Generally relativistical Tetrode-Weyl-Fock-Ivanenko formalism and behaviour of quantum-mechanical particles of spin $1/2$ in the Abelian monopole field.

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Abstract

Some attention in the literature has been given to the case of a particle of spin $1/2$ on the background of the external monopole potential. Certain aspects of this problem are reexamined here. The primary technical ‘novelty’ is that the tetrad generally relativistic method of Tetrode-Weyl-Fock-Ivanenko for describing a spinor particle is exploited. The choice of the formalism to deal with the monopole-doublet problem has turned out to be of great fruitfulness for examining this system. It is matter that, as known, the use of a special spherical tetrad in the theory of a spin $1/2$ particle had led Schrödinger to a basis of remarkable features. In particular, the following explicit expression for momentum operator components had been calculated

\[ J_1 = l_1 + i\sigma^{12}\cos \phi/\sin \theta, \quad J_2 = l_2 + i\sigma^{12}\sin \phi/\sin \theta, \quad J_3 = l_3. \]

This basis has been used with great efficiency by Pauli in his investigation on the problem of allowed spherically symmetric wave functions in quantum mechanics. For our purposes, just several simple rules extracted from the much more comprehensive Pauli’s analysis will be quite sufficient; those are almost mnemonic working regulations. So, one may remember some very primary facts of $D$-functions theory and then produce, almost automatically, proper wave functions. It seems rather likely, that there may exist a generalized analog of such a representation for $J_i$-operators, that might be successfully used whenever in a linear problem there exists a spherical symmetry, irrespective of the concrete embodiment of such a symmetry. In particular, the case of electron in the external Abelian monopole field, together with the problem of selecting the allowed wave functions as well as the Dirac charge quantization condition, completely come under that Schrödinger-Pauli method. In particular, components of the generalized conserved momentum can be expressed as follows

\[ j_{1}^{eg} = l_1 + (i\sigma^{12} - eg)\cos \phi/\sin \theta, \quad j_{2}^{eg} = l_2 + (i\sigma^{12} - eg)\sin \phi/\sin \theta, \quad j_{3}^{eg} = l_3, \]

where $e$ and $g$ are an electrical and magnetic charge, respectively. In accordance with the above regulations, the corresponding electron-monopole wave functions can be constructed like in the purely electron pattern but with a single change

\[ D_{l,m,\pm1/2}(\phi, \theta, 0) \rightarrow D_{l,m,eg\pm1/2}(\phi, \theta, 0). \]
1. Introduction.

While there not exists at present definitive succeeded experiments concerning monopoles, it is nevertheless true that there exists a veritable jungle of literature on the monopole theories. Moreover, properties of more general monopoles, associated with large gauge groups now thought to be relevant in physics. As evidenced even by a cursory examination of some popular surveys (see, for example, [1,2]), the whole monopole area covers and touches quite a variety of fundamental problems. The most outstanding of them are: the electric charge quantization [3-10], $P$-violation in purely electromagnetic processes [11-16], scattering on the Dirac string [17-19], spin from monopole and spin from isospin [20-23], bound states in fermion-monopole system and violation of the Hermiticity property [24-38], fermion-number breaking in the presence of a magnetic monopole and monopole catalysis of baryon decay [39-41].

The tremendous volume of publications on monopole topics (and there is no hint that its raise will stop) attests the interest which they enjoy among theoretical physicists, but the same token, clearly indicates the unsettled and problematical nature of those objects: the puzzle of monopole seems to be one of the still yet unsolved problems of particle physics. In general, there are several ways of approaching the monopole problems. As known, together with geometrically topological way of exploration into them, another approach to studying such configurations is possible; namely, that concerns any physical manifestations of monopoles when they are considered as external potentials. Moreover, from the physical standpoint, this latter method can be thought of as a more visualizable one in comparison with less obvious and more direct topological language.

Some more concrete remarks referring to our further work and designated to delineate its content are to be given. The most attention in the literature has been given to the case of a particle of spin $1/2$ on the background of the external monopole potential: for the Abelian case see, for instance, [42-45]; for the non-Abelian one see [46-54]. For the present work, the Abelian situation only will be treated; on the line developed here a corresponding non Abelian system will be considered in a separate work.

Now, for convenience of the readers, some remarks about the approach and technique used in the work are to be given. The primary technical ‘novelty’ is that, in the paper, the tetrad (generally relativistic) method [55-63] of Tetrode-Weyl-Fock-Ivanenko (TWF1) for describing a spinor particle will be exploited. The choice of the formalism to deal with the monopole-doublet problem has turned out to be of great fruitfulness for examining this system. Taking of just this method is not an accidental step. It is matter that, as known (but seemingly not very vastly), the use of a special spherical tetrad in the theory of a spin $1/2$ particle had led Schrödinger and Pauli [64, 65] to a basis of remarkable features. In particular, the following explicit expression for (spin $1/2$ particle’s) momentum operator components had been calculated

$$J_1 = l_1 + \frac{i\sigma_{12} \cos \phi}{\sin \theta}, \quad J_2 = l_2 + \frac{i\sigma_{12} \sin \phi}{\sin \theta}, \quad J_3 = l_3$$

(1.1)

1Very physicists have contributed to investigation of the monopole-based theories. The wide scope of the field and the prodigious number of investigators associated with various of its developments make it all but hopeless to list even the principal contributors. The present study does not pretend to be a survey in this matter, so I give but a few of the most important references which may be useful to the readers who wish some supplementary material or are interested in more technical developments beyond the scope of the present treatment.
just that kind of structure for $J_i$ typifies this frame in bispinor space. This Schrödinger’s basis had been used with great efficiency by Pauli in his investigation [65] on the problem of allowed spherically symmetric wave functions in quantum mechanics. For our purposes, just several simple rules extracted from the much more comprehensive Pauli’s analysis will be quite sufficient (those are almost mnemonic working regulations). They can be explained on the base of $S = 1/2$ particle case. To this end, using any representation of $\gamma$ matrices where $\sigma^{12} = \frac{1}{2} (\sigma_3 \oplus \sigma_3)$ (throughout the work, the Weyl’s spinor frame is used) and taking into account the explicit form for $\tilde{J}^2, J_3$ according to (1.1), it is readily verified that the most general bispinor functions with fixed quantum numbers $j, m$ are to be (see also in [61])

$$
\Phi_{jm}(t, r, \theta, \phi) = \begin{pmatrix}
    f_1(t, r) D^{j}_{-m,-1/2}(\phi, \theta, 0) \\
    f_2(t, r) D^{j}_{-m,+1/2}(\phi, \theta, 0) \\
    f_3(t, r) D^{j}_{-m,-1/2}(\phi, \theta, 0) \\
    f_4(t, r) D^{j}_{-m,+1/2}(\phi, \theta, 0)
\end{pmatrix}
$$

(1.2)

where $D^{j}_{mm'}$ designates the Wigner’s $D$-functions (the notation and subsequently required formulas according to [66], are adopted). One should take notice of the low right indices $-1/2$ and $+1/2$ of $D$-functions in (1.2), which correlate with the explicit diagonal structure of the matrix $\sigma^{12} = \frac{1}{2} (\sigma_3 \oplus \sigma_3)$. The Pauli criterion allows only half integer values for $j$.

So, one may remember some very primary facts of $D$-functions theory and then produce, almost automatically, proper wave functions. It seems rather likely, that there may exist a generalized analog of such a representation for $J_r$-operators, that might be successfully used whenever in a linear problem there exists a spherical symmetry, irrespective of the concrete embodiment of such a symmetry. In particular, the case of electron in the external Abelian monopole field, together with the problem of selecting the allowed wave functions as well as the Dirac charge quantization condition, completely come under that Schrödinger-Pauli method. In particular, components of the generalized conserved momentum can be expressed as follows (for more detail, see [67])

$$
\begin{align*}
    j^{eg}_1 &= l_1 + \frac{(i\sigma^{12} - eg) \cos \phi}{\sin \theta}, \\
    j^{eg}_2 &= l_2 + \frac{(i\sigma^{12} - eg) \sin \phi}{\sin \theta}, \\
    j^{eg}_3 &= l_3
\end{align*}
$$

(1.3)

where $e$ and $g$ are an electrical and magnetic charge, respectively. In accordance with the above regulations, the corresponding electron-monopole wave functions can be constructed like in the purely electron pattern (1.2) but with a single change

$$
D^{j}_{-m,\pm 1/2}(\phi, \theta, 0) \rightarrow D^{j}_{-m,\pm g \pm 1/2}(\phi, \theta, 0).
$$

(1.4)

The Pauli criterion produces two results: first, $| eg | = 0, 1/2, 1, 3/2, \ldots$ (what is called the Dirac charge quantization condition; second, the quantum number $j$ in (1.4) may take the values $| eg | = -1/2, | eg | = +1/2, | eg | = +3/2, \ldots$ that selects the proper spinor particle-monopole functions.

There exists additional line justified the interest to just the aforementioned approach: the Schrödinger’s tetrad basis and Wigner’s $D$-functions are deeply connected with what is called the formalism of spin-weight harmonics [68-70] developed in the frame of the Newman-Penrose method of light (or isotropic) tetrad. Some relationships between spin-weight and spinor monopole harmonics have already been examined in the literature [71-73], the present work follows the notation used in [67].
2. The Pauli criterion.

Let the $J_i^\lambda$ denote

$$J_1 = (l_1 + \lambda \frac{\cos \phi}{\sin \theta}), \quad J_2 = (l_2 + \lambda \frac{\sin \phi}{\sin \theta}), \quad J_3 = l_3$$ \tag{2.1}

at an arbitrary $\lambda$, as readily verified, those $J_i$ satisfy the commutation rules of the Lie algebra $SU(2)$: $[J_a, J_b] = i \epsilon_{abc} J_c$. As known, all irreducible representations of such an abstract algebra are determined by a set of weights $j = 0, 1/2, 1, 3/2, \ldots$ (dim $j = 2j + 1$). Given the explicit expressions of $J_a$ above, we will find functions $\Phi_{j^m}(\theta, \phi)$ on which the representation of weight $j$ is realized. In agreement with the general approach \[65\], those solutions are to be established by the following relations

$$J_+ \Phi_{j^m} = 0, \quad \Phi_{j^m} = \frac{(j+m)!}{(j-m)! (2j)!} J^{(j-m)} \Phi_{j^m},$$ \tag{2.2}

$$J_\pm = (J_1 \pm iJ_2) = e^{\pm i\phi} \left[ \pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} + \frac{\lambda}{\sin \theta} \right].$$

From the equations $J_+ \Phi_{j^m} = 0$ and $J_3 \Phi_{j^m} = j \Phi_{j^m}$ it follows that

$$\Phi_{j^m} = N_{j^m} e^{i\phi} \sin^j \theta \frac{(1 + \cos \theta)^{j+1}}{(1 - \cos \theta)^{j+1/2}}, \quad N_{j^m} = \frac{1}{\sqrt{2\pi}} \frac{1}{2^j} \sqrt{\frac{(2j+1)(j+m)!}{\Gamma(j+m+1) \Gamma(j-m+1)}}.$$ \tag{2.3}

Further, employing (2.2) we produce the functions $\Phi_{j^m}$

$$\Phi_{j^m} = N_{j^m} e^{i\phi} \sin^m \theta \frac{(1 - \cos \theta)^{j+1/2}}{(1 + \cos \theta)^{j+1/2}} \times \left( \frac{d}{d \cos \theta} \right)^{j-m} [ (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} ] \tag{2.3}$$

where

$$N_{j^m} = \frac{1}{\sqrt{2\pi} 2^j} \sqrt{\frac{(2j+1)(j+m)!}{2(j-m)! \Gamma(j+\lambda+1) \Gamma(j-\lambda+1)}}$$

The Pauli criterion tells us that the $(2j+1)$ functions $\Phi_{j^m}(\theta, \phi), \ m = -j, \ldots, +j$ so constructed are guaranteed to be a basis for a finite-dimension representation, providing that the function $\Phi_{j^m}(\theta, \phi)$ found by this procedure obeys the identity

$$J_- \Phi_{j^m} = 0.$$ \tag{2.4a}

After substituting the function $\Phi_{j^m}(\theta, \phi)$ (in the form given (2.3)) to the (2.4a), the latter reads

$$J_- \Phi_{j^m} = N_{j^m} e^{-i(j+1)\phi} (\sin \theta)^{j+1} \frac{(1 - \cos \theta)^{j+1}}{(1 + \cos \theta)^{j+1/2}} \times \left( \frac{d}{d \cos \theta} \right)^{2j+1} [ (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} ] = 0 \tag{2.4b}$$
which in turn gives the following restriction on \( j \) and \( \lambda \)

\[
\left( \frac{d}{d \cos \theta} \right)^{2j+1} \left[ (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} \right] = 0.
\] (2.4c)

But the relation (2.4c) can be satisfied only if the factor \( P(\theta) \) subjected to the operation of taking derivative \((d/d \cos \theta)^{2j+1}\) is a polynomial of degree \(2j\) in \( \cos \theta \). So, we have (as a result of the Pauli criterion)

1. the \( \lambda \) is allowed to take values \(+1/2, -1/2, +1, -1, \ldots\).

Besides, as the latter condition is satisfied, \( P(\theta) \) takes different forms depending on the \((j - \lambda)\)-correlation:

\[
P(\theta) = (1 + \cos \theta)^{j+\lambda} (1 - \cos \theta)^{j-\lambda} = P^{2j}(\cos \theta), \quad if \quad j = | \lambda |, | \lambda | + 1, ...
\]

or

\[
P(\theta) = \frac{P^{2j+1}(\cos \theta)}{\sin \theta}, \quad if \quad j = | \lambda | + 1/2, | \lambda | + 3/2, ...
\]

so that the second necessary condition resulting from the Pauli criterion is

2. given \( \lambda \) according to 1., the number \( j \) is allowed to take values \( j = | \lambda |, | \lambda | + 1, ... \)

Hereafter, these two conditions: 1 and 2 will be termed as the first and respectively the second Pauli consequences. It should be noted that the angular variable \( \phi \) is not affected (charged) by this Pauli condition; in other words, it is effectively eliminated out of this criterion, but a variable that worked above is the \( \theta \). Significantly, in the contrast to this, the well-known procedure [ ] of deriving the Dirac quantization condition from investigating continuity properties of quantum mechanical wave functions, such a working variable is the \( \phi \).

If the first and second Pauli consequences fail, then we face rather unpleasant mathematical and physical problems\(^2\). As a simple illustration, we may indicate the familiar case when \( \lambda = 0 \); if in those circumstances, the second Pauli condition has failed, then we face the integer and half-integer values of the orbital angular momentum number \( l = 0, 1/2, 1, 3/2, \ldots \). As regards the Dirac electron with the components of the total angular momentum in the form [65]

\[
J_1 = (l_1 + \lambda \frac{\cos \phi}{\sin \theta} \Sigma_3), \quad J_2 = (l_2 + \lambda \frac{\sin \phi}{\sin \theta} \Sigma_3), \quad J_3 = l_3
\]

we have to employ the above Pauli criterion in the constituent form owing to \( \lambda \) changed into \( \Sigma_3 \)

\[
\Sigma_3 = \begin{pmatrix}
+1/2 & 0 & 0 & 0 \\
0 & -1/2 & 0 & 0 \\
0 & 0 & +1/2 & 0 \\
0 & 0 & 0 & -1/2
\end{pmatrix}.
\]

Ultimately, we obtain the allowable set \( J = 1/2, 3/2, \ldots \).

A fact of primary interest to us is that the functions \( \Phi_{jm}^\lambda(\theta, \phi) \) constructed above relate directly to the well-known Wigner \( D \)-functions (bellow we will use the notation according to [66]):

\[
\Phi_{jm}^\lambda(\theta, \phi) = (-1)^{j-m} D_{m,\lambda}^j(\phi, 0)
\] (2.6)

\(^2\)A reader is referred to the Pauli article [65] for more detail about those peculiarities.
Because of the detailed development of $D$-function theory, this relation (2.6) will be of great importance in our further work. Closing this paragraph, we draw attention to that the Pauli criterion (here $\Phi_{\lambda,j,-j}(t,r,\theta,\phi)$ denotes a spherically symmetrical wave function): $J_{-j}(t,r,\theta,\phi) = 0$ affords a condition that is invariant relative to possible gauge transformations. The function $\Phi_{j,m}(t,r,\theta,\phi)$ may be subjected to any gauge transformation. But if all the components $J_i$ vary in a corresponding way too, then the Pauli condition provides the same result on $J$-quantization. In contrast to this, the common requirement to be a single-valued function of spatial points is often applied to producing a criterion on selection of allowable wave functions in quantum mechanics, in general, is not invariant under gauge transformations and can easily be destroyed by suitable gauge one.

3. Electron in a spherically symmetric gravitational field and Wigner $D$-functions

Below we review briefly some relevant facts about the TWFI tetrad formalism. In the presence of an external gravitational field, the starting Dirac equation

$$(i\gamma^\alpha \partial_\alpha - m)\Psi(x) = 0$$

is generalized into [55-63]

$$[i\gamma^\alpha(x)(\partial_\alpha + \Gamma_\alpha(x)) - m] \Psi(x) = 0$$

where $\gamma^\alpha(x) = \gamma^\alpha e^\alpha_{(a)}(x)$, $e^\alpha_{(a)}(x)$ is a tetrad; $\Gamma_\alpha(x) = \frac{1}{2}\sigma^{ab} e^\alpha_{(a)} \nabla_\alpha(e^\alpha_{(b)\beta})$ is the bispinor connection; \(\nabla_\alpha\) is the covariant derivative symbol. In the spinor basis

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix}, \quad \xi(x) = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \quad \eta(x) = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix}, \quad \sigma^a = (I, +\sigma^k), \quad \bar{\sigma}^a = (I, -\sigma^k)$$

where ( $\sigma^k$ are the two-row Pauli spin matrices; $k = 1, 2, 3$) we have two equations

$$i\sigma^a(x)(\partial_\alpha + \Sigma_\alpha(x))\xi(x) = m\eta(x), \quad (3.2a)$$

$$i\bar{\sigma}^a(x)(\partial_\alpha + \bar{\Sigma}_\alpha(x))\eta(x) = m\xi(x) \quad (3.2b)$$

the symbols $\sigma^a(x), \bar{\sigma}^a(x), \Sigma_\alpha(x), \bar{\Sigma}_\alpha(x)$ denote respectively

$$\sigma^a(x) = \sigma^a e^\alpha_{(a)}(x), \quad \bar{\sigma}^a(x) = \bar{\sigma}^a e^\alpha_{(a)}(x),$$

$$\Sigma_\alpha(x) = \frac{1}{2}\Sigma^{ab} e^\alpha_{(a)} \nabla_\alpha(e^\alpha_{(b)\beta}), \quad \bar{\Sigma}_\alpha(x) = \frac{1}{2}\bar{\Sigma}^{ab} e^\alpha_{(x)} \nabla_\alpha(e^\alpha_{(b)\beta}),$$

$$\Sigma^{ab} = \frac{1}{4}(\bar{\sigma}^a\sigma^b - \sigma^b\sigma^a), \quad \bar{\Sigma}^{ab} = \frac{1}{4}(\sigma^a\bar{\sigma}^b - \bar{\sigma}^b\sigma^a).$$

Setting $m$ equal to zero, we obtain the Weyl equations for neutrino $\eta(x)$ and anti-neutrino $\xi(x)$, or Dirac’s equation for a massless particle.
The form of equations (3.1), (3.2) implies quite definite their symmetry properties. It is common, considering the Dirac equation in the same space-time, to use some different tetrads \( e^{\beta}_b(x) \) and \( e^\beta_{(a)}(x) \), so that we have the equation (3.1) and analogous one with a new tetrad mark. In other words, together with (3.1) there exists an equation on \( \Psi'(x) \) where the quantities \( \gamma'^\alpha(x) \) and \( \Gamma^\alpha_\beta(x) \), in comparison with \( \gamma^\alpha(x) \) and \( \Gamma^\alpha_\beta(x) \), are based on another tetrad \( e^{\beta}_b(x) \) related to \( e^\beta_{(a)}(x) \) through some local Lorentz matrix

\[
e^{\beta}_b(x) = L^a_b(x) e^\beta_{(a)}(x). \tag{3.3a}
\]

It may be shown that these two Dirac equations on functions \( \Psi(x) \) and \( \Psi'(x) \) are related to each other by a quite definite bispinor transformation

\[
\xi'(x) = (k(x)) \xi(x), \quad \eta'(x) = +\langle\bar{k}(x)\rangle \eta(x). \tag{3.3b}
\]

Here, \( B(k(x)) = \sigma^a k_a(x) \) is a local matrix from the \( SL(2,C) \) group; 4-vector \( k_a \) is the well-known parametre on this group [74,75]. The matrix \( L^a_b(x) \) from (3.3a) can be expressed as a function of arguments \( k_a(x) \) and \( k^*_a(x) \):

\[
L^a_b(k, k^*) = \bar{\delta}^\alpha_c \left[ -\delta^a_c k^a k^*_n + k^*_c k^* k^a + i \epsilon^a_{c m n} k_n k^*_m \right] \tag{3.3c}
\]

where \( \bar{\delta}^\alpha_c \) is a special Cronecker’s symbol

\[
\bar{\delta}^\alpha_c = \begin{cases} 0, & if \ c \neq b; \\ +1, & if \ c = b = 0; \\ -1, & if \ c = b = 1, 2, 3 \\ \end{cases}
\]

It is normal practice that some different tetrads are used at examining the Dirac equation on the background of a given Riemannian space-time. If there is a need for analysis of the correlation between solutions in such distinct tetrads, then it is important to know how to calculate the corresponding gauge transformations over the spinor wave functions. Firstly, the need for taking into account such a gauge transformation was especially emphasized by Fock V.I. [57]. The first who were interested in explicit expressions for such spinor matrices, were E. Schrödinger [64] and W. Pauli [65]. Thus, Schrödinger found the matrix relating spinor wave functions in Cartesian and spherical tetrads:

\[
x^\alpha = (x^0, x^1, x^2, x^3), \quad ds^2 = [(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2], \quad e^\alpha_{(a)}(x) = \delta^\alpha_a \tag{3.4a}
\]

and

\[
x'^\alpha = (t, r, \theta, \phi), \quad ds'^2 = [dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ], \quad e^\alpha_{(0)} = (1, 0, 0, 0), \quad e^\alpha_{(1)} = (0, 0, 1/r, 0), \quad e^\alpha_{(2)} = (0, 0, 0, \frac{1}{r \sin \theta}), \quad e^\alpha_{(3)} = (0, 1, 0, 0) \tag{3.4b}
\]

the relevant matrix is (where \( \bar{c} \) is the Gibbs parametre on the group \( S0(3,R) \); see for more details in [75])

\[
B = \pm \begin{pmatrix} \cos \theta/2 e^{i\phi/2} & \sin \theta/2 e^{-i\phi/2} \\ -\sin \theta/2 e^{i\phi/2} & \cos \theta/2 e^{-i\phi/2} \end{pmatrix} \equiv B(\bar{c}) = \pm \frac{I - i \bar{c} \bar{c}}{\sqrt{1 - (\bar{c})^2}}
\]
This basis of spherical tetrad will play a substantial role in our further work. Just one (the spherical tetrad’s basis) was used with great efficiency by Pauli [65] when investigating the problem of allowed spherically symmetrical wave functions in quantum mechanics. Now, let us reexamine the problem of free electron in the external spherically symmetric gravitational field (see also in [68-70] about the manner of working on this in the frame of the so-called light tetrad or Newman-Penrose’s formalism), but centering upon some facts which will be of great importance at extending that method on an electron-monopole system.

In particular, we consider briefly a question of separating the angular variables in the Dirac equation on the background of a spherically symmetric Rimanian space-time. As a starting point we take a flat space-time model, so that an original equation (3.1) being specified for the spheric tetrad (see (3.4b)) takes on the form

\[
\left[ i \gamma^0 \partial_t + i (\gamma^3 \partial_r + \frac{\gamma^1 \sigma^{31} + \gamma^2 \sigma^{32}}{r}) + \frac{1}{r} \Sigma_{\theta,\phi} - m \right] \Psi(x) = 0 \quad (3.5a)
\]

where

\[
\Sigma_{\theta,\phi} = \left[ i \gamma^1 \partial_\theta + \gamma^2 \frac{i \partial_\phi + i \sigma^{12}}{\sin \theta} \right]. \quad (3.5b)
\]

We specialize the electronic wave function through substitution (Wigner functions are designated by \( D_{-m,\sigma}(\phi, \theta, 0) \equiv D_\sigma \))

\[
\Psi_{\epsilon jm}(x) = \frac{e^{-i\epsilon t}}{r} \begin{pmatrix}
  f_1(r) D_{-1/2} \\
  f_2(r) D_{+1/2} \\
  f_3(r) D_{-1/2} \\
  f_4(r) D_{+1/2}
\end{pmatrix}. \quad (3.6)
\]

Using recursive formulas (see in [66])

\[
\partial_\theta D_{+1/2} = (a D_{-1/2} - b D_{+3/2}), \quad \partial_\theta D_{-1/2} = (b D_{-3/2} - a D_{+1/2}),
\]

where \( a = (j + 1)/2 \) and \( b = \frac{1}{2} \sqrt{(j - 1/2)(j + 3/2)} \), we find (\( \nu = (j + 1/2)/2 \))

\[
\Sigma_{\theta,\phi} \Psi_{\epsilon jm}(x) = i \nu \frac{e^{-i\epsilon t}}{r} \begin{pmatrix}
  -f_4(r) D_{-1/2} \\
  +f_3(r) D_{+1/2} \\
  +f_2(r) D_{-1/2} \\
  -f_1(r) D_{+1/2}
\end{pmatrix}. \quad (3.7)
\]

Further one gets the following set of radial equations

\[
\begin{align*}
\epsilon f_3 - i \frac{d}{dr} f_3 - i \frac{\nu}{r} f_4 - mf_1 &= 0, \\
\epsilon f_4 + i \frac{d}{dr} f_4 + i \frac{\nu}{r} f_3 - mf_2 &= 0, \\
\epsilon f_1 + i \frac{d}{dr} f_1 + i \frac{\nu}{r} f_2 - mf_3 &= 0, \\
\epsilon f_2 - i \frac{d}{dr} f_2 - i \frac{\nu}{r} f_1 - mf_4 &= 0.
\end{align*} \quad (3.8)
\]
The usual $P$-reflection symmetry operator in the Cartesian tetrad basis is $\hat{\Pi}_C = i\gamma^0 \otimes \hat{P}$, or in a more detailed form

$$\hat{\Pi}_C = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \otimes \hat{P},$$

being subjected to translation into the spherical one $\hat{\Pi}_{sph} = S(\theta, \phi)\hat{\Pi}_C S^{-1}(\theta, \phi)$ gives us the result

$$\hat{\Pi}_{sph} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \otimes \hat{P}.$$ (3.9)

From the equation on proper values

$$\hat{\Pi}_{sph} \Psi_{jm} = \Pi \Psi_{jm} \text{ when } (\hat{P}_D^j_{m,\sigma}(\phi, \theta, 0) = (-1)^j D^j_{m,-\sigma}(\phi, \theta, 0))$$

we get

$$\Pi = \delta (-1)^{j+1}, \ \delta = \pm 1 : \ f_4 = \delta f_1, \ f_3 = \delta f_2$$ (3.10)

so that $\Psi_{ejm\delta}(x)$ is

$$\Psi(x)_{ejm\delta} = \frac{e^{-i\epsilon t}}{r} \begin{pmatrix} f_1(r) D_{-1/2}(\theta, \phi, 0) \\ f_2(r) D_{+1/2}(\theta, \phi, 0) \\ \delta f_2(r) D_{-1/2}(\theta, \phi, 0) \\ \delta f_1(r) D_{+1/2}(\theta, \phi, 0) \end{pmatrix}. \ \ \ \ (3.11)$$

Noting (3.10), we simplify the system (3.8); it is reduced to

$$\left(\frac{d}{dr} + \frac{\nu}{r}\right) f + (\epsilon + \delta m) g = 0, \ \ \ \left(\frac{d}{dr} - \frac{\nu}{r}\right) g - (\epsilon - \delta m) f = 0$$ (3.12)

where instead of $f_1$ and $f_2$ we have employed their linear combinations

$$f = \frac{f_1 + f_2}{\sqrt{2}}, \ \ \ \ g = \frac{f_1 - f_2}{i\sqrt{2}}.$$

It should be useful to notice that the above simplification ($\Psi_{ejm} \rightarrow \Psi_{ejm\delta}$) can also be obtained through the diagonalization of the operator $\hat{K}$ (see in [65]):

$$\hat{K} = -\gamma^0 \gamma^3 \Sigma_{\theta,\phi} = \gamma^0 \gamma^3 \left[ \gamma^1 (\partial_\theta + 1/2) + \frac{\gamma^2}{\sin \theta} \partial_\phi \right].$$ (3.13a)

Actually, from $\hat{K} \Psi_{ejm}(x) = K \Psi_{ejm}$ we produce

$$K = -\delta (j + 1/2), \ \delta = \pm 1 : \ f_4 = \delta f_1, \ f_3 = \delta f_2.$$ (3.13b)

Everything established above for the flat space-time model can be readily generalized into an arbitrary curved space-time with a spherically symmetrical metric $g_{\alpha\beta}(x)$:

$$dS^2 = \left[ e^\nu (dt)^2 - e^\mu (dr)^2 - r^2 ((d\theta)^2 + \sin^2 \theta (d\phi)^2) \right].$$ (3.14a)
and its naturally corresponding diagonal tetrad $e_\alpha^\beta (x)$:

$$
e_\alpha^\beta = (e^{-\nu/2}, 0, 0, 0), \quad e_\alpha^\beta = (0, e^{-\mu/2}, 0, 0),$$

$$
e_\alpha^\beta = (0, 0, \frac{1}{r}, 0), \quad e_\alpha^\beta = (0, 0, 0, \frac{1}{r \sin \theta}). \quad (3.14b)$$

The general covariant Dirac equation can be specified, according to [57], for an arbitrary diagonal tetrad as follows

$$
\left[ i \gamma^a \left( e_\alpha^\beta \partial_\beta + \frac{1}{2} e_\alpha^\beta \right) - m \right] \Psi(x) = 0 \quad (3.15a)
$$

where the $e_\alpha^\beta$ can be computed by means of

$$
e_\alpha^\beta = \frac{1}{\sqrt{-\det g}} \frac{\partial}{\partial x^\beta} \sqrt{-\det g} e_\alpha^\beta. \quad (3.15b)$$

So, for the function $\Phi(x)$ defined by

$$
\Psi(t, r, \theta, \phi) = \exp\left( -\frac{1}{4} (\nu + \mu) \right) \frac{1}{r} \Phi(t, r, \theta, \phi) \quad (3.16a)
$$

we produce the equation

$$
\left[ i \gamma^0 e^{-\nu/2} \partial_t + i \gamma^3 e^{-\mu/2} \partial_r + \frac{1}{r} \Sigma_{\theta, \phi} - m \right] \Phi(t, r, \theta, \phi) = 0. \quad (3.16b)
$$

On comparing (3.16b) with (3.5a), it follows immediately that all the calculations carried out above for the flat space-time case are still valid only with some evident modifications. Thus,

$$\Phi_{jm\delta}(x) = \left( \begin{array}{c} f_1(r, t) D_{-1/2}(\theta, \phi, 0) \\ f_2(r, t) D_{1/2}(\theta, \phi, 0) \\ \delta f_2(r, t) D_{-1/2}(\theta, \phi, 0) \\ \delta f_1(r, t) D_{1/2}(\theta, \phi, 0) \end{array} \right) \quad (3.17a)$$

and instead of (3.12) now we find

$$
\left( e^{-\nu/2} \frac{d}{dr} + \frac{\nu}{r} \right) f + (i e^{-\nu/2} \partial_t + \delta m) g = 0,
$$

$$
\left( e^{-\nu/2} \frac{d}{dr} - \frac{\nu}{r} \right) g - (i e^{-\nu/2} \partial_t - \delta m) f = 0. \quad (3.17b)
$$

4. Electronic wave functions in the external monopole field.

In the literature, the electron-monopole problem has attracted a lot of attention. In particular, the various properties of occurring so-called monopole harmonics were investigated in great detail. Here, we are going to look into this problem in the context of generalized Pauli-Schrödinger formalism reviewed in Sections 2-3. At this we seek to maintain as close connection as possible with the preceding formalism.
For our further purpose it will be convenient to use a monopole Abelian potential in the Schwinger’s form:

$$A^a(x) = (A^0, A^i) = \left(0, g \frac{(\vec{r} \times \vec{n}) \cdot (\vec{r} \, \vec{n})}{r (r^2 - (\vec{r} \, \vec{n})^2)}\right) \quad (4.1a)$$

after translating the $A_\alpha$ to the spherical coordinates and specifying $\vec{n} = (0, 0, 1)$, we get

$$A_0 = 0, \quad A_r = 0, \quad A_\theta = 0, \quad A_\phi = g \cos \theta. \quad (4.1b)$$

Correspondingly, the Dirac equation in this electromagnetic potential takes the form

$$\left[i\gamma^0 \partial_t + i\gamma^3 (\partial_r + \frac{1}{r}) + \frac{1}{r} \Sigma^k_{\theta, \phi} - \frac{mc}{\hbar}\right] \Psi(x) = 0 \quad (4.2a)$$

where

$$\Sigma^k_{\theta, \phi} = \left[i\gamma^1 \partial_\theta + \gamma^2 \frac{i\partial_\phi + (i\sigma^{12} - k) \cos \theta}{\sin \theta}\right] \quad (4.2b)$$

and $k \equiv eg/hc$. As readily verified, the wave operator in (4.2a) commutes with the following three ones

$$J^k_1 = \left[l_1 + \frac{(i\sigma^{12} - k) \cos \phi}{\sin \theta}\right], \quad J^k_2 = \left[l_2 + \frac{(i\sigma^{12} - k) \sin \phi}{\sin \theta}\right], \quad J^k_3 = l_3 \quad (4.3a)$$

which in turn obey the $SU(2)$ Lie algebra. Clearly, this monopole situation come entirely under the Schwinger-Pauli approach, so that our further work will be a matter of simple (quite elementary) calculations.

Thus, corresponding to diagonalization of the $J^k_2$ and $J^k_3$, the function $\Psi'$ is to be initially taken as ($D_{\sigma} \equiv D^j_{-m, \sigma}(\phi, \theta, 0)$)

$$\Psi^k_{e jm}(t, r, \theta, \phi) = \frac{e^{-i \epsilon t}}{r} \begin{pmatrix} f_1 D_{k-1/2}^j \\ f_2 D_{k+1/2}^j \\ f_3 D_{k-3/2}^j \\ f_4 D_{k+3/2}^j \end{pmatrix} \quad (4.3b)$$

Further, noting recursive relations [66]

$$\partial_\theta D_{k+1/2} = (+a \, D_{k-1/2} - b \, D_{k+3/2}),$$

$$-m - (k + 1/2) \cos \theta \sin \theta \, D_{k+1/2} = (-a \, D_{k-1/2} - b \, D_{k+3/2}),$$

$$\partial_\theta D_{k-1/2} = (+c \, D_{k-3/2} - a \, D_{k+1/2}),$$

$$-m - (k - 1/2) \cos \theta \sin \theta \, D_{k-1/2} = (-c \, D_{k-3/2} - a \, D_{k+1/2})$$

where

$$a = \frac{1}{2} \sqrt{(j + 1/2)^2 - k^2}, \quad b = \frac{1}{2} \sqrt{(j - k - 1/2)(j + k + 3/2)},$$

$$c = \frac{1}{2} \sqrt{(j + k - 1/2)(j - k + 3/2)}.$$
we find how the $\Sigma_{\theta, \phi}^k$ acts on $\Psi$:

$$
\Sigma_{\theta, \phi}^k \Psi_{\epsilon jm} = i \sqrt{(j + 1/2)^2 - k^2} \frac{e^{-i\epsilon t}}{r} \begin{pmatrix}
-f_4 D_{k-1/2} + f_3 D_{k+1/2} \\
+f_2 D_{k-1/2} - f_1 D_{k+1/2}
\end{pmatrix}
$$

(4.4)

Hereafter the factor $\sqrt{(j + 1/2)^2 - k^2}$ will be denoted by $\nu$. For the radial $f_i(r)$ we establish

$$
\epsilon f_3 - i \frac{d}{dr} f_3 - i \frac{\nu}{r} f_4 - m f_1 = 0, \quad \epsilon f_4 + i \frac{d}{dr} f_4 + i \frac{\nu}{r} f_3 - m f_2 = 0,
$$

$$
\epsilon f_1 + i \frac{d}{dr} f_1 + i \frac{\nu}{r} f_2 - m f_3 = 0, \quad \epsilon f_2 - i \frac{d}{dr} f_2 - i \frac{\nu}{r} f_1 - m f_4 = 0 . \quad (4.5)
$$

As evidenced by analogy with preceding Sec.3 and also on direct calculation, else one operator can be simultaneously diagonalized together with $\{i \partial_t, \vec{J}_k^2, J_3^k\}$, namely, a generalized Dirac operator

$$
\hat{K}^k = -i \gamma^0 \gamma^3 \Sigma_{\theta, \phi}^k . \quad (4.6a)
$$

From the equation $\hat{K}^k \Psi_{\epsilon jm} = K \Psi_{\epsilon jm}$ we can produce two possible values for this $K$ and the corresponding limitations on $f_i(r)$:

$$
K = -\delta \sqrt{(j + 1/2)^2 - k^2} : \quad f_4 = \delta f_1 , \quad f_3 = \delta f_2 \quad (4.6b)
$$

and in a consequence of this, the system (4.5) is reduced to

$$
(\frac{d}{dr} + \nu \frac{1}{r})f + (\epsilon + \delta m) g = 0 , \quad (\frac{d}{dr} - \nu \frac{1}{r})g - (\epsilon - \delta m) f = 0 . \quad (4.7)
$$

On direct comparing (4.7) with analogous system in Sec.3, we can conclude that these systems are formally similar apart from the difference between $\nu = j + 1/2$ and $\nu = \sqrt{(j + 1/2)^2 - k^2}$.

Now let us pass over to quantization of $k = eg/\hbar c$ and $J$. As a direct result from the first Pauli condition (2.5a) we derive that

$$
\frac{eg}{\hbar c} = \pm 1/2, \pm 1, \pm 3/2, \ldots \quad (4.8a)
$$

which coincides with the Dirac’s quantization, and from the second Pauli consequence it follows immediately that

$$
k = \frac{eg}{\hbar c} = \pm 1/2, \pm 1, \pm 3/2, \ldots \quad \text{and} \quad j = \mid k \mid -1/2, \mid k \mid +1/2, \mid k \mid +3/2, \ldots \quad (4.8b)
$$

The case of minimal allowable value $j_{\text{min.}} = | k | -1/2$ must be separated out and looked into in a special way. For example, let $k = +1/2$, then to the minimal value $j = 0$ there corresponds a wave function in terms of solely $(t, r)$-dependent quantities

$$
\Psi_{\epsilon jm}^{(j=0)}(x) = \frac{e^{-i\epsilon t}}{r} \begin{pmatrix}
f_1(r) \\
f_3(r)
\end{pmatrix} . \quad (4.9a)
$$
At \( k = -1/2 \), in an analogous way, we have

\[
\Psi_{k=-1/2}^{(j=0)}(x) = e^{-iet/r} \begin{pmatrix} 0 \\ f_2(r) \\ 0 \\ f_4(r) \end{pmatrix}.
\] (4.9b)

Thus, if \( k = \pm 1/2 \), then to the minimal allowed values \( J_{\text{min}} \) there correspond the function substitutions which do not depend at all on the angular variables \((\theta, \phi)\); at this point there exists some formal analogy between these electron-monopole states and \( e \)-states with \( l = 0 \) for a boson field of spin zero: \( \Phi_{l=0} = \Phi(r,t) \). However, it would be unwise to attach too much significance to this formal coincidence because such a \((\theta, \phi)\)-independence of \((e-g)\)-states is not a fact invariant under tetrad gauge transformations. In contrast, the relation below (let \( k = +1/2 \))

\[
\sum_{\theta, \phi}^{1/2} \Psi_{k=+1/2}^{(j=0)}(x) = \gamma^2 \cot(\theta) (i\sigma^1 - 1/2) \Psi_{k=+1/2}^{(j=0)} = 0
\] (4.10a)

is invariant under any gauge transformations. The identity (4.10a) holds because all the zeros in the \( \Psi_{k=+1/2}^{(j=0)} \) are adjusted to the non-zeros in \((i\sigma^1 - 1/2)\); and conversely, the non-vanishing constituents in \( \Psi_{k=+1/2}^{(j=0)} \) are canceled out by zeros in \((i\sigma^1 - 1/2)\). Correspondingly, the matter equation (4.2a) takes on the form

\[
\left[ i \gamma^0 \partial_t + i \gamma^3 (\partial_r + \frac{1}{r}) - mc/h \right] \Psi^{(j=0)} = 0.
\] (4.10b)

It is readily verified that both (4.9a) and (4.9b) representations are directly extended to \((e-g)\)-states with \( j = J_{\text{min}} \) at all the other \( k = \pm 1, \pm 3/2, \ldots \). Indeed,

\[
k = +1, +3/2, +2, \ldots: \quad \Psi_{j_{\text{min}}}^{k>0}(x) = e^{-iet/r} \begin{pmatrix} f_1(r) D_{k-1/2} \\ 0 \\ f_3(r) D_{k-1/2} \\ 0 \end{pmatrix};
\] (4.11a)

\[
k = -1, -3/2, -2, \ldots: \quad \Psi_{j_{\text{min}}}^{k<0}(x) = e^{-iet/r} \begin{pmatrix} 0 \\ f_2(r) D_{k+1/2} \\ 0 \\ f_4(r) D_{k+1/2} \end{pmatrix}
\] (4.11b)

and, as can be shown, the relation \( \sum_{\theta, \phi} \Psi_{j_{\text{min}}} = 0 \) still holds. For instance, let us consider in more detail the case of positive \( k \). Using the recursive relations [66]

\[
\partial_\theta D_{k-1/2} = \frac{1}{2} \sqrt{2k-1} D_{k-3/2}, \quad \frac{-m - (k - 1/2) \cos \theta}{\sin \theta} D_{k-1/2} = -\frac{1}{2} \sqrt{2k-1} D_{k-3/2},
\]

we get

\[
i\gamma^1 \partial_\theta \begin{pmatrix} f_1(r) D_{k-1/2} \\ 0 \\ f_3(r) D_{k-1/2} \\ 0 \end{pmatrix} = i \frac{1}{\sqrt{2k-1}} \begin{pmatrix} 0 \\ -f_3(r) D_{k-3/2} \\ 0 \\ +f_1(r) D_{k-3/2} \end{pmatrix}.
\]
\[ \gamma^2 \frac{i \partial_\phi + (i \sigma^{12} - k) \cos \theta}{\sin \theta} \begin{pmatrix} f_1(r) \\ f_3(r) \end{pmatrix} = \frac{i}{2} \sqrt{2k-1} \begin{pmatrix} 0 \\ f_3(r) \end{pmatrix} \]

in a sequence, the identity \( \Sigma_{\theta, \phi} \Psi_{j_{\min}} \equiv 0 \) has been proved. The case of negative \( k \) can be considered in the same way.

Thus, at every \( k \), the \( j_{\min} \)-state’s equation has the same unique form

\[ \left[ i \gamma^0 \partial_t + i \gamma^3 (\partial_r + \frac{1}{r}) - mc/\hbar \right] \Psi_{j_{\min}} = 0 \] (4.11c)

which leads to the radial system

\[ k = +1/2, +1, \ldots : \quad \epsilon f_3 - i \frac{d}{dr} f_3 - m f_1 = 0, \quad \epsilon f_1 + i \frac{d}{dr} f_1 - m f_3 = 0 \] (4.12a)

\[ k = -1/2, -1, \ldots : \quad \epsilon f_4 + i \frac{d}{dr} f_4 - m f_2 = 0, \quad \epsilon f_2 - i \frac{d}{dr} f_2 - m f_4 = 0 \] (4.12b)

These equations are equivalent respectively to

\[ k = +1/2, +1, \ldots : \quad \left[ \frac{d^2}{dr^2} + \epsilon^2 - m^2 \right] f_1 = 0, \quad f_3 = \frac{1}{m} \left( \epsilon + i \frac{d}{dr} \right) f_1 \] (4.13a)

\[ k = -1/2, -1, \ldots : \quad \left[ \frac{d^2}{dr^2} + \epsilon^2 - m^2 \right] f_4 = 0, \quad f_2 = \frac{1}{m} \left( \epsilon + i \frac{d}{dr} \right) f_4 \] (4.13b)

which both end up with the functions \( f = \exp(\pm \sqrt{m^2 - \epsilon^2} \ r) \). This latter, at \( \epsilon < m \), looks as

\[ \exp[-\sqrt{m^2 - \epsilon^2} \ r] \] (4.13c)

which seems to be appropriate to describe a bound state in the electron-monopole system. It should be emphasised that today the \( j_{\min} \) bound state problem remains a still yet question to understand. In particular, the important question faced us is of finding a physical and mathematical criterion on selecting values for \( \epsilon \): whether \( \epsilon < m \), or \( \epsilon = m \), or \( \epsilon > m \); and what value of \( \epsilon \) is to be chosen after specifying an interval above.

Now let us proceed with studying the properties which stem from the \( \theta, \phi \)-dependence of the wave functions. In particular, we restrict ourselves to the \( P \)-parity problem in the presence of the monopole. This problem was investigated in some detail in the literature [11-16,76-85], so our first step is to particularize some relevant facts in accordance with the formalism and notation used in the present paper.

As evidenced by straightforward computation, the well-known purely geometrical bispinor \( P \)-reflection operator does not commute with the Hamiltonian \( \hat{H} \) under consideration. The same conclusion is also arrived at by attempt to solve directly the proper value equation

\[ \hat{\Pi}_{sph.} \Psi_{ijm}^k = \Pi \Psi_{ijm}^k \]

which leads to

\[ (-1)^{j+1} \begin{pmatrix} f_4 D_{-k-1/2} \\ f_3 D_{-k+1/2} \\ f_2 D_{-k-1/2} \\ f_1 D_{-k+1/2} \end{pmatrix} = P \begin{pmatrix} f_1 D_{k-1/2} \\ f_2 D_{k+1/2} \\ f_3 D_{k-1/2} \\ f_4 D_{k+1/2} \end{pmatrix} \]
the latter matrix relation is satisfied only by the trivial substitution \( f_i = 0 \) for all \( i \). The matrix relation above indicates how a required discrete transformation can be constructed (further we will denote it as \( \hat{N}_{sph.} \))

\[
\hat{N}_{sph.} = \hat{\pi} \otimes \Pi_{sph.} \otimes \hat{P}
\]  

(4.14)

where \( \hat{\pi} \) is a special discrete operator changing \( k (= \frac{\text{eg}}{\hbar c}) \) into \(-k : \hat{\pi} F(k) = F(-k)\). Such an operator \( \hat{N}_{sph.} \) commutes with \( \hat{H} \) and \( J_i^k \); besides, from the equation \( \hat{N}_{sph.} \Psi_{ejm} = N\Psi_{ejm} \) it follows

\[
N = \delta (-1)^{j+1} (\delta = \pm 1) : \quad f_4 = \delta f_1, \quad f_3 = \delta f_2.
\]  

(4.15a)

The latter relations are compatible with the above radial system (4.5) and they are transformed into ( \( f(r) \) and \( g(r) \) are already used combinations from \( f_1(r) \) and \( f_2(r) \))

\[
\left( \frac{d}{dr} + \frac{\nu}{r} \right) f + (\epsilon + \delta m) g = 0, \quad \left( \frac{d}{dr} - \frac{\nu}{r} \right) g - (\epsilon - \delta m) f = 0
\]  

(4.15b)

that coincides with (4.7).

We are to say that everything just said about diagonalizing the \( \hat{N}_{sph.} \) is applied only to the cases when \( j > j_{\text{min.}} \). As regards the lower value of \( j \), the situation turns out to be very specific and unexpected. Actually, let \( k = +1/2 \) and \(-1/2 \) (\( j = 0 \)); then we have

\[
\hat{N}_{sph.} \Psi(j = 0) = N \Psi(j = 0) \rightarrow \begin{pmatrix} 0 \\ -f_3 \\ 0 \\ -f_4 \end{pmatrix} = N \begin{pmatrix} f_1 \\ 0 \\ f_3 \\ 0 \end{pmatrix};
\]

\[
\hat{N}_{sph.} \Psi(j = 0) = N \Psi(j = 0) \rightarrow \begin{pmatrix} -f_4 \\ 0 \\ -f_2 \\ 0 \end{pmatrix} = N \begin{pmatrix} 0 \\ f_2 \\ 0 \\ f_4 \end{pmatrix}
\]

respectively. Evidently, they both have no solutions, excluding trivially null ones (and therefore being of no interest). Moreover, as may be easily seen, in both cases a function \( \Phi(x) \), defined by \( \hat{N}_{sph.} \Psi(j = 0) \equiv \Phi(x) \), lies outside a fixed totality of states that are only valid as allowed quantum states of the system under consideration. At greater values of this \( k \), we come to analogous relations: the equation \( \hat{N}_{sph.} \Psi_{j_{\text{min.}}} = N \Psi_{j_{\text{min.}}} \) leads to positive \( k \):

\[
(-1)^{j+1} \begin{pmatrix} 0 \\ f_3 D_{k+1/2} \\ 0 \\ f_1 D_{k+1/2} \end{pmatrix} = N \begin{pmatrix} f_1 D_{k-1/2} \\ 0 \\ f_3 D_{k-1/2} \\ 0 \end{pmatrix};
\]

negative \( k \):

\[
(-1)^{j+1} \begin{pmatrix} f_4 D_{k-1/2} \\ 0 \\ f_2 D_{k-1/2} \\ 0 \end{pmatrix} = N \begin{pmatrix} 0 \\ f_2 D_{k+1/2} \\ 0 \\ f_4 D_{k+1/2} \end{pmatrix}
\]

and the same arguments above may be repeated again.
In turn, as regards the operator $\hat{K}^k$, for the $j_{\text{min}}$ states we get $\hat{K}^k \Psi_{j_{\text{min}}} = 0$; that is, this state represents the proper function of the $\hat{K}$ with the null proper value. So, application of this $\hat{K}$ instead of the $\hat{N}$ has an advantage of avoiding the paradoxical and puzzling situation when $\hat{N}_{\text{sph.}} \Psi(g_{j_{\text{min}}}) \not\in \{\Psi\}$. In a sense, this second alternative (the use of $\hat{K}^k$ instead of $\hat{N}$ at separating the variables and constructing the complete set of mutually commuting operators) gives us a possibility not to attach great significance to the monopole discrete operator $\hat{N}$ but to focus our attention solely on the continual operator $\hat{K}^k$. Indeed, we have described both these alternatives in case either one (first or second) be required.

5. Some additional facts on the monopole system

Now let us consider relationship between $D$-functions used above and the so-called spinor monopole harmonics. To this end one ought to perform two translations: from the spherical tetrad and 2-spinor (by Weyl) frame in bispinor space into, respectively, the Cartesian tetrad and the so-called Pauli’s (bispinor) frame. In the first place, it is convenient to accomplish those translations for a free electronic function; so as, in the second place, to follow this pattern further in the monopole case.

So, subjecting that free electronic function (spherical solution from Sec. 3) to the local bispinor gauge transformation (associated with the tetrad change $e_{\text{sph.}} \to e_{\text{Cart.}}$)

$$
\Psi_{\text{Cart.}} = \left( \begin{array}{cc} U^{-1} & 0 \\ 0 & U^{-1} \end{array} \right) \Psi_{\text{sph.}}, \quad U^{-1} = \left( \begin{array}{cc} \cos \theta/2 e^{-i\phi/2} & -\sin \theta/2 e^{-i\phi/2} \\ \sin \theta/2 e^{i\phi/2} & \cos \theta/2 e^{i\phi/2} \end{array} \right)
$$

and further, taking the bispinor frame from the Weyl 2-spinor form into the Pauli’s

$$
\Psi_{\text{Cart.}}^P = \left( \begin{array}{c} \varphi \\ \xi \end{array} \right), \quad \Psi_{\text{Cart.}} = \left( \begin{array}{c} \xi \\ \eta \end{array} \right), \quad \varphi = \frac{\xi + \eta}{\sqrt{2}}, \quad \chi = \frac{\xi - \eta}{\sqrt{2}}
$$

we get

$$
\varphi = \left[ \frac{f_1 + f_3}{\sqrt{2}} \left( \cos \theta/2 e^{-i\phi/2} \right) D_{-1/2} + \frac{f_2 + f_4}{\sqrt{2}} \left( -\sin \theta/2 e^{-i\phi/2} \right) \cos \theta/2 e^{i\phi/2} \right] D_{+1/2} \quad \text{(A.1a)}
$$

$$
\chi = \left[ \frac{f_1 - f_3}{\sqrt{2}} \left( \cos \theta/2 e^{-i\phi/2} \right) D_{-1/2} + \frac{f_2 - f_4}{\sqrt{2}} \left( -\sin \theta/2 e^{-i\phi/2} \right) \cos \theta/2 e^{i\phi/2} \right] D_{+1/2} \quad \text{(A.1b)}
$$

Further, for the above solutions with fixed proper values of $\Pi$-operator, we produce

$$
\Pi = (-1)^{j+1} : \quad \Psi_{\text{Cart.}}^P = \frac{e^{-i\eta}}{r \sqrt{2}} \left( \begin{array}{c} (f_1 + f_2) \left( \chi_{+1/2} D_{-1/2} + \chi_{-1/2} D_{+1/2} \right) \\ (f_1 - f_2) \left( \chi_{+1/2} D_{-1/2} - \chi_{-1/2} D_{+1/2} \right) \end{array} \right), \quad \text{(A.2a)}
$$

$$
\Pi = (-1)^{j} : \quad \Psi_{\text{Cart.}}^P = \frac{e^{-i\eta}}{r \sqrt{2}} \left( \begin{array}{c} (f_1 - f_2) \left( \chi_{+1/2} D_{-1/2} - \chi_{-1/2} D_{+1/2} \right) \\ (f_1 + f_2) \left( \chi_{+1/2} D_{-1/2} + \chi_{-1/2} D_{+1/2} \right) \end{array} \right) \quad \text{(A.2b)}
$$

where $\chi_{+1/2}$ and $\chi_{-1/2}$ designate the columns of matrix $U^{-1}(\theta, \phi)$ (in the literature they are termed as helicity spinors)

$$
\chi_{+1/2} = \left( \begin{array}{c} \cos \theta/2 e^{-i\phi/2} \\ \sin \theta/2 e^{i\phi/2} \end{array} \right), \quad \chi_{-1/2} = \left( \begin{array}{c} -\sin \theta/2 e^{-i\phi/2} \\ \cos \theta/2 e^{i\phi/2} \end{array} \right). \quad \text{(A.2c)}
$$
Now, using the known extensions for spherical spinors $\Omega_{jm}^{\pm 1/2}(\theta, \phi)$ in terms of $\chi_{\pm 1/2}$ and $D$-functions [66]:

$$\Omega_{jm}^{+1/2} = (-1)^{m+1/2} \sqrt{(2j + 1)/8\pi} \left[ \chi_{+1/2} \ D_{-1/2} + \chi_{-1/2} \ D_{+1/2} \right],$$

$$\Omega_{jm}^{-1/2} = (-1)^{m+1/2} \sqrt{(2j + 1)/8\pi} \left[ -\chi_{+1/2} \ D_{-1/2} + \chi_{-1/2} \ D_{+1/2} \right],$$

we eventually arrive at the common representation of the spinor spherical solutions

$$\Pi = (-1)^{j+1} : \quad \Psi_{\text{Cart.}}^P = \frac{e^{-i\epsilon t}}{r} \begin{pmatrix} -f(r) \Omega_{jm}^{+1/2}(\theta, \phi) \\ g(r) \Omega_{jm}^{-1/2}(\theta, \phi) \end{pmatrix}; \quad (A.3a)$$

$$\Pi = (-1)^j : \quad \Psi_{\text{Cart.}}^P = \frac{e^{-i\epsilon t}}{r} \begin{pmatrix} f(r) \Omega_{jm}^{+1/2}(\theta, \phi) \\ -g(r) \Omega_{jm}^{-1/2}(\theta, \phi) \end{pmatrix}. \quad (A.3b)$$

The monopole situation can be considered in the same way. As a result, we produce the following representation of the monopole-electron functions in terms of ‘new’ angular harmonics

$$N = (-1)^{j+1} : \quad \Psi_{\text{Cart.}}^P = \frac{e^{-i\epsilon t}}{r} \begin{pmatrix} -f(r) \xi_{jm}^{(1)}(\theta, \phi) \\ g(r) \xi_{jm}^{(2)}(\theta, \phi) \end{pmatrix}; \quad (A.4a)$$

$$N = (-1)^j : \quad \Psi_{\text{Cart.}}^P = \frac{e^{-i\epsilon t}}{r} \begin{pmatrix} f(r) \xi_{jm}^{(1)}(\theta, \phi) \\ -g(r) \xi_{jm}^{(2)}(\theta, \phi) \end{pmatrix}. \quad (A.4b)$$

Here, the two column functions $\xi_{jm}^{(1)}(\theta, \phi)$ and $\xi_{jm}^{(2)}(\theta, \phi)$ denote special combinations of $\chi_{\pm 1/2}(\theta, \phi)$ and $D_{-m}e^{i\epsilon t}/e^{\pm 1/2}(\phi, \theta, 0)$:

$$\xi_{jm}^{(1)} = [ \chi_{-1/2} \ D_{k+1/2} + \chi_{+1/2} \ D_{-k-1/2} ] , \quad \xi_{jm}^{(2)} = [ \chi_{-1/2} \ D_{k+1/2} - \chi_{+1/2} \ D_{k-1/2} ] \quad (A.5)$$

compare them with analogous extensions for $\Omega_{jm}^{\pm 1/2}(\theta, \phi)$. These 2-component and $(\theta, \phi)$-dependent functions $\xi_{jm}^{(1)}(\theta, \phi)$ and $\xi_{jm}^{(2)}(\theta, \phi)$ just provide what is called spinor monopole harmonics. Given the known expressions for $\chi$- and $D$-functions, the formulas (A.5) yield the following

$$\xi_{jm}^{(1,2)}(\theta, \phi) = \left[ e^{im\phi} \begin{pmatrix} -\sin\theta/2 \ e^{-i\phi/2} \\ \cos\theta/2 \ e^{+i\phi/2} \end{pmatrix} d_{m,k+1/2}(\cos\theta) \right. \pm$$

$$\left. e^{im\phi} \begin{pmatrix} \cos\theta/2 \ e^{-i\phi/2} \\ \sin\theta/2 \ e^{+i\phi/2} \end{pmatrix} d_{m,k-1/2}(\cos\theta) \right]. \quad (A.6)$$

Here, the signs $+$ (plus) and $-$ (minus) refer to $\xi^{(1)}$ and $\xi^{(2)}$, respectively.

One can equally work whether in terms of monopole harmonics $\xi^{(1,2)}(\theta, \phi)$ or directly in terms of $D$-functions, but the latter alternative has an advantage over the former because of the straightforward access to the "unlimited" $D$-function apparatus; instead of proving and producing just disguised old results. In any case, one should establish existing correlations and relations (as much as possible) between at first sight unrelated matters; namely, the tetrad formalism, special Schrödinger basis, Pauli's investigation
[64,65], D-function apparatus, and spinor (scalar, vector, and so on) harmonics. It should be mentioned that to the above list, we ought to add the so-called formalism (of great popularity) of spin-weight harmonics, which was developed in the light tetrad frame (also known as the Newman-Penrose formalism).

Above, at translating the electron-monopole functions into the Cartesian tetrad and Pauli’s spin frame, we had overlooked the case of minimal \( j \). Returning to it, on straightforward calculation we find (for \( k < 0 \) and \( k > 0 \), respectively)

\[
\text{positive } \kappa : \quad \Psi_{\text{Cart.}}^{k>0} = \frac{e^{-ict}}{\sqrt{2r}} \left( \frac{f_1 + f_3}{f_1 - f_3} \right) \frac{\chi_{1/2}}{\chi_{1/2}} D_{-mk-1/2}^{k-1/2}(\theta, \phi, 0) ; \quad (A.7a)
\]

\[
\text{negative } \kappa : \quad \Psi_{\text{Cart.}}^{k<0} = \frac{e^{-ict}}{\sqrt{2r}} \left( \frac{f_2 + f_4}{f_2 - f_4} \right) \frac{\chi_{-1/2}}{\chi_{-1/2}} D_{-mk+1/2}^{k-1/2}(\theta, \phi, 0) . \quad (A.7b)
\]

Now we pass on to another subject and take up demonstrating how the major facts obtained so far are extended to a curved background geometry (of spherical symmetry). All above, the flat space monopole potential \( A_\phi = g \cos \theta \) preserves its simple form at changing the flat space model into a curved one of spherical symmetry) \( A_\phi = g \cos \theta \rightarrow F_{\theta \phi} = -F_{\phi \theta} = -g \sin \theta \) and the general covariant Maxwell equation in such a curved space yields

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} F_{\alpha \beta} = 0 \rightarrow \frac{\partial}{\partial \theta} \left[ e^{\nu + \mu} r^2 \sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) F_{\theta \phi} \right] = 0 .
\]

So, the monopole potential (for a curved background geometry) is given again as \( A_\phi = g \cos \theta \). In a sequence, the problem of electron in external monopole field (in a curved background) remains, in a whole, unchanged. There are only some new features brought about by curvature, but they do not affect the \((\theta, \phi)\)-aspects of the problem. Thus, we arrive at the following

\[
\kappa = +1, +3/2, +2, \ldots : \quad \Psi_{\text{jmin.}}^{\kappa>0} (x) = \frac{1}{r} \begin{pmatrix} f_1(r, t) D_{k-1/2} \\ 0 \\ f_3(r, t) D_{k-1/2} \end{pmatrix} \quad (A.8a)
\]

from that it follows

\[
ie^{-\nu/2} \partial_t f_1 + ie^{-\mu/2} \partial_r f_1 - mf_3 = 0 , \quad ie^{-\nu/2} \partial_t f_3 - ie^{-\mu/2} \partial_r f_3 - mf_1 = 0 \quad (A.8b)
\]

and further

\[
f_3 = \frac{i}{m} \left( e^{-\nu/2} \partial_t + e^{-\mu/2} \partial_r \right) f_1(r, t),
\]

\[
\left[ ( e^{-\nu/2} \partial_t - e^{-\mu/2} \partial_r ) \left( e^{-\nu/2} \partial_t + e^{-\mu/2} \partial_r \right) + m^2 \right] f_1 = 0 . \quad (A.8c)
\]

The case \( \Psi_{\text{jmin.}}^{\kappa<0} (x) \) can be considered in the same way. Let us discuss several simple examples.

**SPHERICAL GEOMETRY**

In the spherical coordinates

\[
dS^2 = \left[ (dt)^2 - \frac{(dr)^2}{1 - r^2} - r^2((d\theta)^2 + \sin^2 \theta (d\phi)^2) \right] 
\]
the equation for \( f_1 (t, r) \) takes the form
\[
[ (\partial_t - \sqrt{1 - r^2} \partial_r) (\partial_t + \sqrt{1 - r^2} \partial_r) + m^2 ] f_1 = 0 .
\]
Factorizing \( f_1 \) according to \( f_1 = e^{-itf(r)} \) and introducing the variable \( \chi \) by relation \( \sin \chi = r \) (the metric above becomes \( dS^2 = [(dt)^2 - (d\chi)^2 - \sin^2 \chi ((d\theta)^2 + \sin^2 \theta (d\phi)^2)] \)), we get
\[
\left[ \frac{d^2}{d\chi^2} + (m^2 - \epsilon^2) \right] f(r) = 0 , \quad f = \exp(\pm \sqrt{m^2 - \epsilon^2} \chi) .
\]
Here, the variable \( \chi \) lies in the \([0, \pi]\) or \([0, \pi/2]\) intervals according to whether the spherical or elliptic space model is meant. Else one example is

LOBACHEVSKI GEOMETRY

Here, instead of the above there will be

\[
r = \sinh \chi , \quad f_1 = e^{-itf(r)} ,
\]
\[
dS^2 = \left[ (dt)^2 - \frac{(dr)^2}{1 + r^2} - r^2 ((d\theta)^2 + \sin^2 \theta (d\phi)^2) \right] ;
\]
\[
dS^2 = \left[ (dt)^2 - (d\chi)^2 - \sinh^2 \chi ((d\theta)^2 + \sin^2 \theta (d\phi)^2) \right] ;
\]
\[
\left[ \frac{d^2}{d\chi^2} + (m^2 - \epsilon^2) \right] f(r) = 0 , \quad \rightarrow \quad f = \exp(\pm \sqrt{m^2 - \epsilon^2} \chi) .
\]

Now, we pass on another interesting peculiarity that concern properties of the electron current \( J_\alpha (x) \). This current is given by \( J^\alpha (x) = \Psi^+(x) \gamma^0 \gamma^\alpha (x) \Psi(x) \). Noting the wave function substitution

\[
\Psi(t, r, \theta, \phi) = \frac{e^{-(\nu+\mu)/4}}{r} \begin{pmatrix} f_1(t, r) & D_{k-1/2} \\ f_2(t, r) & D_{k+1/2} \\ f_3(t, r) & D_{k-1/2} \\ f_4(t, r) & D_{k+1/2} \end{pmatrix} \quad (A.9a)
\]

for those current components we get

\[
J^t(x) = e^{i(0)} \left[ d_{k-1/2}^2 (\theta) \left( | f_1 |^2 + | f_3 |^2 \right) + d_{k+1/2}^2 (\theta) \left( | f_4 |^2 + | f_2 |^2 \right) \right] ,
\]
\[
J^r(x) = e^{i(3)} \left[ d_{k-1/2}^2 (\theta) \left( | f_1 |^2 - | f_3 |^2 \right) + d_{k+1/2}^2 (\theta) \left( | f_4 |^2 - | f_2 |^2 \right) \right] ,
\]
\[
J^\theta(x) = e^{i(1)} \left[ \left( f_1^* f_2 + f_1 f_2^* \right) + \left( f_3^* f_4 + f_3 f_4^* \right) \right] d_{k-1/2} (\theta) d_{k+1/2} (\theta) ,
\]
\[
J^\phi(x) = -i e^{i(2)} \left[ \left( f_1^* f_2 - f_1 f_2^* \right) - \left( f_3^* f_4 - f_3 f_4^* \right) \right] d_{k-1/2} (\theta) d_{k+1/2} (\theta) \quad (A.9b)
\]

here and in the following, the factor \( r^{-2} e^{-(\nu+\mu)/2} \) is omitted; also we have taken into account the notation

\[
D_\sigma = D_{m,\sigma} (\theta, \phi, 0) = e^{-im\phi} d_{m,\sigma} (\theta) = e^{-im\phi} d_\sigma (\theta) .
\]

Further, for solutions of fixed \( N \)-parity values, the formulas \( A.9b \) result in

\[
N = (-1)^j , (-1)^{j+1} : \quad J^t(x) = e^{i(0)} \left[ d_{k-1/2}^2 (\theta) + d_{k+1/2}^2 (\theta) \right] \left( | f_1 |^2 + | f_3 |^2 \right) ,
\]
\[ J^r(x) = e^{r(0)} \left[ d_{k-1/2}^2(\theta) + d_{k+1/2}^2 \right] (|f_1|^2 - |f_3|^2), \]
\[ J^\theta(x) = 0, \quad J^\phi(x) = -2i e^{\phi(2)} d_{k-1/2} d_{k+1/2} (f_1 f_2 - f_1 f_2^*). \]  
(A.9c)

In turn, for the \( j_{\text{min}} \) states we obtain

\[ k = +1/2, +1, +3/2, \ldots : \quad J^r(x) = e^{r(0)} d_{k-1/2}(\theta) (|f_1|^2 + |f_3|^2), \]
\[ J^\theta(x) = e^{\theta(3)} d_{k-1/2}(\theta) (|f_1|^2 - |f_3|^2), \quad J^\phi(x) = 0; \quad (A.10a) \]

\[ k = -1/2, -1, -3/2, \ldots : \quad J^r(x) = e^{r(0)} d_{k+1/2}(\theta) (|f_4|^2 + |f_2|^2), \]
\[ J^\theta(x) = e^{\theta(0)} d_{k+1/2}(\theta) (|f_4|^2 - |f_2|^2), \quad J^\phi(x) = 0. \]  
(A.10b)

It should be noted that the \( J^\phi \) vanishes at \( j = j_{\text{min}} \). This sharply contrasts with behaviour of \( J^\phi \) component for all remaining values \( j \) and also contrasts with free electronic states (in the absence an external monopole potential).

Finally, let us consider the question of gauge choice for description of the monopole potential. From general considerations we can conclude that, for the problems considered above, it was not basically essential thing whether to use the Schwinger’s form of the monopole potential or to use any other form. Every possible choice could bring about some technical incidental variation in a corresponding description, but this will not affect the applicability of \( D\)-function apparatus to the procedure of separating out the variables \( \theta, \phi \) in the electron-monopole system. For example, in the Dirac gauge the monopole potential is given by

\[ (A_n)^D = \left( 0, g \frac{\vec{n} \times \vec{r}}{r (r + \vec{n} \cdot \vec{r})} \right) \]  
(A.11a)

which, after translating to spherical coordinates, becomes

\[ A_n^D = (A_t = 0, A_r = 0, A_\theta = 0, A_\phi = g(\cos \theta - 1)). \]  
(A.11b)

On comparing \( A_\phi^D \) with \( A_\phi^S \), it follows immediately that we can relate these electron-monopole pictures (\( S \) and \( D \) gauges) by gauge transformation \( S(\phi) = e^{+ik\phi} \):

\[ \Psi^D(x) = S(\phi) \Psi^S, \quad A^D_\beta(x) = A^S_\beta(x) - i (hc/e) S(\phi) \partial_\beta S^{-1}(\phi). \]  
(A.11c)

Simultaneously translating the operators \( \hat{J}_j^D, \hat{K}, \hat{N} \) from \( S \) to \( D \) gauge

\[ \hat{J}_j^D = S \hat{J}_j^S S^{-1}, \quad \hat{K}^D = S \hat{K}^S S^{-1}, \quad \hat{N}^D = S \hat{N}^S S^{-1} \]

we produce

\[ \hat{J}_1^D = \left( l_1 + \frac{\cos \phi}{\sin \theta} (i\sigma^{12} - k(1 - \cos \theta)) \right), \]
\[ \hat{J}_2^D = \left( l_2 + \frac{\sin \phi}{\sin \theta} (i\sigma^{12} - k(1 - \cos \theta)) \right), \]
\[ \hat{J}_3^D = (l_3 - k), \]
\[ \hat{K}^D = -i \gamma^0 \gamma^3 \left( i\gamma^1 \partial_\theta + \gamma^2 \frac{i\partial_\phi + k + (i\sigma^{12} - k) \cos \theta}{\sin \theta} \right), \]
\[ \hat{N}^D = e^{ik(2\phi + \pi)} \hat{N}^S. \]  
(A.11d)
Thus, the explicit forms of the operators vary from one representation to another, but their proper values remain unchanged; any alterations in operators and corresponding modifications in wave functions cancel out each other completely. That is, as it certainly might expected, the complete set of proper values provides such a description that is invariant, by its implications, under any possible $U(1)$ gauge transformations.

Now, let us consider else one variation in $U(1)$ gauge, namely, from Schwinger’s gauge to the Wu-Yang’s. In the Wu-Yang (hereafter, designated as W-Y) gauge, the monopole potential is characterized by two different respective expressions in two complementary spatial regions

$$0 \leq \theta < (\pi/2 + \epsilon) \implies A^{(N)}_{\phi} = g(\cos \theta - 1) ,$$

$$ (\pi/2 - \epsilon) < \theta \leq \pi \implies A_{\phi}(S) = g(\cos \theta + 1) \quad (A.12a)$$

and the transition from the $S.$-basis into $W - Y$’s can be obtained by

$$\Psi^{S}(x) \implies \Psi^{W-Y}(x) = \left\{ \begin{array}{ll}
\Psi^{(N)}(x) = S^{(N)}(\phi) \Psi^{S}(x) , & S^{(N)}(\phi) = e^{ik\phi} \\
\Psi^{(S)}(x) = S^{(S)}(\phi) \Psi^{S}(x) , & S^{(S)}(\phi) = e^{-ik\phi}
\end{array} \right. \quad (A.12b)$$

Correspondingly, for the operators $\hat{J}^{+}_{j}$, $\hat{K}$, $\hat{N}$ we get two different forms in $N$- and $S$-regions, respectively:

$$\hat{J}^{+}_{1} = \left( l_{1} + \frac{\cos \phi}{\sin \theta} (i\sigma^{12} - k(1 \pm \cos \theta)) \right) ,$$

$$\hat{J}^{+}_{2} = \left( l_{2} + \frac{\sin \phi}{\sin \theta} (i\sigma^{12} - k(1 \pm \cos \theta)) \right) , \quad \hat{J}^{D}_{3} = (l_{3} \pm k) ,$$

$$\hat{K}^{\pm} = -i \varepsilon^0 \varepsilon^3 \left( i\varepsilon^1 \partial_{\phi} + \varepsilon^2 \frac{i\partial_{\phi} \mp k + (i\sigma^{12} - k) \cos \theta}{\sin \theta} \right) ,$$

$$\hat{N}^{\pm} = \exp(\mp ik(2\phi + \pi)) \hat{N}^{S} . \quad (A.12c)$$

where the over sign (+ or −) relates to $S.$-region, and the lower one (− or +, respectively) to $N$-region.

It should be noted that only the Schwinger’s $U(1)$ gauge (in virtue of the relation $\hat{j}_{3} = -i\partial_{\phi}$) represents analogue of the Schrödinger’s (tetrad) basis discussed in Sec.2, whereas the Dirac and Wu-Yang gauges are not. The explicit form of the third component of a total conserved momentum $J_{3} = -i \partial_{\phi} \equiv j_{3}^{Schr}$ can be regarded as a determining characteristic, which specifies this basis (and its possible generalizations). The situations in $S.$, $D.$, and $W - Y$ gauges are as follows

$$J_{3}^{S} = l_{3} \, , \quad J_{3}^{D} = (l_{3} - k) \, , \quad J_{3}^{(N)} = (l_{3} - k) \, , \quad J_{3}^{(S)} = (l_{3} + k) \, ,$$

what proves the above assertion.
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