Quantum Newton duality

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Abstract: Newton revealed an underlying duality relation between power potentials in classical mechanics. In this paper, we establish the quantum version of the Newton duality. The main aim of this paper is threefold: (1) first generalizing the original classical Newton duality to more general potentials, including general polynomial potentials and transcendental-function potentials, (2) constructing a quantum version of the Newton duality, including power potentials, general polynomial potentials, transcendental-function potentials, and power potentials in different spatial dimensions, and (3) suggesting a method for solving eigenproblems in quantum mechanics based on the quantum Newton duality provided in the paper. The classical Newton duality is a duality among orbits of classical dynamical systems. The quantum Newton duality provides a duality relation among wave functions and eigenvalues. Our result shows that the Newton duality is not only limited to power potentials, but a more universal duality relation among dynamical systems with various potentials. The key task of this paper is to construct a quantum Newton duality, the quantum version of the classical Newton duality. As applications, we suggest a method for solving potentials from their Newtonianly dual potential: once the solution of a potential is known, the solution of all its dual potentials can be obtained by the duality transformation directly. Using this method, we obtain a series of exact solutions of various potentials. In appendices, as preparations, we solve the potentials which is solved by the Newton duality method in this paper by directly solving the eigenequation.

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1 Introduction

In classical mechanics, Newton revealed a duality between power potentials in his *Principia* (Corollary III of Proposition VII) [1]. Newton’s duality relates the orbits of two power potentials $U(r) = \xi r^{a+1}$ and $V(\rho) = \eta \rho^{A+1}$ by the duality relation $\frac{a+3}{2} \leftrightarrow \frac{2}{A+3}$, $E \leftrightarrow -\eta$, and $\xi \leftrightarrow -E$, where $E$ and $E$ are the energies of the orbits of $U(r)$ and $V(\rho)$, respectively. That is, for these two power potentials $U(r)$ and $V(\rho)$, when $\frac{a+3}{2} = \frac{2}{A+3}$, the energy of a system equals the negative coupling constant of its dual system, and then the orbit of a potential can be obtained from the orbit of its Newtonianly dual potential.

The Newton duality reveals profound dynamical nature of mechanical systems. The original question Newton asked is that given a power law of centripetal attraction, does there exist a dual law for which a body with the same constant of areas will describe the same orbit [1]. The modern formulation of the Newton duality is given by some authors [2–4]. Newton pointed out that between the harmonic-oscillator potential ($r^2$-potential) and the Coulomb potential ($1/r$-potential), there exists such a duality [1].

We will construct a quantum version for the Newton duality. The quantum Newton duality constructed in the present paper includes not only power-potentials, but also more general potentials.

1. Newton himself found the classical Newton duality between power potentials. In this paper, we generalize the Newton duality to a more general case — general polynomial potentials. The general polynomial potential is a linear combination of power potentials with arbitrary real number powers.

2. A quantum version of the classical Newton duality is a main result of this paper. The quantum Newton duality provides a duality relation between wave functions and eigenvalues, just like that the classical Newton duality provides a duality relation between orbits. We will show that by the quantum Newton duality relation between two dual potentials, one can transform the wave function and eigenvalue of one potential to those of its dual potential. Besides power potentials, we provide duality relations for more general kinds of potentials, including general polynomial potentials and transcendental function potentials. Moreover, for power potentials, beyond three dimensions, we also consider the quantum Newton duality of two power potentials in different spatial dimensions. It should be pointed out that if two potentials are classical Newtonianly dual, they are also quantum Newtonianly dual.

3. Another important issue of the present paper is to make the quantum Newton duality serve as a method of solving eigenproblems in quantum mechanics. The quantum Newton duality allows us to transform the eigenfunction and eigenvalue of one potential to the eigenfunction and eigenvalue of its dual potential. That is, if the solution of one potential is known, one can obtain the solution of its dual potential through the duality relation directly. Concretely, the original Newton duality is a duality between two one-term power potentials. A one-term potential has one dual potential; in other words, a potential $V_1$ with its duality $V_2$ constitute a dual pair $(V_1, V_2)$. In this paper, we generalize the Newton duality to general polynomial potentials with arbitrary terms. It will be shown that a $N$-term general polynomial potential $V_1$ has $N$ dual potentials $V_2, V_3, \ldots, V_{N+1}$. 

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These $N + 1$ potentials, which are Newtonianly dual to each other, constitute a dual set $(V_1, V_2, \ldots, V_{N+1})$. The eigenfunction and eigenvalue of the potentials in a dual set are related by the duality relation. In a dual set $(V_1, V_2, \ldots, V_{N+1})$, if a potential, say, $V_i$, is solved, then the solution of the other potentials in the dual set can be immediately obtained by the duality relation. The Newton duality provides us an efficient tool for seeking the solution in mechanics.

To illustrate how this works, we solve some potentials from their exactly solved dual potentials by the duality relation. The potentials solved in this paper belong to the following sets. \( (\alpha r, \beta r, \sigma r) \), and \( (\alpha r, \sqrt{\beta r}, \sigma r) \) are given in section 3.1. An example of a dual set of general polynomial potentials. In sections 3.1-3.7, we provide a main result of this paper, the quantum Newton duality. A general discussion of the quantum Newton duality is given in section 3.4. The original classical Newton duality which is only a duality between power potentials is generalized to general polynomial potentials. In sections 3.1-3.7, we provide a main result of this paper, the quantum Newton duality. A general discussion of the quantum Newton duality is given in section 3.1. In section 3.2, we give a quantum version of the original Newton duality between power potentials. In section 3.3, we provide the quantum Newton duality between power potentials in different spatial dimensions, i.e., a duality between a $n$-dimensional potential and its $m$-dimensional dual potential. In section 3.4, the quantum Newton duality for general polynomial potentials is given. In section 3.5, we discuss the problem of the dual set. In section 3.7, we give an example of the Newton duality between two transcendental-function potentials. In sections 4.1-4.3.4, we show how to solve the eigenproblem by the duality relation. As examples, we consider some sets of dual potentials. Examples of dual sets of one-term potentials, including three-dimensional harmonic-oscillator potentials and three-dimensional Coulomb potentials, \( (\alpha^2 r^2/\beta r, \alpha^2 r^2/\beta r) \), and \( (\alpha^2 r^2/\beta r, \alpha^2 r^2/\beta r) \) are given in section 4.2. Examples of dual sets of two-term potentials, \( (\alpha r^2 + \beta, \beta r^2 + \beta) \), \( (\alpha r^2 + \beta, \beta r^2 + \beta) \), \( (\alpha r^2 + \beta, \beta r^2 + \beta) \), and \( (\alpha r^2 + \beta, \beta r^2 + \beta) \) are given in section 4.3. An example of a dual set of three-term potentials \( (\alpha r^2 + \beta, \beta r^2 + \beta) \), \( (\alpha r^2 + \beta, \beta r^2 + \beta) \), \( (\alpha r^2 + \beta, \beta r^2 + \beta) \) is given in section 4.4. A dual set of transcendental-function potentials \( (\alpha r^2 + \beta, \beta r^2 + \beta) \) is given in 3.7. The conclusion is given in section 5. Moreover, in Appendices (A)-(F), we solve exact solutions for preparations.
2 The classical Newton duality

2.1 The original Newton duality of power potentials: Revisit

The original Newton duality revealed by Newton himself is a duality between two power potentials in classical mechanics. A modern expression of the original classical Newton duality can be found in Refs. [2–4]. For completeness, in this section we give a brief review on the original classical Newton duality. Our expression and proof are somewhat different from the original version.

**Theorem 1** Two power potentials

\[ U(r) = \xi r^{a+1} \quad \text{and} \quad V(\rho) = \eta \rho^{A+1} \]  

are classically Newtonianly dual to each other, if

\[ \frac{a+3}{2} = \frac{2}{A+3}. \]  

The orbit with the energy \( E \) of the potential \( V(\rho) \) can be obtained by performing the replacements

\[ E \rightarrow -\eta, \quad \xi \rightarrow -E \]  

to the orbit with the energy \( E \) of the potential \( U(r) \). As a result, the corresponding transformation of coordinates is

\[ r \rightarrow \rho^{(A+3)/2} \quad \text{or} \quad r^{(a+3)/2} \rightarrow \rho, \]  

\[ \theta \rightarrow A+3 \phi \quad \text{or} \quad \frac{a+3}{2} \theta \rightarrow \phi. \]  

**Proof.** For the potential \( U(r) = \xi r^{a+1} \), the equation of the orbit \( r = r(\theta) \) of the energy \( E \) reads [5]

\[ \frac{d\theta}{dr} = \frac{L/r^2}{\sqrt{2[E - L^2/(2r^2) - \xi r^{a+1}]}} \]  

where \( L \) is the angular momentum.

Substituting the replacements (2.2), (2.3), and (2.4) into the orbit equation (2.7) and rearranging the equation as

\[ \frac{d\left( \frac{2}{A+3} \theta \right)}{d\left( r^{\frac{2}{a+1}} \right)} = \frac{L/\left( r^{\frac{2}{a+1}} \right)^2}{\sqrt{2 \left\{ E - L^2 / \left[ 2 \left( r^{\frac{2}{a+1}} \right)^2 \right] - \eta \left( r^{\frac{2}{a+1}} \right)^{A+1} \right\}}}, \]  

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we can see that this is just the equation of the orbit $\rho(\phi)$ with the energy $E$ of the potential $V(\rho) = \eta \rho^{d+1}$ with a transformation of coordinates

$$r^{\frac{2}{d+1}} \rightarrow \rho \quad \text{and} \quad \frac{2}{d+3} \theta \rightarrow \phi.$$ \hfill (2.9)

It can be seen that the energy of an orbit of a system equals the negative coupling constant of its dual system.

It is worth mentioning that in classical mechanics the angular momenta of these two Newtonianly dual systems are the same.

2.2 General discussions

The original Newton duality is only for power potentials. In this paper, we will generalize Newton’s result to more general cases. To show the possibility of the generalization of the original classical Newton duality, we first give a general discussion on the duality in classical mechanics.

In classical mechanics, the orbit equation of the energy $E$ of the potential $U(r)$ reads

$$\frac{d\theta}{dr} = \frac{L}{r^2}\sqrt{2\left[\frac{E - L^2}{2r^2} - U(r)\right]} \hfill (2.10)$$

Writing the transformation of coordinates on the orbital plane as

$$r \rightarrow g(\rho), \quad \theta(r) \rightarrow f(\rho) \phi(\rho)$$ \hfill (2.11)\hfill (2.12)

gives

$$\frac{d\phi(\rho)}{d\rho} = \frac{L}{r^2}\sqrt{2\left[\frac{E(\frac{g^2(\rho)f(\rho)}{g'(\rho)r^2})^2 - \frac{L^2}{2g^2(\rho)}\left(\frac{g^2(\rho)f(\rho)}{g'(\rho)r^2}\right)^2 - U(\rho)\left(\frac{g^2(\rho)f(\rho)}{g'(\rho)r^2}\right)^2}{1 - \frac{f'(\rho)}{f(\rho)}\phi(\rho)}\right]} \hfill (2.13)$$

Here the only change is a transformation of coordinates, so, of course, Eq. (2.13) must also be an orbit equation.

In order to make Eq. (2.13) still serve as an orbit equation, it is required that

$$\frac{f'(\rho)}{f(\rho)} = 0,$$ \hfill (2.14)

i.e.,

$$f(\rho) = \beta$$ \hfill (2.15)

with $\beta$ a constant. Substituting Eq. (2.15) into Eq. (2.13) gives

$$\frac{d\phi(\rho)}{d\rho} = \frac{L}{r^2}\sqrt{2\left[\beta^2 E\left(\frac{g^2(\rho)}{g'(\rho)r^2}\right)^2 - \beta^2 \frac{L^2}{2g^2(\rho)}\left(\frac{g^2(\rho)}{g'(\rho)r^2}\right)^2 - \beta^2 U(\rho)\left(\frac{g^2(\rho)}{g'(\rho)r^2}\right)^2\right]} \hfill (2.16)$$
Insisting that Eq. (2.16) is still an orbit equation, we must requires that in

\[
\beta^2 E \left( \frac{g^2(\rho)}{g'(\rho)\rho^2} \right)^2 - \beta^2 \frac{L^2}{2g^2(\rho)} \left( \frac{g^2(\rho)}{g'(\rho)\rho^2} \right)^2 - \beta^2 U(\rho) \left( \frac{g^2(\rho)}{g'(\rho)\rho^2} \right)^2,
\]

(2.17)

serving as the centrifugal potential, there must exist a term being proportional to \(1/\rho^2\),

(2.18)

serving as the energy, there must exist a constant term which is independent of \(\rho\).

(2.19)

The choice is not unique, and different choices lead to different dualities.

2.3 General polynomial potentials

In this section, we generalize the classical Newton duality to general polynomial potentials. By general polynomial here we mean a superposition power series containing arbitrary real-number powers. The original Newton duality, a duality between power potentials, is a special case of this general duality since a power potential is nothing but a one-term general polynomial potential.

2.3.1 Generalizing the classical Newton duality to general polynomial potentials

\textbf{Theorem 2} Two general polynomial potentials

\[ U(r) = \xi r^{a+1} + \sum_n \mu_n r^{b_n+1} \quad \text{and} \quad V(\rho) = \eta \rho^{A+1} + \sum_n \lambda_n \rho^{B_n+1} \]

(2.20)

are classically Newtonianly dual to each other, if

\[ \frac{a+3}{2} = \frac{2}{A+3}, \]

(2.21)

\[ \sqrt{\frac{2}{a+3}} (b_n + 3) = \sqrt{\frac{2}{A+3}} (B_n + 3). \]

(2.22)

The orbit with the energy \(E\) of the potential \(V(\rho)\) can be obtained by performing the replacements

\[ E \rightarrow -\eta, \]

(2.23)

\[ \xi \rightarrow -\xi, \]

(2.24)

\[ \mu_n \rightarrow \lambda_n \]

(2.25)

to the orbit with the energy \(E\) of the potential \(U(r)\). As a result, the corresponding transformation of coordinates is

\[ r \rightarrow \rho^{(A+3)/2} \quad \text{or} \quad r^{(a+3)/2} \rightarrow \rho, \]

(2.26)

\[ \theta \rightarrow \frac{A+3}{2} \phi \quad \text{or} \quad \frac{a+3}{2} \theta \rightarrow \phi. \]

(2.27)
Proof. For the potential \( U(r) = \xi r^{a+1} + \sum_n \mu_n r^{b_n+1} \), the orbit \( r = r(\theta) \) of the energy \( E \) reads

\[
\frac{d\theta}{dr} = \frac{L}{r^{2}} \sqrt{2 \left[ E - L^2 / (2r^2) - (\xi r^{a+1} + \sum_n \mu_n r^{b_n+1}) \right]}.
\] (2.28)

Substituting the replacements (2.21), (2.23), (2.24), and (2.25) into the orbit equation (2.28) and rearranging the equation as

\[
\frac{d}{d \left( \frac{2A+3}{r^{A+3}} \right)} \left( \frac{2A+3}{r^{A+3}} \right) = \frac{L}{\left( \frac{2A+3}{r^{A+3}} \right)^2} \sqrt{2 \left\{ E - \frac{L^2}{2r^2} - \eta \left( \frac{2A+3}{r^{A+3}} \right)^{A+1} - \sum_n \lambda_n \left( \frac{2A+3}{r^{A+3}} \right)^{B_n+1} \right\}},
\] (2.29)

we can see that this is just an orbit equation of the energy \( \mathcal{E} \) for the potential \( V(\rho) = \eta \rho^{A+1} + \sum_n \lambda_n \rho^{B_n+1} \) with the transformation of coordinates

\[
r^{\frac{2}{A+3}} \rightarrow \rho \quad \text{and} \quad \frac{2}{A+3} \theta \rightarrow \phi.
\] (2.30)

It is worth showing how the result given by Theorem 2 is reached.

The classical Newton duality for general polynomial potentials given by Theorem 2 can be obtained by choosing the term \( \beta^2 \frac{L^2}{2g^2(\rho)} \left( \frac{g'(\rho)}{g(\rho)} \right)^2 \) in Eq. (2.17) as the centrifugal potential term, i.e.,

\[
\beta^2 \frac{L^2}{2g^2(\rho)} \left( \frac{g'(\rho)}{g(\rho)} \right)^2 = \frac{L^2}{2 \rho^2}
\] (2.31)

to satisfy Condition 2.18. Solving Eq. (2.31) gives

\[
g(\rho) = \rho^\beta,
\] (2.32)

i.e., the transformation of coordinates is

\[
r \rightarrow \rho^\beta \quad \text{or} \quad r^{1/\beta} \rightarrow \rho.
\] (2.33)

With the transformation of coordinates (2.33), Eq. (2.17) becomes

\[
E \rho^{2\beta-2} - \frac{L^2}{2 \rho^2} - \rho^{2\beta-2} U(\rho).
\] (2.34)

To satisfy Condition 2.19, we choose

\[
- \rho^{2\beta-2} U(\rho) = \mathcal{E} - \sum_n \kappa_n \rho^n.
\] (2.35)

Then we have

\[
U(\rho) = - \mathcal{E} \rho^{2-2\beta} + \sum_n \kappa_n \rho^{n+2-2\beta},
\] (2.36)

and Eq. (2.17) becomes

\[
\mathcal{E} = \frac{L^2}{2 \rho^2} + E \rho^{2\beta-2} - \sum_n \kappa_n \rho^n.
\] (2.37)
Applying the transformation of coordinates (2.33), we obtain a pair of dual potentials

\[ U(r) = -E r^{(2-2\beta)/\beta} + \sum_n \kappa_n r^{(n+2-2\beta)/\beta}, \]  

(2.38)

\[ V(\rho) = -E \rho^{2\beta-2} + \sum_n \kappa_n \rho^n. \]  

(2.39)

Rewrite the potential \( U(r) \) given by Eq. (2.38) as

\[ U(r) = \xi r^{a+1} + \sum_n \mu_n r^{b_n+1}, \]  

(2.40)

with

\[ \xi \rightarrow -E, \]  

(2.41)

\[ \beta \rightarrow \frac{2}{a+3}, \]  

(2.42)

\[ b_n \rightarrow \frac{(2+n)(a+3)}{2} - 3, \]  

(2.43)

\[ \kappa_n \rightarrow \mu_n; \]  

(2.44)

rewrite the potential \( V(\rho) \) given by Eq. (2.39) as

\[ V(\rho) = \eta \rho^{A+1} + \sum_n \lambda_n \rho^{B_n+1}, \]  

(2.45)

with

\[ E \rightarrow -\eta, \]  

(2.46)

\[ \beta \rightarrow \frac{A+3}{2}, \]  

(2.47)

\[ B_n \rightarrow n - 1, \]  

(2.48)

\[ \kappa_n \rightarrow \lambda_n. \]  

(2.49)

Then comparing Eq. (2.41) with Eq. (2.46), Eq. (2.42) with Eq. (2.47), Eq. (2.43) with Eq. (2.48), and Eq. (2.44) with Eq. (2.49), we arrive at the duality relation given by Theorem 2.

### 2.3.2 General polynomial potentials consist of two Newton dual potentials

In this section, we consider an interesting special case of the Newton duality of general polynomial potentials. In this case, \( U_1(r) \) and \( V_1(\rho) \) are Newtonianly dual, \( U_2(r) \) and \( V_2(\rho) \) are Newtonianly dual, and their sum \( U(r) = U_1(r) + U_2(r) \) and \( V(\rho) = V_1(\rho) + V_2(\rho) \) are also Newtonianly dual.

**Theorem 3** Two general polynomial potentials

\[ U(r) = \xi r^{a+1} + \mu r^{2\left(\frac{A+3}{2}-1\right)}, \]  

(2.50)

\[ V(\rho) = \eta \rho^{A+1} + \mu \rho^{2\left(\frac{A+3}{2}-1\right)}, \]  

(2.51)
are Newtonianly dual, while $\xi r^{a+1}$ and $\eta r^{a+1}$ are Newtonianly dual and $\mu r^2 \left( \sqrt{\frac{a+2}{2}} - 1 \right)$ and $\mu \rho^2 \left( \sqrt{\frac{A+2}{2}} - 1 \right)$ are Newtonianly dual, if

$$\frac{a + 3}{2} = \frac{2}{A + 3}. \quad (2.52)$$

The proof is similar to the proof of Theorem 2.

3 The quantum Newton duality

3.1 General discussions

The main aim of this paper is to provide a quantum version of the classical Newton duality, including power potentials, general polynomial potentials, and some other potentials.

In quantum mechanics, instead of the orbit, the wave function is used to describe a mechanical system. Therefore, the quantum Newton duality is a duality relation between wave functions. Moreover, the energy of quantum-mechanical bound states is discrete, so we also need a duality relation between discrete bound-state eigenvalue spectra.

Now we first give a general discussion on the quantum Newton duality.

Generally speaking, the existence of the Newton duality between two potentials means that there exists a duality relation which can transform the eigenfunction and eigenvalue of a potential to those of the other potential. In other words, the eigenfunction and eigenvalue of a potential can be achieved from the solution of its dual potential with the help of the duality relation.

Consider two potentials $U(r)$ and $V(\rho)$. The radial equation of $U(r)$ is

$$\frac{d^2 u(r)}{dr^2} + \left[ E - \frac{l (l + 1)}{r^2} - U(r) \right] u(r) = 0 \quad (3.1)$$

with the eigenfunction $u(r)$ and eigenvalue $E$; the radial equation of $V(\rho)$ is

$$\frac{d^2 v(\rho)}{d\rho^2} + \left[ \mathcal{E} - \frac{\ell (\ell + 1)}{\rho^2} - V(\rho) \right] v(\rho) = 0, \quad (3.2)$$

with the eigenfunction $v(\rho)$ and eigenvalue $\mathcal{E}$. Two potentials $U(r)$ and $V(\rho)$ are called a pair of dual potentials, if there exists a duality transform which can transform $u(r)$ and $E$ to $v(\rho)$ and $\mathcal{E}$.

Generally, write the duality transformations of coordinates and eigenfunctions as

$$r \to g(\rho), \quad (3.3)$$
$$u(r) \to f(\rho) v(\rho), \quad (3.4)$$

where the duality transform is described by the functions $g(\rho)$ and $f(\rho)$. Substituting Eqs. (3.3) and (3.4) into the radial equation of $U(r)$, Eq. (3.1), gives

$$v''(\rho) + \left( 2 \frac{f'(\rho)}{f(\rho)} - \frac{g''(\rho)}{g'(\rho)} \right) v'(\rho) + \left( \frac{f''(\rho)}{f(\rho)} - \frac{f'(\rho)}{f(\rho)} \frac{g''(\rho)}{g'(\rho)} \right) v(\rho) + g'(\rho)^2 \left[ E - \frac{l (l + 1)}{g^2(\rho)} - U(g(\rho)) \right] v(\rho) = 0. \quad (3.5)$$
If requiring that Eq. (3.5) is the radial equation of the potential $V(\rho)$ with the eigenfunction $v(\rho)$, the first-order derivative term in Eq. (3.5) must vanish, i.e.,

$$2\frac{f'(\rho)}{f(\rho)} - \frac{g''(\rho)}{g'(\rho)} = 0, \quad (3.6)$$

and the solution of Eq. (3.6) gives a relation between $f(\rho)$ and $g(\rho)$,

$$f(\rho) = \sqrt{g'(\rho)}. \quad (3.7)$$

Then substituting Eq. (3.7) into Eq. (3.5) gives

$$v''(\rho) + \left[\frac{g'(\rho)^2}{2g'(\rho)} + \frac{3g''(\rho)^2}{4g'(\rho)^2} - l(l+1)\frac{g'(\rho)^2}{g^2(\rho)} - g'(\rho)^2 U(g(\rho))\right] v(\rho) = 0. \quad (3.8)$$

If insisting that Eq. (3.8) is still a radial equation, then it must requires that there must exist a term in the coefficient of $v(\rho)$ playing the role of the centrifugal potential, which should be proportional to $1/\rho^2$, \quad (3.9)

there must exist a term in the coefficient of $v(\rho)$ playing the role of the eigenvalue, which should be a constant $E$. \quad (3.10)

Different choices lead different duality relations.

### 3.2 Three-dimensional power potentials

The original Newton duality in *Principia* is the duality between power potentials. In this section, we present a quantum version of the Newton duality for three-dimensional power potentials.

**Theorem 4** Two power potentials

$$U(r) = \xi r^{a+1} \text{ and } V(\rho) = \eta \rho^{A+1} \quad (3.11)$$

are quantum Newtonianly dual to each other, if

$$\frac{a+3}{2} = \frac{2}{A+3}. \quad (3.12)$$

The bound-state eigenfunction of the energy $E$ and the angular quantum number $\ell$ of the potential $V(\rho)$ can be obtained by performing the replacements

$$E \rightarrow -\eta \left(\frac{2}{A+3}\right)^2, \quad (3.13)$$

$$\xi \rightarrow -E \left(\frac{2}{A+3}\right)^2. \quad (3.14)$$
to the bound-state eigenfunction of the energy $E$ and the angular quantum number $l$ of the potential $U(r)$. As a result, the duality relation of the angular momentum between these two dual systems is

$$\ell + \frac{1}{2} \rightarrow \frac{2}{A+3} \left( \ell + \frac{1}{2} \right),$$

(3.15)

the transformation of coordinates is

$$r \rightarrow \rho^{(A+3)/2} \quad \text{or} \quad r^{(a+3)/2} \rightarrow \rho,$$

(3.16)

and the transformation of the eigenfunction is

$$u(r) \rightarrow \rho^{(A+1)/4} v(\rho) \quad \text{or} \quad r^{(a+1)/4} u(r) \rightarrow v(\rho).$$

(3.17)

Proof. The radial equation of the potential $U(r) = \xi r^{a+1}$ reads

$$\frac{d^2 u(r)}{dr^2} + \left[ E - \frac{l(l+1)}{r^2} - \xi r^{a+1} \right] u(r) = 0,$$

(3.18)

where $u(r) = R(r)/r$ with $R(r)$ the radial wave function. Substituting the duality relations (3.12), (3.13), and (3.14) into the radial equation (3.18) and rearrange the equation as

$$\frac{d^2 u(r)}{dr^2} + \left( \frac{2}{A+3} \right)^2 \left[ \mathcal{E} - \frac{(A+3)^2}{r^{2(A+3)}} \left( \ell + \frac{1}{2} \right) - \eta \left( r^{2(A+3)} \right)^{A+1} \right] \frac{1}{r^{2(A+3)}} u(r) = 0,$$

(3.19)

we can see that this is just the radial equation of the potential $V(\rho) = \eta \rho^{A+1}$,

$$\frac{d^2 v(\rho)}{d\rho^2} + \left[ \mathcal{E} - \frac{\ell(\ell+1)}{\rho^2} - \eta \rho^{A+1} \right] v(\rho) = 0,$$

(3.20)

with the duality relations (3.15), (3.16), and (3.17). \qed

The duality relation between the eigenvalues of two dual potentials can be obtained directly.

The positive power attractive potential $U(r) = \xi r^{a+1}$, i.e., $a+1 > 0$ and $\xi > 0$, has only bound states with positive eigenvalues. The duality of $U(r)$ is the potential $V(\rho) = \eta \rho^{A+1}$. The potential $V(\rho)$ is a negative power attractive potential, i.e., $A+1 < 0$ and $\eta < 0$, with negative bound-state eigenvalues.

**Corollary 5** If the eigenvalue of the power potential $U(r) = \xi r^{a+1}$ is

$$E = f(n_r, l, \xi),$$

(3.21)

where $n_r$ is the radial quantum number and $l$ is the angular quantum number, then the eigenvalue of its Newton duality, the negative power potential $V(\rho) = \eta \rho^{A+1}$, is

$$\mathcal{E} = - \left( \frac{A+3}{2} \right)^2 f^{-1} \left( n_r, \frac{2}{A+3} \left( \ell + \frac{1}{2} \right) - \frac{1}{2}, - \left( \frac{2}{A+3} \right)^2 \frac{\eta}{2} \right),$$

(3.22)

where $f^{-1}$ denotes the inverse function of $f$ and $\ell$ is the angular quantum number of the system of $V(\rho)$.
**Proof.** Substituting the relation between the two dual potentials, Eqs. (3.13), (3.14), and (3.15), into the expression of the eigenvalue of the potential $U(r)$, Eq. (3.21), gives

$$-\eta \left( \frac{2}{A+3} \right)^2 = f \left( n_r, \frac{2}{A+3} \left( \ell + \frac{1}{2} \right) - \frac{1}{2} - \mathcal{E} \left( \frac{2}{A+3} \right)^2 \right). \quad (3.23)$$

The eigenvalue $\mathcal{E}$ of the potential $V(\rho)$ given by Eq. (3.22) then can be solved directly.

In the following, we show how the result given by Theorem 4 is reached.

The Newton duality between power potentials is indeed obtained by choosing the term $g' (\rho)^2 / g^2 (\rho)$ in Eq. (3.8) to be proportional to $1/\rho^2$, which will finally become the centrifugal-potential term or a part of the centrifugal-potential term.

Concretely, first, according to Condition 3.9, we need to choose a term in the coefficient of $v(\rho)$ being proportional to $1/\rho^2$ and serving as a centrifugal potential or a part of a centrifugal potential. For the original Newton duality between power potentials, choose

$$\frac{g' (\rho)^2}{g^2 (\rho)} = \frac{\gamma^2}{\rho^2} \quad (3.24)$$

in Eq. (3.8), where $\gamma$ is a constant. Solving Eq. (3.24) gives

$$g(\rho) = \rho^\gamma. \quad (3.25)$$

By Eq. (3.25), the radial equation Eq. (3.8) becomes

$$\frac{d^2 v(\rho)}{d\rho^2} + \left[ -\frac{\gamma^2}{\rho^{2(1-\gamma)}} U(\rho) - \frac{l (l + 1) \gamma^2 - (1 - \gamma^2) / 4}{\rho^2} + \frac{E \gamma^2}{\rho^{2(1-\gamma)}} \right] v(\rho) = 0. \quad (3.26)$$

Clearly, the term $\left[ l (l + 1) \gamma^2 - \frac{1-\gamma^2}{4} \right] / \rho^2$ is the centrifugal-potential term.

Second, we need to consider Condition 3.10, which requires us to choose a term in the coefficient of $v(\rho)$ as a constant term and serve as the eigenvalue $\mathcal{E}$. Now there is only one possible choice:

$$-\frac{\gamma^2}{\rho^{2(1-\gamma)}} U(\rho) = \mathcal{E}, \quad (3.27)$$

which gives

$$U(\rho) = -\frac{\mathcal{E}}{\gamma^2} \rho^{2(1-\gamma)}. \quad (3.28)$$

By Eqs. (3.3) and (3.25), we can see that the transformation of coordinates between two dual potentials is

$$\rho^\gamma \to r. \quad (3.29)$$

By this transformation of coordinates, we arrive at the potential

$$U(r) = -\frac{\mathcal{E}}{\gamma^2} r^{2(1-\gamma)/\gamma}. \quad (3.30)$$
Now by Eqs. (3.26) and (3.28), Eq. (3.8) becomes

$$\frac{d^2 v(\rho)}{d\rho^2} + \left[ \mathcal{E} - \frac{l(l+1)\gamma^2 - (1-\gamma^2)/4}{\rho^2} + \frac{E\gamma^2}{\rho^{2(1-\gamma)}} \right] v(\rho) = 0. \quad (3.31)$$

This is just the radial equation of the potential

$$V(\rho) = -\frac{E\gamma^2}{\rho^{2(1-\gamma)}}. \quad (3.32)$$

The potential $V(\rho)$ given by Eq. (3.32) is the Newton duality of the potential $U(r)$ given by Eq. (3.30).

Rewriting these two potentials as

$$U(r) = \xi r^{a+1} \quad (3.33)$$

with

$$\gamma = \frac{2}{a+3}, \quad (3.34)$$

and

$$E = -\left(\frac{2}{a+3}\right)^2 \xi \quad (3.35)$$

and

$$V(\rho) = \eta \rho^{A+1} \quad (3.36)$$

with

$$\gamma = \frac{A+3}{2}, \quad (3.37)$$

$$E = -\left(\frac{2}{A+3}\right)^2 \eta. \quad (3.38)$$

we then arrive at the duality relations (3.12), (3.13), and (3.14).

The term which is proportional to $1/\rho^2$ play a role of the centrifugal potential, so we have

$$\ell (\ell + 1) = l(l+1)\gamma^2 - \frac{1}{4} (1-\gamma^2)$$

$$= l(l+1) \left(\frac{A+3}{2}\right)^2 - \frac{1}{4} \left[ 1 - \left(\frac{A+3}{2}\right)^2 \right]. \quad (3.39)$$

Then we achieve the replacement of the angular momentum given by Eq. (3.15).

The transformation of coordinates, Eq. (3.16), can be obtained by substituting Eq. (3.37) into Eq. (3.25). Then the transformation of the eigenfunction, by using (3.7), is

$$u(r) \rightarrow \left(\frac{A+3}{2}\right)^{1/2} \rho^{(A+1)/4} v(\rho). \quad (3.40)$$

Dropping the constant $\left(\frac{A+3}{2}\right)^{1/2}$ which does not influence the result of the wave function, we arrive at the transformation (3.17).
By the Newton duality relation, we can obtain the condition of the existence of bound states. The power potential with a positive power, \( r^{a+1} \) \((a > -1)\), has only bound states. The Newton duality of \( r^{a+1} \), by Eq. (3.12), is \( r^{A+1} \) with \( A = \frac{4a}{a+3} - 3 \). The upper bound of \( A \) is \( A_{\text{upper}}^{a=-1} - 1 \) and the lower bound of \( A \) is \( A_{\text{lower}}^{a\to\infty} - 3 \). Therefore, the condition of the existence of bound states of the potential \( r^{A+1} \) is \(-3 < A < -1\), or, equivalently, the condition of the existence of bound states of the potential \( r^\beta \) is \(-2 < \beta < 0\). This agrees with the usual result [6].

It can be seen that if two potentials are classical Newton duality, they are also quantum Newton duality. In classical mechanics, the orbits of two dual potentials can be simply transformed from one to the other. Similarly, in quantum mechanics, the wave functions of two dual potentials can be simply transformed from one to the other.

### 3.3 Arbitrary-dimensional power potentials

In section 3.2, we consider the quantum Newton duality between two three-dimensional power potentials. In this section, we consider the quantum Newton duality between two power potentials in different dimensions: \( n \)-dimensional potential \( U(r) = \xi r^{a+1} \) and \( m \)-dimensional potential \( V(\rho) = \eta \rho^{A+1} \).

**Theorem 6** Two power potentials in different dimensions,

\[
U(r) = \xi r^{a+1}, \quad n\text{-dimensional,} \\
V(\rho) = \eta \rho^{A+1}, \quad m\text{-dimensional,}
\]

are quantum Newtonianly dual to each other, if

\[
\frac{a + 3}{2} = \frac{2}{A + 3}. \tag{3.43}
\]

The bound-state eigenfunction of the energy \( E \) and the angular quantum number \( \ell \) of the \( m \)-dimensional potential \( V(\rho) \) can be obtained by performing the replacements

\[
E \rightarrow -\eta \left( \frac{2}{A + 3} \right)^2, \tag{3.44}
\]

\[
\xi \rightarrow -\xi E \left( \frac{2}{A + 3} \right)^2 \tag{3.45}
\]

to the bound-state eigenfunction of the energy \( E \) and the angular quantum number \( l \) of the \( n \)-dimensional potential \( U(r) \). As a result, the duality relation of the angular momentum between these two dual systems is

\[
l + \frac{n}{2} - 1 = \frac{2}{A + 3} \left( \ell + \frac{m}{2} - 1 \right), \tag{3.46}
\]

the transformation of coordinates is

\[
r \rightarrow \rho^{(A+3)/2} \quad \text{or} \quad r^{(a+3)/2} \rightarrow \rho. \tag{3.47}
\]
and the transformation of the eigenfunction is

\[ u(r) \rightarrow \rho^{(A+1)/4} v(\rho) \quad \text{or} \quad r^{(a+1)/4} u(r) \rightarrow v(\rho). \quad (3.48) \]

**Proof.** In \( n \) dimensions, the radial equation of the potential \( U(r) = \xi r^{a+1} \) reads \[ d^2 u(r) \over dr^2 + \left[ E - \left( l - \frac{3}{2} + \frac{a}{2} \right) \left( l - \frac{1}{2} + \frac{a}{2} \right) - \xi r^{a+1} \right] u(r) = 0, \quad (3.49) \]

where \( u(r) = R(r)/r \) with \( R(r) \) the radial wave function. Perform the transformations \((3.43), (3.44), \) and \((3.45)\) and rearrange the equation as

\[ d^2 u(r) \over dr^2 + \left[ \left( l - \frac{3}{2} + \frac{a}{2} \right) \left( l - \frac{1}{2} + \frac{a}{2} \right) - \eta r^{A+1} \right] u(r) = 0. \quad (3.50) \]

This is just the radial equation of the central potential \( V(\rho) = \eta \rho^{A+1} \) in \( m \) dimensions:

\[ d^2 v(\rho) \over d\rho^2 + \left[ \left( \frac{\ell}{2} + \frac{m}{2} \right) \left( \ell - \frac{1}{2} + \frac{m}{2} \right) - \eta \rho^{A+1} \right] v(\rho) = 0 \quad (3.51) \]

with the replacement of the angular momentum, Eq. \((3.46)\), the transformation of coordinates \((3.47)\), and the transformation of eigenfunctions, Eq. \((3.48)\). \( \Box \)

Note that the influence of the value of the spatial dimension appears only in the centrifugal potential term, because the angular quantum number is different in different dimensions.

In the following, we present a relation between the eigenvalues of two dual potentials in different dimensions.

The \( n \)-dimensional positive-power attractive potential \( U(r) = \xi r^{a+1} \) \((a + 1 > 0 \) and \( \xi > 0)\) has only bound states and positive eigenvalues, i.e., \( E > 0 \). Its \( m \)-dimensional Newton duality \( V(\rho) = \eta \rho^{A+1} \) is a negative-power attractive potential \((A + 1 < 0 \) and \( \eta < 0)\) and has bound-state eigenvalues, i.e., \( E < 0 \).

**Corollary 7** If the eigenvalue of the \( n \)-dimensional positive-power potential \( U(r) = \xi r^{a+1} \) is

\[ E = f(n_r, l, \xi, n), \quad (3.52) \]

where \( n_r \) is the radial quantum number and \( l \) the angular quantum number, then the eigenvalue of the its \( m \)-dimensional Newton duality, the negative-power potential \( V(\rho) = \eta \rho^{A+1} \), is

\[ E = -4 \over (a + 3)^2 \frac{1}{f^{-1}} \left( n_r, \frac{2}{A + 3} \left( \ell + \frac{m}{2} - 1 \right) + 1 + \frac{n}{2}, -\frac{(a + 3)^2}{4} \eta, m \right), \quad (3.53) \]

where \( f^{-1} \) denotes the inverse function of \( f \) and \( \ell \) is the angular quantum number.
Proof. Substituting the relation between the two dual potentials, Eqs. (3.44), (3.45), and (3.46), into the expression of the eigenvalue of the $m$-dimensional potential $U(r)$, Eq. (3.52), gives

$$\frac{(a+3)^2}{4} - \eta = f\left(n_r, \frac{2}{A+3}\left(\ell + \frac{m}{2} - 1\right) + 1 + \frac{n}{2}, -E\frac{(a+3)^2}{4}, n\right).$$

(3.54)

The eigenvalue $E$ of the potential $V(\rho)$ given by Eq. (3.53) then can be solved directly.

3.4 General polynomial potentials

The original Newton duality is only for power potentials in classical mechanics. In section 2.3, we generalize Newton’s original result to the case of general polynomial potentials in classical mechanics. In this section, we provide the quantum version of the generalized Newton duality for general polynomial potentials.

In the following, we will show that the duality among general polynomial potentials.

Theorem 8 Two general polynomial potentials

$$U(r) = \xi r^{a+1} + \sum_n \mu_n r^{b_n+1}$$

and

$$V(\rho) = \eta \rho^{A+1} + \sum_n \lambda_n \rho^{B_n+1}$$

(3.55)

are quantum Newtonianly dual to each other, if

$$\frac{a+3}{2} = \frac{2}{A+3},$$

(3.56)

$$\sqrt{\frac{2}{a+3}} (b_n + 3) = \sqrt{\frac{2}{A+3}} (B_n + 3).$$

(3.57)

The bound-state eigenfunction of the energy $E$ and the angular quantum number $\ell$ of the potential $V(\rho)$ can be obtained by performing the replacements

$$E \rightarrow -\eta \left(\frac{2}{A+3}\right)^2,$$

(3.58)

$$\xi \rightarrow -E \left(\frac{2}{A+3}\right)^2,$$

(3.59)

$$\mu_n \rightarrow \left(\frac{2}{A+3}\right)^2 \lambda_n$$

(3.60)

to the bound-state eigenfunction of the energy $E$ and the angular quantum number $l$ of the potential $U(r)$. As a result, the duality relation of the angular momentum between these two dual systems is

$$l + \frac{1}{2} \rightarrow \frac{2}{A+3} \left(\ell + \frac{1}{2}\right),$$

(3.61)

the transformation of coordinates is

$$r \rightarrow \rho^{\frac{A+3}{2}} \text{ or } r^{\frac{a+3}{2}} \rightarrow \rho,$$

(3.62)
and the transformation of the eigenfunction is

\[ u(r) \rightarrow \rho^{(A+1)/4} v(\rho) \quad \text{or} \quad r^{(a+1)/4} u(r) \rightarrow v(\rho). \]  

(3.63)

**Proof.** The radial equation of the general polynomial potential \( U(r) = \xi r^{a+1} + \sum \mu_n r^{b_n+1} \) reads

\[
\frac{d^2 u(r)}{dr^2} + \left[ E - \frac{l(l+1)}{r^2} - \xi r^{a+1} - \sum_n \mu_n r^{b_n+1} \right] u(r) = 0. \tag{3.64}
\]

Substituting the duality relations (3.56), (3.57), (3.58), (3.59), and (3.60) into the radial equation of \( U(r) \), Eq. (3.64), and rearranging the equation as

\[
\frac{d^2 u(r)}{dr^2} + \left( \frac{2}{A+3} \right)^2 \left[ \mathcal{E} - \frac{(A+3)^2}{2} \frac{\ell(\ell+1)}{(A+3)^2} - \frac{\eta}{(A+3)^2} \right] u(r) = 0,
\]

(3.65)

we can see that this is just the radial equation of the potential \( V(\rho) \) with the replacements (3.61), (3.62), and (3.63):

\[
\frac{d^2 v(\rho)}{d\rho^2} + \left[ \mathcal{E} - \frac{\ell(\ell+1)}{\rho^2} - \frac{\eta A+1}{\rho^2} - \sum \lambda_n \rho^{B_n+1} \right] v(\rho) = 0. \tag{3.66}
\]

By Theorem 8, we can obtained the eigenvalue of \( V(\rho) \).

**Corollary 9** If the eigenvalue of the general polynomial potential \( U(r) = \xi r^{a+1} + \sum \mu_n r^{b_n+1} \) is

\[ E = f(n_r, l, \xi, \mu_n), \]  

(3.67)

where \( n_r \) is the radial quantum number and \( l \) is the angular quantum number, then the eigenvalue of its Newtonianly dual potential \( V(\rho) = \eta \rho^{A+1} + \sum \lambda_n \rho^{B_n+1} \) is

\[ \mathcal{E} = -f^{-1} \left( \frac{n_r}{A+3} \frac{\ell + 1}{2} - \frac{1}{2} \eta \left( \frac{2}{A+3} \right)^2 \right), \]  

(3.68)

where \( f^{-1} \) denotes the inverse function of \( f \) and \( \ell \) is the angular quantum number of the system of \( V(\rho) \).

**Proof.** Substituting the relation between the two dual potentials, Eqs. (3.58), (3.59), (3.60) and (3.61), into the expression of the eigenvalue of the potential \( U(r) \), Eq. (3.67), gives

\[ -\eta \left( \frac{2}{A+3} \right)^2 = f \left( n_r, \frac{2}{A+3} \left( \ell + \frac{1}{2} \right) - \frac{1}{2} \mathcal{E} \right). \]  

(3.69)

The eigenvalue \( \mathcal{E} \) of the potential \( V(\rho) \) given by Eq. (3.68) then can be solved directly. ■

In the following, we show how the result given by Theorem 8 is reached.
The duality between two general polynomial potentials, according to Condition 3.9, is obtained by choosing the term \( g'(\rho)^2 / g^2(\rho) \) in Eq. (3.8) as the centrifugal-potential term or a part of the centrifugal-potential term, i.e.,

\[
g'(\rho)^2/g^2(\rho) = \frac{\gamma^2}{\rho^2} \tag{3.70}
\]

with \( \gamma \) a constant. Solving Eq. (3.70) gives

\[
g(\rho) = \rho^\gamma. \tag{3.71}
\]

Substituting Eq. (3.71) into the radial equation (3.8) gives

\[
\frac{d^2 v(\rho)}{d\rho^2} + \left[ -\frac{\gamma^2}{\rho^{2(1-\gamma)}} U(\rho) - \frac{l(l+1)\gamma^2 - (1-\gamma^2)/4}{\rho^2} + \frac{E\gamma^2}{\rho^{2(1-\gamma)}} \right] v(\rho) = 0. \tag{3.72}
\]

Here the term \( \left[ l(l+1)\gamma^2 - (1-\gamma^2)/4 \right]/\rho^2 \) is the centrifugal-potential term.

According to Condition 3.10, we need a constant term serving as the eigenvalue \( E \). Choosing

\[
-\frac{\gamma^2}{\rho^{2(1-\gamma)}} U(\rho) = E - \sum \kappa_n \rho^n, \tag{3.73}
\]

where \( E \) is the eigenvalue, gives

\[
U(\rho) = -\frac{E}{\gamma^2} \rho^{2(1-\gamma)} + \sum \frac{\kappa_n}{\gamma^2} \rho^{2(1-\gamma)+n}. \tag{3.74}
\]

The transformation of coordinates between two dual potentials, by Eqs. (3.3) and (3.71), is

\[
r \rightarrow \rho^\gamma. \tag{3.75}
\]

By this transformation of coordinates, we arrive at the potential

\[
U(r) = -\frac{E}{\gamma^2} r^{2(1-\gamma)/\gamma} + \sum \frac{\kappa_n}{\gamma^2} r^{(2(1-\gamma)+n)/\gamma}. \tag{3.76}
\]

By Eqs. (3.72) and (3.74), Eq. (3.8) becomes

\[
\frac{d^2 v(\rho)}{d\rho^2} + \left[ E - \frac{l(l+1)\gamma^2 - (1-\gamma^2)/4}{\rho^2} + \frac{E\gamma^2}{\rho^{2(1-\gamma)}} - \sum \kappa_n \rho^n \right] v(\rho) = 0. \tag{3.77}
\]

This is just the radial equation of the potential

\[
V(\rho) = -\frac{E\gamma^2}{\rho^{2(1-\gamma)}} + \sum \kappa_n \rho^n. \tag{3.78}
\]

Rewriting these two potentials as

\[
U(r) = \xi r^{a+1} + \sum \mu_n r^{b_n+1} \tag{3.79}
\]
with
\[ \gamma = \frac{2}{a+3}, \quad (3.80) \]
\[ b_n = \frac{(2+n)(a+3)}{2} - 3, \quad \kappa_n = \left( \frac{2}{a+3} \right)^2 \mu_n, \quad \mathcal{E} = -\left( \frac{2}{a+3} \right)^2 \xi \quad (3.81) \]
and
\[ V(\rho) = \eta \rho^{4+1} + \sum_n \lambda_n \rho^{B_n+1} \quad (3.82) \]
with
\[ \gamma = \frac{A+3}{2}, \quad (3.83) \]
\[ B_n = n-1, \quad \lambda_n = \kappa_n, \quad E = -\eta \left( \frac{2}{A+3} \right)^2, \quad (3.84) \]
we then arrive at the duality relations (3.56), (3.57), (3.58), (3.59), and (3.60).

The term which is proportional to $1/\rho^2$ plays a role of the centrifugal potential, so
\[ \ell (\ell + 1) = l (l + 1) \gamma^2 - \frac{1}{4} (1 - \gamma^2) \]
\[ = l (l + 1) \left( \frac{A+3}{2} \right)^2 - \frac{1}{4} \left[ 1 - \left( \frac{A+3}{2} \right)^2 \right]. \quad (3.85) \]

Then we arrive at the replacement of the angular momentum given by Eq. (3.61).

The transformation of coordinates (3.62) can be obtained by substituting Eq. (3.83) into Eq. (3.71). Then the transformation of the eigenfunction, by using (3.7), is
\[ u(r) \rightarrow \left( \frac{A+3}{2} \right)^{1/2} \rho^{(A+1)/4} v(\rho). \quad (3.86) \]
The constant factor $\left( \frac{A+3}{2} \right)^{1/2}$ does not change the wave function, so the transformation (3.86) can be written as the transformation (3.63).

In the above, we provide a quantum Newton duality between general polynomial potentials. The general polynomial potential is a very general potential. In fact, the general polynomial potential contains all potentials that can be expressed as a series of power potentials with arbitrary real number powers. Therefore, many potentials can be considered as a special case of general polynomial potentials and can be treated by use of the Newton duality of general polynomial potentials.

### 3.5 A $N$-term potential and its $N$ dual potentials

Inspecting the result of the quantum Newton duality among general polynomial potentials, we can see that a $N$-term potential has $N$ dual potentials.
3.5.1 A $N$-term potential and its $N$ dual potentials: general discussion

As a direct result of Theorem 8, we have the following conclusion, which is useful in solving various potentials.

$N + 1$ $N$-term general polynomial potentials

$$\{V_1(r), \cdots, V_P(r), \cdots, V_Q(r), \cdots, V_{N+1}(r)\} \quad (3.87)$$

are quantum Newtonianly dual to each other, if any two potentials of them, say

$$V_P(r) = A_1r^{\alpha_1+1} + \cdots + A_i r^{\alpha_i+1} + \cdots + A_N r^{\alpha_N+1}, \quad (3.88)$$

$$V_Q(r) = B_1 r^{\beta_1+1} + \cdots + B_j r^{\beta_j+1} + \cdots + B_N r^{\beta_N+1}, \quad (3.89)$$

satisfy the relations

$$\frac{\alpha_i + 3}{2} = \frac{2}{\beta_j + 3}, \quad i, j = 1, \cdots, N; \quad (3.90)$$

$$\sqrt{\frac{2}{\alpha_m + 3}} (\alpha_m + 3) = \sqrt{\frac{2}{\beta_n + 3}} (\beta_n + 3), \quad m, n \neq i, j, \quad (3.91)$$

where $P, Q = 1, \cdots, N + 1$ and $m, n = 1, \cdots, N$. The bound-state eigenfunction of the energy $E_Q$ and the angular quantum number $\ell$ of the potential $V_Q(r)$ can be obtained by performing the replacements

$$E_P \rightarrow -B_j \left(\frac{2}{\beta_j + 3}\right)^2, \quad (3.92)$$

$$A_i \rightarrow -E_Q \left(\frac{2}{\beta_j + 3}\right)^2, \quad (3.93)$$

$$A_m \rightarrow B_n \left(\frac{2}{\beta_j + 3}\right)^2 \quad (3.94)$$

to the bound-state eigenfunction of the energy $E_P$ and the angular quantum number $l$ of the potential $V_P(r)$. As a result, the duality relation of the angular momentum between the two dual potentials $V_P(r)$ and $V_Q(r)$ is

$$l_P + \frac{1}{2} = \frac{2}{\beta_j + 3} \left(l_Q + \frac{1}{2}\right), \quad (3.95)$$

the transformation of coordinates is

$$r_P \rightarrow r_Q^{\beta_j+3} \quad \text{or} \quad r_P^{\frac{\beta_j+3}{2}} \rightarrow r_Q, \quad (3.96)$$

and the transformation of the eigenfunction is

$$u_P(r_P) \rightarrow r_Q^{(\beta_j+1)/4} u_Q(r_Q) \quad \text{or} \quad r_P^{(A_i+1)/4} u_P(r_P) \rightarrow u_Q(r_Q). \quad (3.97)$$

This result is in fact a corollary of Theorem 8.

In the following, we illustrate the dualities among $N + 1$ dual potentials.
3.5.2 A 2-term potential and its two dual potentials: examples

Three 2-term general polynomial potentials \( \{ V_1 (r), V_2 (r), V_3 (r) \} \),

\[
V_1 (r) = A_1 r^{\alpha_1+1} + B_1 r^{\beta_1+1},
V_2 (r) = A_2 r^{\alpha_2+1} + B_2 r^{\beta_2+1},
V_3 (r) = A_3 r^{\alpha_3+1} + B_3 r^{\beta_3+1},
\]

are dual to each other, if they satisfy the relation

\[
\frac{\alpha_1 + 3}{2} = \frac{2}{\alpha_2 + 3}, \quad \sqrt{\frac{2}{\alpha_1 + 3}} (\beta_1 + 3) = \sqrt{\frac{2}{\alpha_2 + 3}} (\beta_2 + 3),
\]

\[
\frac{\beta_1 + 3}{2} = \frac{2}{\beta_3 + 3}, \quad \sqrt{\frac{2}{\beta_1 + 3}} (\alpha_1 + 3) = \sqrt{\frac{2}{\beta_3 + 3}} (\alpha_3 + 3).
\]

Their eigenvalues of bound states and the coupling constants have the relation

\[
E_1 = -A_2 \left( \frac{2}{\alpha_2 + 3} \right)^2, \quad A_1 = -E_2 \left( \frac{2}{\alpha_2 + 3} \right)^2, \quad B_1 = B_2 \left( \frac{2}{\alpha_2 + 3} \right)^2,
\]

\[
E_1 = -B_3 \left( \frac{2}{\beta_3 + 3} \right)^2, \quad B_1 = -E_3 \left( \frac{2}{\beta_3 + 3} \right)^2, \quad A_1 = A_3 \left( \frac{2}{\beta_3 + 3} \right)^2
\]

and their angular momenta have the relation

\[
l_1 + \frac{1}{2} = \frac{2}{\alpha_2 + 3}\left( l_2 + \frac{1}{2} \right),
\]

\[
l_1 + \frac{1}{2} = \frac{2}{\beta_3 + 3}\left( l_3 + \frac{1}{2} \right).
\]

Their eigenfunctions are related by the transformations

\[
r_1 \rightarrow r_2^{\alpha_2+3} \quad \text{or} \quad r_1^{\alpha_1+3} \rightarrow r_2,
\]

\[
r_1 \rightarrow r_3^{\beta_3+3} \quad \text{or} \quad r_1^{\beta_1+3} \rightarrow r_3
\]

and

\[
u_1 (r_1) \rightarrow r_2^{(\alpha_1+1)/4} u_2 (r_2) \quad \text{or} \quad r_1^{(\alpha_1+1)/4} u_1 (r_1) \rightarrow u_2 (r_2),
\]

\[
u_1 (r_1) \rightarrow r_3^{(\beta_1+1)/4} u_3 (r_3) \quad \text{or} \quad r_1^{(\beta_1+1)/4} u_1 (r_1) \rightarrow u_3 (r_3).
\]

3.5.3 A 3-term potential and its three dual potentials

Four general polynomial potentials \( \{ V_1 (r), V_2 (r), V_3 (r), V_4 (r) \} \),

\[
V_1 (r) = A_1 r^{\alpha_1+1} + B_1 r^{\beta_1+1} + C_1 r^{\gamma_1+1},
V_2 (r) = A_2 r^{\alpha_2+1} + B_2 r^{\beta_2+1} + C_2 r^{\gamma_2+1},
V_3 (r) = A_3 r^{\alpha_3+1} + B_3 r^{\beta_3+1} + C_3 r^{\gamma_3+1},
V_4 (r) = A_4 r^{\alpha_4+1} + B_4 r^{\beta_4+1} + C_4 r^{\gamma_4+1},
\]

(3.106)
2.3.2

Their eigenvalues of bound states and the coupling constants have the relation

\[
\begin{align*}
\frac{\alpha_1 + 3}{2} &= \frac{2}{\alpha_2 + 3}, \\
\frac{\beta_1 + 3}{2} &= \frac{2}{\beta_3 + 3}, \\
\gamma_1 + 3 &= \frac{2}{\gamma_4 + 3}, \\
\sqrt{\frac{2}{\alpha_1 + 3}} (\beta_1 + 3) &= \sqrt{\frac{2}{\alpha_2 + 3}} (\beta_2 + 3), \\
\sqrt{\frac{2}{\beta_1 + 3}} (\alpha_1 + 3) &= \sqrt{\frac{2}{\beta_3 + 3}} (\alpha_3 + 3), \\
\sqrt{\frac{2}{\gamma_1 + 3}} (\alpha_1 + 3) &= \sqrt{\frac{2}{\gamma_4 + 3}} (\alpha_4 + 3), \\
\sqrt{\frac{2}{\gamma_1 + 3}} (\beta_1 + 3) &= \sqrt{\frac{2}{\gamma_4 + 3}} (\beta_4 + 3). 
\end{align*}
\]

(3.107)

Their eigenvalues of bound states and the coupling constants have the relation

\[
\begin{align*}
E_1 &= -A_2 \left( \frac{2}{\alpha_2 + 3} \right)^2, & A_1 &= -E_2 \left( \frac{2}{\alpha_2 + 3} \right)^2, & B_1 &= B_2 \left( \frac{2}{\alpha_2 + 3} \right)^2, & C_1 &= C_2 \left( \frac{2}{\alpha_2 + 3} \right)^2, \\
E_1 &= -B_3 \left( \frac{2}{\beta_3 + 3} \right)^2, & B_1 &= -E_3 \left( \frac{2}{\beta_3 + 3} \right)^2, & A_1 &= A_3 \left( \frac{2}{\beta_3 + 3} \right)^2, & C_1 &= C_3 \left( \frac{2}{\beta_3 + 3} \right)^2, \\
E_1 &= -C_4 \left( \frac{2}{\gamma_4 + 3} \right)^2, & C_1 &= -C_4 \left( \frac{2}{\gamma_4 + 3} \right)^2, & A_1 &= A_4 \left( \frac{2}{\gamma_4 + 3} \right)^2, & B_1 &= B_4 \left( \frac{2}{\gamma_4 + 3} \right)^2.
\end{align*}
\]

(3.108)

and their angular momenta have the relation

\[
\begin{align*}
l_1 + \frac{1}{2} &= \frac{2}{\alpha_2 + 3} \left( l_2 + \frac{1}{2} \right), \\
l_1 + \frac{1}{2} &= \frac{2}{\beta_3 + 3} \left( l_3 + \frac{1}{2} \right), \\
l_1 + \frac{1}{2} &= \frac{2}{\gamma_4 + 3} \left( l_4 + \frac{1}{2} \right).
\end{align*}
\]

(3.109)

Their eigenfunctions are related by the transformations

\[
\begin{align*}
r_1 \rightarrow \frac{\alpha_2 + 3}{2} r_2, & \quad \text{or} \quad r_1 \rightarrow \frac{\alpha_1 + 3}{2}, \\
r_1 \rightarrow \frac{\beta_3 + 3}{2} r_3, & \quad \text{or} \quad r_1 \rightarrow \frac{\beta_1 + 3}{2}, \\
r_1 \rightarrow \frac{\gamma_4 + 3}{2}, & \quad \text{or} \quad r_1 \rightarrow \frac{\gamma_1 + 3}{2} r_4.
\end{align*}
\]

(3.110)

and

\[
\begin{align*}
u_1 (r_1) \rightarrow r_2^{(\alpha_2 + 1)/4} u_2 (r_2) & \quad \text{or} \quad r_1^{(\alpha_1 + 1)/4} u_1 (r_1) \rightarrow u_2 (r_2), \\
u_1 (r_1) \rightarrow r_3^{(\beta_3 + 1)/4} u_2 (r_3) & \quad \text{or} \quad r_1^{(\beta_1 + 1)/4} u_1 (r_1) \rightarrow u_2 (r_3), \\
u_1 (r_1) \rightarrow r_4^{(\gamma_4 + 1)/4} u_4 (r_4) & \quad \text{or} \quad r_1^{(\gamma_1 + 1)/4} u_1 (r_1) \rightarrow u_4 (r_4).
\end{align*}
\]

(3.111)

3.6 General polynomial potentials consist of two Newton dual potentials

In section 2.3.2, we discuss a special case of the classical Newton duality, in which each term of a potential is Newtonianly dual to the corresponding term of its Newton dual potential. In this section, we consider the quantum version of such a case.
Theorem 10  Two Newtonianly dual general polynomial potentials

\[ U(r) = \xi r^{a+1} + \mu r^2 \left( \sqrt{\frac{a+3}{2}} - 1 \right), \quad (3.112) \]
\[ V(\rho) = \eta \rho^{A+1} + \mu \left( \frac{A+3}{2} \right)^2 \rho^2 \left( \sqrt{\frac{A+3}{2}} - 1 \right), \quad (3.113) \]

in which \( \xi r^{a+1} \) is the Newton duality of \( \eta \rho^{A+1} \) and \( \mu r^2 \left( \sqrt{\frac{a+3}{2}} - 1 \right) \) is the Newton duality of \( \mu \left( \frac{A+3}{2} \right)^2 \rho^2 \left( \sqrt{\frac{A+3}{2}} - 1 \right) \), if

\[ \frac{a+3}{2} = \frac{2}{A+3}. \quad (3.114) \]

The proof is straightforward by the duality relation given above. An example will be given in Sec. 4.3.4.

3.7 The quantum Newton duality between \( U(r) = \xi e^{sr} \) and \( V(\rho) = \frac{\eta}{(\rho \ln \alpha)^2} \)

In this section, we give an example of the Newton duality between two transcendental-function potentials.

Theorem 11  Two potentials

\[ U(r) = \xi e^{sr} \quad \text{and} \quad V(\rho) = \frac{\eta}{(\rho \ln \alpha)^2} \quad (3.115) \]

are quantum Newtonianly dual to each other. The duality relations are

\[ E \rightarrow -\left( \frac{\sigma}{2} \right)^2 \left[ \ell (\ell + 1) + \frac{1}{4} \right], \quad (3.116) \]
\[ \xi \rightarrow -\left( \frac{\sigma}{2} \right)^2 \frac{E}{\alpha^2}, \quad (3.117) \]
\[ l (l + 1) \rightarrow \eta, \quad (3.118) \]

or, equivalently,

\[ E \rightarrow -\left( \frac{2}{\sigma} \right)^2 \alpha^2 \xi, \quad (3.119) \]
\[ \eta \rightarrow l (l + 1), \quad (3.120) \]
\[ \ell (\ell + 1) \rightarrow -\left( \frac{2}{\sigma} \right)^2 E - \frac{1}{4}, \quad (3.121) \]

Here \( E, \xi, \) and \( l \) are the eigenvalues, the coupling constants, and the angular momentum of the \( U(r) \) system; \( \mathcal{E}, \eta, \) and \( \ell \) are those of the \( V(\rho) \) system. As a result, the corresponding transformation of coordinates is

\[ r \rightarrow \frac{2}{\sigma} \ln \alpha \rho \quad \text{or} \quad \frac{1}{\alpha} e^{sr} \rightarrow \rho, \quad (3.122) \]

and the transformation of the eigenfunction is

\[ u(r) \rightarrow \rho^{-1/2} v(\rho) \quad \text{or} \quad e^{sr} u(r) \rightarrow v(\rho). \quad (3.123) \]
Proof. First, we prove the dual transformation from $U(r)$ to $V(\rho)$. The radial equation of the central potential $U(r) = \xi e^{\sigma r}$ reads
\[
\frac{d^2 u(r)}{dr^2} + \left[ E - \frac{l(l+1)}{r^2} - \xi e^{\sigma r} \right] u(r) = 0. \tag{3.124}
\]
Substitute the replacements (3.122) and (3.123) into Eq. (3.124) and rearrange the equation as
\[
\frac{d^2 v(\rho)}{d\rho^2} + \left[ \left( \frac{2}{\sigma} \right)^2 \alpha^2 \xi - \frac{(\frac{2}{\sigma})^2 E - 1/4}{\rho^2} - \frac{l(l+1)}{(\rho \ln \alpha \rho)^2} \right] v(\rho) = 0. \tag{3.125}
\]
Substituting the replacements (3.116), (3.117), and (3.118) into Eq. (3.125) gives the radial equation of the potential $V(\rho)$,
\[
\frac{d^2 v(\rho)}{d\rho^2} + \left[ E - \frac{l(l+1)}{\rho^2} - \frac{\eta}{(\rho \ln \alpha \rho)^2} \right] v(\rho) = 0. \tag{3.126}
\]

Second, we prove the dual transformation from $V(\rho)$ to $U(r)$. The radial equation of the central potential $U(r) = \frac{\xi}{(r \ln \beta r)^2}$ reads
\[
\frac{d^2 u(r)}{dr^2} + \left[ E - \frac{l(l+1)}{r^2} - \frac{\xi}{(r \ln \beta r)^2} \right] u(r) = 0. \tag{3.127}
\]
Substitute the replacements (3.122) and (3.123) into (3.127) and rearrange the equation as
\[
\frac{d^2 v(\rho)}{d\rho^2} + \left[ - \left( \frac{\sigma}{2} \right)^2 l(l+1) - \left( \frac{\sigma}{2} \right)^2 \frac{1}{4} - \frac{\xi}{\rho^2} + \left( \frac{\sigma}{2} \right)^2 \frac{E}{\beta^2} e^{\sigma \rho} \right] v(\rho) = 0. \tag{3.128}
\]
Substituting the replacements (3.119), (3.120), and (3.121) into Eq. (3.128) gives the radial equation of the potential $V(\rho)$,
\[
\frac{d^2 v(\rho)}{d\rho^2} + \left[ E - \frac{l(l+1)}{\rho^2} - \eta e^{\sigma \rho} \right] v(\rho) = 0. \tag{3.129}
\]

In the following, we show how the result given by Theorem 11 is reached.

The duality transformation from $U(r)$ to $V(\rho)$. The duality transformation from $U(r)$ to $V(\rho)$ is obtained by choosing the term $Eg'(\rho)^2$ in Eq. (3.8) to be a part of the centrifugal potential term in Condition 3.9, i.e.,
\[
g'(\rho)^2 = \frac{\gamma^2}{\rho^2}, \tag{3.130}
\]
so
\[
g(\rho) = \gamma \ln \alpha \rho \tag{3.131}
\]
with $\alpha$ an arbitrary constant.
Substituting Eq. (3.131) into Eq. (3.8) gives

\[
\frac{d^2 v(\rho)}{d\rho^2} + \left[ -\frac{\gamma^2}{\rho^2} U(\rho) + \frac{E\gamma^2 + 1/4}{\rho^2} - \frac{l(l+1)}{(\rho \ln \alpha \rho)^2} \right] v(\rho) = 0. \quad (3.132)
\]

Here the term \((E\gamma^2 + 1/4)/\rho^2\) is the centrifugal potential term.

Now we need a term serving as the term of the eigenvalue \(E\) according to Condition 3.10. Choosing

\[-\frac{\gamma^2}{\rho^2} U(\rho) = E\]

(3.133)

gives

\[U(\rho) = -\frac{\rho^2}{\gamma^2} E. \quad (3.134)\]

By Eq. (3.131), we obtain the transformation of coordinates between two dual potentials:

\[r \rightarrow \gamma \ln \alpha \rho. \quad (3.135)\]

Then we have

\[U(r) = -\frac{E}{\alpha^2 \gamma^2} e^{2r/\gamma}. \quad (3.136)\]

By Eqs. (3.132) and (3.134), Eq. (3.8) becomes

\[
\frac{d^2 v(\rho)}{d\rho^2} + \left[ E - \frac{E\gamma^2 - 1/4}{\rho^2} - \frac{l(l+1)}{(\rho \ln \alpha \rho)^2} \right] v(\rho) = 0. \quad (3.137)
\]

This is just the radial equation of the potential

\[V(\rho) = \frac{l(l+1)}{(\rho \ln \alpha \rho)^2}. \quad (3.138)\]

Rewrite these two potentials \(U(r)\) and \(V(\rho)\) as

\[U(r) = \xi e^{\sigma r} \quad (3.139)\]

with

\[\gamma = \frac{2}{\sigma}, \quad (3.140)\]

\[\xi = -\left(\frac{\sigma^2}{2}\right)^2 \frac{E}{\alpha^2} \quad (3.141)\]

and

\[V(\rho) = \frac{\eta}{(\rho \ln \alpha \rho)^2} \quad (3.142)\]

with

\[\eta = l(l+1). \quad (3.143)\]

The term being proportional to \(1/\rho^2\) in Eq. (3.137) is the centrifugal potential, so

\[\ell (\ell + 1) = -E\gamma^2 - \frac{1}{4}. \quad (3.144)\]
By Eqs. (3.140), (3.141), (3.143), and (3.144), we obtain the duality relations (3.116), (3.117), and (3.118). By Eqs. (3.135) and (3.140), we obtain the transformation of coordinates (3.122). Then by Eq. (3.130), we obtain the transformation of the eigenfunction,

\[ u(r) \rightarrow \sqrt{\frac{\gamma}{\rho \sigma}} v(\rho). \]  

(3.145)

Dropping the constant factor, we arrive at the replacement (3.123).

The duality transformation from \( V(\rho) \) to \( U(r) \). First the duality transformation from \( V(\rho) \) to \( U(r) \) is obtained by choosing

\[ g'(\rho)^2 U(\rho) = \frac{\gamma^2}{\rho^2} \] 

(3.146)

in Eq. (3.8) to be a part of the centrifugal potential term in Condition 3.9. This gives

\[ g(\rho) = \int \frac{\gamma}{\rho \sqrt{U(\rho)}} d\rho. \] 

(3.147)

Substituting Eq. (3.147) into Eq. (3.8), we have

\[
\frac{d^2 v(\rho)}{d\rho^2} + \left[ \frac{\gamma^2 E}{\rho^2 U(\rho)} + \frac{1/4 - \gamma^2}{\rho^2} + \frac{3}{16} \frac{U'(\rho)^2}{U(\rho)^2} - \frac{1}{4\rho} \frac{U'(\rho)}{U(\rho)} - \frac{U''(\rho)}{4U(\rho)} - \frac{l(l + 1)}{\rho^2 U(\rho)} \left( \int \frac{1}{\rho \sqrt{U(\rho)}} d\rho \right)^2 \right] v(\rho) = 0.
\] 

(3.148)

Here the term \((1/4 - \gamma^2)/\rho^2 \) is a part of the centrifugal-potential term.

Second, let us choose a term serving as the term of the eigenvalue \( \mathcal{E} \) according to Condition 3.10. Choosing

\[ \frac{1}{\rho^2 U(\rho)} \left( \int \frac{1}{\rho \sqrt{U(\rho)}} d\rho \right)^2 = \alpha^2 \] 

(3.149)

gives

\[ U(\rho) = \frac{e^{-2\alpha \rho}}{\rho^2}. \] 

(3.150)

Then by Eq. (3.147) we obtain

\[ g(\rho) = \frac{\gamma}{\alpha} e^{\alpha \rho}, \] 

(3.151)

so the replacement of coordinates between two dual potentials is

\[ r \rightarrow \frac{\gamma}{\alpha} e^{\alpha \rho}. \] 

(3.152)

Then we have

\[ U(r) = \frac{\gamma^2}{(\ln \frac{2}{\gamma} r)^2}. \] 

(3.153)

By Eqs. (3.147) and (3.150), Eq. (3.8) becomes

\[
\frac{d^2 v(\rho)}{d\rho^2} + \left\{ - \left[ l(l + 1) + \frac{1}{4} \right] \alpha^2 - \frac{\gamma^2}{\rho^2} + \gamma^2 E e^{2\alpha \rho} \right\} v(\rho) = 0
\] 

(3.154)
This is just the radial equation of the potential

\[ V(\rho) = -\gamma^2 E e^{2\rho}. \]  \hfill (3.155)

Rewrite these two potentials \( U(r) \) and \( V(\rho) \) as

\[ U(r) = \frac{\xi}{(r \ln \beta r)^2} \]  \hfill (3.156)

with

\[ \gamma = \frac{\alpha}{\beta} \]  \hfill (3.157)

and

\[ V(\rho) = \eta e^{\sigma \rho} \]  \hfill (3.158)

with

\[ \alpha = \frac{\sigma}{2}, \]  \hfill (3.159)

\[ \gamma^2 = -\frac{\eta}{E}. \]  \hfill (3.160)

The term being proportional to \( 1/\rho^2 \) in Eq. (3.137) is the centrifugal potential, so

\[ \ell (\ell + 1) = \gamma^2. \]  \hfill (3.161)

By Eqs. (3.157), (3.159), (3.160) and (3.161), we obtain the duality relations (3.119), (3.120), and (3.121). By Eqs. (3.157) and (3.159), we obtain the transformation of coordinates (3.122). Then by Eq. (3.151), we obtain the replacement of the eigenfunction,

\[ u(r) \rightarrow \xi^{1/4} e^{\sigma \rho} v(\rho). \]  \hfill (3.162)

Dropping the constant factor, we arrive at the replacement (3.123).

4 The quantum Newton duality as a method for solving eigenproblems

4.1 General discussions

The Newton duality reveals an internal connection among various potentials. Once a solution of a potential is known, one can immediately obtain the solution of its dual potentials by the duality relation. That is to say, the Newton duality can serve as a method for solving eigenproblems.

Concretely, as shown in section 3.5, \( N + 1 \) \( N \)-term general polynomial potentials can form a dual-potential set

\[ \{ V_1(r), V_2(r), \ldots, V_{N+1}(r) \} \]  \hfill (4.1)

in which any two potentials are dual to each other. That is to say, once a solution of a potential in a dual-potential set is known, the solution of the other \( N \) potentials in the set can be immediately obtained by the duality relation.
For one-term potentials, the duality set is \((a r^{\alpha+1}, A r^{\alpha+\frac{1}{2} - 2})\), for two-term potentials, the duality set is \((a r^\alpha + b r^\beta, A_1 r^{\alpha+2} + B_1 r^{\alpha+2(\beta+2)-2}, A_2 r^{\alpha+2(\alpha+2)-2} + B_2 r^{\alpha+2(\beta+2)-2}, A_3 r^{\alpha+2(\alpha+2)-2} + B_3 r^{\alpha+2(\beta+2)-2})\), for three-term potentials, the duality set is \((a r^\alpha + b r^\beta + c r^\gamma, A_1 r^{\alpha+2} + B_1 r^{\alpha+2(\beta+2)-2} + C_1 r^{\alpha+2(\gamma+2)-2}, A_2 r^{\alpha+2(\alpha+2)-2} + B_2 r^{\alpha+2(\beta+2)-2} + C_2 r^{\alpha+2(\gamma+2)-2}, A_3 r^{\alpha+2(\alpha+2)-2} + B_3 r^{\alpha+2(\beta+2)-2} + C_3 r^{\alpha+2(\gamma+2)-2})\), and so on.

In the following, we show how this works.

4.2 Power potentials

First consider some power potentials, including three-dimensional and arbitrary-dimensional harmonic-oscillator potentials and Coulomb potentials, \(r^{2/3}\)-potential and \(1/\sqrt{r}\)-potential, and \(1/r^{3/2}\)-potential and \(r^6\)-potential.

4.2.1 The harmonic-oscillator potential and the Coulomb potential: three dimensions and arbitrary dimensions

To show how to solve a potential from its Newton duality, we first use the harmonic-oscillator potential and the Coulomb potential in three dimensions and in arbitrary dimensions as examples. The harmonic-oscillator potential and the Coulomb potential are Newtonianly dual to each other [2], so we can solve one from another.

The harmonic-oscillator potential and the Coulomb potential: three dimensions

The harmonic-oscillator potential and the Coulomb potential form a duality set

\[
(\xi r^2, \frac{\eta}{\rho}),
\]

i.e., the harmonic-oscillator potential \(U(r) = \xi r^2\) and the Coulomb potential \(V(\rho) = \frac{\eta}{\rho}\) are Newtonianly dual to each other.

In the following, we solve the Coulomb potential from the harmonic-oscillator potential.

The harmonic-oscillator potential. For the harmonic-oscillator potential \(U(r) = \xi r^2\), i.e., the power potential \(U(r) = \xi r^{\alpha+1}\) with \(\alpha = 1\), the radial equation reads

\[
\frac{d^2 u(r)}{dr^2} + \left(E - \frac{l(l+1)}{r^2} - \xi r^2\right) u(r) = 0.
\]

The radial eigenfunction is [8–10] (see Appendix F)

\[
u_l(r) = A_l e^{-\frac{\xi r^2}{2}} \xi^{l+1/4} r^{l+1} N \left(2l+1, 0, \frac{E}{\xi^{1/2}}, 0, \xi^{1/4} r\right),
\]

where \(N(\alpha, \beta, \gamma, \delta, z)\) is the Heun biconfluent function [8, 9]. To be in accordance with the form of the solution of other potentials discussed later, we first express the solution of the harmonic-oscillator potential by the Heun function. Usually, the eigenfunction of the harmonic-oscillator potential is represented by the hypergeometric function. By the relation between the Heun function and the hypergeometric function [8, 9],

\[
N(\alpha, 0, \gamma, 0, z) = \, _1F_1 \left(\frac{1}{2} + \frac{\alpha}{4} - \frac{\gamma}{4}, 1 + \frac{\alpha}{2}, z^2\right),
\]

\[
- 28 -
\]
the eigenfunction (4.4) reduces to
\[ u_l(r) = A_l e^{-\frac{r^2}{\xi}} \frac{\xi^{(l+1)/2}}{\xi^{l+1}} {}_1F_1\left(\frac{l}{2} + \frac{3}{4} - \frac{E}{4\sqrt{\xi}}, \frac{3}{2} + l, \sqrt{\xi} r^2\right). \] (4.6)

In this approach, the eigenvalue is the zero of the function
\[ K_2\left(2l + 1, 0, \frac{E}{\xi^{1/2}}, 0\right), \] i.e., \[ K_2\left(2l + 1, 0, \frac{E}{\xi^{1/2}}, 0\right) = 0, \] (4.7)

where
\[ K_2(\alpha, \beta, \gamma, \delta) = \frac{\Gamma(1 + \alpha)}{\Gamma((\alpha - \gamma)/2) \Gamma(1 + (\alpha + \gamma)/2)} J_{1+(\alpha+\gamma)/2}\left(\frac{\alpha + \gamma}{2}, \frac{3}{2} \alpha - \frac{1}{2} \gamma, \delta - \frac{\alpha - \gamma}{2}\right) \] with
\[ J_\lambda(\alpha, \beta, \gamma, \delta) = \int_0^\infty x^{\lambda-1} e^{-\beta x - x^2} N(\alpha, \beta, \gamma, \delta, x) \, dx. \] (4.8)

In the case of harmonic-oscillator potentials, \( K_2\left(2l + 1, 0, \frac{E}{\xi^{1/2}}, 0\right) \) reduces to \[ K_2\left(2l + 1, 0, \frac{E}{\xi^{1/2}}, 0\right) = \frac{\Gamma\left(\frac{3}{4} + l + \frac{3}{2}\right)}{\Gamma\left(\frac{3}{4} + l - \frac{E}{4\xi^{1/2}}\right)}. \] (4.10)

Therefore, the eigenvalue is the singularities of \( \Gamma\left(\frac{3}{4} + l + \frac{3}{2}\right)\):
\[ E = 2\sqrt{\xi} \left(2n_r + l + \frac{3}{2}\right), \quad n_r = 0, 1, 2, \ldots, \] (4.11)

where \( n_r \) is the radial quantum number.

The reason why, instead of the Hermite polynomial, we use the Heun function to express the eigenfunction of the harmonic-oscillator potential is that the eigenfunction of the Coulomb potential can also be expressed as a Heun function.

The Coulomb potential. The Coulomb potential is a dual potential of the harmonic-oscillator potential. Then the solution of the Coulomb potential can be obtained from the solution of the harmonic-oscillator potential by the duality relation given by Theorem (4) and Theorem (5).

The radial equation of the Coulomb potential is
\[ \frac{d^2 v(\rho)}{d\rho^2} + \left[\mathcal{E} - \frac{\ell(\ell + 1)}{\rho^2} - \frac{\eta}{\rho}\right] v(\rho) = 0. \] (4.12)

By the duality relations (3.12), (3.13), (3.14), and (3.15) given by Theorem (4), we have the following replacements:
\[ a + \frac{3}{2} \rightarrow \frac{2}{A + 3} \quad \text{with} \quad a = 1, \] (4.13)
\[ E \rightarrow -4\eta, \quad \xi \rightarrow -4\mathcal{E}, \quad l + \frac{1}{2} \rightarrow 2\left(\ell + \frac{1}{2}\right). \] (4.14)
By these duality relations, we can obtain the eigenvalue and eigenfunction of the Coulomb potential from the eigenvalue and eigenfunction of the harmonic-oscillator potential, Eqs (4.4) and (4.11).

The duality transformation of coordinates, Eq. (3.16), and the duality relation of radial eigenfunctions, Eq. (3.17), in this case become

\[ r \rightarrow \rho^{1/2} \text{ and } u(r) \rightarrow \rho^{-1/4} v(\rho). \] (4.15)

**Eigenvalue.** Substituting the duality relation of eigenvalues, Eq. (3.23), into the eigenvalue of the harmonic-oscillator potential, Eq. (4.11), gives the eigenvalue of the Coulomb potential:

\[ E = -\frac{\eta^2}{4} \frac{1}{(n_r + \ell + 1)^2}, \quad n_r = 0, 1, 2, \ldots. \] (4.16)

This is just the eigenvalue of the Coulomb potential [12].

**Eigenfunction.** Starting from the eigenfunction of the harmonic-oscillator potential (4.4), by the duality relations, including the replacements (4.14), the transformation of coordinates, and the transformation of eigenfunctions (4.15), we obtain the eigenfunction of the Coulomb potential:

\[ v(\rho) = A\ell e^{-\sqrt{-E}\rho} \left( 2\sqrt{-E}\rho \right)^{\ell+1} \frac{N \left( 4\ell + 2, 0, -\frac{2\eta}{\sqrt{-E}}, 0, (2\rho)^{1/2} (-E)^{1/4} \right)}{\left( 2\rho \right)^{\ell+1} \left( \ell+1, \frac{\eta}{2\sqrt{-E}}, 2(\ell+1), 2\sqrt{-E}\rho \right)}. \] (4.17)

By the relation between the hypergeometric function and the Heun function, Eq. (4.5), the eigenfunction of the Coulomb potential, Eq. (4.17), becomes

\[ v(\rho) = A\ell e^{-\sqrt{-E}\rho} \left( 2\sqrt{-E}\right)^{\ell+1} \rho^{\ell+1} \frac{1}{F_1 \left( \ell+1 + \frac{\eta}{2\sqrt{-E}}, 2(\ell+1), 2\sqrt{-E}\rho \right)}. \] (4.18)

This is just the familiar form of the eigenfunction of the Coulomb potential [12].

It should be noted here that the same procedure can also be applied to solve the harmonic-oscillator potential from the Coulomb potential.

**n-dimensional harmonic-oscillator potentials and m-dimensional Coulomb potentials** In this section, by the duality relation, we calculate the eigenproblem of the m-dimensional Coulomb potential from the solution of the n-dimensional harmonic-oscillator potential.

The **n-dimensional harmonic-oscillator potential.** In n dimensions, the radial equation for the harmonic-oscillator potential \( U(r) = \xi r^2 \), i.e., the power potential \( U(r) = \xi r^{a+1} \) with \( a = 1 \), reads

\[ \frac{d^2 u(r)}{dr^2} + \left[ E - \frac{(l - 3/2 + n/2)(l - 1/2 + n/2)}{r^2} - \xi r^2 \right] u(r) = 0. \] (4.19)

The n-dimensional harmonic-oscillator radial eigenfunction is

\[ u(r) = A l \xi^{l/4+(n-1)/8} r^{l+n/2-1/2} e^{-\sqrt{\xi} r/2} N \left( 2l + n - 2, 0, E, \xi^{1/2}, 0, \xi^{1/4} r \right). \] (4.20)
By the relation between the hypergeometric function and the Heun function, Eq. (4.5), the $n$-dimensional radial harmonic-oscillator eigenfunction, Eq. (4.20), becomes

$$u_l(r) = A_l \xi^{l/2+(n-1)/8} \rho^{l+n/2-1/2} e^{-\sqrt{\xi} r^2/2} {}_1F_1 \left( \frac{l}{2} + \frac{n}{4} - \frac{E}{4\sqrt{\xi}}; l + \frac{n}{2}; \sqrt{\xi} r^2 \right). \quad (4.21)$$

The eigenvalue is the zero of the function $K_2 \left( 2l + n - 2, 0, \frac{E}{\xi^{1/2}} \right)$, i.e., [10]

$$K_2 \left( 2l + n - 2, 0, \frac{E}{\xi^{1/2}} \right) = \frac{\Gamma \left( l + \frac{n}{2} - 3 \right)}{\Gamma \left( \ell + \frac{n}{2} - \frac{E}{4\xi^{1/2}} \right)} = 0. \quad (4.22)$$

Then the eigenvalue of the bound state is the singularities of $\Gamma \left( \frac{3}{4} + \ell + \frac{n}{2} - \frac{E}{4\xi^{1/2}} \right)$:

$$E = 2\sqrt{\xi} \left( 2n_r + l + \frac{n}{2} \right), \quad n_r = 0, 1, 2, \ldots. \quad (4.23)$$

where $n_r$ is the radial quantum number.

The $m$-dimensional Coulomb potential. The $m$-dimensional Coulomb potential is a dual potential of the $n$-dimensional harmonic-oscillator potential. In the following, we solve the solution of the $m$-dimensional Coulomb potential from the solution of the $n$-dimensional harmonic-oscillator potential.

The radial equation of the $m$-dimensional Coulomb potential is

$$\frac{d^2 v(\rho)}{d\rho^2} + \left[ E - \frac{(\ell - 3/2 + m/2) (\ell - 1/2 + m/2)}{\rho^2} - \frac{\eta}{\rho} \right] v(\rho) = 0. \quad (4.24)$$

By the duality relation between potentials in different dimensions, Eqs. (3.44), (3.45), and (3.46), we have the following replacements:

$$E \rightarrow -4\eta, \quad \xi \rightarrow -4\xi, \quad l + \frac{n}{2} - 1 \rightarrow 2 \left( \ell + \frac{m}{2} - 1 \right). \quad (4.25)$$

Substituting the duality relation, Eq. (3.54), into the bound-state eigenvalue of the $n$-dimensional harmonic-oscillator potential, Eq. (4.23), gives the eigenvalue of the $m$-dimensional Coulomb potential:

$$\mathcal{E} = -\frac{\eta^2/4}{n_r + \ell + m/2 - 1/2}, \quad n_r = 0, 1, 2, \ldots. \quad (4.26)$$

The duality transformation of coordinates, Eq. (3.47), and the duality relation of radial eigenfunctions, Eq. (3.48), in this case become

$$r \rightarrow \rho^{1/2} \text{ and } u(r) \rightarrow \rho^{1/4} v(\rho). \quad (4.27)$$

Substituting the duality relations (4.25) and (4.27) into the eigenfunction of the $n$-dimensional harmonic-oscillator, we obtain the eigenfunction of the $m$-dimensional Coulomb potential:

$$v(\rho) = A_\ell \left( 2\sqrt{-\mathcal{E}} \right)^{\ell+m/2-1/2} \rho^{\ell+m/2-1/2} e^{-\sqrt{-\mathcal{E}} \rho} N \left( 4\ell + 2m - 4, 0, -\frac{2\eta}{\sqrt{-\mathcal{E}}}; 0; (-\mathcal{E})^{1/4} (2\rho)^{1/2} \right). \quad (4.28)$$
By the relation between the hypergeometric function and the Heun function, Eq. (4.5), the eigenfunction of the \( m \)-dimensional Coulomb potential, Eq. (4.28), becomes

\[
v (\rho) = A_\ell \left( 2\sqrt{-E} \right)^{\ell-1/2+m/2} \rho^{\ell-1/2+m/2} e^{-\sqrt{-E} \rho} \, _1F_1 \left( \ell + \frac{m}{2} - \frac{1}{2} + \frac{\eta}{2\sqrt{-E}}; 2\ell + m - 1; 2\sqrt{-E} \rho \right) .
\]

(4.29)

This agrees with the result obtained by directly solving the eigenvalue equation of the \( m \)-dimensional Coulomb potential [13].

### 4.2.2 \( r^{2/3} \)-potential and \( 1/\sqrt{r} \)-potential

In this section, we solve the eigenproblem of the \( r^{2/3} \)-potential from the solution of its duality \( 1/\sqrt{r} \)-potential whose eigenproblem has been exactly solved [10].

The duality of the \( r^{2/3} \)-potential

\[
V (\rho) = \eta \rho^{2/3}
\]

(4.30)

is the inverse-square-root potential

\[
U (r) = \frac{\xi}{\sqrt{r}} .
\]

(4.31)

\( 1/\sqrt{r} \)-potential. The radial equation for the inverse-square-root potential (4.31) reads

\[
\frac{d^2 u (r)}{dr^2} + \left[ E - \ell (\ell + 1) r - \frac{\xi}{\sqrt{r}} \right] u (r) = 0.
\]

(4.32)

The eigenproblem is solved exactly in Ref. [10]. The eigenfunction is

\[
u (r) = A_\ell \exp \left( -\sqrt{-E} r + \frac{\xi}{\sqrt{-E} \sqrt{r}} \right) \left[ 2 (-E)^{1/2} r \right]^{l+1} N \left( 4l + 2, -\frac{\sqrt{2} \xi}{(-E)^{3/4}}, \frac{\xi^2}{2 (-E)^{3/2}}, 0, \sqrt{2} (-E)^{1/2} r \right) .
\]

(4.33)

The eigenvalue is the zero of the function \( K_2 \left( 4l + 2, -\frac{\sqrt{2} \xi}{(-E)^{3/4}}, \frac{\xi^2}{2 (-E)^{3/2}}, 0, \sqrt{2} (-E)^{1/2} r \right) = 0 .
\]

(4.34)

\( r^{2/3} \)-potential. The radial equation of \( r^{2/3} \)-potential is

\[
\frac{d^2 v (\rho)}{d\rho^2} + \left[ E - \ell (\ell + 1) \rho^2 - \eta \rho^{2/3} \right] v (\rho) = 0.
\]

(4.35)

The duality relation between these two dual potentials, by Eqs. (3.13), (3.14), and (3.15), are

\[
E \rightarrow -\frac{9}{16} \eta, \quad \xi \rightarrow -\frac{9}{16} E, \quad \ell + \frac{1}{2} \rightarrow \frac{3}{4} \left( \ell + \frac{1}{2} \right) .
\]

(4.36)

Substituting the duality relation between the eigenvalues into the implicit expression of the eigenvalue of the \( r^{2/3} \)-potential, Eq. (4.30), gives an implicit expression of the eigenvalue of the \( r^{2/3} \)-potential (4.31):

\[
K_2 \left( 3\ell + 3, \frac{3\sqrt{6} E}{247}, \frac{3\xi^2}{8 \eta^{3/2}}, 0 \right) = 0 .
\]

(4.37)
The duality transformation of coordinates, Eq. (3.16), and the duality relation of radial eigenfunctions, Eq. (3.17), in this case become

\[ r^{3/4} \rightarrow \rho \quad \text{and} \quad u(r) \rightarrow \rho^{1/6} v(\rho). \tag{4.38} \]

Substituting the duality relations (4.36) and (4.38) into the radial eigenfunction of the \(1/\sqrt{r}\)-potential, Eq. (4.33), gives the radial eigenfunction of the \(r^{2/3}\)-potential:

\[ v(\rho) = A_l \exp \left( -\frac{3}{4} \eta^{1/2} \rho^{4/3} - \frac{3\sqrt{E}}{4\eta^{1/2} \rho^{2/3}} \left( \frac{3}{2} \eta^{1/2} \right)^{3(l+1/2)/4+1/2} \right) \rho^{\ell+1} \times N \left( 3\ell + \frac{3}{2}, -\frac{\sqrt{6E}}{2\eta^{3/4}}, \frac{3E^2}{8\eta^{3/2}}, 0, \frac{\sqrt{6}}{2} \eta^{1/4} \rho^{2/3} \right). \tag{4.39} \]

### 4.2.3 \(1/r^{3/2}\)-potential and \(r^6\)-potential

In this section, we solve the eigenproblem of the \(r^6\)-potential from its duality \(1/r^{3/2}\)-potential. Nevertheless, these two potentials have not been solved in literature. Additionally, we solve an exact solution of the \(1/r^{3/2}\)-potential in Appendix A.

The duality of the \(r^6\)-potential

\[ V(\rho) = \eta \rho^6, \tag{4.40} \]

according to the duality relation (3.12), is the \(1/r^{3/2}\)-potential

\[ U(r) = \frac{\xi}{r^{3/2}}. \tag{4.41} \]

**1/r^{3/2}-potential.** The radial equation of the \(1/r^{3/2}\)-potential (4.31) reads

\[ \frac{d^2 u(r)}{dr^2} + \left[ E - \frac{l(l+1)}{r^2} - \frac{\xi}{r^{3/2}} \right] u(r) = 0. \tag{4.42} \]

From Appendix A, the eigenfunction of the bound state of the \(1/r^{3/2}\)-potential

\[ u(r) = A_l e^{-\sqrt{-E}r^{l+1}} N \left( 4l + 2, 0, 0, -\frac{4\sqrt{2}\xi}{(-E)^{1/4}}, \sqrt{2(-E)^{1/2}} r \right) \tag{4.43} \]

and the eigenvalue of the bound state is given by an implicit expression

\[ K_2 \left( 2(2l+1), 0, 0, -\frac{4\sqrt{2}\xi}{(-E)^{1/4}} \right) = 0. \tag{4.44} \]

**\(r^6\)-potential.** The solution of the \(r^6\)-potential can be obtained by its dual potential, the \(1/r^{3/2}\)-potential.

The radial equation of \(r^6\)-potential (4.40) is

\[ \frac{d^2 v(\rho)}{d\rho^2} + \left[ \mathcal{E} - \frac{l(l+1)}{\rho^2} - \eta \rho^6 \right] v(\rho) = 0. \tag{4.45} \]
The duality relations between these two dual potentials, by Eqs. (3.13), (3.14), and (3.15), are

\[ E \rightarrow -\frac{\eta}{16}, \quad \xi \rightarrow -\frac{\xi}{16}, \quad l + \frac{1}{2} = \frac{1}{4} \left( \ell + \frac{1}{2} \right). \] (4.46)

Substituting the duality relation (3.23) between the eigenvalues of these two dual potentials into the implicit expression of the eigenvalue of the \(1/r^{3/2}\)-potential, Eq. (4.44), gives an implicit expression of the eigenvalue of the \(r^6\)-potential:

\[ K_2 \left( \ell + \frac{1}{2}, 0, 0, \frac{\sqrt{2} \xi}{2\eta^{1/4}} \right) = 0. \] (4.47)

The duality transformation of coordinates, Eq. (3.16), and the duality relation of radial eigenfunctions, Eq. (3.17), in this case become

\[ r \rightarrow \rho^4 \quad \text{and} \quad u(r) \rightarrow \rho^{3/2} v(\rho). \] (4.48)

For eigenfunctions, substituting the duality relations (4.46) and (4.48) into the radial eigenfunction of the \(1/r^{3/2}\)-potential, Eq. (4.43), gives the radial wave function of the \(r^6\)-potential:

\[ v(\rho) = A_l e^{-\frac{\sqrt{2} \eta}{4} \rho^{l+1}} N \left( \ell + \frac{1}{2}, 0, 0, \frac{\sqrt{2} \xi}{2\eta^{1/4}}, \frac{\sqrt{2}}{2} \eta^{1/4} \rho^2 \right). \] (4.49)

4.3 Two-term general polynomial potentials

A two-term general polynomial potential has two dual potentials. That is to say, one can solve two potentials from their dual potential. In this section, we show how to solve two potentials from their two-term dual potential.

4.3.1 Solving \( V(\rho) = \frac{\eta}{\rho} + \frac{\lambda}{\rho^{3/2}} \) and \( V(\rho) = \eta \rho^2 + \lambda \rho^6 \) from their duality \( U(r) = \xi r^2 + \frac{\mu}{r} \)

In this section, taking the dual set of potentials

\[ \left( \xi r^2 + \frac{\mu}{r}, \eta \frac{r}{r}, \xi + \frac{\lambda}{r^{3/2}}, \eta \rho^2 + \lambda \rho^6 \right) \] (4.50)

as an example, we show how to solve not only one potential from a solved potential. Additionally, we solve exactly the \(\xi r^2 + \frac{\mu}{r}\)-potential in Appendix B.

The two-term general polynomial potential

\[ U(r) = \xi r^2 + \frac{\mu}{r} \] (4.51)

has two dual potentials:

\[ V(\rho) = \frac{\eta}{\rho} + \frac{\lambda}{\rho^{3/2}}, \] \hspace{1cm} (4.52)
\[ V(\rho) = \eta \rho^2 + \lambda \rho^6. \] \hspace{1cm} (4.53)
The potential $U (r) = \xi r^2 + \frac{\mu}{r}$ can be exactly solved (see Appendix B). The radial equation of $U (r)$ is

$$\frac{d^2 u (r)}{dr^2} + \left[ E - \frac{l(l+1)}{r^2} - \frac{\xi r^2 - \mu}{r} \right] u (r) = 0.$$  \hspace{1cm} (4.54)

The radial eigenfunction is

$$u (r) = A_l r^{l+1} e^{\sqrt{\xi} r^2/2} N \left( 2l+1, 1, 0, -\frac{E}{\sqrt{\xi}}, -\frac{2i\mu}{\xi^{1/4}}, i^{1/4} \right).$$  \hspace{1cm} (4.55)

and the eigenvalue of bound states can be expressed by an implicit expression:

$$K^2_2 \left( 2l+1, 1, 0, -\frac{E}{\sqrt{\xi}}, -\frac{2i\mu}{\xi^{1/4}} \right) = 0.$$  \hspace{1cm} (4.56)

Solving $V (\rho) = \frac{2}{\rho} + \frac{\lambda}{\rho^{3/2}}$ from $U (r) = \xi r^2 + \frac{\mu}{r}$ The potential (4.52) is one of the two dual potentials of the potential (4.51) with $a = 1$, $b = -2$ and $A = -2$, $B = -5/2$ in the duality relations (3.56) and (3.57). We can solve it from the solution of the potential $U (r)$.

The radial equation of $V (\rho)$ is

$$\frac{d^2 v (\rho)}{d\rho^2} + \left[ E - \frac{l(l+1)}{\rho^2} - \frac{\eta}{\rho} - \frac{\lambda}{\rho^{3/2}} \right] v (\rho) = 0.$$  \hspace{1cm} (4.57)

The dual relations (3.58), (3.59), (3.60), (3.61), (3.62), and (3.63) with $a = 1$ and $b = -2$ read

$$E \rightarrow -4\eta, \quad \xi \rightarrow -4E, \quad \mu \rightarrow 4\lambda,$$

$$l + \frac{1}{2} \rightarrow 2 \left( l + \frac{1}{2} \right),$$  \hspace{1cm} (4.58)

and

$$r \rightarrow \sqrt{\rho} \enspace \text{and} \enspace u (r) \rightarrow \rho^{-1/4} v (\rho).$$  \hspace{1cm} (4.59)

Substituting the above dual relations into the eigenfunction of the potential (4.51), Eq. (4.55), gives the radial eigenfunction of the potential (4.52):

$$v (\rho) = A_l \rho^{\ell+1} e^{\sqrt{-E} \rho} N \left( 4\ell + 2, 1, 0, \frac{2\eta}{\sqrt{-E}}, -\frac{4\sqrt{2i\lambda}}{(-E)^{1/4}}, i (-E)^{1/4} \sqrt{2\rho} \right).$$  \hspace{1cm} (4.60)

Substituting the above replacements into Eq. (4.56) gives an implicit expression of the eigenvalue of the potential (4.52):

$$K^2_2 \left( 4\ell + 2, 1, 0, \frac{2\eta}{\sqrt{-E}}, -\frac{4\sqrt{2i\lambda}}{(-E)^{1/4}} \right) = 0.$$  \hspace{1cm} (4.61)
Solving $V(\rho) = \eta \rho^2 + \lambda \rho^{6}$ from $U(r) = \xi r^2 + \frac{\mu}{r}$ The potential (4.53) is another dual potential of the potential (4.51) with $a = -2$, $b = 1$ and $A = 1$, $B = 5$ in the duality relations (3.56) and (3.57). We can also solve it from the solution of the potential (4.51).

The radial equation of $V(\rho) = \eta \rho^2 + \lambda \rho^{6}$ is

$$\frac{d^2 v(\rho)}{d\rho^2} + \left[ E - \frac{\ell (\ell + 1)}{\rho^2} - \eta \rho^2 - \lambda \rho^{6} \right] v(\rho) = 0. \quad (4.62)$$

The dual relations (3.58), (3.59), (3.60), (3.61), (3.62), and (3.63) with $a = -2$ and $b = 1$ read

$$E \rightarrow -\frac{\eta}{4}, \quad \mu \rightarrow -\frac{\xi}{2}, \quad \xi \rightarrow \frac{\lambda}{4},$$

$$l + \frac{1}{2} \rightarrow \frac{1}{2} \left( \ell + \frac{1}{2} \right), \quad (4.63)$$

and

$$r \rightarrow \rho^2 \quad \text{and} \quad u(r) \rightarrow \rho^{1/2} v(\rho). \quad (4.64)$$

Substituting the dual relations into the eigenfunction of the potential (4.51), Eq. (4.55), gives the radial eigenfunction of the potential (4.53):

$$v(\rho) = A_{\ell} \rho^{\ell+1} e^{\sqrt{\lambda} \rho^4} N \left( \ell + \frac{1}{2}, 0, \frac{\eta}{2 \sqrt{\lambda}}, \frac{i E}{\sqrt{2 \lambda^{1/4}}}, \frac{i}{\sqrt{2}} \right). \quad (4.65)$$

Substituting the replacements into Eq. (4.56) gives an implicit expression of the eigenvalue of the potential (4.53):

$$K_2 \left( \ell + \frac{1}{2}, 0, \frac{\eta}{2 \sqrt{\lambda}}, \frac{i E}{\sqrt{2 \lambda^{1/4}}} \right) = 0. \quad (4.66)$$

4.3.2 Solving $V(\rho) = \frac{\eta}{\rho} + \frac{\lambda}{\sqrt{\rho}}$ and $V(\rho) = \frac{\eta}{\rho^{2/3}} + \lambda \rho^{2/3}$ from their duality $U(r) = \xi r^2 + \mu r$

In this section, we consider a dual set of two-term potentials

$$\left( \xi r^2 + \mu r, \frac{\eta}{\rho} + \frac{\lambda}{\sqrt{\rho}}, \frac{\eta}{\rho^{2/3}} + \lambda \rho^{2/3} \right), \quad (4.67)$$

in which the solution of $U(r) = \xi r^2 + \mu r$ is already known. Additionally, we solve exactly the $\xi r^2 + \mu r$-potential in Appendix D.

The two-term general polynomial potential

$$U(r) = \xi r^2 + \mu r \quad (4.68)$$

has two dual potentials:

$$V(\rho) = \frac{\eta}{\rho} + \frac{\lambda}{\sqrt{\rho}}, \quad (4.69)$$

$$V(\rho) = \frac{\eta}{\rho^{2/3}} + \lambda \rho^{2/3}. \quad (4.70)$$

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The radial equation of the potential (4.68) is

\[ \frac{d^2 u(r)}{dr^2} + \left( E - l(l+1) r^2 - \xi r^2 - \mu r \right) u(r) = 0. \]  \hspace{1cm} (4.71)

The radial eigenfunction is (see Appendix D)

\[ u(r) = A_l r^{l+1} \exp \left( \frac{\sqrt{\xi} r^2 + \frac{\mu}{2\sqrt{\xi}} r}{2} \right) N \left( 2l + 1, \frac{i\mu}{\xi^{3/4}} - \frac{E}{\sqrt{\xi}} - \frac{\mu^2}{4\xi^{3/2}}, 0, i\xi^{1/4} r \right). \]  \hspace{1cm} (4.72)

The eigenvalue of the bound state can be expressed by the implicit expression

\[ K_2 \left( 2l + 1, \frac{i\mu}{\xi^{3/4}} - \frac{E}{\sqrt{\xi}} - \frac{\mu^2}{4\xi^{3/2}}, 0 \right) = 0, \]  \hspace{1cm} (4.73)

i.e., the eigenvalue of \( U(r) \) is the zero of \( K_2 \left( 2l + 1, \frac{i\mu}{\xi^{3/4}} - \frac{E}{\sqrt{\xi}} - \frac{\mu^2}{4\xi^{3/2}}, 0 \right). \)

Solving \( V(\rho) = \frac{\rho}{\rho^4} + \frac{1}{\sqrt{\rho}} \) from \( U(r) = \xi r^2 + \mu r \) The potential (4.69) is a dual potential of the potential (4.68) with \( a = 1, b = 0 \) and \( A = -2, B = -3/2 \) in the duality relations (3.56) and (3.57).

The radial equation of \( V(\rho) = \frac{\rho}{\rho^4} + \frac{1}{\sqrt{\rho}} \) is

\[ \frac{d^2 v(\rho)}{d\rho^2} + \left[ \mathcal{E} - \frac{\ell (\ell+1)}{\rho^2} - \frac{\eta}{\rho} - \frac{\lambda}{\sqrt{\rho}} \right] v(\rho) = 0. \]  \hspace{1cm} (4.74)

The dual relations (3.58), (3.59), (3.60), (3.61), (3.62), and (3.63) with \( a = 1 \) and \( b = 0 \) read

\[ E \to -4\eta, \ \xi \to -4\mathcal{E}, \ \mu \to 4\lambda, \]  
\[ l + \frac{1}{2} \to 2 \left( \ell + \frac{1}{2} \right), \]  \hspace{1cm} (4.75)

and

\[ r \to \sqrt{\rho} \ \text{and} \ u(r) \to \rho^{-1/4} v(\rho). \]  \hspace{1cm} (4.76)

Substituting the dual relations into the eigenfunction of the potential (4.68), Eq. (5.55), gives the eigenfunction of the potential (4.69):

\[ v(\rho) = A_l \rho^{l+1} \exp \left( \frac{\sqrt{-\mathcal{E}} \rho + \frac{\lambda}{\sqrt{-\mathcal{E}}} \sqrt{\rho}}{2} \right) N \left( 4\ell + 2, \frac{\sqrt{2i\lambda}}{(-\mathcal{E})^{3/4}} - \frac{2\eta}{\sqrt{-\mathcal{E}}} - \frac{\lambda^2}{2(-\mathcal{E})^{3/2}}, 0, i(-\mathcal{E})^{1/4} \sqrt{2\rho} \right). \]  \hspace{1cm} (4.77)

Substituting the dual relations into (4.73) gives an implicit expression of the eigenvalue of the potential (4.69):

\[ K_2 \left( 4\ell + 2, \frac{\sqrt{2i\lambda}}{(-\mathcal{E})^{3/4}} - \frac{2\eta}{\sqrt{-\mathcal{E}}} - \frac{\lambda^2}{2(-\mathcal{E})^{3/2}}, 0 \right) = 0. \]  \hspace{1cm} (4.78)
Solving $V(\rho) = \frac{\eta}{\rho^{2/3}} + \lambda \rho^{2/3}$ from $U(r) = \xi r^2 + \mu r$ The potential (4.70) is another dual potential of the potential (4.68) with $a = 0$, $b = 1$ and $A = -5/3$, $B = -1/3$ in the duality relations (3.56) and (3.57).

The radial equation of $V(\rho) = \eta \rho^{2/3} + \lambda \rho^{2/3}$ is

$$\frac{d^2 v(\rho)}{d\rho^2} + \left[ E - \frac{\ell (\ell + 1)}{\rho^2} - \frac{\eta}{\rho^{2/3}} - \lambda \rho^{2/3} \right] v(\rho) = 0. \quad (4.79)$$

The dual relations (3.58), (3.59), (3.60), (3.61), (3.62), and (3.63) with $a = 0$ and $b = 1$ read

$$E \rightarrow -\frac{9}{4} \eta, \mu \rightarrow -\frac{9}{4} \xi, \xi \rightarrow \frac{9}{4} \lambda,$$

$$l + \frac{1}{2} \rightarrow \frac{3}{2} \left( \ell + \frac{1}{2} \right), \quad (4.80)$$

and

$$r \rightarrow \rho^{2/3} \text{ and } u(r) \rightarrow \rho^{-1/6} v(\rho). \quad (4.81)$$

Substituting the dual relations into the eigenfunction of the potential (4.68), Eq. (4.72), gives the radial eigenfunction of the potential (4.70):

$$v(\rho) = A_{\ell} \rho^{\ell+1} \exp \left( \frac{9}{4} \lambda \frac{1}{2} \rho^{1/3} + \frac{-\frac{9}{2} \xi}{2 \left( \frac{9}{4} \lambda \right)^{1/2}} \rho^{2/3} \right) N \left( 3\ell + \frac{3}{2}, -\frac{i\sqrt{6} \xi}{2 \lambda^{3/4}}, \frac{3\eta}{2 \sqrt{\lambda}} - \frac{3 \xi^2}{8 \lambda^{3/2}}, 0, \frac{\sqrt{6}}{2} \lambda^{1/4} \rho^{2/3} \right). \quad (4.82)$$

Substituting the dual relations into Eq. (4.73) gives an implicit expression of the eigenvalue of the potential (4.70):

$$K_2 \left( 3\ell + \frac{3}{2}, -\frac{i\sqrt{6} \xi}{2 \lambda^{3/4}}, \frac{3\eta}{2 \sqrt{\lambda}} - \frac{3 \xi^2}{8 \lambda^{3/2}}, 0 \right) = 0. \quad (4.83)$$

4.3.3 Solving $V(\rho) = \eta \rho^{2/3} + \lambda \rho^{2/3}$ and $V(\rho) = \eta \rho^6 + \lambda \rho^4$ from their duality $U(r) = \frac{\xi}{\sqrt{r}} + \frac{\mu}{r^{3/2}}$

In this section, we consider the dual set of two-term potentials

$$\left( \frac{\xi}{\sqrt{r}}, \frac{\mu}{r^{3/2}}, \eta \rho^{2/3} + \lambda \rho^{2/3}, \eta \rho^6 + \lambda \rho^4 \right), \quad (4.84)$$

in which the solution of $U(r) = \frac{\xi}{\sqrt{r}} + \frac{\mu}{r^{3/2}}$ is already known. Additionally, we solve exactly the $\frac{\xi}{\sqrt{r}} + \frac{\mu}{r^{3/2}}$-potential in Appendix C.

The two-term general polynomial potential

$$U(r) = \frac{\xi}{\sqrt{r}} + \frac{\mu}{r^{3/2}} \quad (4.85)$$
has two dual potentials:

\[ V(\rho) = \eta \rho^{2/3} + \frac{\lambda}{\rho^{1/3}}, \quad \text{(4.86)} \]

\[ V(\rho) = \eta \rho^6 + \lambda \rho^4. \quad \text{(4.87)} \]

The radial equation of the potential (4.85) is

\[ \frac{d^2 u(r)}{dr^2} + \left[ E - \frac{l(l + 1)}{r^2} - \frac{\xi}{\sqrt{r}} - \frac{\mu}{r^{3/2}} \right] u(r) = 0. \quad \text{(4.88)} \]

The radial eigenfunction is (see Appendix C)

\[ u(r) = A r^{l+1} \exp \left( \sqrt{-E} r + \frac{\xi}{\sqrt{-E}} \sqrt{r} \right) \times N \left( 4l + 2, \frac{i \xi \sqrt{2}}{(-E)^{3/4}}, -\frac{\xi^2}{2(-E)^{3/2}}, \frac{-4 \sqrt{2i} \mu}{(-E)^{1/4}}, i \sqrt{2} (-E)^{1/4} \sqrt{r} \right). \quad \text{(4.89)} \]

The eigenvalue of the bound state can be expressed by the implicit expression [14]

\[ K_2 \left( 4l + 2, \frac{i \xi \sqrt{2}}{(-E)^{3/4}}, -\frac{\xi^2}{2(-E)^{3/2}}, \frac{-4 \sqrt{2i} \mu}{(-E)^{1/4}} \right) = 0. \quad \text{(4.90)} \]

**Solving** \( V(\rho) = \eta \rho^{2/3} + \frac{\lambda}{\rho^{1/3}} \) form \( U(r) = \frac{\xi}{\sqrt{-E}} + \frac{\mu}{r^{3/2}} \). The potential (4.86) is a dual potential of the potential (4.85) with \( a = -3/2, b = -5/2 \) and \( A = -1/3, B = -7/3 \) in the duality relations (3.56) and (3.57).

The radial equation of the potential (4.86) is

\[ \frac{d^2 v(\rho)}{d\rho^2} + \left[ \mathcal{E} - \frac{\ell(\ell + 1)}{\rho^2} - \eta \rho^{2/3} - \frac{\lambda}{\rho^{4/3}} \right] v(\rho) = 0. \quad \text{(4.91)} \]

The dual relations (3.58), (3.59), (3.60), (3.61), (3.62), and (3.63) with \( a = -3/2 \) and \( b = -5/2 \) read

\[ E \rightarrow \frac{9}{16} \eta, \quad \xi \rightarrow -\frac{9}{16} \mathcal{E}, \quad \mu \rightarrow \frac{9}{16} \lambda, \]

\[ l + \frac{1}{2} \rightarrow \frac{3}{4} \left( \ell + \frac{1}{2} \right), \quad \text{(4.92)} \]

and

\[ r \rightarrow \rho^{4/3} \quad \text{and} \quad u(r) \rightarrow \rho^{1/6} v(\rho). \quad \text{(4.93)} \]

Substituting the dual relations into the eigenfunction of the potential (4.85), Eq. (4.89), gives the eigenfunction of the potential (4.86):

\[ v(\rho) = A_4 \rho^{\ell+1} \exp \left( \frac{3 \sqrt{\eta}}{4} \rho^{4/3} - \frac{3 \mathcal{E}}{4 \sqrt{\eta} \rho^{2/3}} \right) N \left( 3 \ell + \frac{3}{2}, -\frac{\sqrt{6} \mathcal{E}}{2 \eta^{3/4}}, -\frac{3 \mathcal{E}^2}{8 \eta^{3/2}}, -\frac{3 \sqrt{6} \lambda}{2 \eta^{1/4}}, i \frac{\sqrt{6}}{2} \eta^{1/4} \rho^{4/3} \right). \quad \text{(4.94)} \]

Substituting the dual relations into Eq. (4.90) gives an implicit expression of the eigenvalue of the potential (4.86):

\[ K_2 \left( 3 \ell + \frac{3}{2}, -\frac{\sqrt{6} \mathcal{E}}{2 \eta^{3/4}}, -\frac{3 \mathcal{E}^2}{8 \eta^{3/2}}, -\frac{3 \sqrt{6} \lambda}{2 \eta^{1/4}} \right) = 0. \quad \text{(4.95)} \]
Solving $V (\rho) = \eta \rho^b + \lambda \rho^4$ form $U (r) = \xi r^6 + \lambda r^2$ The potential (4.87) is a dual potential of the potential (4.85) with $a = -5/2$, $b = -3/2$ and $A = 5$, $B = 3$ in the duality relations (3.56) and (3.57).

The radial equation of the potential (4.87) is

$$
\frac{d^2 v (\rho)}{d \rho^2} + \left[ E - \frac{\ell (\ell + 1)}{\rho^2} - \eta \rho^6 - \lambda \rho^4 \right] v (\rho) = 0. \tag{4.96}
$$

The dual relations (3.58), (3.59), (3.60), (3.61), (3.62), and (3.63) with $a = -5/2$ and $b = -3/2$ read

$$
E \to -\frac{1}{16} \eta, \quad \mu \to -\frac{1}{8} \xi, \quad \xi \to \frac{1}{16} \lambda, \quad \ell + \frac{1}{2} \to \frac{1}{4} \left( \ell + \frac{1}{2} \right), \tag{4.97}
$$

and

$$
r \to \rho^4 \quad \text{and} \quad u (r) \to \rho^{3/2} v (\rho). \tag{4.98}
$$

Substituting the dual relations into the eigenfunction of the potential (4.85), Eq. (4.89), gives the eigenfunction of the potential (4.87):

$$
v (\rho) = A_l \rho^{\ell+1} \exp \left( \frac{\sqrt{\eta}}{4} \rho^4 + \frac{\lambda}{4 \sqrt{\eta}} \rho^2 \right) N \left( \ell + \frac{1}{2}, i \frac{\sqrt{2} \xi}{2 \eta^{3/4}}, \frac{\lambda^2}{2 \eta^{3/4}}, i \frac{\sqrt{2} \lambda^2}{8 \eta^{1/4}}, \frac{i \sqrt{2} \xi}{2 \eta^{1/4}} \right). \tag{4.99}
$$

Substituting the dual relations into (4.90) gives an implicit expression of the eigenvalue of the potential (4.87):

$$
K_2 \left( \ell + \frac{1}{2}, i \frac{\sqrt{2} \xi}{2 \eta^{3/4}}, \frac{\lambda^2}{2 \eta^{3/4}}, i \frac{\sqrt{2} \lambda^2}{8 \eta^{1/4}} \right) = 0. \tag{4.100}
$$

4.3.4 Solving $V (\rho) = \eta \rho^b + \lambda \rho^4$ from $U (r) = \xi r^6 + \lambda r^2$

This is an example of the Newton duality discussed in section 3.6. In this case, each term of one potential is Newtonianly dual to the corresponding term of its Newton dual potential. That is, $(\eta \rho^b + \lambda \rho^4, \xi r^6 + \lambda r^2)$, $(\eta \rho^b + \lambda \rho^4, \xi^6)$, and $(\lambda \rho^4, \lambda r^2)$ are all dual sets. In this case, though the potential is a two-term potential, it has only one dual potential, i.e., the dual set contains only two two-term potentials: $(\eta \rho^b + \lambda \rho^4, \xi r^6 + \lambda r^2)$.

The $r^6$-potential is the Newton duality of $r^{-2/3}$-potential and the $r^2$-potential is the Newton duality of $r^{-1}$-potential, while the linear combination of $r^6$-potential and $r^2$-potential is also the Newton duality of the linear combination of $r^{-2/3}$-potential and $r^{-1}$-potential, i.e., the potential

$$
U (r) = \xi r^6 + \lambda r^2 \tag{4.101}
$$

is the Newton duality of

$$
V (\rho) = \frac{\eta}{\rho^{b/2}} + \frac{\mu}{\rho}. \tag{4.102}
$$

The potential $U (r) = \xi r^6 + \lambda r^2$. The potential (4.101) can be exactly solved (see Sec. 4.3.1).
The radial equation of \( U (r) \) is
\[
\frac{d^2 u (r)}{dr^2} + \left[ E - \frac{l (l + 1)}{r^2} - \xi r^6 - \lambda r^2 \right] u (r) = 0. \tag{4.103}
\]

The radial eigenfunction is (see Sec. 4.3.1)
\[
u (r) = e^{\sqrt{\xi} r^{1/4}} N \left( l + \frac{1}{2}, 0, \frac{\lambda}{2 \sqrt{\xi}}, i \frac{\sqrt{2} E}{2 \xi^{1/4}}, i \frac{\sqrt{2}}{2} \xi^{1/4} r^2 \right) r^{l+1}. \tag{4.104}
\]

The eigenvalue of the bound state can be expressed by the implicit expression
\[
K_2 \left( l + \frac{1}{2}, 0, \frac{\lambda}{2 \sqrt{\xi}}, i \frac{\sqrt{2} E}{2 \xi^{1/4}} \right) = 0. \tag{4.105}
\]

The potential \( V (\rho) = \frac{\eta}{\rho^3} + \mu \). The radial equation of the potential (4.102) is
\[
\frac{d^2 v (\rho)}{d\rho^2} + \left[ \mathcal{E} - \frac{\ell (\ell + 1)}{\rho^2} - \frac{\eta}{\rho^{3/2}} - \frac{\mu}{\rho} \right] v (\rho) = 0. \tag{4.106}
\]

The dual relations (3.58), (3.59), (3.60), (3.61), (3.62), and (3.63) with \( a = 5 \) and \( b = 1 \) read
\[
E \rightarrow -16 \eta, \ \xi \rightarrow -16 \xi, \ \lambda \rightarrow 16 \mu,
\]
\[
l + \frac{1}{2} \rightarrow 4 \left( \ell + \frac{1}{2} \right), \tag{4.107}
\]

and
\[
r \rightarrow \rho^{1/4} \text{ and } u (r) \rightarrow \rho^{-3/8} v (\rho). \tag{4.108}
\]

Substituting the dual relations into the eigenfunction of the potential (4.101), Eq. (4.104), gives the eigenfunction:
\[
v (\rho) = e^{\sqrt{-\mathcal{E}} \rho} N \left( 4 \ell + 2, 0, \frac{2 \mu}{\sqrt{-\mathcal{E}}}, -i \frac{4 \sqrt{2} \eta}{(-\mathcal{E})^{1/4}}, i \sqrt{2} (-\mathcal{E})^{1/4} \sqrt{\rho} \right) \rho^{\ell+1}. \tag{4.109}
\]

Substituting the dual relations into the eigenvalue of the potential (4.101), Eq. (4.105), gives an implicit expression of the eigenvalue:
\[
K_2 \left( 4 \ell + 2, 0, \frac{2 \mu}{\sqrt{-\mathcal{E}}}, -i \frac{4 \sqrt{2} \eta}{(-\mathcal{E})^{1/4}} \right) = 0. \tag{4.110}
\]

4.4 Three-term general polynomial potentials

As mentioned above, a three-term general polynomial potentials has three dual potentials. Once the solution of a three-term general polynomial potentials is solved, the solutions of its three dual potentials can be immediately obtained by the duality relation given in Section 3.4.
In this section, we consider a dual set of three-term potentials

\[
\left( \xi r^2 + \frac{\mu}{r} + \kappa r, \frac{\eta}{r^3/2} + \frac{\nu r^6}{r^{1/2}}, \frac{\lambda}{r^{2/3}} \right).
\]

The potential

\[
U(r) = \xi r^2 + \frac{\mu}{r} + \kappa r
\]

has three dual potentials:

\[
V(\rho) = \eta \rho + \frac{\nu}{\rho^{3/2}} + \frac{\lambda}{\rho^{1/2}}, \quad (4.112)
\]

\[
V(\rho) = \eta \rho^2 + \nu \rho^6 + \lambda \rho^4, \quad (4.113)
\]

\[
V(\rho) = \frac{\eta}{\rho^{2/3}} + \nu \rho^{2/3} + \lambda \rho^{4/3}. \quad (4.114)
\]

The radial equation of the potential (4.111) is

\[
\frac{d^2 u(r)}{dr^2} + \left[ E - \frac{l(l+1)}{r^2} - \xi r^2 - \frac{\mu}{r} - \kappa r \right] u(r) = 0. \quad (4.115)
\]

The radial eigenfunction is (see Appendix E)

\[
u(r) = A_l r^{l+1} \exp \left( \sqrt{\xi} r^2 + \frac{\kappa}{2\sqrt{\xi}} r \right) N \left( 2l+1, \frac{i \kappa}{\xi^{3/4}}, -\frac{\kappa^2}{4 \xi^{3/2}}, -\frac{E}{\xi^{1/2}}, -\frac{2}{\xi^{1/2}} \right).
\]

The eigenvalue of the bound states can be expressed by the implicit expression [14]

\[
K_2 \left( 2l+1, \frac{i \kappa}{\xi^{3/4}}, -\frac{\kappa^2}{4 \xi^{3/2}}, -\frac{E}{\xi^{1/2}}, -\frac{2}{\xi^{1/2}} \right) = 0. \quad (4.117)
\]

4.4.1 Solving \( V(\rho) = \eta \rho + \frac{\nu}{\rho^{3/2}} + \frac{\lambda}{\rho^{1/2}} \) from \( U(r) = \xi r^2 + \frac{\mu}{r} + \kappa r \)

The potential (4.112) is a dual potential of the potential (4.111) with \( a = 1, b_1 = -2, b_2 = 0 \) and \( A = -2, B_1 = -5/2, B_2 = -3/2 \) in the duality relations (3.56) and (3.57).

The radial equation is

\[
\frac{d^2 v(\rho)}{d\rho^2} + \left[ \mathcal{E} - \frac{(l+1)(l+2)}{\rho^2} - \frac{\eta}{\rho^{3/2}} - \frac{\nu}{\rho^{1/2}} - \frac{\lambda}{\rho^{1/2}} \right] v(\rho) = 0. \quad (4.118)
\]

The dual relations (3.58), (3.59), (3.60), (3.61), (3.62), and (3.63) with \( a = 1, b_1 = -2, b_2 = 0 \) read

\[
E \rightarrow -4\eta, \quad \xi \rightarrow -4\mathcal{E}, \quad \mu \rightarrow 4\nu, \quad \kappa \rightarrow 4\lambda,
\]

\[
l + \frac{1}{2} \rightarrow 2 \left( l + \frac{1}{2} \right), \quad (4.119)
\]

and

\[
r \rightarrow \sqrt{\rho} \text{ and } u(r) \rightarrow \rho^{-1/4} v(\rho). \quad (4.120)
\]
Substituting the dual relations into the eigenfunction of the potential (4.111), Eq. (4.116), gives the eigenfunction

\[
v(\rho) = A_\ell \rho^{\ell+1} \exp\left(\sqrt{-\mathcal{E}} \rho + \frac{\lambda}{\sqrt{-\mathcal{E}}} \sqrt{\rho}\right) \times N\left(4\ell+2, \frac{i\sqrt{2}\lambda}{(-\mathcal{E})^{3/4}}, -\frac{\lambda^2}{2(-\mathcal{E})^{3/2}} + \frac{2\eta}{(-\mathcal{E})^{1/2}}, -i\frac{4\sqrt{2}\nu}{(-\mathcal{E})^{1/4}}, i(-\mathcal{E})^{1/4} \sqrt{2\rho}\right). \tag{4.121}
\]

Substituting the dual relations into (4.117) gives an implicit expression of the eigenvalue of the potential (4.112):

\[
K_2\left(4\ell+2, \frac{i\sqrt{2}\lambda}{(-\mathcal{E})^{3/4}}, -\frac{\lambda^2}{2(-\mathcal{E})^{3/2}} + \frac{2\eta}{(-\mathcal{E})^{1/2}}, -i\frac{4\sqrt{2}\nu}{(-\mathcal{E})^{1/4}}\right) = 0. \tag{4.122}
\]

4.4.2 Solving \(V(\rho) = \eta \rho^2 + \nu \rho^6 + \lambda \rho^4\) from \(U(r) = \xi r^2 + \frac{\mu}{r} + \kappa r\)

The potential (4.113) is a dual potential of the potential (4.111) with \(a = -2, b_1 = 1,\) and \(b_2 = 0\) and \(A = 1, B_1 = 5, B_2 = 3\) in the duality relations (3.56) and (3.57).

The radial equation of the potential (4.113) is

\[
d^2v(\rho) + \left[\mathcal{E} - \frac{\ell(\ell+1)}{\rho^2} - \frac{\eta}{\rho^2} - \nu \rho^6 - \lambda \rho^4\right]v(\rho) = 0. \tag{4.123}
\]

The dual relations (3.58), (3.59), (3.60), (3.61), (3.62), and (3.63) with \(a = -2, b_1 = 1,\) and \(b_2 = 0\) read

\[
E \rightarrow -\frac{\eta}{4}, \mu \rightarrow -\frac{\mathcal{E}}{4}, \xi \rightarrow \frac{\nu}{4}, \kappa \rightarrow \frac{\lambda}{4},
\]

\[
l + \frac{1}{2} \rightarrow \frac{1}{2}\left(\ell + \frac{1}{2}\right), \tag{4.124}
\]

and

\[
r \rightarrow \rho^2\) and \(u(r) \rightarrow \rho^{1/2}v(\rho).
\]

Substituting the dual relations into the eigenfunction of the potential (4.111), Eq. (4.116), gives the eigenfunction of the potential (4.113):

\[
v(\rho) = A_\ell \rho^{\ell+1} \exp\left(\frac{\sqrt{\mathcal{E}}}{4} \rho^4 + \frac{\lambda}{4\sqrt{\rho^2}}\right) N\left(\ell + \frac{1}{2}, \frac{i\sqrt{2}\lambda}{2\nu^{3/4}}, \frac{\eta}{2\nu^{1/2}}, -\frac{\lambda^2}{8\nu^{3/2}}, i\frac{\sqrt{2}\mathcal{E}}{2\nu^{1/4}}, i\frac{\sqrt{2}}{2\nu^{1/4}}\rho^2\right). \tag{4.126}
\]

Substituting the dual relations into (4.117) gives an implicit expression of the eigenvalue of the potential (4.113):

\[
K_2\left(\ell + \frac{1}{2}, \frac{i\sqrt{2}\lambda}{2\nu^{3/4}}, \frac{\eta}{2\nu^{1/2}}, -\frac{\lambda^2}{8\nu^{3/2}}, i\frac{\sqrt{2}\mathcal{E}}{2\nu^{1/4}}\right) = 0. \tag{4.127}
\]
4.4.3 Solving \( V(\rho) = \frac{\eta}{\rho^{2/3}} + \nu \rho^{2/3} + \frac{\lambda}{\rho^{4/3}} \) from \( U(r) = \xi r^2 + \frac{\mu}{r} + \kappa r \)

The potential (4.114) is a dual potential of the potential (4.111) with \( a = 0, b_1 = 1, b_2 = -2 \) and \( A = -5/3, B_1 = -1/3, B_2 = -7/3 \) in the duality relations (3.56) and (3.57).

The radial equation of the potential (4.114) is

\[
\frac{d^2 v(\rho)}{d\rho^2} + \left[ E - \frac{\ell(\ell+1)}{\rho^2} - \frac{\eta}{\rho^{2/3}} - \nu \rho^{2/3} - \frac{\lambda}{\rho^{4/3}} \right] v(\rho) = 0. \tag{4.128}
\]

The dual relations (3.58), (3.59), (3.60), (3.61), (3.62), and (3.63) with \( a = 0, b_1 = 1, \) and \( b_2 = -2 \) read

\[
E \to -\frac{9}{4} \eta, \quad \kappa \to -\frac{9}{4} \xi, \quad \xi \to \frac{9}{4} \nu, \quad \mu \to \frac{9}{4} \lambda,
\]

\[
l + \frac{1}{2} \to \frac{3}{2} \left( \ell + \frac{1}{2} \right), \tag{4.129}
\]

and

\[
r \to \rho^{2/3} \text{ and } u(r) \to \rho^{-1/6} v(\rho). \tag{4.130}
\]

Substituting the dual relations into the eigenfunction of the potential (4.111), Eq. (4.116), gives the eigenfunction of the potential (4.114):

\[
v(\rho) = A_\ell \rho^{\ell+1} \exp \left( \frac{3\nu^{1/2}}{4} \rho^{1/3} - \frac{3\xi}{4\nu^{1/2}} \rho^{2/3} \right) \times N \left( 3\ell + \frac{3}{2}, -\frac{i\sqrt{6}\xi}{2\nu^{3/4}}, \frac{3\eta}{2\nu^{1/2}} - \frac{3\xi^2}{8\nu^{3/2}}, -i\frac{3\sqrt{6}\lambda}{2\nu^{1/4}}, -i\frac{\sqrt{6}}{2} \right). \tag{4.131}
\]

Substituting the dual relations into (4.117) gives an implicit expression of the eigenvalue of the potential (4.114):

\[
K_2 \left( 3\ell + \frac{3}{2}, -\frac{i\sqrt{6}\xi}{2\nu^{3/4}}, \frac{3\eta}{2\nu^{1/2}} - \frac{3\xi^2}{8\nu^{3/2}}, -i\frac{3\sqrt{6}\lambda}{2\nu^{1/4}} \right) = 0. \tag{4.132}
\]

5 Conclusions and Outlook

The Newton duality reveals the underlying nature of mechanics. In this paper, we provide three results.

1. In classical mechanics, we generalize Newton’s original duality which involves only power potentials to more general kinds of potentials — general polynomial potentials.

2. In quantum mechanics, we provide a quantum Newton duality for power potentials, general polynomial potentials, transcendental-function potentials, and power potentials in different spatial dimensions.

3. Based on the quantum Newton duality, we develop a method for solving eigenproblems of a quantum mechanical system and solve some potentials with this method.

The original Newton duality includes only power potentials. We show that the Newton duality is in fact a more general duality in dynamics: such a duality exists in more general
potentials, including general polynomial potentials and transcendental-function potentials. This inspires us to make a conjecture that the Newton duality exists in all kinds of potentials and every potential has Newton dualities.

In appendices, as preparations, we solve some potentials: \( \frac{\alpha}{r^{3/2}} \), \( \alpha r^2 + \frac{\beta}{r} \), \( \alpha \sqrt{r} + \frac{\beta}{r^{3/2}} \), \( \alpha r^2 + \beta r \), and \( \alpha r^2 + \frac{\beta}{r} + \sigma r \). Starting with these solved potentials, using the method of the Newton duality, we solve their dual potentials: \( \frac{\alpha}{r^{3/2}} \), \( \alpha r^6 \), \( \alpha r^2 + \frac{\beta}{r^{3/2}} \), \( \alpha \sqrt{r} + \frac{\beta}{r^{3/2}} + \sigma r \), \( \alpha r^2 + \beta r^6 + \sigma r^4 \), and \( \frac{\alpha}{r^{3/2}} + \beta r + \frac{\sigma}{r^{1/2}} \).

These potentials are all long-range potentials. The exact solution is important for understanding long-range potentials. The long-range potential is the most important model in quantum mechanics, e.g., the Coulomb potential and the harmonic oscillator potential. Beyond the Coulomb potential and the harmonic oscillator potential, recently, some other long-range power potentials are exactly solved, e.g., the one-dimensional inverse square root potential [15] and the three-dimensional inverse square root potential [10]. The long-range-potential scattering is often a difficult problem. One reason is that different long-range potentials often have different scattering boundary conditions which are determined by the large-distance asymptotic solution. The long-range-potential scattering has been studied for many years [16–22]. Besides scattering of potentials in quantum mechanics, there are also other kinds of long-range scattering, e.g., the scattering on black holes [23, 24]. In a scattering problem, a key task is to seek scattering phase shifts [25, 26]. Many solutions mentioned in the paper are the Heun function, there are also researches on the Heun function [27–30].

In this paper, we discuss the Newton duality in classical mechanics and in quantum mechanics. The quantum version of the Newton duality is essentially the duality for the Schrödinger equation. In future work, we will consider the Newton duality for other dynamical equations, such as the Klein–Gordon equation and the Dirac equation, etc. The Einstein equation of gravity is also a field equation. By considering the Newton duality between Einstein equations, we can reveal the duality relation between spacetime manifolds.

In this paper, we also suggest a method for solving eigen equations. We here only focus on how to obtain an exact solution, in future work, we will try to construct an approximate method based on the Newton duality for eigen equations. Furthermore, we can also consider the Newton duality between field equations, and then consider the Newton duality in quantum field theory.

A The exact solution of \( U (r) = -\frac{\alpha}{r^{3/2}} \)

The inverse fractional power potential \( U (r) = -\alpha/r^{3/2} \) is a long-range potential, which has both bound states and scattering states. In this Appendix, we present the exact solutions of both bound and scattering states for the \( 1/r^{3/2} \)-potential.

In particular, the study of exact solutions is important for the long-range potential scattering. The scattering boundary conditions for different long-range potentials are different, while in short-range potential scattering, the scattering boundary conditions is the
same for all short-range potentials [14, 31]. To solve a scattering problem, one first needs to determine the scattering boundary condition which is determined by the long-distance asymptotic behavior of the solution. The $1/r^{3/2}$-potential is a long-range potential since the inverse-power potential $1/r^s$ with $0 < s < 2$ is identified as a long-range potential [32]. Though $1/r^s$ with $0 < s \leq 1$ and with $1 < s < 2$ are both long-range potentials, they are quite different in scattering: the scattering boundary for $1/r^s$-potential with $0 < s \leq 1$ differs from one potential to another, but the scattering boundary for $1/r^s$-potential with $1 < s \leq 2$ is the same as short-range potentials.

**A.1 The radial equation**

The radial wave function $R_l (r) = u_l (r)/r$ of the potential

$$U (r) = -\frac{\alpha}{r^{3/2}}$$

(A.1)

is determined by the radial equation

$$\frac{d^2 u_l (r)}{dr^2} + \left[ k^2 - \frac{l (l + 1)}{r^2} + \frac{\alpha}{r^{3/2}} \right] u_l (r) = 0$$

(A.2)

with the boundary conditions at $r = 0$ and at $r \to \infty$.

At $r = 0$, both for scattering states and bound states, the boundary condition is $u_l (0) = 0$, or, more stronger, $\lim_{r \to 0} u_l (r)/r^{l+1} = 1$ [22], since the radial wave function $R_l (r)$ at $r = 0$ must be finite.

At $r \to \infty$, the boundary condition of scattering states is that the wave function equals the asymptotic solution at $r \to \infty$, i.e., $u_l (r) \sim u_\infty (r)$; the boundary condition of bound states is that the wave function equals zero, i.e., $u_l (r) \to 0$.

In order to solve the radial equation (A.2), we introduce

$$u_l (z) = A_l e^{-z^2/2} z^{2(l+1)} f_l (z)$$

(A.3)

with

$$z = -\sqrt{-2ikr},$$

(A.4)

where $A_l$ is a constant. Substituting Eq. (A.3) into the radial equation (A.2) gives an equation of $f_l (z)$:

$$zf_l'' (z) + (z^2 + 4l + 3) f_l' (z) + [-4 (l + 1) z - 4\lambda] f_l (z) = 0,$$

(A.5)

where $\lambda = \alpha/\sqrt{-2ik}$. This equation is just the Biconfluent Heun equation [8].

Next, we solve Eq. (A.5) with boundary conditions.

**A.2 The regular solution**

The solution satisfying the boundary condition at $r = 0$ is called the regular solution.

At $r = 0$, as mentioned above, the scattering state and the bound state has the same boundary condition [22],

$$\lim_{r \to 0} \frac{u_l (r)}{r^{l+1}} = 1,$$

(A.6)
because the asymptotic solution of the radial equation (A.2) at $r = 0$ is $u_l (r) \sim r^{l+1}$ (there is also another asymptotic solution $u_l (r) \sim r^{-l}$ but it diverges at $r = 0$) \[33].

The equation of $f_l (z)$, Eq. (A.5), has two linearly independent solutions \[8],

\[
y_l^{(1)} (z) = N (4l + 2, 0, 0, 8\lambda, z), \tag{A.7}
\]

\[
y_l^{(2)} (z) = cN (4l + 2, 0, 0, 8\lambda, z) \ln z + \sum_{n \geq 0} d_n z^{n-2(2l+1)}, \tag{A.8}
\]

where $N (\alpha, \beta, \gamma, \delta, z)$ is the Heun Biconfluent function \[8, 9\]. The constant $c$ is

\[
c = \frac{2}{2l + 1} (\lambda d_{4l+1} + ld_{4l}) \tag{A.9}
\]

and $d_{\nu}$ is determined by the recurrence relation \[8\]

\[
d_{-1} = 0, \quad d_0 = 1, \quad (\nu + 2) (\nu - 4l) d_{\nu+2} - 4\lambda d_{\nu+1} + [-2 (\nu + 1) + 4l + 2] d_{\nu} = 0. \tag{A.10}
\]

To determine the solution with the boundary condition (A.6), we use the expansion of the Heun Biconfluent function near $z = 0$ \[8\],

\[
N (4l + 2, 0, 0, 8\lambda, z) = \sum_{n \geq 0} \frac{A_n}{(4l + 3)_n} \frac{z^n}{n!}, \tag{A.11}
\]

where $(a)_n = \Gamma (a + n) / \Gamma (a)$ is Pochhammer’s symbol and the coefficient $A_n$ is determined by the recurrence relation \[8\]:

\[
A_0 = 1, \quad A_1 = 4\lambda, \quad A_{n+2} = 4\lambda A_{n+1} + 2 (n + 1) (n + 4l + 3) (2l + 2 + n) A_n. \tag{A.12}
\]

It is clear that only $y_l^{(1)} (z)$ satisfies the boundary condition at $r = 0$, Eq. (A.6). Then, by Eqs. (A.3) and (A.4), we obtain the regular solution:

\[
u_l (r) = A_l e^{i\lambda r} (-2i\lambda r)^{l+1} N \left(4l + 2, 0, 0, 8\lambda, -\sqrt{-2i\lambda r}\right). \tag{A.13}
\]

### A.3 The irregular solution

The solution satisfying the boundary condition at $r \to \infty$ is called the irregular solution. Scattering states and bound states have different boundary conditions at $r \to \infty$.

#### A.3.1 The Scattering boundary condition

The scattering boundary condition requires that the wave function equals the asymptotic solution of the radial equation (A.2) at $r \to \infty$. The asymptotic behavior of the wave function at $r \to \infty$ is dominated by the external potential, $-\alpha / r^{3/2}$, rather than the centrifugal potential $l (l + 1) / r^2$ like that in the case of short-range potentials. That is to say, the scattering boundary condition is determined by the external potential $-\alpha / r^{3/2}$.

To determine the scattering boundary condition, we first solve the asymptotic solution of the radial equation (A.2).
Letting
\[ u_l(r) = e^{h(r)}e^{\pm ikr} \] (A.14)
and substituting into the radial equation (A.2) give an equation of \( h(r) \):
\[ h''(r) + h'(r)^2 \pm 2ikh'(r) = \frac{l(l+1)}{r^2} - \frac{\alpha}{r^{3/2}}. \] (A.15)
Only taking the leading contribution into account for we only concentrate on the asymptotic behavior, we have
\[ \pm 2ikh'(r) \sim \frac{-\alpha}{r^{3/2}}. \] (A.16)
Then we have
\[ h(r) \sim \pm \frac{i\alpha}{k\sqrt{r}}. \] (A.17)
Obviously, this contribution vanishes when \( r \to \infty \) and can be dropped out in the asymptotic solution. Then, the large-distance asymptotic solution of Eq. (A.2) reads
\[ u_l(r) = e^{\pm ikr}. \] (A.18)
It can be seen that the asymptotic behavior of the potential \( U(r) = -\alpha/r^{3/2} \) is the same as that of the short-range potential.

The scattering boundary condition, then, can be written as
\[ \lim_{r \to \infty} e^{\pm ikr}u_l(r) = 1. \] (A.19)

A.3.2 The irregular solution

Now we can determine the irregular solution by the scattering boundary condition given above.

Eq. (A.5) with the scattering boundary condition (A.19) has two linearly independent solutions [8]
\[ B^+_l(4l + 2, 0, 0, 8\lambda, z) = z^{-2(l+1)} \sum_{n \geq 0} \frac{a_n}{z^n}, \] (A.20)
\[ H^+_l(4l + 2, 0, 0, 8\lambda, z) = z^{-2(l+1)}e^{z^2} \sum_{n \geq 0} \frac{e_n}{z^n}, \] (A.21)
where \( B^+_l(\alpha, \beta, \gamma, \delta, z) \) and \( H^+_l(\alpha, \beta, \gamma, \delta, z) \) are two other biconfluent Heun functions [8], different from the Heun functions mentioned above. The coefficients in Eqs. (A.20) and (A.21) are given by
\[ a_0 = 1, \quad a_1 = 2\lambda, \]
\[ 2(n+2)a_{n+2} - 2\lambda a_{n+1} + \left[n(n+2) - (2l+1)^2 + 1\right]a_n = 0, \] (A.22)
and
\[ e_0 = 1, \quad e_1 = -2\lambda, \]
\[ 2(n+2)e_{n+2} + 2\lambda e_{n+1} - \left[n(n+2) - (2l+1)^2 + 1\right]e_n = 0. \] (A.23)
By Eq. (A.3), one can directly check that \( B^+_l(\alpha, \beta, \gamma, \delta, z) \) and \( H^+_l(\alpha, \beta, \gamma, \delta, z) \) satisfy the scattering boundary condition (A.19).
A.4 Bound states and scattering states

With the regular solution and the irregular solution, we can construct the solution of bound states and scattering states.

First express the regular solution as a linear combination of the two irregular solutions [8]:

\[
N (4l + 2, 0, 0, 8\lambda, z) = K_1 (4l + 2, 0, 0, 8\lambda) B^+_l (4l + 2, 0, 0, 8\lambda, z) + K_2 (4l + 2, 0, 0, 8\lambda) H^+_l (4l + 2, 0, 0, 8\lambda, z),
\]

(A.24)

where \( K_1 (4l + 2, 0, 0, 8\lambda) \) and \( K_2 (4l + 2, 0, 0, 8\lambda) \) are the coefficients of combination [8].

Using the expansions (A.20) and (A.21), we can rewrite (A.3) as

\[
u_l (r) = A_l K_1 (4l + 2, 0, 0, 8\lambda) e^{ikr} \sum_{n \geq 0} \frac{a_n}{(-\sqrt{-2ikr})^n} + A_l K_2 (4l + 2, 0, 0, 8\lambda) e^{-ikr} \sum_{n \geq 0} \frac{e_n}{(\sqrt{-2ikr})^n}.
\]

(A.25)

A.4.1 The bound state

By analytical continuation \( k \) to whole complex plane, we can consider \( k \) on the positive imaginary axis, i.e.,

\[
k = i\kappa, \quad \kappa > 0.
\]

(A.26)

Notice that on the positive imaginary axis \( \lambda = \alpha/\sqrt{2\kappa} \), Eq. (A.25) becomes

\[
u_l (r) = A_l K_1 (4l + 2, 0, 0, 8\lambda) e^{-\kappa r} \sum_{n \geq 0} \frac{a_n}{(2\kappa r)^{n/2}} + A_l K_2 (4l + 2, 0, 0, 8\lambda) e^{\kappa r} \sum_{n \geq 0} \frac{e_n}{(2\kappa r)^{n/2}}.
\]

(A.27)

For bound states, only the first term contributes because the bound state boundary condition requires \( \nu_l (\infty) \rightarrow 0 \). Then we have

\[
K_2 (4l + 2, 0, 0, 8\lambda) = 0,
\]

(A.28)

where [8]

\[
K_2 (\alpha, \beta, \gamma, \delta) = \frac{\Gamma (1 + \alpha)}{\Gamma ((\alpha - \gamma)/2) \Gamma (1 + (\alpha + \gamma)/2)} J_{1+(\alpha + \gamma)/2} \left( \frac{1}{2} (\alpha + \gamma), \beta, \frac{1}{2} (3\alpha - \gamma), \delta + \frac{1}{2} \beta (\gamma - \alpha) \right),
\]

(A.29)

with

\[
J_\lambda (\alpha, \beta, \gamma, \delta) = \int_0^\infty x^{\lambda-1} e^{-x^2-\beta x} N (\alpha, \beta, \gamma, \delta, x) \, dx.
\]

(A.30)

Eq. (A.28) is just an implicit expression of the bound-state eigenvalue.

The bound-state wave function reads

\[
u_l (r) = Ce^{-\kappa r} \sum_{n \geq 0} \frac{a_n}{(2\kappa r)^{n/2}},
\]

(A.31)

where \( C \) is the normalization constant.
A.4.2 The scattering state

The singularity of the $S$-matrix on the positive imaginary axis corresponds to the eigenvalues of bound states \[34\], so the zero of $K_2 (4l + 2, 0, 0, 8\lambda)$ on the positive imaginary is just the singularity of the $S$-matrix. Considering that the $S$-matrix is unitary, i.e.,

$$S_l = e^{2i\delta_l}, \quad (A.32)$$

we have

$$S_l (k) = \frac{K_2^* (4l + 2, 0, 0, 8i\lambda)}{K_2 (4l + 2, 0, 0, 8\lambda)} = \frac{K_2 (4l + 2, 0, 0, 8i\lambda)}{K_2^* (4l + 2, 0, 0, 8\lambda)}. \quad (A.33)$$

where the relation $K_2^* (4l + 2, 0, 0, 8\alpha/\sqrt{-2ik}) = K_2^* (4l + 2, 0, 0, 8i\alpha/\sqrt{-2ik})$ is used.

The scattering wave function can be constructed with the help of the $S$-matrix. The scattering wave function can be expressed as a linear combination of the radially ingoing wave $u_{in}$ and the radially outgoing wave $u_{out}$, which are conjugate to each other, i.e., \[34\]

$$u_l (r) = A_l \left[ (-1)^{l+1} u_{in} (r) + S_l (k) u_{out} (r) \right]. \quad (A.34)$$

From Eq. (A.25), we have

$$u_{in} = e^{-ikr} \sum_{n \geq 0} e_n \left(-\sqrt{-2ikr} \right)^n, \quad u_{out} = e^{ikr} \sum_{n \geq 0} e_n^* \left(-\sqrt{2ikr} \right)^n. \quad (A.35)$$

Then by Eq. (A.33) we achieve the scattering wave function,

$$u_l (r) = A_l \left[ (-1)^{l+1} e^{-ikr} \sum_{n \geq 0} e_n \left(-\sqrt{-2ikr} \right)^n + \frac{K_2 (4l + 2, 0, 0, 8i\lambda)}{K_2^* (4l + 2, 0, 0, 8\lambda)} e^{ikr} \sum_{n \geq 0} e_n^* \left(-\sqrt{2ikr} \right)^n \right]. \quad (A.36)$$

Taking $r \to \infty$, we have

$$u_l (r) \approx A_l \left[ (-1)^{l+1} e^{-ikr} + \frac{K_2 (4l + 2, 0, 0, 8i\lambda)}{K_2^* (4l + 2, 0, 0, 8\lambda)} e^{ikr} \right]$$

$$= A_l e^{i\delta_l} \sin \left( kr + \delta_l - \frac{l\pi}{2} \right). \quad (A.37)$$

By Eqs. (A.32) and (A.33), we obtain the scattering phase shift

$$\delta_l = - \arg K_2 (4l + 2, 0, 0, 8\lambda). \quad (A.38)$$

B The exact solution of $U (r) = \xi r^2 + \frac{\mu}{r}$

In this appendix, we provide an exact solution of the eigenproblem of the potential

$$U (r) = \xi r^2 + \frac{\mu}{r} \quad (B.1)$$
by solving the radial equation directly. This potential has only bound states. The radial equation of the potential (B.1) reads

$$\frac{d^2 u_l (r)}{dr^2} + \left[ E - \frac{l (l + 1)}{r^2} - \xi r^2 - \frac{\mu}{r} \right] u_l (r) = 0.$$  \hspace{1cm} (B.2)

Using the variable substitution

$$z = i \xi^{1/4} r$$ \hspace{1cm} (B.3)

and introducing $f_l (z)$ by

$$u_l (z) = A_l \exp \left( -\frac{z^2}{2} \right) z^{l+1} f_l (z)$$ \hspace{1cm} (B.4)

with $A_l$ a constant, we convert the radial equation (B.2) into an equation of $f_l (z)$:

$$zf_l'' (z) + (-2z^2 + 2l + 2) f_l' (z) + \left[ \left( \frac{-E}{\sqrt{\xi}} - 2l - 3 \right) z + \frac{i \mu}{\xi^{1/4}} \right] f_l (z) = 0.$$  \hspace{1cm} (B.5)

This is a Biconfluent Heun equation [8].

The choice of the boundary condition has been discussed in Ref. [10].

B.1 The regular solution

The regular solution is a solution satisfying the boundary condition at $r = 0$ [10]. The regular solution at $r = 0$ should satisfy the boundary condition $\lim_{r \to 0} u_l (r)/r^{l+1} = 1$. In this section, we provide the regular solution of Eq. (B.5).

The Biconfluent Heun equation (B.5) has two linearly independent solutions [8]

$$y_l^{(1)} (z) = N \left( 2l + 1, 0, -\frac{E}{\sqrt{\xi}}, -\frac{2i \mu}{\xi^{1/4}}, z \right),$$  \hspace{1cm} (B.6)

$$y_l^{(2)} (z) = cN \left( 2l + 1, 0, -\frac{E}{\sqrt{\xi}}, -\frac{2i \mu}{\xi^{1/4}}, z \right) \ln z + \sum_{n \geq 0} d_n z^{n-2l-1},$$  \hspace{1cm} (B.7)

where

$$c = \frac{1}{2l+1} \left[ \frac{-i \mu}{\xi^{1/4}} d_{2l} - \left( \frac{-E}{\sqrt{\xi}} - 2l - 3 \right) d_{2l-1} \right]$$ \hspace{1cm} (B.8)

is a constant with the coefficient $d_\nu$ given by the following recurrence relation,

$$d_{-1} = 0, \hspace{1cm} d_0 = 1,$$

$$(\nu + 2) (\nu + 1 - 2l) d_{\nu+2} + \frac{i \mu}{\xi^{1/4}} d_{\nu+1} + \left( \frac{-E}{\sqrt{\xi}} - 2v - 1 + 2l \right) d_\nu = 0$$ \hspace{1cm} (B.9)

and $N(\alpha, \beta, \gamma, \delta, z)$ is the biconfluent Heun function [8–10].

The biconfluent Heun function $N \left( 2l + 1, 0, -\frac{E}{\sqrt{\xi}}, -\frac{2i \mu}{\xi^{1/4}}, z \right)$ has an expansion at $z = 0$ [8]:

$$N \left( 2l + 1, 0, -\frac{E}{\sqrt{\xi}}, -\frac{2i \mu}{\xi^{1/4}}, z \right) = \sum_{n \geq 0} \frac{A_n}{(2l+2)_n} \frac{z^n}{n!},$$ \hspace{1cm} (B.10)
where the expansion coefficients is determined by the recurrence relation,

\[ A_0 = 1, \quad A_1 = -i \mu / \xi^{1/4}, \]

\[ A_{n+2} = -i \mu / \xi^{1/4} A_{n+1} - (n + 1) (n + 2l + 2) \left( -E / \sqrt{\xi} - 3 - 2l - 2n \right) A_n, \quad (B.11) \]

and \((a)_n = \Gamma (a + n) / \Gamma (a)\) is Pochhammer’s symbol.

Only \(y_l^{(1)} (z)\) satisfies the boundary condition for the regular solution at \(r = 0\), so the radial eigenfunction reads

\[ u_l (z) = A_l \exp \left( -\frac{z^2}{2} \right) z^{l+1} y_l^{(1)} (z) = A_l \exp \left( -\frac{z^2}{2} \right) z^{l+1} N \left( 2l + 1, 0, -\frac{E}{\sqrt{\xi}}, \frac{2i \mu \xi^{1/4}}{\xi^{1/4}}, z \right). \quad (B.12) \]

By Eq. (B.3), we obtain the regular solution,

\[ u_l (r) = A_l \exp \left( \frac{\xi^{1/2} r^2}{2} \right) r^{l+1} N \left( 2l + 1, 0, -\frac{E}{\sqrt{\xi}}, -\frac{2i \mu \xi^{1/4}}{\xi^{1/4}}, i \xi^{1/4} r \right). \quad (B.13) \]

### B.2 The irregular solution

The irregular solution is a solution satisfying the boundary condition at \(r \to \infty\) [10].

The Biconfluent Heun equation (B.5) has two linearly independent irregular solutions [8]:

\[ B_l^+ \left( 2l + 1, 0, -\frac{E}{\sqrt{\xi}}, -\frac{2i \mu \xi^{1/4}}{\xi^{1/4}}, z \right) = e^{z^2} B_l^+ \left( 2l + 1, 0, -\frac{E}{\sqrt{\xi}}, \frac{2i \mu \xi^{1/4}}{\xi^{1/4}}, -iz \right) = e^{z^2} (-iz)^{1/2} \left( \frac{E}{\sqrt{\xi}} - 2l - 3 \right) \sum_{n = 0}^{\infty} \frac{a_n}{(-iz)^n}, \quad (B.14) \]

\[ H_l^+ \left( 2l + 1, 0, -\frac{E}{\sqrt{\xi}}, -\frac{2i \mu \xi^{1/4}}{\xi^{1/4}}, z \right) = e^{z^2} H_l^+ \left( 2l + 1, 0, -\frac{E}{\sqrt{\xi}}, \frac{2i \mu \xi^{1/4}}{\xi^{1/4}}, -iz \right) = (-iz)^{-\frac{1}{2}} \left( \frac{E}{\sqrt{\xi}} + 2l + 3 \right) \sum_{n = 0}^{\infty} \frac{e_n}{(-iz)^n}, \quad (B.15) \]

with the expansion coefficients given by the recurrence relation

\[ a_0 = 1, \quad a_1 = -\frac{\mu}{2 \xi^{1/4}}, \]

\[ 2 (n + 2) a_{n+2} - \frac{\mu}{\xi^{1/4}} a_{n+1} + \left[ \frac{E^2}{4 \xi} - \frac{(2l + 1)^2}{4} + 1 - \frac{E}{\sqrt{\xi}} + n \left( n + 2 - \frac{E}{\sqrt{\xi}} \right) \right] a_n = 0 \quad (B.16) \]

and

\[ e_0 = 1, \quad e_1 = -\frac{\mu}{2 \xi^{1/4}}, \]

\[ 2 (n + 2) e_{n+2} + \frac{\mu}{\xi^{1/4}} e_{n+1} - \left[ \frac{E^2}{4 \xi} - \frac{(2l + 1)^2}{4} + 1 + \frac{E}{\sqrt{\xi}} + n \left( n + 2 + \frac{E}{\sqrt{\xi}} \right) \right] e_n = 0. \quad (B.17) \]
B.3 Eigenfunctions and eigenvalues

To construct the solution, we first express the regular solution (B.13) as a linear combination of the two irregular solutions (B.14) and (B.15).

The regular solution (B.13), with the relation [8, 10]

\[
N(2l + 1, 0, -E \sqrt{\xi} - \frac{2i\mu}{\xi^{1/4}}, z) = K_1(2l + 1, 0, -E \sqrt{\xi} - \frac{2i\mu}{\xi^{1/4}}) B_1^+ \left(2l + 1, 0, -E \sqrt{\xi} - \frac{2i\mu}{\xi^{1/4}}, z\right) + K_2 \left(2l + 1, 0, -E \sqrt{\xi} - \frac{2i\mu}{\xi^{1/4}}\right) H_1^+ \left(2l + 1, 0, -E \sqrt{\xi} - \frac{2i\mu}{\xi^{1/4}}, z\right)
\]

(B.18)

and the expansions (B.14) and (B.15), becomes

\[
u_l(r) = A_l K_1 \left(2l + 1, 0, -E \sqrt{\xi} - \frac{2i\mu}{\xi^{1/4}}\right) \exp \left(-\frac{\xi^{1/2}r^2}{2}\right) \exp \left(\left(\frac{E}{2\sqrt{\xi}} - \frac{1}{2}\right) \ln \left(\xi^{1/4}\right)\right) \sum_{n \geq 0} \frac{a_n}{(\xi^{1/4})^n}
\]

+ \[A_l K_2 \left(2l + 1, 0, -E \sqrt{\xi} - \frac{2i\mu}{\xi^{1/4}}\right) \exp \left(-\frac{\xi^{1/2}r^2}{2}\right) \exp \left((-\frac{E}{2\sqrt{\xi}} - \frac{1}{2}) \ln \left(\xi^{1/4}\right)\right) \sum_{n \geq 0} \frac{e_n}{(\xi^{1/4})^n},
\]

where \(K_1 \left(2l + 1, 0, -E \sqrt{\xi} - \frac{2i\mu}{\xi^{1/4}}\right)\) and \(K_2 \left(2l + 1, 0, -E \sqrt{\xi} - \frac{2i\mu}{\xi^{1/4}}\right)\) are combination coefficients and \(z = i\xi^{1/4}r\).

The boundary condition of bound states, \(u(r)|_{r \to \infty} = 0\), requires that the coefficient of the second term must vanish since this term diverges when \(r \to \infty\), i.e.,

\[
K_2 \left(2l + 1, 0, -E \sqrt{\xi} - \frac{2i\mu}{\xi^{1/4}}\right) = 0,
\]

(B.20)

where

\[
K_2 \left(2l + 1, 0, -E \sqrt{\xi} - \frac{2i\mu}{\xi^{1/4}}\right) = \frac{\Gamma(2l + 2)}{\Gamma\left(l + \frac{1}{2} + \frac{E}{2\sqrt{\xi}}\right) \Gamma\left(l + \frac{3}{2} - \frac{E}{2\sqrt{\xi}}\right)} \times J_{l + \frac{3}{2} - \frac{E}{2\sqrt{\xi}}} \left(l + \frac{1}{2}, -\frac{E}{2\sqrt{\xi}}, 0, 3l + \frac{3}{2}, -\frac{E}{2\sqrt{\xi}}, -\frac{2i\mu}{\xi^{1/4}}\right)
\]

(B.21)

with

\[
J_{l + \frac{3}{2} - \frac{E}{2\sqrt{\xi}}} \left(l + \frac{1}{2} - \frac{E}{2\sqrt{\xi}}, 0, 3l + \frac{3}{2}, -\frac{E}{2\sqrt{\xi}}, -\frac{2i\mu}{\xi^{1/4}}\right) = \int_0^\infty x^{l + \frac{1}{2} - \frac{E}{2\sqrt{\xi}}} e^{-x} N \left(l + \frac{1}{2} - \frac{E}{2\sqrt{\xi}}, 0, 3l + \frac{3}{2}, -\frac{E}{2\sqrt{\xi}}, -\frac{2i\mu}{\xi^{1/4}}, x\right) dx.
\]

(B.22)

Eq. (B.20) is an implicit expression of the eigenvalue.

The eigenfunction, by Eqs. (B.19) and (B.20), reads

\[
u_l(r) = A_l K_1 \left(2l + 1, 0, -E \sqrt{\xi} - \frac{2i\mu}{\xi^{1/4}}\right) \exp \left(-\frac{\xi^{1/2}r^2}{2}\right) \exp \left(\left(\frac{E}{2\sqrt{\xi}} - \frac{1}{2}\right) \ln \left(\xi^{1/4}\right)\right) \sum_{n \geq 0} \frac{a_n}{(\xi^{1/4})^n}.
\]

(B.23)
C The exact solution of $U(r) = \frac{\xi}{\sqrt{r}} + \frac{\mu}{r^{3/2}}$

In this appendix, we provide an exact solution of the eigenproblem of the potential

$$U(r) = \frac{\xi}{\sqrt{r}} + \frac{\mu}{r^{3/2}} \quad \text{(C.1)}$$

by solving the radial equation directly. This potential has both bound states and scattering states.

The radial equation reads

$$\frac{d^2}{dr^2} u_l(r) + \left[ E - \frac{l(l+1)}{r^2} - \frac{\xi}{\sqrt{r}} - \frac{\mu}{r^{3/2}} \right] u_l(r) = 0. \quad \text{(C.2)}$$

Using the variable substitution

$$z = i\sqrt{2r(-E)^{1/4}} \quad \text{(C.3)}$$

and introducing $f_l(z)$ by

$$u_l(z) = A_l \exp\left(-\frac{z^2}{2} - \frac{\beta}{2} z\right) z^{2(l+1)} f_l(z) \quad \text{(C.4)}$$

with $A_l$ a constant, we convert the radial equation (C.2) into an equation of $f_l(z)$:

$$zf_l''(z) + \left( 4l + 3 - \frac{i\sqrt{2}\xi}{(-E)^{3/4}}z - 2z^2 \right) f_l'(z) + \left[ \left( -\frac{\xi^2}{2(-E)^{3/2}} - 4l - 4 \right) z - \frac{1}{2} \left( -\frac{i4\sqrt{2}\mu}{(-E)^{1/4}} + \frac{i\sqrt{2}\xi}{(-E)^{3/4}} (4l + 3) \right) \right] f_l(z) = 0. \quad \text{(C.5)}$$

This is a Biconfluent Heun equation [8].

The choice of the boundary condition has been discussed in Ref. [10].

C.1 The regular solution

The regular solution is a solution satisfying the boundary condition at $r = 0$ [10]. The regular solution at $r = 0$ should satisfy the boundary condition $\lim_{r\to 0} u_l(r)/r^{l+1} = 1$ for both bound states and scattering states. In this section, we provide the regular solution of Eq. (C.5).

The Biconfluent Heun equation (C.5) has two linearly independent solutions [8]

$$y_l^{(1)}(z) = N \left( 4l + 2, \frac{i\sqrt{2}\xi}{(-E)^{3/4}}, -\frac{\xi^2}{2(-E)^{3/2}}, -\frac{i4\sqrt{2}\mu}{(-E)^{1/4}} z \right), \quad \text{(C.6)}$$

$$y_l^{(2)}(z) = cN \left( 4l + 2, \frac{i\sqrt{2}\xi}{(-E)^{3/4}}, -\frac{\xi^2}{2(-E)^{3/2}}, -\frac{i4\sqrt{2}\mu}{(-E)^{1/4}} z \right) \ln z + \sum_{n\geq 0} d_n z^{-4l-2}, \quad \text{(C.7)}$$

where

$$c = \frac{1}{4l+2} \left[ d_{4l+1} \frac{1}{2} \left( -\frac{i4\sqrt{2}\mu}{(-E)^{1/4}} + \frac{i\sqrt{2}\xi}{(-E)^{3/4}} (4l + 1) \right) - d_{4l} \left( -\frac{\xi^2}{2(-E)^{3/2}} - 4l \right) \right] \quad \text{(C.8)}$$
is a constant with the coefficient \( d_\nu \) given by the following recurrence relation,

\[
d_{-1} = 0, \quad d_0 = 1, \quad (v + 2) (v - 4l) d_{v+2} - \frac{1}{2} \left[ \frac{-i4\sqrt{2}\mu}{(-E)^{1/4}} + \frac{i\sqrt{2}\xi}{(-E)^{3/4}} (2v + 1 - 4l) \right] d_{v+1} + \left[ \frac{-\xi^2}{2 (-E)^{3/2}} - 2v + 4l \right] d_v = 0 \tag{C.9}
\]

and \( N(\alpha, \beta, \gamma, \delta, z) \) is the biconfluent Heun function [8–10].

The biconfluent Heun function \( N \left( 4l + 2, \frac{i\sqrt{2}\xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, \frac{-i4\sqrt{2}\mu}{(-E)^{1/4}}, \frac{-i4\sqrt{2}\mu}{(-E)^{1/4}}, z \right) \) has an expansion at \( z = 0 \) [8]:

\[
N \left( 4l + 2, \frac{i\sqrt{2}\xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, \frac{-i4\sqrt{2}\mu}{(-E)^{1/4}}, z \right) = \sum_{n \geq 0} \frac{A_n}{(4l + 3)_n} \frac{z^n}{n!}, \tag{C.10}
\]

where the expansion coefficients is determined by the recurrence relation,

\[
A_0 = 1, \quad A_1 = \frac{1}{2} \left[ \frac{-i4\sqrt{2}\mu}{(-E)^{1/4}} + \frac{i\sqrt{2}\xi}{(-E)^{3/4}} (4l + 3) \right], \quad A_{n+2} = \left\{ \frac{i\sqrt{2}\xi}{(-E)^{3/4}} (n+1) + \frac{1}{2} \left[ \frac{-i4\sqrt{2}\mu}{(-E)^{1/4}} + \frac{i\sqrt{2}\xi}{(-E)^{3/4}} (4l + 3) \right] A_{n+1} \right. \\
\left. - (n+1) (n+4l+3) \left[ \frac{-\xi^2}{2 (-E)^{3/2}} - 4 - 4l - 2n \right] A_n \right\}, \tag{C.11}
\]

and \((a)_n = \Gamma(a+n)/\Gamma(a)\) is Pochhammer’s symbol.

Only \( y^{(1)}_l(z) \) satisfies the boundary condition for the regular solution at \( r = 0 \), so the radial eigenfunction reads

\[
u_l(z) = A_l \left( -\frac{\xi^2}{2} - \frac{\beta}{2} \right) z^{2(l+1)} y^{(1)}_l(z) = A_l \left( -\frac{\xi^2}{2} - \frac{\beta}{2} \right) z^{2(l+1)} N \left( 4l + 2, \frac{i\sqrt{2}\xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, \frac{-i4\sqrt{2}\mu}{(-E)^{1/4}}, \frac{-i4\sqrt{2}\mu}{(-E)^{1/4}}, z \right). \tag{C.12}
\]

By Eq. (B.3), we obtain the regular solution,

\[
u_l(r) = A_l \left[ (-E)^{1/2} r + \frac{\xi}{(-E)^{1/2} \sqrt{r}} \right] r^{l+1} N \left( 4l + 2, \frac{i\sqrt{2}\xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, \frac{-i4\sqrt{2}\mu}{(-E)^{1/4}}, \frac{-i4\sqrt{2}\mu}{(-E)^{1/4}}, \right). \tag{C.13}
\]

### C.2 Irregular solution

The irregular solution is a solution satisfying the boundary condition at \( r \to \infty \) [10]. The boundary conditions for bound states and scattering states at \( r \to \infty \) are different.
The Biconfluent Heun equation (C.5) has two linearly independent irregular solutions [8]:

\[ B_1^+ (4l + 2, \frac{i \sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{i 4 \sqrt{2} \mu}{(-E)^{1/4}}, z) = \exp \left( \frac{i \sqrt{2} \xi}{(-E)^{3/4}} z + z^2 \right) B_1^+ (4l + 2, \frac{\sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{4 \sqrt{2} \mu}{(-E)^{1/4}}, -iz) = \exp \left( \frac{i \sqrt{2} \xi}{(-E)^{3/4}} z + z^2 \right) \frac{1}{(-iz)^n} \sum_{n=0}^\infty \frac{a_n}{(-iz)^n}, \] (C.14)

\[ H_1^+ (4l + 2, \frac{i \sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{i 4 \sqrt{2} \mu}{(-E)^{1/4}}, z) = \exp \left( \frac{i \sqrt{2} \xi}{(-E)^{3/4}} z + z^2 \right) H_1^+ (4l + 2, \frac{\sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{4 \sqrt{2} \mu}{(-E)^{1/4}}, -iz) = (-iz)^{-l} \frac{1}{(\sqrt{E} - E)^2 + 4l + 4} \sum_{n=0}^\infty \frac{e_n}{(-iz)^n}, \] (C.15)

with the expansion coefficients given by the recurrence relation

\[ a_0 = 1, \quad a_1 = \frac{1}{4} \left[ \frac{4 \sqrt{2} \mu}{(-E)^{1/4}} + \frac{\sqrt{2} \xi}{(-E)^{3/4}} - \frac{\xi^2}{2 (-E)^{3/2}} - 1 \right], \]

\[ 2(n + 2) a_{n+2} + \left[ \frac{\sqrt{2} \xi}{(-E)^{3/4}} \left( \frac{3}{2} - \frac{\xi^2}{4 (-E)^{3/2}} + n \right) - \frac{2 \sqrt{2} \mu}{(-E)^{1/4}} \right] a_{n+1} + \frac{\xi^4}{16 (-E)^4} \left( \frac{(4l + 2)^2}{4} + 1 - \frac{\xi^2}{2 (-E)^{3/2}} + n \left( n + 2 - \frac{\xi^2}{2 (-E)^{3/2}} \right) \right) a_n = 0 \] (C.16)

and

\[ e_0 = 1, \quad e_1 = \frac{1}{4} \left[ \frac{4 \sqrt{2} \mu}{(-E)^{1/4}} + \frac{\sqrt{2} \xi}{(-E)^{3/4}} - \frac{\xi^2}{2 (-E)^{3/2}} + 1 \right], \]

\[ 2(n + 2) e_{n+2} + \left[ \beta \left( \frac{3}{2} + \frac{\xi^2}{4 (-E)^{3/2}} + n \right) + \frac{\delta}{2} \right] e_{n+1} - \frac{\xi^4}{16 (-E)^4} \left( \frac{(4l + 2)^2}{4} + 1 + \frac{\xi^2}{2 (-E)^{3/2}} + n \left( n + 2 + \frac{\xi^2}{2 (-E)^{3/2}} \right) \right) e_n = 0. \] (C.17)

C.3 Bound states and scattering states

C.3.1 The bound state

To construct the solution, we first express the regular solution (C.13) as a linear combination of the two irregular solutions (C.14) and (C.15).
The regular solution (C.13), with the relation [8, 10]

\[ N \left( 4l + 2, \frac{i \sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{i 4 \sqrt{2} \mu}{(-E)^{1/4}}, z \right) \]

\[ = K_1 \left( 4l + 2, \frac{i \sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{i 4 \sqrt{2} \mu}{(-E)^{1/4}}, z \right) \]

\[ + K_2 \left( 4l + 2, \frac{i \sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{i 4 \sqrt{2} \mu}{(-E)^{1/4}}, z \right) \]

and the expansions (C.14) and (C.15), become

\[ u_i (r) = A_1 K_1 \left( 4l + 2, \frac{i \sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{i 4 \sqrt{2} \mu}{(-E)^{1/4}} \right) \]

\[ \times \exp \left( -(-E)^{1/2} r - \frac{\xi}{(-E)^{1/2} \sqrt{r}} \right) \left[ i (-E)^{1/4} \sqrt{2r} \right]^{\xi^2/[4(-E)^{3/2}]} \sum_{n \geq 0} \frac{a_n}{\sqrt{2r} (-E)^{1/4}}^n \]

\[ + A_1 K_2 \left( 4l + 2, \frac{i \sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{i 4 \sqrt{2} \mu}{(-E)^{1/4}} \right) \]

\[ \times \exp \left( -(-E)^{1/2} r + \frac{\xi}{(-E)^{1/2} \sqrt{r}} \right) \left[ i (-E)^{1/4} \sqrt{2r} \right]^{-\xi^2/[4(-E)^{3/2}]} \sum_{n \geq 0} \frac{e_n}{\sqrt{2r} (-E)^{1/4}}^n, \]

where \( K_1 \left( 4l + 2, \frac{i \sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{i 4 \sqrt{2} \mu}{(-E)^{1/4}} \right) \) and \( K_2 \left( 4l + 2, \frac{i \sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{i 4 \sqrt{2} \mu}{(-E)^{1/4}} \right) \)

are combination coefficients and \( z = i \sqrt{2} \xi (-E)^{1/4} \).

The boundary condition of bound states, \( u_i (r) |_{r \to \infty} \to 0 \), requires that the coefficient of the second term must vanish since this term diverges when \( r \to \infty \), i.e.,

\[ K_2 \left( 4l + 2, \frac{i \sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{i 4 \sqrt{2} \mu}{(-E)^{1/4}} \right) = 0. \]  

Eq. (C.20) is an implicit expression of the eigenvalue.

The eigenfunction, by Eqs. (C.19) and (C.20), reads

\[ u_i (r) = A_1 K_1 \left( 4l + 2, \frac{i \sqrt{2} \xi}{(-E)^{3/4}}, -\frac{\xi^2}{2 (-E)^{3/2}}, -\frac{i 4 \sqrt{2} \mu}{(-E)^{1/4}} \right) \]

\[ \times \left[ i (-E)^{1/4} \sqrt{2r} \right]^{\xi^2/[4(-E)^{3/2}]} \sum_{n \geq 0} \frac{a_n}{\sqrt{2r} (-E)^{1/4}}^n. \]  

(C.21)
C.3.2 The scattering state

For scattering states, \( E > 0 \), introduce

\[
E = k^2. \tag{C.22}
\]

The singularity of the \( S \)-matrix on the positive imaginary axis corresponds to the eigenvalue of bound states \([34]\), so the zero of \( K_2 \left( 4l + 2, -\frac{(1+i)\xi}{k^{3/2}}, \frac{i\xi^2}{2k^3}, \frac{4\sqrt{-2i\mu}}{\sqrt{k}} \right) \) on the positive imaginary is just the singularity of the \( S \)-matrix. Considering that the \( S \)-matrix is unitary, i.e.,

\[
S_l = e^{2i\delta_l}, \tag{C.23}
\]

we have

\[
S_l (k) = \frac{K_2 \left( 4l + 2, -\frac{(1+i)\xi}{k^{3/2}}, \frac{i\xi^2}{2k^3}, \frac{4\sqrt{-2i\mu}}{\sqrt{k}} \right)}{K_2 \left( 4l + 2, -\frac{(1+i)\xi}{k^{3/2}}, \frac{i\xi^2}{2k^3}, \frac{4\sqrt{-2i\mu}}{\sqrt{k}} \right)} = \frac{K_2 \left( 4l + 2, -\frac{(1+i)\xi}{k^{3/2}}, -\frac{i\xi^2}{2k^3}, \frac{4\sqrt{2i\mu}}{\sqrt{k}} \right)}{K_2 \left( 4l + 2, -\frac{(1+i)\xi}{k^{3/2}}, -\frac{i\xi^2}{2k^3}, \frac{4\sqrt{2i\mu}}{\sqrt{k}} \right)}. \tag{C.24}
\]

The scattering wave function can be constructed with the help of the \( S \)-matrix. The scattering wave function can be expressed as a linear combination of the radially ingoing wave \( u_{in} (r) \) and the radially outgoing wave \( u_{out} (r) \), which are conjugate to each other, i.e., \([34]\)

\[
u_l (r) = A_l \left[ (-1)^{l+1} u_{in} (r) + S_l (k) u_{out} (r) \right]. \tag{C.25}
\]

From Eq. (C.19), we have

\[
\begin{align*}
  u_{in} (r) &= \exp \left( -ikr + i\frac{\xi}{k} \sqrt{r} \right) (2ikr)^{-i\xi^2/(8k^3)} \sum_{n \geq 0} \frac{\epsilon_n}{(-2ikr)^{n/2}}, \\
  u_{out} (r) &= \exp \left( ikr - i\frac{\xi}{k} \sqrt{r} \right) (-2ikr)^{i\xi^2/(8k^3)} \sum_{n \geq 0} \frac{\epsilon_n^*}{(2ikr)^{n/2}}. \tag{C.26}
\end{align*}
\]

Then by Eq. (C.24), we obtain the scattering wave function,

\[
\begin{align*}
u_l (r) &= A_l \left[ (-1)^{l+1} \exp \left( -ikr + i\frac{\xi}{k} \sqrt{r} \right) (2ikr)^{-i\xi^2/(8k^3)} \sum_{n \geq 0} \frac{\epsilon_n}{(-2ikr)^{n/2}} \right. \\
&\quad + \left. \frac{K_2 \left( 4l + 2, -\frac{(1-i)\xi}{k^{3/2}}, -\frac{i\xi^2}{2k^3}, \frac{4\sqrt{2i\mu}}{\sqrt{k}} \right)}{K_2 \left( 4l + 2, -\frac{(1+i)\xi}{k^{3/2}}, \frac{i\xi^2}{2k^3}, \frac{4\sqrt{-2i\mu}}{\sqrt{k}} \right)} \exp \left( ikr - i\frac{\xi}{k} \sqrt{r} \right) (-2ikr)^{i\xi^2/(8k^3)} \sum_{n \geq 0} \frac{\epsilon_n^*}{(2ikr)^{n/2}} \right]. \tag{C.27}
\end{align*}
\]

Taking \( r \to \infty \), we have

\[
\begin{align*}
u_l (r) \sim A_l \left[ (-1)^{l+1} \exp \left( -ikr + i\frac{\xi}{k} \sqrt{r} \right) (2kr)^{-i\xi^2/(8k^3)} \\
&\quad + \frac{K_2 \left( 4l + 2, -\frac{(1-i)\xi}{k^{3/2}}, -\frac{i\xi^2}{2k^3}, \frac{4\sqrt{2i\mu}}{\sqrt{k}} \right)}{K_2 \left( 4l + 2, -\frac{(1+i)\xi}{k^{3/2}}, \frac{i\xi^2}{2k^3}, \frac{4\sqrt{2i\mu}}{\sqrt{k}} \right)} \exp \left( ikr - i\frac{\xi}{k} \sqrt{r} \right) (2kr)^{i\xi^2/(8k^3)} \right] \\
&= A_l e^{i\delta_l} \sin \left( kr - \frac{\xi}{k} \sqrt{r} + \frac{\xi^2}{8k^3} \ln 2kr + \delta_l - \frac{l\pi}{2} \right). \tag{C.28}
\end{align*}
\]
By Eqs. (C.23) and (C.24), we obtain the scattering phase shift
\[ \delta_l = - \arg K_2 \left( 4l + 2, \frac{(1 + i) \xi}{k^{3/2}}, \frac{i \xi^2}{2k^3}, \frac{4\sqrt{-2\mu}}{\sqrt{k}} \right). \]  
(C.29)

D The exact solution of \( U (r) = \xi r^2 + \mu r \)

In this appendix, we provide an exact solution of the eigenproblem of the potential
\[ U (r) = \xi r^2 + \mu r \]  
(D.1)
by solving the radial equation directly. This potential has only bound states.

The radial equation reads
\[ \frac{d^2 u_l (r)}{dr^2} + \left[ E - \frac{l(l+1)}{r^2} - \xi r^2 - \mu r \right] u_l (r) = 0. \]  
(D.2)
Using the variable substitution
\[ z = i^{l+1/4} \xi r \]  
(D.3)
and introducing \( f_l (z) \) by
\[ u_l (z) = A_l \exp \left( -\frac{z^2}{2} - \frac{\beta}{2} z \right) z^{l+1} f_l (z) \]  
(D.4)
with \( A_l \) a constant, we convert the radial equation (D.2) into an equation of \( f_l (z) \):
\[ z f_l'' (z) + \left( 2l + 1 - \frac{i \mu}{4^{3/4} \xi} \right) f_l' (z) + \left[ -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4^{3/2} \xi^{3/2}} - 2l - 3 \right] z - \frac{i \mu}{4^{3/4} \xi} (l+1) = 0. \]  
(D.5)
This is a Biconfluent Heun equation [8].

The choice of the boundary condition has been discussed in Ref. [10].

D.1 The regular solution

The regular solution is a solution satisfying the boundary condition at \( r = 0 \) [10]. The regular solution at \( r = 0 \) should satisfy the boundary condition \( \lim_{r \to 0} u_l (r) / r^{l+1} = 1 \). In this section, we provide the regular solution of Eq. (D.5).

The Biconfluent Heun equation (D.5) has two linearly independent solutions [8]
\[ y_l^{(1)} (z) = N \left( 2l + 1, \frac{i \mu}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\mu^2}{4^{3/2} \xi^{3/2}}, 0, z \right), \]  
(D.6)
\[ y_l^{(2)} (z) = c N \left( 2l + 1, \frac{i \mu}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4^{3/2} \xi^{3/2}}, 0, z \right) \ln z + \sum_{n \geq 0} d_n z^{n-2l-1}, \]  
(D.7)
where
\[ c = \frac{1}{2l+1} \left[ \frac{i \mu}{\xi^{3/4}} d_{2l} - d_{2l-1} \left( -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4^{3/2} \xi^{3/2}} + 1 - 2l \right) \right]. \]  
(D.8)
is a constant with the coefficient $d_\nu$ given by the following recurrence relation,

$$d_{n-1} = 0, \quad d_0 = 1,$$

$$(v + 2)(v + 1 - 2l) d_{v+2} - \frac{i\mu}{\xi^{3/4}} (v + 1 - l) d_{v+1} + \left( - \frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}} - 2v - 1 + 2l \right) d_v = 0$$  \hspace{1cm} (D.9)

and $N(\alpha, \beta, \gamma, \delta, z)$ is the biconfluent Heun function \cite{8–10}.

The biconfluent Heun function $N \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0, z \right)$ has an expansion at $z = 0$ \cite{8}:

$$N \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0, z \right) = \sum_{n \geq 0} \frac{A_n}{(2l + 2)_n} \frac{z^n}{n!},$$  \hspace{1cm} (D.10)

where the expansion coefficients is determined by the recurrence relation,

$$A_0 = 1, \quad A_1 = \frac{i\mu}{\xi^{3/4}} (l + 1),$$

$$A_{n+2} = \frac{i\mu}{\xi^{3/4}} (n + l + 2) A_{n+1} - (n + 1)(n + 2l + 2) \left( - \frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}} - 3 - 2l - 2n \right) A_n,$$  \hspace{1cm} (D.11)

and $(a)_n = \Gamma (a + n) / \Gamma (a)$ is Pochhammer’s symbol.

Only $y_l^{(1)} (z)$ satisfies the boundary condition for the regular solution at $r = 0$, so the radial eigenfunction reads

$$u_l (z) = A_l \exp \left( - \frac{z^2}{2} - \frac{\beta}{2} z \right) z^{l+1} y_l^{(1)} (z)$$

$$= A_l \exp \left( - \frac{z^2}{2} - \frac{\beta}{2} z \right) z^{l+1} N \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0, z \right).$$  \hspace{1cm} (D.12)

By Eq. (D.3), we obtain the regular solution,

$$u_l (r) = A_l \exp \left( \frac{\xi^{1/2}}{2} r^2 + \frac{\mu}{2\xi^{3/2}} r \right) r^{l+1} N \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0, i\xi^{1/4} r \right).$$  \hspace{1cm} (D.13)

### D.2 The irregular solution

The irregular solution is a solution satisfying the boundary condition at $r \to \infty$ \cite{10}.

The Biconfluent Heun equation (D.5) has two linearly independent irregular solutions \cite{8}:

$$B_l^+ \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0, z \right) = \exp \left( \frac{i\mu}{\xi^{3/4}} z + z^2 \right) B_l^+ \left( 2l + 1, \frac{\mu}{\xi^{3/4}} \frac{E}{\xi^{1/2}} + \frac{\mu^2}{4\xi^{3/2}}, 0, -iz \right)$$

$$= \exp \left( \frac{i\mu}{\xi^{3/4}} z + z^2 \right) (-iz)^l \frac{a_n}{(iz)^n} \sum_{n \geq 0} \frac{a_n}{(iz)^n},$$  \hspace{1cm} (D.14)
To construct the solution, we first express the regular solution (with the expansion coefficients given by the recurrence relation (D.15))

\[ H_l^+ \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0, z \right) = \exp \left( \frac{i\mu}{\xi^{3/4}} z + z^2 \right) H_l^+ \left( 2l + 1, \frac{\mu}{\xi^{1/2}}, \frac{E}{\xi^{1/2}} + \frac{\mu^2}{4\xi^{3/2}}, 0, -iz \right) \]

\[ = (-iz)^{-\frac{1}{4}} \left( \frac{E}{\xi^{1/2}} + \frac{\mu^2}{4\xi^{3/2}} + 2l + 3 \right) \sum_{n \geq 0} \frac{e_n}{(-iz)^n} \]  

with the expansion coefficients given by the recurrence relation

\[ a_0 = 1, \quad a_1 = \frac{\mu}{4\xi^{3/4}} \left( \frac{E}{\xi^{1/2}} + \frac{\mu^2}{4\xi^{3/2}} - 1 \right), \]

\[ 2(n + 2) a_{n+2} + \frac{\mu}{\xi^{3/4}} \left( \frac{3}{2} - \frac{E}{2\xi^{1/2}} - \frac{\mu^2}{8\xi^{3/2}} + n \right) a_{n+1} \]

\[ + \left[ \frac{1}{4} \left( \frac{E}{\xi^{1/2}} + \frac{\mu^2}{4\xi^{3/2}} \right)^2 - \frac{1}{4} (2l + 1)^2 + 1 - \frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}} + n \right] a_n = 0 \]  

(D.16)

and

\[ e_0 = 1, \quad e_1 = -\frac{\mu}{4\xi^{3/4}} (\gamma + 1), \]

\[ 2(n + 2) e_{n+2} + \frac{\mu}{\xi^{3/4}} \left( \frac{3}{2} + \frac{E}{2\xi^{1/2}} + \frac{\mu^2}{8\xi^{3/2}} + n \right) e_{n+1} \]

\[ - \left[ \frac{1}{4} \left( \frac{E}{\xi^{1/2}} + \frac{\mu^2}{4\xi^{3/2}} \right)^2 - \frac{1}{4} (2l + 1)^2 + 1 + \frac{E}{\xi^{1/2}} + \frac{\mu^2}{4\xi^{3/2}} + n \right] e_n = 0. \]  

(D.17)

### D.3 Eigenfunctions and eigenvalues

To construct the solution, we first express the regular solution (D.13) as a linear combination of the two irregular solutions (D.14) and (D.15).

The regular solution (D.13), with the relation [8, 10]

\[ N \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0, z \right) \]

\[ = K_1 \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0 \right) B_l^+ \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0, z \right) \]

\[ + K_2 \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0 \right) H_l^+ \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0, z \right) \]  

(D.18)

and the expansions (D.14) and (D.15), becomes

\[ u_l (r) = A_l K_1 \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0 \right) \exp \left( -\frac{\xi^{1/2}}{2} r^2 - \frac{\mu}{2\xi^{1/2}} r \right) \frac{E}{2^{1/2} \xi^{1/2}} + \frac{\mu^2}{8\xi^{3/2}} + \frac{1}{2} \sum_{n \geq 0} \frac{a_n}{\xi^{1/4} r^n} \]

\[ + A_l K_2 \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0 \right) \exp \left( \frac{\xi^{1/2}}{2} r^2 + \frac{\mu}{2\xi^{1/2}} r \right) \frac{E}{2^{1/2} \xi^{1/2}} - \frac{\mu^2}{8\xi^{3/2}} - \frac{1}{2} \sum_{n \geq 0} \frac{e_n}{\xi^{1/4} r^n}. \]  

(D.19)
where \( K_1 \left( 2l + 1, \frac{i\mu}{\xi^{1/4}}, -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0 \right) \) and \( K_2 \left( 2l + 1, \frac{i\mu}{\xi^{1/4}}, -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0 \right) \) are combination coefficients and \( z = i\xi^{1/4}r \).

The boundary condition of bound states, \( u (r) |_{r \to \infty} = 0 \), requires that the coefficient of the second term must vanish since this term diverges when \( r \to \infty \), i.e.,

\[
K_2 \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0 \right) = 0, \tag{D.20}
\]

where

\[
K_2 \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0 \right) =
\frac{\Gamma (2l + 2)}{\Gamma \left( l + \frac{1}{2} + \frac{E}{2\xi^{1/2}} + \frac{\mu^2}{8\xi^{3/2}} \right) \Gamma \left( l + \frac{3}{2} - \frac{E}{2\xi^{1/2}} - \frac{\mu^2}{8\xi^{3/2}} \right)} \times J_{l + \frac{1}{2} - \frac{E}{2\xi^{1/2}} - \frac{\mu^2}{8\xi^{3/2}}, \frac{i\mu}{2\xi^{3/4}} \left( -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}} - 2l - 1 \right) (2l + 1) + \frac{E}{2\xi^{1/2}} + \frac{\mu^2}{2\xi^{3/2}}, \frac{i\mu}{2\xi^{3/4}} \left( -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}} - 2l - 1 \right) \right) \tag{D.21}
\]

with

\[
J_{l + \frac{1}{2} - \frac{E}{2\xi^{1/2}} - \frac{\mu^2}{8\xi^{3/2}}, \frac{i\mu}{2\xi^{3/4}} \left( -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}} - 2l - 1 \right)} = \int_0^\infty \frac{e^{-x^2}}{x} \left( l + \frac{1}{2} - \frac{E}{2\xi^{1/2}} - \frac{\mu^2}{8\xi^{3/2}}, \frac{i\mu}{2\xi^{3/4}} \left( -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}} - 2l - 1 \right) \right) \tag{D.22}
\]

\[
	imes N \left( l + \frac{1}{2} - \frac{E}{2\xi^{1/2}} - \frac{\mu^2}{8\xi^{3/2}}, \frac{i\mu}{2\xi^{3/4}} \left( -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}} - 2l - 1 \right) \right) \tag{D.23}
\]

Eq. (D.20) is an implicit expression of the eigenvalue.

The eigenfunction, by Eqs. (D.19) and (D.20), reads

\[
u_l (r) = A_l K_1 \left( 2l + 1, \frac{i\mu}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}} - \frac{\mu^2}{4\xi^{3/2}}, 0 \right) \exp \left( -\frac{\xi^{1/2}}{2} r^2 - \frac{\mu}{2\xi^{1/2}} r \right) \sum_{n \geq 0} \frac{a_n}{(\xi^{1/4} r)^n}. \tag{D.24}
\]

**E  The exact solution of** \( U (r) = \xi r^2 + \frac{\mu}{r} + \kappa r \)

In this appendix, we provide an exact solution of the eigenproblem of the potential

\[
U (r) = \xi r^2 + \frac{\mu}{r} + \kappa r \tag{E.1}
\]

by solving the radial equation directly. This potential has only bound states.

The radial equation reads

\[
\frac{d^2}{dr^2} u_l (r) + \left[ E - \frac{l (l+1)}{r^2} - \xi r^2 - \frac{\mu}{r} - \kappa r \right] u_l (r) = 0. \tag{E.2}
\]
Using the variable substitution

\[ z = i\xi^{1/4}r \]  

(E.3)

and introducing \( f_l(z) \) by

\[ u_l(z) = A_l \exp \left( -\frac{z^2}{2} - \frac{\beta}{2}z \right) z^{l+1} f_l(z) \]  

(E.4)

with \( A_l \) a constant, we convert the radial equation (E.2) into an equation of \( f_l(z) \):

\[ zf_l''(z) + \left( 2l + 2 - \frac{ik}{\xi^{3/4}} z - 2z^2 \right) f_l'(z) + \left( -\frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}} - 2l - 3 \right) z + \frac{i\mu}{\xi^{1/4}} - (l + 1) \frac{ik}{\xi^{3/4}} \right) f_l(z) = 0. \]  

(E.5)

This is a Biconfluent Heun equation [8].

The choice of the boundary condition has been discussed in Ref. [10].

#### E.1 The regular solution

The regular solution is a solution satisfying the boundary condition at \( r = 0 \) [10]. The regular solution at \( r = 0 \) should satisfy the boundary condition \( \lim_{r \to 0} u_l(r) / r^{l+1} = 1 \). In this section, we provide the regular solution of Eq. (E.5).

The Biconfluent Heun equation (E.5) has two linearly independent solutions [8]

\[ y_l^{(1)}(z) = N \left( 2l + 1, \frac{ik}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}}, -\frac{\kappa^2}{4\xi^{3/2}}, \frac{-2\mu}{\xi^{1/4}}, z \right), \]  

(E.6)

\[ y_l^{(2)}(z) = cN \left( 2l + 1, \frac{ik}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}}, -\frac{\kappa^2}{4\xi^{3/2}}, \frac{-2\mu}{\xi^{1/4}}, z \right) \ln z + \sum_{n \geq 0} d_n z^{n-2l-1}, \]  

(E.7)

where

\[ c = \frac{1}{2l + 1} \left[ d_{2l} \left( \frac{i\mu}{\xi^{1/4}} + \frac{il\kappa}{\xi^{3/4}} \right) - d_{2l-1} \left( -\frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}} + 1 - 2l \right) \right] \]

is a constant with the coefficient \( d_v \) given by the following recurrence relation,

\[ d_{-1} = 0, \quad d_0 = 1, \]

\[ (v + 2)(v + 1 - 2l) d_{v+2} - \left( -\frac{i\mu}{\xi^{1/4}} + \frac{ik}{\xi^{3/4}} (v + 1 - l) \right) d_{v+1} + \left( -\frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}} - 2v + 1 - 2l \right) d_v = 0 \]  

(E.8)

and \( N(\alpha, \beta, \gamma, \delta, z) \) is the biconfluent Heun function [8–10].

The biconfluent Heun function \( N \left( 2l + 1, \frac{ik}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}}, -\frac{\kappa^2}{4\xi^{3/2}}, \frac{-2\mu}{\xi^{1/4}}, z \right) \) has an expansion at \( z = 0 \) [8]:

\[ N \left( 2l + 1, \frac{ik}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}}, -\frac{\kappa^2}{4\xi^{3/2}}, \frac{-2\mu}{\xi^{1/4}}, z \right) = \sum_{n \geq 0} \frac{A_n}{(2l + 2)_n} \frac{z^n}{n!}, \]  

(E.9)

where the expansion coefficients is determined by the recurrence relation,

\[ A_0 = 1, \quad A_1 = \frac{-i\mu}{\xi^{1/4}} + \frac{ik}{\xi^{3/4}} (l + 1), \]

\[ A_{n+2} = \left[ (n + l + 2) \frac{ik}{\xi^{3/4}} + \frac{-i\mu}{\xi^{1/4}} \right] A_{n+1} - (n + 1)(n + 2l + 2) \left( -\frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}} - 2l - 3 - 2n \right) A_n, \]  

(E.10)
and \((a)_n = \Gamma(a + n)/\Gamma(a)\) is Pochhammer’s symbol.

Only \(y_{l}^{(1)}(z)\) satisfies the boundary condition for the regular solution at \(r = 0\), so the radial eigenfunction reads

\[
    u_l(z) = A_l \exp\left(-\frac{z^2}{2} - \frac{\beta}{2}z\right) z^{l+1} y_{l}^{(1)}(z)
\]

\[
    = A_l \exp\left(-\frac{z^2}{2} - \frac{\beta}{2}z\right) z^{l+1} N \left(2l + 1, \frac{i\kappa}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}, \frac{-i2\mu}{\xi^{1/4}}, z\right).
\]  

(E.11)

By Eq. (E.3), we obtain the regular solution,

\[
    u_l(r) = A_l \exp\left(\frac{\xi^{1/2}}{2} r^2 + \frac{\kappa}{2\xi^{1/2}} r\right) r^{l+1} N \left(2l + 1, \frac{i\kappa}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}, \frac{-i2\mu}{\xi^{1/4}}, i\xi^{1/4} r\right).
\]  

(E.12)

### E.2 The irregular solution

The irregular solution is a solution satisfying the boundary condition for the regular solution at \(r \to \infty\) [10].

The Biconfluent Heun equation (E.5) has two linearly independent irregular solutions [8]:

\[
    B_i^+(2l + 1, \frac{i\kappa}{\xi^{3/4}} - \frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}, \frac{-i2\mu}{\xi^{1/4}}, z)
\]

\[
    = \exp\left(\frac{i\kappa}{\xi^{3/4}} z + z^2\right) B_i^+(2l + 1, \frac{\kappa}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} + \frac{\kappa^2}{4\xi^{3/2}}, \frac{2\mu}{\xi^{1/4}}, -iz)
\]

\[
    = \exp\left(\frac{i\kappa}{\xi^{3/4}} z + z^2\right) (-iz)^{1/2} \left(e^{\frac{z}{2\xi^{1/2}} + \frac{\kappa^2}{4\xi^{3/2}} z} - 2iz\right) \sum_{n \geq 0} \frac{a_n}{(-iz)^n}.
\]  

(E.13)

\[
    H_i^+(2l + 1, \frac{i\kappa}{\xi^{3/4}} - \frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}, \frac{-i2\mu}{\xi^{1/4}}, z)
\]

\[
    = \exp\left(\frac{i\kappa}{\xi^{3/4}} z + z^2\right) H_i^+(2l + 1, \frac{\kappa}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} + \frac{\kappa^2}{4\xi^{3/2}}, \frac{2\mu}{\xi^{1/4}}, -iz)
\]

\[
    = (-iz)^{-1/2} \left(e^{\frac{z}{2\xi^{1/2}} + \frac{\kappa^2}{4\xi^{3/2}} z} + 2iz\right) \sum_{n \geq 0} \frac{e_n}{(-iz)^n}.
\]  

(E.14)

with the expansion coefficients given by the recurrence relation

\[
    a_0 = 1, \quad a_1 = \frac{\mu}{2\xi^{1/4}} + \frac{\kappa}{4\xi^{3/4}} \left(\frac{E}{\xi^{1/2}} + \frac{\kappa^2}{4\xi^{3/2}} - 1\right),
\]

\[
    2(n + 2) a_{n+2} + \left[\frac{\kappa}{\xi^{3/4}} \left(\frac{3}{2} - \frac{E}{2\xi^{1/2}} - \frac{\kappa^2}{8\xi^{3/2}} + n\right) - \frac{\mu}{\xi^{1/4}}\right] a_{n+1}
\]

\[
    + \left[\left(\frac{E}{2\xi^{1/2}} + \frac{\kappa^2}{8\xi^{3/2}} - 1\right)^2 - \frac{(2l + 1)^2}{4}\right] n \left(n + 2 - \frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}\right) a_n = 0
\]

(E.15)
and

\[ e_0 = 1, \quad e_1 = -\frac{\mu}{2\xi^{1/4}} - \frac{\kappa}{4\xi^{3/4}} \left( \frac{E}{\xi^{1/2}} + \frac{\kappa^2}{4\xi^{3/2}} + 1 \right), \]

\[ 2(n+2)e_{n+2} + \left[ \frac{\kappa}{\xi^{3/4}} \left( \frac{3}{2} + \frac{E}{2\xi^{1/2}} + \frac{\kappa^2}{8\xi^{3/2}} + n \right) + \frac{\mu}{\xi^{1/4}} \right] e_{n+1} \]

\[ - \left( \frac{E}{2\xi^{1/2}} + \frac{\kappa^2}{8\xi^{3/2}} + 1 \right)^2 - \frac{(2l+1)^2}{4} + n \left( n + 2 + \frac{E}{\xi^{1/2}} + \frac{\kappa^2}{4\xi^{3/2}} \right) \right] e_n = 0. \quad (E.16) \]

### E.3 Eigenfunctions and eigenvalues

To construct the solution, we first express the regular solution (E.12) as a linear combination of the two irregular solutions (E.13) and (E.14).

The regular solution (E.12), with the relation [8, 10]

\[ N \left( 2l + 1, \frac{ik}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}, \frac{-i2\mu}{\xi^{1/4}}, z \right) \]

\[ = K_1 \left( 2l + 1, \frac{ik}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}, \frac{-i2\mu}{\xi^{1/4}}, \right) \]

\[ + K_2 \left( 2l + 1, \frac{ik}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}, \frac{-i2\mu}{\xi^{1/4}}, \right) \]

\[ (E.17) \]

and the expansions (E.13) and (E.14), becomes

\[ u_1 (r) = A_1 K_1 \left( 2l + 1, \frac{ik}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}, \frac{-i2\mu}{\xi^{1/4}}, \right) \]

\[ \times \exp \left( -\frac{\xi^{1/2}r^2}{2} - \frac{\kappa}{2\xi^{1/4}} r \right) r^{\mu/2} \left( \psi^{1/2} \right) \sum_{n \geq 0} \frac{a_n}{(\xi^{1/4}r)^n} \]

\[ + A_1 K_2 \left( 2l + 1, \frac{ik}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}, \frac{-i2\mu}{\xi^{1/4}}, \right) \]

\[ \times \exp \left( \frac{\xi^{1/2}r^2}{2} + \frac{\kappa}{2\xi^{1/2}} r \right) r^{\mu/2} \left( \psi^{1/2} \right) \sum_{n \geq 0} \frac{e_n}{(\xi^{1/4}r)^n}, \quad (E.18) \]

where \( K_1 \left( 2l + 1, \frac{ik}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}, \frac{-i2\mu}{\xi^{1/4}}, \right) \) and \( K_2 \left( 2l + 1, \frac{ik}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}, \frac{-i2\mu}{\xi^{1/4}}, \right) \) are combination coefficients and \( z = i\xi^{1/4}r \).

The boundary condition of bound states, \( u (r)|_{r \to \infty} \to 0 \), requires that the coefficient of the second term must vanish since this term diverges when \( r \to \infty \), i.e.,

\[ K_2 \left( 2l + 1, \frac{ik}{\xi^{3/4}}, \frac{E}{\xi^{1/2}} - \frac{\kappa^2}{4\xi^{3/2}}, \frac{-i2\mu}{\xi^{1/4}}, \right) = 0. \quad (E.19) \]

Eq. (E.19) is an implicit expression of the eigenvalue.
The eigenfunction, by Eqs. (E.18) and (E.19), reads

\[ u_l(r) = A_l K_1 \left( 2l + 1, \frac{i \kappa}{\xi^{3/4}}, -\frac{E}{\xi^{1/2}}, \frac{\kappa^2}{4 \xi^{3/2}}, \frac{-i 2 \mu}{\xi^{1/4}} \right) \exp \left( -\frac{\xi^{1/2} r^2}{2} - \frac{\kappa}{2 \xi^{1/2}} r^2 + \frac{\kappa^2}{8 \xi^{3/2}} \sum_{n \geq 0} \frac{a_n}{(\xi^{1/4} r)^n} \right). \]  

(E.20)

F The solution of the harmonic-oscillator potential \( U(r) = \xi r^2 \) in terms of the Heun biconfluent function

In this appendix, for consistency with the solutions of other potentials, we solve the harmonic-oscillator potential in terms of the Heun biconfluent function.

The radial equation of the harmonic-oscillator potential \( U(r) = \xi r^2 \),

\[ \frac{d^2}{dr^2} u_l(r) + \left[ E - \frac{l(l+1)}{r^2} - \xi r^2 \right] u_l(r) = 0. \]  

(F.1)

Introducing \( f_l(z) \) by

\[ u_l(z) = A_l e^{-\frac{z^2}{2}} z^{l+1} f_l(z) \]

(F.2)

with

\[ z = \xi^{1/4} r, \]

(F.3)

where \( A_l \) is a constant, we then arrive at an equation of \( f_l(z) \)

\[ f_l''(z) + \frac{-2z^2 + 2(l+1)}{z} f_l'(z) + \left( \frac{E}{\xi^{1/2}} - 2l - 3 \right) f_l(z) = 0. \]  

(F.4)

This is a Biconfluent Heun equation [8].

The choice of the boundary condition has been discussed in Ref. [10].

F.1 The regular solution

The regular solution is a solution satisfying the boundary condition at \( r = 0 \) [10]. The regular solution at \( r = 0 \) should satisfy the boundary condition \( \lim_{r \to 0} u_l(r)/r^{l+1} = 1 \) for both bound states and scattering states. In this section, we provide the regular solution of Eq. (F.4).

The Biconfluent Heun equation (F.4) has two linearly independent solutions [8]

\[ y_l^{(1)}(z) = N \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0, z \right), \]

(F.5)

\[ y_l^{(2)}(z) = \tilde{c} N \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0, z \right) \ln z + \sum_{n \geq 0} d_n z^n - 2l - 1, \]

(F.6)

where

\[ \tilde{c} = \frac{1}{2l + 1} \left\{ -d_{2l-1} \left( \frac{E}{\xi^{1/2}} + 1 - 2l \right) \right\}. \]  

(F.7)
is a constant with the coefficient \(d_\nu\) given by the following recurrence relation,
\[
\begin{align*}
    d_{-1} &= 0, \\
    d_{0} &= 1, \\
    (\nu + 2) (\nu + 1 - 2l) d_{\nu+2} + \left( \frac{E}{\xi^{1/2}} - 2(\nu + 1) + 2l + 1 \right) d_{\nu} &= 0.
\end{align*}
\]
(\(F.8\))

and \(N(\alpha, \beta, \gamma, \delta, z)\) is the biconfluent Heun function [8–10].

The biconfluent Heun function \(N \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0, z \right)\) has an expansion at \(z = 0\) [8]:
\[
N \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0, z \right) = \sum_{n \geq 0} \frac{A_n z^n}{(2l + 2)_n n!},
\]
(\(F.10\))

where the expansion coefficients is determined by the recurrence relation,
\[
\begin{align*}
    A_0 &= 1, \\
    A_1 &= 0, \\
    A_{n+2} &= - (n + 1) (n + 2l + 2) \left( \frac{E}{\xi^{1/2}} - 2l - 3 - 2n \right) A_n
\end{align*}
\]
(\(F.11\), \(F.12\))

and \((a)_n = \Gamma (a + n) / \Gamma (a)\) is Pochhammer’s symbol. In the case of harmonic-oscillator
potential, the recurrence relation among the adjacent three terms reduces to a recurrence
relation between the adjacent two terms.

Only \(y_1^{(1)} (z)\) satisfies the boundary condition for the regular solution at \(r = 0\), so the
radial eigenfunction reads
\[
\begin{align*}
    u_1 (z) &= A_1 e^{-z^2/2} z^{l+1} y_1^{(1)} (z) \\
    &= A_1 e^{-z^2/2} z^{l+1} N \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0, z \right).
\end{align*}
\]
(\(F.13\))

By Eq. (\(F.3\)), we obtain the regular solution,
\[
\begin{align*}
    u_1 (r) &= A_1 e^{-\sqrt{\xi} r^2/2} r^{l+1/4} \frac{1}{r} N \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0, \xi^{1/4} r \right).
\end{align*}
\]
(\(F.14\))

In the case of harmonic-oscillator potential, by the relation between the Heun function
and the hypergeometric function [8, 9],
\[
N (\alpha, 0, \gamma, 0, z) = {}_1 F_1 \left( \frac{1}{2} + \frac{\alpha}{4} - \frac{\gamma}{4}, 1 + \frac{\alpha}{2}, z^2 \right),
\]
(\(F.15\))

the regular solution (\(F.14\)) reduces to
\[
\begin{align*}
    u_1 (z) &= A_1 e^{-\sqrt{\xi} r^2/2} r^{l+1/4} \frac{1}{r} {}_1 F_1 \left( \frac{l}{2} + \frac{3}{4} - \frac{E}{4\sqrt{\xi}}, \frac{3}{2} + l, \sqrt{\xi} r^2 \right).
\end{align*}
\]
(\(F.16\))

F.2 The irregular solution

The irregular solution is a solution satisfying the boundary condition at \(r \to \infty\) [10]. The
boundary conditions for bound states and scattering states at \(r \to \infty\) are different.
The Biconfluent Heun equation (F.4) has two linearly independent irregular solutions [8]:

\[ B^+_l \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0, z \right) = z^{1/2} \left[ \frac{E}{\xi^{1/2}} - 2 - (2l + 1) \right] \sum_{n \geq 0} a_n \frac{z^n}{n!} \] (F.17)

\[ H^+_l \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0, z \right) = z^{-1/2} \left[ \frac{E}{\xi^{1/2}} + 2 + (2l + 1) \right] e^{-z} \sum_{n \geq 0} e_n \frac{z^n}{n!} \] (F.18)

with the expansion coefficients given by the recurrence relation

\[ a_0 = 1, \quad a_1 = 0, \]

\[ 2(n + 2) a_{n+2} + \left[ \frac{1}{4} \left( \frac{E^2}{\xi} - (2l + 1)^2 + 4 \right) - \frac{E}{\xi^{1/2}} + n \left( n + 2 - \frac{E}{\xi^{1/2}} \right) \right] a_n = 0, \] (F.19)

and

\[ e_0 = 1, \quad e_1 = 0, \]

\[ 2(n + 2) e_{n+2} - \left[ \frac{1}{4} \left( \frac{E^2}{\xi} - (2l + 1)^2 + 4 \right) + \frac{E}{\xi^{1/2}} + n \left( n + 2 + \frac{E}{\xi^{1/2}} \right) \right] e_n = 0, \] (F.20)

where the recurrence relation among the adjacent three terms reduces to a recurrence relation between the adjacent two terms.

F.3 Eigenfunctions and eigenvalues

To construct the solution, we first express the regular solution (F.14) as a linear combination of the two irregular solutions (F.17) and (F.18).

The regular solution (F.14) with the relation [8, 10]

\[ N \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0, z \right) = K_1 \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0 \right) B^+_l \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0, z \right) + K_2 \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0 \right) H^+_l \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0, z \right) \] (F.21)

and the expansions (F.17) and (F.18) become

\[ u_l (r) = A_l e^{-\frac{E}{2} r^2} \xi^{(l+1)/4} r^{l+1} N \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0, \xi^{1/4} r \right) \\
= A_l e^{-\frac{E}{2} r^2} K_1 \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0 \right) \xi^{1/4} r^{l+1/2} \sum_{n \geq 0} \frac{a_n}{(\xi^{1/4} r)^{n/2}} \\
+ A_l e^{\frac{E}{2} r^2} K_2 \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0 \right) \xi^{1/4} r^{l+1/2} \sum_{n \geq 0} \frac{e_n}{(\xi^{1/4} r)^{n/2}}, \] (F.22)

where \( K_1 \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0 \right) \) and \( K_2 \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0 \right) \) are combination coefficients and \( \xi^{1/4} r \).
Only the first term satisfies the boundary condition of bound states \( u(r)|_{r \to \infty} \to 0 \), so the coefficient of the second term must vanish, i.e.,

\[
K_2 \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0 \right) = 0.
\]

(F.23)

In the case of harmonic-oscillator potential, \( K_2 \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0 \right) \) reduces to[11]

\[
K_2 \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0 \right) = \frac{\Gamma \left( \frac{3}{4} + \frac{l}{2} \right)}{\Gamma \left( \frac{3}{4} + \frac{l}{2} - \frac{E}{4\xi^{1/2}} \right)}.
\]

(F.24)

The zeros of \( K_2 \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0 \right) \) correspond to the bound state of the Coulomb potential, so the eigenvalue is the singularities of \( \Gamma \left( \frac{3}{4} + \frac{l}{2} - \frac{E}{4\xi^{1/2}} \right) \). Thus

\[
\frac{3}{4} + \frac{l}{2} - \frac{E}{4\xi^{1/2}} = -n, \quad n = 0, 1, 2, \ldots,
\]

(F.25)

then

\[
E = 2 \sqrt{\xi} \left( 2n + l + \frac{3}{2} \right), \quad n = 0, 1, 2, \ldots
\]

(F.26)

The eigenfunction, by Eqs. (F.22) and (F.23), reads

\[
u_l(r) = A_l e^{-\frac{\sqrt{\xi}}{4} r^2} K_1 \left( 2l + 1, 0, \frac{E}{\xi^{1/2}}, 0 \right) \left( \xi^{1/4} r \right)^{E/2} \sum_{n \geq 0} \frac{a_n}{\left( \xi^{1/4} r \right)^{n/2}}.
\]

(F.27)

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