PROPERTIES OF SHAPE-INVARIANT TRIDIAGONAL HAMILTONIANS

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As is known, a nonnegative-definite Hamiltonian $H$ that has a tridiagonal matrix representation in a basis set allows defining forward (and backward) shift operators that can be used to determine the matrix representation of the supersymmetric partner Hamiltonian $H^{(+)}$ in the same basis. We show that if the Hamiltonian is also shape-invariant, then the matrix elements of the Hamiltonian are related such that the energy spectrum is known in terms of these elements. It is also possible to determine the matrix elements of the hierarchy of supersymmetric partner Hamiltonians. Moreover, we derive the coherent states associated with this type of Hamiltonian and illustrate our results with examples from well-studied shape-invariant Hamiltonians that also have a tridiagonal matrix representation.

Keywords: supersymmetry, shape-invariant potential, tridiagonal Hamiltonian, superpotential, raising operator, lowering operator, coherent state

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1. Introduction

With the idea of supersymmetric (SUSY) quantum mechanics, the concept of shape-invariance was proposed by Gendenshtein [1]. The shape-invariance condition requires that the SUSY partner potentials satisfy a condition of the type $V^{(+)}_{\eta}(x) = V(f(\eta))(x) + R(\eta)$, where $f(\eta)$ is a function of $\eta$ and $R(\eta)$ is independent of $x$. This is equivalent to the operator relation $H^{(+)}(\eta) = H(f(\eta)) + R(\eta)$, which is also an integrability condition [1] that was proved sufficient for obtaining exact results (see [2] for a review of supersymmetry, shape-invariance, and exactly solvable potentials).

Applying shape-invariance in SUSY quantum mechanics was proposed in [3], [4] as an effective tool for constructing and studying exactly solvable systems. Indeed, SUSY quantum mechanics and the shape-invariance condition provide an algebraic procedure for determining the entire spectrum of solvable quantum systems without solving a differential equation. Many exactly solvable problems with one-dimensional potentials encountered in quantum mechanics are shape-invariant, and the parameters are related by a translation $f(\eta) = \eta + \delta$ (see [5]). The Lie algebra $so(2, 1)$ is associated with such potentials and is used to recover the spectrum of the model. The factorization method for second-order difference equations, similar to the method developed for second-order differential equations, was systematically studied in [6] for the purpose of solving certain eigenvalue problems. The utility and success of this tool were demonstrated in

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many applications. As a result, the spectral analysis of difference equations became the subject of further
extensive research to enlarge the class of solvable problems beyond those associated with simple Lie algebras
to those associated with a general form of a polynomial algebra [7].

Our purpose here is to develop the necessary tools needed to apply the factorization method to the
Schrödinger equation in its matrix form by analogy with the extensive tools developed for applying the
method to the differential form of the equation. To accomplish this task, we fully use the rich lore of
theoretical results and tools that have been developed for solving the spectral problem of second-order
difference equations. Further, we note that in the framework of the matrix approach, it is easy to pass to
truncation approximations that allow finding approximate numerical solutions of physical problems that
are not yet known to have exact solutions.

We recall that SUSY quantum mechanics is traditionally formulated in terms of the representation of
the system Hamiltonian in the configuration space [8]. One advantage of this representation is related to
the possibility of obtaining the explicit form in the configuration space of the partner potential and the
so-called superpotential. On the other hand, it has been known since the initial development of quantum
mechanics that the matrix representation of a Hamiltonian

\[ H = \sum_{n,m} H_{n,m} \langle \phi_n | H | \phi_m \rangle, \]

then the supersymmetric partner Hamiltonian \( H^{(+)} \) also has a tridiagonal representation in the same basis,

\[ H^{(+)} = \sum_{n,m} H^{(+)}_{n,m} \langle \phi_n | H^{(+)} | \phi_m \rangle, \]

where the coefficients are related by

\[ a_n = c_n^2 + d_n^2, \quad b_n = c_n d_{n+1}, \]

\[ a_n^{(+)} = c_n^2 + d_{n+1}^2, \quad b_n^{(+)} = c_{n+1} d_{n+1}. \]

Here, the two sets of coefficients \((c_n)\) and \((d_n)\) are involved in defining the forward-shift operator \(A\) by
specifying its action on a basis vector as

\[ A | \phi_m \rangle = c_m | \phi_m \rangle + d_m | \phi_{m-1} \rangle. \]

This means that the action of its adjoint \(A^\dagger\) on the basis is now

\[ A^\dagger | \phi_m \rangle = c_m | \phi_m \rangle + d_{m+1} | \phi_{m+1} \rangle. \]

The Hamiltonian and its supersymmetric partner are related to these operators by

\[ H = A^\dagger A, \quad H^{(+)} = AA^\dagger. \]

More explicitly, the parameters \(c_n\) and \(d_n\) can be calculated recursively if the tridiagonal matrix elements
of the Hamiltonian are known or, alternatively, from the relations

\[ c_n^2 = -b_n \frac{P_{n+1}(\varepsilon_0)}{P_n(\varepsilon_0)}, \quad d_{n+1}^2 = -b_n \frac{P_n(\varepsilon_0)}{P_{n+1}(\varepsilon_0)}. \]
where $\varepsilon_0$ is the energy of the ground state. Here, the set $(P_n(\varepsilon))$ is a solution of the three-term recurrence relation
\[ \varepsilon P_n(\varepsilon) = b_{n-1} P_{n-1}(\varepsilon) + a_n P_n(\varepsilon) + b_n P_{n+1}(\varepsilon). \] (1.9)
The energy eigenstate $|\varepsilon\rangle$ has the representation
\[ |\varepsilon\rangle = \sqrt{\Omega(\varepsilon)} \sum_{n=0}^{\infty} P_n(\varepsilon) |\phi_n\rangle. \] (1.10)

The polynomials $P_n$ are orthogonal in the sense that
\[ \int \Omega(\varepsilon) P_n(\varepsilon) P_m(\varepsilon) \, d\varepsilon = \delta_{n,m}, \] (1.11)
where the integration is over the spectrum of the physical system.

Here, we study the consequences of shape-invariance and find additional properties of the abovementioned basic parameters. These properties in turn allow finding physical quantities associated with the Hamiltonian system. In Sec. 2, we study the properties of the parameters listed above, in particular, the set $(c_n, d_n)$, which plays a fundamental role in our analysis. We show that a modified version of the operators $A$ and $A^\dagger$ can be defined, which can be essentially interpreted as lowering and raising operators. This allows finding the full energy spectrum of the system using only the parameters $c_0$, $c_1$, and $d_1$. We hence easily obtain the determination of the supersymmetric partner potential. We also show that the shape-invariance allows completely classifying the set of $c_n$-like and $d_n$-like parameters for the hierarchy of supersymmetric partner Hamiltonians. Although the superpotential concept plays no role in describing the system in our problem setup, we nevertheless provide an expression for a quantity with all the properties of a superpotential. In Sec. 3, we construct an explicit form of coherent states associated with a shape-invariant tridiagonal Hamiltonian and in examples show that it has the expected properties. Section 4 is devoted to some concluding remarks. Throughout, we illustrate our results using cases previously studied and leaving the details to the appendices.

2. Consequences of shape-invariance

2.1. Properties of the matrix elements of shape-invariant Hamiltonians. A given nonnegative-definite Hamiltonian $H$ (with a zero ground state $\varepsilon_0$) with shape-invariance is related to its supersymmetric partner Hamiltonian $H^{(+)}$ by
\[ H^{(+)}(\eta) = H(\eta + \delta) + R(\eta), \] (2.1)
where $\eta$, $\delta$, and $R$ are parameters reflecting properties of the given Hamiltonian. Because the spectrum of $H^{(+)}$ is shifted by $\varepsilon_1$ with respect to the spectrum of $H$, we must set $R(\eta) = \varepsilon_1(\eta)$. Further, if $H$ has tridiagonal representation (1.1) in some basis, then $H^{(+)}$ is also a tridiagonal operator (see Eq. (1.2)). Hence the above symmetry relation translates into a relation between the matrix elements of the Hamiltonians,
\[ a_n^{(+)}(\eta) = a_n(\eta + \delta) + \varepsilon_1(\eta), \quad b_n^{(+)}(\eta) = b_n(\eta + \delta). \] (2.2)

From relations (1.3) and (1.4) of the parameters $(a_n^{(+)}, b_n^{(+)})$ and $(a_n, b_n)$ to the basic coefficients $(c_n, b_n)$, Eqs. (2.2) become
\[ c_n(\eta) + d_{n+1}(\eta) = c_n^{(+)}(\eta + \delta) + d_{n+1}^{(+)}(\eta + \delta) + \varepsilon_1(\eta), \] (2.3)
\[ c_n(\eta) d_{n+1}(\eta) = c_n(\eta + \delta) d_{n+1}(\eta + \delta). \]
We now postulate that the coefficients \( d_n \) are independent of \( \eta \). This indeed holds for many known shape-invariant Hamiltonians such as the harmonic oscillator and Morse Hamiltonians. We then have \( d_n(\eta) = d_n(\eta + \delta) = d_n \) and hence \( c_{n+1}(\eta) = c_n(\eta + \delta) \). Combining the above relations, we obtain

\[
\varepsilon_1(\eta) = [\varepsilon_n^2(\eta) - \varepsilon_{n+1}^2(\eta)] + (d_{n+1}^2 - d_n^2).
\]  

(2.4)

We make two remarks about this result. First, an explicit identification of the parameter \( \eta \) is unnecessary for calculating \( \varepsilon_1(\eta) \). Second, this result is independent of \( n \). Specifically, if we set \( n = 0 \), then we have the simple but useful relation

\[
\varepsilon_1(\eta) = c_0^2(\eta) - c_1^2(\eta) + d_1^2,
\]  

(2.5)

giving \( \varepsilon_1(\eta) \) in terms of only the three parameters \( c_0(\eta) \), \( c_1(\eta) \), and \( d_1 \).

Based on symmetry property (2.1), we can conclude that it is convenient to use the translation operator \( T := e^{-\delta \partial / \partial \eta} \). In essence, if \( K(\eta) \) is an operator, then \( T^\dagger K(\eta) T = K(\eta + \delta) \). Here, of course, \( T^\dagger = e^{\delta \partial / \partial \eta} \).

We intend to take this property into account in our basic setup by replacing \( A \) and \( A^\dagger \) with the operator \( B = TA \) and its adjoint \( B^\dagger = A^\dagger T^\dagger \). Remarkably, we still have \( H(\eta) = A^\dagger A = B^\dagger B \). On the other hand, we have a new version of the supersymmetric partner Hamiltonian, \( \overline{H}(\eta) = BB^\dagger \). It is simply related to \( H(\eta) \),

\[
\overline{H}(\eta) = BB^\dagger = TAA^\dagger T^\dagger = H(\eta - \delta).
\]  

(2.6)

It clearly follows from this relation that \( \overline{H}(\eta) \) also has a tridiagonal matrix representation in the same basis:

\[
\overline{H}(\eta) |\phi_n\rangle = c_n(\eta - \delta)d_n|\phi_{n-1}\rangle + (c_n^2(\eta - \delta) + d_{n+1}^2)|\phi_n\rangle + c_{n+1}(\eta - \delta)d_{n+1}|\phi_{n+1}\rangle.
\]

Comparing \( BB^\dagger |\phi_n\rangle \) and \( B^\dagger B|\phi_n\rangle \) and using the symmetry properties

\[
d_n(\eta) = d_n(\eta + \delta) = d_n, \quad c_{n+1}(\eta) = c_n(\eta + \delta),
\]

we now obtain

\[
[B, B^\dagger]|\phi_n\rangle = ([c_{n-1}^2(\eta) + d_{n+1}^2] - [c_n^2(\eta) + d_n^2])|\phi_n\rangle.
\]  

(2.7)

It is easy to see that the quantity in the right-hand side is just \( \varepsilon_1(\eta - \delta)|\phi_n\rangle \). We thus obtain our major result:

\[
[B, B^\dagger] = \varepsilon_1(\eta - \delta).
\]  

(2.8)

This commutation relation suggests that \( B^\dagger \) and \( B \) can be regarded as raising and lowering operators if we take into account that they do not commute with \( \varepsilon_1(\eta) \). The fact that \( B^\dagger \) and \( B \) do not commute with any function \( f(\eta) \) is given by the two important relations

\[
f(\eta)B^\dagger = B^\dagger f(\eta - \delta), \quad f(\eta)B = Bf(\eta + \delta).
\]

In Table 1, we list results for three shape-invariant Hamiltonians that admit a tridiagonal matrix representation: the kinetic energy operator of the \( l \)th partial wave, the harmonic oscillator Hamiltonian,
The proof of this equality is obtained by induction (see Appendix A). Another version of it is

\[ |\varepsilon_{m+1}(\eta)| = \frac{1}{\sqrt{\sum_{k=0}^{m} \varepsilon_1(\eta + k\delta)}} B^\dagger \frac{1}{\sqrt{\sum_{k=0}^{m-1} \varepsilon_1(\eta + k\delta)}} B^\dagger \times \ldots \frac{1}{\sqrt{\varepsilon_1(\eta) + \varepsilon_1(\eta + \delta)}} B^\dagger \frac{1}{\sqrt{\varepsilon_1(\eta)}} B^\dagger |\varepsilon_0(\eta)\rangle. \]  

(2.9)

The proof of this equality is obtained by induction (see Appendix A). Another version of it is

\[ |\varepsilon_{m+1}(\eta)| = \frac{1}{\sqrt{\sum_{k=0}^{m} \varepsilon_1(\eta + k\delta)}} B^\dagger |\varepsilon_m(\eta)\rangle. \]  

(2.10)

### Table 1

|                        | Kinetic energy | Harmonic oscillator | Morse Hamiltonian |
|------------------------|----------------|---------------------|-------------------|
| $H$                    | $-\frac{1}{2} \frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2}$ | $-\frac{1}{2} \frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} + \frac{\omega^2}{2r^2} - (l + \frac{3}{2})\omega$ | $-\frac{1}{2} \frac{d^2}{dr^2} + V_0(e^{-2\alpha x} - 2e^{-\alpha x}) + \frac{\alpha^2}{2} D^2$ |
| $\phi_n(y)$            | $K_n(y)^{\nu+1/2} e^{-y^2/2} L^\nu_n(y^2)$ | $K_n(y)^{\nu+1/2} e^{-y^2/2} L^\nu_n(y^2)$ | $K_n(y)^{\gamma+1/2} e^{-y^2/2} L^\nu_n(y^2)$ |
| $y$                    | $\lambda r$   | $\lambda r$         | $\frac{\sqrt{2n\lambda}}{\alpha} e^{-\alpha x}$ |
| $K_n$                  | $\sqrt{\frac{(2\lambda n)!}{1(\nu + \nu + 1)!}}$ | $\sqrt{\frac{(2\lambda n)!}{1(\nu + \nu + 1)!}}$ | $\sqrt{\frac{\alpha \lambda n}{1(\nu + \nu + 1)!}}$ |
| Free parameters        | $\lambda$      | $\lambda$           | $\gamma$          |
| Other parameters       | $\nu = l + \frac{1}{2}$ | $\nu = l + \frac{1}{2}$ | $D = \sqrt{2V_0} - \frac{1}{2}$ |
| $c_n^2$                | $\frac{\lambda^2}{\gamma^2} (n + \nu + 1)$ | $\frac{(\lambda - \omega \lambda^2)}{(n + \nu + 1)}(n + \nu + 1)$ | $\frac{\alpha^2}{\gamma^2} (n + \nu + 1)$ |
| $d_n^2$                | $\frac{\lambda^2}{\gamma^2} n$ | $\frac{\lambda^2}{\gamma^2} n$ | $\frac{\alpha^2}{\gamma^2} n(n + \nu + 1)$ |
| $\varepsilon_1(\eta)$ | 0              | $2\omega$           | $\frac{\alpha^2}{\gamma^2} (2D - 1)$ |
| $\eta$                | $\nu$          | $\nu$               | $D$               |
| $\delta$              | 1              | 1                   | $-1$              |
| $\varepsilon_1(\eta - \delta)$ | 0 | $2\omega$ | $\frac{\alpha^2}{2} (2D + 1)$ |
| $\varepsilon^{(+)}_\mu(\eta)$ | 0 | $2(\mu + 1)\omega$ | $\frac{\alpha^2}{2} (\mu + 1)(2D - (\mu + 1))$ |
| $[B, B^\dagger]$      | 0              | $2\omega$           | $\frac{\alpha^2}{2} (2D - 1)$ |
| $V(r)$                | $\frac{l(l+1)}{2r^2}$ | $\frac{l(l+1)}{2r^2} + \frac{\omega^2}{2r^2} - (l + \frac{3}{2})\omega$ | $\frac{\alpha^2}{2} (D + \frac{1}{2})^2 (e^{-2\alpha x} - 2e^{-\alpha x}) + \frac{\alpha^2}{2} D^2$ |
| $V^{(+)}(r)$          | $\frac{(l+1)(l+2)}{2r^2}$ | $\frac{(l+1)(l+2)}{2r^2} + \frac{\omega^2}{2r^2} - (l + \frac{3}{2})\omega$ | $\frac{\alpha^2}{2} (D - \frac{1}{2})^2 (e^{-2\alpha x} - 2e^{-\alpha x}) + \frac{\alpha^2}{2} D^2$ |

Parameters and results for the kinetic energy operator, the harmonic oscillator Hamiltonian, and the Morse Hamiltonian.

2.2. The operator $B^\dagger$ as a raising operator. Here, we clarify the sense of $B^\dagger$ behaving as a raising operator. We show that if $|\varepsilon_0\rangle$ is the ground state and $|\varepsilon_m\rangle$, $m = 1, 2, \ldots$, are the excited states of the considered system, then

\[ |\varepsilon_{m+1}(\eta)| = \frac{1}{\sqrt{\sum_{k=0}^{m} \varepsilon_1(\eta + k\delta)}} B^\dagger \frac{1}{\sqrt{\sum_{k=0}^{m-1} \varepsilon_1(\eta + k\delta)}} B^\dagger \times \ldots \frac{1}{\sqrt{\varepsilon_1(\eta) + \varepsilon_1(\eta + \delta)}} B^\dagger \frac{1}{\sqrt{\varepsilon_1(\eta)}} B^\dagger |\varepsilon_0(\eta)\rangle. \]  

(2.9)
This relation clearly shows that $B^\dagger$ has the sense of a raising operator because it generates higher energy eigenstates not only from the ground state, as in Eq. (2.9), but also from the level directly below. In Sec. 3, we use this property together with the displacement operator $D(z) = e^{zB^\dagger - z^*B}$ to construct coherent states.

As an example in Appendix B, we present explicit calculations showing that $B$ acts as a lowering operator for the harmonic oscillator Hamiltonian.

### 2.3. The energy spectrum

Remarkably, shape-invariance together with minimal additional information allows fully characterizing the system energy spectrum. To see this, we first note that

$$H^{(+)}(\eta - \delta) = H(\eta) + \varepsilon_1(\eta - \delta) \quad (2.11)$$

follows from Eq. (2.1). Because $B^\dagger$ and $B$ are interpreted as raising and lowering operators, the Hamiltonians share the same spectrum except the ground state. Therefore, if $H(\eta)|\varepsilon_m\rangle = \varepsilon_m(\eta)|\varepsilon_m\rangle$, then the above equation gives

$$H^{(+)}(\eta - \delta)|\varepsilon_m\rangle = [H(\eta) + \varepsilon_1(\eta - \delta)]|\varepsilon_m\rangle, \quad (2.12)$$

$$\varepsilon_m^{(+)}(\eta - \delta)|\varepsilon_m\rangle = [\varepsilon_m(\eta) + \varepsilon_1(\eta - \delta)]|\varepsilon_m\rangle.$$ 

Hence,

$$\varepsilon_m^{(+)}(\eta - \delta) = \varepsilon_m(\eta) + \varepsilon_1(\eta - \delta) \quad (2.13)$$

or

$$\varepsilon_m^{(+)}(\eta) = \varepsilon_m(\eta + \delta) + \varepsilon_1(\eta). \quad (2.14)$$

On the other hand, if we write $\varepsilon_m^{(+)}(\eta) = \varepsilon_{m+1}$, then for tridiagonal Hamiltonians with shape-invariance, we have the interesting new result

$$\varepsilon_{m+1}(\eta) = \varepsilon_m(\eta + \delta) + \varepsilon_1(\eta) = \sum_{k=0}^{m} \varepsilon_1(\eta + k\delta). \quad (2.15)$$

We can obtain $\varepsilon_m(\eta)$ and $\varepsilon_m^{(+)}(\eta)$ from $\varepsilon_1(\eta)$, and $\varepsilon_1(\eta)$ is expressed in terms of the parameters $c_0$, $c_1$, and $d_1$. Therefore, the system energy spectrum is also essentially determined by these parameters. On the other hand, we can find a very simple expression for $\varepsilon_m(\eta)$ or $\varepsilon_m^{(+)}(\eta)$ in terms of the parameters $c_0, c_1, \ldots$ and $d_1$. From Eq. (2.5), we obtain

$$\varepsilon_1(\eta + k\delta) = [c_0^2(\eta + k\delta) - c_1^2(\eta + k\delta)] + d_1^2. \quad (2.16)$$

Because $c_{n+1}(\eta) = c_n(\eta + \delta)$,

$$\varepsilon_m^{(+)}(\eta) = \varepsilon_{m+1}(\eta) = \sum_{k=0}^{m} \varepsilon_1(\eta + k\delta) = \sum_{k=0}^{m} ([c_k^2(\eta) - c_{k+1}^2(\eta)] + d_1^2). \quad (2.17)$$

Canceling terms in the last sum, we obtain

$$\varepsilon_m^{(+)}(\eta) = \varepsilon_{m+1}(\eta) = [c_0^2(\eta) - c_{m+1}^2(\eta)] + (m + 1)d_1^2. \quad (2.18)$$

As examples, the parameters in Table 1 can be used to show how the obtained results apply to the listed Hamiltonians. Needless to say, just as the energy spectrum $\{\varepsilon_m\}$ of $H(\eta)$ is related to the potential $V(\eta)$, the energy spectrum $\{\varepsilon_m^{(+)}\}$ of the partner Hamiltonian $H^{(+)}(\eta)$ is related to the partner potential.
\( V^{(+)}(\eta) \). In fact, if the Hamiltonian satisfies shape-invariance condition, (2.1), then in more detail, we can write

\[
H_0(\eta) + V^{(+)}(\eta) = H_0(\eta + \delta) + V(\eta + \delta) + \varepsilon_1(\eta).
\]  
(2.19)

Because the considered Hamiltonian \( H_0(\eta) = -(1/2) d^2/dx^2 \), we must require that \( H_0(\eta) \) be independent of the parameter \( \eta \). Hence, \( H_0(\eta) = H_0(\eta + \delta) \). Therefore, the partner potentials are related by

\[
V^{(+)}(\eta) = V(\eta + \delta) + \varepsilon_1(\eta).
\]  
(2.20)

We verify this relation for the three well-known potentials in Table 1.

2.4. The coefficients \((c_n,d_n)\) associated with the hierarchy of supersymmetric shape-invariant Hamiltonians. For a nonnegative-definite tridiagonal Hamiltonian \( H(\eta) \) (with the ground state energy \( \varepsilon_0(\eta) = 0 \)), the coefficients \( c_n \) and \( d_n \) are related to the coefficients \( a_n \) and \( b_n \) by Eq. (1.3). We ask what must the coefficients \( c^{(+)}_n \) and \( d^{(+)}_n \) of the partner Hamiltonian \( H^{(+)}(\eta) \) be for us to write the analogous relations

\[
a^{(+)}_n = (c^{(+)}_n)^2 + (d^{(+)}_n)^2, \quad b^{(+)}_n = c^{(+)}_n d^{(+)}_{n+1}.
\]  
(2.21)

We already know that

\[
a^{(+)}_n = c^2_n + d^2_{n+1}, \quad b^{(+)}_n = c_{n+1} d_{n+1},
\]  
(2.22)

and we hence derive the answer obtained in [12]: \( c^{(+)}_n = d_{n+1} \) and \( d^{(+)}_n = c_n \). This is a general result. If we now ask the same question regarding the supersymmetric partner \( H^{(+)}(\eta) \) in the next level of the hierarchy, we must be careful in seeking the supersymmetric partner of the Hamiltonian

\[
\tilde{H}^{(+)}(\eta) = H^{(+)}(\eta) - \varepsilon_1(\eta),
\]  
(2.23)

which has the ground state \( \varepsilon^{(+)}_0 = 0 \). General speaking, we do not know the explicit form of the energy \( \varepsilon_1(\eta) \), but \( \varepsilon_1(\eta) \) is given by Eq. (2.4) for shape-invariant Hamiltonians. We show that this allows finding the explicit form of the sought coefficients for the hierarchy of supersymmetric partner Hamiltonians. For this, we note that Eq. (2.23) yields

\[
\tilde{a}^{(+)}_n(\eta) = a^{(+)}_n(\eta) - \varepsilon_1(\eta) = (c^2_n(\eta) + d^2_{n+1}) - (c^2_n(\eta) - c^2_{n+1}(\eta) + (d^2_{n+1} - d^2_n)) = c^2_{n+1}(\eta) + d^2_n,
\]  
(2.24)

\[
\tilde{b}^{(+)}_n(\eta) = b^{(+)}_n(\eta) = c_{n+1}(\eta) d_{n+1}.
\]

It is easy to find that \( \tilde{c}^{(+)}_n \) and \( \tilde{d}^{(+)}_n \) are just \( c_{n+1} \) and \( d_n \). We can now write the explicit coefficients \( c^{(+)}_n \) and \( d^{(+)}_n \) for the supersymmetric partner \( H^{(+)}(\eta) \) of the Hamiltonian \( \tilde{H}^{(+)}(\eta) \), namely, \( c^{(+)}_n = d_{n+1} \) and \( d^{(+)}_n = c_{n+1} \). This immediately leads to the explicit forms of \( a^{(+)}_n \) and \( b^{(+)}_n \): \( a^{(+)}_n = c^2_{n+1} + d^2_{n+1} \) and \( b^{(+)}_n = c_{n+2} d_{n+1} \). Based on shape-invariance, we now verify that

\[
H^{(+)}(\eta) = \tilde{H}^{(+)}(\eta + \delta) + \tilde{R}(\eta), \quad \tilde{R}(\eta) = \varepsilon_2 - \varepsilon_1.
\]  
(2.25)

For this, we first note that \( b^{(+)}_n = \tilde{b}^{(+)}_n(\eta + \delta) \). Further,

\[
\tilde{R}(\eta) = a^{(+)}_n(\eta) - \tilde{a}^{(+)}_n(\eta + \delta) = d^2_{n+1} - d^2_n - [c^2_{n+2}(\eta) - c^2_{n+1}(\eta)] = \varepsilon_1(\eta + \delta),
\]

which exactly coincides with the difference \( \varepsilon_2 - \varepsilon_1 \) obtained from relation (2.15).
If we want to continue the procedure and find the analogous results for the next Hamiltonian in the hierarchy, then we must first consider the intermediary Hamiltonian with zero energy of the ground state and then repeat the procedure described in detail above. The results are summarized in Table 2.

### Table 2

| The Hamiltonian | Ground state energy | $c_n$ | $d_n$ | $a_n$ | $b_n$ |
|----------------|---------------------|-------|-------|-------|-------|
| $H$            | 0                   | $c_n$ | $d_n$ | $c_n^2 + d_n^2$ | $c_n d_{n+1}$ |
| $H^{(+)}$      | $\epsilon_1(\eta)$ | $d_{n+1}$ | $c_n$ | $c_{n+1}^2 + d_{n+1}^2$ | $c_{n+1} d_{n+1}$ |
| $\tilde{H}^{(+)}$ | 0                   | $c_{n+1}$ | $d_n$ | $c_{n+1}^2 + d_n^2$ | $c_{n+1} d_{n+1}$ |
| $H^{(++)}$     | $\epsilon_1(\eta + \delta)$ | $d_{n+1}$ | $c_{n+1}$ | $c_{n+1}^2 + d_{n+1}^2$ | $c_{n+1} d_{n+1}$ |
| $\tilde{H}^{(++)}$ | 0                   | $c_{n+2}$ | $d_n$ | $c_{n+2}^2 + d_n^2$ | $c_{n+2} d_{n+1}$ |
| $H^{(++++)}$   | $\epsilon_1(\eta + 2\delta)$ | $d_{n+1}$ | $c_{n+2}$ | $c_{n+2}^2 + d_{n+1}^2$ | $c_{n+3} d_{n+1}$ |
| $\vdots$       | $\vdots$            | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\tilde{H}^{k(+)}$ | 0                   | $c_{n+k}$ | $d_n$ | $c_{n+k}^2 + d_n^2$ | $c_{n+k} d_{n+1}$ |
| $H^{(k+1)(+)}$ | $\epsilon_1(\eta + k\delta)$ | $d_{n+1}$ | $c_{n+k}$ | $c_{n+k}^2 + d_{n+1}^2$ | $c_{n+k} d_{n+1}$ |

Basic parameters for the hierarchy of supersymmetric partner Hamiltonians

$(\tilde{H}^{(+)} = H^{(+)} - \epsilon_1(\eta), \tilde{H}^{(++)} = H^{(++)} - \epsilon_1(\eta + \delta), \text{ etc.})$.

#### 2.5. Constructing $(c_n, d_n)$ from the spectrum of a shape-invariant Hamiltonian.

Above, we showed how the energy spectrum of a shape-invariant Hamiltonian is derived from the known parameters $c_n$ and $d_n$. Here, we show the reverse: if the energy spectrum of a shape-invariant Hamiltonian is known, then we can recreate a tridiagonal representation of the shape-invariant Hamiltonian taking minimal additional information into account.

From the properties of the hierarchy of shape-invariant Hamiltonians considered in detail in the preceding subsection, we can see that the relation between the set $(c_n, d_n)$ and $\epsilon_1(\eta)$ is analogous to the relation between the set $(c_{n+k}, d_n)$ and $\epsilon_1(\eta + k\delta)$. Therefore, we can write a relation analogous to (2.5),

$$\epsilon_1(\eta + k\delta) = [c_{k}^2(\eta)] - [c_{k+1}^2] + d_k^2.$$  \hspace{1cm} (2.26)

We compare this with (2.4) for $n = k$. We have

$$\epsilon_1(\eta) - \epsilon_1(\eta + k\delta) = [d_{k+1}^2 - d_k^2] - d_1^2.$$  \hspace{1cm} (2.27)

We sum each side of this equation from $k = 0$ to $k = m$ and obtain

$$(m + 1)\epsilon_1(\eta) - \sum_{k=0}^{m} \epsilon_1(\eta + k\delta) = \sum_{k=0}^{m} [d_{k+1}^2 - d_k^2] - (m + 1)d_1^2.$$  \hspace{1cm} (2.28)

Canceling terms in the summation in the right-hand side, we obtain

$$(m + 1)\epsilon_1(\eta) - \epsilon_{m+1}(\eta) = d_{m+1}^2 - (m + 1)d_1^2.$$  \hspace{1cm} (2.29)

We thus obtain an important result: $d_{m+1}$ is determined by the energy spectrum of the Hamiltonian and the parameter $d_1$. More specifically,

$$d_{m+1}^2 = (m + 1)d_1^2 + [(m + 1)\epsilon_1(\eta) - \epsilon_{m+1}(\eta)].$$  \hspace{1cm} (2.30)
Moreover, from (2.18), we know that
\[ c_{m+1}^2(\eta) = c_0^2(\eta) + [(m + 1)d_1 - \varepsilon_{m+1}(\eta)]. \]
(2.31)

From the last two relations, we can find \( a_n \) and \( b_n \) using (1.2) and hence construct a tridiagonal matrix representation of the shape-invariant Hamiltonian \( H(\eta) \) from the details of its energy spectrum and the two parameters \( c_0^2(\eta) \) and \( d_1^2 \).

**2.6. The superpotential for a shape-invariant tridiagonal Hamiltonian.** In the framework of the “differential” approach to supersymmetry [9], a quantity \( W(x) \), called the superpotential, is introduced using a relation reflecting its close connection with the ground state wave function \( \psi_0(x) \) of the quantum system:
\[ W(x) = -\frac{1}{\sqrt{2\psi_0(x)}} \frac{d\psi_0(x)}{dx}. \]
(2.32)

Although our interpretation of the supersymmetry of tridiagonal Hamiltonians does not require the superpotential concept, we still feel the need to show that there exists an analogous quantity with the necessary property of a superpotential. For this, we start from the basic operators \( A \) and \( A^\dagger \) and their representation in some basis and then define two operators \( \tilde{W} \) and \( \tilde{D} \) as
\[ \tilde{W} = \frac{1}{2}(A + A^\dagger), \quad \tilde{D} = \frac{1}{2}(A - A^\dagger). \]
(2.33)

Conversely, we have
\[ A = \tilde{W} + \tilde{D}, \quad A^\dagger = \tilde{W} - \tilde{D}. \]
(2.34)

We note that \( \tilde{W} \) is Hermitian and \( \tilde{D} \) is anti-Hermitian. We also have
\[
H = A^\dagger A = (\tilde{W} - \tilde{D})(\tilde{W} + \tilde{D}) = \tilde{W}^2 - \tilde{D}\tilde{W} + \tilde{W}\tilde{D} - \tilde{D}^2 \equiv H_0 + V,
\]
\[
H^{(+)} = AA^\dagger = (\tilde{W} + \tilde{D})(\tilde{W} - \tilde{D}) = \tilde{W}^2 + \tilde{D}\tilde{W} - \tilde{W}\tilde{D} - \tilde{D}^2 \equiv H_0 + V^{(+)}.
\]

Comparing the form of the Hamiltonian and its supersymmetric partner, we can identify the correspondences
\[ H_0 \mapsto -\tilde{D}^2 = \tilde{D}^\dagger \tilde{D}, \quad V \mapsto \tilde{W}^2 - \tilde{D}\tilde{W} + \tilde{W}\tilde{D}, \quad V^{(+)} \mapsto \tilde{W}^2 + \tilde{D}\tilde{W} - \tilde{W}\tilde{D}, \]
which lead to the correspondence
\[ \tilde{D} \mapsto \frac{1}{\sqrt{2}} \frac{d}{dx}. \]
(2.35)

Comparing relations (2.34) with the analogous relations in “differential” supersymmetry
\[ A = W + \frac{1}{\sqrt{2}} \frac{d}{dx}, \quad A^\dagger = W - \frac{1}{\sqrt{2}} \frac{d}{dx}, \]
(2.36)

we immediately obtain the correspondence \( \tilde{W} \mapsto W \).

We now present some arguments to support the interpretation of \( \tilde{W} \) as the superpotential. From the definition of \( \tilde{W} \) in (2.33), we can write
\[ \tilde{W}|\varepsilon_0\rangle = \frac{1}{2}(A + A^\dagger)|\varepsilon_0\rangle = \frac{1}{2}A^\dagger|\varepsilon_0\rangle \]
(2.37)
because the operator \( A \) annihilates the ground state \( |\varepsilon_0\rangle \) (see Appendix C). This is equivalent to

\[
\tilde{W}|\varepsilon_0\rangle = \frac{1}{2}(A^\dagger - A)|\varepsilon_0\rangle = -\tilde{D}|\varepsilon_0\rangle.
\]

(2.38)

More clearly, if \( \tilde{W} \) is a local function of \( x \), then

\[
\tilde{W}(x)|\varepsilon_0\rangle = -\langle x|\tilde{D}|\varepsilon_0\rangle.
\]

(2.39)

Taking into account that \( D \) has differential form (2.35), we see that expression (2.39) corresponds exactly to (2.32).

In addition, we show that \( \tilde{W} \) is related to the potentials \( V \equiv V^{(-)} \) and \( V^{(+)} \) just as in the “differential” variant of supersymmetry. We start from our correspondence \( V^{(\pm)} \leftrightarrow \tilde{W}^2 \pm \tilde{D}\tilde{W} = \tilde{W}\tilde{D} \). Then for any state \( |\varphi\rangle \),

\[
V^{(\pm)}(x)^{(\pm)}(x)|\varphi(x)\rangle = \langle x|V^{(\pm)}|\varphi\rangle \leftrightarrow \langle x|\tilde{W}^2 \pm \tilde{D}\tilde{W} = \tilde{W}\tilde{D} |\varphi\rangle,
\]

(2.40)

where

\[
\langle x|\tilde{W}^2 \pm \tilde{D}\tilde{W} = \tilde{W}\tilde{D} |\varphi\rangle = \tilde{W}^2(x)|\varphi(x)\rangle \pm \frac{1}{\sqrt{2}} \frac{d}{dx} \tilde{W}(x)\varphi(x)] \mp \tilde{W}(x) \frac{1}{\sqrt{2}} \frac{d\varphi(x)}{dx} = \tilde{W}^2(x)|\varphi(x)\rangle + \frac{1}{\sqrt{2}} \frac{d\tilde{W}(x)}{dx} \varphi(x).
\]

(2.41)

Hence,

\[
V^{(\pm)}(x) = \tilde{W}^2(x) \pm \frac{1}{\sqrt{2}} \frac{d\tilde{W}(x)}{dx}.
\]

(2.42)

This result can be found in the literature if \( \tilde{W} \) is understood as the superpotential.

We apply the obtained results to the Morse Hamiltonian. Its ground state has the explicit form

\[
|\psi_0\rangle = \sqrt{\frac{\alpha}{\Gamma(2D)}} e^{-\zeta/2} e^{-\alpha x}, \quad \zeta = \zeta(x) = e^{-\alpha x}.
\]

(2.43)

Using (2.32), we hence obtain the known result that the superpotential is

\[
W(x) = \frac{\alpha}{2\sqrt{2}} (2D - \zeta(x)).
\]

(2.44)

On the other hand, Eq. (2.39) states that

\[
\tilde{W}(x)|z|\varepsilon_0\rangle = -\langle x|\tilde{D}|\varepsilon_0\rangle = -\frac{1}{2}(A - A^\dagger)|\varepsilon_0\rangle.
\]

As shown in Appendix D, \( |\varepsilon_0\rangle \) can be represented in the form

\[
|\varepsilon_0\rangle = \Lambda_0(0) \sum_{n=0}^{\infty} Q_n(0)|\phi_n\rangle,
\]

(2.45)

where

\[
\Lambda_0(0) = \frac{\Gamma(\gamma + 1/2 + D)}{\sqrt{\Gamma(2\gamma + 1)\Gamma(2D)}} \quad Q_n(0) = \frac{(\gamma + 1/2 - D)_n}{\sqrt{n!} (2\gamma + 1)_n}.
\]

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Hence,
\[
\tilde{W}(x|\varepsilon_0) = -\frac{1}{2} L_0(0) \sum_{n=0}^{\infty} Q_n(0)[d_n(x|\phi_{n-1}) - d_{n+1}(x|\phi_{n+1})].
\]  
(2.46)

A careful calculation gives
\[
\tilde{W}(x|\psi_0(x)) = -\frac{\alpha \sqrt{\alpha L_0(0)}}{2 \sqrt{2}\Gamma(2\gamma + 1)} \zeta^{\gamma+1/2} e^{-\zeta/2} \left[ \zeta - (2\gamma + 1) - 2\zeta \frac{d}{d\zeta} \right] F(\zeta),
\]  
(2.47)

where
\[
F(\zeta) = \sum_{n=0}^{\infty} Q_n(0) L_n^{2\gamma}(\zeta) = \zeta^{D-(\gamma+1/2)} \frac{\Gamma(2\gamma + 1)}{\Gamma(\gamma + 1/2 + D)}.
\]

Taking
\[
\psi_0(x) = \frac{\alpha}{2 \sqrt{2}} \sqrt{\alpha L_0(0)} \zeta^{D} e^{-\zeta/2}
\]
into account, we now obtain the final expression identical to (2.43):
\[
\tilde{W} = \frac{\alpha}{2 \sqrt{2}} (2D - \zeta).  
\]  
(2.48)

3. Coherent states associated with shape-invariant tridiagonal Hamiltonians

We construct the coherent state \(|z\rangle\) by displacing the ground state \(|\varepsilon_0\rangle\):
\[
|z\rangle = e^{z B^\dagger - z^* B} |\varepsilon_0\rangle.  
\]  
(3.1)

We can simplify the exponential operator using the well-known factorization [13]
\[
e^{z B^\dagger - z^* B} = e^{-\frac{1}{2} z^* z (B^\dagger B)} e^{z B^\dagger} e^{-z^* B}.  
\]  
(3.2)

Using commutation relation (2.8), we can represent the coherent state in terms of the energy eigenstates of the Hamiltonian as
\[
|z\rangle = e^{-\frac{1}{2} z^* z (\varepsilon_1(\eta - \delta))} \sum_{n=0}^{\infty} \frac{z^n}{n!} (B^\dagger)^n |\varepsilon_0\rangle.  
\]  
(3.3)

In Sec. 2.2, we already studied how the raising operator acts on various eigenstates of the system. It follows from expression (2.9) that the coherent state can be written as
\[
|z\rangle = e^{-\frac{1}{2} z^* z (\varepsilon_1(\eta - \delta))} \sum_{n=0}^{\infty} \frac{z^n}{n!} \left( \prod_{j=0}^{n-1} \left( \sum_{k=j}^{n-1} \varepsilon_1(\eta + k\delta) \right) \right) |\varepsilon_n\rangle.  
\]  
(3.4)

We now show that these coherent states satisfy the resolution of the identity operator in the quantum Hilbert space \(\mathcal{H}\) on which the Hamiltonian acts:
\[
1_{\mathcal{H}} = \int_C |z\rangle\langle z| \varrho(|z|^2) \frac{dz}{\pi},  
\]  
(3.5)

where \(\varrho(|z|^2)\) is an auxiliary density function to be determined. Replacing \(|z\rangle\) in the rank-1 operator \(|z\rangle\langle z|\) with expression (3.4) and using the polar coordinates \(z = re^{i\theta}\), \(r > 0\), \(\theta \in [0, 2\pi]\), we bring (3.5) to the form
\[
1_{\mathcal{H}} = \int_0^\infty e^{-r^2 \varepsilon_1(\eta - \delta)} \prod_{j=0}^{n-1} \left( \sum_{k=j}^{n-1} \varepsilon_1(\eta + k\delta) \right) \varrho(r^2) dr.  
\]  
(3.6)
More explicitly, we require that the function $\varrho(r^2)$ solve the equation

$$\int_0^{\infty} e^{-r^2 \varepsilon_1(\eta - \delta)} r^{2n+1} \varrho(r^2) \, dr = \frac{(n!)^2}{\prod_{j=0}^{n-1} \left( \sum_{k=j}^{n-1} \varepsilon_1(\eta + k\delta) \right)}$$

(3.7)

which is equivalent to the moment problem

$$\int_0^{\infty} x^n e^{-x \varepsilon_1(\eta - \delta)} \varrho(x) \, dx = \frac{2(n!)^2}{\prod_{j=0}^{n-1} \left( \sum_{k=j}^{n-1} \varepsilon_1(\eta + k\delta) \right)}$$

(3.8)

for the function $x \mapsto e^{-x \varepsilon_1(\eta - \delta)} \varrho(x)$. This equation can be solved using some known integral or by way of a transformation of the type of Mellin [14] or Fourier.

As an example, we consider the harmonic oscillator Hamiltonian, for which $\varepsilon_1(\eta) = 2\omega$. Therefore, the quantity in the left-hand side of (3.8) reduces to

$$\int_0^{\infty} x^n e^{-2\omega x} \varrho(x) \, dx = 2 \frac{n!}{(2\omega)^n} \left( \frac{\alpha^2}{2} \right)^n \Gamma^{2D-n+1} \Gamma^{2D-2n+1}.$$
We give two examples of such a construction. The first example is the harmonic oscillator Hamiltonian. Using the quantities listed in Table 1, we easily find that

\[ Q_n(0) = \prod_{j=0}^{n-1} \left( \frac{-c_j}{d_{j+1}} \right) = \left( \frac{\omega - \lambda^2}{\omega + \lambda^2} \right)^n \sqrt{\frac{\Gamma(n + \nu + 1)}{n! \Gamma(n + 1)}}. \]  

(3.15)

Using the basis in Table 1 associated with the Hamiltonian, we obtain

\[ \psi_0(r) = \Lambda_0(0) x^{l+1} e^{-x^2/2} \sqrt{\frac{2 \lambda}{\Gamma(n + 1)}} \sum_{n=0}^{\infty} t^n L_n^{(\nu)}(x^2), \quad t = \frac{\omega - \lambda^2}{\omega + \lambda^2}, \quad x = \lambda r. \]  

(3.16)

As is known, under the condition \(|t| < 1\), which is satisfied in this case, we can write the sum of the series

\[ \sum_{n=0}^{\infty} t^n L_n^{(\nu)}(u) = (1 - t)^{-\nu - 1} e^{ut/(t-1)}. \]  

(3.17)

Hence,

\[ \psi_0(r) = \Lambda_0(0) \sqrt{\frac{2 \lambda}{\Gamma(n + 1)}} \left( \frac{\omega + \lambda^2}{2 \lambda^2} \right)^{\nu + 1} (\lambda r)^{l+1} e^{-\omega r^2/2}. \]  

(3.18)

On the other hand,

\[ \frac{1}{(\Lambda_0(0))^2} = \sum_{n=0}^{\infty} |Q_n(0)|^2 = \sum_{n=0}^{\infty} L_n^{(\nu)}(0) (t^2)^n = (1 - t^2)^{-\nu - 1} = \left( \frac{4 \lambda^2 \omega}{(\omega + \lambda^2)^2} \right)^{-\nu - 1}. \]  

(3.19)

As a final result, we thus obtain

\[ \psi_0(r) = \sqrt{\frac{2 \sqrt{\omega}}{\Gamma(n + 1)}} (\sqrt{\omega} r)^{l+1} e^{-\omega r^2/2}, \]  

(3.20)

which is indeed the ground state wave function. It is important that the wave function, as expected, is independent of the free scale parameter \(\lambda\), which characterizes the basis and not the physical system.

The second example is the Morse Hamiltonian. The details of constructing its ground state are presented in Appendix D.

### 4. Concluding remarks

We emphasize several important points regarding the supersymmetry properties of shape-invariant tridiagonal Hamiltonians.

First, the basic quantities in our approach are the parameters \((c_n, d_n)\) which are related to the matrix elements of the tridiagonal Hamiltonian by formulas (1.3) and (1.4). Second, for Hamiltonians with shape-invariance, an important derived quantity is the energy \(\varepsilon_1(\eta)\) of the first excited state, which plays a key role in the complete description of the energy spectra of the Hamiltonian and its supersymmetric partner. This quantity is completely determined by the three parameters \(c_0(\eta), c_1(\eta),\) and \(d_1\). Third, the approach adopted here accommodates familiar concepts such as the superpotential, although they play no role in the analysis. Finally, there is an obvious need to establish a class as wide as possible of Hamiltonians with tridiagonal matrix representations in a specific basis. The efforts of Alhaidari et al. [15] in this regard is commendable.
Appendix A

We prove relation (2.9) by induction. For \( m = 0 \), we consider the vector \( B^{(+)}|\varepsilon_0\rangle \). Using (2.8), we obtain

\[
(B^{(+)}B)[B^{(+)}|\varepsilon_0\rangle] = B^{(+)}(BB^{(+)})(|\varepsilon_0\rangle) = B^{(+)}[B^{(+)}B + \varepsilon_1(\eta - \delta)]|\varepsilon_0\rangle = \\
= B^{(+)}\varepsilon_1(\eta - \delta)|\varepsilon_0\rangle = \varepsilon_1(\eta)[B^{(+)}|\varepsilon_0\rangle]
\]

(A.1)

because \( B^{(+)}|\varepsilon_0\rangle = A^{(+)}A|\varepsilon_0\rangle = H|\varepsilon_0\rangle = 0 \). This means that \([B^{(+)}|\varepsilon_0\rangle]\) is proportional to the eigenvector \(|\varepsilon_1\rangle\) with the eigenvalue \( \varepsilon_1(\eta) \). The normalized version of this vector is

\[
|\varepsilon_1(\eta)\rangle = \frac{1}{\sqrt{\varepsilon_1(\eta)}}B^{(+)}|\varepsilon_0\rangle.
\]

(A.2)

To see this, we note that the last equality gives

\[
\langle \varepsilon_1(\eta)|\varepsilon_1(\eta)\rangle = \langle \varepsilon_0\bigg| \frac{B}{\sqrt{\varepsilon_1(\eta)}} \frac{1}{\sqrt{\varepsilon_1(\eta)}} B^{(+)} \bigg| \varepsilon_0 \rangle = \langle \varepsilon_0\bigg| \frac{B}{\varepsilon_1(\eta)} B^{(+)} \bigg| \varepsilon_0 \rangle = \\
= \langle \varepsilon_0\bigg| B^{(+)} \frac{1}{\varepsilon_1(\eta - \delta)} \bigg| \varepsilon_0 \rangle = \langle \varepsilon_0\bigg| B^{(+)}B + \varepsilon_1(\eta - \delta) \frac{1}{\varepsilon_1(\eta - \delta)} \bigg| \varepsilon_0 \rangle = \\
= 1 + \langle \varepsilon_0\bigg| A^{(+)}A \frac{1}{\varepsilon_1(\eta - \delta)} \bigg| \varepsilon_0 \rangle = 1
\]

(A.3)

because \( A \) annihilates the ground state.

We now assume that formula (2.9) holds for \( m \leq n \) and consider its right-hand side for \( m + 1 \). Using the induction hypothesis, we obtain

\[
\frac{1}{\sqrt{\sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta)}} B^{(+)} \frac{1}{\sqrt{\sum_{k=0}^{n} \varepsilon_1(\eta + k\delta)}} \times \ldots \\
\ldots \times \frac{1}{\sqrt{\varepsilon_1(\eta) + \varepsilon_1(\eta + \delta)}} B^{(+)} \frac{1}{\sqrt{\varepsilon_1(\eta)}} B^{(+)}|\varepsilon_0\rangle = \frac{1}{\sqrt{\sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta)}} B^{(+)}|\varepsilon_{n+1}\rangle.
\]

(A.4)

We now show that the vector in the right-hand side is an eigenvector of the Hamiltonian with the eigenvalue \( \varepsilon_{n+2} \). We have

\[
H \left[ \frac{1}{\sqrt{\sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta)}} B^{(+)}|\varepsilon_{n+1}\rangle \right] = \frac{1}{\sqrt{\sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta)}} (B^{(+)}B)|\varepsilon_{n+1}\rangle = \\
= \frac{1}{\sqrt{\sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta)}} B^{(+)}[B^{(+)}B + \varepsilon_1(\eta - \delta)]|\varepsilon_{n+1}\rangle = \\
= \frac{1}{\sqrt{\sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta)}} B^{(+)}[\varepsilon_{n+1}(\eta) + \varepsilon_1(\eta - \delta)]|\varepsilon_{n+1}\rangle.
\]

But in this case,

\[
\varepsilon_{n+1}(\eta) + \varepsilon_1(\eta - \delta) = \sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta) + \varepsilon_1(\eta - \delta) = \sum_{k=-1}^{n+1} \varepsilon_1(\eta + k\delta).
\]

(A.5)
Therefore,

\[
H \left[ \frac{1}{\sqrt{\sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta)}} B^+(\varepsilon_{n+1}) \right] = \frac{1}{\sqrt{\sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta)}} B^+(\sum_{k=-1}^{n+1} \varepsilon_1(\eta + k\delta)) \varepsilon_{n+1} = \\
\frac{\sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta)}{\sqrt{\sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta)}} B^+(\varepsilon_{n+1}) = \\
= \frac{\varepsilon_{n+2}}{\sqrt{\sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta)}} B^+(\varepsilon_{n+1}). \tag{A.6}
\]

It is easy to show that the vector \( \left[ \sqrt{\sum_{k=0}^{n+1} \varepsilon_1(\eta + k\delta)} \right]^{-1} B^+(\varepsilon_{n+1}) \) is indeed normalized. This completes the proof of relation (2.9).

Because equality (2.15) holds, we can bring expressions (2.9) and (2.10) to the simplified form

\[
|\varepsilon_{m+1}(\eta)\rangle = \frac{1}{\sqrt{\varepsilon_{m+1}(\eta)}} B^+ \frac{1}{\sqrt{\varepsilon_{m}(\eta)}} B^+ \cdots \frac{1}{\sqrt{\varepsilon_{1}(\eta)}} |\varepsilon_{0}(\eta)\rangle \tag{A.7}
\]

or

\[
|\varepsilon_{m+1}(\eta)\rangle = \frac{1}{\sqrt{\varepsilon_{m+1}(\eta)}} B^+ |\varepsilon_{m}(\eta)\rangle. \tag{A.8}
\]

Appendix B

In this appendix, we show that for the tridiagonal harmonic oscillator Hamiltonian, \( B = TA \) (where \( T = e^{-\delta \partial^2/\partial \eta^2} \)) is a lowering operator associated with the harmonic oscillator. We recall that the signature property consists in this operator acting on energy eigenstates as

\[
B |\varepsilon_m\rangle = \sqrt{\varepsilon_m} |\varepsilon_{m-1}\rangle. \tag{B.1}
\]

In particular, it annihilates the ground state because \( \varepsilon_0 = 0 \). In fact, because \( \varepsilon_m = 2m\omega \), for the harmonic oscillator, we have

\[
B |\varepsilon_m\rangle = \sqrt{2m\omega} |\varepsilon_{m-1}\rangle. \tag{B.2}
\]

We recall that the basis in which the harmonic oscillator is tridiagonal is given by (see Table 1)

\[
\phi_n(r) = \langle r | \phi_n \rangle = \sqrt{\frac{2\lambda^m}{\Gamma(n + \nu + 1)}} (\lambda r)^{l+1} e^{-\lambda^2 r^2/2} L_l^{(\nu)}(\lambda^2 r^2), \quad \nu \equiv l + \frac{1}{2}. \tag{B.3}
\]

On the other hand, the \( m \)th energy eigenstate \( |\varepsilon_m\rangle \) has the explicit form

\[
\chi_m(r) = \langle r | \varepsilon_m \rangle = \sqrt{\frac{2\sqrt{\omega} m!}{\Gamma(m + \nu + 1)}} (\sqrt{\omega} r)^{l+1} e^{-\omega r^2/2} L_l^{(\nu)}(\omega r^2). \tag{B.4}
\]

We expand the eigenstate in the basis,

\[
|\varepsilon_m\rangle = \sum_{n=0}^{\infty} |\phi_n\rangle \Gamma_{n,m}, \tag{B.5}
\]
and then
\[ \Gamma_{n,m} = \langle \phi_n | \varepsilon_m \rangle = \int_0^\infty \phi_n(r) \chi_m(r) \, dr. \]

Carefully calculating this integral yields the explicit result
\[
\Gamma_{n,m} = \sqrt{\frac{4 \lambda \sqrt{\omega} \, n! \, m!}{\Gamma(n + \nu + 1) \Gamma(m + \nu + 1)}} \times \frac{(\lambda \sqrt{\omega})^{n+1} \Gamma(n + m + \nu + 1)}{2 \, n! \, m!} \times (-1)^m \frac{(m/2 - \lambda^2/2)^{n+m}}{(m/2 + \lambda^2/2)^{n+m+\nu+1}} \cdot 2F_1 \left( -m, -(m + \nu); -(n + m + \nu); \left( \frac{\lambda^2 + \omega}{\lambda^2 - \omega} \right) \right). \tag{B.6}
\]

Taking \( B = TA \) (\( T = e^{-\delta \partial/\partial \eta} \)) and the expansion of the eigenstate in the basis, we obtain
\[
B|\varepsilon_m\rangle = e^{-\delta \partial/\partial \eta} \sum_{n=0}^{\infty} A|\phi_n\rangle \Gamma_{n,m}. \tag{B.7}
\]

Moreover, from (1.5), we know that \( A|\phi_n\rangle = c_n|\phi_n\rangle + d_n|\phi_{n-1}\rangle \), and therefore
\[
B|\varepsilon_m\rangle = e^{-\delta \partial/\partial \eta} \sum_{n=0}^{\infty} [c_n|\phi_n\rangle + d_n|\phi_{n-1}\rangle] \Gamma_{n,m} =
\]
\[
e^{-\delta \partial/\partial \eta} \sum_{n=0}^{\infty} [c_n(\eta) \Gamma_{n,m}(\eta) + d_{n+1}\Gamma_{n+1,m}(\eta)]|\phi_n\rangle. \tag{B.8}
\]

We can now write the quantity in square brackets in the right-hand side explicitly:
\[
c_n(\eta) \Gamma_{n,m}(\eta) + d_{n+1}\Gamma_{n+1,m}(\eta) =
\]
\[
= \sqrt{\frac{2}{\lambda}} \sqrt{\frac{4 \lambda \sqrt{\omega} \, n! \, m!}{\Gamma(n + \nu + 2) \Gamma(m + \nu + 1)}} \frac{(\lambda \sqrt{\omega})^{n+1} \Gamma(n + m + \nu + 1)}{n! \, m!} \times (-1)^m \frac{(m/2 - \lambda^2/2)^{n+m}}{1 \left( m/2 + \lambda^2/2 \right)^{n+m+\nu+1}} \times
\]
\[
\times \left[ (-(n + m + \nu)) \cdot \cdot \cdot \frac{\tau}{\tau - 1} \right] +
\]
\[
+ (n + m + \nu + 1) \cdot \cdot \cdot \frac{\tau}{\tau - 1} \right],
\]
where we introduce the notation \( \tau = (\lambda^2 + \omega)/(\lambda^2 - \omega) \) for brevity and apply the known equality [16]
\[
2F_1(a, b; c; z) = (1 - z)^{-a} \cdot \cdot \cdot \frac{z}{z - 1}. \tag{B.9}
\]

We take one more known equality [16]
\[
(1 - c) \cdot \cdot \cdot \frac{c - 1}{c - 1} \cdot \cdot \cdot \frac{c - a - 1}{c - a - 1} \cdot \cdot \cdot \frac{a_2 F_1(a, b; c; z) + a_2 F_1(a + 1, b; c; z)}{0} = \tag{B.10}
\]
and substitute the parameters \( a = -(n + m), b = -(m + \nu), \) and \( c = -(n + m + \nu) \) in it. We obtain
\[
c_n(\eta) \Gamma_{n,m}(\eta) + d_{n+1}\Gamma_{n+1,m}(\eta) = \sqrt{2m\omega} \Gamma_{n,m-1}(\eta + 1).
\]

With \( \delta = 1 \) and \( \eta \equiv \nu \), expression (B.8) becomes
\[
B|\varepsilon_m\rangle = \sqrt{2m\omega} \sum_{n=0}^{\infty} e^{-\delta \partial/\partial \eta} \Gamma_{n,m-1}(\eta + 1)|\phi_n\rangle = \sqrt{2m\omega}|\varepsilon_{m-1}\rangle = \sqrt{\varepsilon_{m-1}} |\varepsilon_{m-1}\rangle. \tag{B.11}
\]

This shows that the operator \( B = TA \) is indeed the lowering operator for the harmonic oscillator Hamiltonian.

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Appendix C

We verify that $B|\varepsilon_0\rangle = 0$. We use expression (1.10) for the ground state. The operator $A$ then acts on this state as

$$A|\varepsilon_0\rangle = \sqrt{\Omega(\varepsilon_0)} \sum_{n=0}^{\infty} P_n(\varepsilon_0)(c_n|\phi_n\rangle + d_n|\phi_{n-1}\rangle) =$$

$$= \sqrt{\Omega(\varepsilon_0)} \sum_{n=0}^{\infty} (P_n(\varepsilon_0)c_n + P_{n+1}(\varepsilon_0)d_{n+1})|\phi_n\rangle. \quad (C.1)$$

It follows from the relations

$$c_n^2 = -b_n \frac{P_{n+1}(\varepsilon_0)}{P_n(\varepsilon_0)}, \quad d_{n+1}^2 = -b_n \frac{P_n(\varepsilon_0)}{P_{n+1}(\varepsilon_0)}, \quad (C.2)$$

that $A|\varepsilon_0\rangle = 0$. We recall that $B = T_A$, whence it follows that $B|\varepsilon_0\rangle = 0$.

Appendix D

A general coherent state $|z\rangle$ satisfying the equation $A|z\rangle = z|z\rangle$ has an expansion $|z\rangle = \sum_{n=0}^{\infty} \Lambda_n(z)|\phi_n\rangle$ in the basis. Acting on this state with the operator $A$ and taking (1.5) into account, we find that $\Lambda_n(z)$ has the form $\Lambda_n(z) = Q_n(z) \Lambda_0(z)$, where $Q_n(z)$, $n = 0, 1, 2, \ldots$, are defined in (3.13) and $\Lambda_0(z) = \left(\sqrt{\sum_{n=0}^{\infty} |Q_n(z)|^2}\right)^{-1}$ guarantees normalization of the coherent state. The ground state $|\varepsilon_0\rangle$ is a coherent state $|z\rangle$ with $z = 0$. Therefore,

$$|\varepsilon_0\rangle = \Lambda_0(0) \sum_{n=0}^{\infty} Q_n(0)|\phi_n\rangle. \quad (D.1)$$

In this appendix, we consider the calculation of the ground state of the Morse Hamiltonian in detail using the corresponding parameters in Table 1. We easily find that

$$Q_n(0) = \frac{\Gamma(n + \gamma + 1/2 - D)}{\Gamma(\gamma + 1/2 - D)} \sqrt{\frac{\Gamma(2\gamma + 1)}{n! \Gamma(n + 2\gamma + 1)}} = \frac{(\gamma + 1/2 - D)n}{\sqrt{n!} (2\gamma + 1)_n}, \quad n = 0, 1, 2, \ldots. \quad (D.2)$$

In the basis in which the Morse Hamiltonian is tridiagonal, the ground state wave function has the form

$$\psi_0(\zeta) = \Lambda_0(0) \sqrt{\alpha \frac{\Gamma(2\gamma + 1)}{\Gamma(n + 1/2 - D)}} \zeta^{\gamma+1/2} e^{-\zeta^2/2} \sum_{n=0}^{\infty} \frac{\left(\gamma + 1/2 - D\right)_n}{(2\gamma + 1)_n} L_n^{2\gamma}(\zeta). \quad (D.2)$$

On the other hand,

$$\sum_{n=0}^{\infty} \frac{\left(\gamma + 1/2 - D\right)_n}{(2\gamma + 1)_n} L_n^{2\gamma}(\zeta) = \lim_{t \to 1} \sum_{n=0}^{\infty} \frac{\left(\gamma + 1/2 - D\right)_n}{(2\gamma + 1)_n} L_n^{2\gamma}(\zeta)t^n =$$

$$\lim_{t \to 1} (1 - t)^{-\left(\gamma + 1/2 - D\right)} \text{I}_1 F_1 \left(\gamma + \frac{1}{2} - D; 2\gamma + 1; -\frac{t\zeta}{1-t}\right).$$

We now apply the two known relations [16]

$$\text{I}_1 F_1 (a; b; z) = e^z \text{I}_1 F_1 (b - a; b; -z) \quad (D.3)$$
and
\[ \_1F_1(a, c; z) \sim \frac{\Gamma(c)}{\Gamma(a)} e^z z^{a-c} \quad \text{as Re } z \to +\infty. \] (D.4)

As a result, we obtain
\[ \sum_{n=0}^{\infty} \frac{(\gamma + 1/2 - D)n}{(2\gamma + 1)n} L_n^2(\zeta) = \frac{\Gamma(2\gamma + 1)}{\Gamma(\gamma + 1/2 + D)} \zeta^{-(\gamma+1/2)+D}. \] (D.5)

On the other hand,
\[ \sum_{n=0}^{\infty} |Q_n(0)|^2 = \sum_{n=0}^{\infty} \frac{(\gamma + 1/2 - D)n}{n! (2\gamma + 1)n} = \_2F_1\left(\gamma + \frac{1}{2} - D, \gamma + \frac{1}{2} - D; 2\gamma + 1; 1\right). \] (D.6)

Using the Gauss summation formula [16]
\[ \_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad \text{Re}(a + b - c) < 0, \quad c \neq 0, -1, -2, \ldots, \]
we bring (D.6) to the form
\[ \sum_{n=0}^{\infty} |Q_n(0)|^2 = \frac{\Gamma(2\gamma + 1)\Gamma(2D)}{\Gamma(\gamma + 1/2 + D)\Gamma(\gamma + 1/2 + D)}. \] (D.7)

Therefore,
\[ A_0(0) = \frac{1}{\sqrt{\sum_{n=0}^{\infty} |Q_n(0)|^2}} = \frac{\Gamma(\gamma + 1/2 + D)}{\sqrt{\Gamma(2\gamma + 1)\Gamma(2D)}}. \] (D.8)

Combining all the above results, we finally obtain the explicit form of the ground state wave function
\[ \psi_0(\zeta) = \sqrt{\frac{\alpha}{\Gamma(2D)}} \zeta^D e^{-\zeta/2}. \] (D.9)

We again note that the wave function is independent of the free scale parameter \( \gamma \) characterizing the chosen basis.

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