On Reducing the Heun Equation to the Hypergeometric Equation

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Abstract

The reductions of the Heun equation to the hypergeometric equation by polynomial transformations of its independent variable are enumerated and classified. Heun-to-hypergeometric reductions are similar to classical hypergeometric identities, but the conditions for the existence of a reduction involve features of the Heun equation that the hypergeometric equation does not possess; namely, its cross-ratio and accessory parameters. The reductions include quadratic and cubic transformations, which may be performed only if the singular points of the Heun equation form a harmonic or an equianharmonic quadruple, respectively; and several higher-degree transformations. This result corrects and extends a theorem in a previous paper, which found only the quadratic transformations. [See K. Kuiken, “Heun’s equation and the hypergeometric equation”, SIAM Journal on Mathematical Analysis 10 (3) (1979), 655–657.]

Key words: Heun equation, hypergeometric equation, hypergeometric identity, Lamé equation, special function, Clarkson–Olver transformation.

1 Introduction

Consider the class of linear second-order differential equations on the Riemann sphere $\mathbb{C}P^1$ which are Fuchsian, i.e., have only regular singular points \cite{15}. Any such equation with exactly three singular points can be transformed to the hypergeometric equation by appropriate changes of the independent and dependent variables. Similarly, any such equation with exactly four singular points can be transformed to the Heun equation. (See \cite{10}, Chapter 15, \cite{22,26}.)

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Solutions of the Heun equation are much less well understood than hypergeometric functions \[3\]. No general integral representation for them is known, for instance. Such solutions have recently been used in fluid dynamics \[7,25\] and drift–diffusion theory \[8\]. They also arise in lattice combinatorics \[13,17\]. But it is difficult to carry out practical computations involving them. An explicit solution to the two-point connection problem for the general Heun equation is not known \[24\], though the corresponding problem for the hypergeometric equation has a classical solution. Most work on solutions of the Heun equation has focused on special cases, such as the Lamé equation \[10,19\].

Determining which Heun equation solutions are expressible in terms of more familiar functions would obviously be useful: it would facilitate the solution of the two-point connection problem, and the computation of Heun equation monodromies. A significant result in this direction was obtained by Kuiken \[18\]. It is sometimes possible, by performing a quadratic change of the independent variable, to reduce the Heun equation to the hypergeometric equation, and thereby express its solutions in terms of hypergeometric functions. Kuiken’s quadratic transformations are not so well known as they should be. The useful monograph edited by Ronveaux \[22\] does not mention them explicitly, though it lists Ref. \[18\] in its bibliography. One of Kuiken’s transformations was recently rediscovered by Ivanov \[16\], in a disguised form.

Unfortunately, the main theorem of Ref. \[18\] is incorrect. The theorem asserts that a reduction to the hypergeometric equation, by a rational change of the independent variable, is possible only if the singular points of the Heun equation form a harmonic quadruple in the sense of projective geometry; in which case the change of variables must be quadratic. In this paper, we show that there are many alternatives. A reduction may also be possible if the singular points form an equianharmonic quadruple, with the change of variables being cubic. Additional singular point configurations permit changes of variable of degrees 3, 4, 5, and 6. Our main theorem (Theorem \[3.1\]) and its corollaries classify all such reductions, up to affine automorphisms of the Heun and hypergeometric equations. It replaces the theorem of Ref. \[18\].

It follows from Theorem \[3.1\] that in a suitably defined ‘nontrivial’ case, the local Heun function \(H_l\) can be reduced to the Gauss hypergeometric function \(2F_1\) by a formula of the type \(H_l(t) = 2F_1(R(t))\) only if the pair \((d, q/\alpha\beta)\), computed from the parameters of \(H_l\), takes one of exactly 23 values. These are listed in Theorem \[3.7\] A representative list of reductions is given in Theorem \[3.8\]. These theorems should be of interest to special function theorists and applied mathematicians. We were led to our correction and expansion of the theorem of Ref. \[18\] by a discovery of Clarkson and Olver \[6\]: an unexpected reduction of the Weierstrass form of the equianharmonic Lamé equation to the hypergeometric equation. In \[3.8\] we explain how this is a special case of the cubic Heun-to-hypergeometric reduction.
The new reductions are similar to classical hypergeometric transformations. (See [2] Chapter 3; also [10] Chapter 2.) But reducing the Heun equation to the hypergeometric equation is more difficult than transforming the hypergeometric equation to itself, since conditions involving its singular point location parameter and accessory parameter, as well as its exponent parameters, must be satisfied. Actually, the reductions classified in this paper are of a somewhat restricted type, since unlike many classical hypergeometric transformations, they involve no change of the dependent variable. A classification of reductions of the more general type is possible, but is best phrased in algebraic-geometric terms, as a classification of certain branched covers of the Riemann sphere by itself. A further extension would allow the transformation of the independent variable to be algebraic rather than polynomial or rational, since at least one algebraic Heun-to-hypergeometric reduction is known to exist [17]. Extended classification schemes are deferred to one or more further papers.

2 Preliminaries

2.1 The Equations

The Gauss hypergeometric equation is

\[
\frac{d^2 y}{dz^2} + \left( \frac{c}{z} + \frac{a+b-c+1}{z-1} \right) \frac{dy}{dz} + \frac{ab}{z(z-1)} y = 0, \tag{6}
\]

where \(a, b, c \in \mathbb{C}\) are parameters. It and its solution space are specified by the Riemann \(P\)-symbol

\[
P = \begin{pmatrix}
0 & 1 & \infty \\
0 & 0 & a \\
1 - c & c - a - b & b
\end{pmatrix}, \tag{2.1}
\]

where each column, except the last, refers to a regular singular point. The first entry is its location, and the final two are the characteristic exponents of the solutions there. The exponents at each singular point are obtained by solving an indicial equation [15]. In general, each finite singular point \(z_0\) has \(\zeta\) as an exponent if and only if the equation has a local (Frobenius) solution of the form \((z-z_0)^\zeta h(z)\) in a neighborhood of \(z = z_0\), where \(h\) is analytic and nonzero at \(z = z_0\). If the exponents at \(z = z_0\) differ by an integer, this statement must be modified: the solution corresponding to the smaller exponent may have a logarithmic singularity at \(z = z_0\). The definition extends in a straightforward way to \(z_0 = \infty\), and also to ordinary points, each of which has exponents 0, 1.
There are \(2 \times 3 = 6\) local solutions of \((h)\) in all: two per singular point. If \(c\) is not a nonpositive integer, the solution at \(z = 0\) belonging to the exponent zero will be analytic. When normalized to unity at \(z = 0\), it will be the Gauss hypergeometric function \(\, _2F_1(a, b; c; z)\) [10]. This is the sum of a hypergeometric series, which converges in a neighborhood of \(z = 0\). In general, \(\, _2F_1(a, b; c; z)\) is not defined when \(c\) is a nonpositive integer.

The Heun equation is usually written in the form
\[
\frac{d^2u}{dt^2} + \left( \frac{\gamma}{t} + \frac{\delta}{t-1} + \frac{\epsilon}{t-d} \right) \frac{du}{dt} + \frac{\alpha \beta t - q}{t(t-1)(t-d)} u = 0. \tag{5}
\]

Here \(d \in \mathbb{C}\), the location of the fourth singular point, is a parameter \((d \neq 0, 1)\), and \(\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}\) are exponent-related parameters. The \(P\)-symbol is
\[
P \begin{cases} 
0 & 1 & d & \infty \\
0 & 0 & 0 & \alpha ; t \\
1-\gamma & 1-\delta & 1-\epsilon & \beta 
\end{cases} . \tag{2.2}
\]

This does not uniquely specify the equation and its solutions, since it omits the accessory parameter \(q \in \mathbb{C}\). The exponents are constrained by
\[
\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0. \tag{2.3}
\]

This is a special case of Fuchs’s relation, according to which the sum of the \(2n\) characteristic exponents of any second-order Fuchsian equation on \(\mathbb{CP}^1\) with \(n\) singular points must equal \(n - 2\) [21].

There are \(2 \times 4 = 8\) local solutions of \((5)\) in all: two per singular point. If \(\gamma\) is not a nonpositive integer, the solution at \(t = 0\) belonging to the exponent zero will be analytic. When normalized to unity at \(t = 0\), it is called the local Heun function, and is denoted \(Hl(d, q; \alpha, \beta, \gamma, \delta; t)\) [22]. It is the sum of a Heun series, which converges in a neighborhood of \(t = 0\) [22,26]. In general, \(Hl(d, q; \alpha, \beta, \gamma, \delta; t)\) is not defined when \(\gamma\) is a nonpositive integer.

If \(\epsilon = 0\) and \(q = \alpha \beta d\), the Heun equation loses a singular point and becomes a hypergeometric equation. Similar losses occur if \(\delta = 0, q = \alpha \beta\), or \(\gamma = 0, q = 0\). This paper will exclude the case when the Heun equation has fewer than four singular points, since reducing \((h)\) to itself is a separate problem, leading to the classical hypergeometric transformations. The following case, in which the solution of \((5)\) can be reduced to quadratures, will be initially excluded.

**Definition 2.1.** If \(\alpha \beta = 0\) and \(q = 0\), the Heun equation \((5)\) is said to be trivial. Triviality implies that one of the exponents at \(t = \infty\) is zero (i.e., \(\alpha \beta = 0\)), and is implied by absence of the singular point at \(t = \infty\) (i.e., \(\alpha \beta = 0, \alpha + \beta = 1, q = 0\)).
The transformation to \((\mathcal{H})\) or \((h)\) of a linear second-order Fuchsian differential equation with singular points at \(t = 0, 1, d, \infty\) (resp. \(z = 0, 1, \infty\)), and with arbitrary exponents, is accomplished by certain linear changes of the dependent variable, called F-homotopies. (See [10] and [22, §A2 and Addendum, §1.8].)

If an equation with singular points at \(t = 0, 1, d\) has dependent variable \(u\), carrying out the substitution \(\tilde{u}(t) = t^{-\rho}(t - 1)^{-\sigma}(t - d)^{-\tau}u(t)\) will convert the equation to a new one, with the exponents at \(t = 0, 1, d\) reduced by \(\rho, \sigma, \tau\) respectively, and those at \(t = \infty\) increased by \(\rho + \sigma + \tau\). By this technique, one exponent at each finite singular point can be shifted to zero.

In fact, the Heun equation has a group of F-homotopic automorphisms isomorphic to \((\mathbb{Z}_2)^3\), since at each of \(t = 0, 1, d\), the exponents \(0, \zeta\) can be shifted to \(-\zeta, 0\), i.e., to \(0, -\zeta\). Similarly, the hypergeometric equation has a group of F-homotopic automorphisms isomorphic to \((\mathbb{Z}_2)^2\). These groups act on the 6 and 3-dimensional parameter spaces, respectively. For example, one of the latter actions is \((a, b; c) \mapsto (c - a, c - b; c)\), which is induced by an F-homotopy at \(z = 1\). From this F-homotopy follows Euler’s transformation [2, §2.2]

\[
_2F_1(a, b; c; z) = (1 - z)^{c - a - b} _2F_1(c - a, c - b; c; z),
\]

which holds because \(_2F_1\) is a local solution at \(z = 0\), rather than at \(z = 1\).

If the singular points of the differential equation are arbitrarily placed, transforming it to the Heun or hypergeometric equation will require a Möbius (i.e., projective linear or homographic) transformation, which repositions the singular points to the standard locations. A unique Möbius transformation maps any three distinct points in \(\mathbb{CP}^1\) to any other three; but the same is not true of four points, which is why \((\mathcal{H})\) has the singular point \(d\) as a free parameter.

2.2 The Cross-Ratio

The characterization of Heun equations that can be reduced to the hypergeometric equation will employ the cross-ratio orbit of \(\{0, 1, d, \infty\}\), defined as follows. If \(A, B, C, D \in \mathbb{CP}^1\) are distinct, their cross-ratio is

\[
(A, B; C, D) \overset{\text{def}}{=} \frac{(C - A)(D - B)}{(D - A)(C - B)} \in \mathbb{CP}^1 \setminus \{0, 1, \infty\},
\]

which is invariant under Möbius transformations. Permuting \(A, B, C, D\) yields an action of the symmetric group \(S_4\) on \(\mathbb{CP}^1 \setminus \{0, 1, \infty\}\). The cross-ratio is invariant under interchange of \(A, B\) and \(C, D\), and also under simultaneous interchange of the two points in each pair. So each orbit contains no more than \(4! / 4 = 6\) cross-ratios. The possible actions of \(S_4\) on \(s \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}\)
are generated by $s \mapsto 1 - s$ and $s \mapsto 1/s$, and the orbit of $s$ comprises

\[ s, \ 1 - s, \ 1/s, \ 1/(1 - s), \ (s - 1)/s, \ (s - 1)/(1 - s) \]  

(2.6)

which may not be distinct. This is called the cross-ratio orbit of $s$; or, if $s = (A, B; C, D)$, the cross-ratio orbit of the unordered set $\{A, B, C, D\} \subset \mathbb{CP}^1$. Two sets of distinct points $\{A_i, B_i, C_i, D_i\} (i = 1, 2)$ have the same cross-ratio orbit iff they are related by a M"obius transformation.

Cross-ratio orbits generically contain six values, but there are two exceptional orbits: one with three and one with two. If $(A, B; C, D) = -1$, the cross-ratio orbit of $\{A, B, C, D\}$ will be $\{-1, 1/2, 2\}$. The value $-1$ for $(A, B; C, D)$ defines a so-called harmonic configuration: $A, B$ and $C, D$ are said to be harmonic pairs. More generally, if $\{A, B, C, D\}$ has cross-ratio orbit $\{-1, 1/2, 2\}$, it is said to be a harmonic quadruple. It is easy to see that if $C = \infty$ and $A, B, D$ are distinct finite points, then $A, B$ and $C, D$ will be harmonic pairs iff $D$ is the midpoint of the line segment $AB$. In consequence, $\{A, B, \infty, D\}$ will be a harmonic quadruple iff $\{A, B, D\} \subset \mathbb{C}$ comprises collinear, equally spaced points. So, $\{A, B, C, D\} \subset \mathbb{CP}^1$ will be a harmonic quadruple iff it can be mapped by a M"obius transformation to a set consisting of three equally spaced finite points and the point at infinity; equivalently, to the vertices of a square in $\mathbb{C}$.

The cross-ratio orbit containing exactly two values is $\{1/2 \pm i\sqrt{3}/2\}$. Any set $\{A, B, C, D\}$ with this as cross-ratio orbit is said to be an equianharmonic quadruple. $\{A, B, \infty, D\}$ will be an equianharmonic quadruple iff $A, B, D$ are the vertices of an equilateral triangle in $\mathbb{C}$. If $\mathbb{CP}^1$ is interpreted as a sphere via the usual stereographic projection, then by an affine transformation (a special M"obius transformation), this situation reduces to the case when $A, B, \infty, D$ are the vertices of a regular tetrahedron. So, $\{A, B, C, D\} \subset \mathbb{CP}^1$ will be an equianharmonic quadruple iff it can be mapped by a M"obius transformation to the vertices of a regular tetrahedron in $\mathbb{CP}^1$.

Cross-ratio orbits are of two sorts: real orbits such as the harmonic orbit, and non-real orbits such as the equianharmonic orbit. All values in a real orbit are real, and in a non-real orbit, all have a nonzero imaginary part. So, $\{A, B, C, D\}$ will have a specified real orbit as its cross-ratio orbit iff it can be mapped by a M"obius transformation to a set consisting of three specified collinear points in $\mathbb{C}$ and the point at infinity; equivalently, to the vertices of a specified quadrangle (generically irregular) in $\mathbb{C}$. Similarly, it will have a specified non-real orbit as its cross-ratio orbit iff it can be mapped to a set consisting of a specified triangle in $\mathbb{C}$ and the point at infinity; equivalently, to the vertices of a specified tetrahedron (generically irregular) in $\mathbb{CP}^1$.

The cross-ratio orbit of $\{0, 1, d, \infty\}$ will be the harmonic orbit iff $d$ equals $-1, 1/2, \text{ or } 2$, and the equianharmonic orbit iff $d$ equals $1/2 \pm i\sqrt{3}/2$. In contrast, it will be a specified generic orbit iff $d$ takes one of six orbit-specific values.
The cross-ratio orbit of \( \{0, 1, d, \infty\} \) being a specified orbit is equivalent to its being the same as the cross-ratio orbit of some specified quadruple of the form \( \{0, 1, D, \infty\} \), i.e., to there being a Möbius transformation that maps \( \{0, 1, d, \infty\} \) onto \( \{0, 1, D, \infty\} \). The possibilities are

\[
d = D, \quad 1 - D, \quad 1/D, \quad 1/(1 - D), \quad D/(D - 1), \quad (D - 1)/D.
\]

(2.7)

By examination, this is equivalent to \( \{0, 1, d\} \) being mapped onto \( \{0, 1, D\} \) by some affine transformation, i.e., to \( \triangle 01d \) being similar to \( \triangle 01D \). The corresponding affine transformations \( t \mapsto A_1(t) \) are

\[
A_1(t) = t, \quad 1 - t, \quad Dt, \quad (D - 1)t + 1, \quad (1 - D)t + D, \quad D(1 - t).
\]

(2.8)

This interpretation gives a geometric significance to the possible values of \( d \).

For the equianharmonic orbit, in which the six values degenerate to two, the triangle \( \triangle 01D \) may be taken to be any equilateral triangle. For any real orbit, \( \triangle 01D \) must be taken to be degenerate, with collinear vertices. For example, the cross-ratio orbit of \( \{0, 1, d, \infty\} \) being the harmonic orbit is equivalent to \( \triangle 01d \) being similar to \( \triangle 012 \), i.e., to \( \{0, 1, d\} \) consisting of three equally spaced collinear points. This will be the case iff \( d \) equals \(-1\), \( \frac{1}{2} \), or \( 2 \), in agreement with the definition of a harmonic quadruple.

### 2.3 Automorphisms

According to the theory of the Riemann \( P \)-function, any Möbius transformation \( M \) of the independent variable will preserve characteristic exponents. For the hypergeometric equation \( h \), this implies that if \( M \) is one of the \( 3! \) Möbius transformations that permute the singular points \( z = 0, 1, \infty \), the exponents of the transformed equation \( \tilde{h} \) at its singular points \( M(0), M(1), M(\infty) \) will be those of \( h \) at \( 0, 1, \infty \). But if \( M \) is not affine, i.e., \( M(\infty) \neq \infty \), then \( \tilde{h} \) will not in general be a hypergeometric equation, since its exponents at \( M(\infty) \) may both be nonzero. To convert \( \tilde{h} \) to a hypergeometric equation, the permutation must in this case be followed by an F-homotopic transformation of the form \( \tilde{y}(z) = [z - M(\infty)]^{-a}y(z) \) or \( \tilde{y}(z) = [z - M(\infty)]^{-b}y(z) \).

**Definition 2.2.** \( \text{Aut}(h) \), the automorphism group of the hypergeometric equation \( h \), is the group of changes of variable (Möbius of the independent variable, linear of the dependent) which leave \( h \) invariant, up to parameter changes. Similarly, \( \text{Aut}(\mathcal{H}) \) is the automorphism group of \( \mathcal{H} \).

\( \text{Aut}(h) \) acts on the 3-dimensional parameter space of \( h \). It contains the symmetric group \( S_3 \) of permutations of singular points as a subgroup, and the group \( (\mathbb{Z}_2)^2 \) of F-homotopies as a normal subgroup. So \( \text{Aut}(h) \simeq (\mathbb{Z}_2)^2 \rtimes S_3 \), a semidirect product. It is isomorphic to \( S_4 \), the octahedral group \( \mathcal{H} \).
Definition 2.3. Within $\text{Aut}(\mathfrak{h})$, the Möbius automorphism subgroup is the group $\mathcal{M}(\mathfrak{h}) \overset{\text{def}}{=} \{1\} \times S_3$, which permutes the singular points $z = 0, 1, \infty$. The subgroup of affine automorphisms is $\mathcal{A}(\mathfrak{h}) \overset{\text{def}}{=} \{1\} \times S_2$, which permutes the finite singular points $z = 0, 1$, and fixes $\infty$. (It is generated by the involution $z \mapsto 1 - z$.) The F-homotopic automorphism subgroup is $(\mathbb{Z}_2)^2 \times \{1\}$.

The action of $\text{Aut}(\mathfrak{h})$ on the $2 \times 3 = 6$ local solutions is as follows. $|\text{Aut}(\mathfrak{h})| = 2^2 \times 3! = 24$, and applying the transformations in $\text{Aut}(\mathfrak{h})$ to any single local solution yields 24 solutions of $\mathfrak{h}$. Applying them to $2F_1$, for instance, yields the 24 series solutions of Kummer \[9\]. However, the 24 solutions split into six sets of four, since for each singular point $z_0 \in \{0, 1, \infty\}$ there is a subgroup of $\text{Aut}(\mathfrak{h})$ of order $2! \times 2! = 4$, each element of which fixes $z = z_0$ and performs no F-homotopy there; so it leaves each local solution at $z = z_0$ invariant.

For example, the four transformations in the subgroup associated to $z = 0$ yield four equivalent expressions for $2F_1(a, b; c; z)$; one of which is $2F_1(a, b; c; z)$ itself, and another of which appears above in (2.4). The remaining two are $(1 - z)^{-a}2F_1(a, c - b; c; z/(z - 1))$ and $(1 - z)^{-b}2F_1(b, c - a; c; z/(z - 1))$. The five remaining sets of four are expressions for the five remaining local solutions. One that will play a role is the ‘second’ local solution at $z = 0$, which belongs to the exponent $1 - c$. One of the four expressions for it, in terms of $2F_1$, is $\overline{2F_1}(a, b; c; z) \overset{\text{def}}{=} z^{-1-c}2F_1(a - c + 1, b - c + 1; 2 - c; z)$. (2.9) $\overline{2F_1}(a, b; c; z)$ is defined if $c \neq 2, 3, 4, \ldots$. The second local solution must be specified differently if $c = 2, 3, 4, \ldots$, and also if $c = 1$, since in that case, $\overline{2F_1}$ reduces to $2F_1$. (Cf. \[11\] \S 15.5.) When $\overline{2F_1}$ is defined, it may be given a unique meaning by choosing the principal branch of $z^{1-c}$.

The automorphism group of the Heun equation is slightly more complicated. There are 4! Möbius transformations $M$ that map the singular points $t = 0, 1, d, \infty$ onto $t = 0, 1, d', \infty$, for some $d' \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$. The possible $d'$ constitute the cross-ratio orbit of $\{0, 1, d, \infty\}$. Of these 4! transformations, 3! fix $t = \infty$, i.e., are affine. All values $d'$ are obtained by affine transformations, i.e., a mapping is possible iff $\triangle 01d$ is similar to $\triangle 01d'$. (Cf. the discussion in \S [22].) If $M$ is not affine, it must be followed by an F-homotopic transformation of the form $\tilde{u}(t) = [t - M(\infty)]^{-\alpha}u(t)$ or $\tilde{u}(t) = [t - M(\infty)]^{-\beta}u(t)$.

$\text{Aut}(\mathfrak{S})$ acts on the 6-dimensional parameter space of $\mathfrak{S}$. It contains the group $S_4$ of singular point permutations as a subgroup, and the group $(\mathbb{Z}_2)^3$ of F-homotopies as a normal subgroup. So $\text{Aut}(\mathfrak{S}) \simeq (\mathbb{Z}_2)^3 \rtimes S_4$. It turns out to be isomorphic to the Coxeter group $\mathcal{D}_4$ \[12\].

Definition 2.4. Within $\text{Aut}(\mathfrak{S})$, the Möbius automorphism subgroup is the group $\mathcal{M}(\mathfrak{S}) \overset{\text{def}}{=} \{1\} \times S_4$, which maps between sets of singular points of the form $\{0, 1, d', \infty\}$. The subgroup of affine automorphisms is $\mathcal{A}(\mathfrak{S}) \overset{\text{def}}{=} \{1\} \times S_3$. 

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which maps between sets of finite singular points of the form \( \{0, 1, d'\} \), and fixes \( \infty \). The F-homotopic automorphism subgroup is \((\mathbb{Z}_2)^3 \times \{1\}\).

The action of \(\text{Aut}(\mathfrak{H})\) on the \(2 \times 4 = 8\) local solutions is as follows. \(|\text{Aut}(\mathfrak{H})| = 2^3 \times 4! = 192\), and applying the transformations in \(\text{Aut}(\mathfrak{H})\) to any single local solution yields 192 solutions of (5). However, the 192 solutions split into eight sets of 24, since for each singular point \(t_0 \in \{0, 1, d, \infty\}\) there is a subgroup of \(\text{Aut}(\mathfrak{H})\) of order \(2^2 \times 3! = 24\), each element of which fixes \(t = t_0\) and performs no F-homotopy there; so it leaves each local solution at \(t = t_0\) invariant. This statement must be interpreted with care: selecting \(t_0 = d\) selects not a single singular point, but rather a cross-ratio orbit.

The 24 transformations in the subgroup associated to \(t_0 = 0\) yield 23 equivalent expressions for \(Hl(d, q; \alpha, \beta, \gamma, \delta; t)\), one of which, the only one with no F-homotopic prefactor, appears on the right in the identity \([22, 26]\).

\[
Hl(d, q; \alpha, \beta, \gamma, \delta; t) = Hl(1/d, q/d; \alpha, \beta, \gamma, \alpha + \beta - \gamma - \delta + 1; t/d).
\]

(2.10)

(The two sides are defined if \(\gamma\) is not a nonpositive integer.) The remaining seven sets of 24 are expressions for the remaining seven local solutions. One that will play a role is the ‘second’ solution at \(t = 0\), which belongs to the exponent \(1 - \gamma\). One of the 24 expressions for it, in terms of \(Hl\), is \([26]\).

\[
\tilde{Hl}(d, q; \alpha, \beta, \gamma, \delta; t) \overset{\text{def}}{=} t^{1-\gamma} Hl(d, \tilde{q}; \alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, \delta; t), \quad (2.11)
\]

where the transformed accessory parameter \(\tilde{q}\) equals \(q + (1 - \gamma)(\epsilon + d\delta)\). The quantity \(\tilde{Hl}(d, q; \alpha, \beta, \gamma, \delta; t)\) is defined if \(\gamma \neq 2, 3, 4, \ldots\) The second local solution must be specified differently if \(\gamma = 2, 3, 4, \ldots\), and also if \(\gamma = 1\), since in that case, \(\tilde{Hl}\) reduces to \(Hl\). When \(\tilde{Hl}\) is defined, it may be given a unique meaning by choosing the principal branch of \(t^{1-\gamma}\).

In general, transformations in \(\text{Aut}(\mathfrak{H})\) will alter not merely \(d\) and the exponent parameters, but also the accessory parameter \(q\). This is illustrated by (2.10) and (2.11). The general transformation law of \(q\) is rather complicated. Partly for this reason, no satisfactory list of the 192 solutions has appeared in print. The original paper of Heun [14] tabulates 48 of the 192, but omits the value of \(q\) in each. His table also unfortunately contains numerous misprints and cannot be used in practical applications [25, §6.3]. Incidentally, one sometimes encounters a statement that there are only 96 distinct solutions [111122]. This is true only if one uses (2.10) to identify the 192 solutions in pairs.
3 Polynomial Heun-to-Hypergeometric Reductions

We now state and prove Theorem 3.1, our corrected and expanded version of the theorem of Kuiken [18].

The theorem will characterize when a homomorphism of rational substitution type from the Heun equation \((H)\) to the hypergeometric equation \((h)\) exists. It will list the possible substitutions, up to affine automorphisms of the two equations. It is really a characterization of the \(A(H)\)-orbits that can be mapped by homomorphisms of this type to \(A(h)\)-orbits. The possible substitutions, it turns out, are all polynomial.

For ease of understanding, the characterization will be concrete: it will state that \(\Delta 01d\) must be similar to one of five specified triangles of the form \(\Delta 01D\). By the remarks in §2.2, similarity occurs iff \(d\) belongs to the cross-ratio orbit of \(D\), i.e., iff \(D\) can be generated from \(d\) by repeated application of \(d \mapsto 1 - d\) and \(d \mapsto 1/d\). The two exceptional cross-ratio orbits, namely \(\{-1, \frac{1}{2}, 2\}\) (harmonic) and \(\{\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\}\) (equianharmonic), will play a prominent role. It is worth noting that if \(\text{Re } D = \frac{1}{2}\), the orbit of \(D\) is closed under complex conjugation.

For each value of \(D\), the polynomial map from \(\mathbb{CP}^1 \ni t \to \mathbb{CP}^1 \ni z\), which is denoted \(R\), will be given explicitly when \(d = D\). If \(d\) is any other value on the cross-ratio orbit of \(D\), as listed in (2.7), the polynomial map would be computed by composing with the corresponding affine transformation \(A_1\) of \(\mathbb{C}\) that takes \(\Delta 01d\) to \(\Delta 01D\); which is listed in (2.8). So if \(d \neq D\), statements in the theorem dealing with singular points, characteristic exponents, and the accessory parameter must be altered. For example, case 1 of the theorem refers to a distinguished singular point \(d_0\), the mandatory value of which is given when \(d = D\). If \(d \neq D\), its mandatory value would be computed as the preimage of that point under \(A_1\). Similarly, a statement “the exponents of \(t = 0\) must be \(0, 1/2\)”, valid when \(d = D\), would be interpreted if \(d \neq D\) as “the exponents of the preimage of \(t = 0\) under \(A_1\) must be \(0, 1/2\)”. And a statement that \(q/\alpha \beta\) must take some value would be interpreted if \(d \neq D\) as a statement that \(q/\alpha \beta\) must equal the preimage of that value under \(A_1\).

In the statement of the theorem and what follows, \(S \overset{\text{def}}{=} 1 - R\).

**Theorem 3.1.** A Heun equation \((H)\), which has four singular points and is nontrivial (i.e., \(\alpha \beta \neq 0\) or \(q \neq 0\)), can be transformed to the hypergeometric equation \((h)\) by a rational substitution \(z = R(t)\) if and only if \(R\) is a polynomial, \(\alpha \beta \neq 0\), and one of the following two conditions is satisfied.

1. \(\Delta 01d\) is similar to \(\Delta 01D\), for one of the values of \(D\) listed in subcases 1a–1c; each of which is real, so the triangle must be degenerate. Also, the
normalized accessory parameter $q/\alpha\beta$ must equal one of $0, 1, d$, which may be denoted $d_0$. Each subcase lists the value of $d_0$ when $d = D$.

(2) $\Delta 01d$ is similar to $\Delta 01D$, for one of the values of $D$ listed in subcases 2a–2d; each of which is non-real and has real part equal to $\frac{1}{2}$, so the triangle must be isosceles. Each subcase lists the value of $q/\alpha\beta$ when $d = D$.

Besides specifying $D$ and the value of $q/\alpha\beta$ when $d = D$, each subcase imposes restrictions on the characteristic exponent parameters at the singular points $0, 1, d$. The subcases of the ‘real’ case 1 are the following.

(1a) [Harmonic (equally spaced collinear points) case.] $D = 2$. Suppose $d = D$. Then $d_0$ must equal 1, and $t = 0, d$ must have the same characteristic exponents, i.e., $\gamma = \epsilon$. In general, either $R$ or $S$ will be the degree-2 polynomial $t(2 - t)$, which maps $t = 0, d$ to $z = 0$ and $t = 1$ to $z = 1$ (with double multiplicity). There are special circumstances in which $R$ may be quartic, which are listed separately, as subcase 1c.

(1b) $D = 4$. Suppose $d = D$. Then $d_0$ must equal 1, the point $t = 1$ must have characteristic exponents that are double those of $t = d$, i.e., $1 - \delta = 2(1 - \epsilon)$, and $t = 0$ must have exponents $0, 1/2$, i.e., $\gamma = 1/2$. Either $R$ or $S$ will be the degree-3 polynomial $(t - 1)^2(1 - t/4)$, which maps $t = 0$ to $z = 1$ and $t = 1, d$ to $z = 0$ (the former with double multiplicity).

(1c) [Special harmonic case.] $D = 2$. Suppose $d = D$. Then $d_0$ must equal 1, and $t = 0, d$ must have the same characteristic exponents, i.e., $\gamma = \epsilon$. Moreover, the exponents of $t = 1$ must be twice those of $t = 0, d$, i.e., $1 - \delta = 2(1 - \gamma) = 2(1 - \epsilon)$. Either $R$ or $S$ will be the degree-4 polynomial $4[t(2 - t) - \frac{1}{2}]^2$, which maps $t = 0, 1, d$ to $z = 1$ ($t = 1$ with double multiplicity).

The subcases of the ‘non-real’ case 2 are the following.

(2a) [Equianharmonic (equilateral triangle) case.] $D = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. $q/\alpha\beta$ must equal the mean of $0, 1, d$, and $t = 0, 1, d$ must have the same characteristic exponents, i.e., $\gamma = \delta = \epsilon$. Suppose $d = D$. Then $q/\alpha\beta$ must equal $\frac{1}{2} + i\frac{\sqrt{3}}{2}$. In general, either $R$ or $S$ will be the degree-3 polynomial $[1 - t/(\frac{1}{2} + i\frac{\sqrt{3}}{6})]^3$, which maps $t = 0, 1, d$ to $z = 1$ and $t = q/\alpha\beta$ to $z = 0$ (with triple multiplicity). There are special circumstances in which $R$ may be sextic, which are listed separately, as subcase 2d.

(2b) $D = \frac{1}{2} + i\frac{\sqrt{15}}{10}$. Suppose $d = D$. Then $q/\alpha\beta$ must equal $\frac{1}{2} + i\frac{\sqrt{15}}{8}$, $t = d$ must have characteristic exponents $0, 1/3$, i.e., $\epsilon = 2/3$, and $t = 0, 1$ must have exponents $0, 1/2$, i.e., $\gamma = \delta = 1/2$. Either $R$ or $S$ will be the degree-4 polynomial $[1 - t/(\frac{1}{2} + i\frac{\sqrt{15}}{10})][1 - t/(\frac{1}{2} + i\frac{\sqrt{15}}{8})]^3$, which maps $t = d, q/\alpha\beta$ to $z = 0$ (the latter with triple multiplicity) and $t = 0, 1$ to $z = 1$.

(2c) $D = \frac{1}{2} + i\frac{1 + \sqrt{5}}{90}$. Suppose $d = D$. Then $q/\alpha\beta$ must equal $\frac{1}{2} + i\frac{\sqrt{15}}{18}$, $t = d$ must have characteristic exponents $0, 1/2$, i.e., $\epsilon = 1/2$, and $t = 0, 1$ to $z = 1$.
must have exponents 0, 1/3, i.e., $\gamma = \delta = 2/3$. Either $R$ or $S$ will be the degree-5 polynomial $At(t-1) \left[ t - \left( \frac{1}{2} + i\frac{\sqrt{15}}{18} \right) \right]^3$, which maps $t = 0, 1, q/\alpha\beta$ to $z = 0$ (the last with triple multiplicity). The factor $A$ is chosen so that it maps $t = d$ to $z = 1$, as well; explicitly, $A = -i\frac{2025\sqrt{15}}{64}$.

(2d) [Special equianharmonic case.] $D = \frac{1}{2} + i\frac{\sqrt{5}}{2}$. $q/\alpha\beta$ must equal the mean of 0, 1, $d$, and $t = 0, 1, d$ must have characteristic exponents 0, 1/3, i.e., $\gamma = \delta = \epsilon = 2/3$. Suppose $d = D$. Then $q/\alpha\beta$ must equal $\frac{1}{2} + i\frac{\sqrt{5}}{2}$. Either $R$ or $S$ will be the degree-6 polynomial $4 \left\{ [1 - t/(\frac{1}{2} + i\frac{\sqrt{5}}{6})]^3 - \frac{1}{2} \right\}^2$, which maps $t = 0, 1, d, q/\alpha\beta$ to $z = 1$ (the last with triple multiplicity).

**Remark 3.1.1.** The origin of the special harmonic and equianharmonic sub-cases is easy to understand. In subcase 1c, $t \mapsto R(t)$ or $S(t)$ is the composition of the quadratic map of subcase 1a with the map $z \mapsto 4(z - \frac{1}{2})^2$. In subcase 2d, $t \mapsto R(t)$ or $S(t)$ is similarly the composition of the cubic map of subcase 2a with $z \mapsto 4(z - \frac{1}{2})^2$. In both 1c and 2d, the further restrictions on exponents make possible the additional quadratic transformation of $z$, which transforms the hypergeometric equation into itself (see [21 §3.1] and [18]).

**Remark 3.1.2.** $R$ is determined uniquely by the choices enumerated in the theorem. There is a choice of subcase, a choice of $d$ from the cross-ratio orbit of $D$, and a binary choice between $R$ and $S$. The final two choices amount to choosing affine maps $A_1 \in \mathcal{A}(\overline{5})$ and $A_2 \in \mathcal{A}(\overline{6})$, i.e., $A_2(z) = z$ or $1 - z$, which precede and follow a canonical substitution.

In the harmonic case 1a, in which the $\mathcal{A}(\overline{5})$-orbit includes three values of $d$, there are accordingly $3 \times 2 = 6$ possibilities for $R$; namely,

$$R = t^2, \quad 1 - t^2; \quad (2t - 1)^2, \quad 1 - (2t - 1)^2; \quad t(2t - 1), \quad 1 - t(2t - 1), \quad (3.1)$$

corresponding to $d = -1, -1; \frac{1}{2}, \frac{1}{2}; 2, 2$, respectively. These are the quadratic transformations of Kuiken [18]. In the equianharmonic case 2a, in which the orbit includes only two values of $d$, there are $2 \times 2 = 4$ possibilities; namely,

$$R = [1 - t/(\frac{1}{2} \pm i\frac{\sqrt{3}}{6})]^3, \quad 1 - [1 - t/(\frac{1}{2} \pm i\frac{\sqrt{3}}{6})]^3, \quad (3.2)$$

corresponding to $d = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The remaining subcases, with the exception of 1c and 2d, correspond to generic cross-ratio orbits: each value of $D$ specifies six values of $d$. In each of those subcases, there are $6 \times 2 = 12$ possibilities. So in all, there are 56 possibilities for $R$.

**Remark 3.1.3.** The characteristic exponents of the singular points $z = 0, 1, \infty$ of [18] can be computed from those of the singular points $t = 0, 1, d, \infty$ of [51], together with the formula for $R$. The computation relies on Proposition 3.3 below, which may be summarized thus. If $t = t_0$ is not a critical point of the map $t \mapsto z = R(t)$, then the exponents of $z = R(t_0)$ will be the same as those of $t_0$. If, on the other hand, $t = t_0$ is mapped to $z = z_0 \overset{\text{def}}{=} R(t_0)$ with
multiplicity $k > 1$, i.e., $t = t_0$ is a $k - 1$-fold critical point of $R$ and $z = z_0$ is a critical value, then the exponents of $z_0$ will be $1/k$ times those of $t_0$.

For example, in the harmonic case 1a, the map $t \mapsto z$ takes two of $t = 0, 1, d$ to either $z = 0$ or $z = 1$, and by examination, the coalesced point is not a critical value of the map; so the characteristic exponents of those two points are preserved, and must therefore be the same, as stated in the theorem. On the other hand, the characteristic exponents of the third point of the three, $t = d_0$, are necessarily halved when it is mapped to $z = 1$ or $z = 0$, since by examination, $R$ always has a simple critical point at $t = d_0$, i.e., $z \sim \text{const} + C(t - d_0)^2$ for some nonzero $C$. (These statements follow by considering the canonical $d = D$ case.) So if $\delta_0$ denotes the parameter (out of $\gamma, \delta, \epsilon$) corresponding to $t = d_0$, the characteristic exponents of $z = 1$ or $z = 0$ will be $0, (1 - \delta_0)/2$. $R$, being a quadratic polynomial, also has a simple critical point at $t = \infty$, so the characteristic exponents of $z = \infty$ are one-half those of $t = \infty$, i.e., $\alpha/2, \beta/2$. It follows that in the harmonic case, the Gauss parameters $(a, b; c)$ of the resulting hypergeometric equation will be $(\alpha/2, \beta/2; (\delta_0 + 1)/2)$ or $(\alpha/2, \beta/2; (\alpha + \beta - \delta_0 + 1)/2)$.

In the equianharmonic case 2a, the map $t \mapsto z$ takes $t = 0, 1, d$ to either $z = 0$ or $z = 1$; and by examination, the coalesced point is not a critical value of the map; so the characteristic exponents of those three points are preserved, and must therefore be the same, as stated in the theorem. On the other hand, at $t = q/\alpha\beta$, which is mapped to $z = 1$ or $z = 0$, $R$ has, by examination, a double critical point, i.e., $z \sim \text{const} + C(t - q/\alpha\beta)^3$ for some nonzero $C$. So the characteristic exponents of $z = 1$ or $z = 0$, since $t = q/\alpha\beta$ is an ordinary point of the Heun equation and effectively has characteristic exponents $0, 1$, are $0, 1/3$. $R$, being a cubic polynomial, also has a double critical point at $t = \infty$, so the characteristic exponents of $z = \infty$ are one-third those of $t = \infty$, i.e., $\alpha/3, \beta/3$. It follows that in the equianharmonic case, the parameters $(a, b; c)$ of the resulting hypergeometric equation will be $(\alpha/3, \beta/3; 2/3)$ or $(\alpha/3, \beta/3; (\alpha + \beta + 1)/3)$.

**Definition 3.2.** A rational map $R : \mathbb{CP}^1 \to \mathbb{CP}^1$ is said to map the characteristic exponents of the Heun equation (5) to the characteristic exponents of the hypergeometric equation (b) if, for all $t_0 \in \mathbb{CP}^1$, the exponents of $t = t_0$ according to the Heun equation, divided by the multiplicity of $t_0 \mapsto z_0 \overset{\text{def}}{=} R(t_0)$, equal the exponents of $z = z_0$ according to the hypergeometric equation.

For example, if $t_0$ and $z_0$ are both finite, this says that if $z \sim z_0 + C(t - t_0)^k$ to leading order, for some nonzero $C$, then the exponents of $z = z_0$ must be those of $t = t_0$, divided by $k$. If $t = t_0$ is an ordinary point of the Heun equation, then the exponents of $z = z_0$ will be $0, 1/k$. This implies that if $k > 1$, $z = z_0$ must be one of the three singular points of the hypergeometric equation.
Proposition 3.3. A Heun equation of the form (5) can be reduced to a hyper-geometric equation of the form (1) by a rational substitution \( z = R(t) \) of its independent variable only if \( R \) maps exponents to exponents.

Proposition 3.3, which was already used in Remark 3.1.3 above, is a special case of a basic fact in the theory of the Riemann \( P \)-function: if a rational change of the independent variable transforms one Fuchsian equation to another, then the characteristic exponents are transformed multiplicatively. It can be proved by examining the effects of the change of variable on each local (Frobenius) solution. It also follows immediately from Lemma 3.4 below.

This lemma begins the study of sufficient conditions for the existence of a Heun-to-hypergeometric transformation. Finding them requires care, since an accessory parameter is involved. Performing the substitution \( z = R(t) \) explicitly is useful. Substituting \( z = R(t) \) into (5) ‘pulls it back’ (cf. [18]) to

\[
\frac{d^2y}{dt^2} + \left\{ -\frac{\ddot{R}}{R} + \frac{\dot{R}}{R(1-R)}[c - (a + b + 1)\dot{R}] \right\} \frac{dy}{dt} - \frac{ab\ddot{R}^2}{R(1-R)} y = 0. \tag{3.3}
\]

To save space here, \( dR/dt, d^2R/dt^2 \) are written as \( \dot{R}, \ddot{R} \).

Lemma 3.4. The coefficient of the \( dy/dt \) term in the pulled-back hypergeometric equation (5), which may be denoted \( W(t) \), equals the coefficient of the \( du/dt \) term in the Heun equation (5), i.e., \( \gamma/t + \delta/(t - 1) + \epsilon/(t - d) \), if and only if \( R \) maps exponents to exponents. That is, the transformation at least partly ‘works’ if and only if \( R \) maps exponents to exponents.

PROOF. This follows by elementary, if tedious calculations. Suppose that \( R \) maps \( t = t_0 \) to \( z = z_0 \overset{\text{def}}{=} R(t_0) \) with multiplicity \( k \), i.e., to leading order \( R(t) \sim z_0 + C(t - t_0)^k \); if \( t_0 \) and \( z_0 \) are finite, that is. By direct computation, the leading behavior of \( W \) at \( t = t_0 \) is the following. In the case when \( t_0 \) is finite, \( W(t) \sim (1 - k)(t - t_0)^{-1} \) if \( z_0 \neq 0, 1, \infty; [1 - k(1 - c)](t - t_0)^{-1} \) if \( z_0 = 0; [1 - k(c - a - b)](t - t_0)^{-1} \) if \( z_0 = 1 \); and \( [1 - k(a + b)](t - t_0)^{-1} \) if \( z_0 = \infty \).

In the case when \( t_0 = \infty \), \( W(t) \sim (1 + k)t^{-1} \) if \( z_0 \neq 0, 1, \infty; [1 + k(1 - c)]t^{-1} \) if \( z_0 = 0; [1 + k(c - a - b)]t^{-1} \) if \( z_0 = 1 \); and \( [1 + k(a + b)]t^{-1} \) if \( z_0 = \infty \).

This may be restated as follows. At \( t = t_0 \), for finite \( t_0 \), the leading behavior of \( W \) is \( W(t) \sim (1 - k\eta)(t - t_0)^{-1} \), where \( k \) is the multiplicity of \( t_0 \mapsto z_0 \overset{\text{def}}{=} R(t_0) \) and \( \eta \) is the sum of the two characteristic exponents of the hypergeometric equation at \( z = z_0 \); if the coefficient \( 1 - k\eta \) equals zero then \( W \) has no pole at \( t = t_0 \). Likewise, the leading behavior of \( W \) at \( t = \infty \) is \( W(t) \sim (1 + k\eta)t^{-1} \), where \( k \) is the multiplicity of \( \infty \mapsto z_0 \overset{\text{def}}{=} R(\infty) \) and \( \eta \) is the sum of the two exponents at \( z = z_0 \); if the coefficient \( 1 + k\eta \) equals zero then \( W \) has a higher-order zero at \( t = \infty \).
By the definition of ‘mapping exponents to exponents’, it follows that the
leading behavior of $W$ at $t = t_0$, for all $t_0$ finite, is of the form $W(t) \sim (1 - \eta')(t - t_0)^{-1}$, and also at $t_0 = \infty$, is of the form $W(t) \sim (1 + \eta')t^{-1}$, where in both cases $\eta'$ is the sum of the exponents of the Heun equation at $t = t_0$, iff $R$ maps exponents to exponents.

That is, the rational function $W$ has leading behavior
$\gamma t^{-1}$ at $t = 0$, $\delta (t - 1)^{-1}$ at $t = 1$, $\epsilon (t - d)^{-1}$ at $t = d$, $(1 + \alpha + \beta)t^{-1} = (\gamma + \delta + \epsilon)t^{-1}$ at $t = \infty$, and is regular at all $t$ other than $0, 1, d, \infty$, iff $R$ maps exponents to exponents.

The following two propositions characterize when the pulled-back hypergeometric equation (3.3) is, in fact, the Heun equation (5). The first deals with Heun equations that are trivial in the sense of Definition 2.1, and will be used in §6. The second will be applied to prove Theorem 3.1.

**Proposition 3.5.** A Heun equation of the form (5), which is trivial (i.e., $\alpha\beta = 0$ and $q = 0$), will be reduced to a hypergeometric equation of the form (5) by a specified rational substitution $z = R(t)$ of its independent variable if and only if $R$ maps exponents to exponents.

**PROOF.** The ‘if’ half is new, and requires proof. By Lemma 3.4, the coefficients of $dy/dt$ and $du/dt$ agree iff $R$ maps exponents to exponents, so it suffices to determine whether the coefficients of $y$ and $u$ agree. But by triviality, the coefficient of $u$ in (5) is zero. Also, $t = \infty$ has zero as one of its exponents, so all points $t \in \mathbb{CP}^1$ have zero as an exponent. By the mapping of exponents to exponents, $z = \infty$ must also have zero as an exponent, i.e., $ab = 0$. So the coefficient of $y$ in (3.3) is also zero.

**Proposition 3.6.** A Heun equation of the form (5), which has four singular points and is nontrivial (i.e., $\alpha\beta \neq 0$ or $q \neq 0$), will be reduced to a hypergeometric equation of the form (5) by a specified rational substitution $z = R(t)$ of its independent variable if and only if $R$ maps exponents to exponents, and moreover, $R$ is a polynomial, $\alpha\beta \neq 0$, and one of the following two conditions on the normalized accessory parameter $p \equiv q/\alpha\beta$ is satisfied.

(1) $p$ equals one of $0, 1, d$. Call this point $d_0$, and the other two singular points $d_1$ and $d_2$. In this case, $d_0$ must be a double zero of $R$ or $S$, and each of $d_1, d_2$ must be a simple zero of $R$ or $S$.

(2) $p$ does not equal any of $0, 1, d$. In this case, each of $0, 1, d$ must be a simple zero of $R$ or $S$, and $p$ must be a triple zero of either $R$ or $S$.

In both cases, $R$ and $S$ must have no additional simple zeroes or zeroes of order greater than two. Also, if $1 - c$ (the nonzero exponent at $z = 0$) does not equal $1/2$, then $R$ must have no additional double zeroes; and if $c - a - b$ (the
nonzero exponent at \( z = 1 \) does not equal 1/2, then \( S \) must have no additional double zeroes.

Moreover, in both cases no additional double zero, if any, must be mapped by \( R \) to the point \((0, 1)\) which \( p \) is mapped. (So additional double zeroes, if any, must all be zeroes of \( R \), or all be zeroes of \( S \).)

**PROOF.** Like Proposition 3.5, this follows by comparing the pulled-back hypergeometric equation (3.3) to the Heun equation (5). By Lemma 3.4, the coefficients of \(dy/dt\) and \(du/dt\) agree iff \( R \) maps exponents to exponents, so it suffices to characterize when the coefficients of \( y \) and \( u \) agree.

The coefficient of \( y \) in (3.3) is to equal the coefficient of \( u \) in (5). It follows that \( ab = 0 \) is possible iff \( \alpha \beta = 0 \) and \( q = 0 \), which is ruled out by nontriviality. So \( ab \neq 0 \), and equality of the coefficients can hold iff

\[
U \equiv \frac{dR/dt}{R} \frac{dS/dt}{S} = \frac{-(\alpha \beta t - q)/ab}{t(t-1)(t-d)} \equiv \frac{C_0}{t} + \frac{C_1}{t-1} + \frac{C_d}{t-d}, \tag{3.4}
\]

where \( S \equiv 1 - R \), and at least two of \( C_0, C_1, C_d \in \mathbb{C} \) are nonzero.

Both \( R^{-1} dR/dt \) and \( S^{-1} dS/dt \) are sums of terms of the form \( n(t-\lambda)^{-1} \), where \( n \) is a nonzero integer and \( \lambda \) is a zero or a pole of \( R \) or \( S \). Poles are impossible, since \( \lambda \) is a pole of \( R \) iff \( \lambda \) is a pole of \( S \), and there are no double poles on the right-hand side of (3.4). So \( R \) must be a polynomial.

By examining the definition of \( U \) in terms of \( R \) and \( S \), one sees the following is true of any \( \lambda \in \mathbb{C} \): if \( R \) or \( S \) has a simple zero at \( t = \lambda \), then \( U \) will have a simple pole at \( t = \lambda \); if \( R \) or \( S \) has a double zero at \( t = \lambda \), then \( U \) will have an ordinary point (non-zero, non-pole) at \( t = \lambda \), and if \( R \) or \( S \) has a zero of order \( k > 2 \) at \( t = \lambda \), then \( U \) will have a zero of order \( k - 2 \) at \( t = \lambda \).

Most of what follows is devoted to proving the ‘only if’ half of the proposition in the light of these facts, by examining the consequences of the equality (3.4). In the final paragraph, the ‘if’ half will be proved.

There are exactly three ways in which the equality (3.4) can hold.

(0) \( \alpha \beta = 0 \), but due to nontriviality, \( q \neq 0 \). \( U \) has three simple poles on \( \mathbb{C} \), at \( t = 0, 1, d \). It has no other poles, and no zeroes. So each of \( 0, 1, d \) must be a simple zero of either \( R \) or \( S \); also, \( R \) and \( S \) can have no other simple zeroes, and no zeroes of order \( k > 2 \). Except for possible double zeroes, the zeroes of \( R \) and \( S \) are determined. The degree of \( R \) must equal the number of zeroes of \( R \), and also equal the number of zeroes of \( S \), counting multiplicity. But irrespective of how many double zeroes are assigned to
We can prove the main theorem, with the aid of the polynomial manipulation facilities of the MACSYMA computer algebra system.
Proof (of Theorem 3.1) By Proposition 3.6 the preimages of $z = 0, 1$ under $R$ must include $t = 0, 1, d$, and in case 2, $t = p \equiv q/\alpha \beta$. They may also include $l$ (additional) double zeroes of $R$ or of $S$, which will be denoted $t = a_1, \ldots, a_l$. Cases 1,2 of the theorem correspond to cases 1,2 of the proposition, and the subcases of the theorem correspond to distinct choices of $l$.

Necessarily $\deg R = |R^{-1}(0)| = |R^{-1}(1)|$, where the inverse images are defined as multisets rather than sets, to take multiplicity into account. This places tight constraints on $l$, since each of 0, 1, $d$ (and $p$, in case 2) may be assigned to either $R^{-1}(0)$ or $R^{-1}(1)$, but by the proposition, all of $a_1, \ldots, a_l$ must be assigned, twice, to one or the other. In case 1, one of 0, 1, $d$ (denoted $d_0$ in the proposition) has multiplicity 2, and the other two (denoted $d_1, d_2$) have multiplicity 1. It follows that $0 \leq l \leq 2$, with $\deg R = l + 2$. In case 2, each of 0, 1, $d$ has multiplicity 1, and $p$ has multiplicity 3. It follows that $0 \leq l \leq 3$, with $\deg R = l + 3$. Subcases are as follows.

1a) Case 1, $l = 0$, $\deg R = 2$. Necessarily $R^{-1}(0), R^{-1}(1)$ are $\{d_0, d_0\}, \{d_1, d_2\}$, or vice versa. Without loss of generality (w.l.o.g.), assume the latter, and assume $d_0 = 1$. Then $R^{-1}(0) = \{0, d\}$ and $R^{-1}(1) = \{1, 1\}$, i.e., $S^{-1}(0) = \{1, 1\}$ and $S^{-1}(1) = \{0, d\}$. Since $t = 1$ is a double zero of $S$, $S(t) = C(t - 1)^2$ for some $C$. But $S(0) = 1$, which implies $C = 1$, and $S(d) = 1$, which implies $d = 2$. So $S(t) = (t - 1)^2$ and $R(t) = t(2 - t)$. Since $t = 0, d$ are both mapped singly to the singular point $z = 0$ by $R$, their exponents must be those of $z = 0$, and hence must be identical.

1b) Case 1, $l = 1$, $\deg R = 3$. Necessarily $R^{-1}(0), R^{-1}(1)$ are $\{d_0, d_0, d_1\}, \{d_2, a_1, a_1\}$, or vice versa. W.l.o.g., assume the former, and also assume $d_0, d_1, d_2$ equal 1, $d, 0$, respectively. Then $R^{-1}(0) = \{1, 1, d\}$ and $R^{-1}(1) = \{0, a_1, a_1\}$. It follows that $R(t) = (t - 1)^2(1 - t/d)$, where $d$ is determined by the condition that the critical point of $R$ other than $t = 1$ (i.e., $t = a_1$) be mapped to 1. Solving $dR/dt = 0$ yields $a_1 = (2d + 1)/3$, and substitution into $R(a_1) - 1 = 0$ yields $d = 4$ or $-1/2$. But the latter is ruled out: it would imply $a_1 = 0$, which is impossible. So $d = 4$ and $a_1 = 3$. Since $t = 1, d$ are mapped to the singular point $z = 0$, doubly and singly respectively, the exponents of $t = 1$ must be twice those of $z = 0$, and the exponents of $t = d$ must be the same as those of $z = 0$.

1c) Case 1, $l = 2$, $\deg R = 4$. Necessarily $R^{-1}(0)$ and $R^{-1}(1)$ are $\{d_0, d_0, d_1, d_2\}$ and $\{a_1, a_1, a_2, a_2\}$, or vice versa. W.l.o.g., assume the latter, and assume $d_0 = 1$. Then $R^{-1}(0) = \{a_1, a_1, a_2, a_2\}$ and $R^{-1}(1) = \{0, 1, 1, d\}$, i.e., $S^{-1}(0) = \{0, 1, 1, d\}$ and $S^{-1}(1) = \{a_1, a_1, a_2, a_2\}$. So $S(t)$ equals $t(t - 1)^2(t - d)$, where $d$ is determined by the condition that $S$ must have two critical points other than $t = 1$, i.e., $t = a_1, a_2$, which are mapped by $S$ to the same critical value (in fact, to $z = 1$). Computation yields $dS/dt = A(t - 1)[4t^2 - (3d + 2)t + d]$, so $a_1, a_2$ must be the roots of $4t^2 - (3d + 2)t + d$. If the corresponding critical values are $Aw_1, Aw_2$, then $w_1, w_2$ are the roots of the polynomial in $w$ obtained by eliminating $t$.
between \( w - S(t)/A \) and \( 4t^2 - (3d + 2)t + d \). Its discriminant turns out to be proportional to \( (d - 2)^2(9d^2 - 4d + 4)^3 \), so the criterion for equal values is that \( d = 2 \) or \( 9d^2 - 4d + 4 = 0 \). But the latter can be ruled out, since by examination it would result in \( a_1, a_2 \) being equal. So \( d = 2; a_1, a_2 = 1 \pm \sqrt{2}/2; \) and \( S(t) = At(t - 1)^2(t - 2) \) with \( A = -4 \), so that \( S(a_1) = 1 \). Hence \( R(t) = 4 \left[ t(2-t) - \frac{1}{2} \right]^2 \). Since \( t = 0, d \) are mapped simply to \( z = 1 \) and \( t = 1 \) is mapped doubly, the exponents of \( t = 0, d \) must be the same, and double those of \( t = 1 \).

(2a) Case 2, \( l = 0, \deg R = 3 \). Necessarily \( R^{-1}(0), R^{-1}(1) \) are \( \{p, p, p\}, \{0, 1, d\} \), or vice versa. W.l.o.g., assume the former. Then \( R(t) = A(t - p)^3 \) for some \( A \). Since \( t = 0, 1, d \) are to be mapped singly to 1, they must be the vertices of an equilateral triangle, with mean \( p \), so \( A = -1/p^3 \) and \( R = (1-t/p)^3 \). W.l.o.g., take \( d = \frac{1}{2} + i\frac{\sqrt{3}}{2} \), so \( p = \frac{1}{2} + i\frac{\sqrt{3}}{6} \). The exponents at \( t = 0, 1, d \), must be equal, since they all equal the exponents at \( z = 1 \).

(2b) Case 2, \( l = 1, \deg R = 4 \). Assume w.l.o.g. that \( R^{-1}(0) = \{p, p, p, p\} \) and \( R^{-1}(1) = \{0, 1, a_1, a_1\} \), i.e., \( S^{-1}(0) = \{0, 1, a_1, a_1\} \) and \( S^{-1}(1) = \{p, p, p, p\} \). It follows that \( R(t) = (1-t/d)(1-t/p)^3 \), but to determine \( d \) and \( p \), it is best to focus on \( S \). Necessarily \( S(t) = At(t-1)(t-a_1)^2 \), and \( p \) can be a triple zero of \( R \) iff it is a double critical point of \( S \) as well as \( R \). The condition that \( S \) have a double critical point determines \( a_1 \). \( dS/dt = A(t-a_1)[4t^2 - (3 + 2a_1)t + a_1] \), so the polynomial \( 4t^2 - (3 + 2a_1)t + a_1 \) must have a double root. Its discriminant is \( 4a_1^2 - 4a_1 + 9 \), which will equal zero iff \( a_1 = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \). The corresponding value of the double root, i.e., the mandatory value of \( p \), is \( \frac{1}{2} \pm i\frac{\sqrt{3}}{4} \). The requirement that \( S \) map \( p \) to 1 implies \( A = 1/p(p-1)(p-a_1)^2 \). \( d \) is determined as the root of \( R = 1 - S \) other than \( p \); some computation yields \( \frac{1}{2} \pm i\frac{\sqrt{2}}{4} \). W.l.o.g. the ‘±’ in the expressions for \( p \) and \( d \) can be replaced by ‘+’. Since \( t = p \) is an ordinary point and \( R \) maps \( t = p \) triply to \( z = 0, z = 0 \) must have exponents 0, 1/3. Since \( R \) maps \( t = d \) simply to \( z = 0, t = d \) must also have exponents 0, 1/3. Similarly, since \( R \) maps the ordinary point \( t = a_1 \) doubly to \( z = 1, z = 1 \) must have exponents 0, 1/2; so \( t = 0 \) and \( t = 1 \), which are mapped simply to \( z = 1, \) must also.

(2c) Case 2, \( l = 2, \deg R = 5 \). Assume w.l.o.g. that \( R^{-1}(0) = \{p, p, p, 0, 1\} \) and \( R^{-1}(1) = \{d, a_1, a_1, a_2, a_2\} \). Then \( R(t) = At(t-1)(t-p)^3 \), where \( p \) is determined by \( R \) having two critical points other than \( t = p \), i.e., \( t = a_1, a_2 \), which are mapped to the same critical value (i.e., to \( z = 1 \)). \( dR/dt = A(t-p)^2[5t^2 - (2p + 4)t + p] \), so \( a_1, a_2 \) must be the two roots of \( 5t^2 - (2p + 4)t + p \). If the corresponding critical values are \( Aw_1, Aw_2 \), then \( w_1, w_2 \) are the roots of the polynomial in \( w \) obtained by eliminating \( t \) between \( w - R(t)/A \) and \( 5t^2 - (2p + 4)t + p \). Its discriminant turns out to be proportional to \( (p^2 - p + 4)^3(27p^2 - 27p + 8)^2 \), so the criterion for equal values is that \( p^2 - p + 4 = 0 \) or \( 27p^2 - 27p + 8 = 0 \). But the former can be ruled out, since by examination it would result in \( a_1, a_2 \) being equal. The latter is true iff \( p = \frac{1}{2} \pm i\frac{\sqrt{15}}{18} \). W.l.o.g., the plus sign may be
used. This yields \( a_1, a_2 = \frac{1}{2} \pm \frac{2\sqrt{3}}{9} + i\frac{\sqrt{15}}{90} \). From the condition \( R(a_i) = 1 \), it follows that \( A = -\frac{2025\sqrt{15}}{64} \). \( d \) is determined as the root of \( R(t) - 1 \) other than \( a_1, a_2 \); computation yields \( d = \frac{1}{2} + \frac{11\sqrt{15}}{90} \). Since \( t = p \) is an ordinary point mapped triply to \( z = 0, z = 0 \) must have exponents 0, 1/3. Similarly, since \( R \) maps the ordinary points \( t = a_i \) to \( z = 1, z = 1 \) must have exponents 0, 1/2, so \( t = d \), which is mapped singly to it, must also.

(2d) Case 2, \( l = 3 \), deg \( R = 6 \). Necessarily \( R^{-1}(0) \) and \( R^{-1}(1) \) are \( \{p, p, p, 0, 1, d\} \) and \( \{a_1, a_1, a_2, a_2, a_3, a_3\} \), or vice versa. W.l.o.g., assume the latter. Then
\[
R(t) = A(t - a_1)^2(t - a_2)^2(t - a_3)^2
\]
and
\[
S(t) = Bt(t - 1)(t - d)(t - p)^3.
\]
Since \( t = p \) is a triple zero of \( S \), \( R(t) \sim 1 - C(t - p)^3 \) for some nonzero \( C \).

So \( \sqrt{R(t)} \), defined to equal \( +1 \) at \( t = p \), will have a similar Taylor series:
\[
\sqrt{A}(t - a_1)(t - a_2)(t - a_3) \sim 1 - C(t - p)^3/2.
\]
This is possible only if \( a_1, a_2, a_3 \) are the vertices of an equilateral triangle, and \( p \) is their mean. It follows that the roots of \( S \) other than \( t = p \), i.e., \( t = 0, 1, d \), are also the vertices of an equilateral triangle centered on \( p \). W.l.o.g., choose \( d = \frac{1}{2} + \frac{i\sqrt{3}}{2} \) and \( p = \frac{1}{2} + \frac{i\sqrt{2}}{6} \). With a bit of algebra, \( R \) can be rewritten in the form given in the theorem. Since \( t = p \) is an ordinary point and \( R \) maps it triply to \( z = 0, z = 0 \) must have exponents 0, 1/3. Since \( R \) maps \( t = 0, 1, d \) simply to \( z = 0, t = 0, 1, d \) must also have exponents 0, 1/3.

The theorem is proved. \( \square \)

Now that the main theorem is proved, we can proceed to derive explicit Heun-to-hypergeometric reduction formulæ. Theorem 3.7 gives a necessary condition for the existence of a reduction, and Theorem 3.8 presents a representative list.

**Theorem 3.7.** Suppose the Heun equation \([5]\) has four singular points and is nontrivial \((\alpha \beta \neq 0 \text{ or } q \neq 0)\). Then its local solution \( Hl(d, q; \alpha, \beta, \gamma, \delta; t) \) can be reduced to a hypergeometric function \( _2F_1(a, b; c; z) \) by a formula of the type \( Hl(t) = _2F_1(R(t)) \), with \( R \) a rational function, only if its parameters \( d, q; \alpha, \beta \) satisfy \( q = \alpha \beta t \), with \( (d, p) \) equal to one of the following 23 pairs. If a reduction of this type exists, \( R(t) \) will be a polynomial of the stated degree.

\[
\begin{align*}
(1a) \quad & [\text{deg } R = 2 \text{ or } 4]. \quad (-1, 0), \left(\frac{1}{2}, \frac{1}{2}\right), (2, 1). \\
(1b) \quad & [\text{deg } R = 3]. \quad (-3, 0), \left(-\frac{1}{3}, 0\right), \left(\frac{1}{4}, \frac{1}{4}\right), \left(\frac{3}{4}, \frac{3}{4}\right), (4, 1), (4, 1). \\
(2a) \quad & [\text{deg } R = 3 \text{ or } 6]. \quad \left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \frac{1}{2} \pm i\frac{\sqrt{3}}{6}\right). \\
(2b) \quad & [\text{deg } R = 4]. \quad \left(\frac{1}{2} \pm i\frac{\sqrt{2}}{4}, \frac{1}{2} \pm i\frac{\sqrt{2}}{4}\right), \\
& \quad \left(\frac{4}{27} \pm i\frac{10}{27}, \frac{4}{27} \pm i\frac{10}{27}, \frac{23}{27} \pm i\frac{10}{27}, \frac{20}{27} \pm i\frac{10}{27}\right). \\
(2c) \quad & [\text{deg } R = 5]. \quad \left(\frac{1}{2} \pm i\frac{11}{90}, \frac{1}{2} \pm i\frac{11}{90}\right), \\
& \quad \left(\frac{13}{128} \pm i\frac{33}{128}, \frac{95}{128} \pm i\frac{9}{128}, \frac{95}{128} \pm i\frac{9}{128}\right), \left(-\frac{7}{128} \pm i\frac{33}{128}, \frac{33}{128} \pm i\frac{9}{128}\right).
\end{align*}
\]
PROOF. The five subcases are taken from Theorem 3.1 with (1c),(2d) respectively subsumed into (1a),(2a). Theorem 3.1 supplies only a single pair \((D, p_0)\) for each subcase. By the remarks preceding that theorem, \(d\) may be any value on the cross-ratio orbit of \(D\). There are up to 6 distinct possibilities, which were given in (2.7). The value of \(p\) associated to \(d\) is the preimage of \(p_0\) under the corresponding affine map \(t \mapsto A_1(t)\), given in (2.8). \(\square\)

**Theorem 3.8.** Suppose a Heun equation has four singular points and is non-trivial \((\alpha \beta \neq 0\ or\ q \neq 0)\). Then the only reductions of its local Heun function \(Hl\) to \(2F_1\) that can be performed by a rational transformation of the independent variable involve polynomial transformations of degrees 2, 3, 4, 5, and 6. There are seven distinct types, each of which can exist only if \(d\) lies on an appropriate cross-ratio orbit. The following list includes a representative reduction of each type. The ones with real \(d\) (and \(\deg R = 2, 3, 4\)) include

\[
Hl(2, \alpha \beta; \alpha, \beta, \gamma, \alpha + \beta - 2\gamma + 1; t) = 2F_1\left(\frac{\alpha}{2}, \frac{\beta}{2}; \gamma; t(2 - t)\right),
\]

\[
Hl\left(4, \alpha \beta; \alpha, \beta, \frac{1}{2}, \frac{2(\alpha + \beta)}{3}; t\right) = 2F_1\left(\frac{\alpha}{3}, \frac{\beta}{3}; 1 - (t - 1)(1 - t/4)\right),
\]

\[
Hl\left(2, \alpha \beta; \alpha, \beta, \frac{\alpha + \beta + 2}{4}, \frac{\alpha + \beta}{2}; t\right) = 2F_1\left(\frac{\alpha}{4}, \frac{\beta}{4}; 1 - 4\left[t(2 - t) - \frac{1}{2}\right]^2\right),
\]

and the ones with non-real \(d\) (and \(\deg R = 3, 4, 5, 6\)) include

\[
Hl\left(\frac{1}{2} \pm \frac{i\sqrt{3}}{2}, \alpha \beta; \alpha, \beta, \frac{\alpha + \beta + 1}{3};\ t\right) = 2F_1\left(\frac{\alpha}{3}, \frac{\beta}{3}; 1 - \left[1 - t/(\frac{1}{2} \pm \frac{i\sqrt{3}}{6})\right]^3\right),
\]

\[
Hl\left(\frac{1}{2} \pm \frac{i\sqrt{2}}{4}, \frac{\alpha - \alpha}{2} + \frac{i\sqrt{2}}{4}; \frac{\alpha}{2} - \frac{\alpha}{2}, \frac{1}{2}, \frac{1}{2}; t\right) = 2F_1\left(\frac{\alpha}{4}, \frac{1}{6} - \frac{1}{6}; \frac{1}{2}; 1 - \left[1 - t/(\frac{1}{2} \pm \frac{i\sqrt{2}}{4})\right]^3\right),
\]

\[
Hl\left(\frac{1}{2} \pm \frac{i\sqrt{16}}{18}, \frac{\alpha - \alpha}{2} + \frac{i\sqrt{16}}{18}; \frac{5}{6} - \frac{\alpha}{2}, \frac{2}{3}, \frac{2}{3}, t\right) = 2F_1\left(\frac{\alpha}{5}, \frac{1}{6} - \frac{1}{6}; \frac{2}{3}; 1 - \left[1 - t/(\frac{1}{2} \pm \frac{i\sqrt{16}}{18})\right]^3\right),
\]

\[
Hl\left(\frac{1}{2} \pm \frac{i\sqrt{3}}{2}, \alpha - \alpha; \frac{\alpha}{2} \pm \frac{i\sqrt{3}}{6}; \frac{1}{2} - \alpha, \frac{2}{3}, \frac{2}{3}, t\right) = 2F_1\left(\frac{\alpha}{6}, \frac{1}{6} - \frac{1}{6}; \frac{2}{3}; 1 - 4\left[1 - t/(\frac{1}{2} \pm \frac{i\sqrt{3}}{6})\right]^3 - \frac{1}{2}\right)^2.
\]

In the preceding reductions, \(\alpha, \beta, \gamma\) are free parameters. Each of these equalities
holds in a neighborhood of \( t = 0 \) whenever the two sides are defined, e.g., whenever the fifth argument of \( H_l \) and the third argument of \( _2F_1 \) are not equal to a nonpositive integer.

**Remark 3.8.1.** The equalities of Theorem 3.8 hold even if the Heun equation has fewer than four singular points, or is trivial; but in either of those cases, additional reductions are possible. For the trivial case, see §6.

**Remark 3.8.2.** The special harmonic reduction (3.5c) is composite: it can be obtained from the case \( \gamma = (\alpha + \beta + 2)/4 \) of the reduction (3.5a) by applying Gauss’s quadratic hypergeometric transformation \([2, \S 3.1]\)

\[
_2F_1\left( a, b; \frac{a+b+1}{2}; z \right) = _2F_1\left( \frac{a}{2}, \frac{b}{2}; \frac{a+b+1}{2}; 1-4(z-\frac{1}{2})^2 \right) \tag{3.7}
\]
to the right-hand side. The special equianharmonic reduction (3.6d) can be obtained in the same way from the case \( \beta = 1 - \alpha \) of the reduction (3.6a).

One might think that by applying (3.7) to the right-hand sides of the remaining reductions in (3.5a–3.6d), additional composite reduction formulae could be generated. However, there are only a few cases in which it can be applied; and when it can, it imposes conditions on the parameters of \( H_l \) which require that the corresponding Heun equation have fewer than four singular points.

**PROOF.** \( H_l \) and \( _2F_1 \) are the local solutions of their respective equations which belong to the exponent zero at \( t = 0 \) (resp. \( z = 0 \)), and are regular and normalized to unity there. So the theorem follows readily from Theorem 3.1 (3.5a–3.5c) come from subcases 1a–1c, and (3.6a–3.6d) from subcases 2a–2d. In each subcase, the Gauss parameters \((a, b; c)\) of \( _2F_1 \) are computed by first calculating the exponents at \( z = 0, 1, \infty \), in the way explained in Remark 3.1.3.

In some subcases, the polynomial map supplied in Theorem 3.1 must be chosen to be \( S = 1 - R \) rather than \( R \), due to the need to map \( t = 0 \) to \( z = 0 \) rather than to \( z = 1 \), so that the transformation will reduce \( H_l \) to \( _2F_1 \), and not to another local solution of the hypergeometric equation.

The list of Heun-to-hypergeometric reductions given in Theorem 3.8 is representative rather than exhaustive. For each subcase of Theorem 3.1 there is one reduction for each allowed value of \( d \). Each reduction on the above list came from choosing \( d = D \), but any other \( d \) on the cross-ratio orbit of \( D \) may be chosen. The orbit is defined by \( \triangle 01d \) being one of the triangles (at most six) similar to \( \triangle 01D \), i.e., by \( \triangle 01D \) being obtained from \( \triangle 01d \) by an affine transformation \( A_1 \in \mathcal{A}(\mathfrak{S}) \). So for any subcase of Theorem 3.1 and choice of \( d \), the appropriate polynomial map will be \( z = A_2(R_1(A_1(t))) \), where \( A_1 \) is constrained to map \( \triangle 01d \) to \( \triangle 01D \) and is listed in (2.8), \( R_1 \) is the polynomial map given in the subcase, and \( A_2 \in \mathcal{A}(\mathfrak{h}) \), i.e., \( A_2(z) = z \) or \( 1 - z \), is chosen so that \( t = 0 \) is mapped to \( z = 0 \) rather than to \( z = 1 \).
As an example, consider the harmonic subcase 1a of Theorem 3.1 in which $D = 2$, the cross-ratio orbit of $D$ is $\{ -1, \frac{1}{2}, 2 \}$, and the polynomial map is $R_1(t) = t(2 - t)$. Choosing $d = D$ yields the reduction (3.3a). Choosing $d = 1 - D = -1$ yields an alternative reduction of $Hl$ to $2F_1$, namely

$$Hl(-1, 0; \alpha, \beta, \gamma, (\alpha + \beta - \gamma + 1)/2; t) = _2F_1(\alpha/2, \beta/2; (\gamma + 1)/2; t^2),$$

(3.8)
in which $A_1(t) = 1 - t$ according to (2.8), and $A_2(z) = 1 - z$.

It is not difficult to check that in all, exactly 28 Heun-to-hypergeometric reductions can be derived from Theorem 3.1. They exhibit the 23 values of the pair $(d, q/\alpha\beta)$ listed in Theorem 3.7. Of the 28, eleven were given in Theorem 3.8, and (3.8) is a twelfth. With the exception of the two reductions with $(d, q/\alpha\beta) = (-1, 0)$, one of which is (3.8), the 28 split into pairs, each pair being related by the identity (2.10), which takes $d$ to $1/d$.

4 A Generalization

In applied mathematics, it is seldom the case that the four singular points of an equation of Heun type are located at $0, 1, d, \infty$. But the main theorem, Theorem 3.1 may readily be generalized. Consider the situation when three of the four have zero as a characteristic exponent, since this may always be arranged by applying an F-homotopy. There are two cases of interest: either the singular points include $\infty$ and each of the finite singular points has zero as a characteristic exponent; or the location of the singular points is unrestricted. The latter includes the former. They have respective $P$-symbols

$$P\begin{pmatrix} d_1 & d_2 & d_3 & \infty \\ 0 & 0 & 0 & \alpha \\ 1 - \gamma & 1 - \delta & 1 - \epsilon & \beta \end{pmatrix}, \quad P\begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ 0 & 0 & 0 & \alpha \\ 1 - \gamma & 1 - \delta & 1 - \epsilon & \beta \end{pmatrix}.$$ 

(4.1)

In the nomenclature of Ref. [22], they are canonical cases of the natural general-form and general-form Heun equations. They are transformed to the Heun equation (3.5) by the affine and Möbius transformations

$$t = \frac{s - d_1}{d_2 - d_1}, \quad t = \frac{(s - d_1)(d_2 - d_4)}{(d_2 - d_1)(s - d_4)},$$

(4.2)
respectively. Each of the \( P \)-symbols \((4.1)\) is accompanied by an accessory parameter. The equation specified by \((4.1a)\) can be written as
\[
\frac{d^2 u}{ds^2} + \left( \frac{\gamma}{s-d_1} + \frac{\delta}{s-d_2} + \frac{\epsilon}{s-d_3} \right) \frac{du}{ds} + \frac{\alpha \beta s - q'}{(s-d_1)(s-d_2)(s-d_3)} u = 0,
\]
where \( q' \) is the accessory parameter \([22]\). The equation specified by \((4.1b)\) with \( d_4 \neq \infty \) can be written as
\[
\frac{d^2 u}{ds^2} + \left( \frac{\gamma}{s-d_1} + \frac{\delta}{s-d_2} + \frac{\epsilon}{s-d_3} + \frac{1 - \alpha - \beta}{s-d_4} \right) \frac{du}{ds} + \frac{\alpha \beta \prod_{i=1}^{3}(d_4-d_i)}{(s-d_1)(s-d_2)(s-d_3)(s-d_4)} u = 0, \tag{4.4}
\]
where \( q'' \) is the accessory parameter \([22]\).

The impediment to the generalization of Theorem 3.1 to these two equations is the specification of the cases that should be excluded due to their being ‘trivial’, or having fewer than four singular points. The excluded cases should really be specified not in terms of the \textit{ad hoc} parameters \( q' \) and \( q'' \), but rather in an invariant way, in terms of an accessory parameter defined so as to be invariant under affine or Möbius transformations, respectively. Schäfke \([23]\) has defined new accessory parameters of second-order Fuchsian equations on \( \mathbb{C}P^1 \) that are invariant under affine transformations, but no extension to general Möbius transformations seems to have been developed.

In the absence of an invariantly defined accessory parameter, an \textit{ad hoc} approach will be followed. It is clear that \((4.3)\) is trivial, i.e., can be transformed to a trivial Heun equation by an affine transformation, iff \( \alpha \beta = 0, q' = 0 \). Also, it will have fewer than four singular points if \( \gamma = 0, q' = 0 \); or \( \delta = 0, q' = \alpha \beta \); or \( \epsilon = 0, q' = \alpha \beta d \). Likewise, it is fairly clear that \((4.4)\) will be trivial, i.e., can be transformed to a trivial Heun equation by a Möbius transformation, iff \( \alpha \beta = 0, q'' = 0 \). The conditions on the parameters for there to be a full set of singular points are, however, more complicated.

The first generalization of Theorem 3.1 is Corollary 4.1, which follows from Theorem 3.1 by applying the affine transformation \((4.2a)\). It mentions a polynomial transformation, which is the composition of the \( s \mapsto t \) affine transformation with the \( t \mapsto z \) polynomial map of Theorem 3.1. To avoid repetition, Corollary 4.1 simply cites Theorem 3.1 for the necessary and sufficient conditions on the exponent parameters and the accessory parameter.

\textbf{Corollary 4.1.} A natural general-form Heun equation of the canonical type \((4.3)\), which has four singular points and is nontrivial (i.e., \( \alpha \beta \neq 0 \) or \( q' \neq 0 \)),
can be reduced to a hypergeometric equation of the form \([h]\) by a rational substitution \(z = R(s)\) iff \(\alpha \beta \neq 0\), \(R\) is a polynomial, and the Heun equation satisfies the following conditions.

(i) \(\Delta d_1d_2d_3\) must be similar to \(\Delta 01D\), with \(D = 2\) or \(\frac{1}{2} + \frac{i\sqrt{3}}{2}\), or \(D = 4, \frac{1}{2} + \frac{i5\sqrt{3}}{4}\), or \(\frac{1}{2} + \frac{i11\sqrt{15}}{90}\). That is, it must either be a degenerate triangle consisting of three equally spaced collinear points (the harmonic case), or be an equilateral triangle (the equianharmonic case), or be similar to one of three other specified triangles, of which one is degenerate and two are isosceles. (ii) The exponent parameters \(\gamma, \delta, \epsilon\) must satisfy conditions that follow from the corresponding subcases of Theorem 3.1. (iii) The parameter \(q'\) must take a value that can be computed uniquely from the parameters \(\gamma, \delta, \epsilon\) and the choice of subcase.

Example 4.1.1. In the harmonic case, the two endpoints of the degenerate triangle of singular points \(\Delta d_1d_2d_3\) must have equal exponent parameters, and \(q'\) must equal \(\alpha \beta\) times the intermediate point. In this case, \(R\) will typically be a quadratic polynomial. There are two possibilities: \(R\) will map the two endpoints to \(z = 0\) and the intermediate point to \(z = 1\), or vice versa. If the characteristic exponents of the intermediate point are twice those of the endpoints, then \(R\) may be quartic instead: the composition of either possible quadratic polynomial with a subsequent \(z \mapsto 4(z - \frac{1}{2})^2\) or \(z \mapsto 4z(1 - z)\) map.

Example 4.1.2. In the equianharmonic case, all three exponent parameters \(\gamma, \delta, \epsilon\) must be equal, and the accessory parameter \(q'\) must equal \(\alpha \beta\) times the mean of \(d_1, d_2, d_3\). In this case, \(R\) will typically be a cubic polynomial. There are two possibilities: \(R\) will map \(d_1, d_2, d_3\) to \(z = 0\) and their mean to \(z = 1\), or vice versa. If the exponent parameters \(\gamma, \delta, \epsilon\) equal \(\frac{2}{3}\), then \(R\) may be sextic instead: the composition of either possible cubic polynomial with a subsequent \(z \mapsto 4(z - \frac{1}{2})^2\) or \(z \mapsto 4z(1 - z)\) map.

The further generalization of Theorem 3.1 is Corollary 4.2, which follows from Theorem 3.1 by applying the Möbius transformation \((4.2b)\). It mentions a rational substitution, which is the composition of the \(s \mapsto t\) Möbius transformation with the \(t \mapsto z\) polynomial map of Theorem 3.1.

Corollary 4.2. A general-form Heun equation of the canonical type \((4.4)\), which has four singular points and is nontrivial (i.e., \(\alpha \beta \neq 0\) or \(q'' \neq 0\)), can be reduced to a hypergeometric equation of the form \([h]\) by a rational substitution \(z = R(s)\) iff \(\alpha \beta \neq 0\), and the Heun equation satisfies the following conditions.

(i) The cross-ratio orbit of \(\{d_1, d_2, d_3, d_4\}\) must be that of \(\{0, 1, D, \infty\}\), where \(D\) is one of the five values enumerated above. That is, it must be the harmonic orbit, the equianharmonic orbit, or one of three specified generic orbits, one real and two non-real. (ii) The exponent parameters \(\gamma, \delta, \epsilon\) must satisfy conditions that follow from the corresponding subcases of Theorem 3.1. (iii) The parameter \(q''\) must take a value that can be computed uniquely from the pa-
rameters $\gamma, \delta, \epsilon$ and the choice of subcase.

Example 4.2.1. Suppose $d_1, d_2, d_3, d_4$ form a harmonic quadruple, i.e., can be mapped by a M"{o}bius transformation to the vertices of a square in $\mathbb{C}$. Moreover, suppose two of $d_1, d_2, d_3$ have the same characteristic exponents, and are mapped to diagonally opposite vertices of the square. That is, of the three parameters $\gamma, \delta, \epsilon$, the two corresponding to a diagonally opposite pair must be equal. Then provided $q''$ takes a value that can be computed from the other parameters, a substitution $R$ will exist. It will typically be a degree-2 rational function, the only critical points of which are the third singular point (out of $d_1, d_2, d_3$) and $d_4$. Either $R$ will map the two distinguished singular points to $z = 1$ and the third singular point to $z = 0$, or vice versa; and $d_4$ to $z = \infty$. In the special case when the exponents of the third point are twice those of the two distinguished points, it is possible for $R$ to be a degree-4 rational function.

Example 4.2.2. Suppose $d_1, d_2, d_3, d_4$ form an equianharmonic quadruple, i.e., can be mapped by a M"{o}bius transformation to the vertices of a regular tetrahedron in $\mathbb{CP}^1$. Moreover, suppose $d_1, d_2, d_3$ have the same characteristic exponents, i.e., $\gamma = \delta = \epsilon$. Then provided $q''$ takes a value uniquely determined by the other parameters, a substitution $R$ will exist. Typically, $R$ will be a degree-3 rational function, the only critical points of which are the mean of $d_1, d_2, d_3$ with respect to $d_4$, and $d_4$. Either $R$ will map $d_1, d_2, d_3$ to $z = 1$ and the mean of $d_1, d_2, d_3$ with respect to $d_4$ to $z = 0$, or vice versa; and $d_4$ to $z = \infty$. In the special case when the exponents of each of $d_1, d_2, d_3$ equal $0, 1/3$, it is possible for $R$ to be a degree-6 rational function.

Remark 4.2.3. In Example 4.2.2. the concept of the mean of three points in $\mathbb{CP}^1$ with respect to a distinct fourth point was used. A projectively invariant definition is the following. If $T$ is a M"{o}bius transformation that takes $d_4$ ($\neq d_1, d_2, d_3$) to the point at infinity, the mean of $d_1, d_2, d_3$ with respect to $d_4$ is the point that would be mapped to the mean of $Td_1, Td_2, Td_3$ by $T$.

5 The Clarkson–Olver Transformation

The reduction discovered by Clarkson and Olver \cite{6}, which stimulated these investigations, turns out to be a special case of the equianharmonic Heun-to-hypergeometric reduction of \S3. Their reduction was originally given in a rather complicated form, which we shall simplify.

Recall that the Weierstrass function $\wp(u) \equiv \wp(u; g_2, g_3)$ with invariants $g_2, g_3 \in \mathbb{C}$, which cannot both equal zero, has a double pole at $u = 0$ and satisfies

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3). \quad (5.1)$$
Here \(e_1, e_2, e_3\), the zeroes of the defining cubic polynomial, are the finite critical values of \(\wp\), the sum of which is zero; they are required to be distinct. \(\wp\) is doubly periodic on \(\mathbb{C}\), with periods denoted \(2\omega, 2\omega'\). So it can be viewed as a function on the torus \(T \equiv \mathbb{C}/\mathcal{L}\), where \(\mathcal{L} = 2\omega \mathbb{Z} \oplus 2\omega' \mathbb{Z}\) is the period lattice. It turns out that the half-lattice \(\{0, \omega, \omega', \omega + \omega'\} + \mathcal{L}\) comprises the critical points of \(\wp\). The map \(\wp : T \to \mathbb{CP}^1\) is a double branched cover of the Riemann sphere, but \(T\) is uniquely coordinatized by the pair \((\wp, \wp')\).

The modular discriminant \(\Delta \equiv g_3^2 - 27g_3^2 \neq 0\) is familiar from elliptic function theory. If \(g_2, g_3 \in \mathbb{R}\) and \(\Delta > 0\) (the so-called real rectangular case, which predominates in applications), \(\omega, \omega'\) can be chosen to be real and imaginary, respectively. If \(\Delta < 0\) (the less familiar real rhombic case), they can be chosen to be complex conjugates, so that the third critical point \(\omega_2 \equiv \omega + \omega'\) is real.

Clarkson and Olver considered the Weierstrass-form Lamé equation

\[
\frac{d^2\psi}{du^2} - \left[\ell(\ell + 1)\wp(u) + B\right] \psi = 0, \tag{5.2}
\]

which is a Fuchsian equation on \(T\) with exactly one singular point (at \((\wp, \wp') = (\infty, \infty))\) and a single accessory parameter, \(B\). [We have altered their exponent parameter \(-36\sigma\) to \(\ell(\ell + 1)\), to agree with the literature, and have added the accessory parameter.] In particular, they considered the case \(g_2 = 0, g_3 \neq 0, B = 0\). They mapped \(u \in T\) to \(z \in \mathbb{CP}^1\) by the formal substitution

\[
u = \frac{i}{(16g_3)^{1/6}} \int (1 - z)^{1/3} \frac{d\tau}{\sqrt{1 - \tau^3}}, \tag{5.3}
\]

and showed that the Lamé equation is reduced to

\[
z(1 - z) \frac{d^2\psi}{dz^2} + \left(\frac{1}{2} - \frac{7}{6}z\right) \frac{d\psi}{dz} + \frac{\ell(\ell + 1)}{36} \psi = 0. \tag{5.4}
\]

This is a hypergeometric equation with \((a, b; c) = (-\ell/6, (\ell + 1)/6; 1/2)\). It has exponents 0, 1/2 at \(z = 0\); 0, 1/3 at \(z = 1\); and \(-\ell/6, (\ell + 1)/6\) at \(z = \infty\).

In elliptic function theory the case \(g_2 = 0, g_3 \neq 0\) is called equianharmonic, since the corresponding critical values \(e_1, e_2, e_3\) are the vertices of an equilateral triangle in \(\mathbb{C}\). If, for example, \(g_3 \in \mathbb{R}\), then \(\Delta < 0\); and by convention, \(e_1, e_2, e_3\) correspond to \(\omega, \omega_2, \omega'\), respectively. \(e_1\) and \(e_3\) are complex conjugates, and \(e_2\) is real. The triangle \(\triangle 0\omega_2\omega'\) is also equilateral [1, §18.13].

So, what Clarkson and Olver considered was the equianharmonic Lamé equation, the natural domain of definition of which is a torus \(T\) (i.e., a complex elliptic curve) with special symmetries. For the Lamé equation \((5.2)\) to be viewed as a Heun equation on \(\mathbb{CP}^1\), it must be transformed by \(s = \wp(u)\) to its
algebraic form \([15]\). The algebraic form is

\[
\frac{d^2 \psi}{ds^2} + \left( \frac{1/2}{s-e_1} + \frac{1/2}{s-e_2} + \frac{1/2}{s-e_3} \right) \frac{d\psi}{ds} + \frac{[-\ell(\ell+1)/4]s - B/4}{(s-e_1)(s-e_2)(s-e_3)} \psi = 0.
\] (5.5)

This is a special case of \([4.3]\), the canonical version of the natural general-form Heun equation, with distinct finite singular points \(d_1, d_2, d_3 = e_1, e_2, e_3\). Also, \(\alpha, \beta = -\ell/2, (\ell+1)/2\), \(\gamma = \delta = \epsilon = 1/2\), and \(q' = B/4\). It has characteristic exponents \(0, 1/2\) at \(s = e_1, e_2, e_3\), and \(-\ell/2, (\ell+1)/2\) at \(s = \infty\).

Applying Corollary 4.1 to (5.5) yields the following.

**Theorem 5.1.** The algebraic-form Lamé equation (5.5), in the equianharmonic case \(g_2 = 0, g_3 \neq 0\), can be reduced when \(\ell(\ell+1) \neq 0\) to a hypergeometric equation of the form \([h]\) by a rational transformation \(z = R(s)\) iff the accessory parameter \(B\) equals zero. If this is the case, \(R\) will necessarily be a cubic polynomial;

\[
z = 4s^3/g_3, \quad z = 1 - 4s^3/g_3
\] (5.6)

will both work, and they are the only possibilities.

**PROOF.** If \(\ell(\ell+1) \neq 0\), the Heun equation (5.5) is nontrivial in the sense of Definition 2.1, with four singular points; by (5.1), the \(e_i\) are the cube roots of \(g_3/4\), and are the vertices of an equilateral triangle. Since \(\gamma = \delta = \epsilon\), the equianharmonic case of Corollary 4.1 applies, and no other.

The mean of \(e_1, e_2, e_3\) is zero. So the polynomial \(4s^3/g_3\) is the cubic polynomial that maps each singular point to 1, and their mean to zero; \(1 - 4s^3/g_3\) does the reverse. These are the only possibilities for the map \(s \mapsto z\), since the sextic polynomials mentioned in the equianharmonic case of Corollary 4.1 can be employed only if \(\gamma, \delta, \epsilon\) equal 2/3, which is not the case here. \(\Box\)

**Remark.** Corollary 4.1 (equianharmonic case) also applies to the equianharmonic algebraic-form Lamé equation with \(\ell(\ell+1) = 0, B \neq 0\), and guarantees it cannot be transformed to the hypergeometric equation by any rational substitution; since in the sense used above, this too is a nontrivial Heun equation.

**Corollary 5.2.** The Weierstrass-form Lamé equation (5.2), in the equianharmonic case \(g_2 = 0, g_3 \neq 0\), can be reduced when \(\ell(\ell+1) \neq 0\) to a hypergeometric equation of the form \([h]\) by a substitution of the form \(z = R(\wp(u))\), where \(R\) is rational, iff the accessory parameter \(B\) equals zero. In this case,

\[
z = 4\wp(u)^3/g_3, \quad z = 1 - 4\wp(u)^3/g_3
\] (5.7)

will both work, and they are the only such substitutions.
Applying the substitution $z = 1 - 4 \wp(u)^3/g_3$ to the Lamé equation (5.2) reduces it to the hypergeometric equation (5.4), as is readily verified. (The other substitution $z = 4 \wp(u)^3/g_3$ yields a closely related hypergeometric equation, with the singular points $z = 0, 1$ interchanged.) The Clarkson–Olver substitution formula (5.3) contains a multivalued elliptic integral, but it may be inverted with the aid of (5.1) to yield $z = 1 - 4 \wp(u)^3/g_3$. So their transformation fits into the framework of Corollary 5.2.

The most noteworthy feature of the Clarkson–Olver transformation is that it can be performed irrespective of the choice of exponent parameter $\ell$. Only the accessory parameter $B$ is restricted. As they remark, when $\ell = 1, 1/2, 1/4$, or $1/10$, it is a classical result of Schwarz that all solutions of the hypergeometric equation (5.4) are necessarily algebraic [15, §10.3]. This implies that if $B = 0$, the same is true of all solutions of the algebraic Lamé equation (5.5); which had previously been proved by Baldassarri [5], using rather different techniques. But irrespective of the choice of $\ell$, the solutions of the $B = 0$ Lamé equation reduce to solutions of the hypergeometric equation. This is quite unlike the other known classes of exact solutions of the Lamé equation, which restrict $\ell$ to take values in a discrete set [20, §2.8.4]. But it is typical of hypergeometric reductions of the Heun equation. As the theorems of §3 make clear, in general it is possible to alter characteristic exponents continuously, without affecting the existence of a reduction to the hypergeometric equation.

It should be mentioned that the harmonic as well as the equianharmonic case of Corollary 4.1 can be applied to the algebraic-form Lamé equation. One of the resulting quadratic transformations was recently rediscovered by Ivanov [16], in a heavily disguised form. The case of quadratic rather than cubic changes of the independent variable will be considered elsewhere.

### 6 The Seemingly Trivial Case $\alpha \beta = 0, q = 0$

If the Heun equation (5.1) is trivial in the sense of Definition 2.1 it may be solved by quadratures. A basis of solutions is

$$u_1(t) = 1, \quad u_2(t) = \int^t \exp \left[ - \int^v \left( \frac{\gamma}{w} + \frac{\delta}{w-1} + \frac{\epsilon}{w-d} \right) dw \right] dv. \quad (6.1)$$

In the trivial limit, the local Heun function $Hl(d, q; \alpha, \beta, \gamma, \delta; t)$ degenerates to the former, and the solution belonging to the exponent $1 - \gamma$ at $t = 0$, denoted $Hl(d, q; \alpha, \beta, \gamma, \delta; t)$ here, to the latter. In applications, explicit solutions, if any, are what matter. It is nonetheless interesting to examine under what circumstances a trivial Heun equation can be reduced to a hypergeometric equation. This question was first considered by Kuiken [18].
The canonical polynomial substitutions of §3 give rise to many nonpolynomial rational reductions of trivial Heun equations to hypergeometric equations, by composing with certain Möbius transformations. To understand why, recall that Theorem 3.1 characterized, up to affine automorphisms of the two equations, the polynomial substitutions that can reduce a nontrivial Heun equation to a hypergeometric equation. If \( t \) denotes a canonical polynomial substitution, the full set of polynomial substitutions derived from it comprises all \( t \mapsto A_2(R_1(A_1(t))) \), where \( A_1 \in \mathcal{A}(\mathfrak{g}) \) is an affine automorphism of the Heun equation, which maps \( \{0, 1, d\} \) onto \( \{0, 1, D\} \), and \( A_2 \in \mathcal{A}(\mathfrak{h}) \) is an affine automorphism of the hypergeometric equation, which maps \( \{0, 1\} \) onto \( \{0, 1\} \). (The only two possibilities are \( A_2(z) = z \) and \( A_2(z) = 1 - z \).)

In the context of nontrivial Heun equations, Möbius automorphisms that are not affine could not be employed; essentially because, as discussed in §2, moving the point at infinity would require a compensating F-homotopy. But in the trivial case no such issue arises: by Proposition 3.5, the Heun equation is reduced to a hypergeometric equation by a rational substitution of its independent variable, \( z = R(t) \), iff the substitution maps exponents to exponents. And Möbius transformations that are not affine certainly preserve exponents.

**Theorem 6.1.** A Heun equation of the form (5), which has four singular points and is trivial (i.e., \( \alpha \beta = 0 \) and \( q = 0 \)), can be reduced to a hypergeometric equation of the form (1) by any rational substitution of the form \( z = M_2(R_1(M_1(t))) \), where \( z = R_1(t) \) is a polynomial that maps \( \{0, 1, D\} \) to \( \{0, 1\} \), listed (along with \( D \) in one of the seven subcases of Theorem 3.1 and where \( M_1 \in \mathcal{M}(\mathfrak{g}) \), and \( M_1 \in \mathcal{M}(\mathfrak{h}) \). That is, \( M_1 \) maps \( \{0, 1, d, \infty\} \) onto \( \{0, 1, D, \infty\} \), and \( M_2 \) maps \( \{0, 1, \infty\} \) onto \( \{0, 1, \infty\} \). The necessary conditions on characteristic exponents stated in Theorem 3.1 must be satisfied, the conditions on exponents at specified values of \( t \) being taken to refer to the exponents at the preimages of these points under \( M_1 \).

**Remark 6.1.1.** As in the derivation of the \( Hl(t) = {}_2F_1(R(t)) \) reduction formulae listed in Theorem 3.8, the Gauss parameters \( (a, b; c) \) of the resulting hypergeometric equation can be computed by first calculating the exponents at \( z = R(t) = 0, 1, \infty \), using the mapping of exponents to exponents.

The following example shows how such nonpolynomial rational substitutions are constructed. In the harmonic subcase 1a of Theorem 3.1, \( D = 2 \) and the polynomial tranformation is \( t \mapsto z = R_1(t) = t(2 - t) \); the necessary condition on exponents is that \( t = 0, d \) have identical exponents. Consider \( d = -1 \), which is on the cross-ratio orbit of \( D \). \( M_1(t) = (t - 1)/t \) can be chosen; also, let \( M_2(z) = 1/z \). Then the composition

\[
z = R(t) \equiv M_2(R_1(M_1(t))) = t^2/(t^2 - 1)
\]

maps \( t = 0 \) to \( z = 0 \) and \( t = \infty \) to \( z = 1 \) (both with double multiplicity),
and \( t = 1, d \) to \( z = \infty \). This substitution may be applied to any trivial Heun equation with \( d = -1 \) and identical exponents at \( t = 1, d \), i.e., with \( \delta = \epsilon \).

In this example, \( M_1, M_2 \) were selected with foresight, to ensure that \( R \) maps \( t = 0 \) to \( z = 0 \). This makes it possible to regard the substitution as a reduction of \( Hl \) to \( _2F_1 \), or of \( \tilde{Hl} \) to \( \tilde{\_2F_1} \). By calculation of exponents, the reduction is

\[
\tilde{Hl} (-1, 0; 0, \beta, \gamma, (1 + \beta + \gamma)/2; t) = (-1)^{(\gamma-1)/2} _2\tilde{F}_1 (0, (1 - \beta + \gamma)/2; (1 + \gamma)/2; t^2/(t^2 - 1)).
\] (6.3)

The normalization factor \((-1)^{(\gamma-1)/2}\) is present because by convention \( \tilde{Hl}(t) \sim t^{1-\gamma} \) and \( _2\tilde{F}_1(z) \sim z^{1-\epsilon} \) in a neighborhood of \( t = 0 \) (resp. \( z = 0 \)), where the principal branches are meant. The corresponding reduction of \( Hl \) to \( _2F_1 \) is trivially valid (both sides are constant functions of \( t \), and equal unity).

Working out the number of rational substitutions \( z = R(t) \) that may be applied to trivial Heun equations, where \( R \) is of the form \( M_2 \circ R_1 \circ M_1 \), is a useful exercise. There are seven subcases of Theorem 3.1, i.e., choices for the polynomial \( R_1 \). Each subcase allows \( d \) to be chosen from an orbit consisting of \( m \) cross-ratio values: \( m = 3 \) in the harmonic subcases 1a and 1c, \( m = 2 \) in the equianharmonic subcases 2a and 2d, and \( m = 6 \) in the others. In any subcase, the \( 4! \) choices for \( M_1 \) are divided equally among the \( m \) values of \( d \), and there are also \( 3! \) choices for \( M_2 \). So each subcase yields \((4!/m)3!\) rational substitutions for each value of \( d \), but not all are distinct.

To count distinct rational substitutions for each value of \( d \), note the following. \( R \) will map \( t = 0, 1, d, \infty \) to \( z = 0, 1, \infty \). Each of the subcases of Theorem 3.1 has a ‘signature’, specifying the cardinalities of the inverse images of the points \( 0, 1, \infty \). For example, case 1a has signature \( 2; 1; 1 \), which means that of those three points, one has two preimages and the other two have one. (Order here is not significant.) In all, subcases 1a,1b,2b,2c have signature \( 2; 1; 1 \), and the others have signature \( 3; 1; 0 \). By inspection, the number of distinct mappings of \( t = 0, 1, d, \infty \) to \( z = 0, 1, \infty \) consistent with the signature \( 2; 1; 1 \) is 36, and the number consistent with \( 3; 1; 0 \) is 18.

Kuiken [18] supplies a useful list of the 36 rational substitutions arising from the harmonic subcase 1a, but states incorrectly that they are the only rational substitutions that may be applied to a trivial Heun equation. Actually, subcases 1a–1c and 2a–2d give rise to 36, 36, 18; 18, 36, 36, 18 rational substitutions, respectively. By dividing by \( m \), it follows that for each subcase, the number of distinct rational substitutions per value of \( d \) is \( 12, 6, 6; 9, 6, 6, 9 \). Of these, exactly one-third map \( t = 0 \) to \( z = 0 \), rather than to \( z = 1 \) or \( z = \infty \), and consequently yield reductions of \( Hl \) to \( _2F_1 \), or of \( \tilde{Hl} \) to \( \tilde{\_2F_1} \). So for each subcase, the number of such reductions per value of \( d \) is \( 4, 2, 2; 3, 2, 2, 3 \).
For example, the four reductions with $d = -1$ that arise from the harmonic subcase 1a are

$$\tilde{Hl}(-1, 0; 0, \beta, \gamma, (1 + \beta - \gamma)/2; t) = \hat{2F1}(0, \beta/2; (1 + \gamma)/2; t^2)$$ (6.4a)

$$\tilde{Hl}(-1, 0; 0, \beta, \gamma, (1 + \beta + \gamma)/2; t) = (-1)^{(\gamma-1)/2}\hat{2F1}(0, (1 - \beta + \gamma)/2; (1 + \gamma)/2; t^2/(t^2 - 1))$$ (6.4b)

$$\tilde{Hl}(-1, 0; 0, \beta, 1 - \beta, \delta; t) = 4^{-\beta}\hat{2F1}(0, (1 - 2\beta + \delta)/2; 1 - \beta; 4t/(t + 1)^2)$$ (6.4c)

$$\tilde{Hl}(-1, 0; 0, \beta, 1 - \beta, \delta, t) = (-4)^{-\beta}\hat{2F1}(0, (1 - \delta)/2; 1 - \beta; -4t/(t - 1)^2)$$ (6.4d)

The reduction (6.4a), which is the only one of the four in which the degree-2 rational function $R$ is a polynomial, is simply the trivial (i.e., $\alpha = 0$) case of the quadratic reduction (3.8). The reduction (6.4b) was derived above as (6.3), but (6.4c) and (6.4d) are new. They are related by composition with $z \mapsto z/(z - 1)$, i.e., by the involution in $\mathcal{M}(h)$ that interchanges $z = 1$ and $z = \infty$.

Remarkably, many rational reductions of trivial Heun equations to the hypergeometric equation are not derived from the polynomial reductions of Theorem 3.1. The following curious degree-4 reduction is an example. The rational function

$$z = Q(t) \overset{\text{def}}{=} 1 - \left(\frac{t - 1 - i}{t - 1 + i}\right)^4 = \frac{8i t (t - 1)(t - 2)}{(t - 1 + i)^4}$$ (6.5)

takes $t = 0, 1, d \equiv 2, \infty$ to $z = 0$; and $t = 1 \pm i$ to $z = 1, \infty$ (both with quadruple multiplicity). By Proposition 3.3, a trivial Heun equation with $d = 2$ will be reduced by $Q$ to a hypergeometric equation iff $Q$ maps exponents to exponents. This constrains the singular points $t = 0, 1, d, \infty$ to have the same exponents; which by Fuchs’s relation (2.3) is possible only if each has exponents $0, 1/2$; which must also be the exponents of $z = 0$. Also, since $t = 1 \pm i$ are ordinary points of the Heun equation, with exponents $0, 1$, the exponents of the hypergeometric equation at $z = 1, \infty$ must be $0, 1/4$. It follows that on the level of solutions, the reduction is

$$\tilde{Hl}(2, 0; 0, \frac{1}{2}, 0, \frac{1}{2}; t) = (i/4)^{1/2}\hat{2F1}(0, \frac{1}{2}, \frac{1}{2}; \frac{8i t (t - 1)(t - 2)}{(t - 1 + i)^4})$$

(6.6)

where the normalization factor $(i/4)^{1/2}$ follows from the known behavior of the functions $Hl(t)$ and $2F1(z)$ as $t \to 0$ and $z \to 0$.

Each of the preceding rational reductions of a trivial $\tilde{Hl}$ to a $\hat{2F1}$ can be converted to a rational reduction of a nontrivial $Hl$ to a $\hat{2F1}$, by using the
definitions (2.3), (2.11) of $\tilde{Hl}_1, \tilde{2F_1}$. For example, (6.6) implies

$$Hl(2, \frac{3}{4}, \frac{1}{2}, 1, \frac{3}{2}, \frac{1}{2}; t) = (1 - t)^{1/2}(1 - t/2)^{1/2}[1 - t/(1 - i)]^{-2} 2F_1\left(\frac{1}{2}, \frac{3}{4}; \frac{3}{2}; \frac{8it(t - 1)(t - 2)}{(t - 1 + i)^4}\right).$$

The equality (6.7) holds in a neighborhood of $t = 0$ (both sides are real when $t$ is real and sufficiently small). This reduction is not related to the previously derived harmonic reduction (3.3a), in which $d = 2$ also. The pair $(d, q/\alpha\beta)$ here equals $(2, \frac{3}{2})$, which is not listed in Theorem 3.7.

The formula (6.7) is a reduction of a nontrivial $Hl$ to $2F_1$, but of a more general type than has been considered in this paper. The underlying reduction of the Heun equation (5) to the hypergeometric equation (6) includes a linear change of the dependent variable, resembling a complicated F-homotopy, in addition to a rational change of the independent variable.

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