On the Sum of Divisors of Mixed Powers

Jinjiang Li* & Min Zhang†
Department of Mathematics, China University of Mining and Technology*†
Beijing 100083, P. R. China

Abstract: Let \( d(n) \) denote the Dirichlet divisor function. Define
\[
S_k(x) = \sum_{1 \leq n_1, n_2, n_3, n_4 \leq x^{1/k}} d(n_1^2 + n_2^2 + n_3^2 + n_4^k), \quad 3 \leq k \in \mathbb{N}.
\]
In this paper, we establish an asymptotic formula of \( S_k(x) \) and prove that
\[
S_k(x) = C_1(k)x^{3/2+1/k} \log x + C_2(k)x^{3/2+1/k} + O(x^{3/2+1/k-\delta_k+\epsilon}),
\]
where \( C_1(k), C_2(k) \) are two constants depending only on \( k \), with \( \delta_3 = \frac{19}{60}, \delta_4 = \frac{5}{24}, \delta_5 = \frac{19}{120}, \delta_6 = \frac{25}{192}, \delta_7 = \frac{107}{4032}, \delta_k = \frac{1}{k+2} + \frac{1}{2k(k-1)} \) for \( k \geq 8 \).

Keywords: Divisor function; circle method; mixed power; asymptotic formula

Mathematics Subject Classification 2010: 11P05, 11P32, 11P55

1 Introduction and main result

Let \( d(n) \) be the Dirichlet divisor function. Gafurov [2,3] studied the number of divisors of a binary quadratic form and obtained the asymptotic formula
\[
\sum_{1 \leq m, n \leq x} d(m^2 + n^2) = A_1x^2 \log x + A_2x^2 + O(x^{5/3} \log^9 x),
\]
where \( A_1 \) and \( A_2 \) are certain constants. Later this result was improved by Yu [13], who gave the asymptotic formula
\[
\sum_{1 \leq m, n \leq x} d(m^2 + n^2) = A_1x^2 \log x + A_2x^2 + O(x^{3/2+\epsilon}),
\]
for any fixed \( \epsilon > 0 \).
In 2000, Calderón and de Velasco [1] studied the divisors of the quadratic form \( m_1^2 + m_2^2 + m_3^2 \) and established the asymptotic formula

\[
\sum_{1 \leq m_1, m_2, m_3 \leq x} d(m_1^2 + m_2^2 + m_3^2) = \frac{8\zeta(3)}{5\zeta(5)} x^3 \log x + O(x^3). \quad (1.1)
\]

In 2012, Guo and Zhai [5] improved (1.1) to

\[
\sum_{1 \leq m_1, m_2, m_3 \leq x} d(m_1^2 + m_2^2 + m_3^2) = 2C_1 I_1 x^3 \log x + (C_1 I_2 + C_2 I_1) x^3 + O(x^{8/3+\epsilon}),
\]

where

\[
C_1 = \sum_{q=1}^{\infty} \frac{1}{q^4} \sum_{(a,q)=1}^{q} \left( \sum_{h=1}^{q} e\left( \frac{ah^2}{q} \right) \right)^3,
\]

\[
C_2 = \sum_{q=1}^{\infty} \frac{-2 \log q + 2\gamma}{q^4} \sum_{(a,q)=1}^{q} \left( \sum_{h=1}^{q} e\left( \frac{ah^2}{q} \right) \right)^3,
\]

\[
I_1 = \int_{-\infty}^{+\infty} \left( \int_{0}^{1} e(\mu^2 \beta) d\mu \right)^3 \left( \int_{0}^{\infty} e(-\mu \beta) d\mu \right) d\beta
\]

and

\[
I_2 = \int_{-\infty}^{+\infty} \left( \int_{0}^{1} e(\mu^2 \beta) d\mu \right)^3 \left( \int_{0}^{\infty} e(-\mu \beta) \log \mu d\mu \right) d\beta.
\]

Later, Zhao [14] improved the error term \( O(x^{8/3+\epsilon}) \) to \( O(x^{2 \log^7 x}) \). Moreover, Lü and Mu [8] consider the nonhomogeneous case. They proved that, for \( k \geq 3 \), there holds

\[
\sum_{1 \leq n_1, n_2, n_3 \leq x^{1/2}} d(n_1^2 + n_2^2 + n_3^k) = A(k) x^{1+1/k} \log x + B(k) x^{1+1/k} + O(x^{1+1/k-\theta(k)+\epsilon}),
\]

where \( A(k), B(k) \) are two constants depending only on \( k, \theta(3) = 5/42, \theta(4) = 1/16, \theta(5) = 1/40, \theta(k) = 1/(k2^{k-1}) \) for \( 6 \leq k \leq 7 \) and \( \theta(k) = 1/(2k^2(k-1)) \) for \( k \geq 8 \).

In 2014, Hu [6] considered the divisors of the quaternary form \( m_1^2 + m_2^2 + m_3^2 + m_4^2 \) and obtained

\[
\sum_{1 \leq m_1, m_2, m_3, m_4 \leq x} d(m_1^2 + m_2^2 + m_3^2 + m_4^2) = 2C_1 I_1 x^4 \log x + (C_1 I_2 + C_2 I_1) x^4 + O(x^{7/2+\epsilon}), \quad (1.2)
\]

where

\[
C_1 = \sum_{q=1}^{\infty} \frac{1}{q^4} \sum_{(a,q)=1}^{q} \left( \sum_{r=1}^{q} e\left( \frac{ar^2}{q} \right) \right)^4,
\]
\[ C_2 = \sum_{q=1}^{\infty} \frac{-2 \log q + 2\gamma}{q^5} \sum_{a=1}^{q} \left( \sum_{r=1}^{q} e\left( \frac{ar^2}{q} \right) \right)^4, \]

\[ \mathcal{I}_1 = \int_{-\infty}^{+\infty} \left( \int_{0}^{1} e(u^2\lambda)du \right)^4 \left( \int_{0}^{1} e(-u\lambda)du \right) d\lambda \]

and

\[ \mathcal{I}_2 = \int_{-\infty}^{+\infty} \left( \int_{0}^{1} e(u^2\lambda)du \right)^4 \left( \int_{0}^{1} e(-u\lambda) \log udu \right) d\lambda. \]

Later, Liu and Hu [7] improved the error term \( O\left( x^{7/2+\varepsilon} \right) \) to \( O\left( x^3 \log x \right) \).

In this paper, we consider the nonhomogeneous case of the form \( n_1^2 + n_2^2 + n_3^2 + n_4^k \) and establish the following theorem.

**Theorem 1.1** Let

\[ S_k(x) = \sum_{1 \leq n_1, n_2, n_3 \leq x^{1/2}} d(n_1^2 + n_2^2 + n_3^2 + n_4^k), \quad 3 \leq k \in \mathbb{N}. \]

Then we have

\[ S_k(x) = \mathcal{G}_1 \mathcal{J}_1 x^{3/2+1/k} \log x + (\mathcal{G}_1 \mathcal{J}_2 + \mathcal{G}_2 \mathcal{J}_1) x^{3/2+1/k} + O(x^{3/2+1/k-\delta_k+\varepsilon}), \]

where

\[ \delta_3 = \frac{19}{60}, \quad \delta_4 = \frac{5}{24}, \quad \delta_5 = \frac{19}{140}, \quad \delta_6 = \frac{25}{192}, \]

\[ \delta_7 = \frac{457}{4032}, \quad \delta_k = \frac{1}{k+2} + \frac{1}{2k^2(k-1)} \quad \text{for} \quad k \geq 8, \]

\[ \mathcal{G}_1 = \sum_{q=1}^{\infty} \frac{1}{q^3} \sum_{a=1}^{\infty} \left( \sum_{r=1}^{q} e\left( \frac{ar^2}{q} \right) \right)^3 \left( \sum_{r=1}^{q} e\left( \frac{ar^k}{q} \right) \right), \]

\[ \mathcal{G}_2 = \sum_{q=1}^{\infty} \frac{-2 \log q + 2\gamma}{q^3} \sum_{a=1}^{\infty} \left( \sum_{r=1}^{q} e\left( \frac{ar^2}{q} \right) \right)^3 \left( \sum_{r=1}^{q} e\left( \frac{ar^k}{q} \right) \right), \]

\[ \mathcal{J}_1 = \int_{-\infty}^{+\infty} \left( \int_{0}^{1} e(\alpha \mu^2) d\mu \right)^3 \left( \int_{0}^{1} e(\alpha \mu^k) d\mu \right) \left( \int_{0}^{1} e(-\alpha \mu) d\mu \right) d\alpha, \]

\[ \mathcal{J}_2 = \int_{-\infty}^{+\infty} \left( \int_{0}^{1} e(\alpha \mu^2) d\mu \right)^3 \left( \int_{0}^{1} e(\alpha \mu^k) d\mu \right) \left( \int_{0}^{1} e(-\alpha \mu) \log \mu d\mu \right) d\alpha. \]

**Notation.** Throughout this paper, \( x \) always denotes a sufficiently large real number; \( \varepsilon \) always denotes an arbitrary small positive constant, which may not be the same
at different occurrences. \( e(x) = e^{2\pi ix} \); \( f(x) \ll g(x) \) means that \( f = O(g(x)) \). For the sake of brevity, we define

\[
\mathcal{J}_1(\beta) = \left( \int_0^1 e(\beta \mu^2) d\mu \right)^3 \left( \int_0^1 e(\beta \mu^k) d\mu \right) \left( \int_0^1 e(-\beta \mu) d\mu \right)
\]

and

\[
\mathcal{J}_2(\beta) = \left( \int_0^1 e(\beta \mu^2) d\mu \right)^3 \left( \int_0^1 e(\beta \mu^k) d\mu \right) \left( \int_0^1 e(-\beta \mu \log \mu) d\mu \right).
\]

## 2 Preliminary Lemmas

For any \( \alpha \in \mathbb{R} \), define

\[
f_\ell(\alpha) = \sum_{1 \leq n \leq x^{1/\ell}} e(\alpha n), \quad f(\alpha) = \sum_{1 \leq n \leq 4x} d(n)e(\alpha n).
\]

**Lemma 2.1** For any real numbers \( \alpha \) and \( \tau \geq 1 \), there exist integers \( a \) and \( q \), \( (a, q) = 1 \), \( 1 \leq q \leq \tau \), such that

\[
\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \leq \frac{1}{q\tau}.
\]

**Proof.** See C. D. Pan and C. B. Pan [9], Lemma 5.19. \( \blacksquare \)

Let

\[
\log x < 2Q < \tau < x, \quad Q \tau \asymp x, \quad Q \ll x^{2/(k+2)}.
\]

(2.1)

For any \( 1 \leq a < q \leq Q \) with \( (a, q) = 1 \), define

\[
\mathcal{M}(a, q) = \left[ \frac{a}{q} - \frac{1}{q\tau}, \frac{a}{q} + \frac{1}{q\tau} \right]
\]

and

\[
\mathcal{M} = \bigcup_{q \leq Q} \bigcup_{1 \leq a \leq q \atop (a, q) = 1} \mathcal{M}(a, q), \quad m = \left[ \frac{1}{\tau}, \frac{1}{\tau} + \frac{1}{\tau} \right] \setminus \mathcal{M}.
\]

We call \( \mathcal{M} \) the major arc and \( m \) the minor arc. By the definition of \( S_k(x) \) and orthogonality of exponential function, we have

\[
S_k(x) = \int_0^1 f_2^3(\alpha)f_k(\alpha)f(-\alpha)d\alpha
\]

\[
= \left\{ \int_{\mathcal{M}} + \int_{m} \right\} f_2^3(\alpha)f_k(\alpha)f(-\alpha)d\alpha
\]

\[
= S_k(x, \mathcal{M}) + S_k(x, m).
\]

**Lemma 2.2** For any \( a, q \in \mathbb{Z} \) with \( (a, q) = 1 \) and \( q > 0 \), let

\[
S_k(q, a) = \sum_{r=1}^{q} e\left( \frac{aq^k}{q} \right).
\]
Then we have
\[ S_k(q, a) \ll q^{(k-1)/k}. \]

**Proof.** See Vaughan [11], Theorem 4.2. ■

**Lemma 2.3** For \( k \geq 1 \), we have
\[
\int_0^1 e(\beta \mu^k) d\mu \ll \frac{1}{(1 + |\beta|)^{1/k}}, \tag{2.2}
\]
\[
\int_0^3 e(\beta \mu^k) \log \mu d\mu \ll \frac{\log(2 + |\beta|)}{1 + |\beta|}. \tag{2.3}
\]

**Proof.** See Lü and Mu [8], Lemma 1.2. ■

**Lemma 2.4** Suppose that \((a, q) = 1\) and \(\alpha = a/q + \beta\). Then
\[ f_k(\alpha) = V_k(\alpha, q, a) + O(q^{1/2+\epsilon}(1 + x|\beta|)^{1/2}). \]
Moreover, if \(|\beta| \leq \frac{2^{1/k-1}}{2xq}\), then
\[ f_k(\alpha) = V_k(\alpha, q, a) + O(q^{1/2+\epsilon}), \]
where
\[ V_k(\alpha, q, a) = x^{1/k} \frac{S_k(q, a)}{q} \int_0^1 e(x\beta \mu^k) d\mu. \]

**Proof.** See Vaughan [11], Theorem 4.1. ■

**Lemma 2.5** Suppose that \(\alpha = a/q + \beta \in \mathfrak{M}\) and \(Q\tau \leq x, \tau > x^{1/2+\epsilon}\). Then
\[ f(-\alpha) = \frac{x \log x}{q} \int_0^3 e(-\mu x\beta) d\mu + \frac{x}{q} \int_0^3 e(-\mu x\beta) \log \mu d\mu \]
\[ + \frac{-2 \log q + 2\gamma}{q} x \int_0^3 e(-\mu x\beta) d\mu + O(\Delta), \]
where
\[ \Delta = x^\epsilon (q^{1/2}x\tau^{-1} + q^{2/3}x^{1/3}). \tag{2.4} \]

**Proof.** See Guo and Zhai [5], Lemma 7.1. ■

**Lemma 2.6** Suppose that
\[ L(H) = \sum_{i=1}^m A_i H^{a_i} + \sum_{j=1}^n B_j H^{b_j}, \]
where \(A_i, B_j, a_i, b_j\) are positive. Assume that \(H_1 \leq \mathcal{H} \leq H_2\) and there exists some \(\mathcal{H}\) with \(H_1 \leq \mathcal{H} \leq H_2\) and
\[ L(\mathcal{H}) \ll \sum_{i=1}^m A_i H_1^{a_i} + \sum_{j=1}^n B_j H_2^{b_j} + \sum_{i=1}^m \sum_{j=1}^n (A_i^{a_i} B_j^{b_j})^{1/(a_i+b_j)}. \]

The implied constant depends only on \(m\) and \(n\).
Proof. See Srinivasan [10], Lemma 3 or Graham and Kolesnik [4], Lemma 2.4. ■

Lemma 2.7 Suppose that

\[(a, q) = 1, \quad \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2}, \quad \phi(x) = \alpha x^k + \alpha_1 x^{k-1} + \cdots + \alpha_{k-1} x + \alpha_k.\]

Then

\[\sum_{1 \leq x \leq Y} e(\phi(x)) \ll Y^{1+\varepsilon} (q^{-1} + Y^{-1} + qY^{-k})^{1/2^{k-1}}.\]

Proof. See Vaughan [11], Lemma 2.4. ■

Lemma 2.8 Suppose that \(1 \leq j \leq k\). Then

\[
\int_0^1 \left| \sum_{m=1}^Y e(\alpha m^k) \right|^{2^j} d\alpha \ll Y^{2^j-j+\varepsilon}.
\]

Proof. See Vaughan [11], Lemma 2.5. ■

Lemma 2.9 Let \(j\) be an integer with \(j \geq 2\). Suppose that there exist integers \(a, q\) with \(q \geq 1, (a, q) = 1\) such that \(\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2}\) and \(q \leq x\). Then one has

\[f_j(\alpha) \ll x^{1/j+\varepsilon} (q^{-1} + x^{-1/j} + qx^{-1})^{1/2j(j-1)}.\]

Proof. See Wooley [12], Theorem 1.5. ■

3 Proof of Theorem 1.1

It is obvious that

\[S_k(x, \mathcal{M}) = \sum_{1 \leq q \leq Q} \sum_{\substack{\alpha = 1 \atop (a, q) = 1}}^q \int_{\frac{a}{q} - \frac{1}{q^j}}^{\frac{a}{q} + \frac{1}{q^j}} f_2^3(\alpha) f_k(\alpha) f(-\alpha) d\alpha.\]

For \(\alpha = \frac{a}{q} + \beta \in \mathcal{M}\), by Lemma 2.4 and Lemma 2.5, we have

\[
f_2^3(\alpha) f_k(\alpha) f(-\alpha) = \left( V_2(\alpha, q, a) + O(q^{1/2+\varepsilon}) \right)^3 \left( V_k(\alpha, q, a) + O(q^{1/2+\varepsilon}(1 + x|\beta|)^{1/2}) \right)
\]

\[
\times \left( \frac{x \log x}{q} \int_0^3 e(-\mu x \beta) d\mu + \frac{x}{q} \int_0^3 e(-\mu x \beta) \log \mu d\mu
\]

\[
- \frac{2 \log q + 2\gamma}{q} x \int_0^3 e(-\mu x \beta) d\mu + O(\Delta) \right).
\]
where \( \Delta \) is defined in (2.4). Then we have

\[
\int \frac{a+1}{q} f_3^2(\alpha) f_k(\alpha) f(-\alpha) d\alpha = \int \frac{1}{q^7} f_3^2 \left( \frac{a}{q} + \beta \right) f_k \left( \frac{a}{q} + \beta \right) f \left( - \frac{a}{q} - \beta \right) d\beta
\]

\[
= \frac{S_3^2(q, a) S_k(q, a)}{q^5} x^{3/2+1/k} \log x
\]

\[
\times \left( \int \frac{1}{q^7} \left( \int_0^1 e(x\beta u^2) d\mu \right)^3 \left( \int_0^1 e(x\beta v^k) d\mu \right) \left( \int_0^1 e(x\beta w^k) d\mu \right) d\beta \right)
\]

\[
+ \frac{S_3^2(q, a) S_k(q, a)}{q^5} x^{3/2+1/k}
\]

\[
\times \left( \int \frac{1}{q^7} \left( \int_0^1 e(x\beta u^2) d\mu \right)^3 \left( \int_0^1 e(x\beta v^k) d\mu \right) \left( \int_0^1 e(x\beta w^k) \log x d\mu \right) d\beta \right)
\]

\[
+ \left( -2 \log q + 2 \gamma \right) \frac{S_3^2(q, a) S_k(q, a)}{q^5} x^{3/2+1/k}
\]

\[
\times \left( \int \frac{1}{q^7} \left( \int_0^1 e(x\beta u^2) d\mu \right)^3 \left( \int_0^1 e(x\beta v^k) d\mu \right) \left( \int_0^1 e(x\beta w^k) \log x \right) d\beta \right)
\]

\[
\times \left( \int \frac{1}{q^7} \left( \int_0^1 e(x\beta u^2) d\mu \right)^3 \left( \int_0^1 e(x\beta v^k) d\mu \right) \left( \int_0^1 e(x\beta w^k) \log x \right) d\beta \right)
\]

\[
+ O \left( x^{3/2+1/k} q^{-2} + x^{1/k+\varepsilon} q^{1/2-1/k} \right) + O \left( x^{3/2+1/k} q^{-1/k} \right)
\]

\[
+ O \left( x^{3/2+1/k} q^{-2} + x^{1/k+\varepsilon} q^{1/2-1/k} + x^{3/2+1/k} q^{-1/k} \right) + O \left( x^{3/2+1/k} q^{-2} + x^{1/k+\varepsilon} q^{1/2-1/k} + x^{3/2+1/k} q^{-1/k} \right)
\]

\[
+ O \left( x^{3/2+1/k} q^{-2} + x^{1/k+\varepsilon} q^{1/2-1/k} + x^{3/2+1/k} q^{-1/k} + x^{3/2+1/k} q^{-1/k} \right)
\]

\[
+ O \left( x^{3/2+1/k} q^{-2} + x^{1/k+\varepsilon} q^{1/2-1/k} + x^{3/2+1/k} q^{-1/k} + x^{3/2+1/k} q^{-1/k} + x^{3/2+1/k} q^{-1/k} \right)
\]

\[
= \frac{S_3^2(q, a) S_k(q, a)}{q^5} x^{3/2+1/k} \log x \int \frac{1}{q^7} \tilde{3}_1(\beta) d\beta
\]

\[
+ \frac{S_3^2(q, a) S_k(q, a)}{q^5} x^{3/2+1/k} \int \frac{1}{q^7} \tilde{3}_2(\beta) d\beta
\]

\[
+ \left( -2 \log q + 2 \gamma \right) \frac{S_3^2(q, a) S_k(q, a)}{q^5} x^{3/2+1/k} \int \frac{1}{q^7} \tilde{3}_1(\beta) d\beta
\]

\[
+ O \left( x^{3/2+1/k} q^{-2} + x^{1/k+\varepsilon} q^{1/2-1/k} + x^{3/2+1/k} q^{-1/k} \right)
\]

\[
+ O \left( x^{3/2+1/k} q^{-2} + x^{1/k+\varepsilon} q^{1/2-1/k} + x^{3/2+1/k} q^{-1/k} + x^{3/2+1/k} q^{-1/k} + x^{3/2+1/k} q^{-1/k} \right).
\]
Applying Lemma 2.3, we have
\[ \tilde{J}_1(\beta) \ll \frac{1}{(1 + |\beta|)^{5/2 + 1/k}}, \quad \tilde{J}_2(\beta) \ll \frac{\log(2 + |\beta|)}{(1 + |\beta|)^{5/2 + 1/k}}. \]
Therefore, we have
\[
\int_{|\beta| > \frac{m}{\tau}} \tilde{J}_1(\beta) d\beta \ll \int_{|\beta| > \frac{m}{\tau}} \frac{1}{(1 + |\beta|)^{5/2 + 1/k}} d\beta \ll \frac{1}{(1 + \frac{x}{q\tau})^{3/2 + 1/k}} \tag{3.2}
\]
and
\[
\int_{|\beta| > \frac{m}{\tau}} \tilde{J}_2(\beta) d\beta \ll \int_{|\beta| > \frac{m}{\tau}} \frac{\log(2 + |\beta|)}{(1 + |\beta|)^{5/2 + 1/k}} d\beta \ll \frac{\log \left(2 + \frac{x}{q\tau}\right)}{(1 + \frac{x}{q\tau})^{3/2 + 1/k}}. \tag{3.3}
\]
Hence, from (3.1), (3.2) and (3.3), we get
\[
\int_{\frac{m}{q} \leq \alpha \leq \frac{m}{q} - 1} f_2^3(\alpha) f_k(\alpha) f(-\alpha) d\alpha
\]
\[= \tilde{J}_1 S_2^3(q, a) S_k(q, a) q^5 x^{3/2 + 1/k} \log x + \tilde{J}_2 S_2^3(q, a) S_k(q, a) x^{3/2 + 1/k}
\]
\[+ (-2 \log q + 2\gamma) \tilde{J}_1 S_2^3(q, a) S_k(q, a) q^5 x^{3/2 + 1/k} + O \left(x^\epsilon q^{-1} x^{3/2 + 1/k} + x^{3/2 + \epsilon} q^{-2} + x^{1/k + \epsilon} q^{1/2 - 1/k} + x^{3/2 + 1/k + \epsilon} q^{1 - 1/k - 1} + x^{5/6 + 1/k + \epsilon} q^{1 - 2/3} + x^{3/2 + \epsilon} q^{-1/2 - 1} + x^{5/6 + \epsilon} q^{-1/3}\right). \tag{3.4}
\]
By Lemma 2.2, it follows that
\[
\sum_{q > Q} \sum_{(a, q) = 1}^q \frac{S_2^3(q, a) S_k(q, a) q^5}{x^{3/2 + 1/k}} \ll \sum_{q > Q} q^{3/2 - 1/k} \ll Q^{-1/2 - 1/k}. \tag{3.5}
\]
Therefore, from (3.4) and (3.5), we obtain
\[
S_k(x; \mathfrak{m}) = \sum_{1 \leq q \leq Q} \sum_{(a, q) = 1}^q \int_{\frac{m}{q} \leq \alpha \leq \frac{m}{q} - 1} f_2^3(\alpha) f_k(\alpha) f(-\alpha) d\alpha
\]
\[= \mathfrak{S}_1 \tilde{J}_1 x^{3/2 + 1/k} \log x + (\mathfrak{S}_1 \tilde{J}_2 + \mathfrak{S}_2 \tilde{J}_1) x^{3/2 + 1/k}
\]
\[+ O \left(x^{3/2 + 1/k + \epsilon} Q^{-1/2 - 1/k} + x^{3/2 + 1/k + \epsilon} Q^{3/2 - 1/k} + x^{1/k + \epsilon} Q^{3/2 - 1/k}
\]
\[+ x^{3/2 + 1/k + \epsilon} Q^{-1/2 - 1} + x^{5/6 + 1/k + \epsilon} Q^{7/6 - 1/k}
\]
\[+ x^{3/2 + \epsilon} Q^{3/2 - 1} + x^{5/6 + \epsilon} Q^{5/3}\right). \tag{3.6}
\]
It remains to estimate the integral on the minor arc \( \mathfrak{m} \). At this time, for \( \alpha \in \mathfrak{m} \), we have \( Q < q \leq \tau \). We consider four different cases as follows.
Case 1. If $3 \leq k \leq 5$, by noting the fact that $Qe \asymp x$, $Q < x^{2/(k+2)}$ and Lemma 2.7, we have

$$f_2(\alpha) \ll x^{1/2+\varepsilon}(q^{-1} + x^{-1/2} + qx^{-1})^{1/2}$$

$$\ll x^{1/2+\varepsilon}(Q^{-1/2} + x^{-1/4} + q^{1/2}x^{-1/2})$$

$$\ll x^{1/2+\varepsilon}Q^{-1/2}. \quad (3.7)$$

Also, we have

$$\int_0^1 |f(-\alpha)|^2 d\alpha = \sum_{n \leq 4x} d^2(n) \ll x \log^3 x. \quad (3.8)$$

Therefore, it follows from H"older’s inequality, Lemma 2.8 and (3.8) that

$$\mathcal{S}_k(x, m) = \int_m f_2^3(\alpha)f_k(\alpha)f(-\alpha)d\alpha$$

$$\ll \max_{\alpha \in \mathbb{C}^m} |f_2(\alpha)|^2 \left( \int_0^1 |f_2(\alpha)|^4 d\alpha \right)^{1/4} \left( \int_0^1 |f_k(\alpha)|^4 d\alpha \right)^{1/4} \left( \int_0^1 |f(-\alpha)|^2 d\alpha \right)^{1/2}$$

$$\ll x^{1+\varepsilon}Q^{-1} \cdot x^{3/4+1/(2k)+\varepsilon} \ll x^{7/4+1/(2k)+\varepsilon}Q^{-1}. \quad (3.9)$$

Case 2. If $k = 6$, from Lemma 2.7 we have

$$f_6(\alpha) \ll x^{1/6+\varepsilon}(Q^{-1/32} + x^{-1/192} + \tau x^{-1/32})$$

$$\ll x^{1/6+\varepsilon}(Q^{-1/32} + x^{-1/192}). \quad (3.10)$$

Therefore, it follows from Cauchy’s inequality, Lemma 2.8 and (3.8) that

$$\mathcal{S}_6(x, m) = \int_m f_2^3(\alpha)f_6(\alpha)f(-\alpha)d\alpha$$

$$\ll \max_{\alpha \in \mathbb{C}^m} |f_2(\alpha)| \cdot \max_{\alpha \in \mathbb{C}^m} |f_6(\alpha)| \left( \int_0^1 |f_2(\alpha)|^4 d\alpha \right)^{1/2} \left( \int_0^1 |f(-\alpha)|^2 d\alpha \right)^{1/2}$$

$$\ll x^{5/3+\varepsilon}Q^{-17/32} + x^{319/192+\varepsilon}Q^{-1/2}. \quad (3.11)$$

Case 3. If $k = 7$, from Lemma 2.7 we have

$$f_7(\alpha) \ll x^{1/7+\varepsilon}(Q^{-1/64} + x^{-1/448} + \tau x^{-1/64})$$

$$\ll x^{1/7+\varepsilon}(Q^{-1/64} + x^{-1/448}). \quad (3.12)$$

Therefore, it follows from Cauchy’s inequality, Lemma 2.8 and (3.8) that

$$\mathcal{S}_7(x, m) = \int_m f_2^3(\alpha)f_7(\alpha)f(-\alpha)d\alpha$$

$$\ll \max_{\alpha \in \mathbb{C}^m} |f_2(\alpha)| \cdot \max_{\alpha \in \mathbb{C}^m} |f_7(\alpha)| \left( \int_0^1 |f_2(\alpha)|^4 d\alpha \right)^{1/2} \left( \int_0^1 |f(-\alpha)|^2 d\alpha \right)^{1/2}$$

$$\ll x^{23/14+\varepsilon}Q^{-33/64} + x^{735/448+\varepsilon}Q^{-1/2}. \quad (3.13)$$
Case 4. If $k \geq 8$, from Lemma 2.9 we have

\[
\begin{align*}
    f_k(\alpha) & \ll x^{1/k+\varepsilon} (Q^{-1} + x^{-1/k} + \tau x^{-1})^{1/(2k(k-1))} \\
    & \ll x^{1/k+\varepsilon} (Q^{-1/(2k(k-1))} + x^{-1/(2k^2(k-1))}).
\end{align*}
\]

Therefore, it follows from Hölder’s inequality, Lemma 2.8 and \((3.8)\) that

\[
\begin{align*}
    S_k(x, m) = \int_m f_2^3(\alpha)f_k(\alpha)f(-\alpha)d\alpha \\
    & \ll \max_{\alpha \in m} |f_2(\alpha)| \cdot \max_{\alpha \in m} |f_k(\alpha)| \left( \int_0^1 |f_2(\alpha)|^4 d\alpha \right)^{1/2} \left( \int_0^1 |f(-\alpha)|^2 d\alpha \right)^{1/2} \\
    & \ll x^{3/2+1/k+\varepsilon} Q^{-(k^2-k+1)/(2k(k-1))} + x^{3/2+1/k-1/(2k^2(k-1))+\varepsilon} Q^{-1/2}.
\end{align*}
\]

(3.15)

The rest thing that we need to do is to choose optimal parameters $\tau$ and $Q$. By noting the condition \((2.1)\), we can use $xQ^{-1}$ to substitute $\tau$ in \((3.6)\). Then, by a simple calculation, it is easy to use Lemma 2.6 to obtain the desired asymptotic formula of $S_k(x)$. This completes the proof of Theorem 1.1.

Acknowledgement

The authors would like to express the most and the greatest sincere gratitude to Professor Wenguang Zhai for his valuable advice and constant encouragement.

References

[1] C. Calderón, M. J. de Velasco, On divisors of a quadratic form, Bol. Soc. Brasil. Mat., 31 (2000) 81-91.

[2] N. Gafurov, On the sum of the number of divisors of a quadratic form, Dokl. Akad. Nauk Tadzhik., 28 (1985), 371-375.

[3] N. Gafurov, On the number of divisors of a quadratic form, Proc. Steklov Inst. Math., 200 (1993) 137-148.

[4] S. W. Graham, G. Kolesnik, Van der Corput’s Method of Exponential Sums, Cambridge University Press, 1991.

[5] R. T. Guo, W. G. Zhai, Some problems about the ternary quadratic form $m_1^2 + m_2^2 + m_3^2$, Acta Arith., 156 (2) (2012), 101-121.

[6] L. Hu, An asymptotic formula related to the divisors of the quaternary quadratic form, Acta Arith., 166 (2014) 129-140.
[7] H. Liu, L. Hu, *On the number of divisors of a quaternary quadratic form*, Int. J. Number Theory, **12** (05) (2016) 1219-1235.

[8] X. D. Lü, Q. W. Mu, *The Sum of Divisors of Mixed Powers*, Adv. Math. (China), **45** (3) (2016), 357-364.

[9] C. D. Pan, C. B. Pan, *Goldbach Conjecture*, Science Press, Beijing, 1992.

[10] B. R. Srinivasan, *The lattice point problem of many-dimensional hyperboloids II*, Acta Arith., **8** (2) (1962), 173-204.

[11] R. C. Vaughan, *The Hardy-Littlewood Method*, 2nd edn., Cambridge Tracts Math., Vol. 125, Cambridge University Press, 1997.

[12] T. D. Wooley, *Vinogradov’s mean value theorem via efficient congruencing*, Ann. of Math. (2), **175** (3) (2012), 1575-1627.

[13] G. Yu, *On the number of divisors of the quadratic form*, Canadian Math. Bull., **43** (2000) 239-256.

[14] L. Zhao, *The sum of divisors of a quadratic form*, Acta Arith., **163** (2014) 161-177.