ON A QUASILINEAR FULLY PARABOLIC TWO-SPECIES CHEMOTAXIS SYSTEM WITH TWO CHEMICALS

XU PAN AND LIANGCHEN WANG*

School of Science
Chongqing University of Posts and Telecommunications
Chongqing 400065, China

(Communicated by Christina Surulescu)

Abstract. This paper deals with the following two-species chemotaxis system with nonlinear diffusion, sensitivity, signal secretion and (without or with) logistic source

\[
\begin{align*}
\begin{cases}
u_t &= \nabla \cdot (D_1(u) \nabla u - S_1(u) \nabla v) + f_1(u), & x \in \Omega, \quad t > 0, \\
v_t &= \Delta v - v + g_1(w), & x \in \Omega, \quad t > 0, \\
w_t &= \nabla \cdot (D_2(w) \nabla w - S_2(w) \nabla z) + f_2(w), & x \in \Omega, \quad t > 0, \\
z_t &= \Delta z - z + g_2(u), & x \in \Omega, \quad t > 0,
\end{cases}
\end{align*}
\]

under homogeneous Neumann boundary conditions in a bounded domain \(\Omega \subset \mathbb{R}^n\) with \(n \geq 2\). The diffusion functions \(D_i(s) \in C^2([0, \infty))\) and the chemotactic sensitivity functions \(S_i(s) \in C^2([0, \infty))\) are given by

\[
D_i(s) \geq C_{d_i} (1 + s)^{-\alpha_i} \quad \text{and} \quad 0 < S_i(s) \leq C_{s_i} s (1 + s)^{\beta_i - 1} \quad \text{for all} \quad s \geq 0,
\]

where \(C_{d_i}, C_{s_i} > 0\) and \(\alpha_i, \beta_i \in \mathbb{R}\) \((i = 1, 2)\). The logistic source functions \(f_i(s) \in C^0([0, \infty))\) and the nonlinear signal secretion functions \(g_i(s) \in C^1([0, \infty))\) are given by

\[
f_i(s) \leq r_i s - \mu_i s^{k_i} \quad \text{and} \quad g_i(s) \leq s^{\gamma_i} \quad \text{for all} \quad s \geq 0,
\]

where \(r_i \in \mathbb{R}, \mu_i, \gamma_i > 0\) and \(k_i > 1\) \((i = 1, 2)\). With the assumption of proper initial data regularity, the global boundedness of solution is established under the some specific conditions with or without the logistic functions \(f_i(s)\).

Moreover, in case \(r_1 > 0\), for the large time behavior of the smooth bounded solution, by constructing the appropriate energy functions, under the conditions \(\mu_i\) are sufficiently large, it is shown that the global bounded solution exponentially converges to \(\left(\left(\frac{r_1}{\mu_1}\right)^{\frac{k_1}{1-\gamma_1}}, \left(\frac{r_2}{\mu_2}\right)^{\frac{k_2}{1-\gamma_2}}, \left(\frac{r_2}{\mu_2}\right)^{\frac{k_2}{1-\gamma_2}}, \left(\frac{r_1}{\mu_1}\right)^{\frac{k_1}{1-\gamma_1}}\right)\) as \(t \to \infty\).

1. Introduction. In the present work, we deal with the following system, which describes the fully parabolic two-species chemotaxis system with nonlinear diffusion,

---

2020 Mathematics Subject Classification. Primary: 92C17, 35K35; Secondary: 35A01, 35B35.

Key words and phrases. Boundedness, two-species chemotaxis, two chemicals, logistic source, stabilization.

* Corresponding author: Liangchen Wang.
sensitivity, signal secretion and (without or with) logistic source

\[
\begin{align*}
  u_t &= \nabla \cdot (D_1(u)\nabla u - S_1(u)\nabla v) + f_1(u), \quad x \in \Omega, \quad t > 0, \\
  v_t &= \Delta v - v + g_1(w), \quad x \in \Omega, \quad t > 0, \\
  w_t &= \nabla \cdot (D_2(w)\nabla w - S_2(w)\nabla z) + f_2(w), \quad x \in \Omega, \quad t > 0, \\
  z_t &= \Delta z - z + g_2(u), \quad x \in \Omega, \quad t > 0, \\
  \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
  (u, v, w, z)(x, 0) &= (u_0(x), v_0(x), w_0(x), z_0(x)), \quad x \in \Omega,
\end{align*}
\]

with homogeneous Neumann boundary conditions, where \( \Omega \subset \mathbb{R}^n (n \geq 2) \) is a bounded domain, and \( \partial / \partial \nu \) is the derivative of the normal with respect to \( \partial \Omega \). In system (1.1), \( u = u(x, t) \) and \( w = w(x, t) \) represent the densities of two populations, respectively, \( v = v(x, t) \) and \( z = z(x, t) \) denote the concentrations of chemicals produced by populations \( w \) and \( u \), respectively. The nonnegative initial data satisfies

\[
(u_0, v_0, w_0, z_0) \in C^0(\overline{\Omega}) \times C^1(\overline{\Omega}) \times C^0(\overline{\Omega}) \times C^1(\overline{\Omega}).
\]

In this article, the diffusion functions \( D_i(s) \in C^2([0, \infty)) \) and the chemotactic sensitivity functions \( S_i(s) \in C^2([0, \infty)) \) are given by

\[
D_i(s) \geq C_{d_i}(1 + s)^{-\alpha_i} \quad \text{and} \quad 0 < S_i(s) \leq C_{s_i}(s + 1 + s)^{\beta_i - 1} \quad \text{for all } s \geq 0,
\]

where \( C_{d_i}, C_{s_i} > 0 \) and \( \alpha_i, \beta_i \in \mathbb{R} \) (\( i = 1, 2 \)). The signal secretion functions \( g_i(s) \in C^1([0, \infty)) \) satisfy

\[
0 < g_i(s) \leq s^{\gamma_i} \quad \text{for all } s \geq 0
\]

with \( \gamma_i > 0 \) (\( i = 1, 2 \)). The logistic source functions \( f_i(s) \in C^0([0, \infty)) \) satisfy that

\[
f_i(s) \leq r_is - \mu_is^{k_i} \quad \text{for all } s \geq 0,
\]

where \( r_i \in \mathbb{R}, \mu_i > 0 \) and \( k_i > 1 \) (\( i = 1, 2 \)).

The purpose of this paper is to study the effects of logistic sources on the boundedness and asymptotic stability of solutions. Here we need to overcome the difficulties caused by nonlinear terms and cross terms. First, let us review some relevant achievements in this field.

(1) Single-species chemotaxis model

The well-known chemotaxis model for the chemotactic movement of one species [12], it is proposed by Keller and Segel, which describe the aggregation phenomenon of the Dictyostelium discoideum

\[
\begin{align*}
  u_t &= \nabla \cdot (D(u)\nabla u - S(u)\nabla v) + f(u), \quad x \in \Omega, \quad t > 0, \\
  v_t &= \Delta v - v + g(u), \quad x \in \Omega, \quad t > 0.
\end{align*}
\]

In the absence of the logistic source (i.e. \( f(u) \equiv 0 \)), if \( g(u) = u \), the asymptotics of \( \frac{\partial f(u)}{\partial u} \approx u^\gamma \) is critical to distinguish the blow-up and global boundedness: under the condition \( \frac{\partial f(u)}{\partial u} \leq cu^{\gamma - \epsilon} \) for all \( u > 1 \) with some \( c > 0 \), Tao and Winkler [33] obtained the global bounded of solution; while if \( \frac{\partial f(u)}{\partial u} \leq cu^{\gamma + \epsilon} \) for all \( u > 1 \) [45], the solution of system (1.6) blow-up either in infinite time or finite time.

If \( 0 < g(u) \leq u^\gamma \), under the conditions \( D(u) = 1, S(u) = u \) for all \( u > 1 \), the solution of (1.6) is bounded in \( n \geq 2 \) if \( \gamma \in (0, \frac{2}{n}) \) [16]; the second equation of (1.6) is replaced by \( 0 = \Delta v - \frac{1}{\Omega} \int_{\Omega} u^\gamma + w^\gamma \), Winkler [43] proved the radial blow-up solution of (1.6) if \( \gamma > \frac{2}{n} \). If \( \frac{\partial f(u)}{\partial u} \leq C\gamma (1 + u)^\eta \) and \( 0 < \eta \leq 1 \), the global boundedness is obtained under the condition \( \eta < \min \{ 1 + \frac{1}{n}, 1 + \frac{2}{n} - \gamma \} \) in [26].
TWO SPECIES QUASILINEAR CHEMOTAXIS SYSTEM

In case \( f(u) = \mu u(1 - u) \), under the conditions \( D(u) = 1, S(u) = u \) for all \( u > 0 \), the blow-up can be prevented under the condition that arbitrarily small \( \mu > 0 \) in \( n \leq 2 \) \([24, 25]\); in high-dimensional case, the global boundedness of \( (1.6) \) was obtained whenever \( \mu \) is sufficiently large \([15, 34, 48]\).

There are also some another results about system \( (1.6) \), we refer to \([2, 4, 5, 6, 10, 20, 42, 44, 46, 47, 49]\).

(II) Two-species chemotaxis model (Linear)

The following is the two-species and two-stimuli chemotaxis system (simplified version of the model \( (1.1) \)), which describes the spatiotemporal evolution of two populations that reproduce and compete with Lotka-Volterra dynamics, on the other hand, individuals that move in a random diffusion pattern and move toward chemical signals generated by opposing species on the other

\[
\begin{aligned}
\left\{
\begin{array}{ll}
u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla v) + \mu_1 u(1 - u - a_1 w), & x \in \Omega, \quad t > 0, \\
\tau v_t = \Delta v - v + w, & x \in \Omega, \quad t > 0, \\
w_t = \Delta w - \chi_2 \nabla \cdot (w \nabla z) + \mu_2 w(1 - w - a_2 u), & x \in \Omega, \quad t > 0, \\
\tau z_t = \Delta z - z + u, & x \in \Omega, \quad t > 0,
\end{array}
\right.
\end{aligned}
\] (1.7)

- system \( (1.7) \) without logistic sources
  - In case \( \tau = 0 \), the global boundedness and the finite time blow-up of solution to \( (1.7) \) was studied by Tao and Winkler \([35]\) under the different form (attraction-attraction, attraction-repulsion and repulsion-repulsion); the blow-up result was investigated in \([52]\), the solution of \( (1.7) \) blow-up if \( m_1 m_2 - 2\pi(\frac{m_1}{\chi_1} + \frac{m_2}{\chi_2}) > 0 \), where \( m_1 := \int_\Omega u_0 \) and \( m_2 := \int_\Omega w_0 \), and the bounded condition was satisfied if max\( \{m_1, m_2\} < 4\pi \).
  - In case \( \tau = 1 \), Xie and Wang \([50]\) further studied the global boundedness of \( (1.7) \), the boundedness was proved if \( m_1 \) and \( m_2 \) were suitably small.
- system \( (1.7) \) with logistic sources
  - In case \( \tau = 0 \), the global solution was obtained in \([37]\) if \( \mu_i (i = 1, 2) \) were sufficiently large; the stabilization was also obtained if \( \mu_i \) are sufficiently large; the results were further investigated by Wang and Mu \([39]\).
  - In case \( \tau = 1 \), the boundedness was obtained if the all parameters were positive, and the large time behavior of solution to \( (1.7) \) under the conditions that \( \frac{\mu_1}{\chi_1} \) and \( \frac{\mu_2}{\chi_2} \) were large enough by Black \([3]\); Wang et al. \([41]\) further studied this system in \( n \leq 3 \), under the condition that \( \mu_i (i = 1, 2) \) were large enough, the global boundedness and the large time behavior were obtained; recently, the global boundedness of \( n = 3 \) was further studied by Pan et al. \([28]\). There are also some another results about system \( (1.7) \), we refer to \([27, 53, 54, 55, 56]\).

(III) Two-species chemotaxis model (Quasilinear)

For system \( (1.1) \), it has been investigated in \([29]\), the diffusion functions were replaced by \( D_i(s) \geq C_D s^{-m_i} \) \( i = 1, 2 \), the chemotactic sensitivity functions and the signal secretion functions were replaced by \( S_i(s) \leq C_s s^{m_i} \) and \( g_i(s) = s^{\gamma_i} \) \( i = 1, 2 \), respectively. The logistic sources terms \( r_1 u - \mu_1 u^{1+} \) and \( r_2 w - \mu_2 w^{1+} \) were replaced by \( \mu_1 u(1 - w^{1+} - a_1 w) \) and \( \mu_2 w(1 - w^{1+} - a_2 w) \), respectively. For full parabolic case, if some specific and complex conditions are satisfied, then system \( (1.1) \) possessed at least one global weak solution; for parabolic-elliptic-parabolic-elliptic case, \( (1.1) \) possessed at least one global weak solution under some specific conditions. Without logistic source, system \( (1.1) \) has also been studied in \([55]\) for parabolic-elliptic-parabolic-elliptic case.
In contrast to the system above, the majority of the research in the field of multi-species chemotaxis focuses on two-species and one-stimuli chemotaxis model, the global existence and large time behavior have been extensively studied by many authors \([1, 17, 18, 22, 23, 30, 36, 38, 40]\).

**Main challenge of the present paper.**

The purpose of this paper is to study the effects of (without or with) logistic source on the boundedness and asymptotic stability of solutions. It is difficult to overcome the problems caused by the nontrivial effects of nonlinear diffusion, sensitivity, signal secretion and cross terms (without or with) logistic source; yet, we can obtain the boundedness of solution without logistic source by establishing an appropriate prior estimates; with logistic source, by using the variation of maximal Sobolev regularity principle, the global boundedness was obtained under the some implicit and explicit conditions; in the main part, for the large time behavior, until now there have been no results on stabilization of quasilinear multi-population with two chemicals, fortunately, it is possible to properly absorb the undesirable symbolic contributions caused by dissipation, the corresponding results can be obtained by constructing some proper Lyapunov functionals under different parameters conditions. Now, we state our main results in this paper are stated as follows.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) be a smooth bounded domain. Assume that \((1.2) - (1.4)\) hold and \(f_i \equiv 0\). If \(0 < \gamma_1 \leq 1, \alpha_i + \beta_i < 1 + \frac{1}{n}\) \((i = 1, 2)\) and one of the following conditions holds:

(i) \(\alpha_1 > \alpha_2\) and

\[
\begin{align*}
\frac{1}{2}(\alpha_1 + \alpha_2) + \beta_1 + \gamma_1 &< 1 + \frac{2}{n}, \\
\alpha_1 + \beta_1 + \gamma_1 &< 1 + \frac{2}{n}, \\
\alpha_2 + \beta_2 + \gamma_2 &< 1 + \frac{2}{n};
\end{align*}
\]

(ii) \(\alpha_1 = \alpha_2\) and

\[
\alpha_1 + \beta_1 + \gamma_1 < 1 + \frac{2}{n};
\]

(iii) \(\alpha_1 < \alpha_2\) and

\[
\begin{align*}
\frac{1}{2}(\alpha_1 + \alpha_2) + \beta_1 + \gamma_1 &< 1 + \frac{2}{n}, \\
\alpha_2 + \beta_2 + \gamma_2 &< 1 + \frac{2}{n};
\end{align*}
\]

then system \((1.1)\) possesses a global bounded classical solution \((u, v, w, z)\) in the sense that there exists some constant \(C > 0\) satisfies

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{L^\infty(\Omega)} < C \quad \text{for all } t > 0.
\]

The result on global boundedness of solution to \((1.1)\) with the logistic source are given in the following.

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) be a smooth bounded domain. Assume that \((1.2) - (1.5)\) hold. Assume that there exists \(\mu^* = \mu^*(\beta_i, k_i, \gamma_i, C_s, r_i, \mu_i, \Omega) > 0\). If one of the conditions hold:

(i) \(\gamma_2 < k_1 - \beta_1, \gamma_1 < k_2 - \beta_2\);  
(ii) \(\gamma_2 < k_1 - \beta_1, \gamma_1 = k_2 - \beta_2, \mu_2 > \mu^*\);  
(iii) \(\gamma_2 = k_1 - \beta_1, \gamma_1 < k_2 - \beta_2, \mu_1 > \mu^*\);  
(iv) \(\gamma_2 = k_1 - \beta_1, \gamma_1 = k_2 - \beta_2, \mu_i > \mu^* \ (i = 1, 2)\),

then system \((1.1)\) possesses a global bounded classical solution \((u, v, w, z)\) in the
sense that there exists some constant $C > 0$ satisfies
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{L^\infty(\Omega)} < C \quad \text{for all } t > 0. \]

**Remark 1.** Compared to the [29], in the case of no competition, Theorem 1.2 partially generalizes and improves the results.

Next, for the large time behavior of solutions to (1.1), the exponential decay of solution are given in case of $r_i > 0$ ($i = 1, 2$).

**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$). Suppose that $g_i(s) = s^{r_i}$ and $f_i(s) = r_i s - \mu_i s^{k_i}$ ($i = 1, 2$), and (1.3) hold. If $r_i > 0$, $\gamma_i \geq \frac{1}{2}$ and there exists some $\mu_0 = \mu_0(\alpha_i, \beta_i, k_i, \gamma_i, C_d, C_s, r_i, \mu_i, \Omega) > 0$ such that $\mu_i > \mu_0$ ($i = 1, 2$), then whenever the initial data satisfying (1.2) with $u_0, w_0 \neq 0$ and $(u, v, w, z)$ is a global bounded solution of (1.1), there exist $C > 0$ and $\lambda > 0$ such that
\[
\|u(\cdot, t) - u_*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v_*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w_*\|_{L^\infty(\Omega)} + \|z(\cdot, t) - z_*\|_{L^\infty(\Omega)} < Ce^{-\lambda t}
\]
for all $t > 0$, where
\[
u_* = \left(\frac{r_1}{\mu_1}\right)^{\frac{1}{r_1 - 1}}, \quad v_* = \left(\frac{r_2}{\mu_2}\right)^{\frac{1}{r_2 - 1}}, \quad w_* = \left(\frac{r_2}{\mu_2}\right)^{\frac{1}{r_2 - 1}} \quad \text{and} \quad z_* = \left(\frac{r_1}{\mu_1}\right)^{\frac{1}{r_1 - 1}}.
\]

This paper is organized as below. In Section 2, we establish the local-in-time existence of solution to system (1.1), and we also give the variation of maximal Sobolev regularity. In Sections 3 and 4, we obtain the global boundedness results of existence solution of system (1.1), these solutions are obtained through Banach contraction principle and parabolic regularity, we omit it here and refer the reader to [10, 30, 33] for a detailed local existence result in the closely related single-chemical setting.

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a smooth bounded domain. Assume that (1.2) – (1.5) hold, then there exist $t \in (0, T_{\text{max}})$ and $T_{\text{max}} \in (0, \infty]$ such that system (1.1) has a non-negative solution and satisfies
\[
u, w, z \in C(\overline{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\text{max}})). \tag{2.1}
\]
Moreover,
\[\text{either } T_{\text{max}} = \infty, \text{ or } \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \quad \text{as } t \nearrow T_{\text{max}}. \tag{2.2}\]

Next we give the lemma for the variation of maximal Sobolev regularity, refer to [11, Lemma 2.1] and [51, Lemma 2.2].

**Lemma 2.2.** Let $1 \leq m < \infty$, we consider the following system
\[
\begin{cases}
\frac{\partial f_t}{\partial t} = \Delta f_t - f + b, & (x, t) \in \Omega \times (0, T) \\
\frac{\partial f}{\partial v} = 0, & (x, t) \in \partial\Omega \times (0, T) \\
f(x, 0) = f_0(x), & x \in \Omega.
\end{cases}
\]
For each \( f_0 \in W^{2,m}(\Omega) \) and \( b \in L^m(0,T;L^m(\Omega)) \) \((m > n)\), there exists a unique solution

\[
f \in W^{1,m}((0,T);L^m(\Omega)) \cap L^m((0,T);W^{2,m}(\Omega)),
\]

moreover, if \( f(\cdot,t_0) \in W^{2,m}(\Omega) \) in \( t_0 \in [0,T) \) with \( \frac{\partial f(\cdot,t_0)}{\partial \nu} = 0 \) on \( \partial \Omega \), then there exists some constant \( C(m) > 0 \) such that

\[
\int_{t_0}^T \int_\Omega e^{mt} |\Delta f|^m \leq C(m) \int_{t_0}^T \int_\Omega e^{mt} u^m + C(m)e^{mt0} \left( ||f(\cdot,t_0)||L^m(\Omega)^m + ||\Delta f(\cdot,t_0)||L^m(\Omega)^m \right).
\]

3. Boundedness without logistic source. The purpose of this part is to obtain global boundedness of solution to system (1.1) without logistic source. We first establish a series of prior estimates; then we treat the dissipative terms on the right hand side of the inequality by using the Gagliardo-Nirenberg inequality \([8, 21, 32]\); last, we get our final results by controlling the parameter range in the inequality.

Next, we first give some basic estimates of \( u, v, w, z \).

**Lemma 3.1.** Let \( \Omega \subset \mathbb{R}^n (n \geq 1) \) be a smooth bounded domain and \( f_i = 0 \) \((i = 1, 2)\). Assume that (1.2) - (1.4) hold, then we have

\[
\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \text{and} \quad \|w(\cdot,t)\|_{L^1(\Omega)} = \|w_0\|_{L^1(\Omega)} \quad (3.1)
\]

for all \( t \in (0,T_{\text{max}}) \). Assume that \( 0 < \gamma_i \leq 1 \), if \( s_i \in \left[1, \frac{n}{(n\gamma_i - 1)\gamma_i}\right] \), then we obtain

\[
\|v(\cdot,t)\|_{W^{1,s_i}(\Omega)} \leq c_1 \quad \text{and} \quad \|z(\cdot,t)\|_{W^{1,s_2}(\Omega)} \leq c_2 \quad (3.2)
\]

for all \( t \in (0,T_{\text{max}}) \) with some \( c_1 = c_1(s_1, \gamma_1, w_0) > 0 \) and \( c_2 = c_2(s_2, \gamma_2, u_0) > 0 \).

**Proof.** Integrating the first and third equations of (1.1) imply (3.1). Form the Neumann semigroup estimates method in [13, Lemma 1], (3.2) can be obtained. \( \square \)

In the next two lemmas, we give the priori estimates of \( u, v, w, z \), respectively.

**Lemma 3.2.** Let \( p > 1 \). Assume that (1.2) and (1.3) hold and \( f_i \equiv 0 \) \((i = 1, 2)\), then we conclude

\[
\frac{d}{dt} \int_\Omega (1 + u)^p + \frac{2C_{d_1}p(p-1)}{(p-\alpha_1)^2} \int_\Omega |\nabla (1 + u)^{\frac{p-\alpha_1}{2}}|^2 \leq \frac{C_{s_1}^2p(p-1)}{2C_{d_1}} \int_\Omega (1 + u)^{p+\alpha_1+2\beta_1-2} |\nabla v|^2 \quad (3.3)
\]

and

\[
\frac{d}{dt} \int_\Omega (1 + w)^p + \frac{2C_{d_2}p(p-1)}{(p-\alpha_2)^2} \int_\Omega |\nabla (1 + w)^{\frac{p-\alpha_2}{2}}|^2 \leq \frac{C_{s_2}^2p(p-1)}{2C_{d_2}} \int_\Omega (1 + w)^{p+\alpha_2+2\beta_2-2} |\nabla z|^2 \quad (3.4)
\]

for all \( t \in (0,T_{\text{max}}) \).
Proof. Multiplying both sides the first equation of (1.1) by \(p(u + 1)^{p-1}\) and integrating over \(\Omega\) by parts, then using Young’s inequality, we have
\[
\frac{d}{dt} \int_{\Omega} (1 + u)^p = - C_{d_i} p(p - 1) \int_{\Omega} (1 + u)^{p-\alpha_i-2} \nabla u \cdot \nabla v + C_{s_i} p(p - 1) \int_{\Omega} (1 + u)^{p+\beta_i-2} |\nabla u| |\nabla v|
\]
\[
\leq - \frac{C_{d_i} p(p - 1)}{2} \int_{\Omega} (1 + u)^{p-\alpha_i-2} |\nabla u|^2 + \frac{C_{s_i}^2 p(p - 1)}{2C_{d_i}} \int_{\Omega} (1 + u)^{p+\alpha_i+2\beta_i-2} |\nabla v|^2
\]
for all \(t \in (0, T_{\max})\). The first term on the right-hand side of the inequality (3.5) can be expressed as
\[
\frac{C_{d_i} p(p - 1)}{2} \int_{\Omega} (1 + u)^{p-\alpha_i-2} |\nabla u|^2 = \frac{C_{d_i} p(p - 1)}{(p - \alpha_i)^2} \int_{\Omega} |\nabla (1 + u)^{p-\alpha_i}|^2,
\]
thus, we obtain (3.3). Similarly, we obtain (3.4).

Lemma 3.3. Let \(\Omega \subset \mathbb{R}^n (n \geq 1)\) be a smooth bounded domain. Assume that (1.2) and (1.4) hold, \(q_i \geq 1\) \((i = 1, 2)\), then there exits \(C_1 = C_1(q_i, \Omega) > 0\) such that
\[
\frac{1}{q_1} \frac{d}{dt} \int_{\Omega} \nabla v|^{2q_1} + \frac{q_1 - 1}{q_1} \int_{\Omega} |\nabla |\nabla v|^{q_1}|^2 
\]
\[
\leq \left(2(q_1 - 1) + \frac{n}{2}\right) \int_{\Omega} u^2 |\nabla v|^{2(q_1 - 1)} + (C_1 - 2) \int_{\Omega} |\nabla v|^{2q_1}, \tag{3.6}
\]
and
\[
\frac{1}{q_2} \frac{d}{dt} \int_{\Omega} \nabla z|^{2q_2} + \frac{q_2 - 1}{q_2} \int_{\Omega} |\nabla |\nabla z|^{q_2}|^2 
\]
\[
\leq \left(2(q_2 - 1) + \frac{n}{2}\right) \int_{\Omega} u^2 |\nabla z|^{2(q_2 - 1)} + (C_1 - 2) \int_{\Omega} |\nabla z|^{2q_2}, \tag{3.7}
\]
for all \(t \in (0, T_{\max})\).

Proof. Applying the second equation of (1.1), the point-wise identity \(\Delta |\nabla v|^2 = 2 |D^2 v|^2 + 2 \nabla v \cdot \Delta \nabla v\) and the fact \(|\nabla v|^2 \leq n |D^2 v|^2\), we derive
\[
\frac{1}{q_1} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q_1} + \frac{2}{n} \int_{\Omega} |\nabla v|^{2(q_1 - 1)} |\Delta v|^2 + 2 \int_{\Omega} |\nabla v|^{2q_1}
\]
\[
\leq \int_{\Omega} |\nabla v|^{2(q_1 - 1)} |\Delta v|^2 + 2 \int_{\Omega} |\nabla v|^{2(q_1 - 1)} \nabla v \cdot \nabla g_1(w)
\]
\[
= - (q_1 - 1) \int_{\Omega} |\nabla v|^{2(q_1 - 2)} |\nabla \nabla v|^2 + \int_{\Omega} |\nabla v|^{2(q_1 - 1)} \frac{\partial |\nabla v|^2}{\partial v} dS
\]
\[
- 2(q_1 - 1) \int_{\Omega} |\nabla v|^{2(q_1 - 2)} \nabla v |\nabla v|^2 \cdot \nabla v g_1(w) - 2 \int_{\Omega} |\nabla v|^{2(q_1 - 1)} \Delta v g_1(w)
\]
for all \(t \in (0, T_{\max})\). Using the property of boundary integral without the convexity of domain [19, Lemma 4.2] and the trace inequality [9, Proposition 4.22, 4.24] we have
\[
\int_{\partial \Omega} |\nabla v|^{2(q_1 - 1)} \frac{\partial |\nabla v|^2}{\partial v} dS \leq 2\kappa \int_{\partial \Omega} |\nabla v|^{2q_1} dS
\]
\[
\leq \frac{q_1 - 1}{q_1} \int_{\Omega} |\nabla |\nabla v|^{q_1}|^2 + C_1 \int_{\Omega} |\nabla v|^{2q_1}. \tag{3.9}
\]
with some $\kappa_\Omega > 0$, and $C_1 = C_1(q_i)$ is some positive constant. Combining (3.8) and (3.9), using Young’s inequality, which implies

\[
\frac{1}{q_1} \frac{d}{dt} \int_\Omega |\nabla v|^{2q_1} + \frac{2}{n} \int_\Omega |\nabla v|^{2(q_1 - 1)} |\Delta v|^2 + 2 \int_\Omega |\nabla v|^{2q_1} \\
\leq - \frac{q_1 - 1}{2} \int_\Omega |\nabla v|^{2(q_1 - 2)} |\nabla v|^2 + \frac{q_1 - 1}{q_1} \int_\Omega |\nabla v|^{q_1} |\Delta v|^2 + C_1 \int_\Omega |\nabla v|^{2q_1} \\
+ \frac{2}{n} \int_\Omega |\nabla v|^{2(q_1 - 1)} |\Delta v|^2 + \left(2(q_1 - 1) + \frac{n}{2}\right) \int_\Omega |\nabla v|^{2(q_1 - 1)} |\nabla v|^2 \\
= - \frac{q_1 - 1}{q_1} \int_\Omega |\nabla v|^{q_1}|^2 + C_1 \int_\Omega |\nabla v|^{2q_1} + \frac{2}{n} \int_\Omega |\nabla v|^{2(q_1 - 1)} |\Delta v|^2 \\
+ \left(2(q_1 - 1) + \frac{n}{2}\right) \int_\Omega |\nabla v|^{2(q_1 - 1)} |\nabla v|^2.
\]

Thus, according to (3.10) and (1.4) imply (3.6). Similarly, we obtain (3.7).

Before we give the result of main part, we can select the appropriate parameters for Lemma 3.4 below (the proof see in [31, 33]).

**Corollary 1.** In case $0 < \gamma_i \leq 1$, $\alpha_i + \beta_i < 1 + \frac{1}{n}$ ($i = 1, 2$), if

1. $\alpha_1 > \alpha_2$

   \[
   \begin{aligned}
   \alpha_1 + \beta_1 + \gamma_1 &< 1 + \frac{2}{n}, \\
   \frac{1}{2}(\alpha_1 + \alpha_2) + \beta_2 + \gamma_2 &< 1 + \frac{2}{n},
   \end{aligned}
   \]

then there exist $s_i \in \left[1, \frac{n}{(\alpha_\gamma_i - 1)n}\right]$ such that

\[
\gamma_1 - \frac{1}{n} < \frac{1}{s_1} < 1 + \frac{1}{n} - \alpha_1 - \beta_1
\]

and

\[
\frac{1}{2}(\alpha_1 - \alpha_2) + \gamma_2 - \frac{1}{n} < \frac{1}{s_2} < 1 + \frac{1}{n} - \alpha_2 - \beta_2;
\]

2. $\alpha_1 = \alpha_2$

\[
\alpha_i + \beta_i + \gamma_i < 1 + \frac{2}{n},
\]

then there exist $s_i \in \left[1, \frac{n}{(\alpha_\gamma_i - 1)n}\right]$ such that

\[
\gamma_i - \frac{1}{n} < \frac{1}{s_i} < 1 + \frac{1}{n} - \alpha_i - \beta_i;
\]

3. $\alpha_1 < \alpha_2$

\[
\begin{aligned}
\frac{1}{2}(\alpha_2 + \alpha_1) + \beta_1 + \gamma_1 &< 1 + \frac{2}{n}, \\
\alpha_2 + \beta_2 + \gamma_2 &< 1 + \frac{2}{n},
\end{aligned}
\]

then there exist $s_i \in \left[1, \frac{n}{(\alpha_\gamma_i - 1)n}\right]$ such that

\[
\frac{1}{2}(\alpha_2 - \alpha_1) + \gamma_1 - \frac{1}{n} < \frac{1}{s_1} < 1 + \frac{1}{n} - \alpha_1 - \beta_1
\]

and

\[
\gamma_2 - \frac{1}{n} < \frac{1}{s_2} < 1 + \frac{1}{n} - \alpha_2 - \beta_2.
\]
Let $1 < a_i < \min \left\{ \frac{n}{n-2}, \frac{s_i}{1-s_i} \right\}$ and $b_i > \max \left\{ \frac{n}{2}, \frac{1}{2\gamma_i} \right\}$, then there exist $p_* > \max \left\{ 1 + \frac{2\alpha_i}{\gamma_i}, \alpha_1, \alpha_2 \right\}$ and $q_i \geq 2\gamma_i$ such that for all $p > p_*$ and $q_i > q_i$, we have

\[
\frac{n-2}{n} \cdot \frac{p + \alpha_i + 2\beta_i - 2}{p - \alpha_i} < \frac{1}{a_i} < p + \alpha_i + 2\beta_i - 2
\]  

(3.11)

and

\[
1 - \frac{2}{s_i} \cdot \frac{1}{a_i} < 1 - \frac{n - 2}{nq_i},
\]  

(3.12)

and we also have

\[
\frac{n-2}{n} \cdot \frac{2\gamma_1}{p - \alpha_2} < \frac{1}{b_1} < \frac{2}{q_1} \left( 1 - \frac{2}{n} \right) \quad \text{and} \quad \frac{2b_1(q_1 - 1)}{b_1 - 1} > s_1
\]  

(3.13)

and

\[
\frac{n-2}{n} \cdot \frac{2\gamma_2}{p - \alpha_1} < \frac{1}{b_2} < \frac{2}{q_2} \left( 1 - \frac{2}{n} \right) \quad \text{and} \quad \frac{2b_2(q_2 - 1)}{b_2 - 1} > s_2.
\]  

(3.14)

In the following lemma, combining (3.3), (3.4), (3.6) and (3.7), and taking appropriate parameters to establish the uniformly bounded of $\|u\|_{L^p(\Omega)}$ and $\|w\|_{L^p(\Omega)}$.

Lemma 3.4. Let $\Omega \subset \mathbb{R}^n \ (n \geq 2)$ be a smooth bounded domain. Assume that (1.2) - (1.4) and Corollary 1 hold, the logistic source terms $f_i \equiv 0$. If $0 < \gamma_i \leq 1$, $\alpha_i + \beta_i < 1 + \frac{1}{n}$ ($i = 1, 2$) and one of the following conditions holds:

(i) $\alpha_1 > \alpha_2$ and

\[
\begin{cases}
\alpha_1 + \beta_1 + \gamma_1 < 1 + \frac{2}{n}, \\
\frac{1}{2}(\alpha_1 + \alpha_2) + \beta_2 + \gamma_2 < 1 + \frac{2}{n};
\end{cases}
\]

(ii) $\alpha_1 = \alpha_2$ and

\[
\alpha_i + \beta_i + \gamma_i < 1 + \frac{2}{n};
\]

(iii) $\alpha_1 < \alpha_2$ and

\[
\begin{cases}
\frac{1}{2}(\alpha_1 + \alpha_2) + \beta_1 + \gamma_1 < 1 + \frac{2}{n}, \\
\alpha_2 + \beta_2 + \gamma_2 < 1 + \frac{2}{n};
\end{cases}
\]

then for all $p \in (\max \{1, p_*\}, \infty)$ and $q_i \in (q_i, \infty)$, there exists $C = C(p, q_i, \alpha_i, \beta_i, \gamma_i, C_d, C_s, u_0, w_0, \Omega) > 0$ such that

\[
\|u(\cdot, t)\|_{L^p(\Omega)} + \|\nabla v(\cdot, t)\|_{L^{q_1}(\Omega)} + \|w(\cdot, t)\|_{L^p(\Omega)} + \|\nabla z(\cdot, t)\|_{L^{q_2}(\Omega)} \leq C
\]  

(3.15)

for all $t \in (0, T_{\max})$.

Proof. Combining (3.3), (3.4), (3.6) and (3.7) we have

\[
\frac{d}{dt} \int_{\Omega} \left( (1 + u)^{p} + \frac{1}{q_1} |\nabla v|^{2q_1} + (1 + w)^{p} + \frac{1}{q_2} |\nabla z|^{2q_2} \right) - C_2 \int_{\Omega} |\nabla v|^{2q_1}
\]

\[
- C_2 \int_{\Omega} |\nabla z|^{2q_2} + \frac{q_1}{q_1^2} \int_{\Omega} |\nabla |\nabla v|^{q_1}|^2 + \frac{q_2}{q_2^2} \int_{\Omega} |\nabla |\nabla z|^{q_2}|^2
\]
In view of (3.1) and the Gagliardo-Nirenberg inequality [8, 21, 32] we have
\begin{equation}
10XU PAN AND LIANGCHEN WANG
\end{equation}
and
\begin{equation}
C_2 \int_\Omega (1 + w)^{p+\alpha_1+2\beta_1-2} |\nabla v|^2 + C_2 \int_\Omega (1 + w)^{p+\alpha_2+2\beta_2-2} |\nabla z|^2
\end{equation}
for all $t \in (0, T_{\text{max}})$ with $C_2 = \max \left\{ \frac{c_2^2 p(p-1)}{2e_{cl}}, C_1 - 2, 2(q_1 - 1) + \frac{2}{q_2} \right\} > 0$. For convenience, let $g(t) := \int_\Omega \left( (1 + u)^p + \frac{1}{q_1} |\nabla v|^{2q_1} + (1 + w)^p + \frac{1}{q_2} |\nabla z|^{2q_2} \right)$. According to Corollary 1, $a_i, b_i > 1$ ($i = 1, 2$), let $a'_i = \frac{a_i}{a_i - 1} > 1$ and $b'_i = \frac{b_i}{b_i - 1} > 1$, applying Hölder’s inequality to the four terms on the right-hand side of (3.16), which infer
\begin{equation}
\int_\Omega (1 + u)^{p+\alpha_1+2\beta_1-2} |\nabla v|^2 \leq \left( \int_\Omega (1 + u)^{(p+\alpha_1+2\beta_1-2)\alpha_1} \right)^{\frac{1}{\alpha_1}} \left( \int_\Omega |\nabla v|^{2\alpha_1} \right)^{\frac{1}{\alpha_1}},
\end{equation}
\begin{equation}
\int_\Omega (1 + u)^{p+\alpha_2+2\beta_2-2} |\nabla z|^2 \leq \left( \int_\Omega (1 + u)^{(p+\alpha_2+2\beta_2-2)\alpha_2} \right)^{\frac{1}{\alpha_2}} \left( \int_\Omega |\nabla z|^{2\alpha_2} \right)^{\frac{1}{\alpha_2}},
\end{equation}
\begin{equation}
\int_\Omega (1 + w)^{2\gamma_1} |\nabla v|^{2(q_1 - 1)} \leq \left( \int_\Omega (1 + w)^{2\gamma_1 b_1} \right)^{\frac{1}{\beta_1}} \left( \int_\Omega |\nabla v|^{2(q_1 - 1)b'_1} \right)^{\frac{1}{b'_1}},
\end{equation}
and
\begin{equation}
\int_\Omega (1 + u)^{2\gamma_2} |\nabla z|^{2(q_2 - 1)} \leq \left( \int_\Omega (1 + u)^{2\gamma_2 b_2} \right)^{\frac{1}{\beta_2}} \left( \int_\Omega |\nabla z|^{2(q_2 - 1)b'_2} \right)^{\frac{1}{b'_2}}.
\end{equation}
In view of (3.1) and the Gagliardo-Nirenberg inequality [8, 21, 32] we have
\begin{equation}
\left( \int_\Omega (1 + u)^{(p+\alpha_1+2\beta_1-2)\alpha_1} \right)^{\frac{1}{\alpha_1}} = \| (1 + u)^{p-\frac{\alpha_1}{2}} \|_{L^{2(p+\alpha_1+2\beta_1-2)-2(1-\alpha_1)}}^{2(p+\alpha_1+2\beta_1-2)-2(1-\alpha_1)}(\Omega)
\end{equation}
\begin{equation}
\leq C_3 \| \nabla (1 + u)^{p-\frac{\alpha_1}{2}} \|_{L^{2(p+\alpha_1+2\beta_1-2)-2(1-\alpha_1)}}^{2(p+\alpha_1+2\beta_1-2)-2(1-\alpha_1)}(\Omega)
\end{equation}
\begin{equation}
+ C_3 \| (1 + u)^{p-\frac{\alpha_1}{2}} \|_{L^{2(p+\alpha_1+2\beta_1-2)-2(1-\alpha_1)}}^{2(p+\alpha_1+2\beta_1-2)-2(1-\alpha_1)\theta_1}(\Omega)
\end{equation}
\begin{equation}
\leq C_4 \left( \int_\Omega |\nabla (1 + u)^{p-\frac{\alpha_1}{2}}|^{\frac{2}{p-\alpha_1+2\beta_1-2(1-\alpha_1)}}(\Omega) \right)^{\frac{1}{\alpha_1}} + C_4
\end{equation}
with some $C_3 = C_3(\Omega) > 0$ and $C_4 = C_4(p, \alpha_1, \beta_1, u_0, \Omega) > 0$, where $\theta_1 = \frac{p-\frac{\alpha_1}{2}}{\frac{p-\alpha_1+2\beta_1-2(1-\alpha_1)}{2} - \frac{1}{2} - \frac{1}{2}} \in (0, 1)$ is guaranteed by (3.11). In view of (3.2) and the Gagliardo-Nirenberg inequality again we have
\begin{equation}
\left( \int_\Omega |\nabla v|^{2\alpha_1} \right)^{\frac{1}{\alpha_1}} = \| |\nabla v|^{q_1} \|_{L^{2\alpha_1}}^{\frac{2\alpha_1}{1-\alpha_1}}(\Omega)
\end{equation}
\begin{equation}
\leq C_5 \| |\nabla v|^{q_1} \|_{L^{2\alpha_1}}^{2\alpha_1} \| |\nabla v|^{q_1} \|_{L^{2\alpha_1}}^{2(1-\alpha_1)} + C_5 \| |\nabla v|^{q_1} \|_{L^{2\alpha_1}}^{\frac{2\alpha_1}{1-\alpha_1}}(\Omega)
\end{equation}
\begin{equation}
\leq C_6 \left( \int_\Omega |\nabla v|^{q_1} |^{2} \right)^{\frac{1}{2\alpha_1}} + C_6
\end{equation}
with some $C_5 = C_5(\Omega) > 0$ and $C_6 = C_6(q_1, s_1, a', w_0, \Omega) > 0$, where $\delta_1 = \frac{p - \alpha_1 + 2\beta_1 - q_1}{s_1 - 1 + \frac{p - \alpha_2}{2}} \in (0, 1)$ is guaranteed by (3.12). Combining (3.21) and (3.22) with (3.17), there exists a constant $C_7 = C_7(p, \alpha_i, \beta_i, \gamma_i, C_{d_i}, C_{s_i}, w_0, w_0, \Omega) > 0$ fulfilling

$$C_2 \int_\Omega (1 + u)^{p+\alpha_1+2\beta_1-2} |\nabla v|^2 \leq C_7 \left( \int \nabla (1 + u)^{p-\alpha_2} \right)^{\frac{q_1}{q_1-1}} \left( \int |\nabla v|^{q_1} \right)^{\frac{q_1}{q_1-1}} + C_7,$$

(3.23)

similarly, according to (3.11), (3.12) and Lemma 3.1 again, we also have

$$C_2 \int_\Omega (1 + u)^{p+\alpha_2+2\beta_2-2} |\nabla z|^2 \leq C_7 \left( \int \nabla (1 + u)^{p-\alpha_2} \right)^{\frac{q_1}{q_1-1}} \left( \int |\nabla z|^{q_1} \right)^{\frac{q_1}{q_1-1}} + C_7,$$

(3.24)

where

$$\theta_2 = \frac{p - \alpha_2}{s_2 - 1 + \frac{p - \alpha_2}{2}} \in (0, 1) \quad \text{and} \quad \delta_2 = \frac{q_2}{s_2} + \frac{q_2}{p - \alpha_2} - \frac{q_2}{2} \in (0, 1). \quad (3.25)$$

Similarly, in view of (3.13), the Gagliardo-Nirenberg inequality and Lemma 3.1 we derive

$$\left( \int (1 + u)^{2\gamma_1 b_1} \right)^{\frac{1}{b_1}} = \left\| (1 + u)^{p - \alpha_2} \right\|_{\frac{4\gamma_1}{4\gamma_1 + \gamma_0} \frac{4\gamma_1}{4\gamma_1 + b_1} (\Omega)} \leq C_8 \left( \int \nabla (1 + u)^{p - \alpha_2} \right)^{2\gamma_1 b_1} + C_8$$

(3.26)

and

$$\left( \int |\nabla v|^{2(q_1-1)\bar{b}_1'} \right)^{\frac{1}{\bar{b}_1'}} = \left\| |\nabla v|^{q_1} \right\|_{\frac{2(q_1-1)\bar{b}_1'}{L \frac{2(q_1-1)\bar{b}_1'}{q_1}} (\Omega)} \leq C_9 \left( \int |\nabla |\nabla v|^{q_1} \right)^{\frac{q_1-1}{q_1}} + C_9$$

(3.27)

with $C_8 = C_8(p, \alpha_2, \gamma_1, b_1, w_0, \Omega) > 0$ and $C_9 = C_9(q_1, s_1, b_1', w_0, \Omega) > 0$, where

$$\bar{b}_1 = \frac{p - \alpha_2}{s_1 - 1 + \frac{p - \alpha_2}{2}} \in (0, 1) \quad \text{and} \quad \bar{d}_1 = \frac{q_1}{s_1} + \frac{2(q_1 - 1)}{s_1} \in (0, 1). \quad (3.28)$$

Combining (3.26) and (3.27) with (3.19), there exists $C_{10} = C_{10}(p, \alpha_i, \gamma_i, C_{d_i}, C_{s_i}, w_0, w_0, \Omega) > 0$ fulfilling

$$C_2 \int_\Omega (1 + u)^{2\gamma_1 |\nabla v|^{2(q_1-1)}} \leq C_{10} \left( \int \nabla (1 + u)^{p - \alpha_2} \right)^{2\gamma_1 \bar{b}_1} \left( \int |\nabla |\nabla v|^{q_1} \right)^{\frac{q_1-1}{q_1}} + C_{10},$$

(3.29)
similarly, in view of (3.14), the Gagliardo-Nirenberg inequality and Lemma 3.1 imply
\[
C_2 \int_\Omega (1 + u)^{2\gamma_2} |\nabla z|^{2(q_2 - 1)}
\leq C_{10} \left( \int_\Omega \left| \nabla (1 + u)^{\frac{p - \alpha_1}{2}} \right|^2 \right)^{\frac{2\gamma_2}{p - \alpha_1}} \left( \int_\Omega |\nabla |\nabla z|^{q_2}|^2 \right)^{(q_2 - 1)\frac{\delta_2}{q_2}} + C_{10},
\]
(3.30)
where
\[
\delta_2 = \frac{\frac{p - \alpha_1}{2} - \frac{p - \alpha_1}{q_2}}{\frac{1}{n} - \frac{1}{2} + \frac{p - \alpha_1}{2}} \in (0, 1)
\text{ and } \ \delta_2 = \frac{q_2}{n} + \frac{2(q_2 - 1)\delta_2}{q_2} - \frac{q_2}{n} \in (0, 1).
\]
(3.31)
Therefore, (3.16) in conjunction with (3.23), (3.24), (3.29) and (3.30), which infer
\[
y'(t) + \frac{2C_{d_1}p(p - 1)}{(p - \alpha_1)^2} \int_\Omega |\nabla (1 + u)^{\frac{p - \alpha_1}{2}}|^2 + \frac{q_1 - 1}{q_1^2} \int_\Omega |\nabla |\nabla v|^{q_1}|^2
+ \frac{2C_{d_2}p(p - 1)}{(p - \alpha_2)^2} \int_\Omega |\nabla (1 + w)^{\frac{p - \alpha_2}{2}}|^2 + \frac{q_2 - 1}{q_2^2} \int_\Omega |\nabla |\nabla z|^{q_2}|^2
\leq C_7 \left( \int_\Omega \left| \nabla (1 + u)^{\frac{p - \alpha_1}{2}} \right|^2 \right)^{\frac{p + \alpha_1 + 2\theta_1 - 1}{p - \alpha_1}} \left( \int_\Omega |\nabla |\nabla v|^{q_1}|^2 \right)^{\frac{\delta_1}{q_1}} + C_2 \int_\Omega |\nabla v|^{2q_1}
+ C_7 \left( \int_\Omega \left| \nabla (1 + w)^{\frac{p - \alpha_2}{2}} \right|^2 \right)^{\frac{p + \alpha_2 + 2\theta_2 - 1}{p - \alpha_2}} \left( \int_\Omega |\nabla |\nabla z|^{q_2}|^2 \right)^{\frac{\delta_2}{q_2}} + 2C_7
+ C_{10} \left( \int_\Omega \left| \nabla (1 + u)^{\frac{p - \alpha_1}{2}} \right|^2 \right)^{\frac{2\gamma_1}{p - \alpha_1}} \left( \int_\Omega |\nabla |\nabla v|^{q_1}|^2 \right)^{\frac{\delta_1}{q_1}} + C_2 \int_\Omega |\nabla z|^{2q_2}
+ C_{10} \left( \int_\Omega \left| \nabla (1 + w)^{\frac{p - \alpha_2}{2}} \right|^2 \right)^{\frac{2\gamma_2}{p - \alpha_2}} \left( \int_\Omega |\nabla |\nabla z|^{q_2}|^2 \right)^{\frac{\delta_2}{q_2}} + 2C_{10}
\]
for all \( t \in (0, T_{\text{max}}) \). Thus, we can obtain
\[
y'(t) + \frac{C_{d_1}p(p - 1)}{(p - \alpha_1)^2} \int_\Omega |\nabla (1 + u)^{\frac{p - \alpha_1}{2}}|^2 + \frac{q_1 - 1}{q_1^2} \int_\Omega |\nabla |\nabla v|^{q_1}|^2
+ \frac{C_{d_2}p(p - 1)}{(p - \alpha_2)^2} \int_\Omega |\nabla (1 + w)^{\frac{p - \alpha_2}{2}}|^2 + \frac{q_2 - 1}{q_2^2} \int_\Omega |\nabla |\nabla z|^{q_2}|^2
\leq C_2 \int_\Omega |\nabla v|^{2q_1} + C_2 \int_\Omega |\nabla z|^{2q_2} + C_{11}
\]
(3.33)
with \( C_{11} = C_{11}(p, q_1, \alpha_1, \beta_1, \gamma_1, C_{d_1}, C_{d_2}, u_0, w_0, \Omega) > 0 \) if
\[
\frac{p + \alpha_1 + 2\beta_1 - 2}{p - \alpha_1} \theta_1 + \frac{\delta_1}{q_1} < 1 \quad (i = 1, 2)
\]
(3.34)
and
\[
\frac{2\gamma_1 \theta_1}{p - \alpha_1} + \frac{(q_1 - 1)\delta_1}{q_1} < 1 \quad \text{and} \quad \frac{2\gamma_2 \theta_2}{p - \alpha_2} + \frac{(q_2 - 1)\delta_2}{q_2} < 1.
\]
(3.35)
Therefore, in order for the assumptions in (3.34) and (3.35) to be satisfied, let
\[
h_i(q_i) := \frac{p + \alpha_1 + 2\beta_1 - 2}{p - \alpha_1} \theta_1 + \frac{\delta_1}{q_1} = \frac{p + \alpha_1 + 2\beta_1 - 2}{p - \alpha_1} - \frac{1}{2q_1} + \frac{1}{n} - \frac{1}{2} + \frac{q_2}{s}, \quad (i = 1, 2)
\]
(3.36)
and
\[ h_1(q_1) := \frac{2\gamma_1 \tilde{\theta}_1}{p - \alpha_1} + \frac{(q_1 - 1) \tilde{\delta}_1}{q_1} = \frac{\gamma_1 - \frac{1}{2\theta_1}}{\frac{1}{n} - \frac{1}{2} + \frac{p - \alpha_1}{2}} + \frac{q_1 - 1}{2\theta_1} + \frac{1}{2} - \frac{1}{2} \]  
(3.37)

and
\[ h_2(q_2) := \frac{2\gamma_2 \tilde{\theta}_2}{p - \alpha_1} + \frac{(q_2 - 1) \tilde{\delta}_2}{q_2} = \frac{\gamma_2 - \frac{1}{2\theta_2}}{\frac{1}{n} - \frac{1}{2} + \frac{p - \alpha_1}{2}} + \frac{q_2 - 1}{2\theta_2} + \frac{1}{2} - \frac{1}{2} \]  
(3.38)

According to the conditions of (1), (2) and (3) in Corollary 1, we have
\[ h_i(q_i(p)) < 1 \quad \text{and} \quad \tilde{h}_i(q_i(p)) < 1 \]  
(3.39)

with \( q_i(p) := \frac{p - \alpha_i}{s_i} \) (i = 1, 2). Since \( q_i(p) \to +\infty \) as \( p \to \infty \), there exists \( q_i \in (q_i, \infty) \) for all \( p > p_* \) and \( q_i > q_i \), fulfilling
\[ h_i(q_i) < 1 \quad \text{and} \quad \tilde{h}_i(q_i) < 1, \]  
(3.40)

thus, the assumptions in (3.34) and (3.35) are satisfied.

In order for the inequality (3.33) to satisfy the Gronwall inequality, using the Gagliardo-Nirenberg inequality and Lemma 3.1, we arrive at
\[ \int_{\Omega} (1 + u)^p = \left\| (1 + u) \right\|_{L^\frac{p}{p-1} (\Omega)}^{\frac{p-1}{p}} C_{12} \left( \int_{\Omega} \| \nabla (1 + u) \|_{p-1}^{\frac{p-1}{2}} \right) \frac{p-1}{p} + C_{12} \]  
(3.41)

and
\[ \int_{\Omega} (1 + w)^p = \left\| (1 + w) \right\|_{L^\frac{p}{p-2} (\Omega)}^{\frac{p-2}{p}} C_{12} \left( \int_{\Omega} \| \nabla (1 + w) \|_{p-2}^{\frac{p-2}{2}} \right) \frac{p-2}{p} + C_{12} \]  
(3.42)

with \( C_{12} = C_{12}(p, \alpha_1, u_0, w_0, \Omega) > 0 \), where \( \sigma_1 = \frac{p - \alpha_1}{2} - \frac{p - \alpha_1}{2} \in (0, 1) \) and \( \sigma_2 = \frac{p - \alpha_2}{2} - \frac{p - \alpha_2}{2} \in (0, 1) \) are satisfied because of the condition \( p > 1 + \frac{n\alpha_2}{2} \) in Corollary 1. In the same way, we obtain
\[ \left( \frac{1}{q_1} + C_2 \right) \int_{\Omega} |\nabla v|^{2q_1} = \left( \frac{1}{q_1} + C_2 \right) \left\| |\nabla v|^{q_1} \right\|_{L^2(\Omega)} \]  
(3.43)

and
\[ \left( \frac{1}{q_2} + C_2 \right) \int_{\Omega} |\nabla z|^{2q_2} = \left( \frac{1}{q_2} + C_2 \right) \left\| |\nabla z|^{q_2} \right\|_{L^2(\Omega)} \]  
(3.44)
with some $C_{13}, C_{14} > 0$, here $C_{13}, C_{14}$ depend on $p, q_i, C_{d_i}, C_{s_i}, s_i, u_0, w_0, \Omega$, where 
\[ \sigma_1 = \frac{a_i - b_i}{2} \in (0, 1) \] and \[ \sigma_2 = \frac{a_i - b_i}{2} \in (0, 1) \] are satisfied because of the condition $q_i > 1 + \frac{\sigma}{\sigma_1}$ in Corollary 1. Therefore, combining (3.41) – (3.44) with (3.33), which implies

\[
y'(t) + C_{15} \left( \int_\Omega (1 + u)^p \right)^{\frac{\mu - 1}{p - \frac{1}{2}}} + \frac{1}{q_1} \int_{\Omega} |\nabla z|^2 q_1 \\
+ C_{15} \left( \int_\Omega (1 + u)^p \right)^{\frac{\mu - 1}{p - \frac{1}{2}}} + \frac{1}{q_2} \int_{\Omega} |\nabla z|^2 q_2 \leq C_{16}
\]

with some $C_{15} = C_{15}(p, \alpha_i, C_{d_i} u_0, w_0, \Omega) > 0$ and $C_{16} = C_{16}(p, q_i, \alpha_i, \beta_i, \gamma_i, C_{d_i}, C_{s_i}, u_0, w_0, \Omega) > 0$. Thus, according to the ODE comparison principle with (3.45), which implies (3.15).

Now, we can easily prove Theorem 1.1.

**Proof of Theorem 1.1.** According to the ideas in [7, Lemma 2.2 (ii)], we can select $p > \gamma_i n$ to ensure that $\frac{1}{2} + \frac{n}{p}, \frac{2}{p} < 1$ ($i = 1, 2$), combining with Lemma 3.4 and fixing a $p > \max\{1, p_s, \gamma_i n\}$, then we have

\[
\sup_{t \in (0, T_{max})} (\|\nabla u(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla z(\cdot, t)\|_{L^\infty(\Omega)}) \leq C
\]

with $C = C(\alpha_i, \beta_i, \gamma_i, C_{d_i}, C_{s_i}, u_0, w_0, \Omega) > 0$. This along with a standard argument of the Moser-Alakakos iterative technique [33, Lemma A.1] results in

\[
\sup_{t \in (0, T_{max})} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)}) < \infty,
\]

which together with Lemma 2.1 guarantees the global boundedness of solutions to system (1.1). Therefore, we obtain the desired results.

4. **Boundedness with logistic source.** The purpose of this part is to obtain global boundedness of solution to system (1.1) with logistic source. We first establish a prior estimates; then we treat the dissipative terms by applying the variation of maximal Sobolev regularity (refer to [11, Lemma 2.1] and [51, Lemma 2.2]); finally, we get our final results by dividing the parameter range.

**Lemma 4.1.** Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a smooth bounded domain. Assume that (1.2) – (1.5) hold and there exists $\mu^* = \mu^*(\beta_i, k_i, \gamma_i, C_{s_i}, r_i, \mu_i, \Omega) > 0$. If one of the conditions holds:

(i) $\gamma_2 < k_1 - \beta_1, \gamma_1 < k_2 - \beta_2$;

(ii) $\gamma_2 < k_1 - \beta_1, \gamma_1 = k_2 - \beta_2, \mu_2 > \mu^*$;

(iii) $\gamma_2 = k_1 - \beta_1, \gamma_1 < k_2 - \beta_2, \mu_1 > \mu^*$;

(iv) $\gamma_2 = k_1 - \beta_1, \gamma_1 = k_2 - \beta_2, \mu_i > \mu^*$,

then for all $p_i \in (1, \infty) (i = 1, 2)$, there exists $C = C(p_i, \beta_i, \gamma_i, C_{s_i}, r_i, \mu_i, \Omega) > 0$ such that

\[
\|u(\cdot, t)\|_{L^{p_1}(\Omega)} + \|w(\cdot, t)\|_{L^{p_2}(\Omega)} \leq C \quad \text{for all} \ t \in (0, T_{max}).
\]

**Proof.** Define $t_0 := \min \left\{ 1, \frac{1}{2} T_{max} \right\}$ and let $p_i > \max\{1, n(k_i - \beta_i) + 1 - k_i\}$ ($i = 1, 2$) and $p_i = \frac{k_i - \beta_i}{k_i - \beta_i}(p_2 + k_2 - 1) + 1 - k_i$. Multiplying both sides the first equation of
therefore, the last term of inequality (4.2) satisfies
\[
\left(\frac{1}{p_i} - 1\right) \int_\Omega (1 + u)^{p_i} \leq - \int_\Omega (1 + u)^{p_i} - D_i(u)|\nabla u|^2 + \int_\Omega (1 + u)^{p_i - 1} f_i(u)
\]
\[
+ \left(\frac{1}{p_i} - 1\right) \int_\Omega (1 + u)^{p_i - 2} S_i(u) \nabla u \cdot \nabla v
\]
for all \( t \in (t_0, T_{\text{max}}) \). According to (1.5) and using Young’s inequality we derive
\[
\int_\Omega (1 + u)^{p_i - 1} f_i(u) \leq r_i \int_\Omega (1 + u)^{p_i - 1} u - \mu_i \int_\Omega (1 + u)^{p_i - k_i}
\]
\[
\leq r_i \int_\Omega (1 + u)^{p_i - 1} u + \mu_i \int_\Omega (1 + u)^{p_i - 1} - \frac{\mu_1}{2k_i - 1} \int_\Omega (1 + u)^{p_i + k_i - 1}
\]
\[
\leq 2r_i + \int_\Omega (1 + u)^{p_i} - \frac{\mu_1}{2k_i - 1} \int_\Omega (1 + u)^{p_i + k_i - 1} + C_i
\]
with \( C_i = C_i(p_i, \mu_i, r_i, \Omega) > 0 \), where we used \((1 + u)^{k_i} \leq 2^{k_i - 1}(u^{k_i} + 1)\). Define
\[
\varphi_i(u) := \left(1 + \sigma\right)^{p_i - 2} S_i(\sigma) \quad \text{for all } j \geq 0,
\]
the assumption \( p_i > n(k_i - \beta_i) + 1 - k_i \) and the condition \( k_i > \beta_i \) ensure that \( p_i + \beta_i - 1 > 0 \), combining with (1.3), which implies
\[
0 \leq \varphi_i(u) \leq \frac{C_s(p_i - 1)}{p_i + \beta_i - 1} (1 + u)^{p_i + \beta_i - 1} \quad \text{for all } u \geq 0,
\]
therefore, the last term of inequality (4.2) satisfies
\[
\left(\frac{1}{p_i} - 1\right) \int_\Omega (1 + u)^{p_i - 2} S_i(u) \nabla u \cdot \nabla v = \int_\Omega \nabla \varphi_i(u) \cdot \nabla v
\]
\[
\leq \frac{C_s(p_i - 1)}{p_i + \beta_i - 1} \int_\Omega (1 + u)^{p_i + \beta_i - 1} \Delta v
\]
\[
\leq \frac{\mu_1}{2k_i - 1} \int_\Omega (1 + u)^{p_i + k_i - 1} + C_2 \int_\Omega |\Delta v|^{\frac{p_i + k_i - 1}{k_i - \beta_i}}
\]
with some \( C_2 = C_2(p_i, \beta_i, k_i, C_s, \mu_i) > 0 \). Let
\[
m := \frac{p_i + k_i - 1}{k_i - \beta_i}, \quad (i = 1, 2),
\]
in view of \( p_i > n(k_i - \beta) + 1 - k_i \), the need \( m > n \) is satisfied (i.e. the assumption \( m > n \) of Lemma 2.2 is satisfied). Since the diffusion functions \( D_i(u) > 0 \), combining (4.2) – (4.5) and using Young’s inequality because of \( k_i > 1 \), which infer
\[
\left(\frac{1}{p_i} - 1\right) \int_\Omega (1 + u)^{p_i} + \frac{m}{p_i} \int_\Omega (1 + u)^{p_i}
\]
\[
\leq \frac{\mu_1}{2k_i - 1} \int_\Omega (1 + u)^{p_i + k_i - 1} + C_2 \int_\Omega |\Delta v|^m + \left(\frac{m}{p_i} + 2r_i\right) \int_\Omega (1 + u)^{p_i}
\]
\[
- \frac{\mu_1}{2k_i - 1} \int_\Omega (1 + u)^{p_i + k_i - 1} + C_1
\]
\[
\leq - \frac{\mu_1}{2k_i} \int_\Omega (1 + u)^{p_i + k_i - 1} + C_2 \int_\Omega |\Delta v|^m + C_3
\]
with \( C_3 = C_3(p_1, k_1, r_1, \mu_1, m, \Omega) > 0 \). By combining the variation-of-constants formula with (4.6), in view of (1.4) and Lemma 2.2 entails that

\[
\frac{1}{p_1} \int_{\Omega} (1 + u)^{p_1} \\
\leq - \frac{\mu_1}{2k_1} \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)}(1 + u)^{p_1 + k_1 - 1} dx d\tau + C_2 \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)} |\Delta u|^m dx d\tau \\
+ C_3 \int_{t_0}^{t} e^{-m(t-\tau)} dx d\tau + \frac{1}{p} e^{-m(t-t_0)} \int_{\Omega} (1 + u(x, t_0))^{p_1} \\
\leq - \frac{\mu_1}{2k_1} \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)}(1 + u)^{p_1 + k_1 - 1} dx d\tau + C_4 \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)} w^{m \gamma_1} dx d\tau + C_4
\]

(4.7)

with some \( C_4 = C_4(p_1, \beta_1, k_1, C_s, r_1, \mu_1, m, \Omega) > 0 \). Similarly, there exists a constant \( C_5 = C_5(p_2, \beta_2, k_2, C_{s_2}, r_2, \mu_2, m, \Omega) > 0 \) such that

\[
\frac{1}{p_2} \int_{\Omega} (1 + w)^{p_2} \leq - \frac{\mu_2}{2k_2} \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)}(1 + w)^{p_2 + k_2 - 1} dx d\tau \\
+ C_5 \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)} w^{m \gamma_2} dx d\tau + C_5
\]

(4.8)

Combining (4.7) and (4.8), which implies

\[
\frac{1}{p_1} \int_{\Omega} (1 + u)^{p_1} + \frac{\mu_1}{2k_1} \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)}(1 + u)^{p_1 + k_1 - 1} dx d\tau \\
+ \frac{1}{p_2} \int_{\Omega} (1 + w)^{p_2} + \frac{\mu_2}{2k_2} \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)}(1 + w)^{p_2 + k_2 - 1} dx d\tau \\
\leq C_5 \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)}(1 + u)^{m \gamma_2} dx d\tau + C_4 \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)}(1 + w)^{m \gamma_3} dx d\tau + C_6
\]

(4.9)

with \( C_6 = C_6(p_1, \beta_1, k_1, C_s, r_1, \mu_1, m, \Omega) > 0 \). Next, we obtain the global boundedness of solution by dividing it into four cases.

**Case (i).** \( \gamma_2 < k_1 - \beta_1, \gamma_1 < k_2 - \beta_2 \). The two conditions together with (4.5) ensure that \( m \gamma_2 < p_1 + k_1 - 1 \) and \( m \gamma_1 < p_2 + k_2 - 1 \) hold, thanks to Young’s inequality, we have

\[
C_5 \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)}(1 + u)^{m \gamma_2} dx d\tau \leq \frac{\mu_1}{2k_1} \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)}(1 + u)^{p_1 + k_1 - 1} dx d\tau + C_7
\]

(4.10)

and

\[
C_4 \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)}(1 + w)^{m \gamma_3} dx d\tau \leq \frac{\mu_2}{2k_2} \int_{t_0}^{t} \int_{\Omega} e^{-m(t-\tau)}(1 + w)^{p_2 + k_2 - 1} dx d\tau + C_7
\]

(4.11)

with \( C_7 = C_7(p_1, \beta_1, k_1, \gamma_i, C_s, r_1, \mu_1, \Omega) > 0 \). We thus infer from (4.9) – (4.11) that

\[
\frac{1}{p_1} \int_{\Omega} (1 + u)^{p_1} + \frac{1}{p_2} \int_{\Omega} (1 + w)^{p_2} \leq C_8 \text{ for all } t \in (t_0, T_{\text{max}})
\]
with $C_8 = C_8(p_i, \beta_i, k_i, \gamma_i, C_{\theta}, r_i, \mu_i, \Omega) > 0$. Therefore, (4.1) is obtained.

Case (ii). $\gamma_2 < k_1 - \beta_1$, $\gamma_1 = k_2 -\beta_2$. Similarly, the two conditions together with (4.5) ensure that $m\gamma_2 < p_1 + k_1 - 1$ and $m\gamma_1 = p_2 + k_2 - 1$ hold. Thus, there exists $\mu^* := 2k_2C_4 > 0$ such that $\mu_2 \geq \mu^*$, from (4.9) and (4.10) we obtain (4.1).

Case (iii). $\gamma_2 = k_1 - \beta_1$, $\gamma_1 = k_2 -\beta_2$. Cases (iii) and (iv) are treated exactly similar to Case (ii), so we have omitted here. Therefore, this proof is completed. \qed

Proof of Theorem 1.2. The proof here is similar to the Proof of Theorem 1.1, therefore, we can obtain the result of Theorem 1.2 by similar argument. \qed

5. Stabilization. In the main part of this study, we derive the asymptotic behavior of solutions to (1.1) with $r_i > 0$ ($i = 1, 2$), considering the expected nontrivial behavior of the solution, the construction of Lyapunov function is a little difficult. However, different Lyapunov functionals are constructed under different parameters conditions, it is possible to properly absorb the undesirable symbolic contributions caused by dissipation, which is derived primarily from the ideas [1, 7, 41]. To achieve this goal, we first give the following the standard parabolic regularity theory lemma and elementary analysis that are important for proving Theorem 1.3.

Lemma 5.1. Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary. Assume that $(u, v, w, z)$ is a global bounded smooth solution of (1.1), then there exist $\theta \in (0, 1)$ and $c > 0$ such that
\[
\|u\|_{C^{2+\theta,1+\theta/2}(\bar{\Omega} \times [t, t+1])} + \|v\|_{C^{2+\theta,1+\theta/2}(\bar{\Omega} \times [t, t+1])} + \|w\|_{C^{2+\theta,1+\theta/2}(\bar{\Omega} \times [t, t+1])} + \|z\|_{C^{2+\theta,1+\theta/2}(\bar{\Omega} \times [t, t+1])} \leq c \quad \text{for all } t > 1.
\]

Proof. Since $(u, v, w, z)$ is global bounded smooth solution of (1.1), the parabolic gradient Hölder regularity theory [14] and the standard parabolic Schauder theory are applied to equations of (1.1), this lemma can be obtained. \qed

Lemma 5.2. [1, Lemma 3.1] Let $f : (1, \infty)$ is a nonnegative uniformly continuous function, if $\int_1^\infty f(t)dt < \infty$, then we have $f(t) \to 0$ as $t \to \infty$.

Definition 5.3. Assume that $(u, v, w, z)$ is global bounded smooth solution of (1.1) and $\chi_i := \left(\frac{r_i}{\mu_i}\right)^{\frac{1}{\alpha_i}}$, then we define
\[
F(t) := \int_\Omega b_1(u) + a_1 \int_\Omega (v - \chi_1^2)^2 + \int_\Omega b_2(w) + a_2 \int_\Omega (z - \chi_2^2)^2 > 0
\]
for all $t > 0$ with
\[
b_i(s) := s - \chi_i - \chi_i \ln (\chi_i^{-1}s) \geq 0 \quad \text{for all } s > 0 \quad (i = 1, 2)
\]
and
\[
a_1 := C_u^2 \frac{C_u}{4C_d} \cdot \frac{r_1}{\mu_1} \gamma_1^{1/\gamma_1} \quad \text{and} \quad a_2 := C_w^2 \frac{C_w}{4C_d} \cdot \frac{r_2}{\mu_2} \gamma_2^{1/\gamma_2},
\]
where $C_u := (\|u\|_{L^\infty} + 1)^{\alpha_1 + 2\beta_1 - 2} > 0$ and $C_w := (\|w\|_{L^\infty} + 1)^{\alpha_2 + 2\beta_2 - 2} > 0$.

With the above preparation, in the next two lemmas, we can design the suitable Lyapunov functionals with the desired performance in the case of $k_i \geq 2$ and $k_i < 2$ ($i = 1, 2$), respectively.
Lemma 5.4. Assume that (1.2) – (1.5) and Definition 5.3 hold, $r_i > 0$, if $k_i \geq 2$ $(i = 1, 2)$, then the global smooth solution of (1.1) satisfies

$$
\frac{d}{dt} \int_{\Omega} b_1(u) = \int_{\Omega} \frac{u - \chi_1}{u} \left( \nabla \cdot (D_1(u) \nabla u) - \nabla \cdot (S_1(u) \nabla v) + r_1 u - \mu_1 u^{k_1} \right)
$$

$$
= -\chi_1 \int_{\Omega} \frac{D_1(u)|\nabla u|^2}{u^2} + \chi_1 \int_{\Omega} \frac{S_1(u)|\nabla v|^2}{u^2} - \mu_1 \int_{\Omega} (u - \chi_1) \left( u^{k_1-1} - \frac{r_1}{\mu_1} \right)
$$

for all $t > 0$ with

$$
H_1(w) := \left\{ \begin{array}{ll}
4^{1-\gamma_1} \chi_2^{2\gamma_1-2}, & \text{if } \gamma_1 < 1, \\
\gamma_1^2 (w + \chi_2)^{2\gamma_1-2}, & \text{if } \gamma_1 \geq 1
\end{array} \right.
$$

and

$$
H_2(u) := \left\{ \begin{array}{ll}
4^{1-\gamma_2} \chi_1^{2\gamma_2-2}, & \text{if } \gamma_2 < 1, \\
\gamma_2^2 (u + \chi_1)^{2\gamma_2-2}, & \text{if } \gamma_2 \geq 1.
\end{array} \right.
$$

Proof. Multiplying the first equation in (1.1) by $(1 - \frac{u}{w})$ and integrating over $\Omega$ by parts we derive

$$
\frac{d}{dt} \int_{\Omega} b_1(u) = \int_{\Omega} \frac{u - \chi_1}{u} \left( \nabla \cdot (D_1(u) \nabla u) - \nabla \cdot (S_1(u) \nabla v) + r_1 u - \mu_1 u^{k_1} \right)
$$

$$
= -\chi_1 \int_{\Omega} \frac{D_1(u)|\nabla u|^2}{u^2} + \chi_1 \int_{\Omega} \frac{S_1(u)|\nabla v|^2}{u^2} - \mu_1 \int_{\Omega} (u - \chi_1) \left( u^{k_1-1} - \frac{r_1}{\mu_1} \right)
$$

for all $t > 0$, where $b_1(s)$ from (5.2), applying Young’s inequality which implies

$$
\int \frac{S_1(u)|\nabla v|^2}{u^2} \leq \frac{1}{2} \int \frac{D_1(u)|\nabla u|^2}{u^2} + \frac{1}{2} \int \frac{S_1^2(u)|\nabla v|^2}{u^2 D_1(u)}
$$

$$
\leq \frac{1}{2} \int \frac{D_1(u)|\nabla u|^2}{u^2} + \frac{C_1^2}{2C_4} \int (u + 1)^{\alpha_1 + \beta_1 - 2} |\nabla v|^2.
$$

For the term on the right-hand side of (5.7) that have not been processed, according to the assumptions $k_i \geq 2$ $(i = 1, 2)$, we have

$$(u - \chi_1) \left( u^{k_1-1} - \frac{r_1}{\mu_1} \right) = (u - \chi_1) \left( u^{k_1-1} - \chi_1^{k_1-1} \right) \geq \chi_1^{k_1-2} (u - \chi_1)^2,$$

thus we obtain

$$-\mu_1 (u - \chi_1) \left( u^{k_1-1} - \frac{r_1}{\mu_1} \right) \leq -\mu_1 \chi_1^{k_1-2} \int (u - \chi_1)^2 = -\frac{r_1}{\chi_1} \int (u - \chi_1)^2,$$

therefore, combining (5.7) – (5.9) yields

$$
\frac{d}{dt} \int_{\Omega} b_1(u) \leq \frac{C_1^2 \chi_1}{2C_4} \int (u + 1)^{\alpha_1 + \beta_1 - 2} |\nabla v|^2 - \frac{r_1}{\chi_1} \int (u - \chi_1)^2 \quad \text{for all } t > 0.
$$

(5.10)
The same treatment to the $b_2(u)$, and combining it with (5.10), we have

$$
\frac{d}{dt} \int_\Omega \left( b_1(u) + b_2(w) \right) \leq \frac{C_{r_1}^2 \chi_1}{2C_{d_1}} \int_\Omega (u + 1)^{\alpha_1 + 2\beta_1 - 2} |\nabla v|^2 - \frac{\tau_1}{\chi_1} \int_\Omega (u - \chi_1)^2 \\
+ \frac{C_{r_2}^2 \chi_2}{2C_{d_2}} \int_\Omega (w + 1)^{\alpha_2 + 2\beta_2 - 2} |\nabla z|^2 - \frac{\tau_2}{\chi_2} \int_\Omega (w - \chi_2)^2
$$

(5.11)

for all $t > 0$. Multiplying the second equation of (1.1) by $v - \chi_2^{|\gamma_1|}$, integrating over $\Omega$ by parts and applying Young’s inequality we derive

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega (v - \chi_2^{|\gamma_1|})^2 \leq -\int_\Omega |\nabla v|^2 - \int_\Omega (v - \chi_2^{|\gamma_1|})^2 + \int_\Omega (u^{\gamma_1} - \chi_2^{|\gamma_1|})(v - \chi_2^{|\gamma_1|}) \\
\leq -\int_\Omega |\nabla v|^2 - \frac{1}{2} \int_\Omega (v - \chi_2^{|\gamma_1|})^2 + \int_\Omega (u^{\gamma_1} - \chi_2^{|\gamma_1|})^2
$$

(5.12)

for all $t > 0$. For the last term of inequality (5.12), we can handle it under different conditions of $\gamma_i$ $(i = 1, 2)$, as below.

In case $\gamma_1 \in (0, 1)$, if $w(x, t) \leq \frac{\alpha}{\chi_2}$ on $(x, t) \in \Omega \times (0, \infty)$, we infer that

$$
|w^{\gamma_1} - \chi_2^{|\gamma_1|}| \leq |w - \chi_2|^{\gamma_1} = |w - \chi_2|^{\gamma_1 - 1}|w - \chi_2| \leq 2^{1 - \gamma_1} \chi_2^{1 - 1}|w - \chi_2|;
$$

(5.13)

if $w(x, t) > \frac{\alpha}{\chi_2}$ on $(x, t) \in \Omega \times (0, \infty)$, we set $h_1(s) := s^{\gamma_1}$ on $s \in (\frac{\alpha}{\chi_2}, \infty)$, according to the Mean value theorem, there exists some $\theta_1 \in (0, 1)$ satisfies

$$
|w^{\gamma_1} - \chi_2^{|\gamma_1|}| = |h_1(w) - h_1(\chi_2)| \leq h_1'(w - \theta_1 w + \theta_1 \chi_2)|w - \chi_2| \\
\leq h_1' \left( \frac{\chi_2}{2} \right) |w - \chi_2| \\
= \gamma_1 2^{1 - \gamma_1} \chi_2^{1 - 1}|w - \chi_2|,
$$

(5.14)

where we used the fact that $h_1'(s) = \gamma_1 s^{\gamma_1 - 1}$ is a monotone decreasing function on $(\frac{\alpha}{\chi_2}, \infty)$, and we also used $w - \theta_1 w + \theta_1 \chi_2 > \frac{\alpha}{\chi_2}$ if $\frac{\alpha}{\chi_2} > \frac{\alpha}{\chi_2}$.

In case $\gamma_1 \geq 1$, $h_1'(s) = \gamma_1 s^{\gamma_1 - 1}$ is a monotone increasing function, using the Mean value theorem again, there exists some $\theta_1 \in (0, 1)$ satisfies

$$
|w^{\gamma_1} - \chi_2^{|\gamma_1|}| = |h_1(w) - h_1(\chi_2)| \leq h_1'(w - \theta_1 w + \theta_1 \chi_2)|w - \chi_2| \\
\leq \gamma_1 (w + \chi_2)^{\gamma_1 - 1}|w - \chi_2|.
$$

(5.15)

Combining (5.13) - (5.15) to (5.12) and using the definition of $H_1(w)$ in (5.5) yields

$$
\frac{d}{dt} \int_\Omega (v - \chi_2^{|\gamma_1|})^2 + 2 \int_\Omega |\nabla v|^2 + \int_\Omega (w - \chi_2^{|\gamma_1|})^2 \leq \int_\Omega H_1(w)(w - \chi_2)^2
$$

(5.16)

for all $t > 0$, similarly, we have

$$
\frac{d}{dt} \int_\Omega (z - \chi_1^{|\gamma_1|})^2 + 2 \int_\Omega |\nabla z|^2 + \int_\Omega (w - \chi_2^{|\gamma_1|})^2 \leq \int_\Omega H_2(u)(u - \chi_1)^2
$$

(5.17)

for all $t > 0$, where we used the definition of $H_2(u)$ in (5.6). Thus, (5.4) is a result from combining (5.11), (5.16) and (5.17).
Lemma 5.5. Assume that (1.2) – (1.5) and Definition 5.3 hold, \( r_i > 0 \), if \( k_i < 2 \) \((i = 1, 2)\), then the global smooth solution of (1.1) satisfies

\[
\frac{d}{dt} F(t) + 2a_1 \int_\Omega |\nabla v|^2 + 2a_2 \int_\Omega |\nabla z|^2 + a_1 \int_\Omega (v - \chi_2^0)^2 + a_2 \int_\Omega (z - \chi_1^0)^2
\]
\[
+ r_1 \lambda_1^{3-2k_1} \int_\Omega \left( u^{k_1-1} - \chi_1^{k_1-1} \right)^2 + r_2 \lambda_2^{3-2k_2} \int_\Omega \left( u^{k_2-1} - \chi_2^{k_2-1} \right)^2
\]
\[
\leq \frac{C^2_{\alpha_1} \chi_1}{2C_{d_1}} \int_\Omega (u + 1)^{\alpha_1 + 2\beta_1 - 2} |\nabla v|^2 + \frac{C^2_{\alpha_2} \chi_2}{2C_{d_2}} \int_\Omega (w + 1)^{\alpha_2 + 2\beta_2 - 2} |\nabla z|^2
\]
\[
+ a_1 \int_\Omega \bar{H}_1(w) \left( u^{k_2-1} - \chi_2^{k_2-1} \right)^2 + a_2 \int_\Omega \bar{H}_2(u) \left( u^{k_1-1} - \chi_1^{k_1-1} \right)^2
\]
for all \( t > 0 \) with

\[
\bar{H}_1(w) := \begin{cases} 
(1 - \frac{1}{2^{\gamma_1 - 1}})^{2^{\gamma_1 - 2}} 4^{k_2-1-\gamma_1} \lambda_2^{2^{\gamma_1 - 2}k_2 - 2}, & \text{if } \gamma_1 < k_2 - 1, \\
\frac{\gamma_2}{(k_2-1)^2} \left( u^{k_2-1} + \chi_2^{k_2-1} \right)^{2^{\gamma_2 - 2}} & \text{if } \gamma_1 \geq k_2 - 1 
\end{cases}
\]

and

\[
\bar{H}_2(u) := \begin{cases} 
(1 - \frac{1}{2^{\gamma_2 - 1}})^{2^{\gamma_2 - 2}} 4^{k_1-1-\gamma_2} \lambda_1^{2^{\gamma_2 - 2}k_1 - 2}, & \text{if } \gamma_2 < k_1 - 1, \\
\frac{\gamma_2}{(k_1-1)^2} \left( u^{k_1-1} + \chi_1^{k_1-1} \right)^{2^{\gamma_2 - 2}} & \text{if } \gamma_2 \geq k_1 - 1
\end{cases}
\]

Proof: According to (5.7) and (5.8) yields

\[
\frac{d}{dt} \int_\Omega b_1(u) \leq \frac{C^2_{\alpha_1} \chi_1}{2C_{d_1}} \int_\Omega (u + 1)^{\alpha_1 + 2\beta_1 - 2} |\nabla v|^2 - \mu_1 \int_\Omega (u - \chi_1) \left( u^{k_1-1} - \frac{r_1}{\mu_1} \right)
\]

for all \( t > 0 \). According to the assumptions \( k_i < 2 \) \((i = 1, 2)\), we have

\[
(u - \chi_1) \left( u^{k_1-1} - \frac{r_1}{\mu_1} \right) = (u - \chi_1) \left( u^{k_1-1} - \chi_1^{k_1-1} \right) \geq \lambda_1^{2-k_1} \left( u^{k_1-1} - \chi_1^{k_1-1} \right)^2,
\]

thus we obtain

\[
-\mu_1 \int_\Omega (u - \chi_1) \left( u^{k_1-1} - \frac{r_1}{\mu_1} \right) \leq -\mu_1 \lambda_1^{2-k_1} \int_\Omega \left( u^{k_1-1} - \chi_1^{k_1-1} \right)^2
\]
\[
= -r_1 \lambda_1^{3-2k_1} \int_\Omega \left( u^{k_1-1} - \chi_1^{k_1-1} \right)^2.
\]

Therefore, combining (5.21) and (5.22), we derive

\[
\frac{d}{dt} \int_\Omega b_1(u) \leq \frac{C^2_{\alpha_1} \chi_1}{2C_{d_1}} \int_\Omega (u + 1)^{\alpha_1 + 2\beta_1 - 2} |\nabla v|^2 - r_1 \lambda_1^{3-2k_1} \int_\Omega \left( u^{k_1-1} - \chi_1^{k_1-1} \right)^2
\]

for all \( t > 0 \). The same treatment to \( b_2(w) \), and combining it with (5.23), we have

\[
\frac{d}{dt} \int_\Omega \left( b_1(u) + b_2(w) \right)
\]
\[
\leq \frac{C^2_{\alpha_1} \chi_1}{2C_{d_1}} \int_\Omega (u + 1)^{\alpha_1 + 2\beta_1 - 2} |\nabla v|^2 + \frac{C^2_{\alpha_2} \chi_2}{2C_{d_2}} \int_\Omega (w + 1)^{\alpha_2 + 2\beta_2 - 2} |\nabla z|^2
\]
\[
- r_1 \lambda_1^{3-2k_1} \int_\Omega \left( u^{k_1-1} - \chi_1^{k_1-1} \right)^2 - r_2 \lambda_2^{3-2k_2} \int_\Omega \left( u^{k_2-1} - \chi_2^{k_2-1} \right)^2.
\]
for all $t > 0$. From (5.12) yields that

$$\frac{d}{dt} \int_{\Omega} (v - \chi_2^{\gamma_1})^2 \leq -2 \int_{\Omega} \left| \nabla v \right|^2 - \int_{\Omega} (v - \chi_2^{\gamma_1})^2 + \int_{\Omega} (w^{\gamma_1} - \chi_2^{\gamma_1})^2$$

(5.25)

for all $t > 0$. For the last term of inequality (5.25), the treatment is similar to that of the previous Lemma 5.4, we can handle it under different conditions of $\gamma_i$ ($i = 1, 2$), as below.

In case $\gamma_1 \in (0, k_2 - 1)$, if $w(x, t) \leq \frac{\chi_2}{2}$ on $(x, t) \in \Omega \times (0, \infty)$, we infer that

$$|w^{\gamma_1} - \chi_2^{\gamma_1}| = \left| w^{k_2 - 1} - \chi_2^{k_2 - 1} \right|^\frac{\gamma_1}{k_2 - 1}$$

\leq \left| w^{k_2 - 1} - \chi_2^{k_2 - 1} \right|^\frac{\gamma_1}{k_2 - 1} \left| w^{k_2 - 1} - \chi_2^{k_2 - 1} \right|$$

(5.26)

if $w(x, t) > \frac{\chi_2}{2}$ on $(x, t) \in \Omega \times (0, \infty)$, we set $\tilde{h}_1(s) := s^\frac{\gamma_1}{k_2 - 1}$ on $s \in (\frac{\chi_2}{2}, \infty)$, according to the Mean value theorem again, there exists some $\theta_2 \in (0, 1)$ satisfies

$$|w^{\gamma_1} - \chi_2^{\gamma_1}| = \left| \tilde{h}_1 \left( w^{k_2 - 1} \right) - \tilde{h}_1 \left( \chi_2^{k_2 - 1} \right) \right|$$

\leq \frac{\gamma_1}{k_2 - 1} \left| w^{k_2 - 1} - \chi_2^{k_2 - 1} \right| \left| w^{k_2 - 1} - \chi_2^{k_2 - 1} \right|$$

(5.27)

where we used the fact that $\tilde{h}_1'(s) = \frac{\gamma_1}{k_2 - 1} s^\frac{\gamma_1 - 1}{k_2 - 1}$ is a monotone decreasing function on $(\frac{\chi_2}{2}, \infty)$, we also used $w^{k_2 - 1} - \theta_2 w^{k_2 - 1} + \theta_2 \chi_2^{k_2 - 1} > \chi_2^{k_2 - 1}$ if $w > \frac{\chi_2}{2}$.

In case $\gamma_1 \geq k_2 - 1$, $\tilde{h}_1'(s) = \frac{\gamma_1}{k_2 - 1} s^\frac{\gamma_1 - 1}{k_2 - 1}$ is a monotone increasing function, using the Mean value theorem again, there exists some $\theta_2 \in (0, 1)$ satisfies

$$|w^{\gamma_1} - \chi_2^{\gamma_1}| = \left| \tilde{h}_1 \left( w^{k_2 - 1} \right) - \tilde{h}_1 \left( \chi_2^{k_2 - 1} \right) \right|$$

\leq \frac{\gamma_1}{k_2 - 1} \left| w^{k_2 - 1} + \chi_2^{k_2 - 1} \right| \left| w^{k_2 - 1} - \chi_2^{k_2 - 1} \right|$$

(5.28)

Combining (5.26) – (5.28) to (5.25) and using the definition of $\tilde{H}_1(w)$ in (5.19) imply

$$\frac{d}{dt} \int_{\Omega} (v - \chi_2^{\gamma_1})^2 + 2 \int_{\Omega} \left| \nabla v \right|^2 + \int_{\Omega} (v - \chi_2^{\gamma_1})^2 \leq \int_{\Omega} \tilde{H}_1(w) \left( w^{k_2 - 1} - \chi_2^{k_2 - 1} \right)^2$$

(5.29)

for all $t > 0$. Similarly, we have

$$\frac{d}{dt} \int_{\Omega} (z - \chi_1^{\gamma_2})^2 + 2 \int_{\Omega} \left| \nabla z \right|^2 + \int_{\Omega} (z - \chi_1^{\gamma_2})^2 \leq \int_{\Omega} \tilde{H}_2(u) \left( u^{k_1 - 1} - \chi_1^{k_1 - 1} \right)^2$$

(5.30)
for all \( t > 0 \), where we used the definition of \( \tilde{H}_2(u) \) in (5.20). Thus, (5.18) is a result from combining (5.24), (5.29) and (5.30).

**Proof of Theorem 1.3.** Let \( \gamma_i \geq \frac{1}{2} \) \( (i = 1, 2) \), according to the definitions of \( \chi_i := \left( \frac{r_i}{\mu_i} \right)^{\beta_i} \) \( (i = 1, 2) \), since \( u, w \) are global bounded solutions of (1.1), then there exist some \( C_u := \left( \|u\|_{L^\infty} + 1 \right)^{\alpha_1+2\beta_1-2} > 0 \) and \( C_w := \left( \|w\|_{L^\infty} + 1 \right)^{\alpha_2+2\beta_2-2} > 0 \) such that

\[
\frac{C_u^2 \chi_1}{2C_{d_1}} \int_\Omega (u + 1)^{\alpha_1+2\beta_1-2} |\nabla u|^2 + \frac{C_w^2 \chi_2}{2C_{d_2}} \int_\Omega (w + 1)^{\alpha_2+2\beta_2-2} |\nabla z|^2 \\
\leq \frac{C_u^2 C_w}{2C_{d_1}} \cdot \left( \frac{r_1}{\mu_1} \right)^{\beta_1} \int_\Omega |\nabla u|^2 + \frac{C_u^2 C_w}{2C_{d_2}} \cdot \left( \frac{r_2}{\mu_2} \right)^{\beta_2} \int_\Omega |\nabla z|^2, \tag{5.31}
\]

next, we obtain the results of asymptotic stability to global solution by dividing the values of \( k_i \) and \( \gamma_i \) \( (i = 1, 2) \) for eight cases.

**Case 1.** \( k_i \geq 2, \frac{1}{2} \leq \gamma_i < 1 \) \( (i = 1, 2) \). In view of (5.5) and (5.6) and the definitions of \( \chi_i := \left( \frac{\mu_i}{\mu} \right)^{\beta_i} \) \( (i = 1, 2) \), we derive

\[
a_1 \int_\Omega H_1(w)(w - \chi_2)^2 + a_2 \int_\Omega H_2(u)(u - \chi_1)^2 \\
= a_1 4^{1-\gamma_1} \left( \frac{r_2}{\mu_2} \right)^{\beta_2-2} \int_\Omega (w - \chi_2)^2 + a_2 4^{1-\gamma_2} \left( \frac{r_1}{\mu_1} \right)^{\beta_1-2} \int_\Omega (u - \chi_1)^2, \tag{5.32}
\]

for all \( t > 0, a_i \) \( (i = 1, 2) \) are defined in (5.3). In conjunction with (5.4), (5.31) and (5.32) which implies

\[
\frac{d}{dt} F(t) + a_1 \int_\Omega (v - \chi_2)^2 + a_2 \int_\Omega (z - \chi_1)^2 \\
\leq \left( \frac{C_u^2 C_w}{2C_{d_1}} \cdot \left( \frac{r_1}{\mu_1} \right)^{\beta_1} - 2a_1 \right) \int_\Omega |\nabla u|^2 + \left( \frac{C_u^2 C_w}{2C_{d_2}} \cdot \left( \frac{r_2}{\mu_2} \right)^{\beta_2} - 2a_2 \right) \int_\Omega |\nabla z|^2 \\
+ \left( a_1 4^{1-\gamma_1} \left( \frac{r_2}{\mu_2} \right)^{\beta_2-2} - r_2 \left( \frac{r_2}{\mu_2} \right)^{\beta_2-1} \right) \int_\Omega (w - \chi_2)^2 \\
+ \left( a_2 4^{1-\gamma_2} \left( \frac{r_1}{\mu_1} \right)^{\beta_1-2} - r_1 \left( \frac{r_1}{\mu_1} \right)^{\beta_1-1} \right) \int_\Omega (u - \chi_1)^2, \tag{5.33}
\]

for all \( t > 0 \), where \( \frac{C_u^2 C_w}{2C_{d_1}} \cdot \left( \frac{r_1}{\mu_1} \right)^{\beta_1} - 2a_1 \leq 0 \) and \( \frac{C_u^2 C_w}{2C_{d_2}} \cdot \left( \frac{r_2}{\mu_2} \right)^{\beta_2} - 2a_2 \leq 0 \) because of the definitions in (5.3). Thus, we only need to handle the last two terms of inequality (5.33), we infer

\[
\frac{a_1 4^{1-\gamma_1} \left( \frac{r_2}{\mu_2} \right)^{\beta_2-2}}{r_2 \left( \frac{r_2}{\mu_2} \right)^{\beta_2-1}} = \frac{C_u^2 C_w}{4C_{d_1}} \cdot \left( \frac{r_1}{\mu_1} \right)^{\beta_1} \cdot 4^{1-\gamma_1} \left( \frac{r_2}{\mu_2} \right)^{\beta_2-1} \to 0 \text{ as } \mu_i \to \infty \tag{5.34}
\]
because of the condition $\gamma_1 \geq \frac{1}{2}$, thus, there exists $\mu_i > \mu_0 > 0$ ($i = 1, 2$) such that

$$r_2 \left( \frac{r_2}{\mu_2} \right)^{-\frac{1}{\gamma_2 - 1}} > a_1 4^{1-\gamma_1} \left( \frac{r_2}{\mu_2} \right)^{\frac{2\gamma_1 - 2}{\gamma_2 - 1}} + 1. \quad (5.35)$$

Similarly, we have

$$r_1 \left( \frac{r_1}{\mu_1} \right)^{-\frac{1}{\gamma_1 - 1}} > a_2 4^{1-\gamma_2} \left( \frac{r_1}{\mu_1} \right)^{\frac{2\gamma_2 - 2}{\gamma_1 - 1}} + 1 \quad (5.36)$$

with $\mu_i > \mu_0$ ($i = 1, 2$) and $\gamma_2 \geq \frac{1}{2}$. Therefore, combining (5.35) and (5.36) to (5.33) implies that

$$\frac{d}{dt} F(t) + \int_{\Omega} (u - \chi_1)^2 + \int_{\Omega} (w - \chi_2)^2 + a_1 \int_{\Omega} (v - \chi_1^2)^2 + a_2 \int_{\Omega} (z - \chi_1^2)^2 \leq 0 \quad (5.37)$$

with $\mu_i > \mu_0$ and $\gamma_i \geq \frac{1}{2}$ ($i = 1, 2$), where $a_i$ are defined in (5.3) and $\chi_i := \left( \frac{r_i}{\mu_i} \right)^{\frac{1}{\gamma_i - 1}}$ ($i = 1, 2$). According to Lemma 5.2, we get

$$\|u(\cdot, t) - \chi_1\|_{L^2(\Omega)} \to 0 \quad \text{and} \quad \|w(\cdot, t) - \chi_2\|_{L^2(\Omega)} \to 0 \quad \text{as} \ t \to \infty. \quad (5.38)$$

In order to turn this into a respective convergence statement with respect to the norm in $L^\infty(\Omega)$, we invoke the Gagliardo-Nirenberg inequality to find $c_1 > 0$ fulfilling

$$\|\varphi\|_{L^\infty(\Omega)} \leq c_1 \|\varphi\|_{W^{1,\infty}(\Omega)}^{n/(n+2)} \|\varphi\|_{L^2(\Omega)}^{2/(n+2)} \quad \text{for all} \ \varphi \in W^{1,\infty}(\Omega),$$

thus, in view of Lemma 5.1, we obtain

$$u(\cdot, t) \to \chi_1 \quad \text{and} \quad w(\cdot, t) \to \chi_2 \quad \text{in} \ L^\infty(\Omega) \quad \text{as} \ t \to \infty.$$

For the convergence rate of global smooth solution to system (1.1), in view of the Taylor expansion, there exists $\theta(s) \in (0, 1)$ such that

$$b_i(s) := b_i(x) + b'_i(x)(s - x) + \frac{b''_i(\theta s + (1 - \theta) \chi_i)}{2}(s - x)^2$$

$$= \frac{\chi_i}{2(\theta s + (1 - \theta) \chi_i)^2} (s - \chi_i)^2$$

for all $s > 0$, which implies

$$\lim_{s \to \chi_i} \frac{b_i(s)}{(s - \chi_i)^2} = \lim_{s \to \chi_i} \frac{\chi_i}{2(\theta s + (1 - \theta) \chi_i)^2} = \frac{1}{2 \chi_i}. \quad (5.39)$$

Therefore, thanks to the properties in (5.2) and (5.39), we have

$$\frac{1}{4 \chi_1} (u(\cdot, t) - \chi_1)^2 \leq b_1(u(\cdot, t)) \leq \frac{1}{\chi_1} (u(\cdot, t) - \chi_1)^2 \quad (5.40)$$

and

$$\frac{1}{4 \chi_2} (w(\cdot, t) - \chi_2)^2 \leq b_2(w(\cdot, t)) \leq \frac{1}{\chi_2} (w(\cdot, t) - \chi_2)^2 \quad (5.41)$$

for all $t > t_0 > 0$, combining (5.40), (5.41) and (5.37), we derive $\frac{d}{dt} F(t) + c_2 F(t) \leq 0$ for all $t > t_0 > 0$ with $c_2 > 0$. According to the Gronwall inequality implies that
for all $t > t_0 > 0$ with $c_4, c_5 > 0$. Using the Gagliardo-Nirenberg inequality and Lemma 5.1 with (5.42) which implies

$$\|u(\cdot,t) - \left(\frac{r_1}{\mu_1}\right)^{\frac{1}{1-\sigma_1}}\|_{L^\infty(\Omega)} + \|w(\cdot,t) - \left(\frac{r_2}{\mu_2}\right)^{\frac{1}{1-\sigma_2}}\|_{L^\infty(\Omega)} \leq c_6 \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)}^{\frac{1}{1-\sigma_1}} \|u(\cdot,t) - \left(\frac{r_1}{\mu_1}\right)^{\frac{1}{1-\sigma_1}}\|_{L^2(\Omega)} + c_6 \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}^{\frac{1}{1-\sigma_2}} \|w(\cdot,t) - \left(\frac{r_2}{\mu_2}\right)^{\frac{1}{1-\sigma_2}}\|_{L^2(\Omega)} \leq c_7 e^{-\frac{\kappa}{2} t}$$

for all $t > t_0 > 0$ with $c_6, c_7 > 0$. In the same way, applying the Gagliardo-Nirenberg inequality we obtain

$$\|v(\cdot,t) - \left(\frac{r_2}{\mu_2}\right)^{\frac{1}{2-\sigma_2}}\|_{L^\infty(\Omega)} + \|z(\cdot,t) - \left(\frac{r_1}{\mu_1}\right)^{\frac{1}{2-\sigma_1}}\|_{L^\infty(\Omega)} \leq c_8 e^{-\frac{\kappa}{2} t}$$

(5.44)

for all $t > t_0 > 0$ with $c_8 > 0$.

**Case 2.** $k_i \geq 2$, $\frac{1}{2} \leq \gamma_i < 1$, $\gamma_2 \geq 1$ ($i = 1, 2$). In view of (5.5) and (5.6) and the definitions of $\chi_i := \left(\frac{r_i}{\mu_i}\right)^{\frac{1}{1-\gamma_i}}$ ($i = 1, 2$), we derive

$$a_1 \int_\Omega H_1(w)(w - \chi_2)^2 + a_2 \int_\Omega H_2(u)(u - \chi_1)^2 = a_1 4^{1-\gamma_1} \left(\frac{r_2}{\mu_2}\right)^{\frac{2(2-\gamma_2)}{2-\gamma_1}} \int_\Omega (w - \chi_2)^2 + a_2 \gamma_2 \int_\Omega (u + \chi_1)^{2\gamma_2-2}(u - \chi_1)^2 \leq a_1 4^{1-\gamma_1} \left(\frac{r_2}{\mu_2}\right)^{\frac{2(2-\gamma_2)}{2-\gamma_1}} \int_\Omega (w - \chi_2)^2 + a_2 \gamma_2 \left(C_{u1} + \left(\frac{r_1}{\mu_1}\right)^{\frac{1}{1-\gamma_1}}\right)^{2\gamma_2-2} \int_\Omega (u - \chi_1)^2$$

(5.45)

for all $t > 0$ with some $C_{u1} = C_{u1}(\|u\|_{L^\infty,\Omega}) > 0$, $a_i$ ($i = 1, 2$) are defined in (5.3). In conjunction with (5.3), (5.4), (5.31) and (5.45) which implies

$$\frac{d}{dt} F(t) + a_1 \int_\Omega (v - \gamma_1)^2 + a_2 \int_\Omega (z - \chi_2^2)^2 \leq \left(4^{1-\gamma_1} \left(\frac{r_2}{\mu_2}\right)^{\frac{2(2-\gamma_2)}{2-\gamma_1}} - r_2 \left(\frac{r_2}{\mu_2}\right)^{\frac{1}{2-\gamma_2}}\right) \int_\Omega (w - \chi_2)^2$$
with \( \text{Case 3.} \) \( k_i > 0 \) \( \mu_0 \) and \( \gamma_1 \geq \frac{1}{2} \). Similarly, we have

\[
r_1 \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{\gamma_1^{-1}}} > a_2 \gamma_2^2 \left( C_{u_1} + \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{\gamma_1^{-1}}} \right) \geq 2 \gamma_2^{-2} + 1 \tag{5.48}
\]

with \( \mu_i > \mu_0 \) \( (i = 1, 2) \) and \( \gamma_2 \geq 1 \). Therefore, inserting (5.47) and (5.48) into (5.46) implies that

\[
\frac{d}{dt} F(t) + \int_{\Omega} (u - \chi_1)^2 + \int_{\Omega} (w - \chi_2)^2 + a_3 \int_{\Omega} (v - \chi_2^\gamma_2)^2 + a_2 \int_{\Omega} (z - \chi_1^\gamma_1)^2 \leq 0
\]

with \( \mu_i > \mu_0 \) \( (i = 1, 2) \) and \( \gamma_1 \geq \frac{1}{2} \) and \( \gamma_2 \geq 1 \), where \( a_i \) are defined in (5.3) and \( \chi_i := \left( \frac{r_i}{\mu_i} \right)^{\frac{1}{\gamma_i^{-1}}} \) \( (i = 1, 2) \). The rest of part is handled similarly to Case 1, so we omit it here.

**Case 3.** \( k_i \geq 2, \gamma_1 \geq 1, \frac{1}{2} \leq \gamma_2 < 1 \) \( (i = 1, 2) \). Case 3 is exactly the same as Case 2, so we omit it here.

**Case 4.** \( k_i \geq 2, \gamma_i \geq 1 \) \( (i = 1, 2) \). In view of (5.5) and (5.6) and the definitions of \( \chi_i := \left( \frac{r_i}{\mu_i} \right)^{\frac{1}{\gamma_i^{-1}}} \) \( (i = 1, 2) \), we derive

\[
a_1 \int_{\Omega} H_1(w)(w - \chi_2)^2 + a_2 \int_{\Omega} H_2(u)(u - \chi_1)^2
\]

\[
= a_1 \gamma_1^2 \int_{\Omega} (w - \chi_2)^2 + a_2 \gamma_2^2 \int_{\Omega} (u + \chi_1)^2 - (u - \chi_1)^2
\]

\[
\leq a_1 \gamma_1^2 \left( C_{w_1} + \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{\gamma_2^{-1}}} \right) 2 \gamma_2^{-2} \int_{\Omega} (w - \chi_2)^2
\]

\[
+ a_2 \gamma_2^2 \left( C_{u_1} + \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{\gamma_1^{-1}}} \right) 2 \gamma_1^{-2} \int_{\Omega} (u - \chi_1)^2 \tag{5.49}
\]

for all \( t > 0 \) with some \( C_{u_1} = C_{u_1}(\|u\|_{L^\infty}, \Omega) > 0 \) and \( C_{w_1} = C_{w_1}(\|w\|_{L^\infty}, \Omega) > 0 \), \( a_i \) \( (i = 1, 2) \) are defined in (5.3). In conjunction with (5.3), (5.4), (5.31) and (5.49) which implies

\[
\frac{d}{dt} F(t) + a_1 \int_{\Omega} (v - \chi_2^\gamma_2)^2 + a_2 \int_{\Omega} (z - \chi_1^\gamma_1)^2
\]

\[
\leq \left( a_1 \gamma_1^2 \left( C_{w_1} + \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{\gamma_2^{-1}}} \right) - r_2 \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{\gamma_2^{-1}}} \right) \int_{\Omega} (w - \chi_2)^2
\]

\[
+ \left( a_2 \gamma_2^2 \left( C_{u_1} + \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{\gamma_1^{-1}}} \right) - r_1 \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{\gamma_1^{-1}}} \right) \int_{\Omega} (u - \chi_1)^2 \tag{5.50}
\]
for all \( t > 0 \). Thus, we only need to handle the two terms on the right-hand side of inequality (5.50), similarly to (5.34) we have
\[
\begin{align*}
r_1 \left( \frac{r_1}{\mu_1} \right)^{-\frac{1}{\gamma_1 - 1}} &> 2a\gamma_2^2 \left( C_{u1} + \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{\gamma_1 - 1}} \right)^{2}\gamma_2^{-2} + 1 \quad (5.51) \\
r_2 \left( \frac{r_2}{\mu_2} \right)^{-\frac{1}{\gamma_2 - 1}} &> a_1\gamma_1^2 \left( C_{u1} + \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{\gamma_2 - 1}} \right)^{2}\gamma_1^{-2} + 1 \quad (5.52)
\end{align*}
\]
with \( \mu_i > \mu_0 \) and \( \gamma_i \geq 1 \) \((i = 1, 2)\). Therefore, combining (5.50) – (5.52) yields
\[
\frac{d}{dt} F(t) + 1 \int_\Omega (w - \chi_i t)^2 + \int_\Omega (w - \chi_i t)^2 + a_1 \int_\Omega (u - \chi_2) t)^2 + a_1 \int_\Omega (u - \chi_2) t)^2 + a_2 \int_\Omega (z - \chi_2) t)^2 \leq 0
\]
with \( \mu_i > \mu_0 \) and \( \gamma_i \geq 1 \) \((i = 1, 2)\), where \( a_i \) are defined in (5.3) and \( \chi_i \) := \( \left( \frac{r_i}{\mu_i} \right)^{\frac{1}{\gamma_i - 1}} \) \((i = 1, 2)\). The rest of part is handled similarly to Case 1, so we omit it here.

**Case 5.** \( k_1 < 2, \frac{1}{2} \leq \gamma_1 < k_2 - 1, \frac{1}{2} \leq \gamma_2 < k_1 - 1 \) \((i = 1, 2)\). In view of (5.19) and (5.20) and the definitions of \( \chi_i := \left( \frac{r_i}{\mu_i} \right)^{\frac{1}{\gamma_i - 1}} \) \((i = 1, 2)\), we derive
\[
\begin{align*}
a_1 \int_\Omega \tilde{H}_1(u) \left( u^{k_2-1} - \chi_2^{k_2-1} \right)^2 + a_2 \int_\Omega \tilde{H}_2(u) \left( u^{k_1-1} - \chi_1^{k_1-1} \right)^2 \\
= a_1 \left( 1 - \frac{1}{2k_2-1} \right)^{\frac{2\gamma_1}{\gamma_1 - 1} - 2} \int_\Omega \left( u^{k_2-1} - \chi_2^{k_2-1} \right)^2 + a_2 \left( 1 - \frac{1}{2k_1-1} \right)^{\frac{2\gamma_1}{\gamma_1 - 1} - 2} \int_\Omega \left( u^{k_1-1} - \chi_1^{k_1-1} \right)^2 \quad (5.53)
+ a_2 \left( 1 - \frac{1}{2k_1-1} \right)^{\frac{2\gamma_1}{\gamma_1 - 1} - 2} \int_\Omega \left( u^{k_1-1} - \chi_1^{k_1-1} \right)^2
\end{align*}
\]
for all \( t > 0 \), \( a_i \) \((i = 1, 2)\) are defined in (5.3). In conjunction with (5.3), (5.18), (5.31) and (5.53) which implies
\[
\frac{d}{dt} F(t) + a_1 \int_\Omega (w - \chi_2 t)^2 + a_2 \int_\Omega (z - \chi_1 t)^2 \leq \left( a_1 \left( 1 - \frac{1}{2k_2-1} \right)^{\frac{2\gamma_1}{\gamma_1 - 1} - 2} \int_\Omega \left( u^{k_2-1} - \chi_2^{k_2-1} \right)^2 \right) \\
+ \left( a_2 \left( 1 - \frac{1}{2k_1-1} \right)^{\frac{2\gamma_1}{\gamma_1 - 1} - 2} \int_\Omega \left( u^{k_1-1} - \chi_1^{k_1-1} \right)^2 \right) \quad (5.54)
\]
for all \( t > 0 \). Thus, we only need to handle the two terms on the right-hand side of inequality (5.54), we infer
\[
\begin{align*}
a_1 \left( 1 - \frac{1}{2k_2-1} \right)^{\frac{2\gamma_1}{\gamma_1 - 1} - 2} \int_\Omega \left( u^{k_2-1} - \chi_2^{k_2-1} \right)^2 + \frac{r_2 \chi_2^{3-2k_2}}{4C_{u2}^2} \left( 1 - \frac{1}{2k_2-1} \right)^{\frac{2\gamma_1}{\gamma_1 - 1} - 2} \int_\Omega \left( u^{k_1-1} - \chi_1^{k_1-1} \right)^2 \rightarrow 0 \text{ as } \mu_i \rightarrow \infty
\end{align*}
\]
because of the condition \( \gamma_i \geq \frac{1}{2} \), thus, there exists \( \mu_i > \mu_0 > 0 \) \((i = 1, 2)\) such that
\[
r_2 \chi_2^{3-2k_2} > a_1 \left( 1 - \frac{1}{2k_2-1} \right)^{\frac{2\gamma_1}{\gamma_1 - 1} - 2} \int_\Omega \left( u^{k_2-1} - \chi_2^{k_2-1} \right)^2 + 1. \quad (5.55)
\]
Similarly, we have

$$r_1 \chi_1^{3-2k_1} > a_2 \left(1 - \frac{1}{2k_1-1}\right) \frac{\gamma_2 \gamma_1}{\gamma_2 r_1} 4^{k_1-1-\gamma_2} \chi_1^{2\gamma_2-2k_1+1} + 1$$

(5.56)

with $\mu_i > \mu_0$ ($i = 1, 2$) and $\gamma_2 \geq \frac{1}{2}$. In order to obtain the desire result, we have

$$(k_1 - 1)^2 (2 \chi_1)^{2k_1-4} \int_{\Omega} (u - \chi_1)^2 \leq \int_{\Omega} \left(u^{k_1-1} - \chi_1^{k_1-1}\right)^2$$

(5.57)

for all $u \leq 2\chi_1$ and

$$(k_2 - 1)^2 (2 \chi_2)^{2k_2-4} \int_{\Omega} (w - \chi_2)^2 \leq \int_{\Omega} \left(w^{k_2-1} - \chi_2^{k_2-1}\right)^2$$

(5.58)

for all $w \leq 2\chi_2$. Therefore, combining (5.55) - (5.58) to (5.54) implies that

$$\frac{d}{dt} F(t) + (k_1 - 1)^2 (2 \chi_1)^{2k_1-4} \int_{\Omega} (u - \chi_1)^2 + (k_2 - 1)^2 (2 \chi_2)^{2k_2-4} \int_{\Omega} (w - \chi_2)^2$$

$$+ a_1 \int_{\Omega} (v - \chi_1^2)^2 + a_2 \int_{\Omega} (z - \chi_2^2)^2 \leq 0$$

with $\mu_i > \mu_0$ and $\gamma_i \geq \frac{1}{2}$ ($i = 1, 2$), where $a_i$ are defined in (5.3) and $\chi_i := \left(\frac{r_i}{\mu_i}\right)^{\frac{1}{\gamma_i}}$ ($i = 1, 2$). The rest of part is handled similarly to Case 1, so we omit it here.

**Case 6.** $k_i < 2$, $\frac{1}{2} \leq \gamma_i < k_i - 1$, $\gamma_2 \geq \max\{k_1 - 1, \frac{1}{2}\}$ ($i = 1, 2$). In view of (5.19) and (5.20) and the definitions of $\chi_i := \left(\frac{r_i}{\mu_i}\right)^{\frac{1}{\gamma_i}}$ ($i = 1, 2$), we derive

$$a_1 \int_{\Omega} \tilde{H}_1(w) \left(u^{k_2-1} - \chi_2^{k_2-1}\right)^2 + a_2 \int_{\Omega} \tilde{H}_2(u) \left(u^{k_1-1} - \chi_1^{k_1-1}\right)^2$$

$$= a_1 \left(1 - \frac{1}{2k_1-1}\right) \frac{\gamma_2 \gamma_1}{\gamma_2 r_1} 4^{k_1-1-\gamma_1} \chi_2^{2\gamma_1-2k_1+2} \int_{\Omega} \left(u^{k_2-1} - \chi_2^{k_2-1}\right)^2$$

$$+ a_2 \frac{\gamma_2}{(k_1 - 1)^2} \left(k_2 - 1\right)^2 \left(k_1 - 1\right)^2 \int_{\Omega} \left(u^{k_1-1} + \chi_1^{k_1-1}\right)^{\frac{2\gamma_2^{k_2-1}}{r_2}} \left(u^{k_2-1} - \chi_2^{k_2-1}\right)^2$$

(5.59)

$$\leq a_1 \left(1 - \frac{1}{2k_2-1}\right) \frac{\gamma_2 \gamma_1}{\gamma_2 r_1} 4^{k_1-1-\gamma_1} \chi_2^{2\gamma_1-2k_1+2} \int_{\Omega} \left(u^{k_2-1} - \chi_2^{k_2-1}\right)^2$$

$$+ a_2 \frac{\gamma_2^2 C_{u_2}}{(k_1 - 1)^2} \left(k_1 - 1\right)^2 \left(k_1 - 1\right)^2 \int_{\Omega} \left(u^{k_1-1} - \chi_1^{k_1-1}\right)^2$$

for all $t > 0$ with some $C_{u_2} = C_{u_2}(k_1, \gamma_2, \mu_1, r_1, \|u\|_{L^\infty(\Omega)}) > 0$, where we used the fact $(u^{k_1-1} + \chi_1^{k_1-1})^{\frac{2\gamma_2^{k_2-1}}{r_2}} \leq 4^{k_1-1} (u^{2\gamma_2^{k_2-1}} + (r_1 \chi_1)^{2\gamma_2^{k_2-1}})$, $a_i$ ($i = 1, 2$) are defined in (5.3). In conjunction with (5.3), (5.18), (5.31) and (5.59) which implies

$$\frac{d}{dt} F(t) + a_1 \int_{\Omega} (v - \chi_1^2)^2 + a_2 \int_{\Omega} (z - \chi_2^2)^2$$

$$\leq \left( a_1 \left(1 - \frac{1}{2k_2-1}\right) \frac{\gamma_2 \gamma_1}{\gamma_2 r_1} 4^{k_1-1-\gamma_1} \chi_2^{2\gamma_1-2k_1+2} - r_2 \chi_2^{3-2k_2}\right) \int_{\Omega} \left(u^{k_2-1} - \chi_2^{k_2-1}\right)^2$$

$$+ \left(a_2 \frac{\gamma_2^2 C_{u_2}}{(k_1 - 1)^2} \left(k_1 - 1\right)^2 \left(k_1 - 1\right)^2 \int_{\Omega} \left(u^{k_1-1} - \chi_1^{k_1-1}\right)^2 \right)$$

(5.60)
for all $t > 0$. Thus, we only need to handle the two terms on the right-hand side of inequality (5.60), similarly to (5.55) we have

$$r_2 \lambda_2^{3-2k_2} > a_1 \left(1 - \frac{1}{2e_2-1}\right)^{\frac{2\gamma_1}{r_2-1}-2} 4^{k_2-1-\gamma_1} \lambda_2^{2\gamma_1-2k_2+2} + 1 \tag{5.61}$$

with $\mu_i > \mu_0$ ($i = 1, 2$) and $\gamma_1 \geq \frac{1}{2}$. Similarly, we have

$$r_1 \lambda_1^{3-2k_1} > a_2 \gamma_2^2 C_{u2} \left(\frac{r_1}{\mu_1}\right)^{\frac{2\gamma_2-\gamma_1-(k_2-1)}{k_1-1}} + 1 \tag{5.62}$$

with $\mu_i > \mu_0$ ($i = 1, 2$) and $\gamma_2 \geq \frac{1}{2}$. Therefore, combining (5.61), (5.62), (5.57) and (5.58) with (5.60) implies that

$$\frac{d}{dt} F(t) + (k_1 - 1)^2(2\gamma_1)^{2k_1-4} \int_{\Omega} (u - \chi_1)^2 + (k_2 - 1)^2(2\gamma_2)^{2k_2-4} \int_{\Omega} (w - \chi_2)^2$$

$$+ a_1 \int_{\Omega} (v - \chi_1^2)^2 + a_2 \int_{\Omega} (z - \chi_2^2)^2 \leq 0$$

with $\mu_i > \mu_0$ and $\gamma_i \geq \frac{1}{2}$ ($i = 1, 2$), where $a_i$ are defined in (5.3) and $\chi_i := \left(\frac{r_i}{\mu_i}\right)^{\frac{1}{\gamma_i-1}}$ ($i = 1, 2$). The rest of part is handled similarly to Case 1, so we omit it here.

**Case 7.** $k_i < 2$, $\gamma_i \geq \max\{k_i - 1, \frac{1}{2}\}$, $\frac{1}{2} \leq \gamma_2 < k_1 - 1$ ($i = 1, 2$). Case 7 is exactly the same as Case 6, so we omit it here.

**Case 8.** $k_i < 2$, $\gamma_i \geq \max\{k_i - 1, \frac{1}{2}\}$, $\gamma_2 \geq \max\{k_1 - 1, \frac{1}{2}\}$ ($i = 1, 2$). In view of (5.19) and (5.20) and the definitions of $\chi_i := \left(\frac{r_i}{\mu_i}\right)^{\frac{1}{\gamma_i-1}}$ ($i = 1, 2$), similarly to (5.59), we derive

$$a_1 \int_{\Omega} \tilde{H}_1(w) \left(u^{k_2-1} - \chi_2^{k_2-1}\right)^2 + a_2 \int_{\Omega} \tilde{H}_2(u) \left(u^{k_1-1} - \chi_1^{k_1-1}\right)^2$$

$$= a_1 \frac{\gamma_2^2}{(k_2 - 1)^2} \int_{\Omega} \left(u^{k_2-1} + \chi_2^{k_2-1}\right)^{\frac{2\gamma_1}{r_2-1}-2} \left(u^{k_2-1} - \chi_2^{k_2-1}\right)^2$$

$$+ a_2 \frac{\gamma_2^2}{(k_1 - 1)^2} \int_{\Omega} \left(u^{k_1-1} + \chi_1^{k_1-1}\right)^{\frac{2\gamma_2}{r_1-1}-2} \left(u^{k_1-1} - \chi_1^{k_1-1}\right)^2 \tag{5.63}$$

for all $t > 0$ with some $C_{u2} = C_{u2}(k_1, \gamma_2, \mu_1, r_1, \|u\|_{L^{\infty}, \Omega}) > 0$ and $C_{u2} = C_{u2}(k_2, \gamma_1, \mu_2, r_2, \|w\|_{L^{\infty}, \Omega}) > 0$, $a_i$ ($i = 1, 2$) are defined in (5.3). In conjunction with (5.3), (5.18), (5.31) and (5.63) which implies

$$\frac{d}{dt} F(t) + a_1 \int_{\Omega} (v - \chi_1^{2\gamma_1})^2 + a_2 \int_{\Omega} (z - \chi_2^{\gamma_2})^2$$

$$\leq + \left(\frac{a_1 \gamma_1^2 C_{u2}}{(k_2 - 1)^2} \frac{r_2}{\mu_2} \right)^{\frac{2\gamma_1-\gamma_2-(k_2-1)}{r_2-1}} - r_2 \lambda_2^{3-2k_2} \int_{\Omega} \left(u^{k_2-1} - \chi_2^{k_2-1}\right)^2.$$
\[
+ \left( \frac{a_2 \gamma_2^2 C_{u_2}}{(k_1 - 1)^2} \cdot \left( \frac{r_1}{\mu_1} \right)^{\frac{2\gamma_2 - (k_1 - 1)}{k_1 - 1}} - r_1 \chi_1^{3 - 2k_1} \right) \int_{\Omega} \left( u^{k_1 - 1} - \chi_1^{k_1 - 1} \right)^2 \n (5.64)
\]

for all \( t > 0 \). Thus, we only need to handle the two terms on the right-hand side of inequality (5.64), similarly to (5.62) we have

\[
r_2 \chi_2^{3 - 2k_2} > a_1 \gamma_1^2 C_{u_2} \left( \frac{r_2}{\mu_2} \right)^{\frac{2\gamma_2 - (k_2 - 1)}{k_2 - 1}} + 1 \quad (5.65)
\]

and

\[
r_1 \chi_1^{3 - 2k_1} > a_2 \gamma_2^2 C_{u_2} \left( \frac{r_1}{\mu_1} \right)^{\frac{2\gamma_2 - (k_1 - 1)}{k_1 - 1}} + 1 \quad (5.66)
\]

with \( \mu_i > \mu_0 \) and \( \gamma_i \geq \frac{1}{2} \) (\( i = 1, 2 \)). Therefore, combining (5.65), (5.66), (5.57) and (5.58) with (5.64) implies that

\[
\frac{d}{dt} F(t) + (k_1 - 1)^2 (2 \chi_1)^{2k_2 - 4} \int_{\Omega} (u - \chi_1)^2 + (k_2 - 1)^2 (2 \chi_2)^{2k_2 - 4} \int_{\Omega} (w - \chi_2)^2
\]

\[
+ a_1 \int_{\Omega} (v - \chi_2^{\gamma_2})^2 + a_2 \int_{\Omega} (z - \chi_1^{\gamma_1})^2 \leq 0
\]

with \( \mu_i > \mu_0 \) and \( \gamma_i \geq \frac{1}{2} \) (\( i = 1, 2 \)), where \( a_i \) are defined in (5.3) and \( \chi_i := \left( \frac{r_i}{\mu_i} \right)^{\gamma_i/2} \) (\( i = 1, 2 \)). The rest of part is handled similarly to Case 1, so we omit it here.

Thus, the proof of Theorem 1.3 is finished. \( \square \)

Acknowledgments. The authors are very grateful to the anonymous reviewers for their carefully reading and valuable suggestions which greatly improved this work. This work is supported by the Chongqing Research and Innovation Project of Graduate Students (No. CYS20271) and Chongqing Basic Science and Advanced Technology Research Program (No. cstc2017jcyjXB0037).

REFERENCES

[1] X. Bai and M. Winkler, Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics, Indiana Univ. Math. J., 65 (2016), 553–583.

[2] N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci., 25 (2015), 1663–1763.

[3] T. Black, Global existence and asymptotic stability in a competitive two-species chemotaxis system with two signals, Discrete Contin. Dyn. Syst. Ser. B., 22 (2017), 1253–1272.

[4] X. Cao, Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces, Discrete Contin. Dyn. Syst., 35 (2015), 1891–1904.

[5] T. Cieślak and M. Winkler, Global bounded solutions in a two-dimensional quasilinear Keller-Segel system with exponentially decaying diffusivity and subcritical sensitivity, Nonlinear Anal. Real World Appl., 35 (2017), 1–19.

[6] T. Cieślak and M. Winkler, Stabilization in a higher-dimensional quasilinear Keller-Segel system with exponentially decaying diffusivity and subcritical sensitivity, Nonlinear Anal., 159 (2017), 129–144.

[7] M. Ding, W. Wang, S. Zhou and S. Zheng, Asymptotic stability in a fully parabolic quasilinear chemotaxis model with general logistic source and signal production, J. Differential Equations, 268 (2020), 6729–6777.

[8] A. Friedman, Partial Differential Equations, Holt, Rinehart and Winston, New York, 1969.

[9] D. D. Haroske and H. Triebel, Distributions, Sobolev Spaces, Elliptic Equations, European Mathematical Society, Zürich., 2008.

[10] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations, 215 (2005), 52–107.
[11] S. Ishida, K. Seki and T. Yokota, Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains, *J. Differential Equations*, 256 (2014), 2995–3010.

[12] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, 26 (1970), 399–415.

[13] R. Kowalczyk and Z. Szymańska, On the global existence of solutions to an aggregation model, *J. Math. Anal. Appl.*, 343 (2008), 379–398.

[14] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, 1996.

[15] K. Lin and C. Mu, Global dynamics in a fully parabolic chemotaxis system with logistic source, *Discrete Contin. Dyn. Syst.*, 36 (2016), 5025–5046.

[16] D. Liu and Y. Tao, Boundedness in a chemotaxis system with nonlinear signal production, *Appl. Math. J. Chinese Univ. Ser. B*, 31 (2016), 379–388.

[17] M. Mizukami, Boundedness and asymptotic stability in a two-species chemotaxis-competition model with signal-dependent sensitivity, *Discrete Contin. Dyn. Syst. Ser. B.*, 22 (2017), 2301–2319.

[18] M. Mizukami, Improvement of conditions for asymptotic stability in a two-species chemotaxis-competition model with signal-dependent sensitivity, *Discrete Contin. Dyn. Syst. Ser. S.*, 13 (2020), 269–278.

[19] N. Mizoguchi and P. Souplet, Nondegeneracy of blow-up points for the parabolic Keller-Segel system, *Ann. Inst. H. Poincaré Anal. Non Linéaire.*, 31 (2014), 851–875.

[20] T. Nagai, T. Senba and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, *Funkcial. Ekvac.*, 40 (1997), 411–433.

[21] L. Nirenberg, An extended interpolation inequality, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 20 (1966), 733–737.

[22] M. Negreanu and J. I. Tello, Asymptotic stability of a two species chemotaxis system with non-diffusive chemoeffectant, *J. Differential Equations*, 258 (2015), 1592–1617.

[23] M. Negreanu and J. I. Tello, On a two species chemotaxis model with slow chemical diffusion, *SIAM J. Math. Anal.*, 46 (2014), 3761–3781.

[24] K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, Exponential attractor for a chemotaxis-growth system of equations, *Nonlinear Anal.*, 51 (2002), 119–144.

[25] K. Osaki and A. Yagi, Global existence for a chemotaxis-growth system in $\mathbb{R}^2$, *Adv. Math. Sci. Appl.*, 12 (2002), 587–606.

[26] X. Pan and L. Wang, Improvement of conditions for boundedness in a fully parabolic chemotaxis system with nonlinear signal production, *C. R. Mathématique*, (2020), to appear.

[27] X. Pan, L. Wang and J. Zhang, Boundedness in a three-dimensional two-species and two-stimuli chemotaxis system with chemical signalling loop, *Math. Methods Appl. Sci.*, 43 (2020), 9529–9542.

[28] X. Pan, L. Wang, J. Zhang and J. Wang, Boundedness in a three-dimensional two-species chemotaxis system with two chemicals, *Z. Angew. Math. Phys.*, 71 (2020).

[29] G. Ren and B. Liu, Global boundedness and asymptotic behavior in a two-species chemotaxis-competition system with two signals, *Nonlinear Anal. Real World Appl.*, 48 (2019), 288–325.

[30] C. Stinner, J. I. Tello and M. Winkler, Competitive exclusion in a two-species chemotaxis model, *J. Math. Biol.*, 68 (2014), 1607–1626.

[31] X. Tao, S. Zhou and M. Ding, Boundedness of solutions to a quasilinear parabolic-parabolic chemotaxis model with nonlinear signal production, *J. Math. Anal. Appl.*, 474 (2019), 733–747.

[32] Y. Tao and Z.-A. Wang, Competing effects of attraction vs. repulsion in chemotaxis, *Math. Models Methods Appl. Sci.*, 23 (2013), 1–36.

[33] Y. Tao and M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity, *J. Differential Equations*, 252 (2012), 692–715.

[34] Y. Tao and M. Winkler, Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system, *Z. Angew. Math. Phys.*, 66 (2015), 2555–2573.

[35] Y. Tao and M. Winkler, Boundedness vs. blow-up in a two-species chemotaxis system with two chemicals, *Discrete Contin. Dyn. Syst. Ser. B.*, 20 (2015), 3165–3183.

[36] J. I. Tello and M. Winkler, Stabilization in a two-species chemotaxis system with a logistic source, *Nonlinearity*, 25 (2012), 1413–1425.

[37] X. Tu, C. Mu, P. Zheng and K. Lin, Global dynamics in a two-species chemotaxis-competition system with two signals, *Discrete Contin. Dyn. Syst.*, 38 (2018), 3617–3636.
TWO-SPECIES QUASILINEAR CHEMOTAXIS SYSTEM

[38] L. Wang, Improvement of conditions for boundedness in a two-species chemotaxis competition system of parabolic-parabolic-elliptic type, *J. Math. Anal. Appl.*, 484 (2020), 123705.

[39] L. Wang and C. Mu, A new result for boundedness and stabilization in a two-species chemotaxis system with two chemicals, *Discrete Contin. Dyn. Syst. Ser. B.*, 25 (2020), 4585–4601.

[40] L. Wang, C. Mu, X. Hu and P. Zheng, Boundedness and asymptotic stability of solutions to a two-species chemotaxis system with consumption of chemoattractant, *J. Differential Equations*, 264 (2018), 3369–3401.

[41] L. Wang, J. Zhang, C. Mu and X. Hu, Boundedness and stabilization in a two-species chemotaxis system with two chemicals, *Discrete Contin. Dyn. Syst. Ser. B.*, 25 (2020), 191–221.

[42] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, *J. Differential Equations*, 248 (2010), 2889–2905.

[43] M. Winkler, A critical blow-up exponent in a chemotaxis system with nonlinear signal production, *Nonlinearity*, 31 (2018), 2031–2056.

[44] M. Winkler, Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source, *Comm. Partial Differential Equations*, 35 (2010), 1516–1537.

[45] M. Winkler, Does a 'volume-filling effect' always prevent chemotactic collapse?, *Math. Methods Appl. Sci.*, 33 (2010), 12–24.

[46] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pures Appl.*, 100 (2013), 748–767.

[47] M. Winkler, Global existence and slow grow-up in a quasilinear Keller-Segel system with exponentially decaying diffusivity, *Nonlinearity*, 30 (2017), 735–764.

[48] T. Xiang, How strong a logistic damping can prevent blow-up for the minimal Keller-Segel chemotaxis system?, *J. Math. Anal. Appl.*, 459 (2018), 1172–1200.

[49] L. Xie, On a fully parabolic chemotaxis system with nonlinear signal secretion, *Nonlinear Anal. Real World Appl.*, 49 (2019), 24–44.

[50] L. Xie and Y. Wang, Boundedness in a two-species chemotaxis parabolic system with two chemicals, *Discrete Contin. Dyn. Syst. Ser. B.*, 22 (2017), 2717–2729.

[51] C. Yang, X. Cao, Z. Jiang and S. Zheng, Boundedness in a quasilinear fully parabolic Keller-Segel system of higher dimension with logistic source, *J. Math. Anal. Appl.*, 430 (2015), 585–591.

[52] H. Yu, W. Wang and S. Zheng, Criteria on global boundedness versus finite time blow-up to a two-species chemotaxis system with two chemicals, *Nonlinearity*, 31 (2018), 502–514.

[53] Q. Zhang, Competitive exclusion for a two-species chemotaxis system with two chemicals, *Appl. Math. Lett.*, 83 (2018), 27–32.

[54] Q. Zhang, X. Liu and X. Yang, Global existence and asymptotic behavior of solutions to a two-species chemotaxis system with two chemicals, *J. Math. Phys.*, 58 (2017), 111504, 9 pp.

[55] J. Zheng, Boundedness in a two-species quasi-linear chemotaxis system with two chemicals, *Topol. Methods Nonlinear Anal.*, 49 (2017), 463–480.

[56] P. Zheng and C. Mu, Global boundedness in a two-competing-species chemotaxis system with two chemicals, *Acta Appl. Math.*, 148 (2017), 157–177.

Received August 2020; 1st revision December 2020; 2nd revision January 2021.

E-mail address: panx_math@163.com
E-mail address: wanglc@cqupt.edu.cn