COHOMOLOGY RING OF THE FLAG VARIETY VS CHOW
COHOMOLOGY RING OF THE GELFAND-ZETLIN TORIC VARIETY

KIUMARS KAVEH AND ELISE VILLELLA

Abstract. We compare the cohomology ring of the flag variety $F_{\ell n}$ and the Chow cohomology ring of the Gelfand-Zetlin toric variety $X_{GZ}$. We show that $H^\ast(F_{\ell n}, \mathbb{Q})$ is the Gorenstein quotient of the subalgebra $L$ of $A^\ast(X_{GZ}, \mathbb{Q})$ generated by degree 1 elements. We compute these algebras for $n = 3$ to see that, in general, the subalgebra $L$ does not have Poincaré duality.

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This is a preliminary version. Comments are welcome!

INTRODUCTION

Throughout the paper, the base field is assumed to be $\mathbb{C}$. The complete flag variety $F_{\ell n}$ is the variety whose points parameterize complete flags of subspaces in $\mathbb{C}^n$, namely:

$$F = (\{0\} \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n).$$

The variety $F_{\ell n}$ can be identified with the homogeneous space $\text{GL}(n, \mathbb{C})/B$ where $B$ is the subgroup of upper triangular matrices. The geometry of flag variety plays an important role in representation theory of $\text{GL}(n, \mathbb{C})$ and combinatorics related to the permutation group. More generally there is a notion of flag variety for any reductive algebraic group $G$.

We recall that $\dim(F_{\ell n}) = N = n(n-1)/2$. The classes of Schubert varieties form an important $\mathbb{Z}$-basis for $H^\ast(F_{\ell n}, \mathbb{Z})$. Since $F_{\ell n}$ has a paving by affine cells (Schubert cells), it has no odd cohomology. Moreover, $H^\ast(F_{\ell n})$ is generated by degree 2 elements.

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Also its Chow ring $A^\ast(F_\ell, n)$ is isomorphic to $H^\ast(F_\ell, \mathbb{Z})$ where the isomorphism doubles the degree. The famous Borel description states that $H^\ast(F_\ell, \mathbb{Z})$ is isomorphic to the polynomial algebra in $n$ variables quotient by the ideal generated by non-constant symmetric polynomials.

We identify the weight lattice $\Lambda = \Lambda_{GL(n, \mathbb{C})}$ with the additive group $\mathbb{Z}^n$ and the semi-group of dominant weights $\Lambda^\ast = \Lambda_{GL(n, \mathbb{C})}^\ast$ (respectively the positive Weyl chamber $\Lambda^\ast_+$) with the collection of all increasing sequences $\lambda = (\lambda_1 \leq \cdots \leq \lambda_n)$ of integers (respectively real numbers). If $\lambda_1 < \cdots < \lambda_n$ we call $\lambda$ a regular dominant weight. We also denote the weight lattice $\Lambda(SL(n, \mathbb{C}))$ of $SL(n, \mathbb{C})$ by $\Lambda'$. It can be identified with the quotient $\Lambda/\mathbb{Z}(1, \ldots, 1)$.

In their fundamental work [GZ50], Gelfand and Zetlin construct a certain vector space basis $B_\lambda$ for an irreducible representation $V_\lambda$ of $GL(n, \mathbb{C})$ with highest weight $\lambda$, and they explicitly describe the action of $\mathfrak{gl}(n, \mathbb{C}) = \text{Lie}(GL(n, \mathbb{C}))$ on basis elements in $B_\lambda$. The Gelfand-Zetlin basis $B_\lambda$ has the remarkable property that its elements are indexed by the lattice points in a convex polytope $\Delta_\lambda \subset \mathbb{R}^N$, where $N = (n - 1)/2$, called the Gelfand-Zetlin polytope (or GZ polytope) associated to $\lambda$. The defining inequalities of $\Delta_\lambda$ can be explicitly written down. If $\lambda = (\lambda_1 \leq \cdots \leq \lambda_n)$ the polytope $\Delta_\lambda$ is the collection of $(x_{ij} \mid 1 \leq i \leq n - 1, 1 \leq j \leq i) \in \mathbb{R}^N$ satisfying the following array of inequalities:

\[
\begin{array}{cccccc}
\lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\
x_{11} & x_{12} & x_{13} & \cdots & x_{1(n-1)} \\
x_{21} & x_{22} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
x_{(n-1)1} & \cdots & \cdots \\
\end{array}
\]

where each small triangle $a \quad c \quad b$ corresponds to the inequalities $a \leq c \leq b$. For example, if $\lambda = (-1, 0, 1)$, then the Gelfand-Zetlin polytope $\Delta_\lambda$ is given by the following inequalities (see Figure 1):

\[-1 \leq x \leq 0, \quad 0 \leq y \leq 1, \quad x \leq z \leq y.\]

Since there is a one-to-one correspondence between the elements of the Gelfand-Zetlin basis $B_\lambda$ and the lattice points in $\Delta_\lambda$ one immediately sees that:

\[\dim(V_\lambda) = \#(\Delta_\lambda \cap \mathbb{Z}^N).\]

It is well-known that a weight $\lambda$ gives rise to a $GL(n, \mathbb{C})$-linearized line bundle $L_\lambda$ on the flag variety $F_\ell_n$. When $\lambda$ is regular dominant the line bundle $L_\lambda$ is very ample. By the Borel-Weil theorem $H^0(F_\ell_n, L_\lambda) \cong V_\lambda^\ast$ as a $GL(n, \mathbb{C})$-module. Thus, in particular we have:

\[\dim(H^0(F_\ell_n, L_\lambda)) = \#(\Delta_\lambda \cap \mathbb{Z}^N).\]

A general philosophy, suggested in the work of several authors and in particular A. Okounkov [Oko98], is that GZ polytopes play a role for the flag variety similar to that of Newton polytopes for toric varieties. In this direction in [Kav11] the first author obtains a description of $H^\ast(F_\ell_n, Q)$ in terms of volume of GZ polytopes. This description is very similar to the Khovanskii-Pukhlikov description of cohomology ring of a smooth projective toric variety in terms of volume of Newton polytopes. The description in [Kav11] turns out to be equivalent to the Borel description via a theorem of Kostant (see [Kav11] Remark 5.4).
Making the connection between geometry of $\mathcal{F}_{\ell_n}$ and GZ polytopes stronger, in [KST12] the authors make a correspondence between Schubert varieties and certain unions of faces of GZ polytopes. They use this correspondence to give applications in Schubert calculus.

It can be shown that for regular dominant weights $\lambda$, all the polytopes $\Delta_\lambda$ have the same normal fan (Proposition 1.1). We call this common normal fan the Gelfand-Zetlin fan and denote it by $\Sigma_{GZ}$. It is well-known that, for each regular dominant $\lambda$ the pair $(\mathcal{F}_{\ell_n}, L_\lambda)$ can be degenerated, in a flat family with reduced irreducible fibers, to $(X_{GZ}, L_{\Delta_\lambda})$. Here $L_{\Delta_\lambda}$ is the equivariant line bundle on the toric variety $X_{GZ}$ corresponding to the lattice polytope $\Delta_\lambda$ (see [KM05]). Such degenerations have been used to study mirror symmetry for the flag variety and partial flag varieties (see [BCFKvS00]). This motivates the problem of comparing the geometry and topology of $\mathcal{F}_{\ell_n}$ with that of $X_{GZ}$.

The variety $X_{GZ}$ is not smooth and hence its Chow group does not have a ring structure. There is a dual version of the Chow ring, due to Fulton and MacPherson [FM81], that works for singular varieties as well. It is called the operational Chow ring or simply Chow cohomology ring. For a variety $X$ we denote its Chow cohomology ring by $A^*(X)$. In [FS97] it is shown that the Chow cohomology ring of a toric variety $X_{\Sigma}$ is naturally isomorphic to the ring of Minkowski weights on its fan $\Sigma$. A degree $k$ Minkowski weight on a fan $\Sigma$ is an assignment of integers to $k$-dimensional cones in $\Sigma$ which satisfy certain balancing condition. One defines a product of Minkowski weights that makes the collection of all Minkowski weights into a ring (see Section 4, see also [FS97, Kaz03]).

Let $k$ be a field. Given a graded algebra $A = \bigoplus_{i=0}^n A^i$ with $A^0 \cong A^n \cong k$, one can form the largest quotient $A/I$ of $A$ such that $A/I$ has Poincaré duality (Lemma 5.1). We call this the Gorenstein quotient of $A$ and denote it by $\text{Gor}(A)$. The main result of the paper is the following (Theorem 6.1):

**Theorem 1.** The cohomology ring $H^*(\mathcal{F}_{\ell_n}, \mathbb{Q})$ is isomorphic to the Gorenstein quotient of the subalgebra of $A^*(X_{GZ}, \mathbb{Q})$ generated by degree 1 elements.

One key combinatorial ingredient in the proof is the following statement due to Valentina Kiritchenko (Proposition 1.3):
**Proposition 2.** Let $P$ be a polytope whose normal fan is $\Sigma_{GZ}$, then $P = c + \Delta_\lambda$ for some $\lambda \in \Lambda^*$ and $c \in \mathbb{R}^N$.

Another ingredient in the proof of Theorem 1 is an algebra lemma which states that a Poincaré duality algebra $A = \bigoplus_{i=0}^n A^i$ that is finite dimensional as a vector space and is generated (over $A_0$) by $A^1$, is uniquely determined by its top product polynomial $p : A^1 \to A^n \cong A^0$, $p(x) = x^n$ (Lemma 5.2).

In Section 7 we compute $A'(X_{GZ}, \mathbb{Q})$ for $n = 3$ and see directly that its subalgebra generated by degree 1 elements does not have Poincaré duality.

For the sake of completeness and making the manuscript accessible to a wider range of audience we have tried to include more background material and most of the proofs.

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1. SOME FACTS ABOUT GELFAND-ZETLIN POLYTOPES

In this section we prove some basic facts about GZ polytopes. We start with the normal fan to a GZ polytope $\Delta_\lambda$. Recall that the normal fan $\Sigma_\Delta$ of a polytope $\Delta$ is constructed as follows: for each face $F$ let $C_F$ be the face cone of $F$ and let $\sigma_F$ be the dual cone to $C_F$. Then $\Sigma_\Delta = \{ \sigma_F \mid F$ face of $\Delta \}$ (see [CLS11, Section 2.3]).

**Proposition 1.1.** For a regular dominant weight $\lambda$, the normal fan $\Sigma_\lambda$ of $\Delta_\lambda$ is independent of $\lambda$.

**Proof.** Facets of the polytope correspond to single equalities in the array [1], and lower dimensional faces of $\Delta_\lambda$ correspond to multiple equalities in the array. There are two types of equality that can occur, (i) those of the form $x_i = \lambda_j$ and (ii) those of the form $x_{ij} = x_{(i-1)k}$. The second type of equality is clearly independent of $\lambda$, and the first type depends on $\lambda$ but yields faces which get translated when $\lambda$ is varied. For a face $F$ defined by either type of equalities one verifies that the corresponding face cone $C_F$ and hence its dual cone $\sigma_F$ is independent of $\lambda$. This proves the claim. \(\square\)

**Definition 1.2** (Gelfand-Zetlin fan). We refer to the common normal fan of the $\Delta_\lambda$, where $\lambda$ is regular dominant, as the Gelfand-Zetlin fan and denote it by $\Sigma_{GZ}$.

**Proposition 1.3.** Let $P$ be a polytope whose normal fan is $\Sigma_{GZ}$, then $P = c + \Delta_\lambda$ for some $\lambda \in \Lambda^*$ and $c \in \mathbb{R}^N$. Moreover, if $P$ is a lattice polytope then $\lambda$ is a dominant weight and $c \in \mathbb{Z}^N$.

**Proof.** Since normal fan of $P$ is $\Sigma_{GZ}$ then the hyperplanes defining $P$ are parallel to the ones defining $\Delta_\lambda$, for any dominant regular $\lambda$ (as we have already showed the fan is independent of $\lambda$). Let us use $y_{ij}$ (respectively $x_{ij}$) for coordinates of a point in $P$ (respectively a GZ polytope). Recall that there are two types of inequalities defining $\Delta_\lambda$ namely, $\lambda_{j-1} \leq x_{1j} \leq \lambda_j$ and $x_{(i-1),j-1} \leq x_{ij} \leq x_{i-1,j}$. Since the facets of $P$ are parallel to those of a GZ polytope we conclude that the inequalities defining $P$ come in two types as well:

\[
\begin{align*}
\text{(2)} \quad \quad a_i & \leq y_{1i} & \leq b_i & \quad 1 \leq i \leq n - 1 \\
y_{(i-1)j} + a_{ij} & \leq y_{ij} & \leq y_{(i-1)(j+1)} + b_{ij} & \quad 2 \leq i \leq n, \quad 1 \leq j \leq n - i + 1.
\end{align*}
\]

We wish to find $\lambda = (\lambda_1 \leq \cdots \leq \lambda_n)$ and $c = (c_{ij}) \in \mathbb{R}^N$ such that if $x_{ij} = y_{ij} + c_{ij}$ then the inequalities (2) for the $y_{ij}$ are equivalent to the GZ inequalities (1) for the $x_{ij}$.
The first type of inequalities \( a_i \leq y_{1i} \leq b_i \) will tell us what \( \lambda \) to choose, up to the choice of \( \lambda_1 \). Set \( \lambda_1 = a_1 \) and \( \lambda_2 = b_1 \). By induction, if we have

\[
\lambda_1 \leq x_{11} = y_{11} \leq \lambda_2 \leq x_{12} \leq \ldots \leq x_{1i} \leq \lambda_{i+1}
\]

then we can translate \( y_{1(i+1)} \) as follows. We are given \( a_{i+1} \leq y_{1(i+1)} \leq b_{i+1} \) and want to shift \( y_{1(i+1)} \) to some \( x_{1(i+1)} \) satisfying \( \lambda_{i+1} \leq x_{1(i+1)} \). This can be done by setting

\[
x_{1(i+1)} = y_{1(i+1)} + \lambda_{i+1} - a_{i+1},
\]

and this also implies that

\[
\lambda_{i+2} = b_{i+1} + \lambda_{i+1} - a_{i+1},
\]

so that \( \lambda_{i+1} \leq x_{1(i+1)} \leq \lambda_{i+2} \). Thus we can translate the variables \( y_{1i} \) by \( c_{1i} = \lambda_{i+1} - a_{i+1} \) to fit into the first row of an array of the form \( \text{[1]} \). For the following rows, we first need to examine relations occurring in each small diamond \( b \quad c \quad \text{appearing in the GZ array} \quad d \text{[1]} \). When we have equalities \( b = a \) and \( c = a \), then since \( b \leq d \leq c \) we must have \( d = a \). This gives us linear relations among the ray generators in the fan \( \Sigma \) which translate to relations between the \( a_{ij}, b_{ij} \) for our polytope \( P \). Suppose we have translated variables \( y_{11}, \ldots, y_{(j-1)} \) to fit into a triangular array of inequalities, then the diamond relation will allow us to shift \( y_{ij} \) so that the associated inequalities will fit into the array as well. Our goal is to define \( x_{ij} \) so that it fits into the diamond

\[
x_{(i-2)(j+1)} \quad x_{(i-1)j} \quad x_{ij} \quad x_{(i-1)(j+1)}
\]

except in the case \( i = 2 \) where we have \( \lambda_{j+1} \) instead of \( x_{(i-2)(j+1)} \). Now when we consider the face of \( P \) where \( x_{(i-1)j} = x_{(i-2)(j+1)} \) and \( x_{(i-1)(j+1)} = x_{(i-2)(j+1)} \), we have

\[
x_{(i-1)j} + a'_{ij} \leq y_{ij} \leq x_{(i-1)(j+1)} + b'_{ij}
\]

which by the diamond relation becomes two equalities rather than inequalities (here \( a'_{ij} = a_{ij} + y_{(i-1)j} - x_{(i-1)j} \) and \( b'_{ij} = b_{ij} + y_{(i+1)(j+1)} - x_{(i-1)(j+1)} \)). This, together with the relations for this face, gives

\[
x_{(i-2)(j+1)} + a'_{ij} = x_{(i-2)(j+1)} + b'_{ij}
\]

which implies \( a'_{ij} = b'_{ij} \). Now, since \( a'_{ij} = b'_{ij} \), we can define \( x_{ij} = y_{ij} - a'_{ij} \) and the relation

\[
x_{(i-1)j} \leq x_{ij} \leq x_{(i-1)(j+1)}
\]

becomes

\[
x_{(i-1)j} \leq x_{ij} \leq x_{(i-1)(j+1)}.
\]

Therefore \( P = c + \Delta_\lambda \) where \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is as constructed above and \( c = (c_{ij}) = (x_{ij} - y_{ij}) \) encodes the translations. Finally, if \( P \) is a lattice polytope then the \( a_i, b_i, a_{ij}, b_{ij} \) should be integers. This implies that \( \lambda \) and \( c \) are integer vectors as well.

\[\text{Remark 1.4.}\] Proposition \( \text{[13]} \) was suggested to us by Valentina Kiritchenko. The proof presented above is due to the second author.

\[\text{Remark 1.5.}\] Observe that there are \( n + n(n - 1)/2 \) parameters present in \( c + \Delta_\lambda \), but a GZ polytope is cut out by \( n(n - 1) \) facets, one for each ray in \( \Sigma_{GZ}(1) \). The dimension of the space of polytopes with normal fan \( \Sigma_{GZ} \) is hence much smaller than the number of rays in the fan due to the fact that \( \Delta_\lambda \) is not a simple polytope, or equivalently, the fan \( \Sigma_{GZ} \) is not simplicial.
A third useful property of the GZ polytopes is that they behave well with respect to Minkowski addition. We recall that for polytopes \( P \) and \( Q \), we can define the Minkowski sum \( P + Q \) to be the polytope
\[
P + Q = \{ x + y \mid x \in P, y \in Q \}.
\]

**Proposition 1.6.** The assignment \( \lambda \mapsto \Delta_\lambda \) is additive, that is, for any dominant weights \( \lambda, \mu \in \mathbb{Z}^n \) we have:

\[
\Delta_{\lambda + \mu} = \Delta_\lambda + \Delta_\mu,
\]

where the addition on the right is the Minkowski addition of polytopes.

**Proof.** The inclusion \( \Delta_\lambda + \Delta_\mu \subset \Delta_{\lambda + \mu} \) is clear. We need to show the other direction. We prove this by induction on \( n \). Let \( x \in \Delta_{\lambda + \mu} \), our goal is to write \( x = x' + x'' \) with \( x' \in \Delta_\lambda \) and \( x'' \in \Delta_\mu \). We begin with the first row of inequalities, \( \lambda_1 + \mu_1 \leq x_{11} \leq \lambda_2 + \mu_2 \leq \cdots \). This can be reduced to a number of inequalities of the form
\[
0 \leq y \leq a + b
\]
with appropriate definitions of \( a, b, y \). We can then write \( y = y' + y'' \) where \( y' = y_{1 \leq n} \) and \( y'' = y_{1 \leq n} \). Note that \( 0 \leq y' \leq a \), \( 0 \leq y'' \leq b \). Now, in the top row of GZ array, we convert each \( \lambda_i + \mu_i \leq x_{1i} \leq \lambda_{i+1} + \mu_{i+1} \) into \( 0 \leq y \leq a + b \) by taking
\[
y = x_{1i} - \lambda_i - \mu_i, \quad a = \lambda_{i+1} - \lambda_i, \quad b = \mu_{i+1} - \mu_i.
\]

These quantities are all positive and satisfy the desired inequality following directly from our assumption. Thus we can write the \( x_{1i} \) into \( x_{1i} + x_{1i}' \), then continue inductively treating the \( x_{1i} + x_{1i}' \) terms as we did the \( \lambda_i + \mu_i \) to give a splitting for the next row. Thus we have shown that the assignment \( \lambda \mapsto \Delta_\lambda \) is in fact additive. \( \square \)

**Remark 1.7.** Proposition 1.6 shows that the collection of Gelfand-Zetlin polytopes is an example of a linear family of polytopes (as defined in [KV18]). In this regard, Proposition 1.1 is related to [KV18] Proposition 1.3.

Recall that a virtual polytope is a formal difference of two polytopes. The set of virtual polytopes in \( \mathbb{R}^N \) form an infinite dimensional \( \mathbb{R} \)-vector space. For a fan \( \Sigma \) in \( \mathbb{R}^N \) let \( \mathcal{P}(\Sigma) \) denote the subgroup of virtual lattice polytopes in \( \mathbb{R}^N \) generated by polytopes whose normal fan is \( \Sigma \). The group \( \mathcal{P}(\Sigma) \) contains a copy of the additive group \( \mathbb{Z}^N \) as the virtual lattice polytopes whose support function is linear on the whole \( \mathbb{R}^N \).

**Corollary 1.8.** The map \( \lambda \mapsto \Delta_{\mathrm{GZ}} \) gives a homomorphism \( \phi : \Lambda = \Lambda(\mathrm{GL}(n, \mathbb{C})) \rightarrow \mathcal{P}(\Sigma_{\mathrm{GZ}}) \). Moreover, this homomorphism induces an isomorphism \( \phi' : \Lambda' = \Lambda(\mathrm{SL}(n, \mathbb{C})) = \Lambda/\mathbb{Z}(1, \ldots, 1) \rightarrow \mathcal{P}(\Sigma_{\mathrm{GZ}})/\mathbb{Z}^N \).

**Proof.** The first assertion is an immediate corollary of Proposition 1.6. To prove the second assertion note that surjectivity of \( \phi \) follows from Proposition 1.3. The injectivity of \( \phi \) is the content of the following proposition. \( \square \)

**Proposition 1.9.** Suppose for two dominant weights \( \lambda, \lambda' \in \Lambda \) and \( c \in \mathbb{Z}^N \) we have \( c + \Delta_\lambda = \Delta_{\lambda'} \). Then \( \lambda - \lambda' \) is a multiple of \((1, \ldots, 1)\), that is, \( \lambda, \lambda' \) represent the same weight in \( \Lambda' \).

**Proof.** Let the \( x_{ij}, x'_{ij} \) denote the coordinates of points in \( \Delta_\lambda, \Delta_{\lambda'} \) respectively. Also let \( c = (c_{ij}) \). The assumption that \( c + \Delta_\lambda = \Delta_{\lambda'} \) implies that for all \( 1 \leq i \leq n - 1 \), \( \lambda_i \leq x_{1i} \leq \lambda_{i+1} \) if and only if \( \lambda'_i \leq x'_{1i} + c_{1i} \leq \lambda'_{i+1} \). It follows that \( \lambda'_i = \lambda_i + c_{1i} \) and \( \lambda'_{i+1} = \lambda_{i+1} + c_{1i} \), which in turn implies that \( c_{1i} = c_{1(i+1)} \). This finishes the proof. \( \square \)
2. Review of degrees of line bundles on toric and flag varieties

We recall that, for a projective variety $X$ of dimension $d$ embedded into $\mathbb{P}^s$,

$$\deg(X) = \#(X \cap H_1 \cap \ldots \cap H_d),$$

where the $H_i$ are generic hyperplanes in $\mathbb{P}^s$. Alternatively, let $[H]$ be the class of a hyperplane in $\text{Pic}(\mathbb{P}^s) \cong \mathbb{Z}$ and let $[H']$ be the pullback of $[H]$ to $X$ via the embedding, then $\deg(X) = [H']^d$, the self-intersection number of the divisor class $[H']$.

If the embedding $X \to \mathbb{P}^s$ is given by the sections of a very ample line bundle $\mathcal{L}$, that is, $X \to \mathbb{P}(H^0(X, \mathcal{L}^*))$, we will write $\deg(X, \mathcal{L})$ for $\deg(X)$. The asymptotic Riemann-Roch theorem, implies that

$$\deg(X, \mathcal{L}) = d! \lim_{m \to \infty} \frac{\dim H^0(X, \mathcal{L}^{\otimes m})}{m^d}.$$ 

If $\mathcal{L}$ is not very ample, we still define $\deg(X, \mathcal{L})$ as the self-intersection number of the divisor class of $\mathcal{L}$.

In the case $X = X_\Sigma$ is the toric variety of a fan $\Sigma$, we recall that all divisors are linearly equivalent to $T$-invariant divisors which in turn are generated by codimension 1 orbit closures $D_\rho = \overline{O_\rho}$, $\rho \in \Sigma(1)$. Thus an arbitrary $T$-invariant divisor on $X_\Sigma$ can be written in the form $D = \sum_\rho a_\rho D_\rho$. The associated line bundle will be $\mathcal{L} = \mathcal{O}(D)$, and the dimension of $H^0(X, \mathcal{L})$ is equal to the number of lattice points in the polytope $P_D = \{ m \mid \langle m, v_\rho \rangle \leq -a_\rho \}$ where $v_\rho$ is the primitive vector along the ray $\rho$. One can also start with a lattice polytope $P$ normal to the fan of $X_\Sigma$. The support numbers $\{ a_\rho \}_{\rho \in \Sigma(1)}$ of the polytope enable us to define a $T$-invariant divisor $\sum_\rho a_\rho D_\rho$ on $X_\Sigma$, and $P_D = P$. One shows that $D$ is ample that is, $kD$ defines an embedding into projective space for sufficiently large $k \in \mathbb{N}$. We have the following (which is a version of the well-known Bernstein-Kushnirenko theorem):

**Proposition 2.1.** Let $\mathcal{L}_D$ be the line bundle associated to the divisor $D_P$. Then:

$$\deg(X_\Sigma, \mathcal{L}_D) = d! \text{Vol}_d(P).$$

**Proof.** By the asymptotic Riemann-Roch we have:

$$\deg(X_\Sigma, \mathcal{L}_D) = d! \lim_{m \to \infty} \frac{\dim H^0(X_\Sigma, \mathcal{L}_D^{\otimes m})}{m^d} = d! \lim_{m \to \infty} \frac{\#(mP \cap \mathbb{Z}^d)}{m^d} = d! \text{Vol}_d(P).$$

As we are interested in comparing $X_{GZ}$ with the flag variety $G/B$, we also recall some facts about degrees of embeddings for $\mathcal{F}_{\ell_n}$. Recall that to a weight $\lambda$ one associates a line bundle $\mathcal{L}_\lambda$ on $\mathcal{F}_{\ell_n}$. This line bundle satisfies the property

$$\mathcal{L}_\lambda^{\otimes m} = \mathcal{L}_{m\lambda}.$$

An argument similar to the proof of Proposition 2.1 shows the following (see for example [Kav11], Remark 2.4).

**Proposition 2.2.** For any dominant regular weight $\lambda$ we have:

$$\deg(\mathcal{F}_{\ell_n}, \mathcal{L}_\lambda) = N! \text{Vol}_N(\Delta_\lambda),$$

where $N = n(n-1)/2 = \dim(\mathcal{F}_{\ell_n})$.  

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3. Review of Intersection Theory on Toric and Flag Varieties

In this section we recall some basic facts about Chow rings and Chow cohomology rings of toric and flag varieties.

For an algebraic variety $X$ and $1 \leq k \leq n = \dim(X)$, the $k$-th Chow group $A_k(X)$ is the group generated by algebraic $k$-cycles on $X$, that is, formal sums of irreducible $k$-dimensional subvarieties in $X$, modulo rational equivalence. Two $k$-cycles are are called rational equivalent if their difference is the divisor of a rational function on a $(k + 1)$-dimensional subvariety. The total Chow group of $X$ is $A^*(X) = \bigoplus_{k=0}^{n} A_k(X)$. When $X$ is smooth we let $A^k(X) = A_{n-k}(X)$ and $A^*(X) = \bigoplus_{k=0}^{n} A^k(X)$. In this case, the transverse intersection of subvarieties gives a well-defined multiplication on $A^*(X)$ making it into a graded algebra (see [Ful13] Proposition 8.3) called the Chow ring of $X$. More generally, for a commutative ring $R$, one can define the Chow groups $A_k(X, R)$ and the Chow ring $A^*(X, R)$ for $X$ is smooth.

In general, for a smooth variety $X$, the cohomology ring $H^*(X)$ and the Chow ring $A^*(X)$ are different. Nevertheless, for some nice varieties $X$ these algebras are naturally isomorphic.

**Theorem 3.1** ([Ful13] Example 19.1.11). Suppose $X$ has a paving by affine cells then $H^*(X)$ and $A^*(X)$ are naturally isomorphic.

The above theorem in particular applies to smooth toric varieties and the flag variety $F\ell_n$.

When $X = X_\Sigma$ is a smooth complete toric variety, there is a nice description of the Chow ring $A^*(X_\Sigma)$. In this case, for each $k$, the Chow group $A_k(X_\Sigma) = A_{n-k}(X_\Sigma)$ is generated by the orbit closures of codimension $k$. Moreover, one has the following well-known result (see [Ful93] Section 5.2).

**Theorem 3.2.** Let $X_\Sigma$ be a smooth complete toric variety. Let $D_1, \ldots, D_r$ be the codimension 1 orbit closures corresponding to rays $\rho_1, \ldots, \rho_r \in \Sigma(1)$. Then $A^*(X_\Sigma) \cong H^*(X_\Sigma) \cong \mathbb{Z}[D_1, \ldots, D_r]/I$ where $I$ is the ideal generated by the following relations:

1. $D_{i_1} \cdots D_{i_k}$ for all $v_{i_1}, \ldots, v_{i_k}$ not contained in any cone of $\Sigma$ and $u$.
2. $\sum_{i=1}^{d} (u, v_i)D_i$ for all $u \in M$.

There is also a nice description of the ring $A^*(F\ell_n) \cong H^*(F\ell_n)$ due to Borel. For each weight $\lambda$ let $c_1(L_\lambda)$ be the divisor class (Chern class) of the line bundle $L_\lambda$ on $F\ell_n$.

**Theorem 3.3.** We have the following:

1. The map $\lambda \mapsto c_1(L_\lambda)$ gives an isomorphism of $A^1(F\ell_n) = \text{Pic}(F\ell_n)$ with the weight lattice $\Lambda = \Lambda(SL(n, \mathbb{C})) = \Lambda/\mathbb{Z}(1, \ldots, 1)$.
2. $A^*(F\ell_n)$ is generated, as an algebra, by $c_1(L_\lambda), \lambda \in \Lambda$.
3. $A^*(F\ell_n) \cong \text{Sym}(\Lambda^*)/I_W$ where $I_W$ is the ideal generated by non-constant $W$-invariants.

Alternatively, $H^*(F\ell_n, \mathbb{Q})$ can be viewed as the polytope algebra of the Gelfand-Zetlin family (see [Kav11] Corollary 5.3]). There it is shown that $H^*(F\ell_n) \cong \text{Sym}(\Lambda_{\mathbb{Q}})/I$ where $I$ is the ideal of polynomials which, when viewed as differential operators annihilate the volume polynomial of the Gelfand-Zetlin polytope.

We note that the toric variety $X_{GZ}$ is not smooth except when $n = 1, 2$ and hence we need a more general notion of Chow ring that applies to non-smooth varieties as well. For a (not necessarily smooth) variety $X$ in [FMS1], Fulton and MacPherson construct a variant of

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Chow ring called the operational Chow ring or Chow cohomology ring $A^*(X) = \bigoplus_{k=0}^{\infty} A^k(X)$. When $X$ is smooth it coincides with the usual Chow ring. When $X = X_\Sigma$ is a complete toric variety one has $A^k(X_\Sigma) = \text{Hom}(A_k(X_\Sigma), \mathbb{Z})$. Moreover, the ring $A^*(X_\Sigma)$ can be described purely in terms of combinatorial data of Minkowski weights, which are certain integer valued functions on the fan $\Sigma$. The next section is devoted to this combinatorial description.

4. Minkowski weights

In this section we recall the description of the Chow cohomology ring of a toric variety in terms of Minkowski weights (see [FS97], see also [Kaz03]). Let $\Sigma$ be a complete fan. Recall that $\Sigma(k)$ is the set of cones of dimension $k$ in $\Sigma$.

Definition 4.1. A function $c : \Sigma(n-k) \to \mathbb{Z}$ is a Minkowski weight if it satisfies a balancing condition

$$
\sum_{\sigma \in \Sigma(n-k), \sigma \supset \tau} \langle u, n_{\sigma, \tau} \rangle c(\sigma) = 0
$$

where $n_{\sigma, \tau}$ is a lattice point in $\sigma$ which generates $N_\sigma/N_\tau$, the quotient of the lattices spanned by $\sigma$ and $\tau$. The above equation must be satisfied for all $u \in M(\tau)$, the lattice perpendicular to the span of $\tau$.

Let $MW^k$ denote the set of Minkowski weights on cones of codimension $k$. For two Minkowski weights, $c \in MW^p$ and $\hat{c} \in MW^q$, the product $c \cup \hat{c} \in MW^{p+q}$ is given by

$$(c \cup \hat{c})(\gamma) = \sum_{(\sigma, \tau) \in \Sigma(n-p) \times \Sigma(n-q)} m_{\sigma, \tau}^\gamma c(\sigma) \hat{c}(\tau)$$

where $\gamma$ is a cone of codimension $p+q$, and $m_{\sigma, \tau}^\gamma = \lceil N_\sigma + N_\tau \rceil$ and the sum is over all pairs of cones $(\sigma, \tau)$ which both contain $\gamma$ and such that $\sigma$ meets $\tau + v$ for fixed generic vector $v$ (see [FS97] Theorem 4.2).

In [FS97] an isomorphism between the ring of Minkowski weights and the operational Chow ring of a complete toric variety $X_\Sigma$ is given. In fact it is shown that $MW^k \cong A^k(X_\Sigma)$ (see [FS97] Theorem 3.1)). In particular:

$$\text{Pic}(X_\Sigma) \cong A^1(X_\Sigma)$$

Example 4.2 (Hypersimplex). The following is an example of a variety where the ring $MW^*$ is not generated by $MW^1$ (see [FS97] Example 3.5 or [KP08] Example 4.2). We consider the fan $\Sigma_H$ over the cube in $\mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$, then consider the ring of Minkowski weights for the toric variety $X_{\Sigma_H}$. The rays in the fan $\Sigma_H$ will be notated as follows:

$$
\begin{align*}
\rho_1 &= (1,1,1) & \rho_5 &= -\rho_1 \\
\rho_2 &= (1,-1,1) & \rho_6 &= -\rho_2 \\
\rho_3 &= (1,1,-1) & \rho_7 &= -\rho_3 \\
\rho_4 &= (-1,1,1) & \rho_8 &= -\rho_4
\end{align*}
$$

The 2-dimensional spanned by $\rho_1$ and $\rho_2$ will be denoted $\sigma_{ij}$, and similarly the 3-dimensional cone spanned by $\rho_1$, $\rho_2$ and $\rho_3$ will be denoted $\sigma_{ijk}$.

We first show that $MW^1 \cong \mathbb{Z}$. We recall that a weight $c \in MW^1$ is a function on cones of codimension 1, which in this example will have dimension 2. Let

$$c(\sigma_{ij}) = c_{ij}$$
for each cone $\sigma_{ij}$. We will have a relation as in Equation (4) for each ray $\rho_i$. As $\Sigma_H$ is the fan over a cube, without loss of generality we can consider the equation for the ray $\rho_1$ and by symmetry draw conclusions about the relations corresponding to other rays. Each ray is contained in exactly three cones of dimension 2. For $\rho_1$, the balancing condition will involve the cones $\sigma_{12}, \sigma_{13}$ and $\sigma_{14}$. The other ingredients we require are the vectors $n_{\sigma_{1i}, \rho_1}$ for $i = 2, \ldots, 4$. Again, appealing to symmetry, it will be enough to understand $n_{12}$.

Recall $n_{12}$ is the lattice point in $\sigma_{12}$ which generates the lattice $N_{\sigma_{12}}/N_{\rho_1}$, so since the rays are orthogonal, we can take $n_{12} = \rho_2$ and similarly $n_{1i} = \rho_i$. It is enough to consider the balancing equations for $u \in \{(1,0,-1), (0,1,-1)\}$ as these vectors for a basis for the lattice $M(\rho_1)$ orthogonal to $\rho_1$. We obtain:

$$0 = \sum_{i=2}^4 \langle (1,0,-1), \rho_i \rangle c_{11} = 2c_{12} - 2c_{14}$$

$$0 = \sum_{i=2}^4 \langle (0,1,-1), \rho_i \rangle c_{11} = 2c_{12} - 2c_{13}.$$

Thus the balancing equations associated with $\rho_1$ imply that the value of $c$ on all 2-dimensional cones is the same. The symmetry of our cone implies that this same computation can be done for any other ray, and hence the value of $c$ on all 2-dimensional cones is the same and thus $MW^1 \cong \mathbb{Z}$.

To show that $MW^*$ is not generated by $MW^1$, it is enough to show that rank $MW^2 > 1$. To prove this, consider $c \in MW^2$, then $c$ is a function on rays of $\Sigma_H$. As usual, let $c(\rho_i) = c_i$. The balancing condition will correspond to the 0-dimensional origin cone, and is given by

$$\sum_{i=1}^8 c_i \rho_i = 0.$$

As this equation is a 3-dimensional vector equation, our 8 values $\{c_1, \ldots, c_8\}$ must satisfy at most 3 additonal equations, hence rank $MW^2 \geq 5$. It can in fact be shown that these equations are independent and that $MW^2 \cong \mathbb{Z}^5$ and thus cannot be generated by products of elements of $MW^1$.

5. SOME ALGEBRA LEMMAS

Let $A = \bigoplus_{i=0}^n A^i$ be a graded ring over a field $k$ which is finite dimensional as a $k$-vector space and $A^0 \cong A^n \cong k$. Then following [HW17], we call the graded subalgebra of $A$ generated by $k$, the Lefschetz subalgebra and denote it by $L_A$. We recall that $A$ has Poincaré duality if the multiplication maps

$$A^i \times A^{n-i} \to A^n \cong k$$

are non-degenerate for all $i$. Our goal is to compare $A^*(F\ell_n) \cong H^*(F\ell_n)$, which has Poincaré duality, with the algebra $A^*(X_{G2})$, which in general does not. We start by observing how to get a Poincaré duality algebra from a general graded algebra.

**Lemma 5.1.** Let $A = \bigoplus_{i=0}^n A^i$ with $A^0 \cong A^n \cong k$. There exists a homogeneous ideal $I \subset A$ which is minimal with respect to inclusion such that $A/I$ has Poincaré duality.

**Proof.** Consider the ideal $I$ generated by all the homogeneous elements $x \in A$ such that

$$x \cdot A^{n-\deg(x)} = 0.$$
First let us see that the $n$-th graded piece of $I$ is $\{0\}$. Let $z \in I$ with deg $z = n$, then we must have $z = \sum c_i x_i$ where the $x_i$ are generators of $I$ with deg $x_i = d_i$ and deg $c_i = n - d_i$. By assumption the $x_i$ satisfy $x_i A^{n-d_i} = 0$, so $c_i x_i = 0$ for all $i$ and hence $z = 0$. Suppose for contradiction that $A/I$ does not have Poincaré duality, then there is $0 < i < n$ and $0 \neq x \in A^i \setminus I$, such that for all $y \in A^{n-i}$ we have $xy \in I$. As the degrees of $x$ and $y$ are complementary, deg $xy = n$, so these products lie in the $n$-th graded piece of the ideal $I$.

We have shown that the degree $n$ part of $I$ is trivial, and hence $xy \in I$ with deg $xy = n$ implies that $xy = 0$. This implies that $x \in I$ which is a contradiction. Thus $A/I$ does have Poincaré duality. We next show that $I$ is the minimal such homogeneous ideal. Suppose not, then there exists homogeneous ideal $J$ such that $A/J$ has Poincaré duality and also nonzero $x \in (I \setminus J)$.

We call the algebra $A/I$ in Lemma 5.2. the Gorenstein quotient $\text{Gor}(A)$ of $A$. We next recall a useful algebra fact (see [Kav11, Theorem 1.1] and [Eis95, Exercise 21.7]) which we will need later. For the sake of completeness we include a proof here.

**Lemma 5.2.** Let $A = \oplus_{i=0}^{n} A^i$ be a finite dimensional graded algebra over a field $k$ which is generated by $A^1$, satisfies $A^0 \cong k \cong A^n$, and has Poincaré duality.

Fix a basis $\{a_1, \ldots, a_r\}$ for $A^1$, and consider the polynomial $P : k^r \rightarrow k$ defined by

$$P(x_1, \ldots, x_r) = (x_1 a_1 + \cdots + x_r a_r)^n \in A^n \cong k.$$ 

Then we obtain an isomorphism of graded algebras

$$A \cong k[\partial_1, \ldots, \partial_r]/I$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and $I$ is the ideal of polynomials in the operators $\partial_1, \ldots, \partial_r$ which annihilate $P$.

**Proof.** We follow the sketch outlined in [Kav11]. Consider the evaluation homomorphism

$$\Phi : k[t_1, \ldots, t_r] \rightarrow A$$

under which $t_i \mapsto a_i$. Since $A$ is generated by $A^1$, this map is clearly surjective. We aim to show that $\ker \Phi = I$, so that we will have $A \cong k[t_1, \ldots, t_r]/I$. Since $\Phi$ respects the degree, $\ker \Phi$ is a homogeneous ideal, i.e., is generated by homogeneous elements. Now take $f \in k[t_1, \ldots, t_r]$ homogeneous of degree $n$,

$$f(t_1, \ldots, t_r) = \sum_{\beta_1 + \cdots + \beta_r = n} c_{\beta_1, \ldots, \beta_r} t_1^{\beta_1} \cdots t_r^{\beta_r},$$

where $\beta_i$ are nonnegative integers.
Then
\[
\begin{align*}
    f(\partial_1, \ldots, \partial_r) \cdot P &= \left( \sum_{\beta_1 + \ldots + \beta_r = n} c_{\beta_1, \ldots, \beta_r} a_1^{\beta_1} \cdots a_r^{\beta_r} \right) \cdot (x_1 a_1 + \ldots + x_r a_r)^n \\
    &= \left( \sum_{\beta_1 + \ldots + \beta_r = n} c_{\beta_1, \ldots, \beta_r} \partial_1^{\beta_1} \cdots \partial_r^{\beta_r} \right) \cdot \left( \sum_{\alpha_1 + \ldots + \alpha_r = n} \left( a_1^{\alpha_1}, \ldots, a_r^{\alpha_r} \right) \partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r} \cdot (x_1^{\alpha_1} \cdots x_r^{\alpha_r}) \right) \\
    &= \sum_{\beta_1 + \ldots + \beta_r = n} \sum_{\alpha_1 + \ldots + \alpha_r = n} c_{\beta_1, \ldots, \beta_r} a_1^{\alpha_1} \cdots a_r^{\alpha_r} \left( \partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r} \cdot (x_1^{\alpha_1} \cdots x_r^{\alpha_r}) \right) \\
    &= \sum_{\beta_1 + \ldots + \beta_r = n} c_{\beta_1, \ldots, \beta_r} a_1^{\beta_1} \cdots a_r^{\beta_r} \left( \frac{n! \beta_1! \cdots \beta_r!}{\alpha_1! \ldots \alpha_r! (\alpha_1 - \beta_1)! \ldots (\alpha_r - \beta_r)!} \right) (x_1^{\alpha_1 - \beta_1} \cdots x_r^{\alpha_r - \beta_r}) \\
    &= n! f(a_1, \ldots, a_r).
\end{align*}
\]

From this we see that \( f(a_1, \ldots, a_r) = 0 \), i.e. \( f \in \ker \Phi \), if and only if \( f \) annihilates \( P \) so \( f \in I \). It remains to show that the same holds for \( f \) homogeneous of degree \( m < n \). Let \( f(t_1, \ldots, t_r) = \sum_{\beta_1 + \ldots + \beta_r = m} c_{\beta_1, \ldots, \beta_r} t_1^{\beta_1} \cdots t_r^{\beta_r} \). Suppose first that \( f \notin \ker \Phi \), that is, \( f(a_1, \ldots, a_r) \neq 0 \). Since \( A \) has Poincaré duality and \( f(a_1, \ldots, a_r) \in A^m \) there must be some \( a' \in A^{m-n} \) such that \( a' \cdot f(a_1, \ldots, a_r) \neq 0 \). As \( A \) is generated in degree one, there is a homogeneous polynomial \( g \) of degree \( n-m \) which gives this element \( a' \). Then \( gf \) is a nonzero homogeneous polynomial of degree \( n \), and the above computation shows that \( (gf)(\partial_1, \ldots, \partial_r) \cdot P = n! (gf)(a_1, \ldots, a_n) \neq 0 \). Then \( f(\partial_1, \ldots, \partial_r) \cdot P \) cannot be zero, so \( f \) is not in \( I \). Thus we have shown that \( f \) in \( I \) implies \( f \) in \( \ker \Phi \).

Conversely suppose that \( f(a_1, \ldots, a_r) = 0 \), so \( f \in \ker \Phi \). Then
\[
\begin{align*}
    f(\partial_1, \ldots, \partial_r) \cdot P &= \sum_{\beta_1 + \ldots + \beta_r = n} \sum_{\alpha_1 + \ldots + \alpha_r = n} c_{\beta_1, \ldots, \beta_r} a_1^{\alpha_1} \cdots a_r^{\alpha_r} \partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r} \cdot (x_1^{\alpha_1} \cdots x_r^{\alpha_r}) \\
    &= \sum_{\beta_1 + \ldots + \beta_r = n} \sum_{\alpha_1 + \ldots + \alpha_r = n} c_{\beta_1, \ldots, \beta_r} a_1^{\alpha_1} \cdots a_r^{\alpha_r} \left( \frac{\alpha_1! \ldots \alpha_r!}{(\alpha_1 - \beta_1)! \ldots (\alpha_r - \beta_r)!} \right) (x_1^{\alpha_1 - \beta_1} \cdots x_r^{\alpha_r - \beta_r}) \\
    &= \sum_{\gamma_1 + \ldots + \gamma_r = n-m} \sum_{\beta_1 + \ldots + \beta_r = m} c_{\gamma_1, \ldots, \gamma_r} a_1^{\gamma_1} \cdots a_r^{\gamma_r} (\gamma_1 \cdots \gamma_r) (x_1^{\gamma_1} \cdots x_r^{\gamma_r}) \\
    &= f(a_1, \ldots, a_r) \left( \sum_{\gamma_1 + \ldots + \gamma_r = n-m} a_1^{\gamma_1} \cdots a_r^{\gamma_r} (\gamma_1 \cdots \gamma_r) (x_1^{\gamma_1} \cdots x_r^{\gamma_r}) \right) \\
    &= 0,
\end{align*}
\]

thus \( f \) is in the ideal \( I \).
We now use Lemma 5.2 to prove the following key lemma required in the proof of our main result (Theorem 6.1).

**Lemma 5.3.** Suppose $A = \oplus_{i=0}^n A^i$ and $B = \oplus_{i=0}^n B^i$ are $k$-algebras which are finite dimensional $k$-vector spaces and have the following properties:

1. $A^1 \cong A^n \cong B^0 \cong B^n \cong k$.
2. $A$ and $B$ are generated in degree one.
3. $A$ has Poincaré duality.
4. There exists a linear isomorphism $\varphi : A^1 \to B^1$ such that for all $a_1, \ldots, a_n \in A^1$ we have:

$$a_1 \cdot \ldots \cdot a_n = \varphi(a_1) \cdot \ldots \cdot \varphi(a_n)$$

using fixed isomorphisms $A^n \cong k \cong B^n$.

Then $\varphi$ extends to give a $k$-algebra isomorphism $\tilde{\varphi}$ between $A$ and the Gorenstein quotient of $B$.

**Proof.** We apply Lemma 5.2 to $A$ and to the Gorenstein quotient $\text{Gor}(B)$. It is clear that $A$ already satisfies the conditions of Lemma 5.2 so $A \cong k[\partial_1, \ldots, \partial_r]/I$ where $r = \dim_k(A^1)$ and ideal $I$ is the annihilator of the top power polynomial $P$ described in Lemma 5.2.

We need to show that $\text{Gor}(B)$ also satisfies these conditions. First note that $B^0 \cong k \cong B^n$ so the multiplication $B^0 \times B^n \to B^n \cong k$ is already non-degenerate and thus the ideal $I$ in Lemma 5.1 contains neither $B^0$ nor $B^n$. This gives us $\text{Gor}(B)^0 \cong k \cong \text{Gor}(B)^n$. Also, by construction $\text{Gor}(B)$ has Poincaré duality. Finally, $\text{Gor}(B)$ is generated in degree one since $B$ is generated in degree 1. Now consider the map on degree one pieces:

$$A^1 \cong B^1 \xrightarrow{q} \text{Gor}(B)^1,$$

where $q$ is the map in the construction of the Gorenstein quotient. Call this composition $\tilde{\varphi} : A^1 \to \text{Gor}(B)^1$. We claim $\tilde{\varphi}$ is an isomorphism. Since $\varphi$ is an isomorphism and $q$ is surjective, $\tilde{\varphi}$ is surjective and we only need to verify injectivity. Suppose for contradiction that some nonzero $a \in A^1$ has image $\tilde{\varphi}(a) = q(\varphi(a)) = 0$ in $\text{Gor}(B)^1$. Since $\varphi$ is an isomorphism, $\varphi(a) = b$ for some nonzero $b \in B^1$. Then $b$ is in the ideal in Lemma 5.1 so it is a linear combination of $x_i$ satisfying $x_i \cdot B^{n-\deg(x_i)} = 0$. Since $b \in B^1$, the $x_i$ must be in degree 0 or 1. We argued above that $B^0 \cap I = \{0\}$, so we can only have $x_i \in B^1$. It follows that $b \cdot B^{n-1} = 0$. But the assumption (4) then implies that $a \cdot A^{n-1} = 0$ which contradicts that $A$ has Poincaré duality. Thus $\text{Gor}(B)$ satisfies the conditions required for Lemma 5.2 and hence $\text{Gor}(B) \cong k[\partial_1, \ldots, \partial_r]/I$. We have already seen that $A$ is isomorphic to this quotient algebra and thus $A \cong \text{Gor}(B)$. \hfill \Box

6. Main theorem

We now state and prove our main theorem relating the cohomology ring of the flag variety $\mathcal{F}_n$ and the Chow cohomology ring of the toric variety $X_{\text{GZ}}$.

**Theorem 6.1.** For $X_{\text{GZ}}$ the toric variety associated to GZ fan $\Sigma \subset \mathbb{R}^N$ and the flag variety $\mathcal{F}_n$, the Chow cohomology ring $A^*(\mathcal{F}_n, \mathbb{Q})$ can be identified with the Gorenstein quotient of the Lefschetz subalgebra of $A^*(X_{\text{GZ}}, \mathbb{Q})$.

**Proof.** We claim that there is an isomorphism of groups $A^1(\mathcal{F}_n) \cong A^1(X_{\text{GZ}})$. One knows that $A^1(\mathcal{F}_n) = A_{N-1}(\mathcal{F}_n) = \text{Pic}(\mathcal{F}_n) \cong \Lambda(\text{SL}(n, \mathbb{C})) = \Lambda(\text{GL}(n, \mathbb{C}))/\mathbb{Z}(1, \ldots, 1)$.

One also knows that for a complete toric variety $X_\Sigma$, where $\Sigma$ is a fan in $\mathbb{R}^N$, the Chow cohomology group $A^1(X_\Sigma)$ is naturally isomorphic to $\text{Pic}(X_\Sigma)$ (see [FS97, Corollary 3.4]).
Moreover, it is well-known that Pic\((X_\Sigma)\) is isomorphic to the algebra \(\text{PL}(\Sigma, \mathbb{Z}^N)\) of integer piecewise linear functions on \(\Sigma\) modulo integer linear functions. The algebra \(\text{PL}(\Sigma, \mathbb{Z}^N)\) consists of functions on \(\mathbb{R}^N\) that \(f\) are linear on each cone in \(\Sigma\) and have integer values on \(\mathbb{Z}^N\). This algebra contains a copy of \(\mathbb{Z}^N\) as the subgroup of all linear functions from \(\mathbb{Z}^N\) to \(\mathbb{Z}\). Equivalently, \(\text{PL}(\Sigma, \mathbb{Z}^N)\) can be identified with the quotient group \(\mathcal{P}(\Sigma) / \mathbb{Z}^N\). Recall that \(\mathcal{P}(\Sigma)\) is the subgroup of virtual lattice polytopes generated by lattice polytopes whose normal fan is \(\Sigma\). It naturally contains \(\mathbb{Z}^N\) as a subgroup (see paragraph before Corollary 1.8). From Corollary 1.8 it now follows that Pic\((\mathcal{F}_{\ell_n})\) \(\cong \Lambda' \cong \text{PL}(\Sigma, \mathbb{Z}^N) \cong \text{Pic}(X_{GZ})\) as required.

One knows that for an \(N\)-dimensional toric variety \(X_\Sigma\), under the isomorphism \(A^1(X_\Sigma) \cong \text{Pic}(X_\Sigma)\) the top product of an element in \(A^1(X_\Sigma)\) coincides with the self-intersection number of the corresponding divisor in Pic\((X_\Sigma)\). From Propositions 2.1 and 2.2 we now conclude that the this isomorphism respects the multiplication, i.e. it satisfies the assumption (4) in Lemma 5.3. Alternatively this can be deduced from [JY16, Theorem 4.3 and Corollary 4.5]. Applying Lemma 5.3 to \(A = A^*(\mathcal{F}_{\ell_n})\) and \(B = \) the Lefschetz subalgebra of \(A^*(X_{GZ})\) finishes the proof.

\[\Box\]

7. GELFAND-ZETLIN EXAMPLE, \(n = 3\)

In this section we compute the Chow cohomology ring of \(X_{GZ}\) for \(n = 3\) using the Minkowski weights. We consider the GZ polytope of the weight \(\lambda = (-1, 0, 1)\) for ease of computation. The polytope \(\Delta_\lambda\) is defined by the following array of inequalities

\[
\begin{array}{ccc}
-1 & 0 & 1 \\
x & y & z
\end{array}
\]

and has normal fan \(\Sigma_{GZ}\) as in Figure 2. We enumerate the rays as follows:

\[
\begin{align*}
\rho_1 &= (1, 0, 0) & \rho_3 &= (0, 1, 0) & \rho_5 &= (1, 0, -1) \\
\rho_2 &= (-1, 0, 0) & \rho_4 &= (0, -1, 0) & \rho_6 &= (0, -1, 1).
\end{align*}
\]

![Figure 2. Rays of \(\Sigma_{GZ}\) for \(n = 3\).](image)
Likewise, we let $\sigma_{ij}$ denote the 2-dimensional cone spanned by rays $\rho_i$ and $\rho_j$.

$$
\begin{align*}
\sigma_{13} & \quad \sigma_{23} & \quad \sigma_{24} \\
\sigma_{15} & \quad \sigma_{25} & \quad \sigma_{35} & \quad \sigma_{45} \\
\sigma_{16} & \quad \sigma_{26} & \quad \sigma_{36} & \quad \sigma_{46}
\end{align*}
$$

Similarly, the collection of 3-dimensional cones are:

$$
\begin{align*}
\gamma_{135} & \quad \gamma_{235} & \quad \gamma_{245} & \quad \gamma_{1456} \\
\gamma_{136} & \quad \gamma_{236} & \quad \gamma_{246}
\end{align*}
$$

We now determine $MW^k$ for each value $k = 0, \ldots, 3$, as these are the only codimensions in the fan $\Sigma$. We first compute $MW^3$. There is a single cone of codimension 3, namely, the origin. Then a Minkowski weight on $\Sigma(3)$ is a map $0 \to \mathbb{Z}$, and there are no cones $\tau \subset 0$, thus no relations to satisfy. Hence

$$MW^3 \cong \mathbb{Z}.$$ 

(6)

We next determine $MW^2$. A weight $c \in MW^2$ is a function on cones of codimension 2, i.e., on rays $\rho_i$. Let $c(\rho_i) = c_i$, then the single relation coming from the cone $\tau = 0$ which is a subcone of all $\rho_i$ is given by

$$\sum_{i=1}^{6} c_i \rho_i = 0$$

as the positive generator of the lattice $N_{\rho_i}/N_0$ is just the ray $\rho_i$, and the lattice orthogonal to 0 is the entire lattice. Expanding this equation in terms of our basis, we get three relations:

$$
\begin{align*}
c_1 - c_2 + c_5 &= 0 \\
c_3 - c_4 - c_6 &= 0 \\
-c_5 + c_6 &= 0.
\end{align*}
$$

(7)

We see from this that any weight $c \in MW^2$ is determined by its value on three rays, say, $c(\rho_1) = a$, $c(\rho_4) = b$ and $c(\rho_5) = c$, then

$$
\begin{align*}
c(\rho_1) &= a - c & c(\rho_2) &= a & c(\rho_3) &= b + c & c(\rho_5) &= c \\
c(\rho_4) &= b & c(\rho_6) &= c.
\end{align*}
$$

(8)

Thus $MW^2 \cong \mathbb{Z}^3$.

Next, we examine $MW^1$. These are functions on codimension 1 cones $\sigma_{ij}$. Let $c \in MW^1$ and suppose the value on cone $\sigma_{ij}$ is $c(\sigma_{ij}) = c_{ij}$. Then a weight of codimension 1 is given by the data

$$
\begin{align*}
c_{13} & \quad c_{23} & \quad c_{24} \\
c_{15} & \quad c_{25} & \quad c_{35} & \quad c_{45} \\
c_{16} & \quad c_{26} & \quad c_{36} & \quad c_{46}
\end{align*}
$$

subject to relations coming from the rays $\{\rho_i\}$.

First, the relation for $\tau = \rho_1$ involves the cones $\sigma_{13}, \sigma_{15}$ and $\sigma_{16}$. For each of these, we need to compute $n_{\sigma \tau}$, the lattice point in $\sigma$ which generates the one-dimensional lattice $N_\sigma/N_{\rho_1}$. The relation will be a vector equation in the vector space perpendicular to $\rho_1 = (1,0,0)$. We compute:

$$
\begin{align*}
n_{13} &= (0,1,0), & n_{15} &= (0,0,-1), & n_{16} &= (0,-1,1)
\end{align*}
$$

where all vectors are considered modulo $\rho_1$. The relation equation becomes

$$c_{13} \cdot (0,1,0) + c_{15} \cdot (0,0,-1) + c_{16} \cdot (0,1,-1) = 0.$$
which implies
\[ c_{13} = c_{15} = c_{16}. \]

Similar computations for the other rays yield the following results:
\[ c_{13} = c_{15} = c_{16} = c_{25} = c_{26} \]
\[ c_{24} = c_{35} = c_{36} = c_{45} = c_{46} \]
\[ c_{23} = c_{13} + c_{24} \]

For later computations, we will let \( a \) and \( b \) be the generators of \( MW^1 \cong \mathbb{Z}^2 \), that is,
\[ a = c_{13} = c_{15} = c_{16} = c_{25} = c_{26} \]
\[ b = c_{24} = c_{35} = c_{36} = c_{45} = c_{46} \]
\[ c_{23} = a + b. \]

We now examine \( MW^0 \). A weight \( c \in MW^0 \) is a function on top-dimensional cones subject to relations coming from each 2-dimensional cone. Each 2-dimensional cone \( \sigma_{ij} \) separates two top-dimensional cones, and the corresponding relation gives equality between the values of \( c \) on each pair of top-dimensional cones. Hence \( MW^0 \cong \mathbb{Z} \) as the value of \( c \) on each 3-dimensional cone must be the same. In summary, we have the following:
\[ MW^0 \cong \mathbb{Z} \]
\[ MW^1 \cong \mathbb{Z}^2 \]
\[ MW^2 \cong \mathbb{Z}^3 \]
\[ MW^3 \cong \mathbb{Z}. \]

Before understanding the product structure on \( MW^* \), it is already clear that the ring cannot have Poincaré duality as the rank of \( MW^2 \) is greater than \( MW^1 \).

Our next goal is to understand the product structure on \( MW^* (X_{GZ}) \). For weights \( c \in MW^p \) and \( \tilde{c} \in MW^q \), their product is a function on cones of codimension \( p + q \), and its value on a cone \( \gamma \in MW^{p+q} \) is given by
\[
(c \cup \tilde{c})(\gamma) = \sum_{\sigma, \tau \in \Sigma(n-p) \times \Sigma(n-q)} m_{\sigma \tau}^\gamma \cdot c(\sigma) \cdot \tilde{c}(\tau),
\]
where \( m_{\sigma \tau}^\gamma \) is \([N : N_\sigma + N_\tau]\) as long as

(a) \( \sigma, \tau \ni \gamma \)

(b) \( \sigma \) meets \( \tau + v \) for a generic fixed \( v \in N \)

otherwise \( m_{\sigma \tau}^\gamma = 0 \). Recall also that \( \Sigma(n-p) \) is the set of cones in \( \Sigma \) of dimension \( n-p \).

Our goal is to compute products of Minkowski weights in our example to determine whether \( MW^* (X_{GZ}) \) is generated in degree 1. To this end, let \( c, \tilde{c} \in MW^1 (X_{GZ}) \), such that
\[ c : \{\sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{25}, \sigma_{26}\} \mapsto a \]
\[ c : \{\sigma_{24}, \sigma_{35}, \sigma_{36}, \sigma_{45}, \sigma_{46}\} \mapsto b \]
\[ c : \{\sigma_{23}\} \mapsto a + b \]
\[ \tilde{c} : \{\sigma_{13}, \sigma_{15}, \sigma_{16}, \sigma_{25}, \sigma_{26}\} \mapsto \tilde{a} \]
\[ \tilde{c} : \{\sigma_{24}, \sigma_{35}, \sigma_{36}, \sigma_{45}, \sigma_{46}\} \mapsto \tilde{b} \]
\[ \tilde{c} : \{\sigma_{23}\} \mapsto \tilde{a} + \tilde{b}. \]
Then \(c \cup \tilde{c} \in MW^2\) will be evaluated on cones of codimension 2, i.e., rays. From the arguments above, see Equation (8), it is enough to determine the value of this weight on the rays \(\rho_2, \rho_4\) and \(\rho_5\).

We begin by examining \((c \cup \tilde{c})(\rho_2)\) via Equation (9). Recall that this involves looking at all pairs \((\sigma, \tau) \in \Sigma(2) \times \Sigma(2)\) where \(\sigma\) and \(\tau\) both contain \(\rho_2\) where \(\sigma\) meets \(\tau + v\) for a generic fixed \(v \in N\). The cones in \(\Sigma(2)\) which contain \(\rho_2\) are \(\{\sigma_{23}, \sigma_{24}, \sigma_{25}, \sigma_{26}\}\), so \(\sigma, \tau\) will come from this collection. Since all these cones involve \(\rho_2 = (-1, 0, 0)\), we can sketch the relevant cones in the \(yz\)-plane where for example \(\sigma_{23}\) can be viewed as \(\rho_2 = (1, 0)\).

![Figure 3. Intersection of \(\sigma\) and \(\tau + v\)](image)

In Figure 3 we see the cones for \(c\) in blue, and for \(\tilde{c}\) in green using a shift of \(v = (1, 1, 1)\). Then there are two pairs \((\sigma, \tau)\) which meet for this vector \(v\), either \((\sigma, \tau) = (\sigma_{23}, \sigma_{25})\) or \((\sigma, \tau) = (\sigma_{26}, \sigma_{24})\). The last ingredient required to compute this product are the coefficients \(m_{\sigma\tau}\rho_2\) for the sum.

Recall \(m_{\sigma\tau}^+ \) is \(\lfloor N : N_\sigma + N_\tau \rfloor\). In both cases, \(N_\sigma + N_\tau = N\) so \(m_{\sigma\tau}^+ = 1\). Thus we have

\[
(c \cup \tilde{c})(\rho_2) = c(\sigma_{23})\tilde{c}(\sigma_{25}) + c(\sigma_{26})\tilde{c}(\sigma_{24})
\]

\[
= (a + b)\tilde{a} + a\tilde{b}
\]

\[
= a\tilde{a} + b\tilde{a} + a\tilde{b}.
\]

Similar computations for \((c \cup \tilde{c})(\rho_4)\) and \((c \cup \tilde{c})(\rho_5)\) yield:

\[
(c \cup \tilde{c})(\rho_4) = \tilde{b}b
\]

\[
(c \cup \tilde{c})(\rho_5) = \tilde{a}a + \tilde{a}b.
\]

Thus we see that products \(c \cup \tilde{c}\) in fact generate the entire 3-dimensional space \(MW^2\), and hence \(MW^*\) for \(\Sigma_{GZ}\) is generated in degree 1 for the case \(n = 3\).

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