**} + H_Z|^2 = \log |1 + A|^2|C^* + H_Z|^2 = \log |1 + A|^2S^*_Y.
\]
By Jensen’s formula, the entropy rate of \(S^{**}_Y\) is the same as that of \(S^*_Y\). On the other hand, the power of \(C^*\) can be computed as follows:
\[
P^{**}(x) = \int_{-\pi}^{\pi} |C^{**}|^2 \frac{d\theta}{2\pi}
= \int_{-\pi}^{\pi} |C^* + A(C^* + H_Z)|^2 \frac{d\theta}{2\pi}
= \int_{-\pi}^{\pi} |C^*|^2 + 2 \int AC^*(C^* + H_Z) + \int_{-\pi}^{\pi} |A|^2|C^* + H_Z|^2 \frac{d\theta}{2\pi}
= P + 2\gamma x + P_Y x^2,
\]
where \(P_Y = \int S^*_Y > 0\). Therefore, we can choose certain \(x\) such that \(P^{**}(x) < P\), i.e., we can achieve same information rate using less power, which is contradictory to Condition i) of Theorem 1.4.

We are now ready to state the main results of this paper.
Theorem 2.3. Suppose the noise \( \{ Z_i \} \) is not white with the power spectral density \( S_Z(e^{i\theta}) \) taking the form as in (6). Then, the feedback capacity \( C_{FB} \) can be achieved by \( C(z) \) taking the following form:

\[
C(z) = \sum_{i=1}^{l} \sum_{j=1}^{m_i} \frac{y_{ij} z^j}{(1 - x_i z)^j},
\]

(17)

where \( m_i \) are positive integers for all \( i = 1, 2, \ldots, l \) and \( \sum_{i=1}^{l} m_i \leq k \), \( x_i \in \mathbb{C} \) are all distinct and \( |x_i| < 1 \) for all \( i = 1, 2, \ldots, l \), \( y_{ij} \in \mathbb{C} \) for all \( i \) and \( j \). Furthermore, \( C(z) \) is optimal yielding the capacity

\[
C_{FB} = -\log \prod_{i=1}^{l} |x_i|^{m_i}
\]

(18)

if and only if all \( x_i \), \( m_i \) and \( y_{ij} \) satisfy the following four conditions:

i) Power:

\[
\sum_{i=1}^{l} \sum_{j=1}^{m_i} \sum_{p=1}^{l} \sum_{q=1}^{m_p} \frac{y_{ij} y_{pq} \left( z^{j-1} \right)}{(1 - x_i z)^j} \bigg|_{z=x_p} = P,
\]

where, as elsewhere in this paper, the parenthesized superscript means the derivative with respect to \( z \);

ii) Roots: \( x_1, x_2, \ldots, x_l \) are the roots of the function

\[
f(z) \triangleq \sum_{i=1}^{l} \sum_{j=1}^{m_i} \frac{y_{ij} z^j}{(1 - x_i z)^j} + \frac{\prod_{i=1}^{k} (1 + \alpha_i z)}{\prod_{i=1}^{k} (1 + \beta_i z)}
\]

that are strictly inside the unit circle, while the other roots \( r_1^{-1}, r_2^{-1}, \ldots, r_k^{-1} \) are all strictly outside the unit circle;

iii) Strong orthogonality: there exists a real number \( \lambda > 0 \) such that for all \( i = 1, 2, \ldots, l \) and \( j = 1, 2, \ldots, m_i \),

\[
h_{ij}(x_i) = \lambda y_{ij}(j-1)!,
\]

where

\[
h_{ij}(z) \triangleq \sum_{p=j}^{m_i} \frac{C_{p-1}^{j-1}}{(p-1)!(m_i-p)!} \left( \frac{(z - x_i)^{m_i}}{\prod_{s=1}^{l} (z - x_s)^{m_s}} \right)^{(m_i-p)} \times \left( \frac{\prod_{s=1}^{k} (1 - x_s z)^{m_s} (-x_s)^{m_s}}{\prod_{t=1}^{k} (1 - \beta_t z)} \right)^{(p-j)};
\]

iv) Output spectrum: For almost all \( \theta \in [-\pi, \pi) \),

\[
\lambda \geq \frac{1}{S_{Y,e^{i\theta}}^*} = \prod_{j=1}^{l} |x_j|^{2m_j} \frac{\prod_{t=1}^{k} (1 + \beta_t e^{i\theta})}{\prod_{t=1}^{k} (1 - r_t e^{i\theta})}.
\]
Proof. Through a similar argument as in the proof of Theorem 1.2, we first show that any capacity achieving \( C^*(z) \triangleq \sum_{k=1}^{\infty} c_k z^k \) must take the form in (17). First of all, we consider \( S \triangleq |Q|^2 S^*_Y \), which, by straightforward computations, can be rewritten as follows:

\[
S = |Q|^2 |C^* + H_Z|^2 \\
= |Q|^2 C^*(C^* + H_Z) + |Q|^2 \overline{T}_Z (C^* + H_Z) \\
= |Q|^2 C^*(C^* + H_Z) + \overline{T}_Q C^* + |P|^2.
\]

It then follows from Lemma 2.2 and the fact that \( P \) and \( Q \) are both polynomials of degree at most \( k \) that \( S \) must be of the following form:

\[
S(z) = s_{-k} z^{-k} + s_{-k+1} z^{-k+1} + \cdots.
\]

Then, by the fact that \( S = \overline{S} \), we deduce that \( S(z) \) can be written as

\[
S(z) = s_{-k} z^{-k} + s_{-k+1} z^{-k+1} + \cdots + s_{-k+1} z^{k-1} + s_{-k} z^k.
\]

Apparently \( S(z) \) satisfies the Paley-Wiener condition, which means it can be written as

\[
S(z) = \sigma^2 |R(z)|^2
\]

where \( \sigma \) is a positive constant and \( R(z) \) is a \( k \)-th order stable polynomial with \( R(0) = 1 \). It then immediately follows that

\[
|C^* + H_Z|^2 = \sigma^2 \frac{|R|^2}{|Q|^2},
\]

which, together with the canonical representation theorem (see Theorem 2.8 in [12]), implies that \( C^* + H_Z \) must have the following form:

\[
C^*(z) + H_Z(z) = \frac{\prod_{i=1}^{\infty} (1 - x_i^{-1} z) R(z)}{\prod_{i=1}^{\infty} (1 - x_i z) Q(z)}
\]

for some complex numbers \( x_1, x_2, \ldots \) with \( |x_j| < 1 \) for all \( j \) and \( \prod_j (1/|x_j|^2) = \sigma^2 \). By Condition iii) of Theorem 1.4,

\[
\frac{1}{C^* + H_Z} - \lambda C^* = \frac{1 - \lambda S^*_Y + \lambda \overline{H}_Z (C^* + H_Z)}{C^* + H_Z}
\]

is causal, which, together with the fact that \( C^*(z) + H_Z(z) \) has the factor of \( A(z) \triangleq \prod_{i=1}^{\infty} (1 - u_i^{-1} z) \) (for this, see [19]), implies that \( 1 - \lambda S^*_Y \) must also have the factor \( A(z) \). By symmetry, \( 1 - \lambda S^*_Y \) must also have the factor \( A(z^{-1}) \). Since \( 1 - \lambda S^*_Y \) is a polynomial with degree at most \( 2k \), we conclude that

\[
C^*(z) + H_Z(z) = \frac{\prod_{i=1}^{l} (1 - x_i^{-1} z)^{m_i} R(z)}{\prod_{i=1}^{l} (1 - x_i z)^{m_i} Q(z)}
\]

where all \( x_i \) are distinct with \( |x_i| < 1 \), all \( m_i \) are positive integers with \( \sum_{i=1}^{l} m_i \leq k \).
The causality of
\[ \frac{1}{C^* + H_Z} - \lambda \overline{C^*} \]
implies that for any \( k = 1, 2, \ldots \),
\[ \int_{-\pi}^{\pi} \left( \frac{1}{C^*(e^{i\theta})} + H_Z(e^{i\theta}) \right) e^{ik\theta} \frac{d\theta}{2\pi} = 0, \]
which, together with (21), yields
\[ \int_{-\pi}^{\pi} e^{ik\theta} \frac{\prod_{i=1}^{l}(1 - x_i e^{i\theta})^{m_i} Q(e^{i\theta})}{\prod_{i=1}^{l}(1 - x_i^{-1} e^{i\theta})^{m_i} R(e^{i\theta})} \frac{d\theta}{2\pi} = \lambda c^*_k. \]
Rewriting the above integral as a line integral, we have
\[ \oint_{\gamma} z^{k-1} \frac{\prod_{i=1}^{l}(1 - x_i z)^{m_i} Q(z)}{\prod_{i=1}^{l}(1 - x_i^{-1} z)^{m_i} R(z)} \frac{dz}{2\pi i} = \lambda c^*_k, \]
where \( \gamma \) is the unit circle. Denote
\[ h(z) \triangleq \frac{\prod_{i=1}^{l}(1 - x_i z)^{m_i} (-x_i)^{m_i} Q(z)}{R(z)}. \]
It’s easy to check that \( h(z) \) is an analytic function on the unit disk since \( R(z) \) is stable. Via the Heaviside cover-up method, the integrand of the LHS of (22) can be decomposed as
\[ z^{k-1} \sum_{i=1}^{l} \sum_{j=1}^{m_i} \frac{\tilde{h}_{ij}(z)}{(z - x_i)^{j}}, \]
where \( \tilde{h}_{ij}(z) = a_{ij} h(z) \) and
\[ a_{ij} = \frac{1}{(m_i - j)!} \left( \frac{(z - x_i)^{m_i}}{\prod_{s=1}^{l}(z - x_s)^{m_s}} \right)^{(m_i - j)} \bigg|_{z=x_i} \]
is a constant depending on \( x_i \) and \( m_i \). Thus \( \tilde{h}_{ij}(z) \) is also an analytic function on the unit disk for all \( i, j \). Applying Cauchy’s integral formula, we deduce that for any \( k \),
\[ \sum_{i=1}^{l} \sum_{j=1}^{m_i} \left( \tilde{h}_{ij}(z) z^{k-1} \right)^{(j-1)} \bigg|_{z=x_i} = \lambda c^*_k, \]
or equivalently,
\[ \sum_{i=1}^{l} \sum_{j=1}^{m_i} \sum_{p=1}^{\min\{j,k\}} C_{j-1}^{p-1} a_{ij}(h(z))^{(j-p)} (z^{k-1})^{(p-1)} \bigg|_{z=x_i} = \lambda c^*_k. \]
Hence, each $c_k^\star$ takes the following form

$$
= \sum_{i=1}^l \sum_{j=1}^{\min\{m_i,k\}} \tilde{y}_{ij}(k - 1) \cdots (k - j + 1) x_i^{k-j},
$$

(24)

where $\tilde{y}_{ij}$ is a constant independent of $k$, which immediately implies that

$$
C^\star(z) = \sum_{k=1}^\infty c_k^\star z^k
= \sum_{k=1}^\infty \sum_{i=1}^l \sum_{j=1}^{\min\{m_i,k\}} \tilde{y}_{ij}(k - 1) \cdots (k - j + 1) x_i^{k-j} z^k
= \sum_{i=1}^l \sum_{j=1}^{\min\{m_i,k\}} \sum_{\gamma=0}^j \frac{y_{ij} z^j}{(1-x_i z)^j},
$$

(25)

where $y_{ij} \triangleq \tilde{y}_{ij}/(j-1)!$.

We next prove that Conditions i)-iv) are necessary and sufficient for the optimality of $C^\star(z)$, which, given (25), readily follows from Theorem 1.1 and some technical computations.

First of all, Condition i) follows from (25) and Condition i) in Theorem 1.1
Second, it follows from (21) and (25) that
\[
C^*(z) + H_Z(z) = \prod_{i=1}^l (1 - x_i^{-1}z)^{m_i} R(z) \bigg/ \prod_{i=1}^l (1 - x_i z)^{m_i} Q(z) \\
= \sum_{i=1}^l \sum_{j=1}^{m_i} \frac{y_{ij} z^j}{(1 - x_i z)^j} + \frac{P(z)}{Q(z)},
\]
which immediately implies Condition ii).

Condition iii) follows from the fact that the coefficients of each $x_i^{k-j}$ at both sides of (22) are equal. More precisely, by (24), the coefficient of $x_i^{k-j}$ on the right hand side is $(j - 1)! (k - 1) \cdots (k - j + 1) y_{ij}$. On the other hand, via (23), the coefficient of $x_i^{k-j}$ on the left hand side of (22) is as follows:
\[
(k - 1) \cdots (k - j + 1) \sum_{p=j}^{m_i} \frac{C_{p-1}^{j} a_{ip} (h(z))^{(p-j)}}{c^{p-j}} \bigg|_{z=x_i}.
\]
Condition iii) then immediately follows.

Last, Condition iv) follows from Condition iii) of Theorem 1.1 and some technical computations.

Finally, noting the uniqueness of the output PSD $S^*_Y$ corresponding the optimal $C^*(z)$ (Theorem 2.1) and applying Jensen’s formula, we obtain
\[
C_{FB} = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\log S^*_Y(e^{i\theta})}{2\pi} d\theta \\
= \frac{1}{2} \int_{-\pi}^{\pi} \log \prod_{j=1}^l x_j^{-2m_j} \left| \frac{\prod_{t=1}^k (1 - r_t e^{i\theta})}{\prod_{t=1}^k (1 + \beta_t e^{i\theta})} \right|^2 d\theta \\
= - \log \prod_{i=1}^l |x_i|^{m_i}.
\]
The proof of Theorem 2.3 is then complete.

Remark 2.4. By Theorem 2.3, to compute the ARMA($k$) Gaussian feedback capacity, one needs to first find a solution to one of the following systems of rational equations: for some positive $m_1, m_2, \ldots, m_l$ with $\sum_{j=1}^l m_i \leq k$,
\[
\begin{cases}
\sum_{i=1}^l \sum_{j=1}^{m_i} \sum_{p=1}^l \sum_{q=1}^{m_p} y_{ij} y_{pq} \left( \frac{z^{j-1} - 1}{1 - x_i z} \right) \left( \frac{z^{q-1} - 1}{1 - x_i z} \right) = P, \\
\text{f}(x_j) = 0, \quad j = 1, 2, \ldots, l \\
y_{ij} h_{11}(x_1)(j - 1)! = y_{11} h_{ij}(x_i), \quad i = 1, 2, \ldots, l, \quad j = 1, 2, \ldots, m_i,
\end{cases}
\]
such that $|x_i| < 1$ for all $i$ and it also satisfies Condition iv) in Theorem 2.3 to compute the capacity with (18).
3 Examples and Numerical Results

In this section, we give a couple of examples and some numerical results.

**Example 3.1.** When \( k = 1 \), both \( l \) and \( m_1 \) are necessarily 1, and the corresponding system of equations is:

\[
\begin{align*}
\frac{y_{11}}{1-x_1^2} &= P, \\
\frac{y_{11}x_1}{1-x_1^2} + \frac{(1+\alpha_1x_1)(1+\alpha_2x_1)}{(1+\beta_1x_1)(1+\beta_2x_1)} &= 0,
\end{align*}
\]

which immediately gives rise to (9). An elementary analysis (see, e.g., [1] or [21]) will show that Condition iv) of Theorem 2.3 translates to (10), an extra condition \( x_1 \) has to satisfy. It turns out that for this case, \( x_1 \) is unique, which, by (18), yields

\[C_{FB} = -\log |x_1|.
\]

So, Theorem 2.3 recovers Theorem 1.3 as a special case.

**Example 3.2.** When \( k = 2 \), by Theorem 2.3, we have three cases to deal with:

1. \( l = 1 \) and \( m_1 = 1 \): We need to find \( |x_1| < 1 \), \( y_{11} \neq 0 \) such that

\[
\begin{align*}
\frac{y_{11}^2}{1-x_1^2} &= P, \\
\frac{y_{11}x_1}{1-x_1^2} + \frac{(1+\alpha_1x_1)(1+\alpha_2x_1)}{(1+\beta_1x_1)(1+\beta_2x_1)} &= 0,
\end{align*}
\]

and for all \( \theta \in [-\pi, \pi) \),

\[
\frac{x_1(1+\beta_1x_1)(1+\beta_2x_1)(x_1^2-1)}{y_{11}(1-r_1x_1)(1-r_2x_1)} \geq x_1 \left| \frac{(1+\beta_1e^{i\theta})(1+\beta_2e^{i\theta})}{(1-r_1e^{i\theta})(1-r_2e^{i\theta})} \right|^2,
\]

where \( r_1 + r_2 = x_1 - x_1^{-1} - \alpha_1 - \alpha_2 - y_{11} \) and \( r_1r_2 = \alpha_1\alpha_2x_1^2 - \beta_1\beta_2x_1y_{11} \). If such \( x_1 \) exists, we have

\[C_{FB} = -\log |x_1|.
\]

2. \( l = 1 \) and \( m_1 = 2 \): We need to find \( |x_1| < 1 \) and \( y_{11}, y_{12} \neq 0 \) such that

\[
\begin{align*}
\frac{y_{11}^2}{1-x_1^4} + \frac{y_{12}^2}{(1-x_1^2)^2} + \frac{2y_{11}y_{12}x_1}{(1-x_1^2)^2} &= P, \\
\frac{y_{11}x_1}{1-x_1^4} + \frac{y_{12}x_1}{(1-x_1^2)^2} + \frac{(1+\alpha_1x_1)(1+\alpha_2x_1)}{(1+\beta_1x_1)(1+\beta_2x_1)} &= 0,
\end{align*}
\]

and for all \( \theta \in [-\pi, \pi) \),

\[
w(x_1) \geq y_{12}^4 \left| \frac{(1+\beta_1e^{i\theta})(1+\beta_2e^{i\theta})}{(1-r_1e^{i\theta})(1-r_2e^{i\theta})} \right|^2,
\]

where

\[
w(z) \Delta \frac{x_1^2(1-x_1z)^2(1+\beta_1z)(1+\beta_2z)}{(1-r_1z)(1-r_2z)},
\]

and \( r_1 + r_2 = 2x_1 - 2x_1^{-1} - \alpha_1 - \alpha_2 - y_{11} \) and \( r_1r_2 = \alpha_1\alpha_2x_1^4 - \beta_1\beta_2x_1^2y_{11} - \beta_1\beta_2x_1^2y_{12} \). If such \( x_1, y_{11}, y_{12} \) exist, then we have

\[C_{FB} = -\log |x_1|^2.
\]
3. \( l = 2 \) and \( m_1 = 1, m_2 = 1 \): We need to find distinct \(|x_1|, |x_2| < 1\) and \(y_{11}, y_{21} \neq 0\) such that
\[
\begin{align*}
\frac{y_{11}^2}{1-x_1^2} + \frac{y_{21}^2}{1-x_2^2} + 2y_{11}y_{21} & = P, \\
y_{11}x_1 + y_{21}x_2 & = 0, \\
y_{11}x_1^2 + y_{21}x_2^2 & = 0, \\
y_{11}(1-r_1 x_1)(1-r_2 x_2) & = \frac{-y_{21}(1-r_1 x_2)(1-r_2 x_2)}{(1+r_1 x_2)(1+r_2 x_2)(1-x_2)},
\end{align*}
\]
and for all \( \theta \in [-\pi, \pi) \),
\[
\frac{x_1 x_2 (1+\beta_1 x_1) (1+\beta_2 x_2) (1-x_1^2) (1-x_2^2)}{(x_1 - x_2) y_{11} (1-r_1 x_1) (1-r_2 x_2)} \geq x_1^2 x_2^2 \left| \frac{(1+\beta_1 e^{i\theta})(1+\beta_2 e^{i\theta})}{(1-r_1 e^{i\theta})(1-r_2 e^{i\theta})} \right|^2
\]
where \( r_1 + r_2 = x_1 + x_2 - x_1^{-1} - x_2^{-1} - \alpha_1 - \alpha_2 - y_{11} - y_{21} \) and \( r_1 r_2 = \alpha_1 \alpha_2 x_1^2 x_2^2 - \beta_1 \beta_2 x_1^2 x_2 y_{21} - \beta_1 \beta_2 x_1 x_2^2 y_{11} \). If such \( x_1, x_2, y_{11}, y_{21} \) exist, then we have
\[
C_{FB} = -\log |x_1 x_2|.
\]

Complicated as they may look, the systems of equations in (28), (29) and (30) all have finitely many solutions and therefore can be solved numerically. Below, fixing \( P = 1 \) and \( \beta_2 = 0 \), assuming different values for \( \alpha_2, \beta_1 \), we have plotted the values of \( C_{FB} \) against the values of \( \alpha_1 \).

4 Concluding Remarks and Future Work

We have expressed the ARMA\( (k) \) Gaussian feedback capacity as a simple function evaluated at a solution to the rational system (27). Though solving a general polynomial system can be extremely intricate [22], it turns out that the system has only finitely many solutions for \( k = 1, 2 \), which has helped us to recover the known ARMA(1) and obtain the new ARMA(2) Gaussian feedback capacity. Preliminary computations suggests the conjecture that for any \( k \), the system (27) always has finitely many solutions, which would greatly facilitates solving the system (27) and thereby the computation of the more general ARMR\( (k) \) Gaussian feedback capacity.

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Figure 1: Plot of $C_{FB}$ as a function of $\alpha_1$ when $\alpha_2 = 0.1$

Figure 2: Plot of $C_{FB}$ as a function of $\alpha_1$ when $\alpha_2 = 0.4$

Figure 3: Plot of $C_{FB}$ as a function of $\alpha_1$ when $\alpha_2 = 0.7$
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