Weyl Spreading Sequence Optimizing CDMA

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Abstract—This paper shows an optimal spreading sequence in the Weyl sequence class belonging to the extended Frank-Zadoff-Chu (FZC) sequence family for asynchronous CDMA systems. The sequences in this class have the desired property that the order of crosscorrelation is low. We evaluate the upper bound of crosscorrelation and odd crosscorrelation in the class and construct the optimization problem: minimize the upper bound. Then, since the optimization problem is convex programming, we can derive the optimal spreading sequences as the global solution of the problem. We show their Signal to Interference plus Noise Ratio (SINR) in a special case. From this result, we propose how the initial elements are assigned, that is, how spreading sequences are assigned to each users. In an asynchronous CDMA system, we also numerically compare our spreading sequences with other ones, the Gold codes, the sequences of the FZC sequence families, the optimal Chebyshev spreading sequences and the SP sequences in Bit Error Rate. Our spreading sequence which is the global solution has the highest performance in the other spreading sequences.

Index Terms—Spread spectrum communication, Asynchronous CDMA, Nonlinear programming, Spreading sequence, Signal to Interference plus Noise Ratio, Bit error rate

I. INTRODUCTION

Signal to interference plus noise ratio (SINR) is an important index for wireless communication systems. In wireless communication systems, it is the most significant to achieve high capacity. In general, it is necessary and sufficient for achieving high capacity to increase SINR under the condition that the width of the frequency band is constant when the interference noise is approximated as additive white Gaussian Noise (AWGN). Similarly, the performance of wireless communication is evaluated in Bit Error Rate (BER). However, these two are not independent, and it is known that BER decreases as SINR increases since the interference noise is the most important factor relating to the system performance in CDMA systems.

As a wireless communication system, we focus on a code division multiple access (CDMA) system, in particular, an asynchronous CDMA system. It is one of the multiple access systems with which many people can communicate each other at the same time. In CDMA systems, spreading sequences are utilized as codes to multiplex. Each user is assigned a different code and uses it to modulate and demodulate his signal.

In CDMA systems, many methods have been proposed to increase SINR. The one of such methods is based on the blind multiuser detection. On the other hand, improving the receiver with the application of digital implementation of ICA and Maximum Likelihood (ML) estimation are also efficient. However, in particularly, ML estimation method needs a large amount of calculations.

In general, the value of SINR depends on the spreading sequences. In synchronous CDMA systems, it is known that the Welch bound equality (WBE) sequence realizes the maximal capacity. The Welch bound represents the lower bound of the maximum value of crosscorrelation. When the time delays are given and fixed, the way to find the optimal spreading sequence has been suggested. However, in asynchronous CDMA systems, the optimal spreading sequences have not been found. Therefore, asynchronous CDMA systems have been investigated.

In uplink of W-CDMA systems, the current spreading sequence is the Gold code. It is known that the Gold code is optimal in all the binary spreading sequences as well as the Kasami sequence in a sense of the maximum value among all periodic autocorrelation and periodic crosscorrelation. To explore a better sequence for asynchronous CDMA systems, the use of chaotic spreading sequences is proposed. These chaotic spreading sequences have been evaluated in Bit Error Rate. Examples of such spreading sequences have been given in [18]-[22]. In [24], the approach to obtain the capacity of spreading sequences has been proposed.

In [24], Sarwate has shown two kinds of characterized sequences on his limitation. One kind is a set of sequences whose periodic crosscorrelation is always zero. The other kind is a set of sequences whose period autocorrelation is always zero except for only one point, that is, Frank-Zadoff-Chu (FZC) sequences. In [27], the extended set of the FZC sequences, the FZC sequence families are proposed. They have three parameters and their SINR, autocorrelation and crosscorrelation has been investigated.

In this paper, we define the Weyl sequence class, which is a set of sequences generated by the Weyl transformation. This class belongs to the extended FZC sequence families and includes the Sarwate sequences. The sequence in the Weyl sequence class has a desired property that the order of crosscorrelation is low. We evaluate the upper bound of crosscorrelation and construct the optimization problem: minimize the upper bound of crosscorrelation. From the problem, we derive optimal spreading sequences in the Weyl sequence class. We show SINR of them in a special case and compare them with other sequences in Bit Error Rate.

II. WEYL SEQUENCE CLASS

In this section, we define the Weyl sequence class and show their properties. Let $N$ be the length of spreading sequences.
We define the Weyl sequence \( (x_n) \) as the following formula \[ x_n = n\rho + \Delta \mod 1 \quad (n = 1, 2, \ldots, N), \] (1) where \( \rho \) and \( \Delta \) are real parameters. From the above definition, we can assume that the parameters \( \rho \) and \( \Delta \) satisfy \( 0 \leq \Delta < 1 \) and \( 0 \leq \rho < 1 \). The sequences whose \( \rho \) is an irrational number are used in a Quasi-Monte Carlo method \[23\]. We apply this sequence to a spreading sequence. Then, the Weyl spreading sequence \( (w_{k,n}) \) is defined as \[ w_{k,n} = \exp(2\pi j x_{k,n}) \quad (n = 1, 2, \ldots, N), \] (2) where \( k \) is the number of the user and \( j \) is the unit imaginary number. In CDMA systems, the value of \( \Delta_k \) has no effects to Signal to Interference plus Noise Ratio (SINR) since \( \exp(2\pi j\Delta_k) \) is united to the phase term of the signal. Thus, we set \( \Delta_k = 0 \). We call the class which consists of Weyl spreading sequences as the Weyl sequence class. Note that this class is similar to the FZC sequence families \[27\]. The \( n \)-th element of the FZC sequence families is defined as \[ w_{k,n} = \exp(j\pi M_k^p n^q + n^r) \quad (n = 1, 2, \ldots, N), \] (3) where \( M_k \) is an integer that is relatively prime to \( N \) such that \( 1 \leq M_k < N \) and \( p, q, r \) are any real numbers. The triple \( \{p, q, r\} \) specifies the set of sequences. When the triple \( \{p, q, r\} \) is \( \{2, 1, -\infty\} \), we obtain the element of the FZC sequence \[25\] \[ u_{k,n} = (-1)^{\alpha M_k} \exp\left(j\pi M_k^p n^q / N \right). \] (4) The Weyl sequence class is obtained when the triple is \( \{1, 1, -\infty\} \) and \( M_k = \rho_k \cdot 2N/(N + 1) \). Note that \( M_k \) is not always an integer. Thus, the Weyl sequence class belongs to the extended FZC sequence families whose \( M_k \) is a real number.

The element of the Weyl sequence class, \( (w_{k,n}) \) has a desired property that crosscorrelation is low. We define the periodic correlation function \( \theta_{i,k}(l) \) and odd periodic correlation function \( \tilde{\theta}_{i,k}(l) \) as
\[
\theta_{i,k}(l) = C_{i,k}(l) + C_{i,k}(l - N), \tag{5}
\]
\[
\tilde{\theta}_{i,k}(l) = C_{i,k}(l) - C_{i,k}(l - N), \tag{6}
\]
where
\[
C_{i,k}(l) = \sum_{n=1}^{N-1} w_{i,n} w_{k,n} \quad 0 \leq l \leq N - 1, \tag{7}
\]
\[
\sum_{n=1}^{N} w_{i,n} w_{k,n-l} \quad 1 - N \leq l < 0, \tag{8}
\]
\[
0 \quad |l| \geq N
\]
and \( \overline{z} \) is the conjugate of \( z \). The correlation functions \( \theta_{i,k}(l) \) and \( \tilde{\theta}_{i,k}(l) \) have been studied in \[24\] \[10\] \[31\]. When \( i \neq k \), \( \theta_{i,k}(l) \) and \( \tilde{\theta}_{i,k}(l) \) are periodic and odd periodic crosscorrelation functions. It is necessary for achieving high SINR to keep the value of the crosscorrelation functions, \( |\theta_{i,k}(l)| \) and \( |\tilde{\theta}_{i,k}(l)| \) low for all \( 0 \leq l < N \). The absolute value of crosscorrelation functions \( |\theta_{i,k}(l)| \) and \( |\tilde{\theta}_{i,k}(l)| \) have the common upper bound that
\[
|\theta_{i,k}(l)| \leq |C_{i,k}(l)| + |C_{i,k}(l - N)|, \tag{8}
\]
\[
|\tilde{\theta}_{i,k}(l)| \leq |C_{i,k}(l)| + |C_{i,k}(l - N)|. \tag{9}
\]
With the sequences in the Weyl sequence class, we evaluate the absolute value of \( C_{i,k}(l) \) as
\[
|C_{i,k}(l)| = \frac{1 - \exp(2\pi j(N - l)(\rho_k - \rho_i))}{1 - \exp(2\pi j\rho_k - \rho_i))} \tag{10}
\]
\[
= \sqrt{1 - \cos(2\pi j(N - l)(\rho_k - \rho_i)) \cos(2\pi j\rho_k - \rho_i)} \tag{11}
\]
\[
= \frac{|\sin(\pi(N - l)(\rho_k - \rho_i))|}{|\sin(\pi(\rho_k - \rho_i))|} \tag{12}
\]
\[
\leq \frac{1}{2} + m, \tag{13}
\]
where \( m \in \mathbb{Z} \). From the above result, \( |C_{i,k}(l)| \) obeys
\[
|C_{i,k}(l)| = O(1). \tag{14}
\]
Similarly, \( |C_{i,k}(l - N)| \) obeys \( O(1) \). Thus, the upper bound of \( |\theta_{i,k}(l)| \) and \( |\tilde{\theta}_{i,k}(l)| \) is independent of \( N \). For the general spreading sequences, due to the central limit theorem (CLT), the crosscorrelations \( |\theta_{i,k}(l)| \) and \( |\tilde{\theta}_{i,k}(l)| \) become large as \( N \) becomes large. For this reason, compared to the general spreading sequences, the Weyl spreading sequence is expected to have low crosscorrelation.

### III. Optimal Spreading Sequence in Weyl Sequence Class

In this section, we consider an asynchronous binary phase shift keying (BPSK) CDMA system. Our goal is to derive the spreading sequences whose interference noise is the smallest in the Weyl sequence class. Let \( K, T_c \) and \( T_c \) be the number of users, the durations of the symbol and each chip, respectively. In this situation, the user \( i \) despreads the spreading sequences \( (w_{k,n}) \) with the spreading sequence of the user \( i \). \( (w_{i,n}) \). The symbols \( b_{k,-1}, b_{k,0} \in \{-1, 1\} \) denote bits which the user \( k \) send. The transmitted signal of the user \( k \) has time delay \( \tau_k \). From \[22\], we assume that time delay \( \tau_k \) is distributed in \( [0, T_c) \) and satisfies \( b_{k,-1} T_c \leq \tau_k \leq (b_{k,0} + 1) T_c \), where \( b_k \in \{0, 1, \ldots, N - 1\} \) is an integer. Then, the interference noise between the user \( i \) and the user \( k \), \( I_{i,k}(\tau_k) \) is obtained as
\[
I_{i,k}(\tau_k) = \exp(j\phi_k) \left[ (\tau_k - b_{k,-1} T_c) + b_{k,0} C_{i,k}(l - N) \right] \tag{15}
\]
\[
+ ((l_k + 1) T_c - \tau_k) \cdot \left[ b_{k,-1} C_{i,k}(l + 1) + b_{k,0} C_{i,k}(l + 1 - N) \right], \tag{16}
\]
where \( \phi_k \in [0, 2\pi) \) is the phase of user \( k \)’s carrier. With Eq. \[10\], the absolute value of the interference noise \( I_{i,k}(\tau_k) \) is
evaluated as
\[
|I_{i,k}(\tau_k)|\leq (\tau_k - i_k T_e) |(C_{i,k}(l_k) + |C_{i,k}(l_k - N)|)
\]
\[+ ((l_k + 1) T_e - \tau_k)
|C_{i,k}(l_k + 1) + |C_{i,k}(l_k + 1 - N)|]
\]
\[
\leq \frac{2T_e}{\sin(\pi(\rho_i - \rho_k))}
\]
(14)

Thus, we have shown that the upper bound of interference noise between two sequences is inversely proportional to \(\sin(\pi(\rho_i - \rho_k))\). To reduce the interference noise \(I_{i,k}(\tau_k)\), it is necessary to reduce \(2T_e / |\sin(\pi(\rho_i - \rho_k))|\). To eliminate the absolute value function, we introduce the distance between the phases \(\rho_i\) and \(\rho_k\). The distance \(d(\rho_i, \rho_k)\) we propose here is given by
\[
d(\rho_i, \rho_k) = \min(|\rho_i - \rho_k|, 1 - |\rho_i - \rho_k|).
\]
(15)

Note that this \(d\) satisfies the axiom of distance, and
\[
|\sin(\pi(\rho_i - \rho_k))| = \sin(\pi d(\rho_i, \rho_k)),
\]
(16)
\[
0 \leq d(\rho_i, \rho_k) \leq \frac{1}{2}
\]
(17)
if we regard \(\rho = 1\) in the same light as \(\rho = 0\). From Eq. (15), we rewrite Eq. (14) without any absolute value as
\[
|I_{i,k}(\tau_k)| \leq \frac{2T_e}{\sin(\pi d(\rho_i, \rho_k))}.
\]
(18)

We should take into account the whole interference noise in the users. The whole interference noise \(I\) is written as
\[
I = \sum_{i=1}^{K} \sum_{k=1}^{K} I_{i,k}.
\]
(19)

With Eq. (18), it is clear that \(|I|\) has the upper bound:
\[
|I| \leq \sum_{i=1}^{K} \sum_{k=1}^{K} \frac{2T_e}{\sin(\pi d(\rho_i, \rho_k))}.
\]
(20)

Thus, we minimize Eq. (20) and obtain the problem \((P)\)
\[
\min_{\rho_k} \sum_{i=1}^{K} \sum_{k=1}^{K} \frac{1}{\sin(\pi d(\rho_i, \rho_k))}
\]
subject to \(\rho_k \in [0, 1) (1 \leq k \leq K)\).
(21)

This problem is equivalent to that we minimize the sum of the upper bound of \(C_{i,k}(l)\). Thus, the crosscorrelation among all the users is expected to be always low when we solve this problem. From Eq. (16), it is clear that \(d(\rho_i, \rho_k) = d(\rho_k, \rho_i)\). Then, in the problem \((P)\), we count two times the same distance. Thus, we obtain the equivalent problem \((P')\)
\[
\min_{\rho_k} \sum_{i=1}^{K} \sum_{k=1}^{K} \frac{1}{\sin(\pi d(\rho_i, \rho_k))}
\]
subject to \(\rho_k \in [0, 1) (1 \leq k \leq K)\).
(22)

It is not clear if the objective function of the problem \((P')\) is convex since the form of function \(d\) is complicated. To eliminate the function \(d\), we introduce slack variables \(t_{i,k}\) for

\((P)\). Then, the problem \((P')\) is rewritten as
\[
\min_{\rho_k} \sum_{i=1}^{K} \sum_{k=1}^{K} \frac{1}{\sin(\pi t_{i,k})}
\]
subject to \(\rho_k \in [0, 1) (1 \leq k \leq K)\),
\[
|\rho_i - \rho_k| \geq t_{i,k} \ (i \leq K),
\]
\[
1 - |\rho_i - \rho_k| \geq t_{i,k} \ (i < k),
\]
\[
t_{i,k} \geq 0 \ (i < k).
\]
(23)

Without loss of generality, we assume \(\rho_k \leq \rho_{k+1}\). Then, the problem \((P')\) can be rewritten as
\[
\min_{\rho_k} \sum_{i=1}^{K} \sum_{k=1}^{K} \frac{1}{\sin(\pi t_{i,k})}
\]
subject to \(\rho_k - \rho_i \geq t_{i,k} \ (i < k)\),
\[
1 - \rho_k + \rho_i \geq t_{i,k} \ (i < k),
\]
\[
\rho_{k+1} \geq \rho_i \ (1 \leq i \leq K - 1),
\]
\[
\rho_1 \geq 0, \rho_K \leq 1,
\]
\[
t_{i,k} \geq 0 \ (i < k).
\]
(24)

Notice that the objective function and the inequality constraints of the problem \((P)\) are convex. It has been known that convex programming can be solved with the KKT conditions [33]. To write such conditions, we define the variable vector \(z\) as
\[
z = \begin{pmatrix} \rho \ 
\rho T \end{pmatrix},
\]
(25)
where \(\rho \in \mathbb{R}^K\), \(T \in \mathbb{R}^{K(K-1)/2}\), \(z \in \mathbb{R}^{K(K+1)/2}\) and \(z^T\) is the transpose of \(z\). From the KKT conditions, the solution \(z^*\) is a global solution of \((P)\) if \(z^*\) satisfies the following equation:
\[
\nabla f(z^*) + \sum_{i=1}^{K} \lambda_{i,k} \nabla c_{i,k}(z^*) + \sum_{i=1}^{K} \mu_{i,k} \nabla d_{i,k}(z^*) + \sum_{i=1}^{K} \nu_i \nabla e_i(z^*)
\]
\[
+ \sum_{i=1}^{K} \mu_i \nabla g_i(z^*) + \sum_{i=1}^{K} \lambda_i \nabla h_i(z^*) = 0,
\]
(26)

where
\[
f(z) = \sum_{i=1}^{K} \frac{1}{\sin(\pi \alpha_{i,k})},
\]
\[
c_{i,k}(z) = t_{i,k} + \rho_i - \rho_k,
\]
\[
d_{i,k}(z) = t_{i,k} - 1 - \rho_i + \rho_k,
\]
\[
e_i(z) = \rho_i - \rho_{i+1},
\]
\[
g_i(z) = -x_i,
\]
\[
h_i(z) = -t_{i,k},
\]
(27)
and the Lagrange multipliers \(\lambda_{i,k}, \mu_{i,k}, \nu_i, \xi_1, \xi_K\) and \(\alpha_i\) are non-negative real numbers. They have to satisfy the following
In this section, we fix the maximum number of users in a channel, \( K_{\text{max}} \), and assign the spreading sequences \( \tilde{w}_{k,n}(N, \gamma) \) to \( K_{\text{max}} \) users. When \( K_{\text{max}} = N, \) the sequence \( \tilde{w}_{k,n}(N, \gamma) \) is the WBE sequence since the orthogonal condition is satisfied. From the above reason, we define \( K_{\text{max}} = N, \) that is, we consider the following sequences

\[
\tilde{w}_{k,n}(N, \gamma) = \exp \left( 2\pi j n \left( \gamma + \frac{\sigma_k}{N} \right) \right) \quad (n = 1, 2, \ldots, N).
\]

(35)

In this section, we assume that \( \sigma_k \) is a random variable and is uniformly distributed in \( \{0, 1, 2, \ldots, N-1\}. \) In the next section, we consider how to assign \( \sigma_k \) in a systematic approach.

The expression of SINR of the user \( i \) is obtained in (32) as

\[
\text{SINR}_i = \left( \frac{6N^3}{E} \right)^{-1/2} \sum_{k=1}^{K_{\text{max}}} r_{i,k} + \frac{N_0}{2E},
\]

(36)

where

\[
r_{i,k} = \sum_{l=0}^{N-1} \left[ |C_{i,k}(l-N)|^2 + \text{Re}[C_{i,k}(l-N)C_{i,k}(l-N-1)] \right.
\]

\[
+ |C_{i,k}(l-N+1)|^2 + |C_{i,k}(l)|^2
\]

\[
+ \text{Re}[C_{i,k}(l)C_{i,k}(l+1)] + |C_{i,k}(l+1)|^2 \right],
\]

\( E \) is the energy per data bit and \( N_0 \) is the power of Gaussian noise. SINR is the ratio between the variance of a desired signal and the one of a noise signal. In the appendix B, with spreading sequence \( \tilde{w}_{k,n}(N, \gamma) \), we prove that SINR of the user \( i \) is given by

\[
\text{SINR}_i = \left( R_i + \frac{N_0}{2E} \right)^{-1/2},
\]

(38)

where

\[
R_i = \frac{(K-1)}{18N^2} \left( 2(N+1) \right) \left( 2(N-1) \right) \left( 2\pi \gamma + \frac{\sigma_i}{N} \right).
\]

(39)

Equation (38) is obtained when the ratio \( K/N \) is close to 1, that is, the number of users \( K \) is sufficiently large. From Eqs. (38) and (39), the spreading sequence \( \tilde{w}_{i,n}(N, \gamma) \) has different SINR in \( \sigma_i \). Thus, some users have high SINR and other users have low SINR. The lower bound of SINR \( \text{SINR}_i \) is

\[
\text{SINR}_i = \left( \frac{K-1}{6N} + \frac{N_0}{2E} \right)^{-1/2}.
\]

(40)

V. How to Assign \( \sigma_k \)

In this section, we consider how to assign \( \sigma_k \) to the each users. Let us consider the spreading sequences

\[
\tilde{w}_{k,n}(N, \gamma) = \exp \left( 2\pi j n \left( \gamma + \frac{\sigma_k}{N} \right) \right).
\]

From Eq. (29), it is demanded that we assign \( \sigma_k \) at regular interval. However, these sequences cannot be used if the number of users changes. Thus, we have to make the rule to assign \( \sigma_k \) when the number of users changes.
We give the rule to assign \( \sigma_k \) in the situation that the number of users monotonically increases. From the demand of the problem \((P)\), it is desirable that we assign \( \sigma_k \) to each users at regular interval. Thus, it is appropriate to assign them at nearly regular interval in every number of users. We apply the Van der Corput sequence \([33]\) to the method to assign \( \sigma_k \) since the sequence is a regular interval sequence in some situations. For example, the Van der Corput sequence \((v_n)\) is obtained as
\[
(v_n) = \left\{ 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \ldots \right\}.
\]
(41)

In particular, when we take the first eight elements out from \((v_n)\) and sort them, we obtain the sequence
\[
\left\{ 0, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4} \right\}.
\]
(42)

This sequence is a regular interval sequence. We can consider \((v_n)\) as a nearly regular interval sequence.

When the length of spreading sequences \(N\) equals \(2^m\), where \(m > 1\) is an integer, \((v_n)\) is rewritten in terms of \(1/N\). For example, when \(N = 16\), the sequence \((v_n)\) is obtained as
\[
(v_n) = \left\{ 0, \frac{8}{16}, \frac{4}{16}, \frac{12}{16}, \frac{2}{16}, \frac{10}{16}, \frac{6}{16}, \frac{14}{16}, \frac{1}{16}, \ldots \right\}.
\]
(43)

Thus, we propose that we use the \(k\)-th element of \((v_n)\) as \(\sigma_k/N\), that is, the spreading sequences are expressed as
\[
\tilde{w}_{k,n}(N, \gamma) = \exp(2\pi j n (\gamma + v_k)),
\]
where \(v_k\) is the \(k\)-th element of \((v_n)\).

VI. SIMULATION RESULT

In this section, we simulate an asynchronous CDMA communication system and discuss the performance of the spreading sequences obtained by Eqs. (31), (35), and (44). We use two parameters \(\gamma = 1/(2N)\) and \(\gamma = 1/(2K)\), which are independent of the number of users \(K\) and depend on \(K\). We consider a BPSK model. The channel has a white Gaussian noise (AWGN) and no fading signals. In this simulation, we make the following assumptions about the receivers and the channel:

1) the receiver has the perfect synchronization with the desired signal and no knowledge about the time delay of the other signals,
2) there are no fading effects,
3) the time delay \(\tau_k\), the symbols \(b_{k,1}\) and \(b_{k,0}\), and the phase \(\theta_k\) are normally distributed in \([0, T] \), \([-1, 1]\), and \([0, 2\pi]\), where \(T\) is the duration of each symbol,
4) the spreading sequences are uniformly and randomly chosen. With the Weyl spreading sequences, the parameter \(\sigma_k\) is uniformly and randomly chosen,
5) the matched filter is used in the correlation receiver.

The detail of asynchronous CDMA systems is shown in \([19]\), \([22]\), \([32]\).

We measure the average Bit Error Rate that
\[
\text{BER} = \frac{1}{KU} \sum_{k=1}^{K} \sum_{u=1}^{U} \text{BER}_{k,u},
\]
(45)

where \(U\) is the trial numbers, \(u\) is the \(u\)-th trial number and \(\text{BER}_{k,u}\) is the Bit Error Rate of the user \(k\) at the \(u\)-th trial. This section consists of three subsections. In the first subsection, we compare the spreading sequences obtained by Eq. (35) and (44) with other sequences, the Gold codes \([13]\), the optimal Chebyshev spreading sequences \([19]\) and the FZC sequence families \([24]\). In particular, we choose the triple \([p, q, r]\) = \([1.0, 1.0, 1.275]\) as the parameters of the FZC sequence families. This triple is shown in \([27]\) as the optimal parameters when \(N = 31\) with \(N\) being the length. In the second subsection, we compare the spreading sequences obtained by Eq. (35) and (41) with one obtained by Eq. (44). We compare the random assigning approach with the systematic approach. In the final subsection, we compare the spreading sequences obtained by Eq. (35) and (41) with the SP sequences \([34]\).

A. Comparison with Other Sequences

We consider the following spreading sequences:
\[
\tilde{w}_{k,n}(N, \gamma) = \exp(2\pi j n (\gamma + \sigma_k)) \quad (n = 1, 2, \ldots, N),
\]
\[
\tilde{w}_{k,n}(K, \gamma) = \exp(2\pi j n (\gamma + \sigma_k/K)) \quad (n = 1, 2, \ldots, N).
\]

The former sequence is obtained from Eq. (35) and the latter sequence is obtained from Eq. (31). In this section, the “Weyl” spreading sequence is \([\tilde{w}_{k,n}(N, \gamma)]\) and the “Optimal” one is \([\tilde{w}_{k,n}(K, \gamma)]\). Note that the optimal spreading sequences are different in the number of users \(K\). The “Upper Bound” is obtained from Eq. (38).

Figure 1 shows the relation between the number of users and Bit Error Rate (BER) when \(N = 31\) and \(E/N_0 = 25\)(db), where \(E\) is the energy per data bit. In this figure, we set \(\gamma = \frac{1}{2\pi}\), which is independent of the number of users \(K\). The BER of the Weyl spreading sequences is lower than the one of the Gold codes and the optimal Chebyshev spreading sequences. However, it is higher than one of the FZC sequence families. The upper bound is established when the number of the users \(K\) is larger than 20. On the other hands, the BER of the global solution of the problem \((P)\), the Optimal Weyl sequences \([\tilde{w}_{k,n}(K, \gamma)]\) is the lowest. These sequences are dramatically efficient when the number of the users \(K\) is fixed.

Figure 2 shows the relation between the \(E/N_0\)(db) and Bit Error Rate (BER) when \(K = 7\). We used the two cases for \(\gamma\): \(\gamma = \frac{1}{\pi}\) and \(\gamma = \frac{1}{2\pi}\). The former parameter \(\gamma = \frac{1}{\pi}\) is independent of \(K\) and the latter parameter \(\gamma = \frac{1}{2\pi}\) is dependent on \(K\). In this figure, the BER of the Weyl spreading sequences \([\tilde{w}_{k,n}(N, \frac{1}{\pi})]\) is lower than one of the FZC sequence families. This result shows that BER of the Weyl spreading sequences is changed when the value of \(\gamma\) is varied. However, the BER of the optimal sequence, which is the global solution of \((P)\), is independent of \(\gamma\).

B. Comparison with Systematic Approaches

In section V, we discussed how to assign the element \(\sigma_k\) to the user \(k\) and proposed the method to assign. We set the length \(N = 32\) and the parameter \(\gamma = \frac{1}{4\pi}\). We compare two types of
the spreading sequences. Figure 4 shows the relation between sequences whose $K_{\text{max}} = 14$ and one of the SP sequences are the same. Further, the BER of the Weyl spreading sequences whose $K_{\text{max}} = 14$ is the same BER in different $\gamma$. This result and the result of subsection (A) could suggest that the optimal parameter $\gamma$ depends on $N$, $K$ and $K_{\text{max}}$. The BER of the global solutions is lowest and each BER of them is the same in $\gamma = \frac{1}{2N}$ and $\frac{1}{2K}$. By taking into account these, we conclude that the BER of the global solution is independent of $\gamma$.

C. Comparison with SP Sequence

The Song-Park (SP) sequences have been proposed in [34]. We set the length $N = 30$ and the number of the users $K = 7$. Then, the maximum number of the users of the SP sequences is 14. Thus, we compare them with four types of the Weyl spreading sequences. We choose the parameters as $(K_{\text{max}}, \gamma) = \{(30, 1/(2N)), (30, 1/(2K)), (14, 1/(2N)), (14, 1/(2K))\}$. The parameter $K_{\text{max}} = 14$ represents the situation where the maximum number of the users is 14. Thus, the Weyl spreading sequences $\{w_k, 14, \gamma\}$ have the same feature to the SP sequences. Figure 4 shows the relation between the $E/N_0$(db) and BER. The BER of the Weyl spreading sequences whose $K_{\text{max}} = 30$ is higher than one of the SP sequences. However, the BER of the Weyl spreading sequences whose $K_{\text{max}} = 14$ and one of the SP sequences are the same. Further, the BER of the Weyl spreading sequences whose $K_{\text{max}} = 14$ is the same BER in different $\gamma$. This result and the result of subsection (A) could suggest that the optimal parameter $\gamma$ depends on $N$, $K$ and $K_{\text{max}}$. The BER of the global solutions is lowest and each BER of them is the same in $\gamma = \frac{1}{2N}$ and $\frac{1}{2K}$. By taking into account these, we conclude that the BER of the global solution is independent of $\gamma$.

VII. Conclusion

In this paper, we have defined the Weyl sequence class and shown the features of the sequences in the class. We have constructed the optimization problem: minimize the upper bound of the absolute value of the whole interference noise and derive the global solutions. From this solution, we can derive other sequences, the Sarwate’s sequences, the SP sequences. We have evaluated their SINR in the special case and shown the simulation results in an asynchronous CDMA system. From these results, the global solution is dramatically efficient when the number of the users $K$ is fixed. Moreover, performance of the global solution is independent of the parameter $\gamma$.

In the global solution of the problem ($P$), the parameter $\gamma$ is any real number. However, its BER depends on $\gamma$ when we
let the maximum number of users $K_{\text{max}}$ as $N$ or other number, not $K$. The remained issue is to investigate the optimal $\gamma$ and how to assign $\sigma_k$ successfully to the user $k$.

**APPENDIX A**

In this appendix, we prove that the global optimal solutions of $(P)$, $\rho_i^*$ and $t_{i,k}^*$ are given by

$$
\rho_i^* = \gamma + \frac{i - 1}{K} \quad (i = 1, 2, \ldots, K),
$$

$$
t_{i,k}^* = \min \left\{ \frac{|k - i|}{K}, 1 - \frac{|k - i|}{K} \right\},
$$

where $\gamma$ is a real number.

Since the problem $(P)$ is a convex programming, it is necessary and sufficient for the global solution to satisfy the KKT conditions, Eqs. (28)–(29).

When $\rho_i^*$ satisfies Eq. (28), it is clearly that

$$
v_i = 0 \quad (i = 1, 2, \ldots, K - 1),
$$

$$
\alpha_{i,k} = 0 \quad (i < k)
$$

since $e_i(x^*) < 0$ and $h_i(k^*) < 0$. We let $\xi_1 = \xi_K = 0$. Thus, it is sufficient to consider only two kinds of the Lagrange multipliers, $\lambda_{i,k}$ and $\mu_{i,k}$. They satisfy the following equation which is obtained from Eq. (26):

$$
- \sum_{i < k} \frac{\pi \cos(\pi t_{i,k}^*)}{\sin^2(\pi t_{i,k}^*)} \left( \begin{array}{c} 0 \\ e_{i,k} \end{array} \right) + \sum_{i < k} \lambda_{i,k} \left( \begin{array}{c} e_i - e_k \\ e_{i,k} \end{array} \right) + \sum_{i < k} \mu_{i,k} \left( \begin{array}{c} -e_i + e_k \\ e_{i,k} \end{array} \right) = 0,
$$

where $e_i \in \mathbb{R}^K$ have in the $i$-th element and 0 in the others and $e_{i,k} \in \mathbb{R}^{K(K-1)/2}$ have 1 in the $[i(2K - i - 1)/2 + k - K]$-th element and 0 in the others. From Eq. (48), we consider two vector equations. One is the first $K$-dimensional vector equation of Eq. (49) and the other is the last $K(1)/2$-dimensional vector equation. They are expressed as

$$
\sum_{i < k} (\lambda_{i,k} - \mu_{i,k})(e_i - e_k) = 0,
$$

$$
\sum_{i < k} \left( \frac{\pi \cos(\pi t_{i,k}^*)}{\sin^2(\pi t_{i,k}^*)} \lambda_{i,k} - \mu_{i,k} \right) e_{i,k} = 0.
$$

Then, we define $\alpha(t_{i,k}^*)$ as

$$
\alpha(t_{i,k}^*) = \frac{\pi \cos(\pi t_{i,k}^*)}{\sin^2(\pi t_{i,k}^*)}.
$$

Note that $\alpha(t_{i,k}^*) \geq 0$ since $0 < t_{i,k}^* \leq \frac{1}{2}$. From the definition of $t_{i,k}^*$, $\alpha(t_{i,k}^*)$ only depends on the absolute value of difference, $|k - i|$. We therefore rewrite $\alpha(t_{i,k}^*)$ as

$$
\alpha(t_{i,k}^*) = \tilde{\alpha}(k - i).
$$

The variable $\tilde{\alpha}(k - i)$ has the property that

$$
\tilde{\alpha}(k) = \tilde{\alpha}(K - k) \quad (1 \leq k \leq K).
$$

This result is obtained from the definition of $t_{i,k}^*$. We consider the two types of $K$: $K$ is an odd number or $K$ is an even number.

**A. $K$ is an odd number**

For all $i$ and $k$ $(i < k)$, $\rho_i^*$, $\rho_k^*$ and $t_{i,k}^*$ satisfy either only $c_{i,k}(x^*) = 0$ or $d_{i,k}(x^*) = 0$. They satisfy

$$
c_{i,k}(x^*) = 0, d_{i,k}(x^*) < 0, \quad (k - i < K/2),
$$

$$
d_{i,k}(x^*) = 0, c_{i,k}(x^*) < 0, \quad (k - i > K/2),
$$

$$
\lambda_{i,k} = \begin{cases} 
\tilde{\alpha}(k - i) & (k - i < K/2), \\
0 & (k - i > K/2)
\end{cases}
$$

$$
\mu_{i,k} = \begin{cases} 
0 & (k - i < K/2), \\
\tilde{\alpha}(k - i) & (k - i > K/2)
\end{cases}
$$

We consider the $n$-th element of the left side of Eq. (49):

$$
\sum_{n < k} (\lambda_{n,k} - \mu_{n,k}) - \sum_{i < n} (\lambda_{i,n} - \mu_{i,n}) = 0.
$$

We let

$$
\sum_{n < k} \tilde{\alpha}(k - n) + \sum_{n < i} \tilde{\alpha}(n - i) - \sum_{n < k} \tilde{\alpha}(K + n - n) = 0
$$

$$
\sum_{n < k} \tilde{\alpha}(n - n) - \sum_{n < k} \tilde{\alpha}(n - n) = 0.
$$

From Eq. (54), for all the integers $i$ and $k$, the term in summation of the left side of Eq. (50) equals 0. From the above proof, all the Lagrange multipliers satisfy Eq. (28).

**B. $K$ is an even number**

The Lagrange multipliers $\rho_i^*$, $\rho_k^*$ and $t_{i,k}^*$ satisfy

$$
c_{i,k}(x^*) = 0, d_{i,k}(x^*) < 0, \quad (k - i < K/2),
$$

$$
d_{i,k}(x^*) = 0, c_{i,k}(x^*) < 0, \quad (k - i > K/2),
$$

$$
d_{i,k}(x^*) = 0, c_{i,k}(x^*) = 0, \quad (k - i = K/2).
$$

When $k - i = K/2$, they satisfy $c_{i,k}(x^*) = 0$ and $d_{i,k}(x^*) = 0$. Thus, we set

$$
\lambda_{i,k} = \begin{cases} 
\tilde{\alpha}(k - i) & (k - i < K/2), \\
\tilde{\alpha}(k - i)/2 & (k - i = K/2), \\
0 & (k - i > K/2)
\end{cases}
$$

$$
\mu_{i,k} = \begin{cases} 
\tilde{\alpha}(k - i) & (k - i < K/2), \\
\tilde{\alpha}(k - i)/2 & (k - i = K/2), \\
\tilde{\alpha}(k - i) & (k - i > K/2)
\end{cases}
$$
Similar to the case that \( K \) is an odd number, we consider the \( n \)-th element of left side of Eq. (59):

\[
\sum_{n<k}^{\rho} (\lambda_{n,k} - \mu_{n,k}) = \sum_{\rho} (\lambda_{n,n} - \mu_{n,n})
\]

\[
= \sum_{n<k}^{\rho} \lambda_{n,k} - \sum_{\rho} \mu_{n,k} - \sum_{\rho} \lambda_{n,n} + \sum_{\rho} \mu_{n,n}
\]

\[
+ \sum_{n<k}^{\rho} \lambda_{n,k} - \sum_{\rho} \mu_{n,k} - \sum_{\rho} \lambda_{n,n} + \sum_{\rho} \mu_{n,n}
\]

The terms of the difference equaling \( K/2 \) vanish. Therefore, we obtain

\[
\sum_{n<k}^{\rho} (\lambda_{n,k} - \mu_{n,k}) = \sum_{\rho} (\lambda_{n,n} - \mu_{n,n})
\]

where

\[
\sum_{n<k}^{\rho} \lambda_{n,k} - \sum_{\rho} \mu_{n,k} - \sum_{\rho} \lambda_{n,n} + \sum_{\rho} \mu_{n,n}
\]

\[
= \sum_{n<k}^{\rho} \lambda_{n,k} - \sum_{\rho} \mu_{n,k} - \sum_{\rho} \lambda_{n,n} + \sum_{\rho} \mu_{n,n}
\]

\[
- \sum_{\rho} \lambda_{n,n} + \sum_{\rho} \mu_{n,n}
\]

\[
= \sum_{n<k}^{\rho} \lambda_{n,k} - \sum_{\rho} \mu_{n,k} - \sum_{\rho} \lambda_{n,n} + \sum_{\rho} \mu_{n,n}
\]

\[
- \sum_{\rho} \lambda_{n,n} + \sum_{\rho} \mu_{n,n}
\]

\[
= \sum_{n<k}^{\rho} \lambda_{n,k} - \sum_{\rho} \mu_{n,k} - \sum_{\rho} \lambda_{n,n} + \sum_{\rho} \mu_{n,n}
\]

\[
= \sum_{n<k}^{\rho} \lambda_{n,k} - \sum_{\rho} \mu_{n,k} - \sum_{\rho} \lambda_{n,n} + \sum_{\rho} \mu_{n,n}
\]

Thus, we have proven that Eq. (59) equals to 0. It is clearly that the left side of Eq. (50) equals 0 when \( k - i \neq K/2 \). When \( k - i = K/2 \), it follows that

\[
\hat{a}(K/2) = \frac{-\hat{a}(K/2)}{2} - \frac{\hat{a}(K/2)}{2} = 0.
\]

Thus, for all the integer \( i \) and \( k \), Eq. (50) is satisfied.

From the proofs A and B, we have proven that the existence of the Lagrange multipliers which satisfy Eq. (28). Therefore, \( \rho_i^* \) and \( t_i^{*,*} \) are the global solutions of the problem (P).

**Appendix B**

In this appendix, with the spreading sequences \( \{ \tilde{w}_{i,a}(N, \gamma) \} \), we prove that SINR of the user \( i \) is given by

\[
\text{SINR}_i = \left( R_i + \frac{N_0}{2E} \right)^{-1/2},
\]

where

\[
R_i = \frac{(K - 1)}{18N^2} \left\{ 2(N + 1) + (N - 2) \cos \left( 2\pi \left( \frac{\gamma + \sigma_i}{N} \right) \right) \right\}.
\]

We assume that the element \( \sigma_i \) is a random variable uniformly distributed in \( (0, 1, 2, \ldots, N - 1) \). This assumption is fulfilled when the ratio \( K/N \) is close to 1, that is, the number of users is sufficiently large since SINR is not the reciprocal of the average of the interference noise over the users. However, with the spreading sequences \( \{ \tilde{w}_{i,a}(N, \gamma) \} \), they are equivalent when the number of users \( K \) equals \( N \) (see Eq. (73)). Thus, it is conceivable that the assumption is established when the ratio \( K/N \) is close to 1.

The correlation function \( C_{i,k}(l) \) of the spreading sequences in Eq. (51) is

\[
C_{i,k}(l) = \begin{cases} 
-Z_{\sigma_i,\gamma_i} \Phi_{\gamma_i,\sigma_i,k}(l) & 0 \leq l \leq N - 1, \\
Z_{\sigma_i,\gamma_i} \Phi_{\gamma_i,\sigma_i,k}(l) & 1 - N \leq l < 0, \\
0 & \text{otherwise},
\end{cases}
\]

where

\[
Z_{\sigma_i,\gamma_i} = \frac{\exp(2\pi j \frac{\sigma_i}{N})}{1 - \exp(2\pi j \frac{\sigma_i}{N})}.
\]

and

\[
\Phi_{\gamma_i,\sigma_i,k}(l) = \exp(-2\pi j l \left( \gamma + \frac{\sigma_i}{N} \right)) - \exp(-2\pi j l \left( \gamma + \frac{\sigma_i}{N} \right)).
\]

Thus, we obtain the squared absolute value of \( C_{i,k}(l) \):

\[
|C_{i,k}(l)|^2 = \frac{1 - \cos \left( 2\pi \frac{l}{N} \frac{\sigma_i}{\sigma_i} \right)}{1 - \cos \left( 2\pi \frac{\sigma_i}{\sigma_i} \right)}.
\]

On the other hand, the following relations are satisfied:

\[
\sum_{l=0}^{N-1} |C_{i,k}(l - N)|^2 = \sum_{l=0}^{N-1} |C_{i,k}(l - N + 1)|^2 = \sum_{l=0}^{N-1} |C_{i,k}(l)\|^2
\]

\[
= \sum_{l=0}^{N-1} |C_{i,k}(l + 1)\|^2 = \frac{N}{1 - \cos \left( 2\pi \frac{\sigma_i}{\sigma_i} \right)},
\]

and

\[
\sum_{l=0}^{N-1} \text{Re} \left[ C_{i,k}(l - N) \overline{C_{i,k}(l - N + 1)} \right] = \sum_{l=0}^{N-1} \text{Re} \left[ C_{i,k}(l) \overline{C_{i,k}(l + 1)} \right].
\]

In the above equations, we used the assumption \( \sigma_i \neq \sigma_k \). From Eqs. (67) - (69), \( r_{i,k} \) in Eq. (57) is given by

\[
r_{i,k} = \frac{N}{1 - \cos \left( 2\pi \frac{\sigma_i}{\sigma_i} \right) - 4 \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right) + \cos \left( 2\pi \left( \gamma + \frac{\sigma_i}{N} \right) \right).}
\]

When we calculate the sum of Eq. (70), the first term of it is given by

\[
\sum_{k=1}^{N} \frac{4N}{1 - \cos \left( 2\pi \frac{\sigma_i}{\sigma_i} \right) - \sum_{k=i}^{N} \frac{2N}{1 - \cos \left( 2\pi \frac{\sigma_i}{N} \right)}}.
\]
The integer $\sigma_k \in \{0, 1, 2, \ldots, N-1\}$ is a random variable and satisfies $\sigma_k \neq \sigma_i$ when $k \neq i$. Thus, $\sigma_k$ is expressed as

$$\sigma_k = \sigma_i + q \mod N, \quad q \in \{1, 2, \ldots, N-1\}. \tag{72}$$

and we can treat $q$ as a random variable instead of $\sigma_k$. The integer $q$ is uniformly distributed in $\{1, 2, \ldots, N-1\}$. Thus, the average of Eq. (71) is

$$E \left\{ \sum_{k \neq i} \frac{2N}{\sin^2(\frac{\pi \sigma_k - \sigma_i}{N})} \right\} = \sum_{k \neq i} E \left\{ \frac{2N}{\sin^2(\frac{\pi \sigma_k - \sigma_i}{N})} \right\}$$

$$= \sum_{k \neq i} \frac{1}{N-1} \sum_{q=1}^{N-1} \frac{2N}{\sin^2(\frac{\pi \sigma_k - \sigma_i}{N})}$$

$$= \frac{K-1}{N-1} \sum_{q=1}^{N-1} \frac{2N}{\sin^2(\frac{\pi \sigma_k - \sigma_i}{N})} \tag{73}$$

where $E$ is the average over $\sigma_k$.

In [56], it is shown that

$$\frac{1}{3} \sum_{k=1}^{n-1} \frac{1}{\sin^2(\pi \frac{k}{n})} = \frac{n^2 - 1}{3} = \frac{(n-1)(n+1)}{3}. \tag{74}$$

Thus, Eq. (73) is equivalent to

$$\frac{K-1}{N-1} \sum_{q=1}^{N-1} \frac{2N}{\sin^2(\frac{\pi \sigma_k - \sigma_i}{N})} = \frac{2N(N+1)(K-1)}{3}. \tag{75}$$

From the above result, we obtain the following relation

$$E \left\{ \sum_{k \neq i} \frac{4N}{1 - \cos(\pi \frac{\sigma_k - \sigma_i}{N})} \right\} = \frac{2N(N+1)(K-1)}{3}. \tag{76}$$

The average of the second term of the sum of Eq. (70) is given by

$$\sum_{k=1 \atop k \neq i}^{N-1} \frac{\frac{1}{\sin^2(\pi \frac{\sigma_k - \sigma_i}{N})}}{1 - \cos(\pi \frac{\sigma_k - \sigma_i}{N})}$$

$$= \frac{K-1}{N-1} \sum_{q=1}^{N-1} \frac{N \cos(\pi \frac{\sigma_k - \sigma_i}{N})}{1 - \cos(\pi \frac{\sigma_k - \sigma_i}{N})} \tag{77}$$

Note that it is clear that

$$\sum_{q=1}^{N-1} \cos(\pi \frac{\sigma_k - \sigma_i}{N}) = 0. \tag{78}$$

Therefore, Eq. (77) is rewritten as

$$\frac{N}{N-1} \cos(\frac{2\pi}{N} \pi(\gamma + \sigma_i) N) \sum_{q=1}^{N-1} \left\{ \frac{1}{2 \sin^2(\pi \frac{q}{N})} - 1 \right\} \tag{79}$$

$$= \frac{N(N-1)\frac{N}{6} - 1}{\cos(\frac{2\pi}{N} \pi(\gamma + \sigma_i) N)} \tag{80}$$

Thus, the sum of the average of the second term in Eq. (70) is written as

$$\sum_{k=1 \atop k \neq i}^{N-1} \frac{N \cos(\pi \frac{\sigma_k - \sigma_i}{N})}{1 - \cos(\pi \frac{\sigma_k - \sigma_i}{N})} \tag{81}$$

Similarly, we obtain the average of the sum of the third term of Eq. (70):

$$\sum_{k=1 \atop k \neq i}^{N-1} \frac{N \cos(\pi \frac{\sigma_k - \sigma_i}{N})}{1 - \cos(\pi \frac{\sigma_k - \sigma_i}{N})} \tag{82}$$

Finally, we obtain the following relation

$$E \left\{ \sum_{k=1 \atop k \neq i}^{N-1} \frac{N \cos(\pi \frac{\sigma_k - \sigma_i}{N})}{1 - \cos(\pi \frac{\sigma_k - \sigma_i}{N})} \right\} \tag{83}$$

From Eq. (76) and Eq. (82), we arrive at SINR of the user $i$ with the spreading sequence $(\gamma_k)_{\sigma_k}$

$$\text{SINR}_{i} = \left( R_{i} + \frac{N_{0}}{2E} \right)^{-1/2}, \tag{84}$$

where

$$R_{i} = \frac{(K-1)}{18N^2} \left\{ 2(N+1) + (N-2) \cos(2\pi \gamma + \sigma_i N) \right\}. \tag{84}$$

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