Projective normality of special scrolls II.

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Abstract: We study the projective normality of a linearly normal special scroll \( R \) of degree \( d \) and speciality \( i \) over a smooth curve \( X \) of genus \( g \). We relate it with the Clifford index of the base curve \( X \). If \( d \geq 4g - 2i - \text{Cliff}(X) + 1 \), \( i \geq 3 \) and \( R \) is smooth, we prove that the projective normality of the scroll is equivalent to the projective normality of its directrix curve of minimum degree.

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Introduction.

Let \( R \subset P^N \) be a linearly normal special scroll of genus \( g \), speciality \( i \) and degree \( d \). We know that \( R \) has an associated ruled surface \( \pi: S \rightarrow X \) with a linear system \( |H| = |X_0 + b f| \), such that \( R \) is the image of \( S \) by the map defined by \( |H| \) (see [1]). We study the projective normality of \( R \), or equivalently, the normal generation of the invertible sheaf \( O_S(H) \) on \( S \).

In the previous paper [3] we gave the bound \( d \geq 4g - 2i + 1 \) to reduce the problem of the projective normality of the special scroll \( R \) to the problem of the projective normality of the curve of minimum degree.

In this paper we improve this result by using the Clifford index of the base curve \( X \). We will prove the following theorem:

Theorem Let \( R \subset P^N \) be a smooth special linearly normal scroll of genus \( g \), degree \( d \) and speciality \( i \geq 3 \). Let \( X \) be the base curve of the scroll.

If \( d \geq 4g - 2i - \text{Cliff}(X) + 1 \), then:

1. \( R \) has an unique special directrix curve \( X_0 \). Moreover, \( X_0 \) is the curve of minimum degree, it is linearly normal and it has the speciality of \( R \).

2. \( R \) and \( X_0 \) have the same speciality respect to hypersurfaces of degree \( m \).

In particular the scroll is projectively normal if and only if the curve of

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minimum degree is projectively normal.

We will see that this result is optimal. We also study the particular case of $R$ being a cone. In this case the bound $d \geq 4g - 2i - \text{Cliff}(X) + 1$ is equivalent to the bound $d \geq 2g - 2h^1(L) - \text{Cliff}(X)$, that M. Green and R. Lazarsfeld gave in [6] for a line bundle $L$ on a curve.

We refer to [1] for a systematic development of the projective theory of scrolls and ruled surfaces that we will use in this paper and to [2] to study the special scrolls. In the first section we recall some basic facts about the Clifford index of a curve, that we will use along the paper.

1 Preliminaries.

A geometrically ruled surface, or simply a ruled surface, will be a $\mathbb{P}^1$-bundle over a smooth curve $X$ of genus $g > 0$. It will be denoted by $\pi : S = \mathbb{P}(\mathcal{E}_0) \longrightarrow X$. We will suppose that $\mathcal{E}_0$ is a normalized sheaf and $X_0$ is the section of minimum self-intersection that corresponds to the surjection $\mathcal{E}_0 \longrightarrow \mathcal{O}_X(e) \longrightarrow 0$, $\bigwedge^2 \mathcal{E} \cong \mathcal{O}_X(e)$ and $e = -\text{deg}(e)$ (see [3], V, §2 and [1]).

If $|H| = |X_0 + b|$ is a base-point-free linear system on a ruled surface $S$, $|H|$ defines a regular map $\phi_H : S \longrightarrow \mathbb{P}(H^0(\mathcal{O}_S(H)^\vee))$. The image of $S$ is a scroll $R$. If $\phi_H$ is a birational map we say that $S$ and $H$ are the ruled surface and the linear system associated to the scroll $R$. We denote the image of a curve $C \subset S$ by $\overline{C} \subset R$. The curve $X_0$ is the curve of minimum degree of $R$. It is embedded by the linear system $|b + \mathfrak{t}|$ on $X$.

Let $X$ be a smooth curve of genus $g \geq 2$. Let $L$ be a line bundle on $X$. We define the Clifford index of $L$ by:

$$\text{Cliff}(L) = \text{deg}(L) - 2(h^0(L) - 1)$$

The Clifford index of the curve $X$ is defined by:

$$\text{Cliff}(X) = \min\{\text{Cliff}(L)/h^0(L) \geq 2, h^1(L) \geq 2\}$$

From this we have the following formula:

**Lemma 1.1** If $\mathfrak{b}$ is an effective special divisor such that $h^0(\mathcal{O}_X(\mathfrak{b})) \geq 2$ and $h^0(\mathcal{O}_X(\mathfrak{b})) \geq 2$ then

$$h^0(\mathcal{O}_X(\mathfrak{b})) \leq \frac{\text{deg}(\mathfrak{b}) - \text{Cliff}(X)}{2} + 1$$
By the Clifford Theorem, \( \text{Cliff}(X) \geq 0 \) with equality if and only if \( X \) is hyperelliptic; \( \text{Cliff}(X) = 1 \) if and only if either \( X \) is trigonal or a smooth plane quintic. Furthermore, if \( X \) is a general curve of genus \( g \) then \( \gamma = \left[ \frac{g-1}{2} \right] \), and in any event \( \gamma \leq \left[ \frac{g-1}{2} \right] \).

Note that if \( b \) is a divisor such that \( \mathcal{O}_X(b) \) provides the Clifford index of \( X \), the linear system \( |b| \) is base-point-free. In other case, if \( P \) is a base point of \( |b| \), then \( b - P \) is a divisor with a Clifford index less than \( \text{Cliff}(\mathcal{O}_X(b)) \).

2 Projective normality of a special scroll.

**Proposition 2.1** Let \( R \subset \mathbb{P}^N \) be a special linearly normal scroll of genus \( g \), degree \( d \) and speciality \( i \geq 3 \). Suppose that \( R \) is not a cone. Let \( X \) be the base curve of the scroll.

If \( d \geq 4g - 2i - \text{Cliff}(X) + 1 \), then \( R \) has an unique special directrix curve \( X_0 \). Moreover, \( X_0 \) is the curve of minimum degree, it is linearly normal and it has the speciality of \( R \).

**Proof:** Let \( S \) be the ruled surface and \( |H| = |X_0 + b| \) the linear system corresponding to the scroll \( R \). Let \( \gamma = \text{Cliff}(X) \) be the Clifford index of \( X \).

Since \( R \) is special, it has a special directrix curve (see [2]) so the curve \( X_0 \) of minimum degree of the scroll verifies \( \text{deg}(b + \epsilon) \leq 2g - 2 \). Furthermore, we know that:

\[
d - 2g + 2 + i = h^0(\mathcal{O}_S(H)) \leq h^0(\mathcal{O}_X(b + \epsilon)) + h^0(\mathcal{O}_X(b)) \tag{1}
\]

and

\[
i = h^1(\mathcal{O}_S(H)) \leq h^1(\mathcal{O}_X(b + \epsilon)) + h^1(\mathcal{O}_X(b)) \tag{2}
\]

Because \( R \) is not a cone, \( h^0(\mathcal{O}_X(b + \epsilon)) \geq 2 \). We will prove that \( \text{deg}(b) \geq 2g + 1 \) and then we will apply the Proposition 2.3 of [3].

1. Suppose that \( h^1(\mathcal{O}_X(b + \epsilon)) \geq 2 \). Then we can apply the formula of Lemma 1.1 to the divisor \( b + \epsilon \).

If \( \text{deg}(b) \leq 2g \), then we also can apply the Clifford Theorem ([7], page 343) to the divisor \( b \). From the inequality (1) we obtain:

\[
d - 2g + 2 + i \leq \frac{\text{deg}(b + \epsilon) - \gamma}{2} + 1 + \frac{\text{deg}(b)}{2} + 1 = \frac{d - \gamma}{2} + 2
\]

and then \( d \leq 4g - 2i - \gamma \) which contradicts the hypothesis.

2. Suppose that \( h^1(\mathcal{O}_X(b + \epsilon)) \leq 1 \). By hypothesis \( i \geq 3 \), so \( h^1(\mathcal{O}_X(b)) \geq 2 \).
If \( h^0(\mathcal{O}_X(b)) \geq 2 \) we can apply the formula of Lemma 1.1 to the divisor \( b \) and the Clifford Theorem to the divisor \( b + e \). From the inequality (2) we obtain \( d \leq 4g - 2i - \gamma \) which contradicts the hypothesis.

If \( h^0(\mathcal{O}_X(b)) \leq 1 \), we have that:

\[
1 \geq h^0(\mathcal{O}_X(b)) = \deg(b) + 1 + h^1(\mathcal{O}_X(b)) \\
\geq \deg(b) - g + i
\]

Furthermore, by Nagata Theorem \[4\] we know that \( \deg(e) \leq g \), so:

\[
\deg(b + e) \leq \deg(b) + g \leq 2g - i + 1 \tag{3}
\]

On the other hand, from the inequality (1) we have:

\[
d - 2g + 2 + i \leq 1 + \deg(b + c) - g + 1 + h^1(\mathcal{O}_X(b + e)) \leq 1 + \deg(b + e) - g + 1 + 1
\]

and because \( d \geq 4g - 2i - \gamma + 1 \),

\[
\deg(b + e) \geq 3g - i - \gamma
\]

Now, replacing the above expression at inequality (3) we obtain:

\[
2g - i + 1 \geq 3g - i - \gamma \implies \gamma \geq g - 1
\]

but the Clifford index verifies \( \gamma \leq \left\lfloor \frac{g-1}{2} \right\rfloor \).

\[\blacksquare\]

Remark 2.2 The inequality and the condition \( i \geq 3 \) are optimal in the following way:

Given a non hyperelliptic smooth curve \( X \), let \( a \) be a divisor such that \( \mathcal{O}_X(a) \) provides the Clifford index \( \gamma \) of \( X \). Let us consider the ruled surface \( S = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(a - K)) \). The linear system \( |X_0 + Kf| \) on \( S \) is base-point-free and defines a birational map \( \phi_H \). Let \( R \) be the image of \( S \) by the map \( \phi_H \). The degree of \( R \) is \( d = 2g - 2 + \deg(a) \) and the speciality is \( i = 1 + g - \frac{\deg(a) + \gamma}{2} \geq 3 \). From this:

\[
4g - 2i - \gamma = 4g - 2 - 2g + \deg(a) + \gamma - \gamma = d
\]

However, the scroll \( R \) has two special directrix curves: \( X_0 \) and \( X_1 \) defined by the linear systems \( |a| \) and \( |K| \) respectively.

On the other hand, we can also take the ruled surface \( S = X \times \mathbb{P}^1 \) and the linear system \( |X_0 + Kf| \). In this case, the corresponding scroll \( R \) has degree \( d = 4g - 4 \) and speciality 2. Since \( X \) is non hyperelliptic, \( \gamma = \text{Cliff}(X) \geq 1 \) and \( d \geq 4g - 2i - \gamma + 1 \), but the scroll \( R \) has a one dimensional family of special directrix curves.

\[\blacksquare\]
Proposition 2.3 Let $R \subset \mathbb{P}^N$ be a special linearly normal scroll of genus $g$, degree $d$ and speciality $i \geq 3$. Suppose that $R$ is a cone. Let $X$ be the base curve of the scroll.

If the unique singular point of $R$ is the vertex and $d \geq 4g - 2i - \text{Cliff}(X) + 1$, then $R$ is projectively normal.

Proof: We know that $S = \mathbb{P}(|O_X \oplus O_X(-b)|)$ is the ruled surface associated to $R$ and $R$ is given by the linear system $|X_1| = |X_0 + bf|$. Moreover, the degree of $R$ is $d = \deg(b)$ and the speciality is $i = g + h^1(O_X(b))$ (see [1]).

It is clear that $R$ is projectively normal iff $O_X(b)$ is normally generated (see [2]). Since the unique singular point of $R$ is the vertex, the linear system $|b|$ is very ample. Moreover,

$$
\deg(b) = d \geq 4g - 2i - \text{Cliff}(X) + 1 = 2g - 2h^1(O_X(b)) - \text{Cliff}(X) + 1
$$

Thus, we can apply the Green-Lazarsfeld Theorem (see [3]) to the divisor $b$ and we deduce that the cone is projectively normal. ■

Remark 2.4 Note that the condition $d \geq 4g - 2i - \text{Cliff}(X) + 1$ is optimal, because it is equivalent to the inequality $d \geq 2g - 2h^1(O_X(b)) - \text{Cliff}(X) + 1$ for the hyperplane section of the cone. This condition is the best possible for the projective normality of line bundles on curves (see [3]). ■

Theorem 2.5 Let $R \subset \mathbb{P}^N$ be a smooth special linearly normal scroll of genus $g$, degree $d$ and speciality $i \geq 3$. Let $X$ be the base curve of the scroll.

If $d \geq 4g - 2i - \text{Cliff}(X) + 1$, then:

1. $R$ has an unique special directrix curve $X_0$. Moreover, $X_0$ is the curve of minimum degree, it is linearly normal and it has the speciality of $R$.

2. $R$ and $X_0$ have the same speciality respect to hypersurfaces of degree $m$.

In particular the scroll is projectively normal if and only if the curve of minimum degree is projectively normal.

Proof: Let $S$ be the ruled surface and $|H| = |X_0 + bf|$ the linear system corresponding to the scroll $R$. Let $\gamma = \text{Cliff}(X)$ be the Clifford index of $X$.

The first assertion is the Proposition 2.1. From the proof of this Proposition we also know that $\deg(b) \geq 2g + 1$.

To prove the second statement we will apply the Proposition 2.1 of [4]. We will see that:

$$
s(b + c, \ldots, b + c, b, \ldots, b) = 0 \text{ for all } i, \text{ with } 0 \leq i \leq k - 1
$$
Reasoning as in the proof of the Theorem 2.4 of [3] it is sufficient to see that 
\( s(b, b + e) = 0 \) and in particular we only have to prove that (see Lemma 1.5 in [3]):

\[
h^1(O_X(b - (b + e))) < h^0(O_X(b + e)) - 1
\]

We distinguish two cases:

1. Suppose that \( h^1(O_X(-e)) \leq 1 \). It is sufficient to prove that \( h^0(O_X(b + e)) \geq 3 \). But this follows from the smoothness of the scroll:

   (a) If \( h^0(O_X(b + e)) = 0 \) the scroll is a cone.

   (b) \( h^0(O_X(b + e)) = 1 \) can not occur, because \( b + e \) is base-point-free.

   (c) If \( h^0(O_X(b + e)) = 2 \) the directrix curve of minimum degree is a line.

   Since the scroll is not rational, it must be a singular curve of the scroll.

2. Suppose that \( h^1(O_X(-e)) \geq 2 \):

   (a) If \( h^0(O_X(-e)) \leq 1 \), then by Riemann-Roch Theorem \( h^1(O_X(-e)) \leq g - e \). Moreover, we know that \( \text{deg}(b) \geq 2g + 1 \) and from this:

   \[
h^0(O_X(b + e)) - 1 = b - e - g + i \geq 2g + 1 - e - g + i > g - e \geq h^1(O_X(-e))
\]

   (b) If \( h^0(O_X(-e)) \geq 2 \) we can apply the formula of Lemma 1.1 to the divisor \(-e\):

   \[
h^0(O_X(-e)) \leq \frac{e - \gamma}{2} - 1, \text{ or equivalently, } h^1(O_X(-e)) \leq g - \frac{e + \gamma}{2}
\]

   Furthermore,

   \[
h^0(O_X(b + e)) - 1 = \text{deg}(b) - e - g + 1 + i - 1 = \text{deg}(b) - e - g + i
\]

   By hypothesis \( d \geq 4g - 2i - \gamma + 1 \), so \( 2\text{deg}(b) - e \geq 4g - 2i - \gamma + 1 \) and \( \text{deg}(b) \geq \frac{4g - 2i - \gamma + 1 + e}{2} \). Then:

   \[
h^0(O_X(b + e)) - 1 \geq \frac{4g - 2i - \gamma + 1 + e}{2} - e - g + i \geq g - \frac{e + \gamma}{2} + \frac{1}{2} \geq h^1(O_X(-e)) + \frac{1}{2} > h^1(O_X(-e))
\]
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