The existence of a local solution to the $Sp(2)$ master equation for gauge field theory is proven in the framework of perturbation theory and under standard assumptions on regularity of the action. The arbitrariness of solutions to the $Sp(2)$ master equation is described, provided that they are proper. It is also shown that the effective action can be chosen to be $Sp(2)$ and Lorentz invariant (under the additional assumption that the gauge transformation generators are Lorentz tensors).
1 Introduction

It is well known that the quantization methods for gauge theories in Lagrangian and Hamiltonian formalisms can be generalized to include ghost and antighost variables in a symmetric way [1]–[10]. In that case two nilpotent (BRST and anti–BRST) charges appears. Ghost and antighost variables, BRST and anti–BRST charges and the pair of equations for the effective action form $Sp(2)$ doublets. In [2]–[7], the formal proof of the existence of a solution to the $Sp(2)$ master equation and description of the arbitrariness of solutions were given. In those papers, however, the locality of the effective action was assumed as a hypothesis. In [11], the existence of a local solution to the $Sp(2)$ master equation was demonstrated by using the Hamiltonian $Sp(2)$ formalism in the case of finite number of degrees of freedom (it was shown also that the Lagrangian and Hamiltonian $Sp(2)$ formalisms are equivalent).

In the present paper, we shall prove the existence of a local solution to the $Sp(2)$ master equation in the field theory. For the standard master equation of the BV formalism the existence of a local solution, the structure of renormalization, anomalies and related questions have been discussed in literature (see [12], [13], [14] and references therein). The method for solving the locality problem is based on considering instead of equations for functionals, equations for corresponding functions on spaces, which have the fields and their finite order derivatives in an arbitrary spacetime point as coordinates (jet–spaces [15]). We apply these scheme to the $Sp(2)$ master equation. In sec. 2 preliminary of the problem and assumptions on regularity of the action are given. In sec. 3 we give the proof of the existence of a local solution to the $Sp(2)$ master equation. In sec. 4 we discuss the arbitrariness of solutions of the equations and show, under some restrictions on the structure of the solution in the first order in the ghost and gauge introducing fields, that the arbitrariness of solutions is described by special transformations (gauge transformations), which do not change the physical contents of the theory. It is also shown that, under some additional assumptions on gauge generators, there exist $Sp(2)$ symmetric and Lorentz invariant solutions to the $Sp(2)$ master equation.

In what follows $\partial_\mu$ designates the total derivative with respect to $x^\mu$, $\partial_{\mu_1\ldots\mu_k} \equiv \partial_\mu_1 \cdots \partial_\mu_k$, $D_{\mu_1\ldots\mu_k} \equiv D_\mu_1 \cdots D_\mu_k$. By a local finite order differential operator (LFODO), we mean the following

$$ M = \sum_{k=0}^{n} g^{\mu_1\ldots\mu_k} \partial_{\mu_1\ldots\mu_k}, $$

where $g^{\mu_1\ldots\mu_k}$ are functions of $x^\mu$, of the field variables and their finite order derivatives, $n$ is a finite number. Transposed LFODO $M^{ij}$ relative to $M$ is defined as

$$ (M^{Tij} \varphi_j) \psi_i - \varphi_i M^{ij} \psi_j = \partial_\mu j^\mu_i - (M^{-1})^T = (M^T)^{-1}, \quad (M^T)^T = (-1)^{\varepsilon_i + \varepsilon_j} M^{ij}, $$

$$ \varepsilon(\varphi^i) = \varepsilon_i, \quad \varepsilon(M^{ij}) = \varepsilon_i + \varepsilon_j $$

with arbitrary functions $\varphi_i$, $\psi_i$ and local functions $j^\mu_i$. All considerations are performed in the framework of perturbation theory, i.e., in terms of formal power series in field variables and their derivatives. Throughout this article all functions can depend explicitly on $x^\mu$, unless otherwise is stated.
2 Preliminary of the problem

In this section we shall discuss the formulation of the problem and assumptions under which it will be solved. The full set of variables $T^\Sigma$ is divided into the groups $\Phi^A, \Phi^*_{ab}, \Phi_A$, $a, b, c = 1, 2$. The variables $\Phi^A$ are $\Phi^A = (\varphi^i, C_{ab}^i, B^\alpha)$, where $\varphi^i$ are variables of the classical theory, while $C_{ab}^i$ and $B^\alpha$ are respectively ghost, antighost and gauge introducing fields. All variables are ascribed by the Grassman parity $\varepsilon(\Phi^A) = \varepsilon(\Phi_A) = \varepsilon_A, \varepsilon(\Phi^*_{ab}) = \varepsilon_A + 1, \varepsilon(B^\alpha) = \varepsilon_\alpha$, $\varepsilon(C_{ab}^i) = \varepsilon_\alpha + 1$ and the new Grassman parity $\text{ngh}$: $\text{ngh}(\varphi) = 0$, $\text{ngh}(B) = 2$, $\text{ngh}(\Phi) = -2 - \text{ngh}(\Phi)$, $\text{ngh}(\Phi^*_{ab}) = -1 - \text{ngh}(\Phi)$, $\text{ngh}(F) = \text{ngh}(M)$. Moreover, the fields $\varphi^i, B^\alpha, \bar{\varphi}_i, \bar{B}_\alpha$ form $Sp(2)$ singlets. $C_{ab}^i$ and $\varphi^*_i$, $B^\alpha_{ab}, \bar{C}_{aa}$ form respectively $Sp(2)$ doublets and antiduales, $C_{ab|ia}$ transforms as a product of $Sp(2)$ antiduales. In the $Sp(2)$ formalism the effective action $S(\Phi, \Phi^*_a, \bar{\Phi})$ satisfies the $Sp(2)$ master equation

\[
\frac{1}{2}(S, S)^a + \int dx \varepsilon^{ab} \Phi^*_A \frac{\delta}{\delta \Phi_A} S = 0, \quad \varepsilon^{ab} = -\varepsilon^{ba}, \quad \varepsilon_{ab} \varepsilon^{bc} = \delta^c_a, \quad \varepsilon_{12} = \varepsilon^{21} = 1, \quad (1)
\]

and the boundary condition

\[
S|_{\Phi^*_a = \Phi = 0} = S(\varphi).
\]

In $[,]^a$ denotes the doublet of antibrackets

\[
(F, G)^a \equiv \int dx \left( F \frac{\delta}{\delta \Phi^*_A} \frac{\delta}{\delta \Phi^*_a} G - F \frac{\delta}{\delta \Phi^*_a} \frac{\delta}{\delta \Phi^*_A} G \right),
\]

where $S(\varphi)$ is the original classical action, which has a gauge symmetry: $R_i^a \delta S / \delta \varphi_i = 0$. It is assumed that the effective action conserves ngh, has zero Grassman parity and forms $Sp(2)$ singlet. Equation (1) will be solved in the framework of perturbation theory in fields $C_{ab}^i$ and $B^\alpha$:

\[
S = S + \sum_{k=1}^{n} S_k, \quad S_k \sim C^l B^m, \quad l + m = k.
\]

Let us assume that the terms $S_k$ are proved to exist for $k = 1, ..., n$, that is, equation (1) is satisfied in orders $C^l B^{k-l}, k \leq n$. In the $(n+1)$th order in fields $C_{ab}^i$ and $B^\alpha$, we obtain equation for $S_{n+1}$:

\[
W^a S_{n+1} = F^a_{n+1}, \quad F^a_{n+1} = \frac{1}{2} (S_{[n]}, S_{[n]})^a_{n+1}, \quad S_{[n]} = S + \sum_{k=1}^{n} S_k, \quad (2)
\]

where (see next Sec.):

\[
W^a = \int dx \left( (-1)^{\varepsilon_i} L_i \frac{\delta}{\delta \varphi^*_i} - (-1)^{\varepsilon_A} R^A_{\alpha} \varphi^*_{ab} \frac{\delta}{\delta C^*_{ab} \varphi^*_{i}} + ((-1)^{\varepsilon_i} R^i_{\alpha} \bar{\varphi}^i + \varepsilon_{ab} C^*_{ab | i}) \frac{\delta}{\delta B^\alpha} \right)\]

\[
- (-1)^{\varepsilon_u} \varepsilon^{ab} B^\alpha \frac{\delta}{\delta C^{|ab}} + \varepsilon_{ab} \Phi^*_A \frac{\delta}{\delta \Phi^*_A}
\]

3
\[ L_i(x) \equiv \delta S / \delta \varphi_i(x). \]

The operators \( W^a \) are nilpotent, i.e.,
\[ W^a W^b + W^b W^a = 0. \]

\( W^a \) and \( F^a_{n+1} \) form \( Sp(2) \) doublets. The Jacobi identity for doublet of antibrackets or explicit form of \( F^a_{n+1} \) implies the consistency condition
\[ W^a F^b_{n+1} + W^b F^a_{n+1} = 0, \tag{3} \]
for equation (2). One can see that
\[ F^a_{n+1} = W^a Y_{n+1} \tag{4} \]
is a solution to equation (3). If this is the general solution we may choose \( Y_{n+1} \) as \( S_{n+1} \).

We shall show, that under some assumptions formulated below the general solution to equation (4) indeed has the form (5). The main difficulty in solving equation (4) is to obtain a solution belonging to the class of local functionals, i.e., the functionals \( S_k \) should have the form \( S_k = \int dx s_k(\Gamma^\Sigma, \partial \Gamma^\Sigma, ...) \). Consequently, \( F^a_{n+1} \) will be \( F^a_{n+1} = \int dx f^a_{n+1}(\Gamma^\Sigma, \partial \Gamma^\Sigma, ...) \). The operators \( W^a \) on arbitrary local functions are:
\[ W^a Z = \int dx w^a(x(\Gamma^\Sigma, \partial \Gamma^\Sigma, ...)), \]
\[ w^a = \sum_{k=0} (-1)^{\varepsilon_i} D_{\mu_1...\mu_k} L_i \frac{\partial}{\partial(\mu_1...\mu_k \varphi^*_{ia})} - \sum_{k=0} (-1)^{\varepsilon_a} D_{\mu_1...\mu_k} (R^i_{\alpha \varphi^* c}) \frac{\partial}{\partial(\mu_1...\mu_k C^*_a)} + \]
\[ \sum_{k=0} D_{\mu_1...\mu_k} ((-1)^{\varepsilon_i} R^i_{\alpha \varphi^*} + \varepsilon^{ab} C^*_a B^*_{abc}) \frac{\partial}{\partial(\mu_1...\mu_k B^*_{abc})} - (-1)^{\varepsilon_a} \varepsilon_{ab} \sum_{k=0} \partial_{\mu_1...\mu_k} B^a \frac{\partial}{\partial(\mu_1...\mu_k C^*_{ab})} + \]
\[ \varepsilon^{ab} \sum_{k=0} \partial_{\mu_1...\mu_k} \varphi^*_{ic} \frac{\partial}{\partial(\mu_1...\mu_k \varphi^*_i)} + \varepsilon^{ab} \sum_{k=0} \partial_{\mu_1...\mu_k} B^a_\beta \frac{\partial}{\partial(\mu_1...\mu_k B^\alpha)} + \varepsilon^{ab} \sum_{k=0} \partial_{\mu_1...\mu_k} C^*_{abc} \frac{\partial}{\partial(\mu_1...\mu_k C^*_{abc})}, \]
\[ D_\mu \equiv \frac{\partial}{\partial x^\mu} + \sum_{k=0} \partial_{\mu_1...\mu_k} \Gamma^\Sigma \frac{\partial}{\partial(\mu_1...\mu_k \Gamma^\Sigma)}, \tag{5} \]
all quantities in (5) are taken in an arbitrary fixed spacetime point \( x^\mu \). The operators \( w^a \) and \( D_\mu \) commute:
\[ w^a w^b + w^b w^a = [D_\mu, w^a] = [D_\mu, D_\nu] = 0. \]

Equation (3) in terms of nonintegrated densities is
\[ w^a f^b_{n+1} + w^b f^a_{n+1} = D_\mu j^a_{n+1}, \tag{6} \]
where \( j^{ab}_{n+1} = j^{ab}_{n+1}(\Gamma^\Sigma, \partial \Gamma^\Sigma, ...) \) are functions of \( \Gamma^\Sigma \) and their derivatives up to finite order.

To solve equation (5), it is helpful to introduce operators \( \gamma_a \), which contain derivatives \( \partial / \partial L_i, \partial / \partial (R^i_{\alpha \varphi^*_a}) \). To make the introducing of such operators possible we make the standard assumptions on the structure of the action \( S \) and of the gauge generators \( R^i_{\alpha} \), which we will call the regularity assumptions of the theory.

\(^1\) The functions \( s_k \) and \( f^a_{n+1} \) are defined up to a total derivative.

4
1. Gauge transformation generators are LFODO’s:

\[ R^i_{\alpha} = r^i_{\alpha} + r^{i\mu}_{\alpha} \partial_\mu + \ldots + r^{i\mu_1\mu_2\ldots\mu_k}_{\alpha} \partial_{\mu_1\mu_2\ldots\mu_k}, \]  
\[ (7) \]

\( r^{i\mu_1\mu_2\ldots\mu_k} \), \( k = 0, \ldots, t \), are functions of the classical fields and their derivatives up to finite order, \( \varepsilon(R^i_{\alpha}) = \varepsilon(r^{i\mu_1\mu_2\ldots\mu_k}) = \varepsilon_i + \varepsilon_{\alpha} \).

2. Generators \( R^i_{\alpha} \) form an irreducible set in the sense, that any LFODO \( \Lambda^i \), satisfying the equation

\[ \Lambda^i L_i = 0, \]

has the form

\[ \Lambda^i = m^\alpha R^i_{\alpha} + \hat{M}^{ij} L_j, \quad \hat{M}^{ij} A_j = \sum_{k,l} m^{i\mu_1\ldots\mu_k|j\nu_1\ldots\nu_l} \partial_{\nu_1\ldots\nu_l} A_j \partial_{\mu_1\ldots\mu_k}, \]

where \( m^\alpha \) are LFODO’s, \( m^{i\mu_1\ldots\mu_k|j\nu_1\ldots\nu_l} \) are functions of \( \Gamma^m \) and of their derivatives up to a finite order. Furthermore, equations \( R^{i\alpha}_{\Lambda} n^\alpha = 0 \) have the only solution \( n^\alpha = 0 \).

3. Let us consider the space \( J_l(\varphi) \) (jet space) with coordinates \( \varphi_l \equiv (\varphi^i, \partial_\mu \varphi^i, \partial_{\mu_1\ldots\mu_s} \varphi^i) \). Given set of functions \( L_Q \equiv (L_i, D_\mu L_i, \ldots, D_{\mu_1\ldots\mu_s} L_i), Q = (i, i\nu_1, \ldots, i\nu_1\ldots\nu_s) \), such that the order of the highest derivatives of fields \( \varphi^i \) occurring in this set is equal to \( l \). There exist constraints among the functions \( L_Q \):

\[ R^i_{\alpha} L_i = 0, \quad D_\mu (R^i_{\alpha} L_i) = 0, \ldots, \quad D_{\mu_1\ldots\mu_s} (R^i_{\alpha} L_i) = 0, \]  
\[ (8) \]

where \( s_l \) is chosen from the condition that the highest derivatives of fields \( \varphi^i \) occurring in this set be equal to \( l \). Let us rewrite the constraints in the form

\[ \tilde{R}_D^Q L_Q = \tilde{R}_D^I L_I + \tilde{R}_D^{D'} L_{D'} = 0, \]

where \( D = (\alpha, \alpha\nu_1, \ldots, \alpha\nu_1\ldots\nu_s) \), \( \tilde{R}_D^I \) and \( \tilde{R}_D^{D'} \) are matrices, depending on \( \varphi_l \). We assume that for any \( l \) the set \( L_Q \) can be divided into sets of independent functions \( L_I \) ([\( L_I \)] designates the number of \( L_I \))

\[ \text{rank} \left( \frac{\partial L_I}{\partial \varphi_l} \right)_{L_Q=0} = [L_I] \]

and dependent ones \( L_{D'} \), \( \tilde{R}_D^{D'} \) being a nonsingular matrix.

In sections 3 and 4 we will show that under the regularity assumptions of the theory the \( Sp(2) \) master equation has a local solution.

\[ ^2 \text{In } (8), \ R^i_{\alpha} \text{ are given by } (7), \text{ with } \partial_\mu \rightarrow D_\mu. \]

\[ ^3 \text{In general, } s_l \text{ and } t \text{ depend on } i \text{ and } \alpha \text{ respectively. In the present paper, we omit these nonessential questions.} \]
3 The existence of a local solution

One can check that the functional

$$S_1 = \int dx \left( C^{\alpha \beta} R^\alpha_a \varphi^*_{Ia} + \epsilon^{ab} C^*_{ab} B^\alpha + B^\alpha R^\alpha_{a} \tilde{\varphi}_i (1 + \epsilon_i + \epsilon_a) \right)$$

(9)

is a solution to the $Sp(2)$ master equation in the first order in $C^{ab}, B^\alpha$. Let us investigate
the structure of the general solution to equation (9). All functions $\sigma_k, k \leq n$ contain
finite order derivatives of $\Gamma^\Sigma$ up to a finite order. Consequently, the functions $f_{n+1}^a$
also contain finite order derivatives of $\Gamma^\Sigma$. We choose the jet space $J_{{(\Gamma^\Sigma)}} = J_{{(\phi)} \otimes J_{{(C)}}} \otimes \ldots$, \nwhere $l_0^a, l_0^b = \sigma_0, l_0^{C^*} = l_0 = s_0, l_0^{C} = l_0$ in such a way that $f_{n+1}^a, w^a f_{n+1}^b,\nD_{\mu} j^{\mu ab}$ be defined on it. Note, that if we restrict ourselves to the operators $w^a$ and $D_{\mu}$ as
only acting on functions defined on $J_{{(\Gamma^\Sigma)}}$, then all sums in expressions (9) and (10) are
finite. Of course, jet spaces are defined ambiguously: if a set $(l)$ is admissible, then any
other set $(l')$, $l' \geq l, l'_C \geq l_C, \ldots$ is admissible, too.

The operators $w^a$ acting on $J_{{(\Gamma^\Sigma)}}$ are

$$w^a = (-1)^{\varepsilon I} L_I \frac{\partial}{\partial \varphi^*_{Ia}} + (-1)^{\varepsilon \ell} L_{\ell} \frac{\partial}{\partial \varphi^*_{\ell a}} - (-1)^{\varepsilon D} \bar{R}_D^Q \varphi^*_{Qa} \frac{\partial}{\partial \varphi^*_{Da}} + ((-1)^{\varepsilon Q} \bar{R}_D^Q \bar{\varphi}_Q + \epsilon^{bc} C^{*}_{DCb}) \frac{\partial}{\partial \varphi^*_{Da}} -$$

$$(-1)^{\varepsilon a} \epsilon^{ab} \sum_{k=0} \partial_{\mu_1 \ldots \mu_k} B^\alpha \frac{\partial}{\partial (\mu_1 \ldots \mu_k C^{ab})} + \epsilon^{ab} \varphi^*_{Qb} \frac{\partial}{\partial \varphi^*_{Qa}} + \epsilon^{ab} B^*_{Db} \frac{\partial}{\partial B^*_{Da}} + \epsilon^{ab} C^{*}_{Db} \frac{\partial}{\partial C^{*}_{Da}}.$$

After the change of variables

$$(\varphi^*_{Ia}, \varphi^*_{Da} = \bar{R}_D^Q \varphi^*_{Qa}), (\bar{\varphi}_I, \bar{\varphi}_D) \rightarrow (\bar{\varphi}_I, \bar{\varphi}'_D = (-1)^{\varepsilon Q} \bar{R}_D^Q \bar{\varphi}_Q + \epsilon^{bc} C^{*}_{DCb}),$$

with the rest of the variables left unchanged, the operators $w^a$ take the form (the primer
is omitted)

$$w^a = (-1)^{\varepsilon I} L_I \frac{\partial}{\partial \varphi^*_{Ia}} + \epsilon^{ab} \varphi^*_{Ib} \frac{\partial}{\partial \varphi^*_{Ia}} - (-1)^{\varepsilon D} \bar{R}_D^Q \varphi^*_{Qa} \frac{\partial}{\partial \varphi^*_{Da}} + \epsilon^{ab} C^{*}_{Db} \frac{\partial}{\partial C^{*}_{Da}} + \epsilon^{ab} \varphi^*_{Qb} \frac{\partial}{\partial \varphi^*_{Qa}} + \epsilon^{ab} B^*_{Db} \frac{\partial}{\partial B^*_{Da}} -$$

$$(-1)^{\varepsilon a} \epsilon^{ab} \sum_{k=0} \partial_{\mu_1 \ldots \mu_k} B^\alpha \frac{\partial}{\partial (\mu_1 \ldots \mu_k C^{ab})}.$$

Next, we introduce the operators

$$\gamma_a = (-1)^{\varepsilon I} \varphi^*_{Ia} \frac{\partial}{\partial L_I} - \epsilon^{ab} \varphi^*_{Ib} \frac{\partial}{\partial \varphi^*_{Ia}} - (-1)^{\varepsilon D} \bar{R}_D^Q \varphi^*_{Qa} \frac{\partial}{\partial \varphi^*_{Da}} + \epsilon^{ab} \bar{C}^{*}_{Da} \frac{\partial}{\partial C^{*}_{Da}} +$$

$$B^*_{Da} \frac{\partial}{\partial \varphi^*_{Da}} - \epsilon^{ab} B^*_{Db} \frac{\partial}{\partial B^*_{Db}}.$$

Simple calculations give the following relations

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 0, \quad w^a \gamma_b + \gamma_b w^a = \delta^a_b N, \quad w^a N = N w^a, \quad \gamma_a N = N \gamma_a,$$

$$N = L_I \frac{\partial}{\partial L_I} + \varphi^*_{Qb} \frac{\partial}{\partial \varphi^*_{Qb}} + \varphi_Q \frac{\partial}{\partial \varphi^*_{Qb}} + B^*_{Db} \frac{\partial}{\partial B^*_{Db}} + C^{*}_{Db} \frac{\partial}{\partial C^{*}_{Db}} + B_D \frac{\partial}{\partial B^*_{Db}} + \bar{C}^{*}_{Da} \frac{\partial}{\partial C^{*}_{Da}}.$$

6
\[ D_\lambda \text{ can be written as } D_\lambda = \bar{D}_\lambda + \tilde{D}_\lambda, \]

\[ \bar{D}_\lambda = \sum_{k=1}^{n} \partial_{\mu_1 \ldots \mu_k} C^{\alpha \alpha} \frac{\partial}{\partial (\partial_{\mu_1 \ldots \mu_k-1} C^{\alpha \alpha})} + \sum_{k=1}^{n} \partial_{\mu_1 \ldots \mu_k} B^\alpha \frac{\partial}{\partial (\partial_{\mu_1 \ldots \mu_k-1} B^\alpha)}, \]

\[ [w^\alpha, \bar{D}_\lambda] = [\gamma_a, \bar{D}_\lambda] = [N, \bar{D}_\lambda] = 0, \quad [w^\alpha, D_\lambda] = [N, D_\lambda] = 0, \quad [\gamma_a, D_\lambda] \neq 0. \]

Consider equation (3). To solve it we apply the method used in [16] for the standard formalism. Let us expand \( f_{n+1}^a \) and \( j_{n+1}^{\mu ab} \) according to the total number of derivatives of the fields \( C^{\alpha \alpha} \) and \( B^\alpha \):

\[ f_{n+1}^a = \sum_{k=0}^{p} f_{n+1}^{(k)a}, \quad j_{n+1}^{\lambda ab} = \sum_{k=0}^{m} j_{n+1}^{(k)\lambda ab}, \quad D f_{n+1}^{(k)a} = k f_{n+1}^{(k)a}, \quad D j_{n+1}^{(k)\lambda ab} = k j_{n+1}^{(k)\lambda ab}, \]

\[ D = \sum_{k=0}^{p} k \partial_{\mu_1 \ldots \mu_k} C^{\alpha \alpha} \frac{\partial}{\partial (\partial_{\mu_1 \ldots \mu_k-1} C^{\alpha \alpha})} + \sum_{k=0}^{m} k \partial_{\mu_1 \ldots \mu_k} B^\alpha \frac{\partial}{\partial (\partial_{\mu_1 \ldots \mu_k-1} B^\alpha)}, \]

where \( p \) and \( m \) are finite numbers. It is obvious that \( m \) can be put equal to \( p - 1 \). It follows from (3) that

\[ w^\alpha f_{n+1}^{(p)b} + w^\beta f_{n+1}^{(p)\beta} = \bar{D}_\lambda j_{n+1}^{(p-1)\lambda ab}. \]

After applying the results of Appendix 1 to equation (11), we have:

\[ N^2 f_{n+1}^{(p)a} = w^\alpha y_{n+1}^{(p)a} + \bar{D}_\lambda j_{n+1}^{(p-1)\lambda a}, \quad y_{n+1}^{(p)a} = \left( \frac{2}{3} N - \frac{1}{6} \gamma_c w^c \right) \gamma_b j_{n+1}^{(p-1)\lambda ab}. \]

Since \( ngh(f_{n+1}^{(p)a}) = 1 \), \( f_{n+1}^{(p)a} = O(C^d B^{n+1-l}), \quad n \geq 1 \), the functions \( f_{n+1}^{(p)a} \) contain antifields \( \Phi^*_a, \bar{\Phi} \), hence

\[ f_{n+1}^{(p)a} = w^\alpha \left( \frac{1}{N^2} y_{n+1}^{(p)a} \right) + \bar{D}_\lambda \left( \frac{1}{N^2} j_{n+1}^{(p-1)\lambda a} \right). \]

Then, let us write \( f_{n+1}^a \) as

\[ f_{n+1}^a = w^\alpha \left( \frac{1}{N^2} y_{n+1}^{(p)a} \right) + D_\lambda j_{n+1}^{\lambda a} + f_{n+1}^{(p-1)a}. \]

Obviously, \( f_{n+1}^{(p-1)a} \) also obey (4), and the highest summary derivatives of \( C^{\alpha \alpha} \) and \( B^\alpha \) in \( f_{n+1}^{(p-1)a} \) is equal to \( p - 1 \). Going in the same way and taking into account that \( f_{n+1}^a |_{C^{\alpha \alpha}=B^\alpha=0} = 0 \), we arrive at

\[ f_{n+1}^a = w^\alpha y_{n+1} + D_\lambda j_{n+1}^{\lambda a}, \quad F_{n+1} = W^a \int dxy_{n+1} = W^a Y_{n+1} \]

where \( y_{n+1} \) and \( j_{n+1}^{\lambda a} \) are some functions. In Appendix 2 we show that, under additional assumptions the functions \( y_{n+1} \) can be chosen as \( Sp(2) \) and Lorentz scalars. Taking \( S_{n+1} = Y_{n+1} \), we can see that the \( Sp(2) \) master equation is satisfied up to \( n+1 \) order. Accordingly, the existence of a local solution conserving the \( Sp(2) \) symmetry is proven.
4 The arbitrariness of solutions of the $Sp(2)$ master equation

In this section we study the arbitrariness of solutions to the $Sp(2)$ master equation. Before doing that, let us introduce a set of operators which we will call the gauge (G) transformations:

$$K = \exp \left\{ \frac{i\hbar}{2} : \varepsilon_{ab}[\bar{\Delta}^b, [\bar{\Delta}^a, F]]_+ : \right\},$$

(12)

where

$$F = \sum_{n=0}^{\infty} \hbar^n F_n, \quad F_n = \int dx F_{n}^{\Sigma_1 \ldots \Sigma_n} \frac{\delta}{\delta \Gamma_1} \ldots \frac{\delta}{\delta \Gamma_n},$$

$F_{n}^{\Sigma_1 \ldots \Sigma_n}$ are functions of $\Gamma^\Sigma$ and of their finite order derivatives,

$$\bar{\Delta}^a = \int dx (-1)^\varepsilon_{\alpha} \frac{\delta}{\delta \Phi^\alpha} \hat{\Lambda}_{ij}^a \Phi^i \Phi^j + \int dx \frac{i}{\hbar} \varepsilon_{\alpha} \hat{\Lambda}_{i}^a \Phi^i \Phi^\alpha,$$

the symbol $\ldots :$ means that all functionals like $\delta(0)$ must be set equal to zero.

G transformations have the following properties:
1. A product of G transformations is a G transformation.
2. Given a functional $S(\Gamma^\Sigma)$, we construct the functional $S'(\Gamma^\Sigma)$ in accordance with the rule

$$\exp \left[ \frac{i}{\hbar} S' \right] = K \exp \left[ \frac{i}{\hbar} S \right].$$

(13)

If $S$ is a local functional, then $S'$ also is a local functional; if $S$ does not depend on $\hbar$, then $S'$ has the same property; if $S$ satisfies the $Sp(2)$ master equation, then $S'$ satisfies the $Sp(2)$ master equation. Two functionals $S$ and $S'$ are called gauge equivalent if they are related by (13). In [3] it is shown that G transformations do not change the physical contents of the theory. Let us proceed by studying the general solution to the $Sp(2)$ master equation in first order on $C^\alpha_a, B^\alpha$. The general form of the functional, that is first order in $C^\alpha_a, B^\alpha$, conserves $ngh$ and is $Sp(2)$ a scalar is

$$S_1 = \int dx \left( C^\alpha_a \Lambda_{\alpha}^a \varphi^*_i + \varepsilon_{ab} B^\alpha \Lambda_{\beta}^a C_{ab}^\alpha (-1)^\varepsilon_{\beta} + B^\alpha \bar{\Lambda}_{\alpha}^j \varphi^*_j (-1)^{\varepsilon_{\alpha} \varepsilon_{\beta} + 1} \varepsilon_{ab} \hat{\Lambda}_{\alpha}^i \varphi^*_i \varphi^*_j \right),$$

(14)

where $\Lambda_{\alpha}^i, \Lambda_{\beta}^i, \bar{\Lambda}_{\alpha}^i$ are LFODO’s,

$$\hat{\Lambda}_{\alpha}^i A_j B_i = \sum_{k,l} \lambda^{i j k l} \varphi_i \varphi_j \partial_{\nu_1 \ldots \nu_l} A_j \partial_{\mu_1 \ldots \mu_k} B_i,$$

$\lambda^{i j k l}$ being functions, depending on $\varphi^i$ and their finite order derivatives. The G transformation on $S$ with $F = -(-1)^{\varepsilon_{\alpha} \varepsilon_{\beta} \varepsilon_{\gamma} \varepsilon_{\delta} \varepsilon_{\epsilon}} \hat{\Lambda}_{\alpha}^i \varphi^*_i \varphi^*_j B^\alpha$ removes the term $\frac{1}{2} \varepsilon_{ab} \hat{\Lambda}_{\alpha}^i \varphi^*_i \varphi^*_j \varepsilon_{ab} \hat{\Lambda}_{\alpha}^i B^\alpha$ from $S_1$.

The $Sp(2)$ master equation for (14) leads to

$$\Lambda_{\alpha}^i L_i = 0, \quad \bar{\Lambda}_{\alpha}^i = \Lambda_{\alpha}^\beta \Lambda_{\beta}^i$$

---

4 Correct definition of operators like (12) and their properties will be given in [17].

5 In [3], it was postulated that $\hat{\Lambda}_{\alpha}^i = 0$. In fact, we see, that $\hat{\Lambda}_{\alpha}^i$ can be removed by a G transformation.
Due to the regularity assumptions one has:

$$\Lambda^i_\alpha = m^\beta_\alpha R^i_\beta + \hat{M}^{ij}_\alpha L_j, \quad \hat{M}^{ij}_\alpha A_j = \sum_{k,l} m^{ij_1...i_j\alpha}_\alpha j_1...i_j\mu_k | \nabla_\alpha j_1...| i_j\mu_k,\nonumber$$

where $m^{ij}_\alpha$ are LFODO’s. The expression $C^{\alpha\beta} \hat{M}^{ij}_\alpha \varphi^\alpha_{ia} \varphi^\beta_{ja}$ can be compensated by the G transformation with $F = (1/2) C^{\alpha\beta} M^{ij}_\alpha \varphi^\alpha_{ia} \varphi^\beta_{ja}$. Note, that $S|_{\alpha A_2 = \Phi A = 0}$ is a solution to the master equation. We suppose that it is a proper solution, i.e. the equations

$$m^{T\alpha}_\beta C^{\beta\alpha} = 0, \quad \Lambda^{T\alpha}_\beta B^{\beta} = 0$$

have only trivial solutions, hence there exist LFODO’s $m^{-1\alpha}_\alpha, \Lambda^{-1\beta}_\beta$. After the change of variables

$$C^{\alpha\alpha} = (-1)^{\varepsilon_\alpha + \varepsilon_\beta} m^{T\alpha}_\beta C^{\beta\alpha}, \quad C^{\alpha\beta} |_{\alpha} = m^{-1\beta}_\alpha C^{\beta\alpha}, \quad \bar{C}^{\alpha}_\alpha = m^{-1\beta}_\alpha \bar{C}^{\beta\alpha}, \quad B^{\alpha} = m^{T\alpha}_\beta \Lambda^{T\beta}_\gamma B^{\gamma} (-1)^{\varepsilon_\gamma + \varepsilon_\alpha},$$

$B^{\alpha} = \Lambda^{-1\beta}_\alpha m^{-1\beta}_\gamma B^{\gamma}$

(whose conserves the $Sp(2)$ master equation and can be represented as a G transformation) we transform $S_1$ to the form $[\S]$. Given two solutions $S$ and $S'$ to the $Sp(2)$ master equation $[\S]$ with the same boundary condition $S(\varphi)$, we suppose that they are G equivalent up to order $n$. Performing the G transformation on $S'$, we can write

$$S_{[n]} = S'_{[n]}, \quad S'_{n+1} = S_{n+1} + \Delta S_{n+1}$$

where $S_1$ is given by $[\S]$. Then $\Delta S_{n+1}$ satisfies the equation $W^a \Delta S_{n+1} = 0$ or $w^a \Delta s_{n+1} = D_\mu j^\mu_{n+1}$. From the results of section 2 and Appendix 1, we conclude that:

$$\Delta S_{n+1} = W^2 W^1 X_{n+1}.$$ 

$\Delta S_{n+1}$ can be removed by applying the G transformation to $S$ with $F = X_{n+1}$. Applying the induction method, we conclude that the general local solution to the $Sp(2)$ master equation, conserving $Sp(2)$ symmetry and ngh can be represented in the form

$$\exp \left( \frac{i}{\hbar} S \right) = \exp \left\{ \frac{ih}{2} : \varepsilon_{ab} [\Delta^b, [\Delta^a, X]]_+ : \right\} \exp \left( \frac{i}{\hbar} S_c \right),$$

where $S_c$ is a special solution.

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Appendix 1

Here we show how equation (10) may be solved.

Let the operators $w^a$, $\gamma^a$, $N$ and $d$ define an algebra:

$$ w^a w^b + w^b w^a = 0, \quad \gamma^a \gamma^b + \gamma^b \gamma^a = 0, \quad w^a \gamma^b + \gamma^b w^a = \delta^a_b N, \quad [w^a, d] = [\gamma^a, d] = 0. $$

Consider equation

$$ w^a f = d j^a. $$

By using the algebra we have

$$ N^2 f = N (w^2 \gamma_1 f + d \gamma_1 j^2) = d N \gamma_2 j^2 + w^2 w^1 \gamma_1 \gamma_2 f + d w^2 \gamma_1 \gamma_2 j^1 = w^2 w^1 \gamma_1 \gamma_2 f + d \left( \frac{1}{2} (N + \gamma_1 w^a) \gamma_b j^b \right), $$

or

$$ N^2 f = w^2 w^1 y + dj. $$

Now, consider the equation

$$ \sum_{i=1}^{l+1} w^{a_i} f^{a_1 \ldots a_i} = d j^{a_1 \ldots a_{l+1}} \tag{15} $$

for the function $f^{a_1 \ldots a_i}$, symmetric in its indices $a_i$. Multiplying (14) by $w^b \varepsilon_{ba_i+1}$ and $\gamma_{a_{l+1}}$, we arrive at

$$ (l+2)w^2 w^1 f^{a_1 \ldots a_l} = dw^c \varepsilon_{cb} j^{a_1 \ldots a_{l+1}} \tag{16} $$

and

$$ \gamma^b w^b f^{a_1 \ldots a_l} + lN f^{a_1 \ldots a_l} - \sum_{i=1}^{l} w^{a_i} \left( \gamma^b f^{a_1 \ldots a_i b} \right) = d \left( \gamma^b j^{a_1 \ldots a_{l+1}} \right) \tag{17} $$

respectively. After multiplying (17) by $\gamma_c w^c$ and taking into account the expressions

$$ \gamma_c w^c \gamma_b w^b = N \gamma_b w^b + 2 \gamma_1 \gamma_2 w^2 w^1, \quad \gamma_c w^c w^a = w^a \left( \gamma_c w^c - N \right), $$

we have

$$ (l+1)N \gamma_c w^c f^{a_1 \ldots a_l} = \sum_{i=1}^{l} w^{a_i} \left[ \left( \gamma_c w^c - N \right) \gamma^b f^{a_1 \ldots a_i b} \right] + d \left[ \left( \frac{l}{l+2} \gamma_c w^c + \frac{2}{l+2} N \right) \gamma^b j^{a_1 \ldots a_{l+1}} \right], \tag{18} $$

where in derivation of (18) use has been made of the equality $\gamma_1 \gamma_2 w^b \varepsilon_{bc} = (\gamma_b w^b - N) \gamma_c$. Substituting (18) into (17), we obtain

$$ l(l+1)N^2 f^{a_1 \ldots a_l} = \sum_{i=1}^{l} w^{a_i} \left[ ((l+2)N - \gamma_c w^c) \gamma^b f^{a_1 \ldots a_i b} \right] + d \left[ \left( \frac{l(l+3)}{l+2} N - \frac{l}{l+2} \gamma_c w^c \right) \gamma^b j^{a_1 \ldots a_{l+1}} \right] \tag{19} $$

or

$$ N^2 f^{a_1 \ldots a_l} = \sum_{i=1}^{l} w^{a_i} j^{a_1 \ldots a_i} + dj^{a_1 \ldots a_l}. $$

Thus, (14) follows from (19) with $l = 1$, $d \to \bar{D}_{\mu}$, $j^{a} \to j^{a \mu}$. 

10
Appendix 2

Here we show that the functionals $Y_n$ can be chosen $Sp(2)$ and (under additional assumptions) Lorentz scalars. We closely follow [12] in our arguments.

Let $T^\sigma$ be transformation generators of the fields $\Gamma^\Sigma$, defining a semi-simple algebra $g$. Let the action of $T^\sigma$ on any local functional $A = \int dx a$ be:

\[ T^\sigma A = \int dt^\sigma a, \tag{20} \]

\[ t^\sigma = \sum_{k=0}^\infty t^\sigma_{(k)} \Sigma_{\nu_1...\nu_k} \partial_{\mu_1...\mu_k} \Gamma^\Sigma \frac{\partial}{\partial(\partial_{\nu_1...\nu_k} \Gamma^{\Sigma'})}, \tag{21} \]

where $t^\sigma_{(k)} \Sigma_{\nu_1...\nu_k}$ are constant matrices, $t^\sigma$ define the same algebra $g$ and

\[ [t^\sigma, w^a] = \tau_b^{\sigma a} w^b, \quad [t^\sigma, w^2 w^1] = 0, \quad [t^\sigma, D_\mu] = a_\mu^{\sigma \nu} D_\nu, \tag{22} \]

with constant matrices $\tau_b^{\sigma a}$, $a_\mu^{\sigma \nu}$. Next, we consider the functionals

\[ F^a = W^a Y, \quad Y = \int dy, \tag{23} \]

obeying the conditions

\[ T^\sigma F^a = \tau_b^{\sigma a} F^b. \tag{24} \]

From (24), it follows that:

\[ w^a t^\sigma y = D_\mu j^{\mu a \sigma}, \]

with some functions $j^{\mu a \sigma}$. We suppose that there exists an operator $N^{-1}$ defined on the functions $t^\sigma y$. Taking into account the results of Appendix 1, one has:

\[ t^\sigma y = w^2 w^1 z^\sigma + D_\mu j^{\mu \sigma}. \tag{25} \]

Every subspace of the fixed power uniform polynomials in the variables $\Gamma^\Sigma$ and their finite order derivatives defines a finite completely reducible representation of $t^\sigma$. The expansion of $y$ into irreducible representations reads

\[ y = y_0 + \sum_{R \neq 0} y_R, \]

where $y_0$ belongs to the trivial representation. Equation (23) can be represented as follows:

\[ \sum_{R \neq 0} t^\sigma y_R = w^2 w^1 z^\sigma + D_\mu j^{\mu \sigma}. \tag{26} \]

The equation (26) gives

\[ y_R = w^2 w^1 z_R + D_\mu j^{\mu}_R, \]

with some functions $z_R$ and $j^{\mu}_R$, due to the standard arguments. Hence, we obtain

\[ y = y_0 + w^2 w^1 z + D_\mu j^{\mu} \]
Therefore, the results of this Appendix allow us to choose

\[ y \]

\[ m \]

\[ M \]

\[ C \]

\[ \text{the fields } M \]

\[ \text{holds. Accordingly, the generators } \]

\[ \text{of the } \]

\[ \text{Lorentz transformation of the fields } \]

\[ C \]

\[ \text{where the Lorentz transformation matrix } M \]

\[ \text{of the original fields } \phi^i \]

\[ \text{are known. The elements of the matrices for the remaining fields will be defined below. For simplicity, we}
\]

\[ \text{consider classical action } S(\varphi) \]

\[ \text{which does not depend explicitly on } x^\mu. \]

Next, we suppose that the gauge transformation generators are chosen by Lorentz covariant way, i.e.,

\[ m_{\mu\nu} \]

\[ D_\mu = \sum_{k=0}^1 \partial_{\mu_1 \ldots \mu_k} \Gamma^{\Sigma} \frac{\partial}{\partial (\nu_1 \ldots \nu_k \Sigma^\nu)}, \]

hence \[ [m_{\mu\nu}, D_\lambda] = \lambda^{\sigma}_{\lambda \mu \nu} D_\sigma. \]

The Lorentz invariance of the original action leads to \[ m_{\mu\nu} L_i = -M^i_{\mu\nu} L^i. \]

It is natural to define

\[ m_{\mu\nu} \varphi^i_{\alpha a} = -M^i_{\mu\nu} \varphi^i_{\alpha a}, \quad m_{\mu\nu} \varphi^i = -M^i_{\mu\nu} \varphi^i. \]

Next, we suppose that the gauge transformation generators are chosen by Lorentz covariant way, i.e.,

\[ m_{\mu\nu} R^i_{\alpha} = M^i_{\mu\nu} R^i_{\alpha} - M_{\alpha \mu \nu}^i R^i_{\alpha}, \]

where \[ M_{\alpha \mu \nu}^i \]

\[ \text{are generators of a finite dimensional representation of the algebra } o(3, 1) \]

\[ \text{(gauge parameters transform according to this representation). We will suppose, that the}
\]

\[ \text{Lorentz transformation of the fields } C_{ab}, B^\alpha, C^*_{a|\alpha}, B^*_{a\alpha}, C_{ab}, B_\alpha \]

\[ \text{has the following form}
\]

\[ m_{\mu\nu}(C_{ab}, B^\alpha) = M_{\alpha \mu \nu}^a(C_{ab}, B^\alpha), \quad m_{\mu\nu}(C_{a|\alpha}, B^*_{a\alpha}, C_{ab}, B_\alpha) = -M_{\alpha \mu \nu}^a(C_{a|\alpha}, B^*_{a\alpha}, C_{a|b}, B_\alpha) \]

\[ \text{(the fields } C_{ab} \]

\[ \text{and } B^\alpha \]

\[ \text{Lorentz transform in the same way as the gauge transformation parameters) It is easy to check, that}
\]

\[ m_{\mu\nu} \]

\[ \text{and } M_{\mu\nu} \]

\[ \text{define the algebra } o(3, 1) \]

\[ \text{and that the relation}
\]

\[ [u^a, m_{\mu\nu}] = 0 \]

holds. Accordingly, the generators \[ M_{\mu\nu} \]

\[ \text{and } m_{\mu\nu} \]

\[ \text{obey the conditions (24), (21), (22). Therefore, the results of this Appendix allow us to choose } y \]

\[ \text{as a Lorentz scalar.} \]
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