CMV BIORTHOGONAL LAURENT POLYNOMIALS:
CHRISTOFFEL FORMULAS FOR CHRISTOFFEL AND GERONIMUS PERTURBATIONS

GERARDO ARIZNABARRETA\textsuperscript{1,2}, MANUEL MAÑAS\textsuperscript{1}, AND ALFREDO TOLEDANO

\textbf{Abstract.} Quasidefinite sesquilinear forms for Laurent polynomials in the complex plane and corresponding CMV biorthogonal Laurent polynomial families are studied. Bivariate linear functionals encompass large families of orthogonalities like Sobolev and discrete Sobolev types. Two possible Christoffel transformations of these linear functionals are discussed. Either the linear functionals are multiplied by a Laurent polynomial, or are multiplied by the complex conjugate of a Laurent polynomial. For the Geronimus transformation, the linear functional is perturbed in two possible manners as well, by a division by a Laurent polynomial or by a complex conjugate of a Laurent polynomial, in both cases the addition of appropriate masses (linear functionals supported on the zeros of the perturbing Laurent polynomial) is considered. The connection formulas for the CMV biorthogonal Laurent polynomials, its norms, and Christoffel–Darboux kernels, in all the four cases, are given. For the Geronimus transformation, the connection formulas for the second kind functions and mixed Christoffel–Darboux kernels are also given in the two possible cases. For prepared Laurent polynomials, i.e. of the form $L(z) = L_n z^n + \cdots + L_{-n} z^{-n}$, $L_n L_{-n} \neq 0$, these connection formulas lead to quasideterminantal (quotient of determinants) Christoffel formulas for all the four transformations, expressing an arbitrary degree perturbed biorthogonal Laurent polynomial in terms of $2n$ unperturbed biorthogonal Laurent polynomials, their second kind functions or Christoffel–Darboux kernels and its mixed versions. Different curves are presented as examples, like the real line, the circle, the Cassini oval and the cardioid. The unit circle case, given its exceptional properties, is discussed in more detail. In this case, a particularly relevant role is played by the reciprocal polynomial, and the Christoffel formulas provide now with two possible ways of expressing the same perturbed quantities in terms of the original ones, one using only the nonperturbed biorthogonal family of Laurent polynomials, and the other using the Christoffel–Darboux kernels and its mixed versions, as well.

1. Introduction

The study of perturbations of a linear functional $u$ in the space of polynomials is an active area of research in the theory of orthogonal polynomials. When you deal with positive definite measures, this study provides information about the Gaussian quadrature rules [31, 32]. Christoffel perturbations, $\hat{u} = p(x)u$, where $p(x)$ is a polynomial, were studied in 1858 by Christoffel [22] giving explicit formulas relating the corresponding sequences of orthogonal polynomials with respect to two measures. These are called Christoffel formulas, and can be considered a classical result in the theory of orthogonal polynomials which can be found in a number of textbooks, see for example [21, 66, 32]. Explicit relations between the corresponding sequences of orthogonal polynomials have been extensively studied, see [21]. Connection formulas between two families of orthogonal polynomials allow to express any polynomial of a given degree $n$ as a linear combination of all polynomials of degree less than or equal to $n$ in the second family. A remarkable fact regarding the Christoffel finding is that in some cases the number of terms does not grow with the degree $n$ but remain constant, equal to the degree of the perturbing polynomial. See [31, 32].

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DEPARTAMENTO DE FÍSICA TEÓRICA II (MÉTODOS MATEMÁTICOS DE LA FÍSICA), UNIVERSIDAD COMPLUTENSE DE MADRID, CIUDAD UNIVERSITARIA, PLAZA DE CIENCIAS 1, 28040 MADRID, SPAIN

\textit{E-mail addresses:} gariznab@ucm.es, manuel.manas@ucm.es, alfrtole@ucm.es.

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When the perturbed functional is $v$ given by $p(x)v = u$, for $p(x)$ a polynomial, we say that we have a Geronimus transformations. Was Geronimus [39], studying the results of [43] concerning the characterization of classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel), the first who discussed these transformations. Christoffel type formulas in terms of the second kind functions where found in [52]. Let us notice that in [39] no Christoffel type formula was derived. Despite this fact, we will refer to the Christoffel formulas for Geronimus transformations as Christoffel–Geronimus formulas. The problem of given two functionals $u$ and $v$ such that $p(x)u = q(x)v$, where $p(x)$, $q(x)$ are polynomials was analyzed in [69] when $u$, $v$ are positive definite measures supported on the real line and in [26] for linear functionals, see also [72]. Uvarov [69] found Christoffel type formulas, and the addition of a finite number of Dirac masses to a linear functional appears in the framework of the spectral analysis of fourth order linear differential operators with polynomial coefficients and with orthogonal polynomials as eigenfunctions. Geronimus perturbations of degree two of scalar bilinear forms have been very recently treated in [24] and in the general case in [25].

For a positive definite functional the orthogonal polynomials in the unit circle $T$ or Szegö polynomials are monic polynomials $P_n$ of degree $n$ which fulfill $\int_T P_n(z)z^{-k}d\mu(z) = 0$, for $k = 0, 1, \ldots, n - 1$, [63] and also [64]. Orthogonal polynomials on the real line with support on $[-1, 1]$ are connected with the Szegö polynomials [30]. The extension to the unit circle context of the three-term relations and tridiagonal Jacobi matrices of the real scenario, require of Hessenberg matrices and give the Szegö recursion relation, which is expressed in terms of the reciprocal, or reverse, Szegö polynomials $P_1^*(z) := z^2P_1(z^{-1})$ and reflection or Verblunsky coefficients $\alpha_1 := P_1(0), \left(\frac{z}{z_0}, \frac{\alpha_i}{1}\right) = \left(z, \frac{1}{P_{i-1}^*}\right)$. Szegö’s theorem implies for a nontrivial probability measure $d\mu$ on $T$ with Verblunsky coefficients $\alpha_n = 0$ that the corresponding Szegö’s polynomials are dense in $L^2(T, \mu)$ if and only if $\prod_{n=0}^{\infty}(1 - |\alpha_n|^2) = 0$. For an absolutely continuous probability measure Kolmogorov’s density theorem ensures that density in $L^2(T, \mu)$ of the Szegö polynomials holds iff the so called Szegö’s condition $\int_T \log|w(\theta)|d\theta = -\infty$ is fulfilled, [63]. The studies of [48], [49] about the the strong Stieltjes moment problem constitute one of the seeds for the activity about orthogonal Laurent polynomials on the real line. If there is a solution of the moment problem we can find Laurent polynomials $\{Q_n\}_{n=0}^{\infty}$ satisfying $\int_T x^{-n+j}Q_n(x)d\mu(x) = 0$ for $j = 0, \ldots, n - 1$. Laurent polynomials on the unit circle $T$ where discussed in [67], see also [15], [19], [13], [14] where recursion relations, Favard’s theorem, quadrature problems, and Christoffel–Darboux formulae were considered. Despite the set of orthogonal Laurent polynomials being dense in $L^2(T, \mu)$ in general this is not true for the Szegö polynomials, [18] and [15]. The Szegö recursion relation is replaced by a five-term relation similar to the real line situation. Generic orders in the basis used to span the space of orthogonal Laurent polynomials in the unit circle were discussed in [14]. The CMV ( Cantero–Morales–Velázquez) matrices [19], which constitute the representation of the multiplication operator in terms of the basis of orthonormal Laurent polynomials, where discussed in [20] where the connection with Darboux transformations and their applications to integrable systems has been analyzed. Regarding the CMV ordering, orthogonal Laurent polynomials and Toda systems, see §4.4 of [5] where discrete Toda flows and their connection with Darboux transformations were discussed. For a matrix version of this discussion see [8]. As was pointed out in [64] the discovery the CMV ordering goes back to [20]. In [33] spectral transformations of measures supported on the unit circle with real moments are considered and the connection with spectral transformations of measures supported on the interval $[-1, 1]$ using the Szegö transformation is presented. Then, in [30], a Geronimus perturbation of a nontrivial positive measure supported on the unit circle is studied as well as the connection between the associated Hessenberg matrices. Finally, in [35] linear spectral transformations of Hermitian linear functionals using the multiplication by some class of Laurent polynomials are considered and the behavior of the Verblunsky parameters of the perturbed linear functional is found.

Strong links do exist between Hermitian linear functionals in the space of Laurent polynomials with complex coefficients and the theory of orthogonal polynomials on the unit circle. For every positive definite Hermitian linear functional there exists a probability measure supported on a subset of the unit circle giving an integral representation for the functional. In this context, it was at the early 1990’s when two groups found two different kind of Christoffel type formulas. Godoy and Marcellán published two papers, in [40] extensions of the Christoffel determinant type formulas were given for the analogue of the Christoffel transformation, with an
arbitrary degree polynomial having multiple roots, using the original Szegő polynomials and its Christoffel–Darboux kernels. This paper has a distinctive approach from the classical by Christoffel, which allows to express the Christoffel formulas in terms of the Christoffel–Darboux kernel and one polynomial solely. This, is in principle (as you have the Christoffel–Darboux formula at your disposal) simpler than the classical Christoffel formula, in where one needs to evaluate not only one, but a number \( \Theta \) related to the degree of the perturbing polynomial of orthogonal polynomials. Let us notice that this technique relies on the orthogonal decomposition of linear spaces, which is lost in the context of biorthogonality with respect to a sesquilinear form, as we consider in our present paper. Then, in [41], the Geronimus transformation for OPUC was discussed in terms of second kind functions. In particular, in Proposition 3 of that paper a Christoffel–Geronimus formula for a perturbation of degree 2 is given, where no masses are considered. On the other hand, in [47], alternative formulas à la Christoffel, not based on the Christoffel–Darboux kernel [40], were presented in terms of determinantal expressions of the Szegő polynomials and their reverse polynomials, also as Uvarov did in [69], they considered multiplication by rational functions, but no masses at all where discussed in this paper. Finally, in [60], some examples on concrete cases where considered within the biorthogonal scenario.

The transformations considered in this work are also known as Darboux transformations [53]. Indeed, in the context of the Sturm–Liouville theory, Darboux discussed in [23] a dimensional simplification of a geometrical transformation in two dimensions founded previously [55] which can be considered, as we called it today, a Darboux transformation. In [68], in the context of orthogonal polynomials, Darboux transformations are considered in terms of the factorization of the Jacobi matrices, while in [43] the bispectrality and Darboux transformations are discussed. Let us mention that in the theory of Differential Geometry [25] the Christoffel, Geronimus, Uvarov and linear spectral transformations are related to geometrical transformations like the Laplace, Lévy, adjoint Lévy and the fundamental Jonas transformations. See [27] for a discrete version of these geometrical transformations, and its connection with integrable systems. The interested reader may consult the excellent monographs [54] and [59].

The framework for biorthogonality in this paper is that of continuous sesquilinear forms in the space of Laurent polynomials. We base our discussion on the Schwartz’s noyau-distribution [61]. These general non Toeplitz scenarios have been considered in the scalar case, for an orthogonal polynomial approach see [17] while from an integrable systems point of view see [1, 2]. For a linear functional setting for orthogonal polynomials see [50, 51]. The space of Laurent polynomials \( \mathbb{C}(z, z^{-1}) \), with an appropriate topology, is considered as the space of fundamental functions, in the sense of [36, 37], and the corresponding space of generalized functions provides with a linear functional setting for orthogonal polynomials theory. Discrete orthogonality appears when we consider linear functionals with discrete and infinite support [56]. We will consider an arbitrary nondegenerate continuous sesquilinear form given by a generalized kernel \( u_{z_1, z_2} \) with a quasidefinite Gram matrix. This scheme not only contains the more usual choices of Gram matrices like those of Toeplitz type on the unit circle, or those leading to discrete orthogonality but also Sobolev orthogonality, as these first examples correspond to matrices of generalized kernels supported by the diagonal, [46], and our scheme is applicable to the general case, with support off the diagonal as happens in the last Sobolev mentioned case. The quasidefinite condition on the Gram matrix ensures for a block Gauss–Borel factorization and, consequently, for the existence of two biorthogonal families of Laurent polynomials.

Even though we are in a commutative scenario, not like those necessary for the matrix or multivariate orthogonality, we will express our Christoffel formulas in terms of quasideterminants. This is justified for the simplicity that the formulas adopt an also because they will be ready for a future update to more general situations. We refer the reader to [38] and [57] for two complementary expositions on this subject. For our needs, given a \( 2 \times 2 \) block matrix \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \), where \( A \in \mathbb{C}^p \times p \), \( B \), \( C^T \in \mathbb{C}^p \) and \( D \in \mathbb{C} \), the last quasideterminant is the Schur complement [71], that in our case is just the quotient of two determinants \( \Theta_s \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \frac{|A|}{\Theta_s} \).

The scheme based on the Gauss–Borel factorization problem, used here for transformations for matrix orthogonal polynomials or non-Abelian 2D Toda lattices, has been applied by our group also in the following situations

i) In cite [6] and in [9] we consider some extensions of the Christoffel–Darboux formula to generalized orthogonal polynomials [2] and to multiple orthogonal polynomials, respectively.
Matrix orthogonal polynomials, its Christoffel transformations and the relation with non-Abelian Toda hierarchies were studied in [3], and in [4] we extended those results to include the Geronimus, Geronimus–Uvarov and Uvarov transformations.

Multiple orthogonal polynomials and multicomponent Toda [7].

Matrix orthogonal Laurent polynomials on the unit circle, CMV orderings, and non-Abelian lattices on the circle [8].

Multivariate orthogonal polynomials in several real variables and corresponding multispectral integrable Toda hierarchy [10, 11]. Multivariate orthogonal polynomials on the multidimensional unit torus, the multivariate extension of the CMV ordering and integrable Toda hierarchies [12].

1.1. Objectives, results, layout of the paper and perspectives. In the paper we consider a general sesquilinear form in the complex plane determined by a bivariate linear functional, its biorthogonal Laurent families and its behavior under Christoffel and Geronimus perturbations. The Gauss–Borel factorization of the Gram matrix, which we assume to be quasidefinite, leads to connection formulas for the biorthogonal Laurent polynomial families, the corresponding second kind functions and the standard and mixed Christoffel–Darboux kernels. This result allows us for the finding of quasideterminantal Christoffel formulas for the Christoffel transformation as well as for the Geronimus transformation. Let us observe that regarding previous works [40, 41, 47] on the subject we may say that

i) The results of [40, 41, 47] concern sesquilinear forms supported on the diagonal (in fact associated with a positive Borel measure). Our scheme allows for more general biorthogonality and therefore includes Sobolev orthogonality and discrete Sobolev orthogonality.

ii) The interesting version of the Christoffel formula in terms of Christoffel–Darboux kernels given in [40] can not be extended to bivariate linear functionals supported off diagonal.

iii) Regarding Geronimus or Geronimus–Uvarov transformations the papers [41, 47] do not incorporate masses at all. In our paper we include a very general class of masses.

We have considered two possible Christoffel and two possible Geronimus transformations, and found the corresponding Christoffel formulas. These two transformations, when the bivariate linear functional reduces to an univariate linear functional supported on the unit circle, can be made to coincide. When this happens we are lead to two possible Christoffel formulas, giving alternative expressions of the perturbed objects. This is characteristic of the unit circle, and does not happens in other cases, like the Cassini oval or the cardioid.

The layout of the paper is as follows. We now proceed with a reminder on known results regarding CMV Laurent polynomials and biorthogonality. Then, in §2 we perturb a general quasidefinite sesquilinear form by multiplying the corresponding bivariate linear functional with a Laurent polynomial or the complex conjugate of a Laurent polynomial. Assuming quasidefiniteness of the perturbed sesquilinear forms, we derive then, using the Gauss–Borel factorization, connections formulas for the biorthogonal Laurent polynomials and the Christoffel–Darboux kernels, see Propositions 9 and 11. Then, when the perturbing Laurent polynomials are prepared, i.e. their larger positive and smaller negative powers are equal, we derive the Christoffel formulas, see Theorem 1. We use spectral jets because we work in the general setting of generic multiplicities of the zeros of the perturbing polynomials and express all relations in terms of quasideterminants, as they are more compact and ready for future use with matrix or multivariate orthogonalities. We give the expressions for the two biorthogonal families of Laurent polynomials as well as their norms, and for both type of transformations. Then, we discuss the situation that appears when the bivariate linear functional collapses to a univariate linear functional supported on the unit circle. We see that both types of perturbations can be made equal, by choosing one of the polynomials the reciprocal of the other. This gives us two alternative quasideterminantal Christoffel formulas for the perturbed biorthogonal polynomials and norms, see Proposition 12. Some comments are made for other supports different from the unit circle. Section 3 is devoted to the analysis of the Geronimus transformations, which can be considered as the inversion of the previous Christoffel transformations and, as we are working in a linear functional setting, some masses, which are supported in the zeros of the perturbing Laurent polynomials, can be included. In the first place we derive connection formulas for the biorthogonal Laurent polynomials, Cauchy second kind functions, Christoffel–Darboux kernels, and mixed Christoffel–Darboux kernels, see Propositions 16, 18, 19 and
With this at hand we present Theorem 2 where we the Christoffel–Geronimus formulas for both type of perturbations, i.e., dividing by a prepared Laurent polynomial or dividing by the complex conjugate of a prepared Laurent polynomial and the addition of general masses are given. Moreover, we write, for the first time, this type of expressions including the masses. As we commented above, to our best knowledge, this has not been discussed so far for Christoffel–Geronimus formulas in the unit circle or complex plane scenarios. Then, in Theorem 3 we consider the situation in where the bivariate linear functional is just an univariate linear functional with compact support, \( \Omega \subset \mathbb{C} \) of compact support, \( C(0) \). Its complementary set, which is closed, is the support, \( \text{supp} \). As for the Christoffel transformations, two alternative forms of expressing the perturbed biorthogonal Laurent polynomials and their norms. The masses are in this case the most general ones and therefore go beyond the standard masses supported on the diagonal, discussed in Proposition 22, and can be considered as a discrete Sobolev perturbation. There is an Appendix containing some of the proofs.

For perspectives and future work, we want to extend these results to the linear spectral case, where there is a multiplication by a quotient of two prepared Laurent polynomials. Moreover, we have some preliminary results regarding perturbations with not prepared polynomials, that we what to understand better. Finally, a matrix version on these results is needed as well as a discussion for multivariate orthogonal polynomials in the complex plane and in the multidimensional torus.

1.2. CMV biorthogonal Laurent polynomials.

Definition 1 (Sequilinear forms). A sequilinear form \( \langle \cdot, \cdot \rangle \) in the ring of Laurent polynomials \( \mathbb{C}[z, z^{-1}] \) is a continous map \( \langle \cdot, \cdot \rangle : \mathbb{C}[z, z^{-1}] \times \mathbb{C}[z, z^{-1}] \rightarrow \mathbb{C} \), such that for any triple \( L(z), M(z), N(z) \in \mathbb{C}[z, z^{-1}] \) the following conditions are satisfied

\[
\begin{align*}
\text{i)} & \quad \langle AL(z_1) + BM(z_1), N(z_2) \rangle = A \langle L(z_1), N(z_2) \rangle + B \langle M(z_1), N(z_2) \rangle, \quad \forall A, B \in \mathbb{C}, \\
\text{ii)} & \quad \langle L(z_1), AM(z_2) + BN(z_2) \rangle = \langle L(z_1), M(z_2) \rangle A + \langle L(z_1), N(z_2) \rangle B, \quad \forall A, B \in \mathbb{C}.
\end{align*}
\]

Given the ordered biinfinite basis \( \{ z^l \}_{l=-\infty}^{\infty} \) or the semiinfinite CMV basis \( \{ \chi^{(1)}(z) \}_{l=0}^{\infty} \), \( \chi^{(1)}(z) := \begin{cases} z^{l/2}, & \text{l even} \\ z^{-(l+1)/2}, & \text{l odd} \end{cases} \) of \( \mathbb{C}[z, z^{-1}] \) the sesquilinear form is characterized by the corresponding Gram matrix. For example, in the first case we have the bi-infinite Gram matrix \( g = \begin{bmatrix} g_{0,0} & g_{0,1} & \cdots \\ g_{1,0} & g_{1,1} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \) with \( g_{k,l} = \langle (z_1)^{k}, (z_2)^{l} \rangle \) and \( k, l \in \mathbb{Z} \).

Definition 2 (Laurent polynomial spectrum). The zero set of a function \( L(z) \) in \( \mathbb{C}^* := \mathbb{C} \setminus \{0\} \) will be denoted by \( \sigma(L) \) and said to be the spectrum of \( L(z) \).

In this paper we will consider sesquilinear forms constructed in terms of bivariate linear functionals with well-defined support. The space of distributions is the space of generalized functions when the space fundamental functions is the set of complex smooth functions \( L^2[0,1] \) with compact support \( \mathcal{D}^* := C_0^\infty(\mathbb{C}^*) \). The zero set of a distribution \( u \in (\mathcal{D}^*)' \) is the open region \( \Omega \subset \mathbb{C}^* \) whenever for every \( f(z) \) supported on \( \Omega \) we have \( \langle u, f \rangle = 0 \). Its complementary set, which is closed, is the support, \( \text{supp}(u) \), of the distribution \( u \). The distributions of compact support, \( u \in (\mathcal{E}^*)' \), are the generalized functions with fundamental functions \( \mathcal{E}^* = C^\infty(\mathbb{C}^*) \). As \( \mathbb{C}[z, z^{-1}] \subset \mathcal{E}^* \) we deduce that \( (\mathcal{E}^*)' \subset (\mathbb{C}[z, z^{-1}])' \cap (\mathcal{D}^*)' \). For any bivariate linear functional \( u_{z_1,z_2} \) with support \( \text{supp}(u_{z_1,z_2}) \) its projections in the axis \( z_1 \) are denoted by \( \text{supp}_1(u_{z_1,z_2}) \), \( i = 1, 2 \). Sesquilinear forms are constructed in terms of bivariate linear functionals. The space \( \mathbb{C}[z, z^{-1}] \), with a suitable topology is considered as the space of fundamental or test functions and the generalized functions are the continuous linear functionals on this space.

Definition 3. We consider the following sesquilinear forms \( \langle L(z_1), M(z_2) \rangle_u = \langle u_{z_1,z_2}, L(z_1) \otimes M(z_2) \rangle \) with \( L(z), M(z) \in \mathbb{C}[z, z^{-1}] \).

Hence, the following sesquilinear forms \( \langle L(z_1), M(z_2) \rangle_u = \sum_{0 \leq n, m < \infty} \int \frac{a_{n} b_{m} L(z_1) b_{n} m M(z_2)}{a_{n} b_{m} b_{n} m} \ d \mu^{(m,n)}(z_1, z_2) \), for Borel measures \( \mu^{(m,n)}(z_1, z_2) \) in \( \mathbb{C}^2 \), with at least one of them with infinite support, are included in our considerations.
Notice that in the bi-infinite basis \( \{z^n\}_{n \in \mathbb{Z}} \) we have the Gram matrix \( g = [g_{n,m}] \) with \( g_{n,m} = \langle (z_1)^n, (z_2)^m \rangle_u = \langle u_{z_1, z_2}, z_1 \otimes (z_2)^m \rangle \). A bivariate linear functional \( u_{z_1, z_2} \) is supported on the diagonal \( z_1 = z_2 \) if

\[
\langle L(z_1), M(z_2) \rangle_u \quad \sum_{0 \leq n, m \leq \infty} \left( u_{z_1, z_2}^{(n,m)} \right) \frac{\partial^n L}{\partial z^n} \left( \frac{\partial^m M}{\partial z^m} \right)
\]

where \( u_{z_1, z_2}^{(n,m)} \) are univariate linear functionals, i.e., we are dealing with a Sobolev sesquilinear form.

A particularly relevant example is

\[
\langle L(z), M(z) \rangle_u = \langle u_z, L(z) \overline{M(z)} \rangle
\]

that when \( \text{supp} u \subset \mathbb{T} \) gives \( g_{n,m} = \langle u_z, z^{n-m} \rangle \), which happens to be a Toeplitz matrix, \( g_{n,m} = g_{n+1,m+1} \). We will refer to this case as the Toeplitz case.

Following [19,20], we will use the CMV basis \( \{\chi^{(0)}, \chi^{(1)}, \chi^{(2)}, \ldots\} \) with \( \chi^{(1)}(z) = \begin{cases} z^k, & l = 2k, \\ z^{-k-1}, & l = 2k + 1. \end{cases} \)

**Definition 4.** Let us consider

\[
\begin{align*}
\chi_1(z) & := [1, 0, z, 0, z^2, 0, \ldots]^T, \\
\chi_2(z) & := [0, 1, 0, z, 0, z^2, 0, \ldots]^T, \\
\chi_1^*(z) & := z^{-1}\chi_1(z^{-1}) = [z^{-1}, 0, z^{-2}, 0, z^{-3}, 0, \ldots]^T, \\
\chi_2^*(z) & := z^{-1}\chi_2(z^{-1}) = [0, z^{-1}, 0, z^{-2}, 0, z^{-3}, 0, \ldots]^T.
\end{align*}
\]

In terms of which we define the CMV sequences

\[
\chi(z) := \chi_1(z) + \chi_2^*(z) = [1, z^{-1}, z, z^{-2}, \ldots]^T, \\
\chi^*(z) := \chi_1^*(z) + \chi_2(z) = [z^{-1}, 1, z^{-2}, z, z^{-3}, z^2, \ldots]^T.
\]

We also consider the semi-infinite matrix

\[
\gamma := \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix}.
\]

Given a bivariate linear functional \( u_{z_1, z_2} \) and the associated sesquilinear form we consider the corresponding Gram matrix

\[
G = \langle \chi(z_1), \chi(z_2)^\dagger \rangle_u = \langle u_{z_1, z_2}, \chi(z_1) \otimes (\chi(z_2))^\dagger \rangle.
\]

For the Toeplitz scenario this Gram matrix has the following moment matrix form \( G = \langle u_z, \chi(z)(\chi(z))^\dagger \rangle \), \( \text{supp}(u) \subset \mathbb{T} \). If \( u_z \) is a real functional

**Proposition 1.** The semi-infinite matrix \( \gamma \), is unitary \( \gamma^\dagger = \gamma^{-1} \), and has the important spectral properties \( \gamma \chi(z) = z\chi(z) \) and \( \gamma^{-1} \chi(z) = z^{-1} \chi(z) \).

For the truncation of semi-infinity matrices we will use the following notation \( A = \begin{bmatrix} A_{0,0} & A_{0,1} & \ldots & A_{0,1-l} \\
A_{1,0} & A_{1,1} & \ldots & A_{1,1-l} \\
\vdots & \vdots & \ddots & \vdots \\
A_{l-1,0} & A_{l-1,1} & \ldots & A_{l-1,1-l} \end{bmatrix} \) and also, for the corresponding block structure, we will write \( A = \begin{bmatrix} A^{[l]} \\
A^{[l, l+1]} \\
A^{[l, \geq l+1]} \end{bmatrix} \).
In this paper we assume that the Gram matrix $G$ is quasidefinite, i.e., all its principal minors are not zero, so that the following Gauss–Borel $LU$ factorization of $G$ holds
\begin{equation}
G = S_1^{-1}H(S_2^{-1})^\dagger, 
\end{equation}
where $S_1$ and $S_2$ are lower unitriangular matrices and $H$ is a diagonal matrix with no zeros at the diagonal.

**Definition 5.** Let us introduce the following vectors of Laurent polynomials
\begin{equation}
\phi_1(z) := S_1\chi(z), \quad \phi_2(z) := S_2\chi(z).
\end{equation}
Its components $\phi_1(z) = [\phi_{1,0}(z), \phi_{1,1}(z), \ldots]^\top$ and $\phi_2(z) = [\phi_{2,0}(z), \phi_{2,1}(z), \ldots]^\top$ are such that

**Proposition 2** (Biorthogonal polynomials). The following biorthogonality conditions hold
\begin{equation}
\langle \phi_{1,n}(z_1), \phi_{2,m}(z_2) \rangle_u = \delta_{n,m}H_n, \quad n, m \in \{0, 1, 2, \ldots\}
\end{equation}

**Corollary 1.** The orthogonality relations are satisfied
\begin{align*}
\langle \phi_{1,2k}(z_1), (z_2)^l \rangle_u &= 0, \quad -k \leq l \leq k - 1, \\
\langle \phi_{1,2k+1}(z_1), (z_2)^l \rangle_u &= 0, \quad -k \leq l \leq k, \\
\langle (z_1)^l, \phi_{2,2k}(z_2) \rangle_u &= 0, \quad -k \leq l \leq k - 1, \\
\langle (z_1)^l, \phi_{2,2k+1}(z_2) \rangle_u &= 0, \quad -k \leq l \leq k.
\end{align*}

**Definition 6.** The Christoffel–Darboux kernel is
\begin{equation}
K^{[l]}(z_1, z_2) := \sum_{k=0}^{l-1} \frac{\phi_{2,k}(z_1)}{H_k} \phi_{1,k}(z_2) = [\phi_{2}(z_1)]^\dagger \{H^{-1}\}^{[l]}[\phi_{1}(z_2)]^{[l]}.
\end{equation}

**Proposition 3.** The Christoffel–Darboux kernel satisfies the ABC theorem: $K^{[l]}(z_1, z_2) = (\chi^{[l]}(z_1))^\dagger (G^{[l]})^{-1} \chi^{[l]}(z_2)$.

**Proposition 4.** The Christoffel–Darboux kernel satisfies the projection properties
\begin{equation}
\langle \sum_{j=0}^{M} f_j \phi_{1,j}(z_1), K^{[l]}(z_2, z) \rangle_u = \sum_{j=0}^{l-1} f_j \phi_{1,j}(z), \quad \langle K^{[l]}(z, z_1), \sum_{j=0}^{M} f_j \phi_{2,j}(z_2) \rangle_u = \sum_{j=0}^{l-1} f_j \phi_{2,j}(z).
\end{equation}

**Proof.** If we have an expansion of the form $\sum_{j=0}^{M} f_j \phi_{1,j}(z)$ then
\begin{equation}
\langle \sum_{j=0}^{M} f_j \phi_{1,j}(z_1), K^{[l]}(z_2, z) \rangle_u = \sum_{j=0}^{M} f_j \phi_{1,j}(z_1), \sum_{k=0}^{l-1} \phi_{2,k}(z_2)H_k^{-1}\phi_{1,k}(z)\rangle_u = \sum_{j=0}^{l-1} f_j \phi_{1,j}(z).
\end{equation}

Analogously,
\begin{equation}
\langle K^{[l]}(z, z_1), \sum_{j=0}^{M} f_j \phi_{2,j}(z_2) \rangle_u = \sum_{k=0}^{l-1} \phi_{2,k}(z)H_k^{-1}\phi_{1,k}(z_1)\sum_{j=0}^{M} f_j \phi_{2,j}(z_2)\rangle_u = \sum_{j=0}^{l-1} f_j \phi_{2,j}(z).
\end{equation}

This simply says that, when acting on the right $K^{[l+1]}(z_2, z)$ projects over $\Lambda_1 := \mathbb{C}\chi^{[k]}(z)_k=0$ while when acting on the left $K^{[l+1]}(z, z_1)$ projects on $\bar{\Lambda}_1$. Notice that $\Lambda_{2k} = \{1, z^{-1}, z, \ldots, z^{-k}, z^k\}$ and $\Lambda_{2k+1} = \{1, z^{-1}, z, \ldots, z^k, z^{-k-1}\}$. 

\[\square\]
Corollary 2. If \( L(z) \in \Lambda_1 := \mathbb{C}[x^{(k)}(z)]_{k=0}^{-1} \) then \( \left\langle L(z_1), K^{(l)}(z_2, z) \right\rangle_u = L(z) \) and \( \left\langle K^{(l)}(z, z_1), L(z_2) \right\rangle_u = L(z) \).

Definition 7. The Gram partial second kind functions are defined by

\[
\begin{align*}
C_{1,1}(z) & := H(S_2^{-1})^\dagger \chi_1^\dagger (z), & C_{1,2}(z) & := H(S_2^{-1})^\dagger \chi_2(z), \\
(C_{2,1}(z))^\dagger & := (\chi_1^\dagger (z))^\dagger (S_1)^{-1}H, & (C_{2,2}(z))^\dagger & := (\chi_2(z))^\dagger (S_1)^{-1}H.
\end{align*}
\]

Proposition 5. The following relations hold true

\[
\begin{align*}
C_{1,1}(z) & = \left\langle \phi_1(z_1), \frac{1}{z - z_2} \right\rangle_u, & |z| > \sup_{z_2 \in \text{supp}_2 u} |z_2|, \\
C_{1,2}(z) & = -\left\langle \phi_1(z_1), \frac{1}{z - z_2} \right\rangle_u, & |z| < \inf_{z_2 \in \text{supp}_2 u} |z_2|, \\
(C_{2,1}(z))^\dagger & = \left\langle \frac{1}{z - z_1}, (\phi_2(z_2))^\dagger \right\rangle_u, & |z| > \sup_{z_1 \in \text{supp}_1 u} |z_1|, \\
(C_{2,2}(z))^\dagger & = -\left\langle \frac{1}{z - z_1}, (\phi_2(z_2))^\dagger \right\rangle_u, & |z| < \inf_{z_1 \in \text{supp}_1 u} |z_1|,
\end{align*}
\]

in the indicated disks, which are the domains of definition of these functions.

Proof. We can write

\[
\begin{align*}
C_{1,1}(z) & = S_1 G \chi_1^\dagger (z) = \left\langle \phi_1(z_1), (\chi(z_2))^\dagger \right\rangle_u \chi_1^\dagger (z), \\
C_{1,2}(z) & = S_1 G \chi_2(z) = \left\langle \phi_1(z_1), (\chi(z_2))^\dagger \right\rangle_u \chi_2(z), \\
(C_{2,1}(z))^\dagger & = (\chi_1^\dagger (z))^\dagger G(S_2)^\dagger = (\chi_1^\dagger (z))^\dagger \left\langle \chi(z_1), (\phi_2(z_2))^\dagger \right\rangle_u, \\
(C_{2,2}(z))^\dagger & = (\chi_2(z))^\dagger G(S_2)^\dagger = (\chi_2(z))^\dagger \left\langle \chi(z_1), (\phi_2(z_2))^\dagger \right\rangle_u,
\end{align*}
\]

but form the following uniform convergence

\[
\begin{align*}
(\chi(z_2))^\dagger \chi_1^\dagger (z) & = \frac{1}{z - z_2}, & |z| > |z_2|, & (\chi(z_2))^\dagger \chi_2(z) & = -\frac{1}{z - z_2}, & |z| < |z_2|, \\
(\chi_1^\dagger (z))^\dagger \chi_1(z) & = \frac{1}{z - z_1}, & |z| > |z_1|, & (\chi_2(z))^\dagger \chi_1(z) & = -\frac{1}{z - z_1}, & |z| < |z_1|,
\end{align*}
\]

and the continuity of the sesquilinear forms we conclude the result.

Using this result we extend the domain of definition of the Gram functions of the second kind to:

Definition 8. The Cauchy second kind functions are given by

\[
\begin{align*}
C_1(z) & = \left\langle \phi_1(z_1), \frac{1}{z - z_2} \right\rangle_u = \left\langle u_{z_1,z_2}, \phi_1(z_1) \otimes \frac{1}{z - z_2} \right\rangle_u, & z \notin \text{supp}_2 u, \\
(C_2(z))^\dagger & = \left\langle \frac{1}{z - z_1}, (\phi_2(z_2))^\dagger \right\rangle_u = \left\langle u_{z_1,z_2}, \frac{1}{z - z_1} \otimes (\phi_2(z_2))^\dagger \right\rangle_u, & z \notin \text{supp}_1 u.
\end{align*}
\]

Definition 9. The mixed Christoffel–Darboux kernels are

\[
\begin{align*}
K_{C,\phi}^{(l)}(z_1, z_2) & := \sum_{k=0}^{l-1} C_{2,k}(z_1) H_k^{-1} \phi_1, k(z_2) = [C_{2}(z_1))^\dagger [H^{-1})^{\dagger} ([\phi_1(z_2)]^{\dagger}), & z_1 \notin \text{supp}_1 u, \\
K_{\phi,C}^{(l)}(z_1, z_2) & := \sum_{k=0}^{l-1} \phi_2, k(z_1) H_k^{-1} C_{1,k}(z_2) = [\phi_2(z_1))^\dagger [H^{-1})^{\dagger} ([C_1(z_2)]^{\dagger]), & z_2 \notin \text{supp}_2 u.
\end{align*}
\]
Proposition 6. The mixed kernels have the following expressions

\[ K^{[1]}_{C,\phi}(\bar{x}_1, x_2) = \left\langle \frac{1}{\bar{x}_1 - z_1}, K^{[1]}_{\bar{x}_1}(z_2, x_2) \right\rangle_u, \]
\[ K^{[1]}_{\phi,C}(\bar{x}_1, x_2) := \left\langle K^{[1]}_{\bar{x}_1}(z_1), \frac{1}{\bar{x}_2 - z_2} \right\rangle_u. \]

Proof. We recall that \( C_{2,k}(x_1) = \left\langle \frac{1}{x_1 - z_1}, \phi_{2,k}(z_2) \right\rangle \) for \( \bar{x}_1 \not\in \text{supp}_1(u) \), and \( C_{1,k}(x_2) = \left\langle \phi_{1,k}(z_1), \frac{1}{x_2 - z_2} \right\rangle_u \) for \( x_2 \not\in \text{supp}_2(u) \), so that

\[ K^{[1]}_{C,\phi}(\bar{x}_1, x_2) = \sum_{k=0}^{l-1} C_{2,k}(x_1) H_k^{-1} \phi_{1,k}(x_2) = \left\langle \frac{1}{\bar{x}_1 - z_1}, \sum_{k=0}^{l-1} \phi_{2,k}(z_2) H_k^{-1} \phi_{1,k}(x_2) \right\rangle_u, \]
\[ K^{[1]}_{\phi,C}(\bar{x}_1, x_2) = \sum_{k=0}^{l-1} \phi_{2,k}(x_1) H_k^{-1} C_{1,k}(x_2) = \sum_{k=0}^{l-1} \phi_{2,k}(x_1) H_k^{-1} \left\langle \phi_{1,k}(z_1), \frac{1}{\bar{x}_2 - z_2} \right\rangle_u. \]

Hence, the mixed kernels can be thought as the projections of the Cauchy kernels or, equivalently, the Cauchy transforms of the Christoffel–Darboux kernels.

Remark 1. In what follows, we will consider several types of supports that belong to curves in the plane. As examples, we consider curves \( \gamma := \{ z \in \mathbb{C} : \bar{z} = \tau(z) \} \) for some function \( \tau(z) \), here we give some examples

i) If \( \tau(z) = z \) we get that \( \gamma = \mathbb{R} \) and we recover the theory of biorthogonal Laurent polynomials on the real line.

ii) When \( \tau(z) = z^{-1} \) we have that \( \gamma = \mathbb{T} \). This leads to biorthogonal polynomials on the unit circle.

iii) The Cassini oval is the locus of points in the complex plane such that the product of the distance to the two foci remains constant. Let the foci be \((0, a)\) and \((-\alpha, 0)\), where \( a \) is a positive number, then the Cassini oval equation is \( |z - a||z + a| = b^2 \) where \( b \) is a positive number, so that \( \bar{z}^2 = a^2 + \frac{b^4}{z^2 - a^2} \). This implies that \( z = \tau(z) \) with \( \tau(z) := \sqrt{\frac{a^2 z^2 + b^4 - a^4}{z^2 - a^2}} \). Here we must take a determination of the square root, so that we choose the branch to be at the semiaxis \((-\infty, 0)\). The Cassini oval has three different shapes depending on \( a \leq b \): If \( b > a \) we have a closed loop, enclosing the two foci, that cuts the \( x \)-axis at \( \pm \sqrt{a^2 + b^2} \). In this case the function \( \tau(z) \) is analytic but for a branch cut at the imaginary axis of the form \(-\sqrt{\frac{b^4 - a^4}{a^2} i, \sqrt{\frac{b^4 - a^4}{a^2} i}}\). When \( b < a \) we have two closed loops each of which contains one focus. Now the function \( \tau(z) \) is analytic but for a branch cut at \( -\sqrt{\frac{a^4 - b^4}{a^2} i, \sqrt{\frac{a^4 - b^4}{a^2} i}}\).

Finally, for \( a = b \) we have the Bernoulli lemniscate, having the shape of an eight with a double point at the origin, in this case \( \tau(z) \) is analytic at \( \mathbb{C}^* \).

iv) The cardioid is the locus of points in the complex plane that satisfy \((zz - a^2)^2 - 4a^2(z - a)(z - \bar{a}) = 0\), for \( a > 0 \), that is \((\bar{z} - a^2z - 2a)^2 = 4a^3z^2 - 4a^3z^3 \). Therefore, \( \bar{z} = \tau(z) \) with \( \tau(z) := \left(2a^3/2(a - z)^{3/2} + 4a^3(z - 2a)\right)z^{-2} \). This is an analytic function in \( \mathbb{C}^* \) but for \( z \) close to \( z = a \) of the cardioid given by \( e^{\pi/3}(a, +\infty), (a, +\infty), e^{-\pi/3}(a, +\infty) \), and a double pole at the origin \( z = 0 \).

2. Christoffel transformations

We consider two possible types of perturbations of the sesquilinear form, the first is the multiplication by a Laurent polynomial \( L^{[1]}(z_1) \) acting on the first variable. Secondly, we also consider a perturbation but now we multiply by \( L^{[2]}(z_2) \), when the non perturbed bivariate linear functional supports lay on the curve \( z_2 = \tau(z_2) \), this will be a Laurent polynomial on the variable \( \tau(z_2) \).

Definition 10. Given two Laurent polynomials \( L^{(a)}(z) = L_n^{(a)} z^n + \cdots + L_1^{(a)} z - M, \) with \( L_n^{(a)} L_m^{(a)} \neq 0 \) and \( n, m \in \{0, 1, 2, \ldots\}, a \in \{1, 2\} \), let us consider the following two perturbations of the bivariate linear functional

\[ \hat{u}^{(1)}_{z_1, z_2} = L^{(1)}(z_1) u_{z_1, z_2}, \]
\[ \hat{u}^{(2)}_{z_1, z_2} = u_{z_1, z_2} L^{(2)}(z_2). \]
Proposition 7. Christoffel transformations associated with the Laurent polynomials \( L^{(1)}(z) \) and \( L^{(2)}(z) \) imply for the corresponding Gram matrices

\[
\hat{G}^{(1)} = L^{(1)}(\gamma)G, \quad \hat{G}^{(2)} = G(L^{(2)}(\gamma))^\dagger.
\]

Proof. Just follow the following chains of equalities

\[
\hat{G}^{(1)} = \left<u_{z_1,z_2}, \kappa(z_1) \otimes (\kappa(z_2))^\dagger\right>
= \left<u_{z_1,z_2}, L^{(1)}(\kappa(z_1) \otimes (\kappa(z_2))^\dagger\right>
= L^{(1)}(\gamma) \left<u_{z_1,z_2}, \kappa(z_1) \otimes (\kappa(z_2))^\dagger\right>,
\]

\[
\hat{G}^{(2)} = \left<u_{z_1,z_2}, \kappa(z_1) \otimes (\kappa(z_2))^\dagger\right>
= \left<u_{z_1,z_2}, \kappa(z_1) \otimes (\kappa(z_2))^\dagger\right>(L^{(2)}(\gamma))^\dagger.
\]

We assume that the two Gram matrices are quasidefinite, so that the Gauss–Borel factorization is ensured in both cases, i.e.

\[
\hat{G}^{(1)} = (\hat{S}_1^{(1)})^{-1} \hat{H}^{(1)}(\hat{S}_2^{(1)})^{-\dagger}, \quad \hat{G}^{(2)} = (\hat{S}_1^{(2)})^{-1} \hat{H}^{(2)}(\hat{S}_2^{(2)})^{-\dagger}.
\]

2.1. Connection formulas.

Definition 11 (Christoffel connectors). Let us consider a bivariate linear functional and the two Christoffel transformations \( \Pi_{10} \) and \( \Pi_{11} \) associated with the Laurent polynomials \( L^{(a)}(z) \), \( a \in \{1, 2\} \). The connectors are the following semi-infinite matrices

\[
\omega^{(1)} = \hat{S}_1^{(1)} L^{(1)}(\gamma)(S_1)^{-1}, \quad \omega^{(2)} = S_2(S_2)^{-1},
\]

\[
\omega^{(1)} = S_1(\hat{S}_1^{(2)})^{-1}, \quad \omega^{(2)} = S_1(\hat{S}_1^{(2)})^{-1}.
\]

Proposition 8. The connectors fulfill the ligatures \( \hat{H}^{(1)}(\omega^{(2)})^{-\dagger} = \omega^{(1)} H \) and \( \omega^{(2)} \hat{H}^{(2)} = H(\omega^{(2)})^{-\dagger} \).

Proof. From (12) and (13) we get

\[
(\hat{S}_1^{(1)})^{-1} \hat{H}^{(1)}(\hat{S}_2^{(1)})^{-\dagger} = L^{(1)}(\gamma)(S_1)^{-1} H(S_2)^{-\dagger}, \quad (\hat{S}_1^{(2)})^{-1} \hat{H}^{(2)}(\hat{S}_2^{(2)})^{-\dagger} = (S_1)^{-1} H(S_2)^{-\dagger} L^{(2)}(\gamma),
\]

and, consequently, we conclude

\[
\hat{H}^{(1)}(\hat{S}_2^{(1)})^{-\dagger} S_1^{(2)} = (S_1)^{-1} H(S_2)^{-\dagger} (L^{(2)}(\gamma))^\dagger (\hat{S}_2^{(2)})^\dagger,
\]

\[
S_1(\hat{S}_1^{(2)})^{-1} \hat{H}^{(2)} = H(S_2)^{-\dagger} (L^{(2)}(\gamma))^\dagger (\hat{S}_2^{(2)})^\dagger.
\]

Proposition 9. We will use the notation \( \bar{n} \) := \( \max(n, m) \).

i) The connector \( \omega^{(1)} \) is an upper triangular matrix with only \( 2\bar{n} + 1 \) nonzero upper diagonals.

ii) The connector \( \omega^{(2)} \) is a lower unitriangular matrix with only \( 2\bar{n} + 1 \) nonzero lower diagonals.

iii) The connector \( \omega^{(2)} \) is a lower unitriangular matrix with only \( 2\bar{n} + 1 \) nonzero lower diagonals.

iv) The connector \( \omega^{(2)} \) is an upper triangular matrix with only \( 2\bar{n} + 1 \) nonzero upper diagonals.

The connectors are banded semi-infinite matrices that give the relationship between the perturbed and the original biorthogonal families.
Proposition 10 (Connection formulas for the CMV biorthogonal Laurent polynomials). We have the following connection formulas

\[
\omega_1^{(1)} \phi_1(z) = L^{(1)}(z) \hat{\phi}_1^{(1)}(z), \quad \omega_2^{(1)} \hat{\phi}_2^{(1)}(z) = \phi_2(z), \\
\omega_1^{(2)} \phi_1^{(2)}(z) = \phi_1(z), \quad \omega_2^{(2)} \phi_2(z) = L^{(2)}(z) \hat{\phi}_2^{(2)}(z).
\]

Proposition 11 (Connection formulas for the Christoffel–Darboux kernels). For \( l \geq 2n \), the perturbed and original Christoffel–Darboux kernels satisfy

\[
K^{[l]}(z_1, z_2) = L^{[l]}(z_1) (\hat{\phi}^{[l]}(z_1))^\dagger (\hat{\phi}^{[l]}(z_2) - \hat{\phi}^{[l]}(z_1)) \text{diag}(\hat{H}^{(l)}_{1-2n} - 1, \ldots, (\hat{H}^{(1)}_{1-1})^{-1})
\]

(14)

\[
K^{[l]}(z_1, z_2) = L^{[l]}(z_1) (\hat{\phi}^{[l]}(z_1))^\dagger (\hat{\phi}^{[l]}(z_2) - \hat{\phi}^{[l]}(z_1)) \text{diag}(\hat{H}^{(l)}_{1-2n} - 1, \ldots, (\hat{H}^{(1)}_{1-1})^{-1})
\]

(15)

Proof. See Appendix. \(\square\)

2.2. Christoffel formulas. We now require the perturbing polynomials to be of a particular form. Féjer [29] and Riesz [58] found a representation for nonnegative trigonometric polynomials. Any nonnegative trigonometric polynomial \( f(\theta) = a_0 + \sum_{k=1}^{n} (a_k \cos(k\theta) + b_k \sin(k\theta)) \) has the form \( f(\theta) = \|p(z)\|^2 \) where \( p(z) = \sum_{i=0}^{n} p_i z^i \) and \( z = e^{i\theta} \). Following Grenader and Szegő, see 1.12 in [42], this is equivalent to write \( f(\theta) = z^{-n} P(z) \) with \( z = e^{i\theta} \) and \( P(z) \in \mathbb{C}[z] \) is a polynomial with deg \( P(z) = 2n \) such that \( P(z) = P^*(z) \), here we have used the Szegő reciprocal polynomial \( P^*(z) := z^{2n} \overline{P(z^{-1})} \) of \( P(z) \), fulfilling \( z^{-n} P(z) = \|P(z)\| \) for \( z \in \mathbb{T} \). Observe that for \( z \in \mathbb{C}^* \) the function \( L(z) = z^{-n} P(z) \) is not any more a trigonometric polynomial but a Laurent polynomial. These Laurent polynomials are precisely those that as perturbations allow us to find Christoffel formulas. Given a Laurent polynomial its reciprocal is given by \( L_0(z) := L(z^{-1}) \), thus for \( L(z) = z^{-n} P(z) \) we have \( L_0(z) = z^n \overline{P(z^{-1})} = z^{-n} P^*(z) \) and if \( P(z) = P^*(z) \) we find \( L_0(z) = L(z) \); the positivity condition reads: \( f(\theta) = L(z) \) with \( L(z) \) a Laurent polynomial with \( L(z) = L_0(z) \) and \( L(z) = |L(z)| \) for \( z \in \mathbb{T} \).

Definition 12 (Prepared Laurent polynomials). For every \( 2n \)-degree polynomial \( P(z) = P_{2n} z^{2n} + \cdots + P_0 \in \mathbb{C}[z] \) with \( P_0 \neq 0 \), its Féjer–Riesz corresponding Laurent polynomial is given by

\[
L(z) = z^{-n} P(z) = L_n z^n + \cdots + L_{-n} z^{-n}, \quad L_n = P_{2n}, \quad L_{-n} = P_0.
\]

We say that a Laurent polynomial is prepared whenever it is the Féjer–Riesz corresponding Laurent polynomials of an even degree polynomial non vanishing at the origin.

For the consideration of arbitrary multiplicities of the zeros of the perturbing polynomials we need of
Definition 13 (Spectral jets). Given a Laurent polynomial \( L(z) \) with zeros and multiplicities \( \{ \zeta_i, m_i \}_{i=1}^d \) we introduce the spectral jet of a function \( f(z) \) along \( L(z) \) as follows:

\[
\hat{J}^1_f := \left[ f(\zeta_1), f'(\zeta_1), \ldots, \frac{f^{(m_1-1)}(\zeta_1)}{(m_1 - 1)!}, \ldots, f(\zeta_2), f'(\zeta_2), \ldots, \frac{f^{(m_d-1)}(\zeta_d)}{(m_d - 1)!} \right] \in \mathbb{C}^{2m}.
\]

The spectral jet \( \hat{J}^1_{K[1]}(\zeta_1) \) of the kernel \( K[1](\zeta_1, \zeta_2) \) is taken with respect to the variable \( \zeta_2 \) leaving the variable \( \zeta_1 \) as a fixed parameter. However, the spectral jet \( \hat{J}^1_{K[n]}(\zeta_2) \) is taken with respect to the variable \( \zeta_1 \) leaving the variable \( \zeta_2 \) as fixed parameter.

Remark 2. For prepared Laurent polynomials we have that \( (\omega_1^{(1)}), k, k + 2n = L^{(1)}_{(-1)^k n} \) and \( (\omega_2^{(2)}), k, k + 2n = L^{(2)}_{(-1)^k n} \).

Theorem 1 (Christoffel formulas). Given a prepared Laurent polynomial \( L^{(1)}(z) \), let us suppose that \( \Phi^{(1)}_1 := \begin{vmatrix} \frac{\partial L^{(1)}_{\Phi_1}}{\partial \zeta_1} \\ 1 \end{vmatrix} \Phi^{(1)}_1(z) \neq 0 \) and \( l \geq 2m \). Then, given the Christoffel transformation \( [10] \), the perturbed biortogonal polynomials are expressed in terms of the original ones according to the following formulas

\[
\begin{align*}
\Phi^{(1)}_{1,1}(z) &= \frac{L^{(1)}_{(z)}}{L^{(1)}(z)} \Theta \begin{bmatrix} \frac{\partial L^{(1)}_{\Phi_1}}{\partial \zeta_1} & \Phi^{(1)}_1(z) \\ \vdots & \vdots \\ \frac{\partial L^{(1)}_{\Phi_{1,1+2n}}}{\partial \zeta_1} & \Phi^{(1)}_{1,1+2n}(z) \end{bmatrix} = \frac{L^{(1)}_{(z)}}{L^{(1)}(z)} \Phi^{(1)}_1(z), \\
\hat{H}^{(1)}_l &= L^{(1)}_{(-1)^l n} H_l \Theta \text{ with } \Phi^{(1)}_{1,1}(z), \\
\hat{\Phi}^{(1)}_{2,1} &= \frac{\hat{H}^{(1)}_l}{L^{(1)}_{(-1)^l n}} \Theta \begin{bmatrix} \frac{\partial L^{(1)}_{\Phi_{2,1}}}{\partial \zeta_1} & \Phi^{(1)}_{2,1}(z) \\ \vdots & \vdots \\ \frac{\partial L^{(1)}_{\Phi_{2,1,1+2n}}}{\partial \zeta_1} & \Phi^{(1)}_{2,1,1+2n}(z) \end{bmatrix} = \frac{L^{(1)}_{(z)}}{L^{(1)}(z)} \hat{\Phi}^{(1)}_{2,1}.
\end{align*}
\]

Given a prepared Laurent polynomial \( L^{(2)}(z) \), for the Christoffel transformation \( [11] \), whenever \( \Phi^{(2)}_{1,1}(z), \) we find

\[
\begin{align*}
\Phi^{(2)}_{2,1}(z) &= \frac{L^{(2)}_{(-1)^l n}}{L^{(2)}(z)} \Theta \begin{bmatrix} \frac{\partial L^{(2)}_{\Phi_{2,1}}}{\partial \zeta_1} & \Phi^{(2)}_{2,1}(z) \\ \vdots & \vdots \\ \frac{\partial L^{(2)}_{\Phi_{2,1,1+2n}}}{\partial \zeta_1} & \Phi^{(2)}_{2,1,1+2n}(z) \end{bmatrix} = \frac{L^{(2)}_{(z)}}{L^{(2)}(z)} \Phi^{(2)}_{2,1}, \\
\hat{H}^{(2)}_l &= L^{(2)}_{(-1)^l n} \hat{H}_l \Theta \text{ with } \Phi^{(2)}_{2,1}(z).
\end{align*}
\]
Reductions to univariate linear functionals supported on the unit circle.

\[
\phi^{(2)}_{1,1}(z) = \frac{\hat{\phi}^{(2)}_{1,1}}{L^{(2)}_{(-1)^{1-n}}} \Theta^* \begin{bmatrix}
\partial L_{\phi_{2,1+1}}^{(2)} \\
\vdots \\
\partial L_{\phi_{2,1+2n-1}}^{(2)} \\
\partial L_{\phi_{2,1+2n}}^{(2)} \\
\partial L_{K_{(1+1)^{1-n}}}^{(2)}(z)
\end{bmatrix} \begin{bmatrix}
0 \\
\vdots \\
0 \\
1 \end{bmatrix} = -\hat{H}_1 \begin{bmatrix}
\partial L_{\phi_{2,1+1}}^{(2)} \\
\vdots \\
\partial L_{\phi_{2,1+2n-1}}^{(2)} \\
\partial L_{K_{(1+1)^{1-n}}}^{(2)}(z)
\end{bmatrix}.
\]

**Proof.** See Appendix.

We have two possible Christoffel transformations and corresponding Christoffel formulas. For the simplest diagonal situation \( (L(z), M(z))_{u_z} = \langle u_z, L(z)M(z) \rangle \) we have these two possibilities, \( \hat{u}^{(1)}_2 = L^{(1)}(z)u_z \) or \( \hat{u}^{(2)}_2 = L^{(2)}(z)u_z \). If \( \text{supp } u_z \subset \gamma \), where the curve \( \gamma = (z \in C^*: z = \tau(z)) \), we have that the perturbation of type 2 can be described as \( \hat{u}^{(2)}_2 = L^{(2)}(\tau(z))u_z \). For example, if we consider the unit circle \( \gamma = \mathbb{T} \), with \( \tau(z) = z^{-1} \), this Christoffel transformation can be thought, as \( \hat{u}^{(2)}_2 = L^{(1)}_+(z)u_z \). Hence, taking \( L^{(2)}(z) = L^{(1)}_+(z) \) we get that \( \hat{u}^{(1)}_2 = \hat{u}^{(2)}_2 = L^{(1)}(z)u_z \). In such a case, we have \( \hat{C}^{(1)} = \hat{C}^{(2)} = \hat{C} \), and \( \hat{S}^{(1)} = \hat{S}^{(2)} = \hat{S} \), and \( \hat{H}^{(1)} = \hat{H}^{(2)} = \hat{H} \), so that \( \hat{\phi}^{(1)}_1(z) = \hat{\phi}^{(2)}_1(z) = \hat{\phi}_1(z) \) and \( \hat{\phi}^{(1)}_2(z) = \hat{\phi}^{(2)}_2(z) = \hat{\phi}_2(z) \) and we find two alternative forms for the Christoffel formulas. Realize also that if \( P(z) = P_{2n} \prod_{i=1}^{d} (z - \zeta_i)^{m_i} \) then \( L(z) = L_{n} z^{-n} \prod_{i=1}^{d} (z - \zeta_i)^{m_i}, \zeta_i \in \mathbb{C} \setminus \{0\} \), where \( L_{-n} = L_{n} \prod_{i=1}^{d} \zeta_i^{m_i} \) and \( 2n = \sum_{i=1}^{d} m_i \). A prepared Laurent polynomial \( L(z) = L_{n} z^n + \cdots + L_{-n} z^{-n} \) is such that \( L(z) \Lambda_k \subset \Lambda_{k+2n} \) and \( L(z) \Lambda_k \not\subset \Lambda_{k+2n-1} \).

### 2.3 Reductions to univariate linear functionals supported on the unit circle.

**Proposition 12.** For a sesquilinear form given by an univariate linear functional and supported on the unit circle \( \mathbb{T} \), for \( l \geq 2n \), we have the following alternative Christoffel formulas, whenever the involved quasideterminants make any sense,

\[
\hat{\phi}_{1,1}(z) = \frac{L^{(1)}_{(-1)^{1-n}}}{L(z)} \Theta^* \begin{bmatrix}
\partial L_{\phi_{2,1}}^{(1)} \\
\vdots \\
\partial L_{\phi_{2,1+2n}}^{(1)} \\
\partial L_{K_{(1+1)^{1-n}}}^{(1)}(z)
\end{bmatrix} \begin{bmatrix}
\phi_{1,1}(z) \\
\vdots \\
\phi_{1,1+2n}(z) \end{bmatrix} = -\hat{H}_1 \begin{bmatrix}
\partial L_{\phi_{2,1}}^{(1)} \\
\vdots \\
\partial L_{\phi_{2,1+2n}}^{(1)} \\
\partial L_{K_{(1+1)^{1-n}}}^{(1)}(z)
\end{bmatrix}.
\]

\[
\hat{\phi}_{2,1}(z) = \frac{\hat{H}_1}{L_{(-1)^{1-n}}} \Theta^* \begin{bmatrix}
\partial L_{\phi_{2,1}}^{(1)} \\
\vdots \\
\partial L_{\phi_{2,1+2n}}^{(1)} \\
\partial L_{K_{(1+1)^{1-n}}}^{(1)}(z)
\end{bmatrix} \begin{bmatrix}
\phi_{2,1}(z) \\
\vdots \\
\phi_{2,1+2n}(z) \end{bmatrix} = \frac{L_{(-1)^{1-n}}}{L(z)} \Theta^* \begin{bmatrix}
\partial L_{\phi_{2,1}}^{(1)} \\
\vdots \\
\partial L_{\phi_{2,1+2n}}^{(1)} \\
\partial L_{K_{(1+1)^{1-n}}}^{(1)}(z)
\end{bmatrix}.
\]

However, this identification does not hold in general, in which case we have two different Christoffel transformations —multiplying by a Laurent polynomial \( L^{(1)}(z) \), or multiplying by a Laurent polynomial composed with the supporting curve function \( \tau(z) \), \( L^{(2)}(\tau(z)) \)— for example, multiplying by \( L^{(2)}(\sqrt{a^2 + \frac{b^2}{z^2 - a^2}}) \) for the Cassini oval and \( L^{(2)}(2a^{3/2}/\sqrt{(a - z)^3 + a^2z - 2a}) \) for the cardioid. If the original functional is nonnegative and biorthogonality simplifies to orthogonality with \( \phi_{1,1}(z) = \phi_{2,1}(z) =: \phi(z) \) we could consider a Christoffel transformation.
with a prepared polynomial \( L(z) \) which is equal to its reciprocal, \( L_\ast(z) = L(z) \), and such that \( L(z) = |L(z)| \) for \( |z| \in \mathbb{T} \), i.e. a nonnegative trigonometrical polynomial when evaluated over the circle, and find

\[
\hat{\phi}_1(z) = \frac{L_{(-1)^n}}{L(z)} \begin{bmatrix}
\partial_{\phi_1}^{L} & \phi_1(z) \\
\vdots & \vdots \\
\partial_{\phi_{1+2n}}^{L} & \phi_{1+2n}(z)
\end{bmatrix} = \frac{\hat{H}_l}{L_{(-1)^{l+1}n}} \begin{bmatrix}
\partial_{\phi_1}^{L} & 0 \\
\vdots & \vdots \\
\partial_{\phi_{1+2n}}^{L} & 0
\end{bmatrix},
\]

\[
\hat{H}_l = L_{(-1)^{l}n} \begin{bmatrix}
\partial_{\phi_1}^{L} & H_l \\
\partial_{\phi_{1+1}}^{L} & 0 \\
\vdots & \vdots \\
\partial_{\phi_{1+2n}}^{L} & 0
\end{bmatrix} = L_{(-1)^{l+1}n} \begin{bmatrix}
\partial_{\phi_1}^{L} & 0 \\
\partial_{\phi_{1+1}}^{L} & H_l \\
\vdots & \vdots \\
\partial_{\phi_{1+2n}}^{L} & 0
\end{bmatrix}.
\]

Notice that, in terms of \( \hat{t}_1 := \begin{bmatrix}
\partial_{\phi_1}^{L} \\
\partial_{\phi_{1+1}}^{L} \\
\vdots \\
\partial_{\phi_{1+2n-1}}^{L}
\end{bmatrix} \), the last relation gives \( \hat{H}_l = L_{(-1)^{l}n} H_l \hat{t}_1 = L_{(-1)^{l+1}n} H_l \hat{t}_1 \), which implies \( L_{(-1)^{l}n} \hat{t}_1 \hat{t}_1 = L_{(-1)^{l+1}n} \hat{t}_1 \). We also have

\[
\hat{\phi}_1(z) = \frac{L_{(-1)^n}}{L(z)} \begin{bmatrix}
\partial_{\phi_1}^{L} & \phi_1(z) \\
\vdots & \vdots \\
\partial_{\phi_{1+2n}}^{L} & \phi_{1+2n}(z)
\end{bmatrix} = H_l egin{bmatrix}
\partial_{\phi_1}^{L} & 0 \\
\partial_{\phi_1}^{L} & H_l \\
\vdots & \vdots \\
\partial_{\phi_1}^{L} & 0
\end{bmatrix}.
\]

### 3. GERONIMUS TRANSFORMATIONS

For this transformation we need a bivariate linear functional with a well-defined support.

**Definition 14.** Given Laurent polynomials \( L^{(a)}(z) = L^{(a)}_n z^n + \cdots + L^{(a)}_{-m} z^{-m} \), with \( L^{(a)}_n L^{(a)}_m \neq 0 \) and \( n, m \in \{0, 1, 2, \ldots\} \), \( a \in \{1, 2\} \), and \( \sigma(L^{(1)}(z)) \cap \text{supp}_1 u = \emptyset \), \( \sigma(L^{(2)}(z)) \cap \text{supp}_2 u = \emptyset \), we consider two possible families of Geronimus transformations \( \tilde{u}^{(1)}_{z_1, z_2} \) and \( \tilde{u}^{(2)}_{z_1, z_2} \), of the bivariate linear functional \( u_{z_1, z_2} \) characterized by

\[
L^{(1)}(z_1) \tilde{u}^{(1)}_{z_1, z_2} = u_{z_1, z_2},
\]

\[
L^{(2)}(z_2) \tilde{u}^{(2)}_{z_1, z_2} = u_{z_1, z_2}.
\]

Therefore, the perturbed bivariate linear functionals are

\[
\tilde{u}^{(1)}_{z_1, z_2} = \frac{u_{z_1, z_2}}{L^{(1)}(z_1)} + \sum_{i=1}^{d^{(1)}} \sum_{l=0}^{m^{(1)}_{i-1}} \frac{(-1)^{l}}{l!} \delta^{(1)}(z_1 - \xi^{(1)}_i) \otimes \delta^{(1)}(z_2 - \zeta^{(1)}_i),
\]

\[
\tilde{u}^{(2)}_{z_1, z_2} = \frac{u_{z_1, z_2}}{L^{(2)}(z_2)} + \sum_{i=1}^{d^{(2)}} \sum_{l=0}^{m^{(2)}_{i-1}} \frac{(-1)^{l}}{l!} \delta^{(2)}(z_1 - \xi^{(2)}_i) \otimes \delta^{(2)}(z_2 - \zeta^{(2)}_i),
\]

where \( (\xi^{(1)}_i)_z \) and \( (\xi^{(2)}_i)_z \) are univariate linear functionals.

**Proposition 13.** Geronimus transformations associated with the perturbing Laurent polynomials \( L^{(1)}(z) \) and \( L^{(2)}(z) \) imply for the corresponding Gram matrices the following relations

\[
L^{(1)}(\Gamma) \tilde{G}^{(1)} = G,
\]

\[
\tilde{G}^{(2)}(L^{(2)}(\Gamma))^\dagger = G.
\]
Proof. It follows from
\[
G = \left\langle u_{z_1, z_2}, \chi(z_1) \otimes (\chi(z_2))^{\dagger} \right\rangle
\]
\[
= \left\langle \tilde{u}_{z_1, z_2}, L^{(1)}(z_1) \chi(z_1) \otimes (\chi(z_2))^{\dagger} \right\rangle
\]
\[
= L^{(1)}(\Upsilon) \left\langle \tilde{u}_{z_1, z_2}, \chi(z_1) \otimes (\chi(z_2))^{\dagger} \right\rangle,
\]
\[
G = \left\langle \tilde{u}_{z_1, z_2}, \chi(z_1) \otimes (\chi(z_2))^{\dagger} \right\rangle
\]
\[
= \left\langle u_{z_1, z_2}, \chi(z_1) \otimes (\chi(z_2))^{\dagger} \right\rangle L^{(2)}(z_2)
\]
\[
= \left\langle u_{z_1, z_2}, \chi(z_1) \otimes (\chi(z_2))^{\dagger} \right\rangle \left( \Phi^{(2)}(\Upsilon) \right)^{\dagger}.
\]

When both Gram matrices are quasidefinite, the following Gauss–Borel factorizations do exist
\[
(\text{CMV BIORTHOGONAL LAURENT POLYNOMIALS: CHRISTOFFEL AND GERONIMUS TRANSFORMATIONS})
\]
\[
\tilde{G}^{(1)} = (\tilde{S}^{(1)}_1)^{-1} \tilde{H}^{(1)}(\tilde{S}^{(2)}_1)^{-\dagger},
\]
\[
\tilde{G}^{(2)} = (\tilde{S}^{(2)}_1)^{-1} \tilde{H}^{(2)}(\tilde{S}^{(2)}_2)^{-\dagger}.
\]

3.1. Connection formulas.

Definition 15 (Geronimus connectors). Consider a bivariate linear functional and Geronimus transformations associated with two Laurent polynomial \( L^{(a)}(z) \), \( a = 1, 2 \). We define the following connectors
\[
\Omega^{(1)}_1 = S_1 L^{(1)}(\Upsilon)(\tilde{S}_1^{(1)})^{-1}, \quad \Omega^{(2)}_1 = \tilde{S}_2^{(1)}(S_2)^{-1},
\]
\[
\Omega^{(1)}_2 = \tilde{S}_1^{(2)}(S_1)^{-1}, \quad \Omega^{(2)}_2 = S_2 L^{(2)}(\Upsilon)(\tilde{S}_2^{(2)})^{-1}.
\]

Proposition 14. Geronimus connectors satisfy
\[
\Omega^{(1)}_1 \tilde{H}^{(1)} = H(\Omega^{(1)}_2)^{\dagger} \quad \text{and} \quad \tilde{H}^{(2)}(\Omega^{(2)}_2)^{\dagger} = \Omega^{(2)}_1 H.
\]

Proof. According to (25) and (26) one has
\[
L^{(1)}(\Upsilon)(\tilde{S}^{(1)}_1)^{-1} \tilde{H}^{(1)}(\tilde{S}^{(2)}_1)^{-\dagger} = (S_1)^{-1} H(S_2)^{-\dagger},
\]
\[
(\tilde{S}^{(2)}_1)^{-1} \tilde{H}^{(2)}(\tilde{S}^{(2)}_2)^{-\dagger} L^{(2)}(\Upsilon) = (S_1)^{-1} H(S_2)^{-\dagger}.
\]

Therefore, since the factors \( S \) are lower unitriangular matrices we conclude the

Proposition 15. i) The connector \( \Omega^{(1)}_1 \) is an upper triangular matrix with only \( 2n + 1 \) nonzero upper above.

ii) The connector \( \Omega^{(2)}_1 \) is a lower unitriangular matrix with only \( 2n + 1 \) nonzero lower diagonals.

iii) The connector \( \Omega^{(1)}_2 \) is a lower unitriangular matrix with only \( 2n + 1 \) nonzero diagonals below.

iv) The connector \( \Omega^{(2)}_2 \) is an upper triangular matrix with only \( 2n + 1 \) nonzero upper diagonals.

v) We have the formulas
\[
(\Omega^{(1)}_2)_{k, k-2n} = \frac{L^{(1)}_{(-1)^k n}}{H^{(1)}_{k-2n}}, \quad (\Omega^{(2)}_1)_{k, k-2n} = \frac{L^{(2)}_{(-1)^k n}}{H^{(2)}_{k-2n}}.
\]

The connectors are matrices that establish the relationships between the perturbed and original families of biorthogonal Laurent polynomials

Proposition 16 (Connection formulas for the CMV Laurent orthogonal polynomials). The following connection formulas hold
\[
\Omega^{(1)}_1 \phi^{(1)}_1(z) = L^{(1)}(z) \phi_1(z), \quad \Omega^{(2)}_1 \phi_2(z) = \tilde{\phi}_1^{(1)}(z),
\]
\[
\Omega^{(2)}_1 \phi_1(z) = \tilde{\phi}_1^{(2)}(z), \quad \Omega^{(2)}_2 \phi_2(z) = L^{(2)}(z) \phi_2(z).
\]
Definition 16. Given a Laurent polynomial \( L(z) \) we consider \( \delta L(z_1, z_2) := \frac{L(z_1) - L(z_2)}{z_1 - z_2} \) and introduce the completely homogeneous symmetric polynomials \( h_j(z_1, z_2) := (z_1)^j + (z_1)^{j-1}z_2 + \cdots + z_1(z_2)^{j-1} + (z_2)^j \) and their duals \( h_j^*(z_1, z_2) := (z_1z_2)^{-1}h_j((z_1)^{-1}, (z_2)^{-1}) \) with \( j \in \{0, 1, 2, \ldots\} \).

Proposition 17. It is true that \( \delta L(z_1, z_2) = \sum_{j=1}^n L_j h_{j-1}(z_1, z_2) - \sum_{j=1}^m L_j h_{j+1}^*(z_1, z_2) \) and therefore the bivariate Laurent polynomial \( \delta L(z_1, z_2) \) is symmetrical and, fixing one of the variables, is a Laurent polynomial in the other variable of positive maximum degree \( n-1 \) and negative degree \( -m \).

Proof. For \( n \in \{1, 2, \ldots\} \) we deduce that \( (z_1)^j - (z_2)^j = (z_1 - z_2)h_{j-1}(z_1, z_2) \) and \( (z_1)^{-j} - (z_2)^{-j} = -(z_1 - z_2)h_{j+1}^*(z_1, z_2) \).

Proposition 18 (Connection formulas for the Cauchy second kind functions). The following connection formulas are fulfilled

\[
(C_2(z))^{\dagger}(\Omega_2^{(1)})^{\dagger} - L^{(1)}(z)(\tilde{C}_2^{(1)}(z))^{\dagger} = - \left\langle \delta L^{(1)}(z, z_1), (\tilde{C}_2^{(1)}(z_2))^{\dagger} \right\rangle_{\tilde{u}^{(1)}},
\]

\[
(\Omega_1^{(2)})_c(z) - \tilde{C}_2^{(2)}(z) \left( C_2^{(1)}(z) \right)_c = - \left\langle \phi_1^{(2)}(z_1), \delta L^{(2)}(z, z_2) \right\rangle_{\tilde{u}^{(2)}},
\]

\[
(\Omega_1^{(2)})_c(z) = C_1(z),
\]

\[
(\tilde{C}_2^{(2)}(z))^{\dagger}(\Omega_2^{(2)})^{\dagger} = (C_2(z))^{\dagger}.
\]

Proof. See Appendix.

Corollary 3. The following connection formulas

\[
(\Omega_2^{(1)})_{l,1-2n} C_{2,1-2n}(z) + \cdots + (\Omega_2^{(1)})_{l,1} C_{2,1}(z) + C_{2,1}(z) = \Omega_2^{(1)}(z) \tilde{C}_2^{(1)}(z),
\]

\[
(\Omega_1^{(2)})_{l,1-2n} C_{1,1-2n}(z) + \cdots + (\Omega_1^{(2)})_{l,1} C_{1,1}(z) + C_{1,1}(z) = \Omega_2^{(2)}(z) \tilde{C}_2^{(2)}(z),
\]

hold for \( l \geq 2n \).

Proof. It is a consequence of the orthogonality relations in Corollary [T] because they involve

\[
\left\langle \delta L^{(1)}(z, z_1), (\phi_2^{(1)}(z_2))^{\dagger} \right\rangle_{\tilde{u}^{(1)}} = \left\langle \phi_1^{(2)}(z_1), \delta L^{(2)}(z, z_2) \right\rangle_{\tilde{u}^{(2)}} = 0, \quad l \geq 2n.
\]

Definition 17. Let’s define

\[
\Omega_2^{(1)}[n, l] :=
\begin{bmatrix}
(\Omega_2^{(1)})_{l,1-2n} & (\Omega_2^{(1)})_{l,1-2n+1} & (\Omega_2^{(1)})_{l,1-2n+2} & \cdots & (\Omega_2^{(1)})_{l,1-1} \\
0 & (\Omega_2^{(1)})_{l+1,1-2n+1} & (\Omega_2^{(1)})_{l+1,1-2n+2} & \cdots & (\Omega_2^{(1)})_{l+1,1-1} \\
0 & 0 & (\Omega_2^{(1)})_{l+2,1-2n+1} & \cdots & (\Omega_2^{(1)})_{l+2,1-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (\Omega_2^{(1)})_{l+2n-1,1-1}
\end{bmatrix},
\]

\[
\Omega_1^{(2)}[n, l] :=
\begin{bmatrix}
(\Omega_1^{(2)})_{l,1-2n} & (\Omega_1^{(2)})_{l,1-2n+1} & (\Omega_1^{(2)})_{l,1-2n+2} & \cdots & (\Omega_1^{(2)})_{l,1-1} \\
0 & (\Omega_1^{(2)})_{l+1,1-2n+1} & (\Omega_1^{(2)})_{l+1,1-2n+2} & \cdots & (\Omega_1^{(2)})_{l+1,1-1} \\
0 & 0 & (\Omega_1^{(2)})_{l+2,1-2n+1} & \cdots & (\Omega_1^{(2)})_{l+2,1-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (\Omega_1^{(2)})_{l+2n-1,1-1}
\end{bmatrix},
\]

and consider \( \tilde{H}^{(1)}[n, l] := \text{diag}(\tilde{H}^{(1)}_1, \ldots, \tilde{H}^{(1)}_{l+2n-1}) \) and \( \tilde{H}^{(2)}[n, l] := \text{diag}(\tilde{H}^{(2)}_1, \ldots, \tilde{H}^{(2)}_{l+2n-1}) \).
Proposition 19 (Connection formulas for the Christoffel–Darboux kernels). For \( l \geq 2n \), the Christoffel–Darboux kernels and their Geronimus transformations satisfy

\[
\begin{align*}
\mathcal{K}^{[1]}_{1,1}(z_1, z_2) - L^{(1)}(z_2)\mathcal{K}^{[1]}_{1,1}(z_1, z_2) &= -\left[ \mathcal{H}^{(1)}(1, n, l) \right]^{-1} \Omega^{(1)}_{2 \rightarrow 1}(n, l) \\
&= -\left[ \mathcal{H}^{(1)}(1, n, l) \right]^{-1} \Omega^{(1)}_{2 \rightarrow 1}(n, l),
\end{align*}
\]

Proof. See Appendix.

Proposition 20 (Connection formulas for the mixed Christoffel–Darboux kernels). When \( l \geq 2n \), the mixed Christoffel–Darboux kernels fulfill the following connection formulas

\[
\begin{align*}
\mathcal{K}^{(1,1)}_{C,\phi}(x_1, x_2) - \mathcal{K}^{(1)}_{C,X}(x_1, x_2) &= -\left[ \mathcal{H}^{(1)}(1, n, l) \right]^{-1} \Omega^{(2)}_{1 \rightarrow 1}(n, l) \\
&= -\left[ \mathcal{H}^{(1)}(1, n, l) \right]^{-1} \Omega^{(2)}_{1 \rightarrow 1}(n, l),
\end{align*}
\]

Proof. See Appendix.

3.2. Christoffel–Geronimus formulas. The perturbed Cauchy second kind functions read as follows

\[
\begin{align*}
\mathcal{C}^{(1)}_{z,k}(z) &= \left\langle \frac{1}{z - z_1}, \mathcal{H}^{(1)}(z_1) \right\rangle_{L^{(1)}(z_1)}^{-1} + \sum_{i=1}^{d} \sum_{l=0}^{m} \frac{d^l}{l!} \frac{1}{z - z_1} \left( \frac{1}{z - l} \right)_{\zeta = \zeta^{(1)}_{i,k}} \\
\mathcal{C}^{(2)}_{z,k}(z) &= \left\langle \frac{1}{z - z_2}, \mathcal{H}^{(2)}(z_2) \right\rangle_{L^{(2)}(z_2)}^{-1} + \sum_{i=1}^{d} \sum_{l=0}^{m} \frac{d^l}{l!} \frac{1}{z - z_2} \left( \frac{1}{z - l} \right)_{\zeta = \zeta^{(2)}_{i,k}},
\end{align*}
\]

Definition 18. i) For \( a \in \{1, 2\} \) we define

\[
\langle \xi^{(a)}_l, L \rangle := \left[ \langle \xi^{(a)}_{l,1}, L \rangle, \ldots, \langle \xi^{(a)}_{l,m^{(a)}_L-1}, L \rangle \right], \quad \langle \xi^{(a)}, L \rangle := \left[ \langle \xi^{(a)}_1, L \rangle, \ldots, \langle \xi^{(a)}_{d^{(a)}_L}, L \rangle \right].
\]
ii) The expression $L(z) = L_n z^{-m} \prod_{j=1}^{d} (z - \zeta_j)^{m_j}$ with $m_1 + \cdots + m_d = n + m$, allows us to introduce $L_{i}(z) := L_n z^{-m} \prod_{\substack{j=1 \atop j \neq i}}^{d} (z - \zeta_j)^{m_j}$.

Given relations (33) and (32) let’s look at

$$\overline{L}(1)(z) \bar{C}(1)_{2,k}(z) = \left\langle \frac{L^1(z)}{z - z_1}, \phi_{2,k}^{(1)}(z) \right\rangle_{(L^1(z_1))^{-1}} + \sum_{i=1}^{d(1)} \left\langle \xi_i^{(1)}, \phi_{2,k}^{(1)} \right\rangle_{(L^1(z_1))^{-1}} \left[ \begin{array}{c} (z - \zeta_i^{(1)})^{m_i^{(1)} - 1} \\ \vdots \\ 1 \end{array} \right] \left[ \begin{array}{c} (z - \zeta_i^{(2)})^{m_i^{(2)} - 1} \\ \vdots \\ 1 \end{array} \right],$$

$$\overline{L}(2)(z) \bar{C}(2)_{1,k}(z) = \left\langle \phi_{1,k}^{(2)}(z_1), \frac{L^2(z)}{z - z_2} \right\rangle_{u(L^2(z_2))^{-1}} + \sum_{i=1}^{d(2)} \left\langle \xi_i^{(2)}, \phi_{1,k}^{(2)} \right\rangle_{L^2(z_2)} \left[ \begin{array}{c} (z - \zeta_i^{(1)})^{m_i^{(1)} - 1} \\ \vdots \\ 1 \end{array} \right] \left[ \begin{array}{c} (z - \zeta_i^{(2)})^{m_i^{(2)} - 1} \\ \vdots \\ 1 \end{array} \right].$$

We need to evaluate the spectral jets $\partial_{\overline{L}^{(1)} L_i^{(1)} C_{2,k}^{(1)}}$ and $\partial_{\overline{L}^{(1)} L_i^{(2)} C_{i,k}^{(2)}}$. We have to do it as limits since the perturbing polynomial zeros lay on the border of the perturbed functional support.

**Definition 19.** For $i \in \{1, 2\}$, $j \in \{1, \ldots, d(1)\}$, $k \in \{0, \ldots, m^{(1)}_j - 1\}$, let’s use the notation

$$\ell_{j,k}^{(1)} := \frac{1}{k!} \frac{d^k \overline{L}_j^{(1)}(z)}{dz^k} \bigg|_{z = \zeta_j^{(1)}} = \begin{bmatrix} 0 & 0 & 0 & \ell_{j,0}^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ell_{j,1}^{(1)} & \ell_{j,0}^{(1)} \ell_{j,1}^{(1)} \ell_{j,2}^{(1)} \\ \ell_{j,0}^{(1)} & \ell_{j,1}^{(1)} & \ell_{j,2}^{(1)} & \ell_{j,0}^{(1)} \ell_{j,1}^{(1)} \ell_{j,2}^{(1)} \ell_{j,3}^{(1)} \end{bmatrix}, \quad \mathcal{L}^{(1)} := \text{diag}(\mathcal{L}^{(1)}_1, \ldots, \mathcal{L}^{(1)}_{d(1)}).$$

**Proposition 21.** The spectral jets satisfy

$$(40) \quad \partial_{\overline{L}^{(1)} L_i^{(2)} C_{2,k}^{(1)}} = \left\langle \xi^{(2)}, \begin{array}{c} \phi_{1,k}^{(2)} \end{array} \right\rangle \mathcal{L}^{(2)},$$

$$(41) \quad \partial_{\overline{L}^{(1)} L_i^{(2)} C_{i,k}^{(2)}} = \left\langle \xi^{(1)}, \begin{array}{c} \phi_{2,k}^{(1)} \end{array} \right\rangle \mathcal{L}^{(1)}.$$

**Proof.** See Appendix. \qed

**Theorem 2 (Christoffel-Geronimus formulas).** Let’s assume that $n = m$ and $l \geq 2n$ and that Laurent polynomials $L^{(1)}(z)$ and $L^{(2)}(z)$ are prepared. Then, when $\ell_i^{(2)} := \left| \partial_{\overline{L}^{(1)} L_i^{(2)} C_{1,i}^{(1)}} - \left\langle \xi^{(2)}, \phi_{1,i}^{(2)} \right\rangle \mathcal{L}^{(2)} \right| \neq 0$, we have the Christoffel-Geronimus formulas

$$(42) \quad \phi_{i}^{(2)}(z) = \Theta_{*} \begin{bmatrix} \partial_{\overline{L}_{C_{1,i}^{(1)}}}^{(2)} - \left\langle \xi^{(2)}, \phi_{1,i}^{(2)} \right\rangle \mathcal{L}^{(2)} & \phi_{1,i}^{(2)}(z) \\ \vdots & \vdots \end{bmatrix} = \frac{1}{\ell_i^{(2)}} \begin{bmatrix} \partial_{\overline{L}_{C_{1,i}^{(1)}}}^{(2)} - \left\langle \xi^{(2)}, \phi_{1,i}^{(2)} \right\rangle \mathcal{L}^{(2)} & \phi_{1,i}^{(2)}(z) \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \phi_{1,i}^{(2)}(z) \\ \vdots \end{bmatrix}.$$
where the spectral jets of the mixed Christoffel–Darboux kernel and of $\delta L_2$ are taken with respect to the second variable.

Whenever $\tau^{(1)}_l := \left| \begin{array}{c} \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l-2n} \rangle L^{(1)} \\ \vdots \\ \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l-1} \rangle L^{(1)} \end{array} \right| \neq 0$ and $l \geq 2n$ the following Christoffel formulas are satisfied

\begin{align}
\Phi^{(1)}_{2,l}(z) &= \Theta \left[ \begin{array}{c} \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l-2n} \rangle L^{(1)} \\ \vdots \\ \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l} \rangle L^{(1)} \end{array} \right] \\
\Phi^{(1)}_{2,l}(z) &= \Theta \left[ \begin{array}{c} \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l-2n} \rangle L^{(1)} \\ \vdots \\ \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l} \rangle L^{(1)} \end{array} \right]
\end{align}

(43) \quad \mathcal{H}^{(2)}_1 = \frac{H_{l-2n}}{L^{(2)}_{(-1)^m n}} \Theta \left[ \begin{array}{c} \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l-2n} \rangle L^{(1)} \\ \vdots \\ \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l} \rangle L^{(1)} \end{array} \right] = \frac{H_{l-2n}}{L^{(2)}_{(-1)^m n}} \mathcal{H}^{(2)}_1'' \Theta \left[ \begin{array}{c} \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l-2n} \rangle L^{(1)} \\ \vdots \\ \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l} \rangle L^{(1)} \end{array} \right]

(44) \quad \Phi^{(2)}_{2,l}(z) = -\frac{H_{l-2n}}{L^{(2)}_{(-1)^m n}} \Theta \left[ \begin{array}{c} \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l-2n} \rangle L^{(1)} \\ \vdots \\ \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l} \rangle L^{(1)} \end{array} \right]

(45) \quad \Phi^{(1)}_{2,l}(z) = \Theta \left[ \begin{array}{c} \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l-2n} \rangle L^{(1)} \\ \vdots \\ \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l} \rangle L^{(1)} \end{array} \right]

(46) \quad \mathcal{H}^{(1)}_1 = \frac{H_{l-2n}}{L^{(1)}_{(-1)^m n}} \Theta \left[ \begin{array}{c} \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l-2n} \rangle L^{(1)} \\ \vdots \\ \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l} \rangle L^{(1)} \end{array} \right] = \frac{H_{l-2n}}{L^{(1)}_{(-1)^m n}} \mathcal{H}^{(1)}_1'' \Theta \left[ \begin{array}{c} \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l-2n} \rangle L^{(1)} \\ \vdots \\ \frac{\partial L^{(1)}}{\partial x} - \langle \xi^{(1)}, \phi_{2,l} \rangle L^{(1)} \end{array} \right]
where the spectral jet of the Christoffel–Darboux kernels and of $\delta L_1$ is taken with respect to the first variable.

Proof. See Appendix. \hfill \Box

3.3. Reductions to univariate linear functionals supported on the unit circle. As we have discussed, for the two possible Geronimus transformations we have the corresponding Christoffel–Geronimus formulas. As for the Christoffel transformation situation we look at the case $\langle L(z), M(z) \rangle_u = \left\langle u_{z \gamma} L(z) M(z) \right\rangle$ with supp $u_{z \gamma} \subset \gamma$, where the curve $\gamma = \{ z \in \mathbb{C}^+ : \bar{z} = \tau(z) \}$, we have that the perturbation of type 2 can be described as $\hat{u}_{2}^{(2)} = L_{2}^{(2)}(\tau(z)) u_{2}$. For the unit circle $\gamma = T$, with $\tau(z) = z^{-1}$, this Geronimus transformation can be thought, as $L_{2}^{(2)}(z) = u_{2}$ and $L_{2}^{(2)}(z) = L_{2}^{(1)}(z)$, so that taking $L_{(2)}(z) = L_{2}^{(1)}(z)$ we get that $L_{(2)}(z) \hat{u}_{2}^{(2)} = L_{2}^{(1)}(z) \hat{u}_{2}^{(2)} = u_{2}$. Recall that we have the following two mass terms, each coming from one of the two Geronimus transformations we have considered, one generated by $L(z)$ and the other by $L_{*}(z) = L(z^{-1})$, having zeros $\sigma(L)$ = $\{ \zeta_{i} \}_{i=1}^{d}$ and $\sigma(L_{*})$ = $\{ \zeta_{i}^{-1} \}_{i=1}^{d}$, respectively,

$$\mathcal{M}^{(1)} = \sum_{i=1}^{d} \sum_{k=0}^{m_{i}-1} \frac{(-1)^{k}}{k!} \delta^{(k)}(z_{i} - \zeta_{i}) \otimes \hat{\theta}^{(1)}_{i,k} z_{i}, \quad \mathcal{M}^{(2)} = \sum_{i=1}^{d} \sum_{j=1}^{m_{i}-1} \frac{(-1)^{j}}{j!} \delta^{(j)}(z_{i} - \zeta_{i}) \otimes \hat{\delta}^{(1)}(z_{i}^{-1} - \zeta_{i}^{-1}).$$

If we take $(\hat{\theta}^{(1)}_{i,k})_{z_{i}} = \sum_{j=1}^{d} \sum_{l=0}^{m_{i}-1} \frac{(-1)^{j}}{j!} \bar{\mu}_{i,k,l} \delta^{(j)}(z_{i} - \zeta_{i}^{-1})$ and $(\hat{\delta}^{(1)}_{i,j})_{z_{i}} = \sum_{i=1}^{d} \sum_{k=0}^{m_{i}-1} \frac{(-1)^{k}}{k!} \bar{\mu}_{i,k,l} \delta^{(k)}(z_{i} - \zeta_{i})$ we find

\begin{equation}
\mathcal{M}^{(1)} = \mathcal{M}^{(2)} =: \mathcal{M} \quad \text{with}
\end{equation}

$$\mathcal{M} = \sum_{i,j=1}^{d} \sum_{k=0}^{m_{i}-1} \sum_{l=0}^{m_{j}-1} \frac{(-1)^{k+l}}{k!l!} \bar{\mu}_{i,k,l} \delta^{(k)}(z_{i} - \zeta_{i}) \otimes \hat{\delta}^{(1)}(z_{2}^{-1} - \zeta_{2}^{-1}).$$

This mass term is possibly not supported on the diagonal, but it is the most general mass term such that both Geronimus transformations, of an univariate sesquilinear form, are equal. The corresponding mass contribution to the perturbed sesquilinear form is $\left\langle \mathcal{M}, \mathcal{M}^{(1)}(z_{1}) \otimes \bar{\mathcal{M}}(z_{2}) \right\rangle = \sum_{i,j=1}^{d} \sum_{k=0}^{m_{i}-1} \sum_{l=0}^{m_{j}-1} \frac{(-1)^{k+l}}{k!l!} \bar{\mu}_{i,k,l} \mathcal{M}^{(1)}(\zeta_{i}) \mathcal{M}^{(1)}(\zeta_{j}^{-1})$, so that the perturbed sesquilinear form is

$$\left\langle M^{(1)}(z_{1}), M^{(2)}(z_{2}) \right\rangle_{\hat{u}} = \left\langle u_{z \gamma}, \frac{M_{1}(z) M_{2}(z)}{L(z)} \right\rangle + \sum_{i,j=1}^{d} \sum_{k=0}^{m_{i}-1} \sum_{l=0}^{m_{j}-1} \bar{\mu}_{i,k,l} M_{1}^{(1)}(\zeta_{i}) M_{2}^{(1)}(\zeta_{j}^{-1}).$$

As we have

$$(\hat{\theta}^{(1)}_{i,k})_{z_{i}} = \sum_{j=1}^{d} \sum_{l=0}^{m_{i}-1} \frac{1}{l!} \phi^{(j)}((\zeta_{j}^{-1})^{-1}) \bar{\mu}_{i,k,l}, \quad (\hat{\delta}^{(1)}_{i,j})_{z_{i}} = \sum_{i=1}^{d} \sum_{k=0}^{m_{i}-1} \frac{1}{k!} \phi^{(j)}((\zeta_{i})^{-1}) \bar{\mu}_{i,k,l}.$$
we introduce
\[
\Xi := \begin{bmatrix}
\Xi_{1,0|1,0} & \cdots & \Xi_{1,0|m_1-1} & \cdots & \Xi_{1,0|d,0} & \cdots & \Xi_{1,0|d,m_d-1} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\Xi_{1,m_1-1|1,0} & \cdots & \Xi_{1,m_1-1|m_1-1} & \cdots & \Xi_{1,m_1-1|d,0} & \cdots & \Xi_{1,m_1-1|d,m_d-1} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\Xi_{d,0|1,0} & \cdots & \Xi_{d,0|m_1-1} & \cdots & \Xi_{d,0|d,0} & \cdots & \Xi_{d,0|d,m_d-1} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\Xi_{d,m_d-1|1,0} & \cdots & \Xi_{d,m_d-1|m_1-1} & \cdots & \Xi_{d,m_d-1|d,0} & \cdots & \Xi_{d,m_d-1|d,m_d-1}
\end{bmatrix} \in \mathbb{C}^{2n \times 2n}
\]

so that we can we deduce the following expressions in terms of spectral jets \( \langle \xi^{(1)}, \phi \rangle = \mathcal{L}_{\phi}^{L} \Xi^{\dagger} \) and \( \langle \xi^{(2)}, \phi \rangle = \mathcal{L}_{\phi}^{L} \Xi \). We also need the matrix \( \mathcal{L}_\ast \) associated to the reciprocal Laurent polynomial \( \mathcal{L}_\ast(z) \).

**Theorem 3.** Given a sesquilinear form associated with a univariate linear functional supported on the unit circle \( T \) and a Geronimus transformation with a prepared perturbing Laurent polynomial \( \mathcal{L}(z) \), for \( l \geq 2n \), we have the following expressions (whenever the involved quasideterminants do exist)

\[
\begin{align*}
\tilde{\phi}_{1,1}(z) &= \Theta_\ast \left[ \begin{array}{cc}
\mathcal{L}_{C_{1,1}}^{\ast} - \mathcal{L}_{\phi_{1,1}}^{\ast} \Xi \mathcal{L}_\ast & \phi_{1,1}(z) \\
\vdots & \vdots \\
\mathcal{L}_{C_{1,1}}^{\ast} - \mathcal{L}_{\phi_{1,1}}^{\ast} \Xi \mathcal{L}_\ast & \phi_{1,1}(z)
\end{array} \right] = -\Theta_\ast \\
\tilde{H}_1 &= \Theta_\ast \left[ \begin{array}{c}
\mathcal{L}_{C_{1,1}} - \mathcal{L}_{\phi_{1,1}} \Xi \mathcal{L}_\ast \\
\vdots \\
\mathcal{L}_{C_{1,1}} - \mathcal{L}_{\phi_{1,1}} \Xi \mathcal{L}_\ast
\end{array} \right] = \Theta_\ast \\
\tilde{\phi}_{2,1}(z) &= -\Theta_\ast \left[ \begin{array}{c}
\mathcal{L}_{C_{1,1}}^{\ast} - \mathcal{L}_{\phi_{1,1}}^{\ast} \Xi \mathcal{L}_\ast \\
\mathcal{L}_{C_{1,1}}^{\ast} - \mathcal{L}_{\phi_{1,1}}^{\ast} \Xi \mathcal{L}_\ast \\
\vdots \\
\mathcal{L}_{C_{1,1}}^{\ast} - \mathcal{L}_{\phi_{1,1}}^{\ast} \Xi \mathcal{L}_\ast \\
\mathcal{L}_\ast(z) \left( \mathcal{L}_{K_{1,1}}^{\ast}(z) - \mathcal{L}_{K_{1,1}}^{\ast}(z) \Xi \mathcal{L}_\ast \right) + \mathcal{L}_{\phi_{2,1}}^{\ast}(z) \\
\mathcal{L}_\ast(z) \left( \mathcal{L}_{K_{1,1}}^{\ast}(z) - \mathcal{L}_{K_{1,1}}^{\ast}(z) \Xi \mathcal{L}_\ast \right) + \mathcal{L}_{\phi_{2,1}}^{\ast}(z)
\end{array} \right] = \Theta_\ast
\end{align*}
\]

in the first line spectral jets of the Christoffel–Darboux kernels and of \( \delta \mathcal{L} \) are taken with respect to its first variable, while in the third line spectral jets of the Christoffel–Darboux kernel and \( \delta \mathcal{L} \) are taken with respect to the second variable.

3.3.1. **Diagonal masses.** We now discuss what happens when the masses are restricted to be supported on the diagonal.
Definition 20. We define
\[ B_{k,j}^{[i]} := (-1)^k \sum_{j_1+j_2+\cdots+j_{k+1}=j} \frac{k!}{j_1! \cdots j_{k+1}!} (\zeta_i)^{-k-j}, \]
where \( \zeta \) is given by
\[ \zeta = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & B_{1,1}^{[i]} & 0 & \cdots & 0 \\ 0 & B_{2,1}^{[i]} & B_{2,2}^{[i]} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & B_{m_i-1,1}^{[i]} & B_{m_i-1,2}^{[i]} & \cdots & B_{m_i-1,m_i-1}^{[i]} \end{bmatrix}. \]

and given complex numbers \( \Xi_i \) we consider
\[ \Xi_i = \begin{bmatrix} \Xi_0^i & \frac{1}{1!} \Xi_1^i & \frac{1}{2!} \Xi_2^i & \cdots & \frac{1}{(m_i-1)!} \Xi_{m_i-1}^i \\ \frac{1}{1!} \Xi_1^i & \frac{1}{2!} \Xi_2^i & \cdots & 0 \\ \frac{1}{2!} \Xi_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(m_i-1)!} \Xi_{m_i-1}^i & 0 & \cdots & 0 \end{bmatrix}. \]

Proposition 22. For the particular choice \( \Xi = \text{diag}(\Xi_1, \ldots, \Xi_d) \) with \( \Xi_i = \Xi_i^{[i]} B_{i}^{[i]} \) the mass term (48) is of the form
\[ \left\langle M_{1}, M_{2}(z_1) \otimes M_{2}(z_2) \right\rangle = \left\langle \sum_{i=1}^{d} \sum_{l=0}^{m_i-1} \Xi_i^{[i]} \frac{(-1)^{l}}{l!} \delta^{(l)}(z - \zeta_i), M_{1}(z) M_{2,*}(z) \right\rangle. \]

Proof. Using Bell polynomials and the Faà di Bruno formula one can show that \( (M_{*})^{(k)}(\zeta_i) = \sum_{j=1}^{k} M^{(j)}((\zeta_i)^{-1}) B_{k,j}^{[i]} \) with \( B_{k,0}^{[i]} = \delta_{k,0} \). Let us consider the expression
\[ \left\langle \sum_{i=1}^{d} \sum_{l=0}^{m_i-1} \Xi_i^{[i]} \frac{(-1)^{l}}{l!} \delta^{(l)}(z - \zeta_i), M_{1}(z) M_{2,*}(z) \right\rangle = \sum_{i=1}^{d} \sum_{l=0}^{m_i-1} \Xi_i^{[i]} \frac{1}{l!} \sum_{k=0}^{l} \binom{l}{k} M_{1}^{(l-k)}((\zeta_i)^{-1}) M_{2,*}^{(k)}((\zeta_i)^{-1}) \]
\[ = \sum_{i=1}^{d} \sum_{n=0}^{m_i-1} \Xi_i^{[i]} \frac{1}{n!} B_{k,j}^{[i]} M_{1}^{(n)}((\zeta_i)^{-1}) M_{2,*}^{(j)}((\zeta_i)^{-1}) \]
\[ = \sum_{i=1}^{d} \sum_{n=0}^{m_i-1} \Xi_i^{[i]} M_{1}^{(n)}((\zeta_i)^{-1}) \Xi_i^{[i]} \Xi_i^{[i]} M_{2,*}^{(j)}((\zeta_i)^{-1}), \]

where \( \Xi_{i,n,j} = \sum_{k=j}^{m_i-1} \Xi_{i,n,k} \frac{1}{k!} B_{k,j}^{[i]} \). Then, we find that \( \Xi = \text{diag}(\Xi_1, \ldots, \Xi_d) \) with \( \Xi_i^{[i]} B_{i}^{[i]} \).

Finally, if the original functional is nonnegative, i.e., biorthogonality is just orthogonality and we consider a Geronimus transformation with a prepared Laurent polynomial \( L(z) \) which is equal to its reciprocal, \( L_{r}(z) = L(z) \) and \( L_{s}(z) = [L(z)]^{-1} \) for \( z \in \mathbb{T} \), and masses being positive linear functionals, which is achieved only when \( \Xi_i^{[i]} = \xi^{[i]} \delta_{i,0}, \xi^{[i]} > 0 \), we get \( \Xi = \Xi_{i} = \left[ \xi^{[i]} 0 \cdots 0 \right] \in \mathbb{C}^{m_i \times m_i} \) and, consequently, we find \( \Xi L_{*} = \Xi L = \text{diag} \left[ \left[ \begin{array}{c} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{array} \right] \right] \), so that
\[ d_{\Xi}^{[i]} \Xi L_{*} = \left[ 0, \ldots, 0, \phi(\zeta_1)\xi^{[i]} L_{[1]}(\zeta_1), \ldots, 0, \ldots, 0, \phi(\zeta_d)\xi^{[i]} L_{[d]}(\zeta_d) \right]. \]
We conclude that

\[
\tilde{\phi}_1(z) = \Theta_* \begin{bmatrix}
\tilde{g}_C^{l+2n} - \tilde{g}_{\phi_1}^{l+2n} z \tilde{L}_* & \phi_1(z) \\
\vdots & \vdots \\
\tilde{g}_C^{l+1} - \tilde{g}_{\phi_1}^{l+1} z \tilde{L}_* & \phi_1(z)
\end{bmatrix} = -\Theta_* \begin{bmatrix}
\tilde{g}_C^{l+2n} - \tilde{g}_{\phi_1}^{l+2n} z \tilde{L}_* & \phi_1(z) \\
\vdots & \vdots \\
\tilde{g}_C^{l+1} - \tilde{g}_{\phi_1}^{l+1} z \tilde{L}_* & \phi_1(z)
\end{bmatrix}
\]

\[
\tilde{H}_1 = \Theta_* \begin{bmatrix}
\tilde{g}_C^{l+2n} - \tilde{g}_{\phi_1}^{l+2n} z \tilde{L}_* & \phi_1(z) \\
\vdots & \vdots \\
\tilde{g}_C^{l+1} - \tilde{g}_{\phi_1}^{l+1} z \tilde{L}_* & \phi_1(z)
\end{bmatrix} = -\Theta_* \begin{bmatrix}
\tilde{g}_C^{l+2n} - \tilde{g}_{\phi_1}^{l+2n} z \tilde{L}_* & \phi_1(z) \\
\vdots & \vdots \\
\tilde{g}_C^{l+1} - \tilde{g}_{\phi_1}^{l+1} z \tilde{L}_* & \phi_1(z)
\end{bmatrix}
\]

In terms of \( \tau_1 := \begin{bmatrix}
\tilde{g}_C^{l+2n} - \tilde{g}_{\phi_1}^{l+2n} z \tilde{L}_* \\
\vdots \\
\tilde{g}_C^{l+1} - \tilde{g}_{\phi_1}^{l+1} z \tilde{L}_*
\end{bmatrix} \) we can write

\[
\tilde{\phi}_1(z) = \frac{1}{\tau_1} \begin{bmatrix}
\tilde{g}_C^{l+2n} - \tilde{g}_{\phi_1}^{l+2n} z \tilde{L}_* & \phi_1(z) \\
\vdots & \vdots \\
\tilde{g}_C^{l+1} - \tilde{g}_{\phi_1}^{l+1} z \tilde{L}_* & \phi_1(z)
\end{bmatrix} = -\frac{H_{l+2n}}{L_{(-1)^{l+1}n} \tau_1} \begin{bmatrix}
\tilde{g}_C^{l+2n} - \tilde{g}_{\phi_1}^{l+2n} z \tilde{L}_* & \phi_1(z) \\
\vdots & \vdots \\
\tilde{g}_C^{l+1} - \tilde{g}_{\phi_1}^{l+1} z \tilde{L}_* & \phi_1(z)
\end{bmatrix}
\]

\[
\tilde{H}_1 = \frac{H_{l+2n}}{L_{(-1)^{l+1}n}} \tilde{\tau}_{l+1} = \frac{H_{l+2n}}{L_{(-1)^{l+1}n}} \tilde{\tau}_{l+1}.
\]

**Proofs**

**Proof of Proposition 11** We use the relations

\[
(\breve{H}^{(1)})^{-1} \omega_1^{(1)} \phi_1(z_2) = L^{(1)}(z_2)(\breve{H}^{(1)})^{-1} \breve{\phi}_1^{(1)}(z_2),
\]

\[
(\hat{\phi}_2^{(1)}(z_1))^\dagger (\breve{H}^{(1)})^{-1} \omega_1^{(1)} = \hat{\phi}_2^{(1)}(z_1) H^{-1},
\]

which imply

\[
\begin{bmatrix}
(\hat{\phi}_2^{(1)}(z_1))^\dagger \\
(\breve{H}^{(1)})^{-1} \omega_1^{(1)} \\
[\phi_1(z_2)]^{|l|} + [\hat{\phi}_2^{(1)}(z_1)]^\dagger \\
[(\breve{H}^{(1)})^{-1} \omega_1^{(1)}]^{[l], [l]} [\phi_1(z_2)]^{[l], [l]}
\end{bmatrix} = \begin{bmatrix}
(\hat{\phi}_2^{(1)}(z_1))^\dagger \\
(\breve{H}^{(1)})^{-1} \omega_1^{(1)} \\
[\phi_1(z_2)]^{|l|} + [\hat{\phi}_2^{(1)}(z_1)]^\dagger \\
[(\breve{H}^{(1)})^{-1} \omega_1^{(1)}]^{[l], [l]} [\phi_1(z_2)]^{[l], [l]}
\end{bmatrix}.
\]

and, after subtracting and cleaning, you get

\[
\begin{bmatrix}
(\hat{\phi}_2^{(1)}(z_1))^\dagger \\
(\breve{H}^{(1)})^{-1} \omega_1^{(1)} \\
[\phi_1(z_2)]^{[l], [l]} + [\hat{\phi}_2^{(1)}(z_1)]^\dagger \\
[(\breve{H}^{(1)})^{-1} \omega_1^{(1)}]^{[l], [l]} [\phi_1(z_2)]^{[l], [l]}
\end{bmatrix} = L^{(1)}(z_2)(\breve{H}^{(1)})^{-1}[\xi_1, z_2] - K^{[l]}(\xi_1, z_2).
On the other hand, we have \((\phi_2(z_1))^\dagger (\omega_2^{(2)})^\dagger (\hat{H}^{(2)})^{-1} = L^{(2)}(z_1) \hat{\phi}_2^{(2)}(z_1)(\hat{H}^{(2)})^{-1}\) and \((\omega_2^{(2)})^\dagger (\hat{H}^{(2)})^{-1} \hat{\phi}_1^{(2)}(z_2) = H^{-1} \phi_1(z_2)\), and, therefore, we conclude

\[
[(\phi_2(z_1))^\dagger [ (\omega_2^{(2)})^\dagger (\hat{H}^{(2)})^{-1} [ \hat{\phi}_1^{(2)}(z_2) ]^{[1]} + \left[(\phi_2(z_1))^\dagger [ (\omega_2^{(2)})^\dagger (\hat{H}^{(2)})^{-1} [ \hat{\phi}_1^{(2)}(z_2) ]^{[1]} ,
\]

\[
= L^{(2)}(z_1) \hat{\phi}_2^{(2)}(x) [ (\hat{H}^{(2)})^{-1} [ \hat{\phi}_1^{(2)}(z_2) ]^{[1]} ,
\]

\[
[(\phi_2(z_1))^\dagger [ (\omega_2^{(2)})^\dagger (\hat{H}^{(2)})^{-1} [ \hat{\phi}_1^{(2)}(z_2) ]^{[1]} = [(\phi_2(z_1))^\dagger [ H^{-1} [ \hat{\phi}_1^{(2)}(z_2) ]^{[1]} .
\]

Consequently, we arrive to the following expression

\[
[(\phi_1^{(2)}(z_2))^\dagger [ (\hat{H}^{(2)})^{-1} [ \omega_2^{(2)} ]^{[1]} [ \phi_2(z_1) ]^{[1]} = L^{(2)}(z_1) [ (\hat{K}^{(2)})^{[1]}(z_1, z_2) − K^{[1]}(z_1, z_2). \]

\[
\square
\]

**Proof of Theorem** In the first formula of Proposition \(10\) we take the \(l\)-th row and evaluate the spectral jets along \(L(z)\), then \([(\omega_1^{(1)})_{l,1}, \ldots, (\omega_1^{(1)})_{l,1+2n}]\)

\[
\begin{bmatrix}
\partial_{\phi_1}^{(1)} \\
\vdots \\
\partial_{\phi_1}^{(1)} \\
\end{bmatrix}
\]

is \(0\). Since the number of zeros, counting their multiplicities, is \(2n\), we will have a square matrix

\[
\begin{bmatrix}
\partial_{\phi_1}^{(1)} \\
\vdots \\
\partial_{\phi_1}^{(1)} \\
\end{bmatrix} \in \mathbb{C}^{2n \times 2n}
\]

and whenever it is not singular we obtain

\[
[(\omega_1^{(1)})_{l,1}, \ldots, (\omega_1^{(1)})_{l,1+2n−1}] = −(\omega_1^{(1)})_{l,1+2n} \partial_{\phi_1}^{(1)} \left[ \begin{bmatrix}
\partial_{\phi_1}^{(1)} \\
\vdots \\
\partial_{\phi_1}^{(1)} \\
\end{bmatrix}^{-1}
\right].
\]

Using again Proposition \(10\) we get \(17\). To obtain \(18\) we multiply with the vector \([0; 0; \vdots; 0]\). Taking \(14\) and evaluating the spectral jet (on the \(z_2\) variable) along \(L^{(1)}(z)\) of the Christoffel–Darboux kernel \(K^{[1]}(z_1, z_2)\) we obtain

\[
−\partial_{K^{[1]}(z)} \left[ \begin{bmatrix}
\partial_{\phi_1}^{(1)} \\
\vdots \\
\partial_{\phi_1}^{(1)} \\
\end{bmatrix}^{-1}
\right] = \left[ \hat{\phi}_{2,1−2n}(z), \ldots, \hat{\phi}_{2,1−1}(z) \right] \text{diag}((\hat{H}_{1−2n})^{-1}, \ldots, (\hat{H}_{1−1})^{-1})
\]

\[
\times \left[ \begin{bmatrix}
(\omega_1^{(1)})_{1−2n,1} & 0 & 0 & \ldots & 0 \\
(\omega_1^{(1)})_{1−2n+1,1} & (\omega_1^{(1)})_{1−2n+1,1+1} & 0 & \ldots & 0 \\
(\omega_1^{(1)})_{1−2n+2,1} & (\omega_1^{(1)})_{1−2n+2,1+1} & (\omega_1^{(1)})_{1−2n+2,1+2} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(\omega_1^{(1)})_{1−1,1} & (\omega_1^{(1)})_{1−1,1+1} & (\omega_1^{(1)})_{1−1,1+2} & \ldots & (\omega_1^{(1)})_{1−1,1+2n}
\end{bmatrix}
\right],
\]

and multiplying by \([0; 0; \vdots; 1]\), we get

\[
−\partial_{K^{[1]}(z)} \left[ \begin{bmatrix}
\partial_{\phi_1}^{(1)} \\
\vdots \\
\partial_{\phi_1}^{(1)} \\
\end{bmatrix}^{-1}
\right] = \left[ \hat{\phi}_{2,1−1}(z)(\hat{H}_{1−1})^{-1}(\omega_1^{(1)})_{1−1,1+2n}.\right]
Analogously, from the last equation in Proposition 10 we deduce

\[ [(\omega_2^{(2)})_{1,1}, \ldots, (\omega_2^{(2)})_{1,1+2n-1}] = -\langle \omega_2^{(2)} \rangle_{1,1+2n} J^{(1)}_{\phi_1,1} \left[ \begin{array}{c} J^{(1)}_{\phi_1,1} \\ \vdots \\ J^{(1)}_{\phi_1,1+2n-1} \end{array} \right]^{-1}. \]

And using the same equation again we conclude 20. Relation 21 is obtained just as we got 18. Finally, taking 15 and evaluating the spectral jet (on the \(z_1\) variable) along \(L^{(1)}(z)\) of the kernel \(K^{(1)}(z_1, z_2)\), and proceeding as in the proof of 19 we deduce 22.

**Proof of Proposition 18** The Cauchy second kind functions are

\[ \mathcal{C}_1^{(1)}(z) = \left\langle \phi_1^{(1)}(z_1) - \frac{1}{z - z_2}, \phi_1^{(1)}(z_1) \right\rangle_{\tilde{u}^{(1)}}, \quad z \notin \text{supp} \tilde{u}^{(1)}, \]

\[ \mathcal{C}_1^{(2)}(z) = \left\langle \phi_2^{(2)}(z_1) - \frac{1}{z - z_2}, \phi_2^{(2)}(z_1) \right\rangle_{\tilde{u}^{(2)}}, \quad z \notin \text{supp} \tilde{u}^{(2)}, \]

\[ \left( \mathcal{C}_2^{(1)}(z) \right)^\dagger = \left\langle \frac{1}{z - z_1}, \phi_2^{(1)}(z_2) \right\rangle_{\tilde{u}^{(1)}}, \quad z \notin \text{supp} \tilde{u}^{(1)}, \]

\[ \left( \mathcal{C}_2^{(2)}(z) \right)^\dagger = \left\langle \frac{1}{z - z_1}, \phi_2^{(2)}(z_2) \right\rangle_{\tilde{u}^{(2)}}, \quad z \notin \text{supp} \tilde{u}^{(2)}. \]

To prove 28 we proceed as follows

\[ (C_2(z))^{\dagger} (\Omega_2^{(1)})^{\dagger} - L^{(1)}(\bar{z}) (\mathcal{C}_2^{(1)}(z))^{\dagger} = \left\langle \frac{1}{z - z_1}, \phi_2(z_2) \right\rangle_{u^{(1)}} (\Omega_2^{(1)})^{\dagger} - L^{(1)}(\bar{z}) \left\langle \frac{1}{z - z_1}, \phi_2(z_2) \right\rangle_{\tilde{u}^{(1)}} \]

\[ = \left\langle \frac{1}{z - z_1}, \phi_2(z_2) \right\rangle_{\tilde{u}^{(2)}}, \quad z \notin \text{supp} \tilde{u}^{(2)}, \]

For 29 we have

\[ \Omega_2^{(1)} C_1(z) - \mathcal{C}_1^{(2)}(z) L^{(2)}(z) = \Omega_2^{(1)} \left\langle \phi_1(z_1), \frac{1}{z - z_2} \right\rangle_{u^{(1)}} - \left\langle \phi_2^{(2)}(z_1), \frac{1}{z - z_2} \right\rangle_{\tilde{u}^{(2)}}, \quad \frac{L^{(2)}(z)}{\tilde{z} - z_2} \]

\[ = \left\langle \phi_1^{(2)}(z_1), \frac{1}{z - z_2} \right\rangle_{\tilde{u}^{(2)}}, \quad \frac{L^{(2)}(z)}{\tilde{z} - z_2} \]

Relation 30 is a consequence of

\[ \Omega_1^{(1)} \mathcal{C}_1^{(1)}(z) = \left\langle L^{(1)}(z_1) \phi_1(z), \frac{1}{z - z_2} \right\rangle_{\tilde{u}^{(1)}} = \left\langle \phi_1(z), \frac{1}{z - z_2} \right\rangle_{L^{(1)}(z_1) \tilde{u}^{(1)}} = C_1(z), \]

and, finally, to obtain 31 just observe that

\[ \left( \mathcal{C}_2^{(2)}(z) \right)^\dagger (\Omega_2^{(2)})^{\dagger} = \left\langle \frac{1}{z - z_1}, L^{(2)}(z) \left\langle \phi_2(z) \right\rangle_{\tilde{u}^{(2)}}, \quad \left( C_2(z) \right)^\dagger. \]

\[ \square \]
Proof of Proposition 19. From 
\[
(\phi_2(z_1))^\dagger (\Omega_2^{(1)})(\hat{H}^{(1)})^{-1} = (\phi_2(z_1))^\dagger (\hat{H}^{(1)})^{-1},
\]
we deduce that 
\[
\left[\left(\phi_2(z_1)\right)^\dagger\right]^{[l]} \left[\left(\Omega_2^{(1)}\right)^\dagger (\hat{H}^{(1)})^{-1}\right]^{[l]} \left[\phi_1(z_2)\right]^{[l]} = \left[\left(\phi_2(z_1)\right)^\dagger\right]^{[l]} \left[\left(\hat{H}^{(1)}\right)^{-1}\right]^{[l]} \left[\phi_1(z_2)\right]^{[l]},
\]
\[
\left[\left(\phi_2(z_1)\right)^\dagger\right]^{[l]} \left[\left(\Omega_2^{(1)}\right)^\dagger (\hat{H}^{(1)})^{-1}\right]^{[l]} \left[\phi_1(z_2)\right]^{[l]} + \left[\left(\phi_2(z_1)\right)^\dagger\right]^{[l]} \left[\left(\Omega_2^{(1)}\right)^\dagger (\hat{H}^{(1)})^{-1}\right]^{[l]=1} \left[\phi_1(z_2)\right]^{[\geq 1]}
\]
\[= L^{(1)}(z_2) \left[\left(\phi_2(z_1)\right)^\dagger\right]^{[l]} \left[\hat{H}^{-1}\right]^{[l]} \left[\phi_1(z_2)\right]^{[l]},\]
that once subtracted give 
\[
\left[\left(\phi_1(z_2)\right)^\dagger\right]^{[\geq 1]} \left[\left(\hat{H}^{(1)}\right)^{-1} \Omega_2^{(1)}\right]^{[\geq 1]} \left[\phi_2(z_1)\right]^{[l]} = L^{(1)}(z_2) K^{(1)}(\tilde{z}_1, z_2) - \tilde{K}^{(1),[l]}(\tilde{z}_1, z_2),
\]
from where we get (34). On the other hand, we have 
\[
(\hat{H}^{(2)})^{-1} \Omega_2^{(2)} \phi_1(z_2) = (\hat{H}^{(2)})^{-1} \phi_2(z_2),
\]
so that 
\[
\left[\left(\phi_2(z_1)\right)^\dagger\right]^{[l]} \left[\left(\hat{H}^{(2)}\right)^{-1} \Omega_2^{(2)}\right]^{[l]} \left[\phi_1(z_2)\right]^{[l]} = \left[\left(\phi_2(z_1)\right)^\dagger\right]^{[l]} \left[\left(\hat{H}^{(2)}\right)^{-1}\right]^{[l]} \left[\phi_1(z_2)\right]^{[l]},
\]
\[
\left[\left(\phi_2(z_1)\right)^\dagger\right]^{[l]} \left[\left(\hat{H}^{(2)}\right)^{-1} \Omega_2^{(2)}\right]^{[l]} \left[\phi_1(z_2)\right]^{[l]} + \left[\left(\phi_2(z_1)\right)^\dagger\right]^{[l]} \left[\left(\hat{H}^{(2)}\right)^{-1} \Omega_2^{(2)}\right]^{[\geq 1]} \left[\phi_1(z_2)\right]^{[l]}
\]
\[= L^{(2)}(z_1) \left[\left(\phi_2(z_1)\right)^\dagger\right]^{[l]} \left[\hat{H}^{-1}\right]^{[l]} \left[\phi_1(z_2)\right]^{[l]},\]
which lead to 
\[
\left[\left(\phi_2(z_1)\right)^\dagger\right]^{[\geq 1]} \left[\left(\hat{H}^{(2)}\right)^{-1} \Omega_2^{(2)}\right]^{[\geq 1]} \left[\phi_1(z_2)\right]^{[l]} = L^{(2)}(z_1) K^{(1)}(\tilde{z}_1, z_2) - \tilde{K}^{(2),[k]}(\tilde{z}_1, z_2),
\]
and (35) is immediately deduced. 

Proof of Proposition 20. Let’s prove (36). From the relations 
\[
(\Omega_2^{(1)})^\dagger (\hat{H}^{(1)})^{-1} = -\left\langle \delta L^{(1)}(\bar{x}_1, z_1), (\phi_2^{(1)}(z_2))^\dagger \right\rangle_{\tilde{u}^{(1)}} \left(\hat{H}^{(1)}\right)^{-1},
\]
we obtain 
\[
\left[\left(\Omega_2^{(1)}\right)^\dagger (\hat{H}^{(1)})^{-1}\right]^{[l]} \left[\phi_2(z_2)\right]^{[l]} = - L^{(1)}(z_1) \left[\left(\tilde{C}_1^{(1)}(x_1)\right)^\dagger\right]^{[l]} \left[\left(\hat{H}^{(1)}\right)^{-1}\right]^{[l]} \left[\phi_2(z_2)\right]^{[l]},
\]
\[
\left[\left(\Omega_2^{(1)}\right)^\dagger (\hat{H}^{(1)})^{-1}\right]^{[l]} \left[\phi_2(z_2)\right]^{[l]} + \left[\left(\Omega_2^{(1)}\right)^\dagger (\hat{H}^{(1)})^{-1}\right]^{[\geq 1]} \left[\phi_2(z_2)\right]^{[l]}
\]
\[= L^{(1)}(z_2) \left[\left(\Omega_2^{(1)}\right)^\dagger\right]^{[l]} \left[\hat{H}^{-1}\right]^{[l]} \left[\phi_2(z_2)\right]^{[l]},\]
When subtracted we get 
\[
\left[\left(\phi_2(z_2)\right)^\dagger\right]^{[\geq 1]} \left[\left(\hat{H}^{(1)}\right)^{-1} \Omega_2^{(1)}\right]^{[\geq 1]} \left[\phi_2(z_2)\right]^{[l]} + L^{(1)}(\bar{x}_1) \tilde{K}^{(1),[l]}_{\tilde{C},\phi}(\bar{x}_1, x_2)
\]
\[= \left\langle \delta L^{(1)}(\bar{x}_1, z_1), \frac{\tilde{K}^{(1),[l]}(\bar{z}_2, x_2)}{\bar{x}_1 - z_1} \right\rangle_{\tilde{u}^{(1)}} + L^{(1)}(z_2) K^{(1)}_{\tilde{C},\phi}(\bar{x}_1, x_2).
\]
From Corollary 2 and Proposition 17 we obtain 
\[
\left\langle \frac{L^{(1)}(\bar{x}_1) - L^{(1)}(z_1)}{\bar{x}_1 - z_1}, \frac{\bar{K}^{(1),[l]}(\bar{z}_2, x_2)}{\bar{x}_1 - x_2} \right\rangle_{\tilde{u}^{(1)}} = \frac{L^{(1)}(\bar{x}_1) - L^{(1)}(x_2)}{\bar{x}_1 - x_2},
\]
and the desired result follows.

For the proof of (37), we notice that from

\[
(\bar{C}_2^{(2)}(x_1))^\dagger (\bar{\Omega}_1^{(2)})^{-1} = (C_2(x_1))^{-1} H^{-1}, \quad (\bar{\Omega}_1^{(2)})^{-1} \phi_1(x_2) = (\bar{\Omega}_1^{(2)})^{-1} \phi(x_2),
\]

we get

\[
[(\bar{C}_2^{(2)}(x_1))^\dagger (\bar{\Omega}_1^{(2)})^{-1} \phi_1(x_2)]^l + [(\bar{C}_2^{(2)}(x_1))^\dagger (\bar{\Omega}_1^{(2)})^{-1} \phi_1(x_2)]^{l+1} = [(C_2(x_1))^\dagger [H^{-1}]^l \phi_1(x_2)],
\]

\[
[(\bar{C}_2^{(2)}(x_1))^\dagger (\bar{\Omega}_1^{(2)})^{-1} \phi_1(x_2)]^l = [(\bar{C}_2^{(2)}(x_1))^\dagger (\bar{\Omega}_1^{(2)})^{-1} \phi_1(x_2)]^l,
\]

and, consequently, we conclude

\[
K_{\phi,\phi}^{(1)}(x_1, x_2) - [(\bar{C}_2^{(2)}(x_1))^\dagger (\bar{\Omega}_1^{(2)})^{-1} \phi_1(x_2)]^{l+1} = K_{\phi,\phi}^{(1)}(x_1, x_2).
\]

To prove (38) observe that

\[
(\Omega_1^{(2)})^\dagger (\bar{\Omega}_1^{(2)})^{-1} \bar{C}_1^{(2)}(x_2) = H^{-1} C_1(x_2), \quad (\phi_2(x_1))^\dagger (\Omega_1^{(2)})^\dagger (\bar{\Omega}_1^{(2)})^{-1} = (\phi_2(x_1))^\dagger (\bar{\Omega}_1^{(2)})^{-1},
\]

imply

\[
[(\phi_2(x_1))^\dagger [(\Omega_1^{(2)})^\dagger (\bar{\Omega}_1^{(2)})^{-1}]^l \phi_1(x_2)]^l + [(\phi_2(x_1))^\dagger [(\Omega_1^{(2)})^\dagger (\bar{\Omega}_1^{(2)})^{-1}]^{l+1} \phi_1(x_2)]^{l+1} = [(\phi_2(x_1))^\dagger [H^{-1}]^l C_1(x_2)],
\]

\[
[(\phi_2(x_1))^\dagger [(\Omega_1^{(2)})^\dagger (\bar{\Omega}_1^{(2)})^{-1}]^l \phi_1(x_2)]^l = [(\phi_2(x_1))^\dagger (\bar{\Omega}_1^{(2)})^{-1}]^l \bar{C}_1^{(2)}(x_2)
\]

whose difference is

\[
- [(\bar{C}_1^{(1)}(x_2))^\dagger [H^{-1}]^l \phi_1(x_2)]^{l+1} = \tilde{K}_{\phi,\phi}^{(1)}(x_1, x_2).
\]

Finally, just consider

\[
(\bar{\Omega}_1^{(2)})^{-1} \bar{C}_1^{(2)}(x_2) - \bar{L}^{(2)}(x_2) (\bar{\Omega}_1^{(2)})^{-1} \bar{C}_1^{(2)}(x_2) = -(\bar{\Omega}_1^{(2)})^{-1} \left< \phi_1^{(2)}(z_1), \delta L^{(2)}(\bar{x}_2, z_2) \right>_{\bar{\Omega}_1^{(2)}},
\]

\[
(\phi_2^{(2)}(x_1))^\dagger (\bar{\Omega}_1^{(2)})^{-1} \phi_1^{(2)}(x_2) = \bar{L}^{(2)}(x_1) (\phi_2^{(2)}(x_1))^\dagger H^{-1},
\]

that lead to

\[
[(\phi_2^{(2)}(x_1))^\dagger [\bar{\Omega}_1^{(2)}]^{-1} \phi_1^{(2)}(x_2)]^l \] 

\[
= \left< \phi_1^{(2)}(z_1), \delta L^{(2)}(\bar{x}_2, z_2) \right>_{\bar{\Omega}_1^{(2)}},
\]

\[
[(\phi_2^{(2)}(x_1))^\dagger [\bar{\Omega}_1^{(2)}]^{-1} \phi_1^{(2)}(x_2)]^l + [(\phi_2^{(2)}(x_1))^\dagger [\bar{\Omega}_1^{(2)}]^{-1} \phi_1^{(2)}(x_2)]^{l+1} = \bar{L}^{(2)}(x_1) [(\phi_2^{(2)}(x_1))^\dagger [H^{-1}]^l C_1(x_2)],
\]

that once subtracted provide us with the relation

\[
[(\phi_2^{(2)}(x_1))^\dagger [\bar{\Omega}_1^{(2)}]^{-1} \phi_1^{(2)}(x_2)]^{l+1} \] 

\[
= \bar{L}^{(2)}(x_1) K_{\phi,\phi}^{(1)}(x_1, x_2) - \bar{L}^{(2)}(x_2) \tilde{K}_{\phi,\phi}^{(1)}(x_1, x_2)
\]

\[
+ \left< \tilde{K}_{\phi,\phi}^{(2)}(x_1, z_1), \delta L^{(2)}(\bar{x}_2, z_2) \right>_{\bar{\Omega}_1^{(2)}}.
\]
Recall Corollary 2 and Proposition 17 we have

\[ \left\langle K^{(2),2}(\xi_1, z_1), \frac{L^{(2)}(\xi_2) - L^{(2)}(z_2)}{\xi_2 - z_2} \right\rangle_{u^{(2)}} = \frac{L^{(2)}(\xi_1) - L^{(2)}(z_2)}{\xi_1 - z_2}. \]

and we get the result. \(\square\)

**Proof of Proposition 27** To get (40), observe that for \(l = 0, \ldots, m_j^{(2)} - 1\) we have

\[ \frac{d}{d z} \left|_{z = \xi_l^{(2)}} \phi_{1,k}^{(2)}(z_1), \frac{L^{(2)}(z)}{z - z_2} u(L^{(2)}(z_2))^{-1} \right|_{z = \xi_l^{(2)}} = 0 \]

This holds because

\[ \frac{d^r}{d z^r} \left|_{z = \xi_l^{(2)}} \phi_{1,k}^{(2)}(z_1), \frac{L^{(2)}(z)}{z - z_2} u(L^{(2)}(z_2))^{-1} \right|_{z = \xi_l^{(2)}} = 0 \]

for \(r \in \{0, \ldots, m_j^{(2)} - 1\}\), and since \(\text{supp}_2(u) \cap \sigma(L^{(2)}) = \emptyset\), we get the result. Therefore, we conclude that

\[ \frac{d}{d z} \left|_{z = \xi_l^{(2)}} \frac{L^{(2)}(z)C_{1,k}^{(2)}(z)}{L^{(2)}(z)} \right|_{z = \xi_l^{(2)}} = \sum_{i = 1}^{d^{(2)}} \left\langle \xi_i^{(2)}, \phi_{1,k}^{(2)} \right\rangle \frac{d}{d z} \left|_{z = \xi_l^{(2)}} \frac{L^{(2)}(z)}{z - z_2} \right|_{z = \xi_l^{(2)}} \left[ \frac{(z - \xi_i^{(2)})^{m_i^{(2)} - 1}}{(z - \xi_i^{(2)})^{m_i^{(2)} - 2}} \right. \]

For \(l > 0\), we have

\[ \frac{d}{d z} \left|_{z = \xi_l^{(2)}} \frac{L^{(2)}(z)}{z - \xi_l^{(2)}} \right|^{m_i^{(2)}} = \sum_{r = 0}^{m_i^{(2)}} \frac{1}{m!} \frac{d^r}{d z^r} \left|_{z = \xi_l^{(2)}} \frac{L^{(2)}(z)}{z - \xi_l^{(2)}} \right|^{m_i^{(2)} - 1} \]

But, if \(i \neq j\), \(\left( L^{(2)} \right)^{(s)}(\xi_j^{(2)}) = 0 \) for \(s \in \{0, 1, \ldots, m_j^{(2)} - 1\}\), which is our case because \(l \in \{0, 1, \ldots, m_j^{(2)} - 1\}\); when \(i = j\) we get that only terms with \(r = 0\) will survive and, therefore, \(m_i^{(2)} \leq 1\) with

\[ \frac{d}{d z} \left|_{z = \xi_j^{(2)}} \frac{L^{(2)}(z)}{z - \xi_j^{(2)}} \right|^{m_i^{(2)}} = \xi_j^{(2)} \]

To show (41) let’s compute \( \frac{1}{l!} \frac{d}{d z} \left|_{z = \xi_l^{(2)}} \frac{L^{(1)}(z)}{z - z_1} \right| C_{2,k}^{(1)}(z) \) para \(l = 0, 1, \ldots, m_j^{(1)} - 1\). For that aim we evaluate

\[ \frac{d}{d z} \left|_{z = \xi_l^{(2)}} \frac{L^{(2)}(z)}{z - z_1} \phi_{2,k}^{(1)}(z_2) \right| \left( L^{(1)}(z_1) \right)^{-1} u = \left\langle \frac{d}{d z} \left|_{z = \xi_l^{(2)}} \frac{L^{(2)}(z)}{z - z_1} \phi_{2,k}^{(1)}(z_2) \right| \left( L^{(1)}(z_1) \right)^{-1} u \cdot \right\rangle \]

\[ = \sum_{r = 0}^{l \cdot (l - 1)} \frac{l \cdot (l - 1)}{r!} \frac{d^r}{d z^r} \left|_{z = \xi_l^{(2)}} \frac{1}{(\xi_j^{(1)} - z_1)^{l - r + 1}} \phi_{2,k}^{(1)}(z_2) \right| \left( L^{(1)}(z_1) \right)^{-1} u \]
that, remembering that the zeros are not in the support of the linear functional, vanishes. Finally we realize, that

\[
\frac{1}{l!} \frac{d^l}{d\bar{z}^l} \bigg|_{z = \xi^{(1)}_i} L_{[i]}^{(1)}(\bar{z}) (\bar{z} - \xi^{(1)}_i)^m = \sum_{r=0}^{l-1} \binom{l}{r} \frac{1}{r!} \left( \frac{d^r}{d\bar{z}^r} L_{[i]}^{(1)}(\bar{z}) \right) \left|_{z = \xi^{(1)}_i} \frac{m!}{(l-r)!} (\xi^{(1)}_i - \xi^{(1)}_i)^{m-1+r} \right.
\]

\[
= \begin{cases} 
0, & i \neq j, \\
\xi^{(1)}_{[j],l-m}, & i = j.
\end{cases}
\]

\[\square\]

**Proof of Theorem** To show (42) we proceed as follows. From (33) we get

\[
\partial_{[2]}^{L_{[2]}^{(1)}} C_{1,1}^{(2)} = \left[ (\Omega_1^{(2)})_{1,1-2n}, \ldots, (\Omega_1^{(2)})_{1,1-1} \right] \begin{bmatrix} \partial_{L_{[2]}^{(1)}} \omega_{C,1,2n}^{(2)} \\ \vdots \\ \partial_{L_{[2]}^{(1)}} \omega_{C,1,1-1}^{(2)} \end{bmatrix} + \partial_{C,1,1}^{L_{[2]}^{(2)}}.
\]

Using (40) and

\[(\Omega_1^{(2)})_{1,1-2n} \phi_{1,1-2n}(z) + \cdots + (\Omega_1^{(2)})_{1,1-1} \phi_{1,1-1}(z) + \phi_{1,1}(z) = \tilde{\phi}_1^{(2)}(z),\]

see Proposition[16], we deduce that

\[
\partial_{[2]}^{L_{[2]}^{(1)}} C_{1,1}^{(2)} = \left[ (\Omega_1^{(2)})_{1,1-2n}, \ldots, (\Omega_1^{(2)})_{1,1-1} \right] \begin{bmatrix} \langle \tilde{\xi}^{(2)}, \phi_{1,1-2n} \rangle \\ \vdots \\ \langle \tilde{\xi}^{(2)}, \phi_{1,1-1} \rangle \end{bmatrix} L^{(2)} + \langle \tilde{\xi}^{(2)}, \phi_{1,1-1} \rangle L^{(2)},
\]

and, consequently,

\[
\left[ (\Omega_1^{(2)})_{1,1-2n}, \ldots, (\Omega_1^{(2)})_{1,1-1} \right] = -\left( \partial_{C,1,1}^{L_{[2]}^{(2)}} - \langle \tilde{\xi}^{(2)}, \phi_{1,1-1} \rangle L^{(2)} \right)^{-1} \begin{bmatrix} \partial_{C,1,1-2n}^{L_{[2]}^{(2)}} - \langle \tilde{\xi}^{(2)}, \phi_{1,1-2n} \rangle L^{(2)} \\ \vdots \\ \partial_{C,1,1-1}^{L_{[2]}^{(2)}} - \langle \tilde{\xi}^{(2)}, \phi_{1,1-1} \rangle L^{(2)} \end{bmatrix}.
\]

Recalling (49) we conclude with the proof of this Christoffel formula. From this very equation, we obtain

\[(\Omega_1^{(2)})_{1,1-2n} = -\left( \partial_{C,1,1}^{L_{[2]}^{(2)}} - \langle \tilde{\xi}^{(2)}, \phi_{1,1-1} \rangle L^{(2)} \right)^{-1} \begin{bmatrix} \partial_{C,1,1-2n}^{L_{[2]}^{(2)}} - \langle \tilde{\xi}^{(2)}, \phi_{1,1-2n} \rangle L^{(2)} \\ \vdots \\ \partial_{C,1,1-1}^{L_{[2]}^{(2)}} - \langle \tilde{\xi}^{(2)}, \phi_{1,1-1} \rangle L^{(2)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]

and recalling (27) we get (46).

For the proof of (45), realize that the relation (32) gives

\[(\Omega_2^{(1)})_{1,1-2n} \phi_{2,1-2n}(z) + \cdots + (\Omega_2^{(1)})_{1,1-1} \phi_{2,1-1}(z) + \phi_{2,1}(z) = \tilde{\phi}_2^{(1)}(z).\]
Now, taking into account (41) we conclude

\[(51) \quad (\Omega^{(1)}_2)_{1,1-2n, \ldots, (\Omega^{(2)}_2)_{1,1-1} = - \left( \frac{\partial \Omega^{(1)}_2}{\partial \Omega^{(1)}_2} - \langle \xi^{(1)}, \phi_{2,1} \rangle \mathcal{L}^{(1)} \right)^{-1} \left[ \begin{array}{c} \partial \Omega^{(1)}_2 - \langle \xi^{(1)}, \phi_{2,1-2n} \rangle \mathcal{L}^{(1)} \\ \vdots \\ \partial \Omega^{(1)}_2 - \langle \xi^{(1)}, \phi_{2,1-1} \rangle \mathcal{L}^{(1)} \end{array} \right] \]

from where the Christoffel-Geronimus formula follows.

This equation also implies that

\[(52) \quad (\Omega^{(1)}_2)_{1,1-2n} = - \left( \frac{\partial \Omega^{(1)}_2}{\partial \Omega^{(1)}_2} - \langle \xi^{(1)}, \phi_{2,1} \rangle \mathcal{L}^{(1)} \right)^{-1} \left[ \begin{array}{c} 1 \\ \vdots \\ 0 \end{array} \right], \]

and recalling (27) we get (43).

We will prove (47) and (44) simultaneously. Let’s write (36) and (39) as follows

\[
\sum_{k=0}^{L-1} \phi^{(1)}_k(x_1, x_2) (H^{(1)}_k)^{-1} \phi^{(1)}(x_2) = L^{(1)}(x_2) K^{(1)}_{C, \phi}(\bar{x}_1, x_2) - \delta L^{(1)}(\bar{x}_1, x_2)
\]

\[
\sum_{k=0}^{L-1} \phi^{(2)}_k(x_1) (H^{(2)}_k)^{-1} \phi^{(2)}(x_2) = \bar{L}^{(2)}(\bar{x}_1) K^{(1)}_{C, \phi}(\bar{x}_1, x_2) - \delta \bar{L}^{(2)}(\bar{x}_1, x_2)
\]

and let’s compute the spectral jets, with respect to the first and second variables, respectively

\[
\sum_{k=0}^{L-1} (H^{(1)}_k)^{-1} \phi^{(1)}_k(x_2) \frac{\partial (H^{(1)}_k)}{\partial \phi^{(1)}_k} (x_2) = \frac{\partial L^{(1)}(x_2)}{\partial \phi^{(1)}_k} (x_2) - \frac{\partial L^{(1)}}{\partial \phi^{(1)}_k} (x_2)
\]

\[
\sum_{k=0}^{L-1} \phi^{(2)}_k(x_1) (H^{(2)}_k)^{-1} \frac{\partial (H^{(2)}_k)}{\partial \phi^{(2)}_k} (x_2) = \frac{\partial \bar{L}^{(2)}(\bar{x}_1)}{\partial \phi^{(2)}_k} (\bar{x}_1) - \frac{\partial \bar{L}^{(2)}(\bar{x}_1)}{\partial \phi^{(2)}_k} (\bar{x}_1)
\]
From (40) and (41) we deduce

\[
\sum_{k=0}^{l-1} (\tilde{H}^{(1)}_k)^{-1} \Phi^{(1)}_{1,k}(x_2) \left\langle \xi^{(1)}, \Phi^{(1)}_{2,k} \right\rangle \mathcal{L}^{(1)} = L^{(1)}(x_2) J^{(1)}_{\mathcal{C},\Phi}(x_2) - J^{(1)}_L(x_2)
\]

\[
= - \left[ \Phi^{(1)}(x_2), \ldots, \Phi^{(1)}_{1,l+2n-1}(x_2) \right] (\tilde{H}^{(1)}_{n,l})^{-1} \Omega^{(1)}_2[n, l]
\]

Now, from (34) and (35), we get

\[
\sum_{k=0}^{l-1} \Phi^{(2)}_{2,k}(x_1)(\tilde{H}^{(2)}_k)^{-1} \left\langle \xi^{(2)}, \Phi^{(2)}_{1,k} \right\rangle \mathcal{L}^{(2)} = L^{(2)}(x_1) J^{(2)}_{\mathcal{C},\Phi}(x_1) - J^{(2)}_L(x_1)
\]

\[
= - \left[ \Phi^{(2)}_{2,1}(x_1), \ldots, \Phi^{(2)}_{2,l+2n-1}(x_1) \right] (\tilde{H}^{(2)}_{n,l})^{-1} \Omega^{(2)}_1[n, l]
\]

Recalling (5) we get

\[
\left\langle \xi^{(1)}, \tilde{K}^{(1)}(z, x_2) \right\rangle \mathcal{L}^{(1)} = L^{(1)}(x_2) J^{(1)}_{\mathcal{C},\Phi}(x_2) + J^{(1)}_L(x_2)
\]

\[
- \left[ \Phi^{(1)}(x_2), \ldots, \Phi^{(1)}_{1,l+2n-1}(x_2) \right] (\tilde{H}^{(1)}_{n,l})^{-1} \Omega^{(1)}_2[n, l]
\]

\[
\left\langle \xi^{(2)}, \tilde{K}^{(2)}(x_1, z) \right\rangle \mathcal{L}^{(2)} = L^{(2)}(x_1) J^{(2)}_{\mathcal{C},\Phi}(x_1) + J^{(2)}_L(x_1)
\]

\[
- \left[ \Phi^{(2)}_{2,1}(x_1), \ldots, \Phi^{(2)}_{2,l+2n-1}(x_1) \right] (\tilde{H}^{(2)}_{n,l})^{-1} \Omega^{(2)}_1[n, l]
\]
Therefore, we conclude

\[
\begin{align*}
(L^{(1)}(x_2)\left( I^{(1)}_{K_{C,\phi}}(x_2) - \left( (\xi^{(1)}_z)_{\bar{z}}, K^{(1)}(\bar{z}, x_2) \right) L^{(1)} \right) + J^L_{\bar{S}L_1}(x_2) & \right. \\
\left. \left. \begin{bmatrix}
J_{C_{2,1-2n}}^{(1)} - \left( (\xi^{(1)}_z)_{\bar{z}}, \Phi_{2,1-2n}(\bar{z}) \right) L^{(1)} \\
\vdots \\
J_{C_{2,1-1}}^{(1)} - \left( (\xi^{(1)}_z)_{\bar{z}}, \Phi_{2,1-1}(\bar{z}) \right) L^{(1)}
\end{bmatrix}^{-1} \right]
\end{align*}
\]

\[
\left. \begin{align*}
\Phi_{1,1}^{(1)}(x_2), \ldots, \Phi_{1,1+2n-1}^{(1)}(x_2) \right) \left( \bar{H}^{(1)}[n, l] \right)^{-1} \Omega_{2,1}^{(1)}[n, l],
\end{align*}\]

and, in particular, recalling (27), we obtain

\[
\Phi_{1,1}^{(1)}(x_2)L_{-1}^{(1)}n
\]

\[
\left( L^{(1)}(x_2)\left( I^{(1)}_{K_{C,\phi}}(x_2) - \left( (\xi^{(1)}_z)_{\bar{z}}, K^{(1)}(\bar{z}, x_2) \right) L^{(1)} \right) + J^L_{\bar{S}L_1}(x_2) \right).
\]

\[
\Phi_{2,1}^{(2)}(x_1)L_{-1}^{(2)}n
\]

\[
\left( L^{(2)}(x_1)\left( I^{(2)}_{K_{C,\phi}}(x_1) - \left( (\xi^{(2)}_z)_{\bar{z}}, K^{(2)}(\bar{z}, x_1) \right) L^{(2)} \right) + J^L_{\bar{S}L_2}(x_1) \right).
\]

\[
\begin{align*}
\left[ H^{l-2n} \right] & \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
\left[ H^{l-2n} \right] & \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix},
\end{align*}
\]

\[
\Box
\]

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Sur la suite de polynômes orthogonaux associée à la forme

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