$f(R)$ cosmology with torsion

S. Capozziello$^{1,2}$, R. Cianci$^3$, C. Stornaiolo$^2$, S. Vignolo$^3$

$^1$ Dipartimento di Scienze Fisiche, Università “Federico II” di Napoli and $^2$INFN Sez. di Napoli, Compl. Univ. Monte S. Angelo Ed. N, via Cinthia, I- 80126 Napoli (Italy) and $^3$DIPTM Sez. Metodi e Modelli Matematici, Università di Genova, Piazzale Kennedy, Pad. D - 16129 Genova (Italy)

(Dated: April 27, 2013)

$\mathcal{L} = R$-gravity with geometric torsion (not related to any spin fluid) is considered in a cosmological context. We derive the field equations in vacuum and in presence of perfect-fluid matter and discuss the related cosmological models. Torsion vanishes in vacuum for almost all arbitrary functions $f(R)$ leading to standard General Relativity. Only for $f(R) = R^2$, torsion gives contribution in the vacuum leading to an accelerated behavior. When material sources are considered, we find that the torsion tensor is different from zero even with spinless material sources. This tensor is related to the logarithmic derivative of $f(R)$, which can be expressed also as a nonlinear function of the trace of the matter energy-momentum tensor. We show that the resulting equations for the metric can always be arranged to yield effective Einstein equations. When the homogeneous and isotropic cosmological models are considered, terms originated by torsion can lead to accelerated expansion. This means that, in $f(R)$ gravity, torsion can be a geometric source for acceleration.

PACS numbers: 04.20.Cv, 04.20+Fy, 04.20.Gz, 98.80.-k
Keywords: Alternative theories of gravity; torsion; gauge symmetry; cosmology; dark energy

I. INTRODUCTION

CDM has recently assumed the role of a new Cosmological Standard Model giving a coherent picture of the today observed universe [1]. Although being the best fit to a wide range of data, it suffers of several theoretical shortcomings [2] so it fails in tracking cosmic dynamics at every redshift and fails in according observational cosmology to some fundamental theory of physical interactions. Among the defects of this model, there is the lack of final probes, at fundamental level, for dark energy and dark matter candidates (which should be the 95% of the energy-matter content of the universe!) which frustrates the possibility to reduce CDM to some self-consistent scheme, despite of the fact that it is a fair “snapshot” of the present status of the universe. These facts motivate the search for other models, among which alternative theories of gravity that should reproduce the successes of CDM but should be more appropriate in describing the cosmological dynamics [3, 4].

In particular, the large part of dark energy models relies on the implicit assumption that Einstein’s General Relativity (GR) is the correct theory of gravity indeed. Nevertheless, its validity on large astrophysical and cosmological scales has never been tested but only assumed [5], and it is therefore conceivable that both cosmic speed up and missing matter, respectively the dark energy and the dark matter, are nothing else but signals of a breakdown of GR at large scales. In other words, GR could fail in giving self-consistent pictures both at ultraviolet scales (early universe) and at infrared scales (late universe) also it is fairly working at Solar System scales and in the weak field regime.

Staring from these considerations, a different possibility could be to better investigate the gravitational sector and consider, as source of the field equations, only the observed (and probed at fundamental level) baryonic matter and radiation (photons and neutrinos). A choice could be to take into account generic functions $f(R)$ of the Ricci scalar $R$. The goal should be to match observational data without considering exotic dark ingredients, unless these are found by means of experiments at fundamental level [6, 7]. This is the underlying philosophy of the so-called $f(R)$-gravity (see e.g. [4, 8, 9]).

These theories are receiving much attention in cosmology, since they are able to give rise to the accelerating expansion [7] and it is possible to demonstrate that they play a major role also at astrophysical scales (for recent comprehensive reviews see [8, 9]). In fact, modifying the gravity Lagrangian affects the gravitational potential in the low energy limit. Provided that the modified potential reduces to the Newtonian one on the Solar System scale, this implication could represent an intriguing opportunity rather than a shortcoming for $f(R)$ theories. In fact, a corrected gravitational potential could offer the possibility to fit galaxy rotation curves without the need of huge amounts of dark matter [11, 12, 13, 14, 15, 16, 17]. In addition, it is possible to work out a formal analogy between the corrections to the Newtonian potential and the usually adopted galaxy halo models which allow to reproduce dynamics and observations without dark matter [13].

However, extending the gravitational Lagrangian could give rise to several problems. These theories could have instabilities [18], ghost- like behaviors [19], and they have to be matched with the low energy limit experiments which quite fairly test GR.

In summary, it seems that the paradigm to adopt $f(R)$-gravity leads to interesting results at cosmological, galactic and Solar System scales but, up to now, no definite physical criterion has been found to select the final $f(R)$ theory (or a class of theories) capable of matching the data at all scales. Interesting results have been achieved in this line of thinking [20, 21, 22, 23, 24] but the approaches are all phenomenological and are not based on some fundamental principle as the conservation or the invariance of some quantity or some intrinsic symmetry of the theory. In any case, some results are emerging in this direction and the presence of Noether symmetries into dynamics seems to play an important role in selecting physically interesting $f(R)$ theories.
Besides selection criteria, the “full” geometric sector of $f(\mathcal{R})$ gravity has to be investigated considering also the role of torsion. Such an “ingredient” has been firstly considered by Cartan and then by Sciama and Kibble in order to deal with spin in GR (see [27] for a review). Being spin as fundamental as the mass of the particles, torsion was introduced in order to complete the scheme that mass(energy) are the source of curvature and spin is the source of torsion. Unfortunately, torsion in the context of GR, seems not to produce models with observable effects since the gravitational coupling is extremely weak in all the torsion phenomena and only in the very early universe its effect could have been significant.

However, it has been proven that spin is not the only source for torsion. As a matter of facts, torsion can be decomposed in three irreducible tensors, with different properties. In [28], a systematic classification of these different types of torsion and their possible sources was discussed.

In two recent papers, the role of torsion in $f(\mathcal{R})$ gravity has been considered in the framework of metric-affine formalism [29, 30]. The field equations have been discussed in empty space and in presence of perfect fluid matter taking into account the analogy with the Palatini formalism [31]. As a result, the extra curvature and torsion degrees of freedom can be dealt as an effective scalar field of fully geometric origin [29].

From a more formal viewpoint, $f(\mathcal{R})$ gravity with torsion can be studied in the framework of the $J$ -bundles formalism [30]. Such an approach is particularly useful since the components of the torsion and curvature tensors can be chosen as fiber $J$ -coordinates on the bundles and then the symmetries and the conservation laws of the theory can be easily achieved. Also in this case, field equations of $f(\mathcal{R})$-gravity have been studied in empty space and in presence of various forms of matter as Dirac fields, Yang–Mills fields and spin perfect fluid. Such fields enlarge the jet bundles framework and characterize the dynamics.

In this paper, we discuss the cosmological applications of $f(\mathcal{R})$-gravity with torsion, considering the possibility that the whole dark side of the universe (dark matter and dark energy) could be geometrically interpreted by curvature and torsion. Examples in which repulsive gravity and clustered structures could be implemented considering torsion are present in literature [27, 32, 33] but, in that cases, GR has been adopted and the whole dark sector has not been addressed.

The layout of the paper is the following. In Sec.II, we give the definitions and conventions for torsion tensor and construct the in the main geometrical quantities in $U_4$ spacetime. Field equations in vacuum are derived in Sec.III. In Sec.IV, we derive the cosmological equations in vacuum and find out some interesting solutions where cosmological term is given by the trace of torsion tensor. Field equations, in presence of standard fluid matter, are discussed in Sec.V, while, in Sec.VI, also spin is considered. A relevant discussion is devoted to the fact that torsion contribution can be dealt under the standard of a scalar field. In Sec.VII, we consider weak and strong energy conditions in presence of standard fluid matter and find out some interesting solutions where cosmological term is given by the trace of torsion tensor. Field equations, in presence of perfect fluid matter are discussed in Sec.VIII. Discussion and conclusions are drawn in Sec.IX.

**II. TORSION TENSOR AND INVARIANT QUANTITIES IN $U_4$**

A 4D-differential manifold equipped with torsion is defined in a $U_4$ space, while standard torsionless Riemannian manifolds are defined in $V_4$. In literature, there are several definitions for torsion and quantities related to it. For a summary, see [28]. The conventions adopted in this paper are those in [27].

The torsion tensor $S_{\alpha\beta\gamma}$ is defined as the antisymmetric part of connection in a coordinate basis

$$S_{\alpha\beta\gamma} = \frac{1}{2}$$

(1)

a connection with torsion and metric-compatible ($\mathcal{R} \quad \mathcal{g} = 0$) has the form

$$\mathcal{R} + \mathcal{S} + \mathcal{S} = \mathcal{K}$$

(2)

where are the Christoffel symbols and

$$ \mathcal{K} = \mathcal{S} + \mathcal{S} + \mathcal{S} $$

(3)

is the contorsion tensor. Another combination of torsion tensor, often used in the calculations, is the modified torsion defined by the following relation

$$ \mathcal{T} = \mathcal{S} + \mathcal{S} + \mathcal{S} $$

(4)

where

$$ \mathcal{S} = \mathcal{S} $$

(5)
is the torsion trace-vector.

Here, we shall consider a vectorial torsion of the form

\[ S = A \mathbf{A} \]  \hspace{1cm} (6)

Its trace is

\[ S = S \mathbf{A} = \frac{3}{2} A \mathbf{A} \]  \hspace{1cm} (7)

so the vectorial torsion takes the form

\[ S = \frac{2}{3} S \mathbf{A} \mathbf{A} \]  \hspace{1cm} (8)

In the following, we will find convenient to express the field equations in terms of one of these quantities. To this purpose, let us give here the relations among torsion, modified torsion, contorsion, and their respective traces. From Eq.(4), it is

\[ T = 2 S \mathbf{A} \mathbf{A} \left( S \mathbf{A} \right) = 2 S \mathbf{A} \]  \hspace{1cm} (9)

and then

\[ T = 2 S \mathbf{A} \]  \hspace{0.5cm} or \hspace{0.5cm} S = \frac{1}{2} T \]  \hspace{1cm} (10)

where the modified torsion can be expressed in the following form

\[ T = \frac{2}{3} T \mathbf{A} \mathbf{A} \]  \hspace{1cm} (11)

The Riemann tensor can be decomposed in a part depending only on the Christoffel symbols and a part which contains the contorsion tensor. Using Eq.(11) in Eq.(3), we obtain

\[ K = \frac{1}{2} T \mathbf{A} + T \mathbf{A} \]  \hspace{1cm} (12)

Finally, substituting (11) in (12), we get

\[ K = \frac{1}{3} T \mathbf{A} \mathbf{A} T \mathbf{A} \]  \hspace{1cm} (13)

and

\[ K = g \mathbf{A} \mathbf{A} = T \mathbf{A} \]  \hspace{1cm} (14)

Furthermore, starting from the standard definition of the Riemann tensor

\[ R = \mathbf{A} + \mathbf{A} \mathbf{A} \]  \hspace{1cm} (15)

one can insert the decomposition of the connection in the Christoffel and contorsion parts

\[ = \mathbf{K} \]  \hspace{1cm} (16)

In this way, the Riemann tensor results decomposed in a first term obtained by the Christoffel connection and its derivative and a second term given by the contorsion and its derivative.

\[ R = \mathbf{K} + \mathbf{K} \mathbf{A} \mathbf{K} \mathbf{A} \mathbf{K} \mathbf{K} \mathbf{K} \]  \hspace{1cm} (17)

Using the covariant derivative without contorsion \( \mathbf{A} \), the previous expression takes the form

\[ R = R \mathbf{A} + \mathbf{A} \mathbf{K} \mathbf{A} \mathbf{K} \mathbf{K} \mathbf{K} \mathbf{K} \]  \hspace{1cm} (18)
The corresponding Ricci tensor and the curvature scalar are respectively

$$R \quad R = R \left( fg + \frac{2}{3} K \right) + \frac{\partial}{\partial r} K + K K K K K$$

and

$$R \quad R = R \left( fg + \frac{2}{3} T \right) + \frac{1}{3} g K K + \frac{2}{9} g T T + \frac{2}{9} T T T T$$

If torsion is in the vectorial form, the Ricci tensor and the scalar curvature take the forms

$$R = R \left( fg + \frac{2}{3} T \right) \frac{1}{3} g K K + \frac{2}{9} g T T + \frac{2}{9} T T T T$$

which will be widely used in the following discussion.

III. FIELD EQUATIONS IN THE VACUUM

Let us start our considerations deriving the field equations in vacuum. The action for the gravitational part is

$$A = \int g f (R) d^4 x$$

It is instructive to report the variational principle step by step in order to put in evidence the differences with respect to a variation in \(V_4\). It is

$$A = \int g f (R) d^4 x + \int g \frac{\partial}{\partial r} f (R) d^4 x$$

$$= \int g f^0 (R) R \frac{1}{2} g f (R) g \, d^4 x + \int g f^0 (R) g \, d^4 x$$

$$= \int g f^0 (R) R \frac{1}{2} g f (R) g \, d^4 x + \int g f^0 (R) g \, d^4 x + \int g f^0 (R) g \, d^4 x$$

$$= \int g f^0 (R) R \frac{1}{2} g f (R) g \, d^4 x + \int g f^0 (R) g \, d^4 x + \int g f^0 (R) g \, d^4 x$$

In a space equipped with torsion, the following property holds

$$\Theta (P, Q) = P \times Q + \{ r \cdot P \} Q$$

where \(P, Q\) is a vectorial density and \(r = r + 2S\). If the variation \(Q\) is zero on the boundary or at infinity, we find

$$P \times Q = \{ r \cdot P \} Q$$

The variation assumes the form

$$A = \int g f^0 (R) R \frac{1}{2} g f (R) g \, d^4 x$$
Giving the explicit expression for \( r \), we obtain

\[
A = \int \frac{Z}{p} g f^0(R) d^4x + \frac{Z}{p} g g \theta (f^0(R)) + 2f^0(R)g T d^4x
\]

where \( T \) is the above modified torsion tensor. Finally, the variation of the connection, expressed in terms of the metric and the contorsion, is

\[
= \frac{1}{2}g \left( r g + r g + r g + r g \right) + g K : \quad (31)
\]

By substituting Eq. (31) in (30), we obtain

\[
U \frac{g f^0(R)}{K} = g \theta (f^0(R)) + 2f^0(R)g T = 0 \quad \text{(32)}
\]

In general, \( f(R) \) theories, in metric formalism, present fourth order terms in the field equations. In this case, such terms are absorbed in the torsion components and then

\[
\frac{g f^0(R)}{g} = f^0(R)R + \frac{1}{2}g f(R) + r (U U + U) = 0 : \quad (33)
\]

The divergence in Eq. (33) is zero because of Eq. (32) and then the field equations in vacuum are

\[
f^0(R)R + \frac{1}{2}g f(R) = 0 \quad \text{(34)}
\]

and

\[
f^0(R)T + \frac{1}{2}g (f^0(R)) = 0 : \quad (35)
\]

Taking into account the trace of Eq. (34)

\[
f^0(R)R = 2f(R) = 0 ; \quad (36)
\]

and substituting it in Eq. (34), it follows

\[
f^0(R)R + \frac{1}{4}g R = 0 \quad \text{(37)}
\]

The set of Eqs. (36) and (37) is equivalent to Eq. (34).

In general, Eq. (36) is an algebraic or transcendental equation for \( R \) which is satisfied if and only if \( R \) is a constant. This means, from Eq. (35), that torsion is identically zero and then we have. The resulting spacetime may be non-trivial, say a Schwarzschild-de Sitter. The trivial case may be de Sitter or anti-de Sitter, according to the sign of \( R \), i.e.

\[
R = g : \quad (38)
\]

Only for \( f(R) = R^2 \), Eq. (36) is an identity and \( R \) is not necessarily a constant. In this case, torsion is different from zero in the vacuum. By substituting \( f(R) = R^2 \) in Eq. (35), we have

\[
T + \frac{\theta R}{R} \frac{1}{2} R = 0 ; \quad (39)
\]

which can be reduced to

\[
T = T = 3 \frac{\theta R}{2 R} : \quad (40)
\]
This equation, together with
\[
R - \frac{1}{4}g = 0
\]  (41)
is a system of equations for the gravitational field.

We can decompose the curvature terms in Eqs. (41) and (40) in a part given by the Christoffel symbols and a part depending on torsion. The resulting equation is
\[
R (fg) - \frac{1}{4}g R (fg) + \frac{2}{3} T + \frac{1}{6} g T^2 + \frac{2}{9} T T = 0
\]  (42)

where the symbols \(\mathfrak{E}\) and \(R (fg)\) represent, respectively, the covariant derivative and the scalar curvature given by the Christoffel symbols. Similarly, the Ricci scalar \(R\) results
\[
R = R (fg) - \frac{2}{3} T + \frac{2}{3} T T ;
\]  (43)

On the other hand, the vector component of torsion \(T\) satisfies the following differential equation
\[
\frac{3}{2} T = R (fg) - \frac{2}{3} T - \frac{1}{4} R (fg) - \frac{2}{3} T T ;
\]  (44)

which is a propagation equation for \(T\). Finally, applying the contracted Bianchi identities, adding and subtracting \(R (fg) = 4\) in Eq. (42) and taking the covariant divergence with respect to the \(\mathfrak{E}\) derivative, it follows that
\[
\frac{2}{3} g \mathfrak{E} \mathfrak{E} T + \frac{2}{9} \mathfrak{E} (T T) - \frac{1}{4} R - \frac{1}{6} \mathfrak{E} T + \frac{1}{18} (T T) = 0 ;
\]  (45)

which we will be useful for the discussion below.

IV. COSMOLOGY WITH TORSION IN VACUUM

In order to develop our cosmological considerations, let us take into account a Friedmann-Robertson-Walker metric of the type
\[
ds^2 = dt^2 + a^2(t) \left(\frac{a^2}{1 + \frac{3}{4} x^2}\right) dx^2 + dy^2 + dz^2 ;
\]  (46)

The scalar (Christoffel) curvature in such a metric is
\[
R (fg) = 6 \frac{a^2}{a} + \frac{a^2}{a^2} + \frac{k}{a^2} ;
\]  (47)

Being the system dependent only on \(t\), from Eq. (44), only the \(T_0 = T\) component of \(T\) is different from zero. Eqs. (42) and (47), after some algebraic manipulations, give the following cosmological equations
\[
\frac{3}{2} \frac{a^3}{a} + \frac{a^2}{a^2} + \frac{k}{a^2} = \frac{1}{3} T + \frac{1}{3} H T + \frac{1}{9} T^2 ;
\]  (49)

which, in terms of the Hubble parameter \(H\), can be reduced respectively to
\[
3H + 12H H - 6 \frac{k}{a^2} H + T + H T + 3H T = 4H^2 + 2 \frac{k}{a^2} T + 2H T^2 + \frac{2}{9} T^3 ;
\]  (50)

and
The Bianchi identities (45) gives
\[ 3T \ (\ T + 12H \ 2T + 3H \ 3T + 4H \ T^2 + 9H + 36H \ H) \ 2H = 0 \] (51)
while the spatial components reduce to an algebraic identity. Let us find a solution for the system (50), (51) in the physically interesting case \( k = 0 \) corresponding to a spatially flat universe. Substituting \( H \) in (52), it reduces to
\[ \frac{d}{dt} H + \frac{1}{3} T^2 = \frac{2}{3} H + \frac{1}{3} T^2 T ; \] (53)
and then we have two cases
\[ H + \frac{1}{3} T = 0 ; \] (54)
\[ \frac{d}{dt} H + \frac{1}{3} T = \frac{1}{3} H + \frac{1}{3} T^2 T ; \] (55)
In the first case, we have the solution
\[ a(t) = a_0 \exp \left( \frac{1}{3} T t \right) \] (56)
In the second case, we obtain
\[ H = H_0 \exp \left( \frac{T}{3} t - \frac{T}{3} \right) \] (57)
which gives a cosmological expansion driven by torsion:
\[ a(t) = a_0 \exp \left( \frac{T}{3} t + A_0 \exp \left( \frac{T}{3} t \right) \right) \] (58)
being \( H_0, a_0 \) and \( A_0 \) arbitrary integration constants. This situation is particularly interesting in view of considering vectorial torsion as the geometrical source of cosmological accelerated expansion also without assuming an additional spin fluid [33]. It is important to point out at this point that solutions (56) and (58) are derived in the particular case of \( T = \) constant but they can be derived also for \( T = T(t) \). In this general case, \( T \) has to be substituted with \( T = T(t) \) dt. These results are clear indications that geometric torsion can be an effective source for inflation and, in general, accelerated expansion.

V. FIELD EQUATIONS IN PRESENCE OF PERFECT-FLUID MATTER

A more realistic situation is considering, in the action (23), the presence of a perfect fluid matter Lagrangian density. We have
\[ A = P \int g f(R) + L_m \ \text{d}^4x : \] (59)
By the sake of simplicity, let us consider first \( L_m \) not containing torsion terms. The corresponding matter fluid energy-momentum tensor is\[ \text{and} \quad = g \quad \text{its trace. The field equations are then} \]
\[ f^0(R) \ = \frac{1}{2} \ g \ f(R) = \] (60)
and
\[ f^0(R) \ = \frac{1}{2} \ \partial \ (f^0(R)) = 0 ; \] (61)
The trace of Eq. (60)

\[ f^0(R) = 2f(R) = \phi^{(R)} \tag{62} \]
gives a non-linear relation between the curvature scalar \( R \) and which is

\[ R = F(\phi) \tag{63} \]

and then Eq. (60) and (61) can be rewritten as

\[ R = \frac{1}{4}g \left( f^0(F(\phi)) \right) \tag{64} \]

\[ T = \frac{1}{2} \left( \frac{f^0(F(\phi))}{f^0(F(\phi))} \right) T \tag{65} \]

It is worth noting that the equation for torsion is now an “algebraic” expression. Torsion is present as a sort of gravitational coupling being \( f^0(F(\phi)) \). Eq. (64) can be written in an Einstein form by adding and subtracting \( \frac{1}{4}g R \) and then using Eq. (63). We obtain

\[ R = \frac{1}{4}g \left( f^0(F(\phi)) \right) \tag{66} \]

By using Eqs. (42) and (43), we can decompose the Ricci tensor and the curvature scalar in their Christoffel and the torsion dependent terms

\[ R (fg) = \frac{1}{2} g R (fg) = \frac{1}{f^0(F(\phi))} \frac{1}{4} g \frac{1}{4} g F(\phi) + \frac{2}{3} e^e T \]

\[ g \frac{1}{4} g \frac{1}{4} g \frac{1}{4} g F(\phi) + \frac{2}{3} e^e T \tag{67} \]

By taking the trace of Eq. (65)

\[ T = \frac{3}{2} \phi' \tag{68} \]

where we define the auxiliary scalar field

\[ \phi' = f^0(F(\phi)) \tag{69} \]

we obtain

\[ G (fg) = \frac{1}{2} + \frac{3}{2} e^e @ \phi' + \frac{3}{4} g g @ \phi' + \frac{1}{4} g @ \phi' + g V(\phi') \]

\[ + \frac{1}{2} e^e @ \phi' + \frac{1}{4} g @ \phi' \tag{70} \]

The RHS of this equation can be rearranged as the sum of the energy-momentum tensors of standard perfect fluid matter and of a scalar field being

\[ G (fg) = \frac{1}{2} + \frac{3}{2} e^e @ \phi' + \frac{3}{4} g g @ \phi' + \frac{1}{4} g @ \phi' + g V(\phi') \]

\[ + \frac{2}{3} e^e @ \phi' + \frac{2}{3} g @ \phi' \tag{71} \]

where the scalar field effective potential is given by

\[ V(\phi') = \frac{1}{6} (f^0)^2 (\phi') + \frac{1}{2} F(\phi) \tag{72} \]

and the operator \( @ \) is defined, as above, by the covariant derivative without contorsion. This result shows that our approach is fully equivalent to an effective theory with perfect fluid matter and a scalar field with a geometrical interpretation given by Eq. (69) and dynamics given by the self-interaction potential (72). We obtained this picture without adopting conformal transformation but only splitting torsion from Christoffel contributions in the connection. In other words, the result strictly depends on the intrinsic non-linearities of the gravitation interacting with perfect-fluid matter assuming a gravitational Lagrangian density of the form \( \frac{1}{2} \frac{2}{f^0(R)} \) defined on a \( U_4 \) manifold.
VI. FIELD EQUATIONS IN PRESENCE OF MATTER AND SPIN

As it is well known, spin can be one of the sources of torsion. In 1923, Cartan showed that intrinsic angular momentum (after called the spin) could have an important role in a geometric theory of space-time like General Relativity [34]. He showed that spin could originate torsion. This idea was considered also by Sciama [35] and Kibble [36] and, in more recently, by Hehl and his coworkers [27] and Trautmann and his coworkers [37]. The path followed in this stream of research was to extend GR to a theory (Einstein-Cartan-Sciama-Kibble, ECKS, theory) in which spin is the source of torsion. In the limit of zero spin distribution, the theory reduces to standard GR.

In the ECSK theory, spin and torsion are related by an algebraic equation and torsion does not propagate. In this case, only considering a distribution of aligned spins, described by a spin-density tensor $\rho$, it is possible to deal with torsion as a fluid. It is then necessary not only to have fluid of particles with spin, but also the average spin alignment has to be different from zero, in order to produce torsion. To deal with spin-torsion cosmology in the ECKS approach, it is necessary to consider a cosmological fluid where spin alignment is produced, for example, by huge magnetic fields. Such field are difficult to be found in standard cosmological situations as discussed in several works in literature.

In [28], it was discussed in details that spin is not the only source of torsion and that the torsion tensor can be decomposed in three irreducible tensors: (i) the totally antisymmetric torsion, (ii) the vectorial torsion and (iii) the traceless torsion. It was shown, for example, that a spin field generates a totally antisymmetric torsion, while a classical spin fluid, as that introduced by Mathisson [38] and Weyssenhoff [39], generates a traceless torsion.

In the previous section of this paper, we have seen that the vectorial part of torsion can be generated by the logarithmic derivative of a given scalar field $\chi$ of geometric origin.

Considering also spin in the context of this paper leads us to have a combined effect of different sources of torsion. In other words, one has to take into account the vectorial torsion, obtained by the covariant derivative of $\chi$, and the classical spin average effect. If in the action $S_m$ [59], we take into account a material Lagrangian $L_m$ with spin, the field equations have to be modified in the following way

\[
\frac{\partial}{\partial g} \left( \frac{p_g f(R)}{g} \right) = \chi^0 (R) R + \frac{1}{2} g f(R) + \chi \left( U - U + U \right) = \frac{1}{2} \frac{p_g L_m}{g}; \tag{73}
\]

\[
\frac{\partial}{\partial K} \left( \frac{p_g f(R)}{g} \right) = g \theta \chi^0 (R) R + 2 \chi^0 (R) g \chi \tag{74}
\]

Such equations result more complicated than Eqs. (60), (61). In this paper, we will not consider anymore the presence of the spin fluid as one of the source of torsion. The remarks of this section have been necessary in order to point out that torsion can have also a purely geometric origin. In this case, despite of the spin fluid which gives only an algebraic relation, the torsion dynamical field related to the geometric part can greatly contribute to the cosmic dynamics being a very natural source for cosmic speed up.

VII. THE WEAK AND STRONG ENERGY CONDITIONS IN $f (R)$ GRAVITY WITH TORSION

As we said, the $f (R)$ gravitational field equations with torsion can be expressed in the equivalent Einstein-like form

\[
G = \frac{1}{r} \frac{3}{2} \frac{1}{\tau^2} \tag{75}
\]

where $G$ is the effective scalar field energy-momentum tensor in Eq. (71). Such a term has a purely geometric origin and takes in account the nonminimal coupling between standard fluid matter and $f (R)$ gravity. It is important to stress that it gives a negative contribution in RHS of Eq. (75) and it is possible to show that it can naturally give rise to the observed accelerated behavior (due to $\Omega_m$) and the clustering properties (due to $\Omega_s$) of the standard CDM.

In order to consider the properties of $G$ and its relevance for the cosmic evolution, let us recast it as a (perfect) fluid energy-momentum tensor. Following this prescription, after considering Eq. (68), let us define

\[
U = \frac{T}{T+T} \tag{76}
\]

which plays the role of a four-velocity. We are assuming that $T$ is a timelike-vector. Then the energy density is
\[
\frac{1}{r} = \frac{3}{2} \frac{1}{2 - r^2} \quad U \ U = \frac{1}{r} \left( \frac{3}{2} \frac{1}{2 - r^2} \left( \frac{1}{2} \frac{1}{r} \right) + V(r) \right) + \frac{2}{3} \frac{1}{r} \frac{1}{r} \frac{1}{r} \quad (77)
\]
and the pressure
\[
P = \frac{1}{r} p \left( \frac{3}{2} \frac{1}{2 - r^2} \left( \frac{1}{2} \frac{1}{r} \right) + V(r) \right) \frac{2}{3} \frac{2}{3} \frac{1}{r} \frac{1}{r} + \frac{1}{3} (\frac{1}{2} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r}) \quad (78)
\]
For our purposes, it is important to consider the weak and strong energy conditions [40]. The weak energy condition implies that
\[
0 \leq \frac{1}{r} \left( \frac{3}{2} \frac{1}{2 - r^2} \left( \frac{1}{2} \frac{1}{r} \right) + V(r) \right) + \frac{2}{3} \frac{1}{r} \frac{1}{r} + \frac{2}{3} (\frac{1}{2} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r}) \quad (79)
\]
where \( \% = \frac{U \ U}{p} \) and \( p = (g + \frac{U \ U}{r}) \) are respectively the energy density and the pressure for the standard-fluid matter source.

The strong energy condition holds, if the RHS tensor of Eq. (75) satisfies the condition
\[
0 \geq \frac{1}{r \ r} \left( \frac{3}{2} \frac{1}{2 - r^2} \left( \frac{1}{2} \frac{1}{r} \right) + V(r) \right) \frac{2}{3} \frac{1}{2} \frac{1}{r} \frac{1}{r} + + (\frac{1}{2} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r}) \quad (80)
\]
corresponding to
\[
\% + 3p \left( \frac{3}{2} \frac{1}{2 - r^2} \left( \frac{1}{2} \frac{1}{r} \right) + V(r) \right) \frac{2}{3} \frac{1}{2} \frac{1}{r} \frac{1}{r} + + (\frac{1}{2} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r} \frac{1}{r}) \quad (81)
\]
When this condition is satisfied, the universe undergoes a decelerated cosmological expansion, vice versa an accelerated expansion is expected when it is violated. These inequalities strictly depend on the form of the function \( f(R) \) and on the nonminimal coupling of \( f(R) \) with the standard matter given by (see Eqs. (62) and (69)).

In conclusion, we can say that introducing torsion of geometric origin in \( f(R) \) gravity can lead to an accelerated behavior of the universe due to a repulsive nonlinear interaction of the (baryon) matter with itself. A qualitative discussion of this point is given in [29]. There is also the possibility that comparing the effective \( \rho_{\text{eff}} = \frac{1}{r} \) and \( \gamma_{\text{eff}} = \frac{V(r)}{r^2} \) can lead to comparable values for \( \% \) and \( p \). This could be a straightforward way to solve the so-called coincidence problem [1, 2, 3, 4].

VIII. THE COSMOLOGICAL SOLUTIONS IN PRESENCE OF PERFECT FLUID MATTER

In order to determine the cosmological equations in presence of perfect-fluid matter, we can arrange Eq. (70) in the form
\[
R = \frac{1}{r} \left( \frac{1}{4} \frac{1}{r} \right) F(\ ) \left( \frac{3}{2} \frac{1}{2} \frac{1}{r} \frac{1}{r} \frac{1}{r} \right. \left. \frac{1}{r} \frac{1}{r} \right) \quad (82)
\]
We consider the components of the Ricci tensor
\[
R_{00} = \frac{1}{r} \left( \frac{1}{4} \frac{1}{r} \right) \left( \frac{1}{4} \frac{1}{r} \right) \left( \frac{1}{4} \frac{1}{r} \right) F(\ ) \left( \frac{3}{2} \frac{1}{2} \frac{1}{r} \frac{1}{r} \right. \left. \frac{1}{r} \frac{1}{r} \right) \quad (83)
\]
and
\[
R_{11} = \frac{1}{r} \left( \frac{1}{4} \frac{1}{r} \right) \left( \frac{1}{4} \frac{1}{r} \right) F(\ ) \left( \frac{3}{2} \frac{1}{2} \frac{1}{r} \frac{1}{r} \right. \left. \frac{1}{r} \frac{1}{r} \right) \quad (84)
\]
In the homogeneous and isotropic FRW universe, we have

\[ R_0^0 = \frac{3a}{a} ; \tag{85} \]

\[ R_1^1 = \frac{a}{a} + 2\frac{a^2}{a^2} + 2k \frac{a^2}{a^2} ; \tag{86} \]

\[ 0 = \% ; \tag{87} \]

\[ 1 = p = ( 1 )\% \tag{88} \]

\[ 0 = 3p = ( 4 ) 3 \% ; \tag{89} \]

where 0 1 is the adiabatic index of the equation of state. The cosmological equations are then

\[ \frac{3a}{a} = \frac{3}{4} \% + \frac{1}{4} F ( ) \frac{3}{2} r^2 + \frac{3}{2} \frac{r}{r} + \frac{3}{2} \frac{r}{r} ; \tag{90} \]

\[ \frac{a}{a} + 2\frac{a^2}{a^2} + 2k \frac{a^2}{a^2} = \frac{\%}{4} + \frac{1}{4} F ( ) + \frac{1}{2} \frac{r}{r} + \frac{4}{2} \frac{r}{r} \tag{91} \]

By composing these two equations to eliminate the \( a=a \), one obtains

\[ H^2 + \frac{k}{a^2} = \frac{\%}{4} + \frac{1}{12} F ( ) \frac{1}{4} \frac{r}{r}^2 + \frac{1}{2} \frac{r}{r} + \frac{1}{2} \frac{r}{r} ; \tag{92} \]

which, solved with respect to \( H \), gives

\[ H = \frac{r}{2} + \frac{\%}{2} \frac{k}{k} ; \tag{93} \]

with \( \% = 1 \) and

\[ k = \% \frac{1}{3} \frac{1}{F ( )} - \frac{4k}{a^2} \tag{94} \]

In order to derive the evolution of the density \%, let us consider the following equation, derived by combining Eqs. (90) and (91),

\[ H + H^2 = \frac{r}{2} + \frac{r}{2} + \frac{n-k}{4} \frac{r}{2} + \frac{r}{2} \frac{r}{2} + \frac{n-k}{4} \frac{r}{2} + \frac{1}{4} \frac{k}{k} \]

\[ = \frac{\%}{4} \frac{1}{12} F ( ) + \frac{r}{2} + \frac{r}{2} + \frac{1}{2} \frac{r}{r} + \frac{1}{2} \frac{r}{r} \tag{95} \]

This equation simplifies to

\[ \frac{n-k}{4} \frac{r}{2} = \frac{\%}{4} \frac{1}{12} F ( ) + \frac{1}{4} \frac{r}{r} \tag{96} \]

A further simplification comes from writing, explicitly in Eq. (96), the expressions for \( k \) and \( H \). We finally find the evolution equation

\[ \frac{n-k}{4} \frac{r}{k} = \frac{\%}{2} + \frac{k}{a^2} + \frac{\%}{4} \frac{r}{k} \frac{r}{r} \tag{97} \]

These equations can be solved by quadrature. The scale factor \( a(t) \) is obtained by solving Eq. (93).

As an example, let us assume \( F (R) / R^n \) and consider a perfect fluid with equation of state \( p = ( 1 )\% \) in a universe with \( k = 0 \). After some calculations we find, from Eqs. (85) and (89),

\[ F (\%) = \frac{1 - n}{n} \frac{1 - n}{2} = \frac{(4) 3 \%}{n} \frac{1 - n}{2} \tag{98} \]
\[ \dot{\theta} = \frac{(4/3)^n}{n} \cdot \frac{1}{2} \cdot (n-1) \theta^n \]  
\[ \theta_0 = C_1 \theta^{1-n} \]  
with

\[ C_1 = \frac{1}{3} \left( 3 + \frac{(4/3)^n}{n} \right) \cdot \frac{1}{2} \cdot (n-2) \cdot C_2 \]

Solving Eq. (97), we find

\[ \dot{\theta}(t) = \frac{1}{2n} \cdot (n-2) \cdot C_2 \cdot \frac{1}{2} \cdot (t-t_0)^{2n} \]

where \( C_2 = \frac{1}{n} \left( \frac{4}{3} \right)^n \cdot \frac{1}{2} \cdot (n-1) \theta^n \). The explicit expression for \( \dot{\theta}(t) \) is immediately found and

\[ \theta(t) = \frac{(C_1 \cdot C_2 \cdot (n-2))^2}{(t-t_0)^2} \]

The evolution of \( a(t) \) is obtained by integrating Eq. (93). We have solutions for \( n = 4 \), that is

\[ \frac{\dot{a}}{a} = \frac{n}{(t-t_0)} + \frac{(C_1 \cdot C_2 \cdot (n-2))}{(t-t_0)^2} \]

and then

\[ a(t) = A_1(t-t_0) \quad \text{where} \quad n = 1 + \frac{(C_1 \cdot C_2 \cdot (n-2))}{2} \]

This is a power law expansion depending on the parameters \( a_t \) and \( n \). Accelerated/decelerated behaviors are easily achieved by discussing the values of these parameters with respect to the Bianchi identities. As an example, let us consider the case \( n = 4 \). The decelerated behavior is obtained for

\[ 0 < \frac{C_1 \cdot C_2}{2} < 1 \]

otherwise the universe is accelerating. The transition from the decelerated to the accelerated phases is achieved for \( n = 1 \) and then the redshift has to evolve as \( z \sim t^{-1} \). Following [21], it is easy to see that the condition \( [1 + \text{eff}] \) to recover CDM prescriptions is easily recovered depending on the couple of parameters \( fn; g \).

**IX. DISCUSSION AND CONCLUSIONS**

Several issues from cosmology and astrophysics are telling us that the Einstein General Relativity should be revised in order to avoid shortcomings as dark energy and dark matter which, up to now, have not been probed at any fundamental level, but manifest their presence at large scales. Despite of the lack of final experimental evidences, the dark side of the universe should constitute almost the 95% of the whole matter-energy content.

This situation is extremely disturbing and a more “economic” way to solve the problem could be to revise the geometric part of the gravitational interaction extending the GR. In \( f(R) \) gravity, this point of view is considered and, in general, the good results of the Einstein theory are preserved trying, at the same time, to encompass the observational data in some self-consistent scheme [9].

In this paper, we have pushed forward this approach taking also into account the role of torsion in \( f(R) \) cosmology. With respect to the ECKS theory, where torsion couples algebraically to the spin and matter without dynamics, we have put in evidence the fact that torsion fields of purely geometric origin can perfectly mimic the role of scalar fields in cosmological evolution. Such
a feature naturally gives rise to accelerated behaviors as in dark energy models. In general, torsion has no effect in vacuum but, for $\mathbb{R}^2$, it is a source for the field equations leading to de Sitter-like expansions.

In presence of standard fluid matter, torsion and curvature degrees of freedom give rise to nonminimal couplings which determine cosmological evolution. This depends on the form of $f(\mathbb{R})$. Also in these cases, accelerated behaviors are easily achieved. However, also in presence of torsion, $f(\mathbb{R})$-gravity models must satisfy some viability conditions in order to describe cosmic acceleration. Such conditions are summarized [27]. The cosmological models proposed here fully satisfy these compatibility conditions since the presence of torsion acts as a further additive fluid with the only effect to move the extremum of the effective potential which, following the notation in [27], lies at the GR value $\mathbb{R}^2$.

As a final comment, it is worth noticing that we have not used any conformal transformation so the issue to choose between the Einstein and the Jordan frame is avoided. Besides, torsion naturally results as a fundamental scalar field whose origin is perfectly understood.

In a forthcoming paper, we will discuss more realistic $f(\mathbb{R})$ gravity models with torsion confronting them with data.

[1] U. Seljak et al. Phys. Rev. D71, 103515 (2005).
[2] S.M. Carroll, W.H. Press, E.L. Turner, Ann. Rev. Astron. Astroph. 30, 499 (1992).
[3] Peebles P.J.E., Rathra B., Rev. Mod. Phys. 75, 559 (2003); Padmanabhan T., Phys. Rept., 380, 235 (2003).
[4] E.J. Copeland, M. Sami, S. Tsujikawa, Int. J. Mod. Phys. D15, 1753 (2006).
[5] C. M. Will, Living Rev. Relativity 9 (2006), arXiv:gr-qc/0510072.
[6] H. Kleiner, H.-J. Schmidt, Gen. Relativ. Grav. 34 1295 (2002).
[7] S. Capozziello, Int. J. Mod. Phys. D11, 483 (2002).
[8] S. Capozziello, S. Carloni, A. Troisi, Rec. Res. Dev. in Astron. and Astroph. 1, 1 (2003) (arXiv : astro - ph/0303041).
[9] S.D. Odintsov, S. Nojiri Phys. Lett. B 576, 5 (2003).
[10] S. Capozziello, V.F. Cardone, S. Carloni, A. Troisi, Int. J. Mod. Phys. D 12, 1969 (2003).
[11] S.M. Carroll, V. Duvvuri, M. Trodden, M. Turner, Phys. Rev. D 70, 043528 (2004).
[12] S.M. Carroll, V. Duvvuri, M. Trodden, M. Turner, Phys. Rev. D 70, 103503 (2004).
[13] S. Nojiri and S.D. Odintsov, Phys. Rev. D 0307288.
[14] S. Nojiri and S.D. Odintsov, Gen. Rel. Grav. 36, 1765 (2004).
[15] S.Nojiri and S.D. Odintsov, Int. J. Meth. Mod. Phys. 4, 115 (2007).
[16] S. Capozziello and M. Francaviglia, Gen. Rel. Grav. 40, 357 (2008).
[17] S. Capozziello, V.F. Cardone, S. Carloni, A. Troisi, Phys. Lett. A 326, 292 (2004).
[18] M. Milgrom, Astroph. Journ., 270, 365 (1983);
J. Bekenstein, Phys. Rev. D 70, 083509 (2004).
[19] S. Capozziello, V.F. Cardone and A. Troisi JCAP 08, 001 (2006).
[20] S. Capozziello, V.F. Cardone, A. Troisi, Mon. Not. Roy. Astron. Soc. 375, 1423 (2007).
[21] Y. Sobouti, A&A, 464, 921 (2007).
[22] C.Frigerio Martins and P. Salucci, MNRAS 381, 1103, (2007).
[23] S. Mendoza and Y.M. Rosas-Guevara, A&A, 472, 367 (2007).
[24] V. Faraoni, Phys. Rev. D 72, 124005 (2005);
G. Cognola and S. Zerbini, J. Phys. A 39, 6245 (2006);
G. Cognola, M. Gastaldini and S. Zerbini, arXiv: gr - qc/0701138.
[25] K. S. Stelle, Gen. Rel. Grav. 9, 353 (1978).
[26] S. Capozziello, V.F. Cardone, A. Troisi, Phys. Rev. D 71, 043503 (2005).
[27] W. Hu and I. Sawicki, Phys. Rev. D 76, 064004 (2007).
[28] A. A. Starobinsky, JETP Lett. 86, 157 (2007).
[29] S. Nojiri and S. D. Odintsov, Phys. Lett. B 652, 343 (2007).
[30] S. Capozziello, S. Tsujikawa, Phys. Rev. D 77, 107501 (2008).
[31] S. Capozziello, P. Martin-Moruno, C. Rubano, Phys. Lett. B 664, 12 (2008).
[32] S. Capozziello and A. De Felice, Jour. Cosm. Astrop. Phys. 08, 016, 2008 B. Vakili, arXiv: 0804.3449 [gr-qc] (2008).
[33] S. Capozziello, G. Lambiase and C. Stornaiolo, Annals. Phys. (Leipzig) 10, 713 (2001).
[34] S. Capozziello, R. Cianci, C. Stornaiolo, and S. Vignolo, Class. Quant. Grav. 24, 6417 (2008).
[35] S. Capozziello, R. Cianci, C. Stornaiolo, and S. Vignolo, Int. Jou. Geom. Meth. Mod. Phys. 5, 765 (2008).
[36] G. Magnano, M. Ferraris, and M. Francaviglia, Gen.Rel.Grav. 19, 465 (1987).
[37] M. Gasperini, Gen. Rel. Grav. 30, 1703 (1998).
[38] M. Szydlowski and A. Krawiec, Phys. Rev. D 70, 043510 (2004).
[39] E. Cartan, Ann. Sci. Ec. Normale Super. 40, 325 (1923).
[40] D.W. Sciama, On the analog between charge and spin in General Relativity, in Recent Developments in General Relativity, Festschrift for Leopold Infeld, (1962) 415, Pergamon Press, New York.
[36] T.W. Kibble, J. Math. Phys. 2, 212 (1960).
[37] A. Trautmann, Bull. Acad. Pol. Sci., Ser. Sci., Math., Astron. Phys. 20, 895 (1972).
[38] M. Mathisson, Acta Phys. Pol. 6, 163 (1937).
[39] J. Weyssenhoff and A. Raabe, Acta Phys. Pol. 9, 7 (1947).
[40] S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Space-Time, Cambridge Univ. Press, Cambridge (1973).
[41] L. Pogosian and A. Silvestri Phys. Rev. D 77, 023503 (2008).