Existence and Hyers–Ulam Stability of Solutions for a Mixed Fractional-Order Nonlinear Delay Difference Equation with Parameters

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This paper focuses on a kind of mixed fractional-order nonlinear delay difference equations with parameters. Under some new criteria and by applying the Brouwer theorem and the contraction mapping principle, the new existence and uniqueness results of the solutions have been established. In addition, we deduce that the solution of the addressed equation is Hyers–Ulam stable. Some results in the literature can be generalized and improved. As an application, three typical examples are delineated to demonstrate the effectiveness of our theoretical results.

1. Introduction

Fractional differential equations arise naturally in promoting contemporary mathematics development, see [1–14]. Fractional differential theories have been widely applied, especially in dynamics mechanics, heat energy, automation, medicine, traffic signal, and communication engineering. Sometimes differential equations are discretized in order to approximate their solutions. In the past ten years, there has been a significant increase in the quantity of research in discrete fractional calculus. For an extensive collection of more knowledge in this field, we refer the readers to [15–27] and the references therein. In [16], Goodrich investigated a discrete fractional boundary value problem (FBVP) of the form

\[
\begin{cases}
-\Delta^\gamma y(t) = f(t + \nu - 1, y(t + \nu - 1)), \\
y(\nu - 2) = g(y), \\
y(\nu + b) = 0,
\end{cases}
\]

\hspace{1cm} (1)

where \( t \in [0, b]_{\mathbb{N}_0} = \{0, 1, \ldots, b\} \) and \( f: [\nu - 2, \nu + b - 1]_{\mathbb{N}_0} \times \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function, and we let \( g \in C([\nu - 2, \nu + b]_{\mathbb{N}_0}, \mathbb{R}) \) is a given functional, and \( 1 < \nu \leq 2 \). Using the contraction mapping theorem, the Brouwer theorem, and the Krasnosel’skii theorem, the author proved the existence and uniqueness of solution to this problem.

For a fractional difference system, we not only study the existence and uniqueness of its solution but also investigate the stability. The famous Hyers–Ulam problem goes back to the years 1940-1941 when Ulam [28] and Hyers [29] firstly proposed this issue. Many mathematicians had considered the wide scope of this same problem for fractional equations of different types. Such problem may be found in [25, 30–35] and other papers. In [25], the authors firstly considered the following antiperiodic boundary value problem:

\[
\begin{aligned}
\Delta^\alpha_x(t) &= f(t + \alpha - 1, x(t + \alpha - 1)), & t \in [0, b]_{\mathbb{N}_0}, & 1 < \alpha < 2, \\
x(\alpha - 1) + x(b + \alpha) &= 0, \\
\Delta x(\alpha - 1) &= 0, \\
\Delta x(b + \alpha - 1) &= 0,
\end{aligned}
\]

\hspace{1cm} (2)
where $\Delta^n_C$ is a Caputo fractional difference operator, $N_a = [a, a + 1, a + 2, \ldots]$, and $f: [a - 1, b + a]_{N_a} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with respect to the second variable, when $f(\cdot, x(\cdot))$ satisfies Lipschitz condition, that is to say, there exists a constant $L > 0$ such that $|f(t, x) - f(t, y)| \leq L|x - y|$ for each $t \in [a - 1, b + a]_{N_a}$. Then, the boundary value problem (2) has a unique solution if

$$L \leq \frac{3\Gamma(b + a + 1)}{2\Gamma(a + 1)\Gamma(b + 1)} + \frac{b + a}{2a\Gamma(a)} \left( \Gamma(b + a) - \Gamma(a) \right)$$

(3)

holds. Secondly, the authors researched the Hyers–Ulam stability of solutions for the following boundary value problem:

$$\begin{align*}
\Delta^n_a x(t) &= f(t + a - 1, x(t + a - 1)), \quad t \in [0, b]_{N_a}, 1 < a < 2, \\
x(a - 1) &= y(a - 1), \\
x(b + a) &= y(b + a),
\end{align*}$$

(4)

where $y$ is a solution of inequality $|\Delta^n_a y(t) - f(t + a - 1, y(t + a - 1))| \leq \varepsilon$, and let $x$ be a solution of boundary value problem (4). Then, the solution is the Hyers–Ulam stable provided that

$$L \leq \frac{\Gamma(a + 1)\Gamma(b + 1)}{2\Gamma(b + a + 1)}$$

(5)

However, the nonlinear terms in (1), (2), and (4) are too simple to portray the development of things well. We adopt mixed fractional equation which makes the model more generalized, such as in [36–38]. Inspired by the abovementioned articles, in this paper, we are concerned with the existence, uniqueness, and Hyers–Ulam stability of solutions for the following discrete fractional equation:

$$\Delta^\mu y(t) + \lambda f(t + v - 1, \eta y(t + v - 1), \rho \Delta^\beta y(t)) = 0,$$

(6)

where $\lambda$, $\eta$, and $\rho$ are positive real numbers, and $t \in [0, m + 1]_{N_a} = [0, 1, \ldots, m + 1]$, $m \in \mathbb{N}$ is given, $f: [\nu - 1, \nu + m]_{N_a} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $\Delta^\mu$ denotes the fractional difference operator of order $\mu$, and $\Delta^\beta$ is the $\beta$th fractional sum operator, $\nu, \beta \in (1, 2]$ are given. Equation (6) is an important parameters system and also is a mixed fractional system, which is quite different from (1) and (2), and it can enrich the description of the mathematical model. Besides, the most interesting thing is that, in Theorem 2, we find the necessary conditions are only dependent on $\lambda$ (independent on $\eta$ and $\rho$).

Compared with some new achievements in the articles, such as [16–18, 25], the major contributions of our research contain at least the following three:

1. The Hyers–Ulam stability is introduced into the mixed fractional order nonlinear difference equation.
2. The model we are concerned with is more generalized, and some ones in the articles are the special cases of it. Moreover, we provide more ecumenical boundary value conditions in researching Hyers–Ulam stability of solutions for the fractional difference equation. Thus, the comprehensive model is originally discussed in the present paper.
3. A ground-breaking approach based on contraction mapping and the Brouwer theorem is utilized to discuss the existence and uniqueness of the solutions for the mixed fractional-order difference equation. The results established are essentially new.

The following article is organized as follows. In Section 2, we will recall some known results for our consideration. Some lemmas and definitions are useful to our works. Section 3 is devoted to researching the existence and uniqueness of solutions for equation (6). In Section 4, we will investigate the Hyers–Ulam stability of this fractional order difference equation, and then we will come up with the main theorem. To explain the results clearly, we finally provide three examples in Section 5.

2. Preliminaries

In this section, we plan to introduce some basic definitions and lemmas which are useful throughout this paper.

**Definition 1** (see [16, 18]). We define

$$t^\gamma := \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \nu)}$$

(7)

as for any $t$ and $\nu$ for which the right-hand side is defined. Here and in what follows, $\Gamma$ denotes the Gamma function. We also appeal to the common convention that if $t + 1 - \nu$ is a pole of the gamma function and $t + 1$ is not a pole, then $t^\nu = 0$.

**Definition 2** (see [18]). The $v$th fractional sum of a function $f \in N_a \rightarrow \mathbb{R}$, for $v > 0$, is defined to be

$$\Delta_a^{-v} f(t) = \Delta_a^{-v} f(t; a) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-s-1)^{v-1} f(s),$$

(8)

where $t \in [a + \nu, a + \nu + 1, \ldots] =: N_{a+\nu}$. We also define the $v$th fractional difference, where $\nu > 0$ and $0 \leq N - 1 < \nu \leq N$ with $N \in \mathbb{N}$, to be $\Delta_a^{-v} f(t) = \Delta_a^{\nu} \Delta_a^{-\nu} f(t)$, where $t \in N_{a+N^{-\nu}}$.

**Lemma 1** (see [16]). Let $0 \leq N - 1 < \nu \leq N$. Then, $\Delta_a^{\nu} \Delta_a^{-\nu} y(t) = y(t) + C_1 t^{\nu - 1} + C_2 t^{\nu - 2} + \cdots + C_N t^{\nu - N}$, for some $C_i \in \mathbb{R}$, with $1 \leq i \leq N$.

**Lemma 2** (see [18]). Let $\nu \in \mathbb{R}$ and $t, s \in \mathbb{R}$ such that $(t - s)^2$ is well defined, then $\Delta_a(t - s)^2 = -(t - s - 1)^{\nu - 1}$.

**Lemma 3** (see Theorem 2.40 in [15]). Assume that $\mu > 0$ and $N - 1 < \nu < N$, $N \in \mathbb{N}$, then

$$\Delta_a^{\nu} (t - a)^2 = \frac{\Gamma(\mu + 1)}{(\mu - \nu + 1)} (t - a)^{\mu - \nu},$$

(9)
for $t \in [n_{\nu+\varepsilon}, \infty)$. 

Lemma 4 (see [24]). A function $y$ is a solution of the boundary value problem:

$$\Delta_0^\nu y(t) + h(t + \nu - 1) = 0, \quad t \in [0, m + 1]_{\mathbb{N}},$$

$$y(\nu - 2) = u(y),$$

$$y(\nu + m + 1) = g(y),$$

(10)

Proof. For each $0 \leq t < \nu + 1 \leq m + 1$,

$$\Delta_0^\nu G(t, s) = \frac{1}{\Gamma(\nu)} \left[ \frac{t^{\nu-2} - s^{\nu-2}}{(\nu + m + 1)^{\nu-1}} \right].$$

(12)

$$\Delta_0^\nu G(t, s) = \frac{t^{\nu-1} (\nu + m - s)^{\nu-1} - (t - s - 1)^{\nu-1}}{(\nu + m)^{\nu-1}}, \quad 0 \leq s < t + 1 \leq m + 1,$$

(14)

For each $0 \leq t < \nu + 1 \leq m + 1$, we have $\Delta_0 G(t, s) > 0$. Then, $G(t, s)$ is increasing in $t$. We have

$$G(\nu + 1, s) \leq G(t, s) \leq G(s + 1, s).$$

(18)

Therefore, for all $t \in [\nu - 1, \nu + m]_{\mathbb{N}}$, inequality (15) holds.

Lemma 5. Green function $G$ satisfies the inequalities

$$\max\{G(\nu + m, s), G(\nu - 1, s)\} \leq G(t, s) \leq G(s + 1, s),$$

(15)

for any $(t, s) \in [\nu - 1, \nu + m]_{\mathbb{N}} \times [0, m + 1]_{\mathbb{N}}$.

Proof. Let $h \in C([\nu - 1, \nu + m]_{\mathbb{N}}, \mathbb{R})$, $u, g: C([\nu - 1, \nu + m + 1]_{\mathbb{N}}, \mathbb{R}) \rightarrow \mathbb{R}$ are given functionals.

Lemma 6. The function $\bar{a}(t)$ is strictly decreasing in $t$, for all $t \in [\nu - 1, \nu + m + 1]_{\mathbb{N}}$, and $\bar{a}(t) \in [0, 1]$. In addition, the function $\bar{b}(t)$ is strictly increasing in $t$, and $\bar{b}(t) \in [0, 1]$.

Definition 3. We say that equation (6) has the Hyers–Ulam stability if there exists a constant $K > 0$ with the following property. Let $\varepsilon > 0$ be a given arbitrary constant. If a function $x: [0, m + 1]_{\mathbb{N}} \rightarrow \mathbb{R}$ satisfies

$$|\Delta_0^\nu x(t) + \lambda f(t + \nu - 1, \eta x(t + \nu - 1), \rho \Delta_0^\beta x(t))| \leq \varepsilon,$$

(20)

for all $t \in [0, m + 1]_{\mathbb{N}}$, then there exists a solution $y: [0, m + 1]_{\mathbb{N}} \rightarrow \mathbb{R}$ of equation (6) such that $|x(t) - y(t)| \leq K\varepsilon$ for all $t \in [0, m + 1]_{\mathbb{N}}$.

Remark 1. $x$ is a solution of inequality (20) if and only if there exists a function $g: [\nu - 1, m + 1]_{\mathbb{N}} \rightarrow \mathbb{R}$ such that

$$T_1 |g(t + \nu - 1)| \leq \varepsilon, t \in [0, m + 1]_{\mathbb{N}},$$

$$T_2 |\Delta_0^\nu x(t) + \lambda f(t + \nu - 1, \eta x(t + \nu - 1), \rho \Delta_0^\beta x(t)) = g(t + \nu - 1)|, t \in [0, m + 1]_{\mathbb{N}}.$$
Lemma 7. Let \( t \in [0, m + 1]_{\mathbb{N}} \) and \( h: [v-1, v+m]_{\mathbb{N}} \rightarrow \mathbb{R} \) be a continuous function; then, \( y \) is a solution of the fractional difference equation:

\[
\Delta_0^\nu y(t) + h(t + v - 1) = 0,
\]

subject to the fractional boundary value problem

\[
y(t) = \frac{t^{\nu-1}}{p_1} \sum_{s=0}^{m+1} \frac{a_2 (y + m - s)^{\nu-1} + b_2 (y + m - s - \mu)^{\nu-1}}{\Gamma(v-\mu)} + \frac{b_1 (y + m - s - \mu)^{\nu-1}}{\Gamma(v-\mu)} \left[ h(s + v - 1) \right] \]

\[
\times h(s + v - 1) + \frac{t^{\nu-1} \phi_2(x)}{p_1} \left[ h(s + v - 1) \right] \]

\[
\left[ \Delta_0^\nu \phi_1(x) \right] + \frac{t^{\nu-1} \phi_1(x)}{(a_1 + b_1)p_1 \Gamma(v-1)} + \frac{t^{\nu-2} \phi_1(x)}{(a_1 + b_1)p_1 \Gamma(v-1)}
\]

\[
- \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t - s - 1)^{\nu-1} h(s + v - 1),
\]

where \( p_1 = a_2 (v + m + 1)^{\nu-1} + b_2 (v + m + 1 - \mu)^{\nu-1} \frac{\Gamma(v)}{\Gamma(v-\mu)} \neq 0, \)

\[
p_2 = a_2 (v + m + 1)^{\nu-2} + b_2 (v + m + 1 - \mu)^{\nu-2} \frac{\Gamma(v-1)}{\Gamma(v-\mu-1)}
\]

(25)

where \( a_i, b_i \in \mathbb{R}, i = 1, 2, \) and \( 0 < \mu \leq 1, a_1+b_1 \neq 0, \phi_1 \) and \( \phi_2 \) are continuous functionals, and \( x \) is defined as in Definition 3.

Proof. From equation (21), we can get \( \Delta_0^\nu y(t) = -h(t + v - 1), \) and let us multiply both sides of this equation by \( \Delta_0^\nu \); by Definition 2 and Lemma 1, we can get the general solution of equation (21) as follows:

\[
y(t) = \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t - s - 1)^{\nu-1} h(s + v - 1) + C_1 t^{\nu-1} + C_2 t^{\nu-2},
\]

where \( C_1, C_2 \) are constants, and we have

\[
\Delta_0^\nu y(t) = C_1 \Delta_0^\nu t^{\nu-1} + C_2 \Delta_0^\nu t^{\nu-2} - \Delta_0^\nu h(t + v - 1)
\]

\[
= C_1 \frac{\Gamma(v)}{\Gamma(v-\mu)} t^{\nu-1-\mu} + C_2 \frac{\Gamma(v-1)}{\Gamma(v-\mu)} t^{\nu-2-\mu}
\]

\[
+ \frac{1}{\Gamma(v-\mu)} \sum_{s=0}^{t-v} (t - s - 1)^{\nu-1} h(s + v - 1).
\]

Therefore,

\[
y(v - 2) = C_2 (v - 2)^{\nu-2} = C_2 \Gamma(v - 1),
\]

\[
y(v + m + 1) = -\frac{1}{\Gamma(v)} \sum_{s=0}^{m+1} (v + m - s)^{\nu-1} h(s + v - 1) + C_1 (v + m + 1)^{\nu-1}
\]

\[
+ C_2 (v + m + 1)^{\nu-2},
\]

\[
\Delta_0^\nu y(v - 2 - \mu) = C_2 \frac{\Gamma(v-1)}{\Gamma(v-1-\mu)} (v - 2 - \mu)^{\nu-2-\mu}
\]

\[
= C_2 \frac{\Gamma(v-1)}{\Gamma(v-1-\mu)} \Gamma(v-1-\mu),
\]

\[
\Delta_0^\nu y(v + m + 1 - \mu) = C_1 \frac{\Gamma(v)}{\Gamma(v-\mu)} (v + m + 1 - \mu)^{\nu-1-\mu}
\]

\[
+ C_2 \frac{\Gamma(v-1)}{\Gamma(v-1-\mu)} (v + m + 1 - \mu)^{\nu-2-\mu}
\]

\[
- \frac{1}{\Gamma(v-\mu)} \sum_{s=0}^{m+1} (v + m - s - \mu)^{\nu-1} h(s + v - 1).
\]
By the boundary conditions (22) and (23), we can solve $C_1, C_2$ as follows:

$$C_1 = \frac{1}{p_1} \sum_{s=0}^{m+1} \left[ \frac{a_s}{\Gamma(v)} (v + m - s)^{v-1} + \frac{b_s}{\Gamma(v - \mu)} (v + m - s - \mu)^{v-\mu-1} \right]$$

$$\times h(s + v - 1) - \frac{p_t q_s(x)}{(a_1 + b_1) p_t \Gamma(v - 1)} \phi_s(x) + \frac{q_1(x)}{(a_1 + b_1) \Gamma(v - 1)}$$

$$C_2 = \frac{q_1(x)}{(a_1 + b_1) \Gamma(v - 1)}.$$

(29)

Substituting $C_1$ and $C_2$ into (26), then we can obtain (24).

\[ \square \]

3. Existence and Uniqueness of Solutions

In this section, we consider the following boundary value problem (BVP):

$$\begin{align*}
\Delta^\gamma_t y(t) + \lambda f(t + v - 1, y(t + v - 1), \rho \Delta^\beta_s y(t)) &= 0, \quad t \in [0, m + 1]_{\mathbb{N}^*}, \\
y(v - 2) &= u(y), \\
y(v + m + 1) &= g(y),
\end{align*}$$

(30)

**Theorem 1.** Assume that

(T3) $f: [v - 1, v + m]_{\mathbb{N}^*} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and also $u, g: \mathcal{C}([v - 2, v + m + 1]_{\mathbb{N}^*}, \mathbb{R}) \rightarrow \mathbb{R}$ are given functionals.

Denote that $E = \{ y: y \in C([v - 1, v + m]_{\mathbb{N}^*}; \mathbb{R}) \}$ and endowed with the norm $\| y \| = \max_{t \in [v - 1, v + m + 1]_{\mathbb{N}^*}} | y(t) |$. Then, $(E, \| \cdot \|)$ is a Banach space. Define the operator:

$$(Ty)(t) = \bar{a}(t) u(y) + \bar{b}(t) g(y) + \lambda \sum_{s=0}^{m+1} G(t, s) f(s + v - 1, y(s + v - 1), \rho \Delta^\beta_s y(s)),$$

(31)

where $\bar{a}(t)$ and $\bar{b}(t)$ are defined as (12) and (13) and $G(t, s)$ is given as (14). Obviously, $y$ is a solution of (30) if it is a fixed point of the operator $T$.

With all the preparatory works done, we will give the main conclusions. First, we provide the uniqueness result by contraction mapping as follows.

\[ L_1 + L_2 + \frac{\lambda (K_1 \eta \Gamma(\beta + 1) + K_2 \rho (m + 1) \beta)}{\Gamma(\beta + 1) (v + m + 1)^{v-1}} \sum_{s=0}^{m+1} (s + v - 1)^{v-1} (v + m - s)^{v-1} < 1. \]

(34)
Proof. Let $x, y \in E$; then, for each $t \in [v - 1, v + m]_{N_0}$, and by Definition 2 and Lemma 5, we have

\[
\|(T x)(t) - (T y)(t)\| \leq \alpha(t)|u(x) - u(y)| + \beta(t)|g(x) - g(y)|
\]

\[
+ \lambda \cdot \sum_{s=0}^{m+1} G(t, s) \left( K_1 \eta \|x(s) - y(s)\| + K_2 \rho \Delta_0^{-\beta} x(s) - \Delta_0^{-\beta} y(s)\right)
\]

\[
\leq \alpha(t)L_1 \|x - y\| + \beta(t)L_2 \|x - y\| + \lambda \cdot \sum_{s=0}^{m+1} G(s + v - 1, s)
\]

\[
\times \left( K_1 \eta \|x - y\| + K_2 \rho \frac{1}{\Gamma(\beta)} \cdot \sum_{k=0}^{s-\beta} (s - k - 1) \cdot \|x - y\| \right)
\]

\[
\leq \left[ L_1 + L_2 + \lambda \cdot \sum_{s=0}^{m+1} \frac{(s + v - 1)^{v-1} (v + m - s)^{v-1}}{\Gamma(v)(v + m + 1)^{v-1}} \left( K_1 \eta + \frac{K_2 \rho (m + 1)^{-v}}{\Gamma(\beta + 1)} \right) \right] \cdot \|x - y\|
\]

and by condition (34), we get that $T$ is a contraction mapping. Therefore, the Banach fixed-point theorem (see Lemma 7 in [25]) implies that the operator $T$ has a unique fixed point which is a unique solution of (30).

Now, we plan to adopt Brouwer theorem to give the existence result of solutions. \(\square\)

**Theorem 2.** Suppose that there exists a constant $M > 0$ such that when $\|y\| \leq M$, we have

\[
|u(y)| \leq \frac{M}{2 + \lambda \sum_{s=0}^{m+1} (s + v - 1)^{v-1} (v + m - s)^{v-1}} \cdot \left( \Gamma(v)(v + m + 1)^{v-1} \right) \cdot \Gamma(\beta + 1)
\]

\[
|g(y)| \leq \frac{M}{2 + \lambda \sum_{s=0}^{m+1} (s + v - 1)^{v-1} (v + m - s)^{v-1}} \cdot \left( \Gamma(v)(v + m + 1)^{v-1} \right) \cdot \Gamma(\beta + 1)
\]

\[
|f(t + v - 1, \eta y(t + v - 1), \rho \Delta_0^{-\beta} y(t))| \leq \frac{M}{2 + \lambda \sum_{s=0}^{m+1} (s + v - 1)^{v-1} (v + m - s)^{v-1}} \cdot \left( \Gamma(v)(v + m + 1)^{v-1} \right) \cdot \Gamma(\beta + 1)
\]

for each $t \in [0, m + 1]_{N_0}$. Then, (30) has at least one solution $y_0$, satisfying $\|y_0\| \leq M$.

**Proof.** Consider the Banach space $\mathcal{B} := \{ y \in \mathbb{R} : \|y\| \leq M \}$. Let $T$ be the operator defined in (31). It is clear that $T$ is a
continuous operator. Therefore, the main objective in establishing this result is to show that \( T: \mathcal{B} \rightarrow \mathcal{B}, \) that is, whenever \( \|y\| \leq M, \) it follows that \( \|Ty\| \leq M. \) Note that

\[
Ty \leq \max_{t \in [\nu-1, \nu+\nu]} \alpha(t)\|u(y)\| + \max_{t \in [\nu-1, \nu+\nu]} \beta(t)\|g(y)\| + \lambda \max_{t \in [\nu-1, \nu+\nu]} \sum_{t=0}^{m-1} G(t, s) \cdot \left| f\left( s + \nu - 1, \eta y(s + \nu - 1), \rho \Delta_0^\beta y(s) \right) \right|
\]

\[
\leq \|u(y)\| + \|g(y)\| + \lambda \sum_{t=0}^{m-1} G(s + \nu - 1, s) \cdot \left| f\left( s + \nu - 1, \eta y(s + \nu - 1), \rho \Delta_0^\beta y(s) \right) \right|
\]

\[
\leq \frac{\|u(y)\| + \|g(y)\| + \lambda \sum_{t=0}^{m-1} G(s + \nu - 1, s) \cdot \left| f\left( s + \nu - 1, \eta y(s + \nu - 1), \rho \Delta_0^\beta y(s) \right) \right|}{2 + \lambda \sum_{t=0}^{m-1} \left( \Gamma(v) \left( \nu + m - s \right) \right)^{\nu - 1} \left( \Gamma(v) \left( \nu + m + 1 \right) \right)^{-\nu - 1}} + \lambda M
\]

\[
\leq \frac{\|u(y)\| + \|g(y)\| + \lambda \sum_{t=0}^{m-1} G(s + \nu - 1, s) \cdot \left| f\left( s + \nu - 1, \eta y(s + \nu - 1), \rho \Delta_0^\beta y(s) \right) \right|}{2 + \lambda \sum_{t=0}^{m-1} \left( \Gamma(v) \left( \nu + m - s \right) \right)^{\nu - 1} \left( \Gamma(v) \left( \nu + m + 1 \right) \right)^{-\nu - 1}} + \lambda M
\]

Thus, we deduce that \( T: \mathcal{B} \rightarrow \mathcal{B}. \) Consequently, it follows from the Brouwer theorem that there exists a fixed point \( y_0 \) of the map \( T. \) This function \( y_0 \) is a solution of (30). Moreover, \( y_0 \) satisfies \( \|y_0(t)\| \leq M, \) for each \( t \in [\nu - 1, \nu + m]_{\mathbb{N}_0}. \) This completes the proof of the theorem.

4. Hyers–Ulam Stability

In this section, we study the Hyers–Ulam stability of the fractional-order difference system:

\[
\begin{cases}
\Delta_0^\nu y(t) + \lambda f\left( t + \nu - 1, \eta y(t + \nu - 1), \rho \Delta_0^\beta y(t) \right) = 0, & t \in [0, m + 1]_{\mathbb{N}_0}, \\
a_1 y(t - 2) + b_1 \left[ \Delta_0^\nu y(t - 2 - \mu) \right] = \phi_1(x), \\
b_2 y(t + m + 1) + b_2 \left[ \Delta_0^\nu y(t + m + 1 - \mu) \right] = \phi_2(x),
\end{cases}
\]

where \( x \in E \) satisfies (20). According to Lemma 7, we have

\[
y(t) = \frac{\lambda^{\nu - 1}}{p_1} \sum_{s=0}^{m-1} \left[ \frac{a_2}{\Gamma(v)} (v + m - s)^{\nu - 1} + \frac{b_2}{\Gamma(v - \mu)} (v + m - s - \mu)^{\nu - \mu - 1} \right] x
\]

\[
f\left( s + \nu - 1, \eta y(s + \nu - 1), \rho \Delta_0^\beta y(s) \right) + \frac{t^{\nu - 1}}{p_1} \phi_1(x) = \frac{t^{\nu - 1} \phi_1(x)}{(a_1 + b_1) \Gamma(v - 1)} + \frac{t^{\nu - 2} \phi_1(x)}{(a_1 + b_1) \Gamma(v - 1) - \lambda \sum_{s=0}^{m-1} (t - s - 1)^{\nu - 1} f\left( s + \nu - 1, \eta y(s + \nu - 1), \rho \Delta_0^\beta y(s) \right)}
\]

Theorem 3. Assume that \( (T_3) - (T_4) \) hold, and

\[
\frac{\lambda (m + 1)^2}{\Gamma(v + 1)} + \frac{\lambda (m + 1)^{\nu - 1}}{p_1} \left( \frac{a_2}{\Gamma(v + 1)} + \frac{b_2}{\Gamma(v - \mu + 1)} \right)^{\nu - \mu} \leq \frac{1}{K_1 \eta + K_2 \beta (\Gamma(m + 1) \rho)}
\]

(42)
If \( x \in E \) satisfies (20) and \( y \in E \) is a solution of (40), then the fractional difference equation (6) is Hyers–Ulam stable.

Proof. If \( x \in E \) satisfies (20) and due to the Remark 1, we can obtain

\[
\Delta^\gamma_0 x(t) + \lambda f(t + \nu - 1, \eta x(t + \nu - 1), \rho \Delta^\gamma_0 x(t)) = \xi(t + \nu - 1), \quad t \in [0, m + 1]_{\mathbb{N}_0},
\]

\[
|\xi(t + \nu - 1)| \leq \epsilon, \quad t \in [0, m + 1]_{\mathbb{N}_0}, \tag{43}
\]

We can solve equation (43) with the corresponding boundary value conditions (22) and (23) as follows:

\[
x(t) = \frac{\lambda t^{\nu-1}}{p_1} \sum_{s=0}^{m+1} \left[ \frac{a_2}{\Gamma(\nu)} (v + m - s)^{\nu-1} + \frac{b_2}{\Gamma(\nu - \mu)} (v + m - s - \mu)^{\nu-1} \right] \times
\]

\[
f(s + \nu - 1, \eta x(s + \nu - 1), \rho \Delta^\gamma_0 x(s)) + \frac{t^{\nu-1} \varphi_2(x)}{p_1} - \frac{t^{\nu-1} \varphi_2(y(x))}{(a_1 + b_1)p_1 \Gamma(\nu - 1)}
\]

\[
+ \frac{t^{\nu-2} \varphi_1(x)}{a_1 + b_1 \Gamma(\nu - 1)} + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-1} (t - s - 1)^{\nu-1} \xi(s + \nu - 1)
\]

\[
- \frac{\lambda}{\Gamma(\nu)} \sum_{s=0}^{t-1} (t - s - 1)^{\nu-1} f(s + \nu - 1, \eta x(s + \nu - 1), \rho \Delta^\gamma_0 x(s)). \tag{44}
\]

Then, we have

\[
x(t) - y(t) \leq \left| \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-1} (t - s - 1)^{\nu-1} \xi(s + \nu - 1) \right| + \frac{\lambda}{\Gamma(\nu)} \sum_{s=0}^{t-1} (t - s - 1)^{\nu-1} \times
\]

\[
f(s + \nu - 1, \eta y(s + \nu - 1), \rho \Delta^\gamma_0 y(s)) - f(s + \nu - 1, \eta x(s + \nu - 1), \rho \Delta^\gamma_0 x(s))
\]

\[
+ \frac{\lambda t^{\nu-1}}{p_1} \sum_{s=0}^{m+1} \left[ \frac{a_2}{\Gamma(\nu)} (v + m - s)^{\nu-1} + \frac{b_2}{\Gamma(\nu - \mu)} (v + m - s - \mu)^{\nu-1} \right] \times
\]

\[
f(s + \nu - 1, \eta y(s + \nu - 1), \rho \Delta^\gamma_0 y(s)) - f(s + \nu - 1, \eta x(s + \nu - 1), \rho \Delta^\gamma_0 x(s)). \tag{45}
\]
According to Lemma 2, we know
\[
\frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t - s - 1)^{v-1} \xi(s + v - 1) \leq \frac{1}{\Gamma(v)} \frac{t^v}{v} \epsilon \leq \frac{(m + 1)^{\beta}}{\Gamma(v + 1)},
\]
and with the help of condition \((T_4)\), we can obtain
\[
\frac{\lambda}{\Gamma(v)} \sum_{s=0}^{t-v} (t - s - 1)^{v-1} \cdot \left| f(s + v - 1, \eta y(s + v - 1), \rho \Delta^\beta y(s)) - f(s + v - 1, \eta x(s + v - 1), \rho \Delta^\beta x(s)) \right|
\]
\[
\frac{\lambda}{\Gamma(v)} \sum_{s=0}^{t-v} (t - s - 1)^{v-1} \cdot \left| K_1 \eta |x(t) - y(t)| + K_2 \rho \Delta_0^\beta y(t) - \Delta_0^\beta x(t) \right|
\]
\[
\frac{\lambda (m + 1)^{\beta}}{\Gamma(v + 1)} \left( K_1 \eta + K_2 \rho \frac{(m + 1)^{\beta}}{\Gamma(\beta + 1)} \right) \cdot \|x - y\|,
\]
and we obtain
\[
\frac{\lambda t^{v-1}}{\rho_1} \sum_{s=0}^{m+1} \left[ \frac{a_2(y + m - s)^{v-1}}{\Gamma(v)} + \frac{b_2(y + m - s - \mu)^{v-1}}{\Gamma(v - \mu)} \right] \times \left| K_1 \eta + K_2 \rho \frac{(m + 1)^{\beta}}{\Gamma(\beta + 1)} \right) \cdot \|x - y\|
\]
\[
\leq \frac{\lambda t^{v-1}}{\rho_1} \sum_{s=0}^{m+1} \left[ \frac{a_2(y + m - s)^{v-1}}{\Gamma(v)} + \frac{b_2(y + m - s - \mu)^{v-1}}{\Gamma(v - \mu)} \right] \left( K_1 \eta + K_2 \rho \frac{(m + 1)^{\beta}}{\Gamma(\beta + 1)} \right) \cdot \|x - y\|
\]
\[
\leq \frac{\lambda (m + 1)^{\beta}}{\rho_1} \left[ \frac{a_2(y + m + 1)^{v}}{\Gamma(v + 1)} + \frac{b_2(y + m + 1 - \mu)^{v}}{\Gamma(v - \mu + 1)} \right] \left( K_1 \eta + K_2 \rho \frac{(m + 1)^{\beta}}{\Gamma(\beta + 1)} \right) \cdot \|x - y\|.
\]

By (45)–(48), we can conclude that
\[
\|x - y\| \leq \frac{(m + 1)^{\beta}}{\Gamma(v + 1)} \cdot \epsilon
\]
\[
1 - \left( \frac{(m + 1)^{\beta}}{\Gamma(v + 1)} \right)^{p_1 \left( \frac{(a_2(y + m + 1)^{v}}{\Gamma(v + 1)} + \left( \frac{b_2(y + m + 1 - \mu)^{v}}{\Gamma(v - \mu + 1)} \right) \left( K_1 \eta + K_2 \rho \frac{(m + 1)^{\beta}}{\Gamma(\beta + 1)} \right) \cdot \|x - y\| \right)}
\]
\[
(49)
\]
By condition (42) in Theorem 3, we have

\[
\left(\frac{m+1}{m!}\Gamma(m+1)\right)\left(1 - \frac{(\lambda(m+1)^{\nu}/\Gamma(v+1))}{1} + \left(\frac{1}{m+1}\right)^{\nu} \right) > 0.
\]

and we note that the quantity on the left-hand side of the inequality is the constant "\(K\)" in Definition 3. We can deduce that system (40) is Hyers–Ulam stable. \(\square\)

5. Examples

In this section, we will present the following three examples to illustrate our main results.

Example 1. Suppose that \(\lambda = 1/10, \eta = 1/20, \rho = 1, m = 5, v = 5/4, \) and \(\beta = 4/3.\) In addition, let \(u(y) = (1/20)\sin y, g(y) = (1/20)\cos y,\) and \(f(t,x,y) = e^t + (1/10)x + (1/20)y.\) Then, the following boundary value problem (BVP)

\[
L_1 + L_2 + \frac{\lambda(K_1\eta\Gamma(\beta + 1) + K_2\rho(m + 1)\Gamma)}{\Gamma(\nu\Gamma(\beta + 1)(v + m + 1)} \sum_{k=0}^{m+1} (s + v - 1) (y + m - s) ^{\nu-1}
\]

has a unique solution.

Therefore, we deduce from Theorem 1 that problem (51) has a unique solution.

Example 2. Assume that \(\lambda = 1, \eta > 0, \rho > 0, m = 4, v = 3/2, \beta \in (1,2], \) and \(M = 10.\) Suppose that \(u(y) = \sin y, g(y) = e^{-y},\) and \(f(t,x,y) = \cos(tx + y).\) We can deduce that (36)–(38) hold, and the following boundary value problem (BVP)

\[
\left\{
\begin{array}{ll}
\Delta^{(\nu/2)} y(t) + f(t + \frac{1}{2}, \eta)(t + \frac{1}{2})\rho \Delta^\beta y(t) = 0, & t \in [0,5], \\
y\left(\frac{1}{2}\right) = \sin y, \\
y\left(\frac{13}{2}\right) = e^{-y},
\end{array}
\right.
\]

is problem (30). According to Theorem 2, we can obtain that (30) has at least one solution \(y_{t_0},\) and \(\|y_0(t)\| \leq 10.\)

Remark 2. Since there are few papers research solutions of the mixed fractional-order nonlinear difference equation, one can see that all the results in [16–18, 23–25, 36–38] cannot directly be applicable to (51) and (53) to obtain the existence and uniqueness of the solution. These imply that the results in this paper are essentially new.

Example 3. Assume that \(\lambda = 1/5, \eta = 1/10, \rho = 1/5, m = 5, v = 4/3, \beta = 5/3, \mu = 1/3, a_1 = 1, a_2 = 2, b_1 = 2, b_2 = 4,\) and \(f(t,x,y) = t + (1/10)\sin x + (1/20)\cos y.\) We let \(K_1 = 1/10\) and \(K_2 = 1/20.\) System (40) turns into

\[
\left\{
\begin{array}{ll}
\Delta^{(\nu/2)} y(t) + f(t + \frac{1}{3}, \frac{1}{10})(t + \frac{1}{3}) \frac{1}{3} \Delta^\beta y(t) = 0, & t \in [0,6], \\
y\left(\frac{1}{3}\right) = \varphi_1(x), \\
2\gamma\left(\frac{22}{3}\right) + 4\Delta^{(1/3)} y(7) = \varphi_2(x),
\end{array}
\right.
\]

where \(\varphi_1 \) and \(\varphi_2 \) are continuous functionals and \(x \in E\) satisfies

\[
\left|\Delta^{(\nu/2)} x(t) + f(t + \frac{1}{3}, \frac{1}{10})(t + \frac{1}{3}) \frac{1}{3} \Delta^\beta x(t)\right| \leq \varepsilon.
\]

By Mathematica, we note that
Therefore, (42) holds. If \( y \in E \) is a solution of (54), then fractional difference system (54) is Hyers–Ulam stable.

6. Conclusion

In this paper, we are concerned with the nonlinear mixed fractional order difference equations, which are quite different from the related references discussed in the literature. The fractional order difference equation studied in the present paper is more generalized and more practical. By applying the Brouwer theorem and contraction mapping principle and the definition of Hyers–Ulam stability, the easily verifiable sufficient conditions have been provided to determine the existence, uniqueness, and Hyers–Ulam stability of the solutions for the considered equation. Finally, the necessary three typical numerical examples have been presented at the end of this paper to illustrate the effectiveness and feasibility of the proposed criterion. Consequently, this paper shows theoretically and numerically that the proposed method by the authors could be applied to other fractional difference equation of other similar type, such as [39–44].

An interesting extension of our study would be to discuss Ulam–Hyers–Mittag–Leffler stability and finite-time stability for the mixed fractional nonlinear difference equation with time-varying delay terms or fractional stochastic system based on [45, 46]. This topic will be the subject of a forthcoming paper.

Data Availability

The data in this study were mainly collected via discussion during our class and obtained from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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