PSD-throttling on trees

Michael S. Ross*

June 17, 2019

Abstract

PSD-forcing is a coloring process on a graph that colors vertices blue by starting with an initial set $B$ of blue vertices and applying a color change rule (CCR-$Z_+$. The PSD-throttling number is the minimum of the sum of the cardinality of $B$ and the number of the time-steps needed to color the graph (the PSD-propagation time of $B$). Concentration, which is a technique for computing the PSD-throttling number by reducing a tree to a smaller vertex-weighted tree, is introduced and used to determine the PSD-throttling numbers of balanced spiders. It is shown that the PSD-throttling number of a balanced spider does not exceed that of the path of the same order.

Keywords Throttling, concentration, PSD, zero forcing, propagation time, spider

AMS subject classification 05C57, 05C15, 05C50

1 Introduction

The concept of zero forcing has its origins as a bounding parameter in minimum rank/maximum nullity problems from linear algebra and spectral graph theory [1], and also independently in control of quantum systems [5, 10]. However, the study of zero forcing and related parameters, such as propagation time and throttling, has since developed into a topic of interest in its own right, with a purely graph-theoretic interpretation and additional applications, such as graph searching [12]. Throttling, which can be viewed as the optimization of the sum of the resources used to accomplish a task and the time needed to accomplish the task with those resources, was introduced for standard zero forcing by Butler and Young in [6]. The study of throttling was extended to positive semidefinite zero forcing (PSDZF) by Carlson et al. in [7] and to the game of Cops and Robbers by Breen et al. in [4], where it was shown that cop-throttling and PSD-throttling are equivalent on trees. Results from [4] are discussed here in PSD-throttling notation, because the focus of this paper is on PSD-throttling for trees and spiders, which are trees that have exactly one vertex with degree higher than 2. Spiders are usually described in terms of lengths of their legs; e.g. $S(7, 6, 2)$ is a tree on 16 vertices, with one vertex adjacent to three paths of orders 7, 6, and 2, respectively. A balanced spider is one in which every leg has the same length.

In Section 2 we give a sometimes useful variant of the formula for the PSD-throttling number of paths, extend the definition of throttling to include vertex weighted graphs, and

*Department of Mathematics, Iowa State University, Ames, IA 50011, USA (msross@iastate.edu)
introduce the method of concentration for PSD-throttling on trees; this is a technique to reduce a tree with a high degree of symmetry to a smaller vertex-weighted tree. In Section 3, we apply the concentration technique to determine the exact PSD-throttling number of every balanced spider. In Section 4, we show that no balanced spider has a PSD-throttling number that exceeds that of the path of the same order, and demonstrate that there are many examples of unbalanced spiders having PSD-throttling number exactly one more than that of the path of the same order. The remainder of this introduction contains definitions and results from prior work that will be used.

The definitions of any graph theory terms not presented here can be found in [8]. Positive semidefinite (PSD) throttling was defined in [7] from positive semidefinite zero forcing, introduced in [3] and positive semidefinite propagation time, defined in [11]. Let $G$ be a graph, and let $B \subseteq V(G)$ be the set of blue vertices. Let $W_1, \ldots, W_k$ be the sets of white vertices corresponding to the connected components of $G - B$. The PSD color change rule (CCR-Z+) colors $w_i \in W_i$ blue when $w_i$ is the only white neighbor of some $v$ in $G[W_i \cup B]$; in this case we say that $v$ forces $w$ and write $v \rightarrow w$. An initial subset of vertices $B(0) = B \subseteq V(G)$ is colored blue, with all other vertices being colored white. Then the PSD color change rule is applied iteratively, and the set $B(k)$ is defined to be the set of all the vertices that $\bigcup_{i=0}^{k-1} B^{(i)}$ can force independently; the collection of forces that color the vertices in $B(k)$ blue are said to occur during the $k$th time-step. If this process eventually colors all of $V(G)$ blue, the set $B(0)$ is said to be a PSD-forcing set of $G$, and the least $k$ such that $\bigcup_{i=0}^{k-1} B^{(i)} = V(G)$ is the PSD-propagation time of $B$, denoted by $pt_+(G; B)$. If $B \subset V(G)$ is not a PSD-forcing set, then $pt_+(G; B) = \infty$. The minimum cardinality of a PSD-forcing set for $G$ is the positive semidefinite zero forcing number of $G$ and is denoted by $Z_+(G)$. Following [7], the PSD-throttling number of $B$ in $G$ is

$$th_+(G; B) = |B| + pt_+(G; B),$$

and the PSD-throttling number of $G$ is

$$th_+(G) = \min_{B \subseteq V(G)} th_+(G; B).$$

In the event that $th_+(G; B) = th_+(G)$, $B$ is said to be an optimal set.

**Theorem 1.1.** [7] Let $n \geq 1$. Then

$$th_+(P_n) = \left\lceil \sqrt{2n} - \frac{1}{2} \right\rceil.$$  

The proof of Theorem 1.1 given in [7] uses a snaking construction, similar to the one given for the throttling of paths under standard zero-forcing, as given by Butler and Young in [6]. Roughly speaking, it snakes a path through an appropriately sized rectangle, where one dimension is the propagation time, and the other is $|B_0|$. In most cases the paths don’t fit perfectly into the rectangle, and thus we get some variation in $|B_0|$.

**Remark 1.2.** [7] There is an optimal PSD-throttling set $B_0$ the path $P_n$ with $pt_+(P_n; B_0) = q := \left\lfloor \sqrt{\frac{2n}{3}} \right\rfloor$ and $|B_0| \in \{q, q+1, q+2\}$, determined by whichever size is necessary to complete the propagation in the given time.
2 New Tools for PSD-throttling on Trees

In this section we give a useful variant of $\text{th}_+(P_n)$, show that PSD-throttling number is monotonic over connected minors of trees, and introduce the method of concentration—a technique for computing PSD-throttling by reducing a tree to a smaller (vertex) weighted tree—which is especially useful for trees with high degrees of symmetry.

2.1 Paths and Triangles

Lemma 2.1. Let $t \in \mathbb{Z}^+$. Then, the largest path with throttling number $t$ is $P_{n_t}$, where $n_t := \frac{t(t+1)}{2}$ is the $t$-th triangle number. Consequently,

$$\text{th}_+(P_n) = \left\lceil \sqrt{2n - \frac{1}{2}} \right\rceil = \left\lceil \sqrt{2n + \frac{1}{4} - \frac{1}{2}} \right\rceil$$

Proof. Let $n_t$ be the largest integer for which $\text{th}_+(P_{n_t}) = \left\lceil \sqrt{2n_t - \frac{1}{2}} \right\rceil = t$. Then, $n_t$ is the largest integer such that

$$\sqrt{2n_t} \leq t + \frac{1}{2}$$
$$2n_t \leq t^2 + t + \frac{1}{4}$$
$$n_t \leq \frac{t(t+1)}{2} + \frac{1}{8}$$
$$n_t = \left\lfloor \frac{t(t+1)}{2} + \frac{1}{8} \right\rfloor$$
$$n_t = \frac{t(t+1)}{2}.$$ 

Thus $n_t$ is the $t$-th triangle number, and consequently the throttling number of $P_n$ is the ceiling of the inverse triangle number of $n$.

On its own, Lemma 2.1 may appear to be no more than a curiosity. However, this variant of $\text{th}_+(P_n)$ can lead to some elegant cancellation when used alongside other throttling formulae, as in the proof of Lemma 4.3. Further, the triangle numbers will appear once again when we use them to construct a family of super-spiders in Section 4.

2.2 Connected Minor Monotonicity

It was established in [7] that PSD-throttling is subtree monotonic. We extend PSD-throttling monotonicity to all connected minors.

Observation 2.2. Any connected minor of a tree $T$ can be created using only edge contractions.
Theorem 2.3. Let $T$ be a tree, and let $T'$ be a connected minor of $T$. Then,

$$\text{th}_+(T') \leq \text{th}_+(T).$$

That is, positive semidefinite throttling is connected minor monotonic for trees.

Proof. We need consider only edge contraction by Observation 2.2. Let $uv \in E(T)$, let $B \subseteq V(T)$, and let $B'$ be the image of $B$ under the edge contraction $T/uv$. That is, $B'$ contains $B \setminus \{u, v\}$, and contains the new vertex if and only if at least one of $u$ or $v$ are in $B$. Thus, $|B'| \leq |B|$. Next, consider the propagation of $B$ through $T$. If that process forces through edge $uv$, then all subsequent forces in that component will occur one time-step sooner in the propagation of $B'$ through $T/uv$. If the process does not force through edge $uv$, $\text{pt}_+(T/uv; B') \leq \text{pt}_+(T; B)$. Thus, for all $uv \in E(T)$,

$$\text{th}_+(T/uv) \leq \text{th}_+(T).$$

2.3 Concentration, and the PSD-throttling Number of Weighted Graphs

First, we extend the definition of PSD-throttling to (vertex) weighted graphs $(G, w)$ where $w : V(G) \to \mathbb{R}^+$ is a weight function on the vertices of $G$.

Definition 2.4. Let $(G, w)$ be a weighted graph and let $B \subseteq V(G)$ be a PSD-forcing set of $G$. Define $w(B) = \sum_{v \in B} w(v)$. Then, using PSD-forcing for propagation,

$$\text{th}_+(G, w; B) = w(B) + \text{pt}_+(G; B)$$

and the PSD-throttling number of $(G, w)$ is

$$\text{th}_+(G, w) = \min_{B \subseteq V(G)} \text{th}_+(G, w; B).$$

In the event that $\text{th}_+(G, w; B) = \text{th}_+(G, w)$, $B$ is said to be an optimal set for $(G, w)$.

Note that PSD-throttling of weighted graphs is also a generalization of weighted PSD-throttling as defined in [7], wherein the “weighting” takes the form of a scalar $\omega$ multiplied by the size of $B$, rather than weights on individual vertices: The weighted PSD-throttling number of an unweighted graph $G$ is

$$\text{th}_+^{\omega}(G) = \min_{B \subseteq V(G)} (\omega|B| + \text{pt}_+(G; B)).$$

Observation 2.5. When the weight function $w$ is constant, i.e., $w(v) = \omega$ for all $v \in V(G)$, throttling of the weighted graph $(G, w)$ is equal to $\omega$-weighted PSD-throttling of the unweighted graph $G$: $\text{th}_+(G, w) = \text{th}_+^{\omega}(G)$. When the weight function is identically one, the result is ordinary PSD-throttling.

Next, we define a method by which we can use throttling on weighted graphs to simplify throttling of unweighted graphs.
**Definition 2.6.** Let \((T, w)\) be a weighted tree and let \(v \in V(T)\). Then, the components of \(T - v\) are called branches of \(T\) at \(v\). Branches \(T_1, T_2\) of \(T\) at \(v\) are called symmetric when

1. there is an automorphism \(\sigma\) of \(T\) such that for all \(x \in V(T) \setminus (V(T_1) \cup V(T_2)), \sigma(x) = x, \sigma(V(T_1)) = V(T_2),\) and \(\sigma(V(T_2)) = V(T_1)\), and

2. for all \(x \in V(T_i), w(x) = w(\sigma(x))\).

In this case \(\sigma\) is called a symmetry map. A set of vertices \(B\) is symmetric whenever \(B \cap V(T_i) = \sigma_i(B \cap V(T_i))\) for \(i = 2, \ldots, m\).

**Theorem 2.7.** Let \((T, w)\) be an integer weighted tree. Suppose that \(T_v = \{T_1, \ldots, T_m\}\) is a set of pairwise symmetric branches of \(T\) at \(v \in V(T)\) with symmetry maps \(\sigma_i\) between \(V(T_i)\) and \(V(T_1)\) for \(i = 2, \ldots, m\), and that \(w(v) = 1\). Then there is an optimal set \(B\) for \(T\) that is symmetric.

**Proof.** Let \(B\) be an optimal set of \(T\), and define \(B_i = B \cap V(T_i)\) for \(i = 1, \ldots, m\). There are two cases, depending on the role of \(v\).

First, suppose \(v \in B\) or \(v\) can be forced by a vertex not in \(\cup_{i=1}^m V(T_i)\) in at most \(pt_+(T; B)\) time-steps. In particular, this means the PSD-forcing process happens independently in each \(T_i\). Without loss of generality, \(w(B_1) \leq w(B_i)\) for \(i = 2, \ldots, m\). Since forcing in all branches concludes in at most \(pt_+(T; B)\) time-steps, if \(w(B_1) < w(B_i)\) we could replace \(B_i\) by \(\sigma_i(B_1)\) and \(B\) was not optimal. Thus, \(w(B_i) = w(B_1)\) for \(i = 2, \ldots, m\), and we can replace \(B_i\) by \(\sigma_i(B_1)\) to get an optimal starting set that is symmetric.

Next we consider the case where \(v\) is forced by a vertex in some \(T_k\). Let \(H_i = T[V(T_i) \cup \{v\}]\) for \(i = 1, \ldots, m\). We may assume without loss of generality that \(B\) is symmetric across all \(B_i\) where \(i \neq k\). Thus, for the remainder of this proof, assume \(i \in \{1, \ldots, m\}\) with \(i \neq k\). Again, if \(w(B_k) \leq w(B_i)\), we could replace \(B_i\) by \(\sigma_i^{-1}(B_k)\); so we assume \(w(B_k) > w(B_i)\). We may also assume \(pt_+(H_k; B_k) < pt_+(H_i; B_i)\), or else replacing \(B_k\) with \(\sigma_k^{-1}(B_i)\) is cheaper. Now, consider \(B' = (B \setminus B_k) \cup \sigma_i^{-1}(B_k)\). Clearly, since \(B\) was optimal, \(pt_+(H_i; B_i \cup \{v\}) \leq pt_+(T; B)\), as any forcing caused by \(v\) under propagation from \(B\) now occurs sooner. Further, since \(w(B_k) > w(B_i)\) and \(w(v) = 1\), \(w(B') \leq w(B)\). Thus, \(B'\) is an optimal set that is symmetric across the branches of \(v\). \(\square\)

It should be noted that there are cases without the condition “\(w(v) = 1\)” which do not have a symmetric optimal set, as demonstrated with the following example.

**Example 2.8.** Consider the balanced spider \(T = T_{3,7}\) with center vertex \(c\). Let \(A = \{x \in V(T) \mid \text{dist}(c, x) = 1\}\), \(B = \{x \in V(T) \mid \text{dist}(c, x) = 5\}\), and suppose \(T\) is given weight function

\[
    w(x) = \begin{cases} 
        1 & \text{if } \text{dist}(c, x) \in \{1, 5\} \\
        10 & \text{otherwise.}
    \end{cases}
\]

First, note that \(\text{th}_+(T; A) = w(A) + pt_+(T; A) = 3 + 6 = 9\), and thus no starting set containing a weight 10 vertex can be optimal. Thus, the only symmetric starting sets that could be optimal are \(A, B,\) and \(A \cup B,\) with \(\text{th}_+(T; B) = 8,\) and \(\text{th}_+(T; A \cup B) = 8.\) However, if our starting set is \(B \cup \{a\}\) where \(a \in A,\) we get

\[
    \text{th}_+(T; B \cup \{a\}) = 4 + 3 = 7.
\]
Thus, no symmetric set is optimal.

This construction uses a low cost vertex \( v \) near the center \( c \) to force through to vertices in other branches, thereby reducing the overall propagation time. However, if \( w(c) = 1 \), then replacing \( v \) with \( c \) cannot increase the cost, as \( w(v) \) is a positive integer. Further, the time need for \( v \to c \) is the same as \( c \to v \), and the vertices in other branches that \( v \) was forcing get forced from \( c \) sooner, meaning propagation in all other branches finishes in at most the same amount of time.

**Definition 2.9.** Let \((T,w)\) be an integer weighted tree. Suppose \( \{T_1, \ldots, T_m\} \) is a maximal set of pairwise symmetric branches of \( T \) at vertex \( v \in V(T) \), such that \( w(v) = 1 \). A single concentration of \( T \) at \( v \) is the weighted tree \( T' = T - \{T_2, \ldots, T_m\} \) with weight function

\[
  w'(x) = \begin{cases} 
    mw(x) & \text{if } x \in V(T_1) \\
    w(x) & \text{otherwise.}
  \end{cases}
\]

Each graph formed by one or more iterations of this process is called a concentration of \( T \).

**Observation 2.10.** Let \( T' \) be a concentration of a tree \( T \). Then, there is a one-to-one correspondence between subsets of \( V(T') \) and symmetric subsets of \( V(T) \).

**Example 2.11.** Consider \( T \), a full binary tree of height \( h \). Suppose \( w(v) = 1 \) for all \( v \in V(T) \), and let \( c \) denote the root vertex of \( T \). Then, consider each vertex at distance \( h - 1 \) from \( c \). Each has a weight of 1, and has two symmetric branches (just leaves). Performing a concentration then merges each leaf pair, doubling the cost of the vertices in each branch. Then one can move to the vertices at distance \( h - 2 \) from \( c \), each of which has weight 1, and two symmetric branches which are paths. Again, concentrating these paths doubles the weights of the merged vertices. Iterating this concentration process towards \( c \) thus results in a path of length \( h \), with a weight sequence \( 2^0, 2^1, 2^2, \ldots, 2^h \).

**Theorem 2.12.** Let \((T,w)\) be an integer weighted tree, and let \((T',w')\) be a concentration of \( T \). Then,

\[
  \text{th}_+(T) = \text{th}_+(T').
\]

**Proof.** We consider a single concentration \( T' \) of \( T \) at \( v \) with branches \( T_1, \ldots, T_m \) and symmetric maps \( \sigma_i, i = 2, \ldots, m \). Note that \( V(T') = V(T) \setminus (V(T_2) \cup \cdots \cup V(T_n)) \subset V(T) \). By Theorem 2.7, there is an optimal set \( B \) for \( T \) that is symmetric across \( T_1, \ldots, T_m \). Let \( B' \) be the subset of \( V(T') \) corresponding to \( B \). Clearly, \( w(B) = w'(B') \) and \( \text{pt}_+(T; B) = \text{pt}_+(T'; B') \), so \( \text{th}_+(T) = \text{th}_+(T') \). □

The concentration approach can simplify proofs and computations of the PSD-throttling number, especially those with a high degree of symmetry.

### 3 Spiders

In [4], Breen et al. give an algorithm that constructs, for any tree \( T \), an initial coloring set \( B \subseteq V(T) \) such that \( \text{th}_+(T; B) \leq 2\sqrt{n} \). Since [7] noted that all paths have a throttling
number approximately $\sqrt{2}\sqrt{n}$, the authors of [4] posed an interesting question: What is the smallest coefficient $\mu$ such that for all trees $T$, asymptotically $th_+(T) \lesssim \mu\sqrt{n}$? Or, symmetrically, which trees have the highest throttling number across all trees on $n$ vertices?

It was originally thought that balanced spiders could provide a family of examples for which $th_+(T) \approx \mu\sqrt{n}$ with $\mu > \sqrt{2}$. However, we show in this section that this is not possible, after determining the exact value of the throttling number of a balanced spider.

We denote the (unweighted) balanced spider with $\alpha$ legs of order $\beta$ by $T_{\alpha, \beta}$; thus $T_{\alpha, \beta}$ has $\alpha\beta + 1$ vertices. Using prior notation, $T_{\alpha, \beta} = S(\beta, \ldots, \beta)$, with $\alpha$ copies of $\beta$. Note that $T_{\alpha, \beta}$ has $\alpha$ symmetric branches at the center vertex $c$, all of which are paths of order $\beta$. Thus, $T_{\alpha, \beta}$ can be concentrated to a weighted path of order $\beta + 1$, wherein one end vertex (which inherits the label $c$) has weight one, and all other vertices have weight $\alpha$.

**Observation 3.1.** If $s$ vertices in each leg of $T_{\alpha, \beta}$ (i.e. $s$ non-$c$ vertices from the full concentration) are optimally chosen and $c$ is not chosen, the propagation time is $\lceil \frac{\beta + 1 - s}{2s} \rceil$. If $c$ is chosen, the propagation time is $\lceil \frac{\beta - s}{2s + 1} \rceil$.

**Lemma 3.2.** Every (unweighted) balanced spider with at least three legs has an optimal initial set containing the center vertex.

**Proof.** For $\alpha \geq 3$, $\beta, s \geq 1$, define

$$g(\alpha, \beta, s) = \alpha s + \frac{\beta + 1 - s}{2s},$$

which corresponds to the PSD-throttling number when $s$ vertices from each leg of $T_{\alpha, \beta}$ are chosen and $c$ is not. Similarly, for $\alpha \geq 3$, $\beta \geq 1$, $s \geq 0$, define

$$h(\alpha, \beta, s) = 1 + \alpha s + \frac{\beta - s}{2s + 1},$$

corresponding to the PSD-throttling number when $c$ is chosen in addition to the $s$ vertices chosen from each leg. It suffices to show that for every triple $(\alpha, \beta, s)$ with $\alpha \geq 3$, $\beta, s \geq 1$,

$$h(\alpha, \beta, s) \leq g(\alpha, \beta, s) \quad \text{or} \quad h(\alpha, \beta, s - 1) \leq g(\alpha, \beta, s).$$

Observe that for fixed $\alpha$ and $s$, both $h$ and $g$ are linear functions in $\beta$. The slopes are

$$\left. \frac{dg(\alpha, \beta, s)}{ds} \right|_{s} = \frac{1}{2s},$$

$$\left. \frac{dh(\alpha, \beta, s)}{ds} \right|_{s} = \frac{1}{2s + 1} < \frac{1}{2s},$$

$$\left. \frac{dh(\alpha, \beta, s)}{ds} \right|_{s-1} = \frac{1}{2s - 1} > \frac{1}{2s}$$

and the intercepts are

$$g(\alpha, 0, s) = \alpha s + \frac{1 - s}{2s} = \alpha s - \frac{1}{2} + \frac{1}{2s},$$

$$h(\alpha, 0, s) = 1 + \alpha s - \frac{s}{2s + 1} = 1 + \alpha s - \frac{1}{2} + \frac{1}{4s + 2} > \alpha s - \frac{1}{2} + \frac{1}{2s},$$

$$h(\alpha, 0, s - 1) = 1 - \alpha + \alpha s - \frac{s - 1}{2s - 1} = 1 - \alpha + \alpha s - \frac{1}{2} + \frac{1}{4s - 2} < \alpha s - \frac{1}{2} + \frac{1}{2s}.$$
Fix \( \alpha \) and \( s \). Define \( b_0 \) to be the value of \( \beta \) for which \( h(\alpha, \beta, s) = g(\alpha, \beta, s) \), so \( b_0 = -1 + s + 4s^2 \). Then, \( h(\alpha, \beta, s) \leq g(\alpha, \beta, s) \) for \( \beta \geq b_0 \). Once we show that \( h(\alpha, b_0, s - 1) \leq g(\alpha, b_0, s) \), it follows that \( h(\alpha, \beta, s - 1) \leq g(\alpha, \beta, s) \) for \( \beta \leq b_0 \), completing the proof.

\[
g(\alpha, b_0, s) - h(\alpha, b_0, s - 1) = \frac{\alpha(2s - 1) - 4s + 1}{2s - 1}
\]

Since \( 2s - 1 \geq 1 \) it suffices to show that \( 1 + \alpha(2s - 1) - 4s \geq 0 \). Since \( \alpha \geq 3 \) and \( s \geq 1 \),

\[
1 + \alpha(2s - 1) - 4s \geq 1 + 3(2s - 1) - 4s = 1 + 6s - 3 - 4s = 2s - 2 \geq 0.
\]

**Theorem 3.3.** For the balanced spider \( T = T_{\alpha, \beta} \) with \( \alpha \geq 3 \), \( \operatorname{th}_+(T) = 1 + \alpha \hat{s} + t \) where

\[
\hat{s} = \left\lfloor \frac{2\beta + \alpha + 1}{4\alpha} \right\rfloor \quad \text{and} \quad t = \left\lceil \frac{\beta - \hat{s}}{2s + 1} - \frac{1}{2} \right\rceil.
\]

**Proof.** By Lemma 3.2 there is an optimal initial set \( B_0 \) containing the center vertex. Let \( P_T \) be the concentration of \( T \) at the center. Then the value of \( t \) follows from Observation 3.1 and thus \( \operatorname{th}_+(P_T; B_0) = 1 + \alpha s + t \) where \( s \) is the number of vertices of weight \( \alpha \) (that originally came from the legs).

Now, consider the family of real-valued functions

\[
h(\alpha, \beta, s) = 1 + \alpha s + \frac{\beta - s}{2s + 1},
\]

and note that for fixed \( \alpha \geq 3 \) and \( s \) these are linear functions of \( \beta \).

Observe that the sequence of intercepts \( \{h(\alpha, 0, s)\}_{s \in \mathbb{N}} \) is strictly increasing in \( s \), and that the sequence of slopes \( \{h'(\alpha, \beta, s)\}_{s \in \mathbb{N}} \) is strictly decreasing, but is always positive. We consider the sequence \( \{\beta_s\}_{s=0}^\infty \), where \( \beta_s \) is the value for which \( h(\alpha, \beta_s, s - 1) = h(\alpha, \beta_s, s) \), i.e. the point at which increasing to \( s \) vertices will not raise and may lower the PSD-throttling number, which is also the values of \( b \in \mathbb{R} \) at which the linear functions \( h \) intersect.

\[
\begin{align*}
h(\alpha, \beta_s, s - 1) &= h(\alpha, \beta_s, s) \\
1 + \alpha(s - 1) + \frac{\beta_s - (s - 1)}{2(s - 1) + 1} &= 1 + \alpha s + \frac{\beta_s - s}{2s + 1} \\
\frac{\beta_s - s + 1}{2s - 1} &= \alpha + \frac{\beta_s - s}{2s + 1} \\
\frac{\beta_s - s + 1}{2s - 1} &= 2\alpha s + \alpha + \beta_s - s \\
2\beta_s s - 2s^2 + 2s + \beta_s - s + 1 &= 4\alpha s^2 + 2\alpha s + 2\beta_s s - 2s^2 - 2\alpha s - \alpha - \beta_s + s \\
2\beta_s &= 4\alpha s^2 - \alpha - 1 \\
\beta_s &= 2\alpha s^2 - \frac{\alpha + 1}{2}.
\end{align*}
\]

Solving (1) for \( s \) and taking the floor then gives the optimal choice for \( \hat{s} \), given any \( \beta \).

\[
\hat{s} = \left\lfloor \frac{2\beta + \alpha + 1}{4\alpha} \right\rfloor.
\]
Here, we define a continuous variant of the throttling number for balanced spiders, which will be used in the next section. Let

$$t_S(\alpha, \beta) = 1 + \alpha \hat{s} + \hat{t}$$

where \(\hat{s}\) is as defined in Theorem 3.3 and \(\hat{t} = \frac{\beta+1}{2s+1} - \frac{1}{2}\). Note that \(\hat{t}\) is obtained by removing the ceiling from \(t\) in Theorem 3.3.

**Corollary 3.4.** The functions \(t_S(\alpha, \beta)\) are continuous in \(\beta\), and \(\text{th}_+(T_{\alpha, \beta}) = \lceil t_S(\alpha, \beta) \rceil\).

**Proof.** Note that the second statement follows immediately from the fact that \(\alpha\) and \(\hat{s}\) are integers. For the first, note that for all \(\beta\), \(t_S(\alpha, \beta) = h(\alpha, \beta, \hat{s})\), and so \(t_S(\alpha, \beta) = \min_{s \in \mathbb{N}} h(\alpha, \beta, s)\). Finally, recall that when the “optimal” value of \(s\) changes to \(s+1\), it is at the \(\beta\) for which \(h(\alpha, \beta, s) = h(\alpha, \beta, s+1)\), and thus \(t_S(\alpha, \beta)\) must be continuous. \(\square\)

## 4 Super-spiders

In [1], it is shown that the spider \(S(4, 3, 2)\) has a higher throttling number than the path of the same order. Specifically, \(\text{th}_+(S(4, 3, 2)) = 5 = 1 + \text{th}_+(P_{10})\). A computer search of small spiders shows that this is the smallest spider whose PSD-throttling number exceeds that of the path of the same order. This search, which is described in Appendix 1, also produced several thousand spiders that have throttling numbers one more than that of the path of the same order. For example, the PSD-throttling numbers of next few smallest such spiders \(\text{th}_+(S(5, 4, 3, 2)) = \text{th}_+(S(5, 4, 4, 1)) = \text{th}_+(S(6, 4, 4)) = \text{th}_+(S(7, 4, 3)) = 6 > 5 = \text{th}_+(P_{15})\) and \(\text{th}_+(S(6, 5, 4, 4)) = \text{th}_+(S(7, 5, 4, 3)) = 7 > 6 = \text{th}_+(P_{20})\).

In light of this, we define a super-spider as a spider \(S\) for which \(\text{th}_+(S) > \text{th}_+(P_{|V(S)|})\).

It is worth noting that, while there are currently no examples of trees in the literature that exceed the PSD-throttling numbers of their paths by more than one, there are thousands of super-spiders of order at most 75 that exceed the corresponding path’s PSD-throttling number by exactly one. We give an infinite family of super-spiders (the triangle spiders) below.

**Proposition 4.1.** Let \(t \in \mathbb{Z}^+\) with \(t \geq 4\). Then, the spider \(S = S(l_t, l_{t-1}, \ldots, l_2) = S(t, t-1, \ldots, 2)\) on \(n = \frac{\binom{t+1}{2}}{2}\) vertices has PSD-throttling number \(\text{th}_+(S) = t + 1 = 1 + \text{th}_+(P_n)\).

**Proof.** Let \(p = t - i\) be the proposed propagation time of an optimally chosen set \(B\) of vertices. First, observe that all legs \(l_k\) with \(k > p\) must contain a blue vertex if propagation is going to conclude on time. Next, note that leg \(l_p\) cannot be fully forced by any vertex in \(l_{p+1}\), as the distance from \(l_{p+1}\)’s most central vertex and \(l_p\)’s least central vertex is \(p + 1\). Thus, we must either choose a vertex in \(l_p\), or choose the center vertex. As the center vertex will guarantee the forcing of all \(l_k\) with \(k \leq p\), this is clearly an optimal choice. Thus we have \(\text{th}_+(S; B) \geq (i + 1) + p = t + 1\). So \(\text{th}_+(S) \geq t + 1\). Note that choosing \(p = t\) by just coloring the center vertex gives \(\text{th}_+(S) \leq t + 1\). The lemma above then gives \(\text{th}_+(S) = \text{th}_+(P_n) + 1\). \(\square\)

**Remark 4.2.** As an immediate consequence of Proposition 4.1 there is no constant bound on the number of legs a super-spider can have.
Finally, we show that there are no balanced super-spiders. To do so, we will examine the continuous analogues of the throttling number functions for balanced spiders and paths. $t_S(\alpha, \beta)$ is already defined before Corollary 3.4. For paths, recall from Lemma 2.1 that $\text{th}_+(P_{\alpha, \beta+1}) = \left\lceil \sqrt{2(\alpha\beta + 1) + \frac{1}{4} - \frac{1}{2}} \right\rceil$. Thus, we define

$$t_P(\alpha, \beta) := \sqrt{2(\alpha\beta + 1) + \frac{1}{4} - \frac{1}{2}}.$$

Since $t_P(\alpha, \beta) \leq \text{th}_+(P_{\alpha, \beta+1})$, showing that $t_S(\alpha, \beta) \leq t_P(\alpha, \beta)$ is sufficient to show that $\text{th}_+(T_{\alpha, \beta}) \leq \text{th}_+(P_{\alpha, \beta+1})$.

**Lemma 4.3.** Let $\alpha \geq 3$, and suppose $t_S(\alpha, \beta) \leq t_P(\alpha, \beta)$ for some $\beta \geq \beta_1$, where $\beta_1$ is defined in (7). Then $t_S(\alpha, \beta') \leq t_P(\alpha, \beta')$ for all $\beta' \geq \beta$.

**Proof.** Observe that $t_S$ is locally linear in $\beta$ (except at each $\beta_s$), whereas $t_P$ is concave down in $\beta$. Thus, we need only show that the inequality holds for the values $\beta_s$, where each change in slope occurs. We prove this by examining the average rates of change of the respective functions over the intervals $[\beta_s, \beta_{s+1}]$. As $t_S(\alpha, \beta)$ is linear on each $[\beta_s, \beta_{s+1}]$, we know it has slope $\frac{1}{2s+1}$. Computing the average slope of $t_P$ on each interval is a bit trickier.

$$\frac{t_P(\alpha, \beta_{s+1}) - t_P(\alpha, \beta_s)}{\beta_{s+1} - \beta_s} = \frac{\left(\sqrt{2(\alpha\beta_{s+1} + 1) + \frac{1}{4} - \frac{1}{2}}\right) - \left(\sqrt{2(\alpha\beta_s + 1) + \frac{1}{4} - \frac{1}{2}}\right)}{\beta_{s+1} - \beta_s}$$

$$= \frac{\sqrt{2\alpha\beta_{s+1} + \frac{9}{4}} - \sqrt{2\alpha\beta_s + \frac{9}{4}}}{\beta_{s+1} - \beta_s}$$

$$= \frac{4\alpha^2 s^2 + 8\alpha^2 s + 3\alpha^2 + -\alpha + \frac{9}{4} - \sqrt{4\alpha^2 s^2 - \alpha^2 - \alpha + \frac{9}{4}}}{4\alpha s + 2\alpha}$$

$$= \sqrt{16\alpha^2 s^2 + 32\alpha^2 s + 12\alpha^2 - 4\alpha + 9 - \sqrt{16\alpha^2 s^2 - 4\alpha^2 - 4\alpha + 9}}$$

$$= \frac{4\alpha(2s + 1)}{4\alpha(2s + 1)}$$

With a little algebraic manipulation, we see that the average rate of change for $t_S(\alpha, \beta)$ is less than the average rate of change of $t_P(\alpha, \beta)$ when the inequality

$$4\alpha \leq \sqrt{16\alpha^2 s^2 + 32\alpha^2 s + 12\alpha^2 - 4\alpha + 9 - \sqrt{16\alpha^2 s^2 - 4\alpha^2 - 4\alpha + 9}}$$

holds. To establish this condition, $r_{s+1}$ and $r_s$ will be used as shorthand for the two square roots, respectively. Thus we need to show that

$$4\alpha + r_s \leq r_{s+1}^2$$
$$16\alpha^2 + 8\alpha r_s \leq r_{s+1}^2 - r_s^2$$
$$16\alpha^2 + 8\alpha r_s \leq 32\alpha^2 s + 16\alpha^2$$
$$8\alpha r_s \leq 32\alpha^2 s$$
$$r_s^2 \leq 16\alpha^2 s^2$$
$$16\alpha^2 - 4\alpha^2 - 4\alpha + 9 \leq 16\alpha^2 s^2$$
$$9 \leq 4\alpha^2 + 4\alpha$$
Since $\alpha \geq 3$ by hypothesis, the last inequality holds for all balanced spiders. \hfill \Box

One should note that this inequality is true for all $\beta$. However, $\beta_0$ is actually negative, and thus has no context within the problem. Hence the initial restriction $\beta \geq \beta_1$.

**Lemma 4.4.** For all $\alpha \geq 3$, $t_S(\alpha, \beta_2) \leq t_P(\alpha, \beta_2)$.

*Proof.* Note that $\beta_2 = 8\alpha - \frac{\alpha + 1}{2}$. Then we need only show that the difference
\[
t_P(\alpha, \beta_2) - t_S(\alpha, \beta_2) = \sqrt{2\alpha \beta_2 + \frac{9}{4} - \left(1 + 2\alpha + \frac{\beta_2 + \frac{1}{2}}{5}\right)} \nonumber
\]
\[
= \sqrt{2\alpha \left(8\alpha - \frac{\alpha + 1}{2}\right) + \frac{9}{4} - \left(1 + 10\alpha + (8\alpha - \frac{\alpha + 1}{2}) + \frac{1}{2}\right)} \nonumber
\]
\[
= \sqrt{15\alpha^2 - \alpha + \frac{9}{4} - \frac{7}{2}\alpha - 1}\nonumber
\]

is non-negative, which it is for all $\alpha \geq 1$. \hfill \Box

**Lemma 4.5.** For all $\alpha \geq 5$, $t_S(\alpha, \beta_1) \leq t_P(\alpha, \beta_1)$.

*Proof.* Note that $\beta_1 = \frac{3\alpha - 1}{2}$. Then, we need only show that the difference
\[
t_P(\alpha, \beta_1) - t_S(\alpha, \beta_1) = \sqrt{2\alpha \beta_1 + \frac{9}{4} - \left(1 + \alpha + \frac{\beta_1 + \frac{1}{2}}{3}\right)} \nonumber
\]
\[
= \sqrt{2\alpha \left(\frac{3\alpha - 1}{2}\right) + \frac{9}{4} - \left(1 + \alpha + \frac{(3\alpha - 1) + \frac{1}{2}}{3}\right)} \nonumber
\]
\[
= \sqrt{3\alpha^2 - \alpha + \frac{9}{4} - \frac{3}{2}\alpha - 1}\nonumber
\]

is non-negative, which it is for all $\alpha \geq 5$. \hfill \Box

**Theorem 4.6.** There are no balanced super-spiders.

*Proof.* Assume first that $\alpha \geq 5$. By Lemma 4.5, $t_S(\alpha, \beta) \leq t_P(\alpha, \beta)$ for all $\beta \geq \beta_1$. Further, since $t_P(\alpha, 0) = t_S(\alpha, 0) = 1$ for all $\alpha \geq 3$, $t_P$ is concave down in $\beta$, and $t_S$ is linear in $\beta$ over the interval $[0, \beta_1]$, we have that $t_S(\alpha, \beta) \leq t_P(\alpha, \beta)$ for all $\beta \geq 0$. Thus, there are no balanced super-spiders on five or more legs, and we need only demonstrate that there are no balanced super-spiders on three or four legs.

Now, suppose $\alpha \in \{3, 4\}$. Then $t_S(\alpha, \beta_1) \geq t_P(\alpha, \beta_1)$, and $t_S(\alpha, \beta_2) \leq t_P(\alpha, \beta_2)$. Thus, for $\alpha \in \{3, 4\}$, there is a point $b_1$ in the interval $[0, \beta_1]$ where $t_S$ becomes larger than $t_P$, and there is another point $b_2$ in the interval $[\beta_1, \beta_2]$ where $t_P$ becomes larger than $t_S$ (See Figure 1). Since Lemma 4.4 proves there are no balanced super spiders with $\beta \geq \beta_2$, any balanced super-spider must have $\alpha \in \{3, 4\}$, and $\beta \in (b_1, b_2)$.

Suppose $\alpha = 3$. Then $\beta_1 = 4$ and $t_S(3, 3) = t_P(3, 3) = 4$, so $b_1 = 3$. On the other end, $\beta_2 = 22$ and $\frac{47}{3} = t_S(3, 6) < t_P(3, 6) = 5.685$, so $b_2 < 6$. Thus, the only integer candidates for $\beta$ are 4 and 5.

\begin{center}

10

\end{center}
Suppose $\alpha = 4$. Then $\beta_1 = 5.5$ and $t_S(4, 5) = t_P(4, 5) = 6$, so $b_1 = 5$. On the other end, $b_2 = 29.5$ and $7 = t_S(4, 7) < t_P(4, 7) \approx 7.132$, so $b_2 < 7$. Thus, the only integer candidate for $\beta$ is 6.

To summarize, the balanced spiders $T_{3, 4}$, $T_{3, 5}$, and $T_{4, 6}$ are the only candidate balanced super-spiders. However, it is easy to verify that each these has a throttling number equal to that of its correlated path, and thus is not a super-spider.

\[\boxed{}\]

### A Algorithms, Computations, and Data

The overall process is as follows. Given $n$, we first compute the throttling number $t$ of $P_n$. To iterate through all spiders, we observe that the spiders have a one-to-one correspondence with the partitions of the integer $n - 1$ that have at least three parts. Next, we iterate through all possible values for $|B| = s \leq \text{th}_+(P_n)$. Once the spider and proposed starting size are chosen, the recursive sage function below determines if the spider can be fully forced within $p = t - s$ time steps.

```python
def spidthrot(partlist, cbool, s, p):
    # partlist = Partition representing the spider
    # cbool = Boolean, true if center is already colored
    # s = remaining number of choices for starting set
    # p = proposed propagation time
    plist=list(partlist)  # Convert Partition to list
    plist.sort()  # Sort legs
    if (not plist) and (cbool or s>0):
        return true  # Legs empty and can finish
    elif(not plist):  # Legs empty but can’t finish
        return false
    elif plist and s == 0:
        return false  # Legs not empty, can’t choose more; s>=0 for rest
```
return false  
eelif plist[-1] > 2*p +1:    #Longest needs more than 1, can choose more  
    l = plist.pop()  
    l = l - (2*p+1)  
    plist.append(l)  
    return spidthrot(plist, cbool, s-1, p)  
        #Recursive, cover longest, costs 1  
    elif plist[-1] == 2*p +1:    #Longest needs full time, w/o center  
        plist.pop()  
        return spidthrot(plist, cbool, s-1, p)  
    elif plist[-1] == 2*p:       #Covering longest includes center  
        plist.pop()  
        return spidthrot(plist, true, s-1, p)  
    elif plist[-1] > p:         #Covering longest cleans up short legs  
        l = plist.pop()  
        while plist and plist[0]<= 2*p - l:  
            plist.pop(0)  
        return spidthrot(plist, true, s-1, p)  
    else:                       #Have one to spare, and choosing center covers all.  
        return true

If the spider has throttling number at most \( t \), we move on to the next spider. If not, we run the recursion for \( t \leq t' \leq t + k \) (usually \( k = 1 \)), to determine the spider's throttling number, with a special message given if the choice of \( k \) is too small.

Below is a table containing most of what is known about super-spiders. For all \( 1 \leq n \leq 74 \), \( n \) is omitted from the table if there are no super-spiders of order \( n \). The smallest super-spiders for each value of \( t \) are given as tuples.

Note that as the throttling number increases, super-spiders appear sooner (relative to \( n \)). This suggests a potential way to construct a tree with a throttling number higher than path plus one. For example, \( n_{12} = 78 \), but we have a super-spider (with \( \text{th}_\star(S) = 13 \)) on 69 vertices, \( S(15,12,10,9,8,7,7) \). Thus there are 9 vertices one might cleverly place to get a throttling number of 14.
| $n$ | $\theta_+(P_n)$ | # of S-Spiders | Examples |
|-----|-----------------|----------------|---------|
| 10 | 4               | 1              | $(4,3,2)$ |
| 15 | 5               | 4              | $(5,4,3,2), (5,4,4,1), (6,4,4), (7,4,3)$ |
| 20 | 6               | 2              | $(6,5,4,4), (7,5,4,3)$ |
| 21 | 6               | 17             | -Many- |
| 26 | 7               | 3              | $(7,6,5,4,3), (9,6,5,5), (10,6,5,4)$ |
| 27 | 7               | 17             | -Many- |
| 28 | 7               | 62             | -Many- |
| 33 | 8               | 2              | $(9,7,6,5,5), (10,7,6,5,4)$ |
| 34 | 8               | 19             | -Many- |
| 35 | 8               | 77             | -Many- |
| 36 | 8               | 221            | -Many- |
| 41 | 9               | 5              | $(9,8,7,6,5,5), (10,8,7,6,5,4), (12,9,7,6,6), (13,8,7,6,6), (13,9,7,6,5)$ |
| 42 | 9               | 31             | -Many- |
| 43 | 9               | 118            | -Many- |
| 44 | 9               | 330            | -Many- |
| 45 | 9               | 783            | -Many- |
| 49 | 10              | 2              | $(12,9,8,7,6,6), (13,9,8,7,6,5)$ |
| 50 | 10              | 14             | -Many- |
| 51 | 10              | 61             | -Many- |
| 52 | 10              | 210            | -Many- |
| 53 | 10              | 595            | -Many- |
| 54 | 10              | 1399           | -Many- |
| 55 | 10              | 2920           | -Many- |
| 59 | 11              | 4              | $(12,10,9,8,7,6,6), (13,10,9,8,7,6,5), (15,12,9,8,7,7), (16,12,9,8,7,6)$ |
| 60 | 11              | 32             | -Many- |
| 61 | 11              | 131            | -Many- |
| 62 | 11              | 441            | -Many- |
| 63 | 11              | 1201           | -Many- |
| 64 | 11              | 2803           | -Many- |
| 65 | 11              | 5792           | -Many- |
| 66 | 11              | 10986          | -Many- |
| 69 | 12              | 3              | $(15,12,10,9,8,7,7), (16,11,10,9,8,7,7), (16,12,10,9,8,7,6)$ |
| 70 | 12              | 22             | -Many- |
| 71 | 12              | 104            | -Many- |
| 72 | 12              | 380            | -Many- |
| 73 | 12              | 1123           | -Many- |
| 74 | 12              | 2823           | -Many- |
References

[1] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, A. Wangsness). Zero forcing sets and the minimum rank of graphs. *Linear Algebra Appl.*, 428 (2008), 1628–1648.

[2] American Institute of Mathematic workshop Zero Forcing and its Applications. [http://aimath.org/pastworkshops/zeroforcing.html](http://aimath.org/pastworkshops/zeroforcing.html)

[3] F. Barioli, W. Barrett, S. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra Appl.*, 433 (2010), 401–411.

[4] J. Breen, B. Brimkov, J. Carlson, L. Hogben, K.E. Perry, C. Reinhart. Throttling for the game of cops and robbers on graphs. *Discrete Math.*, 341 (2018) 2418–2430.

[5] D. Burgarth and V. Giovannetti. Full control by locally induced relaxation. *Phys. Rev. Lett.* PRL 99 (2007), 100501.

[6] S. Butler, M. Young. Throttling zero forcing propagation speed on graphs. *Australas. J. Combin.*, 57 (2013), 65–71.

[7] J. Carlson, L. Hogben, J. Kritschgau, K. Lorenzen, M.S. Ross, S. Selken, V. Valle-Martinez. Throttling positive semidefinite zero forcing propagation time on graphs. *Discrete Appl. Math.*, in press, [https://doi.org/10.1016/j.dam.2018.06.017](https://doi.org/10.1016/j.dam.2018.06.017).

[8] R. Diestel. *Graph Theory*, 5th edition. Springer, Berlin, 2017.

[9] L. Hogben, M. Huynh, N. Kingsley, S. Meyer, S. Walker, M. Young. Propagation time for zero forcing on a graph, *Discrete Appl. Math.*, 160 (2012), 1994–2005.

[10] S. Severini. Nondiscriminatory propagation on trees. *J. Physics A*, 41 (2008), 482–002 (Fast Track Communication).

[11] N. Warnberg. Positive semidefinite propagation time. *Discrete Appl. Math.*, 198 (2016) 274–290.

[12] Boting Yang. Fast-mixed searching and related problems on graphs. *Theoret. Comput. Sci.* 507 (2013), 100–113.