Coupling effects between elastic and electromagnetic fields from the perspective of conservation of energy

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Abstract Coupling effects among different physical fields reflect the conversion of energies from one field into another substantially. For simple physical processes, their governing or constitutive equations all satisfy the law of conservation of energy (LCE). Then, an analysis is extended to the coupling effects. First, for the linear direct and converse piezoelectric and piezomagnetic effects, their constitutive equations guarantee that the total energy is conserved during the process of energy conversion between the elastic and electromagnetic fields. However, the energies are converted via the work terms, $(\beta_{ijk}E_i)_{,k}v_j$ and $(\gamma_{ijk}H_i)_{,k}v_j$, rather than via the energy terms, $\beta_{ijk}E_i\varepsilon_{jk}$ and $\gamma_{ijk}H_i\varepsilon_{jk}$.

Second, for the generalized Villari effects, the electromagnetic energy can be treated as an extra contribution to the generalized elastic energy. Third, for electrostriction and magnetostriction, both effects are induced by the Maxwell stress. Moreover, their energies are purely electromagnetic and thus both have no converse effects. During these processes, the energies can be converted in three different ways, i.e., via the non-potential forces, via the cross-dependence of the energy terms, and directly via the electromagnetic interactions of ions and electrons. In the end, the general coupling processes which involve elastic, electromagnetic fields and diffusion are also analyzed. The advantages of using this energy formulation are that it facilitates discussion of the conversion of energies and provides better physical insights into the mechanisms of these coupling effects.

Key words piezoelectricity, piezomagnetism, Villari effect, electrostriction, magnetostriction

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1 Introduction

The study of coupling effects between electromagnetic and elastic fields dates back to more than one century[^1[^2]]. The first set of such coupling effects is piezoelectric and piezomagnetic effects. Piezoelectricity was first reported by the Curie brothers in 1881[^3]. Piezomagnetism, the magnetomechanical analogue of piezoelectricity, was experimentally observed in antiferromagnetic fluorides of cobalt and manganese in 1960[^4]. The second set of these effects is elec-
trostriction and magnetostriction. Magnetostriction was reported by Joule in iron in 1842\(^5\). There are two specific effects of magnetostriction called the Matteucci effect\(^6\) and the Wiedemann effect\(^7\). In addition, there is another effect called the Villari effect\(^8\). Among all these effects, it is worth noting that the strains induced for the first set are linearly dependent on the externally applied electromagnetic fields, while those for the second set are quadratic.

The underlying mechanisms of these effects are both important and interesting. Many models and theoretical approaches have been developed over the past century to study them both microscopically and macroscopically. The microscopic understanding mainly focuses on the relations between the crystal structures of materials and their properties, while the macroscopic understanding mainly uses phenomenological models to obtain the constitutive equations of these coupling effects. One major theoretical approach at the macroscopic level is the Lagrangian formulation. Using the Lagrangian formulation, the governing equations are derived to ensure that the first variation of the integral of Lagrangian over a time interval is zero. A detailed theory of piezoelectricity and its application in piezoelectric devices can be found in Ref.\(^1\). However, given the fact that the coupling effects between the electromagnetic and elastic fields essentially arise from the energy conversion between these two fields, it appears that one more important issue is still left unaddressed. That is, how is energy converted between these two fields and how can this process be formulated?

In this paper, the fundamental law of conservation of energy (LCE) is applied to analyze these coupling effects. For brevity, this approach is hereafter called the energy formulation. It is shown that this energy formulation enables the governing equations of these coupling effects to strictly satisfy the LCE. The purpose of this paper is two-fold. First, using this energy formulation, it is to contribute to a straightforward understanding of the issue of conversion of energies and furthermore to provide more physical insights into the mechanisms of these coupling effects. Second, it is to show that the energy formulation can be developed into a general approach to analyze the coupling effects among reversible processes. Organization of this paper is as follows. In Section 2, general discussion on the relations between the governing equations and the LCE for simple processes is provided. In Sections 3 and 4, the piezoelectric and piezomagnetic effects, as well as the Villari effect and its analogue in the electric field, are analyzed, respectively. In Section 5, electrostriction and magnetostriction are studied. In Section 6, general coupling processes involving elastic, electromagnetic fields and diffusion are analyzed. In Section 7, some comments on the energy conversion are provided and a summary is given in the end.

2 Governing equations and the LCE

In this section, discussion is provided to show that for simple processes, not only in classical mechanics but also in thermodynamics and electrodynamics, the governing or constitutive equations must satisfy the LCE.

To begin with, the equations of motion in classical mechanics are discussed. The Lagrangian for a mechanical system is usually defined to be \( L = T(q) - V(q) = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j - V(q_k) \), where \( \dot{q} \) is the generalized velocity and \( m_{ij} \) is the generalized mass, a symmetric tensor. The Lagrangian formulation is a standard approach to determine the equations of motion in a mechanical system\(^9\). Also, Lagrange’s equations or the equations of motion, \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \), lead to the LCE directly. That is, \( \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} \right) \dot{q}_k = \frac{d}{dt}(T + V) = 0 \). This in fact provides an alternative approach to determine the equations of motion for a mechanical system. When the total energy \( E = T + V \) of this system is determined, the time derivative of \( E \) can be written as a linear combination of the generalized velocities \( \dot{q}_k \) as shown above. Then, the coefficients associated with each \( \dot{q}_k \) must be zero, which gives the equations of motion, because the series \( \dot{q}_k \) are linearly independent.

A major disadvantage with the Lagrangian formulation is that it does not offer any physical
insights into the issue of conversion of energies. Fortunately, the energy formulation can achieve this goal handily. The time derivative of $E$ gives \( \frac{dE}{dt} = \frac{d(T+V)}{dt} = (m_i \dot{q}_i - (\frac{\partial}{\partial q_i}) \dot{q}_i) \), where the equations of motion are found by setting the coefficients in the outer bracket to be zero. Note that, the first term in the outer bracket, $m_i \dot{q}_i$, is the inertia force, and a positive product of it with the generalized velocity $\dot{q}_i$ indicates an increase in the kinetic energy $T$, which gives the rate of conversion of the potential energy $V$ into $T$. The second term, $-\frac{\partial}{\partial q_i}$, is the potential force, and a positive product of it with $\dot{q}_i$ indicates a positive work and a decrease in the potential energy $V$, which also gives the rate of conversion of the potential energy $V$ into $T$. Thus, these two terms must cancel each other, which leads to the LCE. It is evident that with this energy formulation, the discussion of conversion of energies becomes straightforward.

The energy formulation can also be extended to continuum mechanics to give Cauchy’s equation of motion. In a continuum medium, the total energy can be defined as $E = \int_V \left( \frac{1}{2} \rho v_i v_i + u^{el}(e_{ij}) \right) dv$, where $\rho$ is the mass density, $v_i$ is the velocity, and $u^{el}(e_{ij})$ is the elastic energy density function of the infinitesimal elastic strain $e_{ij}$. Then,

\[
\frac{dE}{dt} = \frac{d}{dt} \int_V \left( \frac{1}{2} \rho v_i v_i + u^{el}(e_{ij}) \right) dv = \int_V \left( \frac{\partial}{\partial t} \left( \frac{1}{2} \rho v_i v_i \right) + \frac{\partial u^{el}}{\partial e_{ij}} \frac{\partial e_{ij}}{\partial t} \right) dv
\]

where the elastic stress $\sigma_{ij} = \frac{\partial u^{el}}{\partial e_{ij}}$, and $\frac{\partial u^{el}}{\partial e_{ij}} = \frac{1}{2} (v_{i,j} + v_{j,i})$. Thus, to guarantee that the energy is conserved in a continuum medium with a stress-free boundary, the terms in the parenthesis of the volume integral at the last step must be zero, which leads to Cauchy’s equation of motion directly. Here, the discussion of conversion of energies is also straightforward. At the last step, the surface integral indicates that the work done by the surface traction is $t_i = \sigma_{ij} n_j$ and the surface density of work is $t_i v_i$. Within the volume integral, the positive products of both the inertia force $\rho \frac{\partial}{\partial t}$ and the potential force $\sigma_{ij} v_i$ with the velocity $v_i$ indicate the rate of conversion of the elastic potential energy $u^{el}$ into the kinetic energy, and thus they are canceled.

In the following discussion, the energy formulation is extended to study the governing equations in both thermodynamics and electrodynamics. The governing equations in thermodynamics are diffusion-type equations. Recently, the phase-field variational approach (PFVA) has become a popular tool to construct governing equations which abide by the corollary of the second law of thermodynamics, i.e., the total free energy is non-increasing during the evolution of physical fields in an isolated system\textsuperscript{[10–11]}. Suppose that in a thermodynamic system, the chemical free energy density is $f(c)$ and the chemical potential is $\mu = \frac{df}{dc}$. Then,

\[
\frac{d}{dt} \int_V f(c) dv = \int_V \frac{\partial f}{\partial c} \frac{\partial c}{\partial t} dv = \int_V \mu \frac{\partial J_i}{\partial t} dv = - \int_V \mu \frac{\partial J_i}{\partial t} n_i dv + \int_V \mu J_i n_i dv,
\]

where the law of conservation of mass, $\frac{\partial c}{\partial t} = - \frac{\partial}{\partial t} \frac{\partial f}{\partial c}$, is applied in the second step. Then, to guarantee that in an isolated system, the total free energy is non-increasing as time evolves, the compositional flux $J_i$ needs be defined as $J_i = -m \frac{\partial}{\partial t} \frac{\partial f}{\partial c}$, where $m$ is the mobility of diffusion. Considering that during the thermodynamic evolution, the chemical free energy is usually dissipated as heat, a slight modification (by introducing a dissipation function in the above derivation) guarantees that the first law of thermodynamics is also satisfied. Let the rate of change of the free energy dissipation function be $\frac{1}{m} \frac{\partial}{\partial c} f(c)$, which is analogous to that considered in classical mechanics\textsuperscript{[9]}. Then, the variation of the total energy is

\[
\frac{dE}{dt} = \int_V \frac{1}{m} J_i J_i^* dv + \frac{d}{dt} \int_V f(c) dv = - \int_V \mu J_i n_i dv + \int_V \left( \frac{1}{m} J_i^* + \partial_t \mu \right) J_i dv.
\]
Thus, in an isolated system, to guarantee the satisfaction of the LCE, the constitutive flux equation is found to be $J_i^e = -m \partial_t \mu$.

In electrodynamics, the governing equations are a set of Maxwell equations,

$$\partial_t D_i = \rho, \quad (1)$$
$$\xi_{ijk} \partial_j E_k = -\partial_t B_i, \quad (2)$$
$$\partial_t B_i = 0, \quad (3)$$
$$\xi_{ijk} \partial_j H_k = J_i^e + \partial_t D_i, \quad (4)$$

where $D_i$ and $B_i$ are the electric displacement and magnetic induction, respectively, $E_i$ and $H_i$ are the electric and magnetic fields, respectively, $\xi_{ijk}$ is the permutation symbol, and $J_i^e$ is the free electric current density. Among them, Eqs. (1) and (3) are time-independent, i.e., static equations, while Eqs. (2) and (4) are time-dependent, i.e., dynamic equations. Here, $D_i = \epsilon_{ij} E_j$, and $B_i = \mu_{ij} H_j$, where $\epsilon_{ij}$ and $\mu_{ij}$ are the permittivity and permeability, respectively. In electrodynamics, the LCE is called Poynting’s theorem. Beginning with the work done on the free electric current by the electric field $E_i$,

$$J_i^e E_i = (\xi_{ijk} \partial_j H_k - \partial_t D_i) E_i + (-\xi_{ijk} \partial_j E_k - \partial_t B_i) H_i$$
$$= -\partial_t (\xi_{ijk} E_j H_k) - \partial_t (\frac{1}{2} \epsilon_{ij} E_i E_j + \frac{1}{2} \mu_{ij} H_i H_j) = -\partial_t S_i - \partial_t u^e, \quad (5)$$
$$-\partial_t S_i = \partial_t u^e + J_i^e E_i, \quad (6)$$

where $S_i = \xi_{ijk} E_j H_k$ is the Poynting vector, and $u^e = \frac{1}{2} \epsilon_{ij} E_i E_j + \frac{1}{2} \mu_{ij} H_i H_j$ is the electromagnetic energy. At the first step of Eq. (5), the first term on the right-hand side indicates the work done by the electric field, and the second term, analogously, can be considered as the work done by the magnetic field. Since in the expression of the Lorentz force, the magnetic force in fact does no work, this term is zero. According to Eq. (6), the physical content of Poynting’s theorem states that the energy transferred into the boundary of a continuum medium by the Poynting vector (usually from the electric power source) equals the sum of the increase in the electromagnetic energy and the work done on the free electric current by the electric field (which is usually dissipated into heat or converted into other energies via electronic devices). Shown by the above derivation, it is evident that the two dynamic equations of the Maxwell equations must satisfy the LCE.

The Lagrangian formulation has already been extended to electrodynamics, but unfortunately it does not help to clarify the issue of conversion of energies here. Furthermore, the form of the Lagrangian is also puzzling since the electrical and magnetic potentials are assigned with opposite signs. Compared with the LCE in mechanics and thermodynamics, Poynting’s theorem is of a very different form. However, if starting with the time derivative of $u^e$ and the divergence of the Poynting vector as shown in the second step of Eq. (5), the two dynamic equations can still be found by identifying the electric and magnetic work as shown in the first step of Eq. (5). That is, in electrodynamics, the energy formulation can also be used to determine the two dynamic equations of the Maxwell equations via Poynting’s theorem.

In the above discussion, it is argued that the governing equations of uncoupled and simple processes in mechanics, thermodynamics, and electrodynamics can be determined using the energy formulations. However, most natural processes are coupled and complex, resulting in conversion of energies from one field into another. In the following sections, discussion is provided for processes coupled between elastic and electromagnetic fields. The governing equations found with the energy formulations are shown to strictly satisfy the LCE, and furthermore the discussion of conversion of energies becomes straightforward.
3 Linear piezoelectric and piezomagnetic effects

For the direct piezoelectric and piezomagnetic effects which behave linearly, their constitutive equations are

\[ D_i = \varepsilon_{ij} E_j + \beta_{ijk} e_{jk}, \]  
\[ B_i = \mu_{ij} H_j + \gamma_{ijk} e_{jk}, \]

where \( \beta_{ijk} \) and \( \gamma_{ijk} \) are the piezoelectric and piezomagnetic coefficients, respectively. For the converse effect, the constitutive equation is

\[ \sigma_{jk}^{ge} = C_{jklm} e_{lm} + \beta_{ijk} E_i + \gamma_{ijk} H_i = \sigma_{jk}^{el} + \beta_{ijk} E_i + \gamma_{ijk} H_i, \]  

where \( \sigma_{jk}^{ge} \) and \( \sigma_{jk}^{el} \) are the generalized and elastic stress tensors, respectively, and \( C_{jklm} \) is the stiffness tensor. Sometimes \( \sigma_{jk}^{el} \) and \( \sigma_{jk}^{el} \) are chosen as independent variables. Then, using \( e_{jk} = S_{jklm} \sigma_{lm}^{el} \) and introducing the generalized strain tensor \( \epsilon_{st} = S_{stjk} \sigma_{jk}^{el} \) (\( S_{stjk} \) is the compliance tensor), the above constitutive equations can be converted to be dependent on \( \sigma_{jk}^{el} \). These two sets of equations are in fact equivalent and can be reduced to the constitutive equations for simple processes while ignoring the coupling effects. However, using \( e_{jk} \) as independent variables helps to yield a straightforward physical interpretation for each term in the derivations below.

Following the steps in Eq. (5) and substitution of Eqs. (7) and (8) leads to

\[ J^s_i E_i = -\partial_t (\xi_{ij} E_j H_k) - \partial_t \left( \frac{1}{2} (\varepsilon_{ij} E_j E_j + \mu_{ij} H_j H_j) \right) - E_i \beta_{ijk} \partial_t e_{jk} - H_i \gamma_{ijk} \partial_t e_{jk} \]
\[ \Rightarrow -\partial_t S_i = \partial_t u^{st} + J^s_i E_i + E_i \beta_{ijk} \partial_t e_{jk} + H_i \gamma_{ijk} \partial_t e_{jk}. \]  

(10)

On the right-hand side of Eq. (10), it is straightforward to see that the last two terms represent the conversion of energies due to the coupling effects. During the converse effects, the energy transferred by the Poynting vector from the power source is converted into the elastic energy \( u^{el} \). The rate of the increase in \( u^{el} \) is \( E_i \beta_{ijk} \partial_t e_{jk} + H_i \gamma_{ijk} \partial_t e_{jk} \) according to Eq. (10). Thus, in this case, the conservation of the total mechanical energy is

\[ \frac{d}{dt} \int_V \frac{1}{2} \rho v_j v_j dv + \frac{d}{dt} \int_V u^{el} dv + \int_V (E_i \beta_{ijk} \partial_t e_{jk} + H_i \gamma_{ijk} \partial_t e_{jk}) dv = 0. \]  

(11)

Then,

\[ 0 = \int_V \left( \partial_t \left( \frac{1}{2} \rho v_j v_j \right) + \partial_t u^{el} \partial_t e_{jk} + E_i \beta_{ijk} \partial_t e_{jk} + H_i \gamma_{ijk} \partial_t e_{jk} \right) dv \]
\[ = \int_V (\rho \partial_t v_j - (\sigma_{jk}^{el} + \beta_{ijk} E_i + \gamma_{ijk} H_i)_j) v_j dv + \oint_A (\sigma_{jk}^{el} + \beta_{ijk} E_i + \gamma_{ijk} H_i)_j v_j n_k da. \]  

(12)

Thus, the equation of motion and the generalized stress are found to be

\[ \rho \partial_t v_j - (\sigma_{jk}^{el} + \beta_{ijk} E_i + \gamma_{ijk} H_i)_j = 0, \]
\[ \sigma_{jk}^{ge} = \sigma_{jk}^{el} + \beta_{ijk} E_i + \gamma_{ijk} H_i, \]

which is exactly the constitutive equation (9) for the converse effects. Thus, these constitutive equations guarantee satisfaction of the LCE. The terms indicating the conversion of energies are

\[ \int_V ((\beta_{ijk} E_i + \gamma_{ijk} H_i) \partial_t e_{jk}) dv \]
\[ = \int_V (-(\beta_{ijk} E_i + \gamma_{ijk} H_i)_j) v_j dv + \oint_A (\beta_{ijk} E_i + \gamma_{ijk} H_i) v_j n_k da. \]  

(15)
Then, the above equation can be directly used to obtain the rate of conversion of energies for certain electronic devices with specific shapes.

Note that in Eq. (10), due to the lack of terms, $\beta_{ijk}E_{i}e_{jk}$, the variation of energies owing to the piezoelectric and piezomagnetic effects cannot be written as the time derivatives of energy terms such as $\beta_{ijk}E_{i}e_{jk}$ and $\gamma_{ijk}H_{i}e_{jk}$. In contrast, the elastic and electromagnetic energies are explicitly shown in the following two terms, $\partial_{t}U_{el}$ and $\partial_{t}U_{em}$, in the above derivations. Moreover, when an energy term is a function of vectors or tensors, it is usually a positive-definite quadratic function of its independent variables. The elastic and electromagnetic energies are such examples. Hence, whenever the strains switch signs or the electromagnetic fields switch directions, these energies remain positive. However, the terms such as $\beta_{ijk}E_{i}e_{jk}$ and $\gamma_{ijk}H_{i}e_{jk}$ change signs when the strains switch signs or the electromagnetic fields switch directions. These terms are odd functions of $E_{i}$, $H_{i}$, and $e_{jk}$. They introduce a dependence of these energy terms on the directions of the physical fields, which is not reasonable. In addition, the elastic energy comes from the work done by the externally applied forces, and the electromagnetic energy comes from the energy transferred by the Poynting vector from the power source. In the traditional approach, there is no clue about the origin of these piezoelectric and piezomagnetic “energy” terms. More importantly, the phenomenological assumption of these “energy” terms does not help to clarify the following critical problem. That is, how is energy converted between the elastic and electromagnetic fields? Finally, in the traditional approach, the total free energy for the piezoelectric or piezomagnetic materials usually consists of a positive elastic energy term, and a negative electromagnetic energy term with a negative piezoelectric or piezomagnetic “energy” term. Assigning these terms with opposite signs is not justified, since the total free energy should be a summation of all three terms.

Thus, it is argued here that for piezoelectric and piezomagnetic effects, there are no specific energies associated with these two processes. Note that, in classical and fluid mechanics, not all processes are associated with specific energies, e.g., the processes due to friction and viscosity which dissipate the kinetic energy of mechanical movements into heat. These forces, usually called non-potential forces (i.e., they cannot be derived from potential functions), are associated with work which convert energies from one field into another. As a result, the variations of energies owing to the piezoelectric and piezomagnetic effects are in fact work terms, $(\beta_{ijk}E_{i} + \gamma_{ijk}H_{i})e_{jk}$, as shown in Eq. (15), with their non-potential body forces being $(\beta_{ijk}E_{i})_{k}$ and $(\gamma_{ijk}H_{i})_{k}$. The piezoelectric and piezomagnetic effects exemplify the conversion of energies owing to the work done by non-potential forces.

Then, how to understand this conversion of energies via non-potential forces intuitively? Careful examinations of the physical contents of these work terms are still needed. First, Eq. (10) can be rewritten as

$$-\partial_{t}S_{i} = E_{i}J_{i}^{e} + E_{i}\partial_{t}(e_{ij}E_{j} + \beta_{ijk}e_{jk}) + H_{i}\partial_{t}((\mu_{ij}H_{j} + \gamma_{ijk}e_{jk}) = E_{i}J_{i}^{e} + E_{i}\partial_{t}D_{i} + H_{i}\partial_{t}B_{i}.$$  

Evidently, the last step shows that the energy transferred from the power source by the Poynting vector is used to do work by both the electric and magnetic fields. For the electric field, the first term, $E_{i}J_{i}^{e}$, is the work done on the free electric currents. This portion of work is usually converted into Joule heat, used to achieve electromigration and so on. The second term, $E_{i}\partial_{t}D_{i}$, is the work done on the displacement currents. Here, $\partial_{t}D_{i} = J_{i}^{P}$ is the displacement current density. The third term, $H_{i}\partial_{t}B_{i}$, is the work done by the magnetic field. In the absence of piezoelectric and piezomagnetic effects, the second and third work terms are converted into or stored as the electromagnetic energy $u_{em}$. However, when the above two effects are present, extra work is done by both the electric and magnetic fields as shown by $E_{i}\partial_{t}(\beta_{ijk}e_{jk}) + H_{i}\partial_{t}(\gamma_{ijk}e_{jk})$. These terms are the integrand on the left-hand side of Eq. (15), and they are both work terms. Using the converse piezoelectric effect as an example, an externally applied electric field leads to polarization in the piezoelectric materials, and polarization results in displacement currents. Then, work is done on these displacement currents by the electric field. $E_{i}\partial_{t}(\beta_{ijk}e_{jk})$ is actually a
port of this work, i.e., \( E_i J^P_i \), done on the displacement currents by the electric field. During polarization of the piezoelectric materials, besides the electronic polarization, ionic polarization also arises from the deviation of these ions away from their equilibrium positions. This usually causes distortion of the crystal lattice as shown in Fig. 1. Thus, strains and stresses arise which lead to an increase in the elastic energy within the materials. As a result, a port of the work done on the displacement currents by the electric field is converted into the elastic energy of the materials. This is the underlying mechanism of the piezoelectric effects and they arise from the above-mentioned work rather than any specific energy. Both the work done by the materials. This is the underlying mechanism of the piezoelectric effects and they arise from the above-mentioned work rather than any specific energy. Both the work done by \( E_i \) on the free currents \( J^p_i \) and on the portion of displacement currents \( \partial_t (\beta_{jik} e_{ijk}) \) cannot be related to or derived from any specific energies. They are just work done by non-potential forces and help to achieve conversion of energies between different physical fields.

![Fig. 1 A schematic diagram of the work done on the displacement currents \( J^P_i \) by \( E_i \) resulting in a distortion of the crystal lattice during the converse piezoelectric effect](image)

### 4 Generalized Villari effects

In this section, the Villari effect and its analogue in the electric field are analyzed. It is reasonable to assume that there is an analogue in the electric field. That is, the permittivity tensors (similar to the permeability tensors in the Villari effect) also vary under the influence of externally applied stresses. These effects are hereafter called the generalized Villari effects. To facilitate the discussion here, the permittivity and permeability tensors are both treated as functions of strains, i.e., \( \epsilon_{ij}(e_{kl}) \) and \( \mu_{ij}(e_{kl}) \). According to Poynting’s theorem,

\[
J^P_i E_i = -\partial_t S_i - \partial_t \left( \frac{1}{2} \epsilon_{ij}(e_{kl}) E_i E_j + \mu_{ij}(e_{kl}) H_i H_j \right)
\]

\[
= -\partial_t S_i - \epsilon_{ij} E_i \partial_t E_j - \mu_{ij} H_i \partial_t H_j - \frac{1}{2} \left( \frac{\partial \epsilon_{ij}}{\partial e_{kl}} E_i E_j + \frac{\partial \mu_{ij}}{\partial e_{kl}} H_i H_j \right) \partial_t e_{kl}.
\]

Evidently, the last term indicates the conversion of energies between the elastic and electromagnetic fields. Analogous to the analysis in the above section,

\[
0 = \frac{d}{dt} \int_V \frac{1}{2} \rho \sigma_{ij} v_j dv + \frac{d}{dt} \int_V u^{el} dv + \int_V \left( \frac{1}{2} \frac{\partial \epsilon_{ij}}{\partial e_{kl}} E_i E_j + \frac{\partial \mu_{ij}}{\partial e_{kl}} H_i H_j \right) \partial_t e_{kl} dv
\]

\[
= \int_V \left( \rho \partial_t v_j + \frac{\partial u^{el}}{\partial e_{jk}} \partial_t e_{jk} + \frac{\partial u^{em}}{\partial e_{kl}} \partial_t e_{kl} \right) dv
\]

\[
= \int_V \left( \rho \partial_t v_j + \left( \frac{\partial u^{el}}{\partial e_{jk}} + \frac{\partial u^{em}}{\partial e_{jk}} \right) \partial_t e_{jk} \right) dv
\]

\[
= \int_V \left( \rho \partial_t v_j - \sigma_{jk}^{el} + \frac{\partial u^{em}}{\partial e_{jk}} \right) v_j dv + \int_A \left( \sigma_{jk}^{el} + \frac{\partial u^{em}}{\partial e_{jk}} \right) v_j n_k da.
\]

The generalized stress is found to be

\[
\sigma_{jk}^{ge} = \sigma_{jk}^{el} + \frac{\partial u^{em}}{\partial e_{jk}} = \frac{\partial (u^{el} + u^{em})}{\partial e_{jk}},
\]
which guarantees the satisfaction of the LCE. In addition, the electromagnetic energy can be treated as an extra contribution to the generalized elastic energy. In Ref. [13], there were similar approaches of using the elastic energy as an extra contribution to the generalized chemical energy while studying diffusion under the influence of elastic fields. The generalized Villari effects exemplify the conversion of energies owing to a cross-dependence in the energy terms and arise from the dependence of the permittivity and permeability on the elastic stresses or strains. Since the Matteucci effect refers to the arising of a helical anisotropy of the susceptibility of materials under the influence of a torque, it should be classified as a specific example of the generalized Villari effects.

5 Electrostriction and magnetostriction

Electrostriction and magnetostriction are induced by polarization and magnetization of materials, respectively. Both are quadratic effects with their strains $e_{ij}^e$ and $e_{ij}^m$ being

$$e_{ij}^e = Q_{ijkl}^e P_k P_l,$$

(19)

$$e_{ij}^m = Q_{ijkl}^m M_k M_l,$$

(20)

where $Q_{ijkl}^e$ and $Q_{ijkl}^m$ are the electrostriction and magnetostriction coefficients, respectively, and $P_k$ and $M_k$ are the polarization and magnetization vectors, respectively.

Among the mechanisms of polarization in dielectric materials, it is believed that both the ionic and electronic polarizations contribute mostly to electrostriction[2]. Under the influence of an externally applied electric field, electrostriction mainly arises from the displacements of ions away from their equilibrium positions at the lattice sites and the distortion of the electronic distribution around these ions, while for magnetostriction, boundary movements of magnetic domains are believed to be the major cause[14]. Under the influence of an externally applied magnetic field $H_i$, the domains with magnetic moments aligned with $H_i$ will grow. The magnetic moments around the domain boundary will rotate to the direction of $H_i$ and thus help the domain boundaries move forward. Such a process simultaneously results in a dimensional change for the materials. During these processes, there is an increase in energy which apparently comes from the electric power source. According to Eq. (6), this energy is transferred into the bulk of materials via the Poynting vector. The transfer of energy is always accompanied by the transfer of momentum simultaneously. Momentum is also needed during the processes of polarization and magnetization, e.g., when ions are pushed out of their equilibrium positions during ionic polarization and when the magnetic moments start to rotate to help the domain grow. Furthermore, at the macroscopic level, materials under the influence of alternative fields usually vibrate, which indicates a gain of mechanical momentum. Then, where does this momentum (which is substantially the cause of electrostriction and magnetostriction) gained at both the microscopic and macroscopic levels come from? Evidently, it would be beneficial if the law of conservation of momentum in electrodynamics[12] is also examined.

Here, electrostriction is mainly analyzed while magnetostriction is merely treated as an analogue. As we know, charged particles in electromagnetic fields are subject to the Lorentz force. The Lorentz force not only distorts the electronic distribution around ions, but also drives the ions away from their equilibrium positions during ionic polarization. Thus, it results in a transfer of momentum to the ions. According to Newton’s second law, the mechanical momentum $P_M^i$ obtained by all charged particles are

$$\frac{dP_M^i}{dt} = \int_V (\rho E_i + \xi_{ijk} J_k B_k) dv,$$

(21)

where the integrand on the right-hand side is the Lorentz force. In the following discussion, for simplicity, the permittivity $\epsilon$ and permeability $\mu$ are assumed constant. Thus, $D_i = \epsilon E_i$ and
\( B_i = \mu H_i \). Substituting Eqs. (1) and (4) into Eq. (21) yields
\[
\frac{dP_i^M}{dt} = \int_V \left( \delta_j (\epsilon E_j) E_i + \epsilon_{ijk} \xi_{kim} \partial_i H_m - \partial_i (\epsilon E_j) (\mu H_k) \right) dv \\
= \int_V (\epsilon E_i \delta_j E_j - \epsilon \epsilon_{ijk} \xi_{kim} E_j \partial_i E_m + \mu H_i \partial_j H_j \\
- \mu \epsilon_{ijk} \xi_{kim} H_j \partial_i H_m - \partial_i (\epsilon \mu \epsilon_{ijk} E_j H_k)) dv.
\]
Within the integrand of the last step, the field momentum \( P_i^F = \epsilon \mu \epsilon_{ijk} E_j H_k \) is shown in the fifth term, and the other four terms equal the divergence of the Maxwell stress \( T_{ij} \).

\[
T_{ij} = \epsilon \left( E_i E_j - \frac{1}{2} E_n E_n \delta_{ij} \right) + \mu \left( H_i H_j - \frac{1}{2} H_n H_n \delta_{ij} \right) .
\]

Using the divergence theorem, the law of conservation of momentum\(^{[12]}\) is obtained as
\[
\frac{dP_i^M}{dt} + \frac{dP_i^F}{dt} = \oint_A T_{ij} n_j da .
\]

Evidently, as shown by Eq. (23), both the mechanical momentum \( P_i^M \) and the field momentum \( P_i^F \) come from the momentum transferred by the Maxwell Stress. \( P_i^F \) is the momentum associated with the electromagnetic fields, while \( P_i^M \) is the momentum obtained by ions and electrons during polarization and magnetization. As argued above, it is this mechanical momentum \( P_i^M \) that causes dimensional changes, i.e., electrostriction and magnetostriiction. Since \( P_i^M \) is transferred by the Maxwell stress \( T_{ij} \), it is reasonable to argue that both electrostriction and magnetostriiction are induced by \( T_{ij} \). To be more specific, they are induced by a portion of \( T_{ij} \). First, consider a control volume and assume it is vacuum. The mechanical momentum within this volume is zero since there are no particles within. Let \( T_{ij}^0 \) be the Maxwell stress in the vacuum, whose expression is still Eq. (22) but with \( \epsilon \) and \( \mu \) replaced by \( \epsilon_0 \) and \( \mu_0 \) in the vacuum. Then, the Maxwell stress \( T_{ij}^0 \) acting on the surface of the control volume is totally responsible for the transfer of the field momentum \( \epsilon_0 \mu_0 \epsilon_{ijk} E_j H_k \) into the volume. Suppose that the control volume is occupied by the materials. The variation in \( P_i^F \) is \( (\epsilon_i - \epsilon_0 \mu_0) \epsilon_{ijk} E_j H_k \) which can be assumed as a minor change for typical dielectric and paramagnetic substances. However, the mechanical momentum changes from 0 to \( P_i^M \). The difference between \( T_{ij} \) and \( T_{ij}^0 \) can be argued to be roughly the portion of stress which is responsible for the transfer of \( P_i^M \). Since \( \epsilon = \epsilon_0 (1 + \chi_e) \), \( \mu = \mu_0 (1 + \chi_m) \), \( P_i = \epsilon_0 \chi_e E_i \), and \( M_i = \mu_0 \chi_m H_i \) (here \( \chi_e \) and \( \chi_m \) are the electric and magnetic susceptibilities), this portion of stress is
\[
T_{ij}^m = T_{ij} - T_{ij}^0 = \epsilon_0 \chi_e \left( E_i E_j - \frac{1}{2} E_n E_n \delta_{ij} \right) + \mu_0 \chi_m \left( H_i H_j - \frac{1}{2} H_n H_n \delta_{ij} \right) \\
= \frac{1}{\epsilon_0 \chi_e} \left( P_i P_j - \frac{1}{2} P_n P_n \delta_{ij} \right) + \frac{1}{\mu_0 \chi_m} \left( M_i M_j - \frac{1}{2} M_n M_n \delta_{ij} \right) .
\]

Since \( T_{ij}^m \) induces both electrostriction and magnetostriiction, the corresponding strain is
\[
e_{ij} = S_{ijkl} T_{kl}^m = \frac{S_{ijkl}}{\epsilon_0 \chi_e} \left( P_k P_l - \frac{1}{2} P_n P_n \delta_{kl} \right) + \frac{S_{ijkl}}{\mu_0 \chi_m} \left( M_k M_l - \frac{1}{2} M_n M_n \delta_{kl} \right) .
\]

Here, the strains are assumed to be infinitesimal strains. Compared with Eqs. (19) and (20), the electrostriction and magnetostriiction coefficients are found to be \( Q^{e}_{ijkl} = \frac{S_{ijkl}}{\epsilon_0 \chi_e} \) and \( Q^{m}_{ijkl} = \frac{S_{ijkl}}{\mu_0 \chi_m} \). This qualitatively explains the resemblance of \( Q^{e}_{ijkl} \) to \( S_{ijkl} \) in a cubic crystal system\(^{[2]}\). The Wiedemann effect found in 1858, i.e., the torsion of a ferromagnetic rod in a helical magnetic field, is a specific example of magnetostriiction. Previous analysis found that the angle
Permutation of indices 1
updated Poynting’s theorem becomes
strain, arising from torsion, is
that the rod is elastically isotropic. Using Eq. (25), it is straightforward to show that the shear
can be assumed to
by Eq. (25).
the rod. Thus, this result agrees well with the experimental finding and the conclusion from
the previous analysis. Furthermore, the rod is in fact subject to both electrostriction and magnetostriction. The induced longitudinal and circumferential strains can be obtained handily
by Eq. (25).
From the analysis above, the energy associated with electrostriction and magnetostriction can be assumed to be \( \frac{1}{2} T_{ij}^{m} \epsilon_{ij} = \frac{1}{2} S_{ijkl} T_{ij}^{m} T_{kl}^{m} \). Here, \( T_{ij}^{m} \) is of the order of the magnitude of the electromagnetic energy \( u^{em} \) stored within the substance. Since the strains resulting from electrostriction and magnetostriction are usually around 0.1%, the kinetic energy obtained by mechanical vibrations is only roughly one thousandth of \( u^{em} \). Furthermore, according to Poynting’s theorem, this energy (similar to the electromagnetic energy \( u^{em} \)) also comes from the electric power source and is transferred into and stored within the bulk of materials via the Poynting vector. Thus, it should play the same role as \( u^{em} \) in Poynting’s theorem. Then, the updated Poynting’s theorem becomes

\[
J_{i}^{e} E_{i} = -\partial_{t}(\xi_{ijk} E_{j} H_{k}) - \partial_{t}\left( \frac{1}{2}(\epsilon_{ij} E_{i} E_{j} + \mu_{ij} H_{i} H_{j}) + \frac{1}{2} T_{ij}^{m} S_{ijkl} T_{kl}^{m} \right),
\]

For simplicity, the second step of Eq. (24) is used for the substitution of \( T_{ij}^{m} \), and the expansion of \( T_{ij}^{m} \) and \( S_{ijkl} \) leads to the second and fourth equations of the Maxwell equations,

\[
\begin{align*}
\xi_{1jk} \partial_{j} E_{k} &= -\partial_{t} B_{1} - \frac{T_{kl}^{m} \mu_{0} \chi_{m} \mu}{\epsilon} ((S_{11kl} - S_{22kl} - S_{33kl}) \partial_{t} B_{1} + 2 S_{12kl} \partial_{l} B_{2} + 2 S_{13kl} \partial_{l} B_{3}), \\
J_{1}^{e} &= \xi_{1jk} \partial_{j} H_{k} + \partial_{t} D_{1} - \frac{T_{kl}^{m} \mu_{0} \chi_{m} \mu}{\epsilon} ((S_{11kl} - S_{22kl} - S_{33kl}) \partial_{t} D_{1} + 2 S_{12kl} \partial_{l} D_{2} + 2 S_{13kl} \partial_{l} D_{3}).
\end{align*}
\]

Permutation of indices 1 \( \rightarrow \) 2 \( \rightarrow \) 3 leads to the rest two sets of equations. It is evident that these equations are highly nonlinear. Note that, in the derivations above, the strains induced by electrostriction and magnetostriction are assumed to be infinitesimal strains. Thus, Eqs. (27) and (28) are only suitable for metallic or ceramic strictive materials.

Furthermore, note that the energy, \( \frac{1}{2} S_{ijkl} T_{ij}^{m} T_{kl}^{m} \), associated with electrostriction and magnetostriction, in fact only depends on the electromagnetic fields. Thus, it is a pure electromagnetic energy. However, it is this pure electromagnetic energy that results in strains which are pure mechanical responses. In this case, energy associated with electromagnetic fields is directly converted into elastic energy, which is very different from the previously mentioned two cases (conversion of energies via the non-potential forces or the cross-dependence of energy terms). When the electromagnetic fields are applied, the charged particles within the substance are subject to the Lorentz force immediately, which leads to a distribution of the Maxwell stress across the lattice of the substance and induces strains there. As a result, the momentum is directly transferred by the Maxwell stress to the substance lattice. Simultaneously, the energy transferred from the power source by the Poynting vector is directly converted into the elastic energy stored in the lattice. However, in this case, the energy associated with electrostriction and magnetostriction has no dependence on the elastic fields. Furthermore, since there is no inverse mechanisms, by which pure mechanic movements can result in a distribution of the Maxwell stress and induce the Lorentz force in the absence of electromagnetic fields, there are no converse effects for electrostriction and magnetostriction discussed above.
In Ref. [16], it is assumed that the Maxwell stresses and strains can be linearly superposed with the elastic stresses and strains. The tensor product between the Maxwell stresses and the elastic strains, which include compositional strains as well, is the energy giving rise to electromigration. Here, it is argued that the tensor product (or a portion of this product) between the Maxwell stresses and strains is the energy giving rise to electrostriction and magnetostriction. Using Poynting’s theorem, the processes of energy conversions among the above analyzed coupling effects are summarized schematically in Fig. 2 for the sake of clarity.

**Fig. 2** Conversion of energies for the coupling effects between the electromagnetic and elastic fields according to Poynting’s theorem

6 **Formulation of general coupling processes in electronic devices**

In electronic devices which use functionally gradient materials, the electromagnetic, elastic, and compositional fields evolve simultaneously. Thus, it is beneficial to analyze such general coupling processes. In our previous work, the diffusion process under the influence of elastic fields was analyzed[^17]. This analysis is extended here to study the above coupling effects. Assume an electromagnetic energy with $\varepsilon_{ij}$ and $\mu_{ij}$ both dependent on the composition and strains, i.e., $u_{\text{em}} = \frac{1}{2}(\varepsilon_{ij}(c,e_{lm})E_iE_j + \mu_{ij}(c,e_{lm})H_iH_j)$. Consider a binary system with $c$ denoting the composition of one species. Then, the coupling processes cause the variation of the following energies: the kinetic energy $k = \frac{1}{2}\rho v_l v_l$, the elastic energy $u_{\text{el}}(c,e_{lm})$, the chemical free energy $f(c,e_{lm})$ of diffusion, and the electromagnetic energy $u_{\text{em}}$. In addition, a dissipation function $\frac{1}{m} J_i^c J_i^c$ is needed for the diffusion process. Here, the LCE is applied for each process. For the mechanical movements, we have

$$\frac{d}{dt} \int_V (k + u^3 + f + u_{\text{em}}) dv + \int_V (E_i\beta_{ilm}\partial_t e_{ilm} + H_i\gamma_{ilm}\partial_t e_{ilm}) dv$$

$$= \int_V \left( \rho v_l \partial_t v_l + \left( \frac{\partial u_{\text{el}}}{\partial e_{ilm}} + \frac{\partial f}{\partial e_{ilm}} + \frac{\partial u_{\text{em}}}{\partial e_{ilm}} \right) \partial_t e_{ilm} \right) dv + \int_V (E_i\beta_{ilm}\partial_t e_{ilm} + H_i\gamma_{ilm}\partial_t e_{ilm}) dv$$

$$= \int_A \left( \frac{\partial u_{\text{el}}}{\partial e_{ilm}} + \frac{\partial f}{\partial e_{ilm}} + \frac{\partial u_{\text{em}}}{\partial e_{ilm}} + E_i\beta_{ilm} + H_i\gamma_{ilm} \right) v_l n_m da$$

$$+ \int_V \left( \rho \partial_t v_l + \left( \frac{\partial u_{\text{el}}}{\partial e_{ilm}} + \frac{\partial f}{\partial e_{ilm}} + \frac{\partial u_{\text{em}}}{\partial e_{ilm}} + E_i\beta_{ilm} + H_i\gamma_{ilm} \right) v_l \right) dv = 0. \quad (29)$$
As a result, the equations of motion and the generalized stress tensor are found to be
\[
\rho \partial_t v_i = \sigma_{im}^{ge,m} + \frac{\partial u^{el}_{im}}{\partial c_{im}} + \frac{\partial f}{\partial c_{im}} + \frac{\partial u^{em}_{im}}{\partial c_{im}} + \beta_{im} E_i + \gamma_{im} H_i. \tag{30}
\]

Natural boundary conditions can be obtained using the surface integral on the right-hand side of Eq. (29) at the last step. However, usually fixed traction or fixed displacements are applied as boundary conditions. For the diffusion process, we have
\[
- \partial_t (\xi_{ijk} E_j H_k) = \partial_t\left(\frac{1}{2} \epsilon_{ij}(c, \epsilon_{im}) E_i E_j + \frac{1}{2} \mu_{ij}(c, \epsilon_{im}) H_i H_j \right) + J^c_i E_i + E_i \beta_{ilm} \partial_t \epsilon_{lm} + H_i \gamma_{ilm} \partial_t \epsilon_{lm}, \tag{33}
\]
\[
- \partial_t (\xi_{ijk} E_j H_k) = \epsilon_{ij}(c, \epsilon_{im}) E_i \partial_t E_j + \frac{1}{2} \partial_t \epsilon_{ij}(c, \epsilon_{im}) E_i E_j + \mu_{ij}(c, \epsilon_{im}) H_i \partial_t H_j + \frac{1}{2} \partial_t \mu_{ij}(c, \epsilon_{im}) H_i H_j + J^c_i E_i + E_i \beta_{ilm} \partial_t \epsilon_{lm} + H_i \gamma_{ilm} \partial_t \epsilon_{lm}, \tag{34}
\]
\[
0 = \left(\xi_{ijk} \partial_t H_k + \epsilon_{ij} \partial_t E_j + \frac{1}{2} \partial_t \epsilon_{ij} E_j + J^c_i E_i + \mu_{ij} \partial_t H_j + \frac{1}{2} \partial_t \mu_{ij} H_j + \gamma_{ilm} \partial_t \epsilon_{lm}\right) E_i + \left(\xi_{ijk} \partial_t E_k + \mu_{ij} \partial_t H_j + \frac{1}{2} \partial_t \mu_{ij} H_j + \gamma_{ilm} \partial_t \epsilon_{lm}\right) H_i. \tag{35}
\]

As a result, the two dynamic equations of the Maxwell equations are found by setting the terms in the two parenthesis of the above equation equal to zero, and they are
\[
J^c_i = \xi_{ijk} \partial_j H_k - \partial_t D_i + \frac{1}{2} \left(\frac{\partial \epsilon_{ij}}{\partial c} \partial_t c + \frac{\partial \epsilon_{ij}}{\partial \epsilon_{im}} \partial_t \epsilon_{im}\right) E_j. \tag{36}
\]
\[
\xi_{ijk} \partial_j E_k = -\partial_t B_i + \frac{1}{2} \left(\frac{\partial \mu_{ij}}{\partial c} \partial_t c + \frac{\partial \mu_{ij}}{\partial \epsilon_{im}} \partial_t \epsilon_{im}\right) H_j. \tag{37}
\]

Then, the governing equations for the electromagnetic fields are the above two equations with Eqs. (1) and (3). For the boundary conditions, relations on either the normal or tangential components of \(E_i (D_i)\) and \(B_i (H_i)\) are given, respectively, at the boundaries.

Furthermore, the following terms in Eq. (34) can be rewritten as
\[
\frac{1}{2} \partial_t \epsilon_{ij}(c, \epsilon_{im}) E_i E_j + \frac{1}{2} \partial_t \mu_{ij}(c, \epsilon_{im}) H_i H_j = \frac{\partial u^{em}_{im}}{\partial c_{im}} \partial_t \epsilon_{lm} + \frac{\partial u^{em}_{im}}{\partial c} \partial_t c. \tag{38}
\]

Note that the two terms on the right-hand side are shown at the first steps of Eqs. (29) and (31), respectively. This guarantees that energy is conserved during the conversion of energies between the electromagnetic fields with the elastic and compositional fields, respectively. In Ref. [17], it has already been shown that energy is conserved during the conversion of energies between the elastic and compositional fields and thus it is omitted here.
7 Some comments on energy conversions

Energy conversion is very important for coupling processes. According to the analysis above, energy can be converted via three different ways, i.e., work done by non-potential forces, the cross dependence of energy terms, and direct conversion as shown by electrostriction and magnetostriction. Compared with the first two ways, direct conversion is rare but it in fact reflects the substantial feature of energy conversion. In a continuum medium, energy is stored via the electromagnetic interactions of ions and electrons. For example, the elastic energy arises from the variations of interactions among ions on lattice sites in the presence of strains, and so does chemical energy in the presence of compositional changes. As a result, when energy is converted directly, it is converted directly via the interactions of ions and electrons. In fact, for all three ways, energy is substantially converted via the interactions of ions and electrons. During the piezoelectric effect, the externally applied pressures change the lattice distances and thus induce variations of interactions among ions at the lattice sites, and so do the generalized Villari effects.

In thermodynamics, the internal energy is a sum of all specific energies, e.g., the thermal energy, chemical energy, and elastic energy. These energies are in fact all stored via the interactions of ions and electrons within the substance. Thus, it is argued that there should be a general expression for the energy stored via the interactions of ions and electrons at the microscopic level. When this general expression is perturbed by the compositional field, the variation is the chemical energy at the macroscopic level. When perturbed by the elastic or electromagnetic fields, the variation is the elastic or electromagnetic energy, respectively. It is believed that the determination of this general expression of the energy stored via the interactions of ions and electrons at the microscopic level can be quite beneficial for further understanding of energy conversions at the macroscopic level.

8 Conclusions

Coupling processes substantially reflect the conversion of energies from one field into another. According to the LCE, loss of energy in one field must equal the gain of energy in another. This principle is used to analyze the reversible processes coupled between the elastic and electromagnetic fields. It is found that for the direct and converse piezoelectric and piezomagnetic effects which behave linearly, their constitutive equations guarantee that energy is conserved during the conversion of energies. Furthermore, rather than associated with any specific energy terms, these two processes are associated with the work done by non-potential forces. For the generalized Villari effects, it is found that while deriving the equation of motion, the electromagnetic energy can be treated as an extra contribution to the generalized elastic energy and thus energy is converted via the cross dependence of the energy terms. However, for electrostriction and magnetostriction, these effects are induced by the Maxwell stresses and their energy is purely electromagnetic.

General coupling processes in electronic devices which involve elastic, electromagnetic fields and diffusion are also analyzed. Both the equation of motion and the governing equation for diffusion are determined. Using Poynting’s theorem, the two dynamic equations of the Maxwell equations are found with extra terms, taking into account the generalized Villari effects and the compositional dependence of permittivity and permeability. These extra terms guarantee that energy is conserved during the conversion of energies between the electromagnetic fields with the elastic and compositional fields.

Though the Lagrangian formulation is a traditional approach to determine the constitutive equations and to construct governing equations for coupling processes, it does not offer any physical insights into the issue of conversion of energies. The phenomenological assumption of these energy terms, i.e., \( \beta_{ijk} E_i e_{jk} \) and \( \gamma_{ijk} H_i e_{jk} \), with the piezoelectric and piezomagnetic
effects is also unreasonable. In this paper, the LCE is used to construct the governing equations directly for reversible processes coupled between the electromagnetic and elastic fields. With this energy formulation, not only the conservation of energy is guaranteed for these coupling processes, but also the discussion of conversion of energies becomes straightforward. Furthermore, it contributes to better understanding of the underlying physical mechanisms of these processes. This energy formulation can be used to study most coupled reversible processes so that their governing equations can strictly satisfy the first law of thermodynamics.

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