s-Inversion Sequences and $P$-Partitions of Type $B$

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Abstract

Given a sequence $s = (s_1, s_2, \ldots)$ of positive integers, the notion of inversion sequences with respect to $s$, or $s$-inversion sequences, was introduced by Savage and Schuster in their study of lecture hall polytopes. A sequence $(e_1, e_2, \ldots, e_n)$ of nonnegative integers is called an $s$-inversion sequence of length $n$ if $0 \leq e_i < s_i$ for $1 \leq i \leq n$. Let $I_n$ be the set of $s$-inversion sequences of length $n$ for $s = (1, 4, 3, 8, 5, 12, \ldots)$, that is, $s_{2i-1} = 2i - 1$ and $s_{2i} = 4i$ for $i \geq 1$, and let $P_n$ be the set of signed permutations on the multiset $\{1^2, 2^2, \ldots, n^2\}$. Savage and Visontai conjectured that the descent number over $P_n$ is equidistributed with the ascent number over $I_{2n}$. In this paper, we give a proof of this conjecture by using $P$-partitions of type $B$. Lin independently obtained a proof based on recurrence relations. Moreover, we find a set of signed permutations over which the descent number is equidistributed with the ascent number over $I_{2n-1}$. Let $I_n'$ be the set of $s$-inversion sequences of length $n$ for $s = (2, 2, 6, 4, 10, 6, \ldots)$, that is, $s_{2i-1} = 4i - 2$ and $s_{2i} = 2i$ for $i \geq 1$. We also find two sets of signed permutations over which the descent number is equidistributed with the ascent number over $I_n'$, depending on whether $n$ is even or odd.

Keywords: inversion sequence, ascent number, signed permutation, descent number, $P$-partition of type $B$

AMS Subject Classifications: 05A05, 05A15
1 Introduction

The notion of $s$-inversion sequences was introduced by Savage and Schuster [4] in their study of lecture hall polytopes. Let $s = (s_1, s_2, \ldots)$ be a sequence of positive integers. An inversion sequence of length $n$ with respect to $s$, or an $s$-inversion sequence of length $n$, is a sequence $e = (e_1, e_2, \ldots, e_n)$ of nonnegative integers such that $0 \leq e_i < s_i$ for $1 \leq i \leq n$. An ascent of an $s$-inversion sequence $e = (e_1, e_2, \ldots, e_n)$ is defined to be an integer $i \in \{0, 1, \ldots, n-1\}$ such that

$$\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}},$$

where we assume that $e_0 = 0$ and $s_0 = 1$. The ascent number $\text{asc}(e)$ of $e$ is meant to be the number of ascents of $e$.

The generating function of ascent numbers of $s$-inversion sequences can be viewed as a generalization of the Eulerian polynomial for permutations, since the ascent number over $s$-inversion sequences of length $n$ for $s = (1, 2, 3, \ldots)$ is equidistributed with the descent number over permutations on $\{1, 2, \ldots, n\}$, see Savage and Schuster [4]. Savage and Visontai [5] showed that for any sequence $s$ of positive integers and any positive integer $n$, the generating function of ascent numbers of $s$-inversion sequences of length $n$ has only real roots. In particular, by establishing a relation between the generating function of ascent numbers of $s$-inversion sequence for $s = (2, 4, 6, \ldots)$ and the generating function of descent numbers of even-signed permutations, they proved the real-rootedness of the Eulerian polynomial of type $D$ as conjectured by Brenti [1].

We adopt the notation $\{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ for a multiset in which $i$ occurs $m_i$ times for $1 \leq i \leq n$. When $s = (1, 1, 3, 2, 5, 3, \ldots)$, that is, $s_{2i-1} = 2i - 1$ and $s_{2i} = i$ for $i \geq 1$, Savage and Visontai [5] showed that the descent number over permutations on $\{1^2, 2^2, \ldots, n^2\}$ is equidistributed with the ascent number over $s$-inversion sequences of length $2n$. Moreover, they posed an equidistribution conjecture for the descent number of signed permutations on the multiset $\{1^2, 2^2, \ldots, n^2\}$. Recall that a signed permutation on a multiset $\{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ is a permutation on $\{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ for which each element is possibly associated with a minus sign. For example, $31231$ is a signed permutation on the multiset $\{1^2, 2, 3^2\}$, where $i$ is identified with $-i$.

Conjecture 1.1 (Savage and Visontai [5]) The descent number over signed permutations on $\{1^2, 2^2, \ldots, n^2\}$ is equidistributed with the ascent number over $s$-inversion sequences of length $2n$ with $s = (1, 4, 3, 8, 5, 12, \ldots)$, that is, $s_{2i-1} = 2i - 1$ and $s_{2i} = 4i$ for $i \geq 1$.

In this paper, we give a proof of Conjecture [1.1]. Let $P_n$ denote the set of signed permutations on $\{1^2, 2^2, \ldots, n^2\}$, and let $I_n$ denote the set of $s$-inversion sequences of length $n$ for $s = (1, 4, 3, 8, 5, 12, \ldots)$. Moreover, let $P_n(x)$ denote the generating function of descent numbers of signed permutations in $P_n$, and let $I_n(x)$ denote the generating
function of ascent numbers of inversion sequences in $I_n$. Savage and Schuster [4] deduced that
\[
\frac{I_{2n}(x)}{(1-x)^{2n+1}} = \sum_{t\geq 0} (t+1)^n (2t+1)^n x^t.
\]
Equation (1.1)

Using $P$-partitions of type $B$ introduced by Chow [2], we show that $P_n(x)$ satisfies the same relation as $I_{2n}(x)$. Thus $P_n(x) = I_{2n}(x)$, and this proves Conjecture 1.1.

It should be noted that Lin [3] independently found a proof of Conjecture 1.1 by showing that the coefficients of $P_n(x)$ and $I_{2n}(x)$ satisfy the same recurrence relation.

Besides the equidistribution conjectured by Savage and Visontai, we also find a set of signed permutations over which the descent number is equidistributed with the ascent number over $I_{2n-1}$. Let $U_n(x)$ be the generating function of descent numbers of signed permutations on $\{1^2, 2^2, \ldots, (n-1)^2, n\}$, and let $V_n(x)$ be the generating function of descent numbers of signed permutations on $\{1^2, 2^2, \ldots, (n-1)^2, n\}$ such that $n$ always has a minus sign. Similar to relation (1.1) for $I_{2n}(x)$, Savage and Schuster [4] deduced a relation for $I_{2n-1}(x)$. We show that $V_n(x)$ satisfies the same relation as $I_{2n-1}(x)$, which implies that $I_{2n-1}(x) = V_n(x)$.

For the sequence $s = (2, 2, 6, 4, 10, 6, \ldots)$, that is, $s_{2i-1} = 4i - 2$ and $s_{2i} = 2i$ for $i \geq 1$, let $I_n'$ be the set of $s$-inversion sequences of length $n$, and let $I_n'(x)$ denote the generating function of ascent numbers of inversion sequences in $I_n'$. We obtain the equidistributions $I_{2n-1}(x) = U_n(x)$ and $I_{2n}(x) = P_n(x)$.

2 Proof of Conjecture 1.1

In this section, we present a proof of Conjecture 1.1. For $n \geq 1$, we use $F_n$ to denote the forest consisting of $n$ rooted trees each of which has exactly two vertices. We show that the generating function $P_n(x)$ of descent numbers of signed permutations on $\{1^2, 2^2, \ldots, n^2\}$ equals the generating function $G_n(x)$ of descent numbers of linear extensions of $F_n$ with signed labelings under certain conditions. Using the technique of $P$-partitions of type $B$, we deduce that $G_n(x)$ satisfies the same relation (1.1) as $I_{2n}(x)$, which implies that $G_n(x) = I_{2n}(x)$. Thus we reach the conclusion that $P_n(x) = I_{2n}(x)$, and this proves Conjecture 1.1.

Let us begin with an overview of linear extensions of a poset. Let $P$ be a poset on the set $\{v_1, v_2, \ldots, v_n\}$ with order relation $\leq$. As usual, we use the notation $v_i < v_j$ to denote that $v_i \leq v_j$ but $v_i \neq v_j$. A labeling of $P$ is an assignment of positive integers to the elements $v_1, v_2, \ldots, v_n$ such that each positive integer cannot be used more than once. A signed labeling of $P$ is a labeling of $P$ with each label possibly associated with a minus sign. We adopt the notation $(P, w)$ for a signed labeled poset, where $w$ is a signed labeling of $P$. For a signed labeled poset $(P, w)$ and an element $v$ of $P$, we use $w(v)$ to denote the label associated with $v$. 

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In this paper, we will be concerned only with a special type of posets, namely, labeled forests with each tree consisting of at most two vertices. Such a forest will be called a simple forest. When viewed as a poset, a simple forest is endowed with the following order relation. We say that \( u < v \) if \( u \) is a child of \( v \). For example, Figure 2.1 illustrates a simple forest \( P \) along with a signed labeling of \( P \).

![Figure 2.1: A simple forest along with a signed labeling.](image)

Recall that a linear extension of a poset \( P \) is a permutation \( v_1 \), \( v_2 \), \( \cdots \), \( v_n \) of the elements of \( P \) such that \( v_j < v_k \) only if \( j < k \), see Stanley [6]. However, by a linear extension of a signed labeled poset \( (P, w) \) we mean a permutation \( w(v_1)w(v_2) \cdots w(v_n) \) of the labels associated with the elements of \( P \), where \( v_1, v_2, \ldots, v_n \) is a linear extension of \( P \). Let \( \mathcal{L}(P, w) \) denote the set of linear extensions of \( (P, w) \). For example, for the signed labeled forest in Figure 2.1 we have

\[
\mathcal{L}(P, w) = \{3\bar{1}56, 35\bar{1}6, 35\bar{6}1, 53\bar{1}6, 53\bar{6}1, 56\bar{1}\},
\]

where \( \bar{i} \) is identified with \( -i \).

In this section, we shall further restrict our attention to simple forests for which each component is a rooted tree with two vertices. More precisely, let \( F_n \) denote such a simple forest with \( n \) trees \( T_1, T_2, \ldots, T_n \), where \( T_i \) is rooted at \( v_i \) with \( u_i \) being the only child. A signed labeling \( w \) of \( F_n \) is said to be local if it satisfies one of the following conditions:

1. \( w(u_i) = 2i - 1 \) and \( w(v_i) = 2i \);
2. \( w(u_i) = \overline{2i - 1} \) and \( w(v_i) = 2i \);
3. \( w(u_i) = 2i - 1 \) and \( w(v_i) = \overline{2i} \);
4. \( w(u_i) = \overline{2i} \) and \( w(v_i) = 2i - 1 \).

We use \( \mathcal{L}(F_n) \) to denote the set of local signed labelings of \( F_n \). A linear extension of \( F_n \) with a local signed labeling becomes a signed permutation on \( \{1, 2, \ldots, 2n\} \). As will be shown in Theorem 2.1, the generating function \( P_n(x) \) of descent numbers of signed permutations on the multiset \( \{1^2, 2^2, \ldots, n^2\} \) equals the generating function \( G_n(x) \) of descent numbers of linear extensions of \( F_n \) with local signed labelings.
Recall that the descent set of a signed permutation \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \) is defined as
\[
\{ i \mid \sigma_i > \sigma_{i+1}, \ 1 \leq i \leq n-1 \} \cup \{ 0 \mid \text{if } \sigma_1 < 0 \},
\]
see Savage and Visontai [5]. However, for the purpose of this paper, we choose the following alternative definition of the descent set of \( \sigma \):
\[
\{ i \mid \sigma_i > \sigma_{i+1}, \ 1 \leq i \leq n-1 \} \cup \{ n \mid \text{if } \sigma_n > 0 \}. \tag{2.2}
\]
The descent number \( \text{des}_B(\sigma) \) of \( \sigma \) is referred to as the number of elements in the descent set defined by (2.2). In fact, via the bijection
\[
\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \mapsto \sigma' = (-\sigma_n)(-\sigma_{n-1}) \cdots (-\sigma_1),
\]
we see that the descent numbers defined by (2.1) and (2.2) are equidistributed over the set of signed permutations on \( \{1^2, 2^2, \ldots, n^2\} \).

With the above notation, the generating function \( G_n(x) \) can be written as
\[
G_n(x) = \sum_{w \in L(F_n)} \sum_{\sigma \in L(F_n, w)} x^{\text{des}_B(\sigma)}.
\]
We have the following equidistribution property.

**Theorem 2.1** For \( n \geq 1 \), we have
\[
G_n(x) = P_n(x).
\]

**Proof.** Define a map \( \phi \) from the set
\[
\bigcup_{w \in L(F_n)} L(F_n, w) \tag{2.3}
\]
of linear extensions of \( F_n \) with local signed labelings to the set of signed permutations on \( \{1^2, 2^2, \ldots, n^2\} \). Let \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n} \) be a linear extension in \( L(F_n, w) \), where \( w \in L(F_n) \). The construction of \( \phi(\sigma) = \tau = \tau_1 \tau_2 \cdots \tau_{2n} \) can be described as follows. For \( 1 \leq i \leq 2n \), assume that \( \tau_i \) has the same sign as \( \sigma_i \). Moreover, set \( |\tau_i| = |\sigma_i|/2 \) if \( |\sigma_i| \) is even and set \( |\tau_i| = |\sigma_i|+1/2 \) if \( |\sigma_i| \) is odd. Since \( \sigma \) is a signed permutation on \( \{1, 2, \ldots, 2n\} \), it can be easily checked that \( \tau \) is a signed permutation on \( \{1^2, 2^2, \ldots, n^2\} \).

To show that \( \phi \) is a bijection, we construct a map \( \psi \) from the set of signed permutations on \( \{1, 2, \ldots, 2n\} \) to the set in (2.3) and we shall prove that \( \psi \) is the inverse of \( \phi \). Let \( \tau = \tau_1 \tau_2 \cdots \tau_{2n} \) be a signed permutation on \( \{1^2, 2^2, \ldots, n^2\} \). Define \( \psi(\tau) = \sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n} \) by the following procedure. For each \( 1 \leq i \leq n \), assume that \( a_i \) and \( b_i \) (\( a_i < b_i \)) are the two positions of \( \tau \) occupied by \( i \) or \( \bar{i} \). Moreover, \( \sigma_{a_i} \) and \( \sigma_{b_i} \) are determined according to the following cases:
\begin{enumerate}
  \item \( \sigma_{a_i} = 2i - 1 \) and \( \sigma_{b_i} = 2i \) if \( \tau_{a_i} = \tau_{b_i} = i \);
  \item \( \sigma_{a_i} = 2i - 1 \) and \( \sigma_{b_i} = 2i \) if \( \tau_{a_i} = \tau_{b_i} = i \)
  \item \( \sigma_{a_i} = 2i - 1 \) and \( \sigma_{b_i} = 2i \) if \( \tau_{a_i} = \tau_{b_i} = i \);
  \item \( \sigma_{a_i} = 2i \) and \( \sigma_{b_i} = 2i - 1 \) if \( \tau_{a_i} = \tau_{b_i} = i \).
\end{enumerate}

So \( \sigma \) is a signed permutation on \( \{1, 2, \ldots, 2n\} \). Let \( w \) be a signed labeling of \( F_n \) defined by \( w(u_i) = \sigma_{a_i} \) and \( w(v_i) = \sigma_{b_i} \). It is routine to check that \( w \) is a local signed labeling of \( F_n \). It is also straightforward to verify that \( \sigma \) is a linear extension of \( (F_n, w) \).

For any linear extension \( \sigma \) of \( F_n \) with a local signed labeling, by direct verification we see that \( \psi(\phi(\sigma)) = \sigma \). This implies that \( \psi \) is the inverse of \( \phi \), and hence \( \phi \) is a bijection. Finally, by the construction of \( \phi \), it can be seen that \( j \in \{1, 2, \ldots, 2n\} \) is a descent of \( \sigma \) if and only if it is a descent of \( \phi(\sigma) \). This completes the proof.

As the simplest example of the bijection \( \phi \), consider the case \( n = 1 \). For \( F_1 \), there are four local signed labelings and the set of linear extensions of \( F_1 \) with local signed labelings is \( \{12, 12, 12, 21\} \). In this case, we have

\[ \phi(12) = 11, \quad \phi(12) = 11, \quad \phi(12) = 12, \quad \phi(21) = 11. \]

The next theorem shows that \( G_n(x) \) satisfies the same relation as \( I_{2n}(x) \).

**Theorem 2.2** For \( n \geq 1 \), we have

\[ \frac{G_n(x)}{(1 - x)^{2n+1}} = \sum_{t \geq 0} (t + 1)^n (2t + 1)^n x^t. \]

(2.4)

To prove the above theorem, recall the notion of a \((P, w)\)-partition of type \( B \) introduced by Chow \[2\]. Let \( P \) be a poset and \( w \) be a signed labeling of \( P \). A \((P, w)\)-\textit{partition of type }\( B \) is a map \( f \) from \( P \) to the set of nonnegative integers that satisfies the following conditions:

\begin{enumerate}
  \item \( f(u) \geq f(v) \) if \( u \leq v \);
  \item \( f(u) > f(v) \) if \( u < v \) and \( w(u) > w(v) \);
  \item \( f(v) \geq 1 \) if \( w(v) > 0 \).
\end{enumerate}

When \( w \) is a labeling with positive integers, a \((P, w)\)-partition of type \( B \) reduces to an ordinary \((P, w)\)-partition defined by Stanley \[6\]. Substituting each element \( v \in P \) with its label \( w(v) \), a \((P, w)\)-partition of type \( B \) can be viewed as a map from the set of labels of \( P \) to the set nonnegative integers. Chow \[2\] showed that \((P, w)\)-partitions of type \( B \) can be generated by linear extensions of \((P, w)\). For a linear extension \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \) of \((P, w)\), a map \( g \) from \( \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \) to the set of nonnegative integers is called \( \sigma\)-\textit{compatible} if the following conditions are satisfied:
(1) \( g(\sigma_1) \geq g(\sigma_2) \geq \cdots \geq g(\sigma_n) \); 
(2) \( g(\sigma_i) > g(\sigma_{i+1}) \) if \( 1 \leq i \leq n-1 \) and \( \sigma_i > \sigma_{i+1} \); 
(3) \( g(\sigma_n) \geq 1 \) if \( \sigma_n > 0 \).

Notice that for two distinct linear extensions \( \sigma \) and \( \sigma' \) of \( (P, w) \), any \( \sigma \)-compatible map is not \( \sigma' \)-compatible.

The following theorem is due to Chow [2], which will be used to establish a relation between the generating function for the number of \( (P, w) \)-partitions of type \( B \) and the generating function for the descent number of linear extensions of \( (P, w) \).

**Theorem 2.3** (Chow [2]) Let \( P \) be a poset with a signed labeling \( w \). A map \( f \) from \( P \) to the set of nonnegative integers is a \( (P, w) \)-partition of type \( B \) if and only if there exists a linear extension \( \sigma \) of \( (P, w) \) such that \( f \) is \( \sigma \)-compatible.

For a nonnegative integer \( t \), let \( \Omega_P(w, t) \) denote the number of \( (P, w) \)-partitions \( f \) of type \( B \) such that \( f(v) \leq t \) for any \( v \in P \). We have the following relation.

**Theorem 2.4** Let \( P \) be a poset with \( n \) elements, and let \( w \) be a signed labeling of \( P \). Then

\[
\frac{1}{(1-x)^{n+1}} \sum_{\sigma \in \mathcal{L}(P,w)} x^{\text{des}_B(\sigma)} = \sum_{t \geq 0} \Omega_P(w, t) x^t. \tag{2.5}
\]

**Proof.** The proof is analogous to that of Stanley [6] for the case of an ordinary labeling. For a linear extension \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \) of \( (P, w) \), let \( \Omega_\sigma(t) \) denote the number of \( \sigma \)-compatible maps \( g \) such that \( g(\sigma_i) \leq t \) for \( 1 \leq i \leq n \). In view of Theorem 2.3 we see that

\[ \Omega_P(w, t) = \sum_{\sigma \in \mathcal{L}(P,w)} \Omega_\sigma(t). \]

Thus, to prove (2.5) it suffices to show that

\[ \sum_{t \geq 0} \Omega_\sigma(t) x^t = \frac{x^{\text{des}_B(\sigma)}}{(1-x)^{n+1}}. \tag{2.6} \]

To count \( \Omega_\sigma(t) \), we establish a bijection between the set of \( \sigma \)-compatible maps \( g \) with \( g(\sigma_i) \leq t \) for \( 1 \leq i \leq n \) and the set of partitions \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \) with \( \lambda_1 \leq t - \text{des}_B(\sigma) \). Recall that a partition is a sequence \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \) of nonnegative integers such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \). For \( 1 \leq i \leq n \), let \( d_i \) denote the number of descents of \( \sigma \) that are greater than or equal to \( i \), that is, \( d_i = |\{ j | i \leq j \leq n-1, \sigma_j > \sigma_{j+1} \} \cup \{ n \text{ if } \sigma_n \geq 0 \}|. \)
Let \( g \) be a \( \sigma \)-compatible map with \( g(\sigma_i) \leq t \) for \( 1 \leq i \leq n \). It is easily checked that by setting \( \lambda_i = g(\sigma_i) - d_i \) for \( 1 \leq i \leq n \), we are given a partition \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) with \( \lambda_1 \leq t - \text{des}_B(\sigma) \). It can be seen that this procedure is reversible. So we arrive at a bijection. Notice that the number of partitions \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) with \( \lambda_1 \leq t - \text{des}_B(\sigma) \) is equal to 

\[
\binom{n + t - \text{des}_B(\sigma)}{n},
\]

see Stanley \[6\]. It follows that

\[
\Omega_{\sigma}(t) = \binom{n + t - \text{des}_B(\sigma)}{n},
\]

which implies (2.6). This completes the proof.

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. By Theorem 2.4, we aim to prove the following equivalent form of (2.4):

\[
\sum_{w \in L(F_n)} \Omega_{F_n}(w, t) = (t+1)(2t+1)^n. \tag{2.7}
\]

Recall that \( F_n \) consists of \( n \) components \( T_1, T_2, \ldots, T_n \), where \( T_i \) is a tree rooted at \( v_i \) with \( u_i \) being the only child. Keep in mind that the left-hand side of (2.7) equals the number of \((F_n, w)\)-partitions \( f \) of type \( B \) such that \( f(u_i) \leq t \) and \( f(v_i) \leq t \), where \( w \) is a local signed labeling of \( F_n \). Restricting \( f \) to the tree \( T_i \), we obtain a map \( f_i \) from \( T_i \) to the set nonnegative integers. Similarly, restricting \( w \) to \( T_i \) gives a signed labeling \( w_i \) of \( T_i \). Recall that \( w_i \) is given by one of the following assignments:

1. \( w_i(u_i) = 2i - 1 \) and \( w_i(v_i) = 2i \),
2. \( w_i(u_i) = 2i - 1 \) and \( w_i(v_i) = 2i \),
3. \( w_i(u_i) = 2i - 1 \) and \( w_i(v_i) = 2i \),
4. \( w_i(u_i) = 2i \) and \( w_i(v_i) = 2i - 1 \).

Clearly, \( f_i \) is a \((T_i, w_i)\)-partition of type \( B \) satisfying the conditions \( f_i(u_i) \leq t \) and \( f_i(v_i) \leq t \). Conversely, \( f \) can be recovered from \( f_1, f_2, \ldots, f_n \).

For a signed labeling \( w_i \) of \( T_i \) induced by a local signed labeling of \( F_n \), we now compute the number \( \Omega_{T_i}(w_i, t) \) of \((T_i, w_i)\)-partitions \( f_i \) of type \( B \) such that \( f_i(u_i) \leq t \) and \( f_i(v_i) \leq t \). We consider the above four cases.

Case 1: \( w_i(u_i) = 2i - 1 \) and \( w_i(v_i) = 2i \). It is easily seen that in this case \( f_i \) is a \((T_i, w_i)\)-partition of type \( B \) if and only if

\[
0 < f_i(v_i) \leq f_i(u_i) \leq t.
\]
So we have

\[ \Omega_{T_i}(w_i, t) = \binom{t+1}{2}. \]

Case 2: \( w_i(u_i) = 2i - 1 \) and \( w_i(v_i) = 2i \). Similarly, in this case, \( f_i \) is a \((T_i, w_i)\)-partition of type \( B \) if and only if

\[ 0 < f_i(v_i) \leq f_i(u_i) \leq t. \]

Thus,

\[ \Omega_{T_i}(w_i, t) = \binom{t+1}{2}. \]

Case 3: \( w_i(u_i) = 2i - 1 \) and \( w_i(v_i) = 2i \). We see that \( f_i \) is a \((T_i, w_i)\)-partition of type \( B \) if and only if

\[ 0 \leq f_i(v_i) < f_i(u_i) \leq t. \]

This implies that

\[ \Omega_{T_i}(w_i, t) = \binom{t+1}{2}. \]

Case 4: \( w_i(u_i) = 2i \) and \( w_i(v_i) = 2i - 1 \). In this case, \( f_i \) is a \((T_i, w_i)\)-partition of type \( B \) if and only if

\[ 0 \leq f_i(v_i) \leq f_i(u_i) \leq t. \]

Hence,

\[ \Omega_{T_i}(w_i, t) = \binom{t+2}{2}. \]

Combining the above four cases, we see that for any \( 1 \leq i \leq n \), the number of possible configurations of \((T_i, w_i)\)-partitions of type \( B \) equals

\[ 3 \binom{t+1}{2} + \binom{t+2}{2} = (t+1)(2t+1). \]

It follows that

\[ \sum_{w \in L(F_n)} \Omega_{F_n}(w, t) = ((t+1)(2t+1))^n, \]

as required.

3 Signed permutations and \( I_{2n-1} \)

In the previous section, we proved the conjecture of Savage and Visontai on the equidistribution of the descent number over signed permutations and the ascent number over \( s \)-inversion sequences in the set \( I_{2n} \). In this section, we find a set \( V_n \) of signed permutations over which the descent number is equidistributed with the ascent number over
the set $I_{2n-1}$. Recall that $I_{2n-1}$ is the set of $s$-inversion sequences of length $2n-1$ for $s = (1, 4, 3, 8, 5, 12, \ldots)$. For an $s$-inversion sequence $e = (e_1, e_2, \ldots, e_n)$, an ascent of $e$ is defined as an integer $i \in \{0, 1, \ldots, n-1\}$ such that
\[
\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}},
\]
where we assume that $e_0 = 0$ and $s_0 = 1$. The ascent number of $e$ is meant to be the number of ascents of $e$. Define $V_n(x)$ as the generating function of descent numbers of signed permutations in $V_n$. Recall that $I_{2n-1}(x)$ denotes the generating function of ascent numbers of inversion sequences in $I_{2n-1}$. Savage and Schuster [4] showed that
\[
\frac{I_{2n-1}(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^n(2t+1)^{n-1}x^t.
\]
We show that $V_n(x)$ satisfies the same relation (3.1) as $I_{2n-1}(x)$. The proof is similar to that of Conjecture 1.1 So we reach the conclusion that $V_n(x) = I_{2n-1}(x)$.

Let $U_n$ be the set of signed permutations on the multiset \{1$^2$, 2$^2$, \ldots, (n-1)$^2$, n\}. Define $V_n$ to be the subset of $U_n$ consisting of signed permutations such that the element $n$ carries a minus sign. Set
\[
V_n(x) = \sum_{\sigma \in V_n} x^{\text{des}_B(\sigma)}.
\]
We have the following equidistribution property.

**Theorem 3.1** For $n \geq 1$, we have $V_n(x) = I_{2n-1}(x)$.

**Proof.** In view of (3.1), we aim to show that
\[
\frac{V_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^n(2t+1)^{n-1}x^t. \tag{3.2}
\]
Let $F^*_n$ be the forest obtained from $F_{n-1}$ by adding a single vertex $v_n$ as a component. For example, Figure 3.2 illustrates the forest $F^*_n$ for $n = 3$. Write $L(F^*_n)$ for the set
\[
\begin{array}{c}
v_1 & v_2 & \cdot \\
u_1 & u_2 & v_3
\end{array}
\]
Figure 3.2: The forest $F^*_n$ for $n = 3$.
of signed labelings $w$ of $F^*_n$ such that $w(v_n) = -(2n-1)$ and the labels on $F_{n-1}$ form
a local signed labeling of $F_{n-1}$. Let $Q_n(x)$ denote the generating function of descent numbers of linear extensions of $F^*_n$ with signed labelings $w \in L(F^*_n)$, namely,

$$Q_n(x) = \sum_{w \in L(F^*_n)} \sum_{\sigma \in L(F^*_n, w)} x^{\text{des}_B(\sigma)}.$$

Analogous to the bijection $\phi$ in the proof of Theorem 2.1 we can construct a descent preserving map $\phi^*$ from the set

$$\bigcup_{w \in L(F^*_n)} L(F^*_n, w)$$

to the set $V_n$. Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n-1}$ be a linear extension in $L(F^*_n, w)$, where $w \in L(F^*_n)$. Define $\phi^*(\sigma) = \tau = \tau_1 \tau_2 \cdots \tau_{2n-1}$ as follows. For $1 \leq i \leq 2n - 1$, assume that $\tau_i$ has the same sign as $\sigma_i$. Then we set $|\tau_i| = \frac{|\sigma_i|}{2}$ if $|\sigma_i|$ is even and set $|\tau_i| = \frac{|\sigma_i|+1}{2}$ if $|\sigma_i|$ is odd. Clearly, $\tau$ is a signed permutation in $V_n$. Moreover, one can construct the inverse of $\phi^*$, which is analogous to the inverse of $\phi$. This proves that $\phi^*$ is a bijection. So we obtain that

$$V_n(x) = Q_n(x).$$

Thus (3.2) is equivalent to

$$\frac{Q_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} (t+1)^n (2t+1)^{n-1} x^t. \tag{3.4}$$

By Theorem 2.4 the left-hand side of (3.4) can be written as

$$\frac{Q_n(x)}{(1-x)^{2n}} = \sum_{t \geq 0} \sum_{w \in L(F^*_n)} \Omega_{F^*_n}(w, t) x^t.$$

Hence (3.4) is equivalent to

$$\sum_{w \in L(F^*_n)} \Omega_{F^*_n}(w, t) = (t+1)^n (2t+1)^{n-1}. \tag{3.5}$$

The proof of (3.5) is similar to that for (2.7). For completeness, a detailed proof is presented. Recall that $F_{n-1}$ contains $n - 1$ components $T_1, T_2, \ldots, T_{n-1}$, where $T_i$ is a tree rooted at $v_i$ with $u_i$ being the only child. Let $T^*_n$ denote the component consisting of the single vertex $v_n$. By definition, the left-hand side of (3.5) equals the total number of $(F^*_n, w)$-partitions $f$ of type $B$ such that $f(v) \leq t$ for any vertex $v$ of $F^*_n$, where $w$ is a signed labeling belonging to $L(F^*_n)$. Restricting $f$ to $F^*_{n-1}$, we obtain a map $f'$ from $F_{n-1}$ to the set nonnegative integers. While, restricting $f$ to $T^*_n$, we are led to a map $f''$ from $T^*_n$ to the set nonnegative integers. On the other hand, restricting $w$ to $F^*_{n-1}$ gives a local signed labeling $w'$ of $F_{n-1}$, whereas restricting $w$ to $T^*_n$ gives a signed labeling $w''$ of $T^*_n$ such that $w''(v_n) = -(2n-1)$. Obviously, $f'$ is a $(F_{n-1}, w')$-partition of type
satisfying the condition that $f'(v) \leq t$ for any vertex $v$ of $F_{n-1}$, and $f''$ is a $(T^*_n, w'')$-partition of type $B$ such that $f''(v_n) \leq t$. It can be seen that the above procedure is reversible. Hence we get

$$
\sum_{w' \in L(F_{n-1})} \Omega_{F^n_{n-1}}(w', t) = \Omega_{T^*_n}(w'', t). \quad (3.6)
$$

In the proof of Theorem 2.2 it has been shown that

$$
\sum_{w' \in L(F_{n-1})} \Omega_{F^n_{n-1}}(w', t) = ((t + 1)(2t + 1))^{n-1}. \quad (3.7)
$$

To compute $\Omega_{T^*_n}(w'', t)$, we see that $f''$ is a $(T^*_n, w'')$-partition of type $B$ if and only if

$$
0 \leq f''(v_n) \leq t.
$$

Thus

$$
\Omega_{T^*_n}(w'', t) = t + 1. \quad (3.8)
$$

Combining (3.6), (3.7) and (3.8), we are led to (3.5). This completes the proof.

4 Signed permutations and $I'_n$

In this section, we consider equidistributions of the descent number over signed permutations and the ascent number over $s$-inversions sequences for $s = (2, 2, 6, 4, 10, 6, \ldots)$. Recall that the set of such $s$-inversion sequences of length $n$ is denoted by $I'_n$. It turns out that we need to distinguish the parity of $n$.

First, we consider the case for $I'_2n$. Let $I'_2n(x)$ be the generating function of ascent numbers of inversion sequences in $I'_2n$. Savage and Schuster [4] obtained a relation for the generating function of ascent numbers of $s$-inversion sequences for $s = (1, 1, 3, 2, 5, 3, \ldots)$, that is, $s_{2i} = i$ and $s_{2i-1} = 2i - 1$ for $i \geq 1$. This leads to a relation satisfied by $I'_2n(x)$. As will be seen, this relation coincides with the relation (1.1) for $I'_2n(x)$, and so we get $I'_2n(x) = I_{2n}(x)$. Since $I_{2n}(x)$ equals the generating function $P_n(x)$ of descent numbers of signed permutations on $\{1^2, 2^2, \ldots, n^2\}$, we are led to the equidistribution as stated below.

**Theorem 4.1** For $n \geq 1$, we have $P_n(x) = I'_2n(x)$.

To prove the above theorem, we recall two formulas of Savage and Schuster [4] on the generating function of ascent numbers of $s$-inversion sequences of length $n$. For any sequence $s = (s_1, s_2, \ldots)$ of positive integers, let $f^{(s)}_n(t)$ denote the number of sequences $(a_1, a_2, \ldots, a_n)$ of nonnegative integers such that

$$
0 \leq \frac{a_1}{s_1} \leq \frac{a_2}{s_2} \leq \cdots \leq \frac{a_n}{s_n} \leq t. \quad (4.1)
$$
Savage and Schuster [4] deduced that
\[
\frac{1}{(1 - x)^{n+1}} \sum_e x^\text{asc(e)} = \sum_{t \geq 0} f^{(s)}_n(t) x^t, \quad (4.2)
\]
where \(e\) ranges over \(s\)-inversion sequences of length \(n\). For the sequence
\[
s = (1, 1, 3, 2, 5, 3, \ldots),
\]
Savage and Schuster [4] showed that
\[
f^{(s)}_n(t) = (t + 1)^\left\lfloor \frac{n}{2} \right\rfloor \left(\frac{t + 2}{2}\right)^\left\lceil \frac{n}{2} \right\rceil. \quad (4.3)
\]

**Proof of Theorem 4.1**

Let \(s = (2, 2, 6, 4, 10, 6, \ldots)\) and \(s' = s/2 = (1, 1, 3, 2, 5, 3, \ldots)\).

By (4.1), we see that
\[
f^{(s)}_n(t) = f^{(s')}_{2n}(t). \quad (4.4)
\]

Let \(I'_n(x)\) be the generating function of ascent numbers of inversion sequences in \(I'_n\). By (4.2) and (4.4), we obtain that
\[
\frac{I'_n(x)}{(1 - x)^{n+1}} = \sum_{t \geq 0} (t + 1)^\left\lfloor \frac{n}{2} \right\rfloor (2t + 1)^\left\lceil \frac{n}{2} \right\rceil x^t. \quad (4.5)
\]
Replacing \(n\) with \(2n\) in (4.5), we arrive at
\[
\frac{I'_{2n}(x)}{(1 - x)^{2n+1}} = \sum_{t \geq 0} (t + 1)^n(2t + 1)^n x^t. \quad (4.6)
\]

Comparing (4.6) with (1.1), we see that \(I'_{2n}(x)\) satisfies the same relation as \(I_{2n}(x)\). This implies that \(I'_{2n}(x) = I_{2n}(x)\). Since \(P_n(x) = I_{2n}(x)\), we conclude that \(P_n(x) = I'_{2n}(x)\). This completes the proof.

We now consider the case for \(I'_{2n-1}\). Recall that \(U_n\) is the set of signed permutations on \(\{1^2, 2^2, \ldots, (n-1)^2, n\}\). Let \(U_n(x)\) be the generating function of descent numbers of signed permutations in \(U_n\). Replacing \(n\) with \(2n - 1\) in (4.5), we find that
\[
\frac{I'_{2n-1}(x)}{(1 - x)^{2n}} = \sum_{t \geq 0} (t + 1)^{n-1}(2t + 1)^n x^t. \quad (4.7)
\]
It will be shown that \(U_n(x)\) also satisfies relation (4.7). So we have the following equidistribution property for \(I'_{2n-1}\).
Theorem 4.2 For \( n \geq 1 \), we have \( U_n(x) = I_{2n-1}(x) \).

Proof. We proceed to show that

\[
\frac{U_n(x)}{(1 - x)^{2n}} = \sum_{t \geq 0} (t + 1)^{n-1}(2t + 1)^n x^t. \quad (4.8)
\]

As defined in the proof of Theorem 3.1, \( F_n^* \) denotes the forest obtained from \( F_{n-1} \) by adding a single vertex \( v_n \) as a component \( T_n^* \). We use \( L'(F_n^*) \) to stand for the set of signed labelings \( w \) of \( F_n^* \) such that \( w(v_n) = 2n - 1 \) or \( w(v_n) = -(2n - 1) \), and the restriction of \( w \) to \( F_{n-1} \) forms a local signed labeling of \( F_{n-1} \). Let

\[
H_n(x) = \sum_{w \in L'(F_n^*)} \sum_{\sigma \in L(F_n^*, w)} x^{\text{des}_B(\sigma)}. \quad (4.9)
\]

Analogous to the construction of \( \phi^* \) in the proof of Theorem 3.1, we can establish a descent preserving bijection from the set

\[
\bigcup_{w \in L'(F_n^*)} L(F_n^*, w)
\]

to the set \( U_n \). This yields that

\[
H_n(x) = U_n(x).
\]

Therefore, (4.8) is equivalent to

\[
\frac{H_n(x)}{(1 - x)^{2n}} = \sum_{t \geq 0} (t + 1)^{n-1}(2t + 1)^n x^t. \quad (4.10)
\]

By Theorem 2.4, for each signed labeling \( w \in L'(F_n^*) \),

\[
\frac{1}{(1 - x)^{2n}} \sum_{\sigma \in L(F_n^*, w)} x^{\text{des}_B(\sigma)} = \sum_{t \geq 0} \Omega_{F_n^*}(w, t) x^t.
\]

It follows that

\[
\frac{1}{(1 - x)^{2n}} \sum_{w \in L'(F_n^*)} \sum_{\sigma \in L(F_n^*, w)} x^{\text{des}_B(\sigma)} = \sum_{t \geq 0} \sum_{w \in L'(F_n^*)} \Omega_{F_n^*}(w, t) x^t.
\]

In view of the definition of \( G_n(x) \) as given in (4.9), we obtain that

\[
\frac{H_n(x)}{(1 - x)^{2n}} = \sum_{t \geq 0} \sum_{w \in L'(F_n^*)} \Omega_{F_n^*}(w, t) x^t.
\]

Hence (4.10) is equivalent to the following relation

\[
\sum_{w \in L'(F_n^*)} \Omega_{F_n^*}(w, t) = (t + 1)^{n-1}(2t + 1)^n. \quad (4.11)
\]

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The proof of (4.11) is similar to that of (3.5). Let $w_1$ be the signed labeling of $T_n^*$ such that $w_1(v_n) = -(2n - 1)$, and let $w_2$ be the signed labeling of $T_n^*$ such that $w_2(v_n) = 2n - 1$. Then

$$\sum_{w \in L(F_n^*)} \Omega_{F_n}(w, t) = (\Omega_{T_n}(w_1, t) + \Omega_{T_n}(w_2, t)) \sum_{w \in L(F_{n-1})} \Omega_{F_{n-1}}(w, t).$$

(4.12)

In the proof of Theorem 2.2, we have shown that

$$\sum_{w \in L(F_{n-1})} \Omega_{F_{n-1}}(w, t) = ((t + 1)(2t + 1))^{n-1},$$

(4.13)

whereas in the proof of Theorem 3.1, we deduced that

$$\Omega_{T_n}(w_1, t) = t + 1.$$  

(4.14)

Clearly, a map $f$ from $T_n^*$ to the set of nonnegative integers is a $(T_n^*, w_2)$-partition of type $B$ if and only if $0 < f(v_n) \leq t$. Thus

$$\Omega_{T_n}(w_2, t) = t.$$ 

(4.15)

Substituting (4.13), (4.14) and (4.15) into (4.12), we arrive at (4.11). This completes the proof.

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