Representing Simple $d$-Dimensional Polytopes by $d$ Polynomials*

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Abstract

A polynomial representation of a convex $d$-polytope $P$ is a finite set \{\(p_1(x), \ldots, p_n(x)\)\} of polynomials over $\mathbb{R}^d$ such that $P = \{x \in \mathbb{R}^d : p_i(x) \geq 0 \text{ for every } 1 \leq i \leq n\}$. By $s(d, P)$ we denote the least possible number of polynomials in a polynomial representation of $P$. It is known that $d \leq s(d, P) \leq 2d - 1$. Moreover, it is conjectured that $s(d, P) = d$ for all convex $d$-polytopes $P$. We confirm this conjecture for simple $d$-polytopes by providing an explicit construction of $d$ polynomials that represent a given simple $d$-polytope $P$.

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1 Introduction

The Euclidean space of dimension $d \geq 2$ is denoted by $\mathbb{R}^d$. The origin, scalar product, and norm in $\mathbb{R}^d$ are denoted by $o$, $\langle \cdot, \cdot \rangle$, and $|\cdot|$, respectively. In analytic expressions points of $\mathbb{R}^d$ are treated as real column vectors of length $d$. The transposition is denoted by $(\cdot)^\top$.

Let $x$ be a vector variable in $\mathbb{R}^d$. Given a finite set $\mathcal{P}$ of polynomials from $\mathbb{R}[x]$, the sets

\[ S_0 := \{x \in \mathbb{R}^d : p(x) > 0 \ \forall \ p \in \mathcal{P}\} \quad \text{and} \quad S := \{x \in \mathbb{R}^d : p(x) \geq 0 \ \forall \ p \in \mathcal{P}\} \]

are called basic open and basic closed semi-algebraic set represented by $\mathcal{P}$, respectively. Let $s(d, S_0)$ and $s(d, S)$ stand for the least cardinality of a set of polynomials representing $S_0$ and $S$, respectively. It is known that

\begin{align*}
\max_{S_0} s(d, S_0) & = d, \quad (1.1) \\
\max_{S} s(d, S) & = \frac{d(d + 1)}{2}. \quad (1.2)
\end{align*}

This was shown by Bröcker and Scheiderer [Bro84, Sch89, Bro91, BCR98 §6.5, §10.4]; some extensions are given in [ABR96 Chapter 5], and a modified proof is presented in [Mah89] and [BM98]. The known proofs of (1.1) and (1.2) are non-constructive. More precisely, explicit procedures for constructing the sets of polynomials representing a general $S_0$ (resp. $S$) and having cardinality at most $d$ (resp. $d(d + 1)/2$) are not known, since the available proofs are based on some non-constructive existence theorems.

A set $P$ in $\mathbb{R}^d$ is a convex polyhedron if it is a non-empty intersection of a finite number of half-spaces. A convex polyhedron $P \subseteq \mathbb{R}^d$ is said to be a convex polytope if it is bounded and a $d$-polytope if it is bounded and of dimension $d$. In this paper we study the quantity $s(d, P)$, where $P$ is a $d$-polytope. A $d$-polytope is said to be simple if each of its vertices is contained in precisely $d$ facets. By $\text{vert}(P)$ we denote the set of all vertices of $P$. We refer to [Zie95] for the background information on convex polytopes. A set of polynomials representing a convex polyhedron $P$ in $\mathbb{R}^d$ is called a polynomial representation of $P$. Thus, polynomial representations are generalization of $H$-representations, cf. [Zie95] p. 28. In [CH03]

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Let us enumerate known constructive results on \(s(d, P)\), see also the survey \cite{Hen07}. Improving a result of vom Hofe \cite{vH92} Bernig \cite{Ber98} showed that \(s(2, P) = 2\) for every convex polygon \(P\) in \(\mathbb{R}^2\), see Section 2 for more details for that case. For an arbitrary dimension Gr"otschel and Henk \cite{GH03} constructed \(O(d^d)\) polynomials representing a simple \(d\)-polytope and pointed out the lower bound \(s(d, P) \geq d\) for all \(d\)-polytopes \(P\). The smallest known upper bounds for \(s(d, P)\) were given in \cite{BGH05} and \cite{Bos05}. More precisely, in \cite{BGH05}, it was shown that

- \(s(d, P) \leq 2d - 2\) for pointed \(d\)-dimensional cones,
- \(s(d, P) \leq 2d - 1\) for \(d\)-polytopes,
- \(s(d, P) \leq 2d\) for \(d\)-polyhedra.

Each of the above three bounds has a constructive proof. In \cite{BGH05}, Section 1 it was conjectured that \(s(d, P) = d\) for every convex \(d\)-polytope \(P\) in \(\mathbb{R}^d\). The aim of this paper is to confirm this conjecture for the class of simple \(d\)-polytopes, see Theorem 1.1 below. We recall that a \(d\)-polytope is simple if each of its vertices is incident with precisely \(d\) facets. Our construction involves elementary symmetric polynomials defined by

\[
\sigma_l(y) := \sigma_l(y_1, \ldots, y_m) := \sum_{J \subseteq \{1, \ldots, m\}, \# J = l} \prod_{j \in J} y_j,
\]

where \(y := [y_1, \ldots, y_m]^T \in \mathbb{R}^m\) and \(#\) stands for the cardinality. We also put \(\sigma_0(y) := 1\) and \(\sigma_l(y) := 0\) for \(l < 0\) and \(l > m\).

**Theorem 1.1.** Let \(P\) be a simple \(d\)-polytope in \(\mathbb{R}^d\). Then \(s(d, P) = d\). Furthermore, assume that \(P\) has \(m\) facets and is given by affine inequalities \(q_1(x) \geq 0, \ldots, q_m(x) \geq 0\). Then

\[
P = \{ x \in \mathbb{R}^d : p_i(x) \geq 0 \text{ for } 0 \leq i \leq d - 1 \}.
\]

where

\[
p_{d-1}(x) := \sigma_m(q_1(x), \ldots, q_m(x)), \quad \ldots
\]

\[
p_i(x) := \sigma_{m-d+i+1}(q_1(x), \ldots, q_m(x)), \quad \ldots
\]

\[
p_1(x) := \sigma_{m-d+2}(q_1(x), \ldots, q_m(x))
\]

and

\[
p_0(x) := 1 - \sum_{v \in \text{vert}(P)} y_v \left( \frac{1}{d} \sum_{j = 1, \ldots, m, q_j(v) = 0} \left( 1 - \lambda_j q_j(x) \right)^{2k} \right)
\]

with appropriate \(k \in \mathbb{N}\), \(y_v > 0\) and \(\lambda_j > 0\). \(\square\)

We notice that for \(p_0(x)\) from Theorem 1.1, \(p_i(x)\) vanishes on each \(i\)-face of \(P\) for \(i \in \{0, \ldots, d - 1\}\). As a direct consequence of Theorem 1.1 we obtain that the polynomials \(p_i(x), 0 \leq i \leq d - 1\), from Theorem 1.1 represent the interior of \(P\). Thus, there exists a constructive proof of 1.1 for the special case when \(S_0\) is the interior of a simple polytope.

As a consequence of the Positivstellensatz it can be derived that every polynomial \(p(x)\) which is non-negative on \(P\) can be represented by

\[
p(x) = \sum_i f_i(x) \sum_{j=1}^m q_j(x)^{l(j)},
\]
where \( l \) ranges over maps from \( \{1, \ldots, m\} \) to \( \mathbb{N} \cup \{0\} \) and \( f_l \) are non-negative polynomials on \( \mathbb{R}^d \) (see [BCR98] p. 106). In our construction the polynomials are even of a more specific type, namely, such that \( f_l(x) = \text{const} \) for every \( l \). It turns out that it is reasonable to consider the polynomials of these form, see [GH03] p. 487, [Han88], and [PR01]. In fact, such polynomials were also used in the previous papers.

The paper is organized as follows. In Section \( \ref{sec:examples} \) we illustrate the statement of Theorem 1.1 by several examples. In Section \( \ref{sec:proof} \) we give the proof of Theorem 1.1. Estimates which allow to explicitly determine the possible choice of the parameter \( k \) involved in the construction of \( p_0(x) \) are given in Section \( \ref{sec:estimates} \).

## 2 Examples of polynomial representations

Let us illustrate the case \( d = 2 \). This case was completely settled by Bernig. Since convex polygons are simple polytopes, the case \( d = 2 \) is also covered by Theorem 1.1 The polynomial \( p_0(x) \) describes a semi-algebraic set \( \{ x \in \mathbb{R}^d : p_0(x) \geq 0 \} \) which is sufficiently close to \( P \). In [Ber98] it was proved that if \( P \) is a convex \( m \)-gon given by affine inequalities \( q_1(x) \geq 0, \ldots, q_m(x) \geq 0 \), then a strictly concave polynomial \( p_0(x) \) vanishing on each vertex of \( P \) can be constructed such that \( p_0(x) \) together with the polynomial \( p_1(x) := q_1(x) \cdot \cdots \cdot q_m(x) \) form a polynomial representation of \( P \) (see also Fig. \( \ref{fig:bernig} \)).

![Figure 1. Bernig's construction; the region shaded by \( \square \) is \( P \), \( \square \) is \( \{ x \in \mathbb{R}^2 : p_1(x) \geq 0 \} \), \( \infty \) is the boundary of \( \{ x \in \mathbb{R}^2 : p_0(x) \geq 0 \} \) ](image)

We illustrate Theorem 1.1 for the case \( d = 3 \) by some concrete choices of \( P \). For \( J \subset \{0, \ldots, d-1\} \) with \( J \neq \emptyset \) we use the notation \( P_J := \{ x \in \mathbb{R}^d : p_j(x) \geq 0 \text{ for } j \in J \} \). By Theorem 1.1 one has \( P = P_J \) for \( J = \{0, \ldots, d-1\} \).

If \( P \) is a regular tetrahedron with vertices

\[
\begin{align*}
v_1 &:= [1, -1, 1]^\top, & v_2 &:= [-1, 1, 1]^\top, \\
v_3 &:= [1, 1, -1]^\top, & v_4 &:= [-1, -1, -1]^\top,
\end{align*}
\]

then we can choose

\[
\begin{align*}
q_1(x) &:= 1 + x_1 - x_2 + x_3, & q_2(x) &:= 1 - x_1 + x_2 + x_3, \\
q_3(x) &:= 1 + x_1 + x_2 - x_3, & q_4(x) &:= 1 - x_1 - x_2 - x_3.
\end{align*}
\]

In this case

\[
\begin{align*}
p_2(x) &= q_1(x)q_2(x)q_3(x)q_4(x) \\
&= 1 - 2x_1^2 - 2x_2^2 - 2x_3^2 - 8x_1x_2x_3 - 2x_1^2x_2^2 - 2x_1^2x_3^2 - 2x_2^2x_3^2 + x_1^4 + x_2^4 + x_3^4 \\
p_1(x) &= q_1(x)q_2(x)q_3(x) + q_1(x)q_2(x)q_4(x) + q_1(x)q_3(x)q_4(x) + q_2(x)q_3(x)q_4(x) \\
&= 4(1 - x_1^2 - x_2^2 - x_3^2 - 2x_1x_2x_3),
\end{align*}
\]

and thus the boundary of \( P_J \) is the well-known Cayley cubic. Fig. \( \ref{fig:cayley} \) depicts all possible \( P_J \) in a diagram where an arrow is drawn from the image of \( P_{J_1} \) to the image of \( P_{J_2} \) whenever \( J_1 \subset J_2 \). We wish to illustrate the properties of \( p_1(x), p_2(x) \) from Theorem 1.1 rather than the properties of \( p_0(x) \). Therefore, we choose \( p_0(x) \) having a simpler form than in Theorem 1.1, namely \( p_0(x) := 3 - x_1^2 - x_2^2 - x_3^2 \) so that \( P_0 \) is a ball of radius \( \sqrt{3} \).
Now let $P$ be the cube given by $P := \{ x \in \mathbb{R}^3 : |x_i| \leq 1 \text{ for } 1 \leq i \leq 3 \}$. Then we can take
\[
q_1(x) := -x_1, \quad q_2(x) := -x_2, \quad q_3(x) := -x_3, \\
q_4(x) := +x_1, \quad q_5(x) := +x_2, \quad q_6(x) := +x_3.
\]
We have
\[
p_2(x) = (1 - x_1^2)(1 - x_2^3)(1 - x_3^2), \\
p_1(x) = 2(3 - 2x_1^2 - 2x_2^2 - 2x_3^2 + x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2).
\]
We can choose $p_0(x)$ in the same way as for the previous example. The diagram depicting $P_J$ is given in Fig. 3. One can see that the boundary of $P_1$ is a surface sharing some properties with the Cayley cube, namely every vertex of $P$ is the conic double point of the mentioned surface. Thus, for a general simple 3-polytope $P$ the boundary of $P_1$ can be viewed as a generalized Cayley surface assigned to $P$.

Singularities of algebraic surfaces are discussed in [Zar95, Section 5 of Chapter I], [Băd01, Chapters 3,4], and [GP02, Section A.9].

3 The proof

3.1 Preliminaries

In what follows, $P$ is a $d$-polytope in $\mathbb{R}^d$ and $\mathcal{F}_i$ denotes the class of all $i$-faces of $P$. Given $F \in \mathcal{F}_{d-1}$, $u_F$ stands for the outward unit normal of $P$ at the facet $F$. By $\text{diam}(P)$ we denote the diameter of $P$, which is equal to the largest possible distance between two vertices of $P$. With each $F \in \mathcal{F}_{d-1}$ we associate affine functions
\[
q_F(x) := \frac{h(P, u_F) - \langle u_F, x \rangle}{\text{diam}(P)},
\]
where
\[
h(P, u) := \max \{ \langle x, u \rangle : x \in P \}, \quad u \in \mathbb{R}^d.
\]
is the support function of $P$. We have $0 \leq q_F(x) \leq 1$ with $q_F(x) = 0$ for all $x \in F$. In what follows $m$ always denotes the number of facets in $P$.

In many cases we shall consider matrices and vectors indexed by the elements of $\mathcal{F}_{d-1}$ and $\text{vert}(P)$ rather than by segments of natural numbers, which is possible if some linear order on each of these two classes is assumed to be fixed. For example, we introduce the affine mapping

$$q(x) := [q_F(x)]_{F \in \mathcal{F}_{d-1}} = [q_{F_1}(x), \ldots, q_{F_m}(x)]^\top,$$

where $F_1, \ldots, F_m$ is a sequence of all facets of $P$ that determines an order on $\mathcal{F}_{d-1}$. For each $v \in \text{vert}(P)$ we also introduce the set

$$\mathcal{F}_{d-1}^v := \{F \in \mathcal{F}_{d-1} : v \in F\}$$

and the affine functions

$$q_v(x) := [q_v(x)]_{v \in \mathcal{F}_{d-1}^v},$$
$$\bar{q}_v(x) := [q_v(x)]_{v \in \mathcal{F}_{d-1} \setminus \mathcal{F}_{d-1}^v}.$$

### 3.2 Lemmas on $\varepsilon_1, \varepsilon_2, \varepsilon_3$

Given $\varepsilon > 0$ consider the polytope

$$P_\varepsilon := \{x \in \mathbb{R}^d : q_F(x) \geq -\varepsilon \text{ for } F \in \mathcal{F}_{d-1}\},$$

see also Fig. 4.

**Lemma 3.1.** Let $P$ be a simple $d$-polytope. Then there exists an $\varepsilon_1 > 0$ such that $\sigma_i(q(x)) > 0$ for $1 \leq i \leq m - d$ and $x \in P_{\varepsilon_1}$.

**Proof.** By (1.3) we see that $\sigma_i(q(x)) > 0$ for $1 \leq i \leq m - d$ and $x \in P$. In fact, in view of (1.3) the polynomial $\sigma_i(q(x))$ is given as a sum, where each summand is represented as a product of at most $m - d$ polynomials from the class $q_F(x), F \in \mathcal{F}_{d-1}$. But for $x \in P$ all values $q_F(x), F \in \mathcal{F}_{d-1}$, are
non-negative and at most \( d \) values from \( q_F(x) \), \( F \in \mathcal{F}_{d-1} \), vanish. Consequently, at least one of the mentioned summands is strictly positive and hence \( \sigma_i(q(x)) > 0 \). In view of the continuity of \( \sigma_i(q(x)) \) we obtain the assertion. \( \square \)

The non-negative orthant can be represented as the set where all elementary symmetric functions are non-negative. In [Ber98] this statement was derived from the Descartes’ rule of signs (see [BCR98] Proposition 1.2.14] for the statement and a short proof of the Descartes’ rule). Below we give an alternative direct proof.

**Proposition 3.2.** (Bernig, [Ber98] p. 38). Let \( d \geq 2 \). Then

\[
\{ x \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0 \} = \{ x \in \mathbb{R}^d : \sigma_1(x) \geq 0, \ldots, \sigma_d(x) \geq 0 \}.
\]

**Proof.** The inclusion “\( \subseteq \)” is trivial. Let us prove the reverse inclusion. Assume that \( \sigma_i(x) \geq 0 \) for \( 1 \leq i \leq d \). Let \( p(t) := (t + x_1) \cdots (t + x_d) \). By Vieta’s formulas

\[
p(t) = \sum_{i=0}^{n} \sigma_{n-i}(x)t^i,
\]

The polynomial \( p(t) \) is not identically equal to zero. Since all its coefficients are non-negative, it cannot have positive real roots. Thus, all its roots \( -x_j, 1 \leq j \leq d \), are non-positive, and we are done. \( \square \)

We observe that \( \sigma_i(x) = O(|x|^i), 1 \leq i \leq d \), since \( \sigma_i(x) \leq |x|^i \max \{ \sigma_i(u) : u \in \mathbb{R}^d, |u| = 1 \} \). Notice also that for \( x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2} \), and \( z := [x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}]^T \in \mathbb{R}^{n_1+n_2} \) one has

\[
\sigma_i(z) = \sum_{j=-\infty}^{+\infty} \sigma_{i-j}(x)\sigma_j(y), \quad (3.1)
\]

where \( 1 \leq i \leq n_1 + n_2 \). In (3.1) only the items with \( 0 \leq i - j \leq n_1 \) and \( 0 \leq j \leq n_2 \) (equivalently \( \max\{0, n_1 - i\} \leq j \leq \min\{n_2, i\} \) can be non-zero.

Given \( v \in \text{vert}(P) \) and \( \varepsilon > 0 \) we introduce the sets

\[
\Pi_{v, \varepsilon} := \{ x \in \mathbb{R}^d : |q_v(x)|_{\infty} \leq \varepsilon \},
\]

\[
C_v := \{ x \in \mathbb{R}^d : -\sigma_1(q_v(x)) \geq \frac{2}{3}|q_v(x)| \},
\]

see Figs. \( \ref{fig:1} \) \( \ref{fig:2} \).

It can be seen that \( \Pi_{v, \varepsilon} \) is a small polytope enclosing \( v \). The set \( C_v \) is a convex cone with apex at \( v \). This follows from the fact that the function \( \frac{2}{3}|z| - \sigma_1(z), z \in \mathbb{R}^d \), is sublinear (see [Sch93] p.26). Furthermore, \( C_v \cap P = \{ v \} \) and

\[
P \subseteq 2v - C_v = \{ x \in \mathbb{R}^d : \sigma_1(q_v(x)) \geq \frac{2}{3}|q_v(x)| \}.
\]

Notice that \( 2v - C_v \) is the reflection of \( C_v \) with respect to \( v \).

**Lemma 3.3.** Let \( P \) be a simple \( d \)-polytope. Then there exists an \( \varepsilon_2 > 0 \) such that for every \( v \in \text{vert}(P) \)

\[
\{ x \in \Pi_{v, \varepsilon_2} : \sigma_i(q(x)) \geq 0 \text{ for } m - d + 2 \leq i \leq m \} \subseteq P \cup C_v.
\]
Proof. Let \( \varepsilon_1 \) be as in the statement of Lemma 3.1 and let us consider an arbitrary \( v \in \text{vert}(P) \). We have

\[
\sigma_{m-d+1}(q(x)) = \sum_{i=1}^{+\infty} \sigma_i(q_v(x))\sigma_{m-d+1-i}(\bar{q}_v(x)) = \sigma_1(q_v(x))\sigma_{m-d}(\bar{q}_v(x)) + r_1(x) \tag{3.2}
\]

\[
\sigma_{m-d+2}(q(x)) = \sum_{i=2}^{+\infty} \sigma_i(q_v(x))\sigma_{m-d+2-i}(\bar{q}_v(x)) = \sigma_2(q_v(x))\sigma_{m-d}(\bar{q}_v(x)) + r_2(x)
\]

\[
= \frac{1}{2}\sigma_{m-d}(\bar{q}_v(x)) (\sigma_1(q_v(x))^2 - |q_v(x)|^2) + r_2(x)
\]

\[
= g_1(x) (g_2(x)\sigma_1(q_v(x))^2 - |q_v(x)|^2), \tag{3.3}
\]

where the functions

\[
r_1(x) := \sum_{i=2}^{+\infty} \sigma_i(q_v(x))\sigma_{m-d+1-i}(\bar{q}_v(x)), \quad g_1(x) := \frac{1}{2}\sigma_{m-d}(\bar{q}_v(x)) - \frac{r_2(x)}{|q_v(x)|^2},
\]

\[
r_2(x) := \sum_{i=3}^{+\infty} \sigma_i(q_v(x))\sigma_{m-d+2-i}(\bar{q}_v(x)), \quad g_2(x) := \frac{\sigma_{m-d}(\bar{q}_v(x))}{\sigma_{m-d}(\bar{q}_v(x)) - \frac{r_2(x)}{|q_v(x)|^2}}
\]

are such that

\[
r_1(x) = O(|q_v(x)|^2), \quad g_1(x) \rightarrow \frac{1}{2}\sigma_{m-d}(\bar{q}_v(v)) > 0,
\]

\[
r_2(x) = O(|q_v(x)|^3), \quad g_2(x) \rightarrow 1,
\]

as \( x \rightarrow v \). Consequently, we can choose an \( \varepsilon_v \) with \( 0 < \varepsilon_v \leq \varepsilon_1 \) such that for every \( x \in \Pi_{v,\varepsilon_v} \)

\[
q_F(x) > 0 \quad \text{for } F \in \mathcal{F}_{d-1} \setminus \mathcal{F}_d^v, \tag{3.4}
\]

\[
g_1(x) > 0, \tag{3.5}
\]

\[
g_2(x) \leq \frac{9}{4}, \tag{3.6}
\]

\[
|r_1(x)| \leq \frac{1}{3}|q_v(x)|\sigma_{m-d}(\bar{q}_v(x)). \tag{3.7}
\]

From now on, let us assume that \( x \) belongs to \( \Pi_{v,\varepsilon_v} \) and satisfies

\[
\sigma_i(q(x)) \geq 0, \quad m - d + 2 \leq i \leq m. \tag{3.8}
\]

Then

\[
0 \leq \sigma_{m-d+2}(q(x)) \leq g_1(x) \left( \frac{9}{4}\sigma_1(q_v(x))^2 - |q_v(x)|^2 \right),
\]

which implies that the inequality

\[
-\sigma_1(q_v(x)) \geq \frac{2}{3}|q_v(x)|
\]

or the inequality

\[
\sigma_1(q_v(x)) \geq \frac{2}{3}|q_v(x)|
\]

(3.9)
is fulfilled. In the former case we get $x \in C_v$. In the latter case we have
\[
\sigma_{m-d+1}(q(x)) \geq \frac{2}{3} |q_v(x)||\sigma_{m-d}(q_v(x))| + r_1(x) \geq \frac{1}{3} |q_v(x)||\sigma_{m-d}(q_v(x))| \geq 0.
\]
In view of $\varepsilon_v \leq \varepsilon_1$ and (3.4) we get $\Pi_{v,\varepsilon_v} \subseteq \mathcal{P}_v$. Hence, by Lemma 3.1, $\sigma_i(q(x)) \geq 0$ for every $1 \leq i \leq m - d$. Summarizing we get that $\sigma_i(q(x)) \geq 0$ for every $1 \leq i \leq m$. But then, by Proposition 3.2, it follows that $q_\mathcal{F}(x) \geq 0$ for all $F \in \mathcal{F}_{d-1}$, i.e., $x \in \mathcal{P}$. Thus, the assertion is valid by putting $\varepsilon_2 := \min_{v \in \text{vert}(\mathcal{P})} \varepsilon_v$. 

**Lemma 3.4.** Let $P$ be a simple $d$-polytope. Then there exists a scalar $\varepsilon_3 > 0$ such that
\[
\{x \in P_{\varepsilon_3} : \sigma_i(q(x)) \geq 0 \text{ for } m - d + 2 \leq i \leq m \} \subseteq \mathcal{P} \cup \bigcup_{v \in \text{vert}(\mathcal{P})} C_v.
\]

*Proof.* Let us choose scalars $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ with $\varepsilon_2 \leq \varepsilon_1$ as in the statements of Lemmas 3.1 and 3.4, respectively. If $x \in \mathcal{P}$, then $\sigma_{m-d+1}(q(x)) \geq 0$ with equality if and only if $x$ is a vertex of $\mathcal{P}$. This yields that $\sigma_{m-d+1}(q(x)) > 0$ for $x \in \mathcal{P} \setminus \bigcup_{v \in \text{vert}(\mathcal{P})} \Pi_{v,\varepsilon_2}$. In view of the continuity of $\sigma_{m-d+1}(q(x))$, there exists a scalar $\varepsilon_3$ with $0 < \varepsilon_3 \leq \varepsilon_2$ such that
\[
\sigma_{m-d+1}(q(x)) > 0 \quad \text{for } x \in P_{\varepsilon_3} \setminus \bigcup_{v \in \text{vert}(\mathcal{P})} \Pi_{v,\varepsilon_2}.
\]

Then (3.10) is fulfilled for $\varepsilon_3$ as above. In fact, by construction $\varepsilon_3 \leq \varepsilon_2 \leq \varepsilon_1$. Let $x \in P_{\varepsilon_3}$ be such that $\sigma_i(q(x)) \geq 0$ for $m - d + 2 \leq i \leq m$. If $x \in \Pi_{v,\varepsilon_2}$ for some $v \in \text{vert}(\mathcal{P})$, by Lemma 3.3 we conclude that $x \in \mathcal{C}_v \cup \mathcal{P}$. Otherwise, $x \in P_{\varepsilon_3} \setminus \bigcup_{v \in \text{vert}(\mathcal{P})} \Pi_{v,\varepsilon_2}$, and by Lemma 3.1 together with (3.11) we deduce that $\sigma_i(q(x)) \geq 0$ for $1 \leq i \leq m$. Hence, by Proposition 3.2 $q_\mathcal{F}(x) \geq 0$ for $F \in \mathcal{F}_{d-1}$, i.e., $x \in \mathcal{P}$. 

### 3.3 Approximation theorem and conclusion

We introduce the vector $\mathbf{1} := [1, \ldots, 1]^\top$ from $\mathbb{R}^n$, $n \in \mathbb{N}$. The unit $n \times n$ matrix is denoted by $E$. Whenever we use the notations $E$ and $\mathbf{1}$, the sizes of $E$ and $\mathbf{1}$ are clear from the context. Whenever $x$ is a vector from $\mathbb{R}^n$, the notation $x_i$, $i \in \{1, \ldots, n\}$, stands (if not endowed with another meaning) for the $i$-th component of $x$. For $1 \leq \nu \leq +\infty$ the $l_\nu$-norm in $\mathbb{R}^n$ is denoted by $|\cdot|_\nu$. We also use $|\cdot|_\nu$ to denote the $l_\nu$-norm of matrices induced by the vector $l_\nu$-norm. It is not hard to see that for a real matrix $A = [a_{ij}]_{i,j=1}^k$ one has
\[
|A|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}| \leq (n - 1) \max_{1 \leq i,j \leq n} |a_{ij}|,
\]
see, for example, [Lan69, Exercise 9 to Chapter 6]. If $A$ is invertible, $A^{-1}$ denotes the inverse of $A$ and $A^{-\top} := (A^{-1})^\top = (A^\top)^{-1}$.

Given compact sets $X$ and $Y$ in $\mathbb{R}^d$ the *Hausdorff distance* between $X$ and $Y$ is defined to be the quantity
\[
\max \left\{ \min_{x \in X} \max_{y \in Y} |x - y|, \min_{y \in Y} \max_{x \in X} |x - y| \right\}
\]
In what follows, the convergence of subsets of $\mathbb{R}^d$ will be understood with respect to the Hausdorff distance.

Given a vertex $v$ of $P$ by $\deg_P(v)$ we denote the number of facets of $P$ incident to $v$. We put $\deg_P(v) := \max_{v \in \text{vert}(\mathcal{P})} \deg_P(v)$. We also introduce a certain parameter $\gamma$ which is related to the so-called *eccentricity* of a finite point set in a strictly convex position, which was introduced by Bernig, see [Ber98]. We put
\[
\gamma := \max \{ 1 - q_\mathcal{F}(v) : F \in \mathcal{F}_{d-1}, \ v \in \text{vert}(\mathcal{P}) \setminus \text{vert}(F) \}.
\]

The aim of the following theorem is to present a construction of a convex algebraic surface which, on one hand, contains all vertices of a given polytope $P$ and, on the other hand, approximates the boundary of $P$ with any given precision. The proof of Theorem 3.5 is a modification of arguments of Bernig [Ber98, Theorem 3.1.2], who found a construction of a convex algebraic surface containing the vertices of a given $d$-polytope (without imposing however any approximation conditions).

**Theorem 3.5.** Let $P$ a convex $d$-polytope. Then the following statements hold true.
I. For all sufficiently large $k \in \mathbb{N}$ there exist unique scalars $y_{v,k} > 0$, $v \in \text{vert}(P)$, such that the polynomial

$$f_k(x) := \sum_{v \in \text{vert}(P)} y_{v,k} \left( \frac{1}{\deg(v)} \sum_{F \in \mathcal{F}_{d-1}} (1 - q_F(x))^{2k} \right)^{2k}$$

(3.14)

satisfies the conditions $f_k(w) = 1 \forall w \in \text{vert}(P)$. Furthermore, the scalars $y_{k,v}$, $v \in \text{vert}(P)$, can be determined from the equation

$$A_k y_k = 1,$$

(3.15)

where

$$y_k := \left[ y_{v,k} \right]_{v \in \text{vert}(P)},$$

$$A_k := \left[ A_k(w,v) \right]_{w \in \text{vert}(P), v \in \text{vert}(P)} := \left[ \left( \frac{1}{\deg(v)} \sum_{F \in \mathcal{F}_{d-1}} (1 - q_F(w))^{2k} \right)^{2k} \right]_{w \in \text{vert}(P), v \in \text{vert}(P)}.$$

II. The semi-algebraic set

$$S_k := \left\{ x \in \mathbb{R}^d : f_k(x) \leq 1 \right\}$$

converges to $P$, as $k \to +\infty$.

III. For all sufficiently large $k$ and every $v \in \text{vert}(P)$ the equality $S_k \cap C_v = \{v\}$ holds.

Proof. I. For every $v, w \in \text{vert}(P)$ with $v \neq w$ we have

$$A_k(w,v)^{1/2k} = \frac{1}{\deg(v)} \sum_{F \in \mathcal{F}_{d-1}} (1 - q_F(w))^{2k} \leq \frac{\deg(v) - 1 + \gamma^{2k}}{\deg(v)} \leq 1 - \frac{1 - \gamma^{2k}}{\deg(P)}$$

(3.16)

and hence

$$|A_k - E|_{\infty} \leq (n - 1) \max_{v, w \in \text{vert}(P), w \neq v} A_k(w,v) \leq (n - 1) \left( 1 - \frac{1 - \gamma^{2k}}{\deg(P)} \right)^{2k}.$$  (3.17)

The conditions $f_k(w) = 1$ for $w \in \text{vert}(P)$ are equivalent to the system (3.15). By (3.17), $|A_k - E|_{\infty} \to 0$, as $k \to +\infty$, which shows that $A_k$ is invertible for all sufficiently large $k$, and, by (3.15), for every $v \in \text{vert}(P)$ we have $y_{v,k} \to 1$, as $k \to +\infty$. This shows the assertion of Part I.

II. First we notice that $P \subseteq S_k$, because $w \in S_k$ for every $w \in \text{vert}(P)$ and, since $f_k(x)$ is concave, $S_k$ is convex. If $x \in S_k$, then

$$1 \geq f_k(x)^{1/4k^2} \geq \left( \min_{v \in \text{vert}(P)} y_{v,k}^{1/4k^2} \right) \left( \sum_{v \in \text{vert}(P)} \left( \frac{1}{\deg(v)} \sum_{F \in \mathcal{F}_{d-1}} (1 - q_F(x))^{2k} \right)^{2k} \right)^{1/4k^2}$$

$$\geq \left( \min_{v \in \text{vert}(P)} y_{v,k}^{1/4k^2} \right) \deg(P)^{-1/2k} \left( \sum_{v \in \text{vert}(P)} \left( \sum_{F \in \mathcal{F}_{d-1}} (1 - q_F(x))^{2k} \right)^{2k} \right)^{1/4k^2}$$

$$\geq \left( \min_{v \in \text{vert}(P)} y_{v,k}^{1/4k^2} \right) \deg(P)^{-1/2k} \max_{F \in \mathcal{F}_{d-1}} |1 - q_F(x)|,$$

and hence

$$S_k \subseteq \left\{ x \in \mathbb{R}^d : |1 - q_F(x)| \leq \frac{\deg(P)^{1/2k}}{\min_{v \in \text{vert}(P)} y_{v,k}^{1/4k^2}} \forall F \in \mathcal{F}_{d-1} \right\}. $$

(3.18)

But since

$$\frac{\deg(P)^{1/2k}}{\min_{v \in \text{vert}(P)} y_{v,k}^{1/4k^2}} \to 1,$$
as $k \to +\infty$, and $P \subseteq S_k$, we arrive at the assertion of Part II.

III. We assume that $k$ is big enough so that the assertion of Part I is fulfilled, in particular, $y_{v,k} > 0$ for every $v \in \text{vert}(P)$. We have

\[
\frac{1}{4k^2} \nabla f_k(x) = \sum_{v \in \text{vert}(P)} y_{v,k} \left( \frac{1}{\deg(v)} \sum_{F \in \mathcal{F}_d} (1 - q_F(x))^{2k} \right) \left( \frac{1}{\deg(v)} \sum_{F \in \mathcal{F}_d} (1 - q_F(x))^{2k-1} \frac{u_F}{\text{diam}(P)} \right),
\]

and thus, for $w \in \text{vert}(P)$

\[
\frac{1}{4k^2} \nabla f_k(w) = \frac{y_{w,k}}{\deg(w) \cdot \text{diam}(P)} \sum_{F \in \mathcal{F}_d} u_F + u_k^w,
\]

where

\[
u_k^w := \sum_{v \in \text{vert}(P) \setminus \{w\}} y_{v,k} \cdot A_k(w, v) \frac{2k-1}{\deg(v)} \left( \frac{1}{\deg(v)} \sum_{F \in \mathcal{F}_d} (1 - q_F(w))^{2k-1} \frac{u_F}{\text{diam}(P)} \right).
\]

Assume that $x \in C_w$, that is, $-\sigma_1(q_w(x)) \geq \frac{2}{3} |q_w(x)|$. Then

\[
\left\langle \frac{1}{4k^2} \nabla f_k(w), x - w \right\rangle = -\frac{y_{w,k}}{\deg(w)} \cdot \sigma_1(q_w(x)) + \left\langle u_k^w, x - w \right\rangle \geq \frac{2}{3 \deg(w)} y_{w,k} |q_w(x)| + \left\langle u_k^w, x - w \right\rangle \geq \frac{2}{3 \deg(w)} y_{w,k} |q_w(x)| + \left\langle u_k^w, x - w \right\rangle
\]

From (3.19) we see that $\left\langle u_k^w, x - w \right\rangle \leq \beta(k) |q_w(x)|$ with some $\beta(k)$ converging to 0 as $k \to +\infty$. Thus, in view of (3.20), if $k$ is sufficiently large, we get

\[
\left\langle \frac{1}{4k^2} \nabla f_k(w), x - w \right\rangle \geq \frac{1}{3 \deg(P)} |q_w(x)|
\]

for every $x \in \mathbb{R}^d$. Therefore $f_k(w)$ does not vanish, and by this, is an outward normal of $S_k$ at $w$, and moreover all points of $C_w$ distinct from $w$ lie outside $S_k$. \hfill \Box

Theorem 3.5 and also its improved version Theorem 4.7 (given below) deals with approximation and interpolation of a convex polytope by convex semi-algebraic sets, which is also a topic of independent interest. Related results can be found in [Ham63], [Fir74], and [GH03, Lemma 2.6]).

We finish the section with the proof of our main theorem.

Proof of Theorem 4.7. Let $\varepsilon_3$ be as in the assertion of Lemma 3.4. We can construct a strictly concave polynomial $p_0(x) := 1 - f_k(x)$ with $f_k(x)$ as in Theorem 3.5 and sufficiently large $k \in \mathbb{N}$ such that $p_0(x)$ is non-negative on $P$, negative on $C_v \setminus \{v\}$ for each $v \in \text{vert}(P)$ and $\{x \in \mathbb{R}^d : p_0(x) \geq 0\} \subseteq P_{\varepsilon_3}$. Clearly, the assertion of the theorem is fulfilled for this choice of $p_0(x)$.

Let us describe a “brute-force” approach for finding an appropriate $p_0(x)$. We may assume that our input consists of polynomials $q_F(x)$ with $F \in \mathcal{F}_{d-1}$. We proceed as follows.

1. Set $k \leftarrow 1$.
2. Determine the matrix $A_k$ given as in the statement of Theorem 3.5.
3. If $A_k$ is invertible, determine $y_{v,k}$ from (3.15). Otherwise go to Step 6.
4. If all $y_{v,k}$ are positive, set $p_0(x) := 1 - f_k(x)$ with $f_k(x)$ as in (3.14). Otherwise go to Step 6.
5. If the polynomials $p_0(x), \ldots, p_d(x)$ represent $P$, return $p_0(x)$ and stop.
6. Set $k \leftarrow k + 1$ and go to Step 2.

Notice that Step 5 can be implemented. This is a consequence of algorithmic results on the quantifier elimination theorem, see [BPR06, Chapters 1, 12, and 14]. \hfill \Box
4 Choice of parameters

Apparently the algorithm for determination of $p_0(x)$ described in the proof of Theorem 1.1 is highly complex. It serves a theoretical purpose of providing a relatively short confirmation of constructibility statement from Theorem 1.1. In this section we wish to determine $k$ in a more straightforward manner by giving estimates for parameters involved in Lemmas 3.1, 3.3, 3.4 and Theorem 3.5. From the results of this section it is also clear which metric characteristic of $P$ influence $k$.

4.1 Preliminaries

We refer to [HLP88] for standard inequalities. It is known that for every $x \in \mathbb{R}^n$ one has

$$|x|_{\nu_2} \leq |x|_{\nu_1},$$

$$n^{-1/\nu_1} |x|_{\nu_1} \leq n^{-1/\nu_2} |x|_{\nu_2},$$

where $1 \leq \nu_1 \leq \nu_2 \leq +\infty$. Formula (4.2) is the inequality for power means. Hölder’s inequality states that

$$|\langle x, y \rangle| \leq |x|_\mu |x|_\nu,$$

for every $x, y \in \mathbb{R}^n$ and $1 \leq \mu, \nu \leq +\infty$ with $\frac{1}{\mu} + \frac{1}{\nu} = 1$. The special case $\mu = \nu = 2$ yields the Cauchy-Schwarz inequality.

For any non-empty subset $\mathcal{X}$ of $\mathcal{F}_{d-1}$ we put $U_\mathcal{X}$ to be the matrix with $\# \mathcal{X}$ rows $u^\top_F$, where $F \in \mathcal{X}$. We put

$$\alpha(v) := \max \{|U^{-1}_\mathcal{X}|_2 : \mathcal{X} \subseteq \mathcal{F}_{d-1}^{v}, \# \mathcal{X} = d\},$$

$$\alpha := \max_{v \in \text{vert}(P)} \alpha(P, v).$$

The quantity $\alpha(v)$ can be viewed as anisotropy of the vertex $v$ of $P$. If $P$ is simple we put $U_v := U_\mathcal{X}$ where $\mathcal{X} := \mathcal{F}_{d-1}^{v}$. In the case of simple polytopes we have

$$\alpha := \max_{v \in \text{vert}(P)} |U^{-1}_v|_2.$$

We wish to bound $\alpha$ by some further metric characteristics associated with $P$.

**Proposition 4.1.** Let $P$ be a simple $d$-polytope and let $\phi$ stands for the minimum angle between $\text{aff } I$ and $\text{aff } F$, where $F$ and $I$ range over all facets and edges of $P$, respectively, such that $I$ and $F$ have precisely one vertex in common. Then

$$\alpha \leq \frac{\sqrt{d}}{\sin \phi},$$

$$\alpha \leq \frac{\sqrt{d}}{1 - \gamma}.$$  

**Proof.** We use the Frobenius norm of a matrix $A := [a_{ij}]_{i,j=1}^n$, which is defined by

$$|A|_{Fr} := \left( \sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}.$$ 

The norms $|A|_2$ and $|A|_{Fr}$ are known to be related by

$$|A|_2 \leq |A|_{Fr},$$

see [Zha02] p. 50].

Let us introduce vectors $a_v, F \in \mathcal{F}_{d-1}$, which are columns of $U^{-1}_v$, i.e., for $F, G \in \mathcal{F}_{d-1}$ the quantity $\langle a_v, u_F \rangle$ is 1 if $F = G$ and 0 otherwise. Let us fix $v \in \text{vert}(P)$ and $F \in \mathcal{F}_{d-1}$. Since $P$ is simple, there exists a unique $w \in \text{vert}(P) \setminus \text{vert}(F)$ such that $v$ and $w$ are adjacent vertices of $P$. It is easily seen that
We obtain $\langle u, v \rangle = \frac{1}{\sin \phi_{u,F}} \leq \frac{1}{\sin \phi}$. Consequently,

$$|U_v^{-1}|_2^2 \leq |U_v|_F^2 = \sum_{F \in F_{d-1}^v} |a_{v,F}|^2 \leq \frac{d}{\sin \phi},$$

and we get the assertion.

Let us borrow the notations from the statement and the proof of Proposition 4.1. We have

$$1 - \gamma \leq \frac{|v - w| \cdot \sin \phi_{u,F}}{\text{diam}(P)} \leq \sin \phi_{v,F}.$$ 

Since $v \in \text{vert}(P)$ and $F \in F_{d-1}^v$ are chosen arbitrarily, we get $\sin \phi \geq 1 - \gamma$. The assertion follows from (4.4).

By (4.5) we showed that $\alpha$ is bounded by a multiple of $\frac{1}{1 - \gamma}$. However, we can see that for a general simple $d$-polytope $P$ the quantities $\alpha$ and $\frac{1}{1 - \gamma}$ are not of the same order of magnitude, i.e., the converse statement would not be valid. In fact let $P_t$, $t \in \mathbb{N}$, be simple $d$-polytopes that converge to some polytope $P$ which is not simple. Then $\alpha(P_t)$ converges to some finite value, as $t \to +\infty$, however $\frac{1}{1 - \gamma} \to +\infty$.

4.2 Auxiliary statements for $P_\varepsilon$

The normal cone of $P$ at a boundary point $x$ of $P$ is the set

$$N(P, x) := \{ u \in \mathbb{R}^d : \langle x, u \rangle = h(P, u) \}.$$

Lemma 4.2. Let $P$ be a simple $d$-polytope and let $\varepsilon \geq 0$ be such that

$$\varepsilon < \frac{1 - \gamma}{\sqrt{d} \cdot \alpha},$$

(4.7)

For $v \in \text{vert}(P)$ let $v_\varepsilon$ be the point determined by $d$ equalities $q_F(v^\varepsilon) = -\varepsilon$ with $F \in F_{d-1}^v$. Then

$$\text{vert}(P_\varepsilon) = \{ v_\varepsilon : v \in \text{vert}(P) \},$$

(4.8)

$$N(P_\varepsilon, v_\varepsilon) = N(P, v) \quad \forall v \in \text{vert}(P),$$

(4.9)

$$q_F(x) \leq 2 \quad \forall F \in F_{d-1} \forall x \in P_\varepsilon,$$

(4.10)

$$q_F(v^\varepsilon) \geq -\varepsilon \cdot \sqrt{d} \cdot \alpha \quad \forall v \in \text{vert}(P) \forall F \in F_{d-1},$$

(4.11)

$$q_F(v^\varepsilon) \geq 1 - \gamma - \varepsilon \cdot \sqrt{d} \cdot \alpha > 0 \quad \forall v \in \text{vert}(P) \forall F \in F_{d-1} \setminus F_{d-1}^v.$$

(4.12)

Proof. Since

$$q_F(v^\varepsilon) = (h(P, u_F) - (u_F, v^\varepsilon))/\text{diam}(P) = -\varepsilon \quad \forall F \in F_{d-1}^v,$$

$$q_F(v) = (h(P, u_F) - (u_F, v))/\text{diam}(P) = 0 \quad \forall F \in F_{d-1}^v,$$

we obtain

$$(u_F, v^\varepsilon - v) = \varepsilon \cdot \text{diam}(P) \quad \forall F \in F_{d-1}^v.$$

Then

$$v^\varepsilon - v = \varepsilon \cdot \text{diam}(P) \cdot U_v^{-1} \mathbb{1}.$$  \hspace{1cm} (4.13)

For every $v \in \text{vert}(P)$ and $F \in F_{d-1}$ we have

$$q_F(v^\varepsilon) = q_F(v) + \frac{(u_F, v - v^\varepsilon)}{\text{diam}(P)} \geq q_F(v) - \varepsilon \cdot (u_F, U_v^{-1} \mathbb{1}) \geq q_F(v) - \varepsilon \cdot \sqrt{d} \cdot \alpha,$$

which implies (4.11) and (4.12). From (4.12) we deduce (4.9) and $\{ v_\varepsilon : v \in \text{vert}(P) \} \subseteq \text{vert}(P_\varepsilon)$. But since the cones $N(P, v)$ with $v \in \text{vert}(P)$ cover $\mathbb{R}^d$, we obtain that the cones $N(P, v_\varepsilon)$, $v \in \text{vert}(P)$, also cover $\mathbb{R}^d$ and arrive at (4.9) and (4.8).
It remains to show (4.10). Let \( F \in \mathcal{F}_{d-1} \). We choose \( v \in \text{vert}(P) \setminus \text{vert}(F) \) such that \( h(P, -u_F) = \langle v, -u_F \rangle \). Then, in view of (4.8) and (4.9), \( h(P, -u_F) = \langle v, -u_F \rangle \) and we get
\[
q_F(x) = \frac{h(P, u_F) - \langle u_F, x \rangle}{\text{diam}(P)} \leq \frac{h(P, u_F) - \langle u_F, v \rangle}{\text{diam}(P)} + \frac{\langle u_F, v - v \rangle}{\text{diam}(P)} \\
\leq q_F(v) + \frac{|v - v|}{\text{diam}(P)} \leq 1 + \varepsilon \cdot \sqrt{d} \cdot \alpha \leq 2 - \gamma < 2.
\]
arriving at (4.10).

Given \( v \in \text{vert}(P) \) and \( \varepsilon, \delta > 0 \) we introduce the set
\[
P_v^{\varepsilon, \delta} := \{ x \in \mathbb{R}^d : q_F(x) \geq -\varepsilon \text{ for } F \in \mathcal{F}_d^\varepsilon \text{ and } q_F(x) \geq \delta \text{ for } F \in \mathcal{F}_{d-1} \setminus \mathcal{F}_d^\varepsilon \},
\]
see Fig. 7. The polytope \( P_v^{\varepsilon, \delta} \) does not contain the vertex \( v \) of \( P \) and converges to \( P \) as \( \varepsilon, \delta \to 0 \).

![Figure 7.](image)

In the following lemma we use Carathéodory’s theorem. For the special case of convex polytopes it states that every point of a \( d \)-polytope \( P \) can be represented as a convex combination of at most \( d+1 \) vertices of \( P \), see for example [Sch93, Theorem 1.1.4].

**Lemma 4.3.** Let \( P \) be a simple \( d \)-polytope and let \( \varepsilon > 0 \) be such that
\[
\delta := \frac{1 - \gamma}{1 + d} - \varepsilon \cdot \sqrt{d} \cdot \alpha > 0,
\]
then
\[
P_\varepsilon = \bigcup_{v \in \text{vert}(P)} P_v^{\varepsilon, \delta}.
\]

**Proof.** The inclusion “\( \supset \)” is trivial. Let us show the reverse inclusion. Since (4.14) implies (4.7) we can use Lemma 4.2. Let the points \( v_\varepsilon \) with \( v \in \text{vert}(P) \) be defined as in the assertion of Lemma 4.2. We fix an arbitrary \( x \in P_\varepsilon \). By (4.8) and Carathéodory’s theorem, there exist affinely independent vertices \( v_1, \ldots, v_{d+1} \) of \( P \) and non-negative scalars \( \lambda_1, \ldots, \lambda_{d+1} \) such that
\[
x = \sum_{j=1}^{d+1} \lambda_j v_j,
\]
\[
1 = \sum_{j=1}^{d+1} \lambda_j.
\]
Without loss of generality we may assume that \( \lambda_1 \leq \cdots \leq \lambda_{d+1} \). Then \( \lambda_{d+1} \geq \frac{1}{d+1} \). Let us choose an
Thus, $x \in P_{\varepsilon, \delta}$ and the assertion is proved. □

4.3 Choice of $\varepsilon_1, \varepsilon_2, \varepsilon_3$

Lemmas 4.4, 4.5, and 4.6 below are quantitative improvements of Lemmas 3.1, 3.3, and 3.4, respectively.

**Lemma 4.4.** Let $P$ be a simple $d$-polytope and $\varepsilon_1 > 0$ be such that

$$\delta := \frac{1 - \gamma}{1 + d} - \varepsilon_1 \cdot \sqrt{d} \cdot \alpha > 0,$$

Then for every $1 \leq i \leq m - d$ and $x \in P_{\varepsilon_1, \delta}$

$$\sigma_i(q(x)) \geq \left(\frac{m - d}{i}\right)^{\delta} - 2^{i-1} \varepsilon_1 + \left(\frac{m}{i}\right)2^{i-1} \varepsilon_1. \quad (4.16)$$

In particular, $\sigma_i(q(x)) > 0$ when

$$\varepsilon_1 \leq \frac{(1 - \gamma)}{4 \cdot (1 + d)}, \quad (4.17)$$

$$\varepsilon_1 \leq \frac{1 - \gamma}{2 \cdot (1 + d) \cdot \sqrt{d} \cdot \alpha}. \quad (4.18)$$

**Proof.** In view of Lemma 4.3 it suffice to show (4.10) for $x \in P_{\varepsilon_1, \delta}$ for every $v \in \text{vert}(P)$. Let $v$ be fixed. The quantity $\sigma_i(q(x))$ is the sum of the terms of the form $q_{F_i}(x) \cdots q_{F_i}(x)$ with $F_1, \ldots, F_i \in \mathcal{F}_{d-1}$. There are $\binom{m-d}{i}$ terms with all $F_1, \ldots, F_i$ belonging to $\mathcal{F}_{d-1} \setminus F_{d-1}$. Each of these terms is bounded from below by $\delta$. The remaining $\binom{m}{i} - \binom{m-d}{i}$ terms might contain a negative entry $q_{F_i}$, $1 \leq l \leq i$, which is however bounded from below by $-\varepsilon_1$. Since, by (4.10), positive entries are bounded from above by 2 we deduce that each of these $\binom{m}{i} - \binom{m-d}{i}$ terms is bounded from below by $-2^{i-1} \varepsilon_1$. The above remarks imply the assertion of the main. Now let us show the auxiliary part. We have

$$\frac{1 - \gamma}{2 \cdot (1 + d)} \geq \left(\frac{1 - \gamma}{4 \cdot (1 + d)}\right)^{\delta} = \left(\frac{1 - \gamma}{4 \cdot (1 + d)}\right)^{2^i} \geq \left(\frac{1 - \gamma}{4 \cdot (1 + d)}\right)^{m-d} 2^{i-1} \geq \varepsilon_1 2^{i-1},$$

which implies that $\sigma_i(q(x)) > 0$. □

**Lemma 4.5.** Let $P$ be a simple $d$-polytope and let $\varepsilon_2 \geq 0$ be such that $\varepsilon_2 \leq \varepsilon_1$ with $\varepsilon_1$ as in Lemma 4.4 instead of the first two inequalities */

$$\varepsilon_2 \leq \frac{1 - \gamma}{4 \cdot (1 + d)}, \quad (4.19)$$

$$\varepsilon_2 \leq \frac{1 - \gamma}{2 \cdot (1 + d) \cdot \sqrt{d} \cdot \alpha}, \quad (4.20)$$

$$\varepsilon_2 \leq \frac{5(1 - \gamma)^{m-d}}{18 \binom{d}{(d/2)} 2^{m-d}(3m-d - 2m-d)} \quad (4.21)$$
Then for every $v \in \operatorname{vert}(P)$ the inclusion
\[ \{ x \in \Pi_{v, \varepsilon^2} : \sigma_i(q(x)) \geq 0 \text{ for } m - d + 2 \leq i \leq m \} \subseteq C_v \cup P. \]
holds true.

**Proof.** Let us consider an arbitrary $v \in \operatorname{vert}(P)$. We borrow the notations $r_1(x)$, $r_2(x)$, $g_1(x)$, $g_2(x)$ from the proof of Lemma 3.3. Let $x \in \Pi_{v, \varepsilon^2}$, that is
\[ |q_v(x)|_\infty \leq \varepsilon_2. \] (4.22)

We estimate $|r_1(x)|$ as follows:
\[
|r_1(x)| \leq \sum_{i=2}^{+\infty} |\sigma_i(q_v(x))| \cdot |\sigma_{m-d+1-i}(\bar{q}_v(x))| \leq \max_{2 \leq i \leq d} |\sigma_i(q_v(x))| \sum_{i=2}^{+\infty} |\sigma_{m-d+1-i}(\bar{q}_v(x))|
\]
\[
= \max_{2 \leq i \leq d} |\sigma_i(q_v(x))| \sum_{i=0}^{m-d-1} |\sigma_i(\bar{q}_v(x))| \leq \max_{2 \leq i \leq d} \left( \frac{d^i}{i} \right) |\bar{q}_v(x)|^i \sum_{i=0}^{m-d-1} \left( \frac{m-d}{i} \right) |\bar{q}_v(x)|^i
\]
\[
\leq |q_v(x)|^2 \left( \frac{d}{|d/2|} \right) \sum_{i=0}^{m-d-1} \left( \frac{m-d}{i} \right) |\bar{q}_v(x)|^i \leq |q_v(x)|^2 \left( \frac{d}{|d/2|} \right) \sum_{i=0}^{m-d-1} \left( \frac{m-d}{i} \right) |\bar{q}_v(x)|^i
\]
\[
= \left( \frac{d}{|d/2|} \right) (3^{m-d} - 2^{m-d}) |q_v(x)|^2. \]

An analogous estimate for $|r_2(x)|$ is
\[
|r_2(x)| \leq \sum_{i=3}^{+\infty} |\sigma_i(q_v(x))| \cdot |\sigma_{m-d+2-i}(\bar{q}_v(x))| \leq \left( \frac{d}{|d/2|} \right) (3^{m-d} - 2^{m-d}) |q_v(x)|^3. \]

For every $F \in \mathcal{F}_v^{d-1}$ we have
\[
q_F(x) = \frac{h(P, u_F) - \langle u_F, x \rangle}{\operatorname{diam}(P)} \leq \frac{h(P, u_F) - \langle u_F, v \rangle}{\operatorname{diam}(P)} + \frac{\langle u_F, v - x \rangle}{\operatorname{diam}(P)} = q_F(v) + \frac{(v - x)^\top u_F}{\operatorname{diam}(P)}
\]
\[
\geq 1 - \gamma - \left( \frac{(v - x)^\top u_F}{\operatorname{diam}(P)} \right) = 1 - \gamma - \left( \frac{(v - x)^\top U_v^{-\top} u_F}{\operatorname{diam}(P)} \right)
\]
\[
\overset{\text{(1.1)}}{\geq} 1 - \gamma - \left( \frac{(v - x)^\top U_v^{-\top} u_F}{\operatorname{diam}(P)} \right) \leq 1 - \gamma - |q_v(x)|_\infty |U_v^{-\top} u_F|_1
\]
\[
\overset{\text{(1.2) }}{\geq} 1 - \gamma - \sqrt{d} \cdot |q_v(x)|_\infty |U_v^{-\top} u_F|_2 \geq 1 - \gamma - \sqrt{d} \cdot \varepsilon_2 \cdot \alpha
\]
\[
\overset{\text{(1.2) }}{\geq} \frac{2d - 1}{2(d + 1)} (1 - \gamma) \geq \frac{1}{2} (1 - \gamma)
\]

and so
\[
\sigma_{m-d}(\bar{q}_v(x)) \geq \left( \frac{1 - \gamma}{2} \right)^{m-d}. \] (4.23)

It suffices to show that under the given assumptions on $\varepsilon_2$ inequalities (3.4), (3.5), (3.6), (3.7) are fulfilled. Inequality (3.4) was verified above. Inequality (3.6) is verified as follows:
\[
g_1(x) \geq \frac{1}{2} |\sigma_{m-d}(\bar{q}_v(x))| - \frac{|r_2(x)|}{|q_v(x)|_\infty^2} \geq \frac{1}{2} \left( \frac{1 - \gamma}{2} \right)^{m-d} - \left( \frac{d}{|d/2|} \right) (3^{m-d} - 2^{m-d}) \varepsilon_2 \overset{\text{(1.2) }}{> 0}.
\]

Inequality (3.6) is obviously equivalent to the inequality
\[
18 \frac{r_2(x)}{|q_v(x)|^2} \leq 5 |\sigma_{m-d}(\bar{q}_v(x))|
\]
which is shown as follows:

\[
18 \frac{r_2(x)}{|q_v(x)|^2} \leq 18 \frac{|r_2(x)|}{|q_v(x)|^2} \leq 18 \left( \frac{d}{|d/2|} \right) (3^{m-d} - 2^{m-d}) |q_v(x)|_\infty \leq 18 \left( \frac{d}{|d/2|} \right) (3^{m-d} - 2^{m-d}) \varepsilon_2 \\
\leq 5 \left( \frac{1 - \gamma}{2} \right)^{m-d} \leq 5 \sigma_{m-d}(\bar{q}_v(x)).
\]

Finally we show (3.5):

\[
\frac{r_1(x)}{|q_v(x)|} \leq \frac{|r_1(x)|}{|q_v(x)|_\infty} \leq \left( \frac{d}{|d/2|} \right) (3^{m-d} - 2^{m-d}) |q_v(x)|_\infty \leq \left( \frac{d}{|d/2|} \right) (3^{m-d} - 2^{m-d}) \varepsilon_2 \\
\leq 1 \left( \frac{1 - \gamma}{2} \right)^{m-d} \lesssim \frac{1}{3} \sigma_{m-d}(\bar{q}_v(x)).
\]

\[\square\]

**Lemma 4.6.** Let \( P \) be a simple \( d \)-polytope and let

\[
\varepsilon_3 := \frac{1}{d - 1 + \left( \binom{m}{d-1} - d \right) \left( \frac{2(1+d)}{1-\gamma} \right)^{m-d}} \cdot \varepsilon_2
\]

with \( \varepsilon_2 \) satisfying (4.20), (4.21), (4.24).

Then

\[
\{ x \in P_{\varepsilon_3} : \sigma_i(q(x)) \geq 0 \text{ for } m - d + 2 \leq i \leq m \} \subseteq \bigcup_{v \in \text{vert}(P)} C_v \cup P.
\]

**Proof.** Let \( x \in P_{\varepsilon_3} \) be such that inequalities \( \sigma_i(q(x)) \geq 0 \) are fulfilled for \( m - d + 2 \leq i \leq m \). By Lemma 4.3 there exists a \( v \in \text{vert}(P) \) such that \( x \in P_{\varepsilon_3, \delta} \), where

\[
\delta := \frac{1 - \gamma}{1 + d} - \varepsilon_3 \cdot \sqrt{d} \cdot \alpha \geq 0.
\]

If \( x \in P_{\varepsilon_3, \delta} \), then, by Lemma 4.3, \( x \in C_v \cup P \). Otherwise \( x \in P_{\varepsilon_3, \delta} \setminus P_{\varepsilon_3, \delta} \). Let us show that \( \sigma_{m-d+1}(q(x)) \geq 0 \). The magnitude \( \sigma_{m-d+1}(q(x)) \) is the sum of the terms of the form \( q_{F_1}(x) \cdots q_{F_{m-d+1}}(x) \) with pairwise distinct \( F_1, \ldots, F_{m-d+1} \) from \( F_{d-1} \). There are \( d \) such terms with precisely one \( F_l \), \( 1 \leq l \leq m - d + 1 \), belonging to \( F_{d-1} \). The terms with the mentioned property sum up to \( \sigma_1(q_v(x)) \sqrt{d} \sigma_{m-d}(\bar{q}_v(x)) \). Obviously, \( \sigma_{m-d}(\bar{q}_v(x)) \geq \delta^{m-d} \). For \( \sigma_1(q_v(x)) \) we have

\[
\sigma_1(q_v(x)) = \sum_{F \in F_{d-1}} q_F(x)
\]

Let \( F_0 \in F_{d-1} \) be such that \( |q_{F_0}(x)| = |q_v(x)|_\infty \). Then \( |q_{F_0}(x)| > \varepsilon_2 \), and in fact, since \( \varepsilon_2 > \varepsilon_3 \) and \( q_F(x) > -\varepsilon_3 \) for every \( F \in F_{d-1}, \) we even obtain that \( q_{F_0}(x) > \varepsilon_2 \). Consequently, \( \sigma_1(q_v(x)) \geq \varepsilon_2 - (d-1) \varepsilon_3 \).

Now let us estimate the remaining \( \binom{m}{d-1} - d \) terms \( q_{F_1}(x) \cdots q_{F_{m-d+1}}(x) \) with pairwise distinct \( F_1, \ldots, F_{m-d+1} \) from \( F_{d-1} \) such that at least two of the facets \( F_1, \ldots, F_{m-d+1} \) belong to \( F_{d-1} \). If this kind of product \( q_{F_1}(x) \cdots q_{F_{m-d+1}}(x) \) is negative then at least one entry \( q_{F_1}(x), 1 \leq l \leq m - d + 1 \), lies between \( -\varepsilon_3 \) and \( 0 \), while, by (4.10), the remaining \( 2^{m-d} \) entries have absolute value at most \( 2 \).

Summarizing we obtain

\[
\sigma_{m-d+1}(q(x)) \geq (\varepsilon_2 - (d-1) \varepsilon_3) \delta^{m-d} - \left( \binom{m}{d-1} - d \right) 2^{m-d} \varepsilon_3.
\]

Hence \( \sigma_{m-d+1}(q(x)) \geq 0 \) if

\[
\varepsilon_3 \leq \frac{\delta^{m-d}}{(d-1) \delta^{m-d} + \left( \binom{m}{d-1} - d \right) 2^{m-d} \varepsilon_2}
\]

But the latter inequality follows from (4.23). Consequently \( \sigma_{m-d+1}(q(x)) \geq 0 \). But in view of Lemma 4.4 we have \( \sigma_i(q(x)) \geq 0 \) for \( 1 \leq i \leq m - d \). Summarizing we see that \( \sigma_i(q(x)) \geq 0 \) for \( 1 \leq i \leq m \), and therefore, by Proposition 3.2, \( q_F(x) \geq 0 \) for all \( F \in F_{d-1}, \) i.e., \( x \in P \). \( \square \)
4.4 Approximation theorem: quantitative version

By log we denote the binary logarithm.

**Theorem 4.7.** Let $P$ be a convex $d$-polytope and let $\varepsilon > 0$, $k \in \mathbb{N}$, and $f_k(x)$, $A_k$, $y_k$, $S_k$ be defined as in the statement of Theorem 3.5. Then the following statements hold true.

I. If $k$ satisfies

\[
  k \geq \frac{1}{2 \log \frac{1}{\gamma}}, \tag{4.26}
\]
\[
  k \geq 2 \log (4n), \tag{4.27}
\]

then there exist unique positive real scalars $y_{v,k}$, $v \in \text{vert}(P)$, such that the polynomial $f_k(x)$ satisfies the condition $f_k(w) = 1$ for every $w \in \text{vert}(P)$.

II. If $k$ satisfies (4.26), (4.27) and

\[
  k \geq \frac{\log (2 \deg(P))}{2 \log (1 + \varepsilon)}, \tag{4.28}
\]

and $y_k$ is determined from (4.16), then the semi-algebraic set $S_k$ satisfies the inclusions

\[
  P \subseteq S_k \subseteq P_\varepsilon. \tag{4.29}
\]

III. If $P$ is simple and inequalities (4.26) and

\[
  k \geq 3 \log \left(12 n \cdot \sqrt{d} \cdot \alpha \cdot \deg(P)\right), \tag{4.30}
\]

are fulfilled, then $S_k \cap C_v = \{v\}$.

**Proof.** I. The conditions $f_k(w) = 1$ for $w \in \text{vert}(P)$ are equivalent to the system $A_k y_k = 1$. Let us show that under the given assumptions on $k$ the matrix is invertible.

\[
  |A_k - E|_\infty \leq (n - 1) \left(1 - \frac{1}{\deg(P)}\right)^{2k} \leq \frac{1}{4}. \tag{4.29}
\]

Thus, we have shown that $|A_k - E|_\infty < \frac{1}{4}$. It is known that if $|A_k - E|_\infty < 1$, then $A_k$ is invertible and moreover

\[
  A_k^{-1} = \sum_{l=0}^{+\infty} (E - A_k)^l,
\]

see, for example, [Lan69, Theorem 7.1.1]. Consequently,

\[
  |y_k - 1|_\infty = |(A_k^{-1} - E)1|_\infty \leq |A_k^{-1} - E|_\infty = \left|\sum_{l=1}^{+\infty} (E - A_k)^l\right|_\infty \leq |E - A_k|_\infty \left|\sum_{l=0}^{+\infty} (E - A_k)^l\right|_\infty \leq \frac{|E - A_k|_\infty}{1 - |E - A_k|_\infty} \leq \frac{1}{3}
\]

and hence

\[
  \frac{2}{3} \leq y_{v,k} \leq \frac{4}{3}, \quad \forall v \in \text{vert}(P). \tag{4.32}
\]

II. The inclusion $P \subseteq S_k$ was noticed in the proof of Theorem 3.5. In view of (4.18), the inclusion $S_k \subseteq P_\varepsilon$ is a consequence of the following estimates:

\[
  \log \frac{\deg(P)^{1/2k}}{\min_{v \in \text{vert}(P)} y_{v,k}} \leq \log \deg(P)^{2k} 2^{1/k^2} \leq \frac{1}{2k} \log \deg(P) + \frac{1}{4k^2} \leq \frac{1}{2k} \left(1 + \log \deg(P)\right) \leq \frac{1}{2k} \log (2 \deg(P)) \leq \log (1 + \varepsilon). \tag{4.29}
\]
It suffices to show that under the given assumptions inequality \((2.21)\) is fulfilled. We have

\[
\frac{4}{9 \deg(P)} |q_w(x)| - \left\langle \frac{1}{4k^2} \nabla f_k(w), x - w \right\rangle \leq \frac{2}{3 \deg(P)} y_{w, k} |q_w(x)| - \left\langle \frac{1}{4k^2} \nabla f_k(w), x - w \right\rangle 
\]

\[(3.19)\]

\[
\leq \langle u_k^w, w - x \rangle \leq |x - w| \cdot |u_k^w| \leq \frac{4}{3} \cdot |x - w| \cdot \frac{1}{\diam(P)} \sum_{v \in \text{vert}(P) \setminus \{w\}} A_k(w, v) \frac{2k-1}{2k}
\]

\[(3.19)\]

\[
\leq \frac{4}{3} \cdot \sqrt{d} \cdot \alpha \cdot \max_{\# \mathcal{X} = d} \frac{|U_X(w - x)|}{\diam(P)}, \sum_{v \in \text{vert}(P) \setminus \{w\}} A_k(w, v) \frac{2k-1}{2k}
\]

\[(4.20)\]

\[
= \frac{4}{3} \cdot \sqrt{d} \cdot \alpha \cdot |q_w(x)| \cdot \sum_{v \in \text{vert}(P) \setminus \{w\}} A_k(w, v) \frac{2k-1}{2k}
\]

\[(4.20)\]

\[
\leq \frac{4}{3} \cdot \sqrt{d} \cdot \alpha \cdot n \cdot |q_w(x)| \left( 1 - \frac{1}{\deg(P)} \right) \frac{2k-1}{2k} \leq \frac{4}{3} \cdot \sqrt{d} \cdot \alpha \cdot n \cdot \left( 1 - \frac{1}{2 \deg(P)} \right)^{2k-1} |q_w(x)|
\]

\[(4.20)\]

\[
\leq \frac{4}{3} \sqrt{d} \cdot \alpha \cdot n \cdot \deg(P) \cdot \left( \frac{3}{4} \right)^{2k-1} \leq \frac{4}{3} \sqrt{d} \cdot \alpha \cdot n \cdot \deg(P) \cdot \left( \frac{3}{4} \right)^{k} \leq \frac{1}{9 \deg(P)} |q_w(x)|
\]

and we are done. \(\Box\)

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