Unified Treatment of EPR and Bell Arguments in Algebraic Quantum Field Theory

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Abstract. A conjecture concerning vacuum correlations in axiomatic quantum field theory is proved. It is shown that this result can be applied both in the context of EPR-type experiments and Bell-type experiments.
1 Introduction

In the past, two arguments have played major roles in the discussion of the foundations and interpretations of quantum theory. On the one hand there is the incompleteness argument by Einstein, Podolsky, and Rosen (EPR)\cite{EPR}. For EPR a necessary condition for a theory to be complete is that 'every element of the physical reality must have a counterpart in the physical theory' (\cite{EPR}, 777). They consider a setup, for which, they argue, one can find an element of reality that is not represented by anything in the formalism of quantum mechanics. In Bohm’s version of the argument two spin-$\frac{1}{2}$ particles move in opposite directions as a result of a decay or scattering process. The system is described by the singlet state in which the spins of the two particles are perfectly correlated, that is, by measuring the spin of one of the particles in an arbitrary direction the outcome of the same measurement on the other particle can be predicted with probability one. Assuming further that elements of reality relevant to one part of the system cannot be affected by measurements performed at spacelike distance on another part of the system, EPR show that an element of reality at the unmeasured particle must have existed before the measurement on the spacelike separated particle. Since this element of reality has no counterpart in the formalism of quantum mechanics, one can conclude that the quantum mechanical description is incomplete. This ‘streamlined’ version of the EPR argument is set out in more detail in \cite{EPR_1}. In effect the realist can use this argument to refute not only the instrumentalist interpretation but also Bohr’s complementarity interpretation, by assuming a suitably specialised locality condition.

On the other hand, the realist himself runs into difficulties (and this is the second argument) if he assumes a more complete description in terms of hidden variables and, in addition, the appropriate locality condition that any sharp value for an observable cannot be changed into another sharp value by altering the setting of a spacelike separated piece of apparatus \cite{EPR_1}. The realist then faces the problem that the Bell inequality derived under these assumptions is in general violated. Thus, unless one gives up the locality condition, the realist hidden variable interpretation is not tenable.

In algebraic quantum field theory, which provides a relativistic framework of quantum theory, aspects of these arguments can be generalised. It was shown by one of us (MLGR) that perfect correlations, as they appear in Bohm’s version of the EPR-argument, can effectively be demonstrated in quantum field theory, that is, by appropriate choice of (idealised) measurement devices perfect correlations can be approximated arbitrarily closely - even in the vacuum state \cite{MLGR}. On the other hand it was shown by Landau, Summers and Werner that Bell inequalities are violated in the vacuum state, and that a maximal violation can be achieved by conditioning the Bell inequality on the outcome of a third, spacelike separated measurement \cite{LSW}. The object of the present paper is to show, perhaps surprisingly, that these two claims can be derived from a single ‘root’ theorem, a proposal put forward recently by Malament \cite{Malament}, although no proof of the result has hitherto been published.

The paper is organised as follows. First we briefly state some axioms in order to specify our framework. We then proceed to prove the root theorem. The proofs of the lemmata we employed are given in the appendix. In section \ref{sec:3} we show how previous results in the discussion of the two distinct arguments can be recovered as special cases of the root theorem.
2 Axiomatic QFT

In this section we shall review briefly the axiomatic framework and the result known as the Reeh-Schlieder theorem. Although not all of the axioms presented here are needed to prove the Reeh-Schlieder theorem, section 4 requires a more specific framework for the analysis of Bell-type correlations which is why an extended set of axioms is presented here [1] [9].

Let \( \{ O_i \} \) be bounded open regions in Minkowski space. Local observables are identified with hermitean elements in local (non-commutative) von Neumann-algebras \( \mathcal{R}(O_i) \) which are subsets of the set of bounded operators, denoted by \( \mathcal{B}(\mathcal{H}) \), acting on some Hilberspace \( \mathcal{H} \). The axioms we want to impose are then:

(i) **Isotony:** If \( O_1 \subseteq O_2 \), then \( \mathcal{R}(O_1) \subseteq \mathcal{R}(O_2) \).

(ii) **Spacelike commutativity:** If all points in \( O_1 \) are spacelike separated from all points in \( O_2 \), then \( \mathcal{R}(O_1) \subset \mathcal{R}(O_2) \)' , that is, every operator in \( \mathcal{R}(O_1) \) commutes with every operator in \( \mathcal{R}(O_2) \).

(iii) **Poincaré Covariance:** There exists on \( \mathcal{H} \) a strongly continuous unitary representation \( U(P) \) of the (covering group of the) Poincaré group which meshes with the automorphisms of the global algebra \( \mathcal{R} \), in the following sense: to each (active) Poincaré transformation \( g := (\Lambda, a) \) acting on Minkowski spacetime, there is associated an automorphism \( \alpha_g \) of the global algebra \( \mathcal{R} \) with \( \alpha_g[\mathcal{R}(O)] = \mathcal{R}[g(O)] \) and such that \( U(g)AU(g)^{-1} = \alpha_g(A) \) for all \( A \in \mathcal{R}(O) \).

(iv) **Diamond:** Let \( D(O) = D^+(O) \cup D^-(O) \) be the domain of dependence (sometimes called causal shadow) of the spacetime region \( O \). Then \( \mathcal{R}(O) = \mathcal{R}[D(O)] \).

(v) **Weak additivity:** For any non-empty region \( O \), the set \( C'' \) equals the von Neumann algebra generated by the set of all translations of \( \mathcal{R}(O) \), that is, \( C'' = \langle \bigcup_{a \in \mathbb{R}^4} \mathcal{R}(O_a) \rangle'' \).

(vi) **Irreducibility:** \( \mathcal{H} \) has no non-trivial subspace invariant under all elements of all \( \mathcal{R}(O) \), that is, \( C'' = \mathcal{B}(\mathcal{H}) \).

(vii) **Spectrum:** The spectrum of the translation subgroup of the Poincaré group is contained in the closed forward lightcone, that is, the energy-momentum eigenvalues are restricted by \( p^2 \geq 0, p^0 \geq 0 \).

(viii) **Vacuum:** There is a unique state \( \Omega \), called the vacuum, that is invariant under all actions of the Poincaré group: \( \forall g \in \mathcal{P} : U(g)\Omega = \Omega \).

A state \( \psi \) in \( \mathcal{H} \) for which the set \( \{ A\psi : A \in \mathcal{A} \} \) is dense in \( \mathcal{H} \) is called a cyclic vector for \( \mathcal{A} \) with respect to \( \mathcal{H} \). We have now introduced all the necessary concepts in order to state without proof

**Theorem 1 (Reeh and Schlieder [18]):** Let \( O \) be an open bounded set in spacetime. Then \( \Omega \) a is cyclic vector for \( \mathcal{R}(O) \).

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1The future domain of dependence, \( D^+(O) \), is defined as the set of points in spacetime through which every past inextendible causal curve intersects \( O \). For the definition of \( D^-(O) \) interchange ‘past’ with ‘future’.
Note that the theorem does not merely state that the state $\Omega$ is a cyclic vector for the
global algebra $\mathcal{R}$. The latter is certainly what we would expect even in the Schrödinger
approach to quantum field theory.

It is worth noting that the vacuum can be replaced by any state with bounded energy
in the Reeh-Schlieder theorem. Furthermore, the theorem may be further strengthened by
replacing ‘open bounded set’ by ‘open set with non-empty causal complement’. Theorem
1 has an important consequence, expressed in the following (theorem 5.3.2 in Haag [3])

**Theorem 2** If $O$ has non-void causal complement, then $\mathcal{R}(O)$ does not contain any
operator which annihilates the vacuum, that is, if $A\Omega = 0$ for some $A \in \mathcal{R}(O)$, then it follows
that $A = 0$. The vacuum is therefore called a separating vector. $\square$

This property ensures that, if two operators $A_1$ and $A_2$ in $\mathcal{R}(O)$ generate the same
state from the vacuum, then they have to be the same operator. It also shows that there
are no pure local annihilation operator and thus, using the fact that $\mathcal{R}(O)$ is an involutive
algebra, no pure localised creation operator. This supports the view that the concept of a
quantum is a non-local one.

Theorem 2 also implies the (Theorem 2 in Redhead [14], Corollary in Hellweg and Kraus [7])

**Corollary 1** Any possible outcome of any possible measurement procedure will occur with
non-vanishing probability in the vacuum. $\square$

3 The root theorem

The Reeh-Schlieder theorem provides us with a powerful tool in the analysis of non-locality.
It shows that localised causes can have non-local effects, even in the vacuum state. The
following theorem determines more specifically the kind of effects that can be produced by
selective local actions on the vacuum state. In particular, it deals with two open bounded
spacelike separated regions, in each of which measurements can be performed.

**Theorem 3 (Root Theorem)** For any two spacelike separated bounded open regions $O_1$ and $O_2$, for any self-adjoint operator $A$ in $\mathcal{R}(O_2)$, and for any unit vector $\psi$, if $\langle A \rangle_\psi = K$
then, for all $\varepsilon > 0$, there are projection operators $P_1$ and $P_1^\perp$ in $\mathcal{R}(O_1)$ such that

\[
\langle AP_1 \rangle_{\Omega} > (K - \varepsilon)\langle P_1 \rangle_{\Omega}, \quad \text{and} \quad \langle AP_1^\perp \rangle_{\Omega} < (K + \varepsilon)\langle P_1^\perp \rangle_{\Omega},
\]

where $\Omega$ is the vacuum state.

**Remarks:** First, notice that it is not required that $\langle A \rangle_\psi = K$ for all unit states; it
merely has to hold for at least one unit state. Secondly, as $P_1 \neq P_1^\perp$ in general, it is not
claimed that in general $K$ can be approximated arbitrarily closely. Third, we will assume
that $A$ is defined everywhere on the Hilbert space $\mathcal{H}$. By the theorem of Hellinger and
Toeplitz [17], $A$ is then bounded, that is, $\|A\|$ exists and is finite. In particular, for any
$\phi \in \mathcal{H}$ we have $\|A\phi\| \leq \|A\|\|\phi\|$. Also, $\Omega$ can be replaced by any state with bounded energy.
Proof of Theorem 3: Let $A \in \mathcal{R}(O_2)$, and let $\psi$ be a unit vector for which $\langle A \rangle_\psi = K$. By the Reeh-Schlieder theorem, $\psi$ can be approximated arbitrarily closely by elements of the set $\mathcal{R}_1 \Omega = \{ C\Omega : C \in \mathcal{R}(O_1) \}$, that is

$$\forall \varepsilon_1 > 0, \exists \tilde{C}_{\varepsilon_1} \in \mathcal{R}(O_1), \text{ such that } \| \psi - \tilde{C}_{\varepsilon_1} \Omega \| < \varepsilon_1.$$  

It is convenient to choose the norm of the vector $\tilde{C}_{\varepsilon_1} \Omega$ to be equal to 1. The following lemma ensures that such a choice is always possible.

Lemma 1 For all $\varepsilon_2 > 0$ there exists an operator $C_{\varepsilon_2}$ such that:

$$\| \psi - C_{\varepsilon_2} \Omega \| < \varepsilon_2 \quad \text{and} \quad \| C_{\varepsilon_2} \Omega \| = 1.$$  

Intuitively it is clear that the expectation value of $A$ in the state $C_{\varepsilon_2} \Omega$ is close to $K$. This can be proved rigorously (Theorem 4 in [11]):

Lemma 2 For all $\varepsilon_3 > 0$, there exists a $C_{\varepsilon_2}$ such that

$$K - \varepsilon_3 < \langle A \rangle_{C_{\varepsilon_2} \Omega} < K + \varepsilon_3.$$  

From our general assumption of spacelike commutativity (axiom (ii)) we conclude that $A \in \mathcal{R}(O_2)$ and $C_{\varepsilon_2}^\dagger \in \mathcal{R}(O_1)$ commute, that is,

$$\langle A \rangle_{C_{\varepsilon_2} \Omega} = (C_{\varepsilon_2} \Omega, AC_{\varepsilon_2} \Omega) = (\Omega, C_{\varepsilon_2}^\dagger AC_{\varepsilon_2} \Omega) = (\Omega, AC_{\varepsilon_2}^\dagger C_{\varepsilon_2} \Omega).$$

The operator $C_{\varepsilon_2}^\dagger C_{\varepsilon_2} \overset{\text{def}}{=} Q_1$ is clearly self-adjoint and bounded, and in particular it is positive. Hence, we may approximate it arbitrarily closely by a finite sum of projectors with positive coefficients, that is, $\forall \bar{\varepsilon}_4, \exists n \in \mathbb{N}$ and there is a family of projectors $\{ P_i \}_{i=1}^n \subset \mathcal{R}(O_1)$ together with coefficients $\{ \bar{\lambda}'_i \}_{i=1}^n, \bar{\lambda}'_i > 0 \forall i$, such that

$$\| C_{\varepsilon_2}^\dagger C_{\varepsilon_2} - \sum_{i=1}^n \bar{\lambda}'_i P_i \| \overset{\text{def.}}{=} \| Q_1 - \tilde{Q}'_1 \| < \bar{\varepsilon}_4 \quad (3)$$

This just follows from the spectral theorem. Again, for our purposes it will be convenient to impose the requirement that the vacuum expectation value of the approximating operator should equal 1. The following lemma shows that such a choice is possible.

Lemma 3 It is always possible to rescale the coefficients $\bar{\lambda}'_i \rightarrow \lambda'_i = \bar{\lambda}'_i / \langle \tilde{Q}'_1 \rangle_\Omega$ such that $Q'_1 = \sum_{i=1}^n \lambda'_i P_i$ is an approximation to $Q_1$ in the sense of inequality (3), and $\langle Q'_1 \rangle_\Omega = 1$.

We now incorporate the approximation to the operator $Q_1$ into lemma 2 and find

Lemma 4 For all $\varepsilon_5 > 0$ there is an operator $Q'_1 = \sum_{i=1}^n \lambda'_i P_i$ in $\mathcal{R}(O_1)$, such that

$$K - \varepsilon_5 < \langle AQ'_1 \rangle_\Omega < K + \varepsilon_5.$$  

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Now we can write
\[
\langle AQ'_1 \rangle \Omega = \sum_{i=1}^{n} w_i \frac{\langle AP_i \rangle \Omega}{\langle P_i \rangle \Omega}, \quad \text{where} \quad w_i = \lambda_i \langle P_i \rangle \Omega
\]
and therefore \(\sum_{i=1}^{n} w_i = 1\) as we have chosen using lemma 3. Because of the latter property it is clear that at least one element of the set \(\left\{ \frac{\langle AP_i \rangle \Omega}{\langle P_i \rangle \Omega} \right\}\) is greater or equal to \(\langle AQ'_1 \rangle \Omega\). If this was not the case, then
\[
\langle AQ'_1 \rangle \Omega = \sum_{i=1}^{n} w_i \frac{\langle AP_i \rangle \Omega}{\langle P_i \rangle \Omega} < \sum_{i=1}^{n} w_i \langle AQ'_1 \rangle \Omega = \langle AQ'_1 \rangle \Omega,
\]
which is a contradiction. Therefore,
\[
K - \varepsilon_5 < \langle AQ'_1 \rangle \Omega \leq \max \left\{ \frac{\langle AP_i \rangle \Omega}{\langle P_i \rangle \Omega} \right\} \overset{\text{def.}}{=} \frac{\langle AP_{\text{max}} \rangle \Omega}{\langle P_{\text{max}} \rangle \Omega}
\]
A similar argument leads to
\[
K + \varepsilon_5 > \langle AQ'_1 \rangle \Omega \geq \min \left\{ \frac{\langle AP_i \rangle \Omega}{\langle P_i \rangle \Omega} \right\} \overset{\text{def.}}{=} \frac{\langle AP_{\text{min}} \rangle \Omega}{\langle P_{\text{min}} \rangle \Omega}
\]
Finally, we conclude that there are projectors \(P_1 = P_{\text{max}}\) and \(P_1^2 = P_{\text{min}}\) in \(\mathcal{R}(O_1)\), such that
\[
\langle AP_1 \rangle \Omega > (K - \varepsilon_5)\langle P_1 \rangle \Omega, \quad \text{and} \quad \langle AP_1^2 \rangle \Omega < (K + \varepsilon_5)\langle P_1^2 \rangle \Omega
\]
which concludes the proof of theorem 3. \(\square\)

4 Applications

The root theorem holds in a quite general context. The expectation value of any local observable \(A\) belonging to a spacetime region \(O_2\), in any state \(\psi\) which attains a maximum (or minimum) value, can be approximated arbitrarily closely by acting with an appropriate projector in a spacelike separated region on the vacuum state. A special case of such an operator \(A\) is a projector, say \(P_2\), attached to the region \(O_2\). This corresponds to the situation in which a measurement device is prepared to register a particular, single eigenvalue of an observable. Such a configuration has been investigated before by one of the present authors and formulated as (theorem 4’ in reference [14])

**Theorem 4** For any two spacelike separated bounded open regions \(O_1\) and \(O_2\), and for all \(\varepsilon > 0\) and all projectors \(P_2 \in \mathcal{R}(O_2)\), there exists a projector \(P_1 \in \mathcal{R}(O_1)\), such that
\[
\langle P_1 \rangle \Omega \geq \langle P_1 P_2 \rangle \Omega > (1 - \varepsilon)\langle P_1 \rangle \Omega \tag{4}
\]

**Proof of Theorem [3]:** Let \(P_2\) be a projector in \(\mathcal{R}(O_2)\) such that \(P_2 \phi \neq 0\) for some state \(\phi\). Such a state can always be found: as \(\Omega\) is a separating vector it has the required property. Then the state \(\psi = P_2 \phi / \|P_2 \phi\|\) has the property \(\langle P_2 \rangle \psi = 1\). The root theorem
can now be applied for \( K = 1 \) with the given assignments, and so the second inequality in (4) corresponds to inequality (1), while the first one is trivial as \( P_2 \) is a projector.

Theorem 4 says effectively that given a measurement outcome of one for \( P_1 \), we can predict with probability as close to unity as we like the outcome of measuring a projector at spacelike separation. This is all that is needed to run a vacuum version of the EPR argument.

Let us now turn to the discussion of an application of the root theorem to Bell inequalities. Let \( P_i \) and \( Q_i \) be projection operators in some operator algebras \( \mathcal{A} \) and \( \mathcal{B} \), respectively, that act on the same Hilbert space. Let \( A_i = (2P_i - 1) \) and \( B_i = (2Q_i - 1) \) be corresponding contractions. We call the operator \( R = A_1(B_1 + B_2) + A_2(B_1 - B_2) \) the Bell operator and define

\[
\beta(\phi, \mathcal{A}, \mathcal{B}) = \frac{1}{2} \sup \langle R \rangle_{\phi},
\]

the supremum over all self-adjoint contractions \( A_i \in \mathcal{A}, B_i \in \mathcal{B} \), and call \( \beta(\phi, \mathcal{A}, \mathcal{B}) \) the maximal Bell correlation of the pair \( (\mathcal{A}, \mathcal{B}) \). It is not difficult to show that Bell’s inequality is not only violated in quantum theory, but also that there exists an upper bound for the violation, that is, a corresponding ‘Bell inequality’ in the quantum theory (proposition 2, Landau [9]):

**Lemma 5** For any \( C^* \)-algebra \( \mathcal{C} \), any commuting non-abelian subalgebras \( \mathcal{A} \) and \( \mathcal{B} \) and any state \( \phi \) in the dual of \( \mathcal{C} \),

\[
|\beta(\phi, \mathcal{A}, \mathcal{B})| \leq \sqrt{2}
\]

**Remark:** This has to be compared with the Bell inequality for classical observables, expressed as \( \beta(\phi, \mathcal{A}, \mathcal{B}) \leq 1 \).

Note that lemma 5 does not ensure that there is a state \( \phi \) for which the Bell inequality is maximally violated, that is, for which \( \frac{1}{2} \langle R \rangle_{\phi} = \sqrt{2} \); nor is it clear a priori whether, for a given state, there are projectors such that the violation is maximal. However, Landau proved [9] both propositions for a class of von Neumann algebras, and we will formulate them as

**Lemma 6** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two commuting non-abelian subalgebras in a von Neumann algebra \( \mathcal{C} \), which have the Schlieder property. Then

(i) There is a state \( \tilde{\phi} \) on \( \mathcal{C} \) such that \( |\beta(\tilde{\phi}, \mathcal{A}, \mathcal{B})| = \sqrt{2} \)

(ii) In addition we can find contractions \( A_1, A_2 \in \mathcal{A} \) and \( B_1, B_2 \in \mathcal{B} \) for which \( \frac{1}{2} \langle R \rangle_{\tilde{\phi}} = \sqrt{2} \).

So far we have only considered the classical and the general quantum theoretical situation. We now return to the more specific framework of algebraic quantum field theory with its distinct underlying spacetime structure. After some careful preparations, lemma 5 can be formulated for local algebras without difficulty.

The first thing to notice is that, according to axiom (ii), local algebras associated with spacelike separated regions \( O_1 \) and \( O_2 \) commute. Thus, they are natural candidates for
the algebras $\mathcal{A}$ and $\mathcal{B}$ in lemma 3, and we expect to find maximal violation of the Bell inequality for localised algebras. However, it can be shown that if the stronger condition of strictly spacelike separation holds for $O_1$ and $O_2$, then local algebras $\mathcal{R}(O_1)$ and $\mathcal{R}(O_2)$ posses the Schlieder property. This is not true in general for spacelike separated regions. So we can prove

**Corollary 2** Let $\{\mathcal{R}(O)\}$ be a net of localized von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ subject to the axioms (i)-(viii). Then, for any strictly spacelike separated regions $O_1$ and $O_2$ there exists a state $\phi$ on $\mathcal{B}(\mathcal{H})$ such that $\beta(\phi, \mathcal{R}(O_1), \mathcal{R}(O_2)) = \sqrt{2}$ and there are contractions $A_1, A_2 \in \mathcal{R}(O_1)$ and $B_1, B_2 \in \mathcal{R}(O_2)$ for which $\frac{1}{2}(R)_{\phi} = \sqrt{2}$.

**Remark:** Strict spacelike separation is sufficient but not necessary for $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ to have the Schlieder property. For example, any pair of spacelike separated double cone algebras or wedge algebras have the Schlieder property as a result of the general theorem by Driessler (Theorem 3.5. in [3]).

The state $\phi$ for which $\beta(\phi, \mathcal{A}(O_1), \mathcal{A}(O_2)) = \sqrt{2}$ holds is only one among many (typically infinitely many) states for which the Bell inequality is violated [10]. Since the expectation value of $R$ is a *continuous* functional on the Hilbert space we can expect that there are projectors in the algebras $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ and states in $\mathcal{H}$ which violate Bell’s inequality arbitrarily closely to $\sqrt{2}$. (Corollary 2 holds for arbitrarily distant regions $O_1$ and $O_2$. However, for fixed state $\phi$ the degree of violation depends on their spatial separation [5][21]). Recalling from above that, by the Reeh-Schlieder theorem, any state in $\mathcal{H}$ can be approximated arbitrarily closely by a local operation on the vacuum, we can also expect that Bell’s inequality can be violated ‘almost’ maximally by a selective measurement on the vacuum. This idea was developed by Landau [10] and we shall now see how his result follows directly from the root theorem.

Consider three strictly spacelike separated regions in spacetime, $O_1, O_2$ and $O_3$. The regions $O_1$ and $O_2$ contain measurement apparatuses for measuring observables, $P_i, Q_i, i = 1, 2$ (the $P$’s in $O_1$ and the $Q$’s in $O_2$), represented by operators in $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$, respectively. As before, the self-adjoint contractions will then take values in $[-1, +1]$ in all states. In the third region $O_3$ we consider a single projection operator $P_3$. Following Landau we define for each of the four joint measurements of the observables $(A_1, B_1, P_3)$, $(A_1, B_2, P_3)$, $(A_2, B_1, P_3)$ and $(A_2, B_2, P_3)$ in the vacuum state the conditional expectation for each of these, given the result $P_3 = 1$. For example the conditional expectation for the first set of observable is denoted by $\langle A_1 B_1 \rangle_{P_3=1}$ which can be expressed as

$$\langle A_1 B_1 \rangle_{P_3=1} = \frac{\langle A_1 B_1 P_3 \rangle_{\Omega}}{\langle P_3 \rangle_{\Omega}}.$$  

The conditional expectation value of the Bell operator $R$ is accordingly

$$\langle R \rangle_{P_3=1} = \frac{\langle A_1 B_1 P_3 \rangle_{\Omega}}{\langle P_3 \rangle_{\Omega}} + \frac{\langle A_1 B_2 P_3 \rangle_{\Omega}}{\langle P_3 \rangle_{\Omega}} + \frac{\langle A_2 B_1 P_3 \rangle_{\Omega}}{\langle P_3 \rangle_{\Omega}} - \frac{\langle A_2 B_2 P_3 \rangle_{\Omega}}{\langle P_3 \rangle_{\Omega}}.$$  

If the theory supports a common joint conditional distribution for $A_1, A_2, B_1, B_2$, then

$$\frac{1}{2} \left| \langle R \rangle_{P_3=1} \right| \leq 1 \quad \text{(conditional Bell inequality)} \quad (5)$$

would hold. However, in quantum theory, the conditional Bell inequality is in general violated and in addition, starting from the vacuum state it can be violated ‘almost’ maximally, as we see from the following (proposition 1 in Landau [10]).
\textbf{Theorem 5} Let $O_1, O_2, O_3$ be strictly spacelike separated regions and $\varepsilon > 0$. Then there are self-adjoint contractions $A_1, A_2$ in $\mathcal{R}(O_1)$ and $B_1, B_2$ in $\mathcal{R}(O_2)$, and a projector $P_3$ in $\mathcal{R}(O_3)$ such that in the vacuum state
\begin{equation}
\frac{1}{2} \langle R \rangle_{P_3=1} > \sqrt{2} - \varepsilon \tag{6}
\end{equation}

\textbf{Proof of Theorem 5:} Lemma 3 ensures that we can find a state $\tilde{\phi}$ and contractions $A_1, A_2 \in \mathcal{R}(O_1)$ and $B_1, B_2 \in \mathcal{R}(O_2)$, such that $\frac{1}{2} \langle R \rangle_{\tilde{\phi}} = \sqrt{2}$. Now write $\tilde{O} = O_1 \cup O_2$. Then $R \in \mathcal{R}(\tilde{O})$ and we can apply the root theorem for strictly spacelike separated $\tilde{O}$ and $O_3$: given $\frac{1}{2} \langle R \rangle_{\tilde{\phi}} = \sqrt{2}$ theorem 3 ensures that there is a projector $P_3$ in $O_3$ such that inequality (6) holds.

\textbf{Remark:} This theorem can be generalised to arbitrary observables $A_1, A_2$ and $B_1, B_2$ that are subject to $[A_1, A_2] \neq 0 \neq [B_1, B_2]$.

\section{Conclusion}

We have seen that the vacuum has, in a sense, the property of maximal entanglement, viz. in the sense of maximal correlation we have explained. This means that for all kinds of experiments, performed in any bounded region in spacetime, the vacuum can be collapsed into a state which gives a fixed extremal expectation value arbitrarily precisely. This collapse can be achieved by a selective measurement in a spacelike separated region. As a consequence we have shown that in quantum field theory perfect correlations, such as used in the EPR-argument, can be approximated arbitrarily closely, and also that the Bell inequality can be violated ‘almost’ maximally by selective, distant measurements - even in the vacuum state. For a detailed discussion of the philosophical import of these results reference may be made to [13] and [17]. For a comprehensive discussion of the relation between entanglement and correlation see [2].

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\section{Appendix}

\textbf{Proof of Lemma 1:} Choose $0 < \varepsilon_1$ (without loss of generality $\varepsilon_1 < 1$) and also $\|\psi\| = 1$. Then, by Reeh and Schlieder $\|\psi - \tilde{C}_{\varepsilon_1} \Omega\| < \varepsilon_1$ for some operator $\tilde{C}_{\varepsilon_1} \in \mathcal{R}(O_1)$. Thus,
\begin{equation}
\varepsilon_1 > \|\psi - \tilde{C}_{\varepsilon_1} \Omega\| \geq \|\psi\| - \|\tilde{C}_{\varepsilon_1} \Omega\| = 1 - \|\tilde{C}_{\varepsilon_1} \Omega\|
\end{equation}
which implies that $\|\tilde{C}_{\varepsilon_1} \Omega\| > 1 - \varepsilon_1$. Now choose $C_{\varepsilon_2}$ such that $C_{\varepsilon_2} \Omega = \tilde{C}_{\varepsilon_1} \Omega / \|\tilde{C}_{\varepsilon_1} \Omega\|$. So,
\begin{align}
\|\psi - C_{\varepsilon_2} \Omega\| &= \left\|\psi - \frac{\tilde{C}_{\varepsilon_1} \Omega}{\|\tilde{C}_{\varepsilon_1} \Omega\|}\right\| = \frac{1}{\|\tilde{C}_{\varepsilon_1} \Omega\|} \left\|\tilde{C}_{\varepsilon_1} \Omega - \tilde{C}_{\varepsilon_1} \Omega\|\psi\right\| \\
&= \frac{1}{\|\tilde{C}_{\varepsilon_1} \Omega\|} \left\|\tilde{C}_{\varepsilon_1} \Omega - \psi + \psi - \|\tilde{C}_{\varepsilon_1} \Omega\|\psi\right\|
\end{align}
\[
\frac{1}{\|C_{\varepsilon_1}\|} \left[ \|\psi - \tilde{C}_{\varepsilon_1} \Omega\| + \|\psi - \|\tilde{C}_{\varepsilon_1}\Omega\|\psi\|\right] \\
< \frac{1}{1 - \varepsilon_1} \left[ \varepsilon_1 + \|1 - \|\tilde{C}_{\varepsilon_1}\Omega\|\|\psi\|\right] < \frac{2\varepsilon_1}{1 - \varepsilon_1} \overset{\text{def.}}{=} \varepsilon_2.
\]

Clearly \(\|C_{\varepsilon_2} \Omega\| = 1\) and thus, lemma 1 is proved. 

**Proof of Lemma 2.** The reader might verify that for all \(\varepsilon_3 > 0\), the operator \(C_{\varepsilon_2}\) with \(\varepsilon_2 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\varepsilon_3}{1}}\) has the required property. However, it is instructive to follow the calculations that led to this value for \(\varepsilon_2\). For convenience set \(\psi - C_{\varepsilon_2} \Omega = \mu\) such that \(\|\mu\| < \varepsilon_2\). We calculate

\[
K = (\psi, A\psi) = (\mu + C_{\varepsilon_2} \Omega, A\mu + AC_{\varepsilon_2} \Omega) \\
\leq \|\mu\|\|A\mu\| + \|C_{\varepsilon_2} \Omega\|\|A\mu\| + \|C_{\varepsilon_2} \Omega, A\mu\| + (\mu, AC_{\varepsilon_2} \Omega)
\]

by the triangle inequality. Using the (Cauchy-)Schwarz(-Bunjakowski) inequality, we find

\[
K \leq \|\mu\|\|A\mu\| + \|C_{\varepsilon_2} \Omega\|\|A\mu\| + \|A\mu\|\|C_{\varepsilon_2} \Omega\| \\
\leq \|\mu\|^2 \|A\| + \|C_{\varepsilon_2} \Omega\|\|A\| + \|A\mu\|\|C_{\varepsilon_2} \Omega\| + \langle A\rangle C_{\varepsilon_2} \Omega
\]

Thus, \(\langle A\rangle C_{\varepsilon_2} \Omega > K - \varepsilon_3\), where \(\varepsilon_3 = (\varepsilon_2^2 + 2\varepsilon_2)\|A\|\). For the second inequality consider

\[
\langle A\rangle C_{\varepsilon_2} \Omega - K = - (\mu, A\mu) - (C_{\varepsilon_2} \Omega, A\mu) - (\mu, AC_{\varepsilon_2} \Omega) \\
\leq \|\mu\|^2 \|A\| + \|C_{\varepsilon_2} \Omega, A\mu\| + \|\mu\|\|C_{\varepsilon_2} \Omega\|\|A\| + \langle A\rangle C_{\varepsilon_2} \Omega
\]

and thus, lemma 2 is proved. 

**Proof of Lemma 3.** Clearly \(\langle P_i\rangle \Omega \neq 0 \forall i\), and hence \(\langle \tilde{Q}_1' \rangle \Omega \neq 0\). This follows just from the fact that \(\Omega\) is a separating vector as was shown above. Now set \(Q_1' = \tilde{Q}_1'/\langle \tilde{Q}_1' \rangle \Omega\) in order to find

\[
\|Q_1 - Q_1'\| = \|Q_1 - \frac{\tilde{Q}_1'}{\langle \tilde{Q}_1' \rangle \Omega}\| = \frac{1}{\langle \tilde{Q}_1' \rangle \Omega} \|\langle \tilde{Q}_1' \rangle \Omega Q_1 - \tilde{Q}_1'\| \\
= \frac{1}{\langle \tilde{Q}_1' \rangle \Omega} \|Q_1 (\langle \tilde{Q}_1' \rangle \Omega - \langle Q_1 \rangle \Omega) + \langle Q_1 \rangle \Omega Q_1 - \tilde{Q}_1'\| \\
\leq \frac{1}{\langle \tilde{Q}_1' \rangle \Omega} \left[ \|Q_1\|\|\langle \tilde{Q}_1' \rangle \Omega - \langle Q_1 \rangle \Omega\| + \|Q_1\|\|\langle Q_1 \rangle \Omega - 1\| + \|Q_1 - \tilde{Q}_1'\|\right]
\]

where we have used the triangle inequality and the fact that \(Q_1\) is bounded. The first term can be majorized as follows:

\[
\|Q_1\|\|\langle \tilde{Q}_1' \rangle \Omega - \langle Q_1 \rangle \Omega\| \leq \|Q_1\||\|Q_1 - \tilde{Q}_1'\| < \|Q_1\|\varepsilon_4
\]
The second term vanishes identically according to lemma [I], and the third term is smaller than $\tilde{\varepsilon}_4$. Hence

$$\|Q_1 - Q'_1\| < \frac{1}{\langle Q'_1 \rangle_\Omega} [\|Q_1\| + 1] \tilde{\varepsilon}_4 \overset{\text{def}}{=} \varepsilon_4$$

Thus, the lemma is proved as $\langle Q'_1 \rangle_\Omega = 1$, trivially. \hfill \checkmark

**Proof of Lemma 4:** We know already from lemma 2 that for all $\varepsilon_3 > 0$ there is an operator $Q_1$ such that $K - \varepsilon_3 < (\Omega, AQ_1 \Omega) < K + \varepsilon_3$. Thus,

$$K - \varepsilon_3 < (\Omega, A(Q_1 - Q'_1 + Q'_1) \Omega) = (A\Omega, (Q_1 - Q'_1) \Omega) + (\Omega, AQ'_1 \Omega)$$

$$\leq \|A\| \|Q_1 - Q'_1\| + \langle AQ'_1 \rangle_\Omega < \|A\| \varepsilon_4 + \langle AQ'_1 \rangle_\Omega$$

and hence

$$\langle AQ'_1 \rangle_\Omega > K - (\varepsilon_3 + \|A\| \varepsilon_4) \overset{\text{def}}{=} K - \varepsilon_5.$$ 

Similarly

$$\langle AQ'_1 \rangle_\Omega = (\Omega, AQ'_1 \Omega) = (\Omega, A(Q'_1 - Q_1 + Q_1) \Omega)$$

$$= (A\Omega, (Q'_1 - Q_1) \Omega) + (\Omega, AQ_1 \Omega)$$

$$\leq \|A\| \|Q'_1 - Q_1\| + \langle AQ_1 \rangle_\Omega < \|A\| \varepsilon_4 + \langle AQ_1 \rangle_\Omega$$

$$< \varepsilon_4 \|A\| + K + \varepsilon_3 = K + \varepsilon_5$$

which proves lemma 4. \hfill \checkmark

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