SILVAN: Estimating Betweenness Centralities with Progressive Sampling and Non-uniform Rademacher Bounds

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“Sim Sala Bim!” —Silvan, https://en.wikipedia.org/wiki/Silvan_(illusionist)

Betweenness centrality is a popular centrality measure with applications in several domains and whose exact computation is impractical for modern-sized networks. We present SILVAN, a novel, efficient algorithm to compute, with high probability, accurate estimates of the betweenness centrality of all nodes of a graph and a high-quality approximation of the top-k betweenness centralities. SILVAN follows a progressive sampling approach and builds on novel bounds based on Monte Carlo Empirical Rademacher Averages, a powerful and flexible tool from statistical learning theory. SILVAN relies on a novel estimation scheme providing non-uniform bounds on the deviation of the estimates of the betweenness centrality of all the nodes from their true values and a refined characterisation of the number of samples required to obtain a high-quality approximation. Our extensive experimental evaluation shows that SILVAN extracts high-quality approximations while outperforming, in terms of number of samples and accuracy, the state-of-the-art approximation algorithm with comparable quality guarantees.

CCS Concepts: • Information systems → Data mining • Mathematics of computing → Probabilistic algorithms • Theory of computation → Sketching and sampling: Graph algorithms analysis

Additional Key Words and Phrases: Betweenness centrality, Rademacher averages, random sampling

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1 INTRODUCTION
The computation of node centrality measures, which are scores quantifying the importance of nodes, is a fundamental task in graph analytics [44]. Betweenness centrality is a popular centrality measure, defined first in sociology [1, 29], that quantifies the importance of a node as the fraction of shortest paths in the graph that go through the node.

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The computation of the exact betweenness centrality for all nodes in a graph $G = (V, E)$ can be obtained with Brandes’ algorithm [16] in time $O(|V||E|)$ for unweighted graphs and in time $O(|V||E| + |V|^2 \log |V|)$ for graphs with positive weights, which is impractical for modern networks with up to hundreds of millions of nodes and edges. Several works (e.g., [27, 62]) proposed heuristics to improve Brandes’ algorithm, but they do not improve on its worst-case complexity. In fact, for unweighted graphs a corresponding lower bound (based on the Strong Exponential Time Hypothesis) was proved in [11]. The impracticality of the exact computation for modern networks, and the use of betweenness centrality mostly in exploratory analyses of the data, have motivated the study of efficient algorithms to compute approximations of the betweenness centrality, trading precision for efficiency.

Several works [12, 21, 54, 57] have recently proposed sampling approaches to approximate the betweenness centrality of all nodes in a graph. The main idea is to sample shortest paths uniformly at random and use such paths to estimate the betweenness centrality of the nodes. As for all sampling approaches, the main difficulty is then to relate the estimates obtained from the samples with the corresponding exact quantities, providing tight trade-offs between guarantees on the quality of the estimates and the required computational work. [12, 21, 54, 57] all provide rigorous approximations of the betweenness centrality, and [21, 54, 57] rely on tools from statistical learning theory, such as the VC-dimension [67], the pseudodimension [52], or Rademacher Averages [35], which have been successfully used to obtain rigorous approximations for other data mining tasks (e.g., pattern mining [55, 56, 58]). For pattern mining, a recent work [49] has shown that a more advanced tool from statistical learning theory, namely, Monte Carlo Empirical Rademacher Averages (MCERA) [3] (see Section 3.2), leads to improved results, mostly thanks to it data-dependent nature (in contrast to distribution-free tools such as the VC-dimension and the pseudodimension). Indeed, the MCERA was recently used in BAVARIAN [21] to obtain a unifying framework compatible with different estimators of the betweenness centrality.

Our contributions. In this work, we study the problem of approximating the betweenness centralities of nodes in a graph. We propose SILVAN (Estimating betweenness centralities with progressive sampling and Non-uniform Rademacher bounds), a novel, efficient, progressive sampling algorithm to approximate betweenness centralities while providing rigorous guarantees on the quality of various approximations.

- Our first contribution is empirical peeling, a novel technique that we introduce to obtain sharp non-uniform data-dependent bounds on the maximum deviation of families of functions (Section 4.1). Empirical peeling is based on the MCERA and relies on an effective data-dependent approach to partition a family of functions according to their empirically estimated variance; this allows us to fully exploit variance-dependent bounds at the core of the technique. Our algorithm SILVAN (Section 4.2) relies on such novel bounds to provide guarantees on the approximation of the betweenness centrality that are much sharper than the ones obtained by previous works; these new contributions make SILVAN a practical algorithm for obtaining different approximations of the betweenness centrality. In fact, we show that combining the MCERA with empirical peeling allows us to design flexible algorithms with different guarantees (e.g., additive or relative) and for different tasks (e.g., estimating all betweenness centralities or, in Section 4.4, the top-$k$ ones). This is the first work that obtains different types of approximation guarantees based on the MCERA. Most importantly, our approach is general and of independent interest, as it may apply to other problems, even outside of data mining applications.
- We derive a new bound on the sufficient number of samples to approximate the betweenness centrality for all nodes (Section 4.3), that naturally combines with the progressive
sampling strategy of SILVAN by introducing an upper limit to the number of samples required to converge. Our new bound is governed by key quantities of the underlying graph, not considered by previous works, such as the average shortest path length, and the maximum variance of betweenness centrality estimators, significantly improving the state-of-the-art bounds for the task. Our proof combines techniques from combinatorial optimization and key results from theory of concentration inequalities. While previous results were tailored to analyse a specific estimator of the betweenness centrality, our result is general, since it applies to all available estimators of the betweenness centrality. Furthermore, we extend this result to obtain sharper relative deviation bounds from a random sample.

- We perform an extensive experimental evaluation (Section 5), showing that SILVAN improves the state-of-the-art by requiring a fraction of the sample sizes and running times to achieve a given approximation quality or, equivalently, sharper guarantees for the same amount of work. Our experimental evaluation shows that SILVAN’s guarantees, provided by our theoretical analysis, hold with a true approximation error close to its probabilistic upper bound, confirming the sharpness of our analysis. For the extraction of the top-\(k\) betweenness centralities, our algorithm provides faster approximations, using less samples, and with fewer false positives.

2 RELATED WORK

We now review the works on approximating the betweenness centralities that are most relevant to our contributions. In particular, we focus on approaches that provide guarantees on the quality of the approximation, an often necessary requirement.

The first practical sampling algorithm to approximate the betweenness centrality of all nodes with guarantees on the quality of the approximation is presented in \([54]\). Studying the VC-dimension of shortest paths, Riondato and Kornaropoulos \([54]\) proved that when \(O(\log(D)/\delta)/\varepsilon^2\) shortest paths are sampled uniformly at random, the approximations are within an additive error \(\varepsilon\) of the exact centralities with probability \(\geq 1 - \delta\), where \(D\) is the vertex diameter of the graph (or an upper bound to it). While interesting, this result is characterized by the worst-case and distribution-free nature of the VC-dimension, and thus provides an overly conservative bound and cannot be used to design a progressive sampling approach.

The first rigorous progressive sampling algorithm is ABRA \([57]\), which builds on the theory of Rademacher averages and pseudodimension and does not require an estimate of the vertex diameter. However, ABRA leverages a deterministic and worst-case upper bound to the Rademacher complexity (based on Massart’s Lemma \([41, 43, 64]\)); in a different scenario, Pellegrina et al. \([49]\) show it provides conservative results in most cases compared to its Monte Carlo approximation given by the MCERA. In addition, similar to \([54]\), ABRA obtains uniform and variance-agnostic bounds that hold for all nodes in the graph \(G\). The most recent approach is BAVARIAN \([21]\), which addresses some of the limitations of ABRA using the MCERA and variance-aware tail bounds. Leveraging the flexibility and generality of the MCERA, BAVARIAN is compatible with different estimators of the betweenness centrality, but still obtains uniform approximation bounds not sensible to the heterogeneity of the centrality of different nodes of the graph. In contrast, our algorithm SILVAN uses efficiently computable, non-uniform, and variance-dependent bounds for different subsets of the nodes, which lead to a significant reduction of the number of samples and running times required to obtain rigorous guarantees w.r.t. all methods mentioned above.

A different approach has been proposed by KADABRA \([12]\), a progressive sampling algorithm based on adaptive sampling. KADABRA is based on a weighted union bound, using a data-dependent scheme to assign different probabilistic confidences on the estimates of the betweenness centrality of each individual node, achieving improved approximations compared to the algorithm of
Our experimental evaluation shows that KADABRA is the state-of-the-art solution for the task. Our algorithm SILVAN uses a similar intuition and data-dependent approach, but with crucial differences. In particular, KADABRA assigns confidence parameters $\delta_v$ for each $v \in V$, such that the probability that the approximation of the betweenness centrality for node $v$ not being accurate is at most $\delta_v$ for each node $v \in V$, with $\sum_{v \in V} \delta_v \leq \delta$. In contrast, our approach uses Rademacher averages and empirical peeling to obtain variance-dependent approximations that are valid for sets of nodes, exploiting correlations among nodes instead of considering each node individually. As we show in our experimental evaluation, this leads to significant improvements on the approximation quality compared to KADABRA.

Most of existing methods [12, 54, 57] propose variants of their algorithms for the approximation of the top-$k$ betweenness centralities. Our algorithm SILVAN achieves better results for this task as well, thanks to the non-uniform bounding scheme from the use of empirical peeling.

Other papers study different, but related, problems and notions of centralities. de Lima et al. [25] use an approach similar to [54], and based on pseudodimension, to estimate percolation centrality. Chechik et al. [18] propose an algorithm based on probability proportional to size sampling to estimate closeness centralities, which is the inverse of the average distance of a node to all other nodes in a graph, a popular importance measure in the study of social networks. Boldi and Vigna [10] consider closeness and harmonic centralities though HyperLogLog counters [9, 28]. Bergamini et al. [4] propose a new algorithm for selecting the $k$ nodes with the highest closeness centralities in a graph. Mahmoody et al. [39] and Pellegrina [48] study the problem of finding a set of at most $k$ nodes of maximum betweenness centrality, where the betweenness centrality of a set is defined analogously to the betweenness centrality of nodes [34, 69], and present efficient randomized algorithms to obtain rigorous approximations with high probability. Bergamini et al. [5] study the problem of increasing the betweenness centrality of a node by adding new links. Other recent works considered extending the computation of the betweenness centrality to dynamic graphs [6, 7, 33], uncertain graphs [59], and temporal networks [60]. SILVAN uses estimates of the vertex diameter of graphs, for which several approximation approaches have been proposed (e.g., [6, 22, 23, 38]), including some for distributed frameworks [17].

3 Preliminaries

In this section, we introduce the basic notions used in the remainder of the article. We summarize the most important concepts and abbreviations in Table 1.

3.1 Graphs and Betweenness Centrality

Let $G = (V, E)$ be a graph with $n = |V|$ vertices. For ease of exposition, we focus on unweighted directed graphs, however our algorithms can be easily adapted to weighted graphs. For any pair $(u, z)$ of different nodes $(u \neq z)$, let $\sigma_{uz}$ be the number of shortest paths between $u$ and $z$, and let $\sigma_{uz}(v)$ be the number of shortest paths between $u$ and $z$ that pass through (i.e., contain) $v$. Equivalently, $v$ is internal to such shortest paths. The (normalized) betweenness centrality $b(v)$ of a node $v \in V$ is defined as

$$b(v) = \frac{1}{n(n-1)} \sum_{u,z \in V} \frac{\sigma_{uz}(v)}{\sigma_{uz}}.$$  

Intuitively, a node $v$ has high betweenness centrality $b(v)$ if it is traversed by a large fraction of shortest paths of the graph $G$.

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1We adopt the convention $\sigma_{uz}(v)/\sigma_{uz} = 0$ when $\sigma_{uz} = 0$, e.g., when $u = z$. 

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Table 1. Most Important Notation and Abbreviations Used in the Article

| Symbol | Name/meaning |
|--------|--------------|
| $G = (V, E)$ | Graph |
| $n$ | Number $|V|$ of vertices |
| $D$ | Vertex diameter of the graph |
| $\rho$ | Average shortest path length of the graph (see Section 4.3) |
| $b(v)$ | Betweenness Centrality of node $v$ (see (1)) |
| $\mathcal{F}$ | Family of functions (see Sections 3.2 and 4.2.1) |
| $[\mathcal{F}_j, j \in [1, t]]$ | Partition of $\mathcal{F}$ in $t$ subsets (see Section 4.1) |
| $\mathcal{X}$ | Domain of $\mathcal{F}$ |
| $\gamma$ | Probability distribution over the domain $\mathcal{X}$ of $\mathcal{F}$ |
| $m$ | Sample size |
| $\mathcal{S}$ | Collection of $m$ i.i.d. samples from $\mathcal{X}$ drawn according to $\gamma$ |
| $D(\mathcal{F}, \mathcal{S})$ | Supremum Deviation (SD) of $\mathcal{F}$ on $\mathcal{S}$ (see (2)) |
| $R(\mathcal{F}, \mathcal{S})$ | Empirical Rademacher Average (ERA) of $\mathcal{F}$ on $\mathcal{S}$ (see (3)) |
| $c$ | Number of Monte Carlo trials |
| $\sigma$ | Matrix of $c \times m$ independent Rademacher variables |
| $R^*_m(\mathcal{F}, \mathcal{S}, \sigma)$ | $c$-trials Monte Carlo Empirical Rademacher Average (MCERA) of $\mathcal{F}$ on $\mathcal{S}$ using $\sigma$ (see (4)) |

Some of our algorithms rely on the knowledge of an upper bound to the vertex diameter $D$ of a graph $G$, defined as the maximum number of nodes contained in any shortest path of $G$. (If $G$ is unweighted, then the vertex diameter of $G$ is equal to the diameter of $G$ plus 1.)

### 3.2 Rademacher Averages

Rademacher averages are a core concept in statistical learning theory [35] and in the study of empirical processes [14]. We now present the main notions and results used in our work and defer additional details to [14, 43, 64]. Let $\mathcal{X}$ be a finite domain and consider a probability distribution $\gamma$ over the elements of $\mathcal{X}$. Let $\mathcal{F}$ be a family of functions from $\mathcal{X}$ to $[0, 1]$, and let $\mathcal{S} = \{\tau_1, \ldots, \tau_m\}$ be a collection of $m$ independent and identically distributed samples from $\mathcal{X}$ taken according to $\gamma$. Note that while we focus on functions $\in [0, 1]$ for simplicity, all the results of this article apply to functions in a bounded codomain $[a, b]$ by scaling and shifting. For each function $f \in \mathcal{F}$, define its average value over the sample $\mathcal{S}$ as $\mu_S(f) = \frac{1}{m} \sum_{i=1}^{m} f(\tau_i)$ and its expectation, taken w.r.t. $\mathcal{S}$, as $\mu_\gamma(f) = \mathbb{E}_S[\mu_S(f)]$. Note that, by definition, $\mu_S(f)$ is an unbiased estimator of $\mu_\gamma(f)$.

Given $\mathcal{S}$, a key quantity we are interested in bounding is the supremum deviation $D(\mathcal{F}, \mathcal{S})$ of $\mu_S(f)$ from $\mu_\gamma(f)$ among all $f \in \mathcal{F}$, that is

$$D(\mathcal{F}, \mathcal{S}) = \sup_{f \in \mathcal{F}} |\mu_S(f) - \mu_\gamma(f)|. \quad (2)$$

For the task of betweenness centrality approximation, different estimators can be defined with different notions of the domain $\mathcal{X}$, the family $\mathcal{F}$, and the sampling distribution $\gamma$. The simplest example is the rk estimator [54], where $\mathcal{X}$ is the set of shortest path of the graph, $\gamma$ is the categorical distribution over $\mathcal{X}$ (where a shortest path $\pi \in \mathcal{X}$ from $u$ to $z$ has weight $(n(n-1)\sigma_{u,z})^{-1}$), and the functions in $\mathcal{F}$ estimate the betweenness centrality $b(v)$ of the node $v \in V$ as the fraction of shortest paths of $\mathcal{S}$ that traverse $v$ (we give more details on the rk estimator in Section 4.2.1). More formally, $\mathcal{S}$ is a collection of $m$ shortest paths taken at random from $\mathcal{X}$ according to $\gamma$, and the family $\mathcal{F} = \{f_v : v \in V\}$ is composed of indicator functions $f_v : \mathcal{X} \to \{0, 1\}$.
with \( f_v(\tau) = 1 \) [\( v \) is internal to \( \tau \)], and \( \mathbb{E}_\tau[f_v(\tau)] = b(v) \) for all \( v \in V \). SILVAN employs a more refined estimator that we describe in more details in Section 4.2.1.

The Empirical Rademacher Average (ERA) \( \hat{R}(F, S) \) of \( F \) on \( S \) is a key quantity to obtain a data-dependent upper bound to the supremum deviation \( D(F, S) \). Let \( \sigma = < \sigma_1, \ldots, \sigma_m > \) be a collection of \( m \) i.i.d. Rademacher random variables (r.v.’s), each taking value in \( \{-1, 1\} \) with equal probability. The ERA \( \hat{R}(F, S) \) of \( F \) on \( S \) is

\[
\hat{R}(F, S) = \mathbb{E}_\sigma \left[ \sup_{f \in F} \frac{1}{m} \sum_{i=1}^{m} \sigma_i f(\tau_i) \right].
\]

Computing the ERA \( \hat{R}(F, S) \) is usually intractable, since there are \( 2^m \) possible assignments for \( \sigma \) and for each such assignment a supremum over the functions in \( F \) must be computed. A useful approach to obtain sharp probabilistic bounds on the ERA is given by Monte Carlo estimation [3]. For \( c \geq 1 \), let \( \sigma \in \{-1, 1\}^{c \times m} \) be a \( c \times m \) matrix of i.i.d. Rademacher r.v.’s. The \( c \)-trials Monte Carlo Empirical Rademacher Average (c-MCERA) \( \hat{R}_c^m(F, S, \sigma) \) of \( F \) on \( S \) using \( \sigma \) is

\[
\hat{R}_c^m(F, S, \sigma) = \frac{1}{c} \sum_{j=1}^{c} \sup_{f \in F} \frac{1}{m} \sum_{i=1}^{m} \sigma_{j,i} f(\tau_i).
\]

The c-MCERA allows us to obtain sharp data-dependent probabilistic upper bounds to the supremum deviation, as they directly estimate the expected supremum deviation of sets of functions by taking into account their correlation. For this reason, they are often significantly more accurate than other methods [49], such as the ones based on often loose deterministic upper bounds to Rademacher averages (e.g., Massart’s Lemma [41]), or other distribution-free notions of complexity, such as the VC-dimension. In general, the c-MCERA may be hard to compute, due to the supremaums over \( F \) [3]. However, for the case of betweenness centralities, we show in Section 4.2 that all quantities relevant to the c-MCERA can be efficiently and incrementally updated as shortest paths are randomly sampled.

4 SILVAN: EFFICIENT PROGRESSIVE ESTIMATION OF BETWEENNESS CENTRALITIES

In this section, we introduce SILVAN and the techniques at its core.

We start, in Section 4.1, by presenting empirical peeling and the related main technical results, which provide sharp data-dependent non-uniform approximation bounds supporting our algorithms. We then describe, in Section 4.2, our algorithm SILVAN that builds on such improved bounds to obtain an approximation within additive error \( \epsilon \) of the betweenness centrality for all nodes via progressive sampling. We then present, in Section 4.3, improved bounds on the sufficient number of samples to achieve absolute approximations with high probability. These bounds are naturally combined with the progressive sampling scheme of SILVAN. Finally, in Section 4.4, we introduce SILVAN-TopK, an extension of SILVAN to obtain a relative approximation of the \( k \) nodes with highest betweenness centrality.

4.1 Non-uniform Bounds via Empirical Peeling

In this section, we introduce empirical peeling, a new data-dependent scheme based on the c-MCERA to obtain sharp non-uniform bounds to the supremum deviation. The main idea behind empirical peeling is to partition the set of functions \( F \) to obtain the best possible bounds for different subsets of \( F \).

Classical concentration inequalities, such as Bernstein’s and Bennet’s [14], are well suited to control the deviation \( D(\{f\}, S) \) of a single function \( f \), and to derive an approximation whose
accuracy depends on its variance $\text{Var}(f)$. Instead, when simultaneously bounding the deviation of multiple functions belonging to a set $\mathcal{F}$, the accuracy of the probabilistic bound on the supremum deviation $D(\mathcal{F}, S)$ has a strong but natural dependence on the maximum variance $\sup_{f \in \mathcal{F}} \text{Var}(f)$. However, when the variances of the members of $\mathcal{F}$ are highly heterogenous, this leads to a significant loss of accuracy in the approximation of functions with variance much smaller than the maximum (i.e., we obtain a “blurred” approximation of functions $f' \in \mathcal{F}$ with $\text{Var}(f') \ll \sup_{f \in \mathcal{F}} \text{Var}(f)$).

We propose an intuitive solution to achieve a higher granularity in the approximation: we partition $\mathcal{F}$ into $t \geq 1$ subsets $\{\mathcal{F}_j, j \in [1, t]\}$ with $\bigcup_j \mathcal{F}_j = \mathcal{F}$, such that functions with similar variance belong to the same subset $\mathcal{F}_j$; this allows us to control the supremum deviations $D(\mathcal{F}_j, S)$ for each $\mathcal{F}_j$ separately, exploiting the fact that the maximum variance is now computed on each subset $\mathcal{F}_j$ instead on the entire set $\mathcal{F}$. This idea leads to sharp non-uniform bounds that are locally valid for each subset $\mathcal{F}_j$ of $\mathcal{F}$, and it is the main motivation and intuition behind empirical peeling.

The idea of computing a stratification of the set of functions under consideration is at the core of peeling, an important technique in the study of fine properties of empirical processes, extensively studied in statistical learning theory [2, 14, 30, 66]. However, the issue with existing peeling techniques is that the partition $\{\mathcal{F}_j\}$ either relies on strong assumptions about $\mathcal{F}$, or depends on information computed from the sample $S$; this latter approach incurs in non-trivial issues due to the dependency between the bounds to $D(\mathcal{F}_j, S)$ and $\{\mathcal{F}_j\}$, since both are estimated on the same data $S$. For this reason, available methods are often loose (e.g., with bounds featuring very large constants) and thus received scant application in practical scenarios. Instead, the main idea behind empirical peeling is to use an independent sample $S'$ to partition the set $\mathcal{F}$. This simple but effective idea significantly simplifies the analysis as it resolves the above mentioned dependency issues, with minimal additional work (as good partitions are obtained with a sample $S'$ much smaller than $S$).

Before discussing efficient and effective procedures to partition $\mathcal{F}$, we present a result providing the probabilistic guarantees at the core of empirical peeling, in which we assume the partitioning is fixed before drawing the sample $S$. This improved bound (Theorem 4.2 below) on the supremum deviations holds for families $\mathcal{F}$ of functions $f$ with value in $[0, 1]$, building on Monte Carlo Rademacher Averages introduced in Section 3.2. We use this result in SILVAN to obtain sharp data-dependent non-uniform bounds on the approximation of betweenness centrality.

Before stating Theorem 4.2, we remark that it is based on a novel tight probabilistic bound for the concentration of the $c$-MCERA for general families of functions (Theorem 4.1) that scales with the empirical wimpy variance of $\mathcal{F}$ (defined below). Our novel bound proves that the $c$-MCERA is a sub-gaussian random variable [14], therefore it satisfies concentration bounds that are uniformly sharper than state-of-the-art sub-gamma results (i.e., Theorem 11 of [46] and Theorem 5 of [20]).

The empirical wimpy variance $w_{\mathcal{F}}(S)$ of $\mathcal{F}$ on $S$ is defined [14] as

$$w_{\mathcal{F}}(S) = \sup_{f \in \mathcal{F}} \frac{1}{m} \sum_{i=1}^{m} (f(\tau_i))^2.$$  

**Theorem 4.1.** For $c, m \geq 1$, let $\sigma \in \{-1, 1\}^{c \times m}$ be an $c \times m$ matrix of Rademacher random variables, such that $\sigma_{j,i} \in \{-1, 1\}$ independently and with equal probability. Then, with probability $\geq 1 - \delta$ over $\sigma$, it holds

$$\hat{R}(\mathcal{F}, S) \leq \hat{R}_{\sigma}^c(\mathcal{F}, S, \sigma) + \sqrt{\frac{4w_{\mathcal{F}}(S) \ln \left(\frac{1}{\delta}\right)}{cm}}.$$
The proof of Theorem 4.1, deferred to the Appendix, leverages a concentration inequality for functions uniformly distributed on the binary hypercube (see Section 5.2 of [14]). We are now ready to state the main technical result of this Section. (The proof is in Appendix A.1.)

**Theorem 4.2.** Let $\mathcal{F} = \bigcup_{j=1}^{t} \mathcal{F}_j$ be a family of functions with codomain in $[0, 1]$. Let $S$ be a sample of size $m$ taken i.i.d. from a distribution $\gamma$. Denote $\nu_{\mathcal{F}_j}$ such that $\sup_{f \in \mathcal{F}_j} \text{Var}(f) \leq \nu_{\mathcal{F}_j}, \forall j$. For any $\delta \in (0, 1)$, define

$$
\tilde{R}_j = \tilde{R}_m(\mathcal{F}_j, S, \sigma) + \sqrt{\frac{4w_{\mathcal{F}_j}(S) \ln \left( \frac{4t}{\delta} \right)}{cm}},
$$

$$
R_j = \tilde{R}_j + \frac{\ln \left( \frac{4t}{\delta} \right)}{m} + \sqrt{\frac{\left( \frac{\ln \left( \frac{4t}{\delta} \right)}{m} \right)^2}{m}},
$$

$$
\varepsilon_{\mathcal{F}_j} = 2R_j + \sqrt{\frac{2 \ln \left( \frac{4t}{\delta} \right) (\nu_{\mathcal{F}_j} + 4R_j)}{m}} + \frac{\ln \left( \frac{4t}{\delta} \right)}{3m}.
$$

(5)

With probability at least $1 - \delta$ over the choice of $S$ and $\sigma$, it holds $D(\mathcal{F}_j, S) \leq \varepsilon_{\mathcal{F}_j}$ for all $j \in [1, t]$.

From Theorem 4.2, we observe that, since each $\nu_{\mathcal{F}_j}$ strongly affects $\varepsilon_{\mathcal{F}_j}$, as it typically dominates (5), partitioning $\mathcal{F}$ according to different stratifications of $\nu_{\mathcal{F}_j}$ is very beneficial to obtain sharp non-uniform bounds. We remark that recent works based on Monte Carlo Rademacher Averages [21, 49] used bounds that apply to the particular case $t = 1$ (without any partitioning of $\mathcal{F}$) to obtain a uniform variance-dependent bound that can be very loose for most functions as it ignores any heterogeneity of variances within $\mathcal{F}$ (see Theorem 3.2 of [49] and Theorem 3.1 of [21]).

We note that in many cases, appropriate values for variance upper bounds $\nu_{\mathcal{F}_j}$ are not known. The following result upper bounds every supremum variance $\sup_{f \in \mathcal{F}_j} \text{Var}(f)$ of all sets of functions $\{\mathcal{F}_j\}$ using the empirical wimpy variances $w_{\mathcal{F}_j}(S)$. This bound conveniently defines sharp data-dependent values of $\nu_{\mathcal{F}_j}$ that we plug in (5). Our proof is based on the self-bounding properties of the function $w_{\mathcal{F}_j}(S)$, proved by [47].

**Proposition 4.3.** With probability at least $1 - \delta$ it holds, for all $j \in [1, t]$,

$$
\sup_{f \in \mathcal{F}_j} \text{Var}(f) \leq \nu_{\mathcal{F}_j} \equiv w_{\mathcal{F}_j}(S) + \frac{\ln \left( \frac{2}{\delta} \right)}{m} + \sqrt{\frac{\left( \frac{\ln \left( \frac{2}{\delta} \right)}{m} \right)^2}{m}} + \frac{2w_{\mathcal{F}_j}(S) \ln \left( \frac{2}{\delta} \right)}{m}.
$$

(6)

Theorem 4.2 and Proposition 4.3 are easily combined by replacing $4/\delta$ by $5/\delta$ in Theorem 4.2, and $1/\delta$ by $5/\delta$ in (6); with this adjustment, we obtain that both statements hold simultaneously with probability at least $1 - \delta$.

### 4.2 SILVAN

In this section, we introduce SILVAN, our algorithm, based on the contributions of Section 4.1, to compute rigorous approximations of the betweenness centrality of all nodes in a graph.

We first describe, in Section 4.2.1, the algorithm to efficiently sample shortest paths, that is at the core of SILVAN to approximate the betweenness centrality. We then present SILVAN in Section 4.2.2.
4.2.1 Sampling Shortest Paths. SILVAN works by sampling shortest paths in $G$ uniformly at random and using the fraction of shortest paths containing $v$ as an unbiased estimator of its betweenness centrality $b(v)$. The first estimator following this idea was introduced by Riondato and Kornaropoulos [54] (the r$k$ estimator). The idea is to first sample two nodes $u, z$ uniformly at random, and then a uniformly distributed shortest path $\pi$ between $u$ and $z$. With this procedure the probability $\Pr[v \in \pi]$ that a node $v$ is internal to $\pi$ is $\Pr[v \in \pi] = b(v)$. A more refined approach was proposed by [55] (the ab estimator), which considers all shortest paths between $u$ and $z$ instead of only one, approximating the betweenness centrality $b(v)$ as the fraction of such shortest paths passing through $v$. The ab estimator has smaller variance and provides higher quality approximations than the r$k$ in practice [21]; this is because, intuitively, it updates estimations among all nodes involved in shortest paths between $u$ and $z$, and thus, informally, provides “more information per sample.” Computationally, the set $\Pi_{uz}$ of shortest paths between $u$ and $z$, required by both the r$k$ and ab estimators, is obtained in time $O(|E|)$ using a (truncated) BFS, initialized from $u$ and expanded until $z$ is found. For the r$k$ estimator, a faster approach based on a balanced bidirection BFS was proposed and analysed by Borassi and Natale [12]: they show that all information required to sample one shortest path between two vertices $u$ and $z$ is computed in time $O(\sqrt{|E|})$ with high probability on several random graph models, and experimentally on real-world instances. While this approach drastically speeds-up betweenness centrality approximations via the r$k$ estimator [12], an analogous extension of this technique to the ab estimator is desirable but currently lacking.

Our sampling algorithm extends the balanced bidirection BFS to the ab estimator; this allows us to combine superior statistical properties of ab with the much faster balanced bidirection BFS enjoyed by r$k$. Our main idea is that, once the set of all shortest paths $\Pi_{uz}$ between $u$ and $z$ is implicitly computed by the two BFSs, then it is very efficient to sample multiple shortest paths uniformly at random from $\Pi_{uz}$ (while [12] only sampled one shortest path).

SILVAN samples shortest paths with the following procedure:

1. sample two nodes $u, z$ uniformly at random;
2. perform a balanced bidirection BFS starting from $u$ and $z$, until the two BFSs “meet”;
3. sample uniformly at random $[\alpha \sigma_{uz}]$ shortest paths from the set $\Pi_{uz}$ of shortest paths between $u$ and $z$, where $\sigma_{uz} = |\Pi_{uz}|$ is the number of shortest paths between $u$ and $z$ and $\alpha \geq 1$ a positive constant.

It is easy to see that the expected fraction of shortest paths sampled using this procedure containing $v$ is equal to the betweenness centrality $b(v)$ of $v$. In particular, for each node $v \in V$ and a bag of shortest paths $\tau$ obtained from this sampling procedure, define the function $f_v(\tau)$, with $f_v(\tau) = |\tau|^{-1} \sum_{\pi \in \tau} 1[\pi \in \pi]$ where $1[\pi \in \pi] = 1$ if $v$ is internal to the shortest path $\pi \in \tau$, 0 otherwise. Consequently, the set of functions we use for betweenness centrality approximation contains all $f_v$ with $v \in V$, so that $\mathcal{F} = \{f_v, v \in V\}$. By considering a sample $S$ of size $m$ taken as described above, we define the estimate $\hat{b}(v)$ of the betweenness centrality $b(v)$ of $v$ as

$$\hat{b}(v) = \mu_S(f_v) = \frac{1}{m} \sum_{\tau \in S} f_v(\tau).$$

We have that $\hat{b}(v)$ is an unbiased estimator of $\mu_f(f_v) = b(v)$, so that $\mathbb{E}_S[\hat{b}(v)] = b(v)$:

$$\mathbb{E}_S[\hat{b}(v)] = \mathbb{E}_\tau[f_v(\tau)] = \mathbb{E}\left[\frac{1}{|\tau|} \sum_{\pi \in \tau} 1[\pi \in \pi]\right] = \mathbb{E}[1[\pi \in \pi]] = \Pr(\pi \in \pi) = b(v).$$

Regarding $\alpha$, from standard Poisson approximation to the balls and bins model (Chapter 5 of [43]), we obtain that the expected fraction of shortest paths that are not sampled from the set...
Π_{u_2} in step (3) is \( \sigma_{u_2}(1 - 1/\sigma_{u_2})^\alpha \sigma_{u_2} \approx e^{-\alpha} \). Consequently, to ensure that the set of sampled shortest paths well represents \( \Pi_{u_2} \), we set \( \alpha \) to in \( \frac{1}{2} \) where \( \lambda \) is a small value (e.g., in practice, we use \( \lambda = 0.1 \)).

4.2.2 SILVAN Algorithm. We now describe our algorithm SILVAN to compute an accurate approximation of the betweenness centrality. The goal of SILVAN is to achieve an \( \varepsilon \)-approximation (or absolute \( \varepsilon \)-approximation) of the set \( BC(G) = \{ b(v) : v \in V \} \), defined as follows.

**Definition 4.4.** A set \( \tilde{B}C(G) = \{ \tilde{b}(v) : v \in V \} \) is an \( \varepsilon \)-approximation of \( BC(G) = \{ b(v) : v \in V \} \) if it holds, for all \( v \in V \), that \( |b(v) - \tilde{b}(v)| \leq \varepsilon \).

Algorithm 1 describes the algorithm SILVAN to compute an \( \varepsilon \)-approximation of \( BC(G) \) by employing the techniques introduced in Section 4.1. SILVAN can be logically divided into two phases: in the first phase (lines 1–4), SILVAN generates a sample \( S' \) that is used for empirical peeling (Section 4.1) to partition \( \mathcal{F} \) into \( t \) subsets \( \{ \mathcal{F}_j, j \in [1, t] \} \). The second phase (lines 5–15) describes the main operations of the algorithm to approximate the betweenness centrality.

The second phase of SILVAN is based on a progressive sampling approach. At a high level, the algorithm works in iterations, and in iteration \( i \) SILVAN extracts an approximation \( \hat{b}(v) \) of the values \( b(v) \) for all \( v \in V \) from a sample \( S_i \), which is a collection of \( m_i = |S_i| \) randomly sampled bags of shortest paths. The progressive sampling scheme considers samples sizes \( \{m_i\} \) that form an increasing sequence, following a suitable sampling schedule. At the end of each iteration, SILVAN checks whether a suitable stopping condition is satisfied. This stopping condition is based on estimating the \( c \)-MCERA of each partition \( \mathcal{F}_j \) on the sample \( S_i \), and obtaining bounds to the supremum deviation \( D(\mathcal{F}_j, S) \) for each \( \mathcal{F}_j \) (via the empirical peeling technique of Theorem 4.2). When all the deviations are small enough (i.e., all are \( \leq \varepsilon \)), the stopping condition is satisfied and the algorithm reports the achieved approximation. It is important that the stopping condition is satisfied as soon as possible, as each sample is expensive to compute, in particular for large graphs.

**Algorithm 1:** SILVAN

| Line | Description |
|------|-------------|
| 1    | \( S' \leftarrow \text{sampleSPs}(m') \); |
| 2    | \( \{ \mathcal{F}_j, j \in [1, t] \} \leftarrow \text{empiricalPeeling}(\mathcal{F}, S') \); |
| 3    | \( m_1 \leftarrow \text{sufficientSamples}(\mathcal{F}, S', \delta/2) \); |
| 4    | \( \{m_i, \delta_i\} \leftarrow \text{samplingSchedule}(\mathcal{F}, S') \); |
| 5    | **for all the** \( j \in [1, t] \) **do** \( \varepsilon_{F_j} \leftarrow 1 \); |
| 6    | \( i \leftarrow 0; S_0 \leftarrow 0; \sigma \leftarrow \text{empty matrix} \); |
| 7    | **while** not stoppingCond(\( \varepsilon, \{ \varepsilon_{F_j} \}, \hat{m}, m_i \)) **do** |
| 8    | \( i \leftarrow i + 1; d_m \leftarrow m_i - m_{i-1} \); |
| 9    | \( S_i \leftarrow S_{i-1} \cup \text{sampleSPs}(d_m) \); |
| 10   | \( \sigma \leftarrow \text{add columns} \{ \text{sampleRrvs}(d_m, c) \} \text{ to } \sigma \); |
| 11   | \( \tilde{b}, \tilde{r}, \{v_{F_j}\} \leftarrow \text{updateEstimates}(S_i, \sigma, \{F_j\}) \); |
| 12   | **for all the** \( j \in [1, t] \) **do** |
| 13   | \( \tilde{R}_{m_{j}}(\mathcal{F}_j, S_i, \sigma) \leftarrow \frac{1}{2} \sum_{x=1}^{c} \max_{u \in V, v \in F_j}[\tilde{r}(u, x)] \); |
| 14   | \( \varepsilon_{F_j} \leftarrow \text{epsBound}(\tilde{R}_{m_{j}}(\mathcal{F}_j, S_i, \sigma), v_{F_j}, \delta_i) \); |
| 15   | **return** \( \tilde{b} \); |

We start by describing in more details the first phase. In line 1, the sample \( S' \) is generated using the procedure \( \text{sampleSPs}(m') \), which samples uniformly at random \( m' \) bags of shortest paths (following the procedure described in Section 4.2.1), where \( m' \geq 1 \) is given in input.
obtaining $S'$, SILVAN uses the procedure empiricalPeeling (line 2) to partition $F$ into $t$ subsets. empiricalPeeling splits $F$ according to an estimate of the variance of its members (we describe a simple but very effective partitioning scheme in more details at the end of this section). Using $S'$, SILVAN computes an upper bound $\hat{m}$ to the number of samples required to obtain an $\epsilon$-approximation with the procedure sufficientSamples (line 3), which is described in Section 4.3. This bound $\hat{m}$ guarantees that any sample $S$ with size $\geq \hat{m}$ provides an $\epsilon$-approximation with probability $\geq 1 - \delta/2$. Then, the algorithm fixes a sampling schedule given by the values of $m_i$ using the function samplingSchedule (line 4), which also provides a set of values $\delta_i$. The parameter $\delta_i$ sets the confidence of the obtained approximation at iteration $i$, with the constraint $\sum_i \delta_i \leq \delta/2$.

We note that arbitrary schedules can be used, but we discuss in Section 5 a data-dependent scheme that leverages $S'$.

We now describe the second phase of the algorithm. First, in line 5 it initializes all values of $\epsilon_{F_j}$ to 1. Then, in line 6, other variables used by SILVAN are initialized: $i$ is the index of the iteration, while $m_i$ is the size of the sample $S_i$ considered at the $i$th iteration. The algorithm initializes $\sigma$ as an empty matrix, needed at every iteration to compute the c-MCERA (Section 3.2). In line 7, the iterations of SILVAN begin, which terminate according to the stopping condition stoppingCond (defined below). In every iteration of the while loop, SILVAN performs the following operations: first, it increments $i$; at the $i$th iteration, it generates $d_m = m_i - m_{i-1}$ new samples (line 9) using sampleSPs($d_m$), adding them to the set $S_{i-1}$ to obtain $S_i$. Then, it extends $\sigma$ (line 10) adding $d_m$ columns (each composed by c rows), so that $\sigma \in \{0,1\}^{c \times m_i}$ to consider a sample $S_i$ of size $m_i$. Such columns are generated with the procedure sampleRvs($c, d_m$) that samples a $c \times d_m$ matrix in which each entry is a Rademacher r.v. (Section 3.2). SILVAN updates all estimates needed for the approximation (line 11) using the procedure updateEstimates. This procedure uses the sample $S_i$, the matrix $\sigma$, and the partition ($F_j$) to compute three quantities: $\hat{b}$ is a vector of $n$ components containing the estimates $\hat{b}(v)$ of $b(v)$ for all $v \in V$; $\hat{r}$ is a matrix of $n \times c$ components, in which each entry $\hat{r}(v, x)$ is defined as the estimated c-MCERA for the function $f_v$ using the $x$th row $\sigma_{x, .}$ of $\sigma$, such that $\hat{r}(v, x) = \hat{R}_m^c((f_v), S_i, \sigma_{x, .})$; these values are required to compute the c-MCERA of each set $F_i$. Then, the set $\{v_{F_j}\}$ contains probabilistic upper bounds to supremum variances sup$_{f \in F_j} Var(f)$, such that sup$_{f \in F_j} Var(f) \leq v_{F_j}$; we take each $v_{F_j}$ as in (6) (replacing $1/\delta$ by $5/\delta_i$ for reasons discussed below). While we describe updateEstimates as a separate procedure executed after the creation of $S_i$ for ease the presentation, in practice all of these quantities can be updated incrementally as every new sample is added to $S_i$. More precisely, the algorithm increases $\hat{b}(v)$ for all nodes $v \in \pi$ and for all $\pi$ in each sample $\tau$, and similarly $\hat{r}(v, j)$ all $j \in [1, c]$; analogously, it updates $v_{F_j}(S_i)$ for all $j$ as each new sample is obtained. All these operations are done in $O(c|\tau|D)$ time per sample, where $D$ is the vertex diameter of the graph, since $|\tau| \leq D - 2$, $\forall \tau \in \tau$. Furthermore, every sample generated within sampleSPs can be sampled and processed in parallel with minimal synchronization. After $S_i$ is created and processed, the algorithm computes (line 13), for all partitions $F_j$ of $F$, the c-MCERA $\hat{R}_m^c(F_j, S_i, \sigma)$ using the values stored in $\hat{r}$. Then, it computes (line 14) a probabilistic upper bound $\epsilon_{F_j}$ to the supremum deviation $D(F_j, S)$ for each partition $F_j$ using the function epsBound. This function returns $\epsilon_{F_j}$ from (5) replacing $4/\delta$ by $5/\delta_i$. As anticipated previously, we require the values of $\delta_i$ to satisfy the sum $\sum_i \delta_i$ over all possible iterations of the algorithm to be $\leq \delta/2$; we use this property to obtain simultaneous guarantees for all iterations of the algorithm and for all the probabilistic estimates of $v_{F_j}$, as we formally prove with Proposition 4.5 (we discuss in Section 5 valid choices of these parameters). SILVAN continues to iterate until the stopping condition stoppingCond is true: Since we are interested in an $\epsilon$-approximation, stoppingCond checks that $\epsilon \geq \epsilon_{F_j}, \forall j \in [1, t]$, or that $m_i \geq \hat{m}$. When stoppingCond is true, SILVAN returns the approximation $\hat{b}$ (line 15).
The following result establishes the guarantees provided by SILVAN. The proof (Appendix A.2) follows from contributions of Section 4.1 and the samples bound we formally describe in Section 4.3.

**Proposition 4.5.** With probability \( \geq 1 - \delta \), the output \( \tilde{b} \) of SILVAN is an \( \epsilon \)-approximation of \( \text{BC}(G) \).

The following statement gives bounds to the running time and space of SILVAN.

**Proposition 4.6.** Assume that SILVAN concludes at iteration \( i \) after processing \( m_i \) samples. Then, SILVAN has \( O(T_{bhb} + c \delta D(m_i + m') + icn) \) time and \( O(cn + |E|) \) space complexities, where \( \delta = \max_{u, z} \sigma_{uz} \) and \( T_{bhb} \) is the time to run the balanced bidirection BFS algorithm.

Note that in the statement above it holds that \( m_i \leq \hat{m} \), that the maximum number of iterations is bounded (e.g., when the sampling schedule follows a geometric progression as described in Section 5, it holds \( i \in O(\log(\hat{m}/m_1)) \)), and that, in practice, \( T_{bhb} \in O(\sqrt{|E|}) \) [12].

We now describe a simple but effective criteria to partition \( \mathcal{F} \), implementing the empiricalPeeling method. First, we denote with \( \tilde{w}_v \) the estimated wimpy variance of the function \( f_v \) on sample \( S' \) as

\[
\tilde{w}_v = \frac{1}{|S'|} \sum_{\tau \in S'} (f_v(\tau))^2.
\]

We assign each function \( f_v \) for each node \( v \in V \) to the set \( \mathcal{F}_j \) with index \( j = \lceil \log_a (\min(\tilde{w}_v^{-1}, |S'|)) \rceil \) for a constant \( a > 1 \). Doing so, we can implement the procedure empiricalPeeling with the following simple operations:

forall \( f_v \in \mathcal{F} \) do

\[
\begin{align*}
&j \leftarrow \lceil \log_a (\min(\tilde{w}_v^{-1}, |S'|)) \rceil; \\
&\mathcal{F}_j \leftarrow \mathcal{F}_j \cup \{f_v\};
\end{align*}
\]

Intuitively, this allows us to split \( \mathcal{F} \) into (at most) \( t = \lceil \log_a (|S'|) \rceil \) partitions, such that each set \( \mathcal{F}_j \) groups functions with variances in \( [1/a^{j+1}, 1/a^j] \), therefore within a multiplicative factor \( a \). Our main intuition is that the empirical wimpy variances \( w_{\mathcal{F}_j}(S) \) control the accuracy of the bounds on the supremum deviations \( D(\mathcal{F}_j, S) \) (through \( \nu_j \) in Theorem 4.2 and Proposition 4.3); this partitioning scheme fully exploits the non-uniform variance-dependent bounds at the core of SILVAN, since the empirical wimpy variances \( w_{\mathcal{F}_j}(S) \) are approximated by \( w_{\mathcal{F}_j}(S') \) and are \( w_{\mathcal{F}_j}(S') \leq 1/a^j \), which decrease exponentially with \( j \).

Finally, we may observe that the stopping condition of SILVAN requires \( \max_j \nu_{\mathcal{F}_j} \leq \epsilon \) to terminate and obtain an absolute \( \epsilon \)-approximation. While it is true that the deviation bound \( \nu_{\mathcal{F}_j^*} \) of the partition \( \mathcal{F}_j^* \) with maximum variance \( \max_j \nu_{\mathcal{F}_j} = \nu_{\mathcal{F}_j^*} \) is affected by \( \nu_{\mathcal{F}_j^*} \) in (5), the stratification from empirical peeling may still be beneficial to the computation of \( \epsilon \)-approximations. This is a consequence of reducing the values of \( \nu_j \) in (5) by partitioning \( \mathcal{F} \) (since \( \hat{R}_m(\mathcal{F}_j, S, \sigma) \leq \hat{R}_m^c(\mathcal{F}, S, \sigma), \forall j \)), which is particularly important for the partition \( \mathcal{F}_j^* \) with maximum variance in practice (while dividing \( \delta \) by \( t \) is negligible if \( t \) is small).

We empirically evaluate the impact of empirical peeling in SILVAN with an ablation experiment, described in Section 5.1.4.

### 4.3 Upper Bound to the Number of Samples

In this section, we prove new upper bounds to the sufficient number of samples to obtain accurate approximations of betweenness centrality. Section 4.3.1 presents our new bound for absolute \( \epsilon \)-approximations. Our proof is based on a novel connection between key results
from combinatorial optimization [40] and fundamental concentration inequalities [14]. Then, in
Section 4.3.2, we show that the same technique can be applied to guarantee that, from a sample
of a given size, the estimates of the betweenness centrality satisfy tight relative deviation bounds.
In Section 4.3.3, we provide empirical bounds on the average shortest path length, a key param-
eter governing our novel bounds. For ease of presentation, we defer the proofs to the Appendix
(Appendix A.3).

4.3.1 Absolute Approximation. The following result shows an improved bound to the num-
ber of samples to obtain an absolute $\epsilon$-approximation. We obtain a distribution-dependent bound,
since it takes into account the maximum variance of the betweenness centrality estimators.
In addition, our bound scales with the average shortest path length $\rho$ (that we define in (7)),
a key graph characteristic not considered by previous results. A first observation is that the aver-
age shortest path length is equal to the sum of betweenness centrality $b(v)$ over all $v \in V$. For a pair
of nodes $u, z \in V$, denote with $\pi_{uz}$ one of the shortest paths from $u$ to $z$ in the graph,
and define $|\pi_{uz}|$ as the number of internal nodes in $\pi_{uz}$ (ignoring $u$ and $z$). Then, it holds
(see Lemma A.6)

$$\rho = \sum_{v \in V} b(v) = \frac{1}{n(n-1)} \sum_{u, z \in V} |\pi_{uz}|. \quad (7)$$

The key intuition behind our new bounds (Theorems 4.7 and 4.8, formally stated below) is that
the betweenness centrality measure satisfies a form of negative correlation among vertices: the
existence of a node $v$ with high betweenness centrality $b(v)$ constrains the sum of the between-
ness centrality of all other nodes to be at most $\rho - b(v)$; intuitively, this means that the number
of vertices of $G$ with high betweenness centrality cannot be arbitrarily large. Moreover, we
assume that the maximum variance $\max_{v \in V} Var_{\gamma}(f_{v})$ of the betweenness centrality estimators is
at most $\hat{\gamma}$, rather than using the worst-case bound of $\max_{v \in V} Var_{\gamma}(f_{v}) \leq 1/4$. Consequently, the
estimates $\hat{b}(v)$, which can incur in large deviations w.r.t. to their expected values $b(v)$, may not
only be (naïvely) bounded by the number $n = |V|$ of vertices of $G$ but are tightly constrained by
the parameters $\rho$ and $\hat{\gamma}$. Building on this idea, we are able to characterize an upper bound to
the probability of not obtaining an absolute $\epsilon$-approximation from a sample of size $m$, where this
probability is taken over the space of graphs with average shortest path length at most $\rho$, and
maximum estimator variance at most $\hat{\gamma}$. The key technical tool we use to achieve this is to express
this probability as an instance of a Bounded Knapsack Problem [40]; we explicitly optimize this
combinatorial problem through different relaxations, leading to sharp upper bounds. We remark
that taking advantage of these additional constraints on the space of possible graphs is in strong
contrast with the best available results, based on worst-case analyses leading to more conservative
guarantees.

**Theorem 4.7.** Let $F = \{f_{v}, v \in V\}$ be a set of functions from a domain $X$ to $[0, 1]$. Let a distribu-
tion $\gamma$ such that $\mathbb{E}_{\gamma} \{f_{v}(\tau)\} = b(v)$. Define $\hat{\gamma} \in (0, 1/4]$, $\rho \geq 0$ such that

$$\max_{v \in V} Var_{\gamma}(f_{v}) \leq \hat{\gamma}, \text{ and } \sum_{v \in V} b(v) \leq \rho.$$

$^2$We note that our definition of $\rho$ is slightly different from typical notions of “average shortest path length”; in unweighted
graphs the most common definition is the average number of edges of shortest paths. Our definition of $\rho$ corresponds to
the average number of nodes that are internal to the shortest paths of the graph, where the average is taken over all pairs
of nodes. The two notions are in most cases tightly related.
Fix $\delta \in (0, 1)$, $\varepsilon \in (0, 1)$, and define the functions $g(x) = x(1-x)$ and $h(x) = (1+x)\ln(1+x) - x$ for $x \geq 0$. Let $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}$ be

$$\hat{x}_1 = \inf \left\{ x : \frac{1}{2} - \frac{\sqrt{\frac{2\varepsilon^2}{3}}}{9} \leq x \leq \frac{1}{2} g(x)h \left( \frac{\varepsilon}{g(x)} \right) \leq 2\varepsilon^2 \right\}, \quad \hat{x}_2 = \frac{1}{2} - \sqrt{\frac{1}{4} - \hat{\nu}}, \quad \hat{x} = \min\{\hat{x}_1, \hat{x}_2\}.$$ 

Let $S$ be an i.i.d. sample of size $m \geq 1$ taken from $X$ according to $\gamma$ such that

$$m \geq \sup_{x \in (0, \hat{x})} \left\{ \ln \left( \frac{2\rho}{\hat{\nu}x} \right) \frac{g(x)h \left( \frac{\varepsilon}{g(x)} \right)}{\hat{\nu}} \right\}. \quad (8)$$

With probability $\geq 1 - \delta$ over $S$, it holds $D(\mathcal{F}, S) \leq \varepsilon$.

We note that Theorem 4.7 not only applies to the estimator introduced in Section 4.2.1, but to all available (unbiased) estimators of the betweenness centrality, such as the ones employed by BAVARTIAN [21], providing a tighter upper bound to the sufficient number of random samples required by different estimators as well.

To make the bound (8) more interpretable, we make the following observations. First, while the right-hand side (r.h.s.) of (8) is implicit, and difficult to express in closed form, it can be easily computed with a numerical procedure (e.g., $\hat{x}_1$ can be easily obtained with a binary search in the interval $[1/2 - \sqrt{\varepsilon/3 - \varepsilon^2/9}, 1/2]$, exploiting the convexity of $g(x)h(\varepsilon/g(x))$ in $(0, 1)$). Then, we remark that for typical values of the parameters (e.g., $\delta, \varepsilon \leq 0.25 \leq \rho$), the maximum of (8) is attained at $x^* \approx \hat{x} \leq \hat{x}_2$; furthermore, we note that $h(x) \geq x^2/(2(1 + x/3))$ for $x \geq 0$, and $\hat{x}_2 \geq \hat{\nu} = g(\hat{x}_2)$. Combining all these facts, a very accurate approximation of $m$ in (8) is

$$m \approx \frac{2\hat{\nu} + \frac{\varepsilon^2}{\hat{\nu}}}{\varepsilon^2} \left( \ln \left( \frac{2\rho}{\hat{\nu}} \right) + \ln \left( \frac{1}{\delta} \right) \right) \in O \left( \frac{\hat{\nu} + \varepsilon}{\varepsilon^2} \frac{\rho}{\delta^2} \right) \cdot \quad (9)$$

To confirm this intuition, in Figure 17 (in the Appendix), we compare the values of the function maximized in (8) with the estimate (9), for a wide range of values of $\varepsilon, \rho$, and $\hat{\nu}$. In all cases, we observed the estimate given by (9) to be larger but very close to the maximum in (8).

Moreover, we note that since $\rho$ corresponds to (an upper bound to) the average number of internals nodes in shortest paths of $G$, it is immediate to conclude that $\rho$ cannot exceed the vertex diameter $D$ (the maximum number of internal nodes), therefore it trivially holds $\rho \leq D - 2$ (see Lemma A.6). From these observations, it is natural to compare our new bound with the state-of-the-art result based on the VC-dimension presented by Riondato and Kornaropoulos [54]; they show that, for the $r\kappa$ estimator, $O(\ln(D/\delta)/\varepsilon^2)$ samples are enough for obtaining an $\varepsilon$-approximation with probability at least $1 - \delta$. More recently, a similar $O(\ln(D/\delta)/\varepsilon^2)$ bound (see Theorem 5.2 of [21] and Appendix B.1) was extended to other empirical estimators of the betweenness centrality.

Similarly to these results, our new bound of Theorem 4.7 is independent of the size of the graph (e.g., $|V|$ or $|E|$), and essentially recovers them, since $\hat{\nu} \leq 1/4$ and $\rho \leq D - 2$, but provides much tighter results when smaller bounds to $\hat{\nu}$ and $\rho$ are available: it is distribution-dependent, rather than distribution-free. Interestingly, in many real-world graphs the average shortest path length is typically very small (a phenomena observed in small-world networks [68] for which $\rho \in O(\log |V|)$), and often much smaller than the diameter.

In Appendix B.1 and Figure 18 we compare our bound from Theorem 4.7 with the bounds by [21, 54] for several values of the parameters. In all cases, we observed Theorem 4.7 to be significantly sharper than the above mentioned state-of-the-art results.
Since $\rho$ is usually not known in advance, in Section 4.3.3, we prove that, given a (not necessarily tight) upper bound to $D$, $\rho$ can be sharply estimated as the average number of internal nodes of the shortest paths in a sample $S$, resulting in a very efficient data-dependent bound. In addition, $\hat{v}$ can be sharply upper bounded by Proposition 4.3 (defining $\hat{v} \doteq v_F$, with $f = 1$).

As anticipated in Section 4.2, Theorem 4.7 is not only of theoretical interest, but is used in SILVAN to design its sampling schedule (e.g., by upper bounding with $\hat{m}$ the number of samples it needs to process). SILVAN estimates both $\hat{v}$ and $\rho$ using $S'$, and then plug them in Theorem 4.7 to obtain $\hat{m}$; these operations are part of the procedure 

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Theorem 4.7 states that if $\rho$ is known, then

\[
|b(v) - \hat{b}(v)| \leq \sqrt{\frac{2 \min\{g(b(v)), \hat{v}\} \ln \left( \frac{4}{\delta} \min \left\{ \frac{\rho}{\pi(v)}, n \right\} \right)}{m}} + \frac{\ln \left( \frac{4}{\delta} \min \left\{ \frac{\rho}{\pi(v)}, n \right\} \right)}{3m} \leq d_r(b(v)).
\]

Theorem 4.8 yields sharp upper and lower bounds to $b(v)$, for all $v \in V$. A fully empirical version (i.e., that only depends on the empirical estimates $\hat{b}(v)$) is obtained easily (e.g., with a binary search), as evident from the following Corollary.

**Corollary 4.9.** Assume the setting of Theorem 4.8. It holds with probability $\geq 1 - \delta$

$$\min\{x \in [0, \hat{b}(v)] : \hat{b}(v) \leq x + d_r(x)\} \leq b(v) \leq \max\{x \in [\hat{b}(v), 1] : x \leq \hat{b}(v) + d_r(x)\}, \quad \forall v \in V.$$  

We remark that the above bound is significantly more accurate than results obtained using standard tools (i.e., combining Bernstein’s inequality and a union bound over $n$ events) for most interesting values of $b(v)$ (more precisely, when $b(v) \geq 2\rho/n$, since $\ln(4\rho/(b(v)\delta)) \ll \ln(2n/\delta)$).

4.3.3 **Empirical Bounds to the Average Shortest Path Length.** In this section, we present sharp empirical bounds on the average shortest path length, a key quantity involved in the sample bounds introduced in Section 4.3. The first result (Proposition 4.10) is based on the application of Bernstein’s inequality [14], while the second (Proposition 4.11) uses the Empirical Bernstein Bound introduced by Maurer and Pontil [42].

**Proposition 4.10.** Let $z \geq D - 2$, where $D$ is the vertex diameter of the graph $G$. Let an i.i.d. sample $S$ of size $m$, and denote $\hat{\rho} = \sum_{v \in V} \hat{b}(v)$. Then, for a fixed $\delta \in (0, 1)$, it holds with probability $\geq 1 - \delta$

$$\sum_{v \in V} b(v) \leq \rho \doteq \hat{\rho} + \sqrt{\frac{5}{3} \left( \frac{z \ln \left( \frac{1}{\delta} \right)}{m} \right)^2} + \frac{2z\hat{\rho} \ln \left( \frac{1}{\delta} \right)}{m} + \frac{4z \ln \left( \frac{1}{\delta} \right)}{3m}.$$  

The following gives typically slightly sharper bounds than Proposition 4.10, since it involves an empirical estimator $\Lambda(S)$ of the variance of $\hat{\rho}$. 

**ACM Transactions on Knowledge Discovery from Data, Vol. 18, No. 3, Article 52. Publication date: December 2023.**
Proposition 4.11. Assume the setting of Proposition 4.10 with $S = \{\tau_1, \ldots, \tau_m\}$. Define $\Lambda(S)$ as

$$\Lambda(S) = \frac{1}{m(m-1)} \sum_{1 \leq i < j \leq m} \left( \sum_{v \in V} f_v(\tau_i) - \sum_{v \in V} f_v(\tau_j) \right)^2.$$ 

Then, for a fixed $\delta \in (0, 1)$, it holds with probability $\geq 1 - \delta$

$$\rho \leq \tilde{\rho} + \sqrt{\frac{2\Lambda(S) \ln \left( \frac{2}{\delta} \right)}{m}} + \frac{7z \ln \left( \frac{2}{\delta} \right)}{3m}.$$ 

4.4 Top-$k$ Approximation

In this section, we present SILVAN-TopK, an extension of SILVAN to compute high-quality relative approximations of the $k$ most central vertices.

While in some cases additive approximations, for which we guarantee that $|\tilde{b}(v) - b(v)| \leq \epsilon$ for all $v \in V$, are sufficient, in several practical cases relative approximations, for which the desired bound to $|\tilde{b}(v) - b(v)|$ depends on the value $b(v)$, may be more informative. Such approximations are particularly relevant for the problem of estimating the $k$ most central nodes, as the value of their betweenness centrality is typically highly skewed. In such cases, one may prefer to have relative approximations of the type “$\tilde{b}(v)$ is within 10% of the value of $b(v)$” than an additive approximation of the type “$|\tilde{b}(v) - b(v)| \leq 0.01$” that may be either unnecessarily precise for high values of $b(v)$, or uninformative for low values of $b(v)$. Furthermore, the user needs to only fix $k$ and the relative accuracy, a much more natural choice for exploratory analyses, in which the centrality scores of the top-$k$ nodes are unknown. We note that, in statistical learning theory, it is well known that it is not possible to obtain relative approximations efficiently using uniform additive approximations as a proxy [13]; this motivates the development of specialized techniques for the task (e.g., [19, 32, 36]).

In this section, we show that empirical peeling, introduced in Section 4.1, is naturally suited to the problem of computing relative approximations of the set of top-$k$ central nodes, and can do so progressively and adaptively as samples are processed. First, let $v_1, \ldots, v_n$ be the nodes sorted according to their betweenness centrality, such that $b(v_i) \geq b(v_{i+1})$. The set $TOP(k)$ of top-$k$ nodes is defined as $TOP(k) = \{(v_i, b(v_i)) : i \leq k\}$. We now define the relative approximation we are interested in obtaining.

Definition 4.12. For $\eta \in (0, 1)$ and $k \geq 1$, a set $\tilde{TOP}(k) = \{(v, \tilde{b}(v))\}$ with $v \in V, \tilde{b}(v) \in [0, 1]$, is a relative $\eta$-approximation of $TOP(k)$ if all the following hold:

$$\{v : (v, \tilde{b}(v)) \in \tilde{TOP}(k)\} \supseteq \{v : (v, b(v)) \in TOP(k)\},$$ \hspace{1cm} (10)

$$|b(v) - \tilde{b}(v)| \leq \eta b(v), \forall (v, \tilde{b}(v)) \in \tilde{TOP}(k),$$ \hspace{1cm} (11)

$$b(v) \geq b(v_k) \left( \frac{1 - \eta}{1 + \eta} \right)^2, \forall (v, b(v)) \notin TOP(k), (v, \tilde{b}(v)) \in \tilde{TOP}(k).$$ \hspace{1cm} (12)

Informally, (10) ensures that all nodes in $TOP(k)$ are in the approximation; (11) implies that all the estimates in the approximation are close to the true values of the betweenness centrality, within relative accuracy given by $\eta$; (12) guarantees that nodes $v$ not in the set $TOP(k)$ of the top-$k$ nodes are in the approximation only if their betweenness centrality $b(v)$ is not too far from $b(v_k)$, the centrality of the $k$th node.
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We now discuss how to modify SILVAN to obtain an approximation $\tilde{T}\tilde{O}(k)$ of $TOP(k)$ with the aforementioned guarantees. Assume, at the end of some iteration of SILVAN, that we have confidence intervals $CI_v = [\ell(v), u(v)]$ for each $v \in V$, such that $b(v) \in CI_v, \forall v$. Such confidence intervals are derived from bounds on supremum deviations: for a node $v$ such that $f_v \in \mathcal{F}_j$ for some $j \in [1, t]$, and assuming that $\varepsilon_{\mathcal{F}_j} \geq D(\mathcal{F}_j, S)$, we define

\[
\ell(v) = \hat{b}(v) - \varepsilon_{\mathcal{F}_j} \quad \text{and} \quad u(v) = \hat{b}(v) + \varepsilon_{\mathcal{F}_j}.
\]

Naturally, the validity of such confidence intervals is probabilistic, and thus we aim to obtain a relative $\eta$-approximation of $TOP(k)$, with high probability. To verify that a given set $\tilde{T}\tilde{O}(k)$ is a relative $\eta$-approximation of $TOP(k)$, we inspect the confidence intervals $CI_v$ for each candidate $v$ to be included in $\tilde{T}\tilde{O}(k)$.

**Proposition 4.13.** For each $v \in V$, denote the intervals $CI_v = [\ell(v), u(v)]$ with $\ell(v) \leq \hat{b}(v) \leq u(v)$. Let $v_1, \ldots, v_n$ be the sequence of nodes ordered according to $\ell(\cdot)$, such that $\ell(v_i^j) \geq \ell(v_{i+1}^j)$. Define the set $\tilde{T}\tilde{O}(k)$ as $\tilde{T}\tilde{O}(k) = \{v, b(v) : u(v) \geq \ell(v_k^j)\}$, and assume that, for all $(v, b(v)) \in \tilde{T}\tilde{O}(k)$,

\[
\frac{\hat{b}(v)}{1 + \eta} \leq \ell(v) \leq b(v) \leq u(v) \leq \frac{\hat{b}(v)}{1 - \eta}.
\]

Then, $\tilde{T}\tilde{O}(k)$ is a relative $\eta$-approximation of $TOP(k)$.

Building on this result, Algorithm 2 describes our algorithm SILVAN-TopK to compute relative $\eta$-approximations of $TOP(k)$.

**Algorithm 2: SILVAN-TopK**

**Input:** Graph $G = (V, E); c, k \geq 1; \eta, \delta \in (0, 1)$.

**Output:** A relative $\eta$-approximation of top-$k$ central nodes with probability $1 - \delta$.

1. **While not stoppingCondFirst() do** $S' \leftarrow S' \cup \text{sampleSPs}(1)$;

2. $\{m_j, i_j \} \leftarrow \text{empiricalPeeling}(\mathcal{F}, S')$;

3. $\{s_j, \delta_j \} \leftarrow \text{samplingSchedule}(\mathcal{F}, S');$

4. $i \leftarrow 0; S_0 \leftarrow \emptyset; \sigma \leftarrow \text{empty matrix};$

5. **While not stoppingCondTopk() do**

6. $i \leftarrow i + 1; d_m \leftarrow m_i - m_{i-1};$

7. $S_i \leftarrow S_{i-1} \cup \text{sampleSPs}(d_m);$;

8. $\sigma \leftarrow \text{add columns} \{\text{sampleRvrs}(d_m, c)\} \text{to} \sigma;$

9. $\hat{b}, \bar{r}, \{\nu_f\} \leftarrow \text{updateEstimates}(S_i, \sigma, \{\mathcal{F}_j\});$

10. **For all** $j \in [1, t]$ do

11. $\hat{R}_{m_j}^c(\mathcal{F}_j, S_i, \sigma) \leftarrow \frac{1}{c} \sum_{x=1}^c \max_{u \in V, f \in \mathcal{F}_j} \hat{r}(u, x);$\n
12. $\varepsilon_{\mathcal{F}_j} \leftarrow \text{epsBound}(\hat{R}_{m_j}^c(\mathcal{F}_j, S_i, \sigma), \nu_{\mathcal{F}_j})$;

13. **For all** $v : f_v \in \mathcal{F}_j$ do $[\ell(v), u(v)] \leftarrow [\hat{b}(v) - \varepsilon_{\mathcal{F}_j}, \hat{b}(v) + \varepsilon_{\mathcal{F}_j}];$

14. $\tilde{T}\tilde{O}(k) \leftarrow \{v, \hat{b}(v) : u(v) \geq \ell(v_k^j)\};$

15. **Return** $\tilde{T}\tilde{O}(k);$.

As SILVAN, SILVAN-TopK is divided in two phases: in the first phase (lines 1–3) it samples $S'$, uses it for empirical peeling (line 2), and defines the sampling schedule (line 3). Then, it obtains the relative $\eta$-approximation using progressive sampling in the second phase (lines 4–15).
The first phase of SILVAN-TopK, instead of considering a fixed number of samples \(m\) for \(S'\) (as in Algorithm 1), continues to draw shortest paths taken at random until at least \(k\) distinct nodes have been observed at least a constant number of times (therefore, after \(\approx 1/b(v_k)\) samples); when this is verified, the function stoppingCondFirst returns true (line 1) and the generation of \(S'\) stops. Following this scheme, the first phase adapts to the (unknown) value of \(b(v_k)\).

The second phase of SILVAN-TopK is similar to SILVAN. At iteration \(i\), after obtaining bounds \(\mathcal{F}_j\) on supremum deviations \(D(\mathcal{F}_j, S_i)\) from the sample \(S_i\), the algorithm defines the confidence intervals \([\ell(v), u(v)]\) w.r.t. \(b(v)\) (line 13). Then, it creates the set \(\mathcal{T}(k)\) including all vertices with upper bound \(u(v)\) at least \(\ell(v'_k)\), where \(\ell(v'_k)\) is the \(k\)th lower bound (line 14), as defined in Proposition 4.13. To obtain the approximation described by Definition 4.12, SILVAN-TopK outputs \(\mathcal{T}(k)\) when its stopping condition \(\text{stoppingCondTopk}\) verifies that (13) holds for all \((v, \tilde{b}(v)) \in \mathcal{T}(k)\). Note that the algorithm does not need to know \(b(v_k)\) (or \(b(v)\) for any \(v\)), as the left and rightmost inequalities in (13) only depend on empirical quantities. From the probabilistic guarantees implied by Theorem 4.2 and from Proposition 4.13, the following result easily follows.

**Proposition 4.14.** The output of SILVAN-TopK is a relative \(\eta\)-approximation of \(\text{TOP}(k)\) with probability \(\geq 1 - \delta\).

We remark that this general approach can be adapted easily to other definitions of relative approximations (e.g., [19, 36]). As we will show in our experimental evaluation, empirical peeling is essential to achieve relative \(\eta\)-approximations efficiently.

A possibly interesting extension of the schemes of SILVAN and SILVAN-TopK may be to adaptively improve the partition of \(\mathcal{F}\) at every iteration, extending both the sample \(S\) and the independent reservoir sample \(S'\), instead of using the static \(S'\) and \(\{\mathcal{F}_j\}\) fixed in the first phase, at the beginning of the algorithm. For instance, this modified algorithm could take an additional input parameter \(\phi \in (0, 1]\) and could extend \(S'\) at each iteration with \([\phi d_m]\) new samples, using the instruction \(S' \leftarrow S' \cup \text{sampleSPs}([\phi d_m])\). Then, to update the partition \(\{\mathcal{F}_j\}\), it could call the procedure empiricalPeeling at each iteration. Doing so, the algorithm allocates a fraction \(\phi/(1 + \phi)\) of resources to progressively improve the quality of the empirical peeling partitions. We note that, since \(S'\) remains completely independent of \(S\), updating the empirical peeling partitions with this adaptive procedure does not violate any guarantee proved in previous sections. We leave further theoretical and empirical investigation of this idea to future work.

## 5 Experiments

We implemented SILVAN and tested it on several real-world graphs. In our experimental evaluations, we assess the effectiveness of the progressive sampling approach of SILVAN to approximate the betweenness centrality of all nodes, and evaluate the performance of SILVAN-TopK in approximating the top-\(k\) most central nodes. To do so, our goal is to evaluate and compare SILVAN with state-of-the-art methods in terms of: number of samples and running time required to compute accurate approximations; quality of the computed approximation, measured as the maximum deviation between the approximated and exact centrality values, and compared with the guaranteed upper bound.

**Experimental Setup.** We implemented SILVAN in C++. Our implementation of SILVAN, with automated scripts to reproduce all experiments, is available online.\(^3\) We compare SILVAN with KADABRA [12],\(^4\) that has been shown to uniformly and significantly outperform previous

\(^3\)https://github.com/VandinLab/SILVAN
\(^4\)https://github.com/natema/kadabra
methods, and with BAVARIAN [21], the most recent method for betweenness centrality approximation. When referring to BAVARIAN, we consider its variant based on progressive sampling (denoted BAVARIAN-P, see Algorithm 2 and Section 5.2 of [21]), which addresses the same problem solved by SILVAN and KADABRA, and we tested it using all different estimators for the betweenness centrality described in [21] (called rk, ab, and bp). All the code was compiled with GCC 8 and run on a machine with 2.30 GHz Intel Xeon CPU, 512 GB of RAM, on Ubuntu 20.04, with a total of 64 cores. All experiments were performed using multithreading on all threads.

**Graphs.** We tested SILVAN on 7 undirected and 11 directed real-world graphs from SNAP and KONECT, most of them previously analysed by KADABRA [12] and other previous methods [21, 54, 57]. The characteristics of the graphs are described in detail in Table 2.

### 5.1 Absolute Approximation

We first consider the task of computing an absolute ε-approximation to the betweenness centrality of all nodes.

For every graph, we ran all algorithms with parameter ε ∈ {0.01, 0.005, 0.0025, 0.001, 0.0005}, chosen to have comparable magnitude to the betweenness centrality of the most central nodes

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5https://github.com/acdmammoths/Bavarian-code
6http://snap.stanford.edu/data/index.html
7http://konect.cc/networks/
8We note that in the experiments performed in [21], the graph cit-HepPh is considered as undirected. In all our experiments, we treat cit-HepPh as a directed graph, in accordance with the information at https://snap.stanford.edu/data/cit-HepPh.html
(i.e., see col. $\hat{\epsilon}$ of Table 2); this is required to compute meaningful approximations (i.e., an absolute $\epsilon$-approximation is useless when the centralities of the most central nodes are much smaller than $\epsilon$). We fix $\delta = 0.05$, and use $c$ Monte Carlo Rademacher vectors with $c = 25$ for SILVAN and BAVARIAN (note that $c = k$ in [21]). We do not show results for other values of $\delta$, as this parameter has minimal impact on the results, due to the use of exponential tail bounds (see $\delta$ in (5) and (8)). Regarding $c$, we follow [49] and [21], that have shown that sharp bounds are obtained even with a low number of Monte Carlo trials, and that there are minimal improvements using $c > 30$. We ran all algorithms 10 times and report averages $\pm$ stds. We limit the execution time of each run to 6 h; we terminate the algorithm when exceeding this threshold.

For the empirical peeling scheme of SILVAN, we sample $m' = \lceil \log(1/\delta)/\epsilon \rceil$ shortest paths to generate $S'$; we note that $m'$ always results in a very small fraction of the overall samples analysed by SILVAN. Regarding the sampling schedule followed in the second phase, we use $S'$ to identify a minimum number $m_1$ of samples before starting to evaluate the stopping condition. To do so, we perform a binary search to identify the minimum $m_1$ such that (5) (with $R_j = 0$) is not larger than $\epsilon$; this gives an optimistic first guess of the number of samples to process for obtaining an $\epsilon$-approximation. We then increase the sample size $m_i$ with a geometric progression, such that $m_{i+1} = \theta \cdot m_i$, with $\theta = 1.2$ (i.e., we define the sequence $\{m_i\}$ in line 4 as $\{m_1, m_1 = \theta \cdot m_{i-1}\}$).

While a geometric progression is considered to be optimal [53], we note that the procedure samplingSchedule can be implemented with general schedules.

Regarding the choice of the set of confidence parameters $\{\delta_i\}$, we note that using $\delta_i = \delta/2^{i+1}$ is sufficient to guarantee the properties of Proposition 4.5 when SILVAN stops at any iteration, i.e., for any value of $i$, since $N_{\sum_{i=1}^N \delta_i}$ for any $N$. However, the maximum number $N$ of iterations of SILVAN is fixed before the beginning of the second phase: using the geometric progression defined above, it holds $N = \lceil \log(m_1/m_{i})/\log(\theta) + 1 \rceil$. For this reason, we let SILVAN make the less restrictive choice of $\delta_i = \delta/(2N)$, but different schemes that satisfy the constraint $\sum_{i=1}^N \delta_i = \delta/2$ are possible (e.g., that assign less confidence to early iterations).

As described in Section 4.3, SILVAN uses the procedure sufficientSamples to obtain an upper bound $m$ to the number of samples to process to obtain an absolute $\epsilon$-approximation: it does so using $S'$, computing an upper bound $\rho$ to the average shortest path length (Theorem 4.11) and an upper bound $\hat{\nu}$ to the supremum variance of the estimators $\sup_{f \in F} Var(f_i)$ (e.g., using $\hat{\nu} = \nu_{f_i}$ with $i = 1$ in Proposition 4.3). Then, sufficientSamples plugs these estimates in Theorem 4.7 to compute $\hat{m}$. The empiricalPeeling procedure of SILVAN follows the scheme described at the end of Section 4.2 using $a = 2$.

For the progressive sampling schedule of BAVARIAN, we use the same geometric progression parameter $\theta = 1.2$ of SILVAN.

Figure 1 shows the results for this set of experiments comparing SILVAN to KADABRA, while Figure 2 shows the results comparing SILVAN to BAVARIAN for the estimator ab (more results in Figure 9, and analogous plots for rk and bp in Figures 10–13, all in Appendix).

5.1.1 Sample Sizes. In Figures 1(a) and 1(b) we show the ratios between the number of samples required by KADABRA and SILVAN to converge (we sum the number of samples of both phases, for both algorithms) for directed and undirected graphs. We can see that the number of samples needed by SILVAN is always smaller than KADABRA, by at least 20%; for 14 of 18 graphs, SILVAN finished after processing less than half of the samples considered by KADABRA, and may require up to an order of magnitude less samples. By inspecting the graphs’ statistics (Table 2), the largest improvements are obtained for graphs with smallest $\sup_{V \in V} b(v) \leq \hat{\epsilon}$. In fact, the number of samples required by SILVAN (Figure 4(a)) varies significantly among graphs, with strong dependence on $\hat{\epsilon}$. Notice that $\sup_{V \in V} b(v)$ upper bounds the maximum variance $\sup_{V \in V} Var(f_i)$. A potential
Fig. 1. Comparison between the performance of KADABRA and SILVAN for obtaining absolute approximations. (a) Ratios of the number of samples required by KADABRA and the number of samples required by SILVAN for directed graphs (black line drawn at $y = 1$). (b) Same as panel (a) for undirected graphs. (c) Comparison of the running times of KADABRA ($y$ axis) and SILVAN ($x$ axis) for all graphs (black line drawn at $y = x$). (d) Same as panel (c) with axes in log scale. Additional plots in Figure 8.

cause of the gap between SILVAN and KADABRA may depend on the use of the VC-dimension-based bound in the adaptive sampling analysis of KADABRA; such bound is indeed required for its correctness, but it is agnostic to any property of the underlying graph (apart from the vertex diameter) and thus results in overly conservative guarantees in such cases. This confirms the significance of SILVAN’s sharp variance-adaptive bounds. In addition, the fact that SILVAN obtains simultaneous and non-uniform data-dependent approximations for sets of nodes, exploiting correlations among nodes through the use of the $c$-MCERA, leads to refined guarantees.

We now compare SILVAN with BAVARIAN in terms of sample sizes. We remark that the plots for sample sizes only show the results for cases in which BAVARIAN terminates in reasonable time (i.e., in less than 6 h), while figures for running times show a lower bound for such cases. From Figures 2(a) and 2(b), we can see that SILVAN always requires a fraction of the samples needed by BAVARIAN: at most half of the samples for all graphs, up to 1/4 of the samples. We observed analogous results for the rk estimator (Figures 10(c) and 10(d)). The bp estimator resulted to be
Fig. 2. Comparison between the performance of BAVARIAN (ab estimator) and SILVAN for obtaining absolute approximations. (a) Ratios of the number of samples required by BAVARIAN and the number of samples required by SILVAN for directed graphs (black line drawn at y = 1). (b) Same as panel (a) for undirected graphs. (c) Comparison of the running times of BAVARIAN (y axis) and SILVAN (x axis) for directed graphs (axes in logarithmic scale) (black line drawn at y = x). (d) Same as panel (c) for undirected graphs. Additional plots in Figures 9–13.

the most efficient version of BAVARIAN in terms of number of samples; this is not surprising, since one bp’s sample considers all shortest paths starting from a single node, rather than shortest paths between two nodes (however, each sample is potentially much more expensive to compute, as pointed out by [21]). For 11 graphs, the number of samples needed by SILVAN is smaller than BAVARIAN-bp by at least 10% for all values of ε, up to using less than 1/2 of the samples; for 4 graphs and ε ≤ 0.0025, BAVARIAN-bp concludes with 30% less samples (see Figures 12(c) and 12(d)). The results of Section 5.1.2 show that the trade-off between the accuracy and running time for the bp estimator is not advantageous.

Overall, SILVAN obtains high-quality approximations at a fraction of the samples required by state-of-the-art methods; this highlights the significance of SILVAN’s non-uniform approximation approach via empirical peeling and its novel improved bounds on the number of sufficient samples presented in Section 4.3.
5.1.2 Running Times. We now discuss how the reduction in the number of samples impacts the overall running times. We observed that, generally, the running time roughly increases linearly with the sample size (Figure 4(b) shows that the relationship between the sample sizes and the running times of SILVAN is essentially linear). In fact, the time spent on sampling shortest paths is usually the dominating cost of the algorithms.

In Figure 1(c), we compare the running times of SILVAN (x axis) and KADABRA (y axis) (we show ratios and additional plots in Figure 8). While for smaller graphs both SILVAN and KADABRA terminate very quickly (e.g., in < 10 s), for the largest and most demanding graphs the reduction on the number of samples achieved by SILVAN has a sensible and significant impact on the running times, as clearly shown in Figure 1(c). For instance, SILVAN analyses the most demanding graph (wikipedia-a-link-en) in less than 1/3 of the time required by KADABRA when ε ≤ 10^{-3} (see Figure 8(f)). This is a consequence of significantly reducing the required samples, and also reflects the capability of SILVAN to compute the ε-MCERA incrementally as shortest paths are sampled, incurring in a negligible computational overhead.

In Figures 2(c)–2(d), we compare the running times of SILVAN (x axis) and BAVARIAN using the ab estimator (y axis) (additional plots and other estimators in Figures 9–13). Note that we report a lower bound to the running time of BAVARIAN when exceeding 6 h (= 2.16 · 10^{4} s); BAVARIAN exceeded this threshold on most large graphs and for smaller values of ε, while SILVAN never required more than 17 minutes (= 10^{3} s). Overall, we observed SILVAN to be at least one order of magnitude faster than BAVARIAN (apart for one graph and ε = 0.01, where SILVAN is 7 times faster), up to 3 orders of magnitude. We observed very similar results for the r k estimator (Figure 11). SILVAN is also at least one order of magnitude faster than BAVARIAN using the bp estimator for all but the wiki-Vote and ca-GrQc graphs, for which it is > 4 times faster (Figure 13). SILVAN’s improvements are due to both the significant reduction in the number of samples (as discussed previously) thanks to its non-uniform approximation scheme, and from the fact that SILVAN leverages a more efficient algorithm for sampling shortest paths, based on the balanced bidirectional BFS, drastically reducing the computational requirement for the task.

We conclude that SILVAN requires much fewer resources to obtain rigorous approximations of the betweenness centrality of all nodes of the same quality, or, equivalently, sharper guarantees for the same amount of work.

5.1.3 Quality of SILVAN’s Approximations. Finally, we investigated the accuracy of the approximations reported by SILVAN by computing the exact betweenness centrality of all the nodes of six graphs (three undirected and three directed, representative of other instances) and measuring D(\mathcal{F}, S) over all runs. We show these results in Figure 5 (see Appendix). As expected from our theoretical analysis, we always observed D(\mathcal{F}, S) ≤ ε, thus SILVAN is more accurate than guaranteed. However, the gap between D(\mathcal{F}, S) and ε is not large, confirming the sharpness of the guarantees provided by SILVAN. We remark that the exact approach requires several hours on the larger graphs we considered for this set of experiments (e.g., for the com-db1p graph, the exact approach implemented in Networkit [65] requires > 1 hour to terminate using all 64 cores), and does not complete in reasonable time for the largest instances (e.g., [12] reports that ≈ 1 week is necessary for graphs of size similar to the largest of our test set). Instead, SILVAN finishes in at most few minutes for the lowest value of ε (e.g., always less than 20 s for com-db1p), and it is much faster for other cases.

5.1.4 Impact of SILVAN’s Contributions. In this section, we present experimental results to evaluate the impact of empirical peeling (Theorem 4.2) and our novel upper limit to the number of samples (Theorem 4.7), the two most important components of SILVAN to obtain absolute ε-approximations efficiently. To do so, we compared SILVAN with two of its variants. The first
of SILVAN, called SILVAN-$\hat{m}$, uses empirical peeling but sets $\hat{m} = +\infty$ (i.e., ignoring the value returned by the procedure sufficientSamples in line 3 of Algorithm 1). Note that, since the maximum number of iterations of SILVAN-$\hat{m}$ is not known, the algorithm uses $\delta_i = \delta/2^{i+1}$ to guarantee the validity of the computed $\varepsilon$-approximation. The second variant of SILVAN, called SILVAN-$\hat{m}$-ep, also sets $\hat{m} = +\infty$ (and $\delta_i = \delta/2^{i+1}$) and further disables empirical peeling by imposing the number $t$ of partitions to $t = 1$ (i.e., the set of functions $\mathcal{F}$ is not partitioned). We measured both the number of samples and running times using the same values of $\varepsilon$ tested in previous experiments, and show the results for three graphs (com-amazon, com-youtube, and soc-pokec, representative of other instances). All results are shown in Figure 14. We clearly observe that SILVAN-$\hat{m}$-ep is the worst performing approach, both in terms of number of samples and running times. Enabling empirical peeling in SILVAN-$\hat{m}$ yields a significant improvements for all graphs. Finally, we observe that SILVAN is the best approach overall, combining both empirical peeling and our novel refined upper limit to the number of samples.

5.2 Top-k Approximation

We now present experiments on the task of computing relative approximations of the set of top-$k$ most central nodes. SILVAN-TopK is the first method that allows us to approximate the top-$k$ most central nodes with relative and non-uniform bounds via empirical peeling, different from previous methods that focus on additive approximations (or rely on uniform additive approximations as a proxy) [12, 57]; therefore, we compare SILVAN-TopK with KADABRA [12], the best performing approach that allows us to obtain approximations of comparable quality. We recall that the top-$k$ approximation proposed in [12], for a given $k$ and $\varepsilon$, guarantees an absolute $\varepsilon$-approximation of the top-$k$ nodes; however, the confidence intervals for some of the nodes can be relaxed (i.e., be wider than $2\varepsilon$) if they can be ranked correctly with looser accuracy. Instead, SILVAN-TopK guarantees that all nodes are well estimated within the relative accuracy $\eta$. For given $k$ and $\eta$, we first run SILVAN-TopK on all graphs; when finished, we store the maximum absolute deviation $\varepsilon_k$ required to guarantee all properties of Definition 4.12, and we run KADABRA with parameters $k$ and $\varepsilon_k$.

We considered $k \in \{5, 10, 25\}$ and $\eta \in \{0.25, 0.1, 0.05\}$. As in previous experiments, SILVAN-TopK’s empirical peeling follows the procedure described in Section 4.2 with $a = 2$ and follows the same geometric progression for the sample sizes $m_i$ used by SILVAN (such that $m_{i+1} = 1.2 \cdot m_i$). However, since the maximum number $N$ of iterations of SILVAN-TopK is not known in advance, the procedure samplingSchedule of Algorithm 2 returns a sequence $\{\delta_i\}$ with $\delta_i = \delta/2^i$. For all experiments, we report avgs. ± stds over 10 runs.

5.2.1 Sample Sizes. Figure 3 shows the number of samples and running time of both algorithms to obtain their respective top-$k$ approximations for $k = 10$ and $\eta = 0.1$, representative of other cases (we show all combinations of $k$ and $\eta$ in Figure 15 in Appendix).

From Figure 3(a), we see that SILVAN-TopK requires a fraction of the samples of KADABRA, even if offering stronger guarantees (no confidence intervals are relaxed according to the ranking): SILVAN-TopK requires at most $2/3$ of KADABRA’s samples, and finishes after processing less than $1/3$ of the samples for 17 of 18 graphs; for 10 graphs, it needs less than $1/5$ of the samples, and less than $1/10$ for 3 graphs.

5.2.2 Running Times. The reduction of the number of samples, similarly to the case of absolute approximation, significantly impacts the running times. From Figure 3(b), we conclude that, while in cases in which both algorithms conclude very quickly (e.g., in less than 3 s on small graphs) they obtain comparable performances, on larger graphs SILVAN significantly outperforms KADABRA. In fact, for 14 graphs SILVAN-TopK finished in less than half of the time of KADABRA, and it is at least
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Fig. 3. Comparison of number of samples (a) and running times (b) of SILVAN-TopK with KADABRA for obtaining top-$k$ approximations for $k = 10$ and $\eta = 0.1$ (all other combinations shown in Figure 15). Ratios are in green (right y axis), while number of samples and running times are in blue (for SILVAN) and red (for KADABRA) (left y axis).

5 times faster in 6 cases. For three of the most demanding graphs, SILVAN is more than 10 times faster. Overall, SILVAN analyses all graphs in $<0.5$ h, while KADABRA needs $>4$ h. We conclude that SILVAN significantly reduces the computational requirements for the task, potentially allowing much more interactive exploratory analyses.

5.2.3 Quality of SILVAN-TopK’s Approximations. Additionally, we compared the quality of the output of KADABRA w.r.t. SILVAN-TopK when using the same number of samples: we stop KADABRA at the number of samples required by SILVAN-TopK. Figure 6 in the Appendix shows runs taken at random for three graphs, using $k = 10$ and $\eta = 0.1$. From Figure 6 we can see that SILVAN-TopK (in blue) provides much tighter upper and lower bounds than KADABRA (in red); obtaining sharper confidence intervals on the top-$k$ nodes has a drastic effect on the capability of the algorithms to rank nodes correctly. In fact, for the same work, SILVAN-TopK reports much less false positives (e.g., 16 vs. 45 results for com-db1p) and clearly identifies the rank of the most central nodes.

Finally, we evaluate the quality of the relative approximations $\hat{TOP}(k)$ returned by SILVAN-TopK against the ground truth, i.e., the set $TOP(k)$. To do so, we compare the estimated betweenness centrality values obtained by SILVAN-TopK with the corresponding exact values. In particular, we compare the maximum relative deviation $\max_{v \in \hat{TOP}(k)} |\hat{b}(v) - b(v)|/b(v)$ with the parameter $\eta$ given in input to Algorithm 2. We evaluated SILVAN-TopK on six graphs, i.e., the same considered for assessing the quality of SILVAN (for absolute approximations, Section 5.1.3), using $k = 10$ and $\eta \in \{0.25, 0.1, 0.05\}$. Figure 7 shows the results for this experiment. We observe that, as expected from the definition of relative $\eta$-approximation and the theoretical guarantees of SILVAN-TopK (Definition 4.12 and Proposition 4.13), the maximum relative deviation is always smaller than $\eta$, i.e., SILVAN-TopK is more accurate than guaranteed by our analysis. However, the gap between the measured maximum relative deviation and $\eta$ is in most cases not too large, denoting that SILVAN-TopK provides quite sharp guarantees for its approximations.
6 CONCLUSIONS

We introduced SILVAN, a novel progressive sampling algorithm to estimate the betweenness centrality of all nodes in a graph. SILVAN relies on new bounds on supremum deviation of functions, based on the c-MCERA and non-uniform approximation scheme via empirical peeling. We present variants of SILVAN to obtain absolute approximations, and relative approximations for the top-k betweenness centrality. Our experimental results show that SILVAN significantly outperforms state-of-the-art approaches for approximating betweenness centrality with the same guarantees.

There are multiple interesting directions for future work. While in this work we considered various approximations of the betweenness centrality in a static setting, recent works considered extending the problem to dynamic [6, 7, 33], temporal [60], and uncertain graphs [59], or different types of centralities [25], all settings in which we believe the ideas behind our algorithm SILVAN could lead to improved approximations.

Furthermore, the empirical peeling scheme we introduced in this work is general: it can be applied to sets of functions with arbitrary domains, so it can potentially benefit randomized approximation algorithms in other settings, such as interesting [58, 61, 63] and significant pattern mining [31, 50, 51], and sequential hypothesis testing [24, 26].

A APPENDIX

A.1 Proofs of Section 4.1

We first prove Theorem 4.1, our new concentration bound of the c-MCERA toward the ERA, one of our main technical contributions. Our argument is based on the following important result on the concentration of functions uniformly distributed on the binary hypercube.

THEOREM A.1 (Theorem 5.3, [14]). For $k > 0$, let a function $g : \{-1, 1\}^k \rightarrow \mathbb{R}$ and assume that $X$ is uniformly distributed on $\{-1, 1\}^k$. Let $\nu > 0$ be such that

$$\sum_{i=1}^{k} \left( g(x) - g(\bar{x}^i) \right)^2 \leq \nu$$

for all $x = (x_1, \ldots, x_k) \in \{-1, 1\}^k$, where

$$\bar{x}^i = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_k)$$

is a copy of $x$ with the $i$th component multiplied by $-1$, and $(b)_+ = \max\{b, 0\}$ is the positive part of $b \in \mathbb{R}$. Then, the random variable $Z = g(X)$ satisfies, for all $t > 0$,

$$\Pr(Z > \mathbb{E}[Z] + t), \Pr(Z < \mathbb{E}[Z] - t) \leq \exp\left(-t^2/\nu\right).$$

We now restate and prove Theorem 4.1.

THEOREM 4.1. For $c, m \geq 1$, let $\sigma \in \{-1, 1\}^{c \times m}$ be an $c \times m$ matrix of Rademacher random variables, such that $\sigma_{j,i} \in \{-1, 1\}$ independently and with equal probability. Then, with probability $\geq 1 - \delta$ over $\sigma$, it holds

$$\hat{R}(F, S) \leq \hat{R}_m^c(F, S, \sigma) + \sqrt{\frac{4w_{\mathcal{F}}(S) \ln \left( \frac{1}{\delta} \right)}{cm}}.$$  

PROOF. Recall the setting of Theorem A.1. We first observe the equivalence

$$\left( g(x) - g(\bar{x}^i) \right)_+ = \left( g(x) - g(\bar{x}^i) \right) 1\left[g(x) > g(\bar{x}^i)\right]$$

(14)
for the function $g$ as defined in Theorem A.1; consequently, to apply the result, we only have to consider the cases in which $g(x) > g(x')$, as otherwise (14) is equal to 0. We define the function $g : \{-1, 1\}^m \to \mathbb{R}$ as $g(\sigma) = cm\hat{R}_m^c(\mathcal{F}, \mathcal{S}, \sigma)$. For a given $\sigma$, denote with $f_{j}^*$ one of the functions attaining the supremum $\sup_{f \in \mathcal{F}} \{\sum_{i=1}^{m}\sigma_{j,i}f(\tau_{i})\}$. Note that the proof easily extends to the case for which the supremum is not attained (e.g., by considering instead functions that are arbitrarily close to the supremum). We define $\overline{\sigma}_{j,i}$ as a copy of $\sigma$ with the $(j, i)$ component $\sigma_{j,i}$ of $\sigma$ multiplied by $-1$. Thus, $g(\overline{\sigma}_{j,i})$ is evaluated as $g(\sigma)$ but with $\sigma_{j,i}$ having flipped sign. We first prove that $g(\overline{\sigma}_{j,i}) \geq g(\sigma) - 2\sigma_{j,i}f_{j}^*(\tau_{i})$ for all $i \in [1, m]$ and for all $j \in [1, c]$, following ideas developed in [46, 47].

From the definition of $g$ given above, we have

$$
g(\overline{\sigma}_{j,i}) = cm\hat{R}_m^c(\mathcal{F}, \mathcal{S}, \overline{\sigma}_{j,i}) = \sum_{h=1}^{c} \sup_{h \neq j} \left\{ \sum_{t=1}^{m} \sigma_{h,t}f(\tau_{t}) \right\} + \sup_{f \in \mathcal{F}} \left\{ \sum_{t=1}^{m} \sigma_{j,t}f(\tau_{t}) - \sigma_{j,i}f(\tau_{i}) \right\}
$$

$$
= \sum_{h=1}^{c} \sup_{h \neq j} \left\{ \sum_{t=1}^{m} \sigma_{h,t}f(\tau_{t}) \right\} + \sup_{f \in \mathcal{F}} \left\{ \sum_{t=1}^{m} \sigma_{j,t}f(\tau_{t}) - 2\sigma_{j,i}f(\tau_{i}) \right\}
$$

$$
\geq \sum_{h=1}^{c} \sup_{h \neq j} \left\{ \sum_{t=1}^{m} \sigma_{h,t}f(\tau_{t}) \right\} + \sum_{t=1}^{m} \sigma_{j,t}f^*(\tau_{t}) - 2\sigma_{j,i}f^*(\tau_{i})
$$

$$
= \sum_{h=1}^{c} \sup_{h \neq j} \left\{ \sum_{t=1}^{m} \sigma_{h,t}f(\tau_{t}) \right\} - 2\sigma_{j,i}f^*(\tau_{i}) = g(\sigma) - 2\sigma_{j,i}f^*(\tau_{i}).
$$

Therefore, we have that

$$
\sum_{i=1}^{m} \sum_{j=1}^{c} \left( g(\sigma) - g(\overline{\sigma}_{j,i}) \right)^2 \leq \sum_{i=1}^{m} \sum_{j=1}^{c} \left( g(\sigma) - \left( g(\sigma) - 2\sigma_{j,i}f^*(\tau_{i}) \right) \right)^2
$$

$$
= \sum_{i=1}^{m} \sum_{j=1}^{c} \left( 2\sigma_{j,i}f^*(\tau_{i}) \right)^2 \leq \sum_{i=1}^{m} \sum_{j=1}^{c} 4f^*(\tau_{i})^2 \leq 4cm\omega_{\mathcal{F}}.
$$

We apply Theorem A.1 to $Z = g(\sigma)$ with $\nu = 4cm\omega_{\mathcal{F}}$, obtaining

$$
\Pr \left( cm\hat{R}_m^c(\mathcal{F}, \mathcal{S}, \sigma) > cm\hat{R}(\mathcal{F}, \mathcal{S}) + t \right) \leq \exp \left( -\frac{t^2}{4cm\omega_{\mathcal{F}}} \right), \tag{15}
$$

$$
\Pr \left( cm\hat{R}_m^c(\mathcal{F}, \mathcal{S}, \sigma) < cm\hat{R}(\mathcal{F}, \mathcal{S}) - t \right) \leq \exp \left( -\frac{t^2}{4cm\omega_{\mathcal{F}}} \right).
$$

The substitution $\varepsilon = t/(cm)$ in (15) yields the inequality

$$
\Pr \left( \hat{R}(\mathcal{F}, S) > \hat{R}_m^c(\mathcal{F}, \mathcal{S}, \sigma) + \varepsilon \right) \leq \exp \left( -\frac{cm^2\varepsilon^2}{4\omega_{\mathcal{F}}} \right). \tag{16}
$$

The statement follows from imposing the r.h.s. of (16) to be $\leq \delta$ and solving for $\varepsilon$.

The proof of our main result (Theorem 4.2) builds on the combination of the most refined concentration inequalities relating Rademacher averages to the supremum deviation [14], that we now introduce. Let the Rademacher complexity $R(\mathcal{F}, m)$ of a set of functions $\mathcal{F}$ be defined as $R(\mathcal{F}, m) = \mathbb{E}_{\mathcal{S}}[\hat{R}(\mathcal{F}, \mathcal{S})]$. The following central result (e.g., see Lemma 11.4 of [14]) relates $R(\mathcal{F}, m)$ to the expected supremum deviation.
Lemma A.2 (Symmetrization Lemma [14, 43, 64]). Let $Z$ be either
\[
\sup_{f \in \mathcal{F}} \{ \mu_S(f) - \mu_f(f) \} \quad \text{or} \quad \sup_{f \in \mathcal{F}} \{ \mu_f(f) - \mu_S(f) \}.
\]
It holds $E_S[Z] \leq 2R(\mathcal{F}, m)$.

The following gives a variance-dependent bound to the supremum deviation above its expectation, and it is due to Bousquet [15] (see also Theorem 12.5 of [14]).

Theorem A.3 (Theorem 2.3 [15]). Let $\hat{\nu}_\mathcal{F} \geq \sup_{f \in \mathcal{F}} \{ \text{Var}(f) \}$, and $Z$ be either $\sup_{f \in \mathcal{F}} \{ \mu_S(f) - \mu_f(f) \}$ or $\sup_{f \in \mathcal{F}} \{ \mu_f(f) - \mu_S(f) \}$. Then, with probability at least $1 - \lambda$ over $S$, it holds
\[
Z \leq E_S[Z] + \sqrt{\frac{2 \ln \left( \frac{1}{\lambda} \right) (\hat{\nu}_\mathcal{F} + 2E[Z])}{m} + \frac{\ln \left( \frac{1}{\lambda} \right)}{3m}}.
\]

The next result bounds $R(\mathcal{F}, m)$ above its estimate $\hat{R}(\mathcal{F}, S)$, using the self-bounding property of the ERA $\hat{R}(\mathcal{F}, S)$ (see Example 3.12 of [14]).

Theorem A.4 ([14, 45]). With probability $\geq 1 - \lambda$ over $S$, it holds
\[
R(\mathcal{F}, m) \leq \hat{R}(\mathcal{F}, S) + \sqrt{\frac{\ln \left( \frac{4t}{\delta} \right)^2}{m} + \frac{2 \ln \left( \frac{4t}{\delta} \right) \hat{R}(\mathcal{F}, S)}{m} + \frac{\ln \left( \frac{4t}{\delta} \right)}{3m}}.
\]

We are now ready to prove Theorem 4.2, the main technical contribution of Section 4.2.

Theorem 4.2. Let $\mathcal{F} = \bigcup_{j=1}^t \mathcal{F}_j$ be a family of functions with codomain in $[0, 1]$. Let $S$ be a sample of size $m$ taken i.i.d. from a distribution $\gamma$. Denote $v_{\mathcal{F}_j}$ such that $\sup_{f \in \mathcal{F}_j} \text{Var}(f) \leq v_{\mathcal{F}_j}, \forall j$. For any $\delta \in (0, 1)$, define
\[
\hat{R}_j \doteq \hat{R}_m^c(\mathcal{F}_j, S, \sigma) + \sqrt{\frac{4w_{\mathcal{F}_j}(S) \ln \left( \frac{4t}{\delta} \right)}{cm}},
\]
\[
R_j \doteq \hat{R}_j + \frac{\ln \left( \frac{4t}{\delta} \right)}{m} + \left( \frac{\ln \left( \frac{4t}{\delta} \right)}{m} \right)^2 + \frac{2 \ln \left( \frac{4t}{\delta} \right) \hat{R}_j}{m},
\]
\[
\epsilon_{\mathcal{F}_j} \doteq 2R_j + \sqrt{\frac{2 \ln \left( \frac{4t}{\delta} \right) (\nu_{\mathcal{F}_j} + 4R_j)}{m} + \frac{\ln \left( \frac{4t}{\delta} \right)}{3m}}.
\]
With probability at least $1 - \delta$ over the choice of $S$ and $\sigma$, it holds $D(\mathcal{F}_j, S) \leq \epsilon_{\mathcal{F}_j}$ for all $j \in [1, t]$.

Proof. Define the events $E_{1,j}, E_{2,j},$ and $E$ as
\[
E_{1,j} = \text{"}\hat{R}(\mathcal{F}_j, S) > \hat{R}_j\text{"}, \quad E_{2,j} = \text{"}R(\mathcal{F}_j, m) > R_j\text{"}, \quad E_{3,j} = \text{"}\sup_{f \in \mathcal{F}_j} \{ \mu_S(f) - \mu_f(f) \} > \epsilon_{\mathcal{F}_j}\text{"}, \quad E_4 = \text{"}\sup_{f \in \mathcal{F}_j} \{ \mu_f(f) - \mu_S(f) \} > \epsilon_{\mathcal{F}_j}\text{"},
\]
\[
E_j = \text{"}D(\mathcal{F}_j, S) > \epsilon_{\mathcal{F}_j}\text{"}, \quad E = \exists j : D(\mathcal{F}_j, S) > \epsilon_{\mathcal{F}_j}.
\]
Note that the statement is proved if $\Pr(E) \leq \delta$, and that the event $E$ is contained in the event $\bigcup_j E_j$, meaning that $\Pr(E) \leq \Pr(\bigcup_j E_j) \leq \sum_j \Pr(E_j)$. Moreover, each event $E_j$ is contained in the event $\bigcup_j E_{1,j}$, therefore $\Pr(E_j) \leq \sum_j \Pr(E_{1,j})$. To upper bound the probabilities of $E_{1,j}, E_{2,j},$ and $E_{3,j}$, we apply Lemma A.2, and Theorems A.3 and A.4, replacing $\mathcal{F}$ by $\mathcal{F}_j$ and $\lambda$ by $\delta/(4t)$.
(1) \( \Pr(E_{1,j}) \leq \delta/(4t) \) follows from Theorem 4.1 (replacing \( \delta \) by \( \delta/(4t) \));
(2) \( \Pr(E_{2,j}) \leq \delta/(4t) \) follows from Theorem A.4;
(3) From Lemma A.2 and twice the application of Theorem A.3, \( \Pr(E_{3,j}) \) and \( \Pr(E_{4,j}) \) are bounded below \( \lambda = \delta/(4t) \).

The event \( E \) is true with probability at most \( \delta \), since \( \Pr(E) \leq \sum_j \Pr(E_j) \leq \sum_j \Pr(E_{i,j}) \leq \delta \). □

We now prove Proposition 4.3, which shows that \( \sup_{f \in \mathcal{F}} Var(f) \) can be tightly estimated using the empirical wimpy variance \( w_{\mathcal{F}}(S) \). We prove it using the self-bounding properties of the function \( w_{\mathcal{F}}(S) \), proved in [47].

**Proposition 4.3.** With probability at least \( 1 - \delta \), it holds, for all \( j \in [1, t] \),

\[
\sup_{f \in \mathcal{F}_j} Var(f) \leq v_{\mathcal{F}_j} \doteq w_{\mathcal{F}_j}(S) + \frac{\ln \left( \frac{t}{\delta} \right)}{m} + \sqrt{\left( \ln \left( \frac{t}{\delta} \right) \right)^2 + 2 w_{\mathcal{F}_j}(S) \ln \left( \frac{t}{\delta} \right) \frac{1}{m}}.
\]

**Proof.** We use the fact that

\[
\sup_{f \in \mathcal{F}_j} Var(f) = \sup_{f \in \mathcal{F}_j} \{ \mathbb{E}[f^2] - \mathbb{E}[f]^2 \} \leq \sup_{f \in \mathcal{F}_j} \mathbb{E}[f^2],
\]

so we focus on bounding the wimpy variance \( \sup_{f \in \mathcal{F}_j} \mathbb{E}[f^2] \) of \( \mathcal{F}_j \). We use following result. □

**Theorem A.5** (Theorem 7.5.8 [47]). With probability \( \geq 1 - \lambda \) over \( S \) it holds

\[
\sup_{f \in \mathcal{F}} \mathbb{E}[f^2] \leq w_{\mathcal{F}}(S) + \frac{\ln \left( \frac{t}{\delta} \right)}{m} + \sqrt{\left( \ln \left( \frac{t}{\delta} \right) \right)^2 + 2 w_{\mathcal{F}}(S) \ln \left( \frac{t}{\delta} \right) \frac{1}{m}}.
\]

We apply Theorem A.5 to each set \( \mathcal{F}_j \) (using \( \lambda = \delta/t \)), obtaining the statement.

### A.2 Proofs of Section 4.2

We now prove Proposition 4.5, which provides the probabilistic guarantees of SILVAN.

**Proposition 4.5.** With probability \( \geq 1 - \delta \), the output \( \tilde{b} \) of SILVAN is an \( \epsilon \)-approximation of \( BC(G) \).

**Proof.** We prove that the statement is a consequence of the following facts:

(1) the c-MCERA for all \( \mathcal{F}_j \) is computed correctly at line 13;
(2) \( \hat{\epsilon}_j \), at the end of iteration \( i \), are computed, for all \( j \), using the bound of Proposition 4.3 with probability \( \delta_i/5 \) (i.e., in Proposition 4.3, \( \delta \) is replaced by \( \delta_i/5 \));
(3) \( \epsilon_{\mathcal{F}_j} \), at the end of iteration \( i \), are computed, for all \( j \), using the bound of Theorem 4.2 with probability \( \delta_i/5 \) (i.e., in Theorem 4.2 \( \delta/4 \) is replaced by \( \delta_i/5 \));
(4) the values of \( \{ \delta_i \} \) are such that \( \sum_i \delta_i \leq \delta/2 \);
(5) \( \hat{m} \) is computed s.t. \( D(\mathcal{F}, S) \leq \epsilon \) (with \( |S| \geq \hat{m} \)) with probability \( \geq 1 - \delta/2 \);
(6) SILVAN stops at an iteration \( i \) when \( \max_j \epsilon_j \leq \epsilon \) or at the first \( i \) s.t. \( m_i \geq \hat{m} \).

We now prove the statement. Let \( X \) be a random variable equal to the index of the iteration in which the algorithm stops. Let the events \( A_i \) and \( \hat{A} \) be defined as

\[
A_i = \text{“at } i\text{th iteration, } \exists v : f_v \in \mathcal{F}_j, \epsilon_{\mathcal{F}_j} < |b(v) - \tilde{b}(v)|”
\]

\[
\hat{A} = \text{“at the first } i\text{ s.t. } m_i \geq \hat{m}, \exists v : f_v \in \mathcal{F}_j, \epsilon_{\mathcal{F}_j} < |b(v) - \tilde{b}(v)|”
\]
The algorithm is correct if both $\hat{A}$ and $A_X$ are false, thus we want to prove that $\Pr(A_X \cup \hat{A}) \leq \delta$. (Following analogous derivations discussed in [12], we remark that $A_X$ depends on the random variable $X$, and it should not be confused with $A_i$ for a fixed $i$.) It is enough to prove that $\Pr(A_X) + \Pr(\hat{A}) \leq \delta$; since we assume that the fact (5) is true, we already have $\Pr(\hat{A}) \leq \delta/2$. We proceed to show that $\Pr(A_X) \leq \delta/2$ to prove the statement. Assuming facts (1), (2), and (3) are all true, at the end of each iteration $i$, for $i = 1, 2, \ldots$, it holds that $D(F_j, S) \leq \epsilon_j$ for all $j$ with probability $1 - \delta_i$, combining Proposition 4.3 and Theorem 4.2. Consequently, $\Pr(A_i) \leq \delta_i, \forall i \geq 1$. From fact (4), we have that

$$\Pr(A_X) = \sum_i \Pr(A_i \cap "X = i") \leq \sum_i \Pr(A_i) \leq \sum_i \delta_i \leq \delta/2.$$

**Proposition 4.6.** Assume that SILVAN concludes at iteration $i$ after processing $m_i$ samples. Then, SILVAN has $O((T_{bb} + c\delta D)(m_i + m') + icn)$ time and $O(cn + |E|)$ space complexities, where $\delta = \max_u, \sigma_{az}$ and $T_{bb}$ is the time to run the balanced bidirection BFS algorithm.

**Proof.** We first prove the space complexity upper bound. SILVAN keeps the graph in memory with $O(n + |E|)$ space. Then, it keeps, for each node $v$, the values $\{\hat{r}(v, x), \}$, for all $x \in [1, c]$, the estimate $\hat{b}$ of the betweenness centrality of $v$, the index $j \in [1, t]$ of the partition of $v$ (such that $f_j \in F_j$), and the estimate of the wimpy variance $w_{(f_{jv})}(S_j)$ of $v$. All such entries are stored with $O(c)$ space for each of the $n$ nodes, obtaining the $O(cn)$ term.

We now prove the time complexity upper bound. The $O(T_{bb}(m_i + m'))$ term follows by running the balanced bidirection BFS algorithm for each of the $m'$ pairs of nodes $(u, z)$ sampled by SILVAN in the first phase, and the $m_i$ pairs in the second phase. Then, for each bag of shortest path $\tau$ obtained by the balanced bidirection BFS algorithm, SILVAN needs $O(c\delta D)$ time to update all values of $\{\hat{r}(v, x), \hat{b}, w_{(f_{jv})}(S_j)\}$. This is because each shortest path $\pi \in \tau$ from the node $u$ to the node $z$ contains at most $D - 2$ nodes, and since the number $|\tau|$ of sampled shortest paths from $u$ to $z$ is $|\tau| = [\alpha\sigma_{az}] \leq [\alpha\delta]$. The empiricalPeeling procedure needs $O(n)$ operations, setting for each node $v$ the appropriate partition index depending on the value of $w_{(f_{jv})}(S_j')$. All other operations done in the first phase are completed in constant time. Then, at the end of each iteration, SILVAN needs $O(cn)$ time to evaluate the stopping condition, by computing $\hat{S}_m^c(F, S, \sigma)$ and $\epsilon_j$ for all $j$, using the values $\{\hat{r}(v, x)\}$ and $w_{(f_{jv})}(S_j)$. After $i$ iterations, we have a total of $O(icn)$ operations.

**A.3 Proofs of Section 4.3**

We first prove our novel refined upper limit on the number of samples required to achieve an absolute $\epsilon$-approximation of the betweenness centrality of all nodes in a graph.

**Theorem 4.7.** Let $F = \{f_v, v \in V\}$ be a set of functions from a domain $X$ to $[0, 1]$. Let a distribution $\gamma$ such that $\mathbb{E}_{\tau \sim \gamma}[f_\tau(\tau)] = b(\tau)$. Define $\hat{v} \in (0, 1/4]$, $\rho > 0$ such that

$$\max_{v \in \hat{V}} \text{Var}_\gamma(f_v) \leq \hat{v}, \text{ and } \sum_{v \in \hat{V}} b(v) \leq \rho.$$ 

Fix $\delta \in (0, 1)$, $\epsilon \in (0, 1)$, and define the functions $g(x) = x(1 - x)$ and $h(x) = (1 + x)\ln(1 + x) - x$ for $x \geq 0$. Let $\hat{x}_1, \hat{x}_2, \text{ and } \hat{x}$ be

$$\hat{x}_1 = \inf \left\{ x : \frac{1}{2} - \sqrt{\frac{\epsilon}{3} - \frac{\epsilon^2}{9}} \leq x \leq \frac{1}{2} \right\}, \quad \hat{x}_2 = \frac{1}{2} - \sqrt{\frac{1}{4} - \hat{v}}, \quad \hat{x} = \min(\hat{x}_1, \hat{x}_2).$$
Let $S$ be an i.i.d. sample of size $m \geq 1$ taken from $X$ according to $\gamma$ such that

$$m \geq \sup_{x \in [0,1]} \left\{ \ln \left( \frac{2p}{x\delta} \right) \right\} g(x) h \left( \frac{e}{g(x)} \right).$$

With probability $\geq 1 - \delta$ over $S$, it holds $D(F, S) \leq \epsilon$.

**Proof of Theorem 4.7** For a sample $S$ of size $m$, define the events $A$ and $A_v$ as

$$A = \exists v \in V : |b(v) - \hat{b}(v)| > \epsilon$$

$$A_v = |b(v) - \hat{b}(v)| > \epsilon.$$  

From a union bound, we have that

$$\Pr(A) = \Pr \left( \bigcup_{v \in V} A_v \right) \leq \sum_{v \in V} \Pr(A_v).$$

Then, through the application of Hoeffding’s and Bennett’s inequalities [14], Bathia and Davis inequality on variance [8] (which implies $\text{Var}(f_v) \leq g(b(v))$), and from the fact that $\sup_{f_v \in F} \text{Var}(f_v) \leq \hat{\nu}$, it holds, for all $v \in V$,

$$\Pr(A_v) \leq 2 \min \left\{ \exp(-2m\epsilon^2), \exp \left(-m \min \{\hat{\nu}, g(b(v))\} h \left( \frac{\epsilon}{\min \{\hat{\nu}, g(b(v))\}} \right) \right) \right\}.$$  

By defining the functions $H(m, \epsilon) = \exp(-2m\epsilon^2)$, $B(x, m, \epsilon) = \exp(-mxh(\frac{\epsilon}{\hat{\nu}}))$, and $\psi(x)$ (see below), we rewrite

$$\Pr(A) \leq \sum_{v \in V} 2 \min \{H(m, \epsilon), B(\min \{\hat{\nu}, g(b(v))\), m, \epsilon)\} \approx \sum_{v \in V} \psi(b(v)).$$

Since the values of $b(v)$ are not known a priori, it is not possible to directly compute the r.h.s. of (17). However, we show how to obtain a sharp upper bound by leveraging constraints on the possible values of $b(v)$ imposed by $\hat{\nu}$ and $\rho$. To do so, we define an appropriate optimization problem w.r.t. the (unknown) values of $b(v)$. Denote with $m_x$ the number of nodes of $V$ that we assume have $b(v) = x$, for $x \in \mathbb{Q} \cap (0, 1)$ (we can safely ignore nodes $v$ with $b(v) = 0$ or $b(v) = 1$, since $f_v$ is constant, and $\Pr(A_v) = 0$); then, we define the following constrained optimization problem over the variables $m_x$:

$$\max \sum_{x \in \mathbb{Q} \cap (0,1)} m_x \psi(x),$$

subject to

$$\sum_{x \in \mathbb{Q} \cap (0,1)} xm_x \leq \rho,$$

$$0 \leq m_x \leq \frac{\rho}{x}, m_x \in \mathbb{N}.$$  

The first constraint follows from $\sum_{v \in V} b(v) \leq \rho$, while the second set of constraints imposes that $m_x$ are positive integers and that there cannot be more than $\rho/x$ nodes with $b(v) = x$ by definition of $\rho$. Therefore, from (17), the value of the objective function of the optimal solution of this problem upper bounds $\Pr(A)$, as we consider a worst-case configuration of the admissible values of $b(v)$ (i.e., the graph $G$ belongs to the space of all possible graphs with the above mentioned constraints). We recognize this formulation as a specific instance of the Bounded Knapsack Problem [40] over the variables $m_x$, where items with label $x$ are selected $m_x$ times, with unitary profit $\psi(x)$ and weight $x$; each item can be selected at most $\rho/x$ times, while the total knapsack capacity is $\rho$. We are not interested in the optimal solution of the integer problem, but rather in its upper bound.
given by the optimal solution of the continuous relaxation, in which we let \( m_x \in \mathbb{R} \). Informally, such solution is obtained by choosing at maximum possible capacity every item in decreasing order of profit-weight ratio \( \psi(x)/x \) until the total capacity is filled (Chapter 3 of [40]). In our case, from the particular definition of the constraints, it is enough to fully select the item with higher profit-weight ratio to fill the entire knapsack. More formally, let \( x^* \) be

\[
x^* = \arg \max_{x \in (0, 1)} \left\{ \frac{\psi(x)}{x} \right\};
\]

the optimal solution to the continuous relaxation is \( m_{x^*} = \rho/x^* \), \( m_x = 0, \forall x \neq x^* \), while the optimal objective is equal to

\[
\frac{\rho \psi(x^*)}{x^*} \geq \Pr(A).
\]

We note that \( x^* \) always exists, as \( \psi(x)/x \) is a positive, bounded and continuous function in \((0, 1)\).

We now simplify the search of \( x^* \) limiting the range of \( x \). First, we prove that \( x^* \in (0, \hat{x}_2] \). Observe that the function \( B(\min[\hat{v}, g(x)], m, \varepsilon) \) (thus \( \psi(x) \)) is symmetric w.r.t. 1/2 (since \( g(x) = g(1-x) \)); also, we note that

\[
\min[\hat{v}, g(x)] = \begin{cases}
\hat{v}, & \frac{1}{2} - \sqrt{\frac{1}{4} - \hat{v}} \leq x \leq \frac{1}{2} + \sqrt{\frac{1}{4} - \hat{v}}, \\
g(x), & \text{otherwise},
\end{cases}
\]

which means that \( \psi(x) \) is constant for \( x \in \left[ \frac{1}{2} - \sqrt{\frac{1}{4} - \hat{v}}, \frac{1}{2} + \sqrt{\frac{1}{4} - \hat{v}} \right] \). From these observations, we prove by contradiction that it holds that \( x^* \leq \frac{1}{2} - \sqrt{\frac{1}{4} - \hat{v}} = \hat{x}_2 \). Assume that \( x^* > \frac{1}{2} + \sqrt{\frac{1}{4} - \hat{v}} \); then defining \( x' = 1 - x^* \), we have

\[
\frac{\psi(x^*)}{x^*} = \frac{\psi(x')}{1-x'} < \frac{\psi(x')}{x'},
\]

which contradicts the definition of \( x^* \). Now, assume that \( x^* \in (\frac{1}{2} - \sqrt{\frac{1}{4} - \hat{v}}, \frac{1}{2} + \sqrt{\frac{1}{4} - \hat{v}}] \); we have

\[
\frac{\psi(x^*)}{x^*} = \frac{\psi(\hat{x}_2)}{x^*} < \frac{\psi(\hat{x}_2)}{\hat{x}_2},
\]

which is another contradiction; therefore, we conclude that \( x^* \in (0, \hat{x}_2] \). We can write

\[
x^* = \arg \max_{x \in (0, 1)} \left\{ \frac{\psi(x)}{x} \right\} = \arg \max_{x \in (0, \hat{x}_2]} \left\{ \frac{2 \min[H(m, \varepsilon), B(g(x), m, \varepsilon)]}{x} \right\} \geq \min_{x \in (0, \hat{x}_2]} \left\{ \frac{\psi(x)}{x} \right\} .
\]

We now prove that \( x^* \in (0, \hat{x}_1] \). Note that \( \psi_1(x) \) is symmetric around 1/2; furthermore,

\[
\min \left\{ H(m, \varepsilon), B(g(x), m, \varepsilon) \right\} = \begin{cases}
B(g(x), m, \varepsilon), & g(x)h \left( \frac{\varepsilon}{g(x)} \right) \geq 2\varepsilon^2, \\
H(m, \varepsilon), & \text{otherwise}.
\end{cases}
\]

The function \( g(x)h(\varepsilon/g(x)) \) is monotonically increasing for \( 0 < x < 1/2 \) and decreasing for \( 1/2 < x < 1 \). Denote \( h_1(x) = 1 + x - \sqrt{1 + 2x} \) for \( x \geq 0 \). We use the facts that \( 9h_1(x/3) \leq h(x) \leq x^2/2 \) for all \( x \geq 0 \), so we have that \( g(x)h(\varepsilon/g(x)) = 2\varepsilon^2 \) holds for a pair \( x_1 \) and \( x_2 \) such that \( 1/2 - \sqrt{\varepsilon/3 - \varepsilon^2/9} \leq x_1 \leq 1/2, 1/2 \leq x_2 \leq 1/2 + \sqrt{\varepsilon/3 - \varepsilon^2/9} \), and \( 1 = x_1 + x_2 \) (this follows easily from knowing that the inverse of \( h_1 \) over \([0, +\infty)\) is \( h_1^{-1}(x) = x + \sqrt{2x} \)). We can write

\[
\min \left\{ H(m, \varepsilon), B(g(x), m, \varepsilon) \right\} = \begin{cases}
B(g(x), m, \varepsilon), & x \in (0, x_1] \cup [x_2, 1), \\
H(m, \varepsilon), & \text{otherwise}.
\end{cases}
\]
Exploiting the symmetry of $\psi_1(x)$, it is easy to prove that $x^* \in (0, x_1]$ by following a similar argument used above. Then, note that $x_1 \leq \hat{x}_1, \psi_1(x) \leq 2B(g(x), m, \epsilon)$, and $\hat{x} = \min\{\hat{x}_1, \hat{x}_2\}$; we obtain

$$\Pr(A) \leq \sup_{x \in [0, \min\{x_1, x_2\}]} \left\{ \frac{\rho \psi_1(x)}{x} \right\} \leq \sup_{x \in [0, \min\{x_1, x_2\}]} \left\{ \frac{\rho \psi_1(x)}{x} \right\} \leq \sup_{x \in [0, \hat{x}]} \left\{ \frac{\rho 2B(g(x), m, \epsilon)}{x} \right\}.$$ 

We now show that if $m$ is chosen as assumed in the statement, it holds $\Pr(A) \leq \delta$, thus proving the Theorem. We have, for $x \in (0, \hat{x}]$,

$$\frac{\rho 2B(g(x), m, \epsilon)}{x} \leq \delta \quad \text{if} \quad m \geq \frac{\ln \left( \frac{2\rho}{\delta} \right)}{g(x)h\left( \frac{r}{g(x)} \right)},$$

which follows from (8).

**Theorem 4.8.** Let $\mathcal{T}, \gamma, g, \hat{\gamma}$, and $\rho$ as in Theorem 4.7. Denote $S$ as an i.i.d. sample of size $m \geq 1$ from $\gamma$. It holds, with probability $\geq 1 - \delta$ and for all $v \in V$,

$$|b(v) - \tilde{b}(v)| \leq \sqrt{\frac{2 \min\{g(b(v)), \hat{\gamma}\}}{m} \ln \left( \frac{1}{3} \min \left\{ \frac{\rho}{h(b(v))}, n \right\} \right) + \ln \left( \frac{4}{3} \min \left\{ \frac{\rho}{h(b(v))}, n \right\} \right)} \leq d_r(b(v)).$$

**Proof.** Denote the event $A = \{b(v) - \tilde{b}(v) > d_r(b(v))\}$, and the events $A_v = \{b(v) - \tilde{b}(v) > d_r(b(v))\}$, $\forall v \in V$. Our goal is to show that $\Pr(A) \leq \delta$, which proves the statement. First, through a union bound and Bennet’s inequality, we obtain that

$$\Pr(A) = \Pr \left( \bigcup_{v \in \mathcal{V}} A_v \right) \leq \sum_{v \in \mathcal{V}} \Pr(A_v) \leq \sum_{v \in \mathcal{V}} 2B(\min\{\hat{\gamma}, g(b(v))\}, m, d_r(b(v))),$$

with $B(x, m, y) = \exp(-mxh(y/x))$ and $h(x) = (1 + x) \ln(1 + x) - x$ for $x \geq 0$. We define an optimization problem to upper bound (18):

$$\max \sum_{x \in Q, x \geq 0} 2m_x B(\min\{\hat{\gamma}, g(x)\}, m, d_r(x)),$$

subject to

$$\sum_{x \in Q, x \geq 0} x m_x \leq \rho,$$

$$\sum_{x \in Q, x \geq 0} m_x \leq n,$$

$$0 \leq m_x \leq m_x \leq \frac{\rho}{x}, m_x \in \mathbb{R}.$$

This problem is similar to the one introduced in the proof of Theorem 4.7, but with an additional constraint, which imposes that the number of vertices $\sum_{x \in (0, 1)} m_x$ with $b(v) \in (0, 1)$ is at most $n$, where $n$ is the number of vertices $|V|$ of the graph $G$ (note that we ignore vertices with $b(v) = 0$ or $b(v) = 1$, since the corresponding estimator $f_v$, is constant, therefore $\Pr(A_v) = 0$). We will obtain an upper bound to the optimal solution by upper bounding the objective function and through a Lagrangian relaxation. First, we define the function $\psi(x)$ for all $x \geq 0$ as

$$\psi(x) = \frac{\delta}{2 \min \left\{ \frac{\rho}{x}, n \right\} = \left\{ \begin{array}{ll} \frac{\delta}{2n}, \quad x \geq \frac{\rho}{n}, \\ \frac{\delta}{2n}, \quad x < \frac{\rho}{n}. \end{array} \right.$$
We now prove that \(2B(\text{min}\{g(x), \hat{v}\}, m, d_r(x)) \leq \psi(x), \forall x \in (0, 1)\). Recall \(h_1(x) = 1 + x - \sqrt{1 + 2x}\), and note that \(h(x) \geq 9h_1(x/3), \forall x \geq 0\). Therefore, it holds
\[
2B(\text{min}\{g(x), \hat{v}\}, m, d_r(x)) \leq 2 \exp \left( -9m \text{min}\{g(x), \hat{v}\} h_1 \left( \frac{d_r(x)}{3 \text{min}\{g(x), \hat{v}\}} \right) \right).
\]

We observe that the r.h.s. of (20) is equal to \(\psi(x)\). This follows from the fact that the inverse of \(h_1\) over \([0, +\infty)\) is \(h_1^{-1}(x) = x + \sqrt{2x}\), and from the definition of \(d_r(x)\), proving the inequality
\[
2B(\text{min}\{g(x), \hat{v}\}, m, d_r(x)) \leq \psi(x).
\]

Fix any \(\lambda \geq 0\); from a Lagrangian relaxation and the bound we just proved, the optimal solution of the following problem upper bounds the optimal solution of the problem defined in (19)

\[
\max_{x \in Q \cap (0, 1)} \sum_{x \in Q \cap (0, 1)} m_x \psi(x) + \lambda \left( n - \sum_{x \in (0, 1)} m_x \right),
\]

subject to
\[
\sum_{x \in Q \cap (0, 1)} x m_x \leq \rho,
\]
\[
0 \leq m_x \leq \frac{\rho}{x}, m_x \in \mathbb{R}.
\]

Rewriting the objective function, we have
\[
\sum_{x} m_x \psi(x) + \lambda \left( n - \sum_{x} m_x \right) = \sum_{x} m_x \left( \psi(x) - \lambda \right) + \lambda n.
\]

Ignoring the constant \(\lambda n\), we expressed the problem as another Bounded Knapsack Problem formulation with profit \((\psi(x) - \lambda)\) for items with label \(x\); we compute its optimal solution as done in the proof of Theorem 4.7. We fix \(\lambda = \psi(\rho/n) = \delta/(2n)\), and remark that \(\psi(x)\) is increasing with \(x\); therefore, \(\psi(x) \leq \psi(\rho/n)\), \(\forall x \in (0, \rho/n]\), and \(\psi(x) \geq \psi(\rho/n)\), \(\forall x \in [\rho/n, 1)\). Define \(x^*\) as
\[
x^* = \arg \max_{x \in (0, 1)} \left\{ \frac{\psi(x) - \psi(\rho/n)}{x} \right\};
\]
from the observations above it follows that \(x^* \geq \rho/n\) and \(\psi(x^*) \geq \psi(\rho/n)\), since any item with negative profit and positive weight can’t be the optimal solution. The optimal solution is obtained, as done before, by picking the item with label \(x^*\) with maximum profit/weight ratio, and filling the entire knapsack with it. Therefore, it is \(m_{x^*} = \rho/x^*, m_x = 0, \forall x \neq x^*\), with objective
\[
\frac{\rho}{x^*} \psi(x^*) + n\psi(\rho/n) - \frac{\rho}{x^*} \psi(\rho/n) \leq \delta,
\]
proving the statement. \(\square\)

Before proving data-dependent upper bounds to \(\rho\), we show the following straightforward fact.

**Lemma A.6.** For a pair of nodes \(u, z \in V\), denote with \(\pi_{uz}\) one of the shortest paths from \(u\) to \(z\) in the graph, and define \(|\pi_{uz}|\) as the number of internal nodes in \(\pi_{uz}\) (ignoring \(u\) and \(z\)). Let \(\rho = \sum_{v \in V} b(v)\) and \(D\) be the vertex diameter of a graph \(G\). Then, it holds
\[
\rho = \frac{1}{n(n - 1)} \sum_{u, z \in V} |\pi_{uz}| \leq D - 2.
\]
PROOF. From the definition of \( b(v) \) and \( \rho \), we have
\[
\rho = \sum_{v \in V} b(v) = \sum_{v \in V} \frac{1}{n(n-1)} \sum_{u,z \in V} \frac{\sigma_{uz}(v)}{\sigma_{uz}} = \frac{1}{n(n-1)} \sum_{u,z \in V} \sum_{v \in V} \frac{\sigma_{uz}(v)}{\sigma_{uz}}
\]
\[
= \frac{1}{n(n-1)} \sum_{u,z \in V} \left| \frac{\pi_{uz}}{\sigma_{uz}} \right| = \frac{1}{n(n-1)} \sum_{u,z \in V} |\pi_{uz}|
\]
\[
= \frac{1}{n(n-1)} \sum_{u,z \in V, u \neq z} |\pi_{uz}| \leq \frac{1}{n(n-1)} \sum_{u,z \in V, u \neq z} (D - 2) = D - 2. \quad \Box
\]

The result below shows that given a (not necessarily tight) upper bound to \( D \), the average shortest path length \( \rho \) can be sharply estimated as the average number of internal nodes of the shortest paths in a sample \( S \), resulting in a very efficient data-dependent bound.

**Proposition 4.10.** Let \( z \geq D - 2 \), where \( D \) is the vertex diameter of the graph \( G \). Let an i.i.d. sample \( S \) of size \( m \), and denote \( \hat{\rho} = \sum_{v \in V} b(v) \). Then, for a fixed \( \delta \in (0, 1) \), it holds with probability \( \geq 1 - \delta \)
\[
\sum_{v \in V} b(v) \leq \rho \leq \frac{5}{3} \left( z \ln \left( \frac{1}{\delta} \right) \right)^2 + \frac{2z\hat{\rho} \ln \left( \frac{1}{\delta} \right)}{m} + \frac{4z \ln \left( \frac{1}{\delta} \right)}{3m}.
\]

**Proof.** From the definition of \( \hat{\rho} \), we have
\[
\hat{\rho} = \sum_{v \in V} \frac{1}{m} \sum_{\tau \in S} f_{v}(\tau) = \frac{1}{m} \sum_{\tau \in S} \sum_{v \in V} f_{v}(\tau),
\]
with \( \mathbb{E}_S[\hat{\rho}] = \rho \). We recognize \( \hat{\rho} \) as an average of \( m \) independent (bounded) random variables, it holds, for all \( \tau \), that \( 0 \leq \sum_{v \in V} f_{v}(\tau) \leq D - 2 \leq z \), and, from [8], \( \text{Var}(\sum_{v \in V} f_{v}(\tau)) \leq (D - 2 - \rho)\rho \leq (z - \rho)\rho \leq z\rho \). We define the random variables \( X_i \), for \( i \in [1, m] \), as
\[
X_i = \rho - \sum_{v \in V} f_{v}(\tau_i),
\]
noting that \( \mathbb{E}[X_i] = 0 \), \( X_i \leq \rho \leq z \), and \( \mathbb{E}[X_i^2] = \text{Var}(X_i) \leq z\rho \). Therefore, we apply Bernstein’s inequality (Theorem 2.10 [14]) to the sum \( \sum_{i=1}^{m} X_i = m(\rho - \hat{\rho}) \), obtaining
\[
\Pr \left( \rho \geq \hat{\rho} + \sqrt{\frac{2 \ln \left( \frac{1}{\delta} \right)}{m} \rho z + \frac{\ln \left( \frac{1}{\delta} \right) z}{3m}} \right) \leq \delta.
\]
To conclude the proof, we need the following straightforward Lemma.

**Lemma A.7.** Let \( u, v, y \geq 0 \). The fixed point of
\[
r(x) = u + \sqrt{v + yx}
\]
is at
\[
x = u + \frac{y}{2} + \sqrt{\frac{y^2}{4} + uy + v}.
\]

The statement follows by applying Lemma A.7 to the inequality
\[
\rho \leq \hat{\rho} + \sqrt{\frac{2 \ln \left( \frac{1}{\delta} \right)}{m} \rho z + \frac{\ln \left( \frac{1}{\delta} \right) z}{3m}}. \quad \Box
\]
PROPOSITION 4.11. Assume the setting of Proposition 4.10 with \( S = \{\tau_1, \ldots, \tau_m\} \). Define \( \Lambda(S) \) as

\[
\Lambda(S) = \frac{1}{m(m-1)} \sum_{1 \leq i < j \leq m} \left( \sum_{v \in V} f_v(\tau_i) - \sum_{v \in V} f_v(\tau_j) \right)^2.
\]

Then, for a fixed \( \delta \in (0, 1) \), it holds with probability \( \geq 1 - \delta \)

\[
\rho \leq \tilde{\rho} + \sqrt{\frac{2\Lambda(S) \ln \left( \frac{2}{\delta} \right)}{m}} + \frac{7z \ln \left( \frac{2}{\delta} \right)}{3m}.
\]

PROOF. As discussed in the proof of Proposition 4.10, \( \tilde{\rho} \) is an average of \( m \) i.i.d. (bounded from above by \( z \)) random variables. The result follows from Corollary 5 of [42] after scaling \( \tilde{\rho} \) by \( 1/z \). \( \square \)

A.4 Proofs of Section 4.4

In this section, we state and prove the guarantees for SILVAN-TopK.

PROPOSITION 4.13. For each \( v \in V \), denote the intervals \( CI_v = [\ell(v), u(v)] \) with \( \ell(v) \leq \hat{b}(v) \leq u(v) \). Let \( v^f_1, \ldots, v^f_m \) be the sequence of nodes ordered according to \( \ell(\cdot) \), such that \( \ell(v^f_1) \geq \ell(v^f_2) \). Define the set \( \hat{T}\hat{O}\hat{P}(k) = \{(v, \hat{b}(v)) : u(v) \geq \ell(v^f_k)\} \), and assume that, for all \( (v, \hat{b}(v)) \in \hat{T}\hat{O}\hat{P}(k) \),

\[
\frac{\hat{b}(v)}{1 + \eta} \leq \ell(v) \leq b(v) \leq u(v) \leq \frac{\hat{b}(v)}{1 - \eta}.
\]

Then, \( \hat{T}\hat{O}\hat{P}(k) \) is a relative \( \eta \)-approximation of \( TOP(k) \).

PROOF. We first note that, as the \( k \)th most central node \( v_k \) and its centrality \( b(v_k) \) are both unknown, we need a principled way to identify bounds to \( b(v_k) \). Let \( v^f_1, \ldots, v^f_k \) be the sequence of nodes ordered according to \( \ell(\cdot) \), such that \( \ell(v^f_1) \geq \ell(v^f_2) \). Then, let \( v^u_1, \ldots, v^u_k \) be the sequence of nodes ordered according to \( u(\cdot) \), such that \( u(v^u_1) \geq u(v^u_2) \). We have the following relations between \( b(v_k) \), \( u(v^u_k) \), and \( \ell(v^f_k) \).

**Lemma A.8.** It holds \( \ell(v^f_k) \leq b(v_k) \leq u(v^u_k) \).

PROOF. We prove the statement by contradiction. Assume that it holds \( b(v_k) < \ell(v^f_k) \). From the definition of \( \ell(v^f_k) \), that there are \( k \) nodes \( \{v^f_i, i \leq k\} \) such that \( \ell(v^f_i) \geq \ell(v^f_k) > b(v_k) \), and it holds, for all of them, that \( b(v^f_i) \geq \ell(v^f_i) \). This implies that there are \( k \) nodes such that \( b(v^f_i) > b(v_k) \), that is in contradiction with the definition of \( v_k \). \( \square \)

We now continue proving Proposition 4.13. The first guarantee of the relative \( \eta \)-approximation is immediate from Lemma A.8: Since it holds for all \( v \in V \) that \( b(v) \in CI_v \), we have that \( u(v) \geq b(v) \geq \ell(v) \). Therefore, if \( v \in TOP(k) \), then \( b(v) \geq b(v_k) \), thus we have \( u(v) \geq b(v) \geq b(v_k) \geq \ell(v^f_k) \); this means that \( v \) is in \( \hat{T}\hat{O}\hat{P}(k) \) as \( u(v) \geq \ell(v^f_k) \). The second guarantee follows directly from the conditions in (13). We now focus on the third guarantee. Let \( \tilde{v}_1, \ldots, \tilde{v}_k \) be the sequence of nodes in order of \( \hat{b}(\cdot) \), such that \( \hat{b}(\tilde{v}_i) \geq \hat{b}(\tilde{v}_{i+1}) \). From (13), we have that, for all \( \tilde{v}_i \),

\[
\frac{\hat{b}(\tilde{v}_i)}{1 + \eta} \leq \ell(\tilde{v}_i) \leq u(\tilde{v}_i) \leq \frac{\hat{b}(\tilde{v}_i)}{1 - \eta}.
\]
but also that
\[
\frac{\tilde{b}(\tilde{v}_{i+1})}{1 + \eta} \leq \frac{\tilde{b}(\tilde{v}_i)}{1 + \eta}, \quad \text{and} \quad \frac{\tilde{b}(\tilde{v}_{i+1})}{1 - \eta} \leq \frac{\tilde{b}(\tilde{v}_i)}{1 - \eta}.
\]
Notice that $v^i_k \neq v^u_k$ in general; nevertheless, we show it is possible to bound $u(v^u_k) - \ell(v^i_k)$. Considering $\tilde{v}_k$, we have that
\[
\frac{\tilde{b}(\tilde{v}_k)}{1 + \eta} \leq \ell(v^i_k) \leq u(v^u_k) \leq \frac{\tilde{b}(\tilde{v}_k)}{1 - \eta};
\]
the lower bound to $\ell(v^i_k)$ follows from the definition of $v^i_k$, since there are $k$ values $\ell(\tilde{v}_i) \geq \tilde{b}(\tilde{v}_i)/(1 + \eta), i \leq k$; the upper bound to $u(v^u_k)$ follows from the fact that there are $k$ values $u(\tilde{v}_i) \leq \tilde{b}(\tilde{v}_i)/(1 - \eta), i \leq k$. We combine previous inequalities to obtain
\[
b(v) \geq l(v) \geq \frac{\tilde{b}(v)}{1 + \eta} \geq u(v) \frac{1 - \eta}{1 + \eta} \geq \ell(v^i_k) \frac{1 - \eta}{1 + \eta} \geq b(v_k) \left( \frac{1 - \eta}{1 + \eta} \right)^2.
\]

### B ADDITIONAL EXPERIMENTAL RESULTS

This section presents additional experimental results.

#### B.1 Comparison of Bounds to the Sufficient Sample Size

In this section, we compare the state-of-the-art bounds, providing sufficient amounts of samples to obtain an absolute $\varepsilon$-approximation of the betweenness centrality with high probability, with our new bound from Theorem 4.7. The first result, from [54], that we call RK, applies to the r$k$ estimator and is given by
\[
\frac{C'}{\varepsilon^2} \left( d + \log \left( \frac{1}{\delta} \right) \right),
\]
where $d \leq \lceil \log(D - 2) + 1 \rceil$ and $C'$ is an absolute constant. We set $C' = 0.5$ as suggested by [37, 54].

The second bound, from Theorem 5.2 of [21], that we call CWR, applies to all estimators of the betweenness centrality considered in [21] (the r$k$, ab, and bp estimators), and is
\[
\frac{1}{\varepsilon^2} \left( 4 C d + 4 \sqrt{C d \ln \left( \frac{1}{C} \right)} + \frac{\ln \left( \frac{1}{C} \right)}{2} \right),
\]
where $d$ is defined as before, and $C$ is an absolute constant with $C < 262$. In practice, we set $C = 1$.

For our new bound (8), we set $\rho = D - 2$. For all bounds, we use the same value of $\delta = 0.05$, and vary $\varepsilon$, $D$, and $\hat{v}$ (note that (21) and (22) do not depend on $\hat{v}$).

Figure 18 shows the values of all bounds for different values of the parameters. Overall, the CWR bound is the most conservative, due to the larger constants in (22) and the additional squared term w.r.t. (21). Note that the optimistic choice of $C = 1$ in (22) is not guaranteed to provide the desired probabilistic correctness of the result; however, using $C = 262$ would increase the CWR bound by approximately $2$ orders of magnitude.

Overall, our new bound from Theorem 4.7 recovers the RK bound (21) when $\hat{v} = 0.25$ (its maximum value), while it is significantly sharper for smaller values. We remark that the choice $\rho = D - 2$ is quite conservative, as we expect $\rho < D - 2$ is almost all non-trivial cases.
Fig. 4. Resources required by SILVAN for obtaining absolute $\varepsilon$-approximations. (a) Number of samples for SILVAN vs. $\varepsilon$. (b) Running times of SILVAN vs. Number of samples. (c) Running times of SILVAN vs. $\varepsilon$. 

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Fig. 5. Violin plot showing the empirical distribution of the supremum deviation $D(F, S)$ (blue violins) as function of $\epsilon$ (x axis, and black horizontal bars) for three undirected and three directed graphs over 10 runs.
Fig. 6. Comparison of the quality of the output of SILVAN-TopK and KADABRA for top-$k$ approximation for three graphs (with $k = 10$ and $\eta = 0.1$) when using the same resources: KADABRA is stopped after processing the same number of samples required by SILVAN-TopK to stop. The $x$ axis is the rank of the node reported in output by both algorithms, the $y$ axis shows the estimated values $\hat{b}$ of the betweenness centrality (dots and crosses) and upper and lower bounds w.r.t. the exact values $\hat{b}$ (horizontal bars).
Fig. 7. Violin plot showing the empirical distribution of the maximum relative deviation (blue violins) as function of $\eta$ (x axis, and black horizontal bars) for three undirected and three directed graphs over 10 runs.
Fig. 8. Additional figures for comparing the performance of KADABRA and SILVAN for obtaining an absolute $\varepsilon$-approximation. (a) Comparison of the number of samples for KADABRA ($y$ axis) and SILVAN ($x$ axis) for undirected graphs. (b) Analogous to panel (a) for directed graphs. (c) Comparison of the running times of KADABRA ($y$ axis) and SILVAN ($x$ axis) for undirected graphs (axes in logarithmic scales). (e) Analogous to panel (d) for directed graphs. (c) Ratios of the running times of KADABRA and SILVAN for undirected graphs. (d) Analogous to panel (c) for directed graphs.
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Fig. 9. Additional figures for comparing the performance of SILVAN and BAVARIAN-P (ab estimator) for obtaining an absolute $\varepsilon$-approximation. (a) Comparison of the number of samples for BAVARIAN ($y$ axis) and SILVAN ($x$ axis) for undirected graphs (axes in logarithmic scales). (b) Analogous to panel (a) for directed graphs. (c) Comparison of the running times of BAVARIAN ($y$ axis) and SILVAN ($x$ axis) for undirected graphs (axes in logarithmic scales). (d) Analogous to panel (c) for directed graphs. (e) Ratios of the running times of BAVARIAN and SILVAN for undirected graphs. (f) Analogous to panel (e) for directed graphs.

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Fig. 10. Figures comparing the performance of SILVAN and BAVARIAN-P (rk estimator) for obtaining an absolute \( \varepsilon \)-approximation in terms of number of samples. (a) Comparison of the number of samples for BAVARIAN (y axis) and SILVAN (x axis) for undirected graphs (axes in logarithmic scales). (b) Analogous to panel (a) for directed graphs. (c) Ratios of the number of samples for BAVARIAN and SILVAN for undirected graphs. (d) Analogous to panel (c) for directed graphs.
Fig. 11. Figures comparing the performance of SILVAN and BAVARIAN-P (rk estimator) for obtaining an absolute $\varepsilon$-approximation in terms of running time. (a) Comparison of the running times of BAVARIAN (y axis) and SILVAN (x axis) for undirected graphs (axes in logarithmic scales). (b) Analogous to panel (a) for directed graphs. (c) Ratios of the running times of BAVARIAN and SILVAN for undirected graphs. (d) Analogous to panel (c) for directed graphs.
Fig. 12. Figures comparing the performance of SILVAN and BAVARIAN-P (bp estimator) for obtaining an absolute $\varepsilon$-approximation in terms of number of samples. (a) Comparison of the number of samples for BAVARIAN ($y$ axis) and SILVAN ($x$ axis) for undirected graphs (axes in logarithmic scales). (b) Analogous to panel (a) for directed graphs. (c) Ratios of the number of samples for BAVARIAN and SILVAN for undirected graphs. (d) Analogous to panel (c) for directed graphs.
Fig. 13. Figures comparing the performance of SILVAN and BAVARIA-P (bp estimator) for obtaining an absolute ε-approximation in terms of running times. (a) Comparison of the running times of BAVARIAN (y axis) and SILVAN (x axis) for undirected graphs (axes in logarithmic scales). (b) Analogous to panel (a) for directed graphs. (c) Ratios of the running times of BAVARIAN and SILVAN for undirected graphs. (d) Analogous to panel (c) for directed graphs.
Fig. 14. Comparison of number of samples (left column) and running times (right column) between SILVAN and two variants (lines of different colors) for obtaining an absolute $\varepsilon$-approximation on three graphs (one per each row). SILVAN-$\hat{m}$ is a variant of SILVAN setting $\hat{m} = +\infty$ (i.e., ignoring the upper limit to the number of samples given by Theorem 4.7), while SILVAN-$\hat{m}$-ep is a variant of SILVAN setting $\hat{m} = +\infty$ and without empirical peeling (i.e., setting the number $t$ of partitions to $t = 1$ in Theorem 4.2).
Fig. 15. Comparison of running times of SILVAN-TopK with KADABRA for obtaining top-$k$ approximations for all combinations of $k$ and $\eta$. 
Fig. 16. Comparison of number of samples of SILVAN-TopK with KADABRA for obtaining top-$k$ approximations for all combinations of $k$ and $\eta$. 
Fig. 17. Behaviour of the function $m(x) = \frac{\ln(\frac{\rho}{\delta})}{g(x)h(\frac{\rho}{g(x)})}$ w.r.t. $x$, for different combinations of $\epsilon$, $\rho$, and $\hat{\nu}$, with $\delta = 0.05$. $\hat{m}$ is equal to $m$ in (8), i.e., $\hat{m} = \sup_{x \in (0, \hat{x})} m(x)$. $\hat{m}$ is the estimate of (9).
Fig. 18. Comparison of different bounds to the sufficient number of samples to obtain an absolute $\varepsilon$-approximation, as functions of $\varepsilon$. Each plot shows the bounds for a different value of $\rho$ (the average number of internal nodes in shortest paths, see Section 4.3.1). RK is the bound (21) (from [54]), CWR is the bound (22) (see Theorem 5.2 of [21]), and $m$ is the value of (8) from our bound of Section 4.3.1 for different values of $\hat{\nu} \geq \sup_{f \in F} Var(f)$. Note that the blue and green lines are always very close.

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