ORDER-COMPATIBLE PATHS ARE NOT EDGE-UBIQUITOUS

NOTE

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Abstract. We construct a countable graph which, for any two vertices \( u, v \) and any integer \( k \geq 1 \), contains \( k \) edge-disjoint order-compatible \( u \rightarrow v \) paths but not infinitely many.

1. Introduction

“One of the most basic problems in an infinite setting that has no finite equivalent is whether or not ‘arbitrarily many’, in some context, implies ‘infinitely many’.” (Diestel [9]). For example, Halin [9, 10] proved that if a graph contains \( k \) disjoint rays for every integer \( k \), then it contains infinitely many disjoint rays. Substructures of a given type—subgraphs, minors, rooted minors or whatever—of which there must exist infinitely many disjoint copies (for some notion of disjointness) in a given graph as soon as there are arbitrarily (finitely) many such copies are called ubiquitous [9]. Examples of ubiquity results can be found in [1–13].

Usually, ubiquity problems are trivial as soon as the substructures considered are finite. For example, if a graph \( G \) contains \( k \) disjoint \( u \rightarrow v \) paths for every integer \( k \) and some fixed vertices \( u \) and \( v \), we can greedily find infinitely many disjoint \( u \rightarrow v \) paths in \( G \). Similarly, edge-disjoint paths between two fixed vertices are clearly ubiquitous. Interestingly, this changes as soon as we require our edge-disjoint paths to traverse their common vertices in the same order.

Let us call two \( u \rightarrow v \) paths order-compatible if they traverse their common vertices in the same order. Our aim in this paper is to show that edge-disjoint order-compatible paths between two given vertices are not ubiquitous: we shall construct a graph \( G \) that has two vertices \( u \) and \( v \) such that \( G \) contains \( k \) edge-disjoint order-compatible \( u \rightarrow v \) paths for every integer \( k \), but not infinitely many. In fact, the graph \( G \) we construct will have this property for all pairs of vertices:

**Theorem 1.** There is a countable graph that contains \( k \) edge-disjoint pairwise order-compatible paths between every two of its vertices for every \( k \in \mathbb{N} \), but which does not contain infinitely many edge-disjoint pairwise order-compatible paths between any two of its vertices.

2. The counterexample

In this section we prove our main result, Theorem 1. We use the graph-theoretic notation of Diestel’s book [9].

Our counterexample will be an infinite-order whirl, defined below. We shall use an operation called whirling. By whirling an edge \( xy \) we mean turning it into a multigraph whose edges are coloured with black and blue, as follows. First, we subdivide the edge \( xy \) twice, yielding a path \( P = xx'y'y \). Then, we duplicate the edge \( x'y' \) and add the edges \( xy' \) and \( x'y \). Finally, we colour every edge of the path \( P \) black, and every other new edge blue (in particular, the duplicate of the black edge \( x'y' \in P \) is coloured blue); see Figure 1a.

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A whirl of order 0 with endvertices $x$ and $y$ is the edge-coloured graph consisting of the blue edge $xy$. Recursively, a whirl of order $n \in \mathbb{N}_{\geq 1}$ with endvertices $x$ and $y$ is an edge-coloured multigraph $W$ that is obtained from a whirl $W'$ of order $n-1$ with endvertices $x$ and $y$ by whirling all the blue edges of $W'$. Figure 1 depicts whirls of order 1 and 2.

Let us say that a whirling sequence with endvertices $x$ and $y$ is a sequence $W_0, W_1, \ldots$ of whirls $W_n$ of order $n$ with endvertices $x$ and $y$ where every $W_{n+1}$ is obtained from $W_n$ by whirling all the blue edges of $W_n$. In the context of such a sequence we write $E_{n}^{\text{black}}$ for the set of black edges of $W_n$, and we write $W_{n}^{\text{black}}$ for the subgraph $(V(W_n), E_{n}^{\text{black}})$ of $W_n$ formed by its black edges.

A whirl of infinite order with endvertices $x$ and $y$ is a graph that is the union $\bigcup_{n \in \mathbb{N}} W_n^{\text{black}}$ for some whirling sequence $W_0, W_1, \ldots$ with endvertices $x$ and $y$.

Consider any infinite-order whirl $W$ stemming from some whirling sequence $(W_n)_{n \in \mathbb{N}}$. Every finite-order whirl $W_n$ has a Hamilton path that is formed by its blue edges, and we denote this path by $H_n$. Each path $H_{n}$ gives rise to a Hamilton path $H_{n}^{+}$ of $W_{n}^{\text{black}}$ that is obtained from $H_{n}$ by subdividing every edge twice. The infinite-order whirl $W$ is the countable edge-disjoint union of these paths $H_{n}^{+}$. In particular, every infinite-order whirl is infinitely edge-connected. For every edge $st \in H_{n}$ the subgraph $B_{n}^{st} := \bigcup_{k=n}^{\infty} sH_{k}^{t}$ of $W$ is again an infinite-order whirl, stemming from the whirling sequence $sH_{n}t$, $sH_{n}^{+}t \cup sH_{n+1}t$, $sH_{n}^{+}t \cup sH_{n+1}^{+}t \cup sH_{n+2}t, \ldots$.

Lemma 2.1. Suppose that $W$ is an infinite-order whirl that stems from some whirling sequence $W_0, W_1, \ldots$, let any number $n \in \mathbb{N}$ be given, and consider the graph $W' := W - E_{n}^{\text{black}}$. Then the cutvertices of $W'$ are precisely the inner vertices of the blue Hamilton path $H_{n}$ of $W_n$. And the blocks of $W'$ are precisely the infinite-order whirls $B_{n}^{e}$ with $e$ an edge of $H_n$. Moreover, $B_{n}^{e} \cap B_{n}^{f} = e \cap f$ for all edges $e \neq f$ of $H_n$. Therefore, the block graph of $W'$ is obtained from $H_n = v_1v_2 \ldots v_k$ by renaming the end vertices $v_1$ and $v_k$ to $B_{n}^{v_1v_2}$ and $B_{n}^{v_{k-1}v_k}$, and subdividing every edge $e \in v_2H_nv_{k-1}$ once by adding a new vertex $B_{n}^{e}$. □
In order to show that infinite-order whirls form counterexamples that we seek in Theorem 1 we have to verify that they satisfy the following two conditions. On the one hand, they must contain arbitrarily many edge-disjoint pairwise order-compatible paths between every two of their vertices. And on the other hand, they must not contain infinitely many edge-disjoint pairwise order-compatible paths between any two of their vertices. If it were true that infinite edge-connectivity implied the first condition, this would reduce our workload to verifying only the second condition.

Infinite edge-connectivity does in fact imply the first condition. At the end of one of my talks at the Mathematisches Seminar der Universität Hamburg that involved order-compatible paths, Joshua Erde asked: Is there a function $f: \mathbb{N} \to \mathbb{N}$ such that, for every graph $G$ and every two vertices $u$ and $v$ of $G$, the existence of at least $f(k)$ many edge-disjoint $u$–$v$ paths in $G$ implies the existence of $k$ many edge-disjoint pairwise order-compatible $u$–$v$ paths in $G$? The next day, Jakob Kneip answered the question in the affirmative for $f(k) = k$ the identity on $\mathbb{N}$:

**Proposition 2.2** (Kneip). Let $G$ be any graph, let $u$ and $v$ be any distinct two vertices of $G$, and let $n$ be any natural number. If $G$ contains $n$ edge-disjoint $u$–$v$ paths, then $G$ also contains $n$ edge-disjoint pairwise order-compatible $u$–$v$ paths.

**Proof.** Given $G, u, v, n$ we suppose that $G$ contains $n$ edge-disjoint pairwise order-compatible $u$–$v$ paths. Choose a path-system $\mathcal{P}$ of $n$ edge-disjoint $u$–$v$ paths in $G$ that uses as few edges of $G$ as possible. Then the paths in $\mathcal{P}$ are pairwise order-compatible. For this, assume for a contradiction that $P$ and $Q$ are paths in $\mathcal{P}$ such that $P$ traverses two vertices $x$ and $y$ as $x <_P y$ while $Q$ traverses them as $y <_Q x$. Then $uP \cup xQv$ and $uQy \cup yPv$ are connected edge-disjoint subgraphs of $P \cup Q$, so we may choose one $u$–$v$ path in each of the two. Now replacing $P$ and $Q$ with these new paths yields a system of $n$ edge-disjoint $u$–$v$ paths using strictly fewer edges of $G$ than $\mathcal{P}$, since the edges of $xPy$ and $yQx$ are not used by the new paths. □

Finally, we prove our main result:

**Proof of Theorem 1.** Let $W$ be an infinite-order whirl with endvertices $u$ and $v$ that stems from some whirling sequence $(W_n)_{n \in \mathbb{N}}$. Since $W$ is infinitely edge-connected, it follows from Proposition 2.2 that $W$ contains $k$ edge-disjoint pairwise order-compatible paths between every two of its vertices, for every $k \in \mathbb{N}$. Hence it remains to show that $W$ does not contain infinitely many edge-disjoint pairwise order-compatible paths between any two of its vertices.

For this, let any distinct two vertices $x$ and $y$ of $W$ be given, and let $N \in \mathbb{N}$ be minimal with $x, y \in W_N$. Given a path $P \subseteq W$, denote by $M(P)$ the minimal number $M \in \mathbb{N}$ with $E(P) \subseteq E^\text{black}_M$. Since the edge sets of the finite-order whirls $W_n$ are finite, it suffices to show:

*For every $x$–$y$ path $P$ in $W - E^\text{black}_N$, no $x$–$y$ path in $W - E^\text{black}_M(P)$ is order-compatible with $P$."

Now fix any $x$–$y$ path $P$ in $W - E^\text{black}_N$, write $M = M(P)$ (so $M > N$) and let $Q$ be any $x$–$y$ path in $W - E^\text{black}_M$.

First, we show that there is an edge $st \in xH_{M-1}y$ with $sH^+_M M \subseteq P$ (that is, $P$ contains the black subdivision of a blue edge $st \in xH_{M-1}y$). Indeed, by the choice of $M$ the path $P$ must contain an edge $e$ from $E^\text{black}_M$ that is not in $E^\text{black}_{M-1}$.
This edge must belong to $H^+_M - 1$, and hence there is an edge $st \in H_{M-1}$ with $e \in sH^+_M - 1$. Now the subpath $sH^+_M - 1$, as it was obtained from the edge $st$ by subdividing it twice, is induced in $W^\text{black}_M$, and so $P$ must contain it. It remains to show $st \in xH_{M-1}y$. From $x, y \in P \subseteq W - E^\text{black}_N$ and Lemma 2.1 we deduce that $P$ is included in the subgraph $\bigcup \{ B_f^N \mid f \in E(xH_Ny) \}$ of $W - E^\text{black}_N$ which is separated from the rest of $W - E^\text{black}_N$ by $x$ and $y$. Then $sH^+_M - 1 \subseteq P$ is included in that subgraph as well, and $H^+_M - 1$ meets that subgraph precisely in $xH^+_M - 1$. Combined, this gives $sH^+_M - 1 \subseteq xH^+_M - 1$ which implies $st \in xH_{M-1}y$.

Finally, we show that $P$ and $Q$ are not order-compatible. For this, consider any edge $st \in xH_{M-1}y$ satisfying $sH^+_M - 1 \subseteq P$ and write $sH^+_M - 1 = ss'tt'$. Then $P$, as it contains $sH^+_M - 1$, linearly orders the vertices $s, s', t', t$ either as $ss't't$ or $tt'ss$. Now the vertices $s, s', t', t$ are linearly ordered by the $x-y$ path $H_M$ as well, namely as $s't's$. And $Q \subseteq W - E^\text{black}_M$, as a path from $x$ to $y$, by Lemma 2.1 must contain the vertex set of $xH_My$ and linearly order it in the same way $H_M$ does. Therefore, $Q$ is not order-compatible with $P$, completing the proof.

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