Entanglement-Assisted Capacity of Quantum Channels with Side Information

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Abstract

Entanglement-assisted communication over a random-parameter quantum channel with either causal or non-causal channel side information (CSI) at the encoder is considered. This describes a scenario where the quantum channel depends on the quantum state of the input environment. While Bob, the decoder, has no access to this state, Alice, the transmitter, performs a sequence of projective measurements on her environment to encode her message. Dupuis [25, 26] established the entanglement-assisted capacity with non-causal CSI. Here, we establish characterization in the causal setting, and also give an alternative proof technique and further observations for the non-causal setting.

Index Terms
Quantum information, Shannon theory, communication, channel capacity, state information, entanglement assistance.

I. INTRODUCTION

A fundamental task in classical information theory is to determine the ultimate transmission rate of communication. Shannon’s channel coding theorem [58] states that for a given noisy channel, with a transition probability function $p_{Y|X}$, a vanishing probability of error is achievable as long as the transmission rate is lower than the channel capacity, given by $C(p_{Y|X}) = \max_{p_X} I(X;Y)$, where $I(X;Y)$ is the mutual information between the channel input $X$ and output $Y$. For rates above the channel capacity, reliable communication cannot be accomplished.

Various classical settings of practical significance can be described by a channel $p_{Y|X,S}$ that depends on a random parameter $S$ when there is causal or non-causal channel side information (CSI) available at the encoder (see e.g. [39, 41, 15] and references therein). For example, a cognitive radio in a wireless system may be aware of the channel state and network configuration [29, 32, 64], memory storage where the writer knows the fault locations [33, 48], and digital watermarking where the host data is treated as side information (see e.g. [12, 65, 52]). The capacity with causal CSI is given by [59]

$$C_{E,caus}(p_{Y|X,S}) = \max_{p_T} I(T;Y)$$

with $X = T(S)$, where $T : S \rightarrow \mathcal{X}$ is called a Shannon strategy (see also [41, 15]). A channel with non-causal CSI is often referred to as the Gel’fand-Pinsker model [28]. The capacity of this channel is given by

$$C_{E,n-caus}(p_{Y|X,S}) = \max_{p_U|X=S} [I(U;Y) - I(U;S)]$$

where $U$ is an auxiliary random variable.

The field of quantum information is rapidly evolving in both practice and theory [24, 40, 6, 46, 5, 72, 49, 73]. As technology approaches the atomic scale, we seem to be on the verge of the “Quantum Age” [10, 38]. Dynamics can sometimes be modeled by a noisy quantum channel, describing physical evolutions, density transformation, discarding of sub-systems, quantum measurements, etc [47] [67, Section 4.6]. Quantum information theory is the natural extension of classical information theory. Nevertheless, this generalization reveals astonishing phenomena with no parallel in classical communication [30]. For example, two quantum channels, each with zero quantum capacity, can have a nonzero quantum capacity when used together [63]. This property is known as super-activation.

Communication through quantum channels can be separated into different categories. In particular, one may consider a setting where Alice and Bob are provided with entanglement resources [53]. The entanglement-assisted capacity for transmission of classical information over a quantum channel was fully characterized by Bennet et al. [7, 8]. Further work on entanglement-assisted communication can be found e.g. in [35, 37, 23, 60, 17, 68, 54, 2, 11, 3]. As for classical communication without entanglement between the encoder and the decoder, the Holevo-Schumacher-Westmoreland (HSW) Theorem provides an asymptotic (“multi-letter”) formula for the capacity [34, 57], though calculation of such a formula is intractable in general. This is because the Holevo information is not necessarily additive [31]. Shor has shown that the Holevo information is additive for the class of entanglement-breaking channels [62], in which case the HSW theorem provides a single-letter computable formula for the classical capacity. This class includes both classical-quantum channels and quantum-classical channels [67, Section 4.6.7]. A similar difficulty occurs with transmission of quantum information over a quantum channel. A multi-letter formula for the quantum capacity is given in [4, 50, 61, 20], in terms of the coherent information. A computable formula is obtained in the special case where the channel is degradable [21].
The entanglement-assisted capacity of a quantum channel with non-causal CSI was determined by Dupuis [25, 26]. Furthermore, Boche, Cai, and Nötzel [9] addressed the classical-quantum channel with CSI at the encoder without entanglement. The classical capacity was determined given causal CSI, and a multi-letter formula was provided given non-causal CSI. Warisi and Coon [66] used an information-spectrum approach to derive multi-letter bounds for a similar setting, where the side information has a limited rate. Luo and Devetak [51] considered channel simulation with source side information (SSI) at the decoder, and also solved the quantum generalization of the Wyner-Ziv problem [70]. Quantum data compression with SSI is also studied in [22, 71, 36, 19, 18, 14, 13] without entanglement-assistance. Compression with SSI given entanglement assistance was recently considered by Khanian and Winter [45, 42, 44, 43].

In this paper, we consider a quantum channel with either causal or non-causal CSI. The motivation is as follows. Suppose that Alice wishes to send classical information to Bob through a (fully) quantum channel $\mathcal{N}_{SA \to B}$, where $A$ is the transmitter system, $B$ is the receiver system, and $S$ is the transmitter’s environment, which affects the channel as well. Furthermore, suppose that Alice performs a sequence of projective measurements of the environment system $S$, hence the system is projected onto a particular vector $|s\rangle$ with probability $q(s)$. Using the measurement results, Alice encodes her message and sends her transmission through the channel. Where as, Bob, who does not have access to the measurement results, “sees” the average channel $\sum_s q(s)\mathcal{N}_{A \to B}^{(s)}$, where $\mathcal{N}_{A \to B}^{(s)}$ is the projection of the channel onto $|s\rangle$. Assuming Alice’s measurement projects onto orthogonal vectors, the environment system can be thought of as a classical random parameter $S \sim q(s)$. Therefore, we treat the quantum counterpart of the models in [59] and [28], i.e. a random-parameter quantum channel $\mathcal{N}_{SA \to B}$ with CSI at the encoder.

We give a full characterization of the entanglement-assisted classical capacity and quantum capacity with causal CSI, and also give an alternative proof technique and further observations for the non-causal setting. While Dupuis’ analysis with non-causal CSI in [25, 26] is based on the decoupling approach for the transmission of quantum information (qubits), we take a more direct approach. In our analysis, we incorporate the classical binning technique [33] into the quantum packing lemma [37]. Essentially, in the achievability proof, Alice performs classical compression of the parameter sequence, and then transmits both the classical message and the compressed representation using a random phase variation of the superdense coding protocol (see e.g. [37, 67]). The results are analogous to those in the classical case, although, as usual, the quantum analysis is more involved. As observed in [28, 33], the classical optimization (2) can be restricted to mappings from $(U, S)$ to $X$ that are deterministic. In analogy, we observe that optimization over isometric maps suffices for our problem. With causal CSI, quantum operations are applied in a reversed order, and the Shannon strategy in (1) is replaced with a quantum channel.

II. DEFINITIONS AND RELATED WORK

We begin with basic definitions.

A. Notation, States, and Information Measures

We use the following notation conventions. Calligraphic letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ are used for finite sets. Lowercase letters $x, y, z, \ldots$ represent constants and values of classical random variables, and uppercase letters $X, Y, Z, \ldots$ represent classical random variables. The distribution of a random variable $X$ is specified by a probability mass function (pmf) $p_X(x)$ over a finite set $\mathcal{X}$. We use $x^j = (x_1, x_2, \ldots, x_j)$ to denote a sequence of letters from $\mathcal{X}$. A random sequence $X^n$ and its distribution $p_{X^n}(x^n)$ are defined accordingly. For a pair of integers $i$ and $j$, $1 \leq i \leq j$, we write a discrete interval as $[i : j] = \{i, i+1, \ldots, j\}$.

The state of a quantum system $A$ is given by a density operator $\rho$ on the Hilbert space $\mathcal{H}_A$. A density operator is an Hermitian, positive semidefinite operator, with unit trace, $\text{Tr}\{\rho\} = 1$. A state of $\mathcal{H}_A$ is said to be pure if $\rho = \bra{\psi}\psi\rangle$, for some vector $|\psi\rangle \in \mathcal{H}_A$, where $\bra{\psi}$ is the Hermitian conjugate of $|\psi\rangle$. In general, a density operator has a spectral decomposition of the following form,

$$\rho = \sum_{z \in \mathcal{Z}} p_Z(z) |\psi_z\rangle \langle \psi_z|$$

where $\mathcal{Z} = \{1, 2, \ldots, |\mathcal{H}_A|\}$, $p_Z(z)$ is a probability distribution over $\mathcal{Z}$, and $\{|\psi_z\rangle\}_{z \in \mathcal{Z}}$ forms an orthonormal basis of the Hilbert space $\mathcal{H}_A$. The density operator can thus be thought of as an average of pure states. A measurement of a quantum system is any set of operators $\{\Lambda_j\}$ that forms a positive operator-valued measure (POVM), i.e. the operators are positive semi-definite and $\sum_j \Lambda_j = 1$, where $1$ is the identity operator (see [67, Definition 4.2.1]). According to the Born rule, if the system is in state $\rho$, then the probability of the measurement outcome $j$ is given by $p_A(j) = \text{Tr}(\Lambda_j \rho)$.

Define the quantum entropy of the density operator $\rho$ as

$$H(\rho) \triangleq -\text{Tr}[\rho \log(\rho)]$$

which is the same as the Shannon entropy associated with the eigenvalues of $\rho$. We may also consider the state of a pair of systems $A$ and $B$ on the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ of the corresponding Hilbert spaces. Given a bipartite state $\sigma_{AB}$, define the quantum mutual information by

$$I(A;B)_\sigma = H(\sigma_A) + H(\sigma_B) - H(\sigma_{AB}).$$
### Quantum Channel

A quantum channel maps a quantum state at the sender system to a quantum state at the receiver system. Here, we consider a channel that is governed by a random parameter with a particular distribution. Formally, a random-parameter quantum channel is defined as a linear, completely positive, trace preserving map $N_{SA \rightarrow B}$, corresponding to a quantum physical evolution. The channel parameter $S$ can also be thought of as a classical system at state

$$\rho_S = \sum_{s \in S} q(s)|s\rangle\langle s|$$

where $\{|s\rangle\}_{s \in S}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_S$. A quantum channel has a Kraus representation

$$N_{SA \rightarrow B}(\rho) = \sum_j N_j \rho_{SA} N_j^\dagger$$

for all $\rho_{SA}$, where the operators $N_j$ satisfy $\sum_j N_j^\dagger N_j = I$ [67, Section 4.4.1]. The projection on $|s\rangle$ is then given by

$$N_{A \rightarrow B}^{(s)}(\rho) = \sum_j N_j^{(s)} \rho_{N_j^{(s)}}$$

where $N_j^{(s)} = \langle s|N_j|s\rangle$. A quantum channel is called isometric if it can be expressed as $N_{SA \rightarrow B}(\rho) = N\rho_{SA}N^\dagger$ where the operator $N$ is an isometry, i.e. $N^\dagger N = I$ [67, Section 4.6.3].

We assume that both the random parameter state and the quantum channel have a product form. That is, the state of the joint system $S^n = (S_1, \ldots, S_n)$ is $\rho_{S^n} = \rho_{S^n}^{\otimes n}$, and if the systems $A^n = (A_1, \ldots, A_n)$ are sent through $n$ channel uses, then the parameter-input state $\rho_{S^n} \otimes \rho_{A^n}$ undergoes the tensor product mapping $N_{S^nA^n \rightarrow B^n} = N_{SA \rightarrow B}^{\otimes n}$. Therefore, without CSI, the input-output relation is

$$\rho_{B^n} = \sum_{s^n \in S^n} q^n(s^n)N_{A^n \rightarrow B^n}(\rho_{A^n}) = \left(\sum_{s \in S} q(s)N_{A \rightarrow B}^{(s)}\right)^\otimes n (\rho_{A^n})$$

where $q^n(s^n) = \prod_{i=1}^n q(s_i)$ is the joint distribution of the parameter sequence and $N_{A^n \rightarrow B^n} = N_{A \rightarrow B}^{(s_1)} \otimes \cdots \otimes N_{A \rightarrow B}^{(s_n)}$. The sender and the receiver are often referred to as Alice and Bob.
C. Coding

We define a code to transmit classical information provided that the encoder and the decoder share unlimited entanglement. The entangled system pairs are denoted by \((T^n_A, T^n_B)\) = \((T_{A,i}, T_{B,i})\)\(^{n}_{i=1}\). With causal CSI, Alice knows the sequence of past and present random parameters, \(S_1, \ldots, S_i\), at \(i \in [1 : n]\).

Definition 1. A \((2^nR, n)\) entanglement-assisted classical code with causal CSI at the encoder consists of the following: a message set \([1 : 2^nR]\), where \(2^nR\) is assumed to be an integer, a pure entangled state \(\Psi_{T^n_A, T^n_B}\), a sequence of \(n\) encoding maps \((\mathcal{E}_{T^n_{A,i} \rightarrow A, } m \in [1 : 2^nR], s^i \in S^i, \text{for } i \in [1 : n]\), and a decoding POVM \(\{\Lambda_{m^n_{B^n} T_B^n}\}_{m \in [1 : 2^nR]}\). We denote the code by \((\mathcal{E}, \Psi, \Lambda)\).

The communication scheme is depicted in Figure 1. The sender Alice has the systems \(T^n_A, A^n\) and the receiver Bob has the systems \(T^n_B, B^n\), where \(T^n_A\) and \(T^n_B\) are entangled. Alice chooses a classical message \(m \in [1 : 2^nR]\). At time \(i \in [1 : n]\), given the sequence of past and present parameters \(s^i \in S^i\), she applies the encoding channel \(\mathcal{E}^{m,s^i}_{T^n_{A,i} \rightarrow A_i}\) to her share of the entangled state \(\Psi_{T^n_{A,i}, T^n_{B,i}}\), and then transmits the system \(A_i\) over the channel. In other words, Alice uses an encoding channel \(\mathcal{E}^{m,s^n}_{T^n_A \rightarrow A^n}\) of the following form,

\[
\mathcal{E}^{m,s^n}_{T^n_A \rightarrow A^n} \triangleq \mathcal{E}^{m,s_1}_{T^n_{A,1} \rightarrow A_1} \otimes \mathcal{E}^{m,s_2}_{T^n_{A,2} \rightarrow A_2} \otimes \cdots \otimes \mathcal{E}^{m,s_n}_{T^n_{A,n} \rightarrow A_n},
\]

(10)

and transmits the systems \(A^n\) over \(n\) channel uses of \(N_{S \rightarrow A,B}\).

Bob receives the channel output systems \(B^n\), combines them with the entangled system \(T^n_B\), and performs the POVM \(\{\Lambda_{m^n_{B^n} T_B^n}\}_{m \in [1 : 2^nR]}\). The conditional probability of error, given that the message \(m\) was sent, is given by

\[
P_{e|m}(\mathcal{E}, \Psi, \Lambda) = \sum_{s^n \in S^n} q^n(s^n) \text{Tr} \left[(\mathbb{1} - \Lambda_{m^n_{B^n} T_B^n})(\mathcal{E}^{m,s^n}_{A^n \rightarrow B^n} \otimes \mathbb{1})(\mathcal{E}^{m,s^n}_{T^n_A \rightarrow A^n} \otimes \mathbb{1})\Psi_{T^n_A, T^n_B}\right].
\]

(11)

A \((2^nR, n, \varepsilon)\) entanglement-assisted classical code satisfies \(P_{e|m}(\mathcal{E}, \Psi, \Lambda) \leq \varepsilon\) for all \(m \in [1 : 2^nR]\). A rate \(R > 0\) is called achievable if for every \(\varepsilon > 0\) and sufficiently large \(n\), there exists a \((2^nR, n, \varepsilon)\) code. The entanglement-assisted classical capacity \(C_{E,caus}(N)\) is defined as the supremum of achievable rates.

Next, we give a definition of an entanglement-assisted quantum code. A more general definition can be found in [67].

Definition 2. A \((2^nQ, n)\) entanglement-assisted quantum code with causal CSI consists of the following: A quantum state \(\rho_{M^n}\), where \(M\) is a system of dimension \(2^nQ\), a pure entangled state \(\Psi_{T^n_A, T^n_B}\), a sequence of \(n\) encoding channels \(\mathcal{E}^{m,s^n}_{T^n_{A,i} \rightarrow A_i}\), and a decoding channel \(\mathcal{D}^{B^n T_B^n \rightarrow M^n}\).

The sender Alice has the systems \(T^n_A, M, A^n\) and the receiver Bob has the systems \(T^n_B, B^n, M\), where \(T^n_A\) and \(T^n_B\) are entangled. Alice encodes the state \(\rho_{M^n}\) by applying the encoding channel \(\mathcal{E}^{m,s^n}_{M^n, T^n_A \rightarrow A^n}\) to \(\rho_{M^n}\) and to her share of the entangled state \(\Psi_{T^n_A, T^n_B}\), where \(\mathcal{E}^{m,s^n}_{M^n, T^n_A \rightarrow A^n} = \bigotimes_{i=1}^n \mathcal{E}^{m,s_i}_{M_i, T_{A,i} \rightarrow A_i}\), and transmits the system \(A^n\) over \(n\) channel uses of \(N_{S \rightarrow A,B}\). Bob receives the channel output systems \(B^n\), combines them with the entangled system \(T^n_B\), and applies the decoding channel \(\mathcal{D}^{B^n T_B^n \rightarrow M^n}\). The code is said to be a \((2^nQ, n, \varepsilon)\) entanglement-assisted quantum code if the trace distance between the original state and the resulting state at the receiver is bounded by

\[
\frac{1}{2} \left\| \rho_{M^n} - \mathcal{D} \left(\sum_{s^n \in S^n} q^n(s^n)(\mathcal{N}_{A^n \rightarrow B^n} \otimes \mathbb{1})(\mathcal{E}^{m,s^n}_{A^n \rightarrow B^n} \otimes \mathbb{1})\right) \right\|_1 \leq \varepsilon,
\]

(12)

where \(\|\cdot\|_1\) denotes the trace norm. A positive number \(Q > 0\) is said to be an achievable rate if for every \(\varepsilon > 0\) and sufficiently large \(n\), there exists a \((2^nQ, n, \varepsilon)\) code. The entanglement-assisted quantum capacity \(Q_{E,caus}\) is defined as the supremum of achievable rates.

We also discuss the non-causal setting, where Alice has the parameter sequence \(S^n\) a priori, and can thus applies any encoding channel \(\mathcal{E}^{m,s^n}_{M^n, T^n_A \rightarrow A^n}\). In addition, we consider the case where there is CSI at the decoder, i.e. when Bob receives both \(B^n\) and \(S^n\), and performs a POVM \(\{\Lambda_{m^n_{B^n} S^n} T_B^n\}_{m \in [1 : 2^nR]}\). We note that for the decoder, causality is insignificant. We use the respective subscripts ‘E’-, ‘D’ or ‘ED’ to indicate that CSI is available at either the encoder, the decoder, or both, and the subscripts ‘caus’ or ‘n-c’ to indicate whether CSI is available at the encoder in a causal or non-causal manner, respectively. The notation is summarized in the table in Figure 2.

D. Related Work

We briefly review known results for a quantum channel that does not depend on a random parameter, i.e. \(\mathcal{N}^{(s)}_{A \rightarrow B} = \mathcal{N}^{(0)}_{A \rightarrow B}\) for \(s \in S\). Define

\[
C(\mathcal{N}^{(0)}) \triangleq \max_{\phi_{A,A'}} I(A;B)\_\phi
\]

(13)
| Capacity | CSI | none | encoder | decoder | encoder+decoder | encoder (causal) |
|----------|-----|-------|---------|---------|----------------|-----------------|
| Classical | C(\mathcal{N}) | C_{E,n-c}(\mathcal{N}) | C_{D}(\mathcal{N}) | C_{E,D,n-c}(\mathcal{N}) | C_{E,caus}(\mathcal{N}) |
| Quantum   | Q(\mathcal{N}) | Q_{E,n-c}(\mathcal{N}) | Q_{D}(\mathcal{N}) | Q_{E,D,n-c}(\mathcal{N}) | Q_{E,caus}(\mathcal{N}) |

Fig. 2. Notation of channel capacities with and without CSI. The columns correspond to the location where CSI is available, and the rows indicate the type of information capacity – classical or quantum.

with $\rho_{AB} \equiv (\mathbf{1} \otimes \mathcal{N}^{(0)})(|\phi\rangle\langle\phi|_{AA'})$. Next, we give the respective capacity theorems for the entanglement-assisted classical capacity and the entanglement-assisted quantum capacity.

**Theorem 1** (see [7, 8]). The entanglement-assisted classical capacity of a quantum channel $\mathcal{N}^{(0)}_{A\rightarrow B}$ is given by

$$C(\mathcal{N}^{(0)}) = C(\mathcal{N}^{(0)})$$

Given an unlimited supply of entanglement, the teleportation protocol can send a qubit using two classical bits, while the super-dense coding protocol can send two classical bits using one qubit [53]. This implies the following.

**Corollary 2** (see [7, 8]). The entanglement-assisted quantum capacity of a quantum channel $\mathcal{N}^{(0)}_{A\rightarrow B}$ is given by

$$Q(\mathcal{N}^{(0)}) = \frac{1}{2}C(\mathcal{N}^{(0)})$$

**Remark 1.** We note that the setting of a random-parameter quantum channel $\mathcal{N}^{(0)}_{S\rightarrow B}$ without side information is equivalent to that of a channel that does not depend on a state, with $\mathcal{N}^{(0)}_{A\rightarrow B} = \sum_{s \in S} q(s)\mathcal{N}^{(s)}_{A\rightarrow B}$ (see (9)). On the other hand, with side information at the encoder, this equivalence does not hold, as the channel input is correlated with the parameter sequence.

### III. INFORMATION THEORETIC TOOLS

To derive our results, we use the quantum version of the method of types properties and techniques. The basic definitions and lemmas that are used in this paper are given below.

**A. Classical Types**

The type of a classical sequence $x^n$ is defined as the empirical distribution $\hat{P}_{x^n}(a) = N(a|x^n)/n$ for $a \in \mathcal{X}$, where $N(a|x^n)$ is the number of occurrences of the symbol $a$ in the sequence $x^n$. The set of all types over $\mathcal{X}$ is then denoted by $P_n(\mathcal{X})$. The type class associated with a type $\hat{P} \in P_n(\mathcal{X})$ is defined as the set of sequences of that type, i.e.

$$\mathcal{T}(\hat{P}) \equiv \left\{ x^n \in \mathcal{X}^n : \hat{P}_{x^n} = \hat{P} \right\}$$

For a pair of sequences $x^n$ and $y^n$, we give similar definitions in terms of the joint type $\hat{P}_{x^n,y^n}(a,b) = N(a,b|x^n,y^n)/n$ for $a \in \mathcal{X}$, $b \in \mathcal{Y}$, where $N(a,b|x^n,y^n)$ is the number of occurrences of the symbol pair $(a,b)$ in the sequence $(x_i,y_i)_{i=1}^n$. Given a sequence $y^n \in \mathcal{Y}^n$, we further define the conditional type $\hat{P}_{x^n|y^n}(a|b) = N(a,b|x^n,y^n)/N(b|y^n)$ and the conditional type class

$$\mathcal{T}(\hat{P}|y^n) \equiv \left\{ x^n \in \mathcal{X}^n : \hat{P}_{x^n,y^n}(a,b) = \hat{P}_{y^n}(b)(\hat{P}(a|b)) \right\}$$

Given a probability distribution $p_X \in P(\mathcal{X})$, the $\delta$-typical set is defined as

$$\mathcal{A}(\delta,p_X) \equiv \left\{ x^n \in \mathcal{X}^n : \left| \hat{P}_{x^n}(a) - p_X(a) \right| \leq \delta \quad \text{if} \quad p_X(a) > 0 \right\}$$

$$\hat{P}_{x^n}(a) = 0 \quad \text{if} \quad p_X(a) = 0, \quad \forall \ a \in \mathcal{X}$$

The covering lemma is a powerful tool in classical information theory [16].

**Lemma 3** (Classical Covering Lemma [16][27, Lemma 3.3]). Let $X^n \sim \prod_{i=1}^n p_X(x_i)$, $\delta > 0$, and let $Z^n(m)$, $m \in [1 : 2^{nR}]$, be independent random sequences distributed according to $\prod_{i=1}^n p_Z(z_i)$. Suppose that the sequence $X^n$ is pairwise independent of the sequences $Z^n(m)$, $m \in [1 : 2^{nR}]$. Then,

$$\Pr \left( (Z^n(m),X^n) \notin \mathcal{A}(\delta,p_{Z,X}) \left( \text{for all} \ m \in [1 : 2^{nR}] \right) \right) \leq \exp \left( -2^{n(R - I(Z;X) - \epsilon_n(\delta))} \right)$$

where $\epsilon_n(\delta)$ tends to zero as $n \to \infty$ and $\delta \to 0$. 
Let $X^n \sim \prod_{i=1}^n p_X(x_i)$ be an information source sequence, encoded by an index $m$ at compression rate $R$. Based on the covering lemma above, as long as the compression rate is higher than $I(Z;X)$, a set of random codewords, $Z^n(m) \sim \prod_{i=1}^n p_Z(z_i)$, contains with high probability at least one sequence that is jointly typical with the source sequence.

Though originally stated in the context of lossy source coding, the classical covering lemma is useful in a variety of scenarios [27], including the random-parameter channel with non-causal CSI. In this case, the parameter sequence $S^n \sim \prod_{i=1}^n q(s_i)$ plays the role of the “source sequence”.

### B. Quantum Typical Subspaces

Moving to the quantum method of types, suppose that the state of a system is generated from an ensemble $\{p_X(x), |x\rangle\}_{x \in \mathcal{X}}$, hence, the average density operator is

$$\rho = \sum_{x \in \mathcal{X}} p_X(x) |x\rangle \langle x|.$$  \hfill (20)

Consider the subspace spanned by the vectors $|x^n\rangle$, $x^n \in \mathcal{T}(\hat{P})$, for a given type $\hat{P} \in \mathcal{T}_n(\mathcal{X})$. Then, the projector onto the subspace is given by

$$\Pi_{A^n}(\hat{P}) = \sum_{x^n \in \mathcal{T}(\hat{P})} |x^n\rangle \langle x^n|.$$  \hfill (21)

Note that the dimension of the subspace of type class $\hat{P}$ is given by $\text{Tr}(\Pi_{A^n}(\hat{P})) = |\mathcal{T}(\hat{P})|$. By classical type properties [16, Lemma 2.3] (see also [67, Property 15.3.2]),

$$(n+1)^{\mathcal{X}|2^nH(\rho)} \leq \text{Tr}(\Pi_{A^n}(\hat{P})) \leq 2^nH(\rho).$$  \hfill (22)

The projector onto the $\delta$-typical subspace is defined as

$$\Pi^\delta(\rho) = \sum_{x^n \in \mathcal{A}^\rho(p_X)} |x^n\rangle \langle x^n|.$$  \hfill (23)

Based on [55] [53, Theorem 12.5], for every $\varepsilon, \delta > 0$ and sufficiently large $n$, the $\delta$-typical projector satisfies

$$\text{Tr}(\Pi^\delta(\rho) \rho^{\otimes n}) \geq 1 - \varepsilon$$  \hfill (24)

$$2^{-n(H(\rho) + c\delta)} \Pi^\delta(\rho) \leq \Pi^\delta(\rho) \rho^{\otimes n} \Pi^\delta(\rho) \leq 2^{-n(H(\rho) - c\delta)}$$  \hfill (25)

$$\text{Tr}(\Pi^\delta(\rho)) \leq 2^{n(H(\rho) + c\delta)}$$  \hfill (26)

where $c > 0$ is a constant.

To prove achievability for Theorem 1 above, one may invoke the quantum packing lemma [37, 67]. Suppose that Alice employs a quantum codebook that consists of $2^{nR}$ “codewords” $x(m)$, $m \in [1 : 2^{nR}]$, by which she chooses a state from an ensemble $\{p_X(x), \rho_x\}_{x \in \mathcal{X}}$. The proof is based on random codebook generation, where the codewords are drawn at random according to an input distribution $p_X(x)$. To recover the transmitted message, Bob may perform the square-root measurement [34, 57] using a code projector $\Pi$ and codeword projectors $\Pi_x$, $x \in \mathcal{X}$, which project onto subspaces of the Hilbert space $\mathcal{H}$.

The lemma below is a simplified, less general, version of the quantum packing lemma by Hsieh, Devetak, and Winter [37].

**Lemma 4 (Quantum Packing Lemma [37, Lemma 2])**. Let $\sigma_{AB}$ be a joint state on the product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, such that

$$\sigma_B = \sum_{x \in \mathcal{X}} p_X(x) \rho_x$$  \hfill (27)

where $\{p_X(x), \rho_x\}_{x \in \mathcal{X}}$ is a given random ensemble on $\mathcal{H}_B$. Furthermore, suppose that there is a code projector $\Pi$ and codeword projectors $\Pi_x$, $x \in \mathcal{X}$, that satisfy the following

$$\text{Tr}(\Pi_{\rho_x}) \geq 1 - \alpha$$  \hfill (28)

$$\text{Tr}(\Pi_{x\rho_x}) \geq 1 - \alpha$$  \hfill (29)

$$\text{Tr}(\Pi_x) \leq 2^{n(H(\sigma_{AB}) + \alpha)}$$  \hfill (30)

$$\Pi \sigma_A \Pi \leq 2^{-n(H(\sigma_{AB}) + H(\sigma_B) - \alpha)}$$  \hfill (31)

for some $\alpha > 0$. Then, there exist codewords $x(m)$, $m \in [1 : 2^{nR}]$, and a POVM $\{\Lambda_m\}_{m \in [1 : 2^{nR}]}$ such that

$$\text{Tr} \left( \Lambda_m p_{x(m)} \right) \geq 1 - 2^{-n[I(A;B)_0 - R - \varepsilon_n(\alpha)]}$$  \hfill (32)

for all $m \in [1 : 2^{nR}]$, where $\varepsilon_n(\alpha)$ tends to zero as $n \to \infty$ and $\alpha \to 0$.

In our analysis, where there is non-causal CSI at the encoder, we apply the packing lemma such that the quantum ensemble encodes both the message $m$ and a compressed representation of the parameter sequence $s^n$. 
IV. MAIN RESULTS

We give our results on the random-parameter quantum channel \( N_{SA\rightarrow B} \) with CSI at the encoder.

A. Causal Side Information at the Encoder

We begin with our main result on the random-parameter quantum channel with causal CSI. Define

\[
C_{\text{caus}}(N) \triangleq \max_{\theta_{K^A'}, \mathcal{F}^{(s)}_{K\rightarrow A}} I(K; B)_{\omega}
\]  

(33)

where the maximization is over the quantum state \( \theta_{K^A'} \) and the set of quantum channels \( \{\mathcal{F}^{(s)}_{K\rightarrow A}\}_{s \in S} \), with

\[
\omega_{A'A'}^{s} = (\mathcal{F}^{(s)} \otimes 1)(\theta_{K^A'})
\]

(34)

\[
\omega_{ASA'} = \sum_{s \in S} q(s)|s\rangle\langle s| \otimes \omega_{A'A'}^{s}
\]

(35)

\[
\omega_{AB} = (1 \otimes N)(\omega_{ASA'}).
\]

(36)

Before we state the capacity theorem, we give the following lemma.

Lemma 5. The maximization in (33) can be restricted to pure states \( \theta_{K^A'} = \langle \xi_{K^A'} | \xi_{K^A'} \rangle \).

Lemma 5 follows by state purification [67, Exercise 13.4.4]. The proof is given in Appendix A. Now, we give our main result.

Theorem 6. The entanglement-assisted classical capacity of the random-parameter quantum channel \( N_{SA\rightarrow B} \) with causal CSI at the encoder is given by

\[
C_{\text{caus}}(N) = C_{\text{caus}}(N).
\]

(37)

The proof of Theorem 6 is given in Appendix B. To prove achievability, we apply the random coding techniques from [7, 8] to the virtual channel \( \mathcal{M}_{K\rightarrow B} \), defined by

\[
\mathcal{M}(\rho_K) = \sum_{s \in S} q(s)N^{(s)} \left( \mathcal{F}^{(s)}(\rho_K) \right).
\]

(38)

As without side information, a qubit is exchangeable with two classical bits, given unlimited entanglement. This follows by applying the teleportation protocol and the super-dense coding protocol (see [53, Sections 1.3.7, 2.3] and also [67, Chapter 6]). As a consequence, we can characterize the entanglement-assisted quantum capacity as well.

Theorem 7. The entanglement-assisted quantum capacity of the random-parameter quantum channel \( N_{SA\rightarrow B} \) with causal CSI at the encoder is given by

\[
Q_{\text{caus}}(N) = \frac{1}{2} C_{\text{caus}}(N).
\]

(39)

B. Non-Causal Side Information at the Encoder

The entanglement-assisted capacity of a quantum channel with non-causal CSI was determined by Dupuis [25, 26]. Here, we use an alternative proof approach, which yields an equivalent formulation and further observations. Define

\[
C_{E,n-c}(N) \triangleq \max_{\theta_{K^A'}, \mathcal{F}^{(s)}_{K\rightarrow A}} [I(A; B)_{\omega} - I(A; S)_{\omega}]
\]

(40)

where the maximization is over the quantum state \( \theta_{K^A'} \) and the set of quantum channels \( \{\mathcal{F}^{(s)}_{K\rightarrow A}\}_{s \in S} \), with

\[
\omega_{A'A'}^{s} = (\mathcal{F}^{(s)} \otimes 1)(\theta_{K^A'})
\]

(41)

\[
\omega_{ASA'} = \sum_{s \in S} q(s)|s\rangle\langle s| \otimes \omega_{A'A'}^{s}
\]

(42)

\[
\omega_{AB} = (1 \otimes N)(\omega_{ASA'}).
\]

(43)

Before we state the capacity theorem, we give the following lemma.

Lemma 8. The maximization in (40) can be restricted to pure states \( \theta_{K^A'} = \langle \xi_{K^A'} | \xi_{K^A'} \rangle \) and isometric channels \( \mathcal{F}^{(s)}_{K\rightarrow A}(\rho_A) = F^{(s)}\rho_A F^{(s)\dagger} \).

The proof of Lemma 8 is given in Appendix C, using state purification and isometric channel extension. Not only Lemma 8 simplifies the calculation of the formula in (40), but it will also be useful in our proof for the theorem below.
The entanglement-assisted classical capacity of the random-parameter quantum channel $\mathcal{N}_{SA \rightarrow B}$ with non-causal CSI at the encoder is given by

$$C_{E,n-c}(\mathcal{N}) = C_{E,n-c}(\mathcal{N}).$$

Theorem 9 (also in [25, 26]). The entanglement-assisted classical capacity of the random-parameter quantum channel $\mathcal{N}_{SA \rightarrow B}$ with non-causal CSI at the encoder is given by

$$Q_{E,n-c}(\mathcal{N}) = \frac{1}{2} C_{E,n-c}(\mathcal{N}).$$

Theorem 10 (also in [25, 26]). The entanglement-assisted quantum capacity of the random-parameter quantum channel $\mathcal{N}_{SA \rightarrow B}$ with non-causal CSI at the encoder is given by

$$Q_{E,n-c}(\mathcal{N}) = \frac{1}{2} C_{E,n-c}(\mathcal{N}).$$

In [25, 26], Dupuis applied the decoupling approach to prove Theorem 10, and then, obtained the classical capacity theorem, Theorem 9, as a consequence. The decoupling approach shows that qubits can be transmitted by decoupling between the encoder’s reference system and the output system. Here, we have taken a more direct approach and devised a coding scheme for the transmission of classical information.

C. Side Information at the Decoder

In this subsection, we consider a random-parameter quantum channel $\mathcal{N}_{SA \rightarrow B}$ with CSI at the decoder. That is, Bob receives both $B^n$ and $S^n$, and performs a POVM $\{\Lambda_{B^nS^nT_B}^m\}_{m \in [1:2^n]}$. The results in this subsection are a straightforward consequence of the results above.

First, suppose that only Bob is aware of the channel parameter sequence, and define

$$C_D(\mathcal{N}) = \max_{|\phi\rangle, A^A} I(A; B | S)_\rho$$

with

$$\rho_{SAB} \equiv \sum_{s \in S} q(s) |s\rangle \langle s| \otimes (1 \otimes \mathcal{N}^s)|\phi\rangle \langle \phi|_{AA'}.$$ (47)

Corollary 11. The entanglement-assisted classical capacity of the random-parameter quantum channel $\mathcal{N}_{SA \rightarrow B}$ with CSI at the decoder is given by

$$C_D(\mathcal{N}) = C_D(\mathcal{N})$$

and the entanglement-assisted quantum capacity is given by $Q_{D}(\mathcal{N}) = \frac{1}{2} C_D(\mathcal{N})$.

Corollary 11 is a straightforward consequence of Theorem 1, following the observation that the channel parameter $S$ can be thought of as part of the output system in this setting. That is, the capacity of a channel $\mathcal{N}_{SA \rightarrow B}$ with CSI at the decoder is the same as that of a channel $\mathcal{M}'_{A \rightarrow S,B}$ without parameters, where

$$\mathcal{M}'_{A \rightarrow S,B}(\rho_A) = \sum_{s \in S} q(s) |s\rangle \langle s| \otimes \mathcal{N}^s_{A \rightarrow B}(\rho_A).$$

Hence,

$$C_D(\mathcal{N}) = C(\mathcal{M}') = \max_{|\phi\rangle, A^A} I(A; B, S)_\rho = \max_{|\phi\rangle, A^A} I(A; B | S)_\rho$$

with $\rho_{SAB}$ as in (47), where the last equality holds by the chain rule and since $I(A; S)_\rho = 0$ given that the Alice is not aware of the channel parameter.

Now, suppose that both Alice and Bob are aware of the channel parameter sequence. Then, as explained above, the channel parameter $S$ can be thought of as part of the channel output in this case. Thus, the corollary below immediately follows from Theorem 9. Define

$$C_{ED,n-c}(\mathcal{N}) \triangleq \max I(A; B | S)_{\omega}$$

where the maximization is as in (40).

Corollary 12. The entanglement-assisted classical capacity of the random-parameter quantum channel $\mathcal{N}_{SA \rightarrow B}$ with non-causal CSI at both the encoder and the decoder is given by

$$C_{ED,n-c}(\mathcal{N}) = C_{ED,n-c}(\mathcal{N})$$

and the entanglement-assisted quantum capacity is given by $Q_{ED,n-c}(\mathcal{N}) = \frac{1}{2} C_{ED,n-c}(\mathcal{N})$.

Based on our result in Theorem 6, we observe that the same capacity formula is valid for causal CSI as well. To show achievability, set $F^{(s)}$ to be clean, i.e. $F^{(s)}(\rho) = \rho$ for $s \in S$. The converse part follows from Corollary 12, since the capacity with non-causal CSI is always an upper bound on the capacity with causal CSI.
D. Discussion

We give a few remarks on the results above. There is clear similarity between the capacity formulas (2) and (40) given non-causal CSI. In particular, it can be seen that the classical variables $U$ and $X$ in (2) are replaced by the quantum systems $A$ and $A'$ in (40), respectively. For the classical formula (2), as shown in [28, 33], the maximization can be restricted to distributions $p_{U,X|S} = p_U|S p_{X|U,S}$ such that $p_{X|U,S}$ is a 0-1 probability law, based on simple convexity arguments. The property stated in Lemma 8 can thus be viewed as the quantum counterpart.

As for causal CSI, we observe that as in Shannon’s classical proof for a classical channel with causal CSI [59, 41, Section 3.1], our communication scheme can be interpreted as coding for a virtual channel $\mathcal{M}$, where the auxiliary plays the role of the channel input. Another similar trait is that at time $i$, the encoder applies a mapping that depends on the present $s_i$, while ignoring the sequence of past parameters, $s_1, \ldots, s_{i-1}$. In the classical setting, the mapping is the Shannon strategy $T(s_i)$, while in the quantum setting, it is the quantum channel $\mathcal{F}_{K \rightarrow A}$.

The classical capacity formula (1) for a classical channel with causal CSI can also be expressed as in (2), constrained such that $U$ and $S$ are in a product state. Nonetheless, we observe that in the analysis, the causality requirement also dictates that Alice applies the encoding operations in a different order compared to that of our coding scheme with non-causal CSI (see Remark 2).

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APPENDIX A

Proof of Lemma 5

Fix the quantum state $\theta_{K,A'}$ and channels $\mathcal{F}_{K \rightarrow A}^{(s)}$, $s \in S$, such that

$$C_{E,caus}(\mathcal{N}) = I(K;B)_\omega$$

and consider the spectral decomposition,

$$\theta_{K,A'} = \sum_{x \in X} \sum_{z \in Z} p_{X,Z}(x,z)|x\rangle\langle x| \otimes |z\rangle\langle z|$$

where $P_{X,Z}(x,z)$ is a probability distribution, while $\{|x\rangle\}_{x \in X}$ and $\{|z\rangle\}_{z \in Z}$ are orthonormal bases of the Hilbert spaces $\mathcal{H}_K$ and $\mathcal{H}_{A'}$, respectively.

To show that maximizing over pure states is sufficient, we perform purification of the state $\theta_{K,A'}$. Specifically, define the pure state

$$|\xi_{K,A'}\rangle = \sum_{x \in X} \sum_{z \in Z} \sqrt{p_{X,Z}(x,z)} |x\rangle \otimes |\psi_z\rangle \otimes |z\rangle$$

where $J$ is a reference system and $|\psi_z\rangle$ are orthonormal vectors in $\mathcal{H}_J$. Observe that $|\xi_{K,A'}\rangle$ is a purification of the mixed state $\theta_{K,A'}$, namely, $\theta_{K,A'} = \text{Tr}_J(|\xi_{K,A'}\rangle\langle \xi_{K,A'}|)$. Defining $\mathcal{F}_{K \rightarrow A}^{(s)} = (\mathcal{F}_{K \rightarrow A}^{(s)} \otimes \mathbb{I})$ and $\tilde{K} = (K,J)$, we have that $C_{E,caus}(\mathcal{N}) \geq I(\tilde{K};B)$ by the definition in (33). Yet, by the chain rule for the quantum mutual information [67, Theorem 11.7.1], $C_{E,caus}(\mathcal{N}) = I(K;B) \leq I(K,J;B) = I(\tilde{K};B)$. Hence, $C_{E,caus}(\mathcal{N}) = I(\tilde{K};B)$. Thereby, $\theta_{K,A'}$ can be replaced by the pure state $|\xi_{K,A'}\rangle$.

APPENDIX B

Proof of Theorem 6

A. Achievability Proof

We show that for every $\varepsilon_0, \delta_0 > 0$, there exists a $(2^nR, n, \varepsilon_0)$ code for the random-parameter quantum channel $\mathcal{N}_{S,A \rightarrow B}$ with causal CSI, provided that $R < C_{caus}(\mathcal{N}) - \delta_0$. Based on Lemma 5, it suffices to consider a pure entangled state. Hence, let $|\xi_{KB}\rangle$ be a pure entangled state, and $\mathcal{F}_{K \rightarrow A}^{(s)}(\rho_K)$, $s \in S$, be a set of isometric channels. Suppose that Alice and Bob share the joint state $|\xi_{KB}\rangle^\otimes n$. Define the channel $\mathcal{M}_{K \rightarrow B'}$ by

$$\mathcal{M}(\rho_K) = \sum_{s \in S} q(s)\mathcal{N}^{(s)}(\mathcal{F}^{(s)}(\rho_K))$$

(56)
and consider the Schmidt decomposition of the state,
\[
|\xi_{KB}\rangle = \sum_{x \in X} \sqrt{p_X(x)} |x\rangle \otimes |\psi_x\rangle
\]  
(57)
where \(p_X\) is a probability distribution, \(|x\rangle\) is an orthonormal basis of \(\mathcal{H}_A\), and \(|\psi_x\rangle\) are orthonormal vectors in \(\mathcal{H}_B\).

The code construction, encoding and decoding procedures are described below.

1) Code Construction:
(i) Select \(2^{nR}\) independent sequences \(x^n(m)\) at random, each according to \(\prod_{i=1}^n p_X(x_i)\).
(ii) Quantum Operators: Consider the Heisenberg-Weyl operators \(\{\Sigma(a,b) = X(a)Z(b)\}\) of dimension \(D\), given by

\[
X(a) = \sum_{j=0}^{D-1} |a \oplus j\rangle \langle j|
\]  
(58)
\[
Z(b) = \sum_{j=0}^{D-1} e^{2\pi i bj/D} |j\rangle \langle j|
\]  
(59)
for \(a, b \in \{0, 1, \ldots, D - 1\}\), where \(a \oplus j = (a + j) \mod D\) and \(i = \sqrt{-1}\). For every type class \(T_n(t)\) in \(X^n\), define the operators

\[
V_t(a_t, b_t, c_t) = (-1)^{c_t} \Sigma(a_t, b_t), \quad a_t, b_t \in \{0, 1, \ldots, D_t - 1\}, \quad c_t \in \{0, 1\}
\]  
(60)
where \(D_t = |T_n(t)|\) is the size of type class of \(t\). Define the operator

\[
U(\gamma) = \bigoplus_t V_t(a_t, b_t, c_t)
\]  
(61)
with \(\gamma = ((a_t, b_t, c_t)_t)\), and let \(\Gamma\) denote the set of all possible vectors \(\gamma\). Then, choose \(2^{nR}\) vectors \(\gamma(m), m \in [1:2^{nR}]\), uniformly at random.

2) Encoding and Decoding: The coding scheme is depicted in Figure 3. To send a message \(m \in [1:2^{nR}]\), given a parameter sequence \(s^n \in S^n\), Alice performs the following.
(i) Apply the operator \(U(\gamma(m))\) to \(|\xi_{KB} \rangle \otimes n\), which yields

\[
|\varphi_{K^nB^n}^m\rangle \equiv (U(\gamma(m) \otimes 1))|\xi_{KB} \rangle \otimes n.
\]  
(62)
(ii) Then, at time \(i \in [1:n]\), apply the channel \((F(s_i) \otimes 1)\) to \(|\varphi_{K^nB^n}^m\rangle\), and send the system \(A_i\) through the channel.
Bob receives the systems \(B^n\) at state \(\omega_{B^nB^n}\) and decodes the message by applying a POVM \(\{A_m\}_{m \in [1:2^{nR}]}\), which will be specified later.
3) **Code Properties:** First, we write the entangled states as a combination of maximally entangled states over the typical subspaces, and then we can use the following useful identities. For a maximally entangled state \( |\Phi_{AB}\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} |j\rangle_A \otimes |j\rangle_B \), \( \text{Tr}_B (|\Phi_{AB}\rangle \langle \Phi_{AB}|) = \pi_A \) \hspace{1cm} (63)

where \( \pi_A = \frac{1}{D} \sum_{x \in \mathcal{X}} |x\rangle\langle x| \) is the maximally mixed state. Furthermore, for every state \( \rho \) of the system \( A \), \( \frac{1}{D^2} \sum_{a=0}^{D-1} \sum_{b=0}^{D-1} \Sigma(a, b) \rho \Sigma^\dagger (a, b) = \pi_A \) \hspace{1cm} (64)

(see e.g. [7] [67, Exercise 4.7.6]). Another useful identity is the “ricochet property” [37, Eq. (17)]. \( \hspace{1cm} (U \otimes \mathbb{I}) |\Phi_{AB}\rangle = (\mathbb{I} \otimes U^T) |\Phi_{AB}\rangle \). \hspace{1cm} (65)

Now,

\[ |\xi_{K,B}\rangle \otimes^n = \sum_{x^n \in \mathcal{X}^n} \sqrt{p_{X^n}(x^n)} |x^n\rangle \otimes |\psi_{x^n}\rangle \]

where \( p_{X^n}(x^n) = \prod_{i=1}^{n} p_X(x_i) \) and \( |\psi_{x^n}\rangle = |\psi_{x_1}\rangle \otimes |\psi_{x_2}\rangle \otimes \cdots \otimes |\psi_{x_n}\rangle \). As the space \( \mathcal{X}^n \) can be partitioned into type classes, we may write

\[ |\xi_{K,B}\rangle \otimes^n = \sum_{t \in \mathcal{P}_n(\mathcal{X})} \sqrt{P(t)} |\Phi_t\rangle , \]

where

\[ P(t) = d_t \cdot p_{X^n}(x^n_t) , \text{ with } d_t \equiv |\mathcal{T}_n(t)| \]

\[ |\Phi_t\rangle = \frac{1}{\sqrt{d_t}} \sum_{x^n \in \mathcal{T}_n(t)} |x^n\rangle \otimes |\psi_{x^n}\rangle \] \hspace{1cm} (69)

We note that \( P(t) \) is the probability of the type \( \mathcal{T}_n(t) \) for a classical random sequence \( X^n \sim p_{X^n} \).

Now, Alice applies the operator \( U(\gamma(m)) \) to the entangled states. Since the state \( |\Phi_t\rangle \) is maximally entangled, we have by the “ricochet property” (65) that

\[ |\varphi_{K,B}^m\rangle \equiv (U(\gamma(m)) \otimes \mathbb{I}) |\xi_{K,B}\rangle \otimes^n = (\mathbb{I} \otimes U^T(\gamma(m))) |\xi_{K,B}\rangle \otimes^n . \] \hspace{1cm} (70)

That is, Alice’s unitary operations can be reflected and treated as if performed by Bob. Then, Alice applies the channels \( \mathcal{F}(s_i) \) to her share of \( |\varphi_{K,B}^m\rangle \).

Subsequently, Bob receives the systems \( B^n \) at state

\[ \rho_{B^n,B^n} = \sum_{s^n \in S^n} q^n(s^n)(\mathcal{N}(s^n) \otimes \mathbb{I})(\mathcal{F}(s^n) \otimes \mathbb{I})(|\varphi_{K,B}^m\rangle \langle \varphi_{K,B}^m|) \]

\[ = (\mathcal{M} \otimes \mathbb{I}) \left( (\mathbb{I} \otimes U^T(\gamma(m))) |\xi_{K,B}\rangle \langle \xi_{K,B}| \otimes^n (\mathbb{I} \otimes U^*(\gamma(m))) \right) \] \hspace{1cm} (71)

where the last line is due to (70). Since a quantum channel is a linear map, the above can be written as

\[ \rho_{B^n,B^n} = (\mathbb{I} \otimes U^T(\gamma)) \left[ (\mathcal{M} \otimes \mathbb{I})(|\xi_{K,B}\rangle \langle \xi_{K,B}|) \otimes^n \right] (\mathbb{I} \otimes U^*(\gamma)) \]

\[ = (\mathbb{I} \otimes U^T(\gamma)) \omega_{B^n}(\mathbb{I} \otimes U^*(\gamma)) \] \hspace{1cm} (72)

where we have defined

\[ \omega_{B^n} = (\mathcal{M} \otimes \mathbb{I})(|\xi_{K,B}\rangle \langle \xi_{K,B}|) . \] \hspace{1cm} (73)
4) Packing Lemma Requirements: Next, we use the quantum packing lemma. Consider the ensemble \( \left\{ \rho(\gamma) = \frac{1}{|\Gamma|} \rho^{\gamma}_{B^n,B^n} \right\} \), for which the expected density operator is \[
\sigma_{B^n,B^n} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \rho^{\gamma}_{B^n,B^n} . \tag{75}\]

Define the code projector and the codeword projectors by \( \Pi \equiv \Pi^\delta(\omega_{B'}) \otimes \Pi^\delta(\omega_B) \) (76) \( \Pi_\gamma \equiv (\mathbb{I} \otimes U^T(\gamma)) \Pi^\delta(\omega_{B'}) (\mathbb{I} \otimes U^*(\gamma)) \), for \( \gamma \in \Gamma \) (77)

where \( \Pi^\delta(\omega_{B'}) \), \( \Pi^\delta(\omega_B) \) are the projectors onto the \( \delta \)-typical subspaces associated with the states \( \omega_{B'} \), \( \omega_B = \text{Tr}_{B'}(\omega_{B'B'}) \) and \( \omega_B = \text{Tr}_{B'}(\omega_{B'B'}) \), respectively (see (74)). Now, we verify that the assumptions of Lemma 4 hold with respect to the ensemble and the projectors above.

First, we show that \( \text{Tr}(\Pi_\gamma \rho^{\gamma}_{B^n,B^n}) \geq 1 - \alpha \), where \( \alpha > 0 \) is arbitrarily small. Defining \( \tilde{P} = \mathbb{I} - P \), we have that \[
\Pi = (1 - \tilde{P}^\delta(\rho_{B'})) \otimes (1 - \tilde{P}^\delta(\rho_B)) \geq (\mathbb{I} \otimes \mathbb{I}) - (\tilde{P}^\delta(\rho_{B'}) \times \mathbb{1}) - (\mathbb{1} \otimes \tilde{P}^\delta(\rho_B)) \tag{78}\]
hence, 
\[
\text{Tr}(\Pi_\gamma \rho^{\gamma}_{B^n,B^n}) \geq 1 - \text{Tr} \left( (\tilde{P}^\delta(\rho_{B'}) \otimes \mathbb{I}) \rho^{\gamma}_{B^n,B^n} \right) - \text{Tr} \left( (\mathbb{I} \otimes \tilde{P}^\delta(\rho_B)) \rho^{\gamma}_{B^n,B^n} \right) 
= 1 - \text{Tr} \left( \tilde{P}^\delta(\rho_{B'}) \rho^{\gamma}_{B^n,B^n} \right) - \text{Tr} \left( \tilde{P}^\delta(\rho_B) \rho^{\gamma}_{B^n,B^n} \right) . \tag{79}\]
The first trace term in the RHS of (79) equals \( \text{Tr} \left( \tilde{P}^\delta(\rho_{B'}) \omega^{\otimes n}_{B'} \right) \) by (73), and the last term equals \( \text{Tr} \left( \tilde{P}^\delta(\rho_B) \omega^{\otimes n}_B \right) \) by (71) and (74). Therefore, we have by (24) that 
\[
\text{Tr}(\Pi_\gamma \rho^{\gamma}_{B^n,B^n}) \geq 1 - \text{Tr} \left( \tilde{P}^\delta(\rho_{B'}) \omega^{\otimes n}_{B'} \right) - \text{Tr} \left( \tilde{P}^\delta(\rho_B) \omega^{\otimes n}_B \right) 
\geq 1 - 2 \varepsilon . \tag{80}\]

Similarly, the second requirement of the packing lemma holds since 
\[
\text{Tr}(\Pi_\gamma \rho^{\gamma}_{B^n,B^n}) = \text{Tr} \left[ (\mathbb{I} \otimes U^T(\gamma)) \Pi^\delta(\omega_{B'B'}) (\mathbb{I} \otimes U^*(\gamma)) \omega^{\otimes n}_{B'B'} (\mathbb{1} \otimes U^*(\gamma)) \right]
\geq \text{Tr}(\Pi^\delta(\omega_{B'B'}) \omega^{\otimes n}_{B'B'}) \geq 1 - \varepsilon \tag{81}\]
where the second equality follows from the cyclicity of the trace and the fact that \( U^* U^T = (U U^T)^* = \mathbb{1} \) for a unitary operator, and the last inequality is due to (24).

Moving to the third requirement in Lemma 4, 
\[
\text{Tr}(\Pi_\gamma) = \text{Tr} \left( (\mathbb{I} \otimes U^T(\gamma)) \Pi^\delta(\omega_{B'B'}) (\mathbb{I} \otimes U^*(\gamma)) \right) = \text{Tr}(\Pi^\delta(\omega_{B'B'})) \leq 2^{n(H(\omega_{B'n})+\delta)} \tag{82}\]
where the second equality holds by cyclicity of the trace and the last inequality is due to (26). It is left to verify that the last requirement of the packing lemma holds, i.e. \( \Pi \sigma_{B^n,B^n} \Pi \leq 2^{-n(H(\sigma_{B^n})+H(\sigma_B)-\alpha))} \Pi \). To this end, observe that by (72) and (75), 
\[
\sigma_{B^n,B^n} = (\mathcal{M}^{\otimes n} \otimes \mathbb{1}) \tau_{K^n,B^n} \tag{83}\]
where we have defined \( \tau_{K^n,B^n} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \left( \mathbb{1} \otimes U^T(\gamma) \right) \langle \xi_{KB} \rangle \langle \xi_{KB} \rangle^{\otimes n} \left( \mathbb{1} \otimes U^*(\gamma) \right) . \tag{84} \)

Then, by (70) along with (60)-(61), 
\[
\tau_{K^n,B^n} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \left( \mathbb{1} \otimes U^T(\gamma) \right) \left( \sum_t \sqrt{P(t|s^n)} \Phi_t \right) \left( \sum_{t'} \sqrt{P(t'|s^n)} \Phi_{t'} \right) \left( \mathbb{1} \otimes U^*(\gamma) \right) 
= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \left( \sum_t \sqrt{P(t)}(-1)^{c_t(\gamma)} \left( \mathbb{1} \otimes \sum_{a_t(\gamma), b_t(\gamma)} \Phi_t \right) \right) 
\left( \sum_{t'} \sqrt{P(t')}(-1)^{c_{t'}(\gamma)} \left( \mathbb{1} \otimes \sum_{a_{t'}(\gamma), b_{t'}(\gamma)} \Phi_{t'} \right) \right) . \tag{85}\]
For $t' = t$, the expression above becomes

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{t} P(t) (\mathbb{1} \otimes \sum_{a(t),b(t)}^T |\Phi_t\rangle \langle \Phi_t| (\mathbb{1} \otimes \sum_{a(t),b(t)}^{*})$$

$$\sum_{t} P(t) |s^n\rangle \sum_{a_t,b_t} (\mathbb{1} \otimes \sum_{a_t,b_t}^T |\Phi_t\rangle \langle \Phi_t| (\mathbb{1} \otimes \sum_{a_t,b_t}^{*}) = \sum_{t} P(t) \pi_{K^n}^t \times \pi_{B^n}^t \tag{86}$$

with

$$\pi_{K^n}^t = \frac{\Pi_{K^n}(t)}{\text{Tr}(\Pi_{K^n}(t))}, \quad \pi_{B^n}^t = \frac{\Pi_{B^n}(t)}{\text{Tr}(\Pi_{B^n}(t))} \tag{87}$$

where $\Pi_{K^n}(t)$ is the projector of type $t$ as defined in (21). The last equality in (86) follows from (64). On the other hand, for $t' \neq t$,

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{t} \sum_{t' \neq t} \sqrt{P(t)P(t')} \frac{1}{4D_t^2 D_{t'}} \sum (-1)^{c_t+c_{t'}} \sum_{a_t,a_t',b_t,b_{t'}} (\mathbb{1} \otimes \sum_{a_t,b_t}^T |\Phi_{t'}\rangle \langle \Phi_{t'}| (\mathbb{1} \otimes \sum_{a_{t'},b_{t'}}^{*}) = 0. \tag{88}$$

We deduce from (85)-(88) that $\tau_{K^n,B^n} = \sum_t P(t) \pi_{K^n}^t \times \pi_{B^n}^t$. Plugging this into (83) yields

$$\sigma_{B^n,B^n} = \sum_t P(t) \mathcal{M}^\otimes(n(\pi_{K^n}^t) \otimes \pi_{B^n}^t). \tag{89}$$

Now, we use the formula above in order to show that the last requirement in Lemma 4 holds. Consider that

$$\Pi \sigma_{B^n,B^n} = (\Pi^t(\omega_B) \otimes \Pi^t(\omega_B)) \sigma_{B^n,B^n} (\Pi^t(\omega_B) \otimes \Pi^t(\omega_B))$$

$$= \sum_t P(t) \left( (\Pi^t(\omega_B) \mathcal{M}^\otimes(n(\pi_{K^n}^t) \Pi^t(\omega_B))) \otimes (\Pi^t(\omega_B) \Pi_{B^n}(t) \Pi^t(\omega_B)) \right). \tag{90}$$

Using (87), this can be bounded by

$$\Pi \sigma_{B^n,B^n} \leq 2^{-n(H(\omega_B) + \varepsilon_1)} \sum_t P(t) \left( (\Pi^t(\omega_B) \mathcal{M}^\otimes(n(\pi_{K^n}^t) \Pi^t(\omega_B))) \otimes (\Pi^t(\omega_B) \Pi_{B^n}(t) \Pi^t(\omega_B)) \right) \tag{91}$$

with arbitrarily small $\varepsilon_1 > 0$, following (22) and the fact that $\Pi^t(\omega_B) \Pi_{B^n}(t) \Pi^t(\omega_B) \leq \Pi^t(\omega_B)$. By linearity, this can also be written as

$$\Pi \sigma_{B^n,B^n} \leq 2^{-n(H(\omega_B) + \varepsilon_1)} \Pi^t(\omega_B) \left[ \mathcal{M}^\otimes(n(\sum_t P(t) \pi_{K^n}^t)) \right] \Pi^t(\omega_B) \tag{92}$$

(see (68)). Since the expression in the square brackets equals $\omega_{B^n}^{\otimes n}$ (see (74)), we have by (25) that

$$\Pi \sigma_{B^n,B^n} \leq 2^{-n(H(\omega_B) + H(\omega_B) + \varepsilon_1 + \varepsilon_2)} \Pi^t(\omega_B) \otimes \Pi^t(\omega_B) \tag{93}$$

with arbitrarily small $\varepsilon_2 > 0$, where the last equality follows from the definition of $\Pi$ in (76). It follows that all of the requirements of the packing lemma are satisfied.

Hence, by Lemma 4, there exist deterministic vectors $\gamma(m), m \in [1:2^{nR}]$, and a POVM $\{A_m\}_{m \in [1:2^{nR}]}$ such that

$$\text{Tr} \left( A_m \rho_{B^n,B^n} \right) \geq 1 - 2^{-n[I(B';B)_\omega - R - \varepsilon']} \tag{94}$$

for all $m \in [1:2^{nR}]$, where $\varepsilon'$ is arbitrarily small. That is, the probability of error is bounded by $2^{-n[I(B';B)_\omega - R - \varepsilon_n(\alpha)]}$, which tends to zero if

$$R < I(B';B)_\omega - \varepsilon'. \tag{95}$$

Now, consider the systems $S, K_1, A_1, A_1', B_1$ at state

$$\omega_{A_1A_1'} = (\mathcal{F}_{1}^{(s)} \otimes \mathbb{1}) (|\xi_{K_1,A_1'}\rangle \langle \xi_{K_1,A_1'}|) \tag{96}$$

$$\omega_{A_1SA_1'} = \sum_{s \in S} q(s) |s\rangle \langle s| \otimes \omega_{A_1A_1'} \tag{97}$$

$$\omega_{A_1B_1} = (\mathbb{1} \otimes \mathcal{N}) (\omega_{A_1SA_1'}) = (\mathbb{1} \otimes \mathcal{M}) (|\xi_{K_1,A_1'}\rangle \langle \xi_{K_1,A_1'}|) \tag{98}$$
Observe that this is the same relation as in (74) where \( A'_1, K_1 \) and \( B_1 \) take place with \( A, B, \) and \( B' \), respectively, where \( F_{1,s}^{(n)} = (F_{1,s})^T \) for \( s \in S \), due to (68) and the “ricochet property” (65). Thus, the probability of error tends to zero as \( n \to \infty \) provided that \( R < I(K_1; B_1)_{\omega} - \varepsilon' \). This completes the proof of the direct part.

### B. Converse Proof

Consider the converse part. Suppose that Alice and Bob are trying to distribute randomness. An upper bound on the rate at which Alice can distribute randomness to Bob also serves as an upper bound on the rate at which they can communicate.

In this task, Alice and Bob share an entangled state \( \Psi_{T_A T_B}^{\otimes n} \). Alice first prepares the maximally correlated state

\[
\overline{\Psi}_{M M'} \equiv \frac{1}{2n^R} \sum_{m=1}^{2^nR} |m\rangle \langle m| \otimes |\phi_m\rangle \langle \phi_m|.
\]

(99)

locally. We note that since \( M \) and \( M' \) are classical, they can be copied.

Then, at time \( t \in [1 : n] \), Alice applies an encoding channel \( E_{M'}^{(n)}_{T_A A'} \) to the classical system \( M' \) and her share \( T_{Ai} \) of the entangled state \( \Psi_{T_A T_B}^{\otimes n} \). The resulting state is

\[
\omega_{S^n MA_i T_{Bi}}^{(n)} = \sum_{s^n \in S^n} \eta^n(s^n) |s^n\rangle \langle s^n| \otimes \rho_{MA_i T_{Bi}}^{(n)},
\]

with

\[
\rho_{MA_i T_{Bi}}^{(n)} \equiv (1 \otimes E_{M'}^{(n)} \otimes 1)(\overline{\Psi}_{M M'} \otimes \Psi_{T_A T_B}^{\otimes n}).
\]

(100)

for \( i \in [1 : n] \). After Alice sends the systems \( A_i^n \) through the channel, Bob receives the systems \( B_i^n \) at state

\[
\omega_{S^n MA_i T_{Bi}}^{(n)} = \sum_{s^n \in S^n} \eta^n(s^n) |s^n\rangle \langle s^n| \otimes \rho_{MB_i T_{Bi}}^{(n)},
\]

with

\[
\rho_{MB_i T_{Bi}}^{(n)} \equiv (1 \otimes N^{(n)} \otimes 1)(\rho_{M M'}^{(n)}).
\]

(101)

for \( i \in [1 : n] \). Then, Bob performs a decoding channel \( D_{B_i^n T_{Bi}^n} \), producing \( \omega_{S^n M M'}^{(n)} = \sum_{s^n \in S^n} \eta^n(s^n) |s^n\rangle \langle s^n| \otimes \rho_{MM'}^{(n)} \)

with

\[
\rho_{MM'}^{(n)} \equiv (1 \otimes D) \left( \prod_{i=1}^{n} \rho_{MB_i T_{Bi}}^{(n)} \right).
\]

(102)

Consider a sequence of codes \( (E_{M'}^{(n)}, \Psi_{n}, D_{n}) \) for randomness distribution, such that

\[
\frac{1}{2} \| \omega_{M M'}^{(n)} - \overline{\Psi}_{M M'} \|_1 \leq \alpha_n,
\]

(103)

where \( \omega_{MM'} \) is the reduced density operator of \( \omega_{S^n M M'} \) and while \( \alpha_n \) tends to zero as \( n \to \infty \). By the Alon-Fannes-Winter inequality [1, 69] [67, Theorem 11.10.3], this implies that

\[
|H(M | \tilde{M})_\omega - H(M | M')_{\overline{\Psi}}| \leq n\varepsilon_n
\]

(104)

while \( \varepsilon_n \) tends to zero as \( n \to \infty \). Now, observe that \( H(\overline{\Psi}_{M M'}) = H(\overline{\Psi}_M) = H(\overline{\Psi}_{M'}) = nR \), hence \( I(M; \tilde{M})_{\overline{\Psi}} = nR \). Also, \( H(\omega_M) = H(\overline{\Psi}_M) = nR \) implies that \( I(M; M')_{\overline{\Psi}} - I(M; \tilde{M})_{\omega} = H(M | \tilde{M})_{\omega} - H(M | M')_{\overline{\Psi}} \). Therefore, by (104),

\[
nR = I(M; \tilde{M})_{\overline{\Psi}} \\
\leq I(M; \tilde{M})_{\omega} + n\varepsilon_n \\
\leq I(M; T_B, B_n)_{\omega} + n\varepsilon_n
\]

(105)

where the last line follows from (102) and the quantum data processing inequality [53, Theorem 11.5].

As in the classical case, the chain rule for the quantum mutual information states that \( I(A; B, C)_{\sigma} = I(A; B)_{\sigma} + I(A; C | B)_{\sigma} \) for all \( \sigma_{ABC} \) (see e.g. [67, Property 11.7.1]). Hence,

\[
nR \leq I(T_B, M; B_n)_{\omega} + I(M; T_B)_{\omega} - I(T_B; B_n)_{\omega} + n\varepsilon_n \\
\leq I(T_B, M; B_n)_{\omega} + I(M; T_B)_{\omega} + n\varepsilon_n \\
= I(T_B, M; B_n)_{\omega} + n\varepsilon_n
\]

(106)
where the equality holds since the systems $M$ and $T_B$ are in a product state. The chain rule further implies that
\[
I(T_B, M; B^n) \omega = \sum_{i=1}^{n} I(T_B, M; B_i|B^{i-1}) \omega
\]
\[
\leq \sum_{i=1}^{n} I(T_B, M, S^{i-1}, A_i^{i-1}, B_i^{i-1}; B_i) \omega
\]
\[
= \sum_{i=1}^{n} [I(T_B, M, S^{i-1}, A_i^{i-1}; B_i) + I(B_i^{i-1}; B_i|T_B, M, S^{i-1}, A_i^{i-1})] \omega
\]
\[
= \sum_{i=1}^{n} I(T_B, M, S^{i-1}, A_i^{i-1}; B_i) \omega
\]
(107)
where the last line holds since the channel has a product form, i.e. $\mathcal{N}_{S^{i-1}, A_i^{i-1}} = \mathcal{N}_{S} \otimes \mathcal{N}_{A_i^{i-1}} \otimes \mathcal{N}_{S, A_i^{i-1}}$.

Defining $K_i = (M, M', S^{i-1}, A_i^{i-1}, T_A, T_B)$ and a quantum channel $\mathcal{F}^{(s)}_{K_i \rightarrow A_i}$, we have by (106) and (107) that
\[
R - \varepsilon_n \leq \frac{1}{n} \sum_{i=1}^{n} I(K_i; B_i) \omega \leq \max_{\theta_{K, A'}, F_{K, A}^{(s)}; \mathcal{F}^{(s)}_{K, A} \otimes \mathcal{N}_{S, A}} I(K; B) \omega .
\]
(108)
Observe that by (100), $K_i$ and $S_i$ are in a product state as required. This concludes the proof of Theorem 6.

\section*{APPENDIX C

\section*{PROOF OF LEMMA 8

Fix the quantum state $\theta_{K, A'}$ and channels $\mathcal{F}^{(s)}_{K \rightarrow A'}$, $s \in \mathcal{S}$, such that
\[
\mathcal{C}_{E, n} (\mathcal{N}) = I(A; B) \omega - I(A; S) \omega
\]
(109)
and consider the spectral decomposition,
\[
\theta_{K, A'} = \sum_{x \in X} \sum_{z \in Z} \rho_{X, Z}(x, z) |x \rangle \langle x| \otimes |z \rangle \langle z|
\]
(110)
where $P_{X, Z}(x, z)$ is a probability distribution, while $\{|x\rangle\}_{x \in X}$ and $\{|z\rangle\}_{z \in Z}$ are orthonormal bases of the Hilbert spaces $\mathcal{H}_K$ and $\mathcal{H}_{A'}$, respectively. Also, for every $s \in \mathcal{S}$, consider the Kraus representation of each channel
\[
\tilde{\mathcal{F}}^{(s)}_{K \rightarrow A}(\rho_K) = \sum_j F^{(s)}_j \rho_K F^{(s)}_j \dagger
\]
(111)
with $\sum_j F^{(s)}_j \dagger F^{(s)}_j = \mathbb{1}$ (see Subsection II-B).

First, we show that maximizing over pure states is sufficient. To this end, we perform purification of the state $\theta_{K, A'}$. Specifically, define the pure state
\[
|\xi_{K, A'} \rangle = \sum_{x \in X} \sum_{z \in Z} \sqrt{\rho_{X, Z}(x, z)} |x \rangle \otimes |\psi_z \rangle \otimes |z \rangle
\]
(112)
where $J$ is a reference system and $|\psi_z \rangle$ are orthonormal vectors in $\mathcal{H}_J$. Observe that $|\xi_{K, A'} \rangle$ is a purification of the mixed state $\theta_{K, A'}$, namely, $\theta_{K, A'} = \text{Tr}_J (|\xi_{K, A'} \rangle \langle \xi_{K, A'}|)$. Defining $\tilde{\mathcal{F}}^{(s)}_{K \rightarrow A} = (\mathcal{F}^{(s)}_{K \rightarrow A} \otimes \mathbb{1})$ and $\tilde{K} = (K, J)$, we have that
\[
\omega_{A'} = \tilde{\mathcal{F}}^{(s)}_{K \rightarrow A} (|\xi_{K, A'} \rangle \langle \xi_{K, A'}|).
\]
(113)
Then, observe that the mutual information difference $[I(A; B) \omega - I(A; S) \omega]$ depends on the state $\theta_{K, A'}$ and the channels $\mathcal{F}^{(s)}_{K \rightarrow A}$ only through $\omega_{A'}$, and thus, $\theta_{K, A'}$ can be replaced by the pure state $|\xi_{K, A'} \rangle$.

To show that maximizing over isometric channels is sufficient, we use an isometric extension of the channels $\mathcal{F}^{(s)}_{K \rightarrow A}$, for $s \in \mathcal{S}$. Define the isometric channels $\mathcal{F}^{(s)}_{K \rightarrow AE}$ by
\[
\mathcal{F}^{(s)}_{K \rightarrow AE}(\rho_K) = \tilde{\mathcal{F}}^{(s)}_{K \rightarrow A} (\rho_K) \rho_K \tilde{\mathcal{F}}^{(s)}_{K \rightarrow A} \dagger
\]
(114a)
for all $\rho_K$, with
\[
\tilde{\mathcal{F}}^{(s)} = \sum_j F^{(s)}_j \otimes |j \rangle
\]
(114b)
where $E$ is a reference system and $\{|j \rangle\}$ is an orthonormal basis of $\mathcal{H}_E$. Observe that $\mathcal{F}^{(s)}_{K \rightarrow AE}$ is an extension of the quantum channel $\mathcal{F}^{(s)}_{K \rightarrow A}$, namely, $\text{Tr}_E \big( \mathcal{F}^{(s)}_{K \rightarrow AE}(\rho_K) \big) = \mathcal{F}^{(s)}_{K \rightarrow A}(\rho_K)$ for every $\rho_K$. 


Let
\[
\sigma^{s}_{AEA'} = (F_{K\rightarrow AE}^{(s)} \otimes 1)(\sigma_{K\rightarrow A'})
\]  
(115)
\[
\sigma^{s}_{AESA'} = \sum_{s \in S} q(s) |s\rangle \langle s| \otimes \sigma^{s}_{AEA'}
\]  
(116)
\[
\sigma^{s}_{AEB} = \left(1 \otimes 1 \otimes \mathcal{N}\right)(\sigma^{s}_{AESA'}).
\]  
(117)

Based on the definition in (40),
\[
\mathcal{C}_{E,n,c}(\mathcal{N}) \geq I(A, E; B)_{\sigma} - I(A, E; S)_{\sigma}.
\]  
(118)

On the other hand, by the quantum data processing theorem due to Schumacher and Nielsen [56][67, Theorem 11.9.4],
\[
I(A; B)_{\omega} \leq I(A, E; B)_{\sigma}.
\]  
(119)

Furthermore, by (114), the systems \(S\) and \(E\) are in a product state given \(A\), hence \(I(E; S|A)_{\sigma} = 0\). Thus,
\[
I(A; S)_{\omega} = I(A; S)_{\sigma} = I(A; S)_{\sigma} + I(E; S|A)_{\sigma} = I(A, E; S)_{\sigma}
\]  
(120)
where the last equality is due to the chain rule for the quantum mutual information [67, Theorem 11.7.1]. Together, (119) and (120) imply that
\[
\mathcal{C}_{E,n,c}(\mathcal{N}) \leq I(A, E; B)_{\sigma} - I(A, E; S)_{\sigma}.
\]  
(121)

It thus follows that the channel \(F_{K\rightarrow A}^{(s)}\) in (40) can be replaced by its isometric extension \(\overline{F}_{K\rightarrow A_{0}}^{(s)}\), with \(A_{0} = (A, E)\), for \(s \in S\). This completes the proof of the lemma.

**APPENDIX D**

**PROOF OF THEOREM 9**

A. Achievability Proof

We show that for every \(\varepsilon_{0}, \delta_{0} > 0\), there exists a \((2^{nR}, n, \varepsilon_{0})\) code for the random-parameter quantum channel \(N_{SA\rightarrow B}\) with non-causal CSI, provided that \(R < C_{E,n,c}(\mathcal{N}) - \delta_{0}\). Based on Lemma 8, it suffices to consider a pure entangled state and isometric channels. Hence, let \(|\xi_{AB}\rangle\) be a pure entangled state, and \(F_{K\rightarrow A}^{(s)}(p_{K}) = F^{(s)}\rho_{K} F^{(s)^{\dagger}}\), \(s \in S\), be a set of isometric channels. Suppose that Alice and Bob share the joint state \(|\xi_{AB}\rangle^{\otimes n}\). Define
\[
|\varphi^{s}_{AB}\rangle = (F^{(s)} \otimes 1)|\xi_{AB}\rangle
\]  
(122)
and consider the Schmidt decomposition of the state,
\[
|\varphi^{s}_{AB}\rangle = \sum_{x \in X} \sqrt{p_{X}(x)}|s\rangle \otimes |\psi_{x,s}\rangle
\]  
(123)
where \(p_{X|S}\) is a conditional probability distribution, \(\{|\psi_{x,s}\rangle\}\) is an orthonormal basis of \(\mathcal{H}_{A}\), and \(|\psi_{x,s}\rangle\) are orthonormal vectors in \(\mathcal{H}_{B}\). Observe that the quantum entropy of the system \(B\) is the same as the Shannon entropy of the classical random variable \(X\), i.e. \(H(\omega_{SB}) = H(S, X)\) and \(H(\omega_{B}) = H(X)\). Thus,
\[
I(B; S)_{\varphi} = I(X; S).
\]  
(124)

The code construction, encoding and decoding procedures are described below.

1) Code Construction: Encoding is performed in two stages, first classical compression of the parameter sequence \(S^{n}\), and then, application of quantum operators depending on the result in the first stage. The code construction is specified below.

i) Classical Compression: Let \(\overline{R} > R\). We construct \(2^{nR}\) sub-codebooks at random. For every message \(m \in [1 : 2^{nR}]\), choose \(2^{n(\overline{R}-R)}\) independent sequences \(x^{n}(\ell)\) at random, each according to \(\prod_{i=1}^{n} p_{X}(x_{i})\). Then, we have the following sub-codebooks,
\[
\mathcal{B}(m) = \{x^{n}(\ell) : \ell \in [(m - 1)2^{n(\overline{R}-R)} + 1 : m2^{n(\overline{R}-R)}]\}, \text{for } m \in [1 : 2^{nR}].
\]  
(125)

ii) Quantum Operators: Consider the Heisenberg-Weyl operators \(\{\Sigma(a, b) = X(a)Z(b)\}\) of dimension \(D\), given by
\[
X(a) = \sum_{j=0}^{D-1} |a \oplus j\rangle \langle j|
\]  
(126)
\[
Z(b) = \sum_{j=0}^{D-1} e^{2\pi ibj/D} |j\rangle \langle j|
\]  
(127)
for \(a, b \in \{0, 1, \ldots, D - 1\}\), where \(a \oplus j = (a + j) \mod D\) and \(i = \sqrt{-1}\). For every \(s^n \in S^n\) and every conditional type class \(T_n(t|s^n)\) in \(\mathcal{X}^n\), define the operators

\[
V_i(a_t, b_t, c_t) = (-1)^c_t \Sigma(a_t, b_t), \quad a_t, b_t \in \{0, 1, \ldots, D_t - 1\}, \quad c_t \in \{0, 1\}
\]

where \(D_t = |T_n(t|s^n)|\) is the size of type class associated with the conditional type \(t\). Then, define the operator

\[
U(\gamma) = \bigoplus_t V_i(a_t, b_t, c_t)
\]

with \(\gamma = ((a_t, b_t, c_t)_t)\). Let \(\Gamma\) denote the set of all possible vectors \(\gamma\). Then, choose \(2^{nR}\) vectors \(\gamma(\ell), \ell \in [1 : 2^{nR}]\), uniformly at random.

2) Encoding and Decoding: The coding scheme is depicted in Figure 4. To send a message \(m \in [1 : 2^{nR}]\), given a parameter sequence \(s^n \in S^n\), Alice performs the following.

(i) Find a sequence \(x^n(\ell) \in \mathcal{B}(m)\) that is jointly typical with the parameter sequence, i.e. \((s^n, x^n(\ell)) \in \mathcal{A}^n(p_{S,X})\). If there is none, choose an arbitrary \(\ell\).

(ii) Apply the operators \(F(s^{(1)}), F(s^{(2)}), \ldots, F(s^{(n)})\), and \(U(\gamma(\ell))\), which yields

\[
|\varphi_{A^n,B^n}^{x^n} \rangle = (U(\gamma(\ell))F(s^n) \otimes 1)|\xi_{AB} \rangle = (U(\gamma(\ell) \otimes 1)|\varphi_{A^n,B^n}^{x^n})
\]

with \(F(s^n) \equiv F(s^{(1)}) \otimes \cdots \otimes F(s^{(n)})\) and \(|\varphi_{A^n,B^n}^{x^n} \rangle \equiv |\varphi_{A}^{x^{1}} \rangle \otimes \cdots \otimes |\varphi_{A}^{x^{n}} \rangle\) (see (122)).

(iii) Send the systems \(A^n\) through the channel.

Bob receives the systems \(B^n\) at state \(\omega_{B^n|A^n}\) and applies a POVM \(\{A_{\ell}\}_{\ell \in [1 : 2^{nR}]}\), which will be specified later. Once Bob has a measurement result \(\hat{\ell}\), he decodes the message as the corresponding sub-codebook. That is, Bob declares the message to be \(\hat{m} \in [1 : 2^{nR}]\) such that \(x^n(\hat{\ell}) \in \mathcal{B}(\hat{m})\).

3) Code Properties: First, we write the entwined states as a combination of maximally entangled states over the typical subspaces, and then we can use the following useful identities. For a maximally entangled state \(|\Phi_{AB} \rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} |j\rangle_A \otimes |j\rangle_B\),

\[
\text{Tr}_B(|\Phi_{AB} \rangle \langle \Phi_{AB}|) = \pi_A
\]

where \(\pi_A = \frac{1}{D} \sum_{x \in \mathcal{X}} |x\rangle \langle x|\) is the maximally mixed state. Furthermore, for every state \(\rho\) of the system \(A\),

\[
\frac{1}{D^2} \sum_{a=0}^{D-1} \sum_{b=0}^{D-1} \Sigma(a, b) \rho \Sigma^\dagger(a, b) = \pi_A
\]

(see e.g. [7] [67, Exercise 4.7.6]). Another useful identity is the “ricochet property” [37, Eq. (17)],

\[
(U \otimes 1)|\Phi_{AB} \rangle = (1 \otimes U^T)|\Phi_{AB} \rangle.
\]

Now, for every \(s^n \in S^n\),

\[
|\varphi_{A^n,B^n}^{x^n} \rangle = \sum_{x^n \in \mathcal{X}^n} \sqrt{p_X^n(s^n|x^n)} |x^n \rangle \otimes |\psi_{x^n,s^n}\rangle
\]
where \( p_{X|S^n}(x^n|s^n) = \prod_{i=1}^n p_{X|S}(x_i|s_i) \) and \( |\psi_{z^n}\rangle = |\psi_{x_1}\rangle \otimes |\psi_{x_2}\rangle \otimes \cdots \otimes |\psi_{x_n}\rangle \). As the space \( \mathcal{X}^n \) can be partitioned into conditional type classes given \( s^n \), we may write

\[
|\varphi^{s^n}_{A^n,B^n}\rangle = \sum_{t \in \mathcal{T}_n(X)} \sum_{x^n \in \mathcal{T}_n(P|s^n)} \sqrt{p_{X^n|S^n}(x^n|s^n)} |x^n\rangle \otimes |\psi_{z^n,s^n}\rangle
\]

\[
= \sum_{t \in \mathcal{T}_n(X)} \sqrt{p_{X^n|S^n}(x^n_t|s^n)} \sum_{x^n \in \mathcal{T}_n(t|s^n)} |x^n\rangle \otimes |\psi_{z^n,s^n}\rangle
\] (135)

where \( x^n_t \) is any sequence in the conditional type class \( \mathcal{T}_n(t|s^n) \). Therefore, we have that

\[
|\varphi^{s^n}_{A^n,B^n}\rangle = \sum_{t \in \mathcal{T}_n(X)} \sqrt{P(t|s^n)} |\Phi_t\rangle,
\]

where

\[
P(t|s^n) = d_t(s^n) \cdot p_{X^n|S^n}(x^n_t|s^n), \quad \text{with} \quad d_t(s^n) \equiv |\mathcal{T}_n(t|s^n)|
\]

\[
|\Phi_t\rangle = \frac{1}{\sqrt{d_t(s^n)}} \sum_{x^n \in \mathcal{T}_n(t|s^n)} |x^n\rangle \otimes |\psi_{z^n,s^n}\rangle
\] (137)

We note that \( P(t|s^n) \) is the conditional probability of the type \( \mathcal{T}_n(t|s^n) \) for a classical random sequence \( X^n \sim p_{X^n|S^n=s^n} \).

Now, Alice applies the operator \( U(\gamma(\ell)) \) to the entangled states. Since the state \( |\Phi_t\rangle \) is maximally entangled, we have by the “ricochet property” (133) that

\[
|\varphi^{\gamma(\ell),s^n}_{A^n,B^n}\rangle \equiv (U(\gamma(\ell)) \otimes 1)|\varphi^{s^n}_{A^n,B^n}\rangle = (1 \otimes U^T(\gamma(\ell)))|\varphi^{s^n}_{A^n,B^n}\rangle.
\] (138)

By the same considerations, we also have that

\[
|\varphi^{s^n+}_{A^n,B^n}\rangle = (F(s^n) \otimes 1)|\xi_{AB}\rangle \otimes n = (1 \otimes (F(s^n))^T)|\xi_{AB}\rangle \otimes n.
\] (139)

That is, Alice’s unitary operations can be reflected and treated as if performed by Bob.

Bob then receives the systems \( B^n \) at state

\[
\tilde{\rho}^{\gamma(\ell)}_{B^n,B^n} = \sum_{s^n \in \mathcal{S}^n} q^n(s^n) (\mathcal{N}(s^n) \otimes 1) \left( (|\varphi^{\gamma(\ell),s^n}_{A^n,B^n}\rangle \langle \varphi^{\gamma(\ell),s^n}_{A^n,B^n}|) \right)
\]

\[
= \sum_{s^n \in \mathcal{S}^n} q^n(s^n) (\mathcal{N}(s^n) \otimes 1) \left( (1 \otimes U^T(\gamma(\ell)))(|\varphi^{s^n}_{A^n,B^n}\rangle \langle \varphi^{s^n}_{A^n,B^n}|)(1 \otimes U^*(\gamma(\ell))) \right)
\] (140)

where the last line is due to (138). Since a quantum channel is a linear map, the above can be written as

\[
\tilde{\rho}^{\gamma(\ell)}_{B^n,B^n} = (1 \otimes U^T(\gamma)) \left[ \sum_{s^n \in \mathcal{S}^n} q^n(s^n) (\mathcal{N}(s^n) \otimes 1) \left( |\varphi^{s^n}_{A^n,B^n}\rangle \langle \varphi^{s^n}_{A^n,B^n}| \right) \right] (1 \otimes U^*(\gamma))
\]

\[
= (1 \otimes U^T(\gamma)) \omega_{B^n,B^n} \otimes (1 \otimes U^*(\gamma))
\] (142)

where we have defined

\[
\omega_{AB} = (1 \otimes (F(s))^T)|\xi_{AB}\rangle \langle \xi_{AB}|(1 \otimes (F(s))^*)
\] (143)

\[
\omega_{SAB} = \sum_{s \in \mathcal{S}} q(s)|s\rangle \langle s| \otimes \omega_{AB}
\] (144)

\[
\omega_{B^n,B^n} = (\mathcal{N} \otimes 1)(\omega_{SAB}).
\] (145)

4) Packing Lemma Requirements: Next, we use the quantum packing lemma. Consider the ensemble \( \{p(\gamma) = \frac{1}{|\Gamma|}, \tilde{\rho}^{\gamma(\ell)}_{B^n,B^n}\} \), for which the expected density operator is

\[
\sigma_{B^n,B^n} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \tilde{\rho}^{\gamma(\ell)}_{B^n,B^n}.
\] (146)

Define the code projector and the codeword projectors by

\[
\Pi \equiv \Pi^\delta(\omega_B) \otimes \Pi^\delta(\omega_B)
\]

\[
\Pi_{\gamma} \equiv (1 \otimes U^T(\gamma))\Pi^\delta(\omega_B)B\otimes U^*(\gamma)) , \quad \text{for} \gamma \in \Gamma
\] (147)

\[
\Pi_{\gamma} \equiv (1 \otimes U^T(\gamma))\Pi^\delta(\omega_B)B\otimes U^*(\gamma)) , \quad \text{for} \gamma \in \Gamma
\] (148)

where \( \Pi^\delta(\omega_B) \) and \( \Pi^\delta(\omega_B) \) are the projectors onto the \( \delta \)-typical subspaces associated with the states \( \omega_B, \omega_B = Tr_B(\omega_B) \) and \( \omega_B = Tr_B(\omega_B) \), respectively (see (145)). Now, we verify that the assumptions of Lemma 4 hold with respect to the ensemble and the projectors above.
First, we show that $\text{Tr}(\Pi \rho^B_{n^n} \cap B^n) \geq 1 - \alpha$, where $\alpha > 0$ is arbitrarily small. Defining $\bar{P} = \mathbb{I} - P$, we have that
\[
\Pi = (1 - \bar{\Pi}^A(x_B)) \otimes (1 - \bar{\Pi}^B(x_B)) \geq (\mathbb{I} \otimes \mathbb{I}) - (\bar{\Pi}^A(x_B) \times \mathbb{I}) - (\mathbb{I} \otimes \bar{\Pi}^B(x_B))
\] (149)

hence,
\[
\text{Tr}(\Pi \rho^B_{n^n} \cap B^n) \geq 1 - \text{Tr} \left( (\bar{\Pi}^A(x_B) \otimes \mathbb{I}) \rho^B_{n^n} \cap B^n \right) - \text{Tr} \left( (\mathbb{I} \otimes \bar{\Pi}^B(x_B)) \rho^B_{n^n} \cap B^n \right)
\]
\[
= 1 - \text{Tr} \left( \bar{\Pi}^A(x_B) \rho^B_{n^n} \cap B^n \right) - \text{Tr} \left( \bar{\Pi}^B(x_B) \rho^B_{n^n} \cap B^n \right) \quad \text{(150)}
\]
The first trace term in the RHS of (150) equals $\text{Tr} (\bar{\Pi}^A(x_B) \omega_{n^n}^B \cap B^n)$ by (142), and the last term equals $\text{Tr} (\bar{\Pi}^B(x_B) \omega_{n^n}^B \cap B^n)$ by (140) and (144). Therefore, we have by (24) that
\[
\text{Tr}(\Pi \rho^B_{n^n} \cap B^n) \geq 1 - \text{Tr} (\bar{\Pi}^A(x_B) \omega_{n^n}^B \cap B^n) - \text{Tr} (\bar{\Pi}^B(x_B) \omega_{n^n}^B \cap B^n) \\
\geq 1 - 2\varepsilon \quad \text{(151)}
\]

Similarly, the second requirement of the packing lemma holds since
\[
\text{Tr}(\Pi \gamma B^n \cap B^n) = \text{Tr} \left[ (\mathbb{I} \otimes U^T(\gamma)) \Pi \delta (\omega_{B^n} \cap B^n) \right] (1 \otimes U^T(\gamma)) \omega_{B^n} \cap B^n (1 \otimes U^T(\gamma))
\]
\[
= \text{Tr}(\Pi \delta (\omega_{B^n} \cap B^n)) \geq 1 - \varepsilon \quad \text{(152)}
\]

where the second equality follows from the cyclicity of the trace and the fact that $U^T U = (UU^T)^* = \mathbb{I}$ for a unitary operator, and the last inequality is due to (24).

Moving to the third requirement in Lemma 4,
\[
\text{Tr}(\Pi \gamma) = \text{Tr} \left( (\mathbb{I} \otimes U^T(\gamma)) \Pi \delta (\omega_{B^n} \cap B^n) \right) (1 \otimes U^T(\gamma)) \omega_{B^n} \cap B^n (1 \otimes U^T(\gamma)) \leq 2^{n(\delta(\omega_{B^n} \cap B^n) + \delta(\Pi \gamma))} \quad \text{(153)}
\]

where the second equality holds by cyclicity of the trace and the last inequality is due to (26). It is left to verify that the last requirement of the packing lemma holds, i.e. $\Pi \sigma_{B^n, B^n} \Pi \leq 2^{-n(\delta(\omega_{B^n} \cap B^n) + \delta(\Pi \gamma))} \Pi$. To this end, observe that by (141) and (146),
\[
\sigma_{B^n, B^n} = \sum_{A^n \in S^n} q^n(s^n) (N(s^n) \otimes \mathbb{I}) \tau_{A^n, B^n} \quad \text{(154)}
\]

where we have defined
\[
\tau_{A^n, B^n} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} (\mathbb{I} \otimes U^T(\gamma)) \phi_{A^n, B^n} \langle \phi_{A^n, B^n}| (1 \otimes U^T(\gamma)) \phi_{A^n, B^n} \langle \phi_{A^n, B^n}|. \quad \text{(155)}
\]

Then, by (138) along with (128)-(129),
\[
\tau_{A^n, B^n} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} (\mathbb{I} \otimes U^T(\gamma)) \left( \sum_t \sqrt{P(t|s^n)} \langle \Phi_t| \right) \left( \sum_{t'} \sqrt{P(t'|s^n)} \langle \Phi_{t'}| \right) (1 \otimes U^T(\gamma))
\]
\[
= \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \left( \sum_t \sqrt{P(t|s^n)} (1 \otimes \sum_{a_t, b_t} T^T(\gamma)(1 \otimes \sum a_t, b_t) \langle \Phi_t| \right)
\]
\[
\left( \sum_{t'} \sqrt{P(t'|s^n)} (1 \otimes \sum a_t, b_t) \langle \Phi_{t'}| \right). \quad \text{(156)}
\]

For $t' = t$, the expression above becomes
\[
\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_t P(t|s^n) (1 \otimes \sum a_t, b_t) \langle \Phi_t| (1 \otimes \sum a_t, b_t) \rangle = \sum_t P(t|s^n) \pi_{A^n} t \times \pi_{B^n} t \quad \text{(157)}
\]

with
\[
\pi_{A^n} t = \frac{\Pi_{A^n}(t)}{\text{Tr}(\Pi_{A^n}(t))}, \quad \pi_{B^n} t = \frac{\Pi_{B^n}(t)}{\text{Tr}(\Pi_{B^n}(t))} \quad \text{(158)}
\]

where $\Pi_{A^n}(t)$ is the projector of type $t$ as defined in (21). The last equality in (157) follows from (132). On the other hand, for $t' \neq t$,
\[
\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_t \sum_{t' \neq t} \sqrt{P(t|s^n)} P(t'|s^n) \frac{1}{4D_t^2 D_{t'}} \sum_{c_t, c_{t'} \in \{0, 1\}} (-1)^{c_t + c_{t'}} \sum a_t, b_t, b_{t'} (1 \otimes \sum a_t, b_t) \langle \Phi_t| (1 \otimes \sum a_{t'}, b_{t'}) = 0. \quad \text{(159)}
\]
We deduce from (156)-(159) that \( \tau_{A^n,B^n}^{n} = \sum_t P(t|s^n) \pi_A^{t} \otimes \pi_B^{n}. \) Plugging this into (154) yields

\[
\sigma_{B^n,B^n} = \sum_{s^n \in S^n} q^n(s^n) \sum_t P(t|s^n) (\pi_A^{t} \otimes \pi_B^{n})(\pi_A^{t} \otimes \pi_B^{n}). \tag{160}
\]

Now, we use the formula above in order to show that the last requirement in Lemma 4 holds. Consider that

\[
\Pi \sigma_{B^n,B^n} = \sum_{s^n \in S^n} q^n(s^n) \sum_t P(t|s^n) \left( (\Pi^\delta_\omega B^n) \sigma_{B^n,B^n} \right) = \sum_{s^n \in S^n} q^n(s^n) \sum_t P(t|s^n) \left( (\Pi^\delta_\omega B^n) \sigma_{B^n,B^n} \right) \otimes \left( (\Pi^\delta_\omega B^n) \sigma_{B^n,B^n} \right).
\]

Using (158), this can be bounded by

\[
\Pi \sigma_{B^n,B^n} \leq 2^{-n(H(\omega_B)+\varepsilon_1)} \sum_{s^n \in S^n} q^n(s^n) \sum_t P(t|s^n) \left( (\Pi^\delta_\omega B^n) \sigma_{B^n,B^n} \right) \otimes \left( (\Pi^\delta_\omega B^n) \sigma_{B^n,B^n} \right)
\]

with arbitrarily small \( \varepsilon_1 > 0, \) following (22) and the fact that \( \Pi^\delta_\omega B^n \Pi^\delta_\omega B^n \leq \Pi^\delta_\omega B^n. \) By linearity, this can also be written as

\[
\Pi \sigma_{B^n,B^n} \leq 2^{-n(H(\omega_B)+\varepsilon_1)} \sum_{s^n \in S^n} q^n(s^n) \sum_t P(t|s^n) \left( (\Pi^\delta_\omega B^n) \sigma_{B^n,B^n} \right) \otimes \left( (\Pi^\delta_\omega B^n) \sigma_{B^n,B^n} \right)
\]

(see (136)). Since the expression in the square brackets equals \( \omega_0^{2n} \) (see (145)), we have by (25) that

\[
\Pi \sigma_{B^n,B^n} \leq 2^{-n(H(\omega_B)+\varepsilon_1)} \sum_{s^n \in S^n} q^n(s^n) \sum_t P(t|s^n) \left( (\Pi^\delta_\omega B^n) \sigma_{B^n,B^n} \right) \otimes \left( (\Pi^\delta_\omega B^n) \sigma_{B^n,B^n} \right)
\]

with arbitrarily small \( \varepsilon_2 > 0, \) where the last equality follows from the definition of \( \Pi \) in (147). It follows that all of the requirements of the packing lemma are satisfied.

Hence, by Lemma 4, there exist deterministic vectors \( \gamma(\ell), \ell \in [1:2^{n\tilde{R}}] \), and a POVM \( \{\Lambda_\ell\}_{\ell \in [1:2^{n\tilde{R}}]} \) such that

\[
\text{Tr} \left( \Lambda_\ell \rho_{B^n,A^n\omega} \right) \geq 1 - 2^{-n[I(B^n;B^n\omega - \tilde{R}-\varepsilon']}
\]

for all \( \ell \in [1:2^{n\tilde{R}}] \), where \( \varepsilon' \) is arbitrarily small.

5) Error Probability Analysis: Observe that Bob can only decode the message \( m \) correctly if Alice chooses \( \ell \) such that \( \ell \in \mathcal{B}(m) \). Due to the symmetry, we may assume without loss of generality that Alice chose the message \( m = 1 \) and compressed the state sequence using \( \ell = 1 \). Hence, the error event is bounded by the union of the following events

\[
\mathcal{F}_1 = \{(S^n,X^n(\ell')) \notin \mathcal{A}^\ell(p_{S,X}) \text{ for all } \ell' \in [1:2^{n(\tilde{R}-R)}] \}
\]

\[
\mathcal{F}_2 = \{\ell \neq 1\}.
\]

Thus, by the union of events bound

\[
P_{\text{err}}^{(n)}(E,\phi_{KB},\Lambda) \leq \text{Pr}(\mathcal{F}_1 \cup \mathcal{F}_2) \leq \text{Pr}(\mathcal{F}_1) + \text{Pr}(\mathcal{F}_2) = \text{Pr}(\mathcal{F}_1) + \text{Tr} \left( (I - \Lambda_\ell) \rho_{B^n,A^n\omega} \right),
\]

where the conditioning on \( m = 1 \) and \( \ell = 1 \) is omitted for convenience of notation. By the classical covering lemma (see Lemma 3), we have that \( \text{Pr}(\mathcal{F}_1) \leq \exp \left( -2^{n(\tilde{R}-R-I(X:S)-\varepsilon')} \right) \). We also have that \( I(X;S) = I(B;S|\omega) = I(B;S) \omega \) by (124) and (144). Hence, the first term in the RHS of (168) tends to zero as \( n \to \infty \) provided that

\[
R < \tilde{R} - I(B;S)\omega - \varepsilon'. \tag{169}
\]

Based on (165), the second term in the RHS of (168) is bounded by \( 2^{-n[I(B^n;B^n\omega - \tilde{R}-\varepsilon_n(\alpha)], \) which tends to zero if

\[
\tilde{R} < I(B^n;B)\omega - \varepsilon', \tag{170}
\]

for sufficiently large \( n \) and small \( \alpha > 0 \). Therefore, the probability of error tends to zero as \( n \to \infty \) for \( R = R + \varepsilon' \) and \( R < I(B^n;B^n)\omega - I(B;S)\omega - 3\varepsilon'. \)
Now, consider the systems $S, A_1, A_1', B_1$ at state
\[ |\varphi_{A_1 A_1'}\rangle = (F_1^{(s)} \otimes \mathbb{I})|\xi_{A_1 A_1'}\rangle \]  
(171)
\[ \omega_{A_1 S A_1'} = \sum_{s \in S} \gamma(s)|s\rangle\langle s| \otimes |\varphi_{A_1 A_1'}\rangle \]  
(172)
\[ \omega_{A_1 B_1} = (\mathbb{1} \otimes N)(\omega_{A_1 S A_1'}). \]  
(173)
Observe that those are the same relations as in (145) where $A_1', A_1$ and $B_1$ take place with $A, B,$ and $B', \text{ respectively, with } F_1^{(s)} \equiv (F^{(s)}_1)^T \text{ for } s \in S.$ Thus, the probability of error tends to zero as $n \to \infty$ provided that $R < I(A_1; B_1) - I(A_1; S) - 3\varepsilon'.$ This completes the proof of the direct part.

**Remark 2.** At a first glance, it may seem that we can modify the proof above to prove Theorem 6 for causal CSI by simply removing the compression stage of the encoding procedure, and continuing the analysis without conditioning on the state sequence. However, such coding scheme would still violate the causality requirement, since Alice cannot apply the operator $U(\gamma)$ to the entire sequence of input systems (see Figure 4). Instead, in the proof of Theorem 6 in Appendix B, Alice applies the encoding operations in a reversed order, i.e. first $U(\gamma)$ is applied to a sequence of auxiliary systems $K^n$, which do not depend on the state sequence, and only then $F^{(s)}$ are applied (see Figure 3).

**B. Converse Proof**

Consider the converse part. Suppose that Alice and Bob are trying to distribute randomness. An upper bound on the rate at which Alice can distribute randomness to Bob also serves as an upper bound on the rate at which they can communicate. In this task, Alice and Bob share an entangled state $\Psi_{T_A^r T_B^r}$. Alice first prepares the maximally correlated state
\[ \overline{\Phi}_{MM'} = \frac{1}{2^n R} \sum_{m=1}^{2^n R} |m\rangle\langle m| \otimes |\phi_m\rangle\langle \phi_m|. \]  
(174)
locally. Then, Alice applies an encoding channel $\mathcal{E}_n^{\alpha} : T_A^r \rightarrow A^n$ to the classical system $M'$ and her share $T_A^r$ of the entangled state $\Psi_{T_A^r T_B^r}$. The resulting state is $\omega_{S^n A^n M^n T_B^r} = \sum_{s^n \in S^n} q^n(s^n) |s^n\rangle\langle s^n| \otimes \rho_{A^n M^n T_B^r}^{(1)}$ with
\[ \rho_{A^n M^n T_B^r}^{(1)} \equiv (\mathbb{1} \otimes \mathcal{E}_n^{(1)})|\overline{\Phi}_{MM'} \otimes \Psi_{T_A^r T_B^r}. \]  
(175)
After Alice sends the systems $A^n$ through the channel, Bob receives the systems $B^n$ at state $\omega_{S^n A^n M^n T_B^r} = \sum_{s^n \in S^n} q^n(s^n) |s^n\rangle\langle s^n| \otimes \rho_{M_B^n T_B^r}$ with
\[ \rho_{M_B^n T_B^r} \equiv (\mathbb{1} \otimes \mathcal{D})(\rho_{M^n B^n T_B^r}) \]  
(176)
Then, Bob performs a decoding channel $\mathcal{D}_n : B^n A^n T_B^r \rightarrow M'$, producing $\omega_{S^n M'n M} = \sum_{s^n \in S^n} q^n(s^n) |s^n\rangle\langle s^n| \otimes \rho_{M M'}^{(s)}$ with
\[ \rho_{M M'}^{(s)} \equiv (\mathbb{1} \otimes \mathcal{D})(\rho_{M^n B^n T_B^r}) \]  
(177)
Consider a sequence of codes $(\mathcal{E}_n^{(s)}, \Psi_n, \mathcal{D}_n)$ for randomness distribution, such that
\[ \frac{1}{2} \left\| \omega_{M M'} - \overline{\Phi}_{MM'} \right\|_1 \leq \alpha_n, \]  
(178)
where $\omega_{M M'}$ is the reduced density operator of $\omega_{S^n M M'}$ and while $\alpha_n$ tends to zero as $n \to \infty$. By the Alicki-Fannes-Winter inequality [1, 69] [67, Theorem 11.10.3], this implies that
\[ |H(M; \hat{M}) - H(M)| \leq n \varepsilon_n \]  
(179)
while $\varepsilon_n$ tends to zero as $n \to \infty$. Now, observe that $H(\overline{\Phi}_{MM'}) = H(\overline{\Phi}_{M}) = H(\overline{\Phi}_{M'}) = nR,$ hence $I(M; \hat{M}) \geq nR$. Also, $H(M) = H(\overline{\Phi}_{M}) = nR$ implies that $I(M; \hat{M}) - I(M; \hat{M}) = H(M) - H(M) - H(M) \geq nR$. Therefore, by (179),
\[ nR \geq I(M; \hat{M}) \]  
(180)
where the last line follows from (177) and the quantum data processing inequality [53, Theorem 11.5].

As in the classical case, the chain rule for the quantum mutual information states that $I(A; B, C) = I(A; B) + I(A; C | B)$ for all $\sigma_{ABC}$ (see e.g. [67, Property 11.7]). As a straightforward consequence, this leads to the Csis\'zar sum identity,
\[ \sum_{i=1}^{n} I(A^n_{i+1}; B_i | B^{i-1}) = \sum_{i=1}^{n} I(B^{i-1}; A^n_{i+1}) \]  
(181)
for every sequence of systems $A^n$ and $B^n$. Returning to (180), we apply the chain rule and rewrite the inequality as

$$nR \leq I(T^n_B, M; B^n) + I(M; T^n_B) - I(M; T^n_B) + n\varepsilon_n$$

$$\leq I(T^n_B, M; B^n) + I(M; T^n_B) + n\varepsilon_n$$

$$= I(T^n_B, M; B^n) + n\varepsilon_n$$

(182)

where the equality holds since the systems $M$ and $T^n_B$ are in a product state. The chain rule further implies that

$$I(T^n_B, M; B^n) = \sum_{i=1}^n I(T^n_B, M; B_{i-1} | B_i)$$

$$\leq \sum_{i=1}^n I(T^n_B, M, B_{i-1} | B_i)$$

$$= \sum_{i=1}^n I(T^n_B, M, B_{i-1}, S^n_{i+1}; B_i) - \sum_{i=1}^n I(B_i; S^n_{i+1}|T^n_B, M, B_{i-1})$$

$$= \sum_{i=1}^n I(T^n_B, M, B_{i-1}, S^n_{i+1}; B_i) - \sum_{i=1}^n I(B_{i-1}; S_i | T^n_B, M, S^n_{i+1})$$

(183)

where the last line follows from the quantum version of the Csiszár sum identity in (181). Since the systems $S_i$ and $(T^n_B, M, S^n_{i+1})$ are in a product state, $I(B^{i-1}; S_i | T^n_B, M, S^n_{i+1}) = I(T^n_B, M, S^n_{i+1}, B^{i-1}; S_i)$. Defining $K_i = (M, M', S^{i-1}, S^n_{i+1}, T^n_A, T^n_B)$ and a quantum channel $\mathcal{F}_{K_i \rightarrow A}$ such that $A_i = (M, B^{i-1}, S^n_{i+1}, T^n_B)$, we have by (182) and (183) that

$$R - \varepsilon_n \leq \frac{1}{n} \sum_{i=1}^n [I(A_i; B_i) - I(A_i; S_i)] \leq \max_{\theta_{K,A'}} [I(A; B) - I(A; S)].$$

(184)

This concludes the proof of Theorem 9.  \hfill \Box

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