Classifying toric 3-fold codes of dimensions 4 and 5

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1. Introduction
2. A minimum distance formula
3. Two classification theorems
4. Toric codes of dimension 5
Definition
Let $P$ be an integral convex polytope in $\mathbb{R}^m$. The toric code $C_P$ is a vector space $\text{Im}(\epsilon)$, where

$$\epsilon : \mathcal{L}(P) \rightarrow (\mathbb{F}_q)^m,$$

and $\mathcal{L}(P)$ is the set of polynomials

$$\mathcal{L}(P) = \text{Span}\{x^p : p \in P \cap \mathbb{Z}^m\}.$$
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Remark

$\text{dim}(C_P) = |P \cap \mathbb{Z}^m|$
Example

Let

\[ P = \text{Conv}\{(0, 0, 0), (2, 0, 0), (0, 2, 0), (0, 0, 2), (2, 2, 0), (2, 0, 2), (0, 2, 2), (2, 2, 2)\} \]

Then \( \dim(C_P) = 27 \). Each lattice point \( p = (a, b, c) \in P \cap \mathbb{Z}^3 \) can be identified with a monomial \( x^p = x^a y^b z^c \).
Introduction

Figure: Monomials on the cube in $\mathbb{Z}^3$
Hansen introduced toric codes (1998)
Little and Schwarz classified toric surface codes up to dimension 5 (2007)
UM-Dearborn REU classified toric surface codes of dimension 7 (2019)
We will classify codes of dimension 4 beyond the surface.
Definition (Lattice Equivalence)

Two integral convex $n$-polytopes $P_1$ and $P_2$ are lattice equivalent if there exists an affine unimodular transformation $t : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ such that $t(P_1) = P_2$. 

Theorem (White)

Every 3-polytope with 4 integral lattice points of volume $t$ is lattice equivalent to an empty tetrahedron $T(s, t) = \text{Conv}\{ (0, 0, 0), (1, 0, 0), (0, 0, 1), (s, t, 1) \}$, for some $s \in \mathbb{Z}$ where $\gcd(s, t) = 1$. Moreover, $T(s, t)$ is lattice equivalent to $T(s', t)$ if and only if $s' = \pm s (\pm 1) \pmod{t}$. 

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for some \( s \in \mathbb{Z} \) where \( \gcd(s, t) = 1 \).

Moreover, \( T(s, t) \) is lattice equivalent to \( T(s', t) \) if and only if \( s' = \pm s^{(\pm 1)} \) (mod \( t \)).
Figure: Monomials on the empty tetrahedron
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Definition

The *minimum distance* of a code $C$ is

$$d(C) = (q - 1)^m - \max_{0 \neq f \in \mathcal{L}(P)} Z(f)$$

where $Z(f)$ is the number of zeros of $f$. 
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Theorem (Minimum Distance on Empty Tetrahedra)

Let $C$ be a toric code over $\mathbb{F}_q$ on an empty tetrahedron $T(s, t)$. Then

- $d(C) = (q - 1)^3 - (q - 1)(q - 3) - 2(q - 1)$ if $\gcd(t, q - 1) = 1$, and
- $d(C) = (q - 1)^3 - (q - 1)(q - 3) - q \gcd(t, q - 1)$ otherwise.
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Definition (Generator Matrix)
If $C_P$ is of length $n$ and dimension $k$, then a generator matrix $G$ is a $k \times n$ matrix with rows given by $p \in P$ and columns given by $x \in (\mathbb{F}_q^*)^m$.
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Definition (Monomial Equivalence)
Let $C_1$ and $C_2$ be of length $n$ over $\mathbb{F}_q$. Let $G_1$ be a generator matrix for $C_1$. Then $C_1$ and $C_2$ are monomially equivalent if there is an invertible $n \times n$ diagonal matrix $\Delta$ and an $n \times n$ permutation matrix $\Pi$ such that

$$G_2 = G_1 \Delta \Pi$$

is a generator matrix for $C_2$. 
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**Remark (Little and Schwarz)**

*Lattice equivalence implies monomial equivalence.*
Theorem 1

Theorem (Monomial Equivalence Theorem 1)

Let $C_1$ and $C_2$ be toric codes with four lattice points on empty tetrahedra $T(s, t_1)$ and $T(s, t_2)$, respectively, over the field $\mathbb{F}_q$. Then $C_1$ and $C_2$ are monomially equivalent iff $\gcd(t_1, q - 1) = \gcd(t_2, q - 1)$.
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Proof sketch. ($\Rightarrow$) We apply our results for minimum distance. Two codes with different minimum distances cannot be monomially equivalent.
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Proof sketch. ($\Rightarrow$) We apply our results for minimum distance. Two codes with different minimum distances cannot be monomially equivalent.

($\Leftarrow$) We show a permutation between generator matrices exists by demonstrating a bijective correspondence between columns of the two matrices.
Theorem (Monomial Equivalence Theorem 2)

Let $C_1$ and $C_2$ be toric codes over $\mathbb{F}_q$ on empty tetrahedra $T(s_1, t)$ and $T(s_2, t)$, respectively. Then $C_1$ and $C_2$ are monomially equivalent iff either of the following conditions hold true:

1. $s_1 \equiv s_2 \mod \gcd(t, q - 1)$, or
2. $s_1 \equiv \pm s_2^{\pm 1} \mod t$.
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Case 2: We show explicitly that the two polytopes are lattice equivalent.
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Proof sketch. ($\Rightarrow$) Case 1: Using a primitive element, we show that the generator matrices are equivalent.

Case 2: We show explicitly that the two polytopes are lattice equivalent.

($\Leftarrow$) We argue by contradiction that there exists no permutation between generator matrices.
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| Sig.  | Representative                                                                 |
|-------|-----------------------------------------------------------------------------|
| (2,2) | (0,0,0), (1,0,0), (0,1,0), (1,1,0), (0,0,1)                                  |
| (2,1) | (0,0,0), (1,0,0), (-1,0,0), (0,0,1), (s,t,1) \(0 \leq s \leq \frac{t}{2}, \gcd(s, t) = 1\) |
| (3,2) | (0,0,0), (1,0,0), (0,1,0), (0,0,1), (s,t,1) \(0 < s \leq t, \gcd(s, t) = 1\) |
| (3,1) | (0,0,0), (1,0,0), (0,1,0), (0,0,1), (-1,-1,0)                              |

Table: Five point 3-polytope classes of width 1.
**Figure:** Signature (2,1).

**Figure:** Signature (3,2).
Proposition (Minimum distance for 5 dimensions)

1. If $\text{sig}(P) = (2, 1)$, then $d = (q - 1)^3 - 2(q - 1)^2$;
2. If $\text{sig}(P) = (2, 2)$, then $d = (q - 1)^3 - (2q^2 - 5q + 3)$;
3. If $\text{sig}(P) = (3, 1)$ and $P$ has width 1, then
   $$d \geq (q - 1)^3 - (q - 1)(1 + q + 2\sqrt{q});$$ and
4. If $\text{sig}(P) = (3, 2)$, then
   $$d \geq (q - 1)^3 - (q - 1)^2 - (s + t)q$$

Remark

For codes over polytopes of width 2, we can take the minimum distance of the subpolytope empty tetrahedron as a lower bound on $d$, and hope to improve on these.
We have proof of a full classification of toric 3-fold codes of 4 dimensions.

We have some information about the minimum distances of toric 3-fold codes of 5 dimensions.

Further work is needed on the width 2 polytopes to complete the 5 dimension classification.
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