Homologically Twisted Invariants Related to 
(2+1)- and (3+1)-Dimensional State-Sum 
Topological Quantum Field Theories

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Abstract: We outline a general construction applicable to the Turaev/Viro [TV], Crane/Yetter [CY] and generalized Turaev/Viro invariants (cf. [Y1]) of invariants valued in complex-valued functions on $H_{D-2}(M^D, Gr_C)$, where $Gr_C$ is the abelian group of functorial tensor automorphisms on the artinian tortile category used to construct the TQFT.

Introduction

It is the purpose of this note to introduce a construction of invariants of 3- and 4-manifolds which combines state-sum techniques (cf. [TV,CY,Y1,Y2]) with a dependence on (co)homology classes on the manifold.

The basic ingredient is an extra piece of structure on the tensor categories from which the state-sum invariants are constructed: a (necessarily abelian) group of functorial tensor automorphisms, which is used as the coefficient group for homology.

We assume a familiarity with standard references on monoidal and tensor categories, e.g. Mac Lane [M], Saavedra-Rivano [S], Kerler [Ke].

Throughout all manifolds are assumed to be piece-wise linear (or, equivalently, smooth).

A note on terminology: throughout, we adopt the convention of using names originally used by categorists (e.g. monoidal, autonomous, natural automorphism of the identity functor...) to refer to not-necessarily abelian categories with a given structure, and names originally used by algebraic geometers (e.g. tensor, rigid, functorial automorphism,...) to refer to abelian categories with a given structure implemented by (multi)linear functors exact in all variables. We advocate the adoption of this custom by other authors as a way of cutting down the profusion of terminology now afflicting quantum topology.

This work was inspired by unpublished work of Paolo Cotta-Ramusino at the physical level of rigor, and by conversations with Cotta-Ramusino and Louis Kauffman at the XXII International Conference on Differential Geometric Methods in Theoretical Physics, Ixtapa, Mexico. The results were actually obtained while the author was in Ixtapa, so he extends thanks also to the organizers of the conference for their hospitality and financial support.

Functorial Tensor Automorphisms

As noted in the introduction, the basic ingredient in this construction is an extra piece of categorical structure available on the categories from which the Crane/Yetter [CY] and generalized Turaev/Viro [Y1] invariants were constructed:

Definition 1 Let $C$ be a tensor category. A functorial tensor automorphism of $C$ is a natural isomorphism $\phi : 1_C \Rightarrow 1_C$, which satisfies $\phi_{A \otimes B} = \phi_A \otimes \phi_B$ for all objects $A$ and $B$, and $\phi_I = Id_I$, where $I$ is the identity object for the tensor product.

Observe that functorial tensor automorphisms for a subgroup of the abelian group of natural automorphisms of the identity functor on the category.

The categories to which we will apply this notion are particularly nice, in that they satisfy

Definition 2 A $k$-linear abelian category $C$ is semisimple if there exists a set $J$ of objects such that

1. Every object is isomorphic to a finite direct sum of objects in $J$.
2. The objects of \( J \) are simple in the sense that for all \( j \in J \) \( \text{Hom}(j, j) \) is 1-dimensional over \( k \).
3. For \( i, j \in J \) \( i \neq j \) \( \text{Hom}(i, j) = 0 \).

In the case where \( C \) is a tensor category, we require that \( I \in J \).

For a semisimple tensor category \( C \), functorial tensor automorphisms have a relatively simple structure. First, observe that the direct sum decompositions required by semisimplicity imply that any natural transformation \( \phi \) between exact functors from \( C \) to any abelian category is completely determined by its components at objects of \( J \), and by the last two conditions these are determined by a choice of a \( k \)-scalar \( \phi_j \) for each \( j \in J \).

Now for functorial tensor automorphisms, the condition \( \phi_j \otimes k = \phi_j \otimes \phi_k \) imposes additional conditions: whenever \( i \) is a direct summand of \( j \otimes k \), our choice must satisfy \( \phi_j \cdot \phi_k = \phi_i \). Using

**Definition 3** A \( G \)-grading of a semisimple tensor category is a function \( gr : J \rightarrow G \), for \( G \) a group, which satisfies \( gr(j) gr(k) = gr(i) \) whenever \( i \) is a direct summand of \( j \otimes k \). The grading group of \( C \), \( Gr_C \), is the universal group equipped with a \( G \)-grading of \( C \).

The preceding discussion then gives

**Proposition 4** If \( C \) is a semisimple tensor category, there is a canonical isomorphism between the abelian group of functorial tensor automorphisms of \( C \), and the group of \( k \)-characters of \( Gr_C \).

It is left as an exercise to the reader to show that \( Gr_C \) exists, and is unique and independent of the choice of \( J \) up to canonical isomorphism.

Proposition 4 has as a consequence that the group of functorial automorphisms of \( \text{Rep}(U_q(\mathfrak{sl}_2)) \) is \( \mathbb{Z}_2 \), with the generator given by \( \gamma_\rho = (-1)^\rho \), where following the Temperley/Lieb conventions [KL], simple objects are indexed by twice the spin (equivalently by dimension - 1).

Another observation about functorial tensor automorphisms is worth mentioning:

**Proposition 5** If \( H \) is a Hopf algebra (resp. finite dimensional Hopf algebra), then functorial tensor automorphisms of \( \text{Comod}(H) \) (resp. \( \text{Mod}(H) \)) are in canonical bijection with the set co-central co-group-like functionals on \( H \) (resp. group-like elements of \( H \)).

The proof is an elementary exercise in Tannaka-style reconstruction in the comodule case, which dualizes to give the module case for finite-dimensional \( H \).

**State-Sum Invariants**

Recall that the generalized Turaev/Viro invariant of [Y1] is defined by a state summation of the form

\[
TV_C(M) = \sum_\lambda N^{-n_0} \prod_{\text{edges}} \Delta_\lambda(e) \prod_{\text{tetrahedra}} ||\lambda(\sigma)||
\]

where \( \lambda \) ranges over all labellings of a triangulation with ordered vertices of edges by a chosen set of simple generating objects (in an artinian semisimple tortile category \( C \)) and of faces by maps chosen from bases for the spaces of maps from the tensor product of the objects on the “inbound” edges of the boundary to the object on the “outbound” edge, where \( ||\lambda(\sigma)|| \) is one of the generalized 6-j symbols of [Y1] (according to the orientation agreement/disagreement between the ambient orientation and that induced by the ordering on the vertices), and where \( N \) is the sum of the squares of the quantum dimensions of the simple objects and \( \Delta_\rho \) is the quantum dimension of the simple object \( \rho \).

Similarly, the Crane-Yetter invariant [CY] in its Temperley-Lieb formulation (cf. [CKY]) is given by

\[
\begin{align*}
\text{CY}(W) = N^{n_0-n_1} & \sum_{\text{labellings } \lambda \text{ of faces}} \prod_{\sigma} \Delta(\lambda(\sigma)) \\
& \prod_{\text{tetrahedra}} \frac{\Delta(\lambda(\sigma))}{\theta(\lambda(\tau), \lambda(\tau_0), \lambda(\tau_2))\theta(\lambda(\tau), \lambda(\tau_1), \lambda(\tau_3))} \\
& \prod_{\text{4-simplexes}} 15 - j
\end{align*}
\]
To describe the the homologically twisted version of these invariants, we need to let $A$ be a group of functorial tensor automorphisms on the underlying tensor category of the theory. We can then consider homology and cohomology with coefficients in $A$. Because of the quantum flavor of our constructions it will be convenient to write (co)cycles, (co)boundaries, and (co)homology classes multiplicatively.

Although the construction is given in terms of homology classes, it is important to note that by Poincaré duality, we are in fact giving invariants in both the Turaev/Viro and Crane/Yetter cases which can be regarded as depending on a 2-dimensional cohomology class.

Now, given a 1- (resp. 2-)dimensional homology class $[\alpha]$ with coefficients in $A$ in the Turaev/Viro (resp. Crane/Yetter) case, modify the expressions above by replacing the factors of the form $\Delta_{\lambda(\phi)}$ for $\phi$ a 1- (resp. 2-)simplex with $tr(\alpha_{\lambda(\phi)})$, where $tr$ denotes the internal trace on the category, and $\alpha$ is a representative cycle for $[\alpha]$ supported on the 1- (resp. 2-)simplexes of triangulation, $\alpha^\phi \in A$ is the coefficient of $\phi$ in $\alpha$.

More specifically, we make

**Definition 6** The homologically twisted generalized Turaev/Viro invariant of $(M, [\alpha])$ for $M$ a 3-manifold, $[\alpha] \in H_1(M, A)$ is defined by the state-sum

$$TV_{C, A}(M, [\alpha]) = \sum_{\lambda} N^{-n_0} \prod_{\text{edges}} tr(\alpha_{\lambda(e)}) \prod_{\text{tetrahedra}} ||\lambda(\sigma)||$$

where $C$ is a semi-simple tortile category, $A$ is a group of functorial tensor automorphisms on $C$, and the sum ranges over all labellings of an ordered triangulation as described above.

The homologically twisted Crane/Yetter invariant of $(M, [\alpha])$ for $M$ a 4-manifold, $[\alpha] \in H_2(M, Z_2)$ is defined by the state-sum

$$CY(W, [\alpha]) = \sum_{\text{labellings } \lambda \text{ of faces and tetrahedra}} \prod_{\text{faces}} tr(\alpha_{\lambda(\sigma)}) \prod_{\text{tetrahedra}} \frac{\Delta(\lambda(\sigma))}{\theta(\lambda(\tau), \lambda(\tau_0), \lambda(\tau_2)) \theta(\lambda(\tau), \lambda(\tau_1), \lambda(\tau_3))} \prod_{\text{4-simplexes}} 15 - j$$

We now explicitly state and prove in what sense these state-sums are invariants:

**Theorem 7** $TV_{C, A}(M, [\alpha])$ is independent of the ordered triangulation used to construct it, and of the the choice of representative cycle $\alpha$.

**Theorem 8** $CY(M, [\alpha])$ is independent of the ordered triangulation used to construct it, and of the choice of representative cycle $\alpha$.

**proof of Theorems 7 and 8:** The proof proceeds in three stages: first for a fixed triangulation, we show that the state-sum is independent of the choice of representative cycle; second we show that for any instance of the Pachner moves (cf. [P]), we can change the choice of cycle so that the representing cycle is trivial on the interior of the region replaced by the Pachner move; and finally we observe that having shown this, the proof of invariance of the original untwisted invariant (cf. [Y1], [CY]) under the Pachner moves now carries the same result for the twisted version. Let $\delta$ denote the dimension of the homology class used (1 for generalized Turaev/Viro, 2 for Crane/Yetter).

To see that the state-sum is independent of the choices of representative cycle (for a fixed triangulation), it suffices to show that multiplying a cycle by the boundary of a chain supported on a single $(\delta + 1)$-simplex does not change the state-sum. To do this, observe that we can rewrite that state-sum as a sum indexed by labellings of the $\delta$-simplexes only of smaller state-sums indexed by the extensions of the labelling to the
δ-simplexes. Specifically, in the generalized Turaev/Viro case, we have a sum over labellings λ of 1-simplexes of sums of the form

\[ \sum_{\mu \text{ extending } \lambda \text{ to faces}} N^{-\tau_0} \prod_{\text{edges}} tr(\alpha_{\lambda(e)}) \prod_{\text{tetrahedra}} ||\mu(\sigma)|| \]

while in the Crane/Yetter case, we have a sum over labellings λ of 2-simplexes of sums of the form

\[ N^{\tau_0-\tau_1} \sum_{\text{labellings } \mu \text{ extending } \lambda \text{ to tetrahedra}} \prod_{\text{faces } \sigma} tr(\alpha^\tau_{\lambda(\sigma)}) \prod_{\text{tetrahedra } \tau} \frac{\Delta(\lambda(\sigma))}{\theta(\mu(\tau), \lambda(\tau_0), \lambda(\tau_2)) \theta(\mu(\tau), \lambda(\tau_1), \lambda(\tau_3))} \prod_{4\text{-simplexes } v} 15 - j(\mu, v) \]

Now, suppose \( \alpha = \beta \cdot \partial(\xi^a) \) where \( \xi^a \) is the chain supported on a δ-simplex \( \xi \) with coefficient \( a \). We will show in either case that the λ sums for \( \alpha \) and \( \beta \) are equal.

Now, in either case, \( \mu(\xi) \) represents a map in the underlying tensor category from the tensor product of the labels on the inbound faces of the boundary to the tensor product of the labels on the outbound faces of the boundary the cancellation. The cancellation of the components of \( a \) and \( a^{-1} \) follows from the monoidal naturality condition on \( a \) and “Schur’s lemma”. In the generalized Turaev/Viro case (resp. the Crane/Yetter case), we move the occurrences of \( a \) and \( a^{-1} \) from the loop representing the trace to the edge in the generalized 6j-symbol (resp. quantum 15j-symbol) corresponding to the same face (using the graphical Schur’s lemma), they then cancel by monoidal naturality.

That having been shown, our second step follows directly by excision (and the homology long exact sequence) since the region removed and replaced by the Pachner moves is contractible.

Finally, the proofs of invariance under the Pachner moves for generalized Turaev/Viro theory and Crane/Yetter theory involve only the labels interior to the region removed and its replacement, so provided the support of the cycle is outside this region, they carry the result for the twisted case as well.

We can then assemble these invariants into an invariant of the underlying manifold:

**Definition 9** The total twisted generalized Turaev/Viro invariant (resp. the total twisted Crane/Yetter invariant) of a 3-manifold (resp. 4-manifold) is the isomorphism class of the pair

\[ (H_1(M, A), TV_{C,A}(M, -) : H_1(M, A) \rightarrow k) \]

(resp. \( (H_2(W, Z_2), CY(W, -) : \rightarrow C) ) \)

where two pairs \( (B_i, f_i : B_i \rightarrow k) \) for \( i = 1, 2 \) are isomorphic if there is an group isomorphism \( \phi : B_1 \rightarrow B_2 \) such that \( f_1 = f_2(\phi) \).

**A Surgical Version**

In the case of a simply connected 4-manifold, we can describe a related surgical invariant related to the twisted Crane/Yetter invariant in a way analogous to the relationship between the first invariant of Broda [B1] and the original Crane/Yetter invariant [CY]. We restrict ourselves to the simply connected case here to avoid the difficulty of handling homology classes not represented by spheres.

Specifically, recall that for a simply connected 4-manifold, every 2-homology class is represented by a product of 2-spheres (remember we are writing things multiplicatively instead of additively) with various coefficients in \( A \), and that more specifically the homology group is generated by chains with non-trivial coefficient only on a single sphere obtained by taking the core of a 2-handle and attaching the cone on the attaching link into the original 4-ball.
We can then give a surgical invariant of the pair \((W, [\alpha])\) for \([\alpha] \in H_2(W, Z_2)\) as follows:

Recall that it is sensible to evaluate link-diagrams with components labelled by linear combinations of simple object in \(\text{Rep}(U_q(sl_2))\). In particular, consider two such linear combinations:

\[
\omega = \sum_{i=0}^{r-2} \Delta_i i
\]

and

\[
\theta = \sum_{i=0}^{r-2} (-1)^i \Delta_i i.
\]

**Definition 10** Let \(\mathcal{L} = L \cup \mathring{L}\) be a surgery description of a simply-connected 4-manifold as in Kirby [Ki]: \(\mathcal{L}\) is a framed link with two distinguished sublinks—\(L\) which for the attaching curves for a family of 2-handles, and a zero-framed unlink \(\mathring{L}\) which represent cores along which “2-handles will be hollowed out” (equivalent to 1-handle attaching data). Now, choose a family \(B\) of components of \(L\) such that the chains with non-trivial coefficient on each of the associated 2-spheres form a basis for \(H_2(W, Z_2)\). Now, let \(<\mathcal{L} >_{[\alpha]}\) denote the evaluation of \(\mathcal{L}\) with those components of \(B\) with non-trivial coefficient in the representation of \([\alpha]\) labelled \(\theta\), and all other components labelled \(\omega\).

We then define \(B(W, -) : H_2(W, Z_2) \to \mathbb{C}\) by

\[
B(W, [\alpha]) = \frac{<\mathcal{L} >_{[\alpha]}}{N_{[\text{Rep}(U_q(sl_2))]}^{\frac{\nu(\mathcal{L})}{2}|\Lambda|}}
\]

whenever the Kauffman bracket variable is a principle \(4^r\)th root of unity, and by

\[
B(W, [\alpha]) = \frac{<\mathcal{L} >_{[\alpha]}}{2|\Lambda| N_{[\text{Rep}(U_q(sl_2))]}^{\frac{\nu(\mathcal{L})}{2}|\Lambda|}}
\]

whenever the Kauffman bracket variable is a principle \(2^r\)th root of unity for \(r\) odd, where \(N = \sum_{i=0}^{r-2} \Delta_i^2\), \(|\Lambda|\) is the number of components of a link \(\Lambda\), and \(\nu(\mathcal{L})\) is the nullity of the linking matrix. We call the pair \((H_2(W, Z_2), B(W, -) : H_2(W, Z_2) \to \mathbb{C})\) the total twisted Broda invariant of \(W\).

**Theorem 11** The value of \(B(W, [\alpha])\) is invariant under the Kirby moves for 4-manifold data and change of the choice of basis for \(H_2(W, Z_2)\) and thus \(B(W, -) : H_2(W, Z_2) \to \mathbb{C}\) is an invariant of \(W\).

**proof:** Regarding the introduction of attaching data for cancellable 2-handles, note that a cancellable 2-handle is never included in the basis, and thus will always be labelled \(\omega\), as will any 1-handle attaching curve. Broda’s proof then carries this part of the result. For handle-sliding, observe that handles labelled \(\omega\) or \(\theta\) slide over handles labelled \(\omega\) retaining their labels, while they slide over handles labelled \(\theta\) and change labels (\(\omega\)’s become \(\theta\)’s and vice-versa).

But this behaviour is precisely the rewriting of the homology class \([\alpha]\) under the change of basis corresponding to the handle-slide, and we are done.

Alternatively, the following proposition gives a proof of invariance in terms of the twisted Crane/Yetter invariant. \(\square\)

**Proposition 12** For any simply connected 4-manifold \(W\), for the Kauffman bracket variable a principle \(4^r\)th root of unity

\[
B(W, [\alpha])N^\frac{\chi(W)}{2} = CY(W, [\alpha]).
\]
proof: A proof identical to that given for the untwisted invariants (cf. [CKY], [R]) carries this result. In tracing this, the reader should note one subtlety: we really must extend definition the twisted Broda invariant to 2-homology classes represented by sums of spheres in arbitrary (as opposed to simply connected) 4-manifolds, but having done this, the proof proceeds as before. □

Conclusion

It is the custom in papers on quantum topology to end with a series of speculations about the implications of the construction/results obtained. We will forego this custom, save to note that dependence on cohomology classes is a feature of Donaldson’s invariants [D] which was missing from previous attempts to apply combinatorial methods to 4-dimensional differential topology (cf. [B1,B2,CY]).

The author is currently computing the total twisted Broda invariants for Gompf’s [G] examples of homeomorphic but non-diffeomorphic 4-manifolds with simple surgery descriptions.

References

[B1] Broda, B., Surgical invariants of 4-manifolds, preprint (1993).
[B2] Broda, B., A Surgical invariant of 4-manifolds, Proceedings of the Conference on Quantum Topology (D.N. Yetter, ed.), World Scientific, to appear.
[CKY] Crane, L., Kauffman, L. H. and Yetter, D. N., Evaluating the Crane-Yetter invariant, e-preprint [hep-th/9309063] and to appear Quantum Topology (R. Baadhio and L.H. Kauffman eds.), World Scientific.
[CY] Crane, L. and Yetter, D. N., A categorical construction of 4D topological quantum field theories, e-preprint [hep-th/9301062] and to appear in Quantum Topology (R. Baadhio and L.H. Kauffman eds.), World Scientific.
[D] Donaldson, S.K., Polynomial Invariants for Smooth 4-Manifolds, Topology 29 (1990) 257-315.
[G] Gompf, R.E., Nuclei of Elliptic Surfaces, Topology 30 (1991) 479-511.
[KL] Kauffman, L. H. and Lins, S. L. Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds, Princeton University Press, to appear.
[Ke] Kerler, T., Non-Tannakian Categories in Quantum Field Theory, in New Symmetry Principles in Quantum Field Theory (J. Fröhlich et al.), Plenum Press (1992).
[Ki] Kirby, R., The topology of 4-manifolds, SLNM vol. 1374, Springer-Verlag (1989).
[M] Mac Lane, S., Categories for the Working Mathematician, Springer-Verlag (1971).
[P] Pachner, U., P.L. Homeomorphic Manifolds are Equivalent by Elementary Shelling, Euro. J. Comb. 12 (1991) 129-145.
[R] Roberts, J, Skein theory and Turaev-Viro invariants, preprint (1993).
[S] Saavedra-Rivano, N., Categories Tannakiennes, SLNM vol. 265, Springer-Verlag (1972).
[TV] Turaev, V. and Viro, O., State-Sum Invariants of 3-Manifolds and Quantum 6-J Symbols, Topology 31 (1992) 865-902.
[Y1] Yetter, D.N., State-sum invariants of 3-manifolds associated to artinian semisimple tortile categories, Topology and its App., to appear.
[Y2] Yetter, D.N., Triangulations and TQFT’s, to appear in Quantum Topology (R. Baadhio and L.H. Kauffman eds.), World Scientific.