Polynomial approximation avoiding values in sets II

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Abstract

We prove some results on when functions on compact sets $K \subset \mathbb{C}$ can be approximated by polynomials avoiding values in given sets. We also prove some higher dimensional analogues. In particular we prove that a continuous function from a compact set $K \subset \mathbb{R}^n$ without interior points to $\mathbb{R}^n$ can be uniformly approximated by a polynomial mapping avoiding values in any given countable set $A \subset \mathbb{R}^n$, giving a real $n$-dimensional analogue of a recent version of Lavrentiev’s theorem of Andersson and Rousu. We also prove the same result for infinite dimensional Banach spaces.

1 Introduction

In [2], [10] the following version of Lavrentiev’s theorem was proved

**Theorem 1.** Let $A \subset \mathbb{C}$ be any countable set, let $K \subset \mathbb{C}$ be a compact set with connected complement and without interior points, and let $f$ be a continuous function on $K$. Then given any $\varepsilon > 0$ there exists some polynomial $p$ such that $p(z) \notin A$ if $z \in K$ and such that

$$\max_{z \in K} |f(z) - p(z)| < \varepsilon.$$ 

Lavrentiev’s theorem [7] is the special case where $A = \emptyset$ and Theorem 1 also generalizes [1, Theorem 1] where it was proved for $A = \{0\}$. Examples of sets $K \subset \mathbb{C}$ with positive two dimensional Lebesgue measure where the result holds true can be

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1A result which has applications to the Riemann zeta-function.
given by e.g. \([0,1] + iS\) where \(S \subset [0,1]\) is a fat Cantor set. In the case when the compact set \(K\) has measure zero\(^2\) we prove this in a simpler way than in [2],[10] as a consequence of the fact that the difference set\(^3\) \(p(K) - A\) must then also have measure zero so that its complement \((p(K) - A)^c\) is dense in \(\mathbb{C}\). The general idea will also allow us to generalize this result to higher dimensions, and also allow a sharper result when the set \(K\) does not have full dimension. Finally we will adapt the method of [2], [10] which together with the observation that if \(p : \mathbb{R}^n \to \mathbb{R}^n\) is a polynomial mapping then the fact that \(K\) is a compact set without interior points implies that the image \(p(K)\) is also a compact set without interior points. This will allow us to prove a real \(n\)–dimensional analogue of Theorem 1.

2 Sumsets, translates of sets avoiding sets and polynomial approximation

2.1 Some notation

We use the following standard notation for sumsets, difference sets and sums of an element and a set

\[ A \pm B = \{a \pm b : a \in A, b \in B\}, \quad c \pm A = \{c \pm a : a \in A\}, \]

where \(c\) lies in some vector space and \(A,B\) are subsets of said vector space.

2.2 A Lebesgue measure argument

In this and the next subsection we give some conditions (Propositions 1, 2) on when translates of sets avoids sets each of which gives a corresponding result on polynomial approximation avoiding sets (Theorems 2, 3). These Propositions in turn are direct consequences of some elementary results on sumsets (or difference sets), Lemmas 1,2.

Lemma 1. Let \(A,K \subset \mathbb{R}^n\), where \(A\) is countable and \(K\) has \(n\)-dimensional Lebesgue measure 0. Then the Lebesgue measure of \(A \pm K\) is also 0.

\(^2\)in which case it automatically has an empty interior

\(^3\)where the difference set \(A - B = \{a - b : a \in A, b \in B\}\) here adheres to standard terminology in additive combinatorics.
Proof. Let $A = \{a_j\}_{j=1}^{\infty}$. Then

$$A \pm K = \bigcup_{j=1}^{\infty} (a_j \pm K),$$

and by the subadditivity and the translation invariance of the $n$-dimensional Lebesgue measure $\mu$ we have that

$$\mu(A \pm K) = \mu\left(\bigcup_{j=1}^{\infty} (a_j \pm K)\right) \leq \sum_{j=1}^{\infty} \mu(a_j \pm K) = \sum_{j=1}^{\infty} 0 = 0. \tag{3}$$

Our first Proposition which will give a simple proof of a special case of Theorem 1 now follows as a direct consequence of Lemma 1.

**Proposition 1.** Let $\epsilon > 0$, let $K \subset \mathbb{R}^n$ have $n$-dimensional Lebesgue measure 0 and let $A \subset \mathbb{R}^n$ be countable. Then for almost all $\xi \in \mathbb{R}^n$ such that $|\xi| < \epsilon$ then $(\xi + K) \cap A = \emptyset$.

**Proof.** By Lemma 1 the difference set $A - K$ has measure zero and thus also its subset $(A - K) \cap B^\epsilon(0)$ also has measure zero. This means that for almost all $|\xi| < \epsilon$ then $\xi \notin (A - K)$, which in other words means that $(\xi + K) \cap A = \emptyset$. \qed

Since $\mathbb{C}$ may be identified by $\mathbb{R}^2$ we are thus ready to prove the following special case of Theorem 1.

**Theorem 2.** Let $A \subset \mathbb{C}$ be any countable set, let $K \subset \mathbb{C}$ be a compact set with connected complement and with measure 0, and let $f$ be a continuous function on $K$. Then given any $\epsilon > 0$ there exists some polynomial $p$ such that $p(z) \notin A$ if $z \in K$ and such that

$$\max_{z \in K} |f(z) - p(z)| < \epsilon.$$

**Proof.** Since $K$ has measure zero it has no interior points and by Lavrentiev’s theorem there exist some polynomial $q$ such that

$$\max_{z \in K} |f(z) - q(z)| < \frac{\epsilon}{2}. \tag{1}$$

Since $K$ has measure zero and $q$ is a polynomial, then also $q(K)$ has measure zero. By Proposition 1 we have that there exists some $\xi \in \mathbb{C}$ with

$$|\xi| < \frac{\epsilon}{2}, \tag{2}$$

such that $(q(K) + \xi) \cap A = \emptyset$, which in other words means that $p(z) := q(z) + \xi$ does not attain values in $A$ for $z \in K$. Our conclusion follows from (1), (2) and the triangle inequality. \qed
2.3 A Hausdorff dimension argument

We now replace the Lebesgue measure with the Hausdorff and upper box counting dimension. The following holds true

Lemma 2. Suppose that $A, K \in \mathbb{R}^n$. Then

$$\dim_H(A \pm K) \leq \dim_H(A) + \dim_B(K)$$

where $\dim_H$ and $\dim_B$ denotes the Hausdorff dimension and upper box counting dimension\(^4\) respectively.

Proof. From [3, Product formula 7.3, p.99] it follows that

$$\dim_H(A \times K) \leq \dim_H(A) + \dim_B(K).$$

Now let $f : \mathbb{R}^{2n} \to \mathbb{R}^n$ be given by $f(x, y) = x \pm y$ when $x, y \in \mathbb{R}^n$. It is clear that $f(A \times K) = A \pm K$ and that $f$ is Lipschitz. Thus by [3, Corollary 2.4]

$$\dim_H(A \pm K) \leq \dim_H(A \times K).$$

Our conclusion follows from the inequalities (4) and (5). \(\square\)

Proposition 2. Let $\epsilon > 0$, let $K, A \in \mathbb{R}^n$ and let $\dim_B(K) + \dim_H(A) < n$ or $\dim_H(K) + \dim_B(A) < n$. Then for almost all $\xi \in \mathbb{R}^n$ such that $|\xi| < \epsilon$ then $(\xi + K) \cap A = \emptyset$.

Proof. By Lemma 2 the difference set $A - K$ has dimension strictly less than $n$ and thus its $n$-dimensional Lebesgue measure is zero. It follows that also its subset $(A - K) \cap B^\epsilon_\xi(0)$ has measure zero. This means that for almost all $|\xi| < \epsilon$ then $\xi \notin (A - K)$, which in other words means that $(\xi + K) \cap A = \emptyset$. \(\square\)

In a similar way to how Proposition 1 implies Theorem 2, Proposition 2 implies the following result.

Theorem 3. Let $K \subset \mathbb{C}$ be a compact set with connected complement and let $f$ be a continuous function on $K$ and let $A \subset \mathbb{C}$ be a set such that

$$\dim_H(A) + \dim_B(K) < 2,$$

\(\footnote{\text{which coincides with the box/Minkowski dimension when it is defined [3, Chapter 3.1]}}\)
or

\[ \dim_b(A) + \dim_H(K) < 2. \]

Then given any \( \epsilon > 0 \) there exists some polynomial \( p \) such that \( p(z) \notin A \) if \( z \in K \) and such that

\[ \max_{z \in K} |f(z) - p(z)| < \epsilon. \]  

Proof. Since \( K \) does not have full dimension in \( \mathbb{C} \) it has no interior points and by Lavrientiev’s theorem there exist some polynomial \( q \) such that

\[ \max_{z \in K} |f(z) - q(z)| < \frac{\epsilon}{2}. \]  

(6)

Since a polynomial \( q : \mathbb{C} \to \mathbb{C} \) is Lipschitz we have that \( \dim_H(q(K)) \leq \dim_H(K) \) and \( \dim_b(q(K)) \leq \dim_b(K) \). By Proposition 2 we have that there exists some \( \xi \in \mathbb{C} \) with

\[ |\xi| < \frac{\epsilon}{2}, \]  

(7)

such that \((q(K) + \xi) \cap A = \emptyset\), which in other words means that \( p(z) := q(z) + \xi \) does not attain values in \( A \) for \( z \in K \). Our conclusion follows from (6), (7) and the triangle inequality.  \( \square \)

2.4 A without interior points argument

It may be of some interest to prove a proposition on translates of sets avoiding sets corresponding to Theorem 1. We will do this and also show how this gives a new proof of Theorem 1. Indeed we may use the methods of [2], [10] to prove the following proposition which we may as well state and prove for a general Banach space \( \mathcal{B} \).

Proposition 3. Let \( \mathcal{B} \) be a Banach space, \( \epsilon > 0, K \subset \mathcal{B} \) be a compact set without interior points and let \( A \subset \mathcal{B} \) be countable. Then for any \( \epsilon > 0 \) there exists some \( \xi \in \mathcal{B} \) with \( \|\xi\| < \epsilon \) such that \((\xi + K) \cap A = \emptyset\).

We will be able to use Proposition 3 to give a new proof of Theorem 1.

Proof. (We follow the proof of [2, Lemma 2] with suitable modifications.) Let \( A = \{a_j\}_{j=1}^{\infty} \), let \( \xi_0 := 0 \) and let \( 0 < \epsilon_0 < \epsilon \). For \( j = 1, 2, \ldots \) there is, since \( K \) is a compact set without interior points some \( \xi_j \in \mathcal{B} \) such that

\[ d_j := d(K + \xi_j, a_j) > 0, \]  

(8)
and such that
\[ \| \xi_j - \xi_{j-1} \| < \frac{\epsilon_{j-1}}{2}, \]  
(9)
where \( \epsilon_j > 0 \) for \( j \geq 1 \) is defined recursively so that
\[ \epsilon_j < \min \left( \delta_j, \frac{\epsilon_{j-1}}{2} \right). \]  
(10)
By the inequalities (9), (10), and the triangle inequality we find if \( 0 \leq k < l \) that
\[ \| \xi_l - \xi_k \| \leq \| \xi_l - \xi_{k-1} \| + \cdots + \| \xi_{k+1} - \xi_k \| \]
\[ < \frac{\epsilon_{l-1}}{2} + \cdots + \frac{\epsilon_k}{2} < \sum_{j=1}^{l-k} \frac{\epsilon_k}{2^j} < \sum_{j=1}^{\infty} \frac{\epsilon_k}{2^j} = \epsilon_k. \]  
(11)
By (10) and (11) it follows that \( \{ \xi_j \}_{j=1}^\infty \) is a Cauchy-sequence in \( \mathcal{B} \) and converges to an element
\[ \xi := \lim_{j \to \infty} \xi_j. \]  
(12)
By the definition (12) the inequality (11) implies that
\[ \| \xi - \xi_j \| \leq \epsilon_j. \]  
(13)
For \( \xi \) defined by (12), the inequalities (8), (10), (13) and the triangle inequality gives us
\[ d(K + \xi, a_j) \geq d(K + \xi_j, a_j) - \| \xi - \xi_j \| \geq \delta_j - \epsilon_j > 0, \]  
(14)
for all \( a_j \in A \). The conclusion follows by (13) and (14), by recalling that \( \xi_0 = 0 \) and \( 0 < \epsilon_0 < \epsilon \). \( \square \)

In order to give another proof of Theorem 1 we need the following Lemma

**Lemma 3.** Let \( K \subset \mathbb{C} \) be a compact set without interior points and let \( p \) be a polynomial. Then \( p(K) \) is also a compact set without interior points.

We might as well prove the following more general lemma in order to obtain a higher dimensional analogue of Theorem 1 (by identifying \( \mathbb{C} \) with \( \mathbb{R}^2 \) Lemma 3 is a direct consequence of Lemma 4).

**Lemma 4.** Let \( K \subset \mathbb{R}^n \) be a compact set without interior points and let \( p \) be a polynomial mapping \( p : \mathbb{R}^n \to \mathbb{R}^n \). Then \( p(K) \) is also a compact set without interior points.
Proof. Since \( p \) is continuous and \( K \) is compact it is clear that \( p(K) \) is compact. It remains to show that \( p(K) \) is nowhere dense, which since \( p(K) \) is compact is equivalent to the fact that \( p(K) \) has no interior points. The result is true if \( p(z) = Az + b \), where \( b \in \mathbb{R}^n \) and \( A \) is a non-invertible \( n \times n \) matrix. In such a case \( p \) maps \( \mathbb{R}^n \) onto a set of measure zero by Sard's theorem, Lemma 4 holds also for each \( \mathbb{R}^n \) of dimension at most \( n - 1 \). Since \( E \) does not have full dimension, also \( p(K) \subseteq E \) does not have full dimension and is thus nowhere dense. Now let \( p \) be a polynomial mapping \( p : \mathbb{R}^n \to \mathbb{R}^n \) that is not of such type. The mapping \( p \) is locally invertible whenever the Jacobian determinant \( \det(J_p(z)) \) of \( p(z) \) is non-zero. Let \( A = \{ z \in K : \det(J_p(z)) = 0 \} \) and \( p(A) = B \). It is clear that \( B \) is a closed set and since \( p(z) \) is not of the form \( Az + b \) its Jacobian determinant is a non-zero polynomial and its zero set \( C \subset \mathbb{R}^n \) will be a real variety of dimension at most \( n - 1 \). Since \( p \) is a polynomial mapping which is Lipschitz, also \( p(C) \) and \( B = p(A) \subset p(C) \) will have Hausdorff dimension at most \( n - 1 \) and thus be nowhere dense. Let us now assume that \( p(K) \) is not nowhere dense, i.e. there exist some open set \( D \subset p(K) \). Since \( B \) is a nowhere dense set, we may choose some \( w \in D \setminus B \) (we remark that since \( D \) is open and \( B \) closed, then \( D \setminus B \) is open). Let us now consider the equation \( p(z) = w \). If the equation has infinitely many roots for \( z \in K \), it must have some limit point \( z_0 \in K \) (since \( K \) is compact) such that \( p(z_0) = w \) by continuity. At such a point \( z_0 \) the function is not locally invertible and thus \( \det(J_p(z_0)) = 0 \) which implies that \( w \in B \) contradicting our assumption that \( w \in D \setminus B \). Thus we know that the equation \( p(z) = w \) has \( m \) distinct roots\(^5\) for each \( j = 1, \ldots, m \) and that \( \det(J_p(z_j)) \neq 0 \) for \( j = 1, \ldots, m \). By the inverse function theorem the polynomial mapping \( p \) gives homeomorphisms \( f_j \) between a small neighborhood of \( z_j \) for each \( j = 1, \ldots, m \) and a small neighborhood of \( w \). Let \( \varepsilon > 0 \) be sufficiently small such that \( \Theta = B_\varepsilon(w) \) is a subset of \( D \setminus B \) as well as all these small neighborhoods of \( w \). Let \( E_j = f_j^{-1}(\Theta) \). It is clear that

\[
\Theta \cap p(K) = \bigcup_{j=1}^m f_j(E_j \cap K)
\]

is a finite union of nowhere dense sets (“nowhere dense” is a topological property that is preserved by homeomorphisms) and thus nowhere dense\(^6\). This contradicts our assumption that \( \Theta \subset D \subset p(K) \), so \( p(K) \) is nowhere dense. \( \square \)

Remark 1. By replacing our argument for why the set of critical values of \( p \) has measure zero by Sard’s theorem, Lemma 4 hold also for \('C^1'\) functions.

\(^5\)where by Bezout’s theorem \( m \) is bounded from above by \( d_1 \cdots d_n \), where \( d_j \) is the degree of the polynomial \( p_j \), where \( p = (p_1, \ldots, p_n) \)

\(^6\)For a proof see Proposition 7.1 in https://www.ucl.ac.uk/~ucahad0/3103_handout_7.pdf
Proof of Theorem 1. Since $K \subset \mathbb{C}$ is a compact set with connected complement without interior points, Lavrentiev’s theorem gives us a polynomial $q$ such that
\[
\max_{z \in K} |f(z) - q(z)| < \frac{\varepsilon}{2}.
\] (15)

By Lemma 3 we have that $q(K)$ is a compact set without interior points. By Proposition 3 (with $\mathcal{B} = \mathbb{C}$), we have that there exists some $\xi \in \mathbb{C}$ with
\[
|\xi| < \frac{\varepsilon}{2},
\] (16)
such that $(q(K) + \xi) \cap A = \emptyset$, which in other words means that $p(z) := q(z) + \xi$ does not attain values in $A$ for $z \in K$. Our conclusion follows from (15), (16) and the triangle inequality.

3 Higher dimensional results

3.1 Finite dimensional results

One advantage with our approach is that it readily generalizes to several variables, by using higher dimensional analogues of the one variable approximation theorems (Weierstrass theorem, Lavrentiev’s theorem). By the $n$-dimensional version of the Weierstrass approximation theorem\(^7\) we may prove

**Theorem 4.** Let $A \subset \mathbb{R}^n$ be any countable set, let $K \subset \mathbb{R}^n$ be a compact set without interior points, and let $f : K \to \mathbb{R}^n$ be a continuous function. Then given any $\varepsilon > 0$ there exists some polynomial mapping $p$ such that $p(x) \not\in A$ if $x \in K$ and such that
\[
\max_{x \in K} \|f(x) - p(x)\| < \varepsilon.
\]

We remark that we will use Proposition 3 in the proof. If we instead use Proposition 1 (which have a somewhat simpler proof) we obtain the slightly weaker version where we have to assume that $K$ has Lebesgue measure zero.

**Proof.** (We follow the proof of Theorem 2) By the multivariate Weierstrass theorem we may find a polynomial mapping $q : \mathbb{R}^n \to \mathbb{R}^n$ such that
\[
\max_{x \in K} \|q(x) - f(x)\| < \frac{\varepsilon}{2}.
\] (17)

\(^7\)which is an easy consequence of the Stone Weierstrass theorem, see e.g. [9]
Since \( K \subset \mathbb{R}^n \) is a compact set without interior points and \( q : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a polynomial map, then by Lemma 4 also \( q(K) \) is a compact set without interior points. By Proposition 3 we have that there exists some \( \xi \in \mathbb{R}^n \) with

\[
\|\xi\| < \frac{\varepsilon}{2}, \tag{18}
\]

such that \((q(K) + \xi) \cap A = \emptyset\), which in other words means that \( p(x) := q(x) + \xi \) does not attain values in \( A \) for \( x \in K \). Our conclusion follows from (17), (18) and the triangle inequality. \( \square \)

**Theorem 5.** Let \( K \subset \mathbb{R}^n \) and let \( A \subset \mathbb{R}^m \). Furthermore assume that

\[
\overline{\dim}_B(K) + \overline{\dim}_H(A) < m, \tag{19}
\]

or

\[
\dim_H(K) + \overline{\dim}_B(A) < m. \tag{20}
\]

Then given any continuous function \( f : K \rightarrow \mathbb{R}^m \) and any \( \varepsilon > 0 \) there exists some polynomial mapping \( p \) such that \( p(x) \notin A \) if \( x \in K \) and such that

\[
\max_{x \in K} \|f(x) - p(x)\| < \varepsilon.
\]

**Proof.** (We follow the proof of Theorem 4) By the multivariate Weierstrass theorem we may find a polynomial mapping \( q : \mathbb{R}^n \rightarrow \mathbb{R}^m \) such that

\[
\max_{x \in K} \|q(x) - f(x)\| < \frac{\varepsilon}{2}, \tag{21}
\]

Since a polynomial map \( q : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is Lipschitz we have that \( \dim_H(q(K)) \leq \dim_H(K) \) and \( \dim_B(q(K)) \leq \dim_B(K) \). Thus by Proposition 2 we have that for some \( \xi \in \mathbb{R}^m \) with

\[
\|\xi\| < \frac{\varepsilon}{2}, \tag{22}
\]

that \((q(K) + \xi) \cap A = \emptyset\), which in other words means that \( p(x) := q(x) + \xi \) does not attain values in \( A \) for \( x \in K \). Our conclusion follows from (21), (22) and the triangle inequality. \( \square \)
In several complex variables Lavrentiev’s theorem may be generalised to a result of Harvey-Wells\(^8\) [4] that if \(E\) is a totally real\(^9\) submanifold of class \(C^1\) in an open set in \(\mathbb{C}^n\) then for any compact \(K \subset E\) where \(K\) is polynomially convex then any function \(f\) continuous on \(K\) may be uniformly approximated by polynomials on \(K\). We remark that just like the classical Weierstrass theorem is a special case of the Lavrentiev theorem (since an interval \([a, b] \subset \mathbb{R}\) is a compact set with connected complement and without interior points in \(\mathbb{C}\)), the multivariate Weierstrass theorem is a special case of the Harvey-Wells theorem (since \(\mathbb{R}^n\) is a totally real manifold in \(\mathbb{C}^n\) and any compact set \(K \subset \mathbb{R}^n\) is polynomially convex).

**Theorem 6.** Let \(E \subset \mathbb{C}^n\) be a totally real manifold of class \(C^1\) and let \(K \subset E\) be a compact set that is polynomially convex. Then for any continuous function \(f : K \rightarrow \mathbb{C}^m\) and any set \(A \subset \mathbb{C}^m\) with \(\text{dim}_B(K) + \text{dim}_h(A) < 2m\) or \(\text{dim}_B(K) + \overline{\text{dim}}_g(A) < 2m\) there exists some polynomial mapping \(p\) such that

\[
\max_{z \in K} |f(z) - p(z)| < \epsilon.
\]

and such that \(p(z) \notin A\) if \(z \in K\).

**Proof.** (We follow the proof of Theorem 5) By the Harvey-Wells theorem we may find a polynomial map \(q : \mathbb{C}^n \rightarrow \mathbb{C}^m\) such that

\[
\max_{x \in K} \|q(x) - f(x)\| < \frac{\epsilon}{2}. \tag{23}
\]

Thus by identifying \(\mathbb{C}^m\) with \(\mathbb{R}^{2m}\) and by Proposition 2 we have that for some \(\xi \in \mathbb{C}^m\) with

\[
\|\xi\| < \frac{\epsilon}{2} \tag{24}
\]

that \((q(K) + \xi) \cap A = \emptyset\), which in other words means that \(p(x) := q(x) + \xi\) does not attain values in \(A\) for \(x \in K\). Our conclusion follows from (23), (24) and the triangle inequality. \(\square\)

Since \(E \subset \mathbb{C}^n\) is a totally real manifold of class \(C^1\) in particular means that \(\text{dim}_h(E) = \text{dim}_g(E) \leq n\), a sufficient condition in Theorem 6 is that \(\text{dim}_h A < 2m - n\). We obtain the following Corollary.

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\(^8\)This is a sharper version of a result of Hörmander-Wermer [5], see discussion in [8, section 8.].

\(^9\)For the definition see [8, p.115].
Corollary 1. Let $E \subseteq \mathbb{C}^n$ be a totally real manifold of class $\mathcal{C}^1$ and let $K \subseteq E$ be a compact set that is polynomially convex. Then for any continuous function $f : K \to \mathbb{C}^m$ and any set $A \subseteq \mathbb{C}^m$ with $\dim H(A) < 2m - n$ there exists some polynomial function $p$ such that
\[
\max_{z \in K} |f(z) - p(z)| < \varepsilon.
\]
and such that $p(z) \notin A$ if $z \in K$.

3.2 Infinite dimensional results

It is well known that Weierstrass theorem on approximation of continuous functions on compact sets by polynomials generalizes from $\mathbb{R}^n$ to infinite dimensional real Banach spaces \[6\]. Since compact sets in infinite-dimensional Banach spaces are always without interior points the analogue of Lemma 4 become trivial. We are thus ready to prove our main approximation theorem on Banach spaces which is a natural analogue of Theorem 1 and Theorem 6.

Theorem 7. Let $\mathcal{B}$ be an infinite dimensional real Banach space, let $A \subseteq \mathcal{B}$ be a countable set, let $K \subseteq \mathcal{B}$ be a compact set and let $f : K \to \mathcal{B}$ be a continuous function. Then for any $\varepsilon > 0$ there exists some polynomial mapping $p : K \to \mathcal{B}$ such that
\[
\max_{x \in K} \|f(x) - p(x)\| < \varepsilon,
\]
and such that $p(x) \notin A$ if $x \in K$.

Proof. (We follow the proof of Theorem 5) By the Weierstrass approximation theorem for real Banach spaces \[6, Theorem 2.5\] we may find a continuous polynomial map $q : \mathcal{B} \to \mathcal{B}$ such that
\[
\max_{x \in K} \|q(x) - f(x)\| < \frac{\varepsilon}{2}. \tag{25}
\]
Since $K$ is compact and $q$ is continuous, also $q(K)$ is compact. Since $\mathcal{B}$ is infinite dimensional and any compact set in an infinite dimensional Banach space is a compact set without interior points, it follows that $q(K)$ is a compact set without interior points. By Proposition 3 we have that for some $\xi \in \mathcal{B}$ with
\[
\|\xi\| < \frac{\varepsilon}{2} \tag{26}
\]
that $(q(K) + \xi) \cap A = \emptyset$, which in other words means that $p(x) := q(x) + \xi$ does not attain values in $A$ for $x \in K$. Our conclusion follows from (25), (26) and the triangle inequality. \hfill $\square$
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