Exchanging two identical particles

S.J. van Enk
Department of Physics and Oregon Center for Optical, Molecular & Quantum Sciences
University of Oregon, Eugene, OR 97403

The phase factor \((-1)^{2s}\) that features in the exchange symmetry for identical spin-\(s\) fermions or bosons contributes to but is not identical to the phase factor one may observe when one actually exchanges two such particles. The observable phase contains single-particle geometric and dynamical phases as well, induced by both spin and spatial exchange transformations. Extending the analysis to (abelian) anyons by incorporating the Aharonov-Bohm effect shows that the topological phase two anyons pick up under an interchange is at bottom a single-particle effect, in contrast to the fermion/boson exchange phase factor \((-1)^{2s}\), which is a two-particle effect.

A. Questions

Suppose we have a properly symmetrized state of two identical fermions/bosons of the form

\[
|\Psi\rangle = (2)^{-1/2} \left[ |\phi\rangle_1 |\psi\rangle_2 + (-1)^{2s} |\psi\rangle_1 |\phi\rangle_2 \right]
\]  

where \(|\phi\rangle\) and \(|\psi\rangle\) are (single-particle) orthogonal states and \(s\) denotes the spin of the particles. The labels ‘1’ and ‘2’ are mathematical labels for single-particle Hilbert spaces \([1,2]\). What happens if we actually exchange the two particles? As we will find out below, exchanging particles is different from just exchanging the labels 1 and 2 \([3]\). In the latter case we definitely get a factor \((-1)^{2s}\)—that is, under relabeling 1 \(\leftrightarrow\) 2 we get \(|\Psi\rangle \longrightarrow (-1)^{2s} |\Psi\rangle\)—but what phase factor do we get if we apply a physical operation that takes \(|\phi\rangle\) to \(|\psi\rangle\) and vice versa? Does it depend on how we physically implement the exchange? Do we even get merely a phase factor?

B. One-particle vs two-particle phases

Let us consider the case where the two particles are in spatially different states (located in non-overlapping regions \(A\) and \(B\) respectively), so that one can address the particles individually (for example, by focusing two laser beams on the particles). Their spin states are assumed to be arbitrary but pure, and we write

\[
|\phi\rangle = |A\rangle |\sigma\rangle , \\
|\psi\rangle = |B\rangle |\tau\rangle ,
\]

where \(|A\rangle |B\rangle = 0\) but \(|\langle \sigma|\tau\rangle|\) could be any number between zero and unity (we ignore the uninteresting case \(s = 0\) here and assume \(s \geq 1/2\)). Note that we can perfectly well say that the particle that is located in region \(A\) has spin state \(|\sigma\rangle\) but \(not\) that it is particle ‘1’ that is in region \(A\). (Indeed, “classical particles” emerge from quantum theory as approximately localized entities \([2]\), not as entities labeled ‘1’ or ‘2’.)

Now consider a physical operation that swaps the particles’ spatial states such that the particles remain distinguishable during the whole process. One obvious way to accomplish this is to use one and the same unitary operation—let’s represent this by \(\hat{U}\)—for both particles: since they start in orthogonal spatial states they will remain in orthogonal spatial states that way. In that case we can make the operations that swap the spin states different for the two particles, that is, we can implement position-dependent spin transformations. We may thus introduce three unitary operators \(\hat{U}, \hat{V}_A, \hat{V}_B\) such that the total state transformation is determined by

\[
\hat{U} |\phi\rangle = \exp(i\varphi) |\psi\rangle , \\
\hat{U} |\psi\rangle = \exp(i\varphi') |\phi\rangle , \\
\hat{V}_A |\sigma\rangle = \exp(i\varphi_A) |\tau\rangle , \\
\hat{V}_B |\tau\rangle = \exp(i\varphi_B) |\sigma\rangle ,
\]

where \(\hat{V}_A (\hat{V}_B)\) is the spin transformation applied to the spin in region \(A (B)\). The four phases appearing in Eq. \([3]\) can be viewed as sums of dynamical and geometric phases pertaining to single-particle evolutions, as we will see shortly. Since these four phases are acquired by a single particle they differ (conceptually and physically) from the bosonic/fermionic exchange phase (it takes two to exchange).

Substituting relations \([2]\) and \([3]\) into the transformation of the initial state \([1]\) yields

\[
|\Psi\rangle \longrightarrow \exp(i(2s\pi + \varphi_{\text{spatial}} + \varphi_{\text{spin}})) |\Psi\rangle .
\]

The total phase acquired in this process is written as a sum of three terms. Apart from the exchange phase \(2s\pi\) we also have a phase

\[
\varphi_{\text{spatial}} = \varphi + \varphi'
\]

that results from the combination of two spatial swaps

\[
|\phi\rangle \longrightarrow \hat{U} \hat{U} |\phi\rangle = \exp(i(\varphi + \varphi')) |\phi\rangle , \\
|\psi\rangle \longrightarrow \hat{U} \hat{U} |\psi\rangle = \exp(i(\varphi + \varphi')) |\psi\rangle ,
\]

and similarly, there is a phase

\[
\varphi_{\text{spin}} = \varphi_A + \varphi_B ,
\]

resulting from the combined effect of the two swaps of the spin states

\[
|\sigma\rangle \longrightarrow \hat{V}_B \hat{V}_A |\sigma\rangle = \exp(i(\varphi_A + \varphi_B)) |\sigma\rangle \\
|\tau\rangle \longrightarrow \hat{V}_A \hat{V}_B |\tau\rangle = \exp(i(\varphi_A + \varphi_B)) |\tau\rangle .
\]
So, both phases $\varphi_{\text{spatial}}$ and $\varphi_{\text{spin}}$ pertain to single-particle trajectories around a closed loop in state space, the traditional setting for defining dynamical and geometric phases.\(^1\)

A nice example of a spatial swap operation applied to identical particles with different well-defined localized spatial states was recently considered in the case of an ion trap ring. The spatial swap operation in that case would be a spatial rotation. That is, two (identical) ions trapped in a ring around two angular positions that are 180 degrees apart would both be rotated, say clockwise, over an angle $\pi$. The phase factor accompanying two such pure rotations (a pure rotation being represented by a unitary operator of the form $\exp(iL_{n}\theta/\hbar)$ without any additional phase factors, where $L_{n}$ is the component of the orbital angular momentum vector along the rotation axis and $\theta$ is the angle of rotation) is always trivial (i.e., unity) and is, in particular, independent of whether the ions are bosons or fermions.

In contrast, the phase factor accompanying the swap of the spin states depends on what those spin states are. Moreover, it depends on how the two unitary operations are implemented. One way is to choose $\hat{V}_{A} = \hat{V}_{B}$ such that the total spin-dependent phase automatically vanishes, $\varphi_{\text{spin}} = 0$. In that case, the total phase acquired in the particle-swapping process is indeed equal to the exchange phase, and so that phase would indeed be directly observable in an interference experiment.

On the other hand, it is worthwhile to note that if we would rotate both spins in the same way, then a full rotation of the spin around a fixed axis (applying to the case where $|\sigma\rangle$ would be spin “up” along a certain direction and $|\tau\rangle$ would be spin “down” along that same direction) would produce a geometric phase equal to $2\pi$, thus equaling (and canceling) the exchange phase. This fact, that the geometric phase for a full spin rotation equals the exchange phase, was used in Ref.\(^6\) in an attempt to explain the spin-statistics theorem. Here, though, we see that the two phases are conceptually different—one is a two-particle phase, the other a single-particle phase—and they both contribute independently to an observable phase.

### C. Entangled spin states

So far we have considered an example where both the particle in region $A$ and the particle in region $B$ are in pure spin states. We may also consider the case where the particles’ spin states are entangled with each other. For example, consider the two maximally entangled states (that is, we now assume the spin states $|\sigma\rangle$ and $|\tau\rangle$ to be orthogonal)

$$|\Psi_{\pm}\rangle = \frac{1}{2} \left( |A\rangle_{1} |B\rangle_{2} \pm (-1)^{2s} |B\rangle_{1} |A\rangle_{2} \right) \otimes |\sigma\rangle_{1} |\tau\rangle_{2} \pm |\tau\rangle_{1} |\sigma\rangle_{2}.$$  \(9\)

Both “plus” and “minus” states are properly symmetrized. That is, if we simply swap the labels ‘1’ and ‘2’ we get a phase factor $(-1)^{2s}$. If instead we assume we apply the same unitary swap operations to each particle, $\hat{U}$ to the spatial part and $\hat{V} = \hat{V}_{A} = \hat{V}_{B}$ to the spin part, a swap of the particles thus yields

$$|\Psi_{\pm}\rangle \rightarrow \exp(i(2s\pi + \varphi_{\text{spatial}} + \varphi_{\text{spin}})) |\Psi_{\pm}\rangle,$$  \(10\)

which, again, contains more than just the exchange phase.

We could instead apply position-dependent spin transformations, $\hat{V}_{A} \neq \hat{V}_{B}$. In order to calculate the resulting transformation on the two-particle state we have to separate out the four terms appearing in $(9)$, and we have four spin-related phases to keep track of:

$$\hat{V}_{A} |\sigma\rangle = \exp(i\varphi_{A}) |\tau\rangle,$$
$$\hat{V}_{A} |\tau\rangle = \exp(i\varphi_{A}) |\sigma\rangle,$$
$$\hat{V}_{B} |\sigma\rangle = \exp(i\varphi_{B}) |\tau\rangle,$$
$$\hat{V}_{B} |\tau\rangle = \exp(i\varphi_{B}) |\sigma\rangle.$$  \(11\)

The state transformation resulting from a swap of the particles (where both spin and spatial states are interchanged) involves two different combinations of spin-phases now (in contrast to what we saw before in Eq. $(7)$),

$$\varphi_{\alpha} = \varphi_{A} + \varphi_{B},$$
$$\varphi_{\beta} = \varphi_{B} + \varphi_{A}.$$  \(12\)

The full state transformation is then

$$|\Psi_{\pm}\rangle \rightarrow \frac{1}{2} \left\{ \exp(i\varphi_{\alpha})(|B\rangle_{1} |\tau\rangle_{1} |A\rangle_{2} |\sigma\rangle_{2} + (-1)^{2s} |A\rangle_{1} |\sigma\rangle_{1} |B\rangle_{2} |\tau\rangle_{2}) \right.  \left. \pm \exp(i\varphi_{\beta})(|B\rangle_{1} |\sigma\rangle_{1} |A\rangle_{2} |\tau\rangle_{2} + (-1)^{2s} |A\rangle_{1} |\tau\rangle_{1} |B\rangle_{2} |\sigma\rangle_{2}) \right\}.$$  \(13\)

This resulting state is still properly symmetrized, but it is no longer equivalent to the original state, unless $\varphi_{\alpha} = \varphi_{\beta}$.\(^\dagger\)
This shows once again that swapping particles is not at all the same thing as swapping Hilbert space labels. (See also the companion paper [7].)

D. What about anyons?

A brilliant proposal for quantum computing, inherently robust against local errors, is based on anyons [8, 9]. Anyons are quasi particles, emerging in a 2D world (and even in certain 2D models embedded in our 3D world) where we can consistently distinguish clock-wise from counter-clockwise rotations. A simple model for anyons was proposed in [10]: a “particle” consisting of a magnetic flux plus a charged particle, has the property that rotating the composite particle around its own axis gives rise to a nontrivial phase, equal to $q\Phi/\hbar$ in terms of the magnetic flux $\Phi$ and the charge $q$, due to the Aharonov-Bohm effect. Similarly, two such composite “particles” circling around each other once both acquire half that phase, so that their joint state acquires the same phase $q\Phi/\hbar$. That is, just as we saw in Section 13 the total phase here is a sum of two single-anyon phases. It is a topological phase as it does not depend on how the anyons encircle each other (once), nor on how fast they do so.

In the ion-trap experiment described above [5], we may incorporate the Aharonov-Bohm effect by imagining a magnetic field going through the center of the ring, but such that the ions do not experience a magnetic field. The presence of such a field changes the action of a spatial swap operation from Eq. (3) to

$$
\hat{U} |\phi\rangle = \exp(i\varphi + iq\Phi/2\hbar) |\psi\rangle,
$$

$$
\hat{U} |\psi\rangle = \exp(i\varphi' + iq\Phi/2\hbar) |\phi\rangle,
$$

(14)

so that the spatial phase, which was trivial before, now becomes nontrivial,

$$
\varphi_{\text{spatial}} = \varphi + \varphi' + \frac{q\Phi}{\hbar},
$$

(15)

where $\varphi + \varphi'$ is still trivial (a multiple of $2\pi$), but the Aharonov-Bohm phase could have any value. The ions have, of course, not magically turned into anyons here: they still are bosons/fermions but do acquire a nontrivial (topological) phase when their states are swapped.

E. Conclusions

Actually (physically) exchanging identical particles does not so simply and automatically yield the standard bosonic or fermionic exchange phase factor. Additional single-particle geometric and dynamical phase factors are always present as well and the state may even change by more than just a phase factor when particles are physically exchanged. We concur with the conclusion that single-particle Hilbert space labels are different from physical particle labels [1, 2].

Both anyons and classical particles [2] emerge from a quantum description of identical fermions/bosons, and as such do not obey any new fundamental exchange symmetry. Classical particles don’t have any phase, while for anyons their emergent exchange phase is fundamentally a single-particle effect.

We considered just two identical particles here. Exchanging more than two particles does introduce one simple but notable novelty: the group of permutations of more than two elements is not abelian. Even better—and much less simply—the transformations that accompany exchanges of certain types of “non-abelian anyons” (in 2+1D spacetime, for which the braiding group [8] is the relevant group of transformations) are universal for quantum computing [8].

Thanks to Norman Yao, Hartmut Häffner and Tzula Propp for useful comments.

[1] D. Dieks, Synthese 82, 127 (1990).
[2] D. Dieks and A. Lubberdink, Found. Phys. 41, 1051 (2011).
[3] In the position representation, exchanging labels ‘1’ and ‘2’ for the single-particle Hilbert spaces corresponds to exchanging coordinates $x_1$ and $x_2$, which is not the same as exchanging particles. The difference is similar to that between passive and active coordinate transformations.
[4] M. V. Berry, Proc. R. Soc. Lond. A 439, 45 (1984).
[5] C. Roos, A. Alberti, D. Meschede, P. Hauke, and H. Häffner, Phys Rev Lett 119, 160401 (2017).
[6] M. V. Berry and J. M. Robbins, Proc. R. Soc. Lond. A 453, 1771 (1997).
[7] Companion paper: S.J. van Enk, Thermalizing two identical particles, arXiv:1810.05147.
[8] A. Y. Kitaev, Ann. Phys. 303, 2 (2003).
[9] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, Reviews of Modern Physics 80, 1033 (2008).
[10] F. Wilczek, Phys. Rev. Lett. 49, 957 (1982).