Abstract

Suppose that particles are randomly distributed in $\mathbb{R}^d$, and they are subject to identical stochastic motion independently of each other. The Smoluchowski process describes fluctuations of the number of particles in an observation region over time. This paper studies properties of the Smoluchowski processes and considers related statistical problems. In the first part of the paper we revisit probabilistic properties of the Smoluchowski process in a unified and principled way: explicit formulas for generating functionals and moments are derived, conditions for stationarity and Gaussian approximation are discussed, and relations to other stochastic models are highlighted. The second part deals with statistics of the Smoluchowski processes. We consider two different models of the particle displacement process: the undeviated uniform motion (when a particle moves with random constant velocity along a straight line) and the Brownian motion displacement. In the setting of the undeviated uniform motion we study the problems of estimating the mean speed and the speed distribution, while for the Brownian displacement model the problem of estimating the diffusion coefficient is considered. In all these settings we develop estimators with provable accuracy guarantees.

Keywords: Smoluchowski processes, generating functionals, stationary processes, covariance function, nonparametric estimation, kernel estimators.

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1 Introduction

Suppose that we have an infinite number of particles that are randomly distributed in $\mathbb{R}^d$. The particles are subject to the same stochastic movement independently of each other. We are interested in characteristics of the stochastic process that governs the particle movement, e.g., in diffusion coefficients. The natural assumption is that the particles are indistinguishable, and the trajectory of a single particle cannot be tracked over time. In this setting Smoluchowski (1906,1914) assumed that particles perform Brownian motion, and suggested to measure the number of particles (concentration) in a fixed region over time in order to determine unknown movement characteristics. The classical works of Smoluchowski on concentration fluctuations are of fundamental importance in statistical physics: they are the part of the celebrated Einstein–Smoluchowski theory that provided a molecular–kinetic explanation of the Brownian movement. The exposition of probabilistic aspects of Smoluchowski’s theory is given in classical surveys by Chandrasekhar (1943) and Kac (1959, Chapter III, Sections 21-28), and, more recently, in Bingham & Dunham (1997); we refer also to Mazo (2002) for historical background and additional information.
The model. Consider the following model of moving particles in $\mathbb{R}^d$. Let $\Xi := \sum_{j \in \mathbb{Z}} \varepsilon_{\xi_j}$ be a homogeneous Poisson process on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with rate $\lambda \in (0, \infty)$; here $\varepsilon_x$ is the Dirac measure at $x \in \mathbb{R}^d$. The location of $j$-th particle at time $t$ is determined by the equation
\[ X_t^{(j)} = \xi_j + Y_t^{(j)}, \quad j \in \mathbb{Z}, \quad t \geq 0, \tag{1.1} \]
where $\{Y_t^{(j)}, t \geq 0\}$, $j \in \mathbb{Z}$ are independent copies of random process $\{Y_t, t \geq 0\}$ in $\mathbb{R}^d$ with $Y_0 = 0$.

In what follows process $\{X_t, t \geq 0\}$ is referred to as the displacement process. We assume that Poisson process $\Xi$ describing the initial positions of particles and displacement processes $\{Y_t^{(j)}, t \geq 0\}$, $j \in \mathbb{Z}$ are independent. We also denote $\{X_t, t \geq 0\}$ to be a generic location process with $\{X_t^{(j)}, t \geq 0\}$, $j \in \mathbb{Z}$ being the independent copies of $\{X_t, t \geq 0\}$.

Let $B$ be a compact set with non-empty interior in $\mathbb{R}^d$ representing the observation region, and define
\[ N(t) := \sum_{j \in \mathbb{Z}} 1_B(X_t^{(j)}) = \sum_{j \in \mathbb{Z}} 1\{X_t^{(j)} \in B\}, \quad t \geq 0. \tag{1.2} \]
By definition, process $\{N(t), t \geq 0\}$ counts the number of particles in the observation region $B$ over time. In all what follows we refer to random process $\{N(t), t \geq 0\}$ as the Smoluchowski process.

Different particle displacement processes $\{Y_t, t \geq 0\}$ give rise to different versions of the Smoluchowski process. The following two models of displacement are of particular interest.

(a) Undeviated uniform motion. Let $(v_j)_{j \in \mathbb{Z}}$ be a sequence of iid random vectors in $\mathbb{R}^d$ with common distribution function $G$, independent of $\Xi$. Assume that
\[ Y_t^{(j)} = v_j t, \quad j \in \mathbb{Z}, \quad t \geq 0. \tag{1.3} \]
In this model particles move along straight lines with constant velocity which varies from particle to particle according to probability distribution $G$. We refer to $v_j \in \mathbb{R}^d$ as the $j$-th particle velocity, while the Euclidean norm $\|v_j\|$ of $v_j$ is the corresponding particle speed.

(b) Brownian displacement model. Assume that
\[ Y_t^{(j)} = \sigma W_t^{(j)}, \quad j \in \mathbb{Z}, \quad t \geq 0, \tag{1.4} \]
where $\{W_t^{(j)}, t \geq 0\}$, $j \in \mathbb{Z}$ are independent standard $d$-dimensional Brownian motions in $\mathbb{R}^d$, and $\sigma > 0$ is the diffusion coefficient. Under this setting particles perform Brownian motion.

The Smoluchowski process $\{N(t), t \geq 0\}$ associated with Brownian displacement [1.4] was originally considered by Smoluchowski (1906,1914) in the context of his studies on colloidal suspensions. The model of undeviated uniform motion [1.3] goes back to the work of Fürth who applied Smoluchowski’s methods to estimate the average speed of pedestrians from counts of the number of pedestrians in a fixed section of a road [see, Kac (1959, Chapter III, Section 26)]. The term undeviated uniform motion was coined by Lindley (1954).

Related literature. Chandrasekhar (1943, Chapter III) discusses probabilistic properties of the Smoluchowski process. The presentation there does not explicitly state the probabilistic model, and it is tacitly assumed that $\{N(t), t \geq 0\}$ is Markovian even though this is not in general true. The review of probabilistic properties of $\{N(t), t \geq 0\}$ that is most relevant to our work is Kac (1959). In this work expressions for the moment generating functional and finite dimensional distributions of the Smoluchowski process are derived. Kac (1959) also discusses the issue of Markovianity of the Smoluchowski process, considers the undeviated uniform motion model, and explains significance of the probabilistic results for problems of estimating physical entities from count data. Motivated by the Fürth traffic problem and by the work of Rothschild (1953) on measuring motility of spermatozoa, Lindley (1954) considers the problem of estimating expected velocity of the undeviated uniform motion on the real line. This paper relates the expected velocity to the one-sided derivative at zero of the covariance function of the Smoluchowski process, and uses this relation for constructing an estimator. Ruben (1964) suggests a generalization of the Smoluchowski process assuming that the number of particles can be counted in several disjoint
observation regions. Properties of the multivariate Smoluchowski process constructed in this way are studied in the aforementioned work, and it is argued that the use of such multivariate data improves accuracy of estimation procedures. Parameter estimation problems for the multivariate Smoluchowski processes are considered in McDunnough (1979a, 1979b). These papers promote the idea of using standard asymptotic results for estimating covariance functions of discrete time series in conjunction with the delta method in order to derive asymptotic distributions of parameter estimators.

Another strand of research deals with stochastic models that are closely related to the Smoluchowski processes: the $M/G/\infty$ queueing model and the branching processes with immigration. We postpone the detailed discussion of the connection between these models and the Smoluchowski processes to Section 2.6 here we restrict ourselves with brief description and related references.

Bingham & Dunham (1997) discuss the use of the $M/G/\infty$ queueing model in some problems of statistical estimation for Smoluchowski processes. Because the Smoluchowski process is not in general Markovian, Bingham & Dunham (1997) suggest that there are two ways to handle the mathematical difficulties: (a) to assume that the service time distribution $G$ is exponential which leads to the Markovian $M/M/\infty$ model; (b) to work with reduced data when the observations of the busy and idle periods are available, i.e., the process $\{1(N(t) > 0), t \geq 0\}$ is observed. In this paper we demonstrate that problems of statistical estimation for Smoluchowski processes can be solved in the original setting without resorting to restrictive assumptions like (a) or (b).

A Markovian model that can be used as an approximation for the Smoluchowski process is the branching process with immigration. In this context the model has been studied in Ruben (1962) and Ruben (1964). The relationship between branching processes with immigration and Smoluchowski processes is discussed in detail in McDunnough (1978) where necessary and sufficient conditions for Markovianity of the Smoluchowski processes are derived. Some statistical estimation problems for branching processes with immigration are considered in Heyde & Seneta (1972) and Heyde & Seneta (1974); we also refer to Wei & Winnicki (1989, 1990) and Winnicki (1991) for related results.

We note that there exists a considerable body of work dealing with statistical inference for diffusion processes and infinite particle systems under assumption that trajectories of particles are directly observable. The problems of parametric and nonparametric estimation of drift and diffusion coefficients in these models were intensively studied; we refer, e.g., to Genon-Catalot & Jacod (1993), Jacod (2000), Hoffmann (2001), Gobet et al. (2004) and Kutoyants (2004) where further references can be found. However, as it is shown in the present paper, estimation settings based on the count data lead to statistical inverse problems that require completely different techniques and tools. One of the goals of this paper is to develop such techniques.

Finally, it is also worth mentioning that although Smoluchowski’s theory originated in statistical physics, it found numerous applications in diverse areas, e.g., in biology (Rothschild 1953), spectroscopy (Brenner et al. 1978), medicine (Aebersold et al. 1993) and geology (Culling 1985).

The paper contribution. Our goal in this paper is two-fold. First, we introduce a general model of moving particles in $\mathbb{R}^d$ and revisit probability properties of the Smoluchowski processes using original proof techniques. In the existing literature these properties are discussed or mentioned in passing in different sources under disparate and sometimes not fully specified assumptions on the probabilistic model. In contrast, the framework taken in this paper enables us to investigate probability properties of Smoluchowski processes associated with arbitrary displacement models in a unified and principled way.

Second, we are interested in estimation of some functionals of the particle displacement distributions from continuous time observation $\mathcal{N}_T := \{N(t), 0 \leq t \leq T\}$ of the Smoluchowski process. These estimation problems are reduced to estimating functionals of the correlation function of the Smoluchowski process from indirect observations. The resulting statistical ill–posed inverse problems depend on the geometry of the observation region and require the use of special transform methods for constructing the estimators. Although the developed methods are applicable for observation regions of arbitrary shape, in all what follows we focus on the case when $B$ is a Euclidean ball in $\mathbb{R}^d$. In the setting of the undeviated uniform motion we consider problems of estimating the mean speed and the speed distribution, while in the setting of the Brownian displacement our focus is on estimating the diffusion coefficient. In all aforementioned settings we develop estimators with provable accuracy guarantees.
Organization of the paper. The rest of the paper is structured as follows. Section 2 discusses probabilistic properties of the Smoluchowski process. In particular, we derive explicit expressions for the moment generating functional and mixed moments, present conditions for stationarity of the Smoluchowski processes, establish a Gaussian approximation and discuss connections to the M/G/∞ queue and branching processes with immigration. In Section 3 we present results on estimation of the covariance function of the Smoluchowski process; these results play an important role in all subsequent developments. Section 4 deals with estimation of the expected speed and speed distribution for the Smoluchowski process driven by the undeviated uniform motion. Section 5 considers the Brownian displacement model, and studies estimation of the diffusion coefficient. Proofs of results of Sections 3, 4 and 5 are given in Sections 3, 4 and 5 respectively.

Notation. The following notation is used throughout the paper. The $d$-dimensional volume (the Lebesgue content) of a set in $\mathbb{R}^d$ is denoted $\text{vol}\{\cdot\}$. For a set $C \subset \mathbb{R}^d$ and point $x \in \mathbb{R}^d$ we denote
\[
C(x) := \{y \in \mathbb{R}^d : y = z - x, \ z \in C\}.
\]
The covariogram of a compact set $C \subset \mathbb{R}^d$ is the function $g_C : \mathbb{R}^d \to \mathbb{R}$ defined by
\[
g_C(x) := \text{vol}\{C \cap C(x)\}, \ x \in \mathbb{R}^d; \tag{1.5}
\]
see, e.g., Matheron (1975, Section 4.3). The Euclidean norm on $\mathbb{R}^d$ is denoted $|| \cdot ||$.

2 Properties of the Smoluchowski process

In this section we discuss probabilistic properties of Smoluchowski processes. Some of the presented results appeared in different forms and for different settings, e.g., in Lindley (1954), Kac (1959), and Ruben (1964). Our approach to derivation of these results is purely analytic, and our proofs differ from those in the existing literature. It is also worth to emphasize that the results of this section hold for any Smoluchowski’s process, independently of the displacement model.

2.1 Moment generating functional

Let $\Pi_n$ denote the set of all non-empty subsets of $\{1, 2, \ldots, n\}$, and for $\pi \in \Pi_n$ write $\pi^c := \{1, 2, \ldots, n\} \setminus \pi$. The following theorem establishes the moment generating functional of process $\{N(t), t \geq 0\}$ defined in (1.2).

Theorem 2.1 For any $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ and $t_1 \leq t_2 \leq \cdots \leq t_n$ one has
\[
\ln \mathbb{E}\exp\left\{\sum_{k=1}^n \theta_k N(t_k)\right\} = \lambda \sum_{\pi \in \Pi_n} \left[e^{\sum_{k \in \pi} \theta_k} - 1\right] Q_\pi(t_1, \ldots, t_n), \tag{2.1}
\]
where
\[
Q_\pi(t_1, \ldots, t_n) := \int_{\mathbb{R}^d} Q_\pi(x; t_1, \ldots, t_n) \, dx, \tag{2.2}
\]
\[
Q_\pi(x; t_1, \ldots, t_n) := \mathbb{P}\left\{(x + Y_{t_1} \in B, k \in \pi) \cap (x + Y_{t_k} \notin B, k \in \pi^c)\right\}. \tag{2.3}
\]

Remark 2.1 Note that function $Q_\pi(t_1, \ldots, t_n)$ is well defined because the integral in (2.2) is bounded from above by $\mathbb{E}\text{vol}\{\bigcap_{k \in \pi} B(Y_{t_k})\} \leq \text{vol}(B)$ for any $\pi \in \Pi_n$.

Proof: For $j \in \mathbb{Z}$ and $\pi \in \Pi_n$ define events
\[
A^{(j)}_\pi := \{X^{(j)}_{t_k} \in B, k \in \pi\} \cap \{X^{(j)}_{t_k} \notin B, k \in \pi^c\}.
\]
Event $A^{(j)}_\pi$ states that $j$-th particle is inside $B$ at time instances $\{t_k, k \in \pi\}$ and outside $B$ otherwise. Obviously, for fixed $j$ events $A^{(j)}_\pi, \pi \in \Pi_n$ are disjoint. We let $A^{(j)} := \bigcup_{\pi \in \Pi_n} A^{(j)}_\pi$, and $\bar{A}^{(j)}$ the event
complimentary to $A^{(j)}$, i.e., $\bar{A}^{(j)} = \{X_{tk}^{(j)} \notin B, \forall k \in \{1, \ldots, n\}\}$. Since  
\[
\sum_{k=1}^{n} \theta_k N(t_k) = \sum_{j \in \mathbb{Z}} \sum_{k=1}^{n} \theta_k 1\{X_{tk}^{(j)} \in B\},
\]
with the introduced notation we have for any $j \in \mathbb{Z}$  
\[
\sum_{k=1}^{n} \theta_k 1\{X_{tk}^{(j)} \in B\} = \sum_{\pi \in \Pi_n} 1\{A^{(j)}_{\pi}\} \sum_{k \in \pi} \theta_k,
\]
and therefore  
\[
\exp\left\{\sum_{k=1}^{n} \theta_k 1\{X_{tk}^{(j)} \in B\}\right\} = \sum_{\pi \in \Pi_n} 1\{A^{(j)}_{\pi}\} e^{\sum_{k \in \pi} \theta_k} + 1\{\bar{A}^{(j)}\} = \sum_{\pi \in \Pi_n} 1\{A^{(j)}_{\pi}\} [e^{\sum_{k \in \pi} \theta_k} - 1] + 1. \quad (2.4)
\]
Then in view of (2.4)  
\[
\mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} \theta_k N(t_k)\right\} | \Xi\right] = \prod_{j \in \mathbb{Z}} \mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} \theta_k 1\{X_{tk}^{(j)} \in B\}\right\} | \Xi\right] = \prod_{j \in \mathbb{Z}} \left(1 + \sum_{\pi \in \Pi_n} \mathbb{P}\{A^{(j)}_{\pi} \vert \Xi\} [e^{\sum_{k \in \pi} \theta_k} - 1]\right) = \exp\left\{\sum_{j \in \mathbb{Z}} f(\xi_j)\right\},
\]
where  
\[
f(\xi_j) = \ln\left(1 + \sum_{\pi \in \Pi_n} \mathbb{P}\{A^{(j)}_{\pi} \vert \Xi\} [e^{\sum_{k \in \pi} \theta_k} - 1]\right).
\]
Using Campbell’s formula we obtain  
\[
\ln \mathbb{E}\left[\exp\left\{\sum_{k=1}^{n} \theta_k N(t_k)\right\}\right] = \lambda \sum_{\pi \in \Pi_n} [e^{\sum_{k \in \pi} \theta_k} - 1] \int_{\mathbb{R}^d} Q_{\pi}(x)dx,
\]
where function $Q_{\pi}(x) = Q_{\pi}(x; t_1, \ldots, t_n)$ is defined in (2.3).

**Remark 2.2** Theorem [2.7] holds under very general assumptions on the displacement process $\{Y_t, t \geq 0\}$: only independence of $\{Y_t, t \geq 0\}$ and the initial position Poisson point process $\Xi$ is required. In particular, under these conditions the Smoluchowski process $\{N(t), t \geq 0\}$ should not be stationary.

Formula (2.4) implies that for every $t \geq 0$ random variable $N(t)$ has Poisson distribution with parameter  
\[
\rho := \mathbb{E}N(t) = \lambda \int_{\mathbb{R}^d} \mathbb{P}\{x + Y_t \in B\}dx = \lambda \mathbb{E}\text{vol}\{B(Y_t)\} = \lambda \text{vol}\{B\}.
\]
Thus the expectation of $N(t)$ does not depend on properties of displacement process $\{Y_t, t \geq 0\}$. This fact is consistent with the well known result on random displacement of Poisson point process which implies that $\sum_{j \in \mathbb{Z}} \xi_{X^{(j)}}$ is homogeneous Poisson process of intensity $\lambda$ for any $t \geq 0$ [see, e.g., Doob (1953, Section 5, Chapter VIII)]. Since the expectation of $N(t)$ does not bring any information on the particle displacement process, statistical inference for the displacement process should be based on moments of higher order.
It is instructive to specialize formula (2.1) for the case $n = 2$:

$$\frac{1}{\lambda} \ln \mathbb{E} \exp \{ \theta_1 N(t_1) + \theta_2 N(t_2) \}$$

$$= (e^{\theta_1} - 1)Q_{(1)}(t_1, t_2) + (e^{\theta_2} - 1)Q_{(2)}(t_1, t_2) + (e^{\theta_1 + \theta_2} - 1)Q_{(1,2)}(t_1, t_2),$$

where

$$Q_{(1)}(t_1, t_2) = \mathbb{E} \int_{\mathbb{R}^d} \mathbb{1}_{\{x \in B(Y_{t_1}) \cap B(Y_{t_2})\}} dx = \text{vol} \{B\} - \text{Evol} \{B(Y_{t_1}) \cap B(Y_{t_2})\},$$

$$Q_{(2)}(t_1, t_2) = \text{vol} \{B\} - \text{Evol} \{B(Y_{t_1}) \cap B(Y_{t_2})\}, \quad Q_{(1,2)}(t_1, t_2) = \text{Evol} \{B(Y_{t_1}) \cap B(Y_{t_2})\}.$$

Thus

$$\frac{1}{\lambda} \ln \mathbb{E} \exp \{ \theta_1 N(t_1) + \theta_2 N(t_2) \}$$

$$= \text{vol} \{B\} [(e^{\theta_1} - 1) + (e^{\theta_2} - 1)] + \text{Evol} \{B(Y_{t_1}) \cap B(Y_{t_2})\} (e^{\theta_1} - 1)(e^{\theta_2} - 1).$$

This formula should be compared with (III.22.14) in Kac (1959) which was derived using different considerations.

Another interesting consequence of Theorem 2.1 is stated in the next corollary. Denote

$$N(t_1, \ldots, t_n) := \sum_{j \in \mathbb{Z}} \mathbb{1}_{\{X_{t_1}^{(j)} \in B, \ldots, X_{t_n}^{(j)} \in B\}}.$$ 

In words, $N(t_1, \ldots, t_n)$ is the number of particles that were in $B$ at all time instances $t_1, \ldots, t_n$.

**Corollary 2.1** For any $\theta \in \mathbb{R}$, $n$, and $t_1, \ldots, t_n$ one has

$$\ln \mathbb{E} \exp \{ \theta N(t_1, \ldots, t_n) \} = \lambda(e^{\theta} - 1) \int_{\mathbb{R}^d} \mathbb{P} \{x + Y_{t_1} \in B, \ldots, x + Y_{t_n} \in B\} dx$$

**Proof:** The proof is basically coincides with the one of Theorem 2.1. For $j \in \mathbb{Z}$ define the event

$A^{(j)} = \{X_{t_k}^{(j)} \in B, k = 1, \ldots, n\},$ and let $A^{(j)}$ be the complementary event. We have

$$\theta N(t_1, \ldots, t_n) = \theta \sum_{j \in \mathbb{Z}} \mathbb{1}_{\{X_{t_k} \in B, k = 1, \ldots, n\}} = \sum_{j \in \mathbb{Z}} \theta \mathbb{1}_{\{A^{(j)}\}}.$$

Therefore

$$\mathbb{E} \left\{ \exp \{ \theta N(t_1, \ldots, t_n) \} \mid \Xi \right\} = \mathbb{E} \left\{ \exp \left\{ \theta \sum_{j \in \mathbb{Z}} \mathbb{1}_{\{A^{(j)}\}} \right\} \mid \Xi \right\} = \prod_{j \in \mathbb{Z}} \mathbb{E} \left\{ e^{\theta \mathbb{1}_{\{A^{(j)}\}}} \mid \Xi \right\}$$

$$= \prod_{j \in \mathbb{Z}} \left( e^{\theta} - 1 \right) \mathbb{P} \{A^{(j)} \mid \Xi \} + 1.$$

Then application of Campbell’s formula completes the proof. \qed

Corollary 2.1 shows that number of particles that are in $B$ at $n$ time instances $t_1, \ldots, t_n$ is a Poisson random variable with expectation $\lambda Q_{(1,\ldots,n)}(t_1, \ldots, t_n) = \lambda \int_{\mathbb{R}^d} \mathbb{P} \{x + Y_{t_k} \in B, k = 1, \ldots, n\} dx$. In particular, for any $t_1, t_2$

$$N(t_1, t_2) \sim \text{Poisson} \left( \lambda \text{Evol} \{B(Y_{t_1}) \cap B(Y_{t_2})\} \right), \quad \mathbb{E} N(t_1, t_2) = \lambda \text{Evol} \{B(Y_{t_1}) \cap B(Y_{t_2})\}.$$

### 2.2 Stationarity

It follows from Theorem 2.1 that Smoluchowski process $\{N(t), t \geq 0\}$ is strictly stationary if and only if for all $n, t_1, \ldots, t_n$, and $\pi \in \Pi$, one has

$$Q_{\pi}(t_1, \ldots, t_n) := \int_{\mathbb{R}^d} Q_{\pi}(x; t_1, \ldots, t_n) dx = \tilde{Q}_{\pi}(t_1 + \tau, \ldots, t_n + \tau), \quad \forall \tau,$$

(2.5)
2.3 Moments and covariance function

\[ \int_{\mathbb{R}^d} \mathbb{P}\{Y_{t_1} \in B(x), \ldots, Y_{t_n} \in B(x)\} dx = \int_{\mathbb{R}^d} \mathbb{P}\{Y_{t_1+\tau} \in B(x), \ldots, Y_{t_n+\tau} \in B(x)\} dx. \]

An equivalent condition of stationarity is given in McDunnough (1978).

Condition (2.2) holds for a wide class of particle displacement processes \(\{Y_t, t \geq 0\}\). The following two examples are particularly important.

(i) Let \(\{Y_t, t \geq 0\}\) be a strictly stationary process; then \(Q_\pi(x; t_1, \ldots, t_n) = Q_\pi(x; t_1 + \tau, \ldots, t_n + \tau)\) for all \(\tau \in \mathbb{R}, x \in \mathbb{R}^d, \pi \in \Pi_n\), and (2.3) holds trivially. Thus \(\{N(t), t \geq 0\}\) is strictly stationary.

(ii) Let \(\{Y_t, t \geq 0\}\) be a time homogeneous Markov process with transition function \(P_t(x, A)\) on \([0, \infty) \times \mathbb{R}^d\times \mathcal{B}(\mathbb{R}^d)\); then

\[ Q_\pi(x; t_1, \ldots, t_n) = \int_{B_1} \cdots \int_{B_n} \prod_{j=2}^n P_{t_j-t_{j-1}}(y_{j-1}, dy_j) \] \[ \] \[ P_t(x, A)dx := \hat{P}(A), \quad A \in \mathcal{B}(\mathbb{R}^d) \]

defines measure \(\hat{P}\) independent of \(t\) then

\[ \hat{Q}_\pi(x) = \int_{B_1} \cdots \int_{B_n} \prod_{j=2}^n P_{t_j-t_{j-1}}(y_{j-1}, dy_j) \hat{P}(dy_1), \]

and process \(\{N(t), t \geq 0\}\) is strictly stationary. For instance, if \(P_t(x, A) = \int_A q_t(x-y)dy\) for some transition probability density \(q_t\), then (2.6) holds with \(\hat{P}\) being the Lebesgue measure. An important specific case of the discussed setting is when \(\{Y_t, t \geq 0\}\) is a process with independent and stationary increments.

2.3 Moments and covariance function

Theorem 2.1 allows us to calculate the covariance function and the moments of the finite dimensional distributions of \(\{N(t), t \geq 0\}\). These formulas are repeatedly used in the sequel.

For every fixed \(\pi \in \Pi_n\) let \(\Pi_n(\pi) := \{\pi' \in \Pi_n : \pi' \supseteq \pi\}\) be the set of all supersets of \(\pi\) in \(\Pi_n\). Define

\[ U_\pi(t_1, \ldots, t_n) := \frac{1}{\text{vol}(B)} \sum_{\pi' \in \Pi_n(\pi)} Q_{\pi'}(t_1, \ldots, t_n), \quad \pi \in \Pi_n, \]

where \(Q_{\pi}(t_1, \ldots, t_n)\) is given in (2.2). We obviously have

\[ U_\pi(t_1, \ldots, t_n) = \frac{1}{\text{vol}(B)} \sum_{\pi' \in \Pi_n(\pi)} \int_{\mathbb{R}^d} Q_{\pi'}(x; t_1, \ldots, t_n) dx \]

\[ = \frac{1}{\text{vol}(B)} \int_{\mathbb{R}^d} \sum_{\pi' \in \Pi_n(\pi)} \mathbb{P}\left\{ x + Y_{t_j} \in B, j \in \pi' \right\} \mathbb{P}\left\{ x + Y_{t_j} \notin B, j \in (\pi')^c \right\} dx \]

\[ = \frac{1}{\text{vol}(B)} \int_{\mathbb{R}^d} \mathbb{P}\left\{ x \in \bigcap_{j \in \pi} B(Y_{t_j}) \right\} dx = \frac{1}{\text{vol}(B)} \text{vol}\left\{ \bigcap_{j \in \pi} B(Y_{t_j}) \right\} \]

for every \(\pi \in \Pi_n\) events

\(\{x + Y_{t_j} \in B, j \in \pi' \cap \pi\} \cap \{x + Y_{t_j} \notin B, j \in (\pi')^c\}, \quad \pi' \in \Pi_n(\pi)\)

are disjoint and constitute a partition of the sample space.

Although \(U_\pi(t_1, \ldots, t_n)\) is formally defined as a function of \(n\) variables \(t_1, \ldots, t_n\), it follows from (2.8) that in fact \(U_\pi(t_1, \ldots, t_n)\) is a function of \(\{t_j, j \in \pi\}\) only: here \(\pi\) indicates the subset of variables on
which function $U_\pi(t_1, \ldots, t_n)$ actually depends. Therefore with slight abuse of notation when appropriate we will drop subscript $\pi$ and indicate the corresponding variables explicitly, e.g.,

$$U(t_i, t_j, t_k) := U_{(i,j,k)}(t_1, \ldots, t_n), \quad i, j, k \in \{1, \ldots, n\}.$$ 

Several remarks on the properties of function $U_\pi(t_1, \ldots, t_n)$ defined in (2.1) are in order.

**Remark 2.3**

(a) If $\pi$ is a singleton, $\pi = \{k\}$, then it follows from \[2.8\] that $U_{\{k\}}(t_1, \ldots, t_n) = U(t_k) = 1$ for all $k \in \{1, \ldots, n\}$.

(b) $U_\pi(t_1, \ldots, t_n)$ is monotone non-increasing in the following sense: for all $t_1, \ldots, t_n$ one has

$$U_\pi(t_1, \ldots, t_n) \leq U_{\pi'}(t_1, \ldots, t_n), \quad \forall \pi \supseteq \pi', \quad \pi, \pi' \in \Pi_n.$$ 

(c) If $\{N(t), t \geq 0\}$ is strictly stationary then for all $n, t_1, \ldots, t_n$, and $\pi \in \Pi_n$

$$U_\pi(t_1, \ldots, t_n) = U_\pi(t_1 + \tau, \ldots, t_n + \tau), \quad \forall \tau.$$ 

**Proposition 2.1** For any $t_1 \leq t_2 \leq \cdots \leq t_n$ one has

$$\mathbb{E} \left[ \prod_{j=1}^{n} N(t_j) \right] = \sum_{k=1}^{n} \rho^k \prod_{\pi \in \mathcal{P}(n,k)} \prod_{j=1}^{k} U_{\pi_j}(t_1, \ldots, t_n), \quad \text{(2.9)}$$

where $\mathcal{P}(n,k)$ is the set of all partitions $\pi = (\pi_1, \ldots, \pi_k)$ of $\{1, \ldots, n\}$ in $k$ subsets. In particular, for $n = 2$

$$\mathbb{E}[N(t_1)N(t_2)] = \rho^2 + \rho U(t_1, t_2), \quad \text{cov}\{N(t_1), N(t_2)\} = \rho U(t_1, t_2). \quad \text{(2.10)}$$

**Proof:** The proof follows from Theorem \[2.1\] and a multivariate version of the Faà di Bruno formula [see, e.g., Hardy (2006)]. Letting

$$y(\theta_1, \ldots, \theta_n) = \lambda \sum_{\pi \in \Pi_n} \left[ e^{\sum_{k \in \pi} \theta_k} - 1 \right] Q_\pi(t_1, \ldots, t_n)$$
we have $\psi(\theta_1, \ldots, \theta_n) := \mathbb{E} e^{\sum_{k \in \pi} \theta_k N(t_k)} = e^{y(\theta_1, \ldots, \theta_n)}$. Therefore by the Faà di Bruno formula

$$\frac{\partial^n y(\theta_1, \ldots, \theta_n)}{\partial \theta_1 \cdots \partial \theta_n} = e^{y(\theta_1, \ldots, \theta_n)} \sum_{\pi} \prod_{\pi} \prod_{\pi' \supseteq \pi} \frac{\partial^{n|\pi|} y(\theta_1, \ldots, \theta_n)}{\partial \theta_{\pi}},$$

where the sum is over all partitions $\mathcal{P}$ of set $\{1, \ldots, n\}$, the product is over all blocks $\pi$ of partition $\mathcal{P}$, and $|\pi|$ stands for the cardinality of block $\pi$. We obviously have

$$\frac{\partial^{n|\pi|} y(\theta_1, \ldots, \theta_n)}{\partial \theta_{\pi}} = \lambda \sum_{\pi' \in \Pi_n: \pi' \supseteq \pi} e^{\sum_{k \in \pi'} \theta_k} Q_{\pi'}(t_1, \ldots, t_n),$$

and therefore

$$\mathbb{E} \prod_{j=1}^{n} N(t_j) = \left. \frac{\partial^n y(\theta_1, \ldots, \theta_n)}{\partial \theta_1 \cdots \partial \theta_n} \right|_{\theta_1 = \cdots = \theta_n = 0} = \sum_{\pi} \prod_{\pi} \rho U_\pi(t_1, \ldots, t_n)$$

$$= \sum_{k=1}^{n} \rho^k \prod_{\pi \in \mathcal{P}(n,k)} \prod_{j=1}^{k} U_{\pi_j}(t_1, \ldots, t_n),$$

as claimed. \[\square\]
Remark 2.4 Note that process \( \{N(t), t \geq 0\} \) is weakly stationary if and only if \( U(t_1, t_2) \) depends on \( t_1 \) and \( t_2 \) via difference \( t_2 - t_1 \). Because \( \text{vol}\{B(Y_{t_1}) \cap B(Y_{t_2})\} = \text{vol}\{B \cap B(Y_{t_2} - Y_{t_1})\} \), in view of (2.8), \( \{N(t), t \geq 0\} \) is weakly stationary if process \( \{Y_t, t \geq 0\} \) has stationary increments, i.e., \( Y_{t_2} - Y_{t_1} \overset{d}{=} Y_{t_2 - t_1} \) for all \( 0 \leq t_1 \leq t_2 \).

Remark 2.5 If \( \{N(t), t \geq 0\} \) is weakly stationary then by definition

\[
U(t_1, t_2) = \frac{1}{\text{vol}(B)} \mathbb{E}(B(Y_{t_1}) \cap B(Y_{t_2})) = \frac{1}{\text{vol}(B)} \mathbb{E}(B \cap B(Y_{t_1} - t_2)) = H(t_1 - t_2),
\]

and it follows from (2.10) that \( H(t) \) is the correlation function of \( \{N(t), t \geq 0\} \), while \( R(t) = \rho H(t) = \text{cov}(N(s), N(t + s)) \) is the covariance function. It is also worth noting that

\[
R(t) = \rho H(t) = \text{cov}(N(s), N(t + s)) = \lambda \mathbb{E} g_B(Y_t), \quad t \geq 0,
\]

where \( g_B \) is the covariance of set \( B \) [cf. (1.6)].

For weakly stationary \( \{N(t), t \geq 0\} \) we have

\[
\mathbb{E}[N(s + t) - N(s)]^2 = 2\rho[1 - H(t)].
\]

This equation was derived by Smoluchowski (1914); in his terminology function \( 1 - H(t) \) is called the probability after-effect (Wahrscheinlichkeitsnachwirkung) [see Chandrasekhar (1943) and Kac (1959)]. In addition to the correlation function of the Smoluchowski process, the probability after-effect \( 1 - H(t) \) has another interpretation. It is loosely referred in Chandrasekhar (1943) as a probability that a particle somewhere inside \( B \) will have emerged from it during time \( t \). In fact

\[
H(t) = \frac{\int_{\mathbb{R}^d} \mathbb{P}(x + Y_s \in B, x + Y_{s+t} \in B) \, dx}{\int_{\mathbb{R}^d} \mathbb{P}(x + Y_s \in B) \, dx} = \frac{\mathbb{E}[N(s + t)]}{\lambda \text{vol}(B)},
\]

where \( N(s, s + t) \) is the number of particles that are in \( B \) at both time instances \( s \) and \( s + t \).

### 2.4 Probability generating functional

Theorem (2.1) gives the probability generating functional of \( \{N(t), t \geq 0\} \):

\[
\mathbb{E}\prod_{j=1}^n z_j^{N(t_j)} = \exp\left\{ \lambda \sum_{\pi \in \Pi_n} \left[ \prod_{k \in \pi} z_k - 1 \right] Q_\pi(t_1, \ldots, t_n) \right\}.
\]

This expression allows us to calculate finite dimensional distributions of the Smoluchowski process. In particular, for strictly stationary Smoluchowski process \( \{N(t), t \geq 0\} \)

\[
\psi(z_1, z_2) := \mathbb{E}[z_1^{N(s)} z_2^{N(s+t)}] = \exp\{\rho[(z_1 - 1) + (z_2 - 1)] + \rho H(t)(z_1 - 1)(z_2 - 1)\},
\]

and routine differentiation yields: for \( m \geq 0 \) and \( k \geq 0 \)

\[
\mathbb{P}\{N(s) = m, N(s + t) = m + k\} = \frac{1}{m!(m + k)!} \left\{ \frac{\partial^m}{\partial z_1^m} \frac{\partial^{m+k}}{\partial z_2^{m+k}} \psi(z_1, z_2) \right\}_{z_1 = z_2 = 0}
\]

\[
= e^{-\rho m} \sum_{j=0}^m \binom{m}{j} [H(t)]^j [1 - H(t)]^{m-j} \frac{e^{-\rho[1-H(t)]} [\rho(1 - H(t))]^{m+k-j}}{(m + k - j)!}.
\]

(2.11)

and for \( m \geq 0 \) and \( 0 \leq k \leq m \)

\[
\mathbb{P}\{N(s) = m, N(s + t) = m - k\} = \frac{1}{m!(m - k)!} \left\{ \frac{\partial^m}{\partial z_1^m} \frac{\partial^{m-k}}{\partial z_2^{m-k}} \psi(z_1, z_2) \right\}_{z_1 = z_2 = 0}
\]

\[
= e^{-\rho m} \sum_{j=k}^m \binom{m}{j} [H(t)]^j [1 - H(t)]^{m-j} \frac{e^{-\rho[1-H(t)]} [\rho(1 - H(t))]^{j-k}}{(j - k)!}.
\]

(2.12)
Since $N(t)$ is a Poisson random variable with parameter $\rho$ for every $t$ we have

$$
\mathbb{P}\{N(s+t) = m+k|N(s) = m\} = \sum_{j=0}^{m} \binom{m}{j} [H(t)]^j [1 - H(t)]^{m-j} \frac{e^{-\rho[1-H(t)]} \rho(1-H(t))^{m-k-j}}{(m+k-j)!}, \quad k \geq 0, \tag{2.13}
$$

and

$$
\mathbb{P}\{N(s+t) = m-k|N(s) = m\} = \sum_{j=k}^{m} \binom{m}{j} [H(t)]^j [1 - H(t)]^{m-j} \frac{e^{-\rho[1-H(t)]} \rho(1-H(t))^{j-k}}{(j-k)!}, \quad 0 \leq k \leq m. \tag{2.14}
$$

Formulas (2.11) and (2.12) have been derived in Smoluchowski (1914) using combinatorial arguments; see also Chandrasekhar (1943) and Kac (1959). It is worth noting that the expressions on the right hand sides of (2.13) and (2.14) are discrete convolutions of binomial and Poisson distributions.

2.5 Gaussian approximation

The next statement establishes a Gaussian approximation for the finite dimensional distributions of process $\{N(t), t \geq 0\}$. This result is an easy consequence of Theorem 2.1. To state the result we need the following notation. We consider a family of processes $\{N_{\rho}(t), t \geq 0\}$ indexed by parameter $\rho = \lambda \text{vol}(B) > 0$, and for fixed $(t_1, \ldots, t_n) \in \mathbb{R}^n$ let

$$
N_n^\rho := (N_\rho(t_1), \ldots, N_\rho(t_n)), \quad e_n = (1, \ldots, 1) \in \mathbb{R}^n.
$$

**Proposition 2.2** For any $n \geq 1$ and $(t_1, \ldots, t_n) \in \mathbb{R}^n$ one has

$$
\frac{N_n^\rho - \rho e_n}{\sqrt{\rho}} \overset{d}{\rightarrow} N_n(0, \Sigma_H), \quad \rho \to \infty,
$$

where $\Sigma_H$ is the $n \times n$ matrix with elements

$$
[\Sigma_H]_{i,j} := U_{i,j}(t_1, \ldots, t_n) = U(t_i, t_j), \quad i, j = 1, \ldots, n.
$$

**Proof:** The proof is standard; it is based on application of Theorem 2.1. It follows from this theorem that for any $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ one has

$$
\ln \mathbb{E} \exp \left\{ \sum_{k=1}^{n} \theta_k N_\rho(t_k) - \rho \right\} = -\sqrt{\rho} \sum_{k=1}^{n} \theta_k + \ln \mathbb{E} \exp \left\{ \sum_{k=1}^{n} N_\rho(t_k) \frac{\theta_k}{\sqrt{\rho}} \right\}
$$

$$
= -\sqrt{\rho} \sum_{k=1}^{n} \theta_k + \lambda \sum_{\pi \in \Pi_n} \left[ e^{\sum_{k \in \pi} \theta_k} - 1 \right] Q_\pi(t_1, \ldots, t_n)
$$

$$
= -\sqrt{\rho} \sum_{k=1}^{n} \theta_k + \lambda \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{\pi \in \Pi_n} \left( \sum_{k \in \pi} \frac{\theta_k}{\sqrt{\rho}} \right)^l Q_\pi(t_1, \ldots, t_n). \tag{2.15}
$$

In the second sum on the right hand side of (2.15) the term corresponding to $l = 1$ is

$$
\lambda \sum_{\pi \in \Pi_n} \sum_{k \in \pi} \frac{\theta_k}{\sqrt{\rho}} Q_\pi(t_1, \ldots, t_n) = \frac{\lambda \text{vol}(B)}{\sqrt{\rho}} \sum_{k=1}^{n} \theta_k \frac{1}{\text{vol}(B)} \sum_{\pi \in \Pi_n(t_k)} Q_\pi(t_1, \ldots, t_n)
$$

$$
= \sqrt{\rho} \sum_{k=1}^{n} \theta_k U(t_k) = \sqrt{\rho} \sum_{k=1}^{n} \theta_k. \tag{2.16}
$$
where, we remind, \( \Pi_n(\pi) = \{ \pi' \in \Pi_n : \pi' \supseteq \pi \} \), and we took into account the definition of \( U_\pi(t_1, \ldots, t_n) \) [cf. (2.3)]. For \( l = 2 \) we have

\[
\frac{\lambda}{2} \sum_{\pi \in \Pi_n} \sum_{k_1 \in \pi} \sum_{k_2 \in \pi} \frac{\theta_{k_1} \theta_{k_2}}{\rho} Q_\pi(t_1, \ldots, t_n) = \frac{\lambda}{2 \rho} \sum_{k_1 = 1}^{n} \sum_{k_2 = 1}^{n} \theta_{k_1} \theta_{k_2} \sum_{\Pi_n(\{(k_1, k_2)\})} Q_\pi(t_1, \ldots, t_n) \\
= \frac{1}{2} \sum_{k_1 = 1}^{n} \sum_{k_2 = 1}^{n} \theta_{k_1} \theta_{k_2} U(t_{k_1}, t_{k_2}),
\]

and similarly for \( l > 2 \)

\[
\frac{\lambda}{l! \rho^{l/2}} \sum_{\pi \in \Pi_n} \sum_{k_1 \in \pi} \cdots \sum_{k_l \in \pi} \theta_{k_1} \cdots \theta_{k_l} Q_\pi(t_1, \ldots, t_n) \\
= \frac{1}{l! \rho^{l-1}} \sum_{k_1 = 1}^{n} \cdots \sum_{k_l = 1}^{n} \theta_{k_1} \cdots \theta_{k_l} U(t_{k_1}, \ldots, t_{k_l}).
\]

Taking into account that the terms with \( l > 2 \) tend to zero as \( \rho \to \infty \) and combining (2.17), (2.16) and (2.13) we obtain

\[
\lim_{\rho \to \infty} \mathbb{E} \exp \left\{ - \frac{\theta^T \Sigma \theta}{\sqrt{\rho}} \right\} = \exp \left\{ \frac{1}{2} \theta^T \Sigma_H \theta \right\}
\]

for all \( t_1, \ldots, t_n \). The statement of the proposition follows.

**Remark 2.6** Proposition (2.2) demonstrates that convergence of the finite dimensional distributions of Smoluchowski process \( \{N(t), t \geq 0\} \) to the multivariate normal distribution takes place not only if the rate parameter \( \lambda \) tends to infinity, but also if the volume of the observation region \( \text{vol}(B) \) increases without bound.

### 2.6 Related stochastic models

In this section we discuss some stochastic models that are intimately connected with the Smoluchowski processes.

**M/G/∞ queue.** The M/G/∞ queue is one of the most well studied and well understood models in Queueing Theory [see, e.g., Takacs (1962)]. In this model there is an infinite number of servers, customers arrive at time epochs \( \tau_j \in \mathbb{Z} \) of the homogeneous Poisson process of intensity \( \lambda \), obtain service upon arrival and leave the system after the service completion. The service times \( \{s_j\}_{j \in \mathbb{Z}} \) are independent identically distributed random variables with common distribution function \( S \); they are assumed to be independent of the arrival process.

If we assume that the system operates infinite time (it is in a stationary regime) then the number of busy servers in the system at time \( t \) is given by the formula

\[
N(t) = \sum_{j \in \mathbb{Z}} 1\{\tau_j \leq t, \tau_j + s_j > t\}, \quad t \geq 0.
\]

It has been shown in Goldenshluger (2016) that

\[
\frac{1}{\rho} \ln \mathbb{E} \exp \left\{ \sum_{k=1}^{n} \theta_k N(t_k) \right\} = \sum_{k=1}^{n} (e^{\theta_k} - 1) + \sum_{k=1}^{n-1} H(t_k) \sum_{m=k}^{n-1} (e^{\theta_{m-k+1}} - 1)e^{\sum_{j=m-k+2}^{n} \theta_j (e^{\theta_{j+1}} - 1)},
\]

where

\[
\rho := \frac{\lambda}{\int_0^{\infty} [1 - S(x)]dx}, \quad H(t) := \frac{\int_0^t [1 - S(x)]dx}{\int_0^{\infty} [1 - S(x)]dx}.
\]
This formula can be rewritten as
\[ \frac{1}{\rho} \ln \mathbb{E} \exp \left\{ \sum_{k=1}^{n} \theta_k N(t_k) \right\} = \sum_{k=1}^{n} \sum_{l=k}^{n} (e^{\sum_{j=1}^{n} \theta_j} - 1) \tilde{H}_{l,k}, \tag{2.18} \]
where
\[ \tilde{H}_{l,k} := H(|t_l - t_k|) + H(|t_{l-1} - t_{k+1}|) - H(|t_{l-1} - t_k|) - H(|t_l - t_{k+1}|), \quad 1 \leq l, k \leq n, \]
and the term \( H(|t_l - t_k|) \) is interpreted as zero if \( l \notin \{1, \ldots, n\} \) or \( k \notin \{1, \ldots, n\} \). It follows from (2.18) that structure of the moment generating function coincides with that in (2.1). Therefore the process of the number of busy servers in the \( M/G/\infty \) queue is a Smoluchowski process. In fact, it is a version of the Smoluchowski process with displacement determined by the undeformed uniform motion (this process is studied in detail in Section 4).

A particular case of the \( M/G/\infty \) model is the \( M/M/\infty \) queue when the service time distribution is exponential. The \( M/M/\infty \) model is Markovian; it provides an example of the Smoluchowski process with Markov property. We refer to Bingham & Dunham (1997) for further details on the connection between \( M/G/\infty \) queue and the Smoluchowski process.

**Bernoulli–Poisson branching process with immigration.** Branching process with immigration is a model for the size of a population evolving over time. It is a sequence \( \{N_t, t = 0, 1, 2, \ldots\} \) of non-negative integer random variables defined by equation
\[ N_{t+1} = \sum_{j=1}^{N_t} Z_j^{(t+1)} + I_{t+1}, \quad t = 0, 1, \ldots \tag{2.19} \]
where \( N_t \) stands for the size of the population at time \( t \), \( Z_j^{(t+1)} \) is the number of offsprings of the \( j \)th individual existing in the population at time \( t \), and \( I_{t+1} \) is the number of immigrants joining the population at time \( t + 1 \). The non–negative integer random variables \( \{Z_j^{(t)}, j \geq 1\} \) and \( \{I_t, t = 1, 2, \ldots\} \) are independent of each other; they are assumed to be sequences of independent identically distributed random variables.

The following special case of the branching process with immigration is closely related to the Smoluchowski process. If the offspring distribution is Bernoulli with parameter \( 1 - p \) (each individual replaces itself with probability \( 1 - p \) or “dies” with probability \( p \)), and if the immigration distribution is Poisson with parameter \( \lambda \) then (2.19) implies that conditionally on \( N_t = i \), random variable \( N_{t+1} \) is a sum of two independent random variables: binomial random variable with parameters \( i \) and \( 1 - p \) and Poisson random variable with parameter \( \lambda \). Therefore
\[ \mathbb{P}\{N_{t+1} = j | N_t = i\} = \sum_{l=0}^{\min(i,j)} \binom{j}{l} (1 - p)^l p^{j-l} e^{-\lambda} \frac{\lambda^{j-l}}{(j-l)!}, \]
which coincides with (2.13) and (2.14) for a particular choice of parameters \( \lambda \) and \( p \). This fact shows the relationship between the Bernoulli–Poisson branching processes with immigration and the Smoluchowski processes.

The Bernoulli–Poisson branching process with immigration is Markovian while the Smoluchowski process is in general not. We refer to McDumough (1978) where, using connections between these two models, conditions for Markovianity of the Smoluchowski processes are established.

### 3 Covariance function estimation

In this section and from now on we assume that process \( \{N(t), t \geq 0\} \) is strictly stationary. We consider the problem of estimating the covariance function of the Smoluchowski process from continuous time observations \( \{N(t), 0 \leq t \leq T\} \). The results established in this section are repeatedly used in the sequel.
Recall that covariance and correlation functions of \( \{N(t), t \geq 0\} \) are given by
\[
R(t) = \text{cov}\{N(s), N(t+s)\} = \rho H(t),
\]
\[
H(t) = U(t+s,s) = \frac{1}{\text{vol}(B)} \mathbb{E} \text{vol}\{B \cap B(Y_t)\} = \frac{E g B(Y_t)}{\text{vol}(B)},
\]
and for any \( k, t_1, \ldots, t_k \)
\[
U(t_1, \ldots, t_k) = \frac{1}{\text{vol}(B)} \mathbb{E} \text{vol}\{\cap_{j=1}^k B(Y_{t_j})\}.
\]
The covariance and correlation functions are extended to the entire real line by symmetry: \( R(-t) = R(t) \), and \( H(-t) = H(t) \) for every \( t \in \mathbb{R} \).

The standard unbiased estimators of \( R(t) \) and \( H(t) \) are
\[
\hat{R}(t) := \frac{1}{T-t} \int_0^{T-t} (N(s) - \rho)(N(t+s) - \rho) ds, \quad \hat{H}(t) := \hat{R}(t)/\rho.
\]
The next statement establishes an upper bound on the estimation accuracy of \( \hat{R}(t) \).

**Theorem 3.1** For any \( 0 \leq t < T \) and \( 0 \leq s < T \) one has
\[
(T-s)(T-t) \mathbb{E}[\hat{R}(t) - R(t)][\hat{R}(s) - R(s)] = \rho^2 \int_0^{T-s} \int_0^{T-t} \left[ H(\tau_1 - \tau_2)H(\tau_1 - \tau_2 + t - s) + H(\tau_1 - \tau_2 - s)H(\tau_1 - \tau_2 + t) \right] d\tau_1 d\tau_2
\]
\[+ \rho \int_0^{T-s} \int_0^{T-t} U(\tau_1, \tau_1 + t, \tau_2, \tau_2 + s) d\tau_1 d\tau_2.
\]

In particular,
\[
\mathbb{E}[\hat{R}(t) - R(t)]^2 \leq \frac{C(\rho^2 + \rho)}{T-t} \int_0^T H(y) dy,
\]
where \( C \) is an absolute constant.

**Remark 3.1** Theorem 3.1 is valid for any Smoluchowski process, independently of the displacement model.

It is instructive to compare Theorem 3.1 with results on estimation of covariance functions of stationary Gaussian processes. It is well known that for a stationary Gaussian process with correlation function \( H(t) \) the standard estimator of \( H(t) \) is consistent in the mean square sense if and only if \( T^{-1} \int_0^T H^2(t) dt \rightarrow 0 \) as \( T \rightarrow \infty \), and the mean squared estimation error admits an upper bound which is proportional to \( T^{-1} \int_0^T H^2(t) dt \); see, e.g., Yaglom (1987, Section 17). As Theorem 3.1 shows, this is not the case for the Smoluchowski process: here the mean squared error is proportional to \( T^{-1} \int_0^T H(t) dt \).

The difference is due to the presence of the last term on the right hand side of (3.4) which, in general, cannot be bounded in terms of the \( L_2 \)-norm of the correlation function.

## 4 Undeviated uniform motion

In this section we consider estimation problems for the Smoluchowski process (1.1)–(1.2) associated with the undeviated uniform motion; see (1.3). In this setting function \( Q_\pi(t_1, \ldots, t_n) \) in (2.3) is non-zero only for the subsets \( \pi \) consisting of consecutive numbers from \( \{1, \ldots, n\} \), i.e., for the subsets such that \( \pi = \{l, l+1, \ldots, m\} \) for some \( 1 \leq l \leq m \leq n \). Under these circumstances
\[
Q_\pi(x) = \mathbb{P}\{(x + vt_k \in B, k \in \{l, \ldots, m\}) \cap (x + vt_k \notin B, k \notin \{l, \ldots, m\}\}
\]
\[= \mathbb{P}\{x \in B(vt_l) \cap B(vt_m), x \notin B(vt_{l-1}), x \notin B(vt_{m+1})\}
\]
\[= \mathbb{P}\{x \in B(vt_{l-1}) \cap B(vt_{m+1})\} + \mathbb{P}\{x \in B(vt_l) \cap B(vt_{m})\}
\]
\[- \mathbb{P}\{x \in B(vt_{l-1}) \cap B(vt_m)\} - \mathbb{P}\{x \in B(vt_l) \cap B(vt_{m+1})\},
\]

where $B(vt_k)$ is interpreted as the empty set if $k \notin \{1,\ldots, n\}$. The resulting Smoluchowski process is strictly stationary. Indeed, since

$$vol\{B(vt) \cap B(vt')\} = vol\{B \cap B(v(t - \tau))\} = g_B(v(t - \tau)), \ \forall t, \tau,$$

we have for $\pi = \{l, l + 1, \ldots, m\}$ that

$$Q_\pi(t_1, \ldots, t_n) = \mathbb{E}\left[\frac{1}{\vol(B)} \cdot \text{vol}\{B(v(t_{m+1} - t_{l-1})) + g_B(v(t_{m} - t_{l})) - g_B(v(t_{m} - t_{l-1})) - g_B(v(t_{m+1} - t_{l}))\}\right],$$

where $g_B(v(t_i - t_j)) = 0$ whenever $i \notin \{1,\ldots, n\}$ or $j \notin \{1,\ldots, n\}$. It follows that

$$U_\pi(t_1, \ldots, t_n) = \frac{1}{\vol(B)} \cdot \text{vol}\{B(v \max_{j \in \pi} t_j) \cap B(v \min_{j \in \pi} t_j)\} = \frac{E g_B(v(\max_{j \in \pi} t_j - \min_{j \in \pi} t_j))}{\vol(B)},$$

and the covariance function of $\{N(t), t \geq 0\}$ is

$$R(t) = \rho H(t), \ \ \ \ H(t) = \frac{1}{\vol(B)} \cdot \text{vol}\{B \cap B(vt)\} = \frac{1}{\vol(B)} \int_{\mathbb{R}^d} g_B(vt) \, dG(v),$$

where, we recall, $G$ is the velocity distribution function.

The covariance function in (4.1) depends on the geometry of the observation region via covariogram $g_B$. In this section and from now on we will assume that $B$ is the closed Euclidean ball of radius $r > 0$ centered at the origin,

$$B = \{x : \|x\| \leq r\}, \ \ r > 0.$$

Then the covariogram is

$$g_B(x) = \vol\{B \cap B(x)\} = \vol\{B\} \cdot I\left(\frac{dx}{2}, \frac{1}{2}; 1 - \frac{\|x\|^2}{r^2}\right) \cdot \mathbb{1}\{\|x\| \leq 2r\},$$

where $I(a; b; x)$ is the regularized incomplete beta function

$$I(a; b; x) := \frac{1}{B(a, b)} \int_0^x t^{a-1}(1 - t)^{b-1} \, dt, \ \ a > 0, b > 0, \ 0 \leq x \leq 1,$$

and $B(a, b)$ is the beta function. Formula (4.3) is a consequence of the well known expression for the volume of the spherical cap; see, e.g., Li (2011).

The particular form of covariogram $g_B$ in (4.3) has immediate implications on identifiability of certain functionals of $G$ from the Smoluchowski process data. In view of (4.3), $g_B(x)$ depends on $x$ via the Euclidean norm $\|x\|$ only, i.e., $g_B(x) = g_B(\|x\|)$. Therefore formula (4.1) together with Theorem 2.1 imply that all finite dimensional distributions of the corresponding Smoluchowski process $\{N(t), t \geq 0\}$ depend on $v$ via $\|v\|$ only. Thus, if $B$ is a Euclidean ball then only speed distribution (the distribution of $\|v\|$) is identifiable from the count data.

Therefore we deal with the following two estimation problems: given continuous time observations $X_T = \{N(t), 0 \leq t \leq T\}$ of the Smoluchowski process we want to estimate: (a) the mean speed $\mu := \mathbb{E}\|v\| = \int_0^\infty xdF(x)$; and (b) the value $F(x_0)$ of the speed distribution function $F$ at given point $x_0$. The problem of estimating the mean speed $\mu$ from discrete time observations was discussed by Lindley (1954) in the setting of undeviated uniform motion on the real line. This paper established the relationship between the one–sided derivative of the correlation function at zero and the mean speed, and discussed construction of an estimator based on discrete time data, but did not present rigorous analysis of its accuracy. To the best of our knowledge, the problem of estimating the speed distribution has not been studied in the literature.

We adopt the minimax framework for measuring estimation accuracy. Let $\psi = \psi(F)$ be a functional of the speed distribution $F$ such as $\mu = \mathbb{E}\|v\|$ or $F(x_0)$. By an estimator $\hat{\psi}$ of $\psi(F)$ we mean any measurable function of observation $X_T$, and accuracy of $\hat{\psi}$ is measured by the maximal root mean squared error

$$R_T[\hat{\psi}; \mathcal{F}] := \sup_{F \in \mathcal{F}} \left\{ \mathbb{E}[\hat{\psi} - \psi(F)]^2 \right\}^{1/2}$$

on a natural class $\mathcal{F}$ of speed distribution functions $F$. The minimax risk is defined by $R_T^*[\mathcal{F}] := \inf_{\hat{\psi}} R_T[\hat{\psi}; \mathcal{F}]$, where $\inf$ is taken over all possible estimators of $\psi(F)$. The goal is to develop a rate–optimal estimator $\hat{\psi}_r$ of $\psi(F)$ such that $R_T[\hat{\psi}_r; \mathcal{F}] \approx R_T^*[\mathcal{F}]$ as $T \to \infty$. 

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4.1 Correlation function and its properties

In view of (4.2) and (4.3), covariance function of the Smoluchowski process associated with the undeviated uniform motion is given by

\[ R(t) = \rho H(t) = \frac{\rho}{B(d+1, \frac{1}{2})} \int_0^{\infty} \int_0^{\infty} y^{\frac{1}{2}(d-1)}(1-y)^{-1/2} 1\{xt \leq 2r\} dy \, dF(x) \]

\[ = \frac{1}{B(d+1, \frac{1}{2})} \int_0^{1} F\left(\frac{2r \rho}{t}\right)(1-y)^{\frac{1}{2}(d-1)}y^{-1/2} \, dy. \quad (4.4) \]

The next statement establishes some useful properties of the correlation function.

**Lemma 4.1** One has

\[ \int_0^\infty H(t) dt = 2r B\left(\frac{d+1}{2}, \frac{1}{2}\right) \int_0^\infty x^{-1} F(x). \]

Moreover, if \( F \) is absolutely continuous with density \( f \), and \( f(x) \leq Mx^\alpha \) for some \( \alpha > -1 \) and \( 0 \leq x \leq \delta \) then

\[ H(t) \leq \frac{M}{1+\alpha} \left(\frac{2r}{t}\right)^{1+\alpha}, \quad \forall t \geq 2r/\delta, \quad (4.5) \]

and for any \( T \geq 2r/\delta \)

\[ \int_0^T H(t) dt \leq c_0 (r + Mr^{1/(1+\alpha)} \eta_T), \quad \eta_T := \left\{ \begin{array}{ll} T^{-\alpha}, & -1 < \alpha < 0, \\ \ln T, & \alpha = 0, \\ 1, & \alpha > 0, \end{array} \right. \quad (4.6) \]

where constant \( c_0 \) depends on \( \alpha \) and \( \delta \) only.

Lemma 4.1 demonstrates that the Smoluchowski process associated with the undeviated uniform motion exhibits short range dependence if and only if \( \int_0^\infty x^{-1} F(x) < \infty \). By short range dependence (or short memory) we mean the property of integrability of the covariance function, while absence of this property corresponds to the long range dependence (or long memory) [see, e.g., Samorodnitsky (2016, Chapter 6)]. The Smoluchowski process has long memory when \( \int_0^\infty x^{-1} F(x) = \infty \); this is the case, e.g., if \( F \) is absolutely continuous with respect to the Lebesgue measure with density \( f \), and \( f(0) > 0 \). In particular, if \( f \) is bounded and \( f(0) > 0 \) then \( \int_0^T H(t) dt = O(\ln T) \) as \( T \to \infty \). In general, the rate of decay of the correlation function \( H(t) \) as \( t \to \infty \) is determined by the local behavior of \( F \) near zero.

4.2 Mean speed estimation

Now we are in a position to define an estimator of the mean speed \( \mu = \mathbb{E}\|v\| \). As mentioned above, the covariogram \( g_B(x) \) depends on \( x \) via \( \|x\| \) only, \( g_B(x) = \tilde{g}_B(\|x\|) \), where

\[ \tilde{g}_B(y) := \text{vol}\{B\} I\left(\frac{d+1}{2}, \frac{1}{2}; 1 - \frac{y^2}{4r^2}\right) 1\{y \leq 2r\}; \]

see (4.3). Function \( \tilde{g}_B(y) \) is continuous for all \( y \geq 0 \), monotone decreasing and has continuous derivative in interval \((0, 2r)\) which is

\[ \tilde{g}'_B(y) = -\frac{\text{vol}\{B\}}{r B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \left(1 - \frac{y^2}{4r^2}\right)^{\frac{1}{2}(d-1)}, \quad y \in (0, 2r). \]

In view of (4.2) and (4.3) we have \( R(t) = \left(\frac{\rho}{\text{vol}\{B\}}\right) \int_0^\infty \tilde{g}(xt) F(x) \, dx \), \( t \geq 0 \), so that

\[ R'(t) = -\frac{\rho}{r B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \int_0^{2r/t} \left(1 - \frac{t^2x^2}{4r^2}\right)^{\frac{1}{2}(d-1)} x \, dF(x), \quad (4.7) \]

and

\[ R'(0+) = \lim_{t \downarrow 0} R'(t) = -\frac{\rho}{r B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \mathbb{E}\|v\|. \]
The last equation can be used as a basis for constructing an estimator of the mean speed $\mu = E\|v\|$. The main idea is to estimate the one–sided derivative of the covariance function at zero $R'(0+)$. For one–dimensional case and discrete observations this idea has been discussed in Lindley (1954); see also Bingham & Dunham (1997).

Let $K : [0, 1] \rightarrow \mathbb{R}$ be a kernel satisfying the following conditions

$$
\int_0^1 K(x)dx = 0, \quad \int_0^1 xK(x)dx = 1, \quad \int_0^1 x^2|K(x)|dx =: C_K < \infty.
$$

Fix real number $h > 0$, and consider the following estimator of $\mu$:

$$
\hat{\mu}_h = -\frac{xB(d+1, \frac{1}{2})}{\rho h^d} \int_0^h K\left(\frac{t}{h}\right) \hat{R}(t)dt,
$$

where $\hat{R}(t)$ is defined in (4.7). The bandwidth $h$ is a design parameter of the estimator; it will be specified in the sequel.

**Definition 4.1** Let $L > 0$; we say that distribution function $F$ on $[0, \infty)$ belongs to the class $\mathcal{F}(L)$ if it is absolutely continuous with differentiable density $f$, and

$$
\sup_{x > 0} \left\{ (1 + x^{l+2})|f^{(l)}(x)| \right\} \leq L, \quad \int_0^\infty x^{l+2}|f^{(l)}(x)|dx \leq L, \quad l = 0, 1.
$$

**Theorem 4.1** Let $\hat{\mu}_{h_*}$ be the estimator of $\mu = E\|v\|$ defined in (4.7) and associated with bandwidth $h_*$ satisfying $(ln T)^2/T \leq h_* \leq 1/(\sqrt{T}ln T)$; then

$$
\limsup_{T \to \infty} \left\{ \sqrt{T}R_T[\hat{\mu}_{h_*}; \mathcal{F}(L)] \right\} \leq C \left(1 + \frac{1}{\rho}\right)^{1/2} [L(1 + L)r]^{1/2},
$$

where $C$ is a constant depending on $d$ only.

**Remark 4.1** Theorem 4.1 demonstrates that the mean speed $\mu = E\|v\|$ can be estimated from count data with the parametric rate: the maximal root mean squared error of the proposed estimator over class $\mathcal{F}(L)$ converges to zero at the rate $1/\sqrt{T}$ as $T \to \infty$. Therefore $\hat{\mu}_{h_*}$ is a rate–optimal estimator. The definition of $\mathcal{F}(L)$ requires existence and boundedness of the first derivative of $f$ (second derivative of $F$) along with mild tail conditions. The condition $F \in \mathcal{F}(L)$ implies boundedness of the first and second derivatives of the correlation function $H$ and integrability of the first derivative of $H$; these properties are essential in the theorem proof.

### 4.3 Estimation of the speed distribution

In this section we deal with the problem of estimating the value $F(x_0)$ of the speed distribution $F$ at given point $x_0 > 0$. Our construction uses formula (4.7); it will be convenient to rewrite it in the following form:

$$
H(t) = \int_0^\infty w(tx)dF(x),
$$

$$
w(t) := \frac{1\{t \leq 2r\}}{B(d+1, \frac{1}{2})} \int_0^{1-rac{\rho t}{2}} y^{\frac{1}{2}(d-1)}(1-y)^{-\frac{1}{2}}dy, \quad t \geq 0.
$$

The correlation function on the left hand side can be estimated from the data $\mathcal{N}_T = \{N(t), 0 \leq t \leq T\}$. The distribution function $F$ is related to the correlation function via integral operator in (4.11) that should be inverted. To construct the estimator we use a method based on the Laplace and Mellin transforms.
Preliminaries. First we introduce notation and recall some standard facts about the Laplace and Mellin transforms; for details we refer to Widder (1941).

For generic function \( g \) on \( \mathbb{R} \) the bilateral Laplace transform of \( g \) is defined by

\[
\hat{g}(z) := \mathcal{L}[g; z] = \int_{-\infty}^{\infty} g(t) e^{-zt} dt,
\]

and \( \hat{g}(z) \) is an analytic function in the region where the integral converges. In general, the convergence region is a vertical strip in the complex plane, say, \( \hat{\sigma}_g < \text{Re}(z) < \hat{\sigma}_g^+ \) for some \(-\infty \leq \hat{\sigma}_g < \hat{\sigma}_g^+ \leq \infty\). The inverse Laplace transform is given by

\[
g(t) = \frac{1}{2\pi i} \int_{\hat{\sigma}_g - i\infty}^{\hat{\sigma}_g + i\infty} \hat{g}(z) e^{zt} dz = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \hat{g}(s + i\omega) e^{(s+i\omega)t} d\omega, \quad \hat{\sigma}_g < s < \hat{\sigma}_g^+,
\]

where the integration is performed over any vertical line in the convergence region. The Mellin transform of a function \( g \) on \([0, \infty)\) is defined by the integral

\[
\tilde{g}(z) := \mathcal{M}[g; z] = \int_0^{\infty} t^{z-1} g(t) dt
\]

with convergence region \( \tilde{\Sigma}_g := \{ z \in \mathbb{C} : \hat{\sigma}_g < \text{Re}(z) < \hat{\sigma}_g^+ \} \). Then the inversion formula for the Mellin transform is

\[
g(t) = \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} t^{-z} \tilde{g}(z) dz = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \tilde{g}(s + i\omega) t^{-s-i\omega} d\omega, \quad \hat{\sigma}_g < s < \hat{\sigma}_g^+.
\]

The following standard facts about the Mellin transform are repeatedly used in the sequel. If \( g_1 \) and \( g_2 \) are two functions such that the integral \( \int_0^{\infty} g_1(x) g_2 dx \) exists, and if the Mellin transforms \( \tilde{g}_1(1-z) \) and \( \tilde{g}_2(z) \) have a common strip of analyticity then for any line \( \{ z : \text{Re}(z) = c \} \) in this strip

\[
\int_0^{\infty} g_1(x) g_2(x) dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{g}_1(1-z) \tilde{g}_2(z) dz.
\]

In addition, the Parseval identity for the Mellin transform reads as

\[
\int_0^{\infty} g^2(x) x^{2r-1} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{g}(s+i\omega)|^2 d\omega. \tag{4.12}
\]

Estimator construction. Now we proceed with construction of the estimator. Let \( K \) be a kernel satisfying the following condition.

(K) Function \( K : \mathbb{R} \to \mathbb{R} \) be an infinitely differentiable bounded function on \( \mathbb{R} \) such that

\[
\text{supp}(K) = [0, 1], \quad \int_0^1 K(y) dy = 1,
\]

and for given positive integer \( m \)

\[
\int_0^1 K(y) y^j dy = 0, \quad j = 1, \ldots, m.
\]

Condition (K) is standard in nonparametric estimation with kernel methods.

For \( 0 < h < 1/2 \) and \( x_0 > 2h \) define function

\[
\varphi_{x_0, h}(t) := \int_0^t \left[ \frac{1}{h} K \left( \frac{x}{h} \right) - \frac{1}{xh} K \left( \frac{\ln(x/x_0)}{h} \right) \right] dx, \quad t \geq 0. \tag{4.13}
\]

The next statement demonstrates that for small \( h \) function \( \varphi_{x_0, h} \) is a smooth approximation to the indicator function \( 1_{[0,x_0]}(\cdot) \).
Lemma 4.2 Let $K$ be a kernel satisfying condition (K); then $\varphi_{x_0,h}$ possesses the following properties:

$$\text{supp}(\varphi_{x_0,h}) = [0, x_0 e^h], \quad \varphi_{x_0}(t) = 1, \quad h \leq t \leq x_0.$$ 

In addition, $\tilde{\varphi}_{x_0}$ is an entire function, and

$$\tilde{\varphi}_{x_0,h}(z) = \frac{1}{z} \left[ w_0^z K(-z h) - h^z K(z + 1) \right], \quad \forall z \in \mathbb{C}.$$ 

The next step in our construction is to define

$$\psi_{x_0,h}(t) := \frac{1}{2\pi i} \int_{s-i\infty}^{s+i\infty} \frac{\tilde{\varphi}_{x_0,h}(1 - z)}{w(1 - z)} t^{-z} dz$$

$$= \frac{1}{2\pi} \int_{s-i\infty}^{s+i\infty} \frac{t^{-z} B(\frac{d+1}{2}, \frac{d}{2})}{(2r)^{1-z} B(\frac{d+1}{2}, 1 - \frac{1}{r})} \left[ x_0^{1-z} K((z - 1)h) - h^{1-z} K(2 - z) \right] dz, \quad s < 1. \quad (4.14)$$

The special form of the integrand in the second line on the right hand side of (4.14) is a consequence of the following formula for the Mellin transform of function $w$ defined in (4.13):

$$\tilde{w}(z) = \left( \frac{2r}{z} \right)^{\frac{d+1}{2} - \frac{d}{2}} \left( \frac{z}{B(\frac{d+1}{2}, 1 - \frac{1}{r})} \right)^{Re(z) > 0.}$$

This formula is stated in Lemma 4.2 in Section 7.5 and proved there. Note that the integrand in (4.14) is an analytic function in $\{ z : \text{Re}(z) < 1 \}$ so that the integration can be performed over any vertical line in this region. It is also seen that $\psi_{x_0,h}$ is a function on $[0, \infty)$ defined by the inversion formula for the Mellin transform.

Our construction of estimator of $F(x_0)$ utilizes special properties of function $\psi_{x_0,h}(t)$ established in the following lemma.

Lemma 4.3 Under assumption (K) one has

$$\int_0^\infty \left| \frac{\tilde{\varphi}_{x_0,h}(1 - s - i\omega)}{w(1 - s - i\omega)} \right| d\omega < \infty, \quad \forall s < 1. \quad (4.15)$$

Moreover, if $\int_0^\infty |\psi_{x_0,h}(t)| |H(t)| dt < \infty$ then

$$\int_0^\infty \psi_{x_0,h}(t) H(t) dt = \int_0^\infty \varphi_{x_0,h}(x) dF(x), \quad (4.16)$$

where $\varphi_{x_0,h}$ is given by (4.14).

The property (4.16) is of crucial importance for our purposes. In view of Lemma 4.2 the integral on the right hand side of (4.16) approximates the value $F(x_0)$ to be estimated. Therefore the main idea is to estimate the left hand side of (4.16) by plugging in the estimator of the correlation function. Specifically, define

$$\hat{F}_h(x_0) := \int_0^{T/2} \psi_{x_0,h}(t) \hat{H}(t) dt, \quad (4.17)$$

where $\hat{H}(t)$ is an estimator of $H(t)$ defined in (3.3).

Upper bound on the risk. Our current goal is to study the risk of the constructed estimator $\hat{F}_h(x_0)$. For this purpose we first define the functional class of speed distribution functions on which the risk of $\hat{F}_h(x_0)$ is assessed.

Definition 4.2 Let $A > 0$, $\beta > 0$ be fixed real numbers. We say that distribution function $F$ on $[0, \infty)$ belongs to the functional class $\mathcal{H}_\beta(A)$ if $F$ is $\ell := \lfloor \beta \rfloor = \min\{ k \in \mathbb{N} \cup \{ 0 \} : k < \beta \}$ times continuously differentiable and

$$\max_{k=1, \ldots, \ell} |F^{(k)}(x)| \leq A, \quad |F^{(\ell)}(x) - F^{(\ell)}(x')| \leq A |x - x'|^{\beta-\ell}, \quad \forall x, x' \in [0, \infty).$$

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In words, $\mathcal{H}_\beta(A)$ is the class of all distribution functions on $[0, \infty)$ satisfying Hölder’s condition of order $\beta$.

**Definition 4.3** Let $\alpha > -1$, $M > 0$ and $\delta > 0$. We say that distribution function $F$ on $[0, \infty)$ belongs to the functional class $\mathcal{J}_0(M)$ if $F$ is absolutely continuous with density $f$ and

$$f(x) \leq Mx^\delta, \quad \forall 0 \leq x \leq \delta.$$

The functional class $\mathcal{J}_0(M)$ imposes restrictions of the local behavior of the speed density $f$ near the origin. According to Lemma 4.1 this behavior is responsible for the long/short range dependence of the Smoluchowski process associated with the undeviated uniform motion. Note that the definition of the functional class $\mathcal{J}_0(M)$ also involves parameter $\delta$, but we do not indicate it in the notation.

Define also

$$\mathcal{F}_{\alpha,\beta}(A, M) := \mathcal{H}_\beta(A) \cap \mathcal{J}_0(M).$$

**Theorem 4.2** Let $\hat{F}_{h_\alpha}(x_0)$ be the estimator defined in (4.17) and associated with kernel $K$ that satisfies assumption (K) with $m > \beta + 1$, and with bandwidth $h$ that is set to be

$$h^* = \left[A^{-2}(x_0^\beta + 1)^{-2}(1 + \frac{1}{\rho^2})\frac{\tilde{\eta}_T}{T}\right]^{1/2(d+\delta+2)} \cdot \tilde{\eta}_T := (r + Mr^{1/(1+\alpha)})\eta_T,$$

(4.18)

where $\eta_T$ is defined in (4.6). Let

$$\phi_T := \left[A^2(x_0^\beta + 1)^2\frac{2\delta(d+\delta+1)}{2(d+\delta+2)}\left(1 + \frac{1}{\rho^2}\frac{\tilde{\eta}_T}{T}\right)^{\frac{\delta}{2(d+\delta+2)}}ight]$$

then

$$\lim_{T \to \infty} \left\{ \phi_T^{-1} \mathcal{R}_T[\hat{F}_{h_\alpha}(x_0); \mathcal{F}_{\alpha,\beta}(A, M)] \right\} \leq C,$$

where $C$ may depend on $d, \beta$, and $\alpha$ only.

**Remark 4.2** Theorem 4.2 shows that the rates at which the risk of $\hat{F}_{h_\alpha}(x_0)$ converges to zero are the following: $(\ln T/T)^{\beta/(2\beta+\delta+2)}$ if $\alpha > 0$; $(\ln^2 T/T)^{\beta/(2\beta+\delta+2)}$ if $\alpha = 0$; and $(\ln T/T^{1+\alpha})^{\beta/(2\beta+\delta+2)}$ if $-1 < \alpha < 0$. The existence of these three regimes in the rate of convergence is explained by the short/long range dependence of the Smoluchowski process: $\alpha > 0$ corresponds to the short memory, while $-1 < \alpha \leq 0$ results in the long memory with $\alpha = 0$ being the boundary case.

We do not have a formal proof that $\hat{F}_{h_\alpha}(x_0)$ is nearly rate–optimal up to a logarithmic factor; however we conjecture that this is so. Our conjecture is based on the connection to the results obtained recently in a closely related statistical inverse problem. Belomestny & Goldenshluger (2020) considered the problem of density estimation from observations with multiplicative measurement errors. This is a statistical inverse problem with integral operator of type $\hat{f}(w) = \int f(t)w(t)dt$. It was shown there that the achievable estimation accuracy in such problems is determined by smoothness of the function to be estimated and by the ill–posedness index $\gamma$ characterizing the rate of decay of the Mellin transform of function $w$ along vertical lines in the convergence region. In particular, if the function to be estimated satisfies the Hölder condition with index $\beta$, and the rate of decay of the Mellin transform is $|\omega|^{-\gamma}$ as $|\omega| \to \infty$ then the minimax pointwise risk converges to zero at rate $n^{-\beta/(2\beta+2\gamma+1)}$, where $n$ is the sample size in the setting of Belomestny & Goldenshluger (2020). As it is shown in the proof of Theorem 4.2 the ill–posedness index of the inverse problem in (4.10)–(4.11) is $\gamma = (d + 1)/2$. With this definition of $\gamma$, the rate of convergence established in Theorem 1.2 $(\ln T/T)^{\beta/(2\beta+2\gamma+1)}$, matches the one in Belomestny & Goldenshluger (2020) up to a logarithmic factor. That is why we conjecture that $\hat{F}_{h_\alpha}(x_0)$ is nearly rate–optimal for $F(x_0)$.

5 Brownian displacement model

In this section we consider the Smoluchowski process associated with the displacement governed by the Brownian motion, see (1.2). Recall that in this setting $X^{(j)}_t = \xi_j + Y^{(j)}_t$, $Y^{(j)}_t = \sigma W^{(j)}_t$, for $j \in \mathbb{Z}$, $t \geq 0$, where $W^{(j)}_t$ is the standard Brownian motion in $\mathbb{R}^d$, and $\sigma > 0$ is the diffusion coefficient.
With this displacement process Theorem 2.1 holds with
\[ Q_\pi(x) = \int_{B_1} \int_{B_2} \cdots \int_{B_p} \varphi(t_1; x, y_1) \varphi(t_2 - t_1; y_1, y_2) \cdots \varphi(t_p - t_{p-1}; y_{p-1}, y_p) dy_1 dy_2 \cdots dy_p, \]
where \( \pi \in \Pi_n, \varphi(t; x, y) := (\sqrt{2\pi\sigma^2 t})^{-1} \exp\{-||x - y||^2/(2\sigma^2 t)\} \) is the Gaussian kernel, and
\[ B_k := \begin{cases} B, & k \in \pi, \\ B^c := \mathbb{R}^d \setminus B, & k \in \pi^c, \end{cases} \quad k \in \{1, \ldots, p\}. \]
Therefore function \( Q_\pi \) in (2.2) is given by
\[ \tilde{Q}_\pi(t_1, \ldots, t_p) = \int_{B_1} \int_{B_2} \cdots \int_{B_p} \varphi(t_2 - t_1; y_1, y_2) \cdots \varphi(t_p - t_{p-1}; y_{p-1}, y_p) dy_1 dy_2 \cdots dy_p. \]
This formula shows that the Smoluchowski process associated with the Brownian displacement is strictly stationary. As before, in this section we assume that the observation region is the Euclidean ball of radius \( r \) centered in the origin.

5.1 Correlation function and its properties

The correlation function of the Smoluchowski process governed by Brownian displacement is easily calculated using (5.1), (5.2) and (4.3):
\[ H(t) = \frac{1}{\text{vol}(B)} \mathbb{E} \text{vol} \{ B \cap B(\sigma W_t) \} = \mathbb{E} I \left( \left[ \frac{d+1}{d}, \frac{1}{2} \right] : 1 - \frac{\sigma^2 ||W||^2}{4t^2} \right) \]
\[ = \frac{1}{B(d+1, \frac{1}{2})} \mathbb{E} \int_0^1 1 \{ y \leq 1 - 4t^2 y^2 \} y^{(d-1)/2} (1 - y)^{-1/2} dy \]
\[ = \frac{1}{B(d+1, \frac{1}{2})} \int_0^1 \mathbb{P} \{ y \leq 4t^2 y^2 \} (1 - y)^{(d-1)/2} y^{-1/2} dy, \]
where \( \eta \) is a random variable distributed \( \chi^2 \) with \( d \) degrees of freedom. If \( \Gamma(s; x) := \int_x^\infty t^{s-1} e^{-t} dt \) is the upper incomplete Gamma function [see (Abramowitz & Stegun 1965, Chapter 6)] then
\[ \mathbb{P} \{ \eta > x \} = \frac{1}{2d/2 \Gamma(d/2)} \int_x^\infty t^{d/2-1} e^{-t/2} dt = \frac{\Gamma(d/2; x)}{\Gamma(d/2)}, \]
and we obtain
\[ H(t) = 1 - \frac{1}{B(d+1, \frac{1}{2}) \Gamma(d/2)} \int_0^1 \Gamma \left( \frac{d}{2} : \frac{2t^2 y}{\sigma^2} \right) (1 - y)^{(d-1)/2} y^{-1/2} dy \]
\[ = \frac{1}{B(d+1, \frac{1}{2}) \Gamma(d/2)} \int_0^1 \gamma \left( \frac{d}{2} : \frac{2t^2 y}{\sigma^2} \right) (1 - y)^{(d-1)/2} y^{-1/2} dy, \]
where \( \gamma(s; x) := \Gamma(x) - \Gamma(s; x) \) is the lower incomplete Gamma function.

In the next statement we summarize some properties of the correlation function that are useful for our purposes.

Lemma 5.1 (a) The following asymptotic relationships hold:
\[ 1 - H(t) \sim C(d) \frac{\sigma^2 t}{2\pi}, \quad C(d) := \begin{cases} 1, & d = 1, \\ d/(d-1), & d \geq 2, \end{cases} \quad t \to 0; \]
\[ H(t) \sim \frac{\Gamma(d+1)}{\Gamma(d+1) \Gamma(d/2)} \left( \frac{\sigma^2 t}{2\pi} \right)^d, \quad t \to \infty, \]
where \( a \sim b \) means that \( \lim(a/b) = 1 \).
(b) For every $t > 0$

$$H(t) \leq \frac{2^{d/2}\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(d+1)\Gamma\left(\frac{1}{2}\right)} \left(\frac{r^2}{\sigma^2 t}\right)^{d/2}. \quad (5.2)$$

In addition, for $T > 0$

$$\int_0^T H(t) dt \leq \begin{cases} \frac{4r^2}{\sigma^2(d+1)} & d > 2, \\ 1 + \frac{r^2}{2\sigma^2}\ln T & d = 2, \\ 2\frac{\sqrt{2}}{\sqrt{\pi}\sigma} \sqrt{T} & d = 1. \end{cases} \quad (5.3)$$

Remark 5.1 Lemma 5.1 demonstrates that the one–sided first derivative of correlation function $H$ at zero is infinite, i.e., $H'(0+) = \infty$. Also, the rate of decay of $H$ at infinity depends on dimension $d$: we have $H(t) = O(t^{-d/2})$ as $t \to \infty$. This implies that for $d = 1, 2$ the Smoluchowski process has long memory: in these cases $\int_0^\infty H(t) dt = \infty$.

5.2 Estimation of the diffusion coefficient

In this section we consider the problem of estimating diffusion coefficient $\sigma$ from the data $\mathcal{N}_T = \{N(t), 0 \leq t \leq T\}$. Although the problem of estimating the diffusion coefficient from observations of the Smoluchowski process was discussed in the literature [see, e.g., in Kac (1959) and Bingham & Dunham (1997)], we are not aware of specific estimators with provable accuracy guarantees. The goal of this section is to develop such an estimator.

The proposed estimator is based on the following simple idea. Define

$$J(t) := \frac{1}{B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \Gamma\left(\frac{1}{2}\right) \int_0^1 \gamma(d-1) \left(\frac{2r^2y}{t}\right) (1-y)^{(d-1)/2}y^{-1/2} dy, \quad t > 0.$$

Note that $J$ is a fixed known function; it is completely determined by known parameters $r$ and $d$ and can be computed at any point. Observe that $H(t) = J(\sigma^2 t)$, i.e., the problem of estimating $\sigma^2$ is the problem of estimating the scale parameter of function $J$.

Let $\alpha \in (0, \frac{1}{2})$ be a parameter to be specified, and consider the following functional of the correlation function

$$\Psi_\alpha := \Psi_\alpha(H) = \int_0^\infty \frac{H(t)}{t^{1-\alpha}} dt.$$

It follows from (5.2) that the integral on the right hand side is finite for all $d \in \mathbb{N}$. Note also that

$$\Psi_\alpha = \int_0^\infty \frac{J(\sigma^2 t)}{t^{1-\alpha}} dt = \sigma^{-2\alpha} \Psi_\alpha,$$

where

$$J_\alpha := \Psi_\alpha(J) := \int_0^\infty \frac{J(t)}{t^{1-\alpha}} dt = \frac{(2r^2)\alpha\Gamma\left(\frac{d}{2} - \alpha\right)B\left(\frac{1}{2}, \frac{d+1}{2}\right)}{\alpha\Gamma\left(\frac{1}{2}\right)B\left(\frac{d+1}{2}, \frac{1}{2}\right)}.$$

With the introduced notation $\sigma^2 = [J_\alpha/\Psi_\alpha]^{1/\alpha}$, and if $\hat{\Psi}_\alpha$ is an estimator of $\Psi_\alpha$ then a natural estimator of $\sigma^2$ can be defined as $\hat{\sigma}^2 := [\hat{J}_\alpha/\hat{\Psi}_\alpha]^{1/\alpha}$.

We consider the following estimator of $\Psi_\alpha$. Recall that $\rho = \lambda \text{vol}(B)$, and let

$$\hat{H}(t) := \frac{1}{T-t} \int_0^{T-t} [N(s) - \rho][N(s + t) - \rho] ds,$$

where $\{\cdot\}_+ = \max\{\cdot, 0\}$. For parameter $b > 0$ to be specified define

$$\hat{\Psi}_{\alpha, b} := \int_0^b \frac{\hat{H}(t)}{t^{1-\alpha}} dt, \quad (5.4)$$

$$\hat{\sigma}_\alpha := \left[\int_0^\infty \frac{\hat{H}(t)}{t^{1-\alpha}} dt\right]^{1/\alpha}.$$
and the corresponding estimator of $\sigma^2$ is

$$\hat{\sigma}_{\alpha,b}^2 := \left( \frac{J_\alpha}{\Psi_{\alpha,b}} \right)^{1/\alpha}.$$  

The proposed estimator $\hat{\sigma}_{\alpha,b}^2$ depends on two design parameters $\alpha$ and $b$ that are specified in the sequel.

Now we are in a position to present a result on accuracy of the proposed estimator of $\sigma^2$. For $x, y > 0$ we let $\Delta(x, y) := |x - y|/(x + y)$ and note that $\Delta(\cdot, \cdot)$ defines a distance on $\{x \in \mathbb{R} : x > 0\}$. We use $\Delta(\cdot, \cdot)$ as a loss function in the problem of estimating $\sigma^2$; it measures the relative estimation accuracy.

**Theorem 5.1** Let $\hat{\sigma}_\alpha^2 = [J_{\alpha,b}/\hat{\Psi}_{\alpha,b}]^{1/\alpha}$ be the estimator of $\sigma^2$, where $\hat{\Psi}_{\alpha,b}$ is the estimator in (5.4) associated with

$$\alpha := \frac{1}{\ln T}, \quad b := \begin{cases} (T/\ln^2 T)^{1/d}, & d > 2, \\ (T/\ln^3 T)^{1/2}, & d = 2, \\ \sqrt{T/\ln^2 T}, & d = 1, \end{cases}$$

Then

$$\limsup_{T \to \infty} \sup_{\sigma^2, \sigma^2 \leq r^2 T} \left\{ \left[ \left( 1 + \frac{1}{\rho} \right) \left( \frac{r}{\sigma} \right)^{d/2} + \left( \frac{r}{\sigma} \right)^{2d/3} \right]^{-1} \phi_T^{-1} \mathbb{E} \left[ \Delta(\hat{\sigma}^2, \sigma^2)^2 \right] \right\} \leq C,$$

where $C$ is a constant depending on $d$ only, and

$$\phi_T := \begin{cases} \ln^2 T/T, & d > 2, \\ \ln^3 T/T, & d = 2, \\ \ln^2 T/\sqrt{T}, & d = 1. \end{cases}$$

**Remark 5.2** Theorem 5.1 establishes an upper bound on the relative risk of $\hat{\sigma}_\alpha^2$. The main advantage in using the loss function $\Delta$ is that the derived upper bound holds for a broad range of possible values of $\sigma$, and the estimator does not require any prior information on $\sigma$.

Theorem 5.1 demonstrates that the rate of convergence of the squared relative risk coincides with the best achievable rate in estimation of the correlation function $H$ up to a logarithmic factor. In the case $d \geq 2$ the risk converges at the rate that is within logarithmic factors of the parametric rate; thus the estimator is nearly rate-optimal. The slower rate of convergence for $d = 1$ is a consequence of the long range dependence. In fact, the correlation function is non-integrable in the case $d = 2$ too; here, however, this leads to an extra logarithmic factor in the upper bound. We conjecture that logarithmic factors in $T$ appearing in the upper bounds in the cases $d \neq 2$ can be eliminated, and in the case $d = 2$ the degree of the logarithmic factor can be improved.

In the specific case of $d = 1$ we are able to develop another estimator whose squared risk converges to zero at the rate $\ln T/\sqrt{T}$; however, this estimator requires prior information on $\sigma^2$. The estimator construction is based on the first order approximation of the correlation function $H(t)$ near zero that is established in the following lemma.

**Lemma 5.2** Let $d = 1$ then for any $t \leq \sigma^2/(2r^2)$ one has

$$1 - H(t) = \frac{\sigma \sqrt{t}}{\sqrt{2\pi t}} \left[ 1 - \exp \left\{ -\frac{2r^2}{\sigma^2 t} \right\} \right] + \delta(t), \quad |\delta(t)| \leq \exp \left\{ -\frac{2r^2}{\sigma^2 t} \right\}.$$  

Lemma 5.2 suggests the following construction of an estimator of $\sigma^2$. Let $\tau > 0$ be a parameter to be specified and define

$$\hat{\sigma}_\tau := \frac{\sqrt{2\pi \tau}}{\sqrt{T}} [1 - \hat{H}(\tau)]. \quad (5.5)$$

where $\hat{H}$ is the standard estimator of the correlations function.

**Theorem 5.2** Let $\hat{\sigma}_\tau$, be the estimator (5.5) associated with $\tau = \tau_* = 4r^2/(\sigma^2 \ln T)$, where $\bar{\alpha}$ is a constant. Then for every $\sigma \leq \bar{\sigma}$ and sufficiently large $T$ one has

$$\mathbb{E} \left[ |\hat{\sigma}_\tau - \sigma|^2 \right] \leq c \left( 1 + \frac{1}{\rho} \right) \left( \frac{r}{\sigma} \right)^{2\sigma^2 \ln T}/\sqrt{T}. \quad (5.6)$$

where $c$ is an absolute constant.
Remark 5.3

(i) If the upper bound \( \tilde{\sigma} \) on parameter \( \sigma \) is known then the squared risk of estimator \( \tilde{\sigma}_r \) converges to zero at the rate \( \ln T/\sqrt{T} \) as \( T \to \infty \). This can be compared with the result of Theorem 5.1 which establishes the rate \( (\ln T)^2/\sqrt{T} \). The derived upper bound \( \tilde{\sigma}_r \) depends on \( \sigma \), and the accuracy may be poor under conservative choice of \( \tilde{\sigma} \). In contrast, the estimator \( \tilde{\sigma}_r^2 \) does not require any prior information on \( \sigma \).

(ii) In the case \( d \geq 2 \) the first order approximation of \( 1 - H(t) \) near zero is much less accurate, and the risk of the corresponding estimator is much worse than the one established in Theorem 5.1.

6 Proofs for Section 3

6.1 Proof of Theorem 3.1

Recall that

\[
\bar{R}(t) = \tilde{\nu}(t) - \frac{\rho}{T-t} \int_0^{T-t} [N(\tau) + N(\tau + t)] d\tau + \rho^2, \quad R(t) = r(t) - \rho^2 = \rho H(t),
\]

where we denoted \( r(t) := \mathbb{E}[N(s)N(s + t)] \), and \( \tilde{\nu}(t) := (1/(T-t)) \int_0^{T-t} N(s)N(t+s) ds \). We have

\[
\mathbb{E}\bar{R}(t)\bar{R}(t) = \mathbb{E}[\tilde{\nu}(t)\tilde{\nu}(s)]
\]

\[
= \mathbb{E}\tilde{\nu}(t) \left[ \frac{\rho}{T-t} \int_0^{T-t} [N(\tau_2) + N(\tau_2 + s)] d\tau_2 - \rho^2 \right]
\]

\[
- \mathbb{E}\tilde{\nu}(s) \left[ \frac{\rho}{T-t} \int_0^{T-t} [N(\tau_1) + N(\tau_1 + t)] d\tau_1 - \rho^2 \right]
\]

\[
+ \mathbb{E}\left[ \frac{\rho}{T-s} \int_0^{T-s} [N(\tau_2) + N(\tau_2 + s)] d\tau_2 - \rho^2 \right] \int_0^{T-t} [N(\tau_1) + N(\tau_1 + t)] d\tau_1 - \rho^2
\]

\[
=: J_1 - J_2 + J_3 + J_4.
\]

Our current goal is to calculate the terms on the right hand side of the previous formula.

First we compute \( J_1 \). Using (2.3) with \( n = 4 \) we have

\[
J_1 := \mathbb{E}\tilde{\nu}(t)\tilde{\nu}(s) = \frac{1}{(T-t)(T-s)} \int \int \mathbb{E}[N(\tau_1)N(\tau_1 + t)N(\tau_2)N(\tau_2 + s)] d\tau_1 d\tau_2
\]

\[
= \rho^4 + \rho^3[H(t) + H(s)] + \rho^2 H(t)H(s)
\]

\[
+ \frac{\rho^3}{(T-t)(T-s)} \int \int [H(\tau_1 - \tau_2) + H(\tau_1 - \tau_2 - s) + H(\tau_1 - \tau_2 + t) + H(\tau_1 - \tau_2 + t - s)] d\tau_1 d\tau_2
\]

\[
+ \frac{\rho^2}{(T-s)(T-t)} \int \int [U(\tau_1, \tau_1 + t, \tau_2) + U(\tau_1, \tau_1 + t, \tau_2 + s) + U(\tau_1, \tau_2, \tau_2 + s)
\]

\[
+ U(\tau_1 + t, \tau_2, \tau_2 + s) + H(\tau_1 - \tau_2)H(\tau_1 - \tau_2 + t - s) + H(\tau_1 - \tau_2 - s)H(\tau_1 - \tau_2 + t)] d\tau_1 d\tau_2
\]

\[
+ \frac{\rho}{(T-t)(T-s)} \int \int U(\tau_1, \tau_1 + t, \tau_2, \tau_2 + s)d\tau_1 d\tau_2,
\]

where in the above expression the integrals over \( \tau_1 \) are taken from 0 to \( T - t \) and integrals over \( \tau_2 \) are taken from 0 to \( T - s \).
Now we compute $J_2$ and $J_3$. Using $\tau_n$ with $n = 3$ we obtain

$$
\mathbb{E}[\hat{r}(t)N(\tau_2)] = \frac{1}{T-t} \int_{0}^{T-t} \mathbb{E}[N(\tau_1)N(\tau_1 + t)N(\tau_2)]d\tau_1 \\
= \rho^3 + \rho^2 H(t) + \frac{1}{T-t} \int_{0}^{T-t} \left\{ \rho^2 H(\tau_1 - \tau_2) + \rho^2 H(\tau_1 + t - \tau_2) + \rho U(\tau_1, \tau_1 + t, \tau_2) \right\}d\tau_1,
$$

$$
\mathbb{E}[\hat{r}(t)N(\tau_2 + s)] = \frac{1}{T-t} \int_{0}^{T-t} \mathbb{E}[N(\tau_1)N(\tau_1 + t)N(\tau_2 + s)]d\tau_1 \\
= \rho^3 + \rho^2 H(t) + \frac{1}{T-t} \int_{0}^{T-t} \left\{ \rho^2 H(\tau_1 - \tau_2 - s) + \rho^2 H(\tau_1 + t - \tau_2 - s) + \rho U(\tau_1, \tau_1 + t, \tau_2 + s) \right\}d\tau_1,
$$

and similarly

$$
\mathbb{E}[\hat{r}(s)N(\tau_1)] = \frac{1}{T-s} \int_{0}^{T-s} \mathbb{E}[N(\tau_2)N(\tau_2 + s)N(\tau_1)]d\tau_2 \\
= \rho^3 + \rho^2 H(s) + \frac{1}{T-s} \int_{0}^{T-s} \left\{ \rho^2 H(\tau_2 - \tau_1) + \rho^2 H(\tau_2 + s - \tau_1) + \rho U(\tau_2, \tau_2 + s, \tau_1) \right\}d\tau_2,
$$

$$
\mathbb{E}[\hat{r}(s)N(\tau_1 + t)] = \frac{1}{T-s} \int_{0}^{T-s} \mathbb{E}[N(\tau_2)N(\tau_2 + s)N(\tau_1 + t)]d\tau_2 \\
= \rho^3 + \rho^2 H(s) + \frac{1}{T-s} \int_{0}^{T-s} \left\{ \rho^2 H(\tau_2 - \tau_1 - t) + \rho^2 H(\tau_2 + s - \tau_1 - t) + \rho U(\tau_2, \tau_2 + s, \tau_1 + t) \right\}d\tau_2.
$$

Therefore

$$
J_2 = \rho^4 + \rho^3 H(t) \\
+ \frac{\rho^3}{(T-t)(T-s)} \int \left[ H(\tau_1 - \tau_2) + H(\tau_1 - \tau_2 - s) + H(\tau_1 - \tau_2 + t) + H(\tau_1 - \tau_2 + t - s) \right]d\tau_1 d\tau_2,
$$

and

$$
J_3 = \rho^4 + \rho^3 H(s) \\
+ \frac{\rho^3}{(T-t)(T-s)} \int \left[ H(\tau_2 - \tau_1) + H(\tau_2 - \tau_1 + s) + H(\tau_2 - \tau_1 - t) + H(\tau_2 - \tau_1 + s - t) \right]d\tau_1 d\tau_2.
$$

Now we compute $J_4$:

$$
J_4 = \frac{\rho^2}{(T-s)(T-t)} \int \mathbb{E}[ (N(\tau_2) + N(\tau_2 + s))(N(\tau_1) + N(\tau_1 + t)) ]d\tau_1 d\tau_2 - 3\rho^4 \\
= \rho^4 + \frac{\rho^3}{(T-t)(T-s)} \int \left[ H(\tau_1 - \tau_2) + H(\tau_1 - \tau_2 + t) + H(\tau_1 - \tau_2 - t) + H(\tau_1 - \tau_2 + t - s) \right]d\tau_1 d\tau_2.
$$

Now we combine expressions for $J_1, J_2, J_3$ and $J_4$ to get

$$
\mathbb{E}[\hat{R}(t)\hat{R}(s)] = J_1 - J_2 - J_3 + J_4 \\
= \rho^2 H(t)H(s) + \frac{\rho}{(T-t)(T-s)} \int U(\tau_1, \tau_1 + t, \tau_2, \tau_2 + s)d\tau_1 d\tau_2 \\
+ \frac{\rho^2}{(T-s)(T-t)} \int \left[ H(\tau_1 - \tau_2)H(\tau_1 - \tau_2 + t - s) + H(\tau_1 - \tau_2 - s)H(\tau_1 - \tau_2 + t) \right]d\tau_1 d\tau_2.
$$
and finally taking into account that \( R(t)R(s) = \rho^2 H(t)H(s) \) we obtain
\[
\mathbb{E}[\hat{R}(t) - R(t)]|\hat{R}(s) - R(s)|
\]
\[
= \frac{\rho}{(T-t)(T-s)} \int_0^{T-t} \int_0^{T-s} U(\tau_1, \tau_1 + t, \tau_2, \tau_2 + s) \, d\tau_1 d\tau_2
\]
\[
+ \frac{\rho^2}{(T-s)(T-t)} \int_0^{T-t} \int_0^{T-s} [H(\tau_1 - \tau_2)H(\tau_1 - \tau_2 + t - s) + H(\tau_1 - \tau_2 - s)H(\tau_1 - \tau_2 + t)] \, d\tau_1 d\tau_2.
\]
This completes the proof.

7 Proofs for Section 4

7.1 Proof of Lemma 4.1

We have
\[
H(t) = \frac{1}{B(\frac{d+1}{2}, \frac{1}{2})} \int_0^\infty \int_0^1 \left( t \leq \frac{2r\sqrt{1-y}}{x} \right) y^{\frac{d}{2}(d-1)}(1-y)^{-1/2} \, dy \, df(x), \quad (7.1)
\]
\[
= \frac{1}{B(\frac{d+1}{2}, \frac{1}{2})} \int_0^1 F\left( \frac{2r\sqrt{1-y}}{t} \right) y^{\frac{d}{2}(d-1)}(1-y)^{-1/2} \, dy. \quad (7.2)
\]
It follows from (7.1) that
\[
\int_0^\infty H(t) \, dt = \frac{2r B\left(\frac{d+1}{2}, 1\right)}{B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \int_0^\infty \int_0^1 2r^2 y^{\frac{d}{2}(d-1)} \, dy \, df(x) = \frac{2r B\left(\frac{d+1}{2}, 1\right)}{B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \int_0^\infty \frac{dF(x)}{x}
\]
If \( f(x) \leq Mx^\alpha \) for \( 0 < x \leq \delta \) then for \( t \geq 2r/\delta \)
\[
H(t) \leq \frac{1}{B(\frac{d+1}{2}, \frac{1}{2})} \int_0^{2r/t} f(x) \int_0^1 y^{(d-1)/2}(1-y)^{-1/2} \, dy \, dx \leq \frac{M}{1+\alpha} \left( \frac{2r}{t} \right)^{1+\alpha}
\]
Inequality (4.6) is obtained by integration of the above upper bound on \( H(t) \).

7.2 Proof of Theorem 4.1

We begin with a lemma that establishes bounds on the derivatives of the correlation function.

Lemma 7.1 Let \( L > 0 \) be a real number, and assume that speed distribution \( F \) is absolutely continuous with density \( f \).

(i) If \( \sup_x f(x) \leq L \) and \( \int_0^\infty xf(x) \, dx \leq L \) then for any \( t > 0 \)
\[
|H'(t)| \leq \frac{L}{rB\left(\frac{d+1}{2}, \frac{1}{2}\right)} \left\{ 1 \wedge \frac{4r^2}{(d+1)t^2} \right\}, \quad (7.3)
\]
and
\[
\int_0^\infty |H'(t)| \, dt \leq \frac{L}{B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \left[ 1 + 2B\left(\frac{d+1}{2}, 1\right) \right]. \quad (7.4)
\]
(ii) If \( f \) is differentiable and \( \sup_x |f'(x)| \vee \sup_x f(x) \leq L \) then
\[
|H''(t)| \leq 4Lt \left( \frac{(d+2)r}{t^4} + \frac{1}{t^3} \right), \quad \forall t > 0.
\]
(iii) If \( f \) is differentiable and
\[
\sup_x x^3 |f'(x)| \leq L, \quad \int_0^\infty x^3 |f'(x)| \, dx \leq L, \quad \int_0^\infty x^2 f(x) \, dx \leq L
\]
then
\[
|H''(t)| \leq \frac{3dL}{r^2 B(d+1, \frac{1}{2})}, \quad \forall t > 0,
\]
and
\[
\int_0^\infty |H''(t)| \, dt \leq \frac{L}{r} \left( \frac{3d}{B(d+1, \frac{1}{2})} + 2(2d + 3) \right).
\]

**Proof:** (i). Differentiating (4.3) we obtain
\[
H'(t) = -\frac{2r}{B(d+1, \frac{1}{2})} \int_0^1 \left[ f\left( \frac{2r \sqrt{t}}{t} \right) \frac{1}{t^2} \right] \, dy
\]
\[
= -\frac{1}{r B(d+1, \frac{1}{2})} \int_0^{2r/t} x f(x) \left( 1 - \frac{t^2 x^2}{4r^2} \right)^{(d-1)/2} \, dx.
\]
If \( \sup_x f(x) \leq L \) then the first equality yields for every \( d = 1, 2, \ldots \)
\[
|H'(t)| \leq \frac{2rL}{B(d+1, \frac{1}{2})} \int_0^1 \left( 1 - y \right)^{(d-1)/2} \, dy = \frac{2rLB(d+1,1)}{B(d+1, \frac{1}{2})t^2}, \quad \forall t > 0.
\]
On the other hand, it follows from the second equality that
\[
|H'(t)| \leq \frac{1}{r B(d+1, \frac{1}{2})} \int_0^\infty x f(x) \, dx \leq \frac{L}{r B(d+1, \frac{1}{2})}, \quad \forall t > 0.
\]
Combining these two inequalities we come to (7.3). Moreover,
\[
\int_0^\infty |H'(t)| \, dt \leq \frac{L}{r B(d+1, \frac{1}{2})} + \frac{2rLB(d+1,1)}{B(d+1, \frac{1}{2})t^2} \int_0^\infty \frac{dt}{t^2} \leq \frac{L}{B(d+1, \frac{1}{2})} \left[ 1 + \frac{2B(d+1,1)}{B(d+1, \frac{1}{2})} \right].
\]

(ii). The second derivative of \( H \) is
\[
H''(t) = \frac{2r}{B(d+1, \frac{1}{2})} \int_0^1 \left[ f\left( \frac{2r \sqrt{t}}{t} \right) \frac{2r \sqrt{t}}{t^2} + f\left( \frac{2r \sqrt{t}}{t} \right) \frac{2}{t^3} \right] \left( 1 - y \right)^{(d-1)/2} \, dy
\]
\[
= \frac{1}{B(d+1, \frac{1}{2})r t} \int_0^{2r/t} \left[ x^2 f'(x) + 2xf(x) \right] \left( 1 - \frac{t^2 x^2}{4r^2} \right)^{(d-1)/2} \, dx.
\]
From the first equality we have
\[
|H''(t)| \leq \frac{4r^2L B(d+1, \frac{1}{2})}{t^2} + \frac{4rL B(d+1,1)}{t^3 B(d+1, \frac{1}{2})} \leq 4L \left( \frac{(d+2)r}{t^2} + \frac{1}{t^3} \right),
\]
provided that \( \sup_x f(x) \vee \sup_x |f'(x)| \leq L \).

(iii). By the third inequality in (7.5), \( \lim_{x \to \infty} x^2 f(x) = 0 \) and \( \int_0^\infty [2xf(x) + x^2 f'(x)] \, dx = 0 \); therefore
\[
|H''(t)| \leq \frac{1}{B(d+1, \frac{1}{2})r t} \left\{ \int_0^{2r/t} \left[ x^2 f'(x) + 2xf(x) \right] \left[ \left( 1 - \frac{t^2 x^2}{4r^2} \right)^{(d-1)/2} - 1 \right] \, dx \right\}
\]
\[
+ \left\{ \int_0^{2r/t} \left[ 2xf(x) + x^2 f'(x) \right] \, dx \right\}
\]
\[
= \frac{1}{B(d+1, \frac{1}{2})r t} \left\{ \int_0^{2r/t} \left[ x^2 f'(x) + 2xf(x) \right] \left[ \left( 1 - \frac{t^2 x^2}{4r^2} \right)^{(d-1)/2} - 1 \right] \, dx \right\} + \frac{4r^2}{t^2} f\left( \frac{2r}{t^2} \right). \quad (7.7)
\]
If \( d = 1 \) then
\[
|H''(t)| \leq \frac{1}{B(\frac{d+1}{2}, \frac{1}{2})} t^2 f\left(\frac{2r}{t}\right) \leq \frac{L}{2r^2 B(\frac{d+1}{2}, \frac{1}{2})},
\]
where we have used the first inequality in (7.5). If \( d = 2 \) then by the elementary inequality \( 1 - (1-a)^{1/2} \leq a \) for \( a \in [0,1] \) and by (7.5)
\[
|H''(t)| \leq \frac{1}{B(\frac{d+1}{2}, \frac{1}{2})} \left\{ \int_0^{2r/t} \left[ x^2 |f'(x)| + 2 xf(x) \right] \frac{t^2 x^2}{4r^2} dx + \frac{4r^2}{t^2} f\left(\frac{2r}{t}\right) \right\}
\]
\[
\leq \frac{1}{2B(\frac{d+1}{2}, \frac{1}{2})} \int_0^{2r/t} \left[ x^3 |f'(x)| + 2 x^2 f(x) \right] dx + \frac{L}{2r^2 B(\frac{d+1}{2}, \frac{1}{2})} \leq \frac{2L}{r^2 B(\frac{d+1}{2}, \frac{1}{2})}.
\]
Finally, if \( d \geq 3 \) then expanding in Taylor’s series in (7.5) we obtain
\[
|H''(t)| \leq \frac{1}{B(\frac{d+1}{2}, \frac{1}{2})} \left\{ \int_0^{2r/t} \left[ x^2 |f'(x)| + 2 xf(x) \right] \frac{(d-1)x^2 t^2}{2r^2} dx + \frac{4r^2}{t^2} f\left(\frac{2r}{t}\right) \right\}
\]
\[
\leq \frac{d-1}{B(\frac{d+1}{2}, \frac{1}{2})} \int_0^{2r/t} \left[ x^3 |f'(x)| + 2 x^2 f(x) \right] dx + \frac{L}{2r^2 B(\frac{d+1}{2}, \frac{1}{2})} \leq \frac{3dL}{r^2 B(\frac{d+1}{2}, \frac{1}{2})},
\]
where we took into account (7.5). This completes the proof.

Now we proceed to the proof of the theorem.

**Proof of Theorem 4.1:** In the subsequent proof \( c_1, c_2, \ldots \) stand for positive constants that may depend on \( d \) and characteristics of kernel \( K \) only. These constants may be different on different occasions.

We have the following bias–variance decomposition of the mean squared error of \( \hat{\psi}_h \):
\[
\frac{\rho}{r B(\frac{d+1}{2}, \frac{1}{2})} \left[ \mathbb{E} |\hat{\mu}_h - \mu|^2 \right]^{1/2} \leq \left\{ \mathbb{E} \left[ \frac{1}{h^2} \int_0^h K\left(\frac{t}{h}\right) \left[ \hat{R}(t) - R(t) \right] dt \right]^2 \right\}^{1/2}
\]
\[
+ \frac{1}{h^2} \int_0^h K\left(\frac{t}{h}\right) R(t) dt - R'(0+) \right\}. \tag{7.8}
\]
Our current goal is to bound from above the two terms on the right hand side of the above formula.

The bound on the bias is immediate. Expanding \( R(t) \) is Taylor’s series and using Lemma 7.1 we obtain
\[
\left| \frac{1}{h^2} \int_0^h K\left(\frac{t}{h}\right) R(t) dt - R'(0+) \right| \leq \frac{1}{4} h C_K \sup_{0 < y \leq h} |R''(y)| \leq \frac{3dC_K \rho L}{2r^2 B(\frac{d+1}{2}, \frac{1}{2})} = c_1 \rho L h r^{-2}.
\]
We continue with bounding the variance.

**Bound on the variance.** We have
\[
V := \mathbb{E} \left\{ \frac{1}{r^2} \int_0^h K\left(\frac{t}{h}\right) \left[ \hat{R}(t) - R(t) \right] dt \right\}^2
\]
\[
= \frac{1}{r^4} \int_0^h \int_0^h K\left(\frac{t}{h}\right) K\left(\frac{s}{h}\right) \mathbb{E} \left[ \hat{R}(t) - R(t) \right] \left[ \hat{R}(s) - R(s) \right] dt ds. \tag{7.9}
\]
In our derivation of the upper bound on the variance we substitute formula (3.4) given in Theorem 3.1 in (7.9) and bound the resulting terms. The proof proceeds in the following steps.

| Step | Equation |
|------|----------|
| 1. Follow for brevity \( A_1(t_1 - t_2, s, t) := H(t_1 - t_2)H(t_1 - t_2 + t - s) + H(t_1 - t_2 - s)H(t_1 - t_2 + t) \) | |
| 2. Follow for brevity \( A_2(t_1 - t_2, s, t) := U(t_1, t_1 + t, t_2, t_2 + s) = U(t_1 - t_2, t_1 - t_2 + t, 0, s) \), | |

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and remind that $A_1$ and $A_2$ are non-negative functions. Then using Theorem 3.1 we can write

$$(T - t)(T - s)\mathbb{E}[\tilde{R}(t) - R(t)][\tilde{R}(s) - R(s)]$$

$$= \int_0^{T-t} \int_0^{T-s} \left\{ \rho^2 A_1(\tau_1 - \tau_2, t) + \rho A_2(\tau_1 - \tau_2, s) \right\} d\tau_1 d\tau_2$$

$$= \int_0^{T} \int_0^{T} \left\{ \rho^2 A_1(\tau_1 - \tau_2, t) + \rho A_2(\tau_1 - \tau_2, s) \right\} d\tau_1 d\tau_2 + E(t, s),$$

where

$$E(t, s) := \int_0^{T-t} \int_0^{T-s} \left\{ \rho^2 A_1(\tau_1 - \tau_2, t, s) + \rho A_2(\tau_1 - \tau_2, t, s) \right\} d\tau_1 d\tau_2$$

$$+ \int_0^{T-t} \int_0^{T-s} \left\{ \rho^2 A_1(\tau_1 - \tau_2, t, s)d\tau_1 + \rho A_2(\tau_1 - \tau_2, t, s) \right\} d\tau_1 d\tau_2.$$

Note that for $i = 1, 2$ and for $t, s \in [0, h]$ one has

$$\int_0^{T-t} \int_0^{T-s} A_i(\tau_1 - \tau_2, t, s)d\tau_1 d\tau_2 \leq 2\int_0^{T-t} \int_0^{T-s} H(\tau_1 - \tau_2)d\tau_1 d\tau_2 \leq 4h \int_0^{T} H(y)dy,$$

and similarly,

$$\int_0^{T-t} \int_0^{T-s} A_i(\tau_1 - \tau_2, t, s)d\tau_1 d\tau_2 \leq 2\int_0^{T-t} \int_0^{T-s} H(\tau_1 - \tau_2)d\tau_1 d\tau_2 \leq 4h \int_0^{T} H(y)dy.$$

Therefore $|E(t, s)| \leq 8h(\rho^2 + \rho) \int_0^{T} H(y)dy$, and

$$\left| \frac{1}{h^4} \int_0^h \int_0^h K(\frac{t}{h})K(\frac{s}{h})(\frac{E(t, s)}{(T - s)(T - t)}) dt ds \right|$$

$$\leq \frac{c_3(\rho^2 + \rho)}{2h^3T^2} \int_0^h \int_0^h K(\frac{t}{h})K(\frac{s}{h}) dt ds \int_0^{T} H(y)dy \leq \frac{c_4(\rho^2 + \rho)}{hT^2} \int_0^{T} H(y)dy.$$

Thus,

$$V \leq \tilde{V} + \frac{c_4(\rho^2 + \rho)}{hT^2} \int_0^{T} H(y)dy, \quad (7.10)$$

where we denoted

$$\tilde{V} := \frac{1}{h^4} \int_0^h \int_0^h \left[ K(\frac{s}{h})K(\frac{t}{h}) \right] \frac{1}{(T - s)(T - t)}$$

$$\times \int_0^{T} \int_0^{T} \left\{ \rho^2 A_1(\tau_1 - \tau_2, t, s) + \rho A_2(\tau_1 - \tau_2, t, s) \right\} d\tau_1 d\tau_2 dt ds.$$

Denote

$$\tilde{V} := \frac{1}{h^4} \int_0^h \int_0^h K(\frac{s}{h})K(\frac{t}{h}) \frac{1}{T^2} \int_0^{T} \int_0^{T} \left\{ \rho^2 A_1(\tau_1 - \tau_2, t, s) + \rho A_2(\tau_1 - \tau_2, t, s) \right\} d\tau_1 d\tau_2 dt ds. \quad (7.11)$$

Because for $t, s \in [0, h]$

$$\left| \frac{1}{(T - s)(T - t)} - \frac{1}{T^2} \right| \int_0^{T} \int_0^{T} \left\{ \rho^2 A_1(\tau_1 - \tau_2, t, s) + \rho A_2(\tau_1 - \tau_2, t, s) \right\} d\tau_1 d\tau_2 dt ds$$

$$\leq \frac{c_4 h(\rho^2 + \rho)}{(T - h)^2} \int_0^{T} H(y)dy.$$
we have
\[ |\tilde{V} - \tilde{V}| \leq \frac{c_{9}(p^2 + \rho)}{(T-h)^2} \int_0^T H(y) dy. \]
Combining this inequality with (7.10) we obtain
\[ V \leq \tilde{V} + c_6(p^2 + \rho) \left( \frac{1}{h^2T^2} + \frac{h}{(T-h)^2} \right) \int_0^T H(t) dt. \] (7.12)

Our current goal is to bound \( \tilde{V} \) from above [see (7.11)]. Write
\[ \tilde{V} := \rho^2 V_1 + \rho V_2 \] (7.13)

\[ V_i := \frac{1}{h^2} \int_0^h \int_0^h \left( \frac{1}{T} \right) K \left( \frac{h}{T} \right) \frac{1}{T} \int_0^T \int_0^T A_1(\tau_1 - \tau_2, t) dt_1 d\tau_2 d\tau_1 d\tau_2 ds \\
V_2 := \frac{1}{h^2} \int_0^h \int_0^h K \left( \frac{h}{T} \right) \frac{1}{T} \int_0^T \int_0^T A_2(\tau_1 - \tau_2, t) dt_1 d\tau_2 d\tau_1 d\tau_2 ds. \]

2º. Consider first \( V_1 \). Because
\[ A_1(\tau_1 - \tau_2, t) = H^2(\tau_1 - \tau_2) + H(\tau_1 - \tau_2 - s)H(\tau_1 - \tau_2 + t) \]
we have \( V_1 = V_{1,1} + V_{1,2} \) where
\[ V_{1,1} := \frac{1}{h^2} \int_0^h \int_0^h K \left( \frac{h}{T} \right) \frac{1}{T} \int_0^T \int_0^T H(\tau_1 - \tau_2 - s)H(\tau_1 - \tau_2 + t) dt_1 d\tau_2 d\tau_1 d\tau_2 ds, \]
and
\[ V_{1,2} := \frac{1}{h^2} \int_0^h \int_0^h K \left( \frac{h}{T} \right) \frac{1}{T} \int_0^T \int_0^T H^2(\tau_1 - \tau_2) dt_1 d\tau_2 d\tau_1 d\tau_2 ds. \]

By (4.8), \( V_{1,2} = 0 \); therefore we need to bound \( V_{1,1} \) only. To this end, let us introduce the following subsets of \( \mathbb{R}^2 \)
\[ S_1 := \{(\tau_1, \tau_2) : 0 \leq \tau_1 - \tau_2 \leq h\} \cap [0, T]^2, \]
\[ S_2 := \{(\tau_1, \tau_2) : -h \leq \tau_1 - \tau_2 \leq 0\} \cap [0, T]^2, \]
\[ S := [0, T]^2 \setminus (S_1 \cup S_2), \]
and divide the double integral over \( \tau_1 \) and \( \tau_2 \) in three integrals corresponding to these subsets. For the set \( S_1 \) we obtain
\[ V_{1,1}(S_1) := \frac{1}{h^2} \int_0^h \int_0^h K \left( \frac{h}{T} \right) \frac{1}{T} \int_0^T \int_0^T H(\tau_1 - \tau_2 - s)H(\tau_1 - \tau_2 + t) dt_1 d\tau_2 d\tau_1 d\tau_2 ds \]
\[ = \frac{1}{h^2} \int_0^h \int_0^h K \left( \frac{h}{T} \right) \frac{1}{T} \int_0^T \int_0^T H(\tau_1 - \tau_2 - s) \left\{ \frac{1}{T^2} \int_0^h K \left( \frac{h}{T} \right) H(\tau_1 - \tau_2 + t) dt \right\} d\tau_1 d\tau_2 ds. \]

For \( (\tau_1, \tau_2) \in S_1 \) we have \( \tau_1 - \tau_2 + t > 0 \) for all \( t \in [0, h] \) so that \( H(\tau_1 - \tau_2 + t) \) is smooth and can be expanded in Taylor’s series around \( \tau_1 - \tau_2 \):
\[ \frac{1}{h^2} \int_0^h K \left( \frac{h}{T} \right) H(\tau_1 - \tau_2 + t) dt = H'(\tau_1 - \tau_2) + \frac{h}{2!} \int_0^1 y^2 K(y) H''(\tau_1 - \tau_2 + \vartheta y h) dy, \quad \vartheta \in [0, 1]. \]

We have
\[ \left| \frac{1}{h^2} \int_0^h K \left( \frac{h}{T} \right) H(\tau_1 - \tau_2 + t) dt - H'(\tau_1 - \tau_2) \right| \leq \frac{h}{2!} \int_0^1 y^2 |K(y)||H''(\tau_1 - \tau_2 + \vartheta y h)| dy \leq c_9 L h r^{-2}, \]
where in the last inequality we have used (7.6) of Lemma (7.1) Therefore
\[ |V_{1,1}(S_1)| \leq \frac{1}{h^2} \int_0^h \left| K \left( \frac{h}{T} \right) \right| \frac{1}{T} \int_0^T \int_0^T H(\tau_1 - \tau_2 - s)\left\{ |H'(\tau_1 - \tau_2)| + c_9 L h r^{-2} \right\} d\tau_1 d\tau_2 ds \]
\[ \leq \frac{c_7}{h T^2} \int_0^T \int_0^T |H'(\tau_1 - \tau_2)| d\tau_1 d\tau_2 + \frac{c_8 L h}{T^2} \leq \frac{c_9 L}{T} (1 + h r^{-1}), \]
where we again use Lemma 7.1. Similarly,

\[
V_{1,1}(S_2) := \frac{1}{h^3} \int_0^h \int_0^h K \left( \frac{r}{h} \right) \frac{1}{T^2} \int_S H(\tau_1 - \tau_2 - s)H(\tau_1 - \tau_2 + t) d\tau_1 d\tau_2 dt ds
\]

\[
= \frac{1}{h^2} \int_0^h K \left( \frac{r}{h} \right) \frac{1}{T^2} \left( \int_S \left\{ \frac{1}{h^2} \int_0^h K \left( \frac{s}{h} \right) H(\tau_1 - \tau_2 - s) ds \right\} d\tau_1 d\tau_2 dt. \right.
\]

and by the same reasoning as above we obtain the same bound

\[
|V_{1,1}(S_2)| \leq \frac{c_{10} L}{rT} (1 + hr^{-1}).
\]

Now we consider

\[
V_{1,1}(S) := \frac{1}{h^3} \int_0^h \int_0^h K \left( \frac{r}{h} \right) \frac{1}{T^2} \int_s H(\tau_1 - \tau_2 - s)H(\tau_1 - \tau_2 + t) d\tau_1 d\tau_2 dt ds
\]

\[
= \frac{1}{T^2} \int_s \left\{ \frac{1}{h^2} \int_0^h K \left( \frac{s}{h} \right) H(\tau_1 - \tau_2 - s) ds \right\} \left\{ \frac{1}{h^2} \int_0^h K \left( \frac{r}{h} \right) H(\tau_1 - \tau_2 + t) dt \right\} d\tau_1 d\tau_2 ds.
\]

On the set \( S \) functions \( H'(\tau_1 - \tau_2 - \cdot) \) and \( H'(\tau_1 - \tau_2 + \cdot) \) can be expanded to Taylor’s series up to the second order; therefore for \( (\tau_1, \tau_2) \in S \) one has

\[
\left| \frac{1}{h^2} \int_0^h K \left( \frac{s}{h} \right) H(\tau_1 - \tau_2 - s) ds - H'(\tau_1 - \tau_2) \right| = \frac{1}{h} \int_0^1 y^2 K(y) H''(\tau_1 - \tau_2 - \vartheta_1 yh) dy \leq \frac{L}{2} C_K h \max_{y \in [0,1]} |H''(\tau_1 - \tau_2 - y)| \leq \frac{3dC_K L h}{2r^2 B(\frac{dx^2}{2}, \frac{1}{2})} = c_{11} L h r^{-2}
\]

\[
\left| \frac{1}{h^2} \int_0^h K \left( \frac{s}{h} \right) H(\tau_1 - \tau_2 + t) dt - H'(\tau_1 - \tau_2) \right| = \frac{1}{h} \int_0^1 y^2 K(y) H''(\tau_1 - \tau_2 + \vartheta_2 yh) dy \leq \frac{L}{2} C_K h \max_{y \in [0,1]} |H''(\tau_1 - \tau_2 + y)| \leq \frac{3dC_K L h}{2r^2 B(\frac{dx^2}{2}, \frac{1}{2})} = c_{11} L h r^{-2},
\]

where \( \vartheta_1, \vartheta_2 \in [0,1] \) and we have used Lemma 7.1. This yields

\[
|V_{1,1}(S)| \leq \frac{1}{T^2} \int_S \left| H'(\tau_1 - \tau_2) \right|^2 d\tau_1 d\tau_2 \leq \frac{2L}{T^2} \int_S \left| H'(\tau_1 - \tau_2) \right|^2 d\tau_1 d\tau_2 + c_{12} h^2 L^2 h^{-4} \leq c_{13} \left\{ \frac{L^2}{rT} + L^2 h^2 r^{-4} \right\},
\]

where in the last line we have used (7.13) and (7.14). Combining the obtained inequalities for \( |V_{1,1}(S_1)|, |V_{1,1}(S_2)| \) and \( |V_{1,1}(S)| \) we obtain

\[
|V_1| \leq c_{13} \left\{ \frac{L}{rT} \left[ 1 + L + hr^{-1} \right] + L^2 h^2 r^{-4} \right\}. \tag{7.16}
\]

40. Now we consider

\[
V_2 := \frac{1}{h^3} \int_0^h \int_0^h K \left( \frac{r}{h} \right) K \left( \frac{s}{h} \right) \frac{1}{T^2} \int_0^T \int_0^T A_2(\tau_1 - \tau_2, s, t) d\tau_1 d\tau_2 dt ds dt.
\]

Write for brevity \( \tau_{\text{max}} := \max\{0, \tau_1 - \tau_2, \tau_1 - \tau_2 + s, t\} \) and \( \tau_{\text{min}} := \min\{0, \tau_1 - \tau_2, \tau_1 - \tau_2 + s, t\} \), and recall that

\[
A_2(\tau_1 - \tau_2, s, t) = \frac{1}{\text{vol}(B)} \text{Vol}\{B \cap B(v(\tau_1 - \tau_2)) \cap B(v(\tau_1 - \tau_2 + t)) \cap B(vs)\}
\]

\[
= \frac{1}{\text{vol}(B)} \text{Vol}\{B(v(\tau_{\text{max}} - \tau_{\text{min}}))\}.
\]
Note that for \( t, s \in [0, h] \) we have \( \tau_{\text{max}} = \max\{\tau_1 - \tau_2 + s, t\} \) and \( \tau_{\text{min}} = \min\{0, \tau_1 - \tau_2\} \) so that
\[
A_2(\tau_1 - \tau_2, s, t) = H(\tau_{\text{max}} - \tau_{\text{min}}).
\]

Now we proceed by partitioning \([0, T]^2\) in subsets \( S_1, S_2 \) and \( S \) as defined in (7.11)–(7.15). On the set \( S_1 \) we have \( \tau_{\text{max}} = (\tau_1 - \tau_2 + s) \lor t \) and \( \tau_{\text{min}} = 0 \) so that
\[
V_2(S_1) := \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) \frac{1}{T^2} \int_{S_1} \left\{ \frac{1}{h^2} \int_0^h K\left(\frac{h}{T}\right) H((\tau_1 - \tau_2 + s) \lor t) dt \right\} dr_1 dr_2 ds,
\]
and
\[
\left| \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) H((\tau_1 - \tau_2 + s) \lor t) dt \right|
\]
\[
= \left| \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) H(t)(t - \tau_1 - \tau_2) dt + \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) H((\tau_1 - \tau_2 + s) \lor t) dt \right|
\]
\[
\leq \left| \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) H(t) dt \right| + \left| \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) \left| H((\tau_1 - \tau_2 + s) - H(t)) \right| dt \right|
\]
\[
\leq \left| H'(0^+) \right| + c_1 L h r^{-2} + c_2 L r^{-1} \leq c_3 L r^{-1} (1 + h r^{-1}),
\]
where in the last line we have used bounds on \( |H'(t)| \) established in Lemma (5.11) and the fact that \( |\tau_1 - \tau_2 - s - t| \leq 2h \) on the set \( S_1 \). Substituting this bound in (7.17) we obtain
\[
|V_2(S_1)| \leq \frac{1}{h^2} \int_0^h \left| K\left(\frac{s}{h}\right) ds \int_0^T \int_{S_1} c_{16} L r^{-1} (1 + h r^{-1}) dr_1 dr_2 \leq \frac{c_4 L}{r^T} (1 + h r^{-1}).
\]

On the set \( S_2, \tau_{\text{max}} = (\tau_1 - \tau_2 + s) \lor t \) and \( \tau_{\text{min}} = \tau_1 - \tau_2 \); therefore
\[
V_2(S_2) := \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) \frac{1}{T^2} \int_{S_2} \left\{ \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) H((\tau_1 - \tau_2 + s) \lor t - (\tau_1 - \tau_2)) ds \right\} dr_1 dr_2 dt,
\]
and similarly to bounding \( V_2(S_1) \) we have
\[
\left| \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) H((\tau_1 - \tau_2 + s) \lor t - (\tau_1 - \tau_2)) ds \right|
\]
\[
= \left| \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) H(t - (\tau_1 - \tau_2)) dt + \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) H(s) ds \right|
\]
\[
\leq \left| \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) H(s) ds \right| + \left| \frac{1}{h^2} \int_0^h K\left(\frac{s}{h}\right) \left| H(t - (\tau_1 - \tau_2)) - H(s) \right| ds \right|
\]
\[
\leq \left| H'(0^+) \right| + c_1 L h r^{-2} + c_2 L r^{-1} \leq c_3 L r^{-1} (1 + h r^{-1}),
\]
so that we also have
\[
|V_2(S_2)| \leq \frac{c_4 L}{r^T} (1 + h r^{-1}).
\]

On the set \( S \setminus S_1 \) we have \( \tau_{\text{max}} = \tau_1 - \tau_2 + s \) and \( \tau_{\text{min}} = 0 \) so that
\[
V_2(S \setminus S_1) = \frac{1}{h^2} \int_0^h \int_0^h K\left(\frac{s}{h}\right) K\left(\frac{h}{T}\right) \int_{S \setminus S_1} H(\tau_1 - \tau_2 + s) dr_1 dr_2 ds dt ds = 0.
\]

On the set \( S \setminus S_2 \) we have \( \tau_{\text{max}} = t, \tau_{\text{min}} = \tau_1 - \tau_2 \) so that
\[
V_2(S \setminus S_2) = \frac{1}{h^2} \int_0^h \int_0^h K\left(\frac{s}{h}\right) K\left(\frac{h}{T}\right) \int_{S \setminus S_2} H(t - (\tau_1 - \tau_2)) dr_1 dr_2 ds dt ds = 0.
\]

Finally, combining all these bounds we obtain
\[
|V_2| \leq \frac{c_4 L}{r^T} (1 + h r^{-1}).
\]
where in the last equality we took into account that $x/h$ we have for differentiable at zero and compactly supported. Now we compute the Mellin transform of $\tilde{\psi}$. We begin with the proof of (4.15). This condition ensures that the integral in the definition of function $\psi_{x_0,h}$ is absolutely convergent so that $\psi_{x_0,h}$ is well defined. The result of the theorem follows from the above inequality by selecting $\psi$ as stated in the premise of the theorem. Under this choice as $T \to \infty$ the first term on the right hand side dominates the other two terms and leads to the announced result. We also took into account that for bounded densities $\int_0^T H(t)dt \leq O(\ln T)$ as $T \to \infty$. \hfill \blacksquare

7.3 Proof of Lemma 4.2

By condition (K), $K(x/h)$ is supported on $[0,h]$ and $K(\ln(x/x_0)/h)$ is supported on $[x_0,x_0e^h]$. Therefore we have for $h \leq t \leq x_0$

$$\varphi_{x_0,h}(t) = \frac{1}{h} \int_0^t K\left(\frac{t}{h}\right)dx = \frac{1}{h} \int_0^1 K\left(\frac{t}{h}\right)dx = 1, \text{ for } h \leq t \leq x_0,$$

and for $t \geq x_0e^h$

$$\varphi_{x_0,h}(t) = 1 - \int_0^t \frac{1}{xh} K\left(\frac{\ln(x/x_0)}{h}\right)dx = 1 - \int_0^{\ln(t/x_0)/h} K(u)du = 0.$$

This proves the first statement.

Since $K$ is compactly supported, $\tilde{K}$ is an entire function. Also, $\tilde{K}$ is entire because $K$ is infinitely differentiable at zero and compactly supported. Now we compute the Mellin transform of $\varphi_{x_0,h}$:

$$\tilde{\varphi}_{x_0,h}(z) = \int_0^{x_0e^h} t^{z-1} \varphi_{x_0}(t)dt = \int_0^{x_0e^h} t^{z-1} \left(1 \leq t \leq \frac{1}{h} K\left(\frac{x}{h}\right) - \frac{1}{xh} K\left(\frac{\ln(x/x_0)}{h}\right) \right)dxdt = \int_0^{x_0e^h} \left[\frac{1}{z} K\left(\frac{x}{h}\right) - \frac{1}{xh} K\left(\frac{\ln(x/x_0)}{h}\right) \right]dxdt = -\frac{1}{z} \int_0^{x_0e^h} x^{z-1} \left[\frac{1}{h} K\left(\frac{x}{h}\right) - \frac{1}{xh} K\left(\frac{\ln(x/x_0)}{h}\right) \right]dx = -\frac{1}{z} [h^z \tilde{K}(z+1) - x_0^z \tilde{K}(-zh)],$$

where in the last equality we took into account that $x_0e^h > h$. The lemma is proved. \hfill \blacksquare

7.4 Proof of Lemma 4.3

We begin with the proof of (4.15). This condition ensures that the integral in the definition of function $\psi_{x_0,h}$ is absolutely convergent so that $\psi_{x_0,h}$ is well defined.

10. We note that $|\psi_{x_0,h}(t)| \leq I_1(t) + I_2(t)$, where

$$I_1(t) = \frac{t^{-s}x_0^{-s}}{2\pi} \int_{-\infty}^{\infty} \tilde{K}\left((s-1+i\omega)h\right) \left|\frac{1}{w(1-s-i\omega)}\right| \left|(1-s)^2 + \omega^2\right|^{-1/2} d\omega, \quad s < 1,$$

$$I_2(t) = \frac{t^{-s}h^{-s}}{2\pi} \int_{-\infty}^{\infty} \tilde{K}\left((2-s-i\omega)h\right) \left|\frac{1}{w(1-s-i\omega)}\right| \left|(1-s)^2 + \omega^2\right|^{-1/2} d\omega, \quad s < 1,$$

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It suffices to show that the integrals appearing in definitions of $I_1(t)$ and $I_2(t)$ are finite for $s < 1$.

First consider the integral in the definition of $I_1(t)$. Since

$$
\tilde{K}(\sigma + i\omega) = \int_0^1 e^{(\sigma+i\omega)t}K(t)dt,
$$

(7.20)
in view of condition (K) we have for any positive integer $\ell$

$$
|\tilde{K}(\sigma + i\omega)| \leq \min\left\{c_1e^{\rho} \sigma, c_2(\ell)\sigma^\sigma(\sigma^2 + \omega^2)^{-\ell/2}\right\}, \quad \forall \sigma, \omega,
$$

(7.21)
where the second inequality follows from the repeated integration by parts in (7.20). Therefore using Lemma 7.2 with $\sigma = 1 - s$ we obtain for any $-d \leq s < 1$

$$
\int_{-\infty}^{\infty} \left|\tilde{K}((s-1)+i\omega)\right| |(1-s)^2 + \omega^2|^{-1/2}d\omega
\leq \frac{c_3}{(2\pi)^{1-s}} \int_{|\omega| \leq 2} \left|\tilde{K}((s-1)h + i\omega)\right| |\omega|^{(d+1)/2}d\omega
\leq \frac{c_3}{(2\pi)^{1-s}} \left(4e^{(s-1)h} + h^{-(d+3)/2}\right) \int_{|\xi| \geq 2h} \left|\tilde{K}((s-1)h + i\xi)\right| |\xi|^{(d+1)/2}d\xi
\leq \frac{c_4e^{(s-1)h}}{(2\pi)^{1-s}} \left(1 + h^{-(d+3)/2}ight) \left(1 + \int_{|\xi| \geq 2h} |\xi|^{(d+1)/2-\ell}d\xi\right)
\leq \frac{c_6e^{(s-1)h}}{(2\pi)^{1-s}} h^{-(d+3)/2},
$$

where in order to get the penultimate inequality we split the integral into the sets $2h \leq |\xi| \leq 1$ and $|\xi| \geq 1$, and on the first set we use the first inequality in (7.21), while on the second set, the second inequality in (7.21) is used with $\ell > (d+3)/2$.

Now consider the integral appearing in the definition of $I_2(t)$. Similarly to (7.21) it follows from

$$
\tilde{K}(\sigma + i\omega) = \int_0^1 t^\sigma+1+i\omega K(t)dt
$$

that for $\sigma > 0$

$$
|\tilde{K}(\sigma + i\omega)| \leq c_6(\ell) \min\left\{1, (\sigma^2 + \omega^2)^{-\ell/2}\right\}.
$$

(7.22)
Therefore by Lemma 7.2 with $\sigma = 1 - s$ and (7.22) we have for $-d \leq s < 1$

$$
\int_{-\infty}^{\infty} \left|\tilde{K}((2-s)+i\omega)\right| |(1-s)^2 + \omega^2|^{-1/2}d\omega
\leq \frac{c_7}{(2\pi)^{1-s}} \int_{|\omega| \leq 2} \left|\tilde{K}((2-s)+i\omega)\right| |\omega|^{(d+1)/2}d\omega
\leq \frac{c_8}{(2\pi)^{1-s}}.
$$

Combining these inequalities with definitions of $I_1(t)$ and $I_2(t)$ we finally obtain for any $-d \leq s < 1$

$$
|\psi_{x_0,h}(t)| \leq c_9(2\pi)^{s-1}t^{-s}\left(h^{1-s} + x_0^{1-s}e^{(s-1)h}h^{-(d+3)/2}\right), \quad \forall t \geq 0.
$$

(7.23)
2. Now we prove (4.16). In view of (4.11)

$$
\int_0^\infty \psi_{x_0,h}(t)H(t)dt = \int_0^\infty \psi_{x_0,h}(t) \int_0^\infty w(x) dF(x)dt = \int_0^\infty \left[ \int_0^\infty \psi_{x_0,h}(t)w(tx)dt \right] dF(x).
$$

Therefore in order to prove (4.16) it suffices to show that

$$
\int_0^\infty \psi_{x_0,h}(t)w(tx)dt = \varphi_{x_0,h}(x), \quad \forall x \geq 0.
$$

(7.24)
Talking the Mellin transform of the left hand side we obtain

$$
\int_0^\infty x^{s-1} \int_0^\infty \psi_{x_0,h}(t)w(tx)dtdx = \tilde{w}(z) \int_0^\infty t^{-s}\psi_{x_0,h}(t)dt = \tilde{w}(z)\tilde{\psi}_{x_0,h}(1-z).
$$
Note that the convergence region of \( \tilde{w}(z) \) is \( \{ z : \Re(z) > 0 \} \). By definition of \( \psi_{x_0, h} \) in (4.13), the Mellin transform of \( \psi_{x_0, h} \) is \( \psi_{x_0, h}(z) = \tilde{\phi}_{x_0}(1 - z)/\tilde{w}(1 - z) \), and its convergence region is \( \{ z : \Re(z) < 1 \} \). Therefore the function \( \tilde{w}(z)\psi_{x_0, h}(1 - z) \) is analytic in \( \{ z : \Re(z) > 0 \} \), and in this region

\[
\tilde{w}(z)\psi_{x_0, h}(1 - z) = \tilde{\phi}_{x_0, h}(z), \quad \forall z : \Re(z) > 0.
\]

Therefore, by uniqueness of the Mellin transform, (7.24) is fulfilled for all \( x \geq 0 \) which complete the proof of the lemma.

\[\square\]

### 7.5 Proof of Theorem 4.2

First we state a result on the rate of decay of the Mellin transform of \( w \) on vertical lines in the convergence region.

**Lemma 7.2** The Mellin transform of function \( w \) defined in (4.11) is given by

\[
\tilde{w}(z) = \frac{(2r)^z}{z} \frac{B\left(\frac{d+1}{2}, \frac{z+1}{2}\right)}{B\left(\frac{d+1}{2}, \frac{1}{2}\right)}, \quad \Re(z) > 0.
\]

Moreover, for \( \sigma > 0 \) and \( \omega \in \mathbb{R} \) one has

\[
|\tilde{w}(\sigma + i\omega)| \geq \frac{C_1(2r)^\sigma}{\Gamma\left(\frac{d+1}{2}\right)} \Gamma\left(\frac{\sigma+1}{2}\right), \quad \forall |\omega| \leq 2,
\]

\[
|\tilde{w}(\sigma + i\omega)| \geq \frac{C_2(2r)^\sigma}{\Gamma\left(\frac{d+1}{2}\right)} |\omega|^{-(d+1)/2}, \quad \forall |\omega| \geq 2,
\]

where constants \( C_1 \) and \( C_2 \) depend on \( d \) only.

**Proof:** We have

\[
\tilde{w}(z) = \frac{1}{B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \int_0^\infty t^{z-1} \int_0^1 1\{t \leq 2r\sqrt{1-y}\} y^{z/2} (1-y)^{-1/2} dy dt
\]

\[
= \frac{(2r)^z}{z B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \int_0^1 (1-y)^{z/2} y^{z/2} (1-y)^{-1/2} dy = \frac{(2r)^z B\left(\frac{d+1}{2}, \frac{z+1}{2}\right)}{z B\left(\frac{d+1}{2}, \frac{1}{2}\right)},
\]

where the second equality holds only if \( \Re(z) > 0 \) and the last equality holds for \( \Re(z) > -1 \); thus the equality holds for all \( z \) such that \( \Re(z) > 0 \). Furthermore, for \( z = \sigma + i\omega, \sigma > 0, \omega \in \mathbb{R} \) we have

\[
|\tilde{w}(\sigma + i\omega)| = \frac{(2r)^\sigma}{\Gamma\left(\frac{d+1}{2}\right)} \left| \Gamma\left(\frac{\sigma+1}{2}\right) \Gamma\left(\frac{\sigma+1}{2} + \frac{i\omega}{2}\right) \right|.
\]

We use the following well known properties of the Gamma function. [see, e.g., Carlson (1977) and Andrews et al. (1999)]:

(i) function \( \Gamma(z) \) does not have zeros on \( \mathbb{C} \), and it is analytic in \( \mathbb{C} : = \mathbb{C} \setminus \{0, -1, -2, \ldots\} \);

(ii) \( |\Gamma(x + iy)| \leq \Gamma(x) \) for all \( x + iy \in \mathbb{C} \), and \( |\Gamma(x + iy)| \geq \Gamma(x) e^{-\pi|y|/2} \), \( \forall x \geq 1/2, \forall y \in \mathbb{R} \);

(iii) for all \( x_1 \leq x \leq x_2 \) and \( |y| \geq 2 \) there exist constants \( c_1 \leq c_2 \) depending on \( x_1 \) and \( x_2 \) such that

\[
|c_1| e^{-\pi|y|/2} \leq |\Gamma(x + iy)| \leq |c_2| e^{-\pi|y|/2}
\]

By property (ii) since \( (\sigma + 1)/2 > 1/2 \),

\[
\left| \Gamma\left(\frac{\sigma+1}{2} + \frac{i\omega}{2}\right) \right| \geq \Gamma\left(\frac{\sigma+1}{2}\right) e^{-\pi|\omega|/4}, \quad \forall \omega
\]

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which yields

$$|\hat{w}(\sigma + i\omega)| \geq \frac{(2r)^\sigma}{\sqrt{\sigma^2 + \omega^2}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \epsilon^{-\sigma/2}, \quad \forall |\omega| \leq 2.$$ 

If $|\omega| \geq 2$ then we use property (iii):

$$|\hat{w}(\sigma + i\omega)| \geq \frac{(2r)^\sigma}{\sqrt{\sigma^2 + \omega^2}} \frac{c_1|\frac{1}{2}\omega|^{(d+1)/2}}{c_2|\frac{1}{2}\omega|^{(d+1)/2}} \geq \frac{c_3(2r)^\sigma}{\sqrt{\sigma^2 + \omega^2}} |\omega|^{-(d+1)/2}, \quad \forall |\omega| \geq 2,$$

where $c_3$ depends on $d$ only. This completes the proof. 

Now we proceed with the proof of the theorem.

**Proof of Theorem 4.2** Throughout the proof $c_1, c_2, \ldots$ stand for positive constants that may depend on $d$, $\beta$ and $\alpha$ only.

It follows from (7.23) and the bound on $H(t)$ in (15.5) that $\int_0^\infty |\psi_{x_0, h}(t)|H(t)dt < \infty$. Next, we have

$$\int_0^{T/2} \psi_{x_0, h}(t)[H(t) - H(t)]dt + \int_0^{\infty} \psi_{x_0, h}(t)H(t)dt - F(x_0)$$

where in the last line we have used (4.14). Our goal is to derive bounds on the expectation of the squared terms on the right hand side of the above formula.

Let $\epsilon > 0$ be a small number to be specified. Throughout the proof in the definition of $\psi_{x_0, h}$ in (4.14) we put $s = 1 - \epsilon$.

1. By the Cauchy–Schwarz inequality

$$\mathbb{E}\left[\int_0^{T/2} \psi_{x_0, h}(t)|\tilde{H}(t) - H(t)|dt\right]^2 \leq \left(\int_0^{T/2} |\psi_{x_0, h}(t)|^2 dt\right)^{1/2} \left(\int_0^{T/2} \mathbb{E}|\tilde{H}(t) - H(t)|^2 dt\right)^{1/2}.$$ 

In view of Theorem 3.3

$$\int_0^{T/2} \mathbb{E}|\tilde{H}(t) - H(t)|^2 dt \leq \frac{c_1}{2cT^{1-2\epsilon}} \left(1 + \frac{1}{\rho}\right) \int_0^T H(t)dt,$$

which provides an upper bound on the second integral in previous display formula. Moreover, by (4.12),

$$\int_0^{T/2} |\psi_{x_0, h}(t)|^2 t^{2\epsilon-1} dt = \int_0^{T/2} |\psi_{x_0, h}(t)|^2 e^{2(1-\epsilon)\tau} dt$$

$$\leq \frac{1}{2\pi} \int_0^\infty |\tilde{\psi}_{x_0, h}(1 - \epsilon + i\omega)|^2 d\omega = \frac{1}{2\pi} \int_0^\infty \left|\tilde{\psi}_{x_0, h}(\epsilon - i\omega)\right|^2 d\omega$$

$$\leq \frac{1}{2\pi} \int_0^\infty \left|\frac{x_0}{\epsilon^2 + \omega^2} \hat{\tilde{w}}(\epsilon - i\omega)\right|^2 d\omega + \frac{1}{2\pi} \int_0^\infty \left|\frac{h' K(1 + \epsilon - i\omega)}{\epsilon^2 + \omega^2} \hat{\tilde{w}}(\epsilon - i\omega)\right|^2 d\omega =: I_1 + I_2.$$

We proceed with bounding the integrals $I_1$, $I_2$ on the right hand side.

By Lemma 3.2 and (7.21) and using the same reasoning as in the proof of Lemma 4.3 we obtain

$$I_1 \leq c_1\left(\frac{x_0}{2\pi}\right)^2 \left(\int_{|\omega| \leq \epsilon} |\hat{\tilde{K}}((\epsilon - i\omega)h)|^2 d\omega + \int_{|\omega| \geq \epsilon} |\hat{\tilde{K}}((\epsilon - i\omega)h)|^2 |\omega|^{d+1} d\omega\right)$$

$$\leq c_2\left(\frac{x_0}{2\pi}\right)^2 e^{2\epsilon h} + h^{-d-2} \int_{|\xi| \geq 2h} |\hat{\tilde{K}}(-h + i\xi)|^2 |\xi|^{d+1} d\xi$$

$$\leq c_3 e^{2\epsilon h} \left(\frac{x_0}{2\pi}\right)^2 \left[1 + h^{-d-2} \left(1 + \int_{|\xi| \geq 1} \left|\frac{|\xi|^{d+1}}{e^{2\epsilon h^2 + \xi^2}}\right| d\xi\right)\right] \leq c_4 e^{2\epsilon h} \left(\frac{x_0}{2\pi}\right)^2 h^{-d-2},$$

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where we have used inequality (7.21) with $2\ell > d + 2$. Similarly,

$$I_2 \leq c_6 \left(\frac{h}{2r}\right)^{2\ell} \left\{ \int_{|\omega| \leq 2} |\tilde{K}(1 + \epsilon - i\omega)|^2 d\omega + \int_{|\omega| \geq 2} |\tilde{K}(1 + \epsilon - i\omega)|^2 |\omega|^{d+1} d\omega \right\} \leq c_6 \left(\frac{h}{2r}\right)^{2\ell}.$$  

Thus

$$\int_0^{T/2} |\psi_{x_0, h}(t)|^2 t^{-2\ell + 1} dt \leq c_6 e^{2\ell c_6} \left(\frac{x_0}{2r}\right)^{2\ell} h^{-d-2} + c_6 \left(\frac{h}{2r}\right)^{2\ell} \leq c_7 e^{2\ell c_7} \left(\frac{x_0}{2r}\right)^{2\ell} h^{-d-2}.$$  

Combining this inequality with (7.22) we obtain

$$\mathbb{E} \left[ \int_0^{T/2} \psi_{x_0, h}(t) |\hat{H}(t) - H(t)| dt \right]^2 \leq c_8 \left(\frac{x_0}{2r}\right)^{2\ell} e^{2\ell c_8} \left(1 + \frac{1}{2}\right) \int_0^T H(t) dt. \quad (7.27)$$

20. Now we bound the second term on the right hand side of (7.25). It follows from (7.23) applied with $s = 1 - \epsilon$ that

$$|\psi_{x_0, h}(t)| \leq c_1 (2r)^{t-1+\epsilon} \left[h^{x_0 e^{\epsilon h}} - h^{-(d+3)/2}\right].$$

Therefore using a bound on $H(t)$ in (7.35) for sufficiently large $T$ we obtain

$$\int_0^T \psi_{x_0, h}(t) |H(t)| dt \leq c_1 (2r)^{t} \left[h^{x_0 e^{\epsilon h}} - h^{-(d+3)/2}\right] \int_0^T \psi_{x_0, h}(t) dt \leq c_3 M \left(h^{x_0 e^{\epsilon h}} - h^{-(d+3)/2}\right) \leq \frac{c_3 M \left(h^{x_0 e^{\epsilon h}} - h^{-(d+3)/2}\right)}{1 + \alpha - \epsilon} \left(\frac{1}{h^{(d+3)/2}}\right). \quad (7.28)$$

provided that $1 + \alpha - \epsilon > 0$.

30. Now we work with the third term on the right hand side of (7.25). By definition of $\varphi_{x_0, h}$ we have

$$\int_0^{x_{0e^{\epsilon h}}} \varphi_{x_0, h}(x) dF(x) = \int_{-\infty}^{x_0} \varphi_{x_0, h}(x) dF(x) + F(x_0) - F(0) + \int_{x_0 e^{\epsilon h}}^{x_{0e^{\epsilon h}}} \varphi_{x_0, h}(x) dF(x), \quad (7.29)$$

$$\int_0^{x_{0e^{\epsilon h}}} \varphi_{x_0, h}(x) dF(x) = \int_0^{x_0} \int_0^{1} 1(t \leq x) \frac{1}{h} K \left(\frac{t}{h}\right) dt dF(x) = F(x) - \frac{1}{h} \int_0^{x_{0e^{\epsilon h}}} K \left(\frac{t}{h}\right) F(t) dt \quad (7.30)$$

and

$$\int_{x_0}^{x_{0e^{\epsilon h}}} \varphi_{x_0, h}(x) dF(x) = \int_{x_0}^{x_{0e^{\epsilon h}}} \left[ 1 - \int_{x_0}^{x_{0e^{\epsilon h}}} \frac{1}{t h} K \left(\frac{\ln(t/x_0)}{h}\right) \right] dF(x)$$

$$= F(x_0 e^{\epsilon h}) - F(x_0) - \int_{x_0 e^{\epsilon h}}^{x_{0e^{\epsilon h}}} 1(t \leq x) \frac{1}{t h} K \left(\frac{\ln(t/x_0)}{h}\right) dt dF(x)$$

$$= -F(x_0) + \int_{x_0}^{x_{0e^{\epsilon h}}} \frac{1}{t h} K \left(\frac{\ln(t/x_0)}{h}\right) F(t) dt = \int_0^{x_{0e^{\epsilon h}}} K(y) F(x_0 e^{\epsilon h}) dy - F(x_0). \quad (7.31)$$

Combining (7.29), (7.30) and (7.31) we obtain

$$\left| \int_0^{x_{0e^{\epsilon h}}} \varphi_{x_0, h}(x) dF(x) - F(x_0) \right| \leq \left| \frac{1}{h} \int_0^{x_{0e^{\epsilon h}}} K \left(\frac{t}{h}\right) F(t) dt \right| + \left| \int_0^{x_{0e^{\epsilon h}}} K(y) F(x_0 e^{\epsilon h}) dy - F(x_0) \right|.$$  

Because $F \in \mathcal{A}_{h}(A)$ and $F(0) = 0$, expanding in Taylor’s series we have for some $\xi \in (0, h)$

$$\left| \frac{1}{h} \int_0^{x_{0e^{\epsilon h}}} K \left(\frac{t}{h}\right) F(t) dt \right| \leq \frac{h \xi}{\ell} \int_0^{x_{0e^{\epsilon h}}} |K(y)| |F^{(\ell)}(\xi) - F^{(\ell)}(0+)| y^\beta dy \leq c_1 Ah^\beta.$$  

To bound the second term we define function $R_{x_0}(t) = F(x_0 e^{\epsilon h})$; with this notation

$$\left| \int_0^{x_{0e^{\epsilon h}}} K(y) F(x_0 e^{\epsilon h}) dy - F(x_0) \right| = \left| \int_0^{x_{0e^{\epsilon h}}} K(y) [R_{x_0}(y) - R_{x_0}(0)] dy \right|. $$

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Expanding function $R_{x_0}$ in Taylor’s series around 0 we have

$$R_{x_0}(yh) = R_{x_0}(0) + \sum_{j=1}^{\ell-1} \frac{R_{x_0}^{(j)}(0)}{j!} (yh)^j + \frac{1}{\ell!} R_{x_0}^{(\ell)}(\xi)(yh)^\ell, \quad 0 \leq \xi \leq h,$$

and therefore

$$\left| \int_0^1 K(y)[R_{x_0}(yh) - R_{x_0}(0)]dy \right| \leq \frac{h^\ell}{\ell!} \int_0^1 |K(y)||R_{x_0}^{(\ell)}(\xi) - R_{x_0}^{(\ell)}(0)||y^{\ell}dy. \quad (7.32)$$

By the Faà di Bruno formula

$$R_{x_0}^{(\ell)}(t) = \frac{d^\ell}{dt^\ell} F(x_0 e^t) = \sum_{k=1}^{\ell} \frac{\ell!}{j_1! \cdots j_{\ell-k+1}(\ell)!} \left( \frac{x_0}{\ell} \right)^{j_1} \cdots \left( \frac{x_{\ell-k+1}}{\ell - k + 1} \right)^{j_{\ell-k+1}} B_{\ell,k}(x_1, \ldots, x_{\ell-k+1})$$

where $B_{\ell,k}$ is the Bell polynomial of degree $\ell$ in $\ell - k + 1$ variables given by

$$B_{\ell,k}(x_1, \ldots, x_{\ell-k+1}) = \sum_{j_1, \ldots, j_{\ell-k+1} \geq 0} \prod_{i=1}^{\ell-k+1} \frac{j_i!}{j_1! \cdots j_{\ell-k+1}} \frac{x_{\ell-k+1}}{\ell - k + 1} \cdots \frac{x_i}{j_i}$$

the sum is taken over all over all subsets $j_1, \ldots, j_{\ell-k+1}$ of non–negative integers such that $j_1 + \cdots + j_{\ell-k+1} = k$ and $j_1 + 2j_2 + \cdots + (\ell - k + 1)j_{\ell-k+1} = \ell$. In our specific case the Faà di Bruno formula takes the form

$$R_{x_0}^{(\ell)}(t) = \sum_{k=1}^{\ell} c_{k,\ell} F^{(k)}(x_0 e^t)(x_0 e^t)^k,$$

where $c_{k,\ell}$ are coefficients depending on $k$ and $\ell$ only. Therefore

$$|R_{x_0}^{(\ell)}(\xi) - R_{x_0}^{(\ell)}(0)| = \left| \sum_{k=1}^{\ell} c_{k,\ell}x_0^k [F^{(k)}(x_0 e^\xi) e^\ell \xi - F^{(k)}(x_0)] \right|$$

$$\leq \sum_{k=1}^{\ell} c_{k,\ell}x_0^k \left[ e^{\ell \xi}||F^{(k)}(x_0 e^\xi) - F^{(k)}(x_0)|| + |F^{(k)}(x_0)||e^\ell \xi - 1| \right]$$

$$\leq \sum_{k=1}^{\ell-1} c_{k,\ell}x_0^k \left[ e^{kh} A x_0 |e^h - 1| + A e^{kh} - 1 \right] + c_{\ell,\ell}x_0^\ell \left[ A x_0^\beta - A |e^h - 1|^{\beta - \ell} + A |e^{k_h} - 1| \right]$$

$$\leq c_2 A \left[ x_0^\beta h^{\beta - \ell} + h \sum_{k=1}^{\ell-1} x_0^{k+1} \right],$$

where constant $c_2$ depends on $\beta$ only. To obtain the last formula we have used the elementary inequality $e^x - 1 \leq xe^x$, $x \geq 0$, and the fact that $h < 1/2$. Combining this inequality with (7.32) we obtain

$$\left| \int_0^1 K(y)[R_{x_0}(yh) - R_{x_0}(0)]dy \right| \leq c_3 A \left[ x_0^\beta h^{\beta} + h^{\ell+1} \sum_{k=1}^{\ell-1} x_0^{k+1} \right],$$

and finally

$$\left| \int_0^\infty \varphi_{x_0,h}(x) dF(x) - F(x_0) \right| \leq c_4 Ah^\beta (x_0^\beta + 1) + c_5 Ah^{\ell+1} \sum_{k=1}^{\ell-1} x_0^{k+1}. \quad (7.33)$$

4th. Now we are in a position to complete the proof of the theorem. We combine bounds (7.24), (7.25) and (7.33) and set $\epsilon = 1/\ln T$. Then for $T$ large enough we obtain

$$\mathbb{E}[\hat{F}(x_0) - F(x_0)]^2 \leq \frac{c_1 (1 + \frac{1}{T})}{h^{\ell+2} T^2} \int_0^T H(t)dt + \frac{c_2 M}{h^{\ell+3} T^2} + c_3 A^2 h^{2\beta} (x_0^\beta + 1)^2.$$
In view of (4.10) we have

\[ \int_0^T H(t) dt \leq c_T \eta_T, \quad \eta_T = \left[ r + M r^{1+\alpha} \eta_T \right] , \]

and we recall that \( \eta_T \) is 1 for \( \alpha > 0 \); \( \ln T \) for \( \alpha = 0 \), and \( T^{-\alpha} \) for \(-1 < \alpha < 0 \) [see (4.16)]. Then the choice \( h = h_* \) as in (4.18) leads to the announced result.

8 Proofs for Section 5

8.1 Proof of Lemma 5.1

(a) Behavior at zero. If \( d = 1 \) then by straightforward algebra

\[
H(t) = \frac{1}{B(1, \frac{1}{2})} \frac{1}{\Gamma \left( \frac{3}{2} \right)} \int_0^1 \int_0^\infty x^{-1/2} e^{-x} 1 \left( x \leq \frac{2r^2 y}{\sigma^2 t} \right) y^{-1/2} dx dy
\]

\[
= \frac{2 \Gamma \left( \frac{3}{2} \right)}{\Gamma^2 \left( \frac{1}{2} \right)} \int_0^\infty x^{-1/2} e^{-x} \left( 1 - \frac{\sqrt{2} \sigma}{\sqrt{2} r} \right) 1 \left( x \leq \frac{2r^2}{\sigma^2 t} \right) dx
\]

\[
= \frac{2 \Gamma \left( \frac{3}{2} \right)}{\Gamma^2 \left( \frac{1}{2} \right)} \left[ \int_0^\infty x^{-1/2} e^{-x} 1 \left( x \leq \frac{2r^2}{\sigma^2 t} \right) \right] dx + \frac{\sqrt{2} \sigma}{\sqrt{2} r} \exp \left\{ - \frac{2r^2}{\sigma^2 t} - \frac{\sqrt{2} \sigma}{\sqrt{2} r} \right\}
\]

so that

\[
1 - H(t) \sim \frac{\sigma \sqrt{r}}{\sqrt{2} \Gamma \left( \frac{1}{2} \right)} = \frac{\sigma \sqrt{r}}{\sqrt{2} \pi r} \quad \text{as} \ t \to 0.
\]

If \( d \geq 2 \) then the asymptotic approximation of the upper incomplete Gamma function [see, e.g., (Abramowitz & Stegun 1965, 6.5.32)] yields:

\[
1 - H(t) \sim \frac{1}{B \left( \frac{d+1}{2}, \frac{1}{2} \right)} \frac{2r^2}{\sigma^2 t} \frac{1}{\Gamma \left( \frac{1}{2} \right)} \int_0^1 \exp \left( - \frac{2r^2 y}{\sigma^2 t} \right) (1-y)^{\frac{d-1}{2}} \frac{1}{\Gamma \left( \frac{d}{2} \right)} \left[ \int_0 y^{\frac{d-3}{2}} dy \right], \quad t \to 0.
\] (8.1)

The integral on the right hand side is expressed in terms of Kummer’s function [cf. (Abramowitz & Stegun 1965, Chapter 13)] that is defined as follows: for \( a \) and \( b \) satisfying \( \text{Re}(b) > \text{Re}(a) > 0 \)

\[
M(a, b; z) := 1 + \sum_{k=1}^\infty \left( \frac{a + j}{b + j} \right) \frac{z^k}{k!} = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zu} u^{a-1} (1-u)^{b-a-1} du.
\]

With this notation letting \( a = (d-1)/2 \) and \( b = d \) we have

\[
\int_0^1 \exp \left( - \frac{2r^2 y}{\sigma^2 t} \right) (1-y)^{\frac{d-1}{2}} \frac{1}{\Gamma \left( \frac{d}{2} \right)} \left[ \int_0 y^{\frac{d-3}{2}} dy \right] = \frac{\Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{d-1}{2} \right)}{\Gamma \left( \frac{d}{2} \right)} M \left( \frac{d-1}{2}, d; - \frac{2r^2}{\sigma^2 t} \right).
\] (8.2)

By (Abramowitz & Stegun 1965, 13.1.5), \( M(a, b; z) = \frac{\Gamma(a)}{\Gamma(b-a)} (-z)^{-a} \left[ 1 + O(|z|^{-1}) \right] \), \( \text{Re}(z) < 0 \) as \( |z| \to \infty \). Therefore combining (8.2) and (8.1) we obtain

\[
1 - H(t) \sim \frac{\Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{d-1}{2} \right)}{\Gamma \left( \frac{d}{2} \right)} \frac{2r^2}{\sigma^2 t} \frac{1}{\Gamma \left( \frac{d}{2} \right)} \left[ \frac{\Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{d-1}{2} \right)}{\Gamma \left( \frac{d}{2} \right)} \right] = \frac{d}{(d-1) \Gamma \left( \frac{1}{2} \right)} \left( \frac{\sigma \sqrt{r}}{\sqrt{2} \pi r} \right)^{d-2} \quad \text{as} \ t \to 0.
\]

These calculations show that \( H'(0+) = \infty \).

Behavior at infinity. As \( t \to \infty \) we have

\[
H(t) \sim \frac{1}{B \left( \frac{d+1}{2}, \frac{1}{2} \right)} \frac{2r^2 y^{d/2}}{\sigma^2 t} \frac{1}{\Gamma \left( \frac{d}{2} \right)} \left( y^{d-1} \right)^{1/2} \left( 1-y \right)^{-d/2} \frac{1}{\Gamma \left( \frac{1}{2} \right)} \left( \sqrt{2} \pi r \right)^d,
\]

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as claimed.

(b). It follows from [5,1] that
\[
H(t) = \int_0^\infty H(t) e^{-x} \int_0^{\infty} (x \leq \frac{2r^2y}{\sigma^2t}) (1 - y)^{1/2} y^{-1/2} dx dy
\]
\[
\leq \frac{1}{B(\frac{d+1}{2}, \frac{1}{2})} \int_0^1 \int_0^{\infty} \left( 2(\frac{2r^2y}{\sigma^2t}) \right)^{d/2} (1 - y)^{1/2} y^{-1/2} dy
\]
\[
= \frac{\Gamma(\frac{d+1}{2})}{\Gamma(d+1)} \left( \frac{2r^2}{\sigma^2} \right)^{d/2}, \quad \forall t > 0.
\]

We also have
\[
\int_0^t H(t) dt = \frac{1}{B(\frac{d+1}{2}, \frac{1}{2})} \int_0^1 \int_0^{\infty} x^{d-1} e^{-x} \min\{2r^2y/\sigma^2x, T\} (1 - y)^{1/2} y^{-1/2} dx dy.
\]
If \(d > 2\) then
\[
\int_0^t H(t) dt \leq \frac{1}{B(\frac{d+1}{2}, \frac{1}{2})} \int_0^1 \int_0^{\infty} x^{d-1} e^{-x} \frac{2r^2y}{\sigma^2x} (1 - y)^{1/2} y^{-1/2} dx dy
\]
\[
= \frac{2r^2 \Gamma(\frac{d}{2} - 1) B(\frac{d+1}{2}, \frac{3}{2})}{\sigma^2} = \frac{4 \sigma^2}{\sigma^2(d^2 - 4)}.
\]

If \(d = 2\) then
\[
H(t) = \frac{1}{B(\frac{1}{2}, \frac{1}{2})} \int_0^1 \int_0^{\infty} e^{-x} \min\{2r^2y/\sigma^2x, T\} (1 - y)^{1/2} y^{-1/2} dx dy
\]
\[
= \frac{1}{B(\frac{1}{2}, \frac{1}{2})} \int_0^1 \left[ 1 - \exp\left\{ - \frac{2r^2y}{\sigma^2x} \right\} \right] (1 - y)^{1/2} y^{-1/2} dy \leq \frac{r^2}{2\sigma^2}, \quad \forall t > 0,
\]
where we have used the elementary inequality \(1 - e^{-x} \leq x\). Therefore for \(d = 2\)
\[
\int_0^t H(t) dt \leq 1 + \frac{r^2}{2\sigma^2} \int_1^t \frac{dt}{t} = 1 + \frac{r^2}{2\sigma^2} \ln T.
\]

If \(d = 1\) then
\[
H(t) = \frac{1}{B(\frac{1}{2}, \frac{1}{2})} \int_0^1 \int_0^{\infty} e^{-x} \min\{2r^2y/\sigma^2x, T\} y^{-1/2} dx dy
\]
\[
\leq \frac{2}{B(\frac{1}{2}, \frac{1}{2})} \int_0^1 \left( \frac{2r^2}{\sigma^2} \right)^{1/2} y^{-1/2} dy = \sqrt{\frac{2r^2}{\pi \sigma^2}}, \quad \forall t > 0.
\]

This completes the proof.

\[\blacksquare\]

### 8.2 Proof of Theorem 5.1

In the subsequent proof \(c_1, c_2, \ldots\) stand for positive constant that may depend on \(d\) only. The proof is divided in two steps.

(a). First we establish an upper bound on accuracy of estimating the functional \(\Psi_\alpha\). We have
\[
\hat{\Psi}_{\alpha,b} - \Psi_\alpha = \int_0^b \frac{H(t) - H(t)}{t^{1-\alpha}} dt + \int_b^\infty \frac{H(t)}{t^{1-\alpha}} dt.
\]
hence

\[ E[\hat{\Psi}_{\alpha,b} - \Psi_\alpha]^2 \leq 2 \int_0^b \int_0^b E[(\hat{H}(t) - H(t))(\hat{H}(s) - H(s))] dt ds \]

\[ + 2 \left( \int_b^\infty \frac{H(t)}{t^{1-\alpha}} dt \right)^2 = 2I_1 + 2I_2. \]

The bound on \( I_2 \) is readily obtained from (5.2):

\[ \int_b^\infty \frac{H(t)}{t^{1-\alpha}} dt \leq c_1 \left( \frac{r}{\sigma} \right)^d \int_b^\infty \frac{dt}{t^{1+\alpha} + \sigma/2} = \left( \frac{2c_1}{d-2\alpha} \right) \left( \frac{r}{\sigma} \right)^d b^{d-\alpha/2}, \]

so that

\[ I_2 \leq \left( \frac{2c_1}{d-2\alpha} \right)^2 \left( \frac{r}{\sigma} \right)^{2d} b^{-d+2\alpha}. \]

Now we bound \( I_1 \): by the Cauchy–Schwarz inequality, (3.5) and (5.3)

\[ I_1 \leq \left[ \int_0^b \frac{1}{t^{1-\alpha}} \left( E[\hat{H}(t) - H(t)] \right)^{1/2} dt \right]^2 \leq c_1 \left( 1 + \frac{1}{\rho} \right) \int_0^T H(t) dt \left[ \int_0^b \frac{dt}{t^{1-\alpha}(T-t)^{1/2}} \right]^2 \]

\[ \leq c_2 \left( 1 + \frac{1}{\rho} \right) \frac{\beta^2 \alpha}{\alpha^2 T} \int_0^T H(t) dt. \]

Using (8.3) and combining the bounds for \( I_1 \) and \( I_2 \) we obtain the following results.

If \( d = 1 \) then

\[ E[\hat{\Psi}_{\alpha,b} - \Psi_\alpha]^2 \leq c_3 \left( 1 + \frac{1}{\rho} \right) \left( \frac{r}{\sigma} \right)^2 \frac{b^{2\alpha}}{\alpha^2} + \left( \frac{2c_1}{1-2\alpha} \right) \left( \frac{r}{\sigma} \right)^2 b^{-1+2\alpha}. \]

Letting \( b_* = (1-2\alpha)^{-2\alpha^2} \sqrt{T} \) we obtain

\[ E[\hat{\Psi}_{\alpha,b_*} - \Psi_\alpha]^2 \leq c_4 \left[ \left( 1 + \frac{1}{\rho} \right) \left( \frac{r}{\sigma} \right)^2 + \left( \frac{r}{\sigma} \right)^4 \right] \frac{\alpha^{-2+4\alpha}}{(1-2\alpha)^{4\alpha T^{1/2-\alpha}}}. \quad (8.3) \]

If \( d = 2 \) then

\[ E[\hat{\Psi}_{\alpha,b} - \Psi_\alpha]^2 \leq c_5 \left( 1 + \frac{1}{\rho} \right) \frac{b^{2\alpha}}{\alpha^2 T} \left[ 1 + \left( \frac{r}{\sigma} \right)^2 \ln T \right] + \left( \frac{2c_1}{2-2\alpha} \right) \left( \frac{r}{\sigma} \right)^4 b^{-2+2\alpha}. \]

Therefore letting \( b_* = \alpha \sqrt{T/\ln T} \) we obtain for sufficiently large \( T \)

\[ E[\hat{\Psi}_{\alpha,b_*} - \Psi_\alpha]^2 \leq c_6 \left[ \left( 1 + \frac{1}{\rho} \right) \left( \frac{r}{\sigma} \right)^2 + \left( \frac{r}{\sigma} \right)^4 \right] \alpha^{-2+2\alpha} \left( \ln T / T \right)^{1-\alpha}. \quad (8.4) \]

Finally, if \( d > 2 \) then

\[ E[\hat{\Psi}_{\alpha,b} - \Psi_\alpha]^2 \leq c_7 \left( 1 + \frac{1}{\rho} \right) \left( \frac{r}{\sigma} \right)^2 \frac{b^{2\alpha}}{\alpha^2} + \left( \frac{2c_1}{d-2\alpha} \right)^2 \left( \frac{r}{\sigma} \right)^2 b^{-d+2\alpha}, \]

and for \( b_* = (\alpha^2 T)^{1/d} \) we get

\[ E[\hat{\Psi}_{\alpha,b_*} - \Psi_\alpha|^2 \leq c_8 \left[ \left( 1 + \frac{1}{\rho} \right) \left( \frac{r}{\sigma} \right)^2 + \left( \frac{r}{\sigma} \right)^{2d} \right] \left( \frac{1}{\alpha^2 T} \right)^{1-2\alpha}. \quad (8.5) \]

(b) Now we relate error in estimating \( \sigma^2 \) by \( \hat{\sigma}_{\alpha,b} \) to the mean squared error of \( \hat{\Psi}_{\alpha,b} \). By definition

\[ |\hat{\sigma}_{\alpha,b}^2 - \sigma^2| = \left( \frac{\hat{\Psi}_{\alpha,b}}{\hat{\Psi}_{\alpha,b}} \right)^{1/\alpha} \left| \Psi_{\alpha,b}^{1/\alpha} - \Psi_{\alpha,b}^{1/\alpha} \right| \leq \left( \frac{\hat{\Psi}_{\alpha,b}}{\hat{\Psi}_{\alpha,b}} \right)^{1/\alpha} \frac{1}{\alpha} (\hat{\Psi}_{\alpha,b} \land \Psi_{\alpha,b})^{\frac{1}{\alpha} - 1} |\Psi_{\alpha} - \Psi_{\alpha,b}| \]

\[ = \left( \frac{\hat{\Psi}_{\alpha,b}}{\hat{\Psi}_{\alpha}} \right)^{1/\alpha} \left( \Psi_{\alpha,b} \geq \Psi_{\alpha} \right) \frac{1}{\alpha} |\Psi_{\alpha} - \Psi_{\alpha,b}| + \left( \frac{\hat{\Psi}_{\alpha,b}}{\hat{\Psi}_{\alpha}} \right)^{1/\alpha} \left( \Psi_{\alpha,b} < \Psi_{\alpha} \right) \frac{1}{\alpha} |\Psi_{\alpha} - \Psi_{\alpha,b}| \]

\[ \leq \frac{1}{\alpha \Psi_{\alpha}} |\Psi_{\alpha} - \Psi_{\alpha,b}| \left( \sigma^2 \mathbf{1}(\hat{\Psi}_{\alpha,b} \geq \Psi_{\alpha}) + \hat{\sigma}_{\alpha,b}^2 \mathbf{1}(\hat{\Psi}_{\alpha,b} < \Psi_{\alpha}) \right) \leq \frac{1}{\alpha \Psi_{\alpha}} |\Psi_{\alpha} - \Psi_{\alpha,b}| (\sigma^2 + \hat{\sigma}_{\alpha,b}^2), \]

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where in the first line we have used the elementary inequality $|a^{1/\alpha} - b^{1/\alpha}| \leq \frac{1}{\alpha}(a \vee b)^{\frac{1}{\alpha} - 1}|a - b|$ which holds for all $a, b > 0$ and $0 < \alpha \leq 1$. Therefore

$$E[\Delta(\hat{\sigma}_{a,b}^2, \sigma^2)]^2 = \frac{\hat{\sigma}_{a,b}^2 - \sigma^2}{\sigma_{a,b}^2 + \sigma^2} E[\Delta(\hat{\sigma}_{a,b}, \sigma) \cdot \Delta(\hat{\sigma}_{a,b}, \sigma)] \leq \frac{\sigma^{4\alpha}}{\alpha^2 J_\alpha} E[\hat{\Psi}_\alpha - \hat{\Psi}_{a,b}]^2 \leq c_1 \left(\frac{\sigma}{r}\right)^4 E[\hat{\Psi}_\alpha - \hat{\Psi}_{a,b}]^2,$$  

(8.6)

where we have taken into account that

$$\alpha^2 J_\alpha^2 = \left[\frac{(2\pi)^2 \Gamma(\frac{d}{2}) - \sigma^2 B(\frac{d}{2} + \frac{1}{\alpha}, \frac{d+1}{2})}{\Gamma(\frac{d}{2}) B(\frac{d+1}{2}, \frac{1}{2})}\right]^2 \geq c_2 r^{4\alpha}, \quad \forall \alpha \in (0, 1/2).$$

To complete the proof we combine (8.6) with (8.3), (8.4) and (8.5), and set $\alpha_\ast = 1/\ln T$. \hfill \qed

### 8.3 Proof of Lemma 5.2

We have

$$H(t) = \frac{1}{B(1, \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^1 \int_0^\infty x^{-1/2} e^{-x} \left(1 - \frac{2x}{\sigma^2} \right) y^{-1/2} dx dy$$

$$= \frac{2\Gamma(\frac{1}{2})}{\Gamma^2(\frac{1}{2})} \int_0^\infty x^{-1/2} e^{-x} \left(1 - \frac{\sigma \sqrt{x}}{\sqrt{2}r} \right) \left(1 - \frac{2x}{\sigma^2} \right) dx$$

$$= \frac{2\Gamma(\frac{1}{2})}{\Gamma^2(\frac{1}{2})} \left[ \int_0^\infty x^{-1/2} e^{-x} \left(1 - \frac{2x}{\sigma^2} \right) dx + \frac{\sigma \sqrt{t}}{\sqrt{2}r} \exp \left\{ - \frac{2r^2}{\sigma^2} \right\} \right]$$

$$= 1 - \frac{1}{\sqrt{\pi}} \int_{2r/(\sigma^2t)}^\infty x^{-1/2} e^{-x} dx + \frac{\sigma \sqrt{t}}{\sqrt{2}r} \exp \left\{ - \frac{2r^2}{\sigma^2} \right\} - \frac{\sigma \sqrt{t}}{\sqrt{2}r}.$$  

Then for any $t \leq \sigma^2/(2r^2)$ one has

$$1 - H(t) = \frac{\sigma \sqrt{t}}{\sqrt{2}r} \left[1 - \exp \left\{ - \frac{2r^2}{\sigma^2} \right\} \right] + \delta(t), \quad |\delta(t)| \leq \exp \left\{ - \frac{2r^2}{\sigma^2} \right\}. \hfill \qed$$

### 8.4 Proof of Theorem 5.2

It follows from Lemma 5.2 that

$$\left|\sigma - \frac{\sqrt{2\pi r}}{\sqrt{r}}[1 - H(\tau)]\right| \leq \exp \left\{ - \frac{r^2}{\sigma^2 \tau} \right\} \left[\sigma + \frac{\sqrt{r}}{\sqrt{2\pi r}} \right].$$

Therefore

$$|\hat{\sigma}_\tau - \sigma| \leq \frac{\sqrt{2\pi r}}{\sqrt{r}} |H(\tau) - H(\tau)| + \exp \left\{ - \frac{r^2}{\sigma^2 \tau} \right\} \left[\sigma + \frac{\sqrt{r}}{\sqrt{2\pi r}} \right],$$

and

$$E|\hat{\sigma}_\tau - \sigma|^2 \leq c \left\{ \left(1 + \frac{1}{\rho^2}\right) \frac{1}{T} \int_0^T H(t) dt + \exp \left\{ - \frac{r^2}{\sigma^2 \tau} \right\} \left[\sigma^2 + \frac{\tau}{r^2} \right] \right\}$$

$$\leq c \left\{ \left(1 + \frac{1}{\rho^2}\right) \frac{r^3 \sigma^2}{\tau^{3/2} T} + \exp \left\{ - \frac{2r^2}{\sigma^2 \tau} \right\} \left[\sigma^2 + \frac{\tau}{r^2} \right] \right\},$$

where we have used (5.3). Setting for some $\bar{\sigma} > 0$

$$\tau = \tau_\alpha = \frac{4r^2}{\sigma^2 \ln T},$$

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we obtain
\[
E \left| \frac{\hat{\sigma} - \sigma}{\sigma} \right|^2 \leq c \left\{ \left( 1 + \frac{1}{\rho} \right) \left( \frac{\hat{\sigma}}{\sigma} \right)^2 \ln T + \left( \frac{1}{\sqrt{T}} \right)^{\sigma^2/\sigma^2} \left[ 1 + \frac{1}{\sigma^2 \sigma^2 \ln T} \right] \right\}.
\]

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