Band surgeries and crossing changes between fibered links

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Abstract

We characterize cutting arcs on fiber surfaces that produce new fiber surfaces, and the changes in monodromy resulting from such cuts. As a corollary, we characterize band surgeries between fibered links and introduce an operation called generalized Hopf banding. We further characterize generalized crossing changes between fibered links, and the resulting changes in monodromy.

1. Introduction

A fibered link is one whose exterior fibers over $S^1$, so that each fiber is a Seifert surface for the link. Among the many fascinating properties of fibered links is the ability to express their exteriors as a mapping torus, thereby allowing us to encode the three-dimensional information about the link exterior in terms of a surface automorphism. We will refer to this automorphism as the monodromy of the link, or of the surface. This connection yields generous amounts of information, including geometric classification (for example, the link exterior is hyperbolic if and only if the surface automorphism is pseudo-Anosov [30]), topological information (for example, the fiber surface is the unique minimal genus Seifert surface [3]), and methods to deconstruct/reconstruct fibered links [14, 29].

In addition to providing beautiful examples and visualizations of link exteriors, fibered links are deeply connected with important areas of topology, including the Berge Conjecture [19, 23, 25], as well as contact geometry due to Giroux’s correspondence [10] between open books and contact structures on 3-manifolds.

In this paper, we further explore constructions of fibered links in terms of the monodromy. We will generalize a very well-understood and important operation on fiber surfaces known as Hopf plumbing (or its inverse, Hopf de-plumbing): If a fiber surface has a Hopf plumbing summand, then cutting along the spanning arc of the Hopf annulus results in another fiber surface, and this process is called Hopf de-plumbing. It is known, for instance, that any fiber surface of a fibered link in $S^3$ can be constructed from a disk by a sequence of Hopf plumbings and Hopf de-plumbings [11]. Such an arc corresponding to a Hopf plumbing can be characterized in terms of the monodromy. We will characterize all arcs on a fiber surface so that cutting along them gives another fiber surface. This will lead naturally to the construction of a generalized Hopf banding, and we will leverage our results to relate to two other crucially important operations: band surgeries, and generalized crossing changes. We will complete the characterization of band surgeries between fibered links, and (generalized) crossing changes between fibered links.
The paper is organized as follows: In Section 2, we provide definitions and background for the tools we will use. In Section 3, we study the result of cutting a fiber along an arc, and prove the following theorem.

THEOREM 1. Let $L$ be a fibered (oriented) link with fiber $F$ and monodromy $h$ (which is assumed to be the identity on $\partial F$), and suppose that $\alpha$ is a properly embedded arc in $F$. Let $F'$ be the surface obtained by cutting $F$ along $\alpha$, and the resulting (oriented) link $L' = \partial F'$. The surface $F'$ is a fiber for $L'$ if and only if $i_{\text{total}}(\alpha) = 1$ (that is, when $\alpha$ is clean and alternating, or once-unclean and non-alternating).

We also characterize the resulting changes in monodromy; see Corollary 6. (See Section 2 for the definition of $i_{\text{total}}(\alpha)$.)

By [5, 15, 28], it is known that if a coherent band surgery increases the Euler characteristic of a link, then the band can be isotoped onto a taut Seifert surface. Hence, such a band surgery between fibered links corresponds to cutting the fiber surface. When a coherent band surgery changes the Euler characteristic of a link by at least two, such a band surgery is characterized by Kobayashi [20] (see Theorems 8 and 9). In Section 4, we introduce generalized Hopf banding and give a characterization of the remaining case.

THEOREM 2. Suppose that $L$ and $L'$ are links in $S^3$, and $L'$ is obtained from $L$ by a coherent band surgery and $\chi(L') = \chi(L) + 1$.

1. Suppose that $L$ is a fibered link. Then $L'$ is a fibered link if and only if the fiber $F$ for $L$ is a generalized Hopf banding of a Seifert surface $F'$ for $L'$ along $b$.

2. Suppose that $L'$ is a fibered link. Then $L$ is a fibered link if and only if a Seifert surface $F$ for $L$ is a generalized Hopf banding of the fiber $F'$ for $L'$ along $b$.

As an application, we characterize band surgeries on torus links $T(2,p)$, or connected sums of those, that produce fibered links (Corollaries 2 and 4). In the forthcoming paper by Buck, Ishihara, Rathbun, and Shimokawa, we use this to completely characterize an important biological operation: the unlinking of DNA molecules by a recombinase system.

In Section 5, we consider arc-loops, which are loops around arcs, and characterize Dehn surgeries along arc-loops preserving a fiber surface, using results of Ni [24] about surgeries on knots in trivial sutured manifolds.

THEOREM 3. Suppose that $F$ is a fiber surface in a 3-manifold $M$ and $c$ is an $\alpha$-loop (a loop around an arc $\alpha$). Suppose that $\gamma$ is a non-trivial slope on $c$, and that $N(\gamma)$ is the manifold obtained from $M$ via the $\gamma$-surgery on $c$. Then $F$ is a fiber surface in $N(\gamma)$ if and only if

1. $\alpha$ is clean and $\gamma = i_\partial(\alpha) + 1/n$ for some integer $n$, or
2. $\alpha$ is once-unclean and $\gamma = i_\partial(\alpha)$.

(See Section 2 for the definition of $i_\partial(\alpha)$.)

By [20, 28], crossing changes between fibered links with different Euler characteristics are understood. In Section 6, we investigate the remaining case and characterize when fibered links of the same Euler characteristic are related by crossing changes and generalized crossing changes.

THEOREM 4. Suppose a link $L'$ is obtained from a fibered link $L$ in $S^3$ with fiber $F$ by a crossing change, and $\chi(L') = \chi(L)$. Then $L'$ is a fibered link if and only if the crossing change is a Stallings twist or an $\varepsilon$-twist along an arc $\alpha$ in $F$, where $\alpha$ is once-unclean and alternating with $i_\partial(\alpha) = -\varepsilon$. 

Theorem 5. Suppose that \( L \) and \( L' \) are fibered links in \( S^3 \) related by a generalized crossing change with \( \chi(L) = \chi(L') \). Then the generalized crossing change is an \( n \)-twist around an arc \( \alpha \), and one of the following holds:

1. \( \alpha \) is clean and non-alternating;
2. \( n = \pm 2 \), and \( \alpha \) is clean and alternating with \( i_\beta(\alpha) = -n/2 \);
3. \( n = \pm 1 \), and \( \alpha \) is once-unclean and alternating with \( i_\beta(\alpha) = -n \).

The generalized crossing change of (1) in Theorem 5 implies a Stallings twist of type \((0, 1)\) (see Theorem 8). For the generalized crossing change of (2) in Theorem 5, the crossing circle links a plumbed Hopf annulus and the \( \pm 2 \)-twist reverses the direction of twist in the Hopf annulus (see Theorem 10). The resulting changes in monodromy for (3) of Theorem 5 are characterized in Corollary 5.

In Section 7, we give an alternative proof of Theorem 1 using Theorem 3.

2. Preliminaries

2.1. Surfaces

Definition. A Seifert surface \( F \) for a link \( L \) is taut if it maximizes Euler characteristic over all Seifert surfaces for \( L \). We say the Euler characteristic for the link is \( \chi(L) = \chi(F) \) if \( F \) is taut.

Definition (see [13]). Let \( \alpha, \beta \) be two oriented arcs properly embedded in an oriented surface \( F \) which intersect transversely. At a point \( p \in \alpha \cap \beta \), define \( i_p \) to be \( \pm 1 \), depending on whether the orientation of the tangent vectors \((T_p\alpha, T_p\beta)\) agrees with the orientation of \( F \) or not. If \( \alpha \) and \( \beta \) intersect minimally over all isotopies fixing the boundary pointwise, then the following are well-defined.

1. The geometric intersection number, \( \rho(\alpha, \beta) := \sum_{p \in \alpha \cap \beta} |i_p| \), is the number of intersections (without sign) between \( \alpha \) and \( \beta \) in the interior of \( F \).
2. The boundary intersection number, \( i_\partial(\alpha, \beta) := \frac{1}{2} \sum_{p \in \alpha \cap \beta \cap \partial F} i_p \), is half the sum of the oriented intersections at the boundaries of the arcs.
3. The total intersection number, \( i_{\text{total}}(\alpha, \beta) := \rho(\alpha, \beta) + |i_\partial(\alpha, \beta)| \), is the sum of the (unoriented) interior intersections between \( \alpha \) and \( \beta \), and the absolute value of half the sum of the boundary intersections between the two arcs.
4. If \( F \) is a fiber surface with monodromy \( h \), then we define \( \rho(\alpha) := \rho(\alpha, h(\alpha)) \), and \( i_\partial(\alpha) := i_\partial(\alpha, h(\alpha)) \), and \( i_{\text{total}}(\alpha) := i_{\text{total}}(\alpha, h(\alpha)) \).

Definition (see [9]). Let \( F_i \subset M_i \), for \( i = 1, 2 \), be compact oriented surfaces in the closed, oriented 3-manifolds \( M_i \). Then \( F \subset M_1 \# M_2 = M \) is a Murasugi sum of \( F_1 \) and \( F_2 \) if\n\[
M = (M_1 \setminus \text{int}(B_1)) \cup_{S^2} (M_2 \setminus \text{int}(B_2)) \text{ for 3-balls } B_i \text{ with } S^2 = \partial B_1 = \partial B_2, \text{ and, for each } i,
\]
\[
S^2 \cap F_i \text{ is a } 2n\text{-gon, and } (M_i \setminus \text{int}(B_i)) \cap F = F_i.
\]
When \( n = 2 \), this is known as a plumbing of \( F_1 \) and \( F_2 \). Further, when \( n = 2 \) and one of the surfaces, say \( F_2 \), is a Hopf annulus, this is known as a Hopf plumbing.

2.2. Sutured manifolds

Definition (see [7, 12, 27]). A sutured manifold, \((N, \gamma)\), is a compact 3-manifold \( N \), with a set \( \gamma \subset \partial N \) of mutually disjoint annuli, \( A(\gamma) \), and tori, \( T(\gamma) \), satisfying the orientation
conditions below. (We will only consider the case when \( T(\gamma) = \emptyset \).) Call the core curves of the annuli \( A(\gamma) \) the sutures, and denote them by \( s(\gamma) \). Let \( R(\gamma) = \partial N \setminus A(\gamma) \).

1. Every component of \( R(\gamma) \) is oriented, and \( R_+ (\gamma) \) (respectively, \( R_- (\gamma) \)) denotes the union of the components whose normal vectors point out of (respectively, into) \( N \).
2. The orientations of \( R_{\pm} (\gamma) \) are consistent with the orientations of \( s(\gamma) \).

We will often simplify notation and write \( (N, s(\gamma)) \) in place of \( (N, \gamma) \).

**Definition** (see [7, 12, 27]). We say that a sutured manifold \( (N, \gamma) \) is a trivial sutured manifold if it is homeomorphic to \( (F \times I, \partial F \times I) \), for some compact, bounded surface \( F \), with \( R_+ (\gamma) = F \times \{1\}, R_- (\gamma) = F \times \{0\} \), and \( A(\gamma) = \partial F \times I \).

**Definition** (see [7, 12, 27]). Suppose that \( F \) is a Seifert surface for an oriented link \( L \) in a manifold \( M \). Then \( (n(F), L) = (F \times I, \partial F \times \{ \frac{1}{2} \}) \) is a trivial sutured manifold. We call \( (\overline{M \setminus n(F)}, L) \) the complementary sutured manifold.

**Definition** (see [7, 12, 27]). A properly embedded disk \( D \) in \( (N, \gamma) \) is a product disk if \( \partial D \cap A(\gamma) \) consists of two essential arcs in \( A(\gamma) \). A product decomposition is an operation to obtain a new sutured manifold \( (N', \gamma') \) from a sutured manifold \( (N, \gamma) \) by decomposing along an oriented product disk (see [7]). We denote this by

\[
(N, \gamma) \xrightarrow{D} (N', \gamma').
\]

The following definition and theorem are due to Wu [31]. (Wu’s definition is slightly more general, but we will only need the special case described here.)

**Definition** ([31]). Let \( M \) be a 3-manifold with non-empty boundary and let \( \gamma \) be a collection of essential simple closed curves in \( \partial M \). An \( n \)-compressing disk (with respect to \( \gamma \)) is a compressing disk for \( \partial M \) which intersects \( \gamma \) in \( n \) points; also call a compressing disk for \( \partial M \setminus \gamma \) a 0-compressing disk.

**Theorem 6** ([31]). Let \( M \) be a 3-manifold with non-empty boundary, let \( \gamma \) be a collection of essential simple closed curves in \( \partial M \), and let \( J \) be a simple closed curve in \( \partial M \) disjoint from \( \gamma \). Suppose that \( \partial M \setminus \gamma \) is compressible. Let \( M' \) be the result of attaching a 2-handle to \( M \) along \( J \). If \( \partial M' \) is \( n \)-compressible, then \( \partial M \setminus J \) is \( k \)-compressible for some \( k \leq n \).

**Conventions.** Let us establish some informal conventions to aid in visualization.

1. By \( F \times [0, 1] \), we will refer to a product where the \([0, 1] \) component is ‘vertical’, with \( F \times \{0\} \) on the ‘top’, and \( F \times \{1\} \) on the ‘bottom’.
2. Correspondingly, the orientation of \( F \) will be such that \( F \times \{0\} \) corresponds to the ‘positive’ side of \( F \).
3. By \([0, 1] \times D \), we will refer to a product where the \([0, 1] \) component is ‘horizontal’, with \( \{0\} \times D \) on the ‘left’, and \( \{1\} \times D \) on the ‘right’.
4. A fibered link complement will be thought of as arising from a mapping torus \((F \times [0, 1])/h\), where \( h : F \times \{1\} \to F \times \{0\} \), so that the product disk determined by \( \alpha \) and \( h(\alpha) \) will emanate ‘downwards’ from \( \alpha \) in \( F \times \{1\} \), and ‘upwards’ from \( h(\alpha) \) in \( F \times \{0\} \).
3. Cutting arcs in fiber surfaces

In this section, we will give a direct proof of Theorem 1. See Section 7 for an alternative proof using Ni’s result [24]. Let $L$ be a fibered (oriented) link in a manifold $M$ with fiber $F$ and monodromy $h$ (which is assumed to be the identity on $\partial F$), and suppose that $\alpha$ is a properly embedded arc in $F$. Assume that $\alpha$ and $h(\alpha)$ have been isotoped in $F$, fixing the endpoints, to intersect minimally. If the endpoints of $h(\alpha)$ emanate to opposite sides of $\alpha$, then $|\iota_\partial(\alpha)| = 1$. In this case, $\alpha$ is called alternating. Otherwise, $|\iota_\partial(\alpha)| = 0$, and $\alpha$ is called non-alternating. If $\rho(\alpha) = 0$, then $\alpha$ is said to be clean. If $\rho(\alpha) = n > 0$, then $\alpha$ is said to be $n$-unclean (see Figure 1).

Remark 1. If the arc $\alpha$ is fixed by the monodromy, then $h(\alpha)$ can be isotoped to have interior disjoint from $\alpha$, so this is a special case of a clean, non-alternating arc.

Let $F'$ be the surface obtained by cutting $F$ along $\alpha$ and call the resulting (oriented) link $L' = \partial F'$. We now restate Theorem 1.

Theorem 1. The surface $F'$ is a fiber for $L'$ if and only if $i_{\mathrm{total}}(\alpha) = 1$ (that is, when $\alpha$ is clean and alternating, or once-unclean and non-alternating).

Consider the fiber $F$, and a small product neighborhood $n(F) = F \times I$. This is a trivial sutured manifold, $(n(F), \partial F)$. Let $D_-$ be the product disk $\alpha \times I$. Now, because $F$ is a fiber for $L$, the complementary sutured manifold $(M \setminus n(F), \partial F)$ is also trivial. Let $D_+$ be the product disk determined by $\alpha \subset F \times \{1\}$ and $h(\alpha) \subset F \times \{0\}$, properly embedded in $M \setminus n(F)$. As $D_-$ is a product disk for $(n(F), \partial F)$, we may decompose along this disk to get another trivial sutured manifold, namely $(n(F'), \partial F')$.

Recall that the manifold $n(F')$ was obtained by removing a small product neighborhood of $D_-$, say $[0,1] \times D_- = [0,1] \times (\alpha \times [0,1])$, from $n(F)$. Let $B$ be the ball $[-1,2] \times (\alpha \times [-1,1])$. Now, attach to $(n(F'), \partial F')$ the 1-handle $([-1,2] \times (\alpha \times [-1,0]))$, (attached along $([-1,0] \times (\alpha \times \{0\}))$ and $([1,2] \times (\alpha \times \{0\}))$. Call the resulting sutured manifold $(N_1, \partial F')$ (see Figure 2). We will refer to $(F' \times \{1\}) \subset \partial N_1$ as $\partial_- N_1$, and $\partial N_1 \setminus ((F' \times \{1\}) \cup (\partial F' \times I))$ as $\partial_+ N_1$.

Now, we can modify $D_+$ in a new disk $D^+_1$ in $\overline{M \setminus N_1}$ as follows.

1. Let $\overline{D}^+_1 = D^+_1 \setminus B$. Note that $h(\alpha)$ corresponds, through vertical projection in $B$, to arcs in $((\{-1\} \times (\alpha \times \{-0\})) \cup ([1,2] \times (\alpha \times \{-1\})) \cup (\{2\} \times (\alpha \times [-1,0])))$.

2. Extend the subarc $\alpha \times \{1\}$ of $D^+_1$ through vertical projection in $B$ to $\alpha \times \{0\}$.

Lemma 1. The co-core of the 1-handle is the unique non-separating disk in $N_1$ disjoint from the sutures. Furthermore, every separating disk in $N_1$ disjoint from the sutures, and every product disk in $(N_1, \partial F')$, can be made disjoint from the co-core.

Proof. Let $D$ be the co-core of the 1-handle, and suppose that $D'$ is either: (A) a compressing disk disjoint from $\partial F'$, or (B) a product disk for the sutured manifold. Suppose that $l$ is a loop of intersection innermost in $D$. Then $l$ bounds a subdisk $D$ of $D$ and a subdisk $D'$ of $D'$. These two disks co-bound a sphere, which then bounds a 3-ball because $N_1$ is irreducible. This sphere provides a means of isotoping $D'$ to $D$, which reduces the number of loops in $D \cap D'$. Thus, we may eliminate all such loops of intersections, and may suppose that $D \cap D'$ consists only of arcs. In this case, consider an arc $\gamma$ of $D \cap D'$ that is outermost in $D'$. Then $\gamma$ cuts off a subdisk $D'$ from $D$. If $D'$ is a product disk (case (B) above), then $D'$ cannot contain just one point of intersection with the sutures (as the sutures are separating in $\partial N_1$),
and if $\tilde{D}'$ contained both sutures, then $\gamma$ also cuts off another subdisk from $D'$, and we could instead take an arc of $D \cap D'$ that is outermost in this subdisk. So we may assume also that $\tilde{D}'$ contains no points of intersection with the sutures.

Now, let $D_1$ and $D_2$ be the disks $n(D) \cap N_1 \setminus n(D)$. By definition of $N_1$ and $D$, $(N_1 \setminus n(D), \partial F')$ is homeomorphic to the product sutured manifold $(F' \times [0,1], \partial F')$, so we will identify $N_1 \setminus n(D)$ with $F' \times [0,1]$ so that $D_1$ and $D_2$ are contained in $F' \times \{0\}$. Since $D'$ is disjoint from the sutures, $\tilde{D}'$ is boundary parallel in $N_1 \setminus n(D)$, so there exists a disk $\tilde{D}'$ in $\partial(N_1 \setminus n(D))$ so that $\partial \tilde{D}' = \partial D'$, and the 2-sphere $\tilde{D}' \cup \tilde{D}'$ bounds a 3-ball in $N_1 \setminus n(D)$. Without loss of generality, we may assume that $\partial \tilde{D}'$ is disjoint from $D_2$ (and intersects $D_1$ in a single arc).

If $\tilde{D}'$ does not contain $D_2$, then we can isotope $\tilde{D}'$ in $N_1$ along the 3-ball bounded by $\tilde{D}' \cup \tilde{D}'$ to remove the intersection $\gamma$, together with any other intersections between $D$ and $D'$ that give rise to disk components of $D' \setminus D$ whose boundaries have subarcs contained within $\tilde{D}'$. So suppose that $\tilde{D}'$ does contain $D_2$. Consider the arc components of $\partial D' \cap N_1 \setminus n(D)$, whose endpoints will be contained in $\partial D_1 \cup \partial D_2$. For each of $i = 1$ and $i = 2$, let $n_i$ be the number of arcs with both endpoints contained in $\partial D_i$. Since the numbers of endpoints in $D_1$ must be the same as the number of endpoints in $D_2$, we must have $n_1 = n_2$. Further, the existence of $\tilde{D}'$ implies that $n_1 > 0$, and so $n_2 > 0$. Hence, there exists an arc, $\eta$, of $\partial D' \cap N_1 \setminus n(D)$ with both endpoints contained in $\partial D_2$. The arc $\eta$ must be contained in $\tilde{D}'$, so it must cut off a subdisk $D_\eta$ from the annulus $D' \setminus D_2$. By passing to a subarc of $\partial D' \cap N_1 \setminus n(D)$ that is outermost in $D_\omega$, if necessary, then we may assume that $D_\eta$ contains no other subarcs of $\partial D'$. Then, in $N_1$, $\partial D'$ can be slid along $D_\eta$, which extends to an isotopy of $D'$, so as to reduce the number of arc components of $D \cap D'$.

Thus, in all cases, all arcs of intersection between $D$ and $D'$ may be removed by isotopy of $D'$, and we may assume now that $D' \cap D = \emptyset$. Hence, we may isotope $D'$ completely off the 1-handle. This establishes the last statement. Now, since $D'$ cannot be a compressing disk for $F' \times \{0\}$ in $F' \times I$, it must be a boundary parallel to a disk $D_\partial$ in $F' \times \{0\}$, and $\partial D'$ must be essential in $\partial N_1$. Hence $D_\partial$ contains one or both of $D_1$ and $D_2$. If $D'$ is non-separating in $N_1$,
then $D_0$ must contain only one of them, and $D'$ can be seen to be isotopic to $D$, establishing the first statement.

Now, we attach a 2-handle to $N_1$ along a neighborhood of $\partial D'_+$. Call the resulting sutured manifold $(N_2, \partial F')$, and keep track of the attaching annulus, $A = n(\partial D'_+)$, on the one hand thought of as contained in the boundary of $N_1$, and on the other hand considered to be properly embedded in $N_2$.

Recall that $D_+$ was a product disk for the trivial sutured manifold $(M\setminus n(F), \partial F)$, which is homeomorphic to $N_1$. So from the perspective of $(M\setminus n(F), \partial F)$, attaching the 2-handle to $N_1$ along $\partial D'_+$ results in the same manifold as decomposing $(M\setminus n(F), \partial F)$ along $D_+$. In other words, attaching a 2-handle to one submanifold is equivalent to compressing along a disk in the complement. Furthermore, one of the subarcs of the sutures in $(M\setminus N_2, \partial F')$ corresponding to $\alpha$ can be slid along $D'_+$, so that the sutures in $(M\setminus N_2, \partial F')$ agree with the result of this decomposition. Therefore, as a sutured manifold, $(M\setminus N_2, \partial F')$ is the same as the result of decomposing $(M\setminus n(F), \partial F)$ along the product disk $D_+$, and is thus a trivial sutured manifold.

We conclude then that $F'$ is a fiber in a fibration for $L'$ if and only if $(N_2, \partial F')$ is trivial.

**Remark 2.** We remind the reader that $(N_1, \partial F')$ is simply constructed from $(n(F'), \partial F')$ by attaching a 1-handle along $F' \times \{0\}$, and that $(N_2, \partial F')$ is constructed by attaching a 2-handle to $(N_1, \partial F')$.

**Lemma 2.** If $(N_2, \partial F')$ is trivial, then the co-core of the 1-handle in the construction of $(N_1, \partial F')$ intersects the boundary of $D'_+$ exactly once.

**Proof.** Begin with a maximal collection $\mathcal{D}$ of product disks for $(N_1, \partial F')$ that are disjoint from $\partial D'_+$. By Lemma 1, these disks can also be taken disjoint from the 1-handle of $N_1$. Since these will also be product disks for $(N_2, \partial F')$, the result of attaching the 2-handle along $\partial F'$ after decomposing along all the disks of $\mathcal{D}$ will be trivial if and only if $(N_2, \partial F')$ is trivial. Further, the result of decomposing $(N_1, \partial F')$ along the collection $\mathcal{D}$ is still a surface cross an interval, with a 1-handle attached. Thus, without loss of generality, we may assume that the $(N_1, \partial F')$ has no product disks disjoint from $\partial D'_+$.

Now recall that $(N_2, \partial F')$ is trivial if and only if $F'$ is a fiber in a fibration, which means that $M\setminus n(F')$ is homeomorphic to $F' \times I$. But $\partial D'_+$ was essential in $\partial N_1$, since it arose from a product disk for $(M\setminus n(F), \partial F)$. So observe that $D'_+$ cannot be isotoped off the 1-handle, or else $D'_+$ would be a compressing disk for $F' \times \{0\}$ in $M\setminus n(F')$, which is not possible.

If $F'$ is a disk, then $(N_1, \partial F')$ is a solid torus. In this case, $\partial D'_+$ must intersect the co-core of the 1-handle exactly once, else $(N_2, \partial F')$ would be a punctured lens space, and not a trivial sutured manifold.

Let us then assume that $F'$ is not a disk. It follows that there is an essential product disk in $(N_2, \partial F')$. A product disk intersects the sutures in two points, and hence by Theorem 6, there must be a compressing disk $D$ in $(N_1, \partial F')$ with boundary disjoint from $\partial D'_+$, and intersecting the sutures in at most two points. The disk $D$ cannot intersect the sutures in two points, or else it would be a product disk for $(N_1, \partial F')$ disjoint from $\partial D'_+$, contrary to the maximality condition of the initial collection of product disks. Further, it is not possible that $D$ intersects the sutures in just one point, since the sutures are separating in $\partial N_1$. Thus, $D$ is a compressing disk disjoint from $\partial D'_+$ and the sutures.

If $D$ were non-separating in $N_1$, then Lemma 1 says that $D$ would be the co-core of the 1-handle. But then, $\partial D'_+$, being disjoint from $D$, could be isotoped completely off the 1-handle, which we have already ruled out. Hence, $D$ is a separating disk. By Lemma 1,
$D$ may be assumed to be disjoint from the co-core of the 1-handle, and it must be that $\partial D$ is essential in $\partial_1 N_1$, but not in $F' \times \{0\}$. Hence, $D$ is parallel to a disk $D_0$ in $F' \times \{0\}$, which must contain both feet of the 1-handle.

In this case, the region between $D$ and $D_0$, together with the 1-handle, forms a solid torus. Then $\partial D'_+$ cannot be made disjoint from the 1-handle, as noted above, and $\partial D'_+$ cannot intersect the co-core of the 1-handle more than once, lest there be a punctured lens space in a trivial sutured manifold. Hence, $\partial D'_+$ intersects the co-core of the 1-handle exactly once.

**Proof of Theorem 1.** If $F'$ is a fiber for $L'$, then $(N_2, \partial F')$ is trivial. Let $D$ be the co-core of the 1-handle of $N_1$. Then by Lemma 2, $|\partial D \cap \partial D'_+| = 1$. Since $\partial D'_+$ reflects the product disk $D_1$, and therefore the pattern of $\alpha$ and $h(\alpha)$ on $F$, this shows that $\alpha$ must be either alternating and clean, or non-alternating and once-unclean.

Conversely, we know that if $\alpha$ were alternating and clean, then $F'$ would be the fiber of a fibration for $L'$. Thus, it remains to show that if $\alpha$ is non-alternating and once-unclean, then $(N_2, \partial F')$ is trivial. This is shown by observing that, in this case, $\partial D$ and $\partial D'_+$ form a canceling pair. The sutured manifold $(N_2, \partial F')$ is the result of attaching the 1-handle with co-core of $D$ to $(F' \times I, \partial F')$, and then a 2-handle along $\partial D'_+$. As these are canceling handles, disjoint from the sutures, this is equivalent to doing neither, so that $(N_2, \partial F') \cong (F' \times I, \partial F')$, which is clearly a product sutured manifold. This completes the proof of Theorem 1.

**Remark 3.** Observe that if $F'$ is not a fiber surface, this does not necessarily imply that $L'$ is not fibered. It is possible that $L'$ fibers with a different surface as a fiber. We combine our results with those of Kobayashi to address this question when the manifold $M$ is a rational homology 3-sphere in Section 4.

4. Characterization of band surgeries on fibered links

In this section, we will characterize band surgeries. Throughout this section, $L$ and $L'$ are oriented links in a manifold $M$ related by a coherent band surgery along a band $b$. More precisely, $b$ is an embedding $[0,1] \times [0,1] \to M$ such that $b^{-1}(L) = [0,1] \times \{0,1\}$, $b^{-1}(L') = \{0,1\} \times [0,1]$, and $L$ and $L'$ are the same as oriented sets, except on $b([0,1] \times \{0,1\})$. For simplicity, we use the same symbol $b$ to denote the image $b([0,1] \times \{0,1\})$. Since the numbers of components of $L$ and $L'$ differ by 1, the Euler characteristics of $L$ and $L'$ will never be equal, say $\chi(L') > \chi(L)$. By [5, 15, 28], there exists a taut Seifert surface $F$ for $L$ such that $F$ contains $b$.

**Theorem 7 ([5, 15, 28]).** Suppose that $M$ is $S^3$. Then, $\chi(L') > \chi(L)$ if and only if $L$ has a taut Seifert surface $F$ containing $b$.

Suppose that $L$ is a fibered link. Then $F$ is a fiber surface for $L$, and the band $b$ is contained in $F$. For some $a \in (0,1)$, we call $\alpha := b(\{a\} \times [0,1])$ a spanning arc of the band. The surface $F'$, which is obtained by cutting $F$ along $\alpha$, can be regarded as a Seifert surface for $L'$ by moving $F'$ slightly along $b$. Note that $\alpha$ is fixed by the monodromy of $F$ if and only if $F'$ is a split union of two fiber surfaces; that is, $L$ is a connected sum of the components of the split link $L'$. Kobayashi characterized band surgeries in the case where $\chi(L') > \chi(L) + 1$. By Kobayashi [20] and Yamamoto [32], we have the following theorem.

**Theorem 8.** Suppose $M$ is $S^3$. Then the following conditions are equivalent.

1. $\chi(L') > \chi(L) + 1$;
2. $F'$ is a pre-fiber surface;
(3) there exists a disk $D$ such that the intersection of $D$ and $F$ is a disjoint union of $\partial D$ and $\alpha$, and $\partial D$ is essential in $F$ (hence Stallings twists of type $(0,1)$ can be performed);
(4) $\alpha$ is clean and non-alternating, but not fixed by the monodromy.

Here, a pre-fiber surface is defined as a connected surface in a 3-manifold $M$ such that the complementary sutured manifold has pairwise disjoint compressing disks $D^+, D^-$ for the $(+)$-side and $(-)$-side of the surface, respectively, and the result of compressing along $D^+$ and $D^-$ is homeomorphic to a product sutured manifold; see [20] for details.

Moreover, Kobayashi showed the following theorem.

**Theorem 9** ([20]). Suppose that $F$ is a fiber surface and $F'$ is a pre-fiber surface. Then the band $b$ is type $F'$ with respect to $F'$.

See [21] for the definition of type $F$. Kobayashi also characterized pre-fiber surfaces for fibered links in [20, Theorem 3] and for split links in [21, Theorem 3]. In particular, together with Theorem 8 and [20, Theorem 3], Theorem 9 gives a complete characterization of band surgeries between fibered links $L$ and $L'$ with $\chi(L') > \chi(L) + 1$.

**Proof of Theorem 8.** (1)$\Rightarrow$(2) See Kobayashi [20, Theorem 2.1].
(2)$\Rightarrow$(1) If $F'$ is a pre-fiber surface, then, by the definition of pre-fiber surfaces, it is compressible, and so $\chi(L') > \chi(F') = \chi(F) + 1 = \chi(L) + 1$.
(2)$\Leftrightarrow$(3) This follows from Theorem 9 and the definition of type $F$.
(3)$\Leftrightarrow$(4) See Yamamoto [32, Lemma 3.4].
(4)$\Rightarrow$(2) See Kobayashi [20, Proposition 4.5].

For the remaining case, we will characterize band surgeries between $L$ and $L'$ with $\chi(L') = \chi(L) + 1$. In this case, $F$ is a fiber surface. By Theorem 1, $F'$ is then a fiber surface for $L'$ if and only if $\alpha$ is clean and alternating, or once-unclean and non-alternating. We will translate these conditions on the arc $\alpha$ into conditions on the band $b$.

### 4.1. Hopf banding and generalized Hopf banding

First, we show that if a spanning arc of a band surgery is a clean alternating arc, then the band surgery corresponds to a Hopf plumbing. If $F$ is obtained by plumbing of a surface $F''$ and a Hopf annulus $A$, then $F$ is obtained by attaching a band $A \setminus F''$ to $F''$, and so we call $F$ a Hopf banding of $F''$ along $A \setminus F''$. While known for some time, proofs of the following theorem can be found in Sakuma [26], or Coward and Lackenby [4, Theorem 2.3].

**Theorem 10.** Suppose that $F$ is a fiber surface. Then $F$ is a Hopf banding of $F'$ if and only if $\alpha$ is clean and alternating. Additionally, the Hopf annulus of the Hopf banding has a right-handed twist or left-handed twist, depending on whether $i_\partial(\alpha) = 1$ or $i_\partial(\alpha) = -1$, respectively (see Figure 3).

Next we introduce ‘generalized Hopf bandings’, which correspond to once-unclean non-alternating arcs.

**Definition.** Let $\ell$ be an arc in $F'$ such that $\ell$ has a single self-intersection point and $\ell \cap \partial F' = \partial \ell$. Let $b$ be a once-overlapped band over $F'$ such that $b([0,1] \times \{1\})$ is parallel to $\ell$; see Figure 4. If the surface $F$ is obtained by attaching $b$ to $F'$, then we call $F$ a generalized Hopf banding of $F'$ along $b$. 


EXAMPLE. By generalized Hopf banding of a Hopf annulus, we can obtain two different 3-component fibered links (see Figure 5).

Note that, for each arc in $F'$ having a self-intersection point, we have two choices of generalized bandings, depending on the overlapped sides. Moreover, any Hopf banding is a generalized Hopf banding for $\ell$ whose self-intersection point is removable by a homotopy in $F'$. Say $F$ is a strictly generalized Hopf banding of $F'$ along $b$ if $F$ is not a Hopf banding of $F'$ along $b$, but a generalized Hopf banding of $F'$ along $b$. Then we have the following theorem.
THEOREM 11. Suppose that $F$ is a fiber surface. Then $F$ is a strictly generalized Hopf banding of $F'$ along $b$ if and only if a spanning arc $\alpha$ of $b$ is once-unclean and non-alternating.

Proof. Suppose that $F$ is a generalized Hopf banding of $F'$ along a band $b$. Let $b': [0, 1] \times [0, 1] \to F$ be a projection of $b: [0, 1] \times [0, 1] \to M$ into $F'$ and put $I_i := [(i - 1)/5, i/5]$ for $i \in \{1, 2, 3, 4, 5\}$. We may assume that $b'|_{I_4 \times [0, 1]}((4 - t)/5, s)$ for $(s, t) \in [0, 1] \times [0, 1]$, and the self-intersection of $\ell$ is $b'(\frac{3}{10}, \frac{1}{2})$. We also assume that $b(\frac{7}{10}, \frac{1}{2})$ is over $b(\frac{7}{10}, \frac{1}{2})$. Let $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ be arcs in $F$ ($\beta_1, \beta_3, \beta_5 \subset b([0, 1] \times [0, 1])$ and $\beta_2, \beta_4 \subset F'$) defined by the following (see Figure 4):

$$
\begin{align*}
\beta_1 &:= \left\{ b\left(\frac{3 - 3s}{10}, \frac{3 - s}{3}\right) \mid 0 \leq s \leq 1 \right\}, \\
\beta_2 &:= \left\{ b'\left(\frac{3s}{10}, \frac{2 + s}{3}\right) \mid 0 \leq s \leq 1 \right\} \cup \left\{ b'\left(\frac{3 - 3s}{5}, \frac{1}{2}\right) \mid 0 \leq s \leq 1 \right\}, \\
\beta_3 &:= \left\{ b\left(s, \frac{1}{2}\right) \mid 0 \leq s \leq 1 \right\} = b\left([0, 1] \times \left\{\frac{1}{2}\right\}\right), \\
\beta_4 &:= \left\{ b'\left(\frac{5 - s}{5}, \frac{1}{2}\right) \mid 0 \leq s \leq 1 \right\} \cup \left\{ b'\left(\frac{3 - 3s}{10}, \frac{s}{3}\right) \mid 0 \leq s \leq 1 \right\}, \\
\beta_5 &:= \left\{ b\left(\frac{3s}{10}, \frac{1 - s}{3}\right) \mid 0 \leq s \leq 1 \right\}.
\end{align*}
$$

Set $\alpha := b\left(\left\{\frac{3}{10}\right\}, [0, 1]\right)$ and $\beta := \beta_1 \cup \beta_2 \cup \beta_3 \cup \beta_4 \cup \beta_5$. Then $h(\alpha)$ is isotopic to $\beta$ in $F$, since $\beta$ is a proper arc in $F$ with $\partial \beta = \partial \alpha$ and $\alpha \cup \beta$ bounds a disk in the complement of $F$. The endpoints of $\beta$ emanate to the same side of $\alpha$ and $\text{int}(\alpha) \cap \text{int}(\beta) = b(\frac{3}{10}, \frac{1}{2})$. Now $b$ is not the band of a Hopf banding, and so $\alpha$ is not clean alternating by Theorem 10. Therefore, $\alpha$ is once-unclean and non-alternating.
Suppose now that $\alpha$ is once-unclean and non-alternating. Set $\beta := h(\alpha)$. Then $\beta$ is divided into five arcs $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ by cutting along $b([0,1] \times [0,1])$ so that $\beta_i$ connects $\beta_{i-1}$ and $\beta_{i+1}$ for $i \in \{1, 2, 3, 4, 5\}$, $\beta_0 = \beta_5 = \alpha$. We may assume that $\beta_1, \beta_3, \beta_5$ are represented as above. Set $\ell' := \beta_2 \cup \{b(0, (1 + s)/3) \mid 0 \leq s \leq 1\} \cup \beta_4$. The arc $\ell'$ attaches to $\partial F'$ at $\{b(0, (1 + s)/3) \mid 0 \leq s \leq 1\}$. We have an arc $\ell$ with a single self-intersection point by moving $\ell'$ slightly into the interior of $F'$. Since $\beta = h(\alpha)$, $\alpha \cup \beta$ bounds a disk in the complement of $F$. This disk shows that the core $b([0,1] \times \{\frac{1}{2}\})$ of band $b$ is projected into $\ell$ in $F'$. Then the fact that $F$ is a generalized Hopf banding of $F'$ for $\ell$, and not a Hopf banding follows from Theorem 10.

4.2. Generalized Hopf banding for fiber surfaces

It is well known that a Hopf banding is a fiber surface if and only if the original surface is a fiber surface. In general, a resulting surface of a Murasugi sum is a fiber surface if and only if the summands are both fiber surfaces [8, 9]. We have a similar result for generalized Hopf bandings.

**Theorem 12.** Suppose that $F$ and $F'$ are surfaces such that $F$ is a generalized Hopf banding of $F'$. Then $F$ is a fiber surface if and only if $F'$ is a fiber surface.

**Proof.** One direction follows from Theorems 1, 10, and 11.

We will show that if $F'$ is a fiber surface, then the complementary sutured manifold $(\overline{M \setminus n(F')}, \partial F')$ is trivial, and so $F$ is a fiber surface. As in the proof of Theorem 11, $\alpha \cup \beta$ bounds a disk in the complement of $F$. From the disk, we have the product disk $D$ for $(\overline{M \setminus n(F')}, \partial F')$. Note that $n(F)$ is obtained from $n(F')$ by attaching a 1-handle $n(b)$. Since int($\alpha$) intersects int($\beta$) at a point, and $\beta$ emanates away from $\alpha$ in the same direction at both endpoints of $\alpha$, $\partial D$ intersects a co-core of the 1-handle at a point, and so $D$ cancels the 1-handle. Then $(\overline{M \setminus n(F)}, \partial F)$ is decomposed into $(\overline{M \setminus n(F')}, \partial F')$ by $D$, that is,

$$(\overline{M \setminus n(F)}, \partial F) \xrightarrow{D} (\overline{M \setminus n(F')}, \partial F').$$

Since $F'$ is a fiber surface, $(\overline{M \setminus n(F')}, \partial F')$ is a trivial sutured manifold, and so $(\overline{M \setminus n(F)}, \partial F)$ is also trivial. Hence $F$ is a fiber surface.

By Theorems 1, 10, 11, and 12, we have the following theorem.

**Theorem 13.**

1. Suppose that $F$ is a fiber surface, and $b$ is a band in $F$ such that $b \cap \partial F = b([0,1] \times \{0,1\})$. Set $F' := \overline{F \setminus b}$. Then $F'$ is a fiber surface if and only if $F$ is a generalized Hopf banding of $F'$ along $b$.

2. Suppose that $F'$ is a fiber surface and $b$ is a band attached to $F'$, that is, $b \cap F' = b([0,1] \times [0,1]) \subset \partial F'$. Set $F := F' \cup b$. Then, $F$ is a fiber surface if and only if $F$ is a generalized Hopf banding of $F'$ along $b$.

Theorem 7 implies that any coherent band surgery on links in $S^3$ can be regarded as an operation of cutting a taut Seifert surface along the band by considering the band surgery as an operation from one of the links to the other. Then as a translation of Theorem 13, we have proved Theorem 2.
Theorem 2. Suppose that $L$ and $L'$ are links in $S^3$, and $L'$ is obtained from $L$ by a coherent band surgery with $\chi(L') = \chi(L) + 1$.

(1) Suppose that $L$ is a fibered link. Then $L'$ is a fibered link if and only if the fiber $F$ for $L$ is a generalized Hopf banding of a (taut) Seifert surface $F'$ for $L'$ along $b$.

(2) Suppose that $L'$ is a fibered link. Then $L$ is a fibered link if and only if a (taut) Seifert surface $F$ for $L$ is a generalized Hopf banding of the fiber $F'$ for $L'$ along $b$.

It is well known that any automorphism of a surface can be represented by a composition of Dehn twists. Let $F$ be a fiber surface with monodromy $h$. Honda et al. [16] showed the following lemma.

Lemma 3 (16, Lemma 2.5). Suppose that $h$ is a composition of right-hand Dehn twists along circles in $F$. Then $h$ is right-veering. In other words, $i_\partial(\alpha) = 1$ if $h(\alpha)$ is not isotopic to $\alpha$.

Here $h(\alpha)$ is isotopic to $\alpha$ if and only if there exists a 2-sphere $S$ such that $S \cap F = \alpha$. If we assume additionally that $\partial F$ is prime, then any essential arc in $F$ is alternating. We will discuss the case specifically where $\partial F$ is composite in Subsection 4.4.

Suppose a fiber surface $F$ with monodromy $h$ is obtained by plumbing of two surfaces $F_1$ and $F_2$, where $F_1$ is a Hopf annulus with right-hand twist. Let $C$ be a core circle of $F_1$. We denote by $t_C$ the right-hand Dehn twist along $C$. Then $(t_C^{-1} \circ h)|_{F_2}$ is isotopic to the monodromy for $F_2$. Hence if $F$ is obtained from a disk in $S^3$ by successively plumbing Hopf annuli with right-hand twists, then $h$ is a composition of right-hand Dehn twists. By Theorem 2 (1) and Lemma 3, we have the following corollary.

Corollary 1. Let $L$ be an oriented link in $S^3$ with fiber $F$ such that $F$ is obtained from a disk by successively plumbing Hopf annuli with right-hand twists (or by successively plumbing Hopf annuli with left-hand twists). Suppose that $L'$ is a link obtained from $L$ by a coherent band surgery and $\chi(L') = \chi(L) + 1$. Then $L'$ is a fibered link if and only if $F$ is a Hopf banding of a Seifert surface $F'$ for $L'$ along $b$.

Remark 4. Baader and Dehornoy recently proved a similar result in [1].

4.3. Band surgeries on $(2,p)$-torus link

Let $D_1$ and $D_2$ be disjoint disks in a plane. Let $b_1, \ldots, b_p$ be pairwise disjoint bands with left-hand half-twist connecting the two disks. Set $F := D_1 \cup D_2 \cup b_1 \cup \cdots \cup b_p$. Then $F$ is a fiber surface for the $(2,p)$-torus link $T(2,p)$ (with parallel orientation if $p$ is even; see Figure 6). Let
$b$ be a band in $F$ and set $F' := \overline{F \setminus b}$, $L' := \partial F'$. Since $F$ is obtained from $D_1 \cup D_2 \cup b_1 \cup \cdots \cup b_{p-1}$ by plumbing a Hopf annulus with a left-hand twist along $b_{p-1}$ (or its spanning arc), $F$ is obtained from a disk $D_1 \cup D_2 \cup b_1$ by successively plumbing $(p-1)$ Hopf annuli with left-hand twists. Then, by Corollary 1, $L'$ is fibered if and only if $F$ is a Hopf banding of $F'$ along $b$.

**Corollary 2.** Suppose that $L'$ is obtained from $L = T(2,p)$ by a coherent band surgery along $b$, where $p \geq 2$, and $\chi(L') > \chi(L)$. Then $L'$ is fibered if and only if the band $b$ can be moved into $D_1$ (and also $D_2$) so that $F \setminus b$ is connected by an isotopy fixing $L$ as a set. In particular, $L'$ is $T(2,p-1)$ if it is a prime fibered link, and $L'$ is a connected sum $T(2,p_1) \# T(2,p_2)$ of $T(2,p_1)$ and $T(2,p_2)$ if it is a composite fibered link, where $p_1$ and $p_2$ are positive integers with $p_1 + p_2 > 1$ and $p_1 + p_2 = p$. Moreover, the band is unique up to isotopy fixing $L$ as a set if $L' = T(2,p-1)$ or $T(2,p_1) \# T(2,p_2)$ and either $p_1$ or $p_2$ is odd, and there are two bands up to isotopy fixing $L$ as a set if both $p_1$ and $p_2$ are even ($L'$ is a 3-component link), but they are the same up to homeomorphism.

**Remark 5.** By Murasugi [22], $|\sigma(L) - \sigma(L')| \leq 1$ for two links $L, L'$ which are related by a coherent band surgery, where $\sigma$ means the signature. Since $\chi(T(2,p)) = 2 - p$ and $\sigma(T(2,p)) = 1 - p$, it follows that $\chi(L) + 1 = \sigma(L) + 2 \geq \sigma(L') + 1$ if $L = T(2,p)$. Additionally, $\sigma(L') + 1 \geq 1 - b_1(F') = \chi(L')$ for a taut Seifert surface $F'$ for $L'$, where $b_1$ means the first Betti number. Then the assumption $\chi(L') > \chi(L)$ in Corollary 2 becomes $\chi(L') = \chi(L) + 1$. Note that we can regard $T(2,p-1)$ as $T(2,p_1) \# T(2,p_2)$ for $p_1 = p-1$ and $p_2 = 1$, since $T(2,1)$ is trivial.

**Proof.** Suppose that $b$ is contained in $F$, disjoint from $b_1, \ldots, b_p$, and does not separate $F$. We will prove that $F'$ is fibered, and that the band is unique up to the operations mentioned. Say $b$ is contained in $D_1$, and $b$ splits $D_1$ into two disks with $p_1$ bands of $b_1, \ldots, b_p$ ($i = 1, 2$), where $p_1$ and $p_2$ are positive integers with $p_1 + p_2 = p$. Then $L'$ is a connected sum $T(2,p_1) \# T(2,p_2)$ of $T(2,p_1)$ and $T(2,p_2)$ which is a fibered link. Moreover, two such bands in $F$ are related by the monodromy, and sliding along $\partial F$ if the two bands are attached to the same component of $L$. This implies that the band is unique up to isotopies fixing $L$ as a set if either $p_1$ or $p_2$ is odd. If the two bands are attached to different components of $L$, then they are related by the monodromy, sliding along $\partial F$, and an involution. Here we can take a rotation about the horizontal axis in Figure 6 as the involution, so that $D_1$ is mapped to $D_2$, $D_2$ is mapped to $D_1$, and $b_i$ is mapped to itself. This implies that the two bands are the same up to homeomorphism.

Conversely, let $\alpha$ be a clean and alternating arc in $F$. We will show that $\alpha$ can be moved into $D_1$ or $D_2$ so that $\alpha$ is disjoint from $b_1, \ldots, b_p$. This will show that any band producing a fibered link $L'$ can be moved into $D_1$ or $D_2$ by Corollary 1 and Theorem 1. We arrange the bands $b_1, \ldots, b_p$ along an orientation of $\partial D_1$ (or $\partial D_2$). For each $i \in \{1, 2\}$ and $j \in \{1, \ldots, p\}$, let $\delta_{ij} = \partial D_i \cap \partial b_j$ be an arc with the orientation induced by that of $D_j$. Note that $\delta_{ij}$ and $\delta_{ij}$ are isotopic to each other in $F$, but having opposite orientations. It is well known that the monodromy $h$ of $F$ is represented by $t_1 \circ t_2 \circ \cdots \circ t_{p-1}$, where $t_i$ is a Dehn twist along a loop in $F$ passing only once through each of $b_i, D_1, b_{i+1}$, and $D_2$. Then we can see that $h(\delta_{1j})$ is isotopic to $\delta_{2(j+1)}$ (similarly, $h(\delta_{2j})$ is isotopic to $\delta_{1(j+1)}$). The orientation, by sliding along $\partial F$. Let $\hat{h}$ be an automorphism of $F$ such that $\hat{h}(D_1) = D_2, \hat{h}(D_2) = D_1$, and $\hat{h}(b_i) = b_{i+1} (\text{mod } p)$. Then $\hat{h}$ is obtained from $h$ by sliding along $\partial F$. Since $h$ and $\hat{h}(\alpha)$ intersect only at their endpoints with positive signs, $\alpha$ is disjoint from $\hat{h}(\alpha)$. We may assume that $\alpha$ minimizes intersections with $\text{int}(b_1 \cup \cdots \cup b_p)$, and $\partial \alpha$ consists of two points of $(\partial \delta_{11} \cup \cdots \cup \partial \delta_{1p}) \cup (\partial \delta_{21} \cup \cdots \cup \partial \delta_{2p})$. For a contradiction, suppose that $\alpha$ intersects $\text{int}(b_1 \cup \cdots \cup b_p)$. 


Then $\alpha$ is divided into arcs by cutting $F$ along $b_1 \cup \cdots \cup b_p$. Let $\alpha_1, \alpha_2$ be successive such arcs in $D_1, D_2$, respectively, and define the following (see Figure 7):

1. $\partial \alpha_1 = \{x, y\}$, where $x$ is a point in $\delta_i$, and $y$ is a point in $\delta_j$;
2. $\partial \alpha_2 = \{z, w\}$, where $z$ is a point in $\delta_{2j}$, and $w$ is a point in $\delta_{2k}$;
3. A component of $\alpha \cap b_j$ connects $y$ and $z$ in $b_j$.

Set $\beta_1 := \hat{h}(\alpha_1), \beta_2 := \hat{h}(\alpha_2), x' := \hat{h}(x), y' := \hat{h}(y), z' := \hat{h}(z), w' := \hat{h}(w)$. Then $x', y', z', w'$ are points in $\delta_{2(i+1)}$, $\delta_{2(j+1)}$, $\delta_{1(k+1)}$, $\delta_{1(k+1)}$, respectively.

First, we show that $j - i \equiv \pm 1$ (mod $p$) or $k - j \equiv \pm 1$ (mod $p$). Suppose $k - j \neq \pm 1$ (mod $p$). Let $D'_1$ (respectively, $D'_2$) be a disk cut off from $D_1$ by $\beta_2$ (respectively, $D_2$ by $\alpha_2$), where $\partial D'_1 \cap (b_1 \cup \cdots \cup b_p) = (\partial D'_1 \cap \delta_{1(k+1)}) \cup (\delta_{1(k+2)} \cup \cdots \cup \delta_{1j}) \cup (\partial D'_1 \cap \delta_{1(j+1)})$ (respectively, $\partial D'_2 \cap (b_1 \cup \cdots \cup b_p) = (\partial D'_2 \cap \delta_{2j}) \cup (\delta_{2(j+1)} \cup \cdots \cup \delta_{2(k-1)}) \cup (\partial D'_2 \cap \delta_{1k})$). Since $\alpha_1$ is disjoint from $\beta_2$ in $D_1$, the two points $x$ and $y$ both lie in $\partial D'_1$, and so $i \equiv k + 1, k + 2, \ldots, j - 1, j + 1$ (mod $p$). On the other hand, since $\alpha_2$ is disjoint from $\beta_1$ in $D_2$, the two points $x'$ and $y'$ both lie in $\partial D'_2$, and so $i + 1 \equiv j, j + 2, j + 3, \ldots, k - 1, k$ (mod $p$). Then $j - i \equiv \pm 1$ (mod $p$).

Next we show that if $\alpha_1$ is outermost in $D_1$ and $j - i \equiv 1$ (mod $p$), then $\alpha_2$ is outermost in $D_2$ and $k - j \equiv 1$ (mod $p$). Similarly, if $\alpha_2$ is outermost in $D_2$ and $k - j \equiv -1$ (mod $p$), then $\alpha_1$ is outermost in $D_1$ and $j - i \equiv -1$ (mod $p$). Suppose that $\alpha_1$ is outermost in $D_1$ and $j - i \equiv 1$ (mod $p$). Then $\beta_1$ connects a point $x'$ in $\delta_{2(i+1)}$ and a point $y'$ in $\delta_{2(i+2)}$. Recall that $z$ is a point in $\delta_{2(i+1)}$. Since $\alpha_1$ is outermost in $D_1$, $z$ lies in the side of $\beta_1$ containing no arcs of the form $\delta_{2i}$, and so $\alpha_2$ is parallel to $\beta_1$ and is outermost in $D_2$.

Finally, we show that this results in a contradiction. Suppose that $\alpha$ has a subarc $\alpha'$ which is outermost in $D_1$ or $D_2$ and connecting two adjacent bands. By continuing the same argument above, we may assume that the outermost subarc $\alpha'$ of $\alpha$ is outermost in $D_1$ or $D_2$, say $D_2$, and $k - j \equiv 1$ (mod $p$). Since $\hat{h}(\alpha)$ passes through $b_{j+1}$, there exists a subarc $\ell = \ell_1 \cup \ell_2$ of $L$ such that $\ell_1$ and $\ell_2$ are components of $L \cap \partial b_j$ and $L \cap \partial D_2$, respectively, and $\ell_2 \cap b_{j+1} = \partial \alpha' \cap \partial D_{2(j+1)}$ is an endpoint of $\alpha$. Then an arc component of $\alpha \cap \text{int}(b_j)$ is removable by sliding $\alpha$ along $\ell$. In the case where $\hat{h}(\alpha)$ has a subarc which is outermost in $D_1$ or $D_2$ and connecting two adjacent bands, by the same argument, $\hat{h}(\alpha)$ (and likewise $\alpha$) has a removable intersection with $\text{int}(b_1 \cup \cdots \cup b_p)$. This contradicts the assumption that $\alpha$ minimizes intersections with $\text{int}(b_1 \cup \cdots \cup b_p)$. \hfill $\square$

4.4. Band surgeries on composite fibered links

We say that a fiber surface is prime (respectively, composite) if the boundary is a prime link (respectively, a composite link). Suppose $F$ is a composite fiber surface. There exists a 2-sphere $S$ intersecting $F$ in an arc, such that neither surface cut off from $F$ by the arc is a disk. The
resulting surfaces are both fiber surfaces for the summand links. In general, there exist pairwise disjoint 2-spheres $S_1, \ldots, S_m$ such that $\delta_i := S_i \cap F$ is an arc for each $i \in \{1, \ldots, m\}$, and each component of the surface obtained from $F$ by cutting along $\delta_1, \ldots, \delta_m$ is a prime fiber surface. We call a set $\{\delta_1, \ldots, \delta_m\}$ of such arcs a full prime decomposing system for $F$. We remark that if $m = 1$, a full prime decomposing system (an arc in this case) is unique up to isotopy in $F$. On the other hand, there exist several decomposing systems if $m \geq 2$, but the sets of surfaces obtained from $F$ by cutting along decomposing systems are always the same.

Suppose that a fiber surface $F$ with monodromy $h$ is divided into prime fiber surfaces $F_1, \ldots, F_{m+1}$ with monodromies $h_1, \ldots, h_{m+1}$, respectively, by a full prime decomposing system $\{\delta_1, \ldots, \delta_m\}$. Let $\alpha$ be a properly embedded arc in $F$ which intersects $\delta_1 \cup \cdots \cup \delta_m$, and has no removable intersections with $\delta_1 \cup \cdots \cup \delta_m$. The arc $\alpha$ is divided into subarcs $\alpha_1, \ldots, \alpha_n$ successively by cutting along $\delta_1, \ldots, \delta_m$, where $\alpha_i$ is a properly embedded arc in $F_{j_i}$ for each $i \in \{1, \ldots, n\}$ and $\{p_i\} = \alpha_i \cap \alpha_{i+1} \subset \partial \alpha_i \cap \partial \alpha_{i+1}$ for each $i \in \{1, \ldots, n-1\}$. Let $s_i, t_{i+1} = \pm 1$ be the signs at $p_i$ for a pair $(\alpha_i, h_{j_i}(\alpha_i))$ in $F_{j_i}$, and for a pair $(\alpha_i, h_{j_{i+1}}(\alpha_{i+1}))$ in $F_{j_{i+1}}$, respectively. Note that $i_{\rho}(\alpha_i, h_{j_i}(\alpha_i)) = (t_i + s_i)/2$ for each $i \in \{2, \ldots, n-1\}$; see [13]. Recall that $\rho(\alpha)$ is the geometric intersection number between $\alpha$ and $h(\alpha)$ in the interior of $F$, that is, $\rho(\alpha) = \sum_{p \in \alpha \cap h(\alpha) \cap \text{int}(F)} |i_p|$. Then we have the following lemma.

**Lemma 4.** The following inequality holds:

$$\rho(\alpha) \geq \sum_{i=1}^{n} \rho(\alpha_i) + \frac{1}{2} \sum_{i=1}^{n-1} |s_i + t_{i+1}|.$$ 

Here if $h_{j_i}(\alpha_i)$ is isotopic to $\alpha_i$ in $F_{j_i}$, then $(t_i, s_i) = (1, -1)$ or $(-1, 1)$, which minimizes $\sum_{i=1}^{n-1} |s_i + t_{i+1}|$.

**Proof.** Put $\beta := h(\alpha)$ so that $|\text{int}(\alpha) \cap \text{int}(\beta)| = \rho(\alpha)$. Then there exists a disk $D$ in $M$, possibly with self-intersection in the boundary, such that $D \cap F = \partial D = \alpha \cup \beta$. We analyze the intersection of $D$ and the pairwise disjoint spheres $S_1, \ldots, S_m$, where $S_i \cap F = \delta_i$ for each $i \in \{1, \ldots, m\}$. By cut and paste argument, we may assume that the intersection $D \cap (S_1 \cup \cdots \cup S_m)$ consists of arcs. Let $D'$ be an outermost disk cut off from $D$ by $D \cap S_i$ for some $i \in \{1, \ldots, m\}$. Suppose $\partial D' \cap \partial D \subset \alpha$ or $\partial D' \cap \partial D \subset \beta$. Since $F$ is incompressible, $\partial D' \cap \partial D$ is isotopic to $\partial \alpha$ in a subarc of $\delta_i$ joining the endpoints of the arc of $D \cap S_i$. Hence such an arc of intersection $D \cap S_i$ is removable keeping $|\text{int}(\alpha) \cap \text{int}(\beta)|$ constant. After removing such intersections, $D$ is divided into disks $D_1, \ldots, D_n$ by $D \cap (S_1 \cup \cdots \cup S_m)$, where $\partial D_i$ consists of $\alpha_i$, a subarc $\beta_i$ of $\beta$, and two parallel arcs (respectively, a single arc) of $D \cap (S_1 \cup \cdots \cup S_m)$ for each $i \in \{2, \ldots, n-1\}$ (respectively, $i \in \{1, n\}$; see Figure 8). This implies that $h_{j_i}(\alpha_i)$ is isotopic to $\beta_i$. By sliding $\beta_i$ along $\partial F_{j_i}$ in $F_{j_i}$ so that the endpoints of $\beta_i$ coincide with those of $\alpha_i$, we have $\rho(\alpha) = |\text{int}(\alpha) \cap \text{int}(\beta)| = \sum_{i=1}^{n} |\text{int}(\alpha_i) \cap \text{int}(\beta_i)| + \frac{1}{2} \sum_{i=1}^{n-1} |s_i + t_{i+1}| \geq \sum_{i=1}^{n} \rho(\alpha_i) + \frac{1}{2} \sum_{i=1}^{n-1} |s_i + t_{i+1}|$. 

Suppose, additionally, the monodromy $h_j$ of $F_j$ is a composition of all right-hand Dehn twists or, alternatively, all left-hand Dehn twists for each $j \in \{1, \ldots, m+1\}$. Put $\varepsilon_j = \pm$ as follows:

$$\varepsilon_j = \begin{cases} + & (h_j \text{ is a composition of right-hand Dehn twists}), \\ - & (h_j \text{ is a composition of left-hand Dehn twists}). \end{cases}$$

By Lemma 3, $\alpha_i$ is non-alternating $(t_i + s_i = 0)$ if and only if $h_{j_i}(\alpha_i)$ is isotopic to $\alpha_i$ in $F_{j_i}$, and if $\alpha_i$ is alternating $(t_i + s_i \neq 0)$, then $t_i = s_i = \varepsilon_j$. Since $F_{j_i}$ is prime, $h_{j_i}(\alpha_i)$ is isotopic to $\alpha_i$ in $F_{j_i}$ if and only if $\alpha_i$ is parallel to the boundary $\partial F_{j_i}$ in $F_{j_i}$. Partition the set $\{1, \ldots, n\}$ into $A$ and $B$ so that $\alpha_i$ is alternating if $i \in A$, and parallel to the boundary $\partial F_{j_i}$ in $F_{j_i}$ if $i \in B$. Then we have the following by Lemma 4.
THEOREM 14. (1) If the arc \( \alpha \) is clean and alternating, then the set \( A \) consists of an odd number of elements, \( \alpha_i \) is clean and alternating in \( F_j \), and \( \varepsilon_j \), appears as positive and negative alternately in ascending order for \( i \in A \).

(2) If the arc \( \alpha \) is once-unclean and non-alternating, then either:

2-1 the set \( A \) consists of an even number of elements, \( \alpha_i \) is clean and alternating in \( F_j \), except for at most one once-unclean alternating arc \( \varepsilon_j \), appears as positive and negative alternately in ascending order for \( i \in A \); or

2-2 the set \( A \) consists of an odd number of elements, \( \alpha_i \) is clean and alternating in \( F_j \), and \( \varepsilon_j \) appears as positive and negative alternately, except one successive pair in ascending order for \( i \in A \).

Proof. Suppose that \( i \in B \); then \( \rho(\alpha_i) = 0 \), and \((t_i, s_i)\) can be taken as \((-s_{i-1}, s_{i-1})\) (or \((t_{i+1}, -t_{i+1})\)) so that \( s_{i-1} \) and \( t_i \) (or \( s_i \) and \( t_{i+1} \)) cancel. Hence we can ignore the elements of \( B \) when we calculate the number \( \sum_{i=1}^{n} \rho(\alpha_i) + \frac{1}{2} \sum_{i=1}^{n-1} |s_i + t_{i+1}| \). We remark that \( i \rho(\alpha, h(\alpha)) = \varepsilon_{j_i} + \varepsilon_{j_i'} \), where \( i \) and \( i' \) are the first and the last elements of \( A \), respectively.

1) By the definition, the arc \( \alpha \) is clean and alternating if and only if \( \rho(\alpha) = 0 \) and \( i \rho(\alpha, h(\alpha)) = \pm 1 \). By Lemma 4, if \( \rho(\alpha) = 0 \), then \( \alpha_i \) is clean for each \( i \in A \), and \( \varepsilon_{j_i} + \varepsilon_{j_i'} = 0 \) for each pair of successive integers \( i \) and \( i' \) in \( A \). Then (1) of Theorem 14 holds.

2) By the definition, the arc \( \alpha \) is once-unclean and non-alternating if and only if \( \rho(\alpha) = 1 \) and \( i \rho(\alpha, h(\alpha)) = 0 \). By Lemma 4, if \( \rho(\alpha) = 1 \), either: (2-1) \( \alpha_i \) is clean for \( i \in A \), except at most one which is once-unclean, and \( \varepsilon_{j_i} + \varepsilon_{j_i'} = 0 \) for a pair of successive integers \( i \) and \( i' \) in \( A \); or (2-2) \( \alpha_i \) is clean for \( i \in A \), and \( \varepsilon_{j_i} + \varepsilon_{j_i'} = 0 \) for a pair of successive integers \( i \) and \( i' \) in \( A \), except for one pair. Then (2) of Theorem 14 holds.

The following corollary is derived from Theorem 14 by considering the case when \( \varepsilon_1 = \cdots = \varepsilon_{m+1} = + \) or \( \varepsilon_1 = \cdots = \varepsilon_{m+1} = - \).

COROLLARY 3. Suppose that a fiber surface \( F \) is composite, has monodromy which is a composition of all right-hand Dehn twists or, alternatively, all left-hand Dehn twists, and contains an arc \( \alpha \) in \( F \) that is clean and alternating. Then there exists a full prime decomposing system \( \{\delta_1, \ldots, \delta_m\} \) such that \( \alpha \) is disjoint from \( \delta_1 \cup \cdots \cup \delta_m \).

Proof. Let \( \{\delta_1, \ldots, \delta_m\} \) be a full prime decomposing system for \( F \), so that \( F \) is divided into fiber surfaces \( F_1, \ldots, F_{m+1} \) by \( \{\delta_1, \ldots, \delta_m\} \). Suppose that a clean alternating arc \( \alpha \) in \( F \) intersects \( \delta_1 \cup \cdots \cup \delta_m \), has no removable intersections, and is divided into subarcs \( \alpha_{m+1} \), \( \alpha_n \) \( (n \geq 2) \) successively by cutting along \( \delta_1, \ldots, \delta_m \), that is, \( |\alpha \cap (\delta_1 \cup \cdots \cup \delta_m)| = n - 1 \). Then the monodromy \( h_j \) of \( F_j \) is a composition of all right-hand Dehn twists or, alternatively, all left-hand Dehn twists according to whether that of \( F \) is a composition of all right-hand Dehn
twists or all left-hand Dehn twists, respectively. By Theorem 14(1), there exists \( k \in \{1, \ldots, n\} \) such that \( \alpha_k \) is clean and alternating, and any other arc \( \alpha_i \) \((i \in \{1, \ldots, n\} - \{k\})\) is parallel to \( \partial F_i \), in \( F_j \) (that is, the set \( A \) in Theorem 14(1) must be a singleton set \( \{k\} \) in this case).

Without loss of generality, we may assume that \( k \neq n, \alpha_{n-1} \) and \( \alpha_n \) are arcs in \( F_m \) and \( F_{m+1} \), respectively, and \( F_m \cap F_{m+1} = \delta_m \). Since \( \alpha_n \) is parallel to \( \partial F_{m+1} \) in \( F_{m+1} \), it follows that \( \alpha_n \) divides \( F_{m+1} \) into a disk \( D \) and a surface \( F'_{m+1} \) which is homeomorphic to \( F_{m+1} \), and divides \( \delta_m \) into \( a \) and \( b \), where \( a \subseteq \partial F'_{m+1} \) and \( b \subseteq \partial D \). Let \( \delta''_m \) be an arc obtained from \( \alpha_n \cup \alpha \) by pushing slightly into the interior of \( F''_{m+1} \). Then the set \( \{\delta_1, \ldots, \delta_{m-1}, \delta'_m\} \) is a new full prime decomposing system which divides \( F \) into \( F_1, \ldots, F_{m-1}, F'_m, F''_{m+1} \), where \( F'_m = F_m \cup D \), and \( |\alpha \cap (\delta_1 \cup \cdots \cup \delta_{m-1} \cup \delta''_m)| < n - 1 \). By continuing such operations, we obtain a full prime decomposing system for \( F \) which is disjoint from \( \alpha \).

Hence, in Corollary 1, if we assume that \( L \) is composite, then we can take decomposing spheres for \( L \) so that the band of a Hopf banding is disjoint from the decomposing spheres. Then we have the following from Corollary 2.

**Corollary 4.** Suppose that \( L' \) is obtained from \( L = T(2, p)\# T(2, q) \) by a coherent band surgery along \( b \) and \( \chi(L') > \chi(L) \), where \( p, q > 1 \). If \( L' \) is fibered, then \( L' \) is a connected sum \( T(2, p_1)\# T(2, p_2)\# T(2, q) \) or \( T(2, p)\# T(2, q_1)\# T(2, q_2) \), where \( p_1, p_2, q_1, q_2 \) are positive integers with \( p_1 + p_2 = p, q_1 + q_2 = q \). Moreover, for each \( L' = T(2, p_1)\# T(2, p_2)\# T(2, q) \) or \( T(2, p)\# T(2, q_1)\# T(2, q_2) \), the band is unique up to homeomorphisms.

### 5. Dehn surgeries along arc-loops

Let \( L \) be a fibered link in a manifold \( M \) with fiber \( F \), and let \( \alpha \) be an arc in \( F \). There exists a disk \( D \) in \( M \) such that \( D \cap F = \alpha \) and \( \partial D \) is disjoint from \( F \). We call \( c = \partial D \) an \( \alpha \)-loop, or generally an arc-loop. In this section, we will characterize Dehn surgeries along arc-loops preserving \( F \) as a fiber surface, using results of Ni [24] about surgeries on knots in trivial sutured manifolds. In Sections 6 and 7, we will use this information to characterize generalized crossing changes between fibered links and give an alternative proof of Theorem 1, respectively.

**Theorem 15** [24]. Suppose that \( F \) is a compact surface and that \( c \subseteq F \times I \) is a simple closed curve. Suppose that \( \gamma \) is a non-trivial slope on \( c \), and that \( N(\gamma) \) is the manifold obtained from \( F \times I \) via the \( \gamma \)-surgery on \( c \). If the pair \((N(\gamma), (\partial F) \times I)\) is homeomorphic to the pair \((F \times I, (\partial F) \times I)\), then one can isotope \( c \) such that its image on \( F \) under the natural projection \( p : F \times I \to F \) has either no crossing or exactly 1 crossing.

The slope can be determined as follows: Let \( \lambda_b \) be the frame specified by the surface \( F \). When the projection has no crossing, \( \gamma = 1/n \) for some integer \( n \) with respect to \( \lambda_b \); when the minimal projection has exactly 1 crossing, \( \gamma = \lambda_b \).

**Remark 6.** Conversely, the surgeries in the statement of Theorem 15 do not change the homeomorphism type of the pair \((F \times I, (\partial F) \times I)\).

Our first objective will be to relate such a loop \( c \) to an arc-loop.

**Definition.** In \( F \times [0, 1] \), a loop \( c \) is said to be in 1-bridge position (with respect to \( x_1, x_2 \)) if \( c \) is partitioned into arcs, \( \tau, \beta, \nu_1, \nu_2 \), where \( \tau \) is embedded in \( F \times \{ \frac{1}{2} \} \), \( \beta \) is embedded in \( F \times \{ \frac{2}{3} \} \), and \( \nu_i = \{ x_i \} \times [ \frac{1}{2}, \frac{2}{3} ] \) \((i = 1, 2)\).
We extend this definition to a loop $c$ in the complement of a fibered link $L$ in a manifold $M$ with fiber surface $F$ if $c$ is in 1-bridge position in the product structure of the complementary sutured manifold of $F$.

**Definition.** We will say that two loops $c$ and $c'$ in 1-bridge positions are 1-bridge isotopic if there is an isotopy from $c$ to $c'$ so that the curves are in 1-bridge positions throughout the transformation (where the points $x_1, x_2$ may change throughout).

Recall that $p : F \times I \to F$ is the natural projection map defined by $p(x, t) = x$. The 1-bridge crossing number of a loop $c$ in 1-bridge position, $bc_1(c)$, is the minimum number of crossings of $p(c')$ over all 1-bridge positions $c'$ that are 1-bridge isotopic to $c$. The minimum 1-bridge crossing number of a loop $c$ having 1-bridge positions, $mbc_1(c)$, is the minimum of the 1-bridge crossing number over all 1-bridge positions that are (not necessarily 1-bridge) isotopic to $c$. We will show that $mbc_1(c) = bc_1(c)$ for any loop $c$ in 1-bridge position.

**Lemma 5.** Let $c$ and $c'$ be loops in $F \times I$ in 1-bridge positions. If $c$ and $c'$ are isotopic in $F \times I$, then $c$ and $c'$ are 1-bridge isotopic. Hence $mbc_1(c) = bc_1(c) = bc_1(c')$.

**Proof.** First, by shrinking $\beta$ and sliding the $\nu_i$ along with the endpoints of $\beta$, we may assume that $\beta$ is a very short arc in $F \times \{\frac{1}{3}\}$. Observe that $\tau$ can be slid out of the way during this transformation, so that this operation is a 1-bridge isotopy. Let $p : F \times I \to F$ be the projection map defined by $p(x, t) = x$. Then observe further that the number of double points of $c$ under the map $p$ does not change during this transformation. Similarly, we may shrink $\beta'$, and then translate $\beta'$ through $F \times \{\frac{1}{3}\}$ via 1-bridge isotopy so that $\beta = \beta'$ (and therefore also so that $\nu_i = \nu'_i$ for $i = 1, 2$).

Let $s = (x_1, \frac{1}{3})$, one of the endpoints of $\tau$. The projection map induces an isomorphism on fundamental groups, so that $\pi_1(F \times I, s) \cong \pi_1(F, x_1)$ via $p_x$. Then, since $c$ and $c'$ are isotopic in $F \times I$, we have $[c]_{F \times I} = \ell^{-1} * [c']_{F \times I} * \ell$ for some word $\ell \in \pi_1(F \times I, s)$. In fact, up to homotopy in $\pi_1(F \times I, s)$, we can take $\ell$ to be a loop in $F \times \{\frac{1}{3}\}$, based at $s$, containing the arc parallel to $\beta$ in $F \times \{\frac{1}{3}\}$ as a subarc, and never intersecting the arc parallel to $\beta$ in $F \times \{\frac{1}{3}\}$.

We now perform a 1-bridge isotopy of $c'$ by dragging $\beta' \cup \nu'_1 \cup \nu'_2$ along $\ell$. Any time $\ell$ intersects $\tau'$, move $\tau'$ out of the way of the feet of $\nu'_1 \cup \nu'_2$, dragging $\tau'$ along for the duration of the isotopy. Any time $\ell$ intersects itself, the isotopy will eventually run into $\tau'$ a second time, so we simply drag it along in the same way; see Figure 9. Call the result $c'' = \tau'' \cup \beta'' \cup \nu''_1 \cup \nu''_2$. By design, we now have $[c'']_{F \times I} = \ell^{-1} * [c']_{F \times I} * \ell = [c]_{F \times I}$. Thus, $[p(c)]_F = p_x([c]_{F \times I}) = p_x([c'']_{F \times I}) = [p(c'')]_F$. Now, since $\beta = \beta''$, we in fact know that $p(\tau)$ and $p(\tau'')$ are homotopic in $F$, and hence are isotopic in $F$, fixing endpoints (see [2]).

![Figure 9](image-url)  
*Figure 9 (colour online). 1-bridge isotopy along $\ell$.***
isotopy clearly lifts to a 1-bridge isotopy from $c$ to $c''$. Thus, ultimately $c$, $c'$, and $c''$ are all related by 1-bridge isotopy. □

**Lemma 6.** Let $c$ and $c'$ be loops in the complement of a fibered link, $L$, in 1-bridge positions. If $c$ and $c'$ are isotopic in $M \setminus L$, then there exists an integer $k$, so that $c$ and $H^k(c')$ are 1-bridge isotopic, where $H$ is the natural automorphism of $M \setminus L$ induced by the monodromy of the link. Hence $mbc_1(c) = bc_1(c) = bc_1(c')$.

**Proof.** Consider the infinite cyclic cover $\tilde{N} \cong F \times \mathbb{R}$, with covering map $P : \tilde{N} \to N = M \setminus L$ defined by $P(x, t) = (h^k(x), t - k)$, for $t \in [k, k + 1]$. Then the automorphism $H$ lifts to a map $T : \tilde{N} \to \tilde{N}$ defined by $T(x, t) = (x, t + 1)$.

The isotopy from $c$ to $c'$ in $M \setminus L$ lifts to an isotopy from a lift $\tilde{c}$ of $c$ to a lift $\tilde{c}'$ of $c'$, in $\tilde{N}$. By relabeling if necessary, we may take $\tilde{c}'$ to be in $F \times [0, 1] \subset F \times \mathbb{R}$, and $\tilde{c}$ to be in $F \times [k, k + 1]$ for some $k \in \mathbb{Z}_{\geq 0}$. Then let $\tilde{c}' = T^k(\tilde{c})$, so that $\tilde{c}'$ is isotopic to $\tilde{c}$ in $\tilde{N}$, and $\tilde{c}' \subset F \times [k, k + 1]$.

The isotopy from $\tilde{c}'$ to $\tilde{c}$ is supported in a compact region of $\tilde{N}$, so we can restrict our attention to $F \times [m, n]$, for some $m, n \in \mathbb{Z}$, with $m \leq 0 < k + 1 \leq n$.

Now, $\tilde{c}'$ and $\tilde{c}$ can be considered to be in 1-bridge positions in $F \times [m, n]$. Hence, by Lemma 5, $\tilde{c}'$ and $\tilde{c}$ are 1-bridge isotopic in $F \times [m, n] \subset \tilde{N}$. This descends to a 1-bridge isotopy in $M \setminus L$ from $H^k(c')$ to $c$.

Let $F$ be a fiber surface of a fibered link $L$ with monodromy $h$, and let $\alpha$ be a properly embedded arc in $F$ with endpoints $x_1$ and $x_2$. Put $\beta := \alpha \times \{\frac{2}{3}\}$, $\tau := h(\alpha) \times \{\frac{1}{3}\}$, and $\nu_i = (x_i) \times \{\frac{1}{3}, \frac{2}{3}\}$ ($i = 1, 2$). Recall that a loop formed by ‘pushing-off’ $\alpha$ from $F$ is an $\alpha$-loop. Then the loop $c = \tau \cup \nu_1 \cup \beta \cup \nu_2$ in $M \setminus n(L) = (F \times [0, 1])/h$ is an $\alpha$-loop, which is in 1-bridge position.

**Lemma 7.** If $c$ is an $\alpha$-loop, then $\rho(\alpha) = mbc_1(c)$.

**Remark 7.** Observe that every $\alpha$-loop has a 1-bridge position with $\rho(\alpha)$ crossings, but there probably exist loops with 1-bridge positions that are not isotopic to arc-loops.

**Proof.** As noted in the remark above, the curve $c$ is isotopic to a curve $c'$ in 1-bridge position with $\beta = \alpha \times \{\frac{2}{3}\}$ and $\tau$ an arc isotopic to $h(\alpha) \times \{\frac{1}{3}\}$, so as to minimize intersections in the interior between the projections of $\tau$ and $\beta$. From Lemma 6, we know that $mbc_1(c) = bc_1(c')$. Now, $\rho(c')$ has $\rho(\alpha)$ crossings, so it suffices to show that $\rho(\alpha) = bc_1(c')$.

We may slightly shrink the arc $\beta$ so that the endpoints of $\beta$ are in the interior of $F$ without changing the number of crossings in $\rho(c')$. Similar to the argument in Theorem 5.3 in [6], suppose that $\phi_t$ is a 1-bridge isotopy taking $c'$ to $c''$ so that $\rho(c'')$ has $bc_1(c')$ crossings. Call $c_t = \phi_t(c')$, and note that $c_t$ is in 1-bridge position for each $t$, so $c_t$ consists of the union of arcs $\beta_t$, $\tau_t$, $\nu_1(t)$, and $\nu_2(t)$, as usual. By genericity, there are only a finite number of singular times $t_1, \ldots, t_n$ at which $p(c_t)$ has non-transverse self-intersections. In passing from time $t_j - \epsilon$ to $t_j + \epsilon$, one of four things may occur: a bigon is (1) created or (2) destroyed (with boundary corresponding to subarcs of $p(\tau_t)$ and $p(\beta_t)$), or a monogon is (3) created or (4) destroyed (with boundary corresponding to a subarc of $p(c_t)$ containing one of the $\nu_i(t)$).

If ever a bigon or a monogon is created and then destroyed, then certainly we can replace the isotopy with one in which neither operation occurs.

Suppose at the instant $t_j$, a bigon $B_1$ appears and at $t_{j+1}$ a bigon $B_2$ is removed. If the bigons share no vertices in common, then the operations can occur in either order with the same result. On the other hand, suppose that $B_1$ and $B_2$ share a vertex. Then the creation and annihilation operations cancel one another, and neither need to have been performed at all.
Now, suppose at the instant $t_j$, a monogon $M_1$ appears and at $t_{j+1}$ a bigon $B_2$ is removed. If they share no vertices in common, then the operations can occur in either order with the same result. On the other hand, suppose $M_1$ and $B_2$ share a vertex. Then the end result could equally have been accomplished by a single monogon annihilation operation.

Suppose instead that, at the instant $t_j$, a bigon $B_1$ appears and at $t_{j+1}$ a monogon $M_2$ is removed. If they share no vertices, then the operations commute. On the other hand, if $B_1$ and $M_2$ share a vertex, then the result is equivalent to a single monogon creation operation.

Finally, suppose at the instant $t_j$, a monogon $M_1$ appears and at $t_{j+1}$ a monogon $M_2$ is removed. Again, if $M_1$ and $M_2$ do not share a vertex, then the operations commute. If $M_1$ and $M_2$ are distinct, but share a common vertex, then it can be shown that $\alpha$ and $\beta$ are isotopic, in which case $\rho(\alpha) = bc_1(c') = 0$.

Thus, in all events, by means of such re-ordering, or replacements which reduce the number of operations, we can replace the isotopy with one in which all annihilation operations occur in which case the result of a generalized crossing change of order $1$ is removed. Again, if $\alpha$ and $\beta$ are isotopic, so there are no annihilation operations that can occur. Because creation operations increase the number of crossings in the projection, there are no operations performed in the isotopy, so $p(c')$ and $p(c'')$ contain the same number of crossings, and $bc_1(c') = \rho(\alpha)$.

Now we will characterize Dehn surgeries along arc-loops preserving a fiber surface. Since any arc-loop bounds a disk in $M$, we can take the preferred longitude for a surgery slope.

**Theorem 3.** Suppose that $F$ is a fiber surface in $M$ and $c$ is an $\alpha$-loop. Suppose that $\gamma$ is a non-trivial slope on $c$, and that $N(\gamma)$ is the manifold obtained from $M$ via the $\gamma$-surgery on $c$. Then $F$ is a fiber surface in $N(\gamma)$ if and only if

1. $\alpha$ is clean and $\gamma = i_\partial(\alpha) + 1/n$ for some integer $n$, or
2. $\alpha$ is once-unclean and $\gamma = i_\partial(\alpha)$.

**Proof.** Since $F$ is a fiber surface and $c$ is disjoint from $F$, we may assume that $c$ is in $M \setminus n(F) = F \times I$. Theorem 15 tells us that $mbc_1(c) \leq 1$ in the case when $F$ is a fiber surface in $N(\gamma)$. Then, by Lemma 7, $\alpha$ is either clean, or once-unclean with respect to the monodromy for $F$, depending on whether $c$ has zero or one crossings, respectively.

Let $\lambda$ be the preferred longitude on $c$, $\mu$ be a meridian for $c$, and $\lambda_0$ be the ‘black-board’ frame induced by the surface $F$ together with the small bridge, as in [24]. Then $\lambda_0 = \lambda + i_\partial(\alpha) \cdot \mu$. By Theorem 15, the surgery slope $\gamma$ must be $n \cdot \lambda_0 + \mu = n \cdot \lambda + (n \cdot i_\partial(\alpha) + 1) \cdot \mu$ if $c$ has no crossing, and must be $\lambda_0 = \lambda + i_\partial(\alpha) \cdot \mu$ if $c$ has a single crossing.

6. Characterization of generalized crossing changes between fibered links

In this section, we will characterize generalized crossing changes between fibered links. Throughout this section, $L$ and $L'$ are oriented links in a manifold $M$ related by a generalized crossing change. More precisely, there exists a disk $D$ in $M$ such that $L$ intersects $D$ in two points with opposite orientations, and $L'$ is the image of $L$ after $(-1/n)$-Dehn surgery along $c = \partial D$ for some $n \in \mathbb{Z} \setminus \{0\}$. The curve $c$ is called a crossing circle, and we say that $L'$ is the result of a generalized crossing change of order $n$. When $n = \pm 1$, this is just an ordinary
crossing change. In this section, we work mostly in $S^3$, and make note of this in the statements of theorems when it is assumed that $M = S^3$.

In $S^3$, Scharlemann and Thompson [28] showed that, in the case $n = \pm 1$, there exists a taut Seifert surface $F$ for $L$ or $L'$, say $L$, such that $F$ is disjoint from $c$, but intersects $D$ in an arc, and they described surface locally. For $|n| > 1$, Kalfagianni and Lin [18] showed a similar result.

**Theorem 16 (18, 28).** Suppose that $L'$ is obtained from $L$ in $S^3$ by a generalized crossing change of order $n$. Then $\chi(L') \geq \chi(L)$ if and only if $L$ has a taut Seifert surface $F$ such that $F$ is disjoint from $c$, but intersects $D$ in an arc. Moreover, $\chi(L') > \chi(L)$ if and only if $F'$ is a plumbing of a $(-n)$-times twisted annulus $A$ and a surface $F''$ that is disjoint from $D$, and the result $A'$ of $A$ after the twist is compressible.

Suppose now that $L$ is a fibered link. Let $\alpha$ be the arc $D \cap F$ in Theorem 16. Recall that we call $c$ an $\alpha$-loop. We will say that performing the generalized crossing change ($(-1/n)$-Dehn surgery) along $c$ is an $n$-twist along $\alpha$. Here an $\varepsilon$-twist is right- or left-handed if $\varepsilon = 1$ or $-1$, respectively.

As mentioned earlier, the result of a plumbing of two surfaces is a fiber surface if and only if both summands are fiber surfaces [8, 9]. Further, the only fiber annuli are the left- and right-handed Hopf annuli. Thus, by Theorem 10, we can restate the last part of Theorem 16 as follows.

**Theorem 17.** Suppose that $L$ is a fibered link in $S^3$ with fiber $F$, and $L'$ is obtained from $L$ by an $n$-twist along $\alpha$, where $\alpha$ is a properly embedded arc in $F$. Then $\chi(L') > \chi(L)$ if and only if $n = \pm 1$, and $\alpha$ is clean and alternating with $i_{\partial}(\alpha) = -n$.

Moreover, Kobayashi showed that the resulting surface of the $n$-twist along $\alpha$ is a pre-fiber surface [20, Theorem 2], and he also characterized $\alpha$ in the pre-fiber surface [20, Lemma 4.7 and Proposition 8.1]. For the remaining case, we will characterize generalized crossing changes between fibered links $L$ and $L'$ with $\chi(L) = \chi(L')$.

Observe that if a crossing circle is nugatory (that is, bounds a disk in the complement of the link), then any generalized crossing change will not change the link. For the case of knots, Kalfagianni [17] showed the converse holds: if a crossing change on a fibered knot yields a fibered knot that is isotopic to the original, then the crossing circle must be nugatory.

Stallings proved if $F$ is a fiber surface, and the loop $c$ is isotopic into $F$ so that the framing on $c$ induced by $F$ agrees with that of $D$, then the image of $F$ after $\pm 1$-Dehn surgery along $c$ is a fiber surface for the resulting link [29]. This has come to be known as a Stallings twist. Yamamoto proved that twisting along an arc is a Stallings twist if and only if the arc $\alpha$ is clean and non-alternating [32] (see also Theorem 8). (Note that a crossing change is nugatory if and only if the arc $\alpha$ is fixed by the monodromy. In this case, $\alpha$ is clean and non-alternating. Since the crossing circle can be isotoped to a trivial loop in the surface $F$, this can also be considered a special case of a Stallings twist.)

We generalize Yamamoto’s result and characterize exactly when twisting along an arc results in a fiber surface.

**Theorem 18.** Suppose that $F$ is a fiber surface, and $\alpha$ is a properly embedded arc in $F$. Let $F'$ be the resulting surface of an $\varepsilon$-twist along $\alpha$ for $\varepsilon \in \{\pm 1\}$. Then $F'$ is a fiber surface if and only if $\iota_{\text{total}}(\alpha) = 0$ (that is, $\alpha$ is clean and non-alternating) or $\alpha$ is once-unclean and alternating with $i_{\partial}(\alpha) = -\varepsilon$.
Proof. Plumb a Hopf annulus along an arc parallel to the boundary of $F$, with endpoints on either side of $\alpha$, so that the trivial subdisk cut off by this arc contains only one point $p$ of $\partial F$. The result is a new fibered link, together with its fiber $F''$. Observe that the monodromy of $F''$ differs from that of $F$ by just a Dehn twist along the core of the newly plumbed on Hopf annulus, right- or left-handed, depending on the twist of the Hopf annulus.

Now, the result of cutting $F''$ along $\alpha$ is $F'$. So, by Theorem 1, $F'$ is a fiber if and only if $\alpha$ is clean, alternating, or once-unclean, non-alternating in $F''$. The arc $\alpha$ will be clean, alternating in $F''$ precisely when $\alpha$ is clean, non-alternating in $F$ and the sign of the Hopf annulus disagrees with the sign of $i(\alpha, h(\alpha))$ at $p$ in $F$, where $h$ is the monodromy of $F$. The arc $\alpha$ will be once-unclean, non-alternating in $F''$ exactly when either $\alpha$ is clean, non-alternating in $F$ and the sign of the Hopf annulus agrees with the sign of $i(\alpha, h(\alpha))$ at $p$ in $F$, or when $\alpha$ is once-unclean, alternating in $F$, and the sign of the Hopf annulus disagrees with the sign of $i(\alpha, h(\alpha))$ at $p$ in $F$. \hfill \Box

By Theorem 16, we have Theorem 4 as a translation of Theorem 18.

**Theorem 4.** Suppose that a link $L'$ is obtained from a fibered link $L$ in $S^3$ with fiber $F$ by a crossing change, and $\chi(L') = \chi(L)$. Then $L'$ is a fibered link if and only if the crossing change is a Stallings twist or else an $\varepsilon$-twist along an arc $\alpha$ in $F$, where $\alpha$ is once-unclean and alternating with $i_\partial(\alpha) = -\varepsilon$.

In fact, using Theorem 3, we can characterize any generalized crossing change between fibered links of the same Euler characteristic.

**Theorem 5.** Suppose that $L$ and $L'$ are fibered links in $S^3$ related by a generalized crossing change with $\chi(L) = \chi(L')$. Then the generalized crossing change is an $n$-twist around an arc $\alpha$, and one of the following holds:

1. $\alpha$ is clean and non-alternating;
2. $n = \pm 2$, and $\alpha$ is clean and alternating with $i_\partial(\alpha) = -n/2$, or
3. $n = \pm 1$, and $\alpha$ is once-unclean and alternating with $i_\partial(\alpha) = -n$.

**Proof.** Let $F$ be a fiber surface of $L$. By Theorem 16, $c$ is an $\alpha$-loop for some arc $\alpha$ in $F$ and $F$ is a fiber surface after $(-1/n)$-surgery on $c$. Then, by Theorem 3, $\alpha$ is clean and

$$-\frac{1}{n} = i_\partial(\alpha) + \frac{1}{m}$$

for some integer $m$, or $\alpha$ is once-unclean and

$$-\frac{1}{n} = i_\partial(\alpha).$$

If $\alpha$ is clean, then either $i_\partial(\alpha) = 0$, so $\alpha$ is non-alternating, or $n = \pm 2$ and $i_\partial(\alpha) = -n/2$. If $\alpha$ is once-unclean, then $n = \pm 1$ and $i_\partial(\alpha) = -n$. \hfill \Box

**Corollary 5.** If $L$ and $L'$ are related as in Theorem 5, and the twist is around a once-unclean, alternating arc, then the monodromy map changes by composition with $t_a^2 t_b^2 t_c^{-1}$ or $t_a^{-2} t_b^{-2} t_c$, depending not on $n$, but on $i_p(\alpha, h(\alpha))$ at the interior point of intersection between $\alpha$ and $h(\alpha)$, where $t_a$ denotes a Dehn twist about the curve $a$, and $a, b, c$ are the loops formed by resolving the intersection of $\alpha \cup h(\alpha)$ in two ways, as in [24].

**Proof.** This follows from Proposition 1.4 of [24]. \hfill \Box
7. An alternative proof of Theorem 1

In this section, we will give an alternative proof of Theorem 1 using Theorem 3. Let \( L \) be a fibered link in a manifold \( M \) with fiber \( F \), and let \( F' \) be a surface obtained from \( F \) by cutting along an arc \( \alpha \). Let \( c \) be an \( \alpha \)-loop. We consider \( F \) in \( N(0) \) which is obtained from \( M \) by 0-surgery on \( c \). Theorem 3 gives the following necessary and sufficient condition for \( F \) to be a fiber surface in \( N(0) \):

1. \( \alpha \) is clean and \( i_\partial(\alpha) = \pm 1 \), or
2. \( \alpha \) is once-unclean and \( i_\partial(\alpha) = 0 \).

This is the same condition as in Theorem 1. Then Theorem 1 follows from Lemma 8.

**Lemma 8.** The surface \( F' \) is a fiber surface in \( M \) if and only if \( F \) is a fiber surface in \( N(0) \).

**Remark 8.** A statement analogous to Lemma 8 holds replacing fiber surface with taut surface, since tautness is also invariant under product decomposition and reverse operations.

**Proof.** The idea of this proof is based on Proof of Claim 2 in [28]. Take a small neighborhood \( n(c) \) of \( c \) and a small product neighborhood \( n(F) = F \times I \) of \( F \) so that \( n(c) \) and \( n(F) \) are disjoint. Let \( D \) be a disk \( \alpha \times I \) in \( n(F) \) and let \( \beta \) be a loop \( \partial D \) in \( \partial(n(F)) \). By the definition of \( \alpha \)-loop, there exists an annulus \( A \) in \( M \setminus (n(c) \cup n(F)) \) with boundary components \( \beta \) and a longitude \( \lambda \) on \( \partial(n(c)) \). Then \( \beta \) bounds a disk \( D' \) in \( N(0) \setminus n(F) \), the union of the annulus \( A \), and a meridional disk \( D'' \) of the solid torus filled into \( N(0) \). Since \( \beta \) intersects the suture \( \partial F \) at two points, \( D' \) is a product disk for the sutured manifold \( (N(0) \setminus n(F), \partial F) \). Take a small neighborhood \( n(A) \) of \( A \) in \( M \setminus (n(c) \cup n(F)) \) and a longitude \( \lambda' \) on \( \partial(n(c)) \) so that \( n(A) \) and \( \lambda' \) are disjoint. Then there exists a homeomorphism \( f \) from \( M \setminus n(F) \) to \( M \setminus (n(c) \cup n(F) \cup n(A)) \) such that \( f^{-1}(\partial F \setminus n(A)) \subset \partial F \) and \( f(\beta) = \lambda' \). The solid torus filled into \( N(0) \) becomes a 2-handle along \( \lambda' \) for \( M \setminus n(c) \) by removing a neighborhood of the meridian disk \( D'' \). Since \( D' = A \cup D'' \), it follows that a manifold obtained from \( M \setminus n(F) \) by attaching a 2-handle along \( \beta \) is homeomorphic to a manifold obtained from \( N(0) \setminus n(F) \) by deleting a neighborhood of \( D' \), by extending the homeomorphism \( f \) into the 2-handle. On the other hand, a product neighborhood \( n(F') = F' \times I \) is obtained from \( n(F) \) by removing a neighborhood of \( D' \), and so \( M \setminus n(F') \) is obtained from \( M \setminus n(F) \) by attaching a 2-handle along \( \beta \). When attaching to a sutured manifold a 2-handle along a loop which intersects a suture at two points, a suture of the resulting manifold is uniquely determined from the original one by replacing two arcs crossing the attaching annulus with another two arcs on the (non-attaching) boundary of the 2-handle. Then a sutured manifold \( (M \setminus n(F'), \partial F') \) is obtained from \( (N(0) \setminus n(F), \partial F) \) by decomposing along \( D' \).

\[
(N(0) \setminus n(F), \partial F) \xrightarrow{D'} (M \setminus n(F'), \partial F').
\]

Since the triviality of a sutured manifold is invariant under product decomposition and reverse operations, \( F' \) is a fiber surface in \( M \) if and only if \( F \) is a fiber surface in \( N(0) \).

**Corollary 6.** Suppose that \( F \) and \( F' \) are related as above, and \( \alpha \) is once-unclean, non-alternating. Let \( h \) and \( h' \) be monodromies of \( F \) and \( F' \), respectively. Then \( (t^2_a b^2 c^{-1} h) \mid_{F'} = h' \) or \( (t^2_a b^2 c^{-2} t_c h) \mid_{F'} = h' \), depending on \( i_\partial(\alpha, h(\alpha)) \) at the interior point of intersection between \( \alpha \) and \( h(\alpha) \), where \( t_a \) denotes a Dehn twist about the curve \( a \), and \( a, b, c \) are the loops formed by resolving the intersection of \( \alpha \cup h(\alpha) \) in two ways, as in [24].

**Remark 9.** Theorem 11 tells us that \( F \) is a generalized Hopf banding of \( F' \). The loops \( a, b, c \) in Corollary 6 for a generalized Hopf banding are depicted in Figure 10. For an arc \( \ell \) in
Figure 10 (colour online). Loops of Dehn twists for a generalized Hopf banding.

$F'$ with a single self-intersection point, there are two generalized Hopf bandings, depending on which part of the band is in the higher position at the place of overlap. The two monodromies in Corollary 6 correspond to these two surfaces. If the self-intersection point of $\ell$ is removable in $F'$, then the generalized Hopf banding is a Hopf banding. In that case, $b$ is trivial in $F'$ (and so in $F$), $a$ and $c$ are isotopic to each other in $F'$, and $t_a^2 t_b^2 t_c^{-1} = t_a t_a^{-2} t_b^2 t_c^{-1} = t_a^{-1}$, is a Dehn twist along the core of the Hopf annulus.

Proof. Let $h_0$ be the monodromy of $F$ in $N(0)$, that is, $(F \times [0,1]) / h_0$ is homeomorphic to $\overline{N(0) \setminus n(F)}$. By Proposition 1.4 of [24], then, $t_a^2 t_b^2 t_c^{-1} h = h_0$ or $t_a^{-2} t_b^{-2} t_c h = h_0$. In the proof of Lemma 8, recall that the product disk $D'$ of $(\overline{N(0) \setminus n(F)}, \partial F)$ is the boundary of a disk $\alpha \times I$, and $(M \setminus n(F'), \partial F')$ is obtained from $(\overline{N(0) \setminus n(F)}, \partial F)$ by decomposing along $D'$. This implies that $h_0 (\alpha) = \alpha$ and $h_0 |_{F'} = h'$.

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