NON-RADIAL SIGN-CHANGING SOLUTIONS FOR THE
SCHRÖDINGER-POISSON PROBLEM IN THE SEMICLASSICAL LIMIT

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ABSTRACT. We study the following system of equations known as Schrödinger-Poisson problem
\[
\begin{cases}
-\epsilon^2 \Delta v + v + \phi v = f(v) & \text{in } \mathbb{R}^N \\
-\Delta \phi = a_N v^2 & \text{in } \mathbb{R}^N \\
\phi \to 0 & \text{as } |x| \to +\infty
\end{cases}
\]
where \( \epsilon > 0 \) is a small parameter, \( f : \mathbb{R} \to \mathbb{R} \) is given, \( N \geq 3 \), \( a_N \) is the surface measure of the unit sphere in \( \mathbb{R}^N \) and the unknowns are \( v, \phi : \mathbb{R}^N \to \mathbb{R} \).

We prove that the set of sign-changing solutions has a rich structure in the semiclassical limit: we construct non-radial multi-peak solutions with an arbitrary large number of positive and negative peaks which are displaced in suitable symmetric configurations and which collapse to the same point as \( \epsilon \to 0 \). The proof is based on the Lyapunov-Schmidt reduction.

1. Introduction

In this paper we are concerned with the existence of sign-changing solutions to the following nonlinear Schrödinger-Poisson problem
\[
(SP) \quad \begin{cases}
-\epsilon^2 \Delta v + v + \phi v = f(v) & \text{in } \mathbb{R}^N \\
-\Delta \phi = a_N v^2 & \text{in } \mathbb{R}^N \\
\phi(x) \to 0 & \text{as } |x| \to +\infty
\end{cases}
\]
where \( \epsilon \) is a small and positive parameter, \( f : \mathbb{R} \to \mathbb{R} \) is given, \( N \geq 3 \), \( a_N \) is the surface measure of the unit sphere in \( \mathbb{R}^N \) and the unknown is \( (v, \phi) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \).

Systems like \( (SP) \) have been object of many investigations in the last years because of their strong physical meaning. Indeed they appear in quantum mechanics models (see e.g. [7, 8, 17]) and also in semiconductor theory [5, 6, 18, 19]. In [5, 6], for instance, they have been introduced as models describing solitary waves for nonlinear stationary equations of Schrödinger type interacting with an electrostatic field, and are usually known as Schrödinger-Poisson systems. In this context the nonlinear term \( f \) simulates, as usual, the interaction between many particles, while the solution \( \phi \) of the Poisson equation plays the role of a potential determined by the charge of the wave function itself. From another point of view, the interest on this problem stems also from the Slater approximation of the exchange term in the Hartree-Fock model, see [22]. In this framework \( f(u) = u^p \) with \( p = 5/3 \), however, other nonlinearities have been used in different approximations.

In the following we look for bound states to \( (SP) \) in the semiclassical case, namely as \( \epsilon \to 0 \).

While there are many results about existence, multiplicity and behavior of positive solutions to \( (SP) \) (see [1, 4, 9, 11, 12, 13, 16, 20, 21] and references therein), little is known about the existence of solutions \( (v, \phi) \) such that \( v \) is sign-changing.

In [15] the existence of solutions with \( v \) nodal is established in the case \( \epsilon = 1 \), the solutions found are radial and \( v \) has any fixed number of nodal domains. As far as we know, nothing is known.

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about the existence of non-radial sign-changing $v$.

In this paper we give an improvement in this direction. Indeed we construct, for $\epsilon$ small, non-radial solutions to (SP) such that $v$ is nodal, moreover $v$ is multi-peak shaped and its peaks collapse all at a certain point (which we may assume to be 0 by the invariance by translation) as $\epsilon \to 0$ (cluster nodal solutions). We also show that the set of nodal solutions to (SP) has a rich structure, indeed we construct several different solutions having suitable symmetric spatial displacement of the signed peaks. We recall that D’Aprile and Wei in [12] proved the existence of positive cluster solutions to (SP) as $\epsilon$ goes to zero, hence this paper completes the picture about the existence of cluster solutions to (SP).

Before stating the main results we fix the assumptions on $f$ that we will use in the sequel and we recall some known facts.

(1) $f \in C^{1+\sigma}_{loc}(\mathbb{R}) \cap C^2(\mathbb{R})$ with $\sigma \in (0,1)$, $f(0) = f'(0) = 0$ and $f(t) = -f(-t)$.

(2) the problem

\begin{equation}
\begin{aligned}
\Delta w - w + f(w) &= 0 &\text{in } \mathbb{R}^N \\
w &> 0 &\text{in } \mathbb{R}^N \\
\lim_{|x| \to +\infty} w(x) &= 0 \\
w(0) &= \max_{\mathbb{R}^N} w(x)
\end{aligned}
\end{equation}

has a unique solution $w$ which is non degenerate, i.e. denoting by $\mathcal{L} : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ the linearized operator in $w$,

$$\mathcal{L}[u] := \Delta u - u + f'(w)u,$$

then

$$\text{Kernel}(\mathcal{L}) = \text{span} \left\{ \frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_N} \right\}.$$

We recall that $w$ is a critical point of the following energy functional

$$I[w] := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2)dx - \int_{\mathbb{R}^N} F(w)dx$$

where $F(t) = \int_0^t f(s)ds.$

By the well-know result of Gidas, Ni and Nirenberg ([14]), $w$ is radially symmetric and strictly decreasing in $r = |x|$. Moreover, by classical regularity results, the following asymptotic behaviors hold:

\begin{align}
(1.2) \quad w(r), w''(r) &= A_N r^{-\frac{N+1}{2}} e^{-r} \left(1 + O \left( \frac{1}{r} \right) \right), \\
(1.3) \quad w'(r) &= -A_N r^{-\frac{N+1}{2}} e^{-r} \left(1 + O \left( \frac{1}{r} \right) \right),
\end{align}

where $A_N > 0$ is a suitable positive constant.

The class of nonlinearities $f$ satisfying (1)−(2) includes, and it’s not restricted to, the homogeneous nonlinearity $f(v) = |v|^{p-1}v$ with $p \in (1, \frac{N+2}{N-2})$.

In this paper the dimension $N$ is chosen in the interval $[3, 6]$. Under this assumption it is well known that the system (SP) can be reduced into a single equation. Indeed a simple application of the Lax-Milgram theorem ensures the existence of a unique solution of the second equation of (SP), namely the following result holds:
Lemma 1.1. Let $N \in [3, 6]$. For every $f \in L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ there exists a unique solution $\phi[f]$ in $D^{1,2}(\mathbb{R}^N)$ of the equation $-\Delta \phi = a_N f$. Moreover the following representation formula holds:

$$\phi[f](x) := \int_{\mathbb{R}^N} \frac{f}{|x-y|^{N-2}} dy.$$ 

Furthermore the functional $G : H^1(\mathbb{R}^N) \to \mathbb{R}$

$$G(u) := \int_{\mathbb{R}^N} \phi[u^2]|u|^2 dx$$

is $C^1$ and $G'(u)[v] = 4 \int_{\mathbb{R}^N} \phi[u^2]uv dx$.

By Lemma 1.1 we reduce to study the following nonlinear scalar equation in $H^1(\mathbb{R}^N)$

$$(1.4) \quad - \epsilon^2 \Delta v + v + v\phi[v^2] = f(v) \quad \text{in } \mathbb{R}^N.$$ 

We also recall that the solutions of (1.4) are critical points of the $C^2$-functional $J_\epsilon : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$J_\epsilon[v] = \frac{1}{2} \int_{\mathbb{R}^N} (\epsilon^2|\nabla v|^2 + v^2) \, dx - \int_{\mathbb{R}^N} F(v) \, dx + \frac{1}{4} \int_{\mathbb{R}^N} \phi[v^2](x)v^2(x) \, dx.$$ 

We can now state our results. Our first theorem is about the existence of nodal solutions whose form consists of one positive peak centered in 0 surrounded by $k$ negative peaks located near the vertices of a regular polygon, with the number $k$ sufficiently large (see figure 1).

![Figure 1. A configuration with 1 positive peaks at the origin surrounded by 7 negative peaks.](image)

**Theorem 1.2.** Let (f1)and (f2) hold and let $N \in [3, 6]$.

Fix $k \geq 7$ and let $Q_1, Q_2, \ldots, Q_k \in \mathbb{R}^2$ be the vertices of a two-dimensional convex regular polygon centered at 0. Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, there is $r_\epsilon > 0$ and a sign-changing solution $v_\epsilon \in H^1(\mathbb{R}^N)$ to (1.4) of the form

$$v_\epsilon(x) = w \left( \frac{x}{\epsilon} \right) - \sum_{i=1}^{k} w \left( \frac{x - P_i \epsilon}{\epsilon} \right) + \text{h.o.t.}, \quad \text{as } \epsilon \to 0$$

uniformly for $x \in \mathbb{R}^N$.

Here $P_i \epsilon := (r_i Q_i, 0) \in \mathbb{R}^N$, $i = 1, \ldots, k$ and $r_\epsilon \to 0$ as $\epsilon \to 0$. Moreover

$$\lim_{\epsilon \to 0} \frac{r_\epsilon}{\epsilon \log \frac{1}{\epsilon}} = C$$

for some $C > 0$.

More in general we can prove the following result:
Theorem 1.3. Let \( Q_1, \ldots, Q_k \in \mathbb{R}^h \) \((2 \leq h \leq N)\) be the vertices of a convex regular polytope in \( \mathbb{R}^h \) centered at 0 and having radius 1 and side \( s \).

Assume \( s \leq 1 \), \( h > 2 \) or \( s > 1 \), \( h = 2 \). Then there exists \( \epsilon_0 > 0 \) such that for any \( \epsilon \in (0, \epsilon_0) \), there is a sign-changing solution \( v_\epsilon \) to \((1.4)\) with one positive peak centered in 0 and \( k \) negative peaks centered at points \( P_{i\epsilon} := (r_{i\epsilon} Q_i, 0) \in \mathbb{R}^N \), \( i = 1, \ldots, k \), such that \( r_{\epsilon} \to 0 \) as \( \epsilon \to 0 \). Moreover

\[
\lim_{\epsilon \to 0} \frac{r_{\epsilon}}{\epsilon \log \frac{1}{\epsilon^2}} = C
\]

for some \( C > 0 \).

We may also construct solutions having more than two nodal regions, alternating (suitable) convex regular polygons having respectively negative and positive peaks on their vertices. For instance in the following we construct a solution with a positive peak centered in zero surrounded by \( k \) negative peaks and \( h = km \) positive peaks which are respectively on the vertices of two nested polygons, but of course this construction could be generalized to more than two nested polygons (see figure 2).

![Figure 2](image-url)

Figure 2. A configuration with 1 positive peak at the origin surrounded by 8 negative peaks and 16 positive peaks \((k = 8, m = 2)\).

Theorem 1.4. Fix \( k, m \in \mathbb{N} \) such that

\[
k > \min \left\{ 6, \frac{\pi}{m \arcsin 1/4} \right\}
\]

and let \( Q_1, \ldots, Q_{km} \in \mathbb{R}^2 \) be the vertices of a two-dimensional convex regular polygon centered at 0. Then there exists \( \epsilon_0 > 0 \) such that for any \( \epsilon \in (0, \epsilon_0) \) there is \( r_{\epsilon} > 0 \) and a sign-changing solution \( v_{\epsilon} \in H^1(\mathbb{R}^N) \) to \((1.4)\) of the form

\[
v_{\epsilon}(x) = w \left( \frac{x}{\epsilon} \right) - \sum_{i=1}^k w \left( \frac{x - P_{i\epsilon}}{\epsilon} \right) + \sum_{i=1}^{km} w \left( \frac{x - P_{k+i\epsilon}}{\epsilon} \right) + \text{h.o.t.} \quad \text{as} \ \epsilon \to 0
\]

uniformly for \( x \in \mathbb{R}^N \). Here

\[
\mathbb{R}^N \ni P_{\epsilon} := \begin{cases} (r_{\epsilon} Q_{(j-1)m+1}, 0) & j = 1, \ldots, k \\ (2r_{\epsilon} Q_{j-k}, 0) & j = k + 1, \ldots, k + km 
\end{cases}
\]

and \( r_{\epsilon} \to 0 \) as \( \epsilon \to 0 \). Moreover

\[
\lim_{\epsilon \to 0} \frac{r_{\epsilon}}{\epsilon \log \frac{1}{\epsilon^2}} = C
\]

for some \( C > 0 \).
At the end, we construct a solution having $k = m_1 q$ negative and $h = m_2 q$ positive peaks alternated on the vertices of $d = \max\{m_1, m_2\}$ suitable nested regular polygons, with the same sign on corresponding vertices (see figure 3), precisely

**Theorem 1.5.** Let (f1) and (f2) hold and let $N \in [3, 6]$. Fix $q \in \mathbb{N}$, $q \geq 2$ and $(m_1, m_2) \in \mathbb{N} \times \mathbb{N} \setminus \{(1, 1)\}$. Let $Q_1, \ldots, Q_{2q} \in \mathbb{R}^2$ be the vertices of a two-dimensional convex regular polygon centered at 0. Then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ there is $r_\epsilon > 0$ and a sign-changing solution $v_\epsilon \in H^1(\mathbb{R}^N)$ to (1.4) of the form

$$v_\epsilon(x) = \sum_{i=1}^{2q} (-1)^{I(i)} \left( \sum_{j=1}^{m_{I(i)}} w \left( \frac{x - P_{ij}\epsilon}{\epsilon} \right) \right) + \text{h.o.t.}, \quad \text{as } \epsilon \to 0$$

uniformly for $x \in \mathbb{R}^N$.

Here $I(i) := \frac{(-1)^i + 3}{2}$ and the points $P_{ij}\epsilon := \left( \left( 1 + (j - 1) \sin \frac{\pi}{2q} \right) r_\epsilon Q_i, 0 \right) \in \mathbb{R}^N$, $j = 1, \ldots, m_{I(i)}$.

Moreover $r_\epsilon \to 0$ as $\epsilon \to 0$ and

$$\lim_{\epsilon \to 0} \frac{r_\epsilon}{\epsilon \log \frac{1}{\epsilon^2}} = C$$

for some $C > 0$.

![Figure 3. A configuration with 6 negative peaks and 9 positive peaks ($q = 3$, $m_1 = 2, m_2 = 3$).](image-url)

The proof of our results is based on the well known Lyapunov-Schmidt reduction procedure (see [2]). In particular, in order to deal with nodal clustered solutions, we perform the reduction in suitable symmetric settings in the spirit of [11]. We outline here the main ideas. First our approximate solutions are constructed as the sum (with sign) of suitably rescaled $w$ centered at distinct points $P_i \in \mathbb{R}^N$ such that $P_i \to 0$ as $\epsilon \to 0$.

This choice is the most natural. In fact if $v$ is a solution of (1.4) and $P \in \mathbb{R}^N$ then $v_\epsilon(x) := v(\epsilon x + P)$ solves the equation $-\Delta u + u + \epsilon^2 \partial \phi |u|^2 u = f(u)$ which, since $\epsilon \to 0$, can be approximated by problem (1.1). Hence it’s quite natural to take $v \sim \pm w(\frac{x-P}{\epsilon})$ as a solution of (1.4) for $\epsilon$ small.

Moreover, if we take several different fixed points $P_i \in \mathbb{R}^N$, then $v \sim \sum \pm w(\frac{x-P_i}{\epsilon})$ is still a good approximation of a solution $v$ of (1.4) for $\epsilon$ small enough, in spite of the presence of nonlinear terms in the equation. The reason is that, thanks to the exponential decay of $w$, the interactions among peaks centered at different fixed points becomes negligible when $\epsilon \to 0$.

In our case however we are looking for clustered solutions, namely the points $P_i \to 0$ as $\epsilon \to 0$. This means that the interactions among peaks play a role.
Anyway by locating the peaks in suitable symmetric configurations, still we will be able to find a solution of the desired form.

Indeed we recall that the Lyapunov-Schmidt reduction method reduces the problem to find a critical point for a functional defined on a finite-dimensional space (reduced functional). In our case the reduced functional, up to a positive constant, has the form

$$M_\epsilon([P_1,\ldots,P_\ell]) = \epsilon^2 \sum_{i\neq j} \frac{1}{|P_i - P_j|^N} - \sum_{i\neq j, \lambda_i = \lambda_j} w \left( \frac{P_i - P_j}{\epsilon} \right) + \sum_{i\neq j, \lambda_i = -\lambda_j} w \left( \frac{P_i - P_j}{\epsilon} \right) + \text{h.o.t.,}$$

where $\lambda_i = \pm 1$ according to the sign of each peak and the unknowns $P_i$ determine the location of the peaks. We notice that it consists of three main terms: the first term depends on the Poisson potential effect, the second term is due to the interplay between the peaks of the same sign and has a repulsive effect, the third term is due to the interaction between peaks of opposite sign and has an attractive effect.

Observe that the first term $\epsilon^2 \sum_{i\neq j} \frac{1}{|P_i - P_j|^N}$ increases when the points $P_i$ are close to zero, while using the exponential decay of $w$, the interaction term $-\sum_{i\neq j, \lambda_i = \lambda_j} w \left( \frac{P_i - P_j}{\epsilon} \right)$ increases when the mutual distance between the points $P_i$ is big. Hence, if we restrict the functional to suitable symmetric configurations in which the peaks having opposite sign are kept away from each other, then the mutual interaction between opposite peaks, i.e. the third term $+\sum_{i\neq j, \lambda_i = -\lambda_j} w \left( \frac{P_i - P_j}{\epsilon} \right)$, becomes negligible and so we can easily conclude that the equilibrium is achieved for a suitable (symmetric) configuration of the points $P_i$, which is a local maximum for the functional $M_\epsilon$, namely we have produced a sign-changing cluster solution for the problem ($SP$).

We remark that one can find solutions with analogous symmetric configurations as in all the previous results, also in the case of the scalar Schrödinger equation

$$-\epsilon^2 \Delta v + V(|x|)v = |v|^{p-1}v \quad x \in \mathbb{R}^N$$
in presence of a radially symmetric potential $V$ with a local maximum in 0 (see also [11]).

**Notations**

Before going on we establish some notations.

Let us denote by $H^1(\mathbb{R}^N)$ the usual Sobolev space endowed with scalar product and norm

$$(u,v)_\epsilon = \int_{\mathbb{R}^N} (\epsilon^2 \nabla u \nabla v + uv) \, dx; \quad \|u\|_2^2 := \int_{\mathbb{R}^N} (\epsilon^2 |\nabla u|^2 + u^2) \, dx$$
and by $D^{1,2}(\mathbb{R}^N)$ the completion of the space $C^\infty_c(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,2}} := \left( \int_{\mathbb{R}^N} \epsilon^2 |\nabla u|^2 \, dx \right)^{1/2}.$$ 

Moreover let $L^p(\mathbb{R}^N)$ the usual Lebesgue space endowed with the norm

$$|u|_p := \left( \int_{\mathbb{R}^N} |u|^p \, dx \right)^{1/p} \quad p \in [1,\infty) \quad \|u\|_\infty = \sup_{x \in \mathbb{R}^N} |u(x)|.$$ 

In particular, let us denote by $\langle \cdot, \cdot \rangle$ the usual scalar product in $L^2(\mathbb{R}^N)$, namely

$$\langle u, v \rangle := \int_{\mathbb{R}^N} uv \, dx.$$
2. General setting

The Lyapunov-Schmidt reduction will be made around an appropriate set of approximating solutions. Precisely, for any $\ell \in \mathbb{N}$ we define
\[
\Gamma_\epsilon := \left\{ \mathbf{P} = (P_1, \ldots, P_\ell) \in \mathbb{R}^{N\ell} : \beta^2 \epsilon \log \frac{1}{\epsilon^2} < |P_i - P_j| < \epsilon \left( \log \frac{1}{\epsilon^2} \right)^2 \quad \text{for} \quad i \neq j \right\}
\]
where $\beta \in (\sigma, 1)$ is sufficiently close to 1. Let $\mathbf{P} \in \Gamma_\epsilon$ and set $w_{P_i}(x) = w(\frac{x - P_i}{\epsilon})$, $i = 1, \ldots, \ell$. We look for solutions of (1.4) of the form
\[
v_\epsilon(x) := w_{\mathbf{P}}(x) + \psi_\epsilon(x)
\]
where $\psi_\epsilon$ will be a remainder term belonging into a suitable space and the approximating solution $w_{\mathbf{P}}$ is of the form
\[
w_{\mathbf{P}}(x) = \sum_{i=1}^\ell \lambda_i w_{P_i}(x)
\]
with $\lambda_i = \pm 1$ according to the sign of each peak.

In particular we will reduce ourselves to symmetric configurations, finding solutions $v_\epsilon$ with some symmetric properties. Here we show that $\phi[v_\epsilon^2]$ preserves the same symmetry property. Indeed, let $G$ be a group of symmetries of $\mathbb{R}^N$ and let $g \in G$. For $u : \mathbb{R}^N \to \mathbb{R}$ we set
\[
(T_g u)(x) = u(gx).
\]
Let
\[
X := \left\{ u \in H^1(\mathbb{R}^N) : T_g u = u, \ g \in G \right\}
\]
and
\[
Y := \left\{ \phi \in D^{1,2}(\mathbb{R}^N) : T_g \phi = \phi, \ g \in G \right\}.
\]
We remark that $X, Y$ are the subspace of $H^1$ and $D^{1,2}$ respectively invariant under the action (2.6).

**Lemma 2.1.** If $u \in X$ then $\phi[u^2] \in Y$.

**Proof.** Let $u \in X$. To prove that $\phi[u^2] \in Y$ we have to show that $\phi[u^2]$ is invariant under the action (2.6). To this aim let us evaluate
\[
-\Delta(T_g \phi[u^2]) = T_g(-\Delta \phi[u^2]) = T_g(a_N u^2) = a_N u^2(gx) = a_N u^2(x) = -\Delta \phi[u^2]
\]
and then, by the uniqueness of the solution, it follows that $T_g \phi[u^2] = \phi[u^2]$. \qed

Since we look for a solution near $w_{\mathbf{P}}$, a key step is to evaluate $S(w_{\mathbf{P}})$ where
\[
S(v) := \epsilon^2 \Delta v - v + f(v) - \phi[v^2]v.
\]
What we can prove is the following result (for the proof see for instance [12]):

**Lemma 2.2.** Let $\beta \in (\sigma, 1)$. There exists a constant $C > 0$ such that for every $\epsilon > 0$ and $\mathbf{P} = (P_1, \ldots, P_\ell) \in \mathbb{R}^{N\ell}$ with $|P_i - P_j| \geq 2 \beta^2 \epsilon \log \frac{1}{\epsilon}$, for $i \neq j$:
\[
|S_\epsilon[w_{\mathbf{P}}]| \leq C \epsilon^{5/2}(\beta^2 + \sigma) \sum_{i=1}^\ell w_{P_i}^{1-\beta^2}.
\]
3. Energy estimates

Let us fix $\ell \in \mathbb{N}$ and $P = (P_1, \ldots, P_\ell) \in \mathbb{R}^{N\ell}$. In this section we derive the following key result about the interaction among $\ell$ signed bumps displaced in $P$.

**Proposition 3.1.** The following energy estimates hold as $\epsilon \to +\infty$:

$$\left| \epsilon \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla w_P|^2 + w_P^2 \right) - F(w_P) \right| \to +\infty,$$

and

$$\frac{1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w_P^2(x)w_P^2(y)}{|x-y|^{N-2}} dx dy = \epsilon^{N+2} C_1 + \epsilon^{N+2} C_2 + o(1) \sum_{i \neq j} \frac{1}{|P_i - P_j|^N}$$

where $\gamma_0 := \frac{1}{2} \int_{\mathbb{R}^N} f(w) e^x dx$ and $C_1, C_2$ are positive constants.

In order to prove Proposition 3.1, we will need some useful lemmas that we briefly recall here. From [12, Lemma 3.1] one has

**Lemma 3.2.** For $i \neq j$

$$\int_{\mathbb{R}^N} f(w_{P_i}) w_{P_j} = \epsilon^N w \left( \frac{P_i - P_j}{\epsilon} \right) (2\gamma_0 + o(1))$$

where $\gamma_0$ is the same constant defined in Proposition 3.1.

Moreover in [11, pg. 23] it has been proved that

**Lemma 3.3.** Let

$$H(P) := \int_{\mathbb{R}^N} \left[ F\left( \sum_{i=1}^\ell \lambda_i w_{P_i} \right) - \sum_{i=1}^\ell F(w_{P_i}) - \sum_{i \neq j} \lambda_i \lambda_j f(w_{P_i}) w_{P_j} \right] dx,$$

then

$$|H(P)| = \epsilon^N o(1) \sum_{i \neq j} \left( \frac{P_i - P_j}{\epsilon} \right).$$

The following result can be found in [12]:

**Lemma 3.4.** For every $\beta \in \{1, \ldots, N-1\}$ and $g : \mathbb{R}^N \to \mathbb{R}$ such that $(1 + |y|^\beta)g \in L^1 \cap L^\infty$ set

$$\Phi_\beta[g](x) := \int_{\mathbb{R}^N} \frac{1}{|x-y|^{\beta}} g(y) dy.$$

Then there exist two positive constants $C(\beta, g), C'(\beta, g)$ such that

$$\left| \Phi_\beta[g](x) - \frac{C(\beta, g)}{|x|^{\beta}} \right| \leq C'(\beta, g) \frac{|x|^{\beta+1}}{|x|^{\beta+1}}.$$

Finally in order to estimate the energy term coming from the Poisson equation, we will also need the following:

**Lemma 3.5.** There exists a constant $C > 0$ such that for every $P_i, P_j \in \mathbb{R}^N$ and every $x \in \mathbb{R}^N$

$$\phi[w_{P_i}, w_{P_j}](x) \leq \epsilon^2 C.$$

and

$$\phi[w_{P_i}](x) \leq \epsilon^2 C.$$
Proof. We prove (3.9), the proof of (3.10) is similar. If $i = j$ one has
\[
\phi[w^2_i](x) = \int_{\mathbb{R}^N} \frac{w^2_i(y)}{|x-y|^{N-2}} dy = \epsilon^2 \int_{\mathbb{R}^N} \frac{w^2(y)}{|x-P_i|} dy = \epsilon^2 \phi[w^2] \left( \frac{x-P_i}{\epsilon} \right)
\]
where $x \mapsto \phi[w^2](x)$ is bounded by Strauss Lemma since it belongs to $C^2 \cap D_r^{1,2}$. When $i \neq j$ then
\[
\phi[w_i w_j](x) = \int_{\mathbb{R}^N} \frac{w_i(y)w_j(y)}{|x-y|^{N-2}} dy \leq \epsilon \int_{\mathbb{R}^N} \frac{w(y)}{|x-y|^{N-2}} dy + \epsilon \int_{\mathbb{R}^N} \frac{w(y)}{|x-P_i|} dy \leq \epsilon \phi[w](x),
\]
and $x \mapsto w(x), \phi[w](x)$ are bounded ($\phi[w] \in C^2 \cap D_r^{1,2}$). \hfill \Box

Proof of Proposition 3.1. We first prove (3.7). Easy computations show that
\[
\int_{\mathbb{R}^N} |\nabla w|^2 dx + \int_{\mathbb{R}^N} w^2 dx = \sum_{i=1}^\ell \lambda_i |\nabla w_i|^2 + \sum_{i \neq j} \lambda_i \lambda_j \int_{\mathbb{R}^N} w_i w_j dx + \epsilon^{N-2} \ell \int_{\mathbb{R}^N} |\nabla w|^2 dx + \epsilon \ell \int_{\mathbb{R}^N} w^2 dx + \sum_{i \neq j} \lambda_i \lambda_j \int_{\mathbb{R}^N} w_i w_j dx,
\]
\[
\int_{\mathbb{R}^N} F(w) dx = \int_{\mathbb{R}^N} F(\sum_{i=1}^\ell \lambda_i w_i) dx = \epsilon \ell \int_{\mathbb{R}^N} F(w) dx + \sum_{i \neq j} \lambda_i \lambda_j \int_{\mathbb{R}^N} f(w_i)w_j dx + H(\mathbf{P}).
\]
Hence, combining the previous estimates and using Lemma 3.3 we obtain (3.7) and the conclusion follows applying Lemma 3.2.

Next we prove (3.8). An easy computation shows that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w_i(x)w_j(y)}{|x-y|^{N-2}} dx dy = \sum_{i=1}^\ell \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w_i(x)^2 w_j(y)^2}{|x-y|^{N-2}} dx dy
\]
\[
+ \sum_{i \neq j} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w_i(x)^2 w_j(y)^2}{|x-y|^{N-2}} dx dy
\]
\[
+ 2 \sum_{i \neq j, h} \lambda_i \lambda_j \lambda_h \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w_i(x)w_j(x)w_h(y)}{|x-y|^{N-2}} dx dy.
\]
We evaluate each term in the RHS. Indeed
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w_i(x)^2 w_j(y)^2}{|x-y|^{N-2}} dx dy = \epsilon^{N+2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w(x)^2 w(y)^2}{|x-y|^{N-2}} dx dy.
\]
For $i \neq j$, by using Lemma 3.4 twice we estimate
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} w_{P_i}(x)w_{P_j}(y) \frac{y - y}{|x - y|^N} \, dx \, dy = \epsilon^2 \int_{\mathbb{R}^N} w_{P_i}(x)^2 \Psi_N(x)[w^2] \left( \frac{x - P_i}{\epsilon} \right) \, dx
\]
\[
= \epsilon^2 C \int_{\mathbb{R}^N} w_{P_i}(x)^2 \frac{1}{|x - P_i|^N} \, dx + \epsilon^2 O(1) \int_{\mathbb{R}^N} w_{P_j}(x)^2 \frac{1}{|x - P_j|^N} \, dx
\]
\[
= \epsilon^2 C \Psi_N(x)[w^2] \left( \frac{P_i - P_j}{\epsilon} \right) + \epsilon^2 O(1) \Psi_N(x)[w^2] \left( \frac{P_i - P_j}{\epsilon} \right)
\]
\[
= \epsilon^2 C \Psi_N(x)[w^2] \left( \frac{P_i - P_j}{\epsilon} \right)
\]
where $C = C(N - 2, w^2)$ is the positive constant in Lemma 3.4.

Finally for $i \neq j$ and $h, k \in \{1, \ldots, \ell\}$, by using (3.9) in Lemma 3.5 and the exponential decay of $w$ we have
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w_{P_i}(x)w_{P_j}(y)w_{P_k}(y)w_{P_h}(y)}{|x - y|^N} \, dx \, dy = \int_{\mathbb{R}^N} w_{P_i}(x)w_{P_j}(x)\phi[w_{P_k}w_{P_h}](x) \, dx
\]
\[
\leq \epsilon^2 C \int_{\mathbb{R}^N} w_{P_i}(x)w_{P_j}(x) \, dx
\]
\[
\leq \epsilon^2 C \left( \frac{1}{2} \right) \left( \int_{\{x : |x - P_i| \geq \frac{1}{2}|P_i - P_j|\}} w_{P_i}(x) \, dx + \int_{\{x : |x - P_i| \leq \frac{1}{2}|P_i - P_j|\}} w_{P_j}(x) \, dx \right)
\]
\[
\leq \epsilon^2 C \left( \frac{1}{2} \right) \int_{\mathbb{R}^N} w(x) \, dx \leq \epsilon^2 C \left( \frac{1}{2} \right) = \epsilon^2 \Psi_N(x)[1] \left( \frac{P_i - P_j}{\epsilon} \right)
\]

Estimate (3.8) is obtained as a combination of all the previous estimates. \(\square\)

4. The Linearized Problem

First we need the following result based on PDE estimates.

**Lemma 4.1.** Let $\epsilon > 0$, $P \in \Gamma_\epsilon$ and $v \in C^2(\mathbb{R}^N)$ satisfy
\[
|\epsilon^2 \Delta v - (1 + \epsilon^2 \phi[w^2])v| \leq C_0 e^{-\mu \min_{i = 1, \ldots, \ell} \frac{|x - P_i|}{\epsilon}}, \quad \forall x \in \mathbb{R}^N,
\]
\[
v(x) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty
\]
for some $C_0 > 0$. Then there exists $\mu_0$ (independent on $\epsilon, P$ and $v$) such that if $\mu \in (0, \mu_0)$
\[
|v(x)| \leq 2C_0(\ell - 1 + \epsilon^2)e^{-\mu \min_{i = 1, \ldots, \ell} \frac{|x - P_i|}{\epsilon}}, \quad \forall x \in \mathbb{R}^N.
\]

**Proof.** We use a comparison principle. Take a $\chi(t)$ a smooth cut-off function such that
\[
\chi(t) = 1, \quad \text{for} \quad |t| \leq 1, \quad \chi(t) = 0 \quad \text{for} \quad |t| \geq 2, \quad 0 \leq \chi \leq 1.
\]

Now consider the following auxiliary function:
\[
\xi(x) = 2\alpha \sum_{i=1}^{\ell} \left[ e^{-\mu \frac{|x - P_i|}{\epsilon}} + (1 - e^{-\mu \frac{|x - P_i|}{\epsilon}}) \chi \left( \frac{|x - P_i|}{\epsilon} \right) \right].
\]
We have that
\[
\Delta \xi = 2c_0 \sum_i \left[ \frac{\mu^2}{e^2} e^{-\mu \frac{|x-P_i|}{\epsilon}} + \frac{(N-1) \mu}{e^2} \frac{\epsilon}{|x-P_i|} e^{-\mu \frac{|x-P_i|}{\epsilon}} + \frac{\mu^2}{e^2} (1 - e^{-\mu \frac{|x-P_i|}{\epsilon}}) \chi' \left( \frac{|x-P_i|}{\epsilon} \right) \right] \\
+ \frac{(N-1) \mu}{|x-P_i|} e^{-\mu \frac{|x-P_i|}{\epsilon}} \chi \left( \frac{|x-P_i|}{\epsilon} \right),
\]
Hence
\[
e^2 \Delta \xi = 2c_0 \sum_i \left[ \frac{\mu^2}{e^2} e^{-\mu \frac{|x-P_i|}{\epsilon}} + \frac{(N-1) \mu}{e^2} \frac{\epsilon}{|x-P_i|} e^{-\mu \frac{|x-P_i|}{\epsilon}} + \mu^2 (1 - e^{-\mu \frac{|x-P_i|}{\epsilon}}) \chi' \left( \frac{|x-P_i|}{\epsilon} \right) \right] \\
+ \frac{(N-1) \mu}{|x-P_i|} e^{-\mu \frac{|x-P_i|}{\epsilon}} \chi \left( \frac{|x-P_i|}{\epsilon} \right),
\]
Fixed \(x \in \mathbb{R}^N\). We distinguish three cases:

1. There exists \(i \in \{1, \ldots, \ell\}\) such that \(\mu \frac{|x-P_i|}{\epsilon} \leq 1\). Then
   \[
   \xi(x) = 2c_0 + 2c_0 \sum_{j \neq i} e^{-\mu \frac{|x-P_j|}{\epsilon}} \\
   = 2c_0 e^{-\mu \frac{|x-P_i|}{\epsilon}} + 2c_0 \sum_{j \neq i} e^{-\mu \frac{|x-P_j|}{\epsilon}} \\
   \leq 2c_0(e + \ell - 1)e^{-\mu \frac{|x-P_i|}{\epsilon}}
   \]
   and, since,
   \[
   (4.12)
   e^2 \Delta e^{-\mu |x-P|} = \left( \mu^2 - \frac{(N-1) \mu \epsilon}{|x-P|} \right) e^{-\mu \frac{|x-P|}{\epsilon}}
   \]
   for \(x \neq P_i\) we have
   \[
e^2 \Delta \xi - (1 + \phi[w_p^2]) \xi \leq 2c_0 \sum_{j \neq i} (\mu^2 - 1) e^{-\mu \frac{|x-P_j|}{\epsilon}} - 2c_0 \\
   \leq -c_0 e^{-\mu \frac{|x-P_i|}{\epsilon}}
   \]
   provided \(\mu\) is sufficiently small.

2. For all \(i = 1, \ldots, \ell\), \(\mu \frac{|x-P_i|}{\epsilon} \geq 2\). Then
   \[
   \xi(x) = 2c_0 \sum_i e^{-\mu \frac{|x-P_i|}{\epsilon}} \leq 2c_0 \sum_i \epsilon e^{-\mu \frac{|x-P_i|}{\epsilon}}
   \]
   and by (4.12)
   \[
e^2 \Delta \xi - (1 + \phi[w_p^2]) \xi \leq 2c_0 \sum_i (\mu^2 - 1) e^{-\mu \frac{|x-P_i|}{\epsilon}} \\
   \leq -c_0 \sum \epsilon e^{-\mu \frac{|x-P_i|}{\epsilon}} \leq -c_0 e^{-\mu \frac{|x-P_i|}{\epsilon}}
   \]
   provided \(\mu\) is sufficiently small.
3. There exists $i \in \{1, \ldots, \ell\}$ such that $1 < \mu \frac{|x-P_i|}{\epsilon} < 2$. Then

$$\xi(x) = 2c_0 \left[ e^{-\mu \frac{|x-P_i|}{\epsilon}} + (1 - e^{-\mu \frac{|x-P_i|}{\epsilon}}) \chi \left( \frac{|x-P_i|}{\epsilon} \right) + \sum_{j \neq i} e^{-\mu \frac{|x-P_j|}{\epsilon}} \right]$$

$$\leq 2c_0 \left[ 1 + (\ell - 1)e^{-\mu \frac{|x-P_i|}{\epsilon}} \right]$$

$$= 2c_0(e^{\mu \frac{|x-P_i|}{\epsilon}} + \ell - 1)e^{-\mu \frac{|x-P_i|}{\epsilon}}$$

$$\leq 2c_0(e^2 + \ell - 1)e^{-\mu \frac{|x-P_i|}{\epsilon}}$$

and, since

$$\epsilon^2 \Delta \left[(1 - e^{-\mu \frac{|x-P_i|}{\epsilon}}) \chi \left( \frac{|x-P_i|}{\epsilon} \right) \right] = \mu^2(1 - e^{-\mu \frac{|x-P_i|}{\epsilon}}) \chi'' \left( \frac{|x-P_i|}{\epsilon} \right)$$

$$+ (N-1)\mu e^{-\mu \frac{|x-P_i|}{\epsilon}} \chi' \left( \frac{|x-P_i|}{\epsilon} \right) + 2\mu^2e^{-\mu \frac{|x-P_i|}{\epsilon}} \chi \left( \frac{|x-P_i|}{\epsilon} \right)$$

$$= O(\mu^2)e^{-\mu \frac{|x-P_i|}{\epsilon}}$$

we have, by (4.12),

$$\epsilon^2 \Delta \xi - (1 + \phi[w_P^2])\xi \leq O(\mu^2)e^{-\mu \frac{|x-P_i|}{\epsilon}} - \xi$$

$$\leq (O(\mu^2) - 2c_0)e^{-\mu \frac{|x-P_i|}{\epsilon}}$$

$$\leq -c_0e^{-\mu \frac{|x-P_i|}{\epsilon}}$$

for $\mu$ sufficiently small.

Hence, in any case, we have

$$\xi(x) \leq 2c_0(e^2 + \ell - 1)e^{-\mu \min_{i=1,\ldots,N} \frac{|x-P_i|}{\epsilon}}$$

(4.13)

$$\epsilon^2 \Delta \xi - (1 + \phi[w_P])\xi \leq -c_0e^{-\mu \min_{i=1,\ldots,N} \frac{|x-P_i|}{\epsilon}}$$

(4.14)

for any $x \in \mathbb{R}^N$.

By (4.11)

$$\epsilon^2 \Delta v + (1 + \phi[w_P^2])v \leq \epsilon^2 \Delta v - (1 + \phi[w_P^2])v \leq c_0e^{-\mu \min_{i=1,\ldots,N} \frac{|x-P_i|}{\epsilon}}$$

(4.15)

for all $x \in \mathbb{R}^N$. Then, by (4.14) and (4.15) we get

$$\epsilon^2 \Delta (\xi - v) - (1 + \phi[w_P^2])(\xi - v) \leq 0$$

for all $x \in \mathbb{R}^N$.

We claim that $\xi - v \geq 0$ in $\mathbb{R}^N$.

Indeed, if we supposed by contradiction that the minimum point $\bar{x}$ of $\xi - v$ is such that $(\xi - v)(\bar{x}) < 0$, since $\Delta (\xi - v)(\bar{x}) \geq 0$ then

$$\epsilon^2 \Delta (\xi - v)(\bar{x}) - (1 + \phi[w_P^2])(\xi - v)(\bar{x}) > 0.$$  

Analogously we can prove that $v + \xi \geq 0$. Thus $|v| \leq \xi$ and, using (4.13) we can conclude.  

Let $P \in \Gamma_e$. Let us introduce the following functions

$$Z_{P,j} = f'(w_P) \frac{\partial w_P}{\partial x_j}, \quad i \in \{1, \ldots, \ell\}, \quad j \in \{1, \ldots, N\}.$$

Since

$$Z_{P,j} = -\epsilon^2 \Delta \frac{\partial w_P}{\partial x_j} + \frac{\partial w_P}{\partial x_j}$$
after an integration by parts it is immediate to prove that

\[(4.16) \quad \left( v, \frac{\partial w_{P_n}}{\partial x_j} \right)_\epsilon = \langle v, Z_{P_{\epsilon, j}} \rangle \quad \forall v \in H^1(\mathbb{R}^N). \]

Then orthogonality to the functions \( \frac{\partial w_{P_n}}{\partial x} \) in \( H^1(\mathbb{R}^N) \) is equivalent to orthogonality to \( Z_{P_{\epsilon, j}} \) in \( L^2(\mathbb{R}^N) \). Then we easily get

\[(4.17) \quad \langle Z_{P_{\epsilon, j}}, \frac{\partial w_{P_n}}{\partial x_n} \rangle = \left( \frac{\partial w_{P_n}}{\partial x_j}, \frac{\partial w_{P_n}}{\partial x_n} \right)_\epsilon = \begin{cases} \epsilon N - 2 \left\| \frac{\partial w}{\partial x_1} \right\|^2 & \text{for } (i, j) = (m, n) \\ o(\epsilon N - 2) & \text{for } (i, j) \neq (m, n) \end{cases}

\]
as \( \epsilon \to 0 \).

Let \( \mu > 0 \) a sufficiently small number. We introduce the following weighted norm:

\[\|v\|_{s, P} := \sup_{x \in \mathbb{R}^N} e^{\mu \min_{i=1, \ldots, \ell} |x - P_i|} |v(x)|,\]

and the spaces

\[C_{s, P} = \{ v \in C(\mathbb{R}^N) : \|v\|_{s, P} < \infty \}, \quad H^2_{s, P} = H^2(\mathbb{R}^N) \cap C_{s, P}.\]

We consider the following linear problem:

\[(4.18) \quad \begin{cases} L_P[v] = h + \sum_{i, j} c_{i, j} Z_{P_{\epsilon, j}} \\ v \in H^2_{s, P}, \quad \langle v, Z_{P_{\epsilon, j}} \rangle = 0, \quad i = 1, \ldots, \ell, \quad j = 1, \ldots, N. \end{cases}\]

where

\[L_P[v] := \epsilon^2 \Delta v - v + f'(w_P)v - \phi |w_P|v - 2\phi |w_P v|.

**Lemma 4.2.** There exists \( C > 0 \) such that, provided \( \epsilon \) is sufficiently small, if \( P \in \Gamma_\epsilon \), and \( (v, c_{i, j}, h) \) satisfies (4.18) the following holds

\[\|v\|_{s, P} \leq C \|h\|_{s, P}.\]

**Proof.** By contradiction, we assume the existence of a sequence \( \epsilon_n \to 0 \),

\[(\tilde{v}_n, c_{\epsilon_n, i, j}) \in H^2_{s, P_n} \times \mathbb{R}, \quad \tilde{h}_n \in C_{s, P_n}\]
satisfying (4.18) such that

\[\|\tilde{v}_n\|_{s, P_n} > n \|\tilde{h}_n\|_{s, P_n}.\]

Set

\[v_n = \frac{\tilde{v}_n}{\|\tilde{v}_n\|_{s, P_n}}, \quad c_{\epsilon_n, i, j} = \frac{c_{\epsilon_n, i, j}}{\|\tilde{v}_n\|_{s, P_n}}, \quad h_n = \frac{\tilde{h}_n}{\|\tilde{v}_n\|_{s, P_n}}.\]

We obtain that \((v_n, c_{\epsilon_n, i, j}, h_n)\) satisfies (4.18) and

\[\|v_n\|_{s, P_n} = 1, \quad \|h_n\|_{s, P_n} = o(1).\]

Choose \((h', m) \in \{1, \ldots, \ell\} \times \{1, \ldots, N\}\) be such that, up to a subsequence, \( |c_{h', m}^n| \geq |c_{i, j}^n| \) for all \((i, j)\) and \(n\). By multiplying the equation in (4.18) by \( \frac{\partial w_{P_n}}{\partial x_m} \) and integrating on \( \mathbb{R}^N \), we get

\[\sum_{i, j} c_{\epsilon_n, i, j} \langle Z_{P_{\epsilon, j}}, \frac{\partial w_{P_n}}{\partial x_m} \rangle = -\langle h_n, \frac{\partial w_{P_n}}{\partial x_m} \rangle + \langle L_P[v_n], \frac{\partial w_{P_n}}{\partial x_m} \rangle. \]

First let us examine the term (A). By (4.17)

\[(A) = \epsilon_n^{N-2} c_{h', m} \left( \left\| \frac{\partial w}{\partial x_1} \right\|^2 + o(1) \right) .\]
The term (B) can be estimated as

\[ |(B)| = \left| \int_{\mathbb{R}^3} h_n \frac{\partial w_{P_0}}{\partial x_m} \, dx \right| \leq \|h_n\|_{P_n} \int_{\mathbb{R}^3} |\nabla w_{P_0}| \, dx \leq c_n^{N-1} \|h_n\|_{P_n}. \]

Then, as regards the last term (C) we find

\[ |(C)| = \left| \int_{\mathbb{R}^N} L_{P_n}[v_n] \frac{\partial w_{P_0}}{\partial x_m} \, dx \right| \]

\[ = \left| \int_{\mathbb{R}^N} \left( c_n^2 \Delta v_n - v_n + f'(w_{P_n}) v_n - \phi [w_{P_n}^2] v_n - 2\phi [w_{P_n} v_n] w_{P_n} \right) \frac{\partial w_{P_0}}{\partial x_m} \, dx \right| \]

\[ \leq C \|v_n\|_{P_n} \int_{\mathbb{R}^N} \left| f'(w_{P_n}) - f'(w_{P_0}) \right| \frac{\partial w_{P_0}}{\partial x_m} \, dx + Cc_n^{N-1} \|v_n\|_{P_n} \]

\[ \leq Cc_n^{N-1} \|v_n\|_{P_n} \sum_{i \neq j} \int_{\mathbb{R}^N} w_{P_i}^2 w_{P_j} \, dx + Cc_n^{N-1} \|v_n\|_{P_n} \]

Putting together (A), (B) and (C) we get

\[ c_{n,j}^i = o(\epsilon) \quad \forall \ (i, j) \]

by which

\[ \|h_n + \sum_{i,j} c_{n,j}^i Z_{P_n, j}\|_{P_n} = o(1). \]

This implies (4.19)

\[ \|L_{P_n}[v_n]\|_{P_n} = o(1). \]

Fix \( R > 0 \). We claim

\[ \|v_n\|_{L^\infty \left( \cup_{i \in \mathbb{N}} B_{R_n}(P_n) \right)} = o(1). \]

Otherwise, we may assume that

\[ \|v_n\|_{L^\infty \left( \cup_{i \in \mathbb{N}} B_{R_n}(P_n) \right)} \geq c > 0 \]

for some \( R > 0 \). By multiplying the equation in (4.18) by \( v_n \) and integrating by parts we get that the sequence \( v_n(\epsilon_n x + P_n) \) is bounded in \( H^1(\mathbb{R}^N) \). Therefore, possibly passing to a subsequence, \( v_n(\epsilon_n x + P_n) \to v_0 \) weakly in \( H^1(\mathbb{R}^N) \) and a.e. in \( \mathbb{R}^N \) and \( v_0 \) satisfies

\[ \Delta v_0 - v_0 + f'(w)v_0 = 0, \quad |v_0(x)| \leq e^{-\mu |x|}. \]

According to elliptic regularity theory we may assume \( v_n(\epsilon_n x + P_n) \to v_0 \) uniformly on compact sets. Then \( \|v_0\|_{\infty} \geq c \). By the non-degeneracy of \( w \) (assumption (f2)) it follows

\[ v_0 = \sum_{j=1}^N \beta_j \frac{\partial w}{\partial x_j}. \]

On the other hand, for \( j \in \{1, \ldots, N\} \)

\[ 0 = \int_{\mathbb{R}^N} v_n(\epsilon_n x + P_1) Z_{P_1,m}(\epsilon_n x + P_n) \, dx \leq \left( v_n(\epsilon_n x + P_1), \frac{\partial w_{P_n}}{\partial x_m}(\epsilon_n x + P_n) \right) \epsilon \]

\[ \to \left( v_0, \frac{\partial w}{\partial x_m} \right) = a_m \left\| \frac{\partial w}{\partial x_m} \right\|^2. \]

from which it follows \( a_m = 0 \) and hence, in particular, \( v_0 = 0 \), a contradiction. Then the claim follows. We immediately obtain

\[ \|f'(w_{P_n})v_n\|_{P_n} = o(1) \]

and by (4.19)

\[ \|\epsilon_n^2 \Delta v_n - (1 + \phi [w_{P_n}^2])v_n\|_{P_n} = o(1). \]
By the Lemma 4.1 we get
\[ v_n(x) = o(1) e^{-\alpha \min_{i=1, \ldots, t} \frac{|x-P_n|}{n}} \]
which is a contradiction since \( \|v_n\|_{s,P} = 1 \).

**Lemma 4.3.** For \( \epsilon > 0 \) sufficiently small, for \( P \in \Gamma_\epsilon \) and \( h \in C(P) \) there exists a unique pair \((v, c_{i,j}) \in H^2 P \times \mathbb{R}^N\) solving \((4.18)\). Furthermore by Lemma \((4.2)\)
\[ \|v\|_{s,P} \leq C \|h\|_{s,P}. \]

**Proof.** The existence follows from the Fredholm alternative. To this aim, for every \( P \in \Gamma_\epsilon \), let us consider
\[ W := \left\{ v \in H^1(\mathbb{R}^N) : \left( v, \frac{\partial w_P}{\partial x_j} \right)_\epsilon = 0, i = 1, \ldots, \ell, j = 1, \ldots, N \right\}. \]
It is easy to see that \( W \) is a closed subset of \( H^1(\mathbb{R}^N) \). By (4.16) \( v \in W \) solves the equation in (4.18) if and only if
\[ \langle \mathcal{L}_P[v], z \rangle = \langle h, z \rangle \quad \forall \ z \in W. \]
Indeed, once we know \( v \), we can determine the unique \( c_{i,j} \) from the linear system of equations
\[ \langle \mathcal{L}_P[v], \frac{\partial w_P}{\partial x_j} \rangle = \langle h, \frac{\partial w_P}{\partial x_j} \rangle + \sum_{m,n} c_{m,n} \langle Z_{P,m,n}, \frac{\partial w_P}{\partial x_j} \rangle; \]
with \( j = 1, \ldots, N \) and \( i = 1, \ldots, \ell \).

The system (4.20) is equivalent to
\begin{equation}
\begin{aligned}
- \int_{\mathbb{R}^N} f'(w_P) \frac{\partial w_P}{\partial x_j} v \, dx &+ \int_{\mathbb{R}^N} \left( \phi[w_P^2] + 2 \phi[w_P] v \right) \frac{\partial w_P}{\partial x_j} v \, dx = \\
&= - \int_{\mathbb{R}^N} h \frac{\partial w_P}{\partial x_j} \, dx - \sum_{m,n} c_{m,n} \int_{\mathbb{R}^N} Z_{P,m,n} \frac{\partial w_P}{\partial x_j} \, dx
\end{aligned}
\end{equation}

According to (4.17), the coefficient matrix is nonsingular since it is dominated by its diagonal. By standard elliptic regularity, \( v \in L^\infty(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \). Furthermore, using the \( C^{1,\sigma} \) regularity of \( f \) and the exponential decay of \( w \)
\[ \|f'(w_P)v - 2\phi[w_P]w_P - h - \sum_{i,j} c_{i,j} Z_{P,i,j} \|_{s,P} < \infty \]
hence Lemma ?? implies \( \|w\|_{s,P} < \infty \), consequently \((v, c_{i,j}) \) solves (4.18).

Thus it remains to solve (4.20). According to Riesz’s representation theorem, take \( K_P(v), \bar{h} \in W \) such that
\[ (K_P(v), \psi) = - \langle f'(w_P)v, \psi \rangle + \langle \phi[w_P^2]v + 2\phi[w_P]w_P, \psi \rangle \quad \forall \ \psi \in W. \]
\[ (\bar{h}, \psi) = - \langle h, \psi \rangle \quad \forall \ \psi \in W. \]

Then the problem (4.20) consists in finding \( v \in W \) such that
\[ v + K_P(v) = \bar{h}. \]

It is easy to prove that \( K_P \) is a linear compact operator from \( W \) to \( W \).
Using Fredholm’s alternatives, (4.22) has a unique solution for each \( \bar{h} \), if and only if (4.22) has a unique solution for \( \bar{h} = 0 \). Let \( v \in W \) be a solution of \( v + K_P(v) = 0 \) then \( v \) solves the system (4.18) with \( h = 0 \) for some \( c_{i,j} \in \mathbb{R} \). Lemma 4.2 implies \( v \equiv 0 \). □
Let us consider the metric space \( \| \cdot \| \) that is (4.18) with (5.25)\]

Hence (5.23) can be written as \( (5.23) \)

\[
H \quad \text{with} \quad \bar{\Gamma} \in \mathbb{R}^{N^2}
\]

We note that \( P \in \bar{\Gamma} \), and the maps \( P, 1 \) AND 2

\[
(5.24) \quad \| P \|, \quad (5.24) \quad \| P \|
\]

\[
\| \psi \|^2 < \epsilon^2; \quad (\psi, P), \leq \epsilon^{N+2\tau}
\]

and the maps \( P \in \bar{\Gamma}, \mapsto \psi \), where \( P \in H^1(N^2) \) and \( P \mapsto \psi(P) \in \mathbb{R} \) are continuous.

Proof. We note that

\[
S_\epsilon[w P + \psi] = S_\epsilon[w P] + L_\epsilon[\psi] + R[\psi]
\]

with

\[
R[\psi] = [f(w P + \psi) - f(w P)\psi - f(w P)] - [\phi[\psi]^2]w P + \phi[\psi^2] \psi + 2\phi[w P \psi]\psi]
\]

Hence (5.23) can be written as

\[
L_\epsilon[\psi] = \sum_{i,j} c_{i,j} Z_{P,j} - S_\epsilon[w P] - R[\psi],
\]

(5.25) \quad \psi \in H^2_\epsilon, \quad (\psi, Z_{P,j}) = 0, \quad i = 1, \ldots, \ell, \quad j = 1, \ldots, N.

that is (4.18) with

\[
h = -S_\epsilon[w P] - R[\psi].
\]

Let us consider the metric space

\[
B = \{ \psi \in C(\mathbb{R}^N) : \| \psi \|^2 \leq \epsilon^2 \}
\]

equipped with the norm \( \| \cdot \|^2 \). For all \( \psi_1, \psi_2 \in B \)

\[
\| R[\psi_1] - R[\psi_2] \|^2 \leq C \epsilon^{\sigma^2} \| \psi_1 - \psi_2 \|^2
\]

and for all \( \psi \in B \)

\[
\| R[\psi] \|^2 \leq C \epsilon^{\sigma^2} \| \psi \|^2
\]

Moreover, by Lemma 2.2 we have that

\[
\| S_\epsilon[\psi] \|^2 \leq \epsilon^{\beta^2(\sigma^2 + \sigma)}
\]

Thus, for all \( \psi \in B \), let \( (A(\psi), c_{i,j}) \) the unique solution of (4.18) with

\[
h = -S[w P] - R[\psi]
\]

Then we claim that \( A \) maps \( B \) into \( B \) and \( A \) is a contraction. By lemma 4.3 the choice of \( \tau \)

\[
\| A(\psi) \|^2 \leq C - S_\epsilon[w P] - R[\psi] \leq C \left( \epsilon^{\beta^2(\sigma^2 + \sigma)} + \epsilon^{(1+\sigma)^2} \right) \leq C \epsilon^{\tau}
\]

for \( \epsilon \) sufficiently small and so \( A(\psi) \in B \). Moreover \( A(\psi_1) - A(\psi_2) \) solves (4.18) with \( h = -R[\psi_1] + R[\psi_2] \). Then by Lemma 4.3

\[
\| A(\psi_1) - A(\psi_2) \|^2 \leq C \| R[\psi_1] - R[\psi_2] \|^2 \leq \epsilon^2 \| \psi_1 - \psi_2 \|^2
\]
and so, for $\epsilon$ small, $\mathcal{A}$ is a contraction. Thus, by applying the contraction mapping theorem we conclude. It remains to prove the $H^1$ norm estimate of $\psi$. By multiplying (5.25) by $\psi_P$ and integrating by parts we obtain

$$
(\psi_P, \psi_P)_\epsilon = \int_{\mathbb{R}^N} f'(w_P)\psi_P^2 dx - \int_{\mathbb{R}^N} \phi[w_P^2] \psi_P^2 dx - 2 \int_{\mathbb{R}^N} \phi[w_P^2] w_P \psi_P w_P \psi_P dx + (S[w_P], w_P) + (R[w_P], \psi_P) + (\psi_P, \psi_P)_\epsilon
$$

(5.28)

By using the fact that $\psi_P \in \mathcal{B}$, the estimates (5.27) and (5.28) and by making a change of variable we immediately get

$$
(\psi_P, \psi_P)_\epsilon \leq C\epsilon^{N+2}. \tag{5.29}
$$

Then the family $\{\psi_P : P \in \Gamma_\epsilon\}$ is bounded in $H^1$. Now fix $\epsilon > 0$ and consider $\{P_n\} \subset \Gamma_\epsilon$ such that $P_n \to \bar{P} \in \Gamma_\epsilon$. Up to a subsequence, $\psi_{P_n} \rightharpoonup \bar{\psi}$ weakly in $H^1$; on the other hand, choosing $(m, q)$ such that, up to a subsequence $|c_{m,q}(P_n)| \geq |c_{i,j}(P_n)|$ for every $(i, j)$ and $n$, by using (4.17) we have

$$
\left(\psi_{P_n}, \frac{\partial w_{P_n}}{\partial x_q}\right)_\epsilon = \int_{\mathbb{R}^N} f'(w_{P_n})\psi_{P_n} \frac{\partial w_{P_n}}{\partial x_q} dx - \int_{\mathbb{R}^N} \phi[w_{P_n}^2] \psi_{P_n} \frac{\partial w_{P_n}}{\partial x_q} dx - 2 \int_{\mathbb{R}^N} \phi[w_{P_n}^2] w_{P_n} \frac{\partial w_{P_n}}{\partial x_q} dx + (S[w_{P_n}], w_{P_n}) + (R[w_{P_n}], w_{P_n}) \left(\epsilon^{N-2} \left\|\frac{\partial \psi}{\partial x_1}\right\|^2 + o(\epsilon^{N-2})\right)
$$

by which we deduce that the sequence $\{c_{i,j}(P_n)\}$ is bounded too for every $(i, j)$. Assume, without loss of generality $c_{i,j}(P_n) \to \bar{c}_{i,j}$. Then $P, \bar{c}_{i,j}$ solves the equation

$$
\mathcal{L}_{P}(\bar{\psi}) = -S_{\epsilon}[w_{P}] - R[\bar{\psi}] + \sum_{i,j} \bar{c}_{i,j} Z_{P_{i,j}}, \quad (\bar{\psi}, Z_{P_{i,j}}) = 0, \quad \|\bar{\psi}\|_{H^1} \leq \epsilon^\gamma.
$$

Hence, from uniqueness, it follows $\bar{\psi} = \psi_P$ and $\bar{c}_{i,j} = c_{i,j}(P)$. By (5.28) we get

$$
\|\psi_{P_n}\|^2 = \int_{\mathbb{R}^N} f'(w_P)\psi_P^2 dx - \int_{\mathbb{R}^N} \phi[w_P^2] \psi_P^2 dx - 2 \int_{\mathbb{R}^N} \phi[w_P^2] w_P \psi_P w_P \psi_P dx + (S[w_P], \psi_P) + (R[\psi_P], \psi_P) = \|\bar{\psi}\|^2,
$$

hence we deduce $\psi_{P_n} \rightharpoonup \psi_P$ in $H^1$. \hfill $\square$

**Lemma 5.2.** For $\epsilon > 0$ sufficiently small the map $P \in \Gamma_\epsilon \mapsto \psi_P \in H^1$ constructed in Lemma 5.1 is $C^1$.

**Proof.** To prove that the map $P \in \Gamma_\epsilon \mapsto \psi_P \in H^1$ is $C^1$ consider the following map $T : \Gamma_\epsilon \times H^1(\mathbb{R}^N) \times \mathbb{R}^N \to H^1(\mathbb{R}^N) \times \mathbb{R}^N$:

$$
T(P, \psi_P, c_{i,j}) = \begin{pmatrix}
(\epsilon^2 \Delta - 1)^{-1} \left(S_{\epsilon}[w_P + \psi_P] + \sum_{i,j} c_{i,j} \frac{\partial w_{P_i}}{\partial x_j}\right) \\
(\psi_P, \frac{\partial w_{P_i}}{\partial x_j})
\end{pmatrix}
$$

(5.29)

where $v = (\epsilon^2 \Delta - 1)^{-1}(h)$ is defined as the unique solution $u \in H^1$ of $\epsilon^2 \Delta u - u = h$. Since $-\epsilon^2 \Delta \frac{\partial w_{P_i}}{\partial x_j} = \frac{\partial w_{P_i}}{\partial x_j}$ it is immediate that $(\psi, c_{i,j})$ solves the system (5.23) if and only if $T(P, \psi_P, c_{i,j}) = 0$. The thesis will follow by applying the Implicit Function Theorem (see [12]). \hfill $\square$
6. Reduced energy functional

For \( \varepsilon > 0 \) sufficiently small we define the reduced functional
\[
M_\varepsilon : \Gamma_\varepsilon \to \mathbb{R}, \quad M_\varepsilon[P] := \varepsilon^{-N} J_\varepsilon[w_P + \psi_P] - \ell I[w] - \varepsilon^2 C_1,
\]
where \( \psi_P \) has been constructed in Lemma 5.1 and \( C_1 \) is given by Proposition 3.1.

Next proposition contains the key expansion of \( M_\varepsilon \)

**Proposition 6.1.** For \( \varepsilon > 0 \) sufficiently small the following holds:
\[
M_\varepsilon[P] = -(\gamma_0 + o(1)) \sum_{i \neq j} \lambda_i \lambda_j \left( \frac{P_i - P_j}{\varepsilon} \right) + \varepsilon^2 (C_2 + o(1)) \sum_{i \neq j} \frac{1}{|P_i - P_j|^{N-2}} + O(\varepsilon^{2\tau}),
\]
uniformly for \( P \in \Gamma_\varepsilon \), where \( \tau = \beta^4(1 + \sigma) \) is given by Lemma 5.1 and \( C_2 \) are the constants in Proposition 3.1.

**Proof.** An easy computation gives
\[
J_\varepsilon(w_P + \psi_P) = J_\varepsilon(w_P) - \int_{\mathbb{R}^N} S_\varepsilon[w_P] \psi_P dx + \frac{1}{2}(w_P, \psi_P) - \int_{\mathbb{R}^N} (F(w_P + \psi_P) - F(w_P) - f(w_P) \psi_P) dx
\]
\[+ \int_{\mathbb{R}^N} \psi_P^2 \left( \frac{1}{4} \phi(\psi_P^2) + \frac{1}{2} \phi(w_P^2)(x) + \phi(w_P \psi_P) \right) dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w_P(x) \psi_P(x) w_P(y) \psi_P(y)}{|x - y|^{N-2}} dx dy
\]
\[+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w_P^2(x) w_P^2(y)}{|x - y|^{N-2}} dx dy.
\]

By Lemma 2.2 we have
\[
\left\| \int_{\mathbb{R}^N} S_\varepsilon[w_P] \psi_P dx \right\| \leq C_\varepsilon^{\beta^2(\beta^2 + \sigma)} \|\psi_P\|_{\infty} \sum_{i=1}^\ell \int_{\mathbb{R}^N} w_{P_i}^{1-\beta^2} dx \leq C_\varepsilon^\tau \|\psi_P\|_{\infty} \sum_{i=1}^\ell \int_{\mathbb{R}^N} w_{P_i}^{1-\beta^2} dx = \bar{C}_\varepsilon^{N+\tau} \|\psi_P\|_{\infty}.
\]

Moreover, since \( |F(w_P + \psi_P) - F(w_P) - f(w_P) \psi_P| \leq C|\psi_P|^2 \), one can estimate
\[
\left| \int_{\mathbb{R}^N} (F(w_P + \psi_P) - F(w_P) - f(w_P) \psi_P) dx \right| \leq C \|\psi_P\|^2.
\]

It’s also easy to see that
\[
\int_{\mathbb{R}^N} \psi_P^2 \left( \frac{1}{4} \phi(\psi_P^2) + \frac{1}{2} \phi(w_P^2)(x) + \phi(w_P \psi_P) \right) dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w_P(x) \psi_P(x) w_P(y) \psi_P(y)}{|x - y|^{N-2}} dx dy \leq C \|\psi_P\|^2.
\]

Last, similarly as in the proof of (3.8) (using now (3.10) instead of (3.9)), one has
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w_P^2(x) w_P^2(y)}{|x - y|^{N-2}} dx dy \leq \|\psi_P\|_{\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{w_P^2(x) w_P^2(y)}{|x - y|^{N-2}} dx dy \leq C_2^{N} \|\psi_P\|_{\infty} \sum_{i \neq j} \frac{1}{|P_i - P_j|^{N-2}}
\]
\[\leq C_2^{N} \|\psi_P\|_{\infty} \sum_{i \neq j} \log \left( \frac{1}{|P_i - P_j|^{N-2}} \right) \leq C_2^{N} \|\psi_P\|_{\infty}.
\]

Hence by Lemma 5.1 (observe that by our assumptions \( 2 + \tau > 2\tau \)) one obtains
\[
J_\varepsilon(w_P + \psi_P) = J_\varepsilon(w_P) + O(\varepsilon^{N+2\tau}),
\]
and the thesis easily follows from Proposition 3.1. \( \square \)

We recall the following result (whose proof can be found for instance in [12]) that will be useful in the next sections in order to find a critical point (a maximum) of \( M_\varepsilon \) under symmetry assumptions.
Lemma 6.2. Fix a positive constant $C > 0$ and consider the function
\[ \alpha_{\epsilon,C}(\rho) := -\gamma_0 \omega(\rho) + C \epsilon^2 \frac{1}{\rho^{N-2}}, \quad \rho \geq \beta^2 \log \frac{1}{\epsilon^2}, \]
where $\gamma_0$ is the positive constants introduced in Proposition 3.1. Then for $\epsilon > 0$ small enough, $\alpha_{\epsilon,C}$ has a unique maximum point $\rho_\epsilon$. Moreover we have
\[ \rho_\epsilon = \log \frac{1}{\epsilon^2} + \frac{N-1}{2} \log \log \frac{1}{\epsilon^2} + o \left( \log \log \frac{1}{\epsilon^2} \right) \]
and
\[ \alpha_{\epsilon,C}(\rho_\epsilon) = C \epsilon^2 \left( \frac{1}{\log \frac{1}{\epsilon^2}} \right)^{N-2}(1 + o(1)). \]

7. Proof of Theorem 1.2

For every $x \in \mathbb{R}^N$ we set $x = (x_1, \ldots, x_N) = (x_1, x_2, x') = (z, x')$, where $z \in \mathbb{C}$. In this section we prove the existence of a cluster solution to (1.4) having a positive bump in 0 and $k$ negative bumps at the vertices of a regular polygon centered in 0. Precisely we make the following ansatz
\[ v = u_0 - \sum_{i=1}^k w_{P_i} + \psi_\epsilon \]
where $P_i := rQ_i \in \mathbb{R}^N$, $Q_i := \left( e^{2\pi i/(i-1)/k}, 0 \right) = \left( \cos \frac{2\pi i}{k}, \sin \frac{2\pi i}{k}, 0 \right) \in \mathbb{R}^N$, $i = 1, \ldots, k$, and $r > 0$ is such that $P_r := (0, P_1, \ldots, P_k) = (0, rQ_1, \ldots, rQ_k) \in \Gamma_r(\subset \mathbb{R}^{(k+1)N})$. Moreover we look for a solution $v$ satisfying the following symmetry property
\[ v(x_1, x_2, x') = v(z, |x'|) = v(z e^{2\pi i/k}, |x'|). \]
This translates into restricting to work into the following Sobolev space of symmetric functions
\[ X := \{ v \in H^1(\mathbb{R}^N) : v(x_1, x_2, x') = v(z, |x'|) = v(z e^{2\pi i/k}, |x'|) \}. \]
Hence, for every $r > 0$ such that $P_r \in \Gamma_\epsilon$, we set
\[ H^2_{\epsilon, r, s} := H^2_{\epsilon, P_r} \cap X, \quad C_{\epsilon, r, s} := C_{\epsilon, P_r} \cap X, \]
and proceeding as in Sects. 4 and 5 we find for $\epsilon$ small enough a unique solution $(\psi_{P_r}, \epsilon_{ij}(P_r)) \in H^2_{\epsilon, r, s} \times \mathbb{R}^N$ to problem (5.23) (see Lemma 5.1).

To conclude the proof it is sufficient to find for $\epsilon$ small, a critical point $P_r$ of the reduced functional $M_\epsilon$ introduced in Section 6. Indeed one can prove the following (see for instance [12])

Lemma 7.1. Let $P_r \in \Gamma_\epsilon$ be an interior maximum point for $M_\epsilon$. Then provided that $\epsilon > 0$ is small enough, the corresponding function $v_r := u_{P_r} + \psi_{P_r} \in X$ is a critical point of $J_\epsilon$ in $X$.

Hence, if we denote by $G$ the group of the rotation matrix in $\mathbb{R}^{N-2}$, and for every $i \in \mathbb{N}$ and $g \in G$ we define
\[ T_{i,g} : \mathbb{R}^N \to \mathbb{R}, \quad T_{i,g}(y) = T_{i,g}(z, y') = (ze^{2\pi i V/k}, gy'), \]
then, by Lemma 2.1 the functional $J_\epsilon$ is invariant under the action of the group $\{ T_{i,g} : i \in \mathbb{N}, g \in G \}$, namely $J_\epsilon(u(T_{i,g}(x))) = J_\epsilon(u(x))$. Moreover $X = \{ u \in H^1(\mathbb{R}^N) : u(T_{i,g}(x)) = u(x) \}$. So the principle of symmetric criticality ensures that $v_r$ in Lemma 7.1 is also a critical point of $J_\epsilon$ and, consequently, a solution of (1.4).

The remaining part of the section is then devoted to find an interior maximum point of the reduced functional. Let us observe that, under our ansatz, $M_\epsilon$ is a one-variable function and, thanks to Proposition 6.1 and to the assumption $k \geq 7$, it reduces to the following
\textbf{Proposition 7.2.} For $\epsilon > 0$ sufficiently small

\begin{equation}
M_{\epsilon}[r] = (2k + o(1))\alpha_{\epsilon,C_k} \left( \frac{r}{\epsilon} \beta_k \right) + O(\epsilon^{2r})
\end{equation}

uniformly for $r > 0$ such that $P_r \in \Gamma_{\epsilon}$, where $\beta_k := 2\sin \frac{\pi}{k}$, $C_k$ is a positive constant and $\alpha_{\epsilon,C_k}$ is the function defined in Lemma \[6.2\].

\textbf{Proof.} For $r$ such that $P_r \in \Gamma_{\epsilon}$ the reduced functional becomes

\begin{align*}
M_{\epsilon}[r] &= -(\gamma_0 + o(1)) \left( -2k w \left( \frac{r}{\epsilon} \right) + \sum_{i \neq j} w \left( \frac{r}{\epsilon} |Q_i - Q_j| \right) \right) \\
&+ \epsilon^2 C_2 + o(1) \left( \epsilon^2 \left( 2k + \sum_{i \neq j} \frac{1}{|Q_i - Q_j|^2} \right) + O(\epsilon^{2r}) \right) \\
&= -k(\gamma_0 + o(1)) \left( -2w \left( \frac{r}{\epsilon} \right) + \sum_{i=2}^k w \left( \frac{r}{\epsilon} |Q_i - Q_i| \right) \right) \\
&+ \epsilon^2 k(C_2 + o(1)) \left( \epsilon^2 \left( 2 + \sum_{i=2}^k \frac{1}{|Q_i - Q_i|^2} \right) + O(\epsilon^{2r}) \right) \\
&= -k(\gamma_0 + o(1)) \left( -2w \left( \frac{r}{\epsilon} \right) + 2w \left( \frac{r}{\epsilon} \beta_k \right) + \sum_{i=3}^{k-1} w \left( \frac{r}{\epsilon} \beta_k \right) \right) \\
&+ \epsilon^2 k(C_2 + o(1)) \left( \epsilon^2 \left( 2 + \frac{1}{\beta_k^2} + \sum_{i=3}^{k-1} \frac{1}{(\beta_i)^2} \right) + O(\epsilon^{2r}) \right).
\end{align*}

where we set

\[ \beta_k := |Q_2 - Q_1| = |Q_k - Q_1| = 2\sin \frac{\pi}{k} \]

and

\[ \beta_k^i := |Q_i - Q_1| = \sqrt{2} \sqrt{1 - \cos \frac{2\pi(i - 1)}{k}}, \quad i = 3, \ldots, k - 1. \]

Observe that by our choice

\[ \beta_k < \beta_k^i, \quad i = 3, \ldots, k - 1 \]

hence, from \[1.2\], it follows that

\[ w \left( \frac{r}{\epsilon} \beta_k^i \right) = o \left( w \left( \frac{r}{\epsilon} \beta_k \right) \right), \quad i = 3, \ldots, k - 1, \quad \text{as} \quad \frac{r}{\epsilon} \to +\infty. \]

Moreover $\beta_k < 1$ because $k \geq 7$, hence we also have

\[ w \left( \frac{r}{\epsilon} \right) = o \left( w \left( \frac{r}{\epsilon} \beta_k \right) \right), \quad \text{as} \quad \frac{r}{\epsilon} \to +\infty. \]

As a consequence the reduced functional becomes

\begin{equation*}
M_{\epsilon}[r] = -(2k\gamma_0 + o(1))w \left( \frac{r}{\epsilon} \beta_k \right) + \epsilon^2 \left( 2kC_k + o(1) \right) \frac{1}{|\epsilon \beta_k|^2} + O(\epsilon^{2r}),
\end{equation*}

where $C_k := C_2 \left( 1 + \beta_k^{N-2} + \frac{1}{2} \sum_{i=3}^{k-1} \frac{1}{(\beta_i)^2} \right)$.

\[ \square \]

Observe now that $P_r \in \Gamma_{\epsilon}$ if and only if

\[ \begin{cases} 
\beta_k^i \epsilon \log \frac{1}{\epsilon} < r < \epsilon \left( \log \frac{1}{\epsilon} \right)^2, \\
\beta_k^j \epsilon \log \frac{1}{\epsilon} < r |Q_i - Q_j| < \epsilon \left( \log \frac{1}{\epsilon} \right)^2, 
\end{cases} \quad i \neq j, \]
hence, since $\beta_k < 1$ and $\beta_k \leq |Q_i - Q_j| \leq 2$ for $i \neq j$, it is easy to show that $P_r \in \Gamma_\epsilon$ for $r \in R_\epsilon$ where

$$
R_\epsilon := \left\{ r > 0 : \beta_k \log \frac{1}{\epsilon^2} < r < \frac{1}{2} \left( \log \frac{1}{\epsilon^2} \right)^2 \right\}.
$$

Finally next result gives an interior maximum point $r$ for $M_\epsilon$

**Proposition 7.3.** For $\epsilon > 0$ sufficiently small, the following maximization problem

$$
\max\{ M_\epsilon[r] : r \in \bar{R}_\epsilon \}
$$

has a solution $r_\epsilon \in R_\epsilon$. Furthermore

$$
\lim_{\epsilon \to 0} \frac{r_\epsilon \beta_k}{\epsilon \log \frac{1}{\epsilon^2}} = 1.
$$

**Proof.** Since $M_\epsilon$ is continuous in $r$, there exists $r_\epsilon \in \bar{R}_\epsilon$ such that $M_\epsilon[r_\epsilon] = \max_{r \in \bar{R}_\epsilon} M_\epsilon[r]$.

We claim that $r_\epsilon \in R_\epsilon$. We prove this by energy comparison. We first obtain a lower bound for $M_\epsilon[r_\epsilon]$. Let us choose $s_\epsilon := \frac{\rho_\epsilon}{\beta_k}$, where $\rho_\epsilon > 0$ is given in Lemma 6.2. It is easy to see that $s_\epsilon$ belongs to $R_\epsilon$. Indeed, by Lemma 6.2 $\beta_k > \beta^2 \log \frac{1}{\epsilon^2}$ and, for $\epsilon$ small, $\rho_\epsilon < \frac{1}{2} \beta_k \log \frac{1}{\epsilon^2}$. Then by using again Lemma 6.2 and (7.30)

$$
(7.31) \quad M_\epsilon[r_\epsilon] \geq M_\epsilon[s_\epsilon] = (2k + o(1))\alpha_c \epsilon \log \frac{1}{\epsilon^2} + O(\epsilon^{2\gamma}) = c^2(2kC_k + o(1)) \frac{1}{\left( \log \frac{1}{\epsilon^2} \right)^{N-2}}.
$$

We are going to prove that $\frac{r_\epsilon \beta_k}{\epsilon \log \frac{1}{\epsilon^2}} \to 1$ as $\epsilon \to 0$. By contradiction assume that there exists a sequence $\epsilon_n \to 0$ such that $\frac{r_{\epsilon_n} \beta_k}{\epsilon_n \log \frac{1}{\epsilon_n^2}} > 1 + c$. Using once more (7.30)

$$
M_{\epsilon_n}[r_{\epsilon_n}] \leq c^2_n(2kC_k + o(1)) \frac{1}{\left( \frac{r_{\epsilon_n} \beta_k}{\epsilon_n \log \frac{1}{\epsilon_n^2}} \right)^{N-2}} + O(\epsilon_n^{2\gamma}) \leq c^2_n(2kC_k + o(1)) \frac{1}{(1+c)^{N-2}} \frac{1}{\left( \log \frac{1}{\epsilon_n^2} \right)^{N-2}}
$$

which contradicts the (7.31). Now assume the existence of a sequence $\epsilon_n \to 0$ such that $\frac{r_{\epsilon_n} \beta_k}{\epsilon_n \log \frac{1}{\epsilon_n^2}} < 1 - c$. Then by the decay of $w$ (observe that the function $x \mapsto \frac{e^x}{x^2}$ is nondecreasing for $x$ large)

$$
M_{\epsilon_n}[r_{\epsilon_n}] = (2k + o(1)) \left( -\gamma_0 w \left( \frac{r_{\epsilon_n} \beta_k}{\epsilon_n \log \frac{1}{\epsilon_n^2}} \right) + C_k \epsilon_n^2 \frac{1}{\left( \frac{r_{\epsilon_n} \beta_k}{\epsilon_n \log \frac{1}{\epsilon_n^2}} \right)^{N-2}} \right) + O(\epsilon_n^{2\gamma})
$$

$$
= (2k + o(1)) \left( -\gamma_0 A_N e^{-\frac{r_{\epsilon_n} \beta_k}{\epsilon_n \log \frac{1}{\epsilon_n^2}}} + C_k \epsilon_n^2 \frac{1}{\left( \frac{r_{\epsilon_n} \beta_k}{\epsilon_n \log \frac{1}{\epsilon_n^2}} \right)^{N-2}} \right) + O(\epsilon_n^{2\gamma})
$$

$$
\leq (2k + o(1)) \left( -\gamma_0 A_N e^{-\frac{r_{\epsilon_n} \beta_k}{\epsilon_n \log \frac{1}{\epsilon_n^2}}} + C_k \epsilon_n^2 \frac{1}{\left( \frac{r_{\epsilon_n} \beta_k}{\epsilon_n \log \frac{1}{\epsilon_n^2}} \right)^{N-2}} \right) + O(\epsilon_n^{2\gamma})
$$

$$
= c_n^2(2kC_k + o(1)) \left( -\gamma_0 A_N e^{-\frac{r_{\epsilon_n} \beta_k}{\epsilon_n \log \frac{1}{\epsilon_n^2}}} + \frac{1}{\left( \frac{r_{\epsilon_n} \beta_k}{\epsilon_n \log \frac{1}{\epsilon_n^2}} \right)^{N-2}} \right) + O(\epsilon_n^{2\gamma})
$$

$$
\leq c_n^2(2kC_k + o(1)) \frac{1}{\left( \log \frac{1}{\epsilon_n^2} \right)^{N-2}} \left( -\gamma_0 A_N \frac{e^{-2c}}{C_k \left( 1 - \epsilon_n \log \frac{1}{\epsilon_n^2} \right)^{N-2}} + \frac{1}{\left( \beta_k \right)^{N-2}} \right)
$$
which is in contradiction with (7.31) since \(\frac{3nA_N}{c_\epsilon} - \frac{c_\epsilon^{-2r}}{(|z|)} + \frac{1}{|z|} \to 0\) for \(n\) big. \(\square\)

8. Proof of Theorem 1.3

For every \(x \in \mathbb{R}^N\) and \(h \in [2, N]\) we set \(x = (x_1, \ldots, x_N) = (z, x')\), where \(z = (x_1, \ldots, x_h) \in \mathbb{R}^h\) and \(x' = (x_{h+1}, \ldots, x_N) \in \mathbb{R}^{N-h}\).

In this section we prove the existence of a cluster solution to (1.4) having a positive bump in \(0\) and \(k\) negative bumps at the vertices of a regular polytope \(P\) centered in \(0\). The proof is similar to the one of Theorem 1.2 (which is actually a special case of Theorem 1.3 with \(h = 2\)). For this reason we will only sketch it briefly, emphasizing the main differences.

Let \(Q_i := (z_i, 0)\), \(i = 1, \ldots, k\) be the vertices of a convex regular polytope \(P\) in \(\mathbb{R}^h\) centered in \(0\) and having radius 1 and side \(s\), we make now the following ansatz

\[
v = w_0 - \sum_{i=1}^{k} w_{P_i} + \psi_i,
\]

where \(P_i := rQ_i \in \mathbb{R}^N\), and \(r > 0\) is such that \(P_r := (0, P_1, \ldots, P_k) \in \Gamma_\epsilon(\subset \mathbb{R}^{(k+1)N})\).

The Sobolev space of symmetric functions in which to work is now

\[
X := \{v \in H^1(\mathbb{R}^N) : v(z, x') = v(z, |x'|) = v(gz, |x'|) \quad \forall g \in \mathcal{G}\},
\]

where \(\mathcal{G}\) is the Coxeter group of \(\mathbb{R}^h\) associated to \(P\), namely the symmetry group that leaves \(P\) invariant. Observe that \(X = \{u \in H^1(\mathbb{R}^N) : u(T_{g,h}(x)) = u(x), \quad \forall g \in \mathcal{G}, h \in \mathcal{R}\}\), where

\[
T_{g,h} : \mathbb{R}^N \to \mathbb{R}, \quad T_{g,h}(y) = T_{g,h}(z, y') = (gz, hy'),
\]

and \(\mathcal{R}\) is the group of the rotation matrix in \(\mathbb{R}^{N-h}\). Moreover, by Lemma 2.1 the functional \(J_\epsilon\) is invariant under the action of the group \(\{T_{g,h} : g \in \mathcal{G}, h \in \mathcal{R}\}\), namely \(J_\epsilon(u(T_{g,h}(x))) = J_\epsilon(u(x))\).

The reduced functional is again a one-variable function and it reduces to the following

**Proposition 8.1.** For \(\epsilon > 0\) sufficiently small, if \(s < 1\)

\[
(8.32) \quad M_\epsilon[r] = (qk + o(1))\alpha_\epsilon,_{C_P} \left(\frac{r}{\epsilon} s\right) + O(\epsilon^{2r}),
\]

if \(s = 1\) and \(q \neq 2\),

\[
(8.33) \quad M_\epsilon[r] = ((q - 2)k + o(1))\alpha_\epsilon,_{C_P} \left(\frac{1}{\epsilon} s\right) + O(\epsilon^{2r}),
\]

uniformly for \(r > 0\) such that \(P_r \in \Gamma_\epsilon\), where \(q\) denotes the number of vertices \(Q_i\) which are one side away from \(Q_1\), \(C_\mathcal{P}, C'_\mathcal{P}\) are positive constants and \(\alpha_\epsilon,_{C_P}\) is the function defined in Lemma 6.2.

**Remark 8.2.** Under the assumptions of Theorem 1.3 \(q \neq 2\) when \(s = 1\), indeed \(q \geq h > 2\).
Proof. For \( r \) such that \( P_r \in \Gamma_\epsilon \) the reduced functional becomes

\[
M_\epsilon[r] = -(\gamma_0 + o(1)) \left( -2kw \left( \frac{r}{\epsilon} \right) + \sum_{i \neq j} w \left( \frac{r}{\epsilon} |Q_i - Q_j| \right) \right) + \epsilon^2(C_2 + o(1)) \frac{1}{|r|^{N-2}} \left( 2k + \sum_{i \neq j} |Q_i - Q_j|^{N-2} \right) + O(\epsilon^2 r)
\]

\[
= -k(\gamma_0 + o(1)) \left( -2w \left( \frac{r}{\epsilon} \right) + \sum_{i \neq j} w \left( \frac{r}{\epsilon} |Q_i - Q_j| \right) \right) + \epsilon^2k(C_2 + o(1)) \frac{1}{|r|^{N-2}} \left( 2 + \sum_{i \neq j} |Q_i - Q_j|^{N-2} \right) + O(\epsilon^2 r)
\]

where we set

\[
s_i := |Q_i - Q_1|.
\]

From the exponential decay of \( w \) it follows that for \( s_i > s \)

\[
w \left( \frac{r}{\epsilon} s_i \right) = o \left( w \left( \frac{r}{\epsilon} s \right) \right), \quad \text{as} \quad \frac{r}{\epsilon} \to +\infty,
\]

hence

\[
M_\epsilon[r] = -k(\gamma_0 + o(1)) \left( -2w \left( \frac{r}{\epsilon} \right) + qw \left( \frac{r}{\epsilon} s \right) \right) + \epsilon^2k(C_2 + o(1)) \frac{1}{|r|^{N-2}} \left( 2 + q \frac{1}{s^{N-2}} + \sum_{i \neq j} \frac{1}{s_i^{N-2}} \right) + O(\epsilon^2 r).
\]

If \( s < 1 \) we have

\[
w \left( \frac{r}{\epsilon} \right) = o \left( w \left( \frac{r}{\epsilon} s \right) \right), \quad \text{as} \quad \frac{r}{\epsilon} \to +\infty.
\]

As a consequence the reduced functional becomes

\[
M_\epsilon[r] = -(qk\gamma_0 + o(1))w \left( \frac{r}{\epsilon} s \right) + \epsilon^2 (qkC_P + o(1)) \frac{1}{|r|^{N-2}} \left( 2 + q \frac{1}{s^{N-2}} + \sum_{i \neq j} \frac{1}{s_i^{N-2}} \right) + O(\epsilon^2 r),
\]

where \( C_P := C_2 \left( 1 + \frac{3}{4} s^{N-2} + \frac{1}{4} \sum_{i \neq s} \frac{s^{N-2}}{s_i^{N-2}} \right) \).

While if \( s = 1 \) then

\[
M_\epsilon[r] = -((q - 2)k\gamma_0 + o(1))w \left( \frac{r}{\epsilon} \right) + \epsilon^2 (qkC_P + o(1)) \frac{1}{|r|^{N-2}} + O(\epsilon^2 r).
\]

\( \square \)

In order to find an interior maximum \( r \) for \( M_\epsilon \), observe now that \( P_r \in \Gamma_\epsilon \) if and only if

\[
\begin{align*}
\{ & s^2 \epsilon \log \frac{r}{\epsilon} < r < \epsilon (\log \frac{1}{\epsilon})^2 \\
& s^2 \epsilon \log \frac{1}{\epsilon} < r |Q_i - Q_j| < \epsilon (\log \frac{1}{\epsilon})^2 \quad i \neq j.
\end{align*}
\]
Hence, since by construction \( s \leq |Q_i - Q_j| \leq 2 \) for \( i \neq j \), and also by assumption \( s \leq 1 \), it’s easy to show that \( P_r \in \Gamma_\epsilon \), where
\[
R_\epsilon := \left\{ r > 0 : \kappa \log \frac{1}{r^2} < r < \frac{1}{2} \left( \log \frac{1}{\epsilon^2} \right)^2 \right\}.
\]
The following result (which can be proved similarly as Proposition 7.3) concludes the proof

**Proposition 8.3.** Assume \( s \leq 1 \), \( h > 2 \) or \( s < 1 \) \( h = 2 \). For \( \epsilon > 0 \) sufficiently small, the following maximization problem
\[
\max \{ M_r[r] : r \in R_\epsilon \}
\]
has a solution \( r_\epsilon \in R_\epsilon \). Furthermore
\[
\lim_{\epsilon \to 0} \frac{r_\epsilon s}{\epsilon \log \frac{1}{r_\epsilon}} = 1.
\]

### 9. Proof of Theorem 1.4

For every \( x \in \mathbb{R}^N \) we set \( x = (x_1, \ldots, x_N) = (x_1, x_2, x') = (z, x') \), where \( z \in \mathbb{C} \).

Fix \( k, m \in \mathbb{N} \) such that
\[
(9.34) \quad k > \min \left\{ 6, \frac{1}{m} \arcsin 1/4 \right\}
\]

In this section we construct a cluster solution to (1.4) having a positive bump centered at 0, \( k \) negative bumps and \( h := mk \) positive bumps which are respectively on the vertices of two suitable nested regular \( 2q \)-polygons centered in 0.

Again the proof is similar to the ones of Theorems 1.2 and 1.3 hence we briefly sketch it, emphasizing only the main differences.

Let \( Q_i := (e^{2\pi i (i-1)/km}, 0) = (\cos \frac{2\pi (i-1)}{km}, \sin \frac{2\pi (i-1)}{km}, 0) \in \mathbb{R}^N, i = 1, \ldots, km \), we make now the following ansatz
\[
v_\epsilon = wP_0 - \sum_{i=1}^{k} wP_i + \sum_{i=1}^{km} wP_{k+i} + \psi_\epsilon,
\]
where
\[
\mathbb{R}^N \ni P_j := \begin{cases} 0 & j = 0 \\ rQ_{(j-1)m+1} & j = 1, \ldots, k \\ 2rQ_{j-k} & j = k + 1, \ldots, k + km \end{cases}
\]
and \( r > 0 \) is such that \( P_r := (P_0, \ldots, P_{k+km}) \in \Gamma_\epsilon (\subset \mathbb{R}^{(k+km+1)N}) \).

The Sobolev space of symmetric functions in which to work is now
\[
X := \{ v \in H^1(\mathbb{R}^N) : v(x_1, x_2, x') = v(z, |x'|) = v(z e^{2\pi i x/k}, |x'|) \}.
\]
Observe that \( X = \{ u \in H^1(\mathbb{R}^N) : u(T_{i,g}(x)) = u(x), i \in \mathbb{Z}, g \in G \} \), where \( T_{i,g} : \mathbb{R}^N \to \mathbb{R}, T_{i,g}(y) := T_{i,g}(z, y') = (ze^{2\pi i y'/k}, gy') \), and \( G \) is the group of the rotation matrix in \( \mathbb{R}^{N-2} \). Moreover, by Lemma 2.1, the functional \( J_\epsilon \) is invariant under the action of the group \( \{ T_{i,g} : i \in \mathbb{N}, g \in G \} \), namely \( J_\epsilon (u(T_{i,g}(x))) = J_\epsilon (u(x)) \).

The reduced functional is again a one-variable function and, by condition (9.34), we prove that it reduces to the following.
Proposition 9.1. For \( \epsilon > 0 \) sufficiently small
\[
M_\epsilon[r] = (C(k, m) + o(1)) \alpha_{\epsilon, \tilde{C}} \left( \frac{r}{\epsilon} \right) + O(\epsilon^{2r})
\]
uniformly for \( r > 0 \) such that \( P_r \in \Gamma_\epsilon \), where \( s := \min \left\{ 2 \sin \frac{\pi}{k}, 4 \sin \frac{\pi}{mk} \right\} \),
\[
C(k, m) := \begin{cases} 
2k & \text{if } 2 \sin \frac{\pi}{k} < 4 \sin \frac{\pi}{mk} \\
2km & \text{if } 2 \sin \frac{\pi}{k} > 4 \sin \frac{\pi}{mk} \\
2k(1 + m) & \text{if } 2 \sin \frac{\pi}{k} = 4 \sin \frac{\pi}{mk}
\end{cases}
\]
\( \tilde{C} := \tilde{C}(k, m) \) is a positive constant and \( \alpha_{\epsilon, \tilde{C}} \) is the function defined in Lemma 6.2.

Proof. It’s easy to see that (9.34) holds if and only if \( s < 1 \). Moreover by construction
\[
|P_i - P_j| \geq r \geq rs
\]
if either \( i = 0, j = 1, ..., k + km \) or \( i = 1, ..., k, j = k + 1, ..., k + km \). Moreover for \( i, j = 1, ..., k, |i - j| \geq 2 \) one has
\[
|P_i - P_j| > 2r \sin \frac{\pi}{k} \geq rs
\]
and also, for \( i, j = k + 1, ..., k + km \), \( |i - j| \geq 2 \)
\[
|P_i - P_j| > 4r \sin \frac{\pi}{km} \geq rs.
\]
Hence by the exponential decay of \( w \) it follows that
\[
w \left( \frac{P_i - P_j}{\epsilon} \right) = o \left( w \left( \frac{r}{\epsilon} \right) \right) \quad \text{as } \frac{r}{\epsilon} \to +\infty.
\]
As a consequence, for \( r \) such that \( P_r \in \Gamma_\epsilon \), the reduced functional becomes
\[
M_\epsilon[r] = - (\gamma_0 + o(1)) \left[ 2kw \left( \frac{P_i - P_j}{\epsilon} \right) + 2mkw \left( \frac{P_k+1 - P_k+2}{\epsilon} \right) + o \left( w \left( \frac{r}{\epsilon} \right) \right) \right] + \epsilon^2 \frac{1}{N-2} \tilde{C}(k, m) + O(\epsilon^{2r})
\]
\[
= - (\gamma_0 C(k, m) + o(1)) w \left( \frac{r}{\epsilon} \right) + \epsilon^2 (C_2 \tilde{C}(k, m) + o(1)) \frac{1}{s} + O(\epsilon^{2r})
\]
where \( C(k, m) \) is (9.36) and \( \tilde{C}(k, m) \) is a positive constant. \( \square \)

In order to conclude the proof we need to find an interior maximum \( r \) for the reduced functional. By construction \( P_r \) satisfies
\[
sr \leq |P_i - P_j| \leq 4r, \quad i \neq j,
\]
hence, setting
\[
R_\epsilon := \left\{ r > 0 : \text{se log} \frac{1}{\epsilon^2} < r < \frac{1}{4} \left( \frac{1}{\log \frac{1}{\epsilon^2}} \right)^2 \right\},
\]
one has that \( P_r \in \Gamma_\epsilon \) for \( r \in R_\epsilon \); the following result (which can be proved similarly as Proposition 7.3) concludes the proof

Proposition 9.2. For \( \epsilon > 0 \) sufficiently small, the following maximization problem
\[
\max \{ M_\epsilon[r] : r \in R_\epsilon \}
\]
has a solution \( r_\epsilon \in R_\epsilon \). Furthermore
\[
\lim_{\epsilon \to 0} \frac{r_\epsilon s}{\epsilon \log \frac{1}{\epsilon^2}} = 1.
\]
For every $x \in \mathbb{R}^N$ we set $x = (x_1, \ldots, x_N) = (x_1, x_2, x') = (z, x')$, where $z \in \mathbb{C}$. Fix $q \in \mathbb{N}$, $q \geq 2$ and $(m_1, m_2) \in \mathbb{N} \times \mathbb{N} \setminus \{(1, 1)\}$. In this section we construct a cluster solution to (1.4) having $k := m_1 q$ negative bumps and $h := m_2 q$ positive bumps alternated on the vertices of $d := \max\{m_1, m_2\}$ suitable nested regular $2q$-polygons centered in 0, with the same sign on corresponding vertices. Again the proof is similar to the ones of Theorems 1.2-1.3-1.4, hence we briefly sketch it, emphasizing the main differences.

Let $Q_i := (e^{2\pi \sqrt{-1(i-1)/2q}}, 0) = \left(\cos \frac{\pi(i-1)}{q}, \sin \frac{\pi(i-1)}{q}, 0\right) \in \mathbb{R}^N, \ i = 1, \ldots, 2q,$ we make the following ansatz

$$v = \sum_{i=1}^{2q} (-1)^i \left( \sum_{j=1}^{m_{I(i)}} w_{P_{ij}} \right) + \psi,$$

where $P_{ij} := r \left(1 + (j-1) \sin \frac{\pi j}{2q}\right) Q_i, \ i \in \mathbb{R}^N, \ I(i) := (-1)^{i+1} + 3$ and $r > 0$ is such that $P_r := (P_{11}, \ldots, P_{1m_1}, P_{21}, \ldots, P_{2m_2}, \ldots, P_{(2q)1}, \ldots, P_{(2q)m_2}) \in \Gamma_r(\subset \mathbb{R}^{k+h}N)$.

The Sobolev space of symmetric functions in which to work is now

$$X := \{v \in H^1(\mathbb{R}^N) : v(x_1, x_2, x') = v(z, x')\} = v(z, x')$$

Observe that $X = \{u \in H^1(\mathbb{R}^N) : u(T_{t,g}(x)) = u(x), \ i \in \mathbb{N}, \ g \in G\}$ where

$$T_{t,g} : \mathbb{R}^N \rightarrow \mathbb{R}, \quad T_{t,g}(y) = T_{t,g}(z, y') = (ze^{2\pi i/tq}, y'),$$

and $G$ is the group of the rotation matrix in $\mathbb{R}^{N-2}$. Moreover, by Lemma 2.1 the functional $J_\epsilon$ is invariant under the action of the group $\{T_{t,g} : i \in \mathbb{N}, g \in G\}$, namely $J_\epsilon(u(T_{t,g}(x))) = J_\epsilon(u(x))$. The reduced functional is again a one-variable function and, thanks to Proposition 6.1 we prove that it reduces to the following

**Proposition 10.1.** For $\epsilon > 0$ sufficiently small

$$M_\epsilon[r] = (2(k + h) - 4q) + O(\epsilon^2)$$

uniformly for $r > 0$ such that $P_r \in \Gamma_r$, where $s_q := \sin \frac{\pi}{2q}, \ C := C(h, k, q)$ is a positive constant and $\alpha_{\epsilon,C}$ is the function defined in Lemma 6.2.

**Proof.** Let $r$ such that $P_r \in \Gamma_r$. We observe that by construction

$$|P_{ij} - P_{in}| \geq r|Q_i - Q_p| \geq 2rs_q > rs_q \quad \text{if} \ i \neq p,$$

moreover

$$|P_{ij} - P_{in}| = r|j - n|s_q$$

hence, using the exponential decay of $w$, it follows that if either $i \neq p$ or $i = p, \ |j - n| > 1$

$$w \left( \frac{P_{ij} - P_{in}}{\epsilon} \right) = o \left( \frac{r}{\epsilon} \right) \ \text{as} \ \frac{r}{\epsilon} \rightarrow +\infty$$

and so the reduced functional becomes

$$M_\epsilon[r] = -\gamma_0 + o(1) \left( \sum_{i=1}^{2q} (2m_{I(i)} - 2)w \left( \frac{T}{\epsilon} s_q \right) \right) + \epsilon^2 \left( C_2 + o(1) \right) \frac{1}{\left( \frac{r}{\epsilon} s_q \right)} + O(\epsilon^2)$$

for $\tilde{C}, C$ positive constants which depend only on $q, h$ and $k$. \qed
In order to find an interior maximum \( r \) for the reduced functional, observe that by construction \( P_r \) satisfies
\[
s_q r \leq |P_{ij} - P_{pn}| \leq 2(1 + (d - 1)s_q)r, \quad (i,j) \neq (p,n),
\]
where \( d := \max\{m_1, m_2\} \), hence, setting
\[
R_\epsilon := \left\{ r > 0 : s_q \epsilon \log \frac{1}{\epsilon^2} < r < \frac{1}{2(1 + (d - 1)s_q)} \epsilon \left( \log \frac{1}{\epsilon^2} \right)^2 \right\},
\]
one has that \( P_r \in \Gamma_r \) for \( r \in R_\epsilon \).

The following result (which can be proved similarly as Proposition 7.3) concludes the proof

**Proposition 10.2.** For \( \epsilon > 0 \) sufficiently small, the following maximization problem
\[
\max\{M_\epsilon[r] : r \in \bar{R}_\epsilon\}
\]
has a solution \( r_\epsilon \in R_\epsilon \). Furthermore
\[
\lim_{\epsilon \to 0} \frac{r_\epsilon s_q}{\epsilon \log \frac{1}{\epsilon^2}} = 1.
\]

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