A Characterization of Multiplicity-Preserving Global Bifurcations of Complex Polynomial Vector Fields

Kealey Dias

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Abstract The space of degree $d$ single-variable monic and centered complex polynomial vector fields can be decomposed into loci where the integral curves have equivalent global topological structure. We prove that any bifurcation which preserves the multiplicity of equilibrium points can be realized as a composition of a finite number of simpler bifurcations, and these bifurcations are characterized.

Keywords Global bifurcations · Homoclinic orbits · Holomorphic foliations and vector fields · Complex ordinary differential equations

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1 Introduction

Bifurcations, the qualitative change in dynamics produced by varying parameters, are fundamental to the analysis of any family of dynamical systems but are notoriously difficult to describe in any generality. This paper takes a considerable step towards a complete description of the bifurcations of the global topological structure of the integral curves of the complex polynomial vector fields

$$\xi_P = P(z) \frac{d}{dz}, \quad z \in \mathbb{C},$$

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K. Dias
Department of Mathematics and Computer Science
Bronx Community College of the City University of New York
2155 University Avenue
Bronx, NY 10453 USA Tel.: +1-718-289-5030
E-mail: kealey.dias@bcc.cuny.edu
ORCID iD: 0000-0002-4613-6881
or equivalently, the maximal solutions $\gamma(t, z)$ to associated autonomous ordinary differential equation (ODE)

$$
\dot{z} = P(z), \quad \gamma(0, z) = z, \quad z \in \mathbb{C}, \quad t \in \mathbb{R},
$$

(2)

where $P(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0$ is a monic and centered polynomial of degree $d \geq 2$. Namely, we give a complete description of the multiplicity-preserving bifurcations, i.e. bifurcations where the multiplicities of the equilibrium points (the zeroes of $P$) are preserved under small perturbation.

Complex polynomial ODEs of the form (2) are a subset of the $\mathbb{R}^2$ systems

$$
\begin{align*}
\dot{x} &= u(x, y) \\
\dot{y} &= v(x, y),
\end{align*}
$$

(3)

whose global qualitative structure in general remains a fundamental open problem in dynamics. This includes Hamiltonian systems [26] and the existence of centers [21, 19] and cycles [14, 15]. Famously, part of Hilbert’s 16th problem inquires to the number and configurations of limit cycles in the plane, for each degree $d$ polynomial system in two real variables. Even though holomorphic vector fields, which lack limit cycles (e.g. [26, 21]), may seem distant from Hilbert’s 16th problem, experts in this area have shown that the most important perturbations to study are non-holomorphic perturbations of holomorphic polynomial vector fields with centers [1, 18]. Creating a complete description of bifurcations for complex polynomial vector fields would not only answer a fundamental question about holomorphic systems in their own right, but understanding the holomorphic perturbations may reduce the complexity of the analysis of non-holomorphic perturbations.

Complex vector fields are also linked to other areas of mathematics. In particular, the properties of single-variable complex vector fields have been utilized in proving important results in iterated complex dynamics, including the study of parabolic bifurcations (e.g. [25], [4], [23], [8]) and in the proof that there exist quadratic polynomial Julia sets of positive Lebesgue measure [7]. Single-variable complex vector fields are also being used to study higher dimensional complex systems [24], as well as quadratic and Abelian differentials on Riemann surfaces. The integral curves of holomorphic vector fields on Riemann surfaces foliate the Riemann surface, and they are similar in structure to the trajectories of these differentials. Understanding the bifurcations of complex ODEs may contribute to one of these fields in a way that has not yet been explored.

Sections 1 and 2 give basic properties of complex ODEs and introduce parameter space and bifurcations for the systems under consideration. Section 3 describes deformations in rectifying coordinates and proves that these are all that are needed to study the multiplicity-preserving bifurcations. Rank $k$ bifurcations are defined in Section 4, where it is also proved that all multiplicity-preserving bifurcations can be achieved as a composition of rank 1 bifurcations. Finally, Section 5 characterizes the rank 1 bifurcations to show that they cannot be arbitrarily complicated.

1.1 Basic Properties and Terminology

The complex ODE (2) inherits the general properties of the real planar ODEs (3) including existence and uniqueness of solutions, dependence on initial conditions, Hartman-Grobman theorem (linearization), and long-term behavior described by the Poincare-Bendixson theorem.
Equilibrium points  Classification of the local dynamics of holomorphic vector fields near an equilibrium point, a zero $ζ$ of $P$, is known to depend only on the order of the zero and the dynamical residue $Res(1/P,ζ)$ \cite{17,13,6,12,14,15,16,26}. The simple equilibrium points come in three types: sink (attracting), center (rotational), and source (repelling). There cannot be saddle points in the usual sense, since the Cauchy-Riemann equations force the eigenvalues of the Jacobian to be of the form $λ = α ± iβ$, which cannot have real part with opposite sign. Near zeros of multiplicity $m > 1$, there are $m - 1$ attracting and $m - 1$ repelling directions and $2(m - 1)$ elliptic sectors.

No limit cycles  Another difference with real planar vector fields is that there cannot be isolated periodic trajectories, hence no limit cycles, due to the identity theorem of complex analysis: consider a period $T$ solution $γ(t, z₀)$. Since $γ(T, z) - z$ is holomorphic in $z$ and $= 0$ on arc of $γ$, it is identically 0 on a neighborhood $z₀$.

Poles  Another type of singularity for these systems are the poles. The local dynamics near a pole only depends on the order of the pole. Indeed, the phase portrait of a vector field in the neighborhood of a pole of order $m ∈ \mathbb{N}$ is shown in Sverdlove \cite{26} to be topologically equivalent to the phase portrait of the differential equation

$$\dot{z} = \frac{λ}{z^m}, \quad λ \neq 0, \quad m ∈ \mathbb{N}, \quad (4)$$

in a neighborhood of $z = 0$, and Garijo, Gasull, and Jarque \cite{12} extend the result to conformal (biholomorphic) conjugacy. Even though holomorphic vector fields cannot have saddle points in the usual sense, the local behavior of Equation (4) resembles that of a saddle point by the following. It has $m + 1$ attracting and $m + 1$ repelling straight trajectories meeting at $z = 0$, alternating in orientation as we travel around $z = 0$, and the other trajectories of (4) in neighborhood of $z = 0$ are hyperbolic in form \cite{21}.

Separatrices  The maximal trajectories that are incoming to or outgoing from the poles are the separatrices, which determine the global structure of the integral curves up to topological equivalence \cite{22,12}. Polynomials of degree $d > 2$ have a pole of order $d - 2$ at the point at infinity, hence there are $2(d - 1)$ separatrices. Separatrices for monic polynomials have asymptotic directions $ℓ \pi / (d - 1)$, $ℓ = 0, \ldots , 2d - 3$ at infinity. The separatrices come in two types for polynomial vector fields: landing separatrices limiting at an equilibrium point in infinite forward or backward time; and homoclinic separatrices joining $∞$ to itself in finite time. Separatrices $s_ℓ$ are labeled according to their asymptotic directions. Outgoing (from $∞$) separatrices are denoted $s_ℓ$ with odd $k$, incoming separatrices are denoted $s_j$ with even $j$, and homoclinic separatrices are labeled $s_{ℓ, j}$, with odd $k$ and even $j$ corresponding to its outgoing and incoming asymptotic directions. Near infinity, the complement of the separatrices has $2(d - 1)$ connected components, which define $2(d - 1)$ accesses to infinity. The principal points of the prime ends defined by these accesses are called ends $e_ℓ$ and the label is such that the access corresponding to $e_ℓ$ is between the separatrices with labels $ℓ - 1$ and $ℓ$ mod $2(d - 1)$.

Disk Model  This separatrix structure can be represented in a separatrix disk model, by labeling the points $exp \left( \frac{2π ℓ}{2d} \right)$, $ℓ = 0, \ldots , 2d - 3$ on $S^1$ by $s_ℓ$ and joining each pair of points that correspond to the asymptotic directions of a homoclinic separatrix and joining all points that correspond to asymptotic directions of separatrices that land at the same equilibrium point (see Figure\cite{11}.)
1.2 Rectifying Coordinates, Zones, Analytic Invariants

Rectifying Coordinates and Transversals It is well-known that for polynomials, the connected components of \( \mathbb{C} \) minus the separatrix graph (called zones) are isomorphic to half planes, strips, or cylinders via the rectifying coordinates \( \Phi(z) = \int_0^z \frac{dw}{P(w)} \) (called distinguished or natural parameter in the literature on quadratic differentials \([17]\)), under which trajectories are pushed forward to horizontal lines. These coordinates also show that there are a number of geodesics in \( \mathbb{C} \setminus \{ \text{equilibrium points} \} \) in the metric with length element \( \frac{|dz|}{P(z)} \), which join the point at infinity to itself. There are \( s + h \) among these that are singled out: the \( h \) homoclinic separatrices and the \( s \) distinguished transversals \( T_{k,j} \). The distinguished transversals are chosen so that the indices \( k, j \) of the transversals \( T_{k,j} \) never coincide with the indices \( k, j \) of any homoclinic separatrices \( s_{k,j} \) (this can always be done \([9]\)). The transversal graph is the union of all homoclinic separatrices, distinguished transversals, and the point at infinity. For polynomials, this graph is really a "flower," since the point at infinity is the only vertex (pole), and both the homoclinic separatrices and transversals join infinity to itself.

Note that the labelling of the separatrices in rectifying coordinates follows a pattern: if the separatrices are on the upper (resp. lower) boundary of a strip, a half-plane, or a cylinder, then the odd \( k \) and even \( j \) labels satisfy \( k_{i+1} = j_i - 1 \) (resp. \( k_{i+1} = j_i + 1 \)) mod \( 2(d - 1) \), reading left to right (see Figure 1).

Classification A complete set of realizable invariants for the classification of these vector fields is as follows (see \([11]\) and \([5]\) for details). A labelled metric graph, which is the transversal graph, assigning \( h \) real numbers \( \tau = \int_{T_{k,j}} \frac{dw}{P(w)} > 0 \) to the homoclines and \( s \) complex numbers \( \alpha = \int_{T_{k,j}} \frac{dw}{P(w)} \in \mathbb{H} \) to the distinguished transversals. The analytic invariants \( \tau \in \mathbb{R}_+ \) that correspond to homocline separatrices are the Euclidean lengths of the rectified homoclinic separatrices, and the analytic invariants \( \alpha \in \mathbb{H}_+ \) corresponds to the complex numbers that record the height and shear of each strip.

Note that since each homoclinic separatrix and distinguished transversal separates the equilibrium points, they are sums of dynamical residues: \( \tau, \alpha = 2\pi \sum \text{Res}(1/P, \xi) \). Furthermore, the \( s \) transversals and \( h \) homoclinic separatrices subdivide the dynamical plane into \( N \) components, each component containing one equilibrium point (see Figure 1).

2 Parameter Space and Bifurcations

Even when restricting to complex planar systems, a comprehensive study of the bifurcations remains elusive. Researchers have made headway by considering systems involving few parameters, polynomials of low-degree so that the bifurcation diagram can be visualized \([13, 18, 24]\), or describing bifurcations induced by rotations of vector fields \([20]\). The aim of this paper is to make a significant step towards a comprehensive analysis of all global bifurcations for arbitrary degree by characterizing the bifurcations where the multiplicity of the equilibrium points is preserved.

To understand the possible bifurcations, we examine the space \( \mathcal{E}_d \) of degree \( d \) monic, centered complex polynomial vector fields, and decompose it in a dynamically meaningful way: we partition \( \mathcal{E}_d \) into combinatorial classes \( \mathcal{C} \) such that all \( \xi \in \mathcal{C} \) have the same (labeled) separatrix graph. Note that \( \mathcal{E}_d \cong \mathbb{C}^{d-1} \) since the monic, centered polynomials \( P(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0 \) can be parameterized by the coefficients \((a_0, \ldots, a_{d-2}) \in \mathbb{C}^{d-1} \).
Fig. 1 (Left) The disk model of a degree 5 polynomial vector field, with a sink (ζ_3), a source (ζ_1), and three centers (ζ_2, ζ_4, ζ_5). There are two landing separatrices (s_0 and s_3) and three homoclinic separatrices (s_1, s_2, s_5, s_7), labelled by their asymptotic directions at infinity. The dashed curve T_{3,0} is a transversal and is not a trajectory. It is labelled by the ends it connects. (Right) The same vector field in rectifying coordinates. There is one strip, corresponding to the single strip zone, and there are three half-infinite cylinders, corresponding to the three center zones. The vector field is pushed-forward to d/dz in these coordinates, so horizontal lines are trajectories. The boundaries of these zones consist of homoclinic and landing separatrices, together with the ends. There are a number of geodesics that connect infinity to itself (connects ends to ends): the homoclinic separatrices and the transversals (only the distinguished transversal is pictured). Note that the labelling of the separatrices follows a pattern: if the separatrices are on the upper (resp. lower) boundary of a strip (as pictured), a half-plane, or a cylinder, then the odd k and even j labels satisfy k_i + 1 = j_i (resp. k_i + 1 = j_i + 1) mod 2(d - 1), reading left to right. The analytic invariants τ ∈ R^+ that correspond to homoclinic separatrices are the Euclidean lengths of the rectified homoclinic separatrices, and the analytic invariant α ∈ H_+ corresponds to the complex number that records the height and shear of the strip. Note that since each homoclinic separatrix and distinguished transversal separates the equilibrium points, the α, τ are sums of Res(1/P, ζ).

The separatrix structure gives topological equivalence. A change in separatrix structure under small perturbation therefore gives a bifurcation. So understanding bifurcations is about understanding how the classes C fit together in parameter space. This leads us to ask the following question:

**Question 1** For every c_0 ∈ Ξ_d, what are the C that intersect every neighborhood of c_0?

The **structurally stable** vector fields in Ξ_d, i.e. those that do not change qualitatively under small perturbation, are those with neither homoclinic separatrices nor multiple equilibrium points [11, 10] and are of full dimension in parameter space. The bifurcation locus in Ξ_d consists of all vector fields with at least one homoclinic separatrix or multiple equilibrium point (strictly lower than full dimension) [11, 10]. This is easy to see intuitively, as one can imagine a multiple equilibrium point could split into several equilibrium points under small perturbation, and a homoclinic separatrix can be broken under small rotation.

**Landing Separatrices are Stable** Bifurcations can only involve breakings of homoclinics and/or splittings of multiple points due to the following result from [10], which proves that an equilibrium point cannot lose a landing separatrix under small perturbation, as long as its multiplicity is preserved.

**Theorem 1** (Dias, Tan) Let ζ_0 be an equilibrium point for ξ_{c_0} ∈ Ξ_d, and let ζ be an equilibrium point for ξ_c ∈ Ξ_d in a sufficiently small neighborhood of ξ_{c_0} such that mult(ζ^0) = ...
$\text{mult}(\zeta)$ and $\lim_{P \to P_0} \zeta = \zeta^0$. If the separatrix $s_t$ for $\xi_{P_0}$ lands at $\zeta^0$, then the separatrix $s_t$ for $\xi_P$ lands at $\zeta$.

**Multiplicity-preserving Bifurcations** The bifurcations which allow splitting of multiple equilibrium points are more complicated than the multiplicity-preserving bifurcations, and not only because they involve the variation of more parameters. For splitting bifurcations, the possible changes in topological structure may depend on the initial analytic data, in addition to the initial topological data [10]. The bifurcations with this added complexity are to be studied in a future paper; we aim in this paper to analyze the restricted question:

**Question 2** Given any point $c_0 \in \Xi_d$ in the bifurcation locus, what are the possible bifurcations such that the multiplicities of the equilibrium points are preserved?

We will call such bifurcations **multiplicity-preserving bifurcations**.

**Definition 1** The **local multiplicity-preserving set** of $c_0$, denoted $\text{LMP}(c_0)$, is the set of all vector fields $\xi \in \Xi_d$ in a sufficiently small neighborhood of $c_0 \in \Xi_d$ such that $\text{mult}(\zeta_i) = \text{mult}(\zeta_0^i)$.

**Definition 2** The **multiplicity-preserving set** of $c_0$, denoted $\text{MP}(c_0)$, is the set of all combinatorial classes $C$ whose intersection with the local-non splitting set is non-empty.

Note that the notation above differs from that in [10].

**Remark 1** If $\xi_{c_0}$ has at least one homoclinic separatrix, then $\text{MP}(c_0)$ is non-empty.

**Remark 2** Note that since the multiplicities are preserved, the local multiplicity-preserving set can be locally parameterized by the $N$ roots, and in fact can be locally parameterized by $N - 1$ roots due to centering of the polynomial.

### 2.1 Parameterizing a Class $\mathcal{C}$

Within a class $\mathcal{C}$, each vector field has the same zone types (strip, half plane, cylinder) with the same labeling, but $\alpha \in \mathbb{H}_+$ and $\tau \in \mathbb{R}_+$ may vary (see Figure 2). This idea can be formalized with the following theorem from [10].

**Theorem 2 (Dias)** There exists a real analytic isomorphism $\mathbb{H}^s \times \mathbb{R}^h_+ \to \mathcal{C}$, which is $\mathbb{C}$-analytic in the first $s$ coordinates and $\mathbb{R}$-analytic in the last $h$ coordinates. It is the restriction of a holomorphic mapping in $(s + h)$ complex variables: $\mathbb{H}^s \times V^h_+ \to \Xi_d$, where $V_+$ is an epsilon neighborhood of $\mathbb{R}_+$.

In particular, $(\alpha, \tau) \mapsto \xi_{(\alpha, \tau)} \in \Xi_d$ is holomorphic (in $s + h$ variables) in a neighborhood of $(\alpha_0, \tau_0) \in \mathbb{H}^s \times \mathbb{R}^h_+ \subset \mathbb{C}^{s+h}$.

**Corollary 1** Note that by the theorem above, if we allow the $\tau$ to take on complex values, we leave the class $\mathcal{C}$, i.e. a bifurcation occurs. Furthermore, we only needed continuity (we have holomorphicity) in the $\tau, \alpha$ to prove that these are bifurcations.
We briefly summarize the proof for its use in this manuscript. For details, see [10]. Pick a vector field $\xi_0 \in \Xi_d$ with combinatorial class $\mathcal{C}$ and a corresponding set of analytic invariants $(\alpha, \tau)$. This determines a configuration of half planes, strips, and cylinders. Deform the rectified zones (as sets) by piecewise linear mappings by allowing the $\tau$ to take on non-real values. We endow the deformed zones with $d\xi$ and use quasiconformal surgery to show that this corresponds to polynomial vector fields $(\xi_\tau \in \Xi_d)$ which depend holomorphically on the $\tau$.

3 Deformations in Rectifying Coordinates

Corollary [11] notes that deformations in rectifying coordinates correspond to bifurcations in the family $\Xi_d$, in particular, those where homoclinic separatrices break in various ways. We will see in some examples below that the resulting change in separatrix structure can be read from the deformed rectifying coordinates. We will prove in this section that these
deformations "cover" the local multiplicity-preserving set, meaning that every multiplicity-preserving bifurcation can be seen as some deformation in rectifying coordinates.

**Example 1 - A Single Homoclinic Separatrix Breaks** Consider the example depicted in Figure 4. The top row corresponds to a combinatorial configuration (in the disk and in rectifying coordinates) in $\Sigma_3$ which has one sink, one source, and one center. The separatrices $s_2$ and $s_3$ are landing, and there is one homoclinic separatrix $s_{1,0}$. We want to understand the resulting combinatorics when we break the single homoclinic separatrix, i.e. perturb the single real analytic invariant $\tau$ to have non-zero imaginary part. The center becomes a sink or source, and one can see where the separatrices now land by looking in the rectifying coordinates.

**Fig. 4** The top row corresponds to a combinatorial configuration (in the disk and in rectifying coordinates) in $\Sigma_3$ before perturbation. The equilibrium points are one sink, one source, and one center. The separatrices $s_2$ and $s_3$ are landing, and there is one homoclinic separatrix $s_{1,0}$. There is one real analytic invariant $\tau$ corresponding to this homoclinic separatrix. The second and third rows of the figure correspond to two possible combinatorial configurations after perturbing the initial. The second (resp. third) row corresponds to allowing $\tau$ to take on a complex value with positive (resp. negative) imaginary part. In the first case, the perturbed center equilibrium point becomes a source, and in the second case it becomes a sink. Notice that $s_2$ and $s_3$ still land at their respective equilibrium points.

We explain the general situation when exactly one homoclinic separatrix $s_{k,j}$ breaks. A homoclinic separatrix $s_{k,j}$ is on the boundary of exactly two zones. When we allow the single $\tau$ corresponding to $s_{k,j}$ to vary holomorphically, then this causes the separatrices $s_k$ and $s_j$ to land. If $\tau \in \mathbb{H}_+$, the separatrix $s_k$ (resp. $s_j$) now lands at the equilibrium point on the boundary of the zone having $s_{k,j}$ as part of its upper (resp. lower) boundary in rectifying coordinates.
coordinates. If $\tau \in \mathbb{H}_-^-$, then instead of coming back into (resp. outgoing from) infinity, the separatrix $s_k$ (resp. $s_j$) now lands at the equilibrium point on the boundary of the zone having $s_{k,j}$ as part of its lower (resp. upper) boundary in rectifying coordinates. The equilibrium point at which $s_k$ (resp. $s_j$) lands is either a sink (resp. source) or multiple equilibrium point, depending on whether the lower or upper (resp. upper or lower) rectified zone having $s_{k,j}$ on the boundary was a vertical half-strip or strip in the first case, or in the latter case, a half plane.

**Example 2 - A New Homoclinic Separatrix Can Form** If we allow more than one analytic invariant $\tau$ to vary at the same time, more complicated things can happen. In particular, new homoclinic separatrices can form (see Figure 5). An example of a distorted strip whose corresponding perturbed vector field has a homoclinic separatrix $s_{k_1,j_4}$ which was not there before perturbation. The separatrices $s_{j_3}$ and $s_{j_1-1}$ land at the equilibrium points in the other zones which had the homoclinic separatrices $s_{k_4,j_4}$ and $s_{k_2,j_2}$ on the upper or lower boundary before perturbation. The $\tau(s_{k_1,j_4})$ should all be seen as having imaginary part distorted by some small $\varepsilon_i$.

![Fig. 5](image_url) An example of a distorted strip whose corresponding perturbed vector field has a homoclinic separatrix $s_{k_1,j_4}$ which was not there before perturbation. The separatrices $s_{j_3}$ and $s_{j_1-1}$ land at the equilibrium points in the other zones which had the homoclinic separatrices $s_{k_4,j_4}$ and $s_{k_2,j_2}$ on the upper or lower boundary before perturbation. The $\tau(s_{k_1,j_4})$ should all be seen as having imaginary part distorted by some small $\varepsilon_i$.

We now describe the general requirement for a homoclinic separatrix to form under small perturbation. In order to understand this situation, we need to define H-chains. These H-chains turn out to be the structures we need to understand exactly which homoclinic separatrices can form under small perturbation, so we define them here.

**Definition 3** An **H-chain** of length $n$ is a sequence of $n$ consecutive homoclinic separatrices $\{s_{k_i,j_i}\}$, $i = 1, \ldots, n$, i.e. homoclinic separatrices $s_{k_i,j_i}$ such that for each $i$, either $k_{i+1} = j_i + 1$ or $k_{i+1} = j_i - 1$ (see Figure 1).

That is, an H-chain is an oriented product of homoclinic separatrices (see Figure 6).

**Definition 4** A **closed H-chain** of length $n$ is an H-chain in which $s_{k_{i+n},j_{i+n}} = s_{k_i,j_i}$, for all $i = 1, \ldots, n$. 

Remark 3  Note that if an H-chain is not closed, there are two possibilities:
1. There exists an $i$ and there exists an $m < n$ such that $s_{k_{i},m} = s_{k,j}$, or
2. for all $i, m \leq n, s_{k_{i},m} \neq s_{k,j}$.

**Proposition 1** The separatrix $s_{k,j}$ can form under small perturbation if and only if $s_{k,j_{0}}$ and $s_{k_{0},j}$ have an H-chain in common (belong to some H-chain) such that $s_{k,j_{0}}$ is to the left of $s_{k,j}$ in rectifying coordinates.

**Proof** Either $s_{k,j_{0}}$ and $s_{k_{0},j}$ belong to a closed H-chain, in which case we can define an H-chain such that $s_{k,j_{0}}$ is to the left of $s_{k,j}$; if they do not belong to some closed H-chain, then we assume $s_{k,j_{0}}$ is to the left of $s_{k,j}$ in rectifying coordinates. Let the minimal such H-chain be numbered $s_{k,j_{0}} = s_{k_{1},j_{1}}$, $s_{k_{2},j_{2}}$, ..., $s_{k_{n},j_{n}} = s_{k,j}$. For $i = 2, \ldots, n$, there is a sequence $I_{i}$ of length $n-1$ with elements in $\{+, -\}$ corresponding to whether $k_{i+1} = j_{i} \pm 1, i = 1, \ldots, n-1$.

We consider $I_{1}$ not defined. If there are $q$ sign changes in this itinerary, then there are $q+1$ zones that $s_{k}$ needs to pass through to reach $s_{j}$ (see Figure 7). The separatrix $s_{k,j}$ will form if the following conditions on perturbations of the associated $\tau_{i}, i = 1, \ldots, n$ are satisfied: For $i = 1, \ldots, n-1$,

- if $I_{i+1} = +$, then $\sum_{j=1}^{i} \Im(\tau_{j}) < 0$;
- if $I_{i+1} = -$, then $\sum_{j=1}^{i} \Im(\tau_{j}) > 0$;
- and $\sum_{j=1}^{n} \Im(\tau_{j}) = 0$,

(see Figure 7).

If $s_{k,j_{0}}$ and $s_{k_{0},j}$ do not have an H-chain in common, then there is no overlapping sequence of zones through which $s_{k}$ can have access to $s_{j}$. The "only if" part then follows from Proposition 2 below, which proves that any multiplicity-preserving bifurcation can be realized as some deformation in rectifying coordinates.

In general, several homoclinic separatrices can form simultaneously under small perturbation, if the conditions on the partial sums as in Proposition 1 are all satisfied (see Figure 8).
3.1 Deformations in Rectifying Coordinates Cover the Local Multiplicity-preserving Set

We will prove that all multiplicity-preserving bifurcations can be seen by deformations in the rectifying coordinates. That is, we want to prove that varying the $\tau$ covers all multiplicity-preserving bifurcations. Note that one can locally parameterize the local multiplicity-preserving set by $N - 1$ of the roots.

**Proposition 2** The map $(\alpha_1, \ldots, \alpha_s, \tau_1, \ldots, \tau_h) \mapsto (\xi_1, \ldots, \xi_{N-1})$ in local multiplicity-preserving set, is locally surjective.

**Proof** Let $N$ be the number of equilibrium points not counting multiplicity. Because we assume centering, only $N - 1 = s + h$ roots are independent. We need that the map $(\alpha_1, \ldots, \alpha_s, \tau_1, \ldots, \tau_h) \mapsto (\xi_1, \ldots, \xi_{N-1}) \in \text{LMP}(c_0)$ is locally surjective. It is enough to show that the map $(\xi_1, \ldots, \xi_{N-1}) \mapsto (\alpha_1, \ldots, \alpha_s, \tau_1, \ldots, \tau_h)$ is (locally) well-defined and continuous for all $(\xi_1, \ldots, \xi_{N-1}) \in \text{LMP}(c_0)$, the local multiplicity-preserving set of $c_0$. Since the analytic invariants $\alpha, \tau$ are sums of dynamical residues $\text{Res}(1/P, \xi_i)$ (see Figure 1), we can see that this map is a composition of...
\( R = (\text{Res}(1/P, \zeta_1), \ldots, \text{Res}(1/P, \zeta_{N-1})) \) and a linear map. This can be summarized in the diagram below:

\[
\begin{align*}
(\zeta_1, \ldots, \zeta_{N-1}) & \rightarrow (\alpha_1, \ldots, \alpha_s, \tau_1, \ldots, \tau_h) \\
\downarrow_{\text{linear}} & \\
(\text{Res}(1/P, \zeta_1), \ldots, \text{Res}(1/P, \zeta_{N-1}))
\end{align*}
\]

Now suppose \( \zeta_1 \) is a pole of order \( m_1 \) for \( 1/P \). Then

\[
\text{Res}(1/P, \zeta_1) = \lim_{z \to \zeta_1} \frac{d^{m_1-1}}{dz^{m_1-1}} \frac{(z - \zeta_1)^{m_1}}{P(z)} = \lim_{z \to \zeta_1} \frac{d^{m_1-1}}{dz^{m_1-1}} \frac{1}{(z - \zeta_2)^{m_2} \cdots (z - \zeta_{N-1})^{m_{N-1}}} = \text{Poly}(\zeta_1 - \zeta_{\neq 1})
\]

where \( \text{Poly}(\zeta_1 - \zeta_{\neq 1}) \) is a polynomial in \( (\zeta_1 - \zeta_{\neq 1}) \) and \( M_i \) are natural numbers. Note that this is well-defined on the multiplicity-preserving set since \( \zeta_{\neq 1} \neq \zeta_1 \), and since it is rational, it is continuous where defined. \( \square \)

4 All Multiplicity-preserving Bifurcations Can be Achieved as a Composition of Rank 1 Bifurcations

We want to be able to describe all of the multiplicity-preserving bifurcations in a manageable way. We do this by showing that any bifurcation can be realized as the composition of a finite set of simpler bifurcations. In this section, we prove that every multiplicity-preserving bifurcation can be realized as a composition of rank 1 bifurcations. In Section 5 we will characterize the rank 1 bifurcations, showing they cannot be arbitrarily complicated.

4.1 Dimension and Codimension of a Class

Note that Theorem 2 also gives the (real) dimension of a combinatorial class.

**Proposition 3** For a combinatorial class \( \mathcal{C} \in \mathcal{Z}_d \), with \( s \) strips, \( h \) homoclinic separatrices, and \( m^* = \sum (\text{mult}(\zeta_i) - 1) \),

\[
\dim_{\mathbb{R}}(\mathcal{C}) = 2s + h, \quad \text{and} \quad \text{codim}_{\mathbb{R}}(\mathcal{C}) = 2m^* + h.
\]

**Proof** Recall that \( \mathcal{Z}_d \simeq \mathbb{C}^{d-1} \), so \( \dim_{\mathbb{R}}(\mathcal{Z}_d) = 2d - 2 \). A combinatorial class \( \mathcal{C} \) is analytically isomorphic to \( \mathbb{H}^s \times \mathbb{R}^h \), so \( \dim_{\mathbb{R}}(\mathcal{C}) = 2s + h \). Let \( N = \#P^{-1}(0) \), the number of equilibrium points not counting multiplicity. Then \( m^* = d - N \). Putting this together with \( s + h = N - 1 \) gives \( s = d - m^* - 1 - h \). Therefore, the (real) codimension of each class as a subset of parameter space is \( \text{codim}_{\mathbb{R}}(\mathcal{C}) = 2d - 2 - (2s + h) = 2d - 2 - 2(d - m^* - 1 - h) - h = 2m^* + h \). \( \square \)

**Definition 5** The boundary \( \partial \mathcal{C} \) of a combinatorial class \( \mathcal{C} \) is defined in the usual sense: the closure \( \overline{\mathcal{C}} \) in \( \mathcal{Z}_d \), minus \( \mathcal{C} \).
Lemma 1  For \( \mathcal{C} \subset \text{MP}(c_0) \) (necessarily \( \mathcal{C}_0 \cap \partial \mathcal{C} \neq \emptyset \)), then \( \mathcal{C}_0 \subset \partial \mathcal{C} \), where \( \mathcal{C}_0 \) is the class containing \( c_0 \).

Proof A particular vector field \( c_0 \in \Sigma_d \) gives a combinatorial configuration. Then all of \( \mathcal{C}_0 \subset \partial \mathcal{C} \), since by Theorem 2 the combinatorics of the multiplicity-preserving bifurcations do not depend on initial homoclinic length, but only the relative imaginary parts of the \( \tau \).

Corollary 2  It follows that \( \mathcal{C}_0 \) must have strictly less dimension than all such \( \mathcal{C} \).

4.2 Rank \( k \) Bifurcations

Definition 6  A rank \( k \) bifurcation is a bifurcation from \( c_0 \in \mathcal{C}_0 \) to \( c_1 \in \mathcal{C}_1 \) such that \( \dim_{\mathbb{R}}(\mathcal{C}_1) - \dim_{\mathbb{R}}(\mathcal{C}_0) = k \).

Recall from Proposition 3 that \( \dim_{\mathbb{R}}(\mathcal{C}) = 2s + h \) and \( \text{codim}_{\mathbb{R}}(\mathcal{C}) = 2m^* + h \), where \( m^* = \sum (\text{mult}(\zeta_i) - 1) \). For our current purposes (multiplicity-preserving bifurcations), the codimension proves more useful. Using the notation that an index of 0 corresponds to the vector field before bifurcation and the index 1 after, then using Proposition 3 a rank \( k \) bifurcation is when \( 2(m^*_0 - m^*_1) + (h_0 - h_1) = k \). Since we are in the case where the multiplicities do not change, i.e. \( m^*_0 = m^*_1 \), this simplifies to \( (h_0 - h_1) = k \). In words, the number of homoclinic separatrices before bifurcation is \( k \) greater than the number of homoclinic separatrices after bifurcation. Therefore, a rank \( k \) multiplicity-preserving bifurcation is such that \( h \) homoclinic separatrices break and \( h - k \) new homoclinic separatrices form.

Proposition 4  There do not exist rank \( k \leq 0 \) multiplicity-preserving bifurcations.

Proof  More homoclinic separatrices cannot form than those that initially existed (due to fixed number of asymptotic directions and the landing separatrices are stable result of Dias and Tan [10]). Therefore, \( h_1 \leq h_0 \) so there can be no rank \( k < 0 \) multiplicity-preserving bifurcations. There can be no rank \( k = 0 \) multiplicity-preserving bifurcations, since Corollary 2 implies that the dimension of \( \mathcal{C}_0 \) is strictly less than the dimension of \( \mathcal{C}_1 \).

4.3 Compositions of Rank 1 Bifurcations

By Definition 6 a rank 1 bifurcation is a bifurcation from \( c_0 \in \mathcal{C}_0 \) to \( c_1 \in \mathcal{C}_1 \) such that \( \dim_{\mathbb{R}}(\mathcal{C}_1) - \dim_{\mathbb{R}}(\mathcal{C}_0) = 1 \). The multiplicity-preserving, rank 1 bifurcations are the bifurcations such that \( h \) homoclinic separatrices break and \( h - 1 \) new ones form. See some examples in Figure 9.

Proposition 5  Every multiplicity-preserving bifurcation can be realized as a composition of rank 1 bifurcations.

The idea of the proof is to show that for every \( \mathcal{C} \subset \text{MP}(c_0) \), there exists a sequence of classes \( \mathcal{C}_i \), \( i = 1, \ldots, k - 1 \) such that \( \mathcal{C}_i \subset \text{MP}(c_0) \) and \( \partial \mathcal{C} \supset \mathcal{C}_{i+1} \), \( \partial \mathcal{C}_{i+1} \supset \mathcal{C}_i \) and \( \mathcal{C}_i \) to \( \mathcal{C}_{i+1} \) is a rank 1 bifurcation (see Figure 10). Next we prove that if \( \mathcal{C} \subset \text{MP}(c_0) \), then all of \( \mathcal{C} \)’s relevant boundary components are also in the multiplicity-preserving set \( \text{MP}(c_0) \).
Now we can finish the main proof. For every \( \mathcal{C} \subset \text{MP}(c_0) \), there exists a sequence of classes \( \mathcal{C}_i \), \( i = 1, \ldots, k-1 \) such that \( \mathcal{C}_i \subset \text{MP}(c_0) \) and \( \text{dim}(\mathcal{C}_i) = \text{dim}(\mathcal{C}_0) = k \). Next, \( \mathcal{C}_i \subset \text{MP}(c_0) \) by Lemma 2 and item 2. follows from Lemma 1. Note that \( \mathcal{C}_i \subset \partial \mathcal{C} \) implies \( \text{dim}(\mathcal{C}_i) < \text{dim}(\mathcal{C}) \) by Corollary 2.

**Lemma 2** For all \( \mathcal{C} \subset \text{MP}(c_0) \), it holds that every component \( \mathcal{C}_1 \) of \( \partial \mathcal{C} \) with \( c_0 \in \partial \mathcal{C}_1 \), \( \mathcal{C}_1 \subset \text{MP}(c_0) \).

**Proof** This is almost trivially true by the observation that equilibrium points cannot coalesce under small perturbation. We prove the contrapositive. Say \( \mathcal{C}_1 \) intersects the boundary of \( \mathcal{C} \), and they both have \( c_0 \) in the boundary. If \( \mathcal{C}_1 \) is in the splitting set of \( c_0 \), then it means that any vector field in \( \mathcal{C}_1 \) has more zeros than \( c_0 \). But since \( \mathcal{C}_1 \) intersects (or is contained in) the boundary of \( \mathcal{C} \), then there are points of \( \mathcal{C} \) arbitrarily close to some points in \( \mathcal{C}_1 \), and hence the number of zeros for a vector field in \( \mathcal{C} \) must be greater than or equal to the number of zeros of a vector field in \( \mathcal{C}_1 \) (since points can’t join under small perturbation). Hence, the number of zeros in \( \mathcal{C} \) are greater than the number in \( c_0 \), and hence \( \mathcal{C} \) belongs to the splitting set of \( c_0 \). \( \square \)

Now we can finish the main proof.

**Proof (of Proposition 5)** For every \( \mathcal{C} \) with \( c_0 \in \partial \mathcal{C} \) and \( \text{dim}(\mathcal{C}) - \text{dim}(\mathcal{C}_0) = k > 1 \), there exists \( \mathcal{C}_1 \subset \text{MP}(c_0) \) such that:

1. \( \text{dim}(\mathcal{C}) > \text{dim}(\mathcal{C}_1) > \text{dim}(\mathcal{C}_0) \) and
2. \( \mathcal{C}_1 \subset \partial \mathcal{C} \).

Note that \( \mathcal{C}_0 \) cannot fill a dense subset of \( \partial \mathcal{C} \cap V_0 \) by the assumption that \( \text{dim}(\mathcal{C}) - \text{dim}(\mathcal{C}_0) > 1 \), so there must be some other \( \mathcal{C}_1 \) such that \( \partial \mathcal{C} \cap \mathcal{C}_1 \neq \emptyset \) and \( c_0 \in \partial \mathcal{C}_1 \). Next, \( \mathcal{C}_1 \subset \text{MP}(c_0) \) by Lemma 2 and item 2. follows from Lemma 1. Note that \( \mathcal{C}_1 \subset \partial \mathcal{C} \) implies \( \text{dim}(\mathcal{C}_1) < \text{dim}(\mathcal{C}) \) by Corollary 2. \( \square \)
5 Characterization of the Rank 1 Multiplicity-preserving Bifurcations

The result of Proposition 5 is not helpful if the rank 1 bifurcations can be arbitrarily complicated. We characterize the rank 1 bifurcations in this section to show that this is not the case. We prove that all rank 1 bifurcations are of essentially the same type as the examples in Figure 9.

Theorem 3 Every multiplicity-preserving rank 1 bifurcation is of the type where a sequence $s_{k_1,j_1}, s_{k_2,j_2}, \ldots, s_{k_n,j_n}$ of $n \geq 1$ homoclinic separatrices breaks such that for each $i$, $s_{k_i,j_i}$ and $s_{k_i+1,j_i+1}$ are on the boundary of the same zone, and $n-1$ homoclinic separatrices form:

In order to prove this theorem, we will build new objects: homoclinic graphs (H-graphs, for short). These graphs are to be defined so that an admissible path in the H-graph corresponds to a homoclinic separatrix that is able to form under small perturbation, and the vertices of this path correspond to the homoclinic separatrices that must break to form that new homoclinic.

5.1 H-graphs

H-graphs are directed graphs, based on the initial homoclinic separatrix configuration, to help organize the possible multiplicity-preserving bifurcations. They are to be defined such that:

- an admissible path (to be defined) corresponds to a homoclinic separatrix that is able to form under small perturbation, and
- the vertices of this path correspond to the homoclinic separatrices that must break in order for the new homoclinic to form.

E.g. a single admissible path using $k+1$ vertices corresponds to a rank $k$ bifurcation, since $k+1$ homoclinics break and only one new one forms.

Definition 7 An H-graph corresponding to a combinatorial class $\mathcal{C}$ is a directed graph which is embedded in the combinatorial disk model such that:

1. Each homoclinic separatrix for $\mathcal{C}$ has exactly one vertex of the H-graph,
2. Each edge is contained in a connected component of the disk minus the homoclinic separatrices
3. there is an edge directed from $v_1$ to $v_2$ if there is some H-chain containing both corresponding homoclinic separatrices

Remark 4 In the disk model, each connected component of the disk minus the homoclinic separatrices corresponds to a strip, half-plane, or cylinder zone, so leaving a connected component corresponds to leaving a zone.

Remark 5 Note that within a connected component that does not correspond to a cylinder, all of the directed edges of the H-graph in that component must run in the same direction as the H-chain on the boundary. In a component that does correspond to a cylinder (has a closed H-chain on the boundary), some edges in the H-graph may run contrary to the orientation of the H-chain, since one can 'wrap around' the cylinder.
Admissible Paths A homoclinic separatrix can form from homoclinics on the boundary of a single zone. Here, though more homoclinics may break under small perturbation, only two are required to break (see second from left in Figure 9). If a homoclinic forms from an H-chain that involves more than one zone, two homoclinic separatrices from each zone are required to break since in total: the start homoclinic separatrix, the end homoclinic separatrix, and all the homoclinic separatrices in between that join the consecutive zones (one to enter the zone, one to leave the zone). These observations lead us to define admissible paths in the H-graph.

Definition 8 An admissible path is a path in the H-graph such that it respects the orientation of the edges of the H-graph and uses only one edge per connected component.

Simultaneous Paths The H-graph was designed so that paths correspond to single homoclinic separatrices that are able to form. To consider the possibility when more than one homoclinic may form at once, we need to consider simultaneous paths in the H-graph.

Definition 9 A union graph is a union of admissible paths such that:
- No pair of paths can share a start or end vertex, i.e. there cannot be a vertex which is a start vertex for two paths or an end vertex for two paths (see Figure 14).
- For each vertex of the H-graph, there are two connected components (zones) that have it on the boundary. In one connected component, all edges must be incoming to the vertex, and in the other connected component, all edges must be outgoing from that vertex (i.e. each vertex in the union of these paths has a well-defined direction through it).

We explain the need for the two conditions in the definition. If two paths shared a start (resp. end) vertex, it would be the same as saying $s_k \leftrightarrow j_1$ and $s_k \leftrightarrow j_2$ (resp. $s_k \leftrightarrow j_1$ and $s_k \leftrightarrow j_2$) could form simultaneously, but it is not possible for two homoclinic separatrices to share an asymptotic direction. If the vertices do not have a well-defined direction through them, it corresponds to a single $\tau$ taking on a value in $\mathbb{H}_+$ and $\mathbb{H}_-$ simultaneously, which is not possible (see Figures 13 and 14).
Remark 6 We do not claim that every union graph defined as above corresponds to a possible simultaneous formation of homoclinic separatrices, only that a simultaneous formation of homoclinics must be of that form.

Fig. 13 If the vertices do not have a well-defined direction through them, it corresponds to a single $\tau$ taking on a value in $\mathbb{H}_+$ and $\mathbb{H}_-$ simultaneously, which is not possible.

Proposition 6 The union graph contains neither directed nor undirected cycles.

Proof Suppose the union graph does contain a cycle. Pick a vertex $v$ in the cycle and let $e_1$ and $e_2$ be the edges in the cycle that meet at $v$.

Case I: The cycle contains a vertex $v$ such that $e_1$ is incoming to $v$ and $e_2$ is outgoing from $v$. Now $e_1$ and $e_2$ must be on opposite sides of the homoclinic that goes through $v$, otherwise this would violate direction flow (the second condition in Definition 9). Since
Fig. 14 Not allowed: some pair of paths share a start or end vertex (pictured left). Not allowed: two paths contain a common vertex where the paths have opposite direction through it (pictured right).

Fig. 15 (Left) There can not be a cycle in the union graph that contains a vertex $v$ such that $e_1$ is incoming to $v$ and $e_2$ is outgoing from $v$. This would imply that $e_1$ and $e_2$ are on opposite sides of the homoclinic that goes through $v$, otherwise this would violate direction flow. Since each homoclinic must traverse the entire disk, it must cut the cycle somewhere else besides at $v$. We arrive at a contradiction since a homoclinic cannot cross two vertices, nor can it cross an edge in the H-graph. (Right) If the union graph had a cycle, it would have to have an even number of edges with alternating orientation and would have to be entirely contained in a single connected component.

each homoclinic must traverse the entire disk, it must cut the cycle somewhere else besides at $v$. We arrive at a contradiction since a homoclinic cannot cross two vertices, nor can it cross an edge in the H-graph (see left of Figure 15). Therefore, there can be no cycles in the union graph with such a vertex. Note that this eliminates the possibility of a cycle having an odd number of edges or exactly two edges, which would necessarily have such a vertex.

Case II: For all vertices $v$ in the cycle, either both $e_1$ and $e_2$ are incoming to $v$ or both are outgoing from $v$ (otherwise, we would be in the Case I scenario). Note that this implies there are an even number of edges in the cycle, and the orientation of the edges is alternating (see right of Figure 15). Furthermore, any such cycle would have to be entirely contained in a single connected component. Otherwise, a homoclinic would cross its interior, and this is impossible by the same argument as in Case I. Note that the orientation of all homoclinics on the boundary of the connected component is the same (connected component to the left of all homoclinics, or the connected component is to the right of all homoclinics on the boundary). In fact, this connected component must be a cylinder zone with a closed H-chain on the boundary, since the cycle has alternating orientation of edges and therefore must contain edges contrary to the orientation of the H-chain on the boundary (see Remark 5). We will use two edges in the cycle which have ’opposite’ orientation to the orientation of the H-chain on the boundary to arrive at a contradiction. Note that there must exist two such edges in every cycle with an even number of edges greater than 2 or we would be in Case I (see Figure
In particular, any such cycle contains a segment of three edges in the H-graph, where the first and third are contrary to the orientation of the H-chain on the boundary and the second has the same orientation as the H-chain on the boundary (see Figure 16). We will focus on the two contrary edges in such a segment. Note that each of those two edges corresponds to two different new homoclinics that form, since no single path (new homoclinic) can contain two edges from the same connected component. Pick one edge that has opposite orientation to the H-chain, and cut the corresponding cylinder in rectifying coordinates so that the first homoclinic in the H-chain is the starting homoclinic for that edge in the H-graph (see Figure 17). Now in order for the new homoclinic to pass through the first and last edges in the H-chain without leaving the cylinder, the ordered partial sums from left to right of the analytic invariants for the H-chain need to all be greater than \( b \geq 0 \), the height of the new homoclinic relative to the first vertex in the cut cylinder (\( b \) would be determined by the sums of the other analytic invariants in earlier parts of the chain), and the final partial sum needs to be less than or equal to \( b \) (see the top of Figure 17). The same condition must apply to the other opposite oriented edge in the cycle segment. This will lead to a contradiction. Let \( s_{m+1} \) and \( s_1 \) be the corresponding start and end homoclinics for an opposite oriented edge in the H-graph (see Figure 16). Let \( a_{m+1}, \ldots, a_1 \) be the imaginary parts of the corresponding analytic invariants.
In particular, any such cycle contains a segment of three edges in the H-graph, where the first and third are contrary to the orientation of the H-chain on the boundary and the second has the same orientation as the H-chain on the boundary (see Figure 16). For each of the two edges that have opposite orientation to the H-chain, cut the corresponding cylinder in rectifying coordinates so that the first homoclinic in the H-chain is the starting homoclinic for that edge in the H-graph. Now in order for the new homoclinic to pass through the first and last edges in the H-chain without leaving the cylinder, the ordered partial sums from left to right of the analytic invariants for the H-chain need to satisfy a set of inequalities. These two sets of inequalities lead to a contradiction.

After perturbation. For the directed edge $s_{m+1} \rightarrow s_1$ to be realized, we need

\[
a_{m+1} > b \\
a_{m+1} + a_{m+2} > b \\
\vdots \\
a_{m+1} + a_{m+2} + \cdots + a_{m+n+1} > b \\
\vdots \\
a_{m+1} + a_{m+2} + \cdots + a_{m+n+1} + \cdots + a_{m+n+\ell+1} > b \\
\vdots \\
a_{m+1} + a_{m+2} + \cdots + a_{m+n+1} + \cdots + a_{m+n+\ell+1} + \cdots + a_N > b \\
a_{m+1} + a_{m+2} + \cdots + a_{m+n+1} + \cdots + a_{m+n+\ell+1} + \cdots + a_N + a_1 \leq b.
\]

Now consider the other opposite oriented edge in the cycle segment with starting and ending homoclinics $s_{m+n+\ell+1}$ and $s_{m+n+1}$ respectively (see Figure 16). Let $c \geq 0$ be the height of the new homoclinic relative to the first vertex in this cut cylinder. Than for directed edge $s_{m+n+\ell+1} \rightarrow s_{m+n+1}$ to be realized, we need

\[
a_{m+n+\ell+1} > c \\
\vdots \\
a_{m+n+\ell+1} + \cdots + a_N + a_1 > c \\
\vdots \\
a_{m+n+\ell+1} + \cdots + a_N + a_1 + \cdots + a_{m+n} > c \\
a_{m+n+\ell+1} + \cdots + a_N + a_1 + \cdots + a_{m+n} + a_{m+n+1} \leq c.
\]
Note that by the first set of inequalities that \( a_{m+1} + \cdots + a_{m+n+\ell} > b \), and by the second set of inequalities, \( a_{m+n+\ell+1} + \cdots + a_N + a_1 > c \). Then the last inequality in the first set becomes 
\[
(> b) + (> c) \leq b, \quad \text{a contradiction.}
\]

**Proof (of Theorem)**

Now assume throughout the rest of the proof that we are considering rank 1, multiplicity-preserving bifurcations.

In order for a rank 1 bifurcation to occur, we need \( n \) admissible (simultaneous) paths that altogether use \( n + 1 \) vertices. The union graph for rank 1 bifurcations must be connected, since any two (or more) disjoint paths would require at least 4 vertices, corresponding to rank 2 or more. The union graph is therefore a tree since it has no cycles by Proposition 6. The union graph for rank 1 bifurcations must actually be a path itself by the following.

Each homoclinic separatrix has two asymptotic directions: one outgoing to infinity (odd), and one incoming to infinity (even). So if \( n \) homoclinics break, and \( n - 1 \) are to form, we can lose only one odd and one even asymptotic direction. Each leaf of the union tree would correspond to a lost asymptotic direction, since that vertex can only be at the beginning or end of a path, but not both, so there can only be two leaves. Furthermore, all the vertices besides the two leaves in the union of paths must be both the end of a path and the beginning of a (different) path. Since each vertex has a well-defined direction through it, the union path has an orientation. Choose the leaf that is the tail of a directed edge – it is the beginning of a path. Now this path can only consist of one edge, since the second vertex must have a path end there and it cannot come from the other direction because of the orientation of the union path. Now unless the second vertex is also the last vertex of the union path, a second path must start at the second vertex, and end at the third vertex for the same reason. The same argument must hold for subsequent vertices, so all paths must be of length 1. This corresponds exactly to the type of bifurcation as described in the statement of the theorem.

\( \square \)

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**Conflict of interest**

The authors declare that they have no conflict of interest.

**References**

1. Álvarez, M., Gasull, A., Prohens, R.: Topological classification of polynomial complex differential equations with all the critical points of center type. Journal of Difference Equations and Applications 16(5), 411–423 (2010)
2. Andronov, A.A., Leontovich, E.A., Gordon, I.I., Maier, A.G.: Qualitative theory of second-order dynamic systems. Wiley, New York (1973). Original: Nakua, Moscow 1967
3. Benzinger, H.E.: Plane autonomous systems with rational vector fields. Transactions of the American Mathematical Society 326(2), 465–483 (1991)
4. Benzinger, H.E.: Julia sets and differential equations. Proceedings of the American Mathematical Society 117(4), 939–946 (1993)
5. Branner, B., Dias, K.: Classification of polynomial vector fields in one complex variable. Journal of Difference Equations and Applications 16(5), 463–517 (2010)
6. Brickman, L., Thomas, E.S.: Conformal equivalence of analytic flows. Journal of Differential Equations 25, 310–324 (1977)
7. Buff, X., Chéritat, A.: Ensembles de julia quadratiques de mesure de lebesgue strictement positive. C. R. Acad. Sci. Paris 341(11), 669–674 (2005)
8. Buff, X., Tan Lei: Dynamical convergence and polynomial vector fields. Journal of Differential Geometry 77(1), 1–41 (2007)
9. Dias, K.: Enumerating combinatorial classes of complex polynomial vector fields in C. Ergodic Theory and Dynamical Systems 33, 416–440 (2013)
10. Dias, K., Tan Lei: On parameter space of complex polynomial vector fields in C. Journal of Differential Equations 260, 628–652 (2016)
11. Douady, A., Estrada, F., Sentenac, P.: Champs de vecteurs polynomiaux sur C. Unpublished manuscript
12. Garino, A., Gasull, A., Jarque, X.: Normal forms for singularities of one dimensional holomorphic vector fields. Electronic Journal of Differential Equations 2004(122), 1–7 (2004)
13. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer, New York (1983)
14. Hájek, O.: Notes on meromorphic dynamical systems, I. Czech. Math. J. 16(1), 14–27 (1966)
15. Hájek, O.: Notes on meromorphic dynamical systems, II. Czech. Math. J. 16(1), 28–35 (1966)
16. Hájek, O.: Notes on meromorphic dynamical systems, III. Czech. Math. J. 16(1), 36–40 (1966)
17. Jenkins, J.: Univalent Functions. Springer-Verlag (1958)
18. Llibre, J., Schomer, D.: The geometry of quadratic differential systems with a weak focus of third order. Canadian Journal of Mathematics 56(2), 310–343 (2004)
19. Muñoz-Raymundo, J.: Complex structures adapted to smooth vector fields. Math. Ann. 322, 229–265 (2002)
20. Muñoz-Raymundo, J., Valero-Valdés, C.: Bifurcations of meromorphic vector fields on the Riemann sphere. Ergodic Theory and Dynamical Systems 15(6), 1211–1222 (1995)
21. Needham, D., King, A.: On meromorphic complex differential equations. Dynam. Stability Systems 9, 99–121 (1994)
22. Neumann, D.: Classification of continuous flows on 2-manifolds. Proceedings of the American Mathematical Society 48(1), 73–81 (1975)
23. Oudkerk, R.: The parabolic implosion for \( f_\delta(z) = z + z^{\delta+1} + o(z^{\delta+2}) \). Ph.D. thesis, University of Warwick (1999)
24. Rousseau, C., Teyssier, L.: Analytical moduli for unfoldings of saddle-node vector fields. Mosc. Math. J. 8(3), 547–614 (2008)
25. Shishikura, M.: Bifurcation of parabolic fixed points. In: Tan Lei (ed.) The Mandelbrot Set, Theme and Variations. Cambridge University Press, Cambridge (2000)
26. Sverdloove, R.: Vector fields defined by complex functions. Journal of Differential Equations 34, 427–439 (1979)