Number of Eigenvalues of Non-selfadjoint Schrödinger Operators with Dilation Analytic Complex Potentials
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Abstract
In this paper, we give Lieb-Thirring type inequalities for isolated eigenvalues of $d$-dimensional non-selfadjoint Schrödinger operators with complex-valued dilation analytic potentials. In order to derive them, we prove that isolated eigenvalues and their multiplicities are invariant under complex dilation.

Keywords: Schrödinger operator, Lieb-Thirring (type) inequality, complex potential, dilation analytic potential

1 Introduction
In this paper, we consider Schrödinger operators on $\mathbb{R}^d$,

$$H := H_0 + V, \quad H_0 := -\Delta = -\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$$

with complex potentials $V$ which are dilation analytic (see Assumption 2.1 for definition), where the domain $\mathcal{D}(H_0)$ of $H_0$ is the second order Sobolev space $H^2(\mathbb{R}^d)$. We denote the real and the imaginary parts of $z \in \mathbb{C}$ by $\Re z$ and $\Im z$ respectively. We prove that the sum of moments of isolated eigenvalues of $H$ in the half-plane can be estimated by the Lieb-Thirring type inequalities:

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\[
\sum_{\lambda \in \sigma_d(H) \cap [C^\pm \cup (-\infty, 0)]} |\lambda|^\gamma \leq C_{\gamma,d} \int_{\mathbb{R}^d} |V_{\pm i/4}(x)|^{\gamma + d/2} \, dx
\]
for \( d, \gamma \geq 1 \), where \( C^\pm := \{ z \in \mathbb{C} : \pm \Im z > 0 \} \) and \( V_\theta(x) := V(e^{\theta}x) \). Here and hereafter, the formulas which contain \( \pm \) represent two formulas, one for the upper sign and the other for the lower sign. The space of \( L^p \)-functions from the space \( X \) to the space \( Y \) will be denoted by \( L^p(X; Y) \).

### 1.1 Lieb-Thirring Inequality for Real Potential

We recall the standard Lieb-Thirring inequality. We consider the self-adjoint Schrödinger operator

\[
H = H_0 + V
\]
in \( L^2(\mathbb{R}^d) \) defined by the closure of the quadratic form

\[
q_H(u) := \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx + \int_{\mathbb{R}^d} V(x)|u(x)|^2 \, dx, \quad u \in C_0^\infty(\mathbb{R}^d)
\]
where \( \nabla := (\partial_{x_j})_{1 \leq j \leq d} \) with the derivative \( \partial_{x_j} \) in the sense of distributions. If \( V \in L^{\gamma + d/2}(\mathbb{R}^d; \mathbb{R}) \) and

\[
\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots < 0 \tag{1}
\]
are negative eigenvalues of \( H \), the standard Lieb-Thirring inequality:

\[
\sum_{n=1}^{\infty} |\lambda_n|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-(x)^{\gamma + d/2} \, dx, \quad V_- := \max\{\pm V, 0\} \tag{2}
\]
is well-known (e.g., [3], [16], [17], [19], [20] and [21]), here \( L_{\gamma,d} \) is the sharp constant and \( \gamma \) satisfies

\[
\gamma \geq \begin{cases} \frac{1}{2}, & d = 1 \\ \gamma > 0, & d = 2 \\ \gamma \geq 0, & d \geq 3 \end{cases} \tag{3}
\]
If \( \gamma = 0 \), the left-hand side of (2) is the number of negative eigenvalues (1) of \( H \).

For the constant \( L_{\gamma,d} \), it is well-known that

\[
L_{\gamma,d}^{\text{cl}} \leq L_{\gamma,d}
\]
for all $\gamma \geq 0$ and $d \geq 1$, where $L_{\gamma,d}^{cl}$ is the classical constant:

$$L_{\gamma,d}^{cl} := (2\pi)^{-d} \int_{\mathbb{R}^d} (|\xi| - 1)^\gamma d\xi = \frac{\Gamma(\gamma + 1)}{2^d \pi^{d/2} \Gamma(\gamma + d/2 + 1)}.$$

In fact, it is proven that

$$L_{\gamma,d} = L_{\gamma,d}^{cl},$$

(4)

for all $d \geq 1$ if $\gamma \geq 3/2$ (cf. [21], [3], [17]), and

$$L_{\gamma,d} \leq \frac{\pi}{\sqrt{3}} L_{\gamma,d}^{cl}$$

(5)

for all $d, \gamma \geq 1$ ([9]). On the other hand, Helffer and Robert ([13]) proved that

$$L_{\gamma,d}^{cl} < L_{\gamma,d}$$

if

$$\frac{1}{2} \leq \gamma < \frac{3}{2}, \quad d = 1$$

and

$$\gamma < 1, \quad d \geq 2.$$

Moreover, by virtue of the diamagnetic inequality, the standard Lieb-Thirring inequality is satisfied for magnetic Schrödinger operators with magnetic vector potentials $A(x) = (a_j(x))_{j=1}^d \in L^2_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$:

$$H(A) := -\nabla^2 + V$$

where $i := \sqrt{-1}$ (see [20], P.66).

For further information about $L_{\gamma,d}$ and $L_{\gamma,d}^{cl}$, see Remarks of Theorem 12.4 of [19].

Lieb-Thirring inequality is the key ingredient in the proof of the stability of matter by Lieb and Thirring (e.g., see [20]), and is used for obtaining the efficient lower bound for the energies of fermions.

### 1.2 Lieb-Thirring Inequality for Complex Potential

The Lieb-Thirring inequality (2) has recently been extended to Schrödinger operators with complex potentials $V \in L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C})$ for $d, \gamma \geq 1$.

We suppose that $V$ is a complex-valued potential which is $H_0$-compact. Then

$$H = H_0 + V, \quad \mathcal{D}(H) = \mathcal{D}(H_0)$$

is quasi-maximal accretive ([14]), and the spectrum $\sigma(H)$ of $H$ consists of the essential spectrum $\sigma_{\text{ess}}(H) = [0, \infty)$ and the discrete spectrum $\sigma_d(H)$ which consists of isolated eigenvalues of $H$ of finite algebraic multiplicities:

$$m_\lambda := \sup_{k \in \mathbb{N}} \left( \dim \ker[(H - \lambda)^k] \right).$$
This can be seen via analytic Fredholm theory applied to
\[(H-z)^{-1} = (H_0-z)^{-1}(1 + V(H_0-z)^{-1})^{-1},\]
because
- \( F(z) := V(H_0-z)^{-1} \) is an analytic function of \( z \in \mathbb{C} \setminus [0,\infty) \) with values in the space of compact operators, and because
- the inverse of \( 1 - F(z) \) exists for \( z \) with sufficiently large \(|\Im z|\).

Since \( H \) is non-selfadjoint, \( m_\lambda \) is in general different from the geometric multiplicity defined by
\[ g_\lambda := \dim \{ u \in L^2(\mathbb{R}^d) : (H - \lambda)u = 0 \}, \]
and we count the number of eigenvalues according to their algebraic multiplicities.

Then, following estimates for the sum of moments of eigenvalues of \( H \) outside the cone \( \{ z \in \mathbb{C} : |\Im z| < \kappa \Re z \} \) for any positive constant \( \kappa \) are proven:

**Theorem 1.1** (Frank, Laptev, Lieb and Seiringer; [11]). *Let \( d \geq 1 \) and \( \gamma \geq 1 \). Suppose that \( V \in L^{\gamma+d/2}(\mathbb{R}^d;\mathbb{C}) \). Then, for any \( \kappa > 0 \),

\[ \sum_{|\Im \lambda| \geq \kappa \Re \lambda} |\lambda|^\gamma \leq C_{\gamma,d} \left( 1 + \frac{2}{\kappa} \right)^{\gamma+d/2} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2} \, dx \] (6)

where
\[ C_{\gamma,d} := 2^{1+\gamma/2+d/4} L_{\gamma,d} \] (7)
and \( L_{\gamma,d} \) is the constant of [2]. In particular,

\[ \sum_{\Re \lambda \leq 0} |\lambda|^\gamma \leq C_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2} \, dx. \] (8)

Frank et al. ([11]) conjecture that (6) and (8) in Theorem 1.1 hold for \( \gamma \) satisfying (3). On the other hand, observing that (2) is equivalent to

\[ \sum_{n=1}^{\infty} \frac{\text{dist}(\lambda_n; [0,\infty))^{\gamma+d/2}}{|\lambda_n|^{d/2}} \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-(x)^{\gamma+d/2} \, dx \] (9)
where \( \text{dist}(x; \Omega) \) is a distance from a point \( x \) to the domain \( \Omega \) and \( \{ \lambda_n \} \) are negative eigenvalues of \( H \) given by (1), Demuth, Hansmann and Katriel(6) proposed to study the estimate:

\[
\sum_{\lambda \in \sigma_d(H)} \frac{\text{dist}(\lambda; [0, \infty))^{\gamma+d/2}}{|\lambda|^{d/2}} \leq C_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2} \, dx \tag{10}
\]

for the constant \( C_{\gamma,d} \) independent of \( V \in L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C}) \) and for \( \gamma \) and \( d \) satisfying (3). They(5) actually proved by applying (6) that for any \( \gamma \geq 1 \) and \( 0 < \tau < 1 \),

\[
\sum_{\lambda \in \sigma_d(H)} \frac{\text{dist}(\lambda; [0, \infty))^{\gamma+d/2+\tau}}{|\lambda|^{d/2+\tau}} \leq C_{\gamma,d,\tau} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2} \, dx \tag{11}
\]

with constant

\[ C_{\gamma,d,\tau} = (\text{const.}) \cdot \frac{1}{\tau}. \]

Note that this constant blows up as \( \tau \downarrow 0 \) (see also [7] where a similar estimate is obtained).

Related to this problem, Frank and Sabin([12]) proved that if \( d \geq 1 \) and \( V \in L^p(\mathbb{R}^d; \mathbb{C}) \) such that

\[
\begin{cases}
  p = 1, & d = 1, \\
  1 < p \leq \frac{3}{2}, & d = 2, \\
  \frac{d}{2} < p \leq \frac{d+1}{2}, & d \geq 3,
\end{cases}
\]

then

\[
\sum_{\lambda \in \sigma_d(H)} \frac{\text{dist}(\lambda; [0, \infty))^{1-\varepsilon}}{|\lambda|^{(1-\varepsilon)/2}} \leq C_{d,p,\varepsilon} \left( \int_{\mathbb{R}^d} |V(x)|^p \, dx \right)^{(1-\varepsilon)/(2p-d)}
\]

where \( \varepsilon \) is the non-negative number fulfilling the followings:

\[
\begin{align*}
  \varepsilon > 1, & \quad d = 1 \\
  \varepsilon \geq 0, & \quad d \geq 2 \text{ and } \frac{d}{2} \leq p \leq \frac{d^2}{2d-1} \\
  \varepsilon > \frac{(2d-1)p-d^2}{d-p}, & \quad d \geq 2 \text{ and } \frac{d^2}{2d-1} \leq p \leq \frac{d+1}{2}
\end{align*}
\]

\[ .
\]
We mention that Cuenin, Laptev, Safronov etc. studied the eigenvalues of \( H \) which are close to \([0, \infty)\). For example, they proved\(^{(20)}\) that if \( \Re V \geq 0 \) is bounded and \( \Im V \in L^p(\mathbb{R}^d) \) and
\[
\begin{cases}
p \geq 1, & d = 1 \\
p > \frac{d}{2}, & d \geq 2
\end{cases}
\]
then one has
\[
\sum_{\lambda \in \sigma_d(H)} \left( \frac{\Im \lambda}{|\lambda + 1|^2 + 1} \right)_+^p \leq C(d, p) \int_{\mathbb{R}^d} (\Im V)_+^p \, dx
\]
where
\[
C(d, p) := (2\pi)^{-d} \int_{\mathbb{R}^d} \frac{d\xi}{(|\xi|^2 + 1)^{p'}}.
\]

2 Assumption and Main Results

We write \((\cdot, \cdot)\) for the \(L^2(\mathbb{R}^d)\)-inner product and \(\| \cdot \|\) for the \(L^2(\mathbb{R}^d)\)-norm:
\[
(f, g) := \int_{\mathbb{R}^d} f(x)\overline{g(x)} \, dx, \quad \|f\| := (f, f)^{1/2}.
\]

2.1 Dilation Analytic Method for Complex Potential

We consider the 1-parameter unitary group \( U(\theta), \theta \in \mathbb{R}, \) on \(L^2(\mathbb{R}^d)\) defined by
\[
U(\theta)u(x) := e^{i\theta/2}u(e^{\theta}x), \quad u \in L^2(\mathbb{R}^d).
\]
(12)

We now suppose that \( V \) fulfills the following assumption. Recall that \( D(T) \) is the domain of operator \( T \).

**Assumption 2.1.** Let \( d, \gamma \geq 1 \) and \( \alpha > 0 \).

a) \( V \) is the multiplication operator with complex-valued measurable function satisfying \( V \in L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C}) \).

b) \( V \) is \( H_0 \)-compact, that is, \( D(V) \supset D(H_0) = H^2(\mathbb{R}^d) \) and \( V(H_0 + 1)^{-1} \) is compact.

c) The function \( V_\theta(x) := V(e^\theta x) \) originally defined for \( \theta \in \mathbb{R} \) has an analytic continuation into the complex strip
\[
\mathcal{S}_\alpha := \{ z \in \mathbb{C} : |\Im z| < \alpha \}
\]
as an \( L^{\gamma+d/2}(\mathbb{R}^d; \mathbb{C}) \)-valued function.
d) $V_\theta(H_0 - z)^{-1}$ originally defined for $\theta \in \mathbb{R}$ can be extended to $\mathcal{S}_\alpha$ as a $\mathcal{B}(L^2(\mathbb{R}^d))$-valued analytic function.

We remark that b) and d) are results of a) and c) respectively unless $d = 1$ and $1 \leq \gamma < 3/2$ or $d = 2$ and $\gamma = 1$.

In what follows, we fix $d, \gamma \geq 1$. We call $V$ fulfilling Assumption 2.1 the dilation analytic complex potential on $\mathcal{S}_\alpha$ (Hereafter we say ‘$V$ is dilation analytic on $\mathcal{S}_\alpha$’ for simplicity). We define

$$
\begin{align*}
H_0(\theta) &:= e^{-2\theta} H_0, \\
H(\theta) &:= U(\theta) H U(\theta)^{-1} = H_0(\theta) + V_\theta = e^{-2\theta} (H_0 + e^{2\theta} V_\theta)
\end{align*}
$$

for $\theta \in \mathcal{S}_\alpha$. It is obvious that $H_0(\theta)$ and $H(\theta)$ are operator-valued holomorphic functions of type (A) of $\theta \in \mathcal{S}_\alpha$ in the sense of Kato ([14]). Moreover

$$
H(\theta + \theta') = U(\theta') H(\theta) U(\theta')^{-1}; \quad \theta \in \mathcal{S}_\alpha, \; \theta' \in \mathbb{R},
$$

(14)
since this is true for $\theta \in \mathbb{R}$ and both side of (14) are $\mathcal{B}(H^2(\mathbb{R}^d); L^2(\mathbb{R}^d))$-valued analytic functions of $\theta \in \mathcal{S}_\alpha$. In particular, $\sigma(H(\theta))$ is independent of $\mathbb{R}\theta$.

The following result for real-valued potential is well-known. We remark that the same result is satisfied for complex-valued potential.

**Proposition 2.2.** If $V$ is dilation analytic on $\mathcal{S}_\alpha$, one has

$$
\sigma_{\text{ess}}(H(\theta)) = e^{-2\theta} [0, \infty) = \{e^{-2\theta} x : x \in [0, \infty)\}
$$

(15)
for any $\theta \in \mathcal{S}_\alpha$.

**Proof.** Define for fixed $\theta \in \mathcal{S}_\alpha$ that

$$
\tilde{H}(\theta) := H_0 + e^{2\theta} V_\theta = e^{2\theta} H(\theta).
$$

(16)
That $\sigma(\tilde{H}(\theta)) \setminus [0, \infty)$ consists of isolated eigenvalues of finite multiplicities may be proved as previously by the argument using analytic Fredholm theory for $F(z) = -e^{2\theta} V_\theta (H_0 - z)^{-1}$. In particular, $\sigma_{\text{ess}}(\tilde{H}(\theta)) \subset [0, \infty)$. We show that $[0, \infty) \subset \sigma_{\text{ess}}(\tilde{H}(\theta))$. Suppose that there is an open interval $(a, b) \subset [0, \infty)$ such that $(a, b) \subset \mathbb{C} \setminus \sigma(\tilde{H}(\theta))$. Then, $\sigma(H_0) \cap (a, b)$ must be a discrete set by virtue of the argument above where the roles of $\tilde{H}(\theta)$ and $H_0$ are replaced. This is of course impossible and $[0, \infty) \subset \sigma_{\text{ess}}(\tilde{H}(\theta))$. Therefore,

$$
\sigma_{\text{ess}}(\tilde{H}(\theta)) = \sigma_{\text{ess}}(H_0) = [0, \infty),
$$

(17)
and

$$
\sigma_{\text{ess}}(H(\theta)) = e^{-2\theta} \sigma_{\text{ess}}(\tilde{H}(\theta)) = e^{-2\theta} [0, \infty)
$$

by (16) and (17).
2.2 Estimates on Eigenvalues in $\mathbb{C} \setminus [0, \infty)$

The main result in this paper is the following theorem.

**Theorem 2.3.** Suppose $V$ satisfies Assumption 2.1 for some $\alpha > \pi/4$. Let $C_{\gamma,d}$ be the constant given by (7). Then, we have

$$
\sum_{\lambda \in \sigma_d(H) \cap [C^\pm \cup (-\infty,0)]} |\lambda|^{\gamma} \leq C_{\gamma,d} \int_{\mathbb{R}^d} |V_{\pm\pi i/4}(x)|^{\gamma+d/2} \, dx. \quad (18)
$$

We believe that in general the integrals of the right-hand side of (18) are bigger than $\int |V(x)|^{\gamma+d/2} \, dx$.

**Example.** We consider a 1-dimensional complex potential

$$
V(x) = \frac{c}{(1+x^2)^s}, \quad x \in \mathbb{R} \quad (19)
$$

where $c \in \mathbb{C}$ and $1/2 < s < 1$. It is easy to see that $V$ satisfies Assumption 2.1 for any $\alpha > \pi/4$. Then, we obtain

$$
\int_{-\infty}^{\infty} |V_{\pi i/4}(x)|^{3s/2} \, dx = |c|^{3/2} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{1 - \sqrt{2}x^2 + x^4}^{3s/2}}
\geq |c|^{3/2} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{3s/2}}
= \int_{-\infty}^{\infty} |V(x)|^{3/2} \, dx.
$$

as expected. If $V : \mathbb{R}^d \to \mathbb{C}$ is a complex-valued potential like (19), that is, if $V$ satisfies that

$$
|V(x)| \leq \text{const.} \frac{1}{(1+|x|^2)^s}, \quad \frac{1}{2} < s < 1,
$$

it is known (24) that all non-real eigenvalues of $H = -\Delta + V$ are in a disc of a finite radius.

**Corollary 2.4.** Suppose that $V$ satisfies the assumption of Theorem 2.3. Then,

$$
\sum_{\lambda \in \sigma_d(H)} |\lambda|^{\gamma} \leq C_{\gamma,d} \int_{\mathbb{R}^d} \left( |V_{\pi i/4}(x)|^{\gamma+d/2} + |V_{-\pi i/4}(x)|^{\gamma+d/2} \right) \, dx
$$

where $C_{\gamma,d}$ is the constant of (7).
Corollary 2.5. We denote the number of the isolated eigenvalues of $H$ in $\mathbb{C}$ by $N(H; \mathbb{C})$. Suppose that $V$ satisfies the assumption of Theorem 2.3. Then, we have

$$N(H; \mathbb{C}) \leq \tilde{C}_{\gamma,d} \int_{\mathbb{R}^d} \left( |V_{\pi/4}(x)|^{\gamma+d/2} + |V_{-\pi/4}(x)|^{\gamma+d/2} \right) dx$$

where $\tilde{C}_{\gamma,d} := C_{\gamma,d}/\inf_{\lambda \in \sigma_d(H)} |\lambda|^\gamma$.

Proof. The statement follows immediately from Theorem 2.3, since

$$\sum_{\lambda \in \sigma_d(H)} |\lambda|^\gamma \geq \left( \inf_{\lambda \in \sigma_d(H)} |\lambda|^\gamma \right) \sum_{\lambda \in \sigma_d(H)} 1.$$

3 Proof of Theorem 2.3

The estimate (8) of Theorem 1.1 plays an important role in the proof of the main theorem. We begin with the following lemma. Recall that $\sigma(H(\theta))$ is discrete in $\mathbb{C} \setminus \sigma_{\text{ess}}(H(\theta))$ and all $\lambda \in \sigma_d(H(\theta))$ have finite algebraic multiplicities.

Lemma 3.1. Suppose $0 < \alpha < \pi/2$ and $\theta \in \mathcal{S}_\alpha \cap \mathbb{C}^\pm$. Then, for $\lambda \in \mathbb{C}^\pm$, $\lambda \in \sigma_d(H)$ if and only if $\lambda \in \sigma_d(H(\theta))$.

Proof. We prove only the case $\theta \in \mathcal{S}_\alpha \cap \mathbb{C}^+$ and $\lambda \in \mathbb{C}^+$. The other case may be proved similarly. Suppose that $\lambda(\theta) \in \mathbb{C}^+$ is an eigenvalue of $H(\theta)$ for $\theta \in \mathcal{S}_\alpha \cap \mathbb{C}^+$. We show that if $\lambda$ is an eigenvalue of $H(\theta_0)$ for $\theta_0 \in \mathcal{S}_\alpha \cap \mathbb{C}^+$ then $\lambda$ is an eigenvalue of $H(\theta)$ for all $\theta \in \mathcal{S}_\alpha \cap \mathbb{C}^+$. To see this, we recall that $\{H(\theta)\}_{\theta \in \mathcal{S}_\alpha \cap \mathbb{C}^+}$ is an analytic family of type (A) in the sense of Kato. Then, Theorem 1.8 of chapter VII §1.3 (or chapter II §1) of [14] implies that those $\lambda(\theta) \in \sigma_d(H(\theta))$ such that $\lambda(\theta) \rightarrow \lambda$ as $\theta \rightarrow \theta_0$ are given by the branches of one or several analytic functions as Puiseux series. Moreover, as remarked above, $\lambda(\theta)$ is independent of $\Re \theta$. It follows that $\lambda(\theta) = \lambda$ for all $\theta$ near $\theta_0$, and

$$\lambda(\theta) = \lambda, \quad \theta \in \mathcal{S}_\alpha \cap \mathbb{C}^+.$$

The following fact is also important.
Lemma 3.2. Let $V$ be dilation analytic on $\mathcal{S}_\alpha$ with $\alpha > \pi/4$. Suppose that $\lambda \in \mathbb{C}^\pm$ is an eigenvalue of $H$ and $\theta \in \mathcal{S}_\alpha \cap \mathbb{C}^\pm$. Then, the algebraic multiplicity of $\lambda$ as eigenvalue of $H(\theta)$ coincides with that of $\lambda$ as eigenvalue of $H$.

Proof. We define two Riesz projection onto the generalized eigenspace:

$$P_\lambda := \frac{-1}{2\pi i} \oint_{\Gamma_\varepsilon} (H - z)^{-1} \, dz, \quad P_\lambda(\theta) := \frac{-1}{2\pi i} \oint_{\Gamma_\varepsilon} (H(\theta) - z)^{-1} \, dz$$

for $\theta \in \mathcal{S}_\alpha \cap \mathbb{C}^\pm$, where $\Gamma_\varepsilon$ is the contour around $\lambda$ defined by

$$\Gamma_\varepsilon := \{ z \in \mathbb{C} : |z - \lambda| = \varepsilon \}$$

with sufficient small $\varepsilon > 0$ such that $\Gamma_\varepsilon$ does not enclose any other point in $\sigma(H)$ or $\sigma(H(\theta))$ except $\lambda$. Here, notice that $P_\lambda$ is not necessary an orthogonal projection, since $H$ is non-selfadjoint. It is well-known that $P_\lambda, P_\lambda(\theta)$ are projections and

$$P_\lambda^2 = P_\lambda, \quad P_\lambda(\theta)^2 = P_\lambda(\theta)$$

onto the respective generalized eigenspaces of $H$ and $H(\theta)$ associated with the eigenvalue $\lambda$. It follows that

$$m_\lambda = \text{rank} \, P_\lambda, \quad m_{\lambda(\theta)} = \text{rank} \, P_\lambda(\theta).$$

We denote the orthogonal projection onto $M_\lambda(\theta) := P_\lambda(\theta)L^2(\mathbb{R}^d)$ by $P_\lambda^\perp(\theta)$. In order to prove this lemma, we show that the dimension of $M_\lambda(\theta)$ is independent of $\theta$. $P_\lambda(\theta)$ is analytic with respect to $\theta$. It is known that

$$\|P_\lambda^\perp(\theta) - P_\lambda^\perp(\sigma)\| \leq \|P_\lambda(\theta) - P_\lambda(\sigma)\|$$

for any $\theta, \sigma \in \mathcal{S}_\alpha$ (see Theorem 6.35 in Chapter I §6.8 of [14]). Since $P_\lambda(\theta) \to P_\lambda(\sigma)$ as $\theta \to \sigma$ in norm by the analyticity of $P_\lambda(\theta)$ (see (20)),

$$\|P_\lambda^\perp(\theta) - P_\lambda^\perp(\sigma)\| < 1.$$ 

Thus, $P_\lambda^\perp(\theta)$ and $P_\lambda^\perp(\sigma)$ are unitarily equivalent by Theorem 6.32 in Chapter I §6.8 of [14]. Hence

$$\dim M_\lambda(\theta) = \dim M_\lambda(\sigma),$$

and the proof of this lemma is finished by putting $\sigma = 0$. \qed
We denote $\mathbb{R}_\pm := \{ \pm x : x > 0 \}$ and $i \mathbb{R}_\pm := \{ i x : x \in \mathbb{R}_\pm \}$.

Proof of Theorem 2.3 For eigenvalues $\tilde{\lambda}(\theta)$ of $\tilde{H}(\theta) = e^{2\theta} \hat{H}(\theta)$, we write

$$\lambda(\theta) = e^{-2\theta} \tilde{\lambda}(\theta) \quad (22)$$

for the corresponding eigenvalue of $\hat{H}(\theta)$, $\theta \in \mathcal{S}_\alpha$.

We first set $\theta = \pi i / 4$. Lemma 3.1, Lemma 3.2 and (22) imply that

$$e^{\pi i / 2} [\sigma_d(H) \cap (\mathbb{C}^+ \cup \mathbb{R}_-)] = \sigma_d(\tilde{H}(\pi i / 4)) \cap \{ \Re z < 0 \} \cup i \mathbb{R}_- \quad (23)$$

including their multiplicities. We next set $\theta = -\pi i / 4$. We likewise have

$$e^{-\pi i / 2} [\sigma_d(H) \cap (\mathbb{C}^+ \cup \mathbb{R}_-)] = \sigma_d(\tilde{H}(-\pi i / 4)) \cap \{ \Re z < 0 \} \cup i \mathbb{R}_+ \quad (24)$$

including their multiplicities. We write $\mathcal{S}_\pm(\tilde{H}(\pm \pi i / 4))$ for the right-hand side of (23) and (24) respectively, and apply the estimate (8) of Theorem 2.3 to $\tilde{H}(\pm \pi i / 4) = H_0 \pm i V(e^{\pi i / 2} \cdot)$. This implies

$$\sum_{\lambda \in \sigma_d(H) \cap (\mathbb{C}^+ \cup \mathbb{R}_-)} |\lambda|^{\gamma} = \sum_{\tilde{\lambda}(\pm \pi i / 4) \in \mathcal{S}_\pm(\tilde{H}(\pm \pi i / 4))} |\tilde{\lambda}(\pm \pi i / 4)|^{\gamma} \leq C_{\gamma,d} \int_{\mathbb{R}^d} |V(\pm \pi i / 4)(x)|^{\gamma + d / 2} dx.$$  

Since $m_{\lambda(\pi i / 4)} = m_{\tilde{\lambda}(-\pi i / 4)} = m_{\lambda}$ for any $\lambda \in \sigma_d(H) \cap (\mathbb{C}^+ \cup \mathbb{R}_-)$ by virtue of Lemma 3.2 this completes the proof.

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