Further Inequalities for the Weighted Numerical Radius of Operators

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Abstract: This paper deals with the so-called $A$-numerical radius associated with a positive (semi-definite) bounded linear operator $A$ acting on a complex Hilbert space $H$. Several new inequalities involving this concept are established. In particular, we prove several estimates for $2 \times 2$ operator matrices whose entries are $A$-bounded operators. Some of the obtained results cover and extend well-known recent results due to Bani-Domi and Kittaneh. In addition, several improvements of the generalized Kittaneh estimates are obtained. The inequalities given by Feki in his work represent a generalization of the inequalities given by Kittaneh. Some refinements of the inequalities due to Feki are also presented.

Keywords: positive operator; $A$-numerical radius; $2 \times 2$-operator matrix; $A$-adjoint operator

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1. Introduction

Along this work $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ denotes a complex Hilbert space with associated norm $\| \cdot \|$. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators acting on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$. For every operator $T \in \mathcal{B}(\mathcal{H})$, its range is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$, and its adjoint by $T^*$. By $\overline{\mathcal{R}(T)}$, we mean the closure of $\mathcal{R}(T)$ with respect to the norm topology of $\mathcal{H}$. Throughout this paper, we retain the notation $A$ for a nonzero positive operator on $\mathcal{H}$ which clearly induces the following positive semidefinite sesquilinear form

$$\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \langle x, y \rangle_A := \langle Ax, y \rangle.$$  

The seminorm on $\mathcal{H}$ induced by $\langle \cdot, \cdot \rangle_A$ is given by $\| x \|_A = \sqrt{\langle x, x \rangle_A}$, for every $x \in \mathcal{H}$. We remark that $\| \cdot \|_A$ is a norm on $\mathcal{H}$ if and only if $A$ is an injective operator, i.e., $\mathcal{N}(A) = \{0 \}$. In addition, the semi-Hilbert space $(\mathcal{H}, \| \cdot \|_A)$ is complete if, and only if, $\overline{\mathcal{R}(A)}$ is closed in $\mathcal{H}$. Next, when we use an operator, it means that it is an operator in $\mathcal{B}(\mathcal{H})$. For recent contributions related to operators acting on the $A$-weighted space $(\mathcal{H}, \| \cdot \|_A)$, the readers may consult [1–3]. Before we proceed further, we recall that $\langle \cdot, \cdot \rangle_A$ induces on the quotient $\mathcal{H}/\mathcal{N}(A)$ an inner product which is not complete unless $\mathcal{R}(A)$ is closed in $\mathcal{H}$. On the other hand, it was proved in [4] (see also [5]) that the completion...
of $\mathcal{H}/\mathcal{N}(A)$ is isometrically isomorphic to the Hilbert space $\mathcal{R}(A^{1/2})$ endowed with the following inner product
\[
\langle A^{1/2}x, A^{1/2}y \rangle_{\mathcal{R}(A^{1/2})} := \langle P_{\mathcal{R}(A)}x, P_{\mathcal{R}(A)}y \rangle, \quad \forall x, y \in \mathcal{H},
\]
where $P_{\mathcal{R}(A)}$ stands for the orthogonal projection onto $\mathcal{R}(A)$. Notice that the Hilbert space $(\mathcal{R}(A^{1/2}), \langle \cdot, \cdot \rangle_{\mathcal{R}(A^{1/2})})$ will be simply denoted by $\mathcal{R}(A^{1/2})$. Let us emphasize that $\mathcal{R}(A)$ is dense in $\mathcal{R}(A^{1/2})$ (see [6]). More results related involving the Hilbert space $\mathcal{R}(A^{1/2})$ can be found in [6] and the references therein. An application of (1) gives
\[
\langle Ax, Ay \rangle_{\mathcal{R}(A^{1/2})} = \langle x, y \rangle_A, \quad \forall x, y \in \mathcal{H}.
\]

The numerical range and the numerical radius of $T \in \mathcal{B}(\mathcal{H})$ are defined by $W(T) = \{ \langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1 \}$, and $\omega(T) = \sup \{ |\xi| : \xi \in W(T) \}$, respectively. It is well known that the numerical radius of Hilbert space operators plays an important role in various fields of operator theory and matrix analysis (cf. [7–10]). Recently, several generalizations for the concept of $\omega(\cdot)$ have been introduced (cf. [11–13]). One of these generalizations is the so-called $A$-numerical radius of an operator $T \in \mathcal{B}(\mathcal{H})$, which was firstly defined by Saddi in [14] as
\[
\omega_A(T) = \sup \left\{ |\langle Tx, x \rangle_A| : x \in \mathcal{H}, \|x\|_A = 1 \right\}.
\]

For an account of the recent results related to the $A$-numerical radius, we refer the reader to [15–19] and the references therein. If $T = T_{ij}$ is a $2 \times 2$-operator matrix with $T_{ij} \in \mathcal{B}(\mathcal{H})$ for all $i, j \in \{1, 2\}$, then (3) can be written as:
\[
\omega_A(T) = \sup \left\{ |\langle \mathbb{T}x, x \rangle_A| : x \in \mathcal{H} \oplus \mathcal{H}, \|x\|_A = 1 \right\}.
\]

In addition, Zamani defined in [20] the notion of $A$-Crawford number of an operator $T$ as follows:
\[
c_A(T) = \inf \{ |\langle Tx, x \rangle_A| : x \in \mathcal{H}, \|x\|_A = 1 \}.
\]

Zamani used this notion in [20] in order to derive some improvements of inequalities related to $\omega_A(\cdot)$.

Before continuing, let us recall from [21] the concept of $A$-adjoint operator. For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$ is said to be an $A$-adjoint operator of $T$ if $\langle Tx, y \rangle_A = \langle x, Sy \rangle_A$ for all $x, y \in \mathcal{H}$; that is, $S$ is the solution in $\mathcal{B}(\mathcal{H})$ of the equation $AX = T^* A$. This kind of operator equations may be investigated by using the well-known Douglas theorem [22]. Briefly, this theorem says that equation $TX = S$ has a solution $X \in \mathcal{B}(\mathcal{H})$ if and only if $\mathcal{R}(S) \subseteq \mathcal{R}(T)$. This, in turn, equivalent to the existence of some positive constant $\lambda$ such that $\|S^*x\| \leq \lambda \|T^*x\|$ for all $x \in \mathcal{H}$. Furthermore, among its many solutions, there is only one, denoted by $Q$, which satisfies $\mathcal{R}(Q) \subseteq \mathcal{R}(T^*)$. Such $Q$ is called the reduced solution of $TX = S$. Let $\mathcal{B}_A(\mathcal{H})$ and $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ denote the sets of all operators that admit $A$-adjoints and $A^{1/2}$-adjoints, respectively. An application of Douglas theorem shows that
\[
\mathcal{B}_A(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) ; \mathcal{R}(T^* A) \subseteq \mathcal{R}(A) \},
\]
and
\[
\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) ; \exists c > 0 ; \|Tx\|_A \leq c\|x\|_A, \forall x \in \mathcal{H} \}.
\]

We remark that $\mathcal{B}_A(\mathcal{H})$ and $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathcal{B}(\mathcal{H})$ which are neither closed nor dense in $\mathcal{B}(\mathcal{H})$. It is easy to see that the following property is satis-
The main objective of the present paper is to present a few new \( A \)-numerical radius inequalities for \( 2 \times 2 \) operator matrices. In Theorem 1, we obtain a bound for the \( A \)-numerical radius for the \( 2 \times 2 \) operator matrix. By particularization, we deduce an improvement of the second inequality (9). Another bound for \( A \)-numerical radius for the \( 2 \times 2 \) operator matrix is given in Theorem 2. Next, we present an improvement of the Cauchy–Schwarz inequality type using the inner product \( \langle \cdot , \cdot \rangle_A \). This result is used to find a new bound for \( A \)-numerical radius of operator matrix \( S^A T \). Applying the Bohr inequality, we deduce another new bound for the \( A \)-numerical radius for the \( 2 \times 2 \) operator matrix. In addition to these, we aim to establish an alternative and easy proof of the generalized Kittaneh inequalities (9). In addition, several improvements of the first inequality in (9) are established.
2. Main Results

To establish our first main result in the present work, we require the following two lemmas.

**Lemma 1** ([28]). Let \( P, Q, R, S \in \mathbb{B}_A(\mathcal{H}) \). Then, the following assertions hold

(i) \( \omega_A\left[ \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} \right] = \max \{ \omega_A(P), \omega_A(S) \} \).

(ii) \( \left\| \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} \right\|_A = \max \{ \|P\|_A, \|S\|_A \} \).

(iii) \( \begin{pmatrix} P & Q \\ R & S \end{pmatrix}^{\sharp_A} = \begin{pmatrix} P^{\sharp_A} R^{\sharp_A} \\ Q^{\sharp_A} S^{\sharp_A} \end{pmatrix} \).

**Lemma 2** ([14]). Let \( x, y, e \in \mathcal{H} \) be such that \( \|e\|_A = 1 \). Then

\[
2|\langle x, (e, y) \rangle_A| \leq \|x\|_A \|y\|_A. \tag{10}
\]

Now, we can prove the following result, which generalizes Theorem 2.1 in [29].

**Theorem 1.** Let \( P, Q, R, S \in \mathbb{B}_A(\mathcal{H}) \). Then,

\[
\omega^2_A\left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] \leq \max \{ \omega^2_A(P), \omega^2_A(S) \} + \omega^2_A\left[ \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \right] + \omega^2_A\left[ \begin{pmatrix} 0 & QS \\ RP & 0 \end{pmatrix} \right] + \frac{1}{2} \max \{ \|P^{\sharp_A} + Q Q^{\sharp_A}\|_A, \|S^{\sharp_A} S + R R^{\sharp_A}\|_A \}.
\]

**Proof.** Let \( x \in \mathcal{H} \oplus \mathcal{H} \) be such that \( \|x\|_A = 1 \). One has

\[
\left| \langle \begin{pmatrix} P & Q \\ R & S \end{pmatrix} x, x \rangle_A \right|^2 = \left| \langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} x, x \rangle_A + \langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} x, x \rangle_A \right|^2 \leq \left| \langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} x, x \rangle_A \right|^2 + \left| \langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} x, x \rangle_A \right|^2 + 2 \left| \langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} x, x \rangle_A \right| \left| \langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} x, x \rangle_A \right| \leq \omega^2_A\left[ \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} \right] + \omega^2_A\left[ \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \right] + \omega^2_A\left[ \begin{pmatrix} 0 & QS \\ RP & 0 \end{pmatrix} \right] + \frac{1}{2} \max \{ \omega^2_A(P), \omega^2_A(S) \} + \omega^2_A\left[ \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \right] + \frac{1}{2} \max \{ \omega^2_A(P), \omega^2_A(S) \} + \omega^2_A\left[ \begin{pmatrix} 0 & QS \\ RP & 0 \end{pmatrix} \right] + \frac{1}{2} \max \{ \omega^2_A(P), \omega^2_A(S) \} + \omega^2_A\left[ \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \right] + \frac{1}{2} \max \{ \omega^2_A(P), \omega^2_A(S) \} + \omega^2_A\left[ \begin{pmatrix} 0 & QS \\ RP & 0 \end{pmatrix} \right],
\]

\[
= \max \{ \omega^2_A(P), \omega^2_A(S) \} + \omega^2_A\left[ \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \right] + \omega^2_A\left[ \begin{pmatrix} 0 & QS \\ RP & 0 \end{pmatrix} \right].
\]
where the last equality follows by using Lemma 1 (i) and (iii). So, by applying (10) together with the Cauchy–Schwarz inequality, we obtain

\[
\left\langle \left( \frac{P}{S} \right) x, x \right\rangle_H^2
\]
\[
\leq \max \left\{ \omega_A^2(P), \omega_A^2(S) \right\} + \left\langle \left( \frac{P^2A}{S^2A} \right) x, \left( \frac{0}{Q^2A} \right) x \right\rangle_H^2
\]
\[
+ \omega_A^2 \left[ \left( \frac{0}{R} \frac{Q}{0} \right) \right] + \left\langle \left( \frac{P^2A}{S^2A} \right) x, \left( \frac{0}{Q^2A} \right) x \right\rangle_H^2
\]
\[
\leq \max \left\{ \omega_A^2(P), \omega_A^2(S) \right\} + \left\langle \left( \frac{0}{R} \frac{Q}{0} \right) \right\rangle_H^2
\]
\[
+ \omega_A^2 \left[ \left( \frac{0}{R} \frac{Q}{0} \right) \right] + \left\langle \left( \frac{P^2A}{S^2A} \right) x, \left( \frac{0}{Q^2A} \right) x \right\rangle_H^2
\]
\[
= \max \left\{ \omega_A^2(P), \omega_A^2(S) \right\} + \omega_A^2 \left[ \left( \frac{0}{R} \frac{Q}{0} \right) \right] + \omega_A^2 \left[ \left( \frac{0}{Q^2A} \right) \right]
\]
\[
+ \sqrt{\left\langle \left( \frac{P^2A}{S^2A} \right) x, x \right\rangle_H^2 \left( \frac{0}{Q^2A} \right) x, x \right\rangle_H^2}
\]
\[
= \max \left\{ \omega_A^2(P), \omega_A^2(S) \right\} + \omega_A^2 \left[ \left( \frac{0}{R} \frac{Q}{0} \right) \right] + \omega_A^2 \left[ \left( \frac{0}{Q^2A} \right) \right]
\]
\[
+ \sqrt{\left\langle \left( \frac{P^2A}{S^2A} \right) x, x \right\rangle_H^2 \left( \frac{0}{Q^2A} \right) x, x \right\rangle_H^2}
\]

By applying the arithmetic–geometric mean inequality, we obtain

\[
\left\langle \left( \frac{P}{S} \right) x, x \right\rangle_H^2
\]
\[
\leq \max \left\{ \omega_A^2(P), \omega_A^2(S) \right\} + \omega_A^2 \left[ \left( \frac{0}{R} \frac{Q}{0} \right) \right] + \omega_A^2 \left[ \left( \frac{0}{Q^2A} \right) \right]
\]
\[
+ \frac{1}{2} \left\langle \left( \frac{P^2A}{S^2A} \right) x, x \right\rangle_H^2 + \left\langle \left( \frac{0}{Q^2A} \right) x, x \right\rangle_H^2
\]

This implies that

\[
\left\langle \left( \frac{P}{S} \right) x, x \right\rangle_H^2
\]
\[
\leq \frac{1}{2} \left\langle \left( \frac{P^2A}{S^2A} + \frac{Q^2A}{0} \right) x, x \right\rangle_H^2
\]
\[
+ \max \left\{ \omega_A^2(P), \omega_A^2(S) \right\} + \omega_A^2 \left[ \left( \frac{0}{R} \frac{Q}{0} \right) \right] + \omega_A^2 \left[ \left( \frac{0}{Q^2A} \right) \right].
\]
Let \( T \) be such that \( T \geq A \). Then
\[
\langle Tx, x \rangle_A \leq \langle T^n x, x \rangle_A, \quad \forall n \in \mathbb{N}^+.
\] (11)

Now, we are ready to prove the following theorem.

**Theorem 2.** Let \( T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \) be such that \( P, Q, R, S \in \mathcal{B}_A(\mathcal{H}) \). Then,
\[
\omega_A(T) \leq \max\{\omega_A(QR), \omega_A(RQ)\} \max\left\{\|R^{\frac{1}{2}} R + QQ^{\frac{1}{2}}\|_A, \|Q^{\frac{1}{2}} Q + RR^{\frac{1}{2}}\|_A\right\} + 8 \max\{\omega_A^2(P), \omega_A^2(S)\} + 3 \max\{\mu, \nu\},
\]
where \( \mu = \|(R^{\frac{1}{2}} R)^2 + (QQ^{\frac{1}{2}})^2\|_A \) and \( \nu = \|(Q^{\frac{1}{2}} Q)^2 + (RR^{\frac{1}{2}})^2\|_A \).
Proof. Let \( x \in \mathcal{H} \oplus \mathcal{H} \) be such that \( \|x\|_A = 1 \). One has

\[
|\langle Tx, x \rangle_A|^4 = \left| \langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} x, x \rangle_A + \langle \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} x, x \rangle_A \right|^4
\]

\[
\leq \left( \left| \langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} x, x \rangle_A \right| + \left| \langle \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} x, x \rangle_A \right| \right)^4
\]

\[
= \left( \frac{2\left| \langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} x, x \rangle_A \right| + 2\left| \langle \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} x, x \rangle_A \right|}{2} \right)^4
\]

\[
\leq 8 \left( \left| \langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} x, x \rangle_A \right|^4 + \left| \langle \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} x, x \rangle_A \right|^4 \right).
\]

where the last inequality follows by applying the convexity of the function \( f(t) = t^4 \) with \( t \geq 0 \). This implies, by taking Lemma 1 (i) into consideration, that

\[
|\langle Tx, x \rangle_A|^4 \leq 8\omega_A^4 \left[ \langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} x, x \rangle_A \right]^2 + 8\left| \langle \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} x, x \rangle_A \right|^4
\]

\[
= 8 \max \{ \omega_A^4(P), \omega_A^4(S) \} + 8\left| \langle \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} x, x \rangle_A \right|^4.
\]

On the other hand, let \( S = \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \). By using Lemma 3, we obtain

\[
|\langle Tx, x \rangle_A|^4
\]

\[
\leq 8 \max \{ \omega_A^4(P), \omega_A^4(S) \} + 8\left| \langle Sx, x \rangle_A \langle x, S^*_A x \rangle_A \right|^2
\]

\[
\leq 8 \max \{ \omega_A^4(P), \omega_A^4(S) \} + 2\left( 3\|Sx\|_A^2 \|S^*_A x\|_A^2 + \|Sx\|_A \|S^*_A x\|_A \left| \langle Sx, S^*_A x \rangle_A \right| \right)
\]

\[
= 8 \max \{ \omega_A^4(P), \omega_A^4(S) \} + 6\left( \left| S^*_A x \right|^2 \left| S^*_A x \right| \right) \left| \langle Sx, x \rangle_A \right|^2
\]

\[
+ 2\sqrt{\left( \langle S^*_A Sx, x \rangle_A \langle S^*_A x, x \rangle_A \right) \langle S^*_A Sx, x \rangle_A \left| \langle S^*_A x, x \rangle_A \right|}
\]

\[
\leq 8 \max \{ \omega_A^4(P), \omega_A^4(S) \} + 3\left( \left| S^*_A Sx \right|^2 + \left| S^*_A Sx \right| \right) + \left| \langle S^*_A S + S^*_A S, x \rangle_A \right| \left| \langle S^*_A x, x \rangle_A \right|
\]

where the last inequality follows by applying the arithmetic–geometric mean inequality. Now, since \( S^*_A S \) and \( S^*_A S \) are positive, then an application of Lemma 4 with \( n = 2 \) gives

\[
|\langle Tx, A \rangle|^4 \leq 8 \max \{ \omega_A^4(P), \omega_A^4(S) \} + 3\left( \left( \left| S^*_A S \right|^2 + \left| S^*_A S \right| \right) \right)^2 \left| \langle x, x \rangle_A \right|
\]

\[
+ \left| \langle S^*_A S + S^*_A S, x \rangle_A \right| \left| \langle S^*_A x, x \rangle_A \right|
\]

\[
\leq 8 \max \{ \omega_A^4(P), \omega_A^4(S) \} + 3\left( \left| S^*_A S \right|^2 + \left| S^*_A S \right| \right) + \left| \langle S^*_A S + S^*_A S, x \rangle_A \right| \left| \langle S^*_A x, x \rangle_A \right|
\]

\[
+ \left| S^*_A S + S^*_A S \right| \left| \langle S^*_A x, x \rangle_A \right|
\]

A short calculation reveals that

\[
S^*_A S + S^*_A S = \begin{pmatrix} R^*_A S + QR^*_A & 0 \\ 0 & Q^*_A S + RR^*_A \end{pmatrix}, \quad S^2 = \begin{pmatrix} QR & 0 \\ 0 & RQ \end{pmatrix}
\]
and 
\[(S^{\lambda A}S)^2 + (SS^{\lambda A}) = \begin{pmatrix} (R^{\lambda A} R)^2 + (QQ^{\lambda A})^2 & 0 \\ 0 & (Q^{\lambda A} Q)^2 + (RR^{\lambda A})^2 \end{pmatrix}.\]

Hence, by applying Lemma 1 (i) and (ii), we infer that 
\[\|S^{\lambda A}S + SS^{\lambda A}\|_A = \max \{ \|R^{\lambda A} R + QQ^{\lambda A}\|_A, \|Q^{\lambda A} Q + RR^{\lambda A}\|_A \},\]
\[\omega_A(S^2) = \max \{ \omega_A(QR), \omega_A(RQ) \} \text{ and } \| (S^{\lambda A}S)^2 + (SS^{\lambda A}) \|_A = \max \{ \mu, \nu \},\]
where \( \mu = \| (R^{\lambda A} R)^2 + (QQ^{\lambda A})^2 \|_A \) and \( \nu = \| (Q^{\lambda A} Q)^2 + (RR^{\lambda A})^2 \|_A \). So, we obtain 
\[|\langle Tx, x \rangle_A|^4 \leq \max \{ \omega_A(QR), \omega_A(RQ) \} \max \{ \| R^{\lambda A} R + QQ^{\lambda A} \|_A, \| Q^{\lambda A} Q + RR^{\lambda A} \|_A \} + 8 \times \max \{ \omega_A^2(P), \omega_A^2(S) \} + 3 \max \{ \mu, \nu \},\]
This proves the desired by letting the supremum over \( x \in H \oplus H \) be such that \( \|x\|_A = 1 \) in the last inequality.

Next, we present a result which is an improvement of the inequality of Cauchy–Schwarz type,
\[|\langle x, y \rangle_A| \leq \|x\|_A \|y\|_A,\]  
where \( x, y \in H \oplus H \), similar to a result of [32]; thus:

**Lemma 5.** Let \( \lambda \in [0, 1] \). Then
\[|\langle x, y \rangle_A|^2 \leq \lambda \|x\|_A \|y\|_A + (1 - \lambda) |\langle x, y \rangle_A| \|x\|_A \|y\|_A \leq \|x\|_A^2 \|y\|_A^2,\]  
for any \( x, y \in H \oplus H \).

**Proof.** Using inequality (12), we deduce that 
\[|\langle x, y \rangle_A| \leq \lambda \|x\|_A \|y\|_A + (1 - \lambda) |\langle x, y \rangle_A| \leq \|x\|_A \|y\|_A,\]
for any \( x, y \in H \oplus H \) and \( \lambda \in [0, 1] \).

Multiplying by \( \|x\|_A \|y\|_A \) in the above inequality, we have that 
\[|\langle x, y \rangle_A|^2 \leq \lambda \|x\|_A^2 \|y\|_A^2 + (1 - \lambda) |\langle x, y \rangle_A|^2 \|x\|_A^2 \|y\|_A \leq \|x\|_A^2 \|y\|_A^2,\]
and using again inequality (12), we deduce that the inequality of the statement is true.

**Remark 1.** Inequality (13) can be written as:
\[|\langle x, y \rangle_A| \leq \sqrt{\lambda \|x\|_A^2 \|y\|_A^2 + (1 - \lambda) |\langle x, y \rangle_A|^2 \|x\|_A \|y\|_A} \leq \|x\|_A \|y\|_A,\]
for any \( x, y \in H \oplus H \) and \( \lambda \in [0, 1] \).

**Theorem 3.** Let \( T, S \in B_A(H \oplus H) \) and \( \lambda \in [0, 1] \). Then, the inequality
\[\omega_A^2(S^{\lambda A} T) \leq \frac{\lambda}{2} \| (T^{\lambda A} T)^2 + (S^{\lambda A} S)^2 \|_A + \frac{1 - \lambda}{2} \omega_A(S^{\lambda A} T) \| T^{\lambda A} T + S^{\lambda A} S \|_A \]  
holds.

**Proof.** We take the first inequality from Lemma 5:
\[|\langle x, y \rangle_A|^2 \leq \lambda \|x\|_A^2 \|y\|_A^2 + (1 - \lambda) |\langle x, y \rangle_A| \|x\|_A \|y\|_A,\]
for any \( x, y \in \mathcal{H} \oplus \mathcal{H} \) and \( \lambda \in [0, 1] \). Because we need to apply the inequality Hölder–McCarthy for positive operators, it is easy to see that the operators \( T^{\lambda x} T \) and \( S^{\lambda x} S \) are positive. Now, we replace \( x \) and \( y \) by \( T x \) and \( S x \), in the above inequality, and we assume that \( \|x\|_{A} = 1 \); then, we obtain

\[
|\langle S^{\lambda x}Tx, x \rangle_A|^2 = |\langle Tx, Sx \rangle_A|^2 \\
\leq \lambda \|Tx\|^2_A \|Sx\|^2_A + (1 - \lambda) |\langle S^{\lambda x}Tx, x \rangle_A \|Tx\|_A \|Sx\|_A \\
\leq \lambda \langle Tx, Tx \rangle_A \langle Sx, Sx \rangle_A + (1 - \lambda) |\langle S^{\lambda x}Tx, x \rangle_A \sqrt{\langle Tx, Tx \rangle_A \langle Sx, Sx \rangle_A} \\
= \lambda \langle T^{\lambda x}Tx, x \rangle_A \langle S^{\lambda x}Sx, x \rangle_A + (1 - \lambda) |\langle S^{\lambda x}Tx, x \rangle_A \sqrt{\langle T^{\lambda x}Tx, x \rangle_A \langle S^{\lambda x}Sx, x \rangle_A} \\
\leq \frac{\lambda}{4} \left( |\langle T^{\lambda x}Tx, x \rangle_A + \langle S^{\lambda x}Sx, x \rangle_A|^2 + \frac{1 - \lambda}{2} |\langle S^{\lambda x}Tx, x \rangle_A |\langle (T^{\lambda x}T + S^{\lambda x}S)x, x \rangle_A \\
\leq \frac{\lambda}{2} \left( |\langle T^{\lambda x}T \rangle^2 + (S^{\lambda x}S)^2 \rangle_A \langle x, x \rangle_A + \frac{1 - \lambda}{2} |\langle S^{\lambda x}Tx, x \rangle_A |\langle (T^{\lambda x}T + S^{\lambda x}S)x, x \rangle_A. \\
So, we obtain \\
|\langle S^{\lambda x}Tx, x \rangle_A|^2 \\
\leq \frac{\lambda}{2} \left( |\langle T^{\lambda x}T \rangle^2 + (S^{\lambda x}S)^2 \rangle_A \langle x, x \rangle_A + \frac{1 - \lambda}{2} |\langle S^{\lambda x}Tx, x \rangle_A |\langle (T^{\lambda x}T + S^{\lambda x}S)x, x \rangle_A. \\
Taking the supremum over \( x \in \mathcal{H} \oplus \mathcal{H} \) with \( \|x\|_A = 1 \) in the above inequality, we obtain the inequality of the statement. \( \square \)

**Remark 2.** Through various particular cases of \( \lambda \) in Theorem 3, we obtain some results, thus: for \( \lambda = 1 \) in the inequality of (14), we deduce inequality

\[
\omega^2_A(S^{\lambda x}T) \leq \frac{1}{2} \| (T^{\lambda x}T)^2 + (S^{\lambda x}S)^2 \|_A \\
and for \( \lambda = 0 \), we find inequality

\[
\omega^2_A(S^{\lambda x}T) \leq \frac{1}{2} \| T^{\lambda x}T + S^{\lambda x}S \|_A. \\
**Corollary 2.** Let \( T, S \in B_A(H) \) and \( \lambda \in [0, 1] \). Then, the inequality

\[
\omega^2_A(S^{\lambda x}T) \leq \frac{\lambda}{2} \| (T^{\lambda x}T)^2 + (S^{\lambda x}S)^2 \|_A + \frac{1 - \lambda}{2} \omega_A(S^{\lambda x}T) \| T^{\lambda x}T + S^{\lambda x}S \|_A \\
holds.

**Proof.** Let \( S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( T = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \) (or \( T = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} \)). By using Lemma 1 (iii), we obtain

\[
S^{\lambda x}S = \begin{pmatrix} S^{\lambda x}S & 0 \\ 0 & S^{\lambda x}S \end{pmatrix}, \quad T^{\lambda x}T = \begin{pmatrix} T^{\lambda x}T & 0 \\ 0 & T^{\lambda x}T \end{pmatrix} \quad \text{and} \quad S^{\lambda x}T = \begin{pmatrix} S^{\lambda x}T & 0 \\ 0 & S^{\lambda x}T \end{pmatrix}. \\
Therefore, we obtain \( \omega_A(S^{\lambda x}T) = \omega_A(S^{\lambda x}T) \).

Applying relation (ii) from Lemma 1, we find

\[
\| T^{\lambda x}T + S^{\lambda x}S \|_A = \| T^{\lambda x}T + S^{\lambda x}S \|_A.
Using Theorem 3 and the above results, we deduce the inequality of the statement. □

**Theorem 4.** Let $P, Q, R, S \in B_A(H)$. Then,

$$\omega_A^2\left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\right) \leq p \max\left\{\omega_A^2(P), \omega_A^2(S)\right\} + \frac{q}{2} \max\{\omega_A(QR), \omega_A(RQ)\} + \frac{q}{4} \max\{\|QQ^{\sharp_A} + R^{\sharp_A}R\|_{A'}, \|RR^{\sharp_A} + Q^{\sharp_A}Q\|_{A'}\},$$

where $p, q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** Let $x \in H \oplus H$ be such that $\|x\|_A = 1$. We use the classical Bohr inequality [33]

$$|a + b|^2 \leq p|a|^2 + q|b|^2,$$

where $p, q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b \in \mathbb{C}$.

Therefore, we have

$$\left|\left\langle \begin{pmatrix} P & Q \\ R & S \end{pmatrix} x, x \right\rangle_A\right|^2 \leq p\left|\left\langle \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} x, x \right\rangle_A\right|^2 + q\left|\left\langle \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} x, x \right\rangle_A\right|^2 \leq p\omega_A^2\left(\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}\right) + q\omega_A^2\left(\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right) = p \min\{\omega_A^2(P), \omega_A^2(S)\} + q\omega_A^2\left(\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right).$$

However, we have the inequality given by Xu et al. in [28]:

$$\omega_A^2\left(\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}\right) \leq \frac{1}{4} \max\{\|QQ^{\sharp_A} + R^{\sharp_A}R\|_{A'}, \|RR^{\sharp_A} + Q^{\sharp_A}Q\|_{A'}\} + \frac{1}{2} \max\{\omega_A(QR), \omega_A(RQ)\},$$

for all $Q, R \in B_A(H)$.

Therefore, we proved the inequality of the statement. □

Our next goal consists of deriving an alternative and easy proof of the generalized Kittaneh inequalities (9). In all that follows, for any arbitrary operator $T \in B_A(H)$, we write $R_A(T) := \frac{T + T^A}{2}$ and $\Im_A(T) := \frac{T - T^A}{2}$. In order to provide the alternative proof of (9), we require the following two lemmas.

**Lemma 6 ([25]).** Let $T \in B(H)$ be an $A$-selfadjoint operator. Then, $T^{2n} \geq_A 0$ for all $n \in \mathbb{N}^*$.

**Lemma 7 ([20,25]).** Let $T \in B_A(H)$. Then

$$\omega_A(T) = \sup_{\theta \in \mathbb{R}}\|R_A(e^{i\theta}T)\|_A = \sup_{\theta \in \mathbb{R}}\|\Im_A(e^{i\theta}T)\|_A. \quad (15)$$

Now, we are ready to derive our proof in the next result.

**Theorem 5 ([26]).** Let $T \in B_A(H)$. Then,

$$\frac{1}{4}T^{\sharp_A}T + TT^{\sharp_A} \leq \omega_A^2(T) \leq \frac{1}{2}T^{\sharp_A}T + TT^{\sharp_A}.$$
Proof. Let $\theta \in \mathbb{R}$. By making simple computations, we see that
\[
\left[ \Re A(e^{i\theta}T) \right]^2 + \left[ \Im A(e^{i\theta}T) \right]^2 = \frac{1}{2} \left( TT^*A + T^*A T \right). \tag{16}
\]

Since $\Im A(e^{i\theta}T)$ is an $A$-selfadjoint operator, then by Lemma 6, we deduce that $\left[ \Im A(e^{i\theta}T) \right]^2 \geq A_0$. So, in view of (16), we infer that
\[
\frac{1}{2} \left( TT^*A + T^*A T \right) - \left[ \Re A(e^{i\theta}T) \right]^2 = \left[ \Im A(e^{i\theta}T) \right]^2 \geq A_0.
\]

This implies that
\[
\left\langle \left[ \Re A(e^{i\theta}T) \right]^2 x, x \right\rangle_A \leq \frac{1}{2} \left\langle \left( TT^*A + T^*A T \right) x, x \right\rangle_A, \tag{17}
\]
for all $x \in \mathcal{H}$. Moreover, since $\Re A(e^{i\theta}T)$ is an $A$-selfadjoint operator, then by taking into account (17), it can be seen that
\[
\left\| \left[ \Re A(e^{i\theta}T) \right]^2 x \right\|^2_A \leq \frac{1}{2} \left\langle \left( TT^*A + T^*A T \right) x, x \right\rangle_A, \tag{18}
\]
for all $x \in \mathcal{H}$. Furthermore, clearly, we have $TT^*A + T^*A T \geq A_0$. So, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in (18) and then using (4) and (6), we obtain
\[
\left\| \Re A(e^{i\theta}T) \right\|^2_A \leq \frac{1}{2} \left\| TT^*A + T^*A T \right\|_A.
\]

Therefore, by taking the supremum over all $\theta \in \mathbb{R}$ in the above inequality and then applying Lemma 7, we obtain
\[
\omega^2_A(T) \leq \frac{1}{2} TT^*A + T^*A T, \tag{19}
\]
On the other hand, by taking $\theta = 0$ in (16), we obtain
\[
\frac{1}{2} \left( TT^*A + T^*A T \right) = \left[ \Re A(T) \right]^2 + \left[ \Im A(T) \right]^2.
\]
This implies that
\[
\frac{1}{2} \left\| \left( TT^*A + T^*A T \right) \right\|_A = \left\| \left[ \Re A(T) \right]^2 + \left[ \Im A(T) \right]^2 \right\|_A \leq \left\| \left[ \Re A(T) \right]^2 \right\|_A + \left\| \left[ \Im A(T) \right]^2 \right\|_A \leq 2\omega^2_A(T) \quad \text{(by Lemma 7 and (5))}.
\]

Hence, we obtain
\[
\frac{1}{4} \left\| \left( TT^*A + T^*A T \right) \right\|_A \leq \omega^2_A(T). \tag{20}
\]
Hence, the required result follows by combining (20) together with (19). \qed

The following lemma plays a crucial role in proving our next result.

Lemma 8. Let $T, S \in \mathcal{B}(\mathcal{H})$ be $A$-positive operators. Then,
\[
\| T + S \|_A \leq \max\{ \| T \|_A, \| S \|_A \} + \sqrt{\| TS \|_A}.
\]

To prove Lemma 8, we need the following two results.
Lemma 9 ([5,6]). Let \( T \in \mathcal{B}(\mathcal{H}) \). Then \( T \in \mathcal{B}_{A^{1/2}}(\mathcal{H}) \) if and only if there exists a unique \( \bar{T} \in \mathcal{B}(\mathcal{R}(A^{1/2})) \) such that \( Z_AT = \bar{T}Z_A \). Here, \( Z_A : \mathcal{H} \to \mathcal{R}(A^{1/2}) \) is defined by \( Z_Ax = Ax \). Furthermore, the following properties hold

(i) \( \|T\|_A = \|\bar{T}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}, \) for every \( T \in \mathcal{B}_{A^{1/2}}(\mathcal{H}) \).

(ii) \( \bar{T} + \bar{S} = \bar{T} + S \) and \( \bar{T}\bar{S} = \bar{S}\bar{T} \) for every \( T, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H}) \).

Lemma 10 ([34]). Let \( X, Y \in \mathcal{B}(\mathcal{H}) \) be such that \( X \geq 0 \) and \( Y \geq 0 \). Then

\[
\|X + Y\| \leq \max\{\|X\|, \|Y\|\} + \sqrt{\|XY\|}. \tag{21}
\]

Now, we are in a position to prove Lemma 8.

Proof of Lemma 8. Notice first that since \( T \) and \( S \) are \( A \)-positive, then clearly \( T \) and \( S \) are \( A \)-selfadjoint. This implies that \( T, S \in \mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H}) \). Thus, by Lemma 9, there exist two unique operators \( \bar{T} \) and \( \bar{S} \) in \( \mathcal{B}(\mathcal{R}(A^{1/2})) \) such that \( Z_AT = \bar{T}Z_A \) and \( Z_AS = \bar{S}Z_A \). Furthermore, since \( T \geq_A 0 \), then \( \langle ATx, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \). So, by taking (2) into consideration, we see that

\[
\langle Tx, x \rangle_A = \langle ATx, Ax \rangle_{\mathcal{R}(A^{1/2})} = \langle \bar{T}Ax, Ax \rangle_{\mathcal{R}(A^{1/2})} \geq 0, \quad \forall x \in \mathcal{H}.
\]

On the other hand, the density of \( \mathcal{R}(A) \) in \( \mathcal{R}(A^{1/2}) \) yields that

\[
\langle \bar{T}A^{1/2}x, A^{1/2}x \rangle_{\mathcal{R}(A^{1/2})} \geq 0, \quad \forall x \in \mathcal{H}.
\]

Therefore, the operator \( \bar{T} \) is positive on the Hilbert space \( \mathcal{R}(A^{1/2}) \). By using similar arguments, one may prove that \( \bar{S} \) is also positive on \( \mathcal{R}(A^{1/2}) \). So, by applying (21) together with Lemma 9, we observe that

\[
\|T + S\|_A = \|\bar{T} + \bar{S}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))} \\
= \|\bar{T}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))} + \|\bar{S}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))} \\
\leq \max\{\|\bar{T}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}, \|\bar{S}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}\} + \sqrt{\|\bar{T}\bar{S}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}} \\
= \max\{\|T\|_A, \|S\|_A\} + \sqrt{\|TS\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))}} \\
= \max\{\|T\|_A, \|S\|_A\} + \sqrt{\|TS\|_A}.
\]

This proves the desired result. \( \square \)

Now, we are able to establish the next result which provides a refinement of the first inequality in (9). The inspiration for our investigation comes from [35].

Theorem 6. Let \( T \in \mathcal{B}_A(\mathcal{H}) \). Then

\[
\frac{1}{4}\left\|TT^{A^*} + T^{A^*}T\right\|_A \leq \frac{1}{4}\left\{\max\{\|\Re_A(T) + \Im_A(T)\|_A, \|\Re_A(T) - \Im_A(T)\|_A\}\right\}^2 \\
+ \frac{1}{4}\|\Re_A(T) + \Im_A(T)\|_A\|\Re_A(T) - \Im_A(T)\|_A \\
\leq \omega^2_A(T).
\]

Proof. Notice first that a short calculation reveals that

\[
T^{2A}T + TT^{2A} = \left(\Re_A(T) + \Im_A(T)\right)^2 + \left(\Re_A(T) - \Im_A(T)\right)^2. \tag{22}
\]
Moreover, one may immediately check that the operators $\mathcal{R}_A(T)$ and $\mathcal{A}(T)$ are $A$-selfadjoint. Thus, by Lemma 6, we deduce that $(\mathcal{R}_A(T) + \mathcal{A}(T))^2 \geq_A 0$ and $(\mathcal{R}_A(T) - \mathcal{A}(T))^2 \geq_A 0$. Therefore, an application of (22) together with Lemma (8) ensures that

$$\frac{1}{4} \| T^2 A + T T^4 A \|_A = \frac{1}{4} \left\| \left( \mathcal{R}_A(T) + \mathcal{A}(T) \right)^2 + \left( \mathcal{R}_A(T) - \mathcal{A}(T) \right)^2 \right\|_A$$

$$\leq \frac{1}{4} \max \left\{ \left\| \mathcal{R}_A(T) + \mathcal{A}(T) \right\|_A, \left\| \mathcal{R}_A(T) - \mathcal{A}(T) \right\|_A \right\}$$

$$+ \frac{1}{4} \sqrt{\left\| \left( \mathcal{R}_A(T) + \mathcal{A}(T) \right)^2 \left( \mathcal{R}_A(T) - \mathcal{A}(T) \right)^2 \right\|_A}$$

$$\leq \frac{1}{4} \max \left\{ \left\| \mathcal{R}_A(T) + \mathcal{A}(T) \right\|_A, \left\| \mathcal{R}_A(T) - \mathcal{A}(T) \right\|_A \right\}$$

$$+ \frac{1}{4} \| \mathcal{R}_A(T) + \mathcal{A}(T) \|_A \| \mathcal{R}_A(T) - \mathcal{A}(T) \|_A,$$

where the last inequality follows by using (5). In addition, since the operators $\mathcal{R}_A(T) + \mathcal{A}(T)$ and $\mathcal{R}_A(T) - \mathcal{A}(T)$ are $A$-selfadjoint, then by applying (7), we obtain

$$\frac{1}{4} \| T T^4 A + T^2 A \|_A \leq \frac{1}{4} \left( \max \{ \| \mathcal{R}_A(T) + \mathcal{A}(T) \|_A, \| \mathcal{R}_A(T) - \mathcal{A}(T) \|_A \} \right)^2$$

$$+ \frac{1}{4} \| \mathcal{R}_A(T) + \mathcal{A}(T) \|_A \| \mathcal{R}_A(T) - \mathcal{A}(T) \|_A. \quad (23)$$

On the other hand, let $x \in \mathcal{H}$ be such that $\| x \|_A = 1$. Clearly, $T$ can be decomposed as $T = \mathcal{R}_A(T) + i \mathcal{S}_A(T)$. Notice that $\mathcal{R}_A(T)$ and $\mathcal{A}(T)$ are $A$-selfadjoint operators. This implies that $\langle \mathcal{R}_A(T)x, x \rangle_A$ and $\langle \mathcal{A}(T)x, x \rangle_A$ are real numbers. Furthermore, we see that

$$\langle Tx, x \rangle_A^2 = \langle \mathcal{R}_A(T)x, x \rangle_A^2 + \langle \mathcal{S}_A(T)x, x \rangle_A^2$$

$$= \frac{1}{2} \left( \langle \mathcal{R}_A(T)x, x \rangle_A + \langle \mathcal{S}_A(T)x, x \rangle_A \right)^2 + \frac{1}{2} \left( \langle \mathcal{R}_A(T)x, x \rangle_A - \langle \mathcal{S}_A(T)x, x \rangle_A \right)^2$$

$$= \frac{1}{2} \left( \langle \mathcal{R}_A(T) + \mathcal{A}(T)x, x \rangle_A \right)^2 + \frac{1}{2} \left( \langle \mathcal{R}_A(T) - \mathcal{A}(T)x, x \rangle_A \right)^2$$

$$\geq \frac{1}{2} \left( \langle \mathcal{R}_A(T) + \mathcal{A}(T)x, x \rangle_A \right)^2 + \frac{1}{2} \left( \langle \mathcal{R}_A(T) - \mathcal{A}(T)x, x \rangle_A \right)^2.$$

This implies, by taking the supremum over all $x \in \mathcal{H}$ with $\| x \|_A = 1$ in the last inequality, that

$$\frac{1}{2} c_A^2 \left( \mathcal{R}_A(T) + \mathcal{A}(T) \right) + \frac{1}{2} c_A^2 \left( \mathcal{R}_A(T) - \mathcal{A}(T) \right) \leq c_A^2(T).$$

In addition, since $\mathcal{R}_A(T) - \mathcal{A}(T)$ is $A$-selfadjoint, then an application of (6) gives

$$\frac{1}{2} c_A^2 \left( \mathcal{R}_A(T) + \mathcal{A}(T) \right) + \frac{1}{2} \| \mathcal{R}_A(T) - \mathcal{A}(T) \|_A^2 \leq c_A^2(T). \quad (24)$$

Similarly, it can be proved that

$$\frac{1}{2} c_A^2 \left( \mathcal{R}_A(T) - \mathcal{A}(T) \right) + \frac{1}{2} \| \mathcal{R}_A(T) + \mathcal{A}(T) \|_A^2 \leq c_A^2(T). \quad (25)$$

Combining (24) together with (25) gives

$$\frac{1}{2} \max \left\{ \| \mathcal{R}_A(T) - \mathcal{A}(T) \|_A^2, \| \mathcal{R}_A(T) - \mathcal{A}(T) \|_A^2 \right\} \leq c_A^2(T). \quad (26)$$
Hence, by taking (23) into consideration and then using (26), we observe that

\[
\frac{1}{4} \| T T^\sharp T + T^\sharp A \|_A \leq \frac{1}{4} \left( \max \{ \| R_A(T) + \Im A(T) \|_A, \| R_A(T) - \Im A(T) \|_A \} \right)^2 + \frac{1}{4} \| R_A(T) + \Im A(T) \|_A \| R_A(T) - \Im A(T) \|_A \\
\leq \frac{1}{2} \omega_A^2(T) + \frac{1}{4} \| R_A(T) + \Im A(T) \|_A \| R_A(T) - \Im A(T) \|_A \\
\leq \frac{1}{2} \omega_A^2(T) + \frac{1}{4} \sqrt{2} \omega_A(T) = \omega_A^2(T).
\]

This finishes the proof of our result. \( \square \)

By using the inequalities (24) and (25), we derive in the next theorem another improvement of the first inequality in Theorem 5: \( \frac{1}{4} \| T T^\sharp T + T^\sharp A \|_A \leq \omega_A^2(T) \).

**Theorem 7.** Let \( T \in \mathbb{B}_A(\mathcal{H}) \). Then

\[
\frac{1}{4} \| T^\sharp T + T^\sharp A \|_A \leq \frac{1}{4} \| R_A(T) + \Im A(T) \|_A^2 + \frac{1}{4} \| R_A(T) - \Im A(T) \|_A^2 \\
+ \frac{1}{4} c_A^2 \| R_A(T) + \Im A(T) \|_A + \frac{1}{4} \| R_A(T) - \Im A(T) \|_A \\
\leq \omega_A^2(T).
\]

**Proof.** By applying (22), we see that

\[
\frac{1}{4} \| T^\sharp T + T^\sharp A \|_A = \frac{1}{4} \left( \| R_A(T) + \Im A(T) \|_A^2 + \| R_A(T) - \Im A(T) \|_A^2 \right) \\
\leq \frac{1}{4} \left( \| R_A(T) + \Im A(T) \|_A^2 + \| R_A(T) - \Im A(T) \|_A^2 \right) \\
\leq \frac{1}{4} \| R_A(T) + \Im A(T) \|_A^2 + \frac{1}{4} \| R_A(T) - \Im A(T) \|_A^2 \quad \text{(by (5)).}
\]

This implies, through (5), that

\[
\frac{1}{4} \| T^\sharp T + T^\sharp A \|_A \leq \frac{1}{4} \| R_A(T) + \Im A(T) \|_A^2 + \frac{1}{4} \| R_A(T) - \Im A(T) \|_A^2. \tag{27}
\]

On the other hand, it follows from the inequalities (24) and (25) that

\[
\frac{1}{4} \| R_A(T) + \Im A(T) \|_A^2 + \frac{1}{4} \| R_A(T) - \Im A(T) \|_A^2 \\
+ \frac{1}{4} c_A^2 \| R_A(T) + \Im A(T) \|_A + \frac{1}{4} \| R_A(T) - \Im A(T) \|_A \leq \omega_A^2(T). \tag{28}
\]

Combining (27) together with (28) yields the desired result. \( \square \)

Our next result provides also another refinement of the first inequality in Theorem 5.

**Theorem 8.** Let \( T \in \mathbb{B}_A(\mathcal{H}) \). Then

\[
\frac{1}{4} \| T T^\sharp T + T^\sharp A \|_A \leq \frac{1}{2} \max \left\{ \| R_A(T) - \Im A(T) \|_A^2, \| R_A(T) + \Im A(T) \|_A^2 \right\} \\
\leq \max \{ \gamma_A(T), \delta_A(T) \} \\
\leq \omega_A^2(T),
\]
where
\[
\gamma_A(T) = \frac{1}{2} e^2 (\Re_A(T) + \Im_A(T)) + \frac{1}{2} ||\Re_A(T) - \Im_A(T)||_A^2
\]
and
\[
\delta_A(T) = \frac{1}{2} e^2 (\Re_A(T) - \Im_A(T)) + \frac{1}{2} ||\Re_A(T) + \Im_A(T)||_A^2.
\]

**Proof.** Notice first that the third inequality in Theorem 8 follows immediately by applying the inequalities (24) and (25). Moreover, the second inequality holds immediately. So, it remains to prove the first inequality. We recall the following elementary equality

\[
\max\{a, b\} = \frac{1}{2} (a + b + |a - b|), \quad \forall x, y \in \mathbb{R}.
\]

An application of (29) shows that

\[
\max\{\|\Re_A(T) + \Im_A(T)||_A^2, \|\Re_A(T) - \Im_A(T)||_A^2\} = \frac{1}{4} \left( \|\Re_A(T) + \Im_A(T)||_A^2 + \|\Re_A(T) - \Im_A(T)||_A^2 \right)
\]

\[
= \frac{1}{4} \left( \|\Re_A(T) + \Im_A(T)||_A^2 + \|\Re_A(T) - \Im_A(T)||_A^2 \right)
\]

\[
\geq \frac{1}{4} \left( |\Re_A(T) + \Im_A(T)||_A^2 + |\Re_A(T) - \Im_A(T)||_A^2 \right)
\]

\[
= \frac{1}{4} \left( |\Re_A(T) + \Im_A(T)||_A^2 + |\Re_A(T) - \Im_A(T)||_A^2 \right).
\]

where the last equality follows from (22). This immediately proves the first inequality in Theorem 8. Hence, the proof is complete. \(\square\)

**Remark 3.** The inequalities from Theorem 5 given by Feki in [26] represent a generalization of the inequalities given by Kittaneh [27]. In Theorems 6 and 7, we present some refinements of the inequalities due to Feki.

3. Conclusions

The main objective of the present paper is to present a few new \(A\)-numerical radius inequalities for \(2 \times 2\) operator matrices. In Theorem 1, we obtain a bound for the \(A\)-numerical radius for the \(2 \times 2\) operator matrix. We use an inequality of Buzano type (see Lemma 2) to estimate the \(A\)-numerical radius of an operator \(T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}\), where \(P, Q, R, S \in \mathbb{M}_2(\mathcal{H})\). By particularization, we deduce an improvement of the second inequality (9). Another bound for \(A\)-numerical radius for the \(2 \times 2\) operator matrix is given in Theorem 2. Next, we present an improvement of the Cauchy–Schwarz inequality type using the inner product \(\langle \cdot, \cdot \rangle_A\). This result is used to find a new bound for the \(A\)-numerical radius of operator matrix \(S^* A T\). Applying the Bohr inequality, we deduce another new bound for the \(A\)-numerical radius for the \(2 \times 2\) operator matrix. In addition to these, we aim to establish an alternative and easy proof of the generalized Kittaneh inequalities (9). We also give a lemma which plays a crucial role in proving a result concerning to norm \(\| \cdot \|_A\) (see Lemma 8). Finally, we establish some improvements of the well-known inequalities due to Kittaneh (see [27] Theorem 1) and generalized by Feki in [26]. In the future, we will study better estimates of the \(A\)-numerical radius for the \(2 \times 2\) operator matrix and we will study new inequalities.
involving the Berezin norm and Berezin number of bounded linear operators in Hilbert and semi-Hilbert space. We can also define the A-Berezin norm and number.

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**References**

1. Baklouti, H.; Namouri, S. Closed operators in semi-Hilbertian spaces. *Linear Multilinear Algebra* **2021**, 1–12. [CrossRef]
2. Baklouti, H.; Namouri, S. Spectral analysis of bounded operators on semi-Hilbertian spaces. *Banach J. Math. Anal.* **2022**, 16, 12. [CrossRef]
3. Enderami, S.M.; Abtahi, M.; Zamani, A. An Extension of Birkhoff–James Orthogonality Relations in Semi-Hilbertian Space Operators. *Mediterr. J. Math.* **2022**, 19, 234. [CrossRef]
4. de Branges, L.; Rovnyak, J. *Square Summable Power Series*; Holt, Rinehart and Winston: New York, NY, USA, 1966.
5. Feki, K. On tuples of commuting operators in positive semidefinite inner product spaces. *Linear Algebra Appl.* **2020**, 603, 313–328. [CrossRef]
6. Arias, M.L.; Corach, G.; Gonzalez, M.C. Lifting properties in operator ranges. *Acta Sci. Math. (Szeged)* **2009**, 75, 635–653.
7. Bottazzi, T.; Conde, C. Generalized numerical radius and related inequalities. *Oper. Matrices* **2021**, 15, 1289–1308. [CrossRef]
8. Goldberg, M.; Tadmor, E. On the numerical radius and its applications. *Linear Algebra Appl.* **1982**, 42, 263–284. [CrossRef]
9. Chakraborty, B.; Ojha, S.; Birbonshi, R. On the numerical range of some weighted shift operators. *Linear Algebra Appl.* **2022**, 640, 179–190. [CrossRef]
10. Yan, T.; Hyder, J.; Akram, M.S.; Farid, G.; Nonlaopon, K. On Numerical Radius Bounds Involving Generalized Aluthge Transform. *J. Funct. Spaces* **2022**, 2022, 2897323. [CrossRef]
11. Abu-Omar, A.; Kittaneh, F. A generalization of the numerical radius. *Linear Algebra Appl.* **2019**, 569, 323–334. [CrossRef]
12. Sheikholeslami, A.; Khosravi, M.; Sababheh, M. The weighted numerical radius. *Ann. Funct. Anal.* **2022**, 13, 1–15. [CrossRef]
13. Zamani, A.; Wójcik, P. Another generalization of the numerical radius for Hilbert space operators. *Linear Algebra Appl.* **2021**, 609, 114–128. [CrossRef]
14. Arias, M.L.; Corach, G.; González, M.C. A-Normal operators in Semi-Hilbertian spaces. *Aust. J. Math. Anal. Appl.* **2012**, 9, 1–12.
15. Baklouti, H.; Feki, K.; Ahmed, O.A.M.S. Joint numerical ranges of operators in semi-Hilbertian spaces. *Linear Algebra Appl.* **2018**, 555, 266–284. [CrossRef]
16. Bhunia, P.; Paul, K.; Nayak, R.K. On inequalities for A-numerical radius of operators. *Electron. J. Linear Algebra* **2020**, 36, 143–157.
17. Bhunia, P.; Nayak, R.K.; Paul, K. Refinements of A-numerical radius inequalities and their applications. *Adv. Oper. Theory* **2020**, 5, 1498–1511. [CrossRef]
18. Feki, K. Spectral radius of semi-Hilbertian space operators and its applications. *Ann. Funct. Anal.* **2020**, 11, 929–946. [CrossRef]
19. Kittaneh, F.; Sahoo, S. On A-numerical radius equalities and inequalities for certain operator matrices. *Ann. Funct. Anal.* **2021**, 12, 52. [CrossRef]
20. Zamani, A. A-numerical radius inequalities for semi-Hilbertian space operators. *Linear Algebra Appl.* **2019**, 578, 159–183. [CrossRef]
21. Arias, M.L.; Corach, G.; Gonzalez, M.C. Partial isometries in semi-Hilbertian spaces. *Linear Algebra Appl.* **2008**, 428, 1460–1475. [CrossRef]
22. Douglas, R.G. On majorization, factorization and range inclusion of operators in Hilbert space. *Proc. Am. Math. Soc.* **1966**, 17, 413–416. [CrossRef]
23. Arias, M.L.; Corach, G.; Gonzalez, M.C. Metric properties of projections in semi-Hilbertian spaces. *Integral Equ. Oper. Theory* **2008**, 62, 11–28. [CrossRef]
24. Baklouti, H.; Feki, K.; Ahmed, O.A.M.S. Joint normality of operators in semi-Hilbertian spaces. *Linear Multilinear Algebra* **2020**, 68, 845–866. [CrossRef]
25. Feki, K. A note on the A-numerical radius of operators in semi-Hilbert spaces. *Arch. Math.* **2020**, 115, 535–544. [CrossRef]
26. Feki, K. Some numerical radius inequalities for semi-Hilbert space operators. *J. Korean Math. Soc.* **2021**, 58, 1385–1405. [CrossRef]
27. Kittaneh, F. Numerical radius inequalities for Hilbert space operators. *Stud. Math.* **2005**, 168, 73–80. [CrossRef]
28. Xu, Q.; Ye, Z.; Zamani, A. Some upper bounds for the $A$-numerical radius of $2 \times 2$ block matrices. *Adv. Oper. Theory* **2021**, *6*, 1–13. [CrossRef]

29. Bani-Domi, W.; Kittaneh, F. Norm and numerical radius inequalities for Hilbert space operators. *Linear Multilinear Algebra* **2021**, *69*, 934–945. [CrossRef]

30. Bhunia, P.; Paul, K. Some improvements of numerical radius inequalities of operators and operator matrices. *Linear Multilinear Algebra* **2022**, *70*, 1995–2013. [CrossRef]

31. Conde, C.; Feki, K. On some inequalities for the generalized joint numerical radius of semi-Hilbert space operators. *Ricerche Mat* **2021**, *2021*, 1–19. [CrossRef]

32. Alomari, M.W. On Cauchy–Schwarz type inequalities and applications to numerical radius inequalities. *Ricerche Mat* **2021**, *2021*. [CrossRef]

33. Bohr, H. Zur Theorie der fastperiodischen Funktionen I. *Acta Math.* **1924**, *45*, 29–127. [CrossRef]

34. Davidson, K.; Power, S.C. Best approximation in $C^*$-algebras. *J. Reine Angew. Math.* **1986**, *368*, 43–62.

35. Bhunia, P.; Jana, S.; Moslehian, M.S.; Paul, K. Improved inequalities for the numerical radius via Cartesian decomposition. *arXiv 2021*, arXiv:2110.02499.