Finite-lattice form factors in free-fermion models

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Abstract. We consider the general $Z_2$-symmetric free-fermion model on the finite periodic lattice, which includes as special cases the Ising model on the square and triangular lattices and the $Z_n$-symmetric BBS $\tau^{(2)}$-model with $n = 2$. Translating Kaufman’s fermionic approach to diagonalization of Ising-like transfer matrices into the language of Grassmann integrals, we determine the transfer matrix eigenvectors and observe that they coincide with the eigenvectors of a square lattice Ising transfer matrix. This allows us to find exact finite-lattice form factors of spin operators for the statistical model and the associated finite-length quantum chains, of which the most general is equivalent to the $XY$ chain in a transverse field.

Keywords: form factors, integrable spin chains (vertex models), solvable lattice models

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1. Introduction

Consider a two-dimensional $M \times N$ square lattice with spins $\sigma = \pm 1$ living at each of its sites. The most general $\mathbb{Z}_2$-symmetric plaquette Boltzmann weight is given by

$$W(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = a_0 \left( 1 + \sum_{1 \leq i < j \leq 4} a_{ij} \sigma_i \sigma_j + a_4 \sigma_1 \sigma_2 \sigma_3 \sigma_4 \right).$$  \hspace{1cm} (1)

The partition function of the corresponding statistical model can be represented as an integral over the four-component lattice Grassmann field with a quartic interaction [6,7], which disappears if the parameters satisfy the condition

$$a_4 = a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23}. \hspace{1cm} (2)$$

The model (1) can be mapped to the eight-vertex model in an external field [2]. The analog of the free-fermion condition (2) in the vertex picture was obtained earlier in [13].

Below we study the model defined by (1)–(2). It appeared in several equivalent formulations in papers by different authors. Its partition function can be evaluated by a variety of methods for both infinite and finite lattices [4,6,10,13,20]. Elliptic parametrization of the Boltzmann weights and inversion relations for the partition function were established in [3,5]. Grassmann integral representations for the corresponding transfer matrix and its eigenvectors have been found in [21]. Infinite-lattice correlations in a dual model were studied in [24]. The model (1)–(2) includes as special cases the Ising model on the square and triangular lattices, as well as the $\mathbb{Z}_n$-symmetric Baxter–Bazhanov–Stroganov $\tau^{(2)}$-model with $n = 2$.

The transfer matrix $V_{\varepsilon}[\sigma, \sigma']$ is a function of $2N$ spin variables $\sigma_0, \ldots, \sigma_{N-1}$, $\sigma'_0, \ldots, \sigma'_{N-1}$ given by the product of plaquette weights (1) over one lattice row (figure 1):

$$V_{\varepsilon}[\sigma, \sigma'] = \prod_{j=0}^{N-1} W(\sigma_j, \sigma'_j, \sigma'_{j+1}, \sigma_{j+1}). \hspace{1cm} (3)$$

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The subscript \(\varepsilon = \pm 1\) corresponds to periodic and antiperiodic boundary conditions on spin variables in the horizontal direction, i.e. \(\sigma_N = \varepsilon \sigma_0\) and \(\sigma'_N = \varepsilon \sigma'_0\).

The transfer matrix acts on the \(2^N\)-dimensional space of maps \(f : (\mathbb{Z}_2)^N \to \mathbb{C}\) by

\[
(V_\varepsilon f)[\sigma] = \sum_{[\sigma'] \in [\sigma]} V_\varepsilon[\sigma, \sigma'] f[\sigma'].
\]

It commutes with the operators of translation and spin reflection defined by

\[
(T_\varepsilon f)(\sigma_0, \ldots, \sigma_{N-1}) = f(\sigma_1, \ldots, \sigma_{N-1}, \varepsilon \sigma_0), \quad (U f)[\sigma] = f[-\sigma].
\]

The partition function of the model can be written as \(Z = \text{Tr}(V_\varepsilon^M U^{(1-\varepsilon')/2})\), where \(\varepsilon' = \pm 1\) corresponds to periodic and antiperiodic boundary conditions in the vertical direction. To write spin correlation functions, one should introduce spin operators \(\{s_j\}\) with \(j = 0, \ldots, N-1\) defined by \((s_j f)[\sigma] = \sigma_j f[\sigma]\). Then for \(j_1 \leq j_2 \leq \cdots \leq j_n\) we have

\[
\langle \sigma_{j_1, k_1} \cdots \sigma_{j_n, k_n} \rangle = Z^{-1} \text{Tr}(s_{j_1, k_1} \cdots s_{j_n, k_n} V_\varepsilon^M U^{(1-\varepsilon')/2}),
\]

where

\[
s_{j,k} = V_\varepsilon^{j} s_k V_\varepsilon^{-j} = V_\varepsilon^{j} T_\varepsilon^{k} s_0 T_\varepsilon^{-k} V_\varepsilon^{-j}.
\]

The computation of correlation functions therefore reduces to finding matrix elements (form factors) of the spin operator \(s_0\) in the common basis of eigenstates of \(V_\varepsilon\), \(T_\varepsilon\) and \(U\). The present work is devoted to the solution of this problem.

Multiplication of the Boltzmann weight (1) by \(e^{K(\sigma_1 \sigma_2 - \sigma_3 \sigma_4)}\) with any \(K\) does not change the transfer matrix. Thus, up to an overall factor, \(V_\varepsilon\) nontrivially depends on five parameters. The parameter set may be thought of as a five-dimensional projective space with homogeneous coordinates

\[
\kappa = (a_{12} + a_{34})(a_{13} + a_{24}) + (a_{14} + a_{23})(a_4 + 1),
\]

\[
\lambda = (a_{14} - a_{23})(a_4 - 1) - (a_{12} - a_{34})(a_{13} - a_{24}),
\]

\[
\mu = (a_4 + 1)^2 - (a_{12} + a_{34})^2 - (a_{13} - a_{24})^2 + (a_{14} - a_{23})^2,
\]

\[
\rho = 4 (a_{14} a_{23} - a_{13} a_{24}),
\]

\[
\tau = (a_4 + 1)^2 + (a_{12} + a_{34})^2 + (a_{13} + a_{24})^2 + (a_{14} + a_{23})^2,
\]

\[
v = (a_{12} + a_{34})(a_{13} + a_{24}) - (a_{14} + a_{23})(a_4 + 1).
\]

\[
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\]
It will be shown below that this set is foliated by two-dimensional surfaces such that the transfer matrices corresponding to different surface points are mutually commuting and diagonalized in the same basis. Different surfaces are labeled by the triples \((\lambda/\kappa, \mu/\kappa, \varrho/\kappa)\). Each of them contains a curve representing a set of transfer matrices of the Ising model on the triangular lattice, and this curve contains a point corresponding to the transfer matrix of an anisotropic Ising model on the square lattice.

As a consequence, finite-lattice form factors of the free-fermion model (1)–(2) can be obtained by a suitable parametrization from the Ising ones, for which the corresponding formulas were conjectured in [8, 9] and proved in [11, 12]. A more elegant proof based on earlier results of [14, 23] and the Frobenius determinant formula for elliptic Cauchy matrices was recently found in [16].

It is well known that the six- and eight-vertex model transfer matrices commute with the Hamiltonians of the quantum XXZ and XYZ spin-1/2 chains, respectively. Similarly, the diagonal-to-diagonal transfer matrix of the Ising model on a square lattice commutes with the Hamiltonian of the quantum Ising chain [22]. Local Hamiltonians are also known for the Ising model on the triangular and hexagonal lattices [25].

We will show that to every above-described two-parameter family of commuting transfer matrices of the general free-fermion model one can associate a local spin chain Hamiltonian depending on three parameters \(\lambda/\kappa, \mu/\kappa, \varrho/\kappa\). It turns out to be related to the Hamiltonian of the quantum XY chain in a transverse field by a similarity transformation. Simple considerations then allow us to rederive spin form factors of the finite-length XY chain, recently obtained in [15] by the method of separation of variables.

2. Kaufman’s approach and Grassmann integrals

Let us define two operators \(V_{a,p}\) by the relations

\[
V_{\pm} = \frac{1 \pm U}{2} V_a + \frac{1 \mp U}{2} V_p.
\]

Since \(V_{\pm}\) and \(V_{a,p}\) commute with \(U\), the set of eigenstates of \(V_+ (V_-)\) consists of even (odd) under spin reflection eigenvectors of \(V_a\) and odd (respectively even) eigenvectors of \(V_p\).

It was observed in [21] that \(V_{a,p}\) can be naturally represented as \(2N\)-fold Grassmann integrals. To describe this result in more detail, introduce auxiliary Grassmann variables \(\psi_0, \ldots, \psi_{N-1}, \dot{\psi}_0, \ldots, \dot{\psi}_{N-1}\). Our convention will be that ‘dotted’ variables commute with the usual ones and anticommute between themselves. We will also need to consider discrete Fourier transforms of variables of both types

\[
\psi_\theta = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{-i\theta j} \psi_j,
\]

using two sets of quasimomenta: \(\theta_a = \{\pi/N, 3\pi/N, \ldots, 2\pi - \pi/N\}\) and \(\theta_p = \{0, 2\pi/N, \ldots, 2\pi - 2\pi/N\}\). In this notation, one has [21]

\[
V_\nu[\sigma, \sigma'] = \zeta_\nu \int \mathcal{D}\psi \mathcal{D}\dot{\psi} \ e^{S_\nu[\psi, \dot{\psi}]} \prod_{j=0}^{N-1} e^{\sigma_j \psi_j} \prod_{j=0}^{N-1} e^{\sigma'_j \dot{\psi}_j}, \quad \nu = a, p,
\] (10)
where $\zeta_\nu = a_0 N \prod_{\theta \in \theta_\nu} \chi_\theta | 1/2$ and

$$S_\nu [\psi, \dot{\psi}] = \frac{1}{2} \sum_{\theta \in \theta_\nu} \left( \begin{array}{cc} \psi_{-\theta} & \psi_{-\theta} \end{array} \right) \left( \begin{array}{cc} G_{11}(\theta) & G_{12}(\theta) \\ G_{21}(\theta) & G_{22}(\theta) \end{array} \right) \left( \begin{array}{c} \psi_{\theta} \\ \psi_{\theta} \end{array} \right),$$

(11)

$$\chi_\theta = [a_{12} + a_{34} + (a_{13} + a_{24}) \cos \theta] + [(a_{13} - a_{24})^2 + 4a_{14}a_{23}] \sin^2 \theta,$$

(12)

$$\chi_\theta G_{11}(\theta) = 2i \sin \theta [a_{23} + a_{12}a_{24} + a_{13}a_{34} + a_{14}a_{4} - 2(a_{14}a_{23} - a_{13}a_{24}) \cos \theta],$$

(13)

$$\chi_\theta G_{22}(\theta) = 2i \sin \theta [a_{14} + a_{12}a_{13} + a_{24}a_{34} + a_{23}a_{4} - 2(a_{14}a_{23} - a_{13}a_{24}) \cos \theta],$$

(14)

$$\chi_\theta G_{12}(\theta) = \chi_\theta G_{21}(-\theta) = [(a_{12} + a_{34})(a_{4} + 1) - (a_{13} + a_{24})(a_{23} + a_{14})] + [(a_{13} + a_{24})(a_{4} + 1) - (a_{12} + a_{34})(a_{14} + a_{23})] \cos \theta + [(a_{24} - a_{13})(a_{4} - 1) + (a_{12} - a_{34})(a_{23} - a_{14})] i \sin \theta.$$

(15)

The representation (10) was used in [21] to find the eigenvectors of $V_{a,p}$ explicitly in terms of spin variables. One of the reasons to proceed in this way was the difficulties with the well-known Kaufman’s algebraic approach [19] designed for the two-dimensional Ising model. It can be expected on general grounds that the conjugation by $V_{a,p}$ induces linear transformations of the standard Clifford algebra generators. However, in contrast to the Ising case, it was not clear how one can determine the explicit form of these transformations.

This can be done as follows. First look for a representation for the inverse $V^{-1}_{\nu} [\sigma, \sigma']$ in the form (10):

$$V^{-1}_{\nu} [\sigma, \sigma'] = \tilde{\zeta}_\nu \int D\varphi D\dot{\varphi} e^{\tilde{S}_\nu [\varphi, \dot{\varphi}]} \prod_{j=0}^{N-1} e^{\sigma_j \varphi_j} \prod_{j=0}^{N-1} e^{\sigma'_j \dot{\varphi}_j},$$

(16)

where the overall coefficient $\tilde{\zeta}_\nu$ and quadratic action $\tilde{S}_\nu [\varphi, \dot{\varphi}]$ should be chosen so that

$$\sum_{|\sigma'|} V_{\nu} [\sigma, \sigma'] V^{-1}_{\nu} [\sigma', \sigma''] = \delta_{|\sigma|,|\sigma''|} = \prod_{j=0}^{N-1} \frac{1 + \sigma_j \sigma_j''}{2}.$$

(17)

Using (10) and the ansatz (16), the lhs of the last relation can be written as

$$\zeta_\nu \tilde{\zeta}_\nu \sum_{|\sigma'|} \int D\psi D\dot{\psi} D\varphi D\dot{\varphi} e^{S_{\nu} [\psi, \dot{\psi}]} \prod_{j=0}^{N-1} e^{\sigma_j \psi_j} \prod_{j=0}^{N-1} e^{\sigma'_j \dot{\psi}_j} e^{\tilde{S}_\nu [\varphi, \dot{\varphi}]} \prod_{j=0}^{N-1} e^{\sigma_j \varphi_j} \prod_{j=0}^{N-1} e^{\sigma'_j \dot{\varphi}_j}.$$

(18)

Let us explain how this expression can be simplified, since below similar calculations will be made without going into the details.

- Only the even part of $\prod_{j=0}^{N-1} e^{\sigma_j \psi_j} \prod_{j=0}^{N-1} e^{\sigma'_j \dot{\psi}_j}$ gives non-zero contribution to the integral. Since it commutes with any quadratic expression in Grassmann variables of both types, the factor $e^{S_{\nu} [\varphi, \dot{\varphi}]}$ can be put together with $e^{S_{\nu} [\psi, \dot{\psi}]}$. 

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• The summation over intermediate spins can then be easily made:

\[
\sum_{\{\sigma\}} \left( \prod_{j=0}^{N-1} e^{\sigma^j \psi_j} \prod_{j=0}^{N-1} e^{\sigma_j^j \phi_j} \right) = 2^N e^{\psi \phi},
\]

where we use shorthand notation \( \dot{\psi} \phi = \sum_{j=0}^{N-1} \dot{\psi}_j \phi_j \). This transforms (18) into

\[
2^N \zeta_{\nu} \tilde{\zeta}_{\nu} \int \mathcal{D} \psi \mathcal{D} \dot{\psi} \mathcal{D} \phi \mathcal{D} \dot{\phi} e^{S_{\nu}[\psi, \dot{\psi}] + \tilde{S}_{\nu}[\phi, \dot{\phi}]} \prod_{j=0}^{N-1} e^{\sigma^j \psi_j} \prod_{j=0}^{N-1} e^{\sigma_j^j \phi_j}.
\]

• To pull \( e^{\dot{\psi} \phi} \) to the left through \( F[\psi] = \prod_{j=0}^{N-1} e^{\sigma^j \psi_j} \), write the latter factor as a sum of its odd and even parts, which gives \( F[\psi] e^{\dot{\psi} \phi} = e^{\dot{\psi} \phi} F_{\text{even}}[\psi] + e^{-\dot{\psi} \phi} F_{\text{odd}}[\psi] \). In the integral corresponding to the first term, make the change of variables \( \psi \to -\psi \), \( \dot{\psi} \to -\dot{\psi} \). Since \( S[\psi, \dot{\psi}] \) is even, this allows us to rewrite the previous expression as

\[
2^N \zeta_{\nu} \tilde{\zeta}_{\nu} \int \mathcal{D} \psi \mathcal{D} \dot{\psi} \mathcal{D} \phi \mathcal{D} \dot{\phi} e^{S_{\nu}[\psi, \dot{\psi}] + \tilde{S}_{\nu}[\phi, \dot{\phi}]-\dot{\psi} \phi} \prod_{j=0}^{N-1} e^{\sigma^j \psi_j} \prod_{j=0}^{N-1} e^{\sigma_j^j \phi_j}.
\]

Thus, in order to satisfy (17), it is sufficient to choose \( \tilde{\zeta}_{\nu} \) and \( \tilde{S}_{\nu}[\varphi, \dot{\varphi}] \) so that

\[
2^N \zeta_{\nu} \tilde{\zeta}_{\nu} \int \mathcal{D} \psi \mathcal{D} \phi \ e^{S_{\nu}[\psi, \phi] + \tilde{S}_{\nu}[\phi, \dot{\phi}]} = e^{\dot{\psi} \phi}.
\]

In view of (11), it is natural to try to satisfy this relation with \( \tilde{S}_{\nu}[\varphi, \dot{\varphi}] \) diagonalized by discrete Fourier transform. This indeed works and one finds that

\[
\tilde{S}_{\nu}[\varphi, \dot{\varphi}] = \frac{1}{2} \sum_{\theta \in \Theta_{\nu}} (\varphi_{-\theta} \varphi_{-\theta}) G^{-1}(\theta) \left( \begin{array}{c} \dot{\varphi}_\theta \\ \varphi_\theta \end{array} \right),
\]

\[
\tilde{\zeta}_{\nu}^{-1} = 2^N \zeta_{\nu} \left[ \prod_{\theta \in \Theta_{\nu}} \frac{G_{12}(\theta)G_{21}(\theta)}{\det G(\theta)} \right]^{1/2},
\]

where the \( 2 \times 2 \) matrix \( G(\theta) \) is defined by (12)–(15).

The next step is to introduce the Clifford algebra generators \( \{p_j\}, \{q_j\} \) \( (j = 0, \ldots, N - 1) \),

\[
\left( \begin{array}{c} p_j \\ q_j \end{array} \right) [\sigma, \sigma'] = 2^{-N} \prod_{k=0}^{j-1} (1 - \sigma_k \sigma'_k) \prod_{k=j+1}^{N-1} (1 + \sigma_k \sigma'_k) \left( \begin{array}{c} \sigma_j + \sigma'_j \\ i(\sigma'_j - \sigma_j) \end{array} \right),
\]

which satisfy standard anticommutation relations \( \{p_j, p_k\} = \{q_j, q_k\} = 2\delta_{jk}, \{p_j, q_k\} = 0 \). It is easy to verify that these operators can be represented in a form similar to the Grassmann integral representation (10):

\[
\left( \begin{array}{c} p_j \\ q_j \end{array} \right) [\sigma, \sigma'] = 2^{-N} \int \mathcal{D} \eta \mathcal{D} \dot{\eta} \ e^{\eta \dot{\eta}} \left( \begin{array}{c} \eta_j + \dot{\eta}_j \\ i(\eta_j - \dot{\eta}_j) \end{array} \right) \prod_{k=0}^{N-1} e^{\sigma_k \eta_k} \prod_{k=0}^{N-1} e^{\sigma'_k \dot{\eta}_k}.
\]

We especially note that Fourier transforms \( p_\theta, q_\theta \) are obtained by simple replacement of the subscript \( j \) by \( \theta \) in the rhs of (19).

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It then becomes possible to compute the result of conjugation of \( \{p_j\}, \{q_j\} \) by \( V_\nu \) using the representations (10), (16) and (19). After summation over intermediate spin variables one obtains

\[
V_\nu \left( \begin{array}{c} p_\theta \\ q_\theta \end{array} \right) V_\nu^{-1}[\sigma, \sigma'] = 2^N \tilde{\zeta}_\nu \tilde{\zeta}_\nu \int D\psi D\tilde{\psi} D\eta D\tilde{\eta} D\varphi D\tilde{\varphi} 
\times e^{S_\nu[\psi, \tilde{\psi}] + \tilde{S}_\nu[\varphi, \tilde{\varphi}] - \psi_{\eta} + \eta_{\varphi}} \left( \frac{\eta_\theta + \tilde{\eta}_\theta}{i(\eta_\theta - \tilde{\eta}_\theta)} \right) \prod_{j=0}^{N-1} e^{\alpha_{\psi_j, \varphi_j}} \prod_{j=0}^{N-1} e^{\beta_{\psi_j, \varphi_j}}.
\]

The integration over \( \dot{\psi}, \eta, \tilde{\eta}, \varphi \) can be performed in the Fourier basis. Comparing the resulting integral over \( \psi \) and \( \varphi \) with (19), we find that the induced rotation is explicitly given by

\[
V_\nu \left( \begin{array}{c} p_\theta \\ q_\theta \end{array} \right) V_\nu^{-1} = \Lambda(V_\nu) \left( \begin{array}{c} p_\theta \\ q_\theta \end{array} \right),
\]

\[
\Lambda(V_\nu) = \frac{1}{2\chi_\nu G_{12}(\theta)} \left( \begin{array}{cc} \alpha_\theta & i\beta_\theta \\ -i\beta_\theta & \alpha_{-\theta} \end{array} \right),
\]

where

\[
\alpha_\theta = \chi_\theta \left[ 1 - \det G(\theta) + G_{11}(\theta) - G_{22}(\theta) \right],
\]

\[
\beta_\theta = \chi_\theta \left[ 1 + \det G(\theta) + G_{11}(\theta) + G_{22}(\theta) \right].
\]

One can now follow Kaufman’s method and find the eigenstates of \( V_{a,p} \) by diagonalization of the two-dimensional rotations (21).

It is straightforward to check using (12)–(15) that \( \alpha_\theta, \beta_\theta \) can be written in terms of the parameters (4)–(9) as follows:

\[
\alpha_\theta = \tau + 2\nu \cos \theta + 2i\lambda \sin \theta, \quad \beta_\theta = -\varphi e^{2i\theta} + 2\kappa e^{i\theta} - \mu,
\]

and that \( \alpha_\theta \alpha_{-\theta} - \beta_\theta \beta_{-\theta} = 4\chi_\theta^2 G_{12}(\theta) G_{21}(\theta) \). In fact the relations (23) are the origin of the parametrization (4)–(9). Let us further introduce three parameters \( \K_0, \K_x, \K_y \) by

\[
\lambda = \frac{\sinh 2\K_0}{\cosh 2\K_x \sinh 2\K_y}, \quad \mu = \frac{\cosh 2\K_0 + \cosh 2\K_y}{\cosh 2\K_x \sinh 2\K_y}, \quad \nu = \frac{\cosh 2\K_y - \cosh 2\K_0}{\cosh 2\K_x \sinh 2\K_y}.
\]

It will be shown in the next section that the eigenvectors of \( V_\nu \) coincide with the eigenvectors of a non-symmetrized transfer matrix of the anisotropic Ising model on the square lattice:

\[
\exp\left\{ \frac{\K_y - \K_0}{2} \sum_{j=0}^{N-1} s_j s_{j+1} \right\} \exp\left\{ \K_x^* \sum_{j=0}^{N-1} C_j \right\} \exp\left\{ \frac{\K_y + \K_0}{2} \sum_{j=0}^{N-1} s_j s_{j+1} \right\},
\]

where \( \tanh \K_x^* = e^{-2\K_x} \) and the operators \( \{C_j\} \) are defined by

\[
(C_j f)(\ldots, \sigma_{j-1}, \sigma_j, \sigma_{j+1}, \ldots) = f(\ldots, \sigma_{j-1}, -\sigma_j, \sigma_{j+1}, \ldots).
\]

We assume for concreteness that \( \K_0, \K_x, \K_y \) are real and positive and \( \K_x^* < \K_y \). This mimics the ferromagnetic region of Ising parameters.

Define the functions

\[
b_\theta^\pm = \sqrt{(\alpha_\theta - \alpha_{-\theta})^2 + 4\beta_\theta \beta_{-\theta} \pm (\alpha_\theta - \alpha_{-\theta})} \frac{2e^{-i\varphi}}{\beta_\theta}
\]

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satisfying \( b^+_\theta b^-_{\theta} = 1 \). The square roots here and below are fixed by the requirement of positivity of the real part. Determining the eigenvectors of the induced rotations \((21)\), introduce the creation–annihilation operators

\[
2\psi^\dagger_{\theta} = \rho_\theta \left( e^{-i\theta} \sqrt{b^+_{\theta} p_{-\theta}} - i \sqrt{b^-_{\theta} q_{-\theta}} \right), \quad 2\psi_{\theta} = \rho_\theta \left( e^{i\theta} \sqrt{b^-_{\theta} p_{\theta}} + i \sqrt{b^+_{\theta} q_{\theta}} \right),
\]

where

\[
\rho_\theta = \rho_{-\theta} = \sqrt{2} \left[ \sqrt{b^+_{\theta} b^-_{\theta}} + \sqrt{b^-_{\theta} b^+_{\theta}} \right]^{-1/2}.
\]

These operators satisfy canonical anticommutation relations \( \{ \psi^\dagger_{\theta}, \psi^\dagger_{\varphi} \} = \{ \psi_{\theta}, \psi_{\varphi} \} = 0, \{ \psi^\dagger_{\theta}, \psi_{\varphi} \} = \delta_{\theta,\varphi} \). They transform diagonally under conjugation by \( V_\nu \):

\[
V_\nu \left( \begin{array}{c} \psi^\dagger_{\theta} \\ \psi_{\theta} \end{array} \right) V_\nu^{-1} = \left( \begin{array}{cc} e^{-\mathcal{E}_\theta} & 0 \\ 0 & e^{\mathcal{E}_\nu} \end{array} \right) \left( \begin{array}{c} \psi^\dagger_{\theta} \\ \psi_{\theta} \end{array} \right), \quad \theta \in \Theta_\nu,
\]

where

\[
\mathcal{E}_\theta = \ln \frac{(\alpha_\theta - \alpha_{-\theta})^2 + 4\beta_\theta \beta_{-\theta} + \alpha_\theta + \alpha_{-\theta}}{4\chi_\theta G_{12}(\theta)}.
\]

The matrices \( V_\nu \) can therefore be written as

\[
V_\nu = 2^N a_0^N \prod_{\theta \in \Theta_\nu} \chi_\theta G_{12}(\theta)^{1/2} \exp \left\{ -\sum_{\theta \in \Theta_\nu} \mathcal{E}_\theta (\psi^\dagger_{\theta} \psi_{\theta} - \frac{1}{2}) \right\},
\]

where the overall scalar multiple can be fixed, e.g., by the identification of eigenvalues with formulas in \([21]\). In contrast to the Ising case, \( \mathcal{E}_\theta \neq \mathcal{E}_{-\theta} \) and hence the spectrum is (generically) nondegenerate. In particular, the eigenstates of \( V_\nu \) will automatically diagonalize the translation operator.

The left and right eigenvectors of \( V_\nu \) are multiparticle Fock states

\[
\nu \langle \theta_1, \ldots, \theta_k | = \nu \langle \text{vac} | \psi_{\theta_1} \cdots \psi_{\theta_k}, \quad | \theta_1, \ldots, \theta_k \rangle_\nu = \psi^\dagger_{\theta_1} \cdots \psi^\dagger_{\theta_k} \langle \text{vac} | ,
\]

where \( \theta_1, \ldots, \theta_k \in \Theta_\nu \). The corresponding eigenvalue is equal to \( \exp \left\{ \frac{1}{2} \sum_{\theta \in \Theta_\nu} \mathcal{E}_\theta - \sum_{i=1}^k \mathcal{E}_{\theta_i} \right\} \). The vacuum vectors \( | \text{vac} \rangle \) and \( \nu \langle \text{vac} \) are annihilated by all \( \psi_{\theta} \) (respectively \( \psi^\dagger_{\theta} \)) with \( \theta \in \Theta_\nu \) and are normalized as \( \nu \langle \text{vac} | \text{vac} \rangle_\nu = 1 \). Since \( V \) is not symmetric, \( \nu \langle \text{vac} \rangle \) is not necessarily the Hermitian conjugate of \( \langle \text{vac} | \nu \).

Not all of the states \((28)\) are eigenvectors of the full transfer matrix (and the associated quantum spin chain Hamiltonians below). For periodic and antiperiodic boundary conditions on spin variables the number of particles in these states should be even and odd, respectively, and one has

\[
T_+ | \theta_1, \ldots, \theta_{2k} \rangle_{a,p} = e^{-i\sum_{i=1}^{2k} \theta_i} | \theta_1, \ldots, \theta_{2k} \rangle_{a,p},
T_- | \theta_1, \ldots, \theta_{2k+1} \rangle_{a,p} = e^{-i\sum_{i=1}^{2k+1} \theta_i} | \theta_1, \ldots, \theta_{2k+1} \rangle_{a,p}.
\]
We especially note that the transfer matrix eigenvectors and form factors depend only on $K_x$, $K_y$, $K_0$ and are independent of $\tau/\kappa$, $\nu/\kappa$. The latter two variables appear only in the eigenvalues and can be thought of as spectral parameters.

3. Form factors

Consider instead of $V_\nu$ the conjugated transfer matrix $V'_{\nu} = SV_\nu S^{-1}$ with $S = \exp\{(K_0/2)\sum_{j=0}^{N-1}s_j s_{j+1}\}$. The matrix of induced rotation in (20) then becomes $\Lambda(V'_{\nu}) = \Lambda^{-1}(S)\Lambda(V_{\nu})\Lambda(S)$ with

$$
\Lambda(S) = \begin{pmatrix}
\cosh K_0 & i\sinh K_0 e^{-i\theta} \\
-i\sinh K_0 e^{i\theta} & \cosh K_0
\end{pmatrix},
$$

and it can be straightforwardly checked that $\Lambda(V'_{\nu}) = \Lambda(V_{\nu})|_{K_0=0}$. Therefore the orthonormal system of left and right eigenvectors of $V_{\nu}$ can be chosen as follows:

$$\langle \theta_1, \ldots, \theta_k \rangle^S_{\nu} = S^{-1}(\langle \theta_1, \ldots, \theta_k \rangle_{\nu}|_{K_0=0}) = \langle \nu(\theta_1, \ldots, \theta_k) \rangle|_{K_0=0} S_{\nu}.$$ 

These vectors are of course proportional to those in (28).

Since $S$ commutes with the operators $\{s_j\}$, nontrivial spin form factors coincide with those computed for $K_0 = 0$:

$$
\mathcal{F}^{(l)}_{m,n}(\theta, \theta') = \sum_{a} \langle \theta_1, \ldots, \theta_m|s_l|\theta'_1, \ldots, \theta'_n \rangle_x^S = \sum_{a} \langle \theta_1, \ldots, \theta_m|s_l|\theta'_1, \ldots, \theta'_n \rangle_x^{S^*},
$$

where $m$, $n$ are simultaneously even or odd. On the other hand, setting $K_0 = 0$ leads to important simplifications in the formulas of section 2. It implies in particular that $\lambda = 0$, $\alpha = \alpha_\theta$, $b_{\theta}^{\text{def}} = b_{\theta}$ and $\rho_\theta = 1$. Moreover,

$$
b_\theta = \sqrt{(1 - \tanh K_x^* \coth K_y e^{i\theta})(1 - \tanh K_y^* \tanh K_y e^{-i\theta})}/(1 - \tanh K_x^* \coth K_y e^{i\theta})(1 - \tanh K_y^* \tanh K_y e^{-i\theta}),
$$

and thus the Fock states (28) with $K_0 = 0$ coincide with the eigenvectors of the symmetric transfer matrix of the anisotropic Ising model on the square lattice, cf [16]. Let us therefore denote $|\theta_1, \ldots, \theta_k\rangle^{\text{Ising}}_{\nu} = (|\theta_1, \ldots, \theta_k\rangle_{\nu}|_{K_0=0}$. In the symmetric Ising case, the left and right eigenvectors are related by Hermitian conjugation, i.e. $|\theta_1, \ldots, \theta_k\rangle^{\text{Ising}}_{\nu} = (|\theta_1, \ldots, \theta_k\rangle_{\nu})^\dagger$.

To write the corresponding form factors, introduce two functions $\gamma_\theta$, $\nu_\theta$ and two parameters $\xi$, $\xi_T$ defined by

$$
cosh \gamma_\theta = \cosh 2K_x^* \cosh 2K_y - \sinh 2K_x^* \sinh 2K_y \cos \theta, \quad \gamma_\theta > 0,$$

$$
\nu_\theta = \ln \prod_{\theta \in \theta_p} \sinh(\gamma_\theta + \gamma_\theta)\sinh(\gamma_\theta + \gamma_\theta)/2, \quad \xi = [1 - (\sinh 2K_x \sinh 2K_y)^{-2}]^{1/4},
$$

$$
\xi_T = \prod_{\theta \in \theta_p} e^{\nu_\theta/4} \prod_{\theta \in \theta_\nu} e^{-\nu_\theta/4} = \left[ \prod_{\theta \in \theta_p} \prod_{\theta' \in \theta_\nu} \sinh(\gamma_\theta + \gamma_\theta)/2 \right]^{1/4}.
$$

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9
Then, by (30) and, e.g., theorem 6.1 in [16],

$$F_{m,n}^{(l)}(\theta, \theta') = i^{2m-n+n/2} e^{i\xi x} \prod_{j=1}^{m} e^{-i(l-1/2)\theta + v_\gamma_j/2} \prod_{j=1}^{n} e^{i(l-1/2)\theta' - v_\gamma_j'/2} \times \left( \frac{\sinh 2K y}{\sinh 2K x} \right)^{(m-n)/4} \prod_{1 \leq i < j \leq m} \frac{\sin(\theta_i - \theta_j)/2}{\sin(\gamma_i + \gamma_j)/2} \prod_{1 \leq i \leq n, 1 \leq j \leq n} \frac{\sinh(\gamma_i + \gamma_j'/2)}{\sin(\theta_i - \theta_j'/2)}.$$  \hfill (32)

As explained in section 1, this formula allows us to compute (long-distance expansions of) an arbitrary spin correlation function in the statistical model (1)–(2).

4. Quantum spin chain Hamiltonian

The matrix $V_\nu$ clearly commutes (cf (27)) with the operator

$$H_\nu = -\frac{1}{2} \sum_{\sigma \in \sigma_\nu} \left\{ 4\frac{\beta_0 \beta_1}{\rho_0} \psi_{\sigma}^* \psi_\sigma - \sqrt{(\alpha_0 - \alpha_{-1})^2 + 4\beta_0 \beta_{-1}} \right\},$$

which has the following form in terms of $\{p_\sigma\}, \{q_\sigma\}$:

$$H_\nu = -\frac{1}{4} \sum_{\sigma \in \sigma_\nu} \left\{ (\alpha_0 - \alpha_{-1}) (p_{-1} p_\sigma - q_{-1} q_\sigma) - 4i\beta_0 q_{-1} p_\sigma \right\}.$$  

Now using (22) and performing the inverse Fourier transform, one finds that

$$H_\nu = -\sum_{j=0}^{N-1} \left\{ \lambda (p_j p_{j+1} - q_j q_{j+1}) + i\mu q_j p_j - 2ik q_j p_{j+1} + i\rho q_j p_{j+2} \right\},$$

where $p_{j+N} = p_j, q_{j+N} = q_j$ for $\nu = p$ and $p_{j+N} = -p_j, q_{j+N} = -q_j$ for $\nu = a$.

Using the standard realization of $\{p_j\}, \{q_j\}$ in terms of Pauli matrices $\sigma^{x,y,z}$,

$$p_j = \left( \sigma^x \otimes \cdots \otimes \sigma^x \otimes \frac{1}{2} \otimes \cdots \otimes \frac{1}{2} \right)^{j \text{ times}},$$

$$q_j = \left( \sigma^x \otimes \cdots \otimes \sigma^x \otimes \frac{1}{2} \otimes \cdots \otimes \frac{1}{2} \right)^{j \text{ times}},$$

one finds the quantum spin chain Hamiltonian, which commutes with the transfer matrix $V_\nu$ and the operator (24):

$$H_\varepsilon = \frac{1}{2} e U H_a + \frac{1}{2} e U H_p$$

$$= \sum_{j=0}^{N-1} \left\{ 2\kappa \sigma_j^{x} \sigma_{j+1}^{z} + \mu \sigma_j^{x} - i\lambda (\sigma_j^{y} \sigma_{j+1}^{z} + \sigma_j^{z} \sigma_{j+1}^{y}) - \theta \sigma_j^{z} \sigma_{j+1}^{z} \sigma_{j+2}^{z} \right\},$$

where the boundary conditions are $\sigma_j^{x} = \sigma_j^{x}, \sigma_j^{y} = \varepsilon \sigma_j^{y}, \sigma_j^{z} = \varepsilon \sigma_j^{z}$. To put this Hamiltonian in a more familiar form, conjugate it by the matrix $S' = V_x^{1/2} V_y^{1/2} S$ with

$$V_x = \exp \left\{ K_x \sum_{j=0}^{N-1} \sigma_j^{x} \right\},$$

$$V_y = \exp \left\{ K_y \sum_{j=0}^{N-1} \sigma_j^{y} \sigma_{j+1}^{z} \right\}.$$
which gives
\[ H'_c = S'H_xS'^{-1} = \text{const} \sum_{j=0}^{N-1} \left\{ e^{-2K_x \sigma_j^y \sigma_{j+1}^y} + e^{2K_x \sigma_j^z \sigma_{j+1}^z} + 2 \coth 2K_x \sigma_j^z \right\}. \] (33)

One recognizes here the Hamiltonian of the XY chain in a transverse field. If we restrict ourselves to the symmetric Ising transfer matrix by setting \( V_x = V_y^{1/2} V_x V_y^{1/2} \), this reproduces the well-known observation of [26] that \( H'_c \) commutes with the second symmetric Ising transfer matrix \( V_x^{1/2} V_y^{1/2} \).

The orthonormal eigenvectors of \( H'_c \) can therefore be chosen as follows:
\[ |\theta_1, \ldots, \theta_k\rangle^{XY}_{\nu} = e^{\pm \gamma(\theta)/2} V_x^{\pm 1/2} V_y^{\pm 1/2} |\theta_1, \ldots, \theta_k\rangle^{\text{Ising}}_{\nu}, \] (34)
\[ \langle \theta_1, \ldots, \theta_k|^{XY}_{\nu} = e^{\pm \gamma(\theta)/2} V_y^{\pm 1/2} V_x^{\pm 1/2} |\theta_1, \ldots, \theta_k\rangle^{\text{Ising}}_{\nu}, \] (35)
where
\[ e^{-\gamma(\theta)} = \exp \left\{ \frac{1}{2} \sum_{\theta \in \Theta} \frac{\gamma \theta - \sum_{i=1}^{k} \gamma \theta_i}{2} \right\} \] (36)
is the corresponding eigenvalue of \( V_x^{1/2} V_y^{1/2} \). Note that different signs in (34) and (35) give equivalent representations of the same vector, and we have the Hermitian conjugation identities \( \langle \theta_1, \ldots, \theta_k|^{XY}_{\nu} = (|\theta_1, \ldots, \theta_k\rangle^{XY}_{\nu})^{\dagger} \).

The operators \( \sigma_i^x \) commute with the \( Z_2 \)-charge \( U \) and thus have non-zero form factors only between the states of the same type \((a \text{ or } p)\). Since they are bilinear in fermions, their matrix elements are easily computable. The only nontrivial form factors of the XY chain (33) are those of the operators \( \sigma_i^y \) and \( \sigma_i^z \), which map the \( a \)-sector to the \( p \)-sector and vice versa. Define
\[ F_{m,n}^{XY}(\theta, \theta') = \langle \theta_1, \ldots, \theta_m| \sigma_i^x| \theta_1', \ldots, \theta_n\rangle^{XY}_{a} = \left[ \langle \theta_1', \ldots, \theta_n'| \sigma_i^x| \theta_1, \ldots, \theta_m\rangle^{XY}_{a} \right]^*, \quad r = y, z. \]

Then, using different representations for the eigenvectors (34)–(35), one finds that
\[ \pm i \sinh K_x^{*} F_{m,n}^{XY}(\theta, \theta') = \cosh K_x^{*} F_{m,n}^{XY}(\theta, \theta') \] (37)
\[ \times \frac{\sinh \gamma(\theta) - \gamma(\theta')}{i \sinh K_x^{*}} F_{m,n}^{(l)}(\theta, \theta'), \]
\[ \times V_y^{1/2} V_x^{1/2} |\theta_1, \ldots, \theta_m| \sigma_i^x |\theta_1', \ldots, \theta_n'|^{\text{Ising}} = e^{\pm \gamma(\theta) - \gamma(\theta')}/2 \]
\[ \times \frac{\cosh \gamma(\theta) - \gamma(\theta')}{\cosh K_x^{*}} F_{m,n}^{(l)}(\theta, \theta'), \]
\[ = e^{\pm \gamma(\theta) - \gamma(\theta')}/2 F_{m,n}^{(l)}(\theta, \theta'). \]

From these relations one obtains the final finite-length XY form factor formulas:
\[ F_{m,n}^{XY}(\theta, \theta') = \frac{\sinh \gamma(\theta) - \gamma(\theta')}{i \sinh K_x^{*}} F_{m,n}^{(l)}(\theta, \theta'), \] (37)
\[ F_{m,n}^{XY}(\theta, \theta') = \frac{\cosh \gamma(\theta) - \gamma(\theta')}{\cosh K_x^{*}} F_{m,n}^{(l)}(\theta, \theta'), \] (38)
where \( F_{m,n}^{(l)}(\theta, \theta') \) is defined by (32) and \( \gamma(\theta) \) by (36). The same expressions have been recently found in [15] by the method of separation of variables and used to rederive the
asymptotics of the two-point correlation function in the disordered phase without the use of Toeplitz determinants and the Wiener–Hopf factorization method.

Detailed discussion of the XY chain is beyond the scope of the present paper. We would like to mention, however, the works [17, 18], where the time- and temperature-dependent two-point correlation functions of the finite XY chain were expressed in terms of $2N \times 2N$ determinants. In fact equivalent representations can be simply deduced from the observation that spin operators, as well as the densities $e^{-\beta H^v}$, are elements of the Clifford group; the entries of the corresponding determinants are then given by thermally dressed two-particle form factors. We will report on these issues in a future publication.

Our last remark concerns the operators appearing in the Hamiltonians $H_\varepsilon$ and $H'_\varepsilon$. It can be checked that they obey Onsager algebra relations for the standard generators \{A_j\}, \{G_j\}:

$$H_\varepsilon = 2\kappa A_1 + \mu A_0 - 2\lambda G_1 + \rho A_2,$$
$$H'_\varepsilon = \text{const} (e^{2\kappa x} A_1 + e^{-2\kappa x} A_{-1} + 2 \coth 2\kappa y A_0).$$

The similarity transformations with $\exp(\kappa A_0)$, $\exp(\kappa' A_1)$ relating these two Hamiltonians are combinations of elementary Onsager algebra automorphisms described in [1].

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