Quantifying the Imprecision of Accretion Theory and Implications for Multi-Epoch Observations of Protoplanetary Discs

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ABSTRACT
If accretion disc emission results from turbulent dissipation, then axisymmetric accretion theory must be used as a mean field theory: turbulent flows are at most axisymmetric only when suitably averaged. Spectral predictions therefore have an intrinsic imprecision that must be quantified to interpret the variability exhibited by a source observed at different epochs. We quantify contributions to the stochastic imprecision that come from azimuthal and radial averaging and show that the imprecision is minimized for a particular choice of radial averaging, which in turn, corresponds to an optimal spectral resolution of a telescope for a spatially unresolved source. If the optimal spectral resolution is less than that of the telescope then the data can be binned to compare to the theoretical prediction of minimum imprecision. Little stochastic variability is predicted at radii much larger than that at which the dominant eddy turnover time (∼orbit time) exceeds the time interval between observations; the epochs would then be sampling the same member of the stochastic ensemble. We discuss the application of these principles to protoplanetary discs for which there is presently a paucity of multi-epoch data but for which such data acquisition projects are underway.

Key words: accretion, accretion discs; (stars:) planetary systems: protoplanetary discs; turbulence

1 INTRODUCTION
Accretion discs are ubiquitous in astrophysics, often forming around stars and compact objects where angular momentum would inhibit direct gravitational infall of the plasma onto the central object. A long standing theme of research in accretion theory has been to understand how the discs transport angular momentum and remain quasi-steady accretors. This transport likely involves different combination of local and global mechanisms depending on the circumstances. However, there is presently an intellectual gap between the study of the fundamental theory of angular momentum transport in accretion discs via numerical simulations and practical accretion disc models that can be used to compare with observed spectra. For reviews that sample a range of perspectives see Frank et al. (2002); Balbus and Hawley (2003); Hartmann (2009); Spruit (2010); Blackman (2010).

The most commonly used practical accretion disc model is based on local viscous transport of angular momentum in axisymmetric discs and hides the unknown physics of transport into an effective local “turbulent viscosity” (Shakura and Sunyaev 1973; Pringle 1981; Frank et al. 2002). Because microphysical molecular viscosity is typically much too small to drive the observed accretion rates given constraints on surface densities, some kind of enhanced transport is needed. While these mechanisms routinely involve some form of turbulent motion (often with a magnetic origin), the standard practical model equations describing accretion discs do not specify or accommodate the nuanced physics of transport, lumping the processes into a single dimensionless transport coefficient. The assumption that the discs are axisymmetric can then at best apply in a spatially or temporally averaged mean-field sense since turbulence necessarily violates axisymmetry locally and turbulent transport is a stochastic process.

In order to properly compare standard axisymmetric accretion theory with observations, the limitations of the theory must be quantified. The theory has a limited precision which implies that agreement between models and observation can only be approximate. Disagreement within the precision error may be consistent with a stochastic nature of the discs. A key quantity, and the focus of this paper, is the relative precision error (RPE in the luminosity at a given frequency, Blackman 1998) discussed the RPE

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1 Compare the statements ‘babies sleep between one and twenty three hours per day’ and ‘babies sleep 8 ± 1 h per day.’ The first statement has a large RPE, but is almost certain to be accurate. The second statement has a low RPE, but may well be incorrect. However, its low RPE makes the second statement testable
for accretion discs in the context of Advection Dominated Accretion Flows (ADAFs) and thin discs in active galactic nuclei (AGN) and X-ray binaries (XRBs) in which the observation times are often longer than the orbit times at a given radius. In contrast, protoplanetary discs often involve small observational exposure times compared to disc orbit times. In this context, we revisit quantifying the precision of standard accretion disc theory and make predictions for multi-epoch observations. The predictions can help to diagnose whether a systematic variability may also be present.

In section 2, we compute the azimuthal and radial contributions to the luminosity RPE and combine them for a spatially unresolved source. In section 3 we discuss how to minimize the RPE and the role of telescope spectral resolving power. In section 4 we consider two independent contributions. The first comes from a radial contribution to the variation.

2 CALCULATION OF RPE IN LUMINOSITY FROM TURBULENT DISSIPATION

We assume a standard α-disc prescription (Shakura and Sunyaev 1973) for the turbulent viscosity. The characteristic turbulent eddy scales have characteristic velocities $v_{\text{ed}}$ and sizes $l_{\text{ed}}$ so that the effective viscosity satisfies

$$\nu \approx \alpha c s h \approx v_{\text{ed}} l_{\text{ed}} \sim \frac{v_{\text{ed}}^2}{\Omega}$$

(1)

where $c_s$ is the disc sound speed, $h$ is the scale height, and $\Omega$ is the orbital speed. We take $0 < \alpha < 1$ to be a constant over the whole disc. The last similarity in (1) follows from the assumption that the eddy turnover time $\tau_{\text{ed}} \sim \frac{h}{c_s}$ is consistent with simulations of the magnetorotational instability (MRI) (e.g. Balbus and Hawley 2003). For a thin disc in hydrostatic equilibrium, Eq. (1) also implies that $v_{\text{ed}} \sim \alpha^{1/2} c_s$ and $l_{\text{ed}} \sim \alpha^{1/2} h$ (Blackman 1998).

In what follows, we focus on cases for which the required observational exposure time is much less than the eddy turnover time at all radii. We can then regard a measured spectrum as an instantaneous snapshot. This applies to Spitzer observations of protoplanetary discs. The discs range in size from a few tenths of an AU to several tens of AU so that $l_{\text{ed}}$ ranges from days (in the inner disc) to years (in the outer disc). In contrast, a typical Spitzer exposure takes only a few minutes.

Since the sources of interest are spatially unresolved, we calculate the contributions to the RPE by averaging in the azimuthal and in the radial directions for the axisymmetric theory. If the disc is sufficiently optically thick then the photosphere occurs within the first layer of energy containing eddies. An optically thin disc requires additionally averaging over the third dimension of eddies.

2.1 Azimuthal Contribution

Consider an annulus of width equal to one eddy scale centered at a radius $r$ within the disc. The number of eddies producing observable emission in this annulus is given by

$$N_{\text{ed}} \approx \frac{2 \pi r}{l_{\text{ed}}} \left[ 1 + \left( \frac{h}{l_{\text{ed}}} - 1 \right) e^{-\tau_{\text{ed}}} \right],$$

(2)

where $\tau_{\text{ed}}$ is the optical depth. In the limit that $\tau_{\text{ed}} \to 0$, eddies over the entire thickness of the disc contribute to the emission. For $\tau_{\text{ed}} \to \infty$ only the top layer of eddies contribute to the emission.

Over its lifetime, the brightness of each dominant eddy will rise to some maximum and then fall as it cascades to small scales. Let us assume that the rate of brightening and dimming are constant so that the eddy brightens to its peak luminosity $L_0$ at $t_{\text{ed}}/2$ and then falls to zero at $t = t_0$. The probably distribution function for the luminosity during the rise and fall is then

$$p(L_e) dl_e = \frac{1}{L_0} dl_e.$$  

(3)

Then the mean luminosity for each eddy is $\langle L_e \rangle = L_0/2$ and the mean squared luminosity is

$$\langle L_e^2 \rangle = \frac{1}{2} \left[ \int_0^{L_0} \frac{l_e^2}{L_0} dl_e + \int_0^{L_0} \frac{l_e^2}{L_0} dl_e \right] = L_0^2/3.$$  

(4)

The variance is then

$$\sigma^2(L_e) = \langle L_e^2 \rangle - \langle L_e \rangle^2 = \frac{L_0^2}{12}.$$  

(5)

If the value of the mean luminosity of each eddy is normally distributed about $L_0/2$ then the standard deviation of the luminosity per eddy when averaged over all $N_{\text{ed}}$ eddies in a given annulus satisfies $\sigma(L_e)/\sqrt{N_{\text{ed}}}$. Using $L_v$ to indicate the luminosity at frequency $\nu$ measured from the annulus whose peak temperature corresponds to that frequency, the fractional luminosity variation of the entire annulus of $N_{\text{ed}}$ eddies is then

$$\frac{\Delta L_v}{L_v} = \frac{N_{\text{ed}} \Delta L_e}{N_{\text{ed}} \langle L_e \rangle} = \frac{\sigma(L_e)/\sqrt{N_{\text{ed}}}}{L_0/2} = \frac{1}{\sqrt{3 N_{\text{ed}}}}$$

(6)

where we have used Eq. (4), the relations below Eq. (1), and the subscript $\alpha z$ to indicate the azimuthal contribution to the variation. In the optically thick limit Eq. (6) gives $\frac{\Delta L_v}{L_v} = \frac{\alpha^3}{\sqrt{3 \pi}} \left( \frac{\pi}{\Omega} \right)^2$.

2.2 Radial Contribution

In determining the radial contribution to the luminosity RPE we consider two independent contributions. The first comes from a radial smoothing implicit to the mean field theory. The second comes from turbulent velocity fluctuations that add macroscopic fluctuations about the mean orbital velocity at each radius. We address each in turn.

In formulating a mean field theory for accretion discs, one can in principle simply take vertical and azimuthal averages to obtain a theory that depends only on radius without any radial smoothing. However in the commonly employed framework which invokes a turbulent viscosity (Shakura & Sunyaev 1973), the scale of the turbulence is assumed to be smaller than the scale of the mean quantities. The theory is not really resolved on spatial scales below an eddy scale so we therefore include a radial smoothing scale, $\xi$, chosen such that $l_{\text{ed}} < \xi < r_{\text{disc}}$. In addition, telescopes have a finite spectral resolution and a given frequency in practice corresponds to a radius range in the disc, not a precise radius for a spatially unresolved source. We therefore invoke a radial smoothing of mean quantities of the form

$$\bar{X}(r) \approx \left\langle X(r, t) \right\rangle_\xi = \int_\xi X(r + \lambda, t) d\lambda,$$

(7)

where $X$ is an arbitrary quantity to be averaged and $\bar{X}$ is its mean.
The similarity follows from the assumption that eddy statistics are steady in time. Blackman (1998) discussed an additional temporal average but as mentioned above, our present interest is for cases in which the required snapshot observation time is much less than an eddy turnover time at all radii.

The radial contribution to the luminosity variation can be written as

$$\frac{\Delta L_v}{L_v}_{\text{rad}} \approx \frac{r}{L_v} \frac{\partial L_v}{\partial r} \frac{\Delta r}{r}.$$  

We must now calculate the contributions to the right hand side.

For the radial smoothing term of Eq. (9), we use the fact that

$$\frac{(\Delta r)^2}{r^2} \approx \frac{(\Delta r)^2}{r^2} + \left(\frac{\Delta r}{r}\right)^2$$

where first term comes from the blurring induced by the radial smoothing scale $\xi$ and the second term comes from the macroscopic turbulent velocity fluctuations.

For the radial smoothing term of Eq. [9], we use the fact that

$$r - \frac{\xi}{2N_{\xi}^2} < r < r + \frac{\xi}{2N_{\xi}^2}$$

are indistinguishable once $\xi$ is chosen, so that

$$\frac{(\Delta r)_r}{r} = \frac{\xi}{2rN_{\xi}^2},$$

where

$$N_{\xi} = 1 + \left(\frac{\hbar}{l_{\text{ed}}} - 1\right) e^{-\nu}\frac{t_{\text{obs}}}{\tau_{\text{ed}}}$$

and measures the effective number of averaging scales per $\xi$. The role of $\tau_\nu$ the same as that for Eq. (2). In the limit $t_{\text{obs}} << t_{\text{ed}}$ and $\tau_\nu >> 1$, $N_{\xi} = 1$. For the velocity fluctuation contribution to Eq. (9), we note that the dominant contribution to the local velocity at a given radius is $\propto \nu^{-1/2}$ from Keplerian rotation. Thus

$$\frac{(\Delta r)_v}{r} \approx 2 \frac{(\Delta v)_v}{v_0} \approx 2 \frac{v_{\text{ed}}}{v_0} \frac{1}{N_{\xi}^{1/2}},$$

where we have again used the standard result for error on the mean as applied to $N_{\xi}$ eddies each of whose mean velocity is normally distributed with fluctuations of order $v_{\text{ed}}$. The number of eddies over which the radial averaging is performed is given by

$$N_{\xi} = \frac{\xi}{l_{\text{ed}}} \left[1 + \left(\frac{\hbar}{l_{\text{ed}}} - 1\right) e^{-\nu}\frac{t_{\text{obs}}}{\tau_{\text{ed}}}\right],$$

For $\tau_\nu >> 1$ and $t_{\text{obs}}/l_{\text{ed}} \sim 0$, this gives $N_{\xi} = \xi/l_{\text{ed}}$. In this limit, adding (11) and (13) in quadrature, we obtain

$$\frac{(\Delta r)^2}{r^2} \approx \frac{v_{\text{ed}}^2}{v_0^2} + \frac{\xi^2}{4r^2}. $$

If each ring of material radiates around a specific blackbody temperature $T$, the luminosity of an annulus of width $l_{\text{ed}}$ is

$$L_v = 2\pi r B_{v}^{\text{black}}(T(r)) l_{\text{ed}},$$

where

$$B_{v}^{\text{black}}(T) = AT^3$$

is the peak value of the Planck function $B_v$ and $A$ is a constant that comes from evaluating

$$B_v(T) = T^3 \frac{2\kappa^2}{h^2} \frac{x^3}{e^{x} - 1},$$

where

$$x = \frac{h v}{k T} > 2.82$$

and

$$\text{Imprecision of Accretion Theory and Protoplanetary Discs}$$

Table 1. Values for $p$ in Eq. (20) for various disc models (Edgar et al. 2007).

| Disc Model                      | $p$ |
|---------------------------------|-----|
| Viscously Heated, Optically Thick | 9/10 |
| Viscously Heated, Optically Thin | 3/5  |
| Irradiated by central star      | 1/3  |

at $x \equiv h v/k T \sim 2.82$, corresponding to the Wien displacement law. Then

$$\frac{\partial L_v}{\partial r} = 2\pi AT^3 \left[ l_{\text{ed}}^2 T + 3r_{\text{ed}} \frac{\partial T}{\partial r} + rT \frac{dl_{\text{ed}}}{dr} \right].$$

If we assume that

$$T = Cr^{-p},$$

then combining with the scalings below Eq. (1) gives $l_{\text{ed}} = \propto r^{1/2}$ and $\tau_\nu \propto r^{1/2}$. In combination with Eqs (8), (16) and (19), we then obtain

$$\frac{\Delta L_v}{L_v} \approx 1 - 3p + \frac{3}{2} - \frac{p}{2} \frac{\Delta r}{r} = \frac{5}{2} - \frac{7p}{2} \frac{\Delta r}{r}.$$ 

Edgar et al. (2007) calculated $p$ for a variety of disc models, and their values are given in Table 1. Eq. (21) is strictly applicable for the $\tau_\nu > 1$ regime so here $p = 9/10$.

3 TOTAL RPE AND ROLE OF RESOLVING POWER

Eqs. (6) and (21) give the respective azimuthal and radial contributions to the RPE of the predicted luminosity in terms of disc model properties. They must be added in quadrature to get the total luminosity RPE. The choice of $\xi$ can be taken to minimize the RPE but whether this minimum can be compared with data depends on the resolving power of the instruments. We explain these points further below.

3.1 Minimizing the Intrinsic RPE

First we determine the minimum of Eq. (15), which applies in the optically thick $\tau_\nu \gg 1$ and $t_{\text{obs}} = 0$ limit. Differentiating and setting equal to zero gives

$$\frac{\xi_{\text{opt}}}{r} = 2 \left(\frac{v_{\text{ed}}}{v_0}\right)^3 \left(\frac{l_{\text{ed}}}{r}\right)^{\frac{1}{3}}.$$  

The optically thin case gives the same result because the optically thin corrections to $N_{\xi}$ and $N_{\xi}$ just produce an overall multiplicative factor of $l_{\text{ed}}/\hbar$ to both terms in Eq. (15) and thus do not change its minimum.

Choosing $\xi = \xi_{\text{opt}}$ will minimize the RPE of the theory. Increasing $\xi$ would average over more turbulent fluctuations, but at the price of poorer spatial precision. Reducing $\xi$ would increase the random error due to a smaller number of eddies. If $\xi_{\text{opt}} < l_{\text{ed}}$, we must use $\xi = l_{\text{ed}}$ instead.

Substituting Eq. (22) into Eq. (15), gives

$$\frac{\Delta L_v}{L_v} \approx \left(\frac{v_{\text{ed}}}{v_0}\right)^{\frac{1}{3}} \left(\frac{l_{\text{ed}}}{r}\right)^{\frac{1}{3}},$$

where we have used $v_{\text{ed}} = \alpha c h$. Note that $\xi_{\text{opt}}$ of Eq. (22) can only be used to compare with observations if the observations are resolved on scales $\xi \leq \xi_{\text{opt}}$ since only then can the data be binned.
Substituting $l_{cd} = \alpha^{1/2} h$ and $v_{cd} = \alpha^{1/2} v_{r}$ into Eq. (23), and in turn substituting the result into Eq. (21) gives

$$\frac{\Delta L_{\nu}}{L_{\nu}} \bigg|_{rad, opt} \approx \sqrt{\frac{3}{2}} \left( \frac{5}{2} \frac{7p}{2} \frac{\alpha^{1/2}}{\pi} \left( \frac{h}{r} \right) \right)^{1/2}.$$  (24)

Combining results from Eqs. (6) and (24) by adding the radial and azimuthal contributions in quadrature, we have for the net minimum RPE in the $\tau_{\nu} >> 1$ limit

$$\frac{\Delta L_{\nu}}{L_{\nu}} \bigg|_{opt} \approx \left[ \frac{3}{4} (5 - 7p)^2 \alpha \left( \frac{h}{r} \right)^2 + \frac{\alpha^{1/2}}{6\pi} \left( \frac{h}{r} \right) \right]^{1/2},$$  (25)

where in the optically thick limit (Edgar et al. 2007) $p = 9/10$ and $h/r$ is given by

$$\frac{h}{r} = 1.8 \times 10^{-2} \left( \frac{M_*}{10^{-8} M_{\odot} \text{yr}^{-1}} \right)^{1/2} \left( \frac{M_*}{M_{\odot}} \right)^{1/2} \left( \frac{r}{\text{AU}} \right)^{1/20}.$$  (26)

### 3.2 Expressing Total RPE as function of Resolving Power

For a telescope, we can only choose $\xi$ to be as small as the spectral or spatial resolution allows. If the smallest $\xi$ that the telescope can resolve is smaller than $\xi_{opt}$, the data can be binned to create an effective $\xi = \xi_{opt}$ that minimizes the RPE.

We can estimate the radial averaging scale from the spectral resolving power of the telescope. The resolving power is given by

$$R \equiv \frac{\nu}{\Delta \nu}.$$  (27)

From the Wien displacement law, we know that a blackbody spectrum has $\tau_{peak} \propto T_{peak}$, so the effective resolving power is then $R_{ef} \approx T/\Delta T$. From Eq. (30), we then obtain

$$|\xi_{ef}| \approx \frac{r}{pR_{ef}}$$  (28)

Substituting into Eq. (15), we find

$$\frac{\Delta r}{r} \bigg|_{ef} \approx \left( 4\alpha^{3/2} \left( \frac{h}{r} \right)^3 pR_{ef} + \frac{1}{4p^2 R_{ef}^2} \right)^{1/2}.$$  (29)

Using this in (21) we obtain

$$\frac{\Delta L_{\nu}}{L_{\nu}} \bigg|_{rad, ef} \approx \left( 4\alpha^{3/2} \left( \frac{h}{r} \right)^3 pR_{ef} + \frac{1}{4p^2 R_{ef}^2} \right)^{1/2}.$$  (30)

The azimuthal contribution Eq. (6), does not depend on the frequency explicitly so combining it in quadrature with Eq. (30) gives the total effective RPE

$$\frac{\Delta L_{\nu}}{L_{\nu}} \bigg|_{ef} \approx \left[ \frac{5}{2} \frac{7p}{2} \frac{\alpha^{1/2}}{\pi} \left( \frac{h}{r} \right) \right]^{1/2}.$$  (31)

It might seem surprising that increasing $R$ in Eq. (30) can give less precise results but recall that for a fixed $p$ and $r$ there is a one-to-one correspondence between $R_{ef}$ and $\xi$. But as discussed above, there is a value $\xi = \xi_{opt}$ that minimizes the RPE. There is therefore a corresponding value of $R_{ef}$ that minimizes the RPE and the two turn out to be related by $R_{opt} = R_{ef} / \xi_{opt}$. Figure 1 illustrates these basic principles in plots of Eq. (31) for different values of $\alpha$. Note that the minima are less pronounced as $\alpha$ is decreased.

Note that if $R$ of the telescope is sufficiently large and the data are not more coarsely binned by hand, the data could represent a radial resolution less than the eddy scale of the theory. The $\alpha$-disc model based on turbulence becomes ill-defined below such scales. Coarse graining of the data has to be considered carefully before sensibly comparing with a mean field theory.

### 3.3 Optically thin case

Finally, note that for $\tau_{\nu} << 1$ Eqs. (12) and (14) take a different values compared to the $\tau_{\nu} >> 1$ case. The overall calculation of the RPE follows that which led to (31) with the analogous result being

$$\frac{\Delta L_{\nu}}{L_{\nu}} \bigg|_{ef} \approx \left[ \frac{5}{2} \frac{7p}{2} \frac{\alpha^{1/2}}{\pi} \left( \frac{h}{r} \right) \right]^{1/2}.$$  (32)

where in this $\tau_{\nu} << 1$ limit (Edgar et al. 2007) $p = 3/5$ and

$$\frac{h}{r} = 2.8 \times 10^{-2} \left( \frac{M_*}{10^{-8} M_{\odot} \text{yr}^{-1}} \right)^{1/2} \left( \frac{M_*}{M_{\odot}} \right)^{1/2} \left( \frac{r}{\text{AU}} \right)^{1/20}.$$  (33)

To obtain minimum (optimal) value of (32) (the analogue of (29)), we substitute in (32) the value $R_{ef} = R_{opt} = r/p\xi_{opt}$ and use $\xi_{opt} / r$ from Eq. (15) which applies for both the optically thin and thick cases.

### 4 IMPLICATIONS FOR MULTI-EPOCH OBSERVATIONS

A basic application of the RPE described above arises for multi-epoch observations of a single object when the time scale for a snapshot observation is shorter than the eddy time at any radius. Emission from radii at which the time scale between epochs of observation exceeds the eddy turnover time would then be expected to vary stochastically between epochs. In contrast, at radii larger than the that at which the eddy turnover time matches the epoch interval time, successive observations sample the same member of the statistical ensemble and no stochastic variability corresponding to the local energy-dominating eddies would be expected.
disc emission resulting from stellar irradiation is less influenced by the stochasticity of turbulence than emission via turbulent dissipation. Since the local blackbody emission varies with $T^3$, we would expect a much reduced stochastic variability in regions of the disc dominated by emission from reprocessed starlight. As applied to the specific object LLRL 31 therefore, these would only amount to a variability in a subdominant contribution to the total emission above $r_c$.

An absence of observed stochastic fluctuations for $r > r_c$ would support the expectation that the primary emission in those regions is not the result of turbulent dissipation. For $r < r_c$ the absence of stochastic variability would be expected only if the inter-epoch time scale is shorter than the characteristic eddy turnover time at the radius dominating the radiation at the particular frequency measured. Note also that the amplitude of the predicted stochastic variability would not depend on the epoch interval as long as the epoch interval is larger than the orbit time at the radius producing the corresponding emission. Other systematic non-axisymmetry or variable accretion rates would add to the variability and could be distinguished from stochastic effects given enough multi-epoch observations.

For the specific case of LLRL 31, the largest time scale between epochs is of order 4 months which means that according to Fig. 2, the stochastic variability would be expected only for $r < 0.58$ AU. This is well within the dust depleted inner region of this transitional object where the disc is optically thin and where the emission would come from dissipation internal to the disc rather than reprocessed starlight. Thus there would be some stochastic variability because the inter-epoch time is less than the characteristic eddy turnover time at the radius dominating the radiation at the particular frequency measured. The values of $p$ would then flatten to $p = 1/3$ (see Table 1) and irradiation would dominate at all larger radii. Disc emission resulting from stellar irradiation is less influenced by the stochasticity of turbulence than emission via turbulent dissipation. Since the local blackbody emission varies with $T^3$, we would expect a much reduced stochastic variability in regions of the disc dominated by emission from reprocessed starlight. As applied to the specific object LLRL 31 therefore, these would only amount to a variability in a subdominant contribution to the total emission above $r_c$.
bles that arise from stochastic variability alone. In addition, the presence of a single pivot wavelength of 8.5 microns (see Fig 1 of Muzerolle et al. [2009]) at which there is no variability in this source and above which the sign of the variability between epochs changes sign from that at low wavelengths strongly suggests the influence of a global geometric feature such as a thick wall or a warp. In short the dominant variability in LRLL 31 object likely comes from systematic rather than stochastic effects.

5 CONCLUSIONS

For $\alpha$ models of turbulent accretion discs we have derived a predicted RPE in the luminosity as function of frequency, focusing on the case in which the observational exposure times are small compared to other dynamical time scales. We have discussed the implications for interpreting multi-epoch spectral observations of spatially unresolved protoplanetary discs.

The RPE depends on the radial scale of averaging and there exists an optimal scale that minimizes the error which also corresponds to an optimal spectral resolution when the correspondence between peak frequency of emission and radius is made. The data can be binned to compare with the theory of minimum RPE only when the spectral resolution of the telescope exceeds the value $R_{opt}$ which minimizes the RPE. An optimal resolution exists because if the spectral resolution is too high then the instrument ends up sampling noise and if the spectral resolution is too low, then the instrument samples overly coarsely binned regions of the disk.

The stochastic variability expected between multi-epoch observations of discs is an implicit prediction of alpha disc theory and can be directly compared to observed variabilities. Complementarily, the nature of the observed variability can be used to constrain whether or not the behavior of the disc is consistent with luminosity produced by stochastic or systematic processes.

The predicted stochastic variability from $\alpha$ discs has several distinct characteristics that would signature its prevalence:

- The stochastic variability is predicted primarily for regions where the luminosity is dominated by internal turbulent dissipation in the disc. For regions dominated by surface dust-reprocessed starlight, the stochastic variability could be expected to be strongly reduced.

- For regions of the discs dominated by from turbulent dissipation, the strength of the stochastic variability would increase gradually with decreasing emission frequency (and thus increasing radius) down to a critical frequency below which the associated disc radii have orbital times exceeding the inter-epoch observation time. There the expected variability from the RPE would drop sharply.

- For inner regions of the discs in transitional objects, where emission may be dominated by turbulent dissipation but the discs are optically thin, the stochastic variability would be expected to be lower than that expected from the same radii for optically thick turbulent dissipation dominated discs.

In applying the basic ideas herein to the specific case of LRLL 31, we find that systematic variability NOT stochastic variability dominates. This is indeed consistent with the conclusion of Muzerolle et al. [2009]. As more multi-epoch observations of broader samples of discs at different stages in their lifetimes are obtained, the present work may help provide a tool to distinguish stochastic processes in discs from systematic dynamical changes and the time scales on which these occur.

Two overall lessons from this analysis are: (1) the precision of axisymmetric accretion theories can be quantified and deviations

### Table 2

| $r$ (au) | $\alpha$ | $R_{opt}$ | $\frac{\Delta L_{\nu}}{L_{\nu}}$ (%) | $R_{tel}$ | $\frac{\Delta L_{\nu}}{L_{\nu}}$ (%) |
|---------|----------|-----------|-----------------------------------|---------|-----------------------------------|
| 0.1     | 0.001    | 1832      | 0.08754                           | 200     | 0.09226                           |
| 0.01    | 0.01     | 579.3     | 0.2781                            | 200     | 0.2816                            |
| 0.1     | 183.2    | 0.8914    | 200                               | 0.8915  |
| 1       | 0.001    | 1156      | 0.1103                            | 200     | 0.1139                            |
| 0.01    | 365.5    | 0.3514    | 200                               | 0.3531  |
| 0.1     | 115.6    | 1.135     | 200                               | 1.144   |
| 10      | 0.001    | 729.3     | 0.1392                            | 200     | 0.1417                            |
| 0.01    | 230.6    | 0.4449    | 200                               | 0.4451  |
| 0.1     | 72.93    | 1.454     | 200                               | 1.513   |
| 20      | 0.001    | 634.9     | 0.1493                            | 200     | 0.1515                            |
| 0.01    | 200.8    | 0.4780    | 200                               | 0.4780  |
| 0.1     | 63.49    | 1.570     | 200                               | 1.661   |

### Table 3

| $r$ (au) | $\alpha$ | $R_{opt}$ | $\frac{\Delta L_{\nu}}{L_{\nu}}$ (%) | $R_{tel}$ | $\frac{\Delta L_{\nu}}{L_{\nu}}$ (%) |
|---------|----------|-----------|-----------------------------------|---------|-----------------------------------|
| 0.1     | 0.001    | 625.8     | 0.6935                            | 200     | 0.7111                            |
| 0.01    | 249.2    | 1.116     | 200                               | 1.118   |
| 0.1     | 99.19    | 1.835     | 200                               | 1.881   |
| 1       | 0.001    | 557.8     | 0.7355                            | 200     | 0.7510                            |
| 0.01    | 222.1    | 1.186     | 200                               | 1.186   |
| 0.1     | 88.40    | 1.958     | 200                               | 2.030   |
| 10      | 0.001    | 497.1     | 0.7802                            | 200     | 0.7935                            |
| 0.01    | 197.9    | 1.260     | 200                               | 1.260   |
| 0.1     | 78.79    | 2.091     | 200                               | 2.200   |
| 20      | 0.001    | 480.2     | 0.7942                            | 200     | 0.8069                            |
| 0.01    | 191.2    | 1.284     | 200                               | 1.284   |
| 0.1     | 76.10    | 2.133     | 200                               | 2.256   |

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