Default Logic in a Coherent Setting

Giulianella Coletti
Dipartimento Matematica e Informatica
Università di Perugia, 06100 Perugia (Italy)

Romano Scozzafava
Dipartimento Metodi e Modelli Matematici
Università La Sapienza, 00161 Roma (Italy)

Barbara Vantaggi
Dipartimento Metodi e Modelli Matematici
Università La Sapienza, 00161 Roma (Italy)

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Abstract

In this talk – based on the results of a forthcoming paper (Coletti, Scozzafava and Vantaggi 2002), presented also by one of us at the Conference on “Non Classical Logic, Approximate Reasoning and Soft-Computing” (Anacapri, Italy, 2001) – we discuss the problem of representing default rules by means of a suitable coherent conditional probability, defined on a family of conditional events. An event is singled-out (in our approach) by a proposition, that is a statement that can be either true or false; a conditional event is consequently defined by means of two propositions and is a 3–valued entity, the third value being (in this context) a conditional probability.

1 INTRODUCTION

The concept of conditional event (as dealt with in this paper) plays a central role for the probabilistic reasoning. We give up (or better, in a sense, we generalize) the idea of de Finetti of looking at a conditional event \( E|H \), with \( H \neq \emptyset \) (the impossible event), as a 3–valued logical entity looked on as “undetermined” when \( H \) is false: it is true when both \( E \) and \( H \) are true, false when \( H \) is true and \( E \) is false, while we let the third value suitably depend on the given ordered pair \((E, H)\) and not being just an undetermined common value for all pairs. It turns out (as explained in detail in Coletti and Scozzafava 1999) that this function can be seen as a measure of the degree of belief in the conditional event \( E|H \), which under “natural” conditions reduces to the conditional probability \( P(E|H) \), in its most general sense related to the concept of coherence, and satisfying the classic axioms as given by de Finetti (1949), Rényi (1956), Krauss (1968), Dubins
(1975): see Section 2. Notice that our concept of conditional event differs from that adopted, for example, by Adams (1975), Benferhat, Dubois and Prade (1997), Goodman and Nguyen (1988), Schay (1968).

Among the peculiarities (which entail a large flexibility in the management of any kind of uncertainty) of this concept of coherent conditional probability versus the usual one, we recall the following ones:

- due to its direct assignment as a whole, the knowledge (or the assessment) of the “joint” and “marginal” unconditional probabilities $P(E \land H)$ and $P(H)$ is not required;
- the conditioning event $H$ (which must be a possible one) may have zero probability, but in the assignment of $P(E|H)$ we are driven by coherence, contrary to what is done in those treatments where the relevant conditional probability is given an arbitrary value in the case of a conditioning event of zero probability;
- a suitable interpretation of its extreme values 0 and 1 for situations which are different, respectively, from the trivial ones $E \land H = \emptyset$ and $H \subseteq E$, leads to a “natural” treatment of the default reasoning.

In this talk we deal with the latter aspect.

2 COHERENT CONDITIONAL PROBABILITY

The classic axioms for a conditional probability read as follows (given a set $C = \mathcal{G} \times \mathcal{B}_o$ of conditional events $E|H$ such that $\mathcal{G}$ is a Boolean algebra and $\mathcal{B} \subseteq \mathcal{G}$ is closed with respect to (finite) logical sums, with $\mathcal{B}_o = \mathcal{B} \setminus \{\emptyset\}$):

(i) $P(H|H) = 1$, for every $H \in \mathcal{B}_o$,

(ii) $P(\cdot|H)$ is a (finitely additive) probability on $\mathcal{G}$ for any given $H \in \mathcal{B}_o$,

(iii) $P(E \land A|H) = P(E|H)P(A|E \land H)$,
for any $A, E \in \mathcal{G}$, $H, E \land H \in \mathcal{B}_o$.

Conditional probability $P$ has been defined on $\mathcal{G} \times \mathcal{B}_o$; however it is possible, through the concept of coherence, to handle also those situations where we need to assess $P$ on an arbitrary set $C$ of conditional events.

Definition 1 - The assessment $P(\cdot|\cdot)$ on $C$ is coherent if there exists $C' \supset C$, with $C' = \mathcal{G} \times \mathcal{B}_o$, such that $P(\cdot|\cdot)$ can be extended from $C$ to $C'$ as a conditional probability.

A characterization of coherence is given (see, e.g., Coletti and Scozzafava 1996) by the following

Theorem 1 - Let $C$ be an arbitrary finite family of conditional events $E_1|H_1, \ldots, E_n|H_n$ and $A_o$ denote the set of atoms $A_r$ generated by the (unconditional) events $E_1, H_1, \ldots, E_n, H_n$. For a real function $P$ on $C$ the following two statements are equivalent:

(i) $P$ is a coherent conditional probability on $C$;

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(ii) there exists (at least) a class of probabilities \( \{P_0, P_1, \ldots, P_k\} \), each probability \( P_\alpha \) being defined on a suitable subset \( \mathcal{A}_\alpha \subseteq \mathcal{A}_o \), such that for any \( E_i|H_i \in \mathcal{C} \) there is a unique \( P_\alpha \) with

\[
\sum_{A_r \subseteq H_i} P_\alpha(A_r) > 0,
\]

\[
P(E_i|H_i) = \frac{\sum_{A_r \subseteq E_i \cap H_i} P_\alpha(A_r)}{\sum_{A_r \subseteq H_i} P_\alpha(A_r)};
\]

moreover \( \mathcal{A}_\alpha' \subset \mathcal{A}_{\alpha''} \) for \( \alpha' > \alpha'' \) and \( P_{\alpha''}(A_r) = 0 \) if \( A_r \in \mathcal{A}_{\alpha'} \).

According to Theorem 1, a coherent conditional probability gives rise to a suitable class \( \{P_0, P_1, \ldots, P_k\} \) of “unconditional” probabilities.

Where do the above classes of probabilities come from? Since \( P \) is coherent on \( \mathcal{C} \), there exists an extension \( P^* \) on \( \mathcal{G} \times \mathcal{B}^o \), where \( \mathcal{G} \) is the algebra generated by the set \( \mathcal{A}_o \) of atoms and \( \mathcal{B} \) the additive class generated by \( H_1, \ldots, H_n \); then, putting \( \mathcal{F} = \{\emptyset, \Omega\} \), the restriction of \( P^* \) to \( \mathcal{G} \times \mathcal{F}^o \) satisfies (1) with \( \alpha = 0 \) for any \( E_i|H_i \) such that \( P_0(H_i) > 0 \). The subset \( \mathcal{A}_1 \subset \mathcal{A}_0 \) contains only the atoms \( A_r \subseteq H_1^o \), the union of \( H_i \)'s with \( P_0(H_i) = 0 \) (and so on): we proved (see, e.g., Coletti and Scozzafava 1996, 1999) that, starting from a coherent assessment \( P(E_i|H_i) \) on \( \mathcal{C} \), a relevant family \( \mathcal{P} = \{P_\alpha\} \) can be suitably defined that allows a representation such as (1). Every value \( P(E_i|H_i) \) constitutes a constraint in the construction of the probabilities \( P_\alpha (\alpha = 0, 1, \ldots) \); in fact, given the set \( \mathcal{A}_o \) of atoms generated by \( E_1, E_n, H_1, \ldots, H_n \), and its subsets \( \mathcal{A}_\alpha \) (such that \( P_\beta(A_r) = 0 \) for any \( \beta < \alpha \)), with \( A_r \in \mathcal{A}_\alpha \), each \( P_\alpha \) must satisfy the following system \( (S_\alpha) \) with unknowns \( P_\alpha(A_r) \geq 0, A_r \in \mathcal{A}_\alpha \),

\[
(S_\alpha) \begin{cases}
\sum_{A_r \subseteq E_i \cap H_i} P_\alpha(A_r) = P(E_i|H_i) \sum_{A_r \subseteq H_i} P_\alpha(A_r), \\
[\text{if } P_{\alpha-1}(H_i) = 0] \sum_{A_r \subseteq H_i^o} P_\alpha(A_r) = 1
\end{cases}
\]

where \( P_{\alpha-1}(H_i) = 0 \) for all \( H_i \)'s, and \( H_i^o \) denotes, for \( \alpha \geq 0 \), the union of the \( H_i \)'s such that \( P_{\alpha-1}(H_i) = 0 \); so, in particular, \( H_1^o = H_o = H_1 \lor \ldots \lor H_n \).

Any class \( \{P_\alpha\} \) singled-out by the condition (ii) is said to agree with the conditional probability \( P \). Notice that in general there are infinite classes of probabilities \( \{P_\alpha\} \); in particular we have only one agreeing class in the case that \( \mathcal{C} \) is a product of Boolean algebras.

A coherent assessment \( P \), defined on a set \( \mathcal{C} \) of conditional events, can be extended in a natural way to all the conditional events \( E|H \) such that \( E \cap H \) is an element of the algebra \( \mathcal{G} \) spanned by the (unconditional) events \( E_i, H_i, i = 1, 2, \ldots, n \) taken from the elements of \( \mathcal{C} \), and \( H \) is an element of the additive class
spanned by the $H_i$’s. Obviously, this extension is not unique, since there is no uniqueness in the choice of the class $\{P_\alpha\}$ related to condition (ii) of Theorem 1.

In general, we have the following result (see, e.g., Coletti and Scozzafava 1996):

**Theorem 2** - If $C$ is a given family of conditional events and $P$ a corresponding assessment, then there exists a (possibly not unique) coherent extension of $P$ to an arbitrary family $K$ of conditional events, with $K \supseteq C$, if and only if $P$ is coherent on $C$.

Notice that if $P$ is coherent on a family $C$, it is coherent also on $E \subseteq C$.

### 3 ZERO-LAYERS

Given a class $P = \{P_\alpha\}_{\alpha=0,1,\ldots,k}$, agreeing with a conditional probability on $C$, it naturally induces the zero-layer $\circ(H)$ of an event $H$, defined as

$$\circ(H) = \beta \text{ if } P_\beta(H) > 0;$$

if $P_\alpha(H) = 0$ for every $\alpha = 0, 1, \ldots, k$ (obviously, we necessarily have $H \neq H_i$ for every $i = 1, 2, \ldots, n$), then $\circ(H) = k + 1$.

The zero-layer of a conditional event $E|H$ is defined as

$$\circ(E|H) = \circ(E \land H) - \circ(H).$$

Obviously, for the certain event $\Omega$ and for any event $E$ with positive probability, we have $\circ(\Omega) = \circ(E) = 0$ (so that, if the class contains only an everywhere positive probability $P_\alpha$, there is only one (trivial) zero-layer, i.e. $\alpha = 0$), while we put $\circ(\emptyset) = +\infty$. Clearly,

$$\circ(A \lor B) = \min\{\circ(A), \circ(B)\}.$$  
Moreover, notice that $P(E|H) > 0$ if and only if $\circ(EH) = \circ(H)$, i.e. $\circ(E|H) = 0$.

On the other hand, Spohn (see, for example, Spohn 1994, 1999) considers degrees of plausibility defined via a ranking function, that is a map $\kappa$ that assigns to each possible proposition a natural number (its rank) such that

(a) either $\kappa(A) = 0$ or $\kappa(A^c) = 0$, or both;

(b) $\kappa(A \lor B) = \min\{\kappa(A), \kappa(B)\}$;

(c) for all $A \land B \neq \emptyset$, the conditional rank of $B$ given $A$ is $\kappa(B|A) = \kappa(A \land B) - \kappa(A)$.

Ranks represent degrees of “disbelief”. For example, $A$ is not disbelieved iff $\kappa(A) = 0$, and it is disbelieved iff $\kappa(A) > 0$.

**Remark 1** - Ranking functions are seen by Spohn as a tool to manage plain belief and belief revision, since he maintains that probability is inadequate for this purpose. In our framework this claim can be challenged (see Coletti, Scozzafava and Vantaggi 2001), since our tools for belief revision are coherent conditional probabilities and the ensuing concept of zero-layers: it is easy to check that zero-layers have the same formal properties of ranking functions.
4 COHERENT PROBABILITY AND DEFAULT LOGIC

We recall that in Coletti, Scozzafava and Vantaggi (2001) we showed that a sensible use of events whose probability is 0 (or 1) can be a more general tool in revising beliefs when new information comes to the fore, so that we have been able to challenge the claim contained in Shenoy (1991) that probability is inadequate for revising plain belief. Moreover, as recalled in Section 1, we may deal with the extreme value \( P(E|H) = 1 \) also for situations which are different from the trivial one \( H \subseteq E \).

The aim of this Section is to handle, by means of a coherent conditional probability, some aspects of default reasoning (see, e.g., Reiter 1980, Russell and Norvig 1995): as it is well-known, a default rule is a sort of weak implication.

First of all, we discuss briefly some aspects of the classic example of Tweety. The usual logical implication (denoted by \( \subseteq \)) can be anyway useful to express that a penguin (\( \pi \)) is certainly a bird (\( \beta \)), i.e. \( \pi \subseteq \beta \), so that \( P(\beta|\pi) = 1 \); moreover we know that Tweety (\( \tau \)) is a penguin (that is, \( \tau \subseteq \pi \)), and so also this fact can be represented by a conditional probability equal to 1, that is \( P(\pi|\tau) = 1 \).

But we can express as well the statement “a penguin usually does not fly” (we denote by \( \varphi^c \) the contrary of \( \varphi \), the latter symbol denoting “flying”) by writing \( P(\varphi^c|\pi) = 1 \).

(For simplicity, we have avoided to write down explicit a proposition – that is, an event – such as “a given animal is a penguin”, using the short-cut “penguin” and the symbol \( \pi \) to denote this event; similar considerations apply to \( \beta, \tau \) and \( \varphi \).

The question “can Tweety fly?” can be faced through an assessment of the conditional probability \( P(\varphi|\tau) \), which must be coherent with the already assessed ones: by Theorem 1, it can be shown that any value \( p \in [0,1] \) is a coherent value for \( P(\varphi|\tau) \), so that no conclusion can be reached – from the given premises – on Tweety’s ability of flying.

In other words, interpreting an equality such as \( P(E|H) = 1 \) like a default rule (denoted by \( \mapsto \rightarrow \)), which in particular (when \( H \subseteq E \)) reduces to the usual implication, we have shown its nontransitivity: in fact we have \( \tau \mapsto \pi \) and \( \pi \mapsto \varphi^c \), but it does not necessarily follow the further default rule \( \tau \mapsto \varphi^c \) (even if we might have that \( P(\varphi^c|\tau) = 1 \), i.e. that “Tweety usually does not fly”).

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Definition 2 - Given a coherent conditional probability $P$ on a family $C$ of conditional events, a default rule, denoted by $H \rightarrow E$, is any conditional event $E|H \in C$ such that $P(E|H) = 1$.

Clearly, any logical implication $A \subseteq B$ (and so also any equality $A = B$) between events can be seen as a (trivial) default rule.

Remark 2 - By resorting to the systems $(S_\alpha)$ to check the coherence of the assessment $P(E|H) = 1$ (which implies, for the relevant zero-layer, $\circ(E|H) = 0$), a simple computation gives $P_\circ(E^c \land H) = 0$ (notice that the class $\{P_\alpha\}$ has in this case only one element $P_\circ$). It follows $\circ(E^c|H) = 1$, so that $\circ(E^c|H) > \circ(E|H)$.

In terms of Spohn’s ranking functions (we recall – and underline – that our zero-layers are – so to say – “incorporated” into a coherent conditional probability, so that we do not need an “autonomous” definition of ranking!) we could say, when $P(E|H) = 1$, that the disbelief in $E^c|H$ is greater than that in $E|H$.

This conclusion must not be read as $P(E|H) > P(E^c|H)$!

Given a set $\Delta \subseteq C$ of default rules $H_i \rightarrow E_i$, with $i = 1,...,n$, we need to check its consistency, that is the coherence of the “global” assessment $P$ on $C$ such that $P(E_i|H_i) = 1$, $i = 1,...,n$.

We stress that, even if our definition involves a conditional probability, the condition given in the following theorem refers only to logical (in the sense of Boolean logic) relations.

Theorem 3 - Given a coherent conditional probability $P$ on a family $C$ of conditional events, the following two statements are equivalent:

(i) the set $\Delta \subseteq C$ of default rules

$$H_i \rightarrow E_i, \ i = 1,2,...,n,$$

represented by the assessment

$$P(E_i|H_i) = 1, \ i = 1,2,...,n,$$

is consistent;

(ii) for every subset

$$\{H_{i_1} \rightarrow E_{i_1}, \ldots, H_{i_s} \rightarrow E_{i_s}\}$$

of $\Delta$, with $s = 1,2,...,n$, we have

$$\bigvee_{k=1}^s (E_{i_k} \land H_{i_k}) \nsubseteq \bigvee_{k=1}^s (E_{i_k}^c \land H_{i_k}).$$

Proof - We prove that, assuming the above logical relations (2), coherence of $P$ is compatible with the assessment $P(E_i|H_i) = 1$ ($i = 1,2,...,n$), on $\Delta$.

We resort to the characterization Theorem 1: to begin with, put $P(E_i|H_i) = 1$ ($i = 1,2,...,n$), in the system $(S_\alpha)$. The unconditional probability $P_\circ$ can be
obtained by putting \( P_o(A_r) = 0 \) for all atoms \( A_r \subseteq \bigvee_{j=1}^n \left( E_j^c \land H_j \right) \), so for any atom \( A_k \subseteq E_i \land H_i \) which is not contained in \( \bigvee_{j=1}^n \left( E_j^c \land H_j \right) \) – notice that condition (2) ensures that there is such an atom \( A_k \), since \( \bigvee_{j=1}^n (E_j \land H_j) \not\subseteq \bigvee_{j=1}^n (E_j^c \land H_j) \) – we may put \( P_o(A_k) > 0 \) in such a way that these numbers sum up to 1, and we put \( P_o(A_r) = 0 \) for all remaining atoms.

This clearly gives a solution of the first system \((S_o)\). If, for some \( i, E_i \land H_i \subseteq \bigvee_{j=1}^n \left( E_j^c \land H_j \right) \), then \( P_o(E_i \land H_i) = 0 \). So we consider the second system (which refers to all \( H_i \) such that \( P_o(H_i) = 0 \)), proceeding as above to construct the probability \( P_1 \); and so on. Condition (2) ensures that at each step we can give positive probability \( P_o \) to (at least) one of the remaining atoms.

Conversely, consider the (coherent) assignment \( P(E_i | H_i) = 1 \) (for \( i = 1, \ldots, n \)). Then, for any index \( j \in \{1, 2, \ldots, n\} \) there exists a probability \( P_o \) such that \( P_o(H_j) > 0 \) and \( P_o(E_j^c \land H_j) = 0 \). Notice that the restriction of \( P \) to some conditional events \( E_{i_1} | H_{i_1}, \ldots, E_{i_s} | H_{i_r} \) of \( \Delta \) is coherent as well.

Let \( H \) be the first element of an agreeing class, and \( i_k \) an index such that \( P_o(H_{i_k}) > 0 \); then we have \( P_o(E_{i_k} \land H_{i_k}) > 0 \) and \( P_o(E_{i_k}^c \land H_{i_k}) = 0 \). Suppose that \( E_{i_k} \land H_{i_k} \subseteq \bigvee_{k=1}^s (E_{i_k}^c \land H_{i_k}) \); then \( P_o(E_{i_k} \land H_{i_k}) = 0 \). This contradiction shows that condition (2) holds.

**Definition 3** - A set \( \Delta \) of default rules *entails* the default rule \( H \mapsto E \) if the only coherent value for \( P(E | H) \) is 1. In other words, the rule \( H \mapsto E \) is entailed by \( \Delta \) (or by a subset of \( \Delta \)) if every possible extension (cf. Theorem 2) of the probability assessment \( P(E_i | H_{i_r}) = 1 \), \( r = 1 \ldots s \), assigns the value 1 also to \( P(E | H) \).

Going back to the previous example of Tweety, its possible ability (or inability) of flying can be expressed by saying that the default rule \( \tau \mapsto \varphi \) (or \( \tau \mapsto \varphi^c \)) is not entailed by the premises (the given set \( \Delta \)).

### 5 INFEERENCE

Several formalisms for default logic have been studied in the relevant literature with the aim of discussing the minimal conditions that an entailment should satisfy. In our framework this “inferential” process is ruled by the following

**Theorem 4** - Given a set \( \Delta \) of consistent default rules, we have:

**(Reflexivity)**
\[ \Delta \text{ entails } A \mapsto A \text{ for any } A \neq \emptyset; \]

**(Left Logical Equivalence)**
\[ (A = B), (A \mapsto C) \in \Delta \text{ entails } B \mapsto C; \]

**(Right Weakening)**
\[ (A \subseteq B), (C \mapsto A) \in \Delta \text{ entails } C \mapsto B; \]

**(Cut)**
\[ (A \land B \mapsto C), (A \mapsto B) \in \Delta \text{ entails } A \mapsto C; \]
(Cautious Monotonicity) $(A \rightarrow B), (A \rightarrow C) \in \Delta$ entails $A \land B \rightarrow C$;

(Equivalence) $(A \rightarrow B), (B \rightarrow A), (A \rightarrow C) \in \Delta$ entails $B \rightarrow C$;

(And) $(A \rightarrow B), (A \rightarrow C) \in \Delta$ entails $A \rightarrow B \land C$;

(Or) $(A \rightarrow C), (B \rightarrow C) \in \Delta$ entails $A \lor B \rightarrow C$.

Proof - Reflexivity amounts to $P(A|A) = 1$ for every possible event.

Left Logical Equivalence and Right weakening trivially follow from elementary properties of conditional probability.

Cut: from $P(C|A \land B) = P(B|A) = 1$ it follows that

$$P(C|A) = P(C|A \land B)P(B|A) + P(C|A \land B^c)P(B^c|A) =$$

$$= P(C|A \land B)P(B|A) = 1.$$

Cautious Monotonicity: since $P(B|A) = P(C|A) = 1$, we have that

$$1 = P(C|A \land B)P(B|A) + P(C|A \land B^c)P(B^c|A) =$$

$$= P(C|A \land B)P(B|A),$$

hence $P(C|A \land B) = 1$.

Equivalence: since at least one conditioning event must have positive probability, it follows that $A, B, C$ have positive probability; moreover,

$$P(A \land C) = P(A) = P(A \land B) = P(B),$$

which implies $P(A \land B \land C) = P(A) = P(B)$, so $P(C|B) = 1$.

And: since

$$1 \geq P(B \lor C|A) = P(B|A) + P(C|A) - P(B \land C|A) =$$

$$= 2 - P(B \land C|A),$$

it follows $P(B \land C|A) = 1$.

Or: since

$$P(C|A \lor B) =$$

$$= P(C|A)P(A|A \lor B) + P(C|B)P(B|A \lor B) -$$

$$- P(C|A \land B)P(A \land B|A \lor B) =$$

$$= P(A|A \lor B) + P(B|A \lor B)-$$

$$- P(C|A \land B)P(A \land B|A \lor B) \geq 1,$$

we get $P(C|A \lor B) = 1$.

We consider now some “unpleasant” properties (cf., e.g., Lehmann and Magidor, 1992), that in fact do not necessarily hold also in our framework:
(Monotonicity) \((A \subseteq B), (B \rightarrow C) \in \Delta\) entails \(A \rightarrow C\)

(Transitivity) \((A \rightarrow B), (B \rightarrow C) \in \Delta\) entails \(A \rightarrow C\)

(Contraposition) \((A \rightarrow B) \in \Delta\) entails \(B^c \rightarrow A^c\)

The previous example about Tweety shows that Transitivity can fail.

In the same example, if we add the evaluation \(P(\varphi|\beta) = 1\) (that is, a bird usually flies) to the initial ones, the assessment is still coherent (even if \(P(\varphi|\pi) = 0\) and \(\pi \subseteq \beta\)), but Monotonicity can fail.

Now, consider the conditional probability \(P\) defined as follows:

\[
P(B|A) = 1, \quad P(A^c|B^c) = \frac{1}{4};
\]

it is easy to check that it is coherent, and so Contraposition can fail.

Many authors (cf., e.g., again Lehmann and Magidor, 1992) claim (and we agree) that the previous unpleasant properties should be replaced by others, that we express below in our own notation and interpretation: we show that these properties hold in our framework. Since a widespread consensus among their “right” formulation is lacking, we will denote them as cs–(Negation Rationality), cs–(Disjunctive Rationality), cs–(Rational Monotonicity), where “cs” stands for “in a coherent setting”. Notice that, given a default rule \(H \rightarrow E\), to say \((H \rightarrow E) \notin \Delta\) means that the conditional event \(E|H\) belongs to the set \(C \setminus \Delta\).

\textbf{cs–(Negation Rationality)}

If \((A \land C \rightarrow B), (A \land C^c \rightarrow B) \notin \Delta\)
then \(\Delta\) does not entail \((A \rightarrow B)\)

\textbf{Proof} - If \((A \land C \rightarrow B)\) and \((A \land C^c \rightarrow B)\) do not belong to \(\Delta\), i.e. \(P(B|A \land C) < 1\) and \(P(B|A \land C^c) < 1\) imply

\[
P(B|A) = P(B|A \land C)P(C|A) + P(B|A \land C^c)P(C^c|A) < \\
< P(C|A) + P(C^c|A) = 1.
\]

\textbf{cs–(Disjunctive Rationality)}

If \((A \rightarrow C), (B \rightarrow C) \notin \Delta\)
then \(\Delta\) does not entail \((A \lor B \rightarrow C)\)

\textbf{Proof} - Starting from the equalities

\[
P(C|A \lor B) = \\
= P(C|A)P(A|A \lor B) + P(C|A^c \land B)P(A^c \land B|A \lor B)
\]

and

\[
P(C|A \lor B) = 
\]
\[ P(C|B)P(B|A \lor B) + P(C|A \land B^c)P(A \land B^c|A \lor B), \]

since we have \( P(C|A) < 1 \) and \( P(C|B) < 1 \), then \( P(C|A \lor B) = 1 \) would imply (by the first equality) \( P(A|A \lor B) = 0 \) and (by the second one) \( P(B|A \lor B) = 0 \) (contradiction).

**cs–(Rational Monotonicity)**

If \((A \land B \implies C), (A \implies B^c) \notin \Delta\)

then \(\Delta\) does not entail \((A \implies C)\)

**Proof** - If it were \(P(C|A) = 1\), i.e.

\[ 1 = P(C|A \land B)P(B|A) + P(C|A \land B^c)P(B^c|A), \]

we would get either

\[ P(C|A \land B) = P(C|A \land B^c) = 1 \]

or one of the following

\[ P(C|A \land B) = P(B|A) = 1, \]

\[ P(C|A \land B^c) = P(B^c|A) = 1 \]

(contradiction).

In conclusion, let us notice the simplicity of our approach (Occam’s razor...!), with respect to other well-known methodologies, such as, e.g. those given by Adams (1975), Benferhat, Dubois and Prade (1997), Goldszmidt and Pearl (1996), Lehmann and Magidor (1992), Schaub (1998).

### 6 DISCUSSION

Thought-provoking comments of two anonymous reviewers suggested to us to add this further section.

Among coherence–based approaches to default reasoning (in the framework of imprecise probability propagation), that of Gilio (2000) deserves to be mentioned, even if we claim (besides the utmost simplicity of our definitions and results) many important semantic and syntactic differences.

First of all, our framework (see the very beginning of our Introduction) is clearly and rigorously settled: conditional events \(E|H\) are **not** 3-valued entities whose third value is looked on as “undetermined” when \(H\) is false, but they have been defined instead in a way which entails “automatically” (so-to-say) the axioms of conditional probability, which are those **ruling coherence** (the details, as already recalled in the Introduction, are in Coletti and Scozzafava, 1999).

In other words (french words, since we are in France), “tout se tient”, while in the aforementioned paper by Gilio a concept such as \(E|H\) is interpreted sometimes as a 3-valued entity looked on as “undetermined” when \(H\) is false,
sometimes as an ordered pair of events, sometimes as a conditional assertion \( H \models E \) (in the knowledge base).

Moreover, our notions of consistency and entailment are both different from his: in fact he gives a theorem (without proof) connecting the notion of consistency to that of Adams (1975).

The problem is that we do not understand Adams’ framework: in fact he requires probability to be proper (i.e., positive) on the given events, but (since the domain of a probability \( P \) is an algebra) we need to extend \( P \) from the given events to other events (by the way, coherence is nothing but complying with this need). In particular, these “new” events may have zero probability: it follows, according to Adams’ definition of conditional probability in the case of a conditioning event of zero probability, that we can easily get incoherent assessments (see the example below). By the way, in the section “Some preliminaries”, Gilio claims “We can frame our approach to the problem of propagating imprecise conditional probability assessments from the probabilistic logic point of view, see, e.g., Frisch and Haddawy ...”: unfortunately, Frisch and Haddawy definition of conditional probability coincides (for conditioning events which are null) with that of Adams, and so it violates coherence as well!

Not to mention that both Gilio and Adams (and many others: some of them are mentioned at the end of the previous section) base the concept of consistency on that of quasi conjunction, which is a particular conditional event (and our concept of conditional event is different from theirs); moreover we deem that the notion they give of verifiability of a conditional event \( E|H \), that is \( E \land H \neq \emptyset \), is too weak – except in the case \( H = \Omega \) – to express properly the relevant semantics.

Our discussion can be better illustrated by the following (very simple) example:

**Example** - Consider two (logically independent) events \( H_1 \) and \( H_2 \), and put

\[
E_1 = H_1 \land H_2, \quad E_2 = H_1^c \land H_2, \\
E_3 = H_1^c \land H_2^c, \quad E = E_2, \quad H = H_3 = \Omega.
\]

Given \( \alpha \), with \( 0 < \alpha < 1 \), the assessment

\[
P(E_1|H_1) = P(E_2|H_2) = 1, \quad P(E_3|H_3) = \alpha
\]

on \( C = \{E_1|H_1, E_2|H_2, E_3|H_3\} \) is coherent; the relevant probabilities of the atoms are

\[
P(H_1 \land H_2) = P(H_1^c \land H_2^c) = 0, \\
P(H_1^c \land H_2) = \alpha, \quad P(H_1^c \land H_2) = 1 - \alpha,
\]

so that the set \( \Delta \) of default rules corresponding to \( \{E_1|H_1, E_2|H_2\} \) is consistent.

Does \( \Delta \) entail \( E|H \)? A simple check shows that the only coherent assessment for this conditional event is \( P(E|H) = 1 - \alpha \). Then the answer is NO, since we require (in the definition of entailment) that 1 is (the only) coherent extension.

On the contrary, according to Gilio characterization of entailment – that is: \( \Delta \) (our notation) entails \( E|H \) iff \( P(E^c|H) = 1 \) is not coherent – the answer to
the previous question is YES, since the only coherent value of this conditional probability is $P(E^c|H) = \alpha$ (see the above computation).

For any $\epsilon > 0$, consider now the assessment

$$P(E_1|H_1) = 1, \ P(E_2|H_2) = 1 - \epsilon,$$

so that $\{E_1|H_1, E_2|H_2\}$ is consistent according to Adams, as can be easily checked giving the atoms the probabilities

$$P(H_1 \land H_2) = \epsilon, \ P(H_1 \land H_2^c) = 0,$$

$$P(H_1^c \land H_2^c) = 0, \ P(H_1^c \land H_2) = 1 - \epsilon,$$

(notice that the assessment is proper). But for any event $A \subset H_1 \land H_2^c$ we can extend $P$, according to his definition of conditional probability, as

$$P(A|H_1 \land H_2^c) = P(A^c|H_1 \land H_2^c) = 1,$$

which is not coherent!

Finally, there is no mention in Gilio’s paper of Negation Rationality, Disjunctive Rationality, and Rational Monotonicity (and, according to one of the reviewers, these properties do not hold “in default reasoning under coherent probabilities”, while in our setting they have been proved at the end of Section 5).

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