Goldie twisted partial skew power series rings

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Abstract

In this article, we work with unital twisted partial action of $\mathbb{Z}$ on an unital ring $R$ and we introduce the twisted partial skew power series rings and twisted partial skew Laurent series rings. We study primality, semi-primality and prime ideals in these rings. We completely describe the prime radical in partial skew Laurent series rings. Moreover, we study the Goldie property in partial skew power series rings and partial skew Laurent series rings and we describe conditions of the semiprimality of twisted partial skew power series rings.

Introduction

Partial actions of groups have been introduced in the theory of operator algebras as a general approach to study $C^*$-algebras by partial isometries (see, in particular, \cite{12} and \cite{13}), and crossed products classically, as well-pointed out in \cite{10}, are the center of the rich interplay between dynamical systems and operator algebras (see, for instance, \cite{18} and \cite{20}). The general notion of (continuous) twisted partial action of a locally compact group on a $C^*$-algebra and the corresponding crossed product were introduced in \cite{12}. Algebraic counterparts for some notions mentioned above were introduced

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and studied in [9], stimulating further investigations, see for instance, [1], [2], [14] and references therein. In particular, twisted partial actions of groups on abstract rings and corresponding crossed products were recently introduced in [10].

In [4], it was introduced the partial skew polynomial rings and partial skew Laurent of polynomials, and the authors studied prime and maximal ideals. In [5], it was investigated the Goldie property in partial skew polynomial rings and partial skew Laurent of polynomial. In [6], it was introduced the concept of partial skew power series rings and in the authors studied when it is Bezout and distributive.

The authors in [16] and [17], studied the Goldie rank and prime ideals in skew power series ring and skew Laurent series rings with the assumption of noetherianity on the base ring. In this article, we consider twisted partial actions of \( \mathbb{Z} \) and we introduce the twisted partial skew power series rings and twisted partial skew Laurent series rings \( R[[x; \alpha, w]] \) and \( R\langle x; \alpha, w \rangle \), respectively, where \( \alpha \) is a twisted partial action of \( \mathbb{Z} \) on an unital ring \( R \). We study the Goldie property, prime ideals, primality and semiprimality in these rings which generalizes the results presented in [16] and [17].

This article is organized as follows:

In the Section 1, we give some preliminaries and results that will be used during this paper.

In the Section 2, we study the primality and semiprimality of twisted partial skew power series rings and twisted partial skew Laurent series rings. We describe the prime radical of twisted partial skew Laurent series rings and we study the prime ideals of these rings.

In the Section 3, we study the Goldie rank of the twisted partial power series rings and twisted partial skew Laurent series rings and as a consequence we study the Goldie property of these rings. Moreover, we study when the twisted partial skew power series rings is semiprime and we give a description of the prime radical of twisted partial skew power series rings, when the unital twisted partial action of \( \mathbb{Z} \) has enveloping action.

1 Preliminaries

In this section, we recall some notions about twisted partial actions on rings, more details can be found in [9], [10] and [11]. We introduce, in this section, the twisted partial skew power series rings and twisted partial skew Laurent series rings.

From now on, \( R \) will be always an unital ring, unless otherwise stated.
We begin with the following definition that is a particular case of ([11], Definition 2.1).

**Definition 1.1.** An unital twisted partial action of the additive abelian group \( \mathbb{Z} \) on a ring \( R \) is a triple

\[
\alpha = (\{D_i\}_{i \in \mathbb{Z}}, \{\alpha_i\}_{i \in \mathbb{Z}}, \{w_{i,j}\}_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}),
\]

where for each \( i \in \mathbb{Z} \), \( D_i \) is a two-sided ideal in \( R \) generated by a central idempotent \( 1_i \), \( \alpha_g : D_{-i} \to D_i \) is an isomorphism of rings and for each \( (i, j) \in \mathbb{Z} \times \mathbb{Z} \), \( w_{i,j} \) is an invertible element of \( D_i D_{i+j} \), satisfying the following postulates, for all \( i, j, k \in \mathbb{Z} \):

1. \( D_1 = R \) and \( \alpha_1 \) is the identity map of \( R \);
2. \( \alpha_i(D_{-i}D_j) = D_iD_{i+j} \);
3. \( \alpha_i \circ \alpha_j(a) = w_{i,j} \alpha_{i+j}(a) w_{i,j}^{-1} \), for all \( a \in D_{-j}D_{-ji} \);
4. \( w_{i,1} = w_{1,i} = 1 \);
5. \( \alpha_i(a w_{j,k}) w_{i,j+k} = \alpha_i(a) w_{i,j} w_{i+k,j} \), for all \( a \in D_{-i}D_jD_{j+k} \).

**Remark 1.2.** If \( w_{i,j} = 1, 1_{i+j}, \) for all \( i, j \in \mathbb{Z} \), then we have a partial action which is a particular case of ([11], Definition 1.1) and when \( D_1 = R \), for all \( i \in \mathbb{Z} \), we have that \( \alpha \) is a twisted global action.

Let \( \beta = (T, \{\beta_i\}_{i \in \mathbb{Z}}, \{u_{i,j}\}_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}) \) be a twisted global action of a group \( \mathbb{Z} \) on a (non-necessarily unital) ring \( T \) and \( R \) an ideal of \( T \) generated by a central idempotent \( 1_R \). We can restrict \( \beta \) to \( R \) as follows: putting \( D_i = R \cap \beta_i(R) = R \beta_i(R), \) \( i \in \mathbb{Z} \), each \( D_i \) has an identity element \( 1_R \beta_i(1_R) \).

Then defining \( \alpha_i = \beta_i|_{D_{-i}}, \forall i \in \mathbb{Z} \), the items (i), (ii) and (iii) of Definition [11] are satisfied. Furthermore, defining \( w_{i,j} = u_{i,j} 1_R \beta_i(1_R) \beta_{i+j}(1_R), \)
\( \forall i, j \in \mathbb{Z} \), the items (iv), (v) e (vi) of Definition [11] are also satisfied. So, we obtain a twisted partial action of \( \mathbb{Z} \) on \( R \).

The following definition appears in ([11], Definition 2.2).

**Definition 1.3.** A twisted global action \( (T, \{\beta_i\}_{i \in \mathbb{Z}}, \{u_{i,j}\}_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}) \) of a group \( \mathbb{Z} \) on an associative (non-necessarily unital) ring \( T \) is said to be an enveloping action (or a globalization) of a unital twisted partial action \( \alpha \) of \( \mathbb{Z} \) on a ring \( R \) if, there exists a monomorphism \( \varphi : R \to T \) such that, for all \( i \) and \( j \) in \( \mathbb{Z} \):

1. \( \varphi(R) \) is an ideal of \( T \);
2. \( T = \sum_{i \in \mathbb{Z}} \beta_i(\varphi(R)) \);
3. \( \varphi(D_i) = \varphi(R) \cap \beta_i(\varphi(R)) \);
(iv) \( \varphi \circ \alpha_i(a) = \beta_i \circ \varphi(a) \), for all \( a \in D_{-i} \);
(v) \( \varphi(aw_{i,j}) = \varphi(a)u_{i,j} \) and \( \varphi(w_{i,j}a) = u_{i,j} \varphi(a) \), for all \( a \in D_iD_{i+j} \).

In (11, Theorem 4.1), the authors studied necessary and sufficient conditions for an unital twisted partial action \( \alpha \) of a group \( \mathbb{Z} \) on a ring \( R \) has an enveloping action. Moreover, they studied which rings satisfy such conditions.

Suppose that \((R, \alpha, w)\) has an enveloping action \((T, \beta, u)\). In this case, we may assume that \( R \) is an ideal of \( T \) and we can rewrite the conditions of the Definition 1.3 as follows:
(i') \( R \) is an ideal of \( T \);
(ii') \( T = \sum_{i \in \mathbb{Z}} \beta_i(R) \);
(iii') \( D_i = R \cap \beta_i(R) \), for all \( i \in \mathbb{Z} \);
(iv') \( \alpha_i(a) = \beta_i(a) \), for all \( x \in D_{-i} \) and \( i \in \mathbb{Z} \);
(v') \( aw_{i,j} = au_{i,j} \) and \( w_{i,j}a = u_{i,j}a \), for all \( a \in D_iD_{i+j} \) and \( i,j \in \mathbb{Z} \).

Given an unital twisted partial action \( \alpha \) of \( \mathbb{Z} \) on a ring \( R \), we define the twisted partial skew Laurent series rings \( R\langle x; \alpha, w \rangle = \bigoplus_{i \in \mathbb{Z}} D_i x^i \) whose elements are the series
\[
\sum_{j \geq s} a_j x^j, \text{ with } a_j \in D_j
\]
with the usual addition and multiplication defined by
\[
(a_i x^i)(a_j x^j) = \alpha_i(\alpha_i^{-1}(a_i)b_j)w_{i,j}x^{i+j}.
\]

Using the similar techniques of (10, Theorem 2.4), \( R\langle x; \alpha, w \rangle \) is an associative ring whose identity is \( 1_{R,x^0} \). Note that, we have the injective morphism \( \phi : R \to R\langle x; \alpha, w \rangle \), defined by \( r \mapsto rx^0 \) and we can consider \( R\langle x; \alpha, w \rangle \) as an extension of \( R \). Moreover, we consider the twisted partial power series rings as a subring of \( R\langle x; \alpha, w \rangle \) which we denote it by \( R[[x; \alpha, w]] \) whose elements are the series \( \sum_{i \geq 0} b_i x^i \) with sum and multiplication rule defined as before.

Let \( \alpha \) be an unital twisted partial action of a group \( \mathbb{Z} \) on a ring \( R \). An ideal \( S \) of \( R \) is said to be \( \alpha \)-ideal (\( \alpha \)-invariant ideal) if, \( \alpha_i(S \cap D_{-i}) \subseteq S \cap D_i \), for all \( i \geq 0 \) \( (\alpha_i(S \cap D_{-i}) = S \cap D_i \), for all \( i \in \mathbb{Z} \)).

If \( S \) is an \( \alpha \)-ideal (\( \alpha \)-invariant ideal), then we have the ideals
\[
S[[x; \alpha, w]] = \left\{ \sum_{i \geq 0} a_i x^i \ | \ a_i \in S \cap D_i \right\}
\]
\[
(S(x; \alpha, w) = \left\{ \sum_{i \geq m} a_i x^i \ | \ a_i \in S \cap D_i \ m \in \mathbb{Z} \right\}
\]
is an ideal of \(R[[x; \alpha, w]](R(x; \alpha, w))\). Note that, if \(I\) is a right ideal of \(R\), then \(I[[x; \alpha, w]] = \{\sum_{i \geq 0} a_i x^i : a_i \in D_i\}\) and \(I(x; \alpha, w) = \{\sum_{i \geq m} b_i x^i : b_i \in D_i\}\) are right ideals of \(R[[x; \alpha, w]]\) and \(R(x; \alpha, w)\), respectively.

Note that for each \(\alpha\)-invariant ideal \(I\) of \(R\), the unital twisted partial action \(\alpha\) can be extended to an unital twisted partial action \(\overline{\alpha}\) of \(\mathbb{Z}\) on \(R/I\) as follows: for each \(i \in \mathbb{Z}\), we define \(\overline{\alpha}_i : D_{-i} + I \rightarrow D_i + I\), putting \(\overline{\alpha}_i(a + I) = \alpha_i(a) + I\), for all \(a \in D_{-i}\), and for each \((i, j) \in \mathbb{Z} \times \mathbb{Z}\), we extend each \(w_{i, j}\) to \(R/I\) by \(\overline{w}_{i, j} = w_{i, j} + I\).

Moreover, when \((R, \alpha, w)\) has enveloping action \((T, \beta, u)\), then by similar methods presented in Section 2 of [14], we have that \((T/\mathcal{I}^e, \overline{\beta}, \overline{w})\) is the enveloping action of \((R/\mathcal{I}, \overline{\alpha}, \overline{w})\), where \(\mathcal{I}^e\) is the \(\beta\)-invariant ideal such that \(\mathcal{I}^e \cap R = \mathcal{I}\).

We finish this section with some comments about twisted partial actions of finite type that will be necessary in this paper.

The following definition is a particular case of ([3], Definition 4.13).

**Definition 1.4.** Let \(\alpha\) be an unital twisted partial action. We say that \(\alpha\) is of finite type if, there exists a finite subset \(\{s_1, s_2, \cdots, s_n\}\) of \(\mathbb{Z}\) such that

\[
\sum_{i=1}^{n} D_{j+s_i} = R,
\]

for all \(j \in \mathbb{Z}\).

It is convenient to point out that in the same way as in ([14], Proposition 1.2) as proved in [3], we have that an unital twisted partial action \(\alpha\) of \(\mathbb{Z}\) on an unital ring \(R\) with an enveloping action \((T, \beta, u)\) is of finite type if, and only if, there exists \(s_1, \cdots, s_n \in \mathbb{Z}\) such that \(T = \sum_{i=1}^{n} \beta_{s_i}(R)\) and this is equivalent to say that \(T\) has an identity element.

## 2 Primality and semiprimality

In this section, \(\alpha\) will denote an unital twisted partial action of \(\mathbb{Z}\) on an unital ring \(R\), unless otherwise stated. We begin this section with the following proposition, whose proof is standard, and we put it here for the sake of completeness.

**Proposition 2.1.** If \(I\) is an \(\alpha\)-invariant ideal of \(R\), then \(R[[x; \alpha, w]]/I[[x; \alpha, w]] \simeq (R/I)[[x; \overline{\alpha}, \overline{w}]]\). Moreover, the same result holds to \(R(x; \alpha, w)\).
Proof:
We define \( \varphi : \frac{R[[x;\alpha,w]]}{I[[x;\alpha,w]]} \to (\frac{R}{I})[[x;\alpha,w]] \) by \( \varphi(\sum_{i \geq 0} a_i x^i + I[[x;\alpha,w]]) = \sum_{i \geq 0} (a_i + I) x^i. \) We easily have that \( \varphi \) is an isomorphism. So, \( \frac{R[[x;\alpha,w]]}{I[[x;\alpha,w]]} \cong (\frac{R}{I})[[x;\alpha,w]] \)

The following definition firstly appeared in [4] for ordinary partial actions

**Definition 2.2.** Let \( \alpha \) be an unital twisted partial action of \( \mathbb{Z} \) on \( R \) and \( I \) an ideal of \( R \).

(i) \( I \) is \( \alpha \)-prime if, \( I \) is an \( \alpha \)-invariant ideal and for each \( J \) and \( K \) \( \alpha \)-invariant ideals of \( R \) such that \( JK \subseteq I \) implies that either \( J \subseteq I \) or \( K \subseteq I \).

(ii) \( I \) is strongly \( \alpha \)-prime if, \( I \) is \( \alpha \)-invariant and for each ideal \( M \) of \( R \) and \( \alpha \)-ideal \( N \) of \( R \) such that \( MN \subseteq I \) implies that either \( M \subseteq I \) or \( N \subseteq I \).

Let \( a \in R \). Then we define the \( \alpha \)-invariant ideal generated by \( a \) as \( J = \sum_{i \in \mathbb{Z}} R\alpha_i(a_{1-i})R. \)

In the next result, we study necessary and sufficient conditions for \( \alpha \)-primality and strongly \( \alpha \)-primality.

**Lemma 2.3.** (1) Let \( P \) be an \( \alpha \)-invariant ideal of \( R \). The following conditions are equivalent:

(a) \( P \) is \( \alpha \)-prime

(b) For each \( a,b \in R \) such that \( \alpha_j(a_{1-j})R\alpha_i(b_{1-i}) \subseteq P \), for all \( i,j \in \mathbb{Z} \), then either \( a \in P \) or \( b \in P \)

(c) \( R/P \) is \( \overline{\alpha} \)-prime, where \( \overline{\alpha} \) is the extension of twisted partial action \( \alpha \) to \( R/P \)

(2) Let \( P \) be an \( \alpha \)-invariant ideal of \( R \). The following conditions are equivalent:

(a) \( P \) is strongly \( \alpha \)-prime

(b) For each \( a,b \in R \) such that \( aR\alpha_j(b_{1-j}) \subseteq P \), for all \( j \geq 0 \), then either \( a \in P \) or \( b \in P \)

(c) \( R/P \) is strongly \( \overline{\alpha} \)-prime, where \( \overline{\alpha} \) is the extension of twisted partial action \( \alpha \) to \( R/P \).

**Proof:**
(1) \( (a) \implies (b) \)

Let \( a,b \in R \) such that \( \alpha_j(a_{1-j})R\alpha_i(b_{1-i}) \subseteq P \), for all \( i,j \in \mathbb{Z} \). Then, if we fix \( j \) we have that

\[ \alpha_j(a_{1-j}) \sum_{i \in \mathbb{Z}} R\alpha_i(b_{1-i}) R \subseteq P. \]
and consequently, we get

\[ \sum_{j \in \mathbb{Z}} R\alpha_j(a_{1-j})R \sum_{i \in \mathbb{Z}} R\alpha_i(b_{1-i})R \subseteq P. \]

Since the ideals \( \sum_{i \in \mathbb{Z}} R\alpha_i(a_{1-i})R \) and \( \sum_{i \in \mathbb{Z}} R\alpha_i(b_{1-i})R \) are \( \alpha \)-invariant, then, by assumption, we have that either \( \sum_{i \in \mathbb{Z}} R\alpha_i(a_{1-i})R \subseteq P \) or \( \sum_{i \in \mathbb{Z}} R\alpha_i(b_{1-i})R \subseteq P \). So, either \( a \in P \) or \( b \in P \).

(b) \( \Rightarrow \) (a)

Let \( I, J \) be \( \alpha \)-invariant ideals of \( R \) such that \( IJ \subseteq P \), take \( a \in I \) and suppose that there exists \( b \in J \setminus P \). Then, \( \left( \sum_{i \in \mathbb{Z}} R\alpha_i(a_{1-i})R \right) \left( \sum_{j \in \mathbb{Z}} R\alpha_j(b_{1-j})R \right) \subseteq P \). Thus, by assumption, we have that either \( \sum_{i \in \mathbb{Z}} R\alpha_i(a_{1-i})R \subseteq P \) or \( \sum_{j \in \mathbb{Z}} R\alpha_j(b_{1-j})R \subseteq P \).

Hence, \( a \in P \), because \( b \notin P \). So, \( I \subseteq P \).

(a) \( \Rightarrow \) (c)

Let \( a, b \in R \) such that

\[ \pi_j((a + P)(1_{-j} + P))(R/P)\pi_i((b + P)(1_{-i} + P)) = 0, \]

for all \( i, j \in \mathbb{Z} \). Then, \( \alpha_j(a_{1-j})R\alpha_i(b_{1-i}) \subseteq P \), for all \( i, j \in \mathbb{Z} \). Thus, by assumption, we have that either \( a \in P \) or \( b \in P \). So, either \( a + P = 0 \) or \( b + P = 0 \).

(c) \( \Rightarrow \) (a)

Let \( I \) and \( J \) be \( \alpha \)-invariant ideals of \( R \) such that \( IJ \subseteq P \). Thus, \( I\overline{J} = \overline{I} \) in \( R/P \). Hence, by assumption, we have that either \( I = \overline{0} \) or \( J = \overline{0} \). So, either \( I \subseteq P \) or \( J \subseteq P \).

The proof of item (ii) is analogous. \( \square \)

It is convenient to point out that \( R \) is \( \alpha \)-prime (strongly \( \alpha \)-prime) if the zero ideal is \( \alpha \)-prime (strongly \( \alpha \)-prime). Next, we have an easy consequence of Lemma 2.3.

**Lemma 2.4.** (i) Let \( \alpha \) be an unital twisted partial action of \( \mathbb{Z} \) on \( R \). Then \( R \) is \( \alpha \)-prime if, and only if, for each \( a, b \in R \) such that \( \alpha_j(a_{1-j})R\alpha_i(b_{1-i}) = 0 \) for all \( i, j \in \mathbb{Z} \), we have that either \( a = 0 \) or \( b = 0 \).

(ii) Let \( \alpha \) be an unital twisted partial action of \( \mathbb{Z} \) on \( R \). Then, \( R \) is strongly \( \alpha \)-prime if, and only if, for each \( a, b \in R \) such that \( aR\alpha_i(b_{1-i}) = 0 \) for all \( i \geq 0 \), we have that either \( a = 0 \) or \( b = 0 \).
It is convenient to point out that if, \( L \) is a nonzero right ideal of \( R[x; \alpha, w] \), then \( L \cap R[[x; \alpha, w]] \) is a nonzero right ideal of \( R[[x; \alpha, w]] \) because of for each nonzero element \( f \in L \), there exists \( s \geq 0 \) such that \( 0 \neq f1_s x^s \in L \cap R[[x; \alpha, w]] \). Moreover, if a right ideal \( M \) of \( R[x; \alpha, w] \) is such that \( M \cap R[[x; \alpha, w]] = 0 \), then we have that \( M = 0 \). We use these facts without further mention.

In the next result, we study conditions for the primality of \( R[[x; \alpha, w]] \) and \( R[x; \alpha, w] \) which partially generalizes ([17], Propositions 2.5 and 2.7).

**Proposition 2.5.** The following statements hold.
(a) \( R \) is \( \alpha \)-prime if and only if \( R[x; \alpha, w] \) is prime.
(b) \( R[[x; \alpha, w]] \) is prime if and only if \( R \) is strongly \( \alpha \)-prime. In particular, if \( R[[x; \alpha, w]] \) is prime, then \( R \) is \( \alpha \)-prime.
(c) If \( R[[x; \alpha, w]] \) is prime, then \( R[x; \alpha, w] \) is prime.

**Proof:**
(a) Suppose that \( R[x; \alpha, w] \) is prime and let \( I \) and \( J \) be \( \alpha \)-invariant ideals of \( R \) such that \( IJ = 0 \). Then
\[
I(x; \alpha, w)J(x; \alpha, w) \subseteq (IJ)(x; \alpha, w) = 0.
\]
By the fact that \( R(x; \alpha, w) \) is prime, we have that either \( I(x; \alpha, w) = 0 \) or \( J(x; \alpha, w) = 0 \). Hence, either \( I = 0 \) or \( J = 0 \). So, \( R \) is \( \alpha \)-prime.

Conversely, let \( f, g \in R[x; \alpha, w] \) be nonzero elements, suppose that \( fR(x; \alpha, w)g = 0 \) and consider \( m \) and \( n \) the smallest integers such that \( f_m \neq 0 \) and \( g_n \neq 0 \) where \( f = \sum_{i \geq m} f_i x^i \) and \( g = \sum_{i \geq n} g_i x^i \). Note that, for each \( i \in \mathbb{Z} \), \( fD_i x^i g \subseteq fR(x; \alpha, w)g = 0 \) and we have that \( f_m D_i \alpha_i (1_{-i} g_n) = 0 \), for all \( i \in \mathbb{Z} \). Hence, for each \( j \in \mathbb{Z} \), we have that \( \alpha_j (f_m 1_{-j}) R \alpha_i (g_n 1_{-i}) = 0 \), for all \( i \in \mathbb{Z} \).

Consequently, by Lemma (2.5), we have that \( f_m = 0 \) or \( g_n = 0 \), which is a contradiction. So, \( R[x; \alpha, w] \) is prime.

(b) The proof is similar of the item (a).

(c) Let \( I \) and \( J \) be ideals of \( R[x; \alpha, w] \) such that \( IJ = 0 \). Thus,
\[
(I \cap R[[x; \alpha, w]])(J \cap R[[x; \alpha, w]]) = 0.
\]
Since \( I \cap R[[x; \alpha, w]] \) and \( J \cap R[[x; \alpha, w]] \) are ideals of \( R[[x; \alpha, w]] \), then we have that either \( I \cap R[[x; \alpha, w]] = 0 \) or \( J \cap R[[x; \alpha, w]] = 0 \). Hence either \( I = 0 \) or \( J = 0 \). So, \( R[x; \alpha, w] \) is prime.
Remark 2.6. The authors in ([17], Propositions 2.5 and 2.7) used noetherianity to get the equivalences mentioned there. For us to obtain the same equivalences in the case of twisted partial actions, we would need to know if the following question has a positive answer, but until now we do not know.

Are all \( \alpha \)-ideals of \( R \) \( \alpha \)-invariant ideals when \( R \) is Noetherian?

So, if this question has a positive answer we would have that \( R \) is \( \alpha \)-prime \( \iff R[[x;\alpha, w]] \) is prime \( \iff R\langle x;\alpha, w \rangle \) is prime.

The following result is a direct consequence of the last proposition.

**Corollary 2.7.** If \( R \) is a prime ring, then \( R[[x;\alpha, w]] \) is a prime ring.

The proof of the following result is similar to Proposition 2.5 and it partially generalizes ([17], Corollary 2.12)

**Corollary 2.8.** The following statements hold.

(a) Suppose that \( I \) is \( \alpha \)-invariant ideal of \( R \). Then \( I \) is \( \alpha \)-prime if and only if \( I\langle x;\alpha, w \rangle \) is prime.

(b) Suppose that \( I \) is an \( \alpha \)-invariant ideal of \( R \). Then \( I \) is strongly \( \alpha \)-prime if and only if \( I[[x;\alpha, w]] \) is prime.

The following result generalizes ([16], Theorem 3.18) and is a direct consequence of the last corollary.

**Corollary 2.9.** Let \( \alpha \) be an unital twisted partial action of \( \mathbb{Z} \) on \( R \) and \( I \) a strongly \( \alpha \)-prime ideal of \( R \). Then, there exists a prime ideal \( P \) of \( R[[x;\alpha, w]] \) such that \( P \cap R = I \). Moreover, if \( I \) is \( \alpha \)-prime, then there exists a prime ideal \( Q \) of \( R\langle x;\alpha, w \rangle \) such that \( Q \cap R = I \).

In ([17], Proposition 2.11) is used the noetherianity property to prove the result in the case of skew Laurent series rings, but in that proof the assumption was not necessary. The next result generalizes ([17], Proposition 2.11).

**Proposition 2.10.** If \( K \) is a prime ideal of \( R\langle x;\alpha, w \rangle \), then \( K \cap R \) is an \( \alpha \)-prime ideal of \( R \).
Proof: Let $K$ be an prime ideal of $R\langle x; \alpha, w \rangle$. Then, we easily have that $K \cap R$ is an ideal of $R$. We claim that $K \cap R$ is an $\alpha$-prime ideal of $R$. In fact, let $a \in (K \cap R) \cap D_{-i}$, for $i \in \mathbb{Z}$. Then, $1_i x_i a x_i^{-1} \in K$. Thus,

$$1_i x_i a x_i^{-1} = 1_i \alpha_i(a) w_{i,-i} = \alpha_i(a) w_{i,-i} \in K \cap D_i$$

and since $w_{i,-i}$ is an invertible element of $D_i$, we get that $\alpha_i(a) w_{i,-i} w_{i,-i}^{-1} \in (K \cap R) \cap D_i$ and it follows that $\alpha_i((K \cap R) \cap D_{-i}) \subseteq (K \cap R) \cap D_i$. By similar methods, we show that $\alpha_i^{-1}((K \cap R) \cap D_i) \subseteq (K \cap R) \cap D_{-i}$. Consequently, $\alpha_i((K \cap R) \cap D_{-i}) = (K \cap R) \cap D_i$, for all $i \in \mathbb{Z}$ and we have that $K \cap R$ is an $\alpha$-invariant ideal of $R$.

By Proposition 2.1 we have that

$$\Psi : (R/(K \cap R))\langle x; \alpha, w \rangle \to (R\langle x; \alpha, w \rangle)/(K \cap R)\langle x; \alpha, w \rangle$$

defined by $\Psi(\sum_{i \geq s} a_i x_i) = \sum_{i \geq s} a_i x_i + (K \cap R)\langle x; \alpha, w \rangle$ is an isomorphism. Note that $K'((K \cap R)\langle x; \alpha, w \rangle)$ is a prime ideal and we have that $\Psi^{-1}(K'((K \cap R)\langle x; \alpha, w \rangle)) = K = \{ \sum_{i \geq s} (a_i + (K \cap R)) x_i : \sum_{i \geq s} a_i x_i \in K \}$ is a prime ideal in $(R/(K \cap R))\langle x; \alpha, w \rangle$ and $K \cap (R/(K \cap R)) = 0$. Thus, we may assume that $K \cap R = 0$ and in this case we only need to show that $R$ is $\alpha$-prime. In fact, let $I$ and $J$ be $\alpha$-invariant ideals of $R$ such that $IJ = 0$. Hence, $IR\langle x; \alpha, w \rangle J\langle x; \alpha, w \rangle \subseteq I\langle x; \alpha, w \rangle J\langle x; \alpha, w \rangle \subseteq (IJ)\langle x; \alpha, w \rangle = 0 \subseteq K$. By the fact that $K$ is a prime ideal we have that either $IR\langle x; \alpha, w \rangle \subseteq K$ or $J\langle x; \alpha, w \rangle \subseteq K$ and it follows that either $I \subseteq K$ or $J \subseteq K$. So, either $I = 0$ or $J = 0$ and we have that $R$ is $\alpha$-prime.

The following notion appears in [7].

**Definition 2.11.** Let $\alpha$ be an unital twisted partial action of $\mathbb{Z}$ on $R$. Then the $\alpha$-nil radical $N_{\alpha}(R)$ of $R$ is the intersection of all $\alpha$-prime ideals of $R$.

From now on, for a ring $S$ we denote its prime radical by $\text{Nil}_*(S)$.

Now, we are in conditions to describe the prime radical of $R\langle x; \alpha, w \rangle$.

**Proposition 2.12.** Let $\alpha$ be an unital twisted partial action of $\mathbb{Z}$ on $R$. Then $\text{Nil}_*(R\langle x; \alpha, w \rangle) = N_{\alpha}(R)\langle x; \alpha, w \rangle$.

**Proof:** Let $P$ be a prime ideal of $R\langle x; \alpha, w \rangle$. Then, by Proposition 2.10 we have that $P \cap R$ is $\alpha$-prime. Thus, $\text{Nil}_*(R\langle x; \alpha, w \rangle) \supseteq N_{\alpha}(R)\langle x; \alpha, w \rangle$.

On the other hand, let $I$ be an $\alpha$-prime ideal of $R$. Then, by Corollary 2.8 we have that $I\langle x; \alpha, w \rangle$ is prime. Hence, $N_{\alpha}(R)\langle x; \alpha, w \rangle \supseteq \text{Nil}_*(R\langle x; \alpha, w \rangle)$. So, $\text{Nil}_*(R\langle x; \alpha, w \rangle) = N_{\alpha}(R)\langle x; \alpha, w \rangle$. \(\square\)
Proposition 2.13. Let $\alpha$ be an unital twisted partial action of $\mathbb{Z}$ on $R$.

(i) If $R$ is semiprime, then $R\langle x; \alpha, w \rangle$ is semiprime. Moreover, if $R$ is Noetherian and $R\langle x; \alpha, w \rangle$ is semiprime, then $R$ is semiprime.

(ii) Let $I$ be an $\alpha$-invariant ideal of $R$. If $I$ is semiprime, then $I\langle x; \alpha, w \rangle$ is semiprime.

Proof: (i) Assume, by the way of contradiction, that there exists $f = \sum_{i \geq s} f_i x^i$ such that $f R\langle x; \alpha, w \rangle f = 0$, where $f_s \neq 0$. Take any $c \in D_s$ and write $b = \alpha_s^{-1}(c)$, for some $b \in D_{-s}$. Thus, $f b x^{-s} f = 0$ and we have that $f_s \alpha_s^{-1}(b) w_{-s-s} f_s = 0$. Hence, $f_s c w_{-s-s} f_s = 0$ and we get that $f_s D_s f_s = 0$. Since $R$ is a semiprime ring, then $D_s$ is also a semiprime ring. Consequently, $f_s = 0$ because $f_s \in D_s$, a contradiction. So, $R\langle x; \alpha, w \rangle$ is semiprime.

For the second part, since $R$ is Noetherian, then by ([15], Theorem 4.10.30) the prime radical $\text{Nil}_*(R)$ is nilpotent. As a consequence, there exists $n \geq 1$ such that for every $\alpha$-prime ideal $P$ of $R$ we have that $\text{Nil}_*(R)^n \subseteq P$ and it follows that $\text{Nil}_*(R) \subseteq P$, for every $\alpha$-prime ideal of $R$, because of ([14], Remark 3.2) says that $\text{Nil}_*(R)$ is an $\alpha$-invariant ideal of $R$. Hence, we get that $\text{Nil}_*(R) \subseteq \text{Nil}_\alpha(R)$. By assumption and Proposition 2.12 we have that $\text{Nil}_\alpha(R) = 0$ and consequently, $\text{Nil}_*(R) = 0$. So, $R$ is semiprime.

(ii) The proof is similar of the item (i).

\(\square\)

Fron now on, we proceed to give a more close description of the prime ideals of $R[[x; \alpha, w]]$ and $R\langle x; \alpha, w \rangle$. The proof of the next result is similar to ([4], Proposition 2.6).

Proposition 2.14. Let $P$ be a prime ideal of $R[[x; \alpha, w]]$ (resp. $R\langle x; \alpha, w \rangle$). Then we have one of the following possibilities:

(i) $P = Q \oplus \sum_{i \geq 1} D_i x^i$, where $Q$ is a prime ideal of $R$

(resp. $P = Q \oplus \sum_{i \neq 0} D_i x^i$, where $Q$ is a prime ideal of $R$ with $D_j \subseteq Q$, for any $j \neq 0$).

(ii) $1_i x^i \notin P$, for some $i \geq 1$.

It is clear that for any prime ideal $Q$ of $R$, the ideal $Q \oplus \sum_{i \geq 1} D_i x^i$ is a prime ideal of $R[[x; \alpha, w]]$. Thus, we are in the case (i) of Proposition 2.14. If, in addition, $D_j \subseteq Q$, for all $j \neq 0$, it is easy to see that $P = Q \oplus \sum_{i \neq 0} D_i x^i$ is an ideal of $R\langle x; \alpha, w \rangle$ which is obviously prime.
From now on, we proceed to study the case of the item (ii) of the last proposition and we have the following two results.

**Proposition 2.15.** Let $P$ be an ideal of $R\langle x; \alpha, w \rangle$. If $P \cap R$ is $\alpha$-prime and either $P = (P \cap R)\langle x; \alpha, w \rangle$ or $P$ is maximal amongst the ideals $N$ of $R\langle x; \alpha, w \rangle$ with $N \cap R = P \cap R$, then $P$ is prime.

**Proof:** If $P = (P \cap R)\langle x; \alpha, w \rangle$, then the result follows from Corollary 2.8. Now, suppose that $P \neq (P \cap R)\langle x; \alpha, w \rangle$ and let $I, J$ be ideals of $R\langle x; \alpha, w \rangle$ such that $IJ \subseteq P$. Suppose that $I \nsubseteq P$ and $J \nsubseteq P$ and we get that $P \nsubseteq I + P$ and $P \nsubseteq J + P$. Note that $((I + P) \cap R)((J + P) \cap R) \subseteq P \cap R$ because of $(I + P)(J + P) \subseteq P$. By assumption, we have that either $((I+P) \cap R) \subseteq P \cap R$ or $((J+P) \cap R) \subseteq P \cap R$. Thus, either $(I+P) \cap R = P \cap R$ or $(J+P) \cap R = P \cap R$, which contradicts the assumption on $P$. Hence, either $I \subseteq P$ or $J \subseteq P$. So, $P$ is prime. \(\square\)

The proof of the following result is similar to the proof of the last proposition.

**Proposition 2.16.** Let $P$ be an ideal of $R[[x; \alpha, w]]$ such that $1, x^i \notin P$, for some $i \geq 1$ and $P \cap R$ is an $\alpha$-invariant ideal. If $P \cap R$ is prime and either $P = (P \cap R)[[x; \alpha, w]]$ or $P$ is maximal amongst the ideals $N$ of $R[[x; \alpha, w]]$ with $N \cap R = P \cap R$, then $P$ is prime.

We finish this section with the following remark.

**Remark 2.17.** Until now, we do not know if it is true or not the following natural converse of the last two propositions:

(i) If $P$ is a prime ideal of $R\langle x; \alpha, w \rangle$ and $P \neq (P \cap R)\langle x; \alpha, w \rangle$, then $P$ is maximal amongst the ideals $N$ of $R\langle x; \alpha, w \rangle$ with $N \cap R = P \cap R$.

(ii) Let $P$ be an ideal of $R[[x; \alpha, w]]$ such that $1, x^i \notin P$, for some $i \geq 1$ and $P \cap R$ is a strongly $\alpha$-prime ideal of $R$. If $P$ is a prime ideal of $R[[x; \alpha, w]]$ and $P \neq (P \cap R)[[x; \alpha, w]]$, then $P$ is maximal amongst the ideals $N$ of $R[[x; \alpha, w]]$ with $N \cap R = P \cap R$.

3 Goldie twisted partial skew power series rings

In this section, $\alpha$ is an unital twisted partial action of $\mathbb{Z}$ on $R$, unless otherwise stated.
Let $S$ be a ring and $M$ a right $S$-module. We remind that $M$ is uniform if, the intersection of any two nonzero submodules is nonzero, see ([19], pg. 52) for more details. According to ([19], pg. 57) a ring $S$ is right Goldie if satisfies ACC on right annihilator ideals and $S$ does not have an infinite direct sum of right uniform ideals. In this section, we study the Goldie property in twisted partial skew Laurent series rings and twisted partial skew power series rings. We begin with the following lemma that will be important to prove the principal results of this section, which generalizes ([17], Lemma 2.8).

**Lemma 3.1.** Let $V$ be a right simple $R$-module. Then $VR[[x;\alpha,w]]$ is a right $R$-module whose the only submodules are ordered in the form

$$VR[[x;\alpha,w]] \supset V(\sum_{i\geq 1} D_i x^i) \supset V(\sum_{i\geq 2} D_i x^i) \supset \ldots.$$ 

**Proof:** We easily have that $VR[[x;\alpha,w]]$ is a right $R[[x;\alpha,w]]$-module and note that $VR[[x;\alpha,w]] \supset V(\sum_{i\geq 1} D_i x^i) \supset V(\sum_{i\geq 2} D_i x^i) \supset \ldots$

Let $S$ be a $R[[x;\alpha,w]]$-submodule of $VR[[x;\alpha,w]]$ such that $S \neq V \sum_{i\geq 1} D_i x^i$

and $f = \sum_{i\geq 0} v_i x^i$ a nonzero submodule of $S$ with $0 \neq v_0 \in V$. Since $V$ is a simple right $R$-module, then $v_0 R = V$. Thus there exists $a_i \in R$ such that $v_i = v_0 a_i$ for all $i \geq 1$. Let $g = 1 + u_1 x + u_2 x^2 + \ldots$ be an element of $R[[x;\alpha,w]]$ such that

$$fg = (v_0 + v_0 a_1 x + v_0 a_2 x^2 + \ldots)(1 + u_1 x + u_2 x^2 + \ldots)$$

$$= v_0 + (v_0 u_1 + v_0 a_1)x + (v_0 u_2 + \alpha_1(\alpha_1^{-1}(v_0 a_1))u_1)w_{1,1} + v_0 a_2)x^2$$

$$+ (v_0 u_3 + \alpha_1(\alpha_1^{-1}(v_0 a_1))u_2)w_{1,2} + \alpha_2(\alpha_2^{-1}(v_0 a_2))u_1)w_{2,1} + v_0 a_3)x^3 + \ldots$$

If we take $u_1 = -a_1$, $u_2 = -a_2-a_1\alpha_1(u_1-1)w_{1,1}$, $u_3 = -a_3-a_2\alpha_2(u_1-1)u_2-\alpha_1\alpha_1(u_2-1)w_{1,2}$, ..., $u_n = a_n-\alpha_n-\alpha_{n-1}(u_1-1)w_{2,1}-\alpha_{n-1}(u_2-1)w_{1,1}$, ..., then we get that $fg = v_0$.

Note that

$$VR[[x;\alpha,w]] = (v_0 R)R[[x;\alpha,w]] \subseteq v_0 R[[x;\alpha,w]] = fg R[[x;\alpha,w]] \subseteq f R[[x;\alpha,w]]$$

and $f R[[x;\alpha,w]] \subseteq VR[[x;\alpha,w]]$. Hence, $VR[[x;\alpha,w]] = f R[[x;\alpha,w]]$, for all $f \in S$ and it follows that $VR[[x;\alpha,w]] \subseteq SR[[x;\alpha,w]] \subseteq S$. So, $V \sum_{i\geq 1} D_i x^i$

is the unique submodule of $VR[[x;\alpha,w]]$. Finally, following this technique
we get the result.

Next, we study the uniformity of $VR[[x; \alpha, w]]$ and $VR\langle x; \alpha, w \rangle$.

**Proposition 3.2.** Suppose that $V$ is a right simple ideal of $R$. The following statements hold.

(a) $VR[[x; \alpha, w]]$ is uniform as $R[[x; \alpha, w]]$-module.

(b) $VR\langle x; \alpha, w \rangle$ is uniform as $R\langle x; \alpha, w \rangle$-module.

**Proof:**

(a) By Lemma (3.1), the unique submodules of $VR[[x; \alpha, w]]$ are $V \sum_{i \geq m} D_i x^i$, for $m \geq 0$ and note that $V \sum_{j \geq t} D_j x^j \supset V \sum_{j \geq t + s} D_j x^j$, for all $s \geq 0$. Thus, always that $s \geq t$. So, $VR[[x; \alpha, w]]$ is uniform.

(b) Let $L$ be a nonzero submodule of $VR\langle x; \alpha, w \rangle$. Then, $L \cap VR[[x; \alpha, w]]$ is a nonzero submodule of $VR[[x; \alpha, w]]$. Thus, for each nonzero submodules $C$ and $D$ of $VR\langle x; \alpha, w \rangle$, we have that $C \cap VR[[x; \alpha, w]] \neq 0$ and $D \cap VR[[x; \alpha, w]] \neq 0$, and it follows that $(C \cap D) \cap VR[[x; \alpha, w]] = (C \cap VR[[x; \alpha, w]]) \cap (D \cap VR[[x; \alpha, w]]) \neq 0$. Hence, $C \cap D \neq 0$. So, $VR\langle x; \alpha, w \rangle$ is uniform.

According to ([19], 2.2.10) the right Goldie rank of a ring $S$ is $n$ if there exists a direct sum $\bigoplus_{i=1}^{n} I_i$ of uniform right submodules of $S$ such that $\bigoplus_{i=1}^{n} I_i$ is right essential in $S$ and we denote it by $\text{rank}_S$.

In ([17], Theorem 2.8) the authors used the noetherianity to prove it. In next result we replace the noetherianity condition for a weaker condition, that is, Goldie property and it generalizes ([17], Theorem 2.8).

**Theorem 3.3.** If $R$ is semiprime Goldie, then $\text{rank}_R = \text{rank}_R[[x; \alpha, w]] = \text{rank}_R \langle x; \alpha, w \rangle$.

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Proof: By the fact that $R$ is semiprime Goldie we have, by ([19], Theorem 2.3.6), that there exists the classical quotient ring $E$ of $R$ which is semisimple. Note that $\text{rank} R = \text{rank} E$, because of ([19], Lemma 2.2.12). Since $R \subseteq R\langle x, \alpha, w \rangle \subseteq E\langle x, \alpha^*, w^* \rangle$, then

$$\text{rank} E = \text{rank} R \leq \text{rank} R\langle x, \alpha, w \rangle \leq \text{rank} E\langle x, \alpha^*, w^* \rangle,$$

where $\alpha^*$ is the extension of the unital twisted partial action $\alpha$ of $R$ to $E$, see ([3], Theorem 3.12). Let $d = \text{rank} R$ and we may suppose without loss of generality that $R = E$ and $\alpha = \alpha^*$. Then, we can write

$$R = V_1 \oplus \cdots \oplus V_d$$

where $V_i$ is a simple right ideal of $R$, for all $i = 1, \ldots, d$.

Hence,

$$R\langle x, \alpha, w \rangle = V_1 R\langle x, \alpha, w \rangle \oplus \cdots \oplus V_d R\langle x, \alpha, w \rangle.$$

and by Proposition (3.2), item (b), each $V_i R\langle x, \alpha, w \rangle$ is uniform as right $R\langle x, \alpha, w \rangle$-module. So, $\text{rank} R\langle x, \alpha, w \rangle = d$.

By similar methods, we have that $R[[x; \alpha, w]] = V_1 R[[x; \alpha, w]] \oplus \cdots \oplus V_d R[[x, \alpha, w]]$ and by Proposition (3.2) item (b), each $V_i R[[x; \alpha, w]]$ is an uniform submodule of $R[[x; \alpha, w]]$, for all $i = 1, \ldots, d$. So, $\text{rank} R[[x; \alpha, w]] = d$.

Let $S$ be a ring and $a \in S$. The right annihilator of $a$ in $S$ is $\text{Ann}_S(a) = \{x \in S : ax = 0\}$. Moreover, according to ([19], Definition 2.2.4) the singular ideal of $S$ is $Z(S) = \{a \in S : \text{Ann}_S(a) \text{ is right essential in } S\}$.

Now, we are ready to prove the second principal result of this section.

Theorem 3.4. Let $R$ be a semiprime ring. The following conditions are equivalent:

(a) $R$ is Goldie.

(b) $R[[x; \alpha, w]]$ is Goldie.

(c) $R\langle x; \alpha, w \rangle$ is Goldie.

Proof: (a) $\Rightarrow$ (c) By assumption, Theorem 3.3 and by Proposition 2.13 item (i) we have that $\text{rank} R\langle x, \alpha, w \rangle = \text{rank} R < \infty$ and $R\langle x, \alpha, w \rangle$ is semiprime. We claim that $R\langle x, \alpha, w \rangle$ is nonsingular. In fact, let $f \in Z(R\langle x, \alpha, w \rangle)$, where $f = a_j x^j + \ldots + a_0 + a_1 x + \ldots$ and $I$ a nonzero right ideal of $R$. Then $I\langle x, \alpha, w \rangle$ is a right ideal of $R\langle x, \alpha, w \rangle$ and we obtain that $\text{Ann}_{R\langle x, \alpha, w \rangle}(f) \cap
I\langle x;\alpha,w \rangle \neq 0$. Thus, there exists $0 \neq h \in I\langle x;\alpha,w \rangle \cap Ann_{R[x;\alpha,w]}(f)$, i.e., $fh = 0$. We consider, $h = b_{-k}x^{-k} + \ldots + b_0 + b_1x + \ldots$ and suppose without loss of generality that $b_{-k} \neq 0$. Hence, looking at the smallest degree of the product $fh$ we get

$$a_{-j}\alpha_{-j}(1_jb_{-k})w_{-j,k}x^{-j-k} = 0,$$

which implies that $a_{-j}\alpha_{-j}(1_jb_{-k}) = 0$. Consequently,

$$\alpha_{-j}^{-1}(a_{-j})\alpha_{-j}^{-1}(\alpha_{-j}(1_jb_{-k})) = 0 \implies \alpha_{-j}^{-1}(a_{-j})1_jb_{-k} = 0$$

and we have that $\alpha_{-j}^{-1}(a_{-j})b_{-k} = 0$. So, $0 \neq b_{-k} \in Ann_{R}(\alpha_{-j}^{-1}(a_{-j}))$ and we obtain that $Ann_{R}(\alpha_{-j}^{-1}(a_{-j})) \cap I \neq 0$ which concludes that $\alpha_{-j}^{-1}(a_{-j}) \in Z(R)$. By the fact that $R$ is Goldie we have that $\alpha_{-j}^{-1}(a_{-j}) = 0$. Since, $\alpha_{-j}^{-1}$ is an isomorphism, then $a_{-j} = 0$. Now, following the similar methods, we obtain that $f = 0$. Hence, $Z(R\langle x;\alpha,w \rangle) = 0$. Therefore, by ([19], Theorem 2.3.6) we get that $R\langle x;\alpha,w \rangle$ is Goldie.

We need to show that $(c) \Rightarrow (b) \Rightarrow (a)$. In fact, note that

$$R \subset R[[x;\alpha,w]] \subset R\langle x;\alpha,w \rangle$$

and by the fact that $R$ is semiprime and Goldie, we have, by Theorem [33] that $rank_R = rank_{R[[x;\alpha,w]]} = rank_{R\langle x;\alpha,w \rangle}$. Since the chain conditions on right annihilators is inherited by subrings we obtain the desired result. \(\square\)

In the article [3], the authors worked with twisted partial actions of finite type and the rings satisfied some finiteness conditions as Goldie property. But, at that time the authors did not notice such assumption would imply the existence of the enveloping action. So, in the next result, we show that the unital twisted partial actions on algebras with finite Goldie rank that are of finite type, have enveloping action.

**Theorem 3.5.** Let $R$ be a ring with finite uniform dimension and $\alpha$ a twisted partial action of $Z$ on $R$. If $\alpha$ is of finite type, then $\alpha$ has enveloping action.

**Proof:** By assumption, there exists a finite set $\{g_1, \ldots, g_n\}$ of $Z$ such that

$$R = D_{g+g_1} + \ldots + D_{g+g_n},$$

for every $g \in Z$. We claim that $R$ can be written as direct sum of indecomposable rings. In fact, each $D_{g_i}$ has identity $1_{g_i}$ and by similar methods of ([14], Remark 1.11) we can write

$$R = F_1 \oplus \ldots \oplus F_n,$$
where each $F_i$ is an ideal of $D_{g+g_i}$, $i = 1, \ldots, n$, generated by a central idempotent. Now, if each $F_i$ is indecomposable we are done. Next, if there exists $1 \leq j \leq n$ such that $F_j$ is not indecomposable, then we may write $F_j = F^1_j \oplus F^2_j$, and we get

$$R = F_1 \oplus \ldots \oplus F^1_j \oplus F^2_j \oplus \ldots F_n.$$  

Proceeding in this manner with all other decomposable components we may write

$$R = A_1 \oplus \ldots \oplus A_n$$

Now if all $A_i$ are indecomposable, then we are done. If it is not, proceed with similar methods as before. Since $\text{rank} R$ is finite, then the process must stop and we have that $R$ is a direct sum of indecomposable rings where, each one is generated by a central idempotent of $R$. So, by ([11], Theorem 7.2), $(R, \alpha, w)$ has enveloping action. 

Let $\alpha$ be an unital twisted partial action of $\mathbb{Z}$ on $R$ that admits enveloping action $(T, \beta, u)$. Following [9] and [11], we exhibit an explicit Morita context between $R\langle x; \alpha, w \rangle$ and $T\langle x; \beta, u \rangle$ whose restriction to $T[[x; \beta, u]]$ gives also a Morita context between $R[[x; \alpha, w]]$ and $T[[x; \beta, u]]$.

Recall that given two rings $R$ and $S$, bimodules $R_U S$ and $S_V R$ and maps $\theta : U \otimes_S V \to R$ and $\psi : V \otimes_R U \to S$, the collection $(R, S, U, V, \theta, \psi)$ is said to be a Morita context if the array

$$\begin{bmatrix} R & V \\ U & S \end{bmatrix},$$

with the usual formal operations of $2 \times 2$ matrices, is a ring.

The following result is proved in ([11], Theorem 3.6.2), for rings with identity element. Actually, in the proof of the result, it is not used the fact that the rings have identity element and the modules $U$ and $V$ are unital modules. So, we can easily see that the following is true for rings which do not necessarily have identity.

**Theorem 3.6.** Let $(R, S, U, V, \theta, \psi)$ be a Morita context. Then there is an order preserving one-to-one correspondence between the sets of prime ideals $P$ of $R$ with $P \not\supseteq UV$ and prime ideals $P'$ of $S$ with $P' \not\supseteq VU$. The correspondence is given by $P \longmapsto \{s \in S : UsV \subseteq P\}$ and $P' \longmapsto \{r \in R : VrU \subseteq P'\}$. 

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Following the similar ideas of ([9], Section 5), we put \( U = \{ \sum_{i \in \mathbb{Z}} a_i x^i : a_i \in R \text{, for all } i \in \mathbb{Z} \} \) and \( V = \{ \sum_{i \in \mathbb{Z}} a_i x^i : a_i \in \beta_i(R) \text{, for all } i \in \mathbb{Z} \} \). Then, it can easily be seen that \( U T \langle x ; \beta, u \rangle \subseteq U \), \( T \langle x ; \beta, u \rangle V \subseteq V \), \( R \langle x ; \alpha, w \rangle U \subseteq U \) and \( VR \langle x ; \alpha, w \rangle \subseteq V \) (to show the relations recall that \( \beta_j(R) \) is an ideal of \( T \) and \( S_j = \beta_j(S_{-j}) \)). In case we want to consider \( R[[x; \alpha, w]] \) and \( T[[x; \beta, u]] \) we restrict \( U \) and \( V \) to have just power series and we have similar relations.

Thus, we have the Morita contexts \( (R \langle x ; \alpha, w \rangle, T \langle x ; \beta, u \rangle, U, V, \theta, \psi) \) and \( (R[[x; \alpha, w]], T[[x; \beta, u]], U, V, \theta, \psi) \), where \( \theta \) and \( \psi \) are obvious.

The proof of the following lemma is similar to ([4], Lemma 2.2).

**Lemma 3.7.** Let \( P \) be a prime ideal of \( R[[x; \alpha, w]] \). Then, there exists a unique prime ideal \( P' \) of \( T[[x; \beta, u]] \), given by Theorem 3.6, which satisfies \( P' \cap R[[x; \alpha, w]] = P \).

We have the following easy consequence.

**Corollary 3.8.** There is a one-to-one correspondence, via contraction, between the set of all prime ideals of \( R[[x; \alpha, w]] \) and the set of all prime ideals of \( T[[x; \beta, u]] \) which do not contain \( R \).

The next result is important to prove the last main result of this article and it is an easy consequence of Lemma 3.7 and Corollary 3.8.

**Corollary 3.9.** Let \( \alpha \) be an unital twisted partial action of \( \mathbb{Z} \) on \( R \) with enveloping action \( (T, \beta, u) \). Then \( \text{Nil}_*(T[[x; \beta, u]]) \cap R[[x; \alpha, w]] = \text{Nil}_*(R[[x; \alpha, w]]) \).

Based on the last results, we will proceed to describe the prime radical of \( R[[x; \alpha, w]] \) when \( (R, \alpha, w) \) has enveloping action \( (T, \beta, u) \) and for this we need the following result.

**Lemma 3.10.** Let \( \beta \) be a twisted global action of \( \mathbb{Z} \) on a ring \( S \) with cocycle \( u \). Then \( \text{Nil}_*(S[[x; \beta, u]]) \) of \( S[[x; \beta, u]] \) is \( \text{Nil}_*(S[[x; \beta, u]]) = \text{Nil}_*(S) \cap N_\beta(S) \oplus \sum_{i \geq 1} N_\beta(S)x^i \), where \( N_\beta(S) \) is the intersection of all strongly \( \beta \)-prime ideals of \( S \).

**Proof:** We have two classes of prime ideals in \( S[[x; \beta, u]] \), i.e.,

\[ \mathcal{F}_1 = \{ P : \text{prime ideal such that } S[[x; \beta, u]]x \subseteq P \} \]

and

\[ \mathcal{F}_2 = \{ P : \text{prime ideal, such that } S[[x; \beta, u]]x \nsubseteq P \} \].
Note that \( \bigcap_{P \in \mathcal{F}_1} P = \text{Nil}_* (S) \oplus \sum_{i \geq 1} Sx_i \). Now, for each strongly \( \beta \)-prime ideal \( Q \) of \( S \), we have by similar methods of Corollary 2.8 that \( Q[[x; \beta, u]] \) is prime and we easily get that each prime ideal \( P \) of \( \mathcal{F}_2 \) implies that \( P \cap S \) is a strongly \( \beta \)-prime ideal of \( S \). Thus, \( \bigcap_{P \in \mathcal{F}_2} P \supseteq N_\beta (S) \). Hence, \( \text{Nil}_* (S[[x; \beta, u]]) = (\bigcap_{P \in \mathcal{F}_1} P) \cap (\bigcap_{Q \in \mathcal{F}_2} Q) \supseteq (\text{Nil}_* (S) + \sum_{i \geq 1} Sx_i) \cap (N_\beta (S))[[x; \alpha, w]] \supseteq \text{Nil}_* (S) \cap N_\beta (S) \oplus \sum_{i \geq 1} N_\beta (S)x_i.

On the other hand, since for each prime ideal \( L \) of \( S \) we have that \( L \oplus \sum_{i \geq 1} Sx_i \) is a prime ideal of \( S[[x; \beta, u]] \) and in the same way of Corollary 2.8 we have that \( N[[x; \beta, u]] \) is prime for each strongly \( \beta \)-prime ideal \( N \) of \( S \), then \( \text{Nil}_* (S[[x; \beta, u]]) \subseteq (\text{Nil}_* (S) \cap N_\beta (S)) \oplus \sum_{i \geq 1} N_\beta (S)x_i \).

So, \( \text{Nil}_* (S[[x; \beta, u]]) = \text{Nil}_* (S) \cap N_\beta (S) \oplus \sum_{i \geq 1} N_\beta (S)x_i \).

\( \square \)

**Proposition 3.11.** Let \( \alpha \) be an unital twisted partial action with enveloping action \((T, \beta, u)\). Then the prime radical \( \text{Nil}_* (R[[x; \alpha, w]]) \) of \( R[[x; \alpha, w]] \) is \( \text{Nil}_* (R[[x; \alpha, w]]) = (N_\alpha (R) \cap \text{Nil}_* (R)) \oplus \sum_{i \geq 1} (N_\alpha (R) \cap D_i)x_i \), where \( N_\alpha (R) \) is the intersection of all strongly \( \alpha \)-prime ideals of \( R \).

**Proof:** Using the same methods of ([4], Lemma 2.9), we have, for each strongly \( \beta \)-prime ideal \( Q \) of \( T \), that \( Q \cap R \) is a strongly \( \alpha \)-prime ideal of \( R \) and since \( R \) is an ideal of \( T \) we have that \( \text{Nil}_* (T) \cap R = \text{Nil}_* (R) \). Thus, by Corollary 3.9 we easily get that \( \text{Nil}_* (R[[x; \alpha, w]]) = \text{Nil}_* (T[[x; \beta, u]]) \cap R[[x; \alpha, w]] = (\text{Nil}_* (R) \cap N_\alpha (R)) \oplus \sum_{i \geq 1} (\text{Nil}_* (R) \cap D_i)x_i \) \( \square \)

**Remark 3.12.** In the last result, we use the fact that the twisted partial action has an enveloping action, but we do not know if the Proposition 3.11 is true for unital twisted partial actions of \( \mathbb{Z} \) without enveloping action. To solve this problem we need to know if the following result is true:

Let \( P \) be a prime ideal of \( R[[x; \alpha, w]] \) such that \( 1, x^i \notin P \) for some \( i \geq 1 \). Then \( P \cap R \) is strongly \( \alpha \)-prime.
As it happened in the ([5], Example 2.6) we obtain by a similar example that the twisted partial skew power series over semiprime Goldie rings are not necessary semiprime, but, if we input the condition of “finite type” we get the following.

**Theorem 3.13.** Let $\alpha$ be an unital twisted partial action. If $R$ is semiprime Goldie and $\alpha$ is a twisted partial action of finite type, then $R[[x; \alpha, w]]$ is semiprime Goldie.

**Proof:** Since $\alpha$ is of finite type and $\text{rank}(R)$ is finite, then by Theorem (3.5) $\alpha$ has enveloping action $\beta = (B, \{\beta_g\}_{g \in G}, \{u_{(g,h)}\}_{(g,h) \in G \times G})$. In this case, by ([3], Corollary 4.18), $T$ is semiprime Goldie and we claim that $T[[x; \beta, u]]$ is semiprime. In fact, suppose that $\text{Nil}_*(T[[x; \beta, u]])$ is not zero. Then, by ([15], Lemma 10.10.29), $\text{Nil}_*(T[[x; \beta, u]])$ contains a nonzero nilpotent ideal $L$, since by Theorem 3.4 we have that $T[[x; \beta, u]]$ is Goldie. By the fact that $T$ is semiprime we have that $\text{Nil}_*(T[[x; \beta, u]]) = \sum_{i \geq 1} N_\beta(T)x^i$. Now, consider $H = \{0 \neq a \in N_\beta(T) : \exists 0 \neq f \in L \text{ such that } f = ax^j + ... \in L\} \cup \{0\}$. It is not difficult to see that $H$ is a nonzero ideal of $T$ with $\beta_i(H) \subseteq H$, for all $i \in \mathbb{Z}$. Since $L$ is nilpotent, we obtain that $H$ is nilpotent and consequently $H = 0$, because of $T$ is semiprime, which is a contradiction. So, $\text{Nil}_*(T[[x; \beta, u]]) = 0$.

By Corollary 3.9 we have that

$$\text{Nil}_*(T[[x; \beta, u]]) \cap R[[x; \alpha, w]] = \text{Nil}_*(R[[x; \alpha, w]])$$

which implies that $\text{Nil}_*(R[[x; \alpha, w]]) = 0$. Therefore, $R[[x; \alpha, w]]$ is semiprime Goldie. 

\[\square\]

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