Pair Production in Non-SuSy AdS/CFT

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ABSTRACT: We study pair production of particles in presence of an external electric field in a large N Non-supersymmetric Yang Mills theory using the holographic duality. We numerically calculate the inter-quark potential profile and the effective potential to study pair production and analytically find out the threshold electric field beyond which one gets catastrophic pair creation by studying rectangular Wilson loops using the holographic method. We also find out the pair production rate of particles in presence of an external electric field by evaluating circular Wilson loops using perturbative methods. We infer that the vacuum of the Non-SuSy gauge theory is unstable for a range of Non-supersymmetric parameter.
1 Introduction

For the last few decades the AdS/CFT correspondence [1][2][3][4] (relating $\mathcal{N} = 4$ Super-Conformal Yang-Mills in 4 spacetime dimensions to quantum gravity in asymptotic $AdS_5 \otimes S^5$ spaces) and some of its modifications is one of the exemplary ideas in theoretical physics. AdS/CFT chiefly comes from black hole thermodynamics[5] and type IIB string theory [6] is thus inherently supersymmetric in nature. This is a strong-weak duality meaning, strong coupling in the field theory side corresponds to weak coupling in the Quantum Gravity side and vice-versa. However even after so many years, no trace of supersymmetry has been found by experiments and again conformal symmetry is not found quite much in nature. Thus it is necessary to formulate a modification of AdS/CFT without supersymmetry and conformal symmetry yet respecting its string theory/supergravity origins. Such a solution is obtained in [7][8]. This solution for D3 branes has two parameters $\delta$ and $u_0$ and has certain features which makes its an attractive dual for large N QCD studies via holography [9]. Part of what makes AdS/CFT alluring is that when the field theory coupling is high the corresponding coupling in the quantum gravity side is low and thus we are left with classical gravity which is easily computable.

The coupling constant in field theory is usually used as a perturbative parameter and observables are expressed in a series w.r.t. this parameter, this is called pertubative field theory. However there are quite some effects in quantum field theory which cannot be explained as such i.e. non-pertubative effects. Amongst them the Schwinger Effect stands its ground. The vacuum of QED or any gauge theory interacting with charged matter is
full of virtual particles and antiparticles (henceforth $q\bar{q}$). In presence of an external electric field/gauge field this particles get the required energy and become real particles. There is no magic involved in this. In realistic situations the energy of the the real $q\bar{q}$ pairs is obtained from the electric field. Schwinger calculated [10] the pair production rate for this process in U(1) gauge theory and obtained,

$$\Gamma = \frac{(eE)^3}{(2\pi)^3} e^{-\frac{m^2}{eE}} \tag{1.1}$$

The exponential suppression hints that pair productions can be modeled as a tunneling process. Assuming that the virtual $q\bar{q}$ pair has a separation $x$ the potential on a virtual quark in presence on an external electric field is given as

$$V_{eff} = -\frac{\alpha}{x} - eEx + 2m \tag{1.2}$$

Imagine this to be the potential barrier though which the $q\bar{q}$ pairs tunnel out in the opposite direction and become real. For $E \leq \frac{m^2}{e\alpha}$ there exists two zero points of the potential and $V_{eff}$ is positive for intermediate values of $x$. That means there is a potential barrier and quarks have to tunnel out through them justifying the exponential factor stated above. However for $E > \frac{m^2}{e\alpha}$ the potential becomes negative all along and stops putting up a potential barrier, indicating a catastrophic instability of vacuum where the $q\bar{q}$ are produced spontaneously. The value of electric field for which the potential stops putting up a tunneling barrier is called "critical/threshold electric field" $E_c$.

The Schwinger effect in holographic setting was first calculated in [11] (see [12] for an even earlier work) wherein the pair modified pair production rate was found to be

$$\Gamma \sim \exp \left[ -\frac{\sqrt{\lambda}}{2} \left( \sqrt{\frac{E_c}{E}} - \sqrt{\frac{E}{E_c}} \right)^2 \right] ; \quad E_c = \frac{2\pi m^2}{\sqrt{\lambda}} \tag{1.3}$$

This formula matches with the one above for low electric field (much lower than $E_c$). For field much higher than $E_c$ we don’t see a exponential suppression anymore hinting at catastrophic decay. The chief idea of this work was to place the probe brane at a finite position unlike what is done usually (placing the probe brane at the conformal boundary of AdS) and then to calculate the circular Wilson loop. Another approach was pioneered in [13] which calculated the rectangular Wilson loop for virtual $q\bar{q}$ pair and relate it to inter-quark potential and then find the critical electric field from the same. In this work we want to study the Schwinger effect for Non-Supersymmetric gauge theories via holographic methods using both of this methods. For our purpose the virtual $q\bar{q}$ pairs are imagined to be endpoint of a string in the boundary. We calculate the rectangular Wilson loop in space-time direction to find out the inter-quark potential. To account for an external electric field we add an extra term. We analytically find out the critical electric field from the same. We also plot figures to illustrate the tunneling phenomenon. Next we move on to finding out the critical electric field from analysis of the DBI action using the fact that the action should be real valued. Then we move on to finding out the circular Wilson loop. It is impossible to do so.
without any simplification. We thus expand the expressions to first order of the Non-SuSy deformation parameter \((u_0)^4\). Doing so we explicitly find out the profile of circular Wilson Loop up to first order of \((u_0)^4\) from which we find out the pair production rate. We observe that for certain value of Non-Supersymmetric parameter \(\delta\) the theory becomes unstable.

This paper is organized as follows. In section 2 we recap Non-SuSy D3 branes and their decoupling limit from supergravity. We also show that the Non-SuSy solution goes over to usual AdS when appropriate limits are taken. In section 3 we show the derivation of pair production in theory with U(1) gauge field coupled to charged matter. Relevant expression for large N gauge theory is also given. In section 4 we carry on Potential Analysis of virtual \(q\bar{q}\) pairs from which the critical electric field is derived both by analytical and numerical means. In section 5 we use the DBI action and find out the critical electric field using the fact that the action should be real valued. In section 6 we use perturbative analysis to find out the profile for circular Wilson loop when the string ends at a finite position \((u_b)\). Using this we find the critical electric field and pair production rate and make some comments about the later. We close this paper with conclusions in section 7.

2 Non SuSy Dp Branes and their Decoupling Limit

In this section we will take a brief recap of non-supersymmetric Dp brane solutions [14] and show how to recover the BPS Dp brane solutions from them. Then we will state the decoupling limit of Non-SuSy D3 branes by analogy with the BPS case and make sure the the decoupling goes over to the BPS brane decoupling limit when SuSy is restored[7]. In addition we also show by taking suitable co-ordinate transformation that the decoupled throat geometry is actually identical with two parameter solution obtained previously by Constable and Myres in which supersymmetry and conformal symmetry are both broken [8]. We start with the action for ten dimensional type II supergravity which in addition to the string frame metric

\[
S = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-\det g_{\mu\nu}} \left[ R - \frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi - \frac{1}{2(8-p)!} F_{[8-p]}^2 \right]
\]  

(2.1)

We will be looking for solutions of the above using the ansatz,

\[
d s^2_{str} = e^{2A(r)} \left( -dt^2 + dx_i^2 + ... + dx_p^2 \right) + e^{2B(r)} \left( dr^2 + r^2 d\Omega_{8-p}^2 \right)
\]  

(2.2)

\[
F_{[8-p]} = Q \text{Vol}(\Omega_{8-p})
\]  

(2.3)

In the above the metric has an ISO(p,1) x SO(9-p) isometry and represents a magnetically charged p brane in 10 dimensions with magnetic charge \(Q\). It can be shown that the above solution conserves super-symmetry i.e. saturates the BPS bound if [15],

\[
(p + 1)B(r) + (7 - p)A(r) = 0
\]  

(2.4)

Solution of equations of (2.1) compatible with (2.2)-(2.4) leads to usual BPS p branes. We will be looking for solutions with break the condition (2.4), and thus breaks spacetime
supersymmetries. In the rest of the paper we will be concerned with non-supersymmetric D3 brane solution and thus will consider the case where $p = 3$. The non-supersymmetric D3 brane solution is given as,

$$
 ds^2 = \tilde{F}(\rho)^{-\frac{1}{2}} G(\rho)^{\frac{1}{2}} \left[-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right] + \tilde{F}(\rho)^{\frac{1}{2}} G(\rho)^{-\frac{1}{2}} \left[\frac{d\rho^2}{G(\rho)} + \rho^2 d\Omega_5^2\right]
$$

$$
 e^{2\phi} = g_s^2 G(\rho)^{\delta} ; \quad F_{[5]} = \frac{1}{\sqrt{2}}(1 + \star) Q \text{ Vol}(\Omega_5) \quad (2.5)
$$

In the above the functions $\tilde{F}(\rho)$ and $G(\rho)$ are given as,

$$
 \tilde{F}(\rho) = G(\rho)^{\frac{1}{2}} \cosh^2 \theta - G(\rho)^{-\frac{1}{2}} \sinh^2 \theta
$$

$$
 G(\rho) = 1 + \frac{\rho^0_0}{\rho^4} \quad (2.6)
$$

It can be shown that the non-SuSy solution (2.5) violates the condition (2.4) and thus breaks spacetime supersymmetries. In the above $e^{2\phi}$ is the effective string coupling constant and the solution is characterized by six parameters i.e, $\alpha, \beta, \delta, \rho_0, Q$, of which $\rho_0$ has the dimensions of length, $Q$ has dimensions of four volume and others are dimensionless. One should further note from (2.6) that the solution given above has a naked singularity at $\rho = 0$ and the physical region is given by $\rho > 0$. In the context of string theory one hopes that quantum fluctuations modify the behavior of the solution near the singularity point. As $e^{2\phi}$ is the effective string coupling, for the supergravity description to remain valid one needs the parameter $\delta$ to be less or equal to zero so as to make the string coupling small. The parameters of the solutions are not all independent but satisfy some consistency relations like,

$$
 \alpha = \beta \quad , \quad Q = 2\alpha \rho_0^4 \sinh 2\theta \quad , \quad \alpha^2 + \delta^2 = \frac{5}{2} \quad (2.7)
$$

In arbitrary dimensions the solutions and the constraints are a bit complicated and is given in [16]. Just like the BPS D3 brane solution, the non-SuSy solution too is asymptotically flat. One can recover the BPS solution from the non-SuSy solution given above by considering the limits $\rho_0 \to 0$ and $\theta \to \infty$ keeping $\frac{\alpha}{\rho^4_0}(\cosh^2 \theta + \sinh^2 \theta) \to R^4$ =fixed. Under this scaling one has $G(\rho) \to 1$ and $\tilde{F}(\rho) \to 1 + \frac{\rho^4_0}{\theta^4}$ and $Q \to 4R^4$ under which the standard BPS solution is regained.

The decoupling limit is a low energy limit in which interactions between the bulk theory and theory living on the brane vanishes. To work out the decoupling limit and henceforth the throat geometry one needs to make a change of variables in analogy with the BPS D3 brane.

$$
 \rho = \alpha' u \quad , \quad \rho_0 = \alpha' u_0 \quad , \quad \alpha \cosh^2 \theta = \frac{\lambda}{\alpha'^2 u_0^4} \quad , \quad \alpha' \to 0 \quad (2.8)
$$

In the above $u$ and $u_0$ have the dimensions of energy and are kept fixed. From (2.7) and (2.8) it can be shown that $\frac{Q}{\alpha'^2} \gg 1$ implying that the curvature of spacetime in string units
must be very small for the supergravity description to be valid. A justification of the above
decoupling limit is given explicitly in [7] and [16]. Under the above said limit,
\[ G(\rho) \rightarrow G(u) = 1 + \frac{u_0^4}{u^4} \quad \text{is fixed} \quad (2.9) \]
\[ \bar{F}(\rho) \rightarrow \bar{F}(u) = \frac{\lambda}{\alpha'^2} F(u) \quad (2.10) \]
In the above \( F(u) = \frac{1}{\alpha'^2} (G(u) - G(-u)^{-\frac{3}{2}}) \) and the non-SuSy D3 brane throat geometry
in the decoupling limit mentioned above becomes
\[
\begin{align*}
\text{d}s^2 & = \alpha' \sqrt{\lambda} \left[ F(u)^{-\frac{3}{2}} G(u)^{\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + F(u)^{\frac{3}{2}} G(u)^{\frac{1}{4}} \left( \frac{du^2}{G(u)} + u^2 d\Omega_5^2 \right) \right] \\
e^{2\phi} & = g_s^2 G(u)^{\delta} 
\end{align*}
\]
(2.11)
In the above the spacetime co-ordinates has been rescaled as \((t, x^i) \rightarrow \sqrt{\lambda}(t, x^i)\) where \(\lambda\) is the 't hooft coupling. In the limit \(u_0 \rightarrow 0\) one has \(G(u) \rightarrow 1\) and \(F(u) \approx u^4\). In this limit the non-SuSy throat geometry (2.11) goes over to the known AdS3 × S5,
and the effective string coupling becomes constant. To check the relation of solution (2.11)
with that of the previously known one by Constable and Myres [8] which was conjectured
to be dual to some non-supersymmetric field theory, one has to re-write the solution in the
Einstein frame,
\[
\begin{align*}
\text{d}s^2_E & = \alpha' \sqrt{\lambda} \left[ H(u)^{-\frac{3}{2}} G(u)^{\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + H(u)^{\frac{1}{2}} G(u)^{\frac{1}{4}} \left( \frac{du^2}{G(u)} + u^2 d\Omega_5^2 \right) \right] \\
e^{2\phi} & = g_s^2 G(u)^{\delta} 
\end{align*}
\]
(2.12)
In the above the function \(H(u)\) is defined by \(H(u) = G(u)^{\frac{1}{2}} F(u) = G(u)^{\alpha} - 1\). Now one
has to make a co-ordinate transformation like \(u = r \left( 1 + \frac{\omega^4}{\alpha'} \right)^{-\frac{1}{2}} \) where \(\omega^4 = \frac{\omega^4}{\alpha'}\). Under this
transformation, \(G(u) \rightarrow \left( 1 + 2 \frac{\omega^4}{\alpha'} \right)^2\) and \(H(u) \rightarrow \left( 1 + 2 \frac{\omega^4}{\alpha'} \right)^{2\alpha} - 1\). From these relations
and (2.12) one can exactly produce the two parameter family of solutions as found in [8]
in which both supersymmetry and conformal symmetry is broken. The solution in [8] also
exhibits QCD like behavior like running gauge coupling and confinement in the infrared.
The geometry (2.12) exhibits a naked singularity at \(u = 0\), and thus should be corrected
by stringy corrections which should become dominant at low length scales. Moreover the
proper distance (spatial) from the exterior (say \(u = u_0\)) to the interior is finite (which says
that stringy corrections are a must). In holography the proper distance is identified with
mass of the string hanging from the boundary to the interior [17]. To find the same we
have to choose a gauge of the form : \(x_0 = t, u = s\), all others \(= \text{constant}\). With this gauge
the mass is given by
\[
m = \frac{\sqrt{\lambda}}{2\pi} \int_0^{u_0} du \sqrt{\left( 1 + \frac{u_0^4}{u^4} \right)^{2\delta - 1}} = \text{finite for all allowed values of } \delta. \quad (2.13)
\]
The integral can indeed be done in closed form. However the result is very complicated (hypergeometric functions involved), and it is very difficult to invert \( u_b \) in terms of \( m \). Thus we express our results in this work with formula for mass \( m_0 \) of \( \mathcal{N} = 4 \) SYM.

\[
m_0 = \frac{\sqrt{\lambda}}{2\pi} u_b
\]

(2.14)

### 3 Pair Production in presence of External Fields

In this section we will revisit the concept of pair production in presence of external electric fields i.e the "Schwinger Effect". We will demonstrate the effect using euclidean version of the electromagnetic action \[18\] and generalize to large \( N \) gauge theories. The euclideanized version of U(1) gauge theory coupled to a massive complex scalar field is given by

\[
S = \int d^4x \left[ \frac{1}{4} F_{\mu\nu}^2 + |(\partial_\mu + ieA_\mu + i\alpha_\mu^{ex})\phi|^2 + m^2|\phi|^2 \right]
\]

(3.1)

In the above \( A_\mu \) refers to the dynamical U(1) gauge field and \( \alpha_\mu^{ex} \) refers to the external value of (constant) electromagnetic field. The pair production rate, \( \Gamma \) can be written as \[19\]

\[
V \Gamma = -2 \text{Im} \ln \int DAD\phi \ e^{-S}
\]

\[
= -2 \text{Im} \ln \int DA e^{-S_{eff}}
\]

(3.2)

Where, \( S_{eff} = \frac{1}{2} \int d^4xF_{\mu\nu}^2 + \text{tr} \ln \left[ -\left( \partial_\mu + ie\alpha_\mu^{ex} \right)^2 + m^2 \right] \). For leading order calculations one can ignore the coupling of the dynamical gauge field with the scalar field. Thus the expression above reduces to

\[
V \Gamma = -2 \text{Im} \text{tr} \ln \left[ -\left( \partial_\mu + ie\alpha_\mu^{ex} \right)^2 + m^2 \right]
\]

(3.3)

Using the relation, \( \text{tr} \ln(A) = -\int_0^\infty \frac{dT}{T} \text{tr} e^{-AT} \) and evaluating the trace in position basis, one can rewrite the above expression to

\[
V \Gamma = \text{Im} \int_0^\infty \frac{dT}{T} e^{-\frac{m^2}{2T}} \int d^4x \langle x| \exp \left[ -T \left\{ -\left( \partial_\mu + ie\alpha_\mu^{ex} \right)^2 \right\} \right]|x\rangle
\]

(3.4)

Note that the integrand under \( d^4x \) is synonymous to the path integral of a non-relativistic particle under the influence of the Hamiltonian \( H = \frac{1}{2} \left[ P_\mu + e\alpha_\mu^{ex} \right]^2 \). Using quantum mechanical path integral representation \[20\], one can write

\[
V \Gamma = \text{Im} \int_0^\infty \frac{dT}{T} e^{-\frac{m^2}{2T}} \int_{x(0)=x(T)} Dx \exp \left[ -\frac{1}{2} \int_0^T d\tau \dot{x}^2 + ie \oint a_\mu^{ex} dx_\mu \right]
\]

\[
= \text{Im} \int_0^\infty \frac{dT}{T} \int_{x(0)=x(1)} Dx \exp \left[ -\frac{1}{2T} \int_0^1 d\tau \dot{x}^2 - \frac{m^2T}{2} + ie \oint a_\mu^{ex} dx_\mu \right]
\]

(3.5)

Where in the last line we have rescaled \( \tau \rightarrow \frac{1}{T} \tau \). We assume \( m^2 \int_0^1 d\tau \dot{x}^2 \gg 1 \) (a condition signifying heavy mass) and note that the integration over \( T \) has the form of a modified Bessel
function $K_0(x) = \int_0^\infty \frac{dt}{t} \exp \left( -t - \frac{x^2}{4t} \right)$ with the asymptotic behavior, $K_0(x) \simeq \sqrt{\frac{\pi}{2x}} e^{-x}$, for large $x$. Thus the above integral becomes

$$V\Gamma = \text{Im} \int Dx \exp \left[ -S_p \right] \frac{1}{m} \sqrt{\frac{2\pi}{T_0}}$$

(3.6)

In the above , $T_0 = \frac{1}{m} \sqrt{\int d\tau \dot{x}^2}$ and $S_p = m\sqrt{\int d\tau \dot{x}^2} - ie \oint a^\mu dx_\mu$ and $a_0^\mu = -iE_0$ (signifying constant electric field of value $E$ in $x_1$ direction, iota comes in due to euclidean signature). We like to evaluate the above integral by the method of steepest descent. The argument within the exponential is the action for a relativistic particle executing a periodic motion under influence of $a^\mu_\mu$. The equation of motion for it is given by

$$\frac{1}{\sqrt{\int \dot{x}^2}} m \ddot{x}_\mu = eF^\mu_\nu \dot{x}_\nu$$

(3.7)

Keeping in mind the periodic boundary conditions $x_\mu(0) = x_\mu(1)$, $F^\mu_0 = E$ one has the following classical solution

$$x^c_\mu = R(0,0,\cos 2\pi \tau, \sin 2\pi \tau) \quad , \quad R = \frac{m}{eE} \quad , \quad S^c = \frac{\pi m^2}{eE}$$

(3.8)

Using the above values one has $\frac{1}{m} \sqrt{\frac{2\pi}{T_0}} = \frac{\sqrt{eE}}{m}$. Thus decay rate can be approximated as

$$V\Gamma \approx \frac{\sqrt{eE}}{m} e^{-\frac{\pi m^2}{eE}}$$

(3.9)

Ideally one should go around calculating the one loop prefactor and complete the steepest descent process [18],[21], the calculation of which is indeed complicated. The modified prefactor is given by $(eE)^2$. Thus we see that the pair production rate does to zero if the external electric field is switched off. In arbitrary coupling one can no longer neglect the effect of the dynamical field and one has to include contribution from Wilson loops

$$V\Gamma = -2\text{Im} \int_0^\infty \frac{dT}{T} e^{-\frac{m^2 T}{2}} \int Dx \exp \left[ -\frac{1}{2T} \int_0^1 d\tau \dot{x}^2 + ie \oint a^\mu dx_\mu \right] (\exp \left( ie \oint A_\mu dx_\mu \right))$$

(3.10)

The pair production rate gets modified to [18] [22]

$$\Gamma = \frac{(eE)^2}{(2\pi)^3} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2} \exp \left( -n \frac{\pi \nu m^2}{eE} - \frac{e^2}{4} \right)$$

(3.11)

From the above one can work out that the pair production rate is not exponentially suppressed once the value of electric field exceeds the so called critical value $E_c = \frac{4\pi m^2}{e^2}$, beyond which the vacuum becomes unstable.

To implement this argument for AdS/CFT like theories one faces a number of problems. Firstly the field theory in those circumstances is a conformal one and one cannot get a
mass term a priori. Moreover in the dual gauge theory matter fields exists in the adjoint representation of SU(N) gauge group. To evade these issues one uses the Higgs mechanism to break the symmetry group from SU(N+1) → SU(N) ⊗ U(1). Because of this splitting one has 5 massive W bosons transforming in fundamental representation of SU(N) and interacting with the background Yang Mills theory. Now the pair production rate in presence on an external electric field is given by [22] [23]
\[ \Gamma \sim -5N \int Dx \exp \left( -m \int_0^1 d\tau \sqrt{\dot{x}^2} + i \int_0^1 d\tau a_\mu^{(E)} \dot{x}_\mu \right) W[x] \] (3.12)

Where W[x] is the SU(N) Wilson loop and can be calculated by holographic means.

4 Pair Production in Non-supersymmetric Theories via Holography

The ideal way to argue Schwinger effect[11][24] is to calculate the expectation value of circular Wilson loops and relate it to the decay rate. However one can alternatively view the vacuum to be made of virtual q\bar{q} pairs in presence of an attractive potential and study the influence of an external electric field [13]. This basically amounts to calculating the inter-quark potential which one does by considering the rectangular Wilson loop. In doing so one has to make some additional approximations. One considers that the time scale associated with the Wilson loop is much lesser than the length scale. Intuitively, one thinks that the quark anti-quark pairs are separated in the far past and unite in the far future. In holography the Wilson loop is given by following formula [25] [26]
\[ \langle W[C]\rangle = \frac{1}{\text{Vol}} \int_{\partial X = C} D\mathcal{X} \mathcal{D}h_{ab} e^{-S[X,h]} \] (4.1)

S[X, h] is the Wick rotated action of the fundamental string [6] with endpoints ending at contour C situated on the probe brane. In the classical limit (\(\alpha' \to 0\)) the extremal value of the string action dominates and thus the Wilson loop is the extremal area of string world-sheet ending on the contour. To study the rectangular Wilson loop we take the quark anti-quark dipole to be aligned in the \(x_3\) direction. The string action whose on-shell value we are interested with is the Nambu-Goto action \(S_{NG} = \frac{1}{2\pi\alpha'} \int dt ds \sqrt{\det G_{ab}}\) with \(G^{(in)}_{ab} \equiv \alpha' \frac{\partial x^\mu}{\partial s} \frac{\partial x^\nu}{\partial s} \) which has two diffeomorphism symmetries. We exploit those to choose the following gauge
\[ x^0(s, t) = t ; \quad x^3(s, t) = s ; \quad u(s, t) = u(s) ; \quad x^{1,2} = 0 ; \quad \Theta^1(s, t) = \text{constant} \] (4.2)

For present purposes \(x_3 \equiv s\) is assumed to to range between \([-L, L]\) and temporal direction \(x_0 \equiv t\) is ranged between \([-\mathcal{T}, \mathcal{T}]\) with the assumption that \(\mathcal{T} \gg L\). 2L indicates the inter-quark separation on the probe brane with the boundary condition \(u(\pm L) = u_b\) where \(u_b\) indicates the position of the probe brane along the holographic direction. Finally another word about the configuration, it is possible to consider the \(q\bar{q}\) pairs at a velocity in the \(x_2\) direction. However in the present case where the virtual particles in vacuum are modeled
Figure 1. This figure illustrates the setup used. The probe brane is placed at a finite position \( (u_b) \) on the holographic direction as in (1.2). On the probe brane the placement of the Wilson loop is shown in (1.1), arrows indicating the contour of the loop (not the propagation of the string). For adiabatic interactions one can neglect the effects of the dotted lines and the string profile becomes static.

As \( q\bar{q} \) dipoles, such a configuration seems hardly sensible. The induced metric as per the above gauge choice reads (2.11)(4.2),

\[
\frac{1}{\alpha' \sqrt{\lambda}} G_{ab}^{(in)} ds^a ds^b = -F(u(s))^{-\frac{1}{2}} G(u(s))^{\frac{3}{2}} dt^2 + ds^2 \left[ F(u(s))^{-\frac{1}{2}} G(u(s))^{\frac{3}{2}} + \frac{1}{G(u(s))} \left( \frac{du}{ds} \right)^2 \right] \tag{4.3}
\]

From the above we have the determinant of the induced metric to be

\[
-\det G_{ab}^{(in)} = (\alpha' \sqrt{\lambda})^2 \left[ G(u(s))^{\frac{3+3}{4}} \left( \left( \frac{du}{ds} \right)^2 + G(u(s))^{\frac{3}{4}} F(u(s))^{-1} \right) \right] \tag{4.4}
\]

It is not possible to carry on analysis without some simplification. We therefore assume that \( \left( \frac{u_0}{u} \right)^4 \ll 1 \) and with the mentioned, simplify the area i.e. on shell Nambu-Goto action to

\[
S_{ng} = \frac{1}{2\pi \alpha'} \int_{-T/2}^{T/2} dt \int_{-L}^{L} ds \sqrt{-\det G_{ab}^{(in)}} = \frac{\sqrt{\lambda}}{2\pi} T \int_{-L}^{L} ds \sqrt{\left( \frac{du}{ds} \right)^2} \left( 1 + A \frac{u_0^{1/4}}{u} \right) + u^4 \left( 1 + B \frac{u_0^{1/4}}{u} \right) \tag{4.5}
\]
Wherein
\[ B = \frac{\delta + 1}{2} \; ; \; A = \frac{2\delta - 3}{4} \; ; \; A + \frac{5}{4} = B \]  

(4.6)

Crudely speaking, this can be seen as treating the Non-Susy theory as perturbation over the $\mathcal{N}=4$ supersymmetric Yang Mills. Since the expression (4.5) doesn't explicitly depend on the parameter $s$, the corresponding "Hamiltonian", $Q$ is conserved.

\[ Q = -\frac{du}{ds}dL_{ng} + L_{ng} = \frac{u^4 + Bu_0^4}{\sqrt{\left(\frac{du}{ds}\right)^2 \left(1 + A\frac{u^4}{u^4}\right) + u^4 \left(1 + B\frac{u^4}{u^4}\right)}} \]  

(4.7)

A indicated in [27] the fundamental string is assumed to carry charges at two of its endpoints and is otherwise symmetric about its origin. From the above expression we see that $\frac{du}{ds}$ has both positive and negative sign. Appealing to its symmetric nature there exists a point, namely turning point (with string parameter $s_t$) such that

\[ \left(\frac{du}{ds}\right)(u_t) = 0 \]  

(4.8)

Using the above expression in (4.7) the value of the conserved Hamiltonian is found in terms of the turning point

\[ Q = \sqrt{u_t^4 + Bu_0^4} \]  

(4.9)

Putting the above value in (4.7) we get

\[ \frac{du}{ds} = u^2 \sqrt{\frac{(u^4 - u_t^4)(u^4 + Bu_0^4)}{(u_t^4 + Bu_0^4)(u^4 + Au_0^4)}} \]  

(4.10)

The length of the (virtual) dipole can be calculated to be (see figure 1)

\[ L = \int_{-L/2}^{L/2} dx = \int_{u_t}^{u_b} du \frac{\sqrt{u^4 + Au_0^4}}{u^2 \sqrt{(u_t^4 - u_t^4)(u^4 + Bu_0^4)}} \]  

(4.11)

From (4.10) and (4.5) we can find the on-shell value of inter-quark potential,

\[ U_{PE+SE} = \frac{S_{ng}}{T} = \frac{\sqrt{\lambda}}{2\pi} \int_{u_t}^{u_b} du \frac{(u^4 + Au_0^4)(u^4 + Bu_0^4)}{u^2 \sqrt{(u_t^4 - u_t^4)(u^4 + Bu_0^4)}} \]  

(4.12)

Notice from (4.11) that when $u_t \to u_b$, the value of the inter-quark separation becomes small. But as said earlier we are in an approximation where $\left(\frac{u^4}{\lambda}\right) \ll 1$. Thus the calculations in this section are trustable for large inter-quark separation. Now the expression in (4.12) doesn't take the presence of an external electric field into account. Thus we define an effective potential as

\[ V_{eff} = U_{PE+SE} - E.L = (1 - r)E_c.L + G(u_t(L)) \]  

(4.13)
In the above we have assumed the presence on an critical electric field $E_c$, above which the effective inter-quark force becomes repulsive for all values of the inter-quark separation. The quantity $G(u_t)$ is

$$G(u_t) = U_{PE+SE} - E_c L = \int_{u_t}^{u_b} du \frac{u^4 + Au_0^4}{u^2 \sqrt{u^4 - u_t^4}} \left[ \frac{\sqrt{\lambda}}{2\pi} \sqrt{\frac{u^4 + Bu_0^4}{u_t^4 + Bu_0^4}} - E_c \frac{\sqrt{u_t^4 + Bu_0^4}}{u_t^4 + Bu_0^4} \right] \quad (4.14)$$

The parameter $r$ is the ratio of applied electric field to its critical value. The slope of the effective potential is given as

$$\frac{dV_{eff}}{dL} = (1 - r)E_c + \frac{du_t}{dL} \frac{dG(u_t)}{du_t} \quad (4.15)$$

We now proceed to find the value of the electric field. Note, at $u_t = u_b$ the inter-quark separation $(4.11)$ and the inter-quark potential $(4.12)$ vanishes, see figure 2. At the critical value of the electric field $r = 1$, the 1st term of $(4.15)$ ceases to contribute, and the behavior of the inter-quark force will be completely governed by the second term of $(4.15)$. Criticality demands that the potential ceases to put up a tunneling barrier for all values of inter-quark separation, see red line in figure 4. Given that $G(u_t(L))$ vanishes at $L = 0$ we need to show that $G(u_t(L))$ is a monotonically decreasing function with respect to $L$ whose slope vanishes at $L = 0$. (This is because critical electric field is the least one for which pair production happens spontaneously). From $(4.11)$ we have

$$\frac{dL}{du_t} = -\frac{\sqrt{u_t^4 + Au_0^4}}{u_t^2 \sqrt{(u_t^4 + \epsilon)^4 - u_t^4}} + 2 \int_{u_t+\epsilon}^{u_b} du \frac{u^3}{u^2} \frac{\sqrt{(u^4 + Au_0^4)(u^4 + Bu_0^4)}}{((u^4 - u_t^4)^3)} \quad (4.16)$$

Similarly we have from $(4.14)$

$$\frac{dG}{du_t}(u_t) = -\frac{\sqrt{u_t^4 + Au_0^4}}{u_t^2 \sqrt{(u_t^4 + \epsilon)^4 - u_t^4}} \left[ \frac{\sqrt{\lambda}}{2\pi} \sqrt{u_t^4 + Bu_0^4} - E_c \right] + 2u_t^3 \int_{u_t+\epsilon}^{u_b} du \frac{\sqrt{(u^4 + Au_0^4)(u^4 + Bu_0^4)}}{u^2((u^4 - u_t^4)^3)} \left[ \frac{\sqrt{\lambda}}{2\pi} \sqrt{u_t^4 + Bu_0^4} - E_c \right]
\quad (4.17)$$

Thus we get

$$\frac{dV_{eff}}{dL} = (1 - r)E_c + \frac{\sqrt{\lambda}}{2\pi} \sqrt{u_t^4 + Bu_0^4} - E_c \right) \frac{dL}{du_t} \quad (4.18)$$

At threshold condition the slope of the potential should be zero at when inter-quark separation vanishes i.e. $u_t = u_b$. Implementing the same in $(4.18)$ we get

$$E_c = \frac{\sqrt{\lambda}}{2\pi} u_b^2 \sqrt{1 + \frac{\delta + 1}{2} u_0^4}$$

$$= \frac{2\pi}{\sqrt{\lambda} m_0^2} \sqrt{1 + \frac{\lambda^2}{32\pi^4} (\delta + 1) \frac{u_0^4}{m_0^4}} \quad (4.19)$$
Figure 2. This is the graph of $L$ v/s $u_t$. Note that the function is an isomorphism. The values used are $\delta = -0.75$ and $\frac{u_0}{u_B} = 0.01$

We thus have,

$$\frac{dV_{\text{eff}}}{dL} = (1 - r)E_c + \left[\sqrt{\frac{\lambda}{2\pi}}\sqrt{u_t^4} + Bu_0^4 - \sqrt{\frac{\lambda}{2\pi}}\sqrt{u_b^4} + Bu_0^4\right]$$ (4.20)

From figure 2 we see that $L$ increases as $u_t$ decreases using which we can say from (4.20) that $\frac{dV_{\text{eff}}}{dL}$ is a monotonically decreasing function of $L$ at $r = 1$. It can be easily understood that from $r > 1$ the effective potential is totally repulsive. Thus we establish the existence of a critical electric field with value given by (4.19). We see that as $\delta$ switches over $-1$, the critical electric field increases and decreases respectively compared to the supersymmetric value. Not even that, just at $\delta = -1$, the critical field has the same value as that of the supersymmetric theory. Will this kind of behavior remain when one considers higher orders? How much of the calculation in this section should be trusted for small values of inter-quark separation? The answer to this question will be found in the next section.

It so happens that analytical solutions to (4.11),(4.12),(4.13) cannot be found out in a closed form via Mathematica. Thus we resort to numerical methods. Some plots to illustrate the situation are given.

5 DBI Analysis of Critical Electric Field

In this section we look to find out the critical electric field from analysis of the DBI action of the probe brane in presence of an external electric field. In due course we will also answer the question raised in section 4.
Figure 3. This is the graph of $U_{PE}$ v/s $L$. The rest mass has been duly subtracted. Note that for small values of $L$ the graph is approximately linear and for large $L$ coulombic behavior is mimicked. Deviation from usual coulombic behavior is evident. The values used are $\delta = -0.75$, $u_{ub} = 0.01$ and $\lambda = 4\pi^2$.

As earlier we imagine the probe brane situated at $u = u_{b}$ (see figure 1) in the holographic dual with an electric field switched on at the brane position. The DBI action is given as

$$S_{DBI} = \frac{1}{(2\pi)^3 g_s \alpha'} \int_{u=u_{b}} d^4x \sqrt{-\det(P[g]_{\mu\nu} + B_{\mu\nu} + 2\pi \alpha' F_{\mu\nu})}$$

(5.1)

In the above $P[g]_{\mu\nu}$ is the pullback of the curved metric on the probe brane, $B_{\mu\nu}$ is the NS 2-form which is zero in the present case. $F_{\mu\nu}$ is the Faraday tensor which we set to the value $F_{03} = E$, to indicate the presence of an external electric field. Evaluating the above from (2.11) we have

$$P[g]_{\mu\nu} + 2\pi \alpha' F_{\mu\nu} =$$

$$\begin{pmatrix}
-\alpha' \sqrt{\lambda} F(u_{b})^{-\frac{1}{2}} G(u_{b})^{\frac{1}{4}} & 0 & 0 & 2\pi \alpha' E \\
0 & \alpha' \sqrt{\lambda} F(u_{b})^{-\frac{1}{2}} G(u_{b})^{\frac{1}{4}} & 0 & 0 \\
0 & 0 & \alpha' \sqrt{\lambda} F(u_{b})^{-\frac{1}{2}} G(u_{b})^{\frac{1}{4}} & 0 \\
-2\pi \alpha' E & 0 & 0 & \alpha' \sqrt{\lambda} F(u_{b})^{-\frac{1}{2}} G(u_{b})^{\frac{1}{4}}
\end{pmatrix}$$

(5.2)

Thus the DBI action becomes

$$S_{DBI} = \frac{\alpha' \lambda}{(2\pi)^3 g_s} \int_{u=u_{b}} d^4x \sqrt{F(u_{b})^{-\frac{1}{2}} G(u_{b})^{\frac{1}{4}}} \sqrt{F(u_{b})^{-1} G(u_{b})^{\frac{1}{2}}} - \left(\frac{2\pi E}{\sqrt{\lambda}}\right)^2$$

(5.3)
Figure 4. The plot indicates the effective potential (in presence of external electric field) v/s the inter-quark separation. Imagine this to be then potential through which \( q \bar{q} \) tunnels out. The green line indicates \( r=0.25 \), blue line for \( r=0.75 \). The parameter \( r \) is the ratio of the applied field to its threshold value. The red line which exhibits the threshold behavior i.e. no potential barrier stands for \( r=1.0 \) and cyan for \( r=1.75 \) shows catastrophic decay of vacuum. Note that at the threshold/critical value, the slope of the potential vanishes at \( L = 0 \) and is negative for nonzero value of \( L \) which is precisely the conditions we have used to analytically find out the value of \( E_c \). The values used are \( \delta = -0.75 \), \( \frac{u_0}{u_b} = 0.01 \) and \( \lambda = 4\pi^2 \).

Thus we see that (5.3) is not real for all values of the external electric field and there is an upper limit of the same. This limiting value is nothing but the critical electric field

\[
E_c = \frac{\sqrt{\lambda}}{2\pi} F(u_b)^{-\frac{1}{4}} G(u_b)^{\frac{3}{4}}
\]  

(5.4)

The functions \( F(u) \) and \( G(u) \) has been defined before (2.6). One can check that upto \( \mathcal{O}\left(\frac{u_0}{u_b}\right)^4 \), (5.4) reduces to (4.19). However in finding (5.4) we have refrained from using perturbations of any sort and thus (5.4) is the exact value. Let us check the behavior of it with respect to the parameter \( \delta \).

We see from figure 5 that the critical electric field is same as that of its supersymmetric cousin somewhere around \( \delta = -0.85 \), which matches more or less with our perturbative analysis is the last section. For values of \( \delta > 0.85 \), the critical electric field is greater than
the supersymmetric counterpart, and for $\delta < 0.85$, the critical field is lesser. Thus the question raised in the last section is answered in the affirmative. The calculation in section 4 won’t be affected drastically for small values of inter-quark separation. (This is because it is the small separation behavior that decides the critical value.)

6 Holographic Pair Production Rate for Non-supersymmetric Theories

In this section we calculate the pair production rate by using the method of circular Wilson loops. As indicated earlier in (3.12) and (4.1), to find the pair production rate, we need to find the on-shell value of the Nambu Goto action with string endpoints ending on a circular contour at the probe brane ($u = u_b$). For pure AdS the calculation of the same has been presented in [11] [22] [28]. However, it is not possible to find exact solutions to the relevant equation of motions for the present case (2.11). Thus we will resort to perturbative treatments like that of [29] to calculate the circular Wilson loop and hence the decay rate to first order of the non-susy deformation parameter ($u_0^4$). Since the metric (2.11) enjoys circular symmetry we start by making an ansatz

$$x^0 = r(\sigma) \cos \tau \quad ; \quad x^3 = r(\sigma) \sin \tau \quad ; \quad u = u(\sigma)$$

(6.1)

In the above all other co-ordinates have been put to be constants as circular symmetry would imply. The parameter $\tau$ ranges from $(0, 2\pi)$ while the parameter $\sigma$ is still arbitrary. There exists a diffeomorphism invariance of the Nambu-Goto action with which we can set $u = u(\sigma)$ to a function of our choosing. Putting the ansatz (6.1) in (2.11) we have the
induced metric to be
\[ ds^2 = \alpha' \sqrt{\lambda} \left[ F(u)^{-\frac{1}{2}} G(u)^{\frac{1}{2}} \left( \frac{dr}{d\sigma} \right)^2 + F(u)^{\frac{1}{2}} G(u)^{-\frac{1}{2}} \left( \frac{du}{d\sigma} \right)^2 \right] d\sigma^2 + r^2 F(u)^{-\frac{1}{2}} G(u)^{\frac{1}{2}} dr^2 \]

(6.2)

From the above one can get the Nambu-Goto action to be of the form
\[ S_{ng} = \frac{(\alpha' \sqrt{\lambda})}{2\pi \alpha'} \int_0^{2\pi} d\tau \int_0^{\sigma_b} d\sigma \sqrt{r^2 G(u)^{\frac{1}{2}} \left( F(u)^{-1} (r')^2 + G(u)^{-\frac{3}{2}} (u')^2 \right)} \]

(6.3)

For purposes of calculation we expand the function \( F(u) \) and \( G(u) \) in their leading order to the non-supersymmetric deformation parameter and we have,
\[ S_{ng} = \sqrt{\lambda} \int_0^{\sigma_b} d\sigma \sqrt{r^2 \left( 1 + \frac{\delta u_0^4}{2 u^4} \right) (r')^2 u^4 \left( 1 + \frac{1 + \frac{3}{4} u_0^4}{2 u^4} \right) + (u')^2 \left( 1 - \frac{3 u_0^4}{4 u^4} \right) + O \left( \frac{u_0^8}{u^8} \right)} \]

(6.4)

Wherein
\[ B = \delta + 1 \quad ; \quad A = \frac{2 \delta - 3}{4} \quad ; \quad A + \frac{5}{4} = B \]

(6.5)

The above binomial expansion and all the others that follow is simply treating the Non-SuSy theory as a perturbation over the regular \( \mathcal{N} = 4 \) SYM. In this paper we limit ourselves to first order perturbations. Recall that we still had one diffeomorphism invariance left as mentioned before, with the help of which we set \( \frac{du(\sigma)}{d\sigma} = 1 \). Thus (6.4) is simplified to
\[ S_{ng} = \sqrt{\lambda} \int_{u_1}^{u_b} du \sqrt{r^2 \left( \frac{dr}{du} \right)^2 \left( u^4 + Au_0^4 \right) + \frac{r^2}{u^8} \left( u^4 + Bu_0^4 \right)} \]

(6.6)

We would like to find out the function \( r = r(u) \) which extremizes (6.6). Extremizing the same one has to encounter the equation
\[
u^4 (u^4 + Au_0^4) \left( \frac{d\rho}{du} \right)^2 \left( 2(u^4 + Bu_0^4) - u^7 \frac{d\rho}{du} \right) - 4\rho \left( u^3 \frac{d\rho}{du} (ABu_0^8 + 3Bu^4u_0^4 + 2u^8) \right) - (u^4 + Bu_0^4)^2 \right) - 2u^4 (u^4 + Au_0^4)(u^4 + Bu_0^4) \rho \frac{d^2\rho}{du^2} = 0
\]

(6.7)

Where \( \rho = r^2 \). The above equation is very hard to solve in closed form. Thus we adopt perturbative techniques like that of \[23\]. To do so we decompose the solution to (6.7) as \( \rho = \rho_0 + u_0^4 \rho_1 \) in which \( \rho_0 = -\frac{1}{u^7} \), and \( \rho_1 \) indicates the perturbation. From (6.7) the equation for \( \rho_1 \) to the leading order of \( u_0^4 \) is
\[
2u^2 \left( 6(B - A) + 2u^7 \frac{d\rho_1}{du} + u^8 \frac{d^2\rho_1}{du^2} \right) = 0
\]

(6.8)
One can check that the above is solved by
\[ \rho_1(u) = \frac{A - B}{5u^6} + K' \]  

(6.9)

Thus the full solution is
\[ r^2(u) = \rho(u) = u_0^4 K' - \frac{1}{u^2} + \frac{A - B u_0^4}{5u^6} \]
\[ = K - \frac{1}{u^2} + \frac{A - B u_0^4}{5u^6} \]
\[ = K - \frac{1}{u^2} - \frac{1}{4u^6} \]  

(6.10)

Where a redefinition of constant has been made. Now its time to relate the constant K to physical parameters. At \( u = u_b \) the value of \( r \) is the radius of the Wilson loop \( R \). Thus we have,
\[ K = R^2 + \frac{1}{u_b^2} - \frac{A - B u_0^4}{5u_b^6} \]  

(6.11)

From the above we can also find the value of the turning point \( u_t \), since at the turning point the radius \( r(u_t) = 0 \). Thus the equation which determines the turning point is,
\[ K = \frac{1}{u_t^2} \left( 1 - \frac{A - B u_0^4}{u_t^4} \right) \]  

(6.12)

Now we proceed to calculate the on shell value of the Nambu Goto action (6.6) on the solution (6.10). We have,
\[ S_{ng} = \sqrt{\lambda} \int_{u_t}^{u_b} du \sqrt{\frac{1}{4} \left( \frac{d(r^2)}{du} \right)^2 (u^4 + Av_0^4) + (r^2) \left( 1 + B \frac{u_0^4}{u^4} \right)} \]
\[ = \sqrt{\lambda} \int_{u_t}^{u_b} du \sqrt{\frac{1}{4} \frac{4}{u^6} \left( 1 - \frac{3}{5}(A - B) \frac{u_0^4}{u^4} \right)^2 (u^4 + Av_0^4) + \left( 1 + B \frac{u_0^4}{u^4} \right) \left( K - \frac{1}{u^2} + \frac{A - B u_0^4}{5u^6} \right)} \]
\[ = \sqrt{\lambda} \int_{u_t}^{u_b} du \sqrt{K + KB \frac{u_0^4}{u^4} + O(u_0^8)} \]  

(6.13)

We neglect the \( O(u_0^8) \) term in the above. The integral of the remaining part cannot be done in closed form by using Mathematica. So we resort to perturbative methods again, and write
\[ S_{ng} = \sqrt{\lambda} \int_{u_t}^{u_b} du \sqrt{K} \left[ 1 + \frac{B u_0^4}{2u^4} + O(u_0^8) \right] \]
\[ \approx \sqrt{\lambda} \left[ \sqrt{K} u_b \right]_{u_t}^{u_b} - \left[ 2B \sqrt{K} u_0^4 \right]_{u_t}^{u_b} \]  

(6.14)

So far so good, however the reader may agree that working with (6.14) is still daunting given that we now have to substitute the highly non-linear relations (6.11),(6.12) into it.
Happily there is a way out of this mess. Recall that our theme has been to work in the leading order of $u_4^0$ and the last two terms of (6.14) come with a $u_4^b$ of their own. Thus to leading order we may substitute the usual AdS relations (relating $K$ to $u_t$ and $u_b$) in the last term of (6.14), but use the Non-Susy relations (6.11),(6.12) in the first term of the same. Doing so we have

$$S_{ng} = \sqrt{\lambda} \left( \sqrt{(R^2 u_b^2 + 1) - \frac{A - B u_0^4}{5 u_4^b}} - \sqrt{1 - \frac{A - B u_4^b}{5}} \right)$$

$$\approx \sqrt{\lambda} \left( \sqrt{(R^2 u_b^2 + 1) - \frac{A - B u_0^4}{5 u_4^b}} - \sqrt{1 - \frac{A - B u_4^b}{5}} (R^2 u_b^2 + 1)^2 \right)$$

In the second line of the above we have used the usual AdS relations for the term $u_4^t$ to leading order as it is accompanied by a $u_4^b$. Again in the third line we have used a binomial expansion and retained terms of leading order in $u_4^0 u_4^b$. Now, in presence of an electric field the effective action of the string has an extra piece, $S_B = T_0 \int \frac{d\sigma d\tau}{2} F_{\mu\nu} \partial_\mu x^\nu \partial_\tau x^\nu$. Specializing to constant electric field, the contribution of $S_B$ is a pure boundary term with on-shell value $\pi R^2 E$, where $R$ is the radius of the Wilson loop, $E = B_{01}$ and all other components of the $B_{\mu\nu}$ is set to zero. The effective action is given by

$$S_{eff} = S_{ng} + S_B$$

$$= \sqrt{\lambda} \left( \sqrt{x} - 1 + \frac{u_0^4}{u_b^4} \left[ \frac{A + 19B}{10} x^2 - \frac{A - B}{10 \sqrt{x}} - 2B \sqrt{x} - E x + E \right] \right)$$

In the above, $x = R^2 u_b^2 + 1$ and $E = \sqrt{\lambda u_b^2} \pi \mathcal{E}$. Thus the radius $R (x)$ is a free parameter in the expression (6.16). Following [11],[24] the radius should be set to an extremum of (6.16). Instead of extremizing w.r.t. $R$, we extremize the action (6.16) w.r.t. the parameter $y = \sqrt{x}$. Doing so we find,

$$0 = \frac{dS_{eff}}{dy} = \sqrt{\lambda} \left( 1 - 2E y + \frac{u_0^4}{u_b^4} \left[ 2(A + 19B) y^2 + \frac{A - B}{10 y^2} - 2B \right] \right)$$

The radius $R$ should be set to be the solution of (6.17), recall $y = \sqrt{R^2 u_b^2 + 1}$. Thus the value of $y$ in the above equation is constrained and should always be greater than 1. This
is because the radius of the Wilson loop should be a real number. A subtle point is that the range of parameter \( y \) should be restricted to half of the real line, because the radius \( R \) is nonnegative. The critical electric field \( \mathcal{E}_c \) is the one for which the radius \( R = 0 \), i.e. \( y = 1 \). Setting so in the above we see,

\[
1 - 2 \mathcal{E}_c + \frac{u_0^4}{u_b^4} \left[ \frac{2(A + 19B)}{5} + \frac{A - B}{10} - 2B \right] = 0 \tag{6.18}
\]

Thus

\[
E_c = \frac{\sqrt{\lambda}}{2\pi} u_b^2 \left[ 1 + \frac{1}{8 u_0^4} \left( 23 + 24\delta \right) \right]
\]

\[
= \frac{2\pi}{\sqrt{\lambda}} m_0^2 \left[ 1 + \frac{\lambda^2}{128\pi^4} \left( 23 + 24\delta \right) \frac{u_0^4}{m_0^4} \right] \tag{6.19}
\]

We see that like (4.19) and (5.4) the value of the critical electric field is greater than the supersymmetric value for the value \( \delta = \frac{-23}{24} \) and less than the supersymmetric cousin otherwise. Our perturbative analysis has even shown that the value of parameter \( \delta \) for which this phase transition occurs is slightly bigger than -1 as can be seen from the non-perturbative DBI analysis. Now to find the expression of the pair production rate we need to solve (6.17) for \( y \). As can be seen, that is not analytically possible. We thus resort to perturbative treatments again and write,

\[
y = y_0 + \frac{u_0^4}{u_b^4} y_1 \tag{6.20}
\]

\( y_0 \) being the usual AdS solution i.e. \( u_0 = 0 \) in (6.17). The value of \( y_0 \) is \( \frac{1}{2\mathcal{E}_c} \). We put the above relation in the equation in (6.17) to get up to leading order in \( \frac{u_0^4}{u_b^4} \).

\[
y_1 = \frac{1}{2\mathcal{E}_c} \left[ A + 19B \frac{1}{\mathcal{E}^3} + \frac{2(A - B)}{5} \mathcal{E}^2 - 2B \right] \tag{6.21}
\]

Now we put (6.20) and (6.21) in (6.16), i.e. find out the on-shell action. Retaining terms in leading order of \( \frac{u_0^4}{u_b^4} \) leads us to.

\[
\mathcal{S}_{\text{on-shell}}^{\text{eff}} = \frac{\sqrt{\lambda}}{2} \left( \frac{1}{2\mathcal{E}_c} - 2 + 2\mathcal{E}_c + \frac{u_0^4}{u_b^4} \left[ \frac{A + 19B}{80} \frac{1}{\mathcal{E}^4} + \frac{B - A}{10} \mathcal{E} - \frac{2B}{\mathcal{E}} \right] \right)
\]

\[
= \frac{\sqrt{\lambda}}{2} \left( \frac{1}{2\mathcal{E}_c} - 2 + 2\mathcal{E}_c + \frac{u_0^4}{u_b^4} \left[ \frac{40\delta + 39}{320} \frac{1}{\mathcal{E}^4} + \frac{1}{40} \mathcal{E} - \delta + \frac{1}{\mathcal{E}} \right] \right) \tag{6.22}
\]

The pair production rate of quark anti-quark pairs per unit volume per unit time is given by the formula, \( \Gamma \sim e^{-\mathcal{S}_{\text{on-shell}}^{\text{eff}}} \). Note that we are using reduced parameter \( \mathcal{E} = \frac{\pi}{\sqrt{\lambda} u_b^2} \mathcal{E}_c \), in terms of which the pure AdS pair production rate(per unit volume per unit time) is given by [11][24][28].

\[
\Gamma_{\text{susy}} \sim \exp \left[ -\frac{\sqrt{\lambda}}{2} \left( \sqrt{\frac{1}{2} \mathcal{E}} - \sqrt{\frac{1}{2} \mathcal{E}} \right)^2 \right] \tag{6.23}
\]
For the pure AdS/Supersymmetric scenario, the critical electric field is \( E_c = \frac{1}{\sqrt{2}} \) i.e. \( E_c = \sqrt{\frac{\lambda}{\pi^2}} \). Now unlike the supersymmetric case, the pair production rate cannot be brought in closed form. We will have to resort to numerical calculations. We present the plots of pair production rate. Computation of the fluctuation prefactor (i.e. the \( \frac{(eE)^2}{(2\pi)^3} \) term in (3.11)) is somewhat a open question in holography, which is the reason we have plotted \( e^{-S_{onshell}} \) instead of \( \Gamma \).

Physical interpretation of the plots : As commented in the caption, no stark contrast is found between the susy and non-susy case in plot 6. This is the case when the parameter is greater than (somewhere around) \(-0.955\). The reason for this can be seen from (6.22). For low electric field the pair production rate is dominated by the \( \frac{E^4}{u_b} \cdot \frac{405+39}{320} \) term. Above
δ = −0.955 the effective correction at the low electric field limit is positive. At the limit of high electric field limit the correction of pair production rate due to the non-susy deformation parameter is always positive. This is reason that the high electric field limit, the behavior of non-susy pair production rate is same as its supersymmetric cousin for all values of parameters. Startling effects happen when the parameter δ < −0.955, for which the plot is shown in figure 7. Let us recall, the prefactor of the pair production rate is given by field theoretic calculations to be \((eE)^2/(2π)^4\), see (3.11). Although the holographic calculation of the fluctuation prefactor is currently a mystery, it should definitely match with field theoretic calculation for low electric field. For small applied electric fields the production rate shoots up signaling in non-perturbative instability of the vacuum. We say "non-perturbative" because Schwinger effect is by itself a non-perturbative phenomenon. We see that the limit δ = −0.955 approx. is for more interesting than earlier imagined.

Let us end by writing down the pair production rate per unit spatial volume per unit time for non-susy Yang Mills.

\[
\Gamma_{\text{non-susy}} \approx \exp \left[ -\frac{\sqrt{\lambda}}{2} \left( \frac{2\pi m_0^2}{\sqrt{\lambda}} \frac{1}{E} - 2 + \frac{\sqrt{\lambda}}{2\pi m_0^2} E + \frac{\lambda^2 u_0^4}{16\pi m_0^2} \left( \frac{4\pi^4 m_0^8 (40\delta + 39)}{5\lambda^2} \frac{1}{E^4} ight. 
\right.
\right.
\]

\[
\left. + \frac{\sqrt{\lambda}}{160\pi m_0^2} E - \frac{4\pi m_0^2 (\delta + 1)}{\sqrt{\lambda}} \frac{1}{E} \right) \right] \right) \right) \right) \right) \right) \right)
\]

(6.24)

7 Conclusion

In this paper we have studied pair production (Schwinger Effect) in presence of external electric field for Non-susy AdS/CFT using three methods in the literature. In section 4 we have done a potential analysis by calculating rectangular Wilson loops and have analytically calculated the critical electric field below which pair production happens via a tunneling phenomenon (and above which the quark-antiquark potential ceases to put up a potential barrier. We have seen that the critical electric field is higher / lower than its supersymmetric counterpart depending on the value of the non-supersymmetric parameter δ (in that section we have used a metric perturbation to ease up the calculation). We have also confirmed the same from the DBI analysis of the critical electric field in section 5 where no such approximation has been made. Next in section 6 we have performed the analysis for pair production rate for quark-antiquark pairs using circular Wilson loops. Since the relevant equations are rather impossible to solve, we have resorted to perturbative analysis, which can be thought of as perturbation over \(\mathcal{N} = 4\) SYM by a supersymmetry breaking term with coupling constant proportional to \((\frac{u_0}{u_b})^4\) parametrized by δ. We have explicitly found out the profile for circular Wilson loop for Non-susy AdS/CFT up to first order of \(u_0^4\). To our knowledge this is the first time such a solution has been obtained. We proceed to find the on-shell value of the Nambu-Goto action on the profile found and relate it to pair production rate. We see for a regime of allowed value of the parameter δ the pair production rate shoots up as external electric field decreases towards zero signaling that the vacuum of the dual Non-SuSy gauge theory is non-perturbatively unstable for the regime
of the parameter $-\sqrt{\frac{1}{2}} \leq \delta < -\frac{39}{40}$ (apprx). A relevant question is to find this instability from potential analysis and DBI analysis. That issue eludes us at this moment.

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