Fermionic Supersymmetry in Time-Domain

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A supersymmetric theory in the temporal domain is constructed for bi-spinor fields satisfying the Dirac equation. It is shown that using the Dirac matrices basis, it is possible to construct a simple time-domain supersymmetry for fermion fields with time-dependent mass. This theory is equivalent to a bosonic supersymmetric theory in the time-domain. Solutions are presented, and it is shown that they produce probability oscillations between its spinor mass states. This theory is applied to the two-neutrino oscillation problem, showing that the flavour state oscillations emerge from the supersymmetry originated by the time-dependence of the unique mass of the neutrino. It is shown that the usual result for the two-neutrino oscillation problem is recovered in the short-time limit of this theory. Finally, it is discussed that this time-domain fermionic supersymmetric theory cannot be obtained in the Majorana matrices basis, thus giving hints on the Dirac fermion nature of neutrinos.

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I. INTRODUCTION

Supersymmetry is one of the cornerstones of theoretical physics [1], being ubiquitous in almost every branch of physics [2–24].

The standard way to proceed is to construct supersymmetric theories in a space-domain, where the superpartners and supercharge operators take into account the spatial variations of, for example, external potentials. In this work, it is not our aim to focus in those spatial supersymmetries, but instead, to inquire if a temporal version of such theories is possible for massive fields [25].

Recently, García-Meca, Macho Ortiz and Lorrente Sáez [25] have introduced the concept of supersymmetry in time-domain (T-SUSY) for Maxwell equations. This is a supersymmetry occurring in the temporal part of the massless field dynamics, completely uncoupled from the spatial evolution of the field. They studied the applications of T-SUSY in the realm of optics for dispersive media, showing the novel capabilities of this theory to introduce hypothetical materials with new optical features. Besides, they show that this time-domain supersymmetry applies to any field described, in principle, by a d’Alambertian equation. In other words, they developed the bosonic version of the T-SUSY theory.

It is the purpose of this manuscript to show that a T-SUSY theory can also be obtained in relativistic quantum mechanics for fermions described by the Dirac equation. In this case, we show that the simplest T-SUSY theory can be constructed for Dirac fermions with time-dependent mass, i.e, using the Dirac matrices basis.

We completely develop this fermionic T-SUSY theory, finding solutions for different possible time-dependent masses. This theory is equivalent to its bosonic partner, allowing us to obtain a massive fermion field behavior which is analogue to a light-like one.

Besides, this theory produces probability states oscillations, and thus can be used to study the neutrino oscillation problem. In this way, we give a different perspective to the origin of neutrino oscillations through supersymmetry in the time-domain.

We end our proposal by showing the impossibility to implement this temporal supersymmetry in the Majorana basis. Therefore, this theory discerns between Dirac and Majorana neutrinos.

II. T-SUSY FOR DIRAC FERMIONS

Let us consider a bi-spinor field (satisfying the Dirac equation in flat spacetime) with the form

$$
\Psi = \left( \begin{array}{c}
\psi_+ e_+ \\
\psi_- e_-
\end{array} \right),
$$

(1)

in terms of wavefunctions $\psi_+$ and $\psi_-$, and the spinors

$$
e_+ = \left( \begin{array}{c} 1 \\
0
\end{array} \right), \quad e_- = \left( \begin{array}{c} 0 \\
1
\end{array} \right),
$$

(2)

with the properties $e_+ e_+ = 1 = e_- e_-$, and $e_+ e_- = 0 = e_+ e_-$. 

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The simplest form of a T-SUSY theory can be obtained from field propagation in one spatial dimension. In this form, the Dirac equation in flat spacetime for a massive fermion field $\Psi$ reads

$$i\gamma^0 \partial_0 \Psi + i\gamma^1 \partial_1 \Psi = m \Psi,$$

where $m$ is the mass of the field. Besides, $\partial_0$ is a time derivative and $\partial_1$ is a space derivative in an arbitrary direction. As always, Dirac matrices fulfill $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbb{1}_{4\times4}$, where the metric is $\eta^{\mu\nu} = (1, -1, -1, -1)$, and $\mathbb{1}_{4\times4}$ is the identity matrix in this four-dimensional spacetime.

In order to evaluate explicitly the T-SUSY theory for fermions, let us choose the following Dirac matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (4)$$

Now, in order to decouple the space dynamics, let us assume that $\partial_1 \Psi = ik \Psi$ with an arbitrary constant $k$. This $k$ plays the role of the constant momentum of the fermion propagation. In this way, under this representation, we find that Eq. (3) becomes simply

$$Q_\pm \psi_\pm = k \psi_\mp,$$

in terms of supercharge operators

$$Q_\pm = \frac{d}{dt} \mp m, \quad (5)$$

where $d/dt$ is a total time derivative. Eqs. (5) correspond to a set of supersymmetric equations of quantum mechanics, now obtained in the Dirac matrices basis (4). Notice that, then, the simplest T-SUSY theory requires that the mass be the responsible for the origin of the superpotential of this supersymmetry. This occurs when the mass becomes time-dependent, $m = m(t)$.

Eqs. (5) can be used to calculate the equations that each wavefunction $\psi_\pm$ satisfies. They are

$$H_\pm \psi_\pm = k^2 \psi_\pm,$$

where we have defined the super-Hamiltonians $H_\pm$ in terms of superpotentials $W_\pm$ as

$$H_\pm = Q_\pm Q_\mp = -\frac{d^2}{dt^2} + W_\pm,$$

where

$$W_\pm = \mp i \frac{dm}{dt} - m^2,$$

such that $W_- - W_+ = 2i dm/dt$. Thereby, the difference between states lies in the time dependence of the mass.

In general, from Eq. (7), we can write the wavefunctions $\psi_\pm(t)$ as

$$\psi_\pm(t) = \exp \left( i \int dt E_\pm(t) \right),$$

where $E_\pm$ are complex time-dependent functions that satisfy the Ricatti equation

$$\frac{dE_\pm}{dt} - E_\pm^2 + k^2 = W_\pm,$$  \quad (11)

such that $\psi_\pm(0) = 1$. Whenever $dm/dt \neq 0$, then $E_+ \neq E_-$. In this way, for a specific time functionality of the mass, solutions for $E_\pm$ can be found in order to satisfy this T-SUSY description.

All the framework for a standard supersymmetric theory in quantum mechanics can be straightforwardly utilized for this T-SUSY theory. For example, its algebra is developed in Ref. [25]. We can rewrite Eq. (7) in the form

$$\left( \frac{d^2}{dt^2} + \frac{1}{n_\pm^2} \right) \psi_\pm = 0,$$  \quad (13)

where

$$n_\pm(t) = \left(k^2 - W_\pm\right)^{-1/2}.$$  \quad (14)

Eq. (13) is equivalent to the one describing the propagation of light in a medium with supersymmetric refraction indices $n_\pm$. In this T-SUSY theory, both refraction indices are related by

$$n_+ = \left( \frac{1}{n_-^2} + 2i \frac{dm}{dt} \right)^{-1/2}.$$  \quad (15)

This relation shows the equivalence of fermionic T-SUSY with bosonic T-SUSY discussed in Ref. [25].

### III. SIMPLE SOLUTIONS

We can get simple solutions for fermion fields with time-dependent mass.

Let us first consider the simple solution for the massive case, when $k \ll m(t)$. For this case, from Eq. (11), we find

$$E_\pm(t) \approx -m(t).$$  \quad (16)

Therefore, the two mass states have different behavior stemming from just one time-dependent mass.

Similarly, another simple solution can be obtained in the light-mass (ultra-relativistic) case, when $k \gg m(t)$. For this case, $k$ corresponds (approximately) to the energy of the particle. Let us assume that $m(t) = m_0 + \epsilon(t),\quad (17)$

$$E_\pm(t) \approx \mp \epsilon(t).$$  \quad (18)

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with a constant $m_0 \gg \epsilon$, and $E_{\pm}(t) = \sqrt{k^2 + m_0^2 + \delta_{\pm}} \approx k + \delta_{\pm}(t)$, such that $\delta_{\pm}(t) \ll k$. In this case, Eq. \ref{eq:11} becomes

$$\frac{d\delta_{\pm}}{dt} - 2k\delta_{\pm} = \mp \frac{d\epsilon}{dt} - 2m_0 \epsilon. \quad (17)$$

By writing $\delta_{\pm} = \delta_{R\pm} + i \delta_{I\pm}$ (with $\delta_{R\pm}$ and $\delta_{I\pm}$ reals), then we find $d\delta_{R\pm}/dt - 2k\delta_{R\pm} = \mp d\epsilon/dt$, and $d\delta_{I\pm}/dt + 2k\delta_{R\pm} = 2m_0 \epsilon$. This gives the equation

$$\frac{d^2\delta_{R\pm}}{dt^2} + 4k^2 \delta_{R\pm} = \mp \frac{d^2\epsilon}{dt^2} + 4km_0 \epsilon, \quad (18)$$

which allow us to find the solutions

$$\epsilon(t) = \epsilon_0 \cos(\Lambda t),$$

$$\delta_{R\pm}(t) = -\epsilon_0 \left( \frac{\pm A^2 + 4km_0}{A^2 - 4k^2} \right) \cos(\Lambda t),$$

$$\delta_{I\pm}(t) \approx \pm \frac{2kA\epsilon_0}{\Lambda^2 - 4k^2} \sin(\Lambda t). \quad (19)$$

for arbitrary constants $\epsilon_0 \ll m_0$ and $\Lambda$.

Finally, let us consider the propagation of the fermion field in a medium with constant complex refractive index $n_+ = n_R + i n_I$ ($n_R$ and $n_I$ are real constants). In this case, the fermion field behaves analogously to light propagating in a dispersive and absorbing medium, with solution of Eq. \ref{eq:13} given by $E_+ = 1/n_+$, or $\psi_+ = \exp(it/n_+)$. This can only occur for a complex time-dependent mass

$$m(t) = -i\beta^{1/2} \zeta \tan \left( i^{1/2} \zeta t \right), \quad (20)$$

when $n_+^2 = n_+^2 + 1/k^2$, and for a constant $\zeta$ defined by $\zeta = \sqrt{2n_R n_{R+} + 2n_I n_{R+}}$. This mass \ref{eq:20} allows to find the complex non-constant supersymmetric refractive index $n_- = (k^2 + i\zeta^2 - 2i \sec^2(i^{1/2} \zeta t))^{-1/2}$. From Eqs. \ref{eq:5}, we can also obtain that $\psi_- = -i k \exp(i \int dt' m(t')) \int dt' \exp(it'/n_- - i \int dt'' m(t''))$. Complex mass \ref{eq:20} violates current conservation of the Dirac equation. However, this can be re-interpreted in terms of the effective absorbing media. An interesting feature of mass \ref{eq:20} is that it becomes real for a short time interval $t \ll 1/|\zeta|$, where

$$m(t) \approx \zeta^2 t. \quad (21)$$

This solution is the equivalent to the bosonic propagation in a medium studied in Ref. \ref{eq:25}.

**IV. OSCILLATIONS IN T-SUSY**

The above T-SUSY theory allows now to consider the phenomenon of oscillation of states in a different fashion. These superoscillations have their origin in the subyacent supersymmetry due to the temporal dependence of the mass.

Considering the bi-spinor \ref{eq:1}, let us define a bi-spinor with mixed states

$$\Phi = \left( \begin{array}{c} \Phi_a \\ \Phi_b \end{array} \right) = U \Psi, \quad (22)$$

with the spinor wavefunctions $\Phi_a$ and $\Phi_b$ defining new states. Here, $U$ is a unitary matrix with role of mixing states. It can be written as \ref{eq:20} \ref{eq:29}

$$U = \left( \begin{array}{cc} \cos \theta & \sin \theta \xi_{\times 2} \\ -\sin \theta & \cos \theta \xi_{\times 2} \end{array} \right), \quad (23)$$

where $\theta$ is called the mixing angle in vacuum, and $\xi_{\times 2}$ is the $2 \times 2$ identity matrix.

With all the above, the amplitude of mixed states change, from $\Phi_a(t = 0) \rightarrow \Phi_b(t)$, can be calculated to be $\text{Amp}(\Phi_a \rightarrow \Phi_b) = \sin 2\theta [\psi^*_+ (0) \psi_+ (t) - \psi^-_+ (0) \psi^-_-(t)]/2$. Finally, the probability of mixed states change is given by

$$P(\Phi_a \rightarrow \Phi_b) = |\text{Amp}(\Phi_a \rightarrow \Phi_b)|^2 = \sin^2 2\theta e^{-\alpha} (\sin^2 \beta + \sin^2 \rho), \quad (24)$$

where

$$\alpha = \int dt \{ \text{Im} \{E_-\} + \text{Im} \{E_+\} \},$$

$$\beta = \frac{1}{2} \int dt \{ \text{Im} \{E_-\} - \text{Im} \{E_+\} \},$$

$$\rho = \frac{1}{2} \int dt \{ \text{Re} \{E_-\} - \text{Re} \{E_+\} \}. \quad (25)$$

The terms proportional to $\exp(-\alpha)$ and $\sin^2 \beta$ enter in the probability due to the contribution of the amplitude of wavefunctions \ref{eq:10}. Their phases only contribute to the oscillation probability through $\sin^2 \rho$.

In this way, the superoscillations between the mixed states, given by the above transition probability, are due only to the different solutions of Eq. \ref{eq:11}. This occurs if the fermion field has a non-constant mass.

As an example, let us evaluate the above probability \ref{eq:24} with the previous simple example for the solution of a massive fermion, namely $k \ll m$, and $E_+ \approx \mp m$. For this case, $\text{Im} \{E_\pm\} \approx 0$, and $\alpha = 0 = \beta$. Then the probability \ref{eq:24} reduces to

$$P(\Phi_a \rightarrow \Phi_b) \approx \sin^2 2\theta \sin^2 \left( \int dt \, m \right). \quad (26)$$

Thus, the superoscillation for the massive case occurs because the mass can evolve in time.

**V. APPLICATION TO THE TWO-NEUTRINO OSCILLATION PROBLEM**

We now invite the reader to re-think the two-neutrino oscillation problem in terms of the T-SUSY theory developed above. Neutrino oscillations have been studied in
the context of supersymmetry \cite{30,42}, but not in a theory for fermions in time-domain, with time-dependent mass.

The electron and muon flavour eigenstates can now be identified by the bi-spinor \cite{22}. In terms of the T-SUSY theory, the probability that an electron neutrino be observed later as a muon neutrino is given by Eq. (24). Under this T-SUSY theory, the oscillation occurs due to the supersymmetric nature of the neutrino in time-domain, as its mass is not constant. It is always one neutrino, with a unique mass, that it is oscillating between supersymmetric mass states.

Let us model the neutrino oscillation considering the previous simple solution \cite{19} for an ultra-relativistic fermion, with \( m \ll k \). By choosing \( \Lambda = k \), and \( \epsilon_0 = 3m_0^2/(4k) \), we fulfill the conditions \( \epsilon \ll m_0 \) and \( \delta \ll k \). In this case, \( \text{Re}\{E_{\pm}\} \approx k \pm (m_0^2/4k) \cos(kt) \), and \( \text{Im}\{E_{\pm}\} = \mp (m_0^2/2k) \sin(kt) \). Thereby, \( \alpha = 0, \beta(t) = (m_0^2/2k^2)[1 - \cos(kt)] \), and \( \rho(t) = -(m_0^2/4k^2) \sin(kt) \). In this way, the total probability for flavour states change becomes

\[
P(\nu_e \rightarrow \nu_\mu) \approx \sin^2 \frac{2\theta}{2}
\left[ \sin^2 \left( \frac{m_0^2}{2k^2} [1 - \cos(kt)] \right)
+ \sin^2 \left( \frac{m_0^2}{4k^2} \sin(kt) \right) \right].
\]

(27)

This probability is general under the assumption \( m_0 \ll k \). However, for any time scale \( t \ll 1/k \), the probability \cite{27} for the flavour states changes becomes simply

\[
P_{t \ll 1/k}(\nu_e \rightarrow \nu_\mu) \approx \sin^2 \theta \sin^2 \left( \frac{m_0^2 t}{4k} \right),
\]

(28)

which is equivalent to the standard result for the two-neutrino oscillation problem \cite{20,23}, for neutrinos travelling a distance \( t \sim L \), and identifying \( k \) as the neutrino energy.

The above neutrino superoscillations are not produced by different mass states, but because the appearing of the supersymmetry in time-domain due to the single neutrino non-constant mass. The behavior of this neutrino superoscillation for larger times, described by probability of flavour change \cite{27}, is depicted in Fig. 1(a). We show the probability for superoscillations \( \hat{P} = P/\sin^2 2\theta \) as it evolves in time \( t = kt \), increasing in amplitude as the value of \( \hat{m} = m_0/k \ll 1 \) grows.

Probability \cite{27} differs from the approximate probability \cite{25} in a notorious form as the time increases. This is shown in Fig. 1(b), where the ratio between probabilities \cite{27} and \cite{28}, say \( r = P/P_{t \ll 1/k} \), is presented as it evolves in time, for \( \hat{m} = 10^{-3} \). The oscillation of this ratio \( r \) shows that the complete probability neutrino superoscillation \cite{27} can be larger on times \( 0 < \hat{t} < 3.7 \), to later decreases. For long times, it is expected a departure from the standard probability oscillation \cite{28}. In this way, the effect of neutrino flavour oscillation due to a T-SUSY theory could be tested for neutrinos travelling long times or large distances.

\[\text{FIG. 1: (a) Probability } \hat{P} = P/\sin^2 2\theta \text{ (in units of } 10^{-8}) \text{ of flavour change \cite{24} for neutrino superoscillations, in terms of } \hat{m} = m_0/k \text{ (in units of } 10^{-2}) \text{, and } t = kt. \text{ (b) Ratio } r = P/P_{t \ll 1/k} \text{ (between probabilities \cite{27} and \cite{28}) as a function of } \hat{t} \text{ for } \hat{m} = 10^{-3}. \text{ The inset plot shows this ratio (in units of } 10^{-3}) \text{ for } 20 < \hat{t} < 30.\]

VI. DISCUSSION

When the mass of a fermion field is time-dependent, then a supersymmetric in time-domain theory can be consistently defined. This T-SUSY is the time analogue of any non-relativistic supersymmetric quantum mechanical theory. We have presented several solutions for different time-dependent masses. It is remarkable that this fermion theory contains solutions that mimic the light-like behavior studied for the bosonic T-SUSY theory.

One of the main results extracted from this theory is the possibility for oscillation between states due to the temporal changes of fermion mass. These superoscillation was applied to the two-neutrino oscillation problem, obtaining that the oscillation between flavour states can be explained by invoking only the time-dependence of a single neutrino mass (not different masses for different states). In this way, this fermionic T-SUSY theory in-
roduces a different and interesting way to understand the neutrino oscillation problem under the new light of supersymmetry.

In the above framework for the T-SUSY theory, we have used the Dirac matrices basis. One can ask what happens in other bases. We remark that this T-SUSY theory cannot be constructed in the Majorana basis. In fact, considering for example the Majorana matrices $\gamma^0$ and $\gamma^1$ in Eq. (3), then we obtain for the bi-spinor $\psi$ the equations
\[
\frac{d\psi_{\pm}}{dt} = (ik \mp m) \psi_{\mp},
\]
that do not allow us to construct the T-SUSY version of a Majorana fermion. In this way, Majorana supersymmetry is possible only in space-domain. Therefore, if T-SUSY theory can be used to explain the neutrino oscillations, then it would work as an indicator of the Dirac fermionic nature of neutrinos, instead of Majorana fermions \cite{43,46}.

Similarly, the T-SUSY theory presented here is its simplest, yet not trivial, possible version. The extension to other types of physical interactions is straightforward, and it is currently under investigation. However, in the specific case of electromagnetism, through minimal coupling, no supersymmetric modifications are found to the cases presented here.

\footnotesize
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