PUISEUX MONOIDS AND TRANSFER HOMOMORPHISMS

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Abstract. There are several families of atomic monoids whose arithmetical invariants have received a great deal of attention during the last two decades. The factorization theory of finitely generated monoids, strongly primary monoids, Krull monoids, and C-monoids are among the most systematically studied. Puiseux monoids, which are additive submonoids of $\mathbb{Q}_{\geq 0}$ consisting of nonnegative rational numbers, have only been studied recently. In this paper, we provide evidence that this family comprises plenty of monoids with a basically unexplored atomic structure. We do this by showing that the arithmetical invariants of the well-studied atomic monoids mentioned earlier cannot be transferred to most Puiseux monoids via homomorphisms that preserve atomic configurations, i.e., transfer homomorphisms. Specifically, we show that transfer homomorphisms from a non-finitely generated atomic Puiseux monoid to a finitely generated monoid do not exist. We also find a large family of Puiseux monoids that fail to be strongly primary. In addition, we prove that the only nontrivial Puiseux monoid that accepts a transfer homomorphism to a Krull monoid is $\mathbb{N}_0$. Finally, we classify the Puiseux monoids that happen to be C-monoids.

1. Introduction

The study of the phenomenon of non-unique factorizations in the ring of integers $\mathcal{O}_K$ of an algebraic number field $K$ was initiated by L. Carlitz in the 1950’s, and it was later carried out on more general integral domains. As a result, many techniques to measure the non-uniqueness of factorizations in several families of integral domains were systematically developed during the second half of the last century (see [2] and references therein). However, it was not until recently that questions about the non-uniqueness of factorizations were abstractly formulated in the context of commutative cancellative monoids. This was possible because most of the factorization-related questions inside an integral domain are purely multiplicative in essence. The fundamental goal of abstract (or modern) factorization theory is to measure how far is a commutative cancellative monoid from being factorial by using different arithmetical invariants.

At this point, the arithmetical invariants of several families of atomic monoids have been intensively studied. Finitely generated monoids, strongly primary monoids, Krull monoids, and C-monoids are among the most studied. These families of monoids not only have very diverse arithmetical properties, but also have proved to be useful in the study of the factorization theory of less-understood atomic monoids via transfer

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homomorphisms. A monoid homomorphism is said to be transfer if somehow it allows to shift the atomic structure of its codomain back to its domain (see Definition 3.1). Therefore if one is willing to know the factorization invariants of a given monoid, it suffices to find a transfer homomorphism from such a monoid to a better-understood monoid and carry over the desired factorization properties.

Puiseux monoids were recently introduced as a rational generalization of numerical monoids. Many families of atomic Puiseux monoids were explored in [16, 17, 19], and their elasticity was studied in [20]. However, it is still unanswered whether the non-unique factorization behavior in Puiseux monoids is somehow similar to that of some of the monoids whose factorization properties are already well-understood. To give a partial answer to this, we will determine which atomic Puiseux monoids can be the domain of a transfer homomorphism to some of the monoids whose arithmetical invariants have already been studied. In particular, we consider finitely generated monoids, Krull monoids, and C-monoids as our transfer codomains.

The content of this paper is organized as follows. In Section 2, we establish the notation we shall be using later, and we formally present most of the fundamental concepts needed in this paper. Then, in Section 3 we show that homomorphisms between Puiseux monoids can only be given by rational multiplication, which will allow us to characterize the transfer homomorphisms between Puiseux monoids. We also present a family of Puiseux monoids whose members have \( \mathbb{Z} \) as their group of automorphisms. Section 4 is devoted to characterize the Puiseux monoids admitting a transfer homomorphism to some finitely generated monoid. Then, in Section 5 we investigate which Puiseux monoids are strongly primary. Finally, in Section 6 we prove that the only Puiseux monoid that is transfer Krull is the additive monoid \( \mathbb{N}_0 \). We use this information to classify the Puiseux monoids which happen to be C-monoids.

2. Background

To begin with let us introduce the fundamental concepts related to our exposition as an excuse to establish the notation we need. The reader can consult Grillet [21] for information on commutative semigroups and Geroldinger and Halter-Koch [10] for extensive background in non-unique factorization theory of atomic monoids.

Throughout this sequel, we let \( \mathbb{N} \) denote the set of positive integers, and we set \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). For \( X \subseteq \mathbb{R} \) and \( r \in \mathbb{R} \), we set \( X_{\leq r} := \{ x \in X \mid x \leq r \} \); with a similar spirit we use the symbols \( X_{\geq r} \), \( X_{< r} \), and \( X_{> r} \). If \( q \in \mathbb{Q}_{>0} \), then we call the unique \( a, b \in \mathbb{N} \) such that \( q = a/b \) and \( \gcd(a, b) = 1 \) the numerator and denominator of \( q \) and denote them by \( n(q) \) and \( d(q) \), respectively. For each subset \( Q \) of \( \mathbb{Q}_{>0} \), we call the sets \( n(Q) = \{ n(q) \mid q \in Q \} \) and \( d(Q) = \{ d(q) \mid q \in Q \} \) the numerator set and denominator set of \( Q \), respectively.

As usual, a semigroup is a pair \( (S, \ast) \), where \( S \) is a set and \( \ast \) is an associative binary operation in \( S \); we write \( S \) instead of \( (S, \ast) \) provided that \( \ast \) is clear from the
context. However, inside the scope of this paper, a monoid is a commutative cancellative semigroup with identity (cf. the standard definition of monoid). To comply with established conventions, we will be using simultaneously additive and multiplicative notations; however, the context will always save us from the risk of ambiguity. Let $M$ be a monoid written additively. We set $M^\times := M \setminus \{0\}$ and, as usual, we let $M^\times$ denote the set of units (i.e., invertible elements) of $M$. The monoid $M$ is reduced if $M^\times = \{0\}$. For $a, b \in M$, we say that $a$ divides $b$ in $M$ if there exists $c \in M$ such that $b = a + c$; in this case we write $a \mid_M b$. An element $a \in M \setminus M^\times$ is an atom if whenever $a = u + v$ for some $u, v \in M$, either $u \in M^\times$ or $v \in M^\times$. Atoms are the building blocks in factorization theory; this motivates the special notation

$$\mathcal{A}(M) := \{a \in M \mid a \text{ is an atom of } M\}.$$ 

For $S \subseteq M$, we let $\langle S \rangle$ denote the smallest submonoid of $M$ containing $S$, and we say that $S$ generates $M$ if $M = \langle S \rangle$. The monoid $M$ is said to be finitely generated if it can be generated by a finite set. On the other hand, we say that $M$ is atomic if $M = \langle \mathcal{A}(M) \rangle$. A monoid is factorial if every element can be written as a sum of primes. As every prime is an atom, every factorial monoid is atomic.

Let $\rho \subseteq M \times M$ be an equivalence relation on $M$, and let $[a]_{\rho}$ denote the equivalence class of $a \in M$. We say that $\rho$ is a congruence if for all $a, b, c \in M$ such that $(a, b) \in \rho$ it follows that $(ca, cb) \in \rho$. Congruences are precisely the equivalence relations that are compatible with the operation of $M$, meaning that $M/\rho := \{[a]_{\rho} \mid a \in M\}$ is a commutative semigroup with identity (no necessarily cancellative). Two elements $a, b \in M$ are associates, and we write $a \simeq b$, if $a = ub$ for some $u \in M^\times$. Being associates defines a congruence relation $\simeq$ on $M$, and $M_{\text{red}} := M/\simeq$ is called the associated reduced semigroup of $M$.

We say that a multiplicative monoid $F$ is free abelian with basis $P \subset F$ if every element $a \in F$ can be written uniquely in the form

$$a = \prod_{p \in P} p^{v_p(a)},$$

where $v_p(a) \in \mathbb{N}_0$ and $v_p(a) > 0$ only for finitely many elements $p \in P$. The monoid $F$ is determined by $P$ up to canonical isomorphism, so we shall also denote $F$ by $\mathcal{F}(P)$. By the fundamental theorem of arithmetic, the multiplicative monoid $\mathbb{N}$ is free on the set of prime numbers. In this case, we can extend $v_p$ to $\mathbb{Q}_{\geq 0}$ as follows. For $r \in \mathbb{Q}_{\geq 0}$ let $v_p(r) := v_p(n(r)) - v_p(d(r))$ and set $v_p(0) = \infty$.

The free abelian monoid on $\mathcal{A}(M)$, denoted by $\mathbb{Z}(M)$, is called the factorization monoid of $M$, and the elements of $\mathbb{Z}(M)$ are called factorizations. If $z = a_1 \ldots a_n \in \mathbb{Z}(M)$ for some $n \in \mathbb{N}_0$ and $a_1, \ldots, a_n \in \mathcal{A}(M)$, then $n$ is the length of the factorization $z$; the length of $z$ is denoted by $|z|$. The unique homomorphism

$$\phi: \mathbb{Z}(M) \to M \text{ satisfying } \phi(a) = a \text{ for all } a \in \mathcal{A}(M)$$
is called the factorization homomorphism of $M$. Additionally, for $x \in M^\bullet$,

$$Z(x) := \phi^{-1}(x) \subseteq Z(M)$$

is the set of factorizations of $x$. By definition, we set $Z(0) = \{0\}$. Note that the monoid $M$ is atomic if and only if $Z(x)$ is not empty for all $x \in M$. For each $x \in M$, the set of lengths of $x$ is defined by

$$L(x) := \{|z| : z \in Z(x)\}.$$ 

We say that the monoid $M$ is half-factorial if $|L(x)| = 1$ for all $x \in M$. On the other hand, if $L(x)$ is a finite set for all $x \in M$, then we say that $M$ is a BF-monoid. The system of sets of lengths of $M$ is defined by

$$L(M) := \{L(x) \mid x \in M\}.$$ 

The system of sets of lengths is an arithmetical invariant of atomic monoids that has received significant attention in recent years (see [1, 4] and the literature cited there).

A very special family of atomic monoids is that one comprising all numerical monoids, cofinite submonoids of the additive monoid $N_0$. We say that a numerical monoid is proper if it is strictly contained in $N_0$. Each numerical monoid has a unique minimal set of generators, which is finite. Moreover, if $\{a_1, \ldots, a_n\}$ is the minimal set of generators for a numerical monoid $N$, then $A(N) = \{a_1, \ldots, a_n\}$ and $\gcd(a_1, \ldots, a_n) = 1$. As a result, every numerical monoid is atomic and contains only finitely many atoms. The Frobenius number of $N$, denoted by $F(N)$, is the minimum $n \in N$ such that $Z_{\geq n} \subset N$. An introduction to numerical monoids can be found in [7].

An additive submonoid of $Q_{\geq 0}$ is called a Puiseux monoid. Puiseux monoids are a natural generalization of numerical monoids. However, the general atomic structure of Puiseux monoids drastically differs from that one of numerical monoids. Puiseux monoids are not always atomic; for instance, consider $(1/2^n \mid n \in N)$. On the other hand, if an atomic Puiseux monoid $M$ is not isomorphic to a numerical monoid, then $A(M)$ is infinite. The atomic structure of Puiseux monoids has been studied in [17] and [19], where several families of atomic Puiseux monoids were described.

3. HOMOMORPHISMS BETWEEN PUISEUX MONOIDS

In this section we present characterizations of homomorphisms and transfer homomorphisms between Puiseux monoids. Let us start by introducing the concept of a transfer homomorphism, which is going to play a central role in this paper.

**Definition 3.1.** A monoid homomorphism $\theta : M \to N$ is said to be a transfer homomorphism if the following conditions hold:

(T1) $N = \theta(M)N^\times$ and $\theta^{-1}(N^\times) = M^\times$;

(T2) if $\theta(a) = b_1b_2$ for $a \in M$ and $b_1, b_2 \in N$, then there exist $a_1, a_2 \in M$ such that $a = a_1a_2$ and $\theta(a_i) = b_i$ for $i \in \{1, 2\}$.
We proceed to characterize the homomorphisms between Puiseux monoids. This will immediately yield a characterization of those homomorphisms between Puiseux monoids that happen to be transfer homomorphisms.

**Proposition 3.2.** If \( \phi: M \rightarrow N \) be a homomorphism between Puiseux monoids, then the following conditions hold.

1. There exists \( q \in \mathbb{Q}_{\geq 0} \) such that \( \phi(x) = qx \) for all \( x \in M \), i.e., \( \phi \) is given by rational multiplication.

2. The homomorphism \( \phi \) is a transfer homomorphism if and only if it is surjective.

**Proof.** Let us argue first that \( \phi \) is given by rational multiplication. It is clear that if a map \( P \rightarrow P' \) between two Puiseux monoids is multiplication by a rational number, then it is a monoid homomorphism. Thus, it suffices to verify that the only homomorphisms of Puiseux monoids are those given by rational multiplication. To do this, consider the Puiseux monoid homomorphism \( \varphi: P \rightarrow P' \). Because the trivial homomorphism is multiplication by 0, there is no loss in assuming that \( P \neq \{0\} \). Let \( \{n_1, \ldots, n_k\} \) be a minimal set of generators for the additive monoid \( N = P \cap \mathbb{N}_0 \). Notice that \( N \neq \{0\} \) and, therefore, \( k \geq 1 \). The fact that \( \varphi \) is nontrivial implies that \( \varphi(n_j) \neq 0 \) for some \( j \in \{1, \ldots, k\} \). Set \( q = \varphi(n_j)/n_j \), and take \( r \in P^* \) and \( c_1, \ldots, c_k \in \mathbb{N}_0 \) satisfying that \( n(r) = c_1n_1 + \cdots + c_kn_k \). Since \( n_i\varphi(n_j) = \varphi(n_in_j) = n_j\varphi(n_i) \) for each \( i \in \{1, \ldots, k\} \), one obtains

\[
\varphi(r) = \frac{1}{d(r)}\varphi(n(r)) = \frac{1}{d(r)} \sum_{i=1}^{k} c_i\varphi(n_i) = \frac{1}{d(r)} \sum_{i=1}^{k} c_i n_i \frac{\varphi(n_j)}{n_j} = rq.
\]

As a result, the homomorphism \( \varphi \) is just multiplication by \( q \in \mathbb{Q}_{>0} \).

It is easy to see that condition (2) is a direct consequence of condition (1), which completes the proof. \( \Box \)

**Remark 3.3.** A Puiseux monoid \( M \) is said to be increasing (resp., decreasing) if \( M \) can be generated by an increasing (resp., decreasing) sequence of rational numbers. Also, we say that \( M \) is bounded if \( M \) can be generated by a bounded sequence of rational numbers, and it is said to be strongly bounded if it can be generated by a sequence of rational numbers whose numerator set is bounded. Finally, \( M \) is called dense if it contains 0 as a limit point. Although the definitions just given are not algebraic in nature, we should notice that they are all preserved by Puiseux monoid isomorphisms. This explains why all of them have been useful in the study of the atomic structure of Puiseux monoids (see [17] and [19]).

With notation as in Definition 3.1, when \( M \) and \( N \) are reduced, we can restate the first condition above as

\((T1')\ \theta \) is surjective and \( \theta^{-1}(1) = 1 \).
We have already mentioned that a transfer homomorphism allows us to shift the
atomic structure and the arithmetic of length of factorizations from its codomain to
its domain. This property is formally described in the following proposition.

**Proposition 3.4.** [9 Proposition 1.3.2] If \( \theta : M \to N \) is a transfer homomorphism of
atomic monoids, then the following conditions hold:

1. \( a \in \mathcal{A}(M) \) if and only if \( \theta(a) \in \mathcal{A}(N) \);
2. \( M \) is atomic if and only if \( N \) is atomic;
3. \( L_M(x) = L_N(\theta(x)) \) for all \( x \in M \);
4. \( \mathcal{L}(M) = \mathcal{L}(N) \), and so \( M \) is a BF-monoid if and only if \( N \) is a BF-monoid.

For a Puiseux monoid \( M \), let \( \text{Aut}(M) \) denote the group of automorphisms of \( M \). As
we have seen in Proposition 3.2, the set of homomorphisms between Puiseux monoids
is very exclusive. In particular, we might wonder whether \( \text{Aut}(M) \) is always trivial.
However, it is not hard to verify, for instance, that when \( M_1 = \langle 1/2^n \mid n \in \mathbb{N} \rangle \),
multiplication by 1/2 is in \( \text{Aut}(M_1) \). This example might not be the most desirable
because \( M_1 \) fails to be atomic; in fact, \( M_1 \) does not contain any atoms. The next
proposition exhibits a family of atomic monoids whose groups of automorphisms are
nontrivial. First, let us introduce a family of atomic Puiseux monoids whose atomicity
is used in the proof.

For \( r \in \mathbb{Q}_{>0} \), the monoid \( M_r = \langle r^n \mid n \in \mathbb{N} \rangle \) is the *multiplicatively r-cyclic* Puiseux
monoid. If \( n(r), d(r) > 1 \), then \( M_r \) is atomic with \( \mathcal{A}(M_r) = \{ r^n \mid n \in \mathbb{N} \} \) (see [19
Theorem 6.2]).

**Proposition 3.5.** Let \( r \in \mathbb{Q}_{>0} \) such that \( n(r), d(r) > 1 \). If \( M = \langle r^n \mid n \in \mathbb{Z} \rangle \), then
\( \text{Aut}(M) \cong \mathbb{Z} \).

**Proof.** Set \( A = \{ r^n \mid n \in \mathbb{Z} \} \). For \( n \in \mathbb{Z} \), the fact that \( r^n A = A \) implies that
multiplication by \( r^n \) is an endomorphism of \( M \) whose inverse is given by multiplication
by \( r^{-n} \). Thus, multiplication by any integer power of \( r \) is an automorphism of \( M \). To
prove that these are the only elements of \( \text{Aut}(M) \), let us first argue that \( M \) is atomic
with \( \mathcal{A}(M) = A \).

Assume first that \( r < 1 \). Fix \( k \in \mathbb{Z} \), and let us check that \( r^k \in \mathcal{A}(M) \). To do this
notice that the monoid \( \langle r^n \mid n \geq k \rangle \) is the isomorphic image (under multiplication by
\( r^{k-1} \)) of the multiplicatively \( r \)-cyclic Puiseux monoid \( M_r \), which is atomic with set of
atoms \( A = \{ r^n \mid n \in \mathbb{N} \} \). Since \( r \in \mathcal{A}(M_r) \), it follows that \( r \notin \langle r^n \mid n > 1 \rangle \).
Then \( r^k \notin \langle r^n \mid n > k \rangle \). As \( r < 1 \), no atom in \( \{ r^n \mid n < k \} \) divides \( r^k \). Hence \( r^k \notin \langle A \setminus \{ r^k \} \rangle \)
and, therefore, \( r^k \in \mathcal{A}(M) \). As a result, \( \mathcal{A}(M) = A \).

Now suppose that \( r > 1 \). As before, fix \( k \in \mathbb{Z} \). Because \( r > 1 \), proving that
\( r^k \in \mathcal{A}(M) \) amounts to showing that \( r^k \notin \langle r^n \mid n < k \rangle \). Let us assume, by way
of contradiction, that this is not the case. Then \( r^k = a_1 r^{n_1} + \cdots + a_t r^{n_t} \) for some
\( a_1, \ldots, a_t \in \mathbb{N} \) and \( n_1, \ldots, n_t \in \mathbb{N} \) with \( k > n_1 > \cdots > n_t \). As a consequence,
\[ r^{k-n_1+1} \in \langle r^{n_1-n_1+1}, r^{n_2-n_1+1}, \ldots, r \rangle, \] which contradicts the fact that \( r^{k-n_1+1} \in A(M_r) \).

As in the previous case, we conclude that \( A(M) = A \).

By Proposition 3.2, any automorphism of \( M \) is given by rational multiplication. Take \( s \in \mathbb{Q}_{>0} \) such that \( \phi_s \in \text{Aut}(M) \), where \( \phi_s \) consists in left multiplication by \( s \). Because \( \phi_s \) must send atoms to atoms, it follows that \( sr = \phi_s(r) \in A \). Therefore \( s \) must be an integer power of \( r \). Hence \( \text{Aut}(M) \) is precisely \( A \) when seen as a multiplicative subgroup of \( \mathbb{Q} \). As \( A \) is the infinite cyclic group, the proof follows. \( \square \)

4. Finite Transfer Puiseux Monoids

Now we turn to characterize the transfer homomorphisms from Puiseux monoids to finitely generated monoids.

**Definition 4.1.** We say that a Puiseux monoid \( M \) is transfer finite if there exists a transfer homomorphism from \( M \) to a finitely generated monoid.

By the fundamental structure theorem of finitely generated abelian groups, it immediately follows that every finitely generated monoid \( F \) is a submonoid of a group \( T \times \mathbb{Z}^\beta \) for some finite abelian group \( T \) and \( \beta \in \mathbb{N}_0 \). In case of \( F \) being reduced, it can be thought of as a submonoid of \( T \times \mathbb{N}_0^\beta \).

Condition (T2) in the definition of a transfer homomorphism \( \theta: M \to F \) is crucial to transfer the factorization behavior of \( F \) to \( M \). However, the reader might wonder how much the set \( \text{Hom}(M, F) \) will increase if we drop condition (T2). Surprisingly, the set of homomorphisms will remain the same as long as we impose \( F \) to be reduced.

This fact facilitates to classify the Puiseux monoids that happen to be transfer finite, as we will prove in the next theorem. First, notice that if \( \phi: M \to N \) is a monoid homomorphism, then the map \( \phi_{\text{red}}: M_{\text{red}} \to N_{\text{red}} \) defined by \( \phi_{\text{red}}(aM^\times) = \phi(a)N^\times \) is also a monoid homomorphism.

**Theorem 4.2.** Let \( M \) be a nontrivial Puiseux monoid, and let \( F \) be a finitely generated (additive) monoid.

1. If \( \theta: M \to F \) is a homomorphism satisfying \( \theta^{-1}(0) = \{0\} \), then \( M \) is isomorphic to a numerical monoid.
2. The Puiseux monoid \( M \) is transfer finite if and only if it is isomorphic to a numerical monoid.

**Proof.** We argue first part (1). It is easy to see that \( \theta_{\text{red}}: M_{\text{red}} = M \to F_{\text{red}} \) is also a transfer homomorphism. So we can assume, without loss of generality, that \( F \) is reduced. Suppose that \( F \) is a submonoid of \( T \times \mathbb{N}_0^\beta \), where \( T \) is a finite abelian group and \( \beta \in \mathbb{N}_0 \). First, assume, by way of contradiction, that \( \beta = 0 \). In this case, it is not hard to verify that \( \theta(M) \) must be a subgroup of \( T \). If \( \alpha = |\theta(M)| \) and \( r \in M^\times \), then \( \theta(\alpha r) = \alpha \theta(r) = 0 \). This contradicts that \( \theta^{-1}(0) = \{0\} \). Thus, \( \beta \geq 1 \).
Define $\pi: T \times \mathbb{N}_0^\beta \to \mathbb{N}_0^\beta$ by $\pi(t, v) = v$ for all $t \in T$ and $v \in \mathbb{N}_0^\beta$. Let us verify that $\pi(\theta(M))$ is finitely generated. Take $x = (x_1, \ldots, x_\beta) \in \pi(\theta(M))^*$, and let $d = \gcd(x_1, \ldots, x_\beta)$. We shall verify that $\pi(\theta(M)) \subseteq \langle x/d \rangle$. To do so, consider $y = (y_1, \ldots, y_\beta) \in \pi(\theta(M))^*$. Now take $r, s \in M^*$ such that $\pi(\theta(r)) = x$ and $\pi(\theta(s)) = y$, and take $m, n \in \mathbb{N}$ satisfying that $\gcd(m, n) = 1$ and $mr = ns$. Because
\[ mx = \pi(\theta(mr)) = \pi(\theta(ns)) = ny, \]
one finds that $mx_i = ny_i$ for $i = 1, \ldots, \beta$. As $\gcd(m, n) = 1$, it follows that $n$ divides each $x_i$, i.e., $d/n \in \mathbb{N}$. As a result,
\[ y = \frac{m}{n}x = \left(\frac{md}{n}\right)\frac{x}{d} \in \left\langle \frac{x}{d} \right\rangle. \]
Hence $\pi(\theta(M)) \subseteq \langle x/d \rangle$. Because $\langle x/d \rangle$ is isomorphic to $\mathbb{N}_0$, it follows that $\pi(\theta(M))$ is finitely generated.

We show now that $M$ is also finitely generated, which amounts to proving that $\pi \circ \theta: M \to \mathbb{N}_0^\beta$ is injective. First, let us verify that $\pi$ is injective when restricted to $\theta(M)$. As $\theta(M)$ is a submonoid of the reduced monoid $F$, it is also reduced. Suppose that $(t_1, v), (t_2, v) \in \theta(M)$, and let us check that $t_1 = t_2$. If $v = 0$, then $t_1 = t_2 = 0$ because $\theta(M)^* = \{0\}$. Otherwise, there exist $r, s \in M^*$ such that $\theta(r) = (t_1, v)$ and $\theta(s) = (t_2, v)$. Take $m, n \in \mathbb{N}$ such that $mr = ns$. Since
\[ m(t_1, v) = m\theta(r) = n\theta(s) = n(t_2, v) \]
and $v \neq 0$, one finds that $m = n$ and, therefore, $r = s$. This, in turn, implies that $t_1 = t_2$. Hence the restriction of $\pi$ to $\theta(M)$ is injective.

To conclude the proof of part (1), we show that $\theta$ is also injective. Let $r, s \in M$ such that $\theta(r) = \theta(s) \neq 0$. Taking $m, n \in \mathbb{N}$ satisfying $mr = ns$, we have
\[ m\theta(r) = \theta(mr) = \theta(ns) = n\theta(s). \]
Since $\theta(M)$ is reduced, the element $\theta(r)$ must be torsion-free in $T \times \mathbb{N}_0^\beta$. Thus, $m = n$, which implies that $r = s$. As $\theta^{-1}(0) = \{0\}$, it follows that $|\theta^{-1}(a)| = 1$ for all $a \in \theta(M)$. Therefore $\theta$ is injective, leading us to the injectivity of $\pi \circ \theta$. Now that fact that $\pi(\theta(M))$ is finitely generated implies that $M$ is also finitely generated. Hence $M$ must be isomorphic to a numerical monoid.

Finally, let us argue part (2) of the theorem. For the direct implication, assume that the homomorphism $\theta: M \to F$ is a transfer homomorphism. As we did in the proof of part (1), we can assume that $F$ is a reduced. As both $M$ and $F$ are reduced, condition (T1') yields $\theta^{-1}(0) = \{0\}$. Now it follows by part (1) that the Puiseux monoid $M$ is isomorphic to a numerical monoid. For the reverse implication, just take $\theta$ to be the identity map. \qed
Imposing the homomorphism \( \theta: M \to F \) in Theorem 4.2 to satisfy \( \theta^{-1}(0) = \{0\} \) is not superfluous even if \( M \) is atomic. Then next example sheds some light upon this observation.

Example 4.3. Let \( p_1, p_2, \ldots \), be an enumeration of the odd prime numbers, and let \( M = \langle 1/p_n \mid n \in \mathbb{N} \rangle \). It is not hard to verify that \( \mathcal{A}(M) = \{1/p_n \mid n \in \mathbb{N} \} \). This implies that \( M \) is atomic. Now define \( \theta: M \to \mathbb{Z}_2 \) by setting \( \theta(0) = 0 \), \( \theta(r) = 0 \) if \( n(r) \) is even, and \( \theta(r) = 1 \) if \( n(r) \) is odd. It follows immediately that \( \theta \) is a surjective monoid homomorphism. However, \( M \) is not isomorphic to any numerical monoid because it contains infinitely many atoms.

5. Strongly Primary Puiseux Monoids

In this section we investigate which Puiseux monoids are strongly primary. In the case of Puiseux monoids, being strongly primary is equivalent to being finitary. In general, finitary monoids provide a common algebraic framework to study not only the arithmetic of strongly primary monoids but also that one of \( v \)-noetherian \( G \)-monoids (see [10] Section 2.7). The structure of strongly primary monoids was first studied by Satyanarayana in [23] and has received substantial attention in the literature since then (see [15] and references therein). In particular, the class of strongly primary monoids yields multiplicative models for a large class of one-dimensional local domains (see [10], Proposition 2.10.7).

All monoids mentioned in this section are assumed to be reduced.

Definition 5.1. Let \( M \) be a monoid.

1. A submonoid \( S \) of \( M \) is called divisor-closed provided that for all \( x \in M \) and \( s \in S \) the fact that \( x \mid_M s \) implies that \( x \in S \).
2. The monoid \( M \) is called primary if it is nontrivial and its only divisor-closed submonoids are \( \{0\} \) and \( M \).
3. The monoid \( M \) is called finitary if \( M \) is a BF-monoid and there exist \( n \in \mathbb{N} \) and a finite subset \( S \subseteq M^\bullet \) such that \( nM^\bullet \subseteq S + M \).
4. The monoid \( M \) is called strongly primary provided that \( M \) is both primary and finitary.

Let \( M \) be a Puiseux monoid, and let \( M' \) be a nontrivial proper submonoid of \( M \). Notice that for all \( x \in M \setminus M' \) and \( y \in M' \) satisfying that \( x \mid_M y \), the fact that \( x \mid_M n(x)d(y)y \in M' \) immediately implies that \( M' \) is not a divisor-closed submonoid of \( M \). Thus, \( M \) is primary. On the other hand, suppose that \( F \) is a finitely generated monoid, say \( F = \langle S \rangle \) for some finite subset \( S \) of \( F^\bullet \). Then the fact that \( F^\bullet = S + F \) immediately implies that \( F \) is finitary. In particular, every nontrivial finitely generated Puiseux monoid is finitary and, therefore, strongly primary. The next proposition summarizes the observations made in this paragraph.
Proposition 5.2.

1. Every nontrivial Puiseux monoid is primary.
2. Every finitely generated Puiseux monoid is strongly primary.

A Puiseux monoid that is not finitely generated may fail to be strongly primary (see, for example, Proposition [5.7]). However, in Proposition [5.4] and Proposition [5.6] we exhibit two infinite families of non-finitely generated Puiseux monoids that are strongly primary. To argue Proposition 5.4, we will use the following result, which is a weaker version of [16, Proposition 4.5].

Proposition 5.3. If \( M \) is a Puiseux monoid satisfying that 0 is not a limit point of \( M^* \), then \( M \) is a BF-monoid.

The next two propositions introduce two families of (non-finitely generated) strongly primary Puiseux monoids.

Proposition 5.4. Let \( p, q \in \mathbb{N} \) such that \( \gcd(p, q) = 1 \), and let \( \{S_n\} \) be an inclusion-decreasing sequence of numerical monoids. If a function \( f: \mathbb{N} \to \mathbb{N} \) satisfies that \( f(1) = 1 \) and \( q^{f(n+1)-f(n)} - p^n > p \max\{F(S_n), a \mid a \in \mathcal{A}(S_n)\} \) for every \( n \in \mathbb{N} \), then the Puiseux monoid

\[
\left\langle \frac{q^{f(n)}}{p^n}s \mid n \in \mathbb{N} \text{ and } s \in S_n \right\rangle
\]

is strongly primary.

Proof. Set \( M = \langle q^{f(n)}s/p^n \mid n \in \mathbb{N} \text{ and } s \in S_n \rangle \), and for each \( n \in \mathbb{N} \) set \( A_n = \mathcal{A}(S_n) \). First, we argue that \( M \) is a BF-monoid. To do so, observe that for each \( n \in \mathbb{N} \) the fact that \( q^{f(n+1)-f(n)} > p \max A_n \) implies that

\[
\min \frac{q^{f(n+1)}}{p^n+1}A_{n+1} \geq \frac{q^{f(n+1)}}{p^n+1} > \frac{q^{f(n)}}{p^n} \max A_n.
\]

Therefore the generating set \( \cup_{n \in \mathbb{N}}(q^{f(n)/p^n})A_n \) of \( M \) can be listed as an increasing sequence of rational numbers. As \( M \) is generated by an increasing sequence of rational numbers, 0 cannot be a limit point of \( M^* \). Hence \( M \) is a BF-monoid by Proposition 5.3.

We proceed to prove that \( M \) is finitary. Since \( M \cap \mathbb{N}_0 \) is a submonoid of \((\mathbb{N}_0, +)\), it is atomic and \( \mathcal{A}(M \cap \mathbb{N}_0) \) is finite. Take \( S \) to be the finite set \( \mathcal{A}(M \cap \mathbb{N}_0) \cup qa_1 \subseteq M \). We are done once we verify the inclusion \( pM^* \subseteq S + M \). Fix \( n \in \mathbb{N} \) and \( a \in A_{n+1} \). Because \( q^{f(n)}a \in q^{f(n)}S_{n+1} \subseteq q^{f(n)}S_n \in M \cap \mathbb{N}_0 \), the element \( q^{f(n)}a \) is divisible in \( M \) by an element of \( S \). On the other hand, \( (q^{f(n+1)-f(n)} - p^n)a \geq q^{f(n+1)-f(n)} - p^n > F(S_n) \), which implies that \( (q^{f(n+1)-f(n)} - p^n)a q^{f(n)}/p^n \in M \). Thus,

\[
p\left(\frac{q^{f(n+1)}}{p^n+1}\right) = q^{f(n)}a + (q^{f(n+1)-f(n)} - p^n) \frac{q^{f(n)}}{p^n}a \in S + M.
\]
In addition, the fact that \( qa \in qS_{n+1} \subseteq qS_1 \) guarantees \( qa \) is divisible in \( M \) by some element of \( S \). This, in turn, implies that
\[
p(q^{f(1)}/p)a = qa \in S + M.
\]
As a result, for each \( x \in M^\bullet \) it follows that
\[
px \in m(S + M) \subseteq S + M
\]
for some \( m \in \mathbb{N} \). Hence \( pM^\bullet \subseteq S + M \), as desired. \( \square \)

**Example 5.5.** Consider the Puiseux monoid
\[
M = \left\langle \frac{3^{n^2+1}}{2^n}, \frac{5 \cdot 3^{n^2}}{2^n} \right| n \in \mathbb{N} \right\rangle.
\]
Taking \( S_n \) to be the numerical monoid \( \langle 3, 5 \rangle \) for each \( n \in \mathbb{N} \) and defining the function
\[
f: \mathbb{N} \to \mathbb{N} \text{ by } f(n) = n^2,
\]
we can rewrite the Puiseux monoid \( M \) as follows:
\[
M = \left\langle \frac{3^{f(n)}}{2^n}s \right| n \in \mathbb{N} \text{ and } s \in S_n \right\rangle.
\]
Since \( F(S_n) = 7 \) for each \( n \in \mathbb{N} \), it follows that
\[
3^{f(n+1) - f(n)} - 2^n = 3^{2n+1} - 2^n > 14 = 2 \max\{3, 5, F(S_n)\}.
\]
As \( f(1) = 1 \), Proposition 5.4 guarantees that \( M \) is a strongly primary Puiseux monoid.

As in Section 3 for \( r \in \mathbb{Q}_{>0} \) we let \( M_r \) denote the multiplicatively \( r \)-cyclic Puiseux monoid \( \langle r^n | n \in \mathbb{N} \rangle \).

**Proposition 5.6.** For each \( r \in \mathbb{Q}_{>1} \), the Puiseux monoid \( M_r \) is strongly primary.

**Proof.** Since \( r > 1 \), it follows that 0 is not a limit point of \( M_r^\bullet \). Thus, Proposition 5.3 ensures that \( M_r \) is a BF-monoid. On the other hand, it was proved in [19] that
\[
\mathcal{A}(M_r) = \{ r^n | n \in \mathbb{N} \}.
\]
Now we check that \( n(r) \) divides \( n(r)r^j \) in \( M_r \) for every \( j \in \mathbb{N}_0 \). If \( j = 0 \), then \( n(r) | M_r \), \( n(r)r^j \) follows trivially. Hence it suffices to assume that \( j \in \mathbb{N} \).

In this case, the fact that \( n(r)(r - 1) = r(n(r) - d(r)) \) implies that
\[
n(r)r^j - n(r) = n(r)(r - 1) \sum_{i=0}^{j-1} r^i = r(n(r) - d(r)) \sum_{i=0}^{j-1} r^i = (n(r) - d(r)) \sum_{i=0}^{j-1} r^{i+1} \subseteq M_r.
\]
Therefore \( n(r) \) divides \( n(r)r^j \) in \( M_r \). To show now that \( M_r \) is finitary, we take \( n = d(r) \) and \( S = \{ n(r) \} \) and verify that \( nM_r^\bullet \subseteq S + M_r \). For \( q \in M_r^\bullet \), take \( \alpha_1, \ldots, \alpha_t \in \mathbb{N}_0 \) such that \( q = \sum \alpha_i r^i \), and fix \( k \in \{ 1, \ldots, t \} \) such that \( \alpha_k > 0 \). Because \( n(r) \) divides \( n(r)r^{k-1} \) in \( M_r \), there exists \( s \in M_r \) satisfying that \( n(r)r^{k-1} = n(r) + s \). Thus,
\[
nq = d(r)\alpha_k r^k + \sum_{i \neq k} d(r)\alpha_i r^i
\]
\[
= n(r)r^{k-1} + (\alpha_k - 1)d(r)r^k + \sum_{i \neq k} d(r)\alpha_i r^i
\]
\[
= n(r) + (\alpha_k - 1)d(r)r^k + \sum_{i \neq k} d(r)\alpha_i r^i,
\]
which implies that \( nq \in S + M_r \). Since \( nM_r^* \subseteq S + M_r \), one obtains that \( M_r \) is finitary and, therefore, strongly primary. \( \square \)

We conclude this section providing a family of Puiseux monoids that fail to be strongly primary.

**Proposition 5.7.** Let \( \{a_n\} \) be a sequence of positive rational numbers satisfying that \( \gcd(d(a_i), d(a_j)) = 1 \) for any \( i \neq j \). Then the Puiseux monoid \( \langle a_n \mid n \in \mathbb{N} \rangle \) is atomic but not strongly primary.

**Proof.** Set \( M = \langle a_n \mid n \in \mathbb{N} \rangle \). It is not difficult to verify that \( A(M) = \{a_n \mid n \in \mathbb{N} \} \) from the fact that \( \gcd(d(a_i), d(a_j)) = 1 \) for any \( i \neq j \); we leave the details to the reader. This implies, in particular, that \( M \) is atomic.

Suppose, by way of contradiction, that \( M \) is finitary. Choose \( n \in \mathbb{N} \) and a finite subset \( S \) of \( M^* \) such that \( nM^* \subseteq S + M \). Let \( S' \) be a finite subset of \( A(M) \) such that for each \( s \in S \) there is at least one atom in \( S' \) dividing \( s \) in \( M \). After substituting \( S \) by \( S' \), we can assume that \( S \subseteq A(M) \). Since \( nM^* \subseteq S + M \) and \( \inf(S + M) \geq \min S \), it follows that \( 0 \) cannot be a limit point of \( M^* \). Fix \( \epsilon > 0 \) such that \( \epsilon < \inf M^* \). Since \( \gcd(d(a_i), d(a_j)) = 1 \), there exists \( j \in \mathbb{N} \) such that \( d(a_j) > \max\{n, \max d(S)\} \). Because \( nM^* \subseteq S + M \), one can write

\[
na_j = a_s + \sum_{i=1}^{k} \alpha_i a_i
\]

for some \( k \in \mathbb{N} \), \( a_s \in S \), and \( \alpha_i \in \mathbb{N}_0 \). Applying the \( d(a_j) \)-valuation to both sides of (5.2) and using the fact that \( \gcd(d(a_i), d(a_j)) = 1 \), it is not hard to find that \( d(a_j) \) divides \( n - \alpha_j \). Now \( d(a_j) > n \) yields \( n = \alpha_j \). This, along with (5.2), would force \( a_s = 0 \), which contradicts that \( S \subseteq A(M) \). Thus, \( M \) is not strongly primary. \( \square \)

**Remark 5.8.** The previous proposition not only shows that Puiseux monoids are not, in general, strongly primary, but also illustrates that a natural bounding, ordering, or topological restriction under which a Puiseux monoid is guaranteed to be strongly primary is rather unlikely. For example, consider the Puiseux monoids

\[
M_1 = \left\langle \frac{1}{p} \mid p \text{ is prime} \right\rangle, \quad M_2 = \left\langle \frac{p - 1}{p} \mid p \text{ is prime} \right\rangle, \quad \text{and} \quad M_3 = \left\langle \frac{p^2 + 1}{p} \mid p \text{ is prime} \right\rangle.
\]

It is not hard to check that the sets of atoms of \( M_1 \), \( M_2 \), and \( M_3 \) are precisely the generating sets displayed. Therefore \( M_1 \) is strongly bounded, \( M_2 \) is bounded, and \( M_3 \) is not bounded. We can also see that \( M_1 \) is a decreasing Puiseux monoid, while \( M_2 \) is increasing. Furthermore, notice that \( 0 \) is a limit point of \( M_1^* \), but \( 0 \) is not a limit point of \( M_2^* \). Finally, Proposition 5.7 ensures that \( M_1 \), \( M_2 \), and \( M_3 \) are all strongly primary.
6. Puiseux Monoids Are Almost Never Transfer Krull

We dedicate this section to show that the atomic structure of Puiseux monoids almost never can be obtained by transferring back that one of Krull monoids; specifically we shall prove that the existence of a transfer homomorphism from a nontrivial Puiseux monoid to a Krull monoid forces the domain to be isomorphic to \((\mathbb{N}_0,+}\). The we use this information to show that only finitely generated Puiseux monoids admit transfer homomorphisms to C-monoids. Let us start by giving the definition of a Krull monoid.

**Definition 6.1.** A monoid \(K\) is called a **Krull monoid** if there is a monoid homomorphism \(\varphi: K \to D\), where \(D\) is a free abelian monoid and \(\varphi\) satisfies the following two conditions:

1. if \(a, b \in K\) and \(\varphi(a) \mid_D \varphi(b)\), then \(a \mid_K b\);
2. for every \(d \in D\) there exist \(a_1, \ldots, a_n \in K\) with \(d = \gcd\{\varphi(a_1), \ldots, \varphi(a_n)\}\).

With notation as in Definition 6.1, it is easy to see that \(K\) is a Krull monoid if and only if \(K_{\text{red}}\) is a Krull monoid. The basis elements of \(D\) are called the **prime divisors** of \(K\). The abelian group \(\text{Cl}(K) := D/\varphi(K)\) is called the **class group** of \(K\) (see [10, Section 2.3]). As Krull monoids are isomorphic to submonoids of free abelian monoids, Krull monoids are atomic.

The factorization theory of Krull monoids has been significantly studied (see [5, 13] and references therein). The class of Krull monoids contains many well-studied types of monoids, including the multiplicative monoid of the ring of integers of an algebraic number, the Hilbert monoids, and the regular congruence monoids. These and further examples of Krull monoids are presented in [9, Section 5] and [10, Section 2.3].

From the point of view of factorization theory, perhaps the most important family of Krull monoids is that one consisting of block monoids, which we are about to introduce. This is because block monoids capture the essence of the arithmetic of lengths of factorizations in Krull monoids. Let \(G\) be an abelian group and \(\mathcal{F}(G)\) the free abelian monoid on \(G\). An element \(X = g_1 \ldots g_l \in \mathcal{F}(G)\) is called a **sequence over** \(G\). The **length** of \(X\) is defined as

\[
|X| = l = \sum_{g \in G} v_g(X).
\]

For every \(I \subseteq [1, l]\), the sequence \(Y = \prod_{i \in I} g_i\) is called a **subsequence** of \(X\). The subsequences are precisely the divisors of \(X\) in the free abelian monoid \(\mathcal{F}(G)\). The submonoid

\[
\mathcal{B}(G) := \left\{ X \in \mathcal{F}(G) \mid \sum_{g \in G} v_g(X)g = 0 \right\}
\]

of \(\mathcal{F}(G)\) is called the **block monoid** on \(G\), and its elements are referred to as zero-sum sequences or blocks over \(G\) ([10] Section 2.5] is a good general reference on block
monoids). Furthermore, if $G_0$ is a subset of $G$, then the submonoid
\[ \mathcal{B}(G_0) := \{ X \in \mathcal{B}(G) \mid v_g(X) = 0 \text{ if } g \notin G_0 \} \]
of $\mathcal{B}(G)$ is called the restriction of the block monoid $\mathcal{B}(G)$ to $G_0$. For $X \in \mathcal{B}(G_0)$, the support of $X$ in $G_0$ is defined to be
\[ \text{supp}_{G_0}(X) := \{ g \in G_0 \mid v_g(X) > 0 \}. \]
As mentioned before, the relevance of block monoids in the theory of non-unique factorizations lies in the next result.

**Proposition 6.2.** [10, Theorem 3.4.10.3] Let $K$ be a Krull monoid with class group $G$ and let $G_0$ be the set of classes of $G$ which contain prime divisors. Then
\[ \mathcal{L}(K) = \mathcal{L}(\mathcal{B}(G_0)). \]

As a consequence, understanding the arithmetic of lengths of factorizations in Krull monoids amounts to understanding the same in block monoids.

**Definition 6.3.** A Puiseux monoid $M$ is transfer Krull if there exist an abelian group $G$, a subset $G_0$ of $G$, and a transfer homomorphism $\theta : M \to \mathcal{B}(G_0)$.

**Remark:** Our definition of a transfer Krull monoid coincides with the definition given in [3, Section 4]; this is because in the present setting the concepts of a transfer homomorphism and the concept of a weak transfer homomorphism coincide by [3, Lemma 2.3.(3)].

We denote the field of fractions of an integral domain $R$ by $\mathfrak{q}(R)$. For subsets $X, Y$ of $\mathfrak{q}(R)$ we set $(X : Y) := \{ x \in \mathfrak{q}(R) \mid xY \subseteq X \}$. In addition, $R$ is called a Krull domain if $R^\times$ is a Krull monoid. In this case, the divisor class group of $R$, denoted by $\mathcal{C}(R)$, measures the extent to which factorizations in $R$ fail to be unique (see [10, Section 2.10]). Unlike Krull domains/monoids, which have been central objects in commutative algebra since mid-nineteenth century, transfer Krull monoids (which generalize the concept of Krull monoids) were introduced more recently. Let us proceed to present a few examples of transfer Krull monoids.

**Examples of transfer Krull monoids:**

1. Let $H$ be a half-factorial monoid, and let $\theta : H \to \mathcal{B}(\{0\})$ be the map defined by $\theta(h) = 0$ if $h \in \mathcal{A}(H)$ and $\theta(h) = 1$ if $h \in H^\times$. As the map $\theta$ is a transfer homomorphism, it follows that $H$ is a transfer Krull monoid.

2. Let $R$ be a Krull domain, and let $K$ be a subring of $R$ with the same field of fractions. Suppose, in addition, that the following three conditions hold:
   (a) $R = KR^\times$;
   (b) $K \cap R^\times = K^\times$;
   (c) $(K : R)$ is a maximal ideal of $K$ (see [10, Proposition 3.7.5]).
Then the inclusion map $K^\bullet \hookrightarrow R^\bullet$ is a transfer homomorphism and, therefore, $K^\bullet$ is a transfer Krull monoid.

(3) Transfer Krull monoids can also be defined in a non-commutative context (see, for instance, [3]). Let $R$ be a bounded HNP (hereditary Noetherian prime) ring. If every stably free left $R$-ideal is free, then $R^\bullet$ is a transfer Krull monoid (see [24, Theorem 4.4] for details).

There are also many monoids that fail to be transfer Krull. Examples of non-transfer Krull monoids in a non-commutative setting are provided by [6, Proposition 4.11], [12, Corollary 4.4], and [25, Theorem 1.2]. On the other hand, Theorem 6.6 and the next proposition (which follows from [14, Theorem 5.5]) yield examples of non-transfer Krull monoids in a commutative context.

**Proposition 6.4.** Every proper numerical monoid fails to be transfer Krull.

The next lemma will be used in the proof of Theorem 6.6.

**Lemma 6.5.** If $\{a_n\}$ is an infinite sequence of positive integers, then there exists $m \in \mathbb{N}$ such that $a_{m+1} \in \langle a_1, \ldots, a_m \rangle$.

**Proof.** If $\{a_n\}$ is bounded there is a term that repeats infinitely many times, making the conclusion of the lemma obvious. Thus, suppose that $\{a_n\}$ is not bounded. Let $\{a_{n_j}\}$ be a subsequence of $\{a_n\}$ satisfying that

$$a_{n_{j+1}} > \prod_{i=1}^{j} a_{n_i} \tag{6.1}$$

for every $j \in \mathbb{N}$. Now, for each natural number $j$, set $d_j = \gcd(a_{n_1}, \ldots, a_{n_j})$, and notice that $d_{j+1} \mid d_j$ for every $j \in \mathbb{N}$. Therefore $d_{k+1} = d_k$ must hold for some $k$. In particular, $d_k \mid a_{n_{k+1}}$. On the other hand, condition (6.1) ensures that $a_{n_{k+1}}/d_k$ is greater than the Frobenius number of the numerical monoid $\langle a_{n_1}/d_k, \ldots, a_{n_k}/d_k \rangle$. This implies that $a_{n_{k+1}} \in \langle a_{n_1}, \ldots, a_{n_k} \rangle$. The lemma follows by taking $m = n_{k+1} - 1$. \□

Now we are in a position to prove that atomic Puiseux monoids are almost never transfer Krull.

**Theorem 6.6.** If a nontrivial Puiseux monoid is transfer Krull, then it must be isomorphic to $(\mathbb{N}_0, +)$.

**Proof.** Let $M$ be a nontrivial Puiseux monoid that happens to be transfer Krull. As Krull monoids are atomic, $M$ is atomic by Proposition 3.4. Let $G$ be an abelian group, and let $\theta: M \rightarrow B(G_0)$ be a transfer homomorphism, where $G_0$ is a subset of $G$. Because both $M$ and $B(G_0)$ are reduced, $\theta^{-1}(0) = \{0\}$. Assume, by way of
contradiction, that $M$ is not isomorphic to a numerical monoid. Take $X \in \mathcal{B}(G_0)^*$ and $r, s \in M^*$ such that $\theta(r) = \theta(s) = X$. Taking $m, n \in \mathbb{N}$ such that $mr = ns$, one obtains

\begin{equation}
\prod_{g \in G_0} g^{m_{\nu_g}(X)} = \theta(r)^m = \theta(s)^n = \prod_{g \in G_0} g^{n_{\nu_g}(X)}.
\end{equation}

Since $|X| \geq 1$ and $m_{\nu_g}(X) = n_{\nu_g}(X)$ for every $g \in G_0$, it follows that $m = n$, which yields $r = s$. Hence the preimage under $\theta$ of each element of $\mathcal{B}(G_0)^*$ is a singleton. This, along with the fact that $\theta^{-1}(\emptyset) = \{0\}$, implies that $\theta$ is injective. In addition, the same equality (6.2) implies that

$$\text{supp}_{G_0}(\theta(a)) = \text{supp}_{G_0}(\theta(a'))$$

for all $a, a' \in A(M)$. As a consequence, any two elements of $\theta(M^*)$ have the same support, and we can assume, without loss of generality, that $G_0$ is finite. Let $G_0 =: \{g_1, \ldots, g_t\}$ be the common support. List the set $A(M)$ as a sequence $\{a_n\}$, and let $A_n = \theta(a_n)$ for each $n \in \mathbb{N}$. Because $\theta$ is injective, $A_i \neq A_j$ when $i \neq j$. Now, for any pair $(i, j) \in \mathbb{N}^2$, there exist $c_i, c_j \in \mathbb{N}$ such that $c_i a_i = c_j a_j$. For each $n \in \{1, \ldots, t\}$, we can apply $\nu_{g_n} \circ \theta$ to the equality $c_i a_i = c_j a_j$ to get $c_i \nu_{g_n}(A_i) = c_j \nu_{g_n}(A_j)$. After rewriting this equality, one obtains that

\begin{equation}
\frac{\nu_{g_n}(A_i)}{\nu_{g_n}(A_j)} = \frac{c_j}{c_i} = \frac{\nu_{g_1}(A_i)}{\nu_{g_1}(A_j)}
\end{equation}

for each $n \in \{1, \ldots, t\}$. On the other hand, notice that Lemma 6.5 guarantees the existence of $m \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{N}_0$ such that

\begin{equation}
\nu_{g_1}(A_{m+1}) = \sum_{i=1}^{m} \alpha_i \nu_{g_1}(A_i).
\end{equation}

By (6.3), it follows that the equality (6.4) holds when we replace $g_1$ by any other element of $G_0$ (exactly with the same $\alpha_i$’s). As a result, we obtain

$$A_{m+1} = \prod_{j=1}^{[G_0]} g_j^{\nu_{g_j}(A_{m+1})} = \prod_{j=1}^{[G_0]} m_{\nu_{g_j}(A_i)} g_j^{\nu_{g_j}(A_i)} = \prod_{i=1}^{m} \left( \prod_{j=1}^{[G_0]} g_j^{\nu_{g_j}(A_i)} \right) \alpha_i = \prod_{i=1}^{m} A_i^{\alpha_i}.$$ 

This contradicts the fact that $A_{m+1}$ is an atom of the block monoid $\mathcal{B}(G_0)$. Therefore $M$ must be isomorphic to a numerical monoid. Now the direct implication of the proof follows by Proposition 6.4. For the reverse implication, it suffices to notice that $\theta: 1 \mapsto [1_G]$ is an isomorphism from $(\mathbb{N}_0, +)$ to the block monoid $\mathcal{B}(G)$, where $G$ is the trivial group.

\[ \square \]

**Corollary 6.7.** A Puiseux monoid is a Krull monoid if and only if it can be generated by one element.
Perhaps the second most-systematically studied family of atomic monoids is that one comprising the C-monoids. We would like to know under which conditions a Puiseux monoid happens to be a C-monoid.

Any monoid $M$ can be embedded into a quotient group $g(M)$, which is unique up to canonical isomorphism. Let $D$ be a multiplicative monoid with quotient group $g(D)$, and let $M$ be a submonoid of $D$. Two elements $x, y \in D$ are said to be $M$-equivalent provided that $x^{-1}M \cap D = y^{-1}M \cap D$. It can be easily checked that being $M$-equivalent defines a congruence relation on $D$. For each $x \in D$, let $[x]_M$ denote the congruence class of $x$. The set $C^*(M, D) := \{[x]_M | x \in (D \setminus D^\times) \cup \{1\}\}$ is a commutative semigroup with identity, which is called the reduced class semigroup of $M$ in $D$.

**Definition 6.8.** A monoid $M$ is called a C-monoid if it is a submonoid of a factorial monoid $F$ such that $F^\times \cap M = M^\times$ and $C^*(M, F)$ is finite.

With notation as in Definition 6.8, we say that $M$ is a C-monoid defined in $F$. A C-monoid can be defined in more than one factorial monoid; however, there is a canonical way of choosing $F$ (see [9, Theorem 5.6.A.3]). Because C-monoids are submonoids of factorial monoids, they are atomic. The family of C-monoids allows us to study the arithmetic of non-integrally closed Noetherian domains.

Given a multiplicative monoid $M$ with quotient group $g(M)$, we say that $x \in g(M)$ is almost integral over $M$ if there exists $c \in M$ such that $cx^n \in M$ for every $n \in \mathbb{N}$. The subset of $g(M)$ consisting of all almost integral elements over $M$ is denoted by $\widehat{M}$ and called the complete integral closure of $M$.

Let $R$ be an integral domain with field of fractions $q(R)$. An ideal $I$ of $R$ is divisorial if $(R : (R : I)) = I$. The domain $R$ is called a Mori domain if it satisfies the ascending chain condition on divisorial ideals. Finally, for the domain $R$ we set $\widehat{R} = \widehat{R^\times} \cup \{0\}$, where $R^\times$ is the multiplicative monoid of $R$.

**Example 6.9.** If $A$ is a Mori domain, then $R = \widehat{A}$ is a Krull domain. Moreover, if $\mathfrak{f} = (A : R)$ is nonzero and both the quotient ring $R/\mathfrak{f}$ and the class group $C(R)$ are finite, then $A^\times$ is a C-monoid (see [10, Theorem 2.11.9]). More examples of C-monoids can be found in [11] and [22].

The next theorem is used in the proof of Proposition 6.11.

**Theorem 6.10.** [10, Theorem 2.9.11(2)] The complete integral closure of a C-monoid is a Krull monoid.

**Proposition 6.11.** A nontrivial Puiseux monoid is a C-monoid if and only if it is isomorphic to a numerical monoid.
Proof. Let $M$ be a nontrivial Puiseux monoid that is also a C-monoid. Let $\hat{M}$ be the complete integral closure of $M$. Observe first that if $x \in g(M) \cap \mathbb{Q}_{<0}$, then
$$S_{x,r} := \{ r + nx \mid n \in \mathbb{N} \}$$
contains only finitely many positive rational numbers for all $r \in M$. As a result, $|S_{x,r} \cap M| < \infty$ for all $r \in M$, which implies that no negative element of $g(M)$ is almost integral over $M$. Because $\hat{M}$ is a monoid and it is contained in $\mathbb{Q}_{\geq 0}$, it must be a Puiseux monoid. By Theorem 6.10 the monoid $\hat{M}$ is a Krull monoid; in particular, it is transfer Krull. Now Theorem 6.6 ensures that $\hat{M}$ is isomorphic to $(\mathbb{N}_0, +)$. Finally, the fact that $M$ is a submonoid of $\hat{M}$ forces $M$ to be isomorphic to a numerical monoid.

For the reverse implication, it suffices to note that for every proper numerical monoid $N$, any two natural numbers greater than the Frobenius number of $N$ are $N$-equivalent, which implies that $\mathcal{C}^*(N, \mathbb{N}_0)$ is finite. As $\mathbb{N}_0$ is also a C-monoid, the proof follows. □

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