An Accurate, Scalable and Verifiable Protocol for Federated Differentially Private Averaging

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Abstract
Learning from data owned by several parties, as in federated learning, raises challenges regarding the privacy guarantees provided to participants and the correctness of the computation in the presence of malicious parties. We tackle these challenges in the context of distributed averaging, an essential building block of federated learning algorithms. Our first contribution is a scalable protocol in which participants exchange correlated Gaussian noise along the edges of a network graph, complemented by independent noise added by each party. We analyze the differential privacy guarantees of our protocol and the impact of the graph topology under colluding malicious parties, showing that we can nearly match the utility of the trusted curator model even when each honest party communicates with only a logarithmic number of other parties chosen at random. This is in contrast with protocols in the local model of privacy (with lower utility) or based on secure aggregation (where all pairs of users need to exchange messages). Our second contribution enables users to prove the correctness of their computations without compromising the efficiency and privacy guarantees of the protocol. Our verification protocol relies on standard cryptographic primitives like commitment schemes and zero knowledge proofs.

1. Introduction
Individuals are producing ever growing amounts of personal data, which in turn fuel innovative services based on machine learning (ML). The classic centralized paradigm consists in collecting, storing and analyzing this data on a (supposedly trusted) central server or in the cloud, which poses well documented privacy risks for the users. With the increase of public awareness and regulations, we are witnessing a shift towards a more decentralized paradigm where personal data remains on each user’s device, see the recent trend of federated learning (Kairouz et al., 2019). In this setting, users do not trust the central server (if any), or each other, which introduces new issues regarding privacy and security. First, the information shared by the participants during the decentralized training protocol can reveal a lot about their private data (see Melis et al., 2019; Nasr et al., 2019, for inference attacks on federated learning). Formal guarantees such as differential privacy (DP) (Dwork, 2006) are needed to provably mitigate this and convince users to participate. Second, malicious users may send incorrect results to bias the learned model in arbitrary ways (Hayes & Ohrimenko, 2018; Bhagoji et al., 2019; Bagdasaryan et al., 2020). Ensuring the correctness of the computation is crucial to persuade service providers to move to a more decentralized and privacy-friendly setting.

In this work, we tackle these challenges in the context of private distributed averaging. In this canonical problem, the objective is to privately compute an estimate of the average of values owned by many users who do not want to disclose them. Beyond simple data analytics, distributed averaging is of high relevance to modern ML. Indeed, it is the essential primitive used to aggregate user updates in gradient-based distributed and federated learning algorithms (Lin et al., 2020; Stich, 2019; McMahan et al., 2017; Jayaraman et al., 2018; Agarwal et al., 2018). It also allows to train ML models whose sufficient statistics are averages (e.g., linear models and decision trees). Distributed averaging with differential privacy guarantees has thus attracted a lot of interest in recent years. In the strong model of local differential privacy (LDP) (Kasiviswanathan et al., 2008; Duchi et al., 2013; Kairouz et al., 2015; 2016b;a; Chen et al., 2020), each user randomizes its input locally before sending it to an untrusted aggregator. Unfortunately, the best possible error for the estimated average with \( n \) users is a factor of \( O(\sqrt{n}) \) larger than in the centralized model of DP where a trusted curator aggregates data in the clear and perturbs the output (Chan et al., 2012a). To fill this gap, some work has explored relaxations of LDP that make it possible to match the utility of the trusted curator model. This is achieved through the use of cryptographic primitives such as secure aggregation (Dwork et al., 2006; Chan et al., 2012b; Shi...
et al., 2011; Bonawitz et al., 2017; Jayaraman et al., 2018) and secure shuffling (Erlingsson et al., 2019; Cheu et al., 2019; Balle et al., 2019a; Ghazi et al., 2020). Many of these solutions however assume that all users truthfully follow the protocol (they are honest-but-curious). Furthermore, their practical implementation poses important challenges for large number of parties: for instance, the state-of-the-art secure aggregation approach of Bonawitz et al. (2017) requires all $n^2$ pairs of users to exchange messages.

In this context, our contribution is twofold. First, we propose a novel decentralized differentially private averaging protocol which achieves nearly the same order of utility as the trusted curator model while scaling to a large number of users. Our approach, called GOPA, relies on users exchanging (directly or through a server) some correlated Gaussian noise terms along the edges of a network graph so as to mask their private values without affecting the global average. This ultimately canceling noise is complemented by the addition of independent (non-canceling) Gaussian noise by each user. We show that the privacy guarantees of GOPA against the adversary (colluding malicious users) depend on the branching factor of a spanning tree of the subgraph of honest-but-curious users. Remarkably, we establish that our approach can achieve nearly the same privacy-utility trade-off as a trusted curator who would average the values of honest users, provided that the graph of honest-but-curious users is connected and the pairwise-correlated noise variance is large enough. In particular, for $n_H$ honest-but-curious users, the variance of the estimated average is only $n_H/(n_H - 1)$ times larger than with a trusted curator, a fact which goes to 1 as $n_H$ grows. We further show that if the graph is well-connected, the pairwise-correlated noise variance can be significantly reduced. To ensure both scalability and robustness to malicious users, we propose a randomized procedure in which each user communicates with only a logarithmic number of other users while still matching the privacy-utility trade-off of the trusted curator. We prove these guarantees by adapting results from random graph theory on embedding spanning trees in random graphs. Our protocol is also robust to a fraction of users dropping out.

Our second contribution is a procedure to make GOPA verifiable by untrusted external parties, i.e., to provide users with the means of proving the correctness of their computations without compromising the efficiency or the privacy guarantees of the protocol. Our construction relies on commitment schemes and zero knowledge proofs (ZKPs), which are very popular in auditable electronic payment systems and cryptocurrencies. These cryptographic primitives, originally formalized by (Goldwasser et al., 1989), scale well both in communication and computational requirements and are perfectly suitable in our untrusted decentralized setting. We use classic ZKPs to design a procedure for the generation of noise with verifiable distribution, and ultimately to prove the integrity of the final computation (or detect malicious users who did not follow the protocol). Crucially, the privacy guarantees of the protocol are not compromised by this procedure (not even relaxed through a cryptographic hardness assumption), while the integrity of the computation relies on a standard discrete logarithm assumption. In the end, we argue that our protocol offers correctness guarantees that are essentially equivalent to the case where a trusted curator would itself hold the private data of users.

The paper is organized as follows. Section 2 introduces the problem setting. The GOPA protocol is introduced in Section 3 and we analyze its differential privacy guarantees in Section 4. We present our procedure to ensure correctness against malicious behavior in Section 5, and summarize computational and communication costs in Section 6. We discuss the related work in more details in Section 7. Finally, we present some experimental results in Section 8 and conclude with future lines of research in Section 9.

2. Notations and Setting

We consider a set $U = \{1, \ldots, n\}$ of $n \geq 3$ users (parties). Each user $u \in U$ holds a private value $X_u$, which can be thought of as being computed from the private dataset of user $u$. We assume that $X_u$ lies in a bounded interval of $\mathbb{R}$ (without loss of generality, we assume $X_u \in [0, 1]$). The extension to the vector case is straightforward. We denote by $X$ the column vector $X = [X_1, \ldots, X_n]^{\top} \in [0, 1]^n$ of private values. Unless otherwise noted, all vectors are column vectors. Users communicate over a network represented by a connected undirected graph $G = (U, E)$, where $\{u, v\} \in E$ indicates that users $u$ and $v$ are neighbors in $G$ and can exchange secure messages. For a given user $u$, we denote by $N(u) = \{v : \{u, v\} \in E\}$ the set of its neighbors. We note that in settings where users can only communicate with a server, the latter can act as a relay that forwards (encrypted and authenticated) messages between users, as done in secure aggregation (Bonawitz et al., 2017).

The users aim to collaboratively estimate the average value $X_{avg} = \frac{1}{n} \sum_{i=1}^{n} X_i$ without revealing their individual private values. Such a protocol can be readily used to privately execute distributed ML algorithms that interact with data through averages over values computed locally by the participants, but do not actually need to see the individual values.

We give two concrete examples below.

**Example 1** (Linear regression). Let $\lambda \geq 0$ be a public parameter. Each user $u$ holds a private feature vector $\phi_u = [\phi_u^1, \ldots, \phi_u^d] \in \mathbb{R}^d$ and a private label $y_u \in \mathbb{R}$. The goal is to solve a ridge regression task, i.e., find $\theta^* \in \arg\min_{\theta} \frac{1}{n} \sum_{u \in U} (\phi_u^\top \theta - y_u)^2 + \lambda \|	heta\|^2$. $\theta^*$ can be computed in closed form from the quantities $\frac{1}{n} \sum_{u \in U} \phi_u^\top \phi_u$ and $\frac{1}{n} \sum_{u \in U} \phi_u^\top \phi_j$ for all $i, j \in \{1, \ldots, d\}$.

**Example 2** (Federated ML). In federated learning (Kairouz et al., 2019; Balle et al., 2019a; Ghazi et al., 2020). Many of these solutions however assume that all users truthfully follow the protocol (they are honest-but-curious). Furthermore, their practical implementation poses important challenges for large number of parties: for instance, the state-of-the-art secure aggregation approach of Bonawitz et al. (2017) requires all $n^2$ pairs of users to exchange messages.

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Our second contribution is a procedure to make GOPA verifiable by untrusted external parties, i.e., to provide users with the means of proving the correctness of their computations without compromising the efficiency or the privacy guarantees of the protocol. Our construction relies on commitment schemes and zero knowledge proofs (ZKPs), which are very popular in auditable electronic payment systems and cryptocurrencies. These cryptographic primitives, originally formalized by (Goldwasser et al., 1989), scale well both in communication and computational requirements and are perfectly suitable in our untrusted decentralized setting. We use classic ZKPs to design a procedure for the generation of noise with verifiable distribution, and ultimately to prove the integrity of the final computation (or detect malicious users who did not follow the protocol). Crucially, the privacy guarantees of the protocol are not compromised by this procedure (not even relaxed through a cryptographic hardness assumption), while the integrity of the computation relies on a standard discrete logarithm assumption. In the end, we argue that our protocol offers correctness guarantees that are essentially equivalent to the case where a trusted curator would itself hold the private data of users.

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et al., 2019) and distributed empirical risk minimization, each user $u$ holds a private dataset $D_u$ and the goal is to find $\theta^*$ such that $\theta^* = \arg \min_{\theta} \frac{1}{n} \sum_{u \in U} f(\theta; D_u)$ where $f$ is some loss function. Popular algorithms (Lin et al., 2020; Stich, 2019; McMahan et al., 2017; Jayaraman et al., 2018; Agarwal et al., 2018) all follow the same high-level procedure: at round $t$, each user $u$ computes a local update $\theta_u^t$ based on $D_u$ and the current global model $\theta^{t-1}$, and the updated global model is computed as $\theta^t = \frac{1}{n} \sum_u \theta_u^t$.

**Threat model.** We consider two commonly adopted adversary models formalized by Goldreich (1998) and used in the design of many secure protocols. A **honest-but-curious** (honest for short) user will follow the protocol specification, but may use all the information obtained during the execution to infer information about other users. A honest user may accidentally drop out at any point of the execution (in a way that is independent of the private values $X$). On the other hand, a **malicious user** may deviate from the protocol execution (e.g., sending incorrect values or dropping out on purpose). Malicious users can collude, and thus will be seen as a single malicious party (the adversary) who has access to all information collected by malicious users. Our privacy guarantees will hold under the assumption that honest users communicate through secure channels, while the correctness of our protocol will be guaranteed under some form of the Discrete Logarithm Assumption (DLA), a standard assumption in cryptography (see Appendix B for details).

For a given execution of the protocol, we denote by $U^O$ the set of the users who remained online until the end (i.e., did not drop out). Users in $U^O$ are either honest or malicious: we denote by $U^H \subseteq U^O$ those who are honest, by $n_H = |U^H|$ their number and by $\rho = n_H/n$ their proportion with respect to the total number of users. We also denote by $G^H = (U^H, E^H)$ the subgraph of $G$ induced by the set of honest users $U^H$, i.e., $E^H = \{(u, v) \in E : u, v \in U^H\}$. The properties of $G$ and $G^H$ will play a key role in the privacy and scalability guarantees of our protocol.

**Privacy definition.** Our goal is to design a protocol that satisfies differential privacy (DP) (Dwork, 2006), which has become a gold standard in private information release.

**Definition 1** (Differential privacy). Let $\varepsilon > 0, \delta \geq 0$. A (randomized) protocol $A$ is $(\varepsilon, \delta)$-differentially private if for all neighboring datasets $X = [X_1, \ldots, X_n]$ and $X' = [X'_1, \ldots, X'_n]$ differing only in a single data point, and for all sets of possible outputs $O$, we have:

$$Pr(A(X) \in O) \leq e^\varepsilon Pr(A(X') \in O) + \delta. \quad (1)$$

3. Proposed Protocol

In this section we describe our protocol called GOPA (Gossip noise for Private Averaging). The high-level idea of GOPA is to have each user $u$ mask its private value by adding two different types of noise. The first is a sum of pairwise-correlated noise terms $\Delta_{u,v}$ over the set of neighbors $v \in N(u)$ such that each $\Delta_{u,v}$ cancels out with the $\Delta_{v,u}$ of user $v$ in the final result. The second type of noise is an independent term $\eta_u$ which does not cancel out. At the end of the protocol, each user has generated a noisy version $\hat{X}_u$ of its private value $X_u$, which takes the form:

$$\hat{X}_u = X_u + \sum_{v \in N(u)} \Delta_{u,v} + \eta_u. \quad (2)$$

Algorithm 1 presents the detailed steps. Neighboring nodes $\{u, v\} \in E$ contact each other to draw a real number from the Gaussian distribution $\mathcal{N}(0, \sigma_n^2)$, that $u$ adds to its private value and $v$ subtracts. Intuitively, each user thereby distributes noise masking its private value across its neighbors so that even if some of them are malicious and collude, the remaining noise values will be enough to provide the desired privacy guarantees. The idea is reminiscent of uniformly random pairwise masks in secure aggregation (Bonawitz et al., 2017) but we use Gaussian noise and restrict exchanges to the edges of the graph instead of requiring messages between all pairs of users. As in gossip algorithms (Boyd et al., 2006), the pairwise exchanges can be performed asynchronously and in parallel. Additionally, every user $u \in U$ adds an independent noise term $\eta_u \sim \mathcal{N}(0, \sigma_n^2)$ to its private value. This noise will ensure that the final estimate of the average satisfies differential privacy (see Section 4). $\sigma_n^2$ and $\sigma_n^2$ are public parameters of the protocol.

**Utility of GOPA.** The protocol generates a set of noisy values $\hat{X} = [\hat{X}_1, \ldots, \hat{X}_n]$ which are then publicly released. They can be sent to an untrusted aggregator, or averaged in a decentralized way via gossiping (Boyd et al., 2006). In any case, the expected average is given by $\hat{X} = \frac{1}{n} \sum_{u \in U} \hat{X}_u = X_{avg} + \frac{1}{n} \sum_{u \in U} \eta_u$, which has expected value $X_{avg}$ and variance $\sigma_n^2/n$. Recall that the local model of DP, where each user releases a locally perturbed input without communicating with other users, would require $\sigma_n^2 = O(1)$. In contrast, we would like the total amount of independent noise to be of order $O(1/n_H)$ as needed to protect the average of honest users with the standard Gaussian mechanism in the trusted curator model of DP (Dwork & Roth, 2014). We will show in Section 4 that
we can achieve this privacy-utility trade-off by choosing an appropriate variance $\sigma^2_\Delta$ for our pairwise noise terms.

**Dealing with dropout.** A user $u \notin U^O$ who drops out during the execution of the protocol does not actually publish any noisy value (i.e., $X_u$ is empty). The estimated average is thus computed by averaging only over the noisy values of users in $U^O$. Additionally, any residual noise term that a user $u \notin U^O$ may have exchanged with a user $v \in U^O$ before dropping out can be “rolled back” by having $v$ reveal $\Delta_{u,v}$ so it can be subtracted from the result (we will ensure this does not threaten privacy by having sufficiently many neighbors, see Section 4.2). We can thus obtain an estimate of $\sum_{u \in U^O} X_u$ with variance $\sigma^2_n/[U^O]$. Note that even if some residual noise terms are not rolled back, e.g. to avoid extra communication, the estimate remains unbiased (with a larger variance that depends on $\sigma^2_\Delta$). This is a rather unique feature of GOPA which comes from the use of Gaussian noise rather than the uniformly random noise used in secure aggregation (Bonawitz et al., 2017). We discuss strategies to handle drop out in more details in Appendix C.2.

4. Privacy Guarantees

Our goal is to prove differential privacy guarantees for GOPA. We first define the knowledge acquired by the adversary (colluding malicious users) during a given execution of the protocol. It consists of the following: (i) the noisy value $X_u$ of all users $u \in U^O$ who did not drop out, (ii) the private value $X_u$ and the noise $\eta_u$ of the malicious users, and (iii) all $\Delta_{u,v}$’s for which $u$ or $v$ is malicious. We also assume that the adversary knows the full network graph $G$ and all the pairwise noise terms exchanged by dropped out users (since they can be rolled back, as explained in Section 3). The only unknowns are thus the private value $X_u$ and independent noise $\eta_u$ of each honest user $u \in U^H$, as well as the $\Delta_{u,v}$’s exchanged between honest users $\{u,v\} \in E^H$. Letting $N^H(u) = \{v : \{u,v\} \in E^H\}$, from the above knowledge the adversary can subtract $\sum_{v \in N^H(u) \setminus N^H(u)} \Delta_{u,v}$ from $X_u$ to obtain $\hat{X}_u = X_u + \sum_{u \in N^H(u)} \Delta_{u,v} + \eta_u$ for every honest $u \in U^H$. The view of the adversary thus can be summarized by the vector $\hat{X}^H = (\hat{X}_u)_{u \in U^H}$ and the correlation between its elements.

Now, adapting Definition 1 to our setting, for any input $X$ and any possible outcome $\hat{X}$, we need to compare the probability of the outcome being equal to $\hat{X}$ when a (non-malicious) user $v_1 \in U$ participates in the computation with private value $X^{A}_{v_1}$ to the probability of obtaining the same outcome when the value of $v_1$ is exchanged with an arbitrary value $X^{B}_{v_1} \in [0,1]$. Since honest users drop out independently of $\hat{X}$ and do not reveal anything about their private value when they drop out, in our analysis we will fix an execution of the protocol where some set $U^H$ of $n_H$ honest users have remained online until the end of the protocol. For notational simplicity, we denote by $X^A$ the vector of private values $(X^A_u)_{u \in U^H}$ of these honest users in which a user $v_1$ has value $X^{A}_{v_1}$, and by $X^B$ the vector where $v_1$ has value $X^{B}_{v_1}$. $X^A$ and $X^B$ differ in only in the $v_1$-th coordinate, and their maximum difference is 1.

In this section, we show that GOPA can match the privacy-utility trade-off of the trusted curator setting as long as $G^H$ (the subgraph induced by $U^H$) is connected and the variance $\sigma^2_\Delta$ for the pairwise (cancelling) noise is large enough (this depends on the topology of $G^H$).

4.1. Worst and Best Case Graphs

Our first result gives a differential privacy guarantee for the “worst-case” topology.

**Theorem 1** (Privacy guarantee for worst-case graph). Let $X^A$ and $X^B$ be two databases (i.e., graphs with private values at the vertices) which differ only in the value of one user. Let $\varepsilon, \delta \in (0,1)$ and $\theta = \frac{1}{\sigma^2_n n_H} + \frac{n_H}{\sigma^2_\Delta}$. If $G^H$ is connected and the following inequalities hold:

$$\varepsilon \geq \theta/2 + \theta^{1/2},$$

$$\left(\varepsilon - \theta/2\right)^2 \geq 2 \log(2/\delta \sqrt{2\pi}) \theta,$$

then GOPA is $((\varepsilon, \delta), \varepsilon)$-differentially private, i.e., $P(X \mid X^A) \leq e^\varepsilon P(X \mid X^B) + \delta$.

Crucially, Theorem 1 holds as soon as the subgraph $G^H$ of honest users who did not drop out is connected. In order to get a constant $\varepsilon$, inspecting the term $\theta$ shows that the variance $\sigma^2_\Delta$ of the independent noise must be of order $1/n_H$. This is in a sense optimal as it corresponds to the amount of noise required when averaging $n_H$ values in the trusted curator model. It also matches the amount of noise needed when using secure aggregation with differential privacy in the presence of colluding users, where honest users need to add $n/n_H$ more noise to compensate for collusion (Shi et al., 2011). In order to match the privacy-utility trade-off of the trusted curator setting, further inspection of the equations in Theorem 1 shows that the variance $\sigma^2_\Delta$ of the pairwise noise must be large enough. How large it must be actually depends on the structure of the graph $G^H$. Theorem 1 describes the worst case, which is attained when every node has as few neighbors as possible while still being connected, i.e., when $G^H$ is a path. In this case, Theorem 1 shows that the variance $\sigma^2_\Delta$ needs to be of order $n_H$. Recall that this noise cancels out, so it does not impact the utility of the final output. It only has a minor effect on the communication cost (the representation space of reals needs to be large enough to avoid overflows with high probability), and on the variance.

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1Note that if $G^H$ is not connected, we obtain a similar but weaker result for each connected component separately ($n_H$ is replaced by the size of the connected component).
of the final result if some residual noise terms of dropout
users are not rolled back (see Section 3).

Intuitively, in graphs with higher connectivity, it should how-
ever be possible for $\sigma_\Delta$ to be smaller than $O(n_H)$. Before
considering more practical topologies, we illustrate this on
the case of the complete graph, which corresponds to the
best scenario in terms of connectivity. Theorem 2 shows
that for a fully connected $G^H$, $\sigma_\Delta$ can be of order $1/n_H$, which
is a quadratic reduction compared to the path case.

**Theorem 2 (Privacy guarantee for complete graph).** Let $\varepsilon, \delta \in (0, 1)$ and let $G^H$ be the complete graph. If $\varepsilon, \delta, \sigma_\eta$ and $\sigma_\Delta$ satisfy the inequalities (3) and (4) in Theorem 1 with
$$\theta = \frac{1}{n_H \sigma_\eta^2} + \frac{1}{\sigma_\Delta^2} \left( \frac{1}{(k-1) \rho / 3} - 1 + \frac{12 + 6 \log(n_H)}{n_H} \right)$$
then GOPA is $(\varepsilon, \delta)$-DP.

### 4.2. Practical Random Graphs

Our results so far are not fully satisfactory from the practical
perspective, when the number of users $n$ is large. Theorem 1
assumes that we have a procedure to generate a graph $G$ such
that $G^H$ is guaranteed to be connected (despite dropouts and
malicious users), and requires a large $\sigma_\Delta$ of $O(n_H)$. Theorem 2
works with a complete graph, which ensures connectivity and allows smaller $O(1/n_H)$ variance but is
intractable as all $n^2$ pairs of users need to exchange noise.

To overcome these limitations, we propose a simple randomized
procedure to construct a sparse network graph $G$ such that
$G^H$ will be well-connected with high probability, and
prove a DP guarantee for the whole process (random graph
generation followed by GOPA), under much less noise than the
worst-case. The idea is to make each (honest) user select
$k$ other users uniformly at random among all users. Then,
the edge $\{u, v\} \in E$ is created if $u$ selected $v$ or $v$ selected
$u$ (or both). Such graphs are known as random $k$-out or random
$k$-orientable graphs (Bollobás, 2001; Fenner & Frieze, 1982). They have very good connectivity properties (Fenner & Frieze, 1982; Yağan & Makowski, 2013) and are used in creating secure communication channels in distributed
sensor networks (Chan et al., 2003). Note that GOPA can be
conveniently executed while constructing the random $k$-out
graph. We have the following privacy guarantees.

**Theorem 3 (Privacy guarantee for random $k$-out graphs).** Let $\varepsilon, \delta \in (0, 1)$ and let $G$ be obtained by letting all (honest) users randomly choose $k \leq n$ neighbors. Let $k$ and $\rho = n_H/n$ be such that $\rho n \geq 81$, $\rho k \geq 4 \log(2 \rho m/3 \delta)$, $\rho k \geq 6 \log(\rho n/3)$ and $\rho k \geq \frac{3}{2} \frac{\log(2e/\delta)}{\delta}$. If $\varepsilon, \delta, \sigma_\eta$ and $\sigma_\Delta$ satisfy the inequalities (3) and (4) in Theorem 1 with
$$\theta = \frac{1}{n_H \sigma_\eta^2} + \frac{1}{\sigma_\Delta^2} \left( \frac{1}{(k-1) \rho / 3} - 1 + \frac{12 + 6 \log(n_H)}{n_H} \right)$$
then GOPA is $(\varepsilon, 3\delta)$-differentially private.

This result has a similar form as Theorems 1-2 but requires
$k$ to be large enough (of order $\log(\rho n)/\rho$) so that $G^H$ is

| Table 1. Value of $\sigma_\Delta$ needed to ensure $(\varepsilon, \delta)$-DP with trusted curator utility for $n = 10000$, $\varepsilon = 0.1$, $\delta' = 1/n_H^2$, $\delta = 10^3$. depending on the topology, as obtained from Corollary 1. |
|-----------------|-----------------|-----------------|
|                  | $\rho = 1$      | $\rho = 0.5$    |
| Complete         | $1.7$           | $2.1$           |
| $k$-out (Theorem 3) | $44.7$ $(k = 105)$ | $34.4$ $(k = 203)$ |
| $k$-out (simulation) | $34.7$ $(k = 20)$ | $28.4$ $(k = 40)$ |
| Worst-case       | $9655.0$        | $6114.8$        |

**Scaling the noise in practice.** Using these results, we can
precisely quantify the amount of independent and pairwise
noise needed to achieve a desired privacy guarantee depending
on the topology, as illustrated by the following corollary.

**Corollary 1.** Let $\varepsilon, \delta' \in (0, 1)$, and $\sigma_\eta^2 = c^2/n_H^2$, where $c^2 > 2 \log(1.25/\delta')$. Given some $\kappa > 0$, let $\sigma_\Delta = \kappa \sigma_\eta^2$ if $G$ is complete, $\sigma_\Delta = \kappa \sigma_\eta^2 n_H \left( \frac{1}{(k-1) \rho / 3} - 1 + \frac{12 + 6 \log(n_H)}{n_H} \right)$ if it is a random $k$-out graph with $k$ as in Theorem 3, and $\sigma_\Delta = \kappa \sigma_\eta^2 n_H^2 / 3$ for an arbitrary connected $G^H$. Then GOPA is $(\varepsilon, \delta)$-DP with $\delta \geq \kappa (\delta')^{1/\kappa-1}$, where $a = 3.75$ for the $k$-out graph and $1.25$ otherwise.

In Corollary 1, $\sigma_\eta^2$ is set such that after all noisy values are
aggregated, the variance of the residual noise matches that
required by the Gaussian mechanism (Dwork & Roth, 2014)
to achieve $(\varepsilon, \delta')$-DP for an average of $n_H$ values in the
centralized setting. The privacy-utility trade-off achieved by
GOPA is thus the same as in the trusted curator model up
to a small constant in $\delta$, as long as the pairwise variance $\sigma_\Delta^2$
is large enough. As expected, we see that as $\sigma_\Delta^2 \rightarrow \infty$
(that is, as $\kappa \rightarrow \infty$), we have $\delta \rightarrow \delta'$. Given the desired $\delta \geq \delta'$, we can use Corollary 1 to determine a value for $\sigma_\Delta^2$ that is sufficient for GOPA to achieve $(\varepsilon, \delta)$-DP. Table 1 shows a numerical illustration with $\delta$ only a factor 10 larger than $\delta'$. For random $k$-out graphs, we report the values of $\sigma_\Delta$ and $k$ given by Theorem 3, as well as smaller (yet admissible) values obtained by numerical simulation (see Appendix A.5). Although the conditions of Theorem 3 are a bit conservative (constants can likely be improved), they still lead to practical values. Clearly, random $k$-out graphs provide a useful trade-off in terms of scalability and robustness. Note that in practice, one often does not know in advance the exact proportion $\rho$ of users who are honest and will not drop out, so a lower bound can be used instead.

**Proof techniques.** Our analysis starts by deriving an abstract $(\varepsilon, \delta)$-DP result which holds for graphs $G^H$ with an arbitrary topology. We model the knowledge of the adversary as a system of linear equations, which leads to a
covariance structure for noisy values $\hat{X}^H$ that depend on the topology of $G^H$. Requiring the privacy loss random variable to be smaller than $\varepsilon$ with probability $1 - \delta$, we show that the privacy guarantees relate to the ability to “spread” the difference between the two neighboring datasets $X^A$ and $X^B$ over all vertices of $G^H$. We then concretely instantiate this abstract result for specific topologies, which involves identifying a spanning tree of $G^H$ with a large branching factor. While the optimal spanning tree is easy to construct for the worst-case of paths and for complete graphs, the case of random graphs is more involved and requires specific tools. We prove our result by leveraging and adapting results on embedding spanning trees in random graphs (Krivelevich, 2010). Full derivations can be found in Appendix A.

Remark 1. For clarity of presentation, our privacy guarantees protect against an adversary that consists of colluding malicious users. To simultaneously protect against each single honest-but-curious user (who knows his own independent noise term), we can simply replace $n_H$ by $n_H - 1$ in our results. This introduces a factor $n_H/(n_H - 1)$ in the variance, which is negligible for large $n_H$.

5. Correctness Against Malicious Users

While the privacy guarantees of Section 4 hold regardless of the behavior of malicious users, the utility guarantees discussed in Section 3 are not valid if malicious users tamper with the protocol. In this section, we add to our protocol the capability of being audited to ensure the correctness of the computations while preserving privacy. Due to the lack of space, we give a high-level description of our approach and refer to Appendix B and Appendix C for more details.

Objective. While it is impossible to force a user to give the “right” input to the algorithm, this also holds in the centralized setting. Our goal is thus to guarantee that given the input vector $X$, we can identify malicious behavior and generate a truthfully computed $\hat{X}^{avg}$ which excludes any faulty contributions. Concretely, users will be able to prove the following properties:

$$\hat{X}_u = X_u + \sum_{v \in N(u)} \Delta_{u,v} + \eta_u, \quad \forall u \in U,$$  \hspace{1cm} (5)

$$\Delta_{u,v} = -\Delta_{v,u}, \quad \forall \{u, v\} \in E,$$  \hspace{1cm} (6)

$$\eta_u \sim \mathcal{N}(0, \sigma^2_u), \quad \forall u \in U,$$  \hspace{1cm} (7)

$$X_u \in [0, 1], \quad \forall u \in U. \hspace{1cm} (8)$$

Tools for verifying computations. Our approach consists in publishing an encrypted log of the computation using cryptographic commitments and proving that it is performed correctly without revealing any additional information using zero knowledge proofs. Moreover, signing the messages adds non-repudiation. These techniques are popular in a number of applications such as privacy-friendly auditable financial systems like Zcash and Findora. We here give a brief overview of the functionality of these tools and their role in verifying Properties (5)-(8).

Commitments, first introduced by (Blum, 1983), allow users to commit to chosen values while keeping them hidden from others. After the commitment is performed, the committer cannot change its value, but can later reveal it or prove properties of it. For our protocol we use an optimized version (Frank & Großschädl, 2017) of the Pedersen commitment scheme (Pedersen, 1991). A commitment is obtained by transforming the hidden statement using a hard-to-invert injective function Com, so that the recipient cannot know the original statement but can be sure that the committer cannot change the statement and still obtain the same commitment. Pedersen commitments additionally satisfy the homomorphic property, meaning that $\text{Com}(v_1 + v_2) = \text{Com}(v_1) + \text{Com}(v_2)$ for all values $v_1, v_2$. This property facilitates the verification of summations without revealing them, in our case Properties (5) and (6).

To verify other properties than the additive ones, we use another family of cryptographic operations known as Zero Knowledge Proofs (ZKPs). Informally speaking, ZKPs allow a user (prover) to effectively prove a true statement (completeness), but also allow the other user (verifier) to discover with high probability if a cheating prover is trying to prove a statement which is not true (soundness), in such a way that by performing the proof no information other than the proven statement is revealed (zero knowledge). Here, we use classic proof techniques (see e.g., Schnorr, 1991; Chaum & Pedersen, 1993; Fujisaki & Okamoto, 1997) as building blocks. In particular, these building blocks support the verification of (5) and (6), and constitute the bulk of the verification of (7) and (8). The only assumption needed to ensure the validity of the cryptographic building blocks on which we rely is the Discrete Logarithm Assumption (DLA). We refer the interested reader to Appendix B, where we provide a step-by-step presentation of the employed techniques and our own algorithms using them.

Verification protocol. In a setup phase, users agree on a commitment function $\text{Com}$ and generate random seeds. Then, while executing GOPA, users publish commitments of $X_u$ and $\eta_u$ for all $u \in U$ and of $\Delta_{u,v}$ for all $\{u, v\} \in E$. Due to the homomorphic property of the Pedersen commitments, everyone can then verify the summation relations (5) and (6). In a second phase, ZKPs are performed to prove the remaining parts of the verification. We implement the publication of commitments using a public bulletin board so that any party can verify the validity of the protocol, avoiding the need for a trusted verification entity. Users sign their messages so they cannot deny them. More general purpose distributed ledger technology such as blockchain (Nakamoto, 2008) could be used here, but we aim at an application-specific, light-weight and hence
more scalable solution. Since the basic cryptographic operations are integer-based, we use a fixed precision scheme (see Appendix B.5), which is efficient as our computations are mostly additive. More details on our verification protocol, including how to verify consistency over multiple runs on the same/related data, are provided in Appendix C.1.

It is easy to see that the correctness of the computation is guaranteed if Properties (5)-(8) are satisfied. Note that, as long as they are self-canceling and not excessively large (avoiding overflows and additional costs if a user drops out, see Appendix C.2), we do not need to ensure that pairwise noise terms $\Delta_{u,v}$ have been drawn from the prescribed distribution, as these terms do not influence the final result and only those involving honest users affect the privacy guarantees of Section 4. In contrast, Properties (7) and (8) are necessary to prevent a malicious user from biasing the outcome of the computation. Indeed, (7) ensures that the independent noise is generated correctly, while (8) ensures that input values are in the allowed range. The following result summarizes the security guarantees.

**Theorem 4** (Security guarantees of GOPA). **Under the DLA**, a user $u \in U$ that passes the verification procedure proves that $X_u$ was computed correctly. Additionally, $u$ does not reveal any additional information about $X_u$ by running the verification, even if DLA does not hold.

In Appendix C.4, we discuss additional mitigation against attacks which do not target the correctness or privacy of GOPA, but could affect its efficiency. With the above guarantees, we are able to provide essentially the same control over the correctness of computations as in the centralized setting. We discuss this claim in more details in Appendix C.3.

### 6. Computation and Communication Costs

Our cost analysis considers user-centered costs, which is natural as most operations can be performed asynchronously and in parallel. The following statement summarizes our results (concrete non-asymptotic costs are in Appendix D).

**Theorem 5** (Complexity of GOPA). Let $B > 0$ be the desired fixed precision such that the number 1 would be represented as the closest integer to $1/B$. Then, in the private averaging phase, each user $u$ performs $O(|N(u)| + \log(1/B))$ computations, sends $O(|N(u)|)$ messages and transfers $O(|N(u)| + \log(1/B))$ bits. In the verification phase, the protocol proves the correctness of all computations with $O(|N(u)| + \log(1/B))$ computations and one transfer of $O(|N(u)| + \log(1/B))$ bits from the bulletin board. The $\eta_u$'s are drawn from a Gaussian distribution approximated with $1/B$ equiprobable bins and proved with precision $B$.

Unlike other frameworks like fully homomorphic encryption and secure multiparty computation, the cost of computing Pedersen commitments and ZKPs is tractable, even in massive amounts, as seen from production environment benchmarks (Franck & Großschädl, 2017; Bünz et al., 2018).

### 7. Related Work

Distributed averaging is a key subroutine in distributed and federated learning (Lin et al., 2020; Stich, 2019; McMahan et al., 2017; Jayaraman et al., 2018; Agarwal et al., 2018; Kairouz et al., 2019). Therefore, any improvement in the privacy-utility trade-off for averaging implies gains for many ML approaches downstream.

Local differential privacy (LDP) (Kasiviswanathan et al., 2008; Duchi et al., 2013; Kairouz et al., 2015; 2016b:a) requires users to locally randomize their input before they send it to an untrusted aggregator. This very strong model of privacy comes at a significant cost in utility: the best possible error is of order $1/n$ in the trusted curator model while it is of order $1/\sqrt{n}$ in LDP (Chan et al., 2012a; Chen et al., 2020). This limits the usefulness of the local model to industrial settings where the number of participants is huge (Erlingsson et al., 2014; Ding et al., 2017).

Our approach belongs to the recent line of work which attempts to relax the LDP model so as to improve utility without relying on a trusted curator. Secure aggregation schemes can be used to recover the utility of the trusted curator model, see (Dwork et al., 2006; Shi et al., 2011; Bonawitz et al., 2017; Chan et al., 2012b) for protocols and (Jayaraman et al., 2018) for a concrete application to distributed machine learning. These approaches can be made somewhat robust to malicious parties, using e.g. verifiable secret sharing (Dwork et al., 2006). However, they require $\Omega(n)$ communication per party, which is not feasible beyond a few hundred or thousand participants. In contrast, our protocol requires only $O(\log n)$ communication per party, which is not feasible beyond a few hundred or thousand participants.
the functionalities of secure aggregation or secure shuffling.
We are not aware of other work sharing this feature. (Imtiaz et al., 2019) uses correlated Gaussian noise to achieve
trusted curator utility for averaging, but the dependence structure of the noise must be only at the global level (i.e.,
noise terms sum to zero over all users). Generating such
noise actually requires a call to a secure aggregation prim-
itive, which incurs \(\Omega(n)\) communication per party as
discussed above. In contrast, our pairwise-canceling noise
terms can be generated with only \(O(\log n)\) communication.
Furthermore, (Imtiaz et al., 2019) assume honest parties,
while our protocol is robust to malicious participants.

In summary, our protocol provides a unique combination of
two important features: (a) utility of same order as trusted
curator setting, (b) logarithmic communication per user, and
(c) robustness to malicious users.

8. Experiments

Private averaging. We present some numerical simulations
to study the empirical utility of GOPA and in particular
the influence of malicious and dropped out users. We consider
\(n = 10000\), \(\varepsilon = 0.1\), \(\delta = 1/n^2\) and set the values of \(k\), \(\sigma^2\)
and \(\sigma^2_\Delta\) using Corollary 1 so that GOPA satisfies \((\varepsilon, \delta)\)-DP.
Figure 1 (left) shows the utility of GOPA as a function of \(\rho\),
which is the (lower bound on) the proportion of users
who are honest and do not drop out. We see that even for
reasonably small \(\rho\), GOPA achieves a utility of the same
order as a trusted curator that would average the values of
all \(n\) users. In Appendix C.2, we further illustrate the ability
of GOPA to tolerate a small number of residual pairwise
noise terms of variance \(\sigma^2_\Delta\) in the final result. We note that
this feature is rather unique to GOPA and is not possible
with secure aggregation (Bonawitz et al., 2017).

Application to federated SGD. We present some experiments
on training a logistic regression model in a federated
learning setting. We use a binarized version of UCI Housing
data set with standardized features and points normalized to
unit L2 norm to ensure a gradient sensitivity bounded by 2.
We set aside 20% of the points as test set and split the rest
uniformly at random across \(n = 10000\) users so that each
user \(u\) has a local dataset \(D_u\) composed of 1 or 2 points.

We use the Federated SGD algorithm, which corresponds to
FedAvg with a single local update (McMahan et al., 2017).
At each epoch, each user computes a stochastic gradient
with size \(\epsilon = 1\) and \(\delta = 1/(\rho n)^2\) and use advanced composition
to compute the budget allocated to each epoch. The step
size is tuned for each approach, selecting the value with the
highest accuracy after a predefined number \(T\) of epochs.

Figure 1 (middle) shows a typical run of the algorithm for
\(T = 50\) epochs. Local DP is not shown as it diverges unless
the learning rate is overly small. On the other hand, GOPA
is able to decrease the objective function steadily, although
we see some difference with the trusted aggregator (this is
expected since \(\rho = 0.5\)). Figure 1 (right) shows the final
test accuracy (averaged over 10 runs) for different numbers
of epochs \(T\). Despite the small gap in objective function,
GOPA nearly matches the accuracy achieved by the trusted
aggregator, while local DP is unable to learn useful models.

9. Conclusion

We proposed GOPA, a protocol to privately compute aver-
ages over the values of many users. GOPA satisfies DP, can
nearly match the utility of a trusted curator, and is robust to
malicious parties. It can be used in distributed and federated
ML (Jayaraman et al., 2018; Kairouz et al., 2019) in place of
more costly secure aggregation schemes. In future work, we
plan to provide efficient implementations, to integrate our
approach in complete ML systems, and to exploit scaling
to reduce the cost per average. We think that our work is
also relevant beyond averaging, e.g. in the context of robust
aggregation for distributed SGD (Blanchard et al., 2017)
and for computing pairwise statistics (Bell et al., 2020).
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Appendix A. Proofs of Differential Privacy Guarantees

In this appendix, we provide derivations for the differential privacy guarantees stated in the main paper.

Recall that each user $u \in U^O$ who does not drop out generates $\hat{X}_u$ from its private value $X_u$ by adding pairwise noise terms $\Delta_u = \sum_{v \in N(u)} \Delta_{u,v}$ (with $\Delta_{u,v} + \Delta_{v,u} = 0$) as well as independent noise $\eta_u$. All random variables $\Delta_{u,v}$ (with $u < v$) and $\eta_u$ are independent.

We have the system of linear equations

$$\hat{X} = X + \bar{\Delta} + \eta,$$

where $\bar{\Delta} = (\Delta_u)_{u \in U^O}$ and $\eta = (\eta_u)_{u \in U^O}$. An adversary may be able to determine all $X_u$, $\Delta_{u,v}$ and $\eta_u$ for every non-honest user $u$. Such an adversary can hence fill part of the variables in the system of linear equations. Let $X^H = (X_u)_{u \in U^H}$ be the vector of private values restricted to the honest users. Let $\Delta^H_u = \sum_{v \in N^H(u)} \Delta_{u,v}$ and let $\hat{X}^H_u = \hat{X}_u - \sum_{v \in N(u) \setminus N^H(u)} \Delta_{u,v}$. In this way, the adversary should now solve a new system of linear equations:

$$\hat{X}^H = X^H + \bar{\Delta}^H + \eta^H,$$

where $\bar{\Delta}^H = (\Delta^H_u)_{u \in U^H}$ and $\eta^H = (\eta_u)_{u \in U^H}$.

The expectation and covariance matrix of $\hat{X}^H$ are respectively given by:

$$\mathbb{E}[\hat{X}^H] = X^H \text{ and } \Sigma = \Sigma_{\hat{X}^H} = \sigma^2_{\eta} I_{U^H} + \sigma^2_{\Delta} A - \sigma^2_{\Delta} A = \sigma^2_{\eta} I_{U^H} + \sigma^2_{\Delta} L,$$

where $I_{U^H} \in \mathbb{R}^{n_H \times n_H}$ is the identity matrix, $A \in \mathbb{R}^{n_H \times n_H}$ is the adjacency matrix of $G^H$ (the communication graph restricted to honest users), $\Lambda$ is a diagonal matrix with $A_{u,u} = \sum_{v \in U^H} A_{u,v}$, and $L$ is the graph Laplacian matrix of $G^H$.

Now consider the real vector space $Z$ of dimension $n_H + |E^H|$ of all possible values of independent and pairwise noise terms of honest users. For convenience, we will write $(\eta^H, \Delta^H) \in Z$ to denote that the independent noise terms take values $\eta^H = (\eta_u)_{u \in U^H}$ and the pairwise noise terms take values $\Delta^H = (\Delta_{u,v})_{\{u,v\} \in E^H}$. Let $K \in \mathbb{R}^{n_H \times |E^H|}$ denote the oriented incidence matrix of the graph $G^H$ such that $KK^\top = L$ and for any pairwise noise values $\Delta^H \in \mathbb{R}^{|E^H|}$, $K\Delta^H = \bar{\Delta}^H$. For the sake of readability, in the following we drop the superscript $H$ for $(\eta^H, \Delta^H) \in Z$ as it will clear from the context that we work in the space $Z$.

Let

$$\Sigma^{(g)} = \begin{bmatrix} \sigma^2_{\eta} I_{U^H} & 0 \\ 0 & \sigma^2_{\Delta} I_{E^H} \end{bmatrix}.$$ 

We then have a joint probability distribution of independent Gaussians:

$$P((\eta, \Delta)) = C_1 \exp \left( -\frac{1}{2} (\eta, \Delta)^\top (\Sigma^{(g)})^{-1} (\eta, \Delta) \right),$$

where $C_1 = (2\pi)^{-(n_H + |E^H|)/2} |\Sigma^{(g)}|^{-1/2}$.

Consider the following subspaces of $Z$:

$$Z^A = \{ (\eta, \Delta) \in Z \mid \eta + K\Delta = \bar{X}^H - X^A \},$$

$$Z^B = \{ (\eta, \Delta) \in Z \mid \eta + K\Delta = \bar{X}^H - X^B \}.$$ 

Assume that the (only) vertex for which $X^A$ and $X^B$ differ is $v_1$. We can assume without loss of generality that private values are in the interval $[0,1]$, hence $X^A_{v_1} - X^B_{v_1} = 1$. Now choose any $t = (t_\eta, t\Delta) \in Z$ such that $t_\eta + Kt\Delta = X^A - X^B$.

It follows that $Z^A = Z^B + t$, i.e., $Y \in Z^A$ if and only if $Y + t \in Z^B$.

We split the rest of our derivations into four parts. In Section A.1, we first prove an abstract intermediate result (Lemma 1) which gives differential privacy guarantees for Gopa that depend on the particular choice of $t$. Then, in Section A.2, we study the best and worst-case topologies and show how the choice of $t$ can be made intelligently depending on the graph structure, leading to Theorem 1 and Theorem 2. Section A.3.4 presents the derivations for random $k$-out graphs leading to Theorem 3. Finally, Section A.4 shows these results can be instantiated to match the utility of the centralized Gaussian mechanism (Corollary 1).
A.1. Abstract Differential Privacy Result

Let \( t = (t_\eta, t_\Delta) \in \mathbb{Z} \) such that \( t_\eta + K t_\Delta = X^A - X^B \) as defined above. We start by proving differential privacy guarantees which depend on the particular choice of \( t \).

**Lemma 1.** Let \( \varepsilon, \delta \in (0, 1) \). Under the setting introduced above, if

\[
\varepsilon \geq (t^\top \Sigma^{(-g)} t)^{1/2} + t^\top \Sigma^{(-g)} t / 2,
\]

and

\[
\frac{(\varepsilon - t^\top \Sigma^{(-g)} t / 2)^2}{t^\top \Sigma^{(-g)} t} \geq 2 \log \left( \frac{2}{\delta \sqrt{2\pi}} \right),
\]

where \( \Sigma^{(-g)} = (\Sigma^{(g)})^{-1} \), then GOPA is \((\varepsilon, \delta)\)-differentially private, i.e.,

\[
P(\hat{X} \mid X^A) \leq e^\varepsilon P(\hat{X} \mid X^B) + \delta.
\]

**Proof.** It is sufficient to prove that

\[
\left| \log \frac{P((\eta, \Delta))}{P((\eta, \Delta) + t)} \right| \leq \varepsilon
\]

with probability \( 1 - \delta \) over \((\eta, \Delta)\). Denoting \( \gamma = (\eta, \Delta) \) for convenience, we need to prove that with probability \( 1 - \delta \) it holds that \( |\log(P(\gamma)/P(\gamma + t))| \geq \varepsilon \). We have

\[
\left| \log \frac{P(\gamma)}{P(\gamma + t)} \right| = \left| -\frac{1}{2} \gamma^\top \Sigma^{(-g)} \gamma + \frac{1}{2} (\gamma + t)^\top \Sigma^{(-g)} (\gamma + t) \right|
\]

\[
= \left| \frac{1}{2} (2\gamma + t)^\top \Sigma^{(-g)} t \right|.
\]

To ensure Equation (11) hold with probability \( 1 - \delta \), since we are interested in the absolute value, we will show that

\[
P\left(\frac{1}{2} (2\gamma + t)^\top \Sigma^{(-g)} t \geq \varepsilon \right) \leq \delta/2,
\]

which is equivalent to

\[
P(\gamma^{(-g)} t \geq \varepsilon - t^\top \Sigma^{(-g)} t / 2) \leq \delta/2.
\]

The variance of \( \gamma^{(-g)} t \) is

\[
\text{var}(\gamma^{(-g)} t) = \sum_v \text{var} \left( \eta_v \sigma_\eta^{-2} \Delta_v \right) + \sum_e \text{var} \left( \Delta_e \sigma_\Delta^{-2} \epsilon_e \right) = \sum_v \text{var} (\eta_v) \sigma_\eta^{-4} \Delta_v^2 + \sum_e \text{var} (\Delta_e) \sigma_\Delta^{-4} \epsilon_e^2
\]

\[
= \sum_v \sigma_\eta^{-2} \sum_v \sigma_\eta^{-2} \Delta_v^2 + \sum_e \sigma_\Delta^{-4} \sum_e \epsilon_e^2 = \sum_v \sigma_\eta^{-2} \sigma_\eta^{-2} \Delta_v^2 + \sum_e \sigma_\Delta^{-4} \epsilon_e^2 = \sum_v \sigma_\eta^{-2} \sigma_\eta^{-2} \Delta_v^2 + \sum_e \sigma_\Delta^{-2} \epsilon_e^2 = t^\top \Sigma t.
\]

For any centered Gaussian random variable \( Y \) with variance \( \sigma_Y^2 \), we have that

\[
P(Y \geq \lambda) \leq \frac{\sigma_Y}{\lambda \sqrt{2\pi}} \exp \left( -\lambda^2 / 2\sigma_Y^2 \right).
\]

Let \( Y = \gamma^{(-g)} t, \sigma_Y^2 = t^\top \Sigma^{(-g)} t \) and \( \lambda = \varepsilon - t^\top \Sigma^{(-g)} t / 2 \) so satisfying

\[
\frac{\sigma_Y}{\lambda \sqrt{2\pi}} \exp \left( -\lambda^2 / 2\sigma_Y^2 \right) \leq \delta/2
\]

implies (12). Equation (14) is equivalent to

\[
\frac{\lambda}{\sigma_Y} \exp \left( \lambda^2 / 2\sigma_Y^2 \right) \geq 2/\delta \sqrt{2\pi},
\]

and taking logarithms we need

\[
\log \left( \frac{\lambda}{\sigma_Y} \right) + \frac{1}{2} \left( \frac{\lambda}{\sigma_Y} \right)^2 \geq \log \left( \frac{2}{\delta \sqrt{2\pi}} \right).
\]
To make this inequality hold, we require
\[
\log \left( \frac{\lambda}{\sigma_Y} \right) \geq 0
\] (15)
and
\[
\frac{1}{2} \left( \frac{\lambda}{\sigma_Y} \right)^2 \geq \log \left( \frac{2}{\delta \sqrt{2\pi}} \right).
\] (16)

Equation (15) is equivalent to \( \lambda \geq \sigma_Y \). Substituting \( \lambda \) and \( \sigma_Y \) we get
\[
\varepsilon - t^\top \Sigma^{(-g)} t/2 \geq (t^\top \Sigma^{(-g)} t)^{1/2},
\]
which is equivalent to (9). Substituting \( \lambda \) and \( \sigma_Y \) in Equation (16) gives (10).

Essentially, given some \( \varepsilon \), Equation (9) provides a lower bound for the noise (the diagonal of \( \Sigma^{(g)} \)) to be added. Equation (9) also implies that the lefthandside of Equation (10) is larger than 1. Equation (10) may then require the noise or \( \varepsilon \) to be even higher if \( 2 \log(2/\delta \sqrt{2\pi}) \geq 1 \), i.e., \( \delta \leq 0.48394 \).

A.2. Worst and Best Case Topologies

Lemma 1 holds for all vectors \( t = (t_\eta, t_\Delta) \). There exist multiple vectors \( t \) such that \( t_\eta + Kt_\Delta = X^A - X^B \), so we will select the vector \( t \) which allows us most easily to prove differential privacy using the above.

Let us first consider the worst case, which corresponds to Theorem 1.

**Proof of Theorem 1.** Let \( T \) be a spanning tree of the (connected) communication graph \( G^H \). Let \( E^T \) be the set of edges in \( T \). Let \( t \in \mathbb{R}^{n_H + |E^H|} \) be a vector such that:

- For vertices \( u \in U^H, t_u = 1/n_H \).
- For edges \( e \in E^H \setminus E^T, t_e = 0 \).
- Finally, for edges \( e \in E^T \), we choose \( t_e \) in the unique way such that \( t_\eta + Kt_\Delta = (X^A - X^B) \).

In this way, \( t_\eta + Kt_\Delta \) is a vector with a 1 on the \( v_1 \) position and 0 everywhere else. We can find a unique vector \( t \) using this procedure for any communication graph.

It holds that
\[
t_\eta^\top t_\eta = n_H \left( \frac{1}{n_H} \right)^2 = \frac{1}{n_H}.
\] (17)

In both Equations (9) and (10) of Lemma 1, a higher \( t^\top \Sigma^{(-g)} t \) is 'worse' in the sense that a higher \( \varepsilon \) or \( \delta \) is needed. Conversely, for a given \( \varepsilon \) and \( \delta \), \( t^\top \Sigma^{(-g)} t \) must be sufficiently low to satisfy both inequalities. We can see \( t_\Delta^\top \Sigma^2 t_\Delta \) is maximized (thus producing the worst case) if the spanning tree \( T \) is a path \((v_1, v_2, \ldots, v_{n_H})\), in which case \( t_{\{v_i, v_{i+1}\}} = (n_H - i)/n_H \). Therefore,
\[
t_\Delta^\top t_\Delta \leq \sum_{i=1}^{n_H-1} \left( \frac{n_H - i}{n_H} \right)^2 = \frac{n_H(n_H - 1)(2n_H - 1)/6}{n_H^2} = \frac{(n_H - 1)(2n_H - 1)}{6n_H}.
\] (18)

We now plug in the specific values of \( t \) in our general results of Lemma 1. Consider a given \( \delta, \sigma_\eta \) and \( \sigma_\Delta \). Substituting Equation (17) and (18) into Equation (9) yields
\[
\varepsilon \geq \frac{1}{2} \left( \frac{\sigma_\eta^2}{n_H} + \frac{\sigma_\Delta^2(n_H - 1)(2n_H - 1)}{6n_H} \right) + \left( \frac{\sigma_\eta^2}{n_H} + \frac{\sigma_\Delta^2(n_H - 1)(2n_H - 1)}{6n_H} \right)^{1/2},
\]
which is satisfied if
\[
\varepsilon \geq \frac{1}{2\sigma_\eta^2 n_H} + \frac{n_H}{6\sigma_\Delta^2} + \left( \frac{1}{2\sigma_\eta^2 n_H} + \frac{n_H}{3\sigma_\Delta^2} \right)^{1/2}.
\] (19)
Substituting Equation (17) and (18) into Equation (10) yields

\[
\left( \varepsilon - \left( \sigma_{\eta}^{-2} \frac{1}{n_H} + \sigma_{\Delta}^{-2} \frac{(n_H - 1)(n_H - 2)}{6n_H} \right) \right)^2 \geq 2 \log \left( \frac{2}{\delta \sqrt{2\pi}} \right),
\]

which is satisfied if

\[
\left( \varepsilon - \frac{1}{2 \sigma_{\eta}^2 n_H} - \frac{n_H}{6 \sigma_{\Delta}^2} \right)^2 \geq 2 \log \left( \frac{2}{\delta \sqrt{2\pi}} \right) \left( \frac{1}{\sigma_{\eta}^2 n_H} + \frac{n_H}{3 \sigma_{\Delta}^2} \right). \tag{20}
\]

We can observe that in the worst case \( \sigma_{\Delta}^2 \) should be large (linear in \( n_H \)) to keep \( \varepsilon \) small, which has no direct negative effect on the utility of the resulting \( \hat{X} \). On the other hand, \( \sigma_{\eta}^2 \) can be small (of the order \( 1/n_H \)), which means that independent of the number of participants or the way they communicate a small amount of independent noise is sufficient to achieve differential privacy.

However, the necessary value of \( \sigma_{\Delta}^2 \) depends strongly on the network structure. This becomes clear in Theorem 2, which covers the case of the complete graph.

**Proof of Theorem 2.** If the communication graph is fully connected, we can use the following values for the vector \( t \):

- As earlier, for \( v \in U^H \), let \( t_v = 1/n_H \).
- For edges \( \{u,v\} \) with \( v_1 \not\in \{u,v\} \), let \( t_{\{u,v\}} = 0 \).
- For \( u \in U^H \setminus \{v_1\} \), let \( t_{\{u,v_1\}} = 1/n_H \).

Again, one can verify that \( t_\eta + K t_\Delta = X^A - X^B \) is a vector with a 1 on the \( v_1 \) position and 0 everywhere else. In this way, again \( t_\eta t_\eta = 1/n_H \) but now \( t_\Delta t_\Delta = (n_H - 1)/n_H^2 \) is much smaller. With this network structure, substituting into Equation (9) yields

\[
\varepsilon \geq \frac{1}{2} \left( \sigma_{\eta}^{-2} \frac{1}{n_H} + \sigma_{\Delta}^{-2} \frac{(n_H - 1)}{n_H^2} \right) + \left( \sigma_{\eta}^{-2} \frac{1}{n_H} + \sigma_{\Delta}^{-2} \frac{n_H - 1}{n_H^2} \right)^{1/2},
\]

which is satisfied if

\[
\varepsilon \geq \frac{\sigma_{\eta}^{-2} + \sigma_{\Delta}^{-2}}{2n_H} + \left( \frac{\sigma_{\eta}^{-2} + \sigma_{\Delta}^{-2}}{n_H} \right)^{1/2}. \tag{21}
\]

Substituting into Equation (10) yields

\[
\left( \varepsilon - \left( \sigma_{\eta}^{-2} \frac{1}{n_H} + \sigma_{\Delta}^{-2} \frac{n_H - 1}{n_H^2} \right) / 2 \right)^2 \geq 2 \log \left( \frac{2}{\delta \sqrt{2\pi}} \right),
\]

which is satisfied if

\[
\left( \varepsilon - \left( \sigma_{\eta}^{-2} + \sigma_{\Delta}^{-2} \right) / 2n_H \right)^2 \geq 2 \log \left( \frac{2}{\delta \sqrt{2\pi}} \right) \frac{\sigma_{\eta}^{-2} + \sigma_{\Delta}^{-2}}{n_H}. \tag{22}
\]

A practical communication graph will be between the two extremes of the path and the complete graph, as shown by the case of random k-out graphs (see Appendix A.3).

### A.3. Random k-out Graphs

In this section, we will study the differential privacy properties for the case where all users select \( k \) neighbors randomly, leading to a proof of Theorem 3. We will start by analyzing the properties of \( G^H \) (Section A.3.1). Section A.3.2 consists of preparations for embedding a suitable spanning tree in \( G^H \). Next, in Section A.3.3 we will prove a number of lemmas showing that such suitable spanning tree can be embedded almost surely in \( G^H \). Finally, we will apply these results to
proving differential privacy guarantees for GOPA when communicating over such a random $k$-out graph $G$ in Section A.3.4, proving Theorem 3.

In this section, all newly introduced notations and definitions are local and will not be used elsewhere. At the same time, to follow more closely existing conventions in random graph theory, we may reuse in this section some variable names used elsewhere and give them a different meaning.

A.3.1. THE RANDOM GRAPH $G^H$

Recall that the communication graph $G^H$ is generated as follows:

- We start with $n = |U|$ vertices where $U$ is the set of agents.
- All (honest) agents randomly select $k$ neighbors to obtain a $k$-out graph $G$.
- We consider the subgraph $G^H$ induced by the set $U^H$ of honest users who did not drop out. Recall that $n_H = |U^H|$ and that a fraction $\rho$ of the users is honest and did not drop out, hence $n_H = \rho n$.

Let $k_H = \rho k$. The graph $G^H$ is a subsample of a $k$-out-graph, which for larger $n_H$ and $k_H$ follows a distribution very close to that of Erdős–Rényi random graphs $G_p(n_H, 2k_H/n_H)$. To simplify our argument, in the sequel we will assume $G^H$ is such random graph as this does not affect the obtained result. In fact, the random $k$-out model concentrates the degree of vertices more narrowly around the expected value than Erdős–Rényi random graphs, so any tail bound our proofs will rely on that holds for Erdős–Rényi random graphs also holds for the graph $G^H$ we are considering. In particular, for $v \in U^H$, the degree of $v$ is a random variable which we will approximate for sufficiently large $n_H$ and $k_H$ by a binomial $B(n_H, 2k_H/n_H)$ with expected value $2k_H$ and variance $2k_H(1 - 2k_H/n_H) \approx 2k_H$.

A.3.2. THE SHAPE OF THE SPANNING TREE

Remember that our general strategy to prove differential privacy results is to find a spanning tree in $G^H$ and then to compute the norm of the vector $t_\Delta$ that will “spread” the difference between $X^A$ and $X^B$ over all vertices (so as to get a $\sigma_\eta$ of the same order as in the trusted curator setting). Here, we will first define the shape of a rooted tree and then prove that with high probability this tree is isomorphic to a spanning tree of $G^H$. Of course, we make a crude approximation here, as in the (unlikely) case that our predefined tree cannot be embedded in $G^H$ it is still possible that other trees could be embedded in $G^H$ and would yield similarly good differentially privacy guarantees. While our bound on the risk that our privacy guarantee does not hold will not be tight, we will focus on proving our result for reasonably-sized $U$ and $k$, and on obtaining interesting bounds on the norm of $t_\Delta$.

Let $G^H = ([n_H], E^H)$ be a random graph where between every pair of vertices there is an edge with probability $2k_H/n_H$. The average degree of $G^H$ is $2k_H$.

Let $k_H \geq 4$. Let $q \geq 3$ be an integer. Let $\Delta_1, \Delta_2 \ldots \Delta_q$ be a sequence of positive integers such that

$$\left(\sum_{i=1}^{q} \prod_{j=1}^{i} \Delta_j\right) - (\Delta_q + 1) \prod_{j=1}^{q-2} \Delta_j < n_H \leq \sum_{i=1}^{q} \prod_{j=1}^{i} \Delta_j. \tag{23}$$

Let $T = ([n_H], E_T)$ be a balanced rooted tree with $n_H$ vertices, constructed as follows. First, we define for each level $l$ a variable $z_l$ representing the number of vertices at that level, and a variable $Z_l$ representing the total number of vertices in that and previous levels. In particular: at the root $Z_{-1} = 0, Z_0 = z_0 = 1$ and for $l \in [q-2]$ by induction $z_l = z_{l-1}\Delta_l$ and $Z_l = Z_{l-1} + z_l$. Then, $z_{q-1} = (n_H - Z_{q-2})/(\Delta_q + 1), Z_{q-1} = Z_{q-2} + z_{q-1}, z_q = n_H - Z_{q-1}$ and $Z_q = n_H$. Next, we define the set of edges of $T$:

$$E_T = \{\{Z_{l-1} + i, Z_{l-1} + z_{l-1}j + i\} \mid l \in [q] \land i \in [z_{l-1}] \land z_{l-1}j + i \in [z_l]\}.$$

So the tree consists of three parts: in the first $q - 2$ levels, every vertex has a fixed, level-dependent number of children, the last level is organized such that a maximum of parents has $\Delta_q$ children, and in level $q - 1$ parent nodes have in general $\Delta_{q-1} - 1$ or $\Delta_{q-1}$ children. Moreover, for $0 \leq l \leq q - 2$, the difference between the number of vertices in the subtrees
rooted by two vertices in level \( l \) is at most \( \Delta_q + 2 \). We also define the set of children of a vertex, i.e., for \( l \in [q] \) and \( i \in [z_l-1] \),

\[
ch(Z_{l-2} + i) = \{ Z_{l-1} + z_{l-1}j + i \mid z_{l-1}j + i \in [z_l] \}.
\]

In Section A.3.3, we will show conditions on \( n_H, \Delta, k_H \) and \( \delta \) such that for a random graph \( G^H \) on \( n_H \) vertices and a vertex \( v_1 \) of \( G^H \), with high probability (at least \( 1 - \delta \)) \( G^H \) contains a subgraph isomorphic to \( T \) whose root is at \( v_1 \).

### A.3.3. Random Graphs Almost Surely Embed a Balanced Spanning Tree

The results below are inspired by (Krivelevich, 2010). We specialize this result to our specific problem, obtaining proofs which are also valid for graphs smaller than \( 10^{10} \) vertices, even if the bounds get slightly weaker when we drop terms of order \( O(\log(\log(n_H))) \) for the simplicity of our derivation.

Let \( F \) be the subgraph of \( T \) induced by all its non-leaves, i.e., \( F = ([n_H/\Delta], E_F) \) with \( E_F = \{ \{i, j\} \in E_T \mid i, j \leq Z_{q-1} \} \).

**Lemma 2.** Let \( G^H \) and \( F \) be defined as above. Let \( v_1 \) be a vertex of \( G^H \). Let \( n_H \geq k_H \geq \Delta \geq 3 \). Let \( \gamma = \max_{i=1}^{q-1} \Delta_i/k_H \) and let \( \gamma + 4(\Delta + 2)^{-1} + 2n_H^{-1} \leq 1 \). Let \( k_H \geq 4 \log(2n_H/\delta_F(\Delta_q + 2)) \). Then, with probability at least \( 1 - \delta_F \), there is an isomorphism \( \phi \) from \( F \) to a subgraph of \( G^H \), mapping the root 1 of \( F \) on \( v_1 \).

**Proof.** We will construct \( \phi \) by selecting images for the children of vertices of \( F \) in increasing order, i.e., we first select \( \phi(1) = v_1 \), then map children \( \{ 2 \ldots \Delta_1 + 1 \} \) of \( v_1 \) to vertices adjacent to \( v_1 \), then map all children of 2 to vertices adjacent to \( \phi(2) \), etc. Suppose we are processing level \( l \in [q - 1] \) and have selected \( \phi(j) \) for all \( j \in ch(i') \) for all \( i' < i \) for some \( Z_{l-1} < i \leq Z_l \). We now need to select images for the \( \Delta_l \) children \( j \in ch(i) \) of vertex \( i \) (or in case \( l = q - 1 \) possibly only \( \Delta_l - 1 \) children). This means we need to find \( \Delta_l \) neighbors of \( i \) not already assigned as image to another vertex (i.e., not belonging to \( \cup_{0 \leq i' < i} ch(\phi(i')) \)). We compute the probability that this fails. For any vertex \( j \in [n_H] \) with \( i \neq j \), the probability that there is an edge between \( i \) and \( j \) in \( G^H \) is \( 2k_H/n_H \). Therefore, the probability that we fail to find \( \Delta_l \) free neighbors of \( i \) can be upper bounded as

\[
Pr[\text{FAIL}_F(i)] = Pr\left[ \binomial(n_H - Z_l, 2k_H/n_H) < \Delta_l \right] \leq \exp\left( \frac{\left( (n_H - Z_l) \frac{2k_H}{n_H} - \Delta_l \right)^2}{2(n_H - Z_l) \frac{2k_H}{n_H}} \right), \tag{24}
\]

We know that \( n_H - Z_l \geq z_q \). Moreover, \((z_q + \Delta_q - 1)/\Delta_q \geq z_{q-1} \) and \( Z_{q-2} + 1 \leq z_{q-1} \), hence \( 2(z_q + \Delta_q - 1)/\Delta_q \geq Z_{q-2} + Z_{q-1} + 1 = Z_{q-1} + 1 \) and \( 2(z_q + \Delta_q - 1)/\Delta_q + z_q \geq n_H + 1 \). Therefore

\[
z_q(2 + \Delta_q) \geq n_H + 1 - 2(\Delta_q - 1)/\Delta_q \geq n_H - 1.
\]

Therefore,

\[
n_H - Z_l \geq z_q \geq n_H(1 - 2(\Delta_q + 2)^{-1} - n_H^{-1}) \tag{25}
\]

Substituting this and \( \Delta_l \geq \gamma k_H \) in Equation (24), we get

\[
Pr[\text{FAIL}_F(i)] \leq \exp\left( \frac{-\left( (n_H - 2(\Delta_q + 2)^{-1} - n_H^{-1}) \frac{2k_H}{n_H} - k_H \gamma \right)^2}{2n_H(1 - 2(\Delta_q + 2)^{-1} - n_H^{-1}) \frac{2k_H}{n_H}} \right)
\leq \exp\left( \frac{-k_H^2}{4k_H} \left( 2(1 - 2(\Delta_q + 2)^{-1} - n_H^{-1}) - \gamma \right)^2 \right)
\leq \exp\left( \frac{-k_H^2}{4k_H} \right) = \exp\left( \frac{-k_H}{4} \right),
\]

where the latter inequality holds as \( \gamma + 4(\Delta + 2)^{-1} + 2n_H^{-1} \leq 1 \). As \( k_H \geq 4 \log(2n_H/\delta_F(\Delta_q + 2)) \) we can conclude that

\[
Pr[\text{FAIL}_F(i)] \leq \frac{\delta(\Delta_q + 2)}{2n_H}.
\]
The total probability of failure to embed \( F \) in \( G^H \) is therefore given by
\[
\sum_{i=2}^{Z_{q-1}} \text{FAIL}_F(i) \leq (Z_{q-1} - 1) \frac{\delta_F(\Delta_q + 2)}{2n_H} \leq (n_H(2(\Delta_q + 2)^{-1} + n_H^{-1}) - 1) \frac{\delta_F(\Delta_q + 2)}{2n_H} = \frac{2n_H}{\Delta_q + 2} \frac{\delta_F(\Delta_q + 2)}{2n_H} = \delta_F,
\]
where we again applied (25).

Now that we can embed \( F \) in \( G^H \), we still need to embed the leaves of \( T \). Before doing so, we review a result on matchings in random graphs. The next lemma mostly follows (Bollobás, 2001, Theorem 7.11 therein), we mainly adapt to our notation, introduce a confidence parameter and make a few less crude approximations.\(^2\)

**Lemma 3.** Let \( m \geq 27 \) (in our construction, \( m = z_n \)) and \( \zeta \geq 4 \). Consider a random bipartite graph with vertex partitions \( A = \{a_1 \ldots a_m\} \) and \( B = \{b_1 \ldots b_m\} \), where for every \( i, j \in [m] \) the probability of an edge \( \{a_i, b_j\} \) is \( p = \zeta/(\log(m))/m \). Then, the probability of a complete matching between \( A \) and \( B \) is higher then
\[
1 - \frac{em^{-2(\zeta-1)/3}}{1 - m^{-(\zeta-1)/3}}.
\]

**Proof.** For a set \( X \) of vertices, let \( \Gamma(X) \) be the set of all vertices adjacent to at least one member of \( X \). Then, if there is no complete matching between \( A \) and \( B \), there is some set \( X \), with either \( X \subset A \) or \( X \subset B \), which violates Hall’s condition, i.e., \( |\Gamma(X)| < |X| \). Let \( X \) be the smallest set satisfying this property (so the subgraph induced by \( X \cup \Gamma(X) \) is connected). The probability that such sets \( X \) and \( \Gamma(X) \) of respective sizes \( i \) and \( j \) exist is upper bounded by appropriately combining:

- the number of choices for \( X \), i.e., \( 2 \) (for selecting \( A \) or \( B \)) times \( \binom{m}{i} \),
- the number of choices for \( \Gamma(X) \), i.e., \( \binom{m}{i-1} \),
- an upper bound for the probability that under these choices of \( X \) and \( \Gamma(X) \) there are at least \( 2i - 2 \) edges (as the subgraph induced by \( X \cup \Gamma(X) \) is connected), i.e., \( \binom{ij}{i+j-1} \) possible choices of the vertex pairs and \( p^{i+j-1} \) the probability that these vertex pairs all form edges, and
- the probability that there is no edge between any of \( X \cup \Gamma(X) \) and the other vertices, i.e., \( (1 - p)^{i(m-j) + j(m-i)} = \frac{1}{(1 - p)^{m(i+j) - 2ij}} \).

Thus, we upper bound the probability of observing such sets \( X \) and \( \Gamma(X) \) of sizes \( i \) and \( j \) as follows:
\[
\text{FAIL}_B(i, j) \leq \binom{m}{i} \binom{m}{j} \binom{ij}{i+j-1} p^{i+j-1}(1 - p)^{m(i+j) - 2ij} \leq \left( \frac{me}{i} \right)^{i} \left( \frac{me}{j} \right)^{j} \frac{ij}{i+j-1} p^{i+j-1}(1 - p)^{m(i+j) - 2ij}.
\]

Here, in the second line the classic upper bound for combinations is used: \( \binom{m}{i} < \left( \frac{me}{i} \right)^{i} \). As \( 2j \leq i + j - 1 \), we get
\[
\text{FAIL}_B(i, j) \leq \frac{me}{i} \frac{me}{j} \frac{i}{2} p^{i+j-1}(1 - p)^{m(i+j) - 2ij} \leq \frac{m^{i+j} e^{2i+2j-1} i^{j-1}}{j^{2i+j-1}} p^{i+j-1}(1 - p)^{m(i+j) - 2ij}.
\]

\(^2\)In particular, even though Bollobas’s proof is asymptotically tight, its last line uses the fact that \( (e \log n)^3 n^{1-a+a^2/n} = o(1) \) for all \( a < n^{3/2} \). This expression is only lower than 1 for \( n \geq 5.6 \cdot 10^{10} \), and as the sum of this expression over all possible values of \( a \) needs to be smaller than \( \delta_F \), we do not expect this proof applies to graphs representing current real-life datasets.
As $0 < p < 1$, there also holds

$$(1 - p)^{1/p} < 1/e,$$

and therefore

$$(1 - p)^{m(i+j) - 2ij} = (1 - p)^{\frac{1}{e}p(m(i+j) - 2ij)} < (1/e)^{p(m(i+j) - 2ij)}.$$ We can substitute $p = \zeta(\log(m))/m$ to obtain

$$(1 - p)^{m(i+j) - 2ij} < (1/e)^{\zeta(\log(m))/m(m(i+j) - 2ij)} = \left(\frac{1}{m}\right)^{\zeta(m(i+j) - 2ij)/m}.$$ Substituting this and $p < \zeta(\log(m))/m$ into Equation (26), we get

$$FAIL_B(i, j) \leq m^{i+j} e^{2i+j - 1} \left(\frac{\log(m)\zeta}{2}\right)^{i+j-1} \left(\frac{1}{m}\right)^{\zeta(m(i+j) - 2ij)/m} = \frac{me^{i-1}}{j^j} \left(\frac{\log(m)\zeta^2}{2}\right)^{i+j-1} \left(\frac{1}{m}\right)^{\zeta((i+j) - 2ij)/m}.$$ As $j < i < m/2$, $2 \leq i < (i + j) - 2ij/m < i + j$, there follows

$$FAIL_B(i, j) \leq \frac{me^{i-1}}{j^j} \left(\frac{\log(m)\zeta^2}{2}\right)^{i+j-1} \left(\frac{1}{m}\right)^{\zeta((i+j) - 2ij)/m}.$$ Given that $m^{\zeta/3} \geq \zeta(\log(m)e^2/2$ holds for $m \geq 27$ and $\zeta \geq 4$, we get

$$FAIL_B(i, j) \leq \frac{me^{i-1}}{j^j} \left(\frac{\log(m)\zeta^2}{2m^{1/3}}\right)^{i+j-1} m^{-\zeta((i+j) - 2ij/m) + \zeta((i+j-1)/3}} \leq \frac{e^{i-1}}{j^j} m^{-\zeta(\frac{3}{4} + 1/3 - 2ij/m)} \leq \frac{e^{i-1}}{j^j} m^{-\zeta(\frac{3}{4} + 1/3) + 1}.$$ As $\zeta \geq 4$, this implies

$$FAIL_B(i, j) \leq \frac{e^i}{j^j} m^{-\zeta(\frac{3}{4} + 1/3)}.$$ (27) There holds:

$$\sum_{j=1}^{i-1} \binom{i-1}{j} j^j = \sum_{j=1}^{[i/3]} \binom{i}{j} j^j + \sum_{j=[i/3]+1}^{i} \binom{i}{j} j^j \leq \sum_{j=1}^{[i/3]} \binom{i}{j} j^j + \sum_{j=[i/3]+1}^{i} 3^j \sum_{j=1}^{[i/3]} \binom{i}{j} j^j + 3^{[i/3]+1} \frac{3^{i-[i/3]} - 1}{3 - 1} \leq \frac{i}{3} i^{i/3} + \frac{3^{i+1}}{2}.$$
Substituting in Equation (27) gives

$$\text{FAIL}_B = \sum_{i=2}^{m/2} \sum_{j=1}^{i-1} \text{FAIL}_B(i, j)$$

$$< \sum_{i=2}^{m/2} \sum_{j=1}^{i-1} \frac{e}{i} \left( \frac{i}{j} \right)^j m^{-\frac{i}{2} - \frac{1}{4}} < \sum_{i=2}^{m/2} \left( \frac{i^{3/2} + 3^{i/4} + 1}{3} \right) e m^{-\frac{i}{2} - \frac{1}{4}} < \sum_{i=2}^{m/2} \left( i^{3/2} m^{-\frac{i}{2} - \frac{1}{4}} + 3^{i/4} e m^{-\frac{i}{2} - \frac{1}{4}} \right).$$

As $m \geq 27 = 3^3$ we can now write

$$\text{FAIL}_B < \frac{\frac{1}{e} \sum_{i=2}^{m/2} m^{-\frac{(i-1)}{2} - \frac{1}{4}} + m^{(i+1)/3} m^{-\frac{i}{2} - \frac{1}{4}} < \frac{\frac{1}{e} \sum_{i=2}^{m/2} m^{-\frac{(i-1)}{2} - \frac{1}{4}} + m^{-\frac{(i-1)}{2}}}{1 - \left( m^{-\frac{i}{2} - \frac{1}{4}} \right)^i}.$$}

This concludes the proof.

**Lemma 4.** Let $m \geq 27$ and $\delta_B > 0$. Let

$$\zeta = \max \left( 4, 1 + \frac{3 \log(2e/\delta_B)}{2 \log(m)} \right).$$

Consider a random bipartite graph as described in Lemma 3 above. Then, with probability at least $1 - \delta_B$ there is a complete matching between $A$ and $B$.

**Proof.** From the given $\zeta$, we can infer that

$$\zeta - 1 \geq \frac{2 \log(m)}{2 \log(m)} \frac{3 \log(2e/\delta_B)}{2 \log(m)}$$

$$\zeta - 1 \geq \frac{3-2 \log(2e/\delta_B)}{2 \log(m)}$$

$$-\frac{3}{2} (\zeta - 1) \log(m) \leq \log(\delta_B/2e)$$

$$m^{-2(\zeta - 1)/3} \leq \delta_B/2e.$$

We also know that $\zeta \geq 4$ and $m \geq 27$, hence $(\zeta - 1)/3 \geq 1$ and $1 - m^{-2(\zeta - 1)/3} \geq 26/27 \geq 1/2$. We know from Lemma 3 that the probability of having a complete matching is at least

$$1 - \frac{em^{-2(\zeta - 1)/3}}{1 - m^{-2(\zeta - 1)/3}} \geq 1 - \frac{e(\delta_B/2e)}{1/2} = 1 - \delta_B.$$
Proof. Define an auxiliary random bipartite graph $G'$ with sides $A' = \{a'_1 \ldots a'_m\}$ and $B = \{b_1 \ldots b_m\}$. For every $i,j \in [m]$, the probability of having an edge between $a_i$ and $b_j$ in $G'$ is $p' = 1 - (1 - p)^{\frac{1}{\Delta}}$. We relate the distributions on the edges of $G^H$ and $G'$ by requiring there is an edge between $a_i$ and $b_j$ if and only if there is an edge between $a'_i, a'_{i+1}, \ldots, a'_j$ and $b_j$ for all $i' \in [\Delta]$.

From Equation (28) we can derive

$$p' \geq \max \left( 4, 1 + \frac{3 \log(2e/\delta_B)}{2 \log(m)} \right) \log(m)/m. \tag{29}$$

Setting

$$\zeta = \max \left( 4, 1 + \frac{3 \log(2e/\delta_B)}{2 \log(m)} \right),$$

this ensures $p'$ satisfies the constraints of Lemma 4:

$$p' = \zeta \log(m)/m.$$

As a result, there is a complete matching in $G'$ with probability at least $1 - \delta_B$, and hence the required stars can be found in $G^H$ with probability at least $1 - \delta_B$.

**Lemma 6.** Let $\delta_F > 0$ and $\delta_B > 0$. Let $G^H$ and $T$ and their associated variables be as defined above. Assume that the following conditions are satisfied:

(a) $n_H \geq 27(\Delta_q + 2)/\Delta_q$,

(b) $\gamma + 2(\Delta_q + 2)^{-1} + n_H^{-1} \leq 1$,

(c) $k_H \geq 4 \log(2n_H/\delta_F(\Delta_q + 2))$,

(d) $k_H \geq \max \left( 4, 1 + \frac{3 \log(2e/\delta_B)}{2 \log(n_H \Delta_q/(\Delta_q + 2))} \right) \frac{\Delta_q + 2}{2} \log \left( \frac{n_H \Delta_q}{\Delta_q + 2} \right)$,

(e) $\gamma = \max_{l=1}^{q-1} \Delta_l/k_H$.

Let $G^H$ be a random graph where there is an edge between any two vertices with probability $p$. Let $v_1$ be a vertex of $G^H$. Then, with probability at least $1 - \delta_F - \delta_B$, there is a subgraph isomorphism between the tree $T$ defined above and $G^H$ such that the root of $T$ is mapped on $v_1$.

Proof. The conditions of Lemma 2 are clearly satisfied, so with probability $1 - \delta_F$ there is a tree isomorphic to $F$ in $G^H$. Then, from condition (d) above and knowing that the edge probability is $p = 2k_H/n_H$, we obtain

$$p \geq \max \left( 4, 1 + \frac{3 \log(2e/\delta_B)}{2 \log(n_H \Delta_q/(\Delta_q + 2))} \right) \frac{1}{n_H \Delta_q/(\Delta_q + 2)} \log \left( \frac{n_H \Delta_q}{\Delta_q + 2} \right) \Delta_q.$$

Taking into account that $m = n_H \Delta_q/(\Delta_q + 2)$, we get

$$p \geq \max \left( 4, 1 + \frac{3 \log(2e/\delta_B)}{2 \log(m)} \right) \frac{1}{m} \log(m) \Delta_q,$$

which implies the condition on $p$ in Lemma 4. The other conditions of that lemma can be easily verified. As a result, with probability at least $1 - \delta_B$ there is a set of stars in $G^H$ linking the leaves of $F$ to the leaves of $T$, so we can embed $T$ completely in $G^H$.

**A.3.4. Running GOPA on Random Graphs**

Assume we run GOPA on a random graph satisfying the properties above, what can we say about the differential privacy guarantees? According to Lemma 1, it is sufficient that there exists a spanning tree and vectors $t_\eta$ and $t_\Delta$ such that $t_\eta + Kt_\Delta = X^A - X^B$. We fix $t_\eta$ in the same way as for the other discussed topologies (see Section A.2) in order to
achieve the desired $\sigma_\eta$ and focus our attention on $t_\Delta$. According to Lemma 6, with high probability there exists in $G^H$ a spanning tree rooted at the vertex where $X^A$ and $X^B$ differ and a branching factor $\Delta_l$ specified per level. So given a random graph on $n_H$ vertices with edge density $2k_H/n_H$, if the conditions of Lemma 6 are satisfied we can find such a tree isomorphic to $T$ in the communication graph between honest users $G^H$. In many cases (reasonably large $n_H$ and $k_H$), this means that the lemma guarantees a spanning tree with branching factor as high as $O(k_H)$, even though it may be desirable to select a small value for the branching factor of the last level in order to more easily satisfy condition (d) of Lemma 6, e.g., $\Delta_q = 2$ or even $\Delta_q = 1$.

**Lemma 7.** Under the conditions described above,

\[
\sum_l \prod_{i=1}^{l+1} \Delta_i = \frac{1}{\Delta_1} \left( 1 + \frac{1}{\Delta_2} \left( 1 + \frac{1}{\Delta_3} \left( \ldots \frac{1}{\Delta_q} \right) \right) \right) + \frac{(\Delta_q + 2)(\Delta_q + 2 + 2q)}{n_H} \leq \frac{1}{\Delta_1} \left( 1 + \frac{2}{\Delta_2} \right) + O(n_H^{-1}).
\]

**Proof.** Let $q$ be the depth of the tree $T$. The tree is balanced, so in every node the number of vertices in the subtrees of its children differs at most $\Delta_q + 2$. For edges $e$ incident with the root (a level 0 node), $|t_e - \Delta_1^{-1}| \leq n_H^{-1}(\Delta_q + 2)$. In general, for a node at level $l$ (except leaves or parents of leaves), there are $\prod_{i=1}^{l+1} \Delta_i$ vertices, each of which have $\Delta_{l+1}$ children, and for every edge $e$ connecting such a node with a child,

\[
|t_e - \prod_{i=1}^{l+1} \Delta_i^{-1}| \leq (\Delta_q + 2)/n_H.
\]

For a complete tree (of $1 + \Delta + \ldots + \Delta^q$ vertices), we would have

\[
t_\Delta^T t_\Delta = \sum_l \prod_{i=1}^{l+1} \Delta_i = \frac{1}{\Delta_1} \left( 1 + \frac{1}{\Delta_2} \left( 1 + \frac{1}{\Delta_3} \left( \ldots \frac{1}{\Delta_q} \right) \right) \right) = \frac{1}{\Delta_1} \left( 1 + \frac{2}{\Delta_2} \right) + O(n_H^{-1}),
\]

which corresponds to the first term in Equation (30). As the tree may not be complete, i.e., there may be less than $\prod_{i=1}^{l+1} \Delta_i$ leaves, we analyze how much off the above estimate is. For an edge $e$ connecting a vertex of level $l$ with one of its children,

\[
|t_e - \prod_{i=1}^{l+1} \Delta_i| \leq (\Delta_q + 2)/n_H,
\]

and hence

\[
|t_e^2 - \left( \prod_{i=1}^{l+1} \Delta_i \right)^2| \leq \left( t_e^2 - \prod_{i=1}^{l+1} \Delta_i \right) + \left( \prod_{i=1}^{l+1} \Delta_i - \prod_{i=1}^{l+1} \Delta_i \right) \leq t_e(\Delta_q + 2)/n_H + \prod_{i=1}^{l+1} \Delta_i (\Delta_q + 2)/n_H
\]

\[
\leq (\prod_{i=1}^{l+1} \Delta_i + n_H^{-1})(\Delta_q + 2)/n_H + \prod_{i=1}^{l+1} \Delta_i (\Delta_q + 2)/n_H
\]

\[
= (\Delta_q + 2)^2/n_H^2 + 2 \prod_{i=1}^{l+1} \Delta_i (\Delta_q + 2)/n_H.
\]
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Summing over all edges gives

\[
\sum_{l=1}^{q} \left( \prod_{i=1}^{l} \Delta_i \right)^{-1} \leq \sum_{l=1}^{q} z_l \left( (\Delta_q + 2)/n_H^2 + 2 \left( \prod_{i=1}^{l+1} \Delta_i \right)^{-1} (\Delta_q + 2)/n_H \right)
\]

\[
= (\Delta_q + 2)^2/n_H + \sum_{l=1}^{q} 2z_l \left( \prod_{i=1}^{l} \Delta_i \right)^{-1} (\Delta_q + 2)/n_H
\]

\[
\leq (\Delta_q + 2)^2/n_H + \sum_{l=1}^{q} 2(\Delta_q + 2)/n_H
\]

\[
= \frac{(\Delta_q + 2)(\Delta_q + 2 + 2q)}{n_H}.
\]

So if we choose parameters \( \Delta \) for the tree \( T \), the above lemmas provide a value \( \delta \) such that \( T \) can be embedded in \( G^H \) with probability at least \( 1 - \delta \) and an upper bound for \( t^\Delta \) that can be obtained with the resulting spanning tree in \( G^H \).

Theorem 3 in the main text summarizes these results, simplifying the conditions by assuming that \( \Delta_i = \lfloor (k - 1)\rho/3 \rfloor \) for \( i \leq q - 1 \) and \( \Delta_q = 2 \).

**Proof of Theorem 3.** Let us choose \( \Delta_i = \lfloor (k - 1)\rho/3 \rfloor \) for \( i \in [q - 1] \) and \( \Delta_q = 1 \) for some appropriate \( q \) such that Equation (23) is satisfied. We also set \( \delta = \delta_F = \delta_B \).

Then, the conditions of Lemma 6 are satisfied. In particular, condition (a) holds as \( n_H = \rho n \geq 81 = 27(\Delta_q + 2)/\Delta_q \). Condition (e) implies that

\[
\gamma = \min_{i=1}^{q} \Delta_i/k_H = \frac{1}{k_H} \left( \frac{k - 1)\rho}{3} \right).
\]

Condition (b) holds as

\[
\gamma + 2(\Delta_q + 2)^{-1} + n_H^{-1} = \frac{1}{k_H} \left( \frac{(k - 1)\rho}{3} \right) + 2 + n_H^{-1}
\]

\[
\leq \frac{1}{k_H} \left( \frac{(k - 1)\rho}{3} \right) + \frac{2}{3} + n_H^{-1} \leq \frac{1}{3} - \frac{\rho}{3k_H} + \frac{2}{3} + n_H^{-1} = \frac{1}{3} - \frac{1}{3k} + \frac{2}{3} + n_H^{-1}
\]

\[
\leq \frac{1}{3} - \frac{1}{n_H} + \frac{2}{3} + n_H^{-1} = 1.
\]

Condition (d) holds because we know that \( \rho k \geq 6 \log(\rho n/3) \), which is equivalent to

\[
k_H \geq 4 \frac{\Delta + 2}{\Delta} \log \left( \frac{n_H \Delta_q}{\Delta_q + 2} \right),
\]

and we know that \( \rho k \geq \frac{3}{2} + \frac{9}{4} \log(2e/\delta) \), which is equivalent to

\[
k_H \geq \left( 1 + \frac{3 \log(2e/\delta_B)}{2 \log(n_H \Delta_q/(\Delta_q + 2))} \right) \frac{\Delta + 2}{\Delta} \log \left( \frac{n_H \Delta_q}{\Delta_q + 2} \right).
\]

Finally, condition (c) is satisfied as we know that \( \rho k \geq 4 \log(\rho n/3\delta) \). Therefore, applying the lemma, we can with
probability at least \(1 - 2\delta\) find a spanning tree isomorphic to \(T\). If we find one, Lemma 7 implies that

\[
t_{\Delta}^T t_{\Delta} \leq \sum_{l=1}^{q} \left( \prod_{i=1}^{l} \Delta_i \right)^{-1} + \frac{(\Delta_q + 2)(\Delta_q + 2 + 2q)}{n_H}
\]

\[
= \sum_{l=1}^{q} \Delta_1^{-l} + \Delta_1^{-1} \Delta_q^{-1} + \frac{3(3 + 2q)}{n_H} = \Delta_1^{-1} \frac{1}{1 - \Delta_1^{-1}} + \frac{9 + 6q}{n_H}
\]

\[
\leq \frac{1}{\Delta_1 - 1} + \frac{3}{n_H} + \frac{9 + 6q}{n_H} = \frac{1}{[(k - 1)\rho/3] - 1} + \frac{12 + 6q}{n_H}
\]

This implies the conditions related to \(\sigma_\Delta\) and \(t_\Delta\) are satisfied. From Lemma 1, it follows that with probability \(1 - 2\delta\) GOPA is \((\varepsilon, \delta)\)-differentially private, or in short GOPA is \((\varepsilon, 3\delta)\)-differentially private. \(\square\)

### A.4. Matching the Utility of the Centralized Gaussian Mechanism

From the above theorems, we can now obtain a simple corollary which precisely quantifies the amount of independent and pairwise noise needed to achieve a desired privacy guarantee depending on the topology.

**Proof of Corollary 1.** In the centralized (trusted curator) setting, the standard centralized Gaussian mechanism (Dwork & Roth, 2014, Theorem A.1 therein) states that in order for the noisy average \((\frac{1}{n} \sum_{u \in U} X_u) + \eta\) to be \((\varepsilon', \delta')\)-DP for some \(\varepsilon', \delta' \in (0, 1)\), the variance of \(\eta\) needs to be:

\[
\sigma_{gm}^2 = \frac{c^2}{(\varepsilon'n)^2},
\]

(31)

where \(c^2 > 2 \log(1.25/\delta')\).

Based on this, we let the independent noise \(\eta_u\) added by each user in GOPA to have variance:

\[
\sigma_{\eta}^2 = \frac{n^2 \sigma_{gm}^2}{n_H} = \frac{c^2}{(\varepsilon'n_H)^2},
\]

(32)

which, for the approximate average \(\hat{X}_{avg}\), gives a total variance of:

\[
Var \left( \frac{1}{n_H} \sum_{u \in U_H} \eta_u \right) = \frac{1}{n_H} n_H \sigma_{\eta}^2 = \frac{c^2}{(\varepsilon'n_H)^2}.
\]

(33)

We can see that when \(n_H = n\) (no malicious user, no dropout), Equation (33) exactly corresponds to the variance required by the centralized Gaussian mechanism in Equation (31), hence GOPA will achieve the same utility. When there are malicious users and/or dropouts, each honest user needs to add a factor \(n/n_H\) more noise to compensate for the fact that drop out users do not participate and malicious users can subtract their own inputs and independent noise terms from \(\hat{X}_{avg}\). This is consistent with previous work on distributed noise generation under malicious parties (Shi et al., 2011).

Now, given some \(\kappa > 0\), let \(\sigma_\Delta^2 = \kappa \sigma_{\eta}^2\) if \(G\) is the complete graph, \(\sigma_\Delta^2 = \frac{1}{[(k - 1)\rho/3] - 1} + \frac{12 + 6 \log(n_H)}{n_H}\) for the random \(k\)-out graph, and \(\sigma_\Delta^2 = \kappa n_H^2 \sigma_{\eta}^2/3\) for an arbitrary connected \(G^{\text{CH}}\). In all cases, the value of \(\theta\) in Theorems 1, 2 and 3 after plugging \(\sigma_\Delta^2\) gives

\[
\theta = \frac{\varepsilon^2}{c^2} + \frac{\varepsilon^2}{\kappa c^2} = \frac{(\kappa + 1)\varepsilon^2}{\kappa c^2}.
\]

By setting \(\varepsilon = \varepsilon'\), Equation (3) of Theorem 1 implies:

\[
\varepsilon \geq \frac{(\kappa + 1)\varepsilon^2}{2\kappa c^2} + \sqrt{\frac{(\kappa + 1)\varepsilon}{c}}
\]
Table 2: Examples of admissible values for $k$ and $\sigma_\Delta$, obtained by numerical simulation, to ensure $(\varepsilon, \delta)$-DP with trusted curator utility for $\varepsilon = 0.1$, $\delta' = 1/n^2_H$, $\delta = 10\delta'$.

| $n = 100$ | $\rho = 1$ | $k = 3$ | $\sigma_\Delta = 60.8$ |
|           |           | $k = 5$ | $\sigma_\Delta = 41.3$ |
|           | $\rho = 0.5$ | $k = 20$ | $\sigma_\Delta = 26.8$ |
|           |           | $k = 30$ | $\sigma_\Delta = 17.2$ |
| $n = 1000$ | $\rho = 1$ | $k = 5$ | $\sigma_\Delta = 63.4$ |
|           |           | $k = 10$ | $\sigma_\Delta = 41.1$ |
|           | $\rho = 0.5$ | $k = 20$ | $\sigma_\Delta = 45.4$ |
|           |           | $k = 30$ | $\sigma_\Delta = 27.3$ |
| $n = 10000$ | $\rho = 1$ | $k = 10$ | $\sigma_\Delta = 54.6$ |
|           |           | $k = 20$ | $\sigma_\Delta = 34.7$ |
|           | $\rho = 0.5$ | $k = 20$ | $\sigma_\Delta = 55.5$ |
|           |           | $k = 40$ | $\sigma_\Delta = 28.4$ |

For $d^2 = \frac{\kappa}{\kappa + 1}e^2$ we can rewrite the above as $\varepsilon \geq \frac{2^2}{2d^2} + \frac{\varepsilon}{d}$. Since $\varepsilon \leq 1$, this is satisfied if $d - \frac{\varepsilon}{2d} \geq 1$ and in turn when $d \geq 3/2$, or equivalently when $c \geq 3 \sqrt{\frac{\varepsilon + 1}{\kappa}}$. Now analyzing the inequality in Equation (4) we have:

$$
\varepsilon^2 + \frac{\varepsilon}{2} \left( \frac{\varepsilon^2}{c^2} + \frac{\varepsilon^2}{c^2} \right) \geq 2 \log \left( \frac{2}{\delta} \sqrt{2\pi} \right) + \frac{\varepsilon}{c^2} \left( \frac{\varepsilon}{c^2} - \varepsilon \right) \geq \log \left( \frac{2}{\delta} \sqrt{\frac{\varepsilon}{\kappa}} \right).
$$

Again denoting $d^2 = \frac{\kappa}{\kappa + 1}e^2$ we can rewrite the above as

$$
\frac{1}{2} \left( d^2 + \frac{\varepsilon^2}{4d^2} - \varepsilon \right) \geq \log \left( \frac{2}{\delta} \sqrt{\frac{\varepsilon}{\kappa}} \right).
$$

For $d \geq 3/2$ and $\varepsilon \leq 1$, the derivative of $d^2 + \frac{\varepsilon^2}{4d^2} - \varepsilon$ is positive, so $d^2 + \frac{\varepsilon^2}{4d^2} - \varepsilon > d^2 - 8/9$. Thus, we only require $d^2 \geq 2 \log (1.25/\delta)$. Therefore Equation (4) is satisfied when:

$$
\frac{\kappa}{\kappa + 1} \log \left( \frac{1.25}{\delta} \right) \geq \log \left( \frac{1.25}{\delta} \right),
$$

which is equivalent to $\delta \geq 1.25 \left( \frac{\varepsilon}{1.25} \right)^{1/\kappa+1}$. The constant $3.75$ instead of $1.25$ for the random $k$-out graph case is because Theorem 3 guarantees $(\varepsilon, 3\delta)$-DP instead of $(\varepsilon, \delta)$ in Theorems 1 and 2.

A.5. Smaller $k$ and $\sigma_\Delta^2$ via Numerical Simulation

For random $k$-out graphs, the conditions on $k$ and $\sigma_\Delta^2$ given by Theorem 3 are quite conservative. While we are confident that they can be refined by resorting to tighter approximations in our analysis in Section A.3, an alternative option to find smaller, yet admissible values for $k$ and $\sigma_\Delta^2$ is to resort to numerical simulation.

Given the number of users $n$, the proportion $\rho$ of nodes who are honest and do not drop out, and a value for $k$, we implemented a program that generates a random $k$-out graph, checks if the subgraph $G^H$ is connected, and if so finds a suitable spanning tree for $G^H$ and computes the corresponding value for $t^H_\Delta$ needed by our differential privacy analysis (see for instance Appendix A.2). From this, we can in turn deduce a sufficient value for $\sigma_\Delta^2$ using Corollary 1.

Table 2 gives examples of values obtained by simulations for various values of $n$, $\rho$ and several choices for $k$. In each case, the reported $\sigma_\Delta$ corresponds to the worst-case value required across $10^6$ random runs, and the chosen value of $k$ was large enough for $G^H$ to be connected in all runs. This was the case even for slightly smaller values of $k$. Therefore, the values reported in Table 2 can be considered safe to use in practice.
Appendix B. Cryptographic Framework

In this appendix, we provide an extended overview of the cryptographic notions involved in our verification protocol of Section 5. First, we give deeper intuitions on commitments and ZKPs in Appendices B.1 and B.2. Next, using these building blocks, we present ZKPs on the statistical distributions of uniform and Gaussian random variables in Appendix B.3. After that, we describe how to build a source of unbiased public randomness to generate crypto parameters and ZKPs in Appendix B.4. We conclude by discussing further issues on finite precision in Appendix B.5. For simplicity, we sometimes abstract away some of the technical cryptographic details, but refer to the corresponding bibliography for a more advanced treatment.

Public bulletin board. We implement the publication of commitments and proofs using a public bulletin board so that any party can verify the validity of the protocol, avoiding the need for a trusted verification entity. Users sign their messages so they cannot deny them. More general purpose distributed ledger technology such as in Bitcoin transactions (Nakamoto, 2008) could be used here, but we aim at an application-specific, light-weight and hence more scalable solution.

Cryptographic hash functions. We also make use of a hash function \( H : \mathbb{Z} \rightarrow \mathbb{Z}_2^T \) for a number \( T \) such that \( 2^T \) is a few orders of magnitude bigger than any cryptographic security parameter \( p \) we use, e.g., the order of finite groups for commitments (see Appendix B.1), etc. \( T \) is chosen in that way so that numbers uniformly distributed over \( \mathbb{Z}_2^T \) modulo \( p \) are indistinguishable from numbers uniformly distributed over \( \mathbb{Z}_p \). Such function is easy to evaluate, but predicting its outcome or distinguishing it from random numbers is intractable for polynomially bounded algorithms. Practical instances of \( H \) can be found in HMAC (Barker & Kelsey, 2007) and Sponge-based constructions (Bertoni et al., 2010) for pseudo-random number generators (PRNGs) (Yao, 1982; Blum & Micali, 1984).

B.1. Commitment Schemes

We start by defining formally a commitment and its properties. For clarity, we will use bold variables to denote commitments.

**Definition 2** (Commitment Scheme). A commitment scheme consists of a pair of (computationally efficient) algorithms \((\text{Setup}, \text{Com})\). The setup algorithm \( \text{Setup} \) is executed once, with randomness \( t \) as input, and outputs a tuple \( \Theta \leftarrow \text{Setup}(t) \), which is called the set of parameters of the scheme. The algorithm \( \text{Com} \) with parameters \( \Theta \), denoted \( \text{Com}_\Theta \), is a function \( \text{Com}_\Theta : \mathcal{M}_\Theta \times \mathcal{R}_\Theta \rightarrow \mathcal{C}_\Theta \), where \( \mathcal{M}_\Theta \) is called the message space, \( \mathcal{R}_\Theta \) the randomness space, and \( \mathcal{C}_\Theta \) the commitment space. For a message \( m \in \mathcal{M}_\Theta \), the algorithm draws \( r \in \mathcal{R}_\Theta \) uniformly at random and computes commitment \( c \leftarrow \text{Com}_\Theta(m, r) \).

The security of a commitment scheme typically depends on \( t \) not being biased, in particular, it must be hard to guess non-trivial information about \( t \).

We now define some key properties of commitments.

**Property 1** (Hiding Property). A commitment scheme is hiding if, for all secrets \( x \in \mathcal{M}_\Theta \) and given that \( r \) is chosen uniformly at random from \( \mathcal{R}_\Theta \), the commitment \( c_x = \text{Com}_\Theta(x, r) \) does not reveal any information about \( x \).

**Property 2** (Binding Property). A commitment is binding if there exists no computationally efficient algorithm \( A \) that can find \( x_1, x_2 \in \mathcal{M}_\Theta \), \( r_1, r_2 \in \mathcal{R}_\Theta \) such that \( x_1 \neq x_2 \) and \( \text{Com}_\Theta(x_1, r_1) = \text{Com}_\Theta(x_2, r_2) \).

**Property 3** (Homomorphic Property). A homomorphic commitment scheme is a commitment scheme such that \( \mathcal{M}_\Theta, \mathcal{R}_\Theta \) and \( \mathcal{C}_\Theta \) are abelian groups, and for all \( x_1, x_2 \in \mathcal{M}_\Theta \), \( r_1, r_2 \in \mathcal{R}_\Theta \) we have

\[
\text{Com}_\Theta(x_1, r_1) + \text{Com}_\Theta(x_2, r_2) = \text{Com}_\Theta(x_1 + x_2, r_1 + r_2).
\]

Please note that the three occurrences of the ‘+’ sign in the above definition are operations in three difference spaces, and hence may have different definitions and do not necessarily correspond to normal addition of numbers.

**Pedersen commitments.** For our protocol we use the Pedersen commitment scheme, described in (Pedersen, 1991). Let \( p \) and \( q \) be two large primes such that \( q \) divides \( p - 1 \), and a let \( \mathbb{G} \) be cyclic subgroup of order \( q \) of the multiplicative group \( \mathbb{Z}_p^* \). For such group, we have that \( \mathbb{G} = \{ a^i \mod p \mid 0 \leq i < q \} \) for any \( a \in \mathbb{G} \) distinct to 1. In the Pedersen scheme, \( \text{Setup} \) is a function that draws uniformly and independently at random from \( \mathbb{G} \) a pair of elements \( g \) and \( h \) which form the set of parameters \( \Theta = (g, h) \). Additionally, \( \mathcal{M}_\Theta = \mathcal{R}_\Theta = \mathbb{Z}_q \), and \( \mathcal{C}_\Theta = \mathbb{G} \). We will refer to \( g \) and \( h \) as bases. The commitment
We base our proofs in a Pedersen commitment with parameters \((g, h)\) chosen from \(\mathbb{G}\) as described in Appendix B.1. For some building blocks, we use multiple pairs of bases as parameters, where bases belonging to the same pair \(\Theta\) are always assumed to be chosen independently at random from \(\mathbb{G}\). We denote by \(r \leftarrow \mathcal{S}\) the operation of generating a random number \(r\) uniformly distributed over the set \(\mathcal{S}\), and by \(a \leftarrow b\) the action of verifying the equality of \(a\) and \(b\). For an integer \(n\), we denote by \(\langle n \rangle\) its size in bits, i.e. \(\langle n \rangle = \lfloor \log_2 n \rfloor + 1\). Products and exponentiations of members of \(\mathbb{G}\) are implicitly modulo \(p\), where the prime number \(p\) is the order of \(\mathbb{Z}_p^*\), of which \(\mathbb{G}\) is subgroup of. To analyze the computational cost of

\[
\text{function } \text{Com}_{\Theta} \text{ is defined as}
\]

\[
\text{Com}_{\Theta}: \mathbb{Z}_q \times \mathbb{Z}_q \rightarrow \mathbb{G}
\]

\[
\text{Com}_{\Theta}(x, r) = g^x \cdot h^r,
\]

where \((\cdot)\) is the product modulo \(p\) of group \(\mathbb{G}\), \(x\) is the secret and \(r\) is the randomness. For simplification and when \(r\) is not relevant, we relax the notation \(\text{Com}_{\Theta}(x, r)\) to \(\text{Com}_{\Theta}(x)\) and assume \(r\) is drawn appropriately. Pedersen commitments are homomorphic (in particular, \(\text{Com}_{\Theta}(x+y, r+s) = \text{Com}_{\Theta}(x, r) \cdot \text{Com}_{\Theta}(y, s)\)), unconditionally hiding, and computationally binding under the Discrete Logarithm Assumption, which we informally describe below.

**Assumption 1** (Discrete Logarithm Assumption). Let \(\mathbb{G}\) be a cyclic multiplicative group of large prime order \(q\), and \(g\) and \(h\) two elements chosen independently and uniformly at random from \(\mathbb{G}\). Then, there exists no probabilistic polynomial time algorithm \(\mathcal{A}\) that takes as input the tuple \((\mathbb{G}, g, h)\) and outputs a value \(b\) such that \(P(g^b = h)\) is significant on the size of \(q\).

The DLA is a standard assumption widely used in production environments. A more detailed description can be found in Chapter 7 of (Katz & Lindell, 2014).

The Pedersen commitment scheme can be efficiently implemented with elliptic curves, as described and benchmarked by (Franck & Großschädl, 2017). To generate a common Pedersen scheme in our adversary model, the generation of the unbiased random input \(t\) for \(\text{Setup}\) can be done as described in Algorithm 2 of Appendix B.4.

### B.2. Zero Knowledge Proofs (ZKPs)

On top of Pedersen commitments, we use a family of techniques called Zero Knowledge Proofs (ZKPs), first proposed by (Goldwasser et al., 1989). In these proofs, a party \(P\) called the Prover, convinces another party \(V\), the Verifier, about a statement. For our scope and informally speaking, ZKPs\(^3\)

- allow every \(P\) to successfully prove true a statement (completeness),
- allow every \(V\) to discover with arbitrarily large probability any attempt to prove a false statement (soundness),
- guarantee that by performing the proof, no information about the knowledge of \(P\) other than the proven statement is revealed (zero knowledge).

Importantly, the zero knowledge property of our proofs does not rely on any computational hardness assumption.

**Honest-verifier ZKPs.** We describe how our ZKPs can be performed as an interactive protocol between a prover \(P\) and a (virtual) honest verifier \(V\) whose only task in the conversation is to generate unbiased random numbers, that we will call *challenges*. In our proofs, \(P\) must not be able to choose messages that are previous to the challenge adaptively according to it. Properties of ZKPs mentioned above only hold if \(V\) is honest, which does not apply to our malicious adversary model. We show how we overcome this in Appendices B.2.1 and B.4.

#### B.2.1. ZKP Building Blocks

We now briefly review a number of ZKP building blocks present in literature and necessary for our verification protocol. We illustrate the basic principles for a selection of relevant ZKPs, describing their protocol and complexity, and refer to the literature for others.

We base our proofs in a Pedersen commitment with parameters \((g, h)\) chosen from \(\mathbb{G}\) as described in Appendix B.1. For some building blocks, we use multiple pairs of bases as parameters, where bases belonging to the same pair \(\Theta\) are always assumed to be chosen independently at random from \(\mathbb{G}\). We denote by \(r \leftarrow \mathcal{S}\) the operation of generating a random number \(r\) uniformly distributed over the set \(\mathcal{S}\), and by \(a \leftarrow b\) the action of verifying the equality of \(a\) and \(b\). For an integer \(n\), we denote by \(\langle n \rangle\) its size in bits, i.e. \(\langle n \rangle = \lfloor \log_2 n \rfloor + 1\). Products and exponentiations of members of \(\mathbb{G}\) are implicitly modulo \(p\), where the prime number \(p\) is the order of \(\mathbb{Z}_p^*\), of which \(\mathbb{G}\) is subgroup of. To analyze the computational cost of

\(^3\)Strictly speaking, the proofs we will use are called *arguments*, as the soundness property relies on the computational boundedness of the Prover \(P\) through the DLA described above, but as for general reference to the family of techniques we use the term *proofs*. 
the building blocks, we count modular exponentiations which are the dominant operations. For a ZKP, we call the size of a proof to the amount of bits of all the values exchanged between \( P \) and \( V \). The size of every plain number, commitment or challenge exchanged is not bigger than \((p)\) bits.

**Proof of knowledge.** This protocol allows for a prover \( P \) to prove that he knows the value \( x \) and the randomness \( r \) used to compute a commitment \( c_x = g^x \cdot h^r \). The protocol goes as follows:

1. \( P \) draws \( a, b \leftarrow_R \mathbb{Z}_q \), computes \( c_{ab} \leftarrow g^a h^b \) and sends \( c_{ab} \) to \( V \).
2. \( V \) draws a challenge \( t \leftarrow_R \mathbb{Z}_q \) and sends it to \( P \).
3. \( P \) computes \( d \leftarrow a + xt \pmod{q} \), \( e \leftarrow b + rt \pmod{q} \) and sends \( (d, e) \) to \( V \).

To verify the correctness of the proof, \( V \) checks that \( g^d h^e = \tau c_{ab}(c_x)^t \). We denote this proof as \( \text{ZKPBasic}(c_x)\{(x, r) : c_x = g^x h^r\} \),

where we use the convention that values outside the curly brackets are public, while values private to \( P \) and the proven statement are inside. The computational cost of the proof is 2 modular exponentiations for \( P \) and 3 for \( V \). The complete description of the protocol for a single base is by (Schnorr, 1991) and a generalization to many bases (in our case, \( g \) and \( h \)) is due to (Chaum et al., 1988). By the use of the Fiat-Shamir transformation (see the end of Appendix B.2) this and all of our proofs require only 1 message for \( P \) and 1 message for \( V \) (i.e. \( P \) publishes the proof in the bulletin board and \( V \) later downloads it) and the transfer of challenge \( t \) is avoided. Therefore, the proof size is \( 3(p) \) bits.

**Proof of equality.** This protocol allows for a prover \( P \) to prove that he committed to the same private value \( x \) in two different commitments \( c_x = g_1^x \cdot h_1^r \) and \( c_x' = g_2^x \cdot h_2^r \). The pairs of bases \( (g_1, h_1) \) and \( (g_2, h_2) \) might be different. This proof is described by (Chaum and Pedersen, 1993) for a single base and, as for ZKPBasic, a generalization for two pairs of bases can be performed as in (Chaum et al., 1988).

The protocol goes as follows:

1. \( P \) generates \( a, b, c \leftarrow_R \mathbb{Z}_q \), computes \( c_{ab} \leftarrow_R g_1^a h_1^b \) and \( c_{ac} \leftarrow g_2^a h_2^c \) and sends \( (c_{ab}, c_{ac}) \) to \( V \).
2. \( V \) draws a challenge \( t \leftarrow_R \mathbb{Z}_q \) and sends it to \( P \).
3. \( P \) computes \( d \leftarrow a + xt \pmod{q} \), \( e \leftarrow b + rt \pmod{q} \) and \( f \leftarrow c + r't \pmod{q} \) and sends \( (d, e, f) \) to \( V \).

At any time, to verify the proof \( (c_{ab}, c_{ac}, t, d, e, f) \), \( V \) checks that

\[
g_1^d h_1^e = \tau c_{ab}(c_x)^t \quad \text{and} \quad g_2^d h_2^f = \tau c_{ac}(c_x')^t.
\]

If the equality holds, \( P \) proves the statement. We denote this proof as \( \text{ZKPEq}(c_x, c_x')\{(x, r, r') : c_x = g_1^x h_1^r \land c_x' = g_2^x h_2^r \} \).

The proof requires the computation of 4 modular exponentiations for \( P \) and 6 for \( V \). The proof size is of \( 5(p) \) bits.

**Non-interactive ZKPs and logical composition.** As ZKPBasic and ZKPEq above, all of our ZKPs have the “three-move” form of \((m_1, t, m_2)\) where \( m_1 \) and \( m_2 \) are messages sent by the prover \( P \) and \( t \in \mathbb{Z}_q \) is a random challenge generated by a verifier \( V \). Additionally, the probability of \( P \) cheating without being discovered is in \( O(1/q) \). For these type of proofs, by the use of the Fiat-Shamir heuristic (Fiat and Shamir, 1987) which adapts to our cryptographic setting as in (Schnorr, 1991), we replace the interactive role of \( V \) in the generation of the proof. With the use of this heuristic, \( P \) computes itself the challenge \( t \) by the use of a cryptographic hash function \( H \) (in theory, a random oracle) such that \( t = H(m_1 + c) \pmod{q} \), where \( c \) is the sum of all commitments involved in the proven statement. The construction does not compromise any of the properties a ZKP must have (Pointcheval & Stern, 1996), it is suitable for our setting where we do not assume there are honest verifiers, and does not incur into significant additional computational cost.
Additionally, any pair of ZKPs \( \text{ZKP}_1 \) and \( \text{ZKP}_2 \) that we generate can be composed such that we prove the conjunction or a disjunction of its statements (Cramer et al., 1994), in such a way that only one challenge \( t \) is used in the proof. When composing proofs, the generation remains a “three-move” computation with a cost and proof size equal to the sum of that of \( \text{ZKP}_1 \) and \( \text{ZKP}_2 \) and the verifier has the same order of probability of discovering a cheater. We will use these compositions below and refer to them as \( \text{ZKPAnd} \) and \( \text{ZKPOR} \) for AND and OR compositions respectively. By the use of these techniques, the prover \( P \) can generate a single “three-move” proof of logical statements that combine basic blocks using the above operators. Hence, any set of the non-active proofs involved in the correctness of \( \text{GOPA} \) can be combined and generated together at once, by a single one non-active “three-move” protocol, and posted into the bulletin board, that later can be verified by any party.

**Proof of linear relation.** This is a key proof that will be used several times in our verification procedure. Let \( \bar{x} = (x_i)_{i=1}^k \) for some \( k > 0 \) be a vector of private values, \( \bar{c} = (c_i)_{i=1}^k \) a public vector of commitments such that \( c_i \) is a commitment of \( x_i, \bar{a} = (a_i)_{i=1}^k \) a public vector of integer coefficients, and \( b \) a public scalar. The goal of \( P \) is to prove the equality \( \langle \bar{x}, \bar{a} \rangle = b \pmod{q} \), where \( \langle , \rangle \) is the inner product. The proof is simple: first, note that, by the homomorphic property, 
\[
\text{ZKPLinear}(\bar{c}, \bar{a}, b) = \left\{ \langle \bar{x}, \bar{a} \rangle = b \pmod{q} \right\}.
\]
For both \( P \) and \( V \), the proof requires \( k + 5 \) modular exponentiations to compute \( \bar{c}_L \) and \( \bar{c}_R \), and one execution of \( \text{ZKPEq} \). This leads to a final cost of \( k + 5 \) modular exponentiations for \( P \) and \( k + 7 \) for \( V \). Considering initial commitments \( \bar{c} \) are already published in the bulletin board, the proof size is the same as that of \( \text{ZKPEq} \), which is \( 5\langle p \rangle \) bits. Note that, with this proof, users can already prove the correctness of Properties (5) and (6) for the verification of \( \text{GOPA} \).

**Proof of a committed bit.** An instantiation of \( \text{ZKPOR} \) that we will use is the ZKP that a secret \( b \) is a bit, i.e. that \( \{b = 0 \lor b = 1\} \). This instantiation is described by Mao (1998, Section 3.2 therein). It requires 3 modular exponentiations for \( P \) and 4 for \( V \). The size of the proof is of \( 4\langle p \rangle \) bits.

**Proof that a secret lies in a given range (range proof).** Here, the statement to prove is that, for a positive integer \( M < q - 1 \) and a commitment \( c_x \), \( P \) knows the secret \( x \) inside \( c_x \), and that it lies in the range \( \{0, M\} \). This can be proven by committing to the bits \( b_0, b_2, \ldots, b_{\langle M \rangle - 1} \) that are the bit representation of \( x \), i.e. that \( x = \sum_{i=0}^{\langle M \rangle - 1} 2^i b_i \). For \( i \in \{0, \ldots, \langle M \rangle - 1\} \) proof that \( b_i \) is a bit can be done by the instantiation of \( \text{ZKPOR} \) described above, and the composition can be proven with \( \text{ZKPLinear} \). By this, we prove that \( x \in \{0, 2^\langle M \rangle - 1\} \). For the exact range, we commit to \( M - x \) and prove that it also lies in \( \{0, 2^\langle M \rangle - 1\} \). We denote the proof by
\[
\text{ZKPRange}(c_x, \{0, M\}) = \left\{ (x, r) : c_x = g^x r^r \land x \in \{0, M\} \right\}.
\]
Committing to every bit \( b_i \) has a cost of \( \langle M \rangle \) modular exponentiations for \( P \). The cost of proofs that they are bits is dominated by \( 3\langle M \rangle \) modular exponentiations for \( P \) and \( 4\langle M \rangle \) for \( V \). The proof that these bits are the decomposition of \( x \) is dominated by \( \langle M \rangle \) modular exponentiations for both \( P \) and \( V \). As we perform this proof also for \( M - x \) the above costs are duplicated. Finally we prove that the commitments of \( x \) and \( M - x \) indeed satisfy this relation, which has a marginal cost. The total cost of the proof is dominated by \( 10\log_2(M) \) modular exponentiations for \( P \) and the same cost for \( V \). The size of the proof is dominated by \( 10\log_2(M)\langle p \rangle \) bits. With this proof, the verification of Property (7) can be done.

The verification of Property (7) is more elaborate as it involves proving that a value has been drawn from a prescribed statistical distribution. We describe this proof in Appendix B.3, but first introduce additional building blocks that will be needed for this proof.

**Proof of product.** In this proof, \( P \) wants to prove that for three secrets \( a, b, d \) committed in \( c_a = g^a r^r, c_b = g^b r^r \) and \( c_d = g^d r^r \), it holds that \( ab = d \pmod{q} \). This proof can be done in a straightforward way based on \( \text{ZKPBasic} \) and \( \text{ZKPEq} \). We use the fact that, by properties of our cyclic group \( \mathbb{G} \), \( c_a \) is not only a commitment but also a base, and that \( c_d = (c_a)^b h^{\langle r_{1} \rangle - br_1} \), which is a valid commitment for \( b \) using bases \( (c_a, h) \). For the proof, \( P \) performs a \( \text{ZKPBasic}(c_a) \) to prove knowledge of the committed value, and then \( \text{ZKPEq}(c_b, c_d) \) with the pairs of bases \( (g, h) \) and \( (c_a, h) \) to prove
knowledge and equality of the other two commitments. If \( P \) does not know the commitment of a product of \( ab \) hidden in \( c_d \) it would not pass the proof. We denote this proof as

\[
\text{ZKPProd}(c_a, c_b, c_d) \{ (a, r_1, b, r_2, d, r_3) : c_a = g^ah^{r_1} \land c_b = g^bh^{r_2} \\
\land c_d = g^dh^{r_3} \land ab = d \pmod q \}.
\]

The proof, first proposed by (Fujisaki & Okamoto, 1997) has essentially the cost of the proofs mentioned above which is 6 modular exponentiations for \( P \) and 9 for \( V \). The proof size is of \( 8(p) \) bits.

**Proof of modular sum.** Here, \( P \) wants to prove that, for a public modulus \( M < q \), a public value \( t \in [0, M - 1] \) and two secrets \( x, z \in [0, M - 1] \), the relation \( x = z + t \pmod M \) is satisfied. Let \( c_x \) and \( c_z \) be the commitments of \( x \) and \( z \) respectively. For the proof, \( P \) performs \( \text{ZKPRange}(c_x, [0, M - 1]) \), \( \text{ZKPRange}(c_z, [0, M - 1]) \), and we use an auxiliary commitment of number \( b \) which \( P \) proves to be a bit. Then, \( P \) proves the arithmetic relation \( x = z + t - bM \pmod q \) with \( \text{ZKPLinear} \). Because of their range, we have that \( z + t < q \), hence the modulus \( q \) does not interfere and the linear equality holds if and only if the modular sum is satisfied. We denote this proof as

\[
\text{ZKPMod}(c_x, c_z, M, t) \{ (x, r_1, z, r_2) : c_x = g^xh^{r_1} \land x \in [0, M - 1] \\
\land c_z = g^zh^{r_2} \land z \in [0, M - 1] \\
\land x = z + t \pmod M \}.
\]

This proof is inspired by the more general proof of modular sum proposed by (Camenisch & Michels, 1999). Its complexity is given by the composition of the proofs mentioned above, where the dominant term comes from the two range proofs, and amounts to \( 20 \log_2(M) \) modular exponentiations for \( P \) and the same for \( V \). The size of the proof is of \( 20 \log_2(M)(p) \) bits.

**B.3. Zero Knowledge Proofs for Uniform and Gaussian Random Variables**

Given the building blocks in Appendix B.2, we can construct a proof that a random variable follows a uniform or a Gaussian distribution.

**Proof of uniform distribution.** We prove that a private value \( y \) is uniformly distributed over the interval \( [0, M - 1] \) where \( M < q/2 \) is a public value. Note that the following proof should be interactive to make sure that an involved malicious user does not generate many proofs locally and picks one with convenient \( y \) to bias the distribution. For that, we use Algorithm 2 of Appendix B.4 to securely generate the challenge in Step 2. The proof goes as follows:

1. \( P \) draws \( z \leftarrow_R [0, M - 1] \), computes the commitment \( c_x = g^xh^r \) for some \( r \leftarrow_R \mathbb{Z}_q \) and publishes \( c_x \) in the bulletin board.

2. \( P \) generates a challenge \( t \sim \mathcal{U}(\mathbb{Z}_q^r) \) by participating in Algorithm 2 of Appendix B.4

3. \( P \) computes \( y = z + t \pmod M \) and \( c_y \leftarrow g^yh^{r'} \) for some \( r' \leftarrow_R \mathbb{Z}_q \), generates \( \text{ZKPMod}(c_y, c_x, M, t) \) and publishes \( c_y \) together with the proof of modular sum to the bulletin board.

Essentially, we use \( \text{ZKPMod} \) to prove that \( y = z + t \pmod M \) for a public value \( t \) that is guaranteed to be random (see Appendix B.4). Additionally, the private value \( z \) is used to ensure the secrecy of \( y \). By the properties of modular sum, we can ensure that \( y \) is uniformly distributed.

We denote this proof as

\[
\text{ZKPUmform}((c_y), [0, M - 1]) \{ (y, r') : c_y = g^yh^{r'} \land y \sim U([0, M - 1]) \}.
\]

The cost of \( \text{ZKPUmform} \) is dominated by the \( \text{ZKPMod} \) proof. That is of \( 20 \log_2(M) \) modular exponentiations for \( P \) and \( 20 \log_2(M) \) for \( V \). The proof size is of \( 20 \log_2(M)(p) \) bits.
Proof of Gaussian distribution. In our algorithm, every user \( u \) needs to generate a Gaussian distributed number \( \eta_u \), which he does not publish, but for which we need to verify that it is generated correctly, as otherwise a malicious user could bias the result of the algorithm.

Proving the generation of a Gaussian distributed random number is more involved. We will start from an integer \( y' \), which is only known to a party \( P \), for which \( P \) has published a commitment \( \text{Com}_\sigma(y') \) and for which the other agents know it has been generated uniformly randomly from an interval \([0, M - 1]\) (for large \( M \)), as can be done with \( \text{ZKPUniform} \). Then, \( P \) will compute \( x' \) such that \(((2y' + 1)/M) - 1 = \text{erf}(x'/\sqrt{2})\). We know that \( x' \) is normally distributed. The main task is then to provide a ZKP that \( y = \text{erf}(x) \) for \( y = ((2y' + 1)/M) - 1 \) and \( x = x'/\sqrt{2} \). As \( \text{erf} \) is symmetric, we do our analysis for positive values of \( x \) and \( y \), while the extension for the negative case is straightforward.

The error function relates its input and output in a way that cannot be expressed with additive, multiplicative or exponential equations. We therefore approximate \( \text{erf} \) using a converging series. In particular, we will rely on the series

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{l=0}^{\infty} \frac{(-1)^l x^{2l+1}}{(2l+1)!}.
\]

(35)

As argued by (Chevillard & Revol, 2008), this series has two major advantages. First, it only involves additions and multiplications, while other known series converging to \( \text{erf}(x) \) often include multiple \( \exp(-x^2/2) \) factors which would require additional evaluations and proofs. Second, it is an alternating series, which means we can determine more easily in advance how many terms we need to evaluate to achieve a given precision.

Nevertheless, Equation (35) converges slowly for large \( x \). It is more efficient to prove either that

\[
y = \text{erf}(x) \quad \text{or} \quad 1 - y = \text{erfc}(x),
\]

as for \( \text{erfc}(x) = 1 - \text{erf}(x) \) there exist good approximations requiring only a few terms for large \( x \). An example is the asymptotic expansion

\[
\text{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} S_{\text{erfc}}(x) + R_L(x),
\]

(36)

where

\[
S_{\text{erfc}}(x) = \sum_{l=0}^{L-1} \frac{(-1)^l (2l-1)!!}{(2x^2)^l}
\]

(37)

with \( l!! = 1 \) for \( l < 1 \) and \( (2l-1)!! = \prod_{i=1}^{l} (2i-1) \). This series diverges, but if \( x \) is sufficiently large then the remainder

\[
R_L(x) \leq \frac{e^{-x^2} (2L - 1)!!}{x\sqrt{\pi} (2x^2)^L}
\]

(38)

after the first \( L \) terms is sufficiently small to be neglected. So the prover could prove either part of the disjunction with \( \text{ZKPOr} \) depending on whether the \( \text{erf} \) or \( \text{erfc} \) approximations achieve sufficient precision. Developing Equation (38), we can achieve an \( \text{erfc} \) approximation error of

\[
E_{\text{erfc}}(x) \leq \frac{1}{e^{x^2} x\sqrt{\pi}} \frac{(2L - 1)!!}{(2x^2)^L} \leq \frac{1}{e^{x^2} x\sqrt{\pi} L!2^L(2x^2)^L}
\]

\[
\approx \frac{1}{e^{x^2} x\sqrt{\pi} \sqrt{2\pi L/(e2^L)^2} 2^L(2x^2)^L} = \frac{\sqrt{2}}{e^{x^2} x\sqrt{\pi} (e2^L)(L^L)(2x^2)^L}
\]

\[
= \frac{2^L L^L}{e^{x^2} x\sqrt{\pi} e^{L}(2x^2)^L} = \frac{\sqrt{2}}{e^{2x^2} x\sqrt{\pi} e^{L}(2x^2)^L}.
\]

For a fixed \( x \), Equation (38) is minimal with \( L \approx \left|x^2 + 1/2\right| \), so given \( x \) we can achieve an error of

\[
\frac{\sqrt{2}}{e^{2x^2} x\sqrt{\pi} e^{L}(2x^2)^L} \approx \frac{\sqrt{2}}{e^{x^2} x\sqrt{\pi}} \frac{\left|x^2 + 1/2\right|^2}{e^{L}(2x^2)^L} \leq \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{xe^{(2x^2-1/2)}}.
\]
As we use erfc for large values of $x$, we assume $x \geq 1$ which will not affect our reasoning, then and we can simplify the above by

\[
\frac{\sqrt{2}}{\sqrt{\pi}} \frac{(1 + 1/2x^2)^{\lfloor x^2 + 1/2 \rfloor}}{xe^{(2x^2 - 1/2)}} \leq \frac{\sqrt{2}}{\sqrt{\pi}} \frac{3\sqrt{6}/4}{xe^{(2x^2 + 3/2)}} \leq \frac{1}{2} \frac{27}{\pi} \frac{1}{e^{(2x^2 - 1/2)}}.
\]

Approximating $\text{erfc}(x)$ involves approximating $\exp(-x^2)$. The common series $\exp(z) = \sum_{i=0}^{\infty} z^i / i!$ is known to converge quickly. Even if its terms first go up until $z < i$, for larger $z$ where this could slow down convergence one can simply divide $z$ by a constant $a$ (maybe conveniently a power of 2), approximate $\exp(z/a)$ and then compute $(\exp(z/a))^a$, which can be done efficiently. We omit the details here and focus in the following on the more difficult erf function.

In absolute value, terms of the erf approximation never get larger than

\[
\left| \frac{2}{\sqrt{\pi}} \frac{(-1)^i x^{2i+1}}{i!(2i+1)} \right| \leq \frac{2}{\sqrt{\pi}} \frac{x^{2i^2+1}}{i!} \left(2^i \frac{i!}{2i+1} + 1 \right) \approx \frac{2}{\sqrt{\pi}} \frac{x^{2i^2+1}}{i!} \frac{2^i}{2i+1} \frac{i!}{2i+1} \leq \frac{\sqrt{2}x}{\pi xe^{-x^2}(2x^2 - 1)} = \frac{\sqrt{2}x}{\pi(2x^2 - 1)} \leq \frac{e^{x^2}}{\sqrt{2\pi}x^2}.
\]

(39)

Evaluating a term of one of the series (Equation (35) or Equation (37)) is possible by addition, multiplication and a division. We have described above building blocks for providing ZKPs of additions (ZKPLinear) and multiplications (ZKPProd). For the multiplication, please note that one can compute $x^l$ by $x^l = x^{l-1}x$, reusing the result of earlier multiplications. For round-off division, proving that $a/b = c$ is possible by proving that $a - bc + \lfloor |b|/2 \rfloor \in [0, |b|]$ where $a$ and $c$ are private, or by $a^2 - abc + \lfloor |a|/2 \rfloor \in [0, |a|]$ in the case $b$ and $c$ are private. In both cases, the generation of a ZKPRange proof is required.

In the following, we will explicitly consider fixed precision representations. The implied rounding does not cause major problems for several reasons. First, the Gaussian distribution is symmetric, and hence the probability of rounding up and rounding down is exactly the same, making the rounding error a zero-mean random variable. Second, discrete approximations of the Gaussian mechanism such as binomial mechanisms have been studied and found to give similar guarantees as the Gaussian mechanism (Agarwal et al., 2018). Third, we can require the cumulated rounding error to be an order of magnitude smaller than the standard deviation of the noise we are generating, so that any deviation due to rounding has negligible impact.

We will use a fixed precision for all numbers (except for small integer constants), and we will represent numbers as multiples of $\psi$. We assume $1/\psi M$ to be an integer, so as the variable $x$ is a multiple of $1/M$, we can represent $x$ and $x^2$ without rounding. Let

\[
t_i = \frac{x^{2i+1}}{i!}.
\]

We can compute recursively

\[
t_{i+1} = t_i \frac{x^2}{i+1},
\]

where $t_0 = x$. The multiplication with $x^2$ and the division by $i$ increase the relative error by less than $\psi$. If we evaluate the first $L$ terms of the series, and counting the additional division by $2i + 1$, the relative error due to rounding is thus at most $(L + 1)\psi$. Combining this with Equation (39) implies that the maximal absolute error of a term is $(L + 1)\psi e^{x^2}/\sqrt{8\pi x^2}$, and the total error after summing these terms is

\[
E_{\text{erf}} = \frac{L(L + 1)\psi e^{x^2}}{\sqrt{2\pi}x^2}.
\]

(40)

Suppose we want to achieve an approximation where the error on $y$ as a function of $x$ is at most $B$. It is natural to set $M \approx 1/B$. We require that the rounding error is at most $B/2$ and the approximation error is at most $B/2$. Then, if

\[
\frac{B}{2} \geq \frac{1}{2} \sqrt{\frac{27}{\pi}} \frac{1}{e^{(2x^2 - 1/2)}} \geq E_{\text{erf}}(x),
\]


the prover will use the erfc series, else the erf series. As the erf series more strongly constrains our design choices, we will focus first on the erf case.

We hence assume that
\[
\sqrt{\frac{27}{\pi}} \cdot \frac{1}{2e^{(2x^2-1/2)}} \geq \frac{B}{2}
\]
which implies
\[
\sqrt{\frac{27}{\pi}} \frac{1}{B} \geq e^{(2x^2-1/2)},
\]
from which
\[
\sqrt{\frac{27}{\pi}} \frac{1}{B} \geq e^{(2x^2-1/2)}.
\]
This implies
\[
\log(27/\pi)/2 + \log(1/B) \geq 2x^2 - 1/2
\]
and
\[
1.076 + \log(1/B)/0.434 \geq (2x^2 - 1/2).
\]
Finally, the above implies
\[
x_{\text{erf}}^\text{min} = x_{\text{erf}}^\text{max} = \sqrt{\frac{\log(1/B)}{2} + 0.788} \geq x.
\]

The series in Equation (35) is an alternating series too, so to reach an error smaller than $B/2$, it is sufficient to truncate the series when terms get smaller than $B/2$ in absolute value. In particular, we need $L$ terms with
\[
\frac{2x^{2L+1}}{\sqrt{\pi}L!(2L+1)} \leq \frac{B}{2}.
\]
Using again Stirling’s approximation, this means
\[
\frac{2x^{2L+1}}{\sqrt{\pi}\sqrt{2\pi}L(L/e)^L(2L+1)} \leq \frac{B}{2}.
\]
Taking logarithms, we get:
\[
\log(2) + (2L+1)\log(x) - \log(\pi\sqrt{2}) - (L+1/2)\log(L) + L - \log(2L+1) \leq \log(B) - \log(2).
\]
We have that $2\log(2) - \log(\pi\sqrt{2}) - \log(2L+1) \leq 0$, so the above inequality is satisfied if
\[
(2L+1)\log(x) - (L+1/2)\log(L) + L \leq \log(B)
\]
which is equivalent to
\[
(L+1/2)\log(ex^2/L) \leq \log(B),
\]
or written differently:
\[
(L+1/2)\log\left(1 - \frac{L-ex^2}{L}\right) \leq \log(B)
\]
As $\log(1+\alpha) \leq \alpha$, this is satisfied if
\[
-(L+1/2)\frac{L-ex^2}{L} \leq \log(B),
\]
which is equivalent to
\[
(L+1/2)\frac{L-ex^2}{L} \geq \log(1/B).
\]
The above is satisfied if
\[
L - ex^2 \geq \log(1/B).
\]
It follows that we need
\[ L \geq \log(1/B) + ex^2. \]

Substituting the worst case value \( x_{\text{erf}}^{\text{max}} \) of \( x \) from Equation (41), we get
\[ L \geq \left\lceil \left( 1 + \frac{e}{2} \right) \log(1/B) + 0.79e \right\rceil \]
which, approximating, is implied if
\[ L \geq \left\lceil 2.36 \log(1/B) + 2.15 \right\rceil = L_{\text{erf}}. \tag{42} \]

Now we determine what precision is needed. We already know from equation (40) the maximum error due to rounding of \( \text{erf} \). For the erfc case, we have that \( x > x_{\text{erfc}}^{\text{min}} \) and given that
\[ l < L_{\text{erfc}} = \lceil (x_{\text{erfc}}^{\text{min}})^2 + 1/2 \rceil < \lceil x^2 + 1/2 \rceil, \]
the absolute value of every term is never bigger than
\[ \left| \frac{e^{-x^2} (-1)^l (2l + 1)!!}{x^2(2x^2)^l} \right| \leq \frac{e^{-x^2}}{x^2}. \]

Computing a term of \( \text{erfc} \) also requires a rounding error \( (L_{\text{erfc}} + 1)\psi = \lceil (x_{\text{erfc}}^{\text{min}})^2 + 1/2 \rceil \psi \) in the worst case. Hence, summing \( L_{\text{erfc}} \) terms would require an absolute error of
\[ E_{\text{erfc}}^{\text{round}} = \frac{\lceil (x_{\text{erfc}}^{\text{min}})^2 + 1/2 \rceil \psi (x_{\text{erfc}}^{\text{min}})^2 + 1/2}{x^2e^{x^2}} \tag{43} \]
in the worst case. It is easy to see that \( E_{\text{erfc}}^{\text{round}} \) is the error that would constrain more our precision. Therefore we require that
\[ E_{\text{erfc}}^{\text{round}} = \frac{L_{\text{erfc}}(L_{\text{erfc}} + 1)\psi e^{x^2}}{\sqrt{2\pi}x^2} \leq \frac{B}{2} \]
which is satisfied if
\[ \psi \leq \frac{\pi Bx^2}{\sqrt{2}L_{\text{erfc}}(L_{\text{erfc}} + 1)e^{x^2}}, \]
and the above is implied if
\[ \psi \leq \frac{\pi B(x_{\text{erfc}}^{\text{max}})^2}{\sqrt{2}L_{\text{erfc}}(L_{\text{erfc}} + 1)e^{(x_{\text{erfc}}^{\text{max}})^2}}. \]

Hence, by Equations (41) and (42), the magnitude of \( \psi \) is dominated by
\[ \frac{\pi}{\sqrt{2}(2.36)^2 \log^2 (1/B)} \approx 0.282 \frac{B^2}{\log^2 (1/B)}. \tag{44} \]

Typically, one would like the total error (due to approximation and rounding) to be negligible with respect to the standard deviation \( \sigma_\eta \), so one could choose \( B = \sigma_\eta/10^9 |U^H| \).

We now analyze the cost of the ZKP. Recall that given parameter \( M = 1/B \), generating a uniformly distributed random number \( y \) has a cost of \( O(\log_2(1/B)) \) computations for \( P \) and \( V \) and a proof size of \( O(\log(1/B)) \) bits as described in \text{ZKPUniform}. The dominant cost of the computation is in the evaluation of the \( \text{erf} \) or \( \text{erfc} \) series. We neglect lower order costs. For \( \text{erf} \), the number of terms to evaluate is the constant \( L_{\text{erf}} \) dominated by \( 2.36 \log(1/B) \) which the community computes once in advance.

Computing a term requires three proofs of multiplication, one of division by at most \( L_{\text{erf}} \) and another one of at most \( 2L_{\text{erf}} + 1 \). The cost of computing a division ZKP, \( a/b \) or \( b/a \) where \( a \) is the private term of the division is dominated by a range proof and its cost is of \( 10 \log_2(a) \) modular exponentiations for generating the proof, it has the same cost for its verification, and a size of \( 10 \log_2(a) \langle p \rangle \) bits.
The cost of evaluating the series is dominated by $L_{\text{erf}}$ multiplications, additions and divisions. While per term the first two cost $O(1)$ exponentiations, divisions are dominated by $O(\log(L_{\text{erf}}))$ computations. Taking into account leading constants, evaluating a term has a cost dominated by $20 \log_2(\log(1/B))$ modular exponentiations for generating the proof and verifying it, and a proof size of $20 \log_2(\log(1/B)) \langle p \rangle$ bits.

The cost of the erf ZKP is the evaluation of the $L_{\text{erf}}$ terms. By Equation (42), the cost of evaluating a single term is described in the above paragraph and is dominated by $32.6 \log(1/B) \log(\log(1/B))$ modular exponentiations for generating the proof and for verifying it, and a proof size of $32.6 \log(1/B) \log(\log(1/B)) \langle p \rangle$ bits. Similarly, the erfc series requires the evaluation of $L_{\text{erf}}$ terms, each of them dominated by the computation of a division of private denominator smaller or equal to $L_{\text{erf}}$. This has a computational cost dominated by $7 \log(1/B) \log(\log(1/B))$ exponentiations for generating the proof and for verifying it, and a proof size of $7 \log(1/B) \log(\log(1/B)) \langle p \rangle$ bits.

To summarize, the main statement we prove is

\[
\left\{ \left( y \in \left[ 0, \left\lceil \frac{\text{erf}(x)}{\psi} \right\rceil \right] \land y = \text{erf}(x) \right) \lor \left( y \in \left[ \left\lceil \frac{\text{erf}(x)}{\psi} \right\rceil + 1, \frac{1}{\psi} \right] \land 1 - y = \text{erfc}(x) \right) \right\}
\]

(45)

where $y_{\text{erf}} = \text{erf}(x_{\text{erf}})$ is a public constant also computed in advance. The prover first computes $y$ from the execution of ZKPUniform and then the statement of equation (45).

The two range proofs in the statement have a cost of $O(1/\psi) = O(\log(\log(1/B)/B^2))$ modular exponentiations, which is non dominant and therefore neglected. The cost of the proof of the disjunction is equal to the sum of the costs of the involved sub-statements, which is therefore dominated by $39.6 \log(1/B) \log(\log(1/B))$ modular exponentiations for generating the proof, the same cost for its verification, and a size of $39.6 \log(1/B) \log(\log(1/B)) \langle p \rangle$ bits.

For any desired variance $\sigma^2$, we denote the proof of Gaussian distribution by

\[
\text{ZKPNormal}(c_\mathbf{x}, \sigma^2)\{(x, r) : c_\mathbf{x} = g^x h^r \land x \sim \mathcal{N}(0, \sigma^2) \}.
\]

### B.4. Randomness for Setup and ZKPUniform

The generation of challenges in ZKPUniform cannot be made non-interactive as it provides an opportunity for malicious users to execute many generations and choose a particular sample, which would bias the distribution. Here, we provide a decentralized method to generate $n$ challenges with a computational effort of $O(1)$ per user, and only $O(1)$ messages in the network for the majority of the users, and costs logarithmic in $n$ for a negligible portion of users. Additionally, this algorithm can be used to generate the randomness to run the Setup algorithm and initialize the Pedersen commitment scheme shared by all users.

In our procedure, we enumerate users from 1 to $n = |U|$ and use the cryptographic hash function $H$ described at the beginning of Appendix B to generate random numbers over $\mathbb{Z}_q$. We also make use of a commitment function $\text{Com}$, which is instantiated to the Pedersen function $\text{Com}_0$ if its parameters are already initialized, or otherwise to any other commitment function for which a common trusted initialization is not required (see for example Blum, 1983). The procedure is depicted in Algorithm 2.

The “commit-then-reveal” protocol from in lines 2-5 is to avoid users from choosing the value $s_u$ depending on the choice of other users, which could bias the final seed $S = \sum_{u \in U} s_u \mod q$. Such sum is computed in lines 6-12 in a distributed way, with at most $O(\log(n))$ queries and sums for a user in the worst case, but $O(1)$ of these computations per user in for most of the cases. Users that are inactive during the execution of Algorithm 2 do not affect the computation of $S$, but their $s_u$ terms might be ignored. As it is computed from modular sums, $S$ is uniformly distributed over $\mathbb{Z}_q$ if at least one active user is honest. Finally, we compute the set $\{t_u\}_{u=1}^n$ of random numbers in lines 13 and 14, which are unpredictable due to the properties of $H$ and the unpredictability and uniqueness of its input $S + u$.

Note that a user that requests a challenge gets blocked until all users also perform this request. However, every user needs the same amount of random challenges to execute Setup or ZKPUniform. Hence, we perform a first execution of the protocol for Setup, and a second execution for all instances of ZKPUniform, which can be performed in batch. This prevents that users have to wait for a long time after a request, as would happen in a setting where the requests are unbalanced.

Finally, to verify that a user $u$ participated correctly, one needs to check that (1) the commitment $c_u$ was properly computed from $s_u$, (2) all sums $s[u_{\min}, u_{\max}]$ that $u$ published were computed correctly, and (3) the challenge $t_u$ was correctly computed from $S + u$ by the application of $H$. 
We now describe in this appendix all measures to ensure the correctness guarantees described in Section 5 and additional measures for different types of attacks. We present our main verification protocol in Appendix C.1. After that, we describe how to deal with users dropping out in the middle of the computation in Appendix C.2. Next, we compare our correctness

### Algorithm 2: Generation of Public Randomness

1. **Input**: A cryptographic hash function $H$ and a commitment function $Com$.
2. **for all** $u \in \{1, \ldots, n\}$ **do**
3. Draw $s_u \leftarrow R \mathbb{Z}_q$, compute $c_u \leftarrow Com(s_u)$ for some randomness and publish $c_u$ to the bulletin board.
4. **for all** $u \in \{1, \ldots, n\}$ **do**
5. Publish $s[u, u] \leftarrow s_u$ in the bulletin board, such that such that it can be verified that the commitment $c_u$ was computed correctly.
6. **for** $j = 1$ **to** $\lfloor \log_2(n) \rfloor + 1$ sequentially **do**
7. **for all** $i \in \{0, \ldots, \lfloor n/2^j \rfloor \}$ **do**
8. Let $u_{min} = 2^i + 1$ and $u_{max} = \min(2^j(i + 1), n)$.
9. **if** $s[u_{min}, u_{max}]$ is not already published in the bulletin board **then**
10. **for** any active (but only one) $u \in \{u_{min}, \ldots, u_{max}\}$ **do**
11. Let $u_{mid} = u_{min} + 2^j - 1$, query $s[u_{min}, u_{mid}]$ and $s[u_{mid} + 1, u_{max}]$ from the bulletin board. If any of these values is not available, assume it to be 0.
12. Set $s[u_{min}, u_{max}] \leftarrow s[u_{min}, u_{mid}] + [u_{mid} + 1, u_{max}] \mod q$, and publish it to the bulletin board.
13. **for all** $u \in \{1, \ldots, n\}$ **do**
14. Query $S \leftarrow s[1, n]$ and publish $t_u \leftarrow H(S + u)$ to the bulletin board.
15. **Output**: A set of unbiased (pseudo-)random numbers $\{t_u\}_{u=1}^n \in \mathbb{Z}_{2^2}$, one per user.

The cost of the protocol for a user is dominated in the worst case by $\log_2(n)$ sums and $2 \log_2(n)$ messages of $3 \log_2(n) \langle p \rangle$ bits in total, corresponding to the several queries of $s[u_{min}, u_{mid}]$ and $s[u_{mid} + 1, u_{max}]$ and the publication of its sum. The same amount of sums is required as a dominant term for its verification. However, this cost applies to only $O(1)$ users, while each of the rest of the users only perform one commitment, one hash function evaluation, $O(1)$ sums and $O(1)$ messages during execution, and require the same amount of computations for their verification.

### B.5. Further Discussion on the Impact of Finite Precision

In practice, we cannot work with real numbers but only with finite precision approximations, see Appendix B. We provide here a brief discussion of the impact of this on the guarantees offered by the protocol. We emphasize that there is already a large body of work which addresses most of the potential problems which could arise because of finite precision. Here are the main points:

1. Finite precision can be an issue for differential privacy in general, (see e.g. Balcer & Vadhan, 2018, for a study of the effect of floating point representations). Issues can be overcome with some care, and our additional encryption does not make the problem worse (in fact we can argue that encryption typically uses more bits and in our setting this may help).

2. The issue of finite precision has been studied in cryptography. While some operations such as multiplication can cause additional trouble in the context of homomorphic encryption, in our work we use a partially homomorphic scheme with only addition. As a result, we can just represent our floating point numbers with as many bits (after the decimal dot) as our variables also have in computer memory.

3. If bandwidth would dictate that we can transfer only at lower precision compared to the internal representation, a conversion will be needed before encryption (e.g. from 10 digits to 5 digits after the decimal dot). Still, (a) rounding has been well described in numerical computing literature, (b) in our case it would not decrease privacy as all our operations are linear (no sharp differences in behavior if a value changes less than machine precision), and (c) we feel this would not gain much bandwidth: indeed, in practical applications with sums over millions of parties (plus some random encryption bits), one would already require more than 7 digits before the decimal dot, in comparison with which the number of less significant bits gained is relatively small.

### Appendix C. Correctness and Robustness Guarantees

We now describe in this appendix all measures to ensure the correctness guarantees described in Section 5 and additional measures for different types of attacks. We present our main verification protocol in Appendix C.1. After that, we describe how to deal with users dropping out in the middle of the computation in Appendix C.2. Next, we compare our correctness
We design our verification protocol based on the primitives described in Appendix B. Essentially, we rely on ZKPLinear. Verification phase computations of $G(6)$ involves secrets of two different users and verify Properties (5) and (6), $c$ can be proven between committed secrets. As an illustration, consider ridge regression in Example 1. Every user $u$ can provide proofs for properties of the shared Pedersen commitment scheme and run ZKPUniform for the posterior generation of $\eta_u$ and $c_{\eta_u}$, ZKPNormal and ZKPRange to verify Property (7) and (8). Our verification protocol is described in Algorithm 3. At the beginning of the protocol, all users generate parameters $\Theta$ and each user $u$ has published to the bulletin board:

Algorithm 3 Verification phase computations of GOPA

1: Input: All users have jointly generated a commitment scheme with parameters $\Theta$ and each user $u$ has published to the bulletin board:
2: • ZKPUniform for the posterior generation of $\eta_u$ and $c_{\eta_u}$
3: • $c_{\eta_u} = Com_{\Theta}(X_u)$ (before the execution of Algorithm 1)
4: • $c_{\Delta_{u,v}} = Com_{\Theta}(\Delta_{u,v})$ (when exchanging noise with $v \in N(u)$ during Algorithm 1)
5: • $c_{\eta_u} = Com_{\Theta}(\eta_u)$ (after generating $\eta_u$)
6: • $\hat{X}_u$ (at the end of Algorithm 1)
7: • ZKPLinear proofs for Properties (5) and (6)
8: • ZKPNormal and ZKPRange proofs for Properties (7) and (8)
9: for all user $u \in U$ do
10: Verify that the value inside $c_{\eta_u}$ is in $[0, 1]$ (if not, add $u$ to the cheaters list)
11: Verify that values inside $c_{\Delta_{u,v}}, (c_{\Delta_{u,v}})_{v \in N(u)}, c_{\eta_u}$ sum to $\hat{X}_u$ (if not, add $u$ to cheaters list)
12: Verify that the value inside $c_{\eta_u}$ has distribution $\mathcal{N}(0, \eta_u)$ (if not, add $u$ to cheaters list)
13: for all user $v \in N(u)$ do
14: Verify that values inside $c_{\Delta_{u,v}}$ and $c_{\Delta_{v,u}}$ sum to 0 (if not, add $u$ and/or $v$ to cheaters list)

guarantees to that of the centralized setting in Appendix C.3. Finally, we enumerate how to prevent from other kinds of attacks in Appendix C.4.

C.1. GOPA Verification Protocol

We design our verification protocol based on the primitives described in Appendix B. Essentially, we rely on ZKPLinear to verify Properties (5) and (6), ZKPNormal to verify Property (7) and ZKPRange to verify Property (8). Note that Property (6) involves secrets of two different users $u$ and $v$. However, this is not a problem as these pairwise noise terms are known by both involved users, so they can use negated randomnesses $r_{\Delta_{u,v}} = -r_{\Delta_{u,v}}$ in their commitments of $\Delta_{u,v}$ and $\Delta_{v,u}$ such that everybody can verify that $Com_{\Theta}(\Delta_{u,v}, r_{\Delta_{u,v}}) + Com_{\Theta}(\Delta_{v,u}, r_{\Delta_{v,u}}) = Com_{\Theta}(0, 0)$. Users can choose how they generate pairwise noise according to the normal distribution. We just require that both peers agree by signed messages on the generated terms before publishing commitments, so that in the case that one of the two peers cheats, it can be easily discovered.

Our verification protocol is described in Algorithm 3. At the beginning of the protocol, all users generate parameters $\Theta$ for a shared Pedersen commitment scheme and run ZKPUniform to generate the uniformly distributed number that will later be used for ZKPNormal (see Appendix B.3). After that, all generated ZKPs are non-interactive. Our verification protocol requires users to publish in the bulletin board all commitments and ZKPs as soon as the private values involved are generated by the execution of the private averaging phase (Algorithm 1). As soon as a ZKP has been published, other users can start the verification. This can be done for instance at fixed points of the execution, such as after (1) all commitments of private values, (2) all commitments of pairwise noise terms and/or (3) final noisy values are published. Users may perform verification at other arbitrary points of the execution.

By composition of ZKPs, Algorithm 3 allows each user to prove the correctness of his/her computations and preserve completeness, soundness and zero knowledge properties, thereby leading to our security guarantees in Theorem 4. If a user is found to be a cheater, among other measures that can be taken, it is banned from the computation. The rest of the online non-cheating users can continue with the protocol and recover a partial average following the measures taken in the case of drop-outs (see Appendix C.2).

To reduce the verification load, we note that it is possible to perform the verification for only a subset of users by generating random challenges (for example, by the execution Algorithm 2) after users have published the involved commitments. In this case, we obtain probabilistic security guarantees, which may be sufficient for some applications.

Consistency over multiple executions. While the guarantees of Theorem 4 hold with respect to a single execution of GOPA, commitments to private values can be used when multiple executions of the algorithm are performed over the same or related data. In addition to linear relations, products and range proofs, a wide variety of arithmetic relations can be proven between committed secrets. As an illustration, consider ridge regression in Example 1. Every user $u$ can publish commitments $c_{\eta_u} = Com_{\Theta}(\eta_u)$, $c_{\eta'_u} = Com_{\Theta}(\eta'_u)$ for $i \in \{1, \ldots, d\}$ (computed with the appropriately drawn
randomness), and additionally commit to $\phi_i^u y_u$ and $\phi_i^u \phi_i^v$, for $i, j \in \{1, \ldots, d\}$. Then, by the use of ZKPProd, it can be verified that all these commitments are computed coherently, i.e., that the commitment of $\phi_i^u y_u$ is the product of secrets committed in $c_{\phi_i^u}$ and $c_{\phi_i^u}$ for $i \in \{1, \ldots, d\}$, and analogously for the commitment of $\phi_i^u \phi_i^v$ in relation with $c_{\phi_i^u}$ and $c_{\phi_i^u}$, for $i, j \in \{1, \ldots, d\}$. An efficient ZKP to check arbitrary arithmetic relations, i.e, inner products of vectors of commitments, is provided by (Bootle et al., 2016) and further optimized by (Bünz et al., 2018). Additionally, ZKPs can be composed such that AND and OR composition of the proven statements can be generated by ZKPAAnd and ZKPOr, and by (Cramer et al., 1994; Santis et al., 1994) other families of logical monotone formulas.

C.2. Details on Dealing with Dropout

In this section, we give additional details on the strategies for dealing with dropout outlined in Section 3. Here, we consider that a user drops out of the computation if he/she is off-line for a period which is too long for the community to wait until his/her return. This can happen accidentally to honest users, e.g. due to lost network connection. Malicious users may also intentionally drop out to bias the outcome of the computation. Indeed, drop outs affect the outcome of the computation as we rely on the fact that pairwise noise terms $\Delta_{u,v}$ and $\Delta_{v,u}$ cancel out for the correctness and utility of the computation. Finally, a malicious user detected as cheater by the verification procedure of Appendix C.1 and banned from the system may also be seen as a dropout.

We propose a three-step approach to handling dropout:

1. First, as a preventive measure, users should exchange pairwise noise with enough users so that the desired privacy guarantees hold even if some neighbors drop out. This is what we proposed and theoretically analyzed in Section 4.2, where the number of neighbors $k$ in Theorem 3 depends on (a lower bound on) the proportion $\rho$ of honest users who do not drop out. It is important to use a safe lower bound on $\rho$ to make sure users will have enough safety margin to handle actual dropouts.

2. Second, as long as there is time, users attempt to repair as much as possible the problems incurred by dropouts. A user $u$ who did not publish $\hat{X}_u$ yet can just remove the corresponding pairwise noise (and possibly exchange noise with another active user instead). Second, a user $u$ who did publish $\hat{X}_u$ already but has still some safety margin thanks to step 1 can simply reveal the noise exchanged with the user who dropped out, and subtract it from his published $\hat{X}_u$. Finally, in the critical and unlikely case where many more neighbors dropped out than anticipated in step 1, $u$ might be lucky enough to still be able to exchange additional noise with other users who did not publish their noisy value yet. However, if $u$ cannot trust that a sufficient fraction of these users are honest, the only choice may be that user $u$ must drop himself/herself out. Note that this could cause a cascade of drop-outs if other users also do not have sufficient margin, and a significant part of the pairwise noise exchange procedure may need to be repeated. To avoid such problems, in addition to step 1, it can be useful to check which users went off-line just before publishing $\hat{X}$ and to have penalties for users who (repeatedly) drop out at the most inconvenient times.

3. Finally, when circumstances require and allow it, we can ignore the remaining problems and proceed with the algorithm, which will then output an estimated average with slightly larger error. This can be the case for instance when only a few drop outs have not yet been resolved, there is not much time available, and the corresponding pairwise terms $\Delta_{u,v}$ are known to be not too large (e.g., by proving that they were drawn from $\mathcal{N}(0, \sigma_\Delta^2)$ with $\sigma_\Delta^2$ is small enough, or by the use of range proofs).

We present here a simple experiment to show the ability of GOPA to tolerate a small number of residual pairwise noise terms of variance $\sigma_\Delta^2$ in the final result, which may happen in the rare circumstances that the pairwise noise terms of some users who dropped out are not “rolled back” by their neighbors (step 3 above). We consider $n = 10000$, $\rho = 0.5$, $\epsilon = 0.1$, $\delta = 10/(\rho n)^2$ and set the values of $k$, $\sigma_\Delta^2$ and $\sigma_\Delta^2$ using Corollary 1 so that GOPA satisfies $(\epsilon, \delta)$-DP (in particular we set them in the same way as in Table 1). Assume that some users dropped out and that their pairwise noise could not be rolled back. We simulate this by drawing a random $k$-out graph, selecting a certain number of such users at random, counting the number of residual noise terms and computing the variance of the estimated average. Figure 2 reports the mean of this variance over 100 runs (the standard deviation across random runs is negligible). We see that GOPA can tolerate a few hundreds of these “catastrophic drop-outs” while remaining more accurate than local DP. This ability to retain a useful estimate despite residual pairwise noise terms is rather unique to GOPA, and not possible with secure aggregation methods which typically use uniformly random pairwise masks (Bonawitz et al., 2017). We note that this robustness can be optimized by choosing smaller $\sigma_\Delta^2$ (i.e., smaller $\kappa$) and compensating by adding a bit more independent noise according to Corollary 1.
C.3. Comparison of Correctness Guarantees with the Centralized Setting

Consider the case of a centralized computation, where a trusted central party is provided with the raw data. In general, this central party does not know whether the values it received as input are truthful: in many applications, honest outliers occur naturally and it may be hard to distinguish between a honest outlier, a malicious outlier, and a normal looking value with a small amount of malicious bias added. However, the central party (i) can verify some basic properties of the data, such as checking that the inputs are in an appropriate range, and (ii) knows that all of its computations are performed correctly given the received input data. In this sense, our commitments offer the same guarantees.

In some critical cases where reliable data is needed, the central party could require that the data of users be certified by a trusted third party. In such scenarios, solutions exist to provide users with private credentials that have the same zero knowledge logic as our solution, and also deliver to the central party a certificate from a trusted third party that this data is valid (Camenisch & Lysyanskaya, 2001; Brands, 2000).

We can conclude that GOPA is an auditable protocol that, through existing efficient cryptographic primitives, can offer guarantees similar to the automated auditing which is possible for data shared with a central party.

C.4. Robustness Against Attacks on Efficiency

The complexity of our algorithm is bounded in terms of the number of users and the number of neighbors of every user in the communication graph. In practice, the number of neighbors of users is then again bounded by a function of the fraction of malicious users (or a lower bound on this fraction), see Theorem 3. In this appendix, we discuss potential attacks aimed at making the system less efficient and propose mitigation strategies.

C.4.1. Dropout

As discussed in Appendix C.2, users dropping out of the computation can cause some additional work and delay (e.g., to roll back pairwise noise). However, dropping out is bad for the reputation of a user, and users dropping out more often than agreed could be banned from future participation. One can use techniques from identity management (a separate branch in the field of security) to ensure that creating new accounts is not free, and that possibilities to create new accounts are limited, e.g., by requiring to link accounts to unique persons, unique bank account or similar. This also ensures that the risk of being banned outweighs the incentive of the (bounded) delay of the system one could cause by intentionally dropping out.

C.4.2. Flooding with Neighbor Requests

In Section 4, we discuss privacy guarantees for complete, path graphs and random communication graphs. In the case of a complete communication graph, all users exchange noise with all other users. This is slow, but there is no opportunity for malicious users to interfere with the selection of neighbors as the set of neighbors is fixed. In other cases, e.g., for random communication graphs as discussed in Section 4.2, every user needs to select $k$ neighbors. One could wonder whether
Algorithm 4 Secure generation of graph G

1: **Input:** A cryptographic hash function $H$, a Pedersen commitment function $Com_θ$, random graph parameter $k$ and parameter $k'$. 
2: **for all** $u \in U$ **do** 
3: **for all** $i \in \{1, \ldots, k'\}$ **do** 
4: Draw $z_{u,i} \leftarrow_R \{1, \ldots, n\}$, compute and publish commitment $c_{z_{u,i}} \leftarrow Com_θ(z_{u,i})$ to the bulletin board. 
5: All users run together Algorithm 2 of Appendix B.4 to generate a set $\{t_u\}_{u=1}^n$ of (pseudo-)random challenges. 
6: **for all** $u \in U$ **do** 
7: $t_{u,1} \leftarrow t_u$ 
8: **for** $i = 2$ **to** $k'$ **do** 
9: $t_{u,1} \leftarrow H(t_{u,i-1})$ 
10: $v_{u,i} \leftarrow t_{u,i} + z_{u,i}$ mod $n$ 
11: Send neighbor request to the first $k$ neighbors of $\{v_{u,i}\}_{i=1}^{k'}$ which are not duplicated. Send $z_{u,i}$ and the randomness to compute $c_{z_{u,i}}$ in the request to neighbor $v_{u,i}$.

colluding malicious users could select neighbors in such a way that the good working of the algorithm is disturbed, e.g., a single honest user is flooded with neighbor requests and ends up exchanging noise with $O(n)$ other users.

**Detection.** We first stress the fact that detecting such attacks is easy. If all agents randomly select neighbors uniformly at random as they should, then every agent expects to receive $k$ neighbor invitations, perhaps plus a few standard deviations of order $O(\sqrt{k})$. More generally, tail bounds can be used to upper bound the probability of receiving more than a certain number of neighbor invitations. As soon as an agent receives a sufficiently unlikely number of neighbor invitations, we know that with overwhelming probability the user is targeted by malicious. Agents are likely to be malicious if they are found repeatedly in a set of agents having sent an invitation to an agent who received too many invitations, and could be banned when this has happened several times. Therefore, there is a negative reward to participating in such attack repeatedly.

**Prevention.** There are several ways to prevent such attacks. We explain here one possible strategy, which is to let all users use a community-generated random seed for neighbor selection. Then, users can include in the neighbor invitation a ZKP that the addressed neighbor is the $i$-th neighbor in the list of random neighbors generated from that committed seed.

In particular, we can perform the procedure described in Algorithm 4. We use a cryptographic hash function $H$ as described at the beginning of Appendix B which outputs pseudo-random numbers. We define $k'$ to be a slightly higher number accounting for reserved generated neighbors in the case there are duplicated neighbors in the generation, which is negligible for our desired graphs. Essentially, every user $u$ commits to a set of $k'$ secret random numbers $z_{u,i}$ (lines 2–4) and engage together with other users to compute a public seed $t_u$ (line 5), which is used as input of $H$ to compute $k'$ pseudo-random numbers $t_{u,i}$ (lines 6–9). In lines 10–11, for every $i \in \{1, \ldots, k'\}$ and using terms $z_{u,i}$ and $t_{u,i}$, $u$ determines the neighbors such that they are private but chosen at random. For that, $u$ invites as neighbors the users with ID $v_{u,i}$ for $1 \leq i \leq k$ (and possibly for higher values of $i$ between $k$ and $k'$ if reserve neighbors are needed), providing that invited neighbor with the value $z_{u,i}$, the randomness used to compute $c_{z_{u,i}}$ so that user can verify he is indeed the randomly but correctly selected $i$-th neighbor of $u$.

**C.4.3. Other Common Attacks**

This paper describes the algorithm at a high level. We assume it is implemented on a system which is secure according to classic network-related attacks, such as denial-of-service (DoS) attacks where the network is flooded with messages to deprive a particular user of normal network access. Such attacks are located at the network level rather than at the algorithm level. As such, they apply similarly to any distributed algorithm requiring communication over a network. To the extent such attacks can be mitigated, the solutions are on the network level, including (among others) a correct organization of the network and its routers.

Similarly, we assume that all (honest) communication is secure and properly encrypted, referring the reader to the state-of-the-art literature for details on possible implementations.
Appendix D. Complexity of GOPA

The complexity of the protocol in terms of computation and communication was summarized in Theorem 5. This section presents a more detailed analysis of this complexity, relying in particular on the complexity of the zero knowledge proofs discussed in Appendix B and Appendix C.

For computation, we take into account the dominant terms and types of operation in every task. Some computations depend on the fixed precision arithmetic parameter \( \psi \) which is determined at the end of the description of ZKPNormal, in Appendix B.3. For communication, three types of interactions in the network are taken into account: (1) sending a message to another user, (2) publishing a message or (3) querying the bulletin board for a set of publications. We also measure the size in bits of messages, which contain values of the order of \( \langle p \rangle \) bits, where \( p \) is the size of the set \( \mathbb{Z}_p \), of which \( \mathbb{G} \) is subgroup of, defined in Appendix B.1.

The costs break down as follows:

- The initialization of GOPA requires enrolling users for the computation and defining parameters of the Pedersen commitment scheme, which is dominated by one execution of Algorithm 2 (see Appendix B.4). The computation and publication of commitments of the vector \( X \) of private values, as well as the verification that every one of its components is in the range \([0,1]\) (i.e. Property (8)) with ZKPRange has a computational and communication cost that is logarithmic in the size of the range \([0,1/\psi]\). The above then is dominated by 9 \( \log_2(1/\psi) \) modular exponentiations and a messages of 10 \( \log_2(1/\psi) \) \( \langle p \rangle \) bits in total. The cost of verification of these tasks is dominated by the same amount and type of operations and the same total network transfer. Due to Algorithm 2, a negligible amount of users are required to compute extra \( 2 \log_2(n) \) modular sums, \( 3 \log_2(n) \) extra messages of \( 3 \log_2(n) \langle p \rangle \) bits in total in the execution for these tasks, and the same operations and network transfer cost for their verification.

- Interacting with a neighbor to generate a noise term, and publishing a commitment of it requires 2 modular exponentiations and 2 messages per user. Performing the ZKPLinear proof to verify Property (6) for one neighbor requires 7 modular exponentiations and a message of \( 5 \langle p \rangle \) bits (as the linear relation is composed of 2 terms) and 9 modular exponentiations for its verification. We do the above \( |N(u)| \) times, so the final cost is of \( 7|N(u)| \) modular exponentiations \( , 2|N(u)| \) messages of size of \( \langle p \rangle \) bits, and a message for all ZKPs with total size of \( 7|N(u)| \langle p \rangle \) bits. The verification cost is of \( 9|N(u)| \) modular exponentiations.

- Generating the private Gaussian distribution variable \( \eta_u \) and proving Property (7) is done by performing ZKPNormal. The cost, described at the end of Appendix B.3, this requires \( 39.6 \log(1/B) \log(\log(1/B)) \) modular exponentiations and \( 39.6 \log(1/B) \log(\log(1/B)) \) to verify the ZKP. The proof size is of \( 39.6 \log(1/B) \log(\log(1/B)) \langle p \rangle \) bits. The computational and communicational cost of the only interactive proof ZKPUniform, (including Algorithm 2) as part of this proof is negligible as other terms dominate the costs for reasonable values of \( B \).

- Finally, the computation and verification of the proof of Property (5) requires the computation of ZKPLinear over \( |N(u)| + 3 \) terms per user, which has a dominant cost of \( |N(u)| \) modular exponentiations and a message of size of \( 5 \langle p \rangle \) bits. The cost of verification is dominated by \( |N(u)| \) modular exponentiations.

For the final cost, we discard terms of the first bullet, given that modular sums are dominated by modular exponentiations in the same amounts, and terms in order of magnitude of \( \log_2(1/\psi) \) and \( \log_2(n) \) are dominated by \( \log^2(1/B) \) terms for reasonable values of \( B \). Additionally, the cost of signing messages are marginal.

The overall computational cost of the private averaging phase for a user \( u \in U \) is therefore \( 8|N(u)| + 39.6 \log(1/B) \log(\log(1/B)) \) modular exponentiations. The overall communication cost is composed of \( 2|N(u)| \) messages and a total transfer size of \( (7|N(u)| + 39.6 \log(1/B) \log(\log(1/B))) \langle p \rangle \) bits. Additionally, the verification for \( u \) consist on querying the proofs of correctness from the bulletin board with a total transfer size equal to the mentioned above, and requires \( 10|N(u)| + 39.6 \log(1/B) \log(\log(1/B)) \) modular exponentiations.