ON LANDIS CONJECTURE FOR THE FRACTIONAL SCHRÖDINGER EQUATION

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ABSTRACT. In this paper, we study the Landis-type conjecture for the general fractional Schrödinger equation \((-P)^s + q)u = 0\) with fractional power \(s \in (0,1)\), where \(P = \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k\), with \(a_{jk}(x) \approx \delta_{jk}\) as \(|x| \to \infty\). For the differentiable potential \(q\), if a solution decays at a rate \(\exp(-|x|^{1+})\), then this solution vanishes identically. For the non-differentiable potential \(q\), if a solution decays at a rate \(\exp(-|x|^\alpha)\), with \(\alpha > 4s/(4s - 1)\), then this solution must again be trivial. As \(s \to 1\), note that \(4s/(4s - 1) \to 4/3\), which is the optimal exponent for the classical Schrödinger equation. The proof relies on delicate Carleman-type estimates.

1. INTRODUCTION

In this work, we study a Landis-type conjecture for the fractional Schrödinger equation

\[( (-P)^s + q)u = 0 \text{ in } \mathbb{R}^n, \]

with \(s \in (0,1)\) and \(|q(x)| \leq 1\), where \((-P)^s\) can be defined by the following functional integration

\[ (-P)^s u(x) := \frac{1}{\Gamma(-s)} \int_0^\infty (e^{tP} - 1)u(x) \frac{dt}{t^{1+s}} \]

and \(\{e^{tP}\}_{t \geq 0}\) is the heat-diffusion semigroup generated by \(-P\) (see for example [ST10] or [GLX17]). Here, \(P\) is a second order elliptic operator in divergence form, i.e.,

\[ P = \sum_{j,k=1}^{n} \partial_j a_{jk}(x) \partial_k \]

with the ellipticity condition

\[ \lambda |\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(x) \xi_j \xi_k \leq \lambda^{-1} |\xi|^2 \text{ for some constant } 0 < \lambda \leq 1. \]

Assume that \(a_{jk} = a_{kj}\) for all \(1 \leq j, k \leq n\), \(a_{jk}\) are Lipschitz and satisfy

\[ \max_{1 \leq j, k \leq n} \sup_{|x| \geq 1} |a_{jk}(x) - \delta_{jk}(x)| + \max_{1 \leq j, k \leq n} \sup_{|x| \geq 1} |x||\nabla a_{jk}(x)| \leq \epsilon \]

for some sufficiently small \(\epsilon > 0\) and

\[ \max_{1 \leq j, k \leq n} \sup_{|x| \geq 1} |\nabla^2 a_{jk}(x)| \leq C \]

for some positive constant \(C\).

In this paper we prove the following Landis-type conjecture for the fractional Schrödinger equations.
Theorem 1.1. Let \( s \in (0, 1) \) and assume that \( u \in H^s(\mathbb{R}^n) \) is a solution to (1.1) with (1.2), (1.3) and (1.4). We assume that the potential \( q \in C^1(\mathbb{R}^n) \) satisfies \(|q(x)| \leq 1\) and

\[ |x| |\nabla q(x)| \leq 1. \]

If \( u \) further satisfies

\[ \int_{\mathbb{R}^n} e^{ix|\alpha|} |u|^2 \, dx \leq C < \infty \quad \text{for some } \alpha > 1, \]

then \( u \equiv 0 \).

We also have the following result for non-differentiable potential \( q \).

Theorem 1.2. Let \( s \in (1/4, 1) \) and assume that \( u \in H^s(\mathbb{R}^n) \) is a solution to (1.1) with (1.2), (1.3), and (1.4). Now we assume that the potential \( q \) satisfies \(|q(x)| \leq 1\). If \( u \) satisfies

\[ \int_{\mathbb{R}^n} e^{ix|\alpha|} |u|^2 \, dx \leq C < \infty \quad \text{for some } \alpha > \frac{4s}{4s - 1}, \]

then \( u \equiv 0 \).

Remark 1.3. When \( s = \frac{1}{2} \), Theorem 1.1 and Theorem 1.2 still hold without the second derivatives bound (1.4).

Remark 1.4. As in the case \( a_{jk} = \delta_{jk} \) in [RW19], we prove Theorem 1.2 using the same splitting arguments. So we will also have the same restriction \( s \in (1/4, 1) \) due to the subellipticity nature.

The main tool of proving Theorem 1.1 and 1.2 is Carleman estimates. However, due to the non-locality of \((-P)^s\), the techniques here are far more complicated than those for the classical case, i.e., \( s = 1 \). One of the major tricks is to localize \((-P)^s\), which is motivated by Caffarelli-Silvestre’s fundamental work [CS07]. Here we will use the Caffarelli-Silvestre type extension of \((-P)^s\) proved in [ST10] and [Sti10]. After localizing \((-P)^s\), we will derive a Carleman estimate mimicking the one proved in [RS17]. This Carleman estimate enables us to pass the boundary decay to the bulk decay.

We face other difficulties in dealing with \((-P)^s\). Using the Fourier transform, we can easily see that the additivity property \((-\Delta)^{s} (-\Delta)^{\beta} = (-\Delta)^{\alpha + \beta}\) holds and also \((-\Delta)^{s} : \dot{H}^{\beta + s}(\mathbb{R}) \to \dot{H}^{\beta - s}(\mathbb{R})\) is continuous. The fractional Laplacian \((-\Delta)^{s}\) also has the “integration by parts” formula, namely, the Kato-Ponce inequality, see e.g. [GO14]. However, these properties are not trivially extendable to \((-P)^s\). The additivity property cannot be easily proved using Fourier transform, since computing the Fourier symbol of \((-P)^s\) is not a trivial task. Moreover, the continuity of \((-P)^s\) between the Hilbert spaces is not obvious either. To overcome these difficulties, we introduce the Balakrishnan definition [MS01] of \((-P)^s\), see also Section IX.11 of [Yos80]. The equivalence of definitions can be shown by using the heat-diffusion semigroup \(\{e^{tP}\}_{t \geq 0}\). Consequently, the additivity property can be established by the Balakrishnan definition, and the continuity of \((-P)^s : H^{2s}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\) can be obtained by the interpolation of the single operator \(-P\). Here, we shall not interpolate on the family of the operator \((-P)^s\), see also [CM14] for the interpolation theory of the analytic family of multilinear operators. For the case \( s = \frac{1}{2} \), in our proof, we need not have to use the Balakrishnan operator.

For the case when \( a_{jk} \) are smooth, R.T. Seeley [See67] in 1967 showed that the operator \((-P)^s\) is a pseudo-differential operator (or Calderón-Zygmund operator) of order \( 2s \),
and the explicit formula was given. Thus, the theory of the pseudo-differential operator (see e.g. [Tay74]) is applicable for \((-P)^s\). For the fractional Laplacian \((-\Delta)^s\) and for the powers of second-order differential operators (as well as the \(x\)-dependent pseudodifferential generalizations), the boundary value theories have been elaborated in recent years, see e.g. [Gru14, Gru15, Gru16a, Gru16b, Gru19]. In the very recent preprint [Gru19], Grubb calculated explicitly the first few terms in the symbol of \(L^s\) when \(L\) is a second order strongly elliptic differential operator. Our method (see Lemma 2.2) allows a relaxation of the smoothness hypothesis that are needed to apply the theory of the pseudo-differential operator.

The extension of the Carleman estimates from [RW19] to our case is not trivial. We cannot directly employ the arguments in [RW19]. First of all, we write

\[
\sum_{j,k=1}^n a_{jk} \partial_j \partial_k = \Delta + \sum_{j,k=1}^n (a_{jk} - \delta_{jk}) \partial_j \partial_k.
\]

If we directly follows the arguments in [RW19], we will find out that the remainder term has excessive multiplier and weight, and it cannot be absorbed. To deal with this problem, we modify the ideas in [Reg97]. Roughly speaking, we have \(Lu = f\) (in conformal polar coordinate). We then define \(L^+ := L\) and consider a conjugate operator \(L^-\). We then estimate the difference \(\mathcal{D} = ||L^+u||^2 - ||L^-u||^2\) (which indeed contained the commutator structure) and the sum \(\mathcal{S} = ||\varphi^{-1/2}L^+u||^2 + ||\varphi^{-1/2}L^-u||^2\) for some weight \(\varphi \geq 1\). Then we consider \(\mathcal{D} + \tau^{-1}\mathcal{S}\), so the excessive multiplier and weight can be “adjusted”, and finally the “adjusted” remainder term can be absorbed. It is also interesting to mention that the second derivative term in the Carleman estimate should be \(\nabla(\nabla \tilde{u})\) rather than \(\nabla^2 \tilde{u}\), where \(\nabla = (\nabla, \partial_n)\) is the gradient operator on \(\mathbb{R}^{n+1}\), and \(\tilde{u}\) is the Caffarelli-Silvestre type extension of \(u\).

We would like to mention some results in the classical case when \(s = 1\). The Landis conjecture was proposed by E.M. Landis in the 60’s [KL88]. He conjectured that, if \(|q(x)| \leq 1\) and \(|u(x)| \leq C_0\) satisfies \(|u(x)| \leq \exp(-C|x|^{1+})\), then \(u \equiv 0\). Meshkov [Mes92] constructed a complex-valued potential \(q\) and a complex-valued nontrivial \(u\) with \(|u(x)| \leq C \exp(-C|x|^{3/4})\), shows that the conjecture was not true. However, for \(s = 1\) and \(a_{jk} = \delta_{jk}\), he also showed that if \(|u(x)| \leq C \exp(-C|x|^{1+})\), then \(u \equiv 0\) (in qualitative form). In other words, the exponent \(4/3\) is optimal in the complex case. We emphasize that as \(s \to 1\), the exponent \(\frac{4s}{4s-1}\) in Theorem 1.2 tends to 4/3. In the future, perhaps choosing a more complex weight, we guess Theorem 1.2 can be extend to \(s \in (0, 1)\) with exponent \(e(s) \leq \frac{4s}{4s-1}\) and \(e(s) \to 4/3\) as \(s \to 1\). Also, Bourgain and Kenig [BK05] derived a quantitative form of Meshkov’s result, which is based on the Carleman method. We would like to mention Davey’s result [Dav14], which proves the quantitative Landis conjecture for \(s = 1\) and \(a_{jk} = \delta_{jk}\) including the drift term. Following, Lin and Wang [LW14] extend the result for the case \(s = 1\) and for Lipschitz \(a_{jk}\) with \(|\nabla a_{jk}(x)| \leq \lambda |x|^{-1-\epsilon}\) for some \(\epsilon > 0\), which implies our assumption (1.3) for \(|x| \gg 1\). Cassano [Cas18] proved the Landis conjecture for the Dirac equation. In some sense the Dirac operator is the square root of the Laplacian operator, that is, the phenomena are similar when \(s = 1/2\). We would also like to mention that the Calderón problem for the fractional Schrödinger equation was studied in [CLR18, GLX17, RS17].

\footnote{I would like to thank Prof Gerd Grubb for bringing these issues to my attention and for pointing out several related references.}
This paper is organized as follows. In Section 2 we shall state the definition of \((-P)^s\) and prove some regularity results. In Section 3 we show that the decay of \(u\) implies the decay of the Caffarelli-Silvestre type extension \(\tilde{u}\). We then derive Carleman estimate for \((-P)^s\) in Section 4. Finally, we shall prove the qualitative results Theorem 1.1 and Theorem 1.2 in Section 5.

2. CAFFARELLI-SILLESTRE TYPE EXTENSION

First of all, we introduce some notations. Let \(\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+ = \{(x', x_{n+1}) : x_{n+1} \geq 0\}\), and we write \(x = (x', x_{n+1})\) with \(x' \in \mathbb{R}^n\) and \(x_{n+1} \in \mathbb{R}_+\). For \(x_0 \in \mathbb{R}^n \times \{0\}\), we shall denote the half balls in \(\mathbb{R}_+^{n+1}\) and \(\mathbb{R}^n \times \{0\}\) by

\[
B_r^+(x_0) := \{x \in \mathbb{R}_+^{n+1} : |x - x_0| \leq r\} \quad \text{and} \quad B_r^1(x_0) := B_r^+(x_0) \cap (\mathbb{R}^n \times \{0\}).
\]

For sake of convenience, we simply write \(B_r^+(0) = B_r^+\) and \(B_r'(0) = B_r'\). We also define the annulus

\[
A_{r,R}^+ := \{x \in \mathbb{R}_+^{n+1} : r \leq |x| \leq R\} \quad \text{and} \quad A_{r,R} := A_{r,R}^+ \cap (\mathbb{R}^n \times \{0\}).
\]

We also define the following Sobolev space:

\[
\dot{H}^1(\mathbb{R}_+^{n+1}, x_{n+1}^{1-2s}) := \left\{ v : \mathbb{R}_+^{n+1} \to \mathbb{R} : \int_{\mathbb{R}_+^{n+1}} x_{n+1}^{1-2s} |\nabla v|^2 \, dx < \infty \right\},
\]

\[
H^1(\mathbb{R}_+^{n+1}, x_{n+1}^{1-2s}) := \left\{ v : \mathbb{R}_+^{n+1} \to \mathbb{R} : \int_{\mathbb{R}_+^{n+1}} x_{n+1}^{1-2s} (|v|^2 + |\nabla v|^2) \, dx < \infty \right\}.
\]

For \(s \in (0, 1)\), we consider a solution \(\tilde{u}\) of the degenerate elliptic equation

\[
\left[ \partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} P \right] \tilde{u} = 0 \quad \text{in} \ \mathbb{R}_+^{n+1},
\]

\[
\tilde{u} = u \quad \text{on} \ \mathbb{R}^n \times \{0\}.
\]

By Theorem 1.1 of [ST10], then the fractional elliptic operator \((-P)^s\) is given by

\[
(-P)^s u(x') = c_{n,s} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}(x)
\]

for some constant \(c_{n,s} \neq 0\) (see also [Sti10]). Indeed, in p.48 and p.49 of [Sti10], we have

\[
\|\tilde{u}(\cdot, x_{n+1})\|_{L^2(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^n)}.
\]

The following lemma is well-known (see e.g. [Yu17]):

**Lemma 2.1.** The mapping \(u \in \dot{H}^s(\mathbb{R}^n) \mapsto \tilde{u} \in \dot{H}^1(\mathbb{R}_+^{n+1}, x_{n+1}^{1-2s})\) is continuous. Moreover, the mapping

\[
u \in \dot{H}^s(\mathbb{R}^n) \mapsto (-P)^s u \in \dot{H}^{-s}(\mathbb{R}^n)
\]

is also continuous.

Note that

\[
P u = \sum_{j,k=1}^{n} a_{jk} \partial_j \partial_k u + \sum_{j,k=1}^{n} (\partial_j a_{jk}) \partial_k u.
\]

Since \(a_{jk}\) is uniformly Lipschitz, then

\[
\|Pu\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^2(\mathbb{R}^n)}.
\]
Using duality, we have
\[ \|Pu\|_{H^{-1}(\mathbb{R}^n)} = \sup_{\|\phi\|_{H^1(\mathbb{R}^n)}=1} \langle Pu, \phi \rangle = \sup_{\|\phi\|_{H^2(\mathbb{R}^n)}=1} \langle u, P\phi \rangle \]
\[ \leq \|u\|_{L^2(\mathbb{R}^n)} \sup_{\|\phi\|_{H^2(\mathbb{R}^n)}=1} \|P\phi\|_{L^2(\mathbb{R}^n)} \]
(2.4)
\[ \leq C\|u\|_{L^2(\mathbb{R}^n)}. \]

We shall prove the followings:

**Lemma 2.2.** Let \( a_{jk} \) be uniformly Lipschitz. For \( s \neq \frac{1}{2} \), we further assume that \( a_{jk} \) is smooth. We have the inequality
\[ \|(-P)^s u\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^{2s}(\mathbb{R}^n)}. \]
Moreover, we have
\[ \|(-P)^s u\|_{H^{-2s}(\mathbb{R}^n)} \leq C\|u\|_{L^2(\mathbb{R}^n)}. \]

**Remark 2.3.** Using the duality argument as in (2.4), we know that (2.5) and (2.6) are equivalent.

First of all, we prove Lemma 2.2 for the special case \( s = 1/2 \):

**Proof of Lemma 2.2 for** \( s = 1/2 \): Using the conjugate equation, we can obtain (3.28) with \( s = 1/2 \):
\[ -P = C(-P)^{1/2}(-P)^{1/2}. \]

Since \((-P)^{1/2}\) is self-adjoint, then
\[ \| -Pu\|_{H^{-1}(\mathbb{R}^n)} = \sup_{\|\phi\|_{H^1(\mathbb{R}^n)} \neq 0} \frac{\langle Pu, \phi \rangle_{L^2(\mathbb{R}^n)}}{\|\phi\|_{H^1(\mathbb{R}^n)}} \]
\[ \geq \frac{\langle Pu, u \rangle_{L^2(\mathbb{R}^n)}}{\|u\|_{H^1(\mathbb{R}^n)}} \]
\[ = C\frac{\langle (-P)^{1/2}(-P)^{1/2}u, u \rangle_{L^2(\mathbb{R}^n)}}{\|u\|_{H^1(\mathbb{R}^n)}} = C\frac{\langle (-P)^{1/2}u, (-P)^{1/2}u \rangle_{L^2(\mathbb{R}^n)}}{\|u\|_{H^1(\mathbb{R}^n)}}, \]
so
\[ \|(-P)^{1/2}u\|_{L^2(\mathbb{R}^n)} \leq C\| -Pu\|_{H^{-1}(\mathbb{R}^n)}\|u\|_{H^1(\mathbb{R}^n)} \leq C''\|u\|_{H^2(\mathbb{R}^n)}, \]
where the last inequality can be obtained by interpolating the inequalities (2.3) and (2.4).

To prove the general case of Lemma 2.2, we need to introduce the Balakrishnan operator.

### 2.1. The Balakrishnan operator

Now we introduce the Balakrishnan definition of the fractional power of \(-P\).

**Definition 2.4** (Definition 3.1.1 and 5.1.1 of [MS01]). Let \( \alpha \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Re z > 0\} \).

1. If \( 0 < \Re \alpha < 1 \), then \( \text{Dom}((-P)^{\alpha}_B) = \text{Dom}(-P) \) and
   \[ (-P)^{\alpha}_B \phi = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda - P)^{-1}(-P)\phi \, d\lambda. \]
2. If \( \Re \alpha = 1 \), then \( \text{Dom}((-P)^{\alpha}_B) = \text{Dom}((-P)^2) \) and
   \[ (-P)^{\alpha}_B \phi = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{\alpha-1} \left[ (\lambda - P)^{-1} - \frac{\lambda}{\lambda^2 + 1} \right](-P)\phi \, d\lambda + \sin \frac{\alpha \pi}{2}(-P)\phi. \]
(3) If \( n < \Re \alpha < n + 1 \) for \( n \in \mathbb{N} \), then \( \text{Dom}((-P)_B^n) = \text{Dom}((-P)^{n+1}) \) and
\[
(-P)_B^n \phi = (-P)_B^{\alpha-n}(-P)^n \phi.
\]

(4) If \( \Re \alpha = n + 1 \) for \( n \in \mathbb{N} \), then \( \text{Dom}((-P)_B^n) = \text{Dom}((-P)^{n+2}) \) and
\[
(-P)_B^n \phi = (-P)_B^{\alpha-n}(-P)^n \phi.
\]

The following proposition shows that \((-P)_B^s\) and \((-P)^s\) are equivalent.

**Proposition 2.5** (Theorem 6.1.6 of [MS01]). Let \( 0 < s < 1 \). If \( u \in \text{Dom}((-P)_B^s) \), then the strong limit
\[
\lim_{\epsilon \to 0^+} \int_{\epsilon}^\infty (1 - e^{tP})u \frac{dt}{t^{1+s}} \quad \text{exists}
\]
and
\[
(-P)_B^s u = c_s' \lim_{\epsilon \to 0^+} \int_{\epsilon}^\infty (1 - e^{tP})u \frac{dt}{t^{1+s}} \quad \text{for some positive constant } c_s',
\]
where \( \{e^{tP}\}_{t \geq 0} \) is the heat-diffusion semigroup generated by \(-P\).

Here and after, we shall not distinguish between \((-P)^s\) and \((-P)_B^s\). By Theorem 5.1.2 of [MS01], we have
\[
(-P)^\alpha(-P)^\beta = (-P)^{\alpha+\beta}
\]
for all \( \alpha, \beta \in \mathbb{C} \) with \( \Re \alpha > 0 \) and \( \Re \beta > 0 \).

### 2.2. Proof of Lemma 2.2

Now we are ready to prove Lemma 2.2.

**Proof.** First of all, using Lemma 2.1, note that
\[
\|(P)_B^{2s}u\|_{H^{1-2s}(\mathbb{R}^n)} = \|(P)^{2s-1}(-P)u\|_{H^{1-2s}(\mathbb{R}^n)} \leq C\| - Pu \|_{H^{2s-1}(\mathbb{R}^n)} \leq C\| u \|_{H^{2s+1}(\mathbb{R}^n)}
\]
and
\[
\|(P)_B^{2s}u\|_{H^{-1-2s}(\mathbb{R}^n)} = \|(P)(-P)^{2s-1}u\|_{H^{-1-2s}(\mathbb{R}^n)}
\leq C\| (P)^{2s-1}u \|_{H^{1-2s}(\mathbb{R}^n)} \leq C\| u \|_{H^{2s+1}(\mathbb{R}^n)}.
\]
Interpolate these two inequalities, we reach
\[
\|(P)_B^{2s}u\|_{H^{-2s}(\mathbb{R}^n)} \leq C\| u \|_{H^{2s}(\mathbb{R}^n)}
\]
which is a generalization of Lemma 2.1. Thus, using (2.7) and the self-adjointness of \((-P)^s\), we reach
\[
C\| u \|_{H^{2s}(\mathbb{R}^n)} \geq \|(P)_B^{2s}u\|_{H^{-2s}(\mathbb{R}^n)} = \sup_{\phi \neq 0} \frac{\langle (P)_B^{2s}u, \phi \rangle_{L^2(\mathbb{R}^n)}}{\| \phi \|_{H^{2s}(\mathbb{R}^n)}}
\geq \frac{\langle (P)_B^{2s}u, u \rangle_{L^2(\mathbb{R}^n)}}{\| u \|_{H^{2s}(\mathbb{R}^n)}} = \frac{\langle (P)^s, (P)_B^s u, u \rangle_{L^2(\mathbb{R}^n)}}{\| u \|_{H^{2s}(\mathbb{R}^n)}} = \frac{\langle (P)^s u, (P)_B^s u \rangle_{L^2(\mathbb{R}^n)}}{\| u \|_{H^{2s}(\mathbb{R}^n)}},
\]
which is our desired result. \( \square \)
3. Boundary Decay Implies Bulk Decay

First of all, we translate the decay behavior on $\mathbb{R}^n$ to decay behavior which also holds on $\mathbb{R}^{n+1}$.

**Proposition 3.1.** Let $s \in (0, 1)$ and $u \in H^s(\mathbb{R}^n)$ be a solution to (1.1) with (1.2) and (1.3). For $s \neq \frac{1}{2}$, we further assume (1.4). Assume that $|q(x)| \leq 1$ and there exists $\alpha > 1$ such that

$$
\int_{\mathbb{R}^n} e^{x|\alpha |} |u|^2 \, dx \leq C < \infty.
$$

Then there exist constants $C_1, C_2 > 0$ such that the Caffarelli-Silvestre type extension $\tilde{u}(x)$ satisfies

$$
|\tilde{u}(x)| \leq C_1 e^{-C_2 |x|^{\alpha}} \text{ for all } x \in \mathbb{R}^{n+1}.
$$

In order to obtain the interior decay, similar to Proposition 2.3 of [RW19], we need the following three ball inequality.

**Lemma 3.2.** Let $s \in (0, 1)$ and $\tilde{u} \in H^1(B_4^+, x_1^{1-2s})$ be a solution to

$$
\left[ \partial_{n+1} x_1^{1-2s} \partial_{n+1} + x_1^{1-2s} P \right] \tilde{u} = 0 \text{ in } \mathbb{R}^{n+1}
$$

with (1.2). Assume that $r \in (0, 1)$ and $x_0 = (\overline{x}_0, 5r) \in B_2^+$. Then, there exists $\alpha = \alpha(n, s) \in (0, 1)$ such that

$$
\|\tilde{u}\|_{L^\infty(B_2^+, (\overline{x}_0))} \leq C \|\tilde{u}\|^\alpha_{\infty(B_2^+, (\overline{x}_0))} \|\tilde{u}\|^{1-\alpha}_{L^\infty(B_2^+, (\overline{x}_0))}.
$$

**Proof.** As $(\overline{x}_0)_{n+1} = 5r$, this follows from a standard interior $L^2$ three ball inequality together with $L^\infty$-$L^2$ estimates for uniformly elliptic equations. \qed

We also need the following boundary-bulk propagation of smallness estimate:

**Lemma 3.3.** Let $s \in (0, 1)$ and let $\tilde{u} \in H^1(\mathbb{R}^{n+1}_+, x_1^{1-2s})$ be a solution to (2.1) with (1.2) and $q \in L^\infty(\mathbb{R}^n)$. We assume that

$$
\max_{1 \leq j, k \leq n} \|a_{jk} - \delta_{jk}\|_{\infty} + \max_{1 \leq j, k \leq n} \|\nabla' a_{jk}\|_{\infty} \leq \epsilon
$$

for some sufficiently small $\epsilon > 0$. For $s \neq \frac{1}{2}$, we further assume $a_{jk}$ is smooth and

$$
\max_{1 \leq j, k \leq n} \|\nabla' a_{jk}\|_{\infty} \leq C
$$

for some positive constant $C$. Assume that $x_0 \in \mathbb{R}^n \times \{0\}$. Then
(a) There exists $\alpha = \alpha(n, s) \in (0, 1)$ and $c = c(n, s) \in (0, 1)$ such that
\[
\left\| x_{n+1}^{1-2a} \hat{u} \right\|_{L^2(B_{16r}^+(x_0))} \leq C \left[ \left\| x_{n+1}^{1-2a} \hat{u} \right\|_{L^2(B_{16r}^+(x_0))} + r^{1-s} \left\| u \right\|_{L^2(B_{16r}^+(x_0))} \right]^{\alpha} \times \left[ \lim_{x_{n+1} \to 0} x_{n+1}^{-1-2a} \partial_{n+1} \hat{u} \right]_{L^2(B_{16r}^+(x_0))} \times \left[ 1 \right]^{1-\alpha} \times \left[ C \left( \left\| x_{n+1}^{1-2a} \hat{u} \right\|_{L^2(B_{16r}^+(x_0))} + r^{1-s} \left\| u \right\|_{L^2(B_{16r}^+(x_0))} \right) \right]^{\frac{2a}{1+s}} \times \left[ \lim_{x_{n+1} \to 0} x_{n+1}^{-1-2a} \partial_{n+1} \hat{u} \right]_{L^2(B_{16r}^+(x_0))} \times \left[ C \right]^{\frac{2a}{1+s}}.
\]

(b) There exists $\alpha = \alpha(n, s) \in (0, 1)$ and $c = c(n, s) \in (0, 1)$ such that
\[
\left\| x_{n+1}^{2a} \hat{u} \right\|_{L^\infty(B_{16r}^+(x_0))} \leq C r^{-\frac{a}{2}} \left[ r^{s-1} \left\| x_{n+1}^{1-2a} \hat{u} \right\|_{L^2(B_{16r}^+(x_0))} + \left\| u \right\|_{L^2(B_{16r}^+(x_0))} \right]^{\alpha} \times \left[ r^{2s} \left\| x_{n+1}^{-1-2a} \partial_{n+1} \hat{u} \right\|_{L^2(B_{16r}^+(x_0))} + \left\| u \right\|_{L^2(B_{16r}^+(x_0))} \right]^{1-\alpha} \times \left[ C r^{-\frac{a}{2}} \right]^{r^s \left\| qu \right\|_{L^2(B_{16r}^+(x_0))} \left\| u \right\|_{L^2(B_{16r}^+(x_0))}^{\frac{1}{1+s}}}
\]

Using Lemma 3.2 and Lemma 3.3, following the chain-ball argument in [RSW19], we can obtain Proposition 3.1.

Now we want to proof Lemma 3.3.

3.1. Proof of the part (a) of Lemma 3.3 for the case $s \in \left[ \frac{1}{2}, 1 \right)$. We first prove the following extension of the Carleman estimate in Proposition 5.7 of [RS17].

Lemma 3.4. Let $s \in \left[ \frac{1}{2}, 1 \right)$ and let $w \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ with supp$(w) \subset B_{1/2}^+$ be a solution to
\[
\left[ \partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^n a_{jk} \partial_j \partial_k \right] w = f \quad \text{in } \mathbb{R}^{n+1}_+,\]
\[
w = 0 \quad \text{on } \mathbb{R}^n \times \{0\}.
\]

Suppose that
\[
\phi(x) = \phi(x', x_{n+1}) := -\frac{|x'|^2}{4} + 2 \left( -\frac{1}{2-2s} x_{n+1}^{2-2s} + \frac{1}{2} x_{n+1}^2 \right).
\]
We assume that
\[
\max_{1 \leq j, k \leq n} \|a_{jk} - \delta_{jk}\|_\infty + \max_{1 \leq j, k \leq n} \|\nabla' a_{jk}\|_\infty \leq \epsilon
\]
for some sufficiently small \(\epsilon > 0\). For \(s \neq \frac{1}{2}\), we further assume
\[
\max_{1 \leq j, k \leq n} \|(\nabla')^2 a_{jk}\|_\infty \leq C
\]
for some positive constant \(C\). Assume additionally that
\[
\left\| \frac{2s-1}{2} f \right\|_{L^2(\mathbb{R}^n)} + \lim_{x_{x+1} \to 0} \|\Delta' w\|_{L^2(\mathbb{R}^n \times \{0\})} + \lim_{x_{x+1} \to 0} \|\nabla' w\|_{L^2(\mathbb{R}^n \times \{0\})} + \tau \lim_{x_{x+1} \to 0} \|\frac{1-2s}{s} \partial_{n+1} w\|_{L^2(\mathbb{R}^n \times \{0\})} < \infty.
\]
Then there exists \(\tau_0 > 1\) and a constant \(C\) such that
\[
\tau^3 \left( \|e^{\tau \phi} \frac{2s-1}{2} f\|_{L^2(\mathbb{R}^n)} + \|e^{\tau \phi} \frac{1-2s}{s} \nabla w\|_{L^2(\mathbb{R}^n)} \right) \leq C \left( \|e^{\tau \phi} \frac{2s-1}{2} f\|_{L^2(\mathbb{R}^n)} + \|e^{\tau \phi} \Delta' w\|_{L^2(\mathbb{R}^n \times \{0\})} + \tau \lim_{x_{x+1} \to 0} \|\frac{1-2s}{s} \partial_{n+1} w\|_{L^2(\mathbb{R}^n \times \{0\})} \right)
\]
for all \(\tau \geq \tau_0\).

**Proof.** Now we prove the Carleman estimate for \(s \in (\frac{1}{2}, 1)\), as the case \(s = \frac{1}{2}\) is naturally included in our estimates. Let \(\tilde{u} = x_n^{-s+1} w\), we have
\[
\frac{2s-1}{2} f = \frac{2s-1}{2} \partial_{n+1} x_n^{-1-2s} \partial_{n+1} x_n^{-2} + \sum_{j,k=1}^n a_{jk} \partial_j \partial_k \tilde{u} = \Delta \tilde{u} - \frac{1}{4} (2s-1)(2s+1) x_n^{-2} + \sum_{j,k=1}^n (a_{jk} - \delta_{jk}) \partial_j \partial_k \tilde{u}.
\]
Let \(u = e^{\tau \phi} \tilde{u}\), we have
\[
e^{\tau \phi} \frac{2s-1}{2} f = \left[ \Delta + \tau^2 |\nabla \phi|^2 - \frac{(2s-1)(2s+1)}{4} x_n^{-2} - \tau \Delta \phi - 2\tau \nabla \phi \cdot \nabla \right] u
\]
\[
+ \sum_{j,k=1}^n (a_{jk} - \delta_{jk}) \partial_j \partial_k u
\]
\[
- \tau \sum_{j,k=1}^n (a_{jk} - \delta_{jk}) \left[ (\partial_k \phi) \partial_j + (\partial_j \phi) \partial_k \right] u
\]
\[
+ \sum_{j,k=1}^n (a_{jk} - \delta_{jk}) \left[ \tau^2 (\partial_k \phi) (\partial_j \phi) - \tau (\partial_j \partial_k \phi) \right] u.
\]
Write
\[
L^+ = S + A + (I) + (II) + (III),
\]
where

\[ S = \Delta + \tau^2 |\nabla \phi|^2 + \frac{1 - 4s^2}{4} x_{n+1}^{-2}, \]
\[ A = -2\tau \nabla \phi \cdot \nabla - \tau \Delta \phi, \]
\[ (I) = \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \partial_j \partial_k \]
\[ (II) = -\tau \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \left[ (\partial_k \phi) \partial_j + (\partial_j \phi) \partial_k \right] \]
\[ (III) = \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \left[ \tau^2 (\partial_k \phi) (\partial_j \phi) - \tau (\partial_j \partial_k \phi) \right]. \]

We define the conjugate operator \( L^- := S - A + (I) - (II) + (III). \)

Here, we denote

\[ \| \cdot \| = \| \cdot \|_{L^2(\mathbb{R}^{n+1}_+)} \]
\[ \| \cdot \|_0 = \| \cdot \|_{L^2(\mathbb{R}^n \times \{0\})} \]
\[ \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^{n+1}_+)} \]
\[ \langle \cdot, \cdot \rangle_0 = \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^n \times \{0\})} \]

and we omit the notations “\( \lim_{x_{n+1} \to 0} \)” in \( \| \cdot \|_0 \) and \( \langle \cdot, \cdot \rangle_0 \).

We first estimate below the difference

\[ D = \| L^+ u \|^2 - \| L^- u \|^2 = 4 \langle Su, Au \rangle + R, \]

where

\[ R = 4 \langle Su, (II)u \rangle + 4 \langle Au, (I)u \rangle + 4 \langle Au, (III)u \rangle + 4 \langle (I)u, (II)u \rangle + 4 \langle (II)u, (III)u \rangle. \]

First of all, we estimate the commutator term \( \langle Su, Au \rangle \) following the arguments in [RS17].

Note that

\[ (3.1) \quad 2 \langle Su, Au \rangle = \langle [S, A]u, u \rangle + 2\tau \langle Su, (\partial_{n+1} \phi)u \rangle_0 - \langle Au, \partial_{n+1} u \rangle_0 + \langle \partial_{n+1} (Au), u \rangle_0. \]

Observe that \([S, A] = [S, A]_1 + [S, A]_2\), where

\[ [S, A]_1 = [\Delta' + \tau^2 |\nabla' \phi|^2, -2\tau \nabla' \phi \cdot \nabla' - \tau \Delta' \phi], \]
\[ [S, A]_2 = \left[ \partial_{n+1}^2 + \tau^2 (\partial_{n+1} \phi)^2 + \frac{1 - 4s^2}{4} x_{n+1}^{-2}, -2\tau \partial_{n+1} \phi \partial_{n+1} - \tau \partial_{n+1}^2 \phi \right]. \]

The first commutator part reads

\[ [S, A]_1 = 4\tau^3 \sum_{j=1}^{n} (\partial_j \phi)^2 \partial_j^2 \phi - 4\tau \sum_{j=1}^{n} (\partial_j^2 \phi) \partial_j^2, \]

then

\[ (3.2) \quad \langle [S, A]_1 u, u \rangle = -\frac{1}{2} \tau^3 \| x' \| u \|^2 - 2\tau \| \nabla' u \|^2. \]
For the second part of the commutator part is given by
\[
\langle [S, A]_2 u, u \rangle = 4\tau^3 \langle u, (\partial_{n+1}\phi)^2(\partial_{n+1}^2\phi)u \rangle + 4\tau \langle \partial_{n+1}u, (\partial_{n+1}^2\phi)\partial_{n+1}u \rangle \\
- \tau \langle u, (\partial_{n+1}^2\phi)u \rangle + (2s+1)(2s-1)\tau \langle u, x_{n+1}^{-3}(\partial_{n+1}\phi)u \rangle \\
+ 4\tau \langle (\partial_{n+1}^2\phi)\partial_{n+1}u, u \rangle_0.
\]
Note that
\[
(\partial_{n+1}\phi)^2(\partial_{n+1}^2\phi) = 8(x_{n+1}^{1-2s} - x_{n+1})^2((2s-1)x_{n+1}^{-2s} + 1)
\]
and
\[
- \tau \langle u, (\partial_{n+1}^2\phi)u \rangle + (2s+1)(2s-1)\tau \langle u, x_{n+1}^{-3}(\partial_{n+1}\phi)u \rangle \\
= -2\tau(2s-1)(2s+1)^2\|x_{n+1}^{-1-s}u\|^2 + 2\tau(2s-1)(2s+1)\|x_{n+1}^{-1}u\|^2.
\]
So,
\[
\langle [S, A]_2 u, u \rangle = 32\tau^3\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2 + 32(2s-1)\tau^3\|(x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^{-s}u\|^2 \\
+ 8\tau(2s-1)\|x_{n+1}^{-s}\partial_{n+1}u\|^2 - 2\tau(2s-1)(2s+1)^2\|x_{n+1}^{-1-s}u\|^2 \\
+ 8\tau\|\partial_{n+1}u\|^2 + 2\tau(2s-1)(2s+1)\|x_{n+1}^{-1}u\|^2
\]
(3.3) \\
+ 4\tau \langle (\partial_{n+1}^2\phi)\partial_{n+1}u, u \rangle_0.
\]
Combining (3.1), (3.2) and (3.3), we reach
\[
4\langle Su, Au \rangle = 64\tau^3\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2 + 64(2s-1)\tau^3\|(x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^{-s}u\|^2 \\
+ 16\tau(2s-1)\|x_{n+1}^{-s}\partial_{n+1}u\|^2 + 16\tau\|\partial_{n+1}u\|^2 + 4\tau(2s-1)(2s+1)\|x_{n+1}^{-1-s}u\|^2 \\
- 4\tau(2s-1)(2s+1)^2\|x_{n+1}^{-1-s}u\|^2 - \tau^3\|x\|u\|^2 - 4\tau\|\nabla' u\|^2 \\
+ 8\tau \langle (\partial_{n+1}^2\phi)\partial_{n+1}u, u \rangle_0 \\
+ 4\tau \langle Su, (\partial_{n+1}\phi)u \rangle_0 - 2\langle Au, \partial_{n+1}u \rangle_0 + 2\langle \partial_{n+1}(Au), u \rangle_0.
\]
Using integration by parts, we can estimate $R$ from below:
\[
R \geq -C\epsilon \left[ \tau\|x_{n+1}^{1-2s} - x_{n+1})\nabla' u\|^2 + \tau\|\partial_{n+1}u\|^2 + \tau^3\|(x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^{-s}u\|^2 + \tau\|x_{n+1}^{-1}u\|^2 \\
+ \tau\|x_{n+1}^{-2s}\partial_{n+1}u\|_0 + \tau\|x_{n+1}^{-2s}\|x\|\nabla' u\|^2 + \tau^3\|(x_{n+1}^{1-2s} - x_{n+1})^2u\|^2 \right].
\]
Here we would like to highlight some features while estimating the second term of $R$, that is, $\langle Au, (I)u \rangle$. Note that
\[
\langle -2\tau\partial_{n+1}\phi\partial_{n+1}u, (a_{jk} - \delta_{jk})\partial_j\partial_ku \rangle \\
= \tau \langle \partial_{n+1}\phi\partial_{n+1}u, (\partial_ja_{jk})\partial_ku \rangle - \tau \langle \partial_{n+1}^2\phi\partial_ju, (a_{jk} - \delta_{jk})\partial_ku \rangle \\
+ \tau \langle \partial_{n+1}\phi\partial_ju, (\partial_ka_{jk})\partial_{n+1}u \rangle - \tau \langle \partial_{n+1}\phi\partial_ju, (a_{jk} - \delta_{jk})\partial_ku \rangle_0
\]
(3.5) \\
and
\[
\langle -\tau(\partial_{n+1}^2\phi)u, (a_{jk} - \delta_{jk})\partial_j\partial_ku \rangle \\
= \tau \langle (\partial_{n+1}^2\phi)u, \partial_ja_{jk}\partial_ku \rangle + \tau \langle \partial_{n+1}^2\phi\partial_ju, (a_{jk} - \delta_{jk})\partial_ku \rangle \\
= -\frac{\tau}{2} \langle (\partial_{n+1}^2\phi)u, (\partial_j\partial_ka_{jk})u \rangle + \tau \langle \partial_{n+1}^2\phi\partial_ju, (a_{jk} - \delta_{jk})\partial_ku \rangle.
\]
(3.6)
Thus, summing up (3.8) and (3.7), we note that the harmful term $\tau (\partial_{n+1}^2 \phi \partial_j u, (a_{jk} - \delta_{jk}) \partial_k u)$ is cancelled. The term is harmful because $\partial_{n+1}^2 \phi$ has singularity $x_n^{2s}$ for $s \in (1/2, 1)$. However, when $s = 1/2$, $\partial_{n+1}^2 \phi$ has no singularity. In this case, we consider (3.6) rather than (3.7). This is the reason why we can relax the second derivative assumption for the case $s = 1/2$.

So, for sufficiently small $\epsilon > 0$, we reach

$$D \geq 64 \tau^3 \| (x_n^{1-2s} - x_n u) \|^2 + \frac{639}{10} (2s - 1) \tau^3 \| (x_n^{1-2s} - x_n u) x_n^{-s} u \|^2$$

$$+ 16 \tau (2s - 1) \| x_n^{-s} \partial_n u \|^2 + \frac{159}{10} \tau \| \partial_n u \|^2 + \frac{39}{10} (2s - 1) (2s + 1) \| x_n^{-1} u \|^2$$

$$- 4\tau (2s - 1) (2s + 1) \| x_n^{-1-s} u \|^2 - 4\tau \| \nabla' u \|^2 - C \epsilon \tau \| (x_n^{1-2s} - x_n u) \| \nabla' u \|^2$$

$$+ 8\tau \| \partial_n (\partial_n^2 \phi \partial_n u, u) \|^2$$

Using the Hardy inequality (Lemma A.1), we reach

$$\| x_n^{-s} \partial_n u \|^2 \leq \frac{4}{(2s + 1)^2} \| x_n^{-s} \partial_n u \|^2 + \frac{2}{2s + 1} \| x_n^{-\frac{1}{2}-s} u \|^2,$$

thus

$$16 \tau (2s - 1) \| x_n^{-s} \partial_n u \|^2 - 4\tau (2s - 1) (2s + 1) \| x_n^{-1-s} u \|^2$$

$$\geq - 8\tau (2s - 1) (2s + 1) \| x_n^{-\frac{1}{2}-s} u \|^2.$$

Hence,

$$D \geq 64 \tau^3 \| (x_n^{1-2s} - x_n u) \|^2 + \frac{639}{10} (2s - 1) \tau^3 \| (x_n^{1-2s} - x_n u) x_n^{-s} u \|^2$$

$$+ \frac{159}{10} \tau \| \partial_n u \|^2 + \frac{39}{10} (2s - 1) (2s + 1) \| x_n^{-1} u \|^2 - 4\tau \| \nabla' u \|^2$$

$$- C \epsilon \tau \| (x_n^{1-2s} - x_n u) \| \nabla' u \|^2 + 8\tau \| (\partial_n^2 \phi \partial_n u, u) \| + 4\tau (\partial_n^2 \phi \partial_n u, u)$$

$$- 2 \langle Au, \partial_n u \rangle - 2 \langle \partial_n (Au, u) \rangle - \tau \| x_n^{-\frac{1}{2}-s} \partial_n u \|^2 - \tau \| x_n^{-\frac{1}{2}-s} \partial_n u \|^2$$

$$- \tau \| (x_n^{1-2s} - x_n u) \| \nabla' u \|^2 - 8\tau (2s - 1) (2s + 1) \| x_n^{-\frac{1}{2}-s} u \|^2.$$

Next, we estimate the sum

$$\mathcal{S} = \| L^+ u \|^2 + \| L^- u \|^2$$

$$\geq 2 \| Su \|^2 + 2 \| Au \|^2 - C \epsilon \left[ \sum_{j,k=1}^n \| \partial_j \partial_k u \|^2 + \tau^2 \| \nabla' u \|^2 + \tau^4 \| u \|^2 \right].$$
Note that

\[
2\|Su\|^2 - 2\left\| \Delta u + \tau^2 |\nabla \phi|^2 u - \frac{(2s + 1)(2s - 1)}{4} x_n^{-2} u \right\|^2
\]

\[
= 2\left\| \Delta' u + \left( \partial_n^{2} u + \tau^2 |\nabla \phi|^2 u - \frac{(2s + 1)(2s - 1)}{4} x_n^{-2} u \right) \right\|^2
\]

\[
= 2\|\Delta' u\|^2 + 4\langle \Delta' u, \partial_n^2 u \rangle + 4\tau^2 \langle \Delta' u, |\nabla \phi|^2 u \rangle - (2s + 1)(2s - 1) \langle \Delta' u, x_n^{-2} u \rangle
\]

\[
+ 2\left\| \partial_n^{2} u + \tau^2 |\nabla \phi|^2 u - \frac{(2s + 1)(2s - 1)}{4} x_n^{-2} u \right\|^2
\]

\[
\geq 2 \sum_{j,k=1}^{n} \|\partial_j \partial_k u\|^2 + 4\langle \Delta' u, \partial_n^2 u \rangle + 4\tau^2 \langle \Delta' u, |\nabla \phi|^2 u \rangle
\]

\[
+ (2s + 1)(2s - 1) \langle \nabla' u, x_n^{-2} \nabla' u \rangle
\]

\[
\geq 2 \sum_{j,k=1}^{n} \|\partial_j \partial_k u\|^2 + 4\langle \Delta' u, \partial_n^2 u \rangle + 4\tau^2 \langle \Delta' u, |\nabla \phi|^2 u \rangle.
\]

Since

\[
4\langle \Delta' u, \partial_n^2 u \rangle = 4\langle \nabla' \partial_n u, \nabla' \partial_n u \rangle - 4\langle \Delta' u, \partial_n u \rangle_0
\]

and for \( \epsilon_0 > 0 \), we have

\[
4\tau^2 \langle \Delta' u, |\nabla \phi|^2 u \rangle
\]

\[
= \tau^2 \langle \nabla' u, |x_n|^2 u \rangle + 16\tau^2 \langle \Delta' u, (x_n^{1-2s} - x_n^{-1})^2 u \rangle
\]

\[
\geq - \tau^2 (1 + \epsilon_0) \|\nabla' u\|^2 - \tau^2 C \epsilon_0^{-1} \|u\|^2 - 16\tau^2 \|x_n^{1-2s} - x_n^{-1}\| \nabla' u\|^2.
\]

Thus,

\[
\mathcal{I} \geq 2\|Su\|^2 + 2\|Au\|^2 - C\epsilon \left[ \sum_{j,k=1}^{n} \|\partial_j \partial_k u\|^2 + \tau^2 \|\nabla' u\|^2 + \tau^4 \|u\|^2 \right]
\]

\[
\geq 2\|\nabla (\nabla' u)\|^2 - \tau^2 (1 + \epsilon_0) \|\nabla' u\|^2 - \tau^2 C \epsilon_0^{-1} \|u\|^2 - 16\tau^2 \|x_n^{1-2s} - x_n^{-1}\| \nabla' u\|^2
\]

\[
(3.9) \quad - C\epsilon \left[ \sum_{j,k=1}^{n} \|\partial_j \partial_k u\|^2 + \tau^2 \|\nabla' u\|^2 + \tau^4 \|u\|^2 \right] - 4\langle \Delta' u, \partial_n u \rangle_0.
\]
Hence, from (3.8) and (3.9), we have

\[
\left( \tau + s + \frac{1}{2} \right) \| L^+ u \|^2 \\
\geq \tau \mathcal{D} + \frac{2s - 1}{2} \mathcal{J} + \mathcal{J} \\
\geq 64 \tau^4 \| (x_{n+1}^{1-2s} - x_{n+1})u \|^2 + \frac{639}{10} (2s - 1) \tau^4 \| (x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^{-s}u \|^2 + \| Su \|^2 \\
+ \frac{159}{10} \tau^2 \| \partial_{n+1}u \|^2 + \frac{39}{10} \tau^2 (2s - 1)(2s + 1) \| x_{n+1}^{-1}u \|^2 - 4 \tau^2 \| \nabla' u \|^2 \\
- C \varepsilon \tau^2 \| (x_{n+1}^{1-2s} - x_{n+1})\nabla' u \|^2 + (2s - 1) \| \nabla (\nabla' u) \|^2 - \frac{1}{2} \tau^2 (2s - 1)(1 + \varepsilon_0) \| \nabla' u \|^2 \\
- \tau^2 C \varepsilon_0 \| u \|^2 - 8(2s - 1) \tau^2 \| (x_{n+1}^{1-2s} - x_{n+1})\nabla' u \|^2 \\
- C \varepsilon \left[ \sum_{j,k=1}^n \| \partial_j \partial_k u \|^2 + \tau^2 \| \nabla' u \|^2 + \tau^4 \| u \|^2 \right] \\
+ 8 \tau^2 \langle (\partial_{n+1}^2 \phi) \partial_{n+1}u, u \rangle_0 + 4 \tau^2 \langle Su, \partial_{n+1}^2 \phi \rangle_0 - 2 \tau \langle Au, \partial_{n+1}^2 \phi \rangle_0 + 2 \tau \langle \partial_{n+1}^2 (Au), u \rangle_0 \\
- \tau^2 \| x_{n+1}^{1-2s} \partial_{n+1}u \|^2_0 - \tau^2 \| x_{n+1}^{1-2s} |x'| \nabla' u \|^2_0 - \tau^4 \| (x_{n+1}^{1-2s} - x_{n+1}) \frac{1}{2} u \|^2_0 \\
- 8 \tau^2 (2s - 1)(2s + 1) \| x_{n+1}^{1-2s} u \|^2_0 - 2(2s - 1) \langle \Delta' u, \partial_{n+1}u \rangle_0.
\]

Choose \( \varepsilon > 0 \) sufficiently small, and then \( \varepsilon_0 > 0 \) small, \( \tau \) large, hence

\[
\left( \tau + s + \frac{1}{2} \right) \| L^+ u \|^2 \\
\geq \frac{9}{10} (2s - 1) \| \nabla (\nabla' u) \|^2 + \| Su \|^2 + 64 \tau^4 \| (x_{n+1}^{1-2s} - x_{n+1})u \|^2 \\
+ \frac{639}{10} (2s - 1) \tau^4 \| (x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^{-s}u \|^2 + \frac{159}{10} \tau^2 \| \partial_{n+1}u \|^2 - 4 \tau^2 \| \nabla' u \|^2 \\
- \frac{171}{20} (2s - 1) \tau^2 \| (x_{n+1}^{1-2s} - x_{n+1})\nabla' u \|^2 + \frac{39}{10} \tau^2 (2s - 1)(2s + 1) \| x_{n+1}^{-1}u \|^2 \\
+ 8 \tau^2 \langle (\partial_{n+1}^2 \phi) \partial_{n+1}u, u \rangle_0 + 4 \tau^2 \langle Su, \partial_{n+1}^2 \phi \rangle_0 - 2 \tau \langle Au, \partial_{n+1}^2 \phi \rangle_0 + 2 \tau \langle \partial_{n+1}^2 (Au), u \rangle_0 \\
- \tau^2 \| x_{n+1}^{1-2s} \partial_{n+1}u \|^2_0 - \tau^2 \| x_{n+1}^{1-2s} |x'| \nabla' u \|^2_0 - \tau^4 \| (x_{n+1}^{1-2s} - x_{n+1}) \frac{1}{2} u \|^2_0 \\
- 8 \tau^2 (2s - 1)(2s + 1) \| x_{n+1}^{1-2s} u \|^2_0 - 2(2s - 1) \langle \Delta' u, \partial_{n+1}u \rangle_0.
\]  

(3.10)

Since \( \text{supp}(u) \subseteq B_{1/2}^+ \) and \( s > \frac{1}{2} \), thus

\[
0 \leq (x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^s = x_{n+1}^{1-s} - x_{n+1}^{1+s} \leq 1,
\]
and hence
\[
\frac{172}{20} (2s - 1) \tau^2 \| (x_{n+1}^{1-2s} - x_{n+1}) \nabla' u \|^2 \\
- \frac{172}{20} (2s - 1) \tau^2 \left( (x_{n+1}^{1-2s} - x_{n+1}) x^s \Delta' u, (x_{n+1}^{1-2s} - x_{n+1}) x^{-s} u \right) \\
\leq \frac{86}{20} (2s - 1) \delta \| (x_{n+1}^{1-2s} - x_{n+1}) x^s \Delta' u \|^2 + \frac{86}{20} (2s - 1) \tau^4 \delta^{-1} \| (x_{n+1}^{1-2s} - x_{n+1}) x^{-s} u \|^2 \\
\leq \frac{86}{20} (2s - 1) \delta \| \Delta' u \|^2 + \frac{86}{20} (2s - 1) \tau^4 \delta^{-1} \| (x_{n+1}^{1-2s} - x_{n+1}) x^{-s} u \|^2.
\]

Choose \( \delta = \frac{8}{43} \), we reach
\[
\frac{172}{20} (2s - 1) \tau^2 \| (x_{n+1}^{1-2s} - x_{n+1}) \nabla' u \|^2 \\
\leq \frac{8}{10} (2s - 1) \| \Delta' u \|^2 + 23.1125 (2s - 1) \tau^4 \| (x_{n+1}^{1-2s} - x_{n+1}) x^{-s} u \|^2.
\]

Moreover, we have
\[
\frac{41}{10} \tau^2 \langle Su, u \rangle \\
= \frac{41}{10} \tau^2 \| \nabla u \|^2 - \frac{41}{10} \tau^4 \| \nabla \phi | u \|^2 + \frac{41}{5} (2s + 1) (2s - 1) \tau^2 \| x_{n+1}^{-1} u \|^2 + \frac{41}{10} \tau^2 \langle \partial_{n+1} u, u \rangle_0 \\
= \frac{41}{10} \tau^2 \| \nabla u \|^2 - \frac{41}{10} \tau^4 \left( \frac{1}{16} \| u \|^2 + 4 \| (x_{n+1}^{1-2s} - x_{n+1}) u \|^2 \right) + \frac{41}{10} \tau^2 \langle \partial_{n+1} u, u \rangle_0.
\]

So, since \( (x_{n+1}^{1-2s} - x_{n+1}) \geq \frac{1}{2} \), thus
\[
\frac{41}{10} \tau^2 \| \nabla' u \|^2 + \frac{41}{5} (2s + 1) (2s - 1) \tau^2 \| x_{n+1}^{-1} u \|^2 + \frac{41}{10} \tau^2 \langle \partial_{n+1} u, u \rangle_0 \\
\leq \frac{41}{10} \tau^2 \| \nabla u \|^2 + \frac{41}{5} (2s + 1) (2s - 1) \tau^2 \| x_{n+1}^{-1} u \|^2 + \frac{41}{10} \tau^2 \langle \partial_{n+1} u, u \rangle_0 \\
\leq \frac{41}{10} \tau^2 \langle Su, u \rangle + \frac{41}{10} \tau^4 \| u \|^2 + \frac{164}{10} \tau^4 \| (x_{n+1}^{1-2s} - x_{n+1}) u \|^2 \\
\leq \frac{41}{20} \delta \| Su \|^2 + \frac{41}{20} \delta^{-1} \tau^4 \| u \|^2 + \frac{41}{10} \tau^4 \| (x_{n+1}^{1-2s} - x_{n+1}) u \|^2 \\
\leq \frac{41}{20} \delta \| Su \|^2 + \frac{82}{10} \delta^{-1} \tau^4 \| (x_{n+1}^{1-2s} - x_{n+1}) u \|^2 + \frac{41}{40} \tau^4 \| (x_{n+1}^{1-2s} - x_{n+1}) u \|^2 \\
+ \frac{164}{10} \tau^4 \| (x_{n+1}^{1-2s} - x_{n+1}) u \|^2.
\]

Choose \( \delta = \frac{20}{41} \), hence
\[
\frac{41}{10} \tau^2 || \nabla' u ||^2 + \frac{41}{5} (2s + 1) (2s - 1) \tau^2 || x_{n+1}^{-1} u ||^2 + \frac{41}{10} \tau^2 \langle \partial_{n+1} u, u \rangle_0 \\
\leq || Su ||^2 + 34.235 \tau^4 || (x_{n+1}^{1-2s} - x_{n+1}) u ||^2.
\]
Combining (3.10), (3.11) and (3.12), we reach
\[
\left( \tau + s + \frac{1}{2} \right) \| L^+ u \|^2 \\
\geq \frac{1}{10} (2s - 1) \| \nabla (\nabla' u) \|^2 + 29.765 \tau^4 \|(x_{n+1}^{1-2s} - x_{n+1}) u \|^2 \\
+ 40.7875 (2s - 1) \tau^4 \|(x_{n+1}^{1-2s} - x_{n+1}) x_{n+1}^{-s} u \|^2 + \frac{159}{10} \tau^2 \| \partial_{n+1} u \|^2 + \frac{1}{10} \tau^2 \| \nabla' u \|^2 \\
+ \frac{1}{20} (2s - 1) \tau^2 \|(x_{n+1}^{1-2s} - x_{n+1}) \nabla' u \|^2 + 12.1 \tau^2 (2s - 1)(2s + 1) \| x_{n+1}^{-1} u \|^2 \\
+ 8 \tau^2 \langle (\partial_{n+1}^2 \phi) \partial_{n+1} u, u \rangle_0 + 4 \tau^2 \langle Su, (\partial_{n+1} \phi) u \rangle_0 - 2 \tau \langle Au, \partial_{n+1} u \rangle_0 + 2 \tau \langle \partial_{n+1} (Au), u \rangle_0 \\
- \tau^2 \left\| x_{n+1}^{-\frac{1-2}{2}} \partial_{n+1} u \right\|_0^2 - \tau^2 \left\| x_{n+1}^{-\frac{1-2}{2}} \nabla' u \right\|_0^2 - \tau^4 \|(x_{n+1}^{1-2s} - x_{n+1})^{\frac{1}{2}} u \|^2 \bigg) \\
(3.13) - 8 \tau^2 (2s - 1)(2s + 1) \| x_{n+1}^{-\frac{1-2}{2}} u \|^2_0 - 4(2s - 1) \langle \Delta' u, \partial_{n+1} u \rangle_0 + \frac{41}{10} \tau^2 \langle \partial_{n+1} u, u \rangle_0.
\]

Hence, we reach
\[
2 \tau \| L^+ u \|^2 \\
\geq 25 \tau^4 \|(x_{n+1}^{1-2s} - x_{n+1}) u \|^2 + \frac{1}{10} \tau^2 \| \nabla u \|^2 + 12 \tau^2 (2s - 1)(2s + 1) \| x_{n+1}^{-1} u \|^2 \\
+ 8 \tau^2 \langle (\partial_{n+1}^2 \phi) \partial_{n+1} u, u \rangle_0 + 4 \tau^2 \langle Su, (\partial_{n+1} \phi) u \rangle_0 - 2 \tau \langle Au, \partial_{n+1} u \rangle_0 + 2 \tau \langle \partial_{n+1} (Au), u \rangle_0 \\
- \tau^2 \left\| x_{n+1}^{-\frac{1-2}{2}} \partial_{n+1} u \right\|_0^2 - \tau^2 \left\| x_{n+1}^{-\frac{1-2}{2}} \nabla' u \right\|_0^2 - \tau^4 \|(x_{n+1}^{1-2s} - x_{n+1})^{\frac{1}{2}} u \|^2 \bigg) \\
(3.14) - 8 \tau^2 (2s - 1)(2s + 1) \| x_{n+1}^{-\frac{1-2}{2}} u \|^2_0 - 4(2s - 1) \langle \Delta' u, \partial_{n+1} u \rangle_0 + \frac{41}{10} \tau^2 \langle \partial_{n+1} u, u \rangle_0.
\]

Since \( u = e^{\tau \phi} x_{n+1}^{-\frac{1-2s}{2}} w \), we estimate that
\[
\| \nabla u \|^2 \geq \frac{1}{2} \left\| e^{\tau \phi} x_{n+1}^{-\frac{1-2s}{2}} \nabla w \right\|^2 - 2 \tau^2 \left\| e^{\tau \phi} \nabla \phi \right\| x_{n+1}^{-\frac{1-2s}{2}} w \|^2 - 2 \left( \frac{2s - 1}{2} \right)^2 \left\| e^{\tau \phi} x_{n+1}^{-\frac{1-2s}{2}} w \right\|^2 \\
\geq \frac{1}{2} \left\| e^{\tau \phi} x_{n+1}^{-\frac{1-2s}{2}} \nabla w \right\|^2 - 16 \tau^2 \|(x_{n+1}^{1-2s} - x_{n+1}) u \|^2 - (2s - 1)^2 \| x_{n+1}^{-1} u \|^2.
\]

Next, we want to estimate the boundary terms. First of all, we want to show that
\[
(3.15) \quad \| e^{\tau \phi} x_{n+1}^{-2s} w \|_0 \leq C_s \| e^{\tau \phi} x_{n+1}^{1-2s} \partial_{n+1} w \|_0 < \infty.
\]

Indeed, since \( w(x', 0) \equiv 0 \), thus
\[
x_{n+1}^{-2s} w(x', x_{n+1}) = x_{n+1}^{1-2s} \int_0^1 \partial_{n+1} w(x', tx_{n+1}) \, dt = \int_0^1 (tx_{n+1})^{1-2s} \partial_{n+1} w(x', tx_{n+1}) t^{2s-1} \, dt.
\]

Multiplying by \( e^{\tau \phi} \), taking the \( L^2 \)-norm with respect to \( x' \) and using the fact that \( \partial_{n+1} \phi < 0 \) on \( \text{supp}(w) \) gives
\[
\| e^{\tau \phi} x_{n+1}^{-2s} w(\bullet, x_{n+1}) \|_0 \leq \sup_{t \in (0, 1)} \| e^{\tau \phi} (tx_{n+1})^{1-2s} \partial_{n+1} w(\bullet, tx_{n+1}) \|_0 \int_0^1 t^{2s-1} \, dt.
\]

Taking \( x_{n+1} \to 0 \) proves (3.15).
Observe that
\[ 4\tau^2\langle Su, (\partial_{n+1}\phi)u \rangle_0 - 2\tau\langle Au, \partial_{n+1}u \rangle_0 + 2\tau(\partial_{n+1}(Au), u)_0 \]
\[ = 8\tau^2\langle \partial_{n+1}u, \nabla'\phi \cdot \nabla'u \rangle_0 + 4\tau^2\langle (\partial_{n+1}u)^2, \nabla_{n+1}\phi \rangle_0 - 4\tau^2\langle (\partial_{n+1}\phi), |\nabla'u|^2 \rangle_0 \]
\[ + 4\tau^2\langle (\Delta'\phi - \partial^2_{n+1}\phi)u, \partial_{n+1}u \rangle_0 - 2\tau^2\langle (\partial^3_{n+1}\phi)u, u \rangle_0 + 4\tau^4\langle (\partial_{n+1}\phi)|\nabla\phi|^2 u, u \rangle_0 \]
\[ - \tau^2(2s+1)(2s-1)\langle x^-_{n+1}u, (\partial_{n+1}\phi)u \rangle_0 \]
\[ \geq 8\tau^2\langle \partial_{n+1}u, \nabla'\phi \cdot \nabla'u \rangle_0 + 4\tau^2\langle (\partial_{n+1}u)^2, \nabla_{n+1}\phi \rangle_0 + 4\tau^2\langle (\Delta'\phi - \partial^2_{n+1}\phi)u, \partial_{n+1}u \rangle_0 \]
\[ + 4\tau^4\langle (\partial_{n+1}\phi)|\nabla\phi|^2 u, u \rangle_0. \]

Note that (3.15) imply
\[ \partial_{n+1}u = e^{\tau\phi}\left( \frac{1}{x_{n+1}^{2s}}\partial_{n+1} w - \frac{2s-1}{2} x_{n+1}^{-\frac{1}{2s}} w \right) + x_{n+1}^{-\frac{3}{2s}} R \]
\[ \nabla'u = e^{\tau\phi} x_{n+1}^{-\frac{1}{2s}} \nabla'w + x_{n+1}^{s+\frac{1}{2}} R', \]
where \( \|R\|_0 \leq C\tau \) and \( \|R'\|_0 \leq C\tau \).

Hence,
\[ \langle (\partial_{n+1}u, \nabla'\phi \cdot \nabla'u) \rangle_0 = \left| \langle e^{\tau\phi} \left( \frac{1}{x_{n+1}^{2s}}\partial_{n+1} w - \frac{2s-1}{2} x_{n+1}^{-\frac{1}{2s}} w \right), e^{\tau\phi} x_{n+1}^{-\frac{1}{2s}} \nabla'\phi \cdot \nabla'u \rangle_0 \right| \]
\[ = \left| \langle e^{\tau\phi} \left( x_{n+1}^{1-2s}\partial_{n+1} w - \frac{2s-1}{2} x_{n+1}^{-\frac{1}{2s}} w \right), \frac{1}{2} e^{\tau\phi} x' \cdot \nabla'u \rangle_0 \right| \]
\[ \leq \frac{1}{2} \langle (e^{\tau\phi} x_{n+1}^{1-2s}\partial_{n+1} w, e^{\tau\phi} x' \cdot \nabla'u) \rangle_0 + \frac{2s-1}{4} \langle (e^{\tau\phi} x_{n+1}^{-\frac{1}{2s}} w, e^{\tau\phi} x' \cdot \nabla'u) \rangle_0. \]

Using (3.15), we reach
\[ \langle (\partial_{n+1}u, \nabla'\phi \cdot \nabla'u) \rangle_0 \leq \|e^{\tau\phi} x_{n+1}^{1-2s}\partial_{n+1} w\|_0 \|e^{\tau\phi} x' \cdot \nabla'u\|_0. \]

Similarly, using (3.15), we have
\[ \langle (\partial_{n+1}u)^2, \partial_{n+1}\phi \rangle_0 + \langle (\Delta'\phi - \partial^2_{n+1}\phi)u, \partial_{n+1}u \rangle_0 \leq C\|e^{\tau\phi} x_{n+1}^{1-2s}\partial_{n+1} w\|_0^2 \]
\[ + \|e^{\tau\phi} \partial_{n+1}\phi \|_0 \langle (\partial_{n+1}\phi)|\nabla\phi|^2 u, u \rangle_0 \leq C\|e^{\tau\phi} x_{n+1}^{2-4s} w\|_0^2 \to 0. \]

Also,
\[ \langle (\partial_{n+1}\phi)\partial_{n+1}u, u \rangle_0 \leq C\|e^{\tau\phi} x_{n+1}^{1-2s}\partial_{n+1} w\|_0^2 \]
\[ \|x_{n+1}^{1-2s}\partial_{n+1}u\|_0^2 = \left\| e^{\tau\phi} x_{n+1}^{1-2s}\partial_{n+1} w - \frac{2s-1}{2} e^{\tau\phi} x_{n+1}^{-\frac{1}{2s}} w \right\|_0^2 \leq C\|e^{\tau\phi} x_{n+1}^{1-2s}\partial_{n+1} w\|_0^2 \]
\[ \|x_{n+1}^{\frac{1}{2s}} x' \nabla'u\|_0^2 \to 0 \]
\[ \|(x_{n+1}^{1-2s} - x_{n+1})^\frac{1}{2} u\|_0^2 \to 0 \]
\[ \|x_{n+1}^{-\frac{1}{2s}} u\|_0^2 = \|e^{\tau\phi} x_{n+1}^{-\frac{1}{2s}} w\|_0^2 \leq C\|e^{\tau\phi} x_{n+1}^{1-2s}\partial_{n+1} w\|_0^2 \]
\[ \langle (\partial_{n+1}u, u) \rangle_0 \to 0. \]
Finally, we also have
\[
|\langle \Delta' u, \partial_{n+1} u \rangle_0| \leq \frac{2s^4}{n^2} \Delta' u_k^2 + \| x_{n+1} \partial_{n+1} u \|_0^2
\]
\[
= \left\| - \frac{n\tau}{2} e^{\phi} w + \frac{\tau}{4} \right\| x_{n+1} \partial_{n+1} u \|_0^2 + \| x_{n+1} \partial_{n+1} u \|_0^2
\]
\[
\leq C \| e^{\phi} \Delta' u \|_0^2 + C \tau^2 \| e^{\phi} x' \cdot \nabla' w \|_0^2 + C \| e^{\phi} x_{n+1}\partial_{n+1} u \|_0^2
\].

Put them together, we reach
\[
\tau^3 \| u \|_0^2 + \| e^{\phi} x_{n+1} \partial_{n+1} u \|_0^2
\]
\[
\leq C \left( \| L^+ u \|_0^2 + \tau^{-1} \| e^{\phi} \Delta' w \|_0^2 + \tau \| e^{\phi} x' \cdot \nabla' w \|_0^2 + \tau \| e^{\phi} x_{n+1}\partial_{n+1} u \|_0^2 \right),
\]
which is our desired result.

As in [RS17], we introduce the following sets for \( s \in [\frac{1}{2}, 1) \):
\[
C^+_s := \left\{ (x', x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = \left(1 - s \right) \left( r - \frac{|x'|^2}{4} \right) \right\}
\]
\[
C'_{s,r} := C^+_s \cap (\mathbb{R}^n \times \{0\}).
\]

With this notation at hand, we infer the following analogous of the Proposition 5.10 of [RS17]:

**Lemma 3.5.** Let \( s \in [\frac{1}{2}, 1) \). Suppose that \( \tilde{w} \in H^1(\mathbb{R}^{n+1}, x_{n+1}^{1-2s}) \) is a solution to
\[
\left[ \partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^n \partial_j a_{jk} \partial_k \right] \tilde{w} = 0 \quad \text{in} \ \mathbb{R}^{n+1},
\]
\[
\tilde{w} = w \quad \text{on} \ \mathbb{R}^n \times \{0\},
\]
with \( w = 0 \) on \( B'_1 \). We assume that
\[
\max_{1 \leq j,k \leq n} \| a_{jk} - \delta_{jk} \|_\infty + \max_{1 \leq j,k \leq n} \| \nabla' a_{jk} \|_\infty \leq \epsilon
\]
for some sufficiently small \( \epsilon > 0 \). For \( s \neq \frac{1}{2} \), we further assume
\[
\max_{1 \leq j,k \leq n} \| (\nabla')^2 a_{jk} \|_\infty \leq C
\]
for some positive constant \( C \). Then there exists \( \alpha = \alpha(n, s) \in (0, 1) \) such that
\[
\| x_{n+1}^{1-2s} \tilde{w} \|_{L^2(C^+_s, 1/2)} \leq C \| x_{n+1}^{1-2s} \tilde{w} \|_{L^2(C^+_s, 1/2)}^{1-\alpha} \lim_{x_{n+1} \to 0} \| x_{n+1}^{1-2s} \partial_{n+1} \tilde{w} \|_{L^2(C'_s, 1/2)}^{1-\alpha}.
\]

**Proof.** We may assume that \( \| x_{n+1}^{1-2s} \tilde{w} \|_{L^2(C^+_s, 1/2)} > 0 \) and
\[
\| x_{n+1}^{1-2s} \tilde{w} \|_{L^2(C'_s, 1/2)} \geq c_0 \lim_{x_{n+1} \to 0} \| x_{n+1}^{1-2s} \partial_{n+1} \tilde{w} \|_{L^2(C'_s, 1/2)}^{1-\alpha}
\]
for some sufficiently large constant \( c_0 > 0 \). Otherwise the result is trivial.

Let \( \eta \) be a smooth cut-off function satisfies
\[
\eta(x) = \begin{cases} 1 & \text{in} \ C^+_s, 3/16; \\ 0 & \text{in} \ \mathbb{R}^{n+1} \setminus C^+_s, 1/4. \end{cases}
\]
and $|\partial_{n+1}\eta| \leq Cx_{n+1}$ in $\mathbb{R}^{n+1}_+$ with $\partial_{n+1}\eta = 0$ on $\mathbb{R}^n \times \{0\}$. Define $\bar{w} = \eta \bar{w}$. Note that $\bar{w}$ satisfies $\text{supp}(\bar{w}) \subset B^+_{1/2}$ and it solves
\[
\begin{align*}
\left[ \partial_{n+1}x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} a_{jk} \partial_j \partial_k \right] \bar{w} &= f \quad \text{in } \mathbb{R}^n_{n+1}, \\
\bar{w} &= 0 \quad \text{on } \mathbb{R}^n \times \{0\},
\end{align*}
\]
where
\[
f = \partial_{n+1}(x_{n+1}^{1-2s} \partial_{n+1}\eta) \bar{w} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} \partial_j (a_{jk} \partial_k \eta) \bar{w}
+ 2x_{n+1}^{1-2s} \partial_{n+1}\eta \partial_{n+1}\bar{w} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} a_{jk} \partial_j \eta \partial_k \bar{w} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} a_{jk} \partial_j \eta \partial_k \bar{w}
- x_{n+1}^{1-2s} \sum_{j,k=1}^{n} (\partial_j a_{jk}) \partial_k \bar{w}.
\]

Since $\eta$ and $\nabla \eta$ are bounded, together with $|\partial_{n+1}\eta| \leq Cx_{n+1}$, we know that
\[
\|x_{n+1}^{1-2s} f\|^2_{L^2(\mathbb{R}^{n+1}_+)} \leq C\left(\|x_{n+1}^{1-2s} \bar{w}\|^2_{L^2(C^{s,1/4}_{s,1/8})} + \|x_{n+1}^{1-2s} \nabla \bar{w}\|^2_{L^2(C^{s,1/4}_{s,1/8})}\right) < \infty.
\]
Moreover, since $\|w\|_{B^+_1} = 0$ and $\text{supp}(\eta) \subset B^+_1$ on $\mathbb{R}^n \times \{0\}$, then
\[
\lim_{x_{n+1} \to 0} \nabla \bar{w} = 0, \quad \lim_{x_{n+1} \to 0} \Delta \bar{w} = 0 \quad \text{and also } \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \bar{w} = \eta \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \bar{w},
\]
which shows that the function $\bar{w}$ is admissible in Lemma 3.4. So, by the Carleman estimate in Lemma 3.4, there exists $\tau_0 > 1$ such that
\[
\tau^3 \|e^{\tau \phi} x_{n+1}^{1-2s} \bar{w}\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau \|e^{\tau \phi} x_{n+1}^{1-2s} \nabla \bar{w}\|^2_{L^2(\mathbb{R}^{n+1}_+)} 
\leq C\left(\|e^{\tau \phi} x_{n+1}^{\frac{2s-1}{2}} f\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau \lim_{x_{n+1} \to 0} \|e^{\tau \phi} x_{n+1}^{1-2s} \partial_{n+1} \bar{w}\|^2_{L^2(\mathbb{R}^n \times \{0\})}\right)
\]
for all $\tau \geq \tau_0$. Then, for large $\tau_0$, the last term of $f$ was absorbed by the gradient term in the LHS, and we have
\[
\tau^3 \|e^{\tau \phi} x_{n+1}^{1-2s} \bar{w}\|^2_{L^2(\mathbb{R}^{n+1}_+)} \leq C\left(\|e^{\tau \phi} x_{n+1}^{\frac{2s-1}{2}} g\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau \lim_{x_{n+1} \to 0} \|e^{\tau \phi} x_{n+1}^{1-2s} \partial_{n+1} \bar{w}\|^2_{L^2(\mathbb{R}^n \times \{0\})}\right),
\]
where $g = f + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} (\partial_j a_{jk}) \partial_k \bar{w}$.

Let
\[
\phi_- := \inf_{x \in C^{s,1/8}_{s,1/8}} \phi(x) \quad \text{and } \quad \phi_+ := \sup_{x \in C^{s,1/4}_{s,3/16}} \phi(x).
\]
Hence,
\[
\tau^3 e^{2\phi_-} \|x_{n+1}^{\frac{2s-1}{2}} \bar{w}\|^2_{L^2(C^{s,1/8}_{s,1/8})} 
\leq C\left[e^{2\phi_+} \|x_{n+1}^{\frac{2s-1}{2}} g\|^2_{L^2(C^{s,1/4}_{s,1/4})} + \tau \lim_{x_{n+1} \to 0} \|x_{n+1}^{1-2s} \partial_{n+1} \bar{w}\|^2_{L^2(C^{s,1/4}_{s,1/4})}\right].
\]
Dividing by $\tau$, using that $\tau \geq 1$ and by Caccioppoli’s inequality (Lemma $\text{A.6}$), we obtain
$$\|x_{n+1}^{1-2s} \tilde{w}\|_{L^2(C_{s,1/8}^+)} \leq C \left[ e^{\tau(\phi_+ - \phi_-)} \|x_{n+1}^{1-2s} \tilde{w}\|_{L^2(C_{s,1/8}^+)} + e^{-\tau\phi_-} \lim_{x_{n+1} \to 0} \|x_{n+1}^{1-2s} \tilde{\partial}_{n+1} \tilde{w}\|_{L^2(C_{s,1/2}^+)} \right].$$

Observe that
$$-\frac{|x'|^2}{4} \geq \frac{1}{1-s}x_{n+1}^{2-2s} - \frac{1}{8} \quad \text{in } C_{s,1/8}^+,$$
and also since $s \geq \frac{1}{2}$,
$$x_{n+1}^2 \leq \left[ (1-s) \left( \frac{1}{4} - \frac{|x'|^2}{4} \right) \right]^{\frac{1}{1-s}} \leq \frac{1}{8^{\frac{1}{1-s}}} \leq \frac{1}{64}$$
and
$$-\frac{|x'|^2}{4} \leq \frac{1}{1-s}x_{n+1}^{2-2s} - \frac{3}{16} \quad \text{in } C_{s,1/4}^+ \setminus C_{s,3/16}^+,$$
so $\phi_- \geq -\frac{1}{8}$ and $\phi_+ \leq -\frac{16}{64}$, that is, $\phi_+ - \phi_- \leq -\frac{16}{64} < 0$. So, we can choose $\tau$ (which is large) satisfies
$$e^{\tau(\phi_+ - \phi_-)} = \frac{\lim_{x_{n+1} \to 0} \|x_{n+1}^{1-2s} \tilde{\partial}_{n+1} \tilde{w}\|_{L^2(C_{s,1/2}^+)}^{1-\alpha}}{\|x_{n+1}^{1-2s} \tilde{w}\|_{L^2(C_{s,1/2}^+)}^{1-\alpha}} \leq \frac{1}{c_0}$$
for large $c_0$, where $\alpha \in (0, 1)$ will be chosen later. Note that
$$e^{-\tau\phi_-} = \frac{\lim_{x_{n+1} \to 0} \|x_{n+1}^{1-2s} \tilde{\partial}_{n+1} \tilde{w}\|_{L^2(C_{s,1/2}^+)}^{\phi_- - \phi_- (1-\alpha)}}{\|x_{n+1}^{1-2s} \tilde{w}\|_{L^2(C_{s,1/2}^+)}^{\phi_- - \phi_- (1-\alpha)}}.$$\hfill $\Box$

Finally, choosing $\alpha \in (0, 1)$ satisfies $\alpha = \frac{\phi_- - \phi_- (1-\alpha)}{\phi_- - \phi_- (1-\alpha)}$ will implies our desired result.

For our purpose, we only need the following simplified version of the Lemma above:

**Corollary 3.6.** Let $s \in \left[ \frac{1}{2}, 1 \right)$. Suppose that $\tilde{w} \in H^1(\mathbb{R}_+^{n+1}, x_{n+1}^{1-2s})$ is a solution to
$$\left[ \tilde{\partial}_{n+1} x_{n+1}^{1-2s} \tilde{\partial}_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^n \partial_j a_{jk} \partial_k \right] \tilde{w} = 0 \quad \text{in } \mathbb{R}_+^{n+1},$$
$$\tilde{w} = w \quad \text{on } \mathbb{R}^n \times \{0\},$$
with $w = 0$ on $B'_1$. We assume that
$$\max_{1 \leq j,k \leq n} \|a_{jk} - \delta_{jk}\|_\infty + \max_{1 \leq j,k \leq n} \|\nabla' a_{jk}\|_\infty \leq \epsilon$$
for some sufficiently small $\epsilon > 0$. For $s \neq \frac{1}{2}$, we further assume
$$\max_{1 \leq j,k \leq n} \| (\nabla')^2 a_{jk} \|_\infty \leq C$$
for some positive constant $C$. Then there exists $\alpha = \alpha(n, s) \in (0, 1)$, $c = c(n, s) \in (0, 1)$ and a constant $C$ such that
$$\|x_{n+1}^{1-2s} \tilde{w}\|_{L^2(B_1^+)} \leq C \|x_{n+1}^{1-2s} \tilde{w}\|_{L^2(B_2^+)}^{\alpha} \lim_{x_{n+1} \to 0} \|x_{n+1}^{1-2s} \tilde{\partial}_{n+1} \tilde{w}\|_{L^2(B'_2)}^{1-\alpha}.$$
Proof of the part (a) of Lemma 3.3 for $s \in \left[\frac{1}{2}, 1\right)$. In order to invoke the estimate from Corollary 3.6 we split our solution $u$ into two parts $\tilde{u} = u_1 + u_2$, where $u_1$ satisfies

$$
\left[\partial_{n+1}x_{n+1}^{-2s}\partial_{n+1} + x_{n+1}^{-2s}\sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k \right] u_1 = 0 \quad \text{in } \mathbb{R}^{n+1}_+,
$$

$$
u_1 = \zeta u \quad \text{on } \mathbb{R}^n \times \{0\},$$

where $\zeta \in C_0^\infty(B'_1)$ is a smooth cut-off function with $\zeta = 1$ on $B'_8$. So, by (2.2), we have

$$
\int_{\mathbb{R}^n} |u_1(x', x_{n+1})|^2 \, dx' \leq \|u_1\|_{L^2(\mathbb{R}^n \times \{0\})}^2 \leq \|u\|_{L^2(B'_1 \times 0)}^2.
$$

So,

$$
\|\frac{1-2s}{x_{n+1}^{1-2s}} u_1\|_{L^2(B'_1 \times 0)}^2 \leq \int_0^{10} \int_{\mathbb{R}^n} \frac{1-2s}{x_{n+1}^{1-2s}} |u_1(x', x_{n+1})|^2 \, dx' \, dx_{n+1}
$$

$$
\leq \left( \int_0^{10} \frac{1-2s}{x_{n+1}^{1-2s}} \, dx_{n+1} \right) \|u\|_{L^2(B'_1 \times 0)}^2 = C \|u\|_{L^2(B'_1 \times 0)}^2.
$$

(3.16)

Note that $u_2$ satisfies

$$
\left[\partial_{n+1}x_{n+1}^{-2s}\partial_{n+1} + x_{n+1}^{-2s}\sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k \right] u_2 = 0 \quad \text{in } \mathbb{R}^{n+1}_+,
$$

$$
u_2 = u - \zeta u \quad \text{on } \mathbb{R}^n \times \{0\}.$$

Since $u_2 = 0$ on $B'_8$, by Corollary 3.6 there exists $\alpha = \alpha(n, s) \in (0, 1)$, $c = c(n, s) \in (0, 1)$ and a constant $C$ such that

$$
\|\frac{1-2s}{x_{n+1}^{1-2s}} u_2\|_{L^2(B'_1 \times 0)} \leq C \|\frac{1-2s}{x_{n+1}^{1-2s}} u_2\|_{L^2(B'_1 \times 0)}^\alpha \cdot \lim_{x_{n+1} \to 0} \|x_{n+1}^{-1-2s} \partial_{n+1} u_2\|_{L^2(B'_1)}^{1-\alpha}.
$$

(3.17)

Let $\eta$ be a smooth, radial cut-off function with $\eta = 1$ in $B'_2$ and $\eta = 0$ outside $B'_4$. Plug $w = \eta x_{n+1}^{-1-2s} \partial_{n+1} u_2$ into the trace characterization lemma (Lemma A.5), we reach

$$
\lim_{x_{n+1} \to 0} \|x_{n+1}^{-1-2s} \partial_{n+1} u_2\|_{L^2(B'_1)} \leq C \left[ \mu^{1-s} \|x_{n+1}^{-1-2s} \partial_{n+1} u_2\|_{L^2(B'_1)} + \|x_{n+1}^{-2s} \partial_{n+1} u_2\|_{L^2(\mathbb{R}^{n+1}_+)} \right].
$$

(3.18)

We first control the boundary term of (3.18). Since $\eta$ is a bounded multiplier on $H^{2s}(\mathbb{R}^n)$, using duality, we have

$$
\|\eta v\|_{H^{-2s}(\mathbb{R}^n \times \{0\})} = \sup_{\|\varphi\|_{H^{2s}(\mathbb{R}^n \times \{0\})} = 1} \langle v, \eta \varphi \rangle_{L^2(\mathbb{R}^n \times \{0\})} \leq \|v\|_{H^{-2s}(B'_8)} \sup_{\|\varphi\|_{H^{2s}(\mathbb{R}^n \times \{0\})} = 1} \|\eta \varphi\|_{H^{2s}(\mathbb{R}^n \times \{0\})}
$$

$$\leq C \|v\|_{H^{-2s}(B'_8)} \sup_{\|\varphi\|_{H^{2s}(\mathbb{R}^n \times \{0\})} = 1} \|\varphi\|_{H^{2s}(\mathbb{R}^n \times \{0\})} = C \|v\|_{H^{-2s}(B'_8)}.
$$

Plug $v = x_{n+1}^{-1-2s} \partial_{n+1} u_2$, we have

$$
\lim_{x_{n+1} \to 0} \|\eta x_{n+1}^{-1-2s} \partial_{n+1} u_2\|_{H^{-2s}(\mathbb{R}^n \times \{0\})} \leq C \cdot \lim_{x_{n+1} \to 0} \|x_{n+1}^{-1-2s} \partial_{n+1} u_2\|_{H^{-2s}(B'_8)}.
$$

(3.19)
Apply Lemma A.6 (Caccioppoli’s inequality) with zero Dirichlet condition and zero inhomogeneous terms, we have

\begin{equation}
\| x_{n+1}^{\frac{1-2s}{2}} \nabla u_2 \|_{L^2(B_s^+)} \leq C \| x_{n+1} u_2 \|_{L^2(B_s^+)}.
\end{equation}

Also, we have

\[
\| x_{n+1}^{\frac{s-1}{2}} \nabla \left( \eta x_{n+1}^{s-1} \partial_{n+1} u_2 \right) \|_{L^2(R_n^{x+1})} \\
\leq \| x_{n+1} \left( \nabla \eta \right) \partial_{n+1} u_2 \|_{L^2(R_n^{x+1})} + \| x_{n+1} \eta \nabla \partial_{n+1} u_2 \|_{L^2(R_n^{x+1})} \\
+ \| x_{n+1} \eta \partial_{n+1} x_{n+1}^{s-1} \partial_{n+1} u_2 \|_{L^2(R_n^{x+1})}
\leq C \| x_{n+1}^{\frac{s-1}{2}} \partial_{n+1} u_2 \|_{L^2(B_s^+)} + \| x_{n+1}^{\frac{s-1}{2}} \nabla \partial_{n+1} u_2 \|_{L^2(B_s^+)} + \left( \sum_{j=1}^{n} \sum_{k=1}^{n} \partial_j a_{jk} \partial_k u_2 \right)
\leq C \left[ \| x_{n+1}^{\frac{s-1}{2}} \nabla u_2 \|_{L^2(B_s^+)} + \| x_{n+1}^{\frac{s-1}{2}} \nabla \partial_{n+1} u_2 \|_{L^2(B_s^+)} \right],
\]

where the last inequality follows by the boundedness assumptions of \( a_{jk} \). Observe that

\[
0 = \nabla' \left[ \partial_{n+1} \left( x_{n+1}^{1-2s} \partial_{n+1} u_2 \right) + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k u_2 \right]
\]

\[
= \left[ \partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k \right] \left( \nabla' u_2 \right) + x_{n+1}^{1-2s} \sum_{j=1}^{n} \sum_{k=1}^{n} \partial_j \left( \sum_{k=1}^{n} \nabla' a_{jk} \partial_k u_2 \right).
\]

Apply Lemma A.6 (Caccioppoli’s inequality) on \( \nabla' u_2 \) with zero Dirichlet condition and \( f_j = \sum_{k=1}^{n} \nabla' a_{jk} \partial_k u_2 \), since \( \| \nabla' a_{jk} \|_{\infty} \leq c \), we have

\begin{equation}
\| x_{n+1}^{\frac{s-1}{2}} \nabla \partial_{n+1} u_2 \|_{L^2(B_s^+)} \leq C' \| x_{n+1}^{\frac{s-1}{2}} \nabla' u_2 \|_{L^2(B_s^+)} \leq C \| x_{n+1}^{\frac{s-1}{2}} u_2 \|_{L^2(B_s^+)}.
\end{equation}

where the second inequality follows by (3.20). Hence, we reach

\begin{equation}
\| x_{n+1}^{\frac{s-1}{2}} \nabla \left( \eta x_{n+1}^{s-1} \partial_{n+1} u_2 \right) \|_{L^2(R_n^{x+1})} \leq C \| x_{n+1}^{\frac{s-1}{2}} u_2 \|_{L^2(B_s^+)}.
\end{equation}

Plug (3.19), (3.20) and (3.22) into (3.18), and optimizing the result estimate in \( \mu > 0 \) gives

\[
\lim_{x_{n+1} \to 0} \left\| x_{n+1}^{\frac{1-2s}{2}} \partial_{n+1} u_2 \right\|_{L^2(B_s^+)} \leq C \| x_{n+1} u_2 \|_{L^2(B_s^+)} \cdot \lim_{x_{n+1} \to 0} \left\| x_{n+1}^{2-2s} \partial_{n+1} u_2 \right\|_{H^{-2s}(B_s^+)}.
\]

Insert this into (3.17) leads to

\[
\| x_{n+1}^{\frac{s-1}{2}} u_2 \|_{L^2(B_s^+)} \leq C \| x_{n+1}^{\frac{s-1}{2}} u_2 \|_{L^2(B_s^+)} \cdot \lim_{x_{n+1} \to 0} \left\| x_{n+1}^{1-2s} \partial_{n+1} u_2 \right\|_{H^{-2s}(B_s^+)} \leq C \left( \| x_{n+1}^{\frac{s-1}{2}} \tilde{u} \|_{L^2(B_s^+)} + \| x_{n+1}^{\frac{1-2s}{2}} u_1 \|_{L^2(B_s^+)} \right) \frac{1}{\tilde{c}^\alpha} \times \left( \lim_{x_{n+1} \to 0} \left\| x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} \right\|_{H^{-2s}(B_s^+)} + \lim_{x_{n+1} \to 0} \left\| x_{n+1}^{1-2s} \partial_{n+1} u_1 \right\|_{H^{-2s}(B_s^+)} \right)^{1-\alpha},
\]

(3.23)
where \( \tilde{\alpha} = \frac{1-2s}{1+s} \alpha + \frac{2s}{1+s} \). Then we have

\[
\lim_{x_{n+1} \to 0} \| x_{n+1}^{-2s} \partial_{n+1} u_1 \|_{H^{-2s}(\mathbb{R}^n \times \{0\})} = \| (-P)^s u_1 \|_{H^{-2s}(\mathbb{R}^n \times \{0\})} \leq C \| u_1 \|_{L^2(\mathbb{R}^n \times \{0\})} \leq C \| \tilde{u} \|_{L^2(B'_{16})},
\]

(3.24)

where the second inequality follows by Lemma 2.2.

Combining (3.16), (3.23) and (3.24), we reach

\[
\lim_{x_{n+1} \to 0} \| x_{n+1}^{-2s} \partial_{n+1} u \|_{L^2(B_{16}')} + \| \tilde{u} \|_{L^2(B_{16}')} + \| u \|_{L^2(B_{16}')}^{\tilde{\alpha}} \cdot \left( \lim_{x_{n+1} \to 0} \| x_{n+1}^{-2s} \partial_{n+1} u \|_{L^2(B_{16}')} + \| u \|_{L^2(B_{16}')} \right)^{1-\tilde{\alpha}},
\]

(3.25)

which is our desired claim of (a).

Indeed, combining (3.25) with the Caccioppoli’s inequality (Lemma A.6), we reach

\[
\| x_{n+1}^{-2s} \tilde{u} \|_{L^2(B_{16}')} + \| \tilde{u} \|_{L^2(B_{16}')} + \| u \|_{L^2(B_{16}')}^{\tilde{\alpha}} \cdot \left( \lim_{x_{n+1} \to 0} \| x_{n+1}^{-2s} \partial_{n+1} u \|_{L^2(B_{16}')} + \| u \|_{L^2(B_{16}')} \right)^{1-\tilde{\alpha}} \leq C \left( \| x_{n+1}^{-2s} \partial_{n+1} u \|_{L^2(B_{16}')} + \| \tilde{u} \|_{L^2(B_{16}')} \right)^{\tilde{\alpha}},
\]

(3.26)

with \( \tilde{c} = c/2 \). Slightly modify the proof of (3.23), we can obtain the following analogue of Proposition 5.11 of [RS17]:

Lemma 3.7. Let \( s \in [\frac{1}{2}, 1) \) and \( \tilde{w} \) is the Caffarelli-Silvestre type extension of some \( f \in H^\gamma(\mathbb{R}^n) \) as in (2.1), where \( \gamma \in \mathbb{R} \) with \( f|_{C_{s,1}^+} = 0 \). We assume that

\[
\max_{1 \leq j, k \leq n} \| a_{jk} - \delta_{jk} \|_\infty + \max_{1 \leq j, k \leq n} \| \nabla a_{jk} \|_\infty \leq \epsilon
\]

for some sufficiently small \( \epsilon > 0 \). For \( s \neq \frac{1}{2} \), we further assume

\[
\max_{1 \leq j, k \leq n} \| (\nabla')^2 a_{jk} \|_\infty \leq C
\]

for some positive constant \( C \). Then there exists \( C = C(n, s) \) and \( \alpha = \alpha(n, s) \in (0, 1) \) such that

\[
\| x_{n+1}^{-2s} \tilde{w} \|_{L^2(C_{s,1/2}^+)} \leq C \| x_{n+1}^{-2s} \tilde{w} \|^{\frac{\alpha}{2}}_{L^2(C_{s,1}^+)} \cdot \lim_{x_{n+1} \to 0} \| x_{n+1}^{-2s} \partial_{n+1} \tilde{w} \|^{-\alpha}_{H^{-2s}(C_{s,1/2}^+)}.
\]

Proof. Let \( \eta \) be a smooth cut-off function supported in \( C_{s,1/2}^+ \) with \( \eta = 1 \) in \( C_{s,1/4}^+ \). Using this cut-off function, and following the ideas in the proof of (3.23), using Lemma A.4 rather than Lemma A.5, we can obtain the above inequality.

\[
3.2. \ \text{Proof of the part (a) of Lemma 3.3 for the case } s \in (0, 1/2) \text{. Let } \tilde{w} \text{ solves (2.1). If we define } \overline{s} := 1 - s \in (1/2, 1),
\]

(3.27)

\[
v(x) = x_{n+1}^{-1-2s} \partial_{n+1} \tilde{w}(x) \text{ and } f = \lim_{x_{n+1} \to 0} x_{n+1}^{-1-2s} \partial_{n+1} \tilde{w} = c_{n,s}^{-1}(-P)^s u,
\]
then
\[
\left[ \partial_{n+1} x_{n+1}^{1-2\varpi} \partial_{n+1} + x_{n+1}^{1-2\varpi} \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k \right] v = 0 \quad \text{in } \mathbb{R}_{+}^{n+1},
\]
\[
v = f \quad \text{on } \mathbb{R}^n \times \{0\}.
\]

Note that \((-P)^{1-s}(-P)^s\) is identical to \(-P\) up to a constant. Indeed,
\[
(-P)^{1-s}(-P)^s u = c_{n,s}(-P)^s f = c_{n,s} c_{n,\varpi} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2\varpi} \partial_{n+1} v
\]
\[
= c_{n,s} c_{n,\varpi} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2\varpi} \partial_{n+1} (x_{n+1}^{1-2s} \partial_{n+1} \tilde{w})
\]
\[
= -c_{n,s} c_{n,\varpi} P u.
\]
(3.28)

See also Proposition 3.6 of [Sti10] for the general case.

Using this observation, and follows the ideas in the Proposition 5.12 of [RS17], we can obtain an analogue of Lemma 3.7.

**Lemma 3.8.** Let \(s \in (0, 1/2)\) and let \(x_0 \in \mathbb{R}^n \times \{0\}\). Suppose
\[
\left[ \partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k \right] \tilde{w} = 0 \quad \text{in } \mathbb{R}_{+}^{n+1},
\]
\[
\tilde{w} = w \quad \text{on } \mathbb{R}^n \times \{0\},
\]
with \(w = 0\) on \(C^\prime_{\varpi,2}\). We assume that
\[
\max_{1 \leq j,k \leq n} \|a_{jk} - \delta_{jk}\|_\infty + \max_{1 \leq j,k \leq n} \|\nabla a_{jk}\|_\infty \leq \epsilon
\]
for some sufficiently small \(\epsilon > 0\). We further assume
\[
\max_{1 \leq j,k \leq n} \|(\nabla')^2 a_{jk}\|_\infty \leq C
\]
for some positive constant \(C\). Then there exists \(C = C(n, s)\) and \(\alpha = \alpha(n, s) \in (0, 1)\) such that
\[
\|x_{n+1}^{1-2s} \tilde{w}\|_{L^2(C^+_{\varpi,1/2})} \leq C \max\left\{\|x_{n+1}^{1-2s} \tilde{w}\|_{L^2(C^+_{\varpi,2})} , \lim_{x_{n+1} \to 0} \|x_{n+1}^{1-2s} \partial_{n+1} \tilde{w}\|_{H^{-s}(C^+_{\varpi,2})}\right\}^\alpha \lim_{x_{n+1} \to 0} \|x_{n+1}^{1-2s} \partial_{n+1} \tilde{w}\|_{H^{-s}(C^+_{\varpi,2})}^{1-\alpha}.
\]

**Proof.** Let \(v\) and \(f\) as as in (3.27). Let \(\tilde{v}\) be the Caffarelli-Silvestre type extension of \(\eta f\) as in (2.1), where \(\eta\) is a cut-off function satisfies
\[
\eta = \begin{cases} 1 & \text{in } C^+_{\varpi,1}, \\ 0 & \text{outside } C^+_{\varpi,2}, \end{cases}
\]
with \(|\partial_{n+1} \eta| \leq C \tilde{x}_{n+1}^{-1}\). As a consequence, the function \(\overline{v} := v - \tilde{v}\) is the Caffarelli-Silvestre extension of \((1 - \eta)f\) and solves
\[
\left[ \partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k \right] \overline{v} = 0 \quad \text{in } \mathbb{R}_{+}^{n+1},
\]
\[
\overline{v} = 0 \quad \text{on } C^\prime_{\varpi,1}.
\]
Hence, by Lemma 3.7 and since \( \overline{\alpha} = 1 - s \), we have
\[
\| x_{n+1}^{1-s} v \|_{L^2(C_{1/8})} \leq C \| x_{n+1}^{1-s} v \|_{L^2(C_{1/16})}^\alpha \cdot \lim_{n \to 0} \| x_{n+1}^{1-2s} \partial_{n+1} \overline{\omega} \|_{H^{-\overline{\gamma}}(C_{1/16})}^{1-\alpha} = C \| x_{n+1}^{1-s} v \|_{L^2(C_{1/16})}^\alpha \cdot \lim_{n \to 0} \| x_{n+1}^{1-2s} \partial_{n+1} \overline{w} \|_{H^{-\overline{\gamma}}(C_{1/16})}^{1-\alpha}.
\]
Since \( \overline{w} = 0 \) on \( C_{\overline{\gamma},2} \), thus
\[
\lim_{n \to 0} x_{n+1}^{1-2s} \partial_{n+1} v \bigg|_{C_{\overline{\gamma},2}} = \lim_{n \to 0} x_{n+1}^{1-2s} \partial_{n+1} \overline{w} \bigg|_{C_{\overline{\gamma},2}} = - \lim_{n \to 0} \sum_{j,k=1}^n \partial_j a_{jk} \partial_k w \bigg|_{C_{\overline{\gamma},2}} = 0.
\]
Hence,
\[
\lim_{n \to 0} x_{n+1}^{1-2s} \partial_{n+1} v \bigg|_{C_{\overline{\gamma},2}} = \lim_{n \to 0} x_{n+1}^{1-2s} \partial_{n+1} \overline{w} \bigg|_{C_{\overline{\gamma},2}},
\]
and thus
\[
\| x_{n+1}^{1-s} v \|_{L^2(C_{1/8})} \leq C \| x_{n+1}^{1-s} v \|_{L^2(C_{1/16})}^\alpha \cdot \lim_{n \to 0} \| x_{n+1}^{1-2s} \partial_{n+1} \overline{w} \|_{H^{-\overline{\gamma}}(C_{1/16})}^{1-\alpha}.
\]
Since \( \lim_{n \to 0} x_{n+1}^{1-2s} \partial_{n+1} \overline{w} = -c_\overline{\gamma}(-P)(\eta f) = -c_\overline{\gamma}(-P)^{1-s}(\eta f) \), then we have
\[
\lim_{n \to 0} \| x_{n+1}^{1-2s} \partial_{n+1} \overline{w} \|_{H^{-\overline{\gamma}}(C_{1/8})} \leq C \| (P^{-1})^{1-s}(\eta f) \|_{H^{-\overline{\gamma}}(\mathbb{R}^n)} \leq C \| \eta f \|_{H^{1-s}(\mathbb{R}^n)},
\]
where the last inequality follows by Lemma 2.1. Thus,
\[
\| x_{n+1}^{1-s} v \|_{L^2(C_{1/8})} \leq C \| x_{n+1}^{1-s} v \|_{L^2(C_{1/16})}^\alpha \| \eta f \|_{H^{1-s}(\mathbb{R}^n)}.
\]
We first estimate the right hand side of (3.30) by
\[
\| x_{n+1}^{1-s} v \|_{L^2(C_{1/8})} \leq \| x_{n+1}^{1-s} v \|_{L^2(C_{1/16})} + \| x_{n+1}^{1-s} \overline{w} \|_{L^2(C_{1/16})} \leq \| x_{n+1}^{1-s} v \|_{L^2(C_{1/16})}^\alpha + C \| \eta f \|_{H_{\overline{\gamma}}(\mathbb{R}^n)} + C \| \eta f \|_{H^{1-s}(\mathbb{R}^n)} \leq C \left( \| x_{n+1}^{1-s} \overline{w} \|_{L^2(C_{1/16})} + \| \eta f \|_{H^{1-s}(\mathbb{R}^n)} \right) ,
\]
where the second inequality follows by (2.2) and the last inequality follows by the Caccioppoli’s inequality (Lemma A.6). Similarly, we can estimate the left hand side of (3.30) by
\[
\| x_{n+1}^{1-s} v \|_{L^2(C_{1/8})} \geq \| x_{n+1}^{1-s} v \|_{L^2(C_{1/16})} - \| x_{n+1}^{1-s} \overline{w} \|_{L^2(C_{1/16})} \geq \| x_{n+1}^{1-s} \partial_{n+1} \overline{w} \|_{L^2(C_{1/16})} - C \| \eta f \|_{H^{1-s}(\mathbb{R}^n)} \geq C \| x_{n+1}^{1-s} \overline{w} \|_{L^2(C_{1/16})} - C \| \eta f \|_{H^{1-s}(\mathbb{R}^n)},
\]
where the last inequality is by Poincaré inequality. Thus, (3.30) becomes
\begin{equation}
\|x_{n+1}^{1/2} \tilde{\nu}\|_{L^2(C^{+}_{1/2})} \leq C \left[ \|x_{n+1}^{1/2} \tilde{\nu}\|_{L^2(C^{+}_{1/2})} + \|\eta f\|_{H^{1-s}(\mathbb{R}^n)} \right] \|\eta f\|_{H^{1-s}(\mathbb{R}^n)}.
\end{equation}

Next, we estimate the boundary contribution \(\|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})}\). Using the interpolation inequality (Lemma A.4), we have
\[\|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})} = \|\langle D^\beta \eta f\rangle\|_{L^2(\mathbb{R}^n \times \{0\})} \leq C \mu^{1-s} \left( \|x_{n+1}^{2-s} \langle D^\beta \eta v\rangle\|_{L^2(\mathbb{R}^{n+1})} + \|x_{n+1}^{2-s} \nabla \langle D^\beta \eta v\rangle\|_{L^2(\mathbb{R}^{n+1})} \right) + C \mu^{-s} \|\langle D^\beta \eta f\rangle\|_{H^{-s}(\mathbb{R}^n \times \{0\})}.
\]

Using that \(\|\langle D^\beta \eta v\rangle\|_{L^2(\mathbb{R}^{n+1})} \leq \|x_{n+1}^{2-s} \eta v\|_{L^2(\mathbb{R}^{n+1})} + \|x_{n+1}^{2-s} \nabla \eta v\|_{L^2(\mathbb{R}^{n+1})}
\]
\[\|x_{n+1}^{2-s} \nabla \langle D^\beta \eta v\rangle\|_{L^2(\mathbb{R}^{n+1})} \leq \|x_{n+1}^{2-s} \nabla \eta v\|_{L^2(\mathbb{R}^{n+1})} + \|x_{n+1}^{2-s} \nabla \nabla \eta v\|_{L^2(\mathbb{R}^{n+1})},
\]

Using (3.20) and (3.21), we know that
\[\|x_{n+1}^{2-s} \langle D^\beta \eta v\rangle\|_{L^2(\mathbb{R}^{n+1})} + \|x_{n+1}^{2-s} \nabla \langle D^\beta \eta v\rangle\|_{L^2(\mathbb{R}^{n+1})} \leq C \|x_{n+1}^{2-s} \tilde{\nu}\|_{L^2(C^{+}_{1/2})},
\]
hence
\begin{equation}
\|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})} \leq C \left[ \mu^{1-s} \|x_{n+1}^{2-s} \tilde{\nu}\|_{L^2(C^{+}_{1/2})} + \mu^{-s} \|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})} \right].
\end{equation}

Choosing \(\mu > 0\) in (3.32) such that the right contributions become equal, i.e.
\[
\mu = \frac{\|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})}}{\|x_{n+1}^{2-s} \tilde{\nu}\|_{L^2(C^{+}_{1/2})}}.
\]

Here, we note by unique continuation \(\|x_{n+1}^{2-s} \tilde{\nu}\|_{L^2(C^{+}_{1/2})} \neq 0\), unless \(\tilde{\nu}\) vanishes globally. Using this choice of \(\mu > 0\), we reach the multiplicative estimate
\begin{equation}
\|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})} \leq C \|x_{n+1}^{2-s} \tilde{\nu}\|_{L^2(C^{+}_{1/2})} \|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})}.
\end{equation}

Starting from \(\beta = 1 - s\), if we iterate (3.33) for \(k\) times, we reach
\[\|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})} \leq C \|x_{n+1}^{2-s} \tilde{\nu}\|_{L^2(C^{+}_{1/2})} \|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})}.
\]

Choose \(k \in \mathbb{N}\) be the smallest integer such that \(1 - ks < 0\), we reach
\[\|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})} \leq C \|x_{n+1}^{2-s} \tilde{\nu}\|_{L^2(C^{+}_{1/2})} \|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})}.
\]

Inserting (3.34) into (3.31) gives our desired result. □

For our purpose, we only need the following version of inequality:
Corollary 3.9. Let \( s \in (0, 1/2) \) and let \( x_0 \in \mathbb{R}^n \times \{0\} \). Suppose
\[
\left[ \partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^n \partial_j a_{jk} \partial_k \right] \tilde{w} = 0 \quad \text{in} \ \mathbb{R}^{n+1}_+, \\
\tilde{w} = w \quad \text{on} \ \mathbb{R}^n \times \{0\},
\]
with \( w = 0 \) on \( C^u_{n,2} \). We assume that
\[
\max_{1 \leq j,k \leq n} \| a_{jk} - \delta_{jk} \|_\infty + \max_{1 \leq j,k \leq n} \| \nabla' a_{jk} \|_\infty \leq \epsilon
\]
for some sufficiently small \( \epsilon > 0 \). We further assume
\[
\max_{1 \leq j,k \leq n} \| (\nabla')^2 a_{jk} \|_\infty \leq C
\]
for some positive constant \( C \). Then there exists \( C = C(n, s), c = c(n, s) \) and \( \alpha = \alpha(n, s) \in (0, 1) \) such that
\[
\| x_{n+1}^{\frac{1-2s}{2}} \tilde{w} \|_{L^2(B^+_n)} \leq C \max\{ \| x_{n+1}^{\frac{1-2s}{2}} \tilde{w} \|_{L^2(B^+_n)}, \lim_{x_{n+1} \to 0} \| x_{n+1}^{\frac{1-2s}{2}} \partial_{n+1} \tilde{w} \|_{H^{-s}(B^+_2)}, \lim_{x_{n+1} \to 0} \| x_{n+1}^{\frac{1-2s}{2}} \partial_{n+1} \tilde{w} \|_{H^{-s}(B^+_2)} \}
\]
Now, we are ready to prove the part (a) of Lemma 3.3 for the case \( s \in (0, 1/2) \).

Proof of the part (a) of Lemma 3.3 for \( s \in (0, 1/2) \). The case \( s \in (0, 1/2) \) is similar as the case \( s \in (1/2, 1) \). As above, the estimate for \( u_1 \) is a direct consequence of (2.2). For \( u_2 \), we use Corollary 3.9 and the interpolation inequality in Lemma A.5. With this estimate at hand, the analogues of (3.25) and (3.26) follow by combining the estimates of the splitting argument as above. Note that (3.25) become
\[
\| x_{n+1}^{\frac{1-2s}{2}} \tilde{w} \|_{L^2(B^+_n)} + \| x_{n+1}^{\frac{1-2s}{2}} \nabla \tilde{u} \|_{L^2(B^+_n)} \leq C \left( \| x_{n+1}^{\frac{1-2s}{2}} \tilde{u} \|_{L^2(B^+_n)} + \| \tilde{u} \|_{L^2(B^+_n)} \right)^\alpha \times
\]
\[
\times \left( \lim_{x_{n+1} \to 0} \| x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} \|_{L^2(B^+_1)} + \| \tilde{u} \|_{L^2(B^+_1)} \right)^{1-\alpha}
\]
\[
+ C \left( \| x_{n+1}^{\frac{1-2s}{2}} \tilde{u} \|_{L^2(B^+_1)} + \| \tilde{u} \|_{L^2(B^+_1)} \right)^{\frac{2s}{1+s}} \times
\]
\[
\times \left( \lim_{x_{n+1} \to 0} \| x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} \|_{L^2(B^+_1)} + \| \tilde{u} \|_{L^2(B^+_1)} \right)^{\frac{1-\alpha}{1+s}}
\]
\[
(3.35)
\]
This is our desired result. \( \square \)

Finally, combining (3.35) and Lemma A.7 we can immediately obtain the part (b) of Lemma 3.3. 

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**ON LANDIS CONJECTURE FOR THE FRACTIONAL SCHröDINGER EQUATION**

27
4. CARLEMAN ESTIMATE

4.1. A Carleman estimate with differentiability assumption. Modify the arguments in [Reg97], we can prove the following Carleman estimate.

**Theorem 4.1.** Let $s \in (0, 1)$ and let $\tilde{u} \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ with supp$(\tilde{u}) \subset \mathbb{R}^{n+1}_+ \setminus B_1^0$ be a solution to

$$\left[ \partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^n a_{jk} \partial_j \partial_k \right] \tilde{u} = f \quad \text{in } \mathbb{R}^{n+1}_+, \quad \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} = V \tilde{u} \quad \text{on } \mathbb{R}^n \times \{0\},$$

where $x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+$, $f \in L^2(\mathbb{R}^{n+1}_+, x_{n+1}^{2s-1})$ with compact support in $\mathbb{R}^{n+1}_+$, and $V \in C^1(\mathbb{R}^n)$. Assume that

$$\max_{1 \leq j,k \leq n} \sup_{|x'| \geq 1} |a_{jk}(x') - \delta_{jk}(x')| + \max_{1 \leq j,k \leq n} \sup_{|x'| \geq 1} |x'| \left| \nabla' a_{jk}(x') \right| \leq \epsilon$$

for some sufficiently small $\epsilon > 0$. Let further $\phi(x) = |x|^\alpha$ for $\alpha \geq 1$. Then there exists constants $C = C(n, s, \alpha)$ and $\tau_0 = \tau_0(n, s, \alpha)$ such that

$$\tau^3 \left[ \left( e^{\tau \phi} |x|^{2s-1} \right) \|u\|_{L^2(\mathbb{R}^{n+1}_+, \Omega)}^2 + \tau \|e^{\tau \phi} |x|^\frac{\alpha}{2} x_{n+1}^{1-2s} \nabla u\|_{L^2(\mathbb{R}^{n+1}_+, \Omega)}^2 \right]$$

$$+ \tau^{-1} \|e^{\tau \phi} |x|^{-\frac{\alpha}{2}} x_{n+1}^2 \nabla (\nabla' u)\|_{L^2(\mathbb{R}^{n+1}_+, \Omega)}^2 \leq C \left[ \left( e^{\tau \phi} |x|^{2s-1} \right) \|f\|_{L^2(\mathbb{R}^{n+1}_+, \Omega)}^2 + \tau \|e^{\tau \phi} |x|^\frac{\alpha}{2} (|V|^\frac{1}{2} + |x'|^{\frac{1}{2}} |\nabla V|^\frac{1}{2}) u\|_{L^2(\mathbb{R}^n \times \{0\})}^2 \right]$$

for all $\tau \geq \tau_0$. Here, $\nabla' = (\partial_1, \ldots, \partial_n)$ and $\nabla = (\partial_1, \ldots, \partial_n, \partial_{n+1})$.

**Proof of Theorem 4.1.** Write $x = e^t \omega$ with $t \in \mathbb{R}$ and $\omega \in S^n_+$, we have

$$\partial_j = e^{-t}(\omega_j \partial_t + \Omega_j) \quad \text{for all } j = 1, \ldots, n+1.$$  

Since

$$\Omega_k \omega_j = \delta_{jk} - \omega_k \omega_j,$$

so

$$\partial_j \partial_k = e^{-2t} (\omega_j \omega_k \partial_t^2 + \omega_j \Omega_k \partial_t + \omega_k \Omega_j \partial_t + (\delta_{jk} - 2\omega_j \omega_k) \partial_t + \Omega_j \omega_k - \omega_j \Omega_k).$$

Since $\partial_j$ and $\partial_k$ commute, then

$$\Omega_j \Omega_k - \omega_j \Omega_k = \Omega_k \Omega_j - \omega_k \Omega_j,$$

that is, $\Omega_j$ and $\Omega_k$ commute up to some lower order terms. Write $\partial_j \partial_k = \frac{1}{2}(\partial_j \partial_k + \partial_k \partial_j)$, we reach

$$\partial_j \partial_k = e^{-2t} \left( \omega_j \omega_k \partial_t^2 + \omega_j \Omega_k \partial_t + \omega_k \Omega_j \partial_t + (\delta_{jk} - 2\omega_j \omega_k) \partial_t + \frac{1}{2} \Omega_j \Omega_k + \frac{1}{2} \Omega_k \Omega_j - \frac{1}{2} \omega_j \Omega_k - \frac{1}{2} \omega_k \Omega_j \right).$$
Also, the vector fields have the properties
\[ \sum_{j=1}^{n+1} \omega_j \Omega_j = 0 \quad \text{and} \quad \sum_{j=1}^{n+1} \Omega_j \omega_j = n \quad \text{in} \ S^n_+, \]
\[ \sum_{j=1}^{n} \omega_j \Omega_j = 0 \quad \text{and} \quad \sum_{j=1}^{n} \Omega_j \omega_j = n \quad \text{on} \ \partial S^n_+. \]

Using this coordinate,
\[
\begin{align*}
f &= e^{-(1+2s)t} \left[ \omega_{n+1}^{1-2s} \partial_t^2 + \omega_{n+1}^{1-2s} (n - 2s) \partial_t + \sum_{j=1}^{n+1} \Omega_j \omega_{n+1}^{1-2s} \Omega_j \right] \tilde{u} \\
&\quad + e^{-(1+2s)t} \omega_{n+1}^{1-2s} \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \left[ \omega_j \omega_k \partial_t^2 + \omega_j \Omega_k \partial_t + \omega_k \Omega_j \partial_t + \frac{1}{2} \Omega_j \Omega_k + \frac{1}{2} \Omega_k \Omega_j \right] \tilde{u} \\
&\quad + e^{-(1+2s)t} \omega_{n+1}^{1-2s} \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \left[ (\delta_{jk} - 2 \omega_j \omega_k) \partial_t - \frac{1}{2} \omega_j \Omega_k - \frac{1}{2} \omega_k \Omega_j \right] \tilde{u} \quad \text{in} \ S^n_+ \times \mathbb{R}.
\end{align*}
\]

Next, let \( \vec{u} = e^{\frac{n-2s}{2} t} \tilde{u} \) and \( \vec{f} = e^{\frac{n-2s}{2} t} f = e^{\frac{n+2+2s}{2} t} f, \)
\[
\begin{align*}
\vec{f} &= \left[ \omega_{n+1}^{1-2s} \partial_t^2 + \sum_{j=1}^{n+1} \Omega_j \omega_{n+1}^{1-2s} \Omega_j - \omega_{n+1}^{1-2s} \frac{(n - 2s)^2}{4} \right] \vec{u} \\
&\quad + \omega_{n+1}^{1-2s} \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \left[ \omega_j \omega_k \partial_t^2 + \omega_j \Omega_k \partial_t + \omega_k \Omega_j \partial_t + \frac{1}{2} \Omega_j \Omega_k + \frac{1}{2} \Omega_k \Omega_j \right] \vec{u} \\
&\quad + \omega_{n+1}^{1-2s} \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \left[ (\delta_{jk} - (n + 2 - 2s) \omega_j \omega_k) \partial_t - \frac{n + 1 - 2s}{2} \omega_j \Omega_k - \frac{n + 1 - 2s}{2} \omega_k \Omega_j \right] \vec{u} \\
&\quad + \omega_{n+1}^{1-2s} \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \left[ \frac{(n - 2s)^2}{4} \omega_j \omega_k - \frac{n - 2s}{2} (\delta_{jk} - 2 \omega_j \omega_k) \right] \vec{u} \quad \text{in} \ S^n_+ \times \mathbb{R}.
\end{align*}
\]

Also,
\[
\lim_{\omega_{n+1} \to 0} \omega_{n+1}^{1-2s} \Omega_{n+1} \vec{u} = \tilde{V} \vec{u},
\]
where \( \tilde{V} = e^{2st} V. \)

Next, setting \( \vec{u} = \omega_{n+1}^{\frac{1-2s}{2}} e^{\varphi} \vec{u}, \) where \( \varphi(t) = \phi(e^{t} \omega) = e^{\alpha t}, \) we reach
\[
\begin{align*}
\omega_{n+1}^{\frac{2s-1}{2}} e^{\varphi} \vec{f} &= L^+ \vec{u} = (S - A + (I) + (II) + (III)) \vec{u} \quad \text{in} \ S^n_+ \times \mathbb{R},
\end{align*}
\]
where
\[ S = \partial_t^2 + \tilde{\Delta}_\omega + \tau^2 |\varphi'|^2 - \tau \varphi'' - \frac{(n - 2s)^2}{4}, \quad \tilde{\Delta}_\omega = \sum_{j=1}^{n+1} \omega_{n+1}^{2s-1} \Omega_j \omega_{n+1}^{1-2s} \Omega_j \omega_{n+1}^{2s-1} \]
and
\[ A = 2\tau \varphi' \partial_t \]

(1) \[ (I) = \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \left[ \omega_j \omega_k \partial_t^2 + \omega_j \Omega_k \partial_t + \omega_k \Omega_j \partial_t + \frac{1}{2} \Omega_j \Omega_k + \frac{1}{2} \Omega_k \Omega_j \right] \]

(II) \[ (II) = \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \left[ (-2\tau \varphi' \omega_j \omega_k + (\delta_{jk} - (n+1)\omega_j \omega_k)) \partial_t - \left( \tau \varphi' + \frac{n}{2} \right) (\omega_j \Omega_k + \omega_k \Omega_j) \right] \]

(III) \[ (III) = \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \left[ \omega_j \omega_k (\tau^2 |\varphi'|^2 - \tau \varphi'' + (n+1)\tau \varphi' + C_1) + C_2 \right], \]

for some constants C_1 and C_2. Also,
\[ \lim_{\omega_{n+1} \to 0} \omega_{n+1}^{1-2s} \Omega_n \omega_{n+1}^{2s-1} \nabla = \tilde{V} \omega_{n+1}^{2s-1} \nabla \text{ on } \partial S_{n+1} \times \mathbb{R}. \]

We denote the norm and the scalar product in the bulk and the boundary space by
\[ \| \cdot \| := \| \cdot \|_{L^2(S_n^*)} \]
\[ \| \cdot \|_0 := \| \cdot \|_{L^2(\partial S_n^* \times \mathbb{R})} \]
\[ \langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(S_n^*)} \]
\[ \langle \cdot, \cdot \rangle_0 := \langle \cdot, \cdot \rangle_{L^2(\partial S_n^* \times \mathbb{R})} \]

and we omit the notation “\( \lim_{\omega_{n+1} \to 0} \)” in \( \| \cdot \|_0 \) and \( \langle \cdot, \cdot \rangle_0 \).

First of all, we need to prove the ellipticity of \( \tilde{\Delta}_\omega \): 

**Lemma 4.2.** Suppose (4.4) holds, then
\[ \| \tilde{\Delta}_\omega \nabla \|^2 \geq c_0 \sum_{(j,k) \neq (n+1,n+1)} \| \omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \nabla \|^2 \]
\[ - C \left( \sum_{j=1}^{n+1} \| \omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \nabla \|^2 + \| \nabla \|^2 + \| (|\tilde{V}|^\frac{1}{2} + |\nabla \tilde{V}|^\frac{1}{2}) \nabla \omega_{n+1} \|^2 \| \right) \]

**Proof.** Note that
\[ \| \tilde{\Delta}_\omega \nabla \|^2 = \sum_{j=1}^{n} \omega_{n+1}^{\frac{2s-1}{2}} \Omega_j \omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \nabla + \omega_{n+1}^{\frac{2s-1}{2}} \Omega_{n+1}^{1-2s} \Omega_{n+1}^{1-2s} \nabla \]
\[ + C \left( \sum_{j=1}^{n+1} \| \omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \nabla \|^2 + \| \nabla \|^2 + \| (|\tilde{V}|^\frac{1}{2} + |\nabla \tilde{V}|^\frac{1}{2}) \nabla \omega_{n+1} \|^2 \| \right) \].
We first estimate the lower bound of the difference lower order term. So, to estimate the first term, it is suffice to estimate

\[ \int_{ \mathbb{R}^{n+1} } (\Omega_{n+1} v) u \, dx + \int_{ \mathbb{R}^{n+1} } v(\Omega_{n+1} u) \, dx = \int_{ \mathbb{R}^{n+1} } \Omega_{n+1}(uv) \, dx \]

\[ = \int_{ \mathbb{R}^{n+1} } |x| \partial_{n+1} (uv) \, dx - \int_{ \mathbb{S}^n } \int_0^\infty r \omega_{n+1} \partial_t (uv) r^n \, dr \, d\omega \]

\[ = - \int_{ \mathbb{R}^{n+1} \times \{0\} } |x'| uv \, dx' - \int_{ \mathbb{R}^{n+1} } \omega_{n+1} uv \, dx + (n+1) \int_{ \mathbb{S}^n } \int_0^\infty \omega_{n+1} (uv) r^n \, dr \, d\omega \]

\[ = - \int_{ \mathbb{R}^{n+1} \times \{0\} } |x'| uv \, dx' + n \int_{ \mathbb{R}^{n+1} } \omega_{n+1} uv \, dx. \]

Similar integration by parts formula holds for \( \Omega_j \) for \( j = 1, \cdots, n \).

Indeed, by (4.1), we know that for \( j = 1, \cdots, n \), \( \Omega_j \) and \( \omega_{n+1} \) are commute up to some lower order term. So, to estimate the first term, it is suffice to estimate \( \| \sum_{j=1}^n \Omega_j^2 \nu \|_2^2 \). Then the rest is just simply by the integration by parts.

We define \( L^- \) from \( L^+ \) by replacing \( \partial_t \) and \( \Omega_j \) by \(-\partial_t \) and \(-\Omega_j \), that is,

\[ L^- = S + A + (I) - (II) + (III). \]

We first estimate the lower bound of the difference

\[ \mathcal{D} = \| L^+ \nu \|^2 - \| L^- \nu \|^2 = -4 \langle S \nu, A \nu \rangle + R, \]

where

\[ R = 4 \langle S \nu, (I) \nu \rangle - 4 \langle A \nu, (I) \nu \rangle - 4 \langle A \nu, (III) \nu \rangle + 4 \langle (I) \nu, (II) \nu \rangle + 4 \langle (II) \nu, (III) \nu \rangle. \]

Using (4.1) and integration by parts, we can compute

\[ -4 \langle S \nu, A \nu \rangle \geq 4 \tau \| \phi' \|_2^2 \partial_t \nu \|_2^2 - 4 \tau \sum_{j=1}^{n+1} \| \phi'' \|_2^2 \omega_{n+1}^{1-2s} \Omega_j \omega_{n+1}^{2s-1} \nu \|_2^2 + \frac{119}{10} \tau^3 \| \phi' \|_2^2 \| \nu \|_2^2 \]

\[ - 2 \tau \| (|\tilde{V}|^{1/2} + |\partial_t \tilde{V}|^{1/2}) \|_2^2 \omega_{n+1}^{2s-1} \nu \|_0^2. \]

Since

\[ \max_{1 \leq j, k \leq n} |a_{jk} - \delta_{jk}| + \max_{1 \leq j, k \leq n} |\partial_t a_{jk}| + \max_{1 \leq j, k \leq n} |\nabla_{\omega} a_{jk}| \leq \epsilon, \]

using integration by parts, we reach

\[ R \geq - \tau \epsilon C \| \phi' \|_2^2 \partial_t \nu \|_2^2 - \tau \epsilon C \sum_{j=1}^{n+1} \| \phi'' \|_2^2 \omega_{n+1}^{1-2s} \Omega_j \omega_{n+1}^{2s-1} \nu \|_2^2 - \tau^3 \epsilon C \| \phi' \|_2^2 \| \nu \|_2^2 \]

\[ - \tau \epsilon C \| (|\tilde{V}|^{1/2} + |\partial_t \tilde{V}|^{1/2}) \|_2^2 \omega_{n+1}^{2s-1} \nu \|_0^2. \]

Hence, for small \( \epsilon > 0 \) and large \( \tau_0 \), we reach

\[ \mathcal{D} \geq \frac{39}{10} \tau \| \phi'' \|_2^2 \partial_t \nu \|_2^2 - \frac{41}{10} \tau \sum_{j=1}^{n+1} \| \phi'' \|_2^2 \omega_{n+1}^{1-2s} \Omega_j \omega_{n+1}^{2s-1} \nu \|_2^2 + \frac{118}{10} \tau^3 \| \phi' \|_2^2 \| \nu \|_2^2 \]

\[ - C \tau \| (|\tilde{V}|^{1/2} + |\partial_t \tilde{V}|^{1/2}) \|_2^2 \omega_{n+1}^{2s-1} \nu \|_0^2. \]
Next, we estimate the sum

\[ \mathcal{S} = ||| \varphi' ||| - \frac{1}{2} L^* \varphi |||^2 + ||| \varphi' ||| - \frac{1}{2} L - \varphi |||^2 \]
\[ \geq 2 ||| \varphi' ||| - \frac{1}{2} S \varphi |||^2 + 2 ||| \varphi' ||| - \frac{1}{2} A \varphi |||^2 \]
\[ - C \epsilon ||| \varphi' ||| - \frac{1}{2} \partial^2 \varphi |||^2 - C \epsilon \sum_{j=1}^{n} ||| \varphi' ||| - \frac{1}{2} \partial_j \alpha_{n+1} \Omega_{j+1} \varphi |||^2 - C \epsilon \sum_{j=1}^{n} ||| \varphi' ||| - \frac{1}{2} \Omega_{j+1} \Omega_{k+1} \varphi |||^2 \]
\[ - C \epsilon \tau^2 ||| \varphi' ||| - \frac{1}{2} \partial^2 \varphi |||^2 - C \epsilon \tau^2 \sum_{j=1}^{n} ||| \varphi' ||| - \frac{1}{2} \Omega_{j+1} \varphi |||^2 - C \epsilon \tau^4 ||| \varphi' ||| - \frac{1}{2} \varphi |||^2. \]

Observe that

\[ 2 ||| \varphi' ||| - \frac{1}{2} S \varphi |||^2 \geq \frac{19}{10} ||| \varphi' ||| - \frac{1}{2} \partial^2 \varphi + |\varphi'| - \frac{1}{2} \tilde{A} \omega \varphi + \tau^2 |\varphi' - \frac{1}{2} \varphi |||^2 - C \tau^2 ||| \varphi' ||| - \frac{1}{2} \varphi |||^2. \]

For \( \delta \in (0,1) \), write

\[ ||| \varphi' ||| - \frac{1}{2} \partial^2 \varphi + |\varphi'| - \frac{1}{2} \tilde{A} \omega \varphi + \tau^2 |\varphi' - \frac{1}{2} \varphi |||^2 \]
\[ = ||| \varphi' ||| - \frac{1}{2} \partial^2 \varphi |||^2 + (1 - \delta) ||| \varphi' ||| - \frac{1}{2} \tilde{A} \omega \varphi |||^2 + \delta ||| \varphi' ||| - \frac{1}{2} \tilde{A} \omega \varphi |||^2 + \tau^4 ||| \varphi' ||| - \frac{1}{2} \varphi |||^2 \]
\[ + \langle |\varphi'| - \frac{1}{2} \partial^2 \varphi, \tilde{A} \omega \varphi \rangle + \tau^2 \langle \varphi' \partial^2 \varphi, \varphi \rangle + \tau^2 \langle \varphi' \tilde{A} \omega \varphi, \varphi \rangle. \]

Hence, using integration by parts, and apply Lemma 4.2 on the term \( \delta ||| \varphi' ||| - \frac{1}{2} \tilde{A} \omega \varphi |||^2 \), choose \( \delta > 0 \) small, and then choose \( \epsilon > 0 \) small, we reach

\[ \mathcal{S} \geq \frac{19}{10} ||| \varphi' ||| - \frac{1}{2} \partial^2 \varphi |||^2 + \frac{19}{10} \sum_{j=1}^{n} ||| \varphi' ||| - \frac{1}{2} \partial_j \alpha_{n+1} \Omega_{j+1} \varphi |||^2 + \frac{18}{10} ||| \varphi' ||| - \frac{1}{2} \tilde{A} \omega \varphi |||^2 \]
\[ + c_1 \sum_{(j,k) \neq (n+1,n+1)} ||| \varphi' ||| - \frac{1}{2} \omega_{n+1} \Omega_{j+1} \Omega_{k+1} \varphi |||^2 \]
\[ + \frac{9}{10} \tau^4 ||| \varphi' ||| - \frac{1}{2} \partial^2 \varphi |||^2 + \frac{39}{10} \tau^2 ||| \varphi' ||| - \frac{1}{2} \partial^2 \varphi |||^2 - \frac{11}{10} \tau^2 \sum_{j=1}^{n+1} ||| \varphi' ||| - \frac{1}{2} \omega_{n+1} \Omega_{j+1} \varphi |||^2 \]
\[ - C \langle |\tilde{V}| - \frac{1}{2} \partial \tilde{V} + |\omega \tilde{V}| \tilde{V} ||| \varphi' ||| - \frac{1}{2} \partial \tilde{V} + |\tilde{V} | \tilde{V} ||| \varphi' ||| - \frac{1}{2} \tilde{V} |||^2 \]
\[ - C \langle |\tilde{V}| + |\partial \tilde{V} + |\omega \tilde{V}| \tilde{V} ||| \varphi' ||| - \frac{1}{2} \partial \tilde{V} + |\tilde{V} | \tilde{V} ||| \varphi' ||| - \frac{1}{2} \tilde{V} |||^2 \]
\[ - C \langle |\tilde{V}| + |\partial \tilde{V} + |\omega \tilde{V}| \tilde{V} ||| \varphi' ||| - \frac{1}{2} \tilde{V} |||^2. \]
Multiply (4.5) by $\tau$, and sum with (4.6), we reach
\[
(\tau + 1)\|L^1\tau\|^2 \geq \tau D + \mathcal{F}
\]
\[
\geq c_1 \left( \|\varphi'| - \frac{1}{2} \partial_t^2 \tau\|^2 + \sum_{j=1}^n \|\varphi'\| - \frac{1}{2} \omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \partial_t \varpi\|^2 \right.
\]
\[
+ \left. \sum_{(j,k) \neq (n+1,n+1)} \|\varphi'\| - \frac{1}{2} \omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \Omega_k \omega_{n+1}^{\frac{2s-1}{2}} \tau\|^2 \right)
\]
\[
+ \frac{39}{5} \tau^2 \|\varphi'\| \partial_t \varpi\|^2 + \frac{208}{10} \tau^4 \|\varphi'\| \partial_t^2 \tau\|^2 - \frac{11}{10} \tau^2 \sum_{j=1}^{n+1} \|\varphi'\| \omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \tau\|^2
\]
\[
+ \frac{18}{10} \|\varphi'| - \frac{1}{2} \Delta \omega \tau\|^2
\]
\[- C\|\mathcal{L} V\| + \|\partial_t \tilde{V}\| + \|\nabla \omega \tilde{V}\|) |\varphi'| - \frac{1}{2} \partial_t \omega_{n+1}^{\frac{2s-1}{2}} \tau\|^2
\]
\[- C\|\mathcal{L} V\| + \|\partial_t \tilde{V}\| + \|\nabla \omega \tilde{V}\|) |\varphi'| - \frac{1}{2} \nabla \omega_{n+1}^{\frac{2s-1}{2}} \tau\|^2
\]
\[- C\tau^2 \|\mathcal{L} V\| + \|\partial_t \tilde{V}\| + \|\nabla \omega \tilde{V}\|) |\varphi'| - \frac{1}{2} \omega_{n+1}^{\frac{2s-1}{2}} \tau\|^2.
\]
(4.7)

To obtain the full gradient estimate, note that
\[
\frac{12}{10} \tau^2 \sum_{j=1}^{n+1} \|\varphi'\| \omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \partial_t \tau\|^2 + \frac{12}{10} \tau^2 \langle \tilde{V}, \omega_{n+1}^{\frac{2s-1}{2}} \varpi\rangle
\]
\[
= - \frac{12}{10} \tau^2 \langle \varphi', \tilde{\Delta} \omega \tau\rangle \leq \frac{16}{10} \|\varphi'\| - \frac{1}{2} \tilde{\Delta} \omega \tau\|^2 + \frac{144}{100} \tau^4 \|\varphi'\| \tilde{\varpi}\|^2.
\]
(4.8)

Summing (4.7) and (4.8), we reach
\[
\|\varphi'\| \partial_t^2 \tau\|^2 + \sum_{j=1}^n \|\varphi'\| \omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \partial_t \tau\|^2
\]
\[
+ \sum_{(j,k) \neq (n+1,n+1)} \|\varphi'\| \omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \Omega_k \omega_{n+1}^{\frac{2s-1}{2}} \tau\|^2
\]
\[
+ \tau^2 \|\varphi'\| \partial_t \tau\|^2 + \tau^2 \sum_{j=1}^{n+1} \|\varphi'\| \omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \tau\|^2 + \tau^4 \|\varphi'\| \tilde{\varpi}\|^2,
\]
\[
\leq C\|\tilde{f}\|^2 + C\|\mathcal{L} V\| + \|\partial_t \tilde{V}\| + \|\nabla \omega \tilde{V}\| |\varphi'| - \frac{1}{2} \partial_t \omega_{n+1}^{\frac{2s-1}{2}} \tau\|^2
\]
\[
+ C\|\mathcal{L} V\| + \|\partial_t \tilde{V}\| + \|\nabla \omega \tilde{V}\| |\varphi'| - \frac{1}{2} \nabla \omega_{n+1}^{\frac{2s-1}{2}} \tau\|^2
\]
\[
+ C\tau^2 \|\mathcal{L} V\| + \|\partial_t \tilde{V}\| + \|\nabla \omega \tilde{V}\| |\varphi'| - \frac{1}{2} \omega_{n+1}^{\frac{2s-1}{2}} \tau\|^2.
\]
(4.9)

Changing back to the cartesian coordinate, we obtain our result.

4.2. A Carleman estimate without differentiability assumptions. Following the splitting arguments in [RW19], we can proof the following Carleman estimate.
Theorem 4.3. Let $s \in (0, 1)$ and let $\tilde{u} \in H^1(\mathbb{R}_+^{n+1}, x_n^{1-2s})$ with $\text{supp}(\tilde{u}) \subset \mathbb{R}_+^{n+1} \setminus B^+_1$ be a solution to
\[
\left[ \partial_{n+1} x_n^{1-2s} \partial_{n+1} + x_n^{1-2s} \sum_{j,k=1}^n a_{jk} \partial_j \partial_k \right] \tilde{u} = f \quad \text{in } \mathbb{R}_+^{n+1},
\]
where $x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+$, $f \in L^2(\mathbb{R}_+^{n+1}, x_n^{2s-1})$ with compact support in $\mathbb{R}_+^{n+1}$, and $V \in L^\infty(\mathbb{R}^n)$. Assume that
\[
\max_{1 \leq j,k \leq n} \sup_{|x'| \geq 1} |a_{jk}(x') - \delta_{jk}(x')| + \max_{1 \leq j,k \leq n} \sup_{|x'| \geq 1} |x'| |\nabla a_{jk}(x')| \leq \epsilon
\]
for some sufficiently small $\epsilon > 0$. Let further $\phi(x) = |x|^{\alpha}$ for $\alpha \geq 1$. Then there exists constants $C = C(n, s, \alpha)$ and $\tau_0 = \tau_0(n, s, \alpha)$ such that
\[
\tau^3 \|e^{\tau \phi} |x|^{\frac{2s-1}{2}} x_n^{1-2s} \tilde{u}\|_{L^2(\mathbb{R}_+^{n+1})}^2 + \tau \|e^{\tau \phi} |x|^{\frac{2s-1}{2}} x_n^{1+2s} \nabla \tilde{u}\|^2_{L^2(\mathbb{R}_+^{n+1})}
\leq C \left[ \|e^{\tau \phi} x_n^{2s-1} |x| f\|^2_{L^2(\mathbb{R}_+^{n+1})} + \tau^{2-2s} \|e^{\tau \phi} V |x|^{(1-\alpha)s} \tilde{u}\|_{L^2(\mathbb{R}^n \times \{0\})} \right]
\]
for all $\tau \geq \tau_0$.

Proof of Theorem 4.3. As in the proof of Theorem 4.1, we first pass to conformal coordinates. With the notations from there, recall (1.2):
\[
\left[ \omega_n^{\frac{1-2s}{2}} \partial_t^2 + \sum_{j=1}^{n+1} \Omega_j \omega_n^{1-2s} \Omega_j - \omega_n^{1-2s} \left( \frac{n-2s}{4} \right) \right] \tilde{u} + R \tilde{u} = \tilde{f},
\]
where
\[
R = \omega_n^{\frac{1-2s}{2}} \sum_{j,k=1}^n (a_{jk} - \delta_{jk}) \left[ \omega_j \omega_k \partial_t^2 + \omega_j \Omega_k \partial_t + \omega_k \Omega_j \partial_t + \frac{1}{2} \Omega_j \Omega_k + \frac{1}{2} \Omega_k \Omega_j \right] + \omega_n^{\frac{1-2s}{2}} \sum_{j,k=1}^n (a_{jk} - \delta_{jk}) \left[ (\delta_{jk} - (n + 2 - 2s) \omega_j \omega_k) \partial_t - \frac{n + 1 - 2s}{2} \omega_j \Omega_k - \frac{n + 1 - 2s}{2} \omega_k \Omega_j \right] + \omega_n^{\frac{1-2s}{2}} \sum_{j,k=1}^n (a_{jk} - \delta_{jk}) \left[ \left( \frac{n - 2s}{4} \right) \omega_j \omega_k - \frac{n - 2s}{2} (\delta_{jk} - 2 \omega_j \omega_k) \right].
\]
We split $\tilde{\varphi}$ into two parts $\tilde{\varphi} = u_1 + u_2$. Here $u_1$ is a solution to
\[
\left[ \omega_n^{\frac{1-2s}{2}} \partial_t^2 + \sum_{j=1}^{n+1} \Omega_j \omega_n^{1-2s} \Omega_j - \omega_n^{1-2s} \left( \frac{n-2s}{4} \right) - K^2 \tau^2 |\varphi'|^2 \omega_n^{1-2s} \right] u_1 + Ru_1 = \tilde{f} \quad \text{in } S^+_1 \times \mathbb{R},
\]
and
\[
\lim_{\omega_n^{1-2s} \Omega_n \to 0} \omega_n^{1-2s} \Omega_n u_1 = \lim_{\omega_n^{1-2s} \Omega_n \to 0} \omega_n^{1-2s} \Omega_n \tilde{\varphi} \quad \text{on } \partial S^+_1 \times \mathbb{R}.
\]
We remark that by the Lax-Milgram theorem in $H^1(S^+_1 \times \mathbb{R}, \omega_n^{1-2s})$ a unique energy solution to this problem exists.
Choosing test function $\tau^2 e^{2\tau\varphi} |\varphi''|^2 u_1$ in (4.10), for $\delta > 0$, we reach
\[
\tau^2 \|e^{\tau\varphi} e^{\frac{1-2s}{\omega_{n+1}} \partial_t} u_1\|^2 + 2 \|e^{\tau\varphi} \varphi'' \frac{1-2s}{\omega_{n+1}} \nabla S^n u_1\|^2 + \tau^2 \left( n - 2s \right)^2 \|e^{\tau\varphi} \varphi'' \omega_{n+1} u_1\|^2 \\
+ K \tau^4 \|e^{\tau\varphi} \varphi'' \omega_{n+1} u_1\|^2 \\
= - \tau^2 \left( e^{\tau\varphi} \omega_{n+1} \int f, e^{\tau\varphi} |\varphi''|^2 \omega_{n+1} u_1 \right) - \left( \tau e^{\tau\varphi} \varphi'' e^{\frac{1-2s}{\omega_{n+1}} \partial_t} u_1, \tau e^{\tau\varphi} \varphi' \varphi'' e^{\frac{1-2s}{\omega_{n+1}} u_1} \right) \\
+ \tau^2 \left( \int \varphi''^2 \omega_{n+1} \partial_t u_1, \tau e^{\tau\varphi} \varphi' \varphi'' e^{\frac{1-2s}{\omega_{n+1}} u_1} \right) \\
- \tau^2 \left( e^{\tau\varphi} e^{\frac{1-2s}{\omega_{n+1}} \Omega_n \omega_{n+1} u_1}, e^{\tau\varphi} \varphi' \varphi'' e^{\frac{1-2s}{\omega_{n+1}} u_1} \right) \\
\leq \|e^{\tau\varphi} \omega_{n+1} \int f\|^2 + 2 \|e^{\tau\varphi} |\varphi''|^2 \omega_{n+1} u_1\|^2 + \delta \tau^2 \|e^{\tau\varphi} \varphi'' \omega_{n+1} \partial_t u_1\|^2 \\
+ C \tau^2 \|e^{\tau\varphi} \varphi'' \omega_{n+1} \Omega_n \Omega_n \partial_t \Omega_n u_1\|^2 + \tau^2 \|e^{\tau\varphi} \varphi' \varphi'' \omega_{n+1} u_1\|^2 \\
\leq C \|e^{\tau\varphi} \omega_{n+1} \int f\|^2 + C \tau^2 \|e^{\tau\varphi} \varphi'' \omega_{n+1} \Omega_n \Omega_n \partial_t \Omega_n u_1\|^2 + \tau^2 \|e^{\tau\varphi} \varphi' \varphi'' \omega_{n+1} u_1\|^2.
\]
We first choose $\delta > 0$ and $e > 0$ small, and then choose $K > 1$ large, we obtain
\[
\tau^2 \|e^{\tau\varphi} \varphi'' \omega_{n+1} \partial_t u_1\|^2 + 2 \|e^{\tau\varphi} \varphi'' \omega_{n+1} \Omega_n \Omega_n \partial_t \Omega_n u_1\|^2 + \tau^2 \|e^{\tau\varphi} \varphi' \varphi'' \omega_{n+1} u_1\|^2 \\
\leq C \|e^{\tau\varphi} \omega_{n+1} \int f\|^2 + C \tau^2 \|e^{\tau\varphi} \varphi'' \omega_{n+1} \Omega_n \Omega_n \partial_t \Omega_n u_1\|^2 + \tau^2 \|e^{\tau\varphi} \varphi' \varphi'' \omega_{n+1} u_1\|^2.
\]
From Proposition A.2, we have
\[
|\varphi''|^2 e^{2\varphi} \int_{\partial S^n_+} u_1^2 \leq C \tau^2 - 2s |\varphi''|^2 e^{2\varphi} \int_{S^n_+} \omega_{n+1} u_1^2 + C \tau^2 - 2s |\varphi''|^2 e^{2\varphi} \int_{S^n_+} \omega_{n+1} \nabla S^n u_1^2.
\]
Choosing $\tilde{\tau} = e^{\alpha t} \tau$, we reach
\[
|\varphi''|^2 e^{2\varphi} \int_{\partial S^n_+} u_1^2 \leq C \tau^2 - 2s |\varphi''|^2 e^{2\varphi} \int_{S^n_+} \omega_{n+1} u_1^2 + C \tau^2 - 2s |\varphi''|^2 e^{2\varphi} \int_{S^n_+} \omega_{n+1} \nabla S^n u_1^2.
\]
Multiplying with $e^{2\tau\varphi}$, using that $\varphi' = \alpha e^{\alpha t}$ and integrating in the radial direction, thus implies
\[
\tau^2 + 2s \|e^{\tau\varphi} |\varphi''|^2 e^{\varphi} \omega_{n+1} u_1\|^2 \leq C \tau^2 \|e^{\tau\varphi} \varphi' \varphi'' e^{\varphi} \omega_{n+1} u_1\|^2 + C \tau^2 \|e^{\tau\varphi} \omega_{n+1} \varphi' \varphi'' \nabla S^n u_1\|^2.
\]
Put ito (4.11), choosing $\eta > 0$ small, we reach
\[
\tau^2 \|e^{\tau\varphi} \varphi'' \omega_{n+1} \partial_t u_1\|^2 + 2 \|e^{\tau\varphi} \varphi'' \omega_{n+1} \nabla S^n u_1\|^2 + \tau^2 \|e^{\tau\varphi} \varphi' \varphi'' \omega_{n+1} u_1\|^2 \\
\leq C \|e^{\tau\varphi} \omega_{n+1} \int f\|^2 + C \tau^2 - 2s \|e^{\tau\varphi} \varphi'' \omega_{n+1} \Omega_n \Omega_n \partial_t \Omega_n u_1\|^2.
\]
Indeed, $u_2$ satisfies
\[
|\omega_{n+1}^{1-2s} \partial_t^2 + \sum_{j=1}^{n+1} \Omega_j \omega_{n+1}^{1-2s} \Omega_j - \omega_{n+1}^{1-2s} \left( n - 2s \right)^2 |u_2 + R u_2 = -K^2 \tau^2 |\varphi'|^2 \theta_{n+1} u_1 \text{ in } S^n_+ \times \mathbb{R},
\]
\[
\lim_{\omega_{n+1} \to 0} \omega_{n+1}^{1-2s} \Omega_n u_2 = 0 \text{ on } \partial S^n_+ \times \mathbb{R}.
\]
Compare with (4.2), we should put
\[
\tilde{f} = -K^2 \tau |\varphi'|^2 \omega_{n+1}^{1-2s} u_1 \text{ and } V \equiv 0.
\]
in (4.13). Omitting the second derivative terms, we obtain
\[
\tau^3 \|\varphi'\varphi''\|_{L^2}^2 + \tau \|\varphi''\|_{L^2}^2 + \tau \|\varphi'\|_{L^2}^2 \leq C K^4 \tau^4 \|e^{\tau \varphi}\|_{L^2}^2 \omega_{n+1}^{1-2s} u_1^2,
\]
that is,
\[
\tau^3 \|\varphi'\varphi''\|_{L^2}^2 + \tau \|\varphi''\|_{L^2}^2 + \tau \|\varphi'\|_{L^2}^2 \leq C K^4 \tau^4 \|e^{\tau \varphi}\|_{L^2}^2 \omega_{n+1}^{1-2s} u_1^2.
\]
(4.13)

Summing (4.12) and (4.13), since \( \bar{u} = u_1 + u_2 \), we reach
\[
\tau^3 \|e^{\tau \varphi}\varphi'\varphi''\|_{L^2}^2 + \tau \|e^{\tau \varphi}\varphi''\|_{L^2}^2 + \tau \|\varphi'\varphi''\|_{L^2}^2 \leq C \left( \|e^{\tau \varphi}\omega_{n+1}^{1-2s} \bar{f}\|_{L^2}^2 + \tau^2 \|e^{\tau \varphi}\varphi''\|_{L^2}^2 \right).
\]
Finally, plug in the boundary condition
\[
\lim_{\omega_{n+1} \to 0} \omega_{n+1} \Omega_{n+1} \bar{u} = \bar{V} \bar{u},
\]
and changing back to the cartesian coordinate, we obtain our result. \(\square\)

5. Proofs of the qualitative results of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1 Define \( w := \eta_R \bar{u} \), where \( \eta_R \) is radial,
\[
\eta_R(x) = \begin{cases} 
1 & , 2 \leq |x| \leq R, \\
0 & , |x| \leq 1 \text{ or } |x| \geq 2R,
\end{cases}
\]
and satisfies \( \|\nabla \eta_R\| \leq C/R, \|\nabla^2 \eta_R\| \leq C/R^2 \) in \( A_{R,2R}^+ \),
\[
\|\nabla \eta_R\| \leq C/R, \quad \|\nabla^2 \eta_R\| \leq C/R^2 \quad \text{in } A_{R,2R}^+,
\]
\[
\|\nabla \eta_R\| \leq C, \quad \|\nabla^2 \eta_R\| \leq C \quad \text{in } A_{R,2}^+.
\]
Note that
\[
\left[ \partial_{n+1} x_{n+1}^{-1} \partial_{n+1} + x_{n+1}^{-1} \sum_{j,k=1}^n a_{jk} \partial_j \partial_k \right] w = f,
\]
where
\[
f = x_{n+1}^{-1} \sum_{j,k=1}^n (1 - 2s)x_{n+1}^{-1} \partial_{n+1} \eta_R \bar{u} + x_{n+1}^{-1} \sum_{j,k=1}^n a_{jk} \partial_j \partial_k \eta_R \bar{u}
\]
\[
+ 2x_{n+1}^{-1} \sum_{j,k=1}^n (\partial_{n+1} \eta_R)(\partial_{n+1} \bar{u}) + \sum_{j,k=1}^n a_{jk}(\partial_k \eta_R)(\partial_j \bar{u})
\]
\[
- x_{n+1}^{-1} \sum_{j,k=1}^n (\partial_j a_{jk})(\partial_k \bar{u}) \eta_R.
\]
Since \( \eta_R \) is radial, then \( \partial_{n+1} \eta_R = \eta_R \partial_{n+1} |x| = 0 \) on \( \mathbb{R}^n \times \{0\} \). Thus,
\[
\lim_{x_{n+1} \to 0} x_{n+1}^{-1} \partial_{n+1} w = \lim_{x_{n+1} \to 0} x_{n+1}^{-1} \eta_R \partial_{n+1} \bar{u} = c_{n,s} q_R \eta_R u = c_{n,s} q u \quad \text{on } \mathbb{R}^n \times \{0\}.
\]
Note that $w$ is admissible in the Carleman estimate in Theorem 4.1. For $\beta > 1$, since $|q| \leq 1$ and $|x'\|\nabla q| \leq 1$, we have
\[
\tau^3 \| e^{\tau \phi} |x|^{\frac{3\beta}{2}} \frac{1}{2} x_{n+1}^{1-2\beta} w \|_{L^2(\mathbb{R}^n+1)}^2 + \tau \| e^{\tau \phi} |x|^{\frac{1}{2}} x_{n+1}^{1-2\beta} \nabla w \|_{L^2(\mathbb{R}^n+1)}^2
\]
\[
+ \tau^{-1} \| e^{\tau \phi} |x|^{-\frac{2}{2} + 1} x_{n+1}^{1-2\beta} \nabla (\nabla' w) \|_{L^2(\mathbb{R}^n+1)}^2
\]
\[
\leq C \left[ \| e^{\tau \phi} x_{n+1}^{1-2\beta} |x| f \|_{L^2(\mathbb{R}^n+1)}^2 + \tau \| e^{\tau \phi} |x|^{\frac{1}{2}} w \|_{L^2(\mathbb{R}^n \times \{0\})}^2 \right].
\]
(5.1)

Since $1 \leq \frac{|x|}{R} \in A^+_R|2R$ and $1 \leq |x| \in A^+_1(2)$, then
\[
\| e^{\tau \phi} x_{n+1}^{1-2\beta} |x| f \|_{L^2(\mathbb{R}^n+1)}^2 \leq C \left[ R^{-4} \| e^{\tau \phi} x_{n+1}^{1-2\beta} |x| \tilde{u} \|_{L^2(A^+_R|2R)}^2 + R^{-2} \| e^{\tau \phi} x_{n+1}^{1-2\beta} |x| \nabla \tilde{u} \|_{L^2(A^+_R|2R)}^2 \right.
\]
\[
+ \| e^{\tau \phi} x_{n+1}^{1-2\beta} |x| \tilde{u} \|_{L^2(A^+_1(2))}^2 + \| e^{\tau \phi} x_{n+1}^{1-2\beta} |x| \nabla \tilde{u} \|_{L^2(A^+_1(2))}^2 + \| e^{\tau \phi} x_{n+1}^{1-2\beta} \nabla w \|_{L^2(\mathbb{R}^n+1)}^2 \right].
\]

Write $\bar{\phi}(r) = \phi(x) = r^\beta$ with $r = |x|$, note that
\[
R^{-4} \| e^{\tau \phi} x_{n+1}^{1-2\beta} |x| \tilde{u} \|_{L^2(A^+_R|2R)}^2 + R^{-2} \| e^{\tau \phi} x_{n+1}^{1-2\beta} |x| \nabla \tilde{u} \|_{L^2(A^+_R|2R)}^2 \leq C \left[ R^{-2} e^{\tau \bar{\phi}(2R)} \| x_{n+1} u \|_{L^2(A^+_R|2R)}^2 + e^{\tau \bar{\phi}(2R)} \| x_{n+1} \nabla \tilde{u} \|_{L^2(A^+_R|2R)}^2 \right].
\]

Now we estimate $\| x_{n+1}^{1-2\beta} \nabla \tilde{u} \|_{L^2(A^+_R|2R)}^2$. Choose $\xi_R$ satisfies
\[
\xi_R(x) = \begin{cases} 1 & R \leq |x| \leq 2R, \\ 0 & |x| \leq \frac{R}{2} \text{ or } |x| \geq 2R, \end{cases}
\]
with $|\nabla \eta_R| \leq C/R$ for $x \in A^+_R|3R$ or $x \in A^+_2(3R)$. Test $\partial_{n+1} x_{n+1}^{1-2\beta} \partial_{n+1} \tilde{u} + \sum_{j, k=1}^n \partial_j a_{jk} \partial_k \tilde{u} = 0$ by the function $u_R$, we reach
\[
\| x_{n+1}^{1-2\beta} \nabla \tilde{u} \|_{L^2(A^+_R|2R)}^2 \leq C \| x_{n+1}^{1-2\beta} \tilde{u} \|_{L^2(A^+_R|3R)}^2 + R^{-2} \| x_{n+1}^{1-2\beta} \tilde{u} \|_{L^2(A^+_R|3R)}^2.
\]
So,
\[
R^{-4} \| e^{\tau \phi} x_{n+1}^{1-2\beta} |x| \tilde{u} \|_{L^2(A^+_R|2R)}^2 + R^{-2} \| e^{\tau \phi} x_{n+1}^{1-2\beta} |x| \nabla \tilde{u} \|_{L^2(A^+_R|2R)}^2 \leq C \left[ \| x_{n+1} u \|_{L^2(A^+_R|3R)}^2 + R^{-2} e^{\tau \bar{\phi}(2R)} \| x_{n+1} \tilde{u} \|_{L^2(A^+_R|3R)}^2 \right].
\]

By Proposition 8.1, we have
\[
|\tilde{u}(x)| \leq C e^{-C_2 R^\alpha} \text{ for } x \in A^+_R|3R.
\]
So, if we choose $\beta = \alpha - \epsilon$ for some $\epsilon \in (0, \alpha - 1)$, we obtain
\[
\lim_{R \to \infty} \left( R^{-4} \| e^{\tau \phi} x_{n+1}^{1-2\beta} |x| \tilde{u} \|_{L^2(A^+_R|2R)}^2 + R^{-2} \| e^{\tau \phi} x_{n+1}^{1-2\beta} |x| \nabla \tilde{u} \|_{L^2(A^+_R|2R)}^2 \right) = 0.
\]
However, (5.1) writes
\[
\tau^3 \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \tau^2 \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \tau^{-1} \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \tau \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} \geq C \left[ R^{-4} \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} + R^{-2} \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} \right].
\]

So, taking \( R \to \infty \) in (5.1) and choosing large \( \tau \), we reach
\[
\tau^3 \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \tau^2 \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \tau^{-1} \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \tau \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} \leq C \left[ \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+)} \right].
\]

Now we consider the boundary terms. Using Proposition [A.2] we have
\[
\bar{\tau} | \phi'' | e^{2st} \| v \|_{L^2_2(\mathbb{R}^{n+1}_+)} \leq C \left[ \tau^{2-2s} e^{2st} | \phi'' | \| \omega_{n+1}^{-1} v \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \tau^{2-2s} e^{2st} | \phi'' | \| \omega_{n+1}^{-1} \nabla v \|_{L^2_2(\mathbb{R}^{n+1}_+)} \right].
\]

Setting \( e^{2st} = \tau^{-2s} \) (i.e. \( \bar{\tau} = \tau e^t \)), our choice of \( \varphi \) gives
\[
\tau^{2-2s} e^{2st} | \phi'' | = \tau^{2-2s} e^{2st} | \phi'' | \leq \tau^{2-2s} | \phi'' |^2 | \phi'' |.
\]

Hence, we reach
\[
\tau^{2s+1} | \phi'' | \| v \|_{L^2_2(\mathbb{R}^{n+1}_+)} \leq C \left[ \tau^{3} | \phi'' | \| \omega_{n+1}^{-1} v \|_{L^2_2(\mathbb{R}^{n+1}_+)} + \tau | \phi'' | \| \omega_{n+1}^{-1} \nabla v \|_{L^2_2(\mathbb{R}^{n+1}_+)} \right].
\]

Multiply by \( e^{\tau \phi} \) and integrating in the radial variable \( t \), we reach
\[
\tau^{2s+1} \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+ \times \mathbb{R})} \leq C \left[ \tau^{3} \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+ \times \mathbb{R})} + \tau | \phi'' | \| \omega_{n+1}^{-1} \nabla v \|_{L^2_2(\mathbb{R}^{n+1}_+ \times \mathbb{R})} \right],
\]

that is,
\[
\tau^{2s+1} \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+ \times \mathbb{R})} \leq C \left[ \tau^{3} \| e^{\tau \phi} \|_{L^2_2(\mathbb{R}^{n+1}_+ \times \mathbb{R})} + \tau | \phi'' | \| \omega_{n+1}^{-1} \nabla v \|_{L^2_2(\mathbb{R}^{n+1}_+ \times \mathbb{R})} \right].
\]
Similarly, we have
\[
\tau^{2s-1}\| e^{\tau\phi} |x|^{-\frac{\alpha}{2} + \frac{1}{2}} \nabla w \|_{L^2(\mathbb{R}^n \times \{0\})}^2 \\
\leq C\left[ \tau \| e^{\tau\phi} |x|^{\frac{\alpha}{2} + \frac{1}{2}} \nabla w \|_{L^2(\mathbb{R}^n+)}^2 + \tau^{-1} \| e^{\tau\phi} |x|^{-\frac{\alpha}{2} + \frac{1}{2}} x_{n+1} \nabla (\nabla' w) \|_{L^2(\mathbb{R}^{n+1})}^2 \right].
\]

So, for large $$\tau$$, the boundary terms of (5.2) are absorbed, and we reach
\[
\tau^3 \| e^{\tau\phi} |x|^{\frac{3\alpha}{2} - 1} x_{n+1} \|_{L^2(B_0^+ \setminus B_1^+)}^2 + \tau \| e^{\tau\phi} |x|^{\frac{\alpha}{2} + \frac{1}{2}} x_{n+1} \nabla w \|_{L^2(B_0^+ \setminus B_1^+)}^2 \\
\leq \tau^{-1} \| e^{\tau\phi} |x|^{\frac{\alpha}{2} + \frac{1}{2}} x_{n+1} \nabla (\nabla' w) \|_{L^2(\mathbb{R}^{n+1})}^2.
\]

Pulling out the exponential weight in the above estimate yields
\[
\tau^3 e^{\tilde{\phi}(4)} \| x_{n+1}^{\frac{1-2\alpha}{2}} \tilde{u} \|_{L^2(B_0^+ \setminus B_1^+)}^2 + \tau e^{\tilde{\phi}(4)} \| x_{n+1}^{\frac{1-2\alpha}{2}} \nabla \tilde{u} \|_{L^2(B_0^+ \setminus B_1^+)}^2 \\
\leq C \left[ e^{\tilde{\phi}(2)} \| x_{n+1}^{\frac{1-2\alpha}{2}} \tilde{u} \|_{L^2(A_{1,2}^+)}^2 + e^{\tilde{\phi}(2)} \| x_{n+1}^{\frac{1-2\alpha}{2}} \nabla \tilde{u} \|_{L^2(A_{1,2}^+)}^2 \right].
\]

Since $$\tilde{\phi}(4) \geq \tilde{\phi}(2)$$, taking $$\tau \to \infty$$ will lead to a contradiction, unless $$\tilde{u} = 0$$ in $$B_0^+ \setminus B_1^+$$. Finally, by weak unique continuation property, we conclude that $$\tilde{u} \equiv 0$$. \(\square\)

Following exactly the arguments in [RW19], we can obtain Theorem 1.2.

APPENDIX A. Auxiliary Lemmas

A.1. Some interpolation inequalities. We have the following Hardy inequality:

**Lemma A.1** (Hardy, Lemma 4.6 of [RS17]). If $$\alpha \neq \frac{1}{2}$$ and if $$v$$ vanishes for $$x_{n+1}$$ large, then
\[
\| x_{n+1}^{-\alpha} u \|_{L^2(\mathbb{R}^{n+1})}^2 \leq \frac{4}{(2\alpha - 1)^2} \| x_{n+1}^{-\alpha} \partial_{n+1} u \|_{L^2(\mathbb{R}^{n+1})}^2 + \frac{2}{2\alpha - 1} \lim_{x_{n+1} \to 0} \| x_{n+1}^{-\alpha} u \|_{L^2(\mathbb{R}^n \times \{0\})}^2.
\]

**Proof.** Indeed, this follows from a direct integration by parts argument
\[
\| x_{n+1}^{-\alpha} u \|_{L^2(\mathbb{R}^{n+1})}^2 = \int \partial_{n+1} \left[ \frac{x_{n+1}^{1-2\alpha}}{1 - 2\alpha} \right] u^2 \\
= \frac{2}{2\alpha - 1} \int x_{n+1}^{1-2\alpha} u \partial_{n+1} u + \frac{1}{2\alpha - 1} \int \lim_{x_{n+1} \to 0} x_{n+1}^{1-2\alpha} u^2 \\
\leq \frac{1}{2(2\alpha - 1)^2} \| x_{n+1}^{-\alpha} \partial_{n+1} u \|_{L^2(\mathbb{R}^{n+1})}^2 + \frac{1}{2} \| x_{n+1}^{-\alpha} u \|_{L^2(\mathbb{R}^{n+1})}^2 + \frac{1}{2\alpha - 1} \lim_{x_{n+1} \to 0} \| x_{n+1}^{\frac{1}{2} - \alpha} u \|_{L^2(\mathbb{R}^n \times \{0\})}^2,
\]

which gives our desired result. \(\square\)

We shall use the following interpolation inequality in [Rul15, RW19]:
Lemma A.3 (Lemma 4.4 of [RS17]). Let \( n \geq 1 \) and \( 0 < \tilde{s} < 1 \). There is a bounded surjective linear map
\[
T : H^1(\mathbb{R}^n_{+}, x_{n+1}^{1-2\tilde{s}}) \rightarrow H^{\tilde{s}}(\mathbb{R}^n \times \{0\})
\]
so that \( u(\bullet, x_{n+1}) \rightarrow Tu \) in \( L^2(\mathbb{R}^n) \) as \( x_{n+1} \to 0 \).

We need the following interpolation inequality, which appeared in Proposition 5.11 of [RS17]:

Lemma A.4 (Interpolation inequality II(a)). For any \( w \in H^1(\mathbb{R}^n_{+}, x_{n+1}^{1-2\tilde{s}}) \) and any \( \mu > 0 \), the following interpolation inequality holds:
\[
\|w\|_{L^2(\mathbb{R}^n \times \{0\})} \leq C \left[ \mu^{1-s} (\|x_{n+1}^{\frac{2s-1}{2}} w\|_{L^2(\mathbb{R}^n_{+})} + \|x_{n+1}^{\frac{2s-1}{2}} \nabla w\|_{L^2(\mathbb{R}^n_{+})}) + \mu^{-s}\|w\|_{H^{-s}(\mathbb{R}^n \times \{0\})} \right].
\]

Proof. Let \( \langle \bullet \rangle := \sqrt{1 + |\bullet|^2} \). Note that
\[
\|w\|_{L^2(\mathbb{R}^n \times \{0\})} = \left[ \int_{\mathbb{R}^n \times \{0\}} (\langle \xi \rangle^{2-2s} |\tilde{w}|^2 (\langle \xi \rangle^{-2s} |\tilde{w}|^2)^{1-s} d\xi \right]^{\frac{1}{2}} 
\leq (\mu^{1-s}\|w\|_{H^{-s}(\mathbb{R}^n \times \{0\})})^{\tilde{s}} (\mu^{-s}\|w\|_{H^{-s}(\mathbb{R}^n \times \{0\})})^{1-s}
\]
and hence our result follows by Lemma A.3 with \( \tilde{s} = 1-s \).

Slightly modify the proof, we can obtain the following:

Lemma A.5 (Interpolation inequality II(b)). For any \( w \in H^1(\mathbb{R}^n_{+}, x_{n+1}^{1-2\tilde{s}}) \) and any \( \mu > 0 \), the following interpolation inequality holds:
\[
\|w\|_{L^2(\mathbb{R}^n \times \{0\})} \leq C \left[ \mu^{1-s} (\|x_{n+1}^{\frac{2s-1}{2}} w\|_{L^2(\mathbb{R}^n_{+})} + \|x_{n+1}^{\frac{2s-1}{2}} \nabla w\|_{L^2(\mathbb{R}^n_{+})}) + \mu^{-2s}\|w\|_{H^{-2s}(\mathbb{R}^n \times \{0\})} \right].
\]

Proof. Using Lemma A.3 with \( \tilde{s} = 1-s \), we have
\[
\|w\|_{L^2(\mathbb{R}^n \times \{0\})} \leq C \|w\|_{H^{-s}(\mathbb{R}^n \times \{0\})}^{\frac{1-s}{2}} \|w\|_{H^{-2s}(\mathbb{R}^n \times \{0\})}^{\frac{1-s}{2}} 
\leq C \left( \|x_{n+1}^{\frac{2s-1}{2}} w\|_{L^2(\mathbb{R}^n_{+})} + \|x_{n+1}^{\frac{2s-1}{2}} \nabla w\|_{L^2(\mathbb{R}^n_{+})} \right)^{\frac{1-s}{2}} \|w\|_{H^{-2s}(\mathbb{R}^n \times \{0\})}^{\frac{1-s}{2}} 
\leq C \left[ \mu^{1-s} (\|x_{n+1}^{\frac{2s-1}{2}} w\|_{L^2(\mathbb{R}^n_{+})} + \|x_{n+1}^{\frac{2s-1}{2}} \nabla w\|_{L^2(\mathbb{R}^n_{+})}) + \mu^{-2s}\|w\|_{H^{-2s}(\mathbb{R}^n \times \{0\})} \right],
\]
which is our desired result.
A.2. Caccioppoli inequality. We need a generalized the Caccioppoli inequality in Lemma 4.5 of \textit{RS17}.

\textbf{Lemma A.6.} Let $s \in (0, 1)$ and $u \in H^1(B_{2r}^+, x_{n+1}^{1-2s})$ be a solution to

$$\left[ \partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} P \right] \tilde{u} = -x_{n+1}^{1-2s} \sum_{j=1}^n \partial_j f_j \quad \text{in } B_{2r}^+.$$ 

Then there exists a constant $C = C(n, \lambda)$ such that

$$\|x_{n+1}^{1-2s} \nabla \tilde{u}\|_{L^2(B_{r}^+)}^2 \leq C \left[ r^{-2} \|x_{n+1}^{1-2s} \tilde{u}\|_{L^2(B_{r}^+)}^2 + \sum_{j=1}^n \|x_{n+1}^{1-2s} f_j\|_{L^2(B_{r}^+)}^2 + \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}\|_{L^2(B_{2r}^+)}^2 \right].$$

\textbf{Proof.} Let $\eta : B_{2r}^+ \to \mathbb{R}$ be a smooth, radial cut-off function such that $0 \leq \eta \leq 1$, $\eta = 1$ on $B_r^+$, $\text{supp}(\eta) \subset B_{2r}^+$, and $|\nabla \eta| \leq C/r$ for some constant $C$. Note that

\begin{align*}
2 & \sum_{j=1}^n \int_{\mathbb{R}^{n+1}} \left( \frac{\partial^2}{x_{n+1}^{1-2s} \partial_{n+1}} \eta f_j \left( \frac{\partial^2}{x_{n+1}^{1-2s} \partial_{n+1}} \eta \tilde{u} \right) \right) + \sum_{j=1}^n \int_{\mathbb{R}^{n+1}} \left( \frac{\partial f_j}{x_{n+1}^{1-2s} \partial_{n+1}} \eta \right) \left( \frac{\partial^2}{x_{n+1}^{1-2s} \partial_{n+1}} \eta \tilde{u} \right) \\
& = - \sum_{j=1}^n \int_{\mathbb{R}^{n+1}} x_{n+1}^{1-2s} (\partial_j f_j) (\eta^2 \tilde{u}) \\
& = \int_{\mathbb{R}^{n+1}} \left( \partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} + x_{n+1}^{1-2s} \sum_{i,j=1}^n \partial_i a_{ij} \partial_j \tilde{u} \right) (\eta^2 \tilde{u}) \\
& = - \int_{\mathbb{R}^{n+1}} \eta^2 \tilde{u} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} - \int_{\mathbb{R}^{n+1}} (x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}) \partial_{n+1} (\eta^2 \tilde{u}) \\
& = - \int_{\mathbb{R}^{n+1}} \eta^2 \tilde{u} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} - 2 \int_{\mathbb{R}^{n+1}} (x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}) \eta \partial_{n+1} \eta \tilde{u} \\
& = - \int_{\mathbb{R}^{n+1}} \eta^2 \tilde{u} \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} - 2 \int_{\mathbb{R}^{n+1}} x_{n+1}^{1-2s} \sum_{i,j=1}^n a_{ij} (\eta \partial_j \tilde{u}) (\partial_i \eta \tilde{u}) \\
& = - \int_{\mathbb{R}^{n+1}} \eta^2 x_{n+1}^{1-2s} \left( \sum_{i,j=1}^n a_{ij} \partial_j \tilde{u} \partial_i \tilde{u} \right) \\
(A.1) & = - \int_{\mathbb{R}^{n+1}} \eta^2 \tilde{u} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} - 2 \langle \eta \nabla \tilde{u}, \tilde{u} \nabla \tilde{u} \rangle - \|\eta \nabla \tilde{u}\|^2
\end{align*}

where $\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Here we use the notation

$$\langle \bullet, \bullet \rangle = \langle \bullet, \bullet \rangle_{L^2(\mathbb{R}_+^{n+1} \times x_{n+1}^{1-2s} \tilde{A})} \quad \text{and} \quad \| \bullet \| = \| \bullet \|_{L^2(\mathbb{R}_+^{n+1} \times x_{n+1}^{1-2s} \tilde{A})}.$$
By (1.2), indeed
$$\|\eta \nabla \tilde{u}\|^2 \geq \lambda \|x_{n+1}^{\frac{1-2s}{2}} \nabla \tilde{u})^2_{L^2(\mathbb{R}_{n+1}^+)} \geq \lambda \|x_{n+1}^{\frac{1-2s}{2}} \nabla \tilde{u})^2_{L^2(B_1^+)}.$$ 
Also, by (1.2), for $\delta > 0$, we have
$$2\langle \eta \nabla \tilde{u}, \tilde{u} \nabla \eta \rangle \leq \delta \|\eta \nabla \tilde{u}\|^2 + \delta^{-1} \||\tilde{u} \nabla \eta\|^2 \\
\leq \delta \lambda^{-1} \|x_{n+1}^{\frac{1-2s}{2}} \nabla u\|^2_{L^2(\mathbb{R}_{n+1}^+)} + \delta^{-1} \lambda^{-1} \|\nabla \eta x_{n+1}^{\frac{1-2s}{2}} u\|^2_{L^2(\mathbb{R}_{n+1}^+)}.$$ 
Moreover, we have
$$\left| \int_{\mathbb{R}^n \times \{0\}} \lim_{x_{n+1} \to 0} \eta^2 \tilde{u} x_{n+1}^{\frac{1-2s}{2}} \partial_{n+1} \tilde{u} \right| \leq \|\lim_{x_{n+1} \to 0} x_{n+1}^{\frac{1-2s}{2}} \partial_{n+1} \tilde{u} \|_{L^2(B_{2r}^+)} \|\eta^2 \tilde{u}\|_{L^2(B_{2r}^+)}.$$ 
Plug the inequalities above into (A.1), with small $\delta > 0$, we obtain our desired result. \qed

A.3. $L^\infty$-$L^2$ type interior inequality. Following the arguments in Proposition 3.1 of [TX11] (see also Proposition 2.4 of [JLX11], and also Proposition 3.2 in [FF14]), we can obtain the following:

**Lemma A.7.** Let $s \in (0, 1)$ and $u \in H^1(B_{2r}^+, x_{n+1}^{1-2s})$ be a solution to
$$\left[ \partial_{n+1} x_{n+1}^{\frac{1-2s}{2}} \partial_{n+1} + x_{n+1}^{\frac{1-2s}{2}} P \right] \tilde{u} = 0 \quad \text{in} \ B_1^+$$
with (L2). Then there exists a constant $C = C(n, \lambda)$ such that
$$\|\tilde{u}\|_{L^\infty(B_{1/2}^+)} \leq C \left[ \|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B_1^+)} + \|x_{n+1}^{\frac{1-2s}{2}} \nabla \tilde{u}\|_{L^2(B_1^+)} \right].$$

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