Dimensional reduction of quantum fields on a brane

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Abstract

If we restrict a quantum field defined on a regular \( D \) dimensional curved manifold to a \( d \) dimensional submanifold then the resulting field will still have the singularity of the original \( D \) dimensional model. We show that a singular background metric can force the restricted field to behave as a \( d \) dimensional quantum field.

1 Introduction

Quantum fields are defined by their correlation functions. The Lagrangian serves as a heuristic tool for a construction of quantum fields. A reduction of the number of coordinates in the \( D \) dimensional Lagrangian does not mean that if we had a complete \( D \)-dimensional quantum field theory then we could reduce it in any way to a model resembling a quantum field theory in \( d < D \) dimensions. We can see this problem already at the level of a massless free field \( \phi \). The vacuum correlation function of \( \phi(x(1)) \) and \( \phi(x(2)) \) is \( |x(1) - x(2)|^{-D+2} \). If we restrict the field to the hypersurface \( x_D = 0 \) setting in all correlation functions \( x_D(j) = 0 \) then we obtain a quantum field with a continuous mass spectrum in \( d = D - 1 \) dimensions but this will not be the canonical free field in \( d \) dimensions whose two-point function behaves as \( |x(1) - x(2)|^{-D+3} \) at short distances. Nevertheless, it is an attractive idea that the Universe once had more dimensions and subsequently through a dynamical process shrank to a lower dimensional hypersurface. The dynamics could have the form of a gravitational collapse (say a ball collapsing to a disk). At the level of field correlation functions this would mean that we have initially scalar, electromagnetic and gravitational fields in \( D \)-dimensions with their standard canonical singularities which subsequently evolve into fields with \( d < D \) dimensional singularity. We show that such a reduction of dimensions is possible when the metric becomes singular. A similar mechanism is suggested in the brane scheme of refs.[1][2]. In ref.[1]
the authors derive the Green’s function in $D = 5$ dimensional space-time which on the $d = D - 1 = 4$ submanifold has the singularity of the fourdimensional Green’s function. Their model encounters some difficulties when generalized to arbitrary $D$ and $d$ [3]. Some other brane-type models of quantum fields are discussed in refs.[4] [5][6][7]. In this letter we discuss a general metric which has power-law singularity. In general relativity such metrics could describe collapse phenomena [8]. We can obtain metrics with power-law singularities as solutions to higher dimensional supergravity theories [9]. These solutions describe $m$-branes or intersecting m-branes in an $m + n$ dimensional space-time [10]).

2 A quantum field on a D-1 dimensional hypersurface

We consider a submanifold $M_{D-1}$ of a Riemannian manifold $M_D$ whose metric becomes singular near $M_{D-1}$. The metric on $M_D$ close to $M_{D-1}$ (in local coordinates) is described by a "warp factor" $a(x_D)$ which becomes singular either when $x_D \to 0$ or $|x_D| \to \infty$

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = dx_D^2 + a(x_D)^2(dx_1^2 + ... + dx_{D-1}^2)$$

The Green’s function of the minimally coupled scalar field is a solution of the equation

$$\mathcal{A}G = g^{-\frac{1}{2}}\delta$$

where $\mathcal{A}$ is the Laplace-Beltrami operator

$$\mathcal{A} = g^{-\frac{1}{2}}\partial_\mu(g^{\mu\nu}g^{\frac{1}{2}}\partial_\nu)$$

In the metric (1) eq.(2) reads

$$(\partial_D a^{D-1}\partial_D + a^{D-2}\triangle)G = \delta(x_D - x_D')\delta(x - x')$$

where $d = D - 1$, $x = (x_1, ..., x_d)$ and $\triangle$ is the $d$-dimensional Laplacian. This equation is simplified if we introduce the coordinate

$$\eta = \int a^{-d}dx_D$$

Then

$$\left(\partial^2_\eta + a^{2d-2}\triangle\right)G = \delta(\eta - \eta')\delta(x - x')$$

In the paper of Dvali et al [1] $D = 4$ and $a^4(x_D(\eta)) \to \delta(\eta)$.

We discuss in detail the case

$$a(x_D) = |x_D|^\alpha$$

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Then
\[(\partial_D |x_D|^{\alpha d} \partial_D + |x_D|^{\alpha(d-2)} \Delta) G = \delta(x_D - x'_D) \delta(x - x')\] (8)

We define
\[\eta = |1 - \alpha|^{-1} x_D |x_D|^{-\alpha d}\] (9)

then eq.(8) takes the form
\[\left(\frac{\partial^2}{\partial \eta^2} + \kappa |\eta|^{2\nu} \Delta \right) G = \delta(\eta - \eta') \delta(x - x')\] (10)

or in terms of the Fourier transform \(\tilde{G}\) in \(x\)
\[\left(\frac{\partial^2}{\partial \eta^2} - p^2 V(\eta)\right) \tilde{G} = \delta(\eta - \eta')\] (11)

where
\[V(\eta) = \kappa |\eta|^{2\nu}\] (12)

with
\[\nu = \alpha(d - 1)(1 - \alpha d)^{-1}\] (13)

and
\[\kappa = |1 - \alpha|^{-2\nu}\]

Eq.(11) can be solved by means of the Feynman-Kac integral applying the proper time method
\[G(\eta;x;\eta';x') = \frac{1}{2(2\pi)^d} \int_0^\infty d\tau \int d^d p \exp \left( i p (x' - x) \right) \frac{1}{2} \sum_{s=0}^{\infty} \Gamma(\eta - \eta - b(\tau)) \exp \left( -\frac{1}{2} p^2 \int_0^\tau V(\eta + b(s)) ds \right)\] (14)

Here, \(b(s)\) is the Brownian motion \([11]\) defined as the Gaussian process with the covariance
\[E[b(s) b(t)] = \min(s, t)\] (15)

\(E[..]\) denotes an average over the paths of the Brownian motion.

The dimensional reduction is imposed by setting \(\eta = \eta' = 0\). Next, we use the equivalence \(b(s) = \sqrt{\tau} b(\tau)\) which follows from eq.(15). Then, using the scaling invariance of the potential \(V\) (i.e., \(V(\lambda \eta) = \lambda^{2\nu} V(\eta)\)) we have
\[G(0, x; 0, x') = \frac{1}{2(2\pi)^d} \int_0^\infty d\tau \int d^d p \exp \left( i p (x' - x) \right) \frac{1}{2} \sum_{s=0}^{\infty} \Gamma(\sqrt{\tau} b(1)) \exp \left( -\frac{1}{2} \tau^{1+\nu} p^2 \int_0^1 V(\sqrt{b(s)}) ds \right)\] (16)

Changing the variables
\[p = \tau^{-\frac{d-2}{2\nu}} k\]
and
\[\tau = r |x - x'|^{\frac{2}{1+\nu}}\]
we obtain
\[G(0, x; 0, x') = C |x' - x|^{-d + \frac{2}{1+\nu}}\] (17)
with a certain constant \( C \). If \( 0 > \nu > -1 \) then the singularity of the Green’s function is weaker than the one for the \( D \)-dimensional free field. The Green’s function is equal to the Green’s function of the \( d = D - 1 \) dimensional free field if \( \nu = -\frac{1}{2} \) what corresponds to \( \alpha = \frac{1}{2 - d} \). The potential with \( 2\nu = -1 \) has the same scaling dimension as \( V = \delta(\eta) \) applied by Dvali et al [1]. The Hamiltonian with the potential \( V(\eta) = |\eta|^{-1} \) and the path integral (16) require a careful definition if \( 2\nu \leq -1 \) but at least till \( 2\nu \geq -2 \) such a definition (through a regularization and a subsequent limiting procedure) is possible [12]. Eq.(11) with the \( \delta \)-potential (“\( \delta \)-brane”) also involves a particular regularization and its subsequent removal [13]. Let us consider a solution of this problem by means of the proper time method. The heat kernel \( K^{\delta} \) is known exactly for the \( \delta \)-potential [14]. Hence,

\[
G(\eta, x; \eta', x') = \frac{1}{(2\pi)^{D-2}} \int_0^\infty d\tau \int dp \exp \left( i p (x' - x) \right) K^{\delta}(\eta, \eta', \tau) = \frac{1}{2} (2\pi)^{-4} \int_0^\infty d\tau \int dp \exp \left( i p (x' - x) \right) \left( K_0(\eta - \eta', \tau) - 2p^2 \int_0^\infty du \exp(-2p^2 u) K_0(|\eta| + |\eta'| + u, \tau) \right)
\]

where

\[
K_0(\eta, \tau) = (2\pi\tau)^{-\frac{D}{2}} \exp(-\frac{\tau}{'\eta}^2)
\]

is the heat kernel for the Brownian motion.

When \( \eta = \eta' = 0 \) the \( \tau \)-integral of the first term on the r.h.s. of eq.(18) (the one independent of \( p \) ) is infinite (and proportional to \( \delta(x - x') \)) whereas the second integral gives the formula(17) with \( \nu = -\frac{1}{2} \).

Eq.(18) could have been derived as a limiting case of eqs. (6) and(16) when \( a(x_D)2^{d-2} \to \delta(\eta) \). On the Lagrangian level we have

\[
\int dx_D dx \sqrt{g} D^D \partial_D \phi \partial_D \phi = \int d\eta dx \partial_\eta \phi \partial_\eta \phi
\]

and

\[
\int dx_D dx \sqrt{g} g^{jk} \partial_j \phi \partial_k \phi = \int d\eta dx a^{2d-2} \partial_j \phi \partial_j \phi \to \int d\eta dx \delta(\eta) \partial_j \phi \partial_j \phi
\]

Hence, we recover the Lagrangian of Dvali et al [1].

3 A generalization to surfaces of arbitrary dimensions

Let us consider on a \( D = m + n \) dimensional manifold a metric (in local coordinates) which close to the \( n \)-dimensional surface takes the form

\[
ds^2 = |y|^{2\beta} dy^2 + |y|^{2\alpha} dx^2
\]
where \( y \in \mathbb{R}^m \) and \( x \in \mathbb{R}^n \). Eq.(2) for the Green’s function of the Laplace-Beltrami operator reads

\[
\left( \frac{\partial}{\partial y^i} \mid y \mid^{\beta(m-2)+\alpha n} \frac{\partial}{\partial y^i} + \mid y \mid^{\beta m+\alpha(n-2)} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^j} \right) G_E = \delta \tag{23}
\]

We discuss here only a simplified form of eq.(23) which appears when

\[
\beta(m - 2) + \alpha n = 0 \tag{24}
\]

In such a case eq.(23) reads

\[
\left( \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^i} + \mid y \mid^{2\beta-2\alpha} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^j} \right) G_E = \delta \tag{25}
\]

or taking the Fourier transform in \( x \)

\[
\left( \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^i} - \vec{p}^2 V(y) \right) \tilde{G}_E = \delta(y) \tag{26}
\]

We obtain again an equation for the Green’s function of the Schrödinger operator with the potential

\[
V(y) = \mid y \mid^{2\beta-2\alpha} \tag{27}
\]

and the coupling constant \( \vec{p}^2 \). We solve eq.(26) by means of the proper time method

\[
G(y, x; y', x') = \frac{1}{2} (2\pi)^{-n} \int_0^\infty d\tau \int d\vec{p} \exp \left( i \vec{p} \cdot (\vec{x}' - \vec{x}) \right) E[\delta (y' - y - \vec{b}(\tau)) \exp \left( -\frac{1}{2} \vec{p}^2 \int_0^\tau V(\vec{y} + \vec{b}(s)) ds \right)] \tag{28}
\]

where \( \vec{b} \) is the \( m \)-dimensional Brownian motion.

On the brane \( y = y' = 0 \). In such a case using \( \vec{b}(s) = \sqrt{\tau} \vec{b}(\frac{s}{\tau}) \) we have

\[
\int_0^\tau ds V(\vec{b}(s)) = \tau^{1+\beta-\alpha} \int_0^1 V(\vec{b}(s)) ds \tag{29}
\]

Hence, if we change variables

\[
\vec{p} = k \tau^{-\frac{1}{2}(1+\beta-\alpha)}
\]

then

\[
G(0, x; 0, x')G(0, x; 0, x') = \frac{1}{2} (2\pi)^{-n} \int_0^\infty d\tau \sqrt{\tau}^{\beta(1-\alpha-\beta)-m} \int d\vec{k} \exp \left( i \vec{k} \sqrt{\tau}^{1-\beta} (\vec{x}' - \vec{x}) \right) E[\delta (\vec{b}(1)) \exp \left( -\frac{1}{2} \vec{k}^2 \int_0^1 V(\vec{b}(s)) ds \right)] = C \mid x - x' \mid^{-n+\rho} \tag{30}
\]

with a certain constant \( C \) and

\[
\rho = \frac{(2 - m)(1 - \alpha + \beta)^{-1}}{}
\]
For canonical quantum fields in $n$ dimensions we should have $\rho = 2$. This happens if (in addition to eq.(24))

$$\alpha - \beta = \frac{m}{2}$$  \hspace{1cm} (31)

In such a case the potential is

$$V(y) = |y|^{-m}$$  \hspace{1cm} (32)

The potential (32) scales in the same way as the $\delta$-function in $m$-dimensions. This is a singular potential. However, its regularization $V_\epsilon(y) = |y|^{-m-\epsilon}$ for any $\epsilon > 0$ gives a self-adjoint Hamiltonian with the well-defined path integral. As $\epsilon$ can be arbitrarily small the Newton potential on the brane would be indistinguishable from $r^{-1}$ if the brane is $n - 1 = 3$ dimensional. We could again consider the limit $V(y) \rightarrow \delta(y)$ in order to derive the model of Dvali et al [3]. In contradistinction to the case $m = 1$ the models in $m > 1$ dimensions are more complicated. For $m = 2$ and $m = 3$ the relation of the coupling constant $p^2$ in eq.(26) to the parameters appearing in the heat kernel $K^0$ is not so explicit [15]. For $m > 3$ the $\delta$-potential cannot be defined at all [16] [13].

We have discussed only scale invariant metrics. If the metric is not scale invariant but its asymptotic behaviour for $y \rightarrow 0$ is of the form (22) then our results hold true when $-1 \leq \nu \leq 0$ and when applied to the short distance behaviour $|x - x'| \rightarrow 0$ of $G(0,x;0,x')$. If the asymptotic behaviour of the metric for $|y| \rightarrow \infty$ is of the form (22) then our results apply if $\beta \geq \alpha$ to the behaviour of the Green’s functions $G(0,x;0,x')$ for large $|x - x'|$. In such a case $\rho < 2$ in eq.(30), hence $G(0,x;0,x')$ and the gravitational potential decay to zero faster than in the Newton theory (in the $n$ dimensions). Depending on the asymptotic behaviour of the metric tensor $g(x,y)$ we obtain models which lead to a modification of the Newton law either at small or at large distances (some brane models modifying the classical gravity at small or large distances are discussed in [4][7]).

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