chiral spin-wave edge modes in dipolar magnetic thin films

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Based on a linearized Landau-Lifshitz equation, we show that two-dimensional periodic array of ferromagnetic particles coupled with magnetic dipole-dipole interactions supports chiral spin-wave edge modes, when subjected under the magnetic field applied perpendicular to the plane. The mode propagates along a one-dimensional boundary of the system in a unidirectional way and it always has a chiral dispersion within a band gap for spin-wave volume modes. Contrary to the well-known Damon-Eshbach surface mode, the sense of the rotation depends not only on the direction of the field but also on the strength of the field; its chiral direction is generally determined by the sum of the so-called Chern integers defined for spin-wave volume modes below the band gap. Using simple tight-binding descriptions, we explain how the magnetic dipolar interaction endows spin-wave volume modes with non-zero Chern integers and how their values will be changed by the field.

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I. INTRODUCTION

Spin waves are collective propagations of precessional motions of magnetic moments in magnetic materials. Magnonics research investigates how the spin wave propagates in the sub-micrometer length scale and sub-nanosecond time scale. Especially, the propagation of spin waves in periodically nanostructured magnetic materials dubbed as magnonic crystals are of one of its central interests. Owing to the periodic structurings, the spin wave spectrum in magnonic crystal acquires allowed frequency bands of spin wave modes and forbidden-frequency bands dubbed as magnonic band gap. Like in other solid-state engineering such as electronics, photonics and plasmonics, the main application direction is to explore ability of spin waves to carry and process information. Compared to others, magnonics has a much better prospect for miniaturization of the device, because the velocity of a spin wave is typically several orders slower than those of light and electrons in solids.

Recently, the authors proposed a spin-wave analog of integer quantum Hall (IQH) state which has unidirectional edge modes for spin-wave propagation. IQH state is a two-dimensional electron system with broken time-reversal symmetry, which supports unidirectional electric conducting channels along the boundaries (edges) of the system. The number of the unidirectional (chiral) edge modes is determined by a certain kind of topological number defined for bulk electronic states, called as the first Chern integer. Based on a linearized Landau-Lifshitz equation, we have generalized the Chern integer well-established in quantum Hall physics into the context of the spin wave physics, to argue that non-zero Chern integer for spin-wave volume-mode bands results in an emergence of chiral spin-wave edge mode.

The proposed edge mode has a chiral dispersion with a band gap for volume-mode bands, which supports a unidirectional propagation of spin degree of freedom for a frequency within the gap. The sense of rotation and the number of the chiral mode is determined by the topological number for volume-mode bands below the gap, which itself can be changed by closing the band gap. These features allow us control the chiral edge modes in terms of band-gap manipulation, which could realize novel spintronic devices such as spin current splitter and spin-wave logic gates. To have these devices in real experimental systems, however, it is quite important to have a number of actual magnonic crystals, in which spin-wave volume mode bands take various non-zero Chern integers.

From its electronic analogue, it is expected that non-zero Chern integers for spin-wave volume-mode bands result from strong spin-orbit coupled interactions, such as magnetic dipole-dipole interaction. Namely, having an inner product between spin operator and coordinate operator, the magnetic dipolar interaction locks the relative rotational angle between the spin space and orbital space, just in the same way as the relativistic spin-orbit interaction does in electronic systems. As a result of the spin-orbit locking, the complex-valued character in the spin space (i.e. one of the three Pauli matrices) is transferred into wavefunctions in the orbital space. Especially, in the presence of finite out-of-plane ferromagnetic moments in the spin space, the symmetry argument allows the Chern integer for volume-mode bands to have non-zero integer-value. In the recent work, employing a standard plane-wave theory, we have showed that a two-dimensional (x-y) bi-component magnonic crystal under an out-of-plane field (along the z-direction) acquires spin-wave volume-mode bands with non-zero Chern integers, when magnetic dipolar interaction dominates over short-
ranged isotropic exchange interaction. From the state-of-
art nanotechnology, however, it is not easy to synthesize
the proposed bi-component magnonic crystal experi-
mentally. Moreover, the proposed model is not simple enough
to see how magnetic dipolar interaction leads to non-zero
Chern integers for spin-wave volume-mode bands.

In the present paper, we introduce much simpler thin-
film magnetic models, which also support spin-wave vol-
ume modes with non-zero Chern integers and chiral spin-
wave edge modes, under the field normal to the two-
dimensional plane. Based on the models, we show that
the chiral edge modes have frequency-wavelength disper-
sions within a band gap for spin-wave volume modes,
and their chiral directions are determined by a sign of
the Chern integer for a spin-wave volume mode below
the gap. Using a simple tight-binding model composed
of ‘atomic orbitals’, we further argue that the level in-
version between the parity-odd atomic orbital (such as
p-wave orbital) and parity-even atomic orbital (such as
s-wave orbital) leads to a band inversion, which endows
spin-wave volume-mode bands with non-zero Chern in-
tegers. We expect that these findings would give useful
prototype models for future designing of more realistic
magnonic crystals which support topological chiral spin-
wave edge modes.

The organization of the paper is as follows. In the next
two sections, we introduced the models studied in this pa-
per (sec. II and Fig. 1) and formulate our problem and
summarize a calculation procedure of spin-wave band dis-
persions and the topological Chern integers (sec. III). In
section IV, we show how chiral spin-wave edge modes ap-
ppear and how they change their directions on increasing
the field. The results shows that the sense of the rotation
of chiral edge mode is indeed determined by the sign of
the Chern integer defined for the spin-wave volume mode.
In section V, we introduce a tight-binding description of
linearized Landau-Lifshitz equations in the context of the
present models. In sec.VA, we first clarify spin-wave ex-
citations within a unit cell in terms of a total angular mo-
mement variable. Based on the ‘atomic orbitals’ thus ob-
tained, we construct a tight-binding model for a square-
lattice model (sec. VB) and for a honeycomb-lattice model
(sec. VC). Using this tight-binding model, we
explain how a level inversion between the parity-odd atomic orbital (such as
atomic orbital) leads to a band inversion, which endows
spin-wave volume-mode bands with non-zero Chern in-
tegers. We expect that these findings would give useful
prototype models for future designing of more realistic
magnonic crystals which support topological chiral spin-
wave edge modes.

II. MODEL

In this paper, we consider two-dimensional periodic ar-
rays of ferromagnetic islands. We assume that each fer-
romagnetic island behaves as a single spin and ferromag-
netic islands are coupled via magnetic dipolar interac-
tion. In fact, two-dimensional periodic lattice structures
composed of submicrometer-scale ferromagnetic islands
have been fabricated experimentally, in which they con-
firm that each island behaves as a giant single spin under
some circumstances.26,27

To have volume-mode bands with finite Chern integers,
we generally need multiple-band degree of freedom within
a unit cell of magnonic crystal. To this end, we consider
two models: decorated square-lattice model and honey-
comb lattice model (see Fig. 1). A basic building block
of both models is a cluster of ferromagnetic islands. For
the decorated square lattice model, four ferromagnetic
islands form a circle-shape cluster which encompasses a
site of the square lattice. For the decorated honeycomb
lattice model, 3 neighboring islands form a circle which
encompasses either an A-sublattice site or a B-sublattice
site of the honeycomb lattice.

Experimentally speaking, it is also quite likely that
a submicrometer-scale ferromagnetic island has a num-
ber of low-energy excitation modes having different spin
textures within the island. Such modes can be also re-
garded as multiple-band degree of freedom, so that a sys-
 tem with only one ferromagnetic island within a unit cell
of magnonic crystal could also have a chance to provide
volume mode bands with finite Chern integer and asso-
ciated chiral edge modes. We expect that the theoretical
results obtained in the present model study would also
provide useful starting points for further studies on such
systems.28

III. FORMULATION

For the models introduced above, we first determine a
classical spin configuration which minimizes the following
magnetostatic energy;

\[ E = -\frac{1}{2}(\Delta V)^2 \sum_{i\neq j} M_i(r_i) f_{ab}(r_i - r_j) M_j(r_j) \]

\[ + H\Delta V \sum_i M_z(r_i) \]

\[ (1) \]

where \( r_i \) specifies a spatial location of a ferromagnetic
island (classical spin). For simplicity, the norm of each
spin is fixed; \( |M(r_i)| = M_s \). The magnetic dipole-dipole
interaction is given by a 3 by 3 matrix,

\[ f_{ab}(r) = -\frac{1}{4\pi} \left( \frac{\delta_{ab}}{|r|^3} - \frac{3r_ar_b}{|r|^5} \right) \]

\[ (2) \]

with \( a, b = x, y, z \). The summation over \( i, j \) in eq. 1
are taken over all ferromagnetic islands, while the sum-
dipole interaction. We took the spin cluster. In the following, we take this value to be saturated, which assumes to behaves as a single big spin whose moment is fully polarized along the field. For a ferromagnetic island (volume element is $\Delta V$), the spins are coupled via magnetic dipole-dipole interaction. We took $|\mathbf{e}_x| = |\mathbf{e}_y| = 2.4$, $2r = 1.2$, $\Delta V = 1.70$ and $M_s = 1.0$ for the square-lattice case, while $|\mathbf{e}_1| = |\mathbf{e}_2| = |\mathbf{e}_3| = 2.4$, $2r = 1.2$, $\Delta V = 1.0$ and $M_s = 1.0$ for the honeycomb-lattice case. The primitive translational vector $\mathbf{a}_\mu (\mu = x, y)$ are defined as $\mathbf{a}_x = \mathbf{e}_x$ and $\mathbf{a}_y = \mathbf{e}_y$ for the square-lattice model and as $\mathbf{a}_x = \mathbf{e}_1 - \mathbf{e}_2$ and $\mathbf{a}_y = \mathbf{e}_1 - \mathbf{e}_2$ for the honeycomb-lattice case (see text).

$\Delta V$ is a volume element for each ferromagnetic island. From dimensional analysis, one can see that a saturation field and resonance frequency of spin-wave excitations are scaled by $M_s \Delta V/l^3$, where $l$ is a characteristic length scale for the periodic structuring within the two-dimensional plane, e.g. radius ($r$) of the circle-shape $M$-spin cluster. In the following, we take this value to be around $1$; $M_s = 1$, $\Delta V = 1.0$, $2r = 1.2$ for the square-lattice case and $M_s = 1$, $\Delta V = 1.7$, $2r = 1.2$ for the honeycomb-lattice case.

**A. classical spin configuration**

1. square-lattice model

For the decorated square lattice case, we found that every four spins within a circle-shape cluster form a same vortex,

$$M_0(r = r(c_{\theta_j}, s_{\theta_j})) = M_s(-s_{\varphi}s_{\theta_j}, s_{\varphi}c_{\theta_j}, c_{\varphi}),$$  

(4)

with $\theta_j \equiv \frac{2\pi j}{4}$ $(j = 1, \cdots , 4)$ and $(s_{\theta_j}, c_{\theta_j}) \equiv (\sin \theta, \cos \theta)$, such that the classical spin configuration $M_0(r)$ respects the translational symmetries of the square lattice, $M_0(r + \mathbf{a}_\mu) = M_0(r)$ (see Fig. 2). A finite out-of-plane component $(\varphi \neq \frac{\pi}{2})$ is induced by the field. Above the saturation field ($\mathbf{H} > H_c = 1.71$), all the spins become fully polarized along the field ($\varphi = \pi$).

2. honeycomb-lattice model

For the decorated honeycomb lattice case, the classical spin configuration below a saturation field ($\mathbf{H} < H_c = 0.57$) breaks the translational symmetries of the lattice, while that above the field is a fully polarized state respecting the translational symmetries of the honeycomb lattice. For simplicity, we only consider spin-wave excitations above the saturation field for the decorated honeycomb lattice case.
B. linearized Landau-Lifshitz equation

Starting from the classical spin configurations thus obtained, $M_0(r)$, the Landau-Lifshitz equation is linearized with respect to a small transverse fluctuation field $m(r)$, with $M(r) = M_0(r) + m_\perp(r)$ and $m_\perp \perp M_0$. In terms of a rotated frame with a $3 \times 3$ rotational matrix $R(r)$, with which $M_0(r)$ is always pointing along the $z$-direction, $R(r)M_0(r) = M_1e_z$ and $R(r)m_\perp(r) = m_\perp(r)$, the linearized equation of motion for the transverse moments takes the form:

$$-\partial_t m_\mu(r_i) = \epsilon_{\mu
u}\alpha(r_i)m_\nu(r_i) + M_s\Delta V\epsilon_{\mu
u}\sum_{j\neq i}f_{\nu\lambda}(r_i, r_j)m_\lambda(r_j)$$

(5)

where $m \equiv (m_x, m_y, 0)$ and the summation over the repeated indices $\mu, \nu, \lambda$ are taken only over $x, y$ with $\epsilon_{xy} = -\epsilon_{yx} = 1$. The first term in the right hand side includes a demagnetization field and the external field;

$$\alpha(r_i)M_0(r_i) = -\Delta V\sum_{j\neq i}f(r_i - r_j)M_0(r_j) + He_z,$$

where, provided that $M_0(r_j)$ gives a local minimum for the magnetostatic energy Eq. (4), the equality always holds true for a certain scalar function $\alpha(r_i)$. The dipole-dipole interaction in the second term of eq. (5) is given in the rotated frame;

$$f(r_i, r_j) \equiv R(r_i)f(r_i - r_j)R^t(r_j).$$

In terms of $m_{\pm} \equiv m_x \pm im_y$, which are magnon creation/annihilation fields respectively, the equation of motion reduces to a following form:

$$-i\partial_t \sigma_3 \begin{pmatrix} m_+(r_i) \\ m_-(r_i) \end{pmatrix} = \alpha(r_i)M_s \begin{pmatrix} m_+(r_i) \\ m_-(r_i) \end{pmatrix} + M_s\Delta V\sum_{j\neq i}\begin{pmatrix} f_{++}(r_i, r_j) \\ f_{+-}(r_i, r_j) \end{pmatrix}\begin{pmatrix} m_+(r_j) \\ m_-(r_j) \end{pmatrix},$$

(6)

where a 2 by 2 diagonal Pauli matrix $\sigma_3$ takes +1 for the creation field (particle space), while take -1 for the annihilation field (hole space). A Green function $f_{\alpha\beta}(r, r')$ ($\alpha, \beta = \pm$) in the second term takes a form of a certain Hermite matrix in the particle-hole space;

$$\begin{pmatrix} f_{++}(r, r') \\ f_{+-}(r, r') \\ f_{-+}(r, r') \\ f_{- -}(r, r') \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \\ 1 & 1 \\ -i & i \end{pmatrix},$$

with $f_{\alpha\beta}(r, r') = f_{\beta\alpha}(r', r)$. Accordingly, the problem reduces to solving a following generalized eigenvalue problem;

$$\sum_j (H)_{r_i, r_j} \begin{pmatrix} m_+(r_j) \\ m_-(r_j) \end{pmatrix} = \sigma_3 \begin{pmatrix} m_+(r_i) \\ m_-(r_i) \end{pmatrix} \mathcal{E}$$

(7)

with an Hermite matrix $H$,

$$\begin{pmatrix} (H)_{r_i, r_j} = -M_s\alpha(r_i)\delta_{r_i, r_j} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ -M_s\Delta V(1 - \delta_{r_i, r_j}) \begin{pmatrix} f_{++}(r_i, r_j) \\ f_{+-}(r_i, r_j) \end{pmatrix} \begin{pmatrix} f_{++}(r_i, r_j) \\ f_{+-}(r_i, r_j) \end{pmatrix}, \end{pmatrix}$$

(8)

The sum of $j$ is taken over all spins in the systems. Using the Cholesky decomposition, the Hermite matrix can be diagonalized by a paraunitary transformation matrix $T$;

$$HT = \sigma_3 T\mathcal{E}$$

(9)

with a proper normalization condition $T^\dagger \sigma_3 T = \sigma_3$ and a diagonal matrix $\mathcal{E}$.

Now that the saddle point solution respects the translational symmetries, $M_0(r + a_\mu) = M_0(r)$, so does the Green function and the demagnetization coefficient, $f(r + a_\mu, r') = f(r, r' - a_\mu)$ and $\alpha(r + a_\mu) = \alpha(r)$ with the primitive translational vectors $a_\mu (\mu = x, y)$. Moreover, the classical spin configuration eq. (4) is invariant under the simultaneous $C_4$ rotations in the spin space and the lattice space (around z-axis), so that the demagnetization coefficient within a unit cell has no spatial dependence, $\alpha(r_j) = \alpha$. This also holds true for the honeycomb lattice case considered.

With the Born-von Karman boundary condition, the eigenvalue problem reduces to a diagonalization of following Bogoliubov-de Gennes type Hamiltonian for every crystal momentum $k = (k_x, k_y)$;

$$i\partial_t \sigma_3 \begin{pmatrix} u_+, k(r_j) \\ u_-, k(r_j) \end{pmatrix} = \sum_{j=1}^{M_s} (H_k)_{r_i, r_j} \begin{pmatrix} u_+, k(r_j) \\ u_-, k(r_j) \end{pmatrix},$$

(10)

with

$$\begin{pmatrix} (H_k)_{r_i, r_j} = -M_s\alpha\delta_{r_i, r_j} \\ -M_s\Delta V\begin{pmatrix} f_{k, ++}(r_i, r_j) \\ f_{k, +-}(r_i, r_j) \end{pmatrix} \begin{pmatrix} f_{k, ++}(r_i, r_j) \\ f_{k, +-}(r_i, r_j) \end{pmatrix}, \end{pmatrix}$$

and

$$f_{k, \alpha\beta}(r, r') = e^{-i(k(r - r) - b)} \sum_b (1 - \delta_{r, r' - b}) \times f_{\alpha\beta}(r, r') e^{-i(kb),}$$

and

$$m_{\pm}(r + a_\mu) \equiv e^{\pm ika_\mu a_{\pm, k}(r)}.$$

The summation with respect to $j$ (or $r_j$) in the right hand side is taken over a unit cell. For decorated square and honeycomb lattice, $M_U = 4$ and 6 respectively. The summation over the translation vectors $b$ are taken over sufficiently many unit cells in actual numerical calculations, $b = na_x + ma_y$ with $-50 \leq n, m \leq 50$. In terms
of the Cholesky decomposition, the $2M_U \times 2M_U$ BdG Hamiltonian is diagonalized

$$H_k|\psi_j\rangle = \sigma_3|\psi_j\rangle \mathcal{E}_{j,k},$$  \hspace{1cm} (11)

with the normalization condition, $\langle \psi_j | \sigma_3 | \psi_j \rangle = (-1)^\sigma_j$, where $\sigma_j = 0$ for particle bands, $j = 1, \ldots, M_U$, and $\sigma_j = 1$ for hole bands $j = M_U + 1, \ldots, 2M_U$. Provided that the spin-wave Hamiltonian is derived from an energy minimum of the magnetostatic energy Eq. (1), it is guaranteed that eigenvalues for particle bands ($j = 1, \ldots, M$) are positive definite $\mathcal{E}_{j,k} > 0$ for any $k$, while those for the hole bands ($j = M_U + 1, \ldots, 2M_U$) are all negative, $\mathcal{E}_{j,k} < 0$ for any $k$. In fact, this is true for all the cases studied in this paper.

The eigenvalues in the particle bands, $\mathcal{E}_{j,k}$ ($j = 1, \ldots, M_U$), determine wavelength-frequency dispersion relations for all the spin-wave volume-mode bands. An eigenvector, $|\psi_j\rangle$, is a 'Bloch wavefunction' for the corresponding spin-wave volume-mode band. In terms of the Bloch wavefunction, we have calculated the first Chern integer defined for each spin-wave band as

$$\text{Ch}_j \equiv i(-1)^\sigma_j \epsilon_{\mu \nu} \int_{BZ} d^2k \langle \partial_{\mu} \psi_j | \sigma_3 | \partial_{\nu} \psi_j \rangle$$

$$= i \sum_{m=1}^{M_U} \int_{BZ} d^2k \frac{(-1)^\sigma_j (-1)^\sigma_m}{(\mathcal{E}_{j,k} - \mathcal{E}_{m,k})^2} \times \left\{ \langle \psi_j | \frac{\partial H_k}{\partial x} | \psi_m \rangle \langle \psi_m | \frac{\partial H_k}{\partial y} | \psi_j \rangle - (x \leftrightarrow y) \right\}$$ \hspace{1cm} (12)

Contrary to the Chern integer defined for electron’s wavefunction, eq. (12) contains the diagonal Pauli matrix $\sigma_3$ between bra-state and ket-state, which takes $+1$ in the particle space while $-1$ in the hole space. This additional structure comes from the fact that magnon obeys the boson statistics which enforces the respective BdG Hamiltonian such as eqs. (8,10) to be diagonalized in terms of a paraunitary matrix instead of a unitary matrix. Due to this paraunitary character in the particle-hole space, we can also argue that the sum of the magnonic Chern integer over all particle bands always reduce to zero, $\sum_{j=1}^{M_U} \text{Ch}_j = 0$, which leads to the absence of gapless topological chiral spin-wave edge mode.

In the next section, we have calculated spin-wave excitations with the open boundary condition along one direction ($y$-direction) while the periodic boundary condition along the other ($x$-dir.); the frequency-wavelength dispersions for the spin-wave edge modes are obtained as a function of (surface) momentum along the $z$-direction, $k_z$. The dispersions thus obtained allow us to see the propagation direction of the chiral spin-wave edge mode. With changing the strength of the field, we have calculated spin-wave band dispersions for both volume modes and edge modes and the Chern integer for all the volume modes.

![Image](https://example.com/image.png)

**FIG. 3:** (Color online) (a-d) Side-view of wavelength-frequency dispersions of four spin-wave volume-mode bands in the square lattice model under fields ($H$) normal to the 2-d plane; (a) $H = 0.0$, (b) $H = 0.47H_c$, (c) $H = 0.76H_c$ (d) $H = 0.82H_c$, (e) $H = 1.01H_c$, (f) $H = 1.1H_c$, (g) $H = 1.4H_c$ (h) $H = 2.35H_c$ where $H_c = 1.71$. The Chern integer for red/blue-colored spin-wave bands is $-1/1$, while 0 otherwise.

### IV. RESULTS

#### A. square-lattice model

Results for the square-lattice model are summarized in Figs. [3](#) and [5](#). Without the field, the system has no magnetization perpendicular to the plane, so that the spin-wave Hamiltonian respects both time-reversal symmetry, $H_{-k} = H_k^*$, and mirror symmetries, e.g. $H_{(k_x, -k_z)} = H_{(k_x, k_z)}$. The Chern integer for all the four bands are required to be zero by these symmetries (Fig. [3](#)a)), and no chiral spin-wave edge modes are observed (Fig. [4](#)a)). With the field along the $z$-direction, these symmetries are gone.

On increasing the field, there appear a sequence of
band touchings between the lowest spin-wave band and the second lowest one at the Γ-point ($H = 0.24H_c$) and two inequivalent $X$-points ($H = 0.67H_c$). As a result of these band touchings, the Chern integers for the lowest band and the second lowest one become $+1$ and $-1$ respectively for $0.24 < H/H_c < 0.67$ (Fig. 3(b)), $-1$ and $+1$ respectively for $0.67 < H/H_c$ (Fig. 3(c,d)). Correspondingly, there appears a chiral spin-wave edge mode propagating in the clockwise direction for $0.24 < H/H_c < 0.67$, whose dispersion runs across a band gap between these two spin-wave volume-mode bands (Fig. 4(B-1)). When the band gap closes and reopens at $H/H_c = 0.67$, the chiral spin-wave edge mode changes its direction from clockwise to anticlockwise (Fig. 4(C,C-1)). The anticlockwise edge mode remains for a relatively larger range of the field strength, $0.67 < H/H_c < 1.4$ (Fig. 4(D), Fig. 5(A),(B),(A-1),(B-1)).

There is also another sequence of band touchings between the third lowest spin-wave band and the highest one. They appear at $H = 0.71H_c$ ($M$-point), $H = 0.79H_c$ (two inequivalent $X$-points) and $H = 0.85H_c$ (Γ-point). Correspondingly, the first Chern integers for the third lowest band and the highest band become $-1$ and $+1$ for $0.71 < H/H_c < 0.79$ (Fig. 3(c)), $+1$ and $-1$ for $0.79 < H/H_c < 0.85$ (Fig. 3(d)), while $0$ otherwise. They lead to a chiral spin-wave edge mode with anticlockwise propagation ($0.71 < H/H_c < 0.79$; Fig. 4(C,C-1)) and that with clockwise propagation ($0.79 < H/H_c < 0.85$; Fig. 4(D,D-1)) between these two volume-mode bands.

In the limit of strong field, the system becomes effectively time-reversal symmetric, $H^\ast_k = H_{-k}$ (consult also...
Chiral spin-wave edge mode which crosses over the gap (see also sec. IVC). In other words, the band gap and the Chern integers for these two bands reduce to zero. A perturbation analysis presented in sec. IVC), where the Chern integers for all the four spin-wave volume-mode bands reduce to zero and the system does not support any chiral edge spin-wave which crosses band gaps for spin-wave volume-mode bands. Yet there still exist spin wave edge modes, which have parabolic dispersions at their lowest (or highest) frequency levels and thus consist of both right-moving mode and left-moving modes on a same side of the boundary. (Fig. 6(A,B),(D-1)).

B. honeycomb-lattice model

Results for the decorated honeycomb lattice are shown in Fig. 6. Above the saturation field $H \geq H_c = 0.57$, the lowest spin-wave volume-mode band and the second lowest one are always separated by a finite band gap. The Chern integers for these two bands are quantized to $-1$ and $+1$ respectively for $H \geq H_c = 0.57$, and a chiral spin-wave edge mode with the anticlockwise propagation cross the band gap between these two (Fig. 6(A,A-1)). On increasing the field, the band gap becomes smaller but always remains finite (Fig. 6(B,B-1)). Only in the strong field limit, the gap closes and the lowest two bands form two massless Dirac-cone spectra at two inequivalent $K$-points, $k = K$ and $K'$ with $K \cdot e_1 = -K' \cdot e_3 = 2\pi$ and $K \cdot e_3 = K' \cdot e_2 = 0$, where the Chern integers for these two bands reduce to zero (see also sec. IVC). In other words, the band gap and the chiral spin-wave edge mode which crosses over the gap persist even in the presence of large (but finite) field for the decorated honeycomb lattice model.

V. TIGHT-BINDING DESCRIPTIONS

Although we are dealing with a classical spin problem, the calculation procedure so far is totally in parallel with what standard Holstein-Primakoff approximation does in localized quantum spin models based on large $S$ expansion (where $S$ denotes the size of a localized quantum spin). Finding a classical spin configuration of a given localized spin model (on the order of $S^2$; treating spin as a classical spin) corresponds to the minimization of the magnetostatic energy, eq. (1) (sec. IIIA). Reducing a spin model Hamiltonian into a quadratic form in terms of Holstein-Primakoff boson field (on the order of $S$) corresponds to linearizing the Landau-Lifshitz equation, eq. (2), around the classical spin configuration (see sec. IIIB). In fact, we diagonalized a quadratic form of the spin-wave Hamiltonian, eq. (3), to obtain frequency levels of spin-wave modes (sec. IV). A tight-binding (TB) description introduced in this section is one approximate way of doing this diagonalization, which in fact gives useful physical pictures for results obtained in the previous section.

To construct a TB description for Eqs. (6,7), let us first decompose the Hamiltonian defined by eq. (8) into a diagonal part and off-diagonal part with respect to the $M$-spin cluster index;

$$ (H)_{r_i,r_j} = (H_0)_{r_i,r_j} + (H_1)_{r_i,r_j}, $$

with

$$ (H_0)_{r_i,r_j} = -M_s \alpha \delta_{r_i,r_j} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) $$

$$ -M_s \Delta V \eta_{r_i,r_j} \left( \begin{array}{cc} f_{++}(r_i,r_j) & f_{+-}(r_i,r_j) \\ f_{-+}(r_i,r_j) & f_{--}(r_i,r_j) \end{array} \right), $$

$$ (H_1)_{r_i,r_j} = -M_s \Delta V \eta_{[r_i],[r_j]} \left( \begin{array}{cc} f_{++}(r_i,r_j) & f_{+-}(r_i,r_j) \\ f_{-+}(r_i,r_j) & f_{--}(r_i,r_j) \end{array} \right), $$

where $\eta_{r_i,r_j} = 1 - \delta_{r_i,r_j}$, $\eta_{[r_i],[r_j]} = 1 - \delta_{[r_i],[r_j]}$ and $[r_i]$ specifies a $M$-spin cluster in which the spin site $r_i$ are included; the 4-spin cluster for the decorated square lattice case ($M = 4$) and the 3-spin clusters for the honeycomb lattice case ($M = 3$); see Fig. 7.

Suppose that $H_0$ is diagonalized in terms of appropriate ‘atomic orbitals’ localized at each $M$-spins cluster;

$$ H_0 T_0 = \sigma_j T_0 \sigma_j H_0. $$

$\tilde{H}_0$ is a diagonal matrix, whose elements correspond to respective ‘atomic-orbital’ levels; The column of $T_0$ are specified by atomic-orbital index $(j,m)$, cluster index.
\( (n, n') \) and particle-hole index \((\mu, \nu)\). The orthonormal condition for the new basis is given as \( T_{\mu,\nu}^\dagger T_{0} = \sigma_3 \). In terms of the basis, the original eigenvalue problem, Eq. (11), reduces to

\[
(\hat{H}_0 + \hat{H}_1)S = \sigma_3 S \mathbf{E},
\]

with \( T = T_0 S \). The row of \( S \) and the row and the column of \( \hat{H}_1 \) are specified by atomic orbital \((j, m)\), cluster \((n, n')\) and particle-hole \((\mu, \nu)\) indices:

\[
(\hat{H}_1)_{(j, n, \mu|m, n', \nu)} = (T_0^\dagger H_1 T_0)_{(j, n, \mu|m, n', \nu)},
\]

\[
(\hat{H}_0)_{(j, n, \mu|m, n', \nu)} = \delta_{j, m} \delta_{n, n'} \delta_{\mu, \nu} E_{0,j},
\]

where ‘atomic-orbital’ levels \( E_{0,j} \) being positive definite, \( E_{0,j} > 0 \). By its construction, \( \hat{H}_1 \) has a finite matrix element only between atomic orbitals localized at different clusters, which thus stands for inter- or intra-orbital hopping terms between clusters. In terms of the creation/annihilation fields for the \( j \)-th atomic orbital localized at the \( n \)-th cluster, \( \gamma_{j,n}/\gamma_{j,n} \), Eq. (15) takes the form,

\[
E_{0,j} \gamma_{j,n} + \sum_m \sum_{n'} (t_{(j, n, m, n', \mu)}^{(+,+)} \gamma_{m, n'}) + (t_{(j, n, m, n', \nu)}^{(+,-)} \gamma_{m, n'}) = \gamma_{j,n} \mathbf{E},
\]

\[
E_{0,j} \gamma_{j,n} + \sum_m \sum_{n'} (t_{(j, n, m, n', \mu)}^{(-,+)} \gamma_{m, n'}) + (t_{(j, n, m, n', \nu)}^{(-,-)} \gamma_{m, n'}) = -\gamma_{j,n} \mathbf{E}.
\]

with \( \theta_j = \frac{2\pi j}{M} \) \((j = 1, \cdots, M)\) and \((s_\theta, c_\theta) \equiv (\sin \theta, \cos \theta)\).

The saturation field is given as \( H_s/M_s \equiv 6A_0(0) - 2A_1(0) \) with

\[
A_0(0) = \frac{\Delta V}{64\pi r^3} \sum_{j=1}^{M-1} \frac{1}{s_{\theta_j}}, \quad A_1(0) = \frac{\Delta V}{64\pi r^3} \sum_{j=1}^{M-1} \frac{1}{s_{\theta_j}}.
\]

\( \Delta V \) denotes a volume element of each ferromagnetic island (spin) and \( r \) is a radius of the circle. For \( H < H_c \), \( \varphi \equiv \cos^{-1}[-H/H_c] \) and \( \alpha = -4A_0(0) + 2A_1(0) \), while \( \varphi = \pi \) and \( \alpha = -H/M_s + 2A_0(0) \) for \( H > H_c \). Armed with these values, excitation modes are obtained by diagonalizing eq. (13) with \( r_j = r(c_{\theta_j}, s_{\theta_j}) \) and \( \theta_j = \frac{2\pi j}{M} \) \((j = 1, \cdots, M)\). With a proper choice of the \( U(1) \) gauge for \( m_{\pm} \):

\[
R(r_j) = \begin{pmatrix} 1 & c_{\varphi} & s_{\varphi} \\ -s_{\varphi}^* & c_{\theta_j} & -s_{\theta_j} \\ s_{\varphi}^* & s_{\theta_j} & c_{\theta_j} \end{pmatrix},
\]

eq. (13) can be readily diagonalized in terms of the total angular momentum \( n_j \):

\[
H_{n_j} \begin{pmatrix} m_+(n_j) \\ m_-(-n_j) \end{pmatrix} = \sigma_3 \begin{pmatrix} m_+(n_j) \\ m_-(-n_j) \end{pmatrix} \mathbf{E},
\]

with

\[
m_{\pm}(n_j) = \frac{1}{\sqrt{M}} \sum_{j=1}^{M} e^{\pm \frac{2\pi n_j}{M} \varphi} m_{\pm}(\theta_j),
\]

where \( n_j \) is defined modulo \( M \) \((n_j = 0, 1, \cdots, M - 1)\).

\( H_{n_j} \) takes the form of

\[
H_{n_j} \equiv -M_s \sigma_0 - M_s \begin{pmatrix} g_{++}(n_j) & g_{+-}(n_j) \\ g_{-+}(n_j) & g_{--}(n_j) \end{pmatrix},
\]

with \( g_{\pm\pm}(n_j) \equiv 1 + \cos \theta_j \).
field at which

\( n = \) diagonalized, where the demagnetization field for each site

\( n = \) around \( H/H_c \) magnetization field for each site includes not only those from other cluster. The saturation field is around the spins in the same cluster but also those from spins in the other clusters. The saturation field is around \( H_c = 1.26 \). The level inversion between \( n = 1 \) and \( n = 0 \) is around \( H/H_c = 0.80 \), while that between \( n = 3 \) and \( n = 2 \) is around \( H/H_c = 0.83 \). (a-1) ‘Atomic-orbital’ levels as a function of the field calculated from eq. (13) in the decorated square lattice model. Eq. (13) is diagonalized, where the demagnetization field for each site includes not only those from the spins in the same cluster but also those from spins in the other cluster. The saturation field is around \( H_c = 1.71 \), where the level of \( n = 0 \) goes below the others. (b) ‘Atomic-orbital’ levels as a function of the field (single 3-spin cluster case) with \( H_c = 0.32 \).

\[
\begin{align*}
g_{++}(n_j) & \equiv g_{---}(n_j) = -2iB_0(n_j)c_0 \sigma_3 \\
g_{--}(n_j) & \equiv g_{++}(n_j) = \{A_0(n_j)(-2 + 3c_0^z) - A_1(n_j)(1 + c_0^z)\} \sigma_0 \\
& - \{3A_0(n_j)c_0^z + A_1(n_j)(1 - c_0^z)\} \sigma_1,
\end{align*}
\]

and

\[
A_0(n_j) \equiv \frac{\Delta V}{64\pi r^3} \sum_{j=1}^{M-1} e^{iq_jj} \frac{1}{s^j_j},
\]

\[
A_1(n_j) \equiv \frac{\Delta V}{64\pi r^3} \sum_{j=1}^{M-1} e^{iq_jj} \frac{1}{s^j_j},
\]

\[
B_0(n_j) \equiv \frac{i\Delta V}{64\pi r^3} \sum_{j=1}^{M-1} e^{iq_jj} \frac{c_0^j}{s^j_j},
\]

with \( q_j \equiv \frac{2\pi n_j}{M} \). Eigen-frequency with angular momentum \( n_j \) takes the following form in the particle space;

\[
\epsilon_{n_j} = M_s \lambda_{n_j} - 2c_0 M_s B_0(n_j)
\]

with

\[
\lambda_{n_j} \equiv \sqrt{\left[ -\alpha + 2A_0(n_j) + 2A_1(n_j) \right]} \\
\times \sqrt{\left[ -\alpha + 2A_0(n_j) - 6A_0(n_j) c_0^2 + 2A_1(n_j) c_0^2 \right]}.
\]

Figs. 8(a,b) show how the spin-wave excitations for a single cluster with \( M = 3 \) and \( M = 4 \), eq. (23), behave as a function of the field respectively. In either cases, doubly degenerate modes at the zero field, \( n_j = 1 \) and \( n_j = M - 1 \) \( (p_\pi \) -wave orbitals respectively in the square-lattice case; see Fig. 11), are split on increasing the field, while that with \( n_j = 0 \) \( (s \) -wave orbital) decreases its resonance frequency, to reach zero at the saturation field \( H = H_s \). Above the field, the \( s \) -wave atomic-orbital level increases again, to form a quasi-degeneracy with the atomic-orbital level of \( n_j = M - 2 \) in the large field limit;

\[
\epsilon_{n_j = 0} = \epsilon_{n_j = M - 2} + O(1/H).
\]

The atomic-orbital levels shown in Figs. 8(a,b) are those for a single 4 (3) -spin cluster, where the demagnetization field stems only from those spins in the same cluster. Even when demagnetization fields from surrounding clusters are included, which is the case with eq. (13), the field-dependence of the atomic-orbital levels behaves qualitatively in the same way as in Figs. 8(a,b). Namely, the decorated square lattice model and honeycomb lattice model respects the same 4 (3) -fold rotational symmetry as in the single 4 (3)-spins cluster, so that only a value of the saturation field and specific expression for demagnetization field will be modified, e.g. compare Fig. 8(a) with Fig. 8(a-1).

From Figs. 8(a,a-1), notice that there appear a couple of level inversions between different atomic orbitals, such as the one between \( n_j = 0 \) and \( n_j = 1 \) around \( H = 0.80H_c \), and the one between \( n_j = 2 \) and \( n_j = 3 \) around \( H = 0.83H_c \). Now that one of these two atomic orbitals is always either \( p_x + ip_y \) or \( p_x - ip_y \) -like orbital.
FIG. 9: Side-view of spin-wave band dispersions for decorated square-lattice model calculated from the tight-binding Hamiltonian, eqs. (15,16,17) where only nearest-neighbor hopping integrals are included. As for atomic-orbital levels, eq. (17), and the respective wavefunction $T_0$ used in eq. (16), we used those for the single 4-cluster. (a) $H = 0$, (b) $H = 0.66H_c$, (c) $H = 0.71H_c$, (d) $H = 0.79H_c$, (e) $H = 0.85H_c$, (f) $H = 1.4H_c$, where $H_c$ denotes the saturation field for single 4-spin cluster, $H_c = 1.26$; see the caption of Fig. 8(a). At (a),(b) and (f), the lowest spin-wave band and the 2nd lowest one form band touchings at $\Gamma$-point, $X$-points, and $M$-point respectively. At (c),(d) and (e), the highest spin-wave band and the 3rd lowest one form band touchings at $M$-point, $X$-points, and $\Gamma$-point respectively.

while the others are parity even, it is expected from its electronic analogue that these level inversions endow the spin-wave volume-mode bands constructed from these atomic orbitals with non-zero Chern integers. In fact, a similar type of the band inversion plays a vital role in the emergence of nontrivial topological phases in quantum spin Hall insulators.

B. TB model for the square lattice case

To clarify how the level inversion between atomic orbitals leads to spin-wave bands with non-zero Chern integers, we next construct from Eqs. (15,16,17) a tight-binding (TB) model for the decorated square lattice. $H_0$ and corresponding $T_0$ in Eqs. (16,17) are replaced by those for the single 4-spin cluster. As for $H_1$, only the nearest neighbor hopping integrals are included. Such approximations may be justified, because the dipolar interaction decays as $1/R^3$ with $R$ being a distance between two spins; an amplitude of the next nearest neighbor hopping is roughly $2\sqrt{2}$, 8, and $5\sqrt{5}$ times larger than those of the 2nd, 3rd and 4th nearest neighbor hopping integrals respectively. In fact, band dispersions for spin-wave volume-mode bands obtained from this short-ranged TB model show qualitatively good agreements with those shown in the previous section (compare Figs. 9 with Figs. 3). For example, all the sequences of the band touchings described in sec. III are identified near similar field strengths, when scaled by the respective saturation fields; Fig. 9(a-f). Correspondingly, the Chern integers for all the four spin-wave bands take the same sequence of the integer values as found in the previous section; Fig. 10. The comparison also suggests that the nonparabolic characters of a certain band dispersion around the $\Gamma$-point in Fig. 3 stems from long-range hopping integrals in $H_1$, i.e. long-range character of the magnetic dipolar interaction, which is consistent with the similar feature of the forward volume modes.

The sequence of band touchings between the highest and the third lowest spin wave band results from the level inversion between the atomic orbital with $n_J = 1$ and that with $n_J = 0$; Fig. 8(a), while the other sequence between the lowest and 2nd lowest spin wave band is from those with $n_J = 2$ and $n_J = 3$; Fig. 8(a).

To see this, notice first that the atomic orbitals with...
$n_J = 0, 1, 2, 3$ takes $s$-wave, $p_- \equiv p_x - ip_y$, $d_{x^2-y^2}$, and $p_+ \equiv p_x + ip_y$-wave orbitals respectively; Fig. 8(a) with Figs. 9(c,d,e)) and thus can be approximated by

$$H_{01} = \sum_n (\epsilon_{0,0}^0, a_{11}^0, b_{01}^0)$$

Changing its sign from positive to negative; (compare Fig. 8(a)) with real valued $a_{00}$, $a_{11}$ and $b_{01}$. We have ignored (or integrated out within the 2nd order perturbation) those hopping terms which involve $n_J = 2$ and $n_J = 3$ and those between the particle space and the hole space. Such approximations are justified, when the atomic-orbital level of $n_J = 0$ and that of $n_J = 1$ are approximate to each other and when it comes to those spin-wave bands near these levels.

The highest and the third lowest spin-wave bands around $H/H_c = 0.78 \sim 0.82$ are mainly composed of the atomic orbitals with $n_J = 0$ and $n_J = 1$ (compare Fig. 8(a) with Figs. 9(c,d,e)) and thus can be approximately obtained from $H_{01}$. The Hamiltonian in the momentum space takes the form

$$H_{01,k} = \left( \begin{array}{ccc} \frac{\epsilon_0 + \epsilon_1}{2} & -b_{01}(s_{k_x} - is_{k_y}) & \xi_1 + a_{11}(c_{k_x} + c_{k_y}) \\ 2b_{01}(s_{k_x} + is_{k_y}) & \frac{\epsilon_0 + \epsilon_1}{2} & -b_{01}(s_{k_x} - is_{k_y}) \\ b_{01}(s_{k_x} + is_{k_y}) & b_{01}(s_{k_x} - is_{k_y}) & \frac{\epsilon_0 + \epsilon_1}{2} \end{array} \right)$$

with $(c_{k_x}, s_{k_x}) = (\cos k_x, \sin k_x)$, which gives momentum-frequency dispersions for the highest and the second highest spin-wave bands as

$$\epsilon_{0,k} = \frac{\epsilon_0 + \epsilon_1}{2} + (a_{00} + a_{11})(c_{k_x} + c_{k_y}) + \frac{\Delta k}{2},$$

$$\epsilon_{1,k} = \frac{\epsilon_0 + \epsilon_1}{2} + (a_{00} + a_{11})(c_{k_x} + c_{k_y}) - \frac{\Delta k}{2},$$

with

$$\Delta k = \sqrt{[\epsilon_0 - \epsilon_1 + 2(a_{00} - a_{11})(c_{k_x} + c_{k_y})]^2 + 16b_{01}^2(s_{k_x}^2 + s_{k_y}^2)}$$

The atomic-orbital level with $n_J = 0$ and that with $n_J = 1$ are inverted around $H = 0.80H_c$, where $\epsilon_0 - \epsilon_1 > 0$. Changes its sign from positive to negative; (compare Fig. 8(a)) with real valued $a_{00}$, $a_{11}$ and $b_{01}$. We have ignored (or integrated out within the 2nd order perturbation) those hopping terms which involve $n_J = 2$ and $n_J = 3$ and those between the particle space and the hole space. Such approximations are justified, when the atomic-orbital level of $n_J = 0$ and that of $n_J = 1$ are approximate to each other and when it comes to those spin-wave bands near these levels.

The highest and the third lowest spin-wave bands around $H/H_c = 0.78 \sim 0.82$ are mainly composed of the atomic orbitals with $n_J = 0$ and $n_J = 1$ (compare Fig. 8(a) with Figs. 9(c,d,e)) and thus can be approximately obtained from $H_{01}$. The Hamiltonian in the momentum space takes the form

$$H_{01,k} = \left( \begin{array}{ccc} \frac{\epsilon_0 + \epsilon_1}{2} & -b_{01}(s_{k_x} - is_{k_y}) & \xi_1 + a_{11}(c_{k_x} + c_{k_y}) \\ 2b_{01}(s_{k_x} + is_{k_y}) & \frac{\epsilon_0 + \epsilon_1}{2} & -b_{01}(s_{k_x} - is_{k_y}) \\ b_{01}(s_{k_x} + is_{k_y}) & b_{01}(s_{k_x} - is_{k_y}) & \frac{\epsilon_0 + \epsilon_1}{2} \end{array} \right)$$

with $(c_{k_x}, s_{k_x}) = (\cos k_x, \sin k_x)$, which gives momentum-frequency dispersions for the highest and the second highest spin-wave bands as

$$\epsilon_{0,k} = \frac{\epsilon_0 + \epsilon_1}{2} + (a_{00} + a_{11})(c_{k_x} + c_{k_y}) + \frac{\Delta k}{2},$$

$$\epsilon_{1,k} = \frac{\epsilon_0 + \epsilon_1}{2} + (a_{00} + a_{11})(c_{k_x} + c_{k_y}) - \frac{\Delta k}{2},$$

with

$$\Delta k = \sqrt{[\epsilon_0 - \epsilon_1 + 2(a_{00} - a_{11})(c_{k_x} + c_{k_y})]^2 + 16b_{01}^2(s_{k_x}^2 + s_{k_y}^2)}$$

The atomic-orbital level with $n_J = 0$ and that with $n_J = 1$ are inverted around $H = 0.80H_c$, where $\epsilon_0 - \epsilon_1 > 0$.
negative. This means that, on increasing $H$, the lowest and second lowest spin-wave bands around $H \approx 0.83 H_c$ first form a massless Dirac-cone spectrum at $k = (0, 0)$ when $\epsilon_3 - \epsilon_2 = -4(a_{33} - a_{22})$, then two massless Dirac spectra at $k = (\pi, 0)$ and $k = (0, \pi)$ when $\epsilon_3 - \epsilon_2 = 0$, and finally one massless Dirac spectrum at $k = (\pi, \pi)$ when $\epsilon_3 - \epsilon_2 = 4(a_{33} - a_{22})$. The band touching at the $\Gamma$-point is nothing but that in Fig. 10(a), those at the $X$-points are those in Fig. 10(b), and that at $M$-point corresponds to that in Fig. 10(c). Noting that $b_{32}$ has the same sign as $b_{01}$ (see Fig. 10), one can also see from the previous evaluation that the Chern integer for the second lowest (lowest) spin-wave band becomes $-1(+1)$ for $-4(a_{33} - a_{22}) > \epsilon_3 - \epsilon_2 > 0$ and $+1(-1)$ for $0 > \epsilon_3 - \epsilon_2 > 4(a_{33} - a_{22})$, which is consistent with Fig. 10.

C. TB model for the honeycomb lattice case

In the decorated honeycomb lattice model, we have observed in sec. III a finite band gap between the lowest spin-wave band and second lowest one, which are connected by a dispersion of a chiral edge mode. The gap and chiral edge mode persists for a sufficiently large field $H$. Based on a tight-binding model, we will employ a perturbation analysis from the large field limit and argue that the gap closes at two inequivalent $K$-points only in the limiting case ($|H| \to \infty$), where both the time-reversal symmetry and hexagonal spatial symmetry are effectively recovered. More accurately, we will show that an effective spin-wave Hamiltonian in the large field limit respects these two symmetries within the order of $\mathcal{O}(1)$, while it starts to break them from $\mathcal{O}(1/H)$. As a result, within the order of $\mathcal{O}(1)$, the lowest and second lowest spin-wave band compose massless Dirac spectra at the $K$-points. Once the $\mathcal{O}(1/H)$-corrections are included, the time-reversal symmetry is broken and the hexagonal symmetry ($C_{6v}$) reduces to its subgroup symmetry ($C_6$), which leads to a finite band gap at the $K$-points. These symmetry breakings also endow the two bands with a non-zero Chern integer with opposite signs, which results in the emergence of chiral edge mode within the band gap.

The perturbation analysis begins with a tight-binding Hamiltonian for the honeycomb lattice model, eq. (15):

$$H_0 + H_1 = H\sigma_0 + \lambda V_1 + \lambda V_2$$

(31)

where $\sigma_0$ is a 2 by 2 unit matrix in the particle-hole space and both $V_1$ and $V_2$ are on the order of $\mathcal{O}(1)$. For a bookkeeping, we put $\lambda$, which can be set to 1 from the outset [those terms with $\lambda$ are $\mathcal{O}(1)$, those with $\lambda^2$ are $\mathcal{O}(1/H)$ and those with $\lambda^3$ are $\mathcal{O}(1/H^2)$; see below]. $V_1$ consists of on-cluster ‘atomic-orbital’ levels and hopping terms in the excitonic channel, while $V_2$ consists only of those in the Cooper channel;

$$V_1 \equiv H_0 - H\sigma_0 + \begin{pmatrix} t^{(+,+)} & 0 \\ 0 & t^{(-,-)} \end{pmatrix},$$

$$V_2 \equiv \begin{pmatrix} 0 & t^{(+,-)} \\ t^{(-,+)} & 0 \end{pmatrix},$$

(32)

In the large field limit, all the spin-wave excitations reduce to the ferromagnetic resonance (FMR) with its resonance frequency being $H$. Once the $\mathcal{O}(1)$-corrections ($V_1, V_2$) are included, the FMR resonance is expected to split into a couple of spin-wave bands whose bandwidth are at most on the order of unity. To see this situation, let us erase those terms in the Cooper channel within a given order accuracy in $1/H$, and derive an effective Hamiltonian only for the particle space.

With a matrix satisfying $\sigma_3 \rho \sigma_3 = \rho^1$, the transformed Hamiltonian takes the form

$$H_{\text{eff}} = e^{-i\lambda \sigma_3 \rho \sigma_3} (H + \lambda V_1 + \lambda V_2) e^{i\lambda \rho}$$

$$= \left(1 - i\lambda \sigma_3 \rho \sigma_3 - \frac{\lambda^2}{2} \sigma_3 \rho^2 \sigma_3 + \cdots \right)$$

$$\times \left(\begin{array}{cc} H + \lambda V_1 + \lambda V_2 & (1 + i\lambda \rho - \frac{\lambda^2}{2} \rho^2 + \cdots) \\ (1 + i\lambda \rho - \frac{\lambda^2}{2} \rho^2 + \cdots) & H \end{array}\right)$$

$$= H + \lambda V_1 + \lambda V_2 - i\lambda \sigma_3 \rho \sigma_3 H + \lambda \rho H$$

$$- \frac{\lambda^2}{2} \rho^2 \sigma_3 H - \frac{\lambda^2}{2} H \rho^2 + \lambda \rho^2 \sigma_3 H \rho$$

$$- i\lambda^2 \rho \sigma_3 \rho \sigma_3 V_1 + i\lambda^2 V_1 \rho - i\lambda^2 \sigma_3 \rho \sigma_3 V_2$$

$$+ i\lambda^2 V_2 \rho + \mathcal{O}(\lambda^3)$$

(33)

We choose $\rho$ such that all the matrix elements in the Cooper channels will cancel each other within the order of $\mathcal{O}(1)$;

$$V_2 = i\sigma_3 \rho \sigma_3 H - iH \rho.$$ 

Or equivalently,

$$(\rho)_{n,\bar{m}} = i\frac{(V_2)_{n,\bar{m}}}{2H}, \quad (\rho)_{\bar{m},m} = i\frac{(V_2)_{\bar{m},m}}{2H},$$

(33)

where $\bar{m}$ is for the indices in the hole space and $n$ is for those in the particle space; $\sigma_3 |_{n,\bar{m}} = \delta_{n,m}$ and $(\sigma_3 |_{\bar{m},m} = -\delta_{n,m}$. With this rotated frame, all the matrix elements in the Cooper channel are at most on the order of $\mathcal{O}(1/H)$;

$$H_{\text{eff}} = H + \lambda V_1 - \frac{i\lambda^2}{2} (\sigma_3 \rho \sigma_3 V_2 - V_2 \rho)$$

$$- i\lambda^2 (\sigma_3 \rho \sigma_3 V_1 - V_1 \rho),$$

(34)

The last two terms have matrix elements in Cooper channels. When we further rotate in the particle-hole space such that they will be set off by generated terms, these two terms simply result in higher order terms, $\mathcal{O}(1/H^2)$, while the remaining terms being kept intact. We thus drop them by hand, to keep the first four terms as the
effective Hamiltonian. On the substitution of eq. (33) into eq. (34), we then have the effective Hamiltonian to the order of $1/H$ as

$$\langle H_{\text{eff}}^{(2)} \rangle_{n,m} = H \delta_{n,m} + (V_{1})_{n,m} + \frac{1}{2H} \sum_{\mathbf{p}} (V_{2})_{n,\mathbf{p}} (V_{2})_{\mathbf{p},m}. \quad (35)$$

The superscript ‘(2)’ in the left hand side denotes that the effective Hamiltonian is asymptotically exact within the 2nd order in $H$ (or within the first order in $1/H$). The sum with respect to $\mathbf{p}$ is taken only over the hole space. From eq. (33), one can readily see that, once the $O(1)$-corrections $(V_{1})$ are included, the FMR resonance localized at $H$ is split into a couple of spin-wave bands whose bandwidth are at most on the order of unity.

Within the order of $O(1)$, the effective Hamiltonian derived so far is invariant under the time-reversal operation and hexagonal symmetry operations. To see this, let us focus on the first two terms of eq. (35). With the atomic orbital index $(n_{J} = j, m)$ and cluster index $(n, n^\prime)$ being made explicit, they take the following form

$$\langle H_{\text{eff}}^{(1)} \rangle_{(j,n|m,n^\prime)} = \delta_{n,n^\prime} \delta_{m,n^\prime} + t_{(j,n|m,n^\prime)}^{(+,+)}.$$  \quad (36)

In the leading order in $1/H$, the inter/intra-orbital hopping integral between an orbital with $n_{J} = j$ at the $n$-th cluster and that with $n_{J} = m$ at the $n^\prime$-th cluster is given by

$$t_{(j,n|m,n^\prime)}^{(+,+)} = \sum_{\theta_{1}, \theta_{2}} e^{i(j+1)\theta_{1} - i(m+1)\theta_{2}} \frac{1}{6R^{3}}. \quad (36)$$

$\theta_{(l)}$ $(l = 1, 2, 3)$ in the right hand side specifies a spatial location of a ferromagnetic spin within a cluster. Within a cluster which encompasses an $A$-sublattice site or $B$-sublattice site at $(x_{n}, y_{n})$, we take $\theta_{1} \equiv \frac{2\pi}{3}$ or $\pi + \frac{2\pi}{3}$ respectively, such that the location of the ferromagnetic spin is always given by $(x_{n} - r \sin \theta_{1}, y_{n} + \cos \theta_{1})$. $R$ denotes a spatial distance between a ferromagnetic spin specified by $(\theta_{1}, n)$ and that by $(\theta_{1}, n^\prime)$;

$$R = \begin{vmatrix} x_{n} - r \sin \theta_{1} & x_{n^\prime} - r \sin \theta_{1} \\ y_{n} + \cos \theta_{1} & y_{n^\prime} + \cos \theta_{1} \end{vmatrix}. \quad (36)$$

Within the order of $O(1)$, a complex conjugation of hopping integrals can be readily set off by a sign change of the orbital angular momentum $n_{L} \equiv n_{J} + 1$. Namely, the complex conjugate of eq. (35) is transformed to itself by a proper exchange between $n_{L} = +1$ ($n_{J} = 0$) and $n_{L} = 2 \equiv -1 \ (mod \ 3)$ ($n_{J} = 1$). This in combination with eq. (24) indicates that the effective Hamiltonian up to the order of $O(1)$ is invariant under the following time-reversal operation;

$$\langle H_{\text{eff}}^{(1)} \rangle_{(j,n|m,n^\prime)}^{*} = Q f^{(1)} (\langle H_{\text{eff}}^{(1)} \rangle_{(j^\prime,n^\prime|m^\prime,n^\prime)}^{*}) Q^{*}, \quad (37)$$

with a proper basis change

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{\nu,\nu} \ (\nu = 1, 2, 3) \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (38)$$

which exchanges $n_{L} = +1$ ($n_{J} = 0$) and $n_{L} = -1$ ($n_{J} = 1$), while keeps $n_{L} = 0$ ($n_{J} = 2$) intact. It is also invariant under three mirror operations in the hexagonal symmetry, $\sigma_{v,1}, \sigma_{v,2}, \sigma_{v,3}$; $\pi$-rotation $C_{2}$ and $\frac{2\pi}{3}$-rotation $C_{3}$;

$$\langle H_{\text{eff}}^{(1)} \rangle_{(j^\prime,n^\prime|m^\prime,n^\prime)}^{*} = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}.$$  \quad (38)
The first three columns and rows are for the three atomic orbitals encompassing an A-sublattice site, while the latter three are for those encompassing a B-sublattice site. \( \varepsilon_j \) stands for a level for an atomic orbital with \( n_f = j \) (\( j = 0, 1, 2 \)), \( \alpha_0, \alpha_1, \beta, \eta, \gamma_0 \) and \( \gamma_1 \) are NN inter or intra-orbital (effective) transfer integrals, which can be evaluated from eq. (35) up to \( O(1/H) \). It is clear from Eq. (36) that, within the order of \( O(1) \), \( \gamma_0 = \gamma_1, \alpha_0 = \alpha_1, \) and \( \varepsilon_0 = \varepsilon_1 \), which makes the lowest two bands form massless Dirac spectra at the \( K \)-points; Fig. 12(b). A comparison between Fig. 12(a) and Fig. 12(b) suggests that the present NN TB Hamiltonian can qualitatively well reproduce the band structure of the lowest two bands in the large field limit, expect for a non-parabolic band structure of the lowest band near the \( \Gamma \)-point.

Once finite \( \Delta \gamma \equiv \gamma_0 - \gamma_1, \Delta \alpha \equiv \alpha_0 - \alpha_1 \) and \( \Delta \varepsilon \equiv \varepsilon_0 - \varepsilon_1 \) are included, the exchange between \( n_L = +1 \) and \( n_L = -1 \) changes the signs of these terms, so that the time-reversal symmetry is broken and the hexagonal symmetry reduces to the \( C_6 \) symmetry. These symmetry reductions give a finite mass to the Dirac spectra. The mass can be evaluated from 2 by 2 Dirac Hamiltonians for the lowest two spin-wave bands, which can be obtained via \( \mathbf{k} \cdot \mathbf{p} \) perturbation around these \( K \)-points:

\[
H_{k=k+p}^{2x2} = \frac{1}{2} \left( \Delta \varepsilon \sin^2 \frac{\theta}{2} - 3 \Delta \gamma \sin \theta \right) \sigma_3 \\
+ \frac{3}{2} \left( \eta \cos^2 \frac{\theta}{2} - \beta \sin^2 \frac{\theta}{2} \right) \left( p_x \sigma_1 - p_y \sigma_2 \right),
\]

and

\[
H_{k=k'+p}^{2x2} = -\frac{1}{2} \left( \Delta \varepsilon \sin^2 \frac{\theta}{2} - 3 \Delta \gamma \sin \theta \right) \sigma_3 \\
+ \frac{3}{2} \left( \eta \cos^2 \frac{\theta}{2} - \beta \sin^2 \frac{\theta}{2} \right) \left( p_x \sigma_1 - p_y \sigma_2 \right),
\]

with

\[
\tan \theta \equiv \frac{\gamma_1 + \gamma_2}{\varepsilon_0 - \varepsilon_1}.
\]

From these Hamiltonians, the Chern integers for the lowest and second lowest spin-wave bands are evaluated to be \( \sigma \) and \( -\sigma \) respectively with

\[
\sigma \equiv \text{sign} \left( \Delta \varepsilon \sin^2 \frac{\theta}{2} - 3 \Delta \gamma \sin \theta \right).
\]

A substitution of actual numbers into the parameters in the right hand side shows that \( \sigma = 1 \), which is consistent with previous numerical evaluations in Sec.III. The non-zero Chern integers for the lowest two spin-wave bands results in an edge mode with the anticlockwise propagation, which has a chiral dispersion between these two bands.

VI. MICROMAGNETIC SIMULATION

To uphold the existence of proposed chiral spin-wave edge mode by a standard method in the field, we perform a micromagnetic simulation by solving numerically the Landau-Lifshitz-Gilbert equation for the square-lattice model. We calculate magnetization dynamics by employing the 4th order Runge-Kutta method with a time step \( \Delta t = 1 \) ps. Fig. 13 schematically shows an entire system consisting of 4 ferromagnetic nanograins in the unit cell. Although we took the size of the ferromagnetic nanograin to be \( 5 \times 5 \times 5 \) nm\(^3\) as a demonstration, the system is scalable; the simulation does not include any short-range exchange interactions. The saturation magnetization and Gilbert damping coefficient of the ferromagnetic grain are 1.75 J/\( \mu \)m and 1.0 \times 10\(^{\text{-5}} \) respectively. We regard each nanograin as a uniform magnet, to assign single spin degree of freedom to each nanograin. Different ferromagnetic nanograins are coupled with one another through
FIG. 13: (Color online) Schematic view of a simulated system that comprises ferromagnetic nanograin. Although we took the size of the ferromagnetic nanograin to be $5 \times 5 \times 5$ nm$^3$ as a demonstration, the present simulation is scale free. In the beginning of the simulation, we apply a pulse field either at the center (marked by a black triangle) or around the boundary (marked by a black cross).

The magnetic dipolar interaction. The simulated system ($0 < X < L$ and $0 < Y < L$; Fig. 13) includes $25 \times 25$ unit cells. Without the field, the magnetization of each grain lies within the plane due to the dipolar interaction. Under a large out-of-plane DC field ($H_{dc} > 4700$ Oe), the magnetization becomes fully polarized along the $z$-direction. We took $H = 1.02H_c$ in the present simulation.

In order to excite spin wave modes in a broad frequency range, we apply a pulsed magnetic field within a plane with its pulse time $t_p = 1$ ps and its amplitude $H_p = 1.0 \times 10^{-4}$ Oe. The pulse is applied locally at the center and around an edge of the system for the purpose of exciting volume modes and edge modes, respectively. After calculating a time evolution of the magnetization in the system, we take a Fourier transformation of the transverse moment,

$$m_+(X,Y,t) \equiv m_x(X,Y,t) + im_y(X,Y,t),$$

with respect to time:

$$s_+(X,Y,\omega) \equiv \sum_{j=0}^{n-1} m_+(X,Y,j\Delta T) \exp(2\pi i\omega j\Delta T) \quad (38)$$

with $\Delta T = 50$ ps and $n = 1024$. The frequency power spectrum, $\sum_{X,Y} |s_+(X,Y,\omega)|$, obtained by the pulse at the center and that by the pulse at the edge are shown in Fig. 14(a) separately. Spatial distributions of spin wave excitations, $|s_+(X,Y,\omega)|$, for each case with different frequencies $\omega$ are shown in Figs. 14(b)-(g). From them, one can see that the spin-wave volume modes and edge modes are selectively excited, depending on whether the initial pulse field is applied at the center or at the edge respectively. In the case of the pulse field at the center, we observe two band gaps for volume modes; one from 24GHz to 30GHz and the other from 37GHz to 46GHz. In the case of the pulse at the edge, we observed spin-wave edge modes mainly from 24GHz to 42GHz.

The frequency dependence of the intensity of spin-wave excitations (see text). (b)-(g) Spatial distribution of the intensity, which is obtained by the application of the pulse field at center (b,d,f) and the edge (c,e,g); (b,c) $\omega = 10$GHz, (d,e) $\omega = 29$ GHz, and (f,g) $\omega = 31$ GHz.

A key feature of proposed chiral spin-wave edge mode is a unidirectional propagation of spin wave densities, which is clarified by its frequency-wavelength dispersion relation. To obtain such a dispersion relation, we next take a Fourier transformation of the transverse moment with respect to both space and time. In order to compare the result with Figs. 4,5, we integrate the amplitude of the Fourier component with respect to the $Y$-component.
of the momentum:

\[ A(k_x, \omega) = \sum_{k_y} |s_+(k_x, k_y, \omega)|, \]

\[ s_+(k_x, k_y, \omega) = \sum_{X,Y} s_+(X, Y, \omega) \exp(i k_x X) \exp(i k_y Y). \]

A contour plot of \( A(k_x, \omega) \) as a function of \( k_x \) and \( \omega \) gives a dispersion relation for spin-wave modes. Figure 15(a) shows the dispersion relation for the case of applying the pulse field at the center. It resembles those for spin-wave volume modes obtained in the preceding model calculation at the same parameter regime (Fig. 5(e)). Figures 15(b) and (c) show the dispersion relations for the case of applying the pulse field at the edge. To clarify propagation directions of those two spin-wave edge modes running along the opposite boundaries of the system, we take the Fourier transformation only over the upper (or lower) side of the sample \( L/2 < Y < L \) (or \( 0 < Y < L/2 \)); the one for the upper side is shown in Figure 15(b), while the one for the lower is in Figure 15(c). Both figures clearly indicate the existence of two counter-propagating chiral dispersions, each of which runs across any line of \( \omega = \omega_0 \) (25GHz \(<\omega_0<35\)GHz) once and only once. The results also suggest an existence of another spin-wave edge mode from 35GHz to 42GHz, which has a quasi-parabolic dispersion. Both of these edge modes in combination with volume modes shown in Fig. 15(a) are consistent with the dispersion relations for spin-wave modes obtained in the preceding model calculation at the same parameter regime (Fig. 5A),(A-1).

When the Gilbert damping coefficient becomes larger, unidirectional propagations of spin density along the chiral spin-wave edge mode decay faster. Fig. 16 shows spatial distributions of Fourier power spectra of magnetization dynamics, \[ |s_+(X, Y, \omega)|. \] in the presence of larger Gilbert damping term \( (\alpha = 0.001, 0.01) \), where the initial pulse field is applied at the edge (Fig. 13) and the frequency is chosen within the band gap \( (\omega = 29\text{GHz}) \). The results suggest that the coherence length is roughly 25 unit cell size (500nm) for \( \alpha = 0.001 \) and 8 unit cell size (160nm) for \( \alpha = 0.01 \).

VII. SUMMARY, DISCUSSIONS AND OPEN ISSUES

In this paper, we introduced two simple magnetic thin-film models, in which ferromagnetic nanoislands on periodic arrays are coupled with each other via magnetic dipolar interaction. Under the field applied perpendicular to the two-dimensional plane, spin-wave excitations in the systems have a chiral spin-wave edge mode localized at the boundaries of the systems, whose dispersion runs across a band gap for spin-wave volume modes. The sense of the rotation of the chiral edge mode is determined by a sign of the Chern integer for a spin-wave volume-mode band below the band gap.

To have volume-mode bands with finite Chern integer, we generally need multiple-band degree of freedom within a unit cell. To this end, we considered two periodic arrays of ferromagnetic particles; decorated square-lattice model and honeycomb-lattice model. For the decorated square-lattice model, we observed that, on increasing the out-of-plane field, there appears a sequence of band touchings between pairs of neighboring volume-mode bands. Owing to these band touchings, the Chern integers for the volume modes change their signs and, concomitantly, the chiral edge mode changes its sense of rotation from clockwise to anticlockwise or vice versa. For the decorated honeycomb-lattice model, we observed a finite band gap between the lowest spin-wave volume-mode band and second lowest spin-wave band which are connected by a chiral dispersion of an edge mode. Though its sense of rotation being unchanged by the
FIG. 16: (Color online) Spatial-resolved Fourier power spectra of magnetization dynamics in the presence of stronger dissipation. Spatial distribution of the intensity at \( \omega = 29\text{GHz} \), which is obtained by the application of the pulse field at the edge. (a) \( \alpha = 0.001 \) (b) \( \alpha = 0.01 \).

strength of the field in the honeycomb lattice case, the gap and the chiral edge mode persists for a sufficiently large field.

To interpret these results, we next construct tight-binding descriptions for the linearized Landau-Lifshitz equation, in which atomic orbitals such as \( s \)-wave, \( p_{\pm} \)-wave and \( d_{x^2-y^2} \)-wave orbitals are introduced within each unit cell. Among other, complex-valued characters in the \( p_{\pm} \)-wave orbitals break both the time-reversal symmetry and mirror symmetries of the models. These symmetry breakings lead to a non-zero Chern integer for spin-wave volume-mode bands and associated chiral spin-wave edge modes. Using this tight-binding model, we argue that the level inversions among different atomic orbital levels give rise to the so-called \textit{inverted} spin-wave bands with non-zero Chern integers. Our tight-binding analysis for the square-lattice model gives quantitative criteria for the emergence of finite field ranges within which spin-wave volume-mode bands have non-zero Chern integers.

For the decorated honeycomb lattice model, we employ a perturbation analysis, starting from the large field limit. The analysis suggests that the effective Hamiltonian in the large field limit always respects time-reversal symmetry and the hexagonal symmetry within the order of \( O(1) \). Due to the mirror operations in the hexagonal symmetry, the lowest two spin-wave bands form gapless Dirac cone spectra at two inequivalent \( K \)-points. Once \( O(1/H) \)-order corrections are included, however, the time-reversal symmetry is lost and hexagonal symmetry reduces to its abelian subgroup having no mirror symmetries. As a result, the gapless Dirac-cone spectra acquire a finite mass of the order of \( O(1/H) \), which leads to non-zero Chern integer for the two lowest spin-wave bands. This argument explains why the spin-wave volume-mode bands with non-zero Chern integers and associated chiral spin-wave edge mode persists in a very wide range of the field in the decorated honeycomb lattice model.

Since a state-of-the-art sample production does not necessarily guarantee perfect periodic structurings, considering disorder effects associated with the lattice periodicity are experimentally relevant, which can be speculated from well-established knowledges of integer Quantum Hall physics. The effects are two-folded. When the strength of the disorders is smaller than a characteristic frequency scale of the band gap, those volume modes near the band gap become localized due to the disorders, while chiral edge mode itself is free from these weak disorders. As a result, the frequency regime for the chiral spin-wave edge mode becomes even wider than that in the clean limit. When the strength of the disorders is increased to be larger than the scale of the band gap in the clean limit, however, the ‘mobility gap’ closes and reopens. After the reopening the gap, the topological chiral edge mode disappears. The proposed chiral spin-wave edge mode is also robust against the boundary shape; the edge modes persist in almost arbitrary shapes of the boundary, provided that the edge mode in the boundary has no interference with the other mode running along the opposite sample boundary.

It is also a non-trivial issue whether submicrometer-scale ferromagnetic islands behave as a single spin or not. In preceding experimental systems mentioned before, non-isotropic shapes of ferromagnetic islands give rise to strong magnetic dipolar anisotropies, forcing all the spins in each island to point along a same direction. In our model calculations, magnetic anisotropies within each island are not included from the outset. It is interesting to include these magnetic dipolar anisotropies into the present Landau-Lifshitz equation phenomenologically as the ‘single-ion’ type magnetic anisotropies. It is also equally likely that a ferromagnetic island has a couple of low-frequency relevant modes having different spin textures within the island. Such modes can be also utilized as a kind of ‘atomic orbital’, so that a system with only one ferromagnetic island within a unit cell could also have a chance to provide a volume-mode bands with finite Chern integers. Exploring such systems is, however, beyond the scope of the present paper and we leave them for future open issues.

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