Abstract

Let \( d(x, y) \) denote the length of a shortest path between vertices \( x \) and \( y \) in a graph \( G \) with vertex set \( V \). For a positive integer \( k \), let \( d_k(x, y) = \min\{d(x, y), k + 1\} \) and \( R_k\{x, y\} = \{z \in V : d_k(x, z) \neq d_k(y, z)\} \). A set \( S \subseteq V \) is a distance-\( k \)-resolving set of \( G \) if \( S \cap R_k\{x, y\} \neq \emptyset \) for distinct \( x, y \in V \). In this paper, we study the maker-breaker distance-\( k \) resolving game (MB\( k \)RG) played on a graph \( G \) by two players, Maker and Breaker, who alternately select a vertex of \( G \) not yet chosen. Maker wins by selecting vertices which form a distance-\( k \) resolving set of \( G \), whereas Breaker wins by preventing Maker from winning. We denote by \( O_{R,k}(G) \) the outcome of MB\( k \)RG. Let \( \mathcal{M}, \mathcal{B} \) and \( \mathcal{N} \), respectively, denote the outcome for which Maker, Breaker, and the first player has a winning strategy in MB\( k \)RG. Given a graph \( G \), the parameter \( O_{R,k}(G) \) is a non-decreasing function of \( k \) with codomain \( \{-1 = \mathcal{B}, 0 = \mathcal{N}, 1 = \mathcal{M}\} \).

We exhibit pairs \( G \) and \( k \) such that the ordered pair \( (O_{R,k}(G), O_{R,k+1}(G)) \) realizes each member of the set \( \{(\mathcal{B}, \mathcal{N}), (\mathcal{B}, \mathcal{M}), (\mathcal{N}, \mathcal{M})\} \); we provide graphs \( G \) such that \( O_{R,1}(G) = \mathcal{B}, O_{R,2}(G) = \mathcal{N} \) and \( O_{R,k}(G) = \mathcal{M} \) for \( k \geq 3 \). Moreover, we obtain some general results on MB\( k \)RG and study the MB\( k \)RG played on some graph classes.

Keywords: distance-\( k \) metric, \( k \)-truncated metric, resolving set, \( k \)-truncated resolving set, distance-\( k \) resolving set, Maker-Breaker distance-\( k \) resolving game, Maker-Breaker resolving game

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1 Introduction

Games played on graphs have been studied extensively; examples of two-player games include cop and robber game [21], Hex board game [11], Maker-Breaker domination game [5], etc. Erdős and Selfridge [6] introduced the Maker-Breaker game played on an arbitrary hypergraph \( H = (V, E) \) by two players, Maker and Breaker, who alternately select a vertex from \( V \) not yet chosen in the course of the game. Maker wins the game if he can select all vertices of a hyperedge from \( E \), whereas Breaker wins if she is able to prevent Maker from doing so. For further reading on these games, see [2, 15]. The Maker-Breaker resolving game (MBRG) was introduced in [18], and its fractionalization was studied in [26]. As it turns out, on any graph \( G \) that is connected, MBRG fits as the terminal member of a natural family of metric resolving games numbering \( (\text{diam}(G) - 1) \) in strength, where \( \text{diam}(G) \) is the length of a longest path found in \( G \). In this paper, we introduce and study this family, and we call a general member of this family the Maker-Breaker distance-\( k \) resolving game (MB\( k \)RG). As will be made clear shortly, MB\( k \)RG equals MBRG when \( k = \text{diam}(G) - 1 \). But first, we need to set down some basic terminology and notations.
Let \( G \) be a finite, simple, undirected, and connected graph with vertex set \( V(G) \) and edge set \( E(G) \). Let \( k \) be a member of the set \( \mathbb{Z}^+ \) of positive integers. For \( x, y \in V(G) \), let \( d(x, y) \) denote the minimum number of edges in a path linking \( x \) and \( y \) in \( G \), and let \( d_k(x, y) = \min\{d(x, y), k + 1\} \). Thus, \( d_r(\cdot, \cdot) \) is the usual, shortest-path metric on \( G \), and we can call \( d_k(\cdot, \cdot) \) the distance-\( k \) or the \( k \)-truncated metric on \( G \). For distinct \( x, y \in V(G) \), let \( R(x, y) = \{ z \in V(G) : d(x, z) \neq d(y, z) \} \) and \( R_k(x, y) = \{ z \in V(G) : d_k(x, z) \neq d_k(y, z) \} \). A set \( S \subseteq V(G) \) is a resolving set of \( G \) if \( S \cap R(x, y) \neq \emptyset \) for distinct \( x, y \in V(G) \), and the metric dimension \( \dim(G) \) of \( G \) is the minimum cardinality over all resolving sets of \( G \). Similarly, a set \( S \subseteq V(G) \) is a distance-\( k \) resolving set (also called a \( k \)-truncated resolving set) of \( G \) if \( S \cap R_k(x, y) \neq \emptyset \) for distinct \( x, y \in V(G) \), and the distance-\( k \) metric dimension (also called the \( k \)-truncated metric dimension) \( \dim_k(G) \) of \( G \) is the minimum cardinality over all distance-\( k \) resolving sets of \( G \). For an ordered set \( S = \{u_1, u_2, \ldots, u_\alpha\} \subseteq V(G) \) and for a vertex \( v \in V(G) \), the metric code of \( v \) with respect to \( S \) is the \( \alpha \)-vector \( \text{code}_S(v) = (d(v, u_1), d(v, u_2), \ldots, d(v, u_\alpha)) \). We note that \( S \) is a resolving set of \( G \) if and only if the map \( \text{code}_S(\cdot) \) is injective on \( V(G) \). By replacing the metric \( d(\cdot, \cdot) \) by \( d_k(\cdot, \cdot) \) mutatis mutandis, the notion of a distance-\( k \) metric code map \( \text{code}_{S,k}(\cdot) \) on \( V(G) \) and a distance-\( k \) resolving set are analogously defined; note that, again, \( S \subseteq V(G) \) is a distance-\( k \) resolving set of \( G \) if and only if the map \( \text{code}_{S,k}(\cdot) \) is injective on \( V(G) \).

The concept of metric dimension was introduced in \([14, 23]\), and the concept of distance-\( k \) metric dimension was introduced in \([1, 7]\). For further study on distance-\( k \) metric dimension of graphs, see \([10]\), which is a result of merging \([13, 25]\) with some additional results, and \([25]\) is based on \([24]\). Some applications of metric dimension include robot navigation \([19]\), network discovery and verification \([3]\), chemistry \([20]\), and combinatorial optimization \([22]\). For applications of distance-\( k \) metric dimension see \([10, 17]\), where sensors/landmarks are placed at locations (vertices) forming a distance-\( k \) resolving set of a network, with the understanding that the cost of a sensor/landmark increases as its detection range increases. It is known that determining the metric dimension of a general graph is an NP-hard problem \([12, 19]\) and that determining the distance-\( k \) metric dimension of a general graph is an NP-hard problem \([8, 9]\).

Returning to the eponymous games of this paper, and following \([18]\), MBRG (MBRG, respectively) is played on a graph \( G \) by two players, Maker (also called Resolver) and Breaker (also called Spoiler), denoted by \( M^* \) and \( B^* \), respectively. \( M^* \) and \( B^* \) alternately select (without missing their turn) a vertex of \( G \) that was not yet chosen in the course of the game. \( M \)-game (\( B \)-game, respectively) denotes the game for which \( M^* \) (\( B^* \), respectively) plays first. \( M^* \) wins the MBRG (MBRG, respectively) if he is able to select vertices that form a resolving set (a distance-\( k \) resolving set, respectively) of \( G \) in the course of the game, and \( B^* \) wins MBRG (MBRG, respectively) if she stops Maker from winning. We denote by \( O_R(G) \) and \( O_{R,k}(G) \) the outcomes, respectively, of MBRG and MBRG played on a graph \( G \). Noting that there’s no advantage for the second player in the MBRG, it was observed in \([18]\) that there are three realizable outcomes, as follows: \( O_R(G) = M \) if \( M^* \) has a winning strategy whether he plays first or second in the MBRG, \( O_R(G) = B \) if \( B^* \) has a winning strategy whether she plays first or second in the MBRG, and \( O_R(G) = N \) if the first player has a winning strategy in the MBRG. Analogously, we assign to \( O_{R,k}(G) \) an element of \( \{M, B, N\} \) in accordance with each of the aforementioned outcomes.

The authors of \([18]\) studied the minimum number of moves needed for \( M^* \) (\( B^* \), respectively) to win the MBRG provided \( M^* \) (\( B^* \), respectively) has a winning strategy. In MBRG, let \( M_R(G) \) (\( M'_R(G) \), respectively) denote the minimum number of moves for \( M^* \) to win the \( M \)-game (\( B \)-game, respectively) provided he has a winning strategy with \( O_R(G) = M \), and let \( B_R(G) \) (\( B'_R(G) \), respectively) denote the minimum number of moves for \( B^* \) to win the \( M \)-game (\( B \)-game, respectively) provided she has a winning strategy with \( O_R(G) = B \). Suppose rivals \( X \) and \( Y \) compete to gain control over a network \( Z \), with \( X \) trying to install transmitters with limited range at certain nodes to form a distance-\( k \) resolving set of the network, while \( Y \) seeks to sabotage the effort by \( X \). In this scenario, time becomes a matter of natural concern. If \( O_{R,k}(G) = M \), then we denote by \( M_{R,k}(G) \) (\( M'_R(G) \), respectively) the minimum number of moves for \( M^* \) to win the \( M \)-game (\( B \)-game, respectively). If \( O_{R,k}(G) = B \), then we denote by \( B_{R,k}(G) \) (\( B'_R(G) \), respectively) the minimum number of moves for \( B^* \) to win the \( M \)-game (\( B \)-game, respectively). If \( O_{R,k}(G) = N \), then we denote by \( N_{R,k}(G) \) (\( N'_R(G) \), respectively) the minimum number of moves for the first player to win the \( M \)-game (\( B \)-game, respectively).
Now, we recall a bit more terminology and notations. The diameter, \( \text{diam}(G) \), of \( G \) is \( \max \{d(x, y) : x, y \in V(G)\} \). For \( v \in V(G) \), the open neighborhood of \( v \) is \( N(v) = \{u \in V(G) : uv \in E(G)\} \), and the degree of \( v \) is \( |N(v)| \); a leaf (or an end-vertex) is a vertex of degree one, and a major vertex is a vertex of degree at least three. Let \( P_n, C_n \) and \( K_n \) respectively denote the path, the cycle and the complete graph on \( n \) vertices. For positive integers \( s \) and \( t \), let \( K_s,t \) denote the complete bi-partite graph with two parts of sizes \( s \) and \( t \). For \( a \in \mathbb{Z}^+ \), let \( \{a\} \) denote the set \( \{1, 2, \ldots, a\} \).

This paper is organized as follows. In Section 2 we study the parameter \( O_{R,k}(G) \) as a function of \( k \). We exhibit pairs \( G \) and \( k \) such that the ordered pair \((O_{R,k-1}(G), O_{R,k}(G))\) realizes each member of the set \( \{(|B, N), (B, M), (N, M)|\} \), and we define an MB\&RG-outcome-transition number (a jump) as an integer \( k > 1 \) such that \( O_{R,k-1}(G) \neq O_{R,k}(G) \). In Section 3 we examine \( O_{R,k}(G) \) when \( G \) is the Petersen graph, a complete multipartite graph, a wheel graph, a cycle, and a tree with some restrictions.

## 2 The parameter \( O_{R,k}(G) \) as a function of \( k \)

To facilitate our discussion of \( O_{R,k}(G) \) as a function of \( k \), let us assign outcomes \( B, N \), and \( M \) the values of \(-1, 0 \), and \( 1 \), respectively, and hence we have \( B < N < M \). It is clear from the definitions that \( d_{k+1}(x, y) \geq d_k(x, y) \) and, for \( k \geq \text{diam}(G) - 1 \), \( d_k(x, y) = d(x, y) \). Thus, we can identify \( d(x, y) \) with \( d_k(x, y) \) for any \( k \geq \text{diam}(G) - 1 \); we can likewise identify \( O_R(G) \) with \( O_{R,k}(G) \) for any \( k \geq \text{diam}(G) - 1 \). The following observation is instrumental to studying MB\&RG as a function of \( k \).

**Observation 2.1.** On any graph, a distance-\( k \) resolving set is a distance-(\( k + 1 \)) resolving set.

The preceding observation easily leads to the following two observations, which are found in existing literature.

**Observation 2.2.** \( [1] \) Let \( G \) be a connected graph, and let \( k, k' \in \mathbb{Z}^+ \) with \( k < k' \). Then \( \text{dim}_k(G) \geq \text{dim}_{k'}(G) \geq \text{dim}(G) \).

**Observation 2.3.** \( [4] \) Let \( k \in \mathbb{Z}^+ \) and \( G \) be a connected graph.

(a) If \( \text{diam}(G) \in \{1, 2\} \), then \( \text{dim}_k(G) = \text{dim}(G) \) for all \( k \).

(b) More generally, \( \text{dim}_k(G) = \text{dim}(G) \) for all \( k \geq \text{diam}(G) - 1 \).

The following monotonicity result follows readily from the rules of MB\&RG and Observation 2.1.

**Proposition 2.4.** On any graph \( G \), the parameter \( O_{R,k}(G) \) is a non-decreasing function of \( k \), where the codomain is \( \{B = -1, N = 0, M = 1\} \).

It’s worth noting that there are exactly \( \binom{\text{diam}(G)+1}{2} \) monotone functions from \( \{\text{diam}(G) - 1\} \) to \( \{-1, 0, 1\} \). To see this, observe that such a function is but a vector consisting of a string of \(-1\)’s, followed by a string of \(0\)’s, and then followed by a string of \(1\)’s, satisfying \( a + b + c = \text{diam}(G) - 1 \) and \( a, b, c \in \{0\} \cup \mathbb{Z}^+ \), where \( a, b, \) and \( c \) respectively denote the number of \(-1\)’s, \(0\)’s, and \(1\)’s. The number of solutions to the diophantine equation clearly equals the number of monomials of total degree \( \text{diam}(G) - 1 \) in 3 symbols, and the latter is well known to be the combinatorial symbol asserted. Now, we spotlight a few key results among the bounty of consequences yielded by Proposition 2.4.

**Corollary 2.5.** Let \( k \in \mathbb{Z}^+ \) and \( G \) be a connected graph.

(a) If \( \text{diam}(G) \in \{1, 2\} \), then \( O_{R,k}(G) = O_R(G) \) for all \( k \).

(b) Given \( k < k' \), we have \((O_{R,k}(G), O_{R,k'}(G)) \in \{(B, B), (N, N), (M, M), (B, N), (B, M), (N, M)\}; note that \( O_{R,k_0}(G) \) is \( O_R(G) \) for \( k_0 \geq \text{diam}(G) - 1 \).
The monotonicity of $O_{R,k}(G)$ as a function of $k$ with codomain $\{B, N, M\}$ prompts natural questions. Given a graph $G$, what is the range of $O_{R,k}(G)$? Also, where does the function “jump value”? To this end, for $X, Y \in \{B, N, M\}$ with $X < Y$, define the MBkRG-outcome-transition number (the jump) of $G$ from $X$ to $Y$, denoted by $O^T_{X,Y}(G)$, to be the number $\alpha \in [\text{diam}(G) - 1] - \{1\}$ satisfying $O_{R,\alpha}(G) = X$ and $O_{R,\alpha+1}(G) = Y$, when such an $\alpha$ exists; put $O^T_{X,Y}(G) = \emptyset$ otherwise. Clearly, each graph $G$ has no more than two jumps and, if $O^T_{R,M}(G) \neq \emptyset$, then $G$ has only one jump.

Now, we work towards realization results in conjunction with Proposition 2.7 and Corollary 2.8. First, we recall some terminology. Two vertices $u$ and $v$ are called twins if $N(u) - \{v\} = N(v) - \{u\}$; notice that a vertex is its own twin. Hernando et al. [16] observed that the twin relation is an equivalence relation on $V(G)$ and, under it, each (twin) equivalence class induces either a clique or an independent set. The next few results involve twin equivalence classes.

Observation 2.6. Let $x$ and $y$ be distinct members of the same twin equivalence class of $G$.
(a) $[16]$ If $R$ is a resolving set of $G$, then $R \cap \{x, y\} \neq \emptyset$.
(b) $[10]$ If $R_k$ is a distance-$k$ resolving set of $G$, then $R_k \cap \{x, y\} \neq \emptyset$.

Proposition 2.7. Let $G$ be a connected graph of order at least 4.
(a) $[18]$ If $G$ has a twin equivalence class of cardinality at least 4, then $O_R(G) = B$.
(b) $[18]$ If $G$ has two distinct twin equivalence classes of cardinality at least 3, then $O_R(G) = B$.
(c) $[26]$ If $G$ has $k \geq 0$ twin equivalence classes of cardinality 2 and exactly one twin equivalence class of cardinality 3 with $\text{dim}(G) = k + 2$, then $O_R(G) = N$.

Corollary 2.8. Let $k \in \mathbb{Z}^+$, and let $G$ be a connected graph of order at least 4.
(a) If $G$ has a twin equivalence class of cardinality at least 4, then $O_{R,k}(G) = B$ for all $k$.
(b) If $G$ has two distinct twin equivalence classes of cardinality 3, then $O_{R,k}(G) = B$ for all $k$.

Here is a relation between $\text{dim}_{k}(G)$ and $O_{R,k}(G)$, which is analogous to the relation between $\text{dim}(G)$ and $O_R(G)$ obtained in [18].

Observation 2.9. Let $k \in \mathbb{Z}^+$, and let $G$ be a connected graph of order $n \geq 2$.
(a) If $O_{R,k}(G) = M$, then $\text{dim}_{k}(G) \leq \lceil \frac{n}{2} \rceil$.
(b) If $\text{dim}_{k}(G) \geq \lceil \frac{n}{2} \rceil + 1$, then $O_{R,k}(G) = B$.

Proof. Let $k \in \mathbb{Z}^+$, and let $G$ be a connected graph of order $n \geq 2$.
(a) Let $O_{R,k}(G) = M$. Assume, to the contrary, that $\text{dim}_{k}(G) > \lceil \frac{n}{2} \rceil$. In the $B$-game of the MB$kRG$, $M^*$ can occupy at most $\lfloor \frac{n}{2} \rfloor$ vertices, and hence $M^*$ fails to occupy a distance-$k$ resolving set of $G$; thus, $O_{R,k}(G) \neq M$, which contradicts the hypothesis.
(b) Let $\text{dim}_{k}(G) \geq \lceil \frac{n}{2} \rceil + 1$. In the $M$-game of the MB$kRG$, $M^*$ can occupy at most $\lfloor \frac{n}{2} \rfloor$ vertices of $G$; thus, $M^*$ fails to occupy vertices that form a distance-$k$ resolving set of $G$. Since $B^*$ has a winning strategy for the $M$-game in the MB$kRG$, $O_{R,k}(G) = B$. \qed

Analogous to the concept of a pairing dominating set (see [5]) and a pairing resolving set (see [18]), we define a pairing distance-$k$ resolving set and a quasi-pairing distance-$k$ resolving set of a graph.

Definition 2.10. Let $k, \alpha \in \mathbb{Z}^+$ and $G$ be a connected graph. Let $X = \bigcup_{i \in [\alpha]} \{u_i, v_i\}$, where $\bigcup X \subseteq V(G)$ and $|\bigcup X| = 2\alpha$. Let $Z \subseteq V(G)$ be such that $|Z| = \alpha$ and $Z \cap \{u_i, v_i\} \neq \emptyset$ for each $i \in [\alpha]$.
(a) Suppose each $Z$, as defined, is a distance-$k$ resolving set of $G$, then $X$ is called a *pairing distance-$k$ resolving set of $G*.$

(b) Suppose each $Z$, as defined, fails to be a distance-$k$ resolving set of $G$, and there exists a vertex $v \in V(G) - \bigcup X$ such that $Z \cup \{v\}$ is a distance-$k$ resolving set of $G$ for each $Z$, then $X$ is called a *quasi-pairing distance-$k$ resolving set of $G*.$

**Observation 2.11.** [13] If $G$ admits a pairing resolving set, then $O_R(G) = \mathcal{M}.$

**Observation 2.12.** Let $k \in \mathbb{Z}^+$, and let $G$ be a connected graph of order at least two.

(a) If $G$ admits a pairing distance-$k$ resolving set, then $O_{R,k}(G) = \mathcal{M}.$

(b) If $G$ admits a quasi-pairing distance-$k$ resolving set, then $O_{R,k}(G) \in \{\mathcal{M}, \mathcal{N}\}.$

Now, we present the main theorem of this section, which is a collection of realization results (to wit, examples) which shed much light on MBkRG, viewed as a family of $(\text{diam}(G) - 1)$ resolving games played out on a fixed graph $G$; in other words, viewing $O_{R,k}(G)$ as a function of $k.$

**Theorem 2.13.** Let $k \in \mathbb{Z}^+.$

(a) There exist graphs $G$ satisfying $O_{R,k}(G) = \mathcal{M}$ for $k \geq 1.$

(b) There exist graphs $G$ satisfying $O_{R,k}(G) = \mathcal{N}$ for $k \geq 1.$

(c) There exist graphs $G$ satisfying $O_{R,k}(G) = \mathcal{B}$ for $k \geq 1.$

(d) There exists a graph $G$ such that $O_{R,1}(G) = \mathcal{N}$ and $O_{R,k}(G) = \mathcal{M}$ for $k \geq 2.$

(e) There exist graphs $G$ such that $O_{R,1}(G) = \mathcal{B}$ and $O_{R,k}(G) = \mathcal{N}$ for $k \geq 2.$

(f) There exist graphs $G$ such that $O_{R,1}(G) = \mathcal{B}$ and $O_{R,k}(G) = \mathcal{M}$ for $k \geq 2.$

(g) There exist graphs $G$ such that $O_{R,1}(G) = \mathcal{B}, O_{R,2}(G) = \mathcal{N}$ and $O_{R,k}(G) = \mathcal{M}$ for $k \geq 3.$

**Proof.** (a) Let $G$ be a tree obtained from $K_{1, \alpha},$ where $\alpha \geq 3,$ by subdividing exactly $(\alpha - 2)$ edges once. Let $\ell_1, \ell_2, \ldots, \ell_\alpha$ be the leaves and $v$ be the major vertex of $G$ such that $d(v, \ell_{\alpha-1}) = d(v, \ell_\alpha) = 1$ and $d(v, \ell_i) = 2$ for each $i \in [\alpha - 2].$ By Observation 2.6(b), let $s_i$ be the degree-two vertex lying on the $v - \ell_i$ path for each $i \in [\alpha - 2].$ Since $\{\{\ell_{\alpha-1}, \ell_\alpha\}\} \cup \bigcup_{i=1}^{\alpha-2} \{\{s_i, \ell_i\}\}$ is a pairing distance-$k$ resolving set of $G$ for each $k \in \mathbb{Z}^+, O_{R,k}(G) = \mathcal{M}$ for all $k \geq 1$ by Observation 2.12(a).

(b) Let $G$ be a tree obtained from $K_{1, \alpha},$ where $\alpha \geq 4,$ by subdividing exactly $(\alpha - 3)$ edges once. Let $v$ be the major vertex of $G$ and let $\{\ell_1, \ell_2, \ldots, \ell_\alpha\}$ be the set of leaves of $G$ such that $d(v, \ell_{\alpha-2}) = d(v, \ell_{\alpha-1}) = d(v, \ell_\alpha) = 1$ and $d(v, \ell_i) = 2$ for each $i \in [\alpha - 3];$ further, for each $i \in [\alpha - 3],$ let $s_i$ be the degree-two vertex lying on the $v - \ell_i$ path in $G.$ We note that, for any distance-$k$ resolving set $R$ of $G$, $|R \cap \{\ell_{\alpha-2}, \ell_{\alpha-1}, \ell_\alpha\}| \geq 2$ by Observation 2.6(b). Let $X = \{\ell_\alpha, \ell_{\alpha-1}\} \cup \bigcup_{i=1}^{\alpha-3} \{\{s_i, \ell_i\}\}$ and let $Z \subseteq V(G)$ with $|Z| = \alpha - 2$ such that $Z \cap \{\ell_{\alpha-1}, \ell_\alpha\} \neq \emptyset$ and $Z \cap \{s_i, \ell_i\} \neq \emptyset$ for each $i \in [\alpha - 3].$ Since $Z \cup \{\ell_{\alpha-2}\}$ is a distance-$k$ resolving set of $G$ and $Z$ fails to form a distance-$k$ resolving set of $G,$ $X$ is a quasi-pairing distance-$k$ resolving set of $G.$ By Observation 2.12(b), $O_{R,k}(G) \in \{\mathcal{M}, \mathcal{N}\}.$ In the $B$-game, $B^*$ can select two vertices in $\{\ell_\alpha, \ell_{\alpha-1}, \ell_{\alpha-2}\}$ after her second move; thus, $B^*$ wins the $B$-game. Therefore, $O_{R,k}(G) = \mathcal{N}$ for all $k \geq 1.$

(c) Let $G = K_{1, \beta},$ where $\beta \geq 4,$ be the star on $(\beta + 1)$ vertices such that $L(G) = \bigcup_{i=1}^{\beta+1} \{\ell_i\}$ is the set of leaves of $G.$ Since $L(G)$ is a twin equivalence class of cardinality $\beta \geq 4,$ $O_{R,k}(G) = \mathcal{B}$ for all $k \geq 1$ by Corollary 2.8(a).

(d) Let $G$ be a tree obtained from a 3-path given by $v_1, v_2, v_3$ by joining exactly two leaves $\ell_i$ and $\ell'_i$ to each $v_i,$ where $i \in [3].$ First, we show that $O_{R,1}(G) = \mathcal{N}.$ Let $S$ be any distance-1 resolving set of $G;$ for each $i \in [3], S \cap \{\ell_i, \ell'_i\} \neq \emptyset$ by Observation 2.6(b) since $\ell_i$ and $\ell'_i$ are twins in $G.$ We may assume that $S_0 = \{\ell_1, \ell_2, \ell_3\} \subseteq S$ by relabeling the vertices of $G$ if necessary. We note that, for any distinct $i, j \in [3],$ $\text{code}_{S_0, 1}(\ell'_i) = \text{code}_{S_0, 1}(\ell'_j)$ and $R_1(\ell'_i, \ell'_j) = \{v_i, \ell'_i, v_j, \ell'_j\};$ thus, $S \cap \{v_i, \ell'_i, v_j, \ell'_j\} \neq \emptyset.$ So, $|S| \geq 5,$ and
thus $\dim_1(G) \geq 5$. Let $X = \{\{v_2, v_3\}\} \cup \bigcup_{i=1}^{3} \{\{i, i'\}\}$ and let $Z \subseteq V(G)$ with $|Z| = 4$ such that $Z \cap \{v_2, v_3\} \neq \emptyset$ and $Z \cap \{i, i'\} \neq \emptyset$ for each $i \in [3]$. Since $Z \cup \{v_1\}$ is a minimum distance-1 resolving set of $G$, $X$ is a quasi-pairing distance-1 resolving set of $G$. In the $B$-game of the MB1RG, $B^*$ wins since $M^*$ can occupy at most 4 vertices in the course of the game and $\dim_1(G) = 5$. Thus, $O_{R,1}(G) = N'$ by Observation 2.12(b).

Second, we show that $O_{R,k}(G) = M$ for all $k \geq 2$. Since $\bigcup_{i=1}^{3} \{\{i, i'\}\}$ is a pairing distance-$k$ resolving set of $G$ for all $k \geq 2$, $O_{R,k}(G) = M$ for all $k \geq 2$ by Observation 2.12(a).

(c) Let $G$ be a tree obtained from an $\alpha$-path $v_1, v_2, \ldots, v_{\alpha}$, where $\alpha \geq 3$, by attaching exactly three leaves $\ell_i, \ell_i', \ell_i''$ to $v_0$ and attaching exactly two leaves $\ell_i$ and $\ell_i'$ to each $v_i$, where $i \in [\alpha]$. 

First, we show that $O_{R,1}(G) = B$. Let $S$ be any distance-1 resolving set of $G$. By Observation 2.9(b), $|S \cap \{\ell_0, \ell_0', \ell_0''\}| \geq 2$ and $|S \cap \{\ell_i, \ell_i'\}| \geq 1$ for each $i \in [\alpha - 1]$. By relabeling the vertices of $G$ if necessary, we may assume that $S_0 = \bigcup_{i=1}^{3} \{\ell_i\} \cup \{\ell_i', \ell_i''\} \subseteq S$. Note that, for any distinct $i, j \in [\alpha]$, $\text{code}_{S_0}(\ell_i) = \text{code}_{S_0}(\ell_j)$ and $R_1(\ell_i, \ell_j) = \{v_i, \ell_i, v_j, \ell_j\}$; thus, $|S \cap \{v_i, \ell_i, v_j, \ell_j\}| \geq 1$. So, $|S| \geq 2\alpha$ and hence $\dim_1(G) \geq 2\alpha$. Since $\dim_1(G) \geq 2\alpha \geq \left\lceil \frac{3\alpha + 1}{2} \right\rceil + 1 = \left\lceil \frac{\nu(G)}{2} \right\rceil + 1$ for $\alpha \geq 3$, $O_{R,1}(G) = B$ by Observation 2.9(b).

Second, we show that $O_{R,k}(G) = N$ for all $k \geq 2$. Let $k \geq 2$. Let $X = \bigcup_{i=1}^{3} \{\{i, i'\}\}$ and let $Z \subseteq V(G)$ with $|Z| = \alpha$ such that $Z \cap \{i, i'\} \neq \emptyset$ for each $i \in [\alpha]$. Since $Z \cup \{\ell_0, \ell_0', \ell_0''\}$ forms a minimum distance-$k$ resolving set of $G$, $X$ is a quasi-pairing distance-$k$ resolving set of $G$. By Observation 2.12(b), $O_{R,k}(G) \in \{M, N\}$. Note that, in the $B$-game of the MB4RG, $B^*$ has a winning strategy since $B^*$ can occupy two vertices of $\{\ell_0, \ell_0', \ell_0''\}$ after her second move, and thus preventing $M^*$ from occupying vertices that form a distance-$k$ resolving set of $G$. So, $O_{R,k}(G) = N$ for all $k \geq 2$.

(f) Let $G$ be a tree obtained from an $\alpha$-path $v_1, v_2, \ldots, v_{\alpha}$, where $\alpha \geq 4$, by joining exactly two leaves $\ell_i$ and $\ell_i'$ to each $v_i$, where $i \in [\alpha]$.

First, we show that $O_{R,1}(G) = B$. For each $i \in [\alpha]$ and for any distance-1 resolving set $S$ of $G$, $S \cap \{\ell_i, \ell_i'\} \neq \emptyset$ by Observation 2.9(b). We may assume that $S^* = \bigcup_{i=1}^{3} \{\ell_i\} \subseteq S$ by relabeling the vertices of $G$ if necessary. We note that, for any distinct $i, j \in [\alpha]$, $\text{code}_{S^*}(\ell_i) = \text{code}_{S^*}(\ell_j)$ and $R_1(\ell_i, \ell_j) = \{v_i, \ell_i, v_j, \ell_j\}$; thus, $S \cap \{v_i, \ell_i, v_j, \ell_j\}$ and $R_1(\ell_i, \ell_j) = \{v_i, \ell_i, v_j, \ell_j\}$; thus, $S \cap \{v_i, \ell_i, v_j, \ell_j\} \neq \emptyset$. So, $|S| \geq 2\alpha - 1$, and thus $\dim_1(G) \geq 2\alpha - 1$. Since $\dim_1(G) \geq 2\alpha - 1 \geq \left\lceil \frac{3\alpha + 1}{2} \right\rceil + 1 = \left\lceil \frac{\nu(G)}{2} \right\rceil + 1$ for $\alpha \geq 4$, $O_{R,1}(G) = B$ by Observation 2.9(b).

Second, we show that $O_{R,k}(G) = M$ for all $k \geq 2$. Since $\bigcup_{i=1}^{3} \{\{i, i'\}\}$ is a pairing distance-2 resolving set of $G$, $O_{R,2}(G) = M$ by Observation 2.12(a). By Proposition 2.4, $O_{R,k}(G) = M$ for all $k \geq 2$.

(g) Let $G$ be the graph in Figure A where $\alpha \geq 2$. We note the following: (i) for any minimum distance-$k$ resolving set $S$ of $G$ and for each $i \in [\alpha]$, $S \cap \{\ell_i, \ell_i'\} \neq \emptyset$ and $S \cap \{s_i, s_i'\} \neq \emptyset$ by Observation 2.9(b); (ii) if $S_0 = \bigcup_{i=1}^{3} \{\ell_i, s_i\} \subseteq S$, then $\text{code}_{S_0}(\ell_i) = \text{code}_{S_0}(s_i)$ for each $k \geq 1$ and for each $j \in [\alpha]$; (iii) for each $i \in [\alpha]$, $R_1(\ell_i, s_i') = \{\ell_i', s_i', x_i\}$; (iv) for each $i \in [\alpha]$, $R_2(\ell_i', s_i') = \{\ell_i', s_i', x_i, y\}$; (v) for each $i \in [\alpha]$ and for each $k \geq 3$, $R_k(\ell_i', s_i') = \{\ell_i', s_i', x_i, y, z\}$. Let $W = \bigcup_{i=1}^{3} \{\{i, i'\}, \{s_i, s_i'\}\}$.

First, we show that $O_{R,1}(G) = B$. In the $M$-game, $B^*$ can choose exactly one vertex of each pair in $W$ and at least one vertex in $\bigcup_{i=1}^{3} \{x_i\}$. By relabeling the vertices of $G$ if necessary, we may assume that $B^*$ chose the vertices in $\{x_j\} \cup \bigcup_{i=1}^{3} \{\ell_i, s_i'\}$ after her $(2\alpha + 1)$st move. Since all vertices in $\{\ell_i', s_i', x_j\}$, for some $j \in [\alpha]$, are occupied by $B^*$, (iii) implies that $M^*$ fails to occupy vertices that form a distance-1 resolving set of $G$ in the $M$-game of the MB1RG. Thus, $O_{R,1}(G) = B$.

Second, we show that $O_{R,2}(G) = N$. Let $U \subseteq V(G)$ with $|U| = 2\alpha$ such that $U \cap \{\ell_i, \ell_i'\} \neq \emptyset$ and $U \cap \{s_i, s_i'\} \neq \emptyset$ for each $i \in [\alpha]$. Since $U \cup \{y\}$ is a minimum distance-2 resolving set of $G$, $W$ is a quasi-pairing distance-2 resolving set of $G$. By Observation 2.12(b), $O_{R,2}(G) \in \{M, N\}$. In the $B$-game, $B^*$ can select the vertex $y$ and exactly one vertex of each pair in $W$; we may assume that $B^*$ chose the vertices in $\{y\} \cup \bigcup_{i=1}^{3} \{\ell_i, s_i'\}$ after her $(2\alpha + 1)$st move. In order for $M^*$ to occupy vertices that form a distance-2 resolving set of $G$ in the $B$-game, (iv) implies that $M^*$ must select all vertices in $\bigcup_{i=1}^{3} \{x_i\}$ in addition to the vertices in $\bigcup_{i=1}^{3} \{\ell_i, s_i\}$, but this is impossible since $\alpha \geq 2$ and $B^*$ can select at least one vertex in $\bigcup_{i=1}^{3} \{x_i\}$ in her $(2\alpha + 2)$nd move. So, $B^*$ wins the $B$-game of the MB2RG. Thus, $O_{R,2}(G) = N$.

Third, we show that $O_{R,k}(G) = M$ for $k \geq 3$. Since $\bigcup_{i=1}^{3} \{y, z\}$ is a pairing distance-$k$ resolving set of $G$ for $k \geq 3$, $O_{R,k}(G) = M$ for all $k \geq 3$ by Observation 2.12(a).
Example 3.2. Since $3 = \frac{\alpha}{2}$, Observation 3.1 and Corollary 2.5, we make the following observation.

In this section, we consider the outcome of the MB\textsuperscript{k}RG as well as the minimum number of steps needed to reach the outcome of the MB\textsuperscript{k}RG on some graph classes. Taking into consideration of Proposition 2.3 and Corollary 2.5, we make the following observation.

**Observation 3.1.** Let $k \in \mathbb{Z}^+$ and $G$ be a connected graph of order $n \geq 2$.

(a) If $O_{R,k}(G) = \mathcal{M}$, then $\dim_k(G) \leq M_{R,k}(G) \leq M'_{R,k}(G) \leq \lceil \frac{2n}{k} \rceil$, $M_{R,k+1}(G) \leq M'_{R,k}(G)$.

(b) If $O_{R,k}(G) = \mathcal{B}$, then $B'_{R,k}(G) \leq B_{R,k}(G) \leq \lceil \frac{n}{2} \rceil$; moreover, for $k > 1$, $B_{R,k}(G) \geq B_{R,k-1}(G)$ and $B'_{R,k}(G) \geq B'_{R,k-1}(G)$.

(c) If $\text{diam}(G) \in \{1,2\}$ or $k \geq \text{diam}(G) - 1$, then $M_{R,k}(G) = M_{R}(G)$ and $M'_{R,k}(G) = M'_{R}(G)$ if $O_{R}(G) = \mathcal{M}$, and let $B_{R,k}(G) = B_{R}(G)$ and $B'_{R,k}(G) = B'_{R}(G)$ if $O_{R}(G) = \mathcal{B}$.

(d) If $X$ is a pairing distance-$k$ resolving set of $G$ with $|X| = 2 \dim_k(G)$, then $M_{R,k}(G) = M'_{R,k}(G) = \dim_k(G)$.

(e) If $X$ is a quasi-pairing distance-$k$ resolving set of $G$ with $|X| = 2(\dim_k(G) - 1)$ and $O_{R,k}(G) = \mathcal{N}$, then $N_{R,k}(G) = \dim_k(G)$.

Next, we examine the MB\textsuperscript{k}RG on some graph classes. We first consider the Petersen graph and complete multipartite graphs.

**Example 3.2.** Let $k \in \mathbb{Z}^+$.

(a) Let $\mathcal{P}$ denote the Petersen graph. It was shown in [15] that $O_{R}(\mathcal{P}) = \mathcal{M}$ and $M_{R}(\mathcal{P}) = 3 = M'_{R}(\mathcal{P})$. Since $\text{diam}(\mathcal{P}) = 2$, Corollary 2.5(a) and Observation 3.1(c) imply that $O_{R,k}(\mathcal{P}) = \mathcal{M}$ and $M_{R,k}(\mathcal{P}) = 3 = M'_{R,k}(\mathcal{P})$ for all $k$.

(b) For $m \geq 2$, let $G = K_{a_1,a_2,\ldots,a_m}$ be a complete multi-partite graph of order $\sum_{i=1}^{m} a_i$, and let $s$ be the number of partite sets of $G$ consisting of exactly one element. Since $\text{diam}(G) \leq 2$, Corollary 2.5(a) implies $O_{R,k}(G) = O_{R}(G)$ for all $k$, and $O_{R}(G)$ was determined in [15]. So,

$$O_{R,k}(G) = \begin{cases} \mathcal{B} & \text{if } s \geq 4, \text{ or } a_i \geq 4 \text{ for some } i \in [m], \\ & \text{or } s = a_i = 3 \text{ for some } i \in [m], \\ & \text{or } a_i = a_j = 3 \text{ for distinct } i, j \in [m], \\ \mathcal{N} & \text{if } s = 3 \text{ and } a_i \leq 2 \text{ for each } i \in [m], \\ & \text{or } s \leq 2, a_i \leq 3 \text{ for each } i \in [m] \text{ and } a_j = 3 \text{ for exactly one } j \in [m], \\ \mathcal{M} & \text{if } s \leq 2 \text{ and } a_i \leq 2 \text{ for each } i \in [m]. \end{cases}$$
Moreover, we have the following: (i) if $O_{R,k}(G) = M$, then $M_{R,k}(G) = M'_{R,k}(G) = \dim(G)$ by Theorem 4.12 of [18] and Observation (4.11(c)); (ii) if $O_{R,k}(G) = B$, then $B_{R,k}(G) = B'_{R,k}(G) = 2$ by Proposition 3.2 of [18] and Observation (3.7(c)); (iii) if $O_{R,k}(G) = \tilde{N}$, then $N_{R,k}(G) = \dim(G)$ and $N'_{R,k}(G) = 2$.

To see (iii), we note that if $O_{R,k}(G) = \tilde{N}$, then $G$ has exactly one twin equivalence class of cardinality 3, say $Q$, and $G$ admits a quasi-pairing distance-$k$ resolving set, say $X$, with $|X| = 2(\dim_k(G) - 1)$. In the $M$-game, $N_{R,k}(G) = \dim(G)$ by Observations (2.3(a) and (3.7(c)). In the $B$-game, $N'_{R,k}(G) = 2$ by Observation (2.3(b) and the fact that $B^*$ can occupy 2 vertices of $Q$ after her second move.

Next, we consider cycles. It was obtained in [18] that $O_R(C_n) = M$ for $n \geq 4$. We recall some terminology. Following [4], let $M$ be a set of at least two vertices of $C_n$, let $u_i$ and $u_j$ be distinct vertices of $M$, and let $P$ and $P'$ denote the two distinct $u_i - u_j$ paths determined by $C_n$. If either $P$ or $P'$, say $P$, contains only two vertices of $M$ (namely, $u_i$ and $u_j$), then we refer to $u_i$ and $u_j$ as neighboring vertices of $M$ and the set of vertices of $V(P) - \{u_i, u_j\}$ as the gap of $M$ (determined by $u_i$ and $u_j$). The two gaps of $M$ determined by a vertex of $M$ and its two neighboring vertices of $M$ are called neighboring gaps. Note that, $M$ has $r$ gaps if $|M| = r$, where some of the gaps may be empty. The following lemma and its proof are adapted from [4].

**Lemma 3.3.** Let $S \subseteq V(C_n)$, where $n \geq 5$. Suppose $S$ satisfies the following two conditions: (1) every gap of $S$ contains at most 3 vertices, and at most one gap of $S$ contains 3 vertices; (2) if a gap of $S$ contains at least 2 vertices, then its neighboring gaps contain at most 1 vertex. Then $S$ is a distance-1 resolving set of $C_n$.

**Proof.** Let $S \subseteq V(C_n)$ satisfy the conditions of the present lemma, where $n \geq 5$. Let $C_n$ be given by $u_0, u_1, \ldots, u_n, u_0$ and let $v \in V(C_n) - S$. Let $2_{[S]}$ denote the $|S|$-vector with 2 on each entry of $\text{code}_{S,1}(\cdot)$, and all subscripts in this proof are taken modulo $n$.

First, suppose $v = u_i$ belongs to a gap of size 1 of $S$. Then $W_0 = \{u_{i-1}, u_{i+1}\} \subseteq S$ and $\text{code}_{W_0,1}(u_i) = (1, 1) \neq \text{code}_{W_0,1}(u_j)$ for each $u_j \in V(C_n) - (S \cup \{u_i\})$ since $u_j$ cannot be adjacent to both $u_{i-1}$ and $u_{i+1}$ in $C_n$.

Second, suppose $v = u_i$ belongs to a gap of size 2 of $S$ such that $\{u_i, u_{i+1}\} \cap S = \emptyset$ or $\{u_{i-1}, u_i\} \cap S = \emptyset$, say the former. Let $W_1 = \{u_{i-1}, u_{i+2}\} \subseteq S$. If $n = 5$, then $\text{code}_{W_1,1}(u_j) : u_j \in V(C_5) - W_1 = \{(1, 1), (1, 2), (2, 1)\}$. So, suppose $n \geq 6$. By the condition (2) of the present lemma, $\{u_{i-2}, u_{i-3}\} \cap S \neq \emptyset$. If $u_{i-2} \in S$, then $\text{code}_{W_1,1}(u_j) = (1, 2) \neq \text{code}_{W_1,1}(u_j)$ for each $u_j \in V(C_n) - (S \cup \{u_i\})$ since $u_j$ cannot be adjacent to $u_{i-1}$ in $C_n$. If $u_{i-2} \notin S$ and $u_{i-3} \in S$, then we have the following: (i) $\text{code}_{W_1,1}(u_j) = \text{code}_{W_1,1}(u_{i-2}) = (1, 2) \neq \text{code}_{W_1,1}(u_j)$ for each $u_j \in V(C_n) - (S \cup \{u_i, u_{i-2}\})$ since $u_j$ is not adjacent to $u_{i-1}$ in $C_n$; (ii) $\text{code}_{S,1}(u_i) \neq \text{code}_{S,1}(u_{i-2})$ since $d_1(u_i, u_{i-3}) = 2 > 1 = d_1(u_{i-2}, u_{i-3})$.

Third, suppose $v = u_i$ belongs to a gap of size 3 of $S$. If $\{u_{i-1}, u_{i+1}\} \cap S = \emptyset$, then $\text{code}_{S,1}(u_i) = 2_{[S]} \neq \text{code}_{S,1}(u_{i-2})$ for each $u_j \in V(C_n) - (S \cup \{u_i\})$ since $u_j$ is adjacent to at least one vertex of $S$. Now, suppose $\{u_i, u_{i+1}, u_{i+2}\} \cap S = \emptyset$ or $\{u_{i-2}, u_{i-1}, u_i\} \cap S = \emptyset$, say the former; then $W_2 = \{u_{i-1}, u_{i+3}\} \subseteq S$. If $n = 5$, then $\{u_j \in V(C_5) - W_2 = \{(1, 2), (2, 1), (2, 2)\}$. If $n = 6$, then $\text{code}_{W_2,1}(u_j) : u_j \in V(C_6) - W_2 \neq \text{code}_{W_2,1}(u_j)$ for each $u_j \in V(C_n) - (S \cup \{u_i\})$ since $u_j$ is not adjacent to $u_{i-1}$ in $C_n$. If $u_{i-2} \notin S$ and $u_{i-3} \in S$, then $\text{code}_{W_1,1}(u_i) = \text{code}_{W_1,1}(u_{i-2}) = (1, 2) \neq \text{code}_{W_1,1}(u_j)$ for each $u_j \in V(C_n) - (S \cup \{u_i, u_{i-2}\})$ and $d_1(u_i, u_{i-3}) = 2 > 1 = d_1(u_{i-2}, u_{i-3})$; thus, $\text{code}_{S,1}(u_i) \neq \text{code}_{S,1}(u_{i-2})$ for each $u_j \in V(C_n) - (S \cup \{u_i\})$.

As a generalization of Lemma 3.3, we state the following result without providing a detailed proof, where its proof for $k = 1$ is given in Lemma 3.3. The proof for the converse of Lemma 3.4 is provided in [10].

**Lemma 3.4.** Let $k \in \mathbb{Z}^+$ and $S \subseteq V(C_n)$, where $n \geq 2k + 3$. Suppose $S$ satisfies the following two conditions: (1) every gap of $S$ contains at most $(2k + 1)$ vertices, and at most one gap of $S$ contains $(2k + 1)$ vertices; (2) if a gap of $S$ contains at least $(k + 1)$ vertices, then its neighboring gaps contain at most $k$ vertices. Then $S$ is a distance-$k$ resolving set of $C_n$. 

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Proposition 3.5. Let $k \in \mathbb{Z}^+$ and $n \geq 3$. Then

$$O_{R,k}(C_n) = \begin{cases} N & \text{if } n = 3 \text{ and } k \geq 1, \\ M & \text{if } n \geq 4 \text{ is even and } k \geq 1, \\ \mathcal{M} & \text{if } n \geq 5 \text{ is odd and } k \geq 2. \end{cases}$$

Proof. Let $k \in \mathbb{Z}^+$. Let $C_n$ be given by $u_1, u_2, \ldots, u_n, u_1$, where $n \geq 3$. Note that $O_{R,k}(C_3) = N$ by Example 3.2(b). If $n = 4$, then $\{u_1, u_3\}, \{u_2, u_4\}$ is a pairing distance-$k$ resolving set of $C_4$; thus, $O_{R,k}(C_4) = \mathcal{M}$ by Observation 2.12(a). So, suppose $n \geq 5$.

If $n = 2x$ ($x \geq 3$), let $X = \cup_{i=1}^x \{u_{2i-1}, u_{2i}\}$ and $S$ be the set of vertices that are selected by $M^*$ over the course of the MBIRG such that $M^*$ selects exactly one vertex in $\{u_{2i-1}, u_{2i}\}$ for each $i \in [x]$. Then we have the following: (i) every gap of $S$ contains at most 2 vertices; (ii) if a gap of $S$ contains 2 vertices, then its neighboring gaps contain at most 1 vertex. By Lemma 3.3, $S$ is a distance-1 resolving set of $C_n$. Since $X$ is a pairing distance-1 resolving set of $C_n$, $O_{R,1}(C_n) = \mathcal{M}$ by Observation 2.12(a). By Proposition 2.4, $O_{R,k}(C_n) = \mathcal{M}$ for even $n \geq 6$ and for $k \geq 1$.

If $n = 2x + 1$ ($x \geq 2$), let $Y = \cup_{i=1}^x \{u_{2i-1}, u_{2i}\}$. In the B-game of the MBIRG, suppose $B^*$ selects $u_{2x+1}$ (by relabeling the vertices of $C_n$ if necessary) after her first move and let $S'$ be the set of vertices that are selected by $M^*$ such that $M^*$ selects exactly one vertex in $\{u_{2i-1}, u_{2i}\}$ for each $i \in [x]$. Then we have the following: (i) at most one gap of $S'$ contains 3 vertices (when $B^*$ is able to select all vertices in $\{u_1, u_{n-1}, u_n\}$) over the course of the MBIRG and all other gaps of $S'$ contain at most 2 vertices; (ii) if a gap of $S'$ contains exactly one vertex, then its neighboring gaps contain at most 1 vertex. By Lemma 3.4, $S'$ is a distance-2 resolving set of $C_n$. Since $Y$ is a pairing distance-2 resolving set of $C_n$, $O_{R,2}(C_n) = \mathcal{M}$ by Observation 2.12(a). By Proposition 2.4, $O_{R,k}(C_n) = \mathcal{M}$ for odd $n \geq 5$ and for $k \geq 2$. $\square$

Remark 3.6. Let $C_n$ be given by $u_1, u_2, \ldots, u_n, u_1$. We consider the B-game of the MBIRG.

(1) We show that $O_{R,1}(C_5) = \mathcal{M}$. Without loss of generality, suppose $B^*$ selects $u_5$ on her first move. Then $M^*$ selects $u_4$ on his first move and exactly one vertex in $\{u_2, u_3\}$ on his second move. Since the vertices selected by $M^*$ form a distance-1 resolving set of $C_5$, $O_{R,1}(C_5) = \mathcal{M}$.

(2) We show that $O_{R,1}(C_7) = \mathcal{M}$. Let $S$ be the set of vertices that are selected by $M^*$ over the course of the game. Without loss of generality, suppose $B^*$ selects $u_7$ on her first move. Then $M^*$ selects $u_6$ on his first move. If $B^*$ selects a vertex in $\{u_1, u_2\}$ on her second move, then $M^*$ selects $u_3$ on his second move and exactly one vertex of $\{u_4, u_5\}$ on his third move. If $B^*$ selects $u_3$ on her second move, then $M^*$ selects $u_4$ on his second move and exactly one vertex of $\{u_1, u_2\}$ on his third move. If $B^*$ selects $u_4$ on her second move, then $M^*$ selects $u_3$ on his second move and exactly one vertex of $\{u_1, u_2\}$ on his third move. In each case, $S$ satisfies the conditions of Lemma 3.3; thus, $S$ is a distance-1 resolving set of $C_7$ and $O_{R,1}(C_7) = \mathcal{M}$.

Conjecture 3.7. In addition to $C_5$ and $C_7$, we find that $O_{R,1}(C_n) = \mathcal{M}$ through explicit computation. We further conjecture that $O_{R,1}(C_n) = \mathcal{M}$ for all odd $n \geq 5$, but an argument for the general (odd) $n$ eludes us.

Next, we consider wheel graphs. The join of two graphs $G$ and $H$, denoted by $G + H$, is the graph obtained from the disjoint union of $G$ and $H$ by adding additional edges between each vertex of $G$ and each vertex of $H$. Since $diam(G + H) \leq 2$, $O_{R,k}(G + H) = O_R(G + H)$, for all $k \in \mathbb{Z}^+$, by Corollary 2.6(a). Let $d_G(w_i, w_j)$ denote the distance between the vertices $w_i$ and $w_j$ in a graph $G$, and let $d_{G,k}(w_i, w_j)$ denote $d_k(w_i, w_j)$ in $G$.

For the wheel graph $C_n + K_1$, let $V(K_1) = \{v\}$ and let $C_n$ be given by $u_1, u_2, \ldots, u_n, u_1$ such that $v$ is adjacent to each $u_i$, where $i \in [n]$, in $C_n + K_1$. Let $k \in \mathbb{Z}^+$. If $n = 3$, then $O_{R,k}(C_3 + K_1) = O_{R,k}(K_1) = \mathcal{B}$ by Proposition 2.3(a). If $n = 4$, then $\{u_1, u_3\}, \{u_2, u_4\}$ is a pairing distance-$k$ resolving set of $C_4 + K_1$. If $n = 5$, then $\{u_1, u_2\}, \{u_3, u_4\}, \{u_5, v\}$ is a pairing distance-$k$ resolving set of $C_5 + K_1$. So, $O_{R,k}(C_4 + K_1) = O_{R,k}(C_5 + K_1) = \mathcal{M}$, for all $k \geq 1$, by Observation 2.12(a). For $n \geq 6$, let
Let $k \in \mathbb{Z}^+$ and $n \geq 3$. Then (i) $O_{R,k}(C_n + K_1) = O_R(C_n + K_1) = B$; (ii) if $n \in \{4, 5, 6, 7\}$ or $n \geq 8$ is even, then $O_{R,k}(C_n + K_1) = O_R(C_n + K_1) = M$; (iii) if $n \geq 9$ is odd, then $O_{R,k}(C_n + K_1) = O_R(C_n + K_1) \in \{M, N\}$ (because $M^*$ has a winning strategy in the $M$-game of the MBkRG).

Next, we consider some restrictions. We recall some terminology and notations. Fix a tree $T$. A vertex $\ell$ of degree one is called a terminal vertex of a major vertex $v$ if $d(\ell, v) < d(\ell, w)$ for every other major vertex $w$ in $T$. The terminal degree, $\text{ter}_T(v)$, of a major vertex $v$ is the number of terminal vertices of $v$ in $T$, and an exterior major vertex is a major vertex with positive terminal degree. Let $M(T)$ be the set of exterior major vertices of $T$. For each $i \in [3]$, let $M_i(T) = \{w \in M(T) : \text{ter}_T(w) = i\}$, and let $M_4(T) = \{w \in M(T) : \text{ter}_T(w) \geq 4\}$; note that $M(T) = \bigcup_{j=1}^{4} M_j(T)$.

Theorem 3.9. Let $k \in \mathbb{Z}^+$ and $T$ be a tree that is not a path. Further, suppose that $T$ contains neither a degree-two vertex nor a major vertex with terminal degree zero. Then

$$O_{R,k}(T) = \begin{cases} 
\mathcal{B} & \text{if } |M_4(T)| \geq 1 \text{ or } |M_3(T)| \geq 2 \text{ for all } k \geq 1, \\
\mathcal{N} & \text{if } M_4(T) = \emptyset, |M_3(T)| = 1, |M_2(T)| \geq 2 \text{ and } k = 1, \\
\mathcal{M} & \text{if } M_4(T) = \emptyset, |M_3(T)| = 1, |M_2(T)| \in \{0, 1\} \text{ and } k = 1, \\
\mathcal{M} & \text{if } M_4(T) = \emptyset \text{ and } |M_3(T)| = 1 \text{ for all } k \geq 2, \\
\mathcal{M} & \text{if } M_4(T) = \emptyset, |M_2(T)| = 3 \text{ and } k = 1, \\
\mathcal{M} & \text{if } M_4(T) = \emptyset, |M_2(T)| = 2 \text{ and } k = 1, \\
\mathcal{M} & \text{if } M_4(T) = \emptyset \text{ and } M_2(T) \neq \emptyset \text{ for all } k \geq 2.
\end{cases}$$

Proof. Let $k \in \mathbb{Z}^+$, and let $T$ be a tree as described in the statement of the present theorem. Then each vertex of $T$ is either a leaf or an exterior major vertex. Let $M_1(T) = \{u_1, u_2, \ldots, u_z\}$ and $M'(T) = \bigcup_{i=1}^{z} M_i(T) = \{v_1, v_2, \ldots, v_z\}$, where $x \geq 0$ and $z \geq 1$. If $z \geq 1$, then, for each $j \in [x]$, let $m_j$ be the terminal vertex of $u_j$; notice that $m_j u_j \in E(T)$. For each $i \in [z]$, let $\text{ter}_T(v_i) = \sigma_i \geq 2$ and let $\{\ell_{i_1}, \ldots, \ell_{i_{\sigma_i}}\}$ be the set of terminal vertices of $v_i$ in $T$.

First, suppose $|M_4(T)| \geq 1$; then Corollary 2.8(a) implies that $O_{R,k}(T) = \mathcal{B}$ for all $k \geq 1$. Second, suppose $|M_4(T)| \geq 2$; then Corollary 2.8(b) implies that $O_{R,k}(T) = \mathcal{B}$ for all $k \geq 1$.

Third, suppose that $M_4(T) = \emptyset$ and $|M_3(T)| = 1$; then $z = 1 + |M_2(T)|$. We may assume that $\text{ter}_T(v_1) = 3$ by relabeling the vertices of $T$ if necessary. If $z = 1$ (i.e., $M_2(T) = \emptyset$), then $T = K_{1,3}$ and $O_{R,k}(T) = \mathcal{N}$ for all $k \geq 1$ by Example 3.2(b).

Now, suppose that $z \geq 2$ (i.e., $M_2(T) \neq \emptyset$); then $\text{ter}_T(v_i) = 2$ for each $i \in [z - 1]$. For any minimum distance-$k$ resolving set $W$ of $T$, Observation 2.8(b) implies that $|W \cap \{\ell_{z,1}, \ell_{z,2}, \ell_{z,3}\}| \geq 2$ and $|W \cap \{\ell_{i_1,1}, \ell_{i_2,2}\}| \geq 1$ for each $i \in [z - 1]$. By relabeling the vertices of $T$ if necessary, let $W_0 = (\bigcup_{i=1}^{z-1} \{\ell_{i_1,1}\}) \cup \{\ell_{z,1}, \ell_{z,2}, \ell_{z,3}\} \subseteq W$. Then, for any distinct $i, j \in [z]$ and for any distinct $\alpha, \beta \in [x]$, we have the following: (i) code$^0_{\alpha,1}(\ell_{i,2}) = \text{code}_{\alpha,1}(\ell_{j,2}) = \text{code}_{\alpha,1}(m_\alpha) = \text{code}_{\alpha,1}(m_\beta)$; $R_1\{\ell_{i,1}, \ell_{j,2}\} = \{\ell_{i,2}, v_i, \ell_{j,2}, v_j\}$, $R_1\{m_\alpha, m_\beta\} = \{m_\alpha, m_\beta, u_\alpha, u_\beta\}$ if $x \geq 2$, and $R_1\{m_\alpha, m_\beta\} = \{\ell_{i,2}, v_i, m_\alpha, u_\alpha\}$ if $x \geq 1$; (ii) for $k \geq 2$, code$^0_{\alpha,1}(\ell_{i,2}) \neq \text{code}_{\alpha,1}(m_\beta)$; if $k = 1$ and $z = 2$, then the first player has a winning strategy: (i) in the $M$-game, $M^*$ can select $\ell_{2,3}$ and then exactly one vertex of each pair in $\{\{\ell_{1,1}, 1, \ell_{1,2}\}, \{\ell_{1,1}, 2, \ell_{2,2}\}, \{v_1, v_2\}\} \cup \{\{u_{\ell_{1,1},1}\}\}$ thereby, and thus occupying a distance-1 resolving set of $T$ in the course of the MB1RG; (ii) in the $B$-game, $B^*$ can occupy two vertices of $\{\ell_{2,1}, \ell_{2,2}, \ell_{2,3}\}$ after her second move, and thus preventing $M^*$ from occupying vertices that form any distance-1 resolving set of $T$ in the course of the MB1RG. If $k = 1$ and $z \geq 3$, then dim$_1(G) \geq 2z + x \geq 1 + \left\lceil \frac{x(x-1)}{2} \right\rceil + 1 = \left\lceil \frac{x(x-1)}{2} \right\rceil + 1$; thus, $O_{R_1}(T) = B$ by Observation 2.9(b). If $k \geq 2$, then $A = (\bigcup_{i=1}^{z} \{\ell_{i_1,1}, 1, \ell_{2,2}\}) \cup (\bigcup_{i=1}^{z} \{u_{j,1}, m_{j}\})$ is a quasi-pairing distance-$k$ resolving set of $T$, and the first player has a winning strategy in the MBkRG: (i) in the $M$-game, $M^*$ can occupy $\ell_{2,3}$ after his first move and exactly one vertex of each pair in $A$ thereafter; (ii) in the $B$-game, $B^*$ can occupy two vertices of $\{\ell_{z,1}, \ell_{z,2}, \ell_{z,3}\}$ after her second move.
Fourth, suppose that $M_4(T) = \emptyset = M_3(T)$. Since $T$ is not a path, $M_2(T) \neq \emptyset$; notice that $|M_2(T)| = z \geq 2$ in this case. Let $C = \langle \cup_{i=1}^z \{\ell_i, m_i\} \rangle \cup \langle \cup_{j=1}^z \{u_j, m_j\} \rangle$. If $k \geq 2$, then $C$ is a pairing distance-$k$ resolving set of $T$; thus, $O_{R,k}(T) = \mathcal{M}$ by Observation 2.12(a). So, suppose $k = 1$. If $z = 2$, then $C \cup \{v_1, v_2\}$ is a pairing distance-1 resolving set of $T$; thus, $O_{R,1}(T) = \mathcal{M}$ by Observation 2.12(a). If $z = 3$, then $O_{R,1}(T) = \mathcal{N}$: (i) in the $B$-game of the MB1RG, $B^*$ has a winning strategy since $\dim_1(T) \geq x + 5 = \lceil \frac{x+9}{2} \rceil = \lceil \frac{|V(T)|}{2} \rceil$ (via an argument similar to the third case of the present proof); (ii) in the $M$-game of the MB1RG, $M^*$ can select $v_1$ and then exactly one vertex of each pair in $C \cup \{v_2, v_3\}$ thereafter, and thus occupying vertices that form a distance-1 resolving set of $T$. If $z \geq 4$, then $\dim_1(T) \geq 2z + x - 1 \geq \lceil \frac{3z+4x}{2} \rceil + 1 = \lceil \frac{|V(T)|}{2} \rceil + 1$ using a similar argument shown in the third case of the present proof; thus, $O_{R,1}(T) = \mathcal{B}$ by Observation 2.9(b).

We conclude this section with a couple of questions.

**Question 3.10.** Among connected graphs $G$ of a fixed diameter $\lambda$, how many of the $(\lambda + 1)$-ternary-valued, monotone functions on $|\lambda - 1|$ are realized as $O_{R,k}(G)$? Which ones? (We speculate that answers to the questions are more accessible for certain classes of graphs such as trees.)

**Question 3.11.** For a connected graph $G$ and each $k \in [\text{diam}(G) - 1]$, is there an algorithm to determine the time $(M_{R,k}, M'_{R,k}, N_{R,k}, N'_{R,k}, B_{R,k}, B'_{R,k})$ for the Maker or the Breaker to win MB$k$RG?

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**References**

[1] A.F. Beardon and J.A. Rodríguez-Velázquez, On the $k$-metric dimension of metric spaces. *Ars Math. Contemp.* 16 (2019) 25-38.

[2] J. Beck, *Combinatorial Games. Tic-Tac-Toe Theory*. Cambridge University Press, Cambridge, 2008.

[3] Z. Beerliova, F. Eberhard, T. Erlebach, A.Hall, M. Hoffmann, M. Mihalák and L.S. Lam, Network discovery and verification. *IEEE J. Sel. Areas Commun.* 24 (2006) 2168-2181.

[4] P.S. Buczkowski, G. Chartrand, C. Poisson and P. Zhang, On $k$-dimensional graphs and their bases. *Period. Math. Hungar.* 46 (2003) 9-15.

[5] E. Duchêne, V. Gledel, A. Parreau and G. Renault, Maker-Breaker domination game. *Discrete Math.* 343(9) (2020) #111955.

[6] P. Erdös and J.L. Selfridge, On a combinatorial game. *J. Combin. Theory Ser. A* 14 (1973) 298-301.

[7] A. Estrada-Moreno, On the $(k, t)$-metric dimension of a graph. Ph.D. thesis, Universitat Rovira i Virgili, 2016.

[8] A. Estrada-Moreno, I.G. Yero and J.A. Rodríguez-Velázquez, On the $(k, t)$-metric dimension of graphs. *Comput. J.* 64 (2021) 707-720.

[9] H. Fernau and J.A. Rodríguez-Velázquez, On the (adjacency) metric dimension of corona and strong product graphs and their local variants: combinatorial and computational results. *Discrete Appl. Math.* 236 (2018) 183-202.

[10] R.M. Frongillo, J. Geneson, M.E. Lladser, R.C. Tillquist and E. Yi, Truncated metric dimension for finite graphs. *Discrete Appl. Math.* 320 (2022) 150-169.

[11] M. Gardner, *The Scientific American Book of Mathematical Puzzles and Diversions*. Simon & Schuster, New York, 1959, pp. 73-83.
[12] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Freeman, New York, 1979.

[13] J. Geneson and E. Yi, The distance-k dimension of graphs. arXiv:2106.08303v2 (2021) https://arxiv.org/abs/2106.08303

[14] F. Harary and R.A. Melter, On the metric dimension of a graph. *Ars Combin.* 2 (1976) 191-195.

[15] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, *Positional Games*. Birkhäuser/Springer, Basel, 2014.

[16] C. Hernando, M. Mora, I.M. Pelayo, C. Seara and D.R. Wood, Extremal graph theory for metric dimension and diameter. *Electron. J. Combin.* 17 (2010) #R30.

[17] M. Jannesari and B. Omoomi, The metric dimension of the lexicographic product of graphs. *Discrete Math.* 312 (2012) 3349-3356.

[18] C.X. Kang, S. Klavžar, I.G. Yero and E. Yi, Maker-Breaker resolving game. *Bull. Malays. Math. Sci. Soc.* 44 (2021) 2081-2099.

[19] S. Khuller, B. Raghavachari and A. Rosenfeld, Landmarks in graphs. *Discrete Appl. Math.* 70 (1996) 217-229.

[20] D.J. Klein and E. Yi, A comparison on metric dimension of graphs, line graphs, and line graphs of the subdivision graphs. *Eur. J. Pure Appl. Math.* 5(3) (2012) 302-316.

[21] R. Nowakowski and P. Winkler, Vertex-to-vertex pursuit in a graph. *Discret. Math.* 43 (1983) 235-239.

[22] A. Sebő and E. Tannier, On metric generators of graphs. *Math. Oper. Res.* 29 (2004) 383-393.

[23] P.J. Slater, Leaves of trees. *Congr. Numer.* 14 (1975) 549-559.

[24] R.C. Tillquist, Low-dimensional embeddings for symbolic data science. Ph.D. thesis, University of Colorado, Boulder, 2020.

[25] R.C. Tillquist, R.M. Frongillo and M.E. Lladser, Truncated metric dimension for finite graphs. arXiv:2106.14314v1 (2021) https://arxiv.org/abs/2106.14314

[26] E. Yi, Fractional Maker-Breaker resolving game. *Lecture Notes in Comput. Sci.* 12577 (2020) 577-593.