Abstract. In this paper, a notion of cyclotomic (or level $k$) walled Brauer algebras $B_{k,r,t}$ is introduced for arbitrary positive integer $k$. It is proven that $B_{k,r,t}$ is free over a commutative ring with rank $k^{r+t}(r+t)!$ if and only if it is admissible. Using super Schur-Weyl duality between general linear Lie superalgebras $\mathfrak{gl}_{m|n}$ and $B_{2,r,t}$, we give a classification of highest weight vectors of $\mathfrak{gl}_{m|n}$-modules $M_{pq}^{rt}$, the tensor products of Kac-modules with mixed tensor products of the natural module and its dual. This enables us to establish an explicit relationship between $\mathfrak{gl}_{m|n}$-Kac-modules and right cell (or standard) $B_{2,r,t}$-modules over $\mathbb{C}$. Further, we find an explicit relationship between indecomposable tilting $\mathfrak{gl}_{m|n}$-modules appearing in $M_{pq}^{rt}$ and principal indecomposable right $B_{2,r,t}$-modules via the notion of Kleshchev bipartitions. As an application, decomposition numbers of $B_{2,r,t}$ arising from super Schur-Weyl duality are determined.

1. Introduction

Motivated by Brundan-Stroppel’s work on higher super Schur-Weyl duality in [5], we introduced affine walled Brauer algebras $B_{r,t}^{\text{aff}}$ in [19] so as to establish higher super Schur-Weyl duality on the tensor product $M_{pq}^{rt}$ of a Kac-module with a mixed tensor product of the natural module and its dual for general linear Lie superalgebra $\mathfrak{gl}_{m|n}$ over $\mathbb{C}$ under the assumption $r+t \leq \min\{m,n\}$. One of purposes of this paper is to generalize super Schur-Weyl duality to the case $r+t > \min\{m,n\}$. For this aim, we need to establish a bijective map from a level two walled Brauer algebra $B_{2,r,t}$ appearing in [19] to a level two degenerate Hecke algebra $H_{2,r+t}$. This can be done by showing that the dimension of $B_{2,r,t}$ is $2^{r+t}(r+t)!$ over $\mathbb{C}$. We consider this problem in a general setting by introducing a cyclotomic (or level $k$) walled Brauer algebra $B_{k,r,t}$ appearing in [19] to a level two degenerate Hecke algebra $H_{2,r+t}$. This can be done by showing that the dimension of $B_{2,r,t}$ is $2^{r+t}(r+t)!$ over $\mathbb{C}$. We consider this problem in a general setting by introducing a cyclotomic (or level $k$) walled Brauer algebra $B_{k,r,t}$ for arbitrary $k \in \mathbb{Z}^>0$. By employing a totally new method, which is independent of seminormal forms of $B_{k,r,t}$, we prove that $B_{k,r,t}$ is free over a commutative ring $R$ with rank $k^{r+t}(r+t)!$ if and only if it is admissible in the sense of Definition [23]. It is expected that $B_{k,r,t}$ can be used to study the problem on a classification of finite dimensional simple $B_{r,t}^{\text{aff}}$-modules over an algebraically closed field. Details will be given elsewhere.

The establishment of the higher super Schur-Weyl duality [19] enables us to use the representation theory of $B_{2,r,t}$ to classify highest weight vectors of $M_{pq}^{rt}$ (at this point, we would like to mention that purely on the Lie superalgebra side, it seems to be hard to construct highest weight vectors of a given module, which is an interesting problem on its own right). On the other hand, a classification of highest weight vectors of $M_{pq}^{rt}$ also enables us to relate the category of finite dimensional $\mathfrak{gl}_{m|n}$-modules with that of $B_{2,r,t}$, which in turn gives us
an efficient way to calculate decomposition numbers of \( \mathcal{B}_{2,r,t} \) (cf. [18] for quantum walled Brauer algebras). This is the main motivation of this paper. We explain some details below.

It is proven in [19] that \( \text{End}_U(\mathfrak{g}_{m|n})(M^\mathcal{B}_{r,t}^{pq}) \cong \mathcal{B}_{2,r,t} \) if \( r + t \leq \min\{m, n\} \). Since there is a bijection between the dominant weights of \( M^\mathcal{B}_{pq}^{rt} \) and the poset \( \Lambda_{2,r,t} \) in (3.12), and since \( \mathcal{B}_{2,r,t} \) is a weakly cellular algebra over \( \Lambda_{2,r,t} \) in the sense of [11], it is very natural to ask the following problem: whether a \( \mathbb{C} \)-space of \( \mathfrak{g}_{m|n} \)-highest weight vectors of \( M^\mathcal{B}_{pq}^{rt} \) with a fixed highest weight is isomorphic to a cell (or standard) module of \( \mathcal{B}_{2,r,t} \).

We give an affirmative answer to the problem. In sharp contrast to the Lie algebra case, due to the existence of the parity of \( \mathfrak{gl}_{m|n} \), the known weakly cellular basis of \( \mathcal{B}_{2,r,t} \) in [19] cannot be directly used to establish a relationship between \( \mathfrak{gl}_{m|n} \)-highest weight vectors of \( M^\mathcal{B}_{pq}^{rt} \) and right cell modules of \( \mathcal{B}_{2,r,t} \). One has to find new cellular bases of level two Hecke algebra \( \mathcal{H}_{2,r} \) which are different from that in [3]. These new cellular bases of \( \mathcal{H}_{2,r} \), which relate both trivial and signed representations of symmetric groups, are used to construct a new weakly cellular basis of \( \mathcal{B}_{2,r,t} \). Motivated by explicit descriptions of bases of right cell modules for \( \mathcal{B}_{2,r,t} \), we construct and classify \( \mathfrak{gl}_{m|n} \)-highest weight vectors of \( M^\mathcal{B}_{pq}^{rt} \). This leads to a \( \mathcal{B}_{2,r,t} \)-module isomorphism between each \( \mathbb{C} \)-space of \( \mathfrak{gl}_{m|n} \)-highest weight vectors of \( M^\mathcal{B}_{pq}^{rt} \) with a fixed highest weight and the corresponding cell module of \( \mathcal{B}_{2,r,t} \). Based on the above, we are able to construct a suitable exact functor sending \( \mathfrak{gl}_{m|n} \)-Kac-modules to right cell modules of \( \mathcal{B}_{2,r,t} \). This functor also sends an indecomposable tilting module appearing in \( M^\mathcal{B}_{pq}^{rt} \) to a principal indecomposable right \( \mathcal{B}_{2,r,t} \)-module indexed by a pair of so-called Kleshchev bipartitions in the sense of (3.15). It gives us an efficient way to calculate decomposition numbers of \( \mathcal{B}_{2,r,t} \) via Brundan-Stroppel’s result [5] on the multiplicity of a Kac-module in an indecomposable tilting module appearing in \( M^\mathcal{B}_{pq}^{rt} \).

We organize the paper as follows. In section 2, after recalling the definition of \( \mathcal{B}_{r,t}^{\text{aff}} \) over a commutative ring \( R \), we introduce cyclotomic walled Brauer algebras \( \mathcal{B}_{k,r,t} := \mathcal{B}_{r,t}^{\text{aff}}/I \) for arbitrary \( k \in \mathbb{Z}_{>0} \), where \( I \) is the two-sided ideal of \( \mathcal{B}_{r,t}^{\text{aff}} \) generated by two cyclotomic polynomials \( f(x_1) \) and \( g(\bar{x}_1) \) of degree \( k \), which satisfy (2.5)–(2.7). When \( \mathcal{B}_{r,t}^{\text{aff}} \) is admissible in the sense of Definition 2.3, we describe explicitly an \( R \)-basis of \( I \). This enables us to prove that \( \mathcal{B}_{k,r,t} \) is free over \( R \) with rank \( k^{r+t}(r+1)! \) if and only if it is admissible. In section 3, we construct cellular bases of \( \mathcal{H}_{2,r} \) and use them to construct a weakly cellular basis of \( \mathcal{B}_{2,r,t} \). In section 4, higher super Schur-Weyl dualities in [19] are generalized to the case \( r + t > \min\{m, n\} \). In sections 5–6, we classify highest weight vectors of \( M^\mathcal{B}_{pq}^{r0} \) and \( M^\mathcal{B}_{pq}^{rt} \). Based on this, we establish an explicit relationship between indecomposable tilting (resp. Kac) modules for \( \mathfrak{gl}_{m|n} \) and principal indecomposable (resp. cell) right \( \mathcal{B}_{2,r,t} \)-modules via a suitable exact functor. This gives us an efficient way to calculate decomposition numbers of \( \mathcal{B}_{2,r,t} \) arising from the super Schur-Weyl duality in [19].

2. Affine walled Brauer algebras and its cyclotomic quotients

Throughout, we assume that \( R \) is a commutative ring containing \( \Omega = \{\omega_a \mid a \in \mathbb{N}\} \) and identity \( 1 \). In this section, we introduce a level \( k \) walled Brauer algebra \( \mathcal{B}_{k,r,t} \) and prove that
$\mathcal{B}_{k,r,t}$ is free over $R$ with rank $k^{r+t}(r+t)!$ if and only if $\mathcal{B}_{k,r,t}$ is admissible in the sense of Definition 2.3. First, we briefly recall the definition of walled Brauer algebras.

Fix $r,t \in \mathbb{Z}^{>0}$. A **walled $(r,t)$-Brauer diagram** (or simply, a walled Brauer diagram) is a diagram with $(r+t)$ vertices on top and bottom rows, and vertices on both rows are labeled from left to right by $r,...,2,1,\bar{1},\bar{2},...,\bar{t}$, such that every $i \in \{r,...,2,1\}$ (resp., $\bar{i} \in \{\bar{1},\bar{2},...,\bar{t}\}$) on each row is connected to a unique $\bar{j}$ (resp., $j$) on the same row or a unique $j$ (resp., $\bar{\bar{i}}$) on the other row. Thus there are four types of pairs $[i,j]$, $[i,\bar{j}]$, $[\bar{i},j]$ and $[\bar{i},\bar{j}]$. The pairs $[i,j]$ and $[\bar{i},\bar{j}]$ are **vertical edges**, and $[i,j]$ and $[\bar{i},\bar{j}]$ are **horizontal edges**.

The product of two walled Brauer diagrams $D_1$ and $D_2$ can be defined via concatenation. Pasting $D_1$ above $D_2$ and connecting each vertex on the bottom row of $D_1$ to the corresponding vertex on the top row of $D_2$ yields a diagram $D_1 \circ D_2$, called the **concatenation** of $D_1$ and $D_2$. Removing all circles of $D_1 \circ D_2$ yields a unique walled Brauer diagram, denoted $D_3$. Let $n$ be the number of circles appearing in $D_1 \circ D_2$. Then the **product** $D_1 \circ D_2$ is defined to be $\omega_0^n D_3$, where $\omega_0$ is a fixed element in $R$. The **walled Brauer algebra** $\mathcal{B}_{r,t} := \mathcal{B}_{r,t}(\omega_0)$ with defining parameter $\omega_0$ is the associative $R$-algebra spanned by all walled Brauer diagrams with product defined in this way.

Let $\mathcal{S}_r$ (resp. $\mathcal{\bar{S}}_t$) be the symmetric group in $r$ (resp. $t$) letters $r,...,2,1$ (resp. $\bar{1},\bar{2},...,\bar{t}$). It is known that $\mathcal{B}_{r,t}$ contains two subalgebras which are isomorphic to the group algebras of $\mathcal{S}_r$ and $\mathcal{\bar{S}}_t$, respectively. More explicitly, the walled Brauer diagram $s_i$ whose edges are of forms $[k,k]$ and $[\bar{k},\bar{k}]$ except two vertical edges $[i,i+1]$ and $[i+1,i]$ can be identified with the basic transposition $(i,i+1) \in \mathcal{S}_r$, which switches $i$ and $i+1$ and fixes others. Similarly, there is a walled Brauer diagram $\bar{s}_j$ corresponding to $(j,j+1) \in \mathcal{\bar{S}}_t$. Let $e_1$ be the walled Brauer diagram whose edges are of forms $[k,k]$ and $[\bar{k},\bar{k}]$ except two horizontal edges $[1,\bar{1}]$ on the top and bottom rows. Then $\mathcal{B}_{r,t}$ is the $R$-algebra generated by $e_1$, $s_i$, $\bar{s}_j$ for $1 \leq i \leq r - 1$, $1 \leq j \leq t - 1$ such that $s_i$‘s, $\bar{s}_j$’s are distinguished generators of $\mathcal{S}_r \times \mathcal{\bar{S}}_t$ and

$$
e_1^2 = \omega_0 e_1, \quad e_1 s_i e_1 = e_1 s_i, \quad s_i e_1 = e_1 s_i, \quad \bar{s}_j e_1 = e_1 \bar{s}_j \quad (i,j \neq 1),$$

$$e_1 s_i \bar{s}_j e_1 s_i = e_1 s_i \bar{s}_j e_1, \quad s_i e_1 \bar{s}_j e_1 = e_1 \bar{s}_j e_1 s_i. \quad (2.1)$$

Let $H_n^\text{aff}$ be the **degenerate affine Hecke algebra** $[10]$. As a free $R$-module, it is the tensor product $R[y_1,y_2,\cdots,y_n] \otimes R\mathcal{S}_n$ of a polynomial algebra with the group algebra of $\mathcal{S}_n$. The multiplication is defined so that $R[y_1,y_2,\cdots,y_n] \equiv R[y_1,y_2,\cdots,y_n] \otimes 1$ and $R\mathcal{S}_n \equiv 1 \otimes R\mathcal{S}_n$ are subalgebras and $s_i y_j = y_j s_i$ if $j \neq i,i+1$ and $s_i y_i = y_{i+1}s_i - 1, 1 \leq i \leq n - 1$.

Recall that $R$ contains 1 and $\Omega = \{\omega_a \in R \mid a \in \mathbb{N}\}$. The affine walled Brauer algebra $\mathcal{B}_{r,t}^\text{aff}(\Omega)$ (which is $\mathcal{\widehat{B}}_{r,t}$ in [19, §4]) with respect to the defining parameters $\omega_a$’s have been defined via generators and 26 defining relations [19, Definition 2.7]. It follows from [19, Theorem 4.15] that $\mathcal{B}_{r,t}^\text{aff}(\Omega)$ can be also defined in a simpler way as follows: it is an associative $R$-algebra generated by $e_1$, $x_1$, $\bar{x}_1$, $s_i$, $\bar{s}_j$ for $1 \leq i \leq r - 1$, $1 \leq j \leq t - 1$, such that $e_1$, $s_i$‘s, $\bar{s}_j$’s are generators of $\mathcal{B}_{r,t}$ with defining parameter $\omega_0$, and as a free $R$-module,

$$\mathcal{B}_{r,t}^\text{aff}(\Omega) = R[x_1] \otimes \mathcal{B}_{r,t} \otimes R[\bar{x}_1],$$

the tensor product of the walled Brauer algebra $\mathcal{B}_{r,t}$ with two polynomial algebras

$$R[x_r] := R[x_1,x_2,\cdots,x_r], \quad \text{and} \quad R[\bar{x}_t] := R[\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_t].$$
Multiplication is defined as \( R[\mathbf{x}] \cong R[\mathbf{x}] \otimes 1 \otimes 1, R[\mathbf{x}] \cong 1 \otimes 1 \otimes R[\mathbf{x}], \) and \( \mathcal{B}_{r,t} \equiv 1 \otimes \mathcal{B}_{r,t} \otimes 1, \) and \( R[\mathbf{x}] \otimes R\widehat{S}_r \otimes 1 \cong H^\text{aff}_r \otimes 1, 1 \otimes R\widehat{S}_t \otimes R[\mathbf{x}] \cong 1 \otimes H^\text{aff}_t \) and

\[
e_1(x_1 + \bar{x}_1) = (x_1 + \bar{x}_1)e_1 = 0, \quad s_1e_1s_1 = x_1s_1e_1s_1, \quad s_1e_1s_1\bar{x}_1 = \bar{x}_1s_1e_1s_1, \quad (2.2)
\]

\[
s_i\bar{x}_1 = \bar{x}_1s_i, \quad s_i\bar{x}_1 = x_1s_i, \quad x_1(1 + \bar{x}_1) = (1 + \bar{x}_1)x_1, \quad (2.3)
\]

\[
e_1x_i^ke_1 = \omega_ke_1, \quad e_1x_i^ke_1 = \omega_ke_1, \quad \forall k \in \mathbb{Z}^>0, \quad (2.4)
\]

where \( \omega_k \)'s are determined by [19] Corollary 4.3. If \( \omega_k \)'s do not satisfy [19] Corollary 4.3, then \( e_1 = 0 \) and \( \mathcal{B}^\text{aff}_{r,t}(\Omega) \) turns out to be \( H^\text{aff}_r \otimes H^\text{aff}_t \) if \( R \) is a field.

We remark that the isomorphism \( R[\mathbf{x}] \otimes R\widehat{S}_r \otimes 1 \cong H^\text{aff}_r \otimes 1 \) sends \( s_i \)'s (resp. \( x_1 \)) to \( s_i \)'s (resp. \( -y_1 \)), and the isomorphism \( 1 \otimes R\widehat{S}_t \otimes R[\mathbf{x}] \cong 1 \otimes H^\text{aff}_t \) sends \( \bar{s}_j \)'s (resp. \( \bar{x}_1 \)) to \( s_j \)'s (resp. \( -y_1 \)). So, \( x_{i+1} = s_ix_is_i - s_i \) and \( \bar{x}_{j+1} = \bar{s}_j\bar{x}_j\bar{s}_j - \bar{s}_j \) and \( y_{i+1} = s_iy_is_i + s_i \) if all of them make sense.

For the simplification of notation, denote \( \mathcal{B}^\text{aff}_{r,t}(\Omega) \) by \( \mathcal{B}^\text{aff}_{r,t} \). Fix \( u_1, u_2, \ldots, u_k \in R \) for some \( k \in \mathbb{Z}^>0 \). Let \( f(x_1) \in \mathcal{B}^\text{aff}_{r,t} \) be such that

\[
f(x_1) = \prod_{i=1}^{k}(x_1 - u_i). \quad (2.5)
\]

By [19] Lemma 4.2 (or using (2.2)-(2.3)), there is a monic polynomial \( g(\bar{x}_1) \in R[\bar{x}_1] \) with degree \( k \) such that

\[
e_1f(x_1) = (-1)^ke_1g(x_1). \quad (2.6)
\]

If \( R \) is an algebraically closed field, then there are \( \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k \in R \) such that

\[
g(x_1) = \prod_{i=1}^{k}(x_1 - \bar{u}_i). \quad (2.7)
\]

**Definition 2.1.** Let \( R \) be a commutative ring containing 1, \( \Omega = \{\omega \in R \mid a \in \mathbb{N}\} \), and \( u_i, \bar{u}_i, 1 \leq i \leq k \). The cyclotomic (or level \( k \)) walled Brauer algebra \( \mathcal{B}_{k,r,t} \) is the quotient algebra \( \mathcal{B}^\text{aff}_{r,t}/I \), where \( I \) is the two-sided ideal of \( \mathcal{B}^\text{aff}_{r,t} \) generated by \( f(x_1) \) and \( g(x_1) \) satisfying (2.5)-(2.7).

If \( k = 1 \), then \( \mathcal{B}_{k,r,t} \) is \( \mathcal{B}_{r,t} \) with defining parameter \( \omega_0 \). For some special \( u_i, \bar{u}_i \), \( i = 1, 2 \), \( \mathcal{B}_{2,r,t} \) is the level two walled Brauer algebras arising from super Schur-Weyl duality in [19].

**Lemma 2.2.** Let \( f(x_1) \) be given in (2.5). Write \( f(x_1) = x_1^k + \sum_{i=1}^{k}a_ix_1^{k-i} \). Then \( e_1 \) is an \( R \)-torsion element of \( \mathcal{B}_{k,r,t} \) unless

\[
\omega_\ell = -(a_1\omega_{\ell-1} + \cdots + a_k\omega_{\ell-k}) \quad \text{for all} \quad \ell \geq k. \quad (2.8)
\]

**Proof.** Let \( b_\ell = \omega_\ell + a_1\omega_{\ell-1} + \cdots + a_k\omega_{\ell-k} \in R \). By (2.4), \( b_\ell e_1 = e_1f(x_1)x_1^{\ell-k}e_1 \) in \( \mathcal{B}^\text{aff}_{r,t} \) and \( b_\ell e_1 = 0 \) in \( \mathcal{B}_{k,r,t} \). Thus, \( e_1 \) is an \( R \)-torsion element if \( b_\ell \neq 0 \) for some \( \ell \geq k \).

**Definition 2.3.** The algebras \( \mathcal{B}^\text{aff}_{r,t} \) and \( \mathcal{B}_{k,r,t} \) are called admissible if (2.8) holds.

**Lemma 2.4.** Assume \( f(x_1), g(x_1) \in \mathcal{B}^\text{aff}_{r,t} \) satisfying (2.5)-(2.7). If \( \mathcal{B}^\text{aff}_{r,t} \) is admissible, then

1. \( e_1f(x_1)x_1^a e_1 = 0 \) for all \( a \in \mathbb{N} \).
2. \( e_1g(x_1)x_1^a e_1 = 0 \) for all \( a \in \mathbb{N} \).
Lemma 2.7. Let $gr(B)$ with degrees less than or equal to $k$, $i, j, k, \ell$ for all possible as its $M$ \[19, \text{Theorem 4.15}\]

Theorem 2.6. $Elements$ of $M$ are called $regular$ $monomials$ of $B$. $\square$

For each nonnegative integer $f \leq \min\{r, t\}$, let

$$e^f = e_1 e_2 \cdots e_f \text{ for } f > 0 \text{ and } e^0 = 1, \text{ where } e_i = e_{i,i}. \quad (2.10)$$

Set

$$B^f_{r,t} = \{ s_{f,i} s_{f,j} \cdots s_{1,i} s_{1,j} \mid 1 \leq i_1 < \cdots < i_f \leq r, k \leq j_k \}. \quad (2.11)$$

**Definition 2.5.** For $\alpha = (\alpha_1, \cdots, \alpha_t) \in \mathbb{N}^r$ and $\beta = (\beta_1, \cdots, \beta_t) \in \mathbb{N}^t$, let $x^\alpha = \prod_{i=1}^r x_i^{\alpha_i}$, $\bar{x}^\beta = \prod_{j=1}^t \bar{x}_j^{\beta_j}$. Let $M$ be a subset of $B^f_{r,t}$ given by

$$M = \bigcup_{j=0}^{\min\{m,n\}} \{ x^\alpha c^{-1} e^f w d \bar{x}^\beta \mid (\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^t, c, d \in B^f_{r,t}, w \in \mathcal{S}_{r-f} \times \mathcal{S}_{t-f} \}. \quad (2.12)$$

Elements of $M$ are called $regular$ $monomials$ of $B$. $\square$

**Theorem 2.6.** $[19, \text{Theorem 4.15}]$ The affine walled Brauer algebra $B^f_{r,t}$ is free over $R$ with $M$ as its $R$-basis.

We consider $B^f_{r,t}$ as a filtrated $R$-algebra as follows. Let

$$\deg s_i = \deg s_j = \deg e_1 = 0 \text{ and } \deg x_k = \deg \bar{x}_\ell = 1$$

for all possible $i, j, k, \ell$’s. Let $(B^f_{r,t}(k))$ be the $R$-submodule spanned by regular monomials with degrees less than or equal to $k$ for $k \in \mathbb{Z}^{\geq 0}$. Then we have the following filtration

$$B^f_{r,t} \supset \cdots \supset (B^f_{r,t})^{(k)} \supset (B^f_{r,t})^{(0)} \supset (B^f_{r,t})^{(-1)} = 0. \quad (2.13)$$

Let $\text{gr}(B^f_{r,t}) = \bigoplus_{i \geq 0} (B^f_{r,t})^{[i]}$, where $(B^f_{r,t})^{[i]} = (B^f_{r,t})^{(i)}/(B^f_{r,t})^{(i-1)}$. Then $\text{gr}(B^f_{r,t})$ is an associated $\mathbb{Z}$-graded algebra. We will use the same symbols to denote elements in $\text{gr}(B^f_{r,t})$.

**Lemma 2.7.** Let $x'_i = s_{i-1} x'_i - 1 s_{i-1}$, and $\bar{x}'_j = \bar{s}_{j-1} \bar{x}_j - 1 \bar{s}_{j-1}$ for $i, j \in \mathbb{Z}^{\geq 2}$ with $i \leq r$ and $j \leq t$, where $x'_1 = x_1$, and $\bar{x}'_1 = \bar{x}_1$.

1. $x_i = x'_i - L_i$, where $L_i = \sum_{1 \leq j < i} (j, i)$ is the transposition in $\mathcal{S}_r$ which switches $j, i$ and fixes others.
2. $\bar{x}_i = \bar{x}'_i - \bar{L}_i$, where $\bar{L}_i = \sum_{1 \leq j < i} (j, \bar{i})$ is the transposition in $\mathcal{S}_t$ which switches $\bar{j}, \bar{i}$ and fixes others.
3. Any symmetric polynomial of $L_1, L_2, \cdots, L_r$ (resp. $\bar{L}_1, \bar{L}_2, \cdots, \bar{L}_t$) is a central element of $R\mathcal{S}_r$ (resp. $R\mathcal{S}_t$).
Proof. (1)-(2) are trivial and (3) is a well-known result. \qed

The elements $L_i$’s (resp. $L_j$’s) are known as Jucys-Murphy elements of $R\mathcal{S}_r$ (resp. $R\mathcal{S}_t$). Note that $x_ix_j = x_jx_i$ and $\bar{x}_ix_j = \bar{x}_j\bar{x}_i$ for all possible $i, j$. However, $x_i'\bar{x}_j$ (resp. $x_i\bar{x}_j'$) do not commute each other.

Suppose $0 < f \leq \min\{m, n\}$. Denote

$$\vec{i} = (i_1, \ldots, i_f), \quad \vec{j} = (j_1, \ldots, j_f), \quad e_{\vec{i}, \vec{j}} = e_{i_1j_1}e_{i_2j_2}\cdots e_{i_fj_f},$$

where $i_1, i_2, \ldots, i_f$ are distinct numbers in $\{1, 2, \ldots, r\}$, and $j_1, j_2, \ldots, j_f$ are distinct numbers in $\{1, 2, \ldots, t\}$. Then $e_{i_kj_k}$’s commute each other. If $f = 0$, we set $\vec{i} = \vec{j} = \emptyset$ and $e_{\vec{i}, \vec{j}} = 1$.

We always assume that $\mathcal{S}_r$ (resp. $\mathcal{S}_t$) acts on the right of $\{r, \ldots, 2, 1\}$ (resp. $\{1, 2, \ldots, \ell\}$).

Lemma 2.8. Suppose $a \in \mathbb{Z}_{>0}$, $1 \leq i, \ell \leq r$ and $1 \leq j \leq t$.

1. If $w \in \mathcal{S}_r$, then $w\mathbf{f}_s'w^{-1} = \mathbf{f}_s'(w^{-1})$.
2. If $w \in \mathcal{S}_t$, then $w\mathbf{g}_s'w^{-1} = \mathbf{g}_s'(w^{-1})$.
3. $x_i^a\mathbf{f}(x_j') = \mathbf{f}(x_\ell')x_i^av$, where $v = \sum_{b < a} \sum_{h, h_1=1}^{\max\{i, \ell\}} \mathbf{f}(x_h)x_h^{b_1}R\mathcal{S}_r$.
4. $x_i^a\mathbf{f}(x_j') = \mathbf{f}(x_\ell')x_j'^av$, where $v = \sum_{b_1 + b_2 < a, c_1 + c_2 \leq 1} x_j'^{b_1}c_{ij}\mathbf{f}(x_j')x_j'^{c_2}x_j^{b_2}$ for some non-negative integers $b_1, b_2, c_1, c_2$ and $\epsilon = \pm 1$.

Proof. (1)-(2) are trivial. Since $x_2 = x_2' - s_1$ and $x_2x_1 = x_1x_2$, $x_i^a\mathbf{f}(x_j') = \mathbf{f}(x_j')(x_2' - s_1) + \mathbf{f}(x_2')s_1$. \quad (2.15)

Applying the conjugate of $s_i, s_2$ on (2.15) yields (3) for $a = 1$ and $\ell = 1$. If $\ell > 1$, then $x_i'\mathbf{f}(x_\ell') = x_i's_\ell^{-1}\mathbf{f}(x_\ell')s_\ell^{-1} = s_\ell^{-1}x_i's_\ell^{-1}\mathbf{f}(x_\ell')s_\ell^{-1}$. Thus, (3) follows from inductive assumption on $\ell - 1$ and (1) under the assumption $a = 1$. The case $a > 1$ follows by using the previous result on $a = 1$, repeated. Finally, (4) can be checked similarly by induction. We leave the details to the readers. \qed

Proposition 2.9. Let $J_L = \sum_{i=1}^{t} \mathcal{B}^{\text{aff}}_{r,i} g(\bar{x}_j')$ and $J_R = \sum_{i=1}^{r} \mathbf{f}(x_i') \mathcal{B}^{\text{aff}}_{r,i}$. Then

1. $J_L$ is a right $R\mathcal{S}_r \otimes \mathcal{H}^{\text{aff}}_t$-module;
2. $J_R$ is a left $\mathcal{H}^{\text{aff}}_r \otimes R\mathcal{S}_t$-module;
3. if $\mathcal{B}^{\text{aff}}_{r,i}$ is admissible, then $I = J_L + J_R$, where $I$ is the two-sided ideal of $\mathcal{B}^{\text{aff}}_{r,i}$ generated by $\mathbf{f}(x_1)$ and $g(\bar{x}_1)$ satisfying (2.5)–(2.7).

Proof. Obviously, both $J_L$ and $J_R$ are $\mathcal{S}_r \times \mathcal{S}_t$-bimodules. By Lemma 2.8(3), $x_1J_R \subseteq J_L$. Similarly, $J_Lx_1 \subseteq J_L$. This proves (1)–(2). In order to prove (3), it suffices to verify that $J_L + J_R$ is a two-sided ideal of $\mathcal{B}^{\text{aff}}_{r,i}$. If so, since $\{\mathbf{f}(x_1), g(\bar{x}_1)\} \subseteq J_L + J_R$, $I = J_L + J_R$, proving the result.

We claim that $e_1(J_L + J_R) \subseteq J_L + J_R$ and $(J_L + J_R)e_1 \subseteq J_L + J_R$. If so, by (2.3), $(\bar{x}_1 + e_1)\mathbf{f}(x_1) = \mathbf{f}(x_1)(\bar{x}_1 + e_1)$ and hence $\bar{x}_1\mathbf{f}(x_1) \in J_L + J_R$. By (1)–(2), $\bar{x}_1\mathbf{f}(x_1') = s_{1,1}\bar{x}_1\mathbf{f}(x_1)s_{1,i} \in J_L + J_R$, and hence $\bar{x}_1(J_L + J_R) \subseteq J_L + J_R$. Similarly, $(J_L + J_R)x_1 \subseteq J_L + J_R$. Thus the claim implies that $J_L + J_R$ is a two-sided ideal of $\mathcal{B}^{\text{aff}}_{r,i}$. \hfill \box
By symmetry, it remains to prove $e_1(J_L + J_R) \subseteq J_L + J_R$. Obviously, it suffices to verify

$$e_1 J_R \subseteq J_L + J_R. \tag{2.16}$$

By (2.16), $e_1 f_i(x'_i) = f(x'_i)e_1$ for $i \geq 2$. Let $m$ be a regular monomial of $\mathcal{B}_{r,t}^{\text{aff}}$ defined in (2.12). Then $m = x^\alpha e_{ij}^w x^\beta$ for some $w \in \mathcal{S}_r \times \mathcal{S}_t$, $(\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^t$ and some $i, j$. Using induction on $|\alpha|$, we want to prove

$$e_1 f(x_1)m \in J_L + J_R. \tag{2.17}$$

If so, then $e_1 f(x_1)\mathcal{B}_{r,t}^{\text{aff}} \subseteq J_L + J_R$ and hence (2.16) follows.

**Case 1**: $|\alpha| = 0$.

If $f = 0$, then (2.17) follows from (1) and (2.6). Suppose $1 \leq f \leq \min\{r, t\}$. Since $\mathcal{B}_{r,t}^{\text{aff}}$ is admissible, $e_1 f(x_1)m = 0$ if $f$ is a factor of $e_{ij}^\alpha$. Assume that $e_1$ is not a factor of $e_{ij}^\alpha$. If there is an $l$ such that $i_l = p \neq 1$ and $j_l = 1$, by (2),

$$e_1 f(x_1)e_{p,1} = s_{p,2}e_1 f(x_1)s_{1,1} = s_{p,2}e_1 f(x'_2)e_{1,1}p \in J_R.$$ 

Suppose $j_l \neq 1$ for all possible $l$. If there is an $l$ such that $e_{i_l,j_l} = e_{1,p}$ for some $p \neq 1$, then we assume $i_1 = 1$ and $j_1 = p$ without loss of any generality. In this case,

$$e_1 f(x_1)e_{1,p} = (-1)^k \bar{s}_{p,2}e_1 g(\bar{x}_1)\bar{s}_{1,1} = (-1)^k \bar{s}_{p,2}e_1 g(\bar{x}_2)\bar{s}_{1,1} = (-1)^k \bar{s}_{p,2}e_1 g(\bar{x}_1).$$

Since $j_l \neq 1$ for $1 \leq l \leq f$, by [19] Lemma 4.7(2)], $x_1 e_{i_l,j_l} = e_{i_l,j_l}x_1$ and hence

$$g(\bar{x}_1) \prod_{l=2}^{f} e_{i_l,j_l} = \prod_{l=2}^{f} e_{i_l,j_l} g(\bar{x}_1) \in J_L. \tag{2.18}$$

Now, (2.17) follows from (1). Finally, if $\{i_l, j_l\} \cap \{1\} = \emptyset$ for all possible $l$, then (2.17) follows from (1) and the following fact

$$e_1 f(x_1) \prod_{l=1}^{f} e_{i_l,j_l} = \prod_{l=1}^{f} e_{i_l,j_l} e_1 f(x_1) = (-1)^k \prod_{l=1}^{f} e_{i_l,j_l} e_1 g(\bar{x}_1) \in J_L.$$

**Case 2**: $|\alpha| > 0$.

If $a_i 
eq 0$ for some $2 \leq i \leq r$, then $e_1 x_i = x'_i e_1 - e_1 \sum_{j=1}^{i} (j, i)$ and $x_i f(x_1) = f(x_1) x_i$. Let $m'$ be obtained from $m$ by removing $x_i$. Then

$$e_1(1, i) f(x_1)m' = e_1 f(x'_i)(1, i)m' = f(x'_i) e_{1,1}m' \in J_R.$$ 

Now, (2.17) follows from inductive assumption on $|\alpha|$. If $a_i = 0$, $2 \leq i \leq r$, then $x^\alpha = x_1^a$ with $a_1 > 0$. Let $v = e_1 f(x_1)m$. If $j_{\ell} \neq 1$, $1 \leq \ell \leq f$, then by (2.18), Lemma 2.8 and inductive assumption,

$$v = e_1 f(x_1)x_1^a e_{i,j}^w x^\beta = (-1)^k e_1 g(\bar{x}_1)x_1^a e_{i,j}^w x^\beta \equiv (-1)^k e_1 x_1^a g(\bar{x}_1)e_{i,j}^w x^\beta = (-1)^k e_1 x_1^a e_{i,j}^w g(\bar{x}_1) w^\beta \in J_L w^\beta \subseteq J_L + J_R,$$

where the “$\equiv$” is modulo $J_L + J_R$. Finally, if $j_{\ell} = 1$ for some $\ell$, without loss of any generality, we assume $j_1 = 1$. If $i_1 = 1$, by Lemma 2.4, $v = e_1 f(x_1)x_1^a e_{i,j}^w x^\beta = 0$, where
Lemma 2.10. \( \vec{i} = (i_2, \ldots, i_f) \). Then, we assume \( i_1 \neq 1 \). Then
\[
v = e_1 f(x_1)x_1^{\alpha_1} e_{i_1, i} e_{\vec{j}, \vec{k}} e_{\vec{l}, \vec{m}} w x^\beta = e_1 e_{i_1, i} f(x_1)x_1^{\alpha_1} e_{\vec{j}, \vec{k}} e_{\vec{l}, \vec{m}} w x^\beta
\]
\[
= e_1 (1, i) f(x_1)x_1^{\alpha_1} e_{\vec{j}, \vec{k}} e_{\vec{l}, \vec{m}} w x^\beta = e_1 f(x_1')(1, i) x_1^{\alpha_1} e_{\vec{j}, \vec{k}} e_{\vec{l}, \vec{m}} w x^\beta,
\]
\[
= f(x_1') e_1 (1, i) x_1^{\alpha_1} e_{\vec{j}, \vec{k}} e_{\vec{l}, \vec{m}} w x^\beta \in J_R.
\]
This completes the proof of (2.17). \( \square \)

For \( (\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^l \), denote \( f(x')^\alpha = f(x_1)^{\alpha_1} \cdots f(x_r)^{\alpha_r} \) and \( g(\bar{x})^\beta = g(\bar{x}_1)^{\beta_1} \cdots g(\bar{x}_t)^{\beta_t} \).

Let \( \mathbb{N}_k = \{ \alpha \in \mathbb{N}^r \mid \alpha_i \leq k - 1, 1 \leq i \leq r \} \) and \( \mathbb{N}_k^t = \{ \alpha \in \mathbb{N}^l \mid \alpha_i \leq k - 1, 1 \leq i \leq t \} \).

Lemma 2.10. The affine walled Brauer algebra \( \mathcal{B}^{\text{aff}}_{r,t} \) is a free \( R \)-module with \( \mathcal{N} \) as its \( R \)-basis, where
\[
\mathcal{N} = \bigcup_{f=0}^{\min(m,n)} \{ f(x')^{\alpha} e^{\alpha} w x^\beta | (\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^l, (\gamma, \delta) \in \mathbb{N}_k \times \mathbb{N}_k^t, c, d \in \mathcal{D}_{r,t}, w \in \mathcal{S}_{r-f} \times \mathcal{S}_{t-f} \}.
\]

Proof. The result follows from Theorem 2.6 since the transition matrix between \( \mathcal{N} \) and \( \mathcal{M} \) in (2.12) is invertible. \( \square \)

Lemma 2.11. Let \( I \) be the two-sided ideal of \( \mathcal{B}^{\text{aff}}_{r,t} \) generated by \( f(x_1) \) and \( g(\bar{x}_1) \) satisfying (2.5)–(2.7). If \( \mathcal{B}^{\text{aff}}_{r,t} \) is admissible, then \( S \) is an \( R \)-basis of \( I \), where
\[
S = \{ f(x')^{\alpha} x^\gamma e^{\alpha} w x^\beta g(\bar{x})^\beta | (\alpha, \beta) \in \mathbb{N}^r \times \mathbb{N}^l, (\gamma, \delta) \in \mathbb{N}_k \times \mathbb{N}_k^t, \alpha_i + \beta_j \neq 0 \text{ for some } i, j \}.
\]

Proof. Let \( M = \text{span}_R S \). By Lemma 2.10, \( f(x_1) \mathcal{B}^{\text{aff}}_{r,t} \subseteq M \). For any positive integer \( l \) with \( 1 \leq l < i \), by Lemma 2.8(2),
\[
f(x_i') f(x_1') \in \sum_{j=1}^{i-1} f(x_j') \mathcal{B}^{\text{aff}}_{r,t} + f(x_i') D,
\]
such that \( D \in \mathcal{B}^{\text{aff}}_{r,t} \) and the degree of \( D \) is strictly less then \( k \). Thus, \( f(x_i') \mathcal{B}^{\text{aff}}_{r,t} \subseteq M \) which follows from inductive assumption on \( j \) with \( 1 \leq j \leq i - 1 \) and inductive assumption on degrees. This proves \( J_R \subseteq M \). One can check \( J_L \subseteq M \) similarly. By Proposition 2.9(3), \( I = M \).

By abuse of notions, a regular monomial \( m \) in Definition 2.5 is also called a regular monomial of \( \mathcal{B}_{k,r,t} \) if \( 0 \leq \alpha_i, \beta_j \leq k - 1 \) for all \( i, j \) with \( 1 \leq i \leq r \) and \( 1 \leq j \leq t \). Obviously, the number of all such regular monomials is \( k^{r+t}(r+t)! \).

Theorem 2.12. The cyclotomic walled Brauer algebra \( \mathcal{B}_{k,r,t} \) is free over \( R \) with rank \( k^{r+t}(r+t)! \) if and only if \( \mathcal{B}_{k,r,t} \) is admissible.

Proof. Let \( M \) be the \( R \)-submodule of \( \mathcal{B}_{k,r,t} \) spanned by all regular monomials of \( \mathcal{B}_{k,r,t} \). By induction on degrees, it is routine to check that \( M \) is left \( \mathcal{B}_{k,r,t} \)-module (cf. 19, Proposition 4.12) for \( \mathcal{B}^{\text{aff}}_{r,t} \). Since \( 1 \in M \), we have \( M = \mathcal{B}_{k,r,t} \). If \( \mathcal{B}_{k,r,t} \) is not admissible, by Lemma 2.2, \( e_1 \) is an \( R \)-torsion element. Since \( e_1 \in M \), either \( \mathcal{B}_{k,r,t} \) is not free or the rank of \( \mathcal{B}_{k,r,t} \) is strictly less than \( k^{r+t}(r+t)! \). If \( \mathcal{B}_{k,r,t} \) is admissible, by Lemmas 2.10, 2.11, the set of all regular monomials of \( \mathcal{B}_{k,r,t} \) is \( R \)-linear independent. Thus, \( \mathcal{B}_{k,r,t} \) is free over \( R \) with rank \( k^{r+t}(r+t)! \). \( \square \)
3. A weakly cellular basis of $\mathcal{B}_{2,r,t}$

The aim of this section is to construct a weakly cellular basis of $\mathcal{B}_{2,r,t}$ in the sense of [11]. This basis will be used to set up a relationship between $\mathfrak{gl}_{m|n}$-Kac-modules and right cell modules of $\mathcal{B}_{2,r,t}$ in section 6.

Recall that a composition of $r$ is a sequence of non–negative integers $\tau = (\tau_1, \tau_2, \ldots)$ such that $|\tau| := \sum \tau_i = r$. If $\tau_i \geq \tau_{i+1}$ for all possible $i$’s, then $\tau$ is called a partition. Similarly, a $k$-partition of $r$, or simply a multipartition of $r$, is an ordered $k$-tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ of partitions with $|\lambda| := \sum_{i=1}^{k} |\lambda^{(i)}| = r$. Let $\Lambda_k^+(r)$ be the set of all $k$-partitions of $r$. Let $\preceq$ be the dominant order defined on $\Lambda_k^+(n)$ in the sense that $\lambda \preceq \mu$ if and only if

$$
\sum_{h=1}^{\ell-1} |\lambda^{(h)}| + \sum_{j=1}^{i} \lambda_j^{(\ell)} \leq \sum_{k=1}^{\ell-1} |\mu^{(k)}| + \sum_{j=1}^{i} \mu_j^{(\ell)} \text{ for } \ell \leq k \text{ and all possible } i,
$$

(3.1)

where $|\lambda^{(0)}| = 0$. Then $\Lambda_k^+(r)$ is a poset with $\preceq$ as a partial order on it. In this paper, we always assume $k \in \{1, 2\}$.

For each $\lambda \in \Lambda_k^+(r)$, the Young diagram $[\lambda]$ is a collection of boxes arranged in left-justified rows with $\lambda_i$ boxes in the $i$-th row of $[\lambda]$. A $\lambda$-tableau $s$ is obtained by inserting elements $i$, $1 \leq i \leq r$ into $[\lambda]$ without repetition. A $\lambda$-tableau $s$ is said to be standard if the entries in $s$ increase both from left to right in each row and from top to bottom in each column. Let $\mathcal{T}^s(\lambda)$ be the set of all standard $\lambda$-tableaux. Let $t^\lambda \in \mathcal{T}^s(\lambda)$ be obtained from $[\lambda]$ by adding $1, 2, \ldots, r$ from left to right along the rows of $[\lambda]$. Let $t_\lambda \in \mathcal{T}^s(\lambda)$ be obtained from $[\lambda]$ by adding $1, 2, \ldots, r$ from top to bottom along the columns of $[\lambda]$. For example, if $\lambda = (3, 2)$, then

$$
t^\lambda = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & \end{array}, \text{ and } t_\lambda = \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 & \end{array}.
$$

(3.2)

If $\lambda \in \Lambda_k^+(r)$, then the corresponding Young diagram $[\lambda]$ is $([\lambda^{(1)}], [\lambda^{(2)}])$. In this case, a $\lambda$-tableau $s = (s_1, s_2)$ is obtained by inserting elements $i$, $1 \leq i \leq r$ into $[\lambda]$ without repetition. A $\lambda$-tableau $s$ is said to be standard if the entries in $s_i$, $1 \leq i \leq 2$ increase both from left to right in each row and from top to bottom in each column. Let $\mathcal{T}^s(\lambda)$ be the set of all standard $\lambda$-tableaux. Let $t^\lambda \in \mathcal{T}^s(\lambda)$ be obtained from $[\lambda]$ by adding $1, 2, \ldots, r$ from left to right along the rows of $[\lambda^{(1)}]$ and then $[\lambda^{(2)}]$. Let $t_\lambda \in \mathcal{T}^s(\lambda)$ be obtained from $[\lambda]$ by adding $1, 2, \ldots, r$ from top to bottom along the columns of $[\lambda^{(2)}]$ and then $[\lambda^{(1)}]$. For example, if $\lambda = ((3, 2), (3, 1)) \in \Lambda_2^+(9)$, then

$$
t^\lambda = \begin{pmatrix}
1 & 2 & 3 & 6 & 7 & 8 \\
4 & 5 & 9 & & & \\
& & & & & \\
\end{pmatrix}, \text{ and } t_\lambda = \begin{pmatrix}
5 & 7 & 9 & 1 & 3 & 4 \\
6 & 8 & 2 & & & \\
& & & & & \\
\end{pmatrix}.
$$

(3.3)

Recall that $\mathfrak{S}_r$ acts on the right of $1, 2, \ldots, r$. Then $\mathfrak{S}_r$ acts on the right of a $\lambda$-tableau $s$ by permuting its entries. For example, if $\lambda = ((3, 2), (3, 1)) \in \Lambda_2^+(9)$, and $w = s_1s_2$, then

$$
t^\lambda w = \begin{pmatrix}
3 & 1 & 2 & 6 & 7 & 8 \\
4 & 5 & 9 & & & \\
& & & & & \\
\end{pmatrix}.
$$

(3.4)

Write $d(s) = w$ for $w \in \mathfrak{S}_r$ if $t^\lambda w = s$. Then $d(s)$ is uniquely determined by $s$. Let $w_\lambda = d(t_\lambda)$. The row stabilizer $\mathfrak{S}_\lambda$ of $t^\lambda$ for $\lambda \in \Lambda_k^+(r)$ is known as the Young subgroup of
we need the following cellular basis of for any y. Following [3], we define \(\pi \) (resp., on \(S\)), where

\[ I = \sum_{i=1}^{r} \lambda_i R \]

The level two degenerate Hecke \(H_{2,r}\) with defining parameters \(u_1\) and \(u_2\) is \(H^{r\text{aff}} / I\), where \(I\) is the two-sided ideal of \(H^{r\text{aff}}\) generated by \((y_1 - u_1)(y_1 - u_2)\), \(u_1, u_2 \in R\). By definition, \(H_{2,r}\) is an \(R\)-algebra generated by \(s_i, 1 \leq i \leq r - 1\) and \(y_j, 1 \leq j \leq r\) such that

1. \(s_1s_j = s_js_1, 1 < |i - j|\),
2. \(y_iy_\ell = y_\ell y_i, 1 \leq i, \ell \leq r\),
3. \(s_iy_i - y_i y_{i+1} = 1, y_i s_i - s_i y_i = -1, 1 \leq i \leq r - 1\),
4. \(s_j s_j s_j = s_j s_j s_j, 1 \leq j \leq r - 2\),
5. \(s_i^2 = 1, 1 \leq i \leq r - 1\),
6. \((y_1 - u_1)(y_1 - u_2) = 0\).

Following [3], we define \(\pi_\lambda = \pi_a(u_2)\) and \(\bar{\pi}_\lambda = \pi_a(u_1)\) for \(\lambda \in \Lambda_2^+(r)\) with \(|\lambda^{(1)}| = a\), where for any \(u \in R\), \(\pi_0(u) = 1\) and \(\pi_a(u) = \prod_{i=1}^{a}(y_i - u)\) if \(a > 0\). Let

\[ w_a = \begin{pmatrix} 1 & 2 & \cdots & a+1 & a+2 & \cdots & r \\ r-a+1 & r-a+3 & \cdots & r-1 & 2 & \cdots & r-a \end{pmatrix}. \]  

(3.5)

It is well-known that \(w_a s_i = s_{(j)w_a} w_a\), if \(j \neq r - a\). (3.6)

Let \(G_{a,r-a}\) be the Young subgroup with respect to the composition \((a, r - a)\). Then

\[ R G_{a,r-a} w_a = w_a R G_{a,r-a}. \]  

(3.7)

For each composition \(\lambda\) of \(r\), we denote

\[ x_\lambda = \sum_{w \in G_\lambda} w, \quad y_\lambda = \sum_{w \in G_\lambda} (-1)^{\ell(w)} w, \]  

(3.8)

where \(\ell(\cdot)\) is the length function on \(G_\lambda\). Assume \(\lambda \in \Lambda_2^+(r)\) with \(|\lambda^{(1)}| = a\). If we denote \(\mu^{(i)} = (\lambda^{(i)})',\) the conjugate of \(\lambda^{(i)}\) for \(i = 1, 2\), then

\[ w_a x_{\mu^{(2)}} y_{\mu^{(1)}} = y_{\mu^{(1)}} x_{\mu^{(2)}} w_a. \]  

(3.9)

Remark 3.1. When we write \(x_{\mu^{(3)}} y_{\mu^{(1)}}\), then \(x_{\mu^{(2)}}\) (resp., \(y_{\mu^{(1)}}\)) is defined via symmetric group on \((r - a)\) letters \(\{1, 2, \ldots, r - a\}\) (resp., on a letters \(\{r - a + 1, \ldots, r\}\)). Similarly, when we write \(y_{\mu^{(1)}} x_{\mu^{(2)}}\), then \(y_{\mu^{(1)}}\) (resp., \(x_{\mu^{(2)}}\)) is defined via symmetric group on \(a\) letters \(\{1, 2, \ldots, a\}\) (resp., on \(r - a\) letters \(\{a + 1, a + 2, \ldots, r\}\)).

Definition 3.2. For any \(s, t \in T^*(\lambda)\) with \(\lambda \in \Lambda_2^+(r)\), define

1. \(\mathbf{1}_s = d(s)^{-1} \mathbf{1}_d(t)\), where \(\mathbf{1}_d = \pi_\lambda x_{\lambda^{(1)}} y_{\lambda^{(2)}}\),
2. \(\eta_s = d(s)^{-1} \eta_d(t)\), where \(\eta_d = \bar{\pi}_\lambda x_{\lambda^{(1)}} y_{\lambda^{(2)}}\),
3. \(\mathbf{1}_s = d(s)^{-1} \mathbf{1}_d(t)\), where \(\mathbf{1}_d = \pi_\lambda y_{\lambda^{(1)}} x_{\lambda^{(2)}}\),
4. \(\eta_s = d(s)^{-1} \eta_d(t)\), where \(\eta_d = \bar{\pi}_\lambda y_{\lambda^{(1)}} x_{\lambda^{(2)}}\).

It is proven in [3] that \(H_{2,r}\) is a cellular algebra over \(R\) in the sense of [12]. In this paper, we need the following cellular basis of \(H_{2,r}\) so as to construct a new weakly cellular basis of \(B_{2,r,t}\).
Lemma 3.3. The set $S_i$, $i \in \{1, 2, 3, 4\}$, are cellular bases of $\mathcal{H}_{2,r}$ in the sense of [12], where

1. $S_1 = \{ \xi_{st} | \lambda \in \Lambda_2^+(r), s, t \in T^s(\lambda) \}$,
2. $S_2 = \{ \eta_{st} | \lambda \in \Lambda_2^+(r), s, t \in T^s(\lambda) \}$,
3. $S_3 = \{ \xi_{st} | \lambda \in \Lambda_2^+(r), s, t \in T^s(\lambda) \}$,
4. $S_4 = \{ \xi_{st} | \lambda \in \Lambda_2^+(r), s, t \in T^s(\lambda) \}$.

Proof. Let $S = \{ x_{st} | s, t \in T^s(\lambda), \lambda \in \Lambda_2^+(r) \}$ and $x_{st} = d(s)^{-1} \pi_\lambda x_{\lambda(1)} x_{\lambda(2)} d(t)$. It is proven in [3] that $S$ is a cellular basis of $\mathcal{H}_{2,r}$. If we use $y_{\lambda(2)}$ instead of $x_{\lambda(2)}$ in $x_{st}$, we will get $\xi_{st}$. However, for any $s = (s_1, s_2) \in T^s(\lambda)$, $d(s)$ can be written uniquely as $d(s_1)d(s_2)d$ such that $d$ is a distinguished right coset representative of $\mathcal{S}_a \times \mathcal{S}_{r-a}$ in $\mathcal{S}_r$ and $s_1 \in T^s(\lambda(0))$, where $a = |\lambda(1)|$. So, the transition matrix between $S_1$ and $S$ is determined by the transition matrix between the cellular basis $\{ d(s_2)^{-1} x_{\lambda(2)} d(t_2) | \lambda(2) \in \Lambda^+(r-a), s_2, t_2 \in T^s(\lambda(2)) \}$ and $\{ d(s_2)^{-1} y_{\lambda(2)} d(t_2) | \lambda(2) \in \Lambda^+(r-a), s_2, t_2 \in T^s(\lambda(2)) \}$ of $R\mathcal{S}_{r-a}$. Thus, $S_1$ is a basis of $\mathcal{H}_{2,r}$. One can check that $S_1$ is a cellular basis of $\mathcal{H}_{2,r}$ in the sense of [12] by mimicking Dipper-James-Murphy’s arguments in the proof of Murphy basis for Hecke algebras of type $B$ in [8]. We leave the details to the readers. Finally, (2)–(4) can be verified similarly.

By Graham-Lehrer’s results on the representation theory of cellular algebras in [12], one can define right cell modules of $\mathcal{H}_{2,r}$ via the cellular bases $S_i$, $i \in \{1, 2, 3, 4\}$ in Lemma 3.3. The corresponding right cell modules of $\mathcal{H}_{2,r}$ with respect to $S_2$ and $S_4$ are denoted by $\tilde{\Delta}(\lambda)$, and $\tilde{\Delta}(\lambda)$.

For the simplification of discussion, we assume $\mathcal{H}_{2,r}$ is defined over $\mathbb{C}$ in Lemma 3.4.

Lemma 3.4. Suppose $a, b \in \mathbb{N}$. Then

1. $\pi_a(u_2)\mathcal{H}_{2,r}\pi_b(u_1) = 0$ whenever $a + b > r$ and $a, b \in \mathbb{Z}^{>0}$.
2. $\pi_a(u_2)\mathcal{H}_{2,r}\pi_{a-a}(u_1) = \pi_a(u_2)w_a\pi_{a-a}(u_1) \mathcal{C}\mathcal{S}_{r-a,a}$, where $\mathcal{S}_{r-a,a}$ is as in (3.3).
3. $\pi_\lambda\mathcal{H}_{2,r}\eta_\mu' = 0$ if $\lambda, \mu \in \Lambda_2^+(r)$ with $\lambda \triangleright \mu$.
4. $\pi_\lambda\mathcal{H}_{2,r}\eta_\lambda = \text{Span}_\mathbb{C}\{ \pi_\lambda w_\lambda \eta_\lambda \lambda \}$ if $\lambda \in \Lambda_2^+(r)$.
5. $\tilde{\Delta}(\lambda') \cong \pi_\lambda w_\lambda \eta_\lambda \mathcal{H}_{2,r} = 0$.

Proof. (1)–(4) can be proven by arguments similar to those for Hecke algebras of type $B$ in [7]. We only give details for (3) and (5).

If $\lambda \triangleright \mu$, then $|\lambda(1)| \geq |\mu(1)|$. If $|\lambda(1)| > |\mu(1)|$, then $|\mu(1)| \neq r$ and the result follows from (1). When $|\lambda(1)| = |\mu(1)|$, by (2) together with corresponding result for the group algebras of symmetric groups, we have $\lambda(1) \leq \mu(1)$ for $i = 1, 2$ if $\pi_\lambda\mathcal{H}_{2,r}\eta_\mu' \neq 0$. This proves (3).

There is a surjective $\mathbb{C}$-homomorphism from $\phi: \pi_\lambda\mathcal{H}_{2,r} \to \pi_\lambda w_\lambda \eta_\lambda \mathcal{H}_{2,r}$. Let $\mathcal{H}_{2,r}^{\lambda'}$ be the $\mathbb{C}$-submodule spanned by $\{ \eta_{st} | s, t \in T^s(\mu), \mu \triangleright \lambda' \}$. It follows from standard results on cellular algebras that $\mathcal{H}_{2,r}^{\lambda'}$ is a two-sided ideal of $\mathcal{H}_{2,r}$. So, $\pi_\lambda\mathcal{H}_{2,r} + \mathcal{H}_{2,r}^{\lambda'}/\mathcal{H}_{2,r}^{\lambda'}$ is isomorphic to a submodule of $\tilde{\Delta}(\lambda')$. If $\eta_{st} \in \mathcal{H}_{2,r}^{\lambda'}$, we have $\mu \triangleright \lambda'$ which is equivalent to $\lambda \triangleright \lambda'$. By (3), $\pi_\lambda w_\lambda \eta_{st} = 0$ and $\mathcal{H}_{2,r}^{\lambda'} \subset \ker \phi$. So, there is an epimorphism from $\pi_\lambda\mathcal{H}_{2,r} + \mathcal{H}_{2,r}^{\lambda'}/\mathcal{H}_{2,r}^{\lambda'}$ to $\pi_\lambda w_\lambda \eta_\lambda \mathcal{H}_{2,r}$. Mimicking arguments on classical Specht modules for Hecke algebra of type $B$ in [7], we know that $\pi_\lambda w_\lambda \eta_\lambda \mathcal{H}_{2,r}$ has a basis $\{ \pi_\lambda w_\lambda \eta_\lambda d(t) | t \in T^s(\lambda') \}$. So,

$$\dim_\mathbb{C} \tilde{\Delta}(\lambda') = \dim_\mathbb{C} \pi_\lambda w_\lambda \eta_\lambda \mathcal{H}_{2,r} = \# T^s(\lambda'),$$
forcing $\eta_{\lambda}H_{2,r} + H_{2,r}^\oplus / H_{2,r}^\oplus \cong \prod_{\lambda \in \Lambda_2} \eta_{\lambda}H_{2,r} \cong \tilde{\Delta}(\lambda')$. □

Now, we use cellular bases $S_t$ of $H_{2,r}$ in Lemma 3.3 to construct a weakly cellular basis of $B_{2,r,t}$ over an arbitrary field in the sense of [11]. We remark that when we use results on level two degenerate Hecke algebra for $B_{2,r,t}$, we should keep in mind that $x_1, \bar{x}_1 \in B_{2,r,t}$ should be regarded as $-y_1 \in H_{2,r}$ and $H_{2,t}$, respectively. Therefore, we have to use $-u_i$ and $-\bar{u}_i$ instead of $u_i$ and $\bar{u}_i$.

Fix $r, t, f \in \mathbb{Z}^+\!\!\!\!\!\!^0$ with $f \leq \min\{r, t\}$. In contrast to (2.11), we define

$$D_{r,t}^f = \{ s_{t-f+1,i_r-f+1} \cdots s_r i \mid r \geq i_r > \cdots > i_{t-f+1}, i_k \geq k + f - t \}. \quad (3.10)$$

For each $c \in D_{r,t}^f$ as in (3.10), let $\kappa_c$ be the $r$-tuple

$$\kappa_c = (k_1, \ldots, k_r) \in \{0, 1\}^r$$

such that $k_i = 0$ unless $i = i_r, i_{r-1}, \ldots, i_{r-f+1}$. (3.11)

Note that $\kappa_c$ may have more than one choice for a fixed $c$, and it may be equal to $\kappa_d$ although $c \neq d$ for $c, d \in D_{r,t}^f$. Let $N_f = \{f_c \mid c \in D_{r,t}^f\}$. If $\kappa_c \in N_f$, define $x^{\kappa_c} = \prod_{i=1}^r x_i^{k_i}$. In [19], we consider poset $(\Lambda_{2,r,t}, \triangleright)$, where

$$\Lambda_{2,r,t} = \{(f, \lambda, \mu) \mid (\lambda, \mu) \in \Lambda_2^+ (r-f) \times \Lambda_2^+ (t-f), 0 \leq f \leq \min\{r, t\}\}, \quad (3.12)$$

such that $(f, \lambda, \mu) \triangleright (f', \lambda', \mu')$ for $(f, \lambda, \mu), (f', \lambda', \mu') \in \Lambda_{2,r,t}$ if either $f > f'$ or $f = f'$ and $\lambda \triangleright_1 \alpha$, and $\mu \triangleright_2 \beta$, and in case $f = f'$, the orders $\triangleright_1$ and $\triangleright_2$ are dominant orders on $\Lambda_2^+ (r-f)$ and $\Lambda_2^+ (t-f)$ respectively. For each $(f, \mu, \nu) \in \Lambda_{2,r,t}$, let

$$\delta(f, \mu, \nu) = \{(t, c, \kappa_c) \mid t = (t^{(1)}, t^{(2)}) \in T^s(\mu) \times T^s(\nu), c \in D_{r,t}^f \text{ and } \kappa_c \in N_f\}. \quad (3.13)$$

**Definition 3.5.** For any $(s, d, \kappa_d), (t, c, \kappa_c) \in \delta(f, \mu, \nu)$ with $(f, \mu, \nu) \in \Lambda_{2,r,t}$, define

$$C_{(s, d, \kappa_d)(t, c, \kappa_c)} = x^{\kappa_d} d^{-1} e^f_{n_{st}} c^{\kappa_e}, \quad (3.14)$$

where, in contrast to notation $e^f$ in (2.10), we define $e^f = e_{r,t} e_{r-1, t-1} \cdots e_{r-f+1, t-f+1}$ if $f \geq 1$ and $e^0 = 1$, and $n_{st} = n_{s(1)t(1)} n_{s(2)t(2)}$ if $s = (s^{(1)}, s^{(2)})$ and $t = (t^{(1)}, t^{(2)})$ are in $T^s(\mu) \times T^s(\nu)$.

Note that $n_{st}$ in Definition 5.5 are defined via cellular basis elements of $H_{2,r-f}$ and $H_{2,t-f}$ in Lemma 3.3(2) (4). Since $x_i$ and $\bar{x}_j$ do not commute each other, a cellular basis element of $H_{2,r-f}$ is always put on the left. Further, we need to use $x_i, -u_i, -\bar{u}_i$ (resp. $\bar{x}_i, -\bar{u}_i, -u_i$) instead of $-y_i, u_i, \bar{u}_i$ in Lemma 3.3.

**Theorem 3.6.** If $B_{2,r,t}$ is admissible, then the set

$$C = \{C_{(s, d, \kappa_d)(t, c, \kappa_c)} \mid (s, d, \kappa_d), (t, c, \kappa_c) \in \delta(f, \mu, \nu), \forall (f, \lambda) \in \Lambda_{2,r,t}\}$$

is a weakly cellular basis $B_{2,r,t}$ over $R$ in the sense of [11].

**Proof.** Let $S$ be the cellular basis of $H_{2,r-f}$ (resp. $H_{2,t-f}$) for $0 \leq f \leq \min\{r, t\}$ defined in the proof of Lemma 3.3. If we use $S$ instead of the cellular basis $S_2$ of $H_{2,r-f}$ and $S_4$ of $H_{2,t-f}$ in Lemma 3.3, we will obtain the weakly cellular basis of $B_{2,r,t}$ over $R$ in [19] Theorem 6.12] provided that $R = \mathbb{C}$ and $u_1 = -p, u_2 = m - q, \bar{u}_1 = q$ and $\bar{u}_2 = p - n$ with $r + t \leq \min\{m, n\}$. Since $B_{2,r,t}$ is admissible, by Theorem 2.12 the rank of $B_{2,r,t}$ is $2^{r+t}(k+t)!$. As pointed in [19] Remark 6.13, [19] Theorem 6.12 holds over $R$ with arbitrary
parameters $u_1, u_2, \bar{u}_1, \bar{u}_2$ if the rank of $B_{2,r,t}$ is $2^{r+t}(r+t)!$. Thus, $\mathcal{C}$ is an $R$-basis of $B_{2,r,t}$. Further, the weakly cellularity of $B_{2,r,t}$ depends only on cellular bases of $\mathcal{H}_{2,r-f}$ and $\mathcal{H}_{2,t-f}$ and does not depend on the explicit descriptions of cellular bases of $\mathcal{H}_{2,r-f}$ and $\mathcal{H}_{2,t-f}$. (cf. the proof of [19, Theorem 6.12]). So, all arguments for the proof of [19, Theorem 6.12] can be used smoothly to prove that $\mathcal{C}$ is a weakly cellular basis $B_{2,r,t}$ over $R$. □

Suppose $B_{2,r,t}$ is defined over a field $F$. By Theorem 3.6, one can define right cell modules $C(f, \mu, \nu)$ with respect to $(f, \mu, \nu) \in \Lambda_{2,r,t}$ for $B_{2,r,t}$. Let $\phi_{f,\mu,\nu}$ be the corresponding invariant form on $C(f, \mu, \nu)$ and let $D^{f,\mu,\nu} = C(f, \mu, \nu)/\text{Rad} \phi_{f,\mu,\nu}$, where $\text{Rad} \phi_{f,\mu,\nu}$ is the radical of $\phi_{f,\mu,\nu}$. By Graham-Lehrer’s results in [12] (a weakly cellular algebra has similar representation theory of a cellular algebra in [12]), $D^{f,\mu,\nu}$ is either 0 or irreducible and all non-zero $D^{f,\mu,\nu}$ consist of a complete set of pair-wise non-isomorphic irreducible $B_{2,r,t}$-modules. Let $\tilde{\Delta}(\mu)$ (resp. $\Delta(\nu)$) be the cell module of $\mathcal{H}_{2,r-f}$ (resp. $\mathcal{H}_{2,t-f}$) defined via $S_2$ and $S_1$ in Lemma 3.3. Similarly, one has the notations $D^\mu$ and $D^\nu$, respectively.

**Proposition 3.7.** Suppose that $B_{2,r,t}$ is admissible over $F$. For any $(f, \mu, \nu) \in \Lambda_{2,r,t}$, $D^{f,\mu,\nu} \neq 0$ if and only if

1. $D^\mu \neq 0$ and $D^\nu \neq 0$,
2. $f \neq r$ provided $r = t$ and $\omega_0 = \omega_1 = 0$.

**Proof.** The result can be proven by arguments similar to those for Lemmas 7.3–7.4 in [19]. □

**Remark 3.8.** By arguments similar to those for Theorem 3.6, one can lift cellular bases of $\mathcal{H}_{k,r}$ and $\mathcal{H}_{k,t}$ in [2] to obtain a weakly cellular basis of $B_{k,r,t}$ over $R$, provide that $B_{k,r,t}$ is admissible. Further, it is not difficult to prove a result which is similar to Proposition 3.7 for $B_{k,r,t}$ over an arbitrary field $F$ with characteristic $\text{char} F$ either zero or positive. Let $\mathbf{u} = (u_1, \ldots, u_k) \in F^k$ such that $u_i = d_i \cdot 1_F$ and $0 \leq d_i < \text{char} F$ for $1 \leq i \leq k$. Kleshchev [15] has shown that the simple $\mathcal{H}_{k,n}(\mathbf{u})$-modules are labeled by a set of multipartitions which gives the same Kashiwara crystal as the set of $\mathbf{u}$-Kleshchev multipartitions of $n$ in [11]. Thus, the simple $B_{k,r,t}$-modules are labeled by the set $\{(f, \mu, \nu)\}$, where (1) $0 \leq f \leq \min\{r, t\}$, (2) $\mu$’s are Kleshchev multipartitions of $r-f$ with respect to $\mathbf{u}$, (3) $\nu$’s are Kleshchev multipartitions of $t-f$ with respect to $\mathbf{u} := (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k)$, (4) $f \neq r$ if $r = t$ and $\omega_i = 0$ for $0 \leq i \leq k-1$. It is pointed in [3] that one can modify the proof of [9, Theorem 1.1], or [11, Theorem 1.3], to show that when $B_{k,r,t}$ is admissible, the simple $B_{k,r,t}$-modules are always labeled by the $(f, \mu, \nu) \in \Lambda_{k,r,t}$ with $0 \leq f \leq \min\{r, t\}$ and $\mu$ (resp. $\nu$) are Kleshchev multipartitions with respect to $\mathbf{u}$ (resp. $\bar{\mathbf{u}}$) and $f \neq r$ if $r = t$ and $\omega_i = 0$ for $1 \leq i \leq r$. However, we are not claiming that $D^{(f,\mu,\nu)}$ is not 0 for the multipartitions $\mu, \nu$ which Kleshchev [15] uses to label the simple $\mathcal{H}_{k,r-f}(\mathbf{u})$-modules (resp. $\mathcal{H}_{k,t-f}(\bar{\mathbf{u}})$-modules).

We recall the definition of Kleshchev bipartitions over $\mathbb{C}$ as follows (see e.g., [25]), which will be used in sections 5–6. Fix $u_1, u_2 \in \mathbb{C}$ with $u_1 - u_2 \in \mathbb{N}$. Then $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2^+(r)$ is called a **Kleshchev bipartition** [25] with respect to $u_1, u_2$ if

$$
\lambda^{(1)}_{u_1-u_2+i} \leq \lambda^{(2)}_i \quad \text{for all possible } i.
$$

(3.15)
The Cartan subalgebra $h_0$ of $h$

Denote $\xi_{i, j} \in \rho$ where \[
\begin{align*}
\ll(\xi_{i, j}^\rho = (0, 1) \in N | i, j \in \mathbb{N}) \quad \text{and} \quad \ell = \# \{(i, j) \in \mathbb{N}^2 | \xi_{i, j}^\rho + \xi_{j, i}^\rho = 0, 1 \leq i < m, 1 \leq j \leq n \}. 
\end{align*}
\]
Then $\xi$ is called an $\ell$-fold atypical weight if $\ell > 0$. Otherwise, $\xi$ is called a typical weight.

**Example 4.1.** For any $p, q \in \mathbb{C}$, let $\lambda_{pq} = (p, ..., p | -q, ..., -q)$. Then $\lambda_{pq}$ is a typical weight if and only if

$$p - q \notin \mathbb{Z} \quad \text{or} \quad p - q \leq -m \quad \text{or} \quad p - q \geq n.$$  

(4.4)
The current \( q \) should be regarded as \( q + m \) in [5 IV]. In the remaining part of this paper, \( \lambda_{pq} \) is always a typical weight in the sense of (4.4).

Let \( V = \mathbb{C}^{m|n} \) be the natural \( \mathfrak{g} \)-module with natural basis \( \{ v_i | i \in I \} \) such that \( v_i \) has parity \( [v_i] = [i] \). Then the dual space \( V^* \), which has the dual basis \( \{ \bar{v}_i | i \in I \} \), is a left \( \mathfrak{g} \)-module such that

\[
E_{ab} \bar{v}_i = -(-1)^{|a||[a]+[b]|} \delta_{ia} \bar{v}_b \quad \text{for any } (a, b) \in I \times I.
\]

(4.5)

In particular, the weight of \( \bar{v}_i \) is \( -\epsilon_i \). For the simplicity of notation, we set \( W = V^* \).

**Definition 4.2.** Fix \( r, t \in \mathbb{Z}^+ \). Let \( V_{rt} = V \otimes W^t \) and \( M_{pq}^{rt} = V \otimes K_{\lambda_{pq}} \otimes W^t \), where \( K_{\lambda_{pq}} \) is the \( \text{Kac-module} \) [14] with respect to the highest weight \( \lambda_{pq} \) in Example 4.1.

Let \( \pi : M_{pq}^{rt} \rightarrow V_{rt} \) be the projection such that, for any \( v \in M_{pq}^{rt} \), \( \pi(v) \) is the vector obtained from \( v \) by deleting the tensor factor in \( K_{\lambda_{pq}} \). Let \( v_{pq} \) be the highest weight vector of \( K_{\lambda_{pq}} \) with highest weight \( \lambda_{pq} \). Then \( v_{pq} \) is unique up to a scalar. It is well-known (e.g. see [5]) that \( K_{\lambda_{pq}} \) is \( 2^{mn} \)-dimensional with a basis

\[
B = \left\{ b^\sigma := \prod_{i=1}^{m} \prod_{j=1}^{n} P_{m+i,j}^{\sigma_{ij}} v_{pq} | \sigma = (\sigma_{ij})_{i,j=1}^{n,m} \in \{0, 1\}^{n \times m} \right\},
\]

(4.6)

where the products are taken in any fixed order. Define

\[
I(m|n, r) = \{ i | i = (i_r, i_{r-1}, \ldots, i_1), i_j \in I, 1 \leq j \leq r \},
\]

\[
\bar{I}(m|n, t) = \{ j | j = (j_t, j_{t-1}, \ldots, j_1), j_i \in I, 1 \leq i \leq t \}.
\]

(4.7)

If \( (i, b, j) \in I(m|n, r) \times B \times \bar{I}(m|n, t) \), we define

\[
v_{i, b, j} = v_i \otimes v_{i-1} \otimes \cdots \otimes v_1 \otimes b \otimes \bar{v}_{j_1} \otimes \bar{v}_{j_2} \otimes \cdots \otimes \bar{v}_{j_t} \in M_{pq}^{rt}.
\]

(4.8)

**Lemma 4.3.** Let \( B_M = \{ v_i \otimes b \otimes \bar{v}_j | (i, b, j) \in I(m|n, r) \times B \times \bar{I}(m|n, t) \} \). Then \( B_M \) is a basis of \( M_{pq}^{rt} \).

Denote by \( U(\mathfrak{g}) \) the universal enveloping algebra of \( \mathfrak{g} \). Then \( M_{pq}^{rt} \) is a left \( U(\mathfrak{g}) \)-module. Let \( J = J_1 \cup \{0\} \cup J_2 \) with \( J_1 = \{ r, \ldots, 2, 1 \} \) and \( J_2 = \{1, \bar{2}, \ldots, \bar{t}\} \). Then \( (J, \prec) \) is a total ordered set with

\[
r \prec r - 1 \prec \cdots \prec 1 \prec 0 \prec \bar{1} \prec \cdots \prec \bar{t}.
\]

For any \( a, b \in J \) with \( a \prec b \), define \( \pi_{ab} : U(\mathfrak{g})^{\otimes 2} \rightarrow U(\mathfrak{g})^{\otimes (r+t+1)} \) by

\[
\pi_{ab}(x \otimes y) = 1 \otimes \cdots \otimes 1 \otimes^{a-\text{th}} x \otimes 1 \otimes \cdots \otimes^{b-\text{th}} y \otimes 1 \otimes \cdots \otimes 1.
\]

(4.9)

Let \( \Omega \) be a Casimir element in \( \mathfrak{g}^{\otimes 2} \) given by

\[
\Omega = \sum_{i, j \in I} (-1)^{|j|} E_{ij} \otimes E_{ji}.
\]

(4.10)

In [19], we define operators \( s_i, \bar{s}_j, x_1, \bar{x}_1 \) and \( e_1 \) acting on the right of \( M_{pq}^{rt} \) via the following formulae:

\[
s_i = \pi_{i+1,i}(\Omega)|_{M_{pq}^{rt}} (1 \leq i < r), \quad \bar{s}_j = \pi_{j,j+1}(\Omega)|_{M_{pq}^{rt}} (1 \leq j < t),
\]

\[
x_1 = -\pi_{10}(\Omega)|_{M_{pq}^{rt}}, \quad \bar{x}_1 = -\pi_{01}(\Omega)|_{M_{pq}^{rt}}, \quad e_1 = -\pi_{11}(\Omega)|_{M_{pq}^{rt}}.
\]

(4.11)
Then there is an algebra homomorphism $\phi : \mathcal{B}_{2,r,t} \to \text{End}_U(\mathcal{M}_{pq}^r)$ sending the generators $s_i, s_j, x_1, \tilde{x}_1$ and $e_1$ to the operators $s_i, s_j, x_1, \tilde{x}_1$ and $e_1$ as above \cite{19}. In this case, we need to use $-p, m - q$, and $q, p - n$ instead of $u_1, u_2, \bar{u}_1$ and $\bar{u}_2$ respectively in Definition 2.1 for $k = 2$. Further, $\omega_0 = m - n$, $\omega_1 = nq - mp$ and $\omega_a = (m - p - q)\omega_{a-1} - p(q - m)\omega_{a-2}$ for $a \geq 2$ and $\omega_a$'s are determined by \cite{19} Corollary 4.3. Thus, $\mathcal{B}_{2,r,t}$ is admissible in the sense of Definition 2.3. By Theorem 2.12, $\dim_{\mathbb{C}} \mathcal{B}_{2,r,t} = 2^{r+t}(r+t)!$. We will always consider $\mathcal{B}_{2,r,t}$ as above in the remaining part of this paper.

**Theorem 4.4.** \cite{19} Theorem 5.16] Fix $r, t \in \mathbb{Z}^+ \in \mathbb{Z}^+$ with $r + t \leq \min\{m, n\}$. Then $\text{End}_\mathfrak{g}(\mathcal{M}_{pq}^r) \cong \mathcal{B}_{2,r,t}$.

**Theorem 4.5.** \cite{5} IV, Theorem 3.13] If $0 < r \leq \min\{m, n\}$, then $\text{End}_U(\mathcal{M}_{pq}^r) \cong \mathcal{H}_{2,r}$, the level two Hecke algebra with defining parameters $u_1 = -p$ and $u_2 = m - q$.

**Theorem 4.6.** (Super Schur-Weyl duality) Keep the condition \cite{4.4}. The algebra homomorphism $\phi : \mathcal{B}_{2,r,t} \to \text{End}_\mathfrak{g}(\mathcal{M}_{pq}^r)$ is surjective. It is injective if and only $r + t \leq \min\{m, n\}$.

**Proof.** By Theorem 4.4 it suffices to prove that $\phi_1$ is surjective and is not injective if $r + t > \min\{m, n\}$. As in \cite{6} (7.16), the map $\text{flip}_{r,t}$ defined by the following commutative diagram is a $\mathfrak{g}$-module isomorphism

$$
\begin{array}{c}
\text{End}_\mathbb{C}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes (V^*)^{\otimes t}) & \xrightarrow{\text{flip}_{r,t}} & \text{End}_\mathbb{C}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes V^{\otimes t}) \\
\end{array} (4.12)
$$

By Theorem 2.12 for $k = 2$, $\dim_{\mathbb{C}} \mathcal{B}_{2,r,t} = 2^{r+t}(r+t)!$. This implies that the top map is a bijection, and the bottom map is a $\mathfrak{g}$-module isomorphism, which induces an isomorphism between two subspaces $\text{End}_\mathfrak{g}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes (V^*)^{\otimes t})$ and $\text{End}_\mathfrak{g}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes V^{\otimes t})$. Since $\pi_1$ is surjectively mapped to $\text{End}_\mathfrak{g}(V^{\otimes r} \otimes K_{\lambda_{pq}} \otimes (V^*)^{\otimes t})$ by \cite{5} IV, Theorem 3.21, we see that $\phi_1 : \mathcal{B}_{2,r,t} \to \text{End}_\mathfrak{g}(\mathcal{M}_{pq}^r)$ is surjective. Finally, the second assertion follows from the corresponding result for $t = 0$ in \cite{5} IV, Theorem 3.21.

5. Highest weight vectors in $V^{\otimes r} \otimes K_{\lambda_{pq}}$

The aim of this section is to give a classification of highest weight vectors of $M_{pq}^r := V^{\otimes r} \otimes K_{\lambda_{pq}}$ when $r \leq \min\{m, n\}$, where $V$ is the natural representation of $\mathfrak{g} := \mathfrak{gl}_{m,n}$ and $K_{\lambda_{pq}}$ is the Kac-module with highest weight $\lambda_{pq}$ in Example 4.1. This will be done in a few steps. First, by noting that $\mathfrak{g}$-highest weight vectors of $M_{pq}^r$ is in one to one correspondence with the $\mathfrak{g}_0$-highest weight vectors of $V^{\otimes r}$ (cf. \cite{21} Lemmas 5.1-5.2), we are able to
reduce the problem to the Lie algebra case. Secondly, since \( \mathfrak{g}_0 = \mathfrak{gl}_m \oplus \mathfrak{gl}_n \), and \( V^{\otimes r} \) can be decomposed as a direct sum of tensor products of natural representations of \( \mathfrak{gl}_m \) and \( \mathfrak{gl}_n \), we are able to further simplify the problem to the \( \mathfrak{gl}_m \) case.

To begin with, we briefly recall the results on a classification of \( \mathfrak{gl}_m \)-highest weight vectors of \( V^{\otimes r} \), where \( V \) temporarily denotes the natural representation of \( \mathfrak{gl}_m \) over \( \mathbb{C} \). Let \( \{v_i \mid 1 \leq i \leq m\} \) be a basis of \( V \). Obviously, \( V^{\otimes r} \) has a basis \( \{v_i \mid i \in I(m|0, r)\} \), where

\[
v_i = v_{ir} \otimes v_{ir-1} \otimes \cdots \otimes v_{i1}.
\]

We consider a Casimir element \( \Omega \) in \( \mathfrak{gl}_m^{\otimes 2} \) with

\[
\Omega = \sum_{1 \leq i < j \leq m} E_{ij} \otimes E_{ji} \in \mathfrak{gl}_m^{\otimes 2},
\]

which is a special case of (4.10). Define \( s_i = \pi_{i,i+1}(\Omega) \), \( 1 \leq i \leq r - 1 \). Then \( (i, i+1) \in \mathcal{S}_r \) acts on \( V^{\otimes r} \) via \( s_i \). Thus, \( V^{\otimes r} \) is a \( (\mathfrak{gl}_m, \mathbb{C}\mathcal{S}_r) \)-bimodule such that

\[
v_{iw} = v_{i(r+1)w-1} \otimes v_{i(r)w-1} \otimes \cdots \otimes v_{i(1)w-1} \text{ for any } w \in \mathcal{S}_r.
\]

For example, \( v_{i3} \otimes v_{i2} \otimes v_{i1} \cdot s_1 s_2 = v_{i1} \otimes v_{i3} \otimes v_{i2} \). If \( r \leq m \), it is well-known that

\[
\text{End}_{U(\mathfrak{gl}_m)}(V^{\otimes r}) \cong \mathbb{C}\mathcal{S}_r.
\]

**Definition 5.1.** If \( \lambda \in \Lambda^+(r, m) \), the set of partitions of \( r \) with at most \( m \) parts, we define \( v_{\lambda} = v_{i_\lambda} \in V^{\otimes r} \), where \( i_\lambda = (1^{\lambda_1}, 2^{\lambda_2}, \ldots, m^{\lambda_m}) \) and \( k^{\lambda_k} \) denotes the sequence \( k, k, \ldots, k \) with multiplicity \( \lambda_k \).

The following result is well-known, and Lemma 5.3 follows from Lemma 5.2.

**Lemma 5.2.** Suppose \( \lambda \) and \( \mu \) are two compositions of \( r \) and \( \mu' \) is the conjugate of \( \mu \), and \( x_\lambda, y_\mu \) are defined in (3.8). Then \( x_\lambda \mathbb{C}\mathcal{S}_r y_{\mu'} = 0 \) unless \( \lambda \subseteq \mu \).

**Lemma 5.3.** There is a bijection between the set of dominant weights of \( V^{\otimes r} \) and \( \Lambda^+(r, m) \), the set of partitions of \( r \) with at most \( m \) parts. Further, the \( \mathbb{C} \)-space of \( \mathfrak{gl}_m \)-highest weight vectors with highest weight \( \lambda \) has a basis \( \{v_{\lambda}\mathbb{C}\mathcal{S}_r d(t) \mid t \in T^s(\lambda')\} \).

Now, we turn to construct \( \mathfrak{g} \)-highest weight vectors of \( M_{pq}^{r0} \). Since \( r \leq \min\{m, n\} \), there is a bijection between the set of dominant weights of \( M_{pq}^{r0} \) and \( \Lambda_2^+(r) \). Further, if \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2^+(r) \), the corresponding dominant weight of \( M_{pq}^{r0} \) is

\[
\tilde{\lambda} := \lambda_{pq} + \tilde{\lambda},
\]

where

\[
\tilde{\lambda} = (\lambda_1^{(1)}, \ldots, \lambda_m^{(1)} | \lambda_1^{(2)}, \ldots, \lambda_n^{(2)}).
\]

For instance, if \( \lambda = ((3, 1), (2, 1)) \), then \( \tilde{\lambda} = (3, 1, 0, \cdots, 0 | 2, 1, 0, \cdots, 0) \). Recall that \( \Omega \) is a Casimir element in \( \mathfrak{g}^{\otimes 2} \) given in (4.10). Define operators \( s_i, x_1 \) acting on the right of \( M_{pq}^{r0} \) via the following formulae: \( s_i = \pi_{i,i+1}(\Omega) \), \( 1 \leq i \leq r - 1 \) and \( x_1 = -\pi_{10}(\Omega) \). In this case, \( u_1 = -p \) and \( u_2 = m - q \). We remind that Brundan-Stroppel [5] defined \( x_1 \) via \( \pi_{10}(\Omega) \). So, the current \( x_1 \) is \( -x_1 \) in [5]. Recall that \( v_i \otimes v_{pq} = v_i \otimes \cdots \otimes v_{i2} \otimes v_{i1} \otimes v_{pq} \) for any \( i \in I(m|n, r) \) (cf. (4.7)), and \( x'_k = x_k + L_k \) with \( L_k = \sum_{i=1}^{k-1} (i, k) \) (see Lemma 2.7).
Lemma 5.4. [3, Lemma 3.1] Suppose $i \in I(m|n, r)$, and $1 \leq k \leq r$.

1. $v_i \otimes v_{pq}x_k^r = -pv_{pq} \otimes v_i$ if $1 \leq i_k \leq m$.
2. $v_j \otimes v_{pq}x_k^r = -qv_{pq} \otimes v_j + \sum_{j=1}^{m} (-1)^{\Lambda} v_j \otimes (E_{j, k}v_{pq})$ if $m + 1 \leq i_k \leq m + n$, where $j \in I(m|n, r)$ which is obtained from $i$ by using $j$ instead of $i_k$ in $i$. In particular, the weight of $v_j$ is strictly bigger than that of $v_i$.

Definition 5.5. For $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda^+_r(r)$, define $v_\lambda = v_i$ with $i = (i_{\lambda^{(1)}}, i_{\lambda^{(2)}}) \in I(m|n, r)$.

For instance, $v_\lambda = v_i$ if $\lambda = ((3, 1), (2, 1))$, where $i = (1^3, 2, (m + 1)^2, m + 2)$.

Definition 5.6. For any $t \in T^s(\lambda')$, we define $v_t = v_\lambda \otimes v_{pq} w_{\lambda'} d(t)$, where $\eta_{\lambda'}$ is given in Definition 5.2(2).

Theorem 5.7. Suppose $r \leq \min\{m, n\}$. There is a bijection between the set of dominant weights of $M_{pq}^0$ and $\Lambda^+_r(r)$. Further, the $C$-space $V_\lambda$ of $\mathfrak{g}$-highest weight vectors of $M_{pq}^0$ with highest weight $\lambda$ has a basis $\{v_t \mid t \in T^s(\lambda')\}$.

Proof. The required bijection between $\Lambda^+_r(r)$ and the set of dominant weights of $M_{pq}^0$ is the map sending $\lambda$ to $\tilde{\lambda}$ defined in (5.3). We claim that each $v_t$ is killed by $E_{m, m+1}$ and $E_{i, j}$ with $i < j$ and either $i, j \in I_0$ or $i, j \in I_1$. Since $M_{pq}^0$ is $(\mathfrak{g}, \mathcal{H}_r)$-bimodule, we need only consider the case $d(t) = 1$. In this case, $t = t^\prime$.

Denote $|\lambda^{(i)}| = a$. Recall that $w_{\lambda^{(1)}} \in \mathfrak{g}_a$ and $w_{\lambda^{(2)}} \in \mathfrak{g}_{r-a}$ such that $t^{\lambda^{(i)}} w_{\lambda^{(i)}} = t^{\lambda^{(i)}}$ for $i = 1, 2$. Then

$$w_\lambda = w_{\lambda^{(1)}} w_{\lambda^{(2)}} w_a = w_a w_{\lambda^{(2)}} w_{\lambda^{(1)}}.$$ (5.5)

By (3.6) and (5.5),

$$v_t = v_\lambda \otimes v_{pq} w_{\lambda^{(1)}} w_{\lambda^{(2)}} y_{\mu^{(1)}} x_{\mu^{(2)}} w_a \pi_{r-a}(-p),$$

where $\mu^{(i)}$ is the conjugate of $\lambda^{(i)}$ for $i = 1, 2$. By Lemmas 5.2, 5.3, $v_t$ is killed by $E_{i, j}$ with $i < j$ and either $i, j \in I_0$ or $i, j \in I_1$. Since $E_{m, m+1}$ acts on $M_{pq}^0$ via $\sum_{i=1}^{r+1} \otimes_{-1} E_{m, m+1} \otimes 1^\otimes r-1$, we have $E_{m, m+1} v_\lambda \otimes v_{pq} = 0$ if $v_{m+1}$ does not occur in $v_\lambda$. Otherwise, $\lambda^{(2)} \neq \emptyset$ and $r - a \neq 0$. In this case, up to a sign, $E_{m, m+1} v_\lambda \otimes v_{pq}$ is equal to

$$v_j \otimes v_{pq} (1 - s_{a+1} + s_{a+1, a+3} + \cdots + (-1)^{b-a} s_{a+1, b+1}),$$

where $b = a + \lambda^{(2)} - 1$ and $v_j$ is obtained from $v_\lambda$ by replacing $v_{m+1}$ by $v_m$ at the $(a + 1)$-th position. Thus, $j_{a+1} = m$. Let

$$h = (1 - s_{a+1} + s_{a+1, a+3} + \cdots + (-1)^{b-a} s_{a+1, b+1}) w_{\lambda^{(1)}} w_{\lambda^{(2)}} y_{\mu^{(1)}} x_{\mu^{(2)}}.$$ Then $h \in C \mathfrak{g}_a \otimes C \mathfrak{g}_{r-a}$. By (3.6), $hw_a = w_a h_1$ for some $h_1 \in C \mathfrak{g}_{r-a} \otimes C \mathfrak{g}_a$. Since $h_1 \pi_{r-a}(-p) = \pi_{r-a}(-p) h_1$, it is enough to prove $v_j \otimes v_{pq} w_a \pi_{r-a}(-p) = 0$. Up to a sign, $v_j \otimes v_{pq} w_a = v_k \otimes v_{pq}$ for some $k$ such that $v_k = v_m \in V_0$. Since $r - a \neq 0$, $x_1 + p$ is a factor of $\pi_{r-a}(-p)$. By Lemma 5.4(1), $v_j \otimes v_{pq} w_a \pi_{r-a}(-p) = 0$. Thus, $v_t$ is a highest weight vector of $M_{pq}^0$ if $v_t \neq 0$.

Note that any vector of $M_{pq}^0$ can be written as $v = \sum_{b \in B} v_b \otimes b$, where $B$ is a basis of $K_{pq}$ defined in (4.6) and $v_b \in V^\otimes r$. Following [5], $v_b$ is called the $b$-component of $v$. By Lemma 5.4(2) (or the arguments in the proof of [6, Corollary 3.3]), the $v_{pq}$-component of
where $v_\lambda \otimes v_{pq} w_{a} \pi_{r-a}(-p)$ is an element in $M_{pq}$, which acts on $v_\lambda \otimes v_{pq} w_{(2)} x_{\mu(2)}$ as scalar $\prod_{i=1}^{r-a} (p - q - l_i)$, where $\mu = \lambda'$ and $\mu_{(2)(i)}$ is 1 if $i$ is in the $l$-th row and $j$-th column of $t$. Since $\lambda_{pq}$ is typical (cf. (4.4)), and $r \leq \min\{m, n\}$, $\prod_{i=1}^{r-a} (p - q - \mu_{(2)(i)}) \neq 0$. So,

the $v_{pq}$-component of $v_1 = v_\lambda w_{a} w_{(2)} x_{\mu(2)} y_{\mu(1)} d(t)$ (up to a non-zero scalar). (5.6)

By Lemma [5.3], it is a $g_{0, \lambda}$-highest vector of $V^\otimes r$ with highest weight $\lambda$ (cf. (5.4)), forcing $v_1 \neq 0$.

Now, we prove that $\{v_t \mid t \in T^s(\lambda')\}$ is $C$-linear independent. First, consider $V = V_0 \oplus V_1$ as a module for $g_0 = \mathfrak{gl}_m \oplus \mathfrak{gl}_n$. Then $V^\otimes r$ can be decomposed as a direct sum of $V_{i_1} \otimes V_{i_2} \cdots \otimes V_{i_r}$, where $i_j \in \{0, 1\}$. As $g_{0, \lambda}$-modules, $V_{i_1} \otimes V_{i_2} \cdots \otimes V_{i_r} \cong V_{i_1}^\otimes \otimes V_{i_r}^\otimes$ for some non-negative integer $\alpha \leq r$ with $\alpha = \#\{i_j \mid i_j = 0\}$. The corresponding isomorphism is given by acting a unique element $w$ on the right hand side of $V_{i_1}^\otimes \otimes V_{i_r}^\otimes$, where $w$ is a distinguished right coset representatives of $S_\alpha \times S_{r-\alpha}$ in $S_r$. By Lemma 5.3 all $g_{0, \lambda}$-highest weight vectors of $V_{i_1} \otimes V_{i_2} \cdots \otimes V_{i_r}$ with highest weight $\lambda$ are $v_\lambda w_{\lambda(1)} y_{\mu(1)} x_{\mu(2)} d(t_1) d(t_2) w$ for all $t_1 \in T^s(\mu(1))$ and $t_2 \in T^s(\mu(2))$. Therefore, the $C$-space $V_\lambda$ of all $g_{0, \lambda}$-highest weight vectors of $V^\otimes r$ with highest weight $\lambda$ has a basis $\{v_\lambda w_{\lambda(1)} y_{\mu(1)} x_{\mu(2)} d(t_1) d(t_2) w \mid t_1 \in T^s(\mu(1)) \}$, where $\mu = \lambda'$. By (5.6), $\{v_t \mid t \in T^s(\lambda')\}$ is $C$-linear independent. Finally, since there is a one to one correspondence between $g$-highest weight vectors of $M_{pq}^\otimes$ and $g_{0, \lambda}$-highest weight vectors of $V^\otimes r$ (cf. [21] Lemmas 5.1-5.2), and dim $V_\lambda = \#\{v_t \mid t \in T^s(\lambda')\}$, one obtains that $\{v_t \mid t \in T^s(\lambda')\}$ is a basis of $V_\lambda$.

In the remaining part of this section, we want to establish the relationship between $V_\lambda$ with a special cell module of $H_{2, r}$ with respect to $\lambda \in \Lambda_+^+(r)$. This result will be needed in section 6. We go on using $-x_1$ instead of $x_1$ in [5]. In this case, the current $-p$ and $m - q$ are the same as $p$ and $q$ in [5].

**Proposition 5.8.** For any $\lambda \in \Lambda_+^+(r)$, $V_\lambda \cong \mathfrak{gl}_m \mathfrak{gl}_n \cdot V_{2, r}$ as right $H_{2, r}$-modules, where $V_\lambda$ is defined in Theorem [5.7].

**Proof.** By Lemma 3.4(2), $S^\lambda := \mathfrak{gl}_m \mathfrak{gl}_n \cdot H_{2, r}$ has a basis $M = \{v_\lambda w_{\lambda(1)} d(t) \mid t \in T^s(\lambda')\}$. It follows from Theorem 5.7 that there is a linear isomorphism $\phi : V_\lambda \rightarrow S^\lambda$ sending $v_t$ to $v_\lambda w_{\lambda(1)} d(t)$. Obviously, $\phi$ is a right $S_r$-homomorphism. In order to show that $\phi$ is a right $H_{2, r}$-homomorphism, it suffices to prove that

\[ \phi(v_1 x_k) = \phi(v_1) x_k \text{ for } 1 \leq k \leq r. \]  

(5.7)

Denote $a = |\lambda(1)|$. If $1 \leq k \leq r - a$, then $\pi_{(1)} x_k = \pi_{r-a} (m - q) x_k = \pi_{r-a} (m - q) (-p - L_k)$. Since $\phi$ is a right $S_r$-homomorphism, (5.7) holds for $1 \leq k \leq r - a$. If $r - a + 1 \leq k \leq r$, then $x_k = s_{k, r-a+1} x_{r-a+1} - \sum_{j=r-a+1}^{k-1} (j, k)$. By Lemma 3.4(1),

\[ \pi_{(1)} w_{a} \pi_{(1)} x_k = \pi_{(1)} w_{a} \pi_{(1)} \left( -p - \sum_{j=r-a+1}^{k-1} (j, k) \right). \]  

(5.8)

On the other hand, $\pi_{(1)} x_k = x_{k} \pi_{(1)}$ and $v_{\lambda} \otimes v_{pq} w_{(1)} w_{(2)} y_{(1)} x_{(2)} w_a$ is a linear combination of elements $v_i \otimes v_{pq}$, for some $i \in I(m|n, r)$ such that $v_{ij} \in V_0$ for all $r - a + 1 \leq j \leq r$. By

This proposition will be used in the following sections.
Lemma 5.4(1), $x_k$ acts on $v_1 \otimes v_{pq}$ as $-p - L_k$. In order to verify (5.7) for $k \geq r - a + 1$, by (5.8), it remains to show that

$$v_1 \otimes v_{pq}(i, k) \tilde{\pi}_{\lambda'} = 0 \quad \text{for all } i, 1 \leq i \leq r - a. \tag{5.9}$$

Write $v_1 \otimes v_{pq}(i, k) = v_j$ up to a sign. Then $v_j \in V_0$ and $v_j(1, i)\tilde{\pi}_{\lambda'} = 0$ by Lemma 5.4(1). Since $(1, i)\tilde{\pi}_{\lambda'} = \tilde{\pi}_{\lambda'}(1, i)$, and $(1, i)$ is invertible, $v_j\tilde{\pi}_{\lambda'} = 0$, proving (5.9). \hfill \Box

**Corollary 5.9.** Suppose $\lambda \in \Lambda_+^2(r)$. As right $\mathcal{H}_{2,r}$-modules,

$$\text{Hom}_{U(g)}(K_{\lambda'}, M_{pq}^0) \cong \tilde{\Delta}(\lambda') \tag{5.10}$$

where $\tilde{\Delta}(\lambda')$ is the right cell module defined via the cellular basis of $\mathcal{H}_{2,r}$ in Lemma 3.3(2).

**Proof.** For any $g$-highest weight vector $v$ of $M_{pq}^0$ with highest weight $\tilde{\lambda}$, there is a unique $U(g)$-homomorphism $f_v : K_{\lambda'} \rightarrow U(g)v \subset M_{pq}^0$ sending $v_{\lambda}$ to $v$, where $v_{\lambda}$ is the highest weight vector of $K_{\lambda'}$. Further $f_v$ can be considered as a homomorphism in $\text{Hom}_{U(g)}(K_{\lambda'}, M_{pq}^0)$ by composing an embedding homomorphism.

For any $0 \neq f \in \text{Hom}_{U(g)}(K_{\lambda'}, M_{pq}^0)$, $f(v_{\lambda})$ is a highest weight vector of $M_{pq}^0$. By Theorem 5.7, $f(v_{\lambda})$ is a linear combination of $v_t$'s, for $t \in T^r(\lambda')$. So, $f$ can be written as a linear combination of $f_{v_t}$'s. Thus, $\{f_{v_t} \mid t \in T^r(\lambda')\}$ is a basis of $\text{Hom}_{U(g)}(K_{\lambda'}, M_{pq}^0)$. Let $V_{\tilde{\lambda}}$ be defined in Theorem 5.7. Then the linear isomorphism $\phi : \text{Hom}_{U(g)}(K_{\lambda'}, M_{pq}^0) \rightarrow V_{\tilde{\lambda}}$ sending $f_{v_t}$ to $v_t$ for any $t \in T^r(\lambda')$ is a right $\mathcal{H}_{2,r}$-homomorphism. By Lemma 3.3(5) and Proposition 5.8, $V_{\tilde{\lambda}} \cong \tilde{\Delta}(\lambda')$, proving (5.10). \hfill \Box

In the remaining part of this section, we always assume $p - q \leq -m$. If $p - q \geq n$, one can switch roles between $p$ and $q$ (or consider the dual module of $M_{pq}^0$). Without loss of any generality, we assume $p, q \in \mathbb{Z}$.

Let $\lambda \in \Lambda_+^2(r)$ with $r \leq \min\{m, n\}$. Then $\lambda$ corresponds to a dominant weight $\tilde{\lambda}$ defined in (5.3). In particular, $\emptyset = \lambda_{pq}$. Following 5.13 22, we are going to represent a dominant weight $\tilde{\lambda}$ in a unique way by a weight diagram $D_{\lambda}$. First we write (cf. (1.3))

$$\tilde{\lambda}^\rho = \lambda + \rho = (\tilde{\lambda}^L, \ldots, \tilde{\lambda}^m, \tilde{\lambda}^R, \ldots, \tilde{\lambda}^R). \tag{5.11}$$

Denote

$$S(\lambda)_L = \{\tilde{\lambda}^i | i = 1, \ldots, m\}, \quad S(\lambda)_R = \{-\tilde{\lambda}^j | j = 1, \ldots, n\},$$

$$S(\lambda) = S(\lambda)_L \cup S(\lambda)_R, \quad S(\lambda)_B = S(\lambda)_L \cap S(\lambda)_R.$$

**Definition 5.10.** The **weight diagram** $D_{\lambda}$ associated with the dominant weight $\tilde{\lambda}$ is a line with vertices indexed by $\mathbb{Z}$ such that each vertex $i$ is associated with a symbol $D_{\tilde{\lambda}} = \emptyset, <, >$ or $\times$ according to whether $i \notin S(\lambda), i \in S(\lambda)_L \setminus S(\lambda)_R, i \in S(\lambda)_R \setminus S(\lambda)_L$ or $i \in S(\lambda)_B$.  

For example, if $p, q \in \mathbb{Z}$ with $p \leq q - m$, then the weight diagram $D_{\emptyset}$ of $\emptyset = \lambda_{pq}$ is given by

$$\overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overset{\emptyset}{\cdots} \overline{p}$$

where, for simplicity, we have associated vertex $i$ with nothing if $D_{\tilde{\lambda}} = \emptyset$. Note that $\sharp S(\emptyset)_B = 0$, i.e., $\lambda_{pq}$ is typical.
Definition 5.11. Let $\tilde{\lambda}$ as in (5.3), where $\lambda \in \Lambda_2^+(r)$.

1. Let $\tilde{\lambda}\top$ be the unique dominant weight such that $L_{\tilde{\lambda}}$ is the simple submodule of the Kac-module $K_{\tilde{\lambda}\top}$. Then $\tilde{\lambda}\top$ is obtained from $\tilde{\lambda}$ via the unique longest right path (cf. [22 Definition 5.2], [24 Conjecture 4.4]) or via a raising operator (cf. [1]). For example, if $D_\lambda$ is given by

$$\cdots -0 \times 1 \overset{3}{\times} 4 \overset{5}{\times} 6 \overset{7}{\times} 9 < 10 < 11 \cdots$$

then the weight diagram $D_{\lambda\top}$ of $\tilde{\lambda}\top$ is given by

$$\cdots -0 \times 1 \overset{3}{\times} 4 \overset{5}{\times} 6 \overset{7}{\times} 9 < 10 < 11 \cdots$$

where the $\times$’s at vertices 9, 6, 3, 11 in (5.14) are respectively obtained from the $\times$’s at vertices 7, 4, 2, 1 in (5.13) (thus every symbol “×” is always moved to the unique empty place at its right side which is closest to it, under the rule that the rightmost “×” should be moved first, as indicated in (5.14)). Alternatively, $\tilde{\lambda}$ is obtained from $\tilde{\lambda}\top$ via the unique longest left path.

2. Let $\lambda\top \in \Lambda_2^+(r)$ be the unique bipartition such that $\lambda\top = \lambda_{pq} + \tilde{\lambda}\top$ (cf. (5.4) and (5.3)).

Write $p = q - m - k$ for some $k \in \mathbb{N}$. If $\mu = ((\mu_1^L, ..., \mu_{pq}^L), (\mu_1^R, ..., \mu_{pq}^R)) \in \Lambda_2^+(r)$, then $\mu'$ is Kleshchev with respect to $u_1 = -p$, $w_2 = m - q$ (cf. (5.15)) if and only if

$$\mu_i^L \geq \mu_i^R - k \quad \text{for all possible } i.$$  

Following [3 IV], we denote $I_{pq}^+ = \{p - m + 1, p - m + 2, ..., q - m + n\}$. For any $\lambda \in \Lambda_2^+(r)$ and any $j \in I_{pq}^+$, set

$$I_{\geq j}^0(\lambda) = \mathbb{Z}_{\geq j} \cap (I_{pq}^+ \setminus S(\lambda) \cap I_{pq}^+),$$

$$I_{\leq j}^0(\lambda) = \mathbb{Z}_{\leq j} \cap (I_{pq}^+ \setminus S(\lambda) \cap I_{pq}^+),$$

$$I_{> j}^x(\lambda) = \mathbb{Z}_{> j} \cap (I_{pq}^+ \cap S(\lambda) \setminus B),$$

$$I_{< j}^x(\lambda) = \mathbb{Z}_{< j} \cap (I_{pq}^+ \cap S(\lambda) \setminus B).$$

In terms of the above notations, Brundan and Stroppel [3 IV, Lemma 2.6] have proved that the indecomposable tilting module $T_\lambda$ is a direct summand of $M_{pq}^0$ if

$$S(\lambda) \subset I_{pq}^+ \quad \text{and} \quad \#I_{\geq j}^0(\lambda) \geq \#I_{< j}^x(\lambda) \quad \text{for all } j \in I_{pq}^+.$$  

These two conditions on bipartition $\lambda$ (or weight $\tilde{\lambda}$) are equivalent to the following conditions on $\lambda\top$ (which can be seen from (5.13)–(5.14) in case $I_{pq}^+ = \{1, 2, ..., 11\}$):

$$S(\lambda\top) \subset I_{pq}^+ \quad \text{and} \quad \#I_{\geq j}^0(\lambda\top) \geq \#I_{< j}^x(\lambda\top) \quad \text{for all } j \in I_{pq}^+.$$  

Lemma 5.12. Let $\mu \in \Lambda_2^+(r)$ such that $\mu'$ is Kleshchev with respect to $u_1 = -p$, $u_2 = m - q$, where $p = q - m - k$ with $k \in \mathbb{N}$. Then

$$S(\mu) \subset I_{pq}^+ \quad \text{and} \quad \#I_{\geq j}^0(\mu) \geq \#I_{< j}^x(\mu) \quad \text{for all } j \in I_{pq}^+.$$
Corollary 5.13. Suppose $\lambda \in \Lambda_2^+(r)$ such that $(\lambda^{\text{top}})'$ is Kleshchev, where $(\lambda^{\text{top}})'$ is the conjugate of $\lambda^{\text{top}} \in \Lambda_2^+(r)$. Then $T_\lambda$ is a direct summand of $M_{pq}^0$. Further, any indecomposable direct summand of $M_{pq}^0$ is of form $T_\lambda$ for some $\lambda \in \Lambda_2^+(r)$ such that $(\lambda^{\text{top}})'$ is Kleshchev.

Proof. The first assertion follows from [5, IV, Lemma 2.6] and Lemma 5.12. To prove the last assertion, since $r \leq \min\{m, n\}$, by Theorem 4.5 $\text{End}_{U(\mathfrak{gl}_m|n)}(M_{pq}^0) \cong \mathcal{H}_2,r$. So, the number of non-isomorphic indecomposable direct summands of $\mathfrak{gl}_m|n$-module $M_{pq}^0$ is equal to that of non-isomorphic irreducible $\mathcal{H}_2,r$-modules, which is equal to the number of Kleshchev bipartitions in $\Lambda_2^+(r)$. Now, everything is clear. □

Corollary 5.14. Suppose $\lambda \in \Lambda_2^+(r)$ such that $(\lambda^{\text{top}})'$ is Kleshchev. As right $\mathcal{H}_2,r$-modules,

$$\text{Hom}_{U(\mathfrak{g})}(T_\lambda, M_{pq}^0) \cong P^{(\lambda^{\text{top}})'},$$

where $P^{(\lambda^{\text{top}})'}$ is the projective cover of $D^{(\lambda^{\text{top}})'}$ which is the simple head of $\tilde{\Delta}((\lambda^{\text{top}})')$.

Proof. Since $r \leq \min\{m, n\}$ and $(\lambda^{\text{top}})'$ is Kleshchev, by Corollary 5.13 $T_\lambda$ is a direct summand of $M_{pq}^0$, forcing $0 \neq \text{Hom}_{U(\mathfrak{g})}(T_\lambda, M_{pq}^0)$ to be a direct summand of $\mathcal{H}_2,r$. We claim that $\text{Hom}_{U(\mathfrak{g})}(T_\lambda, M_{pq}^0)$ is indecomposable. If not, then the number of indecomposable direct summands of the right $\mathcal{H}_2,r$-module $\mathcal{H}_2,r$ is strictly bigger than $\Sigma_\lambda \ell_\lambda$ if we write $M_{pq}^0$ as $M_{pq}^0 = \oplus_\lambda T_\lambda^{(\mathfrak{g}, \mathcal{H}_2,r)}$ with $\ell_\lambda \neq 0$.

On the other hand, since $M_{pq}^0$ is a right $\mathcal{H}_2,r$-module, we can consider the right exact functor $\mathfrak{F} := M_{pq}^0 \otimes \mathcal{H}_2,r$? from the category of left $\mathcal{H}_2,r$-modules to the category of left $U(\mathfrak{g})$-modules. We have an epimorphism from $\mathfrak{F}(P^\mu)$ to $\mathfrak{F}(\tilde{\Delta}(\mu))$, where $P^\mu$ is any principal indecomposable left $\mathcal{H}_2,r$-module and $\tilde{\Delta}(\mu)$ temporarily denotes the left cell module of $\mathcal{H}_2,r$ defined via the cellular basis of $\mathcal{H}_2,r$ in Lemma 5.3(1) with the simple head $D^\mu$. By Lemma 3.4(5) and Theorem 4.5 $\mathfrak{F}(\tilde{\Delta}(\mu)) \neq 0$, forcing $\mathfrak{F}(P^\mu) \neq 0$. So, $\mathfrak{F}(P^\mu)$ is a direct sum of indecomposable direct summands of $U(\mathfrak{g})$-module $M_{pq}^0$. In particular, $\Sigma_\lambda \ell_\lambda$ is no less than the number of indecomposable direct summands of left $\mathcal{H}_2,r$-module $\mathcal{H}_2,r$. This is a contradiction since the
number of indecomposable direct summands of left \( \mathcal{H}_{2,r} \)-module \( \mathcal{H}_{2,r} \) is equal to that of indecomposable direct summands of right \( \mathcal{H}_{2,r} \)-module \( \mathcal{H}_{2,r} \). So, \( \mathcal{F}(T) \) is a principal indecomposable right \( \mathcal{H}_{2,r} \)-module. Since \( K_{\lambda^{op}} \hookrightarrow T \), Hom\(_{(\mathcal{G})}(T_{\lambda}, M_{pq}^{0}) \rightarrow \text{Hom}_{U(\mathcal{G})}(K_{\lambda^{op}}, M_{pq}^{0}) \).

By Corollary 5.9, \( \text{Hom}_{U(\mathcal{G})}(K_{\lambda^{op}}, M_{pq}^{0}) \cong \tilde{\Delta}((\lambda^{op})') \). Since Hom\(_{(\mathcal{G})}(T_{\lambda}, M_{pq}^{0}) \) is a principal indecomposable right \( \mathcal{H}_{2,r} \)-module, it implies that \( \tilde{\Delta}((\lambda^{op})') \) has the unique simple head, denoted by \( D((\lambda^{op})') \). Thus, Hom\(_{\mathcal{G}}(T_{\lambda}, M_{pq}^{0}) \cong P((\lambda^{op})') \). □

Brundan-Stroppel have already proved that decomposition numbers of \( \mathcal{H}_{2,r} \) arising from super Schur–Weyl duality in \([5]\) can be determined by the multiplicity of Kac-modules in indecomposable tilting modules appearing in \( M_{pq}^{0} \). This result can also be seen via the exact functor Hom\(_{(\mathcal{G})}((?), M_{pq}^{0}) \).

6. Highest weight vectors in \( M_{pq}^{rt} \)

In this section, we classify \( \mathfrak{g} \)-highest weight vectors of \( \mathfrak{gl}_{m|n} \)-module \( M_{pq}^{rt} \) over \( \mathbb{C} \). As an application, we set up explicit relationship between Kac (resp. indecomposable tilting) modules of \( \mathfrak{g} \) and cell (resp. principal indecomposable) modules of \( B_{2,r,t} \). This gives us an efficient way to calculate decomposition numbers of \( B_{2,r,t} \). Throughout, assume \( r, t \in \mathbb{Z}^{\geq 0} \) such that \( r + t \leq \min\{m, n\} \). The case \( t = 0 \) has been dealt with in section 5. By symmetry, one can also classify highest weight vectors of \( M_{pq}^{rt} \) via those in section 5. The following result, which is the counterpart of Lemma 5.4, can be verified directly.

**Lemma 6.1.** Suppose \( i \in I(m|n, r) \), \( j \in I(m|n, t) \) (cf. (4.7)) and \( 1 \leq k \leq t \).

1. \( v_{i} \otimes v_{pq} \otimes \bar{v}_{j} \bar{x}_{k}' = qv_{i} \otimes v_{pq} \otimes \bar{v}_{j} \) if \( 1 + m \leq j_{k} \leq m + n \).
2. \( v_{i} \otimes v_{pq} \otimes \bar{v}_{j} \bar{x}_{k}' = pv_{i} \otimes v_{pq} \otimes \bar{v}_{j} + \sum_{j = m + 1}^{j_{k} - 1} (-1)^{m+n+1} v_{i} \otimes (E_{jjk} v_{pq}) \otimes \bar{v}_{\ell} \) if \( 1 \leq j_{k} \leq m \), where \( \ell \in I(m|n, t) \) which is obtained from \( j \) by using \( j \) instead of \( j_{k} \) in \( j \). In particular, the weight of \( \bar{v}_{\ell} \) is strictly bigger than that of \( \bar{v}_{j} \).

For any integral weight \( \xi = (\xi_{1}, ..., \xi_{m} | \xi_{m+1}, ..., \xi_{m+n}) \) of \( \mathfrak{g} \), let

\[
\xi^{L} = (\xi_{1}^{L}, ..., \xi_{m}^{L}) = (\xi_{1}, ..., \xi_{m}), \quad \text{and} \quad \xi^{R} = (\xi_{1}^{R}, ..., \xi_{m}^{R}) = (\xi_{m+1}, ..., \xi_{m+n}).
\]

We define two bicompositions \( \mu, \nu \) such that all \( \mu_{i}^{(1)}, \mu_{j}^{(2)}, \nu_{i}^{(1)}, \nu_{j}^{(2)} \) are zero except that

1. for \( 1 \leq i \leq m \), \( \mu_{i}^{(1)} = \xi_{i}^{L} \) if \( \xi_{i}^{L} > 0 \) or \( \nu_{i}^{(1)} = -\xi_{i}^{L} \) if \( \xi_{i}^{L} < 0 \).
2. for \( 1 \leq j \leq n \), \( \mu_{j}^{(2)} = \xi_{j}^{R} \) if \( \xi_{j}^{R} > 0 \) or \( \nu_{j}^{(2)} = -\xi_{j}^{R} \) if \( \xi_{j}^{R} < 0 \).

Then both \( \mu \) and \( \nu \) correspond to integral weights of \( \mathfrak{g} \). In particular, \( \xi = \mu - \hat{\nu} \) with

\[
\hat{\nu} = (\nu_{1}^{(1)}, ..., \nu_{i}^{(1)} | \nu_{i}^{(2)}, ..., \nu_{i}^{(2)}) \in \mathfrak{h}^{*}.
\]

Conversely, if \( \mu \) and \( \nu \) are two bicompositions, then \( \xi = \mu - \hat{\nu} \) is a integral weight of \( \mathfrak{g} \). For instance, if \( \xi = (r-4, 1, 0, ..., 0, -1, -(t-5) | 2, 1, 0, ..., 0, -1, -3) \), then \( \mu = ((r-4, 1), (2, 1)) \) and \( \nu = ((t-5, 1), (3, 1)) \) such that \( \xi = \mu - \hat{\nu} \).

**Definition 6.2.** For any \( \lambda = (f, \mu, \nu) \in \Lambda_{2,r,t} \), let \( \bar{\lambda} := \lambda_{pq} + \mu - \hat{\nu} \) and \( \bar{\lambda} := \mu - \hat{\nu} \). Since \( r + t \leq \min\{m, n\} \), both \( \mu \) and \( \nu \) correspond to integral weights of \( \mathfrak{g} \) as above such that

\[
\mu_{i} \nu_{m+1-i} = 0 \quad \text{for} \quad 1 \leq i \leq m \quad \text{and} \quad \mu_{m+j} \nu_{m+n+1-j} = 0 \quad \text{for} \quad 1 \leq j \leq n.
\]
Lemma 6.3. For any \( \mathfrak{g} \)-highest weight \( \Lambda \) of \( M_{pq}^{rt} \), there is a unique triple \( \lambda = (f, \mu, \nu) \in \Lambda_{2,r,t} \) such that \( \Lambda = \bar{\lambda} \).

Proof. By [19, Lemma 5.20], \( \Lambda = \lambda_{pq} + \eta - \zeta \) for some bicompositions \( \eta \) and \( \zeta \) of sizes \( r \) and \( t \) respectively. For \( i \in I \), let \( \xi_i = \min\{\eta_i, \zeta_i\} \) and \( f = \sum_{i \in I} \xi_i \). Then we obtain a weight \( \xi \), and two bicompositions \( \mu := \eta - \xi \) and \( \gamma := \zeta - \xi \) such that \( |\mu| = r - f \), \( |\gamma| = t - f \) and \( \Lambda = \lambda_{pq} + \mu - \gamma \). Set \( \nu = \gamma \), then \( \Lambda = \bar{\lambda} \) and (6.2) is satisfied by definition of \( \xi \). Since \( \Lambda \) is dominant, \( \mu, \nu \) must be bipartitions. Thus \( \Lambda \) corresponds to \( \lambda = (f, \mu, \nu) \in \Lambda_{2,r,t} \). Such a \( \lambda \) is unique.

Definition 6.4. For each \( \lambda = (f, \mu, \nu) \in \Lambda_{2,r,t} \), denote \( v_\lambda = v_1 \otimes v_{pq} \otimes v_2 \), where
\[
i = (i_{\mu(1)}, i_{\mu(2)}, 1, \ldots, 1) \in I(m|n, r), \quad j = (j_{\nu(2)}, j_{\nu(1)}, 1, \ldots, 1) \in I(m|n, t),
\]
such that

1. \( j_{\nu(2)} \) is obtained from \( i_{\nu(2)} \) by using \( m + n - i + 1 \) instead of \( i \) for \( 1 \leq i \leq n \),
2. \( j_{\nu(1)} \) is obtained from \( i_{\nu(1)} \) by using \( m - i + 1 \) instead of \( i \) for \( 1 \leq i \leq m \).

For instance, if \( \lambda = (1, \mu, \nu) \in \Lambda_{2,8,10} \) with \( \mu = ((3, 1), (2, 1)) \) and \( \nu = ((4, 1), (3, 1)) \), then \( i = (1^3, 2, (m + 1)^2, (m + 2), 1) \) and \( j = (m + n)^3, (m + n - 1), m^4, (m - 1), 1) \). Thus,
\[
v_\lambda = v_1 \otimes v_{m+2} \otimes v_{m+1}^{\otimes 2} \otimes v_2 \otimes v_1^{\otimes 3} \otimes v_{pq} \otimes v_{m+2}^{\otimes 3} \otimes v_{m+n-1}^{\otimes 4} \otimes v_{m-1} \otimes v_1.
\]

Definition 6.5. For any \( (f, \mu, \nu) \in \Lambda_{2,r,t} \), define \( v_{\lambda, t, d, \kappa} = v_{\lambda, f} w_{t, \nu, \eta} \tilde{\eta}_{(\nu^o)^o} d(t) \), \( \kappa \in \mathbb{N}_f \), \( d \in \mathbb{D}_{r,t}^{j_f} \) and \( t \in \mathbb{T}^s((\mu^o)^o) \times \mathbb{T}^s((\nu^o)^o) \).

Theorem 6.6. Suppose \( r + t \leq \min\{m, n\} \).

1. There is a bijection between the set of dominant weights of \( M_{pq}^{rt} \) and \( \Lambda_{2,r,t} \).
2. If \( \lambda = (f, \mu, \nu) \in \Lambda_{2,r,t} \), then \( V_{\lambda, \bar{\lambda}} \), the \( C \)-space of all \( \mathfrak{g} \)-highest weight vectors of \( M_{pq}^{rt} \) with highest weight \( \bar{\lambda} \), has a basis \( S := \{v_{\lambda, t, d, \kappa} | t \in \mathbb{T}^s((\mu^o)^o) \times \mathbb{T}^s((\nu^o)^o), \kappa \in \mathbb{N}_f \} \).

Proof. Obviously, (1) follows from Lemma 6.3. To obtain (2), we prove that for each \( \lambda = (f, \mu, \nu) \in \Lambda_{2,r,t} \), \( V_{\lambda, \bar{\lambda}} \) has the required basis in the case either \( f = 0 \) or \( f > 0 \).

Case 1: \( f = 0 \).
By Definition 6.5, \( v_{\lambda, f} = v_1 \otimes v_{pq} \otimes \tilde{v}_1 w_{\mu} d(t_1) \otimes v_{\nu^o} \eta_{(\nu^o)^o} d(t_2) \), where \( i, j \) are defined in Definition 6.4. By Theorem 5.7, \( v_1 \otimes v_{pq} \otimes \tilde{v}_1 w_{\mu} d(t_1) \) can be regarded as a \( \mathfrak{g} \)-highest weight vector of \( M_{pq}^{0t} \). Similarly, \( v_1 \otimes v_{pq} \otimes \tilde{v}_1 w_{(\nu^o)^o} d(t_2) \) can be regarded as a \( \mathfrak{g} \)-highest weight vector of \( M_{pq}^{0t} \). Thus, \( v_{\lambda, f} \) is a \( \mathfrak{g} \)-highest weight vector of \( M_{pq}^{rt} \). The last assertion follows from arguments on counting the dimensions of \( V_{\lambda, \bar{\lambda}} \) and that of \( \mathfrak{g}_0 \)-highest weight vectors of \( V^{rt} := V^{\otimes r} \otimes W^{\otimes t} \) with highest weight \( \mu - \nu \).

Case 2: \( f > 0 \).
For any \( i \in I \),
\[
v_1 \otimes \tilde{v}_i e_1 = (-1)^{|i|} \sum_{j \in I} v_j \otimes \tilde{v}_j.
\]
Thus \( v_t \otimes \bar{v}_\ell \) is unique up to a sign for different \( \ell \)'s. Since \( M_{r,t}^{\ast q} \) is a \((\mathcal{g}, \mathcal{B}_{2r,t})\)-bimodule, we can switch \( v_{r-k} \) and \( \bar{v}_{j-t-k} \) in \( v_\lambda \) with \( i_{r-k} = j_{t-k} \) to \( v_o \) and \( \bar{v}_o \) for any fixed \( o, 1 \leq o \leq m + n \) simultaneously when we consider the action of \( E_{j,\ell} \) on \( i_{r-k} \) (resp. \( j_{t-k} \)) tensor factor of \( v_{\lambda,t,d,\kappa_d} \) for \( 0 \leq k \leq f - 1 \). Let

\[
v_t := v_{i_{r-k}} \otimes \cdots \otimes v_1 \otimes v_{pq} \otimes \bar{v}_{j_1} \otimes \cdots \otimes \bar{v}_{j_{t-k}} w_{\mu,\nu} x_{\alpha(2)} y_{\beta(1)} y_{\beta(2)} \pi_{r-f-a} (-p) \pi_b q d(x), \tag{6.3}
\]

where \( \alpha(i) \) (resp. \( \beta(i) \)) is the conjugate of \( \mu(i) \) (resp. \( \nu(i) \)), \( i = 1, 2 \). Applying Theorem 5.7 to both \( V \otimes_{r-f} \otimes K_{\lambda_{pq}} \) and \( K_{\lambda_{pq}} \otimes W \otimes_{t-f} \) yields \( E_{j,\ell} v_t = 0 \). So, \( E_{j,\ell} v_{\lambda,t,d,\kappa_d} = 0 \) for any \( j < \ell \).

We claim that \( S \) is linear independent, where \( S \) is given in (2). If so, each \( v_{\lambda,t,d,\kappa_d} \neq 0 \), forcing \( v_{\lambda,t,d,\kappa_d} \) to be a \( g \)-highest weight vector of \( M_{r,t}^{\ast q} \) with highest weight \( \lambda \).

Suppose \( i \in I(m|n, r_1 - 1) \) and \( j \in I(m|n, t_1 - 1) \) with \( r_1 \leq r \) and \( t_1 \leq t \) such that there are at least some \( k_0 \in I_0 \) and \( \ell_0 \in I_1 \) satisfying \( k_0, \ell_0 \notin \{i_1, j_0\} \) for all possible \( i, o \)'s. We consider

\[
\sum_{k \in I} v_k \otimes v_i \otimes v_j \otimes \bar{v}_k(x_{r_1} + L_{r_1}) = -\sum_{k \in I}(\prod_{k \in I} \otimes v_i \otimes v_j \otimes \bar{v}_k,
\]

where \( [k] = \sum_{j=1}^{r_1} \binom{r_1}{k} \). So, up to some scalar \( a \), \( \sum_{k=1}^{m+n} v_k \otimes v_i \otimes v_j \otimes \bar{v}_k x_{r_1} \) contains the unique term \( v_{k_0} \otimes v_i \otimes v_j \otimes \bar{v}_k \). In particular, if \( v \neq v_{pq} \), \( \sum_{k \in I} v_k \otimes v_i \otimes v_j \otimes \bar{v}_k x_{r_1} \) does not contribute terms with form \( v_{k_0} \otimes v_{pq} \otimes v_{pq} \otimes v_j \otimes \bar{v}_{k_0} \) for all possible \( i \) and \( j \). If \( v = v_{pq} \), by Lemma 5.4 the previous scalar is \( -p \). Similarly, the coefficient of \( v_{k_0} \otimes v_i \otimes v_{pq} \otimes v_{pq} \otimes v_j \otimes \bar{v}_{k_0} \) is \( -q \). Assume

\[
c \sum_{k \in I} v_k \otimes v_i \otimes v_{pq} \otimes v_j \otimes \bar{v}_k x_{r_1} + d \sum_{k \in I} v_k \otimes v_i \otimes v_{pq} \otimes v_j \otimes \bar{v}_k = 0 \tag{6.4}
\]

for some \( c, d \in \mathbb{C} \). Then \( d = cp - eq \) by considering the coefficients of \( v_{k_0} \otimes v_i \otimes v_{pq} \otimes v_{pq} \otimes v_{pq} \), \( k \in \{k_0, \ell_0\} \) in the expression of LHS of (6.4). If \( c \neq 0 \), then \( p - q = 0 \). This is a contradiction since \( \lambda_{pq} \) is typical in the sense of (1.4). So, \( c = d = 0 \) and hence \( \sum_{k \in I} v_k \otimes v_i \otimes v_{pq} \otimes v_j \otimes v_{pq} \otimes v_j x_{r_1} \) and \( \sum_{k \in I} v_k \otimes v_i \otimes v_{pq} \otimes v_j \otimes v_{pq} \) are linear independent. Now, we assume

\[
\sum_{t, d, \kappa_d} r_{t,d,\kappa_d} v_{\lambda,t,d,\kappa_d} = 0 \text{ for some } r_{t,d,\kappa_d} \in \mathbb{C}. \tag{6.5}
\]

We claim that \( r_{t,d,\kappa_d} = 0 \) for all possible \( t, d, \kappa_d \). If not, then we pick up a \( d \in D_{r,t}^{\text{f}} \) such that

(1) \( r_{t,d,\kappa_d} \neq 0 \),

(2) \( d = s_r f + 1, i_{r-f-1} \), \( i_{r-f-1} \), \( i_{r-f-1} \), \( \cdots \), \( i_r > i_{r-1} > \cdots > i_{r-f+1} \),

(3) \( (i_r, \cdots, i_{r-f+1}) \) is maximal with respect to lexicographic order.

Since \( r + t = \min\{m, n\} \) and \( 0 < f \leq \min\{r, t\} \), we can pick \( f \) pairs \((k_i, \ell_i)\), \( r - f + 1 \leq i \leq r \) such that

(1) \( k_i \in I_0 \), \( \ell_i \in I_1 \), \( k_i > k_j \) and \( \ell_i > \ell_j \) if \( i > j \);

(2) both \( v_{k_i} \) and \( v_{\ell_i} \) are not a tensor factor of \( v_{k_j} \),

(3) both \( \bar{v}_{k_i} \) and \( \bar{v}_{\ell_i} \) are not a tensor factor of \( v_{\ell_j} \).
We consider the terms $v_a \otimes vpq \otimes \tilde{v}_b$'s in the expressions of $v_{\lambda, d, \kappa}$'s in LHS of (6.3) with $r_{t, d, \kappa} \neq 0$ such that either $v_{a_{h}} = v_{h}$ and $\tilde{v}_{b_{l-r+h}} = \tilde{v}_{h}$ or $v_{a_{l}} = v_{l}$ and $\tilde{v}_{b_{l-r+h}} = \tilde{v}_{h}$ for $r - f + 1 \leq h \leq r$. Such terms occur in the expression of $v_{i}^{\otimes f} \otimes v_{t} \otimes v_{i}^{\otimes f} e^{dx_{d}}$, where $v_{t}$ is a linear combination of the terms in $v_{i}$'s (cf. (6.3)) with forms $v_{t} \otimes vpq \otimes \tilde{v}_{y}$. If $v_{a_{h}} = v_{h}$ and $\tilde{v}_{b_{l-r+h}} = \tilde{v}_{h}$, by previous arguments, the coefficient of $v_{a} \otimes vpq \otimes \tilde{v}_{b}$ in $v_{i}^{\otimes f} \otimes v_{t} \otimes v_{i}^{\otimes f} e^{dx_{d}}$ is \( \prod_{h=r}^{r-f+1}(-p)^{\epsilon_{h}} \), where $\epsilon_{h} = 1$ if $\kappa_{h} = 1$ and 0 if $\kappa_{h} = 0$. If $v_{a_{h}} = v_{l}$ and $\tilde{v}_{b_{l-r+h}} = \tilde{v}_{h}$, then the coefficient of $v_{a} \otimes vpq \otimes \tilde{v}_{b}$ in $v_{i}^{\otimes f} \otimes v_{t} \otimes v_{i}^{\otimes f} e^{dx_{d}}$ is \( \prod_{h=r}^{r-f+1}(-q)^{\epsilon_{h}} \), where $\epsilon_{h} = 1$ if $\kappa_{h} = 1$ and 0 if $\kappa_{h} = 0$. By (6.5), \( \sum_{l, d, \kappa} v_{l} = 0 \) for any fixed $\kappa_{d}$. Thus, we can assume that $\kappa_{d} = (0, \cdots, 0) \in \mathbb{N}_{f}$. If we identify $\tilde{v}_{l}$ with its $vpq$-component, then $\tilde{v}_{l}$ can be considered as $g_{0}$-highest weight vectors of $V^{\otimes r-f} \otimes W^{\otimes t-f}$ (cf. arguments in the proof of Theorem 5.7) of the form

$$
\tilde{v}_{l} = v_{i_{r-f}} \otimes \cdots v_{i_{l}} \otimes \tilde{v}_{j_{1}} \otimes \cdots \tilde{v}_{j_{l-f}} w_{p, \nu, \alpha_{a}(\cdot), \alpha_{b}(\cdot), \beta_{j_{1}}(\cdot), \beta_{j_{l-f}}(\cdot)}(1, t).
$$

So, $r_{t, d, \kappa} = 0$, a contradiction. This proves that $S$ is $C$-linear independent. Further, $S$ is a basis of $V_{\lambda}$ since the cardinality of $S$ is $2^{f} |D_{r, t}^{f}| |T^{s}(\mu')| |T^{s}(\nu')|$, which is the dimension of space consisting of $g_{0}$-highest weight vectors of $V^{rt}$ with highest weight $\mu - \tilde{\nu}$.

\[ \square \]

Definition 6.7. Let \( \mathcal{F} = \text{Hom}_{U(g)}(?, M_{pq}^{rt}) \) be the functor from the category of finite dimensional left $g$-modules to the category of right $\mathcal{B}_{2, r, t}$-modules over $C$.

Lemma 6.8. The functor $\mathcal{F}$ is exact.

\[ \square \]

Proof. Since $\lambda_{pq}$ is typical, $M_{pq}^{rt}$ is projective, injective and tilting as left $g$-module (e.g., [5 IV]). So, $\mathcal{F}$ is exact.

\[ \square \]

Proposition 6.9. Suppose $\lambda = (f, \mu, \nu) \in \Lambda_{2, r, t}$. Then $\mathcal{F}(K_{\lambda}) \cong C(f, \mu', (\nu^{\circ})')$, where $\nu^{\circ} = (\nu^{(2)}, \nu^{(1)})$.

\[ \square \]

Proof. By Proposition 3.9 there is an explicit linear isomorphism between $C(f, \mu', (\nu^{\circ})')$ and $V_{\lambda}$, where $V_{\lambda}$ is given in Theorem 6.6. By Proposition 5.8 and [19] Proposition 6.10, this linear isomorphism is a $\mathcal{B}_{2, r, t}$-homomorphism. Thus, $C(f, \mu', (\nu^{\circ})') \cong V_{\lambda}$ as right $\mathcal{B}_{2, r, t}$-modules. Using the universal property of Kac-modules yields $\text{Hom}_{U(g)}(K_{\lambda}, M_{pq}^{rt}) \cong V_{\lambda}$ as $\mathcal{B}_{2, r, t}$-modules (cf. the proof of Corollary 5.9). Now, everything is clear.

In the remaining part of this section, we calculate decomposition matrices of $\mathcal{B}_{2, r, t}$. We always assume that $p \in \mathbb{Z}$. Otherwise, one can use $x_{i} + p_{1}$ instead of $x_{i}$ for any $p_{1} \in \mathbb{C}$ with $p - p_{1} \in \mathbb{Z}$. Since $\lambda$ is typical, we have $p - q \notin \mathbb{Z}$ or $p - q \leq -m$ or $p - q \geq n$. In the first case, by [19] Theorem 5.21], $\mathcal{B}_{2, r, t}$ is semisimple and hence its decomposition matrix is the identity matrix. We assume that $p - q \leq -m$. If $p - q \geq n$, one can switch the roles between $p$ and $q$ (or by considering the dual module of $M_{pq}^{rt}$) in the following arguments.

Suppose $\lambda = (f, \mu, \nu) \in \Lambda_{2, r, t}$. Let $T_{\lambda}$ be the indecomposable tilting module, where $\bar{\lambda} = \lambda_{pq} + \tilde{\lambda} = \lambda_{pq} + \mu - \tilde{\nu}$ (cf. Definition 6.2). It is the projective cover of $L_{\lambda}$, where $L_{\lambda}$ is the simple $g$-module with highest weight $\bar{\lambda}$. It is known that $T_{\lambda}$ has filtrations of Kac-modules. Let $K_{\lambda}^{\text{top}}$ be the unique bottom of $T_{\lambda}$. Then $L_{\lambda}$ is the simple $g$-module of $K_{\lambda}^{\text{top}}$. 


Further, $\lambda^{\text{top}}$ is the dominant weight defined in Definition 5.11(1). Since $M_{pq}^r$ is a tilting module, it can be decomposed into the direct sum of indecomposable tilting modules

$$M_{pq}^r = \bigoplus_{\mu \in P^+} T^\mu_{p+\mu}$$

for some $\mu \in \mathbb{N}$. (6.6)

In the remaining part of this paper, we denote $S$ to be the following finite subset of $P^+$,

$$S := \{ \mu \in P^+ | \ell_\mu \neq 0 \}.$$ (6.7)

Parallel to Corollary 5.13, we have the following.

**Lemma 6.10.** Let $\lambda = (f, \mu, \nu) \in \Lambda_{2,r,t}$ such that $(\lambda^{\text{top}})'$ is Kleshchev, where $\lambda^{\text{top}}$ is defined in Definition 5.11(2). Then $T_{\lambda}$ is a direct summand of $M_{pq}^r$.

**Proof.** We claim that $T_{\lambda}$ is a direct summand in $M_{pq}^{r-f-r^{-f}}$. If so, then

$$v'_1 \otimes T_{\lambda} \otimes v'_f$$

is obviously a tilting submodule in $M_{pq}^r$ which is isomorphic to $T_{\lambda}$. Thus the claim implies the result. Therefore, it suffices to consider the case $f = 0$.

Denote $\nu = \lambda_{pq} - \nu$. Since $p \leq q - m$, the weight diagram $D_{\nu}$ (cf. Definition 5.11) of $\nu$ is obtained from that of $\lambda_{pq}$ in (5.12) by moving the “$>$” at vertex $p - i + 1$ to its left side at vertex $p - i + 1 - \nu_{m-i+1}$ for each $i$ with $1 \leq i \leq m$, and moving the “$<$” at vertex $q - m + j$ to its right side at vertex $q - m + j + \nu_{m-j+1}$ for each $j$ with $1 \leq j \leq n$ (cf. (6.1)). Thus no “$\times$” can be produced, i.e., $\nu$ is typical. Hence $K_\nu$ is a direct summand in $M_{pq}^r$. Thus, it suffices to prove that $T_{\lambda}$ is a direct summand in $V^{\otimes r} \otimes K_\nu \cong M_{pq}^r$, here $\nu$ means direct summand of $M_{pq}^r$. For this, we can apply [5, IV, Lemmas 2.4 and 2.6]. Note from [5, IV, Lemma 2.4] that the action of the functor $F_i$ on $K_n \nu$ defined in [5, IV] only depends on symbols at vertices $i$ and $i + 1$ of the weight diagram $D_{\nu}$ of $\nu^i$ (we remark that symbols $o, x, v, \times$ in [5, IV] are respectively symbols $<, =, < >$ in this paper). Due to condition (6.2), for any $i \in I_{pq} := I_{pq}^+ \setminus \{q - m + n\}$ such that $i$ is involved in a path in the crystal graph in [5, IV, Lemma 2.6], the symbols at vertex $i$ and $i + 1$ in the weight diagram $D_{\nu}$ of $\nu$ are the same as that in the weight diagram $D_{\nu}^o$ of $\lambda_{pq}$. This shows that $T_{\lambda}$ is a direct summand in $V^{\otimes r} \otimes K_\nu$ if and only if $T_{\lambda_{pq}^i} \nu$ is a direct summand in $V^{\otimes r} \otimes K_{\lambda_{pq}^i}$, more precisely, [5, IV, Lemma 2.6] implies

$$F_{i_1} \cdots F_{i_{\ell}} K_{\nu} \cong T_{\lambda}^{\otimes 2\ell} \iff F_{i_1} \cdots F_{i_{\ell}} K_{\lambda_{pq}} \cong T_{\lambda_{pq}^i}^{\otimes 2\ell},$$

where $\ell$ is the number of edges in the given path of the form $\emptyset \times \rightarrow < >$. Thus the result follows from Corollary 5.13.

We remark that there is a bijection between $S$ defined in (6.7) and the set of pair-wise non-isomorphic simple $\mathcal{B}_{2,r,t}$-modules. See [19, Theorem 7.5]. For any $\xi \in S$ as above, parallel to Definition 5.11, we define $\xi^{\text{top}}$ to be the unique dominant weight such that $L_{\xi}$ is the simple submodule of $K_{\xi^{\text{top}}}$.

**Proposition 6.11.** For any $\xi \in S$, there is a unique $(f, \mu, \nu) \in \Lambda_{2,r,t}$ such that $\xi^{\text{top}} = \lambda_{pq} + \mu - \nu$. Further, $\mathfrak{F}(T_{\xi})$ is isomorphic to the projective cover of $D^{\lambda_{pq}^i \nu^i, (\nu^i)}$, where $D^{\lambda_{pq}^i \nu^i, (\nu^i)}$ is the simple head of $C(f, \mu^i, (\nu^i))$. 

\[\]
Proposition 6.13. Suppose $\xi \in P^+$. Then $\mathfrak{F}(L_\xi) = 0$ if $\xi \not\in S$ (cf. (6.7)) and $\mathfrak{F}(L_\xi) \cong D^{f_\mu',(\nu')^\gamma}$ if $\xi \in S$, where $\xi^{\top} = \lambda_{pq} + \mu - \nu$ with $(f, \mu, \nu) \in \Lambda_{2,r,t}$.

Proof. By (6.10), $\mathfrak{F}(L_\xi) = \bigoplus_{\xi \in S} \text{Hom}_F(L_\xi, T_\xi^{\otimes \xi})$. Suppose $0 \neq f \in \text{Hom}_{U(\mathfrak{g})}(L_\xi, T_\xi^{\otimes \xi})$. Then $L_\xi \cong f(L_\xi)$ is a simple submodule of $T_\xi^{\otimes \xi}$. Since $T_\xi$ has the unique simple submodule $L_\xi$,
\[ \mathfrak{F}(L_\xi) = 0 \text{ if } \xi \not\in S. \text{ If } \xi \in S, \text{ then} \]
\[ \mathfrak{F}(L_\xi) = \text{Hom}_{U(g)}(L_\xi, T_\xi^{\oplus \ell_\xi}), \quad (6.8) \]
which is obviously \( \ell_\xi \)-dimensional. Let \( v_1^\xi, ..., v_{\ell_\xi}^\xi \in T_\xi^{\oplus \ell_\xi} \) be the generators of the tilting module \( T_\xi^{\oplus \ell_\xi} \) (then \( v_1^\xi, ..., v_{\ell_\xi}^\xi \) span the generating space, denoted \( V \), of \( T_\xi^{\oplus \ell_\xi} \), and \( v_1^{\prime \xi}, ..., v_{\ell_\xi}^{\prime \xi} \in L_\xi^{\oplus \ell_\xi} \), the corresponding generators of the submodule \( L_\xi^{\oplus \ell_\xi} \) of \( T_\xi^{\oplus \ell_\xi} \). Thus, there exists a unique \( u \in U(g) \) such that
\[ v_i^{\prime \xi} = uv_i^\xi \text{ for } i = 1, ..., \ell_\xi. \quad (6.9) \]
Let \( \tilde{v}_\xi \in L_\xi \) be the generator of the simple module \( L_\xi \). As in the proof of Corollary 5.14 we can define \( f^i : L_\xi \rightarrow T_\xi^{\oplus \ell_\xi} \) to be the \( U(g) \)-homomorphism sending \( \tilde{v}_\xi \) to \( v_i^{\prime \xi} \) for \( i = 1, ..., \ell_\xi \). Then \( (f^1, ..., f^{\ell_\xi}) \) is obviously a basis of \( \mathfrak{F}(L_\xi) \) (cf. (6.8)).

For any \( A \in M_{\ell_\xi} \) (the algebra of \( \ell_\xi \times \ell_\xi \) complex matrices), we can define an element \( \phi_A \in \text{End}_{U(g)}(M_{pq}^{rt}) = B_{2,r,t} \) as follows: \( \phi_A|_{T_\xi^{\oplus \ell_\xi}} = 0 \) if \( \xi \not\in \xi \) and
\[ \phi_A|_{T_\xi^{\oplus \ell_\xi}} : (v_1^\xi, ..., v_{\ell_\xi}^\xi) \mapsto (v_1^\xi, ..., v_{\ell_\xi}^\xi)A \quad (\text{regarded as vector-matrix multiplication}), \quad (6.10) \]
i.e., the transition matrix of the action of \( \phi_A|_{T_\xi^{\oplus \ell_\xi}} \) on the generating space \( V \) of \( T_\xi^{\oplus \ell_\xi} \) under the basis \((v_1^\xi, ..., v_{\ell_\xi}^\xi)\) is \( A \). By the universal property of projective modules, this uniquely defines an element \( \phi_A \in B_{2,r,t} \). Thus we have the embedding \( \phi : M_{\ell_\xi} \rightarrow B_{2,r,t} \) sending \( A \) to \( \phi_A \). Write \( A \) as \( (a_{ij})_{i,j=1}^{\ell_\xi} \). Then by (6.10) and definition of the right action of \( B_{2,r,t} \) on \( M_{pq}^{rt} \), we have
\[ f^i(\tilde{v}_\xi)\phi_A = v_i^{\prime \xi}\phi_A = (uv_i^\xi)\phi_A = u(v_i^\xi\phi_A) = u\sum_{j=1}^{\ell_\xi} a_{ji} v_j^\xi = \sum_{j=1}^{\ell_\xi} a_{ji} v_j^{\prime \xi} = \left( \sum_{j=1}^{\ell_\xi} a_{ji} f^j \right)(\tilde{v}_\xi), \quad (6.11) \]
i.e., the transition matrix of the action of \( \phi_A \) on \( \mathfrak{F}(L_\xi) \) under the basis \((f^1, ..., f^{\ell_\xi})\) is \( A \). Thus \( \phi(M_{\ell_\xi}) \) acts transitively on the \( \ell_\xi \)-dimensional space \( \mathfrak{F}(L_\xi) \) and hence \( \mathfrak{F}(L_\xi) \) is a simple \( B_{2,r,t} \)-module. Finally, since \( L_\xi \rightarrow K_{\xi^{\text{top}}} \), we have \( \mathfrak{F}(K_{\xi^{\text{top}}}) \rightarrow \mathfrak{F}(L_\xi) \). Note that \( D^{f,\mu^{\prime},(\nu^{\prime})^{\prime}} \) is the simple head of \( \mathfrak{F}(K_{\xi^{\text{top}}}) \). Thus, \( \mathfrak{F}(L_\xi) \cong D^{f,\mu^{\prime},(\nu^{\prime})^{\prime}} \).

**Theorem 6.14.** Suppose \((f, \alpha, \beta) \in \Lambda_{2,r,t} \) such that there is a \( \lambda \in S \) (cf. (6.7)) satisfying
\[ \lambda_{\text{top}} = \lambda_{pq} + \alpha - \beta. \]
If \( \mu := (\ell, \gamma, \delta) \in \Lambda_{2,r,t} \), then \( [C(\ell, \gamma^{\prime}, (\delta^{\prime})^{\prime}) : D^{f,\alpha^{\prime},(\beta^{\prime})^{\prime}}] = (T_\lambda : K_\mu) \).

**Proof.** The result follows from Lemma 6.8, Propositions 6.9 and 6.13 together with the BGG reciprocity formula for \( g \). □
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