Study of the $\Upsilon(1S) \to B_c D_s^*$ decay with pQCD approach

Junfeng Sun,$^1$ Yueling Yang,$^1$ Qingxia Li,$^1$ Gongru Lu,$^1$ Jinshu Huang,$^2$ and Qin Chang$^1$

$^1$Institute of Particle and Nuclear Physics,
Henan Normal University, Xinxiang 453007, China

$^2$College of Physics and Electronic Engineering,
Nanyang Normal University, Nanyang 473061, China

Abstract

The $\Upsilon(1S) \to B_c D_s^*$ weak decay is studied with the perturbative QCD approach firstly. It is found that (1) main contributions to branching ratio come from the longitudinal and parallel helicity amplitudes, (2) branching ratio, longitudinal and parallel polarization fractions are sensitive to the wave functions of the $\Upsilon(1S)$ meson, (3) branching ratio for the $\Upsilon(1S) \to B_c D_s^*$ decay can reach up to $10^{-9}$, which might be promisingly measured by the future experiments.
I. INTRODUCTION

The Υ(1S) meson is the ground spin-triplet $S$-wave state of bottomonium (bound state consists of both the bottom quark $b$ and the anti-bottom quark $\bar{b}$) with the quantum number of $I^GJ^{PC} = 0^-1^{--}$ [1]. The Υ(1S) meson lies below the kinematic open-bottom threshold. It is generally believed that the Υ(1S) meson decays mainly through the annihilation of the $b\bar{b}$ pairs into three gluons, one photon, two gluons plus one photon, with branching ratios [1] of some $(81.7\pm0.7)\%$, $(3 + R)B_{\ell\ell}$, $(2.2\pm0.6)\%$, respectively, where $R \approx 10/3$ is the ratio of the hadron production rate to the lepton $\mu^+\mu^-$ pair production rate at the energy scale of $m_\Upsilon(1S)$, and $B_{\ell\ell}$ is the branching ratio for the pure leptonic $\Upsilon(1S) \to \ell^+\ell^-$ decay. One of the prominent features is its narrow width, $\Gamma_{\Upsilon(1S)} = 54.02\pm1.25$ keV [1], while the Υ(1S) meson is about ten times heavier than the nucleon, $m_{\Upsilon(1S)} = 9460.30\pm0.26$ MeV [1]. This fact can be explained by the following argument. The hadronic decay $\Upsilon(1S) \to ggg$ is suppressed by the phenomenological Okubo-Zweig-Iizuka (OZI) rule [2–4]. The branching ratio $B_{\ell\ell}$ is proportional (a) to the square of the electric charge of the bottom quark, $e_b^2$ where $e_b = -1/3$ in the unit of $|e|$; (b) to the square of the electromagnetic coupling constant, $\alpha^2$ where $\alpha \approx 1/128$; and (c) to the energy dependence of the photon propagator, $1/m_{\Upsilon(1S)}^2$ [5].

Besides the above-mentioned strong, electromagnetic and radiative decay mechanisms, the Υ(1S) meson can also decay via the weak interaction within the standard model. As it is well known, over $10^8$ Υ(1S) data samples have been accumulated at Belle [6]. More and more upsilon data samples are hopefully expected at the running LHC and the forthcoming SuperKEKB. Although branching ratio for the Υ(1S) weak decay is tiny, about $2/\tau_B \Gamma_{\Upsilon(1S)} \sim \mathcal{O}(10^{-8})$ where $\tau_B$ is the lifetime of the $B_{u,d}$ meson, there seems to exist a realistic possibility to search for the Υ(1S) weak decay at future experiments. In this paper, we will study the $\Upsilon(1S) \to B_c D_s^*$ weak decay with the perturbative QCD (pQCD) approach [7–9].

Experimentally, branching ratios for the two-body leptonic $\Upsilon(1S)$ decays, some OZI-suppressed hadronic $\Upsilon(1S)$ decays and radiative decays have been measured, but there is still no measurement report on the magnetic dipole transition decay $\Upsilon(1S) \to \gamma\eta_b$ and weak decays for the moment [1]. The $\Upsilon(1S)$ weak decay posses a unique structure due to the Cabibbo-Kobayashi-Maskawa (CKM) matrix properties which predicts the channels with one $B^{(*)}_s$ meson are dominant. The signals for the $\Upsilon(1S) \to B_c D_s^*$ weak decay should, in principle, be easily distinguished from possibly intricate background, due to the facts that
the back-to-back final states with opposite electric charges have definite momentum and energy in the rest frame of the $\Upsilon(1S)$ meson. The identification of a single flavored either $B_c$ or $D^*_s$ meson could be used as an effective selection criterion. Moreover, the radiative decay of the $D^*_s$ meson can provide a useful extra signal and a powerful constraint. Of course, any evidences of an abnormally large branching ratio for the $\Upsilon(1S)$ weak decay might be a hint of new physics.

Theoretically, in recent years, many attractive methods have been fully developed, such as the pQCD approach [7–9], the QCD factorization (QCDF) approach [10–12], soft and collinear effective theory [13–16], and widely applied to accommodate measurements on the $B$ meson weak decays. The $\Upsilon(1S)$ weak decays permit one to cross check parameters obtained from the $B$ meson decay, and to test various phenomenological models. The $\Upsilon(1S)$ weak decays into final states containing one $B_c$ meson are favorable processes due to the CKM factor $V_{cb}$, which also provide an additional occasion to scrutinize the underlying structure of doubly-heavy hadrons, and to improve our understanding on the short- and long-distance contributions in heavy quark weak decay. The semileptonic decays $\Upsilon(1S) \rightarrow B_c\ell^- \bar{\nu}_\ell$ ($\ell = e, \mu, \tau$) have been studied based on the Bauer-Stech-Wirbel model [17]. The two-body nonleptonic decays $\Upsilon(1S) \rightarrow B_cM$ ($M = \pi, K^{(*)}, \rho$) have been investigated recently by employing the factorization scheme, such as the naive factorization approximation [17, 18], the QCD-improved QCDF formulation [19, 20] and the pQCD approach [21, 22]. The $\Upsilon(1S) \rightarrow B_cD^*_s$ weak decay is favored by color and the CKM factor $|V_{cb}V_{cs}|$, so it should, in principle, have relatively large branching ratio among the $\Upsilon(1S)$ weak decays. However, there is still no theoretical study devoted to the $\Upsilon(1S) \rightarrow B_cD^*_s$ weak decays now. In this paper, we will investigate the $\Upsilon(1S) \rightarrow B_cD^*_s$ decay with the pQCD approach to offer a ready reference for the future experiments.

This paper is organized as follows. The theoretical framework and the amplitudes for the $\Upsilon(1S) \rightarrow B_cD^*_s$ decay are presented in section [III]. The numerical results and discussion are given in section [III]. The last section is a summary.
II. THEORETICAL FRAMEWORK

A. The effective Hamiltonian

The effective Hamiltonian responsible for the $\Upsilon(1S) \to B_c D_s^*$ weak decay is

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \left\{ V_{cb} V_{cs}^* \sum_{i=1}^{2} C_i(\mu) \bar{q}_i(\mu) g_{\mu} - V_{tb} V_{ts}^* \sum_{j=3}^{10} C_j(\mu) \bar{q}_j(\mu) \right\} + \text{H.c.}, \quad (1)$$

where the Fermi coupling constant $G_F \simeq 1.166 \times 10^{-5} \text{GeV}^{-2}$, with the Wolfenstein parameterization, the CKM factors are written as [1],

$$V_{cb} V_{cs}^* = + A \lambda^2 - \frac{1}{2} A \lambda^4 - \frac{1}{8} A \lambda^6 (1 + 4 A^2) + \mathcal{O}(\lambda^7),$$
$$V_{tb} V_{ts}^* = - V_{cb} V_{cs}^* A \lambda^{4} (\rho - i \eta) + \mathcal{O}(\lambda^7). \quad (2)$$

The local tree operators $Q_{1,2}$ and penguin operators $Q_{3,\ldots,10}$ are defined below.

- $Q_1 = [\bar{c}_\alpha \gamma_\mu (1 - \gamma_5) b_\alpha] [\bar{s}_\beta \gamma^\mu (1 - \gamma_5) c_\beta]$, \quad (4)
- $Q_2 = [\bar{c}_\alpha \gamma_\mu (1 - \gamma_5) b_\beta] [\bar{s}_\beta \gamma^\mu (1 - \gamma_5) c_\alpha]$, \quad (5)
- $Q_3 = \sum_q [\bar{s}_\alpha \gamma_\mu (1 - \gamma_5) b_\alpha] [\bar{q}_\beta \gamma^\mu (1 - \gamma_5) q_\beta]$, \quad (6)
- $Q_4 = \sum_q [\bar{s}_\alpha \gamma_\mu (1 - \gamma_5) b_\beta] [\bar{q}_\beta \gamma^\mu (1 - \gamma_5) q_\alpha]$, \quad (7)
- $Q_5 = \sum_q [\bar{s}_\alpha \gamma_\mu (1 - \gamma_5) b_\alpha] [\bar{q}_\beta \gamma^\mu (1 + \gamma_5) q_\beta]$, \quad (8)
- $Q_6 = \sum_q [\bar{s}_\alpha \gamma_\mu (1 - \gamma_5) b_\beta] [\bar{q}_\beta \gamma^\mu (1 + \gamma_5) q_\alpha]$, \quad (9)
- $Q_7 = \sum_q \frac{3}{2} e_q [\bar{s}_\alpha \gamma_\mu (1 - \gamma_5) b_\alpha] [\bar{q}_\beta \gamma^\mu (1 + \gamma_5) q_\beta]$, \quad (10)
- $Q_8 = \sum_q \frac{3}{2} e_q [\bar{s}_\alpha \gamma_\mu (1 - \gamma_5) b_\beta] [\bar{q}_\beta \gamma^\mu (1 + \gamma_5) q_\alpha]$, \quad (11)
- $Q_9 = \sum_q \frac{3}{2} e_q [\bar{s}_\alpha \gamma_\mu (1 - \gamma_5) b_\alpha] [\bar{q}_\beta \gamma^\mu (1 - \gamma_5) q_\beta]$, \quad (12)
- $Q_{10} = \sum_q \frac{3}{2} e_q [\bar{s}_\alpha \gamma_\mu (1 - \gamma_5) b_\beta] [\bar{q}_\beta \gamma^\mu (1 - \gamma_5) q_\alpha]$, \quad (13)

where $\alpha$ and $\beta$ are color indices; $q$ denotes all the active quarks at the scale of $\mu \sim \mathcal{O}(m_b)$, i.e., $q = u, d, s, c, b$; and $e_q$ is the electric charge of the $q$ quark in the unit of $|e|$. 


The Wilson coefficients $C_i(\mu)$ summarize the physical contributions above the scale of $\mu$, and could be reliably calculated with the renormalization group improved perturbation theory. The physical contributions below the scale of $\mu$ are included in the hadronic matrix elements (HME) where the local operators sandwiched between initial and final hadron states. To obtain the decay amplitudes, the remaining work is to calculate HME properly.

B. Hadronic matrix elements

Phenomenologically, combining the $k_T$ factorization theorem \cite{24} with the collinear factorization hypothesis, and using the Lepage-Brodsky approach for exclusive processes \cite{25}, HME can be written as the convolution of universal wave functions reflecting the nonperturbative contributions with hard scattering subamplitudes containing the perturbative contributions within the pQCD framework, where the transverse momentum of valence quarks is retained and the Sudakov factor is introduced, in order to regulate the endpoint singularities and provide a naturally dynamical cutoff on the nonperturbative contributions \cite{7–9}. Generally, the decay amplitude can be separated into three parts: the Wilson coefficients $C_i$ incorporating the hard contributions above the typical scale of $t$, the process-dependent scattering amplitudes $T$ accounting for the heavy quark decay, and the universal wave functions $\Phi$ including the soft and long-distance contributions, i.e.,

$$\int dk \, C_i(t) \, T(t, k) \, \Phi(k) e^{-S}, \quad (14)$$

where $k$ is the momentum of valence quarks, and $e^{-S}$ is the Sudakov factor.

C. Kinematic variables

The light cone kinematic variables in the $\Upsilon(1S)$ rest frame are defined as follows.

$$p_{\Upsilon(1S)} = p_1 = \frac{m_1}{\sqrt{2}}(1, 1, 0), \quad (15)$$

$$p_{B_c} = p_2 = (p_2^+, p_2^-, 0), \quad (16)$$

$$p_{D^*_s} = p_3 = (p_3^+, p_3^-, 0), \quad (17)$$

$$k_i = x_i p_i + (0, 0, \vec{k}_{iT}), \quad (18)$$
\[ \epsilon_i^\parallel = \frac{p_i}{m_i} - \frac{m_i}{p_i \cdot n_+} n_+, \]
\[ \epsilon_i^\perp = (0, 0, \vec{1}^i), \]
\[ n_+ = (1, 0, 0), \]
\[ p_i^\pm = (E_i \pm p)/\sqrt{2}, \]
\[ s = 2p_2 \cdot p_3, \]
\[ t = 2p_1 \cdot p_2 = 2m_1 E_2, \]
\[ u = 2p_1 \cdot p_3 = 2m_1 E_3, \]
\[ p = \sqrt{[m_1^2 - (m_2 + m_3)^2] [m_1^2 - (m_2 - m_3)^2]} \]
\[ 2m_1, \]
\[ (26) \]

where \( x_i \) and \( \vec{k}_{iT} \) are the longitudinal momentum fraction and transverse momentum of the valence quark, respectively; \( \epsilon_i^\parallel \) and \( \epsilon_i^\perp \) are the longitudinal and transverse polarization vectors, respectively, satisfying with the relations \( \epsilon_i^2 = -1 \) and \( \epsilon_i \cdot p_i = 0 \); the subscript \( i = 1, 2, 3 \) on variables \( p_i, E_i, m_i, \) and \( \epsilon_i^{\parallel \perp} \) correspond to the \( \Upsilon(1S), B_c, D_s^* \) mesons, respectively; \( n_+ \) is the null vector; \( s, t \) and \( u \) are the Lorentz-invariant variables; \( p \) is the common momentum of final states. The notation of momentum is displayed in Fig 2(a).

**D. Wave functions**

With the notation in [26, 27], the definitions of the diquark operator HME are

\[ \langle 0 | b_i(z) \bar{b}_j(0) | \Upsilon(p_1, \epsilon_1^\parallel) \rangle = \frac{f_{\Upsilon(1S)}}{4} \int dk_1 e^{-ik_1 \cdot z} \left\{ \epsilon_1^\parallel \left[ m_1 \phi_T^\parallel(k_1) - \phi_1 \phi_T^\parallel(k_1) \right] \right\}_{ji}, \]
\[ (27) \]

\[ \langle 0 | b_i(z) \bar{b}_j(0) | \Upsilon(p_1, \epsilon_1^\perp) \rangle = \frac{f_{\Upsilon(1S)}}{4} \int dk_1 e^{-ik_1 \cdot z} \left\{ \epsilon_1^\perp \left[ m_1 \phi_T^\perp(k_1) - \phi_1 \phi_T^\perp(k_1) \right] \right\}_{ji}, \]
\[ (28) \]

\[ \langle B_c(p_2) | \bar{c}_i(z) b_j(0) | 0 \rangle = \frac{i f_{B_c}}{4} \int dk_2 e^{ik_2 \cdot z} \left\{ \gamma_5 \left[ \phi_2 + m_2 \phi_{B_c}(k_2) \right] \right\}_{ji}, \]
\[ (29) \]

\[ \langle D_s^*(p_3, \epsilon_3^\parallel) | c_i(0) \bar{s}_j(z) | 0 \rangle = \frac{f_{D_s^*}}{4} \int_0^1 dk_3 e^{ik_3 \cdot z} \left\{ \epsilon_3^\parallel \left[ m_3 \phi_{D_s^*}(k_3) + \phi_3 \phi_{D_s^*}(k_3) \right] \right\}_{ji}, \]
\[ (30) \]

\[ \langle D_s^*(p_3, \epsilon_3^\perp) | c_i(0) \bar{s}_j(z) | 0 \rangle = \frac{f_{D_s^*}}{4} \int_0^1 dk_3 e^{ik_3 \cdot z} \left\{ \epsilon_3^\perp \left[ m_3 \phi_{D_s^*}(k_3) + \phi_3 \phi_{D_s^*}(k_3) \right] \right\}_{ji}, \]
\[ (31) \]

where \( f_{\Upsilon(1S)}, f_{B_c}, f_{D_s^*} \) are decay constants.

Because of the mass relations, \( m_{\Upsilon(1S)} \simeq 2m_b, m_{B_c} \simeq m_b + m_c, \) and \( m_{D_s^*} \simeq m_c + m_s \) (see Table I), it might assume that the motion of the valence quarks in all participating
mesons is nearly nonrelativistic. The wave functions of the \( \Upsilon(1S) \), \( B_c \), \( D_s^* \) mesons could be approximately described with the nonrelativistic quantum chromodynamics \cite{28-30} and Schrödinger equation. Combining the wave functions of a nonrelativistic isotropic harmonic oscillator potential with their asymptotic forms \cite{26, 27}, we obtain \cite{21},

\[
\phi_{\Upsilon}^v(x) = \phi_{\Upsilon}^T(x) = A x \bar{x} \exp\left\{-\frac{m_b^2}{8 \beta_1^2 x \bar{x}}\right\},
\]

\[
\phi_{\Upsilon}^t(x) = B t^2 \exp\left\{-\frac{m_b^2}{8 \beta_1^2 x \bar{x}}\right\},
\]

\[
\phi_{\Upsilon}^V(x) = C (1 + t^2) \exp\left\{-\frac{m_b^2}{8 \beta_1^2 x \bar{x}}\right\},
\]

\[
\phi_{B_c}(x) = D x \bar{x} \exp\left\{-\frac{\bar{x} m_b^2 + x m_b^2}{8 \beta_2^2 x \bar{x}}\right\},
\]

\[
\phi_{D_s^*}^v(x) = \phi_{D_s^*}^T(x) = E x \bar{x} \exp\left\{-\frac{\bar{x} m_s^2 + x m_s^2}{8 \beta_3^2 x \bar{x}}\right\},
\]

\[
\phi_{D_s^*}^t(x) = F t^2 \exp\left\{-\frac{\bar{x} m_s^2 + x m_s^2}{8 \beta_3^2 x \bar{x}}\right\},
\]

\[
\phi_{D_s^*}^V(x) = G (1 + t^2) \exp\left\{-\frac{\bar{x} m_s^2 + x m_s^2}{8 \beta_3^2 x \bar{x}}\right\},
\]

where \( \bar{x} = 1 - x \); \( t = x - \bar{x} \); \( \beta_i = \xi_i \alpha_s(\xi_i) \) with \( \xi_i = m_i/2 \) based on the NRQCD power counting rules \cite{28}; \( \alpha_s \) is the QCD coupling constant; the exponential function represents the \( k_T \) distribution; parameters \( A, B, C, D, E, F, G \) are the normalization coefficients satisfying the conditions

\[
\int_0^1 dx \phi_{B_c}(x) = 1,
\]

\[
\int_0^1 dx \phi_i(x) = 1, \quad \text{for } i = v, t, V, T,
\]

\[
\int_0^1 dx \phi_i^j(x) = 1, \quad \text{for } i = v, t, V, T.
\]

The shape lines of the normalized distribution amplitudes for the \( \Upsilon(1S) \), \( B_c \), \( D_s^* \) mesons are showed in Fig.\(1\) It is clearly seen that (1) distribution amplitudes for the \( \Upsilon(1S) \), \( B_c \), \( D_s^* \) mesons shrink rapidly to zero at the endpoint \( x \to 0, 1 \) due to the suppression from the exponential functions, (2) although the nonrelativistic model of wave functions is crude, distribution amplitudes Eq.\(\text{(32)}\)-Eq.\(\text{(38)}\) can reflect, at least to some extent, the feature that the valence quarks share momentum fractions according to their masses.
FIG. 1: The distribution amplitudes for the Υ(1S), $B_c$, $D^*_s$ mesons in (a), (b), (c), respectively.

E. Decay amplitudes

The Feynman diagrams for the $\Upsilon(1S) \to B_c D^*_s$ weak decay are shown in Fig. 2. There are two types. One is the emission topology, and the other is annihilation topology. Each type is further subdivided into factorizable diagram where gluon attaches to quarks in the same meson, and nonfactorizable diagrams where gluon connects to quarks between different mesons.

FIG. 2: Feynman diagrams for the $\Upsilon(1S) \to B_c D^*_s$ decay with the pQCD approach, where (a,b) are the factorizable emission diagrams, (c,d) are the nonfactorizable emission diagrams, (e,f) are the nonfactorizable annihilation diagrams, and (g,h) are the factorizable annihilation diagrams.

The amplitude for the $\Upsilon(1S) \to B_c D^*_s$ weak decay is defined as below \[31\],

$$\mathcal{A}(\Upsilon(1S)\to B_c D^*_s) = \mathcal{A}_L(\epsilon^1_1, \epsilon^2_3) + \mathcal{A}_N(\epsilon^1_1, \epsilon^1_3) + i \mathcal{A}_T \varepsilon_{\mu\alpha\beta} \epsilon^\mu_1 \epsilon^\alpha_3 p^\beta_1 p^\beta_3, \quad (42)$$
which is conventionally written as the helicity amplitudes, 

\[ \mathcal{A}_0 = -C_A A_L(\epsilon_1^\parallel, \epsilon_3^\parallel), \]

\[ A_\parallel = \sqrt{2} C_A A_N(\epsilon_1^\perp, \epsilon_3^\perp), \]

\[ A_\perp = \sqrt{2} C_A m_1 p A_T, \]

\[ C_A = i \frac{G_F C_F}{\sqrt{2} N_c} \pi f_{\Upsilon(1S)} f_{B_c} f_{D_s^*}, \]

and the polarization amplitude \( \mathcal{A}_j \) is written as

\[ \mathcal{A}_j = V_{cb} V_{cs}^* \left\{ \left( A_{a,j}^{LL} + A_{b,j}^{LL} \right) a_1 + \left( A_{c,j}^{LL} + A_{d,j}^{LL} \right) C_2 \right\} 
- V_{tb} V_{tq}^* \left\{ \left( A_{a,j}^{LL} + A_{b,j}^{LL} \right) (a_4 + a_{10}) + \left( A_{c,j}^{LL} + A_{d,j}^{LL} \right) (C_3 + C_9) 
+ (A_{e,j}^{LL} + A_{f,j}^{LL}) (C_3 - C_9) \right\} 
+ (A_{g,j}^{LL} + A_{h,j}^{LL}) (a_3 + a_4 - \frac{1}{2} a_9 - \frac{1}{2} a_{10}) 
+ (A_{i,j}^{LR} + A_{j,j}^{LR}) (C_6 - \frac{1}{2} C_8) + (A_{k,j}^{LR} + A_{l,j}^{LR}) (a_5 - \frac{1}{2} a_7) 
+ (A_{m,j}^{SP} + A_{n,j}^{SP}) (C_5 + C_7) + (A_{o,j}^{SP} + A_{p,j}^{SP}) (a_5 - \frac{1}{2} C_7) \right\}, \]

where \( C_F = 4/3 \) and the color number \( N_c = 3 \); for the building blocks \( \mathcal{A}_{i,j}^k \), the first subscript \( i \) corresponds to the indices of Fig.2 the second subscript \( j = L, N, T \) denotes to three different helicity amplitudes; the superscript \( k \) refers to three possible Dirac structures \( \Gamma_1 \otimes \Gamma_2 \) of the four-quark operator \( (\bar{q}_1 \Gamma_1 q_2)(\bar{q}_3 \Gamma_2 q_3) \), namely \( k = LL \) for \( (V - A) \otimes (V - A) \), \( k = LR \) for \( (V - A) \otimes (V + A) \), and \( k = SP \) for \( -2(S - P) \otimes (S + P) \). The explicit expressions of building blocks \( \mathcal{A}_{i,j}^k \) are collected in Appendix A. The parameter \( a_i \) is defined as follows.

\[ a_i = \begin{cases} 
C_i + C_{i+1}/N_c, & \text{for odd } i; \\
C_i + C_{i-1}/N_c, & \text{for even } i. 
\end{cases} \]

### III. NUMERICAL RESULTS AND DISCUSSION

In the rest frame of the \( \Upsilon(1S) \) meson, branching ratio \( (Br) \), polarization fractions \( (f_{0,\parallel,\perp}) \) and relative phases \( (\phi_{\parallel,\perp}) \) between helicity amplitudes \( (\mathcal{A}_{0,\parallel,\perp}) \) for the \( \Upsilon(1S) \rightarrow B_c D_{s}^{*} \) weak decay are defined as

\[ Br = \frac{1}{12\pi} \frac{p}{m_{\Upsilon(1S)}^2 \Gamma_{\Upsilon(1S)}} \left\{ |\mathcal{A}_0|^2 + |\mathcal{A}_\parallel|^2 + |\mathcal{A}_\perp|^2 \right\}, \]
\[ f_{0,||,\perp} = \frac{|A_{0,||,\perp}|^2}{|A_0|^2 + |A_\parallel|^2 + |A_\perp|^2}, \quad (50) \]
\[ \phi_{||,\perp} = \arg(A_{||,\perp}/A_0). \quad (51) \]

\textbf{TABLE I: The numerical values of input parameters.}

| The Wolfenstein parameters\(^a\) | \[ A = 0.814^{+0.023}_{-0.024} \quad \lambda = 0.22537\pm0.00061 \quad \bar{\rho} = 0.117\pm0.021 \quad \bar{\eta} = 0.353\pm0.013 \] |
| --- | --- |
| Mass, width and decay constant | \[ m_b = 4.78\pm0.06 \text{ GeV} \quad m_c = 1.67\pm0.07 \text{ GeV} \quad m_s \simeq 510 \text{ MeV} \quad \Gamma_{\Upsilon(1S)} = 54.02\pm1.25 \text{ keV} \quad m_{\Upsilon(1S)} = 9460.30\pm0.26 \text{ MeV} \quad f_{\Upsilon(1S)} = 676.4\pm10.7 \text{ MeV} \quad m_{B_c} = 6275.6\pm1.1 \text{ MeV} \quad f_{B_c} = 427\pm6 \text{ MeV} \quad m_{D_s^*} = 2112.1\pm0.4 \text{ MeV} \quad f_{D_s^*} = 274\pm6 \text{ MeV} \] |

\(^a\)The relation between parameters \((\rho, \eta)\) and \((\bar{\rho}, \bar{\eta})\) is \[ (\rho + i\eta) = \frac{\sqrt{1 - A^2\lambda^4}(\bar{\rho} + i\bar{\eta})}{\sqrt{1 - \lambda^4}[1 - A^2\lambda^4(\bar{\rho} + i\bar{\eta})]} \].

The input parameters are listed in Table I. If not specified explicitly, we will take their central values as the default inputs. Our numerical results are collected in Table II, where the first uncertainty comes from the CKM parameters, the second uncertainty is from the choice of the typical scale \((1\pm0.1)t_i\) and \(t_i\) is given in Eqs. (A57-A60); the third uncertainty is from the variation of mass \(m_b\) and \(m_c\). The following are some comments.

(1) Branching ratio for the \(\Upsilon(1S) \to B_cD_{s}^*\) decay can reach up to \(\mathcal{O}(10^{-9})\) with the pQCD approach, which might be promisingly measurable at the running LHC and forthcoming SuperKEKB. For example, the \(\Upsilon(1S)\) production cross section in p-Pb collision is about a few \(\mu b\) at the LHCb \([35]\) and ALICE \([36]\) detectors. So, more than \(10^{12}\ \Upsilon(1S)\) data samples could be in principle available per \(ab^{-1}\) data collected by the LHCb and ALICE detectors, corresponding to a few thousands of the \(\Upsilon(1S) \to B_cD_{s}^*\) events.

(2) The contributions to branching ratio mainly come from the longitudinal and parallel polarization helicity amplitudes, \(f_0 + f_\parallel \gtrsim 90\%\), while the perpendicular polarization fraction \(f_\perp\) is generally less than \(10\%\). From Table II, it is seen that the polarization fractions are not sensitive to the input parameters. However, we find that branching ratio, polarization fractions \(f_0\) and \(f_\parallel\), are sensitive to the \(\Upsilon(1S)\) wave functions (see Table II). This might
Sudakov factor to suppress the nonperturbative contributions, which deserve much attention as the discussion in [7–9], there are many factors for this, for example, the choice of the $\alpha_s$ parameter, the region. The contributions to branching ratio from different regions of the pQCD approach is applicable and whether the perturbative calculation is reliable. Therefore, they could be determined experimentally, will improve our understanding on nonfactorizable contributions, factorization mechanism, and the strong dynamics at different energy scales.

TABLE II: Branching ratio, polarization fractions, and relative phases for different cases, where we use the wave functions of Eqs. (32–34) and Eqs. (36–38) for the Υ(1S) and $D_s^*$ meson, respectively in case A; we use the same wave functions for both the transversal and longitudinal polarization Υ(1S) meson in case B, i.e., $\phi^{v,t,V,T}_X$ = Eq. (32); case C for $\phi^{v,t,V,T}_{D_s^*}$ = Eq. (36); case D for $\phi^{v,t,V,T}_X$ = Eq. (32) and $\phi^{v,t,V,T}_{D_s^*}$ = Eq. (36).

|                | case A          | case B          | case C          | case D          |
|----------------|-----------------|-----------------|-----------------|-----------------|
| $10^0 \times Br$ | 1.68±0.12+0.24+0.10 | 2.10±0.14+0.29+0.09 | 1.63±0.11+0.18+0.10 | 2.04±0.14+0.22+0.09 |
| $10^2 \times f_0$ | 35.6±0.0+0.2+1.3 | 47.8±0.0+0.1+1.3  | 35.8±0.0+0.1+1.3  | 48.0±0.0+0.1+1.3  |
| $10^2 \times f_{\parallel}$ | 56.5±0.0+0.1+0.9 | 46.0±0.0+0.1+0.9  | 56.3±0.0+0.1+0.9  | 45.8±0.0+0.1+0.5  |
| $10^2 \times f_{\perp}$ | 7.9±0.0+0.0+0.5 | 6.2±0.0+0.0+0.2   | 7.8±0.0+0.0+0.5   | 6.5±0.0+0.0+0.2   |
| $\phi_{\parallel}$ | $\approx 1.9^\circ$ | $\approx 0.4^\circ$ | $\approx 2.1^\circ$ | $\approx 0.4^\circ$ |
| $\phi_{\perp}$ | $\approx -174.0^\circ$ | $\approx -175.6^\circ$ | $\approx -173.7^\circ$ | $\approx -175.2^\circ$ |

imply that the polarization measurement on the Υ(1S) → $B_cD_s^*$ decay would provide some information on the wave functions and thus the interquark binding forces responsible for the Υ(1S) meson.

(3) The relative phase $\phi_\parallel$ is very small. This is consistent with prediction of the QCD factorization approach [10, 11], where the strong phase arising from nonfactorizable contributions is suppressed by color and $\alpha_s$ for the $a_1$-dominated processes. The relative phases, if they could be determined experimentally, will improve our understanding on nonfactorizable contributions, factorization mechanism, and the strong dynamics at different energy scales.

(4) As it is well known, due to the large mass of final states, the momentum transition in the Υ(1S) → $B_cD_s^*$ decay may be not large enough. One might naturally wonder whether the pQCD approach is applicable and whether the perturbative calculation is reliable. Therefore, it is necessary to check what percentage of the contributions comes from the perturbative region. The contributions to branching ratio from different region of $\alpha_s/\pi$ are showed in Fig. 3 It can be clearly seen that more than 90% contributions to branching ratio come from the $\alpha_s/\pi \leq 0.3$ region, implying that the calculation with the pQCD approach is reliable. As the discussion in [12], there are many factors for this, for example, the choice of the typical scale in Eqs. (57–60), retaining the quark transverse moment and introducing the Sudakov factor to suppress the nonperturbative contributions, which deserve much attention.
FIG. 3: The contributions to the branching ratio from different region of $\alpha_s/\pi$ (horizontal axises), where the numbers over histogram denote the percentage of the corresponding contributions.

and further investigation, but beyond the scope of this paper.

(5) Besides the uncertainties listed in Table II, the decay constants, $f_{\Upsilon(1S)}$, $f_{B_c}$, and $f_{D^*_s}$, can bring about 6% uncertainties to branching ratios. Other factors, such as the models of wave functions, contributions of higher order corrections to HME, relativistic effects, and so on, deserve the dedicated study. Our results just provide an order of magnitude estimation.

IV. SUMMARY

The $\Upsilon(1S)$ weak decay is allowable within the standard model. With anticipation of the potential prospects of the $\Upsilon(1S)$ physics at high-luminosity dedicated heavy-flavor factories, the $\Upsilon(1S) \to B_c D^*_s$ weak decay is studied with the pQCD approach firstly. It is found that (1) the longitudinal plus parallel polarization fractions are main shares, but sensitive to the $\Upsilon(1S)$ wave functions; (2) branching ratio for the $\Upsilon(1S) \to B_c D^*_s$ weak decay can reach up to $O(10^{-9})$, which might be measurable at the future experiments.

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Appendix A: The building blocks of decay amplitudes

For the sake of simplicity, we decompose the decay amplitude Eq. (47) into some building blocks $A^{k}_{i,j}$, where the subscript $i$ corresponds to the indices of Fig[2], the subscript $j$ relates with different helicity amplitudes; the superscript $k$ refers to one of the three possible Dirac structures $\Gamma_1 \otimes \Gamma_2$ of the four-quark operator $(\bar{q}_1 \Gamma_1 q_2)(\bar{q}_1 \Gamma_2 q_2)$, namely $k = LL$ for $(V - A) \otimes (V - A)$, $k = LR$ for $(V - A) \otimes (V + A)$, and $k = SP$ for $-2(S - P) \otimes (S + P)$. The explicit expressions of $A^{k}_{i,j}$ are written as follows.

\[
A^{\text{LL}}_{a,L} = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \phi^*_T(x_1) \\
\phi_{B_e}(x_2) E_a(t_a) H_{ab}(\alpha_e, \beta_a, b_1, b_2) \alpha_s(t_a) \\
\left\{m_1^2 s + m_2 m_b u - (4 m_1^2 p^2 + m_2^2 u) \bar{x}_2 \right\},
\]

(A1)

\[
A^{\text{LL}}_{a,N} = m_1 m_3 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \\
\phi^*_T(x_1) \phi_{B_e}(x_2) E_a(t_a) H_{ab}(\alpha_e, \beta_a, b_1, b_2) \\
\alpha_s(t_a) \left\{2 m_2 \bar{x}_2 - 2 m_2 m_b - t \right\},
\]

(A2)

\[
A^{\text{LL}}_{a,T} = 2 m_1 m_3 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \\
\phi^*_T(x_1) \phi_{B_e}(x_2) E_a(t_a) H_{ab}(\alpha_e, \beta_a, b_1, b_2) \alpha_s(t_a),
\]

(A3)

\[
A^{\text{LL}}_{b,L} = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \\
\phi_{B_e}(x_2) E_b(t_b) H_{ab}(\alpha_e, \beta_b, b_2, b_1) \alpha_s(t_b) \\
\left\{\phi^*_T(x_1) \left[ m_1^2 (s - 4 p^2) \bar{x}_1 + 2 m_2 m_c u - m_2^2 u \right] \\
+ \phi^*_T(x_1) m_1 \left[ (2 m_2 - m_c) - 2 m_2 u \bar{x}_1 \right] \right\},
\]

(A4)

\[
A^{\text{LL}}_{b,N} = m_3 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \\
\phi_{B_e}(x_2) E_b(t_b) H_{ab}(\alpha_e, \beta_b, b_2, b_1) \alpha_s(t_b) \\
\left\{\phi^*_T(x_1) m_1 \left[ 2 m_2^2 - 4 m_2 m_c - t \bar{x}_1 \right] \\
+ \phi^*_T(x_1) m_1 \left[ (m_c - 2 m_2) + 4 m_1^2 m_2 \bar{x}_1 \right] \right\},
\]

(A5)

\[
A^{\text{LL}}_{b,T} = -2 m_3 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \\
\phi_{B_e}(x_2) E_b(t_b) H_{ab}(\alpha_e, \beta_b, b_2, b_1) \alpha_s(t_b) \\
\left\{\phi^*_T(x_1) m_1 \bar{x}_1 + \phi^*_T(x_1) (m_c - 2 m_2) \right\},
\]

(A6)
\[ A_{c,L}^{LL} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \\
\left(\phi_{B_c}(x_2) \phi_{D^T_c}(x_3) E_c(t_c) H_{cd}(\alpha_c, \beta_c, b_2, b_3) \alpha_s(t_c) \right) \]
\[ \delta(b_1 - b_2) \left\{ \phi_T^v(x_1) u \left[ t x_1 - 2 m_2^2 x_2 - s \bar{x}_3 \right] \right. \\
\left. + \phi_T^L(x_1) m_1 m_2 \left[ s x_2 + 2 m_3^2 \bar{x}_3 - u x_1 \right] \right\}, \quad (A7) \]

\[ A_{c,N}^{LL} = \frac{m_3}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \\
\left(\phi_{B_c}(x_2) \phi_{D^T_c}(x_3) E_c(t_c) H_{cd}(\alpha_c, \beta_c, b_2, b_3) \alpha_s(t_c) \right) \]
\[ \delta(b_1 - b_2) \left\{ \phi_T^v(x_1) 2 m_1 \left[ 2 m_2^2 x_2 + s \bar{x}_3 - t x_1 \right] \right. \\
\left. + \phi_T^L(x_1) m_2 \left[ 2 m_3^2 x_1 - t x_2 - u \bar{x}_3 \right] \right\}, \quad (A8) \]

\[ A_{c,T}^{LL} = \frac{2 m_2 m_3}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \\
\int_0^\infty b_3 db_3 \delta(b_1 - b_2) \phi_T^L(x_1) \phi_{B_c}(x_2) \phi_{D^T_c}(x_3) \\
E_c(t_c) H_{cd}(\alpha_c, \beta_c, b_2, b_3) \alpha_s(t_c) (x_2 - \bar{x}_3), \quad (A9) \]

\[ A_{c,L}^{SP} = \frac{m_3}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \\
\left(\phi_{B_c}(x_2) \phi_{D^T_c}(x_3) E_c(t_c) H_{cd}(\alpha_c, \beta_c, b_2, b_3) \alpha_s(t_c) \right) \]
\[ \delta(b_1 - b_2) \left\{ \phi_T^v(x_1) m_1 m_2 \left[ 2 m_2^2 x_1 - t x_2 - u \bar{x}_3 \right] \right. \\
\left. + \phi_T^L(x_1) m_1 \left[ 2 m_2^2 x_2 + s \bar{x}_3 - t x_1 \right] \right\}, \quad (A10) \]

\[ A_{c,N}^{SP} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \\
\left(\phi_{B_c}(x_2) \phi_{D^T_c}(x_3) E_c(t_c) H_{cd}(\alpha_c, \beta_c, b_2, b_3) \alpha_s(t_c) \right) \]
\[ \delta(b_1 - b_2) \left\{ \phi_T^v(x_1) m_1 m_2 \left[ s x_2 + 2 m_3^2 \bar{x}_3 - u x_1 \right] \right. \\
\left. + \phi_T^L(x_1) \left[ m_1^2 s x_1 + (m_2^2 u - s t) x_2 - m_3^2 t \bar{x}_3 \right] \right\}, \quad (A11) \]

\[ A_{c,T}^{SP} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \\
\left(\phi_{B_c}(x_2) \phi_{D^T_c}(x_3) E_c(t_c) H_{cd}(\alpha_c, \beta_c, b_2, b_3) \alpha_s(t_c) \right) \]
\[ \left\{ \phi_T^v(x_1) \left[ (s + t) x_2 + 2 m_3^2 \bar{x}_3 - (t + u) x_1 \right] \right. \\
\left. + \phi_T^L(x_1) 2 m_1 m_2 (x_1 - x_2) \right\} \delta(b_1 - b_2), \quad (A12) \]
\[ A_{dL}^{LL} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \]
\[ \phi_{B_c}(x_2) E_d(t_d) H_{cd}(\alpha_c, \beta_d, b_2, b_3) \alpha_s(t_d) \delta(b_1 - b_2) \]
\[ \left\{ \phi_T^\dagger(x_1) \phi_{D_s}^V(x_3) m_1 m_2 \left[ s x_2 + 2 m_3^2 x_3 - u x_1 \right] \right. \]
\[ + \phi_T^\dagger(x_1) \phi_{D_s}^T(x_3) 4 m_1^2 p^2 (x_3 - x_2) \]
\[ - \phi_T^\dagger(x_1) \phi_{D_s}^T(x_3) m_3 m_c t \right\}, \quad (A13) \]

\[ A_{dL}^{LL} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \]
\[ \phi_{B_c}(x_2) E_d(t_d) H_{cd}(\alpha_c, \beta_d, b_2, b_3) \alpha_s(t_d) \delta(b_1 - b_2) \]
\[ \left\{ \phi_T^\dagger(x_1) \phi_{D_s}^V(x_3) m_2 m_3 \left[ 2 m_1^2 x_1 - t x_2 - u x_3 \right] \right. \]
\[ + \phi_T^\dagger(x_1) \phi_{D_s}^T(x_3) m_1 m_c s \right\}, \quad (A14) \]

\[ A_{dL}^{LL} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \]
\[ \phi_{B_c}(x_2) E_d(t_d) H_{cd}(\alpha_c, \beta_d, b_2, b_3) \alpha_s(t_d) \delta(b_1 - b_2) \]
\[ \left\{ \phi_T^\dagger(x_1) \phi_{D_s}^V(x_3) 2 m_2 m_3 (x_2 - x_3) \right. \]
\[ - \phi_T^\dagger(x_1) \phi_{D_s}^T(x_3) 2 m_1 m_c \right\}, \quad (A15) \]

\[ A_{dL}^{SP} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \]
\[ \phi_{B_c}(x_2) E_d(t_d) H_{cd}(\alpha_c, \beta_d, b_2, b_3) \alpha_s(t_d) \delta(b_1 - b_2) \]
\[ \left\{ \phi_{D_s}^V(x_3) m_c \left[ \phi_T^\dagger(x_1) m_1 s - \phi_T^\dagger(x_1) m_2 u \right] \right. \]
\[ + \phi_T^\dagger(x_1) \phi_{D_s}^T(x_3) m_2 m_3 \left[ 2 m_1^2 x_1 - t x_2 - u x_3 \right] \]
\[ + \phi_T^\dagger(x_1) \phi_{D_s}^T(x_3) m_1 m_3 \left[ 2 m_2^2 x_2 + s x_3 - t x_1 \right] \right\}, \quad (A16) \]

\[ A_{dN}^{SP} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \]
\[ \phi_{B_c}(x_2) E_d(t_d) H_{cd}(\alpha_c, \beta_d, b_2, b_3) \alpha_s(t_d) \delta(b_1 - b_2) \]
\[ \left\{ \phi_{D_s}^V(x_3) m_3 m_c \left[ \phi_T^\dagger(x_1) 2 m_1 m_2 - \phi_T^\dagger(x_1) t \right] \right. \]
\[ + \phi_T^\dagger(x_1) \phi_{D_s}^T(x_3) m_1 m_2 \left( s x_2 + 2 m_3^2 x_3 - u x_1 \right) \]
\[ + \phi_T^\dagger(x_1) \left\{ m_1^2 s x_1 + (m_2^2 u - s t) x_2 - m_3^2 t x_3 \right\} \right\}, \quad (A17) \]

\[ A_{dT}^{SP} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty db_1 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \]
\[
\phi_{B_c}(x_2) E_d(t_d) H_{cd}(\alpha, \beta, b_1, b_2, b_3) \alpha_s(t_d) \delta(b_1 - b_2)
\]
\[
\left\{ \phi_{D_{r}}^{V}(x_3) \left[ \phi_{T}^{V}(x_1) 2 m_1 m_2 (x_1 - x_2) \right. \right.

+ \phi_{T}^{T}(x_1) \left[ 2 m_3^2 x_3 + (s + t) x_2 - (u + t) x_1 \right] \left. \right\} 
\]
\[
+ \phi_{T}^{T}(x_1) \phi_{D_r}^{V}(x_3) 2 m_3 m_c \right\}, \quad (A18)
\]
\[
A^{LL}_{c, L} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 
\phi_{B_c}(x_2) E_e(t_e) H_{ef}(\alpha, \beta, b, b_1, b_2, b_3) \alpha_s(t_e) \delta(b_2 - b_3)
\]
\[
\left\{ \phi_{T}^{V}(x_1) \left[ \phi_{D_r}^{V}(x_3) 2 m_1 m_2 \left( 2 m_2^2 x_2 + s \bar{x}_3 - t x_1 \right) \right. \right.

+ \phi_{D_r}^{T}(x_3) m_1 m_2 \left( s x_2 + 2 m_3^2 \bar{x}_3 - u x_1 \right) \left. \right]\left. \right\} 
\]
\[
+ \phi_{T}^{T}(x_1) m_1 m_b \left[ \phi_{D_r}^{V}(x_3) s + \phi_{D_r}^{T}(x_3) 4 m_2 m_3 \right], \quad (A19)
\]
\[
A^{LL}_{c, N} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 
\phi_{B_c}(x_2) E_e(t_e) H_{ef}(\alpha, \beta, b, b_1, b_2, b_3) \alpha_s(t_e) \delta(b_2 - b_3)
\]
\[
\left\{ \phi_{T}^{V}(x_1) \left[ \phi_{D_r}^{V}(x_3) 2 m_1 m_3 \left( 2 m_2^2 x_2 + s \bar{x}_3 - t x_1 \right) \right. \right.

+ \phi_{D_r}^{T}(x_3) m_1 m_2 \left( s x_2 + 2 m_3^2 \bar{x}_3 - u x_1 \right) \left. \right]\left. \right\} 
\]
\[
+ \phi_{T}^{T}(x_1) m_1 m_b \left[ \phi_{D_r}^{V}(x_3) m_3 t + \phi_{D_r}^{T}(x_3) 2 m_2 u \right], \quad (A20)
\]
\[
A^{LL}_{c, T} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 
\phi_{B_c}(x_2) E_e(t_e) H_{ef}(\alpha, \beta, b_1, b_2, b_3) \alpha_s(t_e) \delta(b_2 - b_3)
\]
\[
\left\{ \phi_{T}^{V}(x_1) \phi_{D_r}^{T}(x_3) 2 m_1 m_2 (x_1 - x_2) \right. \left. \right\} 
\]
\[
+ \phi_{T}^{T}(x_1) 2 m_b \left[ \phi_{D_r}^{V}(x_3) m_3 - \phi_{D_r}^{T}(x_3) 2 m_2 \right], \quad (A21)
\]
\[
A^{LR}_{c, L} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 
\phi_{B_c}(x_2) E_e(t_e) H_{ef}(\alpha, \beta, b_1, b_2) \alpha_s(t_e) \delta(b_2 - b_3)
\]
\[
\left\{ \phi_{D_r}^{V}(x_3) \left[ \phi_{T}^{V}(x_1) 4 m_1^2 p^2 (x_2 - x_1) + \phi_{T}^{T}(x_1) m_1 m_b s \right] \right. \right.

+ \phi_{T}^{T}(x_1) \phi_{D_r}^{V}(x_3) m_2 m_3 \left[ t x_2 + u \bar{x}_3 - 2 m_1^2 x_1 \right] \left. \right\}, \quad (A22)
\]
\[
A^{LR}_{c, N} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^1 b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 
\phi_{B_c}(x_2) E_e(t_e) H_{ef}(\alpha, \beta, b_1, b_2) \alpha_s(t_e) \delta(b_2 - b_3)
\]
\[
\left\{ \phi_{T}^{V}(x_1) \phi_{D_r}^{T}(x_3) m_2 m_3 \left[ u x_1 - s x_2 - 2 m_3^2 \bar{x}_3 \right] \right. \right.

- \phi_{T}^{T}(x_1) \phi_{D_r}^{V}(x_3) m_3 m_b t \right\}, \quad (A23)
\]
\[ A_{c,T}^{LR} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 \]
\[ \phi_{B_x}(x_2) E_e(t_e) H_{ef}(\alpha_1, \beta_1, b_1, b_2) \alpha_s(t_e) \delta(b_2 - b_3) \]
\[ \left\{ \phi_T^V(x_1) \phi_{D_3^*}^V(x_3) 2 m_1 m_2 (x_2 - x_1) \right. \]
\[ + \phi_T^T(x_1) \phi_{D_3^*}^T(x_3) 2 m_3 m_b \}, \]  
\[ (A24) \]

\[ A_{c,L}^{SP} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 \]
\[ \phi_{B_x}(x_2) E_e(t_e) H_{ef}(\alpha_1, \beta_1, b_1, b_2) \alpha_s(t_e) \delta(b_2 - b_3) \]
\[ \left\{ \phi_T^L(x_1) \phi_{D_3}^L(x_3) m_1 m_2 (u x_1 - s x_2 - 2 m_3^2 x_3) \right. \]
\[ + \phi_T^L(x_1) \phi_{D_3^*}^L(x_3) m_1 m_3 (t x_1 - 2 m_3^2 x_2 - s x_3) \]
\[ - \phi_T^T(x_1) m_b \left[ \phi_{D_3^*}^T(x_3) m_2 u + \phi_{D_3}^T(x_3) m_3 t \right] \}, \]  
\[ (A25) \]

\[ A_{c,N}^{SP} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 \]
\[ \phi_{B_x}(x_2) E_e(t_e) H_{ef}(\alpha_1, \beta_1, b_1, b_2) \alpha_s(t_e) \delta(b_2 - b_3) \]
\[ \left\{ \phi_T^V(x_1) m_1 m_b \left[ \phi_{D_3^*}^T(x_3) 2 m_2 m_3 + \phi_{D_3}^T(x_3) s \right] \right. \]
\[ + \phi_T^T(x_1) \left[ \phi_{D_3^*}^V(x_3) m_2 m_3 \left( t x_2 + u \bar{x}_3 - 2 m_1^2 x_1 \right) \right. \]
\[ + \phi_{D_3}^T(x_3) \left\{ (s t - m_2^2 u) x_2 + m_3^2 t \bar{x}_3 - m_1^2 s x_1 \right\} \}], \]  
\[ (A26) \]

\[ A_{c,L}^{SP} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 \]
\[ \phi_{B_x}(x_2) E_e(t_e) H_{ef}(\alpha_1, \beta_1, b_1, b_2) \alpha_s(t_e) \delta(b_2 - b_3) \]
\[ \left\{ \phi_T^T(x_1) \left[ \phi_{D_3}^V(x_3) 2 m_2 m_3 \left( x_2 - \bar{x}_3 \right) \right. \right. \]
\[ + \phi_{D_3}^T(x_3) \left\{ (s + t) x_2 + (u - s) \bar{x}_3 - 2 m_1^2 x_1 \right\} \]
\[ + \phi_{D_3^*}^T(x_3) \left\{ 2 m_1 m_2 \right\}, \]  
\[ (A27) \]

\[ A_{f,L}^{LL} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 \]
\[ \phi_{B_x}(x_2) E_f(t_f) H_{ef}(\alpha_1, \beta_1, b_1, b_2) \alpha_s(t_f) \delta(b_2 - b_3) \]
\[ \left\{ \phi_T^V(x_1) \phi_{D_3}^T(x_3) m_2 m_3 \left[ 2 m_1^2 \bar{x}_1 - t x_2 - u \bar{x}_3 \right] \right. \]
\[ + \phi_T^T(x_1) \phi_{D_3^*}^T(x_3) 4 m_1^2 p^2 (\bar{x}_1 - x_2) \]
\[ - \phi_T^T(x_1) \phi_{D_3^*}^T(x_3) m_1 m_b s \}, \]  
\[ (A28) \]
\[
\mathcal{A}_{j,N}^{LL} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 \\
\phi_{B_s}(x_2) E_f(t_f) H_\phi(\alpha_f, \beta_f, b_1, b_2) \alpha_s(t_f) \delta(b_2 - b_3) \\
\left\{ \phi_T(x_1) \phi_{D^*_s}(x_3) m_1 m_2 (s x_2 + 2 m^2 \bar{x}_3 - u \bar{x}_1) \\
+ \phi_T(x_1) \phi_{D^*_s}(x_3) m_3 m_b t \right\}, \\
\mathcal{A}_{j,T}^{LL} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 \\
\phi_{B_s}(x_2) E_f(t_f) H_\phi(\alpha_f, \beta_f, b_1, b_2) \alpha_s(t_f) \delta(b_2 - b_3) \\
\left\{ \phi_T(x_1) \phi_{D^*_s}(x_3) 2 m_1 m_2 (\bar{x}_1 - x_2) \\
- \phi_T(x_1) \phi_{D^*_s}(x_3) 2 m_3 m_b \right\}, \\
\mathcal{A}_{j,L}^{LR} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 \\
\phi_{B_s}(x_2) E_f(t_f) H_\phi(\alpha_f, \beta_f, b_1, b_2) \alpha_s(t_f) \delta(b_2 - b_3) \\
\left\{ \phi_T(x_1) \phi_{D^*_s}(x_3) u \left[ 2 m^2 x_2 + s \bar{x}_3 - t \bar{x}_1 \right] \\
+ \phi_T(x_1) \phi_{D^*_s}(x_3) m_2 m_3 \left[ t x_2 + u \bar{x}_3 - 2 m^2 \bar{x}_1 \right] \\
+ \phi_T(x_1) m_1 m_2 \left[ \phi_{D^*_s}(x_3) s + \phi_{D^*_s}(x_3) 4 m_2 m_3 \right] \right\}, \\
\mathcal{A}_{j,N}^{LR} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 \\
\phi_{B_s}(x_2) E_f(t_f) H_\phi(\alpha_f, \beta_f, b_1, b_2) \alpha_s(t_f) \delta(b_2 - b_3) \\
\left\{ \phi_T(x_1) \phi_{D^*_s}(x_3) 2 m_1 m_3 \left[ t \bar{x}_1 - 2 m^2 x_2 - s \bar{x}_3 \right] \\
+ \phi_T(x_1) \phi_{D^*_s}(x_3) m_1 m_2 \left[ u \bar{x}_1 - s x_2 - 2 m^2 \bar{x}_3 \right] \\
- \phi_T(x_1) m_6 \left[ \phi_{D^*_s}(x_3) m_3 t + \phi_{D^*_s}(x_3) 2 m_2 u \right] \right\}, \\
\mathcal{A}_{j,T}^{LR} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 \\
\phi_{B_s}(x_2) E_f(t_f) H_\phi(\alpha_f, \beta_f, b_1, b_2) \alpha_s(t_f) \delta(b_2 - b_3) \\
\left\{ \phi_T(x_1) \phi_{D^*_s}(x_3) 2 m_1 m_2 (x_2 - \bar{x}_1) \\
+ \phi_T(x_1) \phi_{D^*_s}(x_3) 2 m_2 \left[ \phi_{D^*_s}(x_3) 2 m_2 - \phi_{D^*_s}(x_3) m_3 \right] \right\}, \\
\mathcal{A}_{j,L}^{SP} = \frac{1}{N_c} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_1 db_1 \int_0^\infty b_2 db_2 \int_0^\infty db_3 \\
\phi_{B_s}(x_2) E_f(t_f) H_\phi(\alpha_f, \beta_f, b_1, b_2) \alpha_s(t_f) \delta(b_2 - b_3) \\
\left\{ \phi_T(x_1) \phi_{D^*_s}(x_3) 2 m_1 m_2 (x_2 - \bar{x}_1) \\
+ \phi_T(x_1) \phi_{D^*_s}(x_3) 2 m_2 \left[ \phi_{D^*_s}(x_3) 2 m_2 - \phi_{D^*_s}(x_3) m_3 \right] \right\}.
\]
\[ \phi_{B_{e}}(x_{2}) E_{f}(t_{f}) H_{e_{f}}(\alpha_{a}, \beta_{f}, b_{1}, b_{2}) \alpha_{s}(t_{f}) \delta(b_{2} - b_{3}) \]
\[ \{ \phi^{g}_{T}(x_{1}) \phi^{D_{z}}_{T}(x_{3}) m_{1} m_{3} \left[ t \bar{x}_{1} - 2 m_{2} x_{2} - s \bar{x}_{3} \right] \]  
\[ + \phi^{g}_{T}(x_{1}) \phi^{D_{z}}_{T}(x_{3}) m_{1} m_{2} \left[ u \bar{x}_{1} - s x_{2} - 2 m_{2} \bar{x}_{3} \right] \]  
\[ - \phi^{g}_{T}(x_{1}) m_{b} \left[ \phi^{D_{z}}_{T}(x_{3}) m_{2} u + \phi^{D_{z}}_{T}(x_{3}) m_{3} t \right] \}, \quad (A34) \]
\[ \mathcal{A}_{j,N}^{SP} = \frac{1}{N_{c}} \int_{0}^{1} dx_{1} \int_{0}^{1} dx_{2} \int_{0}^{1} dx_{3} \int_{0}^{\infty} b_{1} db_{1} \int_{0}^{\infty} b_{2} db_{2} \int_{0}^{\infty} db_{3} \]
\[ \phi_{B_{e}}(x_{2}) E_{f}(t_{f}) H_{e_{f}}(\alpha_{a}, \beta_{f}, b_{1}, b_{2}) \alpha_{s}(t_{f}) \delta(b_{2} - b_{3}) \]
\[ \{ \phi^{g}_{T}(x_{1}) \phi^{D_{z}}_{T}(x_{3}) m_{2} m_{3} \left( t x_{2} + u \bar{x}_{3} - 2 m_{1} \bar{x}_{1} \right) \]  
\[ + \phi^{g}_{T}(x_{3}) \left( (s t - m_{2} u) x_{2} + m_{3} t \bar{x}_{3} - m_{1} s \bar{x}_{1} \right) \]  
\[ + \phi^{V}_{T}(x_{1}) m_{1} m_{b} \left[ \phi^{V}_{D}(x_{3}) 2 m_{2} m_{3} + \phi^{g}_{T}(x_{3}) s \right] \}, \quad (A35) \]
\[ \mathcal{A}_{j,T}^{SP} = \frac{1}{N_{c}} \int_{0}^{1} dx_{1} \int_{0}^{1} dx_{2} \int_{0}^{1} dx_{3} \int_{0}^{\infty} b_{1} db_{1} \int_{0}^{\infty} b_{2} db_{2} \int_{0}^{\infty} db_{3} \]
\[ \phi_{B_{e}}(x_{2}) E_{f}(t_{f}) H_{e_{f}}(\alpha_{a}, \beta_{f}, b_{1}, b_{2}) \alpha_{s}(t_{f}) \delta(b_{2} - b_{3}) \]
\[ \{ \phi^{g}_{T}(x_{1}) \phi^{D_{z}}_{T}(x_{3}) 2 m_{2} m_{3} (x_{2} - \bar{x}_{3}) \]  
\[ + \phi^{g}_{T}(x_{3}) \left( (s + t) x_{2} + (u - s) \bar{x}_{3} - 2 m_{1} \bar{x}_{1} \right) \]  
\[ + \phi^{V}_{T}(x_{1}) \phi^{g}_{T}(x_{3}) 2 m_{1} m_{b} \}, \quad (A36) \]
\[ \mathcal{A}_{g,L}^{LL} = \mathcal{A}_{g,L}^{LR} = \int_{0}^{1} dx_{2} \int_{0}^{1} dx_{3} \int_{0}^{\infty} b_{2} db_{2} \int_{0}^{\infty} b_{3} db_{3} \]
\[ \phi_{B_{e}}(x_{2}) \phi^{V}_{D_{z}}(x_{3}) E_{g}(t_{g}) H_{gh}(\alpha_{a}, \beta_{g}, b_{2}, b_{3}) \]
\[ \alpha_{s}(t_{g}) \left\{ m_{2} t + (4 m_{1}^{2} p^{2} + m_{2} u) x_{2} \right\} \}, \quad (A37) \]
\[ \mathcal{A}_{g,N}^{LL} = \mathcal{A}_{g,N}^{LR} = - \int_{0}^{1} dx_{2} \int_{0}^{1} dx_{3} \int_{0}^{\infty} b_{2} db_{2} \int_{0}^{\infty} b_{3} db_{3} \]
\[ \phi_{B_{e}}(x_{2}) \phi^{V}_{D_{z}}(x_{3}) E_{g}(t_{g}) H_{gh}(\alpha_{a}, \beta_{g}, b_{2}, b_{3}) \]
\[ \alpha_{s}(t_{g}) m_{1} m_{3} \left\{ s + 2 m_{2} x_{2} \right\} \}, \quad (A38) \]
\[ \mathcal{A}_{g,T}^{LL} = \mathcal{A}_{g,T}^{LR} = \int_{0}^{1} dx_{2} \int_{0}^{1} dx_{3} \int_{0}^{\infty} b_{2} db_{2} \int_{0}^{\infty} b_{3} db_{3} \]
\[ \phi_{B_{e}}(x_{2}) \phi^{V}_{D_{z}}(x_{3}) E_{g}(t_{g}) H_{gh}(\alpha_{a}, \beta_{g}, b_{2}, b_{3}) \]
\[ \alpha_{s}(t_{g}) 2 m_{1} m_{3} \}, \quad (A39) \]
\[ A_{h,L}^{LL} = A_{h,L}^{LR} = \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \]
\[ \phi_{B_c}(x_2) E_h(t_h) H_{gh}(\alpha_a, \beta_i, b_2) \alpha_s(t_h) \]
\[ \{ \phi_{D_1}^T(x_3) [2m_2 m_3 (t + u \bar{x}_3) - m_3 m_b t] \]
\[ + \phi_{D_1}^V(x_3) [(4m_1^2 p^2 + m_3^2 t) \bar{x}_3 + m_2 u \]
\[ - 2m_2 m_b u] \}, \quad (A40) \]

\[ A_{h,N}^{LL} = A_{h,N}^{LR} = \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \]
\[ 2m_1 \phi_{B_c}(x_2) E_h(t_h) H_{gh}(\alpha_a, \beta_i, b_3) \alpha_s(t_h) \]
\[ \{ \phi_{D_1}^T(x_3) (2m_2 - m_b) - \phi_{D_1}^V(x_3) m_3 \bar{x}_3 \}, \quad (A41) \]

\[ A_{h,T}^{LL} = A_{h,T}^{LR} = \int_0^1 dx_2 \int_0^1 dx_3 \int_0^\infty b_2 db_2 \int_0^\infty b_3 db_3 \]
\[ 2m_1 \phi_{B_c}(x_2) E_h(t_h) H_{gh}(\alpha_a, \beta_i, b_3) \alpha_s(t_h) \]
\[ \{ \phi_{D_1}^T(x_3) (2m_2 - m_b) - \phi_{D_1}^V(x_3) m_3 \bar{x}_3 \}, \quad (A42) \]

where \( \bar{x}_i = 1 - x_i \); variable \( x_i \) is the longitudinal momentum fraction of the valence quark; \( b_i \) is the conjugate variable of the transverse momentum \( k_T \); and \( \alpha_s(t) \) is the QCD coupling at the scale of \( t \).

The function \( H_i \) are defined as follows [37].

\[ H_{ab}(\alpha_e, \beta, b_i, b_j) = K_0(\sqrt{-\alpha_e b_i}) \{ \theta(b_i - b_j) K_0(\sqrt{\beta b_i}) I_0(\sqrt{\beta b_j}) + (b_i \leftrightarrow b_j) \}, \quad (A43) \]
\[ H_{cd}(\alpha_e, \beta, b_2, b_3) = \{ \theta(-\beta) K_0(\sqrt{-\alpha_e b_2}) + \frac{\pi}{2} \theta(\beta) [i J_0(\sqrt{\beta b_3}) - Y_0(\sqrt{\beta b_3})] \}
\[ \times \{ \theta(b_2 - b_3) K_0(\sqrt{-\alpha_e b_2}) I_0(\sqrt{-\alpha_e b_3}) + (b_2 \leftrightarrow b_3) \}, \quad (A44) \]
\[ H_{ef}(\alpha_a, \beta, b_1, b_2) = \{ \theta(-\beta) K_0(\sqrt{-\alpha_a b_1}) + \frac{\pi}{2} \theta(\beta) [i J_0(\sqrt{\beta b_1}) - Y_0(\sqrt{\beta b_1})] \}
\[ \times \{ \theta(b_1 - b_2) [i J_0(\sqrt{\alpha_a b_1}) - Y_0(\sqrt{\alpha_a b_1})] J_0(\sqrt{\beta b_2}) + (b_1 \leftrightarrow b_2) \}, \quad (A45) \]
\[ H_{hg}(\alpha_a, \beta, b_i, b_j) = \frac{\pi}{4} \{ i J_0(\sqrt{\alpha_a b_j}) - Y_0(\sqrt{\alpha_a b_j}) \}
\[ \times \{ \theta(b_i - b_j) [i J_0(\sqrt{\beta b_i}) - Y_0(\sqrt{\beta b_i})] J_0(\sqrt{\beta b_j}) + (b_i \leftrightarrow b_j) \}, \quad (A46) \]

where \( J_0 \) and \( Y_0 \) (\( J_0 \) and \( K_0 \)) are the (modified) Bessel function of the first and second kind, respectively; \( \alpha_e \) (\( \alpha_a \)) is the gluon virtuality of the emission (annihilation) topological
diagrams; the subscript of the quark virtuality $\beta_i$ corresponds to the indices of Fig.2. The definition of the particle virtuality is listed as follows \[37\].

\[
\alpha_e = x_1^2 m_1^2 + x_2^2 m_2^2 - \bar{x}_1 \bar{x}_2 t, \quad (A47)
\]
\[
\alpha_a = x_2^2 m_2^2 + \bar{x}_3^2 m_3^2 + x_2 \bar{x}_3 s, \quad (A48)
\]
\[
\beta_a = m_1^2 - m_b^2 + \bar{x}_2^2 m_2^2 - \bar{x}_2 t, \quad (A49)
\]
\[
\beta_b = m_2^2 - m_c^2 + \bar{x}_1^2 m_1^2 - \bar{x}_1 t, \quad (A50)
\]
\[
\beta_c = x_1^2 m_1^2 + x_2^2 m_2^2 + \bar{x}_3^2 m_3^2 - x_1x_2 t - x_1 \bar{x}_3 u + x_2 \bar{x}_3 s, \quad (A51)
\]
\[
\beta_d = x_1^2 m_1^2 + x_2^2 m_2^2 + \bar{x}_3^2 m_3^2 - m_c^2 \quad - x_1 x_2 t - x_1 x_3 u + x_2 x_3 s, \quad (A52)
\]
\[
\beta_e = x_1^2 m_1^2 + x_2^2 m_2^2 + \bar{x}_3^2 m_3^2 - m_b^2 \quad - x_1 x_2 t - x_1 \bar{x}_3 u + x_2 \bar{x}_3 s, \quad (A53)
\]
\[
\beta_f = \bar{x}_1^2 m_1^2 + x_2^2 m_2^2 + \bar{x}_3^2 m_3^2 - m_b^2 \quad - \bar{x}_1 x_2 t - \bar{x}_1 \bar{x}_3 u + x_2 \bar{x}_3 s, \quad (A54)
\]
\[
\beta_g = x_2^2 m_2^2 + m_c^2 + x_2 s, \quad (A55)
\]
\[
\beta_h = \bar{x}_3^2 m_3^2 + m_2^2 + \bar{x}_3 s - m_b^2. \quad (A56)
\]

The typical scale $t_i$ and the Sudakov factor $E_i$ are defined as follows, where the subscript $i$ corresponds to the indices of Fig.2.

\[
t_{a(b)} = \max(\sqrt{-\alpha_e}, \sqrt{-\alpha_a}, 1/b_1, 1/b_2), \quad (A57)
\]
\[
t_{c(d)} = \max(\sqrt{-\alpha_e}, \sqrt{|\beta_{c(d)}|}, 1/b_2, 1/b_3), \quad (A58)
\]
\[
t_{e(f)} = \max(\sqrt{\alpha_a}, \sqrt{|\beta_{e(f)}|}, 1/b_1, 1/b_2), \quad (A59)
\]
\[
t_{g(h)} = \max(\sqrt{\alpha_a}, \sqrt{|\beta_{g(h)}|}, 1/b_2, 1/b_3), \quad (A60)
\]

\[
E_i(t) = \begin{cases} 
\exp\{-S_{\gamma}(t) - S_{B_i}(t)\}, & i = a, b \\
\exp\{-S_{\gamma}(t) - S_{B_i}(t) - S_{D_i^+}(t)\}, & i = c, d, e, f \\
\exp\{-S_{B_i}(t) - S_{D_i^+}(t)\}, & i = g, h
\end{cases}, \quad (A61)
\]

\[
S_{\gamma}(t) = s(x_1, p^+_1, 1/b_1) + 2\int_{1/b_1}^{t} \frac{d\mu}{\mu} \gamma_q, \quad (A62)
\]
\[ S_{B_c}(t) = s(x_2, p_2^+, 1/b_2) + 2 \int_{1/b_2}^{t} \frac{d\mu}{\mu} \gamma_q, \quad (A63) \]
\[ S_{D_s^+}(t) = s(x_3, p_3^+, 1/b_3) + 2 \int_{1/b_3}^{t} \frac{d\mu}{\mu} \gamma_q, \quad (A64) \]

where \( \gamma_q = -\alpha_s/\pi \) is the quark anomalous dimension; the explicit expression of \( s(x, Q, 1/b) \) can be found in the appendix of Ref.\[7\].

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