Solvable Models of Magnetic Skyrmions

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Abstract. We give a succinct summary of the recently discovered solvable models of magnetic skyrmions in two dimensions, and of their general solutions. The models contain the standard Heisenberg term, the most general translation invariant Dzyaloshinskii-Moriya (DM) interaction term and, for each DM term, a particular combination of anisotropy and Zeeman potentials. We argue that simple mathematical features of the explicit solutions help understand general qualitative properties of magnetic skyrmion configurations in more generic models.

Keywords: magnetic skyrmions, gauged sigma models

1 Introduction

Magnetic skyrmions are topological solitons in two-dimensional field theories for the magnetisation field \( n \) of a magnetic material [12]. In the continuum version, the energy functional consists of the Dirichlet energy (quadratic in derivatives), a potential which includes anisotropy terms and a Zeeman contribution (no derivatives), and the crucial Dzyaloshinskii-Moriya (DM) interaction term (linear in derivatives) [34]. Such an energy functional has stationary configurations which are stable under Derrick scaling provided the DM term is negative for those configurations.

Magnetic skyrmions have been the subject of intensive experimental and numerical studies in recent years because they combine interesting physics with potential technological applications in magnetic information storage and manipulation [7]. More recently, rigorous analytical studies have established conditions for the existence of solutions as well as energy bounds in different topological sectors [67], and have clarified the interesting way in which the relative energy of skyrmions and anti-skyrmions depends on the DM term [8].

This talk is about the critically coupled models recently proposed in [9] and [10]. These models require a particular choice of potential for any given DM term, but with this choice they can be viewed as a gauged version of the Belavin-Polyakov \( O(3) \) sigma model [11]. In particular, solutions can be obtained explicitly in terms of holomorphic functions to the Riemann sphere \( \mathbb{C} \cup \{\infty\} \). In Fig. 1 we show examples of such solutions in a model with standard DM term \( (\mathbf{n}, \nabla \times \mathbf{n}) \) and the potential \( \frac{1}{2}(1 - n_3)^2 \). They include the axisymmetric
Fig. 1. Magnetic skyrmion solutions in the model with DM term \((n, \nabla \times n)\) and the potential \(\frac{1}{2}(1 - n_3)^2\). The value of the magnetisation vector \(n\), assumed to be of unit length, is shown in terms of the Runge colour sphere: the region near the south pole \(n_3 = -1\) is shown in black, and the ‘vacuum region’ near the north pole \(n_3 = 1\) is shown in white. Elsewhere, the longitudinal angle \(\arctan(n_2/n_1)\) is mapped onto the colours red, green, blue, with intermediate colours interpolating.

Solutions are determined by a choice of a holomorphic function \(h\) and the formula (23). Top from left to right: the Bloch skyrmion with \(h(z) = 0\), the line defect with \(h(z) = \frac{i}{2}z\) and the anti-skyrmion with \(h(z) = z\). Bottom from left to right: the ‘empty bag’ with \(h(z) = 2i/z\), the anti-skyrmion of charge \(Q = 5\) with \(h(z) = z^5\) and the anti-skyrmion of charge \(Q = 4\) with \(h(z) = 1/z^5\).

Skyrmion configuration (which has topological charge \(Q = -1\) in our conventions), a line defect \((Q = 0)\), an anti-skyrmion configuration \((Q = 1)\) as well as bags and multi-(anti)-skyrmion configurations which show qualitative features of the configurations studied numerically in [12] and [13].

This talk is designed to explain the models and the construction of their solutions from holomorphic data as simply and directly as possibly. We sum up the method of solution as a four-step recipe in Sect. 3. For details we refer the reader to the papers [9,10].

2 Magnetic skyrmions and gauged sigma models

2.1 Formulating magnetic skyrmion models as gauged sigma models

The most general energy functional for the magnetisation field \(n : \mathbb{R}^2 \to S^2\) which we consider has the form

\[
E[n] = \int_{\mathbb{R}^2} \left( \frac{1}{2}(\nabla n)^2 + \sum_{a=1}^{3} \sum_{i=1}^{2} D_{ai}(\partial_i n \times n)_a + V(n) \right) dx_1 dx_2, \quad (1)
\]

where \(V\) is a potential which may include a Zeeman term and anisotropy terms, and \(D\) is the spiralization tensor parametrising the DM interaction.
In the following we will use the complex coordinate $z = x_1 + ix_2$ in the plane, and define associated derivatives in the standard way, so $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$. As shown in [10], one can view the expression (1) as the energy functional of a gauged sigma model with a fixed $SU(2)$ background gauge field. To see this, it is convenient to think of an $SU(2)$ gauge field in the plane simply as a pair of vectors $A_1$ and $A_2$ in $\mathbb{R}^3$, one for each Cartesian direction in the plane, which act on the magnetisation field $n \in \mathbb{R}^3$ via the vector product (the commutator of the $su(2)$ Lie algebra) so that the covariant derivative and curvature are

$$D_i n = \partial_i n + A_i \times n, \quad i = 1, 2, \quad F_{12} = \partial_1 A_2 - \partial_2 A_1 + \frac{1}{2}A_1 \times A_2. \quad (2)$$

The energy functional of the gauged non-linear sigma models studied in [10] is

$$E_A[n] = \int_{\mathbb{R}^2} \left( \frac{1}{2}|D_1 n|^2 + \frac{1}{2}|D_2 n|^2 - (F_{12}, n) \right) dx_1 dx_2. \quad (3)$$

In the application to magnetic skyrmions, the gauge field is fully determined by the spiralization tensor: the Cartesian components $A_1$ and $A_2$ are simply the negatives of the column vectors which make up the $3 \times 2$ spiralization matrix $D$.

In symbols

$$(A_i)_a = -D_{ai}, \quad (4)$$

so that the DM term can be written as

$$\sum_{a=1}^{3} \sum_{i=1}^{2} D_{ai}(\partial_i n \times n)_a = \sum_{i=1}^{2} (A_i \times n, \partial_i n). \quad (5)$$

We now need to pick a particular form of the potential $V$ in (1) to obtain a solvable model, namely

$$V_A(n) = \frac{1}{2}|A_1 \times n|^2 + \frac{1}{2}|A_2 \times n|^2 - (n, A_1 \times A_2). \quad (6)$$

With the choice $V = V_A$, the magnetic skyrmion energy (1) equals the energy (3) of the gauged sigma model with the gauge field (4). This is easily checked, and makes use of the fact that, for constant gauge fields, one has $F_{12} = A_1 \times A_2$.

Note that the energy functional (3) is invariant under $SU(2)$ gauge transformations, but that this gauge invariance is broken by the gauge choice (4) to obtain the magnetic skyrme energy functional (1) at critical coupling. The residual symmetries of the critically model are discussed in [9] and [10].

For a simple illustration, consider

$$A_1 = -\kappa e_1, A_2 = -\kappa e_2, \quad (7)$$

where $e_1 = (1, 0, 0)^t$ and $e_2 = (0, 1, 0)^t$ are the first two elements of the canonical frame for $\mathbb{R}^3$, and $\kappa > 0$ is a real parameter. This produces the standard DM term $\kappa(n, \nabla \times n)$ and the potential $V_A = \frac{1}{2\kappa}(1-n_3)^2$. Expanding the square, the potential is seen to be a particular linear combination of an easy-plane anisotropy potential with a Zeeman potential, see [9]. This model with $\kappa = 1$ is the one whose solutions are shown in Fig. 1.
2.2 A Bogomol’nyi equation for gauged sigma models

Returning now to the case of a general gauge field, we use various gauge-theoretical identities [9,10] to write the energy (3) as

\[ E_A[n] = \frac{1}{2} \int_{\mathbb{R}^2} (D_1 n + n \times D_2 n)^2 \, dx_1 \, dx_2 + 4\pi (Q + \Omega_A), \]  

where \( Q \) is the integral expression for the degree of \( n \)

\[ Q[n] = \frac{1}{4\pi} \int_{\mathbb{R}^2} (n, \partial_1 n \times \partial_2 n) \, dx_1 \, dx_2, \]  

and \( \Omega_A \) is a generalised version of what was called total vortex strength in [9]:

\[ \Omega_A[n] = -\frac{1}{4\pi} \int_{\mathbb{R}^2} (\partial_1 (A_2, n) - \partial_2 (A_1, n)) \, dx_1 \, dx_2. \]  

If these integrals are well-defined, they only depend on global properties of \( n \) and on its boundary behaviour. If the latter is kept fixed, the energy is therefore minimised when the square in (8) vanishes, i.e. when the Bogomol’nyi equation holds. This is a gauged version of the Bogomol’nyi equations in the standard Belavin-Polyakov model [11], but with a definite sign:

\[ n \times D_1 n = D_2 n. \]  

The Bogomol’nyi equation implies the variational equation of the energy functional (3), see [10].

In the context of magnetic skyrmions, the equation (11) first appeared in [7] where it was noticed that, for a certain family of potentials \( V \), it characterises the energy minimisers in the \( Q = -1 \) sector of the theory with the standard DM term \( (n, \nabla \times n) \). The role of this equation in critically coupled magnetic skyrmion models for arbitrary degree \( Q \geq -1 \) was observed and explored in [9]. Its role and solvability in the more general gauged sigma model and the associated magnetic skyrmion models is the subject of [10]. Generalised versions of this equation have been studied in differential geometry as vortex equations for maps from Riemann surfaces into Kähler manifolds which permit the action of a Lie group, but in that case the gauge field typically obeys a second, coupled equation [14]. The relation between these vortex equations and the equation (11) with a fixed background as proposed in [10] was clarified in [15].

2.3 Boundary contributions to the energy

As far we are aware it is not known for which class of magnetisation fields \( n \) the general energy functional (1) is well-defined and finite. For the standard DM term \( (n, \nabla \times n) \) and a certain class of potentials \( V \), this question is answered in [6] and [7], where it was also pointed that, for analytical reasons, it is preferable to modify the energy functional by adding the boundary term

\[ E_{\kappa, \infty}[n] = -\kappa \int_{\mathbb{R}^2} (\partial_1 n_2 - \partial_2 n_1) \, dx_1 \, dx_2. \]
Adding this term to the energy effectively modifies the DM term: $\kappa(n, \nabla \times n)$ is replaced by $\kappa(n - e_3, \nabla \times n)$ where $e_3 = (0, 0, 1)^t$, and this is the term considered in \[6,7\].

In the context of gauged sigma models, it was proposed in \[10\] that one should more generally add the boundary term

$$E_{A, \infty}[n] = -4\pi A[n] = \int_{\mathbb{R}^2} (\partial_1(A_2, n) - \partial_2(A_1, n))dx_1dx_2 \quad (13)$$

to obtain a well-defined variational problem. Clearly, \eqref{13} reduces to \eqref{12} for the simple gauge field \(7\).

Adding the term \(13\) to the energy \(3\) has a number of advantages, at least from an analytical point of view. It does not change the Euler-Lagrange equation one obtains for variations which vanish rapidly at infinity, but its inclusion means that one can allow for variation with a slower fall-off. We refer the reader to \[10\] for details. Furthermore, the study of solutions of arbitrary degree in \[9\] shows that the modified energy is well-defined for some solutions for which the unmodified energy integral \(3\) is not.

Geometrically, the unmodified energy \(3\) has a natural interpretation when evaluated on a solution of the Bogomol’nyi equation as the equivariant degree of that solution \[15\]. By contrast, the modified energy evaluated on a solution $n$ of the Bogomol’nyi equation is equal to the integral expression for the degree:

$$E_A[n] + E_{A, \infty}[n] = 4\pi Q[n] \quad \text{if} \quad n \times D_1 n = D_2 n. \quad (14)$$

### 3 Exact magnetic skyrmions

#### 3.1 The general solution in four easy steps

Since the magnetic skyrmion energy functional \(1\) with the potential \(6\) is a particular example of the energy for a gauged sigma model of the form \(3\), we can obtain an infinite family of solutions of the variational equations by solving \(11\). Here we focus on the formula needed for magnetic skyrmions, so for constant gauge fields. In that case, the solution of \(11\) can be obtained via the following recipe. For details we again refer to \[10\].

(I) **Complex coordinate for the magnetisation:** In order to write down the solution, one needs to work in terms of a complex stereographic coordinate for the magnetisation field $n$. It is given by stereographic projection from the south pole, or algebraically by

$$w = \frac{n_1 + in_2}{1 + n_3}. \quad (15)$$

(II) **Complexified gauge field:** Next, one needs to write the gauge field explicitly as an $su(2)$ matrix-valued gauge field on $\mathbb{R}^2$ according to

$$A_i = \sum_{a=1}^{3} A_i^a t_a, \quad i = 1, 2, \quad (16)$$
where \( t_a = -\tfrac{i}{2} \tau_a \), and \( \tau_a \) are the Pauli matrices. In fact we require the complex linear combination

\[
A_\bar{z} = \frac{1}{2} (A_1 + iA_2).
\]

In the case at hand, this is a constant, complex and traceless \( 2 \times 2 \) matrix, so generically an element of \( \text{sl}(2, \mathbb{C}) \).

(III) Solution in complex coordinates: The solution of the Bogomol’nyi equation is given in terms of the exponential

\[
g(\bar{z}) = \exp \left( -\frac{1}{2} (A_1 + iA_2) \bar{z} \right) = \begin{pmatrix} a(\bar{z}) & b(\bar{z}) \\ c(\bar{z}) & d(\bar{z}) \end{pmatrix},
\]

which is a \( 2 \times 2 \) matrix function of \( \bar{z} \) with determinant one. The general solution of \((11)\) in stereographic coordinates is

\[
w(z, \bar{z}) = c(\bar{z}) + d(\bar{z}) f(z) a(\bar{z}) + b(\bar{z}) f(z),
\]

where \( f \) is an arbitrary holomorphic function from \( \mathbb{C} \) into \( \mathbb{CP}^1 \simeq \mathbb{C} \cup \{\infty\} \) (in particular it is allowed to take the value \( \infty \)).

(IV) Translating back into Cartesian coordinates: Substitution of the general solution \((19)\) into the inverse of \((15)\)

\[
n_1 + in_2 = \frac{2w}{1 + |w|^2}, \quad n_3 = \frac{1 - |w|^2}{1 + |w|^2},
\]

yields an explicit (but possibly complicated) formula for the magnetisation field.

The energy density of Bogomol’nyi solutions is either the degree density or the sum of the degree density and the vorticity, depending on the choice of energy functional, see our discussion in Sect. 2.3. Expressions for both directly in terms the stereographic coordinates are given in [9,10].

### 3.2 Examples

Axisymmetric DM terms: As discussed in [10], the DM term is invariant under rotations in the plane and simultaneous rotations of the magnetisation field about a suitable axis if and only if \( A_1 \) and \( A_2 \) are orthogonal and have the same length. In that case \((A_1, A_2, A_1 \times A_2)\) is an oriented and (up to scaling) orthonormal basis of \( \mathbb{R}^3 \). With \( |A_1| = |A_2| = \kappa \), the potential for the solvable model is conveniently expressed in terms of \( \hat{A} := A_1 \times A_2 / \kappa^2 \) as

\[
V_A(n) = \frac{\kappa^2}{2} \left( 1 - (n, \hat{A}) \right)^2 = \frac{\kappa^2}{8} (n - \hat{A})^2.
\]

The DM term \([5]\) and the integrand of the boundary term \([13]\) combine neatly into \( \sum_{a=1}^3 \sum_{i=1}^2 D_{ai} (\partial_n \times (n - \hat{A}))_a \) in this case. For the simple case \([7]\) with
DM term $\kappa(\mathbf{n}, \nabla \times \mathbf{n})$ and potential $V(\mathbf{n}) = \frac{\kappa^2}{2}(1 - n_3)^2$, one checks that the matrix (18) is

$$g = \begin{pmatrix} 1 - \frac{i}{2}\kappa \bar{z} & \bar{h} \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (22)

The solution (19) is best written in terms of the inverse coordinate $v = 1/w$ as

$$v = -\frac{i}{2}\kappa \bar{z} + h,$$  \hspace{1cm} (23)

where $h = 1/f$ is, like $f$, an arbitrary holomorphic map $\mathbb{C} \to \mathbb{C}P^1$. The simplest choice $h = 0$ leads to the Bloch hedgehog skyrmion with $Q = -1$ and the Belavin-Polyakov profile function $\theta(r) = 2 \arctan \left( \frac{r}{\kappa} \right)$, as already noticed in [7]. Many other solutions are discussed in [11], and our Fig. 1 shows the solutions one obtains for different choices of rational functions $h$. It follows from the calculations in [9] that the degree of a skyrmion configuration depends on the parameter $L = \lim_{|z| \to \infty} |(2h)/(\kappa \bar{z})|$. For configurations determined by (23) with rational $h(z) = p(z)/q(z)$, where $p$ and $q$ are polynomials of degree $M$ and $N$, it is

$$Q[\mathbf{n}] = \begin{cases} M & \text{if } L > 1 \\ N & \text{if } L = 1 \\ N - 1 & \text{if } L < 1. \end{cases}$$  \hspace{1cm} (24)

This shows in particular that in this model there are infinitely many solutions of the Bogomol’nyi equation (11) for each integer degree $Q \geq -1$. The modified energy (14) takes the values $4\pi |Q|$ on these solutions.

**Rank one DM interactions:** The spiralization tensor has rank one when $A_1$ and $A_2$ are linearly dependent, so $A_1 \times A_2 = 0$. In this case, the curvature $F_{12}$ vanishes and the gauge field (and therefore the DM interaction) can be removed by an $SU(2)$ gauge transformation. The solvable model with the potential (6) can therefore be mapped into the standard Belavin-Polyakov $O(3)$ sigma model [11]. It follows immediately that solutions of (11) exist for any integer degree $Q$ in these models, and that their energy (14) is $4\pi |Q|$. In particular it follows that skyrmions and anti-skyrmions of equal and opposite degree have the same energy. This result was derived in [8] for $|Q| = 1$ in more general rank one models. Example solutions of solvable rank one models and their properties are discussed in [10] and also [16]. We note that a similar reformulation in terms of a flat gauge field was recently applied to a rather different ferromagnetic model in [17].

4 Conclusion

We have shown that solvable models of magnetic skyrmions exist for any DM interaction term. Even though they require fine-tuning of the potential, their exact solutions shed light on qualitative properties of solutions in more general models. These includes general features of multi-(anti-)skyrmion configurations
such as the appearance of $Q + 1$ maxima in the energy density of certain charge $Q > 0$ configurations (as shown in the $Q = 5$ solution in Fig. [1]), or the deformation of a skyrmion to an anti-skyrmion via a line defect, as shown in the top row of Fig. [1] and discussed in some detail in [9].

The solvable models also shed light on the crucial influence of the DM interaction on the relative energy of skyrmions compared to anti-skyrmions. Our short discussion illustrates the more general findings of [8]. In the axisymmetric models of Sect. 3.2, $Q = -1$ skyrmions have energy $-4\pi$ whereas $Q = 1$ anti-skyrmions have the opposite energy $4\pi$ (this can be reversed by a different choice of solvable model, see [9]). In rank one models, by contrast, skyrmions and anti-skyrmions have the same energy. It was shown in [8] that models with generic spiralization tensors should interpolate between these two extremes, and it would be interesting to explore this in the solvable models with generic DM terms.

To end, we note that the language of gauged non-linear sigma models provides a rare and rather beautiful link between pure mathematics and real physics by connecting the geometry of holomorphic maps and vortex equations as discussed in [14] with magnetic skyrmions. In fact, simply allowing the gauge field to depend non-trivially on space may provide further applications, for example to the study of impurities as discussed in [18] and [10].

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