ELEMENTARY PROOFS OF PALEY–WIENER THEOREMS FOR THE DUNKL TRANSFORM ON THE REAL LINE

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Abstract. We give an elementary proof of the Paley–Wiener theorem for smooth functions for the Dunkl transforms on the real line, establish a similar theorem for $L^2$-functions and prove identities in the spirit of Bang for $L^p$-functions. The proofs seem to be new also in the special case of the Fourier transform.

1. Introduction and overview

The Paley–Wiener theorem for the Dunkl transform $D_k$ with multiplicity $k$ (where $\text{Re}k \geq 0$) on the real line states that a smooth function $f$ has support in the bounded interval $[-R,R]$ if, and only if, its transform $D_k f$ is an entire function which satisfies the usual growth estimates as they are required in the (special) case of the Fourier transform. Various proofs of this result are known, all of which use explicit formulas available in this one-dimensional setting (see Remark 6 for more details). In this paper, however, we present an alternative proof which does not use such explicit expressions, being based almost solely on the formal properties of the transform. Along the same lines, we also obtain a Paley–Wiener theorem for $L^2$-functions for $k \geq 0$. The case $k = 0$ specializes to the Fourier transform, and to our knowledge the proofs of both Paley–Wiener theorems are new even in this case.

In addition, we establish two identities in the spirit of Bang [4, Theorem 1]. These results could be called real Paley–Wiener theorems (although terminology is not yet well-established), since they relate certain growth rates of a function on the real line to the support of its transform. The approach at this point is inspired by similar techniques in [2, 1, 3]. Our results in this direction partially overlap with [7, 6], but the new proofs are considerably simpler, as they are again based almost solely on the formal properties of the transform. We will comment on this in more detail later on, as these results will have been established. For $k = 0$, one retrieves Bang’s result; we feel that the present method of proof, which, e.g., does not use the Paley–Wiener theorem for smooth functions, but rather implies it, is then more direct than that in [4].

The rather unspecific and formal structure of the proofs suggests that the methods can perhaps be put to good use for other integral transforms with a symmetric kernel, both for the Paley–Wiener theorems and the equalities in the spirit of Bang (cf. Remark 6). The structure of the proof is also such that, if certain combinatorial problems can be surmounted, a proof of the Paley–Wiener theorem for the Dunkl transform for invariant balanced compact convex sets in arbitrary dimension might be possible. This would be further evidence for the validity of this theorem for invariant compact convex sets (cf. [13, Conjecture 4.1]), but at the time of writing this higher-dimensional result has not been established.

This paper is organized as follows. In Section 2 the necessary notations and previous results are given. Section 3 contains the Paley–Wiener theorem for smooth functions and can—perhaps—serve as a model for a proof of such a theorem in other contexts. The rest of the paper is independent.

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of this section. Section 4 is concerned with the real Paley–Wiener theorem in the $L^p$-case and the $L^2$-case is settled in Section 5.

2. DUNKL OPERATORS AND THE DUNKL TRANSFORM ON $\mathbb{R}$

The Dunkl operators and the Dunkl transform were introduced for arbitrary root systems by Dunkl [5, 10, 11]. In this section we recall some basic properties for the one-dimensional case of $A_1$, referring to [17, Sections 1 and 2] for a more comprehensive overview and to [8, 9, 11, 12, 14, 16] for details. We suppress the various explicit formulas which are known in this one-dimensional context (as these are not necessary for the proofs), thus emphasizing the basic structure of the problem which might lead to generalizations to the case of arbitrary root systems.

Let $k \in \mathbb{C}$, and consider the Dunkl operator $T_k$

$$T_k f(x) = f'(x) + k \frac{f(x) - f(-x)}{x} \quad (f \in C^\infty(\mathbb{R}), \ x \in \mathbb{R}).$$

Rewriting this as

\begin{equation}
T_k f(x) = f'(x) + k \int_{-1}^{1} f'(tx) \, dt,
\end{equation}

it follows that $T_k$ maps $C^\infty(\mathbb{R}), C_c^\infty(\mathbb{R})$ and the Schwartz space $\mathcal{S}(\mathbb{R})$ into themselves.

If $\text{Re} \ k \geq 0$, as we will assume for the remainder of this section, then, for each $\lambda \in \mathbb{C}$, there exists a unique holomorphic solution $\psi^k_\lambda : \mathbb{C} \to \mathbb{C}$ of the differential-reflection problem

\begin{equation}
\begin{cases}
T_k f = i\lambda f, \\
f(0) = 1.
\end{cases}
\end{equation}

The map $(z, \lambda) \to \psi^k_\lambda(z)$ is entire on $\mathbb{C}^2$, and we have the estimate

\begin{equation}
|\psi^k_\lambda(z)| \leq e^{\|\lambda z\|} \quad (\lambda, z \in \mathbb{C}).
\end{equation}

In view of (3) the Dunkl transform $D_k f$ of $f \in L^1(\mathbb{R}, |w_k(x)| \, dx)$, where the complex-valued weight function $w_k$ is given by $w_k(x) = |x|^{2k}$, is meaningfully defined by

\begin{equation}
D_k f(\lambda) = \frac{1}{c_k} \int_{\mathbb{R}} f(x) \psi^k_\lambda(x) w_k(x) \, dx \quad (\lambda \in \mathbb{R}),
\end{equation}

where

$$c_k = \int_{\mathbb{R}} e^{-\frac{|x|^2}{2}} w_k(x) \, dx \neq 0.$$

We note that $D_0$ is the Fourier transform on $\mathbb{R}$. From (3) we conclude that $D_k f$ is bounded for such $f$, in fact

\begin{equation}
|D_k f(\lambda)| \leq \frac{1}{|c_k|} \int_{\mathbb{R}} |f(x)||w_k(x)| \, dx \quad (\lambda \in \mathbb{R}, f \in L^1(\mathbb{R}, |w_k(x)| \, dx)).
\end{equation}

The Dunkl transform is a topological isomorphism of $\mathcal{S}(\mathbb{R})$ onto itself, the inverse transform $D_k^{-1}$ being given by

$$D_k^{-1} f(x) = \frac{1}{c_k} \int_{\mathbb{R}} f(\lambda) \psi^k_\lambda(x) w_k(\lambda) \, d\lambda = D_k f(-x) \quad (f \in \mathcal{S}(\mathbb{R}), x \in \mathbb{R}).$$

The operator $T_k$ is anti-symmetric with respect to the weight function $w_k$, i.e.,

\begin{equation}
\langle T_k f, g \rangle_k = -\langle f, T_k g \rangle_k,
\end{equation}

for $f \in \mathcal{S}(\mathbb{R})$ and $g \in C^\infty(\mathbb{R})$ such that both $g$ and $T_k g$ are of at most polynomial growth. Here $\langle f, g \rangle_k$ is defined by

$$\langle f, g \rangle_k = \int_{\mathbb{R}} f(x) g(x) w_k(x) \, dx,$$
for functions $f$ and $g$ such that $fg \in L^1(\mathbb{R}, |w_k(x)| dx)$. In particular, (6) yields the intertwining identity

$$D_k(T_k f)(\lambda) = i\lambda(D_k f)(\lambda) \quad (f \in \mathcal{S}(\mathbb{R}), \lambda \in \mathbb{R}).$$

Furthermore, for $\lambda, z, s \in \mathbb{C}$ the symmetry properties $\psi^k_s(z) = \psi^k_s(\lambda)$ and $\psi^k_s(z) = \psi^k_s(sz)$ are valid. Using the first of these and (5), an application of Fubini gives

$$\langle D_k f, g \rangle_k = \langle f, D_k g \rangle_k \quad (f, g \in L^1(\mathbb{R}, |w_k(x)| dx)).$$

If $k \geq 0$, the Plancherel theorem states that $D_k$ preserves the weighted two-norm on $L^1(\mathbb{R}, w_k(x)dx) \cap L^2(\mathbb{R}, w_k(x)dx)$ and extends to a unitary operator on $L^2(\mathbb{R}, w_k(x)dx)$.

### 3. Paley–Wiener theorem for smooth functions

The method of proof in this section is inspired by results of Bang [4]. To be more specific, for $R > 0$ let $\mathcal{H}_R(\mathbb{C})$ denote the space of entire functions $f$ with the property that, for all $n \in \mathbb{N} \cup \{0\}$, there exists a constant $C_{n, f} > 0$ such that

$$|f(z)| \leq C_{n, f}(1 + |z|)^{-n} e^{R|\text{Im} z|} \quad (z \in \mathbb{C}).$$

Then, if $k \geq 0$ and $f \in \mathcal{H}_R(\mathbb{C})$, we will establish that (cf. [4])

$$\sup\{ |\lambda| \mid \lambda \in \text{supp} D_k f \} \leq \liminf_{n \to \infty} \| T^n_k f \|_{L^1}^{1/n} \leq \limsup_{n \to \infty} \| T^n_k f \|_{L^1}^{1/n} \leq R,$$

after which the proof of the Paley–Wiener theorem for smooth functions is a mere formality.

Starting towards the third of these inequalities, we first use (7) to gain control over repeated Dunkl derivatives.

**Lemma 1.** Let $k \in \mathbb{C}$, $f \in C^\infty(\mathbb{R})$ and $n \in \mathbb{N}$. Then

$$T^n_k f(x) = (T^n_{k-1}(f'))(x) + k \int_{-1}^1 t^{n-1} (T^n_{k-1}(f'))(tx) \, dt \quad (x \in \mathbb{R}).$$

**Proof.** For $g \in C^\infty(\mathbb{R})$ and $m \in \mathbb{N} \cup \{0\}$, let $I_{g,m} \in C^\infty(\mathbb{R})$ be defined by

$$I_{g,m}(x) = \int_{-1}^1 t^m g(tx) \, dt \quad (x \in \mathbb{R}).$$

Using (8), we then find

$$(T_k I_{g,m})(x) = \int_{-1}^1 t^{m+1} g'(t_1 x) \, dt_1 + k \int_{-1}^1 \int_{-1}^1 t^{m+1} g'(t_1 t_2 x) \, dt_1 \, dt_2$$

$$= \int_{-1}^1 t^{m+1} \left( g'(t_1 x) + k \int_{-1}^1 g'(t_1 t_2 x) \, dt_2 \right) \, dt_1$$

$$= \int_{-1}^1 t^{m+1} (T_k g)(t_1 x) \, dt_1 = I_{T_k g, m+1}(x).$$

We conclude that $T_k I_{g,m} = I_{T_k g, m+1}$. Since (11) can be written as $T_k f = f' + kI_{f', 0}$ one has

$$T^n_k f = T^n_{k-1}(f') + kI_{T^n_{k-1}(f'), n-1},$$

which is the statement in the lemma.

It follows from Lemma 1 that

$$|T^n_k f(x)| \leq \left( 1 + \frac{2|k|}{n} \right) \sup_{y \in [-|x|,|x|]} |(T^n_{k-1}(f'))(y)| \quad (x \in \mathbb{R}),$$

and induction then yields the following basic estimate, which is more explicit than [7] Prop. 2.1.
Corollary 2. Let \( k \in \mathbb{C}, f \in C^\infty(\mathbb{R}) \) and \( n \in \mathbb{N} \). Then
\[
|T_k^n f(x)| \leq \frac{\Gamma(n + 1 + 2|k|)}{n! \Gamma(1 + 2|k|)} \sup_{y \in [-|x|, |x|]} |f^{(n)}(y)| \quad (x \in \mathbb{R}).
\]

The third inequality in (8) can now be settled.

Proposition 3. Let \( k \in \mathbb{C}, R > 0 \), and suppose \( f : \mathbb{C} \to \mathbb{C} \) is an entire function such that
\[
|f(z)| \leq Ce^{R|\operatorname{Im} z|} \quad (z \in \mathbb{C}),
\]
for some positive constant \( C \). Then, for all \( n \in \mathbb{N}, T_k^n f \) is bounded on the real line, and
\[
\limsup_{n \to \infty} \|T_k^n f\|^{1/n} \leq R.
\]

Proof. We have, for any \( r > 0 \),
\[
f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|\zeta - z| = r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad (z \in \mathbb{C}).
\]
If \(|\zeta - z| = r\), then
\[
|f(\zeta)| \leq Ce^{R(|\operatorname{Im} z|+r)},
\]
implying
\[
|f^{(n)}(z)| \leq C \frac{n!}{r^n} e^{R(|\operatorname{Im} z|+r)} \quad (z \in \mathbb{C}).
\]
Choosing \( r = n/R \), so that \( r > 0 \) if \( n \in \mathbb{N} \), we find
\[
|f^{(n)}(z)| \leq C \frac{n!e^n}{n^n} R^n e^{R|\operatorname{Im} z|} \quad (n \in \mathbb{N}, z \in \mathbb{C}),
\]
whence \( \|f^{(n)}\|_\infty \leq C n! e^{n-n} R^n \), for \( n \in \mathbb{N} \). Combining this with Corollary 2 yields
\[
\|T_k^n f\|_\infty \leq C \frac{n! \Gamma(n + 1 + 2|k|)}{n^n \Gamma(1 + 2|k|)} R^n \quad (n \in \mathbb{N}).
\]
The result now follows from Stirling’s formula. \( \square \)

As to the first inequality in (8), it is actually easy to prove that it holds for the norm \( \| \cdot \|_{k,p} \) in \( L^p(\mathbb{R}, w_k(x)dx) \) for arbitrary \( 1 \leq p \leq \infty \) (not just for \( p = \infty \)), as is shown by the following lemma. It should be noted that this result can be generalized—with different proofs—to complex multiplicities (cf. Lemma 4) and to \( L^p \)-functions for \( k \geq 0 \) (cf. Theorem 10), but we present it here separately nevertheless, in order to illustrate that for the case \( k \geq 0 \), the proof of one of the crucial inequalities (as far as the Paley–Wiener theorem is concerned) is rather elementary and intuitive.

Lemma 4. Let \( k \geq 0, 1 \leq p \leq \infty \) and \( f \in S(\mathbb{R}) \). Then in the extended positive real numbers,
\[
(9) \quad \liminf_{n \to \infty} \|T_k^n f\|_{1/p} \geq \sup \{ |\lambda| : \lambda \in \operatorname{supp} D_k f \}.
\]

Proof. Suppose \( 0 \neq \lambda_0 \in \operatorname{supp} D_k f \) and let \( 0 < \varepsilon < |\lambda_0| \). Define
\[
\phi(\lambda) = D_k f(\lambda) \quad (\lambda \in \mathbb{R}).
\]
Then, with \( \phi \) denoting the conjugate exponent and using (7), we find
\[
\|T_k^{2n} f\|_{k,p} \|D_k \phi\|_{k,q} \geq |\langle T_k^{2n} f, D_k \phi \rangle_k| = |\langle D_k(T_k^{2n} f), \phi \rangle_k|
\]
\[
= \int_{\mathbb{R}} (i\lambda)^{2n} D_k f(\lambda) \phi(\lambda) w_k(\lambda)d\lambda
\]
\[
= \int_{\mathbb{R}} \lambda^{2n} |D_k f(\lambda)|^2 w_k(\lambda)d\lambda
\]
\[
\geq (|\lambda_0| - \varepsilon)^{2n} \int_{|\lambda| \geq |\lambda_0| - \varepsilon} |D_k f(\lambda)|^2 w_k(\lambda)d\lambda.
\]
With \( \psi(\lambda) = D_k(T_kf)(\lambda) \) we similarly get
\[
\|T_k^{2n+1}f\|_{k,p} \geq (|\lambda_0| - \varepsilon)^{2n+1} \int_{|\lambda| \geq |\lambda_0| - \varepsilon} |\lambda||D_kf(\lambda)|^2 w_k(\lambda) d\lambda.
\]
These two estimates together yield
\[
\lim inf_{n \to \infty} \|T_k^n f\|_{k,p}^{1/n} \geq |\lambda_0| - \varepsilon,
\]
and the lemma follows. \( \square \)

Now that (8) has been established, we come to the Paley–Wiener theorem for smooth functions.

Introducing notation, for \( R > 0 \), we let \( C^\infty_{\mathbb{R}}(\mathbb{R}) \) denote the space of smooth functions on \( \mathbb{R} \) with support in \([-R,R]\). Its counterpart under the Dunkl transform, the space \( H_R(\mathbb{C}) \), was defined at the beginning of this section.

**Theorem 5** (Paley–Wiener theorem for smooth functions). Let \( R > 0 \) and \( k \geq 0 \). Then the Dunkl transform \( D_k \) is a bijection from \( C^\infty_{\mathbb{R}}(\mathbb{R}) \) onto \( H_R(\mathbb{C}) \).

**Proof.** If \( f \in C^\infty_{\mathbb{R}}(\mathbb{R}) \), then it is easy to see that \( D_kf \in H_R(\mathbb{C}) \) \( [12, \text{Corollary 4.10}] \). Now assume that \( f \in H_R(\mathbb{C}) \). Using Cauchy’s integral representation as in the proof of Proposition \( \text{3} \) we retrieve the well-known fact that \( f \in S(\mathbb{R}) \). From \( \text{3} \) we infer that \( D_kf \) has support in \([-R,R] \). Since \( D_k^{-1}f(x) = D_kf(-x) \), for \( x \in \mathbb{R} \), the same is true for \( D_k^{-1}f \), as was to be proved. \( \square \)

**Remark 6.**

(1) By holomorphic continuation and continuity, cf. \( \text{13} \), one sees that Theorem \( \text{5} \) also holds in the more general case \( \text{Re } k \geq 0 \). Alternatively, one can use Lemma \( \text{7} \) below instead of Lemma \( \text{4} \) which establishes \( \text{8} \) also in the case \( \text{Re } k \geq 0 \), and then the above direct proof is again valid.

(2) We emphasize that the present proof does not use any explicit formulas for the Dunkl kernel in one dimension, contrary to the alternative methods of proof in \( \text{20} \) (where Weyl fractional integral operators are used), \( \text{12} \) (where asymptotic results for Bessel functions are needed) and \( \text{7} \) (where various integral operators, Dunkl’s intertwining operator and the Paley–Wiener theorem for the Fourier transform all play a role). Also, the fact that a contour shifting argument for the transform is usually not possible (since \( w_k \) generically has no entire extension) is no obstruction. Given this unspecific nature, it is possible that the present method can be applied to other transforms as well, although the symmetry of the kernel—as reflected in \( \text{7} \), which was used in the proof of Lemma \( \text{4} \) and which will again be used in the proof of the alternative Lemma \( \text{7} \) below—is perhaps necessary. The same suggestion applies to the results in the remaining sections of this paper.

### 4. Real Paley–Wiener theorem for \( L^p \)-functions

We will now consider the real Paley–Wiener theorem for \( L^p \)-functions in the spirit of Bang \( \text{4} \). The result is first proved for Schwartz functions in Theorem \( \text{5} \) and subsequently for the general case in Theorem \( \text{10} \).

Let \( \| \cdot \|_{\text{Re } k,p} \) denote the \( L^p(\mathbb{R}, |w_k(x)| dx) \)-norm, for \( 1 \leq p \leq \infty \). Then we have the following generalization of Lemma \( \text{4} \) to complex multiplicities.

**Lemma 7.** Let \( \text{Re } k \geq 0, 1 \leq p \leq \infty \) and \( f \in S(\mathbb{R}) \). Then in the extended positive real numbers,
\[
(10) \quad \lim inf_{n \to \infty} \|T_k^n f\|_{\text{Re } k,p}^{1/n} \geq \sup\{ |\lambda| : \lambda \in \text{supp } D_kf \}.
\]
Proof. Let $0 \neq \lambda_0 \in \text{supp } D_k f$ and choose $\varepsilon > 0$ such that $0 < 2\varepsilon < |\lambda_0|$. Also choose $\phi \in C^\infty_c(\mathbb{R})$ such that supp $\phi \subset [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$, and $(D_k f, \phi)_k \neq 0$. Define $\phi_n(\lambda) = \lambda^{-n} \phi(\lambda)$ and $P_n(x) = x^n$ for $n \in \mathbb{N} \cup \{0\}$. Then

$$(1 + P_N(x))(D_k \phi_n)(x) = \frac{1}{c_k} \int_{\lambda_0 - \varepsilon}^{\lambda_0 + \varepsilon} (1 + (iT_k)^N) (\lambda^{-n} \phi(\lambda)) \psi_n^k(\lambda) w_k(\lambda) d\lambda \quad (N \in \mathbb{N} \cup \{0\}).$$

We fix $N$ such that $N$ is even and $N > 2 \text{Re } k + 1$.

Corollary 2 and the binomial formula imply that

$$\left| (1 + (iT_k)^N) (\lambda^{-n} \phi(\lambda)) \right| \leq C_1 n^N |\lambda_0|^{-n} \quad (n \in \mathbb{N} \cup \{0\}, \lambda \in \mathbb{R}),$$

where $C_1$ is a positive constant. This yields the estimates

$$\|D_k \phi_n\|_{\text{Re } k,q} \leq \|(1 + P_N)^{-1}\|_{\text{Re } k,q} \|(1 + P_N)D_k \phi_n\|_{\infty} \leq \frac{2\varepsilon}{|c_k|} C_1 n^N |\lambda_0|^{-n} \|(1 + P_N)^{-1}\|_{\text{Re } k,q} \leq C_2 n^N |\lambda_0|^{-n}$$

for all $n > N$, where $C_2$ is a positive constant and $q$ is the conjugate exponent.

Using (7), the identity $D_k (P_n \phi_n) = (-i)^n T_k^n D_k \phi_n$ and Hölder’s inequality, we therefore get

$$\|\langle D_k f, \phi \rangle\|_{k,q} = \|\langle D_k f, P_n \phi_n \rangle\|_{k,q} = \|\langle f, D_k (P_n \phi_n) \rangle\|_{k,q} = \|\langle f, T_k^n (D_k \phi_n) \rangle\|_{k,q} \leq \|T_k^n f\|_{\text{Re } k,p} \|D_k \phi_n\|_{\text{Re } k,q} \leq C_2 n^N |\lambda_0|^{-n} \|T_k^n f\|_{\text{Re } k,p},$$

whence

$$\liminf_{n \to \infty} \|T_k^n f\|_{\text{Re } k,p}^{1/n} \geq \liminf_{n \to \infty} (C_2 n^N)^{1/n} |\lambda_0|^{-\varepsilon} \|\langle D_k f, \phi \rangle\|_{k,q}^{1/n} = (|\lambda_0| - \varepsilon),$$

establishing the lemma.

Using the Paley–Wiener theorem, we can extend the inequality in Proposition 3 to the norms $\| \cdot \|_{\text{Re } k,p}$, $1 \leq p \leq \infty$, for $\text{Re } k \geq 0$, and we thus have the following theorem.

**Theorem 8** (real Paley–Wiener theorem for Schwartz functions). Let $\text{Re } k \geq 0$, $1 \leq p \leq \infty$ and $f \in \mathcal{S}(\mathbb{R})$. Then in the extended positive real numbers,

$$\lim_{n \to \infty} \|T_k^n f\|_{\text{Re } k,p}^{1/n} = \sup \{ |\lambda| : \lambda \in \text{supp } D_k f \}.$$

**Proof.** In view of Lemma 4, it only remains to be shown that

$$\limsup_{n \to \infty} \|T_k^n f\|_{\text{Re } k,p}^{1/n} \leq R,$$

if $f \in \mathcal{S}(\mathbb{R})$ is such that $\text{supp } D_k f \subset [-R, R]$ for some finite $R > 0$. Using the inversion formula and the intertwining properties of the transform we have

$$(11) \quad x^N T_k^n f(x) = \frac{i^{N+n}}{c_k} \int_{-R}^{R} T_k^N (P_n D_k f)(\lambda) \psi_n^k(x) w_k(\lambda) d\lambda \quad (n, N \in \mathbb{N} \cup \{0\}),$$

where again $P_n(x) = x^n$. Now Corollary 2 and the binomial formula imply that

$$\|T_k^N (P_n D_k f)\|_{\infty} \leq C_1 n^N R^n,$$

where $C_1$ is a constant depending on $f$ and $N$. Therefore (11) yields that

$$(12) \quad \|(1 + P_N) T_k^n f\|_{\infty} \leq C_2 n^N R^{n+1},$$

where $C_2$ is again a constant depending on $f$ and $N$. We fix $N$ such that $N$ is even and $N > 2 \text{Re } k+1$. Then the observation

$$\|T_k^n f\|_{\text{Re } k,p} \leq \|(1 + P_N)^{-1}\|_{\text{Re } k,p} \|(1 + P_N) T_k^n f\|_{\infty}$$
and \[ (12) \] establish the result. \( \square \)

**Remark 9.** Theorem 8 is new for complex \( k \). For real \( k \), the result can be found in [7], where it is proved using the Plancherel theorem for the Dunkl transform, the Riesz–Thorin convexity theorem and the theory of Sobolev spaces for Dunkl operators.

For \( k \geq 0 \), we will now generalize Theorem 8 to the \( L^p \)-case in Theorem 10 using the structure of \( S(\mathbb{R}) \) as an associative algebra under the Dunkl convolution \( *_k \). We refer to [19] for details on this subject.

Let \( k \geq 0 \), and define a distributional Dunkl transform \( D_k^n f \) of \( f \in L^p(\mathbb{R}, w_k(x)dx) \) by transposition

\[
\langle D_k^n f, \phi \rangle = (f, D_k^* \phi)_k \quad (\phi \in S(\mathbb{R})).
\]

Clearly \( D_k^p \) is injective on \( L^p(\mathbb{R}, w_k(x)dx) \). Furthermore, from loc.cit., one knows that

\[
|\langle D_k^n f, \phi \rangle| \leq \|f\|_p \|D_k \phi\|_\infty \quad (\phi \in S(\mathbb{R})),
\]

we see that \( D_k^n f \) is a tempered distribution. Note also that for \( f \in L^p(\mathbb{R}, w_k(x)dx) \) and \( 1 \leq p \leq 2 \) we have

\[
\langle D_k^n f, \phi \rangle = (D_k f, \phi)_k \quad (\phi \in S(\mathbb{R})),
\]

by (7) and density of \( S(\mathbb{R}) \) in \( L^p(\mathbb{R}, w_k(x)dx) \). Thus, for \( f \in L^p(\mathbb{R}, w_k(x)dx) \) and \( 1 \leq p \leq 2 \), \( D_k^n f \) as defined above corresponds to the distribution \( (D_k f)w_k \), implying that in this case \( \text{supp } D_k^n f = \text{supp } D_k f \).

**Theorem 10** (real Paley–Wiener theorem for \( L^p \)-functions). Let \( k \geq 0 \), \( 1 \leq p \leq \infty \) and \( f \in C^\infty(\mathbb{R}) \) be such that \( T_k^n f \in L^p(\mathbb{R}, w_k(x)dx) \), for all \( n \in \mathbb{N} \cup \{0\} \). Then in the extended positive real numbers,

\[
\lim_{n \to \infty} \|T_k^n f\|_{1/n}^{1/n} = \sup \{\lambda : \lambda \in \text{supp } D_k f\}.
\]

If in addition \( f \in L^s(\mathbb{R}, w_k(x)dx) \), for some \( 1 \leq s \leq 2 \), then the distribution \( D_k^n f \) corresponds to the function \( (D_k f)w_k \), and the support of \( D_k^n f \) in the right hand side is equal to the support of \( D_k f \) as a distribution.

**Proof.** First we note that (10) also holds for \( f \) as above: in the proof of Lemma 7 we just have to change \( \langle D_k f, \cdot \rangle_k \) into \( \langle D_k^n f, \cdot \rangle_k \). Therefore, it only remains to be shown that

\[
(13) \quad \lim_{n \to \infty} \sup \|T_k^n f\|_{1/n}^{1/n} \leq R,
\]

if \( f \) is as in the theorem and such that \( \text{supp } D_k^n f \subset [-R, R] \) for some finite \( R > 0 \). To this end, choose \( \varepsilon > 0 \), and fix a function \( \phi_\varepsilon \in S(\mathbb{R}) \) such that \( D_k^{-1} \phi_\varepsilon = 1 \) on \([-R, R]\) and \( D_k^{-1} \phi_\varepsilon = 0 \) outside \([-R-\varepsilon, R+\varepsilon]\). We have from [19] Proposition 3] that \( D_k^{-1}(\phi \ast_k \psi) = (D_k^{-1} \phi)(D_k^{-1} \psi) \), for all \( \phi, \psi \in S(\mathbb{R}) \). With \( P_n(x) = x^n \), we thus find for arbitrary \( \phi \in S(\mathbb{R}) \) that

\[
(\phi, \ast_k T_k^n \phi)_k = (D_k^n f, D_k^{-1} (\phi \ast_k T_k^n \phi)) = (-i)^n (D_k^n f, P_n(D_k^{-1} \phi)(D_k^{-1} \phi)) = (-i)^n (D_k^n f, P_n(D_k^{-1} \phi)(D_k^{-1} \phi)) = (f, T_k^n \phi)_k.
\]

Furthermore, from loc.cit., one knows that \( \|\phi \ast_k \psi\|_{k,q} \leq 4\|\phi\|_{k,q}\|\psi\|_{k,1} \), for all \( \phi, \psi \in S(\mathbb{R}) \), where \( q \) is the conjugate exponent of \( p \). If we combine these two results with (10) and Hölder’s reverse inequality, we infer that

\[
\|T_k^n f\|_{k,p} = \sup \|\langle T_k^n f, \phi \rangle_k\| = \sup \|\langle f, T_k^n \phi \rangle_k\| = \sup \|\langle f, \phi \ast_k T_k^n \phi \rangle_k\| \leq 4\|f\|_{k,p}\|T_k^n \phi\|_{k,1},
\]

where the supremum is over all functions \( \phi \in S(\mathbb{R}) \) with \( \|\phi\|_{k,q} = 1 \). From Theorem 8 we therefore conclude that

\[
\lim_{n \to \infty} \sup \|T_k^n f\|_{k,p}^{1/n} \leq R + \varepsilon,
\]

proving (13).

The statement on supports was established in the discussion preceding the theorem. \( \square \)
5. Paley–Wiener theorem for $L^2$-functions (for $k \geq 0$)

In this section we assume that $k \geq 0$. For $R > 0$ and $1 \leq p \leq \infty$, we define $L^p_R(\mathbb{R}, w_k(x)dx)$ to be the subspace of $L^p(\mathbb{R}, w_k(x)dx)$ consisting of those functions with distributional support in $[-R, R]$, and we let $H^{p,k}_R(\mathbb{C})$ denote the space of entire functions $f : \mathbb{C} \to \mathbb{C}$ which belong to $L^p(\mathbb{R}, w_k(x)dx)$ when restricted to the real line and which are such that

$$|f(z)| \leq C_f e^{R|\text{Im } z|} \quad (z \in \mathbb{C}),$$

for some positive constant $C_f$.

**Theorem 12** (Paley–Wiener theorem for $L^2$-functions). Let $R > 0$ and $k \geq 0$. Then the Dunkl transform $D_k$ is a bijection from $L^2_R(\mathbb{R}, w_k(x)dx)$ onto $H^{2,k}_R(\mathbb{C})$.

**Proof.** Let $f \in L^2_R(\mathbb{R}, w_k(x)dx)$. Then $f \in L^1(\mathbb{R}, w_k(x)dx)$, and [5] and [8] together with the Plancherel theorem imply that $D_k f \in H^{2,k}_R(\mathbb{C})$. Conversely, let $f \in H^{2,k}_R(\mathbb{C})$. By the Plancherel theorem one has $D^{-1}_k f \in L^2(\mathbb{R}, w_k(x)dx)$. In addition, Proposition 4 and Theorem 10 (with $p = \infty$) show that $	ext{supp } D_k f \subset [-R, R]$. The same is then true for $D^{-1}_k f$, and the result follows. □

**Remark 13.**

1. If $f \in H^{p,k}_R(\mathbb{C})$ and $1 \leq p \leq \infty$, then using Proposition 3 and Theorem 10 as in the proof of Theorem 12 one sees that $D^p_k f$ has support in $[-R, R]$. In particular, for $1 \leq p \leq 2$ with conjugate exponent $q$, we conclude that $D_k$ maps $H^{p,k}(\mathbb{C})$ into $L^q(\mathbb{R}, w_k(x)dx)$. For even functions, when the Dunkl transform can be identified with the Hankel transform, the latter result can be found in [11].

2. As an ingredient for the discussion of the relation with the literature on the Fourier transform, let us make the preliminary observation that, for $R > 0$ and $1 \leq p \leq \infty$, an entire function $f$ is in $H^{p,0}_R(\mathbb{C})$ if, and only if, its restriction to the real line is in $L^p(\mathbb{R}, dx)$ and moreover

$$|f(z)| \leq C_f e^{R|z|} \quad (z \in \mathbb{C}),$$

for some positive constant $C_f$ [2] Theorems 6.2.4 and 6.7.1]. This being said, for $p = 1$ the specialization of the first part of this remark to $k = 0$ therefore proves part of the statement in [5] Theorem 6.8.11], and for $1 < p < 2$ this specialization proves the first statement of [5] Theorem 6.8.13].

3. The aforementioned result about entire functions shows that the specialization to $k = 0$ of Theorem 12 is equivalent to the original Paley–Wiener theorem for the Fourier transform (see [14] or [18] Theorem 19.3]). The present proof seems to be more in terms of general principles than other proofs seen in the literature.

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