INEQUIVALENT COMPLEXITY CRITERIA FOR FREE BOUNDARY MINIMAL SURFACES

ALESSANDRO CARLOTTO AND GIADA FRANZ

Abstract. We obtain a series of results in the global theory of free boundary minimal surfaces, which in particular provide a rather complete picture for the way different complexity criteria, such as area, topology and Morse index compare, beyond the regime where effective estimates are at disposal.

1. Introduction

Free boundary minimal surfaces naturally arise, in Riemannian Geometry, as critical points of the area functional in the category of relative cycles. More precisely, if $(M^n, g)$ is a Riemannian manifold with boundary, and one considers the class of deformations induced by proper diffeomorphisms (that is to say: compactly supported diffeomorphisms that map the boundary $\partial M$ onto itself) the first variation of the area functional at $\Sigma^k$ vanishes if and only if the submanifold in question has zero mean curvature and meets the boundary of the ambient manifold orthogonally: if that is the case $\Sigma$ shall be called a free boundary minimal submanifold (specified to surface when $k = 2$).

In recent years, the series of works by Fraser and Schoen [20], [21], [22] on the relation between free boundary minimal surfaces and the Steklov eigenvalues has breathed new life into the study of these objects, whose investigation goes back almost one century. The theory turns out to be extremely rich already in the simplest case of surfaces in the three-dimensional Euclidean unit ball, where many different examples have been discovered. We refer the reader to the introduction of [2] and to the survey [34] for a gallery of recent existence theorems, but we shall mention here the significant undergoing project, by Q. Guang, M. Li, Z. Wang and X. Zhou (see in particular [35] and [24]) to transfer the min-max theory by Almgren-Pitts to this setting, with the perspective of transposing the impressive results that have been achieved in the closed case (through the efforts of F. Marques, A. Neves, Y. Liokumovich, D. Ketover, K. Irie and A. Song among others).

These striking developments pose a number of challenges. Among those, it is natural to ask how different pieces of information that one can associate to a free boundary minimal surface relate to each other. In this article, we will primarily focus on three sources of data: the Euler characteristic (as a topological descriptor, cf. Section 2.4), the area (as a measure of geometric size) and the Morse index (that plays the role of the most basic analytic invariant one can associate to the surface in question). See Section 2.3 for a precise definition and Appendix A for a discussion on the role of properness in that respect.

The general scope of the present work is to investigate how these different complexity criteria can be compared to each other, under mild positive curvature assumption, i.e. in a regime where effective/quantitative estimates are typically not available. More precisely, the context to have in mind is that of a compact 3-manifold satisfying either of the following two pairs of curvature conditions:

(i) the scalar curvature of $M$ is positive and $\partial M$ is mean convex with no minimal components;
(ii) the scalar curvature of $M$ is non-negative and $\partial M$ is strictly mean convex.

First of all, we consider questions of this sort: ‘Given a Riemannian 3-manifold $(M, g)$ as above, is it possible to construct a monotone function $f$ such that any free boundary minimal surface whose index is bounded by $C$ has area bounded by $f(C)$?’ We can combine two of the main results we present here with other recent advances in the field to fully determine whether each of the six natural
implications that one can associate to the three pieces of data above hold true, thereby obtaining a rather complete description of the scenario in front of us.

\[
\chi(\Sigma) \quad \text{topological bounds}
\]

\[
\text{Remark 1.16 (cf. [22], [32], [33])}
\]

\[
\text{Theorem 1.4}
\]

\[
\text{Remark 1.16 (cf. [22], [32], [33])}
\]

\[
\text{Theorem 1.12}
\]

\[
\text{Theorem 1.12}
\]

\[
\text{index bounds}
\]

\[
\text{ind}(\Sigma)
\]

\[
\text{area bounds}
\]

\[
\mathcal{H}^2(\Sigma)
\]

**FIGURE 1.** A diagram comparing complexity criteria for compact Riemannian 3-manifolds \((M, g)\) satisfying either \(R_g > 0, H^0M \geq 0\) or \(R_g \geq 0, H^0M > 0\).

In particular, we develop a detailed analysis of the topological degenerations that may occur, in the limit, to sequences of free boundary minimal surfaces solely subject to a uniform Morse index bound to ultimately prove that a bound on the index implies a bound on the area, the topology and the total curvature (see Theorem 1.4 for a precise statement). In turn, this theorem implies novel unconditional compactness (Corollary 1.9) and generic finiteness (Corollary 1.10) results. We note that, prior to this work, no (effective or ineffective) counterpart of Theorem 1.4 was known for free boundary minimal surfaces, not even under the stronger curvature assumptions that the Ricci curvature of the ambient manifold be positive, and its boundary strictly convex.

To show that a bound on the topology cannot possibly imply a bound on the area, nor on the index, we build a large class of pathological counterexamples: in Theorem 1.12 we construct, for any smooth 3-manifold \(M\) supporting Riemannian metrics of positive scalar curvature and mean convex boundary and any \(a \geq 0, b > 0\), one such Riemannian metric \(g = g(a, b)\) in a way that \((M, g)\) contains a sequence of connected, embedded, free boundary minimal surfaces of genus \(a\) and exactly \(b\) boundary components, and whose area and Morse index attain arbitrarily large values. In fact, we have some freedom on the geometric boundary conditions we impose, so that a few variants of the construction are actually possible.

Concerning the last two (possible) implications above, we note how a bound on the area cannot possibly imply a bound on the index, nor on the topology (no matter how strong curvature conditions are imposed). To that scope, we examine in Remark 1.16 three different classes of existing examples (due to Fraser-Schoen [22], Kapouleas-Li [32] and Kapouleas-Wiygul [33]): in each case one has a sequence of free boundary minimal surfaces in the unit ball of \(\mathbb{R}^3\) satisfying a uniform bound on the area, but arbitrarily large genus and hence, by Theorem 1.2, arbitrarily large index as well. Roughly speaking, a bound on the area only implies weaker forms of convergence, typically of measure-theoretic character (e.g. in the sense of varifolds, flat chains, currents . . . ), but does not capture finer geometric properties.

This diagram then implicitly defines a hierarchy of conditions, based on the implications that hold or do not hold true. Of course, it is then natural to ask whether pairs of ‘weak conditions’
implies a ‘strong’ one, e.g. prototypically whether a bound on the area and the topology implies a bound on the index. This interesting question has recently been answered, in the affirmative, by V. Lima [36, Theorem B] who adapted to the free boundary setting some remarkable estimates by Ejiri-Micallef [16]: one can bound the Morse index from above by a linear function of the area and the Euler characteristic, with a multiplicative constant only depending on the ambient manifold. Thereby, the picture we obtain is quite complete and exhaustive.

The results we obtain are mostly peculiar of the ambient dimension three, and use a combination of soft tools, including some non-trivial theorems about minimal laminations in compact 3-manifolds.

We will now present the contents of the article in more detail, point out the technical challenges we faced and relate them to the pre-existing results in the literature, some of which played a fundamental role with respect to this project.

1.1. Topological degeneration analysis. A good starting point for our discussion is the following result of Fraser-Li, which is the free boundary analogue of the classical compactness theorem [12, Theorem 1] by Choi-Schoen.

**Theorem 1.1** ([19, Theorem 1.2]). Let \((M^3, g)\) be a compact Riemannian manifold with non-empty boundary. Suppose that \(M\) has non-negative Ricci curvature and strictly convex boundary. Then the space of compact, properly embedded, free boundary minimal surfaces of fixed topological type in \(M\) is compact in the \(C^k\) topology for any \(k \geq 2\).

Afterwards, a general investigation of the spaces of free boundary minimal hypersurfaces with bounded index and volume has been carried through in [3] and [2]. In absence of any curvature assumptions, it is not possible to obtain a strong compactness result as in **Theorem 1.1**, since curvature concentration can occur at certain (isolated) points. However, one can still prove a milder form of subsequential convergence (smooth, graphical convergence with multiplicity \(m \geq 1\) away from finitely many points), and develop an accurate blow-up analysis near the points of bad convergence. This analysis leads to several compactness and finiteness results, among which we want to recall the following theorem.

**Theorem 1.2** ([2, Corollary 4]). Let \((M^3, g)\) be a compact Riemannian manifold with boundary and consider \(I \in \mathbb{N}, \Lambda \geq 0\). Then there exist constants \(a_0 = a_0(M, g, I, \Lambda)\) and \(b_0 = b_0(M, g, I, \Lambda)\) such that every compact, properly embedded, free boundary minimal surface with index bounded by \(I\) and area bounded by \(\Lambda\) has genus bounded by \(a_0\) and number of boundary components bounded by \(b_0\). Furthermore, there exists a constant \(\tau_0 = \tau_0(M, g, I, \Lambda)\) so that the total curvature (i.e. the integral of the square length of the second fundamental form) of any such surface is bounded from above by \(\tau_0\).

**Remark 1.3.** The statement in [2] is more general as it only requires a bound on some eigenvalue of the Jacobi operator instead of an index bound, and it applies to free boundary minimal hypersurfaces in ambient manifolds of dimension \(3 \leq n + 1 \leq 7\).

Here we shall be concerned with the space of free boundary minimal surfaces (inside a three-dimensional Riemannian manifold) with bounded index but without any a priori bound on the area. The analogous task has been carried out, for the case of closed minimal surfaces, in the remarkable article [9] by Chodosh-Ketover-Maximo. Some of the methods developed there are essential for our analysis, and we often rely on the results presented in that article for interior points, although serious technical work (and specific tools) are needed to properly handle the possible degenerations occurring near the boundary of the ambient manifold.

Let us now describe the key steps in this approach. Hence, let us consider a compact Riemannian manifold with non-empty boundary \((M^3, g)\). For the sake of simplicity, let us assume (in the context of this introduction) the following additional property:
(Ψ): If $\Sigma^2 \subset M$ is a smooth, connected, complete (possibly non-compact), embedded surface with zero mean curvature which meets the boundary of the ambient manifold orthogonally along its own boundary, then $\partial \Sigma = \Sigma \cap \partial M$.

We postpone the discussion of this property, its relevance and the (non-trivial) issues that arise when one drops it to Section 2 and Appendix A. For the moment it is important to keep in mind that it prevents the existence of minimal surfaces that touch $\partial M$ in their interior (which would, for instance, create problems for the very definition of index of the surface) and it is implied by simple geometric assumptions (for example property (C) in Section 2, namely that the boundary be mean convex with no minimal component).

Fixed $I \in \mathbb{N}$ and given a sequence of compact, properly embedded, free boundary minimal surfaces $\Sigma^2_j \subset M$ with $\text{ind}(\Sigma_j) \leq I$, we develop our analysis in two steps:

**Macroscopic behavior:** First we prove that, up to subsequence, the surfaces $\Sigma_j$ converge locally smoothly away from a finite set of points $S_\infty$ to a smooth free boundary minimal lamination $L \subset M$, that is a suitable disjoint union of free boundary minimal surfaces (see Definition 3.1). Notice that we cannot expect anything better than a lamination without imposing uniform bounds on the area. Moreover, it holds that the curvature of $\Sigma_j$ is locally uniformly bounded away from $S_\infty$. This tells us that the surfaces are well-controlled away from a finite set of points. Let us remark that the points in $S_\infty$ can belong to the boundary.

The main ingredient here is an extension of the curvature estimate for stable minimal surfaces (cf. [40], [48] and [41] for higher dimensions) to free boundary minimal surfaces with bounded index.

**Microscopic behavior:** The second step consists in carefully studying the local behavior of the surfaces $\Sigma_j$ near the ‘bad’ points in $S_\infty$. In particular, we prove that, for $\varepsilon > 0$ sufficiently small, $\Sigma_j \cap B_\varepsilon(S_\infty)$ contains only a finite number of components where the curvature is not bounded and these components have controlled topology and area.

The way to proceed in the proof is to blow-up the components of $\Sigma_j \cap B_\varepsilon(S_\infty)$ with unbounded curvature at the ‘scale of the curvature’. In this way we obtain a (free boundary) minimal surface in $\mathbb{R}^3$ or in a half-space of $\mathbb{R}^3$ (if initially we were near the boundary of $M$) with index less or equal than $I$. Thanks to [11] (cf. also [10]), we are able to conclude the description of $\Sigma_j$ at small scale, paying attention to prevent data loss in the blow-up.
At this point, due to this precise description of the degeneration, we are able to perform a ‘simplification surgery’ on $\Sigma_j$. Namely, fixing $\varepsilon > 0$ sufficiently small, we modify the surfaces $\Sigma_j$ inside $B_\varepsilon(S_\infty)$ to obtain new surfaces $\tilde{\Sigma}_j$ with the following properties:

- $\tilde{\Sigma}_j$ coincides with $\Sigma_j$ outside $B_\varepsilon(S_\infty)$ (the surgery is performed only near the ‘bad points’);
- the surfaces $\tilde{\Sigma}_j$ have uniformly bounded curvature;
- the topology and the area of $\tilde{\Sigma}_j$ are comparable to those of $\Sigma_j$;
- the surfaces $\tilde{\Sigma}_j$ converge locally smoothly to the lamination $L$ introduced above.

We refer the reader to Theorem 6.1 and Corollary 6.2 for precise statements concerning the description of the topological degeneration and the surgery procedure, respectively. By means of this analysis, we obtain the following result, which shows that it is possible to remove the assumption on the area bound from Theorem 1.2 in the case of ambient manifolds satisfying the mild curvature assumptions mentioned above.

**Theorem 1.4.** Let $(M^3, g)$ be a compact Riemannian manifold with boundary. Moreover assume that

(i) either the scalar curvature of $M$ is positive and $\partial M$ is mean convex with no minimal components;

(ii) or the scalar curvature of $M$ is non-negative and $\partial M$ is strictly mean convex.

Given $I \in \mathbb{N}$, there exist constants $\Lambda_0 = \Lambda_0(M, g, I), \tau_0 = \tau_0(M, g, I), a_0 = a_0(M, g, I)$ and $b_0 = b_0(M, g, I)$ such that for every compact, connected, embedded, free boundary minimal surface $\Sigma^2 \subset M$ with non-empty boundary and with $\text{ind}(\Sigma) \leq I$ we have that its area is bounded by $\Lambda_0$, its total curvature is bounded by $\tau_0$, its genus by $a_0$ and the number of its boundary components by $b_0$.

**Remark 1.5.** Note that the requirement that either of the curvature assumptions hold strictly is actually necessary, for the manifold $S^1 \times S^1 \times I$, endowed with flat metric, contains stable minimal annuli of arbitrarily large area.

**Remark 1.6.** An area bound for free boundary minimal surfaces is required also in the proof of Theorem 1.1 (see [19, Proposition 3.4]). However, the assumption that is made there is a bound on the topology and, most importantly, the method employed in that case is essentially analytic, relying on the aforementioned connection between free boundary minimal surfaces and the first Steklov eigenvalue.

**Remark 1.7.** So far Theorem 1.4 was known only for surfaces with index $I = 0$ or $I = 1$ (see [2, Appendix A]). Indeed, for stable free boundary minimal surfaces the area bound follows from the stability inequality and a similar argument can be applied to surfaces with index 1 based on the well-known Hersch trick.

In order to prove the previous theorem, it turns out that one needs to gain some control on the size of stable subdomains of free boundary minimal surfaces. Prior to this work, this result was known only in the closed case (see [6], based on ideas going back to the work by Schoen-Yau [44]).

**Proposition 1.8.** Let $(M^3, g)$ be a three-dimensional Riemannian manifold with boundary. Denote by $\varrho_0 := \inf_M R_g$ the infimum of the scalar curvature of $M$ and by $\sigma_0 := \inf_{\partial M} H^{\partial M}$ the infimum of the mean curvature of $\partial M$. Assume that

(i) either the scalar curvature of $M$ is uniformly positive ($\varrho_0 > 0$) and $\partial M$ is mean convex ($\sigma_0 \geq 0$) with no minimal components;

(ii) or the scalar curvature of $M$ is non-negative ($\varrho_0 \geq 0$) and $\partial M$ is uniformly strictly mean convex ($\sigma_0 > 0$).

Then every complete, connected, injectively immersed, stable free boundary minimal surface $\Sigma^2 \subset M^3$ that is two-sided and has non-empty boundary is compact, and its intrinsic diameter satisfies the
bound

\[
\text{diam}(\Sigma) := \sup_{x,y \in \Sigma} d_{\Sigma}(x,y) \leq \min \left\{ \frac{2\sqrt{2\pi}}{\sqrt{3\sigma_0}}, \frac{\pi + 8/3}{\sigma_0} \right\}.
\]

Moreover, one has that

\[
0 < \frac{\sigma_0}{2} \mathcal{H}^2(\Sigma) + \sigma_0 \mathcal{H}^1(\partial \Sigma) \leq 2\pi \chi(\Sigma);
\]

in particular, \(\Sigma\) is diffeomorphic to a disc.

It is interesting to note that the variational argument that allows to prove the corresponding estimate in the closed case is not sufficient in the free boundary context and, indeed, this result turns out to be much more delicate. Yet, the key idea remains similar: stable (free boundary) minimal hypersurfaces inherit the ‘positivity’ curvature properties of the ambient manifold (cf. [42] for a striking application of this principle to the proof of the positive mass theorem in all dimensions).

We shall now present two significant consequences of Theorem 1.4. The first one descends by combining the area bound with the geometric compactness result of Theorem 1.1.

**Corollary 1.9.** Let \((M^3, g)\) be a compact Riemannian manifold with boundary. Suppose that \(M\) has non-negative Ricci curvature and that the boundary \(\partial M\) is strictly convex. Then any set of compact embedded free boundary minimal surfaces with uniformly bounded index is compact in the \(C^k\) topology for every \(k \geq 2\).

In addition, one can employ the Baire-type result given in [3, Theorem 9] to obtain the following generic finiteness result.

**Corollary 1.10.** Let \(M^3\) be a compact manifold with boundary. For a generic choice of \(g\) in the class of Riemannian metrics such that \(M\) has positive scalar curvature and \(\partial M\) has strictly mean convex boundary, the space of compact, embedded, free boundary minimal surfaces with index bounded by \(I\) is finite (for any \(I \in \mathbb{N}\)). Hence, the set of all such surfaces (regardless of their Morse index) is countable. Analogous conclusions hold true for \(g\) chosen is a dense subclass of metrics satisfying the curvature conditions (i) or (ii) given above.

**Remark 1.11.** We note that the fact that for a generic choice of the background metric the set of closed minimal surfaces in a compact manifold \((M, g)\) (without boundary) is countable played an essential role in the proof of the generic case of the Yau conjecture by Irie-Marques-Neves [31]. Thus, the conclusion of Corollary 1.10 should be a key building block for the corresponding free boundary result.

### 1.2. Pathological families of free boundary minimal surfaces

Explicit examples, based on equivariant constructions due to Hsiang [30] and Calabi [5] show that the conclusion of the geometric compactness theorem by Choi-Schoen cannot possibly hold in higher dimension or codimension, respectively, no matter how restrictive curvature assumptions one considers on the ambient manifold. A related question, explicitly posed by White in 1984, see [4], is whether strong compactness still holds under weaker curvature conditions but in ambient dimension 3, specifically in the class of compact 3-manifolds of positive scalar curvature. For the closed case, this question was fully answered by Colding and De Lellis in [13] (after earlier, significant contributions by Hass, Norbury and Rubinstein [29]): given any compact 3-manifold \(M\) supporting metrics of positive scalar curvature, and a non-negative integer \(a\), one can construct a metric of positive scalar curvature \(g = g(a)\) so that the ambient manifold \((M, g)\) contains a sequence of pairwise distinct, closed minimal surfaces of genus \(a\) which is not compact in the sense above. In fact, one can analyze the limit behavior of such a sequence in detail: one witnesses convergence to a singular minimal lamination (inside the given ambient manifold) with precisely two singular points, located at two antipodes on a stable minimal sphere. Furthermore, the value of the area and of the Morse index of these surfaces diverge. The
reader is referred to the monograph [15] for basic background and as a reference for the terminology we employ.

Here we discuss, and solve, the corresponding problem in the setting of free boundary minimal surfaces inside smooth compact 3-manifolds with boundary.

**Theorem 1.12.** Let $M$ be a compact, orientable, 3-manifold supporting Riemannian metrics of positive scalar curvature and mean convex boundary and let $a \geq 0$ and $b > 0$ be integers. Then there exists a Riemannian metric $g = g(a,b)$ of positive scalar curvature and totally geodesic boundary such that $(M, g)$ contains a sequence of connected, embedded, free boundary minimal surfaces of genus $a$ and exactly $b$ boundary components, and whose area and Morse index attain arbitrarily large values. Analogous conclusions hold true requiring the ambient manifold to have positive scalar curvature and strictly mean convex boundary.

This shows that such curvature conditions, i.e. $R_g > 0$ in $M$ and $H^\partial M \geq 0$ on its boundary $\partial M$ are in general too weak to ensure any form of geometric compactness. In fact, in each of the above examples we witness not only the lack of smooth single-sheeted subsequential convergence, but even of a milder form of subsequential convergence (namely: smooth, possibly with integer multiplicity $m \geq 1$, away from finitely many points) to a smooth free boundary minimal surface. The limit object is, as above, more pathologic than one would hope for. On the other hand, the reader may want to compare these ‘negative’ results with some of the geometric applications in [2], based on a bubbling analysis: it is shown that (in 3-manifolds satisfying the aforementioned curvature conditions) sequences of free boundary minimal surfaces of fixed topological type and correspondingly low index shall indeed sub-converge in the strongest geometric sense.

**Remark 1.13.** In certain cases, for instance when $b = 1$, it is possible to deform the metric $g$ near the boundary $\partial M$ so that $(M, g)$ has positive scalar curvature, strictly convex boundary and contains a sequence of free boundary minimal surfaces whose limit behavior is as above.

**Remark 1.14.** A simpler variant of the very same construction we shall present in the proof of Theorem 1.12 allows to prove that, when dropping any curvature requirement, these non-compactness phenomena can be made to occur inside any pre-assigned topological 3-manifold.

**Remark 1.15.** A full topological characterization of those compact 3-manifolds that support metrics of positive scalar curvature and mean convex boundary has been obtained by the first author and Chao Li in [8, Theorem A]. Roughly speaking, one can assert that those curvature conditions are only ‘mildly’ restrictive from the topological perspective, as in particular the boundary $\partial M$ can be the disjoint union of closed surfaces whose genera correspond to any pre-assigned string of non-negative integers.

The examples we construct partly rely on earlier work by Colding-De Lellis [13]. Their construction is, in some sense, modular: they build some simple blocks and develop tools to glue such blocks together by means of a suitable wire matching argument. The novel ingredients one needs to prove our results is a new family of building blocks: for any $b \geq 0$ we construct a (conveniently simple) Riemannian metric of positive scalar curvature on the 3-ball so that the resulting 3-manifold contains a sequence of free boundary surfaces of genus 0 and exactly $b$ boundary components. The construction we present is rather different depending on whether $b \geq 2$ or instead $b = 1$, the latter relying on a smoothing lemma by P. Miao employed to the scope of desingularizing a preliminary edgy model. A detailed proof of Theorem 1.12 is provided in Section 9.

To conclude this introduction, we consider three classes of examples that have appeared in recent years in the literature and briefly discuss how they naturally provide counterexamples to the last two implications in Figure 1 we still need to discuss.
Remark 1.16. We remind the reader of the following existence results:

- In [22] A. Fraser and R. Schoen have constructed a sequence $\{\Sigma_k^1\}$ of free boundary minimal surfaces in the unit ball of $\mathbb{R}^3$ having genus 0 and any number $b \geq 2$ of boundary components; it follows from their construction that as one lets $k \to \infty$ the surfaces in question converge (e.g. in the sense of varifolds) with multiplicity two to a flat equatorial disc.

- In [32] N. Kapouleas and M. Li have constructed a sequence $\{\Sigma_k^2\}$ of free boundary minimal surfaces in the unit ball of $\mathbb{R}^3$ having genus $a \geq a_0$ (sufficiently large) and exactly $b = 3$ boundary components; it follows from their construction that as one lets $k \to \infty$ the surfaces in question converge with multiplicity one to the union of an equatorial disc and the critical catenoid.

- In [33] N. Kapouleas and D. Wiygul have constructed a sequence $\{\Sigma_k^3\}$ of free boundary minimal surfaces in the unit ball of $\mathbb{R}^3$ having genus $a \geq a_0$ (sufficiently large) and exactly one boundary component; it follows from their construction that as one lets $k \to \infty$ the surfaces in question converge with multiplicity one to the union of three equatorial discs (sharing a segment and meeting at dihedral angles that are integer multiples of $\pi/6$).

Hence, in each of the three cases we have a sequence of minimal surfaces with uniformly bounded area, unbounded topology and (by Theorem 1.2) with unbounded index. Note that, in fact, the second and third family actually arise via a desingularization procedure.

1.3. Structure of the article. We provide an outline of the contents of this article by means of the following diagram.

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2. Notation and definitions

2.1. Basic notation. Let \((M^3, g)\) be a Riemannian manifold with boundary and let \(\Sigma^2 \subset M\) be an embedded surface \(\Sigma^2 \subset M\). Then we denote by:

- \(D\) the connection on \(M\) and \(\nabla\) the induced connection on \(\Sigma\).
- \(\nu\) a choice of a global unit normal vector field on \(\Sigma\) when \(\Sigma\) is two-sided.
- \(\eta\) the outward unit co-normal vector field to \(\partial \Sigma\).
- \(\hat{\eta}\) the outward unit co-normal vector field to \(\partial M\) (which coincides with \(\eta\) along \(\partial \Sigma\) when \(\Sigma^2\) satisfies the free boundary property).
- \(A(X,Y) = (D_X Y)\perp\) the second fundamental form of \(\Sigma \subset M\) and \(\Pi^M(X,Y) = g(D_X Y, \hat{\eta})\) the second fundamental form of \(\partial M \subset M\). Observe that \(\Pi^M < 0\) if for example \(M\) is the unit ball in \(\mathbb{R}^3\) (and thus \(\partial M\) is the unit sphere).
- \(H^\partial M\) the mean curvature of \(\partial M\), that is \(H^\partial M = -\Pi^M(E_1, E_1) - \Pi^M(E_2, E_2)\) for every choice of orthonormal basis \(\{E_1, E_2\}\) of \(\partial M\). We say that \(\partial M\) is mean convex if \(H^\partial M \geq 0\) and it is strictly mean convex if \(H^\partial M > 0\) (that is for example the case of the unit ball in \(\mathbb{R}^3\)).
- \(\chi(\Sigma), \text{genus}(\Sigma), \text{boundaries}(\Sigma)\) and \(\mathcal{H}^2(\Sigma)\) respectively the Euler characteristic, the genus, the number of boundary components and the area of \(\Sigma\).

See Figure 2 for an example of a free boundary minimal surface in the unit ball \(B^3 \subset \mathbb{R}^3\) with some notation included.

\[
\begin{align*}
M & \quad \eta \\
\Sigma & \quad \nu
\end{align*}
\]

**Figure 2.** Example of free boundary minimal surface with notation.

In certain circumstances, for instance as a result of a blow-up procedure, one needs to work in a half-space of \(\mathbb{R}^3\). In that respect, for \(0 \geq a \geq -\infty\) we shall set \(\Xi(a) := \{x^1 \geq a\}\) and \(\Pi(a) := \{x^1 = a\}\) (its boundary). When \(a = -\infty\), we agree that \(\Xi(a)\) coincides with all \(\mathbb{R}^3\) and \(\Pi(a)\) is empty. We omit the dependence on \(a\) when we are interested in a generic half-space, without caring about the distance of the origin from the boundary.

2.2. Setting. Given a Riemannian manifold \((M^3, g)\) with non-empty boundary, it is customary to say that a surface \(\Sigma^2 \subset M\) is properly embedded if \(\partial \Sigma = \Sigma \cap \partial M\). As anticipated in the introduction, in this article we focus our attention on compact Riemannian manifolds \((M^3, g)\) (with non-empty boundary) that satisfy property \((\mathfrak{P})\), which means that every smooth, connected, complete (possibly non-compact), embedded surface \(\Sigma^2 \subset M\) with zero mean curvature which meets the boundary of \(M\) orthogonally along its own boundary is properly embedded.
Remark 2.1. Notice that, when $\Sigma$ or $M$ are non-compact, there is a possible confusion between proper in the sense stated above, namely $\partial \Sigma = \Sigma \cap \partial M$, and proper in the sense that the inclusion map $\Sigma \hookrightarrow M$ is proper. Unfortunately during the article we have to use both meanings, since we need to consider non-compact free boundary minimal surfaces in our ambient manifold and even free boundary minimal surfaces in non-compact ambient spaces. Therefore, when we talk about property embedded surfaces with boundary we mean that both properties are satisfied. In all cases of possible ambiguity, we try to make the interpretation as explicit and clear as possible.

Thanks to the maximum principle, property (Q) is implied by the following geometric condition on the boundary.

\((C)\): The boundary $\partial M$ is mean convex and has no minimal component.

Remark 2.2. Note that the assumptions of Theorem 1.1, Theorem 1.4, Proposition 1.8 and Corollary 1.9 all imply \((C)\) and thus \((Q)\).

2.3. Free boundary minimal surfaces. Given a Riemannian manifold $(M^3, g)$, we denote by $\mathcal{X}_\partial = \mathcal{X}_\partial(M)$ the linear space of smooth ambient vector fields $X$ such that

\begin{enumerate}[(i)]
  \item $X(x) \in T_x M$ for all $x \in M$,
  \item $X(x) \in T_x \partial M$ for all $x \in \partial M$.
\end{enumerate}

The first variation of the area of a smooth properly embedded surface $\Sigma^2 \subset M$ with respect to the flow $\psi_t$ generated by such a vector field $X$ is given by

$$\frac{d}{dt} \bigg|_{t=0} \mathcal{H}^n(\psi_t(\Sigma)) = \int_{\Sigma} \text{div}_\Sigma(X) \, d\mathcal{H}^2 - \int_{\Sigma} \langle H, X \rangle \, d\mathcal{H}^2 + \int_{\partial \Sigma} \langle X, \eta \rangle \, d\mathcal{H}^1. \tag{2.1}$$

Therefore, $\Sigma$ is a stationary point of the area functional if and only if it has zero mean curvature and meets the ambient boundary orthogonally. In this case $\Sigma$ is called free boundary minimal surface.

Remark 2.3. It is possible to extend this definition also to non-properly embedded surfaces, convening that $\Sigma^2 \subset M$ is a free boundary minimal surface if it has zero mean curvature and meets the boundary of the ambient manifold orthogonally along its own boundary. Note that local limits of properly embedded free boundary minimal surfaces can be non-properly embedded even assuming property (Q). In particular this is the case of Theorem 3.5.

Remark 2.4. In certain circumstances, we shall deal with free boundary minimal surfaces $\Sigma$ whose boundary part is not entirely contained in $\partial M$, that is $\partial \Sigma \setminus \partial M \neq \emptyset$ (cf. also [3, Remark 13]). For example, often we look at surfaces restricted to geodesic balls in the ambient manifold, in that case we talk about edged free boundary minimal surface and we mean that they are free boundary only with respect to the boundary of the ambient contained in the ball, and not with respect to the relative boundary of the ball in question.

Then, given a properly embedded free boundary minimal surface $\Sigma$, one can consider the second variation of the area functional, which can be written as

$$\frac{d^2}{dt^2} \bigg|_{t=0} \mathcal{H}^n(\psi_t(\Sigma)) = \int_{\Sigma} (|\nabla^\perp X^\perp|^2 - (\text{Ric}_M(X^\perp, X^\perp) + |A|^2|X^\perp|^2)) \, d\mathcal{H}^2 + \int_{\partial \Sigma} \Pi^\partial M(X^\perp, X^\perp) \, d\mathcal{H}^1, \tag{2.2}$$

where $X^\perp$ is the normal component of $X$ and $\nabla^\perp$ is the induced connection on the normal bundle $N\Sigma$ of $\Sigma \subset M$. Thus, given a section $V \in \Gamma(N\Sigma)$ of the normal bundle, the second variation along the flow generated by $V$ is equal to the quadratic form

$$Q_\Sigma(V, V) := \int_{\Sigma} (|\nabla^\perp V|^2 - (\text{Ric}_M(V, V) + |A|^2|V|^2)) \, d\mathcal{H}^2 + \int_{\partial \Sigma} \Pi^\partial M(V, V) \, d\mathcal{H}^1.$$
The (Morse) index of $\Sigma$ is defined as the maximal dimension of a linear subspace of $\Gamma(N\Sigma)$ in which $Q^\Sigma$ is negative definite. Under the above assumptions, this number equals the number of negative eigenvalues of the elliptic problem
\[
\begin{cases}
\Delta^\Sigma V + \text{Ric}^\Omega(M)(V,\cdot) + |A|^2 V + \lambda V = 0 & \text{on } \Sigma, \\
\nabla^\eta \eta V = -(\text{II}^\Omega(M)(\cdot,\cdot))^2 & \text{on } \partial \Sigma.
\end{cases}
\]

Observe that, if $\Sigma^2 \subset M^3$ is two-sided, then every vector field $V \in \Gamma(N\Sigma)$ can be written as $V = f\nu$ and $Q^\Sigma(V,V)$ coincides with the quadratic form $Q^\Sigma(f,f)$, defined as
\[
Q^\Sigma(f,f) := \hat{\Sigma}(|\nabla f|^2 - (\text{Ric}^\Omega(M)(\nu,\nu) + |A|^2)f^2) dH^2 + \hat{\partial \Sigma} (f \partial f + \text{II}^\Omega(M)(\nu,\nu)f^2) dH^1,
\]
where $J^\Sigma := \Delta^\Sigma + (\text{Ric}^\Omega(M)(\nu,\nu) + |A|^2)$ is the scalar Jacobi operator of $\Sigma$.

2.4. **Euler characteristic.** Recall that the Euler characteristic of a compact surface $\Sigma$ is equal to
\[
\chi(\Sigma) = \begin{cases} 
2 - 2 \text{genus}(\Sigma) - \text{boundaries}(\Sigma) & \text{if } \Sigma \text{ is orientable,} \\
1 - \text{genus}(\Sigma) - \text{boundaries}(\Sigma) & \text{if } \Sigma \text{ is not orientable.}
\end{cases}
\]

Now let $\Sigma_1, \Sigma_2$ be two compact oriented surfaces with boundary and consider $c_1, c_2$ be two boundary components of $\Sigma_1$ and $\Sigma_2$ respectively. Notice that $c_1$ and $c_2$ are both homeomorphic to $S^1$. As shown in Figure 3, we can glue $\Sigma_1$ and $\Sigma_2$ along $c_1$ and $c_2$ in two ways:

(i) We can attach all $c_1$ to all $c_2$.

(ii) We can attach a segment of $c_1$ to a segment of $c_2$.

![Figure 3. Gluing of two discs via (i) (on the left) and (ii) (on the right).](image)

We can construct an oriented surface $\Sigma$ by gluing $b$ boundary components of $\Sigma_1, \Sigma_2$ as in (i) and $b'$ boundary components as in (ii). Then the Euler characteristic of $\Sigma$ is given by
\[
\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_2) - b'.
\]

Therefore $\Sigma$ has genus equal to $\text{genus}(\Sigma_1) + \text{genus}(\Sigma_2) + b + b' - 1$ and number of boundary components equal to the sum of the boundary components of $\Sigma_1$ and $\Sigma_2$ minus $2b + b'$.

3. **Free boundary minimal laminations**

In this section we want to generalize the definitions and the results about minimal laminations (see for example [6, Definition 2.1] or [37, Definition 2.2]) to the case of manifolds with boundary (see also [25, Section 5]). Indeed, laminations naturally arise as limits of (free boundary) minimal surfaces that are assumed to have uniformly bounded index, but not necessarily uniformly bounded area.
3.1. Definitions and compactness.

**Definition 3.1.** A free boundary minimal lamination $\mathcal{L}$ in a three-dimensional Riemannian manifold $(M, g)$ with boundary $\partial M$ is the union of a collection of pairwise disjoint, connected, embedded free boundary minimal surfaces of $M$. Moreover we require that $\mathcal{L}$ is a closed subset of $M$ and that, for each $x \in M$ one of the following assertions holds:

(i) $x \in M \setminus \partial M$ and there exists an open neighborhood $U$ of $x$ and a local coordinate chart $\varphi : B_2^3(0) \times (0, 1) \subset \mathbb{R}^3 \to U$ such that $\varphi^{-1}(\mathcal{L} \cap U) = B_2^3(0) \times C$ for a closed subset $C \subset (0, 1)$;

(ii) $x \in \partial M$ and there exists a relatively open neighborhood $U$ of $x$ and a local coordinate chart $\varphi : (B_2^3(0) \cap \{x^1 \geq 0\}) \times (0, 1) \subset \mathbb{R}^3 \to U$ such that $\varphi^{-1}(\mathcal{L} \cap U) = (B_2^3(0) \cap \{x^1 \geq 0\}) \times C$ for a closed subset $C \subset (0, 1)$;

(iii) $x \in \partial M$ and there exists an extension $\tilde{M}$ without boundary and an open neighborhood $U$ of $x$ in $\tilde{M}$ such that property (i) is satisfied for the neighborhood $U \ni x$.

Moreover we say that the free boundary minimal lamination is properly embedded if each of its leaves is properly embedded (in both senses, as explained in Remark 2.1).

A schematic representation of the three different situations in Definition 3.1 can be found in Figure 4.

![Figure 4. Definition of lamination in chart.](image)

**Remark 3.2.** Notice that, if we require property (P) on $M$, the case (iii) cannot occur. We have included it in the definition because we need to consider free boundary minimal laminations in half-spaces of $\mathbb{R}^3$ (which do not fulfill (P)) and local limits of free boundary minimal surfaces (or laminations) which a priori can be non-properly embedded (see Remark 2.3 and Theorem 3.5 below).

**Definition 3.3.** We say that a point $p \in L$ of a minimal lamination $\mathcal{L}$ is a limit point if there exists a coordinate chart $(U, \varphi)$ with $p \in U$ as in the previous definition such that $\varphi^{-1}(p) = (t, x)$ and $t$ is an accumulation point for $C$.

**Remark 3.4.** Thanks to the Harnack inequality, if $p$ is a limit point of a lamination $\mathcal{L}$, then the entire leaf through $p$ consists of limit points of $M$. In this case, we shall call it a limit leaf.

As anticipated, we introduce the concept of lamination to gain compactness. Indeed the following theorem holds.

**Theorem 3.5** ([14, Proposition B.1] and [25, Theorem 5.5]). Let $(M^3, g)$ be a complete three-dimensional Riemannian manifold with boundary. Given $x \in M$, let $\mathcal{L}_j \subset B_{2r}(x) \subset M$ for $j \in \mathbb{N}$ be a sequence of free boundary minimal laminations with uniformly bounded curvatures. Then there exists a subsequence which converges in $B_r(x)$ in the $C^{0,\alpha}$ topology for any $\alpha < 1$ to a Lipschitz minimal lamination $\mathcal{L}$ whose (possibly non-properly embedded) leaves have free boundary with respect to $\partial M$. Moreover the leaves of $\mathcal{L}$ are smooth free boundary minimal surfaces and the leafwise convergence is $C^\infty$. In this case, we will say that $\mathcal{L}_j$ locally converges to $\mathcal{L}$ in the sense of laminations.
Remark 3.6. The convergence of a sequence of laminations $L_j$ to a lamination $L$ is leafwise $C^{\infty}$ if, for every sequence of points $x_j \in L_j$ that converges to a point $x \in M$, there exists a leaf $L \subset L_j$ such that $x \in L$ and a neighborhood $U$ of $x$ such that the connected component $L'_j$ of $L_j \cap U$ that contains $x_j$ converges to the connected component of $L \cap U$ that contains $x$ smoothly with multiplicity one (in the sense of graphs).

It is well-known that a two-sided minimal surface having a positive Jacobi field is stable (cf. e.g. [15, Lemma 1.36]). In turn, the existence of a positive Jacobi field can be deduced whenever multi-sheeted convergence occurs. We shall present here a helpful variation on this theme, whose proof is, by now, rather standard (cf. e.g. Theorem 22 in [50], and Proposition 2.1 in [7]).

Lemma 3.7. Let $(M^3, g)$ be a complete three-dimensional Riemannian manifold with boundary and let $\Sigma_j^3 \subset M$ be a sequence of (connected) properly embedded free boundary minimal surfaces that locally converges to a free boundary minimal lamination $L \subset M \setminus S_{\infty}$ away from a finite set of points $S_{\infty}$. Consider a leaf $L \in \mathcal{L}$, then one of the following assertions holds:

1. $L$ has stable universal cover;
2. the convergence of $\Sigma_j$ to $L$ is locally smooth with multiplicity one (in the sense of graphs); namely, for every $x \in L$ there exists a neighborhood $U$ of $x$ in $M$ such that $U \cap (\bigcup_{L \in \mathcal{L}} L') = U \cap L$ and $U \cap \Sigma_j$ converges smoothly to $U \cap L$ with multiplicity one (as graphs).

3.2. Removable singularities. Thanks to Theorem 3.5, in the following section (Section 4) we prove that free boundary minimal surfaces of uniformly bounded index converge, possibly extracting a subsequence, to a free boundary minimal lamination which is smooth away from a finite number of points. The aim of the following propositions is to prove that these singularities are actually removable.

Corollary 3.8. Let $\varphi : \Sigma \to \Xi(0) \setminus \{0\} \subset \mathbb{R}^3$ be a stable, two-sided minimal immersion that has free boundary with respect to $\Pi(0)$ and is complete away from $\{0\}$. Then the closure of $\varphi(\Sigma)$ is a plane or a half-plane.

Proof. Consider the double $\hat{\Sigma}$ of $\Sigma$ (that is to say: the boundaryless surface that is obtained by reflecting $\Sigma$ across $\partial \Sigma$) and let $\tau : \hat{\Sigma} \to \hat{\Sigma}$ be the associated involution. Denoting with $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ the reflection with respect to $\Pi(0)$, we define the map $\hat{\varphi} : \hat{\Sigma} \to \mathbb{R}^3 \setminus \{0\}$ as follows:

$$\hat{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in \Sigma \subset \hat{\Sigma}, \\ \varphi(\tau(x)) & \text{if } x \in \hat{\Sigma} \setminus \Sigma. \end{cases}$$

Observe that $\hat{\varphi}$ is a two-sided minimal immersion that is complete away from $\{0\}$. Moreover, it follows from the discussion presented in [2, Section 2] that $\hat{\varphi}$ is stable. Therefore, the result is a direct consequence of the Bernstein-type theorem by Gulliver-Lawson [27]. \[\square\]

The following result mirrors the one obtained for interior points in Proposition D.3 of [9].

Proposition 3.9. Let $(M^3, g)$ be a complete Riemannian manifold with boundary. Fix $p \in \partial M$ and $\varepsilon_0 > 0$ and consider an embedded minimal surface $\hat{\Sigma} \subset B_{\varepsilon_0}(p) \setminus \{p\}$ that has free boundary with respect to $\partial M$, such that it has stable universal cover. Then $\hat{\Sigma}$ smoothly extends across $p$, i.e. there exists a free boundary minimal surface $\Sigma \subset B_{\varepsilon_0}(p)$ such that $\hat{\Sigma} = \Sigma \setminus \{p\}$.

Remark 3.10. Notice that we are not requiring that $\hat{\Sigma}$ is properly embedded in $B_{\varepsilon_0}(p) \setminus \{p\}$ (in particular it could be non-properly embedded in the sense of maps).

Proof. First notice that we can assume that $p$ belongs to the topological closure of $\hat{\Sigma}$, otherwise the result would be obvious. Taking $\varepsilon_0 > 0$ possibly smaller and using [25, Theorem 1.2], we can assume
that in $B_{\varepsilon_0}(p)$ it holds $|A_{\hat{g}}|(x)d_{\hat{g}}(p,x) \leq C$ for all $x \in \hat{\Sigma}$, for some $C > 0$. Therefore, for any $r_j \to 0$, the surfaces $r_j^{-1}(\hat{\Sigma} - p)$ locally converge (in the sense of laminations), up to subsequence, to a free boundary minimal lamination $L_\infty \subseteq \Xi(0) \setminus \{0\}$ thanks to Theorem 3.5.

Observe that each leaf of $L_\infty$ is complete away from $\{0\}$. We now argue that each leaf also has stable universal cover. Consider a leaf $L \subset L_\infty$; then, by Lemma 3.7, $L$ has stable universal cover or the convergence to $L$ is locally smooth with multiplicity one. However, if the second case occurs, the stability of the universal cover of the surfaces $r_j^{-1}(\hat{\Sigma} - p)$ is inherited by the universal cover of $L$.

Hence, we can apply Corollary 3.8 to obtain that $L_\infty$ consists of parallel planes or half-planes and thus, possibly further restricting the ball, we can improve the curvature estimate to (say)

\[(3.1) \quad |A_{\hat{g}}|(x)d_{\hat{g}}(x,p) \leq \frac{1}{4}\]

for all $x \in \hat{\Sigma} \subset B_{\varepsilon_0}(p) \setminus \{p\}$.

We now want to prove that $L_\infty$ is either a single half-plane $\Delta$ passing through the origin and orthogonal to $\Pi(0)$ or $\Pi(0)$ itself. If not the case, then there would exist another plane or half-plane not passing through the origin which is in the limit of the rescalings $r_j^{-1}(\hat{\Sigma} - p)$; hence one could define $\delta \in (0,\varepsilon_0)$ sufficiently small such that $\hat{\Sigma} \cap (B_{3\delta}(p) \setminus \{p\})$ contains a properly embedded component diffeomorphic to a disc or a half-disc, which intersects $\partial B_{3\delta}(p)$ transversely and does not contain $p$. However, by Corollary C.3, this implies that $\hat{\Sigma}$ itself contains a disc or half-disc (obtained by gluing the previous disc or half-disc with its corresponding ‘trivial’ component of $\Sigma \setminus B_3(p)$), but this is a contradiction since $\hat{\Sigma}$ is connected and its closure contains $p$. Therefore $L_\infty$ consists of only one leaf\(^1\), which is either $\Delta$ or $\Pi(0)$.

We can assume to work in Fermi coordinates (see [2, Section 3]) around $p$ and identify $L_\infty$ (which is $\Delta$ or $\Pi(0)$) with the same plane in coordinates. Hence, let us consider $\delta \in (0,\varepsilon_0/3)$ such that $\hat{\Sigma} \cap (B_{3\delta}(p) \setminus B_3(p))$ intersects $\partial B_{2\delta}(p)$ transversely, is sufficiently smoothly close to the limit half-plane or plane and is a multigraph over it. In particular $\hat{\Sigma} \cap \partial B_{2\delta}(p)$ is the union of injectively immersed curves.

If $\hat{\Sigma} \cap \partial B_{2\delta}(p)$ contains a compact curve, which can be either an $S^1$ or an interval, then $\hat{\Sigma} \cap B_{3\delta}(p)$ is a properly embedded topological punctured disc or half-disc in $B_{2\delta}(p) \setminus \{p\}$, thanks to estimate (3.1) and Corollary C.3. Thus, we can apply Proposition D.1 in [9] or its free boundary analogue (based on [19, Theorem 4.1]), respectively, obtaining that $\hat{\Sigma}$ extends smoothly across $\{p\}$.

The only other situation that can happen is that $\hat{\Sigma} \cap \partial B_{2\delta}(p)$ consists of one or more spiraling curves. Observe that this case can only occur if $L_\infty$ has $\Pi(0)$ as its only leaf since, otherwise, $L_\infty \cap (B_{3\delta}(p) \setminus B_3(p)) = \Delta \cap (B_{3\delta}(p) \setminus B_3(p))$ is simply connected and thus it is not possible to see a spiraling behavior. However, the method to deal with the spiraling situation in the case in which $L_\infty = \{\Pi(0)\}$ is completely analogous to the one in the spiraling case in the proof for interior points in [9, Proposition D.3], therefore we can employ the same argument to conclude.\[\square\]

We are now able to discuss the general case, when one needs to deal with isolated singularities arising when taking limits of free boundary minimal surfaces with bounded index.

**Theorem 3.11.** Let $\{M^3, g\}$ be a compact three-dimensional Riemannian manifold with boundary. Let $\Sigma_j^0 \subset M$ be a sequence of properly embedded free boundary minimal surfaces with uniformly bounded index, which locally converge to a free boundary minimal lamination $\hat{\mathcal{L}} \subset M \setminus S_\infty$ away from a finite set of points $S_\infty$. Then $\hat{\mathcal{L}}$ extends smoothly through $S_\infty$, i.e. there exists a smooth lamination $\mathcal{L} \subset M$ such that $\hat{\mathcal{L}} = \mathcal{L} \setminus S_\infty$. Moreover, given any leaf $L \in \mathcal{L}$, one of the following assertions holds:

1. $L$ has stable universal cover;

\(^1\)Notice that $L_\infty$ cannot contain two intersecting (half-)planes, because this would contradict the embeddedness of $\hat{\Sigma}$.\[\]
(2) the convergence of $\Sigma_j$ to $L$ is locally smooth with multiplicity one (in the sense of graphs); namely, for every $x \in L$ there exists a neighborhood $U$ of $x$ in $M$ such that $U \cap (\bigcup_{j \in \mathcal{L}} L_j) = U \cap L$ and $U \cap \Sigma_j$ converges smoothly to $U \cap L$ with multiplicity one (as graphs).

Proof. We split the proof in two steps: in the first step we show that $\hat{\mathcal{L}}$ extends through $\mathcal{S}_\infty$, while in the second step we prove properties (1) and (2) of the leaves of $\mathcal{L}$.

Step 1. Since the result is local, we can assume to work around a single point $p \in \mathcal{S}_\infty$. First, let us prove that there exists $\varepsilon_0 > 0$ sufficiently small such that each leaf of $\hat{\mathcal{L}}$ in $B_{\varepsilon_0}(p)$ has stable universal cover.

Thanks to Lemma 3.7, either $\hat{L} \in \hat{\mathcal{L}}$ has stable universal cover or the convergence of $\Sigma_j$ to $\hat{L}$ is locally smooth and with multiplicity one. Observe that there can be only a finite number of unstable leaves of the second type, otherwise the uniform bound on the index of $\Sigma_j$ would be violated. If we focus on one such leaf, we can argue as follows. Taking $\varepsilon_0$ possibly smaller, we can assume that $B_{\varepsilon_0}(p)$ is simply connected and thus $\Sigma_j \cap B_{\varepsilon_0}(p)$ is two-sided. Hence, it is easily argued that (by virtue of the multiplicity one convergence) $\hat{L} \cap B_{\varepsilon_0}(p)$ has to be two-sided as well. Now, a straightforward variation of the same argument as in [17, Proposition 1] proves that we can pick $\varepsilon_0$ even smaller, in such a way that $\hat{L} \cap (B_{\varepsilon_0}(p) \setminus \{p\})$ is stable and, in addition, the Jacobi operator has a positive solution in the same domain. Hence, using such a function, we can derive that $\hat{L} \cap (B_{\varepsilon_0}(p) \setminus \{p\})$ has, in fact, stable universal cover.

We have thus proved that there exists $\varepsilon_0 > 0$ such that every leaf of $\hat{\mathcal{L}}$ in $B_{\varepsilon_0}(p) \setminus \{p\}$ has stable universal cover. Then we can apply Proposition 3.9 to obtain that each leaf extends smoothly across $p$. Note that the extended leaves cannot meet at $p$ (since $\hat{\mathcal{L}}$ is assumed to be a laminating in $M \setminus \mathcal{S}_\infty$).

It is only left to prove that $\mathcal{L}$ obtained as union of the extended leaves has the structure of lamination around the point $p$. This fact follows from the proof of Proposition 3.9, where it was shown that that for any sequence of positive numbers $r_j \to 0$ we have that $r_j^{-1}(\mathcal{L} - p)$ converge to a laminating consisting of parallel planes or half-planes.

Step 2. If $L$ is a limit leaf of $\mathcal{L}$ (see Remark 3.4), then $L$ has stable universal cover by standard arguments (cf. also Lemma 3.7). On the other hand, if the convergence of $\Sigma_j$ to $L$ is locally smooth with multiplicity one (in the sense of graphs) away from $\mathcal{S}_\infty$, then the convergence extends across $\mathcal{S}_\infty$ thanks to Lemma D.1 and we end up in case (2).

Therefore, let us assume that $L$ is a limit leaf and that $\Sigma_j$ does not converge to $L$ with multiplicity one. Possibly passing to the double cover, we can assume that $L$ is two-sided (cf. [3, Section 6]). We then show that $L$ admits a positive Jacobi function, which proves that $L$ is stable as well as its universal cover.

Let us consider a regular domain $\Omega \subset L \setminus \mathcal{S}_\infty$. Consider a vector field $X \in \mathfrak{X}_\partial(M)$ that is unitary and normal to $L$ along $L$. Denote by $\Phi(x,t)$ the flow associated to $X$ and, for every $\varepsilon > 0$, define

$$\Omega_\varepsilon := \{ \Phi(x,t) : x \in \Omega, \ |t| < \varepsilon \}.$$  

Observe that we can fix $\varepsilon > 0$ such that the convergence of $\Sigma_j$ to $L$ is smooth in the sense of laminations in $\Omega_\varepsilon$ and such that the only component of $\mathcal{L}$ in $\Omega_\varepsilon$ is $L \cap \Omega_\varepsilon$.

By definition of convergence in the sense of laminations, $\Omega_\varepsilon \setminus \Omega_{\varepsilon/2}$ does not intersect $\Sigma_j$ for every $j$ sufficiently large, since it does not intersect any leaf of $\mathcal{L}$. Then, for every $j$, define

$$u^+_j(x) := \sup \{ t \in (-\varepsilon, \varepsilon) : \Phi(x,t) \in \Sigma_j \}$$  

and

$$u^-_j(x) := \inf \{ t \in (-\varepsilon, \varepsilon) : \Phi(x,t) \in \Sigma_j \}.$$

Note that $-\varepsilon/2 < u^+_j(x) < u^-_j(x) < \varepsilon/2$, since $\Sigma_j \cap (\Omega_\varepsilon \setminus \Omega_{\varepsilon/2}) = \emptyset$ and $\Sigma_j$ does not converge to $L$ with multiplicity one. Furthermore observe that, by Remark 3.6 together with the compactness of $\Omega$,
for every $j$ sufficiently large (depending on $\Omega$) the surfaces $\Sigma_j^\pm := \{\Phi(x,u_j^\pm(x)) : x \in \Omega\} \subset \Sigma_j$ are well-defined smooth surfaces (with boundary) that converge uniformly smoothly to $\Omega$.

At this point, we can go through the very same argument as in the proof of Theorem 5 in [3, Section 6], so we just sketch it briefly.

Fixing $x_0 \in \Omega \setminus \partial M$, define $h_j := u_j^+ - u_j^- > 0$ and $h_j(x) := h_j(x_0)^{-1} h_j(x)$. Then, following exactly the same proof of Claim 1 in [3, Section 6], we have that $h_j$ is bounded in $C^4(\Omega)$ for all $l \in \mathbb{N}$ and converges smoothly to $h \in C^\infty(\Omega)$, solution of

$$
\begin{cases}
J_L(h) = \Delta_L h + (\frac{1}{4}R_g + \frac{1}{2}|A|^2 - K)h = 0 & \text{in } \Omega, \\
\frac{\partial h}{\partial n} = -\Pi(\nu, \nu)h & \text{on } \partial M \cap \Omega.
\end{cases}
$$

Then, by taking an exhaustion of $L \setminus S_\infty$ by domains $\Omega \Subset L \setminus S_\infty$ containing $x_0$, we obtain a function $h \in C^\infty(L \setminus S_\infty)$ solving (3.2) in $L \setminus S_\infty$. Moreover $h(x_0) = 1$ and $h \geq 0$. Finally, thanks to the same proof of Claim 2 in [3, Section 6], it holds that $h$ is uniformly bounded and thus extends to a smooth Jacobi function on all $L$, which is positive everywhere thanks to the maximum principle and the Hopf boundary point lemma. □

### 3.3. Laminations in the Euclidean space.

As anticipated, because of blow-up procedures, we will deal with free boundary minimal surfaces in half-spaces $\Xi$ of (flat) $\mathbb{R}^3$. The first proposition is what we need to transfer the results on the Euclidean space $\mathbb{R}^3$ to the free boundary case and it is just a consequence of Lemma B.1.

**Lemma 3.12.** Let $L \subset \Xi \subset \mathbb{R}^3$ be a free boundary minimal lamination. Then the union of $L$ with its reflection with respect to the plane $\Pi$ is a minimal lamination in $\mathbb{R}^3$.

**Remark 3.13.** We recall that we admit that $L$ could contain the plane $\Pi$ as a leaf.

In the setting of Euclidean half-spaces, Theorem 3.11 reads as follows.

**Proposition 3.14.** Let $g_j$ be a sequence of Riemannian metrics in $\mathbb{R}^3$ that converges, locally smoothly, to the Euclidean metric. Let $\Sigma_j \subset \Xi(a_j) \subset \mathbb{R}^3$ for some $0 \geq a_j \geq -\infty$ be a sequence of properly embedded, edged, free boundary minimal surfaces in $(\Xi(a_j), g_j)$, such that for every $j \in \mathbb{N}$ we have $\Sigma_j \subset \Delta_j$ for a sequence of compact domains $\Delta_j$ exhausting $\Xi(a)$ for $a = \lim_{j \to \infty} a_j$, each being the intersection of a smooth domain of $\mathbb{R}^3$ with $\Xi(a_j)$. Assume that the surfaces $\Sigma_j$ have index bounded by $I \in \mathbb{N}$ and locally converge (in the sense of laminations) to a free boundary minimal lamination $\mathcal{L} \subset \Xi(a) \setminus S_\infty$ away from a finite set of points $S_\infty$. Then $\mathcal{L}$ extends smoothly through $S_\infty$ to a free boundary minimal lamination $\mathcal{L} \subset \Xi(a)$ and the following dichotomy holds:

1. $\mathcal{L}$ consists of parallel planes or half-planes;
2. $\mathcal{L}$ is a complete, non-flat, connected, properly embedded, free boundary minimal surface in $\Xi(a)$ of (positive) index at most $I$ and (a subsequence of) $\Sigma_j$ converges to $\mathcal{L}$ locally smoothly (in the sense of graphs) with multiplicity one.

**Proof.** The first part of the statement follows directly from Theorem 3.11, so let us prove properties (1) and (2). Let us consider a leaf $L \in \mathcal{L}$. By Theorem 3.11, $L$ has stable universal cover or the convergence of $\Sigma_j$ to $L$ is locally smooth with multiplicity one (in the sense of graphs). In the first case, $L$ must be a plane by Corollary 22 in [2]. In the second case, $L$ has index bounded by $I$, otherwise the bound on the index of $\Sigma_j$ would be violated. In particular, all the leaves of $\mathcal{L}$ have bounded index, thus the result is just the free boundary analogue of Corollary B.2 in [9], from which our result follows using Lemma 3.12 and Proposition B.2, which is needed (in particular) to ensure that the reflected minimal lamination still has finite index. □
4. MACROSCOPIC BEHAVIOR

In this section we study the behavior at macroscopic scale of a sequence of free boundary minimal surfaces with bounded index.

4.1. Smooth blow-up sets. Similarly to [9, Section 2.1.1], we first need to define a finite set of 'bad points', away from which a sequence of free boundary minimal surfaces with bounded index is well-controlled.

Definition 4.1. Let \((M^3, g)\) be a compact three-dimensional Riemannian manifold and suppose that \(\Sigma^2_j \subset M\) is a sequence of properly embedded edged free boundary minimal surfaces. A sequence of finite sets of points \(S_j \subset \Sigma_j\) is said to be a sequence of smooth blow-up sets if

(i) The second fundamental form blows up at points in \(S_j\), i.e.
   \[ \liminf_{j \to \infty} \min_{p \in S_j} |A_{\Sigma_j}(p)| = \infty. \]

(ii) Chosen a sequence of points \(p_j \in S_j\), the rescaled surfaces \(\bar{\Sigma}_j = |A_{\Sigma_j}|(p_j)(\Sigma_j - p_j)\) converge (up to subsequence) locally smoothly with multiplicity one (in the sense of graphs) to a complete, non-flat, properly embedded, free boundary minimal surface \(\bar{\Sigma}_\infty \subset \Xi(a)\), for some \(0 \geq a \geq -\infty\), satisfying
   \[ |A_{\bar{\Sigma}_\infty}|(x) \leq |A_{\bar{\Sigma}_\infty}|(0) \]
   for all \(x \in \bar{\Sigma}_\infty\).

(iii) Points in \(S_j\) do not appear in the blow-up limit of other points in \(S_j\), i.e.
   \[ \liminf_{j \to \infty} \min_{p \neq q \in S_j} |A_{\Sigma_j}|(p)d_g(p, q) = \infty. \]

4.2. Curvature estimates. The starting point for our study is a curvature estimate for stable minimal surfaces, here stated in the free boundary setting.

Theorem 4.2 ([25, Theorem 1.2]). Let \((M^3, g)\) be a compact Riemannian manifold with boundary. Then there exists a constant \(C = C(M, g)\) such that, if \(\Sigma^2 \subset M\) is a compact, properly embedded, stable, edged, free boundary minimal surface, then

\[ \sup_{x \in \Sigma} |A_{\Sigma}|(x) \min\{1, d_g(x, \partial \Sigma \setminus \partial M)\} \leq C. \]

Remark 4.3. Actually, in [25] the theorem is stated only for non-edged free boundary minimal surfaces, but it is observed in Remark 1.3 therein that the conclusion still holds in such more general setting.

Remark 4.4. Observe that the second fundamental form multiplied by a distance function is invariant under rescalings. This fact is often useful in our proofs.

We will need the following extension of the previous result.

Lemma 4.5. Let \((M^3, g)\) be a compact Riemannian manifold with boundary and fix \(I \in \mathbb{N}\). Suppose that \(\Sigma^2_j \subset M\) is a sequence of compact, properly embedded, edged, free boundary minimal surfaces with \(\text{ind}(\Sigma_j) \leq I\). Then, up to subsequence, there exist a constant \(C = C(M, g)\) and a sequence of smooth blow-up sets \(S_j \subset \Sigma_j\) with \(|S_j| \leq I\) and

\[ \sup_{x \in \Sigma_j} |A_{\Sigma_j}|(x) \min\{1, d_g(x, S_j \cup (\partial \Sigma_j \setminus \partial M))\} \leq C. \]

Remark 4.6. Notice that the reason to state the lemma with edged free boundary minimal surfaces is to perform the inductive procedure in the proof.
Proof. We proceed by induction on $I \in \mathbb{N}$. If $I = 0$ the statement is exactly Theorem 4.2, thus assume $I > 0$. Note that we can suppose that

$$
\rho_j := \max_{x \in \Sigma_j} |A_{\Sigma_j}|(x) \min\{1, d_g(x, \partial \Sigma_j \setminus \partial M)\} \to \infty,
$$

for otherwise one can take $\mathcal{S}_j = \emptyset$ and there is nothing left to do. Take $q_j \in \Sigma_j$, a point where $\rho_j$ is achieved and define $R_j := d_g(q_j, \partial \Sigma_j \setminus \partial M)/2$.

**Figure 5.** Point picking argument.

Now consider $p_j \in \Sigma_j \cap B_{R_j}(q_j)$ which realizes

$$
\max_{x \in \Sigma_j \cap B_{R_j}(q_j)} |A_{\Sigma_j}|(x) d_g(x, \partial B_{R_j}(q_j) \setminus \partial M)
$$

and define $r_j := d_g(p_j, \partial B_{R_j}(q_j) \setminus \partial M)$ and $\lambda_j := |A_{\Sigma_j}|(p_j)$. Notice that

$$
\lambda_j r_j = |A_{\Sigma_j}|(p_j) d_g(p_j, \partial B_{R_j}(q_j) \setminus \partial M) \geq |A_{\Sigma_j}|(q_j) d_g(q_j, \partial B_{R_j}(q_j) \setminus \partial M)
$$

$$
= |A_{\Sigma_j}|(q_j) R_j = \frac{1}{2} |A_{\Sigma_j}|(q_j) d_g(q_j, \partial \Sigma_j \setminus \partial M) \geq \rho_j \to \infty.
$$

Thus in particular $|A_{\Sigma_j}|(p_j) \min\{1, d_g(p_j, \partial \Sigma_j \setminus \partial M)\} \to \infty$, where we are using that $d_g(p_j, \partial \Sigma_j \setminus \partial M) \geq r_j$, since there are no points of $\partial \Sigma_j \setminus \partial M$ in $B_{R_j}(q_j)$.

Now, we have that $d_g(x, \partial B_{r_j}(p_j) \setminus \partial M) \leq d_g(x, \partial B_{R_j}(q_j) \setminus \partial M)$ for every $x \in B_{r_j}(p_j)$ with equality in $p_j$. Therefore $p_j$ also realizes

$$
\max_{x \in \Sigma_j \cap B_{r_j}(p_j)} |A_{\Sigma_j}|(x) d_g(x, \partial B_{r_j}(p_j) \setminus \partial M).
$$

Hence, we can now perform a blow-up argument around the points $p_j$. In particular, let us define

$$
\Sigma_j := \lambda_j(\Sigma_j - p_j),
$$

a surface in the manifold $M_j := \lambda_j(M - p_j)$ endowed with the metric $g_j$.

Then we have that

$$
|A_{\Sigma_j}|(x) d_g(x, \partial B_{\lambda_j r_j}(0) \setminus \partial M_j) \leq \lambda_j r_j \to \infty
$$

for all $x \in \Sigma_j \cap B_{\lambda_j r_j}(0)$. Note that here $B_{\lambda_j r_j}(0)$ is the ball in $M_j$ with respect to the metric $g_j$. We do not write explicitly the dependence on $j$ since it is always clear from the context.

---

3One can think of $M$ as isometrically embedded in some $\mathbb{R}^K$ and consider the blow-ups in this Euclidean space. Thus, in particular, we have that $g_j(x) = \lambda_j g(x + p_j)$. 

Hence, fixing $R > 0$, for all $x \in \bar{\Sigma}_j \cap B_R(0)$ it holds that
\[
|A_{\Sigma_j}(x)| \leq \frac{\lambda_j r_j}{\lambda_j r_j - R} < 1
\]
as $j \to \infty$. Moreover, observe that $|A_{\Sigma_j}(0)| = 1$ and the domains $B_{\lambda_j r_j}(0) \subset M_j$ do not contain points of $\partial \Sigma_j \setminus \partial M_j$, since $\Sigma_j$ has no points of $\partial \Sigma_j \setminus \partial M$ in $B_{r_j}(p_j)$. By Theorem 3.5 together with Proposition 3.14 (we fall in (2) since the limit lamination cannot be flat), this implies that the surfaces $\Sigma_j$ converge locally smoothly (in the sense of graphs) with multiplicity one to a properly embedded free boundary minimal surface $\Sigma_{\infty} \subset \Xi(a)$ (for some $-\infty < a \leq 0$). Furthermore, the index of $\Sigma_{\infty}$ is strictly positive and less or equal than $I$.

Hence, with a standard argument as in [17, Proposition 1], there exists $R_0 > 0$ such that $\Sigma_{\infty} \cap B_{R_0}(0)$ has index greater than 0 and $\Sigma_{\infty} \setminus B_{R_0}(0)$ is stable. Moreover we can assume, without loss of generality, that $\Sigma_{\infty}$ intersects $\partial B_{R_0}(0)$ transversely and, thanks to Proposition 28 in [2], also that
\[
|A_{\Sigma_{\infty}}|(x) \leq 1/4
\]
on $\Sigma_{\infty} \setminus B_{R_0}(0)$.

Now consider $\Sigma_j' := \Sigma_j \setminus B_{R_0/\lambda_j}(p_j)$. Choosing $j$ sufficiently large, we can assume that $\partial B_{R_0/\lambda_j}(p_j)$ intersects $\Sigma_j$ transversely and that the ball $B_{R_0/\lambda_j}(p_j)$ does not intersect the portion of the boundary of $\Sigma_j$ that is not contained in $\partial M$. Indeed we know that $\lambda_j d_g(p_j, \partial \Sigma_j \setminus \partial M) \to \infty$. Therefore $\Sigma_j'$ is a sequence of manifolds which still fulfills the assumptions of the lemma, but with $\dim(\Sigma_j') \leq I - 1$ for $j$ sufficiently large. Hence, by the inductive hypothesis, up to subsequence there exist a constant $C' > 0$ and a sequence of smooth blow-up sets $S_j' \subset \Sigma_j'$ with $|S_j'| \leq I - 1$ and
\[
|A_{\Sigma_j'}|(x) \min\{1, d_g(x, S_j' \cup (\partial \Sigma_j' \setminus \partial M))\} \leq C'.
\]

We now want to show that $S_j := S_j' \cup \{p_j\}$ is the desired sequence of blow-up sets. The only non-obvious point to check is point (iii) of Definition 4.1, for which it suffices to check that
\[
\lim_{j \to \infty} \inf_{q \in S_j'} |A_{\Sigma_j}|(q) d_g(p_j, q) \leq \lim_{j \to \infty} \inf_{q \in S_j'} |A_{\Sigma_j}|(q) d_g(p_j, q) = \infty.
\]
We first observe that indeed
\[
\lim_{j \to \infty} \inf_{q \in S_j'} |A_{\Sigma_j}|(q) d_g(p_j, q) \geq \lim_{j \to \infty} \inf_{q \in S_j'} |A_{\Sigma_j}|(q) d_g(\partial \Sigma_j' \setminus \partial M, q) = \infty
\]

based on (ii) for $q \in S_j'$ (the associated limit surface is not edged).

However, thanks to (4.1) and (4.2), we derive $|A_{\Sigma_j}|(q) \leq \frac{1}{4} |A_{\Sigma_j}|(p_j)$ for all $q \in S_j'$, provided one takes $j$ sufficiently large; hence $\min_{q \in S_j'} |A_{\Sigma_j}|(p_j) d_g(p_j, q)$ could not be uniformly bounded either.

Now, it remains to check that there exists $C > 0$ such that
\[
|A_{\Sigma_j}|(x) \min\{1, d_g(x, S_j \cup (\partial \Sigma_j \setminus \partial M))\} \leq C
\]
for all $x \in \Sigma_j$. This inequality easily holds on $B_{R_0/\lambda_j}(p_j)$, thus it is sufficient to check it for points $x \in \Sigma_j$. Assume by contradiction that there exists a sequence of points $z_j \in \Sigma_j'$ such that
\[
\lim_{j \to \infty} |A_{\Sigma_j}|(z_j) \min\{1, d_g(z_j, S_j \cup (\partial \Sigma_j \setminus \partial M))\} = \infty.
\]
First observe that $\lim \inf_{j \to \infty} \lambda_j d_g(z_j, p_j) = \infty$, because otherwise $\lambda_j(z_j - p_j)$ would converge to some point $\bar{z} \in \Sigma_{\infty}$ and we would obtain
\[
\lim_{j \to \infty} |A_{\Sigma_j}|(z_j) \min\{1, d_g(z_j, S_j \cup (\partial \Sigma_j \setminus \partial M))\} \leq \lim_{j \to \infty} |A_{\Sigma_j}|(z_j) d_g(z_j, p_j) = \lim_{j \to \infty} |A_{\Sigma_j}|(\lambda_j(z_j - p_j)) d_g(\lambda_j(z_j - p_j), 0) = |A_{\Sigma_{\infty}}|(\bar{z}) d_{\text{eucl}}(\bar{z}, 0) < \infty.
\]
Moreover, since both (4.3) and (4.4) hold and $[S'_j \cup (\partial \Sigma'_j \setminus \partial M)] \setminus [S_j \cup (\partial \Sigma_j \setminus \partial M)] = \partial \Sigma'_j \setminus \partial \Sigma_j$, we have that

$$d_g(z_j, S'_j \cup (\partial \Sigma'_j \setminus \partial M)) = d_g(z_j, \partial \Sigma'_j \setminus \partial \Sigma_j) = d_g(z_j, p_j) - \frac{R_0}{\lambda_j}.$$ 

Therefore, we can conclude that

$$\limsup_{j \to \infty} |A_{\Sigma_j}(z_j)| \min \{1, d_g(z_j, S_j \cup (\partial \Sigma_j \setminus \partial M))\} \leq \limsup_{j \to \infty} |A_{\Sigma_j}(z_j)| d_g(z_j, p_j) \leq \limsup_{j \to \infty} \frac{C'}{d_g(z_j, S'_j \cup (\partial \Sigma'_j \setminus \partial M))} d_g(z_j, p_j) = \limsup_{j \to \infty} \frac{C'}{1 - \frac{R_0}{\lambda_j} d_g(z_j, p_j)} = C',$$

which is a contradiction and concludes the proof.

Given the previous lemma and the tools to handle free boundary minimal laminations developed in Section 3, we can conclude the description of the limit picture at macroscopic scale.

**Corollary 4.7.** Let $(M^3, g)$ be a compact Riemannian manifold with boundary that satisfies (Ψ) and fix $I \in \mathbb{N}$. Suppose that $\Sigma'_j \subset M$ is a sequence of compact, properly embedded, free boundary minimal surfaces with $\text{ind}(\Sigma_j) \leq I$. Then, up to subsequence, there exist a constant $C = C(M, g)$ and a sequence of smooth blow-up sets $S_j \subset \Sigma_j$ with $|S_j| \leq I$ and

$$\sup_{x \in \Sigma_j} |A_{\Sigma_j}(x)| \min \{1, d_g(x, S_j)\} \leq C.$$

Moreover, the sets $S_j$ converge to a set of points $S_{\infty}$ and the surfaces $\Sigma_j$ converge locally smoothly away from $S_{\infty}$ to some smooth free boundary minimal lamination $\mathcal{L} \subset M$ with $\partial L = L \cap \partial M$ for all $L \in \mathcal{L}$.

![Figure 6. Macroscopic description of degeneration.](image)

**Proof.** The first part of the statement is a special case of Lemma 4.5. Then, possibly extracting a further subsequence, we can assume that the sets $S_j$ converge to a set $S_{\infty}$ (of cardinality at most $I$) and, thanks to Theorem 3.5, the surfaces $\Sigma_j$ converge to a free boundary minimal lamination $\hat{\mathcal{L}} \subset M \setminus S_{\infty}$ smoothly away from $S_{\infty}$. However, Theorem 3.11 ensures that the lamination $\hat{\mathcal{L}}$ extends smoothly through $S_{\infty}$; namely, there exists a smooth free boundary minimal lamination $\mathcal{L} \subset M$ such that $\hat{\mathcal{L}} = \mathcal{L} \setminus S_{\infty}$, which concludes the proof observing that each leaf of $\mathcal{L}$ is properly embedded in the sense that $\partial L = L \cap M$ for all $L \in \mathcal{L}$, thanks to property (Ψ).

5. **Microscopic behavior**

In this section we study the behavior of our minimal surfaces at small scales around the points where concentration of curvature occurs, that is to say around the points in $S_{\infty}$ in Corollary 4.7.
5.1. Setting description. We denote by \((\mathcal{N})\) the following set of assumptions:

(i) \(g_j\) is a sequence of metrics on \(M_j^3 := \Xi(a_j) \cap \{|x| < R_j\} \subset \mathbb{R}^3\), with \(0 \geq a_j \geq -\infty\) and \(R_j \to \infty\), locally smoothly converging to the Euclidean metric as \(j \to \infty\).

(ii) \(\Sigma_j^2 \subset M_j\) is a sequence of properly embedded \(\text{edged}\) minimal surfaces that have free boundary with respect to \(\Pi(a_j)\) and \(\partial \Sigma_j \subset \partial M_j\).

(iii) \(\text{ind}(\Sigma_j) \leq I\) for some natural constant \(I > 0\) independent of \(j\).

(iv) \(S_j \subset \Sigma_j \cap B_{1/2}(0)\) is a sequence of non-empty smooth blow-up sets\(^5\) with \(|S_j| \leq I\) and

\[
|A_{\Sigma_j}(x) d_{g_j}(x, \Sigma_j) \bigcup (\partial \Sigma_j \setminus \Pi(a_j))| \leq C
\]

for all \(x \in \Sigma_j\), for some constant \(C > 0\) independent of \(j\).

See Figure 7 for a visualization of the possible situations that can occur.

**Figure 7.** Different possible situations in setting \((\mathcal{N})\).

\[\text{Proposition 5.1.} \quad \text{Let us consider the setting \((\mathcal{N})\). Then, up to subsequence, the smooth blow-up sets } S_j \text{ converge to a finite set of points } S_\infty, \text{ of cardinality at most } I, \text{ and there is a free boundary minimal lamination } L \text{ in } \Pi(a) \text{ (where we assume the existence of } a = \lim_{j \to \infty} a_j \in [-\infty, 0]) \text{ consisting of parallel planes or half-planes and such that } \Sigma_j \text{ locally converges (in the sense of laminations) to } L \text{ away from } S_\infty.\]

**Proof.** First observe that, up to subsequence, we can assume that \(S_j\) converge to a finite set of points \(S_\infty\) of cardinality at most \(I\) and contained in the open unit ball centered at the origin. Then, because of the curvature assumptions we are making, thanks to Proposition 3.14 we gain smooth (subsequential) convergence to a free boundary minimal lamination \(\hat{L}\) in \(\Pi(a)\) (where we assume the existence of \(a = \lim_{j \to \infty} a_j \in [-\infty, 0]\)) consisting of parallel planes or half-planes and such that \(\Sigma_j\) locally converges (in the sense of laminations) to \(\hat{L}\) away from \(S_\infty\).

Later on, we will separately study the components where ‘bad things’ happen (but which are in finite number) and the others. Therefore it will be useful to introduce the following definition.

\[\text{ Proposition 5.1.} \quad \text{Let us consider the setting \((\mathcal{N})\). Then, up to subsequence, the smooth blow-up sets } S_j \text{ converge to a finite set of points } S_\infty, \text{ of cardinality at most } I, \text{ and there is a free boundary minimal lamination } L \text{ in } \Pi(a) \text{ (where we assume the existence of } a = \lim_{j \to \infty} a_j \in [-\infty, 0]) \text{ consisting of parallel planes or half-planes and such that } \Sigma_j \text{ locally converges (in the sense of laminations) to } L \text{ away from } S_\infty.\]

**Proof.** First observe that, up to subsequence, we can assume that \(S_j\) converge to a finite set of points \(S_\infty\) of cardinality at most \(I\) and contained in the open unit ball centered at the origin. Then, because of the curvature assumptions we are making, thanks to Proposition 3.14 we gain smooth (subsequential) convergence to a free boundary minimal lamination \(\hat{L}\) in \(\Pi(a)\) (where we assume the existence of \(a = \lim_{j \to \infty} a_j \in [-\infty, 0]\)) consisting of parallel planes or half-planes and such that \(\Sigma_j\) locally converges (in the sense of laminations) to \(\hat{L}\) away from \(S_\infty\). We now need to rule out alternative (2), namely the possibility that \(L\) consists of a (single) non-flat, two-sided, properly embedded, free boundary minimal surface. However, in this case the convergence must be locally smooth and graphical with multiplicity one (at all points) of such unique leaf: hence this would imply locally uniform curvature estimates for the sequence \(\Sigma_j\), which is in contradiction with the presence of a smooth blow-up set in \((\mathcal{N})\), though. Thus the only possibility is that \(L\) is a lamination in \(\Xi(a)\) of parallel planes, which concludes the proof.

Later on, we will separately study the components where ‘bad things’ happen (but which are in finite number) and the others. Therefore it will be useful to introduce the following definition.

\[\text{Proposition 5.1.} \quad \text{Let us consider the setting \((\mathcal{N})\). Then, up to subsequence, the smooth blow-up sets } S_j \text{ converge to a finite set of points } S_\infty, \text{ of cardinality at most } I, \text{ and there is a free boundary minimal lamination } L \text{ in } \Pi(a) \text{ (where we assume the existence of } a = \lim_{j \to \infty} a_j \in [-\infty, 0]) \text{ consisting of parallel planes or half-planes and such that } \Sigma_j \text{ locally converges (in the sense of laminations) to } L \text{ away from } S_\infty.\]

**Proof.** First observe that, up to subsequence, we can assume that \(S_j\) converge to a finite set of points \(S_\infty\) of cardinality at most \(I\) and contained in the open unit ball centered at the origin. Then, because of the curvature assumptions we are making, thanks to Proposition 3.14 we gain smooth (subsequential) convergence to a free boundary minimal lamination \(\hat{L}\) in \(\Pi(a)\) (where we assume the existence of \(a = \lim_{j \to \infty} a_j \in [-\infty, 0]\)) consisting of parallel planes or half-planes and such that \(\Sigma_j\) locally converges (in the sense of laminations) to \(\hat{L}\) away from \(S_\infty\). We now need to rule out alternative (2), namely the possibility that \(L\) consists of a (single) non-flat, two-sided, properly embedded, free boundary minimal surface. However, in this case the convergence must be locally smooth and graphical with multiplicity one (at all points) of such unique leaf: hence this would imply locally uniform curvature estimates for the sequence \(\Sigma_j\), which is in contradiction with the presence of a smooth blow-up set in \((\mathcal{N})\), though. Thus the only possibility is that \(L\) is a lamination in \(\Xi(a)\) of parallel planes, which concludes the proof.

---

\[\text{Hereafter we denote by } B_r(p) \text{ the ball of center } p \text{ and radius } r \text{ in the metric } g_j, \text{ without specifying } j \text{ when this is clear from the context.}\]

\[\text{The setting in Definition 4.1 is slightly different from the setting here, but the definition can be easily adapted to this context.}\]
Definition 5.2. In the setting (9.1), let us denote by \( \Sigma_j^{(1)} \) the union of the components of \( \Sigma_j \cap B_1(0) \) that contain at least one point in \( S_j \) (informally: the ones with the necks in Figure 7) and with \( \Sigma_j^{(2)} \) the union of the components of \( \Sigma_j \cap B_1(0) \) that do not.

Remark 5.3. It is sufficient to work in \( B_1(0) \), since the information about \( \Sigma_j \) in \( B_1(0)^c \) are then obtained in the applications thanks to suitable Morse-theoretic arguments (see Appendix C).

5.2. Neck components. In this section, we deal with the behavior of \( \Sigma_j^{(1)} \), which is the part of \( \Sigma_j \cap B_1(0) \) that ‘carries the topology’ of the surface \( \Sigma_j \). The components \( \Sigma_j^{(2)} \) are instead well-controlled, in the sense that they have uniformly bounded curvature and they are topological discs.

We postpone the investigation of these properties of \( \Sigma_j^{(2)} \) to the proof Theorem 6.1.

The following proposition is essentially the base case of the induction to prove Proposition 5.5, which is the full description of what happens around the origin along the sequence \( \Sigma_j^{(1)} \).

Lemma 5.4. Let us assume to be in the setting (9.1) with \( |S_j| = 1 \) for all \( j \). Then there exists \( \kappa(I) \geq 0 \) (depending on \( I \)) such that, for \( j \) sufficiently large, the following assertions hold true:

1. The surfaces \( \Sigma_j^{(1)} \) have genus at most \( \kappa(I) \).
2. The surfaces \( \Sigma_j^{(1)} \) intersect \( \Pi(a) \cup \partial B_1(0) \) transversely in at most \( \kappa(I) \) components.

Proof. Let us define \( S_j = \{ p_j \} \), \( S_\infty = \{ p_\infty \} \) and \( \lambda_j := |A_{\Sigma_j}|(p_j) \). By definition of smooth blow-up set, the surfaces \( \Sigma_j := \lambda_j(\Sigma_j - p_j) \) converge up to subsequence to a complete, non-flat, properly embedded, free boundary minimal surface \( \Sigma_\infty \) in \( \Xi(a) \) for some \( 0 \geq a \geq -\infty \). Furthermore \( \Sigma_\infty \) has index at most \( I \). Therefore, by Proposition B.4, the genus, number of ends and number of boundary components of \( \Sigma_\infty \) are all bounded by \( \kappa(I) \).

Take \( R_0 > 0 \) such that \( \Sigma_\infty \) intersects \( \partial B_{R_0}(0) \) transversely and

\[
|A_{\Sigma_\infty}|(x) d_{\mathbb{R}^3}(0, x) < \frac{1}{4}
\]

for \( x \in \Sigma_\infty \setminus B_{R_0}(0) \). Moreover, assume \( R_0 \) sufficiently large such that the genus and the number of boundary components of \( \Sigma_\infty \cap B_{R_0}(0) \) are both bounded by \( \kappa(I) \). Hence observe that, for \( j \) sufficiently large, \( \Sigma_j \cap B_{R_0/\lambda_j}(p_j) \) also has genus and number of boundary components both bounded by \( \kappa(I) \). In order to transfer this information to all \( B_1(0) \) we want to prove that the estimate

\[
|A_{\Sigma_j}|(x) d_{g_j}(p_j, x) < \frac{1}{4}
\]

holds for every \( x \in \Sigma_j \cap (B_1(0) \setminus B_{R_0/\lambda_j}(p_j)) \), for \( j \) sufficiently large.

To this purpose, it is sufficient to prove that there exists \( \delta > 0 \) such that the estimate holds in \( \Sigma_j \cap (B_\delta(p_j) \setminus B_{R_0/\lambda_j}(p_j)) \). Then we deduce the desired estimate using that \( \Sigma_j \) converges (in the sense of laminations) in \( B_1(0) \setminus B_\delta(p_j) \) to a lamination consisting of planes (indeed \( p_\infty \in B_\delta(p_j) \) for \( j \) sufficiently big).

Assume by contradiction that such \( \delta > 0 \) does not exist. Then there exists a sequence \( z_j \in \Sigma_j \setminus B_{R_0/\lambda_j}(p_j) \) such that \( \delta_j := d_{g_j}(p_j, z_j) \to 0 \) and \( |A_{\Sigma_j}|(z_j) \delta_j \geq \frac{1}{4} \). Then consider \( \Sigma_j := \delta_j^{-1}(\Sigma_j - p_j) \), for which we have

\[
|A_{\Sigma_j}|(\delta_j^{-1}(z_j - p_j)) \geq \frac{1}{4}.
\]

Notice that \( \Sigma_j \) must have unbounded second fundamental form at 0, otherwise they would converge to a homothety of \( \Sigma_\infty \), but this is not possible for the choice of \( z_j \) together with (5.1). As a result, possibly extracting a subsequence (which we do not rename) \( \Sigma_j \) still fulfills the assumptions of the setting (9.1) and therefore it converges to a lamination consisting of planes away from 0. However, observe that this implies \( |A_{\Sigma_j}|(\delta_j^{-1}(z_j - p_j)) \to 0 \), which contradicts (5.3) and thus proves (5.2).
Finally note that, for \( j \) sufficiently large, each component of \( \Sigma_j^{(1)} \cap B_1(0) \) intersects (transversely) \( \partial B_{R_0/\lambda_j}(p_j) \), since by definition each component of \( \Sigma_j^{(1)} \) contains \( p_j \) and \( \Sigma_\infty \) (transversely) intersects \( B_{R_0}(0) \). Thus, thanks to Corollary C.4, the genus and the number of boundary components of \( \Sigma_j^{(1)} \) are bounded by \( \kappa(I) \).

We can now proceed and prove the corresponding result for any set \( S_j \) of uniformly bounded cardinality.

**Proposition 5.5.** Assume to be in the setting (III). Then there exists \( \kappa(I) \geq 0 \) such that, for \( j \) sufficiently large, the following assertions hold true:

1. The surfaces \( \Sigma_j^{(1)} \) have genus at most \( \kappa(I) \).
2. The surfaces \( \Sigma_j^{(1)} \) intersect \( \Pi(a) \cup \partial B_1(0) \) transversely in at most \( \kappa(I) \) components.

**Proof.** Let us proceed by induction on \( I > 0 \). The case \( I = 1 \) has been treated in Lemma 5.4, thus let us assume \( I > 1 \). We distinguish two cases and we first consider the case when \( |S_\infty| \geq 2 \). Choose \( \delta > 0 \) sufficiently small such that \( \min_{p,q \in S_\infty} d_p(p,q) \geq 4\delta \). Moreover, we fix one point \( p_\infty \in S_\infty \). Since \( \text{ind}(\Sigma_j \cap B) \geq 1 \) for every connected component \( B \) of \( B_\delta(S_\infty) \), then \( \text{ind}(\Sigma_j \cap B_{\delta}(p_\infty)) \leq I - 1 \).

Now choose positive numbers \( r_j \to 0 \) such that \( S_j \subset B_{r_j/2}(S_\infty) \) and

\[
\liminf_{j \to \infty} \left( r_j \min_{p \in S_j} |A_{\Sigma_j}(p)| \right) = \infty.
\]

Notice that the surfaces \( r_j^{-1}(\Sigma_j - p_\infty) \) still fulfill the assumptions (III) (with blow-up sets that are the rescaled of the blow-up sets \( S_j \)) thanks to the choice of \( r_j \) and thus we can apply the inductive hypothesis to these surfaces. In particular we obtain that the components \( \Sigma_j^{(1)} \cap B_{r_j}(p_\infty) \) have genus at most \( \kappa(I-1) \) and intersect \( \partial B_{r_j}(p_\infty) \) transversely in at most \( \kappa(I-1) \) components.

We now prove that, choosing \( \delta > 0 \) possibly smaller, we have that

\[
|A_{\Sigma_j}(x) d_{\Sigma_j}(x,p_\infty) < \frac{1}{4}
\]

for all \( x \in B_\delta(p_\infty) \setminus B_{r_j}(p_\infty) \). If this is not the case, then there exists a sequence of points \( z_j \in \Sigma_j \) with \( \delta_j := d_{\Sigma_j}(z_j,p_\infty) > r_j \), \( \delta_j \to 0 \), and \( |A_{\Sigma_j}(z_j) \delta_j \geq \frac{1}{4} \). The rescaled surfaces \( \Sigma_j := \delta_j^{-1}(\Sigma_j - p_\infty) \) still satisfy (III) with blow-up set \( \delta_j^{-1}(S_j - p_\infty) \) and therefore they converge away from 0 to a lamination consisting of parallel planes by Proposition 5.1. In particular \( |A_{\Sigma_j}(z_j) \delta_j^{-1}(z_j - p_\infty) \to 0 \), which contradicts the choice of \( z_j \).

As a result, we can apply Corollary C.4 and conclude that also the components \( \Sigma_j^{(1)} \cap B_\delta(p_\infty) \) have genus at most \( \kappa(I-1) \) and intersect \( \partial B_\delta(p_\infty) \) transversely in at most \( \kappa(I-1) \) components.

Now we apply the very same arguments on each ball of radius \( \delta \) (as above) and centered at a point of \( S_\infty \). We gain analogous bounds, hence (keeping in mind that we have uniform curvature estimates away from such balls) we can exploit the topological bounds we have gained to conclude that \( \Sigma_j^{(1)} \) converges graphically smoothly with finite multiplicity in \( B_1(0) \setminus B_\delta(S_\infty) \) to the leaves passing through \( S_\infty \) of the limit lamination \( L \) in Proposition 5.1 and conclude with the basic topological tools presented in Section 2.4.

Therefore, now we have to deal only with the case \( |S_\infty| = 1 \). We can assume \( |S_j| \geq 2 \), since otherwise we could just apply Lemma 5.4. Take \( p_j, q_j \in S_j \) that realize the maximum distance between points in \( S_j \) and define \( r_j := 2d_{\Sigma_j}(p_j,q_j) \to 0 \). The sequence of surfaces \( \Sigma_{r_j} := r_j^{-1}(\Sigma_j - p_j) \) still satisfy the assumptions (III) with index \( I \). Moreover \( |\Sigma_{r_j}| \geq 2 \), therefore we can apply the first part of the proposition to \( \Sigma_{r_j} \), obtaining all the desired information in \( B_{r_j}(p_j) \). However, now we can argue as above to obtain the \( 1/4 \)-curvature estimate in \( \Sigma_j \cap (B_1(0) \setminus B_{r_j}(p_j)) \) and transfer the information to \( B_1(0) \) as we wanted. \( \square \)
6. Global description and simplification

6.1. Global description. We are now ready to put together all the information obtained in Sections 4 and 5 and present a global description of degeneration for a sequence of surfaces with bounded index.

We refer to Theorem 1.17 in [9] for the corresponding description in the closed analogue.

**Theorem 6.1.** There exists a function $\kappa(I) \geq 0$ depending on $I \in \mathbb{N}$ with the following property.

Let $(M^3, g)$ be a three-dimensional Riemannian manifold with boundary that satisfies (3). Let $\Sigma_j^2 \subseteq M$ be a sequence of compact, properly embedded, free boundary minimal surfaces with $\text{ind}(\Sigma_j) \leq I$, for a fixed constant $I \in \mathbb{N}$. Then, up to subsequence, there exists a sequence of points $S_j \subseteq \Sigma_j$ with $|S_j| \leq I$, such that

$$\sup_{x \in S_j} |A_{\Sigma_j}|(x) \min \{1, d_g(x, S_j)\} \leq C,$$

for some constant $C > 0$, and

$$\liminf_{j \to \infty} \min_{p \in S_j} |A_{\Sigma_j}|(p) = \infty.$$

Moreover, the sets $S_j$ converge to a set of points $S_\infty$ and the surfaces $\Sigma_j$ converge locally smoothly away from $S_\infty$ to some smooth free boundary minimal lamination $L \subseteq M$ with $\partial L = L \cap \partial M$ for all $L \in \mathcal{L}$.

Finally, we can precisely describe the behavior of $\Sigma_j$ around the points $S_\infty$. In particular there exists $\varepsilon_0 > 0$ sufficiently small such that $\min_{p,q \in S_\infty} d_g(p, q) \geq 4\varepsilon_0$ and such that for all $\varepsilon \leq \varepsilon_0$ the following properties hold. Let $\Sigma_j^{(1)}$ be the union of the components of $\Sigma_j \cap B_\varepsilon(S_\infty)$ which contain at least one point of $S_j$ and let $\Sigma_j^{(2)}$ be the union of the other components of $\Sigma_j \cap B_\varepsilon(S_\infty)$. Then, for $j$ sufficiently large, we have that

1. (a) No component of $\Sigma_j^{(1)}$ is a topological disc or half-disc.
   (b) The genus of $\Sigma_j^{(1)}$ is bounded by $\kappa(I)$.
   (c) $\Sigma_j^{(1)}$ intersects $\partial B_\varepsilon(S_\infty) \cup \partial M$ transversely in at most $\kappa(I)$ components;
   (d) $\Sigma_j^{(1)}$ has uniformly bounded area, namely

$$\limsup_{j \to \infty} \mathcal{H}^2(\Sigma_j^{(1)}) \leq 4\pi \kappa(I)\varepsilon^2$$

as $\varepsilon \to 0$.

2. (a) Each component of $\Sigma_j^{(2)}$ is a topological disc.
   (b) $\Sigma_j^{(2)}$ has uniformly bounded curvature, that is

$$\limsup_{j \to \infty} \sup_{x \in \Sigma_j^{(2)}} |A_{\Sigma_j}|(x) < \infty.$$

   (c) Each component of $\Sigma_j^{(2)}$ has area uniformly bounded, namely

$$\limsup_{j \to \infty} \sup_{C \subseteq \Sigma_j^{(2)}} \mathcal{H}^2(C) \leq 2\pi \varepsilon^2$$

as $\varepsilon \to 0$.

**Proof.** The first part of the theorem is contained in Corollary 4.7, thus let us prove the second part (that is the properties of $\Sigma_j^{(1)}$ and $\Sigma_j^{(2)}$). The idea (see also Figure 8) is to perform a blow-up procedure choosing the radii in such a way that all the topological information is contained in the unit ball after the dilation. Then, to obtain the description of the topology in such ball, we exploit the local picture carried out in Section 5.

Let $\varepsilon_0 > 0$ be such that $\min_{p,q \in S_\infty} d_g(p, q) \geq 4\varepsilon_0$ and each component of $B_{\varepsilon_0}(S_\infty)$ admits a chart to the unit ball in $\mathbb{R}^3$ (around the points in $M \setminus \partial M$) or in $\Xi(0)$ (around the points in $\partial M$) in which
Let $\mathcal{L}$ be sufficiently close (in the sense of laminations) to a union of parallel planes or half-planes. The choice of such $\varepsilon_0$ is possible by smoothness of $\mathcal{L}$. Moreover, choose $r_j \to 0$ such that $S_j \subset B_{r_j/2}(S_\infty)$ and

$$\liminf_{j \to \infty} \left( r_j \min_{p \in S_j} |A_{\Sigma_j}|(p) \right) = \infty.$$ 

Observe that, fixing a point $p_\infty \in S_\infty$, the rescaled surfaces $r_j^{-1}(\Sigma_j - p_\infty)$ in the ambient manifolds $r_j^{-1}(B_{\varepsilon_0}(p_\infty) - p_\infty)$ (seeing everything in chart) satisfy the setting (9). Therefore, by Proposition 5.5, we obtain that the surfaces $\Sigma_j^{(1)} \cap B_{r_j}(p_\infty)$ have genus at most $\kappa(I)$ and intersect $\partial M \cap \partial B_{r_j}(p_\infty)$ transversely in at most $\kappa(I)$ components.

![Figure 8. Blow-up around the points of degeneration.](image)

Now note that, exactly as in the proof of Proposition 5.5, it is possible to take $\varepsilon_0 > 0$ possibly smaller in such a way that

$$|A_{\Sigma_j}|(x) d_g(x, S_\infty) < \frac{1}{4}$$

for $x \in \Sigma_j \cap (B_{\varepsilon_0}(S_\infty) \setminus B_{r_j}(S_\infty))$, for $j$ sufficiently large. Therefore, thanks to Corollary C.4, we can transfer the information on the genus and the boundary components of $\Sigma_j^{(1)} \cap B_{r_j}(p_\infty)$ to $\Sigma_j^{(1)} \cap B_{\varepsilon_0}(p_\infty)$.

Notice that the surfaces $\Sigma_j^{(1)} \cap B_{\varepsilon_0}(p_\infty)$ locally converge (in the sense of laminations) to the leaf of $\mathcal{L}$ passing through $p_\infty$ away from $p_\infty$. Indeed, the convergence to any other component of $\mathcal{L} \cap B_{\varepsilon_0}(p_\infty)$ is uniformly smooth (in the sense of laminations), but each component of $\Sigma_j^{(1)}$ contains a point where the curvature converges to $\infty$.

Let us now prove that the components of $\Sigma_j^{(2)}$ have uniformly bounded curvature, i.e.

$$\limsup_{j \to \infty} \sup_{x \in \Sigma_j^{(2)}} |A_{\Sigma_j}|(x) < \infty.$$
Assume by contradiction that (possibly after passing to a subsequence) there exists a sequence $z_j \in \Sigma_j^{(2)}$ satisfying
\[
\lambda_j^{(2)} := |A_{\Sigma_j^j}|(z_j) = \sup_{x \in \Sigma_j^{(2)}} |A_{\Sigma_j^j}|(x) \to \infty.
\]
Observe that, by (6.1), this implies that (up to subsequence) the sequence $z_j$ converges to some point $p_\infty \in \mathcal{S}_\infty$. In particular the distance between $z_j$ and $\partial \Sigma_j^{(2)} \setminus \partial M$ is bounded by a positive constant. This means that $\lambda_j^{(2)}(B_{\epsilon_0}(z_j) - z_j)$ is an exhausting sequence of domains of $\Pi(a)$ for some $0 \geq a \geq -\infty$.

Now consider the rescaled surfaces $\bar{\Sigma}_j := \lambda_j^{(2)}(\Sigma_j - z_j)$. Observe that this surfaces have bounded curvature away from a finite set of points (since their curvature is less than the curvature of $\Sigma_j$) and that $|A_{\bar{\Sigma}_j^j}|(0) = 1$. Therefore, by Theorem 3.5 and Proposition 3.14, $\bar{\Sigma}_j$ converges locally smoothly with multiplicity one (in the sense of graphs) to a complete, non-flat, connected, properly embedded, two-sided, free boundary minimal surface $\bar{\Sigma}_\infty$ in $\Xi(a)$, for some $0 \geq a \geq -\infty$.

Now, observe that the limit of the surfaces $\lambda_j^{(2)}(\Sigma_j^{(2)} - z_j)$ must be non-empty since
\[
\limsup_{j \to \infty} \min_{p \in \bar{\Sigma}_j^j} \lambda_j^{(2)}d_g(z_j, p) < +\infty
\]
by (6.1). However, this contradicts the fact that $\lambda_j^{(2)}(\Sigma_j - z_j)$ converges to $\Sigma_\infty$ with multiplicity one since $\lambda_j^{(2)}(\Sigma_j^{(1)} - z_j)$ and $\lambda_j^{(2)}(\Sigma_j^{(2)} - z_j)$ are two different connected components of $\lambda_j^{(2)}(\Sigma_j^{(1)} - z_j)$ and the limit of both of them is non-empty.

Given the bound on the curvature of the components of $\Sigma_j^{(2)}$, the information on their topology follows easily (possibly taking $\epsilon_0$ smaller), using that the convergence of $\Sigma_j^{(2)}$ must be smooth (in the sense of lamination) everywhere in $B_{\epsilon_0}(\mathcal{S}_\infty)$. Therefore each component of $\Sigma_j^{(2)}$ converges smoothly with multiplicity one to a leaf of $\mathcal{L}$ (by simply connectedness of the components of $\mathcal{L} \cap B_{\epsilon}(p_\infty)$).

There are only left to prove the bounds (1d) and (2c) on the area. Fixing $\epsilon \leq \epsilon_0$, the surfaces $\Sigma_j^{(2)}$ intersects $\partial B_{\epsilon}(p_\infty) \cup \partial M$ in at most $\kappa(I)$ components that are simple closed curves or intervals. This implies that $\Sigma_j^{(1)} \cap (B_{\epsilon}(p_\infty) \setminus B_{\epsilon/2}(p_\infty))$ converges graphically smoothly with finite multiplicity bounded by $\kappa(I)$ to the leaf of $\mathcal{L}$ in $B_{\epsilon}(p_\infty)$ passing through $p_\infty$. As a result, choosing $\epsilon_0$ sufficiently small such that all the leaves of $\mathcal{L}$ in $B_{\epsilon}(p_\infty)$ are sufficiently close to discs or half-discs, we have that
\[
\limsup_{j \to \infty} \mathcal{H}^2(\Sigma_j^{(1)} \cap (B_{\epsilon}(\mathcal{S}_\infty) \setminus B_{\epsilon/2}(\mathcal{S}_\infty))) \leq 2\kappa(I)\pi \epsilon^2.
\]
Finally the estimate on $\mathcal{H}^2(\Sigma_j^{(1)} \cap B_{\epsilon}(\mathcal{S}_\infty))$ follows from the monotonicity formula since, for $\epsilon > 0$ sufficiently small, it holds
\[
\mathcal{H}^2(\Sigma_j^{(1)} \cap B_{\epsilon}(\mathcal{S}_\infty)) \leq 2\mathcal{H}^2(\Sigma_j^{(1)} \cap (B_{\epsilon}(\mathcal{S}_\infty) \setminus B_{\epsilon/2}(\mathcal{S}_\infty))).
\]
The area estimate for the components of $\Sigma_j^{(2)}$ is even easier since $\Sigma_j^{(2)}$ converges (in the sense of laminations) to $\mathcal{L}$ everywhere in $B_{\epsilon}(\mathcal{S}_\infty)$ (in particular, each component of $\Sigma_j^{(2)}$ converges smoothly with multiplicity one to a leaf of $\mathcal{L}$ in $B_{\epsilon}(\mathcal{S}_\infty)$, as observed above), thus we omit it. \hfill \square

6.2. Surgery procedure. As a corollary of the degeneration description in Theorem 6.1, we are able to perform surgeries on the surfaces $\Sigma_j$ to obtain new surfaces ‘similar’ to $\Sigma_j$ but with bounded curvature.

**Corollary 6.2.** There exists a function $\tilde{\kappa}(I) \geq 0$ depending on $I \in \mathbb{N}$ with the following property.

Let $(M^3, g)$ be a three-dimensional Riemannian manifold with boundary that satisfies (3). Let $\Sigma_j^0 \subset M$ be a sequence of compact, properly embedded, free boundary minimal surfaces with $\text{ind}(`\Sigma_j`) \leq I$, for a fixed constant $I \in \mathbb{N}$. Then, up to subsequence, there exist a finite set of points $\mathcal{S}_\infty \subset M$ with $|\mathcal{S}_\infty| \leq I$ and $\epsilon_0 > 0$ sufficiently small such that $\min_{p,q \in \mathcal{S}_\infty} d_g(p, q) \geq 4\epsilon_0$ and such that, for
all \( 0 < \varepsilon \leq \varepsilon_0 \) and \( j \) sufficiently large, there exist properly embedded surfaces \( \bar{\Sigma}_j \subset M \) satisfying the following properties:

1. \( \bar{\Sigma}_j \) coincides with \( \Sigma_j \) outside \( B_\varepsilon(S_\infty) \).
2. The curvature of \( \bar{\Sigma}_j \) is uniformly bounded, i.e.
   \[
   \limsup_{j \to \infty} \sup_{x \in \bar{\Sigma}_j} |A_{\bar{\Sigma}_j}(x)| < \infty.
   \]
3. The genus, the number of boundary components, the area and the number of connected components of \( \bar{\Sigma}_j \) are controlled by the ones of \( \Sigma_j \), namely
   \[
   \text{genus}(\Sigma_j) - \kappa(I) \leq \text{genus}(\bar{\Sigma}_j) \leq \text{genus}(\Sigma_j),
   \]
   \[
   \text{boundaries}(\Sigma_j) - \kappa(I) \leq \text{boundaries}(\bar{\Sigma}_j) \leq \text{boundaries}(\Sigma_j),
   \]
   \[
   \mathcal{H}^2(\Sigma_j) - \kappa(I) \leq \mathcal{H}^2(\bar{\Sigma}_j) \leq \mathcal{H}^2(\Sigma_j) + \kappa(I),
   \]
   \[
   |\pi_0(\Sigma_j)| - |\pi_0(\bar{\Sigma}_j)| \leq |\pi_0(\Sigma_j)| + \kappa(I).
   \]
4. The surfaces \( \bar{\Sigma}_j \) locally converge (in the sense of laminations) to the lamination \( \mathcal{L} \) in Theorem 6.1.

**Proof.** The proof is very similar to the proof of the closed analogue [9, Corollary 1.19]. We go through it for completeness. Up to subsequence, we can assume that all the conclusions of Theorem 6.1 hold. Let \( \varepsilon_0 \) be as in the theorem and take \( 0 < \varepsilon \leq \varepsilon_0 \). Consider \( p_\infty \in S_\infty \); then, if \( p_\infty \in M \setminus \partial M \), we can perform the surgery as in [9, Corollary 1.19] (possibly restricting \( \varepsilon_0 \)). Therefore, let us assume that \( p_\infty \in \partial M \). Pick the leaf \( L \) of \( \mathcal{L} \cap B_\varepsilon(p_\infty) \) passing through \( p_\infty \), which satisfies \( \partial L = L \cap \partial M \) hence thanks to property (B). Then fix a diffeomorphism \( \psi : B_\varepsilon(p_\infty) \to B_3(0) \cap \Xi(0) \subset \mathbb{R}^3 \) such that \( \psi \) maps \( B_{\varepsilon/3}(p_\infty) \) diffeomorphically onto \( B_1(0) \cap \Xi(0) \) and \( L \) onto the flat half-disc \( \{ x^3 = 0 \} \cap (B_3(0) \cap \Xi(0)) \).

Consider all the connected components of \( \Sigma_j \cap B_\varepsilon(p_\infty) \) which are converging smoothly to \( L \) in \( B_\varepsilon(p_\infty) \setminus B_{\varepsilon/3}(p_\infty) \). These include all the neck components, called \( \Sigma_j^{(1)} \) in Theorem 6.1, as observed in the proof of the theorem. Notice that the convergence to the leaf \( L \) in \( B_\varepsilon(p_\infty) \setminus B_{\varepsilon/3}(p_\infty) \) is possibly with infinite multiplicity; however, the convergence of the components relative to \( \Sigma_j^{(1)} \) occurs with uniformly bounded multiplicity (for example thanks to the area bound (1d) in Theorem 6.1, or from the proof of (1d) itself).

Let \( \Gamma_j \) be the image through \( \psi \) of the union of the necks components of \( \Sigma_j \cap B_\varepsilon(p_\infty) \) together with the disc components directly above or below a neck component. We are going to perform our surgery on \( \Gamma_j \) leaving invariant its disc components, in this way we are sure that also all the other disc components remain untouched. Let us identify \( L \) with the half-disc \( \{ x^3 = 0 \} \cap (B_3(0) \cap \Xi(0)) \) and let us denote by \( D(2) \) the disc with radius 2 and center 0 in \( L \), that is \( D(2) := L \cap \{ |x| < 2 \} \), and \( A(2,1) \) the annulus between radii 1 and 2 in \( L \), namely \( A(2,1) := L \cap \{ 1 < |x| < 2 \} \). Then choose \( \chi : \mathbb{R} \to [0,1] \) a smooth cutoff function with \( \chi(t) = 1 \) for \( t \geq 7/4 \) and \( \chi(t) = 0 \) for \( t \leq 5/4 \).

We can assume to have only two disc components in \( \Gamma_j \), one at the top and one at the bottom. If this is not the case, we work separately on subsets of the components of \( \Gamma_j \) in this form, eventually adding the extremal disc components if missing. Pick a smooth function \( w_j : D(2) \to \mathbb{R} \) such that

- the graph of \( w_j \) is contained in \( B_3(0) \cap \Xi(0) \) and it lies strictly between the two disc components of \( \Gamma_j \);
- the graph of \( w_j \) intersects transversely \( \Pi(0) \);
- \( w_j \) smoothly converges to 0 as \( j \to \infty \).

Moreover choose real numbers \( \varepsilon_j \to 0 \) such that \( w_j + \varepsilon \) has the same properties for every \( 0 \leq \varepsilon \leq \varepsilon_j \).

Now, since the convergence of \( \Gamma_j \) to \( L \) is smooth graphical with finite multiplicity on \( A(2,1) \), we can find functions \( u_{j,1}, \ldots, u_{j,n(j)} : A(2,1) \to \mathbb{R} \) so that the non-disc components of \( \Gamma_j \) on the cylinder over \( A(2,1) \) are exactly the graphs of \( u_{j,k} \) for \( k = 1, \ldots, n(j) \). Moreover observe that
• \( n(j) \) is uniformly bounded for \( j \to \infty \) for what we said before;
• for all \( h \in \mathbb{N} \) we have \( \sup_{k=1,\ldots,n(j)} \| u_{j,k} \|_{C^h(A(2,1))} \to 0 \) as \( j \to \infty \);
• by embeddedness of \( \Gamma_j \) we can assume \( u_{j,1}(x) < u_{j,2}(x) < \ldots < u_{j,n(j)}(x) \) for all \( x \in A(2,1) \).

We then define
\[
\tilde{u}_{j,k}(x) := \chi(|x|) u_{j,k}(x) + (1 - \chi(|x|)) \left( w_j(x) + \frac{k}{n(j)} \epsilon_j \right),
\]
for all \( x \in A(2,1) \), and the surface \( \tilde{\Gamma}_j \) as the union of the graphs of \( \tilde{u}_{j,k} \) for \( k = 1, \ldots, n(j) \) inside the cylinder over \( D(2) \) and coinciding with \( \Gamma_j \) outside that cylinder. Now just define \( \tilde{\Sigma}_j \) as \( \Sigma_j \) outside \( B_\epsilon(S_\infty) \) and as the preimages of \( \tilde{\Gamma}_j \) through \( \psi \) inside the balls \( B_\epsilon(S_\infty) \). All the properties required to \( \tilde{\Sigma}_j \) are easily fulfilled by construction thanks to the bounds on the genus and the number of boundary components of \( \Sigma_j \) inside \( B_\epsilon(S_\infty) \), proven in Theorem 6.1.

7. Diameter bounds for stable free boundary minimal surfaces

In this section we prove Proposition 1.8, which is the key to derive the area bounds in the case of ambient manifolds satisfying the assumptions (i) or (ii) given in the introduction (see e.g. the statement of Theorem 1.4).

The main idea of the proof is that, under the assumptions of the proposition, the stable minimal surface \( \Sigma \) admits a (complete) conformal metric with non-negative curvature and convex boundary. Moreover, at least one of the two inequalities is strict, that is to say that either the curvature is strictly positive or the boundary is strictly convex. Therefore we expect the size of \( \Sigma \) to be bounded as a consequence of this property. In order to implement this heuristic idea, we employ several different techniques, mainly from [17] and [47] and, of course, from the same result in the closed case [44], in the form proposed in [6, Proposition 2.12].
Proof of Proposition 1.8. Suppose by contradiction that Σ is non-compact. Then by standard arguments as in [18] (cf. also [46, Section 2.2.2]), there exists a positive function ω on Σ such that

\begin{align}
\begin{cases}
J_\Sigma(\omega) = \Delta \omega + \left( \frac{1}{2} R_g + \frac{1}{2} |A|^2 - K \right) \omega = 0 & \text{on } \Sigma \\
\frac{\partial \omega}{\partial \nu} = - \nabla^\partial M(\nu, \nu) \omega & \text{on } \partial \Sigma.
\end{cases}
\end{align}

(7.1)

Recall that we can choose ω strictly positive in all Σ (including on ∂Σ) by the strong maximum principle and the Hopf boundary point lemma.

Consider on Σ the conformal change of metric \( \tilde{g} = \omega^2 g \) where \( g \) (which we denote also by \( \langle \cdot, \cdot \rangle \)) is the metric on Σ that is induced by the ambient metric on \( M \). Then we know (see e.g. [18, pp. 126-127]) that the curvature of (Σ, \( \tilde{g} \)) is given by

\[
\tilde{K} = \omega^{-2} (K - \Delta \log \omega) = \omega^{-2} \left( K - \frac{\omega \Delta \omega - |\nabla \omega|^2}{\omega^2} \right)
\]

(7.2)

Now let τ be a local choice of unit vector tangent to ∂Σ in (Σ, g) and, as usual, let \( \nu \) be the outward unit normal to ∂Σ. After the conformal change of coordinates, \( \tau/\omega \) and \( \eta/\omega \) are an orthonormal basis. Hence we can compute the geodesic curvature of ∂Σ in (Σ, \( \tilde{g} \)) as follows

\[
\tilde{k} = -\tilde{g}(\tilde{\nabla}_\tau/\omega, \tau/\omega, \eta/\omega) = -\omega^{-2} \langle \tilde{\nabla}_\tau/\omega, \tau/\omega, \eta/\omega \rangle = -\omega^{-1} \langle \tilde{\nabla}_\tau, \eta \rangle
\]

(7.3)

\[
= -\omega^{-1} \left( \langle \nabla_\tau, \eta \rangle - \omega^{-1} \frac{\partial \omega}{\partial \eta} \right) = -\omega^{-1} (\nabla^\partial M(\tau, \tau) + \nabla^\partial M(\nu, \nu))
\]

\[
= \omega^{-1} H^\partial M \geq -\omega^{-1} \sigma_0 \geq 0.
\]

Therefore (Σ, \( \tilde{g} \)) is a surface with non-negative Gaussian curvature and convex boundary and, as claimed above, either of the two functions \( \tilde{K} \) or \( \tilde{k} \) is strictly positive.

Lemma 7.1. The surface (Σ, \( \tilde{g} \)) with boundary is complete (as a metric space).

Proof. The proof is analogous to the proof of Theorem 1 in [17] with some precautions to be taken when dealing with the boundary. Consider a point \( x_0 \in \Sigma \setminus \partial \Sigma \) and let \( B_R(x_0) \subset \Sigma \) be the intrinsic ball of center \( x_0 \) and radius \( R > 0 \) with respect to the complete metric \( g \). Then \( \{B_R(x_0)\}_{R>0} \) gives an exhaustion of Σ.

Now consider the shortest geodesic \( \gamma_R \) in the metric \( \tilde{g} \) connecting \( x_0 \) to the closure of \( \partial B_R(x_0) \setminus \partial \Sigma \), which exists by the following argument. Namely let us consider \( \omega_R = \omega + \epsilon_R \) where \( 0 \leq \epsilon_R \leq 1 \) is a smooth function with \( \epsilon_R = 0 \) in \( B_R(x_0) \) and \( \epsilon_R = 1 \) in \( B_{R+1}(x_0) \). Then \( \omega_R \) is bounded from below and thus the metric \( \tilde{g}_R := \omega_R^2 \tilde{g} \) is complete on Σ. Based on the fact, and on the convexity of the boundary (that we just saw above), there exists a length-minimizing geodesic connecting \( x_0 \) to the closure of \( \partial B_R(x_0) \setminus \partial \Sigma \) with respect to the metric \( \tilde{g}_R \). Since this geodesic is obviously contained in \( B_R(x_0) \) it is also length-minimizing with respect to \( \tilde{g} \), since \( \tilde{g} \) coincides with \( \tilde{g}_R \) in \( B_R(x_0) \).

Let us assume \( \gamma_R \) to be parametrized by arclength with respect to \( g \). By standard compactness arguments, we can find a sequence \( R_i \to \infty \) such that \( \gamma_{R_i} \) locally smoothly converge to a curve \( \gamma : [0, \infty) \to \Sigma \) such that \( \gamma(0) = x_0 \), minimizing length with respect to \( \tilde{g} \) between any two of its points, and that is also parametrized by arc length with respect to \( g \). Observe that \( \gamma \) cannot touch the boundary \( \partial \Sigma \) since it starts at an interior point and the boundary \( \partial \Sigma \) is convex.

It is straightforward to note that the same proof goes through, to provide the bound for \( \text{diam}(\Sigma) \) in the case when Σ is a compact, properly embedded free boundary minimal surface: indeed in that case we know that Σ is a (two-sided) disc, and it suffices to take the first eigenfunction of the Jacobi operator \( J_\Sigma \) (subject to the usual oblique boundary condition).
To prove the completeness of \((\Sigma, \hat{g})\) it is now sufficient to prove that \(\gamma\) has infinite length with respect to \(\hat{g}\), that is to say
\[
\int_0^\infty \omega(\gamma(t)) \, dt = \infty;
\]
indeed, by construction, any divergent ray starting from \(x_0\) has length equal or bigger than \(\gamma\). However, to prove this, we can apply exactly the same argument as in the first part of the proof of [17, Theorem 1], since \(\gamma\) is actually contained in \(\Sigma \setminus \partial \Sigma\).

Given the completeness of \((\Sigma, \hat{g})\), we can apply the Gauss-Bonnet theorem on its metric balls as follows. Fix \(x_0 \in \Sigma\) and let consider \(\Omega_r := \hat{B}_r(x_0)\), the metric ball of radius \(r\) and center \(x_0\) in the metric \(\hat{g}\).

**Remark 7.2.** We are now going to apply some results on geodesic balls of a Riemannian manifold taken from [45, Section 4.4], that are stated for complete manifolds without boundary. The following observation clarifies why the same results hold in our case.

Given \(r_0 > 0\), we can regard our manifold as a smooth subdomain of a complete manifold \((\hat{\Sigma}, \hat{g})\) without boundary (the extension depending on \(r_0 > 0\)), such that, thanks to the convexity of \(\partial \Sigma\), for \(r\) close to \(r_0\) the set \(\hat{B}_r(x_0) \subset \Sigma\) is the intersection of the metric ball \(\hat{B}_r(x_0) \subset \hat{\Sigma}\) with respect to the metric \(\hat{g}\) with \(\Sigma\).

Thanks to this remark, we can invoke a classical theorem by Hartman [28] (cf. also Theorem 4.4.1 in [45]) and obtain that the boundary of \(\Omega_r\) is a piecewise smooth embedded closed curve for almost every \(r > 0\). Moreover the length \(l(r)\) of \(\partial \Omega_r\) is differentiable almost everywhere with derivative given by
\[
l'(r) = \int_{\partial \Omega_r \setminus \partial \Sigma} \text{(geodesic curvature of } \partial \Omega_r \setminus \partial \Sigma) + \sum \text{(exterior angles of } \Omega_r)\];

note that
\[
\lim_{r \to \infty} \sup \ l'(r) \geq 0.
\]
since \(l\) only attains positive values.

Let us now pick a radius \(r\) for which \(\partial \Omega_r\) is a piecewise smooth embedded closed curve. Then, by the Gauss-Bonnet theorem on \(\Omega_r\), it holds that
\[
l'(r) = \int_{\partial \Omega_r \setminus \partial \Sigma} \text{(geodesic curvature of } \partial \Omega_r \setminus \partial \Sigma) + \sum \text{(exterior angles of } \Omega_r) = 2\pi \chi(\Omega_r) - \int_{\Omega_r} \tilde{K} \, dH^2 - \int_{\partial \Omega_r \setminus \partial \Sigma} \tilde{k} \, dH^1,
\]
where \(H^1\) and \(H^2\) are the Hausdorff measures with respect to \(\hat{g}\) in \(\Sigma\). Hence, taking the upper limit on both sides and using that \(\tilde{K}, \tilde{k} \geq 0\), we can conclude that
\[
0 \leq 2\pi \lim_{r \to \infty} \sup \chi(\Omega_r) - \int_{\Sigma} \tilde{K} \, dH^2 - \int_{\partial \Sigma} \tilde{k} \, dH^1.
\]
Recalling that \(\chi(\Omega_r) \leq 1\), this implies that both the integrals \(\int_{\Sigma} \tilde{K} \, dH^2\) and \(\int_{\partial \Sigma} \tilde{k} \, dH^1\) are finite.

Now observe that \(dH^1 = \omega dH^1\) and \(dH^2 = \omega^2 dH^2\) by definition of \(\hat{g}\), where \(dH^1\) and \(dH^2\) are the one-dimensional and two-dimensional Hausdorff measures on \((\Sigma, g)\). Hence, applying (7.2) and (7.3), we obtain that
\[
\int_{\Sigma} \tilde{K} \, dH^2 \geq \int_{\Sigma} \frac{\sigma_0}{2\omega^2} \, dH^2 = \int_{\Sigma} \frac{\sigma_0}{2\omega^2} \omega^2 \, dH^2 = \frac{\sigma_0}{2} H^2(\Sigma)
\]
and that
\[
\int_{\partial \Sigma} \tilde{k} \, dH^1 = \int_{\partial \Sigma} \frac{\sigma_0}{\omega} \, dH^1 = \int_{\partial \Sigma} \frac{\sigma_0}{\omega} \omega \, dH^1 = \sigma_0 H^1(\partial \Sigma).
\]
In particular we deduce that
\begin{equation}
0 \leq \frac{\sigma_0}{2} \mathcal{H}^2(\Sigma) + \sigma_0 \mathcal{H}^1(\partial \Sigma) \leq 2\pi \limsup_{r \to \infty} \chi(\Omega_r) \leq 2\pi .
\end{equation}
Notice that this also proves that $\Sigma$ has finite topological type and well-defined Euler characteristic given by $\chi(\Sigma) = \limsup_{r \to \infty} \chi(\Omega_r)$. In particular we have obtained that $\partial \Sigma$ is compact if $\sigma_0 > 0$. However, since a priori we are not in the position to invoke the isoperimetric inequality in [49], we actually need the following lemma.

**Lemma 7.3.** Let $\Sigma^2 \hookrightarrow M^3$ be as in Proposition 1.8 and let $x_0 \in \Sigma$. Then $x_0$ has distance from the boundary of $\Sigma$ bounded by a constant depending only on $\varrho_0 = \inf_M R_g$ and $\sigma_0 = \inf_{\partial M} H^3 M$. Namely it holds
\[
d_S(x_0, \partial \Sigma) \leq \min \left\{ \frac{2\sqrt{2} \pi}{\sqrt{3} \varrho_0}, \frac{4}{3\sigma_0} \right\} .
\]

**Proof.** Let us consider the functional
\[
\tilde{l}(\gamma) := \int_\gamma \omega
\]
on the class of $W^{1,2}$-curves $\gamma$ lying in $\Sigma$ and connecting $x_0$ to a point in $\partial \Sigma$. Note that here with $\int_\gamma \omega$ we mean the integral with respect to the arc parameter of $\gamma$ and thus $\tilde{l}(\gamma)$ coincides exactly with the length of the curve $\gamma$ in the metric $\tilde{g}$ considered above. In particular let $\gamma$ be a curve minimizing this functional. Observe that this curve is smooth and has finite length since $\omega$ is strictly positive on the closure of $\Sigma$.

We now compute the first and second variation of the functional $\tilde{l}$ along $\gamma$. Without loss of generality, we can assume that $\gamma$ is parametrized by arc-length, that is $|\gamma'| = 1$, and it has length $l$. Thus let us choose a variation $\alpha : (-\varepsilon, \varepsilon) \times [0,l] \to \Sigma$ with $\alpha(0, \cdot) = \gamma$, $\alpha(s,0) = x_0$ and $\alpha(s,1) \in \partial \Sigma$. Computing the first variation we obtain\(^6\)
\[
\frac{d}{ds} \tilde{l}(\alpha(s, \cdot)) = \frac{d}{ds} \int_0^l \omega(\alpha(s,t)) \left| \frac{\partial \alpha}{\partial t} \right| (s,t) dt = \int_0^l \frac{d\omega}{d\alpha} \left[ \left| \frac{\partial \alpha}{\partial s} \right| \frac{\partial \alpha}{\partial t} \right] dt + \omega(\alpha) \frac{\langle \nabla \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \rangle}{\left| \frac{\partial}{\partial t} \right|} \cdot
\]
\[
+ \omega(\alpha) \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) \left( \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right) \left| \frac{\partial}{\partial t} \right|^3 .
\]
In particular, evaluating at $s = 0$ and setting $X(t) = \frac{\partial \alpha}{\partial s}(t,0)$, we have that
\[
0 = \frac{d}{ds} \Big|_{s=0} \tilde{l}(\alpha(s, \cdot)) = \int_0^l \left( \frac{d\omega}{d\alpha} \left[ \nabla \frac{\partial \alpha}{\partial s}, X \right] - \omega(\gamma) \nabla \gamma', X \right) dt + \omega(\gamma) \nabla \gamma', X dt + \omega(\gamma) \nabla \gamma', X dt
\]
\[
= \int_0^l \left( \frac{d\omega}{d\alpha} \left[ \nabla \gamma', X \right] - \omega(\gamma) \nabla \gamma', X \right) dt + \omega(\gamma) \nabla \gamma', X dt,
\]
which holds for all variations $\alpha$ as above. Notice that we have used that $0 = \frac{d}{dt} |\gamma'|^2 = 2 \langle \nabla \gamma', \gamma' \rangle$.

\(^6\)Observe that this inequality in the compact case follows directly from the stability inequality applied to a constant function.

\(^7\)We always omit the dependence from $s$ and $t$ if clear from the context.
As a result, since $X(l)$ is tangent to $\partial \Sigma$ (otherwise the image of $\alpha$ would exit from $\Sigma$), we have that
\[
\begin{cases}
\nabla \omega(\gamma) - d\omega(\gamma)[\gamma'] - \omega(\gamma)\nabla_{\gamma'}\gamma' = 0 \\
\gamma'(l) \perp \partial \Sigma.
\end{cases}
\]

We can then compute the second variation for $s = 0$, obtaining
\[
\frac{d^2}{ds^2} \mid_{s=0} \bar{l}(\alpha(s, \cdot)) = \int_0^l \left( \nabla \omega(\alpha) \left| \frac{\partial \alpha}{\partial t} \right|^2 - d\omega(\alpha) \left[ \frac{\partial \alpha}{\partial t} \right] \left[ \frac{\partial \alpha}{\partial t} \right] \right) dt +
\omega(\alpha) \left( \nabla \omega \left| \frac{\partial \alpha}{\partial s} \right|_{s=0} \frac{\partial \alpha}{\partial s} \right) \left( X(l), \nabla \omega \left| \frac{\partial \alpha}{\partial s} \right|_{s=0} \right) \right) .
\]

Now assume that $X(t)$ is of the form $\psi(t)\tau$ where $\tau$ is a unit vector field orthogonal to $\gamma'$. Then we have that
\[
\frac{d^2}{ds^2} \mid_{s=0} \bar{l}(\alpha(s, \cdot)) = \int_0^l (\psi^2 \Delta \omega - \psi^2 \omega'' - \psi \psi' \omega' - \psi \psi'' \omega - \psi^2 K \omega) dt + \omega(l)(\psi \psi' + \psi^2 \Pi^M(\tau, \tau))
\]
\[= \int_0^l \left( \psi^2 J_{\Sigma \omega} - \frac{1}{2} R_g + |A|^2 \omega - \psi \omega'' - \psi \psi' \omega' - \psi \psi'' \omega \right) dt + \omega(l)(\psi \psi' + \psi^2 l \Pi^M(\tau, \tau)).
\]

Let $h : [0, l] \to \mathbb{R}$ be the first (positive) eigenfunction for the eigenvalue problem
\[
\begin{cases}
\psi'' + \omega^{-1} \omega' \psi' + \left( \frac{1}{4} R_g + \frac{3}{2} |A|^2 - \omega^{-1} J_{\Sigma \omega} + \omega^{-1} \omega'' \right) \psi + \lambda \psi = 0 \\
\psi(0) = 0 \\
\psi(l) = -\Pi^M(\tau, \tau) \psi(l).
\end{cases}
\]

Then, since $\frac{d^2}{ds^2} \mid_{s=0} \bar{l}(\alpha(s, \cdot)) \geq 0$ for all variations $\alpha$, we have in particular that
\[
h^{-1} h'' + \omega^{-1} \omega' h' + \frac{1}{2} q_0 + \omega^{-1} \omega'' \leq
\leq h^{-1} h'' + \omega^{-1} \omega' h' + \frac{1}{2} R_g + \frac{1}{2} |A|^2 - \omega^{-1} J_{\Sigma \omega} + \omega^{-1} \omega'' = -\lambda \leq 0.
\]

Therefore, multiplying this inequality by a test function $\xi \in C^\infty([0, l])$ with $\xi(0) = 0$ and integrating by parts, we obtain
\[
0 \geq \int_0^l \left( h^{-1} h'' + \omega^{-1} \omega' h' + \frac{1}{2} q_0 + \omega^{-1} \omega'' \right) \xi^2 dt
\]
\[= \int_0^l \left( \frac{1}{2} (h'' h')^2 + \omega^{-2} (\omega')^2 + \omega^{-1} \omega' h' h' \right) \xi^2 + \frac{1}{2} q_0 \xi^2 - 2(h^{-1} h' + \omega^{-1} \omega') \xi \xi' dt +
+ \left( h^{-1} H h' + \omega^{-1} H \omega' \right) \xi^2 (l)
\]
\[= \int_0^l \frac{1}{2} \left( \frac{d}{dt} (\log(\omega)) \right)^2 \xi^2 dt +
- 2 \int_0^l \frac{d}{dt} (\log(\omega)) \xi \xi' dt - (\Pi^M(\tau, \tau) + H^M(\nu, \nu)) \xi^2 (l)
\]
\[\geq \int_0^l \frac{1}{2} \left( \frac{d}{dt} (\log(\omega)) \right)^2 \xi^2 dt + \frac{1}{2} q_0 \xi^2 + \frac{1}{2} \left( \frac{d}{dt} (\log(\omega)) \right)^2 \xi^2 dt +
- 2 \int_0^l \frac{d}{dt} (\log(\omega)) \xi \xi' dt + \sigma_0 \xi^2 (l),
\]
where we have used the boundary assumptions on $h$ and $ω$, noticing that $ω'(l) = (\nabla ω, ω'(l)) = \frac{dω}{dt}$, since $γ'(l) = η(γ(l))$ ($γ'(l)$ is orthogonal to $∂Σ$ and outward-pointing). Furthermore, we exploited the fact that $−Π^{M}(τ, ν) − Π^{M}(ν, ν) = H^{∂M} ≥ σ_0$.

We can then exploit the inequality

$$2 \left| \frac{d}{dt}(log(ωh)) \xi' \right| \leq \frac{1}{2} \left( h^{-2}(h')^2 + ω^{-2}(ω')^2 \right) ξ^2 + \frac{1}{2} \left( \frac{d}{dt}(log(ωh)) \right)^2 ξ^2 + \frac{4}{3}(ξ')^2$$

to conclude that

$$\frac{1}{2} \varrho_0 \int_0^l ξ^2 dt + σ_0 \xi^2(l) \leq \frac{4}{3} \int_0^l (ξ')^2 dt$$

for all $ξ$ as above. This proves that

$$l ≤ δ_0 := \min \left\{ \frac{2\sqrt{2π}}{\sqrt{σ_0}}, \frac{4}{3σ_0} \right\} < ∞,$$

as in [6, Proposition 2.12] for the first term and for example by taking $ξ(t) = t$ for the second term. As a result we have proven that $Σ$ is contained in the $δ_0$-neighborhood of $∂Σ$ with respect to the intrinsic distance. □

Remark 7.4. Observe that we carried on in parallel the proof of the two cases (i) and (ii), but actually for (i) the proof can be simplified as follows. Pick two points $x_0, y_0 ∈ Σ \setminus ∂Σ$ and consider the curve $γ$ minimizing $I(γ)$ and connecting $x_0$ to $y_0$. Since $γ$ is a minimizing curve in the metric $g$ of $Σ$, then it cannot touch the convex boundary $∂Σ$. Therefore we can follow the very same argument as in [6, Proposition 2.12] to prove that the length of $γ$ in the metric $g$ is bounded by $\frac{2\sqrt{2π}}{\sqrt{σ_0}}$ and this easily concludes the proof.

For what concerns case (ii) we conclude by observing that Lemma 7.3, together with the compactness of $∂Σ$, proves the compactness of $Σ$. The diameter estimate follows by simply combining equation (7.4), with Lemma 7.3 and Remark 7.4. □

8. AMBIENT MANIFOLDS WITH ‘POSITIVE GEOMETRY’

We can now capitalize our efforts and present the proof of Theorem 1.4, which crucially exploits Proposition 1.8.

Proof of Theorem 1.4. We only need to prove the area bound, since all other conclusions then follow from Theorem 1.2. Assume by contradiction that there exists a sequence of connected, compact, properly embedded, free boundary minimal surfaces $Σ_j^2 ⊂ M$ with non-empty boundary and with $ind(Σ_j) ≤ I$ and $H^2(Σ_j) → ∞$. Then, there exists a point $x_0 ∈ M$ such that $H^2(Σ_j ∩ B_r(x_0)) → ∞$ for all $r > 0$.

Denote by $L$ the limit lamination given by Theorem 6.1 and consider the leaf $L ∈ L$ passing through $x_0$. Thanks to Theorem 3.11, $L$ must have stable universal cover (the other case is excluded since the area is diverging around $x_0$), which we can represent by a stable free boundary minimal immersion $φ : Σ → M$. By means of a variation of the pull-back construction presented e.g. in [3, Section 6], we can then reduce to invoking Proposition 1.8 so to conclude that $Σ$ must be a disc, hence the map $φ$ must be an embedding (therefore $L$ is a stable, free boundary minimal surface in $(M, g)$).

Let then $Σ_j$ be the sequence obtained from $Σ_j$ by means of the surgery procedure, as per Corollary 6.2. Observe that the new sequence satisfies uniform curve bounds, and still has diverging area. Also, since Corollary 6.2 provides a uniform bound $k(l) + 1$ on the number of connected components of $Σ_j$, we can select and rename $Σ_j$ so that it is connected for every $j ∈ \mathbb{N}$, and the area concentrates near the point $x_0$. 

Now, fix \( r > 0 \) sufficiently small, and assume to consider a connected component of the intersection \( \tilde{\Sigma}_j \cap B_r(x_0) \) which smoothly converges with multiplicity one to \( L \cap B_r(x_0) \). Since \( L \) is a disc, a standard monodromy argument allows to conclude that, a posteriori, (the whole component) \( \tilde{\Sigma}_j \) converges to \( L \) smoothly with multiplicity one. From this fact, we derive a uniform bound for the areas of \( \tilde{\Sigma}_j \), which is a contradiction. \( \square \)

9. Non-compact families of free boundary minimal surfaces of fixed topology

Theorem 1.12 is proven via a suitable, rather explicit, gluing construction aimed at attaching some elementary blocks. In all cases we shall now list, the word block refers to an ambient manifold together with a sequence of minimal surfaces (closed or having free boundary) satisfying additional requirements.

9.1. The building blocks: Spiraling spheres.

**Lemma 9.1** (cf. [13, Proposition 8]). On \( S^3 \) there exists a Riemannian metric \( g_0 \) of positive scalar curvature such that \((S^3, g_0)\) contains a sequence of minimal spheres \( \Sigma_j \) with arbitrarily large area and index, and converging to a singular lamination \( \mathcal{L} \) whose singular set consists of exactly two points lying on a leaf which is a strictly stable (two-dimensional) sphere. Furthermore, the metric in question coincides with the unit round metric in a neighborhood of two distinct points \( x_0 \) and \( y_0 \), and on both those neighborhoods such minimal spheres can be completed to a local foliation by great spheres parallel at \( x_0 \) and \( y_0 \) respectively.

**Remark 9.2.** An important point here is to clarify what we mean when writing that a local foliation is parallel at a point (cf. [13, Definition 7]). Given \( z \in \Omega \), an open subset of the round 3-sphere, and \( \mathcal{F} \), a local foliation of \( \Omega \) by great spheres, we say that the foliation is parallel at \( z \in F \) (for some \( F \in \mathcal{F} \), to be called the central leaf) if

\[
\sup_{w \in F} d(w, F') = d(z, F')
\]

for any \( F' \in \mathcal{F} \).

The geometric picture this definition captures is easily described. Take \( S^3 \subset \mathbb{R}^4 \) isometrically embedded as unit sphere and consider the foliation of \( S^3 \setminus \{0,0,0,\pm 1\} \) consisting of the 2-spheres obtained by slicing via vertical hyperplanes. Given any point \( x = (x^1, x^2, x^3, 0) \) and any open set \( \Omega \ni x \) then the restriction of the above foliation to \( \Omega \) is parallel at \( x \), and the unique great sphere passing through \( x \) is the central leaf.

**Remark 9.3.** For later use (cf. Remark 9.9 and Section 9.4), it is helpful to introduce some related terminology. We consider the set \( \Omega^+ := \{ z \in \Omega : z^4 \geq 0 \} \) and the corresponding foliation \( \mathcal{F}^+ \) obtained by considering the intersection of each leaf of \( \mathcal{F} \) with the domain \( \Omega^+ \). We will say that \( \mathcal{F}^+ \) is a foliation of \( \Omega^+ \) by half great spheres, parallel at \( x \).

**Remark 9.4.** In the statement of Lemma 9.1 the stability of the limit sphere follows, by now routine arguments, considering the (suitably) normalized distance between adjacent leaves of the lamination \( \mathcal{L} \).

**Remark 9.5.** It follows from the construction (see, specifically, the second paragraph at [13, pp. 30]) that for any open set \( \Omega \) containing the limit sphere, one has that \( \text{ind}(\Sigma_j \cap \Omega) \rightarrow \infty \) as \( j \rightarrow \infty \), where it is understood that one only considers variations that are compactly supported in \( \Omega \). Of course, it is also true that \( H^2(\Sigma_j \cap \Omega) \rightarrow \infty \) as \( j \rightarrow \infty \).
9.2. The building blocks: Minimal tori. We first provide the relevant statement and then mention the key points in the construction, to the extent this is needed in Section 9.4 below to produce, for any given \( b > 1 \), a Riemannian metric of positive scalar curvature on the 3-ball so that the resulting 3-manifold contains a family of free boundary surfaces of genus 0 and exactly \( b \) boundary components.

**Lemma 9.6** (cf. [13, Lemma 12]). On \( S^3 \) there exists a Riemannian metric \( g_1 \) of positive scalar curvature such that:

1. \((S^3, g_1)\) contains a family of minimal tori \( \Pi_\theta \) parametrized by \( \theta \in (-\theta_0, \theta_0) \), for some \( \theta_0 > 0 \);
2. the metric in question coincides with the unit round metric in a neighborhood of given points \( x_1 \) and \( y_1 \) and on both those neighborhoods such minimal tori provide a local foliation by great spheres parallel at \( x_1 \) and \( y_1 \), respectively.

The construction can be schematically described as follows:

- On the (topological) product manifold \([-\pi/2, \pi/2] \times S^1 \times S^1\), one can consider the equivalence relation \( \sim \) given by
  \[
  (-\pi/2, p, q) \sim (-\pi/2, p, q') \quad \forall \ p \in S^1, \ \forall \ q, q' \in S^1,
  \]
  and
  \[
  (\pi/2, p, q) \sim (\pi/2, p', q) \quad \forall \ p, p' \in S^1, \ \forall \ q \in S^1,
  \]
  that corresponds to ‘collapsing vertical (resp. horizontal) fibers on \(-\pi/2 \times S^1 \times S^1\) (resp. on \(\pi/2 \times S^1 \times S^1\)’). Said \( M = M/\sim \), this manifold can be endowed with a smooth Riemannian metric \( \tilde{g} \) so that \((M, \tilde{g})\) is a 3-sphere of positive scalar curvature, and in a neighborhood of \(\{0\} \times S^1 \times S^1\) the metric is isometric to the Riemannian product of \(S^2 \times S^1\). Hence \((M, \tilde{g})\) contains a one-parameter family of **totally geodesic** tori all having two circles in common, say \(\{0\} \times \{0\} \times S^1\) and \(\{0\} \times \{\pi/2\} \times S^1\) (where we are conveniently identifying the round unit \(S^1\) with the interval \([0, 2\pi]\) with endpoints attached).
- At this stage one can perform a **local** modification of the metric \( \tilde{g} \) near the points \((0, \pi/2, 0)\) and \((0, 3\pi/2, 0)\) so to make it round; with the family of tori being locally isometric to a family of standard great spheres in such neighborhoods. The construction is performed by explicitly interpolating between the metric of round \(S^3\) and the product metric of \(S^2 \times S^1\).

**Remark 9.7.** We observe that:

- In \((S^3, g_1)\) the surface
  \[
  \tilde{\Sigma} := \left(\left[-\pi/2, \pi/2\right] \times \left\{\pi/2\right\} \times S^1 \cup \left[-\pi/2, \pi/2\right] \times \left\{3\pi/2\right\} \times S^1\right)/\sim
  \]
  is a totally geodesic 2-sphere, which divides the closed manifold into two (pairwise isometric) three-dimensional balls.
- Denoted by \(\Omega\) one of such balls, for any \( \theta \in (-\theta_0, \theta_0) \) the intersection \(\Xi_\theta := \Pi_\theta \cap \overline{\Omega}\) is checked to be a free boundary minimal annulus.

Hence, the Riemannian manifold \((\overline{\Omega}, g_1)\) contains a family of minimal annuli \(\{\Xi_\theta\}\) parametrized by \( \theta \in (-\theta_0, \theta_0) \), for some \( \theta_0 > 0 \); the metric \( g_1 \) coincides with the unit round metric in a neighborhood of boundary points \( x_1 \) and \( y_1 \) (these points belonging to the two boundary circles of \(\Xi_0\)) and on both those neighborhoods \(\partial \Omega\) is isometric to an equatorial 2-sphere and the minimal annuli provide half of a local foliation by great spheres parallel at \( x_1 \) and \( y_1 \), respectively.

9.3. The building blocks: Free boundary minimal discs. Let us now, instead, move to the free boundary models. We prove this ancillary result.

**Lemma 9.8.** For any \( \varepsilon \in (0, \pi/4)\) there exists a smooth Riemannian metric \( g_2 = g_2(\varepsilon) \) on the closed ball \( \overline{B^3} = ([0, 1 + \pi/2] \times S^2)/\sim \) with coordinates \((r, \omega)\) (where \(\sim\) is the equivalence relation collapsing...
\(\{1 + \pi/2\} \times S^2\) to a point) having positive scalar curvature, and such that the following properties are satisfied:

1. \(g_2\) coincides with the unit round metric of \(S^3\) on the domain \(([1 + \varepsilon, 1 + \pi/2] \times S^2)/\sim\), and coincides with the cylindrical metric on the domain \((0, 1/3] \times S^2)/\sim\);
2. the resulting manifold \((\tilde{B}^3, \tilde{g}_2)\) contains a one-parameter family \(\Delta^{(i)}_\theta\) of embedded free boundary minimal discs (here \(\theta \in (-\theta_0, \theta_0)\) for some \(\theta_0 > 0\));
3. there exist two points \(x_2^{(i)}, y_2^{(i)}\) with \(r(x_2^{(i)}) = r(y_2^{(i)}) = 1/2\) and \(\omega(x_2^{(i)}) = -\omega(y_2^{(i)})\), and open neighborhoods \(\Omega(x_2^{(i)}), \Omega(y_2^{(i)})\) respectively, where \(g_2\) is round (isometric to domains of the unit round metric of \(S^3\)) and the minimal discs above restrict to give two local foliations by great spheres that are parallel at \(x_2^{(i)}, y_2^{(i)}\) respectively.

**Proof.** Consider the cylinder \(I \times S^2\) (where \(I = [0, 1]\) and the sphere \(S^2\) is endowed with the standard unit round metric), fix two antipodal points \(p, q \in S^2\) and consider the family of circles passing through \((0, p)\) and \((0, q)\). Said \(\Gamma\) any of those circles, then \(I \times \Gamma\) is a smooth, free boundary minimal surface \(\Delta\) in \(I \times S^2\) (with two boundary components). Now, cap off \(I \times S^2\) by identifying its upper boundary with the boundary of a hemisphere in \(\mathbb{H}^3\) (the warping factor being a \(C^{1,1}\) function of the radial coordinate). Furthermore, one can attach a two-dimensional half-hemisphere to \(\Delta\) so to get a free boundary minimal surface, which we shall not rename, that is not yet smooth along the connecting circle but has a mild singularity there. Now, since (in the resulting 3-manifold) the mean curvature of the interface on both sides match (for the interface is totally geodesic on both sides), we can apply the smoothing theorem by P. Miao [38] in the simplest possible case (see in particular Theorem 1 therein) to get a smooth metric \(\tilde{g}\) on the closed 3-dimensional ball, that (1) coincides with \(\tilde{g}\) away from a small neighborhood of the gluing interface, and (2) whose scalar curvature is positive. In fact, by the very way the construction is defined, namely by fiberwise convolution, it follows that the metric \(\tilde{g}\) takes the form of a warped product, i.e. we have (in the coordinates \((r, \omega)\) introduced in the statement)

\[
\tilde{g} = dr^2 + f^2(r)gs^2
\]

where

\[
f(r) = \begin{cases} 
1 & \text{if } r \in [0, 1 - \varepsilon] \\
\sin(r - 1 + \pi/2) & \text{if } r \in [1 + \varepsilon, 1 + \pi/2]
\end{cases}
\]

for small \(\varepsilon > 0\). The set \(\Delta\) is then a totally geodesic surface in such ambient manifold, and since the metric has not been modified near the boundary component which has not been capped off one has that the free boundary condition is still fulfilled.

At this stage, let us consider the construction above starting from a one parameter family of circles \(\{\Gamma_\theta\}\) (for \(\theta\) varying in a subset of \(S^1\), say \(\theta \in (-\theta_0, \theta_0)\) for some \(\theta_0 > 0\)) passing through the points \((0, p), (0, q) \in \{0\} \times S^2\) and let \(p', q' \in S^2\) be two antipodal points (on the great circle equidistant from \(p\) and \(q\)) chosen so that \((0, p'), (0, q')\) have small neighborhoods that are foliated by such a family. Let \(\Delta^{(i)}_\theta\) be the corresponding free boundary minimal surfaces.

Since the cylindrical metric has, so far, not been modified away from a small neighborhoods of the gluing interface we can consider the points \(x_2^{(i)} = (1/2, p'), y_2^{(i)} = (1/2, q')\), and open neighborhoods thereof (named \(\Omega(x_2^{(i)}), \Omega(y_2^{(i)})\) respectively) that are foliated by the surfaces \(\Delta^{(i)}_\theta\) as \(\theta\) varies. Hence, we perform a local deformation of the metric in each of these neighborhoods to make it round (for which it is enough to follow, without any modification, the argument given in the second part of Appendix B of [13]). Possibly renaming those open neighborhoods (to be taken slightly smaller than we had originally defined), the metric \(g_2 = g_2(\varepsilon)\), resulting from such local modifications, satisfies all desired properties. \(\Box\)
Remark 9.9. We will also need the following variant of the previous result: the local modifications of the metric are performed at one interior point (as above) and at one boundary point. Thereby, one obtains a 3-manifold \((M_{\text{bdry}}^3, g_{\text{bdry}}^1)\) of positive scalar curvature which contains a family of free boundary minimal discs, still denoted by \(\Delta_{g}^1\), and having two points:

- \(x_{2, \text{bdry}}^{(1)}\) (in the interior) having a full neighborhood where such minimal discs provide a local foliation by great spheres (parallel at \(x_{2, \text{bdry}}^{(1)}\))
- \(y_{2, \text{bdry}}^{(1)}\) (on the boundary) having a half neighborhood where such minimal discs provide a local foliation by half great spheres (parallel at \(y_{2, \text{bdry}}^{(1)}\)).

With respect to the notation employed in the proof above, one can take (for instance)
\[
x_{2, \text{bdry}}^{(1)} = \left(1/2, p', \phi \right), \quad y_{2, \text{bdry}}^{(1)} = \left(0, q', \phi \right).
\]

9.4. The building blocks: Free boundary minimal k-annuli. In order to prove Theorem 1.12, we also need to construct metrics on the closed 3-ball having positive scalar curvature and containing families of free boundary minimal surfaces of genus zero and any pre-assigned number \(b\) of boundary components. So far, this has only been accomplished for \(b = 1\) (in Section 9.3) and for \(b = 2\), as a result of Remark 9.7. To proceed we need the following free boundary analogue of Lemma 11 in [13].

Lemma 9.10. Let \(\delta > 0\) and let \(\Omega_{1}^{+} := B_{\delta}(u_{1})\), \(\Omega_{2}^{+} := B_{\delta}(u_{2})\) be two (relatively) open half-balls of round unit hemispheres \(N_{i}^{+}, N_{i}^{+}\), centered at boundary points \(u_{1}, u_{2}\) respectively. Suppose that \(F_{1}^{+}\) and \(F_{2}^{+}\) are (locally defined) foliations of \(\Omega_{1}\) and, respectively, \(\Omega_{2}\) by half great spheres parallel at \(u_{1}\) and, respectively, \(u_{2}\). Then we can join those hemispheres to obtain a smooth Riemannian manifold with boundary (having the topology of \(B^{3}_{\delta}\) of positive scalar curvature, and (possibly by considering smaller neighborhoods) the leaves of \(F_{1}^{+}\) and \(F_{2}^{+}\) can pairwise be matched to obtain free boundary minimal surfaces in the ambient manifold.

Proof. Let us double each of the two given manifolds with boundary to round spheres \(N_{1}\) and \(N_{2}\) and consider the foliations \(F_{1}\), \(F_{2}\) obtained by extending \(F_{1}^{+}\) and \(F_{2}^{+}\) in the obvious fashion (each leaf of \(F_{i}^{+}\) is extended to an equatorial 2-sphere in \(N_{i}\), for \(i = 1, 2\)). Since our construction is purely local, this does not affect the generality of the argument.

We know (by Lemma 11 in [13]) that one can construct a connected sum \(N_{1} \# N_{2}\) and pairwise match the leaves of the foliations \(F_{1}\) and \(F_{2}\). The connecting neck \(N\), diffeomorphic to \([-\tau, \tau] \times S^{2}\), can be described by means of spherical coordinates
\[
(r, \phi, \theta) \in [-\tau, \tau] \times [0, \pi] \times S^{1}
\]
and each minimal surface that is constructed there is the lift of a graphical curve \(\sigma : [-\sigma_{0}, \sigma_{0}] \to [-\tau, \tau] \times [0, \pi]\) solving a suitable ODE (that is nothing but a geodesic equation in a degenerate metric). In other words, each such minimal surface takes (in the neck) the form
\[
\Sigma := \\{(r, \sigma(r), \theta) : r \in [-\tau, \tau], \theta \in S^{1}\}.
\]

That being said, these coordinates can be chosen so that the condition \(\theta \in S^{1}_{+}\) (for \(S^{1}_{+} \subset S^{1}\) a half-circle that is fixed now and for all) identifies the half-sphere \(\partial \Omega_{1}^{+}\), and the same conclusion holds true for \(\partial \Omega_{2}^{+}\) as well. Now, if we define
\[
N^{+} := \{(r, \phi, \theta) \in [-\tau, \tau] \times [0, \pi] \times S^{1}_{+}\}
\]
the totally geodesic 2-sphere \((\partial N^{+} \setminus \Omega_{1}^{+}) \cup \partial_{1}N^{+} \cup (\partial N_{2}^{+} \setminus \Omega_{2}^{+})\), where \(\partial_{1}N^{+} = [-\tau, \tau] \times [0, \pi] \times \partial S^{1}_{+}\), divides the connected sum into two, mutually isometric balls. If we consider the one, among those, containing \(N_{1}^{+} \setminus \Omega_{1}^{+}\) (which we might call the upper copy) it is straightforward to check that
\[
\Sigma^{+} := \{(r, \sigma(r), \theta) : r \in [-\tau, \tau], \theta \in S^{1}_{+}\}.
\]
is indeed a free boundary minimal surface. This correspondence holds true for any closed minimal surface that is obtained by means of the wire-matching argument by Colding-De Lellis; in particular, the matching is certainly possible for the central leaves and a family of nearby leaves, so the proof is complete.

We shall now present the main, straightforward application of this gluing lemma.

**Lemma 9.11.** Given any \( b \geq 2 \) there exists, on \( \overline{B}^3 \), a Riemannian metric \( g_2^{(b)} \) of positive scalar curvature such that:

1. \( (\overline{B}^3, g_2^{(b)}) \) contains a family of minimal \( b \)-connected domains \( \Delta_\theta^{(b)} \) parametrized by \( \theta \in (-\theta_0, \theta_0) \), for some \( \theta_0 > 0 \);
2. the metric in question coincides with the unit round metric in a neighborhood of given points \( x_2^{(b)} \) (in the interior) and \( y_1^{(b)} \) (on the boundary) and on both those neighborhoods such minimal annuli provide a local foliation by (half-)great spheres parallel at \( x_2^{(b)} \) and \( y_1^{(b)} \), respectively.

**Proof.** When \( b = 2 \) this follows by applying Lemma 9.10 to the blocks \( (M_2^{(1)}_{\text{bdry}}, g_2^{(1)}_{\text{bdry}}) \) near \( y_2^{(1)} \) (see Remark 9.9) and \( (\Omega, g_1) \) near \( x_1 \) (see Remark 9.7). The case \( b > 2 \) follows by simply repeating the operation, namely joining the resulting manifold with further copies of \( (\Omega, g_1) \).

9.5. Construction of the counterexamples.

**Proof of Theorem 1.12.** It is convenient to divide the argument in two steps.

**Step 1.** Given integers \( a \geq 0 \) and \( b > 0 \) as in the statement, let us consider:

- one copy of the Riemannian 3-sphere of positive scalar curvature as per Lemma 9.1, which we shall refer to as \( (M_0, g_0) \) and let \( (x_0, y_0) \) be the pair of points mentioned in that statement;
- \( a \) copies of the Riemannian 3-sphere of positive scalar curvature produced by Lemma 9.6, which we shall refer to as \( (M_1, g_1), \ldots, (M_a, g_a) \), and let \( (x_1, y_1), \ldots, (x_a, y_a) \) be the pairs of points mentioned in that statement, respectively; each \( (M_i, g_i) \) contains a family of minimal tori that provide a foliation by great 2-spheres of suitably small neighborhoods of \( x_i \) and \( y_i \);
- one copy of the Riemannian (closed) 3-ball of positive scalar curvature and totally geodesic boundary produced via Lemma 9.11, which we shall refer to as \( (M^{(b)}, g^{(b)}) \) and let \( (x_2^{(b)}, y_2^{(b)}) \) be the pair of points mentioned in that statement.

![Figure 10](image)

**Figure 10.** Scheme of the construction in the proof of Theorem 1.12 for \( a = 2 \) and \( b = 4 \).

Invoking Lemma 11 in [13], we proceed as follows (see Figure 10). We first attach \( (M_1, g_1) \), near \( y_1 \), to \( (M_1^{i+1}, g_1^{i+1}) \), near \( x_1^{i+1} \), as \( i \) varies from 1 to \( a - 1 \); let \( (M', g') \) be the resulting manifold (of
positive scalar curvature and empty boundary); let \( y' \in M' \) be the point corresponding to \( y_0^1 \in M^1 \), thus with a neighborhood that is foliated by great spheres. Similarly we attach \((M', y')\), near \( y' \), to \((M_0, g_0)\) near \( x_0 \); let \((M'', g'')\) be the resulting manifold (of positive scalar curvature and empty boundary) and let \( y'' \in M'' \) be the point corresponding to \( y_0 \in M_0 \), thus with a neighborhood that is foliated by great spheres. Lastly, we attach \((M'', y'')\), near \( y'' \), to \((M_2^{(b)}, g_2^{(b)})\), near \( x_2^{(b)} \); let \((M''' , g''' )\) be the resulting manifold (of positive scalar curvature and totally geodesic boundary). The manifold \((M''' , g''' )\) is connected, has the topology of a ball, and it contains a sequence of free boundary minimal surfaces of genus \( a \), exactly \( b \) boundary components, that have unbounded area and Morse index (cf. Remark 9.5 above).

Step 2. Let \( M \) be as in the statement: possibly applying Lemma C.1 in [8] we can, and we shall, assume that this manifold comes endowed with a Riemannian metric of positive scalar curvature, and such that \( \partial M \) is strictly mean convex. At that stage we know, by virtue of Theorem 5.7 in [23], that there exists a new metric \( g_T \) on \( M \) still having positive scalar curvature but totally geodesic boundary (in fact this manifold can be doubled to a smooth Riemannian manifold \((M_D, g_D)\) without boundary). Hence, we just observe that one can perform the Gromov-Lawson connected sum of \((M''' , g''' )\) and \((M, g_T)\) so to obtain a compact 3-manifold with positive scalar curvature and totally geodesic boundary. Let us describe this step in some more detail.

Fix a point \( z_D \) on the symmetry locus of \((M_D, g_D)\) (which corresponds to the set \( \partial M \) before the doubling). Similarly, consider a doubled copy \((M''', g'''_D)\) of the manifold \((M''' , g''' )\) we have constructed above, obtained by reflecting along the boundary, and let \( z'''_D \) a point on its symmetry locus but away from the minimal surfaces produced in Step 1. It follows from work of Schoen-Yau (cf. [43]) that one can realize the connected sum \( M_D \# M'''_D \) keeping the scalar curvature positive, with a (normalized) cylindrical neck. Thereby one obtains a compact 3-manifold with boundary, and the neck in question can be constructed so to obtain (by cutting along the symmetry locus of such a manifold) two isometric copies of a compact 3-manifold of positive scalar curvature and totally geodesic boundary, simply denoted by \((M, g)\).

The combination of the two steps above allows to complete the proof of the first assertion of Theorem 1.12. Instead, to obtain strictly positive mean curvature, it suffices to have the previous construction followed by the perturbation argument given in Lemma C.1 in [8].

Concerning the claim in Remark 1.13, it suffices to observe that when \( b = 1 \) one modifies the block \((M_1^{(1)}, g_1^{(1)})\) constructed in Section 9.3 as follows: by the statement of Lemma 9.8, the metric we obtained equals that of the cylinder \( I \times S^2 \) near the boundary sphere, thus we can just consider a smooth warping factor \( f_W : [0, \varepsilon] \times S^2 \to \mathbb{R} \), only depending on the first coordinate, that is monotone decreasing and equals \( f \) on \([\varepsilon/2, \varepsilon] \). For any such choice the boundary is convex (umbilic, with constant mean curvature); furthermore if the derivative of \( f_W \) is small enough then the scalar curvature of the ambient manifold shall still be positive.

Appendix A. Free boundary minimal surfaces and Morse index

Let \((M^3, g)\) be a compact Riemannian manifold with non-empty boundary. Given an embedded surface \( \Sigma^2 \) in \( M \) with \( \partial \Sigma \subset \partial M \), we wish to compare different definitions of free boundary minimality and Morse index when one allows for an (arbitrary) contact set for \( \Sigma \) along \( \partial M \), as in Figure 11. The aim of this section is to analyze this matter in some detail, providing some flavour of the surprisingly subtle nature of the question.

To fix the notation, we will always use the expression free boundary minimal surface to denote a surface with zero mean curvature that meets the boundary of the ambient manifold orthogonally along its own boundary.
Remark A.1. Observe that local minimizers of the area are not necessarily free boundary minimal surfaces with respect to this definition. See Figure 12 for an example of a local minimizer which is not a free boundary minimal surface\(^8\).

![Figure 11. Modified unit ball with non-properly embedded free boundary minimal surface.](image1)

![Figure 12. Minimizer of the area but non-minimal surface.](image2)

Example A.2. A useful example to keep in mind is the following one. Let us denote by \(D\) the horizontal equatorial disc in the three-dimensional unit ball \(B^3\). Given any closed subset \(C \subset D\) with \(C \cap \partial D = \emptyset\), consider a smooth compact ambient manifold \(M_C\) obtained from the ball by removing a portion of the lower half-ball in such a way that \(\partial M_C\) intersects the interior of \(D\) exactly in \(C\). This can be done for every such choice of \(C\). Observe that \(D\) is a non-properly embedded free boundary minimal surface in \(M_C\).

As a particular instance of this example, we consider \(M_C\) where \(C = \bar{B}_{r_0}(0) \subset \mathbb{R}^2\) for any \(0 \leq r_0 < 1\) and we denote it by \(M_{r_0}\) (see Figure 11).

Given a surface \(\Sigma \subset M\), there are several possible families of variations along which we can ‘move’ \(\Sigma\) or, more in general, along which we can compute the first and the second variations of the area at \(\Sigma\). The most natural ones (summarized in Figure 13) are

\[
\begin{align*}
\mathcal{X}_e(M, \Sigma) &:= \{ X \in \mathcal{X}(M) : X(x) \in T_x \partial M \; \forall x \in \partial \Sigma \} \\
\mathcal{X}_i(M, \Sigma) &:= \{ X \in \mathcal{X}(M) : X(x) \in T_x \partial M \; \forall x \in \partial \Sigma, \; g(X(x), \hat{\eta}(x)) < 0 \; \forall x \in \partial M \} \\
\mathcal{X}_\partial(M) &:= \{ X \in \mathcal{X}(M) : X(x) \in T_x \partial M \; \forall x \in \partial M \} \\
\mathcal{X}_c(M, \Sigma) &:= \{ X \in \mathcal{X}(M) : X(x) \in T_x \partial M \; \forall x \in \partial \Sigma, \; \text{supp}(X) \cap \Sigma \subset \Sigma \setminus (\text{int}(\Sigma) \cap \partial M) \}
\end{align*}
\]

Recall that \(\hat{\eta}\) is the outward unit co-normal to \(\partial M\). Moreover observe that \(\mathcal{X}_e(M, \Sigma) \supset \mathcal{X}_i(M, \Sigma) \supset \mathcal{X}_\partial(M) \supset \mathcal{X}_c(M, \Sigma)\). Hereafter, we will write \(\mathcal{X}_*(M, \Sigma)\) to denote any of the previous subsets of \(\mathcal{X}(M)\).

**Proposition A.3.** Let \(M, \Sigma\) be as above and let \(X \in \mathcal{X}_e(M, \Sigma)\). Consider \(\tilde{M}\) to be a compact manifold in which \(M\) embeds as a regular domain and such that \(\Sigma\) is properly embedded in \(\tilde{M}\), namely \(\partial \Sigma = \Sigma \cap \partial \tilde{M}\). Moreover, let \(\tilde{X} \in \mathcal{X}(\tilde{M})\) be a smooth extension of the vector field \(X\) to all \(\tilde{M}\). Then the first variation of the area of \(\Sigma\) with respect to \(\tilde{X}\) does not depend on the choice of the extensions \(\tilde{M}\) and \(\tilde{X}\) and is given by (2.1). Furthermore, if \(\Sigma\) is a free boundary minimal surface, then we can compute the second variation of the area with respect to \(\tilde{X}\) and the result again does not depend on the choice of the extensions and is given by (2.2).

\(^8\)The picture is actually in one dimension less, but it is not difficult to extend it to a surface in a three-dimensional manifold.
**Figure 13.** Visualization of the different possible sets of variations.

*Proof.* The result follows easily from the equations (2.1) and (2.2) for the first and second variations, respectively, applied to $X$. Indeed, the only possible term in the formulas that it is not obviously independent of the extensions is $|\nabla^\bot X^\bot|^2$. However, since the extensions $\tilde{M}$ and $\tilde{X}$ are smooth, the normal derivative of $X^\bot$ is uniquely determined by its value on the inward pointing normal bundle of $\partial M$. □

**A.1. Critical points.** Given Proposition A.3, we can answer the following question: which are the surfaces $\Sigma \subset M$ that are critical points for the area functional with respect to the variations $\mathcal{X}_s(M, \Sigma)$? Namely, for which surfaces $\Sigma \subset M$ the first variation of the area (2.1) is zero for all $X \in \mathcal{X}_s(M, \Sigma)$?

Thanks to (2.1), we can say that:

- If we consider as variations $\mathcal{X}_c(M, \Sigma)$ or $\mathcal{X}_i(M, \Sigma)$, the critical points are exactly all and only the free boundary minimal surfaces.
- In the other two cases $\mathcal{X}_{\partial}(M)$ and $\mathcal{X}_c(M, \Sigma)$, free boundary minimal surfaces are critical points but the other implication is not true. Indeed, being a critical point with respect to these variations does not impose any condition on $\Sigma \cap (\partial M \setminus \partial \Sigma)$. In particular, surfaces as in Figure 12 are critical points.

**A.2. Morse index definition.** Given a free boundary minimal surface $\Sigma$, we can now try to define the Morse index of $\Sigma$ with respect to variations in $\mathcal{X}_s(M, \Sigma)$.

In the cases in which $\mathcal{X}_i(M, \Sigma)$ is a vector space, we can just mime the classical definition of Morse index. Namely, we define the Morse index $\text{ind}_c(\Sigma)$ with respect to the variations in $\mathcal{X}_s(M, \Sigma)$ as the maximal dimension of a linear subspace of $\Gamma(N\Sigma) \cap \mathcal{X}_s(M, \Sigma)$ where the second variation of the area (given by (2.2)) is negative definite. In this way we define $\text{ind}_c(\Sigma)$, $\text{ind}_{\partial}(\Sigma)$ and $\text{ind}_c(\Sigma)$.

**Remark A.4.** The definition of Morse index as $\text{ind}_c(\Sigma)$ is the one given by Guang-Wang-Zhou in [26] and used also in [24].

Observe that $\mathcal{X}_i(M, \Sigma)$ is only a convex cone, thus we cannot apply the standard definition of Morse index to this case. So, let us start giving some terminology.

**Definition A.5.** Given a surface $\Sigma \subset M$, we say that $\Theta \subset \Gamma(N\Sigma) \cap \mathcal{X}_i(M, \Sigma)$ is a linear subcone if it is the intersection of a linear subspace of $\mathcal{X}(M)$ with $\Gamma(N\Sigma) \cap \mathcal{X}_i(M, \Sigma)$. In this case we say that the linear subspace generates the linear subcone $\Theta$. The dimension of a linear subcone $\Theta$ is the minimum dimension of a linear subspace of $\mathcal{X}(M)$ that generates $\Theta$.

There is not a unique way to define the Morse index with respect to the variations $\mathcal{X}_i(M, \Sigma)$ and we do not aim to give a comprehensive discussion about all of them. We present here the one that seems most natural to us.

**Definition A.6.** Given an embedded free boundary minimal surface $\Sigma \subset M$, we define $\text{ind}_i(\Sigma)$ as the maximal dimension of a linear subcone of $\Gamma(N\Sigma) \cap \mathcal{X}_i(M, \Sigma)$ where the second variation of the area is negative definite.
Proposition A.7. Let $\Sigma = D$ be the equatorial disc in the ambient manifold $M = M_{\Gamma_0}$ defined above, for some $0 < \Gamma_0 < 1$. Then $\Sigma$ is a free boundary minimal surface in $M$ and $\text{ind}_c(\Sigma) = \infty$.

Proof. First of all, observe that $\Gamma(\Sigma) \cap \mathcal{X}_c(M, \Sigma)$ can be represented by the set of functions $\{f \in C^\infty(\Sigma) : f \geq 0$ on $\partial B_{\Gamma_0}(0)\}$ and that the second variation variation of the area along any function $f$ in this set can be written as

$$Q_\Sigma(f, f) = \int_\Sigma |\nabla f|^2 \, d\mathcal{H}^2 - \int_{\partial \Sigma} f^2 \, d\mathcal{H}^1.$$  

(A.5)

Now, given any $\Sigma \in \mathbb{N}$, let us consider functions $\rho_1, \ldots, \rho_N \in C^\infty(\Sigma)$ such that

(i) $\text{supp}(\rho_k) \subset B_{\Gamma_0}(0) \subset \Sigma$ for $k = 1, \ldots, N$;
(ii) $\text{supp}(\rho_k) \cap \text{supp}(\rho_h) = \emptyset$ for any $1 \leq k < h \leq N$;
(iii) $\int_\Sigma |\nabla \rho_h|^2 \, d\mathcal{H}^2 < 2\pi/N$ for $k = 1, \ldots, N$;
(iv) there exist $x_1, \ldots, x_N \in \Sigma$ such that $\rho_k(x_k) = 1$ for $k = 1, \ldots, N$.

Then define $\psi_k := 1 - \sum_{h \neq k} \rho_h$ for $k = 1, \ldots, N$ and denote by $\Theta$ the intersection between the linear subspace in $\Gamma(\Sigma)$ generated by $\{\psi_k\}_{k=1,\ldots,N}$ and $\mathcal{X}_c(M, \Sigma)$.

Observe that $\Theta$ is a linear subspace of dimension $N$. We want to prove that $Q_\Sigma$ is negative definite on $\Theta$, which would conclude the proof since $\text{ind}_c(\Sigma) \geq \dim \Theta = N$.

Pick any $\psi := \sum_{k=1}^N a_k \psi_k \in \Theta$, then observe that $\psi(x_k) = a_k$ and thus it must hold that $a_k \geq 0$ for $k = 1, \ldots, N$ by (i). Hence, using (ii) and (iii), together with the fact that $\text{supp}(\rho_k) \cap \partial \Sigma = \emptyset$ for $k = 1, \ldots, N$ by (i), we have that

$$Q_\Sigma(\psi, \psi) = \int_\Sigma \sum_{k=1}^N |\nabla \rho_k|^2 \left(\sum_{h \neq k} a_h\right)^2 \, d\mathcal{H}^2 - \int_{\partial \Sigma} \left(\sum_{k=1}^N a_k\right)^2 \, d\mathcal{H}^1 \leq \left(\sum_{k=1}^N a_k\right)^2 \left(\int_\Sigma \sum_{k=1}^N |\nabla \rho_k|^2 \, d\mathcal{H}^2 - 2\pi\right) < 0,$$

which concludes the proof. \hfill \Box

A.3. Morse index properties. We now present some properties of $\text{ind}_c(\Sigma)$, $\text{ind}_\partial(\Sigma)$ and $\text{ind}_e(\Sigma)$.

Proposition A.8. Given an embedded free boundary minimal surface $\Sigma \subset M$, we have that $\text{ind}_c(\Sigma)$ and $\text{ind}_\partial(\Sigma)$ are well-defined finite numbers and it holds

$$\text{ind}_c(\Sigma) \geq \text{ind}_\partial(\Sigma) = \text{ind}_e(\Sigma).$$

Proof. First observe that, by Proposition A.3, $\text{ind}_c(\Sigma)$ coincides with the Morse index of $\Sigma$ seen as a properly embedded free boundary minimal surface in any extension $\tilde{M}$ of $M$ as in the statement of the proposition. Therefore $\text{ind}_c(\Sigma)$ is a well-defined finite number. Moreover, since $\mathcal{X}_c(M, \Sigma) \subset \mathcal{X}_\partial(M) \subset \mathcal{X}_e(M, \Sigma)$, we have that $\text{ind}_c(\Sigma)$ and $\text{ind}_\partial(\Sigma)$ are well-defined as well, and $\text{ind}_c(\Sigma) \leq \text{ind}_\partial(\Sigma) \leq \text{ind}_e(\Sigma)$.

We only need to prove that $\text{ind}_c(\Sigma)$ actually coincides with $\text{ind}_\partial(\Sigma)$. We first show that $\text{ind}_c(\Sigma)$ is dense in $\text{ind}_\partial(\Sigma)$ with respect to the $H^1(\Sigma)$ norm, which is a consequence of the following lemma thanks to a standard partition of unity argument.

Lemma A.9. Let $\Omega \subset \mathbb{R}^n$ be any open domain and consider $u \in \text{Lip}(\mathbb{R}^n)$ such that $u = 0$ almost everywhere in $\Omega^c$. Then there exists a sequence of functions $u_k \in C^\infty_c(\Omega)$, for $k \in \mathbb{N}$, that converge to $u$ in $H^1(\mathbb{R}^n)$.

Proof. Let $u_+, u_- \in \text{Lip}(\mathbb{R}^n)$ be the positive and the negative parts of $u$, respectively. Moreover, let us define the function $\rho := d(\cdot, \Omega^c)$, for which we have that $\rho \in \text{Lip}(\mathbb{R}^n)$, $\rho = 0$ in $\Omega^c$ and $\rho > 0$ in $\Omega$. Unfortunately this definition is not meaningful as shown in the following proposition.
The functions \( u_1 := \rho + u_+ \), \( u_2 := \rho + u_- \in \text{Lip}(\mathbb{R}^n) \) are zero almost everywhere in \( \Omega^c \) and are strictly positive in \( \Omega \) (in particular they are greater or equal than \( \rho \)). Note that it is sufficient to prove the result for these functions.

Therefore, without loss of generality, let us assume that \( u \geq d(\cdot, \Omega^c) > 0 \) in \( \Omega \). Moreover, to simplify the notation, let us assume that the Lipschitz constant of \( u \) is 1. Then, let us consider \( u_\varepsilon := (u - \varepsilon)_+ \in \text{Lip}(\mathbb{R}^n) \). Observe that \( u_\varepsilon(x) = 0 \) for almost every \( x \in \Omega \) such that \( d(x, \Omega^c) \leq \varepsilon \), as well as for almost every \( x \in \Omega^c \). Furthermore, it holds that

\[
\|u - u_\varepsilon\|^2_{H^1(\mathbb{R}^n)} = \int_{\Omega} |u - u_\varepsilon|^2 dx + \int_{\Omega} |\nabla u - \nabla u_\varepsilon|^2 dx \leq \varepsilon^2 |\Omega| + \int_{\{u < \varepsilon\}} |\nabla u|^2 dx,
\]

which converges to 0 as \( \varepsilon \to 0 \) since \( u \in H^1(\mathbb{R}^n) \) and \( \bigcap_{\varepsilon > 0}\{u < \varepsilon\} = \{u = 0\} \). Thus, let us choose a sequence \( \varepsilon_k \to 0 \) and define \( u_k := u_{\varepsilon_k}^* \varphi_{\varepsilon_k}/2 \), where \( \varphi_{\varepsilon_k}(x) := \varepsilon^{-1}\varphi(x/\varepsilon) \) and \( \varphi \in C_c^\infty(B_1(0)) \) is a bump function with \( \varphi(0) = 1 \), \( \varphi \leq 1 \). Observe that the functions \( u_k \in C_c^\infty(\Omega) \) converge to \( u \) in \( H^1(\mathbb{R}^n) \), which concludes the proof.

Now, let \( \Delta \) be \( (\text{ind}_\partial(\Sigma)) \)-dimensional vector subspace of \( \Gamma(N\Sigma) \cap X(\Sigma) \) in \( M \) such that \( Q_{\Sigma} \) is negative definite. By density of \( X_c(M, \Sigma) \) in \( X(\Sigma) \), we can find an \( (\text{ind}_\partial(\Sigma)) \)-dimensional vector subspace \( \hat{\Delta} \subset \Gamma(N\Sigma) \cap X_c(M, \Sigma) \) such that \( \hat{\Delta} \cap \{V \in X(M) : \|V\|_{H^1} = 1\} \) is as close as we want to \( \Delta \cap \{V \in X(M) : \|V\|_{H^1} = 1\} \) in the \( H^1 \) norm. In particular, we can choose \( \Delta \) such that \( Q_{\Sigma} \) is negative definite in \( \Delta \cap \{V \in X(M) : \|V\|_{H^1} = 1\} \), and so in \( \hat{\Delta} \). Therefore, we have that \( \text{ind}_\partial(\Sigma) = \dim \Delta = \dim \hat{\Delta} \leq \text{ind}_c(\Sigma) \), which concludes the proof.

We conclude computing the Morse indices \( \text{ind}_c(\Sigma) \), \( \text{ind}_\partial(\Sigma) \) and \( \text{ind}_c(\Sigma) \) in a relevant case.

**Proposition A.10.** Let \( \Sigma = D \) be the equatorial disc in the ambient manifold \( M = M_{r_0} \) with \( 0 \leq r_0 < 1 \), defined above. Then \( \Sigma \) is a free boundary minimal surface in \( M_{r_0} \) with \( \text{ind}_c(\Sigma) = 1 \) for all \( 0 \leq r_0 < 1 \) and

\[
\text{ind}_\partial(\Sigma) = \text{ind}_c(\Sigma) = \begin{cases} 1 & \text{if } r_0 < e^{-1}, \\ 0 & \text{if } r_0 \geq e^{-1}. \end{cases}
\]

**Remark A.11.** This means in particular that the free boundary minimal surfaces in Figure 11 are respectively unstable (the one on the left) and stable (the one on the right) with respect to the variations \( \mathcal{X}_\partial(\Sigma) \) and \( \mathcal{X}_c(M, \Sigma) \).

**Proof.** First observe that \( \text{ind}_c(\Sigma) = 1 \), since it coincides with the classical index of the equatorial disc in \( B^3 \). Therefore, we only need to compute \( \text{ind}_\partial(\Sigma) = \text{ind}_c(\Sigma) \), which are equal by Proposition A.8.

By definition, \( \text{ind}_\partial(\Sigma) \) is given by the maximal dimension of a linear subspace of \( \{f \in C^\infty(\Sigma) : f = 0 \text{ on } \partial B_{r_0}(0)\} \) in which \( Q_{\Sigma} \) (given by the formula (A.5)) is negative definite.

Since \( \text{ind}_\partial(\Sigma) \leq \text{ind}_c(\Sigma) = 1 \), it is sufficient to see if \( \text{ind}_\partial(\Sigma) \) is 0 or 1. Namely, if there exists \( f \in C^\infty(\Sigma) \) such that \( f = 0 \text{ on } \partial B_{r_0}(0) \) and \( Q_{\Sigma}(f, f) < 0 \). For any function such function \( f \in C^\infty(\Sigma) \) it holds that

\[
Q_{\Sigma}(f, f) = \int_0^{2\pi} \int_{r_0}^1 (|\partial_r f(r, \theta)|^2 + r^{-2}|\partial_\theta f(r, \theta)|^2) r dr d\theta - \int_0^{2\pi} f(1, \theta)^2 d\theta \geq \int_0^{2\pi} \int_{r_0}^1 |\partial_r f(r, \theta)|^2 r dr d\theta - \int_0^{2\pi} f(1, \theta)^2 d\theta = \int_0^{2\pi} \left( \int_{r_0}^1 |\partial_r f(r, \theta)|^2 r dr - f(1, \theta)^2 \right) d\theta.
\]

However, observe that

\[
|f(1, \theta)| = |f(1, \theta) - f(r_0, \theta)| = \left| \int_{r_0}^1 \partial_r f(r, \theta) dr \right| \leq \left( \int_{r_0}^1 \frac{1}{r} dr \right)^{1/2} \left( \int_{r_0}^1 |\partial_r f(r, \theta)|^2 r dr \right)^{1/2},
\]
thus
\[ f(1, \theta)^2 \leq \ln(r_0^{-1}) \int_{r_0}^{1} |\partial_r f(r, \theta)|^2 r \, dr , \]

which implies that
\[ Q_{\Sigma}(f, f) \geq \left( \frac{1}{\ln(r_0^{-1})} - 1 \right) \int_0^{2\pi} f(1, \theta)^2 \, d\theta \]

with equality if and only if \( f(r, \theta) = c(\ln(r) - \ln(r_0)) \) for some constant \( c \in \mathbb{R} \) (in particular we can choose \( c \) in such a way that \( \|f\|_{L^2(\Sigma)} = 1 \)). Therefore, the index is 1 if and only if \( \ln(r_0^{-1}) < 1 \), that means \( r_0 < e^{-1} \) as we wanted. \( \square \)

Appendix B. Reflecting free boundary minimal surfaces in a half-space

We start recalling a standard reflection lemma, which is useful to transfer information about minimal surfaces in \( \mathbb{R}^3 \) to free boundary minimal surfaces in a half-space. The proof consists in a well-known argument based on elliptic estimates.

Lemma B.1. If \( \Sigma^2 \) is an embedded free boundary minimal surface in \( \Xi \subset \mathbb{R}^3 \) (that is \( \Sigma \) has zero mean curvature and meets \( \Pi \) orthogonally along \( \partial \Sigma \)), then the union of \( \Sigma \) with its reflection with respect to \( \Pi \) is an embedded minimal surface in \( \mathbb{R}^3 \) without boundary.

Here is instead a regularity result for free boundary minimal immersions in a half-space of \( \mathbb{R}^3 \).

Proposition B.2. Let \( \Sigma \subset \Xi \subset \mathbb{R}^3 \) be a complete free boundary minimal injective immersion with finite index and with \( \partial \Sigma = \Pi \cap \Sigma \). Then \( \Sigma \) is two-sided, has finite total curvature and is properly embedded.

Proof. This is a consequence of Theorem B.1 in [9], after applying the reflection principle Lemma B.1 and the argument of [2, Section 2.2], which imply that the reflected surface has finite index. \( \square \)

We further apply [2, Section 2.3] together with [11] to obtain topological information from index bounds for free boundary minimal surfaces in a half-space of \( \mathbb{R}^3 \).

Definition B.3. Given a complete, connected, properly embedded, free boundary minimal surface \( \Sigma^2 \) in \( \Xi \subset \mathbb{R}^3 \), the number of ends of \( \Sigma \) is the number of connected components of \( \Sigma \) outside any sufficiently large compact set. We will denote this number by \( \text{ends}(\Sigma) \).

Observe that the number of ends of a properly embedded free boundary minimal surface in \( \Xi \subset \mathbb{R}^3 \) as above is indeed well-defined (see [2, Remark 26]).

Proposition B.4. Given \( I \geq 0 \), there exists \( \kappa(I) \geq 0 \) such that for every complete, connected, properly embedded, free boundary minimal surface \( \Sigma^2 \) in \( \Xi \subset \mathbb{R}^3 \) of index at most \( I \) one has
\[ 2 \, \text{genus}(\Sigma) + 2 \, \text{ends}(\Sigma) + \text{boundaries}(\Sigma) \leq \kappa(I) , \]
where \( \text{ends}(\Sigma) \) is the number of ends of \( \Sigma \).

Proof. Let \( \overline{\Sigma} \) be the union of \( \Sigma \) with its reflection with respect to \( \Pi \), as above. Then, thanks to [2, Section 2.3] (in particular Equation (2.5) therein), \( \overline{\Sigma} \) is a complete, connected, properly embedded, minimal surface of \( \mathbb{R}^3 \) with index less or equal than \( 2I \). Therefore, using the main estimate in [11], we obtain that
\[ \frac{2}{3} (\text{genus}(\Sigma) + \text{ends}(\Sigma)) - 1 \leq 2I . \]

Now remember that, given two oriented surfaces \( \Sigma_1, \Sigma_2 \) with boundary, the oriented surface obtained by gluing \( b \) boundary components of these two surfaces has genus equal to \( \text{genus}(\Sigma_1) + \text{genus}(\Sigma_2) + b - 1 \) (see Section 2.4). Thus, in our case we have that
\[ 2 \, \text{genus}(\Sigma) + \text{boundaries}(\Sigma) - 1 \leq \text{genus}(\Sigma) . \]
Moreover, by Lemma 29 in [2], it holds that \( \text{ends}(\Sigma) \leq \text{ends}(\tilde{\Sigma}) \). Therefore we can choose \( \kappa(I) \) such that the result holds.

\[ \square \]

**Remark B.5.** An explicit choice of \( \kappa(I) \) is for example \( 3(I + 1) \).

**APPENDIX C. SOME BASIC MORSE-THEORETIC ARGUMENTS**

In this section we collect a few lemmas that will be useful to obtain topological information at intermediate scales. The basic idea is that if a surface fulfills suitable curvature estimates then it is 'locally simple'.

**Lemma C.1.** Let \( g \) be a metric on \( \{|x| \leq 2\} \cap \Xi(a) \subset \mathbb{R}^3 \) sufficiently close to the Euclidean metric, for some \( 0 \geq a \geq -\infty \), and let \( \mathcal{M}^3 := B_1(0) \cap \Xi(a) \subset \mathbb{R}^3 \) be the unit ball with respect to this metric\(^9\).

Fix \( p \in \mathcal{M} \) with \( d_g(0, p) \leq \frac{1}{2} \) and \( r > 0 \) such that \( B_r(p) \subset B_1(0) \). Then, for every smooth properly embedded connected surface \( \Sigma^2 \subset \mathcal{M} \setminus B_r(p) \) such that

(i) \( \Sigma \) is free boundary\(^10\) with respect to \( \Pi(a) \);
(ii) \( \partial \Sigma = \Sigma \cap (\partial (B_1(0) \setminus B_r(p)) \cup \Pi(a)) \);
(iii) \( \Sigma \) intersects \( \partial B_r(p) \) transversely and
(iv) for every \( x \in \Sigma \) it holds

\[ |A_\Sigma|(x)d_g(p, x) \leq \frac{1}{4}, \]

we have that the function \( f : \Sigma \to \mathbb{R} \) given by \( f(x) := d_{\mathbb{R}^n}(p, x)^2 \) has no critical points on \( \Sigma \) and its gradient is never tangent to \( \partial (B_1(0) \setminus B_r(p)) \setminus \Pi(a) \).

**Remark C.2.** Notice that we are not even requiring the minimality of \( \Sigma \).

**Proof.** As long as \( g \) is sufficiently close to the Euclidean metric, we can assume on the one hand that

\[ |A^{\mathbb{R}^n}_\Sigma|(x)d_{\mathbb{R}^n}(p, x) \leq \frac{1}{2}, \]

and on the other hand that \( f \) has gradient pointing (strictly) out of \( \partial B_r(p) \setminus \Pi(a) \) (the latter by virtue of the assumption (iii) above). We further note that

\[ (D^2 f)_x(v, v) = 2|v|^2 - (A^{\mathbb{R}^n}_\Sigma(x)(v, v), x - p), \]

hence

\[ (D^2 f)_x(v, v) \geq 2|v|^2 - |v|^2|A^{\mathbb{R}^n}_\Sigma|(x)d_{\mathbb{R}^n}(p, x)) \geq 2|v|^2(1 - |A^{\mathbb{R}^n}_\Sigma|(x)d_{\mathbb{R}^n}(p, x)) \geq |v|^2, \]

which means that \( f \) has strictly positive Hessian everywhere on \( \Sigma \). This implies, thanks to the free boundary condition, that \( f \) has no critical points, following the modified mountain pass lemma presented in [7, pp. 1003-1005].

Thereby, since \( \Sigma \) is assumed to be connected, \( f \) has gradient pointing (strictly) outwards at \( \partial B^{\mathbb{R}^n}_1(0) \setminus \Pi(a) \). Lastly, observe that, if the metric \( g \) is sufficiently close to the Euclidean one, then \( B_1(0) \) and \( B_r(p) \) are sufficiently smoothly close to the corresponding balls in \( \mathbb{R}^n \), namely \( B^{\mathbb{R}^n}_1(0) \) and \( B^{\mathbb{R}^n}_r(p) \): therefore, possibly taking \( g \) even closer to the Euclidean metric, the gradient of \( f \) is not tangent to \( \partial (B_1(0) \setminus B_r(p)) \setminus \Pi(a) \), since it is not tangent to \( \partial (B^{\mathbb{R}^n}_1(0) \setminus B^{\mathbb{R}^n}_r(p)) \setminus \Pi(a) \). \( \square \)

**Corollary C.3.** Let \((\mathcal{M}^3, g)\) be a compact Riemannian manifold with boundary. Then there exists \( \delta_0 > 0 \) with the following property. Fix \( 0 \leq \delta_2 < \delta_1 \leq \delta_0 \) and \( p \in \partial \mathcal{M} \) and consider an embedded connected minimal surface \( \Sigma^2 \subset B_{\delta_1}(p) \setminus B_{\delta_2}(p) \) such that

(i) \( \Sigma \) is free boundary\(^11\) with respect to \( \partial \mathcal{M} \);

\(^9\)Notice that here \( B_1(0) \) is the metric ball with respect to the metric \( g \), as well as the ball \( B_r(p) \) below.

\(^{10}\)Note that we allow the intersection \( \Sigma \cap \partial \mathcal{M} \) to be empty, thus this condition (i) to be vacuously true.

\(^{11}\)Note that we allow the intersection \( \Sigma \cap \partial \mathcal{M} \) to be empty, thus the condition (i) to be vacuously true.
Then \( \Sigma \) is properly embedded in \( B_{\delta_1}(p) \setminus B_{\delta_2}(p) \) and is either a topological disc or a topological annulus.

Proof. Let us assume to work in Fermi coordinates (see \cite[Section 3]{2}) around \( p \), in particular take \( \varphi : \{ |x| \leq 2 \} \subset \Xi(0) \to M \) a Fermi chart such that \( \varphi(0) = p \), \( B_{\delta_0}(p) \) is contained in the image of \( \varphi \) and \( \varphi^* g \) is as close to the Euclidean metric as required by Lemma C.1 (this can be achieved as soon as \( \delta_0 \) is sufficiently small). Then, thanks to Lemma C.1 (applied for \( p = 0 \), thus with concentric balls), the function \( f(x) := d_{\mathbb{R}^n}(0, x)^2 \) has no critical points on \( \varphi^{-1}(\Sigma) \) and has gradient never tangent to \( \varphi^{-1}(\partial B_{\delta_2}(p) \cup \partial B_{\delta_1}(p)) \). Furthermore observe that the gradient of \( f \) is tangent to \( \Pi(0) \).

If \( \varphi^{-1}(\Sigma) \) does not intersect \( \Pi(0) \), then we can directly apply the standard Morse-theoretic argument given in \cite[Theorem 3.1]{39} to obtain that \( \varphi^{-1}(\Sigma) \) is diffeomorphic to \( S^1 \times [0, 1] \), where \( S^1 \times \{ 0 \} \) corresponds to \( \varphi^{-1}(\Sigma \cap \partial B_{\delta_2}(0)) \) and \( S^1 \times \{ 1 \} \) to \( \varphi^{-1}(\Sigma \cap \partial B_{\delta_1}(0)) \). In this case \( \Sigma \) is a topological annulus. Otherwise, let us consider the \( C^1 \) surface \( \Sigma \) which is the union of \( \varphi^{-1}(\Sigma) \) with its reflection with respect to \( \Pi(0) \). The function \( f(x) = d_{\mathbb{R}^n}(0, x)^2 \) has no critical points on \( \Sigma \) as well. Therefore we can apply Theorem 3.1 in \cite{39} to \( \Sigma \) to obtain this time the same conclusion as before, but for \( \Sigma \). Hence, we obtain patently that \( \Sigma \) is a topological disc.

Corollary C.4. Let \((M^3, g)\) be a compact Riemannian manifold with boundary. Then there exists \( \delta_0 > 0 \) depending on \((M, g)\) with the following property. Fix \( 0 \leq \delta_1 \leq \delta_0 \) and \( q \in M \) and let \( \Sigma^2 \subset B_{\delta_1}(q) \) be a properly embedded minimal surface having free boundary with respect to \( \partial M \), with \( \partial \Sigma = \Sigma \cap (\partial B_{\delta_1}(q) \cup \partial M) \). Assume that there exists \( p \in B_{\delta_0/2}(q) \) and \( \delta_2 > 0 \) such that \( B_{\delta_2}(p) \subset B_{\delta_1}(q) \) and

\[
|A_{\Sigma}|(x) d_g(p, x) \leq \frac{1}{4}
\]

for all \( x \in \Sigma \setminus B_{\delta_2}(p) \). Furthermore suppose that each component of \( \Sigma \) intersects transversely \( \partial B_{\delta_2}(p) \).

If \( \Sigma \cap B_{\delta_2}(p) \) has genus and number of boundary components in \( \partial B_{\delta_2}(p) \cup \partial M \) bounded by \( \kappa \), then the same bounds hold for \( \Sigma \), namely the genus and the number of boundary components in \( \partial B_{\delta_1}(q) \cup \partial M \) are also bounded by \( \kappa \).

Proof. Let \( \varphi : \{ |x| \leq 2 \} \cap \Xi(a) \subset \mathbb{R}^3 \to M \) for some \( 0 \geq a \geq -\infty \) be a Fermi chart\(^{12}\) such that \( B_{\delta_0}(q) \) is contained in the image of \( \varphi \) and \( \varphi^* g \) is as close to the Euclidean metric as required by Lemma C.1. This can be achieved as soon as \( \delta_0 \) is sufficiently small. Note that the origin does not necessarily belong to the boundary \( \Pi(a) \) (in particular the Fermi coordinates are not centered in \( q \), or \( p \)).

If \( p \) belongs to \( \partial M \), then a reflection argument totally similar to the one in the proof of Corollary C.3 concludes the proof; thus assume that \( p \notin \partial M \). Thanks to Lemma C.1, the function \( f(x) := d_{\mathbb{R}^n}(\varphi^{-1}(p), x)^2 \) has no critical points in \( \varphi^{-1}(\Sigma \setminus B_{\delta_2}(p)) \) and its gradient is never tangent to \( \varphi^{-1}(\partial B_{\delta_2}(p) \cup \partial B_{\delta_1}(q)) \setminus \Pi(a) \). Furthermore notice that the gradient of \( f \) is never tangent also to \( \Pi(a) \) and in particular is outward pointing. Therefore, thanks to a flow approach as in \cite[Theorem 3.1]{39}, each component of \( \varphi^{-1}(\Sigma \setminus B_{\delta_2}(p)) \) is diffeomorphic to \( S^1 \times [0, 1] \) or \( [0, 1] \times [0, 1] \), with \( S^1 \times \{ 0 \} \) or \( \{ 0 \} \times [0, 1] \) (in the case of \( \varphi^{-1}(\Sigma \cap \partial B_{\delta_2}(p)) \) \setminus \Pi(a) and \( S^1 \times \{ 0 \} \) or \( \{ 0 \} \times [0, 1] \) corresponding to \( \varphi^{-1}(\Sigma \cap \partial B_{\delta_1}(q)) \) \setminus \Pi(a) \).

Finally observe that, attaching to a surface a component diffeomorphic to \( S^2 \times [0, 1] \) along a boundary component (as in (i) of Section 2.4) leaves the genus and the number of boundary components unchanged by means of the formulas at the end of Section 2.4. The same holds attaching a component diffeomorphic to \( [0, 1] \times [0, 1] \) along a segment as in (ii) of Section 2.4, which concludes the proof.\(^{\Box}\)

\(^{12}\)Actually, the expression ‘Fermi chart’ is necessary only if the boundary \( \Pi(a) \) intersects the domain of the chart; otherwise we are just considering a ‘standard chart’.
Appendix D. Multiplicity one convergence

In this section we prove that convergence to a surface with multiplicity one well-behaves in presence of isolated singularities, which essentially follows from Allard’s regularity theory (see [1]).

Lemma D.1. Let \( \Sigma^2_j \subset M^3 \) be a sequence of connected free boundary minimal surfaces in a three-dimensional complete Riemannian manifold \( M \). Moreover let \( \Sigma^2 \subset M \) be a properly embedded free boundary minimal surface in \( M \). Assume that the sequence \( \Sigma_j \) converges locally smoothly to \( \Sigma \), with multiplicity one away from a finite set of points\(^\text{13}\) \( S \). Then \( \Sigma_j \) converges locally smoothly to \( \Sigma \) everywhere.

Proof. Let \( p \) be a point in \( S \) and let \( r_0 > 0 \) be sufficiently small such that \( B_{4r_0}(p) \) does not contain any other point of \( S \) apart from \( p \). Observe that we can assume that \( p \in \Sigma \). Indeed, if \( p \notin \Sigma \), then (exploiting the properness assumption on \( \Sigma \)) there exists a neighborhood \( U \) of \( p \) such that eventually \( \Sigma_j \cap U = \emptyset \) by connectedness of \( \Sigma_j \).

Fix \( \varepsilon > 0 \) and take \( r_0 \) possibly smaller in such a way that
\[
\frac{\mathcal{H}^2(\Sigma \cap (B_{2r_0}(p) \setminus B_{r_0}(p)))}{\omega_2(4r_0^2 - r_0^2)} < \begin{cases} 
1 + \varepsilon/4 & \text{if } p \in M \setminus \partial M \\
\frac{1}{2}(1 + \varepsilon/4) & \text{if } p \in \partial M . 
\end{cases}
\]
Since the convergence of \( \Sigma_j \) to \( \Sigma \) is smooth and graphical in \( B_{2r_0}(p) \setminus B_{r_0}(p) \), then we can assume that the same estimate holds for every \( j \) sufficiently large substituting \( \varepsilon/4 \) with \( \varepsilon/2 \). By the extended monotonicity formula (cf. [25]) this implies that
\[
\frac{\mathcal{H}^2(\Sigma_j \cap B_r(p))}{\omega_2r_0^2} < \begin{cases} 
1 + \varepsilon/2 & \text{if } p \in M \setminus \partial M \\
\frac{1}{2}(1 + \varepsilon/2) & \text{if } p \in \partial M . 
\end{cases}
\]
Thus, again by the same monotonicity formula, taking \( r_0 > 0 \) possibly smaller, we have
\[
\frac{\mathcal{H}^2(\Sigma_j \cap B_r(p))}{\omega_2r^2} < \begin{cases} 
1 + \varepsilon & \text{if } p \in M \setminus \partial M \\
\frac{1}{2}(1 + \varepsilon) & \text{if } p \in \partial M , 
\end{cases}
\]
for all \( r < r_0 \) and \( j \) sufficiently large. This concludes the proof by virtue of Theorem 17 in [3]. \( \Box \)

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\(^\text{13}\)Here we mean that for every \( x \in M \setminus S \) there exists a neighborhood \( U \) of \( x \) such that \( \Sigma_j \cap U \) converges graphically smoothly to \( \Sigma \cap U \) with multiplicity one. Observe that \( \Sigma_j \cap U \) and \( \Sigma \cap U \) can be possibly empty.
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Alessandro Carlotto:
ETH D-Math, Rämistrasse 101, 8092 Zürich, Switzerland
E-mail address: alessandro.carlotto@math.ethz.ch, alessandro.carlotto@ias.edu

Giada Franz:
ETH D-Math, Rämistrasse 101, 8092 Zürich, Switzerland
E-mail address: giada.franz@math.ethz.ch