The semilinear Klein-Gordon equation
in de Sitter spacetime

Karen Yagdjian*

Abstract

In this article we study the blow-up phenomena for the solutions of the semilinear Klein-Gordon equation \( \Box_g \phi - m^2 \phi = -|\phi|^p \) with the small mass \( m \leq n/2 \) in de Sitter space-time with the metric \( g \). We prove that for every \( p > 1 \) the large energy solution blows up, while for the small energy solutions we give a borderline \( p = p(m,n) \) for the global in time existence. The consideration is based on the representation formulas for the solution of the Cauchy problem and on some generalizations of the Kato’s lemma.

1 Introduction

In this article we study the blow-up phenomena for the solutions of the semilinear Klein-Gordon equation \( \Box_g \phi - m^2 \phi = -|\phi|^p \) with the small mass \( m \leq n/2 \) in de Sitter space-time.

In the model of the universe proposed by de Sitter the line element has the form

\[
ds^2 = - \left( 1 - \frac{2M_{bh}}{r} - \frac{\Lambda r^2}{3} \right) c^2 dt^2 + \left( 1 - \frac{2M_{bh}}{r} - \frac{\Lambda r^2}{3} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).\]

The constant \( M_{bh} \) may have a meaning of the “mass of the black hole”. The corresponding metric with this line element is called the Schwarzschild - de Sitter metric.

The Cauchy problem for the semilinear Klein-Gordon equation in Minkowski spacetime (\( M_{bh} = \Lambda = 0 \)) is well investigated. (See, e.g., [7] and references therein.) In particular, Keel and Tao [7] for the semilinear equation \( u_t - \Delta u = F(u) \), \( u(0,x) = \varphi_0(x) \), \( u_t(0,x) = \varphi_1(x) \) proved that if \( n = 1, 2, 3 \) and \( 1 < p < 1 + 2n \), then there exists a (non-Hamiltonian) nonlinearity \( F \) satisfying \( |D^p F(u)| \leq C|u|^{p-\alpha} \) for \( 0 \leq \alpha \leq |p| \) and such that there is no finite energy global solution supported in the forward light cone, for any nontrivial smooth compactly supported \( \varphi_0 \) and \( \varphi_1 \) and for any \( \varepsilon > 0 \). There is an interesting question of instability of the ground state standing solutions \( e^{i\omega t} \phi_\omega(x) \) for nonlinear Klein-Gordon equation \( \partial_t^2 u - \Delta u + u = |u|^{p-1}u \). Here \( \phi_\omega \) is a ground state of the equation \( -\Delta \phi + (1 - \omega^2)\phi = |\phi|^{p-1}\phi \), while \( 0 < p < 1 < 4/(N - 2) \) and \( 0 \leq |\omega| < 1 \). Ohta and Todorova [9] showed that instability occurs in the very strong sense that an arbitrarily small perturbation of the initial data can make the perturbed solution blow up in finite time.

The Cauchy problem for the linear wave equation without source term on the maximally extended Schwarzschild - de Sitter spacetime in the case of non-extremal black-hole corresponding to parameter values \( 0 < M_{bh} < \frac{\Lambda}{2\sqrt{\Lambda}} \), is considered by Dafermos and Rodnianski [3]. They proved that in the region bounded by a set of black/white hole horizons and cosmological horizons, solutions converge pointwise to a constant faster than any given polynomial rate, where the decay is measured with respect to natural future-directed advanced and retarded time coordinates. The bounds on decay rates for solutions to the wave equation in the Schwarzschild - de Sitter spacetime is a first step to a mathematical understanding of non-linear stability problems for spacetimes containing black holes.

Catania and Georgiev [2] studied the Cauchy problem for the semilinear wave equation \( \Box_g \phi = |\phi|^p \) in the Schwarzschild metric \((3+1)-\)dimensional space-time, that is the case of \( \Lambda = 0 \) in \( 0 < M_{bh} < \frac{\Lambda}{3\sqrt{\Lambda}} \).

*Correspondence: Department of Mathematics, University of Texas-Pan American 1201 W. University Drive, Edinburg, TX 78541-2999, USA; E-mail:yagdjian@utpa.edu.
They established that the problem in the Regge-Wheeler coordinates is locally well-posed in $H^\sigma$ for any $\sigma \in [1, p + 1)$. Then for the special choice of the initial data they proved the blow-up of the solution in two cases: (a) $p \in (1, 1 + \sqrt{2})$ and small initial data supported far away from the black hole; (b) $p \in (2, 1 + \sqrt{2})$ and large data supported near the black hole. In both cases, they also gave an estimate from above for the lifespan of the solution.

In the present paper we focus on the another limit case as $M_{bh} \to 0$ in $0 < M_{bh} < \frac{1}{3\sqrt{\Lambda}}$, namely, we set $M_{bh} = 0$ to ignore completely influence of the black hole. Thus, the line element in de Sitter spacetime has the form

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)c^2 dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

The Friedmann cosmological model [8] leads to the following form for the line element: $$ds^2 = -c^2 dt^2 + c^2 \frac{e^{2ct/R}}{1 - \frac{r^2}{R^2}}(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2).$$ By defining coordinates $x', y', z'$ connected with $r', \theta', \phi'$ by the usual equations connecting Cartesian coordinates and polar coordinates in a Euclidean space, the line element may be written [8, Sec.134]

$$ds^2 = -c^2 dt'^2 + e^{2ct'/R}(dx'^2 + dy'^2 + dz'^2).$$

The new coordinates $r', \theta', \phi', t'$ can take all values from $-\infty$ to $\infty$. Here $R$ is the “radius” of the universe.

In this paper we study blow-up phenomena for semilinear equation by applying the Friedmann cosmological model and by employing the fundamental solutions for some model linear hyperbolic equation with variable speed of propagation. In [16] the Klein-Gordon operator in Robertson-Walker spacetime, that is $S := \partial_t^2 - e^{-2t} \Delta + M^2$, is considered. The fundamental solution $E = E(x, t; x_0, t_0)$, that is solution of $SE = \delta(x - x_0, t - t_0)$, with a support in the forward light cone $D_+(x_0, t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and the fundamental solution with a support in the backward light cone $D_-(x_0, t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, defined by $D_\pm(x_0, t_0) := \{(x, t) \in \mathbb{R}^{n+1} : |x - x_0| \leq \pm(e^{-t_0} - e^{-t})\}$, are constructed. These fundamental solutions have been used to represent solutions of the Cauchy problem and to prove $L^p - L^q$ estimates for the solutions of the equation with and without a source term that provide with some necessary tools for the studying semilinear equations.

In the Robertson-Walker spacetime [5], one can choose coordinates so that the metric has the form

$$ds^2 = -dt^2 + S^2(t)dx^2.$$ In particular, the metric in de Sitter and anti-de Sitter spacetime in the Friedmann coordinates [8] has this form with $S(t) = e^t$ and $S(t) = e^{-t}$, respectively. The matter waves in the de Sitter spacetime are described by the function $\phi$, which satisfies equations of motion. In the de Sitter universe the equation for the scalar field with mass $m$ and potential function $V$ is the covariant Klein-Gordon equation

$$\Box g \phi - m^2 \phi = V'(\phi) \quad \text{or} \quad \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ik} \partial_k \phi \right) - m^2 \phi = V'(\phi),$$

with the usual summation convention. Written explicitly in coordinates in the de Sitter spacetime it, in particular, for $V'(\phi) = -|\phi|^p$ has the form

$$\phi_{tt} + n\phi_t - e^{-2t} \Delta \phi + m^2 \phi = |\phi|^p.$$

In this paper we restrict ourselves with consideration of the semilinear equation for particle with small mass $m$, that is $0 \leq m \leq n/2$. If we introduce the new unknown function $u = e^{\frac{n}{2}t}\phi$, then it takes the form of the semilinier Klein-Gordon equation for $u$ on de Sitter spacetime

$$u_{tt} - e^{-2t} \Delta u - M^2 u = e^{-\frac{n(n-1)}{2}}|u|^p,$$
where non-negative curved mass $M \geq 0$ is defined as follows:

$$M^2 := \frac{n^2}{4} - m^2 \geq 0.$$  

The equation (2) can be regarded as Klein-Gordon equation with imaginary mass. Equations with imaginary mass appear in several physical models such as $\phi^4$ field model, tachyon (super-light) fields, Landau-Ginzburg-Higgs equation and others. To solve the Cauchy problem for semilinear equation we use fundamental solution of the corresponding linear operator. We denote by $L$ the decay of characteristic roots, and without loss of generality we consider the spatial point $x$ with respect to usual scaling and that creates additional difficulties. The phenomenon causes the blow up of the solution. The equation (3) is neither Lorentz invariant nor invariant with respect to usual scaling and that creates additional difficulties.

Thus, the speed of propagation is variable, namely, it is equal to $e^{-t}$. The second-order strictly hyperbolic equation (3) possesses two fundamental solutions resolving the Cauchy problem without source term $f$. They can be written in terms of the Fourier integral operators, which give complete description of the wave front sets of the solutions. Moreover, the integrability of the characteristic roots, $\int_0^{\infty} |\lambda_i(t, \xi)|dt < \infty$, $i = 1, 2$, leads to the existence of the so-called “horizon” for that equation. More precisely, any signal emitted from the spatial point $x_0 \in \mathbb{R}^n$ at time $t_0 \in \mathbb{R}$ remains inside the ball $B_{t_0}(x_0) := \{ x \in \mathbb{R}^n \mid |x - x_0| < e^{-t_0} \}$ for all time $t \in (t_0, \infty)$. In particular, it can cause a nonexistence of the $L^p - L^q$ decay for the solutions. In [13] this phenomenon is illustrated by a model equation with permanently bounded domain of influence, power decay of characteristic roots, and without $L^p - L^q$ decay. The above mentioned $L^p - L^q$ decay estimates are one of the important tools for studying nonlinear problems (see, e.g., [11]). In this paper we show that this phenomenon causes the blow up of the solution. The equation (3) is neither Lorentz invariant nor invariant with respect to usual scaling and that creates additional difficulties.

Operator $G$ is constructed in [16] for the case of the large mass $m \geq n/2$. The analytic continuation of this operator in parameter $M$ into $\mathbb{C}$ allows us to use $G$ also in the case of small mass $0 \leq m \leq n/2$. More precisely, we define the operator $G$ acting on $f(x, t) \in C^\infty(\mathbb{R} \times [0, \infty))$ by

$$G[f](x, t) := \int_0^t \int_{x(e^{-n} - e^{-t})}^{x(e^{-n} - e^{-t})} dy \, f(y, b)(4e^{-b-t})^{-M} \left( (e^{-t} + e^{-b})^2 - (x - y)^2 \right)^{-\frac{1}{2} + M} \times F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - (x - y)^2}{(e^{-b} + e^{-t})^2 - (x - y)^2} \right),$$

where $F(a, b; c; \cdot)$ is the hypergeometric function. (See, e.g., [13].) If $n$ is odd, $n = 2m + 1$, $m \in \mathbb{N}$, then for $f \in C^\infty(\mathbb{R}^n \times [0, \infty))$, we define

$$G[f](x, t) := 2 \int_0^t \int_0^{e^{-b} - e^{-t}} \, dr_1 \left( \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \frac{n-3}{2} \int_{S^{n-1}} f(x + r y, b) \, dS_y \right)_{r=r_1} \times 4(e^{-b} - e^{-t})^{-M} \left( (e^{-t} + e^{-b})^2 - r_1^2 \right)^{-\frac{1}{2} + M} \times F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r_1^2}{(e^{-b} + e^{-t})^2 - r_1^2} \right),$$

where $c_0(n) = 1 \cdot 3 \cdot \ldots \cdot (n - 2)$. Constant $\omega_{n-1}$ is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

If $n$ is even, $n = 2m$, $m \in \mathbb{N}$, then for $f \in C^\infty(\mathbb{R}^n \times [0, \infty))$, the operator $G$ is given by the next expression

$$G[f](x, t) := 2 \int_0^t \int_0^{e^{-b} - e^{-t}} \, dr_1 \left( \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \frac{n-2}{2} \int_{S^{n-1}} f(x + r y, b) \, dS_y \right)_{r=r_1} \times 2^{n-1} \omega_{n-1} c_0^{(n)} \int_{B^+_1(0)} f(x + r y, b) \, dV_y \bigg)_{r=r_1},$$

where $c_0^{(n)} = \int_0^{\infty} e^{-r} r^{n-1} dr$. The equation (3) is strictly hyperbolic. This implies the well-posedness of the Cauchy problem (3) in the different functional spaces. Consequently, the operator is well-defined in those functional spaces.
bounded operator: Then any solution $u \in \phi G W$. We are especially interested in the scale of functions $\Gamma(x)$, where the function $v$ if $M > 1$, then for every given $T > 0$ the operator $G$ can be extended to the bounded operator:

$$G : C([0, T]; L^q(\mathbb{R}^n)) \to C([0, T]; L^q(\mathbb{R}^n)).$$

Consequently the operator $G$ maps

$$G : C([0, \infty); L^q(\mathbb{R}^n)) \to C([0, \infty); L^q(\mathbb{R}^n)),$$

in the corresponding topologies. Moreover,

$$G : C([0, \infty); L^q(\mathbb{R}^n)) \to C^1([0, \infty); D'(\mathbb{R}^n)).$$

Let $u_0 = u_0(x, t)$ be a solution of the Cauchy problem

$$\partial_t^2 u_0 - e^{-2t} \Delta u_0 - M^2 u_0 = 0, \quad u_0(x, 0) = \varphi_0(x), \quad \partial_t u_0(x, 0) = \varphi_1(x). \quad (4)$$

Then any solution $u = u(x, t)$ of the equation (2) which takes initial value $u(x, 0) = \varphi_0(x), \quad \partial_t u(x, 0) = \varphi_1(x)$, solves also integral equation

$$u(x, t) = u_0(x, t) + G[e^{-\frac{n(q-1)}{2}} |u|^p](x, t). \quad (5)$$

Let $\Gamma \in C([0, \infty))$. For every given function $u_0 \in C([0, T]; L^q(\mathbb{R}^n))$ we consider integral equation (5)

$$u(x, t) = u_0(x, t) + G[\Gamma(x) \left( \int_{\mathbb{R}^n} |u(y, \cdot)|^p dy \right)^\beta |u(y, \cdot)|^p](x, t), \quad (6)$$

for the function $u \in C([0, T]; L^q(\mathbb{R}^n)) \cap C([0, T]; L^p(\mathbb{R}^n))$. Here $q' \geq q > 1, p \geq 1$. The last integral equation corresponds to the slightly more general equation than (2), namely, to the nonlocal equation

$$u_{tt} - e^{-2t} \Delta u - M^2 u = \Gamma(t) \left( \int_{\mathbb{R}^n} |u(y, t)|^p dy \right)^\beta |u|^p. \quad (7)$$

If $u_0$ is generated by the Cauchy problem (4), then the solution $u = u(x, t)$ of (6) is said to be a weak solution of the Cauchy problem with the initial conditions

$$u(x, 0) = \varphi_0(x), \quad \partial_t u(x, 0) = \varphi_1(x),$$

for the equation (7). In the present paper we are looking for the conditions on the function $\Gamma$, on constants $M, n, p,$ and $\beta$ that guarantee a non-existence of global in time weak solution, namely, the blow-up phenomena. We are especially interested in the scale of functions $\Gamma(t) = (1 + t)^{d_1} e^{d_0 t}$, where $d_0, d_1 \in \mathbb{R}$. The function $e^{-\frac{n \beta + 1}{2} t}$, with $d_0 = -p(p - 1)/2$ and $d_1 = 0$ is in that scale and represents equation (2) if $\beta = 0$. In particular, we find in the next theorem the upper bound for $d_0$ with an existence of the global solution for small initial data. For equation (7) in that scale the bound is given by $d_0 \geq -M (p(\beta + 1) - 1)$ and $d_1 > 2$ if $M > 0$. 

\[ \times (4e^{-b-t})^{-M} \left( (e^{-t} + e^{-b})^2 - r_t^2 \right)^{-\frac{1}{2} + M} \]

\[ \times F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r_t^2}{(e^{-b} + e^{-t})^2 - r_t^2} \right). \]
Theorem 1.1 Suppose that function $\Gamma \in C^1([0, \infty))$ is either non-decreasing or non-increasing, and if $M > 0$ then
\[
\Gamma(t) \geq ce^{-M(p(\beta+1)-1)t^2+\varepsilon} \quad \text{for all } t \in [0, \infty),
\]
with the numbers $\varepsilon > 0$ and $c > 0$, while for $M = 0$ it satisfies
\[
\Gamma(t) \geq ct^{-1-p(\beta+1)}.
\]

Then, for every $p > 1$, $N$, and $\varepsilon$ there exists $u_0 \in C^\infty(\mathbb{R}^n \times [0, \infty))$ which for any given slice of constant time $t = \text{const} \geq 0$ has a compact support in $x$, such that $u_0(x, 0), \partial_t u_0(x, 0) \in C_0^\infty(\mathbb{R}^n)$, and
\[
\|u_0(x, 0)\|_{C^N(\mathbb{R}^n)} + \|\partial_t u_0(x, 0)\|_{C^N(\mathbb{R}^n)} < \varepsilon
\]
but a global in time solution $u \in C([0, \infty); L^q(\mathbb{R}^n))$ of the equation (6) with permanently bounded support does not exist for all $q \in [2, \infty)$ and $\beta > 1/p - 1$. More precisely, there is $T > 0$ such that
\[
\lim_{t \to T} \int_{\mathbb{R}^n} u(x, t) dx = \infty.
\]
This theorem shows that instability of the trivial solution occurs in the very strong sense, that is, an arbitrarily small perturbation of the initial data can make the perturbed solution blowing up in finite time.

If we allow large initial data, then according to the next theorem, for every $d_0 \in \mathbb{R}$ and $M > 0$ the solution blows up in finite time.

Theorem 1.2 Suppose that function $\Gamma(t) = e^{\gamma t}$, where $\gamma \in \mathbb{R}$ and that the curved mass is positive, $M > 0$. Then, for every $p > 1$ and $n$ there exists $u_0 \in C^\infty(\mathbb{R}^n \times [0, \infty))$ which for any given slice of constant time $t = \text{const} \geq 0$ has a compact support in $x$, such that $u_0(x, 0), \partial_t u_0(x, 0) \in C_0^\infty(\mathbb{R}^n)$ but a global in time solution $u \in C([0, \infty); L^q(\mathbb{R}^n))$ of the equation (6) with permanently bounded support does not exist for all $q \in [2, \infty)$ and $\beta > 1/p - 1$. More precisely, there is $T > 0$ such that
\[
\lim_{t \to T} \int_{\mathbb{R}^n} u(x, t) dx = \infty.
\]
Thus, for every $p > 1$ the large energy classical solution of the Cauchy for equation (1) blows up. We will prove global existence of the small energy solution in a forthcoming paper.

The remaining part of this paper is organized as follows. In Section 2 we prove some auxiliary integral representations for the function $\sinh(t)$ and the linear function via Gauss’s hypergeometric function and multidimensional integrals involving also fundamental solution of the Cauchy problem for wave equation in Minkowski spacetime. In Section 3 we suggest two simple generalizations of Kato’s lemma, which allow us to handle the case of differential inequalities with exponentially decreasing kernels. In Section 4 we complete the proofs of both theorems.

## 2 Integral representations of function $M^{-1} \sinh(M(t - b))$ involving hypergeometric function

In [1, Sec. 2.4] one can find one-dimensional integrals involving hypergeometric function. In this section we present one more example of such integral and also examples of multidimensional integrals appearing in the fundamental solutions for the Klein-Gordon equation in de Sitter spacetime. More examples related to the Tricomi and Gellerstedt equations one can find in [14].

**Proposition 2.1** The function $M^{-1} \sinh(M(t - b))$ with $t \geq b \geq 0$, can be represented as follows:

(i) The one-dimensional integral
\[
\frac{1}{M} \sinh(M(t - b)) = \int_{-e^{-b} - e^{-t}}^{e^{-b} - e^{-t}} (4e^{-b-t})^{-M} \left( (e^{-t} + e^{-b})^2 - z^2 \right)^{-\frac{1}{2} + M} \\
\times F \left( \frac{1}{2} - M, 1 - M; 1; \frac{e^{-b} - e^{-t} - z^2}{(e^{-b} + e^{-t})^2 - z^2} \right) dz;
\]
(ii) If \( n \) is odd, \( n = 2m + 1, m \in \mathbb{N} \), then with \( c_0^{(n)} = 1 \cdot 3 \cdots (n - 2) \),
\[
\frac{1}{M} \sinh(M(t - b)) = 2 \int_0^{e^{-b} - e^{-t}} dr_1 \left( \frac{1}{\sqrt{1 - |y|^2}} \int_{\partial B_1(0)} \frac{1}{r} \frac{1}{\omega_{n-1} c_0^{(n)}} dS_y \right) (4e^{-b} - t)^{-M} \\
\times \left( (e^{-t} + e^{-b})^2 - r_1^2 \right)^{\frac{n}{2} + M} F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r_1^2}{(e^{-b} + e^{-t})^2 - r_1^2} \right).
\]

(iii) If \( n \) is even, \( n = 2m, m \in \mathbb{N} \), then with \( c_0^{(n)} = 1 \cdot 3 \cdots (n - 1) \),
\[
\frac{1}{M} \sinh(M(t - b)) = 2 \int_0^{e^{-b} - e^{-t}} dr_1 \left( \frac{1}{\sqrt{1 - |y|^2}} \int_{\partial B_1(0)} \frac{1}{r} \frac{1}{\omega_{n-1} c_0^{(n)}} dS_y \right) (4e^{-b} - t)^{-M} \\
\times \left( (e^{-t} + e^{-b})^2 - r_1^2 \right)^{-\frac{n}{2} - M} F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r_1^2}{(e^{-b} + e^{-t})^2 - r_1^2} \right).
\]

Here the constant \( \omega_{n-1} \) is the area of the unit sphere \( S^{n-1} \subset \mathbb{R}^n \).

**Proof.** First we consider case (i). According to Theorem 0.3 [16] for every function \( f \in C^\infty(\mathbb{R} \times [0, \infty)) \), which for any given slice of constant time \( t = const \geq 0 \) has a compact support in \( x \), the function
\[
v(x, t) = \int_0^t db \int_{x-(e^{-b}-e^{-t})}^{x+(e^{-b}-e^{-t})} dy \left( (e^{-t} + e^{-b})^2 - (x - y)^2 \right)^{-\frac{n}{2} + M} \\
\times F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - (x - y)^2}{(e^{-b} + e^{-t})^2 - (x - y)^2} \right) f(y, b)
\]
is a unique \( C^\infty \)-solution to the Cauchy problem
\[
\partial_t^2 v - e^{-2t} \Delta v - M^2 v = f, \quad v(x, 0) = 0, \quad \partial_t v(x, 0) = 0 \quad (9)
\]
with \( n = 1 \). It follows
\[
\int_{-\infty}^\infty v(x, t) dx = \int_0^t db \left( \int_{-\infty}^\infty f(x, b) dx \right) \int_{x-(e^{-b}-e^{-t})}^{x+(e^{-b}-e^{-t})} dz \left( (e^{-t} + e^{-b})^2 - z^2 \right)^{-\frac{n}{2} + M} \\
\times F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - z^2}{(e^{-b} + e^{-t})^2 - z^2} \right). \quad (10)
\]

On the other hand, from the linear Klein-Gordon equation (9) and the vanishing initial data, we obtain
\[
\int_{-\infty}^\infty v(x, t) dx - M^2 \int_0^t d\tau \int_{-\infty}^\infty v(x, b) dx = \int_0^t d\tau \int_0^\infty d\tau_1 \int_{-\infty}^\infty v(x, t) dx \\
- \int_0^t d\tau \int_0^\infty d\tau_1 \int_{-\infty}^\infty e^{-2b} \partial_x^2 v(x, b) dx + \int_0^t d\tau \int_0^\infty d\tau_1 \int_{-\infty}^\infty f(x, b) dx,
\]
that is
\[
\int_{-\infty}^\infty v(x, t) dx - M^2 \int_0^t d\tau \int_{-\infty}^\infty v(x, b) dx = \int_0^t d\tau \int_0^\infty d\tau_1 \int_{-\infty}^\infty v(x, t) dx. \quad (11)
\]
Denote

\[ V(t) = \int_0^t d\tau \int_0^\tau db \int_{-\infty}^{\infty} v(x, b) dx, \quad F(t) := \int_0^t \left( \int_{-\infty}^{\infty} f(x, b) dx \right) (t - b) db, \quad (12) \]

then (11) and (12) imply

\[ V_{tt} - M^2 V = F, \quad V(0) = V_t(0) = 0. \]

We easily find

\[ V(t) = \frac{1}{M} \int_0^t F(\tau) \sinh(M(t - \tau)) d\tau. \]

Then (11) implies

\[
\int_{-\infty}^{\infty} v(x, t) dx = \int_0^t \left( \int_{-\infty}^{\infty} f(x, b) dx \right) (t - b) db + M \int_0^t F(\tau) \sinh(M(t - \tau)) d\tau
\]

\[ = \int_0^t \left( \int_{-\infty}^{\infty} f(x, b) dx \right) (t - b) db
\]

\[ + M \int_0^t db \left( \int_{-\infty}^{\infty} f(x, b) dx \right) \left( - \frac{1}{M} (t - b) + \frac{1}{M^2} \sinh(M(t - b)) \right). \]

On the other hand

\[ \int_b^t d\tau (\tau - b) \sinh(M(t - \tau)) = -\frac{1}{M} (t - b) + \frac{1}{M^2} \sinh(M(t - b)) \]

implies

\[ \int_{-\infty}^{\infty} v(x, t) dx
\]

\[ = \int_0^t \left[ \int_{-\infty}^{\infty} f(x, b) dx \right] (t - b) db
\]

\[ + M \int_0^t db \left[ \int_{-\infty}^{\infty} f(x, b) dx \right] \left( - \frac{1}{M} (t - b) + \frac{1}{M^2} \sinh(M(t - b)) \right)
\]

\[ = \int_0^t \left[ \int_{-\infty}^{\infty} f(x, b) dx \right] \left( \frac{1}{M} \sinh(M(t - b)) \right) db. \]

Thus, for the arbitrary function \( f \in C^\infty(\mathbb{R} \times [0, \infty)) \) for all \( t \) due to (10) one has

\[ \int_0^t \left( \int_{-\infty}^{\infty} f(x, b) dx \right) \left( \frac{1}{M} \sinh(M(t - b)) \right) db
\]

\[ = \int_0^t db \left( \int_{-\infty}^{\infty} f(x, b) dx \right) \int_{\frac{1}{e^{-b} - e^{-t}}}^{e^{b} - e^{-t}} dz \left( e^{t} + e^{-b} - z^2 \right)^{-\frac{1}{2} + M}
\]

\[ \times F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{b} - e^{-t})^2 - z^2}{(e^{b} + e^{-t})^2 - z^2} \right). \]

It follows (8). Thus (i) is proved.

To prove case (ii) with \( n \) is odd, \( n = 2m + 1, \ m \in \mathbb{N} \), we use the identity

\[ 1 = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \frac{\omega_{n-1}^{(e)} e_0^{(n)} \int_{S^{n-1}} dS_y}{\omega_{n-1}^{(e)} e_0^{(n)} \int_{S^{n-1}} dS_y}
\]

and take into consideration that the kernel

\[ (4e^{-b-t})^{-M} \left( (e^{-t} + e^{-b})^2 - r_1^2 \right)^{-\frac{1}{2} + M} F \left( \frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{b} - e^{-t})^2 - z^2}{(e^{b} + e^{-t})^2 - z^2} \right)
\]
is an even function of \( r_1 \). In the case of (iii) when \( n \) is even, \( n = 2m, m \in \mathbb{N} \), we apply the identity
\[
1 = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) e^{-n-2} \int_{B^*_1(0)} \frac{1}{\sqrt{1-|y|^2}} dV_y.
\]
The proposition is proven. \( \square \)

If we set in the above integrals \( b = 0 \) then we get integral representations of the function \( \sinh(Mt) \) depending on parameter \( M > 0 \). We also note that both sides of these formulas are translation invariant in \( t \). By passing to the limit as \( M \to 0 \) we arrive at the following corollary.

**Corollary 2.2** The function \( t - b \) with \( t \geq b \geq 0 \), can be represented as follows:

(i) The one-dimensional integral
\[
t - b = \int_{0}^{e^{b}-e^{-t}} \left( (e^{-t} + e^{-b})^2 - z^2 \right)^{-\frac{1}{2}} F\left( \frac{1}{2}, \frac{1}{2} ; 1 ; \frac{(e^{-b} - e^{-t})^2 - z^2}{(e^{b} + e^{-t})^2 - z^2} \right) dz;
\]

(ii) If \( n \) is odd, \( n = 2m + 1, m \in \mathbb{N} \), then with \( c^{(n)}_0 = 1 \cdot 3 \cdot \ldots \cdot (n - 2) \),
\[
t - b = 2 \int_{0}^{e^{b}-e^{-t}} \left( \frac{\partial}{\partial r} \right) \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \frac{r^{n-2}}{\omega_{n-1} c^{(n)}_0} \int_{S^{n-1}} dS_y \left. \right|_{r=r_1} \times \left( (e^{-t} + e^{-b})^2 - r_1^2 \right)^{-\frac{1}{2}} F\left( \frac{1}{2}, \frac{1}{2} ; 1 ; \frac{(e^{-b} - e^{-t})^2 - r_1^2}{(e^{b} + e^{-t})^2 - r_1^2} \right) ;
\]

(iii) If \( n \) is even, \( n = 2m, m \in \mathbb{N} \), then \( c^{(n)}_0 = 1 \cdot 3 \cdot \ldots \cdot (n - 1) \),
\[
t - b = 2 \int_{0}^{e^{b}-e^{-t}} \left( \frac{\partial}{\partial r} \right) \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \frac{r^{n-2}}{\omega_{n-1} c^{(n)}_0} \int_{B^*_1(0)} \frac{1}{\sqrt{1-|y|^2}} dV_y \left. \right|_{r=r_1} \times \left( (e^{-t} + e^{-b})^2 - r_1^2 \right)^{-\frac{1}{2}} F\left( \frac{1}{2}, \frac{1}{2} ; 1 ; \frac{(e^{-b} - e^{-t})^2 - r_1^2}{(e^{b} + e^{-t})^2 - r_1^2} \right).
\]

### 3 The second order differential inequalities

The second order differential inequality with the power decreasing kernel play key role in proving blow-up of the solutions of the semilinear equations. Kato’s lemma [6] allows us to derive from inequality

\[ \ddot{w} \geq bt^{-1-p}w^p, \quad p > 1, \ b > 0, \ t \ \text{large} \]

a boundedness of the life-span of solution with property \( w_t \geq a > 0 \). For the equation in de Sitter spacetime the kernel of the corresponding ordinary differential inequality decreases exponentially:

\[ \ddot{w} \geq be^{-Mt}w^p, \quad p > 1, \ b > 0, \ M > 0, \ t \ \text{large}. \]

There is a global solution to the last inequality. Hence, in order to reach exact conditions on the involving functions we have to generalize Kato’s lemma. It is done in two following lemmas.

**Lemma 3.1** Suppose \( F(t) \in C^2([a,b]), \) and
\[ F(t) \geq 0, \quad \dot{F}(t) \geq 0, \quad \ddot{F}(t) \geq \Gamma(t)F(t)^p \quad \text{for all} \quad t \in [a,b], \]
where \( \Gamma \in C^1([a,\infty)) \) is non-negative function, \( \Gamma(a) > 0, \) and \( p > 1 \). Assume that for all \( t \in [a,b] \) either
\[ \dot{F}(t) \leq 0 \quad \text{or} \quad \Gamma(t) \geq \text{const} > 0. \]

If there exists \( a_1 \in (a,b) \) such that
\[ \frac{1}{\sqrt{p+1}} \int_{a}^{a_1} \Gamma(s)^{1/2} ds > \frac{\sqrt{2}}{p-1} F(a)^{(1-p)/2}, \quad \dot{F}(a)^2 \geq \frac{2}{p+1} \Gamma(a)F(a)^{p+1}, \]

then \( b \) must be finite unless \( \lim_{t \to \infty} F(t) \) is finite.
**Proof.** First we consider the case of $\dot{\Gamma} \leq 0$. The conditions of the lemma imply that derivative of the energy density function is non-negative,

$$\frac{d}{dt} \left( F_t(t)^2 - \frac{2}{p+1} \Gamma(t) F(t)^{p+1} \right) \geq 0 \quad \text{for all} \quad t \in [a, b].$$

We integrate the last inequality and obtain

$$F_t(t)^2 \geq \frac{2}{p+1} \Gamma(t) F(t)^{p+1} + F_t(a)^2 - \frac{2}{p+1} \Gamma(a) F(a)^{p+1} \quad \text{for all} \quad t \in [a, b].$$

In fact, according to the second inequality of the condition (13) we have

$$F_t(t)^2 \geq \frac{2}{p+1} \Gamma(t) F(t)^{p+1} \quad \text{for all} \quad t \in [a, b].$$

Hence,

$$F_t(t) \geq \sqrt{\frac{2}{p+1} \Gamma(t)^{1/2} F(t)^{(p+1)/2}} \quad \text{for all} \quad t \in [a, b].$$

It follows

$$\frac{d}{dt} \left( \frac{2}{1-p} F^{1-(p+1)/2}(t) \right) \geq \sqrt{\frac{2}{p+1} \Gamma(t)^{1/2}} \quad \text{for all} \quad t \in [a, b].$$

Consequently,

$$\frac{2}{1-p} F^{(1-p)/2}(t) - \frac{2}{1-p} F^{(1-p)/2}(a) \geq \sqrt{\frac{2}{p+1}} \int_a^t \Gamma(s)^{1/2} ds.$$

According to the first inequality of the condition (13) there exists $a_1 > a$ such that

$$\frac{2}{1-p} F^{(1-p)/2}(t) \geq \sqrt{\frac{2}{p+1} \int_{a_1}^t \Gamma(s)^{1/2} ds + \sqrt{\frac{2}{p+1} \int_a^{a_1} \Gamma(s)^{1/2} ds}} - \frac{2}{p-1} F^{(1-p)/2}(a)$$

for all $t \in [a_1, b)$. Thus, for large $t$ we get contradiction. The case of uniformly positive function $\Gamma$ follows from Kato’s Lemma [6]. Lemma is proven.

Next we turn to the case of the small energy and exponentially decreasing $\Gamma(t)$.

**Lemma 3.2** Suppose $F(t) \in C^2([a, b])$, and

$$F(t) \geq c_0 A(t), \quad F_t(t) \geq 0, \quad F_{tt}(t) \geq \gamma(t) A(t)^{-p} F(t)^p \quad \text{for all} \quad t \in [a, b],$$

where $A, \gamma \in C^1([a, \infty))$ are non-negative functions and $p > 1, c_0 > 0$. Assume that

$$\lim_{t \to \infty} A(t) = \infty,$$

and that

$$\frac{d}{dt} \left( \gamma(t) A(t)^{-p} \right) \leq 0 \quad \text{for all} \quad t \in [a, b].$$

If there exist $\varepsilon > 0$ and $c > 0$ such that

$$\gamma(t) \geq c A(t) (\ln A(t))^{2+\varepsilon} \quad \text{for all} \quad t \in [a, b],$$

then $b$ must be finite.
The last nonlinear differential inequality does not have a global solution. We note here that the equation

\[ F(t) \geq \frac{1}{2} F(t_1) \quad \text{for all} \quad t \in [a_2, b), \]

for sufficiently large \( a_2 \). Furthermore, according to (16) for the energy density function we have

\[ \frac{d}{dt} \left( F(t)^2 - 2 - \frac{1}{p+1} \gamma(t) A(t)^{-p} F(t)^{p+1} \right) \geq 0 \quad \text{for all} \quad t \in [a_1, b). \]

The last inequality implies

\[ F_i(t)^2 \geq 2 \frac{1}{p+1} \gamma(t) A(t)^{-p} F(t)^{p+1} + F_i(a_1)^2 - 2 \frac{1}{p+1} \gamma(a_1) A(a_1)^{-p} F(a_1)^{p+1} \]

for all \( t \in [a_1, b) \). For sufficiently large \( a_2 \geq a_1 \) using conditions (14), (15), and (17) we derive

\[ \frac{1}{p+1} \gamma(t) A(t)^{-p} F(t)^{p+1} \geq \frac{1}{p+1} c_0^{p+1} \gamma(t) A(t) \]

\[ \geq \frac{1}{p+1} c c_0^{p+1} A(t)^2 (\ln A(t))^{2+\varepsilon} \]

\[ \geq F_i(a_1)^2 - 2 \frac{1}{p+1} \gamma(a_1) A(a_1)^{-p} F(a_1)^{p+1} \]

for all \( t \in [a_2, b) \). Hence,

\[ F_i(t) \geq \sqrt{\frac{1}{p+1} \gamma(t)^{1/2} A(t)^{-p/2} F(t)^{(p+1)/2}} \quad \text{for all} \quad t \in [a_2, b). \]

It follows

\[ F_i(t) \geq \delta_\gamma(t)^{1/2} A(t)^{-p/2} F(t)^{(p-1)/2} (\ln F(t))^{-1-\varepsilon/2} F(t) (\ln F(t))^{1+\varepsilon/2} \]

for all \( t \in [a_2, b) \). But with sufficiently large \( a_2 \geq a_1 \) we obtain

\[ F(t)^{(p-1)/2} (\ln F(t))^{-1-\varepsilon/2} \geq \delta A(t)^{(p-1)/2} (\ln A(t))^{-1-\varepsilon/2} \quad \text{for all} \quad t \in [a_2, b). \]

Hence,

\[ F_i(t) \geq \delta (\gamma(t) A(t)^{-1} (\ln A(t))^{-2-\varepsilon})^{1/2} F(t) (\ln F(t))^{1+\varepsilon/2} \quad \text{for all} \quad t \in [a_2, b) \]

implies

\[ F_i(t) \geq \delta c F(t) (\ln F(t))^{1+\varepsilon/2} \quad \text{for all} \quad t \in [a_2, b). \]

The last nonlinear differential inequality does not have a global solution with \( F > 0 \). Lemma is proven. \( \square \)

**Remark 3.3** We note here that the equation

\[ \tilde{F}(t) = e^{-dt} F(t)^p, \quad d > 0, \]

has a global solution \( F(t) = c_F e^{\frac{d}{(p-1)} t} \), where \( c_F = (d/(p-1))^{2/(p-1)} \), while corresponding \( A(t) = c_A e^{at} \), \( a > 0 \), and \( \gamma(t) = c e^{(p-1) dt} \). The condition (17) implies \( a > d/(p-1) \). On the other hand, the first inequality of (14) holds only if \( a \leq d/(p-1) \).

### 4 Nonexistence of the global solution for the integral equation associated with the Klein-Gordon equation

Since \( G \) is a fundamental solution of the strictly hyperbolic operator, for every given function \( u_0 \in C([0, T]; L^3(\mathbb{R}^n)) \cap C^\infty([0, T] \times \mathbb{R}^n) \) there exist \( T_0 > 0 \) and solution \( u \in C([0, T_0]; L^3(\mathbb{R}^n)) \). Moreover, for every given
one can prove existence of the solution \( u \in C([0,T]; L^q(\mathbb{R}^n)) \) provided that \( \sup_{t \in [0,T]} \|u_0(\cdot, t)\|_{L^q(\mathbb{R}^n)} \) is small enough. Theorem 1.1 shows that the set of such \( T \), in general, is bounded.

**Proof of Theorem 1.1.** Let \( u_0 \in C^\infty([0,\infty) \times \mathbb{R}^n) \) be a function with the permanently bounded support, that is \( \text{supp} \ u_0(\cdot, t) \subset \{ x \in \mathbb{R}^n : |x| \leq \text{constant} \} \) for all \( t \geq 0 \). We denote \( \varphi_0(x) := u_0(x,0) \) and \( \varphi_1(x) := \partial_t u_0(x,0) \). One can find \( u_0 \) such that

\[
\int_{\mathbb{R}^n} u_0(x,t)\,dx = C_0 \cosh(Mt) + C_1 \frac{1}{M} \sinh(Mt) \quad \text{for all} \quad t \geq 0,
\]

where

\[
C_0 := \int_{\mathbb{R}^n} \varphi_0(x)\,dx, \quad C_1 := \int_{\mathbb{R}^n} \varphi_1(x)\,dx.
\]

The solution of the problem (4) with the data \( \varphi_0(x) \) and \( \varphi_1(x) \in C_0^\infty(\mathbb{R}^n) \) exemplifies such function. Indeed, this unique smooth solution obeys finite propagation speed property that implies \( \text{supp} \ u_0(\cdot, t) \subset \{ x \in \mathbb{R}^n : |x| \leq R_0 + 1 - e^{-t} \leq R_0 + 1 \} \) if \( \sup \varphi_0 \), \( \sup \varphi_1 \subset \{ x \in \mathbb{R}^n : |x| \leq R_0 \} \). In order to check (18) for that solution \( u_0 \) we integrate (4) with respect to \( x \) over \( \mathbb{R}^n \) and then solve the initial problem with data (19) for the obtained ordinary differential equation.

Suppose that \( u \in C([0,\infty); L^q(\mathbb{R}^n)) \) with permanently bounded support is a solution to (6) generated by \( u_0 \). According to the definition of the solution, for every given \( T > 0 \) we have

\[
G \left[ \Gamma(\cdot) \left( \int_{\mathbb{R}^n} |u(y,\cdot)|^p\,dy \right)^\beta |u| \right] \in C([0,T]; L^q(\mathbb{R}^n))
\]

and \( u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x) \). Then \( u \in C([0,\infty); L^1(\mathbb{R}^n)) \) and we can integrate equation (6):

\[
\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_0(x,t)\,dx + \int_{\mathbb{R}^n} G \left[ \Gamma(\cdot) \left( \int_{\mathbb{R}^n} |u(y,\cdot)|^p\,dy \right)^\beta |u| \right](x,t)\,dx.
\]

In particular,

\[
\int_{\mathbb{R}^n} u(x,0)\,dx = \int_{\mathbb{R}^n} \varphi_0(x)\,dx, \quad \int_{\mathbb{R}^n} u_t(x,0)\,dx = \int_{\mathbb{R}^n} \varphi_1(x)\,dx.
\]

To evaluate the last term of (20) we apply Proposition 2.1. Consider the case of odd \( n \geq 3 \). Then, for the smooth function \( u = u(x,t) \) we obtain

\[
\int_{\mathbb{R}^n} G[\Gamma(\cdot)|u|^p](x,t)\,dx = \int_{\mathbb{R}^n} dx 2 \int_0^t \frac{e^{-b-t} - e^{-t}}{t} \,db \int \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \frac{r^{n-2}}{\omega_{n-1}\ncol{n} e^{-r}} \,dS_r \\
\times (4e^{-b-t} - M) \left( (e^{-t} + e^{-b})^2 - r_2^2 \right)^{-\frac{1}{2} + M} \times F \left( \frac{1}{2} - M, \frac{1}{2} - M; \frac{1}{2}; \frac{(e^{-b} - e^{-t})^2 - r_1^2}{(e^{-b} + e^{-t})^2 - r_1^2} \right).
\]

Therefore,

\[
\int_{\mathbb{R}^n} G[\Gamma(\cdot)|u|^p](x,t)\,dx = 2 \int_0^t \frac{e^{-b-t} - e^{-t}}{t} \,db \int \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \frac{r^{n-2}}{\omega_{n-1}\ncol{n} e^{-r}} \,dS_r
\]

11
\[ \times \int_{S^{n-1}} \left[ \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p dz \right)^{\beta} \left( \int_{\mathbb{R}^n} |u(x + ry, b)|^p dx \right) \right] dS_y \bigg|_{r=r_1} \]
\[ \times (4e^{-b-t} - M \left( (e^{-t} + e^{-b})^2 - r_1^2 \right)^{-\frac{1}{2} + M} \]
\[ \times F \left( \frac{1}{2} - M, \frac{1}{2} - M; \frac{1}{2}; \frac{1}{(e^{-b} + e^{-t})^2 - r_1^2} \right) \]
implies,
\[ \int_{\mathbb{R}^n} G[\Gamma(\cdot)|u|^{p}] (x, t) \, dx = 2 \int_0^t db \int_0^{e^{-b} - e^{-t}} dr_1 \left( \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \frac{r^{n-2}}{\omega_{n-1} c_0^{(n)}} \right) \left[ \int_{S^{n-1}} \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p dz \right)^{\beta + 1} dS_y \right] \bigg|_{r=r_1} \]
\[ \times (4e^{-b-t} - M \left( (e^{-t} + e^{-b})^2 - r_1^2 \right)^{-\frac{1}{2} + M} \]
\[ \times F \left( \frac{1}{2} - M, \frac{1}{2} - M; \frac{1}{2}; \frac{1}{(e^{-b} + e^{-t})^2 - r_1^2} \right) \]
\[ = \int_0^t \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p dz \right)^{\beta + 1} \frac{1}{M} \sinh(M(t - b)) \, db. \]

Thus, for the solution \( u = u(x, t) \) we have proven
\[ \int_{\mathbb{R}^n} G[\Gamma(\cdot)|u|^{p}] (x, t) \, dx = \int_0^t \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p dz \right)^{\beta + 1} \frac{1}{M} \sinh(M(t - b)) \, db. \]

Hence (20) reads
\[ \int_{\mathbb{R}^n} u(x, t) \, dx \]
\[ = \int_{\mathbb{R}^n} u_0(x, t) \, dx + \int_0^t \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p dz \right)^{\beta + 1} \frac{1}{M} \sinh(M(t - b)) \, db. \]

Taking into account (18) and (19) we derive
\[ \int_{\mathbb{R}^n} u(x, t) \, dx = \frac{1}{2} \left( C_0 + \frac{C_1}{M} \right) e^{Mt} + \frac{1}{2} \left( C_0 - \frac{C_1}{M} \right) e^{-Mt} \]
\[ + \int_0^t \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p dz \right)^{\beta + 1} \frac{1}{M} \sinh(M(t - b)) \, db. \]

We discuss separately two cases: with positive curved mass, \( M > 0 \), and vanishing curved mass, \( M = 0 \), respectively.
In the case of $M > 0$ we obtain
\[
\int_{\mathbb{R}^n} u(x, t) \, dx = C_0 \cosh(Mt) + \frac{C_1}{M} \sinh(Mt)
+ \int_0^t \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p \, dz \right)^{\beta+1} \frac{1}{M} \sinh(M(t-b)) \, db.
\]
Denote
\[
F(t) := \int_{\mathbb{R}^n} u(x, t) \, dx,
\]
then the function $F(t)$ is
\[
F(t) = C_0 \cosh(Mt) + \frac{C_1}{M} \sinh(Mt)
+ \int_0^t \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p \, dz \right)^{\beta+1} \frac{1}{M} \sinh(M(t-b)) \, db.
\]
It follows $F \in C^2([0, \infty))$. More precisely,
\[
\ddot{F}(t) = C_1 \cosh(Mt) + M C_0 \sinh(Mt)
+ \int_0^t \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p \, dz \right)^{\beta+1} \cosh(M(t-b)) \, db,
\]
\[
\ddot{F}(t) = M^2 F(t) + \Gamma(t) \left( \int_{\mathbb{R}^n} |u(z, t)|^p \, dz \right)^{\beta+1}.
\]
In particular, since $\Gamma(t) \geq 0$, we obtain
\[
F(t) \geq C_0 \cosh(Mt) + \frac{C_1}{M} \sinh(Mt) \quad \text{and} \quad \ddot{F}(t) \geq \Gamma(t) \left( \int_{\mathbb{R}^n} |u(z, t)|^p \, dz \right)^{\beta+1}.
\]
On the other hand, since the solution $u = u(x, t)$ has permanently bounded support, then $	ext{supp } u(\cdot, t) \subset \{ x \in \mathbb{R}^n : |x| \leq R \}$ for some positive number $R$. Using the compact support of $u(\cdot, t)$ and Hölder’s inequality we get with $\tau_n$ the volume of the unit ball in $\mathbb{R}^n$,
\[
\left| \int_{\mathbb{R}^n} u(x, t) \, dx \right|^p \leq \left( \int_{|x| \leq R} 1 \, dx \right)^{p-1} \left( \int_{|x| \leq R} |u(x, t)|^p \, dx \right)
= \tau_n \Gamma(t)^{-1/(\beta+1)} R^{n(p-1)} \left( \Gamma(t)^{1/(\beta+1)} \int_{\mathbb{R}^n} |u(x, t)|^p \, dx \right)
= \tau_n \Gamma(t)^{-1/(\beta+1)} R^{n(p-1)} \left( \ddot{F}(t) - M^2 F(t) \right)^{1/(\beta+1)}
\leq \tau_n \Gamma(t)^{-1/(\beta+1)} R^{n(p-1)} \ddot{F}(t)^{1/(\beta+1)}.
\]
Here we assume $\Gamma(t) > 0$. Thus
\[
\ddot{F}(t) \geq \tau_n^{-(\beta+1)} R^{-n(p-1)(\beta+1)} \Gamma(t) |F(t)|^{p(\beta+1)} \quad \text{for all} \quad t \in [0, \infty).
\]
By means of the inequality
\[ MC_0 + C_1 > 0 \]
we conclude that $F(t) \geq 0$ and that
\[ \ddot{F}(t) \geq \delta_0 \Gamma(t) F(t)^{p(\beta+1)} \quad \text{for large} \quad t \quad \text{with} \quad \delta_0 > 0. \]
Hence, for appropriate \( C_0 \) and \( C_1 \) the last inequality together with (21) to (23) implies the following system of the ordinary differential inequalities

\[
\begin{align*}
F(t) & \geq C_0 \cosh(Mt) + \frac{C_1}{M} \sinh(Mt) \quad \text{for all} \quad t \in [a, b), \\
\dot{F}(t) & \geq C_1 \cosh(Mt) + MC_0 \sinh(Mt) \quad \text{for all} \quad t \in [a, b), \\
\ddot{F}(t) & \geq \delta_0 \Gamma(t) F(t)^{p(\beta+1)} \quad \text{for all} \quad t \in [a, b).
\end{align*}
\]

The Lemma 3.1 shows that if \( F(t) \in C^2([0, b)) \) and the energy of particle is large, then \( b \) must be finite.

The conditions of the Lemma 3.1 are fulfilled on \((0, b)\) for the function

\[ \Gamma(t) = \delta_0 e^{\gamma t}, \quad \gamma \in \mathbb{R}, \]

with \( \gamma > 0 \) without any condition on the energy. They are fulfilled with \( \gamma < 0 \) if the kinetic energy and the potential energy are sufficiently large, that is \( C_0 > 0, C_1 > 0, \) and

\[ C_1 \geq \sqrt{\frac{2\delta_0}{p+1} C_0^{(p+1)/2}} \quad \text{and} \quad C_0^{p-1} > \frac{\gamma^2 (p+1)}{\delta_0 (p-1)}. \]

Next we turn to the case of the small energy and exponentially decreasing \( \Gamma(t) \). We apply Lemma 3.2 with \( A(t) = e^{Mt} \) and \( p \) replaced with \( p(\beta+1) \). More precisely, if we set

\[ A(t) = e^{Mt}, \quad \gamma(t) = \Gamma(t)e^{Mp(\beta+1)t}, \]

then the conditions of the last lemma read:

\[ p(\beta+1) > 1 \quad \text{and} \quad \Gamma(t) \leq 0 \quad \text{for all} \quad t \in [0, \infty). \]

The last inequality follows from the monotonicity of \( \Gamma(t) \). By the condition of the theorem, there exist \( \varepsilon > 0 \) and \( c > 0 \) such that

\[ \Gamma(t) \geq ce^{-M(p(\beta+1)-1)t^2+\varepsilon} \quad \text{for all} \quad t \in [a, b), \]

that coincides with (17). The case of \( M > 0 \) is proved.

Now consider the case of \( M = 0 \). Let

\[ C_0 := \int_{\mathbb{R}^n} \varphi_0(x) dx, \quad C_1 := \int_{\mathbb{R}^n} \varphi_1(x) dx, \quad C_1 > 0. \]

Then Corollary 2.2 allows us to write

\[
\int_{\mathbb{R}^n} G[\Gamma(\cdot)|u|](x, t) \, dx = \int_0^t \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p \, dz \right)^{\beta+1} (t-b) \, db.
\]

Hence (20) reads:

\[
\int_{\mathbb{R}^n} u(x, t) \, dx = \int_{\mathbb{R}^n} u_0(x, t) \, dx + \int_0^t \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p \, dz \right)^{\beta+1}.
\]

Now we choose a function \( u_0 \in C^\infty([0, \infty) \times \mathbb{R}^n) \) such that

\[ \int_{\mathbb{R}^n} u_0(x, t) \, dx = C_0 + C_1 t. \]

The solution of the problem (4) with \( M = 0 \) exemplifies such functions. Thus

\[
\int_{\mathbb{R}^n} u(x, t) \, dx = C_0 + C_1 t + \int_0^t \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p \, dz \right)^{\beta+1} (t-b) \, db.
\]
Denote
\[ F(t) := \int_{\mathbb{R}^n} u(x, t) \, dx, \]
then
\[ F(t) = C_0 + C_1 t + \int_0^t \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p \, dz \right)^{\frac{p+1}{p}} (t - b) \, db. \]

It follows \( F \in C^2([0, \infty)) \). More precisely,
\[
\begin{align*}
F(t) &= C_1 + \int_0^t \Gamma(b) \left( \int_{\mathbb{R}^n} |u(z, b)|^p \, dz \right)^{\frac{p+1}{p}} db, \\
\dot{F}(t) &= \Gamma(t) \left( \int_{\mathbb{R}^n} |u(z, t)|^p \, dz \right)^{\frac{p+1}{p}}. \tag{24}
\end{align*}
\]

In particular,
\[ F(t) \geq C_0 + C_1 t \quad \text{and} \quad \dot{F}(t) = \Gamma(t) \left( \int_{\mathbb{R}^n} |u(z, t)|^p \, dz \right)^{\frac{p+1}{p}}. \tag{25} \]

On the other hand according to (24) we obtain
\[
\left| \int_{\mathbb{R}^n} u(x, t) \, dx \right|^p \leq \left( \int_{|x| \leq R} 1 \, dx \right)^{p-1} \left( \int_{|x| \leq R} |u(x, t)|^p \, dx \right)
= \tau_n \Gamma(t)^{-1/(\beta+1)} R^{n(p-1)} \left( \Gamma(t)^{1/(\beta+1)} \int_{\mathbb{R}^n} |u(x, t)|^p \, dx \right)
\leq \tau_n \Gamma(t)^{-1/(\beta+1)} R^{n(p-1)} \dot{F}(t)^{1/(\beta+1)}.
\]

Thus
\[
\dot{F}(t) \geq \tau_n^{-(\beta+1)} R^{-n(p-1)}(\beta+1) \Gamma(t)|F(t)|^{p(\beta+1)}
\]
for all \( t \in [0, \infty) \). By means of the condition \( C_1 > 0 \) we conclude
\[ \dot{F}(t) \geq C \Gamma(t) F(t)^{p(\beta+1)} \quad \text{for large } t \quad \text{with } C > 0. \]

But for appropriate \( C_0 \) and \( C_1 \) one has \( F(t) > 0 \) and the last inequality together with (25) implies
\[
\begin{cases}
F(t) &\geq C_0 + C_1 t \\
\dot{F}(t) &\geq \delta_0 \Gamma(t) F(t)^{p(\beta+1)}
\end{cases}
\text{ for all } t \in [a, b),
\]
\[
\begin{cases}
F(t) &\geq \delta_0 \Gamma(t) F(t)^{p(\beta+1)}
\end{cases}
\text{ for all } t \in [a, b).
\]

The Kato’s Lemma 2 [6] shows that if \( F(t) \in C^2([0, b)) \) and \( \Gamma(t) \geq t^{-1-p(\beta+1)} \) with \( p(\beta+1) > 1 \), then \( b \) must be finite. Theorem is proven. \( \Box \)

**Remark 4.1** In fact, we have proved that any solution \( u = u(x, t) \) with permanently bounded support blows up if either \( MC_0 + C_1 > 0 \) and \( M > 0 \) or \( C_1 > 0 \) and \( M = 0 \).

**Proof of Theorem 1.2.** The case of \( \gamma \geq 0 \) is covered by Theorem 1.1 and implies a blow-up even for the small data. Therefore, we restrict ourselves to the case of \( \gamma < 0 \). Then, with a special choice of \( C_0 \) and \( C_1 \) after arguments have been used in the proof of Theorem 1.1 we arrive at the following system of the ordinary differential inequalities
\[
\begin{cases}
F(t) &\geq C e^{Mt} \\
\dot{F}(t) &\geq C e^{Mt} \\
\ddot{F}(t) &\geq \delta_0 e^{\gamma t} F(t)^{p(\beta+1)}
\end{cases}
\text{ for all } t \in [0, b),
\]
\[
\begin{cases}
F(t) &\geq C e^{Mt} \\
\dot{F}(t) &\geq C e^{Mt} \\
\ddot{F}(t) &\geq \delta_0 e^{\gamma t} F(t)^{p(\beta+1)}
\end{cases}
\text{ for all } t \in [0, b),
\]

where $C > 0$ and $\delta_0 > 0$. We claim that $b < \infty$. Indeed, we check conditions of Lemma 3.1 with

$$\Gamma(t) = \delta_0 e^{\gamma t}.$$ 

The condition (13),

$$\frac{1}{\sqrt{p} + 1} \int_0^a \Gamma(s)^{1/2} ds > \frac{\sqrt{2}}{p - 1} F^{(1-p)/2}(0),$$

reads:

$$\frac{1}{\sqrt{p} + 1} \int_0^a \delta_0^{1/2} e^{\frac{\gamma s}{2}} ds > \frac{\sqrt{2}}{p - 1} C_0^{(1-p)/2}, \quad C_1^2 \geq \frac{2}{p + 1} \delta_0 C_0^{p+1}.$$ 

The first inequality is fulfilled if $C_0$, that is the initial potential energy, is sufficiently large, while the second one is fulfilled if $C_1$, that is the initial kinetic energy, is large enough. Theorem is proven. \square

References

[1] H. Bateman, A. Erdelyi, “Higher Transcendental Functions”, vol. 1,2, McGraw-Hill, New York, 1953.

[2] D. Catania, V. Georgiev, Blow-up for the semilinear wave equation in the Schwarzschild metric, Differential Integral Equations, 19 (2006), 799–830. MR2235896 (2008c:58021)

[3] M. Dafermos, I. Rodnianski, The wave equation on Schwarzschild-de Sitter spacetimes, preprint, arXiv:0709.2766

[4] A. Galstian, $L_p$-$L_q$ decay estimates for the wave equations with exponentially growing speed of propagation, Appl. Anal., 82 (2003), 197–214. MR1970785 (2004b:35323)

[5] S. W. Hawking, G. F. R. Ellis, “The large scale structure of space-time”, Cambridge Monographs on Mathematical Physics, No. 1. Cambridge University Press, London-New York, 1973. xi+391 pp.

[6] T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations. Comm. Pure Appl. Math., 33 (1980), 501–505.

[7] M. Keel, T. Tao, Small data blow-up for semilinear Klein-Gordon equations, Amer. J. Math., 121 (1999), 629–669.

[8] C. Møller, “The theory of relativity”, Clarendon Press, Oxford, 1952. MR0049685

[9] M. Ohta, G. Todorova, Strong instability of standing waves for the nonlinear Klein-Gordon equation and the Klein-Gordon-Zakharov system, SIAM J. Math. Anal., 38 (2007), 1912–1931.

[10] A. Rendall, “Partial differential equations in general relativity”, Oxford Graduate Texts in Mathematics, 16, Oxford University Press, Oxford, 2008. MR2406669

[11] J. Shatah, M. Struwe, “Geometric wave equations”, Courant Lect. Notes Math., 2. New York Univ., Courant Inst. Math. Sci., New York, 1999. MR1674843 (2000i:53135)

[12] L. J. Slater, “Generalized hypergeometric functions”, Cambridge University Press, Cambridge 1966.

[13] K. Yagdjian, Global existence in the Cauchy problem for nonlinear wave equations with variable speed of propagation, New trends in the theory of hyperbolic equations, 301–385, Oper. Theory Adv. Appl., 159, Birkhäuser, Basel, 2005. MR2175919 (2007e:35206)

[14] K. Yagdjian, Global existence for the n-dimensional semilinear Tricomi-type equations, Comm. Partial Diff. Equations, 31 (2006), 907-944. MR2233046 (2007e:35207)

[15] K. Yagdjian, A. Galstian, Fundamental Solutions of the Wave Equation in Robertson-Walker spaces, J. Math. Anal. Appl., 346 (2008), 501–520. MR2433945

[16] K. Yagdjian, A. Galstian, Fundamental Solutions for the Klein-Gordon Equation in de Sitter Spacetime. Comm. Math. Phys., 285 (2009), 293-344.