On the quasisymmetrical classification of infinitely renormalizable maps

II. REMARKS ON MAPS WITH A BOUNDED TYPE TOPOLOGY

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§0 Introduction

This note is a remark to the paper [1]. The aim is to show that the techniques in [1] can also be used to understand the quasisymmetrical classification of infinitely renormalizable maps of bounded type. We will use the same terms and notations as those in [1] without further notices. The result we will prove is the following theorem.

THEOREM 1. Suppose $f$ and $g$ in $\mathcal{U}$ are two infinitely renormalizable maps of bounded type and topologically conjugate. Moreover, suppose $H$ is the homeomorphism between $f$ and $g$. Then $H$ is quasisymmetric.

Since the techniques as well as ideas of the proof are similar to those in [1], we outline the proof in the next section. The reader may refer to [1] and [2] for more details.

§1 The outline of the proof of Theorem 1

We outline the proof of Theorem 1 by several lemmas.

Suppose $f = h \circ Q_t$, for some $t > 1$, in $\mathcal{U}$ is an infinitely renormalizable map of bounded type. We note that $Q_t(x) = -|x|^t$. Let $f_0 = f$. And inductively, let $f_k = \alpha_k \circ f_{k-1}^n \circ \alpha_k^{-1}$ be the renormalization $\mathcal{R}(f_{k-1})$ of $f_{k-1}$ where $\alpha_k$ is the linear rescale from $J_k$ to $[-1, 1]$ and $n_k$ is the return time for any integer $k \geq 1$ (see [1]). We call $J_k$ a restricted interval.

Let $I_0$ be the interval $[-1, 1]$ and $I_k$ be the preimage of $[-1, 1]$ under $\alpha_1 \circ \cdots \circ \alpha_k$ for $k \geq 1$. We note that the set $\{I_k\}^\infty_{k=0}$ forms a sequence of nested intervals. Moreover, one of the endpoints of $I_k$, say $p_k$, is a periodic point of period $m_k = n_1 \cdots n_k$ of $f$ and the orbit $O(p_k)$ of $p_k$ under $f$ stays outside of the interior of $I_k$ (see Figure 1).
Suppose $L_k$ is the image of $I_k$ under $f^{\circ m_k}$ and $T_k$ is the interval bounded by the points $p_k$ and $p_{k+1}$. Let $M_k$ be the complement of $T_k$ in $L_k$. Then $M_k$ is the interval bounded by $p_{k+1}$ and $c_{m_k}$, where $c_{m_k} = f^{\circ m_k}(0)$ (See Figure 2).

**Lemma 1.** There is a constant $C_1 = C_1(f) > 0$ such that

$$C_1^{-1} \leq |M_k|/|I_k| \leq C_1.$$

for all the integers $k \geq 0$.

**Proof.** This lemma is actually proved in [3] by using the techniques such as the smallest interval and shuffle permutation on the intervals.

**Lemma 2.** There is a constant $C_2 = C_2(f) > 0$ such that

$$C_2^{-1} \leq |I_k|/|I_{k-1}| \leq C_2$$

for all the integers $k \geq 0$.

We first prove a more general result, as that in [1], as follows. Let $K = K(t, N, K)$, for fixed numbers $t > 1$, $N \geq 2$ and $K > 0$, be the subspace of renormalizable maps $f = h \circ Q_t$ in $U$ such that $|f(N(h))(x)| \leq K$ for all $x$ in $[-1, 0]$ and all the return times $n_k$ of $\mathcal{R}^{\circ k}(f)$ are less than or equal to $N$.

**Lemma 3.** There is a constant $C_3 = C_3(t, N, K) > 0$ such that

$$C_3^{-1} \leq f(0) = c_1(f) \leq C_3$$

for all $f$ in $K$. 

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Proof. The proof of this lemma is similar to the proof of Lemma 3 in [1] but needs little more work to solve a little more complicated equation.

Remember that $f_k$ is the $k^{th}$-renormalization of $f = f_0$. Let $f_k = h_k \circ Q_i$. We note that the graph of $f_k$ is the rescale of the graph of the restriction of $f^{\circ m_k}$ to $I_k$.

**Lemma 4.** There is a constant $C_4 = C_4(f) > 0$ such that

$$\| (N(h_k))(x) \| \leq C_4$$

for all $x$ in $[-1, 0]$ and all the integer $k \geq 0$.

Proof. It is the a priori bound proved in [3].

**Proof of Lemma 2.** It is now a direct corollary of Lemma 1, Lemma 3 and Lemma 4 for $K = C_4$ and $N = \max_{0 \leq k < \infty}\{n_k\}$.

The set of the nested intervals $\{I_0, I_1, \cdots, I_k, \cdots\}$ gives a partition of $[-1, 1]$ as follows. Let $p_k$ be one of the endpoints of $I_k$ and $O_{k,f}(p_k)$ be the intersection of $I_{k-1}$ and the orbit $O_{k,f}(p_k)$ of $p_k$ under $f$ for $k \geq 1$. Suppose $M_{k-1,1}, \cdots, M_{k-1,n_k+1}$ are the connected components of $I_{k-1} \setminus (O_{k,f}(p_k) \cup I_k)$ for $k \geq 1$. Then the set $\eta_0 = \{M_{k,i}\}$ for $i = 1, \cdots, n_k + 1$ and $k = 1, 2, \cdots$ forms a partition of $[-1, 1]$ (see Figure 3).

![Figure 3](image)

Now we are going to define a Markov map $F$ induced by $f$. Let $F$ be a function of $[-1, 1]$ defined by

$$F(x) = \begin{cases} 
  f(x), & x \in M_{1,1} \cup M_{1,2} \cup \cdots \cup M_{1,n_1+1}, \\
  f^{\circ m_1}(x), & x \in M_{2,1} \cup \cdots \cup M_{2,n_2+1}, \\
  \vdots & \\
  f^{\circ m_{n_2} \cdots n_k}(x), & x \in M_{k,1} \cup \cdots \cup M_{k,n_k+1}, \\
  \vdots & 
\end{cases}$$

It is clearly that $F$ is a Markov map in the sense that the image of every $M_{k,i}$ under $F$ is the union of some intervals in $\eta_0$ (Figure 4).

Let $g_{k,i} = (F|M_{k,i})^{-1}$ for $k = 1, \cdots$, and $i = 1, \cdots, n_k + 1$ be the inverse branches of $F$ with respect to the Markov partition $\eta_0$. Suppose $w = i_0 i_1 \cdots i_{l-1}$ is a finite sequence of the set $\mathcal{I} = \{(k,i), k = 1, \cdots \text{ and } i = 1, \cdots, n_k + 1\}$. We say it is admissible if the range $M_{i_s}$ of $g_{i_s}$ is contained in the domain $F_{i_s-1}(J_{i_{s-1}})$ of $g_{i_{s-1}}$ for $s = 1, \cdots, l - 1$. For an admissible
sequence \( w = i_0i_1 \cdots i_{l-1} \), we can define \( g_w = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{l-1}} \). We use \( D(g_w) \) to denote the domain of \( g_w \) and use \( |D(g_w)| \) to denote the length of the interval \( D(g_w) \).

![Figure 4](image)

**Definition 1.** We say the induced Markov map \( F \) has bounded distortion property if there is a constant \( C_5 = C_5(f) > 0 \) such that

(a) \( C_5^{-1} \leq |M_{k,i}|/|M_{k,i+1}| \leq C_5 \) for \( k = 1, 2, \ldots \) and \( i = 1, \ldots, n_k \),

(b) \( C_5^{-1} \leq |M_{k,i}|/|I_k| \leq C_5 \) for \( k = 1, 2, \ldots \) and \( i = 1, \ldots, n_k + 1 \), and

(b) \( |(N(g_w))(x)| \leq C_5/|D(g_w)| \) for all \( x \) in \( D(g_w) \) and all admissible \( w \).

The reason we give this definition is the following lemma as that in [1].

**Lemma 5.** Suppose \( f \) and \( g \) in \( U \) are two infinitely renormalizable maps of bounded type and \( H \) is the conjugacy between \( f \) and \( g \). If both of the induced Markov maps \( F \) and \( G \) have the bounded distortion property, then \( H \) is quasisymmetric.

**Proof.** It can be proved by almost the same arguments as that we used in the paper [2]. For more details of the proof, the reader may refer to [4].

Now the proof of Theorem 1 concentrates on the next lemma.

**Lemma 6.** Suppose \( f = h \circ Q_t \), for some \( t > 1 \), in \( U \) is an infinitely renormalizable map of bounded type and \( F \) is the Markov map induced by \( f \). Then \( F \) has the bounded distortion
property.

Proof. Let \( I_{k,j} = \left( f^{\circ m_{k-1}} I_{k-1} \right)^{o_j} (I_k) \) for \( j = 0, 1, \cdots, n_k \) and \( \{G_{k,i}\} \) are all the connected components of \( I_k \cup \bigcup_{j=0}^{n_k} I_{k,j} \) (Figure 5).

Each \( M_{k,j} \) is either a single \( G_{k,i} \) or \( I_{k,j} \cup G_{k,i} \) for some \( j \) and some \( i \). By the bounded geometry [3] of \( \{I_{k,j}\} \) and \( \{G_{k,i}\} \), there is a constant \( C_0 > 1 \) such that all the ratios \( |I_{k,j}|/|I_{k,j'}| \), \( |G_{k,i}|/|G_{k,i'}| \) and \( |G_{k,i}|/|I_{k,j}| \) are in the interval \([C_0^{-1}, C_0]\). We note that \( C_6 \) does not depend on \( k \) as well as \( i, i', j \) and \( j' \). This fact and Lemma 3 imply the condition (a) in Definition 1.

\[ \begin{array}{c}
\text{Figure 5} \\
\end{array} \]

The condition (b) in Definition 1 is assured by Lemma 2 and the condition (a). The proof of the condition (c) in Definition 1 is similar to that in [1].

The arguments in Lemma 1 to Lemma 6 give the proof of Theorem 1.

References

[1] Y. Jiang, On quasisymmetrical classification of infinitely renormalizable maps – Maps with Feigenbaum’s topology, preprint in this issue, IMS at SUNY at Stony Brook.

[2] Y. Jiang, Dynamics of certain smooth one-dimensional mappings – II. Geometrically finite one-dimensional mappings, preprint 1991/1, IMS, SUNY at Stony Brook.

[3] D. Sullivan, Bounds, quadratic differentials, and renormalization conjectures, preprint, 1991 and American Mathematical Society Centennial Publications, Volume 2: Mathematics into the Twenty-first Century, to appear.

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