Hidden symmetries for transparent de Sitter space

Garrett Compton and Ian A Morrison

Department of Physics & Engineering, West Chester University, West Chester, PA, 19383, United States of America

E-mail: imorrison@wcupa.edu

Received 17 March 2020, revised 20 April 2020
Accepted for publication 23 April 2020
Published 27 May 2020

Abstract

It is known that odd-dimensional de Sitter space acts as a transparent potential for free fields. Previous studies have explained this phenomena by relating de Sitter free field equations of motion to the time-independent Schrödinger equation with known transparent potentials. In this work we show that de Sitter’s transparency is a consequence of an infinite set of ‘hidden’ symmetries. These symmetries arise from an accidental symmetry for the zero-mode of matter fields, as well as the boost isometry of de Sitter space. For simplicity, we consider the case of massive Klein–Gordon theory. We show that the Noether charges associated with these hidden symmetries distinguish the two linearly-independent solutions of the free field wave equation in the asymptotic past and future of de Sitter. Conservation of these charges requires that the asymptotic behavior of any solution be identical, up to a constant phase, in the future and the past, which is the property of transparency. In the quantized theory, these charges act trivially on particle states belonging to the in/out vacuum Fock space. For particle states constructed from other vacua, the action of the charges generates particles.

Keywords: de Sitter space, QFT in curved spacetime, integrability

1. Introduction

Scattering is a ubiquitous feature of quantum systems. Due to particle/wave duality, quanta behave as waves when encountering a barrier, and thus in general both transmit and reflect. In the context of cosmological inflation, spacetime geometry serves as a time-dependent barrier for quantum fluctuations. Scattering in this context results in cosmological particle production [1]. This particle production is observed, indirectly, in measurements of the cosmic microwave background [2]. Thus, scattering plays a fundamental fundamental role in our understanding of...
inflation. The ubiquity of particle production in curved spacetime poses important challenges to the formulation of scattering matrix theory for cosmology (see, e.g., [3, 4]).

It is therefore quite remarkable that de Sitter spacetime, the maximally symmetric model of an inflating Universe, can behave as a transparent barrier for matter fluctuations. As described in [5], following earlier work in [6], odd-dimensional global de Sitter space serves as a transparent barrier for linearized matter fluctuations. This means that fluctuations of matter fields which are pure positive frequency in the asymptotic past emerge in the asymptotic future also pure positive frequency. This surprising behavior suggests that linearized field theories in odd-dimensional de Sitter contain additional, ‘hidden’ structure which constrains the theory and prevents scattering on cosmological scales.

Indeed, [5] found considerable additional structure. This work recognized that the time-dependent part of linearized equations of motion take the form of the Schrödinger equation with transparent potentials. For example, for scalar field fluctuations, the time-dependent part of the wave equation is equivalent to the Schrödinger equation with a Pöschl–Teller potential [7]. This potential arises in many settings, including supersymmetric quantum mechanics [8, 9], the dynamics of instantons in two-dimensional integrable models [10, 11], the inverse scattering approach to the KdV equation [12–14], and tachyon condensation in gauge theories [15, 16]. For certain values of its parameters, the Pöschl–Teller potential is transparent for all incident waves (for a modern analysis, see, e.g., [17]). Reference [5] showed that the Bogoliubov coefficients which relate fluctuations in the asymptotic past and future of de Sitter may in turn be related to the transmission and reflection coefficients of Pöschl–Teller theory. This provides an explanation for why odd-dimensional de Sitter space acts as a transparent barrier for linearized fields.

In this work we further explore structures which impose transparency on odd-dimensional de Sitter space. For simplicity, we consider the case of massive Klein–Gordon theory on a fixed de Sitter background. We show that, when the spacetime dimension is odd, this theory has an infinite set of ‘hidden’ symmetries. To our knowledge, these symmetries of de Sitter Klein–Gordon theory have not previously appeared in the literature. The Noether charges corresponding to these symmetries distinguish between positive- and negative-frequency solutions to the equation of motion. The conservation of these charges requires that solutions have the same asymptotic behavior, up to a constant phase, in both the past and future asymptotic regions of de Sitter space. Thus, the conservation of these charges enforces transparency. Our investigation serves as an example of how heightened symmetry in models of the early Universe can result in simple asymptotic behavior of matter fields—for another example, see, e.g., [18].

Key ingredients in our study have appeared in the literature before. Mode solution the Klein–Gordon equation are related to one another via what are known as Darboux–Crum transformations [19]. In the context of de Sitter field theory, these transformations have been pointed out previously, and in particular, are an important ingredient in the analysis of [5]. We show that these transformations can be regarded as a consequence of the boost isometry of de Sitter space. Another key ingredient in our analysis is the presence of an accidental time-translation symmetry for the zero angular momentum mode solution to the Klein–Gordon equation in three spacetime dimensions. This too has been noted previously. We show that by combining Darboux–Crum transformations with this charge, one may generate an infinite family of conserved quantities which constrain every de Sitter Klein–Gordon mode, and which are present in all odd dimensions. Our analysis employs techniques used to study hidden and nonlinear supersymmetry in quantum mechanical systems [20–24]; indeed, one may view our analysis as a novel application of these techniques.

We also investigate our family of charges in the context of quantized Klein–Gordon theory. In particular, we examine the action of the charges in the Fock spaces built atop the de
Sitter-invariant Mottola–Allen vacua, or ‘α-vacua’ [25, 26]. In this setting, the statement that de Sitter is transparent may be phrased as saying that the natural ‘in’ vacuum (whose mode functions are pure positive frequency in the asymptotic past) is equivalent to the ‘out’ vacuum (whose mode functions are pure positive frequency in the asymptotic future). Thus, in odd dimensions there is just one ‘in/out’ vacuum [6]. While the existence of the conserved quantities is independent of the choice of vacuum, different vacua provide valuable interpretation for our family of charges. We show that the family of charges act simply in the Fock space of the ‘in/out’ vacuum. Particle states in this basis have wavefunctions which are pure positive frequency in the asymptotic past and future, and thus are eigenfunctions of the charges. In the Fock spaces built from other MA vacua, the wavefunctions are not eigenfunctions of the charges, and thus the action of the charges generates particles.

This paper is organized as follows. We begin in section 2 by reviewing relevant aspects of Klein–Gordon theory in de Sitter space, including the property of transparency in odd dimensions. In section 3 we describe the Darboux–Crum transformations which relate different mode solutions of the Klein–Gordon wave equation. We show that these transformations are related to the boost isometry of de Sitter space. We also show how these Darboux–Crum transformations, in the context of Pöschl–Teller theory, give rise to an infinite family of conserved charges which in turn imply transparency. In section 4 we derive, via the standard Noether procedure of classical field theory, an infinite class of hidden symmetries for Klein–Gordon theory in de Sitter, as well as their associated charges. We show that the presence of these charges implies transparency in the de Sitter context. Then in section 5 we consider the quantized Klein–Gordon theory. We construct explicit expressions for the hidden charges, and we analyze their action on different vacua. We conclude with a discussion in section 6.

2. Klein–Gordon fields in de Sitter

In this section we briefly review relevant aspects of Klein–Gordon theory in dS space, as well as our conventions. For further introduction, see, e.g., [1, 25–27].

We consider \((D = d + 1)\)-dimensional de Sitter spacetime whose line element in global coordinates takes the form
\[
\frac{ds^2}{\ell^2} = -dt^2 + \cosh^2 t d\Omega^2_d = -dt^2 + \cosh^2 t \left( d\theta^2 + \sin^2 \theta d\Omega^2_{d-1} \right). \tag{1}
\]
Here \(\ell\) is the de Sitter radius, \(t \in \mathbb{R}\) is a dimensionless global time coordinate, \(d\Omega_d\) is the line element on \(S^d\), and \(\theta \in [0, \pi)\) is the polar angle on \(S^d\). As is evident from the line element, the topology of global de Sitter is \(\mathbb{R} \times S^d\). The isometries of de Sitter space correspond to rotations and boosts.

We study a real scalar field \(\Phi(x)\) satisfying the Klein–Gordon equation
\[
(\Box - M^2)\Phi(x) = 0, \tag{2}
\]
where \(\Box\) is the d’Alembertian operator on de Sitter space and \(M\) is a positive mass satisfying
\[
M^2 \ell^2 > \frac{d^2}{4}. \tag{3}
\]
For our purposes it is more convenient to consider the rescaled field\(^1\) \(\Psi(x)\) related to \(\Phi(x)\) via
\[
\Phi(x) = \ell^{(2-D)/2} (\cosh t)^{-d/2} \Psi(x). \tag{4}
\]
\(^1\)This rescaling, which is common in cosmological applications, has many convenient features—see, e.g., [28, 29].
We have included factors of the de Sitter radius so as to make $\Psi(x)$ dimensionless. We may expand $\Psi(x)$ in a basis of spherical harmonics on $S^d$,

$$\Psi(x) = \sum_L \Psi_L(t) Y_L(\Omega).$$

(5)

Here we denote the $d$ coordinates parameterizing the $S^d$ by the collective coordinate $\Omega$. The harmonics are labeled by a set of $d$ angular momenta $\vec{L}$ with the total angular momentum denoted by $L$; they satisfy the eigenvalue and orthonormality conditions

$$\triangle_d Y_L(\Omega) = -L(L + d - 1)Y_L(\Omega), \quad \int d\Omega_d Y_{L}(\Omega)Y_{L'}(\Omega) = \delta_{L'L}.$$

(6)

where $\triangle_d$ is the Laplacian on $S^d$ and $\delta_{LL'}$ is the Kronecker delta symbol. Our conventions for spherical harmonics are standard and correspond to those of, e.g., [30]. Upon inserting the mode expansion (5) into (2), one may obtain the equation of motion for the time-dependent fields $\Psi_L(t)$. This equation depends only upon the dimension $d$, mass $M$, and the total angular momentum $L$. It is convenient to keep track of these parameters with the dimensionless quantities

$$\nu := L + \frac{d}{2} = 1, \quad \text{and} \quad k := \sqrt{M^2 l^2 - \frac{d^2}{4}}.$$

(7)

We refer to $\nu$ as the level. Given our restriction to sufficiently massive fields (3) it follows that $k > 0$. For simplicity, when there is no risk of confusion we will simply write $\Psi_L(t)$ as $\Psi_\nu(t)$.

Then the equation of motion for $\Psi_\nu(t)$ is

$$\left[ -\partial_t^2 - \nu(\nu + 1)\text{sech}^2 t - k^2 \right] \Psi_\nu(t) = 0.$$

(8)

A convenient basis of solutions to (8) is

$$\psi_{\nu k}(t) = \frac{1}{\sqrt{2|k|}} e^{-ikt} \text{}_2F_1 \left[ -\nu, \nu + 1; 1 + ik; \frac{1 - \tanh t}{2} \right]$$

$$= \frac{1}{\sqrt{2|k|}} \Gamma(1 + ik)(-1)^{k/2} P_{\nu}^{ik} \tanh t).$$

(9)

Here $\text{}_2F_1[a, b; c; z]$ is the Gauss hypergeometric series, $\Gamma(x)$ is the Gamma function, and $P_{\nu}^k(z)$ is the associated Legendre function [31]. We note that the expressions (9), like the equation of motion (8), are invariant under the replacement $\nu \rightarrow -\nu - 1$. For the values of $\nu$ and $k$ we consider, the complex conjugate $\psi_{\nu k}^*(t) = \psi_{-\nu k}(t)$ provides a second linearly independent solution. Thus, the general solution to (8) may be written

$$\Psi_{\nu}(t) = a\psi_{\nu k}(t) + b\psi_{\nu k}^*(t),$$

(10)

where $a$, $b$, are arbitrary coefficients. The solutions $\psi_{\nu k}(t)$ have been normalized so that the Wronskian satisfies

$$-i[\psi_{\nu k}(t)\partial_t\psi_{\nu k}^*(t) - \psi_{\nu k}^*(t)\partial_t\psi_{\nu k}(t)]_{t=\text{const.}} = 1.$$

(11)

The fact that the Wronskian is conserved follows from the the equation of motion. When written in terms of the Klein–Gordon field $\Phi(t)$, this conserved quantity is known as the Klein–Gordon flux.
The behavior of the solutions in the asymptotic regions of de Sitter is crucial to our study. In the asymptotic future, the solutions behave as

\[ \psi_{\nu k}(t \to +\infty) = \frac{1}{\sqrt{2|\nu|}} e^{-ikt} \left( 1 + O(e^{-2|\nu|}) \right), \tag{12} \]

while in the asymptotic past,

\[ \psi_{\nu k}(t \to -\infty) = \frac{1}{\sqrt{2|\nu|}} \frac{\Gamma(1+ik)\Gamma(i\nu)}{\Gamma(1+ik+\nu)\Gamma(-i\nu)} e^{-ikt} \left( 1 + O(e^{-2|\nu|}) \right) + \frac{1}{\sqrt{2|\nu|}} \frac{\Gamma(1+ik)\Gamma(-ik)}{\Gamma(1+i\nu)\Gamma(-i\nu)} e^{ikt} \left( 1 + O(e^{-2|\nu|}) \right). \tag{13} \]

In particular, we note that \( \psi_{\nu k}(t) \) behaves as the plane wave \( e^{-ikt} \) in the asymptotic future. In the asymptotic past, for generic values of \( k \) and \( \nu \), \( \psi_{\nu k}(t) \) behaves like a linear combination of the two plane waves \( e^{\pm ikt} \).

Given our basis of solutions, it is straightforward to see that the de Sitter background acts as a transparent potential when \( \nu = n \), where \( n \in \mathbb{Z} \). These values for \( \nu \) correspond to when the number of spatial dimensions \( d \) is even, i.e., when the spacetime dimension is odd. For \( \nu = n \) the hypergeometric series in (9) terminates and is thus an \( n \)th order polynomial in \( \tanh t \).

Correspondingly, when \( \nu = n \) the second term in (13) vanishes. Thus, \( \psi_{\nu k}(t) \) has the same behavior (up to a constant phase) in both asymptotic regions. This is the property of transparency: a pure positive-frequency solution in the far past travels through de Sitter and emerges pure positive frequency.

### 3. Darboux–Crum transformations

In the previous section, the transparency of odd-dimensional de Sitter space arose in a rather matter-of-fact manner from examining solutions to the Klein–Gordon equation. It turns out that there is much more structure behind this phenomena. In this section we describe the existence of Darboux–Crum transformations which relate the modes \( \Psi_{\nu}(t) \) at different levels, i.e., different values of \( \nu \). We show that this Darboux–Crum structure is intimately related to the boost isometry of de Sitter space. Then, following [5], we describe the relation between dS Klein–Gordon field theory and supersymmetric quantum mechanics with Pöschl–Teller potentials. This is an ideal setting to demonstrate how the Darboux–Crum structure can, when combined with a single conserved quantity, result in an infinite family of conserved quantities and transparency.

#### 3.1. Darboux–Crum transformations

The equation of motion (8) for \( \psi_{\nu k}(t) \) has the standard form of the time-independent Schrödinger equation, with \( t \) playing the role of position [5]. Indeed, may write this equation as

\[ H_\nu \psi_{\nu k}(t) = k^2 \psi_{\nu k}(t), \tag{14} \]

where \( H_\nu \) is the differential operator

\[ H_\nu := -\partial_t^2 - \nu(\nu + 1)\sech^2 t. \tag{15} \]
In analogy with the Schrödinger equation, we refer to $H_\nu$ as the Hamiltonian at level $\nu$. In this analogy, $t$ plays the role of position and $k^2$ plays the role of energy. The Hamiltonian may usefully be written in terms of the differential operators

$$A^+_\nu = -\partial_t + (\nu + 1) \tanh t,$$

$$A^-_\nu = \partial_t + \nu \tanh t. \tag{16}$$

In terms of these operators, $H_\nu$ may be written variously as

$$H_\nu = A^+_\nu A^-_{\nu+1} - (\nu + 1)^2 = A^+_{\nu-1} A^-_\nu - \nu^2. \tag{18}$$

The operators $A^\pm_\nu$ relate Hamiltonians whose index $\nu$ differs by one:

$$H_{\nu+1} A^+_\nu = A^+_\nu H_\nu, \quad H_\nu A^-_{\nu+1} = A^-_{\nu+1} H_{\nu+1}. \tag{19}$$

The relations (19) are known as intertwining relations, and we will refer to $A^\pm_\nu$ as intertwining operators. The intertwining operators act as raising/lowering operators for solutions in that they change the value of the level $\nu$ by one, i.e.,

$$A^+_\nu \psi_{\nu,k}(t) = (ik + \nu + 1) \psi_{\nu+1,k}(t), \quad A^-_\nu \psi_{\nu,k}(t) = (-ik + \nu) \psi_{\nu-1,k}(t). \tag{20}$$

These relations may be verified explicitly by, e.g., utilizing the well-known recursion relations of Legendre functions [31].

Transformations between classes of solutions of the form (20) are known as Darboux–Crum transformations [19]. We note that, due to our restriction to $\nu \in \mathbb{R}$ and $k > 0$, the transformations (20) do not annihilate any wavefunctions. In addition, when mapping wavefunctions from one level to another, these transformations preserve the spectrum of energy eigenvalues. Thus, these Darboux–Crum transformations provide an isomorphism between PT theories whose level differs by unity. In the language of SUSYQM, these transformations are known as isospectral deformations [8, 9].

3.2. Boost symmetry

Before proceeding to analyze the consequences of the Darboux–Crum transformations just described, we pause here to show that these transformations may be viewed as a consequence of the boost isometry of de Sitter space.

Recall that the isometries of de Sitter space may be described as rotations and boosts. For our global coordinates (1), the former act on the $S^d$ coordinates while preserving the time foliation; the latter involve both time and spatial coordinates thus alter the time foliation. One such boost Killing vector is

$$\xi^\mu \partial_\mu = \cos \theta \partial_t - \tanh \theta \partial_\theta. \tag{21}$$

Under an infinitesimal boost along $\xi^\mu$, the Klein–Gordon field $\Phi(x)$ transforms as

$$\Phi(x) \rightarrow \Phi(x) + \epsilon L^\xi \Phi(x), \tag{22}$$

$^2$In the context of supersymmetric quantum mechanics, it is natural to consider all values of $k$ which result in a normalizable wavefunction. In this case, the Darboux–Crum transformations we describe can introduce or remove bound states, i.e. states with $E < 0$. Thus, in this more general context, the Darboux–Crum transformations are referred to as quasi-isospectral deformations.
where $|\epsilon| \ll 1$ is an infinitesimal constant and $\mathcal{L}_\xi$ denotes the Lie derivative along $\xi^\mu$. Since the boost is an isometry, the transformed field is also a solution to the Klein–Gordon equation; in particular, $\mathcal{L}_\xi \Psi(x)$ is itself a solution. Thus, this infinitesimal boost provides a map between solutions to the Klein–Gordon equation.

The rescaled field $\Psi(x)$ transforms under this boost as

$$
\Psi(x) \rightarrow \Psi(x) + \epsilon \left( -\frac{d}{2} \tanh t \cos \theta + \mathcal{L}_\xi \right) \Psi(x).
$$

(23)

Let us examine how this transformation effects a single mode solution $\Psi_L(t)Y^\Omega_{\ell m}(\Omega)$. Since the boost does not preserve the global time foliation, it mixes the modes of $\Psi(x)$. In particular, the transformation (23) takes a single mode into two terms, one with a total angular momentum raised by one, and one with a total angular momentum lowered by one. Explicitly, denoting the angular momenta as $\vec{L} = (L, \vec{m})$, the second term in (23) is

$$
\delta (\Psi_{L\ell m}(t)Y_{\ell m}(\Omega)) := \left( -\frac{d}{2} \tanh t \cos \theta + \mathcal{L}_\xi \right) (\Psi_{L\ell m}(t)Y_{\ell m}(\Omega))
$$

$$
= b^+_{Lm} \Psi_{L-1,\ell m}(t)Y_{L-1,\ell m}(\Omega) - b^-_{Lm} \Psi_{L+1,\ell m}(t)Y_{L+1,\ell m}(\Omega),
$$

(24)

where the coefficients are reported in [32] to be

$$
b^+_{Lm} = \left[ ik + \left( L + \frac{d}{2} \right) \right] \left[ \frac{(L + m + d - 1)(L - m + 1)}{(2L + d - 1)(2L + d + 1)} \right]^{1/2},
$$

(25)

$$
b^-_{Lm} = \left[ -ik + \left( L - 1 + \frac{d}{2} \right) \right] \left[ \frac{(L + m + d - 2)(L - m)}{(2L + d - 1)(2L + d - 3)} \right]^{1/2},
$$

(26)

where $m = |\vec{m}|$. Using the orthogonality of the spherical harmonics it is possible to to isolate the element of the boost which acts on $\Psi_L(t)$ to raise/lower the value of $L$. For instance,

$$
= \int d\Omega Y^\Omega_{L+1,\ell m}(\Omega) \delta (\Psi_{L\ell m}(t)Y_{\ell m}(\Omega))
$$

$$
= \left( -\partial_t + \left( L + \frac{d}{2} \right) \tanh t \right) \Psi_{L\ell m}(t) = \Psi_{L+1,\ell m}(t).
$$

(27)

We recognize the differential operator after the first equality to be $A^+_L$ (written in terms of $L$ and $d$). In a similar way, one obtains $A^-_L$ by isolating the part of the boost which lowers $L$ by one.

To summarize, the intertwining operators $A^\pm_L$ agree with the action of an infinitesimal boost isometry on the time-dependent part of the mode solutions for $\Psi(x)$. Thus, we regard the Darboux–Crum transformations which map mode solutions at different levels as a consequence of the boost isometry of de Sitter space. We note that the Darboux–Crum structure exists for scalar fields in all spacetime dimensions, and also for other linearized fields on de Sitter, including spin-half, symmetric tensor, and p-form fields [5]. In all these cases, the Darboux–Crum structure may be regarded as a consequence the boost isometry combined with the linear nature of the equation of motion.
3.3. Charges in Pöschl–Teller theory

We now return to our discussion of the Schödinger equation (14) which provides the equation of motion for the mode solutions \( \psi_{\nu k}(t) \). The potential which appears in this equation,

\[
V_{\nu}(t) = -\nu(\nu + 1) \text{sech}^2 t,
\]

is known as the Pöschl–Teller (PT) potential \([7, 17]\). We refer to the family of quantum mechanical theories labeled by \( \nu \) as the PT family. The mode solutions \( \psi_{\nu k}(t) \) provide the wavefunctions corresponding to scattering states for this potential. The intertwining operators \( A_{\nu}^\pm \) map the Hamiltonian and wavefunctions of the theory at level \( \nu \) to those of the theory with level \( \nu \pm 1 \). When \( \nu = n \), where \( n \in \mathbb{Z} \), the potentials (28) are transparent. A simple way to determine this is by calculating the transmission and reflection coefficients of the theory \([8, 9]\). For our purposes, it is more enlightening to examine the role that conserved quantities play in enforcing transparency.

Suppose that at level \( \nu \) (not necessarily an integer) there exists a conserved quantity, i.e., an operator \( Q_{\nu} \) which commutes with the level \( \nu \) Hamiltonian,

\[
[Q_{\nu}, H_{\nu}] = 0.
\]

Using the intertwining operators \( A_{\nu}^\pm \) we may define a conserved quantity in the \( \nu + 1 \) theory as \([20]\)

\[
Q_{\nu + 1} := A_{\nu}^+ Q_{\nu} A_{\nu + 1}^-.
\]

We check explicitly that this commutes with the level \( \nu + 1 \) Hamiltonian:

\[
[Q_{\nu + 1}, H_{\nu + 1}] = Q_{\nu + 1} H_{\nu + 1} - H_{\nu + 1} Q_{\nu + 1}
= A_{\nu}^+ Q_{\nu} A_{\nu + 1}^- H_{\nu + 1} - H_{\nu + 1} A_{\nu}^+ Q_{\nu} A_{\nu + 1}^- \\
= A_{\nu}^+ [Q_{\nu}, H_{\nu}] A_{\nu + 1}^- \\
= 0.
\]

The second equality follows from the definition of \( Q_{\nu + 1} \); the third equality follows from the intertwining relations; and the forth equality follows from the fact that \( Q_{\nu} \) is conserved at level \( \nu \). In a similar manner, we can likewise define a conserved quantity at level \( \nu - 1 \) via

\[
Q_{\nu - 1} := A_{\nu}^- Q_{\nu} A_{\nu - 1}^+.
\]

It follows that through repeated application of (30) or (32) we can construct a conserved quantity at all levels \( \nu + j, j \in \mathbb{Z} \) \([20]\). Thus, the existence of a single \( Q_{\nu} \) implies an infinite family of conserved quantities, one in each level \( \nu + j \). We emphasize that this structure occurs for all real \( \nu \).

The concrete family of conserved quantities relevant to our study occurs for \( \nu = n \). We start in the \( n = 0 \) theory, where the potential \( V_0(t) = 0 \) is trivial. In this theory the linear momentum operator, which we denote

\[
Q_0 = i \partial_t,
\]

We remind the reader that the PT potential (28) is invariant under the relabeling \( \nu \to -\nu - 1 \), and so the levels \( \nu \) and \( -\nu - 1 \) define equivalent theories.
is a conserved quantity. The wavefunctions of this theory are the left- and right-moving plane waves $e^{\pm ikx}$. The eigenvalues of $Q_0$ are $\pm k$, and so eigenfunctions of $Q_0$ must exhibit the same left- or right-moving behavior everywhere. In this way, one may say that the $n = 0$ theory is transparent as a consequence of the existence of the conserved quantity $Q_0$.

At higher levels $n > 0$ the PT potential is non-trivial and the theories do not enjoy conserved linear momentum. However, from $Q_0$ we may construct a conserved quantity at each level $n$, simply by repeatedly applying the procedure (30). The result is a conserved quantity $Q_n$ at each level $n$:

$$Q_n := A_{n-1}^+ \cdots A_0^+ Q_0 A_1^+ \cdots A_n^-.$$  

(34)

The action of $Q_n$ on the wave functions can be determined by using the raising and lowering relations (20); the result is

$$Q_n \psi_{nk}(t) = k \left[ \prod_{j=1}^{n} (k^2 + j^2) \right] \psi_{nk}(t).$$  

(35)

We see that $Q_n$, like $Q_0$, distinguishes between $\pm k$ eigenvalues. Let us see how this affects the asymptotic behavior of the wavefunctions. For $n > 0$ the wave functions are not simple plane waves. Nevertheless, at asymptotically large values of $|t| \gg 1$, the potential is exponentially suppressed, and the wave functions are plane waves up to exponentially suppressed corrections, i.e.,

$$\psi_{nk}(|t| \gg 1) \propto e^{\pm ikt} \left( 1 + O(e^{-2|t|}) \right).$$  

(36)

From this asymptotic behavior we see that the eigenfunctions of $Q_n$ must exhibit the same plane wave behavior in both regions $t \to -\infty$ and $t \to +\infty$. Thus, the presence of $Q_n$ enforces transparency for the non-trivial potential $V_n(t)$. Since there exists a $Q_n$ at each level $n \in \mathbb{Z}$, it follows that all PT theories with level $n \in \mathbb{Z}$ are transparent.

4. Hidden symmetries in dS

We now return to our discussion of Klein–Gordon theory on de Sitter. We focus on the case when the spacetime dimension is odd, so we let $\nu = n$ with $n \in \mathbb{Z}$. We will find that there exist an infinite family of "hidden" Noether symmetries. The conserved quantities corresponding to these symmetries are analogues of the $Q_n$ charges of PT theory described in the previous section. However, we emphasize that our analysis in this section is purely classical.

The action for the Klein–Gordon field $\Phi(x)$ is

$$S[\Phi] = -\frac{1}{2} \int d^Dx \sqrt{-g(x)} \left( g^{\mu\nu}(x) \partial_{\mu} \Phi(x) \partial_{\nu} \Phi(x) + \frac{M^2}{2} \Phi^2(x) \right).$$  

(37)

To bring this into a more useful form, we first replace $\Phi(x)$ for $\Psi(x)$ as in (4), insert the mode expansion for $\Psi(x)$ (5), then integrate over the $S^D$. After these steps the action becomes a sum of terms, each of which is quadratic in a single $\Psi_L(t)$; schematically, we write this as

$$S[\Phi] = \sum_L S_L[\Psi_L].$$  

(38)
The expression for $S[\Psi_L]$ can be tidied up with a bit of algebra, and by dropping a total derivative. Then the expression for $S[\Psi_L]$ involves only $M$, $d$, and $L$, and so is most conveniently written in terms of $k$ and $n$ as

$$S_n[\Psi_n] = \frac{1}{2} \int_{-\infty}^{\infty} dt \left( (\partial_t \Psi_n)^2 - n(n+1) \text{sech}^2 t \Psi_n^2 - k^2 \Psi_n^2 \right),$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dt \, \Psi_n H_n \Psi_n. \tag{39}$$

In the second line we have identified the same differential operator $H_n$ which plays the role of the Hamiltonian in the context of PT theory. We remind the reader that our short hand $\Psi_n(t)$ denotes any mode $\Psi_L(t)$ with total angular momentum $L$.

We start by examining the $n=0$ case, which corresponds to the zero angular momentum mode in $D=3$ dimensions. The action for $\Psi_0(t)$ is explicitly

$$S_0 = \frac{1}{2} \int_{-\infty}^{\infty} dt \left( (\partial_t \Psi_0)^2 - k^2 \Psi_0^2 \right). \tag{40}$$

We recognize this as the action for a free harmonic oscillator; the presence of the de Sitter background has been completely absorbed by the field redefinition (4) exchanging $\Phi(x)$ for $\Psi(x)$. Since there is no explicit time dependence, this action enjoys time-translation symmetry. The infinitesimal transformation

$$\Psi_0 \to \Psi_0 + \delta \Psi_0, \quad \delta \Psi_0 = \epsilon \partial_t \Psi_0, \quad |\epsilon| \ll 1, \tag{41}$$

leaves the action invariant up to a total derivative. Indeed, it is easy to show that

$$\delta S_0 = \frac{1}{2} \epsilon \int_{-\infty}^{\infty} dt \, \partial_t \left( (\partial_t \Psi_0)^2 - k^2 \Psi_0^2 \right) \tag{42}$$

follows simply from algebraic manipulation (that is, $\Psi_0$ need not satisfy any equation of motion). It is remarkable that a degree of freedom enjoys time-translation symmetry on a de Sitter background, which is itself not invariant under time translations.

Through successive Darboux–Crum transformations it is possible to construct a Noether symmetry at each level $n$, built from the $n=0$ time translation symmetry. For the mode $\Psi_n(t)$, consider the variation

$$\delta \Psi_n = \epsilon D_n \Psi_n, \tag{43}$$

where $D_n$ is the derivative operator defined recursively as

$$D_0 = \partial_t, \tag{44}$$

$$D_n = A_{n-1}^+ D_{n-1} A_n^- = A_{n-1}^+ \ldots A_0^+ \partial_t A_1^- \ldots A_n^-, \quad n > 0. \tag{45}$$

Then

$$\delta \Psi_n = \epsilon D_n \Psi_n = \epsilon A_{n-1}^+ D_{n-1} A_n^- \Psi_n. \tag{46}$$

With this variation of the field, the variation of the level $n$ action is

$$\delta S_n[\Psi_n] = \int_{-\infty}^{\infty} dt \, \Psi_n H_n \delta \Psi_n = \epsilon \int_{-\infty}^{\infty} dt \, \Psi_n H_n A_{n-1}^+ D_{n-1} A_n^- \Psi_n. \tag{47}$$
We may use the intertwining relation (19) to replace \( H_n A_n^+ = A_n^+ H_n - 1 \), then integrate by parts so that \( \Psi_n A_n^+ \) may be replaced with \( (A_n^+ \Psi_n) \). The result is

\[
\delta S_n [\Psi_n] = \epsilon \int_{-\infty}^{\infty} dt \left( A_n^+ \Psi_n \right) H_n - 1 \mathcal{D}_n - 1 A_n^+ \Psi_n.
\]

(48)

Upon defining a new field

\[
\chi(t) := A_n^+ \Psi_n(t),
\]

(49)

then \( \delta S_n \) takes the form

\[
\delta S_n [\Psi_n] = \epsilon \int_{-\infty}^{\infty} dt \chi H_n - 1 (\mathcal{D}_n - 1 \chi) = \delta S_{n - 1} [\chi].
\]

(50)

Thus, the variation of the level \( n \) action is equivalent to the variation of the level \( n - 1 \) action. Since \( \delta S_{n - 1} [\chi] \) is a total derivative, it follows by induction that for all \( n \) the variation \( \delta S_n [\chi] \) is also a total derivative. This proves that the transformations

\[
\Psi_n(t) \rightarrow \Psi_n(t) + \epsilon \mathcal{D}_n \Psi_n(t)
\]

(51)

are Noether symmetries.

The conserved quantities associated with these symmetries may be constructed by the usual Noether procedure. Although we are interested in the theory of the real field \( \Psi(x) \), it is convenient to construct expressions for conserved quantities valid for complex fields as well, so that we may evaluate these quantities of our basis solutions \( \psi_{nk}(t) \) which are complex. For the \( n = 0 \) symmetry the conserved quantity is

\[
P_0 := |\partial_t \Psi_0|^2 + k^2 |\Psi_0|^2,
\]

(52)

which we recognize as the usual time translation operator. For \( n > 0 \), the conserved quantity associated to (51) is

\[
P_n := |\partial_t (A_1^- \ldots A_n^- \Psi_n)|^2 + k^2 |A_1^- \ldots A_n^- \Psi_n|^2.
\]

(53)

In essence, this is the conserved quantity \( P_0 \) constructed from a higher-level field \( \Psi_n(t) \) by performing multiple Darboux–Crum transformations to lower the field to level \( n = 0 \).

We now show that the existence of the conserved quantities \( P_n \) implies the property of transparency. Our argument is similar to the one we employed to show that the existence of the conserved quantities \( Q_n \) imply transparency in PT theory. For \( n = 0 \), the conservation of \( P_0 \) implies time translation symmetry which in turn implies that there is no potential, and thus nothing for \( \Psi_0 \) to scatter off. For the \( n > 0 \), we examine asymptotic solutions to the wave equation. Suppose that a level \( n \) solution \( \psi(t) \) is pure positive frequency in the asymptotic future, i.e.,

\[
\psi(t \rightarrow +\infty) = \frac{1}{\sqrt{2|k|}} e^{-ikt} + O \left( e^{-2|t|} \right).
\]

(54)

In the asymptotic past, the solution could, \( a \text{ priori} \), contain both positive and negative frequency branches,

\[
\psi(t \rightarrow -\infty) = \frac{\alpha}{\sqrt{2|k|}} e^{-ikt} + \frac{\beta}{\sqrt{2|k|}} e^{ikt} + O \left( e^{-2|t|} \right).
\]

(55)
Conservation of Klein–Gordon flux, i.e., the Wronskian (11), constrains the coefficients $\alpha$ and $\beta$ to satisfy
\[ |\alpha|^2 - |\beta|^2 = 1. \] (56)
Evaluating $P_n$ on these asymptotic solutions, one obtains
\[ P_n[\psi(t \to +\infty)] = |k| \prod_{j=1}^{n} (k^2 + j^2), \] (57)
and
\[ P_n[\psi(t \to -\infty)] = (|\alpha|^2 + |\beta|^2) |k| \prod_{j=1}^{n} (k^2 + j^2). \] (58)
From these expressions we see that conservation of $P_n$ requires
\[ |\alpha|^2 + |\beta|^2 = 1. \] (59)
The two requirements (56) and (59) are satisfied only for $\beta = 0$ and $|\alpha|^2 = 1$. Thus, conservation of $P_n$ implies that any solution at level $n$ which is pure positive frequency in the future must also be pure positive frequency in the past, which is the property of transparency. Since exists a conserved quantity at each level $n$, i.e., there is a conserved quantity $P_{\vec{L}}$ corresponding to each value of angular momentum $\vec{L}$, it follows that the de Sitter background is transparent.

5. Quantization

In this section we turn to quantized Klein–Gordon theory and examine the family of charges found above in this context. We will see that the charges act trivially in the Fock space generated from the in/out vacuum. In other Fock spaces, the charges create particles, and so no finite-particle eigenstates of the charges exist in these spaces.

To quantize the field $\Psi(x)$ we expand in a basis of solutions
\[ \Psi(x) = \sum_{\vec{L}} \left[ a_{\vec{L}} \psi_{\vec{L}}(t) Y_{\vec{L}}(\Omega) + a_{\vec{L}}^\dagger \psi_{\vec{L}}^*(t) Y_{\vec{L}}^*(\Omega) \right]. \] (60)
In this expression we have restored the angular momenta labels; $\psi_{\vec{L}}(t)$ are the same solutions $\psi_{nk}(t)$, i.e. (9), used throughout our study. For a real field, $a_{\vec{L}}$ and $a_{\vec{L}}^\dagger$ are hermitian conjugates. Upon canonical quantization, the coefficients $a_{\vec{L}}, a_{\vec{L}}^\dagger$ are promoted to creation and annihilation operators which satisfy the canonical commutation relations
\[ [a_{\vec{L}}, a_{\vec{L}}^\dagger] = \delta_{\vec{LL}'}, \quad [a_{\vec{L}}, a_{\vec{L}}^\prime] = [a_{\vec{L}}^\dagger, a_{\vec{L}}^\dagger] = 0. \] (61)
We define a vacuum state $|\Omega\rangle$ as the state for which
\[ a_{\vec{L}} |\Omega\rangle = 0, \quad \forall \vec{L}. \] (62)
Then the 1-particle Fock space built from $|\Omega\rangle$ is
\[ \mathcal{H}_1 := \left\{ |E\rangle = a_{\vec{L}}^\dagger |\Omega\rangle \quad \forall \vec{L} \right\}. \] (63)
As is well known, the canonical quantization procedure described above depends upon
the choice of basis solutions used to expand the field in (60). Different choices for the basis
solutions result in different vacua. The most common choices of vacuum correspond to the
family of de Sitter-invariant vacua known as Mottola–Allen (MA) vacua (or \( \alpha \)-vacua) \([25,26]\).

In many applications, the most logical vacuum is that of the Hartle–Hawking⁴ state which
is in thermal equilibrium with the background geometry \([34]\). In contrast, the modes \( \psi_L(t) \)
given in (9), which are pure positive frequency in the asymptotic future and past, define the de
Sitter-invariant ‘in/out’ vacuum \(|\Omega\rangle\).⁵

The creation and annihilation operators of different vacua are related via Bogoliubov trans-
f ormations. For example, if \( a_L, a_L^\dagger \) denote the creation and annihilation operators of another
MA vacuum \(|\Omega\rangle\), then
\[
a_L = \alpha_L a_L + \beta_L a_L^\dagger,
\]
where the Bogoliubov coefficients satisfy
\[
|\alpha_L|^2 - |\beta_L|^2 = 1.
\]

Explicit formulas for these coefficients may be found in, e.g., \([6]\). When \( \beta_L \neq 0 \), as is the case
for different MA vacua, one vacuum will contain particles relative to another’s particle basis.
Casually, we may view the state \(|\Omega\rangle\) as an infinite-particle state in the particle basis of \(|\Omega\rangle\),
and vice versa. Technically, however, the Bogoliubov transformation which relates these states
may not be implementable as a unitary transformation, and so two MA vacua may not exist in
the same Fock space \([25]\).

We may obtain an expression for the \( P_L \) charges in the Fock space of the in/out vacuum by
inserting the expansion (60) into (53). This yields
\[
P_L = |k| \left( \prod_{j=1}^n (k^2 + j^2) \right) \left( a_L^\dagger a_L + a_L a_L^\dagger \right),
\]
where as usual \( n = L + d/2 - 1 \). It is natural to normal order this operator with respect to
in/out vacuum. This amounts to subtracting the vacuum expectation value:
\[
:\mathcal{P}_L:|\Omega\rangle = P_L - \langle \Omega | P_L |\Omega\rangle = |k| \left( \prod_{j=1}^n (k^2 + j^2) \right) a_L^\dagger a_L.
\]

⁴This state is also known as the Bunch–Davies or Euclidean state. This is the state whose correlation functions may
be obtained by analytic continuation from Euclidean signature. One may also construct this state by considering a
massless scalar field in \( D + 1 \) dimensional Minkowski spacetime, and defining a state on de Sitter as the restriction
of the usual Minkowski vacuum to the de Sitter hyperboloid \([33]\).

⁵While not relevant to our discussion, we note that the in/out vacuum has features which make it undesirable for many
applications. Like all MA vacua excepting the Hartle–Hawking state, correlation functions of this state have ultraviolet
(UV) behavior which differs from the Hadamard form \([35]\), and so differs from that of the usual Minkowski vacuum
\([36]\). The Hadamard form is a necessary ingredient in established approaches to axiomatic quantum field theory in
curved spacetime (see, e.g., \([37, 38]\) and references therein). The non-Hadamard UV behavior of the in/out vacuum
causes various subtleties even for linearized fields (e.g., \([39, 40]\)), and presents significant challenges to formulating
perturbative interactions \([41–43]\). Nevertheless, MA vacua like the in/out vacuum may play an interesting role in
approaches to a dS/CFT correspondence \([6, 44]\).
We see that the normal-ordered charges are proportional to the number operator $N_L = a_L^{\dagger} a_L$ which counts the number of quanta with angular momentum $\vec{L}$. Thus, the $P_L$ simply on 1-particle states; indeed, their action is analogous to that of the $Q_n$ operators in PT theory. The $P_L$ also have trivial co-product on the multi-particle Fock space. The simple action of the $P_L$ in this Fock space can be attributed to the fact that the basis solutions which define particle states have only a single asymptotic behavior in the past and future, and thus define eigenstates of the $P_L$.

An expression for the $P_L$ charges in other Fock spaces may be obtained by transforming the creation and annihilation operators in (66) according to (64), then normal ordering with respect to the new vacuum. The result takes the form

$$\mathcal{P}_L \mathcal{T} = |k| \left( \prod_{j=1}^{n} (k^2 + j^2) \right) \left[ (|\alpha_L|^2 + |\beta_L|^2) \mathcal{P}_L \mathcal{T} + 2\alpha_L^* \beta_L \mathcal{P}_L \mathcal{T} + 2\alpha_L \beta_L^* \mathcal{P}_L \mathcal{T} \right].$$

This expression is sufficient to see that in Fock spaces other than the in/out Fock space, the $P_L$ charges will generate particles, even when acting on the vacuum $|\Omega\rangle$. Thus, there are no finite-particle number eigenstates of $\mathcal{P}_L \mathcal{T}$ in these Fock spaces.

6. Discussion

In this paper we have shown that massive Klein–Gordon theory on an odd-dimensional de Sitter background enjoys an infinite set of symmetries which are ‘hidden’ in the sense that they do not generate isometries nor are they internal symmetries of the field theory. Each symmetry acts on a single Klein–Gordon mode. Correspondingly, there is a Noether charge $P_L$ for each value of angular momentum. Conservation of these charges requires that the solutions to the Klein–Gordon equation have the same asymptotic behavior, up to a phase, in the asymptotic past and future. Upon quantization, the quantum charges $\mathcal{P}_L \mathcal{T}$ act simply on particle states belonging to the Fock space of the in/out vacuum (these are eigenstates of the charges). In the Fock spaces of all other Mottola–Allen vacua, the charges generate particles.

The construction of our family of conserved quantities relies on two ingredients: first, Darboux–Crum transformations which relate field modes, and second, the existence of a conserved quantity $P_0$ which acts only on the zero angular momentum mode. We have shown that the former may be regarded as a consequence of the boost isometry of the de Sitter background. Indeed, an alternative way to construct an infinite family of charges is to boost the charge $P_0$. It is easy to see that boosts do not commute with $P_0$. For instance, if one represents $P_0$ as a flux integral through a surface of constant global time, then an infinitesimal boost acting on this charge deforms the Cauchy surface and defines a new conserved quantity. By repeatedly boosting $P_0$, one can construct an infinite family of charges which act upon all modes of the Klein–Gordon field. This family of charges is not identical to the family we have constructed—our construction results in less cumbersome expressions for charges—but the two families are quite analogous. The upshot is that a conserved quantity constructed from the zero mode alone, when combined with a boost isometry, results in an infinite family of conserved quantities.

We expect quite similar results to hold for other linearized fields on de Sitter. Indeed, much of the groundwork in understanding these cases has already be laid by [5] which demonstrated that spin-half, symmetric tensor, and p-form fields all enjoy the Darboux–Crum structure crucial to our analysis. These authors also show that these fields are governed by transparent potentials when the spacetime dimension is odd. Thus, we expect that all such linearized fields
enjoy an infinite family of conserved charges, at least when the spacetime dimension is odd. It would be interesting to see these charges in detail.

We have confined attention to scalar fields with positive mass such that $k^2 = M^2 \ell^2 - d^2/4 > 0$, i.e., masses greater than the de Sitter scale. Such positive mass fields belong to the principle series of scalar representations of the de Sitter isometry group. Restricting to real $k$ ensures that wavefunctions of the PT potentials are scattering states, i.e., states whose energy is greater than the asymptotic value of the potential. It would be interesting to consider lighter, yet still positive mass, fields satisfying $M^2 \ell^2 > 0$ but $k^2 < 0$, which belong to the complementary series of scalar representations of the de Sitter group. For these fields, the associated wavefunctions in PT theory contain exponential growth far from the potential, and thus are not typically regarded as physical wavefunctions in quantum mechanics. Nevertheless, we expect the charge structure to exist, at least for $k^2 \neq -Z$. When $k^2 \neq -Z$, the eigenvalues of the charges $Q_n$ in quantum mechanics (similarly, $P_L$ in de Sitter) include zero, and we expect exceptional behavior to occur. Indeed, these values of $k$ correspond to bound states in PT theory. Bound states have different asymptotic behavior on either side of the potential well; thus, for these values of $k$ the potential is not transparent. Similarly, it would be interesting to examine the case of a massless field, which corresponds to $k^2 = -d^2/4$.

We close by commenting on how a similar charge structure might arise in an interacting field theory on de Sitter. The key ingredients needed to construct the family of charges were the boost isometry of de Sitter space and a charge which acts on the zero angular momentum mode. So long as the field theory interactions are generally covariant and the background is fixed, the de Sitter boost isometry will be present. It remains, then, to find examples of interacting theories for which the zero angular momentum sector has a conserved quantity. The explicit construction of such an interacting theory is an open challenge.

Acknowledgments

IAM thanks the Kavli Institute for Theoretical Physics for its hospitality during early stages of this project. GC was supported by the West Chester University Summer Undergraduate Research Institute (Summer 2019).

ORCID iDs

Ian A Morrison https://orcid.org/0000-0002-8751-3591

References

[1] Birrell N D and Davies P C W 1982 Quantum Fields in Curved Space (Cambridge, UK: Cambridge University Press) p 340
[2] Mukhanov V 2005 Physical Foundations of Cosmology (Oxford: Cambridge University Press)
[3] Witten E 2001 arXiv:hep-th/0106109
[4] Marolf D, Morrison I A and Srednicki M 2013 Class. Quantum Grav. 30 155023
[5] Lagogiannis P, Maloney A and Wang Y 2011 arXiv:1106.2846
[6] Bousso R, Maloney A and Strominger A 2002 Phys. Rev. D 65 104039
[7] Poschl G and Teller E 1933 Z. Phys. 83 143–51
[8] Cooper F, Khare A and Sukhatme U 1995 Phys. Rep. 251 267–385
[9] Gangopadhyaya A, Mallow J V and Rasinariu C 2017 Supersymmetric Quantum Mechanics (Singapore: World Scientific)
[10] Goldstone J and Jackiw R 1975 Phys. Rev. D 11 1486–498
[11] Dashen R F, Hasslacher B and Neveu A 1975 Phys. Rev. D 12 2443
[12] Grant A K and Rosner J L 1994 J. Math. Phys. 35 2142–56
[13] Dunajski M 2010 Solitons, Instantons, and Twistors (Oxford: Oxford University Press)
[14] Babelon O, Bernard D and Talon M 2003 Introduction To Classical Integrable Systems (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)
[15] Zwiebach B 2000 J. High Energy Phys. JHEP09(2000)028
[16] Minahan J A and Zwiebach B 2000 J. High Energy Phys. JHEP09(2000)029
[17] Barut A O, Inomata A and Wilson R 1987 J. Phys. A 20 4083
[18] Costa R and Morrison I A 2016 J. High Energy Phys. JHEP03(2016)056
[19] Matveev V V and Salle M A 1991 Darboux–Crum Transformations and Solitons (Berlin: Springer)
[20] Correa F and Plyushchay M S 2007 Ann. Phys. 322 2493–500
[21] Correa F, Jakubsky V and Plyushchay M S 2009 Ann. Phys. 324 1078–94
[22] Arancibia A, Mateos Guilarte J and Plyushchay M S 2013 Phys. Rev. D 87 045009
[23] Plyushchay M S 2018 Nonlinear supersymmetry as a hidden symmetry (arXiv:1811.11942)
[24] Evnin O and Nivesvivat R 2017 J. Phys. A 50 015202
[25] Mottola E 1985 Phys. Rev. D 31 754
[26] Allen B 1985 Phys. Rev. D 32 3136
[27] Spradlin M, Strominger A and Volovich I A 2001 arXiv:hep-th/0110007
[28] Cortez J, Mena Marugán G A, Olmedo J and Velhinho J M 2012 Phys. Rev. D 86 104003
[29] Cortez J, Marugán G A M and Velhinho J 2019 arXiv:1912.04203
[30] Higuchi A 1987 J. Math. Phys. 28 1553
[31] Abramowitz M and Stegun I 1972 Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables 10th edn (New York: Dover)
[32] Marolf D and Morrison I A 2009 Class. Quantum Grav. 26 235003
[33] Wyrozumski T 1988 Class. Quantum Grav. 5 1607–13
[34] Gibbons G W and Hawking S W 1977 Phys. Rev. D 15 2738–51
[35] Wald R M 1994 Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics (Chicago, USA: Univ. Pr.) p 205
[36] Brunetti R, Fredenhagen K and Hollands S 2005 J. High Energy Phys. JHEP05(2005)063
[37] Hollands S and Wald R M 2010 Conn. Math. Phys. 293 85–125
[38] Hollands S and Wald R M 2015 Phys. Rep. 574 1–35
[39] Einhorn M B and Larsen F 2003 Phys. Rev. D 67 024001
[40] de Boer J, Jejjala V and Minic D 2005 Phys. Rev. D 71 044013
[41] Einhorn M B and Larsen F 2003 Phys. Rev. D 68 064002
[42] Goldstein K and Lowe D A 2003 Nucl. Phys. B 669 325–40
[43] Goldstein K and Lowe D A 2004 Phys. Rev. D 69 023507
[44] Strominger A 2001 J. High Energy Phys. JHEP10(2001)034