Sphere branched coverings and the growth rate inequality

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Abstract
We show that the growth rate inequality \( \limsup \frac{1}{n} \log(\#\text{Fix}(f^n)) \geq \log d \) holds for branched coverings of degree \( d \) of the sphere \( S^2 \) having a completely invariant simply connected region \( R \) with locally connected boundary, except in some degenerate cases with known counterexamples.

Keywords: low dimensional dynamics, growth rate inequality, Shub’s conjecture, surface endomorphisms, branched coverings, topological dynamics

1. Introduction

This paper deals with the following open problem: let \( f : S^2 \to S^2 \) be a continuous map of degree \( d, \ |d| > 1 \), and let \( N_0f \) denote the number of fixed points of \( f^n \). When does the growth rate inequality \( \limsup \frac{1}{n} \log N_0f \geq \log d \) hold for \( f \)? (This is problem 3 posed in [S].)

It is known that this inequality does not always hold. The simplest example is the map expressed in polar coordinates in \( \mathbb{R}^2 \) as \( (r, \theta) \to (dr, d\theta) \) and extended to the sphere with \( \infty \to \infty \). It has degree \( d \) and just two periodic points. In [IPRX2] other examples are presented, where the nonwandering set is not reduced to the set of periodic points. On the other hand, the inequality is known to hold if \( f \) is a rational map [S2], if \( f \) is \( C^1 \) and preserves the latitude foliation [PS] and [Mis], if the critical points form a two-periodic cycle [IPRX3], and if all periodic orbits are isolated as invariant sets and \( f \) has no sources of degree \( r, \ |r| \geq 1 \) [HR]. Whenever the growth rate inequality holds, we say that \( f \ has the rate.\)

We will work with branched coverings and make the following assumption: there exists a simply connected and completely invariant region \( \bar{R} \), proper subset of the sphere \( S^2 \). These two assumptions (branched covering + completely invariant simply connected region) are strong assumptions, but there will be more, since many examples of maps not having the rate satisfy

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these two assumptions. These examples also satisfy that there are exactly two fixed critical points of multiplicity $d - 1$, one in $R$ and the other one in the boundary of $R$. It follows that $f$ can be thought as a covering map of the open annulus $\mathbb{R}^2 \setminus \{0\}$. Related work in the annulus was developed in the sequel [IPRX1], [IPRX2] and [IPRX3], where other various sets of sufficient conditions for a map to have the rate are given.

A branched covering of the sphere is a continuous map and a local homeomorphism except at a finite critical set $C$. If $c \in C$ then there exists $k > 1$ such that $f$ is locally geometrically conjugate to $p_k$, where $p_k(z) = z^k$. This means that there are local (possibly different) homeomorphisms $\phi$ and $\psi$ such that $\psi f = p_k \phi$. The multiplicity of the critical point is $k - 1$ and the Riemann–Hurwitz formula (valid for general branched coverings of surfaces) states that the sum of the multiplicities of critical points is equal to $2d - 2$, where $d$ is the degree of $f$ (see, for example [Mi]). Furthermore, if a region $R$ is completely invariant and simply connected, then it contains exactly $d - 1$ critical points (counted with multiplicities).

As pointed out before, having exactly one fixed critical point of multiplicity $d - 1$ in the boundary of $R$ may be an obstruction to having the rate. In the case that the boundary of $R$ is locally connected, we show that this is the only one:

**Theorem A.** Let $f$ be a degree $d$ branched covering of the sphere, where $|d| > 1$. Assume that there exists a completely invariant simply connected region $R$ whose boundary is locally connected. Assume moreover that it is not the case that there exists only one critical point in the boundary of $R$ that has multiplicity $d - 1$ and is fixed by $f$. Then $f$ has the rate.

We do not know if the hypothesis on local connectivity is necessary. A main ingredient in the proof of theorem A is theorem 1, that states that $f$ extends continuously to the prime end closure of $R$. This may be a well known result, but we include here a simple proof of a general version not found in our revision of the bibliography. This extension of $f$ induces a map $\tilde{f}$ of the circle that turns to be a degree $d$ covering, despite of the fact that $f$ could have critical points on the boundary of the region $R$. It comes that for every positive $n$, $f^n$ has at least $|d^n - 1|$ fixed points.

As the boundary of $R$ is locally connected, to each periodic point of $\tilde{f}$ corresponds a periodic point of $f$ in the boundary of $R$. However, this correspondence is not injective, so in order to get the rate one has to understand how many different $\tilde{f}$-periodic prime ends correspond to the same point in the boundary of $R$. External rays are simple arcs contained in $R$ having an extreme point in the boundary of $R$, called the landing point of the ray. Of course the use of external rays is a useful technique since the works of Thurston, Douady, Hubbard, Yoccoz and many others since early 80s, and has provided new insights in the study of complex polynomials, or polynomial-like maps. Although we do not necessarily have periodic external rays (the map $f$ is not conjugate to $z \to z^d$ in the region $R$, as occurs for complex polynomial maps) the use of rays landing at the boundary of $R$ will provide the technique to separate regions and prevent the multiplication of different periodic prime ends corresponding to the same point in the boundary of $R$. To formalize this we define an equivalence relation in $S^1$, where points are equivalent if correspond to the same periodic point of $f$ (section 2.2); the goal is to conclude that the number of equivalence classes satisfies the inequality of the rate. Then, in section 3.1 we construct stars, related to the relative positions of the critical points that belong to the complement of $R$. This construction ressembles the so called critical portraits that were used to classify post-critically finite polynomials, see [Poi].

An example to have in mind is when $f$ is a complex polynomial with connected and locally connected Julia set. Then $f$ has a superattracting fixed point at infinity and the region $R$ is its basin of attraction, which is the complement of the filled Julia set. Figure 1 shows different periodic rays landing at the same point, namely the periodic orbit $1/7 \to 2/7 \to 4/7$ is reduced.
Figure 1. Julia set for $f(z) = z^2 - 0.110 + 0.6557i$.

to a point in the boundary of $R$. More figures illustrating this phenomenon can be found in chapter 18 in [Mi].

So, in particular, theorem A is a topological version of the fact that complex polynomials (with connected and locally connected Julia set) have the rate. Following the polynomial analogy, the opposite situation corresponds to the case when all critical points belong to the complement of the filled Julia set. We also give a topological version of the fact that such maps have the rate:

**Theorem B.** Let $f$ be a degree $d$ branched covering of the sphere, where $|d| > 1$. Assume that there exists a simply connected open set $U$ whose closure is disjoint from the set of critical values and such that $f^{-1}(U) \subset U$. Then $f$ has the rate.

2. Equivalence of $m_d$-periodic points

This section begins with the proof that a branched covering of an open disc $R$ that continuously extends to the boundary of $R$ extends to the prime ends closure of $R$ and induces a covering on prime ends. This result will be used to establish a relation between the periodic points of $f$ and those of $m_d$, the map defined in the unit circle as $m_d(z) = z^d$.

2.1. Maps induced in the circle

In this section no dynamics are involved. It contains results for coverings that are related to the well known Caratheodory’s theorem of extension of a homeomorphism of a simply connected plane region $R$ to its prime ends closure (the reader unfamiliar with prime end theory can refer
to [?]). There are no assumptions on the structure of the boundary of $R$. We will show that a branched covering also extends to the prime ends closure of $R$. We recall some definitions first. A crosscut in $R$ is a simple arc whose interior is contained in $R$ and whose extreme points belong to $K$, the boundary of $R$. Fix a point $0$ in $R$. Given a crosscut $c$ not containing $0$ define $N(c)$ as the component of $R \setminus c$ not containing $0$. A sequence of crosscuts $\{c_n\}$ is admissible if its lengths tend to zero and $N(c_{n+1}) \subset N(c_n)$. Two sequences $\{c_n\}$ and $\{c'_m\}$ of admissible crosscuts are equivalent if for every $m$ there exist $n$ and $n'$ such that $N(c'_m) \subset N(c_n)$ and $N(c_m) \subset N(c'_n)$. An equivalence class of admissible crosscuts define a prime end. This construction has the following property proved by Carathéodory: let $R$ be the union of $R$ with the set of prime ends. Then there is a topology on $R$ turning $R$ into a topological closed disc whose interior is homeomorphic to $R$ with its plane topology. Moreover, if $f$ is a homeomorphism of the closure of $R$ in the plane, then $f$ has a continuous extension to $\tilde{R}$. The following is an extension of this result.

**Theorem 1.** Let $f$ be a degree $d$ branched covering of the sphere, and $R$ a simply connected completely invariant region whose complement contains more than one point. Then $f$ extends to a continuous map $\tilde{f}$ in the prime ends closure of $R$. Moreover $f$ restricted to the boundary of $\tilde{R}$ is a degree $d$ covering map.

**Proof.** As the complement of $R$ has more than one point and is connected, we can assume that $R$ is a branched region of the plane $\mathbb{R}^2$. By the assumption that $f$ is a degree $d$ branched covering of $R$, it follows that $f$ has $d - 1$ critical points (counted with multiplicities) in $R$ (Riemann–Hurwitz formula). Take $\gamma$ a simple closed curve in $R$ such that all critical values of $f$ in $R$ lie in $D$, the bounded component of $\mathbb{R}^2 \setminus \gamma$. Then, $f - 1(\gamma)$ is a simple closed curve, and $f|_{A} : A \to f(A)$ is a degree $d$ covering map, where $A$ is the annulus bounded by $\partial R$ and $f - 1(\gamma)$. Let $p$ denote a prime end. It is claimed first that there exists a sequence $\{c_n\}$ of crosscuts defining $p$ such that restricted to $c_n$ is injective for each $n$. First note that the crosscuts can be constructed in such a way that both extreme points have different images, as the critical points in the boundary of $R$ are also branched points. Let $\delta > 0$ be the diameter of $D$. If each $c_n$ is small enough then $f(c_n)$ does not intersect $D$ and has diameter less than $\delta$ (use here that $f$ extends continuously to the closure of $R$). Let $x$ and $y$ be different points in $c_n$ such that $f(x) = f(y)$. Then the image under $f$ of the segment $\alpha$ in $c_n$ joining $x$ and $y$ cannot be homotopically trivial in $f(A)$. But this is absurd by the choice of $\delta$ and the crosscuts $c_n$. This proves the claim.

The claim implies immediately that $\{f(c_n)\}$ is a sequence of crosscuts, and thus defines a prime end $\tilde{f}(p)$. Continuity is obvious by the definition of the topology of $\tilde{R}$ and the continuity of $f$ in the closure of $R$, so it remains to prove the last assertion. Given a prime end $p$ defined by a sequence of crosscuts $\{c_n\}$, let $\beta$ be a simple arc in $R$ joining $\gamma$ with a point in the boundary of $R$ and such that $\beta \cap c_n = \emptyset$ for all $n$. Then the preimage of $\bar{\beta}$ under $\bar{f}$ is the union of $d$ simple arcs each one of which joins $f^{-1}(\gamma)$ with a point in the boundary of $R$ and whose interiors are pairwise disjoint (recall that $f|_{A} : A \to f(A)$ is a covering). Then $R \setminus f^{-1}(\beta)$ has exactly $d$ connected components, restricted to each of which the map $f$ is injective and onto $R \setminus \beta$. It follows that each $c_n$ has an $f$-preimage in each of these components, so $f^{-1}(p)$ contains exactly $d$ points.

Note that no condition was imposed on the boundary of $R$. Note also that $f$ may have critical points in the boundary of $R$, but in any case it is part of the assertion of the theorem that the restriction of $f$ to the boundary of $\tilde{R}$ (that is homeomorphic to the circle) has no critical points. This is illustrated in the following example.

**Example.** The complex polynomial $f(z) = z^2 - 2$ satisfies the hypothesis of the theorem above. The region $R$ is the complement of the interval $[-2, 2]$ in the real axis. The critical
point 0 belongs to the boundary of $R$. However, each point in the open interval represent two different prime ends while the extreme points 2 and $-2$ represent only one prime end. The two prime ends whose impression is the critical point 0 have the same image under the map $f$. This is the reason why $f$ has no critical points.

2.2. Equivalence of $m_d$-periodic points

In this section we will give an equivalence relation in the set of periodic points of $m_d(z) = z^d$. Two points will be equivalent if they are associated to periodic points of $f$ which correspond to the same point in $\partial R$. It is necessary to declare first the association of periodic points of $m_d$ to a periodic point of $f$ and then use the association of this point to a point in the boundary of $R$.

Beginning with the assumptions of theorem A, let $f$ be a degree $d$ branched covering of the sphere and let $R$ be a completely invariant simply connected region. From now on, the point $\infty$ is assumed to belong to $R$.

Claim. There exists a branched covering $g$ of the sphere that coincides with $f$ in a neighborhood of $K$ (the complement of $R$) and has a unique critical point of order $d - 1$ in $R$.

To prove the claim, let $\gamma$ be a closed simple curve separating $\infty$ from $K$ such that all critical values of $f$ in $R$ are contained in the unbounded component of the complement of $\gamma$. This implies that $f^{-1}(\gamma)$ is a closed simple curve such that all critical points of $f$ in $R$ are contained in the unbounded component of the complement of $f^{-1}(\gamma)$. Let $R'$ be the intersection of $R$ with the bounded component of the complement of $\gamma$. Then $R'$ is an annulus, $f^{-1}(R')$ is also an annulus, and $f$ restricted to $f^{-1}(R')$ is a $d : 1$ covering onto $R'$. Let $h$ be a homeomorphism of $S^2$ which is the identity in a neighborhood of $K$ and carries $\gamma$ to $f^{-1}(\gamma)$. Clearly $h$ sends the bounded component of $\gamma$ to the bounded component of $f^{-1}(\gamma)$. It follows that the map $h \circ f$ is a $d : 1$ covering of $f^{-1}(R')$ onto itself. Next, let $U$ be the unbounded component of the complement of $f^{-1}(R')$. As $U$ is a disc, the identification of $U$ with the point $\infty$ is again the sphere, and if $\rho$ is the quotient map associated to this identification, then $g = \rho \circ h \circ f$ satisfies the assertion of the claim.

As the periodic points will be found on the complement of $R$, where the dynamics of $f$ remains unchanged, this assumption does not imply a loss of generality.

A ray in $R$ is a simple arc joining $\infty$ with a point in $K$, the boundary of $R$. For each prime end one can choose a ray intersecting every crosscut defining the prime end. Given different prime ends one can choose corresponding rays whose intersection contains at most the extreme points of the rays. As we are assuming that $K$ is locally connected, each prime end defines a point in $K$ (see, for example, chapter 17 in [Mi]). Moreover, if the prime end $p$ is $f^k$-fixed, so is the landing point for $f^k$. Indeed, in this case, if $c_n$ is a sequence of crosscuts defining $p$, then $f^k(c_n)$ is a sequence of crosscuts defining the same prime end as $c_n$ (see subsection 2.1).

It is not true that the restriction of $f$ to the region $R$ is conjugate or semiconjugate to the map $z \rightarrow z^d$ acting on $C \setminus \overline{D}$. By this reason it is not obvious the existence of periodic rays for $f$ restricted to $R$. This is not needed here, we will use instead that $f|_{S^1}$ is semiconjugate to $m_d(z) = z^d$ acting on $S^1$, this means that there exists a continuous degree one map $h : S^1 \rightarrow S^1$ such that $hf = m_dh$ (see lemma 1 in [IPRX1]). Moreover it is shown there that $h$ is monotonically increasing, meaning that if $\pi : R \rightarrow S^1$ is the universal covering of the circle, then any lift $H$ of $h$ is monotonically increasing. Of course $h$ may have intervals where it is constant, but the fact that it has degree one implies the following: if $x$ and $y$ are different points with the same $f$-image, then $h(x) \neq h(y)$ (see item (2) after definition 2 in the above cited reference).
It is easy to find right inverses of \( h(h\phi_0 = id) \). Of course none of them will be continuous, unless the semiconjugacy \( h \) is actually a conjugacy. Choose a right inverse \( \phi_0 \) of \( h \) such that \( \phi_0 m_d = \tilde{f} \phi_0 \) and \( \phi_0 \) is monotonically increasing.

Now the assumption that \( K \) is locally connected will be used to define a map \( I : \partial \tilde{R} \to K \) where \( I(p) \) is the impression of \( p \), a unique point in \( K \). Note that \( I \) is continuous, surjective and \( fI = \tilde{f}I \). The map \( I \) is not injective since different prime ends may have the same impression in \( K \).

Of course, if two rays \( r_1 \) and \( r_2 \) representing different prime ends \( x_1 \) and \( x_2 \), land at the same point \( y \in K \), then this point separates \( K \). Moreover, the union of these rays with \( y \) separates the whole sphere, and \( I \) sends each component of \( S^1 \setminus \{x_1, x_2\} \) into the closure of a component of \( K \setminus \{y\} \). It may happen as well, that some point in the interior of an arc from \( x_1 \) to \( x_2 \) also has its impression at the point \( y \) (see figure 2).

We turn now into the ideas for the proof of theorem A. Note that \( \tilde{f} \), being a degree \( d \) covering map of the circle, has at least \( d^n - 1 \) points with period that divides \( n \), or which is the same, \( f^n \) has at least \( d^n - 1 \) fixed points: this is obvious for \( m_d \) and follows for \( \tilde{f} \) because of the semiconjugacy. As was explained earlier, to each of the fixed points of \( \tilde{f}^n \) corresponds (taking impressions) a fixed point of \( f^n \). However, as many different points may have the same impression, in order to have the rate one has to control this possibility. Counting will be easier to perform with the map \( m_d \) instead of \( \tilde{f} \), so define the map \( \phi = I\phi_0 \), that satisfies \( \phi m_d = f\phi \), and so it carries periodic points of \( m_d \) into periodic points of \( f \).

**Definition 1.** Two \( m_d \)-periodic points \( x \) and \( y \) in \( S^1 \) are said equivalent if \( \phi(x) = \phi(y) \). This will be denoted as \( x \sim y \).

Of course this is an equivalence relation. Note that \( w_1 \neq w_2 \) and \( \phi(w_1) = \phi(w_2) \), then \( K\setminus\{\phi(w_1)\} \) is disconnected.

**Lemma 1.** Let \( x \) and \( y \) be different points in the boundary of a component of the complement of the closure of \( R \) such that \( f(x) = f(y) \). Then there exist points \( x' \) and \( y' \) in \( S^1 \) having the same image under \( m_d \), such that \( x' \in h(I^{-1}(x)) \), \( y' \in h(I^{-1}(y)) \) and the following property holds:
\( \phi(w_1) = \phi(w_2) \) implies that either \( w_1 \) and \( w_2 \) belong to the same component of \( S^1 \setminus \{ x', y' \} \), or \( \phi(w_1) = \phi(w_2) = \phi(x') \) or \( \phi(w_1) = \phi(w_2) = \phi(y') \).

**Proof.** Let \( r \) be a ray landing at \( z = f(x) \) and \( r_x, r_y \) rays landing at \( x \) and \( y \) respectively such that \( f(r_x) = f(r_y) = r \). Note that \( r_x \) and \( r_y \) are different as points in \( \partial R \), and will be denoted by \( x_0 \) and \( y_0 \) as points in \( S^1 \). Note that \( f(x_0) = f(y_0) \), which implies, as was explained above, that \( h(x_0) \neq h(y_0) \). Let \( x' = h(x_0) \) and \( y' = h(y_0) \).

The assumptions on \( x \) and \( y \) imply that there exists a simple arc \( s \) joining \( x \) and \( y \) in the sphere with the property that the interior of \( s \) does not intersect the closure of \( R \). It follows that \( S^1 \setminus \{ s \cup r_x \cup r_y \} \) has exactly two connected components, and that in both components there are points of \( K \). Note that as \( h \) and \( \phi_0 \) are monotonic, if \( w_1 \) and \( w_2 \) are points in different components of \( S^1 \setminus \{ x', y' \} \), then \( \phi_0(w_1) \) and \( \phi_0(w_2) \) belong to different components of \( S^1 \setminus \{ x_0, y_0 \} \). For \( i = 1, 2 \), let \( r_i \) be rays corresponding to \( \phi_0(w_i) \) not intersecting \( r_x \) nor \( r_y \) within \( R \). The interior of the rays \( r_1 \) and \( r_2 \) must belong to different components of \( S^1 \setminus \{ s \cup r_x \cup r_y \} \). As \( \phi(w_1) = \phi(w_2) \) implies that \( r_1 \) and \( r_2 \) land at the same point in \( K \), this point is necessarily \( x \) or \( y \). \( \square \)

### 3. Proof of Theorem A

The fundamental idea is the following. The map \( m_d \) acting on \( S^1 \) has the rate: indeed, \( m_d^n \) has \( d^n - 1 \) fixed points. Moreover, the image under \( \phi \) of a \( m_d \)-periodic point is \( f \)-periodic. The lemma proved above shows that the points \( x' \) and \( y' \) obtained separate the circle in such a way that (almost always) a point in one component cannot be identified by \( \phi \) with a point in another component. As \( x' \) and \( y' \) have the same image under \( m_d \), we will construct in an abstract setting maximal sets of points separating the circle in pieces such that points in different pieces cannot be identified. The next subsection is devoted to this.

#### 3.1. Stars

The procedure begins with some abstract definitions and properties; in the next subsection the construction is realized for the map \( f \). The general construction is similar, but different, to the critical portraits used in [Poi] to classify polynomial maps.

Throughout the following, the circle is considered with the distance \( \text{dist: arc length divided by } 2\pi \). So the circle has length equal to 1. Then two different points having the same image under \( m_d \) are at a distance \( j/d \) for some integer \( 0 < j < d \).

**Definition 2.** Let \( d \) be an integer greater than one. A \( d \)-star (or simply a star when no confusion can arise) is a subset \( E \) of \( S^1 \) containing at least two points and such that the distance between any two points in \( E \) is equal to \( j/d \) for some integer \( j, 1 \leq j \leq d - 1 \). The multiplicity of a star is \( m(E) = n - 1 \geq 0 \) if \( E \) has \( n \) points. Two stars \( E_1 \) and \( E_2 \) are disjoint if \( E_2 \) is contained in the closure of a component of the complement of \( E_1 \) which obviously implies that \( E_1 \) is as well contained in the closure of a component of the complement of \( E_2 \). In other words, \( E_1 \) and \( E_2 \) are disjoint if at most one component of \( S^1 \setminus E_1 \) intersects \( E_2 \). A cycle of stars is a sequence of pairwise disjoint stars \( \{ E_1, \ldots, E_k \} \) such that there exists points \( x_i : 1 \leq i \leq k \) and \( x_k \in E_i \cap E_{i+1} \) for \( i < k \) and \( x_k \in E_k \cap E_1 \). A set \( E = \{ E_1, \ldots, E_k \} \) is a maximal set of \( d \)-stars if every \( E_i \) is a \( d \)-star, the \( E_i \) are pairwise disjoint, there are no cycles in \( E \) and it is maximal with these properties.

For example, if \( d = 4 \), \( E_1 = \{ 1, i \} \), \( E_2 = \{ i, -1 \} \), \( E_3 = \{ -1, -i \} \), \( E_4 = \{ 1, -1 \} \), \( E_5 = \{ -i, 1 \} \), then \( \{ E_1, E_2, E_3 \} \) is a maximal set of stars, \( \{ E_1, E_2, E_3, E_5 \} \) is a cycle of stars and
Figure 3. Some stars for $d = 4$.

Figure 4. Some stars for $d = 6$.

${\{E_3, E_4\}}$ is a set of disjoint stars which is not maximal (figures 3{(a)}–{(c)) respectively). We have also drawn a maximal set of stars as well as a cycle of stars for $d = 6$ in figure 4.

In our context each critical point will give rise to a star and the set of stars so defined will be maximal.

3.2. Properties of stars

The main result of this subsection is:

**Proposition 1.** Assume that $E = \{E_1, \ldots, E_k\}$ is a set of disjoint $d$-stars with no cycles. Then $E$ is maximal if and only if the sum of the multiplicities of the $E_i$ is equal to $d - 1$.

**Proof.** The proof will be done by induction on $d$. The case $d = 2$ is trivial. Also the case where there is only one star in the set. To reduce $d$ and use the induction hypothesis we will use the following. Assume that $E = \{E_1, \ldots, E_k\}$ is a set of pairwise disjoint $d$-stars without cycles. Let $x_1$ and $x_2$ be consecutive points in $E_1$, meaning that there is an arc $s$ from $x_1$ to $x_2$ which does not intersect $E_1 \setminus \{x_1, x_2\}$. A circle $\tilde{s}$ is obtained from $s$ when the points $x_1$ and $x_2$ are identified. As in the definition of stars the circle has length equal to one, then the quotient $\tilde{s}$ has to be rescaled with the constant $\ell/d$, [where $\ell/d = \text{dist}(x_1, x_2)$] to obtain a circle with length equal to 1. So a $d$-star $E$ in $E$ contained in $s$ can be also considered as an $\ell$-star $E\ell$ contained in $\tilde{s}$. Having these considerations in mind define $E'$ as the set of $E\ell$ such that $E_i$ belongs to $E$ and is contained in the arc $s$. Now use the next lemma 2 which assures that $E'$ is a maximal set of $\ell$-stars if $E$ is maximal.

Following with the proof of the proposition, note that the same can be done in each connected component of $S^1 \setminus E_1$, where $E_1 = \{x_1, \ldots, x_k\}$ is cyclically ordered. The arc $s_i$
(x_i, x_{i+1}) (i is taken to range over 1, . . . , k where x_{k+1} = x_1). Each arc induces a circle \( \delta_i \) and the set of stars in \( E \) that are contained in \( s_i \) is denoted \( E_i^\ell \). By the lemma each \( E_i^\ell \) is maximal if \( E \) is maximal. The converse also holds, and is trivial since the stars are disjoint.

Assume the assertion of the proposition true for every \( \ell < d \), this means that each \( E_i^\ell \) is maximal if and only if the sum of its multiplicities is equal to \( d \text{ dist}(x_i, x_{i+1}) - 1 \) (we use the dot here for multiplication). Next note that the multiplicity of \( E_1 \) is \( k - 1 \), so every \( E_i^\ell \) is maximal if and only if the sum of the multiplicities of all the stars is equal to

\[
(k - 1) + \sum_{i=1}^{k} (d \text{ dist}(x_i, x_{i+1}) - 1) = (k - 1) + (d - k) = d - 1,
\]

since the sum of the the distances \( \text{dist}(x_i, x_{i+1}) \) equals to one. \( \Box \)

**Lemma 2.** If \( E \) is maximal then \( E' \) is a maximal.

**Proof.** Note that \( E' \) is a set of disjoint \( \ell \)-stars without cycles. To prove maximality of \( E' \), assume it is not, which means the existence of a new \( \ell \)-star \( F \) such that \( E' \cup \{F\} \) is a set of disjoint \( \ell \)-stars without cycles. Returning to the original circle \( S_1 \), and multiplying \( F \) by \( d/\ell \) we obtain a new \( d \)-star \( F \) such that \( E \cup \{F\} \) is a set of disjoint \( d \)-stars without cycles, contradicting the maximality of \( E \). \( \Box \)

### 3.3. Construction of stars

Let \( c \) be a critical point of multiplicity \( j \) contained in \( K \); take a ray \( r \) landing at \( f(c) \) and note that there are exactly \( j + 1 \) preimages of \( r \) that are rays landing at \( c \). The images under \( h \) of the prime ends corresponding to these \( j + 1 \) rays is a \( d \)-star.

Let \( R_1 \) be a component of the complement of the closure of \( R \) containing \( k \) critical points counted with multiplicities. Choose any point \( z \) that is neither periodic nor a critical value, in the boundary of \( f(R_1) \) and take a ray \( r \) in \( R \) landing at \( z \). Then \( f^{-1}(r) \) contains \( k + 1 \) rays landing at different points of the boundary of \( R_1 \), denoted \( r_1, \ldots, r_{k+1} \). Let \( E_1 = \{ h(p_1), \ldots, h(p_{k+1}) \} \) (where \( p_i \) is the prime end corresponding to the ray \( r_i \)). As \( f(r) = f'(r) \) it follows that \( h(r_1) \neq h(r_i) \) whenever \( j \neq i \) (this is a general fact for semiconjugacies; for a proof see item (2) after definition 2 in [IPRX1]). Then \( E_1 \) has \( k + 1 \) different points, and all of them have the same image under \( m_{\ell} \); it follows that \( E_1 \) is a \( d \)-star of multiplicity \( k \).

So, with this proceeding, we obtain a star for each critical point in \( K \) and for each component of the complement of \( \overline{R} \) containing critical points.

**Lemma 3.** The union of the stars constructed above is a maximal set.

**Proof.** Recall that there are exactly \( d - 1 \) critical points in the complement of \( R \) counted with multiplicities. It follows that the sum of the multiplicities of the stars constructed is equal to \( d - 1 \), so in view of proposition 1 it suffices to show that the stars are disjoint and have no cycles.

That the stars are disjoint follows by construction and because \( h \) is monotonic.

Assume by contradiction that there exists a cycle \( E_1, \ldots, E_k \). Taking a minimal cycle (one that does not properly contain another cycle) it can be assumed that the points \( x_i \) \( (1 \leq i \leq k) \) giving the cycle are not repeated. It is claimed first that the stars are cyclically ordered. Let \( E_1 \) and \( E_2 \) have the point \( x_1 \) in common. This determines two arcs in \( S^1 \): \( a_1 \) which contains all the points in \( E_1 \), is an arc starting at \( x_1 \) not containing any point in \( E_2 \) and ending in the last point of \( E_2 \), and \( a_2 \), which contains all the points in \( E_2 \) is an arc starting at \( x_1 \) not containing any point in \( E_1 \) and ending in the last point of \( E_2 \). These intervals intersect only at \( x_1 \) except if \( E_1, E_2 \) is
already a cycle, in which case the claim is obvious. Assume that the star $E_3$, having the point $x_2$ in common with $E_2$, has a point in $a_2$. In this case $E_3$ must be contained in $a_2$, and so the point $x_1 \neq x_2$ is also contained in $a_2$. This implies that the subsequent $E_i$ (if any) are all contained in $a_2$, which forces $x_3 \in a_2$, a contradiction since $x_3 \in a_1$. This implies that the whole $E_3 \setminus \{x_2\}$ is contained in the complement of the union of $a_1$ and $a_2$. By a simple induction argument the claim follows.

The assumption that $x_i$ belongs to $E_i$ and to $E_{i+1}$ implies that there are two different rays giving the same image under $h$. As we are assuming that there is a cycle, we have two sequences of rays $r_i$ and $s_i$, $1 \leq i \leq k$, having the following properties (see figure 5)

1. All the rays $r_i$ and $s_i$ are different.
2. The rays are in cyclic order: once $s_1$, $r_1$ and $s_2$ are given, they determine an orientation of $S^1$ such that $s_1 < r_1 < s_2$. With this orientation fixed, the claim above implies that $r_i < s_{i+1} < r_{i+1}$ for every $i$.
3. For each $1 \leq i \leq k - 1$, the image under $h$ of the oriented interval $I_i = (r_i, s_{i+1})$ is a point, since the extreme points have the same image under $h$ and $h$ is monotone. The same assertion holds for the interval $I_k = (r_k, s_1)$.
4. It comes from the construction of stars that for each value of $i$, the image under $\tilde{f}$ of $r_i$ and $s_i$ is the same.

Now use that $\tilde{f}$ is a covering of $S^1$, together with properties (b) and (d) to deduce that $\tilde{f}(I) = S^1$, where $I = \bigcup^k_{i=1} I_i$. On the other hand, since by property (c) the image under $h$ of this union is finite, the equality $h(\tilde{f}(I)) = m_d h(I)$ is contradicted.

This maximal set of stars will be denoted $\mathcal{E}$.

**Remark 1.** Let $c$ be a critical point in $K$ with multiplicity $k - 1$. Then $c$ separates $K$ (recall that $K = \{c\}$ was implicitly excluded in the hypothesis) and the construction above gives rays $s_1, \ldots, s_k$ that land at $c$ and separate the sphere in $k$ components. In general, if $c_1$ is another critical point in the complement of $R$ (not necessarily in $K$) with multiplicity $j - 1$, then associated to $c_1$ we have defined several rays $r_1, \ldots, r_j$ that separate $R$ into $j$ components. Then there are components $W_1$ of $R \setminus \{s_1, \ldots, s_k\}$ and $W_2$ of $R \setminus \{r_1, \ldots, r_j\}$ with disjoint closures. These components correspond to disjoint connected components $V_1$ and $V_2$ of different stars.

The core of the proof begins now. The idea is that it suffices to prove that in some small interval periodic points of large period are not equivalent. Fix a prime number $\nu$ larger than the period of every periodic critical point (if any). Let $x = \{x_1, \ldots, x_\nu\}$ and $y = \{y_1, \ldots, y_\nu\}$ be $m_d$-periodic orbits of period $\nu$. We assume that $|x_1 - y_1| = 1/(d^\nu - 1)$, so this difference is less possible. This implies that there exists some nonnegative integer $\ell$ such that

$$|x_\nu - y_\nu| = \frac{1}{d(d^\nu - 1)} + \frac{\ell}{d}.$$  

But $\ell$ cannot be 0 since the $x_\nu$ and $y_\nu$ are fixed points of $m_d^\nu$ and the distance between such points cannot be less than $1/d^\nu - 1$. Note also that by the choice of $\nu$, $x_\nu$ and $y_\nu$ must be separated by a star: otherwise, taking $y_\nu - y_\nu - 1/(d^\nu - 1)$ it comes that $\{x_\nu, y_\nu\}$ is a star, disjoint from every other star in $\mathcal{E}$, contradicting maximality.

Assume that $\phi(x_1) = \phi(y_1)$, which implies in particular $\phi(x_\nu) = \phi(y_\nu)$. As $x_\nu$ and $y_\nu$ are separated by stars, the unique possibility given by lemma 1 is that $\phi(x_\nu) = \phi(y_\nu) = Y$ for some point $Y$ taken as a landing point of a ray defining the stars. It follows that $Y$ is periodic (as it is the landing point of a periodic point of $m_d$), so the construction of stars implies that $Y$ must be
Figure 5. Proof of lemma 3.

a critical point: denote it \( c \). Moreover, as the period \( \nu \) is prime and larger than the periods of the critical points, it comes that \( c \) is fixed. We collect this discussion in the following lemma:

**Lemma 4.** Let \( x \) and \( y \) be periodic points of \( m_d \) of period \( \nu \) such that \( |x - y| = 1/(d\nu - 1) \) for some large prime number \( \nu \). Then \( \phi(x) = \phi(y) \) implies that there is a fixed critical point \( c \) in the boundary of \( R \) such that \( \phi(x) = \phi(y) = c \).

Assuming that \( \phi(x) = \phi(y) \), the existence of a fixed critical point implies, by the general hypothesis about the critical points, that there must be another critical point \( c' \). Therefore there exist at least two stars, and, as in the remark, we have regions \( W_1 \) and \( W_2 \) with disjoint closures. Recall that these correspond to intervals \( V_1 \) and \( V_2 \) in \( S^1 \).

Now we will choose a point \( z \) in \( V_1 \) and a positive integer \( n \) such that \( m^0_d(z) \in V_2 \) and such that \( m^k_d(z) \) does not intersect stars for every \( k \) between \(-1\) and \( n \). This is possible since \( m_d \) is transitive and stars are finite. Then select an interval \( I \) around \( z \) such that \( m^0_d(I) \) does not intersect
stars for $k$ between $-1$ and $n$. These assumptions imply that $\phi(I) \subset W_1$ and $f^n(\phi(I)) \subset W_2$. Then it is obvious that $\phi(I)$ cannot contain fixed points of $f$. The conclusion is:

Either there are no fixed critical points, and then for every large prime number $\nu$ no consecutive fixed points of $m'_d$ can be equivalent, or there is an interval $I$ such that for every large prime number $\nu$ no pair of consecutive fixed points of $m'_d$ in the interval $I$ can be equivalent.

Note that there are approximately $\epsilon d^\nu$ fixed points of $m'_d$ in the interval $I$ (where $\epsilon$ is the length of $I$) and these points have the property that no pair of consecutive fixed points of $m'_d$ are equivalent. There is still another restriction:

**Lemma 5.** If $x, y, z, t$ are points in $S^1$ such that $x \sim y, z \sim t$ but $x$ and $z$ are not equivalent, then $z$ and $t$ belong to the same component of $S^1 \setminus \{x, y\}$.

**Proof.** Let $r_x$ and $r_y$ be rays landing at the same point $p$. Then, $r_x \cup r_y \cup p$ separate the plane. Now if $z$ and $t$ belong to different connected components of $S^1 \setminus \{x, y\}$ this means, as they are equivalent, that the landing point of $r_z$ and $r_t$ is also $p$, contradicting that $x$ and $z$ are not equivalent.

Notice that there are $[\epsilon d^\nu]$ fixed points of $m'_d$ in the interval $I$, where $[x]$ is the integer part of $x$. Let $n = [\epsilon d^\nu]$. The next abstract result implies that there are many different equivalence classes.

**Lemma 6.** Let $R$ be an equivalence relation in the set $\{1, \ldots, n\}$ such that the following properties hold:

(a) $(i, i + 1) \notin R$ for every $i$, and
(b) if $L$ and $L'$ are different classes, then each one of them is contained in a connected component of the complement of the other.

Then the number of classes is greater than or equal to $[n/2] + 1$.

**Proof.** Assume that the property holds for every number less than $n$ and let $R$ be an equivalence relation in $\{1, \ldots, n\}$. Let $C_1$ be the class of $1$. Denote by $\sigma_1, \ldots, \sigma_k$ the maximal intervals of the complement of $C_1$. Note that $k = n_1$ or $k = n_1 - 1$, where $n_1$ is the number of elements in $C_1$. If $p_i$ denotes the cardinal of $\sigma_i$, then $n_1 + \sum p_i = n$. Say that $i \in E$ if $p_i$ is even and that $i \in O$ if $p_i$ is odd. Let $e$ denote the number of elements in $E$ and $o$ the number of elements in $O$.

By the hypothesis (2) on the classes, note that the number $N_R$ of equivalence classes of $R$ is at least

$$N_R = 1 + \sum_{i \in O} \left(1 + \frac{p_i}{2}\right) + \sum_{i \in O} \left(1 + \frac{p_i - 1}{2}\right).$$

where the first 1 comes for the class $C_1$, and the induction hypothesis was used in each $\sigma_i$. Rearranging terms it comes that $N_R$ satisfies the thesis of the lemma if

$$2e + o - n_1 \geq 2[n/2] - n.$$

Note that $e + o = n_1$ or $e + o = n_1 - 1$, so $2e + o - n_1$ is either $e$ or $e - 1$. So the last equation is valid if $e > 0$ or $e + o = n_1$ since the number on the right-hand side is equal to 0 or $-1$. If $e = 0$ and $e + o = n_1 - 1$ then $n = \sum p_i + n_1$ is an odd number, so both sides are equal to $-1$.

This estimation is better possible: for every $n$ there is an equivalence relation $R$ as above such that the number of classes is equal to $[n/2] + 1$: one class is the set of odd numbers, and every even number constitutes a class.
From this last lemma we obtain theorem A. Indeed, for every prime number \( p \) large enough the number of equivalence classes of fixed points of \( m_j' \) within the interval \( I \) is at least \( \lfloor ed''/2 \rfloor \) and as different equivalence classes correspond to different fixed points of \( f' \), then the rate of \( f \) is not less than
\[
\lim_{\nu} \frac{1}{\nu} \log \lfloor ed''/2 \rfloor = \log d.
\]

4. Proof of theorem B

This section is devoted to the proof of:

**Theorem B.** Let \( f \) be a degree \( d \) branched covering of the sphere, where \( |d| > 1 \). Assume that there exists a simply connected open set \( U \) whose closure is disjoint from the set of critical values and such that \( f^{-1}(U) \subset U \). Then \( f \) has the rate.

The proof relies on Brouwer’s theory for orientation preserving homeomorphisms of the plane. We will use the following result:

**Theorem 2.** [CL] Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be an orientation preserving homeomorphism and let \( K \subset \mathbb{R}^2 \) be an \( f \)-invariant non-separating continuum. Then, there exists \( x \in K \) such that \( f(x) = x \).

An easy proof or Cartwright–Littlewood’s theorem can be found in the single-page paper of Brown [Brou]. Existence of a fixed point under the hypothesis that a compact subset is preserved was already known (Brouwer’s plane translation theorem [Brou]). To prove that the fixed point must belong to the set \( K \) one needs connectedness of such a set (just think of a rational rotation, where every periodic orbit is a compact invariant set disjoint from the set of fixed points).

The proof of theorem B follows:

**Proof.** The hypothesis implies that \( f^{-1}(U) \) has \( d \) connected components denoted \( W_1, \ldots, W_d \), each \( W_i \) closed and contained in \( U \). Besides, \( f(W_i) = \overline{U} \) for all \( i \).

We will construct by induction on \( n \) a collection of \( d^n \) sets \( W^n_i \), indexed with \( a \), sequences of \( n \) elements between 1 and \( d \), where \( W^n_1 = W_i \).

Given \( a = (1, \ldots, d^n) \) let \( a|_n \) be the restriction of \( a \) to the set \( \{1, \ldots, n\} \), i.e., \( a|_n = (a_1, \ldots, a_n) \).

The induction hypothesis: For each \( j \leq n \) and \( a = (1, \ldots, d^n) \) there exists a set \( W^n_j \) satisfying the following properties:

(a) \( W^n_j \) is a compact connected subset of \( W^n_{a'-1} \), where \( a' \) is the restriction of \( a \) to \( \{1, \ldots, j-1\} \), i.e., \( a' = (a_1, \ldots, a_{j-1}) \).

(b) \( W^n_{a'-1} \supset \bigcup_{i=1}^{d-1} W^n_{ai} \) where \( ai \) is equal to \( (a_1, \ldots, a_{j-1}, i) \).

(c) \( f(W^n_{ai}) = W^n_{ai'} \) where \( i' \) is equal to \( (i, a_1, \ldots, a_{j-1}) \).

Given \( a = (a_1, \ldots, a_n) \), let \( a'' = (a_2, \ldots, a_n) \), and define \( W^n_a = f^{-1}(W^n_{a''}) \cap W^n_1 \). As \( W^n_{a''} \) is not empty and contained in \( U \), then it has one \( f \)-preimage in each \( W^n_1 \). This implies the properties (a–c) above for \( j = n \).

Theorem B follows from this when \( n \rightarrow \infty \).
with period $k$, then $K_a$ is $f^k$ invariant. Moreover, there exists a neighborhood $V$ of $K_a$ homeomorphic to a disc such that $f(V) \cap S_j$ is empty for every $j \leq k$, and it follows that $f^k$ restricted to $V$ is a homeomorphism onto its image. One can then extend $f^k|_V$ to a plane homeomorphism and apply Cartwright–Littlewood’s theorem 2 to obtain that $f^k$ has a fixed point in $K_a$. As the sets $K_a$ are disjoint, and there are $d^k$ different sequences in $\{1, \ldots, d\}$ having period $k$, the result follows.

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