MONOMIAL REDUCTION OF KNOT POLYNOMIALS

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For Filip, Peter, and Lukas, who all turned 35 recently

Abstract. For all natural numbers $N$ and prime numbers $p$, we find a knot $K$ whose skein polynomial $P_K(a, z)$ evaluated at $z = N$ has trivial reduction modulo $p$. An interesting consequence of our construction is that all polynomials $P_K(a, N) \pmod p$ with bounded $a$-span are realised by knots with bounded braid index. As an application, we classify all polynomials of the form $P_K(a, 1) \pmod 2$ with $a$-span $\leq 10$.

1. Introduction

The HOMFLY polynomial $P_K(a, z) \in \mathbb{Z}[a^\pm 1, z^\pm 1]$ of links $K$ satisfies the skein relation

$$a^{-1}P_{\beta\sigma_i^{-1}}(a, z) - a P_\beta(a, z) = z P_{\beta\sigma_i^{-1}}(a, z),$$

for all braids $\beta \in B_n$, where $\sigma_i \in B_n$ denotes any standard generator in the braid group $B_n$, and the hat symbol denotes the standard closure of a braid [4]. It is not known whether the property $P_K(a, z) = 1$ characterises the trivial knot. The tables of small knots do not even exclude the possibility that $P_K(a, z)$ modulo any prime number detects the trivial knot.

Theorem 1. For all natural numbers $N$ and prime numbers $p$, there exists a knot $K$ with

$$P_K(a, N) = 1 \pmod p.$$

A similar statement is known for the Jones polynomial, but only for finitely many primes [2]. The integer substitution $z = N$ transforms the HOMFLY polynomial into a single variable polynomial, whose span provides a lower bound for the minimal braid index $b(K)$ of knots, by the inequality of Franks-Williams and Morton [3, 8]:

$$\text{a-span}(P_K(a, z)) \leq 2b(K) - 2.$$

Theorem 2. For all $N, S \in \mathbb{N}$ and prime numbers $p$, there exists $T \in \mathbb{N}$, so that the set of polynomials $P_K(a, N) \pmod p$ with $a$-span $\leq S$ is realised by knots $K$ with $b(K) \leq T$.

The above result is of practical use for listing knot polynomials modulo primes, as we will see in the case $p = 2$. In particular, we obtain the following concrete result.
Corollary 1. All polynomials of the form \( P_K(a, 1) \pmod{2} \) with \( a \)-span \( \leq 10 \) are realised by knots with braid index \( \leq 7 \).

Thanks to the inequality of Franks-Williams and Morton, the polynomials \( P_K(a, 1) \pmod{2} \) with \( a \)-span \( \leq 10 \) include the polynomials of all knots with braid index \( \leq 6 \), hence of all knots with crossing number \( \leq 11 \).

The main number theoretic input for the first theorem is the fact that every prime number divides a Fibonacci number. The second theorem is more of a coincidental by-product of the construction. Detailed proofs are presented on the next couple of pages.

2. HOMFLY MONOMIALS

The skein relation for the HOMFLY polynomial allows for an inductive calculation of the polynomials \( P_{T(2,n)}(a, z) \) for all torus links \( T(2, n) \), defined as closures of the braids \( \sigma_1^n \in B_2 \) with two strands. The result is a polynomial of \( a \)-span two. More precisely, for all \( k \in \mathbb{N} \), we obtain polynomials of the form

\[
\begin{align*}
P_{T(2,2k)}(a, z) &= a_k(z)a^{2k-1} + b_k(z)a^{2k+1}, \\
P_{T(2,2k+1)}(a, z) &= c_k(z)a^{2k} + d_k(z)a^{2k+2},
\end{align*}
\]

for suitable \( a_k, b_k, c_k, d_k \in \mathbb{Z}[z^\pm 1] \) with initial values
\[
a_0 = z^{-1}, \quad b_0 = -z^{-1}, \quad c_0 = 1, \quad d_0 = 0.
\]

A slight rearrangement of the skein relation,
\[
P_{T(2,n+2)}(a, z) = a^2P_{T(2,n)}(a, z) + azP_{T(2,n+1)}(a, z),
\]

yields the following recursive formulas:
\[
\begin{align*}
a_k &= a_{k-1} + zc_{k-1}, \quad b_k = b_{k-1} + zd_{k-1}, \\
c_k &= c_{k-1} + za_k, \quad d_k = d_{k-1} + zb_k.
\end{align*}
\]

Now comes the key observation: after the substitution \( z = N \), and the reduction modulo \( p \), there remain only finitely many possible quadruples
\[
(a_k, b_k, c_k, d_k) \in (\mathbb{Z}/\mathbb{Z}_p)^4.
\]

As a consequence, there exists \( m > 0 \) with
\[
(a_{k+m}, b_{k+m}, c_{k+m}, d_{k+m}) = (a_k, b_k, c_k, d_k) \in (\mathbb{Z}/\mathbb{Z}_p)^4,
\]

for all \( k \in \mathbb{N} \). In particular, \( c_m = 1 \) and \( d_m = 0 \), thus
\[
P_{T(2,2m+1)}(a, N) = a^{2m} \in \mathbb{Z}/\mathbb{Z}_p[a^\pm 1].
\]

Finally, we observe that the symmetry
\[
P_{K^*}(a, z) = P_K(a^{-1}, z)
\]

between the HOMFLY polynomial of a knot \( K \) and its mirror image \( K^* \), and the multiplicative rule for connected sums of knots, \( P_{K_1 \# K_2} = P_{K_1}P_{K_2} \),

implies that the connected sum of knots $T(2, 2m + 1) \# T(2, -2m - 1)$ has trivial polynomial:

$$P_{T(2,2m+1)\#T(2,-2m-1)}(a, N) = 1 \pmod{p}.$$ 

Remarks.

(i) The above argument is inspired by the known fact that every prime number divides a Fibonacci number. In fact, in the special case $N = 1$, we obtain

$$P_{T(2,2k+1)}(a, 1) = F_{2k+2}a^{2k} + F_{2k}a^{2k+2},$$

where $(F_n)$ denotes the Fibonacci sequence starting at $F_0 = 0, F_1 = 1$. The periodicity of the Fibonacci sequence modulo $p$ can be derived from the following explicit result:

$$F_{p-\left(\frac{p}{5}\right)} = 0 \pmod{p},$$

where $\left(\frac{p}{5}\right)$ denotes the Legendre symbol $[6]$.

(ii) The case $N = 0 \pmod{p}$ requires a slight adjustment, because of the term $z^{-1}$. An elementary induction shows

$$P_{T(2,2k-1)}(a, 0) = ka^{2k-2} - (k - 1)a^{2k},$$

for all $k \geq 1$. In particular, $P_{T(2,2p+1)}(a, 0) = a^{2p} \pmod{p}$, from which Theorem 1 follows, as above.

As for Theorem 2, suppose we are given a polynomial $P_K(a, N) \in \mathbb{F}_p[a^{\pm 1}]$ whose $a$-span is $\leq S$. By the above argument, for all $l \in \mathbb{N}$, we have

$$P_{T(2,2lm+1)}(a, N) = a^{2lm}, P_{T(2,-2lm-1)}(a, N) = a^{-2lm}.$$ 

For a suitable choice of $l$ and sign, we can arrange for the minimal $a$-degree of $a^{\pm 2lm}P_K(a, N)$ to be in the range $[0, 1, \ldots, 2m - 1]$. We emphasize that this new polynomial is also the HOMFLY polynomial of a knot $\tilde{K}$, the connected sum of $K$ with a suitable torus knot with two strands. Now there are only finitely many potential polynomials $P_K(a, N)$ modulo $p$ with $a$-span $\leq S$ and minimal degree in the range $[0, 1, \ldots, 2m - 1]$. Choose finitely many knots that realise all these polynomials, and let $T - 1 \in \mathbb{N}$ be the largest minimal braid index among those. Then the initial polynomial $P_K(a, N) \in \mathbb{F}_p[a^{\pm 1}]$ is realised by a knot with braid index $T$, again by adding a suitable torus knot with two strands to one of these finitely many reference knots with braid index $\leq T - 1$.

3. Small span modulo two reductions

The goal of this section is to list all knot polynomials $P_K(a, 1)$ of $a$-span $\leq 10$, with coefficients modulo two. As in the previous section, the main ingredient is a family of knots with monomial reduction $P_K(a, 1) \pmod{2}$. The periodicity of the Fibonacci sequence modulo two being $m = 3$, we obtain:

$$P_{T(2,6l+1)}(a, 1) = a^{6l}, P_{T(2,-6l-1)}(a, 1) = a^{-6l},$$
degree range & knot polynomials \\
\([-4, 6]\) & \(O (1), 3_1 (a^2 + a^4), 3_1^* (a^{-4} + a^{-2}), 4_1 (a^{-2} + a^2), 5_1 (a^4), 6_1 (a^{-2} + 1 + a^4), 6_1^* (a^{-4} + 1 + a^2), 8_3 (a^{-4} + a^{-2} + 1 + a^2 + a^4), 10_3 (a^{-4} + a^{-2} + a^2 + a^4 + a^6), 11_1 (a^{-1} + a^2 + a^6), 11_1 (a^{-2} + a^4 + a^6), 3_1 # 4_1 (1 + a^2 + a^4 + a^6), 3_1 # 6_1^* (a^{-2} + 1 + a^2 + a^6), 3_1 # 11n101 (a^{-4} + 1 + a^4 + a^6), 4_1 # 4_1 (a^{-4} + a^4), 4_1 # 6_1 (a^{-4} + a^{-2} + 1 + a^6)\) \\
\([-2, 8]\) & \(O (1), 3_1 (a^2 + a^4), 4_1 (a^{-2} + a^2), 5_1 (a^4), 6_1 (a^{-2} + 1 + a^4), 7_4 (a^2 + a^6 + a^8), 10_1 (a^{-2} + 1 + a^2 + a^4 + a^8), 11a121 (a^{-2} + a^2 + a^4 + a^6 + a^8), 11n101 (a^{-2} + a^4 + a^6), 11n139 (1 + a^2 + a^8), 3_1 # 3_1 (a^4 + a^8), 3_1 # 4_1 (1 + a^2 + a^4 + a^6), 3_1 # 6_1 (1 + a^4 + a^6 + a^8), 3_1 # 6_1^* (a^{-2} + 1 + a^2 + a^6), 3_1 # 8_3 (a^{-2} + a^8), 3_1 # 4_1 # 4_1 (a^{-2} + 1 + a^6 + a^8)\)

Table 1. Knot polynomials \(P_K(a, 1) \mod 2\) of span \(\leq 10\)

for all \(l \in \mathbb{N}\). Therefore, it is enough to classify all knot polynomials \(P_K(a, 1)\) in the degree ranges \([-4, 6]\), \([-2, 8]\), \([0, 10]\). Indeed, we may discard odd powers of the variable \(a\), since the HOMFLY polynomial of knots is in fact a Laurent polynomial in \(a^2\) (and a polynomial in \(z^2\), for that matter). Moreover, by the mirror symmetry \(K \leftrightarrow K^*\), the polynomials in the range \([0, 10]\) are in correspondence with the polynomials in the range \([-10, 0]\), in turn with the polynomials in the range \([-4, 6]\), by multiplying with \(a^6 = P_{T(2, 7)}(a, 1)\).

A priori, there are \(2^6 = 64\) potential polynomials \(P_K(a, 1) \in \mathbb{F}_2[a^\pm 2]\) in each degree range \([-4, 6]\) and \([-2, 8]\). However, the skein relation for the HOMFLY polynomial implies \(P_K(a, a^{-1} - a) = 1\), for all knots \(K\). This fact can be enhanced to the following statement \([3]\): the difference between the HOMFLY polynomials of two knots is divisible (in \(\mathbb{Z}[a^\pm 2, z^\pm 2]\)) by

\[z^2 - (a^{-1} - a)^2.\]

After the substitution \(z = 1\) and reduction modulo two, the latter becomes \(a^{-2} + 1 + a^2\). An elementary argument, based on polynomial division with remainder, shows that every even degree range of span 10 contains only 16 polynomials \(P_K(a, 1) \in \mathbb{F}_2[a^\pm 2]\) with the additional property that \(P_K(a, 1) - 1\) is divisible by \(a^{-2} + 1 + a^2\). An analogous argument is carried out in \([1]\), where we classify Jones polynomials modulo two of span eight.

With the help of the knot tables by Rolfsen and knotinfo \([9, 7]\), we found 16 polynomials \(P_K(a, 1) \in \mathbb{F}_2[a^\pm 2]\) in each degree range \([-4, 6]\) and \([-2, 8]\).

All the knots can be chosen with braid index \(\leq 6\), as seen in Table 1 (where we use the notation of the cited tables, and \(O\) stands for the trivial knot). By adding one summand of the form \(T(2, \pm(6l + 1))\) to all these knots, and by considering all mirror images of the resulting knots, we obtain all polynomials of span \(\leq 10\). This implies the statement of Corollary \([1]\).
The list of knots obtained in this section suggests the following generalisation.

**Question 1.** Let $p(a) \in \mathbb{F}_2[a^{\pm 2}]$ with $p(a) - 1$ divisible by $a^{-2} + 1 + a^2$. Is there a knot $K \subset S^3$ with $P_K(a, 1) = p(a) \pmod{2}$?

We conclude with a prime reduction of the famous unknot detection question for the HOMFLY polynomial.

**Question 2.** Let $p$ be a prime number. Is there a knot $K \subset S^3$ with $P_K(a, z) = 1 \pmod{p}$?

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