NON-ARCHIMEDEAN HYPERBOLICITY OF THE MODULI SPACE OF CURVES

RUIRAN SUN

AbSTRACT. Let $K$ be a complete algebraically closed non-archimedean valued field of characteristic zero, and let $X$ be a finite type scheme over $K$. We say $X$ is $K$-analytically Borel hyperbolic if, for every finite type reduced scheme $S$ over $K$, every rigid analytic morphism from the rigid analytification $S^\text{an}$ of $S$ to the rigid analytification $X^\text{an}$ of $X$ is algebraic. Using the Viehweg-Zuo construction and the $K$-analytic big Picard theorem of Cherry-Ru, we show that, for $N \geq 3$ and $g \geq 2$, the fine moduli space $\mathcal{M}^{[N]}_{g,K}$ over $K$ of genus $g$ curves with level $N$-structure is $K$-analytically Borel hyperbolic.

1. Introduction

Motivated by conjectures of Green-Griffiths-Lang and higher dimensional generalizations of the Shafarevich problem, there has recently been much work on different notions of hyperbolicity [Lan86, Jav20] and the verification of them on the moduli spaces of smooth projective varieties [KL11, CP15, JSZ]. In this paper, we study the non-archimedean analogue of Borel hyperbolicity introduced in [JK20], and verify it for $\mathcal{M}^{[N]}_{g,K}$, the moduli space of genus $g$ curves with level $N$-structure over a non-archimedean field $K$, where $g > 1$ and $N \geq 3$.

Inspired by Cherry’s work [Ch94], in [JV18] Javanpeykar and Vezzani introduced the non-archimedean analogue of Brody hyperbolicity. Let $X$ be a finite type scheme over $K$, where $K$ is a complete algebraically closed non-archimedean valued field of characteristic zero. Then $X$ is $K$-analytically Brody hyperbolic if, for every finite type connected group scheme $G$ over $K$, every morphism $G^\text{an} \to X^\text{an}$ of rigid analytic varieties is constant, where $G^\text{an}$ and $X^\text{an}$ denote the rigid analytification of $G$ and $X$ respectively. In the aforementioned paper they also proved that $\mathcal{A}^{[N]}_{g,K}$, the fine moduli space of $g$-dimensional principally polarized abelian schemes with level $N$-structure ($N \geq 3$) over $K$, is $K$-analytically Brody hyperbolic if the residue field of $K$ has characteristic $0$. As a direct corollary of the Torelli theorem, the same statement holds for $\mathcal{M}^{[N]}_{g,K}$.

In this paper we extend the latter result by proving the hyperbolicity of $\mathcal{M}^{[N]}_{g,K}$ for more general $K$ (e.g., $K = \mathbb{C}_p$). Theorem A. The variety $\mathcal{M}^{[N]}_{g,K}$ is $K$-analytically Brody hyperbolic for $N \geq 3$ and $g \geq 2$.

There are notable differences between the complex case and non-archimedean case. For example, Berkovich [Ber90, Theorem 4.5.1] proved that Picard’s Little Theorem holds for $\mathbb{G}_{m,K}$, i.e., there is no non-constant rigid analytic morphism from $\mathbb{A}^{1\text{an}}_K$ to $\mathbb{G}_{m,K}^\text{an}$, which is contrary to the complex analytic case. In fact this is the reason that Javanpeykar-Vezzani do not use the naive non-archimedean counterpart of the notion of complex Brody hyperbolicity in their aforementioned paper, since one does not want to regard $\mathbb{G}_{m,K}$ as a “hyperbolic” variety.

It is interesting to look for a suitable notion of non-archimedean hyperbolicity. As we have seen above, $K$-analytically Brody hyperbolic varieties defined in [JV18] are non-archimedean analogue of “groupless varieties” (cf. [JX20, JK20]) and should therefore correspond to hyperbolic varieties by Lang’s Conjecture.

2010 Mathematics Subject Classification. 32Q45, 32A22, 30G06.

Key words and phrases. rigid analytic varieties, hyperbolicity, Higgs bundles, moduli of polarized varieties.
Let $K$ be a complete algebraically closed non-archimedean valued field of characteristic zero, and let $X$ be a finite type scheme over $K$. We say $X$ is $K$-analytically Borel hyperbolic if, for every finite type reduced scheme $S$ over $K$, any morphism $S^{an} \to X^{an}$ is algebraic.

Our main result is the following:

**Theorem B.** If $g \geq 2$, $N \geq 3$, and $K$ is a complete algebraically closed non-archimedean valued field of characteristic zero, then $\mathcal{M}_{g,K}^{[N]}$ is $K$-analytically Borel hyperbolic.

Theorem B plus the characterization of $K$-analytic Brody hyperbolicity in [JV18, Theorem 2.18] imply Theorem A. Details of the proofs are given in Section 5.

Our proof of Theorem B uses the Viehweg-Zuo construction associated to the universal family over $\mathcal{M}_{g,K}^{[N]}$ and the $K$-analytic big Picard theorem of Cherry-Ru (Theorem 4.1). Write the universal family as $V_K \to U_K$ for simplicity. First we show a non-archimedean analogue of [JK20, Theorem 1.5], which helps us to reduce the general case to the case of rigid analytic morphisms from curves. This is done in Section 3. To show the algebraicity of a rigid analytic morphism $f$ from a curve $C_K$ to the base space $U_K$ of the family, we shall use the $K$-analytic big Picard theorem of Cherry-Ru to extend $f$ as a rigid analytic morphism between the compactifications of $C_K$ and $U_K$. Then the algebraicity follows from the rigid GAGA theorem.

As we mentioned, the $K$-analytic big Picard theorem of Cherry-Ru [CR04] plays a crucial role in this paper. Cherry-Ru’s theorem requires the existence of certain symmetric differentials of the base space $U_K$ which do not vanish along the rigid analytic morphism $f$. We shall use the Viehweg-Zuo construction associated to the family $V_K \to U_K$ to produce such a symmetric differential. Basics of the Viehweg-Zuo construction are reviewed in Section 2. However, the Viehweg-Zuo construction is designed for families over complex numbers. So in Section 5, we shall descend the original family to a family over a finitely generated subfield of $K$, base change it to get a complex family, apply the usual Viehweg-Zuo construction to this complex family and produce a symmetric differential on the complex base space, and use the descent argument again to obtain a symmetric differential on $U_K$. The final step is to check that all the requirements in the theorem of Cherry-Ru are satisfied, which concludes the proof.

We should mention that before the paper of Javanpeykar-Vezzani there are works on non-archimedean hyperbolic geometry from the point of view of Nevanlinna theory, for instance, variant non-archimedean analogues of Picard theorems (cf. [Ch94, Ru01, CW02, CR04]).

For families of abelian varieties, one can also use the Viehweg-Zuo construction to produce symmetric differentials on the base spaces, but it requires more work to check that they do not vanish along given rigid analytic morphisms. It is natural to ask:

**Question 1.2.** Is $A_{g,K}^{[N]}$ $K$-analytically Borel hyperbolic for $N \geq 3$?

For the complex case, $A_{g,\mathbb{C}}^{[N]}$ is known to be Borel hyperbolic for $N \geq 3$ by the work of Borel [Bor72] (in fact the name “Borel hyperbolicity” comes from the algebraicity theorem of Borel on the arithmetic quotients of bounded symmetric domains).

**Acknowledgment.** This paper owes a tremendous debt to Ariyan Javanpeykar: for giving me the topic on the non-archimedean hyperbolicity of moduli spaces, for invaluable comments and suggestions he provided and for helping me to improve this paper. I am grateful to Alberto Vezzani for very helpful suggestions on Section 3. I would like to thank Professor Kang Zuo for explaining his celebrated work with Viehweg to me, and for his constant supports and encouragements.
2. Recollections about Viehweg-Zuo construction

We first consider families of smooth projective connected varieties over complex numbers. Let \( \mathcal{M}_h \) be the stack of smooth proper polarized varieties with semi-ample canonical divisor and Hilbert polynomial \( h \) over \( \mathbb{Q} \). Let \( U \) be a smooth quasi-projective variety over \( \mathbb{C} \). Let \( \varphi : U \to \mathcal{M}_h \otimes \mathbb{C} \) be a morphism of stacks which is generically finite onto its image. Note that \( \varphi \) is the classifying map of a smooth family \( V \to U \) of polarized varieties. We denote by \( n \) the fiber dimension, so that \( n = \deg h \).

The next step is to compactify the family. First we find the smooth compactifications \( U \subset Y \) and \( V \subset X \) with the simple normal crossing boundary divisors \( S := Y \setminus U \) and \( \Delta := X \setminus V \). Then \( \pi : V \to U \) extends to a projective morphism \( g : X \to Y \). We use the same notation \( g \) to denote the induced log morphism \( (X, \Delta) \to (Y, S) \). After desingularization process we can assume that \( g \) is a log smooth morphism over a big open subset containing \( U \) (a Zariski open subset is said to be big if its complement has codimension at least 2). After leaving out some codimension-2 subvarieties, we still use the notation \( g : (X, \Delta) \to (Y, S) \), which is a partial compactification of \( \pi : V \to U \).

Viehweg-Zuo construct a graded Higgs bundle \((F, \tau)\) which has a close relation with the deformation theory of the family (see [VZ03, §6] and [VZ02, §4] for details). Recall that \((F, \tau)\) has a bi-graded structure \((\bigoplus_{p+q=n} F^{p,q}, \bigoplus_{p+q=n} \tau^{p,q})\), where

\[
F^{p,q} := R^q g_* T_X^{q,q}(\log S)/\text{torsion},
\]

and the component of the Higgs map \( \tau^{p,q} \) is induced by the edge morphism of a long exact sequence of higher direct image sheaves

\[
\tau^{p,q} : R^q g_* T_X^{q,q}(\log S) \to \Omega_X^q(\log S) \otimes R^{q+1} g_* T_X^{q+1,q+1}(\log S).
\]

We extend \( F^{p,q} \) to the compactification by taking the reflexive hull. The Higgs maps \( \tau^{p,q} \) extend automatically over codimension-2 subvarieties. Hereafter we still use \( Y \) to denote the compactification of \( U \), and \((F, \tau)\) is the graded Higgs bundle over \( Y \).

One can iterate the Higgs map

\[
F^{n,0} \cong \mathcal{O}_Y \xrightarrow{\tau^{n,0}} \Omega_Y^1(\log S) \xrightarrow{\tau^{n-1,1} \circ \text{Id}} \Omega_Y^2(\log S) \otimes F^{n-2,2} \to \ldots
\]

and obtain \( \mathcal{O}_Y \to \Omega_Y^k(\log S)^\otimes k \otimes F^{n-k,k} \) for each \( k \)-th iteration. By the integrable condition of the Higgs map \( \tau \), we have the factorization \( \mathcal{O}_Y \to \text{Sym}^k \Omega_Y^1(\log S) \otimes F^{n-k,k} \), which induces the map

\[
\tau^k : \text{Sym}^k \Omega_Y(\log S) \to F^{n-k,k}.
\]

For \( k = 1 \) it becomes

\[
\tau^1 : T_Y(\log S) \to R^1 g_* T_X(\log S)
\]

which is exactly the (log) Kodaira-Spencer map associated to the family \( g \).

For the given family \( g : (X, \Delta) \to (Y, S) \) we can find a positive integer \( m \), which is called the Griffiths-Yukawa coupling length of the family \( g \), such that the \( m \)-th iteration \( \tau^m \) is a nonzero map and it factors through

\[
\tau^m : \text{Sym}^m \Omega_Y(\log S) \to N^{n-m,m}
\]

where \( N^{n-m,m} := \text{Ker}(\tau^{n-m,m}) \). Taking the image of the dual of \( \tau^m \), we can obtain a subsheaf \( \mathcal{P} \) of \( \text{Sym}^m \Omega_Y^1(\log S) \). Recall that Viehweg defined the bigness for torsion-free sheaves. See Definition 1.1 and Lemma 1.2 in [VZ02] for details. For this subsheaf \( \mathcal{P} \), Viehweg-Zuo proved the following important result:

**Theorem 2.1** (Viehweg-Zuo, [VZ02, Theorem 1.4]). Let \( V \to U \) be a smooth family of polarized varieties with semi-ample canonical divisor. Suppose that the family has maximal variation, i.e., the induced classifying map from \( U \) to the moduli space is generically finite onto its image. Then the subsheaf \( \mathcal{P} \) constructed above
is big in the sense of Viehweg. In particular, one can find an ample invertible sheaf $\mathcal{H}$, some positive integer $\eta$ and a morphism

$$\bigoplus \mathcal{H} \to \text{Sym}^\eta \mathcal{P}$$

which is surjective over some Zariski open subset of $U$.

The subsheaf $\mathcal{P}$ is commonly referred as the Viehweg-Zuo big subsheaf.

3. Testing $K$-analytic Borel hyperbolicity on maps from curves

In this section we shall prove a non-archimedean analogue of Theorem 1.5 in [JK20]. Hereafter we denote $X^{\text{an}}$ as the rigid analytification of a finite type $K$-scheme $X$ [Bo14, §5.4].

**Theorem 3.1** (Testing $K$-analytic Borel hyperbolicity on maps from curves). Let $K$ be an algebraically closed field of characteristic 0 which is complete with respect to some non-archimedean valuation. Let $X$ be a finite type separated scheme over $K$. Then the following are equivalent.

(i) $X$ is $K$-analytically Borel hyperbolic (Definition 1.1).

(ii) For every smooth connected algebraic curve $C$ over $K$, every rigid analytic morphism $C^{\text{an}} \to X^{\text{an}}$ is algebraic.

In the rest of this section we will prove the non-archimedean counterpart of the complex-analytic results in section 2.2 of [JK20]. After establishing those results about the rigid analytification, we shall give a proof of Theorem 3.1 following the line of reasoning in [JK20].

3.1. Some facts about the rigid analytification. Let $X$ be a finite type $K$-scheme and $X^{\text{an}}$ be its rigid analytification. We first study the relation between the ring of regular functions on $X$ and the ring of analytic functions on $X^{\text{an}}$.

**Proposition 3.2** (Analogue of Proposition 2.2 in [JK20]). If $X$ is a finite type integral scheme over $K$ of pure dimension, then the ring $\mathcal{O}(X)$ of global regular functions is integrally closed in the ring $\mathcal{H}(X^{\text{an}})$ of global analytic functions.

**Proof.** To check the integrality we can localize to the case where $X$ is affine. Write $X = \text{Spec} \, A$, so that $\mathcal{O}(X) = A$. Then $A \subset \mathcal{H}(X^{\text{an}})$ is a subring. Now let us consider an element $f \in \mathcal{H}(X^{\text{an}})$ which is integral over $A$. We set $B = A[f]$. The goal is to show that $B = A$.

Let $Y = \text{Spec} \, B$. So we have a finite morphism $\pi : Y \to X$ between $K$-schemes. Since $X$ is irreducible and $B = A[f]$, we know that $Y$ is also irreducible.

We now consider the rigid analytification of $\pi : Y \to X$. We have the following diagram

$$
\begin{array}{ccc}
Y^{\text{an}} & \xrightarrow{s} & Y \\
\downarrow{\pi^{\text{an}}} & & \downarrow{\pi} \\
X^{\text{an}} & \xrightarrow{s} & X.
\end{array}
$$

where $s : X^{\text{an}} \to Y$ is the map between locally ringed spaces induced by the inclusion $B \hookrightarrow \mathcal{H}(X^{\text{an}})$. By the universal property of the rigid analytification functor, we know that $s$ factors through $s^{\text{an}} : X^{\text{an}} \to Y^{\text{an}}$, which is an analytic section of $\pi^{\text{an}}$. 
We claim that \( s^{an} \) is a Zariski closed immersion of rigid analytic varieties. To see this, we only need to notice that \( (\pi^{an})^* : \mathcal{H}(X^{an}) \to \mathcal{H}(Y^{an}) \) is injective and the following composition
\[
\mathcal{H}(X^{an}) \xrightarrow{(\pi^{an})^*} \mathcal{H}(Y^{an}) \xrightarrow{(s^{an})^*} \mathcal{H}(X^{an})
\]
is the identity map (here we use the fact that \( s^{an} \) is a section of \( \pi^{an} \)). So \((s^{an})^*\) is surjective and thus \( s^{an} \) is a Zariski closed immersion.

On the other hand, we know that \( \dim Y^{an} = \dim X^{an} \) as \( \pi^{an} \) is finite. Both \( X^{an} \) and \( Y^{an} \) are irreducible since they are the analytification of irreducible \( \mathcal{X} \)-schemes. Thus \( Y^{an} = X^{an} \). We conclude that \( B = A \) and \( f \in A \).

\[\textbf{Proposition 3.3 (Analogue of Proposition 2.3 in [JK20].)}\]

Let \( \mathcal{X} \) be a normal rigid analytic space; let \( \mathcal{A} \subset \mathcal{X} \) be a proper closed analytic subset. Then the ring \( \mathcal{H}(\mathcal{X}) \) is integrally closed in \( \mathcal{H}(\mathcal{X} \setminus \mathcal{A}) \).

**Proof.** Let \( f \) be an analytic function in \( \mathcal{H}(\mathcal{X} \setminus \mathcal{A}) \) and is integral over \( \mathcal{H}(\mathcal{X}) \), namely one can find \( a_i \in \mathcal{H}(\mathcal{X}) \) such that
\[
f^d + a_{d-1}f^{d-1} + \cdots + a_0 = 0.
\]
Then around each point of \( \mathcal{A} \) the \( a_i \) are bounded, hence so is \( f \), and by the Hebbarditssatz (see [Co99, p. 502] or [Lü74, Theorem 1.6]) it can be extended to an analytic function on all of \( \mathcal{X} \). \(\square\)

\[\textbf{Corollary 3.4 (Analogue of Corollary 2.5 in [JK20].)}\]

Let \( \mathcal{X} \) be a normal integral finite type scheme over \( K \) and let \( \mathcal{A} \subset \mathcal{X}^{an} \) be a proper closed analytic subset. Then the ring of regular functions \( \mathcal{O}(\mathcal{X}) \) is integrally closed in the ring of analytic functions \( \mathcal{H}(\mathcal{X}^{an} \setminus \mathcal{A}) \).

**Proof.** Since \( \mathcal{X} \) is normal, it follows that \( \mathcal{X}^{an} \) is normal. Therefore, the statement follows from Proposition 3.2 and Proposition 3.3. \(\square\)

### 3.2. Specialization lemma for power series.

The proof of Theorem 1.5 in [JK20] needs a “transcendental” specialization lemma for power series [JK20, Lemma 2.7]. We state it here in our situation.

Let \( k \subset K \) be an algebraically closed subfield such that \( K \) has infinite transcendent degree over \( k \). Then for some chosen \( \lambda_1, \ldots, \lambda_n \in K \) which are algebraically independent over \( k \), we can define the following ring homomorphism
\[
\iota = \iota_{\lambda_1, \ldots, \lambda_n} : k[x_1, \ldots, x_{n+1}] \to K[z_1, \ldots, z_n]
\]
by letting \( \iota|_k \) be the inclusion \( k \hookrightarrow K \), sending \( x_j \) to \( z_j \) for \( 1 \leq j \leq n \), and sending \( x_{n+1} \) to the linear polynomial \( \lambda_1 z_1 + \cdots + \lambda_n z_n \). This homomorphism extends naturally to
\[
\iota : k[x_1, \ldots, x_{n+1}] \to K[z_1, \ldots, z_n].
\]

**Lemma 3.5 (Lemma 2.7 in [JK20]).** Let \( g \in k[x_1, \ldots, x_{n+1}] \). If \( \iota(g) \in K[z_1, \ldots, z_n] \) is an algebraic function (i.e. when interpreted as an element of the quotient field \( K((z_1, \ldots, z_n)) \) it is algebraic over the subfield \( K(z_1, \ldots, z_n) \)), then \( g \) is an algebraic function (i.e. \( g \) is an element of \( k((z_1, \ldots, z_{n+1})) \) algebraic over \( k(z_1, \ldots, z_{n+1}) \)).

The proof of [JK20, Lemma 2.7] in fact works for any infinite transcendent degree field extension \( k \subset K \). So we omit the proof here and refer the reader to their paper.

### 3.3. Dimension reduction.

The proof of Theorem 1.5 in [JK20] is by induction on the dimension of the source spaces. So in this subsection we first show a non-archimedean counterpart of Proposition 3.6 (Dimension Lemma) in [JK20]:
Proposition 3.6 (Dimension Lemma). Let $V$ and $X$ be algebraic varieties defined over $K$, where $V$ is normal and has dimension at least two, and let $f : V^\text{an} \to X^\text{an}$ be a rigid analytic map. Suppose that for every closed algebraic subvariety $H \subset V$ of codimension one, the composition

$$
\tilde{H}^\text{an} \nu^\text{an} \to H^\text{an} \hookrightarrow V^\text{an} \xrightarrow{f} X^\text{an}
$$

is an algebraic morphism, where $\nu$ is the normalization of schemes. Then $f$ itself is algebraic.

Proof. We first choose a Zariski open subset $U \subset X$ which admits an embedding $j$ to some affine space $\mathbb{A}^m$. Denote by $\mathcal{A}$ the pull back of the complement $X \setminus U$ via the rigid analytic map $f$. Then we get analytic functions

$$
g = (g_1, \ldots, g_m) : V^\text{an} \setminus \mathcal{A} \xrightarrow{j} U^\text{an} \xrightarrow{j^\text{an}} (\mathbb{A}_K^m)^\text{an}.
$$

The goal is to show that all $g_i$’s are algebraic, i.e. that $g_i$’s are rational functions on $V$. To use the algebraicity assumption in Proposition 3.6, we shall choose a suitable subvariety $H \subset V$ of codimension one.

We consider the Noether normalization $\pi : V \to \mathbb{A}^{n+1}_K$ (here $n+1 = \dim V > 1$). One can choose a countable algebraically closed subfield $\mathbb{K}$ such that $\pi$ is defined over $\mathbb{K}$. Next we define the “$k$-generic hyperplane” of $\mathbb{A}^{n+1}_K$

$$
P := \{ (z_1, \ldots, z_{n+1}) \in \mathbb{A}^{n+1}_K \mid z_{n+1} = \lambda_1 z_1 + \cdots + \lambda_n z_n \}
$$

for $\lambda_i \in k$ algebraically independent over $k$. Then the induced homomorphism between complete local rings $\hat{O}_{\mathbb{A}^{n+1}_K,0} \to \hat{O}_{P,0}$ is exactly the map $\iota = \iota_{\lambda_1, \ldots, \lambda_n}$ studied in Lemma 3.5.

Now we choose $H \subset V$ to be some irreducible component of $\pi^{-1}(P)$, which has codimension one in $V$. Denote by $\hat{H}$ the normalization. By our algebraicity assumption, the restriction of $g_i$’s on $\hat{H}^\text{an} \setminus \mathcal{A}$, which we denote by $h_i$’s, are in fact algebraic.

We can choose $\pi$ at the beginning such that it is étale over $0 \in \mathbb{A}^{n+1}_K$ (and choose a preimage $\bar{0}$ of 0 such that $\bar{0} \notin \mathcal{A}$). Then we have the following commutative diagram

$$
\begin{array}{ccc}
(V_{\bar{0}}, \bar{0}) & \xleftarrow{\pi} & (\hat{H}, \bar{0}) \\
\downarrow {\pi} & & \downarrow {\pi|_{\hat{H}}} \\
(\mathbb{A}^{n+1}_K,0) & \xrightarrow{\iota} & (P,0)
\end{array}
$$

which induces the following homomorphisms of complete local rings

$$
\hat{O}_{V_{\bar{0}}^{\text{an}},\bar{0}} \xrightarrow{\pi^*} \hat{O}_{\hat{H},\bar{0}}
$$

$$
\mathbb{k}[x_1, \ldots, x_{n+1}] \xrightarrow{\iota^*} \mathbb{K}[z_1, \ldots, z_n].
$$

Note that via $\pi^*$ we can identify the germs of $g_i$ in $\hat{O}_{V_{\bar{0}}^{\text{an}},\bar{0}}$ as formal power series in $\mathbb{k}[x_1, \ldots, x_{n+1}]$. Here we use the property of the analytification functor $\hat{O}_{V^{\text{an}},\bar{0}} \cong \hat{O}_{V^{\text{an}},\bar{0}}$. The image $\iota(g_i) \in K[z_1, \ldots, z_n]$ exactly corresponds to the germ of $h_i$, which is algebraic by our assumption. Thus by Lemma 3.5 we know that $g_i$ is integral over $k(x_1, \ldots, x_{n+1})$. After shrinking $V$ to some Zariski open subset, we can assume that the finite morphism $\pi : V \to \mathbb{A}^{n+1}_K$ is étale onto its image, and thus the integrality of $g_i$ over $k(x_1, \ldots, x_{n+1})$ implies that $g_i$ is integral over $\hat{O}(V_{\bar{0}})$ as an element in $\mathcal{H}(V_{\bar{0}}^{\text{an}} \setminus \mathcal{A})$. From Corollary 3.4 we know that $\hat{O}(V_{\bar{0}})$ is integrally closed and thus $g_i$ is a regular function on $V$.

Proof of Theorem 3.1. We only need to show that (ii) $\implies$ (i). Using the Dimension Lemma (Proposition 3.6) and the induction on the dimension of the source space, one easily obtains the $K$-analytic Borel hyperbolicity of $X$.

$\blacksquare$
4. Tools from non-archimedean Nevanlinna theory

To show Borel hyperbolicity one needs certain extension theorem for analytic maps. We shall recall here the rigid big Picard theorem of Cherry-Ru, which can be regarded as a non-archimedean counterpart of Lu’s extension theorem [Lu91, §4, Lemma 3].

Let $K$ be an algebraically closed field of characteristic 0 which is complete with respect to some non-trivial non-archimedean valuation $|·|_K$. Typical examples of $K$ are $\mathbb{C}_p$ and $\mathbb{C}((t))$. Denote by $A^1[r_1, r_2] := \{ z \in K : r_1 \leq |z|_K < r_2 \}$ the annulus.

**Theorem 4.1** (Cherry-Ru [CR04, Theorem 6.1]). Let $X$ be a smooth projective variety over $K$. Let $D$ be a simple normal crossing divisor on $X$. Let $f : A^1(0, R) \to X^\text{an} \setminus D^\text{an}$ be a rigid analytic morphism. If there exists a section $\omega$ in $H^0(X, \Omega^s_X(\log D)^{\otimes s})$ for some natural number $s$ such that $f^*\omega \neq 0$ and $\omega$ vanishes along an ample divisor $A$ on $X$ (so $s \geq 1$), then $f$ extends to a rigid analytic morphism from $A^1[0, R]$ to $X^\text{an}$.

We shall use the map (2.1) to produce some symmetric differential vanishing along an ample divisor, as required in Cherry-Ru’s extension theorem.

5. From complex to $K$-coefficients

Let $K$ be a complete algebraically closed non-archimedean valued field of characteristic zero. Let $U_K$ be a smooth quasi-projective variety over $K$ which carries a morphism of stacks $U_K \to \mathcal{M}_h \otimes K$ generically finite onto its image. We denote by $\pi_K : V_K \to U_K$ the induced family of polarized varieties with maximal variation of moduli. We shall study the algebraicity of a rigid analytic morphism $f$ from $S^\text{an}$, the rigid analytification of a finite type reduced scheme over $K$, to $U_K^\text{an}$.

By Theorem 3.1, it suffices to study these rigid analytic morphisms $f$ whose source spaces $C_K^\text{an}$ are curves. To verify the algebraicity, we need to show that $f$ extends to a rigid analytic morphism between compactified spaces $\bar{C}_K^\text{an}$ and $\bar{U}_K^\text{an}$, and then apply the rigid GAGA theorem.

The first step is to construct the graded Higgs bundle used in the Viehweg-Zuo construction. Like families over complex numbers, we find smooth compactification $g_K : (X_K, \Delta_K) \to (Y_K, S_K)$ of $\pi_K : V_K \to U_K$. We define the graded Higgs bundle $(F_K, \tau_K)$ in the same manner (cf. Section 2). Note that all these constructions are purely algebraic, which is independent of the field of definition. We also have the iteration of Higgs maps

$$\tau_K^k : \text{Sym}^k T_{Y_K}(-\log S_K) \to F_K^{n-k,k}$$

for $k = 1, 2, \ldots, n$.

The next step is to descend everything to a finitely generated subfield of $K$. Notice that $X_K, Y_K$ are finite type $K$-schemes, the sheaves $F^{p,q}$ are coherent $\mathcal{O}_{Y_K}$-sheaves, and $\tau_K^{p,q}$ are morphisms between them. Therefore one can find a subfield $L \subset K$ which is finitely generated over $\mathbb{Q}$, and an $L$-model $(\mathcal{X}, \Delta_L) \to (\mathcal{Y}, S)$ of the family, as well as the $L$-model $(F, \tau)$ of the graded Higgs bundle. Since $L$ is finitely generated over $\mathbb{Q}$, it can be embedded into $\mathbb{C}$ as a subfield. By base change we obtain a family $g_L : (X_L, \Delta_L) \to (Y_L, S_L)$ over complex numbers, the graded Higgs bundle $(F_L, \tau_L)$, as well as the iteration of Higgs maps

$$\tau_L^k : \text{Sym}^k T_{Y_L}(-\log S_L) \to F_L^{n-k,k}.$$ 

Now we apply the result of Viehweg-Zuo. Notice that the classifying map $\varphi_L : U_L \to \mathcal{M}_h \otimes \mathbb{C}$ is generically finite onto its image since its $L$-model is.

Let $m$ be the Griffiths-Yukawa coupling length of the complex family $g_L$ with maximal variation. Then by Theorem 2.1, $\tau_L^{m}$ factors through

$$\text{Sym}^m T_{Y_L}(-\log S_L) \to \mathcal{P}_L^d,$$
where $\mathcal{P}_C$ is big in the sense of Viehweg. Thus one can find an ample invertible sheaf $\mathcal{H}_C$, some positive integer $\eta$ and a morphism

$$\text{Sym}^n\mathcal{P}_C \to \bigoplus \mathcal{H}_C^\vee$$

which is injective over some Zariski open subset. The composed map $\text{Sym}^mT_{Y_K}(-\log S_K) \to \bigoplus \mathcal{H}_C^\vee$ gives us plenty of symmetric differentials vanishing along some ample divisor, just as required in Theorem 4.1. But before applying Cherry-Ru’s extension theorem we have to first “transform” those symmetric differentials back to the non-archimedean field $K$.

Note that although Viehweg-Zuo used some transcendental methods from Hodge theory to derive the bigness of $\mathcal{P}_C$, the objects $\tau^*_C$ and $\mathcal{H}_C$ are purely algebraic. That means, we can enlarge the finitely generated subfield $L$ such that the ample invertible sheaf $\mathcal{H}_C$ as well as the morphism $\text{Sym}^mT_{Y_K}(-\log S_K) \to \bigoplus \mathcal{H}_C^\vee$ are also defined over $L$. So the ample invertible sheaf $\mathcal{H}_C$ as well as the map can be base changed to the original compactified base space $Y_K$ over $K$ via the field embedding $L \subset K$.

Therefore we obtain the composed map over $K$

$$(5.1) \quad \text{Sym}^mT_{Y_K}(-\log S_K) \to \text{Sym}^n\mathcal{P}_K \to \bigoplus \mathcal{H}_K^\vee$$

where the second map is injective over a Zariski open subset. Note that $\mathcal{H}_K$ is still ample.

**Definition 5.1.** We say a rigid analytic morphism $f : C^n_K \to U^n_K$ has maximal length of Griffiths-Yukawa coupling if the following composed map

$$(5.2) \quad T^m_{C^K} \xrightarrow{df^\otimes m} f^*\text{Sym}^mT_{Y_K}(-\log S_K) \to f^*P_K^\vee$$

is nonzero.

**Proposition 5.2.** Let $V_K \to U_K$ be a smooth family of polarized varieties with semi-ample canonical divisor.

Assume that the induced classifying map from $U_K$ to the moduli space is generically finite onto its image. Then any rigid analytic morphism $f : C^n_K \to U^n_K$ with maximal length of Griffiths-Yukawa coupling is algebraic.

**Proof.** Since the composed map (5.2) is nonzero, one can find a copy of $\mathcal{H}_K^\vee$ in the direct sum appearing in the diagram (5.1) such that the composition

$$T^m_{C^K} \xrightarrow{df^\otimes m} f^*\text{Sym}^mT_{Y_K}(-\log S_K) \to f^*\mathcal{H}_K^\vee$$

is nonzero. In this way we have found a symmetric differential $\omega \in \Gamma(Y_K, \text{Sym}^m\Omega^1_{Y_K}(\log S_K) \otimes \mathcal{H}_K^\vee)$ which vanishes along the ample divisor associated to $\mathcal{H}_K$, and the pull back $f^*\omega \neq 0$. Therefore, by Theorem 4.1, we can extend $f$ over all the points of $C^n_K \setminus C^{\text{can}}_K$. Now the algebraicity follows from the rigid GAGA theorem [FvP, Theorem 4.10.5].

Now we can state our main theorem.

**Theorem 5.3.** Let $V_K \to U_K$ be a smooth family of polarized varieties with semi-ample canonical divisor. Suppose that the induced classifying map from $U_K$ to the moduli space is quasi-finite. If the Griffiths-Yukawa coupling length of the induced complex family $V_C \to U_C$ is one, then $U_K$ is $K$-analytically Borel hyperbolic.

**Proof.** By Theorem 3.1, we consider a rigid analytic morphism $f$ from the rigid analytification of a smooth quasi-projective curve $C^K$ to $U_K$. By replacing $U_K$ by the Zariski closure of $f(C^K)$, one can assume that the image of $f$ is Zariski dense. The family restricted to the new base has maximal variation as we assume that the classifying map is quasi-finite. After desingularizing the base space by Hironaka’s theorem, we can assume that $U_K$ is smooth, and the family still has maximal variation. The rigid analytic morphism $f$ can
be lifted to the smooth model since its image is not contained in the center of birational modifications. Now since the Griffiths-Yukawa coupling length $m = 1$, the diagram (5.2) becomes

$$\begin{align*}
T^{an}_{C,K} & \xrightarrow{df} f^*T_{Y_K}(-\log S_K) \rightarrow f^*\mathcal{P}^N_{Y_K}.
\end{align*}$$

Note that the second map factors through from the Kodaira-Spencer map. Since the family has maximal variation, it is generically injective. Combining with the Zariski density of the image of $f$, we know that $f$ has maximal length of Griffiths-Yukawa coupling. Then we apply Proposition 5.2. □

**Proof of Theorem B.** For $N \geq 3$, $\mathcal{M}^{[N]}_{g,K}$ is a fine moduli space and so we have a universal family over it. By [VZ02, Theorem 1.4, ii)], the Griffiths-Yukawa coupling length of the family can be bounded from above by its fiber dimension, which is one for family of curves. Now we apply Theorem 5.3. □

**Proof of Theorem A.** By [JV18, Theorem 2.18], we know that $\mathcal{M}^{[N]}_{g,K}$ is $K$-analytically Brody hyperbolic if and only if every rigid analytic morphism $\mathbb{G}^{an}_{m,K} \rightarrow \mathcal{M}^{[N],an}_{g,K}$ is constant and, for every abelian variety $B$ over $K$ with good reduction over $O_K$, every morphism $B \rightarrow \mathcal{M}^{[N]}_{g,K}$ is constant.

We first check the statement about abelian varieties. Since one only needs to consider morphisms $B \rightarrow \mathcal{M}^{[N]}_{g,K}$, we are able to find some finitely generated subfield such that the morphism is defined over it. After embedding this finitely generated subfield into $\mathbb{C}$ and the base change of the morphism, we get a morphism $B_{\mathbb{C}} \rightarrow \mathcal{M}^{[N]}_{g,\mathbb{C}}$, which has to be constant by the hyperbolicity of $\mathcal{M}^{[N]}_{g,\mathbb{C}}$. This forces the original morphism to be constant.

Next we check the statement about rigid analytic morphisms $\mathbb{G}^{an}_{m,K} \rightarrow \mathcal{M}^{[N],an}_{g,K}$. Since we have already proved that $\mathcal{M}^{[N]}_{g,K}$ is $K$-analytically Borel hyperbolic, every such rigid analytic morphism is actually the rigid analytification of some morphism $\mathbb{G}_{m,K} \rightarrow \mathcal{M}^{[N]}_{g,K}$. Then using the descent argument again, we obtain a morphism $\mathbb{G}_{m,\mathbb{C}} \rightarrow \mathcal{M}^{[N]}_{g,\mathbb{C}}$, which is forced to be constant by the hyperbolicity of $\mathcal{M}^{[N]}_{g,\mathbb{C}}$. □

**Remark 5.4.** It is worthwhile to mention that over complex numbers the moduli stack $\mathcal{M}_h$ of polarized complex smooth projective varieties with semi-ample canonical divisor and Hilbert polynomial $h$ is proven to be Borel hyperbolic recently in [DLSZ]. One might expect the $K$-analytic Borel hyperbolicity to hold for more general moduli stacks over $K$.

**References**

[Ber90] Berkovich, Vladimir G. _Spectral theory and analytic geometry over non-Archimedean fields_, (American Mathematical Society, Providence, RI, 1990).

[Bor72] Armand Borel, “Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem.” _J. Differential Geometry_ (1972) vol. 6: 543–560. URL [http://projecteuclid.org/euclid.jdg/1214430642].

[Bo14] Bosch, Siegfried. _Lectures on formal and rigid geometry_, vol. 2105 of Lecture Notes in Mathematics, (Springer-Verlag, 2014). URL [https://doi.org/10.1007/978-3-319-04417-0].

[CP15] Frédéric Campana and Mihai Păun. Orbifolds generic semi-positivity: an application to families of canonically polarized manifolds. _Ann. Inst. Fourier (Grenoble)_, 65(2):835–861, 2015.

[Ch04] Cherry, William. “Non-Archimedean analytic curves in abelian varieties.” _Math. Ann._ (1994) (300): 393–404. URL [https://doi.org/10.1007/BF01450493].

[CR04] Cherry, William and Ru, Min. “Rigid analytic Picard theorems.” _Amer. J. Math._ (2004) (126): 873–889. URL [http://muse.jhu.edu/journals/american_journal_of_mathematics/v126/126.4cherry.pdf].

[CW02] Cherry, William and Wang, Julie Tzu-Yueh. “Non-Archimedean analytic maps to algebraic curves.” In “Value distribution theory and complex dynamics (Hong Kong, 2000),” vol. 303 of _Contemp. Math._, 7–35, (Amer. Math. Soc., Providence, RI 2002). URL [https://doi.org/10.1090/conm/303/05235].

[Co99] Conrad, Brian. “Irreducible components of rigid spaces.” _Ann. Inst. Fourier (Grenoble)_ 49 (1999), no. 2, 473–541. URL [http://www.numdam.org/item?id=AIF_1999__49_2_473_0].

[DLSZ] Ya Deng, Steven Lu, Ruiran Sun and Kang Zuo. “Picard theorems for moduli spaces of polarized varieties.” (2020). arXiv:1911.02973v3.
Fresnel, Jean; van der Put, Marius. Rigid analytic geometry and its applications. Progress in Mathematics, 218. Birkhäuser Boston, Inc., Boston, MA, 2004. xii+296 pp. ISBN: 0-8176-4206-4. URL https://doi.org/10.1007/978-1-4612-0041-3.

Ariyan Javanpeykar. “The Lang-Vojta conjectures on projective pseudo-hyperbolic varieties.” (2020). CRM Short Courses Springer, to appear. arXiv:2002.11981.

Ariyan Javanpeykar and Ljudmila Kamenova. “Demailly’s notion of algebraic hyperbolicity: geometricity, boundedness, moduli of maps.” Math. Zeit. (2020) URL https://doi.org/10.1007/s00209-020-02489-6.

Ariyan Javanpeykar and Robert A. Kucharczyk. “Algebraicity of analytic maps to a hyperbolic variety.” Math. Nachrichten, (2020) (293): 1–15. URL https://doi.org/10.1002/mana.201900098.

Ariyan Javanpeykar, Ruiran Sun and Kang Zuo. “The Shafarevich conjecture revisited: Finiteness of pointed families of polarized varieties.” (2020). arXiv:2005.05933.

Ariyan Javanpeykar and Alberto Vezzani. “Non-archimedean hyperbolicity and applications.” (2018). arXiv:1808.09980.

Ariyan Javanpeykar and Junyi Xie. “Finiteness properties of pseudo-hyperbolic varieties.” (2020). IMRN, to appear. arXiv:1909.12187v2.

S.J. Kovács and M. Lieblich. Erratum for Boundedness of families of canonically polarized manifolds: a higher dimensional analogue of Shafarevich’s conjecture. Ann. of Math. (2), 173(1):585–617, 2011.

Lang, Serge. “Hyperbolic and Diophantine analysis,” Bull. Amer. Math. Soc. (N.S.), (1986) vol. 14 (2): 159–205. URL https://doi.org/10.1090/S0273-0979-1986-15426-1.

Steven Shin-Yi Lu. “On meromorphic maps into varieties of log-general type.” In “Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989),” vol. 52 of Proc. Sympos. Pure Math., 305–333, (Amer. Math. Soc., Providence, RI1991).

Lütkebohmert, Werner. “Der Satz von Remmert-Stein in der nichtarchimedischen Funktionentheorie.” Math. Z. 139 (1974), 69–84. URL https://doi.org/10.1007/BF01194146.

Ru, Min. “A note on $p$-adic Nevanlinna theory.” Proc. Amer. Math. Soc. (2001) (129): 1263–1269. URL https://doi.org/10.1090/S0002-9939-00-05680-X.

Eckart Viehweg and Kang Zuo. “Base spaces of non-isotrivial families of smooth minimal models.” In “Complex geometry (Göttingen, 2000),” 279–328, (Springer, Berlin2002).

Eckart Viehweg and Kang Zuo. “On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds.” Duke Math. J. (2003) vol. 118 (1): 103–150. URL http://dx.doi.org/10.1215/S0012-7094-03-11815-3.