On the symmetries of the modified Emden-type equation

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For an autonomous system the symmetries of the Lagrangian are embedded in the symmetries of the differential equation. Recently, it has been found that the modified Emden-type equations follow from non-standard Lagrangian functions which involve neither the kinetic energy term nor the potential function. By working with one such Lagrangian we have calculated the Lagrangian symmetries and explicitly demonstrated that, as in the case of standard Lagrangian functions, the variational symmetries of the non-standard Lagrangian are also included in the Lie symmetries of the differential equation.

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I. INTRODUCTION

The modified Emden-type equation, also called Painlevé-Ince equation [1], is given by

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0, \quad x = x(t).$$  \hspace{1cm} (1)

Here the dots over \( x \) denote differentiation with respect to time \( t \) and \( \alpha \) and \( \beta \) are arbitrary parameters. The frictional coefficient is positive and satisfies the inequality \( 0 < \alpha < 1 \). This dissipative-like equation received serious attention from both mathematicians and physicists for more than a century because, on the one hand, it appears in a number of mathematical problems including the realization of univalued functions as defined by second-order differential equations [2] and, on the other hand, it plays a role in a number of applicative contexts [2]. The equation cannot be solved analytically for any values of \( \alpha \) and \( \beta \). However, for \( \beta = \frac{2}{9} \), Eq. (1) is integrable. In this case we write the modified Emden-type equation as

$$\ddot{x} + 3k\dot{x} + k^2 x^3 = 0$$  \hspace{1cm} (2)

It is straightforward to verify that Eq. (2) can be linearized by using the coordinate transformation

$$y(t) = e^{k\int x(t)dt}$$  \hspace{1cm} (3)

and also that it follows from a Lagrangian [2].

$$L = \frac{1}{\dot{x} + kx^2}.$$  \hspace{1cm} (4)

For any autonomous differential equation as in Eq. (1) or (2) the standard Lagrangian is defined by \( L = T - V \), where \( T \) is the kinetic energy of the system modeled by the equation and \( V \), the corresponding potential function. The Lagrangian in Eq. (3) is not of this form. Thus it is often called the non-standard Lagrangian [2].

So far the standard Lagrangians are concerned there exists a vast amount of literature on the computation of Lagrangian symmetries of physical systems using the so-called Noether’s theorem which states that the symmetries of a variational problem yield conservation laws. The Lie point symmetry of a differential equation is a set of transformations of the dependent and independent variables that leave the equation unchanged [2]. The number of variational symmetries is less than the number of Lie symmetries of an equation. In fact, the Lagrangian symmetries form a subset of the set of Lie symmetries of the differential equation.

In this work we shall employ the Noether’s theorem [3] to study the variational symmetries of Eq. (2) for the non-standard Lagrangian in Eq. (4) and demonstrate that these symmetries are also embedded in the set of corresponding Lie symmetries. The approach to be followed by us has an old root in the classical mechanics literature. For example, as early as 1951, Hill [4] provided a simplified account of the Noether’s theorem by considering the infinitesimal transformations of the dependent and independent variables characterizing a classical field. In classical mechanics the variational symmetry of a system having degrees of freedom is a consequence of the invariance of the action functional

$$A = \int_{t_1}^{t_2} L(q, \dot{q}, t) \, dt$$  \hspace{1cm} (5)

under the infinitesimal transformations of independent and dependent variables such that

$$\delta t = \epsilon\xi(\tilde{q}, t) \quad \text{and} \quad \delta q_i = \epsilon\eta_i(\tilde{q}, t).$$  \hspace{1cm} (6)

These transformations are generated by the vector field

$$X = \xi(\tilde{q}, t) \frac{\partial}{\partial t} + \sum_{i=1}^{n} \eta_i(\tilde{q}, t) \frac{\partial}{\partial q_i}.$$  \hspace{1cm} (7)

Here \( \tilde{q} = (q_1, q_2, ..., q_n) \) and \( \tilde{q} = \frac{\partial}{\partial q_i} = (\tilde{q}_1, \tilde{q}_2, ..., \tilde{q}_n) \) represent the generalized coordinates and velocities of the system. The prolonged infinitesimal generator of order \( m \) corresponding to \( X \) in Eq. (7) is written as [2]

$$X^{(m)} = X + X_1 + X_2 + ... + X_m.$$  \hspace{1cm} (8)
with
\[ X^{(1)} = X + X_1 = X + \sum_{i=1}^{n} \left[ \eta_i(q_i, t) - \xi(q_i, t) \right] \frac{\partial}{\partial q_i}, \] (8a)
and
\[ X^{(2)} = X + X_1 + X_2 = X^{(1)} + \sum_{i=1}^{n} \eta_i \frac{\partial}{\partial q_i}, \] (8b)
and so on. Clearly, \( X^{(1)} \) and \( X^{(2)} \) are the first and second prolongations of \( X \). In writing (8b) we used
\[ \eta_i = \left[ \dot{\eta}_i(q_i, t) - 2\xi(q_i, t)\dot{q}_i - \xi(q_i, t)\dot{q}_i \right]. \] (9)
The action functional remains invariant under those point transformations whose constituents \( \xi \) and \( \eta \) satisfy
\[ \frac{dI}{dt} + \sum_{i=1}^{n} (\dot{\xi}_i - \eta_i) \left( \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt} \frac{\partial L}{\partial q_i} \right) = 0 \] (10)
with \( I \) given by
\[ I = \sum_{i=1}^{n} (\xi \dot{x}_i - \eta_i) \frac{\partial L}{\partial x_i} - \xi L + f(x, t), \] (11)
where the gauge function \( f(q_i, t) \) satisfies the differential equation
\[ \frac{df(q_i, t)}{dt} = \xi L + \xi \frac{\partial L}{\partial t} + \sum_{i=1}^{n} \left( \eta_i \frac{\partial L}{\partial q_i} + (\dot{\eta}_i - \dot{\xi}) \frac{\partial L}{\partial q_i} \right). \] (12)

Along the trajectories of the system, the Euler-Lagrange equations hold good such that the second term in (10) is zero. Thus \( I \) given in (11) is a conserved quantity. This invariant quantity together with the differential equation satisfied by the gauge function is commonly stated as the Noether’s theorem.

In section 2 we work out the variational symmetries of the modified Emden-type equation by the use of non-standard Lagrangian (4) and present results for the corresponding conserved quantities. In section 3 we present results for Lie symmetries of the equation with and demonstrate how the variational symmetries are embedded in the symmetries of the differential equation. Finally, we make some concluding remarks in section 4.

II. NOETHER’S SYMMETRY ANALYSIS

To study the variational or Noether’s symmetries of (2) we begin with (11) which for our (1+1) dimensional system reads
\[ I = (\xi \dot{x} - \eta) \frac{\partial L}{\partial \dot{x}} - \xi L + f(x, t). \] (13)
We substitute the value of from (4) in (13) and take the time derivative of the resulting expression to write
\[ 2k \eta + p^3 f_x + 2kp^2 f_x + k^2 px^4 f_x + n_{ix} - 2p^2 \xi_x - kpx^2 \xi_x + p^2 f_t + 2kpx^2 f_t + k^2 x^4 f_t + n_t - 2p \xi_t - kx^2 \xi_t. \] (14)
Here the suffixes on \( f, \eta \) and \( \xi \) denote partial derivatives with respect to the appropriate variables and \( p = \dot{x} \). In deriving (14) we have made use of (2). Equation (14) can be globally satisfied for any particular choice of provided the sum of \( p \) independent terms, the coefficients of linear, quadratic and cubic terms in \( p \) vanish separately. This viewpoint leads to four determining equations for the gauge function \( f \) and infinitesimal generators \( \xi \) and \( \eta \). In particular we have
\[ 2k \eta + k^2 x^4 f_t + n_t - kx^2 \xi_t = 0, \] (15a)
\[ k^2 x^4 f_x + n_x - kx^2 \xi_x + 2kx^2 f_t - 2\xi_t = 0, \] (15b)
\[ 2kf_x - 2\xi_x + f_t = 0 \] (15c)
and
\[ f_x = 0. \] (15d)
Solving (15a) – (15d), we find the following
\[ f(x, t) = a(t) = \frac{1}{4} c_0 k^2 t^4 - c_2 k t^3 + c_3 t + c_4 t^2, \] (16a)
\[ \xi(x, t) = c_0 \left( k^2 t^3 x - k^2 t^2 \right) + c_1 + c_2 \left( t - \frac{3}{2} k t^2 x \right) + \frac{1}{2} c_4 x + c_5 t x \] (16b)
and
\[ \eta(x, t) = c_0 (1 - 2 k t x + \frac{3}{2} k^2 t^2 x^2 - \frac{1}{2} k^3 t^3 x^3) + c_2 (2 x - 3 k t x^2 + \frac{3}{2} k^2 t^2 x^3) - \frac{c_4}{4} \right) k x^3 + c_5 (x^2 - k t x^3), \] (16c)
where \( c_i \) are arbitrary constants. We, therefore, have a five-parameter set of solutions from which we can construct five linearly independent group generators given by
\[ X_1 = (k^2 t^3 x - k t^2) \frac{\partial}{\partial t} + (2 - 4 k t x + 3 k^2 t^2 x^2 - k^3 t^3 x^3) \frac{\partial}{\partial x}, \] (17a)
\[ X_2 = \frac{\partial}{\partial t}, \] (17b)
\[ X_3 = (2t - 3kt^2 x) \frac{\partial}{\partial t} + (4x - 6ktx^2 + 3k^2 t^2 x^3) \frac{\partial}{\partial x} \, , \quad (17c) \]

\[ X_4 = x \frac{\partial}{\partial t} - kx^3 \frac{\partial}{\partial x} \quad (17d) \]

and

\[ X_5 = tx \frac{\partial}{\partial t} + (x^2 - ktx^3) \frac{\partial}{\partial x} \, . \quad (17e) \]

These operators generate a five-parameter Lie group and satisfy the closure property. The commutation relations between these symmetry generators are given in Table 1.

The first integrals corresponding to these generators can be found from (13), (16) and (17), and we have

\[ I_1 = \frac{(1 - ktx)(1 + kt(pt - x) + k^2 t^2 x^2)}{(p + kx^2)^2} + a(t) \, , \quad (18a) \]

\[ I_2 = - \frac{(2p + kx^2)}{(p + kx^2)^2} + a(t) \, , \quad (18b) \]

\[ I_3 = \frac{p(-2t + 3kt^2 x) + x(2 - 4ktx + 3k^2 t^2 x^2)}{(p + kx^2)} + a(t) \, , \quad (18c) \]

\[ I_4 = - \frac{x}{(p + kx^2)} + a(t) \quad (18d) \]

and

\[ I_5 = \frac{x^2 - 2ptx - 2ktx^3}{(p + kx^2)^2} + a(t) \, . \quad (18e) \]

\[ M(t, x, \dot{x}, \ddot{x}) = 0 \text{ possesses Lie point symmetries of the form (5) provided} \quad (19) \]

\[ X^{(2)} = \frac{x \partial}{\partial x} + (x^2 - ktx^3) \frac{\partial}{\partial x} \, , \quad (19) \]

where \( X^{(2)} \) is the second-order prolongation of the vector field \( X \) obtained by specializing (7) to \((1 + 1)\) degrees of freedom. In our case

\[ M(t, x, \dot{x}, \ddot{x}) = \ddot{x} + 3kx\dot{x} + k^2 x^3 \, . \quad (20) \]

As in the case of (15), we can construct from (5), (19) and (20) a set of first-order partial differential equations and the solutions of which lead to linearly independent Lie point symmetries. In particular, we have the following eight-parameter Lie point symmetries.

\[ X^L_1 = t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} \, , \quad (21a) \]

\[ X^L_2 = \frac{\partial}{\partial t} \quad (21b) \]

\[ X^L_3 = ktx \frac{\partial}{\partial t} + (-2x + 2ktx^2 + k^3 t^2 x^3) \frac{\partial}{\partial x} \, , \quad (21c) \]

\[ X^L_4 = ktx \frac{\partial}{\partial t} - kx^3 \frac{\partial}{\partial x} \, , \quad (21d) \]

\[ X^L_5 = tx \frac{\partial}{\partial t} + (x^2 - ktx^3) \frac{\partial}{\partial x} \, , \quad (21e) \]

\[ X^L_6 = (2t^2 - ktx^3) \frac{\partial}{\partial t} + (2tx - 3kt^2 x^2 + k^2 t^3 x^3) \frac{\partial}{\partial x} \, , \quad (21f) \]

\[ X^L_7 = (3kt^2 - k^2 t^2 x) \frac{\partial}{\partial t} + (2 - 3k^2 t^2 x^2 + k^3 t^3 x^3) \frac{\partial}{\partial x} \, , \quad (21g) \]

\[ X^L_8 = (-2ktx^3 + k^2 t^2 x) \frac{\partial}{\partial t} + (4t - 6ktx + 4k^2 t^2 x^2 - k^3 t^3 x^3) \frac{\partial}{\partial x} \, , \quad (21h) \]

The superscript \( L \) on \( X \) has been used merely to indicate that the vector fields in (21) are Lie symmetry generators of (2). Ideally, all Noether’s symmetries should lie inside the point Lie symmetries. But looking at (17) and (21) we see that all the generators of the Lagrangian symmetries are not of the same form as those associated with invariance of the equation. However, it is straightforward to verify that variational symmetries which do not appear in (21) can always be expressed as linear combinations of Lie symmetries. For example, we can write

\[ X_1 = X^L_1 - 2kX^L_2 \quad \text{and} \quad X_3 = 2X^L_1 - 3X^L_2 \]
IV. CONCLUDING REMARKS

In this work we calculated the variational symmetries of the modified Emden-type equation in (2) corresponding to the non-standard Lagrangian (4) and explicitly demonstrated that, as with symmetries of the standard Lagrangians, the Noether’s symmetries of (4) are also embedded in the Lie symmetries of (2). In this context we note that (2) also follows from another non-standard Lagrangian

\[ L = \sqrt{2\ddot{x} + kx^2} \]  

(22)

It is of interest to note that the Lagrangians in (4) and (22) are not connected by a gauge term. Yet, both of them, via the Euler-Lagrangian equation, yield the same equation of motion. Such Lagrangians are called the alternative Lagrangians. The presence of alternative Lagrangians in a physical system has deep consequences in physical theories [10]. For example, there can arise ambiguities in the association of symmetries with conservation laws. Moreover, the same physical system, via alternative Lagrangian descriptions, can lead to entirely different quantum mechanical systems. Thus it remains an interesting curiosity to compute the Noether’s symmetries using (22) and compare them with the results presented in this work.

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