Nature of Ground State Incongruence in Two-Dimensional Spin Glasses

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Abstract

We rigorously rule out the appearance of multiple domain walls between ground states in 2D Edwards-Anderson Ising spin glasses (with periodic boundary conditions and, e.g., Gaussian couplings). This supports the conjecture that there is only a single pair of ground states in these models.

A fundamental problem in spin glass physics is the multiplicity of infinite-volume ground states in finite-dimensional short-ranged systems, such as the Edwards-Anderson (EA) Ising spin glass. In 1D, there is no frustration and only a single pair of (spin-reversed) ground states. In the mean-field Sherrington-Kirkpatrick (SK) model, there are presumed to be (in some suitably defined sense) infinitely many ground state pairs (GSP’s). One conjecture, in analogy with the SK model, is that finite D realistic models with frustration have infinitely many GSP’s; for a review, see [1, 2]. A different conjecture, based on droplet-scaling theories [3, 4, 5], is that there is only a single GSP in all finite D. In 2D and 3D, the latter scenario has received support from recent simulations, some [6, 7] based on “chaotic size dependence” [8] and some [9, 10] using other techniques.

In this paper, we provide a significant analytic step towards a resolution of this problem in 2D, by ruling out the presence of multiple domain walls between ground states. We anticipate that the ideas and techniques introduced here will ultimately yield a solution to the problem of ground state multiplicity in two dimensions, and that at least some of them may prove to be useful in higher dimensions as well. Though our result is more general, we confine our attention to the nearest-neighbor EA Ising spin glass, with Hamiltonian

\[ \mathcal{H}_J(\sigma) = -\sum_{\langle x,y \rangle} J_{xy} \sigma_x \sigma_y , \]  

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where \( J \) denotes a specific realization of the couplings \( J_{xy} \), the spins \( \sigma_x = \pm 1 \) and the sum is over nearest-neighbor pairs \( \langle x, y \rangle \) only, with the sites \( x, y \) on the square lattice \( \mathbb{Z}^2 \). The \( J_{xy} \)'s are independently chosen from a mean zero Gaussian (or any other symmetric, continuous distribution with unbounded support) and the overall disorder measure is denoted \( \nu(J) \).

A ground state is an infinite-volume spin configuration whose energy (governed by Eq. (1)) cannot be lowered by flipping any finite subset of spins. That is, all ground state spin configurations must satisfy the constraint

\[
\sum_{\langle x, y \rangle \in C} J_{xy}\sigma_x\sigma_y \geq 0 \tag{2}
\]

along any closed loop \( C \) in the dual lattice. In any \( L \times L \) square \( S_L \) (centered at the origin) with, e.g., periodic b.c.’s, there is (with probability one) only a single finite-volume GSP (the spin configurations of lowest energy subject to the b.c.). An infinite-volume ground state can be understood as a limit of finite-volume ones: consider the ground state \( \sigma^{(L_0, L)} \) inside any given \( S_{L_0} \), but with b.c.’s imposed on \( S_L \) and \( L \gg L_0 \). An infinite-volume ground state (satisfying Eq. (2)) is generated whenever, for each (fixed) \( L_0 \), \( \sigma^{(L_0, L)} \) converges to a limit as \( L \to \infty \) (for some sequence of b.c.’s, which may depend on the coupling realization). If many infinite-volume GSP’s exist, then a sequence as \( L \to \infty \) of finite-volume GSP’s with coupling-independent b.c.’s will generally not converge to a single limit (i.e., \( \sigma^{(L_0, L)} \) continually changes as \( L \to \infty \)), a phenomenon we call chaotic size dependence [11]. So a numerical signal of the existence of many ground states is that the GSP in \( S_L \) with periodic b.c.’s varies chaotically as \( L \) changes [9, 10, 11].

It is important to distinguish between two types of multiplicity. The symmetric difference \( \alpha \Delta \beta \) between two GSP’s \( \alpha \) and \( \beta \) is the set of all couplings that are satisfied in one and not the other. A domain wall (always defined relative to two GSP’s) is a cluster (in the dual lattice) of the couplings satisfied in one but not the other state. So \( \alpha \Delta \beta \) is the union of all of their domain walls, and may consist of a single one or many. Two distinct GSP’s are incongruent [13] if \( \alpha \Delta \beta \) has nonvanishing density in the set of all bonds; otherwise the two are regionally congruent. Incongruent GSP’s can in principle have one or more positive density domain walls, or instead infinitely many of zero density.

If there are multiple GSP’s, the interesting, and physically relevant, situation is the existence of incongruent states. Regional congruence is of mathematical interest, but to see it would require a choice of b.c.’s carefully conditioned on the coupling realization \( J \). It is not currently known how to choose such b.c.’s. Numerical treatments that look for multiple GSP’s implicitly search for incongruent ground states, and it is the question of their existence and nature in 2D that we treat here.

To state our result precisely, we introduce the concept of a metastate. For spin glasses, this was proposed in the context of low temperature states for large finite volumes [14] (and shown to be equivalent to an earlier construct of Aizenman and Wehr [15]), and its properties were further analyzed in [16, 17]. In the current context, a (periodic b.c.) metastate is a measure on GSP’s constructed via an infinite sequence of squares \( S_L \) with both the \( L \)’s and the (periodic) b.c.’s coupling-independent. Roughly speaking, the metastate here provides
the probability (as $L \to \infty$) of various GSP's appearing inside any fixed $S_{L_0}$. It is believed (but not proved) that different sequences of $L$'s yield the same (periodic b.c.) metastate.

If there are infinitely many (incongruent) GSP's, a metastate should be dispersed over them, giving their relative likelihood of appearance in typical large volumes. If there is no incongruence, the metastate would be unique and supported on a single GSP, and that GSP will appear in most (i.e., a fraction one) of the $S_L$'s.

We now state the main result of this paper. It shows that if more than a single GSP is present in the periodic b.c. metastates, then two distinct GSP's cannot differ by more than a single domain wall. After we present the proof of this statement, we will discuss why this result supports the existence of only a single GSP in 2D.

**Theorem.** In the 2D EA Ising spin glass with Hamiltonian (1) and couplings as specified earlier, two infinite-volume GSP's chosen from the periodic b.c. metastates are either the same or else differ by a single, non-self-intersecting domain wall, which has positive density.

We sketch the proof of this theorem in several steps; a full presentation will be given elsewhere. First, some elementary properties of (zero-temperature) domain walls:

**Lemma 1.** A 2D domain wall is infinite and contains no loops or dangling ends.

**Proof.** A domain wall between two spin configurations is a boundary separating regions of agreement from disagreement and thus cannot have dangling ends. To rule out loops, note that the sum $\sum_{<xy>} J_{xy} \sigma_x \sigma_y$ along any such loop must have opposite signs in the two GSP's, violating Eq. (2), unless the sum vanishes. But this occurs with probability zero because the couplings are chosen independently from a continuous distribution.

We now construct a periodic b.c. metastate $\kappa_J$, which will provide a measure on the domain walls between GSP's (that appear in $\kappa_J$). As in construction II of [20] (but at zero temperature), consider for each square $S_L$, two sets of variables, the couplings $J^{(L)}$ (chosen from, e.g., the Gaussian distribution) and the bond variables $\sigma_x^{(L)} \sigma_y^{(L)}$ for the GSP $\sigma^{(L)}$. Consider fixed sets of both random variables as $L \to \infty$; by compactness, there exists a subset of $L$'s along which the joint distribution converges to a translation-invariant infinite-volume (joint) measure. This limit distribution is supported on $\kappa_J$ is supported on (infinite-volume) GSP's for that $J$.

A metastate $\kappa_J$ yields a measure $D_J$ on domain walls. This is done by taking two (replica) GSP's from $\kappa_J$ to obtain a configuration of (unions of) domain walls (i.e., the set of domain walls one would see from two GSP's chosen randomly from $\kappa_J$). If one then integrates out the couplings, one is left with a translation-invariant measure $D$ on the domain wall configurations themselves.

This leads to important percolation-theoretic features of domain walls between GSP's in $\kappa_J$. Some of these are stated in the following:

**Lemma 2.** Distinct 2D GSP’s $\alpha$ and $\beta$ from $\kappa_J$ must (with probability one) be incongruent and the domain walls of their symmetric difference $\alpha \Delta \beta$ must be non-intersecting, non-branching paths, that together divide $\mathbb{Z}^2$ into infinite strips and/or half-spaces.

**Proof.** This lemma, from [21], uses a technique introduced in [22]. First we note that by the translation-invariance of $D$, any “geometrically defined event”, e.g., that a bond belongs
to a domain wall, either occurs nowhere or else occurs with strictly positive density. This immediately yields incongruence. Suppose now that an intersection/branching occurs at some site \(z\) (in the dual lattice). Then there are at least three (actually four) infinite paths in \(\alpha \Delta \beta\) that start from \(z\), and they cannot intersect in another place, because that would form a loop, violating Lemma 1. But then translation-invariance implies a positive density of such \(z\)'s. The tree-like structure of \(\alpha \Delta \beta\) implies that in a square with \(p\) such \(z\)'s, the number of distinct such paths crossing its boundary is at least proportional to \(p\). Since \(p\) scales like \(L^2\), there is a contradiction as \(L \to \infty\), because the number of distinct paths cannot be larger than the perimeter, which scales like \(L\). Similar arguments complete the proof.

The picture we now have for \(\alpha \Delta \beta\) is a union of one or more infinite domain walls (each of which divides the plane into two infinite disjoint parts) that neither branch, intersect, nor form loops, and that mostly remain within \(O(1)\) distance from one another. We now begin a lengthy argument to show that there in fact cannot be more than a single domain wall.

The first step is to introduce the notion of a “rung” between adjacent domain walls. A rung \(R\) in \(\alpha \Delta \beta\) is a path of bonds in the dual lattice connecting two distinct domain walls, and with only the first and last sites in \(R\) on any domain wall. So each of the couplings in \(R\) is satisfied in both \(\alpha\) and \(\beta\) or unsatisfied in both. The energy \(E_R\) of \(R\) is defined to be

\[
E_R = \sum_{\langle xy \rangle \in R} J_{xy} \sigma_x \sigma_y ,
\]

with \(\sigma_x \sigma_y\) taken from \(\alpha\) (or equivalently, \(\beta\)). It must be that \(E_R > 0\) (with probability one) for the following reason. Suppose that a rung could be found with negative energy; by translation-invariance (and arguments somewhat like those used for Lemma 2), there would then be an infinite set of rungs with negative energy connecting some two domain walls. Consider the “rectangle” that is bounded by two such rungs and the connecting domain wall pieces. The sum of \(J_{xy} \sigma_x \sigma_y\) along the couplings in the two domain wall pieces would be positive in one of \(\alpha, \beta\) and negative in the other; hence, the loop formed by the boundary of this rectangle would violate Eq. (2) in \(\alpha\) or \(\beta\), leading to a contradiction.

However, we can impose a more serious constraint on \(E_R\); namely that it must be bounded away from zero for all \(R\) between two fixed domain walls. To explain this, we first consider a single arbitrary bond \(b\), an \(S_L\) large enough to contain \(b\), a coupling realization \(J^{(L)}\) and the corresponding GSP \(\alpha^{(L)}\). Now let \(J_b\) vary with all other couplings fixed. It is easy to see that there will be a transition value \(K_b^{(L)}\) (which is a function of all the couplings in \(J^{(L)}\) except \(J_b\)) beyond which \(\alpha^{(L)}\) ceases to have minimum energy and is replaced by some \(\alpha^{b,(L)}\), related to \(\alpha^{(L)}\) by a droplet flip. The symmetric difference \(\alpha^{(L)} \Delta \alpha^{b,(L)}\) consists of a domain wall (the boundary of the droplet) passing through \(b\) with exactly zero total energy when \(J_b = K_b^{(L)}\). The droplet boundary may or may not reach the boundary of \(S_L\). In other words, as \(J_b\) varies from \(-\infty\) to \(+\infty\), there are exactly two GSP’s (\(\alpha^{(L)}\) and \(\alpha^{b,(L)}\)) that appear, one when \(J_b\) is below \(K_b^{(L)}\) and one when it is above.

What happens when \(L \to \infty\)? As in the construction of metastates, we obtain a translation-invariant infinite-volume joint probability distribution on \(J\) (the couplings \(J_b\),
α (a GSP for J), K (transition values \( K_b \) for J, \( \alpha \)) and \( \alpha^* \) \( (\alpha^b \)'s for J, \( \alpha, K \)). In this limit: J is chosen from the usual disorder distribution \( \nu \), then \( \alpha \) from the metastate \( \kappa_{J, \alpha} \) and finally K and \( \alpha^* \) from some measure \( \kappa_{J, \alpha} \). The symmetric difference \( \alpha \Delta \alpha^b \) may consist of a single finite loop or else of one or more infinite disconnected paths, but in all cases some part must pass through \( b \). The lack of dependence of \( K_b^{(L)} \) on \( J_b \) implies that even after \( L \to \infty \), \( K_b \) and \( J_b \) are independent random variables; this independence leads to the next two lemmas.

**Lemma 3.** With probability one, no coupling \( J_b \) is exactly at its transition value \( K_b \).

*Proof.* From the independence of \( J_b \) and \( K_b \), and the continuity of the distribution of \( J_b \), it follows that there is probability zero that \( J_b - K_b = 0 \), much like in the proof of Lemma 1.

**Lemma 4.** The rung energies \( E_{R'} \) between two fixed (adjacent) domain walls cannot be arbitrarily small; i.e., there is zero probability that \( E' \), the infimum of all such \( E_{R'} \)'s, will be zero.

*Proof.* Were this not so, there would be (by translation-invariance arguments) an infinite set of rungs \( R' \) with \( E_{R'} \leq \epsilon \), for any \( \epsilon > 0 \). That implies (by the “rectangular” construction below Eq. (3)) that each \( J_b \) along the two domain walls would be at the transition value \( K_b \), either for \( \alpha \) or for \( \beta \), violating Lemma 3.

The next lemma relates the location of the droplet boundary, \( \alpha \Delta \alpha^a \), when \( \alpha^a \) replaces \( \alpha \), to the “flexibility” of \( a \). The flexibility \( F_a \) of a bond \( a \) (in a \( (J, \alpha, K, \alpha^* \) configuration) is defined as \( |J_a - K_a| \); the larger the flexibility, the more stable is \( \alpha \) under changes of \( J_a \).

**Lemma 5.** If \( F_b > F_a \), then there is zero probability that \( \alpha \Delta \alpha^a \) passes through \( b \).

*Proof.* For finite \( L \), this is an elementary consequence of the fact that for \( e = a \) or \( b \), \( F_e^{(L)} \equiv |J_e - K_e^{(L)}| \) is the minimum, over all droplets whose boundary passes through \( e \), of the droplet flip energy cost. After \( L \to \infty \), such a characterization of \( F_e \) may not survive, but what does survive is that \( \alpha \Delta \alpha^a \) does not go through \( b \).

The next lemma completes our proof that for GSP’s \( \alpha \) and \( \beta \) chosen from \( \kappa_{J, \alpha} \), \( \alpha \Delta \beta \) cannot consist of more than a single domain wall, since otherwise there would be an immediate contradiction with Lemma 4. For the proof, we need the notion of “super-satisfied”. It is easy to see that a coupling \( J_{xy} \) is satisfied in every ground state if \( |J_{xy}| > \min \{ M_x, M_y \} \), where \( M_x \) is the sum of the three other coupling magnitudes \( |J_{zx}| \) touching \( x \), and \( M_y \) is defined similarly. Such a coupling \( J_{xy} \), called super-satisfied, clearly cannot be part of any domain wall.

**Lemma 6.** There is zero probability that \( E' > 0 \).

*Proof.* Suppose \( E' > 0 \) (with positive probability); we show this leads to a contradiction. First we find, as in Fig. 1, a rung \( R \) with \( E_R - E' = \delta \) strictly less than the flexibility values (for both \( \alpha \) and \( \beta \)) of two couplings \( b_1, b_2 \) along the “left” of the two domain walls, \( b_1 \) “above” and \( b_2 \) “below” the rung. Such an \( R \), \( b_1 \) and \( b_2 \) must exist by Lemma 3 (and translation-invariance arguments).

But we also want a situation, as in Fig. 1, where all the (dual lattice) non-domain-wall couplings that touch the left domain wall between \( b_1 \) and \( b_2 \) (other than the first coupling \( J_a \) in \( R \)) are super-satisfied, and remain so regardless of changes of \( J_a \). How do we know that such a situation will occur (with non-zero probability)? If necessary, one can first adjust the signs and then increase the magnitudes (in an appropriate order) of these (ten) couplings, so
Figure 1: A rung $R$ with $E_R = E' + \delta$. The dots are sites in $\mathbb{Z}^2$, and couplings are drawn in the dual lattice. Two domain walls are solid lines and $R$ is the dashed line. The couplings $b_1$ and $b_2$ have flexibility $> \delta$. The ten dotted line couplings are super-satisfied.

that they first become satisfied and then super-satisfied. This can be done in an “allowed” way because of our assumption that the distribution of individual couplings has unbounded support. Also, this can be done without causing a replacement of either $\alpha$ or $\beta$, without changing $E_R$, without decreasing any other $E_{R'}$ and without decreasing the flexibilities of $b_1$ or $b_2$. Starting from a positive probability event, such an (allowed) change of finitely many couplings in $\mathcal{J}$ yields an event which still has non-zero probability.

Next, suppose we move $J_a$ toward its transition value $K_a$ by an amount slightly greater than $\delta$. The geometry (of Fig. 1) and Lemma 5 forbid the replacement of either $\alpha$ or $\beta$, because it is impossible, under the conditions given, for $\alpha\Delta\alpha^a$ or $\beta\Delta\beta^a$ to connect to the left end of bond $a$. But this move reduces $E_R$ below $E_{R'}$ for any $R'$ not containing $a$, contradicting translation-invariance.

This completes the proof of the theorem: if distinct $\alpha, \beta$ occur, they differ by at most a single domain wall. Although this does not yet rule out many ground states in the 2D periodic b.c. metastate, it greatly simplifies the problem by ruling out all but one possibility about how GSP’s may differ.

We expect, though, that these single domain walls do not exist. There are reasonable arguments and conjectures indicating that this is so, and that even if they do exist, it remains unlikely that there exists an infinite multiplicity of states. We will discuss these in turn.

First, we note that although, for technical reasons, we have not extended our proof to rule out single domain walls, our previous results indicate that it is natural to expect that the “pseudo-rungs” that connect sections of the domain wall that are close in Euclidean distance, but greatly separated in distance along the domain wall, can have arbitrarily low (positive) energies. If these “pseudo-rungs” also connect arbitrarily large pieces of the domain wall containing some fixed bond (and we emphasize that these properties are not yet rigorously proved), then single domain walls would be ruled out in a similar manner as above. The
consequence would be that the periodic b.c. metastate in the 2D EA Ising spin glass with Gaussian couplings is supported on a single GSP.

In the unlikely event that single positive-density domain walls do appear, our theorem could still rule out an infinite multiplicity of GSP’s in 2D. This would be a consequence of the following conjecture (which presents an interesting problem in the topology of random curves):

**Conjecture:** There exists no translation-invariant measure on infinite sequences \((a_1, a_2, \ldots)\) of distinct bond configurations on \(\mathbb{Z}^2\) such that each \(a_i\) and each \(a_i \Delta a_j\) is a single, doubly-infinite, self-avoiding path.

The above conjecture, if true, would rule out the presence of infinitely many distinct GSP’s \(\alpha_0, \alpha_1, \ldots\) (in one or more metastates for a given \(J\)) since taking \(a_i = \alpha_0 \Delta \alpha_i\) would contradict the conjecture.

These considerations, taken together, make it appear unlikely that an infinite multiplicity of GSP’s, constructed from periodic (or antiperiodic \([17]\)) boundary conditions, can exist for the 2D EA Ising spin glass with Gaussian (or similar) couplings.

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