Abstract. We obtain $C^2$ a priori estimates for solutions of the nonlinear second-order elliptic equation related to the geometric problem of finding a strictly locally convex hypersurface with prescribed curvature and boundary in a space form. Under the assumption of a strictly locally convex subsolution, we establish existence results in $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$ by using degree theory arguments.

1. Introduction

In this paper, we stay in $(n+1)$ dimensional space form $N^{n+1}(K)$ ($n \geq 2$) with constant sectional curvature $K = 0$, 1 or $-1$, which can be modeled as follows. In Euclidean space $\mathbb{R}^{n+1}$, fix the origin 0 and let $S^n$ denote the unit sphere centered at 0. Choose the spherical coordinates $(z, \rho)$ in $\mathbb{R}^{n+1}$ with $z \in S^n$. Define the new metric on $\mathbb{R}^{n+1}$ by

$$\bar{g} = d\rho^2 + \phi^2(\rho) \sigma$$

where $\sigma$ is the standard metric on $S^n$ induced from $\mathbb{R}^{n+1}$. Then $(\mathbb{R}^{n+1}, \bar{g})$ is a model of $N^{n+1}(K)$ for $K = 0$ if we choose $\phi(\rho) = \rho$ where $\rho \in [0, \infty)$, for $K = 1$ if $\phi(\rho) = \sin(\rho)$ where $\rho \in [0, \pi/2)$, and for $K = -1$ if $\phi(\rho) = \sinh(\rho)$ where $\rho \in [0, \infty)$, which correspond to the Euclidean space $\mathbb{R}^{n+1}$, the upper hemisphere $S^n_+$ and the hyperbolic space $\mathbb{H}^{n+1}$ respectively. Let $V = \phi(\rho) \frac{\partial}{\partial \rho}$ be the conformal Killing field in $N^{n+1}(K)$. It is well known that $V$ is the position vector field in Euclidean space.

Given a disjoint collection $\Gamma = \{\Gamma_1, \ldots, \Gamma_m\}$ of closed smooth embedded $(n-1)$ dimensional submanifolds, a smooth symmetric function $f$ of $n$ variables and a smooth positive function $\psi$ defined on $N^{n+1}(K)$, it is a fundamental question in differential geometry to seek a strictly locally convex hypersurface $\Sigma$ with the prescribed curvature

\begin{equation}
(1.1) \quad f(\kappa[\Sigma]) = \psi(V)
\end{equation}

and boundary

\begin{equation}
(1.2) \quad \partial \Sigma = \Gamma
\end{equation}

where $\kappa[\Sigma] = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of $\Sigma$ at $V$ with respect to the outward unit normal $\nu$. We call a hypersurface $\Sigma$ strictly locally convex if all its principal curvatures $\kappa_i > 0$ everywhere in $\Sigma$.

Equation (1.1) arises in various geometric problems. If we do not impose boundary condition (1.2) and consider closed hypersurfaces, there is a vast literature in this direction. When requiring the convexity of the hypersurfaces, the Gauss
curvature case was studied by Oliker [22] while the most current breakthrough is due to Guan-Ren-Wang [17], where the authors studied convex hypersurfaces with prescribed Weingarten curvature in $\mathbb{R}^{n+1}$ for general $\psi$ depending on both $V$ and $\nu$. For starshaped compact hypersurfaces, we refer the readers to [2] for the introductory material, and see Jin-Li [15] for Weingarten curvature in hyperbolic space, [2] [21] for Weingarten curvature in elliptic space, Spruck-Xiao [24] for scalar curvature in space forms for general $\psi$, Chen-Li-Wang [6] for Weingarten curvature in warped product spaces for general $\psi$.

For the Dirichlet problem, important examples include the classical Plateau problem concerning the mean curvature as well as the corresponding problem for Gauss curvature (see [3] [13] [12] [14]). The Dirichlet problem in the general setting [1.1]–[1.2] was first studied by Caffarelli-Nirenberg-Spruck [5] for vertical graphs over strictly convex domains in $\mathbb{R}^n$ with constant boundary data. Since then, there have been significant progresses, among which, we mention Guan-Spruck [15] and Trudinger-Wang [28] for general locally convex hypersurfaces in $\mathbb{R}^{n+1}$ which may not be graphs, Su [26] for strictly locally convex radial graphs in $\mathbb{R}^{n+1}$ and Cruz [7] for starshaped radial graphs with prescribed Weingarten curvature in $\mathbb{R}^{n+1}$.

As in [15], the curvature function $f$ is assumed to be defined on the open symmetric convex cone $\Gamma^+_n \equiv \{ \lambda \in \mathbb{R}^n | \lambda_i > 0, i = 1, \ldots, n \}$ satisfying the fundamental structure conditions
\begin{equation}
(1.3) \quad f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0 \quad \text{in} \quad \Gamma^+_n, \quad i = 1, \ldots, n
\end{equation}

(1.4) $f$ is concave in $\Gamma^+_n$

(1.5) $f > 0$ in $\Gamma^+_n$, $f = 0$ on $\partial \Gamma^+_n$

In addition, $f$ is assumed to satisfy the technical conditions
\begin{equation}
(1.6) \quad \sum f_i(\lambda)\lambda_i \geq \sigma_0 \quad \text{on} \quad \{ \lambda \in \Gamma^+_n | \psi_0 \leq f(\lambda) \leq \psi_1 \}
\end{equation}

for any $\psi_1 > \psi_0 > 0$, where $\sigma_0$ is a positive constant depending only on $\psi_0$ and $\psi_1$, and for any $C > 0$ and any compact set $E \subset \Gamma^+_n$ there exists $R = R(E, C) > 0$ such that
\begin{equation}
(1.7) \quad f(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + R) \geq C \quad \forall \lambda \in E
\end{equation}

Examples satisfying (1.3)–(1.7) include a large family $f = \sum f_i$ where
\begin{equation}
f_i = S_n^{\frac{1}{n}} \prod_{l=1}^{N_i - 1} \left( c_i + \sum_{k=1}^{n-1} c_{i,k} S_{n,k}^{\frac{1}{n-1}} \right)^{\frac{1}{N_i}}
\end{equation}

where $c_i, c_{i,k} \geq 0$ are constants, $c_i + \sum_k c_{i,k} > 0$ for each $i$, $S_k$ is the $k$th elementary symmetric function, $S_0 = 1$ and $S_{k,l} = S_k / S_l (0 \leq l < k \leq n)$. However, the pure curvature quotient $S_{n,k}^{1/(n-k)}$ does not satisfy (1.7).

In this paper, we are interested in strictly locally convex hypersurfaces embedded in $N^{n+1}(K)$ which can be represented as radial graphs over a domain in $\mathbb{S}^n$. Assuming $\Gamma$ to be the boundary of a smooth positive radial graph $\varphi$ in $N^{n+1}(K)$ defined on a smooth domain $\Omega \subset \mathbb{S}^n$, we thus have $\Gamma = \{(z, \varphi(z)) | z \in \partial \Omega \}$ and look for a smooth strictly locally convex radial graph $\Sigma = \{(z, \rho(z)) | z \in \Omega \}$ satisfying the Dirichlet problem
\begin{equation}
(1.8) \quad f(\kappa[\rho]) = \psi(z, \rho) \quad \text{in} \quad \Omega
\end{equation}
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\[ \rho = \varphi \text{ on } \partial \Omega \]

where \( \kappa[\rho] \) denotes the principal curvatures of the graph of \( \rho \) and we use the same \( \psi \) for the smooth positive function on the right hand side. For technical purposes, we assume that

\[ \Omega \text{ does not contain any hemisphere.} \]

We obtain the following \( C^2 \) estimates:

**Theorem 1.11.** Under assumption (1.3)–(1.7) and (1.10), suppose \( \Gamma \) can span a \( C^2 \) positive radial graph \( \overline{\rho} \) in \( N^{n+1}(K) \) which is strictly locally convex in a neighborhood of \( \Gamma \). Then for any \( C^4 \) strictly locally convex radial graph \( \rho \) satisfying (1.8)–(1.9) with \( \rho \leq \overline{\rho} \) in \( \Omega \), we have

\[
\| \rho \|_{C^2(\overline{\Omega})} \leq C
\]

where \( C \) depends only on \( \Omega, \| \psi \|_{C^2}, \| \overline{\rho} \|_{C^1(\overline{\Omega})}, \| \varphi \|_{C^4(\overline{\Omega})}, \inf \psi, \inf_{\partial \Omega} \overline{\rho} \) and the convexity of \( \overline{\rho} \).

We remark that for \( C^2 \) estimates, it is necessary in Theorem 1.11 to assume \( \overline{\rho} \) to be strictly locally convex near its boundary. To establish existence results, as in \([13, 11, 12, 14, 15, 26]\), we further require that \( \overline{\rho} \) is a strictly locally convex subsolution. Since there are topological obstructions to the existence of strictly locally convex hypersurfaces spanning a given \( \Gamma \) (see \([23]\)), the existence of a subsolution allows the arbitrary geometry of \( \Gamma \). Using Theorem 1.11, we can prove the following existence results in \( \mathbb{R}^{n+1} \) and \( \mathbb{H}^{n+1} \).

**Theorem 1.12.** Under assumption (1.3)–(1.7) and (1.10), assume in addition that there exists a smooth strictly locally convex radial graph \( \overline{\rho} \) satisfying

\[
(1.13) \quad f(\kappa[\overline{\rho}]) \geq \psi(z, \overline{\rho}) \quad \text{in } \Omega
\]

\[
\overline{\rho} = \varphi \quad \text{on } \partial \Omega
\]

Then there exists a smooth strictly locally convex radial graph \( \Sigma = \{ (z, \rho(z)) \mid z \in \Omega \} \) in space form \( N^{n+1}(K) \) where \( K = 0 \) or \( -1 \) satisfying the Dirichlet problem (1.8)–(1.9) with \( \rho \leq \overline{\rho} \) in \( \overline{\Omega} \) and uniformly bounded principal curvatures

\[
0 < K_0^{-1} \leq \kappa_i \leq K_0 \quad \text{on } \Sigma,
\]

where \( K_0 \) is a uniform positive constant depending only on \( \Omega, \| \psi \|_{C^2}, \| \overline{\rho} \|_{C^1(\overline{\Omega})}, \| \varphi \|_{C^4(\overline{\Omega})}, \inf \psi, \inf_{\partial \Omega} \overline{\rho} \) and the convexity of \( \overline{\rho} \).

In Euclidean space \( \mathbb{R}^{n+1} \), Theorem 1.12 was proved in \([13]\) for constant Gauss curvature assuming the existence of a strictly locally convex strict subsolution and was extended in \([11]\) for general \( \psi \) depending also on the gradient term. These existence results are established via the theory of Monge-Ampère type equations on \( S^n \). The linearized operators may have nontrivial kernels, which call for extra efforts for the proof of existence since one can not directly use continuity method. In \([13]\), the authors established the existence results for equations with \( \partial \psi / \partial \nu \leq 0 \) by monotone iteration approach. In \([11]\) the author rederived \( C^2 \) estimates for a wider class of equations which allows the application of degree theory to the proof of existence for general \( \psi \) (the proof also need the existence result in \([13]\)). In \([12]\), Guan obtained the existence results for Monge-Ampère equations with general \( \psi \) over smooth bounded domains in \( \mathbb{R}^n \) by assuming the existence of a subsolution (improving the results in \([3]\) where the authors assumed the strict convexity of
the domain) and stated that the strict subsolution assumption in [13, 11] can be weakened to a subsolution. More recently, Su [26] proved Theorem 1.12 in $\mathbb{R}^{n+1}$ assuming the existence of a strict subsolution, where the author reformulated (1.8) in a form with invertible linearized operator and thus continuity method and degree theory can be directly applied without extra $C^2$ estimates. In our paper, we will generalize this idea in space forms and weaken the strict subsolution condition into a subsolution.

This paper is organized as follows: in section 2, we reformulate equation (1.8) in two different ways: one is used for deriving $C^2$ boundary estimates in section 3 and the other is for proving existence through degree theory arguments in section 5. Section 4 is devoted to global $C^2$ estimates.

2. Strictly locally convex radial graphs in space forms

Throughout this paper, we focus on hypersurface $\Sigma \subset N^{n+1}(K)$ that can be represented as a smooth radial graph over a smooth domain $\Omega \subset S^n$, i.e. $\Sigma$ can be expressed as

$$\Sigma = \{ (z, \rho(z)) | z \in \Omega \subset S^n \}$$

The range for $\rho = \rho(z)$ is $(0, \rho^K_{01})$ where

$$\rho^K_{01} = \begin{cases} \infty, & \text{if } K = 0 \text{ or } -1 \\ \pi, & \text{if } K = 1 \end{cases}$$

(2.1)

First recall the related geometric objects on $\Sigma$. Following the notations in [25], let $\nabla'$ denote the covariant derivatives with respect to some local orthonormal frame $e_1, \ldots, e_n$ on $S^n$, and we will reserve $\nabla$ for the covariant derivatives with respect to some local orthonormal frame $E_1, \ldots, E_n$ on $\Sigma$ in section 4 for global curvature estimates. The induced metric, its inverse, unit normal, and second fundamental form on $\Sigma$ are given respectively by

$$g_{ij} = \phi^2 \delta_{ij} + \rho_i \rho_j$$

(2.2)

$$g^{ij} = \frac{1}{\phi^2} (\delta_{ij} - \frac{\rho_i \rho_j}{\phi^2 + |\nabla'\rho|^2})$$

(2.3)

$$\nu = -\nabla'\rho + \phi^2 \frac{\partial \phi}{\phi^2}$$

(2.4)

$$h_{ij} = \frac{\phi}{\sqrt{\phi^2 + |\nabla'\rho|^2}} \left( -\nabla'_{ij}\rho + \frac{2\phi'}{\phi} \rho_i \rho_j + \phi \phi' \delta_{ij} \right)$$

(2.5)

where $\rho_i = \rho_{e_i} = \nabla'_{e_i} \rho = \nabla_i' \rho$, $\rho_{ij} = \nabla'_{e_j} \nabla_i' \rho = \nabla_{e_j} \nabla_i' \rho = \nabla_{e_j} \nabla_i' \rho = \nabla_{e_j} \nabla_i' \rho$, and higher order covariant derivatives are interpreted in this manner. Thus $\nabla' \rho = \rho_{e_k} e_k$. In section 4, the covariant derivatives are taken with respect to $E_1, \ldots, E_n$ if without extra explanations. For example, $\rho_i = \nabla_{E_i} \rho$.

The principal curvatures $\kappa_1, \ldots, \kappa_n$ of the radial graph $\rho$ are the eigenvalues of the symmetric matrix $\{a_{ij}\}$:

$$a_{ij} = \gamma^{ik} h_{kl} \gamma^{lj}$$
where \( \{ \gamma_{ik} \} \) and its inverse \( \{ \gamma_{ik} \} \) are given respectively by

\[
\gamma_{ik} = \frac{1}{\phi} (\delta_{ik} - \frac{\rho_i \rho_k}{\sqrt{\phi^2 + |\nabla' \rho|^2 (\phi + \sqrt{\phi^2 + |\nabla' \rho|^2})})
\]

(2.6)

\[
\gamma_{ik} = \phi \delta_{ik} + \frac{\rho_i \rho_k}{\phi + \sqrt{\phi^2 + |\nabla' \rho|^2}}
\]

(2.7)

In fact, \( \{ \gamma_{ik} \} \) is the square root of the metric, i.e., \( \gamma_{ik} \gamma_{kj} = g_{ij} \).

**Definition 2.8.** A hypersurface \( \Sigma \) is strictly locally convex if all its principal curvatures are positive, i.e. \( \kappa_i > 0 \) for \( i = 1, \ldots, n \); or, equivalently, the symmetric matrix \( \{ a_{ij} \} \) (or \( \{ h_{ij} \} \)) is positive definite everywhere in \( \Omega \).

A \( C^2 \) function \( \rho \) is strictly locally convex if the hypersurface \( \Sigma \) represented by \( \rho \) is strictly locally convex.

For simplicity, throughout this paper \( a_{ij} > 0 \) (or \( \geq 0 \)) means that the symmetric matrix \( \{ a_{ij} \} \) is positive definite (or positive semi-definite); and \( a_{ij} \geq b_{ij} \) means that the symmetric matrices \( \{ a_{ij} \} \) and \( \{ b_{ij} \} \) satisfy \( a_{ij} - b_{ij} \geq 0 \).

We remark that a strictly locally convex hypersurface with boundary may not be convex globally; it locally lies on one side of its tangent plane at any point, which may be very complicated in general. However, in this paper, we are only concerned with those which can be represented as radial graphs over some domain of \( \mathbb{S}^n \).

Now we will change \( \rho \) into other variables in order to derive a priori estimates in section 3, and to prove the existence in section 5.

### 2.1. Transformation for deriving a priori estimates.

Set

\[
\rho = \zeta(u) = \begin{cases} 
\frac{1}{u}, & \text{if } K = 0 \\
\arccot u, & \text{if } K = 1 \\
\frac{1}{2} \ln \left(\frac{u + 1}{u - 1}\right), & \text{if } K = -1 
\end{cases}
\]

(2.9)

According to (2.1), the range for \( u \) is \( (u^K_L, \infty) \) where

\[
u^K_L = \begin{cases} 0, & \text{if } K = 0 \text{ or } 1 \\
1, & \text{if } K = -1 
\end{cases}
\]

(2.10)

The formulas (2.2), (2.3), (2.6), (2.7) and (2.5) can be expressed in terms of \( u \),

\[
g_{ij} = \phi^2 \delta_{ij} + \zeta^2(u) u_i u_j
\]

(2.11)

\[
g^{ij} = \frac{1}{\phi^2} \left( \delta_{ij} - \frac{\zeta^2(u) u_i u_j}{\phi^2 + \zeta^2(u)|\nabla' u|^2} \right)
\]

(2.12)
Hence
\[
\gamma_{ik} = \frac{1}{\phi} \left( \delta_{ik} - \frac{\zeta^2(u) u_i u_k}{\sqrt{\phi^2 + \zeta^2(u)|\nabla' u|^2}} \right),
\]
\[
= \begin{cases} 
\frac{u (\delta_{ik} - \frac{1}{u^2 + |\nabla' u|^2} (u + \sqrt{u^2 + |\nabla' u|^2})),}{u_{ik}} & \text{if } K = 0 \\
\sqrt{1 + u^2} (\delta_{ik} - \frac{1}{u^2 + |\nabla' u|^2} (\sqrt{1 + u^2} + \sqrt{1 + u^2 + |\nabla' u|^2})),}{u_{ik}} & \text{if } K = 1 \\
\sqrt{u^2 - 1} (\delta_{ik} - \frac{1}{u^2 - 1 + |\nabla' u|^2} (\sqrt{u^2 - 1 + u^2 - 1 + |\nabla' u|^2})),}{u_{ik}} & \text{if } K = -1 
\end{cases}
\]
(2.14)
\[
\gamma_{ik} = \frac{\zeta^2(u) u_i u_k}{\phi + \sqrt{\phi^2 + \zeta^2(u)|\nabla' u|^2}}.
\]
\[
h_{ij} = \frac{\phi}{\sqrt{\phi^2 + \zeta^2|\nabla' u|^2}} (-\zeta'(u) \nabla'_{ij} u + \phi \phi' \delta_{ij})
\]
\[
= \frac{-\zeta'(u) \phi}{\sqrt{\phi^2 + \zeta^2|\nabla' u|^2}} (\nabla'_{ij} u + u \delta_{ij})
\]
(2.15)
\[
= \begin{cases} 
\frac{1}{\sqrt{u^2 + u^2|\nabla' u|^2}} (\nabla'_{ij} u + u \delta_{ij}), & \text{if } K = 0 \\
\frac{1}{\sqrt{(1 + u^2)^2 + (1 + u^2)|\nabla' u|^2}} (\nabla'_{ij} u + u \delta_{ij}), & \text{if } K = 1 \\
\frac{1}{\sqrt{(u^2 - 1)^2 + (u^2 - 1)|\nabla' u|^2}} (\nabla'_{ij} u + u \delta_{ij}), & \text{if } K = -1 
\end{cases}
\]
Hence
(2.16)
\[
a_{ij} = \frac{-\zeta'(u) \phi}{\sqrt{\phi^2 + \zeta^2|\nabla' u|^2}} \gamma_{ik} (\nabla'_{kl} u + u \delta_{kl}) \gamma_{lj}
\]
It is easy to see that \(\Sigma\) (or \(u\)) is strictly locally convex if and only if
(2.17)
\[\nabla'_{ij} u + u \delta_{ij} > 0 \quad \text{in} \quad \Omega\]

2.2. Transformation for proving existence.

Set
(2.18)
\[
u = \eta(v) = \begin{cases} e^v, & \text{if } K = 0 \\
\sinh v, & \text{if } K = 1 \\
\cosh v, & \text{if } K = -1 
\end{cases}
\]
According to (2.10), the range for \(v\) is \((v^K_L, \infty)\) where
(2.19)
\[v^K_L = \begin{cases} -\infty, & \text{if } K = 0 \\
0, & \text{if } K = 1 \text{ or } -1 
\end{cases}
\]
The formula (2.13) and (2.15) become
(2.20)
\[
\gamma_{ik} = \eta'(v) \left( \delta_{ik} - \frac{v_i v_k}{\sqrt{1 + \sqrt{v^2}(1 + \sqrt{v^2})}} \right)
\]
we have
\[(2.21)\]
\[h_{ij} = \frac{1}{\eta^2(v)\sqrt{1 + |\nabla v|^2}} \left( \eta'(v)\nabla_i v + \eta(v)v_i v_j + \eta(v)\delta_{ij} \right)\]

Denoting
\[w = \sqrt{1 + |\nabla v|^2}\]
we have
\[(2.22)\]
\[a_{ij} = \frac{1}{w} \left( \eta'(v)\nabla_i v + \eta(v)\delta_{ij} - \frac{v_i v_k}{w(1 + w)} \right)\]
\[= \frac{1}{w} \left( \eta(v)\delta_{ij} + \eta'(v)\nabla_i v \nabla_j v \right)\]

From (2.22) we see that \(\Sigma\) (or \(\varphi\)) is strictly locally convex if and only if
\[(2.23)\]
\[\eta(v)\delta_{ij} + \eta'(v)\nabla_i v \nabla_j v > 0 \quad \text{in } \Omega\]

### 2.3. Reformulation of equation (1.8) under transformation (2.9).

Under transformation (2.9), the Dirichlet problem (1.8) is equivalent to
\[(2.24)\]
\[f(\kappa[u]) = \psi(z, u) \quad \text{in } \Omega\]
\[(2.25)\]
\[u = \varphi \quad \text{on } \partial\Omega\]

where we still use \(\psi\) for the function on the right hand side, and \(\varphi\) for the boundary value. Denote \(A[u] = \{a_{ij}\}\) where \(a_{ij}\) is given by (2.10). With the function \(F\) defined by \(F(A) = f(\lambda(A))\) where \(\lambda(A)\) denotes the eigenvalues of \(A\), \(\kappa[u] = (\kappa_1, \ldots, \kappa_n) = \lambda(A[u])\) and \(G\) given by
\[G(r, p, u) = F(A(r, p, u))\]
where \(A(r, p, u)\) is obtained from \(A[u]\) with \((r, p, u)\) in place of \((\nabla^2 u, \nabla' u, u)\), equation (2.24) can be written in the following form
\[(2.26)\]
\[G(\nabla^2 u, \nabla' u, u) = \psi(z, u) \quad \text{in } \Omega\]

We next recall some properties of the function \(F\) and \(G\). We use the notation
\[F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A), \quad F^{ijkl}(A) = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}(A)\]
\[G^{ij}(r, p, u) = \frac{\partial G}{\partial r_{ij}}(r, p, u), \quad G^{ij}(r, p, u) = \frac{\partial G}{\partial p_i}(r, p, u), \quad G_u(r, p, u) = \frac{\partial G}{\partial u}(r, p, u)\]
\[\psi_u(z, u) = \frac{\partial \psi}{\partial u}(z, u)\]

The matrix \(\{F^{ij}(A)\}\) is symmetric with eigenvalues \(f_1, \ldots, f_n\). In view of (1.3), \(F^{ij}(A) > 0\) whenever \(\lambda(A) \in \Gamma_+^n\), while (1.4) implies that \(F\) is a concave function of \(A\), i.e., the symmetric matrix \(F^{ijkl}(A) \leq 0\) whenever \(\lambda(A) \in \Gamma_+^n\).

The function \(G\) satisfies structure conditions similar to \(F\). In fact, from (2.10) we have
\[(2.27)\]
\[G^{ij} = \frac{\partial G}{\partial u_{ij}} = \frac{\partial F}{\partial a_{kl}} \frac{\partial a_{kl}}{\partial u_{ij}} = \frac{-\phi'\zeta(u)}{\sqrt{\phi^2 + \zeta^2(u)|\nabla' u|^2}} F^{kl}_{ijkl} \eta^2 v_i v_j\]
So the symmetric matrix $G^{ij} > 0$ if and only if $F^{ij} > 0$, which in particular implies that equation (2.26) is elliptic for strictly locally convex solutions. Again from (2.16) we have
\[
\frac{\partial^2 G}{\partial u_{ij} \partial u_{kl}} = \frac{\partial a_{pq}}{\partial u_{ij}} \frac{\partial^2 F}{\partial a_{pq} \partial u_{rs}} \frac{\partial a_{rs}}{\partial u_{kl}}
\]
which implies that $G$ is concave with respect to $\{u_{ij}\}$ for strictly locally convex $u$.

In section 3, we will need the linearized operator associated with equation (2.26) for deriving second order boundary estimates,
\[
(2.28) \quad L = G^{ij} \nabla_i' + G^i \nabla'_i
\]
We will also need the following expressions of $G^s$ and $G_u$.

**Lemma 2.29.** Denote $w = \sqrt{\phi^2 + \zeta'^2(u) \nabla'u^2}$. Then
\[
(2.30) \quad G^s = -\frac{2\zeta'^2(u) \gamma^i s u_q + \phi \gamma^q u_i u_q}{w(\phi + w)} F^{ij} a_{ij} = \frac{\zeta'^2 u_s}{w^2} F^{ij} a_{ij}
\]
\[
(2.31) \quad G_u = -2 \left( \frac{\phi \zeta' \gamma^i u_i u_q}{w^2} + \frac{\zeta'^2 u_q u_q}{w^2} \right) F^{ij} a_{ij} + \left( \frac{\phi' \zeta'}{\phi} - \frac{\phi' \zeta'}{u^2} + \frac{\partial^2 \zeta''}{\zeta'} \frac{u^2}{\phi} \right) F^{ij} a_{ij} - \frac{\phi' \zeta'}{w} F^{ij} g^{ij}
\]

**Proof.** We first prove (2.30). Note that
\[
(2.32) \quad G^s = \frac{\partial F}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial u_s} = F^{ij} \left( 2 \frac{\partial \gamma^i}{\partial u_s} h_{kl} \gamma^{lj} + \gamma^i \frac{\partial h_{kl}}{\partial u_s} \gamma^{lj} \right)
\]
where
\[
(2.33) \quad \frac{\partial \gamma^i}{\partial u_s} = -\gamma^i p \frac{\partial \gamma_{pq}}{\partial u_s} \gamma^q k
\]
Direct calculations from (2.14) and (2.18) yield
\[
(2.34) \quad \frac{\partial \gamma_{pq}}{\partial u_s} = \frac{\zeta'^2(u) (\delta_{ps} u_q + \delta_{qs} u_p)}{\phi + w} - \frac{\zeta'^2(u) u_p u_q u_s}{(\phi + w)^2 w} = \frac{\zeta'^2(u) (\delta_{ps} u_q + \phi u_p \gamma^q u_s)}{\phi + w}
\]
and
\[
(2.35) \quad \gamma^i p u_p = \frac{u_i}{w}
\]
Besides, from (2.15) and (2.16) we have
\[
(2.36) \quad \gamma^i k \frac{\partial h_{kl}}{\partial u_s} \gamma^{lj} = -\frac{\zeta'^2(u) u_s}{w^2} a_{ij}
\]
Taking (2.33)–(2.36) into (2.32), the formula (2.30) is proved.

The formula (2.31) can be proved similarly. In fact,
\[
(2.37) \quad G_u = \frac{\partial F}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial u} = F^{ij} \left( 2 \frac{\partial \gamma^i}{\partial u} h_{kl} \gamma^{lj} + \gamma^i \frac{\partial h_{kl}}{\partial u} \gamma^{lj} \right)
\]
where
\[
\frac{\partial \gamma^i}{\partial u} = -\gamma^i p \frac{\partial \gamma_{pq}}{\partial u} \gamma^q k
\]
From (2.14) we have
\[
\frac{\partial \gamma_{ik}}{\partial u} = \phi' \zeta' \delta_{ik} + \frac{2 \zeta'' u_{ik}}{\phi + w} - \frac{\zeta''(u) u_{ik}}{(\phi + w)^2} \left( \phi' \zeta' + \frac{\phi' \zeta'' + \zeta''|\nabla' u|^2}{w} \right)
\]
\[
= \phi' \zeta' \delta_{ik} + \frac{\zeta' u_{ik}}{\phi + w} \left( 2 \zeta'' - \frac{\zeta'}{\phi + w} \left( \phi' \zeta' + \frac{\phi' \zeta'' + \zeta''|\nabla' u|^2}{w} \right) \right)
\]
\[
= \phi' \zeta' \delta_{ik} + \frac{\zeta' u_{ik}}{\phi + w} \left( 2 \zeta'' - \frac{\zeta''}{\phi + w} \left( \phi' \zeta' + \phi' \zeta'' \right) \right)
\]
\[
= \phi' \zeta' \delta_{ik} + \frac{\zeta' u_{ik}}{\phi + w} \left( \frac{\phi' \zeta''}{w} \right)
\]
In view of (2.13), the above formula becomes
(2.38)
\[
\frac{\partial \gamma_{ik}}{\partial u} = \phi' \zeta' \gamma_{ik} + \frac{\zeta'' u_{ik}}{w}
\]
Direct calculation from (2.15) yields
(2.39)
\[
\frac{\partial h_{ij}}{\partial u} = \left( \frac{\phi' \zeta''}{w} + \frac{\phi' \zeta''}{w^3} \right) (\nabla' u_{ij} + u \delta_{ij}) - \phi' \zeta' \delta_{ij}
\]
Inserting (2.38) and (2.39) into (2.37) and in view of (2.16) and (2.35) we obtain (2.31).

**Corollary 2.40.** Suppose that we have the $C^1$ bounds for strictly locally convex solutions $u$ of (2.24):
\[
0 < u^L \leq u \leq u^C \leq C_0, \quad |\nabla' u| \leq C_1 \quad \text{in} \quad \Omega
\]
Then
\[
|G^s| \leq C \quad \text{and} \quad |G_u| \leq C(1 + \sum G^{ii})
\]

**Proof.** Note that \{F^{ij}(A)\} and $A$ can be diagonalized simultaneously by an orthonormal transformation. Consequently, the eigenvalues of the matrix \{F^{ij}(A)\}, which is not necessarily symmetric, are given by
\[
\lambda(\{F^{ij}(A)\} A) = (f_1 \kappa_1, \ldots, f_n \kappa_n)
\]
In particular we have
\[
F^{ij} a_{ij} = \sum f_i \kappa_i
\]
In addition, for a bounded matrix $B = \{b_{ij}\}$, i.e. $|b_{ij}| \leq C$ for all $1 \leq i, j \leq n$ we have
\[
|b_{ik} F^{ij} a_{kj}| \leq C \sum f_i \kappa_i
\]
Thus from (2.30) and (2.31) we have
\[
|G^s| \leq C \sum f_i \kappa_i \quad \text{and} \quad |G_u| \leq C \left( \sum f_i \kappa_i + \sum f_i \right)
\]
Finally, by the concavity of $f$ and $f(0) = 0$ we can derive that $\sum f_i \kappa_i \leq \psi \leq C$. Also, in view of (2.27) we have $\sum f_i \leq C \sum G^{ii}$. Hence the corollary is proved. □
2.4. Reformulation of equation (2.24) under transformation (2.18).

Under transformation (2.18), the Dirichlet problem (2.24)–(2.25) has the following form

\begin{align}
(2.41) & \quad f(\kappa[v]) = \psi(z, v) \quad \text{in } \Omega \\
(2.42) & \quad v = \varphi \quad \text{on } \partial \Omega
\end{align}

where we still use \( \psi \) for the right function and \( \varphi \) for the boundary value. At this time, \( \kappa[v] = (\kappa_1, \ldots, \kappa_n) = \lambda(A[v]) \) and \( A[v] = \{a_{ij}\} \) with \( a_{ij} \) given by (2.22). Define \( G \) by

\[ G(r, p, v) = F(A(r, p, v)) \]

where \( A(r, p, v) \) is obtained from \( A[v] \) with \( (r, p, v) \) in place of \( (\nabla'^2 v, \nabla' v, v) \). Therefore equation (2.41) is equivalent to

\begin{align}
(2.43) & \quad G(\nabla'^2 v, \nabla' v, v) = \psi(z, v) \quad \text{in } \Omega \\
& \quad v = \varphi \quad \text{on } \partial \Omega
\end{align}

The function \( G \) has similar properties as \( F \). Denote

\begin{align}
G^{ij}(r, p, v) = \frac{\partial G}{\partial r_{ij}}(r, p, v), \quad G^i(r, p, v) = \frac{\partial G}{\partial p_i}(r, p, v), \quad G_v(r, p, v) = \frac{\partial G}{\partial v}(r, p, v)
\end{align}

By (2.22), we can see that equation (2.43) is elliptic for strictly locally convex \( v \), and \( G \) is concave with respect to \( \{v_{ij}\} \) for strictly locally convex \( v \).

Under the transformation \( \rho = \zeta(u) \) and \( u = \eta(v) \), the condition (1.13) becomes

\begin{align}
(2.44) & \quad \begin{cases}
G(\nabla'^2 v, \nabla' v, v) \geq \psi(z, v) \quad \text{in } \Omega \\
v = \varphi \quad \text{on } \partial \Omega
\end{cases}
\end{align}

3. A priori estimates

In this section we derive the a priori \( C^2 \) estimates for strictly locally convex solutions \( u \) to the Dirichlet problem (2.26)–(2.25) with \( u \geq \underline{u} \) in \( \Omega 

(3.1) & \quad \|u\|_{C^2(\overline{\Omega})} \leq C
\]

The \( C^1 \) bound follows directly from the convexity of the radial graph \( u \) with \( u \geq \underline{u} \) in \( \Omega \) and \( u = \underline{u} \) on \( \partial \Omega \). In section 4, we will derive global curvature estimates, which is equivalent to the global estimates for \( |\nabla'^2 u| \) on \( \overline{\Omega} \) from its bound on the boundary \( \partial \Omega \). Therefore in this section we focus on the boundary estimate

(3.2) & \quad |\nabla'^2 u| \leq C \quad \text{on } \partial \Omega
\]

The estimate (3.1) as well as \( u \geq C_0^{-1} > u_K \) imply an upper bound for all the principal curvatures of the radial graphs in view of (2.16). By assumption (1.5), the principal curvatures admit a uniform positive lower bound. We thus have

(3.3) & \quad 0 < K_0^{-1} \leq \kappa_i \leq K_0 \quad \text{in } \Omega
\]

which in turn implies the uniform ellipticity of the linearized operator. Consequently we have the \( C^{2, \alpha} \) estimates by Evans-Krylov theory [8, 19]

(3.4) & \quad \|u\|_{C^{2, \alpha}(\overline{\Omega})} \leq C
\]

and the higher-order regularity by classical Schauder theory.
3.1. \( C^1 \) estimates. The \( C^1 \) estimates for the case \( K = 0 \) is established in \([13]\). The method turns out to work in space forms. Here we provide the proof for the sake of completeness.

**Lemma 3.5.** Under assumption \([13,10]\), for any strictly locally convex function \( u \) satisfying \( u \geq u_0 \) in \( \Omega \) and \( u = u_0 \) on \( \partial \Omega \) we have

\[
\begin{align*}
u_L^K &< C_0^{-1} \leq u \leq C_0, \quad |\nabla' u| \leq C_1 \quad \text{in} \quad \overline{\Omega} \\
\end{align*}
\]

where \( C_0 \) depends only on \( \Omega, \sup_{\partial \Omega} u \) and \( \inf_{\Omega} u; \) \( C_1 \) depends in addition on \( \sup_{\partial \Omega} |\nabla' u| \). Here \( u_L^K \) is defined as in \([2,10]\).

**Proof.** Assume that \( \sup_{\overline{\Omega}} u \) is achieved at \( P \in \Omega \). There exists \( Q \in \partial \Omega \) and a geodesic in \( \Omega \) joining from \( P \) to \( Q \), with a total length \( l \leq \frac{\pi}{2} - \epsilon \) for some \( \epsilon > 0 \). Since \( u \) is strictly locally convex, if we use arc length \( s \) as the parameter of the geodesic, we have

\[
\begin{align*}
u'' + u > 0 \quad \text{for} \quad 0 \leq s \leq l \\
\end{align*}
\]

in view of \([2.17]\). Consequently,

\[
\left( \frac{u}{\cos s} \right)' \cos^2 s = (u'' + u) \cos s > 0 \quad \text{for} \quad 0 \leq s \leq l
\]

which in turn gives

\[
\left( \frac{u}{\cos s} \right)' \cos^2 s \geq u'(0) = 0
\]

Therefore

\[
u(P) \leq \frac{u(Q)}{\cos l} \leq \frac{\sup_{\partial \Omega} u}{\cos(\frac{\pi}{2} - \epsilon)} = \frac{\sup_{\partial \Omega} u}{\cos(\frac{\pi}{2} - \epsilon)}
\]

A lower bound for \( u \) can be seen directly from

\[
u \geq u \geq \inf_{\Omega} u > u_L^K \quad \text{in} \quad \Omega
\]

For the gradient estimate, note that by \([2.17]\) we have

\[
\begin{align*}
\Delta' u + nu > 0 \quad \text{in} \quad \Omega \\
\end{align*}
\]

where \( \Delta' \) is the Laplace-Beltrami operator on \( S^n \). Let \( \bar{u} \) be the solution of

\[
\begin{align*}
\Delta' \bar{u} + nC_0 = 0 \quad \text{in} \quad \Omega, \\
\bar{u} = u \quad \text{on} \quad \partial \Omega
\end{align*}
\]

By comparison principle, we have \( \bar{u} \leq u \leq \pi \) in \( \overline{\Omega} \). Since the tangential derivatives of \( u \) on \( \partial \Omega \) are known, we obtain

\[
\begin{align*}
|\nabla' u| \leq C_1 \quad \text{on} \quad \partial \Omega
\end{align*}
\]

Now we estimate the gradient \( \nabla' u \) on \( \overline{\Omega} \). Consider the test function

\[
w = \sqrt{u^2 + |\nabla' u|^2}
\]

Assume \( w \) attains its maximum at \( z_0 \in \Omega \). Choose a local orthonormal frame \( e_1, \ldots, e_n \) around \( z_0 \). At \( z_0 \), there holds

\[
w w_i = (u_{ik} + u \delta_{ik}) u_k = 0, \quad i = 1, \ldots, n
\]

By \([2.17]\) we have \( \nabla' u(z_0) = 0 \) and hence

\[
\sup_{\Omega} |\nabla' u| \leq w(z_0) \leq \sup_{\Omega} u
\]
We thus obtain the estimate
(3.9) \[ |\nabla' u| \leq C_1 \text{ in } \overline{\Omega}. \]

3.2. Boundary estimates for second derivatives.

Consider any fixed point \( z_0 \in \partial \Omega \). Choose a local orthonormal frame field \( e_1, \ldots, e_n \) around \( z_0 \) on \( \Omega \), obtained by parallel translation of a local orthonormal frame field on \( \partial \Omega \) and the interior, unit, normal vector field to \( \partial \Omega \), along the geodesics perpendicular to \( \partial \Omega \) on \( \Omega \). We assume that \( e_n \) is the parallel translation of the unit normal field on \( \partial \Omega \).

Since \( u = \varphi \) on \( \partial \Omega \)
\[ \nabla'_{\alpha \beta}(u - \varphi) = -\nabla'_{n}(u - \varphi) \Pi(e_\alpha, e_\beta), \quad \alpha, \beta < n \text{ on } \partial \Omega \]
where \( \Pi \) denotes the second fundamental form of \( \partial \Omega \). It follows that
(3.10) \[ |\nabla'_{\alpha \beta} u(z_0)| \leq C, \quad \alpha, \beta < n \]

Let \( \rho(z) \) and \( d(z) \) denote the distances from \( z \in \Omega \) to \( z_0 \) and \( \partial \Omega \) on \( S^n \), respectively. Set
\[ \Omega_\delta = \{ z \in \Omega : \rho(z) < \delta \} \]
Choose \( \delta_0 > 0 \) sufficiently small such that \( \rho \) and \( d \) are smooth in \( \Omega_{\delta_0} \), on which, we have
\[ |\nabla' d| = 1, \quad -C I \leq \nabla'^2 d \leq C I, \]
\[ |\nabla' \rho| = 1, \quad I \leq \nabla'^2 \rho^2 \leq 3I, \]
where \( C \) only depends on \( \delta_0 \) and the geometric quantities of \( \partial \Omega \), and
\[ \nabla'^2 u + u I \geq 4 c_0 I \]
for some constant \( c_0 > 0 \), seeing that \( u \) is strictly locally convex in a neighborhood of \( \partial \Omega \) and in view of (2.17).

For the mixed tangential-normal and pure normal second derivatives at \( z_0 \), we use the following barrier function
\[ \Psi = Av + B\rho^2 \]
where
\[ v = u - \underline{u} + \epsilon d - \frac{N}{2} d^2 \]

Direct calculation shows (recall that the linear operator \( L \) is defined by (2.28))
(3.11)
\[ Lv = (G^{ij} \nabla'_{ij} + G^i \nabla' i)(u - \underline{u} + \epsilon d - \frac{N}{2} d^2) \]
\[ = G^{ij} \nabla'_{ij} (u - \underline{u} - \frac{N}{2} d^2) + \epsilon G^{ij} \nabla'_{ij} d + G^i \nabla' i (u - \underline{u} + \epsilon d - \frac{N}{2} d^2) \]
\[ \leq G^{ij} \left( \nabla'_{ij} u - \left( \nabla'_{ij} (\underline{u} + \frac{N}{2} d^2) - 2c_0 \delta_{ij} \right) \right) - 2c_0 \sum G^{ii} + C c_0 \sum G^{ii} + C(1 + \epsilon + N\delta) \]

here we have applied Corollary (2.40). Note that
(3.12)
\[ G^{ij} \left( \nabla'_{ij} u - \left( \nabla'_{ij} (\underline{u} + \frac{N}{2} d^2) - 2c_0 \delta_{ij} \right) \right) \leq G(\nabla'^2 u, \nabla' u, u) - G\left( \nabla'^2 (\underline{u} + \frac{N}{2} d^2) - 2c_0 I, \nabla' u, u \right) \]
by the concavity of $G(\nabla^2 u, \nabla^t u, u)$ with respect to $\nabla^2 u$. Also, the fact that
\[
\nabla^2 (u + \frac{N}{2} d^2) - 2c_0 I + u I
\]
implies that
\[
(3.13)
\]
Variables (3.14) therefore becomes
\[
2c_0 I - CN\delta I + N\nabla^t d \otimes \nabla d := \mathcal{H}
\]
implies that
\[
(3.15)
\]
where $\tilde{c}$ is a positive constant depending only on $C_0$ and $C_1$.

By (3.11)–(3.13) we have
\[
(3.14)
\]
Note that $\mathcal{H} = \text{diag}(2c_0 - CN\delta, \ldots, 2c_0 - CN\delta, 2c_0 - CN\delta + N)$. By assumption (1.7) we can choose $N$ sufficiently large and $\epsilon, \delta$ sufficiently small ($\delta$ depends on $N$) such that
\[
C_1 \leq c_0, \quad CN\delta \leq c_0, \quad -\tilde{c} F(\mathcal{H}) + C + 2c_0 \leq -1,
\]
the inequality (3.14) therefore becomes
\[
(3.15)
\]
We then require $\delta \leq \frac{2c_0}{N}$ to guarantee that
\[
v \geq 0 \quad \text{in} \quad \Omega_\delta
\]

For later use, we will need
\[
(3.16)
\]
which is a direct consequence of (3.15). We also need to estimate $L(\nabla_k u)$. For this, first apply the formula
\[
\nabla_{ij} (\nabla_k u) = \nabla_k \nabla_{ij} u + \Gamma^l_{ik} \nabla_{jl} u + \Gamma^l_{jk} \nabla_{il} u + \nabla_k \Gamma^l_{ij} u_l
\]
to compute
\[
(3.17)
\]
By (2.27) and (2.16) we have
\[ G^{ij} \Gamma^l_{ik}(\nabla^l_j u + u \delta_{jl}) = F^{st} \gamma^{is} \gamma^{jt} \Gamma^l_{ik} \cdot \gamma_{jp} \gamma_{ql} = (\gamma^{is} \Gamma^l_{ik} \gamma_{ql}) F^{st} a_{tq} \]
The term \( G^{ij} \Gamma^l_{jk} \nabla^l_i u \) can be evaluated similarly. Taking the covariant derivative of (2.20) and applying Corollary 2.40 we have
\[ |G^{ij} \nabla^l_i u + G^i \nabla^l_k u| \leq C + |(\psi_u - G_u) u_k| \leq C(1 + \sum G^{ii}) \]
From all these above, (3.17) can be estimated as
\[ |L(\nabla^l_i u)| \leq C(1 + \sum G^{ii}) \]
For fixed \( \alpha < n \), choosing \( B \) sufficiently large such that
\[ \Psi \pm \nabla^\alpha (u - \varphi) \geq 0 \quad \text{on} \quad \partial \Omega \]
From (2.14) and (3.15) we have
\[ L(\Psi \pm \nabla^\alpha (u - \varphi)) \leq A(-c_0 \sum G^{ii} - 1) + BC(1 + \sum G^{ii}) \]
Choosing \( A \) sufficiently large such that
\[ L(\Psi \pm \nabla^\alpha (u - \varphi)) \leq 0 \quad \text{in} \quad \Omega \]
By the maximum principle
\[ \Psi \pm \nabla^\alpha (u - \varphi) \geq 0 \quad \text{in} \quad \Omega \]
which implies
\[ |\nabla^\alpha_n u(z_0)| \leq C \]
It remains to estimate the double normal derivative \( \nabla^\alpha_n u \) on \( \partial \Omega \). In view of (3.7), it suffices to derive an upper bound
\[ \nabla^\alpha_n u \leq C \quad \text{on} \quad \partial \Omega \]
The following proof is motivated by an idea of Trudinger [27]. For this, we want to prove that
\[ M := \min_{z \in \partial \Omega} \min_{\xi \in T_{(z_0, \partial \Omega)}, \xi = 1} (\nabla^\alpha \xi u + u) \geq c_1 \]
for some constant \( c_1 > 0 \). Assume that \( M \) is achieved at \( z_1 \in \partial \Omega \) in the direction of \( \xi_1 \). Let \( e_1, \ldots, e_n \) be the local orthonormal frame field around \( z_1 \) on \( \Omega \subset S^n \) as before. Without loss of generality, we may assume that \( e_1(z_1) = \xi_1 \). Now we have
\[ M = \nabla\xi_1 e_1 u(z_1) + u(z_1) = \nabla^l_1 u(z_1) + u(z_1) \]
\[ = (\nabla^l_1 u(z_1) + u(z_1)) - (u - \underline{u})_n(z_1) \Pi(e_1, e_1)(z_1) \]
We may assume that \( (u - \underline{u})_n(z_1) \Pi(e_1, e_1)(z_1) > \frac{1}{2} (\nabla^l_1 u(z_1) + u(z_1)) \), for, otherwise we are done.
Since \( \Pi(e_1, e_1)(z) \) is continuous and \( (u - \underline{u})_n \) is bounded, we have
\[ \Pi(e_1, e_1)(z) \geq \frac{1}{2} \Pi(e_1, e_1)(z_1) \geq c_2 > 0 \]
on \[ \Omega_\delta = \{ z \in \Omega | \text{dist}_{S^n}(z_1, z) < \delta \} \]
when \( \delta > 0 \) is sufficiently small.
Now we consider
\[ \Phi = \frac{\nabla^l_1 u + \varphi - M}{\Pi(e_1, e_1) - (u - \varphi)_n} \]
Note that \( \Phi \geq 0 \) on \( \partial \Omega \cap \Omega_3 \). This is because on \( \partial \Omega \)
\[
\nabla'_{11}(u - \varphi) = -(u - \varphi)_n \Pi(e_1, e_1)
\]
and consequently
\[
\nabla'_{11}\varphi + \varphi - (u - \varphi)_n \Pi(e_1, e_1) = \nabla'_{11}u + u \geq M
\]
Also by (3.18)
\[
(3.21) \quad L(\Phi) = L\left(\frac{\nabla'_{11}\varphi + \varphi - M}{\Pi(e_1, e_1)} + \varphi_n\right) - Lu_n \leq C(1 + \sum G^{ii})
\]
Now choose \( B \) large such that \( \Psi + \Phi \geq 0 \) on \( \partial \Omega_3 \). In view of (3.16) and (3.21) we then choose \( A \) sufficiently large such that \( L(\Psi + \Phi) \leq 0 \) in \( \Omega_3 \). Since \( (\Psi + \Phi)(z_1) = 0 \), it follows that \( (\Psi + \Phi)_n(z_1) \geq 0 \) and hence
\[
u_{nn}(z_1) \leq C
\]
Together with (3.11) and (3.19), we obtain a bound \( |\nabla'^2 u(z_1)| \leq C \), and by (2.16), a bound for all the principle curvatures of the radial graph at \( z_1 \). By (3.8), the principle curvatures at \( z_1 \) admit a uniform positive lower bound. This in turn yields a positive lower bound for the eigenvalues of \( \nabla'^2 u(z_1) + u(z_1)I \), which implies (3.20).

By (3.20) and Lemma 1.2 of [4] there exists \( R > 0 \) depending on the bounds in (3.10) and (3.19) such that if \( u_{nn}(z_0) \geq R \) and \( z_0 \in \partial \Omega \), then the eigenvalues \( (\lambda_1, \ldots, \lambda_n) \) of \( \nabla'^2 u(z_0) + u(z_0)I \) satisfy
\[
c_1^2 \leq \lambda_{\alpha} \leq C, \quad \alpha = 1, \ldots, n - 1, \quad \lambda_n \geq \frac{R}{2}
\]
it follows that
\[
\nabla'^2 u(z_0) + u(z_0)I \geq X^{-1}AX
\]
where \( X \) is an orthogonal matrix and
\[
A = \text{diag}\left(\frac{c_1}{2}, \ldots, \frac{c_1}{2}, \frac{R}{2}\right)
\]
Consequently at \( z_0 \),
\[
G(\nabla'^2 u, \nabla'u, u)(z_0)
\]
\[
= \left(\frac{-\phi'\zeta(u)}{\sqrt{\phi^2 + \zeta^2|\nabla'u|^2}}\right) F\left(g^{-1/2}\left(\nabla'^2 u(z_0) + u(z_0)I\right) g^{-1/2}\right)
\]
\[
\geq \left(\frac{-\phi'\zeta(u)}{\sqrt{\phi^2 + \zeta^2|\nabla'u|^2}}\right) F\left(g^{-1/2}X^{-1}AX g^{-1/2}\right) = \left(\frac{-\phi'\zeta(u)}{\sqrt{\phi^2 + \zeta^2|\nabla'u|^2}}\right) F(\Lambda^{1/2} X g^{-1} X^{-1} \Lambda^{1/2})
\]
\[
\geq \left(\frac{-\phi'\zeta(u)}{\sqrt{\phi^2 + \zeta^2|\nabla'u|^2}}\right) F(\Lambda^{1/2} X \frac{1}{\phi^2 + \zeta^2(u)|\nabla'u|^2} I X^{-1} \Lambda^{1/2})
\]
\[
= \left(\frac{-\phi'\zeta(u)}{\sqrt{\phi^2 + \zeta^2|\nabla'u|^2}}\right) F(\Lambda) \geq \varepsilon F(\Lambda)
\]
By (1.7), when \( R \) is sufficiently large, \( G(\nabla'^2 u, \nabla'u, u)(z_0) > \psi(z, u)(z_0) \), which is a contradiction to equation (2.20). Hence \( \nabla'^{nn}u \leq R \) on \( \partial \Omega \) and (3.2) is proved.
4. Global curvature estimates

The ideas for deriving global $C^2$ a priori estimates for starshaped compact or convex hypersurfaces can be found in [18, 25, 17] (see also [5] for vertical graphs). For strictly locally convex hypersurfaces, we synthesize the ideas in [18, 25] to estimate from above for the largest principal curvature $\kappa_{\text{max}} = \max_{1 \leq i \leq n} \kappa_i$ of $\Sigma$, which, together with (3.6), (3.2) and (2.5) implies an estimate for $\|\rho\|_{C^2}(\Omega)$.

**Theorem 4.1.** Under assumption (1.3), (1.4) and (1.6), let $\Sigma = \{(z, \rho(z)) | z \in \Omega \subset S^n \} \subset N^{n+1}(K)$ be a strictly locally convex $C^4$ hypersurface satisfying equation (1.1) for some positive $C^2$ function $\psi$ defined on $N^{n+1}(K)$. Suppose in addition that $0 < C^{-1}_0 \leq \rho(z) \leq C_0$ and $|\nabla'\rho| \leq C_1$ on $\Omega$ where $C_0, C_1$ are uniform positive constants and $\rho^K_1$ is given by (2.1). Then there exists a constant $C$ depending only on $C_0, C_1, \|\psi\|_{C^2}$ and $\inf \psi$ such that

$$\max_{z \in \Omega} \kappa_i(z) \leq C (1 + \max_{z \in \partial\Omega} \kappa_i(z))$$

**Proof.** Since $\kappa_i > 0$ for all $i$ on $\Sigma$, it suffices to estimate from above for the largest principal curvature $\kappa_{\text{max}}$ of $\Sigma$. To construct a test function, we will make use of the following ingredients:

$$\Phi(\rho) = \int_0^\rho \phi(r) \, dr$$

and the support function

$$u = \hat{g}(V, \nu) = \langle V, \nu \rangle$$

We note that the support function $u$ has a positive lower bound (see (2.4) for the expression of $\nu$),

$$u = \langle \phi(\rho) \frac{\partial}{\partial \rho}, \frac{-\nabla'\rho + \rho^2 \frac{\partial}{\partial \rho}}{\sqrt{\rho^4 + \rho^2|\nabla'\rho|^2}} \rangle \geq 2a > 0$$

Now define the test function as

$$\Theta = \ln \kappa_{\text{max}} - \ln(u - a) + \beta \Phi$$

Assume $\Theta$ achieves its maximum value at $x_0 = (z_0, \rho(z_0)) \in \Sigma$. Choose a local orthonormal frame $E_1, \ldots, E_n$ around $x_0$ on $\Sigma$ such that $\nabla_i(x_0) = \kappa_i \delta_{ij}$, where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of $\Sigma$ at $x_0$. We may assume $\kappa_1 = \kappa_{\text{max}}(x_0) \geq 1$. Then, at $x_0$,

$$\frac{h_{11}}{h_{11}} - \frac{u_i}{u - a} + \beta \Phi_i = 0$$

$$\frac{h_{11} \phi}{h_{11}} \left( \frac{h_{11}^2}{h_{11}} - \frac{u_{ii}}{u - a} + \left( \frac{u_i}{u - a} \right)^2 \right) + \beta \Phi_{ii} \leq 0$$

We will need the Codazzi equation and Gauss equation in space forms, which are

$$\nabla_i h_{ij} = \nabla_j h_{ii}$$

and

$$h_{i\ell i} = h_{ii\ell} + \kappa_i \kappa_\ell^2 - \kappa_\ell^2 \kappa_i + K(\kappa_i - \kappa_\ell)$$
In the rest of this section all computations are evaluated at $x_0$. Under the local orthonormal frame $E_1, \ldots, E_n$, equation (1.1) appears as
\[(4.6)\quad F(h) = f(\lambda(h)) = \psi(V) \quad \text{where} \quad h = \{h_{ij}\}\]
Covariantly differentiate (4.6) twice
\[(4.7)\quad F^{ii}h_{ii} = \phi' d_V \psi(E_t)\]
\[(4.8)\quad F^{ii}h_{i111} + F^{ij,k}h_{ij1}h_{kl1} \geq -C\kappa_1\]
Here we have used the property of the conformal Killing field $V$,
\[\nabla_E_i V = \phi' E_i\]
Combining (4.3), (4.5) and (4.8) we have
\[(4.9)\quad -\frac{1}{\kappa_1} F^{ij,k}h_{ij1}h_{kl1} - \frac{1}{\kappa_1^2} \sum_{i \in I} f_i h_{111}^2 - \sum f_i k_{i1}^2 + (\kappa_1 - K) \sum f_i k_i + K \sum f_i - C - \frac{1}{u-a} \sum f_i u_i^2 + \beta \sum F^{ii} \Phi_{ii} \leq 0\]

Now we partition $\{1, \ldots, n\}$ into two parts,
\[I = \{ j : f_j \leq 2f_1 \}, \quad J = \{ j : f_j > 2f_1 \}\]
By (2.12), for any $\epsilon > 0$
\[(4.10)\quad \frac{1}{\kappa_1} \sum_{i \in I} f_i h_{111}^2 \leq C(1 + \epsilon^{-1}) \beta^2 f_1 \sum \Phi_i^2 + \frac{(1 + \epsilon)}{(u-a)^2} \sum f_i u_i^2\]
Taking (4.10), (4.7) as well as the following equations (see Lemma 2.2 and Lemma 2.6 in [16] for the proof)
\[\Phi_i = \phi(\rho) \rho_i, \quad \Phi_{ii} = \phi' - u \kappa_i\]
\[u_{ii} = \phi(\rho) \sum_{m} \rho m h_{iim} + \phi'(\rho) \kappa_i - u \kappa_i^2\]
into (4.9) yields
\[(4.11)\quad -\frac{1}{\kappa_1} F^{ij,k}h_{ij1}h_{kl1} - \frac{1}{\kappa_1^2} \sum_{i \in J} f_i h_{111}^2 + \frac{\alpha}{u-a} \sum f_i k_{i1}^2 - C\beta^2(1 + \epsilon^{-1}) f_1 + (\kappa_1 - K - \frac{\phi'}{u-a}) \sum f_i k_i + (\beta \phi' + K) \sum f_i - C - \frac{C}{a} \leq 0\]
Using an inequality due to Andrews [11] and Gerhardt [9] as well as applying (4.4)
\[\frac{a^2}{2(u-a)^2} \sum f_i k_{i1}^2 - C\beta^2(1 + \frac{1}{a^2}) f_1 + (\kappa_1 - 1 - \frac{\phi'}{u-a}) \sigma_0 - C - \frac{C}{a} \leq 0\]
Finally, note that $\sum f_i \kappa_i^2 \geq f_1 \kappa_1^2$. A uniform upper bound for $\kappa_1$ follows easily from the above inequality. Consequently, we obtain a uniform upper bound for $\kappa_{\max}$ on $\Sigma$.

\[ \square \]

5. **Existence**

In this section, we will use classical continuity method and the degree theory developed by Y. Y. Li [20] to prove the existence of solution to the Dirichlet problem (2.43)–(2.42). For this, we assume the existence of a strictly locally convex subsolution $v$, i.e., $v$ satisfies (2.44). We may also assume that $v$ is not a solution of (2.43), for otherwise we are done. Now consider the following two auxiliary equations.

\begin{equation}
\begin{cases}
G(\nabla^2 v, \nabla v, v) = (t \epsilon + (1 - t) \frac{\psi(z)}{\xi(v)}) \xi(v) & \text{in } \Omega \\
v = \underline{v} & \text{on } \partial \Omega
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases}
G(\nabla^2 v, \nabla v, v) = t \psi(z, v) + (1 - t) \epsilon \xi(v) & \text{in } \Omega \\
v = \underline{v} & \text{on } \partial \Omega
\end{cases}
\end{equation}

where $t \in [0, 1]$, $\psi(z) = G(\nabla^2 v, \nabla v, v)(z)$, $\epsilon$ is a small positive constant such that

\begin{equation}
\psi(z) > \epsilon \xi(v) \text{ in } \Omega
\end{equation}

and $\xi(v) = e^{2v}$ if $K = 0$ while $\xi(v) = \sinh v$ if $K = -1$. Hereinafter, we only consider the case when $K = 0$ or $K = -1$. We follow the route of [20] to give the proof.

**Lemma 5.4.** Let $\psi(z)$ be a positive function defined on $\Omega$. For $z \in \Omega$ and a strictly locally convex function $v$ near $z$, if

\[ G(\nabla^2 v, \nabla v, v)(z) = F(a_{ij}[v])(z) = f(\kappa[v])(z) = \psi(z) \xi(v)(z) \]

then

\[ G_v(\nabla^2 v, \nabla^v, v)(z) - \psi(z) \xi'(v)(z) < 0 \]

**Proof.** From (2.22) we have

\[ \frac{\partial a_{ij}}{\partial v} = \frac{1}{w} \left( \eta'(v) \delta_{ij} + \eta(v) \gamma_{ik} \gamma_{lj} \right) \]

\[ = \frac{\eta'^2(v) - \eta^2(v)}{w \eta'(v)} \delta_{ij} + \eta(v) \frac{\eta'(v)}{\eta'(v)} a_{ij} = K \frac{\eta'(v)}{w \eta'(v)} \delta_{ij} + \eta(v) a_{ij} \]

Therefore

\[ G_v = \frac{K}{w \eta'(v)} \sum f_i + \frac{\eta(v)}{\eta'(v)} F^{ij} a_{ij} = \frac{K}{w \eta'(v)} \sum f_i + \frac{\eta(v)}{\eta'(v)} \sum f_i \kappa_i \]

Since $\sum f_i \kappa_i \leq \psi(z) \xi(v)(z)$ by the concavity of $f$ and $f(0) = 0$,

\[ G_v(\nabla^2 v, \nabla^v, v) - \psi(z) \xi'(v) = \frac{K}{w \eta'(v)} \sum f_i + \frac{\eta(v)}{\eta'(v)} \frac{\xi'(v)}{\xi(v)} \sum f_i \kappa_i < 0 \]

\[ \square \]
Lemma 5.5. For any \( t \in [0, 1] \), the Dirichlet problem (5.1) has at most one strictly locally convex solution \( v \) with \( v \geq 0 \).

Proof. Let \( v \) be a strictly locally convex solution of (5.1). It suffices to prove that \( v \geq 0 \) in \( \Omega \). If not, then \( v - v \) achieves a positive maximum at some \( z_0 \in \Omega \). We have

\[
\phi(z_0) > v(z_0), \quad \nabla^2 \phi(z_0) = \nabla' v(z_0), \quad \nabla^2 \phi(z_0) \leq \nabla^2 v(z_0)
\]

We claim that the deformation \( v[s] := s\phi + (1 - s)v \) for \( s \in [0, 1] \) is strictly locally convex near \( z_0 \). To prove this, we use the second expression in (5.22) for \( a_{ij} \) and verify at \( z_0 \) we have

\[
\eta(v[s]) \delta_{ij} + \eta'(v[s]) \tilde{\gamma}^{ij} \left( s \nabla_{kl} \phi + (1 - s) \nabla_{kl} v \right) \tilde{\gamma}^{ij} > 0
\]

In fact, when \( K = 0 \),

\[
\eta(v[s]) \delta_{ij} + \eta'(v[s]) \tilde{\gamma}^{ik} \left( s \nabla_{kl} \phi + (1 - s) \nabla_{kl} v \right) \tilde{\gamma}^{lj}
\]

\[
= \eta(v[s]) \left( \delta_{ij} + \tilde{\gamma}^{ik} \left( s \nabla_{kl} \phi + (1 - s) \nabla_{kl} v \right) \tilde{\gamma}^{lj} \right)
\]

\[
\geq \eta(v[s]) \left( \delta_{ij} + \tilde{\gamma}^{ik} \nabla_{kl} \phi \tilde{\gamma}^{lj} \right)
\]

\[
= \frac{\eta(v[s])}{\eta(\phi)} \left( \eta(\phi) \delta_{ij} + \eta'(v[s]) \tilde{\gamma}^{ij} \nabla_{kl} \phi \tilde{\gamma}^{ij} \right) > 0
\]

When \( K = -1 \), note that

\[
\left( \frac{\eta}{\eta'} \right)'(v) = -\frac{1}{\sinh^2 v} < 0
\]

Therefore

\[
\eta(v[s]) \delta_{ij} + \eta'(v[s]) \tilde{\gamma}^{ik} \left( s \nabla_{kl} \phi + (1 - s) \nabla_{kl} v \right) \tilde{\gamma}^{lj}
\]

\[
\geq \eta(v[s]) \delta_{ij} + \eta'(v[s]) \tilde{\gamma}^{ik} \nabla_{kl} \phi \tilde{\gamma}^{lj}
\]

\[
= \eta'(v[s]) \left( \frac{\eta(v[s])}{\eta(\phi)} - \frac{\eta(v)}{\eta(\phi)} \right) \delta_{ij} + \eta'(v[s]) \left( \eta(\phi) \delta_{ij} + \eta'(v) \tilde{\gamma}^{ik} \nabla_{kl} \phi \tilde{\gamma}^{lj} \right) > 0
\]

Hence (5.1) is proved.

Now we can define a differentiable function of \( s \in [0, 1] \):

\[
a(s) := G(\nabla^2 v[s], \nabla' v[s], v[s]) - (te + (1 - t)\psi(z)) \xi(v[s]) \bigg|_{z_0}
\]

Note that

\[
a(0) = G(\nabla^2 v, \nabla' v, v) - (te + (1 - t)\psi(z_0)) \xi(v) = 0
\]

and by (5.3)

\[
a(1) = G(\nabla^2 v, \nabla' v, v) - (te + (1 - t)\psi(z_0)) \xi(v) = t(\psi - \epsilon \xi(v)(z_0)) \geq 0
\]

So there exists \( s_0 \in [0, 1] \) such that \( a(s_0) = 0 \) and \( a'(s_0) \geq 0 \), i.e.

\[
G(\nabla^2 v[s_0], \nabla' v[s_0], v[s_0]) = (te + (1 - t)\psi(z_0)) \xi(v[s_0])
\]
and

\[ G^{ij}(\nabla^2 v[s_0], \nabla' v[s_0], v[s_0])\nabla'_{ij}(\bar{u} - v)(z_0) + G^i(\nabla^2 v[s_0], \nabla' v[s_0], v[s_0])\nabla_i(\bar{u} - v)(z_0) \]

\[ + (G^i(\nabla^2 v[s_0], \nabla' v[s_0], v[s_0]) - (t\epsilon + (1 - t)\frac{\psi(z_0)}{\xi(\bar{u})})\xi^i(v[s_0]))(\bar{u} - v)(z_0) \geq 0 \]

However, by (5.9), (5.8) and Lemma 5.4 the above expression should be strictly less than 0, which is a contradiction. \( \square \)

**Lemma 5.10.** Let \( v \) be a strictly locally convex solution of (5.2). If \( v \geq \bar{u} \) in \( \Omega \), then \( v > \bar{u} \) in \( \Omega \) and \( n(v - \bar{u}) > 0 \) on \( \partial \Omega \), where \( n \) is the interior unit normal to \( \partial \Omega \).

**Proof.** By (2.44) and (5.3) we know that \( \bar{u} \) is a strict subsolution of (5.2) when \( t \in [0, 1) \), while it is a subsolution but not a solution of (5.2) when \( t = 1 \). It is relatively easy to prove the conclusion when \( t \in [0, 1) \), following the ideas in [26].

For the case \( t = 1 \):\[
\begin{cases}
G(\nabla^2 v, \nabla' v, v) = \psi(z, v) & \text{in } \Omega \\
v = \bar{u} & \text{on } \partial \Omega
\end{cases}
\]

we will make use of the maximum principle which was originally discovered in [24], while more precisely stated for our purposes in [10] (see section 1.3, p. 212). Because the maximum principle and Hopf lemma there are designed for domains in Euclidean spaces, we need to rewrite the above equation in a local coordinate system of \( \mathbb{S}^n \). For convenience, we first transform the above equation back under the transformation (2.18) into a form as (2.26):

\[
\begin{cases}
G(\nabla^2 u, \nabla' u, u) = \psi(z, u) & \text{in } \Omega \\
u = \bar{u} & \text{on } \partial \Omega
\end{cases}
\]

Recall that \( G(\nabla^2 u, \nabla' u, u) = F(A[u]) \) where \( A[u] = \{ \gamma^{ik} h_{kl} \gamma^{lj} \} \). Since at this time we do not use local orthonormal frame on \( \mathbb{S}^n \), but rather a local coordinate system of \( \mathbb{S}^n \), \( \gamma^{ik} \) and \( h_{kl} \) will appear differently (comparing with (2.19) and (2.15)). Also, condition (2.44) (i.e. (1.13)) can be rewritten as

\[
\begin{cases}
G(\nabla^2 u, \nabla' u, u) \geq \psi(z, u) & \text{in } \Omega \\
u = \varphi & \text{on } \partial \Omega
\end{cases}
\]

Note that \( \varphi \) is not a solution of (5.11).

(i) We first show that if a strictly locally convex solution \( u \) of (5.11) satisfies \( u \geq u \) in \( \Omega \), then \( u > u \) in \( \Omega \). Let \( N \notin \Omega \) be the north pole of \( \mathbb{S}^n \). Take the radial projection of \( \mathbb{S}^n \setminus \{ N \} \) onto \( \mathbb{R}^n \times \{-1\} \subset \mathbb{R}^{n+1} \) and let \( \tilde{\Omega} \) be the image of \( \Omega \). We thus have a coordinate system \((x_1, \ldots, x_n)\) on \( \mathbb{R}^n \times \{-1\} \cong \mathbb{R}^n \). The metric on \( \mathbb{S}^n \), its inverse, and the Christoffel symbols are given respectively by

\[
\sigma_{ij} = \frac{16}{\mu^2} \delta_{ij}, \quad \mu = 4 + \sum x_i^2, \quad \sigma^{ij} = \frac{\mu^2}{16} \delta_{ij}
\]

\[
\Gamma^k_{ij} = -\frac{2}{\mu} (\delta_{ik} x_j + \delta_{jk} x_i - \delta_{ij} x_k)
\]
Consequently, the metric on $\Sigma$, its inverse and the second fundamental form on $\Sigma$ are given respectively by (c.f. [25])

$$g_{ij} = \phi^2 \sigma_{ij} + \zeta^2(u) u_i u_j$$

$$g^{ij} = \frac{1}{\phi^2} \left( \sigma^{ij} - \frac{\zeta^2(u) u^i u^j}{\phi^2 + \zeta^2(u) |\nabla u|^2} \right), \quad u^i = \sigma^{ik} u_k$$

$$h_{ij} = \frac{-\zeta'(u) \phi}{\sqrt{\phi^2 + \zeta^2(u) |\nabla u|^2}} (\nabla_i u + u \sigma_{ij})$$

The entries of the symmetric matrices $\{\gamma_{ik}\}$ and $\{\gamma^{ik}\}$ depend only on $x_1, \ldots, x_n$, $u$ and the first derivatives of $u$.

Now, setting $\tilde{u} = \mu u$ and by straightforward calculation we have

$$\nabla_i u + u \sigma_{ij} = \frac{1}{\mu} \tilde{u}_i - \frac{2}{\mu^2} (\tilde{u} - \sum_k x_k \tilde{u}_k)$$

and (5.11) can be transformed into the following form:

$$\begin{cases}
\tilde{G}(D^2 \tilde{u}, D\tilde{u}, \tilde{u}, x_1, \ldots, x_n) = F(A \left[ \frac{\tilde{u}}{\mu} \right]) = \tilde{\psi}(x_1, \ldots, x_n, \tilde{u}) & \text{in } \tilde{\Omega} \\
\tilde{u} = \mu u & \text{on } \partial \tilde{\Omega}
\end{cases}$$

where $\tilde{u}_i = \frac{\partial \tilde{u}}{\partial x_i}$, $D\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)$, $\tilde{u}_{ij} = \frac{\partial^2 \tilde{u}}{\partial x_i \partial x_j}$ and $D^2 \tilde{u} = \{\tilde{u}_{ij}\}$.

In view of (5.12) and (2.24) we know that

$$\frac{\partial \tilde{G}}{\partial \tilde{u}_{ij}} = \frac{\partial F}{\partial a_{kl}} \frac{\partial a_{kl}}{\partial \tilde{u}_{ij}} = \frac{1}{\mu} \frac{\partial G}{\partial u_{ij}}$$

Also, the function $\tilde{u} = \mu u$ satisfies

$$\begin{cases}
\tilde{G}(D^2 \tilde{u}, D\tilde{u}, \tilde{u}, x_1, \ldots, x_n) \geq \tilde{\psi}(x_1, \ldots, x_n, \tilde{u}) & \text{in } \tilde{\Omega} \\
\tilde{u} = \mu u & \text{on } \partial \tilde{\Omega}
\end{cases}$$

Hence we can apply the Maximum Principle (see p. 212 of [10]) to conclude that $\tilde{u} > \bar{u}$ in $\tilde{\Omega}$, which immediately yields $u > \underline{u}$ in $\Omega$.

(ii) To prove $\mathbf{u}(u - \bar{u}) > 0$ on $\partial \Omega$, we pick an arbitrary point $z_0 \in \partial \Omega$ and assume $z_0$ to be the north pole of $S^n \subset \mathbb{R}^{n+1}$. We introduce a local coordinate system about $z_0$ by taking the radial projection of the upper hemisphere onto the tangent hyperplane of $S^n$ at $z_0$ and identifying this hyperplane to $\mathbb{R}^n$. Denote the coordinates by $(y_1, \ldots, y_n)$ and assume that the positive $y_n$-axis is the interior normal direction to $\partial \Omega \subset S^n$ at $z_0$. In this coordinate system, the metric on $S^n$, its inverse, and the Christoffel symbols are given respectively by (see [22] [13])

$$\sigma_{ij} = \frac{1}{\mu^2} \left( \delta_{ij} - \frac{y_i y_j}{\mu^2} \right), \quad \mu = \sqrt{1 + \sum y_i^2}$$

$$\sigma^{ij} = \mu^2 (\delta_{ij} + y_i y_j)$$

$$\Gamma^k_{ij} = -\delta_{ik} y_j + \delta_{jk} y_i \mu^2$$

The metric $g_{ij}$, its inverse $g^{ij}$ and the second fundamental form $h_{ij}$ on $\Sigma$ have the form as above. The entries of the symmetric matrices $\{\gamma_{ik}\}$ and $\{\gamma^{ik}\}$ depend only on $y_1, \ldots, y_n$, $u$ and the first derivatives of $u$. 
Now set \( \tilde{u} = \mu u \). By straightforward calculation we have

\[
(5.13) \quad \nabla'_{ij} u + u \sigma_{ij} = \mu^{-1} \tilde{u}_{ij}
\]

and equation (5.11) can be transformed into an equation defined in an open neighborhood of 0 on \( \mathbb{R}^n \), which is the radial projection of a neighborhood of \( z_0 \) on \( \mathbb{S}^n \):

\[
\tilde{G}(D^2 \tilde{u}, D \tilde{u}, \tilde{u}, y_1, \ldots, y_n) = F \left( A \left[ \frac{\tilde{u}}{\mu} \right] \right) = \tilde{\psi}(y_1, \ldots, y_n, \tilde{u})
\]

where \( \tilde{u}_i = \frac{\partial \tilde{u}}{\partial y_i} \), \( D \tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n) \), \( \tilde{u}_{ij} = \frac{\partial^2 \tilde{u}}{\partial y_i \partial y_j} \) and \( D^2 \tilde{u} = \{ \tilde{u}_{ij} \} \). In view of (5.13) and (2.27) we know that

\[
\frac{\partial \tilde{G}}{\partial \tilde{u}_{ij}} = \frac{\partial F}{\partial a_{kl}} \frac{\partial a_{kl}}{\partial \tilde{u}_{ij}} = \frac{1}{\mu} \frac{\partial G}{\partial u_{ij}}
\]

Applying Lemma H (see p. 212 of [10]) we find that \( (\tilde{u} - \bar{u})_n(0) > 0 \) and equivalently \( n(u - \bar{u})(z_0) > 0 \). \( \square \)

**Theorem 5.14.** For any \( t \in [0, 1] \), the Dirichlet problem (5.1) has a unique strictly locally convex solution.

**Proof.** Uniqueness is proved in Lemma 5.5. We prove the existence using the standard continuity method. Recall that \( u \) and \( v \) are related by the transformation (2.18). Hence the \( C^2 \) estimate (3.1) established in section 3 and 4 is equivalently the \( C^2 \) bound for strictly locally convex solutions \( v \) of (5.11) with \( v \geq \bar{u} \). Besides, the uniform upper and positive lower bounds of the principal curvatures imply that equation (5.11) is uniformly elliptic for strictly locally convex solutions \( v \) with \( v \geq \bar{u} \).

We can then apply Evans-Krylov theory [8, 19] to obtain

\[
(5.15) \quad \| v \|_{C^{2,\alpha}(\bar{\Omega})} \leq C
\]

Here we note that \( C \) is independent of \( t \).

Let \( C^{2,\alpha}_0(\bar{\Omega}) \) be the subspace of \( C^{2,\alpha}(\bar{\Omega}) \) given by

\[
C^{2,\alpha}_0(\bar{\Omega}) := \{ w \in C^{2,\alpha}(\bar{\Omega}) \mid w = 0 \text{ on } \partial \Omega \}
\]

Obviously,

\[
\mathcal{U} := \left\{ w \in C^{2,\alpha}_0(\bar{\Omega}) \left| \mu + w \text{ is strictly locally convex} \right. \right\}
\]

is an open subset of \( C^{2,\alpha}_0(\bar{\Omega}) \). Construct a map \( \mathcal{L} : \mathcal{U} \times [0, 1] \to C^{\alpha}(\bar{\Omega}) \) by

\[
\mathcal{L}(w, t) = G(\nabla^2 (\bar{u} + w), \nabla' (\bar{u} + w), \bar{u} + w) - \left( tc + (1 - t) \frac{\psi(z)}{\xi(\bar{u})} \right) \xi(\bar{u} + w)
\]

Set

\[
S = \{ t \in [0, 1] \mid \mathcal{L}(w, t) = 0 \text{ has a solution in } \mathcal{U} \}
\]

First note that

\[
\mathcal{L}(0, 0) = G(\nabla^2 \bar{u}, \nabla' \bar{u}, \bar{u}) - \frac{\psi(z)}{\xi(\bar{u})} \xi(\bar{u}) = 0
\]

hence \( 0 \in S \) and \( S \neq \emptyset \).
Besides, we also have the estimate (see the expression in (2.22))

\[ \|v\|_{C^{2,\alpha}([0,1])} < C_{4} \]

(5.18)

By Lemma 5.14, \( L_{w}|_{(w_{0},t_{0})} \) is invertible. Hence by implicit function theorem, a neighborhood of \( t_{0} \) is also contained in \( S \).

\( S \) is closed in \([0,1] \). Let \( t_{i} \) be a sequence in \( S \) converging to \( t_{0} \in [0,1] \) and \( w_{i} \in U \) be the unique solution associated with \( t_{i} \) (the uniqueness is guaranteed by Lemma 5.5, i.e. \( L(w_{i},t_{i}) = 0 \)). By Lemma 5.3, \( w_{i} \geq 0 \). Then by (5.15), we see that \( v_{i} := \overline{w} + w_{i} \) is a bounded sequence in \( C^{2,\alpha}(\Omega) \). Possibly passing to a subsequence \( v_{i} \) converges to a strictly locally convex solution \( v_{0} \) of (5.1) as \( i \to \infty \). Obviously \( w_{0} := v_{0} - \overline{w} \in U \) and \( L(w_{0},t_{0}) = 0 \). Thus \( t_{0} \in S \). □

**Theorem 5.17.** For any \( t \in [0,1] \), the Dirichlet problem (5.2) has a strictly locally convex solution. In particular, (2.13) - (2.12) has a strictly locally convex solution.

**Proof.** The \( C^{2,\alpha} \) estimate for strictly locally convex solutions \( v \) of (5.2) with \( v \geq \overline{w} \) can be established in view of (2.18) and (5.1), which in turn yields \( C^{4,\alpha} \) estimate by classical Schauder theory

\[ \|v\|_{C^{4,\alpha}(\overline{\Omega})} < C_{4} \]

Besides, we also have the estimate (see the expression in (2.22))

\[ C_{2}^{-1} I < \tilde{A}[v] := \{ \psi'(v)\nabla^{2}_{ij}v + \psi(v)v_{i}v_{j} + \psi(v)\delta_{ij} \} < C_{2}I \quad \text{in} \quad \overline{\Omega} \]

(5.19)

Here we note that \( C_{2} \) and \( C_{4} \) are independent of \( t \).

Let \( C_{0}^{4,\alpha}(\overline{\Omega}) \) be the subspace of \( C^{4,\alpha}(\overline{\Omega}) \) defined by

\[ C_{0}^{4,\alpha}(\overline{\Omega}) := \{ w \in C^{4,\alpha}(\overline{\Omega}) | w = 0 \text{ on } \partial\Omega \} \]

and consider the bounded open subset

\[ \mathcal{O} := \left\{ w \in C_{0}^{4,\alpha}(\overline{\Omega}) | \begin{array}{l} w > 0 \text{ in } \Omega, \quad \nabla^{2}_{ij}w > 0 \text{ on } \partial\Omega, \\
C_{2}^{-1}I < \tilde{A}[w] < C_{2}I \quad \text{in} \quad \overline{\Omega} \\
\|w\|_{C^{4,\alpha}(\overline{\Omega})} < C_{4} + \|w\|_{C^{4,\alpha}(\overline{\Omega})} \end{array} \right\} \]

Construct a map \( \mathcal{M}_{t}(w) : \mathcal{O} \times [0,1] \to C^{2,\alpha}(\overline{\Omega}) \)

\[ \mathcal{M}_{t}(w) = \mathcal{G}(\nabla^{2}(\overline{w} + w), \nabla^{2}(\overline{w} + w), \overline{w} + w) - t \psi(z, \overline{w} + w) - (1 - t) \varepsilon \xi(\overline{w} + w) \]

Let \( v^{0} \) be the unique solution of (5.1) at \( t = 1 \) (the existence and uniqueness are guaranteed by Theorem 5.14 and Lemma 5.5). Note that \( v^{0} \) is also the solution of (5.2) when \( t = 0 \). Set \( w^{0} = v^{0} - \overline{w} \). By Lemma 5.5, we have \( w^{0} \geq 0 \) in \( \Omega \), which in turn implies that \( w^{0} > 0 \) in \( \Omega \) and \( \nabla^{2}_{ij}w^{0} > 0 \) on \( \partial\Omega \) by Lemma 5.10. Also note that \( v^{0} \) satisfies (5.15) and (5.19). Thus, \( w^{0} \in \mathcal{O} \). From Lemma 5.10, (5.15) and (5.19) we observe that \( \mathcal{M}_{t}(w^{0}) = 0 \) has no solution on \( \partial\mathcal{O} \) for any \( t \in [0,1] \). Besides, \( \mathcal{M}_{t} \) is uniformly elliptic on \( \mathcal{O} \) independent of \( t \). Hence the degree of \( \mathcal{M}_{t} \) on \( \mathcal{O} \) at \( 0 \)

\[ \text{deg}(\mathcal{M}_{t}, \mathcal{O}, 0) \]
is well defined and independent of $t$. Therefore we only need to compute $\deg(\mathcal{M}_0, \mathcal{O}, 0)$.

Note that $\mathcal{M}_0(w) = 0$ has a unique solution $w^0 \in \mathcal{O}$, and the Fréchet derivative of $\mathcal{M}_0$ with respect to $w$ at $w^0$ is a linear elliptic operator from $C^{4,\alpha}_0(\Omega) \to C^{2,\alpha}_0(\Omega)$,

$$
\mathcal{M}_{0,w}|_{w^0}(h) = G^{ij}(\nabla^2 v^0, \nabla v^0, v^0) \nabla^2_{ij} h + G^i(\nabla^2 v^0, \nabla v^0, v^0) \nabla^i h
+ (\tilde{G}^i_v(\nabla^2 v^0, \nabla v^0, v^0) - \epsilon \xi'(v^0)) h
$$

(5.20)

By Lemma 5.1

$$
G_v(\nabla^2 v^0, \nabla v^0, v^0) - \epsilon \xi'(v^0) < 0 \quad \text{in} \quad \Omega
$$

Hence $\mathcal{M}_{0,w}|_{w^0}$ is invertible. By the degree theory in [20]

$$
\deg(\mathcal{M}_0, \mathcal{O}, 0) = \deg(\mathcal{M}_{0,w}, B_1, 0) = \pm 1 \neq 0
$$

where $B_1$ is the unit ball in $C^{4,\alpha}_0(\Omega)$. Thus

$$
\deg(\mathcal{M}_1, \mathcal{O}, 0) \neq 0 \quad \text{for all} \quad t \in [0, 1]
$$

which implies that the Dirichlet problem (5.2) has at least one strictly locally convex solution for any $t \in [0, 1]$. In particular, $t = 1$ solves the Dirichlet problem (2.43).

\[ \square \]

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