The String Calculation of QCD Wilson Loops on Arbitrary Surfaces

Stephen G. Naculich† and Harold A. Riggs‡

†Department of Physics  ‡Department of Physics
Bowdoin College  Brandeis University
Brunswick, ME 04011  Waltham, MA 02254

naculich@polar.bowdoin.edu  hriggs@binah.cc.brandeis.edu

Abstract

Compact string expressions are found for non-intersecting Wilson loops in SU($N$) Yang-Mills theory on any surface (orientable or nonorientable) as a weighted sum over covers of the surface. All terms from the coupled chiral sectors of the $1/N$ expansion of the Wilson loop expectation values are included.

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The recent interpretation of SU(\(N\))-gauge-theory Wilson loops on a variety of surfaces as string amplitudes\(^1-3\) is a potentially useful prelude to finding a string picture of two-dimensional hadrons, and, one hopes, to finding a generalization to four dimensions and real hadrons. This interpretation divides the \(1/N\) expansion of the exact expressions for such Wilson loops into two coupled chiral sectors. If the terms that couple the sectors are neglected, the geometric interpretation is reasonably straightforward.\(^3\) While the structure of SO(\(N\)) (and Sp(\(N\))) gauge-theory Wilson loops is similar to a single chiral SU(\(N\)) sector in that it involves a single sum over Young tableaux, it also includes features similar to those that couple the SU(\(N\)) chiral sectors. Nevertheless, a compact and explicit string expression is available for the entire \(1/N\) expansion of non-intersecting SO(\(N\)) and Sp(\(N\)) Wilson loops, one that makes their geometric nature transparent.\(^4\)

In this letter we obtain analogous compact and explicit string expressions for non-intersecting SU(\(N\))-gauge-theory Wilson loops on arbitrary (orientable or nonorientable) surfaces, including all the terms that couple the chiral sectors. These expressions exhibit the complete set of open string maps which contribute to the Wilson loop expectation values in the full coupled theory.

We begin with the analysis of non-intersecting Wilson loops as sums over partition functions on surfaces with boundary. Let \(Z(\mathcal{S}; U_1, \ldots, U_b)\) denote the partition function on an open surface \(\mathcal{S}\) with gauge-field holonomies \(U_1, \ldots, U_b\) on the \(b\) components of the boundary of \(\mathcal{S}\). Gauge invariance implies that such partition functions are expandable on a complete basis of characters of the gauge group

\[
\chi_{\hat{R}S}^{\hat{\chi}} \left( U \right),
\]

with \(R\) and \(S\) denoting arbitrary tableaux and \(\hat{R}S\) denoting the bitableau\(^5\) formed from them by adjoining the right-justified tableau \(R\) with dots in each cell to the ordinary left-justified tableau \(S\). As long as the column lengths of \(R\) and \(S\) are small relative to \(N\) there will be no overcounting of representations. In fact, the only terms that contribute to the \(1/N\) expansion are those in which \(R\) and \(S\) are finite-cell tableaux\(^3\) (finite relative to \(N\), which is taken to be large). The advantage of this basis is that the expansion coefficients in

\[
Z(\mathcal{S}; U_1, \ldots, U_b) = \sum_{R, S} Z(\mathcal{S}; \{\hat{R}S_1, \ldots, \hat{R}S_b\}) \prod_{j=1}^{b} \chi_{\hat{R}S}^{\hat{\chi}}(U_j)
\]

have exactly calculable values for any surface \(\mathcal{S}\). The topologically unique surface with \(b\) boundary curves, Euler characteristic \(\mathcal{E} = 2 - q - b\), and given orientability class may be constructed by gluing together \(2h + q' + b - 2\) three-holed spheres with \(h \geq 0\) handles and \(q' \geq 0\) cross-caps, given that \(2h + q' = q\). The surface is orientable if \(q' = 0\), nonorientable otherwise. Any such construction allows one to evaluate the partition function,\(^6-8\)

\[
Z(\mathcal{S}; \hat{R}S, \ldots, \hat{R}S) = \epsilon^{q'} (\dim \hat{R}S)^{\mathcal{E}} \exp \left( \frac{-\lambda AC_2(\hat{R}S)}{2N} \right),
\]
where $A$ is the area of $S$, $\sqrt{\lambda/N}$ is the gauge coupling constant, $\dim R_{S}$ denotes the dimension of the representation $R_{S}$, and $C_{2}(R_{S})$ is its quadratic Casimir. The factor

$$
\epsilon = \delta_{R_{S}}(-1)^{(r+s)(N-1)} = \delta_{R,R_{S}}
$$

only appears for nonorientable surfaces.

To make the string interpretation apparent, we adopt the double symmetric-group-transform basis

$$
\mathcal{V}_{\kappa,\lambda}(U) \equiv \sum_{R \in \mathcal{Y}_{r}} \chi_{R}(\kappa)\chi_{S}(\lambda)\chi_{R_{S}}^{SU}(U)
$$

in which $\kappa$ ($\lambda$) denotes a conjugacy class of $S_{r}$ ($S_{s}$), the symmetric group of permutations of $r$ ($s$) elements, and $\mathcal{Y}_{r}$ ($\mathcal{Y}_{s}$) denotes the set of Young tableaux with $r$ ($s$) cells. Due to the orthogonality and completeness of the symmetric group characters,

$$
\sum_{R \in \mathcal{Y}_{r}} \chi_{R}(\kappa_{1})\chi_{R}(\kappa_{2}) = C_{\kappa_{1}\kappa_{2}}\delta_{\kappa_{1}\kappa_{2}}, \quad \sum_{\kappa \in \mathcal{K}_{r}} \frac{1}{C_{\kappa}}\chi_{R}(\kappa)\chi_{R'}(\kappa) = \delta_{R,R'},
$$

in which $C_{\kappa} = r!/|\kappa|$ for $\kappa \in \mathcal{K}_{r}$, the set of conjugacy classes of $S_{r}$, with $|\kappa|$ denoting the number of permutations in class $\kappa$), the basis (5) is also orthogonal and complete

$$
\int dU \mathcal{V}_{\kappa_{1},\lambda_{1}}(U)\mathcal{V}_{\kappa_{2},\lambda_{2}}(U^{-1}) = C_{\kappa_{1}\lambda_{1},\kappa_{2}\lambda_{2}}\delta_{\kappa_{1}\kappa_{2}}\delta_{\lambda_{1}\lambda_{2}}\delta_{r_{1},r_{2}}\delta_{s_{1},s_{2}}.
$$

Here $\kappa_{i} \in \mathcal{K}_{r_{i}}$ and $\lambda_{i} \in \mathcal{K}_{s_{i}}$. The notation $\kappa \cdot \lambda$, for $\kappa \in \mathcal{K}_{r}$ and $\lambda \in \mathcal{K}_{s}$, indicates the conjugacy class in $S_{r} \oplus S_{s}$ composed of the outer product of elements of $\kappa$ and $\lambda$. Note that $C_{\kappa \cdot \lambda} = C_{\kappa}C_{\lambda}$.

For an arbitrary surface $S$ the partition function has the expansion

$$
Z(S; U_{1}, \ldots, U_{b}) = \sum_{r} \sum_{\kappa_{i} \in \mathcal{K}_{r_{i}}, \lambda_{i} \in \mathcal{K}_{s_{i}}} \prod_{j=1}^{b} \mathcal{V}_{\kappa_{j},\lambda_{j}}(U_{j}).
$$

The only nonvanishing contributions occur when $\kappa_{j} \in \mathcal{K}_{r}$ and $\lambda_{j} \in \mathcal{K}_{s}$ for all $j$ for given $r$ and $s$.

For $S$ orientable, Gross and Taylor have argued that $Z(S; \{\kappa_{1}, \lambda_{1}\}, \ldots, \{\kappa_{b}, \lambda_{b}\})$ has a natural interpretation as a weighted sum over $(r+s)$-sheeted covers of $S$, in which the classes $\kappa_{1}, \ldots, \kappa_{b} \in \mathcal{K}_{r}$ describe the boundary covering for the $r$ covering sheets with the same orientation as $S$, and $\lambda_{1}, \ldots, \lambda_{b} \in \mathcal{K}_{s}$ describe the boundary covering for the $s$ sheets with the opposite orientation as $S$. (The fact that they use a basis of permutations rather than conjugacy classes only leads to a slight difference as to whether the sheets that end on a boundary in one orientation sector are considered to be identical or not.)
For $\mathcal{S}$ nonorientable, one may show that the coefficients in expansion (8) have the form of a weighted sum over orientable covers of $\mathcal{S}$. One begins with the observation that all orientable covers of $\mathcal{S}$ (with $b$ boundary components) are covers of $\tilde{\mathcal{S}}$, the orientable double cover of $\mathcal{S}$, which has $2b$ boundary components. Thus, each orientable cover of $\mathcal{S}$ necessarily has an even number $2s$ of sheets, with boundary coverings specified by the conjugacy classes $\{\kappa_1, \lambda_1\}, \ldots, \{\kappa_b, \lambda_b\}$. (Here, $\kappa_j$ and $\lambda_j$ specify the coverings of two boundary components of $\tilde{\mathcal{S}}$ which correspond to a single boundary component of $\mathcal{S}$.) Using this set of covers, a calculation analogous to that in ref. 4 shows that $Z(\mathcal{S}; \{\kappa_1, \lambda_1\}, \ldots, \{\kappa_b, \lambda_b\})$ is a weighted sum over orientable covers of $\mathcal{S}$. From the relation between the expansion coefficients for any $\mathcal{S}$, $Z(\mathcal{S}; \{\kappa_1, \lambda_1\}, \ldots, \{\kappa_b, \lambda_b\}) = \sum_{R,S} Z(\mathcal{S}; \hat{R}S, \ldots,  \hat{R}S) \prod_{i=1}^b C_{\kappa_i, \lambda_i}^{-1} \chi_R(\kappa_i) \chi_S(\lambda_i) \delta_{r_i, r_s} \delta_{s_i, s_s}$, (9)

one sees that the self-conjugacy of the contributing representations (4) for $\mathcal{S}$ nonorientable leads to the covering constraint $r_i = s_i = r = s$ for all $i$. Even though $\mathcal{S}$ is nonorientable, near the $b$ boundaries a given local orientation in $\mathcal{S}$ can be lifted to the sheets of each connected (orientable, even-sheeted) component of the cover. The orientation lifted to the sheets that correspond to the boundary condition $\lambda_j$ will be opposite that lifted to the sheets corresponding to $\kappa_j$. Therefore, the two-sector structure of the boundary conditions remains associated with orientability even when $\mathcal{S}$ is nonorientable. However, the two boundary sectors are tied together in that every connected component of the cover has boundaries corresponding to both $\lambda_j$ and $\kappa_j$. In fact, these results allow one to consistently sew together covers of nonorientable surfaces with boundary with covers of orientable surfaces with boundary.

With the string interpretation of the coefficients $Z(\mathcal{S}; \{\kappa_1, \lambda_1\}, \ldots, \{\kappa_b, \lambda_b\})$ in hand, we first consider a Wilson loop on a homologically trivial curve on an orientable or nonorientable surface $\mathcal{S}$ (i.e., the curve divides $\mathcal{S}$ into two surfaces, $\mathcal{S}_a$ and $\mathcal{S}_b$). Instead of the standard Wilson loop expectation value associated with the gauge group representation $\hat{R}S$, $W_{\hat{R}S} = \int dU \ Z(\mathcal{S}_a; U) \chi_{\hat{RS}}(U) Z(\mathcal{S}_b; U^{-1})$, (10)

we will take the symmetric-group transforms of such Wilson loops

$W_{\kappa, \lambda} = \sum_{R \in Y_r, S \in Y_s} \chi_R(\kappa) \chi_S(\lambda) W_{\hat{RS}}$ (11)

as the fundamental objects amenable to a string interpretation. In order to evaluate

$W_{\kappa, \lambda} = \int dU \ Z(\mathcal{S}_a; U) \mathcal{V}_{\kappa, \lambda}(U) Z(\mathcal{S}_b; U^{-1})$

$= \sum_{\mu_a, \nu_a} Z(\mathcal{S}_a; \{\mu_a, \nu_a\}) Z(\mathcal{S}_b; \{\mu_b, \nu_b\}) \int dU \mathcal{V}_{\mu_a, \nu_a}(U) \mathcal{V}_{\kappa, \lambda}(U) \mathcal{V}_{\mu_b, \nu_b}(U^{-1})$ (12)
we need to calculate the product

$$V_{\mu,\nu}(U)V_{\kappa,\lambda}(U) = \sum_{R_1, S_1} \chi_{R_1}(\mu)\chi_{S_1}(\nu)\chi_{R_2}(\kappa)\chi_{S_2}(\lambda) \sum_{R_3, S_3} N_{R_1, R_2, S_3} \hat{R}_3 S_3 \chi^{SU}_{R_3 S_3}(U),$$

(13)

where $N_{R_1, R_2, S_3}$ is the multiplicity of the representation $\hat{R}_3 S_3$ in the SU($N$) tensor product of $\hat{R}_1 S_1$ and $\hat{R}_2 S_2$. If $N$ is sufficiently large, this multiplicity can be written as a sum of Littlewood-Richardson coefficients

$$N_{R_1, R_2, S_3} \hat{R}_3 S_3 = \sum_{\alpha, \beta} L_{(R_1/\alpha)(R_2/\beta)} \hat{R}_3 L_s(S_1/\alpha)(S_2/\beta) S_3,$$

(14)

where

$$(R/\alpha) \equiv \sum_D L_{\alpha D}^R D,$$

(15)

so that

$$N_{R_1, R_2, S_3} \hat{R}_3 S_3 = \sum_{\alpha, \beta, D_1, D_2, E_1, E_2} L_{\alpha D_1} L_{\beta D_2} L_s \hat{R}_3 L_s(S_1/\alpha)(S_2/\beta) S_3 L_{\alpha D_2}^R D,$$

(16)

Using this and the identities

$$\chi_R(\kappa_1 \cdot \kappa_2) = \sum_{R_1, R_2} L_{R_1 R_2}^R \chi_{R_1}(\kappa_1)\chi_{R_2}(\kappa_2)$$

(17)

and

$$\sum_{R \in \mathcal{Y}_{r_1 + r_2}} \chi_R(\kappa) L_{R_1 R_2} = \sum_{\kappa_1 \in \mathcal{K}_{r_1}} \sum_{\kappa_2 \in \mathcal{K}_{r_2}} \chi_{R_1}(\kappa_1)\chi_{R_2}(\kappa_2) \delta_{\kappa_1 \cdot \kappa_2}$$

(18)

in expansion (13), we find that

$$V_{\mu,\nu}(U)V_{\kappa,\lambda}(U) = \sum_{\mu_1, \mu_2, \nu_1, \nu_2} C_{\mu_1, \nu_1, \mu_2, \nu_2} \chi_{\mu_1}(\kappa_1)\chi_{\mu_2}(\kappa_2) \delta_{\mu_1 \cdot \mu_2} \delta_{\nu_1 \cdot \nu_2} \chi_{\nu_1}(\kappa_1)\chi_{\nu_2}(\kappa_2)$$

(19)

After further simplification, equation (12) becomes

$$W_{\kappa, \lambda} = C_{\kappa, \lambda} \sum_{\kappa_e, \kappa_o} \delta_{\kappa_e \cdot \kappa_o} \delta_{\lambda_e \cdot \lambda_o} \sum_{\nu_e} \sum_{\pi_o} C_{\pi_e, \pi_o} Z(S_a; \{\lambda_e \cdot \nu_e, \kappa_o \cdot \pi_o\}) Z(S_b; \{\kappa_e \cdot \nu_e, \lambda_o \cdot \pi_o\})$$

(20)

With the coefficients $Z(S)$ interpreted as weighted sums over (orientable) covers of the orientable or nonorientable surfaces $S_a$ and $S_b$, this formula provides a compact and geometrically transparent expression for the Wilson loop as a sum over maps from surfaces $W$ with boundary to $S$, with the boundary of each $W$ mapped to the curve in $S$ on which the Wilson loop is defined. For each pair of divisions $\kappa = \kappa_e \cdot \kappa_o$ and $\lambda = \lambda_e \cdot \lambda_o$ into subcycle conjugacy classes, we sew the orientation-preserving
covers of $S_a$ (with $\lambda_e \cdot \pi_e$ describing the boundary covering) to the orientation-preserving covers of $S_b$ (with boundary covering $\kappa_e \cdot \pi_e$) by letting the $r_e$ sheets of $\lambda_e$ end on the Wilson loop on one side, by letting the $s_e$ sheets of $\kappa_e$ end on the Wilson loop on the other side, and by sewing the $p_e$ sheets corresponding to $\pi_e$ on either side together in all possible ways, as described in section 3 of ref. 4. Similarly we sew the orientation-reversing covers of $S_a$ (with boundary covering $\kappa_o \cdot \pi_o$) to the orientation-reversing covers of $S_b$ (with boundary covering $\lambda_o \cdot \pi_o$) by letting the $r_o$ sheets of $\lambda_o$ end on the Wilson loop on one side, by letting the $s_o$ sheets of $\kappa_o$ end on the Wilson loop on the other side, and by sewing the $p_o$ sheets corresponding to $\pi_o$ on either side together in all possible ways, as was done in the other sector. The presence of the factor $C_{r,\lambda}$ simply means that we should consider sheets that end on the Wilson loop as distinct rather than identical when computing the proper weight to attach to each surface.

From equation (20), one easily deduces the relation $r_a - s_a = r_b - s_b + r_\lambda - r_e$ between the number of orientation-preserving ($r_a$) and orientation-reversing ($s_a$) sheets over $S_a$, and the number of orientation-preserving ($r_b$) and orientation-reversing ($s_b$) sheets over $S_b$, in terms of $r_\lambda$ and $r_e$, where $\kappa \in S_{r_\lambda}$ and $\lambda \in S_{r_e}$.

The special case $W_{r,a}$

\[ W_{r,0} = C_r \sum_{\kappa, \kappa_o} \delta_{r, \kappa, \kappa_o} \sum_{p_e=0}^{\kappa_e} \sum_{p_o=0}^{\kappa_o} C_{\pi_e, \pi_o} Z(S_a; \pi_e, \kappa_o, \pi_o) Z(S_b; \kappa_e, \pi_e, \pi_o) \]  

(21)

illustrates the structural parallel with the result\(^4\)

\[ W_r = C_r \sum_{\kappa_1, \kappa_2} \delta_{r, \kappa_1, \kappa_2} \sum_{\pi} \sum_{p} C_{\pi} Z(S_a; \kappa_1, \pi) Z(S_b; \kappa_2, \pi). \]  

(22)

for the gauge groups $SO(N)$ and $Sp(N)$.

For a homologically nontrivial curve which cuts the closed surface $S_c$ into an open surface $S$ with two boundaries, the $SU(N)$ Wilson loop expectation value is

\[ W_{r,\lambda} = \int dU \ Z(S; U, U^{-1}) \mathcal{W}_{r,\lambda}(U) \]

\[ = C_{r,\lambda} \sum_{\kappa, \kappa_o} \delta_{r, \kappa, \kappa_o} \delta_{\lambda, \lambda_e} \lambda_o \sum_{p_e=0}^{\kappa_e} \sum_{p_o=0}^{\kappa_o} C_{\pi_e, \pi_o} Z(S; \{\lambda_e \cdot \pi_e, \kappa_o \cdot \pi_o\}, \{\kappa_e \cdot \pi_e, \lambda_o \cdot \pi_o\}). \]

(23)

Again, this provides a compact prescription for sewing covers of $S$ together along the two boundaries to form maps from string worldsheets $W$ to $S_c$ with the boundary of $W$ mapped to the Wilson loop. Note that this Wilson loop vanishes unless $r$ and $\lambda$ are conjugacy classes of the same symmetric group and have subdivisions into cycles of the same length, since $\lambda_e$ ($\lambda_o$) must have the same length as $\kappa_e$ ($\kappa_o$).
For a nonorientable curve on a nonorientable surface $S_n$, the Wilson loop expectation value is given by

$$W_{\kappa,\lambda} = \int dU \ Z(S; U^2) V_{\kappa,\lambda}(U) = \sum_{\pi} \sum_{\rho} Z(S; \{\pi, \rho\}) M_{\{\kappa, \lambda\}, \{\pi, \rho\}}$$  \hspace{1cm} (24)$$

where

$$M_{\{\kappa, \lambda\}, \{\pi, \rho\}} = \sum_{P,Q,R,S} \chi_{P}(\pi) \chi_{Q}(\rho) \chi_{R}(\kappa) \chi_{S}(\lambda) \left( N_{\hat{P}Q,\hat{P}Q}^+ - N_{\hat{P}Q,\hat{P}Q}^- \right)$$  \hspace{1cm} (25)$$

is an integer that plays the same combinatorial role for sewing together covers of a single boundary (that is to be glued to itself to form $S_n$ from $S$) as $C_{\pi, \pi_o}$ does for sewing together the covers over two glued-together boundaries.

The problem of finding analogous compact formulae for intersecting Wilson loops that include the coupling terms in a way that makes the geometry transparent remains a challenge.

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References

[1] D. Gross, *Nucl. Phys.* **B400** (1993) 161; D. Gross and W. Taylor, *Nucl. Phys. B400* (1993) 181

[2] J. Minahan, *Phys. Rev.* **D47** (1993) 3430

[3] D. Gross and W. Taylor, *Nucl. Phys. B403* (1993) 395

[4] S. Naculich, H. Riggs, and H. Schnitzer, ‘The String Calculation of Wilson Loops in Two-Dimensional Yang-Mills Theory,’ Brandeis preprint BRX-TH-355, hepth#9406100

[5] R. King, *J. Math. Phys.* **12** (1971) 1588

[6] A. Migdal, *Sov. Phys. JETP* **42** (1975) 413; B. Rusakov, *Mod. Phys. Lett.* **A5** (1990) 693

[7] E. Witten, *Comm. Math. Phys.* **141** (1991) 153

[8] M. Blau and G. Thompson, *Int. Jour. Mod. Phys.* **A7** (1992) 3781

[9] G. Robinson, *Representation Theory of the Symmetric Group*, Mathematical Expositions **12** (University of Toronto Press, 1961)