Mid-concavity of survival probability for isotropic Lévy processes*

Tadeusz Kulczycki†

Abstract

Let $X$ be a symmetric, pure jump, unimodal Lévy process in $\mathbb{R}$ with an infinite Lévy measure. We prove that for any fixed $t > 0$ the survival probability $P^x(\tau_{(-a,a)} > t)$ is nondecreasing on $(-a,0]$, nonincreasing on $[0,a)$ and concave on $(-a/2,a/2)$, where $a > 0$ and $\tau_{(-a,a)}$ is the first exit time of the process $X$ from $(-a,a)$. We also show a similar statement for sets $(-a,a) \times F \subset \mathbb{R}^d$.

Keywords: Lévy process; exit time; survival probability; concavity; first eigenfunction.

AMS MSC 2010: 60G51.

Submitted to ECP on September 28, 2015, final version accepted on March 13, 2016.

1 Introduction

The main purpose of this paper is to investigate the monotonicity and concavity properties of the survival probability for some Lévy processes in $\mathbb{R}^d$. Let $\tau_D = \inf \{ t \geq 0 : X_t \notin D \}$ be the first exit time of an open, nonempty set $D \subset \mathbb{R}^d$ of the process $X$. We first formulate our result in one-dimensional setting.

**Theorem 1.1.** Let $X$ be a symmetric, pure jump, unimodal Lévy process in $\mathbb{R}$ with an infinite Lévy measure. Let $D = (-a,a)$, where $a > 0$. Put $\psi^D_t(x) = P^x(\tau_D > t)$ for $t \geq 0$ and $x \in \mathbb{R}$. Then for any $t > 0$ the function $x \to \psi^D_t(x)$ is nondecreasing on $(-a,0]$, nonincreasing on $[0,a)$ and concave on $(-a/2,a/2)$.

The next theorem is the generalization of the above result to higher dimensions.

**Theorem 1.2.** Let $X$ be an isotropic, pure jump, unimodal Lévy process in $\mathbb{R}^d$, $d \geq 2$ with an infinite Lévy measure. Let $D = (-a,a) \times F$, where $a > 0$ and $F \subset \mathbb{R}^{d-1}$ be a bounded Lipschitz domain. Put $\psi^D_t(x) = P^x(\tau_D > t)$ for $t \geq 0$, $x \in \mathbb{R}^d$ and let $e_1 = (1,0,\ldots,0) \in \mathbb{R}^d$. Then for any $t > 0$ and $\tilde{x} \in \{0\} \times F$ the function $y \to \psi^D_t(ye_1 + \tilde{x})$ is nondecreasing on $(-a,0]$, nonincreasing on $[0,a)$ and concave on $(-a/2,a/2)$.

In Section 4 we will apply these results to obtain analogous properties of first eigenfunctions for the related Dirichlet eigenvalue problem.

**Remark 1.3.** The property that the function $x \to \psi^1_{(-a,a)}(x)$ or $y \to \psi^1_{(-a,a) \times F}(ye_1 + \tilde{x})$ is concave on $(-a/2,a/2)$ is called mid-concavity (see Definition 1.1 in [2]).

*The research was supported in part by NCN grant no. 2011/03/B/ST1/00423.
†Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wyb. Wyspianskiego 27, 50-370 Wrocław, Poland.
E-mail: Tadeusz.Kulczycki@pwr.edu.pl
Mid-concavity of survival probability for isotropic Lévy processes

The above results for isotropic \( \alpha \)-stable processes in \( \mathbb{R}^d \) (where \( \alpha \in (0, 2) \)) and intervals \( (-a, a) \) or hyperrectangles \( \prod_{i=1}^d (-a_i, a_i) \) are well known. They were proved by R. Bañuelos, T. Kulczycki and P. Méndez-Hernández in [2]. Indeed, the methods from [2] allow to extend these results for intervals or hyperrectangles to arbitrary subordinated Brownian motions in \( \mathbb{R}^d \).

The main novelty of the results in this paper is that they concern arbitrary isotropic pure jump, unimodal Lévy processes in \( \mathbb{R}^d \) with an infinite Lévy measure. The method used in the proof of Theorems 1.1, 1.2 is completely different than the method used in [2]. The key idea of the proof of Theorems 1.1, 1.2 is probabilistic. A very important step in this proof is the use of some results concerning the so-called difference processes which were introduced in [8].

The proof in [2] is analytical. The main idea in [2] (for \( D = (-a, a) \)) is to prove some properties of

\[
\int_{-a}^a \ldots \int_{-a}^a \prod_{i=1}^n p_t(x_{i-1} - x_i) \, dx_1 \ldots dx_n
\]

for gaussian kernels \( p_t(x) \) and then use subordination to show monotoncity and midconcavity for

\[
x \to P^x(\tau_D > t) \quad \text{for } x \to P^x(X_{t_1} \in D, \ldots, X_{t_n} \in D)
\]

(1.1)

The results for \( P^x(\tau_D > t) \) in [2] follows by a limiting procedure.

Note that in this paper we do not study properties of the function (1.1) but we study only properties of \( P^x(\tau_D > t) \).

Very recently many researchers have been studying convexity properties of solutions of equations involving fractional Laplacians see [1], [4], [6], [7], [10]. In particular, concavity properties of the first eigenfunction for the Dirichlet eigenvalue problem on an interval for the fractional Laplacians have been studied in [1] and [6]. In this paper, using a probabilistic approach, we obtain concavity properties of the first eigenfunction for the Dirichlet eigenvalue problem on an interval for much more general nonlocal operators, namely generators of the isotropic unimodal Lévy processes.

The paper is organized as follows. In Section 2 we present notation and collect some known facts needed in the rest of the paper. Section 3 contains proofs of Theorems 1.1, 1.2. In Section 4 we present regularity results of the first eigenfunction for the related Dirichlet eigenvalue problem.

A very interesting open issue is a question about concavity properties of a survival probability for a Lévy process on an arbitrary convex domain in \( \mathbb{R}^d \) when \( d \geq 2 \). The question seems to be very hard (cf. [7, Open problem]). A natural conjecture is that for any isotropic unimodal Lévy process in \( \mathbb{R}^d, d \geq 2 \), any convex domain \( D \subset \mathbb{R}^d \) and any \( t > 0 \) the survival probability \( P^x(\tau_D > t) \) is quasiconcave on \( D \) as a function of \( x \).

2 Preliminaries

For \( x \in \mathbb{R}^d \) and \( r > 0 \) we let \( B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\} \). A Borel measure on \( \mathbb{R}^d \) is called isotropic unimodal if on \( \mathbb{R}^d \setminus \{0\} \) it is absolutely continuous with respect to the Lebesgue measure and has a finite radial, radially nonincreasing density function (such measures may have an atom at the origin).

A Lévy process \( X = (X_t, t \geq 0) \) in \( \mathbb{R}^d \) is called isotropic unimodal if its transition probability \( p_t(dx) \) is isotropic unimodal for all \( t > 0 \). When additionally \( X \) is a pure-jump process then the following Lévy-Khintchine formula holds for \( t > 0 \) and \( \xi \in \mathbb{R}^d \),

\[
E^0 e^{i\xi X_t} = \int_{\mathbb{R}^d} e^{i\xi x} p_t(dx) = e^{-t\psi(\xi)} \quad \text{where} \quad \psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi x)) \nu(dx).
\]
ψ is the characteristic exponent of X and ν is the Lévy measure of X. E₀ is the expected value for the process X starting from 0. Recall that a Lévy measure is a measure concentrated on R^d \{0\} such that \( \int_{R^d} (|x|^2 \wedge 1) \nu(dx) < \infty \). Isotropic unimodal pure-jump Lévy measures are characterized in [12] by unimodal Lévy measures ν(dx) = ν(x) dx = ν(|x|) dx.

Unless explicitly stated otherwise in what follows we assume that X is a pure-jump isotropic unimodal Lévy process in R^d with (isotropic unimodal) infinite Lévy measure ν. Then for any t > 0 the measure \( p_t(dx) \) has a radial, radially nonincreasing density function \( p_t(x) = p_t(|x|) \) on R^d with no atom at the origin. However, it may happen that \( p_t(0) = \infty \), for some t > 0. As usual, we denote by \( P^x \) and \( E^x \) the probability measure and the corresponding expectation for the process starting from \( x \in \mathbb{R}^d \).

Let \( D \subset \mathbb{R}^d \) be an open, nonempty set. We define a killed process \( X^D_t \) by \( X^D_t = X_t \) if \( t < \tau_D \) and \( X^D_t = \partial \) otherwise, where \( \partial \) is some point adjoined to \( D \). The transition density for \( X^D_t \) on \( D \) is given by

\[
pD(t, x, y) = p_t(x - y) - E^x(p_{t-\tau_D}(X(\tau_D) - y), t > \tau_D), \quad x, y \in D, \quad t > 0, \tag{2.1}
\]

that is for any Borel set \( A \subset \mathbb{R}^d \) we have

\[
P^x(X^D_t \in A) = \int_A pD(t, x, y) \, dy, \quad x \in D, \quad t > 0.
\]

We have \( pD(t, x, y) = pD(t, y, x), \quad x, y \in D, \quad t > 0 \). We define the Green function for \( X^D_t \) by

\[
G_D(x, y) = \int_0^\infty pD(t, x, y) \, dt, \quad x, y \in D,
\]

\( G_D(x, y) = 0 \) if \( x \notin D \) or \( y \notin D \).

Let \( D \subset \mathbb{R}^d \) be an open, nonempty set. The distribution \( P^x(X(\tau_D) \in \cdot) \) is called the harmonic measure with respect to X. The harmonic measure for Borel sets \( A \subset (\overline{D})^c \) is given by the Ikeda-Watanabe formula [5],

\[
P^x(X(\tau_D) \in A) = \int_A \int_D G_D(x, y) \nu(y - z) \, dy \, dz, \quad x \in D. \tag{2.2}
\]

When \( D \subset \mathbb{R}^d \) is a bounded, open Lipschitz set then we have [11], [9],

\[
P^x(X(\tau_D) \in \partial D) = 0, \quad x \in D. \tag{2.3}
\]

It follows that for every such set \( D \) the Ikeda-Watanabe formula (2.2) holds for any Borel set \( A \subset D^c \). Let \( D \subset \mathbb{R}^d \) be an open, nonempty set. For any \( s > 0, x \in D, z \in (\overline{D})^c \) put

\[
h_D(x, s, z) = \int_D pD(s, x, y) \nu(y - z) \, dy. \tag{2.4}
\]

By the Ikeda-Watanabe formula [5] for any Borel sets \( A \subset (0, \infty), B \subset (\overline{D})^c \) we have

\[
P^x(\tau_D \in A, X(\tau_D) \in B) = \int_A \int_B h_D(x, s, z) \, dz \, ds, \quad x \in D. \tag{2.5}
\]

If (2.3) holds then we can take \( B \subset D^c \) in (2.5).

3 The monotonicity and midconcavity

We will prove both Theorems 1.1, 1.2 simultaneously. Let X be an isotropic, pure jump, unimodal Lévy process in R^d, d ≥ 1 with an infinite Lévy measure. Let \( D = (-a, a) \times F \), where \( a > 0 \) and \( F \subset \mathbb{R}^{d-1} \) is a bounded Lipschitz domain (or \( D = (-a, a) \) when \( d = 1 \)). Put \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d \). Note that for any \( x \in D \) we have \( P^x(X(\tau_D) \in \partial D) = 0 \).

The key point in this section is the following result.
Mid-concavity of survival probability for isotropic Lévy processes

**Proposition 3.1.** Let $U = (b, c) \times F$ (or $U = (b, c)$ when $d = 1$), where $-a \leq b < c \leq a$. Put $l(U) = b$, $r(U) = c$, $m(U) = (b + c)/2$, $U_\pm = (b, m(U)) \times F$, $U_+ = (m(U), c) \times F$, $H_\pm(U) = (-\infty, m(U)) \times \mathbb{R}^{d-1}$, $H_+(U) = (m(U), \infty) \times \mathbb{R}^{d-1}$ (or $U_\pm = (b, m(U))$, $U_+ = (m(U), c)$, $H_-(U) = (\infty, m(U))$, $H_+(U) = (m(U), \infty)$ when $d = 1$). For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let

$$T_U(x) = x + 2e_1(m(U) - x_1),$$

(this is the reflection with respect to the hyperplane $x_1 = m(U)$, or with respect to a point $m(U)$ when $d = 1$). For any $s > 0$, $x \in U_+$, $z \in (U)^c$ put

$$f^U_s(x, z) = \int_{U_+} (p_U(s, x, y) - p_U(s, T_U(x), y)) (\nu(y - z) - \nu(T_U(y) - z)) \, dy.$$

For any $x \in U_+$, $t > 0$ we have

$$\psi^D_t(x) - \psi^D_t(T_U(x)) = \int_U \int_0^t f^U_s(x, z) \psi^D_{t-s}(z) \, ds \, dz, \quad (3.1)$$

$$f^U_s(x, z) \geq 0 \quad \text{for} \quad s > 0, \ z \in H_+(U) \setminus \overline{U_+}, \quad (3.2)$$

$$f^U_s(x, z) \leq 0 \quad \text{for} \quad s > 0, \ z \in H_-(U) \setminus \overline{U_-}. \quad (3.3)$$

**Proof.** By the strong Markov property and (2.5) for any $x \in U$, $t > 0$ we have

$$\psi^D_t(x) = P^x(\tau_D > t)$$

$$= P^x(\tau_U > t) + E^x \left( \tau_U \leq t, \left[ p^{(\tau_U)}(\tau_D > t - s) \right]_{s=\tau_U} \right)$$

$$= \psi^U_t(x) + \int_U \int_0^t p_U(s, x, y) \int_{U_+} \psi^D_{t-s}(z) \nu(y - z) \, dz \, ds \, dy.$$

It follows that

$$\psi^D_t(x) - \psi^D_t(T_U(x)) = \psi^U_t(x) - \psi^U_t(T_U(x)) \quad (3.4)$$

$$+ \int_U \int_0^t (p_U(s, x, y) - p_U(s, T_U(x), y)) \int_{U_+} \psi^D_{t-s}(z) \nu(y - z) \, dz \, ds \, dy. \quad (3.5)$$

For any $x \in U_+$, $t > 0$ by the symmetry of the process $X$ and the definition of $T_U(x)$ we have

$$\psi^U_t(x) = \psi^U_t(T_U(x)). \quad (3.6)$$

For any $x \in U_+$, $t > 0$, $z \in (U)^c$ we also have

$$\int_U (p_U(s, x, y) - p_U(s, T_U(x), y)) \nu(y - z) \, dy$$

$$= \int_{U_+} (p_U(s, x, y) - p_U(s, T_U(x), y)) \nu(y - z) \, dy$$

$$+ \int_{U_-} (p_U(s, x, y) - p_U(s, T_U(x), y)) \nu(y - z) \, dy$$

$$= \int_{U_+} (p_U(s, x, y) - p_U(s, T_U(x), y)) \nu(y - z) \, dy$$

$$+ \int_{U_+} (p_U(s, T_U(y)) - p_U(s, T_U(x), T_U(y))) \nu(T_U(y) - z) \, dy$$

$$= \int_{U_+} (p_U(s, x, y) - p_U(s, T_U(x), y)) (\nu(y - z) - \nu(T_U(y) - z)) \, dy,$$
Mid-concavity of survival probability for isotropic Lévy processes

(where in the last equality we used $p_{U}(s, T_{U}(x), T_{U}(y)) = p_{U}(s, x, y)$). Applying this, (3.4-3.5) and (3.6) we get (3.1).

Note that $p_{U}(s, x, y) - p_{U}(s, T_{U}(x), y)$ is the transition density of the so-called difference process (with respect to the hyperplane $x_1 = m(U)$ or the point $m(U)$ when $d = 1$) killed on exiting $U_+$ (see Section 4 in [8] for more details). By (19) in [8] and the first formula after the proof of Lemma 4.3 in [8] we obtain that $p_{U}(s, x, y) - p_{U}(s, T_{U}(x), y) \geq 0$ for any $s > 0, x, y \in U_+$. By unimodality of $\nu(x)$ we obtain that $\nu(y - z) - \nu(T_{U}(y) - z) \geq 0$ for any $y \in U_+, z \in H_+(U) \setminus U_+$ and $\nu(y - z) - \nu(T_{U}(y) - z) \leq 0$ for any $y \in U_+, z \in H_-(U) \setminus U_+$. This gives (3.2) and (3.3).

Now we will show our main results.

**Proof of Theorems 1.1, 1.2.** First we study monotonicity of $\psi^D$. Fix $\bar{x} \in \{0\} \times F$ and $-a < x_1' < x_1'' \leq 0$. Put $x'' = x''_1 e_1 + \bar{x}, x''' = x'''_1 e_1 + x''$ (or $x'' = x'_1, x''' = x''_1$ when $d = 1$). Let $b = -a$, $c = (x''_1 + x''_1)/2, c = -a + 2(x_{s} - (-a)) = a + x'_1 + x''_1, U = (b, c) \times F$ (or $U = (b, c)$ when $d = 1$). Note that $m(U) = x_{s}$ and $T_{U}(x'') = x''$. By (3.1) for any $t > 0$ we get

$$
\psi^D(x'') - \psi^D(x') = \psi^D(x'') - \psi^D(T_{U}(x''))
= \int_{U} \int_{0}^{t} f_{U}^{(x''), z} \psi^D_{t-s}(z) ds dz
= \int_{D \setminus U} \int_{0}^{t} f_{s}^{(x''), z} \psi^D_{t-s}(z) ds dz.
$$

Note that $D \setminus U = (c, a) \times F$ (or $D \setminus U = (c, a)$ when $d = 1$) and $c = a + x'_1 + x''_1 > m(U) = (x'_1 + x''_1)/2$ so $D \setminus U \subset H_+(U) \setminus U_+$. This, (3.7-3.9) and (3.2) give $\psi^D(x'') \geq \psi^D(x')$. It follows that the function $y \rightarrow \psi^D(ye_1 + \bar{x})$ (or $y \rightarrow \psi^D(y)$ when $d = 1$) is nondecreasing on $(-a, 0]$. By symmetry of the process $X$ and the domain $D$ the function $y \rightarrow \psi^D(ye_1 + \bar{x})$ (or $y \rightarrow \psi^D(y)$ when $d = 1$) is nonincreasing on $[0, a]$.

Now we will study midconcavity of the function $\psi^D$. Fix $\bar{x} \in \{0\} \times F$ and $-a/2 < x'_1 < x''_1 < x''_1 \leq 0$ such that $x''_1 - x''_1 = x''_1 - x''_1$. Put $x'' = x'_1 e_1 + \bar{x}, x''' = x''_1 e_1 + \bar{x}, x''' = x'''_1 e_1 + \bar{x}$ (or $x'' = x'_1, x''' = x''_1$ when $d = 1$). As above, let $b = -a, x_{s} = (x'_1 + x''_1)/2, c = -a + 2(x_{s} - (-a)) = a + x'_1 + x''_1, U = (b, c) \times F$ (or $U = (b, c)$ when $d = 1$). We have $m(U) = x_{s}$ and $T_{U}(x''') = x''$. Note that $l(U) = -a$ and $r(U) = c = a + x'_1 + x''_1 \in (0, a)$ (because $-a/2 < x'_1 < x''_1 < 0$).

Let $v_1 = x''_1 - x'_1$ and $v = v_1 e_1$. Put

$$
W = U + v.
$$

Note that $W = (b + v_1, c + v_1) \times F$ (or $W = (b + v_1, c + v_1)$ when $d = 1), m(W) = m(U) + v_1, l(W) = l(U) + v_1, r(W) = r(U) + v_1 = a + 2x''_1 \in (0, a)$.

We have

$$
T_{W}(x) = x + 2e_1(m(W) - x_1).
$$

It follows that

$$
T_{W}(x''') = x''' + 2e_1(m(W) - x''')
= x''' + 2e_1 \left( \frac{x''_1 + x''_1}{2} - x''_1 \right)
= x''' + e_1(x''_1 - x''_1)
= x''' = x''.
$$
Mid-concavity of survival probability for isotropic Lévy processes

Using this and Proposition 3.1 applied to $W$ we get for any $t > 0$

$$
\psi_t^D(x'') - \psi_t^D(x'') = \psi_t^D(T_W(x'')) - \psi_t^D(T_W(x'''))
= \int_{W^c} \int_0^t f_s^W(x'', z) \psi_{t-s}^D(z) \, ds \, dz.
$$

(3.10)

(3.11)

Note that $W_0 = U_0 + v$ and $x'' = x'' + v$. Using this and the definition of $f_s^W(x, z)$ we get for any $s > 0$, $z \in W^c$

$$
f_s^W(x'', z) = \int_{W_+} (p_W(s, x'', y) - p_W(s, T_W(x''), y)) (\nu(y - z) - \nu(T_W(y) - z)) \, dy
= \int_{U_+ + v} (p_W(s, x'' + v, y) - p_W(s, T_W(x'' + v), y))
\times (\nu(y - z) - \nu(T_W(y) - z)) \, dy.
$$

(3.12)

(3.13)

Using substitution $q = y - v$ this is equal to

$$
\int_{U_+} (p_W(s, x'' + v, q + v) - p_W(s, T_W(x'' + v), q + v))
\times (\nu(q + v - z) - \nu(T_W(q + v) - z)) \, dq.
$$

(3.14)

(3.15)

For any $s > 0$, $q \in U_+$ we have

$$
p_W(s, x'' + v, q + v) = p_{U_+}(s, x'' + v, q + v) = p_U(s, x'', q).
$$

By the definition of $T_W$ and the equality $m(W) = m(U) + v_1$ we get

$$
T_W(x'' + v) = x'' + v + 2c_1(m(W) - x''_1 - v_1) = x'' + 2c_1(m(U) - x''_1) + v = T_U(x'').
$$

Hence for any $s > 0$, $q \in U_+$ we obtain

$$
p_W(s, T_W(x'' + v), q + v) = p_{U_+}(s, T_U(x''), v + q + v) = p_U(s, T_U(x''), q).
$$

By similar arguments as above for any $q \in U_+$ we get $T_W(q + v) = T_U(q) + v$. Hence for any $q \in U_+$ and $z \in W^c$ we obtain

$$
\nu(q + v - z) - \nu(T_W(q + v) - z) = \nu(q - (z - v)) - \nu(T_U(q) - (z - v)).
$$

Using this, (3.12-3.15) and (3.16-3.17) we get for any $s > 0$, $z \in \langle W \rangle^c$

$$
f_s^U(x'', z) = \int_{U_+} (p_U(s, x'', q) - p_U(s, T_U(x''), q)) (\nu(q - (z - v)) - \nu(T_U(q) - (z - v))) \, dq
= f_s^U(x'', z - v).
$$

Using this, the fact that $W^c = U^c + v$ and (3.10-3.11) we get for any $t > 0$

$$
\psi_t^D(x'') - \psi_t^D(x'') = \int_{W^c} \int_0^t f_s^U(x'', z - v) \psi_{t-s}^D(z) \, ds \, dz
= \int_{U^c} \int_0^t f_s^U(x'', z) \psi_{t-s}^D(z + v) \, ds \, dz.
$$

(3.18)

(3.19)

Put

$$
L(U) = (-\infty, l(U)) \times F \subset H_-(U), \quad R(U) = (r(U), \infty) \times F \subset H_+(U),
$$

ECP 21 (2016), paper 29. http://www.imstat.org/ecp/
Mid-concavity of survival probability for isotropic Lévy processes

(1)\n
\( \psi^D_t(x'') - \psi^D_t(x') \)
\[ = \int_{L(U)} \int_0^1 f^U_s(x'', z) \psi^D_{t-s}(z) \, ds \, dz + \int_{R(U)} \int_0^1 f^U_s(x'', z) \psi^D_{t-s}(z) \, ds \, dz \]
\[ = I + II. \]

\[ \psi^D_t(x'') - \psi^D_t(x') = \int_{L(U)} \int_0^1 f^U_s(x'', z) \psi^D_{t-s}(z) \, ds \, dz + \int_{R(U)} \int_0^1 f^U_s(x'', z) \psi^D_{t-s}(z) \, ds \, dz \]
\[ = III + IV. \]

Since \( I \) is defined by
\[ \int_{L(U)} \int_0^1 f^U_s(x'', z) \psi^D_{t-s}(z) \, ds \, dz. \]
\[ \int_{R(U)} \int_0^1 f^U_s(x'', z) \psi^D_{t-s}(z) \, ds \, dz \]
\[ = III + IV. \]

Recall that \( \frac{\lambda}{2} < x_1 < x_1'' < x_1''' \leq 0, \) where \( x_1' = x_1''' - x_1'' \)
\[ \psi^D_t(x'') \geq \psi^D_t(x''') - \psi^D_t(x'). \]

By the symmetry we obtain that \( y \rightarrow \psi^D_t(\gamma x_1 + \bar{x}) \) (or \( y \rightarrow \psi^D_t(y) \) when \( d = 1 \)) is concave on \((-\frac{\lambda}{2}, 0].\)

4 The shape of the first eigenfunction

Let us recall that \( X \) is a pure-jump isotropic unimodal Lévy process in \( \mathbb{R}^d \) with an infinite Lévy measure \( \nu, \) \( \psi \) is the characteristic exponent of \( X \) and \( p_t \) is its transition density. In this section we additionally assume that

\[ \lim_{|x| \to \infty} \frac{\psi(x)}{\log |x|} = \infty. \]  \hspace{1cm} (4.1)

This guarantees that for any \( t > 0 \) the function \( p_t \) is continuous and bounded on \( \mathbb{R}^d. \)

Let \( D \subset \mathbb{R}^d \) be a bounded open set. The condition (4.1) and formula (2.1) imply that for any fixed \( t > 0, \) \( x \in D \) the function \( y \rightarrow p_D(t, x, y) \) is continuous on \( D. \) Since \( p_D(t, x, y) = p_D(t, y, x), \) \( t > 0, x, y \in D \) we obtain that for any fixed \( t > 0, y \in D \) the function \( x \rightarrow p_D(t, x, y) \) is continuous on \( D. \) The transition operator \( P^D_t \) for the killed process \( X^D_t \) is defined by

\[ P^D_t f(x) = \int_D p_D(t, x, y) f(y) \, dy, \quad x \in D, t > 0. \]

Now we introduce the Dirichlet eigenvalue problem on \( D \) for the Lévy process \( X. \) Such problem is well known in the literature see e.g. [3]. \( \{P^D_t\}_{t \geq 0} \) forms a strongly continuous semigroup on \( L^2(D). \) Since \( p_D(t, x, y) \leq p_D(x-y), \) \( \|p_t\|_{\infty} < \infty \) and \( D \subset \mathbb{R}^d \) is bounded we obtain that for any \( t > 0 \) the operator \( P^D_t \) is a Hilbert-Schmidt operator.

From the general theory of semigroups there exists an orthonormal basis \( \{\varphi_n\}_{n=1}^{\infty} \) in \( L^2(D) \) and a corresponding sequence

\[ 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots, \quad \lim_{n \to \infty} \lambda_n = \infty, \]
Mid-concavity of survival probability for isotropic Lévy processes

such that for any \( n \in \mathbb{N}, t > 0, x \in D \) we have

\[
P^D_t \varphi_n(x) = e^{-\lambda_n t} \varphi_n(x).
\] (4.2)

\( \lambda_1 \) has multiplicity one and we may assume that \( \varphi_1 > 0 \) on \( D \). By properties of \( p_D(t, x, y) \) all eigenfunctions \( \varphi_n \) are bounded and continuous on \( D \). It is well known that

\[
p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y), \quad t > 0, x, y \in D.
\]

It follows that for any \( t > 0 \) and \( x \in D \) we have

\[
P^x(\tau_D > t) = \int_{D} p_D(t, x, y) \, dy = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \int_{D} \varphi_n(y) \, dy.
\]

Hence for any \( x \in D \) we have

\[
\lim_{t \to \infty} e^{\lambda_1 t} P^x(\tau_D > t) = \varphi_1(x) \int_{D} \varphi_1(y) \, dy.
\]

Using this and Theorems 1.1, 1.2 we immediately obtain the following results.

**Corollary 4.1.** Let \( X \) be a symmetric, pure jump, unimodal Lévy process in \( \mathbb{R} \) satisfying (4.1) with an infinite Lévy measure. Let \( D = (-a, a) \), where \( a > 0 \). Let \( \varphi_1 \) be the first eigenfunction of the spectral problem (4.2) on \( D \) for the process \( X \). Then for any \( t > 0 \) the function \( x \to \varphi_1(x) \) is nondecreasing on \((-a, 0]\), nonincreasing on \([0, a)\) and concave on \((-a/2, a/2)\).

**Corollary 4.2.** Let \( X \) be an isotropic, pure jump, unimodal Lévy process in \( \mathbb{R}^d, d \geq 2 \) satisfying (4.1) with an infinite Lévy measure. Let \( D = (-a, a) \times F \), where \( a > 0 \) and \( F \subset \mathbb{R}^{d-1} \) be a bounded Lipschitz domain. Let \( \varphi_1 \) be the first eigenfunction of the spectral problem (4.2) on \( D \) for the process \( X \). Let \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d \). Then for any \( t > 0 \) and \( \tilde{x} \in \{0\} \times F \) the function \( y \to \varphi_1(y e_1 + \tilde{x}) \) is nondecreasing on \((-a, 0]\), nonincreasing on \([0, a)\) and concave on \((-a/2, a/2)\).

**References**

[1] R. Bañuelos, R. D. DeBlassie, On the First Eigenfunction of the Symmetric Stable Process in a Bounded Lipschitz Domain, *Potential Anal.* 42 (2015), 573–583. MR-3306696

[2] R. Bañuelos, T. Kulczycki, P. J. Méndez-Hernández, On the shape of the ground state eigenfunction for stable processes, *Potential Anal.* 24 (2006), 205–221. MR-2217951

[3] Z.-Q. Chen, R. Song, Two-sided eigenvalue estimates for subordinate processes in domains, *J. Funct. Anal.* 226 (2005), 90–113. MR-2158176

[4] A. Greco, Fractional convexity maximum principle, preprint (2015), https://www.ma.utexas.edu/mp_arc/c/14/14-73.pdf

[5] N. Ikeda, S. Watanabe, On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes, *J. Math. Kyoto Univ.* 2 (1962), 79–95. MR-0142153

[6] M. Kaßmann, L. Silvestre, On the superharmonicity of the first eigenfunction of the fractional Laplacian for certain exponents, preprint (2014), http://math.uchicago.edu/~luis/preprints/cfe.pdf

[7] T. Kulczycki, On concavity of solutions of the Dirichlet problem for the equation \((-\Delta)^{1/2} \varphi = 1\) in convex planar regions, *J. Eur. Math. Soc. (JEMS)*, to appear, arXiv:1405.3846
Mid-concavity of survival probability for isotropic Lévy processes

[8] T. Kulczycki, M. Ryznar, Gradient estimates of harmonic functions and transition densities for Lévy processes, *Trans. Amer. Math. Soc.* **368** (2016), no. 1, 281–318. MR-3413864

[9] P. W. Millar, First passage distributions of processes with independent increments, *Ann. Probab.* **3** (1975), 215–233. MR-0368177

[10] M. Novaga, B. Ruffini, Brunn-Minkowski Inequality for the 1-Riesz Capacity and Level Set Convexity for the 1/2-Laplacian, *J. Convex Anal.* **22** (2015), 1125–1134. arXiv:1401.4322

[11] P. Sztonyk, On harmonic measure for Lévy processes, *Probab. Math. Statist.* **20** (2000), 383-390. MR-1825650

[12] T. Watanabe, The isoperimetric inequality for isotropic unimodal Lévy processes, *Z. Wahrschein. Verw. Gebiete* **63** (1983), 487-499. MR-0705619

**Acknowledgments.** I would like to thank the referee for a careful reading and helpful comments about this paper.
Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)
- Secure publication (LOCKSS\textsuperscript{1})
- Easy interface (EJMS\textsuperscript{2})

Economical model of EJP-ECP

- Non profit, sponsored by IMS\textsuperscript{3}, BS\textsuperscript{4}, ProjectEuclid\textsuperscript{5}
- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund\textsuperscript{6} (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

\textsuperscript{1}LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
\textsuperscript{2}EJMS: Electronic Journal Management System http://www.vtex.lt/en/ejms.html
\textsuperscript{3}IMS: Institute of Mathematical Statistics http://www.imstat.org/
\textsuperscript{4}BS: Bernoulli Society http://www.bernoulli-society.org/
\textsuperscript{5}Project Euclid: https://projecteuclid.org/
\textsuperscript{6}IMS Open Access Fund: http://www.imstat.org/publications/open.htm