DUAL FLOWS IN HYPERBOLIC SPACE AND DE SITTER SPACE

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Abstract. We consider contracting flows in \((n+1)\)-dimensional hyperbolic space and expanding flows in \((n+1)\)-dimensional de Sitter space. When the flow hypersurfaces are strictly convex we relate the contracting hypersurfaces and the expanding hypersurfaces by the Gauß map. The contracting hypersurfaces shrink to a point \(x_0\) in finite time while the expanding hypersurfaces converge to the maximal slice \(\{\tau = 0\}\). After rescaling, by the same scale factor, the rescaled contracting hypersurfaces converge to a unit geodesic sphere, while the rescaled expanding hypersurfaces converge to slice \(\{\tau = -1\}\) exponential fast in \(C^\infty(S^n)\).

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1. Introduction

In a recent paper [7] a pair of dual flows was considered in \(S^{n+1}\). The one flow is the contracting flow

\[
\dot{x} = -F\nu,
\]
while the other is an expanding flow
\begin{equation}
\dot{x} = F^{-1} \nu,
\end{equation}
where $F \in C^\infty(I_+)$ and $\tilde{F}$ is its inverse
\begin{equation}
\tilde{F}(\kappa_i) = \frac{1}{F(\kappa_i^{-1})}.
\end{equation}
There is a Gauss map for the pair $(S^{n+1}, S^{n+1})$, which maps closed, strictly convex hypersurfaces $M$ to their polar sets $\tilde{M}$, cf. [5, Chapter 9]. Gerhardt [7] proved, that the flow hypersurfaces of (1.1) and (1.2) are polar sets of each other, if the initial hypersurface have this property. Under the assumption that $F$ is symmetric, monotone, positive, homogeneous of degree 1, $F$ strictly concave (cf. 3.1) and $\tilde{F}$ concave, it is proved in [7] that the contracting flows contract to a round point and the expanding flows converge to an equator such that after appropriate rescaling, both flows converge to a geodesic sphere exponential fast.

The Gauss map exists also for the pair $(\mathbb{H}^{n+1}, N)$, where $\mathbb{H}^{n+1}$ is the $(n + 1)$-dimensional hyperbolic space and $N$ is the $(n + 1)$-dimensional de Sitter space, cf. [5, Chapter 10]. We prove in this work similar results as in [7] by using this duality. Let $M(t)$ resp. $\tilde{M}(t)$ be solutions of the contracting flows
\begin{equation}
\dot{x} = -F \nu
\end{equation}
in $\mathbb{H}^{n+1}$ resp. the dual flows
\begin{equation}
\dot{x} = -\tilde{F}^{-1} \nu
\end{equation}
in $N$, where $\tilde{F}$ is the inverse of $F$ defined by (1.3). We impose the following assumptions.

1.1. Assumption. Let $F \in C^\infty(I_+)$ be a symmetric, monotone, 1-homogeneous and concave curvature function satisfying the normalization
\begin{equation}
F(1, \ldots, 1) = 1.
\end{equation}
We assume further, either
\begin{enumerate}
\item $F$ is concave and $\tilde{F}$ is concave and the initial hypersurface $M_0$ is horoconvex (i.e. all principal curvatures $\kappa_i \geq 1$),
or
\item $\tilde{F}$ is convex and $M_0$ is strictly convex.
\end{enumerate}
We now state our main results

1.2. Theorem. We consider curvature flows (1.4) and (1.5) under assumption 1.1 with initial smooth hypersurfaces $M_0$ and $\tilde{M}_0$, where $\tilde{M}_0$ is the polar hypersurface of $M_0$. Then the both flows exist on the maximal time interval $[0, T^*)$ with finite $T^*$. The hypersurfaces $\tilde{M}(t)$ are the polar hypersurfaces of $M(t)$ and vice versa during the evolution. The contracting flow
hypersurfaces in \( \mathbb{H}^{n+1} \) shrink to a point \( x_0 \) while the expanding flow hypersurfaces in \( N \) converge to a totally geodesic hypersurface which is isometric to \( \mathbb{S}^n \). We may assume the point \( x_0 \) is the Beltrami point by applying an isometry such that the hypersurfaces of the expanding flow are all contained in \( N \) and converge to the coordinate slice \( \{ \tau = 0 \} \).

Viewing \( \mathbb{H}^{n+1} \) and \( N \) as submanifolds of \( \mathbb{R}^{n+1} \) and by introducing polar coordinates in the Euclidean part of \( \mathbb{R}^{n+1} \) centered in \( (0, \ldots, 0) \in \mathbb{R}^{n+1} \), we can write flow hypersurfaces in \( \mathbb{H}^{n+1} \) resp. \( N \) as graphs of functions \( u \) resp. \( u^* \) over \( \mathbb{S}^n \). Let \( \Theta = \Theta(t, T^*) \) be the solution of (1.4) with spherical initial hypersurface and exitence interval \( [0, T^*) \). Then the rescaled functions

\[
\tilde{u} = u\Theta^{-1}
\]

and

\[
w = u^*\Theta^{-1}
\]

are uniformly bounded in \( C^\infty(\mathbb{S}^n) \). The rescaled principal curvatures \( \kappa_i\Theta \) as well as \( \tilde{\kappa}_i\Theta^{-1} \) are uniformly positive, where \( \tilde{\kappa}_i \) are the principal curvatures of \( \tilde{M}(t) \).

If the curvature function \( F \) is further strictly concave or \( F = \frac{1}{n}H \), then the rescaled functions (1.7) resp. (1.8) converge to the constant functions \( 1 \) resp. \(-1 \) in \( C^\infty(\mathbb{S}^n) \) exponentially fast.

Let us review some results concerning the contracting flows in \( \mathbb{H}^{n+1} \). Under the assumption that the initial hypersurface is strictly convex and satisfies the condition \( \kappa_i H > n \) for each \( i \), Huisken [11] proved that the flow (1.4) with \( F = H \) converges in finite time to a round sphere. Andrews [2] proved similar results for a general class of curvature function with argument \( \kappa_i - 1 \). Makowski [13] proved the contracting flow with a volume preserving term exists for all times and converges to a geodesic sphere exponentially fast.

The key ingredient treating the contracting flow is the pinching estimates. Under assumption 1.1 (1) it follows by a similar calculation as in [13], while Gerhardt [8] proved the pinching estimates under assumption 1.1 (2).

The elementary symmetric polynomials are defined by

\[
H_k(\kappa_1, \ldots, \kappa_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}, \quad 1 \leq k \leq n.
\]

Examples of curvature functions \( F \) satisfying assumption 1.1 (1) (up to normalization condition (1.6)) are

- the power means \( (\frac{1}{n} \sum \kappa_i^r)^{1/r} \) for \( |r| \leq 1 \),
- \( \sigma_k = H_k^{1/k} \) for \( 1 \leq k \leq n \),
- the inverse \( \tilde{\sigma}_k \) of \( \sigma_k \) for \( 1 \leq k \leq n \),
- \( (H_k/H_l)^{1/(k-l)} \) for \( 0 \leq l < k \leq n \),
- \( H_n^{\alpha_n}H_{n-1}^{\alpha_{n-1}} \cdots H_2^{\alpha_2}H_1^{\alpha_1} \) for \( \alpha_i \geq 0 \) and \( \sum \alpha_i = 1 \).
For a proof see [3, Chapter 2]. Moreover, the curvature functions in the above list are all strictly concave with exception of the mean curvature (cf. Section 3)

Examples of convex curvature functions $\bar{F}$, which is used in assumption 1.1 (2) (up to normalization condition (1.6)) are (cf. [5, Remark 2.2.13])

- the mean curvature $H$,
- the length of the second fundamental form $|A| = \left(\sum_i \kappa_i^2\right)^{1/2}$,
- the complete symmetric functions
  $$\gamma_k(\kappa_1, \ldots, \kappa_n) = \left(\sum_{|\alpha|=k} \kappa_1^{\alpha_1} \kappa_2^{\alpha_2} \cdots \kappa_n^{\alpha_n}\right)^{1/k} \text{ for } 1 \leq k \leq n.$$ 

Note that for convex $\bar{F}$ under assumption 1.1 (2), $F$ is of class $(K)$ and homogeneous of degree 1, hence strictly concave. (cf. [5, Definition 2.2.1, Lemma 2.2.12, 2.2.14], [7, Lemma 3.6])

2. Setting and General Facts

We now review some general facts about hypersurfaces from [5, Chapter 1]. Let $N$ be a $(n+1)$-dimensional dimensional semi-Riemannian manifold and $M$ be a hypersurface in $N$. Geometric quantities in $N$ will be denoted by $(\bar{g}_{\alpha\beta}), (\bar{R}_{\alpha\beta\gamma\delta})$, etc., where greek indices range from 0 to $n$. Quantities in $M$ will be denoted by $(g_{ij}), (h_{ij})$ etc., where latin indices range from 1 to $n$.

Generic coordinate systems in $N$ resp. $M$ will be denoted by $(x^\alpha)$ resp. $(\xi^i)$. Covariant differentiation will usually be denoted by indices, only if ambiguities are possible, by a semicolon, e.g. $h_{ij;}$.

Let $x : M \hookrightarrow N$ be a spacelike hypersurface (i.e. the induced metric is Riemannian) with a differentiable normal $\nu$, which is always supposed to be normalized, and $(h_{ij})$ be the second fundamental form, and set $\sigma = \langle \nu, \nu \rangle$.

We have the Gauß formula

$$x^\alpha_{ij} = -\sigma h_{ij} \nu^\alpha,$$

the Weingarten equation

$$\nu^\alpha = h_k^i x_k^\alpha,$$

the Codazzi equation

$$h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_j^\gamma x_k^\delta,$$

and the Gauß equation

$$R_{ij;kl} = \sigma \{h_{ik} h_{jl} - h_{il} h_{jk}\} + \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_j^\gamma x_k^\delta.$$

Let us review some properties of $\mathbb{H}^{n+1}$ and $N$, cf. [5, Section 10.2]. We label the coordinates in the $(n+2)$-dimensional Minkowski space $\mathbb{R}^{n+1,1}$ as $x = (x^a), 0 \leq a \leq n + 1$, where $x^0$ is the time function. Recall that the hyperbolic space $\mathbb{H}^{n+1}$ and de Sitter space $N$ are the subspaces of $\mathbb{R}^{n+1,1}$ defined by

$$\mathbb{H}^{n+1} = \{ x \in \mathbb{R}^{n+1,1} : \langle x, x \rangle = -1, x^0 > 0 \},$$
(2.6) \( N = \{ x \in \mathbb{R}^{n+1} : \langle x, x \rangle = 1 \} \).

Introduce polar coordinates in the Euclidean part of \( \mathbb{R}^{n+1} \) centered in \((0, \ldots, 0) \in \mathbb{R}^{n+1} \) such that the metric in \( \mathbb{R}^{n+1} \) is expressed as

\[
(2.7) \quad ds^2 = -dx^0{}^2 + dr^2 + r^2 \sigma_{ij} d\xi^i d\xi^j,
\]

where \( \sigma_{ij} \) is the spherical metric.

By viewing \( H^{n+1} \) as

\[
(2.8) \quad H^{n+1} = \{ (x^0, r, \xi^i) : r = \sqrt{|x^0|^2 - 1}, x^0 > 0, \xi \in S^n \},
\]

and by setting

\[
(2.9) \quad \varrho = \text{arccosh} x^0,
\]

\( H^{n+1} \) has coordinates \((\varrho, \xi^i)\) and the metric

\[
(2.10) \quad ds^2_{H^{n+1}} = d\varrho^2 + \sinh^2 \varrho \sigma_{ij} d\xi^i d\xi^j.
\]

Similarly,

\[
(2.11) \quad N = \{ (x^0, r, \xi^i) : r = \sqrt{1 + |x^0|^2}, x^0 \in \mathbb{R}, \xi \in S^n \},
\]

and by setting the eigentime

\[
(2.12) \quad \tau = \text{arsinh} x^0,
\]

\( N \) has coordinates \((\tau, \xi^i)\) and the metric

\[
(2.13) \quad ds^2_N = -d\tau^2 + \cosh^2 \tau \sigma_{ij} d\xi^i d\xi^j.
\]

3. Strictly concave curvature functions

For \( \xi, \kappa \in \mathbb{R}^n \), we write \( \xi \sim \kappa \), if there is \( \lambda \in \mathbb{R} \) such that \( \xi = \lambda \kappa \).

3.1. Definition. Let \( F \in C^2(\Gamma) \) be a symmetric, monotone, 1-homogeneous and concave curvature function. We call \( F \) strictly concave (in non-radial directions), if

\[
(3.1) \quad F_{ij} \xi^i \xi^j < 0 \quad \forall \xi \neq \kappa \text{ and } \xi \neq 0,
\]

or equivalently, if the multiplicity of the zero eigenvalue for \( D^2 F(\kappa) \) is one for all \( \kappa \in \Gamma \).

Note since \( F \) is homogeneous of degree 1, \( \kappa \in \Gamma \) is an eigenvector of \( D^2 F(\kappa) \) with zero eigenvalue. In [7, Chapter 3] it is proved that \( \sigma_k, 2 \leq k \leq n \) and the inverses \( \sigma_k \) of \( \sigma_k, 1 \leq k \leq n \) are strictly concave. In [12, Chapter 2] it is proved that \( Q_k = H_{k+1} / H_k, 1 \leq k \leq n - 1 \) are strictly concave in \( \Gamma_+ \). We consider the rest of the concave and inverse concave curvature functions listed on page 3.

3.2. Lemma. The curvature functions

\[
(3.2) \quad F = \left( \frac{1}{n} \sum_k \kappa^r_k \right)^{1/r} - 1 \leq r < 1
\]

are strictly concave in \( \Gamma_+ \).
Proof. Note that $F$ converges locally uniformly to $\sigma_n = (\kappa_1 \cdots \kappa_n)^{1/n}$ as $r \to 0$ and $\sigma_n$ is strictly concave. Furthermore, for $-1 \leq r < 1$ and $r \neq 0$,

\begin{equation}
\frac{\partial F}{\partial \kappa_i} = n^{-1/r} \left( \sum_{l} \kappa_l^r \right)^{\frac{1}{r} - 1} \kappa_i^r \left( \sum_{l} \kappa_l^r \right)^{1/r - 1} \kappa_i^r \kappa_i^r - \sum_{l} \kappa_l^r \delta_{ij},
\end{equation}

where $\sigma_n$ is strictly concave. Furthermore, for $-1 < r < 1$ and $r \neq 0$,

\begin{equation}
\frac{\partial^2 F}{\partial \kappa_i \partial \kappa_j} = n^{-1/r} (1 - r) \left( \sum_{l} \kappa_l^r \right)^{\frac{1}{r} - 2} \kappa_i^r \kappa_i^r - \sum_{l} \kappa_l^r \delta_{ij}).
\end{equation}

Consider $\eta$ such that $F_{ij} \eta^j = 0$. Since $r \neq 1$,

\begin{equation}
\eta_i = \left( \sum_{l} \kappa_l^r \right)^{-1} \kappa_i^r \eta^j \kappa_j.
\end{equation}

Knowing that $F$ is concave for $|r| \leq 1$ we conclude that $F$ is strictly concave for $-1 \leq r < 1$. \hfill \Box

3.3. Lemma. Let $f^\alpha$ be concave in $\Gamma_+$ for all $1 \leq \alpha \leq k$ and strictly concave in $\Gamma_+$ for at least one index in $1 \leq \alpha \leq k$. Let $\varphi$ be strictly monotone increasing and concave in $\Gamma_+$, then

\begin{equation}
F(\kappa_1, \cdots, \kappa_n) = \varphi(f^1(\kappa_1, \cdots, \kappa_n), \cdots, f^k(\kappa_1, \cdots, \kappa_n))
\end{equation}

is strictly concave in $\Gamma_+$.

Proof. Let $0 \neq \xi \in \mathbb{R}^n$ and $\xi \neq \kappa$, then

\begin{equation}
F_{ij} \xi^i \xi^j = \varphi_n f^\alpha_{ij} \xi^i \xi^j + \varphi_{\alpha \beta} f^\alpha_{ij} f^\beta_{ij} \xi^i \xi^j < 0,
\end{equation}

since by assumption

\begin{equation}
\varphi_\alpha > 0, \quad \varphi_{\alpha \beta} \leq 0, \quad f^\alpha_{ij} \xi^i \xi^j \leq 0
\end{equation}

and

\begin{equation}
f^\alpha_{ij} \xi^i \xi^j < 0 \text{ for at least one } 1 \leq \alpha \leq k. \hfill \Box
\end{equation}

Note that the weighted geometric mean

\begin{equation}
\varphi(f^1, \cdots, f^k) = (f^1)^{\alpha_1} \cdots (f^k)^{\alpha_k} \quad \text{with} \quad \sum_i \alpha_i = 1
\end{equation}

is a strictly monotone increasing and concave function. Knowing that $H_{k+1}/H_k, 1 \leq k \leq n - 1$ are strictly concave in $\Gamma_+$, we conclude that

\begin{equation}
(H_{k}/H_l)^{1/(k-l)} = (H_{l+1}/H_l)^{1/(k-l)} \cdots (H_k/H_{k-1})^{1/(k-l)} \quad 0 \leq l < k \leq n
\end{equation}

and

\begin{equation}
H_1^{\alpha_1} H_2^{\alpha_2 - \alpha_1} \cdots H_2^{\alpha_k - \alpha_2} H_n^{1 - \alpha_k} = \left( \frac{H_1}{H_0} \right)^{\alpha_1} \left( \frac{H_2}{H_1} \right)^{\alpha_2} \cdots \left( \frac{H_n}{H_{n-1}} \right)^{\alpha_n}
\end{equation}
with \( \alpha_i \geq 0, \sum_i \alpha_i = 1 \) and \( \alpha_1 \neq 1 \) are strictly concave in \( \Gamma^+ \).

4. Polar sets and dual flows

We state some facts about Gauß maps for \((H^{n+1}, N)\), cf. [5, Section 10.4].

4.1. Theorem. Let \( x : M_0 \to M \subset H^{n+1} \) be a closed, connected, strictly convex hypersurface. Consider \( M \) as a codimension 2 immersed submanifold in \( \mathbb{R}^{n+1,1} \), such that
\[
(4.1) \quad x_{ij} = g_{ij} x - h_{ij} \bar{x},
\]
where \( \bar{x} \in T_x(\mathbb{R}^{n+1,1}) \) is the representation of the exterior normal vector \( \nu = (\nu^i) \) of \( M \) in \( T_x(H^{n+1}) \). Then the Gauß map
\[
(4.2) \quad \bar{x} : M_0 \to N
\]
is the embedding of a closed, spacelike, achronal, strictly convex hypersurface \( \bar{M} \subset N \). Viewing \( \bar{M} \) as a codimension 2 submanifold in \( \mathbb{R}^{n+1,1} \), its Gaussian formula is
\[
(4.3) \quad \bar{x}_{ij} = -\bar{g}_{ij} \bar{x} + \bar{h}_{ij} x,
\]
where \( \bar{g}_{ij}, \bar{h}_{ij} \) are the metric and second fundamental form of \( \bar{M} \) and \( x \) is the embedding of \( M \) which also represents the future directed normal vector of \( \bar{M} \). The second fundamental form \( \bar{h}_{ij} \) is defined with respect to the future directed normal vector, where the time orientation of \( N \) is inherited from \( \mathbb{R}^{n+1,1} \). Furthermore, there holds
\[
(4.4) \quad \bar{h}_{ij} = h_{ij},
\]
\[
(4.5) \quad \bar{\kappa}_i = \kappa_i^{-1}.
\]

\( \square \)

We prove in the following that the duality is also valid in case of curvature flows.

4.2. Lemma. Let \( \Phi \in C^\infty(\mathbb{R}^+) \) be strictly monotone, \( \dot{\Phi} > 0 \), and let \( F \in C^\infty(\Gamma^+_* \mathbb{R}_+ \) be a symmetric, monotone, 1-homogeneous curvature function such that \( F|_{\Gamma^+_* \mathbb{R}_+} > 0 \) and such that the flows
\[
(4.6) \quad \dot{x} = -\Phi(F) \nu
\]
in \( H^{n+1} \) resp.
\[
(4.7) \quad \dot{x} = -\Phi(\bar{F}^{-1}) \bar{\nu}
\]
in \( N \) with initial strictly convex hypersurfaces \( M_0 \) resp. \( \bar{M}_0 \) exist on maximal time intervals \([0, T^*) \) resp. \([0, \bar{T}^*) \), where \( \nu \) and \( \bar{\nu} \) are the exterior normal resp. past directed normal. The flow hypersurfaces are then strictly convex. Let \( M(t) \) resp. \( \bar{M}(t) \) be the corresponding flow hypersurfaces, then \( T^* = \bar{T}^* \) and \( M(t) = \bar{M}(t) \) for all \( t \in [0, T^*) \).

Proof. The arguments are similar to those in [7, Section 4] with combination with the results from [5, Section 10.4]. Since there holds
\[(x, x) = 1, \langle \dot{x}, x \rangle = 0, \langle x_j, x \rangle = 0, \langle \tilde{x}, x \rangle = 0,\]
(see [5, Lemma 10.4.1] for the last identity) we can consider the flow (4.6) as flow in \(\mathbb{R}^{n+1,1}\)
\[(4.8) \quad \dot{x} = -\Phi \tilde{x},\]
and we have the decomposition
\[(4.9) \quad T_x(\mathbb{R}^{n+1,1}) = T_x(\mathbb{H}^{n+1}) \oplus \langle x \rangle.\]
Furthermore, we conclude from
\[(4.10) \quad \langle \dot{\tilde{x}}, x_j \rangle = \Phi_j, \quad \langle \dot{\tilde{x}}, \tilde{x} \rangle = 0, \quad \langle \dot{\tilde{x}}, x \rangle = \Phi,\]
from the Weingarten equation (see [5, Lemma 10.4.3, 10.4.4])
\[(4.11) \quad x_j = \tilde{h}^k_j \tilde{x}_k,\]
and from (4.10) that
\[(4.12) \quad \dot{x} = \Phi x + \Phi^m x_m = \Phi x + \Phi^m \tilde{h}^k_m \tilde{\tilde{x}}_k,\]
where
\[(4.13) \quad \tilde{\tilde{V}}_2 = 1 - |D \tilde{u}|^2 = 1 - \frac{1}{\cosh^2 \tilde{u}} \sigma^{ij} \tilde{u}_i \tilde{u}_j.\]
Note that \(\tilde{v}\) in (4.15) is the future directed normal
\[(4.14) \quad (\tilde{v}^n) = \tilde{v}^{-1}(1, \tilde{\tilde{u}}),\]
where
\[(4.15) \quad \tilde{\tilde{u}}^i = \frac{1}{\cosh^2 \tilde{u}} \sigma^{ij} \tilde{u}_j.\]
Thus it holds in view of (4.15)
\[
\frac{\partial \tilde{u}}{\partial t} = \frac{d \tilde{u}}{dt} - \tilde{u}_i \tilde{x}^i = \Phi \tilde{v} - \frac{1}{\Phi} |D \tilde{u}|^2 - \Phi \tilde{h}_{ik} \delta^i_k \tilde{u}_i = \Phi \tilde{v}.
\]
This is exactly the scalar curvature equation of the flow equation (4.22)
\[
\dot{x} = -\Phi \tilde{v},
\]
where \( \tilde{v} \) in (4.22) is the future directed normal and
\[
\Phi = \Phi(F) = \Phi(F^{-1}).
\]
Now \( \tilde{h}_{ij} \) in \( N \) is defined with respect to the future directed normal. By adapting the convention in [5, p.307] we switch the light cone in \( N \) and by defining \( \tau = -\arcsinh x^0 \) in (2.12) we still derive the flow (4.22) in \( N \), where \( \tilde{v} \) is now the past directed normal and the second fundamental form is defined with respect to this normal. The rest of the proof is identical to [7, Theorem 4.2].

From now we shall employ this duality by choosing
\[
\Phi(r) = r.
\]
Note that the expanding flows in \( \mathbb{H}^{n+1} \) was already considered in [6] and its non-scale-invariant version in [14].

5. Pinching estimates

We consider the contracting flow
\[
\begin{align*}
\dot{x} &= -F \nu, \\
x(0) &= M_0
\end{align*}
\]
in \( \mathbb{H}^{n+1} \) with initial smooth and strictly convex hypersurfaces \( M_0 \), where \( \nu \) is the exterior normal vector.

Under the assumptions of Theorem 1.2 the curvature flow (5.1) exists on a maximal time interval \([0, T^*)\), \( 0 < T^* \leq \infty \), cf. [5, Theorem 2.5.19, Lemma 2.6.1].

5.1. Theorem. Let \( M(t) \) be a solution of the flow (5.1) in \( \mathbb{H}^{n+1} \). If the initial hypersurface \( M_0 \) in \( \mathbb{H}^{n+1} \) satisfies
\[
\kappa_i > 1,
\]
then this condition will also be satisfied by the flow hypersurfaces \( M(t) \) during the evolution.
It was proved in [1] that the tensor $S$, homogeneous of order 2, satisfies the equation
\begin{equation}
\dot{S}_{ij} = h_{ij} - g_{ij}
\end{equation}
which is in (5.3) $S_{ij} = h_{ij} - g_{ij}$. The proof is similar to those in [4, Theorem 3.2], where the notation $C_t$ denotes the part is based on the computation in [4, Theorem 3.2] implies that $S_{ij}$ is a constant, depending on the $C_t^0 \equiv 0$.

Proof. The tensor $S_{ij}$ satisfies the equation
\begin{equation}
\dot{S}_{ij} - F^{kl}S_{ij;kl} = F^{kl}h_{rk}h^r_i + 2Fh_i^k + F^{kl,rs}h_{kl,r}h_{rs,j}
\end{equation}
where $\dot{N}_{ij} = F^{kl,rs}h_{kl,r}h_{rs,j}$. At every point where $h_{ij} \eta^j = \eta$ there holds
\begin{equation}
N_{ij}\eta^i \eta^j \equiv \{F^{kl}h_{rk}h^r_i - 2F + F^{kl}g_{kl}\} [\eta]^2 \geq 0.
\end{equation}
It was proved in [3, Theorem 3.3, Lemma 4.4] that
\begin{equation}
\dot{N}_{ij}\eta^i \eta^j + \sup_{\Gamma} 2F^{kl}\{2\Gamma^r_i S_{ir,k} \eta^j - \Gamma^r_i \Gamma^s_k S_{rs} \} \geq 0,
\end{equation}
where only the inverse concavity of $F$ was used. Andrews’ maximum principle in [3, Theorem 3.2] implies that $S_{ij} > 0$ during the evolution. □

In the next step we use a constant rank theorem to allow the condition $\kappa_t \geq 1$ in the proof of the succeeding Lemma 5.4.

5.2. Lemma. Let $M(t)$ be a solution of the flow (5.1) in $\mathbb{H}^{n+1}$ and assume that the tensor $S_{ij}$ satisfies $S_{ij} \geq 0$ on the hypersurfaces $M(t)$ for $t \in [0, T^*)$, then $S_{ij}$ is of constant rank $l(t)$ for every $t \in (0, T^*)$.

Proof. The proof is similar to those in [15, Theorem 3.2], where the main part is based on the computation in [4, Theorem 3.2]. For $\epsilon > 0$, let
\begin{equation}
W_{ij} = S_{ij} + \epsilon g_{ij}.
\end{equation}
Let $l(t)$ be the minimal rank of $S_{ij}(t)$. For a fixed $t_0 \in (0, T^*)$, let $x_0 \in M(t_0)$ be the point such that $S_{ij}(t_0, x_0)$ attains its minimal rank at $x_0$. Set
\begin{equation}
\phi(t, \xi) = H_{l+1}(W_{ij}(t, \xi)) + H_{l+2}(W_{ij}(t, \xi))
\end{equation}
where $H_l$ is the elementary symmetric polynomials of eigenvalues of $W_{ij}$, homogeneous of order $l$. A direct computation shows
\begin{equation}
F^{kl}W_{ij;kl} - \dot{W}_{ij} = -F^{kl}h_{rk}h^r_i W_{ij} - F^{kl}g_{kl}W_{ij} + 2Fh_i^k W_{kj}
\end{equation}
\begin{equation}
- F^{kl,rs}W_{kl,r}W_{rs,j} + 2F\epsilon g_{ij}
\end{equation}
\begin{equation}
- (1 - \epsilon)\{F^{kl}h_{rk}h^r_i - 2F + F^{kl}g_{kl}\} g_{ij}.
\end{equation}
As in [4], we consider a neighborhood $(t_0 - \delta, t_0) \times \mathcal{O}$ around $(t_0, \xi_0)$. We use the notation $h = O(f)$ if $|h(\xi)| \leq C f(\xi)$ for every $(t, \xi) \in (t_0 - \delta, t_0] \times \mathcal{O}$, where $C$ is a constant, depending on the $C^{1,1}$ norm of the second fundamental form on $(t_0 - \delta, t_0] \times \mathcal{O}$, but independent of $\epsilon$. It was proved in [4, Corollary 2.2] that $\phi$ is in $C^{1,1}$. And as in [4], let $G = \{n-l+1, n-l+2, \ldots, n\}$
and \( B = \{1, \ldots, n - l\} \). We choose the coordinates such that \( h_{ij} = \kappa_i \delta_{ij} \) and \( g_{ij} = \delta_{ij} \). In view of \([4, (3.14)]\), in such coordinates \( \phi^{ij} \) is up to \( O(\phi) \) non-negative in \( O \) and we have

\[
F^{kl} \phi_{,kl} - \dot{\phi} \leq \phi^{ij} \{ - F^{kl} h_{rk} h_{lj} W_{ij} - F^{kl} g_{kl} W_{ij} + 2 F h_{ik} W_{kj} \\
+ 2 F g_{ij} - F^{kl,rs} W_{kl,i} W_{rs,j} \} + F^{kl} \delta^{ij,rs} W_{ij,kl} W_{rs,kl} + O(\phi).
\]

We can choose \( O \) small enough, such that \( \epsilon = O(\phi) \) as in \([4, (3.8)]\). It was proved in \([4, (3.14)]\) that \( \phi_{ii} = O(\phi) \) for \( i \in G \) and since \( W_{ii} \leq \phi \) for \( i \in B \), we infer that

\[
F^{kl} \phi_{,kl} - \dot{\phi} \leq - \phi^{ij} F^{kl,rs} W_{kl,i} W_{rs,j} + F^{kl} \phi^{ij,rs} W_{ij,kl} W_{rs,kl} + O(\phi).
\]

Using the inverse concavity of \( F \) and proceed as in \([4, \text{Theorem 3.2}]\), we conclude

\[
F^{kl} \phi_{,kl} - \dot{\phi} \leq C \{ \phi + |D\phi| \},
\]

where \( C \) is a constant independent of \( \epsilon \) and \( \phi \). Taking \( \epsilon \to 0 \), the strong maximum principle for parabolic equations yields

\[
H_{l(t_0)+1}(S_{ij}(t, \xi)) \equiv 0 \quad \forall (t, \xi) \in (t_0 - \delta, t_0) \times O.
\]

Since \( M(t_0) \) is a closed hypersurface, \( S_{ij}(t_0, \xi) \) is of constant rank \( l(t_0) \) on \( M(t_0) \).

Note that the proof of the Lemma 5.1 implies, if the initial hypersurface satisfies \( \kappa_i \geq 1 \), then this condition remains true during the evolution. Furthermore, every closed hypersurface in \( \mathbb{H}^{n+1} \) contains a point on which holds \( \kappa_i > 1 \). Thus we conclude

5.3. Corollary. Let \( M(t) \) be a solution of the flow (5.1) in \( \mathbb{H}^{n+1} \). If the initial hypersurface \( M_0 \) in \( \mathbb{H}^{n+1} \) satisfies \( \kappa_i \geq 1 \), then \( \kappa_i > 1 \) for every \( t \in (0, T^*) \).

5.4. Lemma. Let \( M(t) \) be a solution of the flow (5.1) in \( \mathbb{H}^{n+1} \) under assumption 1.1 (1), then there exists a uniform positive constant \( \epsilon > 0 \) such that

\[
\kappa_1 \geq \epsilon \kappa_n
\]

during the evolution, where the principal curvatures are labeled as

\[
\kappa_1 \leq \cdots \leq \kappa_n.
\]

Proof. The proof is similar to \([13, \text{Lemma 4.2}]\). By Replacing \( M_0 \) by \( M(t_0) \) for a \( t_0 \in (0, T^*) \) as initial hypersurface, we can assume that \( \kappa_i > 1 \) on \( M_0 \). Let \( F \) be a concave and inverse concave curvature function, then

\[
T_{ij} = h_{ij} - g_{ij} - \epsilon (H - n) g_{ij}
\]
satisfies the equation
\[
\dot{T}_{ij} - F^{kl} T_{ij;kl} = F^{kl} h_{rk} h_{r}^{i} \{ h_{ij} - \epsilon H g_{ij} \} - 2F h_{kl}^{k} \{ h_{kj} - \epsilon H g_{kj} \} \\
+ 2K_{N} F g_{ij} - 2nK_{N} F g_{ij} - K_{N} F^{kl} g_{kl} \{ h_{ij} - \epsilon H g_{ij} \} \\
- 2F(\epsilon n - 1)h_{ij} + F^{kl;r} h_{kl;ij} h_{rs;ij} - \epsilon F^{kl;r} h_{kl;ij} h_{rs;ij} g^{pq} g_{ij} \\
\equiv N_{ij} + \tilde{N}_{ij},
\]
where \(\tilde{N}_{ij} = F^{kl;r} h_{kl;ij} h_{rs;ij} - \epsilon F^{kl;r} h_{kl;ij} h_{rs;ij} g^{pq} g_{ij}\).

At every point where \(T_{ij} \eta = 0\) there holds
\[
N_{ij} \eta^{\iota} \eta^{\iota} = F^{kl} h_{rk} h_{r}^{i} (1 - \epsilon n) |\eta|^{2} + 2F h_{ij} (\epsilon n - 1) \eta^{\iota} \eta^{\iota} \\
+ \{ F^{kl} g_{kl} - 2F \} (1 - \epsilon n) |\eta|^{2} - 2F (\epsilon n - 1) h_{ij} \eta^{\iota} \eta^{\iota} \\
= (1 - \epsilon n) \sum_{i} F_{i} (\kappa_{i}^{2} - 2\kappa_{i} + 1) |\eta|^{2} \geq 0.
\]

It is proved in [1, Theorem 4.1] (see also the modification in [13, Theorem B.2]) that
\[
\tilde{N}_{ij} \eta^{\iota} \eta^{\iota} + \sup_{r} 2F^{kl} \{ 2\Gamma^{r}_{i} T_{ir} \eta^{i} - \Gamma^{r}_{i} \Gamma^{s}_{i} T_{rs} \} \geq 0,
\]
We can choose \(\epsilon > 0\) sufficiently small, such that \(T_{ij} \geq 0\) on \(M_{0}\), then the Andrews’ maximum principle [3, Theorem 3.2] implies \(T_{ij} \geq 0\) and hence
\[
\kappa_{1} \geq 1 \geq \epsilon (H - n)
\]
during the evolution. \(\square\)

The following pinching results is due to Gerhardt. By using [8, Theorem 1.1] and the duality result Lemma 4.2 we obtain

5.5. **Theorem.** Let \(M(t)\) be a solution of the flow (5.1) in \(\mathbb{H}^{n+1}\) under the assumption 1.1 (2), then there exists a uniform constant \(\epsilon > 0\) such that
\[
\kappa_{1} \geq \epsilon \kappa_{n}
\]
during the evolution.

6. **Contracting flows - convergence to a point**

Fix a point \(p_{0} \in \mathbb{H}^{n+1}\), the hyperbolic metric in the geodesic polar coordinates centered at \(p_{0}\) can be expressed as
\[
ds^{2} = dr^{2} + \sinh^{2} r \sigma_{ij} dx^{i} dx^{j},
\]
where \(\sigma_{ij}\) is the canonical metric of \(S^{n}\).

Geodesic spheres with center in \(p_{0}\) are totally umbilic. The induced metric, second fundamental form and the principal curvatures of the coordinate slices \(S_{r} = \{ x^{0} = r \} \) are given by
\[
\dot{g}_{ij} = \sinh^{2} r \sigma_{ij}, \quad \dot{h}_{ij} = \frac{1}{2} \dot{g}_{ij} = \coth r \ddot{g}_{ij}, \quad \bar{k}_{i} = \coth r,
\]
respectively. See [5, (1.5.12)].

6.1. Lemma. Consider (5.1) with initial hypersurface \( x(0) = S_{r_0} \), then the corresponding flow exists in a maximal time interval \([0, T^*)\) with \( T^* \) finite and will shrink to a point. The flow hypersurfaces \( M(t) \) are all geodesic spheres with the same center and their radii \( \Theta = \Theta(t) \) solve the ODE

\[
\begin{align*}
\dot{\Theta} &= -\coth \Theta, \\
\Theta(0) &= r_0.
\end{align*}
\]  

(6.3)

Proof. We set

\[
\begin{align*}
x^0(t, \xi) &= \Theta(t), \\
x^i(t, \xi) &= x^i(0, \xi).
\end{align*}
\]  

(6.4)

In view of [5, (1.5.7)] the exterior normal of a geodesic sphere is \((1,0,\ldots,0)\).

Using that \( F(\bar{h}_{ij}) = \coth \Theta \), we see that \( x \) in (6.4) solves the flow equation (5.1). Now the solution of (6.3) is given by

\[
\cosh \Theta = (\cosh r_0) e^{-t}.
\]  

(6.5)

Thus the spherical flow exists only for a finite time \([0, T^*)\). Note that (6.5) can be rewritten as

\[
\Theta = \arccosh e^{(T^*-t)}.
\]  

(6.6)

Next we want to prove that the flow (5.1) shrinks to a point. Using the inverse of the Beltrami map, \( \mathbb{H}^{n+1} \) is parametrizable over \( B_1(0) \) yielding the metric (cf. [5, Section 10.2])

\[
ds^2 = \frac{1}{(1-r^2)^2} dr^2 + \frac{r^2}{1-r^2} \sigma_{ij} d\xi^i d\xi^j.
\]  

(6.7)

Define the variable \( q \) by

\[
q = \arctanh r = \frac{1}{2} (\log(1+r) - \log(1-r)),
\]  

(6.8)

then

\[
ds^2 = dq^2 + \sinh^2 q \sigma_{ij} d\xi^i d\xi^j.
\]  

(6.9)

Let

\[
ds^2 = dr^2 + r^2 \sigma_{ij} d\xi^i d\xi^j
\]  

be the Euclidean metric over \( B_1(0) \). Define

\[
d\tau = \frac{1}{r\sqrt{1-r^2}} dr, \quad d\hat{\tau} = r^{-2} dr,
\]  

(6.10)
we have further
\[
\begin{align*}
    ds^2 &= \frac{r^2}{1-r^2} \{d\tau^2 + \sigma_{ij} d\xi^i d\xi^j\} \equiv e^{2\psi} \{d\tau^2 + \sigma_{ij} d\xi^i d\xi^j\}, \\
    ds^2 &= r^2 \{d\tilde{\tau}^2 + \sigma_{ij} d\xi^i d\xi^j\} \equiv e^{2\tilde{\psi}} \{d\tilde{\tau}^2 + \sigma_{ij} d\xi^i d\xi^j\}.
\end{align*}
\]  
(6.12)

An arbitrary closed, connected, strictly embedded hypersurface \( M \subset \mathbb{H}^{n+1} \) bounds a convex body and we can write \( M \) as a graph in geodesic polar coordinates.

(6.13)  \[ M = \text{graph } u = \{ \tau = u(x) : x \in \mathbb{S}^n \}. \]

\( M \) can also be viewed as a graph \( \tilde{M} \) in \( B_1(0) \) with respect to the Euclidean metric

(6.14)  \[ \tilde{M} = \text{graph } \tilde{u} = \{ \tilde{\tau} = \tilde{u}(x) : x \in \mathbb{S}^n \}. \]

Writing \( \tilde{u} = \varphi(u) \), then there holds (see [5, (10.2.18)])

(6.15)  \[ \dot{\varphi}^2 = 1 - r^2. \]

The same argument as in [7, Lemma 6.1] yields

6.2. Lemma. Let \( M(t) \) be a solution of (5.1) on a maximal time interval \([0, T^*)\) and represent \( M(t) \), for a fixed \( t \in [0, T^*) \), as a graph in polar coordinates with center in \( x_0 \in M(t) \)

(6.16)  \[ M(t) = \text{graph } u(t, \cdot), \]

then

(6.17)  \[ \inf_M u(t) \leq \Theta(t, T^*) \leq \sup_M u, \]

where the solution of the spherical flow \( \Theta(t, T^*) \) is given by (6.6).  \( \square \)

6.3. Lemma. Let \( x_0 \in \tilde{M}(t) \) be as above and represent \( M(t) \) in Euclidean polar coordinates (6.10), then there exists a constant \( c_0 = c_0(M_0) < 1 \) such that the estimate

(6.18)  \[ r \leq c_0 \]

holds for any \( t \in [0, T^*) \).

Proof. The argument is similar to those in [7, Lemma 6.3, Remark 6.5]. Looking at the scalar flow equation for a short time interval, we conclude that the convex bodies \( \tilde{M}(t) \subset \mathbb{H}^{n+1} \) are decreasing with respect to \( t \). Furthermore, \( \tilde{M}_0 \) is strictly convex. Thus \( \varphi \) is uniformly bounded and the claim follows from the relation

(6.19)  \[ r = \tanh \varphi = 1 - \frac{2}{e^{2\varphi} + 1}. \]

\( \square \)

Denote \( h_{ij} \) resp. \( \tilde{h}_{ij} \) the second fundamental forms and \( \kappa_i \) resp \( \tilde{\kappa}_i \) the principal curvatures of \( M \) with respect to the ambient metric \( \tilde{g}_{\alpha\beta} \) resp. \( \tilde{g}_{\alpha\beta} \).
6.4. Lemma. The principal curvatures $\tilde{\kappa}_i$ of $\hat{M}(t)$ are pinched, i.e., there exists a uniform constant $c$ such that

$$\tilde{\kappa}_n \leq c\tilde{\kappa}_1,$$

where the $\tilde{\kappa}_i$ are labeled as

$$\tilde{\kappa}_1 \leq \cdots \leq \tilde{\kappa}_n.$$

Proof. The $h_{ij}$ and $\tilde{h}_{ij}$ are related through the formula (see [5, (10.2.33)])

$$\tilde{h}_{ij} = (1 - r^2)h_{ij}v,$$

where

$$v^2 = 1 + \sigma^{ij}u_iu_j,$$

$$\tilde{v}^2 = 1 + \dot{\sigma}^{ij}u_iu_j.$$

Because of Lemma 6.3 there exists $0 < \delta < 1$ such that

$$r^2 \leq 1 - \delta,$$

and thus

$$\delta v^2 \leq \tilde{v}^2 \leq v^2,$$

$$\delta h_{ij} \leq \tilde{h}_{ij} \leq \delta^{-1} h_{ij}.$$

Furthermore, there holds

$$g_{ij} = \frac{r^2}{1 - r^2} \{u_iu_j + \sigma^{ij}\},$$

$$\tilde{g}_{ij} = \frac{\dot{r}^2}{r^2} \{\dot{\sigma}^{ij}u_iu_j + \sigma^{ij}\}.$$

and we conclude

$$\delta^2 g_{ij} \leq \tilde{g}_{ij} \leq g_{ij}.$$

Now the claim follows from the maximum-minimum principle. \qed

For $\hat{M}(t) \subset \mathbb{H}^{n+1}$, the inradius $\rho_-(t)$ and circumradius $\rho_+(t)$ of $\hat{M}(t)$ are defined by

$$\rho_-(t) = \sup \{r : B_r(y) \text{ is enclosed by } \hat{M}(t) \text{ for some } y \in \mathbb{H}^{n+1}\},$$

$$\rho_+(t) = \inf \{r : B_r(y) \text{ encloses } \hat{M}(t) \text{ for some } y \in \mathbb{H}^{n+1}\}.$$

Now, choose $x_0 \in \hat{M}(t)$ to be the center of the inball of $\hat{M}(t) \subset \mathbb{H}^{n+1}$ and let $x_0$ be the center of the geodesic polar coordinates. Note that the center of the Euclidean inball is also $x_0$. Let $\rho_-(t)$ resp. $\rho_+(t)$ be the inradius resp. circumradius of $\hat{M}(t) \subset \mathbb{H}^{n+1}$, and let $\tilde{\rho}_-(t)$ resp. $\tilde{\rho}_+(t)$ be the inradius resp. circumradius of $\hat{M}(t) \subset \mathbb{R}^{n+1}$.

6.5. Lemma. Let $B_{\rho_-(t)}(x_0) \subset \hat{M}(t)$ be a geodesic inball, then there exist positive constants $c$ and $\delta$, such that

$$\hat{M}(t) \subset B_{4c\rho_-(t)}(x_0) \quad \forall t \in [T^* - \delta, T^*).$$
Proof. The pinching estimates in the Euclidean ambient space (6.20) and [1, Theorem 5.1, Theorem 5.4] imply
\begin{equation}
\tilde{\rho}_+(t) \leq c \tilde{\rho}_-(t)
\end{equation}
with a uniform constant $c$, hence $\tilde{M}(t)$ is contained in the Euclidean ball $B_{\tilde{\rho}}(0)$,
\begin{equation}
\tilde{M}(t) \subset B_{\tilde{\rho}}(0), \quad \tilde{\rho}(t) = 2\epsilon \tilde{\rho}_-(t).
\end{equation}
Furthermore, we deduce from Lemma 6.2 that
\begin{equation}
\inf_{M(t)} \tilde{u} \leq \tilde{\Theta} \leq \sup_{M(t)} \tilde{u},
\end{equation}
where $M(t) = \text{graph } \tilde{u}$ is a representation of $M(t)$ in Euclidean polar coordinates. We conclude further
\begin{equation}
\tilde{\rho}(t) = 2\epsilon \tilde{\rho}_-(t) \leq 2\epsilon \tilde{\Theta}.
\end{equation}
Choose now $\delta > 0$ small such that
\begin{equation}
2\epsilon \tilde{\Theta}(t, T^*) \leq 1 \quad \forall t \in [T^* - \delta, T^*).
\end{equation}
Now it holds for
\begin{equation}
\rho(t) = \text{arctanh } \tilde{\rho}(t)
\end{equation}
\begin{equation}
\tilde{M}(t) \subset B_{\rho(t)}(x_0) \subset \mathbb{H}^{n+1}.
\end{equation}
Since
\begin{equation}
\tilde{\rho}(t) \leq 1,
\end{equation}
we conclude further
\begin{equation}
\tilde{\rho} \leq \rho \leq 2\tilde{\rho}, \quad \tilde{\rho}_- \leq \rho_-.\n\end{equation}
Thus
\begin{equation}
\rho \leq 2\tilde{\rho} = 4\epsilon \rho_- \leq 4c\rho_-
\end{equation}
and the claim follows. \(\Box\)

6.6. Lemma. During the evolution the flow hypersurfaces $M(t)$ are smooth and uniformly convex satisfying a priori estimates in any compact subinterval $[0, T] \subset [0, T^*)$.

Proof. Let $0 < T < T^*$ be fixed. From (6.31) and (6.33) we infer
\begin{equation}
\epsilon \tilde{\Theta}(T, T^*) \leq \tilde{\rho}_-(T).
\end{equation}
Since
\begin{equation}
\Theta(T, T^*) = \text{arctanh } \tilde{\Theta}(T, T^*), \quad \rho_-(T) = \text{arctanh } \tilde{\rho}_-(T),
\end{equation}
and $\tilde{\rho}_-(T)$, $\tilde{\Theta}(T, T^*)$ are uniformly bounded from above by 1 we infer that
\begin{equation}
0 < \frac{\epsilon}{2} \Theta = \frac{\epsilon}{2} \text{arctanh } \tilde{\Theta} \leq c\tilde{\Theta} \leq \text{arctanh } (c\tilde{\Theta}) \leq \rho_-(T).
\end{equation}
Let \( x_0 \in \hat{M}(T) \) be the center of an inball and introduce geodesic polar coordinates with center \( x_0 \). This coordinate system will cover the flow in \( 0 \leq t \leq T \).

Writing the flow hypersurfaces as graphs \( u(t, \cdot) \) of a function we have

\[
0 < c^{-1} \leq u \leq c.
\]

And since \( M(t) \) are convex,

\[
v^2 = 1 + \sinh^{-2} u \sigma^{ij} u_i u_j
\]

is uniformly bounded. Under assumption 1.1 (1) we have \( \kappa_i \geq 1 \). And under assumption 1.1 (2) it is proved in [8, Lemma 4.4] that

\[
\frac{1}{n} \tilde{\kappa}_n \leq \tilde{F} \leq c
\]

in \( N \) or equivalently, \( \kappa_i \geq c \) in \( \mathbb{H}^{n+1} \). The proof of uniform boundedness of \( \kappa_i \) from above is similar to those in [7, Theorem 6.6]. Since \( \tilde{F} \) is concave, we may first apply the Krylov-Safonov and then the parabolic Schauder estimates to obtain the desired a priori estimates. □

In view of Lemma 6.1, 6.2, 6.5 and 6.6, the flow (5.1) shrinks in finite time to a point \( x_0 \).

7. The rescaled flow

In view of Lemma 6.2 and 6.5 we can choose \( \delta > 0 \) small and define

\[
t_\delta = T^* - \delta,
\]

such that

\[
\hat{M}(t_\delta) \subset B_{8\rho_-(t_\delta)}(x_0) \quad \forall x_0 \in \hat{M}(t_\delta),
\]

and

\[
8\rho_-(t_\delta) \leq 8c\Theta(t_\delta, T^*) < 1.
\]

Fix now a \( t_0 \in (t_\delta, T^*) \) and let \( B_{\rho_-(t_0)}(x_0) \) be an inball of \( \hat{M}(t_0) \). Choose \( x_0 \)

to be the center of a geodesic polar coordinate system, then the hypersurfaces \( M(t) \) can be written as graphs

\[
M(t) = \text{graph } u(t, \cdot) \quad \forall t_\delta \leq t \leq t_0,
\]

such that

\[
\rho_-(t_0) \leq u(t_0) \leq u(t) \leq 1.
\]

7.1. Lemma. Let

\[
\chi = \frac{v}{\sinh u} \equiv v\eta(r),
\]

if \( \chi_i = 0 \), then \( u_i = 0 \).
Proof. Note that
\begin{equation}
(7.7) \quad \eta(r) = \frac{1}{\sinh r}
\end{equation}
solves the equation
\begin{equation}
(7.8) \quad \dot{\eta} = -\frac{\bar{H}}{r} \eta,
\end{equation}
hence the proof is same as those in [7, Lemma 7.1]. \qed

Similar to [7, Lemma 7.2, Corollary 7.3] we obtain
\begin{itemize}
\item [7.2. Lemma] There exists a uniform constant \(c > 0\) such that
\begin{equation}
(7.9) \quad \Theta(t, T^*) F \leq c \quad \forall t \in [t_\delta, T^*),
\end{equation}
and that the rescaled principal curvatures \(\bar{k}_i = k_i \Theta\) satisfy
\begin{equation}
(7.10) \quad \bar{k}_i \leq c \quad \forall t \in [t_\delta, T^*).
\end{equation}
\end{itemize}
\begin{itemize}
\item [7.3. Lemma] Let \(t_1 \in [t_\delta, T^*)\) be arbitrary and let \(t_2 > t_1\) be such that
\begin{equation}
(7.11) \quad \Theta(t_2, T^*) = \frac{1}{2} \Theta(t_1, T^*).
\end{equation}
Let \(x_0 \in \hat{M}(t_2)\) be the center of an geodesic inball and introduce polar coordinates around \(x_0\) and write the hypersurface \(M(t)\) as graphs
\begin{equation}
(7.12) \quad M(t) = \text{graph} u(t, \cdot).
\end{equation}
Define \(\vartheta\) by
\begin{equation}
(7.13) \quad \vartheta(r) = \sinh r,
\end{equation}
and
\begin{equation}
(7.14) \quad \varphi = \int_{r_2}^{u} \vartheta^{-1},
\end{equation}
where \(r_2 = \Theta(t_2, T^*)\). Then \(\varphi(t, \cdot)\) is uniformly bounded in \(C^2(\mathbb{S}^n)\) for any \(t_1 \leq t \leq t_2\) independent of \(t_1, t_2\). Furthermore, let \(\Gamma_{ij}^k\) and \(\tilde{\Gamma}_{ij}^k\) be the Christoffel symbols of the metrics \(g_{ij}\) and \(\sigma_{ij}\) respectively, then the tensor \(\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k\) is also uniformly bounded independent of \(t_1, t_2\).
\end{itemize}
\begin{itemize}
\item [Proof] As in [7, Lemma 7.4], we conclude from Lemma 6.2 and Lemma 6.5 that there exists a uniform constant \(c > 1\), independent of \(t_1, t_2\), such that
\begin{equation}
(7.15) \quad c^{-1} \Theta(t_2, T^*) \leq u(t, \xi) \leq c \Theta(t_2, T^*) \quad \forall t \in [t_1, t_2].
\end{equation}
Note that
\begin{equation}
(7.16) \quad \varphi = \{\log \sinh(\frac{u}{r_2}) - \log \cosh(\frac{u}{r_2})\}_{r_2}^{u},
\end{equation}
thus we derive the \(C^0\)-estimates
\begin{equation}
(7.17) \quad |\varphi| \leq \log c.
\end{equation}
\end{itemize}
As in the proof of [7, Lemma 7.5], an upper bound for the principal curvatures of the slices \( \{ x^0 = \text{const} \} \) intersecting \( M(t) \) satisfies
\[
(7.18) \quad \bar{\kappa} \leq \frac{\sup \cosh u(0, \cdot)}{\sinh u_{\min}} \leq \frac{c}{u_{\min}},
\]
and from [5, (2.7.83)] we infer that the uniformly boundedness of \( v \).
\[
(7.19) \quad v \leq e^{\bar{\kappa}(u_{\max} - u_{\min})} \leq e^{c(u_{\max})},
\]
concluding further that
\[
(7.20) \quad |D\varphi|^2 = v^2 - 1 \leq c.
\]
Define
\[
(7.21) \quad \bar{\varphi}^i = \sigma^{ik} \varphi_k,
\]
where
\[
(7.22) \quad \varphi^i = \sigma_{\ell}^i \varphi^\ell.
\]
Due to the boundedness of \( v \) the metrics \( \bar{g}_{ij} \) and \( \sigma_{ij} \) are equivalent, thus we can raise the indices of \( \varphi_{ij} \) by \( \bar{g}_{ij} \) and by employing the relation [6, (3.26)]
\[
(7.23) \quad h^i_j = v^{-1} \vartheta^{-1} \left\{ -\sigma^{jk} + v^{-2} \varphi^i \varphi^j \right\} \varphi_{jk} + \dot{\vartheta} \delta^i_j,
\]
we infer
\[
(7.24) \quad \bar{g}^{ik} \varphi_{jk} = -v \vartheta h^i_j + \dot{\vartheta} \delta^i_j,
\]
concluding further from (7.10)
\[
(7.25) \quad \| \varphi_{ij} \|^2 \leq c(v^2 \vartheta^2 |A|^2 + n \dot{\vartheta}^2)
\]
is bounded from above for all \( t \in [t_1, t_2] \). We choose coordinates such that \( \bar{F}^i_{ij} \) in a fixed point vanishes. Denote the covariant derivative with respect to \( \sigma_{ij} \) by a colon. In such coordinates
\[
(7.26) \quad \Gamma^k_{ij} = \frac{1}{2} g^{km}(g_{mi,j} + g_{mj,i} - g_{ij,m}).
\]
From
\[
(7.27) \quad g^{ij} = \vartheta^{-2} \bar{g}^{ij}
\]
we compute
\[
(7.28) \quad g^{km} g_{m;i,j} = \bar{g}^{km} (\varphi_{m;j} \varphi^i + \varphi_{i;j} \varphi^m + 2 \cosh u \varphi^m \varphi_i + \sigma_{m;i} \varphi^m).
\]
Using the estimates for \( \varphi \) proved before, we conclude that \( \Gamma^k_{ij} - \bar{F}^k_{ij} \) are uniformly bounded independent of \( t_1 \) and \( t_2 \).

Define a new time parameter as
\[
(7.29) \quad \tau = - \log \Theta,
\]
then
\[
(7.30) \quad \frac{dt}{d\tau} = \Theta \frac{\sinh \Theta}{\cosh \Theta}.
\]
In the following we denote the differentiation with respect to \( t \) by a dot and differentiation with respect to \( \tau \) by a prime.

7.4. **Lemma.** The rescaled quantity \( \tilde{F} = F \Theta \) satisfies the inequality

\[
\sup_{M(t_1)} \tilde{F} \leq c \inf_{M(t_2)} \tilde{F}
\]

with a uniform constant \( c > 0 \).

**Proof.** \( \tilde{F} \) satisfies the equation

\[
\tilde{F}' = F \Theta^2 \frac{\sinh \Theta}{\cosh \Theta} - \tilde{F},
\]

and from the evolution equation of \( F \) in [7, (2.8)] we conclude further

\[
\tilde{F}' + \tilde{F} - \{ F^{ij} F_{ij} + F^{ij} h_ik h_j^k F + K_N F^{ij} g_{ij} F \} \Theta^2 \frac{\sinh \Theta}{\cosh \Theta} = 0.
\]

We consider the non-trivial term in (7.33)

\[
\tilde{F}' - \{ F^{ij} F_{ij} \Theta^2 \frac{\sinh \Theta}{\cosh \Theta} \}
\]

In view of (7.27), the pinching estimate and the boundedness of \( v, \Theta^2 F^{ij} \) and \( \sigma^{ij} \) are equivalent and hence uniformly positive definite. Furthermore,

\[
F_{ij} = \bar{F}_{ij} - \{ \Gamma^k_{ij} - \bar{\Gamma}^k_{ij} \} \bar{F}_k.
\]

Hence we conclude from Lemma 7.3 that \( \tilde{F} \) satisfies a uniform parabolic equation of the form

\[
\tilde{F}' - a^{ij} \tilde{F}_{ij} + b^i \tilde{F}_i + c \tilde{F} = 0
\]

in the cylinder \([\tau_1, \tau_2] \times S^n\), where \( \tau_1 = - \log \Theta(t, T^*) \), with uniformly bounded coefficients. The statement follows then from the parabolic Harnack inequality. □

7.5. **Corollary.** The rescaled principal curvatures \( \tilde{\kappa}_i = \kappa \Theta \) are uniformly bounded from below.

**Proof.** Consider a point \((t, \xi)\) in \( M(t) \) such that

\[
u(t, \xi) = \sup_{M(t)} u.
\]

In view of [5, (1.5.10)], it holds in \((t, \xi)\)

\[
h_{ij} \geq \bar{h}_{ij}, \quad g_{ij} = \bar{g}_{ij}, \quad \kappa_i \geq \bar{\kappa}_i = \frac{\cosh u}{\sinh u}.
\]

where we denote the quantity of the slices \( \{ x^0 = \text{const} \} \) with a bar. In view of (7.15)

\[
\sup_{M(t)} \tilde{F} \geq F(\tilde{\kappa}_i(t, \xi)) \geq F \left( \frac{\cosh u(t, \xi)}{\sinh u(t, \xi)} \Theta(t, T^*) \right) \geq c > 0.
\]

The statement follows from the pinching estimates and Lemma 7.4. □
Let \( x_0 \in \mathbb{H}^{n+1} \) be the point the flow hypersurfaces are shrinking to and introduce geodesic polar coordinates around it. Write \( M(t) = \text{graph } u(t, \cdot) \) and let
\[
\tilde{u}(\tau, \xi) = u(t, \xi) \Theta(t, T^* \tau)^{-1},
\]
\[
\tau_\delta = -\log \Theta(t_\delta, T^* \tau), \quad Q(\tau_\delta, \infty) = [\tau_\delta, \infty) \times S^n.
\]
Using the same argument as in \[7, Lemma 7.9, Lemma 7.10\] we conclude that

7.6. Lemma. The quantities \( v \) and \( |D\tilde{u}| \) are uniformly bounded from above and \( \tilde{u} \) is uniformly bounded from below and above in \( Q(\tau_\delta, \infty) \). \( \square \)

Let
\[
\varphi = -\int_0^{\Theta(0, T^* \tau)} \vartheta^{-1},
\]
then
\[
\varphi_i = \vartheta^{-1} u_i, \quad \varphi_{ij} = \vartheta^{-1} u_{ij} - \cosh u \vartheta^{-2} u_i u_j,
\]
and
\[
\vartheta^{-2} |D^2 u|^2 + |D\tilde{u}|^4 \cosh^2 u - 2\vartheta^{-1} |D^2 u||D\tilde{u}|^2 \cosh u \leq |D^2 \varphi|^2.
\]
Since \( |D^2 \varphi| \) and \( |D\tilde{u}| \) are bounded, we conclude that the \( C^2 \)-norm of \( \tilde{u} \) is uniformly bounded, where the covariant derivatives of \( \tilde{u} \) and \( \varphi \) are taken with respect to \( \sigma_{ij} \). From \[5, Remark 1.5.1, Lemma 2.7.6\] we conclude that
\[
\frac{\sinh \Theta}{\cosh \Theta} F v = \Phi(x, \tau, \tilde{u}, \tilde{u} e^{-\tau}, D\tilde{u}, D^2 \tilde{u}),
\]
where \( \Phi \) is a smooth function with respect to its arguments, and
\[
\Phi^{ij} = \frac{\partial \Phi}{\partial (-\tilde{u}_{ij})} = \frac{\sinh \Theta}{\cosh \Theta}, \quad \Phi^{ij, kl} = \frac{\partial \Phi}{\partial (-\tilde{u}_{ij})}. 
\]
Hence by applying first the Krylov and Safonov, then the Schauder estimates, we deduce (cf. \[5, Remark 2.6.2\])

7.7. Theorem. The rescaled function \( \tilde{u} \) satisfies the uniformly parabolic equation
\[
\tilde{u}' = -\Phi + \tilde{u}
\]
in \( Q(\tau_\delta, \infty) \) and \( \tilde{u}(\tau, \cdot) \) satisfies a priori estimates in \( C^\infty(S^n) \) independently of \( \tau \).
8. Convergence to a sphere

The aim of this section is to prove that \( \tilde{u} \) converges exponentially fast to the constant function 1 if \( F \) is strictly concave or \( F = \frac{1}{n} H \). Comparing the proof in [7, Section 8], we should handle a term stemming from the negative curvature of the ambient space \( K_N < 0 \).

8.1. Lemma. There exists a positive constant \( C \) such that

\[
(8.1) \quad F^{kl}g_{kl}|A|^2 - FH \leq C \sum_{i<j} (\kappa_i - \kappa_j)^2.
\]

Proof. The proof is similar to [7, Lemma 8.2]. Let

\[
(8.2) \quad \varphi = F^{kl}g_{kl}|A|^2 - FH.
\]

Denote the partial derivatives of \( \varphi \) with respect to \( \kappa_i \) by \( \varphi_i \), then

\[
(8.3) \quad \varphi_j = \sum_{i=1}^{n} F_{ij}|A|^2 + \sum_{i=1}^{n} 2F_{ij}\kappa_j - F_j H - F,
\]

\[
\varphi_{jk} = \sum_{i=1}^{n} F_{ijk}|A|^2 + \sum_{i=1}^{n} 2F_{ij}\kappa_k + \sum_{i=1}^{n} 2F_{ik}\kappa_j + 2\delta_{jk} \sum_{i=1}^{n} F_i - F_{jk}H - F_j - F_k.
\]

Therefore

\[
(8.5) \quad \varphi(\kappa_1, \ldots, \kappa_n) = 0, \quad \varphi_j(\kappa_1, \ldots, \kappa_n) = 0 \quad \forall j = 1, \ldots, n.
\]

by using the Euler’s homogeneous relation and the normalization (1.6). Furthermore, \( \varphi_{jk} \) are uniformly bounded from above, since \( \varphi_{jk} \) are homogeneous of grad 0 and \( \frac{\kappa_i}{|A|} \) are compactly contained in the defining cone. The statement follows by an argument using Taylor’s expansion up to the second order similar to those in [7, Lemma 8.2]. \( \square \)

We want to estimate the function

\[
(8.6) \quad f_\sigma = F^{-\alpha}(|A|^2 - nF^2),
\]

where

\[
(8.7) \quad \alpha = 2 - \sigma,
\]

and \( 0 < \sigma < 1 \) small. For simplicity we drop the subscript \( \sigma \) of \( f_\sigma \). In the following we always assume that \( F \) satisfies the assumption 1.1.

By Lemma 8.1 we have the following inequality corresponding to [7, Lemma 8.3].
8.2. Lemma. Let $F$ be strictly concave, then there exist uniform constants $\epsilon > 0$ and $C > 0$, such that

\begin{equation}
-F^{ij} f_{ij} + 2^2 F^{ij} h^k_i h^j_k f \leq \alpha F^{-1} F^{ij} f_{ij} f + 2(\alpha - 1) F^{-1} F^{ij} f_{ij} f - 2 \{h^{ij} - F_n F^{ij}\} F^{-\alpha} f_{ij} - 2c^2 |DA|^2 F^{-\alpha} + 2Cf.
\end{equation}

Corresponding to [7, Lemma 8.5] we have

8.3. Lemma. Let $F$ be strictly concave, then there exist positive constants $C$ and $c$ such that for any $\delta > 0$ and any $0 \leq t < T^*$

\begin{equation}
\epsilon^2 \int_M F^{ij} h^k_i h^j_k f^p \leq \{\delta^{-1} c(p - 1) + c\} \int_M F^{ij} f_{ij} f^{p-2} + \{\delta c(p - 1) + c\} \int_M |DA|^2 F^{-\alpha} f^{p-1} + 2C \int_M f^p.
\end{equation}

Parallel to [7, Lemma 8.6] we have

8.4. Lemma. Let $F$ be strictly concave, then there exist $C_1 > 0$ and $\sigma_0 > 0$ such that for all

\begin{equation}
P \geq 4c\epsilon^{-2}, \quad \sigma \leq \min\left(\frac{1}{4} c^{-1} \epsilon^3 p^{-1/2}, \sigma_0\right),
\end{equation}

the estimate

\begin{equation}
\|f\|_{p, M} \leq C_1 \quad \forall t \in [0, T^*)
\end{equation}

holds, where $C_1 = C_1(M_0, p)$ and $\sigma_0 = \sigma_0(F, M_0)$.

Proof. Multiply [7, (8.30)] with $pf^{p-1}$ and integrate by parts, and note that

\begin{equation}
d\mu_t = \mu_t \, dx \text{ on } M_t,
\end{equation}

where

\begin{equation}
\frac{d}{dt} \mu_t = \frac{d}{dt} \sqrt{g} = \frac{1}{2} \mu_t g^{ij} g_{ij} = -FH\mu_t,
\end{equation}

thus

\begin{equation}
\frac{d}{dt} \int_M f^p = p \int_M f^{p-1} f' - \int_M HF f^p,
\end{equation}

and

\begin{equation}
\frac{d}{dt} \int_M f^p + \frac{1}{2} p(p - 1) \int_M F^{ij} f_{ij} f^{p-2} + \epsilon^2 p \int_M |DA|^2 F^{-\alpha} f^{p-1} \leq \sigma p \int_M F^{ij} h^k_i h^j_k f^p + 4Cp \int_M f^p.
\end{equation}

By choosing

\begin{equation}
c_0 = \frac{1}{4} c, \quad \sigma \leq \min(\epsilon^3 p^{-1/2} c_0^{-1}, \sigma_0), \quad \delta = \epsilon p^{-1/2},
\end{equation}

the estimate

\begin{equation}
\|f\|_{p, M} \leq C_1 \quad \forall t \in [0, T^*)
\end{equation}

holds.
and by using (8.9), the right-hand side of inequality (8.15) can be estimated from above by

\[
\begin{align*}
\epsilon_p^{1/2} c_0^{-1} \left( \epsilon^2 \int_M F^{ij} h_k h_j f^p \right) + 4Cp \int_M f^p \\
\leq \epsilon_p^{1/2} c_0^{-1} \left\{ \delta^{-1} c(p - 1) + \epsilon \right\} \int_M F^{ij} f_i f_j f^{p-2} \\
+ \epsilon_p^{1/2} c_0^{-1} \left\{ \delta c(p - 1) + \epsilon \right\} \int_M |DA|^2 F^{-\alpha} f^{p-1} + \{ 2C\epsilon_p^{1/2} c_0^{-1} + 4Cp \} \int_M f^p \\
= c_0^{-1} \{ p(p - 1)c + \epsilon_p^{1/2} c \} \int_M F^{ij} f_i f_j f^{p-2} \\
+ c_0^{-1} \{ \epsilon^2 (p - 1)c + \epsilon_p^{1/2} c \} \int_M |DA|^2 F^{-\alpha} f^{p-1} + \{ 2C\epsilon_p^{1/2} c_0^{-1} + 4Cp \} \int_M f^p \\
\leq \frac{1}{4} p(p - 1) \int_M F^{ij} f_i f_j f^{p-2} + \frac{1}{2} \epsilon^2 (p - 1) \int_M |DA|^2 F^{-\alpha} f^{p-1} + 5Cp \int_M f^p.
\end{align*}
\]

From (8.15), (8.17) we conclude that

\[
\begin{align*}
\frac{d}{dt} \int_M f^p & \leq 5Cp \int_M f^p, \\
\int_M f^p & \leq \int_M f^p 
|_{t=0} \cdot \exp(5CpT^*), \\
\| f \|_{p} & = \left( \int_M f^p \right)^{\frac{1}{p}} \leq e^{5CpT^*} (|M_0| + 1) \sup_{0 \leq \sigma \leq 1/2} \sup_{M_0} f_\sigma.
\end{align*}
\]

To proceed further, we use the Stampacchia iteration scheme as in the Huisken’s paper [10, Theorem 5.1], as well as [11, Theorem 5.1]. Note that \( \mathbb{H}^{n+1} \) is simply connected and has constant sectional curvature \( K_N = -1 \), thus the Sobolev inequality in [9, Theorem 2.1] has the form

\[
8.5. \textbf{Lemma.} \text{ Let } v \text{ be a nonnegative Lipschitz function on } M, \text{ then there exists a constant } c = c(n) > 0, \text{ such that}
\]

\[
\left( \int_M |v|^{\frac{n}{n-1}} \right)^{n-1} \leq c \int_M |Dv| + \int_M H|v|.
\]

Corresponding to [7, Theorem 8.7], we have

\[
8.6. \textbf{Theorem.} \text{ Let } F \text{ be strictly concave or } F = \frac{1}{n} H, \text{ then there exist constants } \delta > 0 \text{ and } c_0 > 0, \text{ such that}
\]

\[
|A|^2 - nF^2 \leq c_0 F^{2-\delta}.
\]
Proof. As in the proof of [10, Theorem 5.1] let $f_{\sigma,k} = \max(f_{\sigma} - k, 0)$ for all $k \geq k_0 = \sup_{M_0} f_{\sigma}$ and denote by $A(k)$ the set where $f_{\sigma} > k$. We obtain with $v = f_{\sigma,k}^{p/2}$ for $p \geq 4e^{-2}$,

$$\frac{d}{dt} \int_{A(k)} v^2 + \int_{A(k)} |Dv|^2 \leq \sigma p \int_{A(k)} H^2 f_{\sigma}^p + 5Cp \int_{A(k)} f_{\sigma}^p \leq C(p) \int_{A(k)} H^2 f_{\sigma}^p.$$  \hfill (8.23)

By applying Lemma 8.5 we can bound $f_{\sigma}$ for $\sigma$ small as in the proof of [10, Theorem 5.1]. The case $F = \frac{1}{n}H$ is proved in [11, Lemma 5.1].

8.7. Lemma. Let $F$ be strictly concave or $F = \frac{1}{n}H$ and $\tilde{M}(\tau)$ be the rescaled hypersurfaces, then there are constants $c, \delta > 0$ such that

$$\int_{\tilde{M}} |D\tilde{A}|^2 \leq ce^{-\delta \tau} \quad \forall \tau_0 \leq \tau < \infty,$$  \hfill (8.24)

where

$$\tau_0 = -\log \Theta(0,T^*), \quad |D\tilde{A}|^2 = \Theta^2 g^{ij} h^k_{ij} \Theta h^l_{k;ij} \Theta.$$  \hfill (8.25)

Proof. Choose

$$f = F^{-2} \{ |A|^2 - nF^2 \}. \hfill (8.26)$$

From Theorem 8.6 we infer

$$f \leq c_0 F^{-\delta} \leq c \Theta^\delta = ce^{-\delta \tau} \quad \forall \tau \geq \tau_0,$$  \hfill (8.27)

and from Theorem 7.7 we obtain

$$|D^m A| \leq c |A| \quad \forall m \geq 1.$$  \hfill (8.28)

Integrating inequality (8.8) over $M$, using integration by parts and using relation (8.28), we infer

$$2e^2 \int_M |DA|^2 F^{-2} \leq c \int_M f.$$  \hfill (8.29)

Hence (8.24) follows by rescaling (8.29). \qed

Using the same proof of [7, Lemma 8.10] we have

8.8. Lemma. There are positive constants $c$ and $\delta$ such that for all $\tau \geq \tau_0$

$$\tilde{F}_{\text{max}} - \tilde{F}_{\text{min}} \leq ce^{-\delta \tau},$$  \hfill (8.30)

and

$$\|D\tilde{F}\| \leq ce^{-\delta \tau}.$$  \hfill (8.31)
8.9. Lemma. There are positive constants $c$ and $\delta$ such that for all $\tau \geq \tau_0$

\begin{equation}
|\tilde{D}u| \leq ce^{-\delta \tau},
\end{equation}

where

\begin{equation}
|\tilde{D}u|^2 = \sigma^{ij} \tilde{u}_i \tilde{u}_j.
\end{equation}

Proof. As in the proof of [7, Lemma 8.12], we let

\begin{equation}
\varphi = \log \tilde{u}, \quad w = \frac{1}{2}|D\varphi|^2,
\end{equation}

then

\begin{equation}
\varphi' = -e^{-\varphi} \tilde{F} \Theta^{-1} \frac{\sinh \Theta}{\cosh \Theta} v + 1.
\end{equation}

Differentiate now (8.35) with respect to $\varphi^k D_k$ we obtain

\begin{equation}
w' = 2e^{-\varphi} w \tilde{F} \Theta^{-1} \frac{\sinh \Theta}{\cosh \Theta} v - e^{-\varphi} \tilde{F} \Theta^{-1} \frac{\sinh \Theta}{\cosh \Theta} v^{-1} \sinh^{-2} u u^2 w_k \varphi^k + R_1 + R_2,
\end{equation}

where

\begin{equation}
R_1 = -e^{-\varphi} \frac{\sinh \Theta}{\cosh \Theta} F k \varphi^k,
\end{equation}

\begin{equation}
R_2 = e^{-\varphi} \tilde{F} \frac{\sinh \Theta}{\cosh \Theta} v^{-1} |D\varphi|^{-4} \sinh^{-2} u \{ u^3 \cosh u - u^2 \sinh u \} \geq 0.
\end{equation}

In view of (8.31) $R_1$ decays exponentially. Thus the function

\begin{equation}
w_{\text{max}} = \sup_{M(\tau)} w
\end{equation}

is Lipschitz and satisfies

\begin{equation}
w_{\text{max}}' \geq 2e^{-\varphi} w \tilde{F} \Theta^{-1} \frac{\sinh \Theta}{\cosh \Theta} v - ce^{-\delta \tau}
\end{equation}

for almost every $\tau \geq \tau_0$. Using the same argument as in [7, Lemma 8.12] we conclude that

\begin{equation}
w_{\text{max}}(\tau) \leq \frac{c}{\delta} e^{-\delta \tau} \quad \forall \tau \geq \tau_0.
\end{equation}

The same arguments of [7, Corollary 8.13] and the interpolation inequalities for the $C^m$-norms (cf. [6, Corollary 6.2]) yield

8.10. Theorem. Let $F$ be strictly concave or $F = \frac{1}{n} H$, then the rescaled function $\tilde{u}$ converges in $C^\infty(S^n)$ to the constant function 1 exponentially fast.

\[ \square \]

8.11. Lemma. Let $F$ be strictly concave or $F = \frac{1}{n} H$, then there exist positive constants $c$ and $\delta$ such that

\begin{equation}
|\tilde{F}(\tau, \cdot) - 1| \leq ce^{-\delta \tau} \quad \forall \tau \geq \tau_0.
\end{equation}
Proof. Observe that for \( \tau_1 \) sufficiently large we have

\[
|\frac{\sinh \Theta}{\cosh \Theta} - \Theta| \leq c \Theta^2 \quad \forall \tau \geq \tau_1.
\]

The rest of the proof is identical to [7, Lemma 8.16]. □

9. Inverse curvature flows

Let \( M(t) \) be the flow hypersurfaces of the direct flow in \( \mathbb{H}^{n+1} \) and write \( M(t) = \text{graph } u(t, \cdot) \) with respect to the geodesic polar coordinates centered in the point where the direct flow shrinks to. By applying an isometry we may assume that the point \( x_0 \) is the Beltrami point. The polar hypersurfaces \( M(t)^* \) are the flow hypersurfaces of the corresponding inverse curvature flow in the de Sitter space. Write \( M(t)^* = \text{graph } u^*(t, \cdot) \) over \( \mathbb{S}^n \).

9.1. Lemma. The functions \( u, u^* \) satisfy the relations

\[
\begin{align*}
\max u &= -\min u^* \quad \forall t \in [t_0, T^*), \\
\min u &= -\max u^* \quad \forall t \in [t_0, T^*).
\end{align*}
\]

Proof. We use the relation [5, (10.4.65)]

\[
\tilde{x}^0 = \frac{r}{\sqrt{1 - r^2}}.
\]

and note that by comparing [5, (10.2.5)] and the metric in the eigentime coordinate system in \( N(2.13) \) we infer that

\[
\cosh^2 u^* = 1 + |\tilde{x}^0|^2.
\]

From (6.8) we infer that

\[
r = \tanh u.
\]

Since we have switched the light cone such that the uniformly convex slices are contained in \( \{ \tau < 0 \} \), we deduce that

\[
\max u^* = -\arcsinh(\tilde{v} \sinh u) = -\arcsinh \tilde{v}.
\]

In a point where \( u^* \) attains its minimum, there holds \( v = 1 \) in view of Lemma 7.1. Thus \( u = -u^* \) and \( u \) attains its maximum in such a point. This proves (9.1). The proof of (9.2) is similar. □

9.2. Corollary. There exists a positive constant \( c \) such that

\[
-c \leq w \equiv u^* \Theta^{-1} \leq -c^{-1} \quad \forall t \in [t_0, T^*).
\]

Define \( \vartheta(u) = \cosh(u) \) and \( \hat{g}_{ij} = \vartheta^2 \sigma_{ij} \). We prove in the following that \( w \) is uniformly bounded in \( C^\infty(\mathbb{S}^n) \). For simplicity, we write in the following \( u \) instead \( u^* \) for the graphs of the flow hypersurfaces in the de Sitter space. The proof of \( C^1 \)-estimates of \( w \) is similar to [5, Theorem 2.7.11].
9.3. Lemma. There exists a positive constant $c$ such that
\begin{equation}
|Dw|^2 \equiv \sigma^{ij}w_iw_j \leq c \quad \forall t \in [t_\delta, T^*].
\end{equation}

Proof. Since
\begin{equation}
\|Du\|^2 \equiv g^{ij}u_iu_j = v^{-2}g^{ij}u_iu_j \equiv v^{-2}\|Du\|^2,
\end{equation}
we first estimate $\|Du\|\Theta^{-1}$. Let $\lambda$ be a real parameter to be specified later
and define
\begin{equation}
G = \frac{1}{2} \log(\|Du\|^2\Theta^{-2}) + \lambda u\Theta^{-1}.
\end{equation}
There is $x_0 \in S^n$ such that
\begin{equation}
G(x_0) = \sup_{S^n} G,
\end{equation}
and thus in $x_0$
\begin{equation}
0 = G_i = \|Du\|^{-2}u_iu^j + \lambda u_i\Theta^{-1},
\end{equation}
where the covariant derivatives are taken with respect to $g_{ij}$ and
\begin{equation}
u^i = g^{ij}u_j = v^{-2}g^{ij}u_j.
\end{equation}
Since
\begin{equation}
h_{ij}v^{-1} = -u_{ij} - \dot{\vartheta}\sigma_{ij},
\end{equation}
we infer that
\begin{equation}
\lambda\|Du\|^{-4}\Theta^{-4} = -u_{ij}w^iw^j\Theta^{-3}
\end{equation}
\begin{equation}
= v^{-1}h_{ij}w^iw^j\Theta^{-3} + \dot{\vartheta}\sigma_{ij}w^iw^j\Theta^{-3}.
\end{equation}
By considering the dual flow in the hyperbolic space, we conclude that $h_{ij} > 0$. Furthermore,
\begin{equation}
\dot{\vartheta}\sigma_{ij}w^iw^j\Theta^{-3} = (\dot{\vartheta}\Theta^{-1})\vartheta^{-1}v^{-2}\|Du\|^2\Theta^{-2}.
\end{equation}
By applying [5, Theorem 2.7.11] directly, we conclude that $v^{-2}$ is uniformly bounded. Note $\vartheta\Theta^{-1} \leq c$. Let $c_0$ be an upper bound for $(\dot{\vartheta}\Theta^{-1})\vartheta^{-1}v^{-2}$ and by choosing $\lambda < -c_0$ we conclude that $\|Du\|\Theta^{-1}$ can not be too large in $x_0$. Thus $\|Du\|\Theta^{-1}$ is uniformly bounded from above. We conclude that
\begin{equation}
\sigma^{ij}w_iw_j = \|Du\|^2\Theta^{-2}\vartheta^2v^2
\end{equation}
is uniformly bounded. \hfill \Box

9.4. Lemma. There exists a positive constant $c$ such that for all $m \geq 2$
\begin{equation}
|D^m w|^2 \leq c \quad \forall t \in [t_\delta, T^*].
\end{equation}
Proof. Let \((\hat{h}^{ij}) = (h_{ij})^{-1}\) be the inverse of the second fundamental form in \(H^{n+1}\) and \(\bar{h}_{ij}\) the second fundamental form in \(N\). We consider the mixed tensor
\[
(9.19) \quad \hat{h}^j_i = g_{ik} \hat{h}^k_j, \quad \bar{h}^j_i = \bar{g}^{kj} \bar{h}_{ki},
\]
where \(g_{ij}\) and \(\bar{g}_{ij} = h^k_i h_{kj}\) are the metrics of hypersurfaces in \(H^{n+1}\) resp. \(N\).

From the relation
\[
(9.20) \quad \bar{\kappa}_i = \kappa_i^{-1},
\]
we infer that
\[
(9.21) \quad \bar{h}^j_i = \hat{h}^j_i.
\]
From Theorem 7.7 we infer that \(h^j_i \Theta\) are uniformly bounded in \(C^\infty(S^n)\) and due to Lemma 7.2 and Corollary 7.5 there are constants \(c_1, c_2 > 0\) such that
\[
(9.22) \quad 0 < c_1 \delta_j^i \leq h^j_i \Theta \leq c_2 \delta_j^i,
\]
and thus \(\hat{h}^j_i \Theta^{-1} = \bar{h}^j_i \Theta^{-1}\), as the inverse of \(h^j_i \Theta\), are uniformly bounded in \(C^\infty(S^n)\). We switch now our notation by considering the quantities in \(N\) without writing a tilde. Denote the covariant derivatives with respect to \(\bar{g}^{ij}\) resp. \(\sigma^{ij}\) by a semicolon resp. a colon. In view of [5, Remark 1.6.1, Lemma 2.7.6] we have
\[
(9.23) \quad v^{-1} h_{ij} = -v^{-2} u_{ij} - \dot{\theta} \partial \sigma_{ij}
\]
\[
= -v^{-2} \{ u_{ij} - \frac{1}{2} g^{km} ((\dot{\theta})^j m \sigma_{mj} + (\dot{\theta})^i k \sigma_{mj} - (\dot{\theta})^i m \sigma_{mj}) u_k \} - \dot{\theta} \partial \sigma_{ij}.
\]
Therefore,
\[
(9.24) \quad u_{ij} = -v h_{ij} + 2 \dot{\theta}^{-1} \dot{\theta} u_i u_j - \dot{\theta} \partial \sigma_{ij},
\]
By considering the dual flow in hyperbolic space, we infer that
\[
(9.25) \quad |A| \Theta^{-1} \leq c,
\]
and note that
\[
(9.26) \quad \bar{g}^{ij} \leq \bar{g}^{ij} + v^{-2} \bar{a}^i \bar{a}^j = g^{ij},
\]
where
\[
(9.27) \quad \bar{a}^i = \bar{g}^{ij} u_j,
\]
we conclude that
\[
(9.28) \quad \sigma^{ik} \sigma^{jl} h_{ij} h_{kl} \leq c |A|^2.
\]
In view of \(\bar{\theta} \Theta^{-1} \leq c\) we conclude that \(|D^2 w|^2\) is uniformly bounded. Contract (9.24) with \(g^{ij}\) we conclude further
\[
(9.29) \quad - g^{ij} w_{ij} - \bar{g}^{-2} \bar{g} \bar{\theta} v^{-2} |Dw|^2 + v H \Theta^{-1} + n \bar{\theta}^{-1} \bar{\theta} \Theta^{-1} = 0.
\]
Since \( v \) is uniformly bounded, (9.29) is a uniformly elliptic equation in \( w \) with bounded coefficients. A bootstrapping procedure with Schauder theory yields for all \( m \in \mathbb{N} \)
\[
|w|_{m,S^n} \leq c_m \quad \forall t \in [0, T^*)
\]
\( \square \)

From Lemma 8.10 and preceding results in Section 9 we conclude

9.5. Theorem. Let the geodesic polar coordinates \((\tau, \xi)\) of \( N \) be specified in Section 2. Represent the inverse curvature flow (1.5) in \( N \) as graphs over \( S^n \), \( M(t)^* = \text{graph } u^*(t,\cdot) \), where the curvature function \( \tilde{F} \) satisfies the assumption 1.1. Then \( u^* \) converges to the constant function 0 in \( C^\infty(S^n) \). The rescaled function \( w = u^*\Theta^{-1} \) are uniformly bounded in \( C^\infty(S^n) \). When the curvature function \( F \) of the corresponding contracting flow is strictly concave or \( F = \frac{1}{n}H \), then \( w(\tau,\cdot) \) converges in \( C^\infty(S^n) \) to the constant function \(-1\) exponentially fast. \( \square \)

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