Radii problems for Ma-Minda Starlikeness

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Abstract For the standard Ma-Minda class $S^*(\psi)$ of univalent starlike functions, we derive $S^*(\psi)$-radii for some well-known special functions. In addition, we obtain the set of extremal functions for the classical problem

$$\max_{f \in S^*(\psi)} |\Phi (\log (f(z)/z))| \quad \text{or} \quad \max_{f \in S^*(\psi)} \Re \{\Phi (\log (f(z)/z))\},$$

where $\Phi$ is a non-constant entire function. Moreover, we prove certain results on convolution and radius estimates for the case when $\psi(\mathbb{D})$ is starlike.

Keywords Starlike functions · Radius problems · Bessel functions · Struve and Lommel functions · Legendre polynomials of odd degree

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1 Introduction

Let $A_0$ be the collection of analytic functions of the form $p(z) = 1 + \sum_{n=2}^{\infty} p_n z^n$ and $A$ consists of analytic functions, $f$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ defined in the unit disk $\mathbb{D} := \{z : |z| < 1\}$. The Carathéodory class, $P$ consists of functions $p \in A_0$ with $\Re p(z) > 0$. The class of univalent functions in $A$ is denoted by $S$. In 1992, Ma and Minda [31]
introduced and studied the following classes of starlike and convex univalent functions:

\[ S^*(\psi) := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \psi(z) \right\} \]  (1.1)

and

\[ C(\psi) := \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \psi(z) \right\}, \]

where \( \psi \in P \) is univalent, \( \psi(D) \) is symmetric with respect to real axis and starlike with respect to 1. The symbol \( \prec \) denotes the usual subordination. Note that \( S^*(1+\frac{z}{1-z}) \) reduces to the well-known class \( S^* \) of starlike functions.

Today a good amount of literature exists for the different choices of \( \psi \) in (1.1). For example, one may see [14, 28, 29, 33, 34, 36]. We also introduced and studied (see, [29]) the class of cardioid starlike functions:

\[ S^*_{\varphi} := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \varphi(z) := 1 + E_{1,1} \right\}. \]

where \( E_{\alpha,\beta} \) be the normalized form of the Mittag-Leffler function, also see [3]:

\[ E_{\alpha,\beta}(z) = z + \sum_{n \geq 2} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n, \quad (z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \beta \neq 0, -1, \ldots). \]

Several type of radius problems have been studied in [16, 14, 28, 29, 33, 34, 36]. Let us now recall that

**Definition 1** For the subfamilies \( G_1 \) and \( G_2 \) of \( A \), we say that \( r_0 \) is the \( G_1 \)-radius of the class \( G_2 \), if \( r_0 \in (0, 1) \) is largest number such that \( r^{-1}f(rz) \in G_1 \), \( 0 < r \leq r_0 \) for all \( f \in G_2 \).

Enormous interest in the radius problems regarding special functions started from the work of Brown [11, 12], Wilf [39], and Kreyszig and Todd [27]. Recently, the radii of starlikeness and convexity of some normalized special functions were studied widely for certain Ma-Minda sub-classes as they can be represented as Hadamard factorization under certain conditions. See the work on Bessel functions [1, 37], Struve functions [1, 3], Wright functions [6], Lommel functions [1, 4] and Legendre polynomials of odd degree [9]. We also refer to see [10, 15]. For more on recent radius problems, see [18, 19, 20, 21, 22, 23].

**Bessel Function:** The Bessel function \( J_{\beta} \) of first kind of order \( \beta \in \mathbb{C} \) is a particular solution of the homogeneous Bessel differential equation

\[ z^2w''(z) + zw'(z) + (z^2 - \beta^2)w(z) = 0 \]

and have the following series expansion:

\[ J_{\beta}(z) := \sum_{n \geq 1} \frac{(-1)^n}{n! \Gamma(n + \beta + 1)} \left( \frac{z}{2} \right)^{2n+\beta}, \]
where $z \in \mathbb{C}$ and $\beta \notin \mathbb{Z}^-$. Let us consider the following three normalized functions expressed in terms of $\mathcal{J}_\beta(z)$

$$
\begin{align*}
  f_\beta(z) &= \left(2^{\beta} \Gamma(\beta + 1)\mathcal{J}_\beta(z)\right)^{1/\beta} = z - \frac{1}{3(\beta+1)}z^3 + \cdots, \quad \beta \neq 0
  \\
g_\beta(z) &= 2^{\beta} \Gamma(\beta + 1)z^{1-\beta}\mathcal{J}_\beta(z) = z - \frac{1}{3(\beta+1)}z^3 + \cdots
  \\
h_\beta(z) &= 2^{\beta} \Gamma(\beta + 1)z^{1-\beta/2}\mathcal{J}_\beta(\sqrt{z}) = z - \frac{1}{4(\beta+1)}z^2 + \cdots.
\end{align*}
$$

(1.2)

Since the zeros of $\mathcal{J}_\beta$ are real if $\beta > 0$, therefore using the Weierstrass decomposition, we have for $\beta > 0$:

$$
\mathcal{J}_\beta(z) := \frac{z^\beta}{2^\beta \Gamma(\beta + 1)} \prod_{n \geq 1} \left(1 - \frac{z^2}{j_{\beta,n}^2}\right),
$$

where $j_{\beta,n}$ is the $n$-th positive zero of $\mathcal{J}_\beta$ and satisfies $j_{\beta,n} < j_{\beta,n+1}$ for $n \in \mathbb{N}$. Thus we have

$$
z\frac{\mathcal{J}_\beta'(z)}{\mathcal{J}_\beta(z)} = \beta - \sum_{n \geq 1} \frac{2z^2}{j_{\beta,n}^2 - z^2}.
$$

(1.3)

**Struve function:** The Struve function $\mathbf{H}_\beta$ of first kind is a particular solution of the second-order inhomogeneous Bessel differential equation

$$
z^2 w''(z) + zw'(z) + (z^2 - \beta^2)w(z) = \frac{4(\frac{\pi}{2})^{\beta+1}}{\sqrt{\pi} \Gamma(\beta + \frac{1}{2})},
$$

and have the following form:

$$
\mathbf{H}_\beta(z) := \frac{\left(\frac{\pi}{4}\right)^{\beta+1} \sqrt{\pi} \Gamma(\beta + \frac{1}{2})}{\Gamma(\beta + 1/2)} \, _1F_2\left(1; \frac{3}{2}; \beta + \frac{3}{2}; -\frac{z^2}{4}\right),
$$

where $-\beta - \frac{1}{2} \notin \mathbb{N}$ and $_1F_2$ is a hypergeometric function. Since it is not normalized, so we consider the following normalized functions involving $\mathbf{H}_\beta$:

$$
\begin{align*}
  U_\beta(z) &= \left(\sqrt{\pi}2^\beta(\beta + \frac{1}{2})\mathbf{H}_\beta(z)\right)^{-\frac{1}{\beta}}
  \\
  V_\beta(z) &= \sqrt{\pi}2^\beta z^{-\beta} \Gamma(\beta + \frac{1}{2})\mathbf{H}_\beta(z)
  \\
  W_\beta(z) &= \sqrt{\pi}2^\beta z^{1-\beta} \Gamma(\beta + \frac{1}{2})\mathbf{H}_\beta(\sqrt{z}).
\end{align*}
$$

(1.4)

Moreover, for $|\beta| \leq \frac{1}{2}$, it has the Hadamard factorization given by

$$
\mathbf{H}_\beta(z) = \frac{z^{\beta+1}}{\sqrt{\pi}2^\beta \Gamma(\beta + 1/2)} \prod_{n \geq 1} \left(1 - \frac{z^2}{z_{\beta,n}^2}\right),
$$

(1.5)

where $z_{\beta,n}$ is the $n$-th positive root of $\mathbf{H}_\beta$ such that $z_{\beta,n+1} > z_{\beta,n}$ and $z_{\beta,1} > 1$ and also from (1.4), we obtain

$$
z\frac{\mathbf{H}_\beta'(z)}{\mathbf{H}_\beta(z)} = (\beta + 1) - \sum_{n \geq 1} \frac{2z^2}{z_{\beta,n}^2 - z^2}.
$$

(1.6)
**Lommel function:** The Lommel function \( \mathcal{L}_{u,v} \) of first kind is a particular solution of the second-order inhomogeneous Bessel differential equation
\[
z^2 w''(z) + z w'(z) + (z^2 - v^2)w(z) = z^{u+1},
\]
where \( u \pm v \notin \mathbb{Z}^- \) and is given by
\[
\mathcal{L}_{u,v} = \frac{z^{u+1}}{(u-v+1)(u+v+1)} {}_1F_2 \left( 1; \frac{u-v+3}{2}, \frac{u+v+3}{2}; -\frac{z^2}{4} \right),
\]
where \( \frac{1}{2}(-u \pm v - 3) \notin \mathbb{N} \) and \( {}_1F_2 \) is a hypergeometric function. Since it is not normalized, so we consider the following normalized functions involving \( \mathcal{L}_{u,v} \):
\[
\begin{cases}
f_{u,v}(z) = ((u-v+1)(u+v+1)\mathcal{L}_{u,v}(z))^{1/2}, \\
g_{u,v}(z) = (u-v+1)(u+v+1)z^{-u}\mathcal{L}_{u,v}(z), \\
h_{u,v}(z) = (u-v+1)(u+v+1)\frac{z^2}{2}\mathcal{L}_{u,v}(\sqrt{z}).
\end{cases}
\]
Authors in \([1,4]\) obtained the radius of starlikeness for the following normalized functions expressed in terms of \( \mathcal{L}_{u,v} \):
\[
\begin{aligned}
f_{u-\frac{1}{2},\frac{1}{2}}(z), & \quad g_{u-\frac{1}{2},\frac{1}{2}}(z) \quad \text{and} \quad h_{u-\frac{1}{2},\frac{1}{2}}(z),
\end{aligned}
\]
where \( 0 \neq u \in (-1, 1) \).

**Legendre polynomial:** The Legendre polynomials \( P_n \) are the solutions of the Legendre differential equation:
\[
((1-z^2)P_n'(z))' + n(n+1)P_n(z) = 0,
\]
where \( n \in \mathbb{Z}^+ \) and using Rodrigues' formula, \( P_n \) can be represented in the form:
\[
P_n(z) = \frac{1}{2^n n!} \frac{d^n(z^2-1)^n}{dz^n}
\]
and it also satisfies the geometric condition \( P_n(-z) = (-1)^n P_n(z) \). Moreover, the odd degree Legendre polynomials \( P_{2n-1} \) have only real roots which satisfy
\[
0 = z_0 < z_1 < \cdots < z_{n-1} \quad \text{or} \quad -z_1 > \cdots > -z_{n-1}.
\]
Thus, the normalized form is as follows:
\[
\mathcal{P}_{2n-1}(z) := \frac{P_{2n-1}(z)}{P_{2n-1}(0)} = z + \sum_{k=2}^{2n-1} a_k z^k = a_{2n-1} z \prod_{k=1}^{n-1} (z^2 - z_k^2).
\]

At this conjunction, motivated from the work \([1,4,6,9,10,21,37]\) it is natural to consider the radius problem:

**Problem 1.1** Find the \( S^*(\psi) \)-radii for the normalized functions given in (1.2), (1.4), (1.8) and (1.10).
That is, $S^*(\psi)$-radius and $C(\psi)$-radius of $g \in A$ is defined as follows:

$$r_0(g) = \sup\{r \in (0, r_0) : \frac{zg'(z)}{g'(z)} \in \psi(D), z \in D_{r_0}\}$$

and

$$r_0(g) = \sup\{r \in (0, r_0) : 1 + \frac{2g''(z)}{g'(z)} \in \psi(D), z \in D_{r_0}\}.$$
Theorem 2.1 (Bessel function $J_\beta$) Let $\beta > 0$. Then the $S^*(\psi)$-radii $r_\psi(f_\beta)$, $r_\psi(g_\beta)$ and $r_\psi(h_\beta)$ of the functions $f_\beta$, $g_\beta$ and $h_\beta$ as given by (1.2) are the smallest positive root of the following equations, respectively:

(i) $rJ'_\beta(r) - \beta(1-r_1)J_\beta(r) = 0$;
(ii) $rJ'_\beta(r) - (\beta - r_1)J_\beta(r) = 0$;
(iii) $\sqrt{r}J'_\beta(\sqrt{r}) - (\beta - 2r_1)J_\beta(\sqrt{r}) = 0$.

The radii are sharp when $\psi(-1) = 1 - r_1$, where $r_1$ is radius of largest disk inside $\psi(D)$.

Theorem 2.2 (Struve function $H_\beta$) Let $|\beta| \leq 1/2$. Then the $S^*(\psi)$-radii $r_\psi(U_\beta)$, $r_\psi(V_\beta)$ and $r_\psi(W_\beta)$ of the functions $U_\beta$, $V_\beta$ and $W_\beta$ as given by (1.4) are the smallest positive root of the following equations, respectively:

(i) $rH'_\beta(r) - (1 - r_1)(\beta + 1)H_\beta(r) = 0$;
(ii) $rH'_\beta(r) - ((1 + \beta) - r_1)H_\beta(r) = 0$;
(iii) $\sqrt{r}H'_\beta(\sqrt{r}) - (1 + \beta - 2r_1)H_\beta(\sqrt{r}) = 0$.

The radii are sharp when $\psi(-1) = 1 - r_1$, where $r_1$ is radius of largest disk inside $\psi(D)$.

For the convenience of notations, functions defined in (1.8) are written as $f_u, g_u$ and $h_u$, respectively.

Theorem 2.3 (Lommel function $\mathcal{L}_{u,\nu}$) Let $0 \neq u \in (-1,1)$ and write $\mathcal{L}_{u,\nu}^{-1}(z) =: \mathcal{L}_{\nu}(z)$. Then the $S^*(\psi)$-radii $r_\psi(f_u)$, $r_\psi(g_u)$ and $r_\psi(h_u)$ of the functions $f_u$, $g_u$ and $h_u$ given by (1.8) are the smallest positive root of the following equations, respectively:

(i) $\begin{cases} 2r\mathcal{L}_{\nu}^u(r) - (2u + 1)(1 - r_1)\mathcal{L}_{\nu}(r) = 0, & \text{for } u \in (-\frac{1}{2},1) \\
2r\mathcal{L}_{\nu}^u(r) - (2u + 1)(1 + r_1)\mathcal{L}_{\nu}(r) = 0, & \text{for } u \in (-1,-\frac{1}{2}) \end{cases}$;
(ii) $2r\mathcal{L}_{\nu}^u(r) - (2u + 1 - 2r_1)\mathcal{L}_{\nu}(r) = 0$;
(iii) $2\sqrt{r}\mathcal{L}_{\nu}^u(\sqrt{r}) - (2u + 1 - 4r_1)\mathcal{L}_{\nu}(\sqrt{r}) = 0$.

The radii are sharp when $\psi(-1) = 1 - r_1$, where $r_1$ is radius of largest disk inside $\psi(D)$.

Theorem 2.4 (Legendre polynomials $\mathcal{P}_n$) The $S^*(\psi)$-radius $r_\psi(P_{2n-1}) \in (0,z_1)$ of the normalized odd degree Legendre polynomial is the smallest positive root of the following equation:

$rP'_{2n-1}(r) - (1 - r_1)P_{2n-1}(r) = 0$.

The radii are sharp when $\psi(-1) = 1 - r_1$, where $r_1$ is radius of largest disk inside $\psi(D)$.

The following result covers many celebrated and newly introduced classes:

Corollary 2.1 Let $\alpha = r_1$ be the radius of the largest disk $\{w : |w - 1| < \alpha\}$ inside $\psi(D)$, where
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\[ \alpha = \min \left\{ 1 - \frac{1-A^2}{1+B^2}, 1 - \frac{1-B^2}{1+A^2} \right\} = \frac{A-B}{1+AB} \text{ when } \psi(z) = \frac{z+B}{1+AB}, \text{ where } -1 \leq B < A \leq 1; \]

\[ \alpha = \sqrt{2 - 2\sqrt{2} + \sqrt{-2 + 2\sqrt{2}} \text{ when } \psi(z) = \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1-z}{1+z}};} \]

\[ \alpha = \sqrt{2} - 1 \text{ when } \psi(z) = \sqrt{1+z}; \]

\[ \alpha = e - 1 \text{ when } \psi(z) = e^z; \]

\[ \alpha = 2 - \sqrt{2} \text{ when } \psi(z) = z + \sqrt{1+z^2}; \]

\[ \alpha = \frac{e}{e-1} \text{ when } \psi(z) = \frac{z}{1+e^z}; \]

\[ \alpha = 1 + \sin z \text{ when } \psi(z) = 1 + \sin z; \]

\[ \alpha = 1 - e^{z^{-1}-1} \text{ when } \psi(z) = e^{z^{-1}-1}; \]

\( \text{for the domains bounded by the conic sections (see [22]) } \Omega_\kappa := \{ w = u + iv : u^2 > \kappa^2(u-1)^2 + \kappa^2 v^2; \kappa \in [0, \infty) \}, \text{ we have } \)

\[ \alpha = \frac{1}{\kappa + 1}, \]

where the boundary curve of \( \Omega_\kappa \) for fixed \( \kappa \) is represented by the imaginary axis \( (\kappa = 0) \), the right branch of a hyperbola \( (0 < \kappa < 1) \), a parabola \( (\kappa = 1) \) and an ellipse \( (\kappa > 1) \). The univalent Carathéodory functions mapping \( \mathbb{D} \) onto \( \Omega_\kappa \) is given by

\[ \psi(z) := \psi_\kappa(z) = \begin{cases} 
\frac{1+z}{1-z} & \text{for } \kappa = 0; \\
1 + \frac{2}{\sqrt{1-\kappa^2}} \sinh^2(A(\kappa) \arctanh(\sqrt{2})) & \text{for } \kappa \in (0, 1); \\
1 + \frac{2}{\sqrt{1-\kappa^2}} \log(\sqrt{1+\kappa^2}) & \text{for } \kappa = 1; \\
1 + \frac{2}{2K(t)} \sin^2 \left( \frac{\pi}{2K(t)} F \left( \frac{\sqrt{2}}{\sqrt{1+t}} , t \right) \right) & \text{for } \kappa > 1,
\end{cases} \]

where \( A(\kappa) = (2/\pi) \arccos(\kappa), \) \( F(w,t) = \int_0^w \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}} \) is the Legendre elliptic integral of the first kind, \( K(t) = F(1,t) \) and \( t \in (0, 1) \) is chosen such that \( \kappa = \cosh(\pi K'(t)/2K(t)). \)

Then Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4 hold true for the class \( S^*(\psi) \) for the above choices of \( \psi \), respectively. The radii are sharp.

**Remark 2.1** Part (i) of the Corollary 2.1 generalizes several results given in [11], [12], [13], [14] for radius of starlikeness of order \( \eta \in [0, 1) \).

**Remark 2.2** In the above corollary part (iv), Theorem 2.1 simplify the results [10] Theorem 4.2, p. 119, [10] Theorem 4.3, p. 120 and [10] Theorem 4.4, p. 122.

**Remark 2.3** It is worth to mention that the function of the form \( \psi(z) = 1 + \alpha z \) serve as an extremal to the Problem 1.1 which indeed reduces a lot of calculation.

We shall see that in view of the Remark 2.3 it is sufficient to study the above general cases with the help of the class \( S^*_\psi \) for the simplicity.
Proof (Proof of Theorem 2.1) Here we have $r_1 = 1/e$. We now prove the first part. Using the representation (1.3) and equation (1.2), we get

$$
\frac{zf'_\beta(z)}{f_\beta(z)} = \frac{zJ'_\beta(z)}{\beta J_\beta(z)} = 1 - \frac{1}{\beta} \sum_{n \geq 1} \frac{2z^2}{j^2_\beta,n - z^2}, \quad (2.1)
$$

Further using a result [17, Lemma 3.2, p. 10] and from (2.1), $|z| = r < j_\beta,1$ we obtain

$$
\left| \frac{zf'_\beta(z)}{f_\beta(z)} - a \right| \leq \frac{2}{\beta} \sum_{n \geq 1} \frac{j^2_\beta,n r^2}{j^4_\beta,n - r^4}, \quad (2.2)
$$

where $a := 1 - \frac{1}{e} \sum_{n \geq 1} \frac{r^4}{j^2_\beta,n - r^4}$ and $j_\beta,n$ denotes the $n$-th positive zero of the Bessel function $J_\beta$. Also a simple calculation shows that $a \leq 1$. Thus for the disk (2.2) to lie inside $\wp(D)$, we need only to consider that $1 - \frac{1}{e} < a < 1 + \frac{1}{e}$, and so by Lemma 1.1, we have

$$
\frac{2}{\beta} \sum_{n \geq 1} \frac{j^2_\beta,n r^2}{j^4_\beta,n - r^4} \leq a - 1 + \frac{1}{e} = \frac{1}{e} - \frac{2}{\beta} \sum_{n \geq 1} \frac{r^4}{j^2_\beta,n - r^2}, \quad (2.3)
$$

or equivalently,

$$
\frac{2}{\beta} \sum_{n \geq 1} \frac{r^2}{j^2_\beta,n - r^2} - \frac{1}{e} \leq 0. \quad (2.4)
$$

Also using (2.1), (2.3) can be written as $e \frac{r J'_\beta(r)}{\beta J_\beta(r)} + 1 - e \geq 0$. Note that in view of Lemma 1.1, we can also obtain (2.3) directly from (2.1). Moreover, in (2.3) we only need to replace $1/e$ by $r_1$ in view of Assumption 1.1 for a given $\psi$ for the general proof. Hence, without any loss of generality, we further proceed in general settings. Now let us consider the strictly decreasing continuous function

$$
\Psi(r) := \frac{1}{e} - \frac{2}{\beta} \sum_{n \geq 1} \frac{r^2}{j^2_\beta,n - r^2} = r_1 - \frac{2}{\beta} \sum_{n \geq 1} \frac{r^2}{j^2_\beta,n - r^2}, \quad r \in (0, j_\beta,1).
$$

Then $\lim_{r \to 0} \Psi(r) = r_1 = 1/e > 0$ and $\lim_{r \to j_\beta,1} \Psi(r) = -\infty$. Also $\Psi'(r) < 0$, since $r < j_\beta,1$. So we may assume $r_\psi(f_\beta)$ be the unique root of $\Psi(r) = 0$ in $(0, j_\beta,1)$ such that $f_\beta$ is $S^*(\psi)$ in $|z| < r_\psi(f_\beta)$.

Let us denote $r_{f_\beta} = r_\psi(f_\beta)$. Then from (2.1) and (2.3), we see that

$$
\frac{r_{f_\beta} f'_\beta(r_{f_\beta})}{f_\beta(r_{f_\beta})} = 1 - r_1, \quad (2.4)
$$

where $r_1$ depends on $\psi(D)$ in view of Assumption 1.1 and hence, $f_\beta$ belongs to $S^*(1 + r_1 z)$ in $|z| < r_{f_\beta}$.
Now let \( r_1 \in (0, 1] \) such that \( w_1 := \{ w : |w - 1| < r_1 \} \) is the maximal disk inside \( \psi(\mathbb{D}) \). Let us write

\[
F_{\beta}(z) = \frac{zf_{\beta}'(z)}{f_{\beta}(z)}.
\]

Since a function \( f(z) \in S^*(\psi) \) if and only if \( e^{-i\pi} f(e^{it}z) \in S^*(\psi) \) for all \( t \in \mathbb{R} \). Therefore, using (2.2) and (2.4) with \( z = r_{\beta} \), along with \( \psi(-1) = 1 - \alpha \), the maximality of the disk \( w_{\alpha} \) implies that \( F_{\beta}|\{z | \leq r\} \) do not lie inside \( \psi(\mathbb{D}) \) for \( r \geq r_{\beta} \) for some suitable rotation of \( f_{\beta} \). Hence, \( f_{\beta} \) belongs to \( S^*(\psi) \) in \( |z| < r_{\beta} \) and the radius \( r_{\beta} \) is sharp.

Proof of other parts follows similarly.

**Proof of Theorem 2.2** From (1.4) and (1.6), by logarithmic differentiation, we get

\[
\begin{aligned}
\frac{zU_{\beta}'(z)}{U_{\beta}(z)} &= \frac{1}{\beta + 1} + \sum_{n \geq 1} \frac{2z^2}{\beta_n - r} \\
\frac{zV_{\beta}'(z)}{V_{\beta}(z)} &= -\beta + \sum_{n \geq 1} \frac{2z^2}{\beta_n - r} \\
\frac{zW_{\beta}'(z)}{W_{\beta}(z)} &= \frac{1 - \beta}{2} + \sum_{n \geq 1} \frac{2z^2}{\beta_n - r}.
\end{aligned}
\tag{2.5}
\]

Now applying the inequality \( |x| - |y| \leq |x - y| \) and using the Assumption 1.1 in (2.5), we see that \( U_{\beta}, V_{\beta} \) and \( W_{\beta} \) belongs to \( S^*(\psi) \), respectively whenever

\[
\begin{aligned}
\left| \frac{zU_{\beta}'(z)}{U_{\beta}(z)} - 1 \right| &\leq \frac{1}{\beta + 2} \sum_{n \geq 1} \frac{2z^2}{\beta_n - r} \leq r_1, \\
\left| \frac{zV_{\beta}'(z)}{V_{\beta}(z)} - 1 \right| &\leq \sum_{n \geq 1} \frac{2z^2}{\beta_n - r} \leq r_1, \\
\left| \frac{zW_{\beta}'(z)}{W_{\beta}(z)} - 1 \right| &\leq \sum_{n \geq 1} \frac{z}{\beta_n - r} \leq r_1
\end{aligned}
\tag{2.6}
\]

holds, where \( |z| = r < z_{\beta,1} \). Now to find the largest positive radius for which (2.6) holds. Let us consider the strictly increasing continuous functions

\[
\Psi_1(r) := \frac{1}{\beta + 1} \sum_{n \geq 1} \frac{2r^2}{\beta_n - r^2} - r_1, \quad \Psi_2(r) := \sum_{n \geq 1} \frac{2r^2}{\beta_n - r^2} - r_1
\]

and

\[
\Psi_3(r) := \sum_{n \geq 1} \frac{r}{\beta_n - r} - r_1.
\]

Since \( \lim_{r \to 0} \Psi_i(r) < 0 \), \( \Psi_i'(r) > 0 \) for \( i = 1, 2 \) and \( \lim_{r \to 2z_{\beta,1}} \Psi_3(r) > 0 \), there exist the unique positive roots, \( r_0(U_{\beta}), r_0(V_{\beta}) \in (0, z_{\beta,1}) \) and \( r_0(W_{\beta}) \in (0, z_{\beta,1}) \) for \( \Psi_i \), respectively so that the inequalities in (2.6) holds in \( |z| < r_0(U_{\beta}), |z| < r_0(V_{\beta}) \) and \( |z| < r_0(W_{\beta}) \), respectively. Further using (2.5) in \( \Psi_i(r) = 0 \), respectively, we obtain the desired equations. Further, following the proof of Theorem 2.1 sharpness of the radii follows.
Proof (Proof of Theorem 2.2) We prove the first part. Let \( 0 \neq u \in (0,1) \). Then using a result from \([8]\) (also see \([4, \text{Lemma 1, p. 3358}]\)), we can write the Lommel function \( \mathcal{L}_{\frac{u}{2}, \frac{1}{2}} \) as follows:

\[
\mathcal{L}_{\frac{u}{2}, \frac{1}{2}}(z) = \frac{z^{u+\frac{1}{2}}}{u(u+1)} F_2 \left( 1; \frac{u+2}{2}, \frac{u+3}{2}, \frac{z^2}{4} \right) = \frac{z^{u+\frac{1}{2}}}{u(u+1)} \phi_0(z), \tag{2.7}
\]

where

\[
\phi_0(z) = \prod_{n \geq 1} \left( 1 - \frac{z^2}{z_{u,0,n}^2} \right),
\]

and \( z_{u,0,n} \) is the simple and real \( n \)-th positive root of \( \phi_0 \). Also \( z_{u,0,n} \in (n\pi, (n+1)\pi) \) which ensures \( z_{u,0,n} > z_{u,0,1} > \pi > 1 \). Now with this representation, after logarithmic differentiation, from (1.7) we get

\[
\frac{zf'_u(z)}{f_u(z)} = \frac{z\mathcal{L}'_{\frac{u}{2}, \frac{1}{2}}(z)}{(u+\frac{1}{2})\mathcal{L}_{\frac{u}{2}, \frac{1}{2}}(z)} = 1 - \frac{1}{u+\frac{1}{2}} \sum_{n \geq 1} \frac{2z^2}{z_{u,0,n}^2 - z^2}.
\]

Using the triangle inequality and the Assumption 1.1 we have \( f_u \in S^*(\psi) \) provided

\[
T(r) := \frac{1}{u+\frac{1}{2}} \sum_{n \geq 1} \frac{2r^2}{z_{u,0,n}^2 - r^2} - r_1 \leq 0
\]

holds for \( |z| = r < z_{u,0,1} \), where \( T(r) \) is a strictly increasing continuous function in \( (0, z_{u,0,1}) \). Since \( \lim_{r \to 0} T(r) < 0, \lim_{r \to z_{u,0,1}} T(r) > 0 \) and \( T'(r) > 0 \), there exists a root \( r_0(f_u) \in (0, z_{u,0,1}) \) so that \( f_u \in S^*(\psi) \) in \( |z| < r_0(f_u) \). Now for the case \( u \in (-1,0) \), we proceed as in the case when \( u \in (0,1) \), just replacing \( u \) by \( u + 1 \) and \( \phi_0 \) by \( \phi_1 \), where

\[
\phi_1(z) = _1F_2 \left( 1; \frac{u+1}{2}, \frac{u+2}{2}; \frac{z^2}{4} \right) = \prod_{n \geq 1} \left( 1 - \frac{z^2}{z_{u,1,n}^2} \right)
\]

and \( z_{u,1,n} \) be the \( n \)-th positive root of \( \phi_1 \). Proof for the part (ii) and (iii) follows in a similar fashion as in the proof of Theorem 2.2 by applying the Assumption 1.1 on the following two equations, respectively using \( ||x| - |y|| \leq |x - y| \):

\[
\frac{zg'_u(z)}{g_u(z)} = -u + \frac{1}{2} + \frac{z\mathcal{L}'_{\frac{u}{2}, \frac{1}{2}}(z)}{(u+\frac{1}{2})\mathcal{L}_{\frac{u}{2}, \frac{1}{2}}(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{z_{u,0,n}^2 - z^2}
\]

and

\[
\frac{zh'_u(z)}{h_u(z)} = \frac{3 - 2u}{4} + \frac{\sqrt{2}z\mathcal{L}'_{\frac{u}{2}, \frac{1}{2}}(\sqrt{z})}{2\mathcal{L}_{\frac{u}{2}, \frac{1}{2}}(\sqrt{z})} = 1 - \sum_{n \geq 1} \frac{z}{z_{u,0,n}^2 - z^2}
\]

where \( z_{u,0,n} \) is the \( n \)-th positive root of the function \( \phi_0 \). Further, following the proof of Theorem 2.2, sharpness of the radii follows.
Proof (Proof of Theorem 2.4) From (1.10), after logarithmic differentiation, we obtain
\[
\frac{z\mathcal{P}'_{2n-1}(z)}{\mathcal{P}_{2n-1}(z)} = 1 - \sum_{k=1}^{n-1} \frac{2z^2}{z_k^2 - z^2}.
\] (2.8)

Now applying Assumption 1.1 on (2.8), we have \(\mathcal{P}_{2n-1}(z) \in S^*(\psi)\) whenever
\[
\left| \frac{z\mathcal{P}'_{2n-1}(z)}{\mathcal{P}_{2n-1}(z)} - 1 \right| \leq \sum_{k=1}^{n-1} \frac{2r^2}{z_k^2 - r^2} \leq r_1,
\] (2.9)
where \(|z| = r < z_1\) and \(z_k\) satisfies the condition given in (1.9). Now let us consider the strictly increasing continuous function
\[
T(r) := \sum_{k=1}^{n-1} \frac{2r^2}{z_k^2 - r^2} - r_1, \quad r \in (0, z_1).
\]

We have to show that \(T(r) \leq 0\) in \(|z| \leq r < z_1\) so that (2.9) holds. Since \(\lim_{r \to 0} T(r) < 0\), \(\lim_{r \to z_1} T(r) > 0\) and \(T'(r) > 0\), there exists a unique positive root \(r_\psi(\mathcal{P}_{2n-1}) \in (0, z_1)\) of \(T(r)\) such that \(\mathcal{P}_{2n-1} \in S^*(\psi)\) in \(|z| < r_\psi(\mathcal{P}_{2n-1})\). Further, following the proof of Theorem 2.1, sharpness of the radii follows.

3 An extremal problem for the class \(S^*(\psi)\): the region of variability

In 1961, Goluzin [24] obtained the set of extremal functions \(f(z) = z/(1 - xz)^2\), \(|x| = 1\) for the problem of maximization of the quantity \(\Re \Phi(\log(f(z)/z))\) or \(\Phi(\log(f(z)/z))\) over the class \(S^*\), where \(\Phi\) is a non-constant entire function. In 1973, MacGregor [32] proved the result for the class \(S^*(\alpha) := \{f \in \mathcal{A} : \Re(zf'(z)/f(z)) > \alpha, \alpha \in [0, 1]\}\). Later on Barnard [7] discussed this for Bounded starlike functions. Now, we present the result for the Ma-Minda class:

**Theorem 3.1** Suppose \(\Phi\) is a non-constant entire function and \(0 < |z_0| < 1\) and assume that the class \(S^*(\psi)\) is closed. Then maximum of either
\[
\Re \Phi \left( \log \frac{f(z_0)}{z_0} \right) \quad \text{or} \quad \Phi \left( \log \frac{f(z_0)}{z_0} \right)
\] (3.1)

for functions in the class \(S^*(\psi)\) is attained only when the function is of the form
\[
f(z) = z \exp \int_0^z \frac{\psi(t) - 1}{t} \, dt,
\] (3.2)
where \(|\zeta| = 1\).
Proof Since the class $S^*(\psi)$ is compact, therefore the problem under consideration has a solution. Moreover, in view of a result of Goluzin [24], in (3.1) it suffices to consider the continuous functional

$$\Re \Phi \left( \frac{\log f(z_0)}{z_0} \right).$$

Let $f \in S^*(\psi)$. Then using a result from [31], $f(z)/z \prec f_0(z)/z =: F(z)$, where $f_0(z) = z \exp \int_0^z \frac{\psi(t)-1}{t} dt$ or equivalently $\log(f(z)/z) \prec \log F(z)$. Thus,

$$g(z) = \Phi \left( \frac{\log f(z)}{z} \right) \prec \Phi(\log F(z)) = G(z).$$

Note that $G$ is also non-constant as is $\Phi$. Thus for each $r \in (0, 1)$ by subordination principle, we obtain

$$g(D_r) \subset G(D_r) = \Omega.$$ 

Since $G$ is also an open map, therefore there exists a point $z_1$ where $|z_1| = r$ and $G(z_1) = w_1$ such that among finitely many $w_1$, for one suitable $w_1$, we have

$$\Phi \left( \frac{\log f(z_0)}{z_0} \right) = w_1,$$

where $f$ is the solution for the extremal problem. Now by the well known Lindelöf Principle, we have

$$\Phi \left( \frac{\log f(z)}{z} \right) = \Phi(\log F(xz)), \quad (3.3)$$

that is, if $f$ is the desired solution, then (3.3) holds for some $x$, $|x| = 1$. Since $\Phi$ is non-constant analytic function, so we may write

$$\Phi(w) = c_0 + c_n w^n + c_{n+1} w^{n+1} + \cdots; \quad c_n \neq 0.$$

If we set $\log(f(z)/z) = \alpha_1 z + \alpha_2 z^2 + \cdots$ and $\log(F(z)) = \beta_1 z + \beta_2 z^2 + \cdots$, then from (3.3), comparing the coefficients, we get $c_n \alpha_1^n = c_n \beta_1^n$. Or equivalently, $\alpha_1^n = \beta_1^n$, which in particular implies that $|\alpha_1| = |\beta_1|$. Since $\log(f(z)/z) \prec \log F(xz)$, $|\alpha_1| = |\beta_1|$ is possible only if $\log(f(z)/z) = \log F(xyz)$ for some $|y| = 1$. Therefore, we conclude that

$$f(z) = z \exp \int_0^z \frac{\psi(t)-1}{t} dt,$$

where $|u| = 1$ if $f$ is a solution to the extremal problem.

Remark 3.1 Note that the analogous result for the class $C(\psi)$ also holds.
Now as an application of Theorem 3.1, we obtain the result due to MacGregor [32]:

**Corollary 3.1.** Suppose $\Phi$ is a non-constant entire function and $0 < |z_0| < 1$. Then the maximum of the expression $|z|^{2-2\alpha}$ for functions in the class $S^\ast(\alpha)$ is attained only when the function is of the form $f(z) = z(1 - \zeta z)^2 - 2\alpha$, $|\zeta| = 1$.

**Proof.** If $f \in S^\ast(\alpha)$, then $f(z)/z \prec 1/(1 - z)^{2-2\alpha}$ and the result follows.

**Corollary 3.2.** Suppose $\Phi$ is a non-constant entire function and $0 < |z_0| < 1$. Then the maximum of the expression $|z|^{2-2\alpha}$ for functions in the class $S^\ast_\psi$ is attained only when the function is of the form $f(z) = z \exp(e^\zeta z - 1)$, $|\zeta| = 1$.

**Proof.** If $f \in S^\ast_\psi$, then $f(z)/z \prec \exp(e^z - 1)$ and the result follows.

### 4 Convolution Radius: A Case Study For Starlike domains

Note that if the function $\psi$ considered in the Ma-Minda class is a starlike function but not convex, then the following classical theorem doesn’t hold.

**Theorem 4.1.** [31] Let $\psi(D)$ be convex, $g \in C$ and $f \in S^\ast(\psi)$. Then $f \ast g \in S^\ast(\psi)$.

For instance, for $\psi(z) = 1 + ze^z, z + \sqrt{1 + z^2}, e^{e^z - 1}$ and $1 + 4z/3 + 2z^2/3$ Theorem 4.1 is not valid. Therefore, we need to modify Theorem 4.1 to accommodate such cases to further derive various radius problems. But first we need to recall an important result due to Ruscheweyh and Sheil-Small:

**Lemma 4.1.** ([35], p. 126) Suppose that either $g \in C$, $h \in S^\ast$ or else $g, h \in S^\ast_{1/2}$. Then for any analytic function $G$ in $D$, we have

$$
\frac{g \ast h G(z)}{g \ast h(z)} \in \overline{coG}(D),
$$

where $\overline{coG}(D)$ is the closed convex hull of $G(D)$.

Keenly observing the proof of Lemma 4.1 we see that the unit disk $D$ can be replaced by the sub-disk $D_r := \{z : |z| < r\}$, where $0 < r \leq 1$ and consequently, we obtain the following modified result. Since the proof is similar, so it is omitted here.

**Lemma 4.2.** Suppose either $g \in C$, $h \in S^\ast$ or else $g, h \in S^\ast_{1/2}$. Then for any analytic function $G$ in $D_r$, we have $(g \ast h G(z))/(g \ast h(z)) \in \overline{coG}(D_r)$, where $r \in [0, 1]$.

This immediately gives the following fundamental result
Theorem 4.2 (Improvement of Theorem 4.1) Let $r_0$ be the radius of convexity of $\psi$. If $g \in C$ and $f \in S^*(\psi)$. Then $f \ast g \in S^*(\psi)$ in $|z| < r$, where $r = \min\{r_0, 1\}$.

Corollary 4.1 Let $f \in S^*_\psi$ and $g \in C$. Then $f \ast g \in S^*_\psi$ in $D_{r_0}$, where $r_0 = (3 - \sqrt{3})/2$ is the radius of convexity of $\psi$.

Now consider the operators $F_i : \mathbb{A} \to \mathbb{A}$ defined by

$$F_1(f)(z) = f \ast g_1(z) = zf'(z)$$

$$F_2(f)(z) = f \ast g_2(z) = \frac{1}{2}(f(z) + zf'(z))$$

$$F_3(f)(z) = f \ast g_3(z) = \frac{k+1}{z^k} \int_0^z t^{k-1} f(t) dt, \quad \Re k > 0,$$

where $g_3(z) = \sum_{n=1}^{\infty} (k+1)/(k+n) z^n$, $g_2(z) = (z - z^2)/(1 - z^2)$ and $g_1(z) = z/(1 - z)^2$. Note that the function $g_1$ is convex in $|z| < 2 - \sqrt{3}$, $g_2$ is convex in $|z| < 1/2$ while $g_3 \in C$. The above defined operators were introduced by Alexander, Livingston and Bernardi, respectively. Now we obtain the following result, where $S^*_{SG} := S^*(\frac{2}{\sqrt{5} + 1}), S^*_C := S^*(1 + 4z/3 + 2z^2/3), S^*_C := S^*(e^{e^{-1}})$ and $S^*_C := S^*(1 + \sin z)$.

Corollary 4.2 Let $F_i$, $i = 1$ to 3 be the operators as defined above.

(i) Let $f \in S^*_C$. Then $F_i(f) \in S^*_C$ in $D_{r_0}$, where $r_1 = 2 - \sqrt{3}$, $r_2 = (3 - \sqrt{5})/2$ and $r_3 = (3 - \sqrt{5})/2$.

(ii) Let $f \in S^*_C$. Then $F_i(f) \in S^*_C$ in $D_{r_0}$, where $r_1 = 2 - \sqrt{3}$, $r_2 = 1/2$ and $r_3 = 1/2$.

(iii) Let $f \in S^*_C$. Then $F_i(f) \in S^*_C$ in $D_{r_0}$, where $r_1 = 2 - \sqrt{3}$, $r_2 = 0.345$ and $r_3 = 0.345$.

(iv) Let $f \in S^*_C$. Then $F_i(f) \in S^*_C$ in $D_{r_0}$, where $r_1 = 2 - \sqrt{3}$, $r_2 = 1/2$ and $r_3 = 1$.

The radii are sharp.

In 2010, Ali et al. [2] dealt with the problem of finding $S^*(\psi)$-radii of the convolution $f \ast g$, between two starlike functions. In fact, they showed that if $f, g \in S^*$ and $h_\rho(z) = f \ast g(\rho z)/\rho$, then $h_\rho \in SC^*$ for $0 \leq \rho \leq (\sqrt{5} - 2)/(\sqrt{2} - 1) \approx 0.09778$. They used the property of the function $\psi$ being convex. Now using Theorem 4.2 we can obtain the result even for the case when $\psi(D)$ is starlike. Here, we have shown the usability of the radius of convexity of $\psi$.

Theorem 4.3 Let $f, g \in S^*$ and $h_\rho(z) := f \ast g(\rho z)/\rho$. Then

(i) $h_\rho \in S^*_C$ for $0 \leq \rho \leq (2e - \sqrt{4e^2 - 2e + 1})/(2e - 1) \approx 0.0957,$

(ii) $h_\rho \in S^*_C$ for $0 \leq \rho \leq (3 - \sqrt{7})/2 \approx 0.177124,$

(iii) $h_\rho \in S^*_C$ for $0 \leq \rho \leq (\sqrt{\sin 1^2 + 2\sin 1 + 4 - 2})/(2 + \sin 1) \approx 0.185835,$
(iv) \( h_\rho \in S_\rho^* \) for \( 0 \leq \rho \leq (2e - \sqrt{3e^2 + e^2}/c)/(e + e^{1/c}) \approx 0.122919 
(v) \( h_\rho \in S_{5\rho}^* \) for \( 0 \leq \rho \leq (\sqrt{7e^2 + 6e + 3} - 2(1 + e))/(3e + 1) \approx 0.108309. 

The constants are best possible.

Proof We only prove first part and rest part’s proof also follow in a similar fashion.

(i) Let \( H(z) = z + \sum_{n=2}^{\infty} a_n z^n = (z(1 + z))/(1 - z^3) \). It is easy to see that
\[
\frac{|zH'(z)|}{H(z)} - \frac{1 + r^2}{1 - r^2} \leq \frac{4r}{1 - r^2}, \quad |z| = r < 1. \quad (4.1)
\]

Now by Lemma 1.1, the disk \( (4.1) \) lies inside the cardioid \( \varphi(\mathbb{D}) \), provided
\[
\frac{4r}{1 - r^2} \leq 1 + \frac{r^2}{(1 - r^2) - 1 + \frac{1}{e}}
\]
which in turn gives \( r \leq r_0 := (2e - \sqrt{4e^2 - 2e + 1})/(2e - 1) \). Define the function \( h : \mathbb{D} \to \mathbb{C} \) by \( h(z) := f(z) * g(z) \). Then \( h(z) = F(z) * G(z) * H(z) \), where \( F \) and \( G \) are, respectively defined as \( zF'(z) = f(z) \) and \( zG'(z) = g(z) \). Since \( f, g \in S^* \), it follows that \( F * G \in \mathbb{C} \). Also, \( H(r_0 z)/r_0 \in S_{\rho'}^* \). Hence, using Theorem 4.2 we have
\[
F(z) * G(z) * H(\rho_0 z)/\rho_0 \in S_{\rho'}^*.
\]

where \( \rho_0 = \min\{r_0, r_c\} = r_0 \) and \( r_c = (3 - \sqrt{5})/2 \) is the radius of convexity of \( \varphi \). For \( z = -\rho_0 \), \( zH'(z)/H(z) = (1 + 4z + z^2)/(1 - z^2) = 1 - 1/e \), which implies that \( \rho_0 \) is sharp.

Remark 4.1 It is worthy to mention that in Theorem 4.3, we need \( r_c \), radius of convexity. However, the sharp radius of convexity for the class \( S^*(\psi) \) is an open problem.

Theorem 4.4 Let \( f, g \in S^* \) and \( h_\rho(z) := f * g(\rho z)/\rho \). Then \( h_\rho(z) \in S^* \left( \frac{1 + AB}{1 + B^2} \right) \) for
\[
0 \leq \rho \leq \frac{2(B^2 - 1) + \sqrt{4(1 - B^2)^2 + (A - B)^2}}{A - B} := \rho_0,
\]
where \(-1 < B < A \leq 1\).

Proof Since for the function \( p(z) \prec (1 + Az)/(1 + Bz) \), we have
\[
\left| p(z) - \frac{1 - AB}{1 - B} \right| \leq \frac{A - B}{1 - B^2}. \quad (4.2)
\]

Therefore, for the disk \( (4.1) \) to lie inside the disk \( (4.2) \), we must have
\[
\frac{1 - AB}{1 - B^2} - \frac{A - B}{1 - B^2} - \frac{1 + r^2}{1 - r^2} - \frac{1 - AB}{1 - B^2} + \frac{A - B}{1 - B^2} \quad \text{and} \quad \frac{4r}{1 - r^2} \leq \frac{A - B}{1 - B^2}.
\]
which upon simplification hold for \( r \leq r_0 = \sqrt{(A - B)/(2 + A + B)} \) and \( r \leq \rho_0 \) respectively, where \( \rho_0 \) is the smallest positive root of the following equation

\[
(A - B)r^2 + 4(1 - B^2)r - (A - B) = 0.
\]

Since \( \min\{r_0, \rho_0\} = \rho_0 \) and the class \( S^*\left(\frac{1+z^k}{1+z}\right) \) is closed under convolution with convex functions, now the result follows in a similar way as in the part \((i)\) of Theorem 4.3.

5 Some Sufficient conditions for \( S^* (\psi) \) and Further Radius Results

In this section, we determine the sufficient conditions for the functions \( z/(1 + \sum_{k=1}^{\infty} a_k z^k) \), \( z/(1 + z^k)^n \) and certain other types of functions to be in \( S^* (\psi) \). For the clarity, here we set \( r_a = R_a \).

**Theorem 5.1** Let \( f(z) = z/(1 + \sum_{k=1}^{\infty} a_k z^k) \). If the coefficients of \( f \) satisfy

\[
|1 - a| + \sum_{k=1}^{\infty} (R_a + |1 - a - k|)|a_k| \leq R_a,
\]

where \( a \) and \( R_a \) are as defined in the Assumption [1.2]. Then \( f \in S^* (\psi) \).

**Proof** For \( f(z) = z/(1 + \sum_{k=1}^{\infty} a_k z^k) \), we have

\[
\left| \frac{zf'(z)}{f(z)} - a \right| = \left| 1 - a - \frac{\sum_{k=1}^{\infty} k a_k z^k}{1 + \sum_{k=1}^{\infty} a_k z^k} \right|.
\]

Thus by Assumption [1.1], \( f \in S^* (\psi) \), if \( \left| 1 - a - \frac{\sum_{k=1}^{\infty} k a_k z^k}{1 + \sum_{k=1}^{\infty} a_k z^k} \right| \leq R_a \). The above inequality holds whenever

\[
|1 - a| + \sum_{k=1}^{\infty} |1 - a - k||a_k|r^k \leq R_a (1 - \sum_{k=1}^{\infty} |a_k|r^k)
\]

or equivalently, \( |1 - a| + \sum_{k=1}^{\infty} (|1 - a - k| + R_a)|a_k|r^k \leq R_a \). Letting \( r \) tends to 1\(^-\), completes the proof.

**Theorem 5.2** Let \( f(z) = z/(1 + z^k)^n \), where \( n, k \in \mathbb{Z}^+ \) are fixed. Then \( f \in S^* (\psi) \) in

\[
|z| < \left( \frac{R_a - |1 - a|}{R_a + |1 - a - kn|} \right)^{1/k},
\]

where \( a \) and \( R_a \) are as defined in the Assumption [1.1].
Proof For \( f(z) = z/(1 + z^k)^n \), we have
\[
\left| \frac{zf'(z)}{f(z)} - a \right| = \left| 1 - a \frac{knz^k}{1 + z^k} \right|.
\]
Thus by Assumption 1.1, \( f \in S^*(\psi) \), if
\[
\left| 1 - a - \frac{knz^k}{1 + z^k} \right| < R_a.
\]
This inequality holds whenever \(|1-a|+|1-a-\kappa n||z|^k < R_a(1-|z|^k)\) which upon simplification yields that
\[
|z|^k < \frac{R_a - |1-a|}{R_a + |1-a-\kappa n|}.
\]
Hence the result follows.

Theorem 5.3 Let \( p(z) \) be a polynomial such that \( p(0) = 1 \) and \( \deg p(z) = m \). Let \( R = \min\{|z|: p(z) = 0, z \neq 0\} \). Then the function
\[
f(z) = z(p(z))^{\beta/m} \in S^*(\psi)
\]
in
\[
|z| < \frac{R(R_a - |1-a|)}{|\beta| + R_a - |1-a|},
\]
where \( a \) and \( R_a \) are as defined in the Assumption 1.1.

Proof Assume that \( z_k, (k = 1, 2, \ldots, m) \) are zeros of the polynomial \( p(z) \). For the function \( f(z) = z(p(z))^{\beta/m} \), we have
\[
\frac{zf'(z)}{f(z)} = 1 + \frac{\beta}{m} \sum_{k=1}^{\infty} \frac{z}{z-z_k}
\]
or equivalently,
\[
\frac{zf'(z)}{f(z)} - a = 1 - a + \frac{\beta}{m} \sum_{k=1}^{\infty} \left( \frac{z}{z-z_k} + \frac{r^2}{R^2 - r^2} - \frac{r^2}{R^2 - r^2} \right).
\]
Thus by Assumption 1.1, \( f \in S^*(\psi) \) whenever
\[
(|1-a| - R_a)(R^2 - r^2) + |\beta|(Rr + r^2) < 0,
\]
which is satisfied if \(|z| = r < R(R_a - |1-a|)/(|\beta| + R_a - |1-a|)\).

Kuroki and Owa [30] introduced and studied the class \( S(\alpha, \beta) \) of functions \( f \), which satisfy the condition \( zf'(z)/f(z) < p_{\alpha, \beta}(z) \), where
\[
p_{\alpha, \beta}(z) := 1 + \frac{\beta - \alpha}{\pi} \log \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha} z}}{1 - z},
\]
\( \alpha < 1, \beta > 1 \) and \( p_{\alpha, \beta} \) maps \( \mathbb{D} \) onto the convex domain \( \{ w \in \mathbb{C} : \alpha < \Re w < \beta \} \). Note that if \( \alpha \not\in (0, \infty) \) then this class also contains non-univalent functions, and univalent starlike if \( 1 > \alpha \geq 0 \).
Remark 5.1 Note that we can extend Theorem 5.4 \((\psi(z) \neq (1 + z)/(1 - z))\) and Theorem 5.5 for \(S^*(\psi)\)-radius if we replace the radius \(1/e\) by \(r_1\), where \(r_1\) is given by the Assumption 1.1.

We now conclude this section some results explicitly for the class \(S^*_\psi\).

**Theorem 5.4** Let \(f \in S(\alpha, \beta)\). Then \(f \in S^*_\psi\) in \(D_{r_0}\), where \(r_0\) is the least positive root of the equation

\[
\frac{\beta - \alpha}{\pi} \left( \log \left( 1 + \sqrt{2(1 + \cos(2\pi \frac{1 - \alpha}{\beta - \alpha}))} + \arctan \frac{r}{1-r} \right) \right) - \frac{1}{e} = 0. \tag{5.1}
\]

**Proof** Consider the analytic function \(p_{\alpha, \beta}(z) := 1 + \frac{\beta - \alpha}{\pi} i \log q(z)\), where

\[
q(z) = \frac{1 - cz}{1 - z} \quad \text{and} \quad c = \exp \left( 2\pi i \frac{1 - \alpha}{\beta - \alpha} \right).
\]

Note that \(q(z)\) is a bilinear transformation, maps \(D_r\) onto the disk:

\[
|q(z)| \leq \left| 1 + \frac{1 + |c|r + r^2}{1 - r^2} \right|,
\]

which implies

\[
|q(z)| \leq \frac{1 + |c|r + r^2}{1 - r^2},
\]

and therefore,

\[
\log |q(z)| \leq \log \left( \frac{1 + |c|r + r^2}{1 - r^2} \right). \tag{5.2}
\]

For any \(\delta \in \mathbb{C}\) with \(|\delta| = 1\), we have \(1 + \delta z < 1 + z\). So to maximize \(|\arg(1 + \delta z)|\), it suffices to consider \(|\arg(1 + z)|\). Now for \(|z| = r\), we have

\[
|\arg(1 + z)| \leq \arctan \frac{r}{1 - r}. \tag{5.3}
\]

Hence to apply Lemma 1.1 we need to maximize \(|p_{\alpha, \beta}(z) - 1|\), that is,

\[
|p_{\alpha, \beta} - 1| = \left| \frac{\beta - \alpha}{\pi} \log |q(z)| + i \arg \left( \frac{1 - cz}{1 - z} \right) \right|. \tag{5.4}
\]

Using (5.2) and (5.3) in (5.4), we see that

\[
|p_{\alpha, \beta} - 1| \leq \frac{\beta - \alpha}{\pi} \left( \log \left( \frac{1 + |c|r + r^2}{1 - r^2} \right) + 2 \arctan \frac{r}{1 - r} \right) \leq \frac{1}{e}
\]

holds in \(|z| < r_0\) whenever \(r_0\) is the smallest positive root of (5.1).

Note that if we choose \(\alpha = 1 + \frac{\delta - \pi}{2 \sin \delta}\) and \(\beta = 1 + \frac{\delta}{2 \sin \delta}\), where \(\pi/2 \leq \delta < \pi\), then \(S(\alpha, \beta)\) reduces to the class \(V(\delta)\) introduced by Kargar et al. [26].
Corollary 5.1 Let $f \in \mathcal{V}(\delta)$. Then $f \in \mathcal{S}_\nu^*$ in $\mathbb{D}_{r_3}$, where $r_3$ is the least positive root of the equation

$$\frac{1}{2\sin \delta} \left( \log \frac{1 + \sqrt{2(1 + \cos(2(\pi - \delta)))r + r^2}}{1 - r^2} + 2 \arctan \frac{r}{1 - r} \right) - \frac{1}{e} = 0.$$ 

Now we consider the following class introduced in [33]:

$$\mathcal{S}_\lambda := \left\{ f \in \mathcal{A} : \frac{f(z)}{z} \in P_\lambda \right\}, \tag{5.5}$$

where $P_\lambda := \{ p \in \mathcal{A}_0 : \Re(e^{i\lambda}p(z)) > 0, \ -\pi/2 \leq \lambda \leq \pi/2 \}$ denotes the class of tilted Carathéodory functions [38]. Note that $P_0$ reduces to $\mathcal{P}$, the class of Carathéodory functions. For the function $p \in P_\lambda$, upper bound on the quantity $zp'(z)/p(z)$ is given by the following lemma that will be used for our next result:

Lemma 5.1 [38] If $p \in P_\lambda$, then $|zp'(z)/p(z)| \leq M(\lambda, r)$, where

$$M(\lambda, r) = \begin{cases} \frac{2r \cos \lambda}{r^2 - 2rp \sin \lambda + r^2} & \text{for } r < |\tan \frac{\delta}{2}|; \\ \frac{2r}{r^2 - 2rp \sin \lambda + r^2} & \text{for } r \geq |\tan \frac{\delta}{2}|. \end{cases}$$

The equality holds for some point $z = re^{i\theta}, r \in (0, 1)$ if and only if $p(z) = p_\lambda(yz)$, where $p_\lambda(z) = \frac{1 + e^{-z\lambda}}{1 - z}$ and $y = e^{i(\theta_0 - \theta)}$ with

$$\theta_0 = \begin{cases} \frac{\delta}{2} + \lambda & \text{for } r < \tan \frac{\delta}{2}; \\ -\frac{\delta}{2} + \lambda & \text{for } r < \tan \frac{\delta}{2}; \\ \arcsin \left( \frac{r^2 - 1}{2r} \right) + \lambda & \text{for } r \geq |\tan \frac{\delta}{2}|. \end{cases}$$

Next, we determine the largest radius $r$ such that the function $F(z) := f(z)g(z)/z \in \mathcal{S}_\nu^*$ in $|z| < r$, whenever $f, g \in \mathcal{S}_\lambda$.

Theorem 5.5 Let $c_\lambda = \cos \lambda, s_\lambda = \sin \lambda$ and $t_\lambda = |\tan(\lambda/2)|$. If $f, g \in \mathcal{S}_\lambda$, then $F \in \mathcal{S}_\nu^*$ in $\mathbb{D}_{r_3}$, where

$$r_0 := \begin{cases} 2ec_\lambda + |s_\lambda| + \sqrt{(4e^2 - 1)c_\lambda + 4e|s_\lambda|c_\lambda}, & \text{if } r < t_\lambda; \\ \sqrt{4e^2 + 1 - 2e}, & \text{if } r \geq t_\lambda. \end{cases}$$

Proof Since $f, g \in \mathcal{S}_\lambda$, it follows that the functions $p(z) = f(z)/z$ and $q(z) = g(z)/z$ belong to the class $P_\lambda$ such that $F(z) = zp(z)q(z)$. Thus

$$\frac{zF'(z)}{F(z)} - 1 = \frac{zp'(z)}{p(z)} + \frac{zq'(z)}{q(z)}.$$ 

Now from Lemma 5.1, we obtain

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| \leq 2M(\lambda, r).$$
Therefore, using Lemma 1.1, we conclude that if $2M(\lambda, r) \leq 1/e$, then $F \in S^*_\omega$.

Since $2M(\lambda, r) \leq 1/e$ holds whenever $\frac{2rc_\lambda}{r^2 - 2|s_\lambda|r + 1} \leq \frac{1}{2e}$ if $r < t_\lambda$, and $\frac{2r}{1 - r^2} \leq \frac{1}{2e}$ if $r \geq t_\lambda$; or equivalently

$$r^2 - 2(|s_\lambda| + 2ec_\lambda)r + 1 \geq 0, \quad \text{if} \quad r < t_\lambda$$

and

$$r^2 + 4er - 1 \leq 0, \quad \text{if} \quad r \geq t_\lambda,$$

respectively. Hence the result follows with $r_0$ as given in the hypothesis. Further, for the functions

$$f(z) = g(z) = \frac{z(1 + e^{-2i\lambda}y)}{1 - yz},$$

sharpness hold in view of Lemma 5.1.

**Conflict of interest**

The authors declare that they have no conflict of interest.

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