A COUNTEREXAMPLE TO A CONJECTURE ABOUT POSITIVE SCALAR CURVATURE

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Abstract. In his article in Proc. Amer. Math. Soc. 138 (2010), no. 5, 1621–1632, S. Chang conjectures that a closed smooth manifold $M$ with non-spin universal covering admits a metric of positive scalar curvature if and only if a certain homological condition is satisfied. We present a counterexample to this conjecture, based on the counterexample to the unstable Gromov-Lawson-Rosenberg conjecture given in the second author’s article in Topology 37 (1998), no. 6, 1165–1168.

1. The Result

We give a counterexample to the following conjecture stated by Chang as [1, Conjecture 1], and attributed there to Rosenberg and Weinberger.

Conjecture 1.1. Suppose that $M$ is a compact oriented manifold such that its universal covering does not admit a spin structure, with fundamental group $\Gamma$ and of dimension $n \geq 5$. Let $f : M \to B\Gamma$ be the composition of the classifying map $c : M \to B\Gamma$ of the universal covering of $M$, and the natural map $B\Gamma \to B\Gamma$. Denote by $[M]$ the fundamental class of $M$ in $H_n(M)$. Then $M$ admits a metric of positive scalar curvature if and only if $f_*[M]$ vanishes in $H_n(B\Gamma)$.

Here $B\Gamma$ is the classifying space for the group $\Gamma$ and $B\Gamma$ is the quotient of the universal space for proper actions, i.e. the quotient $EG/\Gamma$, where $EG$ is a proper $\Gamma$-space such that for every finite subgroup $F \leq \Gamma$ the fixed point set $EG^F$ is contractible (in particular, non-empty), but such that $EG^H = \emptyset$ for all other subgroups $H \leq \Gamma$, compare [1, p. 1623]. Our counterexample is based on the counterexample to the Gromov-Lawson-Rosenberg conjecture given in [5]. There, a 5-dimensional connected closed spin manifold $M$ with fundamental group $\Gamma = \mathbb{Z}^4 \oplus \mathbb{Z}/3$ is constructed, whose Rosenberg index vanishes but which nevertheless does not admit a metric of positive scalar curvature. By taking the connected sum of this manifold $M$ with a simply connected non-spin manifold $N$, we obtain a totally non-spin manifold $X$ which has the same fundamental group as $M$. One has $B\Gamma = T^4 \times B\mathbb{Z}/3$ and analogously $B\Gamma = T^4$ by [1] (1) and (4), p. 1624. In particular, $H_n(B\Gamma) = 0$ for $n \geq 5$, so that the condition on $f_*[X]$ from Conjecture 1.1 is satisfied in the

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case at hand. The argument in [5] relies on the following observation and we will also make significant use of this result.

**Lemma 1.2.** Let $X$ be a topological space and set for $n \in \mathbb{N}, n \geq 2$

$$H^+_n(X) := \{ f_*[M] \in H_n(X) ; f : M^n \to X \text{ and } M \text{ admits a metric with scal} > 0 \}.$$ Then for any class $u \in H^1(X)$ the map $u \cap : H_n(X) \to H_{n-1}(X)$, $x \mapsto u \cap x$

maps $H^+_n(X)$ into $H^+_{n-1}(X)$ if $3 \leq n \leq 8$.

**Proof.** See [5, Corollary 1.5] for $3 \leq n \leq 7$ and [3, Theorem 4.4] for $n = 8$. \hfill \Box

Our result reads now as follows.

**Proposition 1.3.** Let $M$ be the manifold constructed in [5] (we recall its construction in Section 2) and $N$ a simply connected manifold of dimension 5 which admits no spin structure. Then the manifold $X := M \# N$ has non-spin universal covering and admits no metric with positive scalar curvature.

This result is part of the first named author’s thesis [4].

2. The Proof

**Proof of Proposition 1.3.** First of all, if $X$ is constructed as above, we have already noted that it has non-spin universal covering. To obtain an explicit simply connected non-spin 5-manifold $N$, one can start with $\mathbb{C}P^2 \times S^1$, which is non-spin as $\mathbb{C}P^2$ is, and then do surgery on the embedded $S^1$ to obtain the simply connected $N$. Because this surgery does not touch the embedded $\mathbb{C}P^1$ with its non-spin normal bundle, the resulting $N$ remains a non-spin manifold. In order to see that $X$ admits no metric of positive scalar curvature, we use the same argument as in [5].

To begin with, we choose the model $BG = T^4 \times B\mathbb{Z}/3$. Recall,

$$H_n(T^d) \cong \mathbb{Z}^d(n) , \quad d(n) = \left( \begin{array}{c} d \\ n \end{array} \right)$$

and

$$H_n(B\mathbb{Z}/k\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n = 0; \\ \mathbb{Z}/k\mathbb{Z}, & n \text{ odd}; \\ 0, & n \text{ even}. \end{cases}$$

Together with the Künneth formula this gives

$$H_k(B\Gamma) = \bigoplus_{p_1 + \cdots + p_5 = k} H_{p_1}(X_1) \otimes \cdots \otimes H_{p_5}(X_5).$$

Here we have written $T^4 = X_1 \times \cdots \times X_4$ as product of four copies of $S^1$, and $X_5$ for $B\mathbb{Z}/3$.

Fix a basepoint $x = (x_1, \ldots, x_5) \in B\Gamma$ and let $p : S^1 \to B\mathbb{Z}/3$ be a map which induces an epimorphism on $\pi_1$ as in [5], as well as $f_j : X_j \to B\Gamma$ the map which includes $X_j$ identically and basepoint-preserving. We denote by $[*] \in H_0(B\Gamma)$ the canonical generator. Next, choose for each $1 \leq j \leq 4$ generators $g_j \in H_1(X_j)$ and elements $g^*_j \in H^1(X_j)$ with $\langle g^*_j, g_j \rangle = 1$, and let $g_5 \in H_1(X_5)$ be $p_*[S^1]$ where $[S^1]$
is the standard generator for $H_1(S^1)$. Introduce the elements $v_j := (f_j)_*(g_j) \in H_1(B\Gamma)$ for $j = 1, \ldots, 5$ as well as $a_1, \ldots, a_4 \in H^1(B\Gamma)$ with

$$a_1 := (pr_1)^*(g_1^*) \times 1 \times 1 \times 1,$$
$$a_2 := 1 \times (pr_2)^*(g_2^*) \times 1 \times 1 \times 1,$$
$$a_3 := 1 \times 1 \times (pr_3)^*(g_3^*) \times 1 \times 1,$$
$$a_4 := 1 \times 1 \times 1 \times (pr_4)^*(g_4^*) \times 1.$$

Finally, set

$$w := v_1 \times \cdots \times v_4 \times v_5 \in H_5(B\Gamma)$$

and

$$z := [\ast] \times [\ast] \times [\ast] \times v_4 \times v_5 \in H_2(B\Gamma).$$

By the Künneth formula, $w \neq 0$ and $z \neq 0$. Furthermore,

$$(*) \quad z = a_1 \cap (a_2 \cap (a_3 \cap w)) \in H_2(B\Gamma).$$

For example one has

$$a_3 \cap w = \left((1 \times 1 \times (pr_3)^*(g_3^*)) \times (1 \times 1) \times \left((v_1 \times v_2 \times v_3) \times (v_4 \times v_5)\right)\right)$$

$$= \left((1 \times 1 \times (pr_3)^*(g_3^*)) \cap (v_1 \times v_2 \times v_3) \times (1 \times 1) \cap (v_4 \times v_5)\right)$$

$$= \left((1 \cap v_1) \times (1 \cap v_2) \times (pr_3)^*(g_3^*) \cap v_3\right) \times (1 \cap v_4) \times (1 \cap v_5)$$

$$= v_1 \times v_2 \times (pr_3)^*(g_3^*) \times v_3 \times v_4 \times v_5$$

$$= v_1 \times v_2 \times [\ast] \times v_4 \times v_5,$$

because of $(pr_3)^*(g_3^*) \cap (pr_3)^*(g_3) = (g_3^*, g_3)[\ast]$. Let $f : T^5 \to T^4 \times B\mathbb{Z}/3$ be given by $f = (f_1 \times f_2 \times f_3 \times f_4) \times (f_5 \circ p)$ and choose $(g_1 \times \cdots \times g_4) \times [S^1] =: [T^5]$ as fundamental class for $T^5$. Then $f_*[T^5] = w$. As in [5] one can construct a bordism in $\Omega_5^{\text{bin}}(B\Gamma)$ from $f$ to a map $g : M \to B\Gamma$ which induces an isomorphism of fundamental groups. This defines the manifold $M$. Now let $N$ be any simply connected closed non-spin manifold of dimension 5 and set $X := M \# N$.

Finally, assume that $X$ admits a metric of positive scalar curvature. Then consider the map $h : M \sqcup N \to B\Gamma$ on the disjoint union of $M$ and $N$, which equals $g$ on $M$ and sends $N$ to a point. One has $h_*[M \sqcup N] = g_*[M] = w$ and since $M \sqcup N$ is bordant to $M \# N$, it follows that $w \in H_5^+(X)$. But then it follows from [6] as well as Lemma [1,2] that $w$ is mapped to $z$ under the following composition

$$H_5^+(B\Gamma) \xrightarrow{a_1 \cap \cdot} H_4^+(B\Gamma) \xrightarrow{a_2 \cap \cdot} H_3^+(B\Gamma) \xrightarrow{a_3 \cap \cdot} H_2^+(B\Gamma).$$

Hence $z = k_*[S^2]$ for some $k : S^2 \to B\Gamma$ since $S^2$ is the only orientable surface which admits a metric of positive scalar curvature. On the other hand, $\pi_2(B\Gamma) = 0$ so that $k$ is null homotopic. This implies $z = 0$, which is a contradiction. \qed

**Remark 2.1.** The method described in this note produces a counterexample to Conjecture 1.1 with fundamental group $\Gamma$ whenever $\Gamma$ satisfies the following homological conditions:

- for $5 \leq m \leq 8$ there is a homology class $[M] \in H_m(B\Gamma; \mathbb{Z})$ represented by an $m$-dimensional closed oriented manifold $M$ (with surgeries one can then arrange that $\pi_1(M) = \Gamma$)
• there are classes $\alpha_1, \ldots, \alpha_{m-2} \in H^1(B\Gamma; \mathbb{Z})$ such that $\alpha_1 \cap \cdots \cap (\alpha_{m-2} \cap [M]) \neq 0 \in H_2(B\Gamma; \mathbb{Z})$
• under the map $H_m(B\Gamma) \to H_m(B\Gamma)$ the class $[M]$ is sent to 0.

Note that this condition is similar, indeed much easier than the general homological condition for counterexamples to the Gromov-Lawson-Rosenberg condition derived in [2]. Unfortunately, its structure requires the group $\Gamma$ to contain non-trivial torsion, to allow for a kernel of the map $H_*(B\Gamma) \to H_*(B\Gamma)$ (in contrast to [2]).

The assumption on $H^1(B\Gamma; \mathbb{Z})$ we have to make is very strong; it has to have rank at least $m-2$. In particular, the method tells us nothing about finite groups. Indeed, whether metrics with positive scalar curvature exist on general manifolds with finite fundamental group $(\mathbb{Z}/p\mathbb{Z})^k$ for $p$ odd is completely open (in the totally non-spin case as well as in the spin case) and seems the first obstacle for a full understanding of this problem. Progress in this direction will require a completely new set of ideas.

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