Calculation of a three-layer plate by the finite element method taking into account the creep of the filler

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Abstract. In the article the derivation of the resolving equations for the calculation of a three-layer plate taking into account the creep of the middle layer by the finite element method is given. Rectangular finite elements are used. The problem reduces to a system of linear algebraic equations. An example of calculating a three-layer plate hinged on the contour and loaded with a uniformly distributed load is considered. A comparison of the results with the solution based on the finite difference method is presented.

1 Introduction

Three-layer structures are widely used in many industries, including aircraft construction, shipbuilding, civil and industrial engineering, etc. Such structures, as a rule, consist of two outer layers with high mechanical characteristics (steel, aluminium, fiberglass) and a lightweight filler located between them. Porous polymers (foams) are widely used as a filler. For polymers in addition to elastic properties, a pronounced rheology is characteristic. The calculation of three-layered structures taking the creep into account is considered in [1-3]. Resolving equations for a triangular finite element of a three-layer plate and shell are given in [1-2]. In the present paper rectangular finite elements with higher accuracy will be considered.

2 Derivation of resolving equations

The rectangular finite element of the three-layer plate is shown in Fig. 1-2. Each node of this element has 5 degrees of freedom: displacements in the plane of the upper layer \( u_i^+ \) and \( v_i^+ \), displacements in the plane of the lower layer \( u_i^- \) and \( v_i^- \), as well as deflection \( w_i \).

\[
\begin{align*}
\mathbf{u}^{+(-)} &= N_1 u_1^{+(-)} + N_2 u_2^{+(-)} + N_3 u_3^{+(-)} + N_4 u_4^{+(-)} \\
v^{+(-)} &= N_1 v_1^{+(-)} + N_2 v_2^{+(-)} + N_3 v_3^{+(-)} + N_4 v_4^{+(-)} \\
w &= N_1 w_1 + N_2 w_2 + N_3 w_3 + N_4 w_4,
\end{align*}
\]

where \( N_1, N_2, N_3, N_4 \) – form functions.

\[
\begin{align*}
N_1 &= \frac{1}{ab} \left( \frac{a}{2} - x \right) \left( \frac{b}{2} + y \right) \\
N_2 &= \frac{1}{ab} \left( \frac{a}{2} + x \right) \left( \frac{b}{2} - y \right) \\
N_3 &= \frac{1}{ab} \left( \frac{a}{2} + x \right) \left( \frac{b}{2} + y \right) \\
N_4 &= \frac{1}{ab} \left( \frac{a}{2} - x \right) \left( \frac{b}{2} + y \right)
\end{align*}
\]

where \( a, b \) – dimensions of the finite element.

The coordinates \( x \) and \( y \) in formulas (2) are measured from the centre of the finite element.

The deformation vector of a finite element is written as:

\[
\mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_x^+ & \varepsilon_y^+ & \gamma_{xy} & \varepsilon_x^- & \varepsilon_y^- & \gamma_{xy} & \gamma_{zx} & \gamma_{zy} \end{bmatrix}^T
\]

where \( \varepsilon_x^+, \varepsilon_y^+, \gamma_{xy} \) – deformations of the lower layer, \( \varepsilon_x^-, \varepsilon_y^-, \gamma_{xy} \) – deformations of the upper layer, \( \gamma_{zx}, \gamma_{zy} \) – deformations of the filler.
In the technical theory of three-layer plates, the relationship between displacements and deformations has the form:
\[
\begin{align*}
\varepsilon^{(+)}_x &= \frac{\partial u^{(+)}(-)}{\partial x}; \\
\varepsilon^{(+)}_y &= \frac{\partial v^{(+)}(-)}{\partial y}; \\
\gamma^{(+)}_{xy} &= \frac{\partial u^{(+)}(-)}{\partial y} + \frac{\partial v^{(+)}(-)}{\partial x}; \\
\gamma^m_{xy} &= \frac{v^+ - u^+}{h} + \frac{\partial w}{\partial x}; \\
\gamma^m_{yz} &= \frac{v^+ - v^-}{h} + \frac{\partial v}{\partial y}. 
\end{align*}
\]

Substituting (1) in (3), we obtain the following relationship between the node displacements and deformations in the matrix form:
\[
\{u\} = [B]\{U\},
\]
where \(\{U\}\) — vector of nodal displacements.

\[
\{U\} = \begin{bmatrix} \{p_1\} \\
\{p_2\} \\
\{p_3\} \\
\{p_4\} \end{bmatrix}, \quad \{p_j\} = \begin{bmatrix} u^+_j \\
v^+_j \\
u^-_j \\
v^-_j \end{bmatrix}.
\]

Matrix \([B]\) has a size of 8x20. Non-zero elements of the matrix \([B]\) are defined as follows:
\[
\begin{align*}
B_{1,1} &= B_{3,2} = B_{4,3} = B_{6,4} = B_{7,5} = \frac{\partial N_1}{\partial x}; & B_{1,6} &= B_{3,7} = \\
B_{4,8} &= B_{6,9} = B_{7,10} = \frac{\partial N_2}{\partial x}; & B_{1,11} &= B_{3,12} = B_{4,13} = \\
B_{6,14} &= B_{7,15} = \frac{\partial N_3}{\partial x}; & B_{1,16} &= B_{3,17} = B_{4,18} = B_{6,19} = \\
B_{7,20} &= \frac{\partial N_4}{\partial x}; & B_{2,2} &= B_{3,5} = B_{5,4} = B_{6,3} = B_{8,5} = \frac{\partial N_1}{\partial y}; \\
B_{2,7} &= B_{3,6} = B_{5,9} = B_{6,8} = B_{8,10} = \frac{\partial N_2}{\partial y}; & B_{2,12} = \\
B_{3,11} &= B_{5,14} = B_{6,13} = B_{8,15} = \frac{\partial N_3}{\partial y}; & B_{2,17} &= B_{3,16} = \\
B_{5,19} &= B_{6,18} = B_{8,20} = \frac{\partial N_4}{\partial y}; & B_{7,1} &= B_{8,2} = -B_{7,3} = \\
&= -B_{8,4} = \frac{N_1}{h}; & B_{7,6} &= B_{8,7} = -B_{7,8} = -B_{8,9} = \frac{N_2}{h}; \\
&= -B_{8,14} = \frac{N_3}{h}; & B_{7,16} &= B_{8,17} = \\
&= -B_{7,18} = -B_{8,19} = \frac{N_4}{h}.
\end{align*}
\]

The partial derivatives of the form functions are written as:
\[
\begin{align*}
\frac{\partial N_1}{\partial x} &= -\frac{1}{ab}\left(\frac{b}{2} - y\right); & \frac{\partial N_2}{\partial x} &= \frac{1}{ab}\left(\frac{b}{2} - y\right); \\
\frac{\partial N_3}{\partial x} &= \frac{1}{ab}\left(\frac{b}{2} + y\right); & \frac{\partial N_4}{\partial x} &= \frac{1}{ab}\left(\frac{b}{2} + y\right); \\
\frac{\partial N_1}{\partial y} &= -\frac{1}{ab}\left(\frac{a}{2} + x\right); & \frac{\partial N_2}{\partial y} &= \frac{1}{ab}\left(\frac{a}{2} + x\right); \\
\frac{\partial N_3}{\partial y} &= \frac{1}{ab}\left(\frac{a}{2} - x\right); & \frac{\partial N_4}{\partial y} &= \frac{1}{ab}\left(\frac{a}{2} - x\right).
\end{align*}
\]

The resolving equations will be obtained using the variational principle of Lagrange. The potential energy of deformation of a three-layer plate considering creep is defined as follows:
\[
W = \frac{1}{2}\int_A \left(\sigma^{(s)}_{xx} \varepsilon^{(s)}_x + \sigma^{(s)}_{yy} \varepsilon^{(s)}_y + \tau^{(s)}_{xy} \gamma^{(s)}_{xy} + \frac{\gamma^{(s)}_{mx}}{\gamma^{(s)}_{m}} \gamma^{(s)}_{m} + \frac{\gamma^{(s)}_{my}}{\gamma^{(s)}_{m}} \gamma^{(s)}_{m} \right) dA, \\
\]
where \(\gamma^{(s)}_{mx}, \gamma^{(s)}_{my}\) — creep deformations of the filler, \(A\) — area of the finite element.

Expression (4) can be rewritten as:
\[
W = \frac{1}{2}\int_A \left(\{N\}^T \{\varepsilon\} - \{\varepsilon^s\} \right) dA = \frac{1}{2}\int_A \{N\}^T \{B\} \{U\} - \{\varepsilon^s\} dA,
\]
where \(\{\varepsilon^s\} = [0 \, 0 \, 0 \, 0 \, 0 \, 0 \, \gamma^{(s)}_{mx} \gamma^{(s)}_{my}]^T\) — creep strain vector,
\[
\{N\}^T = \begin{bmatrix} N^+_x & N^+_y & N^+_{xy} & N^-_x & N^-_y & N^-_{xy} & Q_{xx} & Q_{xy} \end{bmatrix}
\]
— vector of internal forces, \(N^{(+/-)}_x = \sigma^{(+/-)}_x \cdot t^{(+/-)}_x\), \(N^{(+/-)}_{xy} = \sigma^{(+/-)}_{xy} \cdot t^{(+/-)}_{xy}\), \(Q_{xx} = \tau^{(+/-)}_{m} \cdot h\), \(Q_{xy} = \tau^{(+/-)}_{m} h\).

The relationship between deformations and internal forces has the form:
\[
\{N\} = [D]\{\varepsilon\} = [D]\{\varepsilon^s\} = [D]\{B\} \{U\} - \{\varepsilon^s\},
\]
where \([D]\) — block matrix of elastic constants.

\[
[D] = \begin{bmatrix} [D^+] & [D^-] & [D^m] \end{bmatrix},
\]

\[
[D^+] = \begin{bmatrix} E & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & (1-v)/2 \end{bmatrix}, \quad [D^-] = \begin{bmatrix} E \cdot \varepsilon^{(-)}_x & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & (1-v)/2 \end{bmatrix}, \quad [D^m] = G_m \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix},
\]

\(G_m\) — shear modulus of the filler.
Substituting (6) in the expression for the potential energy, we get:

\[ W = \frac{1}{2} \left( \{ \{ U \} \}^T \{ \{ B \} \} \{ B \} \{ \{ D \} \} \{ D \} \{ \{ U \} \} - \{ \{ U \} \}^T \{ \{ B \} \} \{ D \} \{ \{ U \} \} \right) - \left( \{ \{ \varepsilon^* \} \}^T \left( \{ \{ D \} \} \{ \{ B \} \} \{ \{ U \} \} \right) + \left( \{ \{ \varepsilon^* \} \}^T \{ \{ D \} \} \{ \{ \varepsilon^* \} \} \right) dA = 0, \]

where \( A = \{ \{ U \} \}^T \{ \{ F \} \}, \{ \{ F \} \} = \) vector of external nodal forces.

Differentiating the total energy with respect to the vector of nodal displacements, we obtain:

\[ \frac{\partial T}{\partial \{ U \}} = \{ K \} \{ U \} - \{ F \} - \{ F^* \} = 0, \]

where \( \{ K \} = \{ \{ B \} \}^T \{ \{ D \} \} \{ D \} \{ \{ B \} \} \) — stiffness matrix,

\( \{ F^* \} = \{ \{ B \} \}^T \{ \{ D \} \} \{ \{ \varepsilon^* \} \} \) — the contribution of creep strains to the right-hand side of the system of linear algebraic FEM equations.

Exact expressions for the matrix \( \{ K \} \) and vector \( \{ F^* \} \) coefficients were obtained using functions for the symbol variables of the Matlab package. The vector \( \{ F^* \} \) has the form:

\[ \{ F^* \} = \begin{pmatrix} -\frac{G_m h}{2} \left( \alpha y_{m+}^{m+} + \beta y_{m-}^{m+} \right) \\ \frac{G_m h}{2} \left( \alpha y_{m+}^{m-} + \beta y_{m+}^{m-} \right) \\ -\frac{G_m h}{2} \left( \alpha y_{m+}^{m+} + \beta y_{m-}^{m+} \right) \\ \frac{G_m h}{2} \left( \alpha y_{m+}^{m-} + \beta y_{m+}^{m-} \right) \\ \frac{G_m h}{2} \left( \alpha y_{m+}^{m+} + \beta y_{m-}^{m+} \right) \\ \frac{G_m h}{2} \left( \alpha y_{m+}^{m-} + \beta y_{m+}^{m-} \right) \end{pmatrix}, \]

where \( \{ F_i \} = \frac{a b G_m}{4} \begin{pmatrix} y_{m+}^{m+} \\ y_{m-}^{m+} \\ y_{m+}^{m+} \\ y_{m-}^{m+} \\ y_{m+}^{m+} \\ y_{m-}^{m+} \end{pmatrix}. \]

The total energy represents the difference between the potential energy of deformation and the potential of external forces:

\[ T = W - A, \]

The calculation was carried out by the step method, the creep strains at the time \( t + \Delta t \) were determined as follows:

\[ \gamma^*_i (t + \Delta t) = \gamma^*_i (t) + \frac{\partial \gamma^*_i}{\partial t} \Delta t. \]

3 Results and discussion

A three-layer rectangular hinged plate (Fig. 3) was calculated with the following initial data: plate thickness \( h = 8 \text{ cm} \), modulus of elasticity of sheaths \( E = 2 \cdot 10^5 \text{ MPa} \), Poisson’s coefficient of sheaths \( \nu = 0.3 \), sheath thickness \( t^+ = t^- = 1.5 \text{ mm} \), shear modulus of the filler \( G_m = 2.5 \text{ MPa} \), dimensions of plate \( a = b = 3 \text{ m} \). The uniformly distributed over the area load \( q = 2 \text{ kPa} \) acts on the plate. As the law of creep, the equation of the linear theory of heredity was used:

\[ G_m \gamma_i = \tau_i + \int_{-\infty}^{t} \tau(t) \, dt, \quad i = (xz, yz). \]
This method is also used in [4-9]. The following boundary conditions were assumed:

\[ x = 0, x = a : \quad w = 0, \quad v^+ = v^- = 0; \]
\[ y = 0, y = b : \quad w = 0, \quad u^+ = u^- = 0. \]

Fig. 4 shows the curve of growth of the deflection in the center of the plate. The solid line corresponds to the solution obtained by the authors using the finite element method, the dashed line is the solution by the finite difference method according to the method described in [10]. When \( t = 0 \) the results are the same and at \( t \to \infty \) the difference is 1.83%.

Stresses in the skins and filler do not change during creep. The stress \( \sigma_x^+ \) and \( \tau_{xy}^+ \) distribution in the lower skin is shown in Fig. 5 - 6. The stresses in the upper skin under the boundary conditions (10) in absolute value coincide with the stresses in the lower skin. The distribution of tangential stresses in the filler is shown in Fig. 7.

Fig. 4. Graph of growth of deflection in the center of the plate: solid line – finite element method, dashed line – finite differences method.

Fig. 5. Distribution of normal stresses in the lower skin.

Fig. 6. The distribution of tangential stresses in the bottom skin.

Fig. 7. Distribution of tangential stresses in the filler.

4 Conclusions

The equations obtained are applicable for arbitrary creep laws of the filler, including nonlinear ones. The correctness of the equations and the reliability of the results are confirmed by a comparison with the solution on the basis of the finite difference method. It is established that under linear creep law the stresses in the shells and filler do not change during the creep process.

References

1. A. Chepurnenko, L. Mailyan, B. Jazyev, Procedia Engineering, 165, pp. 990 – 994 (2016)
2. V. Andreev, B. Yaziev, A. Chepurnenko, S. Litvinov, Vestnik MGSU, 7, pp. 17-24 (2015)
3. B. Yaziev, A. Chepurnenko, S. Litvinov, S. Yaziev, Bulletin of the Dagestan State Technical University, 2, pp.47-55 (2014)
4. S. Litvinov, L. Trush, S. Yaziev, Procedia Engineering, 150, pp. 1686–1693 (2016)
5. S. Litvinov, E. Klimenko, I. Kulinich, S. Yazyeva, International Polymer Science and Technology, 2, pp. 23-25 (2015)
6. S. Litvinov, L. Trush, A. Dudnik, Engineering Bulletin of the Don, 2, 2016. http://www.ivdon.ru/en/magazine/archive/n2y2016/3560

7. V. Andreev, A. Chepurnenko, B. Yazyev, Advanced Materials Research, 1004-1005, pp. 257-260 (2014)

8. V. Andreev, E. Barmenkova, Applied Mechanics and Materials, 204-208, pp. 3596-3599 (2012)

9. V. Andreev, A. Avershyev, WIT Transactions on the Built Environment. Fluid Structure Interaction VII, pp. 123-132 (2013)

10. V. Andreev, B. Yazyev, A. Chepurnenko, Advanced Materials Research, 900, pp. 707-710 (2014)