Homogeneous HKT and QKT manifolds

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Abstract

We present the construction of a large class of homogeneous KT, HKT and QKT manifolds, $G/K$, using an invariant metric on $G$ and the canonical connection. For this a decomposition of the Lie algebra of $G$ is employed, which is most easily described in terms of colourings of Dynkin diagrams of simple Lie algebras. KT structures on homogeneous spaces are associated with different colourings of Dynkin diagrams. The colourings which give rise to HKT structures are found using extended Dynkin diagrams. We also construct homogeneous QKT manifolds from homogeneous HKT manifolds and show that their twistor spaces admit a KT structure. Many examples of homogeneous KT, HKT and QKT spaces are given.

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1 Introduction

A hermitian manifold $M$ with complex structure $I$ and metric $B$ admits a Kähler structure with torsion (KT) provided that

$$\nabla I = 0 ,$$

where $\nabla$ is the connection

$$\nabla = \bar{\nabla} + H ,$$

$\nabla$ is the Levi-Civita connection of $B$ and $H$ is the torsion, which is a three-form on $M$. The condition (1) implies that the holonomy of the connection $\nabla$ is a subgroup of $U(d), \dim(M) = 2d$. A KT structure is a generalization of a Kähler structure (K) and reduces to the latter whenever the torsion $H$ vanishes.

A hyper-complex tri-hermitian manifold $M$ with complex structures $\{I_r; r = 1,2,3\}$ and metric $B$ admits a hyper-Kähler structure with torsion (HKT) provided that

$$\nabla I_r = 0 .$$

The holonomy of the connection $\nabla$ is a subgroup of $Sp(d), \dim(M) = 4d$.

In the same way an almost quaternionic tri-hermitian manifold $M$ with tangent bundle endomorphisms $\{J_r; r = 1,2,3\}$ and metric $B$ admits a quaternionic Kähler structure with torsion (QKT) provided that

$$\nabla J_r = -2\epsilon_r^{st} A_s J_t$$

and

$$N_D(J_r) = 0 ,$$

where $A$ is a $Sp(1)$-connection and $N_D(J_r)$ is the Nijenhuis type tensor with respect to the covariant derivative $D$ of $A$. The holonomy of the connection $\nabla$ is a subgroup of $Sp(d) \cdot Sp(1), \dim(M) = 4d$. We remark that the definitions of QKT and of quaternionic manifolds both utilize the existence of the endomorphisms $\{J_r; r = 1,2,3\}$. However, in the former case the relevant connection is metric with torsion whereas in the latter case the connection is torsion free. A consequence of the conditions (1), (3), (4)
and (5) is that the torsion $H$ is a (1,2)- and (2,1)-form with respect to the endomorphisms $I$, $I_r$ and $J_r$, respectively. If the torsion $H$ is closed, we say that the KT, HKT and QKT structures are strong, otherwise we say that they are weak [2]. In what follows, the KT, HKT and QKT structures which we shall consider are of the weak type except those which are on group manifolds.

The properties of KT, HKT and QKT structures closely resemble those of K, HK and QK ones, respectively. In particular, HKT [2] and QKT [1] geometries admit twistor constructions with twistor spaces which have similar properties to those of HK [3] and QK [4, 5] manifolds. Many examples of manifolds admitting KT and HKT structures are known. In particular, it was shown in [1] that all $2k$- and some $4k$-dimensional compact Lie groups admit strong KT and HKT structures. The simplest such manifold with an HKT structure is the Hopf surface $S^1 \times S^3$. In physics, KT, HKT and QKT manifolds arise as target spaces of two-dimensional supersymmetric sigma models with Wess-Zumino term [7, 8], for which the torsion is proportional to the Wess-Zumino term. Another application of these geometries is in the context of black-holes, where the moduli spaces of a class of black-hole supergravity solutions are HKT manifolds. Homogeneous K [9, 10, 11] and QK [12] manifolds have been investigated, and they have found many applications in physics in the context of sigma models and in supergravity theories [13].

The present paper is dedicated to the investigation of KT, HKT and QKT structures on homogeneous spaces $G/K$ which generalizes the construction of KT and HKT structures on group manifolds. Specifically, we show that the complex homogeneous spaces of [14] and hyper-complex homogeneous spaces of [15] admit KT and HKT structures, respectively. In addition, we find homogeneous QKT spaces associated to a class of HKT ones. In order to do this we use an invariant metric on $G$ and the canonical connection on $G/K$. Our method to construct homogeneous KT, HKT and QKT spaces is based in part on the colouring of Dynkin diagrams of simple Lie algebras. One of the advantages of this approach is that it gives a simple description of a decomposition of the Lie algebra of $G$ which is employed
in the definition of the homogeneous KT, HKT and QKT structures. It also allows us to compile lists of such spaces. This includes the example of $S^1 \times S^3$, which admits an HKT structure, as well as a QK structure.

This paper is organized as follows: In section two, we set up our notation. In section three, we show that all homogeneous complex manifolds admit KT structures and compile a list of such manifolds using Dynkin diagrams. In section four, we give the construction of homogeneous HKT manifolds. In section five, we illustrate this construction employingDynkin diagrams and present many examples of homogeneous HKT manifolds. In section six, we investigate homogeneous QKT manifolds. In section seven, we show that the twistor spaces of QKT manifolds admit a KT structure, and in section eight we give our conclusions.

2 Lie groups and homogeneous spaces

Let $G$ be a semi-simple, compact Lie group with compact Lie algebra $\mathfrak{g}$ and $\mathfrak{g}^c = \mathfrak{g} \otimes \mathbb{C}$. Let $\mathfrak{h}$ be a Cartan sub-algebra of $\mathfrak{g}$, $H$ be the associated maximal Abelian subgroup of $G$ and $\Delta$ be the root system of $\mathfrak{g}^c$ with respect to $\mathfrak{h}^c$. Then (see for example [16, p. 165])

$$\mathfrak{g}^c = \mathfrak{h}^c \oplus_{\alpha \in \Delta} \mathfrak{g}^c(\alpha),$$

where the root subspaces of $\mathfrak{g}^c$ are

$$\mathfrak{g}^c(\alpha) = \{ g_\alpha \in \mathfrak{g}^c : [h, g_\alpha] = \alpha(h) g_\alpha , \forall \ h \in \mathfrak{h}^c \}.$$ (7)

We denote by $\Delta^+$ ($\Delta^-$) the space of positive (negative) roots of $\mathfrak{g}^c$. The choice of $\Delta^+$ is in one-to-one correspondence with the choice of a Weyl chamber in $\mathfrak{h}^c$ [16, p. 458]. Any root of $\mathfrak{g}^c$ can be written as the sum of simple roots, $\Delta^s := \{ \alpha_i : i = 1, \ldots, l = \text{rank}(\mathfrak{g}) \}$, with positive integer coefficients [16, p. 177]. Furthermore, the highest root $\psi$ in $\Delta^+$ is unique for every simple Lie algebra.

For each linear function $\alpha$ on $\mathfrak{h}^c$ there exists a unique element $\bar{h}_\alpha$ of $\mathfrak{h}^c$ such that the Killing metric $B$ induces an inner product on the root space

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1In what follows, $l^c$ will denote the complexification, $l^c = l \otimes \mathbb{C}$, of the vector space $l$. 

Then the commutation relations of $g_c$ in the Chevalley basis are

$$
[h_\alpha, e_\beta] = 2 \frac{\alpha \cdot \beta}{\alpha \cdot \alpha} e_\beta ,
$$

$$
[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta} + \delta_{\alpha,-\beta} h_\alpha ,
$$

where \{$e_\alpha$\} are the step operators, \{$h_\alpha$\} are the generators of the Cartan sub-algebra and the structure constants $N_{\alpha,\beta}$ are integers. In particular,

$$
N_{\alpha,\beta} = \pm (p + 1) ,
$$

where $-p \leq n \leq q$ for the maximal $\alpha$-string, $\beta + n\alpha$, containing $\beta$. We remark that

$$
N_{\alpha,\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha} = -N_{\beta,\alpha} = -N_{-\alpha,-\beta} .
$$

The Killing metric in the Chevalley basis \{$e_\alpha, h_\alpha$\} is

$$
B = \begin{pmatrix}
\frac{2}{\alpha \cdot \alpha} \delta_{\alpha,-\beta} & 0 \\
0 & \frac{2}{\alpha_i \cdot \alpha_j} A_{ij}
\end{pmatrix} ,
$$

where $\alpha_i \in \Delta^s$ and

$$
A_{ij} = 2 \frac{\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_j}
$$

is the Cartan matrix of $g_c$. It is well known that the Cartan matrix can be represented pictorially by a Dynkin diagram. In the next sections we shall make extensive use of Dynkin diagrams in our construction of homogeneous KT, HKT and QKT manifolds.

A semi-simple complex Lie algebra $g_c$ admits a unique, up to an isomorphism, compact real form $g$ [16, p. 181 and 426]

$$
E_\alpha^+ = i(e_\alpha + e_{-\alpha}) , \ E_\alpha^- = (e_\alpha - e_{-\alpha}) , \ H_\alpha = -ih_\alpha
$$

for $\alpha \in \Delta^+$. Then the commutation relations are

$$
[H_\alpha, E_\beta^\pm] = \pm 2 \frac{\alpha \cdot \beta}{\alpha \cdot \alpha} E_\alpha^\pm ,
$$

$$
[E_\alpha^\pm, E_\beta^\mp] = \mp N_{\alpha,\beta} E_{\alpha+\beta} - N_{\alpha,-\beta} E_{\alpha-\beta} ,
$$

$$
[E_\alpha^\pm, E_\beta^\pm] = + N_{\alpha,\beta} E_{\alpha+\beta} \mp N_{\alpha,-\beta} E_{\alpha-\beta} \pm 2 \delta_{\alpha,\beta} H_\alpha ,
$$

for $\alpha \in \Delta^+$. Then the commutation relations are
where the structure constants $N_{\alpha,\beta}$ are given in (11). In the above expression of the commutators we have assumed that, if $\alpha - \beta \in \Delta$, then $\alpha - \beta \in \Delta^+$. But if $\alpha - \beta \in \Delta^-$, then $\beta - \alpha \in \Delta^+$ and the commutators can be easily re-expressed.

Let $g$ be a reductive Lie algebra, i.e. $g = \tilde{u} \oplus \tilde{h}_0$ is the direct sum of the semi-simple Lie algebra $\tilde{u}$ and the Abelian Lie algebra $\tilde{h}_0$ [19, p. 326]. Every compact Lie group has a reductive Lie algebra. An invariant metric on the compact Lie group $G$ restricted on the semi-simple sub-algebra $\tilde{u}$ is proportional to the Killing metric on $\tilde{u}$ and the Abelian Lie algebra $\tilde{h}_0$ is the orthogonal complement of $\tilde{u}$ in $g$.

There is a basis $\{E^+_\alpha, E^-_\alpha, H_{\alpha i}, U_a\}$ in $g$ such that any invariant metric $B$ on $g$ is

$$B = \begin{pmatrix}
\frac{4}{\alpha - \alpha'} \delta_{\alpha \beta} & 0 & 0 & 0 \\
0 & \frac{4}{\alpha - \alpha'} \delta_{\alpha \beta} & 0 & 0 \\
0 & 0 & 2 \frac{2}{\alpha_j - \alpha_j} A_{ij} & 0 \\
0 & 0 & 0 & c_a \delta_{ab}
\end{pmatrix}, \quad (19)
$$

where $\{c_a\}$ are positive real constants and $\{U_a\}$ is a basis in the ideal $h_0$.

Let $M = G/K$ be a homogeneous manifold with $G$ a compact Lie group and $K$ a closed and connected subgroup of $G$. A homogeneous space is reductive [19, p. 190] if there exists a subspace $m$ of $g$ such that

$$g = m + k, \quad [k, m] \subset m. \quad (20)$$

If $g$ is equipped with an invariant metric $B$, then we define

$$m = k^\perp \quad (21)$$

and the decomposition of $g$ is orthogonal, $g = m \oplus k$. The invariant metric on $g$ induces an $\text{ad}(G)$-invariant metric on $m$, the canonical normal metric [20] or normal homogeneous metric [21], which in turn induces an invariant metric on $G/K$, the standard homogeneous metric [21]. We remark, that a homogeneous space is symmetric [19, p. 226] if

$$[m, m] \subset k. \quad (22)$$
Let \( k \) be a sub-algebra of \( g \) associated with a reductive homogeneous space. The set of roots \( \Delta \) of \( g \) decomposes as \( \Delta = \Delta_m \cup \Delta_k \) with \( \Delta_m \cap \Delta_k = \emptyset \), where \( \Delta_k \) is the root space of \( k \) and \( \Delta_m \) is the complement of \( \Delta_k \) in \( \Delta \). A decomposition of \( \Delta_m \) in terms of positive and negative root subspaces is compatible with the decomposition of \( g = m \oplus k \) if

\[
\Delta_m^+ \cup \Delta_m^- = \Delta_m, \quad \Delta_m^+ \cap \Delta_m^- = \emptyset,
\]

\[
\alpha, \beta \in \Delta_m^+, \quad \alpha + \beta \in \Delta_m \quad \Rightarrow \quad \alpha + \beta \in \Delta_m^+,
\]

\[
\alpha \in \Delta_m^+, \quad \beta \in \Delta_k, \quad \alpha + \beta \in \Delta \quad \Rightarrow \quad \alpha + \beta \in \Delta_m^+.
\] (23) (24) (25)

The first two conditions above are straightforward to understand, whereas the last condition implies the \( \text{ad}(K) \)-invariance of \( \Delta_m^+ \). The choice of a positive root system in \( \Delta_m \) is in one-to-one correspondence with the choice of a Weyl chamber in \( h_m \). Let \( \{t_m; m = 1, 2, \ldots, \dim(M)\} \) and \( \{t_a; a = 1, 2, \ldots, \dim(K)\} \) be a basis in \( m \) and \( k \), respectively. We write

\[
g^{-1} dg = e^m t_m + \omega^a t_a,
\]

where \( \{e^m\} \) is the frame on \( G/K \), \( \{\omega^a\} \) is the canonical connection on \( G/K \), and \( g \in G \). Then \( [t_m, t_n] = f_{mn} t_l + f_{mn}^a t_a \) and similarly for the rest of the structure constants. Using the canonical normal metric on \( m \), we find that \( f_{mn} \) is totally anti-symmetric and hence

\[
H = -\frac{1}{3!} f_{lmn} e^l \wedge e^m \wedge e^n
\]

is a left-invariant three-form on \( G/K \). In addition,

\[
dH = -\text{tr}(F \wedge F) = -\frac{1}{4} f_{[lm} a_{n]a} e^l \wedge e^m \wedge e^n \wedge e^a.
\] (27) (28) (29) (30)

Our convention for p-forms is \( w_p := \frac{1}{p!} w_{[a_1, \ldots, a_p]} e^{a_1} \wedge \ldots \wedge e^{a_p} \).
The homogeneous tensors on \( G/K \) are determined by their values on \( T_{eK}(G/K) \) which we have identified with \( m \) \cite[p. 193 and 201]{ref}; \( e \) is the identity in \( G \).

In particular, an endomorphism \( I \) of \( m \) with the properties

\[
I^2(X) = -X, \\
[I(X), I(Y)]_m - [X, Y]_m - I([I(X), Y]_m) - I([X, I(Y)]_m) = 0, \\
I([Z, X]_m) = [Z, I(X)]_m
\]

induces a homogeneous complex structure on \( G/K \), where \( X, Y \in m, Z \in k \) and \([·, ·]_m\) is the restriction of the commutator on \( g \) to \( m \). We remark that homogeneous tensors on \( G/K \) are covariantly constant with respect to the canonical connection. These tensors are uniquely determined by their restriction in \( m \) and they correspond to \( \text{ad}(K) \)-invariant tensors on \( m \).

Every hermitian manifold \( M \) admits a connection of the form (2) with respect to which the complex structure \( I \) is covariantly constant \cite[24]{ref}. The torsion of this connection is

\[
H_{mn\sigma} = \frac{3}{2} I_m \tilde{m} I_n \tilde{n} \hat{I}_\sigma \partial_{[\tilde{m}\tilde{n}\tilde{\sigma}]}.
\]

In particular, every hermitian homogeneous space admits a homogeneous KT structure with respect to the canonical connection.

## 3 Homogeneous KT spaces

We shall show that all homogeneous, closed, simply connected complex spaces, i.e. C spaces \cite{ref}, admit a KT structure. Let \( K \) be a closed and connected subgroup of a compact semi-simple group \( G \) whose semi-simple part coincides with the semi-simple part of the centralizer of a toral subgroup \( H_1 \) of \( G \). The Lie algebra, \( k \), of \( K \) is reductive, i.e. \( k^c = u^c + h^c_0 \), where \( u^c \) and \( h^c_0 \) are semi-simple and Abelian ideals, respectively. We decompose the Cartan sub-algebra \( h^c \) of \( g^c \) as

\[
h^c = \left\{ h_1, \ldots, h_{a_1}, \overbrace{h_{a_1+1}, \ldots, h_{a_2}}^{k_m^c}, \overbrace{h_{a_2+1} \ldots, h_{a_3}}^{k^c}, \overbrace{h^c_0}^{k^c_0}, \overbrace{h^c}^{k^c_x} \right\}.
\]
where \( h_0^c \) is the Cartan sub-algebra of \( u^c \), \( 1 \leq a_1 \leq a_2 \leq a_3 = \text{rank}(g) \) and

\[
h_1^c := h_m^c + h_0^c
\]

is the Lie algebra of the toral subgroup \( H_1 \) of \( G \). It is clear that

\[
\Delta_k = \{ \alpha \in \Delta : \alpha(h) = 0 ; \forall h \in h_1^c \} .
\]

Relations similar to (35) and (36) hold also for the associated real Lie algebras and Lie groups.

An M space is a homogeneous space of the form \( G_s/K \), where \( G_s \) is a compact, simply connected simple Lie group and the subgroup \( K \) is the semi-simple part of the centraliser of a toral subgroup \( T \subset G_s \). Since any toral subgroup of \( G_s \) is contained in a maximal torus, M spaces correspond to decompositions of the form (35) for which \( G = G_s \) is simple and \( h_0^c = \{0\} \). The Lie algebra of the torus \( T \) is \( h_1^c \). All C spaces are fibre decomposition spaces of a product of M spaces with a torus as fibre. Thus we have the following classification of C spaces in terms of M spaces [14]:

**Theorem 1** Let \( G \) be a simply connected, semi-simple, compact group with a decomposition as in (35). There is a one-to-one correspondence between C spaces, \( G/K \), and even-dimensional spaces of the form

\[
\left( M^1 \times \ldots \times M^r \right) / H_0 ,
\]

where the \( (M^i) \)'s, \( 1 \leq i \leq r \), are M spaces and \( H_0 \) is a toral subgroup of \( G \) with Lie algebra \( h_0^c \).

So far we have only considered simply connected spaces. But a version of Theorem 1 also holds for spaces with finite fundamental group.

In what follows we shall consider even-dimensional homogeneous \( G/K \) spaces of the form

\[
\left( M^0 \times M^1 \times \ldots \times M^r \right) / H_0 ,
\]

where all the \( M^i \)'s for \( 1 \leq i \leq r \) are M spaces as above, \( M^0 \) is a toral group and \( H_0 \) is a toral subgroup of \( G \). Choosing an invariant metric on \( G \), we
can arrange for the decomposition $g = k \oplus m$ to be orthogonal. Using this, we shall show that all the above spaces are KT manifolds.

All M spaces can be explicitly constructed from the Dynkin diagrams of simple groups. There are four infinite series of Lie algebras, $A_r$, $B_r$, $C_r$ and $D_r$ for \( \{ r = 1, 2, \ldots \} \), and five exceptional Lie algebras, $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$. To identify the sub-algebra $k^c$ of $g^c$, one takes the Dynkin diagram of $g^c$ and colours a subset of its vertices. The, possibly disconnected, sub-diagram consisting of the coloured vertices and the connecting lines between them is itself a Dynkin diagram of $k^c$. Then $m^c$ is defined using (21) and (37). We remark, that the above embeddings of $k^c$ in $g^c$ are regular \[26\], i.e.

$$\Delta_k \subseteq \Delta, \quad h_k \subseteq h^c. \quad (40)$$

To illustrate the construction, we take $g = E_8$ and colour its Dynkin diagram as follows:

![Dynkin diagram of E8](image)

**Fig. 1:** A coloured Dynkin diagram of $E_8$

Then $k^c$ is $D_4 \oplus A_1$ and $h_1^c$ is spanned by

$$h_1 = 2h_{\alpha_1} + h_{\alpha_3} - h_{\alpha_5} - 2h_{\alpha_6}, \quad (41)$$

$$h_2 = -2h_{\alpha_1} + h_{\alpha_2} + 2h_{\alpha_4} + 3h_{\alpha_5} + 4h_{\alpha_6}, \quad (42)$$

$$h_3 = 2h_{\alpha_7} + h_{\alpha_8}, \quad (43)$$

where the ordering of the simple roots is shown in Fig. 1. It is easy to check that the semi-simple part of the centralizer of the generators $h_1$, $h_2$ and $h_3$ in $E_8$ is indeed the sub-algebra $D_4 \oplus A_1$. The M space associated to this decomposition is $E_8/(SO(8) \times SU(2))$, which is 217-dimensional. This is not a complex space, but

$$\frac{E_8 \times U(1)^b}{SO(8) \times SU(2) \times U(1)^a} \quad (44)$$
are complex spaces for $a = \{0, 1, 2, 3\}$ and $b = \{1, 0, 1, 0\}$. For the homogeneous spaces (14), the Lie algebra of $U(1)^a$ is spanned by a linear combination of the generators (41) - (43) and the generator of $U(1)^b$.

To show that all homogeneous, complex compact spaces admit KT structures, we introduce an endomorphism $I$ on $m^c$. This can be done in two steps. First we define the action of $I$ on the step operators $\{e_\alpha\}$ in $m^c$ as
\begin{equation}
I(e_\alpha) = i\epsilon_\alpha e_\alpha ,
\end{equation}
where $\epsilon_\alpha$ are real constants. The requirement that $I^2 = -1$ (31) and that $I$ in $m^c$ is induced by an endomorphism in $m$ constrains the constants $\epsilon_\alpha$ to be
\begin{equation}
\epsilon_\alpha \in \{1, -1\} , \quad \epsilon_{+\alpha} = -\epsilon_{-\alpha} .
\end{equation}
The invariance of $I$ implies that
\begin{equation}
\alpha \in \Delta_m , \quad \beta \in \Delta_k , \quad \alpha + \beta \in \Delta_m \Rightarrow \epsilon_\alpha = \epsilon_{\alpha + \beta} .
\end{equation}
Finally, the integrability condition (32) for $I$ leads to
\begin{equation}
\alpha, \beta \in \Delta_m , \quad \alpha + \beta \in \Delta_m , \quad \epsilon_\alpha = \epsilon_\beta \Rightarrow \epsilon_\alpha = \epsilon_\beta = \epsilon_{\alpha + \beta} .
\end{equation}
Comparing the relations (46) - (48) with the relations (23) - (25), we note that they have the same structure. Thus there is a one-to-one correspondence between the set of Weyl chambers of $h^c_m$ and the set of constants $\{\epsilon_\alpha\}$, which define $I$ on the set of step operators $\{e_\alpha\}$ in $m^c$. This correspondence is obvious if we assign to each $\epsilon_\alpha$ a value $\pm 1$, depending on whether the root $\alpha$ is positive or negative with respect to a chosen Weyl chamber. We note that the choice of a Weyl chamber for the definition of the positive roots is independent of the choice of a Weyl chamber for the definition of the endomorphism (15).

It remains to define the endomorphism $I$ in the Cartan sub-algebra $h^c_m$. The dimension of $h^c_m$ is even by assumption, i.e. $\dim(h^c_m) = 2\ell$. We begin by taking a basis in $h^c_m$ and mapping it to the $2\ell$ generators of $\mathbb{R}^{2\ell}$, say $\{h_i, h_{\ell+i} : 1 \leq i \leq \ell\}$. There is a $4\ell^2$ parameter freedom in doing this. Then we define $I$ on $h^c_m$ as
\begin{equation}
I(h_i) = h_{\ell+i} , \quad I(h_{\ell+i}) = -h_i .
\end{equation}
One can verify that the endomorphisms (45) and (49) induce homogeneous complex structures on $m$.

The conditions for the hermiticity of the canonical normal metric with respect to the complex structures (45) and (49) are

\begin{align*}
B(E_\alpha^+, E_\alpha^+) &= B(E^-_\alpha, E^-_\alpha), \quad \text{(50)} \\
B(H_i, H_j) &= B(H_{\ell+i}, H_{\ell+j}), \quad \text{(51)} \\
B(H_i, H_{\ell+j}) &= -B(H_j, H_{\ell+i}). \quad \text{(52)}
\end{align*}

Comparing (50) with (19) we see that the first condition is automatically satisfied, whereas the conditions (51) and (52) place constraints on the metric restricted to the Cartan sub-algebra. There are at most $\ell(\ell + 1)$ constraints, which can be satisfied by tuning the same number of parameters in the definition of the complex structure (49). This leads to a family of complex structures which are compatible with the canonical normal metric and have at least $\ell(3\ell - 1)$ parameters:

**Theorem 2** All homogeneous spaces of the form $(M^0 \times M^1 \times \ldots \times M^r)/H_0$ (39) admit infinitely many KT structures.

We remark that in particular all compact, even-dimensional group spaces with a reductive Lie algebra are KT spaces.

In the construction of KT spaces employing Dynkin diagrams the question arises whether differently coloured Dynkin diagrams of a simple algebra $g$ lead to equivalent M spaces. It turns out that Dynkin diagrams which are related by (i) outer automorphisms and (ii) special Weyl transformations of $G$, which preserve the notation of positivity in $m$, should be considered equivalent. This result is similar to that found in [11] for K spaces. A list of M spaces, which takes the equivalence of differently coloured Dynkin diagrams into account, can be found in [11], Tables 3 and 4. They are really tables of simple homogeneous K spaces. But since these homogeneous K spaces and the homogeneous KT spaces have the same complex structures on the space of step operators, they are also lists of M spaces with a KT

\(^{\text{3}}\)The original table of M spaces by Wang [14] is not complete.
structure. Note that these K and KT spaces are equipped with different metrics. In the case of KT spaces the normal canonical metric $B$ is used, whereas in the case of K spaces the metric $\tilde{B}$ is constructed from the symplectic structure on the co-adjoint orbit, the Kirillov-Kostant-Souriau form, and the complex structure. Only in the case of KT spaces with $H \subset K$, where $H$ is a maximal torus of $G$, it is possible to define non-compact duals for compact KT spaces in the same way as for compact K spaces \[11\]. It is interesting to note that a homogeneous space $M$, which is Kähler with respect to the metric $\tilde{B}$, also admits a KT structure with respect to the metric $B$ and with $B = \tilde{B}$ if and only if $M$ is symmetric.

4 Homogeneous HKT spaces

Our main task in this section is to find the subclass of the homogeneous KT spaces which admit HKT structures. In order to do this we shall use a decomposition of the group spaces $G \times U(1)$, $m \in \mathbb{N}$, where $G$ is a semi-simple Lie group with Lie algebra $g$. Let us set $g^c_1 = g^c$ and $\Delta_1 = \Delta$. We choose a highest root $\psi_1$ in $\Delta_1$ and define the three-dimensional complex sub-algebra isomorphic to $\mathfrak{sl}(2)^c$ as $d^c_1 = \text{span}\{e_{\psi_1}, e_{-\psi_1}, h_{\psi_1}\}$. Next, we find the centralizer $b^c_1$ of $d^c_1$ in $g^c_1$ and define the compliment $f^c_1$ of $b^c_1 \oplus d^c_1$ in $g^c_1$. Then the first level of the decomposition of $g^c$ is

$$g^c_1 = b^c_1 \oplus d^c_1 \oplus f^c_1. \quad (53)$$

For the second level of the decomposition of $g^c$, we set $g^c_2 := b^c_1$ and decompose $g^c_2$ using the same procedure as for $g^c_1$ in the first level, and so on. This process of decomposing $g^c_k$’s is continued until a $b := b_n$ is found which is Abelian. This indicates that the decomposition of $g^c$ is completed. In the same way it is possible to decompose the associated compact, real Lie algebra $\mathfrak{g}$. In this case, the three-dimensional sub-algebras $d_k$ are isomorphic to $\mathfrak{su}(2)$. Thus a reductive Lie algebra $\mathfrak{g}$ can be decomposed as \[14\]

$$\mathfrak{g} = b \bigoplus_{k=1}^{n} d_k \bigoplus_{k=1}^{n} f_k, \quad (54)$$

where $b$ is Abelian, the $d_k$’s are highest root subspaces isomorphic to $\mathfrak{su}(2)$ and the $f_k$’s are (possibly empty) subspaces of $\mathfrak{g}$. We remark that the Cartan
sub-algebra of \( g \) is contained in \( b \oplus \bigoplus_{k=1}^n d_k \).

To define a hyper-complex structure on \( G \times U(1) \), we take an integer \( m \) such that the dimension of \( b \oplus m u(1) \) is equal to the maximal level \( n \) in the decomposition of \( g \). Then we choose a basis \( \{ U_k; k = 1, 2, \ldots, n \} \) in \( b \oplus u(1) \) and define an isomorphism \( \phi_k : \mathfrak{su}(2) \mapsto d_k \). The endomorphisms \( I_r \) on \( b \oplus \bigoplus_{k=1}^n (d_k \oplus u(1)_k) \) are

\[
I_r(U_k) = \phi_k(Y_r), \quad (55)
\]
\[
I_r(\phi_k(Y_s)) = -\delta_{rs} U_k + \epsilon_{rs}^t \phi_k(Y_t), \quad (56)
\]

where \( \{ Y_1, Y_2, Y_3 \} \) is a basis in \( \mathfrak{su}(2) \), and on \( f_k \) are

\[
I_r(E^\pm_{\beta}) = [E^\pm_{\beta}, \phi_k(Y_r)], \quad (57)
\]

where the \( E^\pm_{\beta} \)'s are step operators in \( f_k \). These endomorphisms preserve the levels of the decomposition \( g \) and depend on the choice of the basis in \( b \oplus u(1) \), i.e. there are \( n^2 \) inequivalent choices of \( I_r \) in \( g \oplus u(1) \). We remark that different choices of the isomorphisms \( \phi_k \) are related to different choices of the constants \( \epsilon_\alpha \) in the definition of the complex structure (46).

To investigate the endomorphisms \( I_r \) in more detail, we take without loss of generality \( (\phi_k(Y_1), \phi_k(Y_2), \phi_k(Y_3)) = (H_{\psi_k}, E^+_{\psi_k}, E^-_{\psi_k}) \). On each subspace \( d_k \oplus (d_k \oplus u(1)_k), 1 \leq k \leq n \), the endomorphism \( I_1 \) is defined as

\[
I_1(E^\pm_{\psi_k}) = \pm E^\mp_{\psi_k}, \quad I_1(U_k) = H_{\psi_k}, \quad I_1(H_{\psi_k}) = -U_k \quad (58)
\]

and on \( f_k \) as

\[
I_1(E^\pm_{\beta}) = \pm 2 \frac{\psi_k \cdot \beta}{\psi_k \cdot \psi_k} E^\mp_{\beta}. \quad (59)
\]

We remark that if we set \( h_k := iH_{\psi_k}, h_{\ell+k} := iU_k \) and \( n = \ell \), then \( I_1 \) is the same as \( I \) given in (12) and (13). However the other two endomorphisms, \( I_2 \) and \( I_3 \), are not of the same form because they interchange step operators with Cartan sub-algebra generators. On each subspace \( d_k \oplus u(1)_k, 1 \leq k \leq n \), the endomorphism \( I_2 \) is defined as

\[
I_2(E^-_{\psi_k}) = H_{\psi_k}, \quad I_2(H_{\psi_k}) = -E^-_{\psi_k}, \quad I_2(U_k) = E^+_{\psi_k}, \quad I_2(E^+_{\psi_k}) = -U_k \quad (60)
\]
and on $f_k$ as

$$I_2(E^\pm_\beta) = N_{\psi_k,-\beta}E^\pm_{\psi_k-\beta}, \quad I_2(E^\pm_{\psi_k-\beta}) = -N_{\psi_k,-\beta}E^\pm_{\beta}.$$  \hspace{1cm} (61)

Note that if $E^\pm_\beta$ is in $f_k$, then $E^\pm_{\psi_k-\beta}$ is also in $f_k$. We can assume without loss of generality that $N_{\psi_k,-\beta}$ is positive, since if it is negative we can exchange $\beta$ with $\psi_k - \beta$. The endomorphism $I_3$ can be expressed in a similar way.

The endomorphisms $I_r$ equip $g \oplus \mathfrak{u}(1)$ with a hyper-complex structure provided that

$$(N_{\psi_k,-\beta})^2 = 1, \quad 2 \frac{\psi_k \cdot \beta}{\psi_k \cdot \psi_k} = +1$$ \hspace{1cm} (62)

for $1 \leq k \leq n$, roots $\beta$ with $E^\pm_\beta \in f_k$ and highest roots $\psi_k$ in $\Delta_k$. The conditions (62) are satisfied in the basis (16) - (18). The first follows from the definition of $\psi_k$ as a highest root in $\Delta_k$ and relation (11), the second is derived using the Bianchi identity $[[e_{\psi_k}, e_{-\psi_k}], e_{-\beta}] + \text{cyclic} = 0$. The vanishing of the Nijenhuis tensors can be verified by direct computation, as in the last section, or by using an argument by Samelson [23] as generalized in [15]. The hyper-complex structures are homogeneous by construction.

The hyper-complex structures on group spaces $G \times U(1)$ induce hyper-complex structures on homogeneous spaces $(G/K) \times U(1)$ using the fact that the hyper-complex structures on $G \times U(1)$ are constructed for every level of the decomposition (53) separately. Thus we decompose $g$ as before but instead of continuing until we find a $b_n$ which is Abelian, we stop at some level $l$, $1 \leq l \leq n$. Let us denote the semi-simple part of $b_l$ as $\tilde{B}_l$. The Lie algebra $k$ comprises $\tilde{B}_l$ and possibly some of the Abelian ideal of $b_l$, i.e. $\tilde{B}_l \subseteq k \subseteq b_l$. The integer $m$ is fixed by the condition that the dimension of $(b_l/k) \oplus \mathfrak{u}(1)$ is equal to the level $l$ of the decomposition. This proves [15]:

**Theorem 3** Let $G$ be any closed, semi-simple, simply connected, compact Lie group, $G_l$ a subgroup of $G$ as defined above for some integer $l$, $1 \leq l \leq n$, $\tilde{B}_l$ the semi-simple part of $B_l$ and $K$ a subgroup of $G$ with $\tilde{B}_l \subseteq K \subseteq B_l$, then there exists an integer $m$ with $0 \leq m \leq l$, such that $(G/K) \times U(1)$ admits infinitely many hyper-complex structures.

Note that all group manifolds which admit hyper-complex structures [6], [15] are included in Theorem 3 for $K = \tilde{B}_l = \{1\}$. Note also that
every Abelian group of dimension $4d$ is naturally hyper-complex. To see this one maps the $4d$ $u(1)$-generators into $\mathbb{H}^d$, where $\mathbb{H}$ are the quaternions. There is a freedom in doing this of $4d$ parameters. Then the action of the hyper-complex structure on $\mathbb{H}^d$ is generated by the natural action of the quaternions. Thus if $(G/K) \times U(1)$ is hyper-complex, then $(G/K)^{m \times 4d} \times U(1)$ is hyper-complex as well.

A more general class of HKT spaces can be constructed which are of the form $(G^{m \times 4d} \times U(1))/K$. It is understood that taking the coset with $K$ mixes the $U(1)$ generators with the generators of $H$ that lie in $B_l$. It follows that the resulting HKT space may not be of the form of Theorem 3.

To show that these hyper-complex homogeneous spaces admit HKT structures, we consider the conditions under which $B$ is tri-hermitian. It turns out, that the conditions for $B$ to be tri-hermitian on the subspace $\bigoplus_{k=1}^{l} f_k$ are the same as those of (62). The conditions for $B$ to be tri-hermitian on the subspace $\bigoplus_{k=1}^{l} (d_k \oplus u_k)$ are

$$B(E^+_{\psi_j}, E^+_{\psi_k}) = B(E^-_{\psi_j}, E^-_{\psi_k}) = B(H_{\psi_j}, H_{\psi_k}) = B(U_j, U_k) = B(U_j, U_k)$$

for $1 \leq j, k \leq l$; all other components of $B$ must vanish. Using (19) we find

$$B(E^+_{\psi_j}, E^+_{\psi_k}) = B(E^-_{\psi_j}, E^-_{\psi_k}) = \frac{4}{\psi_j \cdot \psi_j} \delta_{j,k}.$$ (64)

In the next section it will become apparent, that the generators $H_{\psi_j}$ are mutually orthogonal by construction. The generators $E^{\pm}_{\psi_j}$ and $H_{\psi_j}$ are of equal length for each $j = 1, 2, \ldots, l$ because they span a standard basis of $\mathfrak{su}(2)$, for which this is true. Thus it is left to determine the Cartan sub-algebra generators $U_j$, such that

$$B(U_j, U_k) = \frac{4}{\psi_j \cdot \psi_j} \delta_{j,k}.$$ (65)

This imposes at most $\frac{l(l+1)}{2}$ constraints on the complex structures, which leave at least $\frac{l(l-1)}{2}$ inequivalent HKT structures, constructed from the $l^2$ hyper-complex structures:

**Theorem 4** The homogeneous manifolds $(G/K) \times U(1)$ of Theorem 3 with more than one level of decomposition (54) admit infinitely many inequivalent HKT structures, and finitely many otherwise.
5 Homogeneous HKT spaces and extended Dynkin diagrams

The decomposition (54) can be most easily described using extended Dynkin diagrams. This is so because the addition of the highest root to the extended Dynkin diagram introduces a natural colouring in the standard Dynkin diagram, which is associated with homogeneous HKT spaces. The use of Dynkin diagrams also enables us to compile lists of homogeneous HKT spaces.

The extended Dynkin diagram of a simple Lie algebra $g$ is found by adding one vertex to the Dynkin diagram of $g$ which represents the highest root. A list of the extended Dynkin diagrams of simple Lie algebras is given in Table 1 (see e.g. [27]). The simple roots $\{\alpha_i\}$ are marked by coloured and uncoloured vertices whereas the highest root $\psi$ is marked by a crossed vertex. In Table 1, we also include the symmetry groups $\bar{\Gamma}$ of the uncoloured extended Dynkin diagrams. Coloured Dynkin diagrams which are related by outer automorphisms of the uncoloured extended Dynkin diagram lead to equivalent HKT structures.
Table 1: The decomposition \([53]\) of the extended Dynkin diagrams of simple Lie groups and their symmetry group \(\hat{\Gamma}\).
The extended Dynkin diagram of $A_r$ has the property that the highest root is connected to two simple roots and its decomposition is different from that of the other simple Lie algebras. So it will be treated separately. The decomposition of the remaining simple Lie algebras involves the following steps:

- In the first level we set $g_1 := g$ and colour the extended Dynkin diagram as in Table 1. The highest root subspace $d_1$ is isomorphic to $su(2)$ spanned by $\{E_{\psi_1}^\pm, H_{\psi_1}\}$.
- Using (16) - (18), we find that the centralizer, $b_1$, of $d_1$ in $g_1$ is isomorphic to the sub-algebra associated with the coloured sub-diagram of the extended Dynkin diagram.
- The subspace $f_1$ consists of all step operators $E_{\alpha}^\pm$ of roots $\alpha$, other than $\psi_1$, which have a non-zero expansion coefficient associated with the simple root marked with an uncoloured vertex in the diagram.
- Next we set $g_2 := b_1$ and distinguish two cases, depending on whether $g_2$ is simple or not. In the former case, we use the extended Dynkin diagram of $g_2$ from Table 1 and repeat the decomposition as for $g_1$ above. If $g_2$ is not simple, it is always of the form $A_1 \oplus \tilde{g}_2$, where $\tilde{g}_2$ is simple. We can proceed by taking the highest root $\psi_2$ of $g_2$ either in $A_1$ or in $\tilde{g}_2$. In the former case we find $d_2 = A_1$, $f_2 = \emptyset$, and $b_2 = \tilde{g}_2$. In the latter case, we use the extended Dynkin diagram of $\tilde{g}_2$ from Table 1 and repeat the decomposition as for $g_1$ above.

This process is continued until either a $b_n$ is found which is Abelian or a $b_k$ is found which is equal to $A_r$ for $r > 1$. So it remains to describe the decomposition of the Lie algebra $A_r$ for $r > 1$ using Dynkin diagrams. For this we use the following steps:

- In the first level we set $g_1 := g$ and identify the highest root subspace $d_1$ as above. Then we colour the extended Dynkin diagram of $A_r$ as in Table 1.
- The centralizer, $b_1$, of $d_1$ in $g_1$ is equal to $A_{r-2} + u(1)$, where the $u(1)$-generator commutes with the highest root step operators $E_{\psi_1}^\pm$ and satisfies a certain orthogonality condition which we shall explain later. The coloured part of the Dynkin diagram is that of a $A_{r-2}$ sub-algebra.
• The subspace $f_1$ consists of all step operators $E^\pm_\alpha$ for roots $\alpha$, other than $\psi_1$, which have non-zero expansion coefficients associated with at least one of the two simple roots marked with an uncoloured vertex in the diagram.

• In the second level we set $g_2 = A_{r-2} + u(1)$ and repeat the decomposition as above.

We note that $b_2$ is equal to $A_{r-4} + u(1) + u(1)$, where the two $u(1)$-generators are linear independent and commute with $E^\pm_\psi_1$ and $E^\pm_\psi_2$. This process is continued until a $b := b_n$ is found which is Abelian. The $n = \lceil r/2 \rceil$ $u(1)$-generators in $b$ commute with all highest root subspace $d_k$ for $1 \leq k \leq n$.

Let us consider in detail how to choose the $u(1)$-generators in $b$ arising from the above decomposition of $A_r$ under the additional assumption of the tri-hermiticity of the metric $B$. As we have seen these are the additional conditions necessary to find HKT structures. The set of roots of $A_r$ can be taken as $\Delta = \{ \epsilon_p - \epsilon_q; 1 \leq p \neq q \leq r+1 \}$, the set of positive roots as $\Delta^+ = \{ \epsilon_p - \epsilon_q; 1 \leq p < q \leq r+1 \}$ and the set of simple roots as $\Delta^s = \{ \alpha_i = \epsilon_i - \epsilon_{i+1}; 1 \leq i \leq r \}$, where $\{ \epsilon_p; p = 1, \ldots, r+1 \}$ is an orthonormal basis in $\mathbb{R}^{r+1}$ (for details see [7, p. 860]). The metric $B$ is

$$B(E^+_\alpha, E^+_\alpha) = B(E^-_\alpha, E^-_\alpha) = B(H_\alpha, H_\alpha) = \frac{4}{\alpha \cdot \alpha} = 4(r + 1) \quad (66)$$

for all $\alpha \in \Delta^+$ and with all other components of $B$ vanishing, except $B(H_{\alpha_i}, H_{\alpha_{i+1}}) = -2(r + 1)$. For $r = 2n$ the number of levels in the decomposition (54) is $n$, and the $n$ highest roots $\psi_k$ are $\epsilon_k - \epsilon_{2n-k}$, $k = 1, 2, \ldots, n$.

The $n$ $u(1)$-generators $U_k$ are constrained by

(i) $\{ U_k, H_{\psi_k}; k = 1, 2, \ldots, n \}$ span $\mathfrak{h}$,

(ii) the $U_k$’s commute with all step operators associated with highest roots, i.e.

$$[U_k, E^\pm_\psi] = 0 \quad (67)$$

for all $k, l = 1, 2, \ldots, n$ and

(iii) $B$ is tri-hermitian on the Cartan sub-algebra with respect to the complex structures of the previous section, i.e.

$$B(U_k, U_l) = 4(2n + 1)\delta_{k,l}, \quad B(U_k, H_\psi) = 0 \quad (68)$$
for all \(k, l = 1, 2, \ldots, n\). We remark that the last condition implies that the \(u(1)\) generators are orthogonal to \(d_l\).

To find a solution to all these conditions we expand

\[
U_k = (c_k)^k H_{\alpha_k} + (c_k)^{k+1} H_{\alpha_k} + \cdots + (c_k)^{2n-k} H_{\alpha_{2n-k}} \tag{69}
\]

in terms of Cartan sub-algebra generators of simple roots which lie in the subspaces \(b_k\), where the coefficients are real numbers. Condition (67) implies

\[
(c_k)^k = -(c_k)^{2n-k}, (c_k)^{k+1} = -(c_k)^{2n-k-1}, \ldots, (c_k)^n + = -(c_k)^{n+1}, \tag{70}
\]

whereas condition (68) determines the \((c_j)^k\) to be

\[
(c_k)^l = [2(n-l)+1](c_k)^n, \quad (c_k)^n = \frac{1}{\sqrt{4(n-k+1)^2 - 1}} \tag{71}
\]

for \(1 \leq k \leq n\) and \(k \leq l \leq n - 1\). In the next section we shall argue, that the general solution of the \(U_k\)'s is generated from the special solution (71) applying an orthogonal transformation.

A similar analysis is possible for the case of \(A_r\) for \(r = 2n - 1\). The Cartan sub-algebra splits in \(n\) highest root generators and \(n-1\) generators \(\{U_{\tilde{k}}; \tilde{k} = 1, 2, \ldots, n - 1\}\). The expansion coefficients of the \(U_{\tilde{k}}\)'s are

\[
(c_{\tilde{k}})^{\tilde{k}} = -(c_{\tilde{k}})^{2n-\tilde{k}}, (c_{\tilde{k}})^{\tilde{k}+1} = -(c_{\tilde{k}})^{2n-\tilde{k}-1}, \ldots, (c_{\tilde{k}})^{n-1} = -(c_{\tilde{k}})^{n+1}, \tag{72}
\]

\[
(c_{\tilde{k}})^{n-1} = 0, \tag{73}
\]

\[
(c_{\tilde{k}})^l = [n-l](c_{\tilde{k}})^{n-1}, \quad (c_{\tilde{k}})^{n-1} = \frac{1}{\sqrt{(n-\tilde{k})(n-\tilde{k}+1)}} \tag{74}
\]

for \(1 \leq \tilde{k} \leq n - 1\) and \(\tilde{k} \leq l \leq n - 2\).

As an example, we take \(A_r = A_4\). Implementing the above decomposition, we find

\[
H_{\psi_1} = H_{\alpha_1} + H_{\alpha_2} + H_{\alpha_3} + H_{\alpha_4}, \tag{75}
\]

\[
H_{\psi_2} = H_{\alpha_2} + H_{\alpha_3}, \tag{76}
\]

\[
U_1 = \frac{1}{\sqrt{15}} (3H_{\alpha_1} + H_{\alpha_2} - H_{\alpha_3} - 3H_{\alpha_4}), \tag{77}
\]

\[
U_2 = \frac{1}{\sqrt{3}} (H_{\alpha_2} - H_{\alpha_3}). \tag{78}
\]

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We note, that all $u(1)$-generators in $b$ arise from the decomposition of a sub-algebra $b_k = A_r$ for some $r > 1$. Thus there are two types of reductive group spaces admitting an HKT structure, those that have a non-trivial Abelian part $b$ and those that have a trivial one. The former are associated with the algebras $A_r, D_{2r+1}$ and $E_6$ and are of the form $G^\oplus U(1)$ for $0 < m < \text{rank}(g)$. The later are associated with the algebras $B_r, C_r, D_{2r}, E_7, E_8, F_4$ and $G_2$ and are of the form $G^\oplus U(1)$ for $m = \text{rank}(g)$.

In the previous section to prove the tri-hermiticity of $B$ we have used the fact that the Cartan sub-algebra generators of the highest root spaces $d_k$ are mutually orthogonal. This is easily verified using Dynkin diagrams because the associated crossed vertices of the highest roots of the different levels are unconnected in the extended Dynkin diagrams.

Homogeneous HKT spaces associated with reductive Lie algebras $g$, as homogeneous KT spaces, are of the form

$$M = \left( M^0 \times M^1 \times \ldots \times M^r \right) / T ,$$

where $M^0$ and $T$ are toral groups and $M^i$ are homogeneous HKT spaces associated with simple groups. Let $g = b^0 \oplus \bigoplus g^i$ be a reductive Lie algebra, where $b^0$ is Abelian and $g^i, 1 \leq i \leq r$, are simple ideals. The homogeneous space $M$ can be constructed in the following steps:

- We first decompose the simple ideals $g^i$ separately and determine the HKT spaces $M^i$, as in the beginning of the section.
- We set $b = b^0 \oplus \bigoplus b^i$, where $b^i$ is the Abelian part of the decomposition of the simple ideal $g^i$. Then we divide out some part of $b$ say $t \subset b$. In particular $t$ can be a linear combination of $u(1)$-generators of $b^0$ and $b^i$’s.
- Since for the construction of a HKT structure every highest root subspace is paired with a $u(1)$ generator, we add $c u(1)$ generators such that $b + c - a$ is zero or positive and divisible by four, where $a = \text{dim}(t)$ and $b = \text{dim}(b^0)$. Then $M^0$ in (79) is spanned by $b + c u(1)$-generators. Our results are summarized as:
### Table 2: Homogeneous hyper-complex and HKT spaces \((G/K) \times U(1)\) of dimension \(d\) for simple groups \(G\).
Theorem 5  All homogeneous HKT spaces constructed in Theorem 4 are of the form
\[ \{ M^0 \times M^1 \times \ldots \times M^r \}/T , \]  
(80)
where \( M^i = \{ G^i/K^i \} \times U(1) \) for \( 1 \leq i \leq r \) are homogeneous HKT spaces listed in Table 2 for simply-connected, simple compact groups \( G^i \), \( M^0 \) is a toral group and \( T \) is a toral subgroup of \( B = M^0 \times U(1) \) for \( m = \sum_{i=1}^r m_i \).

6 Homogeneous QKT spaces

The goal of this section is to construct homogeneous QKT spaces. For this we shall use techniques similar to those for constructing homogeneous quaternionic spaces in [15]. We shall show that these homogeneous quaternionic spaces admit QKT structures.

We begin with a homogeneous HKT manifold, \( G/K \), where \( G \) is a group with reductive Lie algebra, \( \mathfrak{g} = \tilde{\mathfrak{u}} \oplus \mathfrak{m} \). Our QKT spaces are of the form \( G/(K \times \Phi(U(2))) \), where \( \Phi \) is an embedding of the group \( U(2) \) in \( G \). This embedding is chosen such that

(i) \( \Phi(U(2)) \) centralizes \( K \) in \( G \),
(ii) \( \Phi(U(2)) \) is a hyper-complex sub-manifold of \( G/K \) and
(iii) the left action of \( \Phi(U(2)) \) on \( G/K \) induces an \( SO(3) \) rotation on the three complex structures.

Let us take the embedding \( \Phi \) at the Lie algebra level as
\[ \Phi(u(2)) := \text{span}\{ U, \phi(Y_1), \phi(Y_2), \phi(Y_3) \} \]
(81)
for the basis vectors \( U := \sum_{k=1}^l U_k \) and \( \phi(Y_r) := \sum_{k=1}^l \phi_k(Y_r) \), where the rest of the notation can be found in section four. The centre of \( U(2) \) under the embedding \( \Phi \) must be a closed subgroup of the maximal torus of \( G \). For \( l > 1 \), this places a rationality condition on \( U \) which can be satisfied for only a dense subset of the hyper-complex structures on \( G/K \). The homogeneous spaces \( G/(K \times \Phi(U(2))) \) admit a quaternionic structure provided that the
hyper-complex structure on $G/K$ has been chosen such that $U$ is rational. As we have seen the hyper-complex structures on $G/K$ are parameterized by $l^2$ real free parameters whereas those that lead to quaternionic structures on $G/(K \times \Phi(U(2)))$ are parameterized by $l^2$ rational free parameters. Different hyper-complex structures on $G/K$ give rise to different embeddings of $U$ in $G$, which may lead to quaternionic spaces with distinct topological structures. We summarize the above as [15]:

**Theorem 6** A compact homogeneous space of the form $G/(K \times \Phi(U(2)))$ admits quaternionic structures if the homogeneous space $G/K$ is hypercomplex and $\Phi$ is an appropriate embedding of $U(2)$ in $G$.

Let $g = m \oplus k$ be the orthogonal decomposition of $g$ with respect to the invariant metric on $g$ associated to the homogeneous HKT space $G/K$. Then the quaternionic space of the form $G/(K \times \Phi(U(2)))$ admits QKT structures provided that

(i) the invariant metric on $g$ decomposes orthogonally as $g = \tilde{m} \oplus \tilde{k}$, where $\tilde{k} = k \oplus \Phi(u(2))$,

(ii) the invariant metric on $\tilde{m}$ is tri-hermitian with respect to the almost quaternionic structures $J_r$, where $J_r$ are endomorphisms of $\tilde{m}$ associated with the quaternionic structure on $G/(K \times \Phi(U(2)))$,

(iii) the torsion [29] is (2,1) and (1,2) with respect to $J_r$ and

(iv) the conditions for the metric on $m$ to be tri-hermitian are compatible with the rationality condition for the embedding of the centre of $u(2)$ in $g$.

It is sufficient to discuss the decomposition of the invariant metric of $g$ on the subspace $\bigoplus_{k=1}^l (d_k \oplus u(1)_k)$, since $\Phi(u(2)) \subset \bigoplus_{k=1}^l (d_k \oplus u(1)_k)$. We take $u(2)_k = \text{span}\{U_k, \phi_k(Y_1), \phi_k(Y_2), \phi_k(Y_3)\}$, and relabel $U_k = T^k_0$ and $\phi_k(Y_r) = T^k_r$, $r = 1, 2, 3$. Let $\Phi(u(2)) = \text{span}\{K_a; a = 0, 1, 2, 3\}$, then

$$K_a = T^1_a + T^2_a + \cdots + T^l_a, \quad M^a = \sum_{q=1}^n \frac{T^q_a}{B(T^q_a, T^q_a)} - \frac{nT^{n+1}_a}{B(T^{n+1}_a, T^{n+1}_a)}$$

(82)
is an orthogonal basis in $\bigoplus_{k=1}^{l} \mathfrak{u}(2)_k$, where $1 \leq n \leq l - 1$. This can be used to induce an invariant metric in the complement of $\Phi(\mathfrak{u}(2))$ as required.

As we have seen, $\mathfrak{m} = \Phi(\mathfrak{u}(2)) \oplus \tilde{\mathfrak{m}}$. From $I_r(\Phi(\mathfrak{u}(2))) \subset \mathfrak{u}(2)$ it is easy to deduce that $I_r(\tilde{\mathfrak{m}}) \subset \tilde{\mathfrak{m}}$. Then the endomorphisms $J_r$ of $\tilde{\mathfrak{m}}$ associated with the quaternionic structure on $G/(K \times \Phi(U(2)))$ are defined as $J_r = I_r|\tilde{\mathfrak{m}}$. Hence the hermiticity of $B$ with respect to $J_r$ is implied by the hermiticity of $B$ with respect to $I_r$.

The endomorphisms $J_r$ are sections of a rank three associated bundle $V$ of a principal $\Phi(Sp(1))/\mathbb{Z}_2$-bundle over $G/(K \times \Phi(U(2)))$. Moreover they are
covariantly constant with respect to the canonical connection $\omega$ on $G/(K \times \Phi(U(2)))$. The connection of the bundle $V$ is given by the $Sp(1)$ component of $\omega$, i.e. the connection $B_r$ in (3) is the $\Phi(so(3))$ component of the canonical connection:

**Theorem 7** All homogeneous spaces of the form $(M^0 \times M^1 \times \ldots \times M^r)/(T \times \Phi(U(2)))$ admit QKT structures if the homogeneous spaces $(M^0 \times M^1 \times \ldots \times M^r)/T$ admit HKT structures.

We remark that the topological product $M_1 \times M_2$ of two QKT manifolds $N^1$ and $N^2$ does not admit a QKT structure. Nevertheless there is a notion of a topological product of two QKT manifolds $N^1$ and $N^2$, which we call join and denote by $N^1 \ast N^2$. This product is similar to that of two quaternionic manifolds given in [15]. This operation has been used implicitly in the proof of the above theorem.

A simple example of a manifold with a QKT structure is $U(2)$. For this, we observe that $U(2)$ is an HKT manifold. Then we take the product $U(2) \times U(2)$ and embed $U(2)$ diagonally in $U(2) \times U(2)$. Applying the above theorem, we conclude that $M = (U(2) \times U(2))/U(2)$ admits a QKT structure. In fact the torsion vanishes and $M$ is a QK manifold. However, $M$ is diffeomorphic to $U(2)$, so $U(2)$ admits both an HKT and a QK structure. Along the same lines it is possible to define a QKT structure on the homogeneous space $(\hat{\times}U(2))/U(2)$, which is diffeomorphic to $\hat{\times}U(2)$ and which also admits an HKT structure.

In Table 3 we list all eight-dimensional homogeneous HKT spaces and their associated four-dimensional homogeneous QKT spaces (up to possibly finite coverings). We remark that the embedding of $U(1)$ in $SU(3) \times U(1)$ is parametrized by a rational number, which gives rise to HKT spaces with different topology. However, they lead to the same QK space $\mathbb{C}P^2$. 

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Our construction of homogeneous QKT spaces includes that of QK spaces. In fact, only the first level of the above decomposition of $G$, i.e. $l = 1$ is required for the construction of homogeneous QK spaces. In particular, the Wolf spaces are found in this way from the decomposition of simple Lie algebras. Note that compact Wolf spaces have non-compact duals, as there are no Cartan sub-algebra generators left in $\tilde{m}$, but this is not so for generic homogeneous QKT spaces.

### 7 Twistor spaces for homogeneous QKT spaces

The holonomy of a QKT manifold $M$ is a subgroup of $Sp(d) \cdot Sp(1)$. Therefore, its tangent bundle is associated to a principal $Sp(d) \cdot Sp(1)$-bundle. The complexification of the tangent space of $M$ can be split as $T^c M = T_{2d} \otimes T_2$ with the first sub-bundle associated with $Sp(d)$ and the second one associated with $Sp(1)$. This structure group can be lifted to $Sp(d) \times Sp(1)$ provided that the second Witney class of $M$ vanishes, i.e. if $M$ is a spin manifold. The twistor space $Z$ of a QKT manifold \cite{1} can be defined as the projectivization of $T_2$ and it has been shown that it is a complex manifold. Using the conventions and notation of \cite{1}, we show:

**Theorem 8** The twistor space $Z$ for any QKT manifold of dimension bigger...
than four admits a KT structure.

To prove this theorem it is sufficient to show that there is a non-degenerate (1,1) form on $Z$ with respect to the complex structure in $Z$. Such a form is

$$\Omega = 2i \left( E_1^2 \wedge E_2^1 + E^{a2} \wedge E^{a1} \eta_{ab} \right),$$  \hspace{1cm} (84)$$

where $\{E_1^2, E^{a2}\}$ is a basis of (1,0)-forms, $\{E_2^1, E^{a1}\}$ is a basis of (0,1)-forms and $\eta$ is the invariant symplectic form of $Sp(d)$. The metric and the torsion of the KT structure on $Z$ can be determined from the complex structure and $\Omega$.

In [1] it was also shown that if the exterior derivative $dH$ of the torsion $H$ on $M$ is (2,2) with respect to all endomorphisms $J_r$ and if a certain non-degeneracy condition is met, then the twistor space is a Kähler manifold but with respect to a different metric from the one given above. For homogeneous QKT manifolds one can show:

**Theorem 9** Homogeneous QKT spaces with $dH$ a (2,2) form are four-dimensional.

To prove this we use

$$[f_r, f_s] = 2\epsilon_{rs}^t f_t, \quad [f_r, J_s] = 2\epsilon_{rs}^t J_t, \quad [J_r, J_s] = 2\epsilon_{rs}^t J_t,$$  \hspace{1cm} (85)$$

where $f_r$ is the representation of the $sp(1)$-part of $k$ on $m$. We choose a frame compatible with respect to one of the almost complex structures, say $J_1$, i.e. $(J_1)_\mu^\nu = i\delta_\mu^\nu$. Using the above relations, we find $(J_2)_\mu^\nu = (J_3)_\mu^\nu = 0$. Putting these into the second equation in (85), we find that $(f_1)_\mu^\nu = 0$, $(f_2)_\mu^\rho = i(J_3)_\mu^\rho$ and $(f_3)_\mu^\rho = -i(J_2)_\mu^\rho$ and from the third equation in (85) we deduce that $(J_2)_\mu^\rho = -i(J_3)_\mu^\rho$. The (4,0)-part of $dH$ is automatically zero as it can be seen by direct computation, whereas the (3,1)-part is

$$(dH)^{[3,1]}_{[\mu\nu,\rho\phi]} = (f_r)_{[\mu\nu} (f_r)_{\rho\phi]} = (f_2)_{[\mu\nu} (J_2)_{\rho\phi]} + i(f_3)_{[\mu\nu} (J_2)_{\rho\phi]}.$$  \hspace{1cm} (86)$$

For this term to vanish, $(f_2)_{\mu\nu} + if_3(\mu\nu = 0$ or $d = 1$. Assuming the former we deduce from the third relation in (85) that $(f_1)_\mu^\nu = i\delta_\mu^\nu$. So the structure
constant $f_1$ are proportional to the complex structure $J_1$. In a similar way, we can show that $f_r \sim J_r$. This implies that the $Sp(1)$ part of the curvature is proportional to the endomorphisms of the QKT manifold. But it has been shown in [1] that such QKT manifolds have vanishing torsion. Thus the only QKT spaces with non-zero torsion, whose exterior derivative is $(2,2)$ with respect to all $J_r$’s, are four dimensional.

The twistor space $Z$ of the homogeneous QKT space $M = G/\left(\Phi(U(2)) \times K\right)$ is of the form $Z = G/\left(\Phi(U(1) \times U(1)) \times K\right)$, where $U(1) \times U(1) \subset U(2)$. The complex structure on the twistor space is induced by the complex structure in the HKT space $G/K$ which is invariant under the right action of $\Phi(U(1) \times U(1))$; the orbit of this action on the space of complex structures is one-dimensional.

We have seen that our homogeneous QKT manifolds $M$ admit by construction the fibration $\Phi(U(2)) \to G/K \to M$, where $G/K$ is an HKT manifold. It turns out that every QKT manifold admit such a fibration. This fibration can be constructed along the same lines as the one over quaternionic manifolds. In order to do this, we remove the zero section of $T_2$ and compactify each fibre to a Hopf surface. To find a fibration of QKT manifolds which reduces to that of the homogeneous ones above, we further have to twist with a $U(1)$ bundle. It seems likely that the resulting spaces admit an HKT structure.

8 Conclusions

We have investigated a class of KT, HKT and QKT structures on homogeneous spaces $G/K$ using an invariant metric on $G$ and the canonical connection. Our construction was based on an orthogonal decomposition of $G$ which can be most easily understood using Dynkin diagrams. Lists of KT, HKT and QKT spaces were compiled. We have also studied the twistor spaces of homogeneous QKT spaces and have found that they admit a KT structure.

As we have mentioned, these geometries have appeared in the context of sigma models and string theory. Group spaces that admit an HKT structure
are vacua of string theory. One reason for this is that these HKT geometries have torsion which is a harmonic three-form with respect to the invariant metric. This is no longer the case for our homogeneous HKT manifolds. The exterior derivative of the torsion of these spaces can be written as the trace of the square of the curvature of the canonical connection. This is reminiscent of the condition for the cancellation of the gravitational anomaly of the heterotic string at one loop in the sigma model perturbation theory. However, since there is no ‘classical’ torsion which is a closed three form associated with this geometry. The only way to make sense of this is to assume that the one loop anomaly cancellation condition is exact and that the string tension has a particular value for this background. It may be interesting to investigate this further in the future. In connection with M-theory, it is worth pointing out that our HKT eight-dimensional manifolds are closely associated with some of the Rubin-Freud spaces. In particular, most eight-dimensional homogeneous HKT spaces are of the form $M_{(8)} = M_{(7)} \times U(1)$ (see Table 3), where $M_{(7)}$ are Freud-Rubin spaces $^{28, 29}$, which are special seven-dimensional Einstein spaces.

It would be of interest to develop techniques to construct systematically non-homogeneous KT, HKT and QKT spaces. Some examples of KT and HKT spaces are known but all of them are non-compact. In particular, the QKT spaces which satisfy the requirements of the theorem in $^{1}$ will lead to the construction of HK manifolds.

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