We discuss a recently presented boosted Kerr black hole solution which had already been used by other authors. This boosted metric is based on wrong assumptions regarding asymptotic inertial observers and moreover the performed boost is not a proper Lorentz transformation. This note aims to clarify some of the issues when boosting black holes and the necessary care in order to interpret them. As it is wrongly claimed that the presented boosted Kerr metric is of Bondi-Sachs type, we recall out some of the necessary requirements and difficulties, when the casting the Kerr metric into a metric with a surface forming null coordinate.

I. INTRODUCTION

Boosted black holes are relevant in gravitational physics. For example, the final black hole remnant of a binary black hole merger is in general boosted with respect to the rest frame of the two initial black holes. This property has important bearing for gravitational wave physics as it gives rise for an additional observable in gravitational wave astronomy – the gravitational wave memory [6, 7], which is the permanent displacement of test masses after the passage of a gravitational wave. This memory effect can be decomposed into two parts - an ordinary or linear memory effect related to a boost and a null memory effect related to the loss of energy of the radiating system by massless particles (electromagnetic radiation [8, 9], neutrinos [10] or gravitons [11, 12]). In particular, the extraction of physical observables like gravitational radiation [11, 12], linear and angular momentum [13–15] at null infinity needs to be done in a generalization of an inertial frame. These frames at null infinity are tied to a particular null tetrad and called Bondi frames. The corresponding coordinates are the Bondi coordinates. Bondi frames are in general related to one another by transformations of the Bondi-Metzner-Sachs (BMS) group [16], which include the infinite dimensional subgroup of supertranslations. These supertranslations relate different cross sections (“cuts”) of null infinity with each other. Their existence prevents the single out a canonical Poincare sub-group at null infinity. However for stationary metrics, like the Kerr metric, there exists a canonical way to set a preferred Poincare subgroup based in the notion of good cuts [17] or its generalization through nice sections [18].

Since a boost in Special Relativity is done with respect to observers in inertial frames, it is clear that an asymptotic boost in an asymptotically flat spacetime ought be done with respect to an associated Bondi frames. Notably, an expression for the Kerr metric approaching a Bondi frame is not known in an explicit closed analytic form. One of the reason is that the principal null directions of the Kerr solution are twisting. Meaning they do not generate null surfaces. Therefore, it is not a simple task to construct a Bondi-like coordinate system. For the asymptotic analysis, a way to approach a Bondi frame for the Kerr metric at null infinity was archived in [19] by introduction of a set of hyperboloidal coordinates. These coordinates are defined with respect to hypersurfaces that are null at null infinity and spacelike in its neighborhood.

Recently, an algorithm to construct boosted Kerr black hole solutions was presented in the peer-reviewed references [20, 21]. In the first work [20], the author presents a simplified analysis, where the Kerr black hole is boosted along z-axis, only. The subsequent article [21] covers the general boost in arbitrary directions. In both situations, the author claims that these solutions represent boosted Kerr metrics as “seen” by an asymptotic inertial observer. The proposed mechanism seems to be simple. Thus, making it favorable to use, if physical effects of moving rotating black holes ought to be studied. Indeed, follow up work of other authors [22, 23] using these metrics seems to validate them.

We analyze the metric presented in [21] in greater detail and clarify some of the issues arising from a misunderstanding of the meaning of an asymptotic inertial observer. As the mechanism for the boost in [21] uses the same (but more sophisticated) techniques, the faulty assumptions are taken over from [20] to [21]. Therefore, the main results of [21] can be questioned from the same
grounds. We will further show that for the metric presented in [20] (and consequently also for the proposed extension in [21]), it can not be deduced that it is the coordinate representation of a boosted Kerr metric with respect to an asymptotic Lorentzian observer. In particular, the discussed metrics contain an incomplete piece of a Lorentz transformation in a certain sense. More precisely, the coordinate representation of the ‘boosted Kerr metrics’ in [20, 21] only make use of an angular coordinate transformation of the original Kerr metric that could be thought as associated to an asymptotic Lorentzian observer. However, the additional transformations of the timelike and radial coordinates are yet missing. Therefore, the chosen coordinates do not represent adapted coordinates with respect to an inertial observer. Consequently, care must be taken in the interpretation of the ‘boosted’ Kerr metrics of [20, 21], because without the necessary care it can give rise to wrong results with respect to the physics related to moving black holes as measured by asymptotic inertial observers. For example, physical effects of a boosted rotating Kerr black hole (with respect to the proper asymptotic observer) do not differ at leading order from those of a boosted Schwarzschild black hole. This is clear, because for large values of the (proper) radial coordinate $r$, the effects of the spin of the black hole enter at higher order of a $1/r^n$ expansion than those resulting from the mass. There exist several ways to present a boosted Schwarzschild black hole in the literature. Some (e.g. [24]) use properly adapted coordinates to asymptotic inertial observers, while other make usage of non-inertial coordinates, as for example in terms of Newman-Unti coordinates, which in general do not conform an inertial (Bondi) frame [22]. In the last case, extra work and significant machinery is needed in order to extract physical information (see e.g. [14] or [26]). Another effect that cannot be reproduced by [20, 21] in a straightforward way is the fact that the comparison of an un-boosted Kerr black hole in its distant past with its boosted version of it in its distant future gives rise a gravitational wave memory and a corresponding supertranslation [7].

II. FAULTY POINTS IN THE BOOSTED SOLUTION

Here we point out the inconsistencies in [21] that do not capture the physics of asymptotic Lorentz transformation. In particular, we show that the mentioned solution can be easily obtained from a simple coordinate transformation in the angular directions applied to the original Kerr metric.

With respect to coordinates $\tilde{x}^\alpha = (\tilde{u}, \tilde{r}, \tilde{\theta}, \tilde{\phi})$, the outgoing Eddington-Finkelstein form of the Kerr metric is given by [27]

$$
\begin{align*}
\text{ds}^2 &= \left( r^2 + a^2 \cos^2 \theta \right) \left( \text{d}\tilde{u}^2 + \sin^2 \theta \text{d}\tilde{\phi}^2 \right) \\
&- 2 \left( \frac{\text{d}r + a \sin \theta \text{d}\tilde{\phi}}{\tilde{r}^2 + a^2 \cos^2 \theta} \right) \left( \text{d}\tilde{u} + a \sin \theta \text{d}\tilde{\phi} \right) \\
&- \left( 1 - \frac{2m\tilde{r}}{\tilde{r}^2 + a^2 \cos^2 \theta} \right) \left( \text{d}\tilde{u} + a \sin \theta \text{d}\tilde{\phi} \right)^2,
\end{align*}
$$

where $m$ is the mass and $a$ the specific angular momentum. In [21], the most general ‘boosted’ version of this metric with respect to coordinates $(u, r, \theta, \phi)$ is presented as (eq. (27) in [21])

$$
\begin{align*}
\text{ds}^2 &= \frac{r^2 + \Sigma^2}{\mathcal{K}^2} \left( \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2 \right) + \left( \frac{r^2 - 2mr + \Sigma^2}{r^2 + \Sigma^2} \right) \left[ \text{d}u - 2\mathcal{L} \cot \left( \frac{\theta}{2} \right) \text{d}\phi \right]^2 \\
&- 2 \left[ \text{d}u - 2\mathcal{L} \cot \left( \frac{\theta}{2} \right) \text{d}\phi \right] \left\{ \text{d}r + \frac{a}{\mathcal{K}^2} \left[ -n_1 \sin^2 \theta + (n_2 \cos \phi + n_3 \sin \phi) \sin \theta \cos \phi \right] \text{d}\phi + \frac{a}{\mathcal{K}^2} \left( n_2 \sin \phi - n_3 \cos \phi \right) \text{d}\theta \right\},
\end{align*}
$$

where

$$
\begin{align*}
\mathcal{K} &= A + B(\hat{x}^i n_i), \quad A^2 - B^2 = 1 \\
\Sigma &= \frac{B + A(\hat{x}^i n_i)}{A + B(\hat{x}^i n_i)} \\
\mathcal{L} &= \left( \frac{1 - \cos \theta}{\sin \theta} \right) \left( \frac{a}{2B^2} - \frac{\int \Sigma \sin \theta \text{d}\theta}{\mathcal{K}} \right),
\end{align*}
$$

with the general direction of the boost $n_i = (n_1, n_2, n_3)$ that is subject to $\delta_{ij} n_i n_j = 1$, the rapidity $\zeta$ to determine $A = \cosh \zeta$ and $B = \sinh \zeta$, and $\hat{x}^i = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)$. In [21, page 4] it is claimed

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2 Note, here are some corrections to the original form in [23]. The corrections are pointed out by Kerr himself in [23]. In particular, the positive sense of rotation is used. Moreover, in Kerr’s original paper the “advanced” time is called $u$ [23]. Kerr’s original paper should be corrected using $u \to -u$ and $a \to -a$.

3 Note, some slight change in notation to be in tune with standard notation for the Kerr metric; to obtain (3) in [23] make the following substitutions: $a \to \omega$, $A \to a$, $B \to b$. 
that “For $n_2 = 0 = n_3$ and $B = 0$ the metric (27) is the original Kerr metric in retarded Bondi-Sachs-type coordinates.” In addition, in [21] is also claimed that “The derivation and interpretation of this solution will be framed in the Bondi-Sachs (BS) characteristic formulation of gravitational wave emission in general relativity, where we have a clear and complete derivation of physical quantities and its conservation laws...”. Both statements are not true: regarding the former, an expression for the Kerr metric in explicit closed form in Bondi-Sachs-type coordinates is not known. Concerning the latter, a retarded Bondi coordinate system is characterized by a surface forming null coordinate \( \hat{u} \) such that null hypersurfaces \( \hat{u} = \text{const} \) are generated by a null geodesic congruence \( \ell_\mu = (d\hat{u})_\mu \) reaching future null infinity \( J^+ \). Consequently, \( g^{\hat{u}\hat{u}} = 0 \) is a necessary condition be satisfied by the coordinates. It is easy to see that this is not the case for the coordinates used [2]. An equivalent statement for the existence of such one-form \( \ell_\mu \) is that for defining a metric be of Bondi-Sachs type, it has to obey the conditions \( g_{rr} = g_{r\theta} = g_{r\phi} = 0 \) [12], which are violated in [2] by the the presence of term \( g_{r\phi} \). What the author wishes to say is that then the Kerr metric in its out-going Eddington-Finkelstein form is recovered.

If the parameter \( a = 0 \), the metric (11) reduces to the Schwarzschild solution expressed in outgoing-null polar coordinates (Eddington-Finkelstein):

\[
d s^2 = - \left( 1 - \frac{2m}{r} \right) d\hat{u}^2 - 2d\hat{u}d\hat{r} + \hat{r}^2 \left( d\hat{\theta}^2 + \sin^2 \hat{\theta} d\hat{\phi}^2 \right),
\]

Hereafter, we concentrate on the presentation in [20] since all of our arguments can be extended to show the invalidity of [21] for general “boosts” with using the proper adaptations.

For large values of \( \hat{r} \) on hypersurfaces \( \hat{u} = \text{const} \) the takes the form

\[
d s^2 = \left( \hat{r}^2 + a^2 \cos^2 \hat{\theta} \right) \left( d\hat{\theta}^2 + \sin^2 \hat{\theta} d\hat{\phi}^2 \right) - 2 \left( \hat{d} + a \sin^2 \hat{\theta} \hat{d} \hat{\phi} \right) \left( \hat{d} \hat{r} - a \sin \hat{\theta} \hat{d} \hat{\phi} \right) - \left( \hat{d} + a \sin \hat{\theta} \hat{d} \hat{\phi} \right)^2 + O \left( \frac{m}{\hat{r}} \right)
\]

which is a flat metric as can be shown by calculating the (vanishing) components of the Riemann tensor at leading order.

Next, we recall: given the standard Minkowski metric \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) in Cartesian coordinates \( \hat{x}^\mu = (\hat{t}, \hat{x}^i) \), its coordinate representation for an inertial observer in outgoing null coordinates in a rest frame follows from the coordinate transformation, \( \hat{t} = \hat{u} + \hat{r}, \hat{r}^2 = 2 \delta_{ij} \hat{x}^i \hat{x}^j, \hat{x}^i = (\hat{r} \sin \hat{\theta} \cos \hat{\phi}, \hat{r} \sin \hat{\theta} \sin \hat{\phi}, \hat{r} \cos \hat{\theta}) \) and has the form

\[
d s^2 = -d\hat{u}^2 - 2d\hat{u}d\hat{r} + \hat{r}^2 \left( d\hat{\theta}^2 + \sin^2 \hat{\theta} d\hat{\phi}^2 \right),
\]

see e.g. [19, 30] for a recent discussion regarding boosted black holes and inertial frames. Metric (8) is the inertial metric \( \eta_{\mu\nu} \) in outgoing polar null coordinate. If a general metric in outgoing null coordinates approaches the particular form of (8) at large distances from the source, it is said that the asymptotic observer is in a Bondi frame [11, 12, 31].

It is obvious that the leading order term of (7) is certainly not such Minkowski metric for \( a \neq 0 \). That is, if \( a \neq 0 \), the coordinates used in (7) do not correspond those of an inertial observer. However, setting \( a = 0 \), i.e. considering a non-rotating Kerr black hole a.k.a. the Schwarzschild black hole, (7) corresponds to the metric of an asymptotic inertial null metric in coordinates. Hereafter, we start considering the procedure of [20] assuming \( a = 0 \) and show that even in this case the resulting boosted Schwarzschild metric is not properly boosted with respect to an asymptotic observer in the associated inertial Bondi coordinates.

The “boosted” Schwarzschild metric of [20] (equation (23) in [20] with \( a = 0 \) is

\[
ds^2 = \frac{r^2 (d\theta^2 + \sin^2 \theta d\phi^2)}{(A + B \cos \theta)^2} - 2dudr - \left( 1 - \frac{2m}{r} \right) du^2.
\]

The first thing to note is that this metric is easily obtained by a simple change of only one of the angular coordinates in (6). This is achieved by setting \( a = 0 \) in (11) and performing the coordinate transformation

\[
\hat{u} = u \, , \, \hat{r} = r \, , \, \hat{\phi} = \phi.
\]

where \( A^2 - B^2 = 1 \). According to [20], the functions \( A \) and \( B \) relate to the boost velocity \( \beta \) like \( \beta = B/A \) and the rapidity parameter \( \zeta \) like \( A = \cosh \zeta \) and \( B = \sinh \zeta \). Moreover, it is never mentioned in [20] that their “boosted” Kerr metric in their equation (23) can be easily obtained applying the same transformation (11) to the Kerr metric (11), which is reproduced here for completeness

\[
ds^2 = \frac{r^2 + \Sigma}{(A + B \cos \theta)^2} (d\theta^2 + \sin^2 \theta d\phi^2)
\]

\[
- 2 \left[ du + \frac{a \sin^2 \theta}{(A + B \cos \theta)^2} d\phi \right] \left[ dr - \frac{a \sin \theta d\phi}{(A + B \cos \theta)^2} \right]
\]

\[
- \left( 1 - \frac{2mr}{r^2 + \Sigma} \right) \left[ du + \frac{a \sin \theta d\phi}{(A + B \cos \theta)^2} \right]^2;
\]

where \( \Sigma = a(B + A \cos \theta)(A + B \cos \theta)^{-1} \). In other words, despite the claim of [20] that the ‘boosted’ Kerr metric (11) is obtained as an exact stationary analytic solution, we remark that it is just the original Kerr metric in different angular coordinates. We further note and demonstrate below that (11) is not a proper asymptotic Lorentz transformation, since a Lorentz transformation does not
only change the angular coordinates, but also the temporal and radial coordinates. In particular, the asymptotic Lorentz transformation maps one asymptotic inertial metric \( \eta_{\mu\nu}(\bar{x}^\alpha) \) to another asymptotic inertial metric \( \eta_{\mu\nu}(x^\alpha) \).

It means that for large values of \( r \), any asymptotically flat metric in Bondi coordinates \( \{u, r, \theta, \phi\} \) transforms to \( \{\bar{u}, \bar{r}, \theta, \phi\} \) under the BMS group (an in particular under a Lorentz subgroup) like

\[
- d\bar{u}^2 + 2d\bar{u}d\bar{r} + \bar{r}^2(d\bar{\theta}^2 + \sin^2 \bar{\theta}d\bar{\phi}^2) + \mathcal{O}(1/\bar{r}) = - d\bar{u}^2 - 2dudr + r^2(d\theta^2 + \sin^2 \theta d\phi^2) + \mathcal{O}(1/r).
\]

(12)

For simplicity, consider a boost in \( z \) direction at large distances. Let \( \{\bar{t}, \bar{x}, \bar{y}, \bar{z}\} \) the un-boosted Cartesian coordinates, \( \{t, x, y, z\} \) the boosted Cartesian coordinates and \( (r, \theta, \phi) \) be the associated spherical coordinates in the boosted system with \( r^2 = \delta_{ij}x^ix^j \) and \( x^i = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta) \).

Taking \( \bar{v}^ \mu \bar{\rho}_\mu \) as tangent vector to the world lines of the un-boosted observers, corresponding boosted observers are tangent to \( v^ \mu = \gamma(1, \beta^\nu) \) with \( \gamma = -\beta^\nu \beta_\nu = (1 - \delta_{ij}\beta^i \beta^j)^{-1/2} \), so that the Lorentz transformation for the coordinates \( x^\mu \to \bar{x}^\mu \) and the radial functions \( r \to \bar{r} \) are given by

\[
\begin{align*}
\bar{r}^2 &= r^2 + 2\mu \bar{u} + \mu^2 + \beta^r \beta^\mu \mu^\mu = \frac{1}{1 + \beta^r} \left( r^2 + 2\mu \bar{u} + \mu^2 + \beta^r \beta^\mu \mu^\mu \right) \tag{13} \\
\end{align*}
\]

\[
\begin{align*}
\bar{r}^2 &= \bar{r}^2 + 2\bar{u} + \bar{u}^2 + \beta^\theta \beta^\phi \phi^\phi = \frac{1}{1 + \beta^\theta} \left( \bar{r}^2 + 2\bar{u} + \bar{u}^2 + \beta^\theta \beta^\phi \phi^\phi \right) \tag{14} \\
\end{align*}
\]

For a (inverse) boost in \( z \)-direction with \( \beta^r = \beta^\phi = 0 \) and \( \beta^\theta = \beta \), we find the relations

\[
\bar{u} = \gamma \left[ u + (r + 1) + \beta \cos \theta \right] \\
\bar{r} = r \left[ 1 + \gamma^2 \left( \frac{u}{r} + 1 + \beta \cos \theta \right) \right] + \left( \frac{u}{r} + 1 \right) \tag{15} \\
\cos(\bar{\theta}) = \frac{z}{\bar{r}} \\
= \frac{\gamma \cos \theta + \beta(\frac{u}{r} + 1)}{\sqrt{1 + \gamma^2 (\frac{u}{r} + 1)^2} - (\frac{u}{r} + 1)} \tag{16} \\
\bar{\phi} = \arctan \frac{\bar{u}}{\bar{r}} = \arctan \frac{u}{r}, \tag{17}
\]

between the un-boosted and boosted versions of the null coordinates \( \{\bar{u}, \bar{r}, \bar{\theta}, \bar{\phi}\} \to \{u = t - r, r, \theta, \phi\} \). For large distances (keeping \( u, \theta \) and \( \phi \) fixed) \( u, \theta, \phi \) reduce to

\[
\begin{align*}
\bar{u} &= \frac{u}{K(\theta)} + O\left(\frac{1}{r}\right) ; \quad \bar{r} = K(\theta) r + O(r^0) \tag{18} \\
\cos(\bar{\theta}) &= \frac{\beta + \cos(\theta)}{1 + \beta \cos(\theta)} + O\left(\frac{1}{r}\right) \tag{19} \\
\end{align*}
\]

with \( K(\theta) = \gamma(1 + \beta \cos \theta) \). Note, that the first part of \( K(\theta) \) is the commonly known relativistic aberration formula. Relations \( (18) \) and \( (20) \) are the asymptotic Lorentz transformation for a boost along the \( z \)-axis. This transformation is a subset of a larger transformation, which conform the BMS group. In fact, the BMS group is obtained in a more general framework by requiring a corresponding asymptotic behavior of the metric components when they are expressed in a Bondi system \( [11, 12, 31] \), and also in a geometrical way (see for example \( [33] \)).

It is not difficult to check that this Lorentz transformation applied to \( (11) \) with \( a = 0 \) maps the metric of an asymptotic inertial observer in coordinates \( \bar{x}^\mu \) to the metric of an asymptotic inertial observer in coordinates \( x^\mu \), as required by \( (12) \). The main point, we stress here, is that to make a Lorentz boost, a transformation in the \( \bar{u} \) and \( \bar{r} \) coordinates is needed. However, Eqs.(5) do not contain this part of the Lorentz transformation. Therefore, despite the claims of \( [20] \), the metric presented in that reference is not a properly boosted Kerr metric with respect to the adapted coordinates of an asymptotic inertial frame, since the needed transformations are not even completely carried-out in the Schwarzschild limit. More generally, discarding supertranslations, BMS transformations in a neighborhood of null infinity can be written in terms of stereographic coordinates (whose relation to the standard spherical coordinates is \( \zeta = e^{\phi} \cot(\frac{\phi}{2}) \)) as

\[
\begin{align*}
\bar{u} &= \frac{u}{K(\zeta, \bar{\zeta})} + O\left(\frac{1}{r}\right) ; \quad \bar{r} = K(\zeta, \bar{\zeta}) r + O(r^0), \tag{21} \\
\bar{\zeta} &= \frac{a\zeta + b}{c\zeta + d} + O\left(\frac{1}{r}\right), \tag{22}
\end{align*}
\]

where \( \{a, b, c, d\} \) are four complex parameters subject to the constraint \( ac - bd = 1 \) and \( K(\zeta, \bar{\zeta}) \) is given by \( \beta \)

\[
K(\zeta, \bar{\zeta}) = \frac{(a\zeta + c)(\bar{a}\bar{\zeta} + \bar{c}) + (b\zeta + d)(\bar{b}\bar{\zeta} + \bar{d})}{1 + \zeta \bar{\zeta}} \tag{23}
\]

We remark that the “generally boosted” Kerr metric presented in \( (21) \) can also be obtained from the Kerr metric \( (11) \) via the particular angular transformation \( (22) \) associated to a general boost. However, as mentioned above, even in that situation this transformation is not sufficient to express the metric in a Bondi system. Extra transformations are necessary, because for a Bondi system \( u \) must be a null surface forming coordinate, i.e. \( u = \text{const} \) should define surfaces generated by null vector fields reaching \( \mathcal{J}^+ \). This is not the case for the \( u \) coordinate present in \( (20, 21) \).

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5. To check this map, expressions for the \( O(r^{-1}, r^0) \) terms in \( (19) \) and \( (20) \) are also needed. They can be easily found from \( (19) \) and \( (20) \).

6. In fact, the BMS group is defined at null infinity and is given only for the part of the transformation for the null coordinate and the angular coordinates charting null infinity. The exact transformation of the radial coordinate depends of the kind of radial coordinate, which may be e.g. an area distance coordinate or an affine parameter.
In the Schwarzschild case of (9), the \( u = \) constant hypersurfaces are indeed null surfaces reaching null infinity. Nonetheless, the coordinates are not realizing a Bondi coordinate system either. In fact, (9) is expressed in a so called Newman-Unti coordinates (NU)\(^{33}\). More precisely, in terms of stereographic angular coordinates the metric is a particular case of a more general family of metrics known as Robinson-Trautman geometries given by\(^{22}\)

\[
ds^2 = r^2 \left[ \frac{d\zeta d\bar{\zeta}}{(P_0 V)^2} - 2 du dr - \left(1 - \frac{2m}{r} + \frac{V_u}{V} r \right) du^2, \right.
\]

with \( P_0 = 1 + \zeta \bar{\zeta} \), \( V = V(u, \zeta, \bar{\zeta}) \) and \( m = m(u) \). These metrics belong to the class of Robinson-Trautman solutions defined by the property that they admit a geodesic, shear-free and twist-free but expanding null congruence. Regarding (9), we have \( m_u = 0 \) and

\[
V = A + B \cos \theta = A + B \frac{\zeta \bar{\zeta} - 1}{1 + \zeta \bar{\zeta}},
\]

showing that also \( V_u = 0 \). Moreover, the coordinates \( \{u, r, \zeta, \bar{\zeta}\} \) correspond to a Bondi system only if \( V = 1 \) (rest frame). Note, we are not saying that the metric (24) could not be interpreted as a boosted black hole; what we are saying is that if these NU coordinates are used we must yet to relate it to a Bondi system in order to extract physical quantities. For example, as discussed in \(^{13}\), the total linear momentum \( P^\alpha \) for the metric (24) can be computed in a non-Bondi system from the formula

\[
P^\alpha = \int \frac{m}{V^3} \hat{e}^\alpha dS^2
\]

with \( dS^2 \) the surface element of a unit sphere and

\[
\hat{e}^\alpha = \left(1, \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}}, \frac{\zeta - \bar{\zeta}}{i(1 + \zeta \bar{\zeta})}, \frac{\zeta \bar{\zeta} - 1}{1 + \zeta \bar{\zeta}} \right).
\]

Note that this expression was also correctly used in \(^{24}\) to compute the four-momentum of its metrics.

However, some of the analysis carried out on the metrics \(^{20, 21}\) is misleading. For example, the location of the horizon for the ‘boosted’ metric (11) is measured to take the same value as in the Kerr metric. This was interpreted as being a consequence that a boost does not change null surfaces. It is true that boosts do not distort null surfaces, but its coordinate representation for an asymptotic boosted inertial observer, however, would be in general different. The reason why the coordinate location of the horizon for the ‘boosted’ Kerr metric (11) takes the same value as in the Kerr metric is because the radial coordinate was not changed by the coordinate transformation (c.f. (10)). Notwithstanding, it is well-known that the shapes of the boosted vs. unboosted horizon is coordinate dependent (see, e.g \(^{36, 37}\)). We note, if we were to attempt a similar procedure as in \(^{24, 21}\) for the location of a photon sphere \( S_{ph} \) in the boosted Schwarzschild metric (9), we would find it placed at the same radial coordinate \( r = 3m \) as in the un-boosted black hole, even when for this case the surface \( S_{ph} \) is not a null hypersurface.\(^7\) Again, it is only because of we are not properly transforming the radial and timelike coordinates.

In \(^{20}\), it is claimed that “The boosted Kerr geometry also presents an ergosphere...”; this is not surprising at all because \(^{20}\)’s metric is the Kerr metric after the coordinate transformation (10). The coordinate expression for the ergosphere of \(^{20, 21}\) shows a most complex dependence from the angular coordinates. Again, the relevant expression is analyzed by using the un-boosted (Kerr) radial coordinate and the ‘boosted’ angular coordinates. That is, there is again no proper use of the associated ‘boosted’ radial coordinate.

In any case, the geometrical definition of the ergosphere of the Kerr black hole is given by the set of points, where the (global) timelike Killing vector \( \partial_r \) becomes a null vector. This is a geometrical (coordinate-independent) definition. However, it is clear that for the analysis of ergosphere of a boosted Kerr black hole by an asymptotic observer, associated inertial coordinates \( \{u, r, \theta, \phi\} \) should be used instead of the mixed set of coordinates \( \{\bar{u}, \bar{r}, \theta, \phi\} \) like in \(^{20, 21}\).

We also stress the well known fact that Kerr’s original metric does not approach the Minkowski metric of an inertial observer for large radii (also seen in \(^{7}\)). Hence, it ought not be used for the discussion of physical effects resulting from a comparison of boosted and un-boosted black holes in the asymptotic regime. In fact, to unambiguously define a boost, an inertial observer needs to be able to singled out, so that it is clear with respect to which rest frame the boost is performed. Henceforth, one wishes to cast the black hole metric \( g_{\mu\nu} \) to be boosted into a form like \( g_{\mu\nu} = \eta_{\mu\nu} + \epsilon g_{\mu\nu} \) where \( \eta_{\mu\nu} \) is the Minkowski metric and \( \epsilon g_{\mu\nu} \) is a function of the coordinates. Such a representation of \( g_{\mu\nu} \) can be obtained two different ways: (i) a linearization and (ii) finding a Kerr-Schild representation of the black hole metric. The linearisation (i) covers three branches. One realisation of (i) is the introduction of a “smallness” parameter \( \epsilon \) measuring the deviation from flat spacetime (i.e. \( 0 \approx \epsilon \approx |\bar{g}_{\mu\nu}| \) for every component of \( \bar{g}_{\mu\nu} \)). The second realisation is the assumption that at given distance from the black hole an inertial observer is introduced and Fermi normal coordi-
On top of that, Kerr-Schild metric have the property that the metric is written as \( g_{\mu\nu} = \eta_{\mu\nu} + H k_{\mu} k_{\nu} \), where \( H \) is a scalar function and \( k_{\mu} \) is a null vector with respect to \( \eta_{\mu\nu} \) and \( g_{\mu\nu} \). Such ansatz was, in fact, first used by Trautman in the study of radiative spacetimes \([4]\) and it was crucial for finding of the Kerr solution \([28]\). In particular, it had recently been pointed out that the spacetimes of the Schwarzschild and Kerr black hole in Kerr-Schild form have not only one inertial frame serving as a background spacetime to define a boost, but two \([19, 30]\). These two Minkowski backgrounds are tied to the outgoing and outgoing principal null directions of the respective metric in Kerr-Schild form. The inertial coordinates of these Minkowski backgrounds transform between each other via a non-linear coordinate transformation. Indeed it was shown in \([7, 19, 30]\) that for the correct value of the boost memory at future null infinity, the discussion of the boost must be done in the Minkowski background of the ingoing formulation.

For a Schwarzschild/Kerr black hole which is initially at rest and then ejected with mass \( m \) and velocity \( \beta \) along the \( z \)-axis, the boost memory at null infinity is \([4, 7, 15, 30]\)

\[
\Delta \sigma = \frac{4\gamma m \beta^2 \sin^2 \theta}{1 - \beta \cos \theta}.
\]  

The supertranslation \( \alpha \) relating the retarded time cuts \( u = \infty \) and \( u = -\infty \) at null infinity is \([7]\)

\[
\alpha = 4m\gamma (1 - \beta \cos \theta) \ln(1 - \beta \cos \theta) \tag{29}
\]

Above relations \([25] \text{ and } [28] \) can by no means be reproduced from expression \([11]\).

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9 An brief remark about this fact can already be found the in Boyer-Lindquist paper [41].
