Abstract. Let $X$ denote a metric Lie group diffeomorphic to $\mathbb{R}^3$ that admits an algebraic open book decomposition. In this paper we prove that if $\Sigma$ is an immersed surface in $X$ whose left invariant Gauss map is a diffeomorphism onto $S^2$, then $\Sigma$ is an embedded sphere. As a consequence, we deduce that any constant mean curvature sphere of index one in $X$ is embedded.

1. Introduction.

Let $Y$ denote a simply connected, homogeneous Riemannian three-manifold, and assume that $Y$ is not isometric to a Riemannian product $S^2(\kappa) \times \mathbb{R}$, where $S^2(\kappa)$ is the two-sphere with a metric of constant Gaussian curvature $\kappa > 0$. Then, $Y$ is isometric to a metric Lie group $X = (G, \langle \cdot, \cdot \rangle)$, i.e., a simply connected, three-dimensional Lie group $G$ equipped with a left invariant metric $\langle \cdot, \cdot \rangle$.

In this paper we consider immersed oriented surfaces in metric Lie groups, and we study them in terms of their left invariant Gauss map, which we define next.

Let $\psi: \Sigma \hookrightarrow X$ be an immersed oriented surface in a metric Lie group $X$, and let $N: \Sigma \to TX$ denote its unit normal vector field. Note that for any $x \in X$, the left translation $l_x: X \to X$ is an isometry of $X$. Thus, for every $p \in \Sigma$ there exists a unique unit vector $G(p)$ in the tangent space $T_eX$, such that
\[
(dl_\psi(p))_e (G(p)) = N(p), \quad \forall p \in \Sigma,
\]
where $e$ denotes the identity element of $X$.

Definition 1.1. We call the map $G: \Sigma \to S^2 = \{v \in T_eX \mid |v| = 1\}$ the left invariant Gauss map of the oriented surface $\psi: \Sigma \hookrightarrow X$.

Note that if $X$ is the Euclidean space $\mathbb{R}^3$ endowed with its usual abelian Lie group structure, the left invariant Gauss map is just the usual Gauss map for oriented surfaces in $\mathbb{R}^3$. In this situation, it is well known that if $\Sigma$ is a surface in $\mathbb{R}^3$ whose Gauss map is a diffeomorphism onto $S^2$, then $\Sigma$ bounds a strictly convex domain of $\mathbb{R}^3$; in particular, $\Sigma$ is an embedded topological sphere.

This embeddedness property does not hold in the general context of metric Lie groups. For instance, for certain metric Lie groups $X$ diffeomorphic to $S^3$, there exist immersed, non-embedded spheres in $X$ whose left invariant Gauss maps are diffeomorphisms onto $S^2$ (see Remark 3.5). It is then natural to investigate for which metric Lie groups $X$ it is true that any sphere in $X$ whose left invariant Gauss map is a diffeomorphism must be embedded.

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In this paper we give an affirmative answer to the embeddedness question above for a certain class of metric Lie groups, namely, for those admitting algebraic open book decompositions (see Definition 2.1).

**Theorem 1.2.** Let $X$ be a metric Lie group which admits an algebraic open book decomposition, and let $f: S \to X$ be an immersion of a sphere whose left invariant Gauss map $G: S \to S^2$ is a diffeomorphism. Then $f(S)$ is an embedded sphere (i.e., $f$ is an injective immersion).

We point out that the metric Lie groups which admit an algebraic open book decomposition were classified in the lecture notes [5] by the first and third authors; see Section 2 in this paper for more details. This class includes the hyperbolic three-space $\mathbb{H}^3$, the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$, the Riemannian Heisenberg space $\text{Nil}_3$ and the solvable Lie group $\text{Sol}_3$ with any of its left invariant metrics. We note that Theorem 1.2 was previously known only in the particular case $X = \mathbb{R}^3$.

Theorem 1.2 has an interesting application to the context of constant mean curvature surfaces. Indeed, if $\Sigma$ is an immersed oriented sphere of constant mean curvature in a metric Lie group $X$, and if $\Sigma$ has index one for its stability operator, it was proved in [4] that the left invariant Gauss map of $\Sigma$ is a diffeomorphism (this property was previously proved by Daniel and Mira in [3] for the case where $X$ is the three-dimensional Thurston geometry for $\text{Sol}_3$). Thus, the next corollary follows immediately from Theorem 1.2:

**Corollary 1.3.** Let $X$ be a metric Lie group which admits an algebraic open book decomposition, and let $\Sigma$ be an immersed constant mean curvature sphere of index one in $X$. Then $\Sigma$ is embedded.

Corollary 1.3 was previously known in the following cases:

1. If $X$ has constant curvature, then all constant mean curvature spheres in $X$ are totally umbilical round spheres (by Hopf’s classical theorem); in particular they are embedded and have index one.
2. If $X$ has isometry group of dimension 4, Abresch and Rosenberg [1, 2] proved that constant mean curvature spheres in $X$ are rotational spheres. If, additionally, $X$ is diffeomorphic to $\mathbb{R}^3$, then all such spheres have index one [7, 9] and are embedded. We note that all homogeneous manifolds diffeomorphic to $\mathbb{R}^3$ that have a 4-dimensional isometry group also admit a metric Lie group structure with an algebraic open book decomposition (see Remark 2.8).
3. If $X$ is the Lie group $\text{Sol}_3$ endowed with its standard maximally symmetric left invariant metric, then the statement in Corollary 1.3 was proved by Daniel and Mira in [3]. We note that our proof here is completely different from the approach in [3]. As a matter of fact, the proof in [3] uses that the standard left invariant metric in $\text{Sol}_3$ admits planes of reflectional symmetry, a property that is not true for arbitrary left invariant metrics on the Lie group $\text{Sol}_3$.

**Remark 1.4.** In view of Theorem 4.1 in [4], Corollary 1.3 represents an advance towards proving the following conjecture: *any constant mean curvature sphere in a homogeneous manifold diffeomorphic to $\mathbb{R}^3$ is embedded.*

In Section 2 we review several aspects of the geometry of metric Lie groups that we will need. In particular, we will explain the classification of metric Lie groups that admit an algebraic open book decomposition. Section 2 can be seen as introductory material; for a more complete introduction to the geometry of surfaces in metric Lie groups, we refer to [5] and [6].

In Section 3 we prove a more detailed version of Theorem 1.2, which gives relevant information (beyond embeddedness) about the geometry of spheres whose left invariant Gauss maps are diffeomorphisms (see Theorem 3.4).
2. Metric Lie groups and algebraic open book decompositions.

We next consider metric Lie groups that admit an algebraic open book decomposition. This notion is an extension to metric Lie groups of the usual notion of a pencil of half-planes in \( \mathbb{R}^3 \).

**Definition 2.1.** Let \( X \) be a metric Lie group and \( \Gamma \subset X \) a 1-parameter subgroup. An algebraic open book decomposition of \( X \) with binding \( \Gamma \) is a foliation \( \mathcal{B} = \{ L(\theta) \}_{\theta \in [0, \pi)} \) of \( X - \Gamma \) such that the sets

\[
H(\theta) = L(\theta) \cup \Gamma \cup L(\pi + \theta)
\]

are two-dimensional subgroups of \( X \), for all \( \theta \in [0, \pi) \). We will call \( L(\theta) \) the leaves and \( H(\theta) \) the subgroups of the algebraic open book decomposition \( \mathcal{B} \).

Observe that this definition only depends on the Lie group structure of \( X \), and not on its left invariant metric.

The metric Lie groups that admit an algebraic open book decomposition were classified by the first and third authors in [5]. All of them are semidirect products. We will next review some properties of metric semidirect products in order to present this classification.

### 2.1. Semidirect products.

Consider a semidirect product which is a Lie group \( (\mathbb{R}^3 \equiv \mathbb{R}^2 \times \mathbb{R}, +) \), where the group operation \( * \) is expressed in terms of some real \( 2 \times 2 \) matrix \( A \in \mathcal{M}_2(\mathbb{R}) \) as

\[
(p_1, z_1) * (p_2, z_2) = (p_1 + e^{z_1 A} p_2, z_1 + z_2);
\]

here \( e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k \) denotes the usual exponentiation of a matrix \( B \in \mathcal{M}_2(\mathbb{R}) \). We will use the notation \( \mathbb{R}^2 \rtimes_A \mathbb{R} \) to denote such a Lie group.

Suppose \( X \) is isomorphic to \( \mathbb{R}^2 \rtimes_A \mathbb{R} \), where

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Then, in terms of the coordinates \( (x, y) \in \mathbb{R}^2, z \in \mathbb{R} \), we have the following basis \( \{ F_1, F_2, F_3 \} \) of the space of right invariant vector fields on \( X \):

\[
F_1 = \partial_x, \quad F_2 = \partial_y, \quad F_3(x, y, z) = (ax + by) \partial_x + (cx + dz) \partial_y + \partial_z.
\]

In the same way, a left invariant frame \( \{ E_1, E_2, E_3 \} \) of \( X \) is given by

\[
E_1(x, y, z) = a_{11}(z) \partial_x + a_{21}(z) \partial_y, \quad E_2(x, y, z) = a_{12}(z) \partial_x + a_{22}(z) \partial_y, \quad E_3 = \partial_z,
\]

where

\[
e^{zA} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}.
\]

In terms of \( A \), the Lie bracket relations are:

\[
[E_1, E_2] = 0, \quad [E_3, E_1] = aE_1 + cE_2, \quad [E_3, E_2] = bE_1 + dE_2.
\]

Observe that \( \text{Span}\{E_1, E_2\} \) is an integrable two-dimensional distribution of \( X \), whose integral surfaces define the foliation \( \mathcal{F} = \{ \mathbb{R}^2 \rtimes_A \{ z \} \mid z \in \mathbb{R} \} \) of \( \mathbb{R}^2 \rtimes_A \mathbb{R} \). All the leaves of \( \mathcal{F} \) are complete and intrinsically flat with respect to the metric given in the following definition.

**Definition 2.2.** We define the canonical left invariant metric on the semidirect product \( \mathbb{R}^2 \rtimes_A \mathbb{R} \) to be that one for which the left invariant frame \( \{ E_1, E_2, E_3 \} \) given by (2.4) is orthonormal. Equivalently, it is the left invariant extension to \( X = \mathbb{R}^2 \rtimes_A \mathbb{R} \) of the inner product on the tangent space \( T_eX \) at the identity element \( e = (0, 0, 0) \) that makes \( (\partial_x)_e, (\partial_y)_e, (\partial_z)_e \) an orthonormal basis.
2.2. Unimodular semidirect products. A semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is unimodular if and only if \( \text{trace}(A) = 0 \). We next provide four examples of unimodular metric semidirect products. It was proved in [5] that if $X$ is a unimodular metric Lie group that is not isomorphic to $\text{SU}(2)$ or to $\text{SL}(2, \mathbb{R})$, then $X$ is isomorphic and (up to rescaling) isometric to one of these four examples.

**Example 2.3.** The abelian group $\mathbb{R}^3$. If the matrix $A$ is zero, the metric Lie group $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is the abelian group $\mathbb{R}^3$ endowed with a left invariant metric. All such left invariant metrics are isometric to the usual Euclidean metric of $\mathbb{R}^3$.

**Example 2.4.** The Heisenberg group. If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ gives a metric Lie group isomorphic to the Heisenberg group $\text{Nil}_3$. Up to rescaling, $\text{Nil}_3$ with an arbitrary left invariant metric is isometric to $X$. We note that all left invariant metrics on $\text{Nil}_3$ have a four-dimensional isometry group.

**Example 2.5.** The group $\text{Sol}_3$. If $A = \begin{pmatrix} 0 & c \\ 1/c & 0 \end{pmatrix}$, $c \geq 1$, then $X_c = \mathbb{R}^2 \rtimes_A \mathbb{R}$ gives a metric Lie group isomorphic to the Lie group $\text{Sol}_3$. Specifically, for each $c \geq 1$ the metric Lie group $X_c$ (with its canonical metric) is isomorphic to $\text{Sol}_3$. Conversely, up to rescaling, all left invariant metrics in $\text{Sol}_3$ are recovered by this model. We note that $c = 1$ corresponds to the Thurston geometry model of $\text{Sol}_3$, i.e., to the left invariant metric in $\text{Sol}_3$ with a maximal isometry group. Every left invariant metric of $\text{Sol}_3$ has an isometry group of dimension three.

**Example 2.6.** The group $\tilde{\text{E}}(2)$. Suppose $A = \begin{pmatrix} 0 & -c \\ 1/c & 0 \end{pmatrix}$, $c \geq 1$. Then $X_c = \mathbb{R}^2 \rtimes_A \mathbb{R}$ gives a metric Lie group isomorphic to $\tilde{\text{E}}(2)$, the Lie group defined as the universal cover of the Euclidean group of orientation-preserving rigid motions of the plane. These metric Lie groups do not admit algebraic open book decompositions, so we will not be more specific about them here.

2.3. Non-unimodular groups. The cases $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ with $\text{trace}(A) \neq 0$ correspond to the simply connected, three-dimensional, non-unimodular metric Lie groups. In these cases, up to the rescaling of the metric of $X$, we may assume that $\text{trace}(A) = 2$. This normalization in the non-unimodular case will be assumed from now on throughout the paper. After an orthogonal change of the left invariant frame (see Section 2.5 in [5] for details), we may express the matrix $A$ uniquely as

\[
(2.7) \quad A = A(a, b) = \begin{pmatrix} 1 + a & -(1 - a)b \\ (1 + a)b & 1 - a \end{pmatrix}, \quad a, b \in [0, \infty).
\]

The canonical basis of the non-unimodular metric Lie group $X$ is, by definition, the left invariant orthonormal frame $\{E_1, E_2, E_3\}$ given in (2.4) by the matrix $A$ in (2.7). In other words, every non-unimodular metric Lie group is isomorphic and isometric (up to possibly rescaling the metric) to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric, where $A$ is given by (2.7). If $A = I_2$ where $I_2$ is the identity matrix, then we get a metric Lie group that we denote by $\mathbb{H}^3$, which is isometric to the hyperbolic three-space with its standard constant curvature $-1$ metric and where the underlying Lie group structure is isomorphic to that of the set of similarities of $\mathbb{R}^2$. Under the assumption that $A \neq I_2$, the determinant of $A$ determines uniquely the Lie group structure. This number is the Milnor $D$-invariant of $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$:

\[
(2.8) \quad D = (1 - a^2)(1 + b^2) = \det(A).
\]

Assuming $A \neq I_2$, given $D \in \mathbb{R}$, one can solve (2.8) for $a = a(D, b)$, producing a related matrix $A(D, b)$ by equation (2.7), and the space of canonical left invariant metrics
on the corresponding non-unimodular Lie group structure is parameterized by the values of $b \in [m(D), \infty)$, where

$$m(D) = \begin{cases} 
\sqrt{D-1} & \text{if } D > 1, \\
0 & \text{otherwise.}
\end{cases}$$

In particular, the space of simply connected, three-dimensional, non-unimodular metric Lie groups with a given $D$-invariant is two-dimensional (one-dimensional after identification by rescaling of the metric). See Figure 1 for a representation of these metric Lie groups in terms of $(D, b)$.

2.4. Classification of algebraic open book decompositions. The following result was proved by the first and third authors, see Theorem 3.6 in [5].

**Proposition 2.7.** Let $X$ be a metric Lie group. The following properties are equivalent:

1. $X$ admits an algebraic open book decomposition.
2. $X$ is isomorphic to one of the following Lie groups: $\mathbb{R}^3$, Nil$_3$, Sol$_3$ or a non-unimodular Lie group with $D$-invariant $D \leq 1$.

As a matter of fact, the results in Section 3 of [5] also classify the algebraic open book decompositions and prove that when $X$ admits such an algebraic open book decomposition, then every two-dimensional subgroup of $X$ is a subgroup of some open book decomposition of $X$.

It follows from this classification that if $X$ admits an algebraic open book decomposition $\mathcal{B}$, then $X$ is isomorphic to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ for some $A \in \mathcal{M}_2(\mathbb{R})$, and we may assume that the binding $\Gamma$ associated to $\mathcal{B}$ is the 1-parameter subgroup $\{(x, 0, 0) \mid x \in \mathbb{R}\}$ and the $(x, y)$-plane $P_0 = \mathbb{R}^2 \rtimes_A \{0\}$ is one of the subgroups of $\mathcal{B}$.

Note that exchanging the matrix $A$ in Example 3.5 in [5] by its transpose, we do not change the Lie group structure and the binding $\Gamma = \{(0, y, 0) \mid y \in \mathbb{R}\}$ in that example changes to $\Gamma' = \{(x, 0, 0) \mid x \in \mathbb{R}\}$. 
admits a different metric Lie group structure, isomorphic to the one of invariant $D \tilde{H}$.

The spaces $X$ in all cases, $H$ depends on the Lie group structure of $X$.

The left invariant Gauss map (Definition 1.1). Note that the definition of $X$ a metric Lie group be an immersed oriented surface in a metric Lie group $X$, and let $G: \Sigma \rightarrow S^2$ denote its left invariant Gauss map (Definition 1.1). Note that the definition of $G$ depends on the left invariant metric chosen on $X$; however, we next explain that the property of such a Gauss map being a diffeomorphism is independent of the left invariant metric, i.e., it only depends on the Lie group structure of $X$.

To be more precise, suppose that $f: \Sigma \rightarrow X$ is an oriented smooth surface, and let $G(2, T_eX)$ be the set of oriented two-dimensional planes in $T_eX$ passing through the origin of $T_eX$. Then, we can consider two naturally defined Gauss maps

$$G_{\text{left}}: \Sigma \rightarrow G(2, T_eX), \quad G_{\text{right}}: \Sigma \rightarrow G(2, T_eX),$$

defined by left (resp. right) translating the respective oriented tangent spaces of $\Sigma$ to $G(2, T_eX)$. Note that these Gauss maps are defined without any reference to a left invariant metric on $X$. Now, once we choose a left invariant metric, we can define $S^2 \subset T_eX$ to be the set of tangent unit vectors to $X$ at $e$, which is naturally diffeomorphic to $G(2, T_eX)$ just by taking for each vector $v \in S^2$ the oriented two-plane orthogonal to $v$ (using the orientation of $T_eX$ and the chosen left invariant metric). This shows that the left invariant Gauss map $G: \Sigma \rightarrow S^2$ introduced in Definition 1.1 with respect to this metric is a diffeomorphism if and only if $G_{\text{left}}$ has the same property, and therefore proves our claim.

The left invariant Gauss map is useful to detect two-dimensional subgroups of metric Lie groups:

**Lemma 2.9.** (3, Lemma 3.9) Let $f: \Sigma \rightarrow X$ be an immersed oriented surface in a metric Lie group $X$, and let $G: \Sigma \rightarrow S^2$ denote its left invariant Gauss map. Then $G$ is constant if and only if $f(\Sigma)$ is contained in a left coset of a two-dimensional subgroup of $X$.

### 3. Algebraic open book decompositions and embeddedness of spheres.

The main objective of this section is to prove Theorem 3.4, which in particular implies the embeddedness result stated in Theorem 1.2. As a first step, we will prove a transversality lemma. For this purpose, we note that a (simply connected) metric Lie group $X$ admits some two-dimensional subgroup $\Sigma$ if and only if $X$ is not isomorphic to $SU(2)$, or equivalently, if and only if $X$ is diffeomorphic to $\mathbb{R}^3$ (see Theorem 3.6 in [5]).

**Lemma 3.1** (Transversality Lemma). Let $f: S \rightarrow X$ be an immersed oriented sphere in $X$ whose left invariant Gauss map $G$ is a diffeomorphism. Suppose that $X$ admits a two-dimensional subgroup $\Sigma$. Then:

1. The quotient space $X/\Sigma = \{g\Sigma \mid g \in X\}$ consisting of the left cosets of $\Sigma$ is diffeomorphic to $\mathbb{R}$.

2. The set of left cosets of $\Sigma$ which intersect $f(S)$ can be parameterized by the interval $[0, 1]$, i.e., $\{g(t)\Sigma \mid t \in [0, 1]\}$ are these cosets.

3. Each of the left cosets $g(0)\Sigma$ and $g(1)\Sigma$ intersects $f(S)$ at a single point.
(4) For every \( t \in (0, 1) \), \( g(t)\Sigma \) intersects \( f(S) \) transversely in a connected, immersed closed curve, and the preimage under \( f \) of this curve is a simple closed curve in \( S \).

(5) Consider the smooth map
\[
\Pi^\Sigma_{\text{left}} : X \to X/\Sigma \cong \mathbb{R}, \quad \Pi^\Sigma_{\text{left}}(g) = g\Sigma.
\]

Then, \( \Pi^\Sigma_{\text{left}} \circ f \) is a Morse function with exactly two critical points, where one critical point has index 0 and the other one has index 2.

**Proof.** Since \( X \) admits a two-dimensional subgroup, then \( X \) is not isomorphic to \( SU(2) \) and so it is diffeomorphic to \( \mathbb{R}^3 \). In this case, the set of left cosets \( X/\Sigma \) can be smoothly parameterized by \( \mathbb{R} \). By compactness and connectedness of \( f(S) \), we can parameterize those left cosets that intersect \( f(S) \) by \( t \in [0, 1] \mapsto g(t)\Sigma \). After identifying \( X/\Sigma \) with \( \mathbb{R} \), we can view \( \Pi^\Sigma_{\text{left}} : X \to \mathbb{R} \) as the related smooth quotient map. The critical points of \( \Pi^\Sigma_{\text{left}} \circ f \) are those points of \( S \) where the value of \( G \) is one of the two unit normal vectors to \( \Sigma \). Since \( G \) is bijective, then \( \Pi^\Sigma_{\text{left}} \circ f \) has at most two critical points. On the other hand, \( \Pi^\Sigma_{\text{left}} \circ f \) has at least two critical points: a maximum and a minimum. From here, the proof of the first four statements in the lemma is elementary.

It follows from items (3) and (4) that \( \Pi^\Sigma_{\text{left}} \circ f \) has exactly two critical points, one where it takes on its minimum value and one where it takes on its maximum value. Suppose that \( \Pi^\Sigma_{\text{left}} \circ f \) had a degenerate critical point \( p \), which is say the global minimum of \( \Pi^\Sigma_{\text{left}} \circ f \).

We next describe how to perturb \( f \) to an immersion \( f_t : S \hookrightarrow X \) for \( t \geq 0 \) small, so that \( f_0 = f \) and \( \Pi^\Sigma_{\text{left}} \circ f_t \) has at least three critical points when \( t \neq 0 \). To do this, take a diffeomorphism \( \phi : X \to \mathbb{R}^3 \) satisfying:

A. \( \phi(f(p)) = (0, 0, 0) \) and \( \phi(g\Sigma) = \mathbb{R}^2 \times \{ \Pi^\Sigma_{\text{left}}(g) \} \), for every \( g \in X \).

B. Near \((0, 0, 0)\), \( (\phi \circ f)(S) \) is expressed as the graph of a function \( z = z(x, y) \geq 0 \) such that \( z(0, 0) = z_0(0, 0) = z_y(0, 0) = z_y(0, 0) = z_{x y}(0, 0) = 0 \) and \( z_{x y}(0, 0) \geq 0 \).

Then, the family of diffeomorphisms \( \{ Q_t : \mathbb{R}^3 \to \mathbb{R}^3 \mid t \in \mathbb{R} \} \) given by \( Q_t(x, y, z) = (x, y, z - t^2 x^2) \) satisfies:

(a) \( f_t = \phi^{-1} \circ Q_t \circ \phi \circ f \) is a smooth 1-parameter of family of immersions of \( S \) into \( X \) with \( f_0 = f \).

(b) \( \Pi^\Sigma_{\text{left}} \circ f_t \) has a critical point at \( p \) with \( (\Pi^\Sigma_{\text{left}} \circ f_t)(p) = 0 \) (because \( dQ_t(0, 0, 0) \) is the identity).

(c) For \( t \neq 0 \) sufficiently small, near the point \((0, 0, 0)\), the surface \( (\phi \circ f)(S) \) can be expressed as the graph of a function \( z_t = z_t(x, y) \), where \( z_t(1/n, 0) \) is negative for \( n \in \mathbb{N} \) sufficiently large. Therefore, for \( t > 0 \) sufficiently small, the global minimum value of \( \Pi^\Sigma_{\text{left}} \circ f_t \) is negative.

(d) For \( t > 0 \) sufficiently small, the global maximum value of \( \Pi^\Sigma_{\text{left}} \circ f_t \) is close to 1. Therefore, for \( t > 0 \) sufficiently small, \( \Pi^\Sigma_{\text{left}} \circ f_t \) must have at least three critical points.

By the openness of the subspace of smooth immersed spheres in \( X \) whose left invariant Gauss maps are diffeomorphisms, for \( t > 0 \) sufficiently small, the left invariant Gauss map of \( f_t : S \hookrightarrow X \) is a diffeomorphism. But the existence of more than two critical points of \( \Pi^\Sigma_{\text{left}} \circ f_t \) given in item (d) above contradicts that items (3) and (4) must hold for the perturbed immersion \( f_t \). Hence, \( \Pi^\Sigma_{\text{left}} \circ f \) is a Morse function with exactly two critical points, where one critical point (the minimum) has index 0 and the other one (the maximum) has index 2. This completes the proof of item (5), and the lemma is proved.

**Remark 3.2.** Let \( \Sigma \) be a two-dimensional subgroup and consider the two oriented tangent planes \( \pm T_\Sigma \Sigma \in G(2, T_xX) \) of \( \Sigma \) at the identity element. Let \( f : S \hookrightarrow X \) be an immersed oriented sphere in \( X \) such that each of \( T_\Sigma \Sigma \), \( -T_\Sigma \Sigma \) has exactly one preimage in \( S \) by the map \( G_{\text{left}} : S \to G(2, T_xX) \introduced in (2.10). Suppose that these preimages are regular points of \( G_{\text{left}} \). Then the conclusions of Lemma 3.1 hold with small modifications in the proofs.
Next, we investigate some additional properties of immersed spheres in metric Lie groups $X$ that admit an algebraic open book decomposition. Recall that, as explained in Subsection 2.5, the property of the Gauss map being a diffeomorphism only depends on the Lie group structure of $X$ (and not on the metric). Therefore, in the next two results we do not lose generality by assuming that $X$ is equipped with the canonical metric of $\mathbb{R}^2 \times_A \mathbb{R}$, where $A$ is given in one of the Examples 2.3, 2.4, 2.5 or in equation (2.7).

**Lemma 3.3.** Let $X$ be a metric Lie group which admits an algebraic open book decomposition $B$ with binding $\Gamma$. If $f : S \leftrightarrow X$ is an immersion of a sphere whose left invariant Gauss map in $X$ is a diffeomorphism, then for every $x \in X$, the set $f^{-1}(x \Gamma)$ contains at most two points. Furthermore, if $f^{-1}(x \Gamma)$ consists of two points, then $x \Gamma$ is transverse to $f$.

**Proof.** As explained in Subsection 2.4, we can assume that $X$ is a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, that the binding $\Gamma$ is the 1-parameter subgroup \{$(x, 0, 0) \mid x \in \mathbb{R}$\} and that the $(x, y)$-plane is one of the subgroups of $B$.

To prove that given $x \in X$ the set $f^{-1}(x \Gamma)$ contains at most two points, we proceed by contradiction: suppose that this assertion fails to hold. We first find a contradiction in the case $f$ is analytic. In this case, there is some $x \in X$ for which the compact analytic set $f^{-1}(x \Gamma)$ has more than two points; note that this set has a finite number $k \geq 3$ of elements since $f(f^{-1}(x \Gamma))$ is a finite (compact) analytic set of the non-compact curve $x \Gamma$ and $f$ is a finite-to-one mapping. After left translating $f(S)$ by $x^{-1}$ (this does not change the left invariant Gauss map of $f$), we can assume $\Gamma = x \Gamma$.

We claim that by choosing some $a \in X$ arbitrarily close to the identity element $e$, the left translated binding $a \Gamma$ is transverse to $f(S)$ and the cardinality of $f^{-1}(a \Gamma)$ is an even integer greater than or equal to four. This fact can be seen as follows. Since the $(x, y)$-plane is a subgroup $H(\theta_0)$ of $B$ and $f^{-1}(H(\theta_0))$ has more than one point, then the Transversality Lemma 3.1 implies that $H(\theta_0)$ intersects $f(S)$ transversely along a connected, analytic immersed closed curve $\alpha$. Again by Lemma 3.1, $f^{-1}(\alpha)$ is a connected simple closed curve in $S$ which covers the curve $\alpha$ (considered to be an immersion of $S^1$) with fixed integer multiplicity $m \geq 1$. Suppose for the moment that $m = 1$. Since $\alpha$ is analytic, the self-intersection set of $\alpha$ is finite. Since the $x$-axis $\Gamma \subset H(\theta_0)$ intersects $\alpha$ in at least three points (we are using that $m = 1$ here), then, elementary transversality theory ensures that for some $t \neq 0$ sufficiently small and for $a(t) = (0, t, 0) \subset H(\theta_0) \subset \mathbb{R}^2 \times_A \mathbb{R}$, $a(t) \Gamma$ intersects $f(S)$ transversely in an even number of points greater than 2, all of which are disjoint from the self-intersection set of $\alpha$. Moreover, under these conditions $f^{-1}(a(t) \Gamma)$ contains at least four points. This completes the proof of the claim if $m = 1$. If $m > 1$, then for some $t \neq 0$ sufficiently small, $a(t) \Gamma$ intersects $f(S)$ transversely in at least two points, with all of these intersection points being disjoint from the self-intersection set of $\alpha$. Therefore, $f^{-1}(a(t) \Gamma)$ contains at least $2m \geq 4$ points and our claim also holds in this case.

By the discussion in the previous paragraph and after left translating $f(S)$ by $a^{-1}$, we may assume that $\Gamma$ is transverse to $f(S)$ and $f^{-1}(\Gamma) = \{p_1, \ldots, p_{2n}\} \subset S$ with $n$ an integer, $n \geq 2$. Now consider the normal variational vector field $\partial_B$ to the leaves of the product foliation $\mathcal{B} = \{L(\theta) \mid \theta \in [0, 2\pi]\}$ (these $L(\theta)$ are the topological open halfplanes that appear in Definition 2.1), which is defined in $X - \Gamma$. Since each leaf $L(\theta)$ intersects $S$ transversely (again by the Transversality Lemma 3.1), then the pullback by $f$ of the tangential component of the restriction of $\partial_B$ to $f(S) - \Gamma$, defines a vector field $\partial^S_B$ on $S - f^{-1}(\Gamma) = S - \{p_1, \ldots, p_{2n}\}$, and $\partial^S_B$ has no zeros in $S - f^{-1}(\Gamma)$. The fact that the immersion $f$ is transverse to the binding $\Gamma$ of $B$ implies that the index of $\partial^S_B$ at each of the points $p_j$ is $+1$. Then, by the Poincaré-Hopf index theorem, the Euler characteristic of $S$ would be $2n \geq 4$, which is false since the Euler characteristic of $S$ is 2. This is the desired
contradiction in the case that \( f \) is analytic; thus, given \( x \in X \), the set \( f^{-1}(x \Gamma) \) contains at most two points in this case.

We next prove that given \( x \in X \) the set \( f^{-1}(x \Gamma) \) contains at most two points in the smooth case for \( f \). Arguing again by contradiction, suppose that for some \( x \in X \), the set \( f^{-1}(x \Gamma) \) has at least three points, \( p_1, p_2, p_3 \). As before, we can assume \( x \Gamma = \Gamma \). By the Transversality Lemma 3.1, there exist pairwise disjoint compact disks \( D_1, D_2, D_3 \subset S \) such that \( p_i \in \text{Int}(D_i) \) and \( f(D_i) \) intersects the \((x,y)\)-plane in smooth compact arcs \( \beta_i \), \( i = 1, 2, 3 \). By perturbing \( f \) slightly in \( \text{Int}(D_i) \), we can assume that

(A) Each \( \beta_i \) intersects \( \Gamma \) transversely at some point.

(B) The left invariant Gauss map of the perturbed \( f \) is a diffeomorphism.

Since we can approximate \( f \) by analytic immersions with the properties (A) and (B), then we contradict the previously proved analytic case for \( f \). This contradiction finishes the proof of the first statement of the lemma.

It remains to show that if \( f^{-1}(x \Gamma) \) consists of two points \( p_1, p_2 \) for some \( x \in X \), then \( x \Gamma \) is transverse to \( f \). If not, then \( x \Gamma \) intersects tangentially to \( f(S) \) in at least one point say, \( f(p_1) \). After an arbitrarily small smooth perturbation \( \hat{f} \) of \( f \) in a small compact disk neighborhood \( D_1 \) of \( p_1 \) that is disjoint from \( p_2 \), we can suppose that \( \hat{f}|_{D_1} \) is injective and intersects \( x \Sigma \) transversely in an embedded arc, and this arc intersects \( x \Gamma \) in at least two points. It follows that \( \hat{f}^{-1}(x \Gamma) \cap D_1 \) contains at least 2 points and \( p_2 \in \hat{f}^{-1}(x \Gamma) \cap (S - D_1) \), which means that \( \hat{f}^{-1}(x \Gamma) \) contains at least 3 points. As in the previous paragraph, the left invariant Gauss map of \( f \) can be assumed to be a diffeomorphism, which contradicts the previously proved statement that the set \( \hat{f}^{-1}(x \Gamma) \) contains at most 2 points. This contradiction completes the proof that if \( f^{-1}(x \Gamma) \) consists of two points \( p_1, p_2 \) for some \( x \in X \), then \( x \Gamma \) is transverse to \( f \).

\[\text{THEOREM 3.4.}\] Let \( X \) be a metric Lie group which admits an algebraic open book decomposition \( \mathcal{B} \) with binding \( \Gamma \). Let \( \Pi : X \rightarrow X/\Gamma \cong \mathbb{R}^2 \) be the related quotient map to the space of left cosets of \( \Gamma \). If \( f : S \hookrightarrow X \) is an oriented immersion of a sphere whose left invariant Gauss map is a diffeomorphism, then:

1. \( \mathcal{D} = \Pi(f(S)) \) is a compact embedded disk in \( \mathbb{R}^2 \) and \( f(S) \cap \Pi^{-1}(\text{Int}(\mathcal{D})) \) consists of two components \( C_1, C_2 \) such that \( \Pi|_{C_j} : C_j \rightarrow \text{Int}(\mathcal{D}) \) is a diffeomorphism for \( j = 1, 2 \).

2. \( f(S) \) is an embedded sphere (i.e., \( f \) is an injective immersion).

\[\text{PROOF.}\] As explained in Section 2.4, we can assume without loss of generality that \( X \) is a semidirect product \( \mathbb{R}^2 \rtimes_A \mathbb{R} \) equipped with its canonical metric, the binding \( \Gamma \) is \( \{ (x, 0, 0) \mid x \in \mathbb{R} \} \) and \( P_0 = \mathbb{R}^2 \rtimes_A \{ 0 \} \) is one of the subgroups of \( \mathcal{B} \). It also follows from the classification in [5, Theorem 3.6] of the algebraic open book decompositions in metric Lie groups, that the \( 2 \times 2 \)-matrix \( A \) can be chosen so that its position \( a_{21} \) vanishes. This implies that given \( z \in \mathbb{R} \), the matrix \( e^{tA} \) is upper triangular, which gives that each left translate of \( \Gamma \) that intersects \( P_z = \mathbb{R}^2 \rtimes_A \{ z \} \) corresponds to a straight line in \( P_z \) that is of the form \( \Gamma_{y,z} = \{ (x, y, z) \mid x \in \mathbb{R} \} \). Let \( z_0 < z_1 \) be the numbers such that \( f(S) \) is contained in the region \( \mathbb{R}^2 \rtimes_A [z_0, z_1] \) and intersects each of the planes \( P_{z_0}, P_{z_1} \) at single points \( p_0, p_1 \), respectively (see Lemma 3.1). After a left translation, we can suppose \( z_0 = 0 \).

Orient the binding \( \Gamma = \{ (x, 0, 0) \mid x \in \mathbb{R} \} \) by the usual orientation on \( \mathbb{R} \). Also orient the sphere \( S \), orient the planes \( P_z \) by \( E_3 = \partial_z \) and the space \( \mathbb{R}^2 \rtimes_A \mathbb{R} \) by the ordered triple \( E_1, E_2, E_3 \) given in (2.4). By Lemma 3.1, each of the oriented planes \( P_z, z \in (0, z_1) \), is transverse to \( f \) and so \( f^{-1}(P_z) \) is a smooth, embedded, oriented Jordan curve \( \alpha_z \) in \( S \), where the orientation is the homological one arising from the ordered intersection \( P_z \cap f(S) \). Note that with respect to the induced metric, each plane \( P_z \) is intrinsically flat and it is foliated by the collection of parallel lines \( \Gamma_{y,z} = \{ (x, y, z) \mid x \in \mathbb{R} \} \) as \( y \in \mathbb{R} \)
two different algebraic open book decompositions $M$ of $\Gamma \times \{z\}$, each corresponding to finitely many horizontal translates of the interval $\{z\}$ together with two points $\Pi(p_0), \Pi(p_1)$ is the compact embedded disk $D = \Pi(f(S))$. By Lemma 3.3, the interior of each of the intervals $I_z$ lifts through the local diffeomorphism $\Pi$ to two open connected arcs $I_z^1, I_z^2 \subset f(\alpha_z)$ with common extrema, where the superindices $j = 1, 2$ are consistently defined for all $z$, so that the velocity vector to $I_z^1$ at its starting extremum (resp. ending extremum) points at the direction of $\partial_x$ (resp. $-\partial_x$), and the velocity vector to $I_z^2$ at any of its (interior) points is never in the direction of $\pm \partial_x$, see Figure 2. For $j = 1, 2$ fixed, let $C_j$ be the open disk in $f(S)$ given by the union of the arcs $I_z^j$ as $z$ varies. Thus, we have proved that $f(S) \cap \Pi^{-1}(\text{Int}(D))$ decomposes as a disjoint union of the disks $C_1, C_2$ with common boundary $\partial C_1 = \partial C_2$, each $C_j$ is $\Pi$-graphical onto $\text{Int}(D)$, and $\Pi|_{C_j}$ is a diffeomorphism from $C_j$ onto $\text{Int}(D)$. This completes the proof of item (1) of the theorem.

To prove item (2), first assume that $X = \mathbb{R}^2 \ltimes_A \mathbb{R}$ for some diagonal matrix $A \in \mathcal{M}_2(\mathbb{R})$. Then, as shown in Section 3 of [5] (see in particular [5, Example 3.4]), $X$ admits two different algebraic open book decompositions $B_1, B_2$ with respective orthogonal bindings $\Gamma_1 = \{(x,0,0) \mid x \in \mathbb{R}\}$, $\Gamma_2 = \{(0,y,0) \mid y \in \mathbb{R}\}$. By our previous arguments, for $z \in (0, z_1)$, the immersed curve $\alpha_z$ can be also expressed as a “bigraph” over appropriately chosen intervals in the lines $\{(0, y, z) \mid y \in \mathbb{R}\}$, $\{(x, 0, z) \mid x \in \mathbb{R}\}$, which implies that each such $\alpha_z$ is embedded. Since all of the curves $\alpha_z$ are embedded, then $f(S)$ is an embedded sphere. This proves that item (2) holds provided that $X = \mathbb{R}^2 \ltimes_A \mathbb{R}$ for some diagonal matrix $A \in \mathcal{M}_2(\mathbb{R})$.

Finally assume that $X$ admits an algebraic open book decomposition, but is not isomorphic to $\mathbb{R}^2 \ltimes_A \mathbb{R}$ for some diagonal matrix $A \in \mathcal{M}_2(\mathbb{R})$. By Theorem 3.6 in [5], $X$ is isomorphic either to $\text{Nil}_3$ or to the simply connected, three-dimensional Lie group with $D$-invariant $D = 1$ which is not isomorphic to $\mathbb{H}^3$. First consider the case where the underlying Lie group $G$ is the one with $D$-invariant $D = 1$ which is not isomorphic to $\mathbb{H}^3$, and fix the related matrix to be $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $X = \mathbb{R}^2 \ltimes_A \mathbb{R}$ is isomorphic to $G$. 

![Figure 2. Proof of Theorem 3.4; the immersed curve $f(\alpha_z) = f(S) \cap P_z$ consists of two Jordan arcs $I_z^1, I_z^2$ with the same extrema.](image)
Given $n \in \mathbb{N}$, let $A(n) = \left( \begin{array}{cc} 1/n^2 & 1 \\ 1 & 1 \end{array} \right)$ and let $X_n = \mathbb{R}^2 \times A(n) \mathbb{R}$ be the associated metric Lie group. Since $\lim_{n \to \infty} A(n) = A$, then in the associated $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$-coordinates on these spaces $X_n$, compact balls in these coordinates for $X_n$ converge smoothly as Riemannian manifolds to the related compact balls in these coordinates for $X$. Since $X_n$ is isomorphic to a non-unimodular Lie group different from the Lie group structure of $SU(2)$ whose left invariant Gauss map is a diffeomorphism. This is a contradiction that completes the proof of the theorem when $n$ is large enough, the left invariant Gauss map of the immersed sphere $n$ converges smoothly as $n$ large enough, the left invariant Gauss map of the immersed sphere $f(S)$ in $X_n$ is a diffeomorphism (for any left invariant metric on $X_n$). Since $B_n$ is a diagonal matrix, then $f : S \hookrightarrow X_n$ is an embedding by the previous paragraph. Finally, suppose that $f : S \hookrightarrow X$ is not an embedding. Then, a small perturbation $f_\varepsilon$ of $f : S \hookrightarrow X$ in the same ambient space can be assumed to intersect itself transversely at some point, and to still have the property that its left invariant Gauss map is a diffeomorphism. This clearly implies by the previous arguments that $f_\varepsilon : S \hookrightarrow X_n$ is not an embedding but its left invariant Gauss map is a diffeomorphism. This is a contradiction that completes the proof of the theorem when the Lie group structure of $X$ is the one with $D$-invariant $D = 1$ which is not isomorphic to $\mathbb{H}^3$.

Finally, if $X$ is isomorphic to $\text{Nil}_3$, then one can do a similar argument as above, taking into account that for every non-unimodular Lie group different from $\mathbb{H}^3$ there exists a sequence of left invariant metrics on it such that the corresponding sequence of metric Lie groups converges to $\text{Nil}_3$ with its standard metric, see the last paragraph in Section 2.8 of [5] for details on this argument. Now the proof is complete. \hfill \Box

**Remark 3.5.** Let $X$ be a homogeneous manifold diffeomorphic to $S^3$. Then, we can view $X$ as the Lie group $SU(2)$ endowed with a left invariant metric. We show next that there exist immersed spheres in $X$ whose left invariant Gauss map is a diffeomorphism to $S^3$, but that are not embedded, i.e., they self-intersect. Note that this statement is independent of the left invariant metric chosen on $SU(2)$ for $X$; see Subsection 2.5.

A metric Lie group $X = (SU(2), g_0)$ where $g_0$ is a left invariant metric on $SU(2)$ with a four-dimensional isometry group is usually called a Berger sphere. In [8], Torralbo showed that in some Berger spheres there exist rotationally symmetric constant mean curvature spheres that are non-embedded. On the other hand, the authors proved with Ros in [4] that if $S$ is a constant mean curvature sphere in a homogeneous manifold diffeomorphic to $S^3$, then the left invariant Gauss map of $S$ is a diffeomorphism to $S^2$.

These two facts together prove the claimed existence of non-embedded spheres in $SU(2)$ whose left invariant Gauss map is a diffeomorphism.

**Remark 3.6.** Let $\ast$ denote the group multiplication of a metric Lie group $X$ that admits an algebraic open book decomposition and let $\circ$ be the “opposite” multiplication: $a \circ b = b \ast a$, $a, b \in X$. Then the map $a \ast (a) = a^{-1}$ gives a Lie group isomorphism between $(X, \ast)$ and $(X, \circ)$. Let $f : S \hookrightarrow X$ be an immersed oriented sphere in $(X, \ast)$. Then, using the notation in (2.10), the right invariant Gauss map $G_{\text{right}}$ of $f$ in $(X, \ast)$ is the left invariant Gauss map $G_{\text{left}}$ of the immersion $f : S \hookrightarrow (X, \circ)$. Since the algebraic open book decompositions of $(X, \circ)$ are the same as the algebraic open book decompositions of $(X, \ast)$, then Theorem 3.4 implies that if $f : S \hookrightarrow X$ is an immersion of a sphere whose right invariant Gauss map is a diffeomorphism, then $f(S)$ is an embedded sphere.
Remark 3.7. In view of Theorem 1.2 and the previous remark, we have the following natural open problem: *Let $X$ be a Lie group diffeomorphic to $\mathbb{R}^3$, and let $S$ be an immersed sphere in $X$ whose left (or right) invariant Gauss map is a diffeomorphism to $G(2, T_e X)$. Is $S$ an embedded sphere?*

By work in [4], this problem is closely related to the conjecture mentioned in the introduction: *Any constant mean curvature sphere in a homogeneous manifold diffeomorphic to $\mathbb{R}^3$ is embedded.*

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