CONVEXIFICATION NUMERICAL METHOD FOR A COEFFICIENT INVERSE PROBLEM FOR THE RADIATIVE TRANSPORT EQUATION

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Abstract. An \((n + 1)\)–D coefficient inverse problem for the radiative stationary transport equation is considered for the first time. A globally convergent so-called convexification numerical method is developed and its convergence analysis is provided. The analysis is based on a Carleman estimate. In particular, convergence analysis implies a certain uniqueness theorem. Extensive numerical studies in the 2-D case are presented.

1. Introduction

The stationary radiative transfer equation (RTE) governs the propagation of the radiation field in media with absorbing, emitting and scattering radiation. RTE has broad applications in optics, including diffuse optical tomography [17], astrophysics, atmospheric science and other applied disciplines. For example, in the single particle emission tomography the coefficient we reconstruct is the emission coefficient [29, formula (2.1)]. The RTE we consider here is a more general one since we introduce an integral operator in it.

For the first time, we develop in this paper a globally convergent numerical method for a Coefficient Inverse Problem (CIP) of the recovery of a spatially distributed coefficient of RTE in the \((n + 1)\)–D case, \(n \geq 1\). All CIPs are both nonlinear and ill-posed. Numerical methods for inverse source problems for RTE were developed in [11, 12, 13, 31]. In the case of single particle emission tomography, i.e. when the kernel of the integral operator in RTE is the identical zero, an inversion formula for the inverse source problem was first derived in [30] and tested numerically in [16]. Inverse problems of [11, 12, 13, 16, 30, 31] are linear ones.

Our CIP is formally determined, i.e. the number of free variables in the data equals the number of free variables in the unknown coefficient. Our data are incomplete, i.e. the source runs along an interval of a straight line and the data are measured only at a part of the boundary \(\partial \Omega\) of the domain of interest \(\Omega\). This is unlike the classical case of X-ray tomography when the source runs all around \(\Omega\) and the data are measured on the whole boundary \(\partial \Omega\).

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A significant attention has been paid to the questions of uniqueness and stability of CIPs for RTE, see, e.g. [2, 14, 21, 28, 32] and the references cited therein. We refer here only to a limited number of works on this topic since our interest in this paper is in a numerical development.

Our goal is to construct a globally convergent numerical method for our CIP, to conduct its convergence analysis and to confirm our method via numerical experiments. To achieve this goal, we develop here a new version of the so-called convexification method [26]. The convexification constructs a least squares cost functional, which is strictly convex on a convex set in an appropriate Hilbert space. The diameter of this set is fixed and is an arbitrary one. We prove existence and uniqueness of the minimizer of that functional on that set and estimate convergence rate of minimizers to the true solution depending on the level of the noise in the data. As a by-product, we obtain a certain uniqueness result for our CIP, see Theorem 4.3 below. Also, we establish the global convergence of the gradient descent method of the minimization of our functional. Recall that only local convergence of this method can be proven in the non-convex case. In addition, we present numerical experiments in the 2-D case.

We call a numerical method for a CIP globally convergent if a theorem is proven, which claims that this method delivers at least one point in a sufficiently small neighborhood of the solution of this CIP without an advanced knowledge of that neighborhood, also, see [26, Definition 1.4.2] for a similar statement. In other words, a good approximation for the true solution can be obtained without a knowledge of a good first guess about this solution.

Conventional numerical methods for CIPs are based on the minimizations of least squares cost functionals, see, e.g. [9, 15]. However, these functionals are usually non-convex. This means, in turn that they typically have multiple local minima and ravines. On the other hand, since any gradient-like optimization method can stop at any point of a local minimum, then this technique needs to have a good first guess about the solution. Besides, there is no rigorous guarantee of neither the existence of the global minimizer nor that such a minimizer, if it exists, is indeed close to the true solution. These considerations have prompted the first author to work on the developments of the convexification technique with the first publications [24, 20].

The key element of the convexification is the presence of the so-called Carleman Weight Function (CWF) in the above mentioned functional. The presence of the CWF ensures the strict convexity property of that functional on that bounded set. The CWF is the function, which is used as the weight in the Carleman estimate for the corresponding differential operator, see, e.g. books [5, 26] for Carleman estimates. In particular, we demonstrate in our numerical experiments that the absence of the CWF leads to a significant deterioration of the numerical solution.

The convexification uses the idea of the so-called Bukhgeim-Klibanov method (BK), which is based on Carleman estimates. BK was originally introduced in the field of CIPs in 1981 [8] only for the proofs of uniqueness theorems for multidimensional CIPs. The work [8] has generated many publications of many authors since then, see, e.g. books [5, 26], papers [14, 21, 22, 28] for a few of such publications as well as references cited therein. The majority of currently known publications about BK is also dedicated to the issues of uniqueness and stability of CIPs. Unlike these,
it was originally proposed in [24, 20] to use the idea of BK for the construction of the above mentioned globally strictly convex cost functionals for CIPs.

While initially only analytical results for the convexification were derived, an active exploration of this method for numerical studies of CIPs has started from the work [11], which has removed some obstacles for real computations, see, e.g., publications [19, 27, 26] for a few examples out of many. In particular, [19] and [26] Chapter 10 demonstrate the performance of the convexification for experimentally collected data for backscatter microwaves. An updated version of the convexification, which combines Carleman estimates with the contraction principle, can be found in [3, 4, 7].

In our approach, we use a certain approximate mathematical model. Our model amounts to the truncation of a Fourier-like series with respect to a special orthonormal basis in $L_2(a, b)$ as well as to the so-called “partial finite differences”, see subsection 3.3 for details. This basis was originally introduced in [23], also see [26, Chapter 6]. We do not know how to prove convergence of our method in the case when the number of terms of this series $N \to \infty$. Therefore, the only way to verify the validity of this model is via numerical simulations. In this regard, we note that truncations of Fourier-like series with respect to the same basis were done for a variety of CIPs in [19, 27, 31] and in [26] Chapters 7, 10, 12. In each of these cases, the validity of the corresponding approximate mathematical model was verified numerically well. Similar cases of truncated Fourier series without proofs of convergence at $N \to \infty$ can be observed for inverse problems considered by some other authors, see, e.g. [16, 18]. And successful numerical verifications also took place in these references.

Philosophically, the situation of our approximate mathematical model being verified in numerical studies is somewhat similar with the well known situation of the Huygens-Fresnel diffraction theory in optics. This theory is not yet derived rigorously from the Maxwell’s system. This is why it is stated in section 8.1 of the classical textbook of Born and Wolf [6] that “because of mathematical difficulties, approximate mathematical models must be used in most cases of practical interest. Of these the theory of Huygens and Fresnel is by far most powerful and adequate for the treatment of the majority of problems encountered in experimental optics.”

The conclusion we draw from this is that in the case of significant challenges for a certain applied mathematical problem, it is reasonable to introduce an approximate mathematical model. However, this model must be verified numerically.

All functions considered below are real valued ones. Let $B$ be a Banach space of functions and $s \geq 2$ be an integer. Below $B^s_s = B \times B \times \ldots \times B$, $s$ terms, is the space of $s$–dimensional vector functions generated by $B$ and with the obvious extension of the $B$–norm.

In section 2 we pose forward and inverse problems and prove existence and uniqueness theorem for the solution of the forward problem. In section 3 we describe our transformation procedure. In section 4 we introduce our convexification functional and formulate five theorems. These theorems are proven in section 5. In section 6 we present our numerical results.
2. Statements of Forward and Inverse Problems

For \( n \geq 1 \), points in \( \mathbb{R}^{n+1} \) are denoted below as \( x = (x_1, x_2, \ldots, x_{n-1}, y) \in \mathbb{R}^{n+1} \).

Let numbers \( A, a, b, d > 0 \), where

\[
1 < a < b.
\]

Define the rectangular prism \( \Omega \subset \mathbb{R}^{n+1} \) and parts \( \partial_1 \Omega, \partial_2 \Omega, \partial_3 \Omega \) of its boundary \( \partial \Omega \) as:

\[
\Omega = \{ x : -A < x_1, \ldots, x_n < A, a < y < b \},
\]

\[
\partial_1 \Omega = \{ x : -A < x_1, \ldots, x_n < A, y = a \},
\]

\[
\partial_2 \Omega = \{ x : -A < x_1, \ldots, x_n < A, y = b \},
\]

\[
\partial_3 \Omega = \{ x_i = \pm A, y \in (a, b), i = 1, \ldots, n, \}.
\]

Let \( \Gamma_d \) be the line where the external sources are,

\[
\Gamma_d = \{ x_\alpha = (\alpha, 0, \ldots, 0) : \alpha \in [-d, d] \}.
\]

Hence, \( \Gamma_d \) is a part of the \( x_1 \)–axis. It follows from (2.1) and (2.2) that \( \Gamma_d \cap \Omega = \emptyset \).

Let the points of external sources \( x_\alpha \) run along \( \Gamma_d \), \( x_\alpha \in \Gamma_d \). Let \( \varepsilon > 0 \) be a sufficiently small number. To avoid dealing with singularities, we model the \( \delta(x) \)–function as:

\[
f(x) = \sum_{\varepsilon} \frac{C_{\varepsilon}}{\varepsilon} \exp\left(-\frac{|x|^2}{\varepsilon^2} - \frac{|x|}{\varepsilon}\right),
\]

where the constant \( C_{\varepsilon} \) is chosen such that

\[
\int_{|x| < \varepsilon} \frac{C_{\varepsilon}}{\varepsilon^2 - |x|^2} dx = 1.
\]

Hence, the function \( f(x - x_\alpha) = f(x_1 - \alpha, x_2, \ldots, x_n, y) \in C^\infty(\mathbb{R}^{n+1}) \) plays the role of the source function for the source \( x_\alpha \). We choose \( \varepsilon \) so small that

\[
f(x - x_\alpha) = 0, \forall x \in \overline{\Omega}, \forall x_\alpha \in \Gamma_d.
\]

Let \( u(x, \alpha) \) denotes the steady-state radiance at the point \( x \) generated by the source function \( f(x - x_\alpha) \). We assume that the function \( u(x, \alpha) \) is governed by the RTE of the following form [17]:

\[
\frac{\nu(x, \alpha) \cdot \nabla_x u(x, \alpha)}{a(x)} + a(x) u(x, \alpha) = \int_{\Gamma_d} K(x, \alpha, \beta) u(x, \beta) d\beta + f(x - x_\alpha), \ x \in \mathbb{R}^{n+1}, x_\alpha \in \Gamma_d.
\]

The kernel \( K(x, \alpha, \beta) \) of the integral operator in (2.10) is called the “phase function”.

\[
K(x, \alpha, \beta) \geq 0, x \in \overline{\Omega}; \ \alpha, \beta \in [-d, d],
\]

see [17]. In addition, we assume that

\[
K(x, \alpha, \beta) \in C^2(\overline{\Omega} \times [-d, d]^2).
\]
In equation (2.10),

\[ a(x) = \mu_a(x) + \mu_s(x), \]

where \( \mu_a(x) \) and \( \mu_s(x) \) are the absorption and scattering coefficients respectively. As stated in section 1, \( a(x) \) is the emission coefficient [20 formula (2.1)]. We assume that

\[ \mu_a(x), \mu_s(x) \geq 0, \mu_a(x) = \mu_s(x) = 0, x \in \mathbb{R}^{n+1} \backslash \Omega, \]

\[ \mu_a(x), \mu_s(x) \in C^2\left(\mathbb{R}^{n+1}\right). \]

For two arbitrary points \( x, z \in \mathbb{R}^{n+1} \) let \( L(x, z) \) be the line segment connecting these points and let \( ds \) be the element of the euclidean length on \( L(x, z) \). In (2.10) \( \nu(x, \alpha) \) denotes the unit vector, which is parallel to \( L(x, x_{\alpha}) \).

\[ \nu(x, \alpha) = \frac{x - x_{\alpha}}{|x - x_{\alpha}|} = \frac{(x_1 - \alpha, x_2, ..., x_n, y)}{\sqrt{(x_1 - \alpha)^2 + x_2^2 + ... + x_n^2 + y^2}} \]

Denote

\[ D^{n+1} = \{(x_1, ..., x_n, y) \in \mathbb{R}^n \times [0, b]\}, \quad D_a^{n+1} = D^{n+1} \cap \{y > a\}. \]

**Forward Problem.** Let (2.1) - (2.10) hold. Find the function \( u(x, \alpha) \in C^1\left(\overline{D^{n+1}} \times [-d, d]\right) \) satisfying equation (2.10) and the initial condition

\[ u(x_{\alpha}, x_{\alpha}) = 0 \quad \text{for} \quad x_{\alpha} \in \Gamma_d. \]

**Coefficient Inverse Problem (CIP).** Let (2.1) - (2.10) hold. Let the function \( u(x, \alpha) \in C^1\left(\overline{D^{n+1}} \times [-d, d]\right) \) be the solution of the Forward Problem. Assume that the coefficient \( a(x) \) of equation (2.10) is unknown. Determine the function \( a(x) \), assuming that the following function \( g(x, \alpha) \) is known:

\[ g(x, \alpha) = u(x, \alpha), \forall x \in \partial \Omega \setminus \partial_1 \Omega, \forall \alpha \in (-d, d). \]

First, we formulate and prove an existence and uniqueness theorem for the solution of the forward problem. We refer to [31, 32] for some other existence and uniqueness results for the forward problem for equation (2.10). Unlike Theorem 2.1, the positivity of the function \( u \) was not discussed in these references. On the other hand, we need this property of \( u \) for our numerical method.

**Theorem 2.1.** Assume that (2.1), (2.3), (2.6) - (2.8) and (2.11) - (2.16) hold. Then there exists unique solution \( u(x, \alpha) \in C^1\left(\overline{D^{n+1}} \times [-d, d]\right) \) of equation (2.10) with the initial condition (2.18). In addition, the following inequality holds:

\[ u(x, \alpha) \geq m > 0 \quad \text{for} \quad (x, \alpha) \in \overline{\Omega} \times [-d, d], \]

\[ m = \min_{(x, \alpha) \in \partial_1 \Omega \times [-d, d]} \left[ \int_{L(x, x_{\alpha})} f(x(s) - x_{\alpha}) \, ds \right] > 0. \]

In addition, there exists a sufficiently large number \( X > A \) such that

\[ u(x, \alpha) = 0 \quad \text{for} \quad |x_1|, ..., |x_n| > X, \alpha \in (-d, d). \]
Proof. Equation (2.10) can be rewritten as
\begin{equation}
D_\nu u(x, \alpha) + a(x)u(x, \alpha) = \mu_s(x) \int_{r_d} K(x, \alpha, \beta)u(x, \beta) d\beta + f(x - x_\alpha),
\end{equation}
where \(D_\nu\) is the directional derivative of \(u\) in the direction of the vector \(\nu\). The first line of (2.23) can be treated as the first order linear ordinary differential operator along the line segment \(L(x, x_\alpha)\). Denote
\begin{equation}
p(x, \alpha) = \int_{L(x, x_\alpha)} a(x(s)) ds.
\end{equation}
Obviously, \(L(x, x_\alpha) = \{z(t) = (\alpha + t(x_1 - \alpha), tx_2, ..., tx_n, ty), t \in (0, 1)\}\). Hence, \(ds = |x - x_\alpha| dt\) in (2.24). We obtain
\begin{equation}
p(x, \alpha) = |x - x_\alpha| \int_0^1 a(\alpha + t(x_1 - \alpha), tx_2, ..., tx_n, ty) dt.
\end{equation}
Since by (2.13) and (2.15) \(a \in C^1(\mathbb{R}^{n+1})\), then (2.16), (2.25) and elementary calculations imply that
\begin{equation}
D_\nu p(x, \alpha) = a(x),
\end{equation}
where \(D_\nu\) is the operator of the directional derivative in the direction of the vector \(\nu(x, \alpha)\). Consider now the integration factor \(c(x, \alpha)\),
\begin{equation}
c(x, \alpha) = e^{p(x, \alpha)}.
\end{equation}
Then (2.26) and (2.27) imply
\begin{equation}
D_\nu c(x, \alpha) = a(x) c(x, \alpha).
\end{equation}
Multiply both sides of (2.23) by \(c(x, \alpha)\). We obtain
\begin{equation}
c(x, \alpha)D_\nu u(x, \alpha) + c(x, \alpha)a(x)u(x, \alpha) =
\end{equation}
\begin{equation}
= c(x, \alpha)\mu_s(x) \int_{r_d} K(x, \alpha, \beta)u(x, \beta) d\beta + c(x, \alpha)f(x - x_\alpha).
\end{equation}
Using (2.28), we obtain
\begin{equation}
cD_\nu u + cau = D_\nu (cu) - uD_\nu c + cau = D_\nu (cu) - cau + cau = D_\nu (cu).
\end{equation}
Since by (2.7) - (2.9), (2.13) and (2.14) \(c(x, \alpha) = 1\) for all points \(x\) where \(f(x - x_\alpha) \neq 0\), then \(c(x, \alpha)f(x - x_\alpha) = f(x - x_\alpha)\). Hence, (2.20) - (2.23), (2.14) and (2.30) imply that problem equation (2.23) is equivalent with
\begin{equation}
D_\nu ((cu)(x, \alpha)) = c(x, \alpha)\mu_s(x) \int_{r_d} K(x, \alpha, \beta)u(x, \beta) d\beta + f(x - x_\alpha).
\end{equation}
Integrating (2.31) over the line \(L(x, x_\alpha)\) and using the initial condition (2.15), we obtain
\begin{equation}
u(x, \alpha) = \frac{1}{c(x, \alpha)} \int_{L(x, x_\alpha)} c(x(s), \alpha)\mu_s(x(s)) \left( \int_{r_d} K(x(s), \alpha, \beta)u(x(s), \beta) d\beta \right) ds+
\end{equation}
(2.32) \[ + \frac{1}{c(x,\alpha)} \int_{L(x,x_\alpha)} f(x(s) - x_\alpha) ds. \]

Assume now that in (2.32) \( x \in \{ y \in (0, a) \} \). Hence, by (2.22) \( x \notin \Omega \). Hence, (2.14), (2.24), (2.27) and (2.32) imply

\[ u(x,\alpha) = \int_{L(x,x_\alpha)} f(x(s) - x_\alpha) ds, \ x \in \{ y \in (0, a) \}. \]

This and (2.3) imply

\[ u(x,\alpha) = u_0(x,\alpha) = \int_{L(x,x_\alpha)} f(x(s) - x_\alpha) ds, \ (x,\alpha) \in \partial_1 \Omega \times (-d, d), \]

(2.35)

\[ u_0(x,\alpha) \geq m, \]

where the number \( m \) is defined in (2.21). In addition, (2.22), (2.31) and (2.32) imply that (2.22) holds for a sufficiently large number \( X > A \). It follows from the above construction that problem (2.32)–(2.34) is equivalent with problem (2.18), (2.29).

Let now \( x \in D^{n+1}_a \), where the set \( D^{n+1}_a \) is defined in (2.17). Hence, similarly with (2.24) and (2.25), for any appropriate function \( \varphi(z) \)

\[ \int_{L(x,x_\alpha)} \varphi(z) \int_{L(x,x_\alpha)} f(x(s) - x_\alpha) ds, \ (x,\alpha) \in \partial_1 \Omega \times (-d, d), \]

(2.36)

\[ u(x_1, x_2, \ldots, y, \alpha) = \]

\[ = \frac{x - x_\alpha}{y c(x)} \int_a^y (c_\mu) (x(z), \alpha) \left( \int_{\Gamma_d} K(x(z), \alpha, \beta) u(x(z), \beta) d\beta \right) dz + u_0(x,\alpha), \]

(2.37)

\[ x(z) = \left( \frac{x_1 - \alpha}{y} z, \frac{x_2}{y} z, \ldots, \frac{x_n}{y} z, z \right), x \in D^{n+1}_a, \alpha \in (-d, d). \]

Thus, by (2.22), we have obtained the \( \alpha \)-dependent family of integral equations (2.22), (2.30), (2.37) of the Volterra type in the bounded domain

\[ D^{n+1}_{a,b,X} = \{ x = (x_1, \ldots, x_2, y) : |x_1|, |x_2|, \ldots, |x_n| < X, y \in (a, b) \} . \]
Furthermore, any solution of \((2.36), (2.37)\) satisfies \((2.22)\). Since problem \((2.33), (2.36), (2.37)\) is equivalent with equation \((2.29)\) with the initial condition \((2.18)\), then it is sufficient to solve problem \((2.33), (2.36), (2.37)\). It follows from the well known classical results about Volterra equations that there exists unique function \(u(x, \alpha) \in C \left( \overline{D_{a,b,X}} \times [-d, d] \right)\) satisfying \((2.36), (2.37)\), and this function can be obtained via the following iterative process:

\[
\begin{align*}
\forall n &= 1, 2, \ldots \\
|u_n(x, \alpha)| &\leq \frac{n}{k!} \left( M_1(y - a)^k \right) &\in D_{a,b,X}^{n+1} \times [-d, d],
\end{align*}
\]

where \(n = 0, 1, \ldots\) It follows from \((2.38)\) that for \(n = 0, 1, \ldots\)

\[
|\nabla x u_n(x, \alpha)| = |\partial_\alpha u_n(x, \alpha)| \leq \frac{n}{k!} \left( M_2(y - a)^k \right)
\]

for \((x, \alpha) \in D_{a,b,X}^{n+1} \times [-d, d]\) and \(n = 0, 1, \ldots\) Here the number \(M_1 = M_1(a, b, d, X, \|a(x)\|_{C(\Omega)}, \|K\|_{C(\Omega \times [a,b]^2)}, \|f\|_{C^1(|x| \leq \varepsilon)})\) > 0 depends only on listed parameters. Next, \((2.11), (2.34), (2.35), (2.38), (2.39)\) imply that estimate \((2.34)\) holds. Finally, it follows from \((2.12), (2.15)\) and \((2.33)\) that functions \(u_n(x, \alpha)\) can be differentiated once with respect to \(x_1, \ldots, x_n, y, \alpha\) and, similarly with \((2.39)\), the following estimates hold for

\[
|\nabla_x u_n(x, \alpha)| = |\partial_\alpha u_n(x, \alpha)| \leq \frac{n}{k!} \left( M_2(y - a)^k \right)
\]

3. Transformation

3.1. An integral differential equation without the unknown coefficient \(a(x)\). The first step of the convexification is a transformation of the above CIP to a boundary value problem for a certain PDE of the second order, in which the unknown coefficient \(a(x)\) is not involved. By \((2.20)\) we can introduce a new function \(w(x, \alpha)\),

\[
w(x, \alpha) = \ln u(x, \alpha), \quad (x, \alpha) \in \overline{\Omega} \times [-d, d].
\]

Substituting \((3.1)\) in \((2.10)\) and \((2.19)\) and using \((2.34)\), we obtain

\[
\begin{align*}
\nu(x, \alpha) \cdot \nabla_x w(x, \alpha) + a(x) &\quad= e^{-w(x, \alpha)} \mu_\alpha(x) \int_{\Gamma_d} K(x, \alpha, \beta) e^{w(x, \beta)} d\beta, x \in \Omega, \alpha \in (-d, d), \\
w(x, \alpha) |_{\partial\Omega} &\quad= \ln g_1(x, \alpha),
\end{align*}
\]
Hence, (2.1)-(2.6) imply that with a constant $P$ an integral differential equation with the derivatives up to the second order,

$$w(x, \alpha) = \frac{\partial}{\partial \alpha} \left[ e^{-w(x, \alpha)} \int_{\Gamma_d} K(x, \alpha, \beta) e^{w(x, \beta)} d\beta \right], x \in \Omega, \alpha \in (-d, d).$$

We have $$\nu(x, \alpha) = (\nu_1(x, \alpha), \nu_2(x, \alpha), ..., \nu_{n+1}(x, \alpha))$$, where by (2.10)

$$\nu_1(x, \alpha) = \frac{x_1 - \alpha}{\sqrt{(x_1 - \alpha)^2 + x_2^2 + ... + x_n^2 + y^2}}$$

$$\nu_k(x, \alpha) = \frac{x_k}{\sqrt{(x_1 - \alpha)^2 + x_2^2 + ... + x_n^2 + y^2}}, k = 2, ..., n,$$

$$\nu_{n+1}(x, \alpha) = \frac{y}{\sqrt{(x_1 - \alpha)^2 + x_2^2 + ... + x_n^2 + y^2}}.$$

Hence, (2.1)-(2.6) imply that with a constant $M_3 = M_3(A, d) > 0$ depending only on numbers $A, d$ the following estimates are valid:

$$\left| \frac{\partial \nu_k(x, \alpha)}{\nu_{n+1}(x, \alpha)} \right|, \frac{\nu_k(x, \alpha)}{\nu_{n+1}(x, \alpha)} \leq \frac{M_3}{a}, k = 1, ..., n, x \in \Omega, \alpha \in (-d, d).$$

Dividing (3.5) by $\nu_{n+1}(x, \alpha)$, we obtain

$$\frac{\partial \nu_k(x, \alpha)}{\nu_{n+1}(x, \alpha)} \Delta w_y(x, \alpha) + \frac{\partial \nu_{n+1}(x, \alpha)}{\nu_{n+1}(x, \alpha)} w_y(x, \alpha)$$

$$+ \sum_{k=1}^n \nu_k(x, \alpha) \frac{\partial \nu_{n+1}(x, \alpha)}{\nu_{n+1}(x, \alpha)} \Delta w_{x_k}(x, \alpha) + \sum_{k=1}^n \frac{\partial \nu_k(x, \alpha)}{\nu_{n+1}(x, \alpha)} w_{x_k}(x, \alpha)$$

$$- \frac{\mu_s(x)}{\nu_{n+1}(x, \alpha)} \frac{\partial}{\partial \alpha} \left[ e^{-w(x, \alpha)} \int_{\Gamma_d} K(x, \alpha, \beta) e^{w(x, \beta)} d\beta \right] = 0, x \in \Omega, \alpha \in (-d, d),$$

$$w(x, \alpha)|_{\partial \Omega} = g_1(x, \alpha).$$

The new equation (3.7) does not contain the unknown coefficient $a(x)$. We need now to solve problem (3.7), (3.8).

3.2. **An orthonormal basis in $L_2(-d, d)$**. We now describe the orthonormal basis in $L_2(-d, d)$ of [23] and [20] section 6.2.3, which was mentioned in section 1. Consider the set of linearly independent functions $\{a^s e^a \}_{s=0}^\infty$. This set is complete in the space $L_2(-d, d)$. Applying the Gram-Schmidt orthonormalization procedure to this set, we obtain the orthonormal basis $\{\Psi_s(\alpha)\}_{s=0}^\infty$ in $L_2(-d, d)$. The function $\Psi_s(\alpha)$ has the form

$$\Psi_s(\alpha) = P_s(\alpha)e^\alpha, \forall s \geq 0,$$

where $P_s(\alpha)$ is a polynomial of the degree $s$. Even though the Gram-Schmidt orthonormalization procedure is unstable for the infinite number of elements, it is still stable for a non large number of elements, and this was observed in numerical
It was proven in [23], [26, section 6.2.3] that

\[ \{ \text{property of the orthonormal basis} \] (3.10)

that the first row of the analog W

\[ \text{function} \]

(3.15)

where

Therefore, we now need to find the vector function W(x) of coefficients w_s(x), where

(3.13)

\[ W(x) = (w_0, ..., w_{N-1})^T(x). \]

Substituting (3.12) in (3.7), we obtain

\[
\begin{align*}
N-1 & \sum_{s=0} \partial_y w_s(x) \Psi_s'(\alpha) + \frac{\partial_{\alpha} \nu_{n+1}(x,\alpha)}{\nu_{n+1}(x,\alpha)} \sum_{s=0}^{N-1} \partial_y w_{sy}(x) \Psi_n(\alpha) \\
& + \sum_{s=0}^{N-1} \sum_{i=1}^{n} \frac{\nu_i(x,\alpha)}{\nu_{n+1}(x,\alpha)} \partial_{x_i} w_s(x) \Psi_s'(\alpha) + \sum_{s=0}^{N-1} \sum_{i=1}^{n} \frac{\partial_{\alpha} \nu_i(x,\alpha)}{\nu_{n+1}(x,\alpha)} \partial_{x_i} w_s(x) \Psi_s(\alpha) \\
& - \frac{\mu_s(x)}{\nu_{n+1}(x,\alpha)} \times \\
& \times \partial_{\alpha} \left[ \exp \left( - \sum_{s=0}^{N-1} w_s(x) \Psi_s(\alpha) \right) \int_{\Gamma_d} K(x,\alpha,\beta) \exp \left( \sum_{s=0}^{N-1} w_s(x) \Psi_s(\beta) \right) d\beta \right] \\
& = 0, x \in \Omega, \alpha \in (-d, d). \end{align*}
\]

Using (3.8), (3.12) and (3.14), we obtain the boundary condition for the vector function W(x),

(3.15)

\[ W(x)|_{\partial \Omega} = P(x) = (p_0, ..., p_{N-1})^T(x), \]
CONVEXIFICATION FOR A CIP FOR THE RTE 11

(3.16) \[ p_s(x) = \int_{-d}^{d} \ln[g_1(x, \alpha)] \Psi_s(\alpha) d\alpha, n = 0, \ldots, N - 1. \]

Multiply sequentially equation (3.14) by functions \( \Psi_k(\alpha), k = 0, \ldots, N - 1 \) and integrate with respect to \( \alpha \in (-d, d) \). We obtain

(3.17) \( (MN + A_{n+1}(x)) W_y(x) + \sum_{i=1}^{n} A_i(x) W_{x_i}(x) + F(W(x), x) = 0, \ x \in \Omega, \)

where \( A_{n+1} \in C_{N^2}(\Omega) \) and \( A_i \in C_{N^2}(\Omega), i = 1, \ldots, n \) are \( N \times N \) matrices, the \( N-D \) vector function

(3.18) \( F(s, x) \in C^2(\mathbb{R}^{N+n+1}) \)

is generated by the operator in the fourth line of (3.14). Here, (3.18) follows from (2.12), (2.13), (2.15), (3.5) and (3.14). Obviously the vector function \( F(W(x), x) \) is nonlinear with respect to \( W(x) \). By (3.6) and (3.19)

(3.19) \[ \|A_i(x)\|_{C_{N^2}(\Omega)} \leq \frac{C_1}{a}, \]

where the number \( C_1 = C_1(A, d, N, b) > 0 \) depends only on listed parameters.

It follows from (2.1) and (3.19) that there exists such a number \( a_0 > 1 \) that the matrix

(3.20) \( D_N(x) = (MN + A_{n+1}(x)) = MN(I + M_{-1}^{N}A_1(x)) \)

is invertible with the inverse matrix

(3.21) \( D_N^{-1}(x), \forall a \geq a_0 = a_0(A, d, N) > 1, \ \forall x \in \Omega, \)

where the number \( a_0(A, d, N) \) depends only on listed parameters. Thus, the matrix \( D_N^{-1}(x) \) exists if the domain \( \Omega \) is located sufficiently far from the line of sources \( \Gamma_d \).

Remarks 3.2:

1. To work with the convexification method, we need to assume below that the derivatives \( W_{x_i} \) in (3.17) are written in finite differences, unlike the \( y- \)derivative, and the grid step size \( h \geq h_0 > 0 \) with a fixed constant \( h_0 \). The latter assumption is a practical one since one does not allow the grid step sizes tend to zero in practical computations.

2. The assumptions that the sign “\( \approx \)” is replaced with the sign “\( = \)” in (3.12), (3.14) and (3.17) as well as the assumption of item 1 form our approximate mathematical model, see section 1 for a discussion of the issue of approximate mathematical models. Our specific model is verified computationally in the 2D case in section 6.

3.4. Partial finite differences. Let \( m > 1 \) be an integer. Consider \( n \) partitions of the interval \((-A, A)\), see (2.2):

(3.22) \[ -A = x_{i,0} < x_{i,1} < \ldots < x_{i,m} = A, x_{i,j+1} - x_{i,j} = h, j = 0, \ldots, m - 1, i = 1, \ldots, n. \]

We assume that

(3.23) \[ h \geq h_0 = \text{const} > 0. \]
Define the discrete subset \( \Omega_h \) of the domain \( \Omega \) as:

\[
\Omega_h = \{ x_{i,j} \}_{(i,j)=(1,0)}^{(n,m)} ,
\]

(3.24)

\[
\Omega = \Omega_h \times (a, b) = \left\{ (x_{i,j}, y) : \{ x_{i,j} \}_{(i,j)=(1,0)}^{(n,m)} \in \Omega_h, y \in (a, b) \right\} .
\]

(3.25)

We denote \( x^h = \{ (x_{i,j}, y) : x_{i,j} \in \Omega_h, y \in (a, b) \} \). By (3.26), (3.27), the boundary \( \partial \Omega_h \) of the domain \( \Omega_h \) is:

\[
\partial \Omega_h = \partial_1 \Omega_h \cup \partial_2 \Omega_h \cup \partial_3 \Omega_h ,
\]

\[
\partial_1 \Omega_h = \Omega_h \times \{ y = a \} , \quad \partial_2 \Omega_h = \Omega_h \times \{ y = b \} ,
\]

\[
\partial_3 \Omega_h = \{ (x_{i,0}, y), (x_{i,m}, y) : y \in (a, b), i = 1, ..., n \} .
\]

We exclude derivatives as defined in the next paragraph.

Let the vector function \( Q(x) \in C_N(\Omega) \). Denote

\[
Q^h_{ij}(y) = Q(x_{i,j}, y), \quad i = 1, ..., n; j = 0, ..., m; y \in (a, b) ,
\]

\[
Q^h(x) = \{ Q^h(x_{1,j}, x_{2,j}, ..., x_{n,j}, y) \} : y \in (a, b) .
\]

Thus, \( Q^h(x) \) is an \( N - D \) vector function of discrete variables \( x_{i,j} \in \Omega_h \) and continuous variable \( y \in (a, b) \). Note that the boundary terms at \( \partial_3 \Omega_h \) of this vector function, which correspond to \( Q(x) |_{\partial_3 \Omega_h} = \{ Q^h_{i,0}(y) \} \cup \{ Q^h_{i,m}(y) \} , i = 1, ..., n \). For two vector functions \( Q^{(1)}(x) = \left( Q^{(1)}_1(x), ..., Q^{(1)}_N(x) \right)^T \) and \( Q^{(2)}(x) = \left( Q^{(2)}_1(x), ..., Q^{(2)}_N(x) \right)^T \) their scalar product \( Q^{(1)}(x) \cdot Q^{(2)}(x) \) is defined as the scalar product in \( \mathbb{R}^N \) and \( (Q^{(1)}(x))^2 = Q^{(1)}(x) \cdot Q^{(1)}(x) \). Respectively

\[
Q^{(1)h}(x^h) \cdot Q^{(2)h}(x^h) = \sum_{k=1}^{N} \sum_{(i,j)=(1,1)}^{(1,m-1)} Q^{(1)h}_k(x_{i,j}, y)Q^{(2)h}_k(x_{i,j}, y) ,
\]

(3.26)

\[
(Q^h(x^h))^2 = Q^h(x^h) \cdot Q^h(x^h) , \quad |Q^h(x^h)| = \sqrt{Q^h(x^h) \cdot Q^h(x^h)} .
\]

(3.27)

We will use formulas (3.26), (3.27) everywhere below without further mentioning. We exclude \( j = 0 \) and \( j = m \) here since we work below with finite difference derivatives as defined in the next paragraph.

We define finite difference derivatives of \( Q^h(x^h) \) with respect to \( x_1, ..., x_n \) only at interior points of the domain \( \Omega_h \) with as:

\[
\partial_{x_k} Q^h(x^h) = Q^h_{x_k}(x^h) = \{ Q^h(x_{1,j}, x_{2,j}, ..., x_{n,j}, y) \}_{x_k} =
\]

\[
= \left\{ \frac{Q^h(x_{1,j}, ..., x_{k-1,j}, x_{k,j+1}, x_{k+1,j}, ..., x_{n,j}, y) - Q^h(x_{1,j}, ..., x_{k-1,j}, x_{k,j-1}, x_{k+1,j}, ..., x_{n,j}, y)}{2h} \right\} .
\]

(3.28)

The second line of (3.28) should be adjusted in an obvious fashion for \( k = 1 \) and \( k = n \). Also, in that line \( j = 1, ..., m - 1 \). We now define discrete analogs of spaces \( C_N(\Omega), H_N(\Omega) \) and \( L_N(\Omega) \) as:

\[
C_N^h(\Omega) = \left\{ Q^h(x^h) : \| Q^h(x^h) \|_{C_N(\Omega)} = \max_{y \in [a, b]} \left( \max_{i=1, n, j=0, m} |Q^h_{ij}(y)| \right) < \infty \right\} ,
\]
By embedding theorem $H^{1,h}_N(\Omega^h) = \left\{ Q^h(x^h) : \|Q^h(x^h)\|_H^{1,h}_N(\Omega^h) = \sum_{(i,j,k)=(1,1,1)}^b \int_a^b \left[(Q^h_{ij}(y))^2 + (Q^h_{ik}(x^h))^2 + (\partial_y Q^h_{ij}(y))^2\right] dy < \infty \right\}$,

\[ H^{1,h}_N(\Omega^h) = \{ Q^h(x^h) : Q^h(x^h) \in H^{1,h}_N(\Omega^h) \} \cap H^{1,h}_N(\Omega^h) \]

\[ L^2_N(\Omega^h) = \left\{ Q^h(x^h) : \|Q^h(x^h)\|_{H^{1,h}_N(\Omega^h)} = \sum_{(i,j)=(1,1)}^b (Q^h_{ij}(y))^2 \right\} \]

By embedding theorem $H^{1,h}_N(\Omega^h) \subset C^{2,h}_N(\Omega^h)$ and

\[ \|Q^h(x^h)\|_{C^{2,h}_N(\Omega^h)} \leq C_2 \|Q^h(x^h)\|_{H^{1,h}_N(\Omega^h)}, \forall Q^h \in H^{1,h}_N(\Omega^h), \]

where the number $C_2 = C_2(h_0, A, a, b, \Omega) > 0$ depends only on listed parameters, and the number $h_0$ is defined in (3.23).  

Remark 3.3. Since we work with finite difference derivatives (3.28) only at interior points of the discrete domain $\Omega^h$, then in any differential operator below we use $W^h(x^h) = \{ W^h_{ij}(y) \}_{(i,j)=(1,1)}^{b,b}, y \in (a,b)$, i.e., boundary points $x_{i,0}$ and $x_{i,m}$ are involved only in finite difference derivatives $W^h_{i,j}(x^h)$ at $x_{i,1}$ and $x_{i,m-1}$ Points $x_{i,0}$ and $x_{i,m}$ are not included in (3.20) for the same reason.

Using (3.22), we now rewrite problem (3.15) in the form of finite differences with respect to $x_1, \ldots, x_n$ as:

\[ D^h_N(x^h) W^h_y(x^h) + \sum_{i=1}^n A^h_i(x^h) W^h_{x_i}(x^h) + F^h(W^h(x^h), x^h) = 0, x^h \in \Omega^h, \]

\[ W^h(x^h)|_{\partial \Omega^h} = P^h(x^h). \]

In (3.30), $N \times N$ matrices $A^h_i(x^h) \in C_N(\Omega^h)$, the matrix $D^h_N(x^h) \in C(\Omega^h)$ is the discrete analog of the matrix $D_N(x)$ defined in (3.20) by (3.19) and (3.20)

\[ \|D^h_N\|_{C_N(\Omega^h)} \leq C_3, \forall a \geq a_0 = a_0(A,d,N) > 1. \]

The vector function $F^h(W^h(x^h), x^h)$ the discrete analog of the vector function $F$ in (3.17), and by (3.18) $F^h(W^h(x^h), x^h)$ is twice continuously differentiable with respect to components of $W^h(x^h)$. By (3.21) there exists the inverse matrix $(D^h_N)^{-1}(x^h)$ and

\[ \| (D^h_N)^{-1} \|_{C_N(\Omega^h)} \leq C_3, \forall a \geq a_0 = a_0(A,d,N) > 1. \]

Furthermore, (3.19) and (3.33) imply for $i = 1, \ldots, n$

\[ \| (D^h_N)^{-1} A^h_i(x^h) \|_{C_N(\Omega^h)} \leq C_3, \forall a \geq a_0 = a_0(A,d,N) > 1, \]

Here and everywhere below $C_3 = C_3(A,d,N,a,b,h_0,\|K\|_{C_N([-d,d]^2)}) > 0$ denotes different constants depending only on listed parameters.
The following formulas are discrete analogs of (3.12) and (3.13):

\[
w^h(x^h, \alpha) = \sum_{n=0}^{N-1} w^h_n(x^h) \Psi_n(\alpha), \quad \partial_\alpha w^h(x^h, \alpha) = \sum_{n=0}^{N-1} w^h_n(x^h) \Psi'_n(\alpha),
\]

(3.36) \( W^h(x^h) = (w^h_0, \ldots, w^h_{N-1})^T(x^h). \)

Suppose that we have found the vector function \( W^h(x^h) \) in (3.36). Then, to find the discrete analog \( a^h(x^h) \) of the unknown coefficient \( a(x) \), we use (3.12) and (3.35) as:

\[
a^h(x^h) = -\frac{1}{2d} \int_{-d}^{d} \nu(x^h, \alpha) \cdot \nabla_{x^h} w^h(x^h, \alpha) d\alpha + \frac{1}{2d} \int_{-d}^{d} \left( e^{-w^h(x^h, \alpha)} \mu_x(x^h) \int_{-d}^{d} K(x^h, \alpha, \beta) e^{w^h(x^h, \beta)} d\beta \right) d\alpha, x^h \in \Omega^h.
\]

4. Convexification Method for Problem (3.30), (3.31)

Let \( R > 0 \) be an arbitrary number. Define the set \( B(R, P^h) \) as

(4.1) \[ B(R, P^h) = \left\{ W^h \in H^1_{N}^h(\Omega^h) : W^h(x^h) \big|_{\partial \Omega^h} = P^h(x^h), \| W^h \|_{H^1_{N}^h(\Omega^h)} < R \right\}, \]

where \( P^h(x^h) \) is the boundary condition in (3.31). Consider matrices \( G^h_i(x^h) \) and the vector function \( \Phi^h(W^h(x^h), x^h) \),

\[
G^h_i(x^h) = \left( (D^h_N)^{-1} A^h_i \right)(x^h),
\]

(4.3) \[ \Phi^h(W^h(x^h), x^h) = (D^h_N(x^h))^{-1} F^h(W^h(x^h), x^h). \]

By (3.18), (3.33), (3.34), (4.1)–(4.3) and the multidimensional analog of Taylor formula (84)

(4.4) \[ \| G^h_i(x^h) \|_{C^2_N(\Omega^h)} \leq C_3, \forall a \geq a_0 > 1, \quad i = 1, \ldots, n, \]

(4.5) \[ \| \Phi^h(W^h(x^h), x^h) \|_{C^1_N(\Omega^h)} \leq C_3, \forall W^h \in B(R, P^h), \]

(4.6) \[ \Phi^h(W^h_1(x^h), x^h) = \Phi^h(W^h_1(x^h), x^h) + \Phi_1(W^h_1(x^h), x^h)(W^h_2 - W^h_1)(x^h) \]

+ \Phi_2(W^h_1(x^h), x^h), \forall W^h_1, W^h_2 \in B(R, P^h), x^h \in \Omega^h; \]

where the vector function \( \Phi_1(W^h_1(x^h), x^h) \) is independent on \( W^h_2(x^h) \), the vector function \( \Phi_2(W^h_1(x^h), x^h) \) is nonlinear with respect to \( (W^h_2 - W^h_1)(x^h) \), both these vector functions are continuous with respect to their variables for \( x^h \in \Omega^h \) and the following estimates hold for all \( W^h_1, W^h_2 \in B(R, P^h) \) and for all \( x^h \in \Omega^h \)

(4.7) \[ |\Phi_1(W^h_1(x^h), x^h)| \leq C_3, \forall W^h_1 \in B(R, P^h), x^h \in \Omega^h, \]

(4.8) \[ |\Phi_2(W^h_1(x^h), W^h_2(x^h), x^h)| \leq C_3 (W^h_2 - W^h_1)^2(x^h), \forall x^h \in \Omega^h, \]

also, see (3.27).
Lemma 4.1. Let $A$ be a $k \times k$ matrix which has the inverse $A^{-1}$. Then there exists a number $\mu = \mu(A) > 0$ such that $\|Ax\|^2 \geq \mu \|x\|^2$, $\forall x \in \mathbb{R}^k$, where $\|\cdot\|$ is the euclidean norm.

We omit the proof since this lemma is well known.

Corollary 4.1. The following inequality holds:

$$\left( D_N^h(x^h) W_y^h(x^h) + \sum_{i=1}^n A_i^h(x^h) W_{z_i}^h(x^h) \right)^2 \geq C_3 \left( W_y^h(x^h) + \sum_{i=1}^n G_i^h(x^h) W_{z_i}^h(x^h) \right)^2, \forall x^h \in \Omega^h. $$

Proof. Denote

$$Y(x^h) = W_y^h(x^h) + \sum_{i=1}^n G_i^h(x^h) W_{z_i}^h(x^h) + \Phi^h(W^h(x^h), x^h).$$

We have

$$\left( D_N^h(x^h) W_y^h(x^h) + \sum_{i=1}^n A_i^h(x^h) W_{z_i}^h(x^h) \right)^2 = \left( D_N^h(x^h) Y(x^h) \right)^2. $$

The rest of the proof follows immediately from (4.12), (4.13) and Lemma 4.1. □

Introduce the following weighted cost functional $J_\lambda(W^h)$:

$$J_\lambda(W^h) = \left\| D_N^h W_y^h + \sum_{i=1}^n A_i^h W_{z_i}^h + F^h(W^h(x^h), x^h) \right\|_{L_N^{2,h}(\Omega^h)}^2 e^{\lambda y}. $$

Minimization Problem. Minimize functional (4.11) on the set $B(R, P^h)$.

Theorems 4.1-4.5 are our analytical results about the functional $J_{\lambda,\gamma}(W^h)$.

Theorem 4.1 (Carleman estimate). Assume that the number $a \geq a_0$, as in (3.33). Then there exists a sufficiently large number $\lambda_0 = \lambda_0(A, d, N, a, b, h_0) \geq 1$ depending only on listed parameters such that the following Carleman estimate holds

$$\left\| \left( W_y^h(x^h) + \sum_{i=1}^n G_i^h(x^h) W_{z_i}^h(x^h) \right) e^{\lambda y} \right\|_{L_N^{2,h}(\Omega^h)}^2 \geq W_y^h e^{\lambda y} \left\| L_N^{2,h}(\Omega^h) \right\|^2 + \lambda^2 \left\| W^h e^{\lambda y} \right\|_{L_N^{2,h}(\Omega^h)}^2, \forall W^h \in H_{N,0}^{1,h}(\Omega^h), \forall \lambda \geq \lambda_0. $$

Theorem 4.2 (the central analytical result). The following three assertions hold:

1. The functional $J_\lambda(W^h)$ in (4.11) has the Fréchet derivative $J'_\lambda(W^h) \in H_{N,0}^{1,h}(\Omega^h)$ at any point $W^h \in B(R, P^h)$ and for any value of the parameter $\lambda \geq 0$.

This function satisfies the Lipschitz condition

$$\|J'_\lambda(W^h) - J'_\lambda(W_2^h)\|_{H_{N,0}^{1,h}(\Omega^h)} \leq C_4 \| W_1^h - W_2^h \|_{H_{N,0}^{1,h}(\Omega^h)}, \forall W_1^h, W_2^h \in B(R, P^h) $$

for all $\lambda \geq 0$, where the number $C_4 > 0$ depends on the same parameters as ones in $C_3$ as well as on $\lambda$.

Assume that the number $a \geq a_0$, as in (3.33). Then:
2. There exists a sufficiently large number \( \lambda_1 \)
\begin{equation}
(4.14) \quad \lambda_1 = \lambda_1 (R, A, d, N, a, b, h_0) \geq \lambda_0 \geq 1
\end{equation}
depending only on listed parameters such that the functional \( J_\lambda (W^h) \) in (4.11) is strictly convex on the set \( B(R, P^h) \), i.e. the following inequality holds:
\begin{equation}
(4.15) \quad J_\lambda (W^h_2) - J_\lambda (W^h_1) - J'_\lambda (W^h_1) (W^h_2 - W^h_1) \geq C_3 e^{2\lambda a} \| W^h_2 - W^h_1 \|^2_{H_{1,h}^N(\Omega^h)},
\end{equation}
\begin{equation}
(4.16) \quad \forall \lambda \geq \lambda_1, \forall W^h_1, W^h_2 \in B(R, P^h).
\end{equation}

3. For each \( \lambda \geq \lambda_1 \) there exists a unique minimizer \( W^h_{\min, \lambda} \) of \( J_\lambda (W^h) \) on the set \( B(R, P^h) \). Furthermore, the following inequality holds:
\begin{equation}
(4.17) \quad J'_\lambda (W^h_{\min, \lambda}) (W^h - W^h_{\min, \lambda}) \geq 0, \quad \forall W^h \in B(R, P^h).
\end{equation}

Theorem 4.3 follows immediately from (4.15) and (4.16). This is a certain uniqueness result for our CIP, which is obtained as a by-product. A further discussion of the uniqueness issue is outside of the scope of this paper.

**Theorem 4.3.** Assume that the number \( a \geq a_0 \), as in (4.33). Then there exists at most one pair of functions \((W^h, a^h) \) in \( H^1_N(\Omega^h) \times L^2_N(\Omega^h) \) satisfying conditions (4.19)-(4.23).

We now estimate the accuracy of the minimizer \( W^h_{\min, \lambda} \) depending on the level of the noise \( \delta > 0 \) in the data. Following the concept of Tikhonov for ill-posed problems [33], we assume the existence of the exact solution
\begin{equation}
(4.18) \quad W^{h*} \in B(R, P^{h*})
\end{equation}
of problem (4.30)-(4.31) with the exact, i.e. noiseless data \( P^{h*} \), i.e. for \( x^h \in \Omega^h \)
\begin{equation}
(4.19) \quad D^h_N(x^h) W^{h*}_{g^h}(x^h) + \sum_{i=1}^n A^h_i(x^h) W^{h*}_{x_i}(x^h) + F^h(W^{h*}(x^h), x^h) = 0,
\end{equation}
\begin{equation}
(4.20) \quad W^{h*}(x^h) |_{\partial \Omega^h} = P^{h*}(x^h).
\end{equation}

Suppose that there exists a vector function \( S^h \in H^1_N(\Omega^h) \) such that
\begin{equation}
(4.21) \quad S^h(x^h) |_{\partial \Omega^h} = P^h(x^h), \quad \| S^h \|_{H^1_N(\Omega^h)} < R.
\end{equation}
Let \( S^{h*} \in H^1_N(\Omega^h) \) be such a vector function that
\begin{equation}
(4.22) \quad S^{h*}(x^h) |_{\partial \Omega^h} = P^{h*}(x^h), \quad \| S^{h*} \|_{H^1_N(\Omega^h)} < R.
\end{equation}
We assume that
\begin{equation}
(4.23) \quad \| S^h - S^{h*} \|_{H^1_N(\Omega^h)} < \delta.
\end{equation}

**Theorem 4.4.** Assume that the number \( a \geq a_0 \), as in (4.33). Suppose that conditions (4.19)-(4.23) hold. Also, consider the number \( \lambda_2 \),
\begin{equation}
(4.24) \quad \lambda_2 = \lambda_2 (2R, A, d, N, a, b, h_0),
\end{equation}
where \( \lambda_1 (R, A, d, N, a, b, h_0) \) is the number in (4.14). Let \( W^h_{\min, \lambda_2} \) be the minimizer of functional (4.11) on the set \( B(R, P^h) \), which was found in Theorem 4.2. Let \( \alpha \in (0, R) \) be a number. Suppose that (4.15) is replaced with
\begin{equation}
(4.25) \quad W^{h*} \in B(R - \alpha, P^{h*}) \quad \text{and} \quad C_3 \delta < \alpha.
\end{equation}
Then the vector function \( W^h_{\min,\lambda_2} \) belongs to the open set \( B\left(R, P^h\right) \) and the following accuracy estimate holds:

\[
\| W^h_{\min,\lambda_2} - W^{h*} \|_{H^1_N(\Omega^h)} \leq C_3 \delta.
\]

Consider now the gradient descent method of the minimization of functional \( J(\x,\n) \) on the set \( B\left(R, P^h\right) \). Let \( W^n_0 \in B\left(R/3, P^h\right) \) be an arbitrary point of this set. We treat it as the starting point of the latter method. The sequence of this method is:

\[
W^n = W^{n-1} - \gamma J'_\lambda \left( W^{n-1} \right), \quad n = 1, 2, \ldots,
\]

where \( \gamma > 0 \) is a small number and \( \lambda \) is the same as in (4.17). Note that since by Theorem 4.2 functions \( J'_\lambda \left( W^{n-1} \right) \in H^1(\Omega^h) \), then all vector functions \( W^n_0 \) have the same boundary conditions \( P^h \).

**Theorem 4.5.** Let conditions of Theorem 4.4 hold, except that (4.25) is replaced with

\[
W^{h*} \in B\left((R - \alpha)/3, P^{h*}\right) \quad \text{and} \quad C_3 \delta < \alpha/3.
\]

Then there exists a sufficiently small number \( \gamma > 0 \) and a number \( \theta = \theta(\gamma) \in (0, 1) \) such that in (4.27) all functions \( W^n_0 \in B\left(R, P^h\right) \) and the following convergence estimates hold:

\[
\| W^n - W^h_{\min,\lambda_2} \|_{H^1_N(\Omega^h)} \leq \theta^n \| W^h_0 - W^h_{\min,\lambda_2} \|_{H^1_N(\Omega^h)},
\]

\[
\| W^n - W^{h*} \|_{H^1_N(\Omega^h)} \leq C_3 \delta + \theta^n \| W^h_0 - W^h_{\min,\lambda_2} \|_{H^1_N(\Omega^h)},
\]

\[
\| a^n_\theta - a^{h*} \|_{L^2_N(\Omega^h)} \leq C_3 \delta + \theta^n \| W^h_0 - W^h_{\min,\lambda_2} \|_{H^1_N(\Omega^h)},
\]

where \( a^n(\x^h) \) and \( a^{h*}(\x^h) \) are functions which are obtained from \( W^n_0 \) and \( W^{h*} \) respectively via (3.35) and (3.37).

**Remarks 4.1:**

1. Estimates (4.29)–(4.31) guarantee the global convergence of the gradient descent method (4.27) since \( R > 0 \) is an arbitrary number and the starting point \( W^h_0 \) is an arbitrary point of the set \( B\left(R/3, P^h\right) \), see section 1 for our definition of the global convergence. Note that any gradient-like method converges only locally for a non-convex functional, i.e. it needs a good first guess about the correct solution.

2. Although the above results are valid only for sufficiently large values of the parameter \( \lambda \), we have established computationally in section 6 that \( \lambda = 5 \) works quite well. It was computationally established in a number of works on the convexification for a variety of CIPs that values \( \lambda \in [1, 3] \) work well numerically, see, e.g. [26, Chapters 7-10], [27] and references cited therein.

3. An analogy to the issue of item 2 is that any asymptotic theory basically states that if a parameter \( X \) is sufficiently large/small, then a certain ‘nice’ formula provides a good approximation for something. In practice, however, given a specific problem with specific values of its parameters, only numerical experiments can clarify which exactly values of \( X \) are so sufficiently large/small that this ‘nice’ formula indeed provides that good approximation.
5. Proofs

We use in this section (3.26)-(3.28) and Remark 3.3 without further mentioning.

5.1. Proof of Theorem 4.1. In this proof, \( W^h \in H_{N,0}^{1,h}(\Omega^h) \) is an arbitrary function. It follows from (3.26) and (3.28) that

\[
\|W^h e^{\lambda y}\|^2_{L_N^2,\lambda^h} \leq C_3 \|W^h e^{\lambda y}\|^2_{L_N^2,\lambda^h}, \quad \forall W^h \in H_{N,0}^{1,h}(\Omega^h).
\]

(5.1)

By (5.1) and (5.2),

\[
\left\| \left( \sum_{i=1}^{n} C_i^h (x^h) W^h_{x_i} (x^h) \right) e^{\lambda y} \right\|^2_{L_N^2,\lambda^h} \leq C_3 \|W^h e^{\lambda y}\|.
\]

Using Cauchy-Schwarz inequality and (5.2), we obtain

\[
\left\| \left( W^h_y (x^h) + \sum_{i=1}^{n} C_i^h (x^h) W^h_{x_i} (x^h) \right) e^{\lambda y} \right\|^2_{L_N^2,\lambda^h} \geq C_{3,1} \|W^h e^{\lambda y}\|^2_{L_N^2,\lambda^h} - C_{3,2} \|W^h e^{\lambda y}\|^2_{L_N^2,\lambda^h}.
\]

(5.3)

It makes sense to use two numbers \( C_{3,1}, C_{3,2} > 0 \) here, both depend on the same parameters as ones in \( C_3 \). Consider now the first term in the second line of (5.3). Introduce a new vector function \( V^h (x^h) = W^h (x^h) e^{\lambda y} \). Then

\[
(V^h_y (x^h) - \lambda V^h (x^h)) e^{-\lambda y}, \quad (W^h_y (x^h))^2 e^{2\lambda y} = (V^h_y (x^h) - \lambda V^h (x^h))^2 \geq -2\lambda V^h (x^h) V^h (x^h) + \lambda^2 (V^h (x^h))^2 = \partial_y (\lambda V^h (x^h))^2 + \lambda^2 (V^h (x^h))^2.
\]

Hence,

\[
\|W^h e^{\lambda y}\|^2_{L_N^2,\lambda^h} \geq \sum_{(i,j)=(1,1)}^{(n,m-1)} [ -\lambda (W^h (x^h))^2 |_{y=b} + \lambda (W^h (x^h))^2 |_{y=a} ] + \lambda^2 \|W^h e^{\lambda y}\|^2_{L_N^2,\lambda^h}.
\]

(5.4)

Since \( W^h (x^h) |_{y=b} = 0, \forall W^h \in H_{N,0}^{1,h}(\Omega^h) \), then (5.4) implies

\[
\|W^h e^{\lambda y}\|^2_{L_N^2,\lambda^h} \geq \lambda^2 \|W^h e^{\lambda y}\|^2_{L_N^2,\lambda^h}.
\]

(5.5)

Adding the term \( \|W^h e^{\lambda y}\|^2_{L_N^2,\lambda^h} \) to both sides of (5.5) and then dividing by "2", we obtain for all \( \lambda > 0 \)

\[
\|W^h e^{\lambda y}\|^2_{L_N^2,\lambda^h} \geq \frac{1}{2} \|W^h e^{\lambda y}\|^2_{L_N^2,\lambda^h} + \frac{\lambda^2}{2} \|W^h e^{\lambda y}\|^2_{L_N^2,\lambda^h}.
\]

(5.6)

Hence, taking \( \lambda_0^2 \geq 4C_{3,2}/C_{3,1}, \lambda \geq \lambda_0 \) and using (3.3) and (5.6), we obtain the target estimate (4.12) of this theorem. □
5.2. **Proof of Theorem 4.2.** Let $W^h_1 (x^h), W^h_2 (x^h) \in \overline{B(R, P^h)}$ be two arbitrary points. Denote $v^h (x^h) = W^h_2 (x^h) - W^h_1 (x^h)$. By (4.11)
\[(5.7)\]
$v^h \in H^{1, h}_{N, 0} (\Omega^h)$.
By (4.11)
\[(5.8)\]
\[J_\lambda (W^h_2) = J_\lambda (W^h_1 + v^h) = \sum_{(i,j)=(n,m-1)} \times \int_a^b \left( D_N^h (W^h_{1y} + v^h_{1y}) + \sum_{i=1}^n A^h_i (W^h_{1x_i} + v^h_{x_i}) + F^h (W^h + v^h, x^h) \right)^2 e^{2\lambda y} dy. \]
Consider the integrand in the second line of (5.8) without, however, the term $e^{2\lambda y}$. By (4.2), (4.3), (4.9) and (4.10)
\[(5.9)\]
\[\left( D_N^h (W^h_{1y} + v^h_{1y}) + \sum_{i=1}^n A^h_i (x^h) (W^h_{1x_i} + v^h_{x_i}) + F^h (W^h + v^h, x^h) \right)^2 \]
\[= \left[ D_N^h \left( (W^h_{1y} + v^h_{1y}) + \sum_{i=1}^n G^h_i (W^h_{1x_i} + v^h_{x_i}) + \Phi^h (W^h_1 + v^h, x^h) \right) \right]^2. \]
Consider the term $\Phi^h (W^h_1 + v^h, x^h)$. By (4.6)-(4.8) we have for all $x^h \in \Omega^h$
\[(5.10)\]
\[\Phi^h (W^h_1 + v^h, x^h) = \Phi^h (W^h_1, x^h) + \Phi_1 (W^h_1, x^h) v^h (x^h) + \Phi_2 (W^h_1, W^h_1 + v^h, x^h), \]
\[(5.11)\]
\[|\Phi_1 (W^h_1, x^h)| \leq C_3, \ |\Phi_2 (W^h_1, W^h_1 + v^h, x^h)| \leq C_3 (v^h (x^h))^2. \]
It follows from (5.10) and (5.11) that the second line of (5.9) is:
\[(5.12)\]
\[\left[ D_N^h \left( (W^h_{1y} + v^h_{1y}) + \sum_{i=1}^n G^h_i (W^h_{1x_i} + v^h_{x_i}) + \Phi^h (W^h_1 + v^h, x^h) \right) \right]^2 \]
\[= \left[ D_N^h \left( W^h_{1y} + \sum_{i=1}^n G^h_i W^h_{1x_i} + \Phi^h (W^h_1, x^h) \right) \right]^2 \]
\[+ 2 D_N^h \left( W^h_{1y} + \sum_{i=1}^n G^h_i W^h_{1x_i} + \Phi^h (W^h_1, x^h) \right) \]
\[\cdot D_N^h \left( v^h_{1y} + \sum_{i=1}^n G^h_i v^h_{x_i} + \Phi_1 (W^h_1, x^h) v^h (x^h) \right) \]
\[+ 2 D_N^h \left( W^h_{1y} + \sum_{i=1}^n G^h_i W^h_{1x_i} + \Phi^h (W^h_1, x^h) \right) \cdot (D_N^h \Phi_2 (W^h_1, W^h_1 + v^h, x^h)) \]
\[+ \left[ D_N^h \left( v^h_{1y} + \sum_{i=1}^n G^h_i v^h_{x_i} + \Phi_1 (W^h_1, x^h) v^h (x^h) + \Phi_2^h (W^h_1, W^h_1 + v^h, x^h) \right) \right]^2. \]
The linear with respect to $v^h$ term in (5.12) is the scalar product of the third and fourth lines of this equality. We denote the latter as $Lin \left( v^h (x^h), x^h \right)$. The sum of fifth and sixth lines of (5.12) contains only nonlinear terms with respect to $v^h$. 

**CONVEXIFICATION FOR A CIP FOR THE RTE**
Thus, Just as in the proof of Theorem 4.1, it again makes sense to use two positive
(5.14) 
\[ \| Nonlin (v^h (x^h), x^h) \| \geq C_{3,3} \left( \varepsilon_y^h + \sum_{i=1}^{n} G_i^h e_x^h \right)^2 - C_{3,4} (v^h (x^h))^2, \forall x^h \in \Omega^h. \]

Just as in the proof of Theorem 4.1, it again makes sense to use two positive constants \( C_{3,3}, C_{3,4} > 0 \) here, both depend on the same parameters as ones involved in \( C_3 \). Taking into account (5.8), (5.9) and the second line of (5.12), we obtain

\[ \int_\Omega Nonlin (v^h (x^h), x^h) e^{2\lambda_y} dy. \]

It follows from (4.4)-(4.6), (5.7) and from the third and fourth lines of (5.12), which form \( Lin (v^h (x^h), x^h) \), that the first term in the second line of (5.14) is a bounded linear functional with respect to \( v^h \) mapping \( H_{N,0}^{1,h} (\Omega^h) \) in \( \mathbb{R} \). We denote this functional as \( \tilde{J}_\lambda (W^h) \) by Riesz theorem there exists unique function \( J'_\lambda (W^h) \in H_{N,0}^{1,h} (\Omega^h) \) such that

\[ \tilde{J}_\lambda (W^h) = \left( J'_\lambda (W^h), v^h \right), \forall v^h \in H_{N,0}^{1,h} (\Omega^h), \]

where \( (, ) \) is the scalar product in \( H_{N,0}^{1,h} (\Omega^h) \). It follows from (5.3), (5.5)-(5.7) and the fifth and sixth lines of (5.12) combined with (5.11) that for \( \| v^h \|_{H_{N,0}^{1,h} (\Omega^h)} < 1 \)

\[ \left| \sum_{(i,j) = (n,m-1)}^{b} \int_a Nonlin (v^h (x^h), x^h) e^{2\lambda_y} dy \right| \leq C_{3} e^{2\lambda_y} \| v^h \|^2_{H_{N,0}^{1,h} (\Omega^h)}. \]

Hence, (5.14) implies that

\[ J'_\lambda (W^h) = O \left( \| v^h \|_{H_{N,0}^{1,h} (\Omega^h)} \right), \text{ as } \| v^h \|_{H_{N,0}^{1,h} (\Omega^h)} \to 0. \]

Thus, \( J'_\lambda (W^h) : H_{N,0}^{1,h} (\Omega^h) \to \mathbb{R} \) is the Fréchet derivative of the functional \( J_\lambda (W^h) \) at the point \( W^h \). We omit the proof of estimate (4.13) since this proof is completely similar with the proof of Theorem 3.1 of [1].

We now prove (4.15). Using (5.13)-(5.15), we obtain

\[ J_\lambda (W^h_1 + v^h) - J_\lambda (W^h_1) - J'_\lambda (W^h_1), v^h \geq \]

\[ \geq C_{3,3} \left( \varepsilon_y^h + \sum_{i=1}^{n} G_i^h e_x^h \right)^2 e^{2\lambda_y} - C_{3,4} e^{2\lambda_y} \| v^h \|^2_{L_{N,0}^{1,h} (\Omega^h)}. \]

Let the number \( \lambda_0 \) be the one chosen in Theorem 4.1. Recalling (5.7) and using Theorem 4.1, we estimate the second line of (5.10) for all \( \lambda \geq \lambda_0 \),

\[ C_{3,3} \left( \varepsilon_y^h + \sum_{i=1}^{n} G_i^h e_x^h \right)^2 e^{2\lambda_y} - C_{3,4} e^{2\lambda_y} \| v^h \|^2_{L_{N,0}^{1,h} (\Omega^h)} \geq \]
\[ \geq C_3 \left( \left\| v_y e^{\lambda y} \right\|^2_{L^2_\infty(\Omega^h)} + \lambda^2 \left\| v^h e^{\lambda y} \right\|^2_{L^2_\infty(\Omega^h)} \right) - C_{3,4} \left\| v^h e^{\lambda y} \right\|^2_{L^2_\infty(\Omega^h)}. \]

Choosing \( \lambda_1 \geq \lambda_0 \) so large that \( \lambda_1^2 C_3/2 > 2C_{3,4} \) and using (5.17), we obtain

\[ (5.18) \quad C_{3,3} \left( \left\| v_y e^{\lambda y} \right\|^2_{L^2_\infty(\Omega^h)} + \lambda^2 \left\| v^h e^{\lambda y} \right\|^2_{L^2_\infty(\Omega^h)} \right) \]

\[ \geq C_3 \left( \left\| v_y e^{\lambda y} \right\|^2_{L^2_\infty(\Omega^h)} + \lambda^2 \left\| v^h e^{\lambda y} \right\|^2_{L^2_\infty(\Omega^h)} \right). \]

Combining (5.16) with (5.18), we obtain (4.16), (4.18).

Finally, given (4.15) and (4.16), existence and uniqueness of the minimizer \( W_{\min, \lambda} \) of the functional \( J_\lambda (W^h) \) on the set \( B(R, P^h) \) as well as estimate (4.17) follow immediately from a combination of Lemma 2.1 with Theorem 2.1 of [1]. \( \square \)

5.3. **Proof of Theorem 4.4.** Denote

\[ (5.19) \quad B_0 (2R) = \left\{ V^h \in H^{1,h}_{N,0} (\Omega^h) : \left\| V^h \right\|_{H^1_{N,h}(\Omega^h)} < 2R \right\}. \]

Consider the vector functions \( V^h, V^{h*} \)

\[ (5.20) \quad V^h = W^h - S^h, \forall V^h \in B (R, P^h) : V^{h*} = W^{h*} - S^{h*}. \]

By (4.18), (4.21), (4.22), (5.19) and (5.20)

\[ (5.21) \quad V^h, V^{h*} \in B_0 (2R). \]

Consider now the functional \( I_\lambda (V^h) \),

\[ (5.22) \quad I_\lambda (V^h) : B_0 (2R) \rightarrow \mathbb{R}, I_\lambda (V^h) = J_\lambda (V^h + S^h). \]

An obvious analog of Theorem 4.2 holds for \( I_\lambda (V^h) \). However, it follows from (5.21) that we need to replace \( R \) with \( 2R \) in (4.14) in this case, i.e., we need to use now the number \( \lambda_2 \) in (4.14). Let \( V^{h*}_{\min, \lambda_2} \) be the minimizer of \( I_{\lambda_2} (V^h) \) on the set \( B_0 (2R) \).

Consider \( I_{\lambda_2} (V^{h*}) = J_{\lambda_2} (V^{h*} + S^h) \). By (4.16), (4.18) and (5.22)

\[ I_{\lambda_2} (V^{h*}) - I_{\lambda_2} (V^{h*}_{\min, \lambda_2}) - \lambda_2 \left( V^{h*} - V^{h*}_{\min, \lambda_2} \right) \geq C_3 \left\| V^{h*} - V^{h*}_{\min, \lambda_2} \right\|_{H^1_{N,h}(\Omega^h)}. \]

By (4.17) \( -\lambda_2 \left( V^{h*} - V^{h*}_{\min, \lambda_2} \right) \geq 0 \). Since \( -I_{\lambda_2} (V^{h*}_{\min, \lambda_2}) \leq 0 \) as well, then the latter estimate implies

\[ (5.23) \quad \left\| V^{h*} - V^{h*}_{\min, \lambda_2} \right\|_{H^1_{N,h}(\Omega^h)}^2 \leq C_3 I_{\lambda_2} (V^{h*}). \]

Next, by (5.20) and (5.22)

\[ (5.24) \quad I_{\lambda_2} (V^{h*}) = J_{\lambda_2} (W^{h*} + S^h) = J_{\lambda_2} (W^{h*} + (S^h - S^{h*})). \]

By (4.19) the right hand side of (4.11) equals zero if \( W^h \) is replaced with \( W^{h*} \). Hence, using (4.24), (4.28) and (5.20), we obtain \( J_{\lambda_2} (W^{h*} + (S^h - S^{h*})) \leq C_0 \delta^2 \). Hence, (5.23) and (5.24) lead to:

\[ (5.25) \quad \left\| V^{h*} - V^{h*}_{\min, \lambda_2} \right\|_{H^1_{N,h}(\Omega^h)} \leq C_3 \delta. \]

Using again (4.23), (5.20), triangle inequality and (5.25), we obtain

\[ (5.26) \quad \left\| W^{h*} - W^{h*}_{\min, \lambda_2} \right\|_{H^1_{N,h}(\Omega^h)} \leq C_3 \delta. \]
Here $\tilde{W}^{h}_{\min, \lambda_{2}} = W^{h}_{\min, \lambda_{2}} - S^{h}$. By (4.25), (5.20) and the triangle inequality $\|\tilde{W}^{h}_{\min, \lambda_{2}}\|_{H_{N}^{1,h}(\Omega^{h})} < R$. Hence, (4.1), (4.21), (5.19) and (5.21) imply that

$$\tilde{W}^{h}_{\min, \lambda_{2}} \in B \left( R, P^{h} \right).$$

Now, let $W^{h}_{\min, \lambda_{2}}$ be the minimizer of the functional $J_{\lambda_{2}} \left( W^{h} \right)$ on the set $B \left( R, P^{h} \right)$, which is claimed by Theorem 4.2. Let $Q^{h}_{\min, \lambda_{2}} = W^{h}_{\min, \lambda_{2}} - S^{h}$. Then $Q^{h}_{\min, \lambda_{2}} \in B_{0} \left( 2R \right)$. Hence,

$$J_{\lambda_{2}} \left( \tilde{W}^{h}_{\min, \lambda_{2}} \right) = J_{\lambda_{2}} \left( W^{h}_{\min, \lambda_{2}} + G \right) \leq J_{\lambda_{2}} \left( Q^{h}_{\min, \lambda_{2}} + G \right) = \min_{B \left( R, P^{h} \right)} J_{\lambda_{2}} \left( W^{h} \right).$$

Hence, by (5.27) $\tilde{W}^{h}_{\min, \lambda_{2}}$ is also a minimizer of the functional $J_{\lambda_{2}} \left( W^{h} \right)$ on the set $B \left( R, P^{h} \right)$. However, since such a minimizer is unique by Theorem 4.2, then $\tilde{W}^{h}_{\min, \lambda_{2}} = W^{h}_{\min, \lambda_{2}} \in B \left( R, P^{h} \right)$. Hence, estimate (5.26) holds when $\tilde{W}^{h}_{\min, \lambda_{2}}$ is replaced with $W^{h}_{\min, \lambda_{2}}$. The latter immediately implies (4.20). □

5.4. **Proof of Theorem 4.5.** Consider again the minimizer $W^{h}_{\min, \lambda_{2}}$ of the functional $J_{\lambda_{2}} \left( W^{h} \right)$ on the set $B \left( R, P^{h} \right)$. Then (4.28) and Theorem 4.4 imply that $W^{h}_{\min, \lambda_{2}} \in B \left( R/3, P^{h} \right)$. To prove (4.29), we now modify the proof of Theorem 6 of [25]: to adapt it for our specific case.

We set $\lambda = \lambda_{2}$, where $\lambda = \lambda_{2}$ is defined in (4.24). It obviously follows from (4.15) that

$$\left( J'_{\lambda_{2}} \left( W^{h}_{1} \right) - J'_{\lambda_{2}} \left( W^{h}_{2} \right) \right) \left( W^{h}_{1} - W^{h}_{2} \right) \geq C_{3} \left\| W^{h}_{1} - W^{h}_{2} \right\|_{H_{N}^{1,h}(\Omega^{h})}^{2}$$

for all $W^{h}_{1}, W^{h}_{2} \in B \left( R, P^{h} \right)$. Consider the operator $T \left( W^{h} \right) : B \left( R, P^{h} \right) \rightarrow H_{N}^{1,h}(\Omega^{h})$,

$$T \left( W^{h} \right) = W^{h} - \gamma J'_{\lambda_{2}} \left( W^{h} \right).$$

Using (4.13) and (5.28), a contraction property of the operator $X \left( W^{h} \right)$, it was proven in [1, Theorem 2.1] and in [26, page 97] that there exists a sufficiently small number $\gamma > 0$ and a number $\theta = \theta \left( \gamma \right) \in (0, 1)$ such that

$$\left\| T \left( W^{h} \right) - T \left( W^{h}_{2} \right) \right\|_{H_{N}^{1,h}(\Omega^{h})} \leq \theta \left\| W^{h}_{1} - W^{h}_{2} \right\|_{H_{N}^{1,h}(\Omega^{h})}$$

for all $W^{h}_{1}, W^{h}_{2} \in B \left( R, P^{h} \right)$. However, it was not proven in [1, 26] that the operator $T$ maps $B \left( R, P^{h} \right)$ in itself.

Since $W^{h}_{\min, \lambda_{2}}$ is an interior point of the set $B \left( R, P^{h} \right)$, then $J'_{\lambda_{2}} \left( W^{h}_{\min, \lambda_{2}} \right) = 0$. Hence, by (5.29)

$$W^{h}_{\min, \lambda_{2}} = T \left( W^{h}_{\min, \lambda_{2}} \right).$$

Consider now the vector function $W^{h}_{1} = T \left( W^{h}_{0} \right) = W^{h}_{0} - \gamma J'_{\lambda_{2}} \left( W^{h}_{0} \right)$. Since by Theorem 4.2 the function $J'_{\lambda_{2}} \left( W^{h}_{0} \right) \in H_{N}^{1,h}(\Omega^{h})$, then $W^{h}_{1} \mid_{\partial \Omega^{h}} = W^{h}_{0} \mid_{\partial \Omega^{h}} = P^{h}$. Next by (5.30), (5.31) and triangle inequality

$$\left\| W^{h}_{1} - W^{h}_{\min, \lambda_{2}} \right\|_{H_{N}^{1,h}(\Omega^{h})} = \left\| T \left( W^{h}_{0} \right) - T \left( W^{h}_{\min, \lambda_{2}} \right) \right\|_{H_{N}^{1,h}(\Omega^{h})} \leq \theta \left\| W^{h}_{0} - W^{h}_{\min, \lambda_{2}} \right\|_{H_{N}^{1,h}(\Omega^{h})} \leq \frac{2R}{3} < \frac{2R}{3}.
Hence, (1.25), (1.26) and triangle inequality imply \( \|W_1^h\|_{H^1_N(\Omega^h)} < R \). Hence, by (4.1) \( W_1^h \in B(R, P^h) \). Similarly by (5.32)

\[
\|W_2^h - W_{\text{min}, \lambda_2}^h\|_{H^1_N(\Omega^h)} \leq \theta \|W_1^h - W_{\text{min}, \lambda_2}^h\|_{H^1_N(\Omega^h)} \leq \frac{2R}{3}.
\]

Hence, applying again triangle inequality, we obtain \( W_2^h \in B(R, P^h) \). Continuing this process, we obtain (4.20).

To prove (4.30), we note that by triangle inequality, (4.20) and (4.29)

\[
\|W_n^h - W_{\text{min}, \lambda_2}^h\|_{H^1_N(\Omega^h)} \leq \|W_{\text{min}, \lambda_2}^h - W_{\text{min}, \lambda_2}^h\|_{H^1_N(\Omega^h)} + \|W_n^h - W_{\text{min}, \lambda_2}^h\|_{H^1_N(\Omega^h)} \leq \frac{2R}{3}.
\]

The proof of estimate (1.31) follows immediately from (3.35)- (3.37) and (1.30). □

6. Numerical Studies

In order not to introduce new and complicated notations, we slightly abuse below some notations of the previous sections. Nevertheless, the substance is always clear from the context presented below.

6.1. Numerical implementation. We have conducted our numerical studies in the 2-D case. Below \( x = (x, y) \) and, according to (2.2) and (2.6)

(6.1)

\[
\Omega = \{x: x \in (-A, A), y \in (a, b), A = 1/2, a = 1, b = 2, \}
\]

\[
\Gamma_d = \{x_\alpha = (\alpha, 0): \alpha \in [-d, d]\}, d = 1/2.
\]

As to the kernel \( K(x, \alpha, \beta) \) of the integral operator in (2.10), we work below with the 2D Heney-Greenstein function [17]:

(6.2)

\[
K(x, \alpha, \beta) = H(\alpha, \beta) = \frac{1}{2d} \left[ \frac{1 - g^2}{1 + g^2 - 2g \cos(\alpha - \beta)} \right], \quad g = \frac{1}{2}.
\]

Here, \( g = 1/2 \) means an anisotropic scattering, which is half ballistic with \( g = 0 \) an half isotropic scattering with \( g = 1 \) [11] [12] [13]. We take the same function \( f(x) \) as one in (2.4), (2.8) with \( \varepsilon = 0.05 \).

We assume that

(6.3)

\[
\mu_s(x) = \begin{cases} 5, & x \in \Omega, \\ 0, & x \in \mathbb{R}^2 \setminus \Omega. \end{cases}
\]

Given (6.3), we use formula (2.13) for the coefficient function \( a(x) \) and we take in this formula

(6.4)

\[
\mu_a(x) = \begin{cases} c = \text{const.} > 0, & \text{inside the tested inclusion}, \\ 0, & \text{outside the tested inclusion}. \end{cases}
\]

We perform the numerical tests with a variety of the value of the parameter \( c = 5, 10, 15, 20, 30 \), see below. Therefore, by (6.3) and (6.4)

(6.5)

\[
\text{inclusion/background contrast} = 1 + \frac{c}{5}.
\]

From the Physics standpoint \( \mu_s(x) = 5 \) in the domain \( \Omega \) means that an average particle scatters every 1/5 of the unit. Hence, by (6.1) and (6.3) the maximal average number of scattering events for a particle emitted from a point \( x_n \in \Gamma_d \) and entering \( \Omega \) is around 5 before this particle leaves \( \Omega \) [11] [12] [13]. This might
happen in optics before the true diffusion occurs, e.g. in the case of the so-called “snake photons” [10].

In computational results below \( c \neq \text{const.} \) inside of computed shapes of letters. Hence, we set for computed inclusion/background contrasts we replace (6.5) with:

\[
\text{(6.6) computed inclusion/background contrast} = 1 + \frac{1}{5} \max_{\Omega} \mu_a(x).
\]

To solve the forward problem (2.10), (2.18), we have solved integral equation (2.36) with the function \( u_0 \) taken from (2.34). To do this, we have used the discrete form of (2.34), (2.36) and the trapezoidal rule. The discretization steps with respect to \( x, y, \alpha \) where \( h_x = h_y = h_\alpha = 1/40 \). The discretized integral equation (2.36) was solved as a linear system using the Matlab function ‘\('. Thus, the solution of this forward problem has provided computationally simulated data for the inverse problem to us.

To minimize the convexification functional \( J_\lambda (W^h) \) in (4.11), we have written in the finite differences form not only the \( x \)-derivative as in section 4 but the \( y \)-derivative as well. Also, integrals with respect to \( \alpha \) were written in the discrete form using the trapezoidal rule. The mesh sizes were different from ones for the forward problem. They were:

\[
\text{(6.7) } h_x = h_y = h_\alpha = h = 1/20.
\]

Then we have minimized the resulting functional \( J_{\lambda, \text{dis}} (W^h) \),

\[
\text{(6.8) } J_{\lambda, \text{dis}} (W^h) = \| (D^h y W^h + A^h W^h + F^h (W^h (x^h), x^h)) e^{\lambda y} \|_{L^2_N(\Omega^h)}^2
\]

in its fully discrete form with respect to the values of the vector function \( W^h \) at the grid points. Vector functions and matrices in (6.8) are full analogs of those in (4.11) with the only difference that they are fully discrete in the above sense, rather than ‘partially’ discrete as in sections 4,5. The same is true for the norm \( \| \cdot \|^2_{L^2_N(\Omega^h)} \).

It follows from (6.1) and (6.7) that we had total \( 20 \times 20 \times N \) unknown parameters in our minimization procedure. To solve the minimization problem, we have used the Matlab’s built-in function ‘\text{fminunc}’ with the quasi-newton algorithm. The iterations of the function ‘\text{fminunc}’ were stopping when the following inequality occurred at the iteration number \( k \):

\[
|J_{\lambda, \text{dis}} (W^h_k)| < 10^{-2}.
\]

By (3.14) our technique requires computations of first derivatives with respect to \( \alpha \). Note, however, that (3.8) and (3.12) imply that it is not necessary to calculate the \( \alpha \)-derivative of the boundary data, which is an advantage, since boundary data are noisy. The derivatives \( \partial_\alpha \) of functions \( \Psi_s (\alpha) \) and the function \( K(x, \alpha, \beta) \) were calculated via finite differences.

We have introduced the random noise in the boundary data \( g_1(x, \alpha) \) in (3.4) on the boundary \( \partial \Omega \),

\[
\text{(6.9) } g_1(x, \alpha) = g_1(x, \alpha) (1 + \sigma \zeta_x).
\]

Here \( \zeta_x \) is the uniformly distributed random variable in the interval \([0, 1]\) depending on the point \( x \in \partial \Omega \) with \( \sigma = 0.03 \) and \( \sigma = 0.05 \), which correspond respectively to 3% and 5% noise level.

To solve the minimization problem, we need to provide the starting \( W^h_0(x^h) \) for iterations. Due to the global convergence property of our method, the vector
function $W_h^0(x^h) = (w_h^{1,0}(x^h), \ldots, w_h^{N-1,0}(x^h))^T$ should not have any information about the exact solution $W^{h*}(x^h)$. On the other hand, due to (4.1), we should have $W_h^0(x^h) |_{\partial\Omega} = P_h(x^h)$. Therefore, in all numerical tests below we choose the starting point as the discrete version of the following vector function:

$$w_{s,0}(x, y) = \frac{1}{2} \left( \frac{(A - x)}{2A} w_s(-A, y) + \frac{(x + A)}{2A} w_s(A, y) \right) + \frac{1}{2} \left( \frac{(b - y)}{a - b} w_s(x, a) + \frac{(y - a)}{b - a} w_s(x, b) \right), s = 0, \ldots, N - 1. \tag{6.10}$$

Expression (6.10) represents the average of linear interpolations inside of the square $\Omega$ with respect to $x$–direction and $y$–direction of the boundary condition for $w_s(x, y)$.

6.2. Numerical results. Recall that we reconstruct the coefficient $a(x) = \mu_s(x) + \mu_a(x)$, where functions $\mu_s(x)$ and $\mu_a(x)$ are given in (6.3) and (6.4) respectively. Our results for Tests 1-4 are for noiseless data and the results for Test 5 are for noisy data as in (6.9).

To demonstrate a good performance of our technique, we intentionally test it for rather complicated shapes of inclusions, which are non convex and have voids. More precisely, our inclusions are letters ‘A’, ‘Ω’ and two letters jointly ‘SZ’. ‘SZ’ stands for ‘Shenzhen’, the city where the campus of the Southern University of Science and Technology, the workplace of the third and fourth authors, is located. Thus, in our tests the coefficients $\mu_a(x)$ in (6.4) have shapes of those letters located inside of the $1 \times 1$ square $\Omega$ defined in (6.1).

Test 1. We test the letter ‘A’ with $c = 5$ in (6.4). This is our reference case. More precisely, we use this test to figure out optimal values of parameters $N$ and $\lambda$. As soon as optimal parameters are selected, we use them then for all other tests.

First, we select $N$. To do this, we solve the forward problem (2.10), (2.18) for the case when the functions $\mu_s(x)$ and (6.3) and (6.4) respectively and in $c = 5$. Hence, by (6.5) the inclusion/background contrast is 2:1 in this case. Next, we calculate norms $\|w_s(x)\|_{L^2(\Omega)}$ and compare them. We have observed that the $L_2(\Omega)$−norm of the function $w_s(x)$ decreases very rapidly when the number $s$ is growing. More precisely, we have obtained that

$$\sum_{s=3}^{11} \frac{\|w_s(x)\|_{L^2(\Omega)}}{\sum_{s=0}^{11} \|w_s(x)\|_{L^2(\Omega)}} = 0.0084,$$

which means less than 1%. The values of norms $\|w_s(x)\|_{L^2(\Omega)}$ for $s = 0, \ldots, 11$ are displayed in Table 1 and Figure 1. One can observe that starting from $s = 3$, these norms are much less than those for $s = 0, 1, 2$. We conclude therefore, that we should take in our tests

$$N = 3. \tag{6.12}$$

Next, given the optimal value of $N = 3$ in (6.12), we select the optimal value of the parameter $\lambda$ of the Carleman Weight Function $e^{2\lambda y}$ in (6.8). To do this, we test the same letter ‘A’ with $c = 5$ inside of it for values of the parameter
Table 1. The $L_2(\Omega)$–norms of functions $w_s(x)$, $s = 0, 1, ..., 11$ for the reference Test 1 with $c = 5$ in (6.4).

| $s$ | 0   | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|-----|
| $\|w_s(x)\|_{L_2}$ | 5.7122 | 1.6383 | 0.1630 | 0.0118 | 0.0091 | 0.0077 |

Figure 1. The decrease with respect to $s$ of the $L_2(\Omega)$–norms of functions $w_s(x)$, $s = 0, 1, ..., 11$ for the reference Test 1 with $c = 5$ in (6.4).

\[ \lambda = 0, 1, 2, 3, 4, 5, 6, 7. \]

Our numerical results are presented on Figure 2. We observe that the images have a low quality for $\lambda = 0, 1$. Then the quality is improved and is stabilized at $\lambda = 5$. Thus, we treat $\lambda = 5$ as the optimal value of this parameter. We use this value in all subsequent tests. The value $\lambda = 5$ tells us that even though our theorems 4.1–4.5 require sufficiently large values of the parameter $\lambda$, the computational practice shows that a reasonable value of $\lambda$ can be chosen. The same observation was made in all previous works on the convexification of this research group, see items 2 and 3 in Remarks 4.1 in the end of section 4.

Now we want to demonstrate numerically again that $N = 3$ is indeed a good choice of $N$ for our optimal value of $\lambda = 5$. Taking $\lambda = 5$, we test the same letter ‘A’ as above with $c = 5$ in it, but for $N = 1, 2, 3, 5, 7, 12$. The results are displayed in Figure 3. One can observe that reconstructions have a low quality for $N = 1, 2$. Next, the reconstructions are basically the same for $N = 3, 5, 7, 12$. However, the computational cost increases very rapidly with the increase of $N$. Thus, using also Table 1, Figure 1 and (6.12), we conclude that to balance between the reconstruction accuracy and the computational cost, we should use $N = 3$. Thus, in all subsequent computations we use

$N = 3, \lambda = 5.$

Test 2. We test the reconstruction of the coefficient $a(x)$ with the shape of the letter ‘A’ where the function $\mu_a(x)$ is given in (6.4) with different values of the parameter $c = 15, 20, 30$ inside of the letter ‘A’. Thus, by (6.6) the inclusion/background contrasts now are respectively $4 : 1, 5 : 1$ and $6 : 1$. Our computational results for this test are displayed on Figure 4. One can observe that the quality of these images is good for all four cases, although it slightly deteriorates.
Figure 2. The reconstructed coefficient $a(x)$, where the function $\mu_a(x)$ is given in (6.4) with $c = 5$ inside of the letter ‘A’. The goal of here is to test different values of the parameter $\lambda = 0, 1, 2, 3, 4, 5, 6, 7$ for $N = 3$ as in (6.12). The value of $\lambda$ can be seen on the top side of each square. The images have a low quality for $\lambda = 0, 1$. Then the quality is improved and is stabilized at $\lambda = 5$. Thus, we select $\lambda = 5$ as an optimal value of this parameter for all follow up tests.

for $c = 20$ and $c = 30$. The computed inclusion/background contrast is accurate, see (6.6).

Test 3. We test the reconstruction of the coefficient $a(x)$ with the shape of the letter ‘Ω’ where the function $\mu_a(x)$ is given in (6.4) with $c = 5$ inside of the letter ‘Ω’. Results are presented on Figure 5. We again observe an accurate reconstruction.

Test 4. We test the reconstruction of the coefficient $a(x)$ with the shape of two letters ‘SZ’ where the function $\mu_a(x)$ is given in (6.4) with $c = 5$ inside of each of these two letters and $\mu_a(x) = 0$ outside of each of these two letters. In this test, $N = 3, \lambda = 5$ as in (6.13). Results are presented on Figure 6. The image quality is lower than one for the case of the single letter ‘Ω’ on Figure 5. Nevertheless, the quality is still good and the computed inclusion/background contrasts (6.6) are accurate in both letters.

Test 5. In this test we use noisy data as in (6.9) with $\sigma = 0.03$ and $\sigma = 0.05$, i.e. with 3% and 5% noise level. We test the reconstruction of the coefficient $a(x)$ with the shape of either the letter ‘A’ or the letter ‘Ω’, where the function $\mu_a(x)$ is given in (6.4) with $c = 5$ inside of each of these two letters. Again, $N = 3, \lambda = 5$ as in (6.13). The results are shown in Figure 7. One can observe accurate
Figure 3. The reconstructed coefficient $a(x)$, where the function $\mu_a(x)$ is given in (6.3) with $c = 5$ inside of the letter ‘A’. We took the optimal value of the parameter $\lambda = 5$ (see Figure 2) and have tested different values of the parameter $N = 1, 2, 3, 5, 7, 12$. A low quality can be observed for $N = 1, 2$. The reconstructions are basically the same for $N = 3, 5, 7, 12$. However, the computational cost increases very rapidly with the increase of $N$, which is explained by (6.11), Table 1 and Figure 1. We conclude, therefore, that to balance between the reconstruction accuracy and the computational cost, we should use $N = 3$ as in (6.12). Thus, we use below $\lambda = 5$ and $N = 3$.

reconstructions in all four cases. In particular, the inclusion/background contrasts (6.10) are reconstructed accurately.

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Figure 4. Test 2. Exact (left) and reconstructed (right) coefficient $a(x)$ for $c = 10, 15, 20, 30$ inside of the letter ‘A’ as in (6.4) for $N = 3, \lambda = 5$, see (6.13). Thus, by (6.6) the inclusion/background contrasts now are respectively $4 : 1$, $5 : 1$ and $6 : 1$. The image quality remains basically the same for all these values of the parameter $c$, although some deterioration of this quality can be observed for $c = 20$ and $c = 30$. The computed inclusion/background contrasts (6.6) are accurate.

Figure 5. Test 3. Exact (left) and reconstructed (right) coefficient $a(x)$ for the case when the function $\mu_a(x)$ is given in (6.4) with $c = 5$ inside of the letter ‘Ω’. The reconstruction is accurate.

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Figure 6. Test 4. Exact (left) and reconstructed (right) coefficient $a(x)$ for the case when the function $\mu_0(x)$ is given in (6.4) with $c = 5$ with the shape of two letters ‘SZ’. In (6.4) $c = 5$ inside of each of these two letters and $\mu_0(x) = 0$ outside of each of these two letters. Here $N = 3, \lambda = 5$ as in (6.13). The image quality is lower than one for the case of the single letter $\Omega$ on Figure 5. Nevertheless, the quality is still good and the computed inclusion/background contrasts (6.6) are accurate in both letters.

Figure 7. Reconstructed coefficient $a(x)$ with the shape of letters ‘A’ (top) and ‘Ω’ (bottom) with $c = 5$ from noise polluted observation data as in (6.9) with $\sigma = 0.03$ (left) and $\sigma = 0.05$ (right), i.e. with 3% (left) and 5% (right) noise level. Here $N = 3$ and $\lambda = 5$ as in (6.13). One can observe accurate reconstructions in all four cases. In particular, the inclusion/background contrasts (6.6) are reconstructed accurately.

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ABSTRACT.

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