A MODIFIED HARD THERMAL LOOP PERTURBATION
THEORY

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Abstract

Based on the external perturbation that disturbs the system only slightly from its equilibrium position we make the Taylor expansion of the pressure of a quark gas. It turns out that the first term was used in the literature to construct a Hard Thermal Loop perturbation theory (HTLpt) within the variation principle of the lowest order of the thermal mass parameter. Various thermodynamic quantities within the 1-loop HTLpt encountered overcounting of the leading order (LO) contribution and also required a separation scale for soft and hard momenta. Using same variational principle we reconstruct the HTLpt at the first derivative level of the pressure that takes into account the effect of the variation of the external source through the conserved density fluctuation. This modification markedly improves those quantities in 1-loop HTLpt in a simple way instead of pushing the calculation to a considerably more complicated 2-loop HTLpt. Moreover, the results also agree with those obtained in the 2-loop approximately self-consistent Φ-derivable Hard Thermal Loop resummation. We also discuss how this formalism can be extended for the higher order contributions

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I. INTRODUCTION

The HTL resummation developed by Braaten and Pisarski has been used to calculate various thermodynamic quantities in the literature based on two methods. The 2-loop approximately Φ-derivable approach was developed by Blaizot et al., which produces a correct LO and plasmon effects for thermodynamic quantities (e.g., entropy density, number density etc.) and also for quark number susceptibility (QNS). On the other hand the HTLpt using variational principle through the lowest order of the thermal mass parameter was developed for pressure at the 1-loop level by Andersen et al., which is at present pushed to the 2-loop and 3-loop level. However, the 1-loop HTLpt pressure has a bad perturbative LO content, in the sense of severe over-inclusion of the effect of order \( g^2 \) as the HTL action is accurate only for soft momenta and for hard ones only in the vicinity of light cone. Such problem is cured (or at least pushed to higher orders) only after going to 2-loop level in HTLpt, which is indeed a considerably more involved calculation. A very recent 1-loop QNS calculation from pressure in HTLpt had, obviously, the problem of over-inclusion of the order \( g^2 \). Moreover, it required an ad hoc separation scale to distinguish between hard and soft momenta. On the other hand the calculation of QNS in Ref. dealt with the imaginary part of the charge-charge correlator in the vector channel and required to show the charge conservation. It also encountered some technical difficulties and over-inclusion problem in order \( g^2 \) as discussed in Ref. Also the Landau damping (LD) contribution was discussed but ignored.

The equation of state (EOS) of strongly interacting matter at nonzero baryon density and high temperature is a subject of great interest for wide spectrum of physicists. Also QNS is a topical quantity in view of the ongoing efforts towards understanding the actual nature of the QGP as QNS plays an important role in locating the critical end point in QCD phase diagram. As it stands the LO thermodynamic quantities and QNS in HTL approximation led to different results. This requires a detailed analysis of the leading order quantities within the HTLpt before extending it to the higher orders. In view of this we do not aim at higher orders calculations, rather we intend in this article to sort out the problems in 1-loop HTLpt, which produced different results from that of the Φ-derivable approach within the HTL resummation, and finally arrive at a consistent result despite the use of different approaches.
The paper is organised as follows. In Sec. II we briefly discuss some generalities on fluctuations, correlation functions and susceptibilities based on external disturbance to a physical system. In Sec. III we modify the HTLpt based on the external disturbance. In Sec. IV the HTL thermodynamics in presence of the quark chemical potential and then QNS in LO are obtained. We also checked, both numerically and analytically, the perturbative content of LO HTL QNS in Sec. V. Finally, we conclude in Sec. VI.

II. GENERALITIES

A. Fluctuation and Susceptibility:

Let $O_\alpha$ be a Heisenberg operator where $\alpha$ may be associated with a degree of freedom in the system. In a static and uniform external field $F_\alpha$, the (induced) expectation value of the operator $O_\alpha(0, \vec{x})$ is written \cite{12} as

$$\phi_\alpha \equiv \langle O_\alpha(0, \vec{x}) \rangle \ = \ \frac{\text{Tr} \left[ O_\alpha(0, \vec{x}) e^{-\beta(H+H_{ex})} \right]}{\text{Tr} \left[ e^{-\beta(H+H_{ex})} \right]} = \frac{1}{V} \int d^3x \ \langle O_\alpha(0, \vec{x}) \rangle \ ,$$

where the translational invariance is assumed, $V$ is the volume of the system and $H_{ex}$ is given by

$$H_{ex} = -\sum_\alpha \int d^3x \ O_\alpha(0, \vec{x}) F_\alpha \ .$$

The (static) susceptibility $\chi_{\alpha \sigma}$ is defined as the rate with which the expectation value changes in response to an infinitesimal change in external field,

$$\chi_{\alpha \sigma}(T) = \left. \frac{\partial \phi_\alpha}{\partial F_\sigma} \right|_{F=0} = \beta \int d^3x \ \left\langle O_\alpha(0, \vec{x}) O_\sigma(0, \vec{0}) \right\rangle \ ,$$

where $\langle O_\alpha(0, \vec{x}) O_\sigma(0, \vec{0}) \rangle$ is the two point correlation function with operators evaluated at equal times. There is no broken symmetry as

$$\langle O_\alpha(0, \vec{x}) \rangle \big|_{F \to 0} = \langle O_\sigma(0, \vec{0}) \rangle \big|_{F \to 0} = \langle O_\sigma(0, \vec{0}) \rangle \big|_{F \to 0} = 0 \ .$$

B. Thermodynamic Relations:

The pressure is defined as

$$P = \frac{T}{V} \ln Z \ ,$$

(5)
where $T$ is temperature, $V$ is the volume and $\mathcal{Z}$ is the partition function of a quark-antiquark gas. The entropy density is defined as

$$S = \frac{\partial P}{\partial T}.$$  

(6)

The number density for a given quark flavour can be written as

$$\rho = \frac{\partial P}{\partial \mu} = \frac{1}{V} \frac{\text{Tr} \left[ \mathcal{N} e^{-\beta(H-\mu \mathcal{N})} \right]}{\text{Tr} \left[ e^{-\beta(H-\mu \mathcal{N})} \right]} = \frac{\langle \mathcal{N} \rangle}{V},$$

(7)

with $\mathcal{N}$ is the quark number operator and $\mu$ is the chemical potential. If $\mu \to 0$, the quark number density vanishes due to CP invariance.

The QNS is a measure of the response of the quark number density with infinitesimal change in the quark chemical potential, $\mu + \delta \mu$, at $\mu \to 0$. Under such a situation the variation of the external field, $F_\alpha$, in (2) can be identified as the quark chemical potential $\mu$ and the operator $O_\alpha$ as the temporal component ($J^0$) of the external vector current, $J_\sigma(t, \vec{x}) = \bar{\psi} \Gamma_\sigma \psi$, where $\Gamma_\sigma$ is in general a three point function. Then the QNS for a given quark flavour follows from (3) as

$$\chi(T) = \frac{\partial \rho}{\partial \mu} \bigg|_{\mu=0} = \frac{\partial^2 P}{\partial \mu^2} \bigg|_{\mu=0} = \int d^4x \, \left\langle J_0(0, \vec{x}) J_0(0, \vec{0}) \right\rangle = -\lim_{p \to 0} \text{Re} \Pi_{00}^R(0, p),$$

(8)

where the number operator, $\mathcal{N} = \int J_0(t, \vec{x}) \, d^3x = \int \bar{\psi}(x) \Gamma_0 \psi(x) d^3x$ and $\Pi_{00}^R(\omega_p, p)$ is the retarded time-time component of the Fourier transformed vector correlator $\Pi_{\sigma\nu}(\omega_p, \vec{p})$ with an external momentum $P = (\omega_p, |\vec{p}| = p)$. To write (8) in such a compact form we have used the fluctuation-dissipation theorem and the quark number conservation [12, 13], $\lim_{\vec{p} \to 0} \text{Im} \Pi_{00}^R(\omega_p, p) \propto \delta(\omega_p)$.

III. MODIFICATION ON HARD THERMAL LOOP PERTURBATION THEORY

The HTL Lagrangian density for quark including HTL correction term [14] is written as

$$\mathcal{L}_{\text{HTL}} = \mathcal{L}_{\text{QCD}} + \delta \mathcal{L}_{\text{HTL}}$$

$$= \bar{\psi} i \gamma_\mu D^\mu \psi + m_q^2 \bar{\psi} \gamma_\mu \left\langle \frac{R^\mu}{i R \cdot D} \right\rangle \psi,$$

(9)

where $R = (1, \vec{r})$ is a light like four-vector, $\psi$ and $\bar{\psi}$ are the fermionic fields, $D$ is the covariant derivative, $\langle \rangle$ is the average over all possible directions over loop momenta. The second term is gauge invariant, nonlocal and can generate $N$-point HTL functions [1], which
are inter-related through Ward identities. Now $m_q$, is the quark mass in a hot and dense medium, which depends on the strong coupling $g$, temperature $T$ and the chemical potential $\mu$. Despite these facts $m_q$ is treated in (9) as a parameter much like the rest mass of a quark and a HTLpt has been developed [4] around this rest mass (i.e., $m_q^0$) by reorganising the HTL terms where $m_q^2$ was treated as the order of $(gT)^0$ for a hot system. In this way the effect of $m_q^2$ is taken into account in higher orders much like a variational principle. For a hot and dense system we will also treat the mass parameter as the order of $(gT)^0$ and $(g\mu)^0$, and reorganise the HTL term based on the variation of the external source and setting it zero at the end. In this way the effect of $m_q^2$ is taken into account to the higher order variations of the external source and thus to the response of the system.

We note that the covariant derivative usually contains background field or any source, $j$ depending upon the physical requirement. To motivate the perspective we define the covariant derivative $D^\mu$ as

$$D^\mu = [\mathcal{D}^\mu - i\delta^\mu \delta j] = [\tilde{\mathcal{D}}^\mu - i\delta^\mu \delta j].$$

We note that $\mathcal{D}$ contains gauge coupling and $\tilde{\mathcal{D}}^\mu = \mathcal{D}^\mu - i\delta^\mu j$, and $\delta j$ is an infinitesimal change in external source to which the response of the system can be calculated, as discussed in Sec.II. Later it can be identified with a variation of some physical quantity depending upon the requirement of the system under consideration.

Now, expanding the second term in (9), we can write as

$$\mathcal{L}_{HTL}(j + \delta j) = \bar{\psi} \left( i\tilde{\mathcal{D}} + \Sigma \right) \psi + \delta j \bar{\psi} \Gamma_0 \psi + \delta j^2 \bar{\psi} \frac{\Gamma_{00}}{2} \psi + \mathcal{O}(\delta j^3)$$

$$= \mathcal{L}_{HTL}(j) + \delta j \bar{\psi} \Gamma_0 \psi + \delta j^2 \bar{\psi} \frac{\Gamma_{00}}{2} \psi + \mathcal{O}(\delta j^3) \quad (11)$$

where the various $N$-point functions in coordinate space are generated as

$$\Sigma = m_q^2 \left\langle \frac{\mathcal{R}}{i \mathcal{R} \cdot \mathcal{D}} \right\rangle, \quad \Gamma_0 = \delta^\mu \gamma_\mu - m_q^2 \left\langle \frac{\mathcal{R} \mathcal{R}_\mu \delta^\mu}{(i \mathcal{R} \cdot \mathcal{D})^2} \right\rangle, \quad \Gamma_{00} = 2m_q^2 \bar{\psi} \left\langle \frac{\mathcal{R} \mathcal{R}_\mu \mathcal{R}_\nu \delta^\mu \delta^\nu}{(i \mathcal{R} \cdot \mathcal{D})^3} \right\rangle \quad (12)$$

where these functions can easily be transformed into momentum space [15]. We now note that these $N$-point HTL functions in (12) are also inter-related by Ward identities. In HTL-approximation the 2-point function, $\Sigma \sim gT$ (quark-self energy) is of the same order as the tree level one, $S_0^{-1}(K) \sim K \sim gT$ (in the weak coupling limit $g << 1$), if the external momenta are soft, i.e., of the order of $gT$. The 3-point function is given by $g\Gamma_\nu = g(\gamma_\nu + \delta \Gamma_\nu)$,
where \( \delta \Gamma \) is the HTL correction. The 4-point function, \( g^2 \Gamma_{\nu \sigma} \), is higher order and does not exist at the bare perturbation theory and only appears within the HTL approximation \([1, 14]\).

Now considering the HTL Lagrangian in \([11]\), we can write the partition function \([15]\) as
\[
Z[\beta; j + \delta j] = \int D[\bar{\psi}] D[\psi] D[A] e^{i \int d^4 x \mathcal{L}_{HTL}(\psi, \bar{\psi}; j + \delta j)},
\]  
(13)

where \( \beta = 1/T \), is the inverse of the temperature and \( A \) is a background gauge field.

The pressure can be written as
\[
P[\beta; j + \delta j] = \frac{1}{V} \ln Z[\beta; j + \delta j],
\]
(14)

where the four-volume, \( V = \beta V \) with \( V \) is the three-volume.

Expanding \( P \) in Taylor series around \( \delta j \) one can write
\[
P[\beta; j + \delta j] = P[\beta; j] + \delta j P'[\beta; j + \delta j]|_{\delta j \to 0} + \frac{\delta j^2}{2} P''[\beta; j + \delta j]|_{\delta j \to 0} + \cdots .
\]  
(15)

The first derivative of \( P \) w.r.t. \( j \) is related to the conserved density in \([1]\) whereas the second derivative is related to the conserved density fluctuation in \([3]\). The above expansion in \([15]\) is very important for a resummed perturbation theory. We now note that a HTLpt was developed in Ref. \([4]\) by considering the first term in \([15]\) with \( j = 0 \), which caused an over-inclusion of the LO pressure. This was cured by going into two-loop level in HTLpt \([5]\), which is of-course a very involved in nature. As we will see below this could easily be corrected if one constructs a HTLpt at the first derivative level of \( P \) in \([13]\) where the effect of the variation of external field is taken into account.

Now \( P' \) can be obtained as
\[
\left. \frac{\partial P[\beta; j + \delta j]}{\partial j} \right|_{\delta j \to 0} = \frac{i}{V Z[\beta; j]} \int D[\bar{\psi}] D[\psi] D[A] \int d^4 x \bar{\psi}(x) \Gamma_0[j] \psi(x) \exp(i \int d^4 x \mathcal{L}_{HTL}(\psi, \bar{\psi}; j)) \int D[\bar{\psi}] D[\psi] D[A] \exp(i \int d^4 x \mathcal{L}_{HTL}(\psi, \bar{\psi}; j)).
\]  
(16)

where we have used \([12]\). The full HTL quark propagator in presence of uniform \( j \) can be written as
\[
S_{\alpha \sigma}[j](x, x') = \frac{\int D[\bar{\psi}] D[\psi] D[A] \bar{\psi}_\sigma(x) \bar{\psi}_\sigma(x') \exp(i \int d^4 x \mathcal{L}_{HTL}(\psi, \bar{\psi}; j))}{\int D[\bar{\psi}] D[\psi] D[A] \exp(i \int d^4 x \mathcal{L}_{HTL}(\psi, \bar{\psi}; j))}. \]
(17)

We now note that this full HTL propagator, \( S[j] \), is indeed difficult to calculate and we would approximate it by 1-loop HTL resummed propagator \([1, 14]\), \( S^*[j] \) and also other HTL functions below.
Now using (17) and performing the traces over the colour, flavour, Dirac and coordinate indices in (16) one can write
\[ \partial P[\beta; j + \delta j] \bigg|_{\delta_j=0} = -i \int \frac{d^4K}{(2\pi)^4} \text{tr} \left[ S^*[j](K) \Gamma_0[j](K, -K; 0) \right] , \]
where 'tr' indicates the trace over the colour, flavour and Dirac indices.

Similarly, we obtain \( P'' \) as
\[ \frac{\partial^2 P[\beta; j + \delta j]}{\partial j^2} \bigg|_{\delta_j=0} = -N_c N_f T \int \frac{d^3k}{(2\pi)^3} \times \sum_{k_0} \text{Tr} \left[ S^*[j](K) \Gamma_0[j](K, -K; 0) S^*[j](K, -K; 0) \right. \]
\[ \left. + S^*[j](K) \Gamma_0[j](K, -K; 0, 0) \right] , \]
where \( N_f \) is the number of massless flavours, \( N_c \) is the number of colour and 'Tr' indicates the trace over only the Dirac matrices. We have also used an identity based on unitarity of \( S^*[j] \) as
\[ \frac{\partial S^*[j](K)}{\partial j} = -\partial^{-1}[j](K) S^*[j](K) = -S^*[j](K) \Gamma_0[j](K, -K; 0) S^*[j](K) \]
(20)

Now, if we identify \( j \) as the quark chemical potential \( \mu \), and \( \delta j \) as its change \( \delta \mu \), then (19) would represent the QNS as
\[ \chi(\beta) = \frac{\partial \rho}{\partial \mu} \bigg|_{\mu=0} = \frac{\partial^2 P}{\partial \mu^2} \bigg|_{\mu=0} = -N_c N_f T \int \frac{d^3k}{(2\pi)^3} \times \sum_{k_0=(2n+1)\pi iT} \text{Tr} \left[ S^*[K) \Gamma_0(K, -K; 0) S^*[K) \Gamma_0(K, -K; 0) - S^*[K) \Gamma_00(K, -K; 0, 0) \right] , \]
where the temporal correlation functions at the external momentum \( P = (\omega, |\vec{p}|) = 0 \), is related to the thermodynamic derivatives\(^1\). The first term in the second line of (21) is a 1-loop self-energy whereas the second term corresponds to a tadpole in HTLpt with effective \( N \)-point HTL functions.

Now, (18) represents the LO net quark number density in presence of uniform external field \( \mu \) as
\[ \rho(\beta, \mu) = \frac{\partial P}{\partial \mu} = -i \int \frac{d^4K}{(2\pi)^4} \text{tr} \left[ S^*[\mu](K) \Gamma_0[\mu](K, -K; 0) \right] \]

\(^1\) As already discussed Ref. [8] dealt with the definition of QNS that involves the static limit of the imaginary part of the dynamical charge-charge correlator. If one uses the number conservation directly, viz., the imaginary part of the charge-charge correlator is proportional to \( \delta(\omega_p) \), then it becomes equal to (21) in which charge conservation is in-built by construction.
\[
\begin{align*}
= N_c N_f T \int \frac{d^3k}{(2\pi)^3} \sum_{k_0=(2n+1)\pi T+\mu} \text{Tr} [S^*(K) \Gamma_0(K, -K; 0)] .
\end{align*}
\] (22)

\[
\frac{\partial}{\partial \mu} \left[ \begin{array}{c}
\end{array} \right] \equiv \begin{array}{c}
\end{array}
\]

FIG. 1: 1-loop Feynman diagram in HTLpt for quark number density, \( \rho_q \) that originates with the variation of \( \mu \) of the 1-loop HTLpt pressure. The dashed line represents the background field. The solid blobs are 1-loop resummed HTL \( N \)-point functions.

\[
\frac{\partial}{\partial \mu} \left[ \begin{array}{c}
\end{array} \right] \equiv \begin{array}{c}
\end{array}
\]

FIG. 2: Same as Fig.1 but for the lowest order bare perturbation theory.

\[
\frac{\partial}{\partial \mu} \left[ \begin{array}{c}
\end{array} \right] \neq \begin{array}{c}
\end{array}
\]

FIG. 3: The \( \neq \) sign indicates that it is not the correct diagram in the right hand side (rhs) as the \( \mu \) variation is not taken into account properly. The diagram in rhs actually corresponds to \( \rho_q \) that was obtained in Ref. [7].

When \( \mu \to 0 \), the net quark density in (22) would vanish as there is no broken CP symmetry, which becomes consistent with (4). Also, (22) constitutes a LO HTLpt in the first derivative level of \( P \) (see Fig. 1) similar to the usual perturbation theory where the bare \( N \)-point functions (see Fig. 2) are automatically replaced by the 1-loop resummed HTL \( N \)-point functions. This suggests that HTL resummation technique provides a consistent
perturbative expansion if one goes beyond the lowest order perturbation theory. In contrast Ref. [7] did not employ the variation of the external source as done in (15), which leads to Fig. 3 with a bare vertex for the calculation of the net quark number density. This resulted in overcounting of the LO QNS. It also required an ad hoc separation scale to distinguish between soft and hard momenta.

Below we briefly outline some of the essential quantities in HTL resummation [1], which are required to compute (22). The resummed HTL propagator in 1-loop approximation for momentum $K$ is given as

$$S^\star(K) = -\frac{\gamma_0 - \vec{\gamma} \cdot \hat{k}}{2D_+(k_0, k)} - \frac{\gamma_0 + \vec{\gamma} \cdot \hat{k}}{2D_-(k_0, k)}, \quad (23)$$

with

$$D_\pm(k_0, k) = -k_0 \pm k + \frac{m_q^2}{k} \left[ \frac{1}{2} \left( 1 \mp \frac{k_0}{k} \right) \ln \frac{k_0 + k}{k_0 - k} \mp 1 \right], \quad (24)$$

$$m_q^2 = \frac{g^2}{6} \left( T^2 + \mu^2 \right). \quad (25)$$

where $g^2 = 4\pi\alpha_s$, $\alpha_s$ is the strong coupling. Now, the zeros of $D_\pm$ describe [17] the in-medium propagation or quasiparticle (QP) dispersion of a particle excitation with energy $\omega_+$ having chirality to helicity ratio +1, and of a mode called plasmino with energy $\omega_-$ having chirality to helicity ratio −1. In addition, $D_\pm$ contains a discontinuous part corresponding to Landau Damping (LD) due to the presence of Logarithmic term in (24). Using these general properties of the quark propagator one can obtain the in-medium spectral function for quarks.

The pole part of the spectral function can be written as

$$\varrho_\pm(\omega, k) = \frac{\omega^2_\pm - k^2}{2m_q^2} \delta(\omega - \omega_\pm) + \frac{\omega^2_\pm - k^2}{2m_q^2} \delta(\omega + \omega_\pm), \quad (26)$$

as $D_+$ has poles at $\omega_+$ and $-\omega_-$ whereas those of $D_-$ are at $\omega_-$ and $-\omega_+$.

For $k_0^2 < k^2$, there is a discontinuity in $\ln \frac{k_0 + k}{k_0 - k}$ as $\ln y = \ln |y| - i\pi$, which leads to the spectral function, $\beta_\pm(\omega, k)$, corresponding to the discontinuity in $D_\pm(k_0, k)$ as

$$\beta_\pm(\omega, k) = -\frac{1}{\pi} \text{Disc} \frac{1}{D_\pm(k_0, k)} = -\frac{1}{\pi} \text{Im} \left. \frac{1}{D_\pm(k_0, k)} \right|_{k_0 \to \omega_+ + i\epsilon}$$

$$= \frac{m_q^2}{2k} \left( \pm \frac{\omega}{k} - 1 \right) \Theta(k^2 - \omega^2) \left[ \omega \mp k - m_q^2 \left( \pm 1 - \frac{\omega}{2k} \ln \frac{k + \omega}{k - \omega} \right) \right]^2 + \left[ \frac{m_q^2}{2k} \left( 1 \mp \frac{\omega}{k} \right) \right]^2. \quad (27)$$
The zero momentum limit of the 3-point HTL function can be obtained from the Ward identity \[15, 16\] as

\[
\Gamma_0(K, -K; 0) = \frac{\partial}{\partial k_0} \left( S^{*, -1}(K) \right) = a_0 + b \gamma_0 \cdot \hat{k},
\]

where

\[
a \pm b = -D_\pm(k_0, k),
\]

with

\[
D_\pm = \frac{D_\pm}{k_0 \mp k} - \frac{2m_q^2}{k_0^2 - k^2}.
\]

IV. THERMODYNAMICS AND QUARK NUMBER SUSCEPTIBILITY

We first obtain the net quark density \(\rho(T, \mu)\), which is then used to obtain various thermodynamic quantities, viz., pressure, entropy density and QNS.

A. Free case

In free case the number density can be written from (22) as

\[
\rho_f(T, \mu) = N_c N_f T \int \frac{d^3 k}{(2\pi)^3} \sum_{k_0=(2n+1)\pi T + \mu} \text{Tr}[S_f(K) \gamma_0],
\]

where the 3-point function is \(\Gamma_0 = \gamma_0\) and the 2-point function is the free quark propagator for momentum \(K\) is given as

\[
S_f(K) = -\frac{\gamma_0 - \gamma \cdot \hat{k}}{2d_+(k_0, k)} - \frac{\gamma_0 + \gamma \cdot \hat{k}}{2d_-(k_0, k)},
\]

with

\[
d_\pm = -k_0 \pm k.
\]

Using (32) in (31) and performing the trace over Dirac matrices, we get

\[
\rho_f(T, \mu) = 2N_c N_f T \int \frac{d^3 k}{(2\pi)^3} \sum_{k_0=(2n+1)\pi T + \mu} \left[ \frac{1}{k_0 - k} + \frac{1}{k_0 + k} \right].
\]

For evaluating the frequency sum in (34), we use the standard technique of contour integration \[16\] as

\[
\frac{1}{2\pi i} \oint_C \left[ \frac{1}{k_0 - k} + \frac{1}{k_0 + k} \right] \beta \tanh \left( \frac{\beta(k_0 - \mu)}{2} \right) dk_0 = \frac{\beta}{2} \frac{1}{2\pi i} \times (-2\pi i) \sum \text{Residues}.
\]
It can be seen that the first term of (35) has a simple pole at $k_0 = k$ whereas the second term has a pole at $k_0 = -k$. After calculating the residues of those two terms, the number density becomes

$$\rho^f(T, \mu) = -N_c N_f \int \frac{d^3 k}{(2\pi)^3} \left[ \tanh \frac{\beta (k - \mu)}{2} - \tanh \frac{\beta (k + \mu)}{2} \right]$$

$$= 2 N_c N_f \int \frac{d^3 k}{(2\pi)^3} \left[ n(k - \mu) - n(k + \mu) \right], \quad (36)$$

where $n(x) = 1/(e^{\beta x} + 1)$, is the Fermi distribution function.

Now, the pressure is obtained by integrating the first line of (36) w.r.t. $\mu$ as

$$P^f(T, \mu) = 2 N_f N_c T \int \frac{d^3 k}{(2\pi)^3} \left[ \beta k + \ln \left( 1 + e^{-\beta (k - \mu)} \right) + \ln \left( 1 + e^{-\beta (k + \mu)} \right) \right], \quad (37)$$

where the first term is the zero-point energy that generates a usual vacuum divergence [16].

The entropy density in free case can be written from pressure as

$$S^f(T, \mu) = \frac{\partial P^f}{\partial T} = 2 N_c N_f \int \frac{d^3 k}{(2\pi)^3} \left[ \ln \left( 1 + e^{-\beta (k - \mu)} \right) + \ln \left( 1 + e^{-\beta (k + \mu)} \right) \right.$$

$$\left. + \frac{\beta (k - \mu)}{e^{\beta (k - \mu)} + 1} + \frac{\beta (k + \mu)}{e^{\beta (k + \mu)} + 1} \right] = N_c N_f \left( \frac{7\pi^2 T^3}{45} + \frac{\mu^2 T}{3} \right). \quad (38)$$

The QNS is obtained as

$$\chi^f(T) = \frac{\partial}{\partial \mu} \left[ \rho^f(T, \mu) \right] \big|_{\mu = 0} = 4 N_c N_f \beta \int \frac{d^3 k}{(2\pi)^3} n(k) (1 - n(k)) = N_f T^2. \quad (39)$$

B. HTLpt Case

Using (23), (28) in (22) and then performing the trace over Dirac matrices, the quark number density in HTLpt becomes

$$\rho^{HTL}(T, \mu) = 2 N_c N_f T \int \frac{d^3 k}{(2\pi)^3} \sum_{k_0 = (2n+1)\pi T + \mu} \left[ \frac{D'_+}{D_+} + \frac{D'_-}{D_-} \right]$$

$$= 2 N_c N_f T \int \frac{d^3 k}{(2\pi)^3} \sum_{k_0} \left[ \frac{1}{k_0 - k} + \frac{1}{k_0 + k} - \frac{2m^2}{k_0^2 - k^2} \left( \frac{1}{D_+} + \frac{1}{D_-} \right) \right]. \quad (40)$$

Apart from the various poles due to QPs in (40) it has LD part as $D_{\pm}(k_0, k)$ contain Logarithmic terms which generate discontinuity for $k_0^2 < k^2$, as discussed earlier. Equation (40) can be decomposed in individual contribution as

$$\rho^{HTL}(T, \mu) = \rho^{OP}(T, \mu) + \rho^{LD}(T, \mu). \quad (41)$$
1. **Quasiparticle part (QP)**

The pole part of the number density can be written as

\[
\rho_{\text{QP}}(T, \mu) = 2N_c N_f T \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\pi i} \oint_{C'} \left[ \frac{1}{k_0 - k} + \frac{1}{k_0 + k} - \frac{2m_q^2}{k_0^2 - k^2} \left( \frac{1}{D_+} + \frac{1}{D_-} \right) \right] \times \frac{\beta}{2} \tanh \frac{\beta(k_0 - \mu)}{2} dk_0.
\]

In general residues for various poles in third and fourth terms in (42) can be obtained as

\[
\text{Res} \left\{ \frac{2m_q^2}{k_0^2 - k^2} \frac{1}{D_\pm} \right\} \bigg|_{k_0 = \omega_\pm, -\omega_\pm} = -1 ; \quad \text{Res} \left\{ \frac{2m_q^2}{k_0^2 - k^2} \frac{1}{D_\pm} \right\} \bigg|_{k_0 = \pm k} = 1,
\]

where \( D_\pm(k_0 = \pm k) = \pm \frac{m_q^2}{k^2} \).

1. The first two terms in (42) give the same contribution as free case in (36).

2. The third term has four simple poles at \( k_0 = \omega_+, -\omega_-, k, -k \). After performing the contour integration the third term can be written as

\[
\frac{1}{2\pi i} \oint_{C'} \frac{2m_q^2}{k_0^2 - k^2} \frac{1}{D_+} \frac{\beta}{2} \tanh \frac{\beta(k_0 - \mu)}{2} dk_0 = -\frac{\beta}{2} \left[ -\tanh \frac{\beta(\omega_+ - \mu)}{2} + \tanh \frac{\beta(\omega_- + \mu)}{2} + \tanh \frac{\beta(k - \mu)}{2} - \tanh \frac{\beta(k + \mu)}{2} \right].
\]

3. The fourth term has four simple poles at \( k_0 = \omega_-, -\omega_+, -k, k \). After performing the contour integration the fourth term can be written as

\[
\frac{1}{2\pi i} \oint_{C'} \frac{2m_q^2}{k_0^2 - k^2} \frac{1}{D_-} \frac{\beta}{2} \tanh \frac{\beta(k_0 - \mu)}{2} dk_0 = -\frac{\beta}{2} \left[ -\tanh \frac{\beta(\omega_- - \mu)}{2} + \tanh \frac{\beta(\omega_+ + \mu)}{2} + \tanh \frac{\beta(k - \mu)}{2} - \tanh \frac{\beta(k + \mu)}{2} \right].
\]

Using (36), (44) and (45) in (42) one can obtain the HTL quasiparticle contributions to the quark number density as

\[
\rho_{\text{QP}}(T, \mu) = -N_c N_f \int \frac{d^3k}{(2\pi)^3} \left[ \tanh \frac{\beta(\omega_+ - \mu)}{2} + \tanh \frac{\beta(\omega_- - \mu)}{2} - \tanh \frac{\beta(k - \mu)}{2} 
\right.

\[
- \tanh \frac{\beta(\omega_+ + \mu)}{2} - \tanh \frac{\beta(k + \mu)}{2} \right] \right]
\]

\[= 2N_c N_f \int \frac{d^3k}{(2\pi)^3} \left[ n(\omega_+ - \mu) + n(\omega_- - \mu) - n(k - \mu) - n(\omega_+ + \mu) \right].
\]
\[-n(\omega_- + \mu) + n(k + \mu)\] , \hspace{1cm} (46)

which agrees with that of the two-loop approximately self-consistent \(\Phi\)-derivable HTL resummation of Blaizot et al. \[2, 3\].

Now, the pressure is obtained by integrating the first line of (46) w.r.t. \(\mu\) as

\[
P_{QP}(T, \mu) = 2N_cN_fT \int \frac{d^3k}{(2\pi)^3} \left[ \ln \left(1 + e^{-\beta(\omega_+ - \mu)}\right) + \ln \left(\frac{1 + e^{-\beta(\omega_- - \mu)}}{1 + e^{-\beta(k - \mu)}}\right) + \ln \left(1 + e^{-\beta(\omega_+ + \mu)}\right) + \ln \left(\frac{1 + e^{-\beta(\omega_- + \mu)}}{1 + e^{-\beta(k + \mu)}}\right) + \beta \omega_+ + \beta(\omega_- - k) \right]. \hspace{1cm} (47)
\]

This agrees with the form given for quasiparticle contribution by Andersen et al. [4] considering the first term\(^2\) in (14) for \(\mu = 0\). Both quasiparticles with energies \(\omega_+\) and \(\omega_-\) generate \(T\) dependent ultra-violate (UV) divergences in LO HTL pressure, which is an artefact of 1-loop HTL approximation [2, 4]. At very high \(T\), \(\omega_\pm \rightarrow k\) and (47) reduces to free case as obtained in [37].

The corresponding HTL QP entropy density in LO can be obtained as

\[
S_{QP}(T, \mu) = \frac{\partial P_{QP}}{\partial T} = 2N_cN_f \int \frac{d^3k}{(2\pi)^3} \left[ \ln \left(1 + e^{-\beta(\omega_+ - \mu)}\right) + \ln \left(\frac{1 + e^{-\beta(\omega_- - \mu)}}{1 + e^{-\beta(k - \mu)}}\right) + \ln \left(1 + e^{-\beta(\omega_+ + \mu)}\right) + \ln \left(\frac{1 + e^{-\beta(\omega_- + \mu)}}{1 + e^{-\beta(k + \mu)}}\right) + \beta \omega_+ + \beta(\omega_- - k) \right].\hspace{1cm} (48)
\]

which agrees with that of the 2-loop approximately self-consistent \(\Phi\)-derivable HTL resummation of Blaizot et al. [2].

The QNS in LO due to HTL QP can also be obtained from (16) as

\[
\chi_{QP}(T) = \left. \frac{\partial}{\partial \mu} \left[ \beta_{QP} \right] \right|_{\mu=0} = 4N_cN_f \beta \int \frac{d^3k}{(2\pi)^3} \left[ n(\omega_+)(1 - n(\omega_+)) + n(\omega_-)(1 - n(\omega_-)) - n(k)(1 - n(k)) \right] \hspace{1cm} (49)
\]

where the \(\mu\) derivative is performed only to the explicit \(\mu\) dependence. Obviously (19) agrees exactly with that of the 2-loop approximately self-consistent \(\Phi\)-derivable HTL resummation.

\(^2\) We note that the expression for QP pressure in one-loop HTLpt [4] was obtained by adding and subtracting the free gas pressure. However, in our formalism the correct LO form comes out naturally and no addition and subtraction is required as in Ref. [4]. This is because the fluctuation of the conserved density is appropriately taken into consideration in the present formalism.
of Blaizot et al. The above thermodynamical quantities in LO due to HTL quasiparticles with excitation energies \( \omega_\pm \) are similar in form to those of free case but the hard and soft contributions are clearly separated out and one does not need an ad hoc separating scale as used in Ref. [7].

2. Landau Damping part (LD)

The LD part of the quark number density follows from (40) and (27) as

\[
\rho^{LD}(T, \mu) = N_c N_f \int \frac{d^3k}{(2\pi)^3} \int_{-k}^{k} d\omega \frac{-2m_q^2}{\omega^2 - k^2} \pi [\beta_+(\omega, k) + \beta_-(\omega, k)] \tanh \frac{\beta(\omega - \mu)}{2}
\]

\[
= N_c N_f \int \frac{d^3k}{(2\pi)^3} \int_{-k}^{k} d\omega \left( \frac{2m_q^2}{\omega^2 - k^2} \right) \beta_+(\omega, k) [n(\omega - \mu) - n(\omega + \mu)] .
\]

(50)

One can obtain the pressure due to LD contribution by integrating (50) w.r.t. \( \mu \) as

\[
P^{LD}(T, \mu) = N_c N_f T \int \frac{d^3k}{(2\pi)^3} \int_{-k}^{k} d\omega \left( \frac{2m_q^2}{\omega^2 - k^2} \right) \beta_+(\omega, k) \ln \left( 1 + e^{-\beta(\omega - \mu)} \right)
\]

\[+ \ln \left( 1 + e^{-\beta(\omega + \mu)} \right) + \beta \omega \],

(51)

which has UV divergence like Andersen et al [4] and can be removed using the appropriate prescription therein.

The corresponding LD part of entropy density can be obtained as

\[
S^{LD}(T, \mu) = N_c N_f \int \frac{d^3k}{(2\pi)^3} \int_{-k}^{k} d\omega \left( \frac{2m_q^2}{\omega^2 - k^2} \right) \beta_+(\omega, k) \ln \left( 1 + e^{-\beta(\omega - \mu)} \right)
\]

\[+ \ln \left( 1 + e^{-\beta(\omega + \mu)} \right) + \frac{\beta(\omega - \mu)}{e^{\beta(\omega - \mu)} + 1} + \frac{\beta(\omega + \mu)}{e^{\beta(\omega + \mu)} + 1} \].

(52)

Also the LD part of the QNS becomes

\[
\chi_1^{LD}(T) = \frac{\partial}{\partial \mu} [\rho_1^{LD}(T, \mu)] \bigg|_{\mu=0} = 2N_c N_f \beta \int \frac{d^3k}{(2\pi)^3} \int_{-k}^{k} d\omega \left( \frac{2m_q^2}{\omega^2 - k^2} \right)
\]

\[\times \beta_+(\omega, k) n(\omega) (1 - n(\omega)) ,
\]

(53)

where the \( \mu \) derivative is again performed only to the explicit \( \mu \) dependence. It is also to be noted that the LD contribution is of the order of \( m_q^4 \). The LD contribution can
not be compared with that of the 2-loop approximately self-consistent \Phi\text{-derivable HTL resummation of Blaizot et al.} as it does not have any closed form for the final expression. The numerical values of both the QNS agree very well.

It is clearly evident that the various LO thermodynamic quantities in 1-loop HTL\text{pt} can be obtained within this modified formalism at ease instead of pushing the calculation to a more involved approaches. Below we demonstrate the correct inclusion of the perturbative content of the order \(g^2\) to the QNS in HTL\text{pt} in a strict perturbative sense by comparing with the usual perturbation theory.

V. QNS IN PERTURBATIVE LEADING ORDER (\(g^2\))

In conventional perturbation theory, for massless QCD the QNS has been calculated up to order \(g^4\log(1/g)\) at \(\mu = 0\) as

\[
\frac{\chi_p}{\chi_f} = 1 - \frac{1}{2} \left(\frac{g}{\pi}\right)^2 + \sqrt{1 + \frac{N_f}{6} \left(\frac{g}{\pi}\right)^3} - \frac{3}{4} \left(\frac{g}{\pi}\right)^4 \log \left(\frac{1}{g}\right) + \mathcal{O}(g^4) .
\] (54)

We note that for all temperatures of relevance the series decreases with temperature and approaches the ideal gas value from the above, which is due to the convergence problem of the conventional perturbation series.

Nevertheless, the perturbative LO, \(g^2\), contribution is also contained in HTL approximation through the \(N\)-point HTL functions. In the left panel of Fig. 4 we display the LO HTL QNS and LO perturbative QNS scaled with free one as a function of \(m_q/T\), the effective strong coupling. In the weak coupling limit both approach unity whereas the HTL case has a little slower deviation from the ideal gas value than the LO of the conventional perturbative one. The latter could be termed as an improvement over the conventional perturbative results. The results are in very good agreement with that of Ref. [3].

Next we consider a ratio as

\[
R \equiv \frac{\chi_{htl} - \chi_f}{\chi_{p(g^2)} - \chi_f} ,
\] (55)

which measures the deviation of interaction of \(\chi_{htl}\) from that of pQCD to order \(g^2\). In the right panel of Fig. 4 we display this ratio as a function of \(m_q/T\), which approaches unity in the weak coupling limit indicating the correct inclusion of order \(g^2\) in our approach in a strictly perturbative sense. This comes from the \(\omega_+\) branch of the HTL dispersion
FIG. 4: (Color online) Left panel: The ratio of 2-flavour HTL to free quark QNS and also that of LO perturbative one as a function of $m_q/T$. Right panel: The interaction measure $R$ as a function of $m_q/T$.

relations, $\omega_+(k) \approx k + m_q^2/k$, at hard momentum scale, i.e., $k \sim T$. With this one can now trivially show by expanding the QP contribution from (49) in Sect. IV that becomes

$$\chi_q^{QP} = \chi_f (1 - g^2/2\pi^2 + \cdots), \quad (56)$$

which agrees with that in (54). The LD contribution is of the order of $m_q^4$.

We note that the HTL resummation technique provides a consistent perturbative expansion for gauge theories at finite temperature and/or density. As discussed going beyond the lowest order bare perturbation theory for quark number density, we use the HTL resummed propagator and quark-gluon vertices in Fig. [1]. The resummed HTL quark propagators correspond to static external quarks (valence quark). In 1-loop HTLpt (viz., Fig. [1]) there is no dynamical quark (no quark loop) and in this sense 1-loop HTLpt is comparable with the quenched approximation of lattice QCD [19]. The inclusion of dynamical quark loops requires one to consider the higher-order diagrams within HTLpt in which HTL resummed gluon propagators (containing quark loops) will show up. This could be taken care through (11) as it contains the covariant derivative with gauge coupling and the calculation is in progress.
VI. CONCLUSION

In the literature the HTL resummation have been used through various approaches to calculate the thermodynamic quantities and also the response of the system, viz., the QNS to an external perturbation, \textit{i.e.}, the quark chemical potential. This led to different results in LO indicating the sensitivity of the methods. In this paper we revisited the thermodynamic quantities and in particular the QNS in LO within HTLpt to arrive at similar results within the various HTL approaches. For this purpose we modified the existing HTLpt \cite{4} at the first derivative level of pressure by incorporating an infinitesimal variation to an external source, viz., the quark chemical potential that disturbs the system only slightly. We show that the various thermodynamic quantities and the QNS in LO order agree with those of the two-loop approximately self-consistent $\Phi$-derivable HTL resummation approach \cite{2,3} existing in the literature. Our calculation also shows that the soft and hard momenta get separated out naturally and one does not require any ad hoc separating scale as in \cite{7}. All the thermodynamic quantities turned out to be dependent on the chemical potential automatically due to the method employed. We also discussed that our formalism can also be extended for higher-order calculations.

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