Quantum fractionalism: the Born rule as a consequence of the complex Pythagorean theorem

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Abstract

Everettian Quantum Mechanics, or the Many Worlds Interpretation, has no Born rule and lacks, so far, a valid explanation for quantum probabilities. Their values are shown to coincide with projection factors from the complex line of a quantum state to the eigenspaces of an observable. Such factors describe how Lebesgue measures contract under orthogonal projections, and, like probabilities, they add up to 1. This corresponds to a generalized Pythagorean theorem for complex spaces: the 2-dimensional measure of a subset of the complex line equals the sum of the measures of its projections on all eigenspaces.

From two simple, even if unorthodox, physical assumptions, we obtain that Everett’s theory must be augmented to include a continuum infinity of quantum universes, each with a number of classical worlds. Also, in a quantum measurement each projection factor gives the fraction of worlds in all universes with the corresponding result. This fraction is associated to a probability, in both the frequentist and Bayesian views of the concept, thus solving the probability problem of Everett’s theory. This opens the possibility of solving its preferred basis problem as well, and may also help settle questions about the nature of probability in general.

Keywords: foundations of quantum mechanics, Everett interpretation, many worlds interpretation, Born rule, interpretations of probability

1 Introduction

Since the early days of Quantum Mechanics, its probabilistic nature has baffled many physicists, most notoriously Albert Einstein. This motivated alternatives to the Copenhagen interpretation of Quantum Mechanics (CQM), like hidden-variables theories, which have not been so successful. Nowadays most physicists accept the theory as intrinsically probabilistic, but well known problems remain, regarding quantum measurements and the quantum-classical transition [Aul00, WZ14]. They have gained new

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relevance as experimental and theoretical developments, such as Quantum Computation and Quantum Information Theory [NC10], have extended the limits of quantum theory from the microscopic world to increasingly larger scales.

An alternative to CQM which is particularly well suited to be applied to macroscopic systems is the Many Worlds Interpretation, or Everettian Quantum Mechanics (EQM) [EI57, DG73]. It rejects the Measurement Postulate (and, with it, the Born rule), and reinterprets quantum measurements in terms of quantum entanglements, with all possible results actually happening, but entangled to different states of the observer, in a quantum superposition of distinct branches or worlds. As EQM is deterministic, it faces the problem of explaining the apparent randomness of quantum experiments, and the probability values observed.

Many attempts have been made to obtain the Born rule from more fundamental principles. The most famous is Gleason’s theorem [Gle57], but it relies on a hypothesis which cannot be justified, in EQM, before we know how probabilities can emerge in the theory. Deutsch and Wallace’s use of decision theory [Deu99, Wall12] has gained some acceptance among Everettians, but their hypotheses are also problematic [Man19a].

We present a new approach to the Born rule in EQM, identifying its probability values as projection factors, which describe the contraction of 2-dimensional Lebesgue measures in a ray (the complex line of a quantum state) when orthogonally projected onto the eigenspaces of an observable. A Complex Pythagorean Theorem [Man19b] says the measure of a subset of a ray equals the total measure of its projections on all eigenspaces, corresponding to the condition of unit total probability.

Such projection factors satisfy Kolmogorov’s axioms, but this is not enough to ensure they are indeed probabilities. This brings the question of what are probabilities, a notoriously difficult problem even in classical settings, for which none of the available answers (such as the frequentist and Bayesian interpretations) is fully satisfactory [Gil00, Mel05].

To properly link such factors to probabilities in EQM, we adopt two hypotheses which, despite going against some common ideas in Physics, are quite simple and defensible:

- each element of a ray represents a distinct physical state, even if they are experimentally indistinguishable;
- the existence of our Universe is not random, i.e. there is a physical reason for it to exist as it is.

From these we obtain that there must exist a continuum infinity of identical universes, which, as EQM is deterministic, evolve in exactly the same way. When a quantum experiment is carried out in one of them, it is also performed in all others, producing the same worlds in all of them, so in the end there is a continuum infinity of worlds for each result. And we show that in any set of universes, of finite nonzero Lebesgue measure, the relative amount (also in terms of the Lebesgue measure) of worlds with a given result equals the corresponding projection factor.

So the fraction of all worlds with a given result coincides with the probability value that would have been attributed to it by the Born rule. As such fractions can be linked to probabilities, in both the frequentist
and Bayesian interpretations, the probability problem of EQM is solved. This in turn fixes an important flaw in a proposed solution to its preferred basis problem [Man18b, Wal12].

Wallace [Wal12] has argued that EQM provides the appropriate setting for solving longstanding problems in the philosophy of probability. Even if we take issue with his use of decision theory to obtain the Born rule, we agree with this assessment. Our results do provide a better interpretation for the concept of probability, at least for quantum experiments, which we call quantum fractionalism. And if, as we discuss, classical probabilities can be traced back to quantum origins, this might provide an answer as to what is the nature of probability in general.

Section 2 reviews the problems of the Copenhagen and Everettian quantum theories, and of the frequentist and Bayesian interpretations of probability. In section 3 we show the probability values of the Born rule correspond to projection factors, and present the Complex Pythagorean Theorem. In section 4 we discuss the physical assumptions we will need. Section 5 links projection factors to the relative amounts of worlds corresponding to the results of a quantum measurement. In section 6 we show they can indeed be interpreted as quantum probabilities, and discuss whether this might also hold for classical probabilities. We conclude with a few remarks in section 7.

2 Preliminaries

In this section we review Everett’s theory in its modern form, which brings decoherence into the fold, the problems of CQM that motivated it, and the new ones it brings. We also briefly review the frequentist and Bayesian concepts of probability and their difficulties.

2.1 Problems of the Copenhagen interpretation

Despite its experimental success and wide acceptance, CQM has well known theoretical problems [Aul00, WZ14].

One is the measurement problem: even if it reflects accurately what is observed in experiments, the Measurement Postulate is ambiguous. It sets measurements apart from all other quantum processes, as only in them evolution governed by Schrödinger’s equation gives way to the probabilistic collapse of the quantum state. But there is no clear definition of which characteristics a process must have to trigger such change and count as a quantum measurement. If it is an issue of a macroscopic system interacting with a microscopic one, how big is macroscopic enough? Does it have to involve conscious beings or can machines cause the collapse of a wavefunction? If the microscopic system, measuring device and observer are just agglomerates of particles interacting according to quantum laws, there ought to be an explanation of what changes their evolution from one governed by a deterministic equation to a probabilistic one. Also, how exactly does the collapse of the wavefunction occur?

This problem is connected to that of the quantum-classical transition. What is the range of applicability of Quantum Mechanics? It works for
microscopic systems, but what happens as the number of particles grows? Does it gradually turn into Classical Mechanics, as is the case with Relativity as velocities decrease? Some quantum relations turn into classical ones if we take averages and let $\hbar \to 0$, but not everything transitions well. In the usual view, macroscopic quantum superpositions should not happen, lest we observe Schrödinger cats, but it is not clear what might eliminate them as systems get bigger. Decoherence has been suggested as a solution [Sch07], but even if it eliminates interference between components of a macroscopic superposition it does not explain the disappearance of all but one of them. This problem has led some to consider Quantum Mechanics valid only for microscopic systems, with a more universal theory being needed to connect quantum and classical physics.

For a long time most physicists have brushed aside such difficulties, believing that these were philosophical questions of little physical relevance, that they had been settled in the famous Bohr-Einstein debate, or that these were minor flaws in an otherwise very precise theory, which would end up being fixed. But as quantum theory reaches its centenary the problems remain, and gain increasing relevance as new theoretical and experimental advances allow us to explore the limits of the theory in ways that were inconceivable a few decades ago.

2.2 Everettian Quantum Mechanics

In [EI57, DG73], H. Everett III took a fresh look at what would happen if quantum theory was applied to macroscopic systems and measurements were regular quantum processes, proposing what became known as the Many Worlds Interpretation, or Everettian Quantum Mechanics. In it, quantum theory is universally valid, for small and large systems alike, but the Measurement Postulate is rejected, with all systems evolving deterministically at all times according to Schrödinger’s equation. He noted that even though this leads to macroscopic superpositions, it does not contradict our classical experience, as it also explains why observers do not perceive them.

In EQM, quantum measurements split the Universe into “branches” or “worlds”, with all possible results happening in some of them. There is no collapse of the quantum state, but when an observer interacts with the outcome of an experiment he splits into distinct versions of himself, each seeing only the result of his branch, as if the collapse had happened. This may seem like an almost mystical departure from conventional quantum theory, but it is a direct consequence of the usual formalism (minus the Measurement Postulate) applied to macroscopic systems. Linearity of Schrödinger’s equation and algebraic properties of the tensor product lead naturally to branching.

Let us detail such process. In the Everettian view, a measurement is just quantum entanglement of the measuring device with whatever is being measured. More precisely, a measuring device for a basis \{\ket{i}\} of a system is any apparatus, in a quantum state \ket{D}, which interacts in such a way that, if the system is in state \ket{i}, the composite state evolves as

$$\ket{i} \otimes \ket{D} \longrightarrow \ket{i} \otimes \ket{D_i},$$
where $|D_i⟩$ is a new state of the device, registering result $i$ (or simply reflecting in some way the fact that it interacted with $|i⟩$). Linearity of Schrödinger’s equation implies that, if the system is in a superposition $|Ψ⟩ = \sum_i c_i |i⟩$, the composite state evolves as

$$|Ψ⟩ \otimes |D⟩ = \left( \sum_i c_i |i⟩ \right) \otimes |D⟩ \rightarrow \sum_i c_i |i⟩ \otimes |D_i⟩ . \quad (1)$$

This final state is to be accepted as an actual quantum superposition of macroscopic states. But it will not be perceived as such by an observer looking at the device, as, by the same argument, his state $|O⟩$ will evolve into a superposition,

$$\left( \sum_i c_i |i⟩ \otimes |D_i⟩ \right) \otimes |O⟩ \rightarrow \sum_i c_i |i⟩ \otimes |D_i⟩ \otimes |O_i⟩ , \quad (2)$$

with $|O_i⟩$ being a state in which he saw result $i$. The interpretation is that he has split into different versions of himself, each seeing a result. By linearity, the components $|i⟩ \otimes |D_i⟩ \otimes |O_i⟩$ evolve independently, and interference is negligible if they are distinct enough, as tends to be the case with macroscopic systems. Each $|O_i⟩$ evolves as if the initial state had been $|i⟩ \otimes |D⟩ \otimes |O⟩$, so he does not feel the splitting or the existence of his other versions, and to him it is as if the system collapsed to $|i⟩$. Each component is called a world or branch, and this evolution of one world into a superposition of many is called branching. So in EQM all possible results of a measurement do happen, but in different worlds.

The problems of CQM disappear in EQM, but new ones come along, such as the preferred basis and probability problems, described below.

### 2.2.1 The preferred basis problem

The preferred basis problem is how to find a natural way to decompose a macroscopic quantum state into branches behaving like the classical reality we observe (even if not all of them, and not all the time).

Decoherence is seen as a possible mechanism through which a (quasi-) classical world might emerge from a quantum universe [JZK+03, Sch07, Zur02], and Wallace [Wal12] proposed a solution to the problem combining EQM with the decoherent histories formalism [GMH90, GMH93]. But this requires solving first the probability problem [Bak07, DT15], as the probabilistic interpretation of the quantum state norm is used to justify approximations in decoherence. Wallace claims such norm can be used as a measure of significance even without probabilities, but in [Man18b] we contest his arguments. As an alternative, we have suggested that branches with tiny norms can not exist as independent worlds, due to interference from larger ones. Hanson [Han03] has presented a similar idea, with different justifications, but there is still work to do before either proposal can solve the problem.

Anyway, it seems plausible that a branch decomposition, if it can be obtained, should be along these lines. An important characteristic of such decoherence based approach is that the decomposition is not clear-cut or unique. In EQM measurements lose their special status, becoming
just interactions that produce quantum entanglement. But particles get entangled all the time, and if each such process counts as a measurement of a particle by another then branching becomes a pervasive phenomenon, with each world splitting all the time into a myriad of others. Also, lots of branches will be nearly identical, differing only in the states of a few particles, which is not enough to ensure they would evolve with negligible interference.

As Wallace [Wal12] argues, these difficulties can be solved by a coarse-graining of similar branches, which provides more stability to the branch decomposition and ensures the resulting worlds are distinct enough to have negligible interference. This, however, introduces some arbitrariness into the decomposition, as the resulting worlds would depend on the chosen fineness of grain.

2.2.2 The probability problem

When measuring $|\Psi\rangle = \sum_i c_i |i\rangle$, in EQM, any result with $c_i \neq 0$ is obtained with certainty, in some world(s). The probability problem is to reconcile this with experiments, which suggest results are probabilistic and follow the Born rule.

**Born Rule.** In CQM, the probability of obtaining result $i$ when measuring a normalized state $|\Psi\rangle = \sum_i c_i |i\rangle$ in an orthonormal basis $\{|i\rangle\}$ is

$$p_{\Psi,i} = |c_i|^2 = |\langle i | \Psi \rangle|^2.$$ (3)

The problem has a qualitative aspect: how can probabilities emerge in a deterministic theory? In classical mechanics processes can appear random due to ignorance of details, but in EQM we must explain the apparent randomness even if the quantum state and its evolution are perfectly known. Wallace [Wal12] defends an operational and functional definition of probability, via decision theory and Bayesian inference.

Vaidman [Vai98] claims there is a *self-locating uncertainty* in the time between processes (1) and (2), as branching has already happened, but each version of $|O\rangle$ is still ignorant as to which branch he is in. But for this, the mathematical identity

$$\left( \sum_i c_i |i\rangle \otimes |D_i\rangle \right) \otimes |O\rangle = \sum_i c_i |i\rangle \otimes |D_i\rangle \otimes |O\rangle,$$

applied to the state at the beginning of (2), must have a very strong physical interpretation: that process (1) has already caused the whole Universe to branch, so that in this state there are already several versions of the observer, even though they are all identical.

We think it is more plausible to view branches as local macroscopic superpositions, which spread as new systems interact with previously branched ones. So at the beginning of (2) only the device has branched, and there is still a single version of the observer, who only branches once each $|D_i\rangle$ affects him in a different way. In other words, if branching is the result of measurement, which is just entanglement, the observer does not branch until he gets entangled with the device. If after (1) he is asked to guess what was the result of the measurement in his branch, he should
say the question is meaningless, as relative to him there is still only a single branch (which nevertheless contains a local superposition of device states, with all results, but which has not affected him yet).

Another aspect of the problem is quantitative: accounting for probability values. The idea that a measurement with \( n \) results produces \( n \) branches may seem natural for EQM, but it would lead to the same probability \( \frac{1}{n} \) for all results, in disagreement with experiments. Wallace [Wal12] dismisses this by saying there is no good way to count branches. A measurement with 2 results can lead to more than 2 branches, as unpredictable extraneous interactions can cause additional branchings, with an uneven and shifting distribution between the results. Coarse-graining reduces and stabilizes the number of branches, but makes it somewhat arbitrary, depending on the chosen fineness of grain.

Everett [EI57] proved that if a measure can be attributed to branches, and satisfies some hypotheses (like being preserved under finer decompositions), it must agree with the values in (3). And, as the number of measurements tends to infinity, the total measure of branches deviating from the Born rule tends to 0. But for finite experiments this only means branches deviating beyond a given error have small measure, and without a probabilistic interpretation (to avoid circularity) this does not mean they are any less relevant. The same problem affects a similar idea by Graham [Gra73], as well as Wallace’s use of decoherence to solve the preferred basis problem.

Gleason’s theorem [Gle57] also implies the Born rule, assuming the probability of a branch does not depend on what other branches the decomposition basis has (i.e. if a state decomposes as \( \psi = \psi_1 + \psi_2 \) or \( \psi = \psi_1 + \psi_3 + \psi_4 \), depending on the basis chosen, the probability for \( \psi_1 \) should be the same in both). Though such hypothesis seems reasonable if one has the Born rule in mind, it is not natural for probability measures in general (it is violated by a counting measure, for example).

Until we know how probabilities can emerge in EQM, we cannot assume they satisfy Everett’s or Gleason’s hypotheses, so their arguments are not enough to solve the probability problem. Other attempts [AL88, BHZ06, Zur05] have been made to obtain the right probabilities, without much success. An idea that has gained some acceptance is the use of decision theory to explain the Born rule in terms of subjective probabilities [Deu99, SBKW10, Wal12], but it faces many difficulties [Man19a].

2.3 Philosophies of probability

Though we all seem to have, in most cases, an intuitive understanding of what probabilistic statements mean, it is notoriously difficult to describe precisely what probability is. Formally it can be defined as a function in an event space satisfying Kolmogorov’s axioms, but this is not enough to connect such functions to how probabilities are actually used in daily life or science (e.g., areas of regions in a unit square satisfy the axioms, but are not probabilities per se). In [Gil00, Mel05] one can find analyses of different approaches to probability, and in [Wal12] Wallace discusses them in connection with EQM. Here we provide just a brief sketch of the main interpretations and their problems.
Perhaps the most intuitive concept of probability is the frequentist one. In this interpretation, saying each result of a fair die has probability $\frac{1}{6}$ means that, as the die is cast an increasingly large number of times, the relative frequency of each result will tend to $\frac{1}{6}$.

Making precise sense of this is trickier than it seems. Even if we were to throw the die 6 million times, we cannot say each result will occur 1 million times, give or take a thousand, only that it is highly probable this will happen. So even if relative frequencies can be used to measure probabilities, there is a circularity in using them to define probability.

Another problem with such concept is that it depends on the possibility of repeating some test an arbitrary number of times, under the same conditions. But the same die thrown 6 million times might get damaged along the way, breaking its symmetry. Worse yet, frequentism does not apply to one time events: it cannot explain what it means to talk about the probability that a given candidate will win next election, or that stock prices will rise tomorrow.

Another approach to probabilities is the Bayesian one. Roughly speaking, in this interpretation if someone says the probability of getting result 6 in a die is $\frac{1}{6}$ all that is meant is that he would not be willing to bet on such result at odds worse than 5 to 1. Another person, believing the die is loaded, might accept worse odds, and for him the probability would be different. Instead of being an objective concept, in the Bayesian view probability only describes the credence one has about something, based on the information he has. As more data becomes available, a process of Bayesian updating allows probability values to be properly adjusted.

Such subjective view of probabilities is useful when frequentism fails, as in the election or stock market examples, but it may be harder to apply in cases where probabilities seem to have a more objective nature (e.g. the decay probability of an atom). In truth, the distinction between objective and subjective probabilities is not as irreconcilable as it seems. It can be argued that, under certain rational constraints, and with enough information, they should not differ significantly.

Still, it seems odd to define the probability of an atom decaying in terms of one’s willingness to bet on it. If he bases his decision on what Physics tells are the objective probabilities, it brings back the problem of what these mean. If, instead, we define objective probabilities to be the values to which his subjective probabilities converge through Bayesian updating, as he learns from a large number of such bets, it also leads to the question of what is it that he is learning about.

As discussed, in EQM a crude frequentist attempt to get probabilities by world counting would give wrong values, but fortunately it cannot work since the number of worlds is ill defined. Wallace claims Bayesianism works better in EQM than in the classical case, with rational constraints forcing subjective probabilities to agree with the Born rule [Wal12]. But even though his proof is formally sound [Man18a], the concepts and axioms on which it is based are quite problematic [Man19a]. As we will show, EQM may indeed provide the best setting for a good probability concept, but it will be through other means.
3 Complex Pythagorean Theorem

Some known generalizations of the Pythagorean theorem [CB74, LL90] involve areas, volumes or higher dimensional measures, usually relating the squared measure of a set to the squared measures of its orthogonal projections on a complete set of mutually orthogonal subspaces. There are similar generalizations for complex spaces, in which, for dimensional reasons, the measures are not squared. Here we present a particular case, referring to [Man19b] for details and more general formulations.

Let $H$ be a complex Hilbert space, with Hermitian product $\langle \cdot, \cdot \rangle$. Complex vector spaces have underlying real ones, with twice the dimension, and the real part of $\langle \cdot, \cdot \rangle$ gives a real inner product. As $C$-orthogonality (with respect to $\langle \cdot, \cdot \rangle$) implies $R$-orthogonality (with respect to $\text{Re}\langle \cdot, \cdot \rangle$), orthogonal projections with respect to both products coincide.

A complex line is a 1-dimensional complex subspace $L \subset H$. The complex line of a nonzero $v \in H$ is $Cv = \{cv : c \in \mathbb{C}\}$. As it is isometric to a real plane, we measure its subsets using the 2-dimensional Lebesgue measure (roughly speaking, the area), which we denote by $|\cdot|$.

**Definition.** The projection factor of a complex line $L \subset H$ on a complex subspace $W \subset H$ is

$$\pi_{L,W} = \frac{|P(U)|}{|U|},$$

where $U \subset L$ is any Lebesgue measurable subset with $0 < |U| < \infty$, and $P : L \to W$ is the orthogonal projection.

As $P$ is linear, $\pi_{L,W}$ does not depend on the choice of $U$ (this is one of the defining properties of the Lebesgue measure [Rud86]).

**Proposition 3.1.** Given $v \in H$ and a complex subspace $W \subset H$,

$$\|Pv\|^2 = \|v\|^2 \cdot \pi_{Cv,W},$$

(4)

where $P : Cv \to W$ is the orthogonal projection.

**Proof.** As $P$ is $C$-linear, the square of sides $v$ and $iv$ in $Cv$ projects to the square of sides $Pv$ and $iPv$ in $W$. \hfill \square

**Definition.** An orthogonal partition of $H$ is a collection $\{W_i\}$ of mutually orthogonal closed complex subspaces such that $H = \bigoplus_i W_i$.

**Theorem 3.2** (Complex Pythagorean Theorem). For any nonzero $v \in H$ and any orthogonal partition $H = \bigoplus_i W_i$,

$$\sum_i \pi_{Cv,W_i} = 1.$$

(5)

So, for any measurable set $U \subset Cv$,

$$\sum_i |P_i(U)| = |U|,$$

where $P_i : Cv \to W_i$ is the orthogonal projection.

**Proof.** Follows from Proposition 3.1, as $\|v\|^2 = \sum_i \|P_i v\|^2$. \hfill \square
Contrary to the usual Pythagorean theorem, measures are not squared in this complex version. The reason is that 1 complex dimension corresponds to 2 real ones, both contracting by the same factor when projected on complex subspaces.

When $v$ is a quantum state vector and $W$ is an eigenspace of some observable, (4) shows $\pi_{Cv,W}$ has the same value as the probability given by the Born rule for the corresponding eigenvalue. And (5) corresponds to the condition of unit total probability. Of course, this does not allow us to interpret projection factors as probabilities yet, specially in EQM, which has no Born rule. Reaching such interpretation will require a few more steps.

4 Physical hypotheses

To proceed we will need some physical assumptions which may seem like a radical deviation from traditional physical views, but are actually quite simple. And, as we will show, they might lead to a better understanding of both Quantum Mechanics and Probability. This should be reason enough to at least consider their plausibility.

4.1 Distinct physically equivalent states

Let $H$ be the Hilbert space of a quantum system. Since, by quantum laws, any observable property of $\Psi \in H$ also holds for any other $c\Psi$ ($c \in \mathbb{C}^*$), these are usually considered different mathematical representations of the same physical state.

This allows the preferred use of normalized states, which simplify some formulas, but are not physically special in any way. Claims that normalization is necessary to have total probability 1 are unfounded: without it, instead of $|\Psi\rangle = \sum_i c_i |i\rangle$ we can simply write $\Psi = \sum_i \psi_i$, where $\psi_i = c_i |i\rangle$, and replace (3) with

$$p_{\Psi,i} = \frac{\langle \psi_i | \Psi \rangle^2}{\| \psi_i \|^2 \| \Psi \|^2} = \frac{\| \psi_i \|^2}{\| \Psi \|^2}. \quad (6)$$

Also, normalization does not eliminate all redundancy in the mathematical representation of quantum states, as those differing by a phase factor are also physically equivalent.

As there is no canonical way to pick a representative in each equivalence class, many physicists prefer to describe a quantum state by a ray $R_\Psi = \{ c\Psi : c \in \mathbb{C}^* \}$, instead of a single vector. The set of rays forms a projective Hilbert space, whose rich geometry provides nice interpretations to concepts like the Berry phase [BZ17]. But linear combinations of rays are not defined, making it hard to even state the superposition principle. Some replace it by a decomposition one [Boy89]: instead of writing $|\Psi\rangle = \sum_i c_i |i\rangle$, one says $R_\Psi$ decomposes in the rays $R_i = \{ c |i\rangle : c \in \mathbb{C}^* \}$, with probabilities given by the squared cosine of the angular distance between $R_\Psi$ and $R_i$. Such complications cause even proponents of such approach to often end up writing sums of quantum states.
In any case, it is generally agreed that having infinitely many Hilbert space vectors describing the same quantum state is only a redundancy of the mathematical formalism, with no physical implications.

We will take a different view, assuming different elements of \( R_\Psi \) represent distinct physical states, even if no experiment can tell them apart. One might say that experimentally indistinguishable states can not be considered different, but a similar argument would imply all electrons are not only identical particles, but actually a single particle present in multiple locations. Besides, even if one can not distinguish these states experimentally, we will show the assumption that they are not the same has important physical consequences, leading to the Born rule.

4.2 A continuum infinity of universes

Without the measurement postulate, Everettian Quantum Mechanics becomes fully deterministic, and with it all fundamental laws of Physics, so there is no randomness anymore in the evolution of the Universe. Taking this a step further, we assume the initial conditions of the Universe (if there was a beginning) were also not random, but determined by some unknown physical law.

This may seem too unorthodox: in Physics one is usually free to choose the initial conditions of a problem, with determinism governing evolution only after such starting point. But applying this paradigm to the Universe is problematic: if its initial conditions were not determined by Physics, how were they chosen?

In EQM we can not appeal to random quantum fluctuations as a way for the Universe to spontaneously come into existence. As the theory is not probabilistic, such fluctuations would have to be reinterpreted as a quantum superposition of all possible beginnings, which would evolve into a superposition of all possible universes, and the initial superposition state would still require an explanation.

Our hypothesis can be made more precise by assuming there is some special Big Bang or seed state \( \Psi_0 \), which not only can, but actually must, give rise to an universe. As any other state in the ray \( R_{\Psi_0} \) is physically equivalent to \( \Psi_0 \), it must also give rise to another universe, so we end up with a whole ray of baby universes. Determinism ensures they evolve in synchrony: if at time \( t \) an universe is in state \( \Psi(t) \), the states of all universes will form the ray \( R_{\Psi(t)} \).

Even if there was no Big Bang and our Universe has always existed, it is enough to assume its existence is not random, i.e. that there is some unknown physical reason for it to exist in some specific state at some given time. The argument can also be readily adapted to accommodate a “block universe” relativistic perspective.

Of course, our present knowledge does not allow us to talk with any certainty about what it even means to say a universe exists, or why it exists; if there can be more than one; whether it had a beginning; what is the real nature of time and evolution; or if a whole universe can be described by a quantum state.

But these difficulties are not specific to EQM or to our approach, and if Physics is ever to address such questions we must advance hypotheses.
and see if they lead to reasonable conclusions. A strong point in favor of our assumptions is that they will lead to the Born rule. It is conceivable that the fresh perspective they provide might even lead to new insights into those questions.

## 5 Relative amounts of worlds

Next we show that projection factors give, in EQM, the relative amounts of worlds corresponding to the results of a quantum measurement.

Suppose the Universe is in a state $\Psi$, with branch decomposition $\Psi = \sum_i \psi_i$ in terms of mutually orthogonal states $\psi_i$. In the usual Everettian view, there is a single quantum Universe, composed of (quasi-)classical worlds corresponding to the $\psi_i$'s. As discussed above, we assume instead that there is actually a continuum of identical universes, one for each state in the ray $R_\Psi$. Then for each $i$ there is a continuum $R_i = \{c\psi_i : c \in \mathbb{C}\}$ of physically equivalent worlds, one in each universe.

We want to show that (6) gives, in some sense, the relative amount of worlds of type $i$. Quantifying amounts of worlds requires a measure, and with continuum infinities of them a counting measure makes even less sense than before. One might consider counting rays instead of worlds, but Wallace’s argument applies: the number of rays is ill defined, depending on the choice of coarse-graining in the branch decomposition.

Each ray is isometric to a 2-dimensional Euclidean space (minus the origin), and its points are all equivalent. So the natural measure to use is the 2-dimensional Lebesgue one, $|\cdot|$, which is invariant by the group of transformations that preserve the metric.$^1$

As a ray has infinite measure, we consider first a subset of universes $U \subset R_\Psi$ with $0 < |U| < \infty$. The set of worlds of type $i$ in universes of $U$ is $W_{U,i} = P_i(U)$, where $P_i : \mathbb{C}\Psi \rightarrow \mathbb{C}\psi_i$ is the orthogonal projection, and the set of all worlds in all universes of $U$ is a disjoint union $W_U = \bigcup_i W_{U,i}$. By the Complex Pythagorean Theorem, $|W_U| = \sum_i |W_{U,i}| = |U|$. So the fraction $f_{U,i}$ (as measured by $|\cdot|$) of worlds of type $i$ in $U$ is

$$f_{U,i} = \frac{|W_{U,i}|}{|W_U|} = \frac{|P_i(U)|}{|U|} = \pi_{\mathbb{C}\Psi, \mathbb{C}\psi_i},$$

which is independent of $U$. So, even if the amounts of worlds in the $R_i$'s can not be directly compared using their full measures, which are infinite, their relative amounts are well defined by the following quantifiers.

**Definition.** The fraction of worlds of type $i$ in the universes of $R_\Psi$ is $f_{\Psi,i} = \lim_{r \rightarrow \infty} f_{U_{r,i}}$, where $U_r = \{\Psi \in R_\Psi : \|\Psi\| < r\}$.

The density of worlds of type $i$ in the universes of $R_\Psi$ is the number $\delta_{\Psi,i}$ such that $|W_{U,i}| = \delta_{\Psi,i} \cdot |W_U|$ for any measurable subset $U \subset R_\Psi$.

Both are equivalent, as $f_{\Psi,i} = \delta_{\Psi,i} = \pi_{\mathbb{C}\Psi, \mathbb{C}\psi_i}$, and which one to use is a matter of linguistic preference. From now on, when referring to relative amounts of worlds in expressions like “in nearly all worlds”, “in almost no world”, etc., we mean the corresponding value of these quantifiers is close to 1, to 0, etc. The percentage of worlds of type $i$ in $R_\Psi$ is $f_{\Psi,i} \cdot 100\%$.

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$^1$Positive multiples of $|\cdot|$ also satisfy this, but would lead to the same results.
By Proposition 3.1, these quantifiers have the same value as (6),

\[ f_{\psi,i} = \delta_{\psi,i} = \frac{\|\psi_i\|^2}{\|\Psi\|^2}. \]

So it seems what we perceive as probabilities in quantum experiments may be just the relative amounts of worlds corresponding to each result. In section 6 we further investigate this relation.

**Example.** Let \( \Psi = c_1\psi_1 + c_2\psi_2 \), where \( \psi_1 \) and \( \psi_2 \) are normalized orthogonal states, and \( c_1, c_2 \in \mathbb{C} \) with \( |c_1|^2 + |c_2|^2 = 1 \). Any set of universes \( U \subseteq R_\Psi \) decomposes into a set \( W_1 \cup W_2 \) of worlds (Figure 1), with an amount of worlds \( |W_1| = |c_1|^2 \cdot |U| \) in \( R_{\psi_1} \), and \( |W_2| = |c_2|^2 \cdot |U| \) in \( R_{\psi_2} \).

Hence \( |W_1 \cup W_2| = |W_1| + |W_2| = |U| \), so that the fraction of worlds in \( R_{\psi_1} \) is \( f_{\psi_1} = \frac{|W_1|}{|W_1 \cup W_2|} = |c_1|^2 \), and in \( R_{\psi_2} \) is \( f_{\psi_2} = \frac{|W_2|}{|W_1 \cup W_2|} = |c_2|^2 \).

![Figure 1: Complex Pythagorean Theorem, \( |U| = |W_1| + |W_2| \).](image)

### 6 Quantum Fractionalism

Our last step is to show that relative amounts of worlds can really be interpreted as probabilities.

Let us review the different Everettian points of view in an example. When an observer \( |O\rangle \) measures, in the orthonormal basis \( \{|\uparrow\rangle, |\downarrow\rangle\} \), an electron spin in state \( c_1 |\uparrow\rangle + c_2 |\downarrow\rangle \), with \( |c_1|^2 + |c_2|^2 = 1 \) (for simplicity), the process leads to an entangled state

\[ c_1 |\uparrow\rangle |O_\uparrow\rangle + c_2 |\downarrow\rangle |O_\downarrow\rangle. \]

A naive world counter would claim there are 2 worlds, one with each result, and make wrong predictions.

Wallace, on the other hand, says worlds emerge through decoherence, so the final state must take into account the many ways the observer can

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\(^2\)This is just to reframe our result in the usual quantum notation.
get entangled with the environment $|E\rangle$, and we actually have

$$\sum_i c_{1i} |\uparrow\rangle |O_{\uparrow i}\rangle |E_i\rangle + \sum_j c_{2j} |\downarrow\rangle |O_{\downarrow j}\rangle |E_j\rangle,$$  \hspace{1cm} (7)

with $\sum_i |c_{1i}|^2 = |c_1|^2$ and $\sum_j |c_{2j}|^2 = |c_2|^2$. A coarse-graining is necessary to separate each sum into worlds having negligible interference. As the number of worlds in each one depends on the chosen fineness of grain, it is not an objective feature, and cannot be used to claim EQM leads to wrong statistics.

In our view, (7) is just a representative of an actual continuum infinity of indistinguishable universes. As each one decomposes into worlds, we get many continua of worlds for each result, but in such a way that the total fraction (or density, if one prefers) of those with $|\uparrow\rangle$ is $f_\uparrow = |c_1|^2$, and those with $|\downarrow\rangle$ have $f_\downarrow = |c_2|^2 = 1 - f_\uparrow$.

If the experiment is repeated $N$ times, the total fraction of worlds with $n$ ups and $N-n$ downs will be $\binom{N}{n} \cdot f_\uparrow^n \cdot (1 - f_\uparrow)^{N-n}$, corresponding to a binomial distribution with parameter $p = f_\uparrow$. Thus, for large $N$, the distribution of world fractions, in terms of the frequency $f = \frac{n}{N}$ of ups, becomes sharply peaked at $f = f_\uparrow$, with variance $\sigma^2 = \frac{f_\uparrow (1 - f_\uparrow)}{N}$. Even though every possible sequence of results does occur in some world (actually, in continuum infinities of them), in nearly 99.7% of all worlds (as measured by their fractions or densities) the frequency of results will deviate from the Born rule by at most $3\sigma$.

So, when observed frequencies are used to measure probabilities, in quantum experiments, what is actually being measured are the fractions with which worlds branch at each run of the experiment. We call quantum fractionalism (or densitism) this interpretation of quantum probabilities as being in fact branching fractions or densities. It provides a physically objective concept of probability, that works equally well for single or multiple runs, encompassing both the frequentist and Bayesian views (at least for quantum experiments).

For example, if I bet a single measurement of $0.6 |\uparrow\rangle + 0.8 |\downarrow\rangle$ will result up, saying I have a 36% “chance” of winning has a concrete meaning: after universes branch, I will have won in 36% of all resulting worlds. If 10000 measurements like this are performed, every sequence of results will happen in an infinity of worlds, but in 99.7% of them the relative frequency of ups will be close to 36% ($\pm 3\sigma \approx 1.4\%$).

Likewise, the half-life of caesium-137 being around 30 years means that, after such period, in nearly all (fractionwise) worlds approximately half the atoms of a sample of this material will have decayed. But there will also be an infinity of worlds (albeit representing an extremely low fraction or density) in which all atoms have decayed, and another in which none have. How should we interpret this?

As in CQM these are real possibilities, only extremely unlikely, having in EQM a tiny fraction of worlds in which they happen might not be so strange. Still, perhaps due to the human tendency to equate very low probabilities with impossibility, it may seem less than satisfying that all sorts of unbelievable events should always take place in an infinity of worlds. A possible way out is that, as suggested in [Man18b], branches
with norm orders of magnitude lower than the rest might not form stable classical worlds, as interference from larger ones precludes macroscopic causality in them. So, in the quantum case, extremely low probability (fraction or density, to be precise) might indeed equal impossibility.

6.1 Relation to classical probabilities

Can quantum fractionalism tell us anything about probabilities in classical settings? In EQM, our “classical” world is in essence quantum mechanical, so it may seem that classical probabilities should admit the same interpretation as quantum ones. But the apparent classicality of a system depends on the absence of a crucial ingredient for quantum probabilities: branching.

When a fair die is cast, does the Universe branch so that each result happens in 1/6 of all worlds? Not necessarily: if one gently releases the die just above a table there is no reason to expect the Universe will branch into all 6 results. Even a good roll of the die is usually considered a purely classical process, in the sense that given sufficiently detailed data about its initial state (but not so precise as to require a quantum description), one could in principle predict the result. If such classical determinism is an actual feature of the process then there is no branching, and probabilities due to ignorance about initial conditions (assuming they are classically well defined) bear no relation to quantum fractionalism.

On the other hand, branching is basically a gradual amplification of entangled quantum superpositions from a microscopic to a macroscopic scale, until components are distinct enough to have negligible interference. As the position and velocity with which the die is thrown depend on signals the hand receives from the brain, as a result of a myriad of neuronal chemical reactions, it is conceivable that quantum superpositions from such reactions might accumulate, generating a superposition of signals, which the hand+die system amplifies into macroscopically distinct results.

One might argue that quantum effects can not play any significant role in biological systems: as these are not isolated decoherence should eliminate quantum superpositions almost immediately. But in our case this is not a problem, as in EQM decoherence does not destroy components of a quantum superposition, it only separates them into independent worlds, which is precisely what we need.

Let us examine this in more detail. Chemical reactions are messy, especially in biological systems, depending on which molecules are present, in which quantities, which reagent molecules come in contact, and how they interact. One might predict the result statistically, which in EQM translates into a superposition of all possibilities. Each possible set $M$ of molecular reactions generates a certain signal $S_M$, leaving the brain in a different state and releasing some amount of heat into the environment, so the result of all possible reactions can be described as a superposition

$$\sum_M c_M |S_M\rangle |E_M\rangle , \quad (8)$$

where $E_M$ is the state of the environment (taken to include the brain).
For each $M$ there will be others with only a few distinct reactions, so their signals, and their effects on the environment, might be nearly identical. Even a coarse-graining might not be enough to separate (8) into independent branches, so we call this a pre-branching state: an almost continuous superposition of similar states, but whose differences can be amplified by some apparatus (in our case, the hand+die system) into macroscopically distinct branches.

As the superposition of signals reaches the hand, it is translated into a superposition of slightly different movements, each imparting to the die some initial data $I$. This produces a superposition of initial states $|D_I\rangle$ for the die, entangled to states $|E_I\rangle$ of the environment (including the hand), which can be approximated by an integral

$$\int_R k(I) |D_I\rangle |E_I\rangle dI$$

over a small region $R$ of phase space, with $|k(I)|^2$ giving the so called initial “probability” distribution of the die. Unless $R$ is too small (as when the die is released just above the table), this region will spread through phase space as the die bounces and rolls, covering nearly uniformly all 6 results. So, once decoherence does its job and we group the emerging worlds having each die result, each group should correspond to a fraction of approximately $\frac{1}{6}$ of all worlds.

Other “random” classical processes might also be traced back to quantum origins [AP14], with ignorance about initial conditions meaning just lack of entanglement of the observer with the components of the pre-branching state. In other words, a process is deemed random if its initial conditions are not well controlled, allowing cumulative quantum effects (possibly of an unknown origin, lying well in the past) to produce pre-branchings covering the admissible initial region of phase space, with the initial “probability” distribution being simply the density (or fraction) distribution of pre-branches.

Hence classical probabilities might indeed correspond to quantum ones. If a “random” process is repeated a large number of times with the same initial fraction distribution, the same argument as before tells us that in the vast majority of worlds the relative frequency of results will agree with classical predictions for the corresponding probability distribution.

The Bayesian interpretation also becomes clearer in terms of quantum “probabilities”. Saying there is a 60% chance it will rain tomorrow (or stocks will rise, or candidate $X$ will be elected) does not mean this will necessarily happen in 60% of worlds. But it is a subjective estimate, based on one’s incomplete knowledge about the initial distribution and its evolution, about a perfectly objective physical fact: the percentage of worlds in which this will happen due to a cumulative effect of quantum interactions between particles in the atmosphere (or chemical reactions in investors’ or voters’ brains). The meaning of Bayesian updating also becomes more concrete: it improves one’s knowledge about branching fractions.
7 Final remarks

Assuming our hypotheses are valid, quantum probabilities can be interpreted, in EQM, in terms of the relative amounts of worlds with each result, and the same might also be true for classical probabilities. With the probability problem solved, Wallace’s use of decoherence to solve the preferred basis problem can be justified. Hence the main objections to EQM are eliminated.

Popper’s falsifiability principle has also been invoked against EQM, with the claim that it makes no testable predictions differing from CQM. But as more sophisticated experiments test quantum effects at ever larger scales, and the original Everettian view is supplemented by new ideas, it may come a time when this is no longer true. Anyway, if they do make the same predictions, any test ever made of CQM is also a test of EQM. Historical antecedence does not make CQM more falsifiable than EQM.

Some critics also wield Occam’s razor against EQM, and they will certainly not like the idea of a continuum of identical universes, each with lots of distinct worlds. But objections against unnecessarily complex explanations only apply if a simpler theory can explain the same facts, while also being theoretically sound, which is not the case with CQM. Simply claiming a theory with infinitely many universes is needlessly complex is like attacking modern astronomy for requiring billions of galaxies to coherently explain observations.

Our proposal does rely on unorthodox assumptions. But the history of Quantum Mechanics is full of ideas (e.g. Bohr’s atomic model) which did not fit in with the physical knowledge of the time. Some even turned out to be wrong, but provided the seed for other developments, until there were enough new ideas to form a whole new paradigm. That problems of the Copenhagen interpretation still remain a century later suggests another paradigm change may be necessary, and new solutions challenging old conceptions should deserve serious consideration. Even if they turn out to be not quite right, they might still provide us with clues about what to look for.

On a final note, we observe that some authors have considered the use of real or quaternionic Hilbert spaces in Quantum Mechanics [FJSS62, Stu60]. Our work shows that, for dimensional reasons, a complex space may be critical to give the correct probabilities.

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