Explicit formula of a new class of $q$-Hermite-based Apostol-type polynomials and generalization

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Abstract: The present article deals with a recent study of a new class of $q$-Hermite-based Apostol-type polynomials introduced by Waseem A. Khan and Divesh Srivastava. We give their explicit formula and study a generalized class depending in any $q$-analogue generating function.

Keywords: $q$-Hermite-based Apostol-type polynomials, $q$-analogue Cauchy product, $f_q$-Hermite-based Apostol-type polynomials and numbers.

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1 Introduction

Throughout this work, $\mathbb{C}$ designates the field of complex numbers, $\mathbb{N}$ indicates the set of positive integers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. First we recall some concepts related to $q$-calculus, which we need in the development of this article. Let $(a, q) \in \mathbb{C}$ such that $|q| < 1$. The $q$-analog of $a$ is given by

$$[a]_q = \frac{1 - q^a}{1 - q},$$

and the $q$-factorial function is defined by

$$[n]_q! = \prod_{m=0}^{n-1} [m]_q = \frac{(q; q)_n}{(1 - q)^n}$$

1 - 2q
with \((q; q)_n = \prod_{m=1}^{n} (1 - m^q)\). The corresponding \(q\)-binomial coefficient is given by the relation
\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}.
\] (3)

Finally the \(q\)-exponential generating function is defined by
\[
e_q(t) = \sum_{n \geq 0} \frac{t^n}{[n]_q!} = \sum_{n \geq 0} \frac{(1 - q)^n}{(q; q)_n} t^n.
\] (4)

According to these notations, the \(q\)-Hermite polynomials \(H_{n,q}(x)\) are defined by means of the generating function (see [4, 5])
\[
F_q(x, t) = \sum_{n \geq 0} (-1)^n q^{(n)} \frac{e_q(x t) t^{2n}}{[2n]_q!} = \sum_{n \geq 0} H_{n,q}(x) \frac{t^n}{[n]_q!}.
\] (5)

Recently, Waseem A. Khan and Divesh Srivastava (see [5, 12–14]) introduced the generalized \(q\)-Hermite-based Apostol-type polynomials \(H^{(\alpha)}_{n,q}(x; a, b; \lambda; \mu, \nu)\) by means of the generating function
\[
\left(\frac{2^{\mu} t^\nu}{\lambda e_q(t) + a^b}\right)^\alpha \sum_{n \geq 0} (-1)^n q^{(n)} \frac{e_q(x t) t^{2n}}{[2n]_q!} = \sum_{n \geq 0} H^{(\alpha)}_{n,q}(x; a, b; \lambda; \mu, \nu) \frac{t^n}{[n]_q!}
\] (6)

with \(\alpha \in \mathbb{N}^*, \lambda, a, b \in \mathbb{C}\) and \(|t| < |\log(-\lambda)|\). Letting \(x = 0\) in the definition (6):
\[
H^{(\alpha)}_{n,q}(a, b; \lambda; \mu, \nu) = H^{(\alpha)}_{n,q}(0; a, b; \lambda; \mu, \nu)
\]
are so called \(q\)-Hermite-based Apostol-type numbers of order \(\alpha\) and generated by the function
\[
\left(\frac{2^{\mu} t^\nu}{\lambda e_q(t) + a^b}\right)^\alpha \sum_{n \geq 0} (-1)^n q^{(n)} \frac{t^{2n}}{[2n]_q!} = \sum_{n \geq 0} H^{(\alpha)}_{n,q}(a, b; \lambda; \mu, \nu) \frac{t^n}{[n]_q!}.
\] (7)

Other interesting links about \(q\)-Hermite-based Apostol-type numbers, \((p; q)\)-analogue type of Frobenius Genocchi numbers and polynomials and \(q\)-analogue of Hermite poly-Bernoulli numbers and polynomials are illustrated in the works [6–11] of Waseem A. Khan et al.

## 2 Explicit formula of generalized \(q\)-Hermite-based Apostol-type polynomials

The generalized \(q\)-Apostol type polynomials \(F^{(\alpha)}_{n,q}(x; a, b; \lambda)\) of order \(\alpha \in \mathbb{N}^*\) are defined by means of the generating function
\[
\left(\frac{2^{\mu} t^\nu}{\lambda e_q(t) + a^b}\right)^\alpha e_q(x t) = \sum_{n \geq 0} F^{(\alpha)}_{n,q}(x; a, b; \lambda) \frac{t^n}{[n]_q!}
\] (8)

and the generalized \(q\)-Apostol type numbers \(F^{(\alpha)}_{n,q}(a, b; \lambda) = F^{(\alpha)}_{n,q}(0; a, b; \lambda)\) are given by the generating function
\[
\left(\frac{2^{\mu} t^\nu}{\lambda e_q(t) + a^b}\right)^\alpha = \sum_{n \geq 0} F^{(\alpha)}_{n,q}(a, b; \lambda) \frac{t^n}{[n]_q!}.
\] (9)
Based on Cauchy product of series (see [1]); the \( q \)-analog Cauchy product of formal \( q \)-analog generating functions

\[
\sum_{n \geq 0} a_n \frac{t^n}{[n]_q!} \text{ and } \sum_{n \geq 0} b_n \frac{t^n}{[n]_q!}
\]
is given by the following relation

\[
\left( \sum_{n \geq 0} a_n \frac{t^n}{[n]_q!} \right) \left( \sum_{n \geq 0} b_n \frac{t^n}{[n]_q!} \right) = \sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k}_q a_k b_{n-k} \frac{t^n}{[n]_q!}.
\]  \( (10) \)

Regarding the generating function of generalized \( q \)-Hermite-based Apostol-type polynomials; \( H_F^{(\alpha)}(x; a, b; \lambda, \mu, \nu) \) follows from \( q \)-analog Cauchy product of \( \left( \frac{q^{\nu t} e^{\lambda x}}{x q(t + a^q)} \right)^\alpha \) and \( F_q(x, t) \). By means of identity (10) we have

\[
H_F^{(\alpha)}(x; a, b; \lambda, \mu, \nu) = \sum_{k=0}^{n} \binom{n}{k}_q F_{k,q}^{(\alpha)}(a, b; \lambda) H_{n-k,q}(x). \]  \( (11) \)

To get explicit formula of \( H_F^{(\alpha)}(x; a, b; \lambda, \mu, \nu) \) we must compute the corresponding explicit formulae of numbers \( F_{n,q}^{(\alpha)}(a, b; \lambda) \) and polynomials \( H_{n,q}(x) \).

### 2.1 Explicit formula of \( q \)-Hermite polynomials

\( q \)-Hermite polynomials follow from \( q \)-analog Cauchy product of

\[
e_q(x t) \text{ and } F_q(t) = \sum_{n \geq 0} (-1)^n q^{\binom{n}{2}} \frac{t^n}{[n]_q!}.
\]

Explicitly we have the following theorem.

**Theorem 2.1.**

\[
H_{n,q}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k q^{\binom{k}{2}} \binom{n}{2k}_q x^{n-2k}.
\]  \( (12) \)

**Proof.** First let the sequence \( a_n \) be given by

\[
a_n = \frac{1}{2} (1 + (-1)^n) (-1)^{\lfloor \frac{n}{2} \rfloor} q^{\binom{\lfloor \frac{n}{2} \rfloor}{2}}.
\]

Then

\[
F_q(t) = \sum_{n \geq 0} a_n \frac{t^{2n}}{[2n]_q!}
\]

and

\[
F_q(x, t) = \left( \sum_{n \geq 0} a_n \frac{t^n}{[n]_q!} \right) \left( \sum_{n \geq 0} x^n \frac{t^n}{[n]_q!} \right).
\]

Thus

\[
F_q(x, t) = \sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k}_q a_k x^{n-k} \frac{t^n}{[n]_q!},
\]
\[
\sum_{k=0}^{n} \binom{n}{k} q^k a_k x^{n-k} = \sum_{k=0}^{\left\lfloor \frac{n}{2k} \right\rfloor} (-1)^k q^k \binom{n}{2k} q^{n-2k}
\]
and the result follows. \(\square\)

### 2.2 \(\alpha\)-power \(q\)-analog generating function

To compute the explicit formula of \(\alpha\)-power \(q\)-analog generating function, we must revisit some advanced studies in this area. Consider the formal generating function \(f(t) = \sum_{n \geq 0} a_n t^n\) with the coefficients \(a_n\) are numbers or polynomials and the first term \(a_0 \neq 0\). Then \(f^\alpha(t)\) is a generating function too, with hint of umbral calculus we noted in [3] that

\[
f^\alpha(t) = \sum_{n \geq 0} \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} a_{\alpha_1} \cdots a_{\alpha_n} t^n.
\]  

(13)

In the general case \(\alpha \in \mathbb{C}^*\); an improvement of this result is given in our recent work [2], where

\[
f^\alpha(t) = a_0^\alpha + \sum_{n \geq 1} \sum_{k=1}^{n} \binom{\alpha}{k} \left( \begin{array}{c} k \\ k_1, \ldots, k_n \end{array} \right) a_0^{\alpha-k} a_{k_1}^{k_1} \cdots a_{k_n}^{k_n},
\]  

(14)

\(s_n(k)\) is the set of all \((k_1, \ldots, k_n) \in \mathbb{N}^n\) satisfying conditions \(k_1 + \cdots + k_n = k\) and \(k_1 + 2k_2 + \cdots + nk_n = n\). It is obvious to remark that \(k_j = 0\) for \(j \geq n-k+1\) and \(s_n(k)\) reduces to \((n-k+1)\)-uplet \((k_1, \ldots, k_{n-k+1})\). We conclude that

\[
f^\alpha(t) = a_0^\alpha + \sum_{n \geq 1} \sum_{k=1}^{n} (\alpha)_{k} a_0^{\alpha-k} B_{n,k} \left( \frac{1}{n!} \right) \left( \frac{1}{t^n} \right) a_{n-k+1}^{t^n}.
\]  

(15)

\(B_{n,k}\) are exponential partial Bell polynomials given by the expression

\[
B_{n,k} (x_1, \ldots, x_{n-k+1}) = \frac{n!}{k!} \sum_{s_n(k)} \left( \begin{array}{c} k \\ k_1, \ldots, k_{n-k+1} \end{array} \right) \prod_{r=1}^{n-k} \left( \frac{x_r}{r!} \right)^{k_r}
\]  

(16)

and defined by means of the generating function

\[
\frac{1}{k!} \left( \sum_{m \geq 1} \sum_{m \geq k} x_m z^m m^k m! \right) = \sum_{n \geq k} B_{n,k} (x_1, \ldots, x_{n-k+1}) \frac{z^n}{n!}.
\]  

(17)

Stirling numbers \(S_2(n, k)\) obtained by the function

\[
\frac{1}{k!} \left( e^t - 1 \right)^k = \sum_{n \geq 0} S_2(n, k) \frac{t^n}{n!}
\]  

(18)

are special case of \(B_{n,k}\) and we have \(B_{n,k} (1, \ldots, 1) = S_2(n, k)\). Consequently these polynomials admit the following formulation

\[
S_2(n, k) = \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} j^n.
\]  

(19)

According to exponential partial Bell polynomials, the explicit formula of \(q\)-analog generating function \(f_q(t) = \sum_{n \geq 0} b_n \frac{t^n}{[n]_q!}\) is given by the following theorem.
Proposition 2.3. According to Theorem 2.2 it follows that
\[ c_n = \sum_{k=1}^{n} (-\alpha)_k (\lambda + a^b)^{-\alpha-k} \frac{[n]_q!}{n!} \lambda^k B_{n,k} \left( \frac{r!b_r}{(q, q)_r} \right) (1-q)^n \frac{t^n}{[n]_q!}. \] (21)

Proof. The series expansion of \( \lambda e_q(t) + a^b \) is
\[ \lambda e_q(t) + a^b = \lambda + a^b + \sum_{n \geq 1} \lambda \frac{t^n}{[n]_q!}. \]
Then
\[ (\lambda e_q(t) + a^b)^{-\alpha} = (\lambda + a^b)^{-\alpha} + \sum_{n \geq 1} \sum_{k=1}^{n} (-\alpha)_k (\lambda + a^b)^{-\alpha-k} \frac{[n]_q!}{n!} \lambda^k B_{n,k} \left( \frac{r!b_r}{(q, q)_r} \right) (1-q)^n \frac{t^n}{[n]_q!}. \]
Furthermore \( c_0 = (\lambda + a^b)^{-\alpha} \) and for \( n \geq 1; \)
\[ c_n = \sum_{k=1}^{n} (-\alpha)_k (\lambda + a^b)^{-\alpha-k} \frac{[n]_q!}{n!} \lambda^k (1-q)^n B_{n,k} \left( \frac{r!}{(q, q)_r} \right). \]
Corollary 2.3.1. We have \( F_{n,q}^{(a)} (a, b; \lambda) = 0 \) for \( n < \nu \alpha \), \( F_{n,q}^{(a)} (a, b; \lambda) = 2^{\mu \alpha} (\lambda + a^b)^{-\alpha} [\nu \alpha]_q! \) and for \( n > \nu \alpha \):

\[
F_{n,q}^{(a)} (a, b; \lambda) = 2^{\mu \alpha} [n]_q! \sum_{k=1}^{n-\nu \alpha} \frac{(-\alpha)_k (\lambda + a^b)^{-\alpha-k}}{(n-\nu \alpha)!} (1-q)^{n-\nu \alpha} \lambda^k B_{n-\nu \alpha, k} \left( \frac{r^l}{(q, q)_r} \right). 
\]

(22)

Proof. We have

\[
\left( \frac{2^{\mu \nu}}{\lambda e_q(t) + a^b} \right)^\alpha = 2^{\mu \alpha} t^{\nu \alpha} \left( \frac{1}{\lambda e_q(t) + a^b} \right)^\alpha.
\]

Then,

\[
\left( \frac{2^{\mu \nu}}{\lambda e_q(t) + a^b} \right)^\alpha = 2^{\mu \alpha} t^{\nu \alpha} \left( \lambda + a^b \right)^{-\alpha} + \sum_{n \geq 1} c_n \frac{t^n}{[n]_q!}.
\]

Furthermore,

\[
\left( \frac{2^{\mu \nu}}{\lambda e_q(t) + a^b} \right)^\alpha = 2^{\mu \alpha} \left( \lambda + a^b \right)^{-\alpha} t^{\nu \alpha} + 2^{\mu \alpha} \sum_{n \geq \nu \alpha+1} c_{n-\nu \alpha} \frac{t^n}{[n-\nu \alpha]_q!}.
\]

Finally,

\[
\sum_{n \geq 0} F_{n,q}^{(a)} (a, b; \lambda) \frac{t^n}{[n]_q!} = 2^{\mu \alpha} \left( \lambda + a^b \right)^{-\alpha} [\nu \alpha]_q! \frac{t^{\nu \alpha}}{[\nu \alpha]_q!} + 2^{\mu \alpha} \sum_{n \geq \nu \alpha} \frac{[n]_q! c_{n-\nu \alpha}}{[n-\nu \alpha]_q!} \frac{t^n}{[n]_q!}.
\]

Then \( F_{n,q}^{(a)} (a, b; \lambda) = 0 \) for \( n < \nu \alpha \), \( F_{n,q}^{(a)} (a, b; \lambda) = 2^{\mu \alpha} (\lambda + a^b)^{-\alpha} [\nu \alpha]_q! \) and for \( n \geq \nu \alpha \) we have

\[
F_{n,q}^{(a)} (a, b; \lambda) = 2^{\mu \alpha} \frac{[n]_q!}{[n-\nu \alpha]_q!} c_{n-\nu \alpha}.
\]

Substitute the value of \( c_{n-\nu \alpha} \) to get the desired result. \( \square \)

We have already found the necessary tools for computing the explicit formula of \( q \)-Hermite-based Apostol-type polynomial.

Theorem 2.4. If \( \lambda + a^b \neq 0 \) we have \( H_{F_{n,q}^{(a)}} (x; a, b; \lambda; \mu, \nu) = 0 \) for \( n < \nu \alpha \) and for \( n \geq \nu \alpha \):

\[
H_{F_{n,q}^{(a)}} (x; a, b; \lambda; \mu, \nu) = 2^{\mu \alpha} \left( \frac{n^\nu \alpha}{\lambda} \right)_q \sum_{l=0}^{n-\nu \alpha} (-1)^l q^{(l)}(n-\nu \alpha) \frac{(n-\nu \alpha)!}{2l} \right)_q x^{n-\nu \alpha-2l}
\]

\[
+ 2^{\mu \alpha} \sum_{l=1}^{n} \binom{n}{k}_q \frac{(-1)^l q^{(l)}}{2l} \left[ k! \right] (1-q)^{k-\nu \alpha} (-1)^l \lambda^j (-\lambda)_j
\]

\[
\times q^{(l)}(\lambda + a^b)^{-\nu \alpha} \left( \frac{r^l}{(q, q)_r} \right) B_{k-\nu \alpha, j} \left( \frac{r^l}{(q, q)_r} \right) x^{n-\nu \alpha-2l},
\]

where \( \sum_1 \) is the triple sum \( \sum_{k=\nu \alpha}^{n} \sum_{j=1}^{n-\nu \alpha} \sum_{l=0}^{\left[ \frac{n-k}{2} \right]} \).

Proof. Since

\[
H_{F_{n,q}^{(a)}} (x; a, b; \lambda; \mu, \nu) = \sum_{k=\nu \alpha}^{n} \binom{n}{k}_q F_{k,q}^{(a)} (a, b; \lambda) H_{n-k,q}(x)
\]

98
and
\[ H_{n-k,q}(x) = \sum_{l=0}^{[\frac{n-k}{2}]} (-1)^l q^\binom{l}{2} \binom{n-k}{2l} q x^{n-k-2l}. \]

Then
\[ H^{(\alpha)}_{n,q}(x; a, b; \lambda, \mu, \nu) = 2^{\mu\nu} (\lambda + a^b)^{-\alpha}[\nu\alpha]_q! \binom{n}{\nu\alpha}_q H_{n-\nu\alpha,q}(x) \]
\[ + \sum_{k=\nu\alpha+1}^{n} \binom{n}{k} F^{(\alpha)}_{k,q}(a, b; \lambda) H_{n-k,q}(x) \]
and the desired result follows. \[\square\]

**Remark 2.5.** In the case \( \lambda + a^b = 0 \) and \( \lambda \neq 0 \); the result is totally different. We write
\[ \lambda e_q(t) + a^b = t \sum_{n \geq 0} \frac{t^n}{[n+1]_q[1]} \lambda \sum_{n \geq 0} \frac{t^n}{[n+1]_q[1]} \lambda. \]

We consider \( \nu \geq 1 \), then we will have
\[ \left( \frac{2^\mu \nu}{\lambda e_q(t) + a^b} \right)^\alpha = \left( \frac{2^\mu}{\lambda} \right)^\alpha \nu^{\alpha - \alpha} \left( \frac{1}{\sum_{n \geq 0} \frac{t^n}{[n+1]_q[1]}} \right)^\alpha. \]

But
\[ \left( \frac{1}{\sum_{n \geq 0} \frac{t^n}{[n+1]_q[1]}} \right)^\alpha = 1 + \sum_{n \geq 1} \sum_{k=1}^{n} (-\alpha)_k [n]_q! \frac{n!}{n!} B_{n,k} \left( \frac{r!}{[n+1]_q(q,q)_r} \right) \times (1 - q)^n \frac{t^n}{[n]_q}. \]

Then
\[ \left( \frac{2^\mu \nu}{\lambda e_q(t) + a^b} \right)^\alpha = \left( \frac{2^\mu}{\lambda} \right)^\alpha \nu^{\alpha - \alpha} + \left( \frac{2^\mu}{\lambda} \right)^\alpha \sum_{n \geq 1} \sum_{k=1}^{n} (-\alpha)_k [n]_q! \frac{n!}{n!} \]
\[ \times B_{n,k} \left( \frac{r!}{[n+1]_q(q,q)_r} \right) \left( 1 - q \right)^n \frac{t^{n+\nu\alpha - \alpha}}{[n]_q}. \]

and
\[ \left( \frac{2^\mu \nu}{\lambda e_q(t) + a^b} \right)^\alpha = \left( \frac{2^\mu}{\lambda} \right)^\alpha \nu + \left( \frac{2^\mu}{\lambda} \right)^\alpha \sum_{n \geq 1} \sum_{k=1}^{n-c} (-\alpha)_k [n-c]! \frac{(1 - q)^{n-c}}{(n-c)!} \]
\[ \times B_{n-c,k} \left( \frac{r!}{[n-c+1]_q(q,q)_r} \right) t^n, \]

where \( c = \nu\alpha - \alpha \). Let us write
\[ \left( \frac{2^\mu \nu}{\lambda e_q(t) + a^b} \right)^\alpha = \sum_{n \geq 0} d_n \frac{t^n}{[n]_q}. \]

Then \( d_n = 0 \) for \( n < c \), \( d_c = \left( \frac{2^\mu}{\lambda} \right)^\alpha [c]_q! \), and for \( n \geq c + 1 \) we have
\[ d_n = \left( \frac{2^\mu}{\lambda} \right)^\alpha \sum_{k=1}^{n-c} (-\alpha)_k [n]_q! \frac{r!}{(n-c)!} B_{n-c,k} \left( \frac{r!}{[n-c+1]_q(q,q)_r} \right) \times (1 - q)^{n-c}. \]
By means of the identity (11) we will have \( H_{n,q}^{(a)}(x; a, b; \lambda; \mu, \nu) = \sum_{k=c}^{n} \binom{n}{k}_q d_k H_{n-k,q}(x) \). Finally for \( n \geq c \)

\[
H_{n,q}^{(a)}(x; a, b; \lambda; \mu, \nu) = \left( \frac{2^\alpha}{\lambda} \right)^a \left[ \frac{\alpha}{q} \right]_q! \sum_{l=0}^{\left\lfloor \frac{n-c}{2} \right\rfloor} (-1)^l q^{(l)}(2l)_q x^{n-c-2l} + \\
\left( \frac{2^\alpha}{\lambda} \right)^a \sum_{k=0}^{n-c} \sum_{j=1}^{k-c} \sum_{l=0}^{\left\lfloor \frac{n-k}{2} \right\rfloor} \binom{n}{k}_q \binom{n-k}{2l}_q (-1)^l q^{(l)}(-\alpha)_j [k]_q! B_{k-c,j} \left( \frac{r!}{(k-c+1)q(q)_r} \right) x^{n-k-2l}
\]

3 Generalized \( f_q \)-Hermite-based Apostol-type polynomials

Let \( \alpha \neq 0 \) be a complex number and \( \beta \) real number. We consider the formal \( q \)-analog generating function

\[
f_q(t) = \sum_{n \geq 0} b_n \frac{t^n}{[n]_q}.
\]

with the condition that \( b_0 \) is different from zero. A natural generalization of \( q \)-Hermite-based Apostol-type polynomials is given by the following definition

**Definition 3.1.** The \( f_q \)-Hermite-based Apostol-type polynomials \( H_{n,f_q}^{(a)}(x; a, b; \lambda; \mu, \nu) \) are given by the generating function

\[
\beta t^m f_q(t) e_q(x t) \sum_{n \geq 0} (-1)^n q^{(n)}(2) \frac{t^{2n}}{[2n]_q!} = \sum_{n \geq 0} H_{n,f_q}^{(a)}(x; a, b; \lambda; \mu, \nu) \frac{t^n}{[n]_q}.
\]

(23)

Thereafter the \( f_q \)-Hermite-based Apostol-type numbers \( F_{n,f_q}^{(a)}(\beta; c) = F_{n,f_q}^{(a)}(0; \beta; c) \) are given by the generating function

\[
\beta t^m f_q(t) \sum_{n \geq 0} (-1)^n q^{(n)}(2) \frac{t^{2n}}{[2n]_q!} = \sum_{n \geq 0} F_{n,f_q}^{(a)}(\beta; c) \frac{t^n}{[n]_q},
\]

(24)

For \(-\alpha \in \mathbb{N}^*, \beta = 2^{\alpha}, m = \alpha \) and \( f_q(t) = \lambda e_q(t) + a^b; F_{n,f_q}^{(a)}(x; \beta) = H_{n,q}^{(a)}(x; a, b; \lambda; \mu, \nu). \)

Polynomials \( F_{n,f_q}^{(a)}(x; \beta; c) \) follows from \( q \)-analog Cauchy product of generating functions

\[
e_q(x t) \sum_{n \geq 0} (-1)^n q^{(n)}(2) \frac{t^{2n}}{[2n]_q!} = \sum_{n \geq 0} H_{n,q}(x) \frac{t^n}{[n]_q},
\]

and

\[
\beta t^m f_q(t) = \sum_{n \geq 0} b_n \frac{t^n}{[n]_q}.
\]

The closed formula of polynomial \( F_{n,f_q}^{(a)}(x; \beta; c) \) is established in the following theorem.

**Theorem 3.2.**

\[
F_{n,f_q}^{(a)}(x; \beta; c) = \beta b_0 \binom{n}{m}_q \frac{[\frac{n-m}{2}]}{m!} \sum_{l=0}^{\left\lfloor \frac{n-m}{2} \right\rfloor} (-1)^l q^{(l)}(2l)_q x^{n-m-2l} + \\
\beta \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{k}_q \binom{n-k}{2l}_q (-\alpha)_j [k]_q! B_{k-m,j} \left( \frac{r! b_r}{(k-m)!} (1-q)^{k-m} (-1)^l q^{(l)}(2l)_q \right) x^{n-k-2l},
\]

where \( \sum_2 \) is the triple sum \( \sum_{r=m+1}^{n} \sum_{j=1}^{k-m} \sum_{l=0}^{\left\lfloor \frac{n-k}{2} \right\rfloor} \).
Proof. Since

\[ f^\alpha_q(t) = b_0^\alpha + \sum_{n \geq 1} \sum_{k=1}^{n} \frac{(\alpha)_k b_0^{a-k}}{n!} B_{n,k} \left( \frac{r!b_r}{(q,q)_r} \right) (1-q)^n t^n, \]

then

\[ t^m f^\alpha_q(t) = b_0^\alpha [m]_q! t^m + \sum_{n \geq m+1} \sum_{k=1}^{n-m} \frac{(\alpha)_k b_0^{a-k}[n]_q!}{(n-m)!} B_{n-m,k} \left( \frac{r!b_r}{(q,q)_r} \right) (1-q)^{n-m} t^n [n]_q!. \]

Writing

\[ \beta t^m f^\alpha_q(t) = \sum_{n \geq 0} c_n \frac{t^n}{[n]_q!}, \]

then \(c_n = 0\) for \(n < m\), \(c_m = \beta b_0^\alpha [m]_q!\) and for \(n > m\) we have

\[ c_n = \beta \sum_{k=1}^{n-m} \frac{(\alpha)_k b_0^{a-k}[n]_q!}{(n-m)!} B_{n-m,k} \left( \frac{r!b_r}{(q,q)_r} \right) (1-q)^{n-m} \]

Thereafter in means of \(q\)-analogue Cauchy product of generating functions we have

\[ F_{n,f}^\alpha(x; \beta; c) = \sum_{r=m}^{n} \binom{n}{k} \gamma_H_{n-k}(x) \]

and the desired result follows.

\[ \square \]

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