Continuous space modeling in new economic geography with a quasi-linear log utility function

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Abstract

We consider the extension of the mathematical models presented by Pflüger (2004) and Gaspar et al. (2018) in new economic geography to continuous space, and investigate the behavior of its solution mathematically. The model is a system of nonlinear integral and differential equations describing the market equilibrium and the time evolution of the spatial distribution of population density. The homogeneous stationary solution with evenly distributed population is shown to be unstable. Furthermore, it is shown numerically that the destabilized homogeneous stationary solution eventually forms spiky spatial distributions. The number of the spikes decreases as the preference for variety increases or the transport cost decreases.

Keywords new economic geography · continuous racetrack economy · self-organization · spatial patterns · number of cities

JEL classification: R12, R40, C63, C68

1 Introduction

The core-periphery model proposed by Krugman (1991) is a fundamental model in new economic geography. Many mathematical studies have been conducted on the model and its extensions (Various theoretical studies based on the core-periphery model are detailed in Fujita et al. (2001)). However, studying the behavior of their solutions is still difficult due to the strong nonlinearity of those models especially in the case of more than three multiple regions. Meanwhile, some mathematical models that are more tractable to analytical study have been proposed (See Ottaviano et al. (2002), Forslid and Ottaviano (2003) for example. Fujita and Thisse (2013), Satou et al. (2011) and Zeng and Takatsuka (2016) systematically introduce the mathematical models in the field including such tractable ones.). Among such

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tractable models, Pfüger (2004) and Gaspar et al. (2018) are particularly useful in this study. Pfüger (2004) has greatly simplified the core-periphery model by using a quasi-linear log utility function of consumers. Gaspar et al. (2018) have extended Pfüger’s model to the discrete multi-region case. In this paper, we discuss the extension of Pfüger (2004) and Gaspar et al. (2018) to continuous space.

The contributions made in this paper are as follows. Firstly, we construct the model in continuous space. This could facilitate theoretical studies of spatial economy consisting of so many regions that they can be considered a continuum. We propose the model on $n$-dimensional manifolds, but the concrete behavior of the solutions is considered on one-dimensional continuous periodic space (continuous racetrack model). The continuous racetrack model would be convenient for investigating the self-organization of spatial structures under completely symmetric conditions where addressable regions are not predetermined. Secondly, we show that a homogeneous steady-state solution of the racetrack model is unstable, so the flat-earth distribution is not sustainable. Thirdly, in the racetrack model, we show numerically that the density of mobile workers almost completely agglomerate to a finite number of regions that should be called “cities”, and that the number of the cities thus formed increases with the transport cost increases or the preference for variety weakens.

The rest of the paper is organized as follows. Section 2 derives the model. Section 3 investigates the stability of the homogeneous stationary solution. Section 4 shows numerical results on the large-time behavior of the solutions of the racetrack model. Section 5 concludes.

2 The model

In this section, we introduce the model which we handle in this paper. We first review the Dixit-Stiglitz framework (Dixit and Stiglitz (1977)). Next, we apply it to construct a model on continuous space.

2.1 Dixit-Stiglitz framework

As in Pfüger (2004), we use use the quasi-linear utility function of consumers defined as

$$U = \mu \ln M + A, \quad \mu \in [0, 1).$$ \hspace{1cm} (1)

Here, $M$ and $A$ represent a composite index of the consumption of manufacturing goods, and the consumption of the agricultural good, respectively. The composite index $M$ is defined by

$$M = \left[ \int_0^n q(i) \frac{x_{-1}}{x_1} di \right]^{\frac{\sigma}{\sigma - 1}}, \quad \sigma \in (1, \infty),$$ \hspace{1cm} (2)
where \( q(i) \) represents the consumption of a variety \( i \in [0, n] \) of manufactured goods. The parameter \( \sigma > 1 \) represents the elasticity of substitution between any two varieties. Note that the closer the value of \( \sigma \) is to 1, the stronger the consumer’s preference for variety becomes. Let the agricultural good be the numéraire, then the budget constraint of each consumer is

\[
A + \int_0^n p(i)q(i)di = Y, \tag{3}
\]

where \( p(i) \) and \( Y \) represent the price of a variety \( i \) and the consumer’s income, respectively. By maximizing the utility function (1) under (2) and (3), we see that

\[
q(j) = \mu \frac{p(j)^{1-\sigma}}{G^{1-\sigma}}, \tag{4}
\]

for each \( j \in [0, n] \). Here, \( G \) denotes price index for manufacturing goods and it satisfies

\[
G^{1-\sigma} = \int_0^n p(i)^{1-\sigma}di. \tag{5}
\]

By using (2), (3), (4) and (5), we obtain

\[
M = \frac{\mu}{G}, \quad A = Y - \mu.
\]

Here, we assume that \( A > 0 \) for all consumers. A sufficient condition for this assumption is discussed later in Subsection 2.4. Finally, the indirect utility of the consumer is

\[
V = \mu(\ln \mu - 1) + Y - \mu \ln G.
\]

### 2.2 Extension to continuous space

Let us assume that the economic regions are continuously distributed on a \( n \)-dimensional manifold \( \mathcal{N} \), where \( n \geq 1 \). In the following, the regions are indexed by continuous variables \( x, y \in \mathcal{N} \). Let \( \phi(x) \) be the population density of immobile workers at a region \( x \in \mathcal{N} \), and \( \lambda(x) \) be the population density of mobile workers at \( x \in \mathcal{N} \). The total population of the immobile and mobile workers are given by \( \Phi \) and \( \Lambda \), respectively. It means that

\[
\int_{\mathcal{N}} \phi(x)dx = \Phi, \quad \int_{\mathcal{N}} \lambda(x)dx = \Lambda.
\]

All manufactured goods produced in the same region \( x \) are assumed to have the same price \( p(x) \). As in Pflüger (2004), we assume the iceberg transport cost, that is, we have to ship \( T(x, y) \geq 1 \) units of manufacturing good from a region \( x \in \mathcal{N} \) in order to deliver one unit of it to a region \( y \in \mathcal{N} \). Then, the price at \( y \) of manufactured goods produced at \( x \), denoted by \( p(x, y) \), is given by \( p(x, y) = p(x)T(x, y) \). In general, the price indices take different
values in different regions. By (5), the price index at $x$ denoted by $G(x)$ satisfies
\[ G(x)^{1-\sigma} = \int_N n(y)(p(y)T(x,y))^{1-\sigma} dy, \]  
where $n(x)$ denotes the density of the number of available varieties of manufactured goods at $x$. Here, the integral is over the manifold $N$, and thus $dy$ is the volume element of the manifold. Note that each manufacturing firm at region $x$ has to ship $T(x,y)$ times amount of demand in each region $y$, then the output $q(x)$ of the firm at $x$ is calculated by using (4) as
\[ q(x) = \mu \int_N \left[ \frac{p(x,y)^{-\sigma}}{G(y)^{1-\sigma}} \times T(x,y) \times (\phi(y) + \lambda(y)) \right] dy \]
\[ = \mu p(x)^{-\sigma} \int_N T(x,y)^{1-\sigma} (\phi(y) + \lambda(y)) G(y)^{\sigma-1} dy. \]  
Let us consider firms’ pricing behavior. All manufacturing firms producing the quantity $q(x)$ at $x$ face the same costs that are expressed by
\[ c_f + c_m q(x) \]  
where $c_f$ and $c_m$ is the fixed and marginal costs, respectively. Then, the profit earned by each firm at $x$ is
\[ \Pi(x) = p(x)q(x) - (c_f + c_m q(x)). \]  
The firms are assumed to maximize their profits. The first-order condition of optimality \(\frac{d\Pi(x)}{dp(x)} = 0\) yields
\[ p(x) = \frac{\sigma}{\sigma - 1} c_m. \]  
Putting (10) into (9), we see that
\[ \Pi(x) = \frac{c_m q(x)}{\sigma - 1} - c_f. \]  
Due to the zero profit condition and (10), we see that
\[ c_f = \frac{1}{\sigma} p(x)q(x). \]  
We follow the assumption in Forslid and Ottaviano (2003). That is, firms need $F$ units of mobile workers as the fixed input and $(\sigma - 1)/\sigma$ units of immobile workers as the marginal inputs, i.e., $c_f = Fw(x)$ and $c_m = (\sigma - 1)/\sigma$ in (8). Here, $w(x)$ denotes nominal wage for mobile workers at $x$. The nominal wage for immobile workers is assumed to be fixed to 1. Then, by (10) and (11), we obtain
\[ p(x) = 1, \]  
\[ q(x) = \sigma Fw(x). \]
\[ G(x)^{1-\sigma} = \int_{\mathcal{N}} n(y) T(x, y)^{1-\sigma} dy, \quad (14) \]
\[ w(x) = \frac{\mu}{\sigma F} \int_{\mathcal{N}} T(x, y)^{1-\sigma} (\phi(y) + \lambda(y)) G(y)^{\sigma-1} dy. \quad (15) \]

By assuming that a single manufacturing firm is engaged only in the production of a single variety, we see that
\[ n(x) = \frac{\lambda(x)}{F}. \]

The real wage of mobile workers at \( x \) is defined by
\[ \omega(x) = w(x) - \mu \ln G(x), \quad (16) \]
and the average real wage is defined by \( \frac{1}{\Lambda} \int_{\mathcal{N}} \omega(y) \lambda(y) dy \). Then, the temporal change in the mobile population density at each region is assumed to be governed by the differential equation in time
\[ \frac{\partial \lambda(t, x)}{\partial t} = \gamma \left[ \omega(t, x) - \frac{1}{\Lambda} \int_{\mathcal{N}} \omega(y) \lambda(y) dy \right] \lambda(t, x), \quad (17) \]
where \( \gamma > 0 \) represents an adjustment speed of agglomeration. Then, note that the conservation of population \( \frac{d}{dt} \Lambda = 0 \) holds.

The entire system of these equations (14)-(17), with the explicit statement that the functions \( w, G, \omega, \) and \( \lambda \) depend not only on the spatial variable \( x \in \mathcal{N} \) but also on the time variable \( t \in [0, \infty) \), can be shown as follows.

\[ \begin{cases} 
  w(t, x) = \frac{\mu}{\sigma F} \int_{\mathcal{N}} T(x, y)^{1-\sigma} (\phi(y) + \lambda(t, y)) G(t, y)^{\sigma-1} dy, \\
  G(t, x)^{1-\sigma} = \frac{1}{F} \int_{\mathcal{N}} \lambda(t, y) T(x, y)^{1-\sigma} dy, \\
  \omega(t, x) = w(t, x) - \mu \ln G(t, x), \\
  \frac{\partial \lambda(t, x)}{\partial t} = \gamma \left[ \omega(t, x) - \frac{1}{\Lambda} \int_{\mathcal{N}} \omega(y) \lambda(t, y) dy \right] \lambda(t, x) 
\end{cases} \quad (18) \]

with an initial condition \( \lambda(0, x) = \lambda_0(x) \).

In the following, we investigate the properties of the model \([18]\) in detail, assuming that the manifold is one-dimensional circumference with radius \( r > 0 \) denoted by \( S \). In addition, the transportation cost function \( T(x, y) \) is assumed to be
\[ T(x, y) = e^{\tau|x-y|}, \quad \tau > 0 \quad (19) \]
where \( |x-y| \) means the shorter distance between \( x \) and \( y \) on \( S \). Here, \( |x-y| = |y-x| \) holds for any \( x, y \in S \), so the function \( T(x, y) \) is symmetric with
respect to the variables on \( S \); \( T(x, y) = T(y, x) \). Moreover, the immobile population density is assumed to be homogeneous;

\[
\phi(x) \equiv \varphi = \Phi/(2\pi r) \text{ on } S. \tag{20}
\]

In the following of the paper, we simply refer to \((18)\) on \( S \) with the conditions \((19)\) and \((20)\) as *racetrack model*. In the analysis of the racetrack model, the crucial parameters \( \sigma \) and \( \tau \) frequently appear in the form of multiplication by \((\sigma - 1)\tau\). Therefore, it is useful to introduce the variable

\[
\alpha = (\sigma - 1)\tau
\]

to improve the following discussion’s perspective. By definition, \( \alpha \geq 0 \).

### 2.3 Notes on specific calculation on \( S \)

#### 2.3.1 Identification of a function on \( S \) with that on \([-\pi, \pi]\)

A point \( x \in S \) corresponds one-to-one to a point \( \theta \in [-\pi, \pi] \), so

\[
x = x(\theta).
\]

Therefore, the function \( f \) on \( S \) can be written as

\[
f(x) = f(x(\theta)) =: \tilde{f}(\theta),
\]

by a periodic function \( \tilde{f} \) on \([-\pi, \pi]\). Note that the line element \( dx \) of \( S \) is expressed by \( d\theta \) as \( dx = rd\theta \). Then, the integral over \( S \) of the function \( f \) is calculated by

\[
\int_S f(x)dx = \int_{-\pi}^{\pi} f(x(\theta)) \, rd\theta = \int_{-\pi}^{\pi} \tilde{f}(\theta) \, rd\theta.
\]

In the following, we just identify the function \( \tilde{f} \) with the corresponding function \( f \) and denote it as \( f \) simply, when no confusion can arise.

#### 2.3.2 Distance function

The distance \( |x - y| \) between \( x(\theta'), y(\theta) \in S \) is computed as

\[
|x - y| = r \times \min \left\{ |\theta' - \theta|_{\text{abs}}, 2\pi - |\theta' - \theta|_{\text{abs}} \right\},
\]

where \( |\cdot|_{\text{abs}} \) is a function that returns the absolute value of its argument.
2.4 Sufficient condition for positive agricultural demand

We assume that \( A = Y - \mu > 0 \). This assumption always holds as for the immobile workers because their nominal wages are \( 1 > \mu \). On the other hand, the following theorem claims that a large enough immobile population guarantees that the mobile workers’ nominal wage to be greater than \( \mu \).

**Theorem 1.** In the racetrack model, if

\[
\Phi > \frac{\sigma \Lambda \alpha \pi r}{1 - e^{-\alpha \pi r}}
\]

holds, then \( w(x) > \mu \) for any \( x \in S \).

**Proof.** In the racetrack model, the nominal wage of the mobile workers at \( x \) is given by

\[
w(x) = \frac{\mu}{\sigma \mathcal{F}} \int_S (\phi(y) + \lambda(y)) G(y)^{\sigma-1} e^{-\alpha |x-y|} \, dy.
\]

An estimation

\[
G(x)^{\sigma-1} > \frac{F}{\Lambda}
\]

follows from

\[
G(x)^{1-\sigma} = \frac{1}{\mathcal{F}} \int_S \lambda(y) e^{-\alpha |x-y|} \, dy < \frac{\Lambda}{\mathcal{F}}.
\]

Then, we can estimate

\[
w(x) = \frac{\mu}{\sigma \mathcal{F}} \int_S (\phi(y) + \lambda(y)) G(y)^{\sigma-1} e^{-\alpha |x-y|} \, dy
\]

\[
> \frac{\mu}{\sigma \Lambda} \int_S (\phi(y) + \lambda(y)) e^{-\alpha |x-y|} \, dy
\]

\[
> \frac{\mu \Phi}{\sigma \Lambda 2 \pi r} \int_S e^{-\alpha |x-y|} \, dy = \frac{\mu \Phi (1 - e^{-\alpha \pi r})}{\sigma \Lambda \alpha \pi r}.
\]

(23)

Here, we use

\[
\int_S e^{-\alpha |x-y|} \, dy = \frac{2(1 - e^{-\alpha \pi r})}{\alpha},
\]

(24)

which follows from the techniques for calculation in Subsection 2.3. Applying (23) to (22), we see that \( w(x) > \mu \) follows from (21).

\[
\square
\]

3 Stationary solution

In the section, we consider a homogeneous stationary solution of the race-track model and investigate its stability.
3.1 Homogeneous stationary solution

The racetrack model has a homogeneous stationary solution \( \bar{\lambda} \equiv \Lambda / (2\pi r) \). In this case, the nominal wage \( \bar{w} \) and price index \( \bar{G} \) are also spatially homogeneous and satisfy

\[
\bar{w} = \frac{\mu(\bar{\phi} + \bar{\lambda})}{\sigma \bar{\lambda}},
\]

\[
\bar{G}^{1-\sigma} = \frac{\bar{\lambda}}{F} \int_S e^{-\alpha|x-y|} dy = \frac{2\bar{\lambda}(1 - e^{-\alpha r\pi})}{F\alpha},
\]

and then, the real wage \( \bar{\omega} \) is also homogeneous. Here, note that \( \bar{G} \) does not depend on \( x \in S \) by (24).

3.2 Stability of the stationary solution

3.2.1 Linearized equations

Let small perturbations added to the population density of mobile workers, nominal wage, price index, and real wage be \( \Delta \lambda(t,x), \Delta w(t,x), \Delta G(t,x), \Delta \omega(t,x) \), respectively. Note that \( \int_S \Delta \omega(t,x) dx = 0 \) when linearizing the differential equation. Let us define the Fourier coefficients \( \hat{f}_n \) of a function \( \tilde{f}(\theta) \), \( \theta \in [-\pi, \pi] \) identified with \( f(x), x \in S \).

\[
\hat{f}_n = \int_S \tilde{f}(\theta)e^{-in\theta} d\theta.
\]
Here, \( i^2 = -1 \) and \( n = 0, \pm 1, \pm 2, \ldots \). Then, for \( \theta' \in [-\pi, \pi] \), the Fourier expansion of the small perturbations are

\[
\Delta \lambda(t, \theta') = \frac{1}{2\pi} \sum_n \hat{\lambda}_n(t)e^{in\theta'},
\]

(26)

\[
\Delta w(t, \theta') = \frac{1}{2\pi} \sum_n \hat{\omega}_n(t)e^{in\theta'},
\]

(27)

\[
\Delta G(t, \theta') = \frac{1}{2\pi} \sum_n \hat{G}_n(t)e^{in\theta'},
\]

(28)

\[
\Delta \omega(t, \theta') = \frac{1}{2\pi} \sum_n \hat{\omega}_n(t)e^{in\theta'},
\]

(29)

where, \( e^{in\theta'} \) is the \( n \)-th Fourier mode. Substituting (26)-(29) into the linearized system (25) yields

\[
\sum_n \hat{\omega}_n e^{in\theta'} = \frac{\mu(\sigma - 1)(\bar{\omega} + \bar{\lambda})G^{\sigma - 2}}{F\sigma} \int_{-\pi}^{\pi} \hat{G}_n e^{in\theta} e^{-\alpha r|\theta' - \theta|} r d\theta
\]

\[
= \frac{\mu(\sigma - 1)(\bar{\omega} + \bar{\lambda})G^{\sigma - 2}}{F\sigma} \sum_n \hat{G}_n e^{in\theta'}
\]

\[
+ \frac{\mu G^{\sigma - 1}}{F\sigma} \sum_n \hat{\lambda}_n E_n e^{in\theta'},
\]

(30)

\[
\sum_n \hat{G}_n e^{in\theta'} = \frac{\bar{G}^\sigma}{(1 - \sigma)F} \sum_n \hat{\lambda}_n E_n e^{in\theta'},
\]

(31)

\[
\sum_n \hat{\omega}_n e^{in\theta'} = \sum_n \hat{\omega}_n e^{in\theta'} - \frac{\mu}{G} \sum_n \hat{G}_n e^{in\theta'},
\]

(32)

and

\[
\sum_n \frac{d}{dt} \left( \hat{\lambda}_n \right) e^{in\theta'} = \gamma \bar{\lambda} \sum_n \hat{\omega}_n e^{in\theta'}.
\]

(33)

In (30) and (31), the coefficient \( E_n \) is given as

\[
E_n = \frac{2\alpha r^2(1 - (-1)^n)e^{-\alpha r\pi}}{n^2 + \alpha^2 r^2},
\]

and it satisfies \( \int_{-\pi}^{\pi} e^{in\theta} e^{-\alpha r|\theta' - \theta|} r d\theta = E_n e^{in\theta'} \).

\[\footnote{Here, the functions \( \Delta \lambda, \Delta w, \Delta G, \) and \( \Delta \omega \) on \( S \) are identified with the corresponding periodic functions \( \Delta \tilde{\lambda}, \Delta \tilde{w}, \Delta \tilde{G}, \) and \( \Delta \tilde{\omega} \) on \([-\pi, \pi]\), respectively (See Subsection 2.3).} \]
From (30)-(33) the following equations
\begin{align}
\dot{w}_n &= \mu \left( \sigma - 1 \right) \left( \varphi + \lambda \right) G^{\sigma - 2} E_n \hat{G}_n + \mu \sigma G^{\sigma - 1} E_n \dot{\lambda}_n, \\
\dot{G}_n &= \frac{G}{1 - \sigma} E_n \hat{\lambda}_n, \\
\dot{\omega}_n &= \dot{w}_n - \mu \sigma G \hat{G}_n, \\
\frac{d}{dt} \dot{\lambda}_n &= \gamma \lambda \dot{\omega}_n,
\end{align}
(34)
for each \( n \) are obtained. Solving the first three equation of (34) gives
\begin{align}
\dot{\omega}_n &= \mu \sigma G \sigma - 1 E_n F \left\{ 2 \sigma - 1 \sigma - 1 \right\} \left( \varphi + \lambda \right) G^{\sigma - 1} E_n \dot{\lambda}_n
\end{align}
for each \( n \). Then, the fourth equation of (34) becomes
\begin{align}
\frac{d}{dt} \dot{\lambda}_n &= \gamma \lambda \dot{\omega}_n
\end{align}
for each \( n \). This means that if
\begin{align}
J_n := \mu \sigma G^{\sigma - 1} E_n \left\{ 2 \sigma - 1 \sigma - 1 \right\} \left( \varphi + \lambda \right) G^{\sigma - 1} E_n
\end{align}
(35)
is positive (negative), then the \( n \)-th mode is unstable (stable). Obviously, since \( \mu \sigma G^{\sigma - 1} E_n > 0 \), the sign of \( J_n \) is determined by the value in the braces of (35).

3.2.2 No black hole condition

Let us consider a closed economy with infinitely weak preference for variety or infinitely high transportation cost. We can prove that when \( \sigma \to \infty \) or \( \tau \to \infty \), i.e., \( \alpha \to \infty \),
\begin{align}
G^{\sigma - 1} E_n \to \frac{F}{\lambda} = \frac{2\pi F r}{\Lambda}.
\end{align}
for any number \( n \). Therefore, the coefficient (33) satisfies
\begin{align}
J_n \to \frac{\mu 2\pi r}{\Lambda} \left\{ 2 \sigma - 1 \sigma - 1 \right\} \left( \frac{\Phi + \Lambda}{\sigma \Lambda} \right)
\end{align}
when \( \alpha \to \infty \) for any number \( n \). Here, we use \( \lambda = \frac{\Lambda}{2\pi r} \) and \( \varphi = \frac{\Phi}{2\pi r} \). Therefore, if \( \frac{\sigma}{\sigma - 1} > \frac{\Phi}{\Lambda} \) holds, then \( J_n > 0 \), so the agglomeration occurs in spite of closed economy. Then, to eliminate such a situation where the centripetal forces are too strong, so-called the no black hole condition given by
\begin{align}
\frac{\sigma}{\sigma - 1} < \frac{\Phi}{\Lambda}
\end{align}
(36)
must be satisfied.
3.2.3 Infinite number of unstable modes

As discussed above, for any number \( n \), if the preference for variety is sufficiently weak or the transport cost is sufficiently high, the \( n \)-th Fourier mode can be stabilized as long as (36) holds. Meanwhile, no matter how weak the preference for variety is (or how high the transport cost is), if the wave number \( n \) is sufficiently large, the Fourier modes with wave numbers above \( n \) are unstable. This fact can be written as the following theorem.

**Theorem 2.** For any \( \sigma > 1 \) and any \( \tau > 0 \), there is an integer \( \tilde{n} \) such that

\[
J_n > 0 \quad \text{for any} \quad |n| > |\tilde{n}|.
\]

The proof is straightforward by using (35). In fact,

\[
J_n = \frac{\mu \sigma^{-1} E_n}{F} \left\{ \frac{2\sigma - 1}{\sigma(\sigma - 1)} - \frac{\alpha^2 r^2 (\tilde{\sigma} + \tilde{\lambda}) (1 - (-1)^{|n|} e^{-\alpha r \pi})}{\sigma \tilde{\lambda} (n^2 + \alpha^2 r^2) (1 - e^{-\alpha r \pi})} \right\},
\]

In the braces of the above, the second term converges to zero as \( |n| \to \infty \), so one can see that \( J_n > 0 \) for sufficiently large \( |n| \) because the first term \((2\sigma - 1)/(\sigma(\sigma - 1))\) is positive. Obviously, this proof makes use of the mathematical properties of the continuous space model. That is to say, in the continuous space model, there are an infinite number of the Fourier modes, so the wave number \( n \) can be as large as desired.

Theorem 2 simply means that the homogeneous stationary solution \( \bar{\lambda} = \Lambda/(2\pi r) \) is always unstable in the racetrack model.

4 Numerical simulations

In this section, we perform numerical computations of the racetrack model. The integral operator is discretized by the trapezoidal rule 2. The differential equation is then discretized using the 4-th order explicit Runge-Kutta method modified to satisfy the conservation of population.

4.1 Settings and overview

Let us describe basic settings for the simulation. We identify \( S \) with the interval \([-\pi, \pi]\) with mod \( 2\pi \). Then, the variable \( x \) in the interval is discretized into \( I \) nodal points \( x_i = -\pi + (i - 1)\Delta x, \ i = 1, 2, \ldots, I \), where \( \Delta x = 2\pi/I \). Temporal variable \( t \in [0, \infty) \) is discretized by \( t_n = (n - 1)\Delta t, \ n = 1, 2, \ldots, \) with \( \Delta t > 0 \). In our simulation, we set \( I = 256 \) and \( \Delta t = 0.01 \).

Let \( f \) be a function on \([0, \infty) \times S \). Then, we denote \( f(t_n, x_i) = f^n_i \) and

\[\text{2}\]Under the periodic boundary condition, applying the trapezoidal rule is equivalent to approximating the integral by a simple Riemann sum as described in [37] below.
\[ f^n = [f_1^n, f_2^n, \ldots, f_I^n]. \]

Note that approximating an integral \( \int_{-\pi}^{\pi} f(t, x) dx \) by the trapezoidal rule, we obtain

\[
\int_{-\pi}^{\pi} f(t, x) dx \approx \sum_{i=1}^{I} f(t, x_i) \Delta x. \tag{37}
\]

For an approximated solution \( \lambda^n = [\lambda_1^n, \lambda_2^n, \ldots, \lambda_I^n] \), the explicit Runge-Kutta scheme that discretizes the differential equation gives a candidate for \( \lambda^{n+1} \) denoted by \( \nu^{n+1} \). Let us write this relationship as

\[
\nu^{n+1} = f(\lambda^n), \quad \lambda^{n+1} = \nu^{n+1} / \| \nu^{n+1} \|_1
\]

for \( n = 0, 1, 2, \ldots \).

This explicit method gives a sequence of \( I \)-dimensional vectors \( \lambda^0, \lambda^1, \lambda^2, \ldots \). The calculation is performed until the maximum norm \( \| \lambda^{t+1} - \lambda^t \|_\infty \) becomes smaller than \( \epsilon = 10^{-10} \), and the numerical solution thus obtained is considered to be an approximation of a stationary solution \( \lambda^*(x) \). Initial values \( \lambda_0(x) \) are set by adding small perturbations to the homogeneous state \( \overline{\lambda} \equiv 1/(2\pi) \). These small perturbations are randomly generated in each simulation.

In the simulation, the elasticity of substitution \( \sigma > 1 \) and the transportation cost coefficient \( \tau > 0 \) are considered control parameters. Other parameters \( \Lambda, \Phi, F, \mu \) and \( \gamma \) are fixed at \( \Lambda = 1.0, \Phi = 10.0, F = 1.0, \mu = 0.6 \) and \( \gamma = 1.0 \), respectively. To guarantee the positive agricultural demand, we do not rely on the condition (21) but instead, we check that \( w(x) > \mu \forall x \) is satisfied at each time step. The following figures 1 and 2 show the approximated stationary solutions \( \lambda^*(x) \) obtained in the above way\(^3\). In any case, the stationary solutions are extremely non-uniform in space, having several spikes. For each pair of control parameters \( (\sigma, \tau) \), we perform several computations. Then, the number of spikes varied slightly depending on the initial distributions, so the largest number of the spikes in each simulation is referred to as the maximum number in the following. Additionally, it is noteworthy that the location of the spikes depends on the initial distributions much more sensitively than the number of the spikes.

\(^3\)In the figures, the actual computed values are indicated by the blue dots. The dashed lines are just the interpolation for the plot.
4.2 Change in transportation cost

First, we focus on the effects of the transportation cost on the behavior of the solution of the racetrack model, so we only change $\tau$ here.

The stationary states for each value of $\tau$ are shown in Fig.1. Here, we fix $\sigma = 5.0$ and vary $\tau$ to $0.05, 0.1, 0.2, 0.3, 0.35,$ and $0.41$. When $\tau$ is sufficiently small, a single spike is formed, and the mobile population density in other regions is almost zero. As the value of $\tau$ increases, the maximum number of the spikes increases, reaching 6 at $\tau = 0.41$.

Figure 1: Stationary solutions $\lambda^*(x)$ for $\sigma = 5.0$
4.3 Change in preference for variety

Next, we focus on the effects of the preference for variety of consumers on the behavior of the solution of the racetrack model, so we only change $\sigma > 1$ here.

The stationary states for each $\sigma$ are shown in Fig. 2. Here, we fix $\tau = 0.2$ and vary $\sigma$ to $2.0, 3.0, 5.0, 7.0, 8.0,$ and $9.3$. When the value of $\sigma$ is sufficiently small, a single spike is formed, and the mobile population density in other regions is almost zero. As the value of $\sigma$ increases, the maximum number of the spikes increases, reaching $6$ at $\sigma = 9.3$.

![Figure 2: Stationary solutions $\lambda^*(x)$ for $\tau = 0.2$](image)
5 Concluding remarks

In this paper, we consider the extension of the tractable new economic geography models proposed by Pflüger (2004) and Gaspar et al. (2018) to continuous space. Then, we investigate the behavior of the solution of the racetrack model. The homogeneous stationary solution is shown to be unstable. Furthermore, it is shown numerically that the destabilized solution eventually forms peculiar spatial distributions of mobile workers with several spikes that should be called cities. Then, the number of the cities decreases as the preference for variety intensifies or the transport cost decreases. These behavior of the solution is qualitatively consistent with those obtained from the investigation of the original core-periphery model on one-dimensional continuous periodic space (See Ohtake and Yagi (2021)), which suggests the validity of using the simplified model of this paper in theoretical studies of new economic geography.

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