Chiral and scale anomalies of non local Dirac operators

E. Ruiz Arriola and L.L. Salcedo

Departamento de Física Moderna
Universidad de Granada
E-18071 Granada, Spain
(March 28, 2022)

Abstract

The chiral and scale anomalies of a very general class of non local Dirac operators are computed using the ζ-function definition of the fermionic determinant. For the axial anomaly all new terms introduced by the non locality are shown to be removable by counter terms and such counter terms are also explicitly computed. It is verified that the non local Dirac operators have the standard minimal anomaly in Bardeen’s form.

PACS numbers: 11.10.Lm, 11.30.Rd, 11.15.-q, 12.39.Fe
I. INTRODUCTION

Local field theories provide the traditional setup where the implementation of space-time symmetries becomes rather simple. On the other hand, effective theories are not necessarily local, although an appropriate choice of degrees of freedom can make them local \[1\]. An outstanding example is QCD in the domain of low hadronic energies, where light quarks and gluons are dressed by the interaction making the effective theory look highly nonlocal \[2\] in terms of these degrees of freedom and hence a sort of dynamical perturbation theory would be needed \[3\]. This has produced a wealth of work mainly based on Dyson-Schwinger equations properly constrained by the relevant Ward and Slavnov-Taylor identities \[4\]. In this respect anomalies provide an interesting playground to study the interplay between low and high energies both in the local and in the nonlocal case. They are triggered by the ultraviolet regulators which unavoidably violate some classical symmetries but their physical effect is formulated as a low energy theorem. The question whether or not the nonlocal interaction can be implemented without spoiling the anomaly has been previously discussed \[5–7\] for some specific processes like e.g. $\pi^0 \rightarrow 2\gamma$, $\gamma \rightarrow 3\pi$ and $2K \rightarrow 3\pi$ and regarding the chiral anomaly. In this paper we study the question for all processes and both chiral and scale anomalies. Rather than computing specific processes one by one we just prove that the new terms generated by the non locality can be subtracted by adding suitable counterterms.

II. NON LOCAL DIRAC OPERATORS

We will consider Dirac fermions in the flat $D$-dimensional Euclidean space-time $\mathbb{R}^D$ endowed with internal degrees of freedom collectively referred to as “flavor”. The class of Dirac operators to be considered here is

$$D = D_L + M.$$  \hspace{1cm} (1)

The term $D_L$, the local component of $D$, is a standard Dirac operator

$$D_L = \gamma_\mu P_\mu + Y$$ \hspace{1cm} (2)

The Dirac gamma matrices are anti-Hermitian (we will follow the conventions of \[8\]), $P_\mu = i\partial_\mu$ and $Y$ is an arbitrary matrix-valued function in flavor and Dirac spaces. It will be convenient to regard $Y$ as a function of the position operators $X_\mu$, defined by $X_\mu \psi(x) = x_\mu \psi(x)$, so that $Y$ is a multiplicative operator in the Hilbert space of fermions.

$$(Y \psi)(x) = Y(x)\psi(x).$$ \hspace{0.5cm} (3)

The term $M$ is a purely non local, more precisely bilocal, operator also with arbitrary structure in flavor and Dirac spaces,

$$(M \psi)(x) = \int d^Dy M(x,y)\psi(y).$$ \hspace{0.5cm} (4)

By purely non local we mean that $M$ is softer in the ultraviolet sector than any multiplicative operator, that is, the distribution $M(x,y)$ is less singular than the Dirac delta $\delta(x-y)$. More
restrictive assumptions on $M$ will be made below. Further restrictions on the form of $Y$ and $M$ come from imposing hermiticity of the associated Hamiltonian in Minkowski space (that is, the hermiticity of $\gamma^0 D$). In even dimensions, the Euclidean Dirac operator can be split into the components with and without $\gamma_5$, $D_+$ and $D_-$ respectively, then unitarity requires $D_\pm^{\dagger} = \pm \gamma_5 D_\pm \gamma_5$. In the odd dimensional case, $D_+$ and $D_-$ corresponds to an even or odd number of Dirac matrices, respectively, and the unitarity condition becomes $D_\pm^{\dagger} = \pm D_\pm$.

Many of the concepts used for standard local Dirac operators apply directly to the nonlocal case. We define a symmetry as any transformation of $\psi(x)$ and $\bar{\psi}(x)$ that can be compensated by a corresponding transformations of the external fields $Y$ and $M$ (within the class of fields considered) so that the action $\int d^D x \bar{\psi} D \psi$ remains invariant. Presently, we will consider chiral transformations in two and four dimensions. Scale transformations will be treated in section [V]. Chiral transformations are defined as in the local case, namely,

$$D \rightarrow e^{i\beta - i\alpha \gamma_5} D e^{-i\beta - i\alpha \gamma_5},$$

where $\alpha(x)$ and $\beta(x)$ are Hermitian matrices in flavor space only, regarded as multiplicative operators on the fermionic wave functions. The particular cases $\alpha = 0$ and $\beta = 0$ correspond to vector and axial transformations, respectively. In the infinitesimal case

$$\delta D = \delta V D + \delta A D = [i\beta, D] - \{i\alpha \gamma_5, D\}.$$

Because the chiral transformations are local, both $D_L$ and $M$ transform covariantly separately, that is,

$$\delta D_L = [i\beta, D_L] - \{i\alpha \gamma_5, D_L\}, \quad \delta M = [i\beta, M] - \{i\alpha \gamma_5, M\}.$$

Note that the bilocal structure of $M$ implies that local factors at each side of the operator are taken at different points, i.e. $M(x, x') \rightarrow e^{i\beta(x) - i\alpha(x') \gamma_5} M(x, x') e^{-i\beta(x') - i\alpha(x) \gamma_5}$. The effective action of the fermions in presence of the external fields $Y$ and $M$ is defined as in the local case, namely

$$W(D) = -\log \int D \bar{\psi} D \psi \exp \left\{ -\int d^D x \bar{\psi} D \psi(x) \right\} = -\text{Tr} \log D.$$

Here, Tr stands for trace over all degrees of freedom and some renormalization of the ultraviolet divergences is understood. The (consistent) anomaly is defined as the variation of the effective action under infinitesimal chiral transformations. Since we will be considering a $\zeta$-function renormalization of $W$, there will be no vector anomaly,

$$\delta_V W = 0, \quad \delta_A W = A_A.$$

Correspondingly, the same current conservation formulas valid for the local case can be written here,

$$0 = \int d^D x \langle \bar{\psi} (x) [i\beta, D] \psi(x) \rangle_Q \quad \text{(10)}$$

$$-A_A = \int d^D x \langle \bar{\psi} (x) \{i\alpha \gamma_5, D\} \psi(x) \rangle_Q. \quad \text{(11)}$$
(The symbol $\langle \rangle_Q$ stands for quantum vacuum expectation value.) In particular the term $\gamma_\mu P_\mu$ in $D$ in the right-hand side yields, after integration by parts, the divergence of the fermionic vector and axial currents whereas the other terms in $D$, local and non local, represent the explicit chiral symmetry breaking due to the external fields. On the other hand, the left-hand side shows the anomalous breaking of the axial current conservation. As will be shown below, the effective action can be renormalized so that only the local fields $Y$ contribute to the anomaly, and moreover, only the standard minimal Bardeen’s axial anomaly needs to be retained.

For the purpose of doing detailed calculations we will assume that the non local operator $M$ admits an expansion in inverse powers of $P_\mu$ for large $P_\mu$ of the form

$$M = M_\mu \frac{P_\mu}{P^2} + M_{\mu\nu} \frac{P_\mu P_\nu}{P^4} + M_{\mu\nu\rho} \frac{P_\mu P_\nu P_\rho}{P^6} + \cdots \quad (12)$$

The coefficients $M_{\mu_1...\mu_n}$ are multiplicative operators and they are completely symmetric under permutation of indices. For convenience the $P_\mu$ has been put at the right. It should be noted that this choice does not exhaust all possible non local operators. For instance, for each given $k = 0, 1, 2, \ldots$, the class of operators

$$M^{(k)} = \sum_n \sum_{\mu,\alpha} M^{\alpha_1...\alpha_{2k}} \frac{P_{\mu_1} \cdots P_{\mu_n} P_{\alpha_1} \cdots P_{\alpha_{2k}}}{P^{2n}} \quad (13)$$

includes all classes with lower index $k$ as particular cases. Our choice $k = 0$ is the simplest one but it is still non trivial and enjoys the essential property of being closed under chiral transformations.\footnote{This choice is also realistic since it accommodates the operator product expansion estimate of the quark self-energy $\Sigma(p^2) \sim p^2 \log p^2 / p^2$ with $d$ the anomalous dimension of the quark condensate $\bar{\psi}\psi$ (see ref.\footnote{4})}

The better way to obtain the transformation properties of $M$ is by introducing a family of operators associated to $M$ as

$$\tilde{M}(p) = e^{ipX} M e^{-ipX} \quad (14)$$

where the momentum $p_\mu$ is just a constant c-number. Effectively, $\tilde{M}(p)$ corresponds to make the replacement $P_\mu \rightarrow P_\mu + p_\mu$ in $M$. The function $\tilde{M}(p)$ admits an expansion in inverse powers of $p_\mu$ similar to that in eq. (12), namely

$$\tilde{M}(p) = \tilde{M}_\mu \frac{P_\mu}{p^2} + \tilde{M}_{\mu\nu} \frac{P_\mu P_\nu}{p^4} + \cdots \quad (15)$$

The two lowest coefficients are given by

$$\tilde{M}_\mu = M_\mu, \quad \tilde{M}_{\mu\nu} = M_{\mu\nu} + t_{\mu\nu\rho\sigma} M_\rho P_\sigma, \quad (16)$$

where we have introduced $t_{\mu\nu\rho\sigma} = \delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho} - \delta_{\mu\nu} \delta_{\rho\sigma}$. It should be noted that the coefficients $\tilde{M}_{\mu_1...\mu_n}$ are not multiplicative operators. One useful property of $\tilde{M}(p)$ is that
it transforms covariantly under chiral transformations. Indeed, if \( \mathbf{M}_\Omega = \Omega_1 \mathbf{M} \Omega_2 \) for two multiplicative operators \( \Omega_1, \Omega_2 \),

\[
\tilde{\mathbf{M}}_\Omega(p) = e^{ipX} \mathbf{M} e^{-ipX} = \Omega_1 \tilde{\mathbf{M}}(p) \Omega_2 .
\]

As a consequence, the coefficients are also chiral covariant

\[
\delta \tilde{\mathbf{M}}_{\mu_1 \ldots \mu_n} = [i \beta, \tilde{\mathbf{M}}_{\mu_1 \ldots \mu_n}] - \{i \alpha \gamma_5, \tilde{\mathbf{M}}_{\mu_1 \ldots \mu_n}\}.
\]

From here it is immediate to derive the transformation of the original coefficients \( \mathbf{M}_{\mu_1 \ldots \mu_n} \).

For the two lowest order coefficients one finds

\[
\delta \mathbf{M}_\mu = [i \beta, \mathbf{M}_\mu] - \{i \alpha \gamma_5, \mathbf{M}_\mu\},
\]

\[
\delta \mathbf{M}_{\mu \nu} = [i \beta, \mathbf{M}_{\mu \nu}] - \{i \alpha \gamma_5, \mathbf{M}_{\mu \nu}\} + t_{\mu \nu \rho \sigma} \mathbf{M}_\rho (\partial_\sigma \beta + \partial_\sigma \alpha \gamma_5).
\]

In general, the variation of each coefficients involves those of lower order. This shows that the class of non local operators considered carries a representation of the chiral group.

There is another essential requirement which is also satisfied by the particular class of non local operators \( \mathbf{M} \) considered, namely, it is closed under Hermitian conjugation (see section IV) and so the coefficients can be chosen so as to satisfy the unitarity requirement stated above. Another remark is that the classes of operators corresponding to set to zero the first \( n \) coefficients in \( \mathbf{M} \) are also closed under chiral transformations and Hermitian conjugation.

### III. THE AXIAL ANOMALY

In order to compute the axial anomaly, we will adopt the \( \zeta \)-function renormalization prescription combined with an asymmetric Wigner transformation. This method, as well as several of its applications, is presented in great detail in [8]. Since the techniques required in the present non local case are an immediate extension of those used in that reference here we will emphasize only the new issues introduced by the non locality. The \( \zeta \)-function effective action is given by [10][11]

\[
W(D) = -\frac{d}{ds} \text{Tr} \left( D^s \right)_{s=0},
\]

where \( s = 0 \) is to be understood as an analytical extension on \( s \) from the ultraviolet convergent region \( \text{Re}(s) < -D \). The key point is that for sufficiently negative \( s \) there are no ultraviolet divergences and formal operations become justified. By construction, the \( \zeta \)-function renormalized effective action is invariant under all symmetry transformations associated to similarity transformations of \( D \), thus in particular it is vector gauge invariant.

On the other hand the axial anomaly takes the form

\[
\mathcal{A}_A = \text{Tr} \left( 2i \alpha \gamma_5 D^s \right)_{s=0}.
\]

The operator \( D^s \) can be obtained from

\[
D^s = -\int \frac{dz}{2\pi i} \frac{z^s}{D - z}.
\]
where the integration path $\Gamma$ starts at $-\infty$, follows the real negative axis, encircles the origin $z = 0$ clockwise and goes back to $-\infty$. Using the Wigner transformation technique [8], the anomaly can be written as (a similar expression holds for the effective action)

$$A_A = -\int \frac{d^Dp}{(2\pi)^D} \int \frac{dz}{2\pi i} z^s \text{tr} \langle 0 | 2i\alpha \gamma_5 \frac{1}{D(p) - z} | 0 \rangle \bigg|_{s=0}. \quad (23)$$

Here $\text{tr}$ stands for trace over Dirac and flavor degrees of freedom, $|0\rangle$ is the zero momentum state normalized as $\langle x | 0 \rangle = 1$, thus $P_\mu |0\rangle = \langle 0 | P_\mu = 0$. Further

$$\bar{D}(p) = e^{ipX} D e^{-ipX} = \hat{\phi} + D_L + \bar{M}(p). \quad (24)$$

The integration over $z$ should be performed first, since it defines the operator $D^s$, then the integral over $p$ which corresponds to take the trace over space-time degrees of freedom and finally $s$ is to be analytically extended to $s = 0$. The simplest way to proceed is to introduce a mass term, i.e., to apply the formula to the Dirac operator $D + m$ and then make an expansion in powers of $D_L + \bar{M}(p)$, letting $m \to 0$ at the end. In this way the following expression is derived

$$A_A = \sum_{N \geq 0} \int \frac{d^Dp}{(2\pi)^D} \int \frac{dz}{2\pi i} z^s \text{tr} \langle 0 | 2i\alpha \gamma_5 \left( D_L + \bar{M}(p) \right) \left( (\hat{\phi} + z - m)(D_L + \bar{M}(p)) \right)^N \frac{(p^2 + (z - m)^2)^{N+1}}{p^2 + (z - m)^2} | 0 \rangle \bigg|_{s=0, m=0}. \quad (25)$$

This formula has been simplified using the cyclic property for the trace in Dirac space.

From the expression it is clear that for sufficiently large $N$ the integrals become ultraviolet convergent. When this happens $s$ can be set to zero directly and $\Gamma$ no longer encloses any singularity thus the integral vanishes. Using this insight we can expand $\bar{M}(p)$ in inverse powers of $p_\mu$ and keep only the divergent terms. Using eq. (15), the anomaly can be written as a sum of monomials each given as a product of $D_L$ and the various coefficients $\bar{M}_{\mu_1...\mu_n}$ raised to different powers. The canonical dimension of the monomial, $g$, is obtained noting that the canonical dimension of $D_L$ is 1 and that of $M_{\mu_1...\mu_n}$ is $n + 1$ (the various factors of $m$, $z$ and $p_\mu$ do not count to compute $g$). Doing the usual dimensional analysis, it is easily established that after integration each monomial comes with a factor $m^\gamma$ where $\gamma$ is the degree of divergence of the term and further $\gamma = D - g + s$. As noted above, terms with a negative degree of divergence vanish. On the other hand, the terms with a positive degree of divergence also vanish after taking $m = 0$, since $m$ is raised to a positive power. In summary, the only contributions which need to be retained are those logarithmically divergent which correspond to monomials with scale dimension $g = D$. In two dimensions, the only relevant terms are those of the form $D_L^2 + \bar{M}_\mu$, whereas in four dimensions they are $D_L^4 + D_L^2 \bar{M}_\mu + D_L \bar{M}_{\mu\nu} + \bar{M}_\mu^2 + \bar{M}_{\mu\nu\rho}$. A further restriction comes from Euclidean rotational invariance on $p_\mu$. As a consequence, it is immediate to see that the terms $\bar{M}_\mu$ in two dimensions and $\bar{M}_{\mu\nu\rho}$ in four dimensions cancel.

After an angular average over $p_\mu$, the indicated integrals on $p_\mu$ and $z$ can be carried out directly with the integral $I_1$ given in [8]. The result for the two dimensional anomaly is

$$A_A = \langle 2i\alpha \gamma_5 D_L^2 \rangle, \quad (26)$$
and in four dimensions

$$A_A = \langle 2\alpha \gamma_5 \left( \frac{1}{2} D_L^4 + \frac{1}{12} D_L \{\gamma_\mu, D_L\}^2 D_L \right. \\
+ \frac{1}{8} \left( \mathcal{M}_\mu \{\gamma_\mu, D_L\} D_L + D_L \{\gamma_\mu, \mathcal{M}_\mu\} D_L + D_L \{\gamma_\mu, D_L\} \mathcal{M}_\mu \right) \\
+ \frac{1}{4} \mathcal{M}_\mu^2 + \frac{1}{4} \{D_L, \mathcal{M}_\mu\} \rangle \rangle$$

(27)

The notation $\langle f \rangle$ stands for

$$\langle f \rangle = \frac{1}{(4\pi)^{D/2}} \text{tr} \langle 0 | f(X) | 0 \rangle.$$

(28)

Observe that, even for non local Dirac operators, the anomaly is a local polynomial of dimension $D$ constructed with $P_\mu$ and the external fields $Y$ and $M_{\mu_1 \ldots \mu_n}$. This is a general property of all anomalies since only ultraviolet divergent terms can contribute to them. This puts a restriction to the number of non local coefficients that can appear in the axial anomaly, namely, only coefficients of dimension at most $D$ can be relevant. In terms of the kernel $M(x,y)$ in eq. (4), it implies that non local components in the Dirac operator with kernels which are piecewise continuous with jump discontinuities do not contribute to the anomaly. The detailed calculation shows that actually the coefficients with dimension $D$ ($\mathcal{M}_\mu$ in two dimensions and $\mathcal{M}_{\mu\nu\rho}$ in four dimensions) do no have a contribution either. In particular, in two dimensions there is no non local contribution to the axial anomaly.

The expressions found for the anomaly can be put in a more usual form, in terms of vector and axial fields, scalar fields, etc, by making two observations. First, after taking the Dirac trace, the operators that appear there are actually multiplicative, that is, all $P_\mu$ appear inside commutators only. The simplest way to see this is by formally replacing every $P_\mu$ by $P_\mu + a_\mu$ where $a_\mu$ is a constant c-number, and checking that all $a_\mu$-dependence cancels. Second, for a multiplicative operator $f(X)$,

$$\langle f(X) \rangle = \frac{1}{(4\pi)^{D/2}} \int d^D x \text{tr} f(x).$$

(29)

Since the regularization preserves vector gauge invariance, the axial anomaly is also invariant. In our expression for the anomaly, this is a direct consequence of the operators there being multiplicative. Indeed, any operator $f$ constructed with the gauge covariant blocks $D_L$ and $\mathcal{M}_{\mu_1 \ldots \mu_n}$ is also covariant, i.e., $f \rightarrow \Omega f \Omega^{-1}$. If in addition $f$ is multiplicative $\langle f \rangle$ is invariant. Note that $\langle \rangle$ is not a trace and so the cyclic property does not hold for arbitrary non multiplicative operators.

**IV. ESSENTIAL AXIAL ANOMALY**

Presumably due to its topological connection [12], the axial anomaly is a very robust quantity. It is not affected by higher order radiative corrections [13], and remains unchanged at finite temperature and density [14]. It gets no contributions from scalar and pseudo scalar fields [15], tensor fields [16,17] or internal gauge fields, i.e, transforming homogeneously
under gauge transformations \[18,19\]. In all known cases, the anomaly only affects the imaginary part of the effective action and only involves vector and axial fields. The counter terms can always be chosen so that the axial anomaly adopts the minimal or Bardeen’s form \[15\]. Not surprisingly, the new terms introduced in the anomaly by the non local component of the Dirac operator are also unessential, that is, they can be removed by adding a suitable local and polynomial counter term to the effective action. In other words, all new terms can be derived as the axial variation of an action which is a polynomial constructed with the external fields \(Y\) and \(M_{\mu_1...\mu_n}\) and their derivatives. The canonical dimension of the polynomial can be at most \(D\).

The general proof that the anomaly can always be brought to Bardeen’s form is as follows. Let \(Y_i(x)\) denote the various external fields which specify the Dirac operator. Under a variation of them

\[
\delta D = \delta Y_i \frac{\partial D}{\partial Y_i}.
\]

For convenience, we consider \(Y_i\) as multiplicative operators and put them at the right. The operators \(\partial D / \partial Y_i\) are simply constants, in the sense that they do not depend on \(Y_i\), and are just numbers or matrices if \(Y_i\) refers to a local degree of freedom and contain \(P_\mu\) when \(Y_i\) is related to a non local term of \(D\). Each external field defines a consistent current through the variation of the effective action

\[
\delta W = \int d^D x \sum_i \text{tr} \left( \delta Y_i(x) J_i(x) \right).
\]

At a formal level, the currents would be just \(J_i(x) = -\langle x| (\partial D / \partial Y_i) D^{-1}|x\rangle\). However this expression is ultraviolet divergent. Within the \(\zeta\)-function prescription the renormalized consistent currents are given by

\[
J_i(x) = -\frac{d}{ds} \left( s \langle x| \frac{\partial D}{\partial Y_i} D^{s-1}|x\rangle \right)_{s=0}.
\]

It is also possible to define the currents in a chirally covariant manner \[20\]. One such definition \[21\] corresponds to take the finite part, as \(\epsilon\) goes to zero, of

\[
J_i^c(x) = -\int_\epsilon^\infty d\lambda \langle x| \frac{\partial D}{\partial Y_i} D^\dagger e^{-\lambda D D^\dagger}|x\rangle.
\]

(Recall that \(\alpha(x)\) is Hermitian and so \(D^\dagger\) transform as \(D^{-1}\) under axial transformations.) Due to the presence of an essential axial anomaly, the consistent and covariant currents cannot (all of them) coincide. If the covariant currents were consistent they could be integrated to yield a chiral invariant effective action \[21\]. However, because both set of currents correspond to same formal definition, they must coincide in their ultraviolet convergent terms, and thus they can only differ by a local polynomial of dimension at most \(D - 1\) \[20,8\]

\[
J_i(x) = J_i^c(x) + Q_i(x).
\]

Here, \(J_i\) is consistent, \(J_i^c\) is covariant and \(Q_i\) is a polynomial. Note that the arguments used to reach this relation hold in particular for the class of non local Dirac operators considered.
in this work. This relation is already sufficient to show that the essential anomaly only contains vector and axial fields [8]. Indeed, let us separate the Dirac operator $D$ into two components $D_0$ and $N$ both transforming covariantly under chiral transformations and such that $D_0$ contains the term $\gamma_\mu P_\mu$. Consequently, the vector and axial fields should also be in $D_0$ since they mix with $\gamma_\mu P_\mu$ under chiral transformations. Also note that $D_0$ is a valid Dirac operator whereas $N$ by itself is not. The change in the effective action due to passing from $D_0$ to $D = D_0 + N$ can be obtained by integrating the consistent current along the path $D_t = D_0 + tN$ with $0 \leq t \leq 1$. The result does not depend on the particular interpolating path since integrability conditions are satisfied. The contribution from the covariant current will be invariant and does not change the anomaly. The contribution from the polynomial current will be a polynomial. This latter term is responsible for the change in the anomaly introduced by adding $N$. Since this change derives from a polynomial it is removable by counter terms. Therefore taking $D_0 = \gamma_\mu (P_\mu + V_\mu + A_\mu \gamma_5)$ yields an anomaly involving only vector and axial fields. This anomaly will be minimal by using a chiral covariant renormalization for the real part of the effective action, such as the $\zeta$-function prescription applied to $-\frac{1}{2} \text{tr} \log(DD^\dagger)$.

The actual construction of the counter terms can be done using the method in ref. [8]. In order to keep the reasoning straight we will skip some technicalities and use a rather symbolic notation. The current can be written as $J = \partial W / \partial D$. Under vector and axial transformations, the covariant current transforms as $J^c \rightarrow e^{-i\beta} J^c e^{i\beta}$ and $J^c \rightarrow e^{i\alpha \gamma_5} J^c e^{i\alpha \gamma_5}$ respectively so that $\langle NJ^c \rangle$ remains invariant in both cases. (Recall that $N$ transforms as $e^{i\beta} Ne^{-i\beta}$ and $e^{-i\alpha \gamma_5} Ne^{-i\alpha \gamma_5}$ respectively.) Infinitesimally this implies

$$0 = \bar{\delta}_V J^c := \delta_V J^c - [i\beta, J^c], \quad (35)$$

$$0 = \bar{\delta}_A J^c := \delta_A J^c - \{i\alpha \gamma_5, J^c\}, \quad (36)$$

where we have introduced the covariant vector and axial variations $\bar{\delta}_V$ and $\bar{\delta}_A$ respectively. On the other hand, since the polynomial current $Q$ accounts for changes in the anomaly induced by changes in $D$, it should satisfy the following two equations

$$\bar{\delta}_V Q = 0, \quad \bar{\delta}_A Q = \frac{\partial A_A}{\partial D}. \quad (37)$$

Once these equations are solved, the counter terms needed to remove the extra contribution coming from $N$ are

$$W_{ct} = (4\pi)^{D/2} \int_0^1 dt \langle N Q_t \rangle, \quad (38)$$

where $Q_t$ is the polynomial current corresponding to $D_t = D_0 + tN$. (Recall that we are using a schematic notation. In an actual calculation one has to distinguish each of the external fields which define $D$ and their associated currents as discussed below.)

The right hand side of the second eq. (37) can be computed from the known anomaly, eqs. (26,27). For instance, in two dimensions

$$\frac{\partial A_A}{\partial D_L} = \frac{1}{4\pi} \{2i\alpha \gamma_5, D_L\}. \quad (39)$$
The cyclic property has been used since it turns out to be justified in this case. In eq. (37) \(Q\) is the unknown. If the anomaly were the variation of a polynomial action, an immediate solution would be given by the corresponding polynomial current. Because the anomaly contains an essential part, such polynomial action does not exist. Remarkably, at a formal level there is a polynomial action from which the axial anomaly derives [8], namely,

\[
W_0 = -\langle \frac{1}{2} D^2_L\rangle
\]

in two dimensions and

\[
W_0 = -\langle \frac{1}{24} D^4_L + \frac{1}{24} D^2_L \gamma_\mu D^2_L \gamma_\mu + \frac{1}{12} D^3_L \gamma_\mu D_L \gamma_\mu \\
+ \frac{1}{8} (\gamma_\mu \tilde{M}_\mu D^2_L + \gamma_\mu D^2_L \tilde{M}_\mu + \gamma_\mu D_L \tilde{M}_\mu D_L + \tilde{M}_\mu^2 + \{D_L, \tilde{M}_\mu\})(41)\rangle
\]

in four dimensions. In this context “formal level” means that the correct anomaly is obtained from \(W_0\) if the cyclic property is used. Actually, the operators involved in \(W_0\) are not multiplicative and so the cyclic property does not hold (it does hold in Dirac and flavor spaces but not for differential operators in coordinate space). Even so, the action \(W_0\) has a unique well-defined current \(Q_0 = \partial W_0/\partial D\), which is obtained from (using the cyclic property to put all \(\delta D\) together)

\[
\delta W_0 = (4\pi)^{D/2} \langle \delta D Q_0 \rangle .
\]

\(Q_0\) is well-defined in the sense that it is unchanged under cyclic permutations of the operators in \(W_0\). For instance, in two dimensions, \(\delta W_0 = -\langle \delta D_L D_L \rangle\) and so

\[
Q_0 = -\frac{1}{4\pi} D_L .
\]

Because \(W_0\) formally gives the anomaly, \(Q_0\) is a solution of eq. (37). For instance, in two dimensions, \(\delta A Q_0\) is easily computed and gives precisely the right hand side of eq. (39).

This solution is nevertheless formal because \(Q_0\) is not a multiplicative operator in general and \(Q\) must be multiplicative. This must be solved by subtracting to \(Q_0\) a polynomial current \(Q_1\) which transforms covariantly (i.e., \(\delta A Q_1 = 0\)) and such that \(Q = Q_0 - Q_1\) is multiplicative. Again, using the two dimensional case as an example, one can consider a new local Dirac operator \(\tilde{D}_L = -D^\dagger_L\). The minus sign ensures that it is an admissible Dirac operator, i.e., there is a corresponding \(Y\). The adjoint implies that it transforms as a covariant current under axial transformations. Then it can immediately be checked that the covariant polynomial current

\[
Q_1 = -\frac{1}{4\pi} \tilde{D}_L,
\]

has the same non multiplicative part as \(Q_0\) in eq. (41). (In this case this is trivial to see. As noted above, the best way to make this check in general is to replace \(P_\mu\) by \(P_\mu + a_\mu\) and see that \(Q_0 - Q_1\) is \(a_\mu\)-independent.)

To consider the non local case in detail, we introduce the currents as
\[ \delta W = (4\pi)^{D/2} \langle \delta Y J_L + \delta M_\mu J_\mu + \frac{1}{2} \delta M_{\mu\nu} J_{\mu\nu} + \cdots \rangle. \] (46)

In four dimensions, the calculation proceeds by computing the non multiplicative polynomial formal currents

\[ \delta W_0 = (4\pi)^2 \langle \delta Y Q^0_L + \delta M_\mu Q^0_\mu + \frac{1}{2} \delta M_{\mu\nu} Q^0_{\mu\nu} \rangle. \] (47)

The polynomials \( Q_0 \) are well defined and have the correct axial transformation but they are non multiplicative. In order to construct the corresponding \( Q_1 \), let us introduce the Dirac operator

\[ \bar{D} = -D^\dagger. \] (48)

Such operator belongs to the same class as \( D \), that is, it can be put as \( \bar{D}_L + \bar{M} \) where \( \bar{M} \) has the form given in eq. (12). The corresponding coefficients are easily computed by noting that

\[ \bar{M}(p) = e^{ipX} \bar{M} e^{-ipX} = -(\bar{M}(p))^\dagger. \] (49)

And so, \( \bar{M}_{\mu_1\ldots\mu_n} = -\bar{M}^\dagger_{\mu_1\ldots\mu_n} \). For the two lowest coefficients one finds

\[ \begin{align*}
\bar{M}_\mu &= -M^\dagger_\mu, \\
\bar{M}_{\mu\nu} &= -M^\dagger_{\mu\nu} - t_{\mu\nu\rho\sigma} [P_{\rho}, M^\dagger_{\sigma}].
\end{align*} \] (50)

The form of \( \bar{D} \) is such that axially it transforms as a covariant current. The currents \( Q_1 \) are defined by suitably replacing some of the \( D \) in \( Q_0 \) by \( \bar{D} \), so that \( Q_1 \) transforms covariantly. For instance, terms of the form \( \bar{M}_{\mu_1}, \gamma_\mu \bar{M}_L D^L, P_\mu D^L \) or \( \gamma_\mu D_L^2 \) in \( Q_0 \) correspond, respectively, to \( \bar{M}_{\mu_1}, \gamma_\mu \bar{M}_L \bar{D}_L, P_\mu \bar{D}_L \) and \( \gamma_\mu \bar{D}_L D^L \) in \( Q_1 \). It is an exercise to check that \( Q = Q_0 - Q_1 \) is multiplicative.

There is a technical subtlety in checking that \( Q_1 \) is actually an axial covariant current. This is because the coefficients \( M_{\mu_1\ldots\mu_n} \) are multiplicative but not axially covariant. Therefore, the covariance of a current does not directly correspond to the equation \( \delta A^c J^c_{\mu_1\ldots\mu_n} = 0 \). In order to derive the correct equations, the simplest method is to introduce the currents \( \tilde{J} \) corresponding to the covariant quantities \( \bar{M} \),

\[ \delta W = (4\pi)^{D/2} \langle \delta Y J_L + \delta \bar{M}_\mu \tilde{J}_\mu + \frac{1}{2} \delta \bar{M}_{\mu\nu} \tilde{J}_{\mu\nu} + \cdots \rangle. \] (51)

They can be related to the currents \( J \) by comparing with eq. (46). For the two lowest orders

\[ \begin{align*}
\tilde{J}_\mu &= J_\mu - \frac{1}{2} t_{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma} + \cdots, \\
\tilde{J}_{\mu\nu} &= J_{\mu\nu} + \cdots.
\end{align*} \] (52)

Note that the currents \( \tilde{J} \) are covariant (up to anomaly) but not multiplicative due to the presence of \( P_\mu \). The covariant part satisfies \( \delta A^c \tilde{J}^c = 0 \). For the polynomial part in four dimensions, the relations become
\[ \hat{Q}_\mu = Q_\mu - \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \epsilon^\nu \hat{Q}_{\rho \sigma}, \]
\[ \hat{Q}_{\mu \nu} = Q_{\mu \nu}. \]  

A straightforward calculation shows that \( \delta \hat{Q} = 0 \) and so the \( \hat{Q}_1 \) constructed above have the correct axial transformation. This completes the construction of the counter terms in the non local case.

**V. TRACE ANOMALY**

The scale transformation \( \psi(x) \rightarrow e^{-\alpha_S(D-1)/2} \psi(e^{-\alpha_S}x) \) can be compensated by a corresponding transformation in \( D \), namely,

\[ Y(x) \rightarrow e^{-\alpha_S}Y(e^{-\alpha_S}x), \quad M_{\mu_1...\mu_n}(x) \rightarrow e^{-\alpha_S(n+1)}M_{\mu_1...\mu_n}(e^{-\alpha_S}x). \]  

Therefore, scale transformations are a symmetry of our class of non local Dirac operators. The linear operator which produces the scale transformation \( \psi \rightarrow \Omega_S \psi \) can be written as \( \Omega_S = e^{-\alpha_S((D-1)/2+X)} \), and thus the Dirac operator transforms as \( D \rightarrow \Omega_S^{-1}D\Omega_S^{-1} \). Infinitesimally it implies

\[ \delta_S D = -\alpha_S(D - i[X_p P_\mu, D]). \]  

The corresponding trace anomaly, within the \( \zeta \)-function method is

\[ A_S = \delta_S W = \alpha_S \text{Tr}(D^*)_{s=0}. \]  

The calculation is entirely similar to that of the axial anomaly. In two dimensions one finds

\[ A_S = \alpha_S \langle D_L^2 + \frac{1}{4}(\gamma_\mu, D_L)^2 + \gamma_\mu M_\mu \rangle, \]  

and in four dimensions

\[ A_S = \alpha_S \left( \frac{1}{2} D_L^4 + \frac{1}{12} (D_L \gamma_\mu, D_L)^2 + (\gamma_\mu, D_L)^2 + (\gamma_\mu, M_\mu) \right) \]
\[ + \frac{1}{96} (\gamma_\mu, D_L)^2 (\gamma_\nu, D_L)^2 + (\gamma_\mu, D_L)(\gamma_\nu, D_L)^2 + (\gamma_\mu, D_L)(\gamma_\nu, M_\mu) \]
\[ - \frac{1}{24} (\gamma_\mu \tilde{M}_\nu, M_\delta \gamma_\mu \tilde{M}_\delta + M_\nu \gamma_\nu \tilde{M}_\mu \gamma_\mu \tilde{M}_\mu + \tilde{M}_\mu D_L \gamma_\mu D_L) \]
\[ + \delta_{\mu \nu \alpha \beta} \left( \frac{1}{36} (\gamma_\mu \tilde{M}_\nu \gamma_\alpha D_L \gamma_\beta + \gamma_\mu D_L \gamma_\alpha \tilde{M}_\nu \gamma_\beta D_L + \gamma_\mu D_L \gamma_\alpha D_L \gamma_\beta \tilde{M}_\nu \gamma_\nu \right) \]
\[ + \frac{1}{24} \gamma_\mu \tilde{M}_\nu \gamma_\alpha M_\beta + \frac{1}{24} (\gamma_\alpha \tilde{M}_\mu \nu \gamma_\beta D_L + \frac{1}{12} \gamma_\mu \tilde{M}_\nu \alpha \beta) \right). \]  

Where \( \delta_{\mu \nu \alpha \beta} = \delta_{\mu \alpha} \delta_{\nu \beta} + \delta_{\mu \alpha} \delta_{\nu \beta} + \delta_{\mu \beta} \delta_{\nu \alpha} \). The result is again a local polynomial of dimension \( D \) in the external fields and their derivatives. Unlike the axial case, the coefficients \( M_\mu \) in two dimensions and \( M_{\mu \nu \alpha} \) in four dimensions do contribute to the scale anomaly.
Because scale and chiral transformations commute (in a properly defined sense), the crossed variations $\delta_S A_{V,A}$ and $\delta_{V,A} A_S$ coincide and they vanish since the axial anomaly is scale invariant. Thus the scale anomaly must be chiral invariant. The vector gauge invariance of the previous expressions is easy to check noting that the operators inside $\langle \rangle$ are multiplicative. Axial invariance is much more involved in general. In the two dimensional case, it is immediate to see that $\langle \gamma_\mu M_\mu \rangle$ is invariant. In four dimensions it is relatively easy to check that the trace anomaly is axially invariant in the particular case of $M_\mu = 0$, which, as noted previously, defines a class of operators invariant under chiral and scale transformations.

The scale anomaly is already minimal. It can be modified by adding polynomial counter terms of dimension smaller than $D$ but this would add terms of the same type to the scale anomaly.

**ACKNOWLEDGMENTS**

This work is supported in part by funds provided by the Spanish DGICYT grant no. PB95-1204 and Junta de Andalucía grant no. FQM0225.
REFERENCES

[1] S. Weinberg, Physica 96 A (1979) 327.
[2] See e.g. W. Marciano and H. Pagels, Phys. Rep. 3 (1978) 137 and references therein.
[3] H. Pagels and S. Stokar, Phys. Rev. D 20 (1979) 2947.
[4] For a review see e.g. C. D. Roberts and A. G. Williams, Prog. Part. Nucl. Phys. 33 (1994) and references therein.
[5] C. D. Roberts, R. T. Cahill and J. Prashifka, Ann. Phys. (N.Y.) 188 (1988) 20.
[6] B. Holdom, J. Terning and K. Verbeek, Phys. Lett. B 232 (1989) 351.
[7] R. D. Ball and G. Ripka, in Proceedings of the International Conference on Many Body Physics, World Scientific, 1994. C. Fiolhais, M. Fiolhais, C. Sousa and J. N. Urbano Eds. p. 127. (see also hep-ph/9312260).
[8] L.L. Salcedo and E. Ruiz Arriola, Ann. Phys. (N.Y.) 250 (1996) 1.
[9] H. D. Politzer, Nucl. Phys. B 117 (1976) 397.
[10] R.T. Seeley, Amer. Math. Soc. Proc. Symp. Pure Math. 10 (1967) 288.
[11] S.W. Hawking, Comm. Math. Phys. 55 (1977) 133.
[12] L. Álvarez-Gaumé and P. Ginsparg, Ann. Phys. (N.Y.) 161 (1985) 423.
[13] S.L. Adler and W.A. Bardeen, Phys. Rev. 182 (1969) 1517.
[14] A. Gómez Nicola and R.F. Álvarez-Estrada, Int. J. Mod. Phys. A 9 (1994) 1423.
[15] W.A. Bardeen, Phys. Rev. 184 (1969) 1848.
[16] T.E. Clark and S.T. Love, Nucl. Phys. B 223 (1983) 135.
[17] J. Minn, J. Kim and C. Lee, Phys. Rev. D 35 (1987) 1872.
[18] J. Bijnens and J. Prades, Phys. Lett. B 320 (1993) 130.
[19] E. Ruiz Arriola and L.L. Salcedo, Nucl. Phys. A 590 (1995) 703.
[20] W.A. Bardeen and B. Zumino, Nucl. Phys. B 244 (1984) 421.
[21] H. Leutwyler, Phys. Lett. B 152 (1985) 78.