Coherent States in Field Theory

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Abstract

Coherent states have three main properties: coherence, overcompleteness and intrinsic geometrization. These unique properties play fundamental roles in field theory, especially, in the description of classical domains and quantum fluctuations of physical fields, in the calculations of physical processes involving infinite number of virtual particles, in the derivation of functional integrals and various effective field theories, also in the determination of long-range orders and collective excitations, and finally in the exploration of origins of topologically nontrivial gauge fields and associated gauge degrees of freedom.

1 Introduction

In the past thirty-six years, the developments and applications of coherent states have been made tremendous progress. Yet, the idea of creating a coherent state for a quantum system was conceived well before that. In fact, back in 1926, Schrödinger first proposed the idea of what is now called “coherent states” \(I\) in connection with the quantum states of classical motion for a harmonic oscillator. In other words, the coherent states were invented immediately after the birth of quantum mechanics. However, between 1926 and 1962, activities in this field remained almost dormant, except for a few works in condensed matter physics [2, 3, 4] and particle physics [1, 5] in 50’s. It was not until some thirty five years after Schrödinger’s pioneering paper that the first modern and systematic application to field theory was made by Glauber and Sudarshan [7, 8] and launched this fruitful and important field of study in theoretical as well as experimental physics.

I became interested in the subject of coherent states about fifteen years ago. On the occasion of Prof. Sudarshan visiting Suzhou of China (1984), I listened for the first time in life a topic on coherent states presented by Prof. Sudarshan. As a second-year graduate student at that time, I was looking for some research problem on collective excitations in strongly interacted many-body systems (particularly in nuclear physics). Prof. Sudarshan’s lecture inspired me to think whether under constraint(s) of dynamical symmetries collective excitations can be described in terms

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of coherent states, as a result of multi-particle correlations (coherence). Later on I realized, this is indeed a very active subject covering problems from condensed matter physics to nuclear and particle physics. Of course, these coherent states have no longer the simple but beautiful form Glauber and Sudarshan proposed for light beams. Actually, these states are generated by complicated collective composite operators of particle-particle pairs or particle-hole pairs. Their mathematical structure were already developed in early 70’s by Perelomov and Gilmore [9, 10] based on the theory of Lie groups. Newdays, the concept of coherent states has been extensively investigated. Many methods based on coherent states have also been developed for various theoretical problems. Nevertheless, the original development of coherent states in quantum electromagnetic field (or more precisely, in the study of quantum optical coherence) has made tremendous influence in physics.

One can find that a large body of the literature on coherent states has appeared. This vast literature was exhaustively collected, catalogued and classified by Klauder and Skagerstam [11]. About the mathematical usefulness of coherent states as a new tool to study the unitary representations of Lie groups has been described in a well expository book by Perelomov [12]. A review article on the theory of coherent states and its applications that cover subjects of quantum mechanics, statistical mechanics, nonlinear dynamics and many-body physics has also been presented by author and his collaborators [13]. In this article, I will only concentrate on the topic of coherent states in field theory. As usual, it is not my intention to give a complete review about coherent states in field theory. An extensive review on coherent states in field theory and particle physics may be found in [14]. I will rather like to present here a discussion on whether one can formulate field theory in terms of coherent states such that the new formulation may bring some new insights to the next development of field theory in the new millennium. Coherent state can become a useful and important subject in physics because of its three unique properties: the coherence, the overcompleteness and the intrinsic geometrization. These unique properties, in certain contents, are fundamental to field theory. I will select some typical topics in field theory that can be efficiently described by coherent states based on these properties. These topics include the productions of coherent states in field theory, the basic formulation of quantum field theory in terms of coherent state functional integrals, the spontaneously symmetry breaking described from coherent states, and the effective field theories derived from coherent states. Also, I will “sprinkle” discussions about the geometrical phases of coherent states and their interpretation as gauge degrees of freedom in field theory, a subject which has still received increasing importance in one’s attempt to understand the fundamental of nature.

2 Photon coherent states

I may begin with the simplest coherent state of photons, or more generally speaking, bosons. Such a set of coherent states has been described in most of quantum mechanics text books and are familiared to most of physicists. It is indeed the most
popular coherent state that has been used widely in various fields. The coherent state of photons can describe not only the coherence of electromagnetic field, but also many other properties of bosonic fields. It is the basis of modern quantum optics [15], and it also provides a fundamental framework to quantum field theory, as one will see later.

By means of optical coherence, one may consider the \( n \)-th order correlation function of electromagnetic field:

\[
G_n(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n}) = \text{tr}\{\rho E^-(x_1) \cdots E^-(x_n) E^+(x_{n+1}) \cdots E^+(x_{2n})\},
\]

where \( x_i \) is the time-space coordinates, \( \rho \) denotes the density operator, and \( E^\pm(x_i) \) represent the electric field operators with positive and negative frequency. For simplification, the polarization of electric field is fixed. According to Glauber [7] the complete coherence of a radiation field is that all of the correlation functions satisfy the following factorization condition:

\[
G_n(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n}) = E^\ast(x_1) \cdots E^\ast(x_n) E(x_{n+1}) \cdots E(x_{2n}).
\]

This condition implies electric field operators must behave like classical field variables. It may also indicate the electric field operator should have its own eigenstates with the corresponding classical field variables as its eigenvalues:

\[
E^+(x_i)|\phi\rangle = E(x_i)|\phi\rangle, \quad \langle \phi|E^-(x_i) = \langle \phi|E^\ast(x_i).
\]

Moreover, the density operator must also be expressed in terms of the eigenstates \( |\phi\rangle \). Obviously, the conventional Fock space in quantum theory does not obey the above condition.

This is actually a nontrivial problem, because it requires a complete description of classical motions in terms of quantum states. Meantime, the operator \( E^\pm(x_i) \) itself is not a Hermitian operator. The eigenstate problem of a nonhermitian operator is unusual in quantum mechanics. Fortunately, such quantum states have already been constructed by Schrödinger soon after his invention of quantum mechanics in 1926. In order to answer the question how microscopic dynamics transits to macroscopic world, Schrödinger looked for quantum states which follow precisely the corresponding classical trajectories all the time, and meantime, the states must also be the exact solution of quantum dynamical equation (i.e., the Schrödinger equation). But only for harmonic oscillator, such states were constructed [1]:

\[
\phi_2(x) \sim \exp\left\{-\frac{1}{2}(x + z)^2\right\},
\]

where \( z \) is a complex variable. These states are actually the Gaussian wave packets centered on the classical trajectory \( z = (x + ip) \), \( x \) and \( p \) are the position of harmonic oscillator in the phase space that satisfies classical equations of motion. One can show that Eq. (4) is also an exact solution of Schrödinger equation. The classicality of Gaussian wave packets are manifested by the minimum uncertainty relationship:

\[
\Delta x^2 \Delta p^2 = \frac{\hbar^2}{4} \quad \text{and} \quad \Delta p = \Delta x.
\]
In other words, the wave packets governed by the Hamiltonian of harmonic oscillator follow classical trajectories and do not spread in time.

Glauder and Sudarshan discovered [7, 8] that such a wave packet is a superposition of Fock states. It is also an eigenstate of $E^+(x)$. In quantum field theory, electromagnetic field consists of infinite harmonic oscillating modes (photons). Explicitly, the Hamiltonian of quantum electromagnetic field (in Coulomb gauge) is given by

$$H = -\frac{1}{2} \int d^3x \{E^2 + B^2\},$$

(6)

where $E$ and $B$ are the electric and magnetic fields. The electromagnetic field can be expressed by the vector potential $A$: $E = -\partial A/\partial t, B = \nabla \times A$. It is convenient to expand the vector potential in terms of plane waves (Fourier series)

$$A(x, t) = \int \frac{d^3k}{\sqrt{2(2\pi)^3 2\omega_k}} \sum_\lambda \{ a^\lambda_k \varepsilon^\lambda(k)e^{-i k x} + a^\lambda_k\dagger \varepsilon^{\lambda\ast}(k)e^{i k x}\},$$

(7)

where $\varepsilon^\lambda(k)$ is the polarization vector of electromagnetic field, and $(a^\lambda_k, a^{\lambda\dagger}_k)$ are the creation and annihilation operators,

$$[a^\lambda_k, a^{\lambda\dagger}_{k'}] = \delta_{\lambda\lambda'}\delta_{kk'}, \quad [a^\lambda_k, a^\lambda_{k'}] = [a^{\lambda\dagger}_k, a^{\lambda\dagger}_{k'}] = 0.$$  

(8)

Then the Hamiltonian of electromagnetic field can be deduced to

$$H = \sum_{k\lambda} \omega_k (a^{\lambda\dagger}_k a^\lambda_k + 1/2),$$

(9)

which means that the electromagnetic field consists of infinite individual electromagnetic modes, i.e., photons. Each photon corresponds to a harmonic oscillator.

In the particle number representation, the Gaussian wave packet can be written as

$$|z\rangle = \exp\left(-\frac{1}{2} |z|^2\right) \exp(z a\dagger)|0\rangle.$$

(10)

where $|0\rangle$ is the vacuum state: $a|0\rangle = 0$. From the above expression, it is easy to show that the wave packet is also an eigenstate of the annihilation operator $a$:

$$a|z\rangle = z|z\rangle.$$  

(11)

Thus, the quantum state describing the optical coherence of electromagnetic field can be expressed by

$$|\{z^\lambda_k\}\rangle = \exp \left\{-\frac{1}{2} \int d^3k \sum_\lambda |z^\lambda_k|^2\right\} \exp \left\{ \int d^3k \sum_\lambda z^\lambda_k a^{\lambda\dagger}_k\right\}|0\rangle,$$

(12)

which is an eigenstate of the positive frequency part of the electric field operator,

$$E^+(x)|\{z^\lambda_k\}\rangle = \mathcal{E}(x)|\{z^\lambda_k\}\rangle, \quad \mathcal{E}(x) = i \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\sqrt{\omega_k}}{2} \sum_\lambda z^\lambda_k \varepsilon^\lambda(k)e^{-i(\omega_k t - k \cdot x)}.$$  

(13)
Besides, the above state has another very important property: it supports the following resolution of identity:

$$\int |\{z^{\lambda}_k\}\rangle\langle\{z^{\lambda}_k\}| \prod_{k\lambda} \frac{dz^\lambda_k dz_k^{\lambda*}}{2\pi} = I. \quad (14)$$

In other words, these states in the complex space (in terms of the variable $z$) form a complete set of states (more precisely speaking, it is overcomplete because of the continuity of these states). This complete set is certainly very different from the set of Fock states. Because of the overcompleteness and the analyticity of these states, one can expand the density operator by (12) in a diagonal form (the so-called P-representation [7]):

$$\rho = \int P(\{z^{\lambda}_k\})|\{z^{\lambda}_k\}\rangle\langle\{z^{\lambda}_k\}| \prod_{k\lambda} dz^\lambda_k dz_k^{\lambda*},$$

$$\text{tr} \rho = \int P(\{z^{\lambda}_k\}) \prod_{k\lambda} dz^\lambda_k dz_k^{\lambda*} = 1. \quad (15)$$

where $P(z)$ is a weight function. In terms of these states [12], the factorization criterion of coherent light beams is automatically satisfied. Glauber named such states the coherent states. To be more specific, one may call them the “photon coherent states”. Physically, the photon coherent states have a well-defined phase for each mode. Therefore, coherent light beams can be completely described in quantum mechanics in terms of photon coherent states. For those who wish to have more detailed discussion on physical consequences of the photon coherent states in quantum optics, please refer to the excellent book by Klauder and Sudarshan [15].

3 Coherent states and $S$-Matrix

As we have seen, the photon coherent state was introduced by the requirement of optical coherence. Here, I may ask a more general question, namely, how are photon coherent states generated in field theory? In field theory, all physical quantities are derivable from the vacuum-to-vacuum transition amplitude in the presence of external sources. It can show that the final state in such processes is a coherent state if there is no other interactions except for a linear interaction with the external field.

To be specific, one may consider the electromagnetic field interacting with a classical source:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu,$$  \hspace{1cm} (16)

where the classical source is a conserved current: $\partial^\mu j_\mu = 0$. In the Feynman gauge, the equation of motion for the electromagnetic field is given by

$$\partial_\mu F^{\mu\nu} = \Box A^\nu = j^\nu. \quad (17)$$

A general solution of the above equation is

$$A^\mu(x) = A_0^\mu(x) + \int d^4 y \Delta(x - y) j^\mu(y), \quad (18)$$
where $A_0^\mu(x)$ is the solution of free field, and $\Delta(x - y)$ is the Green function determined by $\Box_x \Delta(x - y) = \delta^4(x - y)$. If one assumes that the interaction is switched on adiabatically in a finite time interval, then

$$A^\mu(x) = A^\mu_{in}(x) + \int d^4y \Delta_{ret}(x - y) j^\mu(y)$$

$$= A^\mu_{out}(x) + \int d^4y \Delta_{adv}(x - y) j^\mu(y), \quad (19)$$

where the retarded and advanced Green functions are given by

$$\Delta_{ret}(x) = -\frac{1}{(2\pi)^4} \int d^4p \frac{e^{-ipx}}{(p_0 \pm i\epsilon)^2 - p^2}, \quad (20)$$

and $A^\mu_{in}$ and $A^\mu_{out}$ are free fields describing the photon field before and after its interaction with the classical course $j^\mu$. The corresponding photon states are the in- and out-states (denoted by $|\rangle_{in}$ and $|\rangle_{out}$, respectively). The in- and out-states form two complete sets of free states constructed as a Fock space by the free field operators $A^\mu_{in}$ and $A^\mu_{out}$. Therefore, there must exist a unitary transformation $S$ (namely $S$-matrix) to connect these two complete sets:

$$A^\mu_{out} = S A^\mu_{in} S^\dagger, \quad |\rangle_{out} = S^\dagger |\rangle_{in}. \quad (21)$$

From (19), one can see that

$$A^\mu_{out}(x) = A^\mu_{in}(x) + \int d^4y [\Delta_{ret}(x - y) - \Delta_{adv}(x - y)] j^\mu(y)$$

$$= A^\mu_{in}(x) + A^\mu_{cl}(x), \quad (22)$$

and $A^\mu_{cl}(x)$ is a c-number (classical) field generated by the classical current $j^\mu(x)$. Notice that $\Delta_{adv}(x) - \Delta_{ret}(x) = \Delta(x)$ which relates to the commutator of free fields $[A^\mu_{in}(x), A^\nu_{in}(y)] = [A^\mu_{out}(x), A^\nu_{out}(y)] = -ig^{\mu\nu} \Delta(x - y)$. One may check that the $S$-matrix can be written as

$$S = \exp \{-i \int d^4x A^\mu_{in} \cdot j(x)\} = \exp \{-i \int d^4x A^\mu_{out} \cdot j(x)\}. \quad (23)$$

If we start at time $-\infty$ from the vacuum state $|0\rangle_{in}$, the final state after the free field $A^\mu(x)$ interacted with the classical current $j^\mu(x)$ becomes a coherent state:

$$|0\rangle_{out} = \exp \{i \int d^4x A^\mu_{in}(x) \cdot j(x)\} |0\rangle_{in}. \quad (24)$$

In terms of the Fourier expansion,

$$A^\mu(x) = \int \frac{d^3k}{\sqrt{2(2\pi)^3|k|}} \sum_\lambda \{a^\lambda_k \varepsilon^\lambda_\mu(k) e^{-ikx} + a^{\dagger \lambda}_k \varepsilon^{\dagger \lambda}_\mu(k) e^{ikx}\}. \quad (25)$$

The final state can be expressed as

$$|0\rangle_{out} = \exp \{-\frac{1}{2} \int d^3k \sum_\lambda |z_\lambda|^2\} \exp \{\int d^3k \sum_\lambda z_\lambda \lambda_k \} |0\rangle_{in} = \{|z_\lambda\rangle\}, \quad (26)$$

$$\sum_\lambda |z_\lambda|^2 = 1.$$
where $z^\lambda_k = e^{\lambda(k) \cdot j(k)}$, and $j^\mu(k)$ is the Fourier transform of the classical current $j^\mu(x)$. This is the same photon coherent states introduced by Glauber in the study of quantum optical coherence.

Indeed, one can derive similarly the photon coherent state for the laser beams (discussed in the previous section) from a more microscopic picture. The Hamiltonian in quantum optics that describes the interaction between $N$ atoms and the electromagnetic field can be written as:

$$H = \sum_k \omega_k a_k^\dagger a_k + \sum_i \epsilon_i \sigma_0^i + \sum_{k,i} \gamma_{ki}(t) \left\{ \frac{\sigma_+^i}{\sqrt{N}} a_k + \frac{\sigma_-^i}{\sqrt{N}} a_k^\dagger \right\},$$

(27)

where $\gamma_{ki}$ are the coupling coefficients between atoms and electromagnetic field. One of the crucial assumptions made in the construction of the above Hamiltonian for laser beams is that each of the $N$ atoms, labeled by the index $i$, is a two-level system and therefore its dynamical variables are the usual Pauli operators $\{\sigma_0^i, \sigma_+^i, \sigma_-^i\}$. Furthermore, the atomic variables can be treated as a classical source (i.e. the spin operators $\sigma^i$ can be regarded as $c$-numbers), and the coupling strength $\gamma_{ki}$ are identical for all the atoms (i.e. $\gamma_{ki} = \gamma_k$). Then, Eq. (27) is reduced to

$$H = \sum_k \omega_k a_k^\dagger a_k + \sum_i \epsilon_i \langle \sigma_0^i \rangle + \sum_{k,i} \gamma_{ki}(t) \left\{ \frac{\langle \sigma_+^i \rangle}{\sqrt{N}} a_k + \frac{\langle \sigma_-^i \rangle}{\sqrt{N}} a_k^\dagger \right\}$$

$$= \sum_k \omega_k a_k^\dagger a_k + \sum_k \left[ \lambda_k(t) a_k^\dagger + \lambda_k^*(t) a_k \right] + \text{const.}$$

(28)

This corresponds to the electromagnetic field interacting with an external time dependent source. The general solution of the Schrödinger equation for this Hamiltonian is the photon coherent states. This provides the microscopic picture how the photon coherent state is generated and why it becomes the fundamental of quantum optics.

Soon after the development of coherent states in optical coherence, it was found that the photon coherent state also plays an important role in solving the infrared divergence in quantum electrodynamics for electron scatterings [16, 17, 18, 19, 20] (also see the review by Papanicolaou [21]). As it is well known, the matrix element in quantum electrodynamics for the scattering of an initial state containing a finite number of electrons and photon into a similar final state has a logarithmical infrared divergence for the small momentum $k$ [22]. This is because in an actual scattering experiment, electromagnetic fields interact with the source particles so that an infinite number of soft photons are emitted. These emitted soft photons form a coherent state to the final state, as we have discussed. To be more specific, consider a single electron scattering. The source particle can be represented by a classical current. The Fourier transform of the classical current is given by

$$j^\mu(k) = \frac{ie}{\sqrt{2(2\pi)^3 |k|}} \left( \frac{p_f^\mu}{p_f \cdot k} - \frac{p_i^\mu}{p_i \cdot k} \right),$$

(29)

where $p_{i,f}$ are the electron’s momentum in the initial and final states. If one sums the cross sections over all possible final states containing any number of soft photons
with momenta below the threshold of observability [by using the photon coherent state (26) with the above classical source], the infrared divergence is canceled. This gives a beautiful solution to the infrared problem in quantum electrodynamics.

Moreover, one can also show that if the matrix element for scatterings are calculated with the initial and final states containing infinite number of soft photons by the photon coherent state, the infrared divergences are canceled order by order at matrix element level (not only in cross sections) [16]. The photon coherent state may also use to remove the similar infrared problem in quantum gravity, as noticed by Weinberg [23]. These are perhaps the second important applications of the photon coherent state in field theory. In addition, one has also attempted to use coherent states to treat infrared divergences in non-abelian gauge theory [24]. However, in the non-abelian gauge theory, the infrared divergence is much more complicated [25, 26]. It contains two-type infrared divergences, the massless soft infrared divergence and collinear divergence. It is not clear whether one can construct some non-abelian coherent states to handle both the soft and collinear infrared divergences.

4 Functional integrals in field theory

When the quantum fields interact with quantum fields rather than classical external fields, the $S$-matrix (or the time-evolution operator) does not generate coherent states from the incoming vacuum. In such cases, coherent states are useful in the derivation of functional integral in field theory. Quantum field theory can be reformulated in terms of coherent states not only because of its classicality and being eigenstates of the annihilation operator. As we have mentioned, the coherent states are overcomplete:

$$\int |z\rangle\langle z| \frac{dzdz^*}{2\pi} = I.$$  \hspace{1cm} (30)

All these three properties (the classicality, the eigenstates of the positive frequency part of field operator and the overcompleteness) together allow one to reformulate quantum field theory in terms of a functional integral. Actually, the content of this section can be found in many text books, but for completeness I will repeat these discussions here.

The ordinary path integral of quantum mechanics developed by Feynman [27] can be obtained from the evolution operator by writing the evolution operator as

$$U(t_f,t_0) = \exp \{-iH(t_f - t_0)\} = \lim_{N \to \infty} \left( \exp \{-iH\frac{t_f - t_0}{N}\} \right)^N$$  \hspace{1cm} (31)

and then inserting a resolution of the identity in terms of the position states

$$\int dx |x\rangle\langle x| = I$$  \hspace{1cm} (32)

between the terms of above product. This results in the familiar path integral of quantum mechanics,

$$\langle x'(t_f)|x(t_0)\rangle = \langle x'|U(t_f,t_0)|x\rangle = \int [dx(t)] \exp \left\{ i \int_{t_0}^{t_f} dt \mathcal{L}(x(t),\dot{x}(t)) \right\},$$  \hspace{1cm} (33)
where $\mathcal{L}$ is a classical Lagrangian which generally has a form of

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2} \left( \frac{dx}{dt} \right)^2 - V(x), \quad (34)$$

and $[dx(t)] \equiv \prod_{t_0 \leq t \leq t_f} dx(t)$ is a functional measure of the path integration $[28]$. Instead of using the basis of the position eigenstates $[32]$, we may use the coherent state basis and insert the resolution of identity $[30]$ between the terms of product $[31]$. Then a phase space formulation of path integrals can be obtained as was first proposed by Klauder $[6, 30]$ (More detailed derivation will be given later in the application to field theory),

$$\langle z'(t_f) | z(t_0) \rangle = \langle z'| U(t_f, t_0) | z \rangle = \int [dx(t)] \frac{dp(t)}{2\pi} \exp \left\{ i \int_{t_0}^{t_f} dt \mathcal{L}(x(t), p(t)) \right\}, \quad (35)$$

with

$$\mathcal{L}(x, p) = \langle z| i \frac{d}{dt} | z \rangle - \langle z| H | z \rangle = \frac{1}{2} \left( \frac{dx}{dt} - x \frac{dp}{dt} \right) - \mathcal{H}(x, p). \quad (36)$$

where $z = (x + ip)/\sqrt{2}$ and $z^* = (x - ip)/\sqrt{2}$, with the initial and final positions $x(t_0)$ and $x(t_f)$ fixed. This derivation of Feynman’s path integral is particularly useful in obtaining a functional integral of quantum field theory.

To derive a functional integral of quantum field theory, we may start with a neutral scalar field $\phi(x)$, for simplification. The Lagrangian density of a neutral field is given by

$$\mathcal{L} = \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial t} \right)^2 - m^2 \phi^2 \right] - V(\phi), \quad (37)$$

where $V(\phi)$ represents a self-interacting potential, such as $\frac{\lambda}{4!} \phi^4(x)$. The canonical momentum density conjugate to $\phi(x, t)$ is determined by $\pi(\phi, t) = \partial \mathcal{L}/\partial \dot{\phi}(x, t)$. Then the canonical quantization leads to

$$[\phi_{op}(x, t), \pi_{op}(x', t)] = i\delta^3(x - x'), \quad (38)$$

In the plane-wave expansion, one has

$$\phi_{op}(x, t) = \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{1}{\omega_k} \left\{ a_k e^{-ikx} + a_k^\dagger e^{ikx} \right\}, \quad (39)$$

$$\pi_{op}(x, t) = -i \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{\omega_k}{2} \left\{ a_k e^{-ikx} - a_k^\dagger e^{ikx} \right\},$$

and the quantum Hamiltonian can be written as

$$H(t) = \int d^3 x : \left\{ \frac{1}{2} [\pi_{op}^2 + (\nabla \phi_{op})^2] + m^2 \phi_{op}^2 \right\} + V(\phi_{op}) : \quad (40)$$
denotes the normal ordering with respect to the creation and annihilation operators $a_k^\dagger$ and $a_k$. Now, one can define the scalar field coherent state as

$$
|\phi\pi\rangle = \exp\left\{i \int d^3x [\pi(x)\phi_{op}(x) - \phi(x)\pi_{op}(x)]\right\}|0\rangle
$$

$$
= \exp\left\{-\frac{1}{2} \int d^3k \, \kappa_k^2\right\} \exp\left\{\int d^3k \, \kappa_k \phi_k^\dagger\phi_k\right\}|0\rangle,
$$

(41)

from which a functional integral of field theory can be derived explicitly. Note that the coherent state in field theory is defined at a given instant time $t$ over the whole space $\{x\}$.

Since the Hamiltonian formalism of field theory is the same as in quantum mechanics, one can directly calculate the time Green’s function defined as the matrix element of the evolution operator in coherent state basis,

$$
G(t_f, t_0) = \langle \phi'\pi'|U(t_f, t_0)|\phi\pi\rangle = \langle \phi'\pi'|T \exp\left\{-i \int_{t_0}^{t_f} dtH(t)\right\}|\phi\pi\rangle,
$$

(42)

where $T$ is the time-ordering operator. One may slice the time interval $t_f - t_0$ into $N$ equal segments: $\varepsilon = (t_f - t_0)/N$ so that in the sense of $N \to \infty$, the evolution operator can be written as a subsequently multiplication of the evolution operator in the interval $\varepsilon$:

$$
U(t_f, t_0) = \exp\left\{-i\varepsilon H(t_n)\right\} \exp\left\{-i\varepsilon H(t_{n-1})\right\} \ldots \exp\left\{-i\varepsilon H(t_2)\right\} \exp\left\{-i\varepsilon H(t_1)\right\}.
$$

(43)

Using the same procedure as in the derivation of Feynman’s path integral in quantum mechanics, one should insert the resolution of identity,

$$
\int [d\phi(x)] \left[\frac{d\pi(x)}{2\pi}\right] |\phi\pi\rangle \langle \phi\pi| = I,
$$

(44)

at each interval point, where $[d\phi(x)] \equiv \prod_{-\infty < x < \infty} d\phi(x)$, etc. are defined over the whole space. Then

$$
G(t_f, t_0) = \lim_{N \to \infty} \int \left(\prod_{i=1}^{N-1} [d\phi_i(x)] \left[\frac{d\pi_i(x)}{2\pi}\right]\right) \prod_{i=1}^{N} \langle \phi_i\pi_i| \exp\left\{-i\varepsilon H(t_i)\right\}|\phi_{i-1}\pi_{i-1}\rangle.
$$

(45)

Up to the first order in $\varepsilon$,

$$
\langle \phi_i\pi_i| \exp\left(-i\varepsilon H(t_i)\right) |\phi_{i-1}\pi_{i-1}\rangle \approx \langle \phi_i\pi_i|\phi_{i-1}\pi_{i-1}\rangle \exp\left(-i\varepsilon \frac{\langle \phi_i\pi_i|H(t_i)|\phi_{i-1}\pi_{i-1}\rangle}{\langle \phi_i\pi_i|\phi_{i-1}\pi_{i-1}\rangle}\right).
$$

(46)

Note that the coherent state $|\phi\pi\rangle$ is normalized. In the limit of $\varepsilon \to 0$ (i.e. $N \to \infty$),

$$
\langle \phi_i\pi_i|\phi_{i-1}\pi_{i-1}\rangle = 1 - \langle \phi_i\pi_i|(|\phi_i\pi_i) - |\phi_{i-1}\pi_{i-1}\rangle
$$

$$
\approx \exp\left\{i\varepsilon \langle \phi_i\pi_i|\Delta|\phi_i\pi_i\rangle\right\},
$$

(47)
where $\Delta |\phi_i\pi_i\rangle \equiv |\phi_i\pi_i\rangle - |\phi_{i-1}\pi_{i-1}\rangle$. Then, the Green’s function becomes

$$G(t_f, t_0) = \lim_{N \to \infty} \int \left( \prod_{i=1}^{N-1} [d\phi_i(x)] \frac{d\pi_i(x)}{2\pi} \right) \times \exp i \sum_{i=1}^{N} \varepsilon \left\{ \langle \phi_i\pi_i | i \frac{\Delta |\phi_i\pi_i\rangle}{\varepsilon} - \langle \phi_i\pi_i | H(t_i) |\phi_i\pi_i\rangle \right\}$$

$$= \int [d\phi(x)] \frac{d\pi(x)}{2\pi} \exp \left\{ i \int_{t_0}^{t_f} dt \left[ \langle \phi\pi | i \frac{d}{dt} |\phi\pi\rangle - \langle \phi\pi | H |\phi\pi\rangle \right] \right\}$$

$$= \int [d\phi(x)] \frac{d\pi(x)}{2\pi} \exp \left\{ i \int_{t_0}^{t_f} dt \int d^3x \left[ \frac{1}{2} (\dot{\phi} - \phi \dot{\pi}) - \mathcal{H}(x) \right] \right\}, \quad (48)$$

with

$$\mathcal{H} = \frac{1}{2} [(\nabla \phi)^2 + m^2 \phi^2] + V(\phi). \quad (49)$$

As we see that the coherent state gives a natural derivation of path integrals in field theory.

In field theory, the correlations between $n$ fields are defined by the $n$-point Green functions,

$$G^{(n)}(x_1, \cdots, x_n) = \langle 0 | T(\phi(x_1) \cdots \phi(x_n)) | 0 \rangle, \quad (50)$$

which can be determined from the generating functional $W(J)$ which is defined as the vacuum-to-vacuum amplitude in the presence of external current $J(x)$:

$$W(J) = \langle 0 | U(-\infty, \infty) | 0 \rangle_J. \quad (51)$$

This generating functional can then be expressed in terms of the time Green’s function $G(t_f, t_0)$ by adding a term $\int d^3x J(x) \phi(x)$ in the exponent and then taking $t_0 \to -\infty$ and $t_f \to \infty$:

$$W(J) = \int [d\phi(x)] \frac{d\pi(x)}{2\pi} \exp \left\{ \int d^4x \left[ \frac{1}{2} (\pi \dot{\phi} - \phi \dot{\pi}) - \mathcal{H}(x) + J(x) \phi(x) \right] \right\}$$

$$= \int [d\phi(x)] \exp \left\{ \int d^4x \left[ L(x) + J(x) \phi(x) \right] \right\} = \exp i Z(J), \quad (52)$$

and $Z(J)$ is a functional partition function in quantum field theory. The $n$-point Green’s functions containing only the connected graphs is given by,

$$G^{(n)}_c(x_1, \cdots, x_n) = \left. \frac{(-i)^n}{W(J)} \frac{\delta^n W(J)}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} = \left. \frac{(-i)^{n-1}}{W(J)} \frac{\delta^n Z(J)}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0}. \quad (53)$$

Taking the stationary phase approximation of $W(J)$ naturally results in the classical equations of motions [25, 30]. On the other hand, after integrating out the $\pi(x)$ field, the functional integrals $W(I)$ and $Z(J)$ become covariant. Now, all physical quantities in field theory are, in principle, derivable from $W(J)$ or $Z(J)$ in a covariant form, which are standard in text books. I should not repeat these discussion here.

The above formulation is only for bosonic fields. Field theory that describes the real world must also involve fermion (matter) fields. To formulate a similar
functional integral for fermionic fields, one needs to introduce fermion coherent states. Similarly, one may try to construct such a coherent state as an eigenstate of the fermion annihilation operator:

\[ c_i |\xi\rangle = \xi_i |\xi\rangle, \quad \{c_i, c_j^\dagger\} = \delta_{ij}. \]  

(54)

However, since the fermion creation and annihilation operators satisfy the anticommutation relationship, the eigenvalue \(\xi_i\) of the annihilation operator is its classical analogy which cannot be an ordinary number. The quantum-classical corresponding principle simply requires that \(\xi_i\) must be anticommute:

\[ \xi_i \xi_j = -\xi_j \xi_i, \quad \xi_i \xi_j^* = \xi_j^* \xi_i, \quad \xi_i^2 = 0. \]  

(55)

The numbers satisfies the above relations are called Grassmann numbers. Functions of Grassmann numbers are given by

\[ f(\xi_i) = f^0 + f^{(1)}_i \xi_i + f^{(2)}_{ij} \xi_i \xi_j + \cdots, \]  

(56)

and the Grassmann integrals are defined as

\[ \int d\xi = 0, \quad \int d\xi \xi = 1. \]  

(57)

Using these properties, the fermionic coherent state can be written explicitly as

\[ |\xi\rangle = \exp\left\{ -\frac{1}{2} \sum_i \xi_i^* \xi_i \right\} \exp\left\{ \xi_i c_i^\dagger \right\} |0\rangle, \]  

(58)

where \(|0\rangle\) is the fermion vacuum state: \(c_i |0\rangle = 0\). For the fermionic coherent states, the resolution of identity can be written similarly as

\[ \int \prod_i d\xi_i^* \xi_i |\xi\rangle \langle \xi| = I. \]  

(59)

Based on these properties of fermionic coherent states, the functional integral of fermion fields is rather easy to derive [31]. Consider a fermion field \(\psi\) coupling with a scalar boson field \(\phi\), the Lagrangian is

\[ \mathcal{L}(\bar{\psi}, \psi, \phi) = \bar{\psi}(i \not\partial - m) \psi + \frac{1}{2}[(\partial \phi)^2 - m^2 \phi^2] - g \phi \bar{\psi} \psi. \]  

(60)

Following the same procedure, one obtains the functional integral

\[ W(\bar{\zeta}, \zeta, J) = \int [d\bar{\xi}] [d\xi] [d\phi] \exp\left\{ \int d^4 x \left[ \mathcal{L}(\bar{\xi}, \xi, \phi) + \bar{\zeta} \xi + \bar{\zeta} \xi + J \phi \right] \right\}, \]  

(61)

where the fermionic sources \(\zeta^*, \zeta\) are also Grassmann numbers. In fact, Schwinger first introduced such a generating functional for fermion fields, in order to derive the fermion field Green’s functions [3].

These results can be extended to quantum electrodynamics and quantum chromodynamics, although the later will be more complicated because of non-abelian
gauge fields. In most text books, the discussions on coherent states in field theory are restricted usually in contents of the above formulation. One may derive from such a formulation almost everything about perturbative field theory, such as Feynman rules, the perturbation expansion, and renormalization analysis, etc. At this point, the functional integral of quantum field theory in terms of coherent states is actually nothing special. It is the standard formulation that one can also obtain from other methods. If the field theory can be treated perturbatively, one can always solve the theory in one or other ways, based on the developments of field theory in the last fifty years. The challenge in field theory we faced today (or in the past three decades since the theory of the strong interaction, namely quantum chromodynamics, was proposed) is in the nonperturbation section. There is no a systematic approach in field theory that one can used to completely solve a nonperturbation problem, such as the vacuum structure in non-abelian gauge theory, or bound state problem in strongly coupling systems. In the next few sections, I will try to illustrate some specific problems to see if the generalized coherent states developed later can play some useful roles to the nonperturbation field theory of strongly interacted systems.

5 Squeezed Coherent States and Quantum Fluctuations

The first example I go to discuss is how one may use squeezed states to study the low energy quantum fluctuations in strong interaction theory. Different from what is discussed in the previous section where the content may be found in text books, here I should point out that the formulation presented in this section has actually not been completed yet, and more work remains for further investigations.

The squeezed states are a generalization of photon coherent states. Again, the squeezed states first attracted the attention in quantum optics. In the early development, the principal potential applications of squeezed states are in the field of optical communications and “quantum nondemolition experiments” designed for the detection of gravity waves. Later on, because of the “capacity” of treating quantum fluctuations, squeezed states have been used in various subjects, such as quantum measurement theory, quantum nonlinear dynamics, molecular dynamics, dissipative quantum mechanics as well as in quantum gravity and condensed matter physics.

In quantum optics, the uncertainty principle places a damper on the enthusiasm with which quantum engineers approach the problem of coding and transmitting information by optical means. Specifically, the quantum noise inherent in light beams places a limit on the information capacity of an optical beam. Since the uncertainty principle is a statement about areas in phase space, noise levels in different quadratures are statements about intersections of uncertainty ellipses with these axes. Any procedure which can deform or squeeze the uncertainty circle to an ellipse can in principle be used for noise reduction in one of the quadratures. Such squeezing does not violate the uncertainty principle; rather, it places the larger
uncertainty in a quadrature not involved in the information transmission process. A typical procedure for squeezing the error ellipse involves applying a classical source to drive two photon emission and absorption processes in much the same way that single photon processes can be used to generate a coherent state of the electromagnetic field.

For simplification, one may consider a basic Hamiltonian describing two photon processes in a single mode

\[ H = \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + f(t) \hat{a}^1 \hat{a}^2 + f^*(t) \hat{a}^2. \]  

(62)

Then, the squeezed state can be obtained by directly solving the time dependent Schrödinger equation. If the initial state is the photon vacuum, a general solution is

\[ |\beta\rangle = \exp \left( \frac{1}{2} \beta \hat{a}^1 - \frac{1}{2} \beta^* \hat{a}^2 \right) |0\rangle e^{i\varphi}. \]  

(63)

This is the squeezed states generated by Eq.(62). If one defines the photon’s position and momentum coordinates \((x, p)\) in terms of the creation and annihilation operators as in the previous section, one can then find that

\[ \Delta x^2 = \langle \beta | x^2 | \beta \rangle = \frac{1}{2} \cosh |\beta| + \frac{\beta}{|\beta|} \sinh |\beta|^2, \]

\[ \Delta p^2 = \langle \beta | p^2 | \beta \rangle = \frac{1}{2} \cosh |\beta| - \frac{\beta}{|\beta|} \sinh |\beta|^2, \]  

(64)

and \(\Delta x \neq \Delta p\) but \(\Delta x \Delta p \geq \frac{1}{2}\) (here we set \(\hbar = 1\)). While the vacuum has a circle uncertainty \((\Delta x = \Delta p)\) in phase space. This shows that the operator \(D_{sq}(\beta) = \exp \left( \frac{1}{2} \beta \hat{a}^1 - \frac{1}{2} \beta^* \hat{a}^2 \right)\) squeezes the uncertainty circle of a wave packet into an ellipse so that quantum fluctuation (noise) can be reduced in one of the quadratures.

In general, it is desirable to squeeze a field coherent state which can be generated by

\[ H = \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + f_2(t) \hat{a}^1 \hat{a}^2 + f^*_2(t) \hat{a}^2 + f_1(t) \hat{a}^1 + f^*_1(t) \hat{a}. \]  

(65)

The sequence in which the processes of coherent state formation and squeezing occur is governed by the time dependence of the functions \(f_2(t)\) and \(f_1(t)\). The general form of the state at the time \(t\) can be expressed (apart from a phase factor) by

\[ |z\beta\rangle = \exp \left( za^\dagger - z^* a \right) \exp \left( \frac{\beta}{2} \hat{a}^1 - \frac{\beta^*}{2} \hat{a}^2 \right) |0\rangle, \]  

(66)

where the complex variables \(z\) and \(\beta\) are functions of the time \(t\) in general. Eq. (66) is usually called squeezed coherent state. The physical process of the squeezed coherent states can be understood as follows: by first squeezing the vacuum (the wave packet) by the two photon excitations, and then displacing it as a photon coherent state by the external source.

Using group theory, one can show that the squeezed coherent states must also form a overcomplete set of states.

\[ \int \frac{dz dz^* df dg}{2\pi 2\pi} |z\beta\rangle \langle z\beta| = I \]  

(67)
where the variables $f$ and $g$ are introduced by Jackiw and Kerman \[37\] to characterize quantum fluctuations (noise) of the position and momentum:

$$\Delta x^2 = f, \quad \Delta p^2 = \frac{1}{4f} + 4fg^2,$$

(68)

These two variables relate to the squeezing parameter $\beta$ by (64). By the completeness, one can also derive a path integral of quantum mechanics in the squeezed coherent state representation:

$$\langle z'(t_f)\beta'(t_f)|z(t_0)\beta(t_0)\rangle = \int \left[ \frac{dx(t)}{2\pi} \right] \left[ \frac{dp(t)}{2\pi} \right] \left[ \frac{df(t)}{2\pi} \right] \left[ \frac{dg(t)}{2\pi} \right] \exp \left\{ i \int_{t_0}^{t_f} dt \left[ \frac{1}{2}(p\dot{x} - x\dot{p}) - f\dot{g} - \mathcal{H}_{\text{eff}} \right] \right\},$$

(69)

where the effective Hamiltonian is the matrix element of the Hamiltonian operator $[H = p^2/2 + V(x)]$ in the squeezed coherent state:

$$\mathcal{H}_{\text{eff}}(x, p, f, g) = \frac{1}{2}p^2 + \frac{1}{8f} + 2fg^2 + \exp \left( \frac{f}{2} \frac{\partial^2}{\partial x^2} \right) V(x).$$

(70)

The expression of path integral in squeezed coherent states shows that $f$ and $g$ which characterize quantum fluctuations become a pair of conjugate variables. The extremal values of the exponent in the path integral leads to the following generalized equations of motion:

$$\frac{dx}{dt} = \frac{\partial \mathcal{H}_{\text{eff}}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}_{\text{eff}}}{\partial x},$$

$$\frac{df}{dt} = \frac{\partial \mathcal{H}_{\text{eff}}}{\partial g}, \quad \frac{dg}{dt} = -\frac{\partial \mathcal{H}_{\text{eff}}}{\partial f}.$$  

(71)

Physically, the equations of motion for $(x, p)$ determine the time evolution of the center of wave packets and those for $f, g$ characterize the time evolution of the quantum fluctuations (quadratures). Therefore, the variables $(f, g)$ describe the squeezing and spreading of quadratures in times, which provides a classical-like dynamical theory for the controlling of quantum noise and signal.

It is worth pointing out that although the concept of squeezed state first attracted the attention in quantum optics, squeezed state itself was introduced much earlier by Valatin and Bulter in the study of superfluidity [4, 38]. Similar to the BCS state of superconductivity (which I will discuss later), Valatin’s superfluid ground state is defined as

$$|\{\beta_k\}\rangle = \exp \left\{ \sum_k (z_k a_k^\dagger - z_k^* a_k) \right\} \exp \left\{ \sum_k \frac{1}{2}(\beta_k a_k^\dagger a_{-k}^\dagger - \beta_k^* a_k a_{-k}) \right\} |0\rangle$$

$$= \exp \left\{ \sum_k (z_k a_k^\dagger - z_k^* a_k) \right\} |\{\beta_k\}\rangle$$

(72)

which is the standard form of squeezed coherent states one currently used. In many-body picture, such states have two consequences. The squeezed operator
exp \left\{ \sum_k \frac{1}{2} (\beta_k a_k^\dagger a_{-k} - \beta_k^* a_k a_{-k}) \right\} acting on the trivial vacuum generates a canonical transformation of quasiparticles:

$$\alpha_k = \cosh |\beta_k| a_k - \frac{\beta_k}{|\beta_k|} \sinh |\beta_k| a_{-k}^\dagger, \quad \alpha_k \{ \beta_k \} = 0. \quad (73)$$

Then using the quasiparticle vacuum to generate a bosonic coherent state. With such a state, one may develop a microscopic theory of superfluid helium, in which the normal and superfluid states become a direct analogy of noise and signal in partially coherent radiation fields.

We now use the squeezed coherent state to formulate a possible theory that may be useful in addressing the low energy quantum fluctuations in field theory. Let us consider again the neutral scalar field ($\phi^4$ theory) as an example. One can define the squeezed coherent state of the field $\phi$ as [39]

$$| \Psi \rangle = \mathcal{N} \exp \left\{ i \int d^3 x \left[ \pi(x) \phi_{op}(x) - \phi(x) \pi_{op}(x) \right] \right\} \times \exp \left\{ \int d^3 x d^3 y \left[ \phi_{op}(x) D(x, y) \phi_{op}(y) \right] \right\} |0\rangle \quad (74)$$

where $\mathcal{N}$ is a normalization constant. This squeezed coherent state is also defined at a given instant time so that $t = t_x = t_y$. One can show that:

$$\langle \Psi | \phi_{op}(x) | \Psi \rangle = \phi(x), \quad \langle \Psi | \pi_{op}(x) | \Psi \rangle = \pi(x),$$

$$\langle \Psi | \phi_{op}(x) \phi_{op}(y) | \Psi \rangle = \phi(x) \phi(y) + \Phi(x, y),$$

$$\langle \Psi | \pi_{op}(x) \pi_{op}(y) | \Psi \rangle = \pi(x) \pi(y) + \frac{1}{4} \Phi^{-1}(x, y)$$

$$+ 4 \int d^3 x' d^3 y' \pi(x', y') \Phi(x', y') \Pi(y', y), \quad (75)$$

where

$$D(x, y) \equiv \frac{1}{2} [\Phi_0^{-1}(x, y) - \Phi^{-1}(x, y)] + 2i \Pi(x, y),$$

$$\Phi_0(x, y) = \langle 0 | \phi_{op}(x) \phi_{op}(y) | 0 \rangle. \quad (76)$$

The squeezed coherent states of bosonic field are also overcomplete, namely

$$\int [d\phi(x)] \frac{d\pi(x)}{2\pi} [d\phi(x, y)] \frac{d\Pi(x, y)}{2\pi} | \Psi \rangle \langle \Psi | = I. \quad (77)$$

Following the similar procedure discussed in the previous section, one can derive the functional integral $W(J)$ in the squeezed coherent state representation:

$$W(J) = \int [d\phi(x)] \frac{d\pi(x)}{2\pi} [d\phi(x, y)] \frac{d\Pi(x, y)}{2\pi}$$

$$\times \exp \left\{ \int d^4 x \left[ \frac{1}{2} (\pi \dot{\phi} - \dot{\phi} \pi) - \Phi \dot{\Pi} - \mathcal{H}_{eff}(x) + J(x) \phi(x) \right] \right\}, \quad (78)$$
where
\[ H_{\text{eff}} = \frac{1}{2} \{ \pi^2(x) + (\nabla \phi)^2 + m^2 \phi^2 \} + \exp \left( \frac{\Delta(x)}{2} \frac{\partial^2}{\partial \phi^2} \right) V(\phi) \]
\[ + \frac{1}{8} \Phi^{-1}(x, x) + 2 \int d^3x' d^3y' \Pi(x, x') \Phi(x', y') \Pi(y', x) \]
\[ + \frac{1}{2} \lim_{x \to y} (\nabla_x \nabla_y \Phi(x, y)) + m^2 \Phi(x, x) \]  
(79)

and \( \Delta(x, t) = \lim_{x \to y} [\Phi(x, y) - \Phi_0(x, y)] \). Here I have not integrated out the conjugate momentum \( \{ \pi(x), \Pi(x) \} \) to obtain a covariant functional integral. The physical picture of this new formulation is that besides the original field variable \( \phi(x), \pi(x) \) in the Hamiltonian formulation, quantum fluctuations, characterized by \( \Phi(x, y) \) and \( \Pi(x, y) \), are introduced as new dynamical field variables. These new dynamical field variables describe the low energy excitations (i.e. the composite particles) of strongly interacted systems. Similarly, taking the extreme value of the exponent in (78), one can find the equation of motion that determine the classical-like solution \( \phi_0 \):
\[ (\Box - m^2) \phi_0 + \exp \left( \frac{\Delta(x)}{2} \frac{\partial^2}{\partial \phi^2} \right) V'(\phi_0) = 0, \]
(80)

which is coupled with the composite field \( \Phi \). The equation of motion for its quantum fluctuations by \( \Phi \) is much more complicated.

Usefulness of the squeezed state functional integral is that one can derive an effective theory for the low energy composite particle fields coupling with the original fields. Here I may propose a procedure how to develop such an effective theory. First, one can determine the “classical” ground state by minimizing the effective Hamiltonian with respect to variables \( \phi, \pi \) as well as \( \Phi, \Pi \), which results in \( \phi_0, (\pi_0 = 0) \) and \( \Phi_0, (\Pi_0 = 0) \). Then, expanding the effective Lagrangian near \( (\phi_0, \pi_0, \Phi_0, \Pi_0) \) up to the second-order, namely only keeping the quadratic terms in \( (\delta \phi, \delta \pi, \delta \Phi, \delta \Pi) \). Quantum effects become the time dependent fluctuations about the classical ground states. The corresponding linearized equations of motion determine the dispersions of quasiparticles and composite particles, denoted by \( \omega_k \) and \( \gamma_k \), respectively. For strong interaction field, usually \( \omega_k > \gamma_k \) (due to the spontaneously symmetry breaking). Thus, in the low energy scale \( (\omega_k > \mu \geq \gamma_k) \), the composite particles and the quasiparticles are decoupled. Only the composite particles are kept with the high order corrections (as a perturbation) to form the low energy effective Lagrangian. In the intermediate energy scale \( (\mu \sim \omega_k) \), both the composite particles and quasiparticles become active and coupled each other. The effective theory is then determined by the Lagrangian of the composite particles coupled with quasiparticle degrees of freedom. In a rather high energy scale \( (\mu >> \omega_k) \), \( \Phi \) and \( \Pi \) should spread averagely over the entire space-time space such that only the original field variables remain. Thus, the theory returns back to the original one in high energy region.

The above procedure is different from the conventional procedure of constructing a low energy effective theory. In the conventional approach, the low energy effective theory is constructed by separating the field variables into the low energy and high
energy parts. Then, using the functional integral discussed in the previous section to integrate out the high energy part. The resulting Lagrangian is an effective Lagrangian for the low energy physics. The advantage of the conventional approach is that one can use the powerful renormalization group analysis of Wilson \[40\] to extract universal scaling properties contained in the theory, without explicitly solving the theory itself. However, in reality, physical degrees of freedoms must also be very different in different energy scales. A typical example is the strong interaction in which the degrees of freedom are quarks and gluons in high energy region. But in low energy region, the degrees of freedom become hadrons which are composite particles of quarks and gluons. Thus, in the conventional approach, the effective theory cannot catch the right physical degrees of freedom. Therefore, beyond the critical phenomena, the conventional approach may have its limitation in applications. The squeezed coherent state formulation of the functional integral may provide a new method for the developments of effective field theory, although at this point a lot of work remains for further investigations.

A potential application of the squeezed coherent states in field theory is the Yang-Mills gauge theory, especially the color SU(3) gauge theory in quantum chromodynamics. Of course, the situation in quantum chromodynamics is much more complicated. Because of the nonlinear properties in non-abelian gauge theory, the conventional functional integral is already quite complicated. However, the conventional functional integral in non-abelian gauge theory is only useful for the derivation of covariant Feynman rules and the analysis of renormaliability. In other words, it is only useful for perturbation calculations. As it is well-known, the difficulty of QCD lies in its nonperturbation domain, where quantum fluctuation must be strong. Furthermore, the field strength of non-abelian gauge contains the single as well as double gauge boson emissions and absorptions:

\[
F^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} - g f^{abc} A^{b}_{\mu} A^{c}_{\nu}. \tag{81}
\]

The squeezed coherent states may be the natural quantum states describing non-abelian gauge fields. If one can complete and extend the above formulation to the non-abelian gauge theory, then it may be able to develop a low energy effective theory for non-abelian gauge fields. I believe that such a low energy effective theory should be capable in dealing with gluon condensation, gluball states as well as low energy gluon dynamics.

### 6 Spin Coherent States and Non-Linear Sigma Model

So far, I have only discussed bosonic-type coherent states in field theory. However, coherent states, in terms of the language of group theory, are embedded in a topologically nontrivial geometrical space which involve a deep implication in physics \[\Xi\]. The simplest coherent state carried a topologically nontrivial geometrical space is the spin coherent state. The most attractive property in spin coherent states is that its topological structure naturally induces Dirac’s magnetic monopole \[\Omega\].
Meanwhile, the spin coherent state representation of the path integral for a multi-spin system gives a realistic realization of Non-Linear Sigma Model which is an important field theory model in condensed matter physics and particle physics. Thus, before I go to discuss the general physics implication containing in the geometrical structure of coherent states, it may be useful to illustrate first the spin coherent state in details.

Let us start with a simple example: a spin-1/2 particle in a varying magnetic field: \( B(t) = (B_x(t), B_y(t), B_z(t)) \), described by the Hamiltonian,

\[
H(t) = -\mu S \cdot B(t). \tag{82}
\]

Here \( \mu \) is the particle’s magnetic moment, \( \vec{S} = (S_x, S_y, S_z) \) the spin operator that satisfies the usual angular momentum commutation relationship. The evolution of system governed by (82) can be determined by the Schrödinger equation, whose general solution can be written as

\[
|\psi(t)\rangle = \alpha(t) |\downarrow\rangle + \beta(t) |\uparrow\rangle. \tag{83}
\]

Substituting (83) into Schrödinger equation, it is easy to determine the time dependence of the parameters \( \alpha(t) \) and \( \beta(t) \). However, in order to derive the spin coherent state, here, I may only concentrate on the structure of the state (83). The normalization of (83) results in a constraint on the parameters \( \alpha(t) \) and \( \beta(t) \):\[
|\alpha(t)|^2 + |\beta(t)|^2 = 1. \tag{84}
\]

If I parameterize \( \beta = \sin \frac{\theta}{2} e^{-i\varphi} \), Eq. (83) can be expressed as

\[
|\psi(t)\rangle = \left( \cos \frac{\theta(t)}{2} + \sin \frac{\theta(t)}{2} e^{-i\varphi(t)} S^+ \right) |\downarrow\rangle e^{i\varphi(t)}, \tag{85}
\]

the raising and lowering spin operators \( S^\pm \) are defined by \( S^\pm = S_x \pm iS_y \), and \( S^+ |\downarrow\rangle = |\uparrow\rangle \), \( S^- |\uparrow\rangle = |\downarrow\rangle \). Furthermore, one can easily show that

\[
\left( \cos \frac{\theta}{2} + \sin \frac{\theta}{2} e^{-i\varphi} S^+ \right) |\downarrow\rangle = \exp \left\{ \frac{\theta}{2} e^{-i\varphi} S^+ - \frac{\theta}{2} e^{i\varphi} S^- \right\} |\downarrow\rangle \equiv |\theta\varphi\rangle. \tag{86}
\]

Then, (83) can be simply expressed as

\[
|\psi(t)\rangle = |\theta(t)\varphi(t)\rangle e^{i\varphi(t)}. \tag{87}
\]

The state \( |\theta\varphi\rangle \) is a standard expression of the spin coherent states.

As I mentioned a very important property of the spin coherent state is that \( |\theta\varphi\rangle \) is embedded in a topologically nontrivial geometrical space, i.e., a two-dimensional sphere \( S^2 \). This can be verified directly from (83). Since \( S_i = \sigma_i / 2 \) where \( \sigma_i \) is Pauli matrix,

\[
D_s(\eta) = \exp \left\{ \eta S^+ - \eta^* S^- \right\} = \exp \begin{bmatrix} 0 & \eta \\ -\eta^* & 0 \end{bmatrix} \]

\[
= \begin{bmatrix} \cos |\eta| & \frac{\eta}{|\eta|} \sin |\eta| \\ -\frac{\eta^*}{|\eta|} \sin |\eta| & \cos |\eta| \end{bmatrix} = \begin{bmatrix} x_0 & x \\ -x^* & x_0 \end{bmatrix} \tag{88}
\]
and $x_0$ is real while $x = x_1 + ix_2$. Also, $D_s(\eta)$ is a unitary operator which leads to
\[ x_0^2 + |x|^2 = x_2^0 + x_1^2 + x_2^2 = 1. \tag{89} \]
In other words, the parameter space of $D_s(\eta)$ is a two-sphere $S^2$ with $\eta = \frac{\theta}{2} e^{-i\varphi}$. Therefore, the spin coherent states are one-to-one corresponding to the points on $S^2$ except for the north pole where it is ambiguous since all values of $\varphi$ correspond to the same point.

However, in defining the above spin coherent states, there is an ambiguity. For example, one can also define the spin coherent states as
\[ |\theta\varphi\rangle' = \left( \cos \frac{\theta}{2} e^{i\varphi} + \sin \frac{\theta}{2} S^+ \right) |\downarrow\rangle, \tag{90} \]
which are also one-to-one corresponding to the points in $S^2$ but except for the south pole. These two spin coherent states are related simply by a phase factor,
\[ |\theta\varphi\rangle' = e^{i\varphi} |\theta\varphi\rangle. \tag{91} \]
Geometrically, these two coherent states define the two “patches” of $S^2$. Since these two states are only different by a phase factor, quantum mechanically, they must be equivalent. This implies that there is a gauge degree of freedom in the spin coherent states.

To see clearly the physical implication induced by the topological structure of spin coherent states, one can construct the path integral of a quantum spin system. The spin coherent state also obeys the overcompleteness:
\[ \int d\mu(\theta\varphi) |\theta\varphi\rangle\langle\theta\varphi| = I, \tag{92} \]
where $d\mu(\theta\varphi) = \sin \theta d\theta d\varphi/2\pi$ is an invariant measure on $S^2$. Then, it is easy to show that the path integral of quantum mechanics for $H = H(\vec{S})$ is given by
\[ \langle\theta'(t_f)\varphi'(t_f)|\theta(t_0)\varphi(t_0)\rangle = \int [d\mu(\theta\varphi)] \exp \left\{ i \int_{t_0}^{t_f} dt \left[ \langle\theta\varphi| \frac{d}{dt} |\theta\varphi\rangle - \langle\theta\varphi| H(\vec{S}) |\theta\varphi\rangle \right] \right\}. \tag{93} \]
In this path integral, the first term in the exponent,
\[ \omega[\theta\varphi] \equiv \int_{t_0}^{t_f} dt \langle\theta\varphi| i \frac{d}{dt} |\theta\varphi\rangle = \int_{\varphi_0}^{\varphi_f} \frac{1}{2} (1 - \cos \theta) d\varphi, \tag{94} \]
is pure geometric that only depends on the trajectory over the sphere, but not on its explicit time dependence. For a closed path, $\omega[\theta\varphi]$ is actually a Berry phase of the spin history \cite{12}. Therefore, $\omega[\theta\varphi]$ is a gauge invariant one-form defined on the sphere $S^2$:
\[ \omega[\theta\varphi] = \int_{\varphi_0}^{\varphi_f} A_\varphi d\varphi = \int_{\varphi_0}^{\varphi_f} A(\mathbf{n}) d\mathbf{n}, \tag{95} \]
where $\mathbf{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ is a unit vector, and $A(\mathbf{n})$ is a unit vector potential. Compare with (94) and (95), one can find that
\[ A_\varphi = \frac{1 - \cos \theta}{2 \sin \theta} \dot{\varphi}. \tag{96} \]
This vector potential has one singularity at the south pole. It is this singularity where the Dirac string which carries the magnetic monopole flux enters the sphere. Hence, $A^a$ is nothing but a gauge potential of Dirac’s magnetic monopole. Similarly, for the spin coherent state $|\theta \varphi\rangle$, the corresponding gauge potential is

$$A^b = \frac{1 + \cos \theta}{2 \sin \theta} \hat{\varphi}. \tag{97}$$

$A^a$ and $A^b$ define the two non-singular patches of the monopole section. Their difference is a pure U(1) gauge in the overlapping equatorial region, $S^1$,

$$A^b = A^a + d\varphi = A^a - ig^{-1}dg. \tag{98}$$

where $g = e^{i\varphi} \in U(1)$.

The existence of the above gauge degrees of freedom can be understood clearly by looking at the general definition of coherent states based on group theory \[9, 10\]. In group theory, quantum states of a spin system form a unitary representation $V^s$ of the SU(2) group, here $s$ is an arbitrary spin. Choosing a fixed state, such as the lowest-weight state $|ss_z\rangle = |s - s\rangle \in V^s$, one can define spin coherent states as

$$|g\rangle_s = g|s - s\rangle, \quad g \in SU(2). \tag{99}$$

In general, $g = \exp(i\alpha S_x) \exp(i\beta S_y) \exp(i\gamma S_z) = \exp(i\theta e^{-i\varphi} S^+ - \frac{\theta}{2} e^{i\varphi} S^-) \exp(i\gamma' S_z)$. Note that this decomposition, called the Baker-Campbell-Hausdorff formula, is unique. As a result, one can rewrite the above spin coherent states as

$$|g\rangle_s = \exp(i\theta e^{-i\varphi} S^+ - \frac{\theta}{2} e^{i\varphi} S^-) \exp(i\gamma' S_z)|s, -s\rangle = |\theta \varphi\rangle e^{i\chi}. \tag{100}$$

where

$$|\theta \varphi\rangle = \exp(i\theta e^{-i\varphi} S^+ - \frac{\theta}{2} e^{i\varphi} S^-)|s, -s\rangle \tag{101}$$

is the standard definition of spin coherent state for an arbitrary spin $s$ \[13\]. Spin $s = 1/2$ discussed above is a special case. As one can see, apart from a phase factor, the spin coherent states can be generated by a unitary spin rotational operator acting on the fixed state $|s - s\rangle$. The unitary operator $\exp(i\theta e^{-i\varphi} S^+ - \frac{\theta}{2} e^{i\varphi} S^-)$ is the coset representation of the space $SU(2)/U(1) \simeq S^2$. Therefore the sphere $S^2$ determines the topological structure of spin coherent states. The magnetic monopole potential $A^a$ defines a $U(1)$ fibre bundle over this sphere $S^2$. Meanwhile, the spin coherent states contain an arbitrary phase $\chi$. In quantum mechanics, a quantum state is specified up to a phase factor, namely, physics is invariant for different choices of such a phase factor so that this phase factor is usually ignored. However, when quantum states are embedded in a topologically nontrivial space, this phase freedom is indeed the associated gauge degrees of freedom of the fibre bundle over the space. In spin coherent states, the phase $\chi$ is just the gauge degree of freedom that connects different choices of magnetic monopole potentials. Ignoring (or fixing) this phase factor corresponds to a gauge fixing.
Furthermore, the topological properties of the spin coherent states also play an important role in the study of spin dynamics. A typical example is the Heisenberg model in condensed matter physics. Heisenberg model is used to understand quantum magnetism of strongly correlated electron systems. The model Hamiltonian considered here is very simple:

\[ H_J = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \]  

(102)

which describes a many-spin (each spin = \( s \)) system with the nearest neighbor exchange interaction. “Classically”, the ground state of the above Hamiltonian is easily determined. When \( J > 0 \), the minimum energy is given by the state in which the nearest-neighbor spins are always anti-alignment. These states are called in literatures the Néel states. Correspondingly, the system is an antiferromagnet. If \( J < 0 \), the ground state is simply given by the state with all spin aligned in the same direction, which is a ferromagnetic state. These consequences can be obtained explicitly by taking the spin coherent state

\[ |\{\theta_i,\phi_i\}\rangle = \prod_i \exp \left\{ \frac{i}{2} e^{-i\phi_i} S_i^+ - \frac{i}{2} e^{i\phi_i} S_i^- \right\} |s - s\rangle \]  

(103)

as a trial wave function and minimizing the model Hamiltonian

\[ \delta(|\{\theta_i,\phi_i\}\rangle H_J |\{\theta_i,\phi_i\}\rangle) = \delta(J s^2 \sum_{\langle i,j \rangle} \{\cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j \cos(\phi_i - \phi_j)\}) = 0. \]  

(104)

The resulting ground state is given by

\[
\begin{cases} 
J > 0 & \rightarrow \theta_{i+1} = \pi - \theta_i, \phi_{i+1} = \phi_i + \pi \rightarrow \text{antiferromagnet} \\
J < 0 & \rightarrow \theta_{i+1} = \theta_i, \phi_{i+1} = \phi_i \rightarrow \text{ferromagnet}
\end{cases}
\]  

(105)

An important concept one can obtain from the above result is that the ground state spontaneously breaks the global spin rotational symmetry. As one can check the model Hamiltonian is invariant under global spin rotational transformations: \( T = \exp(i\vec{\alpha} \cdot \mathbf{S}) \), where \( \mathbf{S} = \sum_i \mathbf{S}_i \). While the ground state energy does not change when all the spins in (103) are globally rotated. This leads to a SO(3) degeneracy of the ground states, namely these ground states have a lower symmetry than the Hamiltonian. Such a situation is called the spontaneously symmetry breaking. Quantum mechanically, it leads to gapless spin-wave excitations in the Heisenberg model, and Goldstone bosons in general.

The quantum dynamics of interacting spins can be studied from the time-evolution of the system at zero-temperature. The time-evolution is determined by the Green’s function which is defined by the matrix element of the evolution operator between two spin coherent states:

\[ G(t_f, t_0) = \int \prod_i [d\mu(\theta_i, \phi_i)] \exp \left\{ iS[\theta_i(t), \phi_i(t)] \right\}, \]  

(106)
here, the effective action is given by
\[
S[\theta_i(t), \varphi_i(t)] = \int_{t_0}^{t_f} dt \left[ \langle \{\theta_i, \varphi_i\} | \frac{d}{dt} | \{\theta_i, \varphi_i\} \rangle - \langle \{\theta_i, \varphi_i\} | H_J | \{\theta_i, \varphi_i\} \rangle \right] \\
= \int_{t_0}^{t_f} dt \left[ s \sum_i (1 - \cos \theta_i) \frac{d\varphi_i}{dt} \right. \\
- Js^2 \sum_{\langle i,j \rangle} [\cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j \cos(\varphi_i - \varphi_j)] \right] \tag{107}
\]

Note that the thermal dynamics can be obtained in a similar form. The partition function can be expressed in terms of a spin coherent state path integral as well:
\[
\mathcal{Z}(\beta) = \langle \{\theta_i, \varphi_i\} | \exp \left\{ -\beta H_J \right\} | \{\theta_i, \varphi_i\} \rangle \\
= \int [d\mu(\beta_i(\tau) | \varphi_i(\tau))] \exp \left\{ -S[\theta_i(\tau), \varphi_i(\tau)] \right\} \tag{108}
\]
with
\[
S[\theta_i(\tau), \varphi_i(\tau)] = \int_0^\beta d\tau \left[ -is \sum_i (1 - \cos \theta_i) \frac{d\varphi_i}{d\tau} \right. \\
+ Js^2 \sum_{\langle i,j \rangle} [\cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j \cos(\varphi_i - \varphi_j)] \right], \tag{109}
\]

where \( \tau \) is an imaginary time from 0 to \( \beta = 1/kT \).

Eq. (107) shows that spin dynamics is induced by the geometrical phase \( \omega = \int dt \sum_i s(1 - \cos \theta_i) \dot{\varphi}_i \), which contains the time derivative and therefore leads to the equation of motion for \( \{\theta_i(t), \varphi_i(t)\} \). It also shows that the magnetic monopole potential \( A_{\varphi_i} \) is actually the conjugate momentum of \( \varphi_i \) in spin dynamics. If one defines the generalized position and momentum coordinates by
\[
q_i = \varphi_i, \quad p_i = s(1 - \cos \theta_i), \tag{110}
\]
and expands the effective action \( S[\theta_i(t), \varphi_i(t)] \) around the ground state \( \{q_i^0, p_i^0, q_{i+1}^0, p_{i+1}^0\} \) [given by Eq. (105)]:
\[
q_i = q_i^0 + \delta q_i, \quad p_i = p_i^0 + \delta p_i, \\
q_{i+1} = q_{i+1}^0 + \delta q_{i+1}, \quad p_{i+1} = p_{i+1}^0 + \delta p_{i+1}, \tag{111}
\]
up to the quadratic terms, one can determine explicitly the dispersions of spin-wave excitations [14, 45].

However, since \( p_i \) is related to the magnetic monopole potential, the conjugate coordinates \( \{q_i, p_i\} \) used above are indeed gauge dependent quantities. To explore the possible topological effects in spin dynamics, it is better to use a gauge invariant formulation. This may be done by using a global notation \( \mathbf{n} \) (a unit vector) to represent the spin direction, without specifying the parameterization of the sphere by \( \theta, \varphi \). Then, the Green’s function can be expressed as:
\[
G(t_f, t_0) = \int [d\mu(\mathbf{n}_i)] \exp \left\{ i \int_{t_0}^{t_f} dt \left[ s \sum_i \mathbf{A}_i \cdot \dot{\mathbf{n}}_i - Js^2 \sum_{\langle i,j \rangle} \mathbf{n}_i \cdot \mathbf{n}_j \right] \right\}. \tag{112}
\]
Taking the continuum limit
\[ n_i \rightarrow c_i n(x_i), \quad (113) \]
where \( c_i = 1 \left( e^{i \pi_i} \cdot \pi \right) \) for the ferromagnet (antiferromagnet), \( |n(x_i)| = 1 \), and \( \pi = (\pi, \cdots, \pi) \), \( x_i \in \mathbb{R}^d \). Then the Green’s function can be expressed as
\[ G(t_f, t_0) = \int [d\mu(n(x))] \exp \left\{ i \int d^{d+1}L(n) \right\}, \quad (114) \]
where \( L(n) \) is an effective Lagrangian. In the low energy (long wave-length) limit, it is reduced to the Lagrangian of the **Non-Linear Sigma Model** in \( d+1 \)-dimensional space \([46] \),
\[ L(n) = \frac{1}{2g} \partial_\mu n \cdot \partial^\mu n + \cdots \quad (115) \]
where “\( \cdots \)” dnotes the high order derivatives. This Lagrangian ensures the existence of gapless spin-wave excitations. Such a Non-Linear Sigma Model has been widely studied in condensed matter physics, including the problems in quantum magnetism, quantum Hall effect and disorder dynamics.

### 7 Generalized Coherent States and Nonabelian Gauge Fields

In this last section, I will discuss the generalized coherent states and their potential applications in field theory. Generalization of coherent states is based on group theory developed by Perelomov and also Gilmore \([9, 10] \). The spin coherent state discussed in the previous section is an example of such generalization. Actually, Glauber had pointed out in his seminal paper \([7] \) that the photon coherent states can be constructed starting from any one of three mathematical definitions.

- **Definition 1.** The coherent states \( |z\rangle \) are eigenstates of the annihilation operator \( a \):
  \[ a|z\rangle = z|z\rangle. \quad (116) \]

- **Definition 2.** The coherent states \( |z\rangle \) are quantum states with a minimum uncertainty relationship:
  \[ \Delta x^2 \Delta p^2 = \frac{\hbar^2}{4} \quad (117) \]

- **Definition 3.** The coherent states \( |z\rangle \) can be obtained by applying a displacement operator \( D(z) \) on the ground state of harmonic oscillator:
  \[ |z\rangle = D(z)|0\rangle, \quad D(z) = \exp(za^\dagger - z^*a). \quad (118) \]

We have analyzed these definitions and pointed out \([Zha90] \) that the generalization of eigenstates of the lowering operator is not always possible. Indeed, the adoption of this definition to generalize the coherent state concept has two major drawbacks: a). Coherent states cannot be defined in Hilbert spaces of finite dimensionality.
in this way, as we have seen for the spin systems. b). The states so defined do not correspond to physically realizable states, except under special circumstances that the commutator of the annihilation operator (or lowering step operator) and its hermitian adjoint is a multiple of the identity operator. Therefore, under this condition one restricts oneself to the bosonic field. As a result, the generalization based on definition 1 to other dynamical systems is not always applicable.

On the other hand, the generalization based on the definition 2 is by no means unique. The bosonic (photon) coherent states are the minimum uncertainty states essentially because they are non-spreading wave packets. Although the minimum uncertainty states are physically very interesting, the generalization along this direction has several limitations: a). These coherent states can only be constructed for the classically integrable systems in which there exists a set of canonical coordinates and momenta such that the respective Hamiltonians can be reduced to quadrature. This condition requires a flatness condition on the operator algebra which reduce the commutation relations to those of the standard bosonic creation and annihilation operators. b). The wave packets with the minimum uncertainty are not unique. Different ones may have different properties. Also, such states may be incomplete, or even if they are complete it is not certain that the standard form of a resolution of unity exists. Thus, minimum uncertainty states appear to have few, if any, useful properties.

In literatures, the realization of generalized coherent states are indeed achieved based on displacement operators. The basic theme of this development was to intimately connect coherent states with dynamical symmetry groups of a physical problem. Since all physical problems formulated in quantum theory have a dynamical group (although sometimes the group may be too large to be useful), an important outcome of this recognition is that coherent states can be generalized to all the quantum problems.

I should outline here a generalization procedure how an arbitrary coherent state can be generated by displacement operators. Consider a set of operators \( \{T_i\} \) closed under commutation:

\[
[T_i, T_j] = T_i T_j - T_j T_i = \sum_k C_{ij}^k T_k,
\]

That is, \( \{T_i\} \) span a algebra \( g \), and \( C_{ij} \) in (119) are structure constants of \( g \). If \( g \) is a semisimple Lie algebra, it is more convenient to write \( \{T_i\} \) in terms of the standard Cartan basis \( \{H_i, E_\alpha, E_\alpha^\dagger = E_{-\alpha}\}\):

\[
[H_i, H_j] = 0, \quad [H_i, E_\alpha] = \alpha_i E_\alpha, \\
[E_\alpha, E_{-\alpha}] = \alpha_i H_i, \quad [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha + \beta}.
\]

In quantum theory, for such a given set of closed operators \( \{T_i\} \), the quantum states are described by a Hilbert space \( V^\Lambda \) which is a representation of \( g \). Let \( G \) be the covering group of \( g \). The Hilbert space \( V^\Lambda \) carries a unitary irreducible representation \( \Gamma^\Lambda \) of \( G \). One may choose a normalized state \( |\phi_0\rangle \) in the Hilbert space \( V^\Lambda \) as a fixed state. Then the generalized coherent state is generated by an element
\( g \in G \) acting on the fixed state \( |\phi_0\rangle \).

\[
|g\rangle_G = g|\phi_0\rangle. \tag{121}
\]

In group theory, every element \( g \in G \) can be uniquely decomposed into a product of two group elements: \( g = kh \), here one should require \( h \in H \) such that

\[
h|\phi_0\rangle = |\phi_0\rangle e^{ix}, \tag{122}
\]

and \( H \) is the maximum subgroup of \( G \) that leaves the fixed state invariant up to a phase factor. While \( k \) is an operator of the coset space \( G/H \). If \( G \) is a semisimple Lie group and \( |\phi_0\rangle \) is the lowest weight state, \( k \) can be generally written as

\[
k \equiv D_G(\eta) = \exp \left\{ \sum_{\alpha > 0} (\eta_\alpha E_\alpha - \eta^*_\alpha E^*_{-\alpha}) \right\} \in G/H. \tag{123}
\]

This operator \( D_G(\eta) \) is usually called a displacement operator of \( G \), which gives a coset representation of \( G/H \). As a result,

\[
|g\rangle_G = D_G(\eta)|\phi_0\rangle e^{ix} = |\Phi(Z)\rangle e^{ix}, \tag{124}
\]

Perelomov and Gilmore [9, 10] define the state \( |\Phi(Z)\rangle \) as the generalized coherent states of \( G \):

\[
|\Phi(Z)\rangle = D_G(\eta)|\phi_0\rangle = \mathcal{N}(Z) \exp \left\{ \sum_{\alpha > 0} Z_\alpha E_\alpha \right\} |\phi_0\rangle, \tag{125}
\]

and \( \mathcal{N}(Z) \) is a normalized constant. The generalized coherent states defined in such a way have two important properties

- The set of the generalized coherent states satisfies:

\[
\int d\mu(Z)|\Phi(Z)\rangle \langle \Phi(Z)| = I, \tag{126}
\]

where \( d\mu(Z) \) is the \( G \)-invariant Haar measure on \( G/H \).

- The generalized coherent states are one-to-one corresponding to the points in the coset space \( G/H \) except for some singular points, such as the north pole or south pole of the two-sphere in spin coherent states. Therefore, the generalized coherent states are embedded into a topologically nontrivial geometrical space.

Systems discussed in the previous sections are only some simple examples of the generalized coherent states. The harmonic oscillator admits a dynamical group \( H_4 \), called Heisenberg-Weyl group. The photon coherent states are obtained via a one-to-one correspondence with the geometrical coset space \( H(4)/U(1) \times U(1) \) (a complex plane) by the displacement operator \( D(z) \in H(4)/U(1) \times U(1) \). The two-photon processes has a \( SU(1,1) \) dynamical group. The squeezed states are obtained by the displacement (squeezed) operator \( D_{sq}(\beta) \in SU(1,1)/U(1) \) (a two-dimensional hyperboloid space). And the spin coherent states discussed in the previous section are generated by the displacement operator \( D_s(\theta \varphi) \in SU(2)/U(1) \) (a two-dimensional sphere).
I should emphasize here that the phase $\chi$ in the group-theoretical coherent state (124) is the $H$-gauge degrees of freedom over the coset space $G/H$. All the three sets of coherent states discussed in the previous sections contain an $U(1)$ gauge, but only the sphere (spin) carries a nontrivial fibre bundle so that the gauge degrees of freedom become important. To obtain a non-abelian gauge, one must consider the generalized coherent states of a group $G$ whose rank is larger than one such that $H$ can be a non-abelian group.

To examine non-abelian gauge degrees of freedom in the generalized coherent states, one may extend the path integral to the generalized coherent state representation. The Green's function is now defined as the matrix element of the evolution operator in the generalized coherent states:

$$G(t_f, t_0) = \langle \Phi'(Z)|T \exp\{ -i \int_{t_0}^{t_f} H(t) dt \}|\Phi(Z) \rangle.$$  \hfill (127)

Following the same procedure as it has been done in the previous sections that divides the time interval $t_f - t_0$ into $N$ intervals, each with $\varepsilon = (t_f - t_0)/N$, then inserts the resolution of identity (126) at each interval point, and finally lets $N$ go to infinity, the Green's function can be expressed as a generalized coherent state path integral.

$$G(t_f, t_0) = \lim_{N \to \infty} \int \left[ \prod_{i=1}^{N-1} d\mu_i(Z) \right] \prod_{i=1}^{N} \langle \Phi_i(Z)| \exp\{ -i\varepsilon H(t_i) \}|\Phi_{i-1}(Z) \rangle,$$

where

$$S[Z(t)] = \int_{t_0}^{t_f} dt \left\{ \langle \Phi(Z(t))|i \frac{d}{dt}\Phi(Z(t)) \rangle - \langle \Phi(Z(t))|H(t)|\Phi(Z(t)) \rangle \right\} \hfill (129)$$

is an effective action in the generalized coherent state representation. This path integral is defined over the coset space $G/H$. The effective action contains two terms. The second term is the matrix element of Hamiltonian operator in the coherent states, which determines the static properties of the classical Hamiltonian. The first term is pure geometric, and it is indeed a Berry phase [47, 48] that describes quantum fluctuations, and also determines the time-evolution of the system,

$$\omega[G/H] = \int_{\Gamma \in G/H} \langle \Phi(Z)|d|\Phi(Z) \rangle = \int_{\Gamma \in G/H} A \cdot d\hat{\Omega}, \hfill (130)$$

where $A$ is a gauge vector potential defined over the coset space $G/H$, and $\hat{\Omega}$ is a unit vector in $G/H$. One can then define the gauge connection,

$$F \equiv \langle d\Phi(Z)|d\Phi(Z) \rangle = \sum_{\alpha\alpha'} \omega_{\alpha\alpha'} dZ_\alpha \wedge dZ_{\alpha'}, \hfill (131)$$

and $\omega_{\alpha\alpha'}$ is the Berry curvatures:

$$\omega_{\alpha\alpha'} = \left\langle \frac{\partial \Phi(Z)}{\partial Z_\alpha} \bigg| \frac{\partial \Phi(Z)}{\partial Z_{\alpha'}} \right\rangle - \left\langle \frac{\partial \Phi(Z)}{\partial Z_{\alpha'}} \bigg| \frac{\partial \Phi(Z)}{\partial Z_\alpha} \right\rangle. \hfill (132)$$
When the rank of $G$ is larger than one, the associated gauge potential $A$ is non-abelian with gauge group $\leq H$. From the above generalized coherent state path integral, one can study the so-called geometric quantization [50, 51] and classical gauge equations of motion in quantum mechanics [52, 53].

This path integral formulation and the associated gauge potentials have potential applications in condensed matter physics and particle physics. This is because classical semisimple Lie groups can be generated by bilinear operators of bosonic and fermionic creation and annihilation operators. The bilinear operators describe the basic collective excitations in strongly correlated or strongly coupled systems. Therefore, the above formalism can be applied directly to various realistic physical problems.

Specifically, the SU(n) group can be generated by the particle-hole pairs: $\{a_i^\dagger a_j; 1 \leq i, j \leq n\}$, and the corresponding generalized coherent state is given by

$$\{|\{Z_{ij}\}\rangle = \mathcal{N}(Z) \exp \left\{ \sum_{ij} Z_{ij} a_i^\dagger a_j \right\}|m\rangle. \quad (133)$$

where $a_i^\dagger, a_i$ can be either bosonic or fermionic creation and annihilation operators, and $|m\rangle$ contains $m$ particles in the lowest states, $m < n$. For bosonic system, the coherent states are defined on the coset space $\{Z_{ij}\} \in \text{SU}(n)/\text{SU}(n-1) \times \text{U}(1)$. For fermionic space, the coset space $\{Z_{ij}\}$ is SU(n)/SU(n-m) × U(m). The spin coherent state of the Heisenberg model discussed in the last section is a special case where the spin operators take the form:

$$\vec{S}_i = \frac{1}{2} \sum_{\alpha\beta} a_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} a_{i\beta}, \quad (134)$$

$\sigma$ is the Pauli matrix, and $\alpha, \beta$ denote the spin index of electrons. Let $Z_{ii} = \tan \frac{\theta}{2} e^{-i\varphi}$, the spin coherent state (103) can be reduced to the form of (133) with $Z_{ij} = 0$ for $i \neq j$ and $\mathcal{N}(Z) = 1/(1 + |Z_{ii}|^2)^s$. Correspondingly, the geometrical space SU(n)/SU(n-m) × U(m) is reduced to $\{Z_{ii}\} \in \prod_i \text{SU}(2)/\text{U}(1)$.

The Sp(2n+1) group can be realized by bosonic particle-particle and particle-hole pairs: $\{a_i^\dagger, a_i, a_j^\dagger, a_j, a_i^\dagger a_j^\dagger - \frac{1}{2} \delta_{ij}\}$. Similarly, one can write down the most general coherent state for SO(2n):
and its geometrical space is the coset space \( \{ Z_{ij} \} \in SO(2n)/U(n) \). A typical example of the above coherent states is the BCS superconducting state in which only special fermionic pairs, i.e., Cooper pairs \( c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \), are considered \[54\]:

\[
|\text{BCS}\rangle = \frac{1}{\sqrt{1 + |h_k|^2}} \exp \left\{ \sum_k h_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \right\} |0\rangle. \tag{137}
\]

Since \( \{ c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger, c_{-k\uparrow}^\dagger c_{k\downarrow}^\dagger, c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + c_{-k\uparrow}^\dagger c_{k\downarrow}^\dagger -1 \} \) span a \( \text{su}(2) \) algebra, the geometrical space of the above BCS states is indeed the same as the spin coherent state in Heisenberg model, i.e., \( \{ h_k \} \in \prod_k \otimes SU_k(2)/U_k(1) \). Therefore, the BCS state carries a \( U(1) \) gauge degree of freedom. But physically, the superconductivity is very different from the ferro- and antiferromagnetism. This is because the Heisenberg model has a global spin rotational symmetry, while the BCS Hamiltonian only has a global \( U(1) \) symmetry. In the Heisenberg model, the spontaneously breaking of spin rotational symmetry leads to the spin-wave excitations which can be described by the Non-Linear Sigma Model derived from the spin coherent state path integral, as we have discussed in the previous section. In the BCS theory, the spontaneously breaking of the \( U(1) \) symmetry for the pairing coherence gives pair excitations which can be described by Ginzberg-Landau theory. The Ginzberg-Landau theory should also be derivable from the path integral of BCS coherent states.

The above general fermionic pairing coherent states can also be applied to systems other than the conventional BCS superconductivity. For example, if I take the triplet pairs:

\[
\bar{T}^+(k) = \frac{1}{2} \sum_{\alpha,\beta} c_{k\alpha}^\dagger (i\sigma_2)_{\alpha\beta} c_{-k\beta}^\dagger, \quad \bar{T}(k) = \frac{1}{2} \sum_{\alpha,\beta} c_{-k\alpha} (-i\sigma_2)_{\alpha\beta} c_{k\beta}^\dagger \tag{138}
\]

together with the charge and spin operators:

\[
Q(k) = \frac{1}{2} \sum_{\alpha} (c_{k\alpha}^\dagger c_{k\alpha} + c_{-k\alpha}^\dagger c_{-k\alpha}) - 1, \quad \bar{S}(p) = \frac{1}{2} \sum_{\alpha,\beta} (c_{k\alpha}^\dagger i\sigma_2_{\alpha\beta} c_{k\beta} + c_{-k\alpha}^\dagger i\sigma_2_{\alpha\beta} c_{-k\beta}), \tag{139}
\]

which generates a \( SO(5) \) group, I can construct a generalized coherent state for the triplet pairing for superfluid \( ^3\text{He} \):

\[
|\text{SF}\rangle = \mathcal{N}(Z) \exp \left\{ \sum \{ \bar{Z}_k \cdot \bar{T}^+(k) \} \right\} |0\rangle. \tag{140}
\]

Its coset space is \( \{ \bar{Z}_k \} \in \prod_k \otimes SO_k(5)/U_k(2) \). This \( SO(5) \) coherent state carries a non-abelian \( SU(2) \) gauge. One can use this \( SO(5) \) generalized coherent state to study non-abelian gauge fields and low energy effective theory for superfluid \( ^3\text{He} \) atoms \[53\]. Recently, I also constructed a generalized pairing state to include the singlet and triplet pairs, \[54\]:

\[
|ZW\rangle = \mathcal{N}(Z) \prod_k \exp \left\{ Z_1(k) c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + Z_2(k) c_{k\downarrow}^\dagger c_{-k\uparrow}^\dagger + Z_3(k) c_{k\downarrow}^\dagger c_{k\uparrow}^\dagger + Z_4(k) c_{-k\downarrow}^\dagger c_{-k\uparrow}^\dagger + Z_5(k) c_{k\downarrow}^\dagger c_{-k\uparrow}^\dagger + Z_6(k) c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \right\} |0\rangle, \tag{141}
\]
for the study of high $T_c$ superconductivity and the close proximity between the Mott insulating antiferromagnetic order and $d$-wave superconducting order in cuprates\cite{57}. Here the coset space is $\prod_k \otimes \text{SO}_k(8)/\text{U}_k(4)$. Under the constraint of non-double occupied sites, possible gauge group contains in the above coherent pairing states may be $\text{SU}(2) \times \text{U}(1)$ or a larger one up to $\text{U}(4)$. This may open a new window for the study of the dynamical mechanism of high $T_c$ superconductivity.

If one takes the continuum limit in the coordinate space and lets $t_0 \to -\infty$, $t_f \to \infty$, then the path integral based on the generalized coherent states can be expressed as

$$G = \int [d\mu(\hat{\Omega}(x))] \exp \left\{ i \int d^{d+1}x \left\{ A(x) \cdot \hat{\Omega}(x) - \mathcal{H}[\hat{\Omega}(x)] \right\} \right\},$$

(142)

where $x$ is a coordinate in the $d+1$ dimensional Minkowski space. If the Hamiltonian has a symmetry $S \subset G$, and the static classical ground states (which can be obtained by minimizing $\mathcal{H}$ with respect to the coherent state parameters) spontaneously breaks this symmetry, then one can use the saddle-point expansion to derive a general Non-Linear Sigma Model defined on $G/H$,

$$G = \int [d\mu(\hat{\Omega}(x))] \exp \left\{ i \int d^{d+1}x \left\{ \frac{1}{2g} \partial_\mu \hat{\Omega}(x) \cdot \partial^\mu \hat{\Omega}(x) + \cdots + \Theta_{\text{Top}} \right\} \right\},$$

(143)

to describe the low energy physics in the long-wave length limit, where $\{ \cdots \}$ denotes the higher order derivatives in the Non-Linear Sigma Model, and $\Theta_{\text{Top}}$ is a topological phase, corresponding to a Wess-Zumino-Witten topological term\cite{58,59,60} that is induced by the gauge degrees of freedom contained in the generalized coherent states and/or a Chern-Simons term of topological gauge fields over the coset space $G/H$\cite{61}. There may exist many potential applications of such a Non-linear Sigma Model in real physical problems, such as quantum Hall effect\cite{60,62}, the high $T_c$ superconductivity\cite{63,64}, the disorder systems\cite{65} in condensed matter physics. It is also possible to apply such theory to quantum chromodynamics in particle physics when quantum chromodynamics is formulated in lattices\cite{66,67}, and quantum gravity\cite{68}, etc.

8 Summation

In summation, as I have emphasized throughout the article, coherent states possess three unique properties that are fundamental to field theory. The property of coherent behavior uniquely describes the processes involving infinite number of virtual particles. The coherent excitations obtained from coherent states also give the essential physical picture of long-range orders induced by strong correlations. The property of overcompleteness provides a reformulation and generalization of the functional integral in field theory, in which quantum fluctuations of composite operators are included as new low energy dynamical field variables. One may thereby be able to determine the dynamical degrees of freedom in different energy scales and to derive the corresponding effective theory. The property of topologically nontrivial geometrical structure in generalized coherent states allows one to
explore the origin of gauge fields and associated gauge degrees of freedom. In this article, I have not touched the recently development of coherent states in terms of superalgebras and quantum groups. These topics may also be very important in the modern development of field theory, such as in supersymmetry, superstring and conformal field theory. Nevertheless, in my personal opinions, understanding the origin as well as the nature of gauge degrees of freedom in physics is perhaps the most fundamental problem in field theory.

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