A note on the first-order flexes of smooth surfaces which are tangent to the set of all nonrigid surfaces

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Abstract. We prove that first-order flexes of smooth surfaces in Euclidean 3-space, which are tangent to the set of all nonrigid surfaces, can be extended to second-order flexes.

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1. Introduction

The notion of an infinitesimal flex of a smooth surface in $\mathbb{R}^3$ is classical in the theory of surfaces (see, for example, [4,7,9], and references therein). It is useful also for the study of polyhedral surfaces and frameworks (see, for example, [2,3], and references given there).

In [1], we suggested to consider first-order flexes of polyhedral surfaces subject to the additional condition “to be tangent to a subset of the configuration space consisting of all nonrigid polyhedral surfaces combinatorially equivalent to the polyhedral surface under study.” This condition is of interest because, on the one hand, it narrows the set of first-order flexes and, on the other hand, it holds true for any flexible polyhedral surface. In [5], S. J. Gortler, M. Holmes-Cerfon and L. Theran proved that [1] leads to a novel interpretation of the known sufficient condition for rigidity of frameworks called “prestress stability.”

Let us explain the essence of this new condition by the example of a polyhedral surface $P$, shown on the left-hand side of Fig. 1, obtained by an additional triangulation of one of the faces of nondegenerate tetrahedron $T \subset \mathbb{R}^3$. A
The nontrivial first-order flex of $P$ is shown schematically in the central part of Fig. 1, where the arrow is perpendicular to the additionally triangulated face of $T$ and depicts the velocity vector of the corresponding vertex under the first-order flex; the velocities of the remaining vertices are equal to zero. On the right-hand side of Fig. 1, the arrow lies in the plane of the additionally triangulated face of $T$ and represents the velocity of the corresponding vertex under an infinitesimal deformation, which is a tangent vector to the set of all first-order nonrigid polyhedral surfaces combinatorially equivalent to $P$. The velocities of the remaining vertices are again equal to zero. It is clear from Fig. 1 that the requirements “to be an infinitesimal flex of a polyhedral surface” and “to be an infinitesimal deformation tangent to the set of all nonrigid triangulated polyhedral surfaces combinatorially equivalent to the given one” describe completely different infinitesimal deformations.

In this note, we show that if a smooth surface in $\mathbb{R}^3$ is a smooth point of the set of nonrigid surfaces, then every its first-order flex tangent to the set of nonrigid surfaces can be extended to a second-order flex.

2. Definitions and notation

Following [7] and [4], we recall the standard definition of a higher-order flex of a smooth surface.

**Definition 2.1.** Let $S$ be a smooth boundary free surface in $\mathbb{R}^3$ with position vector $\mathbf{x}$, $n$ be a positive integer, and $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(n)}$ be smooth vector fields on $S$ (which are not supposed to be tangent to $S$). We say that the
Definition 2.1 leads to the following relation $d\mathbf{x}_t^2 - d\mathbf{x}^2 = o(t^n)$ for the first fundamental forms or, equivalently, $(d\mathbf{x}_t - d\mathbf{x})(d\mathbf{x}_t + d\mathbf{x}) = o(t^n)$ for differentials. Replacing $\mathbf{x}_t$ according to (2.1), we get

$$d(t\xi^{(1)} + t^2\xi^{(2)} + \cdots + t^n\xi^{(n)}) d(\mathbf{x} + t\xi^{(1)} + t^2\xi^{(2)} + \cdots + t^n\xi^{(n)}) = o(t^n)$$

and, thus, obtain the following equations

$$\begin{align*}
dx \, d\xi^{(1)} &= 0, \\
dx \, d\xi^{(2)} + d\xi^{(1)} \, d\xi^{(1)} &= 0, \\
&\vdots \\
dx \, d\xi^{(n)} + \sum_{j=1}^{n-1} d\xi^{(j)} \, d\xi^{(n-j)} &= 0.
\end{align*}$$

Note that the presence of factors 2 in (2.1) simplifies (2.2)–(2.4). This simplification was proposed by E. Rembs in [7] in 1932. Since that time, the expression (2.1) has been the standard notation for the $n$th-order flex.

Equation (2.2) means that $\mathbf{x} + 2t\xi^{(1)}$ is a first-order flex of S. In local coordinates $u, v$ on $\mathbf{S}$, (2.2) is equivalent to $(\mathbf{x}_u \, du + \mathbf{x}_v \, dv) (\xi^{(1)}_u \, du + \xi^{(1)}_v \, dv) = 0$ and, thus, to the following system of three partial differential equations

$$\mathbf{x}_u \cdot \xi^{(1)}_u = 0, \quad \mathbf{x}_u \cdot \xi^{(1)}_v + \mathbf{x}_v \cdot \xi^{(1)}_u = 0, \quad \mathbf{x}_v \cdot \xi^{(1)}_v = 0,$$

where · stands for the scalar product in $\mathbb{R}^3$. A first-order flex $\mathbf{x} + 2t\xi^{(1)}$ of $\mathbf{S}$ is called trivial if it is generated by a smooth family of isometries of $\mathbb{R}^3$. $\mathbf{S}$ is called first-order rigid if every its first-order flex is trivial; otherwise, $\mathbf{S}$ is called first-order nonrigid.

Equations (2.2) and (2.3) mean that $\mathbf{x} + 2t\xi^{(1)} + 2t^2\xi^{(2)}$ is a second-order flex of $\mathbf{S}$. It is called an extension of the first-order flex $\mathbf{x} + 2t\xi^{(1)}$.

Let us revil the reason for the traditional interest of geometers in infinitesimal flexes of smooth surfaces. The problem of whether a given surface $\mathbf{x}$ admits a finite flex (i.e., does there exist a family of surfaces $\mathbf{x}(t)$ such that $\mathbf{x}(0) = \mathbf{x}$ and, for each $t$, $\mathbf{x}(t)$ is isometric to $\mathbf{x}$ in the intrinsic metric, but is not obtained from $\mathbf{x}$ by an isometry of $\mathbb{R}^3$) is nonlinear. In local coordinates, this problem is reduced to solving a system of nonlinear partial differential equations corresponding to the fact that the coefficients of the first fundamental form of surface $\mathbf{x}(t)$ are independent of $t$, namely, $(\mathbf{x}(t))_u \cdot (\mathbf{x}(t))_u = \mathbf{x}_u \cdot \mathbf{x}_u$, $(\mathbf{x}(t))_u \cdot (\mathbf{x}(t))_v = \mathbf{x}_u \cdot \mathbf{x}_v$, $(\mathbf{x}(t))_v \cdot (\mathbf{x}(t))_v = \mathbf{x}_v \cdot \mathbf{x}_v$. Using the concept of higher-order flexes, the solution to the latter problem can be written in a form of an infinite series, whose partial sum is given by (2.1), and whose
terms satisfy an infinite system of linear partial differential equations, the first $n$ equations of which are given by (2.2)–(2.4). Thus, the original problem is simplified, because linear equations are easier to solve than nonlinear ones. It is well known that if the homogeneous equation (2.2) has trivial solutions only then surface $x$ does not admit finite flexes, see, e.g., [4,7]. Therefore, it is important to investigate the remaining possibility, when the homogeneous equation (2.2) has a nontrivial solution. In this case, the linear inhomogeneous equations (2.3)–(2.4) cannot be solvable for any right-hand side. For example, conditions on $d\xi^{(1)} d\xi^{(1)}$, under which (2.3) has a solution, are exactly the conditions for the extendibility of a first-order flex $x + 2t\xi^{(1)}$ to a second-order flex $x + 2t\xi^{(1)} + 2t^2\xi^{(2)}$, and conditions on

$$
\sum_{j=1}^{n-1} d\xi^{(j)} d\xi^{(n-j)},
$$

for which (2.4) has a solution, are exactly the conditions for the extendibility of an $(n-1)$th-order flex $x + 2t\xi^{(1)} + 2t^2\xi^{(2)} + \cdots + 2t^{n-1}\xi^{(n-1)}$ to an $n$th-order flex $x + 2t\xi^{(1)} + 2t^2\xi^{(2)} + \cdots + 2t^n\xi^{(n)}$. In linear algebra, functional analysis, and the theory of integral equations, several general conditions for the solvability of an inhomogeneous linear equations are known for the case, when the linear operator of this equation has a nonzero kernel. Probably, the most famous of those conditions is provided by the Fredholm alternative. In short, according to the Fredholm alternative, the inhomogeneous linear equation $Lf = g$ with a compact linear operator $L$ has a solution $f$ if and only if its right-hand side $g$ is orthogonal to every solution $F$ of the adjoint homogeneous equation $L^* F = 0$. Here $L^*$ is the adjoint operator for $L$. Unfortunately, partial differential operators do not enjoy the property of being compact, and the Fredholm alternative is not applicable for them. Therefore, geometers develop specific conditions for the solvability of the equations (2.3)–(2.4), calling them the conditions of the extendibility of the first-order flex to a second-order flex (in the case of the equation (2.3)) and of the extendibility of the $(n-1)$th-order flex to an $n$th-order flex (in the case of the equation (2.4)). More details about such specific conditions can be found, e.g., in [6,8], and references therein.

In this note, we develop a novel approach to the problem of the extendibility of first-order flexes to second-order flexes, which is rather different from the traditional approach. The following definition plays a central role in our study.

**Definition 2.2.** Let $S$ be a smooth boundary free surface in $\mathbb{R}^3$ with position vector $x$. We say that a first-order flex $x + 2t\xi^{(1)}$ of $S$ is tangent to the set of all nonrigid smooth surfaces if the following conditions hold true:

(i) there is a smooth family $\{S(r)\}_{r \in (-1,1)}$ of boundary free nonrigid smooth surfaces in $\mathbb{R}^3$ such that $S(0) = S$, i.e., $x(0) = x$, where $x(r)$ is position vector of $S(r)$;
(ii) there is a smooth family \( \{2\xi^{(1)}(r)\}_{r \in (-1,1)} \) of vector fields such that, for every \( r \in (-1,1) \), \( x(r) + 2t\xi^{(1)}(r) \) is a first-order flex of \( S(r) \) and
\[
\frac{d}{dr} \bigg|_{r=0} x(r) = 2\xi^{(1)}(0) \quad \text{and} \quad \xi^{(1)}(0) = \xi^{(1)}. \tag{2.5}
\]

Conditions (i) and (ii) of Definition 2.2 mean that \( S \) lies on the curve \( \{S(r)\}_{r \in (-1,1)} \), located in the set of nonrigid surfaces, and \( 2\xi^{(1)} \) is the velocity vector of the point \( x(r) \) moving along this curve at \( S \). That is, Definition 2.2 is consistent with the standard for classical differential geometry point of view on the tangent vector to a surface as on the velocity vector of a point moving along a curve lying on the surface.

3. Main result

**Theorem 3.1.** Let \( S \) be a smooth boundary free surface in \( \mathbb{R}^3 \) with position vector \( x \). And let \( x + 2t\xi^{(1)} \) be a first-order flex of \( S \), which is tangent to the set of all nonrigid smooth surfaces. Then the first-order flex \( x + 2t\xi^{(1)} \) can be extended to a second-order flex of \( S \).

**Proof.** Let \( \{S(r)\}_{r \in (-1,1)} \) and \( \{2\xi^{(1)}(r)\}_{r \in (-1,1)} \) be the smooth families from Definition 2.2. Since, for every \( r \in (-1,1) \), \( x(r) + 2t\xi^{(1)}(r) \) is a first-order flex of \( x(r) \), we have \( dx(r) \, d\xi^{(1)}(r) = 0 \). Differentiating this equality at \( r = 0 \) and taking into account (2.5), we get
\[
0 = \frac{d}{dr} \bigg|_{r=0} \left[ dx(r) \, d\xi^{(1)}(r) \right]
= d \left[ \frac{d}{dr} \bigg|_{r=0} x(r) \right] \, d\xi^{(1)}(0) + dx(0) \, d \left[ \frac{d}{dr} \bigg|_{r=0} \xi^{(1)}(r) \right]
= 2 \left[ d\xi^{(1)} \, d\xi^{(1)} + dx \, d\xi^{(2)} \right], \tag{3.1}
\]
where we have putten by definition
\[
2\xi^{(2)} = \frac{d}{dr} \bigg|_{r=0} \xi^{(1)}(r).
\]
It follows from (3.1) that \( dx \, d\xi^{(2)} + d\xi^{(1)} \, d\xi^{(1)} = 0 \) and, thus, the deformation \( x + 2t\xi^{(1)} + 2t^2\xi^{(2)} \) is a second-order flex of \( S \). On the other hand, \( x + 2t\xi^{(1)} + 2t^2\xi^{(2)} \) is obviously an extension of the first-order flex \( x + 2t\xi^{(1)} \). \( \square \)

Theorem 3.1 shows that, for a smooth surface \( S \) in \( \mathbb{R}^3 \) which is a smooth point of the set \( \mathcal{S} \) of all nonrigid surfaces, the condition that its first-order flex is tangent to \( \mathcal{S} \) implies that this first-order flex can be extended to a second-order flex. We cannot prove a similar statement in the case when \( S \) is not a smooth point of \( \mathcal{S} \), since we know nothing about the structure of \( \mathcal{S} \). For example, we do not know what its dimension (or codimension) is, nor what is
the structure of the set of its nonsmooth points. We can only hope that these issues will be clarified in the future.

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Declaration

Conflict of interest The author states that he does not have any conflicts of interest to declare.

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