SOME REMARKS ON THE QUANTUM DOUBLE

S. Majid

Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW

Abstract We recall the abstract theory of Hopf algebra bicrossproducts and double cross products due to the author. We use it to develop some less-well known results about the quantum double as a twisting, as an extension and as \( q \)-Lorentz group.

Keywords: quantum double – quantum group gauge theory – twisting – \( q \)-Lorentz group – \( q \)-Minkowski space

1 Introduction

The quantum double Hopf algebra \( D(H) \) was introduced by Drinfeld[1]. Since then it has been extensively studied by the author in [2][3][4][5][6][7] as well as by Reshetikhin and Semenov-Tian-Shanskii in [8]. It was also proposed as \( q \)-Lorentz group in [9] and connected with an R-matrix approach[10] in our paper [6, Sec. 4]. Here we collect some of the main results and develop two less well known ones. They demonstrate the power of some algebraic techniques for Hopf algebras relating to cross products and cross coproducts. We write formulae over \( \mathbb{C} \) but in fact these abstract constructions work over a general field too. We use the summation convention and also the standard abstract notation \( \Delta h = h_{(1)} \otimes h_{(2)} \) for the quantum group coproduct. This stands for a sum of terms in the tensor product.

2 Quantum double as a double cross product

The definition of the quantum double that we use is the algebraic form due to the author[2] as an example of a double cross product. If \( H \) is a finite dimensional Hopf algebra or quantum group, its quantum double \( D(H) \) is built on the vector space \( H^* \otimes H \) with product

\[
(a \otimes h)(b \otimes g) = b_{(2)}a \otimes h_{(3)}g\langle Sh_{(1)}, b_{(3)}\rangle\langle h_{(1)}, b_{(3)}\rangle
\]  

(1)
and tensor product unit, counit and coproduct. It is a quasitriangular Hopf algebra with universal R-matrix \( R = (f^a \otimes 1) \otimes (1 \otimes e_a) \). This is equivalent to Drinfeld’s definition \[1\] with generators and relations.

The first thing to notice is that the Hopf algebra construction here works perfectly well if \( H \) is not finite-dimensional: in this case it is really a function of two Hopf algebras \( A, H \) which are dually paired by a map \( \langle \ , \, \rangle : H \otimes A \to \mathbb{C} \). We call this \( D(A, H, \langle \ , \, \rangle) \) the *generalised quantum double*. Moreover, it is not necessary to have Hopf algebras either. All this works at the bialgebra level if we replace \( \langle S( ) \, , \, \rangle \) by the *convolution inverse map* \( \langle \ , \, \rangle^{-1} \). The linear maps from \( H \otimes A \to \mathbb{C} \) form an algebra using the coproduct of \( H \otimes A \), and we need our pairing invertible in this algebra \[11\].

Next, we introduce a general construction of which this generalised quantum double is an example. This is the concept of a *double cross product*. We consider two bialgebras or Hopf algebras which act on each other by maps

\[
\triangleleft : H \otimes A \to H, \quad \triangleright : H \otimes A \to A
\]

which are coalgebra homomorphisms (one says that \( H \) is a right \( A \)-module coalgebra and \( A \) is a left \( H \)-module coalgebra). We suppose further that these actions are a *matched pair* in that they obey the compatibility conditions \[2\]

\[
(hg)\triangleleft a = (h\triangleleft a_1)\triangleright (g_2 \triangleleft a_2), \quad 1\triangleleft a = \epsilon(a)
\]

\[
h\triangleright (ab) = (h_1 \triangleright a_1)\triangleleft (h_2 \triangleright a_2)\triangleright b, \quad h\triangleright 1 = \epsilon(h)
\]

\[
h_{(1)} \triangleleft a_{(1)} \otimes h_{(2)} \triangleright a_{(2)} = h_{(2)} \triangleleft a_{(2)} \otimes h_{(1)} \triangleright a_{(1)}
\]

In this case, one has the theorem \[2\] that there is a *double cross product bialgebra* \( A\triangleright \triangleleft H \) built on the vector space \( A \otimes H \) with product

\[
(a \otimes h)(b \otimes g) = a(h_{(1)} \triangleright b_{(1)}) \otimes (h_{(2)} \triangleleft b_{(2)}) g
\]

and tensor product unit, counit and coproduct. It is clear that \( A, H \) are subalgebras and that \( A\triangleright \triangleleft H \) factorises into them. The theorem is:

**Theorem 2.1** cf.\[3\]. If a bialgebra \( X \) factorises in the sense that there are sub-bialgebras

\[
A \hookrightarrow X \leftarrow H
\]

such that \( \cdot \circ (i \otimes j) : A \otimes H \to X \) is an isomorphism of vector spaces, then \( A, H \) are a matched pair as above and \( X \cong A \triangleright \triangleleft H \).
Proof This is based on [2, Sec. 3.2] but in a cleaner form. For this reason we give the full details. Since $\cdot \circ (i \otimes j)$ is a linear isomorphism we have a well-defined linear map $\Psi : H \otimes A \to A \otimes H$ defined by

$$j(h)i(a) = \cdot \circ (i \otimes j) \circ \Psi(h \otimes a).$$

Associativity in $X$ and that $i, j$ are algebra maps tells us that

$$\cdot \circ (i \otimes j) \circ \Psi(hg \otimes a) = j(hj(g)i(a)) = j(h)j(g)i(a)$$

$$= j(h) \cdot \circ (i \otimes j) \circ \Psi(g \otimes a) = \cdot \circ (i \otimes j) \circ (id \otimes \cdot) \circ \Psi_{12} \circ \Psi_{23}(h \otimes g \otimes a)$$

$$- \cdot \circ (i \otimes j) \circ \Psi(h \otimes ab) = j(h)i(ab) = j(h)i(a)i(b)$$

$$= \cdot \circ (i \otimes j) \circ \Psi(h \otimes a)i(b) = \cdot \circ (i \otimes j) \circ (\cdot \otimes id) \circ \Psi_{23} \circ \Psi_{12}(h \otimes a \otimes b)$$

so we conclude from this and by a similar consideration of the identity element in $X$ that

$$\Psi \circ (\cdot \otimes id) = (id \otimes \cdot) \circ \Psi_{12} \circ \Psi_{23}, \quad \Psi(1 \otimes a) = a \otimes 1$$

$$\Psi \circ (id \otimes \cdot) = (\cdot \otimes id) \circ \Psi_{23} \circ \Psi_{12}, \quad \Psi(h \otimes 1) = 1 \otimes h$$

(4)

where the suffices refer to the tensor factor on which $\Psi$ acts. We see that the products in $H$ and $A$ respectively ‘commute’ with $\Psi$ in a certain sense. This much works just at the algebra level (factorisations of algebras are of this form). Note that the braided tensor product of algebras [12] is an example of such an algebra factorisation. Its introduction was motivated by this part of the double cross product theorem.

Next, we use the counits $\epsilon$ to define linear maps $\lhd : H \otimes A \to H$ and $\rhd : H \otimes A \to A$ by

$$\rhd = (id \otimes \epsilon) \circ \Psi, \quad \lhd = (\epsilon \otimes id) \circ \Psi.$$

Applying $id \otimes \epsilon$ to the first line of (4) and $\epsilon \otimes id$ to the second tells us that $\rhd$ is a left action and $\lhd$ is a right action as required. Applying $\epsilon \otimes id$ to the first and $id \otimes \epsilon$ to the second tells us that

$$(hg) \lhd a = (h \circ \Psi(g \otimes a)), \quad 1 \lhd a = \epsilon(a)$$

$$h \rhd (ab) = (\Psi(h \otimes a) \rhd b), \quad h \rhd 1 = \epsilon(h).$$

This much works for algebras equipped with homomorphisms $\epsilon$. Next we use that $i, j$ are coalgebra maps to deduce that $\cdot \circ (i \otimes j) : A \otimes H \to X$ and $\cdot \circ (j \otimes i) : H \otimes A \to X$ are coalgebra maps too. The inverse of $\cdot \circ (i \otimes j)$ is also a coalgebra map hence the ratio $\Psi$ is also a coalgebra map. I.e., we conclude that

$$\Delta_{A \otimes H} \circ \Psi = (\Psi \otimes \Psi) \circ \Delta_{H \otimes A}, \quad (\epsilon \otimes \epsilon) \circ \Psi(h \otimes a) = \epsilon(a)\epsilon(h).$$
Now we apply $\text{id} \otimes \epsilon \otimes \epsilon \otimes \text{id}$ both sides of the first of these to conclude that

$$h_{(1)} \triangleright a_{(1)} \otimes h_{(2)} \triangleleft a_{(2)} = \Psi(h \otimes a)$$

which means that our results above prove the first two of (2). Applying instead the map $\epsilon \otimes \text{id} \otimes \text{id} \otimes \epsilon$ gives by contrast

$$h_{(1)} \triangleleft a_{(1)} \otimes h_{(2)} \triangleright a_{(2)} = \tau \circ \Psi(h \otimes a)$$

where $\tau$ is usual transposition. This proves the third of (2). Likewise, applying instead $\epsilon \otimes \text{id} \otimes \epsilon \otimes \text{id}$ gives that $\triangleleft$ is a coalgebra map, while applying $\text{id} \otimes \epsilon \otimes \text{id} \otimes \epsilon$ gives that $\triangleright$ is a coalgebra map too. Hence we have all the conditions needed for a matched pair in the sense of (2). Finally, looking again at the equation for $j(h)i(a)$ above tells us now that $\cdot(i \otimes j)$ becomes an isomorphism between the corresponding double cross product $A \bowtie H$ and our original bialgebra $X$. □

The quantum double fits as an example of this theory as follows: given two paired bialgebra or Hopf algebras we define

$$h\triangleleft a = h_{(2)} \langle h_{(1)}, a_{(1)} \rangle^{-1} \langle h_{(2)}, a_{(2)} \rangle, \quad h\triangleright a = a_{(2)} \langle h_{(1)}, a_{(1)} \rangle^{-1} \langle h_{(2)}, a_{(3)} \rangle$$

and check that they obey the conditions (2) for a matched pair $(A^\text{op}, H, \triangleleft, \triangleright)$. We recover the generalised quantum double as

$$D(A, H, \langle \ , \rangle) = A^\text{op} \bowtie H$$

which is how the generalised quantum double was first introduced in [11]. In the converse direction, one knows that the double factorises [1], and can use the above theorem to deduce $\triangleleft, \triangleright$.

Next, we make a trivial and completely cosmetic change: we denote $A^\text{op}$ in the above generalised quantum double by $A$. Then of course this new $A$ is no longer dually paired with $H$: instead we denote the same linear map $\langle \ , \rangle$ as a skew pairing $\sigma : H \otimes A \to \mathbb{C}$. It obeys the axioms

$$\sigma(hg \otimes a) = \sigma(h \otimes a_{(1)})\sigma(g \otimes a_{(2)}), \quad \sigma(h \otimes ab) = \sigma(h_{(1)} \otimes b)\sigma(h_{(2)} \otimes a).$$

(5)

which are just the axioms for $\langle \ , \rangle$ in terms of the new $A$ which has the opposite product to the old one which was dually paired. Then

$$D(A, H, \sigma) \equiv A \bowtie H$$

(6)

$$(a \otimes h)(b \otimes g) = ab_{(2)} \otimes h_{(2)}g\sigma^{-1}(h_{(1)} \otimes b_{(1)})\sigma(h_{(3)} \otimes b_{(3)})$$
with tensor product unit and counit is a double cross product by
\[ h \triangleleft a = h_{(2)} \sigma^{-1}(h_{(1)} \otimes a_{(1)}) \sigma(h_{(3)} \otimes a_{(2)}) \]
\[ h \triangleright a = a_{(2)} \sigma^{-1}(h_{(1)} \otimes a_{(1)}) \sigma(h_{(2)} \otimes a_{(3)}). \]

Finally, it is well-known in quantum groups that every construction has a dual one. It is an easy exercise to cast the above into their dual form (and not a new result to do so!). We list the result here for completeness. Firstly, the general construction is a double cross coproduct. We consider two bialgebras or Hopf algebras coacting on each other by maps
\[ \alpha : A \to A \otimes H, \quad \beta : H \to A \otimes H \]
which are algebra homomorphisms (comodule algebras). We require further that the coactions are matched in the sense
\[
\begin{align*}
(\Delta \otimes \text{id}) \circ \alpha(a) &= ((\text{id} \otimes \beta) \circ \alpha(a_{(1)})) (1 \otimes \alpha(a_{(2)})) \\
(\text{id} \otimes \Delta) \circ \beta(h) &= (\beta(h_{(1)}) \otimes 1) ((\alpha \otimes \text{id}) \circ \beta(h_{(2)})) \\
\alpha(a)\beta(h) &= \beta(h)\alpha(a).
\end{align*}
\]

In this case, one has the theorem that there is a bialgebra \( H \triangleright A \) with tensor product algebra structure and counit, and
\[ \Delta(h \otimes a) = h_{(1)} \otimes \alpha(a_{(1)})\beta(h_{(2)}) \otimes a_{(2)}. \]

In the Hopf algebra setting this is a Hopf algebra. The double cross coproduct comes equipped with bialgebra or Hopf algebra surjections
\[ H \xleftarrow{p} H \triangleright A \xrightarrow{q} A \]
into which factors the double cross coproduct decomposes by \((p \otimes q) \circ \Delta\). Conversely, and bialgebra decomposing like this is a double cross coproduct.

Our generalised quantum double example \(\mathfrak{D}\) becomes a generalised quantum codouble associated to a \textit{skew copairing} between bialgebras or Hopf algebras \(A, H\). This is an element \(\sigma \in A \otimes H\) such that
\[
\begin{align*}
(\Delta \otimes \text{id})\sigma &= \sigma_{13}\sigma_{23}, \\
(\text{id} \otimes \Delta)\sigma &= \sigma_{13}\sigma_{12},
\end{align*}
\]
which is just the dual of \(\mathfrak{F}\): We suppose \(\sigma\) is invertible. Then
\[ D^*(H, A, \sigma) = H \triangleright A \]
\[ \Delta(h \otimes a) = \sigma_{23}^{-1}\Delta_{H \otimes A}(h \otimes a)\sigma_{23}, \quad S(h \otimes a) = \sigma_{21}(Sh \otimes Sa)\sigma_{21}^{-1} \]
with tensor product counit and algebra structure is a double cross coproduct by coactions
\[ \alpha(a) = \sigma^{-1}(a \otimes 1)\sigma, \quad \beta(h) = \sigma^{-1}(1 \otimes h)\sigma. \]

This general construction includes the dual of Drinfeld’s quantum double as \( D(H)^* = H^{\text{cop}} \bowtie H^* \). As well as being the dual of our generalised quantum double, the resulting examples \((10)\) of double cross coproducts associated to a skew copairing were found independently in \([8]\) by another route as a ‘twisted product’. The double cross coproduct construction itself is however, much more general than this.

3 Quantum double as a twisting, I

If \( A \) is a dual quasitriangular bialgebra (such as \( A(R) \) according to results in \([13, \text{Sec. 3}]\)) then the universal-R-matrix functional \( \mathcal{R} : A \otimes A \to \mathbb{C} \) as in \([14][15]\) obeys \((5)\). Hence we have a double cross product \([11]\)
\[ D(A, A, \mathcal{R}) = A \bowtie A \]
which we identified\([3]\) as a natural construction for the \( q \)-Lorentz group. It is not in general isomorphic to Drinfeld’s double, i.e. we use the more general double \((3)\) coming out of the theory of double cross products. There is however, a Hopf algebra homomorphism \([3, \text{Sec. 4}]\)
\[ A \bowtie A \to D^*(H), \quad a \otimes b \mapsto (\text{id} \otimes a_{(1)} \otimes b_{(1)})(\mathcal{R}_{12}^{-1}\mathcal{R}_{31}) \otimes a_{(2)}b_{(2)} \]
if \( A \) is dual to \( H \). It is an isomorphism in some (factorisable) cases such as \( SU_q(2) \bowtie SU_q(2) \). In matrix generators \( s, t \) for the latter, the above map is \([6]\)
\[ s \otimes 1 \mapsto S^+ \otimes t, \quad 1 \otimes t \mapsto S^- \otimes t \]
where \( I^\pm \) are the matrix FRT generators \([14]\) of \( U_q(su_2) \). Its codouble as an algebra is the tensor product \( U_q(su_2) \otimes SU_q(2) \). The same formula works for general \( U_q(g) \).

As explained in \([1]\) this is just a dual version of a corresponding result in enveloping algebra form which is essentially due to \([8]\). Thus, if \( H \) is a strict quantum group with universal R-matrix (such as \( U_q(g) \)) then \( \sigma = \mathcal{R} \) is a skew copairing and we have an example of a generalised codouble or double cross coproduct
\[ D^*(H, H, \mathcal{R}) = H \bowtie H \]
recovering the ‘twisted square’ introduced in another framework in \([8]\). These authors showed that
\[ D(H) \to H \bowtie H, \quad a \otimes h \mapsto \left((\text{id} \otimes a)(\mathcal{R}_{31}^{-1}\mathcal{R}_{23})\right)(\Delta h) \]
is a homomorphism of Hopf algebras, and argued that it should be an isomorphism if \( Q = R^{-1} R_{21}^{-1} \) is invertible when regarded as a linear map \( H^* \rightarrow H \) by evaluation on its first tensor factor (a version of the factorisable case). The required inverse map was not given in [8] and appears here I think for the first time (being obtained in fact from the bosonisation theory in Section 6). It is

\[
(h \otimes g) \mapsto (Q^{-1}(hSg_{(1)}))_{(1)}(R^{-1}, (Q^{-1}(hSg_{(1)}))_{(2)}) \otimes R_{21}^{-1}g_{(2)}.
\]

According to [6, Sec. 4] the example \( U_q(su_2) \bowtie U_q(su_2) \) is the correct construction for the enveloping algebra of the q-Lorentz group dual to [10] and is isomorphic to \( D(U_q(su_2)) \) by the above maps. In matrix \( 1^\pm \) generators the isomorphism from the quantum double to the twisted square is

\[
t \otimes 1 \mapsto 1^- \otimes 1^+, \quad 1 \otimes 1^\pm \mapsto 1^\pm \otimes 1^\pm.
\]

We also introduced in [6] a \( * \)-structure in \( A \bowtie A \) given by \( (a \otimes b)^* = b^* \otimes a^* \) in the case when \( R \) is of real-type in the sense in [6]. It is not a new result to repeat this in the dual form for \( H \bowtie H \) where it immediately comes out as

\[
(h \otimes g)^* = R_{21}(g^* \otimes h^*)R_{21}^{-1} : R^{* \otimes *} = R_{21}.
\]

It is also clear from the form of the product of \( A \bowtie A \) and \( H \bowtie H \) that they are examples of dual-twisting and twisting respectively in the more modern sense by a non-Abelian 2-cocycle [17], a theory due to Drinfeld in [18] and restricted to Hopf algebras in [19]. Briefly, a 2-cocycle on a quantum group is \( \chi : A \otimes A \rightarrow \mathbb{C} \) such that

\[
\chi(b_{(1)} \otimes c_{(1)})\chi(a \otimes b_{(2)}c_{(2)}) = \chi(a_{(1)} \otimes b_{(1)})\chi(a_{(2)}b_{(2)} \otimes c),
\]

\[
\chi(1 \otimes a) = \epsilon(a).
\]

The other side \( \chi(a \otimes 1) = \epsilon(a) \) follows. Given such a cocycle, the new product

\[
a \hat{\cdot} b = \chi(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)}\chi(a_{(3)} \otimes b_{(3)})
\]

defines a new bialgebra \( \hat{A} \). Similarly for the antipode and any dual quasitriangular structures of \( \hat{A} \). It is clear from the form (6) of \( A \bowtie A \) that it is the twisting by

\[
\chi((a \otimes b) \otimes (c \otimes d)) = \epsilon(a)R^{-1}(b \otimes c)\epsilon(d)
\]

as a 2-cocycle on \( A \otimes A \). In the braided geometrical approach to space time \( SU_q(2) \otimes SU_q(2) \) acts covariantly on a q-Euclidean space and this cocycle twists it to \( SU_q(2) \bowtie SU_q(2) \) acting on q-Minkowski space, i.e. it is physically a quantum Wick rotation [20].
Likewise, a 2-cocycle for a quantum group $H$ is $\chi \in \mathcal{H} \otimes \mathcal{H}$ such that

$$
\chi_{23}(\text{id} \otimes \Delta)\chi = \chi_{12}(\Delta \otimes \text{id})\chi, \quad (\epsilon \otimes \text{id})\chi = 1
$$

(13)

and means that $\tilde{H}$ with $\tilde{\Delta} = \chi(\Delta)\chi^{-1}$ etc., is also a quantum group. This twisting is even more clear for $H \triangleright H$ where the coproduct (11) is obviously a twisting by a 2-cocycle $\chi = R_{23}^{-1}$ for $H \otimes H$. So this picture of the q-Lorentz enveloping algebra is likewise an immediate corollary of its description introduced by the author in [6] as the twisted square $U_q(su_2) \triangleright U_q(su_2)$. Some of this has subsequently been reiterated by other authors following [6].

4 Quantum double as a twisting, II

Now we look at a more well-known topic which is the quantum double of the Borel subalgebra $U_q(b-)$ say of $U_q(sl_2)$. It is known that $D(U_q(b_+))$ projects onto $U_q(sl_2)$ and this is indeed how the quasitriangular structure of the latter was deduced [1]. Likewise for general $U_q(g)$.

We begin by recalling this calculation with a modern proof based on q-geometry. For $U_q(b_-)$ we take the Hopf algebra $H$ generated by $X, g, g^{-1}$ say and

$$
gX = q^2 Xg, \quad \Delta X = X \otimes 1 + g \otimes X, \quad \Delta g = g \otimes g
$$

$$
\epsilon X = 0, \quad \epsilon g = 1, \quad Sg = g^{-1}, \quad SX = -g^{-1}X.
$$

We show that when $q^2 \neq 0, 1$ it is a self-dual Hopf algebra in the sense that it has a Hopf algebra pairing with itself given by

$$
\langle g, g \rangle = q^2, \quad \langle g, X \rangle = 0, \quad \langle X, g \rangle = 0, \quad \langle X, X \rangle = \frac{1}{1 - q^{-2}}.
$$

This is essentially due to Drinfeld in [1], but here is a new proof. We define a right action of $H$ on itself by

$$
\phi(X, g) \triangleright g = \phi(q^2 X, q^2 g), \quad \phi(X, g) \triangleright X = \lambda \partial_q \phi(X, g), \quad \lambda = \frac{1}{1 - q^{-2}}
$$

where $\phi(X, g) = \sum \phi_{ab} X^a g^b$ is assumed to have powers of $X$ to the left and

$$
\partial_q X^m g^n = [m; q^2] X^{m-1} g^n; \quad [m; q^2] = \frac{1 - q^{2m}}{1 - q^2}
$$

is a q-derivative. Since $\partial_q$ lowers the degree by 1, it is clear that $(\phi \triangleright g) \triangleright X = q^2 (\phi \triangleright X) \triangleright g$ for all $\phi \in H$, so $\triangleright$ indeed defines a right action of our Hopf algebra on itself. Moreover, one has

$$
(\phi \psi) \triangleright g = (\phi \triangleright g)(\psi \triangleright g), \quad (\phi \psi) \triangleright X = (\phi \triangleright X) \psi + (\phi \triangleright g)(\psi \triangleright X)
$$

(14)
for all normal-ordered polynomials $\phi(X,g), \psi(X,g)$. Note that the product $\phi \psi$ must be normal-ordered using the q-commutation relations between $X,g$ before we can apply the definition of $\triangleleft$. The action of $g$ on products is clear since the normal ordering process commutes with scaling of the generators by $q^2$. The action of $X$ is easily verified on monomials as

$$
(X^m g^n X^k g^l) \triangleleft X = (q^{2nk} X^{m+k} g^{n+l}) \triangleleft X = q^{2nk} \lambda [m+k; q^2] X^{m+k-1} g^{n+l}
$$

$$
= q^{2nk} \left( \lambda [m; q^2] + q^{2m} \lambda [k; q^2] \right) X^{m+k-1} g^{n+l}
$$

$$
= \lambda [m; q^2] X^{m-1} g^n X^k g^l + q^{2(m+n)} X^m g^n \lambda [k; q^2] X^k g^l
$$

$$
= ((X^m g^n) \triangleleft X) X^k g^l + ((X^m g^n) \triangleleft g) (X^k g^l) \triangleleft X
$$

from the definition of $\triangleleft$ and the relations of $H$. This result (14) implies that the product of $H$ is covariant (a module algebra) under the action $\triangleleft$.

Given this action $\triangleleft$, we now define

$$
\langle h, a \rangle = \epsilon(h \triangleleft a)
$$

and deduce that it obeys half the axioms of a pairing from our module algebra property (14). But since the resulting pairing

$$
\langle X^m g^n, X^k g^l \rangle = \epsilon ((X^k g^l) \triangleleft X^m) \triangleleft g^n) = \delta_{m,k} \lambda^m [m; q^2] q^{2nl}
$$

is symmetric in the two arguments of $\langle \ , \ \rangle$, we also deduce the other half by symmetry.

This proof also has the merit that the nondegeneracy of the pairing is clear, at least when $q$ is generic. This is because $\langle \phi(X,g), X^m g^n \rangle = \lambda^m (\partial_q^m \phi)(0, q^{2n})$ vanishing for all $m, n$ implies that $\phi$ vanishes too. Reducing non-degeneracy to the q-derivative or higher-dimensional braided-derivative[21] appears to be a useful strategy of proof which I have not seen elsewhere.

It is now an easy matter to compute $D(U_q(b_-)) = H^{op} \triangleleft H$ as generated by $H = U_q(b_-)$ as above and $H^{op} = U_q(b_+)$ with generators $\tilde{X}, \tilde{g}, \tilde{g}^{-1}$ say. The result from (1) is

$$
gX = q^2 X g, \quad g\tilde{X} = q^{-2} \tilde{X} g, \quad X\tilde{X} - q^{-2} \tilde{X} X = \frac{gq-1}{q-1},
$$

$$
\Delta X = X \otimes 1 + g \otimes X, \quad \Delta \tilde{X} = \tilde{X} \otimes 1 + \tilde{g} \otimes \tilde{X}
$$

with the corresponding counit and antipode. The generators $g, \tilde{g}$ are group-like.

We want to make two comments about this quantum double. The first working over formal powerseries $\mathbb{C}[[t]]$ in a deformation parameter $t$, and the second in Section 5 working over $\mathbb{C}$. In the first case we change variables

$$
X = q^{-\frac{H}{2}} X_-, \quad \tilde{X} = q^{-\frac{H}{2}} X_+, \quad g = q^{-H} q^C, \quad \tilde{g} = q^{-H} q^{-C}
$$
to new generators $X_\pm, H, C$ then we get the usual algebra relations of $U_q(sl_2)$ along with $C$
central and primitive and
\[ \Delta X_\pm = X_\pm \otimes q^{\frac{\mu}{2}} + q^{-\frac{\mu}{2}}q^C \otimes X_\pm. \]

Following Drinfeld one can now derive the universal R-matrix of $U_q(sl_2)$ from the one of
\[ D(U_q(b_-)) \]
after projecting by $C = 0$.

On the other hand, we can consider the 1-dimensional quantum group $U_q^{-1}(\mathbb{C})$ with generator $C$
and
\[ \Delta C = C \otimes 1 + 1 \otimes C, \quad R = q^{-\frac{1}{2}}C \otimes C \]
This is called the quantum line in \cite{22}. On the Hopf algebra $U_q(sl_2) \otimes U_q^{-1}(\mathbb{C})$ we define
\[ \chi = q^{-\frac{1}{2}}C \otimes H \]
and check that it is a 2-cocycle. Then $[C \otimes H, ( ) \otimes X_\pm] = \pm 2C( ) \otimes X_\pm$ tells us that the

coproducts of $X_\pm$ in the double are just the conjugation by $\chi$ of the usual coproduct of $U_q(sl_2)$,
i.e. we have
\[ D(U_q(sl_2)) \cong U_q(sl_2) \otimes U_q^{-1}(\mathbb{C}) \]
as twisted by this cocycle. With a bit more work we have this as a twisting of quasitriangular
Hopf algebras.

5 Quantum double as a cocycle bicrossproduct

Now we recall another aspect of abstract Hopf algebra theory, which is the notion of a cocycle
cross product. Thus a (right) cocycle action of a Hopf algebra $H$ on an algebra $A$ is linear maps
\[ \triangleright: A \otimes H \rightarrow A, \quad \chi: H \otimes H \rightarrow A \]
where
\[ 1 \triangleright h = 1 \epsilon(h), \quad (ab) \triangleright h = (a \triangleright h_{(1)})(b \triangleright h_{(2)}) \]
\[ a \triangleright 1 = a, \quad \chi(h_{(1)} \otimes g_{(1)})(a \triangleright h_{(2)} \otimes g_{(2)}) = (a \triangleright (h_{(1)}g_{(1)}))\chi(h_{(2)} \otimes g_{(2)}) \]
\[ \chi(h_{(1)}g_{(1)} \otimes f_{(1)})(\chi(h_{(2)} \otimes g_{(2)}) \triangleright f_{(2)}) = \chi(h \otimes g_{(1)}f_{(1)})\chi(g_{(2)} \otimes f_{(2)}) \]
\[ \chi(1 \otimes h) = \chi(h \otimes 1) = \epsilon(h) \]
for all $h, g, f \in H$ and $a, b \in A$. In this situation, there is a right cocycle cross product algebra
$H \triangleright \chi A$ on the vector space $H \otimes A$ with product
\[ (h \otimes a)(g \otimes b) = h_{(1)}g_{(1)} \otimes \chi(h_{(2)} \otimes g_{(2)})(a \triangleright g_{(3)})b. \]
That this product is associative follows from the assumptions (17). A cohomological picture of such cocycle cross coproducts is due to [23]. They can be interpreted as trivial quantum principal bundles in quantum group gauge theory [24], with $\chi$ a new quantum number not present classically.

In the dual setting we have of course the notion of a cocycle coaction. So a left cocycle coaction of a Hopf algebra $A$ on a coalgebra $H$ for example means maps

$$\beta : H \rightarrow A \otimes H, \quad \psi : H \rightarrow A \otimes A$$

obeying some axioms dual to those above. See [11]. We will not need the most general form with the dual-cocycle $\psi$. In this case we have a usual cross coproduct: if $\beta(h) = h^{(1)} \otimes h^{(2)}$ is a coaction of $A$ on the coalgebra $H$ and respects its structure in the sense

$$(\text{id} \otimes \Delta) \circ \beta(h) = h^{(1)} \otimes h^{(2)} \otimes h^{(3)}, \quad (\text{id} \otimes \epsilon) \circ \beta = \epsilon$$

(a comodule coalgebra) then there is a cross coalgebra $H \prec A$ built on $H \otimes A$ with

$$\Delta(h \otimes a) = h^{(1)} \otimes h^{(2)} \otimes a^{(1)} \otimes a^{(2)} \otimes a^{(3)}, \quad \epsilon(h \otimes a) = \epsilon(h)\epsilon(a).$$

(19)

(20)

Putting these ideas together we suppose next that $H$ acts on $A$ by a cocycle action as above and $A$ coacts back on $H$. We can demand conditions on these such that the resulting cross product algebra and coalgebra fit together to form a bicrossproduct Hopf algebra $H \triangleright A$. This was introduced for general Hopf algebras by the author in [2] (without cocycles) and [11] (with cocycles), with some examples in [2] [25] and [26] respectively. We refer to these papers for the precise compatibility conditions which ensure that the bicrossproduct is a Hopf algebra. A general feature of the bicrossproduct is that

$$A \hookrightarrow H \triangleright A \twoheadrightarrow H$$

by the obvious inclusion and the obvious projection via $\epsilon$. There is a theorem that every extension of Hopf algebras like this which is invertible in a certain sense (the extension should be cleft and cocleft) is of this bicrossproduct type, possibly with cocycles.

For a recent application of this theory of extensions to physics, see [27]. We want to note only that that quantum double of $H = U_q(b_-)$ in the form computed in the last section, is exactly such an extension

$$\mathbb{C}Z \hookrightarrow D(U_q(b_-))^p \twoheadrightarrow U_q(sl_2).$$
Here \( p \) is defined by

\[
p(X) = F \equiv q^{-\frac{H}{2}} X_-, \quad p(\bar{X}) = E \equiv q^{-\frac{H}{2}} X_+, \quad p(g) = p(\bar{g}) = q^{-H}
\]  \hspace{1cm} (21)

where the \( E, F, q^{\pm H} \) can be considered as generating a version of \( U_q(sl_2) \). It is a sub-Hopf algebra of the usual form since we do not use \( q^{\pm H} \) themselves. With this form of \( U_q(sl_2) \) understood, we have \( p \) as a surjection. Next, it is clear that the invertible element \( K = \bar{g} g^{-1} \) generates \( CZ \subset D(H) \) as a commutative sub-Hopf algebra. Moreover, it is clear that \( D(H) \cong U_q(sl_2) \otimes CZ \) as vector spaces by \( X^m \bar{X}^n g^k K^l \mapsto F^m E^n q^{-Hk} \otimes K^l \). Finally, one verifies that this isomorphism respects multiplication by \( K \) from the right, and the coaction of \( U_q(sl_2) \) from the left in the manner required for a Hopf algebra extension.

This and some technical cleftness conditions which one can also verify means that our quantum double must be a cocycle bicrossproduct. It is

\[
D(U_q(b_-)) \cong U_q(sl_2) \bowtie \chi CZ, \quad \beta\left( \begin{array}{c} E \\ F \\ q^{-H} \end{array} \right) = \begin{cases} K \otimes E \\ 1 \otimes F \\ 1 \otimes q^{-H} \end{cases}, \quad \chi(E \otimes F) = \frac{1 - K}{1 - q^{-2}}
\]

with \( \chi = \epsilon \otimes \epsilon \) on the other generators of \( U_q(sl_2) \). The action \( \triangleleft \) in the bicrossproduct and the dual-cocycle \( \psi \) are both trivial. We identify \( X = F \otimes 1, \bar{X} = E \otimes 1, g = q^{-H} \otimes 1 \) and \( K \equiv 1 \otimes K \). Then the product from (18) as modified by the cocycle \( \chi \) gives

\[
\bar{X}X = (E \otimes 1)(F \otimes 1) = E_{(1)} F_{(1)} \otimes \chi( E_{(2)} \otimes F_{(2)})
\]

\[
= EF \otimes 1 + g^2 \otimes \chi(E \otimes F) = EF \otimes 1 + g^2 \otimes \frac{1 - K}{1 - q^{-2}}
\]

\[
= q^2 FE \otimes 1 + \frac{1 - g^2}{1 - q^{-2}} \otimes 1 + g^2 \otimes \frac{1 - K}{1 - q^{-2}}
\]

\[
= q^2 (F \otimes 1)(E \otimes 1) + \frac{1 - g^2 \otimes K}{1 - q^{-2}} = q^2 \bar{X}X + \frac{1 - g^2 K}{1 - q^{-2}}
\]

which are the relations (16) for the quantum double. The coalgebra structure (20) as modified by \( \beta \) is

\[
\Delta \bar{X} = \Delta (E \otimes 1) = E \otimes 1 \otimes 1 \otimes 1 + q^{-H} \otimes \beta(E) \otimes 1 = \bar{X} \otimes 1 + gK \otimes \bar{X}
\]

as required in (16). This completes our bicrossproduct description of \( D(U_q(b_-)) \). We see that as an algebra it is a central extension of \( U_q(sl_2) \) by the cocycle \( \chi \), and as a coalgebra it is a cross coproduct. Since \( CZ = \mathbb{C}(S^1) \) we can think of this as a quantum principle bundle over \( S^1 \) with fibre \( SU_q(2)^* \) in the geometrical picture of [24]. For a full geometrical picture one should consider the \( * \)-structure also, with the above as the chiral part of the full picture.
6 Quantum double as a bosonisation

For completeness, we conclude here with a brief mention of another abstract result on the quantum double already published in [3][5][7]. This applies to the case when $A$ is a dual quasitriangular Hopf algebra or bialgebra (such as $A(R)$). According to the theory of transmutation [12] it has a braided group version $B$ which lives in the braided category of representations of the corresponding quasitriangular Hopf algebra $H$ (such as $U_q(g)$). For example, the transmutation of the quantum matrices $A(R)$ is just the braided matrices $B(R)$ introduced in this way in [28] with matrix generators $u$ and relations $R_{21}u_1Ru_2 = u_2R_{21}u_1R$. It turns quantum $2 \times 2$ matrices $M_q(2)$ into braided (hermitian) $2 \times 2$ matrices $BM_q(2)$ or $q$-Minkowski space. Likewise at the Hopf algebra level it turns $SU_q(2)$ into the braided group $BSU_q(2)$ which is the mass-shell in $q$-Minkowski space [7].

On the other hand, there is a general theory of bosonisation [29] which turns any braided group $B$ in such a representation category into an ordinary quantum group $B \rtimes H$. For the algebra we make a cross product as in (18) but with a left action (say) rather than a right one, and without a cocycle. By definition, $B$ is covariant under $H$ and we use this action $\triangleright$ for the cross product. For the coproduct we use the induced left coaction $\triangleright$

$$\beta : B \rightarrow H \otimes B, \quad \beta(b) = R_{21}\triangleright b$$

and make the cross coproduct (20). This is a generalisation to braided groups of the Jordan-Wigner transformation for superalgebras.

When applied to braided planes, this bosonisation gives us $q$-Poincaré Hopf algebras [3]. On the other hand, when applied to the above-mentioned braided group obtained from $A$ dual to $H$, it gives the quantum double [3][5]

$$B \rtimes H \cong D(H).$$

Cross product algebras can also be considered as quantisation of position functions $B$ by momentum quantum group $H$, which is the interpretation developed in [3]. We can also consider the quantum double in this form as a trivial quantum principal bundle [24]. For example, $BSU_q(2) \rtimes U_q(su_2)$ is a bundle over the mass-shell in $q$-Minkowski space with fibre $SU_q(2)^*$. 

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