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Dynamics and strategies evaluations of a novel reaction-diffusion COVID-19 model with direct and aerosol transmission

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Abstract

The COVID-19 epidemic has infected millions of people and cast a shadow over the global economic recovery. To explore the epidemic’s transmission law and provide theoretical guidance for epidemic prevention and control. In this paper, we investigate a novel SEIR-A reaction-diffusion COVID-19 system with direct and aerosol transmission. First, the solution’s positivity and boundedness for the system are discussed. Then, the system’s the basic reproduction number is defined. Further, the uniform persistence of disease when $R_0 > 1$ is explored. In addition, the system equilibrium’s global stability based on $R_0$ is demonstrated. Next, the system’s NSFD scheme is investigated and the discrete system’s positivity, boundedness, and global properties are studied. Meantime, global sensitivity analysis on threshold $R_0$ is investigated. Interestingly, the effects of three strategies, including vaccination, receiving treatment, and wearing a mask, are evaluated numerically. The results suggest that the above three strategies can effectively control the peak and final scale of infection and shorten the duration of the epidemic. Finally, theoretical simulations and instance predictions are used to give several key indicators of the epidemic, including threshold $R_0$, peak, time to peak, time to clear cases, and final size. The instance prediction results are as follows: (1) The basic reproduction numbers of Yangzhou and Putian in China

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are $R_0 = 2.5107$ and $R_0 = 1.8846$, respectively. (2) This epidemic round in Yangzhou will peak at 56 new daily confirmed cases on the 9th day (August 5), and Putian will peak at 37 new daily confirmed cases on the 6th day (September 15). (3) The final scale of infections in Yangzhou and Putian reached 570 and 205 cases, respectively. (4) The Yangzhou epidemic is expected to be completely cleared on the 25th day (August 21). In addition, the Putian epidemic will continue for 15 days and be cleared on September 24. The analysis results mean that we should improve our immunity by actively vaccinating, reducing the possibility of aerosol transmission by wearing masks. In particular, people should maintain proper social distance, and the government should strengthen medical investment and COVID-19 project research.

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1. Introduction

An unexplained pneumonia epidemic was discovered in December 2019 [1]. Subsequently, WHO named the Corona Virus Disease 2019 (COVID-19), and the International Committee on the Taxonomy of Viruses named the novel coronavirus (SARS-CoV-2) [2,3]. Note that COVID-19 is a more harmful epidemic than SARS and MERS. At present, although strict prevention and control measures have been taken, the epidemic is still spreading widely around the world [4]. As of September 27, 2021, a total of 232,667,774 cases have been confirmed worldwide, and 4,759,441 cases have died, the case fatality rate is approximately 2.1% [5]. The sudden epidemic has brought immeasurable losses to global economic development [6].

It should be noted that direct transmission such as coughing, sneezing, and face-to-face communication is not the only way for the COVID-19 virus to spread [7], aerosol transmission is also an essential way of transmission [8]. Aerosol means small respirable particles smaller than 5–10 μm, these aerosol-like virus particles can not only be suspended in the air but also spread over long distances [9]. Susceptible individuals will immediately become exposed individuals when they come into contact with aerosols carrying the virus. Symptoms appear within 2–14 days after infection, and the average incubation period is 5.2 days [10,11]. To avoid infection as much as possible, we must do some simple personal protection.

Wearing masks, vaccinating, and receiving treatment are commonly used to win the battle against the epidemic. Common masks include cloth masks (protection efficiency 20%–50%), medical masks (protection efficiency 70%–90%), and N95 masks (protection efficiency greater than or equal to 95%) [12–14]. There is no doubt that masks significantly reduce virus transmission’s risk and are the first barrier to preventing direct and aerosol transmission. In addition, vaccination also plays a vital role in preventing infectious diseases [15,16]. Through vaccination, it can effectively resist COVID-19 virus infection [17]. In particular, once we are infected with the virus, active treatment is the most direct and effective way to restore health and reduce community transmission [18]. It is worth noting that choosing the right strategy is of primary importance in disease control. However, quantitatively evaluating the effectiveness of masks, vaccination, and treatment strategies remain to be resolved.

Mathematical modeling has a long history and plays an essential role in revealing disease’s internal transmission law and predicting the development trend of species populations [19–25]. In recent years, studying the dynamic behavior of diseases through mathematical
modeling methods has been widely used. In [26], the authors studied a cell-to-cell and virus-to-cell spreading virus dynamics model in the host and studied the global properties of the corresponding equilibrium points. Authors in [27] considered an infectious disease model with maturity delay and incubation period for the problem of childhood diseases and discussed the asymptotic behavior of the model under all feasible equilibrium states. Authors in [28] proposed a novel cholera epidemic model, which calculated the threshold conditions of the cholera epidemic, and analyzed the system’s local and global properties. In [29], the authors developed an SEIR-SEI malaria epidemic model and analyzed the corresponding equilibrium’s global stability. In [30], the authors proposed a fractional pneumonia model, demonstrated the solution’s existence by fixed-point theory, and investigated model variable’s uniqueness. In addition, the development of the COVID-19 model has also received extensive attention in recent years.

Since the outbreak of the epidemic, scholars have established various models to explore the development of epidemic and made positive contributions to control the disease. Authors in [31] investigated a COVID-19 mathematical model and analyzed the optimal control problem for three strategies. In [32], the authors developed an epidemiological network model that can describe the epidemic’s dynamics. In [33], the authors proposed a SIR Filippov system to explore the impact of vaccination, media reports, and treatment strategies on the COVID-19 epidemic. Authors in [34] discussed traveling wave solution’s existence of the SIR system with spatial diffusion and age structure. The above scholars have made outstanding contributions to the study of the COVID-19 epidemic model. However, all the work mentioned above ignores the fact that COVID-19 can be transmitted through aerosols. Therefore, this paper aims to study the internal laws of the epidemic and evaluate the effectiveness of COVID-19 prevention and control.

According to the transmission mechanism of infectious diseases, we investigate a novel COVID-19 reaction-diffusion system with the direct and aerosol transmission, analyze the theoretical results of the model, obtain the extinction and persistence threshold of COVID-19 and discuss the key factors that affect the disease epidemic. In addition, to numerically verify the system’s positivity, boundedness, and other corresponding dynamic properties. The NSFD scheme has been widely used to study the relevant properties of the system [35–37]. Inspired by them, we propose an NSFD scheme that maintains dynamic consistency with the continuous system and use this scheme for simulation in the numerical simulation part. At the same time, the reaction-diffusion SEIR-A model is used to fit the development of the epidemic situation in Yangzhou, China, and Putian, China, to verify that our model can combine theory with practice and predict the COVID-19 epidemic.

The paper structure is as follows: In Section 2, we propose a COVID-19 reaction-diffusion system with direct and aerosol transmission to characterize the epidemic and spread of COVID-19. In Section 3, we discuss the system’s dynamic properties, such as the solution’s positivity and boundedness, threshold $\mathcal{R}_0$, the equilibrium’s local and global stability, and the epidemic’s persistence. In Section 4, we propose the NSFD scheme of the system Eq. (3.1), discuss the boundedness and positivity of the discrete system Eq. (4.1), and perform a global sensitivity analysis on threshold $\mathcal{R}_0$. Furthermore, numerical simulations verify the accuracy of the theoretical calculation results. Section 5 evaluates the effects of vaccination, masks, and medical treatment on COVID-19. In Section 6, we apply the SEIR-A reaction-diffusion system to predict the epidemic in Yangzhou and Putian, China, respectively. A brief summary and research outlook are given in Section 7.
2. An SEIR-A model with vaccination, masks and medical treatment

In this section, to comprehensively evaluate the effectiveness of vaccination, mask-wearing, and treatment strategies and analyze the impact of aerosol transmission and individual spread on the COVID-19 epidemic. Establishing a mathematical model that can characterize the above factors is very interesting. Through model theory and numerical analysis, we hope to obtain several critical indicators of COVID-19: the basic reproduction number, peak value, peak time, epidemic inflection point, final scale, and case clearing time. Before modeling, we give the following assumptions:

- Vaccinated susceptible and recovered infected individuals will get lifelong immunity against COVID-19;
- Both exposed and infected individuals are infectious to susceptible individuals;
- Infected individuals may die of disease, but exposed individuals will not die of disease;
- The birth rate equals the natural death rate;
- The transmission methods of COVID-19 mainly include direct and aerosol transmission.

Based on the above assumptions, we draw the COVID-19 transmission flow chart shown in Fig. 1. Then we propose an SEIR-A system with direct and aerosol transmission. Among them, this reaction-diffusion model is mainly composed of diffusion term and reaction term, we use the diffusion terms $D_1 \Delta S$, $D_2 \Delta E$, $D_3 \Delta I$, $D_5 \Delta R$ and $D_4 \Delta A$ to denote individuals movement and aerosol diffusion, respectively. Reaction term $\sigma S$ indicates vaccination. Reaction items $(1 - c_2)Sh(E)$, $(1 - c_1)Sg(I)$, and $(1 - c_1)Sf(A)$ represent the protective effect of wearing a mask. Reaction term $(1 + b)\gamma I$ stands for that the treatment will speed up the patient’s recovery and shorten the period of infection. In addition, other reaction terms also characterize
corresponding reactions. Therefore, we can derive the following Eq. (2.1):

\[
\begin{align*}
\frac{\partial S(x, t)}{\partial t} &= D_1 \Delta S + \Lambda - (1 - c_1)Sh(E) - (1 - c_1)Sg(I) - (1 - c_2)Sf(A) - \mu S - \sigma S, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial E(x, t)}{\partial t} &= D_2 \Delta E + (1 - c_1)Sh(E) + (1 - c_1)Sg(I) + (1 - c_2)Sf(A) - \mu E - \theta E, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial I(x, t)}{\partial t} &= D_3 \Delta I + \theta E - \mu I - \rho I - (b + 1)\gamma I, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial A(x, t)}{\partial t} &= D_4 \Delta A + \alpha(I + E) - \xi A, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial R(x, t)}{\partial t} &= D_5 \Delta R + (b + 1)\gamma I - \mu R + \sigma S, \quad x \in \Omega, \quad t > 0,
\end{align*}
\]

with

\[
(S(x, 0), E(x, 0), I(x, 0), A(x, 0), R(x, 0)) = (\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x), \phi_5(x)),
\]

for \( x \in \Omega, \)

and satisfy

\[
\frac{\partial S}{\partial n} = \frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial A}{\partial n} = \frac{\partial R}{\partial n} = 0, \quad x \in \partial \Omega, \quad t > 0,
\]

here \( S(x, t), E(x, t), I(x, t) \) and \( R(x, t) \) stand for the density of individuals in each compartment at time \( t \) and location \( x \), respectively. \( A(x, t) \) represents the density of COVID-19 virus in the air. The total population of the region is \( N = S + E + I + R \). All parameters of system Eq. (2.1) are non-negative. The parameter \( \Lambda \) represents the recruitment rate (assume \( \Lambda = \mu N \)). \( \mu \) describes the birth or natural death rate. \( c_1, c_2 \) represent the effective protection rate of the mask. \( \theta \) stands for the conversion rate from exposed to infected individuals. \( \sigma \) denotes the vaccination rate of COVID-19. \( \gamma \) is the recovery rate without treatment. \( \rho \) stands for the virus-induced average fatality rate. \( b \) is the medical treatment level. Moreover, the average infection period has been shortened from \( 1/\gamma \) to \( 1/(1 + b)\gamma \) through medical treatment. \( \alpha \) stands for the rate from \( E, I \) into \( A \). \( \xi \) is the decay rate of pathogen in the air. \( \Delta \) denotes the Laplace operator and \( D_i (i = 1, 2, \ldots, 5) \) represent the corresponding diffusion coefficient.

In addition, \( f, g \) and \( h \) fulfill the following assumptions

\( (H_1): g(0) = h(0) = f(0) = 0 \) and \( h(E) > 0, f(A) > 0, g(I) > 0 \) for \( E, I, A > 0 \);

\( (H_2): g'(I) > 0, h'(E) > 0, f'(A) > 0 \) and \( h''(E) \leq 0, f''(A) \leq 0 \) for \( E, I, A \geq 0 \).

Such as the classical bilinear incidence rate \( Sh(E) = \beta_1 SE \), standard incidence rate \( Sg(I) = \frac{S}{N} I \), and saturation incidence rate \( Sf(A) = \frac{\beta_3 SA}{\alpha + A} \) satisfy the above assumptions, where \( \alpha, \beta_1, \beta_2, \beta_3, N \) are all numbers greater than zero.

3. Dynamical behavior of COVID-19 model

Note that the first four equations of system Eq. (2.1) have nothing to do with \( R(x, t) \). The corresponding reduced system is as follows

\[
\begin{align*}
\frac{\partial S(x, t)}{\partial t} &= D_1 \Delta S + \Lambda - (1 - c_1)Sh(E) - (1 - c_1)Sg(I) - (1 - c_2)Sf(A) - \mu S - \sigma S, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial E(x, t)}{\partial t} &= D_2 \Delta E + (1 - c_1)Sh(E) + (1 - c_1)Sg(I) + (1 - c_2)Sf(A) - \mu E - \theta E, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial I(x, t)}{\partial t} &= D_3 \Delta I + \theta E - \mu I - \rho I - (b + 1)\gamma I, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial A(x, t)}{\partial t} &= D_4 \Delta A + \alpha(I + E) - \xi A, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial R(x, t)}{\partial t} &= D_5 \Delta R + (b + 1)\gamma I - \mu R + \sigma S, \quad x \in \Omega, \quad t > 0,
\end{align*}
\]
with

\[(S(x, 0), E(x, 0), I(x, 0), A(x, 0)) = (\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x)), \text{ for } x \in \Omega, \tag{3.2}\]

and satisfy

\[
\frac{\partial S}{\partial n} = \frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial A}{\partial n} = 0, \quad x \in \partial \Omega, \quad t > 0. \tag{3.3}
\]

3.1. Positivity and boundedness

The COVID-19 virus’ epidemic in the population is less different, but there is a clear difference from the spread of aerosols. Therefore, we suppose \( D_1 = D_2 = D_3 = D_4 = \mathbf{D} \neq D_4 \), and exhibit that the existential uniqueness of the solution. Next, we give some symbols.

Assume \( X := C(\Omega, \mathbb{R}^4) \) is a Banach space and have the supremum form \( \| \cdot \|_X \), then denote \( X^+ := C(\Omega, \mathbb{R}^4_+) \). Thus \((X, X^+)\) represents a strongly ordered space. Suppose that \( T_1(x), T_2(x), T_3(x), T_4(x) : C(\Omega, \mathbb{R}) \to C(\Omega, \mathbb{R}) \) are the \( C_0 \) semigroups related to \( D\Delta - \mu - \sigma, D\Delta - \theta - \mu, D\Delta - \mu - (1 + b)\gamma - \rho \) and \( D\Delta_4 - \xi \) depend on Eq. (3.3), respectively. Obviously, for every \( \phi \in C(\Omega, \mathbb{R}) \), and \( t \geq 0 \), we have

\[
T_1(t) \phi(x) = e^{-(\mu + \sigma)t} \int_{\Omega} \phi(s) \Gamma_1(x, t, s)ds,
\]

\[
T_2(t) \phi(x) = e^{-(\mu + \theta)t} \int_{\Omega} \phi(s) \Gamma_2(x, t, s)ds,
\]

\[
T_3(t) \phi(x) = e^{-(\mu + (1 + b)\gamma + \rho)t} \int_{\Omega} \phi(s) \Gamma_3(x, t, s)ds,
\]

and

\[
T_4(t) \phi(x) = e^{-\xi t} \int_{\Omega} \phi(s) \Gamma_4(x, t, s)ds,
\]

where \( \Gamma_1 \) and \( \Gamma_4 \) represent the Green functions related to \( D\Delta \) and \( D_4 \Delta \) depend on Eq. (3.3), respectively. From Martin and Smith [38], we obtain that \( T_i(t) : C(\Omega, \mathbb{R}) \to C(\Omega, \mathbb{R})(i = 1, 2, 3, 4) \) stands for strongly positive and compact.

For all initial value \( \phi \in X^+ \) and \( \phi \in \Omega \), then \( F = (F_1, F_2, F_3, F_4) : X^+ \to X \) give by

\[
F_1(\phi)(x) = \Lambda - (1 - c_1)h(\phi_2(x, 0))\phi_1(x, 0) - (1 - c_1)g(\phi_3(x, 0))\phi_1(x, 0)
\]
\[
- (1 - c_2)f(\phi_4(x, 0))\phi_1(x, 0),
\]

\[
F_2(\phi)(x) = (1 - c_1)h(\phi_2(x, 0))\phi_1(x, 0) + (1 - c_1)g(\phi_3(x, 0))\phi_1(x, 0)
\]
\[
+ (1 - c_2)f(\phi_4(x, 0))\phi_1(x, 0),
\]

\[
F_3(\phi)(x) = \theta \phi_2(x, 0),
\]

\[
F_4(\phi)(x) = \alpha(\phi_3(x, 0) + \phi_2(x, 0)).
\]

Hence, system Eqs. (3.1)–(3.3) can be written as

\[
\mathbf{Y}(x, t) = \mathbf{T}(t) \phi(x) + \int_0^t \mathbf{T}(t - s)F(\mathbf{Y}(x, s))ds,
\]

here \( \mathbf{T}(t) = \text{diag} \ (T_1(t), T_2(t), T_3(t), T_4(t)) \) and \( \mathbf{Y}(x, t) = (S(x, t), \ E(x, t), \ I(x, t), \ A(x, t)) \). Then, we get some results as follows.
Theorem 3.1. Given any \( \phi \in \mathbb{X}^+ \), the reaction-diffusion system Eq. (3.1)-(3.3) has only solution \( \Psi(t) \) with \( \Psi(0, \phi) = \phi \); and \( \Psi(t) : \mathbb{X}^+ \rightarrow \mathbb{X}^+ \)

\[
\Psi(t) \phi = (S(\cdot, t, \phi), E(\cdot, t, \phi), I(\cdot, t, \phi), A(\cdot, t, \phi)), \quad \forall x \in \overline{\Omega}, \; t \geq 0,
\]

generated by the system is point dissipative.

Proof. Given every \( \phi \in \mathbb{X}^+ \) and \( h > 0 \), one has

\[
\lim_{h \to 0^+} \frac{1}{h} \text{dist}(hF(\phi) + \phi(0), \mathbb{X}^+) = 0.
\]

Through Corollary 4 of reference [38], we get that \( \Psi(t, \phi) \) represents a unique mild solution of Eqs. (3.1)-(3.3) on \([0, \tau_\phi)\) with \( \Psi(0, \phi) \) and \( \Psi(t, \phi) \in \mathbb{X}^+ \), where \( t_\phi \leq +\infty \). Then, we prove that the solution is global. Consider the first to three equations of system Eq. (3.1), one has

\[
\frac{\partial(S(x,t) + I(x,t) + E(x,t))}{\partial t} = D\Delta (S + I + E) + \Lambda - \mu(S + I + E) - \sigma S - \rho I - (1 + b)\gamma I
\]

\[
\leq D\Delta (S + I + E) + \Lambda - \mu(S + I + E).
\]

By comparison principle, for small enough positive number \( \varepsilon \) and there is a \( t^* > 0 \), then for \( \forall \; t \geq t^* \), such that

\[
S(x,t) + I(x,t) + E(x,t) \leq \frac{\Lambda}{\mu} + \varepsilon, \quad \text{uniformly for} \quad \forall x \in \overline{\Omega}.
\]

Hence,

\[
S(\cdot, t) \leq \frac{\Lambda}{\mu} + \varepsilon, \quad I(\cdot, t) \leq \frac{\Lambda}{\mu} + \varepsilon \quad \text{and} \quad E(\cdot, t) \leq \frac{\Lambda}{\mu} + \varepsilon.
\]

This implies that \( S, I \) and \( E \) are uniformly bounded.

Similarly, according to the fourth equation of system Eq. (3.1), one has

\[
\begin{cases}
\frac{\partial A(x,t)}{\partial t} \leq D_4 \Delta A + 2\alpha \left( \frac{\Lambda}{\mu} + \varepsilon \right) - \xi A, & x \in \Omega, t > t^*, \\
\frac{\partial A}{\partial n} = 0, & x \in \partial \Omega, t > 0.
\end{cases}
\]

(3.4)

According to the comparison principle, one has

\[
\limsup_{t \to +\infty} A(x,t) \leq \frac{2\alpha \left( \frac{\Lambda}{\mu} + \varepsilon \right)}{\xi}, \quad \text{uniformly for} \quad \forall x \in \overline{\Omega}.
\]

Hence, there is \( \overline{t^*} \geq t^* \) such that

\[
A(x,t) \leq \frac{2\alpha \left( \frac{\Lambda}{\mu} + 2\varepsilon \right)}{\xi}, \quad \text{for} \; \forall t \geq \overline{t^*},
\]

this means that \( A(x,t) \) is ultimately bounded. Therefore \( \Psi(t) : \mathbb{X}^+ \rightarrow \mathbb{X}^+ \) is point dissipative. \( \square \)
3.2. Basic reproduction number

This subsection applies the theory of the basic reproduction numbers for the system to obtain the system’s threshold.

System Eq. (3.1) always has a DFE \( P^0(S_0, 0, 0, 0) \), where \( S_0 = \frac{\Lambda}{\mu + \sigma} \). And there exists \( P^*(S^*, E^*, I^*, A^*) \) give by

\[
\begin{align*}
\Lambda - (1 - c_2)Sf(A) - (1 - c_1)Sg(I) - (1 - c_1)Sh(E) - \mu S - \sigma S &= 0, \\
(1 - c_2)Sf(A) + (1 - c_1)Sg(I) + (1 - c_1)Sh(E) - \mu E - \theta E &= 0, \\
\theta E - \mu I - (1 + b)\gamma I - \rho I &= 0, \\
\alpha(I + E) - \xi A &= 0, \\
(E(x, 0), I(x, 0), A(x, 0), S(x, 0)) &= (\phi_2(x), \phi_3(x), \phi_4(x), \phi_1(x)), \quad \text{for } x \in \Omega,
\end{align*}
\]  

(3.5)

where

\[
S^* = \frac{\Lambda - (\mu + \theta)E^*}{\mu + \sigma}, \quad I^* = \frac{\theta}{\mu + \rho + (b + 1)\gamma}E^*, \quad A^* = \left(\frac{\theta}{\mu + \rho + (b + 1)\gamma} + 1\right)\frac{\alpha}{\xi}E^*.
\]

(3.6)

Furthermore, according to the theory in [39], we rewrite system Eq. (3.1) as

\[
\frac{\partial P_i}{\partial t} = D_i \Delta P_i + \mathcal{F}_i(x, P) - \mathcal{V}_i(x, P), \quad i = 1, 2, 3, 4,
\]

where \( P = (E, I, A, S)^T \).

Specifically, system Eqs. (3.1)–(3.3) can be rewritten as

\[
\begin{align*}
\frac{\partial E(x, t)}{\partial t} &= D_2 \Delta E + (1 - c_1)Sh(E) + (1 - c_1)Sg(I) + (1 - c_2)Sf(A) - \mu E - x\theta E, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial I(x, t)}{\partial t} &= D_3 \Delta I + \theta E - \mu I - \rho I - (1 + b)\gamma I, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial A(x, t)}{\partial t} &= D_4 \Delta A + \alpha(I + E) - \xi A, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial S(x, t)}{\partial t} &= D_1 \Delta S + \Lambda - (1 - c_1)Sh(E) - (1 - c_1)Sg(I) - (1 - c_2)Sf(A) - \mu S - \sigma S, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial A}{\partial n} = \frac{\partial S}{\partial n} &= 0, \quad x \in \partial \Omega, \quad t > 0, \\
(E(x, 0), I(x, 0), A(x, 0), S(x, 0)) &= (\phi_2(x), \phi_3(x), \phi_4(x), \phi_1(x)), \quad x \in \Omega.
\end{align*}
\]

(3.7)

Therefore, we get

\[
\mathcal{F}_i(x, P^0(x)) = \begin{pmatrix}
(1 - c_1)Sh(E) + (1 - c_1)Sg(I) + (1 - c_2)Sf(A) \\
0 \\
0 \\
0
\end{pmatrix}
\]

and

\[
\mathcal{V}_i(x, P^0(x)) = \begin{pmatrix}
(\mu + \theta)E - D_2 \Delta E \\
\mu I + \rho I + (1 + b)\gamma I - \theta E - D_3 \Delta I \\
\xi A - \alpha(E + I) - D_4 \Delta A \\
(1 - c_1)Sh(E) + (1 - c_1)Sg(I) + (1 - c_2)Sf(A) - \Lambda + \mu S + \sigma S - D_1 \Delta S
\end{pmatrix}.
\]
\[ F(x), V(x) \] are denoted as
\[
F(x) = \left( \frac{\partial \mathcal{J}_i(x, P^0(x))}{\partial u_j} \right)_{1 \leq i, j \leq 3} \quad \text{and} \quad V(x) = \left( \frac{\partial \mathcal{J}_j(x, P^0(x))}{\partial u_i} \right)_{1 \leq i, j \leq 3},
\]
respectively, where \( P^0(x) = (0, 0, 0, S_0) \). Thus
\[
F(x) = \begin{pmatrix} (1 - c_1)S_0h' (0) & (1 - c_1)S_0g' (0) & (1 - c_2)S_0f' (0) \\ 0 & 0 & 0 \end{pmatrix},
\]
and
\[
V(x) = \begin{pmatrix} \theta + \mu - k^2D_2 & 0 & 0 \\ -\theta & (1 + b)\gamma + \mu + \rho - k^2D_3 & 0 \\ -\alpha & -\alpha & \xi - k^2D_4 \end{pmatrix},
\]
where \( k \) is the wavenumber. \( F(x) \) stands for a 3 \( \times \) 3 continuous and nonnegative matrix function, \(-V(x)\) represents a 3 \( \times \) 3 continuous and cooperative matrix function.

It follows from [39] that the distribution of total new infectious is defined as
\[
\int_0^\infty F(x) \mathcal{T}(t) \phi (x) dt,
\]
Further, we define
\[
[\mathcal{L} (\phi)] (x) = \int_0^\infty F(x)[\mathcal{T}(t) \phi] (x) dt = F(x) \int_0^\infty [\mathcal{T}(t) \phi] (x) dt,
\]
where \( \mathcal{L} \) represents a positive and continuous operator that maps the initial infection \( \phi (x) \) distribution to the total infective members produced during the infection period. From the idea of the next generation operators in [39–41], the basic reproduction number’s spectral radius \( \mathcal{L} \) is defined as
\[
\mathcal{R}_0 \triangleq r(\mathcal{L}).
\]

By simple computation, one has
\[
\mathcal{R}_0 = \frac{(1 - c_1)\Delta h (0)}{(\theta + \mu)} + \frac{(1 - c_1)\Delta \theta g (0)}{(\theta + \mu)(\sigma + \mu)(\mu + (b + 1)\gamma + \rho)} + \frac{(1 - c_2)\Delta \alpha (\theta + \mu + \rho + (1 + b)\gamma) f (0)}{(\theta + \mu)(\sigma + \mu)(\rho + \mu + (b + 1)\gamma)\xi}
\]
\[ \triangleq \mathcal{R}_{0_1} + \mathcal{R}_{0_2} + \mathcal{R}_{0_3}, \]
where the biological meaning of quantity \( \mathcal{R}_{0_1}, \mathcal{R}_{0_2} \) and \( \mathcal{R}_{0_3} \) can be explained as the contribution of exposed individuals, infected individuals and aerosol to the basic reproduction number, respectively.

**Theorem 3.2.** When \( \mathcal{R}_0 < 1 \), the reaction-diffusion system Eq. (3.1) has only \( P^0 (S_0, 0, 0, 0) \); while \( \mathcal{R}_0 > 1 \), there is a unique \( P^* (S^*, E^*, I^*, A^*) \), where
\[
S^* = \frac{\Lambda - (\mu + \theta)E^*}{\sigma + \mu}, \quad I^* = \frac{\theta}{\rho + \mu + (1 + b)\gamma}E^*
\]
and
\[ A^* = \left( \frac{\theta}{\mu + \rho + (1 + b)\gamma} + 1 \right) \frac{\alpha}{\xi} E^*. \]

**Proof.** It is evident that \( P^0 \) is unique when \( R_0 < 1 \). Therefore, we only need to prove the part for \( R_0 > 1 \). We know from Eq. (3.5) that
\[
S = \frac{\Lambda - (\mu + \theta)E}{\mu + \sigma}, \quad I = \frac{\theta}{\rho + \mu + (b + 1)\gamma} E \quad \text{and} \quad A = \left( \frac{\theta}{\mu + \rho + (1 + b)\gamma} + 1 \right) \frac{\alpha}{\xi} E.
\]

From the second equation of Eq. (3.4), one has
\[
J(E) = (1 - c_2) \left( \frac{\Lambda - (\mu + \theta)E}{\mu + \sigma} \right) f\left( \frac{\alpha}{\xi} \left( 1 + \frac{\theta}{\mu + (1 + b)\gamma + \rho} \right) E \right)
\]
\[
+ (1 - c_1) \left( \frac{\Lambda - (\mu + \theta)E}{\mu + \sigma} \right) g\left( \frac{\theta}{\mu + (1 + b)\gamma + \rho} \right) - (\theta + \mu)E
\]
\[
+ (1 - c_1) \left( \frac{\Lambda - (\mu + \theta)E}{\mu + \sigma} \right) h(E).
\]

Since \( J(0) = 0 \).

Adding the first two equations of Eq. (3.5), one has
\[
J\left( \frac{\Lambda}{\mu + \theta} \right) = - (\mu + \theta)E = - \Lambda < 0.
\]

Further
\[
J'(0) = \frac{(1 - c_2)\Delta \alpha(\theta + \mu + (1 + b)\gamma + \rho) f'(0)}{(\mu + \sigma)(\mu + (1 + b)\gamma + \rho)\xi} + \frac{(1 - c_1)\Delta \theta g'(0)}{(\mu + \sigma)(\mu + (1 + b)\gamma + \rho)}
\]
\[
+ (1 - c_1) \Delta h'(0) - (\mu + \theta)
\]
\[
= \left( \frac{(1 - c_2)\Delta \alpha(\theta + \mu + (1 + b)\gamma + \rho) f'(0)}{(\mu + \sigma)(\mu + (1 + b)\gamma + \rho)\xi} + \frac{(1 - c_1)\Delta \theta g'(0)}{(\mu + \sigma)(\mu + (1 + b)\gamma + \rho)}
\]
\[
+ (1 - c_1) \Delta h'(0) - 1 \right)(\mu + \theta)
\]
\[
= (R_0 - 1)(\mu + \theta) > 0.
\]

This implies that \( J(E) = 0 \) exists at least one positive root \( E^* \in \left( 0, \frac{\Lambda}{\mu + \theta} \right) \), which means that the positive equilibrium of Eq. (3.1) exists. Next, we prove \( E^* \) is unique.

Based on Eq. (3.6) and the following facts
\[
(1 - c_1)S^*h(E^*) + (1 - c_1)S^*g(I^*) + (1 - c_2)S^*f(A^*) = (\mu + \theta)E^*.
\]

In the following, according to assumption \((H_2)\), we get
\[
f'(A)A \leq f(A), \quad h'(E)E \leq h(E) \quad \text{and} \quad g'(I)I \leq g(I), \quad \text{for} \quad E, I, A \geq 0.
\]

Therefore, it follows from Eqs. (3.8) and (3.9) that
\[
J'(E^*) = \frac{(1 - c_2)\Delta (\theta + \mu + (1 + b)\gamma + \rho) \alpha}{(\mu + \sigma)(\mu + \rho + (1 + b)\gamma)\xi} f'\left( \frac{(\theta + \mu + \rho + (1 + b)\gamma)\alpha}{(\mu + \rho + (1 + b)\gamma)\xi} E^* \right)
\]

10067
\[
- \frac{(1-c_2)(\mu + \theta)}{\mu + \sigma} f\left(\frac{(\theta + \mu + (1+b)\gamma + \rho)\alpha}{(\mu + (1+b)\gamma + \rho)\xi} E^*\right) \\
- \frac{(1-c_2)(\mu + \theta)(\theta + \mu + \rho + (1+b)\gamma)\alpha}{(\mu + (1+b)\gamma + \rho)\xi} E^* f' \\
\left(\frac{(\theta + \mu + (1+b)\gamma + \rho)\alpha}{(\mu + (1+b)\gamma + \rho)\xi} E^*\right) \\
+ \frac{(1-c_1)(\mu + \theta)}{(\mu + (1+b)\gamma + \rho)\xi} g\left(\frac{\theta}{\mu + (1+b)\gamma + \rho} E^*\right) \\
- \frac{(1-c_1)(\mu + \theta)}{(\mu + (1+b)\gamma + \rho)\xi} h(E^*) \\
- \frac{(1-c_1)(\mu + \theta)}{\mu + \sigma} h(E^*) - \frac{(1-c_1)(\mu + \theta)}{\mu + \sigma} E^* h'(E^*) - (\mu + \theta)
\]

\[
= - \frac{\mu + \theta}{\mu + \sigma} \left(1-c_2\right) f\left(\frac{(\mu + \theta + (1+b)\gamma + \rho)\alpha}{(\mu + (1+b)\gamma + \rho)\xi} E^*\right) \\
+ (1-c_1) \left(g\left(\frac{\theta}{\mu + (1+b)\gamma + \rho} E^*\right) + h(E^*)\right) \\
+ \left(1-c_2\right)(\theta + \mu + (1+b)\gamma + \rho)\alpha \left(\frac{\Delta - (\mu + \rho)E^*}{\mu + (1+b)\gamma + \rho} E^*\right) f'\left(\frac{(\rho + \theta + \mu + (1+b)\gamma)\alpha}{(\rho + \mu + (1+b)\gamma)\xi} E^*\right) \\
+ \frac{(1-c_1)\theta}{\mu + (1+b)\gamma + \rho} \left(\frac{\Delta - (\mu + \rho)E^*}{\mu + (1+b)\gamma + \rho} E^*\right) g\left(\frac{\theta}{\mu + (1+b)\gamma + \rho} E^*\right) \\
+ (1-c_1) \left(\frac{\Delta - (\mu + \rho)E^*}{\mu + (1+b)\gamma + \rho} E^*\right) h'(E^*) - (\theta + \mu)
\]

\[
= - \frac{\mu + \theta}{\mu + \sigma} \left(1-c_2\right) f\left(\frac{(\rho + \mu + \theta + (1+b)\gamma)\alpha}{(\mu + (1+b)\gamma + \rho)\xi} E^*\right) \\
+ (1-c_1) \left(g\left(\frac{\theta}{\mu + (1+b)\gamma + \rho} E^*\right) + h(E^*)\right) \\
+ \left(1-c_2\right) \frac{S^*}{E^*} \left(\frac{(\rho + \mu + \theta + (1+b)\gamma)\alpha E^*}{(\mu + (1+b)\gamma + \rho)\xi E^*} f'\left(\frac{(\mu + \theta + (1+b)\gamma + \rho)\alpha E^*}{\mu + (1+b)\gamma + \rho} \xi E^*\right)\right) \\
- f\left(\frac{(\mu + \theta + (1+b)\gamma + \rho)\alpha E^*}{(\mu + (1+b)\gamma + \rho)\xi E^*}\right) \\
+ (1-c_1) \frac{S^*}{E^*} \left(\frac{\theta E^*}{\mu + (1+b)\gamma + \rho} E^* g\left(\frac{\theta E^*}{\mu + (1+b)\gamma + \rho} E^*\right) - g\left(\frac{\theta}{\rho + \mu + (1+b)\gamma} E^*\right)\right) \\
+ (1-c_1) \frac{S^*}{E^*} (h'(E^*)E^* - h(E^*)) < 0.
\]

Suppose there exists another positive equilibrium \(P^{**}(S^{**}, E^{**}, I^{**}, A^{**})\), then we have \(J'(E^{**}) > 0\), but this contradicts the inequality Eq. (3.9). \[\square\]
3.3. Local stability of the equilibria

Let \( 0 = \mu_1 < \mu_2 < \cdots < \mu_i < \cdots \) be the eigenvalues of \( -\Delta \) on \( \Omega \) with the homogeneous Neumann boundary conditions. \( U(\mu_i) \) be the eigenfunction space dependent on \( \mu_i \), and \( \{ \sigma_{ij} : j = 1, 2, 3, \ldots, \dim U(\mu_i) \} \) be an orthonormal basis of \( U(\mu_i) \). \( \mathbb{Z} \) can be decomposed as

\[
\mathbb{Z} = \bigoplus_{i=1}^{\infty} \mathbb{Z}_i \quad \text{and} \quad \mathbb{Z}_i = \bigoplus_{i=1}^{\dim U(\mu_i)} \mathbb{Z}_{ij},
\]

where \( \mathbb{Z} = \left\{ (E, I, A, S) \in [C'(\Omega)]^4 : \frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial A}{\partial n} = \frac{\partial S}{\partial n} = 0 \text{ on } \partial \Omega \right\} \), \( \mathbb{Z}_{ij} = \{ a\sigma_{ij} : a \in \mathbb{R}^4 \} \). Then, we show the equilibrium’s local stability as follows.

**Theorem 3.3.** When \( R_0 < 1 \), the DFE \( P^0 \) of the reaction-diffusion system Eq. (3.1) is locally asymptotically stable.

**Proof.** Consider the linearization of the reaction-diffusion system Eq. (3.1) at \( P^0 \):

\[
\frac{\partial \mathbb{Y}(x, t)}{\partial t} = \mathbb{D} \Delta \mathbb{Y}(x, t) + \mathbb{A}(P^0) \mathbb{Y}(x, t),
\]

where \( \mathbb{Y} = (S, E, I, A), \mathbb{D} = \text{diag} \{ D_1, D_2, D_3, D_4 \} \) (here \( D_1 = D_2 = D_3 = D \)), and

\[
\mathbb{A} = \begin{pmatrix}
-\mu - \sigma & -(1 - c_1)S_0h'(0) & -(1 - c_1)S_0g'(0) & -(1 - c_2)S_0f'(0) \\
0 & (1 - c_1)S_0h'(0) - (\mu + \theta) & (1 - c_1)S_0g'(0) & (1 - c_2)S_0f'(0) \\
0 & \theta & -\mu - (1 + b)\gamma - \rho & 0 \\
0 & \alpha & \alpha & -\xi
\end{pmatrix}.
\]

Define \( \mathcal{L}\mathbb{Y} = \mathbb{D} \Delta \mathbb{Y} + \mathbb{A}(P^0) \mathbb{Y} \), \( \mathbb{Z}_i(i \geq 1) \) is invariant under \( \mathcal{L} \). \( \lambda \) represents an eigenvalue of \( \mathcal{L} \) if and only if it is an eigenvalue of a matrix \( -\mathbb{D} \mu_i + \mathbb{A}(P^0) \) with \( i \geq 1 \), where there is an eigenvalue in \( \mathbb{Z}_i \). That is, \( \lambda \) must be the root of the following characteristic equation

\[
\text{det} \left( \lambda \mathcal{I} + \mathbb{D} \mu_i - \mathbb{A}(P^0) \right) = 0.
\]

Here \( \mathcal{I} \) stands for the identity matrix. Thus, the characteristic equation at \( P^0 \) can be specifically written as

\[
[\lambda + (\mu + \sigma + u_iD)][\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3] = 0. \tag{3.11}
\]

Clearly, \( \lambda_1 = -(\mu + \sigma + u_iD) \) is an eigenvalue of Eq. (3.11). Therefore, the other three eigenvalues are the roots of the following equation

\[
\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0.
\]

Since \( R_0 = R_{01} + R_{02} + R_{03} < 1 \), we obtain that

\[
A_1 = -(1 - c_1)S_0h'(0) + (\mu + \theta) + u_iD + \mu + (1 + b)\gamma + \rho + u_iD + \xi + u_iD_4 \\
= (\mu + \theta)[1 - R_{01}] + \mu + (1 + b)\gamma + \rho + \xi + 2u_iD + u_iD_4 > 0,
\]

\[
A_2 = [(\mu + \theta) + u_iD - (1 - c_1)S_0h'(0)][\mu + (1 + b)\gamma + \rho + u_iD + \xi + u_iD_4] \\
+ [\mu + (1 + b)\gamma + \rho + u_iD][\xi + u_iD_4] - \alpha(1 - c_2)S_0f'(0) - \theta(1 - c_1)S_0g'(0)
\]

\[
= [(\mu + \theta)(1 - R_{01}) + u_iD][\mu + (1 + b)\gamma + \rho + u_iD] \\
+ [(\mu + \theta)(1 - R_{01}) + u_iD][\xi + u_iD_4] \\
+ [\mu + (1 + b)\gamma + \rho + u_iD][\xi + u_iD_4] - (\mu + \theta)[\mu + (1 + b)\gamma + \rho]R_{02}
\]

10069
\[-\frac{(\mu + \theta)(\mu + (1 + b)\gamma + \rho)\xi}{\mu + \theta + (1 + b)\gamma + \rho}\mathcal{R}_{03}\]
\[> \{(\mu + \theta)(1 - \mathcal{R}_{01}) + u_iD\}[\xi + u_iD_4] + (\mu + \theta)[\mu + (1 + b)\gamma + \rho]\]
\[+ u_i\xi + \mu + (1 + b)\gamma + \rho + u_iD][(\xi + u_iD_4) + (\mu + \theta)[\mu + (1 + b)\gamma + \rho]]\]
\[(1 - \mathcal{R}_{01} - \mathcal{R}_{02})\]
\[+(\mu + \theta)\xi(1 - \mathcal{R}_{01} - \mathcal{R}_{03}) > 0,\]

\[A_3 = [(\mu + \theta) - (1 - c_1)S_0h'(0) + u_iD][\mu + (1 + b)\gamma + \rho + u_iD](\xi + u_iD_4)\]
\[\alpha\theta(1 - c_2)S_0f'(0)\]
\[\alpha\mu + (1 + b)\gamma + \rho + u_iD](1 - c_2)f'(0) - (\xi + u_iD_4)\theta(1 - c_1)S_0g'(0)\]
\[= [(\mu + \theta)(1 - \mathcal{R}_{01}) + u_iD][\xi + u_iD_4] - \alpha[\mu + \theta + (1 + b)\gamma + \rho](1 - c_2)S_0f'(0)\]
\[\alpha u_iD(1 - c_2)S_0f'(0) - (\xi + u_iD_4)\theta(1 - c_1)S_0g'(0)\]
\[(\mu + \theta)(1 - \mathcal{R}_{01})[(b + 1)\gamma + \mu + \rho]\xi + (\theta + \mu)(1 - \mathcal{R}_{01})u_iD_4[(1 + b)\gamma + \mu + \rho]\]
\[+(\mu + \theta)(1 - \mathcal{R}_{01})\xi u_iD\]
\[+(\mu + \theta)(1 - \mathcal{R}_{01})u_iD_4 + u_iD[\mu + (1 + b)\gamma + \rho + u_iD](\xi + u_iD_4)\]
\[\xi(\theta + \mu)(b + 1)\gamma + \mu + \rho\]
\[\mathcal{R}_{02} = u_iD(\theta + \mu)[(1 + b)\gamma + \mu + \rho]\]
\[\mathcal{R}_{03} = (\mu + \theta)\xi(1 - \mathcal{R}_{01} - \mathcal{R}_{02} - \mathcal{R}_{03}) + (\mu + \theta)u_iD_4[\mu + (1 + b)\gamma + \rho](1 - \mathcal{R}_{01} - \mathcal{R}_{02})\]
\[+(\mu + \theta)\xi u_iD(1 - \mathcal{R}_{01} - \mathcal{R}_{03}) + (\mu + \theta)(1 - \mathcal{R}_{01})u_iD_4\]
\[+ u_iD[\mu + (1 + b)\gamma + \rho + u_iD](\xi + u_iD_4) > 0.\]

We can verify that \(A_1A_2 - A_3 > 0\). Then, by applying the theorem of Routh-Hurwitz, we can assert that every eigenvalue of Eq. (3.11) has negative real part. Hence, when \(\mathcal{R}_0 < 1\), the DFE \(P^\theta\) of the reaction-diffusion system Eq. (3.1) is locally asymptotically stable. □

Similarly, we prove the following result.

**Theorem 3.4.** When \(\mathcal{R}_0 > 1\), \(P^\ast\) of the reaction-diffusion system Eq. (3.1) is locally asymptotically stable.

**Proof.** Consider the linearization of the reaction-diffusion system Eq. (3.1) at \(P^\ast\):
\[
\frac{\partial Y(x, t)}{\partial t} = D\Delta Y(x, t) + BY(x, t),
\]

obviously,
\[
B = \begin{pmatrix}
  b_{11} & -(1 - c_1)S^*h'(E^*) & -(1 - c_1)S^*g(I^*) & -(1 - c_2)S^*f'(A^*) \\
  b_{21} & (1 - c_1)S^*h'(E^*) - (\mu + \theta) & (1 - c_1)S^*g(I^*) & (1 - c_2)S^*f'(A^*) \\
  0 & \theta & -[\mu + (1 + b)\gamma + \rho] & 0 \\
  0 & \alpha & \alpha & -\xi 
\end{pmatrix},
\]

where \(b_{11} = -((1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*) + \mu + \sigma)\), \(b_{21} = (1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)\).
Similarly, \( \lambda \) must be a root of
\[
\det (\lambda I + \mathbb{D}\mu_i - A(P^*)) = 0.
\]
Therefore, the characteristic equation at \( P^* \) can be specifically denoted as
\[
\lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4 = 0,
\]
(3.12)
where \( B_1, B_2, B_3 \) and \( B_4 \) are given below.

From Eqs. (3.5) and (3.9), we obtain
\[
\alpha(1 - c_2)S^*f'(A^*) \leq \alpha(1 - c_2)S^*f(A^*) \frac{A^*}{(\mu + \theta)[\mu + (1 + b)\gamma + \rho] \xi (1 - c_2)f(A^*)}
\]
\[
\theta(1 - c_1)S^*g'(I^*) \leq \theta(1 - c_1)S^*g(I^*) \frac{g(I^*)}{I^*} \frac{I^*}{g(A^*)} \frac{g(I^*)}{f(A^*)} (1 - c_2) + g(I^*) (1 - c_1) + h(E^*)(1 - c_1),
\]
\[
(1 - c_1)S^*h'(E^*) \leq (1 - c_1)S^*h(E^*) \frac{h(E^*)}{E^*} \frac{h(E^*)}{h(E^*)} (1 - c_1) + g(I^*) (1 - c_1) + f(A^*)(1 - c_2).
\]
Therefore,
\[
B_1 = (1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*) + \mu + \sigma + u_D + (\mu + \theta)
\]
\[-(1 - c_1)S^*h'(E^*) + u_D + [\mu + (1 + b)\gamma + \rho + u_D] + \xi + u_D D_4
\]
\[\geq (1 - c_1)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*) + \mu + \sigma + 2u_D D
\]
\[+ [\mu + (1 + b)\gamma + \rho + u_D]
\]
\[+ (\mu + \theta) \frac{(1 - c_2)f(A^*) + (1 - c_1)g(I^*)}{(1 - c_1)h(E^*) + (1 - c_1)g(I^*) + (1 - c_2)f(W^*)} + \xi + u_D D_4 > 0,
\]
\[
B_2 = [(\mu + \theta) - (1 - c_1)S^*h'(E^*) + u_D][(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)]
\]
\[+ \mu + \sigma + u_D D + \mu + (1 + b)\gamma + \rho + u_D D + \xi + u_D D_4]
\[+ [\mu + (1 + b)\gamma + \rho + u_D][\xi + u_D D_4]
\]
\[+ [(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*) + \mu + \sigma + u_D D]
\]
\[+ [\mu + (1 + b)\gamma + \rho + u_D D + \xi + u_D D_4]
\]
\[+ [(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + h(E^*))](1 - c_1)S^*h'(E^*) - \alpha(1 - c_2)S^*f(A^*)
\]
\[\geq u_D [\mu + (1 + b)\gamma + \rho]
\]
\[+ \left[ (\mu + \theta) \frac{(1 - c_2)f(A^*) + (1 - c_1)g(I^*)}{(1 - c_1)h(E^*) + (1 - c_1)g(I^*) + (1 - c_2)f(A^*)} + u_D \right]
\]
\[\cdot[(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*) + \mu + \sigma + 2u_D D + \xi + u_D D_4]
\]
\[+ u_D D_4[\mu + (1 + b)\gamma + \rho + u_D D + \xi + u_D D_4]
\]
\[+ [(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + h(E^*))]S^*h'(E^*) (1 - c_1)
\]
\[+ [(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*) + \mu + \sigma + u_D D]
\]
\[+ [\mu + (1 + b)\gamma + \rho + u_D D + \xi + u_D D_4]
\]
\[+ [\mu + (1 + b)\gamma + \rho] \xi \left[ 1 - \frac{(1 - c_2)f(A^*)}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)} \right]
\]
10071
Y. Tu, X. Meng, S. Gao et al.  
Journal of the Franklin Institute 359 (2022) 10058–10097

\[ + \frac{(\mu + \theta)[\mu + (1 + b)\gamma + \rho](1 - c_2)f(A^*)}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)} > 0, \]

\[ B_3 = [(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*) + \mu + \sigma + 2u_D + (\mu + \theta) \]

\[ - (1 - c_1)S^* f'(E^*)] \times [\mu + (1 + b)\gamma + \rho + u_D][\xi + u_D] - \alpha(1 - c_2)S^* f'(A^*) \]

\[ [\mu + \sigma + u_D + \mu + (1 + b)\gamma + \rho + u_D + \theta] \]

\[ + [(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*) + \mu + \sigma + u_D] \]

\[ [(\mu + \theta) - (1 - c_1)S^* f'(E^*)] \times [\mu + (1 + b)\gamma + \rho + u_D][\xi + u_D] \]

\[ + S^* (1 - c_1)h'(E^*)[(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)] \]

\[ [\mu + (1 + b)\gamma + \rho + u_D + \xi + u_D] \]

\[ - \frac{(\theta + \mu)[(1 + b)\gamma + \mu + \rho][\xi + u_D]}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)} \]

\[ + [(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*) + \mu + \sigma + u_D] \]

\[ \times \left[ \frac{(\mu + \theta)[(1 - c_2)f(A^*) + (1 - c_1)g(I^*)]}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)} \right] \]

\[ - \frac{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)} \]

\[ + [(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)] \]

\[ [\mu + (1 + b)\gamma + \rho + u_D + \xi + u_D] \]

\[ + \mu + \sigma + 2u_D] \]

\[ + [(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*) + \mu + \sigma + u_D] \]

\[ \times \left[ \frac{(\mu + \theta)[(1 - c_2)f(A^*) + (1 - c_1)g(I^*)]}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)} \right] \]

\[ - \frac{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)} \]

\[ + [(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)] \]

\[ [\mu + (1 + b)\gamma + \rho + u_D + \xi + u_D] \]

\[ + \mu + \sigma + 2u_D] \]

\[ + [(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*) + \mu + \sigma + u_D] \]
Y. Tu, X. Meng, S. Gao et al.  
Journal of the Franklin Institute 359 (2022) 10058–10097

\[
\times \left[ \frac{(\mu + \theta)(1 - c_2)f(A^*)}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)} + \frac{(\mu + \theta)(1 - c_1)g(I^*)}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)} \right]
\]

\[
- \frac{(\mu + \theta)[\mu + (1 + b)\gamma + \rho](1 - c_1)g(I^*)}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)}[(\mu + \sigma + u_iD) + (\xi + u_iD_4)]
\]

\[
+ (1 - c_1)S^h(E^*)[(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)]
\]

\[
[\mu + (1 + b)\gamma + \rho + u_iD + \xi + u_iD_4]
\]

\[
= \frac{(\mu + \theta)\xi}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)}[(\mu + (1 + b)\gamma + \rho + u_iD + \mu + \sigma + u_iD)]g(I^*)(1 - c_1)h(E^*) + (1 - c_1)h(E^*)]
\]

\[
+ (1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)
\]

\[
[\mu + (1 + b)\gamma + \rho + u_iD + \xi + u_iD_4]
\]

\[
+ \frac{(\mu + \theta)[\mu + (1 + b)\gamma + \rho](\mu + \sigma + u_iD + \xi + u_iD_4)[(1 - c_2)f(A^*) + (1 - c_1)h(E^*)]}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)}
\]

\[
\frac{h(E^*)(1 - c_1)g(I^*)(1 - c_1) + f(A^*)}{(1 - c_2)}(1 - c_2)
\]

\[
+ \frac{(1 - c_1)h(E^*) + (1 - c_1)g(I^*) + (1 - c_2)f(A^*)}{(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)}
\]

\[
+[\mu + (1 + b)\gamma + \rho + u_iD][\xi + u_iD_4]
\]

\[
+ [\mu + (1 + b)\gamma + \rho + \xi][((1 - c_2)f(A^*) + (1 - c_1)g(I^*) > 0,
\]

and

\[
B_4 = [\mu + (1 + b)\gamma + \rho + u_iD][\xi + u_iD_4][((1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)]
\]

\[
+ [\mu + (1 + b)\gamma + \rho + u_iD][\xi + u_iD_4][(1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)]
\]

\[
- (1 - c_1)S^h(E^*)[\mu + (1 + b)\gamma + \rho + u_iD][\xi + u_iD_4][\mu + \sigma + u_iD]
\]

\[
- (1 - c_2)\sigma^Sf'(A^*)[\sigma + \mu + u_iD][\sigma + \mu + u_iD][(\theta + \mu + u_iD)[\theta + \mu + \rho + (1 + b)\gamma + u_iD]
\]

\[
- (1 - c_1)\theta S^g(I^*)[\xi + u_iD_4][\mu + \sigma + u_iD]
\]

\[
\geq [\mu + (1 + b)\gamma + \rho + u_iD][\xi + u_iD_4][(1 - c_2)f(A^*) + (1 - c_1)g(I^*)]
\]

\[
+ (1 - c_1)h(E^*) + \mu + \sigma + u_iD)[\mu + \theta + u_iD]
\]

\[
- [\mu + \theta][\mu + (1 + b)\gamma + \rho][\xi + u_iD_4][\mu + \sigma + u_iD][\mu + \sigma + u_iD][\mu + \theta + u_iD]
\]

\[
- (1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)]
\]

\[
[\mu + \theta + u_iD][\xi + u_iD_4][\mu + \sigma + u_iD][\mu + \sigma + u_iD]
\]

\[
- (1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*)]
\]

\[
- (1 - c_2)f(A^*) + (1 - c_1)g(I^*) + (1 - c_1)h(E^*) + \mu + \sigma + u_iD]
\]

\[
(1 - c_2)f(A^*)
\]

\[
(1 - c_2)f(A^*) + (1 - c_1)h(E^*)
\]

10073
\[
\begin{align*}
= \mu + (1 + b)\gamma + \rho]u_i D[(1 - c_1) f(A^*) + (1 - c_1) g(I^*) + (1 - c_1) h(E^*) + \mu + \sigma + u_i D] \\
+ \mu + (1 + b)\gamma + \rho]u_i D[2 (1 - c_2) f(A^*) + (1 - c_1) g(I^*) + (1 - c_1) h(E^*)] + \mu + \sigma + u_i D] \\
+ u_i D[1 (1 - c_1) f(A^*) + (1 - c_1) g(I^*) + (1 - c_1) h(E^*)] \mu + \sigma + u_i D \\
+ [\mu + (1 + b)\gamma + \rho + u_i D] \xi(\mu + \theta)[(1 - c_1) g(I^*) + (1 - c_1) h(E^*)] \\
+ [\mu + (1 + b)\gamma + \rho]u_i D[\mu + \sigma + u_i D](\mu + \theta)(1 - c_1) f(A^*) + (1 - c_1) g(I^*) + (1 - c_1) h(E^*)] \\
+ u_i D[\mu + \sigma + u_i D](\mu + \theta) h(E^*)(1 - c_1) + g(I^*)(1 - c_1) + f(A^*)(1 - c_2) > 0.
\end{align*}
\]

By a direct calculation, we get that

\[
H_1 = B_1 > 0, \\
H_2 = B_1 B_2 - B_3 > 0, \\
H_3 = \begin{pmatrix} B_1 & B_2 & 0 \\ 1 & B_2 & B_4 \\ 0 & B_1 & B_3 \end{pmatrix} = B_3 H_2 - B_1 B_4 > 0, \\
H_4 = B_4 H_3 > 0.
\]

It follows from the theorem of the Routh-Hurwitz that every eigenvalue of Eq. (3.12) has negative real part. Hence, when \( R_0 > 1 \), the reaction-diffusion system’s equilibrium \( P^* \) is locally asymptotically stable. □

### 3.4. Uniform persistence of COVID-19 when \( R_0 > 1 \)

This subsection studies the uniform persistence of the reaction-diffusion system Eq. (3.1). Linearizing system at \( P_0(0, 0, 0) \). We get the linear reaction-diffusion system for \( E, I \) and \( A \) as follows:

\[
\begin{align*}
\frac{\partial E(x, t)}{\partial t} &= D A E + (1 - c_1) S_0 h'(0) E + (1 - c_1) S_0 g'(0) I + (1 - c_2) S_0 f'(0) A - (\mu + \theta) E, \quad x \in \Omega, \ t > 0, \\
\frac{\partial I(x, t)}{\partial t} &= D A I + \theta E - \mu I - \rho I - (1 + b) \gamma I, \quad x \in \Omega, \ t > 0, \\
\frac{\partial A(x, t)}{\partial t} &= D A A + \alpha(I + E) - \xi A, \quad x \in \Omega, \ t > 0, \\
\frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial A}{\partial n} = 0, \quad x \in \partial \Omega, \ t > 0.
\end{align*}
\]

(3.13)

Clearly, system Eq. (3.13) is a cooperative system. Suppose that \( E(x, t) = e^{i\mathcal{P}_2(x)}I(x, t) = e^{i\mathcal{P}_3(x)} \) and \( A(x, t) = e^{i\mathcal{P}_4(x)} \), thus system Eq. (3.13) yields to

\[
\begin{align*}
\lambda \phi_2(x) &= D A \phi_2(x) + (1 - c_1) S_0 h'(0) \phi_2(x) + (1 - c_1) S_0 g'(0) \phi_3(x) \\
&\quad + (1 - c_2) S_0 f'(0) \phi_4(x) - (\theta + \mu) \phi_2(x), \quad x \in \Omega, \\
\lambda \phi_3(x) &= D A \phi_3(x) + \theta \phi_2(x) - \mu \phi_2(x) - \rho \phi_3(x) - (1 + b) \gamma \phi_3(x), \quad x \in \Omega, \\
\lambda \phi_4(x) &= D A \phi_4(x) + \alpha(\phi_2(x) + \phi_3(x)) - \xi \phi_4(x), \quad x \in \Omega, \\
\frac{\partial \phi_2}{\partial n} = \frac{\partial \phi_3}{\partial n} = \frac{\partial \phi_4}{\partial n} = 0, \quad x \in \partial \Omega.
\end{align*}
\]

(3.14)
It follows from Theorem 7.6.1 in [42] that Eq. (3.14) has a principal eigenvalue \( \lambda_0(S_0(x)) \) with a positive eigenfunction \( \phi(x) = (\phi_2(x), \phi_3(x), \phi_4(x)) \).

Similar to the argument of [39] and [43], we obtain the lemmas as follow.

**Lemma 3.1.** \((R_0 - 1)\) and the principal eigenvalue \( \lambda_0(S_0(x)) > 0 \) have the same sign, and \( P^0 \) is asymptotically stable when \( R_0 < 1 \).

**Lemma 3.2.** Assume \( \mathcal{Y}(\cdot, t, \phi) \) represent the solution of the reaction-diffusion system Eqs. (3.1)–(3.3) with \( \mathcal{Y}(\cdot, 0, \phi) = \phi \in X^+ \). Further,

1. for given all \( \phi \in X^+ \) and \( \forall x \in \Omega, t > 0 \), we know \( S(x, t, \phi) > 0 \), and exists a positive number \( \eta \), such that

\[
\liminf_{t \to +\infty} S(x, t, \phi) \geq \eta, \quad \text{uniformly for } \ x \in \overline{\Omega};
\]

2. when there exists \( t_1 > 0 \), such that

\[
E(\cdot, t_1, \phi) \not\equiv 0 \quad \text{or} \quad I(\cdot, t_1, \phi) \not\equiv 0 \quad \text{or} \quad A(\cdot, t_1, \phi) \not\equiv 0,
\]

then for all \( x \in \overline{\Omega}, t > t_1 \), one has

\[
(E(x, t, \phi), I(x, t, \phi), A(x, t, \phi)) > 0.
\]

Next, similar to the method of [44], we show that COVID-19 is persistent when \( R_0 > 1 \).

**Theorem 3.5.** Assume that \( \mathcal{Y}(x, t, \phi) \) is the reaction-diffusion system Eqs. (3.1)–(3.3)’s solution with \( \mathcal{Y}(\cdot, 0, \phi) = \phi \in X^+ \). When \( R_0 > 1 \), there is a positive number \( \delta \), such that for all \( \phi \in X^+ \) with \( \phi_2 \not\equiv 0 \) or \( \phi_3 \not\equiv 0 \) or \( \phi_4 \not\equiv 0 \), we get

\[
\liminf_{t \to +\infty} E(x, t, \phi) \geq \delta, \quad \liminf_{t \to +\infty} I(x, t, \phi) \geq \delta, \quad \liminf_{t \to +\infty} A(x, t, \phi) \geq \delta, \quad \text{uniformly for } \forall x \in \overline{\Omega}.
\]

**Proof.** We have \( \lambda_0(S_0(x)) > 0 \) when \( R_0 > 1 \). Given any sufficient small positive number \( \varrho \in (0, \varrho^*) \). Let \( \lambda_0(\varrho) \) be the principal eigenvalue yields to

\[
\begin{aligned}
\lambda \phi_2(x) &= D \Delta \phi_2(x) + (1 - c_1) \left( \frac{\Lambda}{\mu + \sigma} - \varrho \right) h'(\varrho) \phi_2(x) + (1 - c_1) \left( \frac{\Lambda}{\mu + \sigma} - \varrho \right) g'(\varrho) \phi_3(x) \\
&\quad + (1 - c_2) \left( \frac{\Lambda}{\mu + \sigma} - \varrho \right) f'(\varrho) \phi_4(x) - (\mu + \theta) \phi_2(x), \quad x \in \Omega, \\
\lambda \phi_3(x) &= D \Delta \phi_3(x) + \theta \phi_2(x) - \mu \phi_3(x) - (1 + b) \gamma \phi_3(x) - \rho \phi_3(x), \quad x \in \Omega, \\
\lambda \phi_4(x) &= D \Delta \phi_4(x) + \alpha (\phi_2(x) + \phi_3(x)) - \xi \phi_4(x), \quad x \in \Omega, \\
\frac{\partial \phi_2}{\partial n} = \frac{\partial \phi_3}{\partial n} = \frac{\partial \phi_4}{\partial n} = 0, \quad x \in \partial \Omega.
\end{aligned}
\]

Obviously, we have \( \lim_{\varrho \to 0^+} \lambda_0(\varrho) = \lambda_0(S_0(x)) \). Thus, we can fix a \( \varrho_0 \in (0, \varrho^*) \), such that \( \lambda_0(\varrho_0) > 0 \).

Define

\[
\mathcal{W} = \{ \phi \in X^+ : \phi_2 \not\equiv 0 \quad \text{or} \quad \phi_3 \not\equiv 0 \quad \text{or} \quad \phi_4 \not\equiv 0 \}
\]

and

\[
\partial \mathcal{W} := X^+ \setminus \mathcal{W} = \{ \phi \in X^+ : \phi_2 \equiv 0, \quad \phi_3 \equiv 0 \quad \text{and} \quad \phi_4 \equiv 0 \}.
\]

10075
We know from Lemma 3.2 that $\mathbb{W}$ stands for the forward invariant set of $\Psi(t)$. Then, we set

$$\mathbb{M}_\beta := \{ \phi \in \partial \mathbb{W} : \Psi(t) \phi \in \partial \mathbb{W}, \quad \forall t \geq 0 \}.$$

For any $t > 0$ and $\phi \in \mathbb{M}_\beta$, we know $\Psi(t) \phi \in \partial \mathbb{W}$. Therefore $(E(\cdot, t, \phi), I(\cdot, t, \phi), A(\cdot, t, \phi)) \equiv 0$ for all $t \geq 0$. According to S-equation of the reaction-diffusion system Eq. (3.1) that

$$\begin{cases}
\frac{\partial S(x, t)}{\partial t} = D\Delta S + \Lambda - \mu S - \sigma S, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial S}{\partial n} = 0, \quad x \in \partial \Omega, \quad t > 0.
\end{cases}$$

According to Smith [42] and Thieme [45], we know $\lim_{t \to +\infty} S(\cdot, t, \phi) = \frac{\Lambda}{\mu + \sigma}$ for $\forall x \in \Omega$.

Next, we proceed by contradiction to prove that

$$\limsup_{t \to +\infty} \| \Psi(t) \phi - P^0 \| \geq \varrho_0, \quad \text{for all } \phi \in \mathbb{W}. \quad (3.15)$$

If Eq. (3.15) doesn’t hold, then for some $\phi_0 \in \mathbb{W}$, we have

$$\limsup_{t \to +\infty} \| \Psi(t) \phi_0 - P^0 \| < \varrho_0.$$

Without loss of generality, we obtain

$$\| \Psi(t) \phi_0 - P^0 \| < \varrho_0, \quad \forall \varrho_0 > 0.$$

Then, there exists $t_1 > 0$, when $t \geq t_1$ such that

$$\frac{\Lambda}{\mu + \sigma} - \varrho_0 < S(x, t, \phi_0) < \frac{\Lambda}{\mu + \sigma} + \varrho_0, \quad E(x, t, \phi_0) < \varrho_0, \quad I(x, t, \phi_0) < \varrho_0, \quad A(x, t, \phi_0) < \varrho_0, \quad x \in \Omega.$$

It follows from assumption $(H_2)$ and inequality Eq. (3.9) that $I(x, t, \phi_0), E(x, t, \phi_0)$ and $A(x, t, \phi_0)$ satisfy

$$\begin{cases}
\frac{\partial E(x, t)}{\partial t} \geq D\Delta E + (1 - c_1) \left( \frac{\Lambda}{\mu + \sigma} - \varrho_0 \right) h'(\varrho_0)E + (1 - c_1) \left( \frac{\Lambda}{\mu + \sigma} - \varrho_0 \right) g'(\varrho_0)I \\
+ (1 - c_2) \left( \frac{\Lambda}{\mu + \sigma} - \varrho_0 \right) f'(\varrho_0)A - (\mu + \theta)E, \quad x \in \Omega, \quad t \geq t_1, \\
\frac{\partial I(x, t)}{\partial t} \geq D\Delta I + \theta E - \mu I - \rho I - (1 + b)\gamma I, \quad x \in \Omega, \quad t \geq t_1, \\
\frac{\partial A(x, t)}{\partial t} \geq D\Delta A + \alpha(E + I) - \xi A, \quad x \in \Omega, \quad t \geq t_1, \\
\frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial A}{\partial n} = 0, \quad x \in \partial \Omega, \quad t \geq t_1.
\end{cases} \quad (3.16)$$

There exists a positive vector $\phi = (\phi_2, \phi_3, \phi_3)^T$, let $\phi_{\lambda_0(\varrho_0)} = (\phi_{2, \lambda_0(\varrho_0)}, \phi_{3, \lambda_0(\varrho_0)}, \phi_{4, \lambda_0(\varrho_0)})$ be the positive eigenfunction associated with $\lambda_0(\varrho_0)$. From the reaction-diffusion equation’s
comparison theorem, we get the comparison system of Eq. (3.16) as follows

\[
\begin{align*}
\frac{\partial U_1(x, t)}{\partial t} &= D\Delta U_1 + (1 - c_1)\left(\frac{\Lambda}{\mu + \sigma} - \varrho_0\right)h'(\varrho_0)U_1 + (1 - c_1)\left(\frac{\Lambda}{\mu + \sigma} - \varrho_0\right)g'(\varrho_0)U_2 + (1 - c_2)\left(\frac{\Lambda}{\mu + \sigma} - \varrho_0\right)f'(\varrho_0)U_3 - (\mu + \theta)U_1, \quad x \in \Omega, \ t \geq t_1, \\
\frac{\partial U_3}{\partial t} &= D_4\Delta U_3 + \alpha(U_1 + U_2) - \xi U_3, \quad x \in \Omega, \ t \geq t_1, \\
\frac{\partial U_1}{\partial n} = \frac{\partial U_2}{\partial n} = \frac{\partial U_3}{\partial n} = 0, \quad x \in \partial \Omega, \ t \geq t_1.
\end{align*}
\]

(3.17)

For any \( t > t_1 \) and \( x \in \overline{\Omega} \), we know:

\[
(U_1(x, t), \ U_2(x, t), \ U_3(x, t)) = \left(e^{\lambda_0(\varrho_0)t}\phi_{2, \lambda_0(\varrho_0)}(x), \ e^{\lambda_0(\varrho_0)t}\phi_{3, \lambda_0(\varrho_0)}(x), \ e^{\lambda_0(\varrho_0)t}\phi_{4, \lambda_0(\varrho_0)}(x)\right)
\]

is a solution of Eq. (3.17). Since \( E(x, t, \varphi_0) > 0, I(x, t, \varphi_0) > 0, A(x, t, \varphi_0) > 0 \) for \( t > 0 \) and \( x \in \overline{\Omega} \), there is \( \zeta_2 > 0 \), such that:

\[
(E(x, t, \varphi_0), I(x, t, \varphi_0), A(x, t, \varphi_0)) \geq \zeta_2 e^{\lambda_0(\varrho_0)t}\phi_{\lambda_0(\varrho_0)}(x), \quad \text{for } \forall t > t_1.
\]

Since \( \lambda_0(\varrho_0) > 0 \), we obtain:

\[
\lim_{t \to +\infty} (E(x, t, \varphi_0), I(x, t, \varphi_0), A(x, t, \varphi_0)) = \infty.
\]

This is contradicting with the boundedness of the solutions. Therefore, the formula Eq. (3.15) holds.

It is clear that every forward orbit of \( \Psi(t) \) in \( \mathbb{M}_\varrho \) converges to \( P^0 \). \( P^0 \) is an isolated invariant set in \( \mathbb{X}^+ \) and \( W^s(P^0) \cap \mathbb{W} = \emptyset \), and it is acyclic in \( \mathbb{M}_\varrho \). This implies that the system’s solution is uniformly persistent, that is Eq. (3.15) holds. \( \square \)

### 3.5. Global stability

In this subsection, We prove the equilibria’s stability for the reaction-diffusion system Eq. (3.1) by designing \( \Phi(x) = x - 1 - \ln x \), it is clear that \( \Phi(x) \geq 0 \) for \( \forall x > 0 \). Next, we obtain the global stability as follows.

**Theorem 3.6.** When \( R_0 < 1 \), the DFE \( P^0 \) of the reaction-diffusion system Eq. (3.1) is globally asymptotically stable.

**Proof.** From Lemma 3.1, we know \( \lambda_0(S_0(x)) < 0 \) when \( R_0 < 1 \). That means there is a sufficiently small \( \bar{\varepsilon}_0 > 0 \) such that \( \lambda_0(S_0(x) + \bar{\varepsilon}_0) < 0 \). Then, according to S-equation of systems Eq. (3.1) that

\[
\frac{\partial S(x, t)}{\partial t} \leq D_1 \Delta S + \Lambda - \mu S - \sigma S, \quad \text{for } x \in \Omega, \ t > 0.
\]

Furthermore, there is a \( \bar{t}_0 > 0 \), such that

\[
S(x, t) \leq S_0(x) + \bar{\varepsilon}_0, \quad \text{for } x \in \overline{\Omega}, \ t \geq \bar{t}_0.
\]

10077
Then by (H₂), we obtain that
\[
\begin{align*}
\frac{\partial E(x, t)}{\partial t} & \leq D_2 \Delta E + (1 - c_2)(S_0(x) + \bar{e}_0)f(A) + (1 - c_1)(S_0(x) + \bar{e}_0)g(I) \\
+ (1 - c_1)(S_0(x) + \bar{e}_0)h(E) - (\mu + \theta)E, & \text{for } x \in \Omega, \ t \geq \bar{t}_0, \\
\frac{\partial I(x, t)}{\partial t} & \leq D_3 \Delta I + \theta E - \rho I - [(1 + b)\gamma + \mu]I, & \text{for } x \in \Omega, \ t \geq \bar{t}_0, \\
\frac{\partial A(x, t)}{\partial t} & \leq D_4 \Delta A + \alpha E + I - \xi A, & \text{for } x \in \Omega, \ t \geq \bar{t}_0, \\
\frac{\partial E(x, t)}{\partial n} = \frac{\partial I(x, t)}{\partial n} = \frac{\partial A(x, t)}{\partial n} = 0, & \text{for } x \in \partial \Omega, \ t \geq \bar{t}_0.
\end{align*}
\]

Let \( \phi(x) = (\phi_2(x), \ \phi_3(x), \ \phi_4(x)) \) is eigenfunction of system Eq. (3.14) corresponding to the principal eigenvalue \( \lambda_0(S_0(x) + \bar{e}_0) < 0 \). There exists a \( \zeta_1 > 0 \) such that 
\( \zeta_1(\phi_2(x), \ \phi_3(x), \ \phi_4(x)) \geq (E(x, \bar{t}_0)), I(x, \bar{t}_0)), A(x, \bar{t}_0)) \). Further, we get 
\( \zeta_1(\phi_2(x), \ \phi_3(x), \ \phi_4(x))e^{\gamma_0(S_0(x) + \bar{e}_0)(t-\bar{t}_0)} \geq (E(x, t), I(x, t), A(x, t)), \) for \( x \in \Omega, \ t \geq \bar{t}_0 \).

Therefore,
\[
\lim_{t \to \infty} E(x, t) = 0, \ \lim_{t \to \infty} I(x, t) = 0 \text{ and } \lim_{t \to \infty} A(x, t) = 0, \text{ uniformly for } x \in \bar{\Omega},
\]
which makes S-equation of the reaction-diffusion system Eq. (3.1) asymptotically
\[
\frac{\partial S(x, t)}{\partial t} = D_1 \Delta S + \Lambda - \mu S - \sigma S.
\]

Furthermore, it follows from Thieme [45] and Guo et al. [46] that
\[
\lim_{t \to \infty} S(x, t) = S_0(x), \text{ uniformly for } x \in \bar{\Omega}.
\]

Hence, \( P^0 \) of the reaction-diffusion system Eqs. (3.1)–(3.3) is globally asymptotic stable. \( \Box \)

Next, to prove the global asymptotic stability of \( \lambda^* \), we propose the assumption
\[
(H_3) : \Phi\left(\frac{A^*(E + I)}{A(E^* + I^*)}\right) - \Phi\left(\frac{E + I}{E^* + I^*}\right) \geq 0.
\]

Then, we get the result as follows.

**Theorem 3.7.** Suppose that \( (H_3) \) holds and \( R_0 > 1 \), then the reaction-diffusion system’s endemic equilibrium \( \lambda^* \) is globally asymptotically stable.

**Proof.** We define Lyapunov function
\[
M(t) = \int_\Omega S(x, t)M(S(x, t)) + E*M(E(x, t)) + a_1M(I(x, t)) + a_2M(A(x, t))dx,
\]
where
\[
a_1 = \frac{(1 - c_2)S^*f(A^*) + (1 - c_1)S^*g(I^*)}{[\mu + (1 + b)\gamma + \rho]I^*} \quad \text{and} \quad a_2 = \frac{(1 - c_2)S^*f(A^*)}{\xi A^*}.
\]

Since
\[
\Lambda = (\mu + \sigma)S^* + (1 - c_1)S^*h(E^*) + (1 - c_1)S^*g(I^*) + (1 - c_2)S^*f(A^*),
\]
\[
(\mu + \theta)E = \frac{(1 - c_2)S^*f(A^*)E}{E^*} + \frac{(1 - c_1)S^*g(I^*)E}{E^*} + \frac{(1 - c_1)S^*h(E^*)E}{E^*},
\]

10078
\[ \theta E^* = [\mu + (1 + b)\gamma]I^*, \]
\[ \alpha(E^* + I^*) = \xi A^*. \]

By simple derivation, we get
\[ \frac{dM(t)}{dt} = \int_\Omega \left[ \left( 1 - \frac{S^*}{S} \right) \frac{\partial S}{\partial t} + \left( 1 - \frac{E^*}{E} \right) \frac{\partial E}{\partial t} + \frac{(1 - c_2)S^* f(A^*) + (1 - c_1)S^* g(I^*)}{[\mu + (1 + b)\gamma + \rho]I^*} \left( \frac{1}{I^*} \right) \frac{\partial I^*}{\partial t} \right] dx. \]
\[ \quad + \frac{(1 - c_2)S^* f(A^*)}{\xi A^*} \left( 1 - \frac{A^*}{A} \right) \left| \frac{\partial A^*}{\partial t} \right| dx. \]

\[ = \int_\Omega \left[ \left( 1 - \frac{S^*}{S} \right) |\Delta S| + (\sigma + \mu)S^* - (\sigma + \mu)S + S^* f(A^*)(1 - c_2) + S^* g(I^*)(1 - c_1) \right. \]
\[ \quad + (1 - c_1)S^* h(E^*) - Sh(E)(1 - c_1) - Sg(I)(1 - c_1) - Sf(A)(1 - c_2)] \]
\[ \quad + \left( 1 - \frac{E^*}{E} \right) |\Delta E| + Sh(E)(1 - c_1) + Sg(I)(1 - c_1) + Sf(A)(1 - c_2) - (\mu + \theta)E \]
\[ \quad + \frac{(1 - c_2)S^* f(A^*) + (1 - c_1)S^* g(I^*)}{[\mu + (1 + b)\gamma + \rho]I^*} \left( 1 - \frac{I^*}{I} \right) |\Delta I + \theta E - (\mu + (1 + b)\gamma)I| \]
\[ \quad + \frac{(1 - c_2)S^* f(A^*)}{\xi A^*} \left( 1 - \frac{A^*}{A} \right) |\Delta A + \alpha(I + E) - \xi A| \right] dx. \]

Applying the Divergence Theorem and Neumann boundary conditions yields to
\[ \int_\Omega \Delta S dx = \int_\Omega \Delta E dx = \int_\Omega \Delta I dx = \int_\Omega \Delta W dx = 0, \]
and
\[ \int_\Omega \Delta S \frac{dS}{S} dx \geq 0, \int_\Omega \Delta E \frac{dE}{E} dx \geq 0, \int_\Omega \Delta I \frac{dI}{I} dx \geq 0, \int_\Omega \Delta A \frac{dA}{A} dx \geq 0. \]

Thus, we get
\[ \frac{dM(t)}{dt} = \int_\Omega - \frac{\mu + \sigma}{S} (S - S^*)^2 dx - (1 - c_2)S^* f(A^*) \int_\Omega \left[ \Phi \left( \frac{S^*}{S} \right) + \Phi \left( \frac{f(A)SE^*}{f(A^*)E^*} \right) + \Phi \left( \frac{E^*}{I^*} \right) \right] dx \]
\[ + \Phi \left( \frac{I^*}{I} \right) + \Phi \left( \frac{A^*(E + I)}{A(E + I)^*} \right) - \Phi \left( \frac{E + I}{E^* + I^*} \right) + \Phi \left( \frac{A^*}{A^*} \right) - \Phi \left( \frac{f(A)}{f(A^*)} \right) \].
\[+\Phi\left(\frac{I}{I^*}\right) - \Phi\left(\frac{g(I)}{g(I^*)}\right)\] 
\[-(1 - c_1)S^* h(E^*) \int_\Omega \left[ \Phi\left(\frac{S^*}{S}\right) + \Phi\left(\frac{h(E)SE^*}{h(E^*)S^*E}\right) + \Phi\left(\frac{E}{E^*}\right) - \Phi\left(\frac{h(E)}{h(E^*)}\right) \right] dx.\]

By assumption \((H_2)\) and \((H_3)\), one has
\[
\Phi\left(\frac{g(I)}{g(I^*)}\right) - \Phi\left(\frac{I}{I^*}\right) = \frac{g(I)}{g(I^*)} - \frac{I}{I^*} + \ln\left(\frac{I g(I^*)}{I^* g(I)}\right) \leq \frac{g(I)}{g(I^*)} - \frac{I}{I^*} + \frac{I}{I^*} \frac{g(I^*)}{g(I)} - 1
= \left(\frac{g(I)}{g(I^*)}\right) \left(1 - \frac{g(I^*)}{g(I)}\right) \leq 0,
\]

Similarly, we get
\[
\Phi\left(\frac{h(E)}{h(E^*)}\right) - \Phi\left(\frac{E}{E^*}\right) \leq \left(\frac{h(E)}{h(E^*)}\right) \left(1 - \frac{h(E^*)}{h(E)}\right) \leq 0,
\]
\[
\Phi\left(\frac{f(A)}{f(A^*)}\right) - \Phi\left(\frac{A}{A^*}\right) \leq \left(\frac{f(A)}{f(A^*)}\right) \left(1 - \frac{f(A^*)}{f(A)}\right) \leq 0,
\]
and
\[
\Phi\left(\frac{A^* (E + I)}{A (E^* + I^*)}\right) - \Phi\left(\frac{E + I}{E^* + I^*}\right) \geq 0.
\]

It is obvious that
\[
\frac{dM(t)}{dt} \leq \int_\Omega -\frac{\mu + \sigma}{S} (S - S^*)^2 dx - (1 - c_2)S^* f(A^*) \int_\Omega \left[ \Phi\left(\frac{S^*}{S}\right) + \Phi\left(\frac{S f(A) E^*}{S^* f(A^*) E}\right) + \Phi\left(\frac{E I^*}{E^* I}\right) \right. \
+ \left. \Phi\left(\frac{I}{I^*}\right)\right] dx - (1 - c_1)S^* g(I^*) \int_\Omega \left[ \Phi\left(\frac{S^*}{S}\right) + \Phi\left(\frac{g(I) S^* E}{g(I^*) S^* E}\right) + \Phi\left(\frac{I E}{I^* E}\right) \right] dx \leq 0.
\]

Clearly, the largest invariant subset of \(\left\{\frac{dM(t)}{dt} = 0\right\}\) is the singleton \(P^*\). Furthermore, we know that \(P^*\) is globally asymptotically stable. \(\Box\)

4. The NSFD scheme of system Eqs. (3.1)–(3.3)

In this section, we use the NSFD scheme to discretize the continuous system Eqs. (3.1)–(3.3) and study the equilibria’s positivity, boundedness, and numerically verify the global stability of continuous system.

System Eq. (3.1) is defined in \(\Omega = [a, b]\), \(\Delta t\) and \(\Delta x = \frac{b - a}{M}\) stand for the time and space stepsize, respectively, where \(M\) is a positive integer. Each mesh point is symbolized by \((x_n, t_k)\) with \(t_k = k\Delta t\) and \(x_n = a + n\Delta x\). For convenience, we write
$S(x_n, t_k), E(x_n, t_k), I(x_n, t_k)$ and $W(x_n, t_k)$ approximately as $S^k_n, E^k_n, I^k_n$ and $W^k_n$, respectively. Based on the above rules, we get the discrete model with NSFD scheme as follows

$$\begin{align*}
\frac{S_{n+1}^k - S_n^k}{\Delta t} &= D \frac{S_{n+1}^{k+1} - 2S_n^{k+1} + S_{n-1}^{k+1}}{2\Delta x} + \Lambda - (1 - c_2)S_n^{k+1}f(A_n^k) - (1 - c_1)S_n^{k+1}g(I_n^k) \\
&\quad - (1 - c_1)S_n^{k+1}h(\beta_n^k) - (\mu + \sigma)S_n^{k+1}, \\
\frac{E_{n+1}^k - E_n^k}{\Delta t} &= D \frac{E_{n+1}^{k+1} - 2E_n^{k+1} + E_{n-1}^{k+1}}{2\Delta x} + (1 - c_2)S_n^{k+1}f(A_n^k) + (1 - c_1)S_n^{k+1}g(I_n^k) \\
&\quad + (1 - c_1)S_n^{k+1}h(\beta_n^k) - (\mu + \theta)E_n^{k+1}, \\
\frac{I_{n+1}^k - I_n^k}{\Delta t} &= D \frac{I_{n+1}^{k+1} - 2I_n^{k+1} + I_{n-1}^{k+1}}{2\Delta x} + \theta E_n^{k+1} - [\mu + (1 + b)\gamma + \rho]I_n^{k+1}, \\
\frac{A_{n+1}^k - A_n^k}{\Delta t} &= D \frac{A_{n+1}^{k+1} - 2A_n^{k+1} + A_{n-1}^{k+1}}{2\Delta x} + \alpha(E_n^{k+1} + I_n^{k+1}) - \xi A_n^{k+1},
\end{align*}$$

(4.1)

where $k \in \mathbb{N}$ and $n \in \{0, 1, 2, \ldots, M\}$, with

$$\begin{align*}
S_0^0 &= \varphi_1(x_n) > 0, & E_0^0 &= \varphi_2(x_n) > 0, & I_0^0 &= \varphi_3(x_n) > 0, & A_0^0 &= \varphi_4(x_n) > 0, \\
\text{and the discrete boundary condition satisfies}
\end{align*}$$

(4.2)

$$\begin{align*}
S_{-1}^k &= S_0^k, & S_M^k &= S_{M+1}^k, & E_{-1}^k &= E_0^k, & E_M^k &= E_{M+1}^k, \\
I_{-1}^k &= I_0^k, & I_M^k &= I_{M+1}^k, & A_{-1}^k &= A_0^k, & A_M^k &= A_{M+1}^k.
\end{align*}$$

(4.3)

4.1. Positivity and boundedness

It is obvious that the discrete system Eq. (4.1) has the same equilibria as the system Eq. (3.1). Next, we show the positivity and boundedness of system Eqs. (4.1)-(4.3) as follows.

**Theorem 4.1.** For every given $\Delta x > 0$ and $\Delta t > 0$, the discrete system’s solution is bounded and non-negative for any $k \in \mathbb{N}$. 10081
Proof. According to S-equation of system Eq. (4.1), one has

\[ \mathcal{A}^k S^{k+1} = S^k + \Gamma \Delta t, \quad (4.4) \]

where \( \Gamma = (\mu N, \mu N, \cdots, \mu N)^T \) and \( \mathcal{A}^k \) is the following \((M+1) \times (M+1)\) dimensional matrix

\[
\mathcal{A}^k = \begin{pmatrix}
da_0^k & a & 0 & \cdots & 0 & 0 & 0 \\
 a & a_1^k & a & \cdots & 0 & 0 & 0 \\
 0 & a & a_2^k & \cdots & 0 & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & a_{M-2}^k & a & 0 \\
 0 & 0 & 0 & \cdots & a & a_{M-1}^k & a \\
 0 & 0 & 0 & \cdots & 0 & a & a_M^k
\end{pmatrix},
\]

\[
a = -\frac{D \Delta t}{(\Delta x)^2}, \quad a_0^k = 1 + \frac{D \Delta t}{(\Delta x)^2} + (f(A_0^k) + g(t_0^k) + h(E_0^k)) \Delta t + \mu + \sigma, \quad a_M^k = 1 + \frac{D \Delta t}{(\Delta x)^2} + (f(A_M^k) + g(t_M^k) + h(E_M^k)) \Delta t + \mu + \sigma \quad \text{and} \quad a_i^k = 1 + \frac{2D \Delta t}{(\Delta x)^2} + (f(A_i^k) + g(t_i^k) + h(E_i^k)) \Delta t + \mu + \sigma \quad \text{with} \quad i = 1, 2, 3, \cdots, M - 1. \]

Clearly, \( \mathcal{A}^k \) represents a strictly diagonally dominant matrix. Therefore, the above Eq. (4.4) can be expressed as \( S^{k+1} = (\mathcal{A}^k)^{-1}(S^k + \Gamma \Delta t) > 0 \).

According to E-equation of system Eq. (4.1) that

\[ \mathcal{B} E^{k+1} = E^k + \Delta t \quad Q^{k+1}, \quad (4.5) \]

where \( Q^{k+1} = [S_0^{k+1}((1 - c_2)f(A_0^k) + (1 - c_1)g(t_0^k) + (1 - c_1)h(E_0^k)), S_1^{k+1}((1 - c_2)f(A_1^k) + (1 - c_1)g(t_1^k) + (1 - c_1)h(E_1^k)), \cdots, S_M^{k+1}((1 - c_2)f(A_M^k) + (1 - c_1)g(t_M^k) + (1 - c_1)h(E_M^k)) ]^T \) and \( \mathcal{B} \) is the following \((M+1) \times (M+1)\) dimensional matrix

\[
\mathcal{B} = \begin{pmatrix}
b_1 & b_3 & 0 & \cdots & 0 & 0 & 0 \\
b_3 & b_2 & b_3 & \cdots & 0 & 0 & 0 \\
0 & b_3 & b_2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & b_2 & b_3 & 0 \\
0 & 0 & 0 & \cdots & b_3 & b_2 & b_3 \\
0 & 0 & 0 & \cdots & 0 & b_3 & b_1
\end{pmatrix},
\]

with \( b_1 = 1 + \frac{D \Delta t}{(\Delta x)^2} + \Delta t(\mu + \theta), b_2 = 1 + \frac{2D \Delta t}{(\Delta x)^2} + \Delta t(\mu + \theta) \) and \( b_3 = -\frac{D \Delta t}{(\Delta x)^2} \). Hence, \( \mathcal{B} \) stands for a \( M \)-matrix, then from Eq. (4.5) we have

\[ E^{k+1} = \mathcal{B}^{-1}(I + \Delta t Q^{k+1}). \]

In the same way, according to I-equation of system Eq. (4.1), one has

\[ \mathcal{C} I^{k+1} = I^k + \theta \Delta t \quad E^{k+1}, \quad (4.6) \]

10082
where

$$
C = \begin{pmatrix}
    c_1 & c_3 & 0 & \cdots & 0 & 0 & 0 \\
    c_3 & c_2 & c_3 & \cdots & 0 & 0 & 0 \\
    0 & c_3 & c_2 & \cdots & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & c_2 & c_3 & 0 \\
    0 & 0 & 0 & \cdots & c_3 & c_2 & c_3 \\
    0 & 0 & 0 & \cdots & 0 & c_3 & c_1 \\
\end{pmatrix},
$$

with $c_1 = 1 + \frac{D\Delta t}{(\Delta x)^2} + \Delta t[\mu + (1 + b)\gamma + \rho]$, $c_2 = 1 + \frac{2D\Delta t}{(\Delta x)^2} + \Delta t[\mu + (1 + b)\gamma + \rho]$ and $c_3 = -\frac{D\Delta t}{(\Delta x)^2}$. According to the fact that $C$ is a $M$-matrix, and from Eq. (4.6) we get

$$I^{k+1} = C^{-1}(I^k + \theta \Delta tE^{k+1}).$$

Similarly, from A-equation of system Eq. (4.1), one has

$$\mathcal{D} A^{k+1} = A^k + \alpha \Delta t \left(E^{k+1} + I^{k+1}\right), \quad (4.7)$$

where

$$
\mathcal{D} = \begin{pmatrix}
    d_1 & d_3 & 0 & \cdots & 0 & 0 & 0 \\
    d_3 & d_2 & d_3 & \cdots & 0 & 0 & 0 \\
    0 & d_3 & d_2 & \cdots & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & d_2 & d_3 & 0 \\
    0 & 0 & 0 & \cdots & d_3 & d_2 & c_3 \\
    0 & 0 & 0 & \cdots & 0 & d_3 & d_1 \\
\end{pmatrix},
$$

with $d_1 = 1 + \frac{D_4\Delta t}{(\Delta x)^2} + \Delta t\xi$, $d_2 = 1 + \frac{2D_4\Delta t}{(\Delta x)^2} + \Delta t\xi$ and $d_3 = -\frac{D_4\Delta t}{(\Delta x)^2}$. Applying the fact that $\mathcal{D}$ is a $M$-matrix, then from Eq. (4.7) we have

$$A^{k+1} = \mathcal{D}^{-1}[A^k + \alpha \Delta t(E^{k+1} + I^{k+1})].$$

The discrete system Eq. (4.1) has a clear biological significance, and all its parameters are positive. Then, the solution of the discrete system Eq. (4.1) remains positive for all $k \in \mathbb{N}$.

From now on, we show the solution’s boundedness. First, we set a sequence $\{Q^k\}$

$$Q^k = \sum_{n=0}^{M} (S_n^k + E_n^k + I_n^k).$$

From the first three equations of the discrete system Eq. (4.1), one has

$$Q^{k+1} - Q^k = \Delta t\mu N(M + 1) - \Delta t(\mu + \sigma)\sum_{n=0}^{M} S_n^{k+1} - \Delta t\mu \sum_{n=0}^{M} E_n^{k+1}$$

$$- \Delta t[\mu + (1 + b)\gamma + \rho] \sum_{n=0}^{M} I_n^{k+1}.$$
\[ \limsup_{k \to +\infty} Q^k \leq (M + 1)N. \]

From A-equation of the discrete system Eq. (4.1), one has

\[ \sum_{n=0}^{M} A_{n+1}^k = \frac{\alpha \Delta t}{1 + \xi \Delta t} \sum_{n=0}^{M} (E_{n+1}^k + I_{n+1}^k) + \frac{1}{1 + \xi \Delta t} \sum_{n=0}^{M} A_{n+1}^k \]

\[ \leq (M + 1)N \left( \frac{\alpha \Delta t}{1 + \xi \Delta t} + \frac{1}{1 + \xi \Delta t} \right) \sum_{n=0}^{M} A_{n+1}^k. \]

Similarly, one has

\[ \limsup_{k \to +\infty} \sum_{n=0}^{M} A_{n+1}^k \leq \frac{\alpha (M + 1)N}{\xi}. \]

Therefore, we get that the discrete system’s solution is bounded. \(\square\)

### 4.2. Sensitivity analysis

Sensitivity analysis can determine disease prediction’s robustness to parameter values (due to statistical uncertainties in data and errors in selected corresponding parameter values) [47]. Before taking large-scale targeted actions to deal with the spread of COVID-19, we are very urgent to understand how different factors affect the spread and epidemic of diseases. Thus, we analyze the global sensitivity of \(R_0\) of the COVID-19 reaction-diffusion model to assess the relative importance of system parameters to disease transmission and control [47]. The results obtained will provide theoretical guidance for relevant epidemic prevention departments to plan targeted intervention strategies.

The sensitivity index describes the state variable’s relative change when the parameter changes. These indices will explain the relative importance of the parameter to the epidemic and spread of the disease. That is, the following definition is available. Suppose the specific form of the general incidence rate is \(f(A) = \beta_1A, g(I) = \beta_2I, h(E) = \beta_3E\). Furthermore, all parameter values of the system Eq. (3.1) are shown in Table 1.

**Definition 4.1.** ([47]) The normalized forward sensitivity index of a variable \(u\), that depends differentiable on a parameter \(p\), is denoted as

\[ \gamma^u_p \triangleq \frac{\partial u}{\partial p} \times \frac{p}{u}. \]

We get the sensitivity index of \(R_0\) to the system’s all parameters in Table 2 and draw the following sensitivity analysis diagram of \(R_0\).

**Fig. 2** shows the global sensitivity analysis of \(R_0\). Clearly, the parameters \((\mu, N, \alpha, \beta_1, \beta_2, \beta_3)\) are positively correlated with \(R_0\), while the parameters
Table 1
Parameter values for system Eq. (3.1).

| Parameter | Value | Unit  | Reference | Parameter | Value | Unit  | Reference |
|-----------|-------|-------|-----------|-----------|-------|-------|-----------|
| \(\alpha\) | 0.02  | day\(^{-1}\) | [48]      | \(\mu\)   | 0.02  | day\(^{-1}\) | [49]      |
| \(\beta_1, \beta_2, \beta_3\) | 0.04, 0.05, 0.03 | day\(^{-1}\) | [11, 50, 51] | \(\theta\) | 0.196 | day\(^{-1}\) | [11, 52] |
| \(N\)     | 1     | -     | Assume    | \(c_1, c_2\) | 0.2-0.95 | -     | [53]      |
| \(b\)     | 0.1-1 | -     | Estimation| \(\gamma\) | 0.0476 | day\(^{-1}\) | [54, 55] |
| \(\sigma\) | 0.05  | day\(^{-1}\) | [56]      | \(\xi\)   | 0.02  | day\(^{-1}\) | [48]      |
| \(\rho\)  | 0.021 | day\(^{-1}\) | [5]       |           |       |       |           |

Fig. 3. The numerical result (a), (c) and (d) exhibit that the dependence of \(R_0\) of system Eq. (3.1) on the mask protection rate \(c\), medical treatment level \(b\) and the vaccination rate \(\sigma\). In addition, the numerical result (b) gives a regional distribution map of the extinction and persistence of COVID-19.

\((c_1, c_2, b, \xi, \theta, \sigma, \gamma, \rho)\) are negatively correlated with \(R_0\). Further, we know that \(R_0\) is very sensitive to the parameters \((c_1, \beta_1, \beta_2, \gamma, \sigma, c_2, \alpha, \xi, \theta, b, \beta_3)\), but not sensitive to the parameters \((\mu, N, \rho)\).

In order to further study the influence of mask protection rate and vaccination rate on disease development, we fix other parameters unchanged and only study the impact of changes in mask protection rate \(c\), medical treatment level \(b\) and vaccination rate \(\sigma\) on the threshold \(R_0\). We obtained the dependence of the basic regeneration number \(R_0\) of the system Eq. (3.1) on mask protection rate \(c\), medical treatment level \(b\) and vaccination rate \(\sigma\) through numerical
Fig. 4. When $R_0 = 4.2135$, the numerical simulation results of system Eq. (3.1). $P^*$ is globally asymptotically stable. The above subfigures show spatiotemporal distributions of (a) Susceptible, (b) Exposed, (c) Infected, (d) Free COVID-19 virus in the air, (e) Recovered, (f) The state of each position $x$ at $t = t_{end}$, respectively.

Table 2
Sensitivity index of $R_0$.

| Parameter | Description                                      | Sensitivity index          |
|-----------|--------------------------------------------------|---------------------------|
| $\alpha$  | The pathogen shedding rate from $E, I$ into $A$  | 0.3637                    |
| $\beta_1, \beta_2, \beta_3$ | Effective infection rate from $E, I$ and $A$ to $S$ | 0.4879, 0.4602, 0.3245        |
| $N$       | Total population of the region                   | 0.05673                   |
| $b$       | mathematical treatment                            | -0.3588                   |
| $c_1, c_2$| Effective protection rate of mask                 | -0.6837, -0.4144          |
| $\mu$     | The birth or natural death rate                   | 0.05554                   |
| $\theta$  | Conversion rate from exposed to infected individuals | -0.3596                  |
| $\sigma$  | Vaccination rate of COVID-19                     | -0.4458                   |
| $\xi$     | The decay rate of pathogen in the air            | -0.3805                   |
| $\gamma$  | Recovery rate of COVID-19 without treatment      | -0.4522                   |
| $\rho$    | The virus-induced average fatality rate          | -0.06904                  |
Fig. 5. When $R_0 = 0.7023$, the numerical simulation results of system Eq. (3.1), $R^0$ is globally asymptotically stable. The above subfigures show spatiotemporal distributions of (a) Susceptible, (b) Exposed, (c) Infected, (d) Free COVID-19 virus in the air, (e) Recovered, (f) The state of each position $x$ at $t = t_{end}$, respectively.

Simulation in Fig. 3(a), (c) and (d). The numerical result (b) gives a regional distribution map of the extinction and persistence of COVID-19 dependent on mask protection rate $c$ and vaccination rate $\sigma$. That is, the COVID-19 extinction area map is given by $R_0 < 1$, and the persistent area map is given by $R_0 > 1$. In addition, $R_0 = 1$ is the critical condition for disease extinction and persistence. Therefore, a suitable combination of different mask protection rates $c$, medical treatment level $b$ and vaccination rates $\sigma$ can effectively control the COVID-19 epidemic. The determination of these keys sensitive parameters provides a theoretical basis for relevant departments to formulate targeted COVID-19 intervention strategies.

Based on the numerical results obtained above, we give the following suggestions: (1) The media should increase efforts to urge the public to wear qualified protective masks and develop a good habit of wearing masks outdoors; (2) Actively vaccinate against COVID-19, improve the coverage of vaccination and establish an immune barrier among the population to reduce the risk of a large-scale outbreak; (3) Increase medical investment, improve medical standards,
and continuously improve existing treatment programs to promote COVID-19 recovery of infected persons; (4) Speed up the attenuation of the COVID-19 virus in the air by spraying disinfectant and inhibit the production of the virus; (5) Increase the intensity of nucleic acid testing and find exposed persons as soon as possible to reduce the risk of transmission. The above five suggestions positively affect reducing the basic reproduction number $\mathcal{R}_0$ and effectively controlling the epidemic.

### 4.3. An example of numerical simulation

In this subsection, we verify the previous theoretical results with a numerical example. Assume the corresponding discrete scheme $f(A^k_n) = \beta_1 A^k_n, g(I^k_n) = \beta_2 I^k_n, h(E^k_n) = \beta_3 E^k_n$ and consider the recovery class $R(x, t)$. Thus, system Eqs. (4.1)–(4.3) can be written as

$$
\frac{S_{n+1}^k - S_n^k}{\Delta t} = D \frac{S_{n+1}^k - 2S_n^k + S_{n-1}^k}{(\Delta x)^2} + \Lambda - (1 - c_2) \beta_1 S_n^k A_n^k - (1 - c_1) \beta_2 S_n^k I_n^k - (1 - c_1) \beta_3 S_n^k E_n^k - (\mu + \sigma) S_n^k,
$$

$$
\frac{E_{n+1}^k - E_n^k}{\Delta t} = D \frac{E_{n+1}^k - 2E_n^k + E_{n-1}^k}{(\Delta x)^2} + (1 - c_2) \beta_1 S_n^k A_n^k + (1 - c_1) \beta_2 S_n^k I_n^k + (1 - c_1) \beta_3 S_n^k E_n^k - (\mu + \theta) E_n^k,
$$

$$
\frac{I_{n+1}^k - I_n^k}{\Delta t} = D \frac{I_{n+1}^k - 2I_n^k + I_{n-1}^k}{(\Delta x)^2} + \theta E_n^k - [\mu + \rho + (1 + b) \gamma] I_n^k,
$$

$$
\frac{A_{n+1}^k - A_n^k}{\Delta t} = D_4 \frac{A_{n+1}^k - 2A_n^k + A_{n-1}^k}{(\Delta x)^2} + \alpha (E_{n+1}^k + I_{n+1}^k) - \xi A_{n+1}^k,
$$

$$
\frac{R_{n+1}^k - R_n^k}{\Delta t} = D \frac{R_{n+1}^k - 2R_n^k + R_{n-1}^k}{(\Delta x)^2} + (1 + b) \gamma I_{n+1}^k - \mu R_{n+1}^k + \sigma S_{n+1}^k,
$$

where $k \in \mathbb{N}, n \in \{0, 1, 2, \cdots, M\}$ and $\Delta x = 0.5, \Delta t = 0.01$. The initial data is selected as $S(x, 0) = 0.8, \ E(x, 0) = I(x, 0) = A(x, 0) = R(x, 0) = e^{-x}, \ \text{for} \ x \in [0, 3], \ (4.10)$ and satisfies

$$
S_{-1}^k = S_0^k, \ S_M^k = S_{M+1}^k, \ E_{-1}^k = E_0^k, \ E_M^k = E_{M+1}^k, \ I_{-1}^k = I_0^k, \ I_M^k = I_{M+1}^k,
$$

$$
A_{-1}^k = A_0^k, \ A_M^k = A_{M+1}^k, \ R_{-1}^k = R_0^k, \ R_M^k = R_{M+1}^k.
$$

Similarly, we calculate $\mathcal{R}_0$ of the discrete system Eq. (4.9) as

$$
\mathcal{R}_0 = \frac{(1 - c_1) \Lambda \beta_3}{(\sigma + \mu) (\theta + \mu)} + \frac{(1 - c_1) \Lambda \theta \beta_2}{(\sigma + \mu) (\theta + \mu) ((1 + b) \gamma + \mu + \rho)} + \frac{(1 - c_2) \Lambda \alpha (\theta + \mu + \rho + (1 + b) \gamma) \beta_1}{(\sigma + \mu) (\theta + \mu) ((1 + b) \gamma + \mu + \rho) \xi}.
$$

First, we choose $\alpha = 0.02, \ \beta_1 = \beta_2 = \beta_3 = 0.3, \ \gamma = 0.0476, \ \rho = 0.021, \ b = 0.5, \ \mu = 0.02, \ \sigma = 0.05, \ \theta = 0.196, \ c_1 = c_2 = 0.7, \ \xi = 0.02$ and $D = D_4 = 0.1$. Then, $\mathcal{R}_0 = 4.2135 > 1$ and it follows from Theorem 3.7 and Fig. 4 that $P^0$ is globally asymptotically stable. From Fig. 6(a), we can more clearly know that COVID-19 is persistent.
The numerical result (a) shows that infectious disease COVID-19 is persistent when $R_0 = 4.2135$, while the numerical result (b) shows that infectious disease COVID-19 is extinct when $R_0 = 0.7023$ (for $x = 1$).

Next, we set $\beta_1 = \beta_2 = \beta_3 = 0.05 \text{ day}^{-1}$ and keep other parameters unchanged. We get $R_0 = 0.7023 < 1$, and it follows from Theorem 3.6 and Fig. 5 that $P^0$ is globally asymptotically stable. Through Fig. 6(b), we know that COVID-19 has been effectively controlled. Exposed individuals, infected individuals, and the free COVID-19 virus in the air will all become extinct, and ultimately only susceptible individuals and recovered individuals will remain.

To illustrate the advantages of our NSFD scheme, we compare the algorithm with the following SFD algorithm:

$$
\begin{align*}
\frac{S_{n+1} - S_n}{\Delta t} &= D \frac{S_{n+1} - 2S_n + S_{n-1}}{(\Delta x)^2} + \frac{1}{\Delta t} \left( \frac{1}{\Delta t} \right) (1 - c_2) \beta_1 S_n I_n - (1 - c_1) \beta_2 S_n R_n,
\frac{E_{n+1} - E_n}{\Delta t} &= D \frac{E_{n+1} - 2E_n + E_{n-1}}{(\Delta x)^2} + (1 - c_2) \beta_1 S_n I_n + (1 - c_1) \beta_2 S_n R_n,
\frac{I_{n+1} - I_n}{\Delta t} &= D \frac{I_{n+1} - 2I_n + I_{n-1}}{(\Delta x)^2} - [\mu + \sigma + (1 + b)\gamma] I_n,
\frac{A_{n+1} - A_n}{\Delta t} &= D \frac{A_{n+1} - 2A_n + A_{n-1}}{(\Delta x)^2} + \alpha (E_n + I_n) - \xi A_n,
\frac{R_{n+1} - R_n}{\Delta t} &= D \frac{R_{n+1} - 2R_n + R_{n-1}}{(\Delta x)^2} + (1 + b)\gamma I_n - \mu R_n + \sigma S_n.
\end{align*}
$$

The NSFD scheme shown in Eq. (4.9) differs from the SFD scheme shown in Eq. (4.11) in maintaining the positivity, boundedness, global properties, and CPU-time of the continuous system Eq. (2.1). Except for the inconsistent NSFD and SFD schemes, other conditions such as time $t$, space $x$, time step $\Delta t$, space step $\Delta x$, initial value, and parameter values are completely consistent. Table 3 shows the results as follows:

5. Evaluate the impact of vaccines, masks, and medical treatment on COVID-19

About two years have passed since the outbreak of COVID-19 at the end of 2019, but due to the complexity of the epidemic, and COVID-19 has a great impact on society. Hence, it is
crucial to prevent the COVID-19 epidemic. However, the effective and common prevention is to vaccinate susceptible populations to curb the spread and prevalence of diseases. Although a wide variety and enough supply of vaccines make mass vaccination possible, having a vaccine and whether the public is willing to vaccinate are entirely different things. Their concerns mainly include the following two aspects: (1) Is the COVID-19 vaccine safe? (2) Can vaccination effectively control the spread of diseases? Concerns about vaccine safety are unnecessary because all vaccine developments and production have strict and scientific specifications. Next, we mainly address the second concern of the public. That is, whether vaccination can effectively control the epidemic.

We combine the reaction-diffusion COVID-19 system Eq. (3.1) to simulate the change in the density of the final scale of infection and infected individuals under the background of $\sigma = 0, 10\%, 30\%, 50\%, 70\%$ when the population of the local area is $N = 5000$. The numerical results in Fig. 7(a) and (b) show that timely vaccination and expansion of vaccination coverage can not only control the epidemic’s peak and final scale, but the epidemic can also be controlled more quickly. Ultimately achieve the goal of reducing the harm of the epidemic and the cost of epidemic prevention.

During the COVID-19 pandemic, the public still disputes whether to use masks to prevent the epidemic. However, with the further development of the epidemic, more and more people have found that wearing masks have a positive effect on preventing the susceptible population from contracting COVID-19 and controlling the epidemic.

Common masks on the market are mainly divided into the following three categories: cloth masks (protection efficiency 20% – 50%), medical masks (protection efficiency 70% – 90%), and N95 masks (protection efficiency greater than or equal to 95%). Next, we conduct related simulations on the system Eq. (3.1) to explore the impact of different types of masks used
Fig. 8. Numerical results (a) and (b) show that the better the mask protection efficiency, the smaller the peak number of infected patients, the earlier the peak appears, the faster the number of infected patients will be cleared, and the smaller the size of infected patients when medical level $b=0.5$ and the COVID-19 vaccination rate $\sigma = 0$ (for $x = 1$).

by the public on the spread of COVID-19. Under the premise of the medical level coefficient $b = 0.5$ and the vaccination rates $\sigma = 0$. We simulate the change in the density of infected patients and the final infection size when the protection efficiency of masks is $c_1 = c_2 = 0, 20\%, 50\%, 80\%, 95\%$, respectively. Our research (see Fig. 8(a) and (b)) shows that although cloth masks are not as good as medical masks and N95 masks, they are better than no mask. In addition, we also found that the higher the protection efficiency of masks, the smaller the peak of the COVID-19 and the final size of infection, and the earlier the peak. Therefore, wearing masks by the public, especially with high protection efficiency, plays a vital role in effectively controlling the epidemic.

Facing the sudden epidemic, although all countries attach great importance to it, the level of medical care directly affects the development of the local epidemic. It cannot be ignored that the number of hospitals, the number of medical personnel, and the supply of medical resources all determine the local medical treatment level. To explore how the level of medical care in a region affects the development of the epidemic, under $\sigma = 0, c_1 = c_2 = 70\%$, and other parameters remain unchanged, we combined the reaction-diffusion model Eq. (3.1) to simulate the change in the density of infected people and the final scale when the medical treatment level coefficients are $b = 0, 0.2, 0.4, 0.6, 0.8$, respectively. The numerical results in Fig. 9(a) and (b) show that the lower the local medical level, the higher the peak of the epidemic and the final scale of infection. In addition, the epidemic lasts longer, which is not conducive to the reasonable and orderly allocation of medical resources and the effective control of the epidemic. The results show that the SEIR-A reaction-diffusion model can accurately predict the density of new cases, the cumulative density of cases, the peak of the epidemic, the time of clearing cases, and the final scale of infection in Yangzhou, China.

6. Data fitting for COVID-19 in Yangzhou, China and Putian, China by using the SEIR-A reaction-diffusion model

This section aims to demonstrate that our SEIR-A reaction-diffusion model can accurately predict the epidemic law of COVID-19. To this end, we use the system Eq. (2.1) to fit the COVID-19 epidemic that occurred in Yangzhou, China and Putian, China in 2021, respectively.
Fig. 9. Numerical results (a) and (b) show that the higher the medical level, the smaller the peak number of infected patients, the earlier the peak appears, the faster the number of infected patients will be cleared, and the smaller the size of infected patients when the COVID-19 vaccination rate \( \sigma = 0 \) and mask protection efficiency \( c_1 = c_2 = 70\% \) (for \( x = 1 \)).

Table 4
COVID-19 case data in Yangzhou, China.

| Date | New cases | Cumulative cases | Date | New cases | Cumulative cases | Date | New cases | Cumulative cases |
|------|-----------|------------------|------|-----------|------------------|------|-----------|------------------|
| 7/28 | 2         | 2                | 8/13 | 18        | 528              | 8/29 | 0         | 570              |
| 7/29 | 4         | 6                | 8/14 | 18        | 546              | 8/30 | 0         | 570              |
| 7/30 | 10        | 16               | 8/15 | 6         | 552              | 8/31 | 0         | 570              |
| 7/31 | 12        | 28               | 8/16 | 3         | 555              | 9/1  | 0         | 570              |
| 8/1  | 26        | 54               | 8/17 | 6         | 561              | 9/2  | 0         | 570              |
| 8/2  | 40        | 94               | 8/18 | 3         | 564              | 9/3  | 0         | 570              |
| 8/3  | 32        | 126              | 8/19 | 2         | 566              | 9/4  | 0         | 570              |
| 8/4  | 36        | 162              | 8/20 | 1         | 567              | 9/5  | 0         | 570              |
| 8/5  | 58        | 220              | 8/21 | 1         | 568              | 9/6  | 0         | 570              |
| 8/6  | 52        | 272              | 8/22 | 0         | 568              | 9/7  | 0         | 570              |
| 8/7  | 36        | 308              | 8/23 | 0         | 568              | 9/8  | 0         | 570              |
| 8/8  | 38        | 346              | 8/24 | 1         | 569              | 9/9  | 0         | 570              |
| 8/9  | 48        | 394              | 8/25 | 0         | 569              | 9/10 | 0         | 570              |
| 8/10 | 54        | 448              | 8/26 | 1         | 570              | 9/11 | 0         | 570              |
| 8/11 | 37        | 485              | 8/27 | 0         | 570              | 9/12 | 0         | 570              |
| 8/12 | 25        | 510              | 8/28 | 0         | 570              | 9/13 | 0         | 570              |

Table 4 shows the daily new cases and cumulative cases in Yangzhou, China, from July 28, to September 13. We set the parameters \( \beta_1 = 1.3 \times 10^{-7} \), \( \beta_2 = \beta_3 = 2.4 \times 10^{-7} \), \( \sigma = 0.25 \), \( b = 0.3 \), \( \theta = 0.196 \), \( c_1 = c_2 = 0.7 \), \( \alpha = 0.02 \), \( \xi = 0.02 \), \( \gamma = 0.06 \) and diffusion coefficient \( D = D_4 = 2.26 \). Since there were no deaths in the COVID-19 outbreak in Yangzhou, China, we have \( \mu = 0 \). In addition, based on the total population of Yangzhou, China, which is about 4,460,000, and the actual initial infection data, we select the following initial data:

\[
S(x, 0) = 4459996, \quad E(x, 0) = 3, \quad I(x, 0) = 1, \quad W(x, 0) = 2 + 4 \times \exp(-x) \quad \text{and} \quad R(x, 0) = 0.
\]

Similarly, Table 5 exhibits the daily new cases and cumulative cases in Putian, China, from September 10, to September 25. Other parameters remain unchanged, we only adjust the effective infection rate \( \beta_1 = \beta_2 = 1.18 \times 10^{-6} \), \( \beta_3 = 9.1 \times 10^{-7} \), and select the diffusion coefficient \( D = 4.2, D_4 = 2.26 \). Considering that the total population of Putian in China is
Table 5
COVID-19 case data in Putian, China.

| Date  | New cases | Cumulative cases | Date  | New cases | Cumulative cases | Date  | New cases | Cumulative cases |
|-------|-----------|------------------|-------|-----------|------------------|-------|-----------|------------------|
| 9/10  | 1         | 1                | 9/16  | 28        | 158              | 9/22  | 3         | 200              |
| 9/11  | 19        | 20               | 9/17  | 21        | 179              | 9/23  | 4         | 204              |
| 9/12  | 15        | 35               | 9/18  | 4         | 183              | 9/24  | 1         | 205              |
| 9/13  | 24        | 59               | 9/19  | 7         | 190              | 9/25  | 0         | 205              |
| 9/14  | 33        | 92               | 9/20  | 5         | 195              |       |           |                  |
| 9/15  | 38        | 130              | 9/21  | 2         | 197              |       |           |                  |

Fig. 10. Numerical results (a) and (b) show the daily new cases and cumulative cases prediction in Yangzhou, respectively.

Fig. 11. Numerical results (a) and (b) show the daily new cases and cumulative cases prediction in Putian, respectively.

about 3,070,000, initial values in Putian city are selected as:

\[ S(x, 0) = 3069996, \quad E(x, 0) = 3, \quad I(x, 0) = 1, \quad W(x, 0) = 2 + 4 \times \exp(-x) \quad \text{and} \quad R(x, 0) = 0. \]

Based on the above initial conditions and parameter values, the data prediction results in Figs. 10 and 11 are obtained through numerical simulation. The model-fitting results in Fig. 10(a) and (b) are as follows: (1) The basic reproduction number of Yangzhou in China is \( R_0 = 2.5107 \); (2) This epidemic round in Yangzhou will peak at 56 new daily confirmed cases on the 9th day (August 5); (3) The final scale of infections in Yangzhou will reach
570 cases; (4) The Yangzhou epidemic is expected to be completely cleared on the 25th day (August 21). Similarly, the model-fitting results in Fig. 11(a) and (b) exhibit as follows: (1) The basic reproduction numbers of Putian in China is $R_0 = 1.8846$; (2) This epidemic round in Yangzhou will peak at 37 new daily confirmed cases on the 6th day (September 15); (3) The final scale of infections in Putian will reach 205 cases; (4) The Putian epidemic is expected to be completely cleared on the 15th day (September 24). These two actual battles have well verified the practicability and effectiveness of the SEIR-A reaction-diffusion model in COVID-19 prediction.

Suppose we can accurately grasp new cases per day, the cumulative cases, and the number of deaths due to illness, at the epidemic’s beginning. In that case, by setting reasonable initial conditions, infection rate and selecting appropriate diffusion coefficients $D$ and $D_4$, we can predict the COVID-19 epidemic more accurately and provide constructive opinions for epidemic prevention and control work.

7. Conclusion and outlook

This paper proposes a novel COVID-19 reaction-diffusion system with direct and aerosol transmission. $R_0$ is obtained, and $R_0$ is the threshold condition for predicting the disease’s persistence and extinction. The theoretical results show that when $R_0 < 1$, the reaction-diffusion system’s disease-free equilibrium $P^0$ is globally asymptotically stable. However, when $R_0 > 1$, then the reaction-diffusion system’s endemic equilibrium $P^*$ is globally asymptotically stable. Furthermore, we use the NSFD scheme to discretize the continuous system, and it can maintain the boundedness, positivity and global properties of the original continuous system. Finally, theoretical simulations and instance predictions are used to give several key indicators of the epidemic, including $R_0$, peak, time to peak, time to clear cases, and final size. The instance prediction results are as follows: (1) The basic reproduction numbers of Yangzhou and Putian in China are $R_0 = 2.5107$ and $R_0 = 1.8846$, respectively. (2) This epidemic round in Yangzhou will peak at 56 new daily confirmed cases on the 9th day (August 5), and Putian will peak at 37 new daily confirmed cases on the 6th day (September 15). (3) The final scale of infections in Yangzhou and Putian reached 570 and 205 cases, respectively. (4) The Yangzhou epidemic is expected to be completely cleared on the 25th day (August 21). In addition, the Putian epidemic will continue for 15 days and be cleared on September 24.

In addition, we conducted a sensitivity analysis of parameters and evaluated the impact of vaccines, masks, and medical treatment. Furthermore, the SEIR-A reaction-diffusion model was used to predict the COVID-19 in Yangzhou and Putian China, respectively. Based on the above results, we have reached the conclusions on the control of COVID-19 as follows: (1) Ignoring aerosol transmission will underestimate the risk of COVID-19 transmission. (2) The government needs to vaccinate key areas and critical populations as soon as possible to increase the vaccination rate further. (3) The prevention and control of the epidemic must adhere to prevention first, combine prevention and control, increase medical investment, improve medical standards, and continuously improve existing treatment programs to promote the recovery of infected persons. (4) The news media should give full play to the role of publicity, comprehensively carry out publicity and education of epidemic prevention and control knowledge, and guide the public to develop the habit of consciously wearing masks.

In the future, we still have the following problems that need exploration: (1) In a heterogeneous environment, the endemic equilibrium’s global stability when $R_0 > 1$ requires further theoretical demonstration. (2) At present, mathematical models can accurately predict single-
peak infectious diseases, but multi-peak infectious disease prediction is still a challenging problem. Here we propose a conjecture: using the time-varying infection rate or segmented infection rate to predict the multi-peak disease problem, and verifying the validity of the conjecture is our future work.

Data availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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