A tensorial Lax pair equation and integrable systems in relativity and classical mechanics

Kjell Rosquist
RGGR group, Chimie-Physique, CP231, ULB Campus Plaine, Brussels, Belgium
and
Department of Physics, Stockholm University, Box 6730, 113 85 Stockholm, Sweden

ABSTRACT
It is shown that the Lax pair equation \( \dot{L} = [L, A] \) can be given a neat tensorial interpretation for finite-dimensional quadratic Hamiltonians. The Lax matrices \( L \) and \( A \) are shown to arise from third rank tensors on the configuration space. The second Lax matrix \( A \) is related to a connection which characterizes the Hamiltonian system. The Toda lattice system is used to motivate the definition of the Lax pair tensors. The possible existence of solutions to the Einstein equations having the Lax pair property is discussed.

1 Introduction

A characteristic feature in the study of integrable systems is the existence of a pair of matrices (Lax pair), \( L, A \), satisfying the Lax equation

\[
\dot{L} = [L, A].
\]  

(1)

It follows from the Lax equation that the quantities \( I_k = k^{-1} \text{Tr} L^k \) are constants of the motion. If sufficiently many of the \( I_k \) are independent and in involution (i.e. \([I_k, I_\ell] = 0\) for all \( k, \ell \)) then the system is said to be integrable in the Liouville sense. In this letter we give a new characterization of the Lax pair property in terms of a tensorial equation valid for any finite-dimensional quadratic Hamiltonian system. In the new formulation the matrix \( A \) is related to a certain dynamical connection. Geometric interpretations of the Lax pair property have appeared before \cite{2, 3}. However, the approach proposed in this contribution is more direct in the sense that it relates the Lax pair property to configuration space tensors rather than phase space tensors.

We shall consider finite-dimensional Hamiltonians with a quadratic kinetic energy, \( H = \frac{1}{2} h^{\alpha\beta} p_\alpha p_\beta + V(q) \). In typical problems in classical mechanics, the kinetic metric is positive definite and constant. In general \( h^{\alpha\beta} \) has an arbitrary signature and depends on the configuration variables \( q^\alpha \). Our approach is to transform the system to a geometric form whereby the orbits will correspond to geodesics in a certain dynamical geometry. We look for a way to write the Lax pair equation in a covariant form with respect to that dynamical geometry. As a guide in this search we shall use the three particle Toda lattice with free ends. Its Hamiltonian is given by

\[
H = \frac{1}{2} \left( p_1^2 + p_2^2 + p_3^2 \right) + V_3 + V_1,
\]

\[
V_1 = a_1^2, \quad V_3 = a_3^2,
\]

\[
a_1 = e^{q_2 - q_3}, \quad a_3 = e^{q_1 - q_2}.
\]  

(2)

*to appear in the proceedings of the 7th Marcel Grossman Meeting on General Relativity
†gr-qc/9410011
The system has a Lax matrix formulation $\dot{\mathbf{L}} \equiv \{\mathbf{L}, H\} = [\mathbf{L}, \mathbf{A}]$ where

$$
\mathbf{L} = \begin{pmatrix} p_1 & a_3 & 0 \\ a_3 & p_2 & a_1 \\ 0 & a_1 & p_3 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & a_3 & 0 \\ -a_3 & 0 & a_1 \\ 0 & -a_1 & 0 \end{pmatrix}.
$$

(3)

The Lax matrix gives rise to the three invariants

$$
I_1 = \text{Tr} \mathbf{L} = p_1 + p_2 + p_3, \\
I_2 = \frac{1}{2} \text{Tr} \mathbf{L}^2 = H, \\
I_3 = \frac{1}{3} \text{Tr} \mathbf{L}^3 = I_1 I_2 - \frac{1}{6} I_1^3 + \hat{I}_3,
$$

(4)

where

$$
\hat{I}_3 = p_1 p_2 p_3 - V_1 p_1 - V_3 p_3.
$$

(5)

These invariants are in involution and the system is therefore Liouville integrable. We mention that the invariants $I_k$ can be interpreted as coadjoint invariants of the solvable Lie group of real upper triangular matrices with unit determinant [4, 5, 6]. In order to obtain a tensorial formulation of the Lax pair equation we formulate the Toda system geometrically in terms of a certain dynamical metric.

### 2 The dynamical geometry

We now discuss how the dynamical metric may be defined. For any given Hamiltonian system which is not explicitly time dependent ($\partial H/\partial t = 0$) let us reparametrize the time variable according to the recipe $dt \to N^{-1} dt$, $H \to \mathcal{H}_N = NH [7, 8]$, where $N$ is some function on the phase space. Hamilton’s equations with respect to the new Hamiltonian are equivalent to the original equations of motion at zero energy ($H = 0$). To transform the dynamics for a nonzero value of the energy one must first subtract that value from the original Hamiltonian before making the time reparametrization. Suppose now that we are dealing with a Hamiltonian of the type $H = \frac{1}{2} h^{\alpha \beta} p_\alpha p_\beta + V(q)$. The special choice $N = [2(E - V)]^{-1}$ leads to the Jacobi time gauge in which the potential energy of the reparametrized Hamiltonian, $\hat{\mathcal{H}}_N$, is just a constant. Subtracting that constant we obtain the Jacobi Hamiltonian $\hat{H}_J = \frac{1}{4}(E - V)^{-1} h^{\alpha \beta} p_\alpha p_\beta$. The dynamics is now equivalent to geodesic motion in the Jacobi geometry given by $ds^2 = 2(E - V) h^{\alpha \beta} dq^\alpha dq^\beta$. This procedure was used by Rosquist and Pucacco [9] to give a unified derivation of invariants at both fixed and arbitrary energy generalizing Darboux’s century old condition for 2-dimensional Hamiltonians admitting a second quadratic invariant. However, the method does suffer from a drawback in that the resulting Jacobi geometry is energy dependent. That is to say that the full dynamics is not represented by a single geometry but rather corresponds to a family of geometries parametrized by the energy. One way of avoiding this drawback is to use a suitable canonical transformation to transform the original Hamiltonian to a geometric form (see e.g. Ashtekar et al. [10]). However, such a transformation is not always possible to find and in general it destroys the conformal flatness of the original problem. We shall now give another method to transform a Hamiltonian to a geometric form which represents the dynamics in terms of a single geometry.
The transformation we are about to describe is a variation of the time reparametrization scheme. In fact, we shall work in an extended Hamiltonian framework [4]. The key point in this approach is to introduce an additional configuration space variable \( q^{n+1} \) together with an associated canonical momentum \( p_{n+1} \). In this extended phase space the dynamics is given by a constrained Hamiltonian \( H(q_1, \ldots, q^{n+1}, p_1, \ldots, p_{n+1}) = 0 \). The constraint is usually chosen as \( \mathcal{H} = H - p_{n+1} \) where \( p_{n+1} = E \). This gives an extended Hamiltonian which is no longer quadratic in the momentum variables. However, as pointed out by Lanczos [4], other choices of extended Hamiltonian are allowed. For a system at positive energy we exploit this freedom to take \( \mathcal{H} = H - \frac{1}{2}p_{n+1}^2 \) as our extended Hamiltonian. This means that the additional momentum is related to the energy by \( p_{n+1} = \sqrt{2E} \).

### 3 Geometrical formulation of the Toda lattice

Before applying the above time reparametrization scheme to the Toda Hamiltonian we first adapt the coordinate to the linear symmetry by performing the orthogonal transformation

\[
q^1 = \frac{1}{\sqrt{3}}q^1 + \frac{1}{\sqrt{3}}q^2 + \frac{1}{\sqrt{3}}q^3, \quad p_1 = \frac{1}{\sqrt{3}}p_1 + \frac{1}{\sqrt{3}}p_2 + \frac{1}{\sqrt{3}}p_3, \\
q^2 = -\sqrt{\frac{2}{3}}q^2 + \frac{1}{\sqrt{3}}q^3, \quad p_2 = -\frac{2}{\sqrt{3}}\bar{p}_2 + \frac{1}{\sqrt{3}}\bar{p}_3, \\
q^3 = -\frac{1}{\sqrt{2}}q^1 + \frac{1}{\sqrt{6}}q^2 + \frac{1}{\sqrt{3}}q^3, \quad p_3 = -\frac{1}{\sqrt{6}}\bar{p}_1 + \frac{1}{\sqrt{6}}\bar{p}_2 + \frac{1}{\sqrt{3}}\bar{p}_3.
\]

The Hamiltonian then becomes

\[
H = \frac{1}{2}(\bar{p}_1^2 + \bar{p}_2^2 + \bar{p}_3^2) + 2e^{\sqrt{3}q^3}\cosh\left(\sqrt{6}q^2\right).
\]

The coordinate \( q^3 \) is now cyclic with constant conjugate momentum \( \bar{p}_3 \). The cubic invariant becomes

\[
\tilde{I}_3 = \sqrt{6}\bar{I}_3 + \frac{5\sqrt{2}}{18}\bar{p}_3^3 - \frac{1}{\sqrt{3}}\bar{p}_3H,
\]

where

\[
\tilde{I}_3 = -\frac{1}{18}\bar{p}_3^3 + \frac{1}{\sqrt{6}}\bar{p}_1^2\bar{p}_2 + \frac{1}{\sqrt{3}}e^{\sqrt{3}q^3}\sinh(\sqrt{6}q^2)\bar{p}_1 - \frac{4}{3}e^{\sqrt{3}q^3}\cosh(\sqrt{6}q^2)\bar{p}_2.
\]

We can then define an extended Hamiltonian by

\[
\mathcal{H} = \frac{1}{2}(-\bar{p}_0^2 + \bar{p}_1^2 + \bar{p}_2^2) + 2e^{\sqrt{3}q^3}\cosh\left(\sqrt{6}q^2\right) = 0,
\]

where the new momentum is given by \( \bar{p}_0 = \sqrt{2E - \bar{p}_3^2} \). The next step is to reparametrize the time leading to the Jacobi time gauge Hamiltonian

\[
\mathcal{H}_N = \frac{1}{2}N(-\bar{p}_0^2 + \bar{p}_1^2 + \bar{p}_2^2),
\]

\[
N^{-1} = 2V = 4e^{\sqrt{3}q^3}\cosh\left(\sqrt{6}q^2\right).
\]

Although the time reparametrized system is equivalent to the original the system, old invariants are not necessarily invariants of the new system. Linear invariants remain invariant after time reparametrization and the new Hamiltonian is itself a quadratic invariant. However, the cubic invariant has no immediately obvious counterpart in the new system.
A prescription to transform invariants to the new time gauge is discussed by Rosquist and Pucacco [9]. According to that reference the new invariant is given by

\[ J = \tilde{I}_3 + R \mathcal{H}_N \]

if we can find a phase space function \( R \) which satisfies the equation

\[ \{ R, \mathcal{H}_N \} = \{ \tilde{I}_3, \mathcal{N}^{-1} \} + \mathcal{H}_N \{ R, \mathcal{N}^{-1} \}. \]

(13)

Straightforward computation shows that

\[ \{ \tilde{I}_3, \mathcal{N}^{-1} \} = -\frac{2\sqrt{2}}{3} e^{\sqrt{2}x} \left[ 2\tilde{p}_x \tilde{p}_y \cosh(\sqrt{6}y) + \sqrt{3} \left( \tilde{p}_x^2 - \tilde{p}_y^2 \right) \sinh(\sqrt{6}y) \right]. \]

(14)

It turns out that the function \( R \) can be identified with the linear part of the the cubic invariant in the old time gauge. More precisely we have the relations \( \{ \tilde{I}_3, \mathcal{N}^{-1} \} = \{ R, \mathcal{H}_N \} \) and \( \{ R, \mathcal{N} \} = 0 \) where \( R = -2\tilde{I}_3^t \) and \( \tilde{I}_3^t \) stands for the linear part of \( \tilde{I}_3 \). This shows that \( R \) satisfies equation (13). Therefore the cubic invariant can be expressed in the Jacobi time gauge as

\[ J = -\frac{1}{18} \tilde{p}_3^3 + \frac{1}{6} \tilde{p}_1^2 \tilde{p}_2 + \frac{1}{12} \left[ -\sqrt{3} \tanh(\sqrt{6}q^2) \tilde{p}_1 + \tilde{p}_2 \right] \left( -\tilde{p}_0^2 + \tilde{p}_1^2 + \tilde{p}_2^2 \right). \]

(15)

For the purposes of this discussion the important thing to notice about this expression is that it is homogeneous in the momenta. This fact will be used to motivate a crucial step in the derivation of the tensorial Lax pair equation.

### 4 The tensorial Lax pair equation

The basic assumption used to derive the covariant Lax pair equation is that we have a system which can be described by a purely kinetic Hamiltonian

\[ H = \frac{1}{2} g^{\alpha\beta}(q) p_\alpha p_\beta. \]

(16)

The equations of motion can then be interpreted as geodesic motion on a Riemannian or pseudo-Riemannian configuration space with the metric

\[ ds^2 = g_{\alpha\beta} dq^\alpha dq^\beta. \]

(17)

As noted above any Hamiltonian which is quadratic in the momenta can be transformed to the geometric form (16). Our aim is to obtain a tensorial form of the Lax equation which is covariant with respect to the geometry defined by the metric (17). Since the right hand side of (1) involves matrix multiplication we must define the matrices \( L \) and \( A \) in such a way that matrix multiplication corresponds directly to contraction of indices. This can be achieved if the components of \( L \) and \( A \) correspond to mixed pairs of indices, i.e. one covariant and one contravariant index. Choosing the row index to be contravariant and the column index to be covariant we can then write

\[ L = (L^\alpha_\beta), \quad A = (A^\alpha_\beta). \]

(18)

At this point we use some properties of the Toda lattice. First, in the sequence of invariants \( I_k = k^{-1} \text{Tr} L^k \), the first one is linear and homogeneous while the second one is equal to
the Hamiltonian itself, $I_2 = \frac{1}{2} \text{Tr} L^2 = H$. Moreover, for geometric Hamiltonians we expect the invariants to be homogeneous in the momenta as in the example of the Toda lattice. These facts suggest that the elements of $L$ itself should be homogeneous first order polynomials in the momenta. With this assumption we have

$$L_{\alpha \beta} = L_{\alpha \beta}^{\gamma} p_\gamma ,$$  \hfill (19)

where $L_{\alpha \beta}^{\gamma}$ is some third rank geometric object which we shall later be able to interpret as a tensor. Taking the time derivative of (19) we obtain

$$\dot{L}_{\alpha \beta} = \left( L_{\alpha \beta}^{(\mu \gamma \nu)} g^{\nu} + L_{\alpha \beta}^{\gamma} \Gamma^{(\mu \gamma \nu)} \right) p_\mu p_\nu ,$$  \hfill (20)

where $\Gamma_{\alpha \beta \gamma}$ is the connection of the metric $g$. Thus the left hand side of the Lax equation (1) is quadratic and homogeneous in the momenta. Therefore the right hand side must also be quadratic and homogeneous. It then becomes natural to assume that the second Lax matrix $A$ is also a linear and homogenous function of the momenta. We then have

$$A_{\alpha \beta} = A_{\alpha \beta}^{\gamma} p_\gamma ,$$  \hfill (21)

where $A_{\alpha \beta}^{\gamma}$ is some geometric object the nature of which will be specified below. The right hand side of the Lax equation can now be written

$$([L, A])_{\alpha \beta} = (L_{\alpha \gamma}^{\mu} A_{\beta \nu}^{\gamma} - A_{\alpha \gamma}^{\mu} L_{\beta \nu}^{\gamma}) p_\mu p_\nu .$$  \hfill (22)

Using the above definitions it follows that the Lax equation (1) can be written as

$$(\dot{L} - [L, A])_{\alpha \beta} = T_{\alpha \beta}^{\mu \nu} p_\mu p_\nu = 0 ,$$  \hfill (23)

where the object $T_{\alpha \beta}^{\mu \nu}$ is a function on the configuration space which can be read off from (20) and (22). Remarkably, $L_{\alpha \beta}^{\gamma}$ and $T_{\alpha \beta}^{\mu \nu}$ can be interpreted as tensorial objects related by the equation

$$T_{\alpha \beta}^{\mu \nu} = L_{\alpha \beta}^{(\mu \nu)} ,$$  \hfill (24)

if we make the identification

$$A_{\alpha \beta}^{\gamma} = \Gamma_{\alpha \beta}^{\gamma} .$$  \hfill (25)

The result follows directly from the formula for the covariant derivative of $L_{\alpha \beta}^{\gamma}$. If the Lax equation is to be satisfied identically for arbitrary initial data the coefficients of the quadratic momentum terms in (23) must vanish separately. This leads to the equation

$$L_{\alpha \beta(\gamma ; \delta)} = 0 .$$  \hfill (26)

Actually, the above identification is not the most general ansatz which leads to a tensorial form of the Lax equation. A more general ansatz is to assume that

$$A_{\alpha \beta}^{\gamma} = \Gamma_{\alpha \beta}^{\gamma} + B_{\alpha \beta}^{\gamma} ,$$  \hfill (27)

where $B_{\alpha \beta}^{\gamma}$ are the components of a tensor. This leads us to the main result of this contribution, namely that with this more general ansatz the Lax equation can be written in the tensorial form

$$L_{\alpha \beta(\gamma ; \delta)} = L_{\alpha \mu(\gamma} B_{\beta \delta)}^{\mu} - B_{\alpha \mu(\gamma} L_{\beta \delta)}^{\mu} ,$$  \hfill (28)
From the form of the Toda lattice system one can guess that the antisymmetric case $B_{\alpha\beta\gamma} = B_{[\alpha\beta]\gamma}$ is of particular interest. The above equation can then be split into symmetric and antisymmetric parts by decomposing the Lax tensor itself in symmetric and antisymmetric parts, $L_{\alpha\beta\gamma} = S_{\alpha\beta\gamma} + R_{\alpha\beta\gamma}$ where $S_{\alpha\beta\gamma} = S_{(\alpha\beta)\gamma}$ and $R_{\alpha\beta\gamma} = R_{(\alpha\beta)\gamma}$. The antisymmetric part then satisfies the sourceless equation, $R_{\alpha\beta(\gamma\delta)} = 0$ while the symmetric Lax tensor equation can be written

$$S_{\alpha\beta}^{\gamma(\delta)} = -2S_{\alpha}^{\mu(\gamma}B_{\beta)\mu}^{\delta)}.$$ 

(29)

The Lax pair equation (28) is in fact a generalization of the Killing vector equation. Indeed any Lax pair tensor $L^{\alpha\beta\gamma}$ with nonvanishing trace on the first two indices gives rise to the Killing vector $\xi^{\alpha} = L^{\beta\alpha}$. Lax pair tensors $L^{\alpha\beta\gamma}$ are also generalizations of third rank Killing-Yano tensors (for a discussion of Killing-Yano tensors in a general relativistic context see Dietz and Rüdiger [11]). This follows immediately from the observation that the Killing-Yano equations are identical with the sourceless Lax pair equation. Killing-Yano tensors, however, are by definition 3-forms. Lax pair tensors, on the other hand, have no a priori symmetry restrictions.

At this point an obvious first problem is to analyze known integrable systems like e.g. the Toda lattice [12] and the Calogero models [13] to see if they fit in this framework. The above discussion of the Toda system makes it very reasonable to believe that at least the Toda lattice can be given a tensorial Lax pair formulation. A preliminary investigation has indicated that $B^{\alpha\beta\gamma} \neq 0$ for the Toda system.

A possible application in general relativity is to look for physically reasonable spacetimes which admit a Lax pair tensor, for example models of gravitational collapse. The existence of a Lax pair would imply integrability of the geodesic equations and thus provide a way to analyze spacetime singularities analytically. It is an open problem if there exists nontrivial integrable solutions to the Einstein equations having the Lax pair property.

It is clear from the results of this contribution that one can define nontrivial integrable spacetimes which are not necessarily solutions of the Einstein equations. An example is provided by the Hamiltonian (11) and its associated metric

$$ds^2 = 4e^{\sqrt{2}q^1} \cosh \left( \sqrt{6} q^2 \right) \left[ -(dq^0)^2 + (dq^1)^2 + (dq^2)^2 \right].$$

(30)

This defines a 3-dimensional integrable spacetime having one linear, one quadratic and one cubic invariant. One can easily extend this construction to 4-dimensional spacetimes for example by considering a four particle Toda lattice.
References

[1] P. Lax, Commun. Pure Appl. Math. 21, 467 (1968).

[2] A. M. Perelomov, *Integrable systems of classical mechanics and Lie algebras* (Birkhäuser, Basel, Switzerland, 1990).

[3] S. De Filippo, G. Marmo, and G. Vilasi, Phys. Lett. B 117, 418 (1982).

[4] B. Kostant, Adv. Math. 39, 195 (1979).

[5] M. Adler, Invent. Math. 50, 219 (1979).

[6] W. Symes, Invent. Math. 59, 13 (1980).

[7] C. Lanczos, *The variational principles of mechanics* (Dover, New York, USA, 1986).

[8] C. Uggla, K. Rosquist, and R. T. Jantzen, Phys. Rev. D 42, 404 (1990).

[9] K. Rosquist and G. Pucacco, in preparation (unpublished).

[10] A. Ashtekar, R. Tate, and C. Uggla, Int. J. Mod. Phys. D 2, 15 (1993).

[11] W. Dietz and R. Rüdiger, Proc. Roy. Soc. Lond. A 375, 361 (1981).

[12] M. Toda, *Theory of nonlinear lattices* (Springer, Berlin, Heidelberg, New York, 1981).

[13] F. Calogero, J. Math. Phys. 12, 419 (1971).