Pion Chiral Symmetry Breaking in the Quark-Level Linear Sigma Model and Chiral Perturbation Theory

Michael D. Scadron

Physics Department, University of Arizona, Tucson, AZ 85721, USA

Frieder Kleefeld† and George Rupp‡

Centro de Física das Interacções Fundamentais (CFIF), Instituto Superior Técnico, Av. Rovisco Pais, P-1049-001 LISBOA, Portugal

(Dated: March 26, 2022)

Chiral symmetry breaking (ChSB) is reviewed to some extent within the quark-level-linear-sigma-model (QLLσM) theory and standard chiral perturbation theory (ChPT). It is shown, on the basis of several examples related to the pion, as a well-known Goldstone boson of chiral symmetry breaking, that even the non-unitarized QLLσM approach accounts, to a good approximation, for a rather simple, self-consistent, linear, and very predictive description of Nature. On the other hand, ChPT — even when unitarized — provides a highly distorted, nonlinear, hardly predictive picture of Nature, which fits experiment only at the price of a lot of parameters, and requires a great deal of unnecessary theoretical effort. As the origin of this distortion, we identify the fact that ChPT, reflecting only direct ChSB by nonvanishing, current-quark-mass values, does not — contrary to Quantum Chromodynamics (QCD) and the QLLσM — contain any mechanism for the spontaneous generation of the dynamical component of the constituent quark mass. This leads to a very peculiar picture of Nature, since the strange current quark mass has to compensate for the absence of nonstrange dynamical quark masses. We thus conclude that standard ChPT — contrary to common wisdom — is unlikely to be the low-energy limit of QCD. On the contrary, a chiral perturbation theory derived from the QLLσM, presumably being the true low-energy limit of QCD, is expected instead to provide a distortion-free description of Nature, which is based on the heavy standard-model Higgs boson as well as light scalar mesons, as the source of spontaneous generation of current and dynamical quark masses, respectively.

PACS numbers: 14.40.Aq, 11.30.Rd, 11.30.Qc, 14.40.Cs

I. INTRODUCTION

The fundamental laws of physics all respect some exact symmetries. In Nature, however, we are mostly — and fortunately — facing symmetries which are broken. In fact, if the electromagnetic, weak, strong, and gravitational interactions obeyed exactly the same symmetries, the observable world would be a dull place. It is precisely the mismatch between the symmetries respected individually by the different interactions that gives rise to the amazing diversity and beauty of Nature at high, intermediate, and particularly low energies [1].

In this paper we shall focus on the chiral symmetry underlying the theory of strong interactions, as they interfere with chiral-symmetry-breaking (ChSB) electroweak interactions. For convenience, our attention will be concentrated on observables associated with the gold-plated test particle of ChSB, i.e., the pion [2], which, being a Goldstone boson, would be massless in the so-called chiral limit (CL), that is, the limit of strong interactions without electroweak interactions.

As will be discussed in more detail below, a convenient quantity to discuss and measure the amount of ChSB in the pion is not only its mass \(m_\pi\), but also the underlying (nonstrange) constituent quark mass \(m_{\text{con}}\). The latter can be additively decomposed as \(m_{\text{con}} = m_{\text{dyn}} + m_{\text{cur}}\) [3] into a bulk part called dynamical quark mass \(m_{\text{dyn}}\) associated with chiral-symmetric strong interactions, and a small correction called current quark mass \(m_{\text{cur}}\), being nonzero only in the presence of ChSB electroweak interactions. Hence, in the CL we have \(m_\pi \rightarrow 0\), \(m_{\text{cur}} \rightarrow 0\), \(m_{\text{dyn}} \rightarrow m_{\text{CL}}\), and \(m_{\text{con}} \rightarrow m_{\text{CL}}\), where \(m_{\text{CL}}\) may be called chiral-limiting quark mass.

Thus, it becomes clear that a self-consistent picture of ChSB will only be achieved through a complete understanding of electroweak and strong interactions, especially their interplay. Unfortunately, and maybe somewhat surprisingly, neither the experimental nor the theoretical situation is even close to settled, despite the seemingly overwhelming success of the present-day standard model of particle physics (SMPP), as we explain next.

In the first place, despite the consensus on the mechanism for generating the tiny up/down current quark masses (~\(10^1\) MeV) in the Lagrangian density of the SMPP, the corresponding heavy scalar \((m_H > 10^5\) MeV), the Higgs boson, remains undetected.

Secondly, there is substantial experimental evidence [1] for the existence of a complete nonet of light \((m < 1\) GeV) scalar mesons, which can be considered responsible for the dynamical generation of the surprisingly

*Electronic address: scadron@physics.arizona.edu
†Electronic address: kleefeld@cfif.ist.utl.pt
‡Corresponding author; Electronic address: george@ist.utl.pt
large $m_{\text{dyn}}$ ($\sim 300$ MeV). However, our present master-\-ing of QCD [4], the commonly accepted theory underlying strong interactions, does not allow to predict the existence of light scalar mesons, not to speak of their multiplet structure and pole positions. But at least one can gain, e.g., on the basis of QCD-inspired variational chiral quark models [5], Bethe-Salpeter equations (BSE) [6–9], or Dyson-Schwinger equations (DSE) [9, 10], some understanding about the dynamical generation of $m_{\text{dyn}}$, once small ChSB values for $m_{\text{cur}}$ are given. Unfortunately, most DSE approaches are Euclidean, whereas it is not clear whether a Wick rotation from Minkowski [11, 12] to Euclidean space will yield identical results.

Finally, chiral perturbation theory (ChPT) (see e.g. Ref. [13]), the mainstream low-energy effective approach to strong interactions, is only capable of recovering the light scalar mesons when it is unitarized [14, 15] by hand. However, its exact relation to QCD remains unexplained, despite well-known claims made by the large ChPT community. Moreover, ChPT relies upon a strongly distorted description of strong interactions featuring spontaneous ChSB, in particular respecting PCAC. Then, the long-standing issue of ChSB is dynamically generated by Lévy [21] and Cabibbo & Maiani [22]. Finally, Delbourgo and Scadron [23] managed to dynamically generate the QLLσM self-consistently in one-loop order, which considerably increased its predictive power. For instance, the famous chiral-limiting NJL result $m_\sigma = 2m_{\text{CL}}$ is reproduced, i.e., in the chiral limit the mass of the lightest scalar meson equals twice the dynamical quark mass, with the pion being massless, of course, in the same limit.

Clearly, the simplicity and linearity of the QLLσM is owing to the inclusion of the $\sigma$ meson, now experimentally confirmed [1], as an elementary field being the chiral partner of the pion [24, 25] whereby in loop order both are self-consistently recovered as $0^-$ and $0^{++}$ $\bar{q}q$ states, respectively. In contrast, ChPT requires unitarization, i.e., infinite order, to resuscitate the $\sigma$ [14, 15]. As we shall demonstrate through several examples, the QLLσM is doing a much better job, even without unitarization, in reproducing a large variety of low-energy observables than ChPT, with only a tiny fraction of the effort needed. Finally, we shall present arguments why the QLLσM may even constitute a full-fledged, asymptotically free theory of strong interactions.

In the present paper, we shall pass in review the mechanism of ChSB for a variegation of low-energy observables and relations, employing the QLLσM as compared to standard ChPT, namely: pion and nucleon sigma terms (Sec. II), meson charge radii (Sec. III), Goldberger-Treiman relations (Sec. IV), Goldberger-Treiman discrepancies (Sec. V), constituent quark mass via baryon magnetic moments (Sec. VI), effective current quark and ChSB pion, kaon, $\eta_8$ masses (Sec. VII), ground-state scalar $\bar{q}q$ nonet (Sec. VIII), $I = 0$ $\pi\pi$ scattering length (Sec. IX), and the process $\gamma\gamma \to \pi^0\pi^0$ (Sec. X). A summary with discussion and conclusions is presented in Sec. XI.

II. $\sigma_{\pi\pi}$ AND $\sigma_{\pi N}$ CHIRAL BREAKING TERMS

The pion $\sigma$ term is considered a $c$-number ($t$-independent). It is defined at $q^2 = t = m_\pi^2$ via PCAC as (with $f_\pi \simeq 93$ MeV for $m_\pi^0$), the Particle Data Group (PDG) 2004 [1] takes $f_\pi \simeq (92.42 \pm 0.26)$ MeV; ETC stands for “Equal-Time Commutator”

$$\sigma_{\pi\pi} \delta_{ij} = \langle \pi_i | Q_5^5, i\partial A_5^\pi^\text{ETC} | \pi_j \rangle \simeq \frac{1}{f_\pi} \langle 0 | \partial A_5^\pi(0) | \pi_j \rangle = m_\pi^2 \delta_{ij} \ .$$

The analogue $\pi N$ $\sigma$ term is defined as [26]

$$\sigma_{\pi N} \bar{u}_N u_N = \langle N | Q_5^5, i\partial A_5^\pi | N \rangle \ ,$$

taken also as a $c$-number in the 1960’s.

Both the nonvanishing values of Eqs. (1) and (2) characterize $SU(2) \times SU(2)$ ChSB.

In the language of the LσM, Eqs. (1) and (2) are represented by “nonquenched” $(NQ)$ $\sigma$ tadpole graphs, shown in Fig. 1, with LσM couplings $2g_{\sigma\pi\pi} = (m_\pi^2 - m_\sigma^2)/f_\pi \simeq m_\pi^2/f_\pi$, $g_{\sigma NN} \simeq m_N/f_\pi$. Both tadpole “heads” are generated by the ChSB Hamiltonian ($H_{\text{SS}}$ means semi-strict Hamiltonian) implying

$$\sigma_{\pi N}^{\text{NQ}} = \left( \frac{m_\pi}{m_\sigma} \right)^2 m_N \simeq 40 \text{ MeV} \ ,$$

for an averaged nucleon mass \( m_N = 938.9 \text{ MeV} \) and chiral-limiting \( \sigma \) mass \( m_{\sigma}^{\text{CL}} \) of 650.8 MeV (see Sec. VIII and Ref. [23]), predicting an “on-shell” broad \( \sigma \) mass of [27–29] \( m_\sigma \simeq 665 \text{ MeV} \), for \( (m_{\sigma}^{\text{CL}})^2 \simeq m_\sigma^2 - m_\sigma^2 \).

The perturbative “quenched” \( \pi N \) \( \sigma \) term found by GMOR [30], or via the APE collaboration [31], is about

\[
\sigma_{\pi N}^{\text{GMOR}} = \frac{m_\pi + m_\Sigma - 2m_N}{2} \left( \frac{m_\pi^2}{m_K^2 - m_\pi^2} \right) \simeq 26 \text{ MeV} ,
\]

so the net \( \pi N \) \( \sigma \) term in the L\( \sigma \)M is the sum

\[
\sigma_{\pi N}^{\text{L\( \sigma \)M}} = \sigma_{\pi N}^{\text{GMOR}} + \sigma_{\pi N}^{\text{HOChPT}} \simeq (26+40) \text{ MeV} = 66 \text{ MeV} .
\]

While the Adler–Weinberg [32, 33] \( \pi N \) low-energy theorems have large background terms, the Cheng–Dashen (CD) [34] theorem at \( t = 2m_\pi^2 \) has a very small background term (for the isospin-even \( \pi N \) amplitude), viz.

\[
\mathcal{F}^+(\nu = 0, t = 2m_\pi^2) = \frac{\sigma_{\pi N}}{f_\pi^2} + \mathcal{O}(m_\pi^4) \simeq 1.020 \, m_\pi^{-1} ,
\]

as measured by Koch and Hohler [35], corresponding to (for \( f_\pi \approx 93 \text{ MeV} \))

\[
\sigma_{\pi N} = (1.018) \frac{f_\pi^2}{m_\pi} \simeq 63 \text{ MeV} ,
\]

where the small \( \mathcal{O}(m_\pi^4) \) CD term in Eq. (6) is 0.002 \( m_\pi^{-1} \).

Or instead one can work in the infinite-momentum frame (IMF), which suppresses the tadpoles and predicts [36] (requiring squared baryon masses with cross term \( 2m_N \sigma_{\pi N} \))

\[
\sigma_{\pi N} = \frac{m_\pi^2 + m_\Sigma^2 - 2m_N^2}{2m_N} \left( \frac{m_\pi^2}{m_K^2 - m_\pi^2} \right) \simeq 63 \text{ MeV} .
\]

Note that the data in Eq. (7) or the IMF value in Eq. (8) of 63 MeV are compatible with the L\( \sigma \)M value in Eq. (5) of 66 MeV.

However, ChPT leads to a more complicated scheme. By 1982, the review by Gasser and Leutwyler preferred \( \sigma_{\pi N} \simeq 24 \) to 35 MeV [37]. In 1984, their revised ChPT [38] ruled out some L\( \sigma \)M in Appendix B “... as a realistic alternative to QCD...” and the “scalar form factor” ChPT scheme of 1990 [39] used doubly-subtracted dispersion relations (adding 6 new parameters) via a low-mass-Higgs decay \( H \to \pi \pi \), yet to be measured. Then, in 1991, ChPT obtained [40] a non-c-number \( \pi N \) \( \sigma \) term with

\[
\sigma_{\pi N}(t = 0) \approx 45 \text{ MeV} ,
\]

and a larger \( \pi N \) \( \sigma \) term at \( t = 2m_\pi^2 \) of 60 MeV as (HOChPT stands for “higher-order ChPT”) [41] (in MeV)

\[
\sigma_{\pi N}(2m_\pi^2) = \sigma_{\pi N}^{\text{GMOR}}(25) + \sigma_{\pi N}^{\text{HOChPT}(10)}
\]

\[
+ \sigma_{\pi N}^{\text{HOChPT}(15)} \simeq 60 \text{ MeV} ,
\]

because the latter “three pieces happen to have the same sign as \( \sigma_{\pi N}^{\text{GMOR}} \)” [41]. Note that the \( \sigma_{\pi N} \) term has been measured as \( \leq 2 \text{ MeV} \) [16]. Such a “happening” suggests that ChPT in Eq. (10) is physically less significant and much more complicated than Eqs. (5), (7), or (8) above.

### III. MESON CHARGE RADII

Using vector and axial-vector form factors, the PDG tables on pages 499 and 621 of Ref. [1] state that the pion and kaon charge radii are measured as

\[
r_{\pi^+} = (0.672 \pm 0.008) \text{ fm} ,
\]

\[
r_{K^+} = (0.560 \pm 0.031) \text{ fm} .
\]

In fact, the vector-meson-dominance (VMD) and L\( \sigma \)M schemes give nearby values [42] (with \( \hat{m} \equiv (m_u + m_d)/2 \) and \( hc \simeq 197.3 \text{ MeV} \))

\[
r_{\pi^+}^{\text{VMD}} = \sqrt{3}/m_\rho \simeq 0.623 \text{ fm} ,
\]

\[
r_{\pi^+}^{\text{L\( \sigma \)M}} = 1/\hat{m}_{\text{CL}} \simeq 0.63 \text{ fm} ,
\]

\[
r_{K^+}^{\text{VMD}} = \sqrt{3}/m_K \simeq 0.54 \text{ fm} ,
\]

\[
r_{K^+}^{\text{L\( \sigma \)M}} = 2/(\hat{m} + m_s)_{\text{CL}} \simeq 0.51 \text{ fm} .
\]

Note that \( \hat{m}_{\text{CL}} \simeq 313 \text{ MeV} \) as we shall point out later in Sec. VI. Then the \( \bar{q}q \) pion is very tightly bound, almost a fused \( \bar{q}q \) meson.

Moreover, on page 498 of Ref. [1], the pion vector and axial-vector form factors are observed at \( q^2 = 0 \) as

\[
F^\pi_V(0) = 0.017 \pm 0.008 ,
\]

\[
F^\pi_A(0) = 0.0116 \pm 0.0016 .
\]
Using $f_\pi \simeq 93$ MeV, the theoretical CVC estimates of these form factors are [43] (see also Ref. [42])

$$F_V(0) = \frac{m_{\pi^+}}{8\pi^2 f_\pi} \simeq 0.0190 ,$$ (19)

$$F_A(0) = \frac{m_{\pi^+}}{12\pi^2 f_\pi} \simeq 0.0127 ,$$ (20)

reasonably near the data in Eqs. (17) and (18).

Sakurai's [44] vector-meson-coupling universality (VMU) [45] suggests $g_\rho \approx g_{\pi\pi\pi} \sim 6$, whereas the LoM predicts [46] that VMU receives a correction of $\frac{2}{3} g_{\rho\pi\pi}$ via the mesonic $\pi\sigma\pi$ loop, giving $g_{\rho\pi\pi} = g_\rho + \frac{1}{6} g_{\rho\pi\pi}$ or

$$\frac{g_{\rho\pi\pi}}{g_\rho} = \frac{6}{5} ,$$ (21)

close to the data [1, 42]

$$g_{\rho\pi\pi} \simeq 5.95 , \quad g_\rho \simeq 4.96 .$$ (22)

These couplings in (22) follow from the newly measured decay rates [1] $\Gamma(\rho\pi\pi) = 150.3$ MeV and $\Gamma(\rho\rho\rho) = 7.02$ keV (neglecting errors), as in Eqs. (11) and (12) of Ref. [42].

Although VMD and the LoM reasonably match charge-radii data, the one-loop-order ChPT prediction gives [47]

$$\langle r_\pi^2 \rangle = 12L_0/f_\pi^2 ,$$ (23)

which does not uniquely predict data, since $L_0$ is a ChPT low-energy-constant (LEC) parameter.

Whereas ChPT does not explain the pion or kaon charge radii, the LoM continues to match data.

Furthermore, two-loop ChPT [48, 49] suggests that the pion scalar-form-factor radius obeys

$$\langle r_\pi^2 \rangle = (0.61 \pm 0.04) \text{ fm}^2 .$$ (24)

This value is debated [50, 51] as

$$\langle r_\pi^2 \rangle = (0.75 \pm 0.07) \text{ fm}^2 .$$ (25)

We in turn claim that the average of Eqs. (24) and (25) is 75% greater than the square of the VMD and LoM pion charge radius values in Eqs. (13) and (14), the latter two being compatible with data in Eq. (11). Moreover, the scalar-form-factor basis of Eqs. (24) and (25) has neither been verified experimentally, nor theoretically via VMD or the LoM, in contrast with the vector & axial-vector form factors in Eqs. (17–20), or Eqs. (29,30) to follow.

If we followed the discussion of Gasser and Leutwyler in Ref. [38], the identity

$$\frac{f_\pi(m_\pi^2)}{f_\pi(0)} - 1 = \frac{1}{6} m_\pi^2 \langle r_\pi^2 \rangle + \frac{13}{192} \pi^2 m_\pi^2 + O(m_\pi^4)$$ (26)

would hold. From Eq. (40) in the upcoming Sec. V, we know that, to a good approximation,

$$\frac{f_\pi(m_\pi^2)}{f_\pi(0)} - 1 = \frac{m_\pi^2}{8\pi^2 f_\pi^2} + O(m_\pi^4) .$$ (27)

This would imply, in combination with Eq. (26) and

$$\langle r_\pi^2 \rangle \simeq 1/\hat{m}^2 = 1/(g f_\pi e)^2 \simeq N_c/(4\pi^2 f_\pi^2) ,$$

$$\langle r_\pi^2 \rangle = \frac{6}{m_\pi^2} \left( \frac{m_\pi^2}{8\pi^2 f_\pi^2} - \frac{13}{192} \pi^2 m_\pi^2 \right)$$

$$= \frac{11}{24} \frac{3}{4\pi^2 f_\pi^2} \simeq \frac{11}{24} \langle r_\pi^2 \rangle \simeq 0.19 \text{ fm}^2 ,$$ (28)

which is clearly in strong conflict with the ChPT results Eqs. (24) and (25).

In order to extend the LoM to $SU(3)$, the $K^+ \rightarrow e^+\nu\gamma$ form factor sum is measured, according to page 621 in Ref. [1], as

$$|F_K^V(0) + F_K^A(0)| = 0.148 \pm 0.010 .$$ (29)

The $SU(3)$ LoM pole terms add up to [42, 52]

$$|F_K^V(0) + F_K^A(0)|_{\text{LoM}} = 0.109 + 0.044 = 0.153 ,$$ (30)

compatible with data in Eq. (29). The quark-loop prediction for the pion charge radius was initially found in Refs. [53] via quark (and meson) loops in a LoM framework, but not via ChPT. The above-measured vector and axial-vector form factors in Eqs. (17), (18), and (29) have not been extended to the mentioned ChPT scalar form factors, as was suggested in deriving Eq. (9), and will be suggested in Eqs. (43) and the nonlinear $a_{00}$ in Secs. V, IX, respectively.

**IV. GOLDBERGER-TREIMAN RELATIONS**

First we note that the pion-quark coupling in the QLLoM is [23] $g_{eqq} = 2\pi/\sqrt{3} \simeq 3.6276$, also following from infrared QCD [54] and the $Z = 0$ compositeness condition [55]. Then the pion Goldberger-Treiman relation (GTR) gives the nonstrange constituent quark mass as

$$f_\pi g = \hat{m} = 93 \text{ MeV} \times \frac{2\pi}{\sqrt{3}} \simeq 337.4 \text{ MeV} .$$ (31)

The nonrelativistic quark model predicts [1], via magnetic dipole moments (see Sec. VI, Eq. (57)), practically the same value, viz.

$$\hat{m} = \frac{1}{2} (m_u + m_d) \simeq 337.5 \text{ MeV} ,$$ (32)

with $m_d - m_u \simeq 4$ MeV following from the kaon [1, 56] mass difference or the $\Sigma^- - \Sigma^+$ baryon mass difference, yielding the constituent masses

$$m_u \simeq 335.5 \text{ MeV} , \quad m_d \simeq 339.5 \text{ MeV} .$$ (33)

As for the strange quark, the data ratio for [1] $f_K/f_\pi \simeq 1.22$ gives

$$\frac{m_s}{\hat{m}} \simeq 2 \frac{f_K}{f_\pi} - 1 \simeq 1.44$$ (34)

$$\Rightarrow m_s \simeq 1.44 \hat{m} \simeq 486.0 \text{ MeV} .$$ (35)
Then the kaon GTR
\[ f_K g = \frac{1}{2} (m_s + \hat{m}) = 411.75 \text{ MeV} \] (36)
predicts, via \( g = 2\pi/\sqrt{3} \),
\[ f_K = \frac{1}{2} (m_s + \hat{m}) / (2\pi/\sqrt{3}) = 113.50 \text{ MeV} , \] (37)
giving the ratio
\[ \frac{f_K}{f_\pi} = \frac{113.50}{93} \approx 1.22 , \] (38)
a second test of \( g = 2\pi/\sqrt{3} \) (Eq. (31) being the first test). Alternatively, from Eq. (32) with \( f_\pi \approx 93 \) MeV,
\[ 2f_Kg - \hat{m} = m_s \approx 485.7 \text{ MeV} , \] (39)
very close to Eq. (35), a third test of \( g = 2\pi/\sqrt{3} \).

ChPT appears not to alter these GTRs, nor does so PCAC, even though both were first obtained theoretically via the LoM (not with quarks, but nucleons as fermions [19]). Recall that ChPT rules out some kind of LoM in Appendix B of Ref. [38], which is an indirect mark against ChPT, because our GTR-LoM chiral scheme in Eqs. (31)–(39) above is a close match with data.

V. GOLDBERGER-TREIMAN DISCREPANCIES

A once-subtracted dispersion relation (containing no additional parameters) predicts (for \( f_\pi = (92.42 \pm 0.26) \) MeV [1]) the \( q^2 \) variation of \( f_\pi \) as \[ f_\pi (m_N^2) / \frac{f_\pi(0)}{1 - \frac{m_N^2}{8\pi f_\pi^2} \left( 1 + \frac{m_N^2}{10m_N^2} \right)} \approx 2.84\% . \] (40)
The dominant 2.79\% \( O(m_N^2) \) term in Eq. (40) was first obtained in Refs. [58]. The smaller 0.05\% \( O(m_N^2) \) term was found in Ref. [29], having been anticipated numerically in Ref. [57]. In any case, Eq. (40) is a Taylor series in \( m_N^2, m_N^4 \), needing only two terms. Alternatively, we invoke data to explain the GT discrepancy as [27–29]
\[ \Delta = 1 - \frac{m_N g_A}{f_\pi g_N N} = (2.07 \pm 0.57)\% , \] (41)
using \( m_N = 938.92 \text{ MeV} \), \( |g_A/g_N| = 1.2695 \pm 0.0029 [1] \), \( f_\pi = (92.42 \pm 0.26) \) MeV [1], and \( g_{\pi NN} = 13.17 \pm 0.06 [59] \). Prior 1971 measurements gave \( g_{\pi NN} \approx 13.40 \) and \( \Delta \approx 3.8\% \). The analysis of Sec. II yields
\[ \frac{\sigma_{\pi N}}{2m_N} = \frac{63}{2 \times 938.92} \approx 3.35\% . \] (42)

All of the above analyses suggest a 3\% chiral-breaking effect.

However, the ChPT scalar form factors predict
\[ \frac{f_\pi (m_\pi^2)}{f_\pi(0)} - 1 \sim \begin{cases} 6.7 \% & \text{via one loop } [60], \\ 7.2 \% & \text{via two loops } [49], \\ 8.2 \% & \text{via two loops } [51]. \end{cases} \] (43)
Even though there is a two-loop debate as to how ChPT should proceed, none of these 6.7\%, 7.2\%, 8.2\% predictions (based on scalar form factors) are near the 3\% chiral-breaking model-independent predictions of Eqs. (40), (41), or (42).

VI. CONSTITUENT QUARK MASS VIA BARYON MAGNETIC MOMENTS

The nonrelativistic valence (V) quark model has axial-vector spin components of the nucleon [61]
\[ \Delta_{uV} = \frac{4}{3}, \quad \Delta_{dV} = -\frac{1}{3}, \quad \Delta_{sV} = 0 , \] (44)
with total spin \( \Sigma_V = \Delta_{uV} + \Delta_{dV} + \Delta_{sV} = 1 \). Although \( \Delta_u - \Delta_d \approx 1.27 [1] \) is about 30\% lower than the valence value from Eq. (44), good valence predictions from the nucleon magnetic moments (m.m.) stem from [61]
\[ \mu_p(udd) = \mu_u \Delta_{uV} + \mu_d \Delta_{dV} + \mu_s \Delta_{sV} , \] (45)
\[ \mu_n(ddu) = \mu_d \Delta_{uV} + \mu_u \Delta_{dV} + \mu_s \Delta_{sV} , \] (46)
predicting the ratio
\[ \frac{\mu_p}{\mu_n} = \frac{\mu_u \Delta_{uV} + \mu_d \Delta_{dV}}{\mu_u \Delta_{uV} + \mu_d \Delta_{dV}} = \frac{9}{-6} = -1.5 \] (47)
(using quark m.m. values \( \mu_u = e/3m_u, \mu_d = -e/6m_d \), close to the observed [1] ratio 2.793/(−1.913) ≈ −1.46. Also, the nucleon m.m. difference [1, 61] is
\[ \mu_p - \mu_n = (\mu_u - \mu_d)(\Delta_{uV} - \Delta_{dV}) = 4.706 \frac{e}{2m_N} , \] (48)
predicting the average constituent quark mass (for \( \hat{m} \approx m_u, m_d \))
\[ \hat{m} = m_N \frac{5/3}{4.706} \approx 332.5 \text{ MeV} , \] (49)
for \( \Delta_{uV} - \Delta_{dV} = 5/3 \), near \( \hat{m} \approx m_p/\mu_p \approx 336 \text{ MeV} \), and near the GT mass in Eq. (31), or the anticipated m.m. quark mass in Eq. (32).
As a matter of fact, assuming \( \Delta s = 0 \), the sum \( \mu_p + \mu_n \) predicts a larger value \( \hat{m} \approx 355.6 \text{ MeV} \), which reduces to Eqs. (31,32) only if there is a slight strangeness component in the nucleon, viz.
\[ \Delta s \approx \frac{337.5 - 355.6}{337.5} \approx -5.4\% . \] (50)
This result is compatible with recent data [16] finding $-\Delta s \leq 6\%$.

In a similar manner, the $\Sigma$ to $N$ m.m. valence-difference ratio is predicted as

$$\frac{\mu_{\Sigma^+} - \mu_{\Sigma^-}}{\mu_p - \mu_n} = \frac{\Delta uV - \Delta sV}{\Delta uV - \Delta dV} = \frac{4/3}{5/3} = 0.800$$

(51)
only $4\%$ higher than the observed $1$ difference ratio $0.769$. Moreover, the present $\beta$-decay value $[1]$

$$\Delta u - \Delta d = g_A = 1.2695 \pm 0.0029,$$

(52)
and the $\lambda_s$ component found from various hyperon semileptonic weak decays give

$$\Delta u + \Delta d - 2\Delta s = g_A \left[ \frac{3f - d}{f + d} \right] \approx 0.584$$

(53)
(for the empirically determined ratio $[62] (d/f)_A \approx 1.74$).

As we shall now see, Eq. (53) requires a slight $\Delta s \neq 0$, i.e., $\Delta s = -5.7\%$.

Lastly, adding and subtracting Eqs. (52,53) leads to $[63]$

$$\Delta u - \Delta s \approx 0.927, \quad \Delta d - \Delta s \approx -0.343.$$  

(54)
Then the latter four equations uniquely predict

$$\Delta u \approx 0.87, \quad \Delta d \approx -0.40, \quad \Delta s \approx -0.057, \quad \Delta \Sigma \approx 0.41,$$

(55)
together with the good valence results Eqs. (47,48,50) and the constituent quark mass Eq. (49). This quark spin pattern (55) is compatible with the QCD predictions $[64]$

$$\Delta u = 0.85 \pm 0.03, \quad \Delta d = -0.41 \pm 0.03, \quad \Delta s = -0.08 \pm 0.03, \quad \Delta \Sigma = 0.37 \pm 0.07.$$  

(56)

Also, dynamical tadpole leakage is similar in spirit to the QLLM, but now with axial-vector $f_1(1285)$, $f_1(1420)$ mixing $[65]$. This generates quark spins close to Eqs. (55,56), with $\Delta s = -6.0\%$, near the values $-5.7\%$ in Eq. (53) and $-5.4\%$ in Eq. (50).

Given this consistent pattern of quark spins with approximate average quark mass in Eq. (49), for $\hat{m} = (m_u + m_d)/2$, $\mu_u = e/3m_u$, $\mu_d = -e/6m_d$ and leading-order $\hat{m} = m_p/\mu_p \approx 336.0$ MeV, and further folding in $m_d - m_u \approx 4$ MeV, we obtain (in MeV) a quadratic equation for $\hat{m}_{\text{con}}$, viz.

$$\hat{m}_{\text{con}}^2 - \frac{m_p}{2.792847} \left( \hat{m}_{\text{con}} + \frac{14}{9} \right) = 0,$$

(57)
whose only positive solution is $\hat{m}_{\text{con}} \approx 337.5$ MeV, as anticipated in Eq. (32). The latter is near Eq. (49), nearer still to $m_p/\mu_p \approx 336.0$ MeV, and even closer to the GTR quark mass in Eq. (31). As mentioned in the Introduction, this bulk constituent quark mass can be decomposed into its CL dynamically generated part $m_{\text{dyn}}$ and its chiral-broken current-quark-mass component $m_{\text{cur}}$:

$$m_{\text{con}} = m_{\text{dyn}} + m_{\text{cur}}.$$  

(58)
Recall that $m_{\text{dyn}}$ was first estimated $[6]$, via quark-dressing and binding equations, as $m_{\text{dyn}} \sim 300–320$ MeV, or ideally

$$m_{\text{dyn}} = \frac{m_N}{3} \approx 313 \text{ MeV}.$$  

(59)
This scale matches infrared QCD $[3, 6, 66]$, for $\alpha_s \approx 0.5$ at a 1 GeV cutoff,

$$m_{\text{dyn}} = \left[ \frac{4\pi}{3} \alpha_s (-\bar{q}q) \right]^{\frac{\lambda}{2}} \approx 313 \text{ MeV},$$

(60)
for $(-\bar{q}q) \approx (245 \text{ MeV})^3$. Also, from Sec. III, recall that

$$m_{\text{dyn}} = r_{\pi}^{-1} \approx 313 \text{ MeV},$$

(61)
for VMD pion charge radius $r_{\pi} \approx 0.63$ fm, near data $[1]$.

As for the ChSB current-quark-mass scale, the ChPT review of 1982 $[37]$ takes

$$\hat{m}_{\text{cur}} \approx \left( \frac{f_{\pi} m_{\pi}}{2} \right)^2 \approx 5.5 \text{ MeV},$$

(62)
for $f_{\pi} \approx 93$ MeV and $(-\bar{q}q) \approx (245 \text{ MeV})^3$. Reference $[37]$ implies this relation is due to GMOR $[30]$, but Eq. (62) and the $f_{\pi}$ scale are hard to find in GMOR, which focuses instead on the good-bad structure of $\bar{q}\lambda_i q$ and $\bar{q}\gamma_5\lambda_i q$ matrix elements $[67]$. In fact, Ref. $[68]$ stresses on page 462 that this GMOR SU(3) assumption (leading directly to our Eq. (62)) may not be correct. More recently, Ref. $[66]$ noted that, if Eq. (62) were true and if $\langle N|\bar{s}s|N \rangle = 0$ (in fact, $\Delta s \approx -0.057$ is compatible with data $[16]$), then it is easy to show that the $\pi N$ $\sigma$-term should be in theory

$$\sigma_{\pi N}^q = \frac{3(m_{\pi} - m_{\lambda})}{\hat{m}_{\text{cur}} - 1} \approx \frac{606 \text{ MeV}}{(m_{\pi} - m_{\lambda})_{\text{cur}} - 1}.$$  

(63)
If we follow recent ChPT with $[40]$ $\sigma_{\pi N}^{\text{ChPT}} = 45$ MeV, then the latter equation requires $(m_{\pi}/\hat{m}_{\text{cur}}) \approx 14.47$, and so $m_{\text{cur}} \approx 80$ MeV for $(m_{\pi}/\hat{m}_{\text{cur}}) \approx 5.5$ MeV. “If this were true then a significant fraction of the nucleon’s mass would be due to strange quarks — in contradiction with the quark model” $[66]$. Moreover, such a result would be in conflict with the usual ChPT ratio $[37]$ $(m_{\pi}/\hat{m}_{\text{cur}}) \approx 25$. We shall return to current quarks in the next section.

VII. EFFECTIVE CURRENT QUARK AND CHIRAL-SYMMETRY-BREAKING PION, KAON, $\eta_s$ MASSES

Following Ref. $[69]$, we use QCD to express a momentum-dependent dynamical quark mass as

$$m_{\text{dyn}}(p^2) = \frac{m_{\text{dyn}}^3}{p^2},$$

(64)
so that \( m_{\text{dyn}} \) at \( p^2 = m_{\text{dyn}}^2 \) is, as required by consistency,

\[
m_{\text{dyn}}(p^2 = m_{\text{dyn}}^2) \equiv m_{\text{dyn}}.
\]

Furthermore, we take the nonstrange constituent quark mass \( m_{\text{con}} \) as 337.5 MeV from Secs. IV, VI and compute the effective current quark mass as (avoiding any good-
bad assumptions besides Eq. (62), but keeping the de-
composition Eq. (58), and also Eqs. (59–61))

\[
\delta \hat{m} = m_{\text{con}} - \frac{m_{\text{dyn}}^2}{\langle \pi^0 \rangle} \approx 68.3 \text{ MeV},
\]

with \( \delta \hat{m} \to 0 \) when \( m_{\text{con}} \to m_{\text{dyn}} \) in the CL, as one expects for the current quark mass. Note that \( \delta \hat{m} \approx 68.3 \) MeV is very near the chiral-breaking \( \sigma_{\pi N} \approx 63–66 \) MeV of Sec. II, also as expected!

Then for the \( q \bar{q} \) pion, its mass is predicted from Eq. (66) as

\[
m_{\pi} = \delta \hat{m} + \delta \hat{m} \approx 136.6 \text{ MeV},
\]

midway between the observed [1] \( m_{\pi^0} = 134.98 \) MeV and \( m_{\pi^\pm} = 139.57 \) MeV.

Instead, the ChPT prediction (62) suggesting \( m_{\text{cur}} \approx \)

5 MeV and \( m_{d, \text{cur}} \approx 9 \) MeV, or taking the ChPT scheme of Ref. [70] having finite-volume effects, do not predict a pion mass anywhere in the region of 140 MeV. The QLLOM assumes standard pion \( \hat{q} \hat{q} \) states \( |\pi^+\rangle = \langle \bar{d}u \rangle \),

\( \langle \pi^-\rangle = \langle \bar{u}d \rangle \), \( \langle \pi^0 \rangle = \langle (\bar{u}u - \bar{d}d)/\sqrt{2} \rangle \), and takes the pion as a tightly bound \( \hat{q} \hat{q} \) Nambu-Goldstone “fused” nonstrange meson (verified via the QLLOM, VMD, and measured meson charge radius in Sec. III). The observed proton charge radius [1] \( R_p = (0.870 \pm 0.008) \) fm suggests the proton is a “touching” quark-pyramid \( u\bar{d} \) state [25] and there is minimal \( \langle ss \rangle \) \( N \), either from data [16], phe-
nomenology [71], or from the magnetic-moment scheme of Sec. VI.

Instead, we return to studying the nonstrange quark mass \( \hat{m}_{\text{cur}} \) in Eq. (66) (also see Ref. [72]) successfully predicts the pion mass at \( m_{\pi} = 2 \delta \hat{m} = 136.6 \) MeV in Eq. (67).

Moreover, \( \hat{m}_{\text{cur}} \) is sometimes called the current mass via neutral PCAC (for a review, see Ref. [58]). Using quark structure functions, one predicts the pion mass (squared) as

\[
m_{\pi}^2 = 2 \hat{m}_{\text{cur}}^2 \bar{h} , \quad \bar{h} = \frac{5}{2}.
\]

\[
\hat{m}_{\text{cur}} = \frac{m_{\pi}}{\sqrt{5}} \approx 62.4 \text{ MeV},
\]

for \( m_{\pi^\pm} \approx 139.57 \) MeV, invoking the spectator-helicity rule [67]. Note from Eqs. (68) that here \( m_{\pi}^2 \propto \hat{m}_{\text{cur}}^2 \), whereas GMOR [30] and ChPT take \( m_{\pi}^2 \propto \hat{m}_{\text{con}} \). Fubini and Furlan [73] suggested that the r.h.s. of Eqs. (68) can behave as either \( \hat{m}_{\text{cur}} \) or \( m_{\text{cur}}^2 \), depending on the renor-
malization scale. At a 1 GeV scale we can write

\[
m_{\pi}^2 = \langle \pi \mid H_{\text{ChSB}} \mid \pi \rangle \sim \hat{m}_{\text{cur}} \langle \pi \mid \bar{u}u + \bar{d}d \mid \pi \rangle, \quad (69)
\]

or roughly, from Eq. (69) given Eq. (58) and \( m_{\text{dyn}} \approx 313 \) MeV,

\[
m_{\pi}^2 \sim \hat{m}_{\text{cur}} (m_{\text{dyn}} + \hat{m}_{\text{cur}}) \Rightarrow \hat{m}_{\text{cur}} \approx 53 \text{ MeV},
\]

taking \( m_{\text{dyn}} \approx 313 \) MeV as in Sec. VI.

Alternatively, we can take the baryon \( d/f \) ratio or structure-function integrals as [67]

\[
\frac{d}{f} = \frac{\bar{f}_u - 2 \bar{f}_d + \bar{f}_s}{\bar{f}_u - \bar{f}_s} = \frac{3}{5} \frac{m_{\pi}^2 - m_{\pi^N}^2 - m_{\pi^N}^2}{m_{\pi}^2 - m_{\pi^N}^2} \approx -0.28,
\]

\[
\frac{\bar{f}_d}{\bar{f}_u} \approx 0.64, \quad \bar{f}_s = 0 \quad (\text{near } \bar{f}_s \approx -0.057). \quad (71)
\]

In fact, the spectator-helicity rule [67] is slightly altered to (using squared baryon masses as in Eq. (71), rather than the \( \bar{f}_u = 7.9, \ldots \) as found in Eq. 39 of Ref. [67])

\[
\bar{h} = \frac{5}{2}, \quad \bar{f}_u \approx 7.64, \quad \bar{f}_d \approx 4.85,
\]

\[
\bar{f}_d \approx 0.635, \quad \bar{f}_u + \bar{f}_d \approx 12.49 \quad \text{for } \bar{f}_s = 0. \quad (72)
\]

Note that the latter ratio is compatible with \( \bar{f}_d/\bar{f}_u = 0.64 \) in Eq. (71), and the latter sum implies the Jaffe–Llewellyn-Smith form [74] for the current quark mass (squared) is

\[
\hat{m}_{\text{cur}}^2 = \frac{m_{\pi} \sigma_{\pi N}}{\bar{f}_u + \bar{f}_d} = \frac{938.9 \text{ MeV} \times 63 \text{ MeV}}{12.49} = (68.8 \text{ MeV})^2, \quad (73)
\]

very near the predicted effective current quark mass of 68.3 MeV in Eq. (66). Thus, we deduce in several independent ways (Eqs. (68), (70), (73)) that \( \hat{m}_{\text{cur}} \approx 62.4 \) MeV, \( \approx 53 \) MeV, \( \approx 68.8 \) MeV, all near \( \delta \hat{m} \approx 68.3 \) MeV from Eq. (66), but not near the ChPT value \( \hat{m}_{\text{cur}} \approx 5.5 \) MeV in Eq. (62).

Also, following Ref. [68], we have

\[
\frac{m_{\pi}^2 - m_{\pi^N}^2}{m_{\pi}^2} = \frac{\bar{f}_d (\hat{m}_{\text{cur}}^2 - m_{\pi^N}^2)}{m_{\pi}^2} = \frac{0.5417 \text{ GeV}^2}{m_{\pi}^2} \approx \frac{0.8556 \text{ GeV}^2}{m_{\pi}^2}. \quad (74)
\]

This leads to the ratio \( \bar{f}_d/\bar{f}_u \approx 0.633 \) (independent of the \( (\hat{m}_{\text{cur}}^2 - m_{\pi^N}^2) \) scale), near the value 0.64 found above. Moreover, if we sum the two equations in Eq. (74) and use \( \bar{f}_u + \bar{f}_d \approx 12.49 \) from Eq. (72), we get \( (\hat{m}_{\text{cur}}^2 - m_{\pi^N}^2) \approx (1.3973/12.49) \text{ GeV}^2 \approx 0.1119 \text{ GeV}^2 \). Substituting then the value \( \hat{m}_{\text{cur}} = 0.0683 \text{ GeV} \) from Eq. (66) yields the consistent ratios

\[
\left( \frac{m_{\pi}}{\hat{m}_{\text{cur}}} \right)^2 \approx \sqrt{\frac{0.1119}{(0.0683)^2}} + 1 = 5.00 \quad (75)
\]
from $(f_u + f_d)$ above,
\[
\left( \frac{m_\pi}{m} \right)_{\text{cur}} \approx \sqrt{\frac{0.1125}{0.0683^2} + 1} = 5.01
\] (76)
from $(f_u - f_d)$ above,
\[
\left( \frac{m_\pi}{m} \right)_{\text{cur}} = \sqrt{\frac{2m_K^2}{m_\pi^2} - 1} = 5.00
\] (77)
from the PCAC $K$-to-$\pi$ ratio [72], for $m_\pi^2 = 13m_\sigma^2$,
\[
\left( \frac{m_\pi}{m} \right)_{\text{cur}} = \frac{2m_K}{m_\pi} - 1 = 6.21
\] (78)
from the light cone [75], and
\[
\left( \frac{m_\pi}{m} \right)_{\text{cur}} = \sqrt{\frac{3\Delta m_N^2}{m_N\sigma_N} + 1} = 4.93
\] (79)
from [72] $\Delta m_N^2 = 0.46$ GeV$^2$, being the nucleon mass shift (squared) from the baryon-octet $SU(3)$ value of $(1.158$ GeV$)^2$. The average ratio from Eqs. (75-79) is 5.23, which when combined with $\hat{m}_{\text{cur}} = 68.3$ MeV implies $m_{\text{cur}} = 357$ MeV, very near the analogue of Eq. (66), i.e., $m_{\text{cur}} = m_\sigma\frac{m_\text{dyn}}{m_\sigma\text{con}} = m_\sigma\text{con} = 356$ MeV, for $m_\sigma\text{con} = 486$ MeV and $m_\text{dyn} = 313$ MeV. Note that the ChPT ratio $(m_\pi/m)_{\text{cur}}$ is exactly 25 scaled to $\hat{m}_{\text{cur}}$ implies $\hat{m}_{\text{cur}} \approx 5.5$ MeV predicts $m_{\text{cur}} \approx 137.5$ MeV, leading to an inconsistent picture of Nature.

Finally, we extend the $\pi\pi\sigma$ term $\sigma_{\pi\pi} \approx m_\pi^2$ from Eq. (1) to the three ChSB meson $\sigma$ term
\[
\sigma_{\pi\pi} , \sigma_{KK} , \sigma_{\eta\eta} = m_\pi^2 \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right)
\] (80)
as found in Refs. [76]. Note that all three vanish in the CL, with $m_\pi^2 \to 0$. To obtain the chiral-broken $\eta$ mass, we invoke the Gell-Mann–Okubo (GMO) value (see also Eqs. (A23))
\[
m_{\eta_8}^{\text{GMO}} = \sqrt{\frac{4m_K^2 - m_\pi^2}{3}} \approx \sqrt{17} m_\pi \approx 563$ MeV,
\] (81)
using $m_K^2 \approx 13m_\pi^2$, and the average ChSB pion mass 136.6 MeV from Eq. (67). Since the observed [1] $m_\eta$ is 547.75 MeV, $m_{\eta_8}^{\text{GMO}} \approx 563$ MeV from Eq. (81) is about 2.8% greater than $m_\eta$, similar to the 3% GTR discrepancies in Sec. V, but much lower than the 6.7–8.2% ChPT predictions in Eq. (43). Moreover, folding the GOM relation into the three meson $\sigma$ terms of Eq. (80), one finds $3\sigma_{\eta_8} - 4\sigma_{KK} + \sigma_{\pi\pi} = 0$, as anticipated in Eqs. (80) and (81), giving a consistent pattern $(3/3) - (4/2) + 1 = 0$.

VIII. GROUND-STATE SCALAR $\bar{q}q$ NONET

It is well-known and commonly accepted that there exists (within a $U(3) \times U(3)$ flavor scheme) a nonet of pseudoscalar mesons ($\pi(137), K(496), \eta(548), \eta'(958)$) that play the role of Goldstone bosons associated with ChSB. What is not so well-known and accepted is that there is also experimental evidence [1] for a corresponding light scalar-meson nonet [7, 77] $f_0(600)$ ($\sigma, K_0^*(800)$ ($\kappa), f_0(980), a_0(985)$. The members of the latter nonet, being much too light to be accommodated as naive (unquenched) quark-model states, are rather to be interpreted as the chiral partners (see e.g. Ref. [24]) of the former pseudoscalar mesons. In this spirit, it is useful to define the following field matrices $S(x)$ and $P(x)$, for $U(3) \times U(3)$ flavor nonets of scalar and pseudoscalar mesons, respectively:
\[
S = \left( \begin{array}{c} \sigma_{uu} \\ a_0^+ \\ \sigma_{dd} \\ \sigma_{ss} \\ \sigma_{uu}^* \\ \sigma_{dd}^* \\ \sigma_{ss}^* \end{array} \right) , \quad P = \left( \begin{array}{c} \eta_{uu} \\ \pi^+ \\ \sigma_{ud} \\ K^0 \end{array} \right)
\] (82)

For later convenience, we also define $\sigma_n \equiv (\sigma_{uu} + \sigma_{dd})/\sqrt{2}, \quad \kappa_n \equiv (\sigma_{uu} - \sigma_{dd})/\sqrt{2} \approx a_0^0, \quad \eta_n \equiv (\eta_{uu} + \eta_{dd})/\sqrt{2}, \quad \eta_3 \equiv (\eta_{uu} - \eta_{dd})/\sqrt{2} \approx \pi^0$.

The still somewhat controversial status of the lightest members of the ground-state scalar-meson nonet, namely the $\sigma(600)$ and the $\kappa(800)$, is due to both experimental and theoretical difficulties. On the one hand, the amplitudes in elastic $\pi\pi$ and $K\pi$ scattering rise very slowly from threshold upwards, due to nearby Adler zeros [78, 79] just below threshold. On the other hand, the theoretical description of light scalar-meson resonances turns out to be quite cumbersome, owing to the large unitarization effects, which demand a manifestly nonperturbative treatment.

In a field-theoretic framework, the most efficient and obvious approach to the light scalars is indubitably the QLL [23, 25, 27, 42], which contains from the outset all the relevant (experimen-
tal) tree level important nonperturbative features of strong interactions are included. The interaction Lagrangian of (anti)quarks and (pseudo)scalar mesons in the QLL$\sigma$M is given by $\mathcal{L}_{\text{int}}(x) = \sqrt{2} \bar{g} \bar{q}(x) (S(x) + iP(x) \gamma_5) q(x)$, with $g$ the strong coupling constant given by $|g| \approx 2\pi/\sqrt{3}$, as follows from one-loop dynamical generation [23, 80]. By integrating out (anti)quarks, the QLL$\sigma$M Lagrangian can be converted into an effective $U(3) \times U(3)$ LeM meson Lagrangian displaying strong similarity with the well-known “traditional” $U(3) \times U(3)$ LeM Lagrangian [21, 22, 27, 81, 82], briefly summarized in Appendix A.

The QLL$\sigma$M (without (axial-)vector mesons) in the CL ($m_\pi^2 = m_K^2 = 0$) predicts [6, 7, 27] the following NJL-like [20] mass relations and chiral-limiting scalar-meson masses:
\[
\begin{align*}
m_{\sigma_n}^2 &= \left(2m_{\text{CL}}\right)^2 \simeq (626 \text{ MeV})^2, \\
m_{\sigma_n}^2 &= \left(2m_{\text{CL}}\right)^2 \simeq (901 \text{ MeV})^2, \\
m_{\kappa_n}^2 &= \left(2m_{\text{CL}}\right)(2m_{\sigma_n})^2 \simeq (751 \text{ MeV})^2.
\end{align*}
\]
with \( \tilde{m}_{\text{CL}} = 313 \text{ MeV} \), \( m^\text{CL}_\sigma = 1.44 \times \tilde{m}_{\text{CL}} = 450.72 \text{ MeV} \). Beyond the CL, the leading contributions to the scalar-meson masses due to quark loops will lead to expressions equivalent to Eqs. (83–85), but now with non-CL quark masses, i.e. (see e.g. Ref. [25]),
\[
\begin{align*}
m^2_\sigma &\simeq (2\tilde{m})^2 = (675 \text{ MeV})^2, \\
m^2_{s_\sigma} &\simeq (2m_\sigma)^2 = (972 \text{ MeV})^2, \\
m^2_{f_\sigma} &\simeq (2\tilde{m})(2m_\sigma) = (810 \text{ MeV})^2,
\end{align*}
\]
with \( \tilde{m} = 337.5 \text{ MeV} \), \( m_\sigma = 1.44 \times \tilde{m} = 486.0 \text{ MeV} \), which predictions lie impressively close to the experimental values.\(^1\)

A realistic nonstrange-strange mixing angle of \( \phi_n \simeq \pm 18^\circ \) [84] implies, with the help of Eqs. (A16) and (A17), the following values for \( m_\sigma \) and \( m_{f_\sigma} \):
\[
\begin{align*}
m^2_\sigma &= \frac{1}{2} \left( m^2_{\sigma_n} + m^2_{\sigma_s} - \frac{m^2_{\sigma_n} - m^2_{\sigma_s}}{\cos(2\phi_S)} \right) \\
&= (630.8 \text{ MeV})^2, \\
m^2_{f_\sigma} &= \frac{1}{2} \left( m^2_{\sigma_s} + m^2_{\sigma_n} + \frac{m^2_{\sigma_n} - m^2_{\sigma_s}}{\cos(2\phi_S)} \right) \\
&= (1001.3 \text{ MeV})^2.
\end{align*}
\]

Of course, the latter mass predictions are still subject, in principle, to further corrections stemming from nonzero pseudoscalar-meson masses, as can be seen from Eqs. (A9–A12) predicted by the “traditional” \( \sigma \)M approach discussed in the Appendix. However, as noticed there, several uncertainties persist within such a formalism, which make it hard to establish a one-to-one correspondence with the more straightforward QL\( \sigma \)M.

Comparing now with the experimental scalar-meson masses, we see that the non-CL QL\( \sigma \)M results do a very good job. Starting with the \( \sigma \) meson, the PDG tables [1] very cautiously list the \( f_0(600) \) with an extremely wide mass range of 400–1200 MeV, but giving a large number of recent experimental findings or analyses tending to converge around a typical mass value of about 600 MeV. Also theoretical coupled-channel models reproducing the \( \sigma \) meson masses, we see that the non-CL QL\( \sigma \)M approach discussed in the Appendix. However, as noticed there, several uncertainties persist within such a formalism, which make it hard to establish a one-to-one correspondence with the more straightforward QL\( \sigma \)M framework is entirely compatible with the above values of 972 MeV and 1001.3 in Eqs. (87) and (90), respectively. So are the \( f_0(980) \) pole positions of \( (998 \pm 4) - i \times (17 \pm 4) \) MeV and \( (994 - i \times 14) \) MeV given in the very recent analysis of Ref. [89].

Concerning the \( a_0(985) \), the coupled-channel approach of Ref. [77], which includes the crucial \( \eta \pi \) and \( K \bar{K} \) channels, predicts a reasonable pole mass of (968 – 4i) 28 MeV, to be compared to (1036 – i \times 84) MeV in the experimental analysis in Ref. [89]. As already noted in Ref. [25], we consider the approximate mass degeneracy of the \( a_0(985) \) and \( f_0(980) \) purely accidental, even though the theoretical situation in a QL\( \sigma \)M/\( \sigma \)M framework is still unsettled, due to the apparent relevance of meson loops [25]. Namely, were we to neglect meson contributions to the \( a_0(985) \) mass in Eq. (A9), then it would be approximately degenerate with the \( \sigma_n \). On the other hand, if we simply evaluate Eq. (A9) with \( \tilde{m} = 337.5 \text{ MeV} \), \( \xi = 1 \), and \( \phi_P = (41.2 \pm 1.1)^\circ \) [82, 92], we obtain a surprisingly accurate mass prediction of 1012 MeV. The more realistic coupling ratio \( \xi = 0.9064 \) (see our discussion in the Appendix) would even imply \( m_{\sigma_n} \approx 990 \text{ MeV} \).

An alternative, more empirical way to estimate scalar-meson masses is by using quadratic mass differences motivated by the IMF. The resulting equal-splitting laws have a remarkable predictive power (see e.g. Ref. [25], Appendix C), being fully compatible with the foregoing QL\( \sigma \)M predictions.

As for the \( I = 1/2 \) scalar \( \kappa \) meson, seemingly doomed not so long ago [85], despite our old theoretical predictions [7, 77], it was first experimentally rehabilitated by the E791 collaboration [86], providing a mass value of
\[
m_{\kappa} = (797 \pm 19) \text{ MeV}.
\]

This is to be compared to 810 MeV from Eq. (88), 800 MeV from Refs. [7, 87], as well as the pole positions (727 – i \times 263) MeV [77] and (714 – i \times 228) MeV [88]. Note that the latter two coupled-channel model calculations both give a peak cross section around 800 MeV. On the other hand, a very recent analysis [89] of \( K \pi \) data from different high-statistics experiments yields the pole position
\[
m_{\kappa} = (760 \pm 20 \pm 40) - i \times (420 \pm 45 \pm 60) \text{ MeV}.
\]

As a final remark on the \( \kappa \), we should stress the importance of unitarization [82, 90] effects, which are capable of e.g. shifting the Lagrangian mass parameter in Eq. (A13) from 1128 MeV to a pole value of \( \sim 700–800 \) MeV, besides dynamically generating extra states [77, 88, 91]. However, in the latter reference, the authors introduce by hand a rather unphysical Adler zero, which apparently moves the dynamically generated \( \kappa \) pole far away from the physical region [79].

Next we discuss the \( f_0(980) \). The mass of \( (980 \pm 10) \) MeV listed by the PDG [1] is entirely compatible with the above values of 972 MeV and 1001.3 in Eqs. (87) and (90), respectively. So are the \( f_0(980) \) pole positions of \( (998 \pm 4) - i \times (17 \pm 4) \) MeV and \( (994 - i \times 14) \) MeV given in the very recent analysis of Ref. [89].

\(^1\) The PDG [1] further states that the nonstrange process \( a_1(1260) \rightarrow \pi \pi \) is the dominant mode, and that \( a_1(1260) \rightarrow f_0(980) \pi \) is not seen, presumably because the \( f_0(980) \) is (approximately) a scalar \( \bar{s}s \) state [83, 84].
with a specific $L\sigma M$, as shown in Appendix B of Ref. [38].

Therefore, non-unitarized ChPT is at odds with the entire ground-state scalar $\bar{q}q$ nonet\(^2\), as summarized in this section, if indeed the $L\sigma M$ of Ref. [38] possesses similar properties as the QLL$\sigma M$ discussed above.

**IX. $I = 0 \pi\pi$ Scattering Length**

Using the $L\sigma M$, PCAC, and crossing symmetry, Weinberg originally predicted [33] for the $I = 0$, $J = 0 \pi\pi$ scattering length

$$a_{00} = \frac{7}{32\pi f_\pi^2} m_\pi \approx 0.159 \text{ m}^{-1} ,$$

(93)

for $m_{\pi^+} = 139.57$ MeV, $f_\pi \approx 92.42$ MeV [1].

Stated slightly differently, and following V. de Alfaro et al. (see Ref. [19]), when working at the soft point $s = m_{\pi^+}^2$, one sees that the net $\pi\pi$ amplitude “miraculously vanishes” (via chiral symmetry), i.e.,

$$M'^{\text{set}}_{\pi\pi} = M^{\text{contact}}_{\pi\pi} + M^{\text{pole}}_{\pi\pi} = \lambda + 2g_\pi^2 (m_{\pi^+}^2 - m_\sigma^2) = 0 ,$$

(94)

since $g_{\pi\pi} = (m_{\pi^+}^2 - m_\sigma^2) / (2f_\pi) = \lambda f_\pi$ from the L$\sigma M$ Lagrangian. Thus, the contact term “chirally eats” the $\sigma$ pole at $s = m_{\pi^+}^2$, yielding the famous amplitude “Adler zero” slightly below the $\pi\pi$ threshold [78]. Crossing symmetry then extends Eq. (94) to [93]

$$M^{abcd}_{\pi\pi}(s, t, u) = A(s, t, u) \delta^{ab} \delta^{cd} + B(s, t, u) \delta^{ac} \delta^{bd} + C(s, t, u) \delta^{ad} \delta^{bc} ,$$

(95)

with

$$A^{\text{LL} \sigma M}(s, t, u) = -2\lambda \left( 1 - \frac{2\lambda f_\pi^2}{m_\sigma^2 - s} \right) = \frac{m_{\pi^+}^2 - m_\sigma^2}{m_\sigma^2 - s} \frac{s - m_{\pi^+}^2}{f_\pi^2} ,$$

(96)

so that the $I = 0$ S-wave amplitude $A + B + C$ generates a 23\% enhancement of Eq. (93):

$$a_{00} \big|_{\text{LL} \sigma M} = \frac{7 + \varepsilon}{1 - 4\varepsilon} \frac{m_\pi}{32\pi f_\pi^2} + O(\varepsilon^2) \approx 1.23 \frac{7}{32\pi f_\pi^2} m_\pi \approx 0.20 \text{ m}^{-1} ,$$

(97)

for $s = 4m_\pi^2$, $t = u = 0$, $\varepsilon = m_\pi^2 / m_\sigma^2 \approx 0.045$. Equation (97) is compatible with the recently measured ($K_{\pi\pi}$) E865 data [94]

$$a_{00} = 0.216 \pm 0.013 \text{ m}^{-1} .$$

(98)

However two-loop ChPT in Refs. [49, 51] predicts $a_{00} = 0.22 m_\pi^{-1}$ or $a_{00} = 0.23 m_\pi^{-1}$, using Roy equations instead of the L$\sigma M$. But we suggest Eqs. (93–97) tell the whole L$\sigma M$ story (and ChPT with very many parameters adds nothing new). We would like to stress that — even though Eq. (97) has been derived without taking into account $t$-channel vector-meson-exchange contributions — the result will change at most very slightly when including such contributions, as they are “chirally shielded” in $I = 0$ S-wave $\pi\pi$ scattering close to the $\pi\pi$ threshold, due to the presence of an extra contact term [95].

It is worth pointing out that Weinberg’s result Eq. (93), amounting to leading-order non-unitarized ChPT, is obtained from the L$\sigma M$ result Eq. (97) by taking the limit $m_\sigma \to \infty$. This limit essentially implies a nonlinear-$\sigma$-model framework, which unfortunately has been dominating theoretical descriptions of $\pi\pi$ scattering during the past three decades, despite its technical complications, deviations from experimental data, and the mounting experimental evidence for a light scalar-meson nonet.

**X. $\gamma\gamma \to \pi^0\pi^0$**

Assuming that the S-wave process $\gamma\gamma \to \pi^0\pi^0$ proceeds via an intermediate $\sigma$ resonance ($\gamma\gamma \to \sigma \to \pi^0\pi^0$), with $m_\sigma \simeq 700$ MeV, Ref. [96] anticipated that the cross section would be very small ($< 10$ nb). In fact, this was later confirmed [97] with Crystal-Ball data. Then, in 1991, Ref. [98] studied $a_1 \to \pi(\pi\pi)_S$, and used the Dirac identity

$$\frac{1}{p - m} - \frac{1}{p - m} \gamma_5 = -\gamma_5 \left( \frac{1}{p - m} - \frac{1}{p - m} \gamma_5 \right)$$

(99)

to verify that the sum of the quark-box and -triangle diagrams vanishes. The same holds for $\gamma\gamma \to \sigma \to \pi^0\pi^0$ [99]. The amplitude for the latter process would even vanish exactly in the CL. In the QLL$\sigma M$, this “chiral shielding”, already mentioned above, has been manifestly built in.

However, already in the abstract of the 1993 ChPT paper Ref. [100], it was stated that “the one-loop ChPT computation does not fit data, even close to threshold, because unitarity effects are important, even at very low energies”. In effect, the latter authors were arguing to unitarize ChPT, which later turned out [14, 15] to reinstate the broad $\sigma$ pole, and presumably even recovers the QLL$\sigma M$ theory. But then the nonperturbative effects of the $\sigma$ meson with a finite mass would have to be taken into account, too, in contrast with non-unitarized ChPT briefly discussed at the end of Sec. IX.

\(^2\) Very recently [15], Roy equations have been used to extract a $\sigma$ pole from low-energy $\pi\pi$ scattering data, in the framework of ChPT. However, the given pole position, especially the claimed very small error on it, is highly questionable in view of the neglect of the $K\pi$ channel.
Since this rate is given by infrared QCD, the QLL of strong interactions—ChPT and also ignoring very serious convergence problems [104]—that ChPT and low-energy QCD, while alleging that the QLLSM may seem far-fetched. However, the upshot is that the bulk of the predictions of the QLLSM does not depend on the (non-) Hermiticity of the Lagrangian, whereas certain observables can only be understood if the quark-meson coupling is taken (close to) imaginary. This might also provide a clue why quarks are not observed on mass-shell at low energies, which is supported by very recent DSE [111] and lattice [112] calculations. As an example of a process requiring a non-Hermitian QLLSM Lagrangian, consider the experimentally measured negative transition-form-factor ratio

$$\frac{f_{K^+\pi^0}(0)}{f_{K^+\pi^0}(0)} = -0.125 \pm 0.023 \ [1],$$

characterizing the semileptonic decay $K^+ \rightarrow \pi^0 e^+ \nu_e$, which can only be reproduced with an imaginary coupling [80].

The obvious advantage of describing strong interactions with a generalized QLLσM theory is its similarity [113] with the mechanism for spontaneous symmetry breaking in the electroweak sector. This allows to treat strong and electroweak interactions in many hadronic processes on an equal footing, and with a minimum of parameters, in sharp contrast with ChPT. For instance, the just mentioned decay $K^+ \rightarrow \pi^0 e^+ \nu_e$ is then dominantly described by a $W$-emission graph and a $\kappa$-exchange diagram [80]. From a more fundamental point of view, the manifest finite divergence of the axial vector current underlying the QLLσM theory displays an instability of the effective action describing strong interactions in the phase of broken chiral symmetry. It yields an instability of strongly interacting Goldstone bosons of ChSB like the pion due to electroweak decays. Nevertheless, the sum of the effective actions describing strong and electroweak interactions must be stable. This imposes rigid constraints on the overall sum of one-point functions, which will intimately relate the parameters of strong and electroweak interactions to one another.

To conclude, it seems amazing that the Higgs scalar of the SMPP, with an estimated mass of order $10^3$ MeV, takes responsibility for the tiny nonstrange current quark masses and the ensuing nonvanishing pion mass, while the scalar $\sigma$ meson with a mass of about $5 \times 10^2$ MeV spontaneously generates considerably heavier dynamical quark masses, moreover in such a way that the sum of the nonstrange current and dynamical quark masses, i.e., the constituent quark masses, make up in the end practically all of the proton’s mass. Thus, let us make ours the constituent quark masses, make up in the end practically all of the proton’s mass.
ACKNOWLEDGMENTS

We are indebted to E. van Beveren for useful discussions and suggestions. This work was supported by the Fundação para a Ciência e a Tecnologia of the Ministério da Ciência, Tecnologia e Ensino Superior of Portugal, under contract POCTI/FP/FNU/50328/2003 and grant SFRH/BPD/9480/2002.

APPENDIX A: THE “TRADITIONAL” \(U(3) \times U(3)\) LINEAR \(\sigma\) MODEL

1. Lagrangian, mass parameters, and mixing angles

The Lagrangian of the “traditional” \(U(3) \times U(3)\) \(\sigma\)M before spontaneous symmetry breaking is given by [21, 22, 27, 81, 82]

\[
\mathcal{L} = \frac{1}{2} \left[ \text{tr}(\partial_{\mu} \Sigma_{+}) (\partial^{\mu} \Sigma_{-}) - \frac{1}{2} \mu^{2} \text{tr}[\Sigma_{+} \Sigma_{-}] \right]
- \frac{\lambda}{2} \left[ \text{tr}[\Sigma_{+} \Sigma_{+}] \Sigma_{+} - \frac{\lambda'}{4} \left( \text{tr}[\Sigma_{+}] \right)^{2} \right]
+ \frac{\beta}{2} \left( \text{det}[\Sigma_{+}] + \text{det}[\Sigma_{-}] \right) + \text{tr}[CS],
\]

(A1)

with \(\Sigma_{\pm}(x) \equiv S(x) \pm iP(x)\) (see Eq. (82)). It is convenient to define also \(\sigma_{n} \equiv (\sigma_{u} + \sigma_{d})/\sqrt{2}, \quad \sigma_{s} \equiv (\sigma_{u} - \sigma_{d})/\sqrt{2}, \quad \sigma_{3} \equiv (\sigma_{u} - \sigma_{d})/\sqrt{2} \approx -\eta, \quad \eta_{n} \equiv (\eta_{u} + \eta_{d})/\sqrt{2}, \quad \eta_{3} \equiv (\eta_{u} - \eta_{d})/\sqrt{2} \approx \eta_{1}\). Scalar and pseudoscalar meson masses are then determined after (isospin-symmetric) spontaneous symmetry breaking \(S \to S - D\), with \(D \equiv \text{diag}(a, b, b), \quad f_{\pi} = \sqrt{2}a, \quad f_{K} = (a + b)/\sqrt{2}, \quad \beta = -\beta_{3}\), and \(X \equiv a/b\), by [82]

\[
m_{a_{u}} = m_{n}^{2} + 6\lambda a^{2} + \beta, \\
m_{a_{d}} = m_{n}^{2} + 6\lambda + 4\lambda' a^{2} - \beta, \\
m_{a_{s}} = m_{n}^{2} + 6\lambda + 2\lambda' b^{2}, \\
m_{s} = m_{n}^{2} + 2\lambda(a^{2} + b^{2} + ab) + \beta X, \\
m_{\pi} = m_{n}^{2} + 2\lambda a^{2} - \beta, \\
m_{\eta_{0}} = m_{n}^{2} + 2\lambda a^{2} + \beta, \\
m_{\eta_{3}} = m_{n}^{2} + 2\lambda b^{2}, \\
m_{K} = m_{n}^{2} + 2\lambda(a^{2} + b^{2} - ab) - \beta X,
\]

(A2)

with \(\mu^{2} = \mu^{2} + \lambda' (2a^{2} + b^{2})\), and

\[
\sin(2\phi_{P}) = 2\sqrt{2}\beta X/(m_{\pi}^{2} - m_{\eta}^{2}), \quad \sin(2\phi_{S}) = 2\sqrt{2}(-\beta X + 2\lambda ab)/(m_{\eta_{0}}^{2} - m_{\sigma}^{2}).
\]

(A3)

where \(\sigma = \sigma_{n} \cos \phi_{S} - \sigma_{s} \sin \phi_{S}, \quad f_{0} = \sigma_{n} \sin \phi_{S} + \sigma_{s} \cos \phi_{S}, \quad \eta = \eta_{n} \cos \phi_{P} - \eta_{s} \sin \phi_{P}, \quad \eta' = \eta_{n} \sin \phi_{P} + \eta_{s} \cos \phi_{P}\). On the basis of Eqs. (A2) and the GTRs \(f_{xy} = \tilde{m} \text{ and } f_{xy} = (\tilde{m} + m_{s})/2\), it is straightforward to derive the \(\sigma\)M predictions

\[
m_{a_{0}}^{2} = m_{\eta_{0}}^{2} + 2(\lambda/g^{2}) \tilde{m}^{2}, \\
m_{a_{2}}^{2} = m_{\eta_{2}}^{2} + 2(\lambda/g^{2}) m_{s}^{2} (1 + \lambda'/2\lambda), \\
m_{\pi}^{2} = m_{n}^{2} + 2(\lambda/g^{2}) \tilde{m} m_{s} + 2\beta X.
\]

(A4)

Inspired by the important relation \(\lambda/g^{2} \approx 2\), which was obtained by Delbourgo and Scadron [23] on the basis of one-loop dynamical generation of the QLL\(\sigma\)M, we define the coupling ratio \(\xi \equiv \lambda/(2g^{2}) \approx 1\), which allows us, also using Eq. (A3), to write Eqs. (A5–A8) in the more convenient form

\[
m_{a_{0}}^{2} \approx \xi (2\tilde{m})^{2} + m_{\eta_{0}}^{2} \cos^{2} \phi_{P} + m_{\eta}^{2} \sin^{2} \phi_{P}, \quad (A9) \\
m_{a_{2}}^{2} \approx \xi (2\tilde{m})^{2} (1 + \lambda'/\lambda) + m_{s}^{2}, \quad (A10) \\
m_{\pi}^{2} \approx \xi (2m_{s})^{2} (1 + \lambda'/2\lambda), \\
+ m_{\eta}^{2} \sin^{2} \phi_{P} + m_{\eta_{0}}^{2} \cos^{2} \phi_{P}, \quad (A11) \\
m_{\eta_{2}}^{2} \approx \xi (2\tilde{m}) (2m_{s}) \\
+ m_{K}^{2} + (m_{\eta_{0}}^{2} - m_{\eta_{3}}^{2}) \sin(2\phi_{P})/\sqrt{2}. \quad (A12)
\]

Note that Eqs. (A2) unambiguously imply for the mass parameter of the \(\kappa\) meson in the \(U(3) \times U(3)\) \(\sigma\)M

\[
m_{K}^{2} = (f_{K}/f_{\pi}) m_{K}^{2} - m_{\pi}^{2} = \frac{(f_{K}/f_{\pi}) - 1}{(1842 MeV)^{2}} \approx 1128 MeV^{2}, \quad (A13)
\]

with \(f_{K}/f_{\pi} \approx 1.22\), and the isospin-averaged masses \(m_{\pi} = 138.0 \text{ MeV}, \quad m_{K} = 495.0 \text{ MeV}\). Note that, as hinted already in Sec. VIII, unitarization will then split such a single “bare” \(\kappa\) state into the pair of physical resonances \(\kappa(800)\) and \(K_{b}(1430)\) [88]. Nevertheless, employing the value for \(m_{\kappa}\) from Eq. (A13), we may use Eq. (A12), \(m_{\eta} = 547.75 \text{ MeV}, \quad m_{\eta_{0}} = 957.78 \text{ MeV}, \quad \tilde{m} = 1.44 \times \tilde{m}\) to determine the pseudoscalar mixing angle \(\phi_{P}\) as a function of \(\tilde{m}\):

\[
\sin(2\phi_{P}) \approx \sqrt{2} \left( \frac{m_{K}^{2} - m_{\pi}^{2} - \xi (2\tilde{m}) (2m_{s})}{m_{\eta_{0}}^{2} - m_{\eta}^{2}} \right). \quad (A14)
\]

The next step is to determine the scalar mixing angle \(\phi_{S}\) as a function of \(\tilde{m}\) and some given value of the coupling ratio \(\lambda'/\lambda\) (which is experimentally known to be very small [82]), via the identity

\[
\tan(2\phi_{S}) \approx \frac{\sqrt{2} \xi (2\tilde{m}) (2m_{s}) \lambda'/\lambda - \sin(2\phi_{P}) (m_{\eta_{0}}^{2} - m_{\eta}^{2})}{m_{\sigma}^{2} - m_{\sigma_{n}}^{2}}, \quad (A15)
\]

where \(m_{\sigma}^{2}\) and \(m_{\sigma_{n}}^{2}\) are given by Eqs. (A10) and (A11), respectively. Given \(\phi_{S}\), we can finally write \(m_{\sigma}\) and \(m_{f_{0}}\) as

\[
m_{\sigma}^{2} = \frac{1}{2} \left( m_{\sigma}^{2} + m_{\sigma_{n}}^{2} - m_{\sigma_{n}}^{2} - m_{\sigma_{n}}^{2} \sin(2\phi_{S}) \right), \quad (A16) \\
m_{f_{0}}^{2} = \frac{1}{2} \left( m_{\sigma}^{2} + m_{\sigma_{n}}^{2} - m_{\sigma_{n}}^{2} - m_{\sigma_{n}}^{2} \sin(2\phi_{S}) \right). \quad (A17)
\]

The importance of the aforementioned “traditional” \(U(3) \times U(3)\) \(\sigma\)M lies in the fact that it has many features of a yet to be determined effective action, constructed on the basis of the QLL\(\sigma\)M by integrating out
quarks and disregarding vector and axial-vector mesons. Furthermore, it provides a mechanism that simultaneously — and almost quantitatively — explains mixing for pseudoscalar as well as for scalar mesons. Empirically we notice that the model, with its limitations (no (axial) vector mesons, no two-loop effects), prefers a coupling ratio \( \xi \approx 0.9064 \), for \( X^{-1} = m_s/m = 1.44 \) and \( \tilde{m} = 337.5 \text{ MeV} \), in order to match the experimentally favored value \( \phi_{\text{phys}} = (41.2 \pm 1.1)° \) [82, 92] (compare also to Refs. [115] and [116], for newer and older experimental/theoretical findings for \( \phi_{\text{phys}} \), respectively), and \( \phi_{\text{phys}} \approx (18 \pm 2)° \) [84, 117]. On the other hand, the one-loop-generated value \( \xi = 1 \), by the same token, rather small, almost chiral-limiting quark-mass values. In contrast with the theoretical uncertainties in \( \phi_{\text{phys}} \) stemming from the mentioned limitations of the “traditional” \( U(3) \times U(3) \) LoM, there is at least one way to justify, in a quite model-independent fashion, the theoretical predictions of the “traditional” \( U(3) \times U(3) \) LoM for \( \phi_{\text{phys}} \), to be explained next.

We recall the pseudoscalar mass matrix of \( \eta \) and \( \eta' \), the diagonalization of which led to Eqs. (A3) and (A14):

\[
\begin{pmatrix}
    m_\eta^2 + 2\beta \\
    \sqrt{2}\beta X \\
    \sqrt{2}\beta X
\end{pmatrix} = 2m^2_K - m^2_\eta + \beta X^2 \quad (A18)
\]

For \( (\beta + 2\lambda\beta^2)(1 - X)^2 = 0 \), this reduces to the mass matrix

\[
\begin{pmatrix}
    m_\eta^2 + 2\beta \\
    \sqrt{2}\beta X \\
    \sqrt{2}\beta X
\end{pmatrix} = 2m^2_K - m^2_\eta + \beta X^2 \quad (A19)
\]

motivated and discussed in Refs. [117–119], the diagonalization of which implies in a “model-independent” way:

\[
\beta = \frac{(m_\eta^2 - m_\eta')^2(m_\eta^2 - m_\eta'^2)}{4(m^2_K - m^2_\eta)} \approx 0.2792 \text{ GeV}^2, \quad (A20)
\]

\[
X = \sqrt{\frac{2(m_\eta^2 - 2m^2_K + m^2_\eta')(2m^2_K - m^2_\eta - m^2_\eta')}{(m_\eta^2 - m_\eta'^2)(m_\eta^2 - m^2_\eta')}} \approx 0.7776 = 1/1.2860, \quad (A21)
\]

\[
\phi_{\text{phys}} = \arctan\left(\frac{(m_\eta^2 - 2m^2_K + m^2_\eta')(m_\eta^2 - m^2_\eta)}{(2m^2_K - m^2_\eta - m^2_\eta')(m_\eta^2 - m^2_\eta')2}\right) \approx 42.1°. \quad (A22)
\]

Notice that this \( \phi_{\text{phys}} \) is fully compatible with the optimal experimental value of \( (41.2 \pm 1.1)° \). Recalling now the definitions \( \eta \equiv (\eta_uu + \eta_dd - 2\eta_s\bar{s})/\sqrt{3} \) and \( \eta_s \equiv (\eta_uu + \eta_dd - 2\eta_s\bar{s})/\sqrt{6} \), concludes, using Eqs. (A19), (A20), (A21), and (A22), with the following “model-independent” predictions:

\[
m^2_\eta = m^2_\eta + 2\beta \simeq (760.0 \text{ MeV})^2, \quad (A23)
\]

\[
m^2_\eta' = 2m^2_K - m^2_\eta + \beta X^2 \simeq (799.9 \text{ MeV})^2, \quad (A24)
\]

\[
m^2_\eta = \frac{2m^2_K + m^2_\eta}{3} + \frac{\beta X^2}{3} \simeq (942.2 \text{ MeV})^2, \quad (A25)
\]

\[
m^2_\eta' \simeq \frac{4m^2_K - m^2_\eta}{3} - \frac{2\beta(X - 1)^2}{3} \simeq (574.1 \text{ MeV})^2. \quad (A26)
\]

Note that indeed [117] \( m_\eta^2 + m_\eta'^2 = 1.21744 \text{ GeV}^2 \approx m_\eta^2 + m_\eta'^2 = 1.21733 \text{ GeV}^2 \approx m_\eta^2 + m_\eta'^2 = 1.21737 \text{ GeV}^2 \), as should be in our mixing scheme employing squared masses.

2. Remark on the CL of the “traditional” \( U(3) \times U(3) \) LoM

In the CL, \( m_\pi \to m_\pi^{\text{CL}} = 0 \) and \( m_K \to m_K^{\text{CL}} = 0 \) should hold. Hence, the “traditional” \( U(3) \times U(3) \) LoM predicts, due to Eq. (A13), in the CL

\[
(m_\pi^{\text{CL}})^2 \left(\frac{f_K^{\text{CL}}}{f_\pi^{\text{CL}}} - 1\right) = \frac{f_\pi^{\text{CL}}}{f_\pi^{\text{CL}}} \left(m_\pi^{\text{CL}}\right)^2 - \left(m_\pi^{\text{CL}}\right)^2 \quad \approx 0, \quad (A27)
\]

implying \( (\beta + 2\lambda\beta^2)(1 - X)^2 \text{CL} = 0 \), i.e., either exact flavor symmetry \( f_\pi^{\text{CL}} = f_\pi^{\text{CL}} \) (\( m_\pi^{\text{CL}} = m_\pi^{\text{CL}} \leftrightarrow X_\pi^{\text{CL}} = 1 \)) or \( m_\pi^{\text{CL}} = 0 \), which is not very desirable according to our foregoing discussion.

We thus learn that the “traditional” \( U(3) \times U(3) \) LoM — despite its success in describing mixing angles — cannot be (without changes e.g. in the ’t Hooft determinant, or further extensions like the inclusion of (axial-)vector mesons) the effective \( U(3) \times U(3) \) meson LoM that would result from the QLLLoM by integrating out quarks.

Fortunately, we know that the mass matrices Eq. (A18) and (A19) describing the \( \eta - \eta' \) system are rather insensitive to likely changes. Hence, their predictions remain reliable even in the CL, where they reduce to [117–119]

\[
\left(\frac{2\beta}{\sqrt{2}\beta X}ight)^{\text{CL}} = \beta_{\text{CL}} \left(\frac{\sqrt{2}}{X_{\text{CL}}} \right) \left(\frac{\sqrt{2}}{X_{\text{CL}}} \right), \quad (A28)
\]

with eigenvalues \( (m_\eta^{\text{CL}})^2, (m_\eta'^{\text{CL}})^2 \in \{0, \beta_{\text{CL}}(2 + X_{\text{CL}}^2)\} \).

[1] S. Eidelman et al. [Particle Data Group], Phys. Lett. B 592, 1 (2004).
[2] A. Marques, Cesar Lattes — a short account of his life (http://www.cbpf.br/∼hadron05/talks/sexta/plenarias/)
[3] V. Elias and M. D. Scadron, Phys. Rev. D 30, 647
M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. D 26, 341 (1982).

J. A. Oller, E. Oset, and J. R. Pelaez, Phys. Rev. D 83, 074005 (2011) [arXiv:hep-ph/0304067]; R. E. Karlsen, M. D. Scadron, and A. Bramon, Phys. Rev. D 80, 014010 (2004) [arXiv:hep-ph/0312187].

M. D. Scadron, C. Foudas, and M. D. Scadron, Modern Phys. Lett. A 10, 251 (1995) [arXiv:hep-ph/9910242].

A. van Beveren, F. Kleefeld, G. Rupp, and M. D. Scadron, Mod. Phys. Lett. A 17, 1673 (2002) [arXiv:hep-ph/0204139].

M. D. Scadron, C. Foudas, and M. D. Scadron, Phys. Rev. D 69, 014010 (2004) [Erratum-ibid. D 69, 059901 (2004)] [arXiv:hep-ph/0309109].

G. E. Hite, R. J. Jacob, and M. D. Scadron, Phys. Rev. D 82, 074005 (2011) [arXiv:hep-ph/0903050].

M. D. Scadron, C. Foudas, and M. D. Scadron, Phys. Rev. D 83, 074005 (2011) [arXiv:hep-ph/0304067].

R. E. Karlsen, M. D. Scadron, and A. Bramon, Phys. Rev. D 80, 014010 (2004) [arXiv:hep-ph/0312187].

M. D. Scadron, C. Foudas, and M. D. Scadron, Phys. Rev. D 83, 074005 (2011) [arXiv:hep-ph/0304067].

R. Delbourgo and M. D. Scadron, Int. J. Mod. Phys. A 13, 657 (1998) [arXiv:hep-ph/9807504].

Y. S. Surovtsev, D. Krupa, and M. Nagy, Eur. Phys. J. A 15, 409 (2002) [arXiv:hep-ph/0204007].

M. Nagy, M. D. Scadron, and G. E. Hite, Acta Phys. Slov. 54, 427 (2004) [arXiv:hep-ph/0406009].

M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

S. Cabasino et al. [APE Collaboration], Phys. Lett. B 258, 202 (1991).

S. L. Adler, Phys. Rev. 137, B1022 (1965).

S. Weinberg, Phys. Rev. Lett. 17, 616 (1966).

T. P. Cheng and R. F. Dashen, Phys. Rev. Lett. 26, 594 (1971).

R. Koch, Z. Phys. C 15, 161 (1982); G. Höhler, “Pion-Nucleon Scattering At Low-Energies: Pion-Nucleon Sigma Term”, TKP 83-4, Jan 1983, 31pp, Preprint from Landolt-Boernstein, New Series, Vol.I/9b, Pt. 2., Ed. H. Schopper, Springer-Verlag.

G. Clement, M. D. Scadron, and J. Stern, J. Phys. G 17, 199 (1991); Z. Phys. C 60, 307 (1993); S. A. Coon and M. D. Scadron, J. Phys. G 18, 1923 (1992).

J. Gasser and H. Leutwyler, Phys. Rept. 87, 77 (1982).

J. Gasser and H. Leutwyler, Annals Phys. 158, 142 (1984).

J. F. Donoghue, J. Gasser, and H. Leutwyler, Nucl. Phys. B 343, 341 (1990).

J. Gasser, H. Leutwyler, and M. E. Sainio, Phys. Lett. B 253, 252 (1991); Phys. Lett. B 253, 260 (1991).

H. Leutwyler, “Nonperturbative methods”, BUTP-92-42 Rapporteur talk given at 26th Intern. Conf. on High Energy Physics (ICHEP 92), Dallas, TX, 6–12 Aug. 1992.

M. D. Scadron, F. Kleefeld, G. Rupp, and E. van Beveren, Nucl. Phys. A 724, 391 (2003) [arXiv:hep-ph/0211275].

V. G. Vaks and B. L. Ioffe, Nuovo Cim. 10, 342 (1958).

J. J. Sakurai, Annals Phys. 11, 1 (1960); Currents and Mesons — Chicago Lectures in Physics, The University of Chicago Press, 1973.

D. Djukanovic, M. R. Schindler, J. Gegelia, G. Japaridze, and S. Scherer, Phys. Rev. Lett. 93, 122002 (2004) [arXiv:hep-ph/0407239].

A. Bramon, Riazuddin, and M. D. Scadron, J. Phys. G 24, 1 (1998) [arXiv:hep-ph/9709274].

J. F. Donoghue and B. R. Holstein, Phys. Rev. D 40, 2378 (1989).

G. Colangelo, J. Gasser, and H. Leutwyler, Phys. Rev. Lett. 86, 5008 (2001) [arXiv:hep-ph/0103063].

B. Ananthanarayan, I. Caprini, G. Colangelo, J. Gasser, and H. Leutwyler, Phys. Lett. B 602, 218 (2004) [arXiv:hep-ph/0409222].

Let us quote here from the Introduction of this paper: “Early work on the scalar form factors of the pion (…) was motivated by the search for a very light Higgs particle. Unfortunately, the outcome of this search was negative: nature is kind enough to let us probe the vector and axial currents, but allows us to experimentally explore only those scalar and pseudoscalar currents that are connected with flavor symmetry breaking.” We believe this very statement speaks volumes on the physical relevance of the pion scalar form factor.

F. J. Yndurain, Phys. Lett. B 612, 245 (2005) [arXiv:hep-ph/0501104].

J. R. Pelaez and F. J. Yndurain, Phys. Rev. D 68, 074005 (2003) [arXiv:hep-ph/0304067]; 69, 114001 (2004) [arXiv:hep-ph/0312187].

A. Bramon and M. D. Scadron, Europhys. Lett. 19, 663 (1992); R. E. Karlsen, M. D. Scadron, and A. Bramon, Mod. Phys. Lett. A 8, 97 (1993).

R. Tarrach, Z. Phys. C 2, 221 (1979); S. B. Gerasimov,
[103] M. Schumacher, Prog. Part. Nucl. Phys. 55, 567 (2005) [arXiv:hep-ph/0501167]; M. I. Levchuk, A. I. L’vov, A. I. Milstein, and M. Schumacher, arXiv:hep-ph/0511193.

[104] T. Hatsuda, Phys. Rev. Lett. 65, 543 (1990).

[105] F. Kleefeld, AIP Conf. Proc. 660, 325 (2003) [arXiv:hep-ph/0211460].

[106] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).

[107] Proc. First Workshop on “Pseudo-Hermitian Hamiltonians in Quantum Physics”, 16–17 June 2003, Villa Lanna, Prague, Czech Republic, published in Czech. J. Phys. 54, 1–156 (2004).

[108] Proc. Second Workshop on “Pseudo-Hermitian Hamiltonians in Quantum Physics”, 14–16 June 2004, Villa Lanna, Prague, Czech Republic, published in Czech. J. Phys. 54, 1005–1148 (2004).

[109] Proc. Third Workshop on “Pseudo-Hermitian Hamiltonians in Quantum Physics”, 20–22 June 2005, Koç University, Istanbul, Turkey, published in Czech. J. Phys. 55, 1049–1192 (2005).

[110] F. Kleefeld, J. Phys. A: Math. Gen. 39, L9 (2006) [arXiv:hep-th/0506142].

[111] R. Alkofer, W. Detmold, C. S. Fischer, and P. Maris, Phys. Rev. D 70, 014014 (2004) [arXiv:hep-ph/0309077].

[112] M. S. Bhagwat and P. C. Tandy, arXiv:nucl-th/0601020; S. Furui and H. Nakajima, arXiv:hep-lat/0511045; P. O. Bowman, U. M. Heller, D. B. Leinweber, M. B. Parappilly, A. G. Williams, and J. b. Zhang, Phys. Rev. D 71, 054507 (2005) [arXiv:hep-lat/0501019].

[113] N. A. Törnqvist, Phys. Lett. B 619, 145 (2005) [arXiv:hep-ph/0504204].

[114] E. Farhi and R. Jackiw, Dynamical Gauge Symmetry Breaking: A Collection Of Reprints, World Scientific, Singapore (1982).

[115] R. Escribano and J. M. Frere, JHEP 0506, 029 (2005) [arXiv:hep-ph/0501072]; A. Aloisio et al. [KLOE Collaboration], Phys. Lett. B 541, 45 (2002) [arXiv:hep-ex/0206010]; A. Bramon, R. Escribano, and M. D. Scadron, Eur. Phys. J. C 7, 271 (1999) [arXiv:hep-ph/9711229].

[116] A. Bramon, R. Escribano, and M. D. Scadron, Phys. Lett. B 403, 339 (1997) [arXiv:hep-ph/9703313]; 503, 271 (2001); C. Amsler, Rev. Mod. Phys. 70, 1293 (1998) [arXiv:hep-ex/9708025]; C. Amsler et al. [Crystal Barrel Collaboration], Phys. Lett. B 294, 451 (1992).

[117] D. Kekez, D. Klabučar, and M. D. Scadron, J. Phys. G 26, 1335 (2000) [arXiv:hep-ph/0003234]; D. Klabučar, D. Kekez, and M. D. Scadron, J. Phys. G 27, 1775 (2001) [arXiv:hep-ph/0101324].

[118] H. F. Jones and M. D. Scadron, Nucl. Phys. B 155, 409 (1979); M. D. Scadron, Phys. Rev. D 29, 2076 (1984).

[119] D. Kekez and D. Klabučar, arXiv:hep-ph/0212286; arXiv:hep-ph/0512064; D. Kekez, D. Klabučar, and M. D. Scadron, Int. J. Mod. Phys. A 20, 6189 (2005) [arXiv:hep-ph/0512123].