Tunnelling with a Negative Cosmological Constant

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ABSTRACT

The point of this paper is see what light new results in hyperbolic geometry may throw on gravitational entropy and whether gravitational entropy is relevant for the quantum origin of the universes. We introduce some new gravitational instantons which mediate the birth from nothing of closed universes containing wormholes and suggest that they may contribute to the density matrix of the universe. We also discuss the connection between their gravitational action and the topological and volumetric entropies introduced in hyperbolic geometry. These coincide for hyperbolic 4-manifolds, and increase with increasing topological complexity of the four manifold. We raise the questions of whether the action also increase with the topological complexity of the initial 3-geometry, measured either by its three volume or its Matveev complexity. We point out, in distinction to the non-supergravity case, that universes with domains of negative cosmological constant separated by supergravity domain walls cannot be born from nothing. Finally we point out that our wormholes provide examples of the type of Perpetual Motion machines envisaged by Frolov and Novikov.
Introduction

There has been great interest recently in calculations of the semi-classical tunneling rates for the production of pairs of black holes in quantum gravity [1]. A notable feature of these results is light they have thrown upon the Bekenstein-Hawking entropy

\[ S_{\text{Bekenstein-Hawking}} = \frac{1}{4} A \]

of non-extreme event horizons.

It has been found that the probability of creating near-extreme black holes compared with extreme, solitonic, holes for which \( S_{\text{Bekenstein-Hawking}} = 0 \) is increased by a factor

\[ \exp\left(\frac{1}{4}A\right) = \exp(S_{\text{Bekenstein-Hawking}}). \]

This lends further support to the interpretation of \( S_{\text{Bekenstein-Hawking}} \) as a purely gravitational contribution to the total thermodynamic entropy of any system containing black holes.

Similar results hold for the entropy \( S_{\text{Cosmological}} \) of Cosmological Horizons. The Euclidean action of \( S^4 \) and \( S^2 \times S^2 \) is given by

\[ I_{\text{euc}} = -S_{\text{Cosmological}} = \frac{1}{4} A_C \]

An interesting question is whether there are other circumstances in which one may associate entropy with other types of gravitational fields. In particular it is tempting to apply the idea of entropy to the initial conditions of the universe. In this paper I shall seek to do so using hyperbolic geometry this by considering semi-classical tunneling models in which the universe is "born from nothing". This is a situation which resembles rather closely the case of pair production and so one may adopt similar methods. The spatial sections \( \Sigma \) of the Lorentzian spacetimes \( M_L \) that I shall consider are hyperbolic 3-manifolds. These are not simply connected and the fundamental group \( \pi_1(\Sigma) \) contains elements of infinite order (the first Betti-number \( b_1(\Sigma) > 0 \)). Thus in a sense one may speak of the creation of wormholes. However it should be stressed that we are not considering connected sums of copies of \( S^1 \times S^2 \) so probably one shouldn’t think of these as Wheeler type wormholes, for which there is evidence that they have an associated entropy. We shall be concerned, in part, with the question of whether this other type of wormhole has an associated entropy.
**Real Tunnelling Geometries**

Current models of the quantum origin of the universe begin with a “real tunneling geometry” [2], that is a solution of the classical Einstein equations which consists of a Riemannian manifold $M_R$ and Lorentzian manifold $M_L$ joined across a totally geodesic spacelike surface $\Sigma$. Each connected component $\Sigma^i$ of the surface $\Sigma$ serves both as a Cauchy surface for a totally disjoint Lorentzian universe $M_L^i$ and a connected component $\partial M_R^i$ of the boundary $\partial M_R$ of the Riemannian manifold $M_R$. In cosmology $\Sigma$ is taken to be closed (that is compact without boundary) and in accordance with the No Boundary Proposal [3] one usually takes the Riemannian manifold $M_R$ to be connected, orientable and compact with sole boundary $\Sigma$.

One sometimes sees semi-classical calculations of topology changing amplitudes with a boundary which is not totally geodesic. For example one might remove a number of solid 4-balls out of flat 4-torus $T^4$ or a round four sphere $S^4$. In the former case the euclidean action $I_{\text{euc}}$ comes entirely from the boundary term and is given by

$$I_{\text{euc}} = \pm \frac{3\pi}{4} R^2$$

where the minus sign gives the action of the 4-ball $B^4$ of radius $R$ and the plus sign gives the action of $T^4 - B^4$ the 4-torus with a 4-ball removed. Removing more 4-balls will evidently increase the action. Consistent with this Carlip [4] has shown that the wave function, considered as a function of both the metric and the second fundamental form is stationary if second fundamental form vanishes. One might anticipate that taking a non-vanishing second fundamental form would lead to a higher classical euclidean action and hence a lower probability. This is certainly consistent with the examples above. Thus in what follows I shall assume that the dominant contributions do indeed come from manifolds with totally geodesic boundaries. I will comment on the physical significance of having more than one boundary component in a later section.

Given this set up one one may pass to the double $2M_R = M_R^+ \cup M_R^-$ by joining two copies of $M_R$ across $\Sigma$. This is a closed orientable Riemannian manifold admitting a reflection map, that is an orientation reversing involution, $\theta$ say which fixes the totally geodesic submanifold $\Sigma$ and permutes the two portions $M_R^\pm$. The involution $\theta$ plays a crucial role in the quantum theory because it allows one to formulate the requirement of “Reflection Positivity” (see [5] for details).
Conversely if one is interested in finding a compact Riemannian manifold $M_R$ with totally geodesic boundary one may start with given a closed orientable Riemannian manifold $M$ and ask whether it admits an orientation reversing involution $\theta$ which fixes an embedded hypersurface $\Sigma_\theta$ (i.e. one without self-intersections). If so then the hypersurface is necessarily two sided and totally geodesic. One may then cut the manifold along $\Sigma_\theta$. There are now two possibilities. If $\Sigma_\theta$ separates $M$ then it will, as it were, fall into two disjoint isometric pieces $M^\pm_R$. This happens in all the cases considered in [2] including the archetypal case when $M$ is the round metric on $S^4$ and $\theta$ is reflection in an equator.

If, on the other hand, $\Sigma_\theta$ does not separate then cutting $M$ along $\Sigma_\theta$ will result in a single connected manifold $M_R$ whose boundary $\partial M_R$ is totally geodesic and consists of two disjoint copies $\Sigma^\pm_\theta$ of $\Sigma_\theta$. In this case the involution $\theta$ will act on $M_R$ permuting the two portions of the boundary $\Sigma^\pm_\theta$. One may of course now join two copies of $M_R$ together across $\partial M = \Sigma^+_\theta \sqcup \Sigma^-_\theta$ to obtain the closed double $M' = 2M_R$ on which some other involution $\theta'$ acts. Clearly $M'$ is a double cover of the original closed manifold $M$. In the case that $\Sigma_\theta$ fails to separate the boundary $\partial M_R$ is never connected even though the fixed point set $\Sigma_\theta$ may be connected. The construction we have just given with its two possible variants really only requires a two sided totally geodesic hypersurface $\Sigma$. It need not necessarily be the fixed point set of an involution. Given $\Sigma$ we may always cut the manifold $M$ along it. However finding a totally geodesic hypersurface may be quite hard. The easiest way to do so in practice is to look for the fixed point set of an involution.

Note that there is a connection between the failure of $\Sigma$ to separate and the topology of $M$ [6]. If a two-sided hypersurface $\Sigma$, totally geodesic or not, fails to separate then it cannot bound and thus it represents an non-trivial homology class in $H^{n-1}(M; \mathbb{R})$, where $n$ is the dimension of $M$. It follows from Hodge duality that the first Betti number $b_1(M)$ of $M$ cannot vanish. It is well known that if the Ricci-tensor of $M$ is non-negative then the first Betti number must vanish. This fact was used in [2] to argue that in this case any boundary must be connected: the birth of disjoint Lorentzian universes is not allowed. Put another way: if the Ricci tensor is non-negative then the assumption made in the No Boundary Proposal that there is only one boundary is redundant: it follows from the compactness of $M_R$. It was also pointed out that if the Ricci tensor is not non-negative then it is easy to find examples with two boundary components. In the explicit examples
considered in [2] the failure of $\Sigma$ to separate was not encountered because they had positive cosmological constant. I shall comment later on the possible significance of manifolds with more than one boundary component.

Perhaps because a closed Riemannian manifold with negative Ricci curvature cannot admit a Killing vector field there are few explicitly known examples. The simplest case to consider the case when the metric is of constant curvature. This gives rise to a locally isotropic F-R-W “ $k = -1$ ” universe after tunnelling and so is of obvious cosmological interest. A closed 4-manifold of constant negative curvature, also referred to as a hyperbolic metric is of the form $H^4/\Gamma$ where $\Gamma \subset O(4,1)$ acts properly discontinuously and has no parabolic elements. As we shall see in more detail later, it is a theorem [7] that any Einstein metric on the same manifold must also be of constant negative curvature so the instantons are unique. This should be contrasted with the more frequently studied case of the 4-sphere. It is not known whether it admits Einstein metrics other than the round one. However the proofs of the cosmic no hair cosmic theorem [8] indicate that the round metric is the only Einstein metric with a hypersurface orthogonal circle action.

To date the only explicit attempts to construct tunnelling geometries known to me have been due Ding, Maeda and Siino [9]. They glued together 12 eight-cells (4-polytopes bounded by 8 congruent hexahedra) to obtain a non-compact hyperbolic manifold $M_{D-M-S}$ of finite volume with 16 totally geodesic boundary hyperbolic components. However the manifold $M_{D-M-S}$ is non-compact and has cusps. The boundary $\partial M_{D-M-S}$ is also non-compact. They also discuss a similar construction with 16-cells and 24-cells.

One might wonder whether the cusps are essential. One knows that Anti-De Sitter spacetime is semi-classically stable [10]. This may be proved using the fact that this spacetime is supersymmetric, admitting Killing spinors [10]. One might think that this would rule out the spontaneous creation of closed universes, without cusps. However one the identifications needed to produce a closed universe are presumably incompatible with the existence of Killing spinors and so perhaps. This is equivalent to asking whether the No Boundary proposal is compatible with a negative cosmological constant. supersymmetry is not relevant in this situation. One might also ask whether the creation of a single universe is possible. This is equivalent to asking whether the No Boundary proposal is compatible with a negative cosmological constant.
New Examples

To answer these questions one needs to examine more examples. One without without cusps is provided by taking $M = M_{\text{Davis}}$ where $M_{\text{Davis}}$ is a compact orientable hyperbolic manifold which is obtained by suitably identifying the 120 dodecahedric faces of a certain hyperbolic Coxeter 4-polytope $X^4 \subset H^4$ [11]. One has

$$M_{\text{Davis}} = H^4 / K$$

where $K \subset G_4$ and $G_4$ is the Coxeter group generated by reflections in the faces of $X^4$. Translating $X^4$ under the action of $K$ gives a tesselation of $H^4$ by identical regular polytopes – a so-called non-euclidean honeycomb. The group $K$ is a subgroup of $G_4$ which acts freely on $H^4$ and which is generated by reflections which identify opposite faces of $X^4$.

In what follows we reproduce Davis’s description, adhering to his notation. Basically ” it’s all done by mirrors ”. We may think if we wish of $H^4$ as the mass shell or future spacelike hyperboloid $Q$ in five-dimensional Minkowski spacetime $\mathbb{R}^{4,1}$. Timelike hyperplanes intersect $H^4$ in hyperbolic planes and each such plane is totally geodesic. A reflection is a reflection in a timelike hyperplane and is contained in $O_+(4,1)$ the group of time-orientation preserving Lorentz transformations in $\mathbb{R}^{4,1}$. The faces $D$ of the polytope $X^4$ are of course planar. Let $r_D$ be reflection in the face $D$. These reflections generate $G_4$. The polytope $X^4$ is centro-symmetric so join the centre of the face $D$ to the centre $x_4$ of the polytope by a geodesic and let $s_D$ be reflection across the orthogonal hyperplane through $x_4$. Clearly $s_D$ takes $D$ to the opposite face $-D$. Let $t_D = r_Ds_D$. Now $s_D \in G_3 \subset G_4$ where $G_3$ is the stabilizer of the origin $x_4$. Thus $t_D$ also takes $D$ to its opposite face and belongs to $G_4$. The group it generates is $K$. Acting on $X^4$, $t_D$ takes it to the polytope adjacent to $D$ in the tesselation. Davis shows that $G_4$ is the semi-direct product of $G_3$ and $K$ and that $K$ acts freely on $H^4$. All elements of $K$, being the products of an even number of reflections, preserve orientation and so the quotient $M_{\text{Davis}} = H^4 / K$ is orientable. Since $K$ is a normal subgroup of $G_4$ the quotient $G_3 = G_4 / K$ acts on $M_{\text{Davis}}$. Thus $s_D$ is an orientation reversing isometry of $M_{\text{Davis}}$ which fixes a connected totally geodesic 2-sided hypersurface called $M^3$ by Davis and $\Sigma$ here. Moreover, as he points out, the the complement of $\Sigma$ in $M_{\text{Davis}}$ is obviously connected, or in other words $\Sigma$ does not separate. In fact $1 \leq b_1(M_{\text{Davis}}) \leq 60$. The Euler characteristic is 26. The second Betti number $b_2(M_{\text{Davis}}) = 2(12 + b_1(M_{\text{Davis}}))$ is therefore even,
which is consistent with the fact that the Hirzebruch signature \( \tau = b_2^+ - b_2^- \) must vanish because the involution \( s_D \) will pull back self-dual harmonic forms to anti-self-dual harmonic forms.

The upshot of all of this is that cutting \( M_{\text{Davis}} \) along \( \Sigma \) will give a manifold with a totally geodesic boundary with two connected components.

Ratcliffe and Tschantz have given examples of non-compact hyperbolic 4-manifolds of finite volume [12]. We shall refer to the simplest example as \( M_{\text{Ratcliffe-Tschantz}} \). It is obtained by identifying the faces of a 24-cell. The vertices of this polytope lie on the absolute at infinity. They correspond to the following 24 lightlike vectors in \( \mathbb{R}^{4,1} \): 

\[
(\pm 1,0,0,0,1), (0,\pm 1,0,0,1), (0,0,\pm 1,0,1), (0,0,0,\pm 1,1), \\
(\pm 1/2,\pm 1/2,\pm 1/2,\pm 1/2,1).
\]

The polytope is invariant under \( O_\uparrow(3,1;\mathbb{Z}) \) the group of integer valued Lorentz transformations preserving the time orientation. The congruence 2 subgroup \( \Gamma \subset O_\uparrow(3,1;\mathbb{Z}) \) consisting of integer valued Lorentz transformations congruent modulo 2 to the identity is torsion free (i.e. has no subgroups of finite order) and thus acts freely on \( H^4 \). One has

\[
M_{\text{Ratcliffe-Tschantz}} = H^4/\Gamma
\]

Evidently the 24-cell is invariant under the reflection \( \theta : \mathbb{R}^{4,1} \to \mathbb{R}^{4,1} \) sending \((X^1, X^2, X^3, X^4, X^0, ) \) to \((-X^1, X^2, X^3, X^4, X^0, ) \). The reflection \( \theta \) normalizes \( \Gamma \) in \( O_\uparrow(3,1;\mathbb{Z}) \) and therefore descends to the quotient \( H^4/\Gamma \). Clearly \( \theta \) fixes a connected totally geodesic submanifold \( \Sigma_\theta \) in \( M_{\text{Ratcliffe-Tschantz}} \) which does not separate. Cutting \( M_{\text{Ratcliffe-Tschantz}} \) along \( \Sigma_\theta \) therefore yields a manifold \( M_R \) with two boundary components.
Topological and Volumetric Entropies and The Einstein Action

Formally one may attempt to evaluate the functional integral over all Riemannian metrics on closed 4-manifolds in Euclidean quantum gravity is dividing the metrics into conformal equivalence classes. In each equivalence class find a representative with with constant Ricci scalar. The integral is then split into an integral over the conformal deformations of that representative and an integral over conformal equivalence classes. The integral over conformal deformations must be treated differently because the Euclidean action:

\[ I_{eu} = -\frac{1}{16\pi} \int_M \sqrt{|g|} d^4x \left( R - 2\Lambda \right) \]

in those directions is not bounded below [13]. If the Ricci scalar \( R \) is scaled to take the constant value \( 4\Lambda \) the euclidean action is proportional to its volume \( V = \text{vol}(M, g) \):

\[ I_{eu} = -\frac{1}{8\pi} \Lambda V. \]

If the cosmological constant is positive then it is not excluded that \( 2M_L \) is locally static, with a Killing horizon of total area \( A_C \). If all connected components of the Killing horizons have the same surface gravity \( \kappa \) one may analytically continue to obtain a closed Einstein manifold admitting a reversible circle action. It then it follows form the Einstein equations that

\[ \frac{1}{8\pi} \Lambda V = \frac{1}{4} A_C. \]

The only two known cases known \( S^4 \) with \( \Sigma \equiv S^3 \) and one component with area

\[ A_C = \frac{12\pi}{\Lambda} \]

and \( S^2 \times S^2 \) with \( \Sigma \equiv S^1 \times S^2 \) and two equal components with total area

\[ A_C = \frac{8\pi}{\Lambda}. \]

If the cosmological constant is negative then no static Lorentzian Einstein metric can be analytically continued to give a closed Riemannian manifold and so Hamiltonian methods cannot be used in a straightforward way to relate the the action to
the gravitational entropy*. However for hyperbolic manifolds (i.e. those admitting a metric, call it $g_0$, of constant negative curvature) the 4-volume and hence the action is known to be related to the topological entropy $h_{\text{top}}(g)$ of the geodesic flow on the unit tangent bundle. This suggests that there might be some connection between topological entropy and gravitational entropy.

Recall [7] that the definition of $h_{\text{top}}(g)$ is

$$h_{\text{top}}(g) = \lim_{L \to +\infty} \frac{1}{L} \log \left( \# \{ \gamma : l_g(\gamma) \leq L \} \right)$$

where $l_g(\gamma)$ is the length of the periodic geodesic $\gamma$ with respect to the metric $g$. One may also define a volumetric entropy $h_{\text{vol}}(g)$ by

$$h_{\text{vol}}(g) = \lim_{L \to +\infty} \frac{1}{L} \log (\text{vol}(B(x, L)))$$

where $\text{vol}(B(x, L))$ is the volume, with respect to the metric $g$ of a ball of radius $L$ centred at the point $x$ in the universal covering space $\tilde{M}$ of the manifold $M$.

In other parts of physics or mathematics one thinks of entropy as a convex function on a space $S$ of mixed "states". For this to make sense the space of states $S$ must be a convex set. This is certainly true for a classical probability distribution on a finite set when $S$ a simplex, and for its quantum mechanical generalization, the set of density matrices for a finite dimensional Hilbert space. The set of Riemannian metrics $\text{Riem}(M)$ a compact manifold, unlike the set of Lorentzian metrics is certainly a convex set. * It is therefore very striking that Robert [14] has shown that the volumetric entropy is a convex function of the metric $g$.

It is known that topological entropy is always smaller than volumetric entropy:

$$h_{\text{vol}}(g) \leq h_{\text{top}}(g)$$

and that if $g$ has negative curvature then $h_{\text{vol}}(g) = h_{\text{top}}(g)$. In the case of a hyperbolic metric $g_0$ one has

$$h_{\text{vol}}(g_0) = h_{\text{top}}(g_0) = \sqrt{-3\Lambda}.$$  

* We shall describe in more detail the Lorentzian sections of hyperbolic manifolds in more detail in a later section.

* Of course this will not remain true once we have taken the quotient by the action of the diffeomorphism group $\text{Diff}(M)$.
Note that unlike gravitational entropy which has the dimensions of area these "entropies" have the dimensions of inverse length. Moreover Anti-De-Sitter spacetime has no gravitational entropy. This should be contrasted with De-Sitter space which in the guise of the 4-sphere has gravitational entropy but has no topological or volumetric entropy. Thus it seems, superficially at least, that these two concepts of entropy are physically unrelated. On the other hand, mathematically, gravitational entropy is related to the gravitational action. In fact topological and volumetric entropy are also related to the volume of the manifold and thus all three entropies are therefore related to the action. The connection between event horizons and hyperbolic geometry will be expanded upon in a later section.

To see this connection in more detail recall from [15] that if the Ricci curvature has a positive lower bound:

\[ R_{\alpha\beta}v^\alpha v^\beta \geq |\Lambda|g_{\alpha\beta}v^\alpha v^\beta \]

then Bishop’s theorem tells us that

\[ \text{vol}(M,g) \leq \frac{24\pi^2}{\Lambda^2} \]

with equality if and only if \( g \) is the round metric on \( S^4 \). Thus the round 4-sphere has the largest volume and hence the lowest action among all metrics with positive cosmological constant.

On the other hand if one has a negative lower bound for the Ricci curvature:

\[ R_{\alpha\beta}v^\alpha v^\beta \geq -|\Lambda|g_{\alpha\beta}v^\alpha v^\beta \]

then one may apply Bishop’s comparison theorem to a ball in the universal cover \( \tilde{M} \) to obtain an upper bound for the volumetric entropy and hence an upper bound for the topological entropy:

\[ h_{\text{top}}(g) \leq h_{\text{vol}} \leq \sqrt{-3\Lambda}. \]

Moreover if a closed 4-manifold \( M \) admits a metric \( g_0 \) of constant negative curvature, and if \( g \) is any other metric on \( M \). One has [7]

\[ \text{vol}(M,g)h_{\text{vol}}(g)^4 \geq \text{vol}(M,g_0)h_{\text{vol}}(g_0)^4. \]
If the Ricci-curvature has a negative lower bound it follows that the volume is always greater than that of the hyperbolic metric on $M$:

$$\text{vol}(M, g) \geq \text{vol}(M, g_0).$$

Now if the metric $g$ is an Einstein metric the Gauss-Bonnet theorem tells us that the volume $V = \text{vol}(M, g_0)$ is given in terms of the Euler characteristic $e(M)$ by

$$V = \frac{12}{\Lambda^2} \pi^2 e(M) - \frac{1}{32\pi^2} \int_M C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \sqrt{g} d^4x,$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor.

Thus for a hyperbolic metric the volume

$$V = \frac{12}{\Lambda^2} \pi^2 e(M).$$

Moreover it follows that for any other Einstein metric on $M$ with the same cosmological constant that it’s volume is bounded above and below by the same value. This can only happen if the Weyl tensor vanishes and hence it must have constant curvature. This is the uniqueness theorem of Besson et al. referred to earlier.

The conclusion is that hyperbolic metrics have the largest volumetric and topological entropy among all metrics on the same manifold and among metrics on the same manifold with Ricci-curvatures having a negative lower bound they have the least volume. Finally and most importantly physically: among all metrics with constant Ricci scalar $R = 4\Lambda$ having a negative lower bound for the Ricci-curvature the hyperbolic metric has the least action. Note that hyperbolic metrics are locally homogenous but not globally so. In this respect at least, it seems reasonable to think of them as having high entropy.

The relationship between euclidean action and volume given above follows directly and straightforwardly if one assumes that the cosmological constant is a fixed constant which remains constant under Wick rotation. If, however, the cosmological constant is related by duality to the vacuum expectation value of a closed four-form $F_{\alpha\beta\gamma\delta}$ the derivation is more subtle. Thus if we add to the Lorentzian action a term
\[-\frac{1}{48} \int d^4 x F_{\alpha \beta \gamma \delta} F^{\alpha \beta \gamma \delta} \sqrt{-g}.

The sign in the action is chosen so that \( F \) contributes positively to the energy. It is the sign which would arise naturally if it were a dynamical four-form in higher dimensions. Thus if \( c \) is a constant, \( \eta_{\alpha \beta \gamma \delta} \) the covariantly constant volume form on necessarily orientable manifold and

\[ F_{\alpha \beta \gamma \delta} = c \eta_{\alpha \beta \gamma \delta}, \]

then the Lorentzian field equations obtained by varying with respect to the metric \( g_{\alpha \beta} \) contain a positive cosmological term with

\[ \Lambda = \frac{1}{4} c^2. \]

That is they have as a solution De-Sitter spacetime. On the other hand if a term

\[ + \frac{1}{48} \int d^4 x \sqrt{g} F_{\alpha \beta \gamma \delta} F^{\alpha \beta \gamma \delta} \]

is added to the Riemannian action the field equations contain a negative cosmological term, i.e. would have \( H^4 \) as a solution with \( F_{\alpha \beta \gamma \delta} \) real. It may ultimately be significant that if the manifold one is working is non-orientable then one cannot induce a cosmological term in this way.

The Riemannian solution may be obtained from the Lorentzian one by making both the time and the integration constant \( c \) pure imaginary. On the other hand just analytically continuing in the time would give a purely imaginary four-form and hence \( S^4 \) as a solution. The question then arises: what is the correct instanton solution and what is Euclidean action of that solution? A similar question arises when considering the action of electrically charged black holes. In that case experience with black hole thermodynamics supports the idea that the action should be evaluated for a purely imaginary electric field on the Riemannian section. This procedure may be justified \textit{a priori} in that case because one wants to evaluate a partition function at fixed real chemical potential. An \textit{a posteriori} justification is that the results so obtained are consistent with electric-magnetic duality.
In the cosmological context Hawking in an attempt to explain the smallness of the cosmological constant took the $S^4$ solution and evaluated its action [16]. Thus amounts to following the electromagnetic case and allowing a purely imaginary four-form. It would preclude using hyperbolic metrics. This procedure was later criticised by Duff [17]. If one follows the Duff procedure one finds that that the Euclidean volume for a hyperbolic manifold with a real four-form would be proportional to the volume with a negative proportionality constant. Since there exist hyperbolic manifolds with arbitrarily large 4-volume this procedure would appear to lead to the unsatisfactory result that these are not suppressed in the path integral. Of course the conclusions above depend upon choosing the negative sign in the Lorentzian action of the four-form. There appear to be good reasons for this in the context of superstring and hence supergravity theory but if the signs are reversed the conclusions above would be reversed.

In the saddle point approximation one usually only considers the classical action of the saddle point and compares the actions of different saddle points. In the case of hyperbolic manifolds the discussion above shows that this reduces to comparing their Euler characteristics.

The closed hyperbolic manifold with the the lowest known Euler characteristic is the Davis manifold [11] for which $e(M_{\text{Davis}}) = 26$. Thus because the totally hyperbolic hypersurface does not separate the action of the Davis instanton is therefore

$$I_{\text{euc}}(M_{\text{Davis}}) = \frac{39\pi}{|\Lambda|}.$$

For hyperbolic manifolds with cusps the volume is given by the same formulae. It is known that all positive integer values occur. The lowest volume and hence presumably lowest action manifold with cusps is the example of Ratcliffe and Tschantz [12] obtained by identifying the faces of a 24-cell. This has $e(M_{\text{Ratcliffe-Tschantz}}) = 1$.

$$M_{\text{Ratcliffe-Tschantz}} = H^4 / \Gamma$$

Evidently the 24-cell is invariant under reflection $\theta : \mathbb{R}^{4,1} \to \mathbb{R}^{4,1}$ sending $(X^1, X^2, X^3, X^4, X^0,)$ to $(-X^1, X^2, X^3, X^4, X^0, )$. The reflection $\theta$ normalizes $\Gamma$ in $O_4(3, 1; \mathbb{Z})$ and therefore descends to the quotient $H^4 / \Gamma$. Clearly $\theta$ fixes a connected totally geodesic submanifold $\Sigma_\theta$ in $M_{\text{Ratcliffe-Tschantz}}$ which does not separate. Cutting $M_{\text{Ratcliffe-Tschantz}}$ along $\Sigma_\theta$ therefore yields a manifold $M_R$ with two boundary components.
The volumes of the three solutions calculated numerically by Ding, Saeda and Siino [9] corresponding to 12 8-cells, 4 16 cells and 6 24 cells give effective Euler characteristics of 6.2017219, 2.6666776 and 26.993285 respectively. The fact they are not integral is puzzling. One would expect the last number to equal 6.

The example constructed from Ratcliffe-Tschantz manifold above definitely has lower in action (assuming that cusps do not contribute), since it is built up from just one 24-cell.

**Product Examples**

Real tunneling solutions of the Einstein equations 4-manifolds with negative cosmological constant $\Lambda$ may also be obtained by taking the metric product of a closed 2-dimensional manifold of genus $g$ with constant curvature $-\frac{1}{|\Lambda|}$ with a compact 2-dimensional manifold with constant curvature $-\frac{1}{|\Lambda|}$ with a geodesic boundary. These metrics after tunneling give homogeneous but anisotropic cosmological models of the form of products of two-dimensional anti-de-Sitter spacetime with a closed 2-dimensional manifold of genus $g$ with constant negative curvature. In general product metrics of this type are relevant to the possibility of spontaneous compactification. In the present case one has in mind compactification form 4 to 2 spacetime dimensions.

Using the Gauss-Bonnet theorem one finds [18] that the euclidean action is given in terms of the Euler characteristic and the cosmological constant by

$$I_{euc} = \frac{\pi}{2|\Lambda|} \epsilon(M),$$

where the expression for the Euler number is the product of the Euler numbers of the factors. Thus

$$I_{euc} = \frac{2\pi}{|\Lambda|} (g - 1)(g_{\text{eff}} - 1),$$

and where the area $A$ of the 2-manifold with boundary is given by

$$A = \frac{2\pi(g_{\text{eff}} - 1)}{|\Lambda|}.$$

The action of a product is smaller by a factor 3 than for a hyperbolic manifold with the same cosmological constant and a Euler characteristic. This is curious
because, naively at least it indicates that anisotropic universes should be formed with higher probability than anisotropic universes.

The lowest action case is when both factor manifolds have the lowest possible genus. Thus set $g = 2$ and think of a pretzel as a suitably sized regular octagon in the hyperbolic plane with opposite edges identified in the opposite sense. One may cut the pretzel along a geodesic joining the mid-points of a pair of opposite edges. This geodesic will not separate and so $g_{\text{eff}} = 2$, and hence

$$I_{\text{euc}} = \frac{2\pi}{|\Lambda|}.$$

If one takes a separating geodesic the action would be at least halved. In either case it is much less than the Davis example. This is because the latter has such a high Euler characteristic.

**Disconnected Boundaries and Density Matrices**

In this section I wish to discuss the physical significance of more than one boundary component. If the components are not isometric then the obvious interpretation is that they give tunneling amplitudes between different three-manifolds. From the Riemannian point of view it is not obvious which components are to be taken to lie in the future and which to lie in the past.

If we can identify some of the boundary components however a different interpretation is possible, as pointed out in a slightly different context by Hawking and Page [19]. Suppose for simplicity we have two isometric boundary components. We may glue the four manifold together across them to obtain a closed manifold containing a totally geodesic hypersurface $\Sigma$ which does not separate. This is the case for the Davis manifold $M_{\text{Davis}}$ for example.

Hawking and Page suggested that one now focus on the “probability $\text{Prob}(\Sigma)$ for the occurrence of the Riemannian 3-manifold $\Sigma$”. This may be expressed as a functional integral over all metrics on all closed 4-manifolds containing $\Sigma$:

$$\text{Prob}(\Sigma) = \sum_{M} \sum_{g} \exp -I_{\text{euc}}(g).$$

The sum decomposes into a sum of terms of the following three kinds:

1. Manifolds $M$ which are separated into two diffeomorphic halves $M_{\pm}$.
ii Manifolds for which $\Sigma$ separates $M$ into two halves $M_1$ and $M_2$ which are not diffeomorphic.

iii Manifolds for which $\Sigma$ does not separate.

Terms of the first and second kind have an obvious interpretation in terms of the Hartle-Hawking type pure state $\Psi_{\text{Hartle-Hawking}}(\Sigma)$

$$\Psi_{\text{Hartle-Hawking}}(\Sigma) = \sum_{M=\partial \Sigma} \sum_{g} \exp - I_{\text{euc}}(g).$$

If one might thinks of the then as the diagonal element of a factorized density matrix $\rho_{\text{H-H}}$:

$$\rho_{\text{Hartle-Hawking}} = \Psi_{\text{Hartle-Hartle}}(\Sigma) \otimes \Psi_{\text{Hartle-Hawking}}(\Sigma)$$

then the remaining terms are a measure of the extent to which the "density matrix of the universe $\Sigma"$ fails to factorize.

It is clearly tempting to argue that the Davis manifold $M_{\text{Davis}}$ represents a semi-classical contribution to the non-factorizable part of the density matrix of the universe in a theory with a negative cosmological constant. It makes even more interesting the question of whether there are hyperbolic 4-manifolds with a single connected boundary.

Another viewpoint is that if we only are interested in a connected 3-manifold $\Sigma$ we should include in the functional integral all manfold which bound $\Sigma$ regardless of whether they have other boundary components or not. Then we have to sum over the 3-metrics on the other boundary components. Presumably this gives a mixed state for the universe.

This is similar to the use of the formalism of density matrices applied to case of a connected boundary in the case of spaces with event horizons, as has been done recently by Barvinsky, Frolov and Zelnikov [20]. Consider the Schwarzshild case. The boundary, for which $\Sigma$ has the topology of an Einstein-Rosen bridge $S^2 \times \mathbb{R}$, is given by values of the imaginary Killing time $\tau = 0$ and $\tau = 4\pi$. These give the two halves $\Sigma_{\pm} \equiv S^2 \times \mathbb{R}_{\pm}$ of the bridge on either side of the throat $r = 2M$. If one is not interested in what happens on one side $\Sigma_-$ of the horizon one should sum over all metrics on $\Sigma_-$. 

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Hyperbolic Geometry and Event Horizons

In this section I would like to explore in more detail the possible relationship between hyperbolic geometry and gravitational entropy. We saw above that any relationship between topological entropy, volumetric entropy and gravitational entropy is at best an indirect one. Nevertheless there does appear to be a common thread: the fact that if the curvature is negative then geodesics diverge exponentially fast and the the volume of a ball increases exponentially with the radius. If the manifold $M$ is compact or possibly, as in the case of the fundamental domain of the modular group $H^2/SL(2;\mathbb{Z})$, merely of finite volume it is well known that this exponential divergence leads to ergodic behaviour of the geodesics. The topological and volumetric entropies $h_{\text{top}}(M)$ and $h_{\text{vol}}(M)$ were originally introduced to make the relation more quantitative.

Consider now a static spacetime $2M_L$ with an event horizon. The Lorentzian metric takes the form

$$ds^2 = -V^2dt^2 + g_{ij}dx^idx^j$$

where the positive function $V$ and the 3-metric $g_{ij}$ on the spatial section $\Sigma$ depend only on the spatial variables $x^i$. For simplicity we assume that the event horizon is non-degenerate and has single connected component. The generalization of the following remarks to more than one component is straight forward. Near the event horizon the function $V$ tends to zero on some 2-dimensional totally geodesic submanifold $B$ of the 3-manifold $\Sigma$:

$$V \to \kappa^2s^2 + \ldots$$

where $s$ is proper distance from $B$ along $\Sigma$ with respect to the metric $g_{ij}$, and $\kappa$, the surface gravity is constant over $B$. Let us introduce the optical or Fermat metric:

$$ds_0^2 = f_{ij}dx^idx^j = V^{-2}g_{ij}dx^idx^j.$$ 

Clearly with respect to the optical metric the horizon $B$ is at infinite distance. If $B$ has the intrinsic geometry of a 2-sphere then the optical metric approaches the standard hyperbolic metric on $H^3$ with radius equal to $\frac{1}{\kappa}$. If the original spacetime is De-Sitter spacetime then the optical metric is exactly that of hyperbolic three-space. Even if the the intrinsic geometry of the horizon $B$ is not exactly spherically symmetric, locally as one approaches the horizon, the optical geometry approaches,
exponentially fast with respect to optical distance, the geometry near infinity in hyperbolic 3-space.

This universal feature of the optical geometry of event horizons (which has been noticed before by many people) is very striking and it is tempting to try to relate it to the thermodynamic properties of event horizons. Clearly there is no direct relationship between horizons and ergodicity because there is no question of making identifications of the optical manifold to render it compact or of finite volume.

What one can say however is that the classical loss of information about sources which approach the event horizon which is the subject of the classical No-Hair theorems may be seen in this language as the loss of information about sources which recede to infinity in hyperbolic space. The hyperbolic geometry leads to an exponential decrease in the multipole moments observed at finite points of hyperbolic space. This is a simple consequence of the exponential divergence of geodesics. In other words looking at black holes is rather like doing astronomy in a static hyperbolic spacetime. Since the classical No Hair theorems are rather well understood in conventional terms it does not seem worthwhile here translating them in detail, line for line, into the language of hyperbolic geometry but it is clear that this could be done.

At the quantum mechanical level the exponential increase of the optical volume as one approaches infinity is closely related to the fact the taking into account the thermal corrections to the classical entropy of a black hole in equilibrium at its Hawking temperature $T = \frac{\kappa}{2\pi}$ gives rise to a an infinite contribution corresponding to a gas of massless particles in equilibrium at the Hawking temperature. Different authors have attached different significance to this fact. The large number of states near the horizon is also believed by some to account for the loss of information during gravitational collapse. This large number of states is of course directly related to the infinite optical volume.

Thus again it seems that there is probably no deep connection between the entropies used in hyperbolic geometry and the gravitational entropy of event horizons. The simple underlying geometrical reason why both concepts are useful is the exponential divergence of geodesics and of volumes. There does not seem however to be a physical connection and certainly it does not seem possible to identify these entropies physically.
In the next section I shall suggest that it may be more useful to think of $h_{\text{top}}(M)$ as a measure of complexity rather than entropy.

**Entropy Action and Complexity**

One often thinks, intuitively at least, that the increase of entropy of an isolated macroscopic system is associated with an increase of disorder. One has in mind the fact that if it is isolated an initially complex system almost always evolve into a much simpler system, the time reverse is rather improbable. It is tempting therefore to attempt to relate entropy to some measure of the complexity of a system so that systems with the largest entropy have the least complexity. One feature of order complexity or order is spatial inhomogeneity. Thus any definition of complexity should presumably have the property that it is low for homogeneous systems. Similarly one expects macroscopic systems with the largest entropy to be spatially homogeneous, and this is certainly true for ordinary systems of particles under the influence of short range forces. It is not always true in the presence of gravity which is a long range field because it tends to favour inhomogeneous systems such as stars. For this reason it is sometimes felt desirable to include a contribution to the total entropy of a macroscopic system due to gravity which would reflect the tendency towards inhomogeneity.

There are many problems with connecting in any precise sense entropy and complexity. Firstly one needs a quantitative measure of complexity. One such measure is Kolmogorov's algorithmic complexity. The is related to the shortest computer programme required to specify the system. More generally one might hope to quantify the amount of information or data need to specify the system. Indeed many people identify entropy with information although in the case of black hole this often seems to lead to more confusion than enlightenment, not least because of a failure to specify what is meant by information. If one has in mind a probability distribution, or in quantum mechanics a density matrix then indeed ordinary thermodynamic or Gibbs entropy $S_G$

\[ S_G = -\text{Tr}\rho \log \rho / \text{Tr}\rho \]

and the Shannon information $H_S$ gained if one discovers precisely what state we are in are effectively the same thing. For complex systems on the other hand one may need to give a great deal of information or data to specify them. An ensemble of
systems, each of which is individually complex may therefore have a large amount of information carrying capacity. Thus two polarization states of a gravitational wave have more information carrying capacity than one and the greater the bandwidth of the gravitational waves one considers the greater the information carrying capacity. Thus the maximum entropy of an ensemble of complex systems should be large.

The difficulties in making the idea of gravitational entropy in general precise are well known. They include the problem that thermodynamic entropy is usually associated not with a single system but a class or ensemble of systems. In the case of a spacetime with an horizon the ensemble is often thought of in some sense as all spacetimes which are identical outside the event horizon. It is thus reasonable to attempt to identify $S_G$ with $S_{\text{Bekenstein-Hawking}}$ or $S_C$.

A gravitational wave on the other hand, provided its amplitude, phase and polarization state is known should presumably have no entropy associated with it. Quantum-mechanically one thinks of it as a single coherent state not a density matrix. By extension one would anticipate that a general gravitational field without an event horizon should not possess thermodynamic entropy. Nevertheless such a field may be very complex. It seems reasonable therefore, if only in the interest of conceptual clarity, to shift ground somewhat and try to consider how one might define the complexity of gravitational fields and then afterwards to see whether complexity is related to other quantities such as the area of event horizons, the Weyl tensor, or the euclidean action. In fact Penrose has tried to relate gravitational entropy to the Weyl tensor for some time and Dzhunushaliev tries to equate Kolmogorov’s algorithmic complexity to the classical euclidean action $I$. It is known that for black holes of area $A$ and Bekenstein-Hawking entropy $S$

$$I_{\text{euc}} = -S_{\text{Bekenstein-Hawking}} = -\frac{1}{4}A.$$  

Let us turn then to the question of how might we define the complexity of a gravitational field. One approach might be to ask roughly speaking how many equations are needed to specify the gravitational field? A homogeneous spacetime $M = G/H$ like De-Sitter spacetime is clearly requires rather little information in this sense. The same is true of the 4-sphere. They also contain little information in a slightly different sense. They do not contain much information about the equations they satisfy. This is because they are stationary points of any local diffeomorphism-invariant action functional constructed from just the metric and its
derivatives. They are examples of what Bleeker [23] has called critical metrics. He showed that if $M, g$ is a closed Riemannian manifold whose metric $g$ is critical then $M = G/H$ where the isotropy subgroup $H$ of the isometry group $G$ acts irreducibly on the tangent space. In four dimensions the only critical metrics are (up to an constant multiple) the round metric on $S^4$ and the Fubini-Study metric on $\mathbb{CP}^2$. Of them, $S^4$ can provide a real tunneling geometry. Thus if we assumed the universe began in a state of least complexity and we took this to mean that the double $2M_R$ was a critical metric we are led to De-Sitter spacetime as the Lorentzian portion $M_L$. Now the 4-sphere has by Bishop’s theorem the least Einstein action among metrics with constant positive scalar curvature and Ricci-curvature bounded below by a non-negative multiple of the metric In particular it has least action among all Einstein metrics with positive scalar curvature. This at least goes in the same direction as Dzhunushaliev. The vanishing of the Weyl tensor is also consistent with Penrose’s idea in this case. However we shall see shortly that if we consider hyperbolic metrics then while Dzhunushaliev’s idea still seems to work there are problems with that of Penrose. 

It seems natural to think of closed hyperbolic 4-manifolds as having higher complexity, however we define it, than $S^4$, the complexity presumably increasing with increasing Euler number. They are certainly not critical metrics in Bleeker’s

* Although there may be global problems with the definition of the action functional, one could apply Bleeker’s idea to Lorentzian metrics. For Lorentzian 4-metrics there is no analogue of $\mathbb{CP}^2$ so one would be left with the maximally symmetric spacetimes among homogeneous examples. However one also acquires an additional case: the Ricci-flat p-p waves. The structure of the curvature tensor means that they satisfy any covariant equations constructed from the metric, Riemann tensor and its covariant derivatives. Moreover any invariant built from the Riemann tensor, including the square of the Weyl tensor, vanishes. The cosmological constant must vanish of course, though a generalization exists for negative cosmological constant. It is rather striking that these metrics are also supersymmetric since they admit a covariantly constant spinor field. As noted above these metrics are quite complex and an ensemble may carry entropy. It seems therefore that for lorentzian metrics the square of the Weyl tensor cannot be taken in a simple unqualified way as a measure of either entropy or complexity. Neither does it say much about criticality in Bleeker’s sense.
sense since although they are locally homogeneous there is no global isometry group. The Einstein action is also larger than that of $S^4$ and increases with Euler number. This agrees in spirit with Dzhunushaliev’s proposal but presents a problem for Penrose’s suggestion because all of these metrics have vanishing Weyl tensor. In this sense at least the Weyl curvature hypothesis would seem to require supplementing in order to render it unambiguous.\footnote{We saw above that the Einstein action of an Einstein metric is proportional to its Euler number. It is perhaps worth pointing out here that for Kähler 4-manifolds with constant Ricci scalar $4\Lambda$ regardless of whether they satisfy the Einstein equations one has
\[
\frac{\Lambda V}{8\pi} \geq \frac{9\pi\tau(M)}{4\Lambda},
\]
where $\tau(M)$ is the Hirzebruch signature of $M$. One has equality for the case of constant holomorphic sectional curvature and the trivial case of $S^2 \times H^2$. If the cosmological constant is positive one is led to $\mathbb{C}P^2$ with the homogeneous Fubini-Study metric for which $\tau(M) = 1$. Thus although they cannot serve as real tunneling geometries, and although we have as yet no way of associating gravitational entropy with them, we see that Kähler metrics are consistent with the general idea that complexity (in this case topological complexity measured by the Hirzebruch signature) and Einstein action increase together and that critical metrics are associated with the smallest possible action.}

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Entropy and Complexity of Initial Data

Rather than thinking of the 4-metric one might prefer to think of the entropy or the complexity of the initial data specifying the spacetime. In the present case this is just the hyperbolic 3-metric induced on $\Sigma = \partial M_R$ with vanishing second fundamental form. One obvious approach to defining the entropy of initial data is to consider the areas of any black hole or cosmological horizons apparent horizons on the hypersurface $\Sigma$. Because $\Sigma$ is totally geodesic these are minimal 2-surfaces lying in $\Sigma$. Black hole horizons correspond to "stable" minimal surfaces, i.e. those whose second variation is non-negative. The known cosmological horizons (for positive $\Lambda$) have just one negative mode (i.e. the Hessian of the second variation is negative on a one-dimensional subspace, or put in another way the Morse index is one.) In the case of static solutions with positive $\Lambda$ the apparent horizons coincide with event horizons and Killing horizons and they are totally geodesic submanifolds. As mentioned above their area is directly related to the action and to gravitational entropy. For non-static time-symmetric initial data it follows from the second variation that the topology of a connected black hole apparent horizons must be that of a 2-sphere and the that its area $A$ is bounded above by

$$A \leq \frac{4\pi}{\Lambda}.$$ 

Again if $\Lambda$ is positive it seems very likely (following unpublished work with S-T Yau, G T Horowitz and S W Hawking ) that the area $A$ of an index one apparent cosmological horizon is bounded by:

$$A \leq \frac{12\pi}{\Lambda}.$$ 

In the case of negative $\Lambda$ one does not expect cosmological horizons and none of the proofs above go through. The topology of stable minimal surfaces does not seem to be restricted to that of a 2-sphere and even if it is, judging by the Schwarzschild-Anti-De-Sitter solution, there is no upper bound to its area. It seems therefore that for negative cosmological constant the area of any apparent horizons is not necessarily and interesting quantity to relate to entropy, action or complexity . Nevertheless not a great deal seems to be known about it. More information would clearly be desirable.
By contrast Hayward and Twamley have suggested, in the context of suggestions that our universe should be spatially closed with hyperbolic sections, that one take the 3-volume $\text{vol}(\Sigma)$ normalized to a radius of unity as a measure of the complexity of the initial data and they expressed the feeling that metrics with high complexity should be less probable than those with low complexity. I shall comment on this point later. Before doing so it may be helpful to recall why the volume $\text{vol}(\Sigma)$ might be regarded as a measure of complexity. It is known from the work of Jørgensen and Thurston that the set of volumes is a well-ordered closed subset of the real line and that the number of closed hyperbolic 3-manifolds with the same volume is finite (but may be arbitrarily large). Moreover Matveev and Fomenko have conjectured [27] that the volume $\text{vol}(\Sigma)$ grows with $d(\Sigma)$, where $d(\Sigma)$ is a topological invariant taking integer values which is additive under connected sum:

$$d(\Sigma_1 \# \Sigma_2) = d(\Sigma_1) + d(\Sigma_2),$$

and which was introduced by Matveev to study 3-manifolds and which he calls complexity. I shall call it the Matveev-invariant. It vanishes for $S^3$, $\mathbb{R}P^3$ and the Lens-space $L_{3,1}$. Matveev’s invariant exceeds 8 for hyperbolic manifolds and takes the value 9 for $Q_1$, the hyperbolic manifold with smallest known volume.

Clearly it would be of great interest to know how the volume $\text{vol}(\partial M)$ of the totally geodesic boundary of a compact hyperbolic 4-manifold $M$ varies with the 4-volume $\text{vol}(M)$. If they increase together one would have some sort of vindication of the idea that Euclidean action and complexity are related.

Of course an alternative viewpoint might be to regard the "entropy" $h_{\text{top}}$ as giving a measure of the complexity of the Riemannian manifold $2M$. Then Dzhuushaliev’s conjecture amounts to relating $h_{\text{top}}$ to Kolmogorov’s algorithmic complexity.
Matter Entropy

The calculations so far have been concerned with a vacuum gravitational field and hence the entropy or complexity has been purely gravitational. In a dynamical situation it is of course only the total entropy which is expected to increase. In cosmology one anticipates a sort of competition between gravity and matter in which the natural tendency of the matter to homogenize, erasing structure and complexity, is offset by the tendency of gravity to produce inhomogeneities and hence complexity by such mechanisms as the Jean’s instability.

The inclusion of matter has three effects. Firstly, and most obviously, assuming that the typical wavelengths of the matter are large compared with the radius of curvature, the local contribution to the entropy from the matter must be included in the total entropy. Secondly, and more subtly, the effect of the matter on the background gravitational field, and in particular the volume of space, must be taken into account. Thirdly, if the matter temperature is very low, non-local Casimir-type effects due to the geometry of spacetime will affect the entropy of the matter. Under this heading I would include effects due to horizons.

A rather simple but illuminating model, which ignores the third effect, is obtained by considering a perfect radiation fluid with pressure equal to one third of the energy density $\rho \ [28]$. As before the cosmological constant $\Lambda$ is considered fixed. The Friedman equation tells us that for a F-L-R-W universe the scale factor $a(t)$ satisfies

$$\dot{a}^2 = -k + \frac{\Lambda a^2}{3} + \frac{8\pi}{3} \frac{\rho_0}{a^2}$$

where

$$\rho = \frac{\rho}{a^4}.$$ 

The constant $\rho_0$ is related to the conserved total matter entropy $S_{\text{matter}}$:

$$S_{\text{matter}} \propto \frac{\Lambda}{a^4} \text{vol}(\Sigma)$$

where the constant of proportionality depends upon the composition of the matter. In general there exist initial data with arbitrarily large (or small) entropy. However if $k$ and $\Lambda$ are both positive the matter entropy $S_{\text{matter}}$ of initial data admitting a moment of time symmetry is bounded above by the value it takes for ESU, the
Einstein Static Universe [28] . The least value of $S_{\text{matter}}$ under these circumstances is of course zero which corresponds to the empty De-Sitter universe.

One may consider inhomogeneous time symmetric initial data. The Einstein Static Universe turns out to be a local maximum of the matter entropy functional $S_{\text{matter}}$ [28] . For fixed volume it as always entropically favourable, by Jensen’s inequality for the matter to be homogeneous. If the volume is allowed to vary this still remains true for radiation. It is not true however for soft equations of state. If ESU is unstable to the Jeans instability then it is not a local maximum for the matter entropy functional $S_{\text{matter}}$.

The situation when both $k$ and $\Lambda$ are negative is different. There is no upper bound for $S_{\text{matter}}$ for time symmetric initial data, even though the volume $\text{vol}(\Sigma)$ is bounded.

One might be tempted to regard $e^{S_{\text{matter}}}$ as providing an estimate of probability of creating matter with these initial data. In that case if $k$ and $\Lambda$ are both positive one would assign greatest probability to the ESU. If $k$ and $\Lambda$ are both negative there is no data of greatest probability. However these estimates would ignore the gravitational contribution. The gravitational action is difficult to estimate because there are no non-singular Riemannian solutions $\{M_R, g_R\}$ with a single boundary component. This is because the Riemannian matter entropy current $S_{\text{matter}}^\alpha$ is divergence free and orthogonal to the boundary $\Sigma = \partial M_R$. Therefore it must have a singular point in the interior of any Riemannian solution. At this point the matter density is infinite. If one ignores this singularity problem the gravitational contribution to the action would be proportional to the 4-volume $\text{vol}(M_R)$ since there is no boundary term. For the case when both $k$ and $\Lambda$ are both positive matter and and gravity contribute to the probability with the same sign and this would favour overwhelmingly the Einstein Static Universe which has infinite 4-volume. For the case when both $k$ and $\Lambda$ are both negative matter and and gravity contribute to the probability with the opposite sign. It is not clear to me which gives the larger effect but in any event it seems likely that the probability will be peaked around universes resembling Lemaitre’s primordial atom.
Cusps and Extreme Black Holes

The physical role of cusps is rather obscure. However they have been encountered before in connection with extreme black holes with non-zero horizon area. Near the horizon these metrics are typically well approximated by the Robinson-Bertotti solution. This is an exact solution of the Einstein-Maxwell equations the euclidean section of which is the metric product $H^2 \times S^2$. The radii of curvature are equal in the Einstein-Maxwell case but in more general examples this is not true.

Now imaginary time translations consist of translations along a set of horocycles of the $H^2$ factor. Thinking of $H^2$ as the interior of the unit disc the horocycles are the set of circles passing through a point $p$ at infinity, i.e. a point $p$ on the unit circle. The constant imaginary time surfaces are geodesics orthogonal to the set of horocycles and correspond to circles through $p$ which are orthogonal to the unit circle. If one makes an identification in imaginary time with a given period one takes just the part of the manifold between two of these geodesics and then identify them. The result, $H^2/\mathbb{Z}$ is a Beltrami type pseudo-sphere, i.e. a 2-manifold of constant negative curvature with topology $S^1 \times \mathbb{R}$ and a cusp at infinity. For visualization purposes it may be isometrically embedded in three-dimensional euclidean space as a surface of revolution looking like an infinitely long horn. The surface of revolution is obtained by rotating a tractrix curve about its asymptote. The four manifold is just the product $H^2/\mathbb{Z} \times S^2$ of the Beltrami pseudo-sphere $H^2/\mathbb{Z}$ with a standard 2-sphere $S^2$.

It seem that the generic spherically symmetric extreme black hole with a regular horizon which is identified in imaginary time has this structure. It is tempting, therefore, to regard the cusps encountered in hyperbolic 4-manifolds as generalizations of this phenomenon. If this viewpoint is correct it should be possible to find an analogue among the three-dimensional black holes.
Supergravity Domain Walls.

If the cosmological “constant” is not really constant but merely approximately so in a region where a scalar field is close to one of its vacuum values one may have domain walls. The simplest case is when the cosmological constant vanishes in two symmetric vacua. In the thin wall approximation each domain corresponds to the interior of a timelike hyperboloid in flat Minkowski spacetime. The complete spacetime is obtained by gluing two such interiors back to back across the hyperboloid which represents the history of the domain wall. The spatial cross sections are diffeomorphic to the 3-sphere $S^3$ and consist of two flat 3-balls glued back to back. The Riemannian section is obtained by gluing together two flat 4-balls to obtain a 4-sphere which is almost everywhere flat. There is just a ridge of curvature separating the two domains which corresponds to the the history in imaginary time of the domain wall. The Riemannian section $2M_R$ is invariant under $SO(4)$, and the Lorentzian section $2M_L$ under $SO(3,1)$. The nucleation hypersurface $\Sigma$ is compact and is just the two flat 3-balls glued back to back. The domain walls are repulsive and this makes possible the simultaneous nucleation of black hole pairs [29].

There is clearly a similar construction for negative cosmological constant. One approach would be glue back to back two hyperbolic 4-balls of finite radius. This would give a 4-sphere which has constant negative curvature almost everywhere. Again the Riemannian section $2M_R$ is invariant under $SO(4)$, and the Lorentzian section $2M_L$ under $SO(3,1)$. The nucleation hypersurface $\Sigma$ is compact and is just the two hyperbolic 3-balls glued back to back.

Another type of domain wall which is static and has Poincaré $E(2,1)$ invariance is also possible and has arisen in supergravity theories [30]. Locally the metric takes the form:

$$ds^2 = A(z) \left( -dt^2 + (dx^1)^2 + (dx^2)^2 + (dz)^2 \right).$$

If

$$A(z) = -\frac{3}{\Lambda z^2}$$

with $\Lambda < 0$, then we obtain one half of Anti-De-Sitter spacetime $ADS_4$. The horospheric coordinates $(t, x, y, z)$ make manifest the Poincaré subgroup of the full Anti-De-Sitter group:

$$E(2,1) \subset SO(3,2).$$
If we take $0 < z < -\infty$ then Spacelike Infinity is $z = 0$ and $z = +\infty$ is a null surface through which one may continue the solution to obtain the complete $AdS_4$ spacetime. The Euclidean section is obtained by setting $t = i\tau$ with $\tau$ real. One then obtains the generalized upper half space model of Hyperbolic 4-space

$$H^4 \equiv \mathbb{R}_+ \times \mathbb{R}^3$$

where $\mathbb{R}_+ \times \mathbb{R}^3 = (\tau, x, y, z)$ with $-\infty < \tau < +\infty$, $-\infty < x < +\infty$, $-\infty < y < +\infty$, and $0 < z < +\infty$. This construction thus makes manifest the Euclidean subgroup of the full De-Sitter group:

$$E(3) \subset SO(4, 1).$$

For the simplest static supergravity domain walls $A(z)$ is an even function of $z$ which is bounded at $z = 0$ and which tends at large $z$ to $-\frac{3}{4\Lambda^2\tau}$. The spacetime looks like two copies of Anti-De-Sitter spacetime glued together across spatial infinity $z = 0$. Looking globally one discovers that certain points must be omitted. The Riemannian section $2M_R$ evidently consist of two copies of the upper half space $\mathbb{R}_+ \times \mathbb{R}^3$ glued across the ideal boundary or absolute at $z = 0$. Both geometrically and topologically this similar but is not the same as the example of two hyperbolic 4-balls glued back to back described earlier. Geometrically this is clear because the isometry groups are different, $E(3)$ as opposed to $SO(4)$. Topologically we now have $\mathbb{R}^4$ rather than $S^4$ because the points $\tau^2 + x^2 + y^2 \to \infty$ are not included. In other words there is some sort of cusp present.

Physically the most important difference between the two examples is that while the more or less conventional $SO(4)$-invariant case has has finite 4-volume and finite action the $E(3)$-invariant supergravity examples have infinite volume and infinite action. This is not just because the nucleation hyper-surfaces $\Sigma$ are of finite or infinite volume respectively. In the supergravity case the nucleation surface $\Sigma$ may be taken to be given by $\tau = 0$. We could make this have finite volume by taking by periodically identifying $x$ and $y$. Even if we did that there would be (at zero temperature) be no justification for making the range of $\tau$ finite. Since $\frac{\partial}{\partial \tau}$ is a Killing vector it follows that the action integrand must be independent of $\tau$ and hence the action integral over $\tau$ will diverge.

Thus it sees clear, by contrast with the accelerating domain walls, that static domain wall spacetimes of this type cannot spontaneously appear from nothing. In
fact we have not used any special properties of supergravity in this discussion. In particular we have not made use of the fact that typically examples arising in supergravity satisfy Bogomol’nyi bounds, are partially supersymmetric and admit Killing spinors. Nevertheless our conclusion is precisely what one would have anticipated of such spacetimes.

Lorentzian Sections and Perpetual Motion Machines

One way of describing the associated Lorentzian manifolds of the Davis example (and others like it) is to use the Gaussian coordinate system constructed from the timelike geodesics orthogonal to $\Sigma$. The result is two connected copies of a F-R-W model with compact spatial sections of constant negative curvature diffeomorphic to $\Sigma$, with scale factor $a(T) = \cos(\sqrt{3|\Lambda|}T)$, where $T$ is propertime measured along the geodesics:

$$ds^2 = -dT^2 + a^2(T)g_{ij}(x)dx^i dx^j$$

where locally $g_{ij}$ is the standard metric on hyperbolic 3-space $H^3$. These two universes begin at $T = 0$ and collapse to a Big Crunch at $T = \frac{\pi}{2} \sqrt{\frac{3}{|\Lambda|}}$. This would be a mere coordinate singularity if we were considering the covering space, Anti-De-Sitter spacetime, but in our case it is a true singularity by virtue of the spatial identifications needed to make $\Sigma$ closed.

Thus if $|\Lambda|$ were Planck size, and so the damping effect of the action rather small, these topological fluctuations might not last very long. However even if $|\Lambda|$ were initially large and hence the initial universe rather small, one could imagine in a more realistic model that a small universe formed initially in this way might be blown up to macroscopic size by some subsequent inflationary process in which the effective cosmological constant became positive. This might mean that the non-trivial topology could have observational consequences. However for this to happen the final size would have to be of the order of the present Hubble radius and there is no obvious reason why this should be the case. Indeed one usually expects an inflationary period to overshoot so that the characteristic size would be vastly greater than the present Hubble radius. For what it is worth, observational searches for the indications of large scale topology have been rather negative and the results of COBE give rather stringent limits on the size of such structures (see [31] for a recent review of such models and of the observational situation). For a
recent, and moore optimistic view of the cosmological significance of this type of model see [32].

There is another, and in some ways rather more interesting way of describing the Lorentzian sections. Since the metric is locally that of Anti-De-Sitter spacetime which is globally static with Killing time coordinate $t$ say we have a locally static metric. It is not however globally static because the identifications made to compactify $M_R$ and hence $\Sigma$ do not commute with the time-translations generated by $\frac{\partial}{\partial t}$. This sort of situation has been discussed by Frolov and Novikov [33] in connection with wormholes and time travel. It has a number of interesting consequences among which is the fact that although energy is locally conserved it is not globally conserved. We can examine this phenomenon in detail in our case.

Locally, in each connected component of $M^i_L$, we may express the metric as

$$ds^2 = -V^2(x)dt^2 + g_{ij}(x)dx^i dx^j$$

where $g_{ij}$ has the same significance as before and where the metric function $V^2$ is a solution of

$$\nabla^2 V^2 + \Lambda V = 0.$$ 

The connected initial hypersurface $\Sigma^i$ for $M^i_L$ may be taken to be at $t = 0$. In other words we have embedded $\Sigma^i$ into a $t = 0$ hypersurface of Anti-DeSitter spacetime. Thus inside a sufficiently small ball centred on some point $p \in \Sigma^i$ in spherical coordinates one has (choosing units such that $\Lambda = -3$)

$$g_{ij} dx^i dx^j = \frac{dr^2}{1 + r^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

and

$$V^2 = 1 + r^2.$$ 

Now this coordinate system cannot be extended to arbitrary large radii because eventually it would take us outside $\Sigma^i$. We may think of $\Sigma^i$ sitting inside $H^3$ as compact subset with a boundary, the points of which are suitably identified. However there are points which must be identified at which the metric function $V$, which gives the length of the timelike Killing field $\frac{\partial}{\partial t}$, does not take the same value. Thus clearly the action of time translations cannot be smoothly extended over all of $M_L$. Normally one thinks of $V$ as the energy per unit mass of a particle at rest with
respect to the Killing field $\frac{\partial}{\partial t}$. Energy conservation demands that as one passes around a closed curve one should get back to the same value of the potential energy. In the present situation that cannot happen. The wormholes that have been created can act as Perpetual Motion Machines of the Second Kind. This is precisely the phenomenon described by Frolov and Novikov. Note however that in our case the wormholes cannot be used as Time Machines. There are no closed timelike curves in $M_L$. In fact the coordinate function $T$ will serve as a time function.

One may see this more explicitly in the non-compact example of Ratcliffe and Tschantz. The 24-cell lies in the hyperboloid $\tilde{M}_R$ given by $X^0 > 0$,

$$(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 - (X^4)^2 = 1.$$  

The totally geodesic surface $\Sigma$ lies in the hyperboloid $\tilde{\Sigma}$ obtained by setting $X^1 = 0$. If we introduce an extra timelike coordinate $Y^1$, the 3-dimensional hyperboloid $\tilde{\Sigma}$ may obtained by setting $Y^1 = 0$ in the Anti-DeSitter hyperboloid $\tilde{M}_L$ given by

$$(X^0)^2 + (Y^1)^2 - (X^2)^2 - (X^3)^2 - (X^4)^2 = 1.$$  

The tilde indicates that the relevant spaces are (non-universal) covering spaces of $2M_R$, $\Sigma$ and $2M_L$. The two quadrics $\tilde{M}_R$ and $\tilde{M}_L$ are real slices of the complex quadric $\tilde{M}^C$

$$(Z^0)^2 - (Z^1)^2 - (Z^1)^2 - (Z^1)^2 - (Z^1)^2 = 1.$$  

which intersect in $\tilde{\Sigma}$. In fact it is not really necessary to consider complexifying $X^2, X^3, X^4$ so lets keep them real. The symmetry group $G_\Sigma$ preserving $\Sigma$ is a subgroup of $O\uparrow(3,1;\mathbb{Z}) \subset O\uparrow(4,1;\mathbb{Z})$ the set of integral Lorentz transformations of $(X^0, X^1, X^2, X^3, X^4)$ space acting on $H^4$ which leave invariant $X^1 = 0$. The same group $G_\Sigma$ acts by isometries on the Lorentzian section $\tilde{M}_L$ $\text{Ad} - S_4$ as a subgroup $O(3,1;\mathbb{Z}) \subset O(3,2;\mathbb{Z})$ of the set of integral Lorentz transformations of $(X^0, Y^1, X^2, X^3, X^4)$ space which leave invariant $Y^1 = 0$. Now the time translation group $SO(2) \subset SO(3,2)$ of $\text{Ad} - S_4$ generated by $\frac{\partial}{\partial t}$ corresponds to rotations of the $(X^0, Y^1)$ 2-plane keeping the $(X^2, X^3, X^4)$ coordinates fixed. It is clear that $G_\Sigma$ does not commute with the action of time translations.
Attractors and Eschatology

In other parts of physics one frequently encounters the claim that there a connection between basins of attraction and states of high entropy, since both are related to apparently reversible behaviour. The usual thermodynamic example is the concept of an equilibrium state. In the gravitational context this is the intuition behind the various formulations of dynamical No-Hair and area increase theorems for event horizons. I have nothing new to say about this. It is of interest however to ask about other gravitational attractors. In the cosmological case this amounts to examining the solutions of the Einstein equations at very late times. In that connection it is of interest to recall that in the context of homogeneous Bianchi cosmological vacuum models there is some evidence that a particular pp-wave, the so-called Lukash solution is an attractor. The restriction to spatial homogeneity is probably not necessary and on the grounds that this is just one type of gravitational wave one might conjecture that all pp-waves have this property. This would be analogous to the behaviour in an asymptotically flat Minkowski spacetime that ultimately all radiation is outgoing. In the cosmological context it is well known that the fact that the universe is not a closed system but permits the escape of radiation is why the Universe is not yet in a state of Heat Death. One may ask whether it ever will be. Whatever the answer it is perhaps fitting to finish this paper with the observation that, while in the long run we as individuals may be dead if the cosmological constant vanishes then the universe will ultimately tend to a supersymmetric state. SUSY will live forever!

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