JORDAN MATING IS ALWAYS POSSIBLE FOR
POLYNOMIALS

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Abstract. Suppose \( f \) and \( g \) are two post-critically finite polynomials
of degree \( d_1 \) and \( d_2 \) respectively and suppose both of them have a finite
super-attracting fixed point of degree \( d_0 \). We prove that one can always
construct a rational map \( R \) of degree
\[
D = d_1 + d_2 - d_0
\]
by gluing \( f \) and \( g \) along the Jordan curve boundaries of the immediate
super-attracting basins. The result can be used to construct many rational
maps with interesting dynamics.

1. Introduction

Polynomial mating was an operation proposed by Douady and Hubbard
to understand the dynamics of rational maps. Very roughly speaking, for
two post-critically finite polynomials \( P \) and \( Q \) of degree \( d \geq 2 \) with both the
Julia sets being connected, we may glue \( f \) and \( g \) along the Julia sets to get a
topological map \( F \). We say \( f \) and \( g \) are matable if \( F \) is a branched covering
map of the two sphere to itself, and moreover, \( F \) is topologically conjugate to
some rational map. Noting that the Julia set is the boundary of the immediate
super-attracting basin of the infinity, the idea can be naturally extended
to the situation of rational maps. Suppose \( f \) and \( g \) are two post-critically
finite rational maps both of which have a simply connected immediate super-
attracting basin of degree \( d_0 \geq 2 \) such that there are no other critical orbits
which intersect the immediate basins. Then one may construct a topological
map by gluing \( f \) and \( g \) along the attracting basin boundaries and then copy
this gluing for all the pre-images of the attracting basins. As in the case of
polynomial mating, we say \( f \) and \( g \) are matable if \( F \) is a branched covering of
the two sphere to itself, and moreover, \( F \) is topologically conjugate to some
rational map \( G \).

A particularly important case is that both the super-attracting basins are
Jordan domains (Noting that all bounded immediate attracting basins of
polynomials are Jordan domain \([1]\)). In this case, no pinching happens when
gluing \( f \) and \( g \) along the Jordan boundary and the topological map is al-
ways a branched covering of the two sphere to itself. Let us describe this

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topological construction as follows. Let $D_f$ and $D_g$ denote the two Jordan super-attracting basins and $D^c_f, D^c_g$ be there complements respectively. Let $\phi : D_f \to \Delta$ and $\psi : D_g \to \Delta$ be the holomorphic isomorphism which conjugate $f$ and $g$ to $z \mapsto z^{d_0}$. Then for each $1 \leq k \leq d_0 - 1$,

\begin{equation}
\Phi = \phi^{-1} \left( \frac{e^{2k\pi i/(d-1)}}{\psi} \right) : \partial D_g \to \partial D_f
\end{equation}

is a homeomorphism which reverses the orientation. We can extend it to a homeomorphism of the sphere so that it maps $\overline{D_g}$ to $\overline{D_f}$ and $\overline{D^c_g}$ to $\overline{D^c_f}$. Now we glue $D^c_g$ and $D^c_f$ by identifying the points $x$ and $\Phi(x)$. It is clear that $D^c_g \bigcup_{x \sim \Phi(x)} D^c_f$ is a topological two sphere. Define

\begin{equation}
F : D^c_g \bigcup_{x \sim \Phi(x)} D^c_f \to D^c_g \bigcup_{x \sim \Phi(x)} D^c_f
\end{equation}

by setting

\begin{equation}
F(z) = \begin{cases} 
  f(z) & \text{for } z \in D^c_f \text{ and } f(z) \in D^c_f, \\
  \Phi^{-1} \circ f(z) & \text{for } z \in D^c_f \text{ and } f(z) \in D_f, \\
  g(z) & \text{for } z \in D^c_g \text{ and } f(z) \in D^g, \\
  \Phi \circ g(z) & \text{for } z \in D^c_g \text{ and } f(z) \in D_g.
\end{cases}
\end{equation}

Since by assumption no other critical orbits of $f$ and $g$ enter into $D_f$ and $D_g$ respectively, the way of extending $\Phi : \partial D_g \to \partial D_f$ dose not affect the combinatorially equivalent class of $F$. In the case that $F$ has no Thurston obstruction, we have a rational map $G$ which is combinatorially equivalent to $F$ (One can actually prove that $G$ is topologically conjugate to $F$). Unlike the usual mating, whose Julia sets is the disjoint union of $J_f$ and $J_g$ with those points in a ray equivalent class being identified, the Julia set of $G$ contains infinitely many copies of $J_f$ and $J_g$. To get $J_G$, one may start from $J_g \bigcup_{x \sim \Phi(x)} J_f$, and then iteratively fill the pre-images of $D_f$ and $D_g$ by copies of $J_g$ and $J_f$ respectively. To make a distinction with the usual mating, we call such mating a Jordan mating.

In contrast to the usual mating, for which there exist cubic polynomials which are topologically matable but not matable, Jordan mating is always possible for two polynomials. We will actually prove a stronger result.

**Definition.** For $d_0 \geq 2$, let $\mathcal{R}_{d_0}$ denote the family of all post-critically finite rational maps which have a marked immediate super-attracting basin $D$ which is a Jordan domain and of degree $d_0$ such that all the other critical orbits do not intersect $D$.

**Main Theorem.** Let $d_0 \geq 2$. Suppose $f, g \in \mathcal{R}_{d_0}$ such that at least one of them is a polynomial. Then $f$ and $g$ can be mated into a rational map $R$ of degree

$$D = d_1 + d_2 - d_0$$
with \( d_1 \) and \( d_2 \) being the degrees of \( f \) and \( g \) respectively. In particular, Jordan mating is always possible for polynomials.

Since the topological map \( F \) in our case is always a branched covering of the sphere to itself, all we need to do is to show that \( F \) has no Thurston obstructions. Up to now there is no general way to check if a given topological map has Thurston obstructions or not, although many tools and ideas have been developed [1] [3] [6] [7] [8]. The idea of our proof is to associate each non-peripheral curve a quantity which is monotonically increasing as we iterate the topological map. This property will lead us to get a Levy cycle from an irreducible Thurston obstruction. We then show that such a Levy cycle can be deformed into a Levy cycle of the rational map \( g \), which is a contradiction. Our argument relies essentially on the assumption that one of the two rational maps is a polynomial.

**Question 1.** Is the Jordan mating always possible for rational maps in \( \mathbb{R}_{d_0} \)?

2. **Examples**

In this section we give two examples of Jordan mating. Let \( f \) be a cubic polynomial which has a degree two super-attracting fixed point at the origin so that the other finite critical point \( c \) belongs to the boundary of the super-attracting basin, and moreover, \( f^2(c) = f(c) \). Let \( g \) be a cubic polynomial which has a degree two super-attracting fixed point at the origin so that the other finite critical point \( c \) belongs to the boundary of the super-attracting basin, and moreover, \( g^3(c) = g^2(c) \neq g(c) \). Let \( h \) be a post-critically finite cubic rational map so that it has a degree two super-attracting fixed point at \( \infty \) and the Julia set is a Sierpinski carpet.

![Figure 1: The Julia set for \( f \)](image-url)
Figure 2: The Julia set for $g$

Figure 3: The Julia set for $h$

Figure 4: Jordan mating of $f$ and $g$
3. Proof of the main theorem

The reader may refer to [2,5] for the details of the Thurston’s theory for characterization of post-critically finite rational maps. Throughout the paper we use \( \hat{\mathbb{C}}, \mathbb{C}, \mathbb{C}^*, \mathbb{T} \) and \( \mathbb{D} \) to denote the Riemann sphere, the complex plane, the puncture complex plane, the unit circle and the unit disk respectively.

Assume that \( f, g \in \mathcal{R}_{d_0} \) with \( f \) being a polynomial. Then there are \( d_0 - 1 \) ways to glue \( f \) and \( g \) along the boundary of the marked attracting basin, see (1.1). Let \( F \) be one of such topological maps. All we need to do is to show that \( F \) has no Thurston obstructions. We may identify the boundaries of the two marked immediate attracting basins with \( \mathbb{T} \). We may assume that \( F : \mathbb{T} \rightarrow \mathbb{T} \) is given by \( z \mapsto z^{d_0} \), and up to combinatorial equivalence, \( F = f \) outside \( \mathbb{T} \) and \( F = g \) inside \( \mathbb{T} \). When a post-critical point \( x \in P_F \) belongs to the forward orbit of some critical point of \( f \), we write \( x \in P_f \), and similarly, if it belongs to the forward orbit of some critical point of \( g \), we write \( x \in P_g \).

In particular, we have

\[
P_F = P_f \cup P_g.
\]

Suppose \( \gamma \) is a non-peripheral curve of \( F \). In the following we only concern those \( \gamma \) so that \( \gamma \cap \mathbb{T} \neq \emptyset \). By homotopy rel \( P_F \) we may assume that \( \gamma \cap \mathbb{T} \) is a finite set. Then \( \gamma \cap \mathbb{T} \) consists of finitely many curve segments. Let \( \sigma \) be any of such curve segments. We call \( \sigma \) of type \( P \) (polynomial type) if it is outside \( \mathbb{T} \), otherwise, we call it of type \( R \) (rational type). Let \( I \subset \mathbb{T} \) be the arc so that \( \partial I = \partial \sigma \) and \( I \) is homotopic to \( \sigma \) rel \( \partial \sigma \) in \( \mathbb{C}^* \). Let \( D(\sigma) \) be the union of \( I \) and the bounded domain bounded by \( \sigma \) and \( I \).

**Lemma 3.1.** Suppose \( \gamma \) is non-peripheral curve in \( \hat{\mathbb{C}} - P_F \) and \( \eta \) is a non-peripheral component of \( F^{-1}(\gamma) \). Suppose \( \sigma \) is a type \( P \) curve segment of \( \eta \) and \( x \in D(\sigma) \cap P_F \). Then there is some type \( P \) curve segment \( \tau \) of \( \gamma \) such that \( F(x) \in D(\tau) \).
Figure 6: curve segments of polynomial type

Proof. By assumption the orbit of the critical points of $F$ which belongs to $\hat{\mathbb{C}} \setminus \mathbb{D}$ does not enter $\mathbb{D}$. So $F(x) \notin \mathbb{D}$. Since the action of $F$ on the outside of $T$ is given by the polynomial $f$, the image of $D(\sigma)$ is bounded whose boundaries is a subset of the union of $T$ and finitely many curve segments of type $P$ and $R$ of $\gamma$. Since $F(x) \notin \mathbb{D}$, it follows that there is some type $P$ curve segment $\tau$ of $\gamma$ so that $F(x) \in D(\tau)$.

For $x \in P_f$ and $\gamma$ a non-peripheral curve in $\hat{\mathbb{C}} - P_F$, let $\Sigma_x(\gamma)$ denote the set of all the type $P$ curve segments $\sigma$ of $\gamma$ so that $x \in D(\sigma)$. Let $N(\Sigma_x(\gamma))$ denote the number of the elements in $\Sigma_x(\gamma)$. Let

$$N(\Sigma_x(\gamma)) = \min_{\gamma'} N(\Sigma_x(\gamma'))$$

where min is taken over all non-peripheral curves $\gamma'$ which are homotopic to $\gamma$ in $\hat{\mathbb{C}} - P_F$. We need more notations.

- Let $O$ denote the set of periodic points in $P_f$.
- Let $\Sigma$ denote the class of non-peripheral curves $\gamma$ so that there is $x \in O$ with $N(\Sigma_x(\gamma)) > 0$ and let $\Pi$ denote the class of other non-peripheral curves.
- Let $\Lambda$ denote the class of non-peripheral curves $\gamma$ so that $\Sigma_x(\gamma) = 0$ holds for any $x \in P_f$.

Lemma 3.2. There is some $n$ large such that for any $\gamma \in \Pi$, if $\eta$ is a non-peripheral component of $F^{-n}(\gamma)$, then $\eta \in \Lambda$.

Proof. Since every critical point of $f$ is eventually periodic, we have an integer $n \geq 1$ such that $f^n(x) \in O$ for all $x \in P_f$. Take an arbitrary $\gamma \in \Pi$ and let $\eta$ be a non-peripheral component of $F^{-n}(\gamma)$. If $\eta \notin \Lambda$, then there would be a type $P$ curve segment of $\eta$, say $\sigma$, such that $D(\sigma) \cap P_f \neq \emptyset$. By applying Lemma 3.1 $n$ times, we would have some type $P$ curve segment of $\gamma$, say $\tau$, such that $D(\tau) \cap O \neq \emptyset$. This implies that $\gamma \in \Sigma$ which contradicts the assumption that $\gamma \in \Pi$. 
Now suppose $F$ has an obstruction. Then by [5] $F$ has a canonical Thurston, say $\Gamma$, which consists of all homotopy classes of the non-peripheral curves whose length go to zero as we iterate the Thurston pull back induced by $F$. We claim that $\Gamma \cap \Lambda = \emptyset$. Let us prove the claim. Suppose $\Gamma \cap \Lambda \neq \emptyset$. Then $\Gamma \cap \Lambda$ must be $F$-stable by Lemma 3.1. Since the length of every curve in $\Gamma \cap \Lambda$ goes to zero as we iterate the Thurston pull back, the transformation matrix associated to $\Gamma \cap \Lambda$ must have an eigenvalue $\geq 1$. By the definition of $\Lambda$, one can deform the curves in $\Gamma \cap \Lambda$ so that it is a stable family of $g$ and with the same transformation matrix. This is a contradiction because $g$ has no obstruction. This, together with Lemma 3.2, implies that $\Gamma \cap \Pi = \emptyset$. We thus have

**Lemma 3.3.** If there is an obstruction for $F$, then there must be one which consists of curves in $\Sigma$ whose length go to zero as we iterate the Thurston pull back induced by $F$.

**Lemma 3.4.** Let $\gamma$ be a non-peripheral curve. Let $x \in P_f$. Then
\[ \sum_{\eta} N(\Sigma_x(\eta)) \leq N(\Sigma_{f(x)}(\gamma)) \]
where the sum is taken over all the non-peripheral components of $F^{-1}(\gamma)$. In particular,
\[ \sum_{\eta} N(\Sigma_x([\eta])) \leq N(\Sigma_{f(x)}([\gamma])). \]

**Proof.** Let
\[ \Sigma_x = \bigcup_{\eta} \Sigma_x(\eta) \]
where the union is taken over all the non-peripheral components of $F^{-1}(\gamma)$. We may introduce an order in $\Sigma_x$: $\sigma < \sigma'$ if and only if $D(\sigma) \subset D(\sigma')$. Similarly we introduce an order in $\Sigma_{f(x)}(\gamma)$ by setting $\tau < \tau'$ if and only if $D(\tau) \subset D(\tau')$.

Now for each $\sigma \in \Sigma_x$, as in the proof of Lemma 3.1 $F(\sigma) - T$ has at least one component in $\Sigma_{f(x)}(\gamma)$. Let $M(\sigma)$ denote the maximal one among these elements. It is sufficient to show that
\[ \sigma < \sigma' \implies M(\sigma) < M(\sigma'). \]
But this follows from the polynomial property: as we make $D(\sigma)$ larger, the polynomial image of $D(\sigma)$ will become larger, and therefore, $M(\sigma)$ will become strictly larger. See Figure 7 for an illustration.

**Corollary 3.5.** Suppose $\gamma$ is a non-peripheral curve and $\eta$ is a non-peripheral component of $F^{-1}(\gamma)$. Then for any $x \in P_f$, we have
\[ N(\Sigma_x([\eta])) \leq N(\Sigma_{f(x)}([\gamma])). \]
Now let us prove the main theorem. Suppose $F$ has an obstruction and let $\Gamma$ be an obstruction guaranteed by Lemma 3.3. We may assume that it is an irreducible one. That is, for any $\gamma, \eta \in \Gamma$, there is an $n \geq 1$ such that $\eta$ is homotopic to a component of $f^{-n}(\gamma)$. Now let us prove $\Gamma$ is a Levy cycle.

Claim: for each $\gamma \in \Gamma$, there exists exactly one non-peripheral component of $F^{-1}(\gamma)$. Suppose this were not true. Then we would have two non-peripheral components $\gamma_1 \neq \eta_1$ of $F^{-1}(\gamma)$ and two sequences:

(3.1) \[ \gamma = \gamma_0 \rightarrow \gamma_1 \rightarrow \gamma_2 \rightarrow \cdots \rightarrow \gamma_l \rightarrow \gamma_{l+1} = \gamma \]

and

(3.2) \[ \gamma = \eta_0 \rightarrow \eta_1 \rightarrow \eta_2 \rightarrow \cdots \rightarrow \eta_k \rightarrow \eta_{k+1} = \gamma, \]

where $\gamma_{i+1}$ is homotopic to some component of $f^{-1}(\gamma_i)$, $0 \leq i \leq l + 1$, and $\eta_{j+1}$ is homotopic to some component of $f^{-1}(\eta_j)$, $0 \leq j \leq k$. Since $\gamma \in \Sigma$, we have some $x \in \mathcal{O}$ such that $N(\Sigma_x([\gamma])) > 0$. Let $y \in \mathcal{O}$ such that $x = F(y)$. Let $p$ be the period of the $x$. Repeating (3.1) $p$ times,

$\gamma_0 \rightarrow \gamma_1 \rightarrow \gamma_2 \rightarrow \cdots \rightarrow \gamma_l \rightarrow \gamma_0 \rightarrow \cdots \rightarrow \gamma_0 \rightarrow \gamma_1 \rightarrow \gamma_2 \rightarrow \cdots \rightarrow \gamma_l \rightarrow \gamma_0$

Apply Corollary 3.5 to the above sequence, we get

$N(\Sigma_x([\gamma])) \geq N(\Sigma_y([\gamma_1])) \geq N(\Sigma_x([\gamma]))$,

which implies that

$N(\Sigma_y([\gamma_1])) = N(\Sigma_x([\gamma]))$.

Similarly, we may repeat (3.2) $p$ times and then apply Corollary 3.5, we get

$N(\Sigma_x([\gamma])) \geq N(\Sigma_y([\gamma_1])) \geq N(\Sigma_x([\gamma]))$,

which implies that

$N(\Sigma_y([\eta_1])) = N(\Sigma_x([\gamma]))$.

But by Lemma 3.4 and $\gamma_1 \neq \eta_1$, we also have

$N(\Sigma_x([\gamma])) \geq N(\Sigma_y([\gamma_1])) + N(\Sigma_y([\eta_1]))$. 

Figure 7: $\sigma < \sigma' \implies M(\sigma) < M(\sigma')$
This implies that \( N(\Sigma_\gamma(\gamma)) = 0 \). This contradicts the assumption that \( \Sigma_\gamma(\gamma) > 0 \). The Claim has been proved. Now for any non-peripheral curve \( \gamma \), let \( K(\gamma) \) denote the number of the type \( P \) curve segments of \( \gamma \) and let
\[
K([\gamma]) = \min K(\eta)
\]
where \( \min \) is taken over all the non-peripheral curves which are homotopic to \( \gamma \). From the claim above, it follows that \( \Gamma \) must be a Levy cycle for \( F \). Let \( \Gamma = \{ \gamma_1, \cdots, \gamma_n \} \) so that
\[
K(\gamma_1) = \min_{1 \leq i \leq n} K([\gamma_i])
\]
and for \( 1 \leq i \leq n-1 \), \( \gamma_{i+1} \) is the unique non-peripheral component of \( F^{-1}(\gamma_i) \). Since the image of a type \( P \) curve segment contains at least one type \( P \) curve segment, we must have
\[
K(\gamma_1) = \cdots = K(\gamma_n).
\]
So all type \( P \) curve segments of \( \gamma_i, 1 \leq i \leq n \), are non-trivial in the sense that \( D(\sigma) \cap P_F \neq \emptyset \). So for any type \( P \) curve segment \( \sigma \) of \( \gamma_{i+1} \), there is a type \( P \) curve segment \( \sigma' \) which is homotopic to \( \sigma \) (Figure 8 illustrates the meaning that two type \( P \) curve segments are homotopic), such that \( \sigma' \) is mapped homeomorphically to some type \( P \) curve segment \( \tau \) of \( \gamma_i \). We thus get a cycle of type \( P \) curve segments \( \{ \sigma_i \}, 1 \leq i \leq m \) with \( n|m \), such that for each \( \sigma_i \), there is a type \( P \) curve segment \( \mu_{i+1} \) which is homotopic to \( \sigma_{i+1} \) so that \( F : \mu_{i+1} \to \sigma_i \) is a homeomorphism. Since \( f \) is post-critically finite which expands the orbifold metric, it follows that \( D(\sigma_i) \) contains exactly one point in \( P_F \), say \( x_i \), which lies in \( I_i = D(\sigma_i) \cap \mathbb{T} \), and moreover, \( \{ x_i \} \) is a periodic cycle. But one may then deform each \( D(\sigma_i) \) into a small neighborhood of \( x_i \).

Figure 8: Two homotopic type P curve segments
In this way $\Gamma = \{\gamma_i\}$ becomes into a Levy cycle of $g$. This is impossible. The proof of the main theorem is completed.

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**References**

[1] B. Bielefeld, Y. Fisher, and J. Hubbard, *The classification of critically preperiodic polynomials*, J. Amer. Math. Soc. 5 (1992), no. 4, 721–762.

[2] A. Douady and J. Hubbard, *A proof of Thurston’s topological characterization of rational functions*, Acta Math. 171 (1993), no. 2, 263–297.

[3] M. Rees, *Realization of matings of polynomials as rational maps of degree two*, manuscript, 1986.

[4] P. Roesch and Y. Yin, *The boundary of bounded polynomial Fatou components*, C. R. Math. Acad. Sci. Paris 346 (2008), no. 15-16, 877–880.

[5] K. Pilgrim, *Canonical Thurston obstructions*, Adv. Math. 158 (2001), no. 2, 154–168.

[6] M. Shishikura and L. Tan, *A family of cubic rational maps and matings of cubic polynomials*, Experiment. Math. 9 (2000), no. 1, 29–53.

[7] L. Tan, *Matings of quadratic polynomials*, Ergodic Theory Dynam. Systems 12 (1992), no. 3, 589–620.

[8] D. P. Thurston, *A positive characterization of rational maps*, Ann. Math. 192 (2020), no. 1, 1–46.