Tunneling through sphalerons: 
the O(3) $\sigma$-model on a cylinder

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Abstract:
We construct all instantons for the O(3) $\sigma$-model on a cylinder, known not to exist on a finite time interval. We show that the widest instantons go through sphalerons. A re-interpretation of moduli-space transforms the scale parameter $\rho$ to a boundary condition in time. This may give a handle on the $\rho \to 0$ divergent instanton gas.

1 Introduction

Due to asymptotic freedom, the large distance behavior of SU($N$) gauge theories must be treated non-perturbatively. A convenient method is to put the model in a finite spatial box of length $L$ and calculate the low-lying energy eigenstates as function of $L$. Obviously the wave functionals of such states are concentrated at small $L$ in the vacua of classical configuration space. At larger $L$ they can spread out over low energy barriers between these vacua. This spreading causes the breakdown of conventional perturbation theory.

This picture has been used [1, 2] for reduction of the field theory to a finite dimensional system. The remaining degrees of freedom are expected to correspond to the set of vacua (called the vacuum valley) and suitable paths over the barriers between vacua. One of the requirements on such paths is that they cross a barrier at its lowest point. This point, a sphaleron [3, 4], is defined through a mini-max procedure. One first finds the maximal energy on a path connecting two vacua. Then one minimizes this maximum over the space of all such paths. In local terms a sphaleron is a saddle point of the energy functional with exactly one unstable mode.

Well-known paths between two vacua are instantons [5, 6], interpreting Euclidean time as a path parameter. In the WKB-approximation the tunneling amplitude is dominated by paths near instantons. Therefore it seems natural to assume that amongst the instanton paths there is one which crosses a barrier at its lowest point, i.e. it goes through a sphaleron. Let us assume that the set of instantons (called the instanton moduli-space) has a scale parameter $\rho$. Since all instantons have the same Euclidean action, a dimensional argument

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easily shows that the maximal energy along the tunneling path goes as \(1/\rho\). Hence it is likely that instantons with maximal width, as set by \(L\), go through a sphaleron.

The latter assumption was made in [3] for the following reasons. First, in a direct approach it is often easier to find instantons than sphalerons. Then, if the assumption holds, one can find all sphalerons via instantons. Second, for small \(L\) (high barriers) the spreading of a low energy wave function is determined by tunneling. For large \(L\) (low barriers) it is determined by classically allowed motion through a sphaleron. If the assumption does not hold, there is some range of \(L\) values in which the most important paths between vacua change dramatically.

The conjecture that a sphaleron lies on an instanton path is in general not true. For example, consider the two-dimensional potential \(V(q_1, q_2) = (q_1^2 - 1)^2((q_2^2 - 1)^2(2 + q_2^2) + 1) + (1 + b^2/q_2^2)^{-1}\) for small values of the parameter \(b\). It has vacua at \((\pm 1, 0)\) with zero energy and (to leading order in \(b\)) sphalerons at \((0, \pm 1)\) with energy 2. However, for small enough \(b\) the instanton does not go through a sphaleron. Instead, it goes straight through the saddle point at \((0, 0)\), which has one unstable mode and energy 3. It is even possible that an instanton goes through several saddle points with one unstable mode (we think of a one-dimensional example), in which only the one with highest energy would be a candidate sphaleron.

The above clearly shows that in field theory the conjecture needs to be checked. For SU(2) gauge theory on a space-time \(T^3 \times [-\frac{1}{2}T, \frac{1}{2}T]\) we [7] have developed a numerical procedure to find the widest instantons (for \(T \to \infty\)). Subsequent numerical investigation [8] has shown that the widest instanton goes through a saddle point with one unstable mode, like for SU(2) on a space-time \(S^3 \times \mathbb{R}\) [2]. It is likely these saddle points are sphalerons. For the two-dimensional O(3) \(\sigma\)-model, which is studied in the present paper, this can be proven rigorously.

It is well-known that this \(\sigma\)-model shares with the four-dimensional gauge theories features like renormalizability and asymptotic freedom. Another similarity is still more important to us now. Both models, when put in a spatial cube with periodic boundary conditions, have vacuum valleys of nonzero dimension. This can increase the number of instantons, as they must have endpoints in the vacuum valley. In this context we also mention the absence of instantons on compact space-times \(T^3 \times T^1\) and \(T^1 \times T^1\), respectively [3, 10, 11]. This does not rule out the existence of instantons on an infinite time interval. It merely suggests the vacuum valley cannot be reached in a finite amount of time.

The outline of this paper is as follows: in section 2 we will set up a convenient formalism, which will be used in section 3 to derive all static solutions of the Euler-Lagrange equations. In particular we find the vacuum valley and the sphaleron solutions. Section 4 is devoted to constructing all instanton solutions and interpreting their moduli-space. A scale parameter will emerge that relates the instanton field at \(t = +\infty\) to that at \(t = -\infty\). We conclude in section 5 by putting our results in perspective with respect to the four-dimensional SU(2) case.
2 The O(3) $\sigma$-model in general coordinates

The action for the O(3) $\sigma$-model on a cylinder reads

$$S[\vec{n}] = \frac{1}{2} \int d^2 x \ |\partial_\mu \vec{n}(x)|^2, \quad \vec{n}(x) \in \mathbb{R}^3, \quad |\vec{n}(x)|^2 = 1,$$

where the integration runs over space-time $\{(x_2, x_1) \in T^1 \times \mathbb{R}\}$. We use overall scale invariance to fix the length of the spatial 1-torus $T^1$ (the circle) to be $2\pi$. So $\vec{n}(x + 2\pi \hat{e}_2) = \vec{n}(x)$. The metric on space-time is Euclidean, and we use the summation convention over repeated indices throughout.

By definition, $\vec{n}(x) \in S^2$. If we use coordinates $v^i$ ($i = 1, 2$) and a metric $g_{ij}$ on $S^2$, eq. (1) can be rewritten as

$$S[v] = \frac{1}{2} \int d^2 x \ g_{ij}(x) \partial_\mu v^i(x) \partial_\mu v^j(x).$$

In this paper we will use both spherical coordinates $(\vartheta, \varphi)$ and stereographic projection $(u_1, u_2)$ which will be paired as $u \equiv u_1 + i u_2$, with complex conjugate $\bar{u} = u_1 - i u_2$ (c.f. [9, 10]):

$$\vec{n} = \begin{pmatrix} \sin \vartheta \cos \varphi \\ \sin \vartheta \sin \varphi \\ \cos \vartheta \end{pmatrix} = \frac{1}{1 + |u|^2} \begin{pmatrix} u + \bar{u} \\ (u - \bar{u})/i \\ |u|^2 - 1 \end{pmatrix},$$

(hence $u = \cot \frac{1}{2} \vartheta e^{i\varphi}$).

In section 3 we will need the Euler-Lagrange equations and the Hessian of eq. (2). A straightforward computation [12] gives (for vanishing $\delta v(x_1 \to \pm \infty, x_2)$):

$$S[v + \delta v] = S[v] + S^{(1)}[v, \xi] + S^{(2)}[v, \xi] + O(\xi^3),$$

$$S^{(1)}[v, \xi] = - \int d^2 x \ (D_\mu \partial_\nu v) \xi^\nu,$$

$$S^{(2)}[v, \xi] = \int d^2 x \ \xi^i \mathcal{H}_{ij}[v] \xi^j, \quad \mathcal{H}_{ij}[v] = -\frac{1}{2} (D_\mu D_\nu v)_{ij} + \frac{1}{2} R_{ijkl} \partial_\mu v^k \partial_\nu v^l.$$

Here $\xi = \delta v + O(\delta v^2)$ is defined in such a way that it transforms covariantly; see [12] for details. Since the action is a scalar, this guarantees that $\mathcal{H}$ is a tensor, as can be verified from the explicit form. Furthermore, $D_\mu$ is the covariant derivative and $R$ is the Riemann tensor, both at the point $v$:

$$(D_\mu \lambda)_i = \partial_\lambda \lambda_i - \Gamma_{ik}^j \partial_\mu \lambda^k, \quad (\text{for any vector } \lambda)$$

$$\Gamma_{kiij} = \frac{1}{2} \left( \partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij} \right), \quad (\partial_i \equiv \partial/\partial v^i)$$

$$R_{ijkl} = \partial_j \Gamma_{ik}^l - \partial_k \Gamma_{ij}^l + \Gamma_{im}^l \Gamma_{kj}^m - \Gamma_{ik}^m \Gamma_{mj}^l.$$

The Eulerian action, eq. (2), is precisely half the total volume of $v(T^1 \times \mathbb{R}) \subset S^2$. This action also naturally occurs in string theory (see e.g. [13]). Therefore, extremizing the action (which amounts to putting $D_\mu \partial_\nu v = 0$, eq. (4)) gives a geodesic surface (in affine parametrization) on the space $S^2$.

Finally, it is important to introduce the winding number which measures the number of times $T^1 \times \mathbb{R}$ is wrapped around $S^2$ by $v$ (and therefore is invariant under continuous deformations of $v$):

$$Q[v] = -\frac{1}{16} \int d^2 x \ \varepsilon_{\mu \nu} \vec{n} \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n}) = \frac{1}{16} \int d^2 x \ \varepsilon_{\mu \nu} \partial_\mu \vec{n} \cdot (\vec{n} \times \partial_\nu \vec{n}) = \frac{1}{16} \int d^2 x \ \varepsilon_{\mu \nu} g_{ij} \partial_\mu v^i \lambda^j_\nu,$$
where $\lambda^j_\nu$ is the vector in the tangent space of $S^2$ (at the point $v$) corresponding to $\vec{n} \times \partial_\nu \vec{n}$; up to orientation $\lambda_\nu$ is defined by $g_{ij} \lambda^j_\nu \lambda^i_\nu = g_{ij} \partial_\nu v^i \partial_\nu v^j$, $g_{ij} \lambda^j_\nu \partial_\nu v^i = 0$ (no summations over $\nu$). From this follows the well-known formula \[ S = \tilde{S}_\pm \mp 4\pi Q, \quad \tilde{S}_\pm[v] = \frac{1}{2} \int d^2 x \ g_{ij}(\partial_\mu v^i \pm \epsilon_{\mu \nu \rho} \lambda^i_\nu) (\partial_\mu v^j \pm \epsilon_{\mu \rho \lambda^j_\nu}). \] (7)

We define an instanton to be a minimal action configuration in the sector $Q = +1$. Therefore it has action $4\pi$ and satisfies the instanton equation

$$\partial_\mu v^i = \epsilon_{\mu \nu \rho} \lambda^i_\nu. \quad (8)$$

3 Sphalerons, vacua and other static solutions

In this section we study the potential, or energy functional, which is the action (2) restricted to space, i.e. without time-dependence and without integration over time:

$$V[v] = \frac{1}{2} \int T_1 dx^2 g_{ij}(\partial_2 v^i \partial_2 v^j). \quad (v = v(x_2))$$

This is the geodesic action for a curve $v(x_2)$ on $S^2$. Therefore all static solutions of the Euler-Lagrange equations, being extrema of eq. (11), are big circles on $S^2$ (affinely parametrized by $x_2$). Remembering the requirement $v(x_2 + 2\pi) = v(x_2)$ we conclude that the most general static solution reads (after an SO(3) rotation)

$$\vec{n}_k(x_2) = \begin{pmatrix} \cos(kx_2) \\ \sin(kx_2) \\ 0 \end{pmatrix}, \quad k \in \mathbb{Z}. \quad (10)$$

One easily computes the energy

$$V[\vec{n}_k] = \pi k^2. \quad (11)$$

In particular $\vec{n}_0$ is a classical vacuum. Due to SO(3) symmetry the vacuum valley is isomorphic to $S^2$.

Now we turn to the sphalerons. For this we have to determine the Hessian of the energy functional, eq. (2), at the field $\vec{n}_k$, eq. (10). This is easy in spherical coordinates where eq. (10) reads $\vartheta_k(x_2) = \frac{\pi}{2}$, $\varphi_k(x_2) = kx_2$ and the metric is given by $(g_{ij}) = \text{diag}(1, \sin^2 \vartheta)$. Substituting these formulas in the static version of eq. (11) we obtain

$$\mathcal{H}[\vec{n}_k] = \frac{1}{2} \begin{pmatrix} -\partial_2^2 - k^2 & 0 \\ 0 & -\partial_2^2 \end{pmatrix}. \quad (12)$$

One immediately sees that $k = \pm 1$ is the only solution with exactly one unstable mode ($\delta \vartheta(x_2) = 1$, $\delta \varphi(x_2) = 0$). So the sphaleron solutions are given by $\vec{n}_1$ and SO(3) transformations thereof. In particular $k \to -k$ can be undone by a rotation over $\pi$ around any vector $\vec{n}_k(x_2)$ ($x_2$ fixed). Thus, sphaleron moduli-space is isomorphic to SO(3).

One can verify that SO(3) rotations are responsible for the 3 zero-modes of the Hessian. Note that the sphaleron is invariant under an $x_2$-translation in combination with a specific SO(2) $\subset$ SO(3) rotation (in the case of eq. (10) around the 3-axis). Therefore spatial translations do not give new sphaleron solutions. Also one can check that the discrete symmetries $x_2 \to -x_2$ and $\vec{n} \to -\vec{n}$ leave the sphaleron moduli-space invariant.

2At this point we do not prove that these saddle points are true sphalerons in the mini-max sense. This proof can be constructed easily with the results of the next section.
4 Instanton solutions

4.1 Construction of the solutions

In stereographic projection [5, 10] the instanton equation (8) is particularly simple:

\[ \partial_z u = 0. \]  

(13)

Here we have introduced complex coordinates on space-time, \( z = x_1 + ix_2, \bar{z} = x_1 - ix_2, \) with derivatives \( \partial_z = \frac{1}{2} (\partial_1 - i\partial_2), \partial_{\bar{z}} = \frac{1}{2} (\partial_1 + i\partial_2). \) The construction of instantons reduces to finding all analytic functions on \( T^1 \times \mathbb{R} \) with topological charge \( Q = 1. \)

Substitution of eq. (3) in eq. (6) gives, for any \( u \) satisfying eq. (13),

\[ Q[u] = \frac{1}{2} \int_{T^1 \times \mathbb{R}} I R d^2x \left| \frac{\partial_z u}{1 + |u|^2} \right|^2 = -\frac{1}{2} \int_{T^1 \times \mathbb{R}} I R d^2x \partial_{\bar{z}} \left( \frac{1}{1 + |u|^2} \frac{\partial_z u}{u} \right). \]

(14)

Handling carefully the poles of \( \frac{1}{1 + |u|^2} \frac{\partial_z u}{u} \), i.e. the zeros of \( u \), one finds

\[ Q[u] = -\frac{1}{2\pi} \left( \int_{x_1=\infty} - \int_{x_1=-\infty} \right) dx_2 \frac{1}{1 + |u|^2} \frac{\partial_z u}{u} + \sum_i n_i, \]

(15)

where the \( x_2 \) integrations run from 0 to \( 2\pi \) and \( i \) runs over the zeros of \( u \), of degree \( n_i \in \mathbb{N} \). In order to simplify this formula, we observe that both the kinetic and the potential term in the Lagrangian are semi-positive definite. Hence any finite-action configuration, in particular an instanton, must approach a point in the vacuum valley for \( x_1 \to \pm \infty \):

\[ \lim_{x_1 \to \pm \infty} u(z) = u_{\pm}. \]

In the derivation below we will assume that \( 0 < |u_{\pm}| < \infty \), as can always be achieved by an \( SO(3) \) rotation\(^3\). Under this assumption eq. (15) reduces to \( Q[u] = \sum_i n_i. \)

Now we are fully prepared to determine all instanton solutions. The strategy is first to find a class of solutions and then to prove no solutions exist outside this class. In order to construct a solution observe that

1. Since \( e^{z+2\pi i} = e^z \), any \( u = h(e^z) \) (\( h \) single-valued and analytic) is a function on \( T^1 \times \mathbb{R} \) satisfying the instanton equation.

2. Under the above assumption any instanton can have only one zero \( z_1 \) of degree \( n_1 = 1. \)

A class of functions satisfying all requirements is

\[ u_{a,b,c,d}^{\text{inst}}(z) = \frac{c + de^z}{a + be^z}, \]

(16)

with certain restrictions on the complex coefficients \( a, b, c, d \); note that \( u_+ = -d/b, u_- = -c/a \) and \( z_1 = \ln(-c/d) \) (which is unique on \( T^1 \times \mathbb{R} \)). So in order to satisfy the assumption \( 0 < |u_{\pm}| < \infty \), we must require \( a, b, c, d \neq 0. \) Also we must demand \( a/b \neq c/d \) as otherwise the zero of \( u_{a,b,c,d}^{\text{inst}} \) cancels against its pole and we have a trivial solution with \( Q = 0. \)

\(^3\)This is equivalent to changing coordinates on \( S^2 \) by shifting the pole used in stereographic projection.
To prove that all instanton solutions are of the form (16) is easy: suppose \( \tilde{u} \) is an instanton. We can still assume \( 0 < |\tilde{u}| < \infty \), so \( \tilde{u} \) can have only one zero of degree 1, say at \( z = \tilde{z}_1 \). Now define a function \( f \) by

\[
\tilde{u}(z) = u^\text{inst}_{a,b,c,d}(z)f(z).
\]

By choosing \(-c/a = \tilde{u}_-, -d/b = \tilde{u}_+, c/d = -e^{-\tilde{z}_1}\) and imposing the instanton requirements \( \partial \bar{z}\tilde{u} = 0 \), \( Q[\tilde{u}] = 1 \) and \( \tilde{u}(z + 2\pi i) = \tilde{u}(z) \), that are already met by \( u \), we see that \( f \) has to satisfy

\[
\begin{align*}
\partial \bar{z}f &= 0 \\
\lim_{\text{Re}(z) \to \pm \infty} f(z) &= 1 \\
f(z + 2\pi i) &= f(z) \\
f(z) &\neq 0.
\end{align*}
\]

Thus, \( 1/f \) is analytic and bounded on \( \mathbb{C} \). By Liouville’s theorem this implies, using the second condition in eq. (18), that \( f(z) = 1 \). This completes the proof.

Finally we drop the assumption \( 0 < |u| < \infty \). Taking into account the boundary terms in eq. (15), one finds that the only restriction on \((a, b, c, d)\) is that \( u^\text{inst}_{a,b,c,d}(z) \) is not constant, corresponding to

\[
ad - bc \neq 0.
\]

So the class of instantons, eq. (19), is precisely the set of conformal mappings of \( e^z \).

We end this part by noting that by the same line of argument each multi-instanton with topological charge \( Q \) can be written as \( \prod_{n=1}^{Q} u^\text{inst}_{a_n,b_n,c_n,d_n} \). For (multi-)anti-instantons one substitutes \( z \to \bar{z} \).

### 4.2 Physical interpretation of the moduli-space

It is clear that the moduli-space has six real dimensions: \((a, b, c, d) \in \mathbb{C}^4 \) with the projective character \( u^\text{inst}_{ga,gb,gc,gd} = u^\text{inst}_{a,b,c,d} \) for any \( g \in \mathbb{C}\backslash\{0\} \) (while the set of coefficients not satisfying eq. (19) has zero measure). A physical parametrization of this moduli-space is \((-c/a, -d/b, \ln(-c/d))\), corresponding to the starting point \( u_- \) in the vacuum valley \( S^2 \), the end point \( u_+ \) and a space-time translation parameter, which one can show to be in \( 1 - 1 \) relation with the instanton position (cf. eq. (22) below). The disadvantage of this parametrization is that it leaves unclear what kind of manifold the moduli-space is; the parametrization is singular for \( u_- = u_+ \), since the requirement of eq. (19) is not met in this case. Note that this means that even for \( T \to \infty \) periodic boundary conditions do not admit instanton solutions. We will see below that \( u_- \to u_+ \) corresponds to \( \rho \to 0 \), where \( \rho \) is the instanton scale parameter.

For a better description of moduli-space it is necessary first to consider the transformation of instantons under space-time translations and \( SO(3) \) rotations. We will see that most instantons are not invariant under these symmetries which therefore give rise to five of the six dimensions of moduli-space. The interesting sixth parameter, not related to a symmetry of the action, will play the role of a scale parameter, related to the geodesic distance between the points \( u_\pm \) in the vacuum valley \( S^2 \).

From eq. (16) it is trivial to see that under a translation \( z \to z + z_0 \), \((a, b, c, d) \to (a, be^{z_0}, c, de^{z_0})\). Note that eq. (19) is therefore invariant under this shift, as it should be. Any continuous symmetry of the action must be present in the instanton moduli-space.
The effect of an SO(3) rotation is more difficult to derive, because while it acts linearly on $\vec{n}$, $\vec{n}$ and $u$ are related non-linearly by eq. (3). Nevertheless, one can show that $(a, b, c, d)$ again transform linearly. After some effort one sees that the rotation

$$\vec{n}(x) \rightarrow R\vec{n}(x), \quad R = e^{\alpha_a L^a}, \quad L^a_{ij} = -\varepsilon_{aij}, \quad (\alpha_a \in \mathbb{R})$$

(20)
duces

$$\left(\begin{array}{c} a \\ b \\ c \\ d \end{array}\right) \rightarrow \tilde{R} \left(\begin{array}{c} a \\ b \\ c \\ d \end{array}\right), \quad \tilde{R} = e^{\alpha_a L^a}, \quad \tilde{L}^a = -\frac{i}{2} \sigma^a \otimes 1, \quad (\text{i.e. } \tilde{L}^1 = -\frac{i}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}) \text{ etc.}$$

(21)

where $\sigma^a$ are the Pauli-matrices. Since $\pm(a, b, c, d)$ are identified, this is a representation of SO(3). Notice that only $(a, c)$ and $(b, d)$ mix. Hence both $|a|^2 + |c|^2$ and $|b|^2 + |d|^2$ are rotationally invariant, as is eq. (13).

Using the projective character of moduli-space, and an SO(3) rotation $\tilde{R}$, it is always possible to bring $u_{ab,c,d}$ (ad $\neq bc$) to the form $u_{-1,0,d}$ with $\tilde{c} \in \mathbb{R}$, $\tilde{d} \in \mathbb{C}\setminus\{0\}$. If $\tilde{c} > 0$, this fixes $\tilde{R}$ completely. If $\tilde{c} = 0$, then $\tilde{R}$ is only unique up to a factor $e^{\alpha_3 L^3}$. This can be fixed by requiring $\tilde{d} = |\tilde{d}|$. Therefore, we can parametrize $u_{ab,c,d}$ uniquely by $\tilde{R}$, $|\tilde{d}|$ and $\tilde{c}e^{i\phi}$ ($\tilde{c} \geq 0$). Here $\phi = \text{Arg}(|\tilde{d}|) (\phi \in [0, 2\pi])$ if $\tilde{c} > 0$ and $\phi$ is undetermined if $\tilde{c} = 0$. We conclude that $(\tilde{c}, \phi)$ can be viewed as polar coordinates on $\mathbb{R}^2$. Furthermore, $|\tilde{d}| > 0$ due to eq. (19), so $\ln|\tilde{d}| \in \mathbb{R}$.

Thus the instanton moduli space is isomorphic to $\text{SO}(3) \times \mathbb{R}^2 \times \mathbb{R}$. The discrete symmetry transformations $\vec{n} \rightarrow -\vec{n}$ (in stereographic coordinates $u \rightarrow -1/\bar{u}$), $x_2 \rightarrow -x_2$ or $x_1 \rightarrow -x_1$ make an instanton solution anti-analytic, and therefore are transformations from instanton moduli-space into anti-instanton moduli-space. This should be compared to the sphaleron moduli space which is invariant under such discrete transformations.

Note that $de^z = e^{z+\ln|\tilde{d}|+i\phi}$. So after an SO(3) rotation and a translation in space-time $T^1 \times \mathbb{R}$, any instanton solution can be brought to the form

$$u^\text{inst}_c(z) \equiv c + e^{z+\ln(\sqrt{1+c^2})+i\pi}. \quad (c \geq 0)$$

(22)

The factor $e^{\ln(\sqrt{1+c^2})+i\pi} = -\sqrt{1+c^2}$ centers the instanton around $z = 0$ (see below). From the above equation it follows that $\lim_{x_1 \to \infty} |u^\text{inst}_c(z)| = \infty$ and $\lim_{x_1 \to -\infty} u^\text{inst}_c(z) = c$, hence (using eq. (3)) this instanton ‘tunnels’ from $(\sin \vartheta_-, 0, \cos \vartheta_-)$ to $(0, 0, 1)$, with $\cot \frac{\pi}{2} \vartheta_- = c$. Note that $\vartheta_-$ is the geodesic distance between these vacua.

Let us determine the instanton size $\rho$ as function of $c = 2\text{arccot} \vartheta_-$. Substituting eq. (22) into the Lagrangian density, which for an instanton in stereographic coordinates is $4\pi$ times the integrand in eq. (14) (see eqs. (23)), one obtains

$$\mathcal{L}[u^\text{inst}_c](x_1, x_2) = 4 \frac{|\partial_z u^\text{inst}_c|^2}{(1 + |u^\text{inst}_c|^2)^2} = \frac{1}{(\sqrt{1+c^2} \cosh x_1 - c \cos x_2)^2}. \quad (23)$$

This function is plotted in fig. 11 for different values of $c$. From the formula it is clear that $u^\text{inst}_c$ is centered at $z = 0$. Now consider the potential along the instanton path,

$$V[u^\text{inst}_c](x_1) = \frac{1}{4} \int_0^{2\pi} dx_2 \mathcal{L}[u^\text{inst}_c](x_1, x_2). \quad (24)$$
Figure 1: The Lagrangian densities of three instantons, eq. (22), with from left to right \( c = 0, 0.25, 0.5 \).

The factor \( \frac{1}{2} \) comes from the fact that the kinetic energy, \( \frac{1}{2} \int T dx g_{ij}(v) \partial_i v^j \partial_i v^j \), is equal to the potential energy, eq. (9) (this ‘self-duality’ follows from eq. (8)). We see that the potential is maximal at \( x_1 = 0 \) where it satisfies

\[
V_c^\text{max} \equiv V[u_c^{\text{inst}}](0) = \pi \sqrt{1 + c^2}.
\]

(25)

Since all instantons have equal action, it is natural to define the instanton size

\[
\rho(c) \equiv \frac{\pi}{V_c^\text{max}} = \frac{1}{\sqrt{1 + c^2}}.
\]

(26)

Note however that for small \( c \) there are two different scales; the shape of \( \mathcal{L}[u_c^{\text{inst}}](x_1, x_2) \) is anisotropic (see fig.1). Only for \( c \gg 1 \) and \( x_1^2 + x_2^2 \ll 1 \) the boundary effects disappear and \( \mathcal{L}[u_c^{\text{inst}}](x_1, x_2) \approx \frac{4c^2}{(1 + c^2(x_1^2 + x_2^2))^2} \) becomes rotationally invariant.

The relationship between instantons and sphalerons is now also clear. From eqs. (25,11) we see that only \( u_c^{\text{inst}} = 0 \) can go through a sphaleron at the time of maximal \( V[u_c^{\text{inst}}] \) (i.e. \( x_1 = 0 \)). Indeed this does happen, since eq. (22) gives \( u_c^{\text{inst}} = 0(x_1 = 0, x_2) = -e^{ix_2} \), which up to a rotation is just the sphaleron solution \( \vec{n}_1 \), eq. (11), in stereographic coordinates. As the instanton is a self-dual solution of the equations of motion, it has to follow streamlines of the energy functional. This explains why \( \partial_1 u_c^{\text{inst}}|_{x_1=0} = -\exp(ix_2) \) corresponds to the unstable mode of the sphaleron solution.

Finally note that \( u_c^{\text{inst}} = 0 \), unlike \( u_c^{\text{inst}} > 0 \), is a point in instanton moduli-space that is symmetric under a joint SO(2) rotation \( e^{\alpha_3 L_3} \) and a spatial translation \( x_2 \to x_2 - \alpha_3 \). The sphaleron has of course the same symmetry, as mentioned in section 3. Also a natural correspondence between sphaleron moduli-space SO(3) and the subspace of widest instantons emerges. From the paragraph above eq. (22) it follows that the latter is isomorphic to \( \text{SO}(3) \times \mathbb{R} \), \( \mathbb{R} \) corresponding to time translations.

### 5 Conclusions

We have proven that the O(3) \( \sigma \)-model on a space-time \( T^1 \times \mathbb{R} \) admits instantons. The moduli-space is 6-dimensional (\( \text{SO}(3) \times \mathbb{R}^2 \times \mathbb{R} \)): 3 parameters for \( \text{SO}(3) \), 2 for scaling and
spatial translations, 1 for time translations. It is possible to ‘tunnel’ between any different points \( \vec{n}_\pm \) in the vacuum valley \((S^2)\), but this gives no independent parameters. Three parameters describing \( \vec{n}_\pm \) can be removed by an SO(3) rotation, while the fourth (the geodesic distance between the points) depends uniquely on the scale parameter \( \rho \). Instantons with maximum scale, as set by the extent of spatial \( T^1 \), satisfy \( \vec{n}_+ = -\vec{n}_- \). These, and only these, instantons go through sphalerons. None of the exotic possibilities sketched in the introduction take place in this model. On the other hand, \( \rho \to 0 \) corresponds to \( \vec{n}_+ \to \vec{n}_- \). Exact equality cannot be reached. We think this peculiar size-dependence is important for improving on instanton gas calculations as in [14]. By doing a proper convolution with the vacuum wave function at \( t \to \pm \infty \) it might be possible to remove the well-known UV divergence for \( \rho \to 0 \), which was also encountered in recent numerical studies [11].

Our results do not admit a straightforward generalization to SU(2) gauge theory on a space-time \( T^3 \times \mathbb{R} \). In that model the vacuum valley is isomorphic to \( T^3 \) [15], which can be parametrized by three Polyakov lines \( P_i \). Instantons again must have endpoints, \( P_i^\pm \), in the vacuum valley. For the special case \( P_i^+ = -P_i^- \) it has already been known for some time that an 8-dimensional moduli-space exists. This anti-periodic situation corresponds to time-like twist [16], which has been analyzed [17] on any space-time \([ -\frac{T}{2}, \frac{T}{2} ] \times T^3 \). For \( T \to \infty \) it is very likely [4] that the moduli-space includes a scale parameter. This is not the case for the O(3) \( \sigma \)-model on a space-time \( T^1 \times \mathbb{R} \). We have just proven that anti-periodic boundary conditions, \( \vec{n}_+ = -\vec{n}_- \), fix the instanton size. It would be nice to understand the cause of such different behavior between two models that are so similar in other respects. This might be a starting point for finding new instanton parameters in SU(2) gauge theory on \( T^3 \times \mathbb{R} \), by relaxing the condition \( P_i^+ = -P_i^- \).

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\footnote{We also note that no instantons exist with anti-periodic boundary conditions in finite time \( T \), \( \vec{n}(x_1 + T, x_2) = -\vec{n}(x_1, x_2) \) (while the winding number \( Q \) is still a well-defined integer object). The reason is simple: in stereographic coordinates anti-pbc read \( u(z + T) = -1/\bar{u}(z) \), which is incompatible with the instanton equation \( \partial_z \bar{u} = 0 \).}
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