ELIMINATION IDEALS AND BÉZOUT RELATIONS

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ABSTRACT. Let $k$ be an infinite field and $I \subset k[x_1, \ldots, x_n]$ be a non-zero ideal such that $\dim V(I) = q \geq 0$. Denote by $(f_1, \ldots, f_s)$ a set of generators of $I$. One can see that in the set $I \cap k[x_1, \ldots, x_{q+1}]$ there exist non-zero polynomials, depending only on these $q+1$ variables. We aim to bound the minimal degree of the polynomials of this type, and of a Bézout (i.e. membership) relation expressing such a polynomial as a combination of the $f_i$. In particular we show that if $\deg f_i = d_i$ where $d_1 \geq d_2 \geq \cdots \geq d_s$, then there exist a non-zero polynomial $\phi(x) \in k[x_1, \ldots, x_{q+1}] \cap I$, such that $\deg \phi \leq d_s \prod_{i=1}^{n-q-1} d_i$.

1. Introduction

Let $I \subset k[x_1, \ldots, x_n]$ be a non-zero ideal such that $\dim V(I) = q \geq 0$. Using Hilbert Nullstellensatz we can easily see, that in the elimination ideal $I \cap k[x_1, \ldots, x_{q+1}]$ there exist non-zero polynomials. It is interesting to know the minimal degree of the polynomials in this ideal. Here, performing a generic change of coordinates, and continuing the approach presented in [1], we get a sharp estimate for the degree of such a minimal polynomial (and also for a corresponding generalized Bezout identity), in terms of the degrees of generators of the ideal $I$. Then, using a deformation arguments we solve the stated problem. We show that if $\deg f_i = d_i$ where $d_1 \geq d_2 \geq \cdots \geq d_s$, then there exist polynomials $g_j \in k[x_1, \ldots, x_n]$ and a non-zero polynomial $\phi(x) \in k[x_1, \ldots, x_{q+1}]$ such that

(a) $\deg g_j f_j \leq d_s \prod_{i=1}^{n-q-1} d_i$,
(b) $\phi(x) = \sum_{j=1}^{s} g_j f_j$.

Note that our result works also in the case $\dim V(I) = -1$ (i.e. in the case when $V(I) = \emptyset$) if we put $k[x_0] := k$ (however our result in this case is a little bit worse than these in [1, 2]). Hence, from this point of view, we can treat our result as a generalization of the Effective Nullstellensatz.

2. Main Result

In this section we present a geometric construction and establish degree bounds, relying on generic changes of coordinates. Let us recall (see [1]) two important tools that we will use in the proof of the main theorem of this section.

Theorem 1. (Perron Theorem) Let $\mathbb{L}$ be a field and let $Q_1, \ldots, Q_{n+1} \in \mathbb{L}[x_1, \ldots, x_n]$ be non-constant polynomials with $\deg Q_i = d_i$. If the mapping $Q = (Q_1, \ldots, Q_{n+1}) : \mathbb{L}^n \to \mathbb{L}^{n+1}$ is generically finite, then there exists a non-zero polynomial $W(T_1, \ldots, T_{n+1}) \in \mathbb{L}[T_1, \ldots, T_{n+1}]$ such that

(a) $W(Q_1, \ldots, Q_{n+1}) = 0$,
(b) $\deg W(T_1^{d_1}, T_2^{d_2}, \ldots, T_{n+1}^{d_{n+1}}) \leq \prod_{j=1}^{n+1} d_j$.

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Lemma 2. Let $\mathbb{K}$ be an algebraic closed field and let $k \subset \mathbb{K}$ be its infinite subfield. Let $X \subset \mathbb{K}^m$ be an affine algebraic variety of dimension $n$. For sufficiently general numbers $a_{ij} \in k$ the mapping

$$
\pi : X \ni (x_1, \ldots, x_m) \mapsto \left( \sum_{j=1}^m a_{1j}x_j, \sum_{j=2}^m a_{2j}x_j, \ldots, \sum_{j=n}^m a_{nj}x_j \right) \in \mathbb{K}^n
$$

is finite. □

In the sequel for a given ideal $I \subset k[x_1, \ldots, x_n]$ by $V(I)$ we mean the set of algebraic zeros of $I$, i.e., the zeroes of $I$ in $\mathbb{K}^n$, where $\mathbb{K}$ is an algebraic closure of $k$. Now we can formulate our first main result:

Theorem 3. Let $k$ be an infinite field and let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be polynomials such $\deg f_i = d_i$ where $d_1 \geq d_2 \geq \cdots \geq d_s$ . Assume that $I = (f_1, \ldots, f_s) \in k[x_1, \ldots, x_n]$ is a non-zero ideal, such that $V(I)$ has dimension $q \geq 0$. If we take a sufficiently general system of coordinates $(x_1, \ldots, x_n)$, then there exist polynomials $g_j \in k[x_1, \ldots, x_n]$ and a non-zero polynomial $\phi(x) \in k[x_1, \ldots, x_{q+1}]$ such that

(a) $\deg g_j f_j \leq d_s \prod_{i=1}^{s-q-1} d_i$,

(b) $\phi(x) = \sum_{j=1}^s g_j f_j$.

Proof. Let $\mathbb{K}$ be the algebraic closure of $k$. Take $F_{n-q} = f_s$ and $F_i = \sum_{j=1}^s \alpha_{ij} f_j$ for $i = 1, \ldots, n - q - 1$, where $\alpha_{ij} \in k$ are sufficiently general. Take $J = (F_1, \ldots, F_{n-q})$. Then $\deg F_{n-q} = d_s$ and $\deg F_i = d_i$ for $i = 1, \ldots, n - q - 1$. Moreover, $V(J)$ has pure dimension $q$ and $J \subset I$. The mapping

$$
\Phi : \mathbb{K}^n \times \mathbb{K} \ni (x, z) \mapsto (F_1(x)z, \ldots, F_{n-q}(x)z, x) \in \mathbb{K}^{n-q} \times \mathbb{K}^n
$$

is a (non-closed) embedding outside the set $V(J) \times \mathbb{K}$. Take $\Gamma = \text{cl}(\Phi(\mathbb{K}^n \times \mathbb{K}))$. Let $\pi : \Gamma \to \mathbb{K}^{n+1}$ be a generic projection defined over the field $k$. Define $\Psi := \pi \circ \Phi(x, z)$. By Lemma 2, we can assume that

$$
\Psi = \left( \sum_{j=1}^{n-q} \gamma_{1j} F_j z + l_1(x), \ldots, \sum_{j=n-q}^{n-q} \gamma_{n-qj} F_j z + l_{n-q}(x), l_{n-q+1}(x), \ldots, l_{n+1}(x), \right),
$$

where $l_1, \ldots, l_{n+1}$ are generic linear forms. In particular we can assume that $l_{n-q+i}, i = 1, \ldots, q + 1$ is the variable $x_i$ in a new generic system of coordinates (of $\mathbb{K}^n$).

Apply Theorem 1 to $L = k(z)$, and to the polynomials $\Psi_1, \ldots, \Psi_{n+1} \in L[x]$. Thus there exists a non-zero polynomial $W(T_1, \ldots, T_{n+1}) \in L[T_1, \ldots, T_{n+1}]$ such that

$$
W(\Psi_1, \ldots, \Psi_{n+1}) = 0 \quad \text{and} \quad \deg W(T_1, \ldots, T_{n+1}) \leq d_s \prod_{j=1}^{n-q-1} d_j,
$$

where $k = n - q$. Since the coefficients of $W$ are in $k(z)$, there is a non-zero polynomial $\tilde{W} \in k[T_1, \ldots, T_{n+1}, Y]$ such that

(a) $\tilde{W}(\Psi_1(x, z), \ldots, \Psi_{n+1}(x, z), z) = 0$,

(b) $\deg_T \tilde{W}(T_1, \ldots, T_{n+1}) \leq d_s \prod_{j=1}^{n-q-1} d_j$, where $\deg_T$ denotes the degree with respect to the variables $T = (T_1, \ldots, T_{n+1})$.

Note that the mapping $\Psi = (\Psi_1, \ldots, \Psi_{n+1}) : \mathbb{K}^n \times \mathbb{K} \to \mathbb{K}^{n+1}$ is locally finite outside the set $V(J) \times \mathbb{K}$. Consider $\mathbb{K}^{n+1}$ as a product $\mathbb{K}^{n-q} \times \mathbb{K}^{q+1}$, and let us consider in this product coordinates $(y_{q+2}, \ldots, y_{n+1}, y_1, \ldots, y_{q+1})$. Hence $\Psi$ restricted to $V(J) \times \mathbb{K}$ coincides
with the mapping: $(x, z) \mapsto (l_1(x), ..., l_{n-q}(x), x_1, ..., x_{q+1})$ (recall that we consider a new generic system of coordinates). Let $\phi' = 0$ describes the image of the projection 

$$
\pi : V(J) \ni x \mapsto (x_1, ..., x_{q+1}) \in \mathbb{K}^{q+1}.
$$

Put $S = \{ T \in \mathbb{K}^{n+1} : \phi'(T) = 0 \}$. Hence $V(J) \times \mathbb{K}$ is contained in $\Psi^{-1}(S)$. Consequently the mapping $\Psi$ is proper outside the hypersurface $S$ and thus the set of non-properness of the mapping $\Psi$ is contained in the $S$.

Since the mapping $\Psi$ is finite outside $S$, for every $H \in k[x_1, ..., x_n, z]$ there is a minimal polynomial $P_H(T, Y) \in k[T_1, ..., T_{n+1}][Y]$ such that $P_H(\Psi_1, ..., \Psi_{n+1}, H) = \sum_i b_i(\Psi_1, ..., \Psi_{n+1})H^{r-i} = 0$ and the coefficient $b_0$ satisfies $\{ T : b_0(T) = 0 \} \subset S$. In particular $b_0$ depends only on variables $x_1, ..., x_{q+1}$. Moreover, $P_H$ describes a hypersurface given by parametric equation $(\Psi_1, ..., \Psi_{n+1}, H)$ and by Gröbner base computation we see that we can assume $P_H(T, Y) \in k[T_1, ..., T_{n+1}][Y]$. Now set $H = z$.

We have

$$
\deg_T P_s(T_1^{d_1}, T_2^{d_2}, ..., T_n^{d_n}, T_{n+1}, Y) \leq d_s \prod_{j=1}^{n-q-1} d_j
$$

and consequently we obtain the equality $b_0(x_1, ..., x_{q+1}) + \sum_{i=1}^{n-q} F_i g_i = 0$, where $\deg F_i g_i \leq d_s \prod_{j=1}^{n-q-1} d_j$. Set $\phi = b_0$. By the construction the polynomial $\phi$ has zeros only on the image of the projection

$$
\pi : V(J) \ni x \mapsto (x_1, ..., x_{q+1}) \in \mathbb{K}^{q+1}.
$$

\begin{proof} \end{proof}

Remark 4. Simple application of the Bezout theorem shows that our bound on the degree of $\phi$ is sharp.

Corollary 5. Let $k$ and $I$ and system of coordinates be as above. If $V(I)$ has pure dimension $q$ and $I$ has not embedded components, then there is a polynomial $\phi_1 \in k[x_1, ..., x_{q+1}]$ which describes the image of the projection

$$
\pi : V(I) \ni x \mapsto (x_1, ..., x_{q+1}) \in \mathbb{K}^{q+1}
$$

such that

(a) $\phi_1 \in I$,

(b) $\deg \phi_1 \leq d_s \prod_{i=1}^{n-q-1} d_i$.

Proof. The set $V(J) = q$ has pure dimension $q$. Consequently $\pi(V(J))$ and $\pi(V(I))$ are hypersurfaces. Moreover, by Gröbner bases computation the set $\pi(V(I))$ is described by a polynomial $\psi$ from $k[x_1, ..., x_{q+1}]$. Let $\phi$ be a polynomial as above which vanishes exactly on $\pi(V(J))$. Let $\phi_1$ be a product of all irreducible factors of $\phi$ (over the field $k$) which divides $\psi$. Hence $\phi = \phi_1 \phi_2$, $\phi_1, \phi_2 \in k[x_1, ..., x_{q+1}]$, where $\phi_2$ does not vanish on any component of $V(I)$. Let $I = \bigcap I_s$ be a primary decomposition of $I$. Consequently $\phi_1 \in I_j$ for every $s$ (by properties of primary ideals) and consequently $\phi_1 \in I$. But $\phi_1$ describes the image of the projection

$$
\pi : V(I) \ni x \mapsto (x_1, ..., x_{q+1}) \in \mathbb{K}^{q+1}.
$$
3. A DEFORMATION ARGUMENT

In this section, we improve Theorem 3 by releasing the necessity of a generic change of coordinates, so conditions (a) and (b) will be satisfied in the initial system of coordinates.

**Theorem 6.** Let $k$ be an infinite field and let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be polynomials such that $\deg f_i = d_i$ where $d_1 \geq d_2 \geq \ldots \geq d_s$. Assume that $I = (f_1, \ldots, f_s) \subset k[x_1, \ldots, x_n]$ is a non zero ideal, such that $V(I)$ has dimension $q \geq 0$. There exist polynomials $g_j \in k[x_1, \ldots, x_n]$ and a non-zero polynomial $\phi \in k[x_1, \ldots, x_{q+1}]$ such that

(a) $\deg g_j f_j \leq d_s \prod_{i=1}^{q-1} d_i$,

(b) $\phi = \sum_{j=1}^{s} g_j f_j$.

**Proof.** We use Theorem 3, but over the field $\mathbb{L} = k(t)$. We consider a new generic change of coordinates using generic values $a_{i,j}$ in the infinite field $k$, together with the inverse change of coordinates

$$X_i = x_i + t \sum_{j=i+1}^{n} a_{i,j} x_j; \quad x_i = X_i + t \sum_{j=i+1}^{n} b_{i,j}(t) X_j,$$

where $b_{i,j}(t) \in k[t]$.

As in the proof of Theorem 3, we obtain some polynomials $G_j \in \mathbb{L}[X_1, \ldots, X_n]$ and a non-zero polynomial $b_0 \in \mathbb{L}[X_{n-q}, \ldots, X_n]$ such that, after chasing the denominators,

$$b_0(X,t) = \sum_{j=1}^{n-q} G_j(X,t) \bar{F}_j(X,t),$$

where $b_0(X,t), G_j(X,t), \bar{F}_j(X,t) \in k[t][X_1, \ldots, X_n]$.

We cannot just simplify this equality by $t$ and then set $t = 0$, because we cannot exclude the possibility that there will be a remaining factor $t^p$ in the left hand side with $p$ strictly positive. To rule out this possibility, we need to perform several reduction steps. Consider the sub-module $M = \{H(x) = (H_1(x), \ldots, H_{n-q}(x))\}$ of $k[x]^{n-q}$ formed by the relations (first syzygies) between the polynomials $F_1(x), \ldots, F_{n-q}(x)$. To each element $H(x)$ in $M$ corresponds a change of coordinates a relation $\bar{H}(X,t)$ between the polynomials $\bar{F}_1(X,t), \ldots, \bar{F}_{n-q}(X,t)$, such that $\bar{H}(X,t) - H(X)$ is divisible by $t$. Re-writing in $(x,t)$, we obtain that

$$b_0(X,t) = \sum_{j=1}^{n-q} (G_j(X,t) - \bar{H}_j(X,t)) \bar{F}_j(X,t).$$

We may assume that in the previous equality $b_0(X,t)$ has the form $b_0(X,t) = t^p(\phi(x) + t\phi_1(x,t))$; notice that the $x-$ degree of $\phi(x)$ is bounded by the $X-$degree of $b_0(X,t)$.

Each reduction step will produce a similar equality (with the same degree in $x$ bounds) but with a strictly smaller power $p$.

Assume $p > 0$ and let $t = 0$, we obtain a non trivial relation $0 = \sum_{j=1}^{n-q} G_j(x,0) F_j(x)$, hence $H = (G_1(x,0), \ldots, G_{n-q}(x,0))$, a non trivial element of $M$. Notice that the $x-$ degree of $G_j(x,0)$ is bounded by the $X-$degree of $G_j(X,t)$. To which we associate its $\bar{H}$ as above with the same degree bound in $X$ (equivalently in $x$ by linearity) and notice that now $\sum_{j=1}^{n-q} (G_j(X,t) - \bar{H}_j(X,t)) \bar{F}_j(X,t)$ vanishes for $t = 0$, hence admits a factor $t$. We can simplify the two sides of the previous equality by $t$ and obtain

$$t^{p-1}(\phi(x) + t\phi_1(x,t)) = \sum_{j=1}^{n-q} (G_j(X,t) - \bar{H}_j(X,t)) \bar{F}_j(X,t).$$
After at most \( p \) such reduction steps, we get rid of the initial factor \( t^p \) and setting \( t = 0 \), we obtain the announced equality with the announced bounds. \( \square \)

References

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