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Irreducible modules with highest weight vectors over modular Witt and special Lie superalgebras

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Abstract: Let $F$ be an arbitrary field of characteristic $p > 2$. In this paper we study irreducible modules with highest weight vectors over Witt and special Lie superalgebras of $F$. The same irreducible modules of general and special linear Lie superalgebras, which are the 0-th part of Witt and special Lie superalgebras in certain $\mathbb{Z}$-grading, are also considered. Then we establish a certain connection called a $P$-expansion between these modules.

Keywords: modular Lie superalgebra, graded module, irreducibility, highest weight

MSC: 17B10, 17B50, 17B70

1 Introduction

Let $F$ be an arbitrary field of characteristic $p > 2$ and $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the residue class ring mod 2. Throughout this paper we have assumed that all vector spaces, linear mappings and tensor products are over the underlying base field $F$. Assume that $x$ is a $\mathbb{Z}_2$-homogeneous element and $d(x)$ is the $\mathbb{Z}_2$-degree of $x$, if $d(x)$ occurs in an expression.

In 1967, Rudakov and Shafarevich [1] described all the irreducible representations of $\mathfrak{sl}(2)$ over an algebraically closed field $F$ of characteristic $p > 2$. They demonstrated that in addition to the $p$-representations known since 1930s, all of which possess a highest and lowest weight and are labeled by one integer, there are other representations that form a variety of dimension 3. They described the $g$-modules not possessing a $p$-structure for Lie algebras $g$ with Cartan matrix. In 1974, Rudakov [2] described irreducible $g$-modules, where $g$ is a simple Lie algebras of vector fields over $\mathbb{C}$, for modules dual to modules of (formal) tensor fields. For a review of similar results and the importance of this particular type of module, we refer the readers to the papers [3, 4]. In the 1980s, Krylyuk [5, 6] studied the highest weight modules over the algebras of vector fields of series $W$ and $S$ possessing a $p$-structure. Shu [7] discussed the representations of Cartan type Lie algebras in characteristic $p > 2$ from the viewpoint of reducing rank. Zhang [8] constructed the simple $L$-modules with nonsingular characters and some simple modules with singular characters, where $L$ is a restricted simple Lie algebra of Cartan type.

Since the classification of all the finite-dimensional simple complex Lie superalgebras was done by Kac [9], the problems of constructing a unified representation theory for all the types of simple Lie superalgebras...
has become more important than ever. Kac obtained essential results for the highest weight representations of classical Lie superalgebras [10, 11]. Most of Kac’s results can also be extended to the remaining classical series of Lie superalgebras [12–14], while the representations of Lie superalgebras of Cartan type have been studied in [15, 16]. Recent work on the representation theory of modular Lie superalgebras of Cartan type can also be found in [17–19].

The structure of gradation plays a critical role in the research of Lie algebras and superalgebras. Shen [20–22] introduced an important notion which is called the mixed product and realized the graded modules over Lie algebras of Cartan type. The method of the mixed product can also be applied to Lie superalgebras of Cartan type over fields of characteristic zero [23]. In the case of modular Lie superalgebras, Zhang [24] has obtained the \( \mathbb{Z} \)-graded modules over finite-dimensional Hamiltonian Lie superalgebras.

This paper generalizes some of Shen’s results in [20–22]. A brief summary of the relevant concepts in generalized Witt and special modular Lie superalgebras is presented in Section 2. Section 3 gives some properties of the graded modules over modular Lie superalgebras. In Section 4, the certain connection which is called a P-expansion between irreducible highest weight representations of generalized Witt and special modular Lie superalgebras, and the same irreducible highest weight representations of general linear Lie superalgebras \( \mathfrak{gl}(m, n) \) and special linear Lie superalgebras \( \mathfrak{sl}(m, n) \), is established.

## 2 Generalized Witt and special modular Lie superalgebras

In addition to the standard notation \( \mathbb{Z}, \mathbb{N} \) and \( \mathbb{N}_0 \) is used for the set of positive integers and the set of nonnegative integers, respectively. Generally, let \( m, n \) denote fixed integers in \( \mathbb{N} \setminus \{1, 2\} \). For \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m \), we put \( |\alpha| := \sum_{i=1}^m \alpha_i \). Following [25], let \( \mathcal{O}(m) \) denote the divided power algebra over \( \mathbb{F} \) with an \( \mathbb{F} \)-basis \( \{x(\alpha) | \alpha \in \mathbb{N}_0^m\} \). For \( \varepsilon_i := (\delta_{i1}, \ldots, \delta_{im}) \), we abbreviate \( x(\varepsilon_i) \) to \( x_i \), \( i = 1, 2, \ldots, m \), where \( \delta_{ij} \) is Kronecker delta.

Let \( \Lambda(n) \) be the exterior superalgebra over \( \mathbb{F} \) in \( n \) variables \( x_{m+1}, x_{m+2}, \ldots, x_{m+n} \) and \( \mathcal{O}(m, n) \) denote the tensor product \( \mathcal{O}(m) \otimes \mathcal{O}(n) \). Clearly, \( \mathcal{O}(m, n) \) is an associative superalgebra with a \( \mathbb{Z}_2 \)-gradation induced by the trivial \( \mathbb{Z}_2 \)-gradation of \( \mathcal{O}(m) \) and the natural \( \mathbb{Z}_2 \)-gradation of \( \Lambda(n) \). Moreover, \( \mathcal{O}(m, n) \) is super-commutative. For \( g \in \mathcal{O}(m), f \in \mathcal{O}(n) \), we write \( gf \) for \( g \otimes f \). The following formulae hold in \( \mathcal{O}(m, n) \):

\[
\begin{align*}
    x(\alpha) x(\beta) & = \binom{\alpha + \beta}{\alpha} x(\alpha + \beta), \quad \text{for } \alpha, \beta \in \mathbb{N}_0^m, \\
    x_i x_j & = -x_j x_i \quad \text{for } i, j = m + 1, \ldots, m + n, \\
    x(\alpha) x_j & = x_j x(\alpha) \quad \text{for } \alpha \in \mathbb{N}_0^m, j = m + 1, \ldots, m + n,
\end{align*}
\]

where \( \binom{\alpha + \beta}{\alpha} := \prod_{i=1}^m \binom{\alpha_i + \beta_i}{\alpha_i} \). Put \( Y_0 := \{1, 2, \ldots, m\}, Y_1 := \{m + 1, m + 2, \ldots, m + n\} \) and \( Y := Y_0 \cup Y_1 \). Set

\[
    \mathbb{B}_k := \{(i_1, i_2, \ldots, i_k) | m + 1 \leq i_1 < i_2 < \cdots < i_k \leq m + n\}
\]

and \( \mathbb{B} := \bigcup_{k=0}^n \mathbb{B}_k \), where \( \mathbb{B}_0 := \emptyset \). For \( u = (i_1, i_2, \ldots, i_k) \in \mathbb{B}_k \), set \( |u| := k \), \( |\emptyset| := 0 \), \( x^0 := 1 \), \( x^u := x_{i_1} x_{i_2} \cdots x_{i_k} \) and \( x^0 := x_{m+1} x_{m+2} \cdots x_{m+n} \). Clearly, \( \{x(\alpha) x^u | \alpha \in \mathbb{N}_0^m, u \in \mathbb{B}\} \) constitutes an \( \mathbb{F} \)-basis of \( \mathcal{O}(m, n) \). Let \( D_1, D_2, \ldots, D_{m+n} \) be the linear transformations of \( \mathcal{O}(m, n) \) such that

\[
    D_r(x(\alpha) x^u) = \begin{cases} 
        x(\alpha - r) x^u, & r \in Y_0, \\
        x(\alpha) \partial(x^u)/\partial x^r, & r \in Y_1,
    \end{cases}
\]

where \( \partial/\partial x^r \) is the superderivation of \( \Lambda(n) \) such that \( \partial x^r/\partial x^s = \delta_{rs} \) for \( r, s \in Y_1 \). For more details on superderivations for Lie superalgebras, the reader is referred to [9, 26]. \( D_1, D_2, \ldots, D_{m+n} \) are superderivations of the superalgebra \( \mathcal{O}(m, n) \). Let

\[
    W(m, n) := \left\{ \sum_{r \in Y} f_r D_r \mid f_r \in \mathcal{O}(m, n) \right\}.
\]
Then \( W(m, n) \) is a Lie superalgebra, which is contained in \( \text{Der}(\mathcal{O}(m, n)) \), where \( \text{Der}(\mathcal{O}(m, n)) \) denotes the superderivation space of \( \mathcal{O}(m, n) \). Obviously, \( d(D_i) = \tau(i) \), where

\[
\tau(i) := \begin{cases} 0, & i \in Y_0, \\
1, & i \in Y_1. \end{cases}
\]

An easy verification shows that

\[
[fD, gE] = fD(g)E - (-1)^{d(D_i)d(gE)}gE(f)D + (-1)^{d(D_i)d(g)}fg[D, E]
\]

for \( f, g \in \mathcal{O}(m, n) \), \( D, E \in \text{Der}(\mathcal{O}(m, n)) \). In particular, the following formula holds in \( W(m, n) \):

\[
[fD_r, gD_s] = fD_r(g)D_s - (-1)^{d(D_r)d(gD_s)}gD_s(f)D_r
\]

for \( f, g \in \mathcal{O}(m, n) \), \( r, s \in Y \).

Let \( t := (t_1, t_2, \ldots, t_m) \in \mathbb{N}^m \), \( \pi := (\pi_1, \pi_2, \ldots, \pi_m) \), where \( \pi_i := p^i - 1, i \in Y_0 \). Let \( A := \{ a \in \mathbb{N}_{0}^{m} | a_i \leq \pi_i, i = 1, 2, \ldots, m \} \). Then

\[
\mathcal{O}(m, n, t) := \text{span}_{\mathbb{F}} \left\{ x^{(a)}x^{u} | a \in A, u \in \mathbb{B} \right\}
\]

is a finite-dimensional subalgebra of \( \mathcal{O}(m, n) \) with a natural \( \mathbb{Z} \)-gradation \( \mathcal{O}(m, n, t) = \bigoplus_{r=1}^{\xi} \mathcal{O}(m, n, t)_{r} \) by putting

\[
\mathcal{O}(m, n, t)_{r} := \text{span}_{\mathbb{F}} \left\{ x^{(a)}x^{u} | a + |u| = r \right\}, \quad \xi := |\pi| + n.
\]

Set

\[
W(m, n, t) := \left\{ \sum_{r \in Y} f_r D_r | f_r \in \mathcal{O}(m, n, t) \right\}.
\]

Then \( W(m, n, t) \) is called the generalized Witt modular Lie superalgebra and it is a subalgebra of \( W(m, n) \). In particular, it is a finite-dimensional simple Lie superalgebra (see [27]). Clearly, \( W(m, n, t) \) is a free \( \mathcal{O}(m, n, t) \)-module with basis \( \{ D_r | r \in Y \} \). Note that \( W(m, n, t) \) possesses a standard \( \mathbb{F} \)-basis \( \left\{ x^{(a)}x^{u} D_r | a \in A, u \in \mathbb{B}, r \in Y \right\} \).

Let \( r, s \in Y \) and \( D_{rs} : \mathcal{O}(m, n, t) \to W(m, n, t) \) be a linear mapping such that

\[
D_{rs}(f) = (-1)^{r(tr)\langle\alpha\rangle}D_{r}(f)D_{s} - (-1)^{r(tr)\langle\alpha\rangle}D_{s}(f)D_{r},
\]

where \( f \in \mathcal{O}(m, n, t) \) and \( r, s \in Y \). Then the following equation holds:

\[
[D_{k}, D_{rs}(f)] = (-1)^{r(k)r}D_{rs}(D_{k}(f)), \quad f \in \mathcal{O}(m, n, t); k, r, s \in Y.
\]

Put

\[
S(m, n, t) := \text{span}_{\mathbb{F}} \left\{ D_{rs}(f) | f \in \mathcal{O}(m, n, t); r, s \in Y \right\}.
\]

Then \( S(m, n, t) \) is called the special modular Lie superalgebra. \( S(m, n, t) \) is also a finite-dimensional simple Lie superalgebra (see [27]).

Let \( \text{div} : W(m, n, t) \to \mathcal{O}(m, n, t) \) be the divergence such that

\[
\text{div} \left( \sum_{r \in Y} f_r D_r \right) = \sum_{r \in Y} (-1)^{r(tr)\langle\alpha\rangle}D_{r}(f_{r}).
\]

It follows that

\[
\text{div}[D, E] = D\text{div}(E) - (-1)^{d(D)d(E)}E\text{div}(D) \quad \text{for any } D, E \in W(m, n, t).
\]

Then \( \text{div} \) is a superderivation from \( W(m, n, t) \) to \( \mathcal{O}(m, n, t) \). Following [27], put

\[
\mathcal{S}(m, n, t) := \{ D \in W(m, n, t) | \text{div}(D) = 0 \}.
\]
Then \( S(m, n, t) \) is contained in \( \bar{S}(m, n, t) \) and \( \bar{S}(m, n, t) \) is a subalgebra of \( W(m, n, t) \). The \( \mathbb{Z} \)-gradation of \( O(m, n, t) \) induces naturally \( \mathbb{Z} \)-gradation structures of \( W(m, n, t) = \bigoplus_{i=1}^{\ell} W(m, n, t)_i \) and \( S(m, n, t) = \bigoplus_{i=1}^{\ell} S(m, n, t)_i \), where

\[
W(m, n, t)_i := \text{span}_F \left\{ fD_r \mid r \in Y, f \in O(m, n, t)_{i+1} \right\}, \\
S(m, n, t)_i := \text{span}_F \left\{ D_{rs}(f) \mid r, s \in Y, f \in O(m, n, t)_{i+2} \right\}.
\]

In addition, \( \bar{S}(m, n, t) \) is also a \( \mathbb{Z} \)-graded subalgebra of \( W(m, n, t) \). For convenience, \( W(m, n, t), S(m, n, t), \bar{S}(m, n, t) \) and \( O(m, n, t) \) will be denoted by \( W, S, \bar{S} \) and \( O \), respectively.

### 3 Graded modules over modular Lie superalgebras

Let \( \mathfrak{gl}(m, n) = \mathfrak{gl}(m, n)_0 \oplus \mathfrak{gl}(m, n)_1 \) be the general linear Lie superalgebra of all \( s \times s \) matrices over \( F \) (see [9]), where \( s = m + n \). Set \( \bar{\mathfrak{gl}}(m, n, t) = O \oplus \mathfrak{gl}(m, n) \). Define the operation \([ , ]\) in \( \bar{\mathfrak{gl}}(m, n, t) \) as follows:

\[
[a \otimes x, b \otimes y] = (-1)^{(i)(d(b))} ab \otimes [x, y],
\]

where \( a, b \in O, x, y \in \mathfrak{gl}(m, n) \). Then \( \bar{\mathfrak{gl}}(m, n, t) \) is a Lie superalgebra. For \( A \in W \), define \( A \otimes 1 \in \text{End}(\bar{\mathfrak{gl}}(m, n, t)) \) by

\[
(A \otimes 1)(a \otimes x) = A(a) \otimes x, \quad a \in O, x \in \mathfrak{gl}(m, n).
\]

Let \( P \in \mathfrak{gl}(m, n)_0 \) be an \( s \times s \) invertible matrix. Suppose that \( A = \sum_{j=1}^{s} a_jD_j \in W_\alpha \), where \( \alpha \in \mathbb{Z}_2 \). Let

\[
\tilde{A} = \sum_{k,j=1}^{s} (-1)^{(k)(j)(a)(b)} (D_k a_j) \otimes P^{-1} E_{kj} P,
\]

where \( E_{kj} \) is an \( s \times s \) matrix whose \((i, l)\)-entry is \( \delta_{ki}\delta_{lj} \). Then \( \tilde{A} \in \bar{\mathfrak{gl}}(m, n, t)_\alpha \). By virtue of the definition of superderivation we have

\[
D_k(ab) = (D_k a)b + (-1)^{(k)(d(a))} a(D_k b),
\]

where \( a, b \in O, k = 1, \ldots, s \).

Let \( A = \sum_{j=1}^{s} a_jD_j \in W_\alpha \) and \( B = \sum_{j=1}^{s} b_jD_j \in W_\beta \), where \( \alpha, \beta \in \mathbb{Z}_2 \). Then

\[
[A, B] = \sum_{j=1}^{s} q_jD_j, \quad \text{where} \quad q_j = \sum_{i=1}^{s} (a_i(D_j b_j) - (-1)^{a\beta}(b_j(D_j a_i))).
\]

Using the formulae from (2) to (6), a direct calculation shows the following proposition.

**Proposition 3.1 ([24], Formula (7)).** Suppose that \( A \in W_\alpha \) and \( B \in W_\beta \), where \( \alpha, \beta \in \mathbb{Z}_2 \). If \( C = [A, B] \), then

\[
\tilde{C} = [\tilde{A}, \tilde{B}] + (A \otimes 1)(B) - (-1)^{a\beta}(B \otimes 1)(\tilde{A}).
\]

Suppose that \( L \) is a subalgebra of \( \mathfrak{gl}(m, n) \) and \( L(P) = \{ P^{-1}AP \mid A \in L \} \). Then \( L(P) \) is a subalgebra of \( \mathfrak{gl}(m, n) \). Let \( \Omega = \Omega_0 \oplus \Omega_1 \) where

\[
\Omega_\alpha = \{ A \in W_\alpha \mid \tilde{A} \in O \otimes L(P) \}, \quad \alpha \in \mathbb{Z}_2.
\]

If \( A, B \in \Omega \), then \( [\tilde{A}, \tilde{B}] \in \Omega \). The formula (7) shows that \( \Omega \) is a subalgebra of modular Witt Lie superalgebras \( W \). The subalgebra \( \Omega \) is called the \( P \)-expansion of \( L \) into \( W \). Then the \( P \)-expansion of \( \mathfrak{gl}(m, n) \) into \( W \) is exactly \( W \).

The special linear Lie superalgebra \( \mathfrak{sl}(m, n) = \{ A \in \mathfrak{gl}(m, n) \mid \text{str}(A) = 0 \} \) is a subalgebra of \( \mathfrak{gl}(m, n) \) (see [9]). Let \( \Omega \) be the \( P \)-expansion of \( \mathfrak{sl}(m, n) \) into \( W \). If \( A = \sum_{j=1}^{s} a_jD_j \in W \), then, for \( \alpha \in \mathbb{Z}_2 \),

\[
A \in \Omega_\alpha \iff \tilde{A} \in (O \otimes \mathfrak{sl}(m, n))_\alpha.
\]
\[
\begin{aligned}
&\Leftarrow \sum_{k=1}^{s} \sum_{j=1}^{s} (-1)^{r(j)} (D_k a_j) \otimes E_{kj} - (-1)^a \sum_{k=m+1}^{s} \sum_{j=1}^{s} (D_k a_j) \otimes E_{kj} \\
&\quad \in (0 \otimes s(m, n))_a \\
&\Rightarrow \sum_{k=1}^{s} (D_k a_k) \otimes E_{kk} - (-1)^a \sum_{k=m+1}^{s} (D_k a_k) \otimes E_{kk} \\
&\quad \in (0 \otimes s(m, n))_a \\
&\Rightarrow \sum_{k=1}^{s} (-1)^{r(k)d(a_k)} (D_k a_k) \otimes E_{11} \in (0 \otimes s(m, n))_a \\
&\Rightarrow \sum_{k=1}^{s} (-1)^{r(k)d(a_k)} (D_k a_k) = 0 \quad \text{and } D_k a_k \in \mathcal{O}_a \\
&\Rightarrow A \in \mathcal{S}_a.
\end{aligned}
\]

Hence \(\Omega_a = \mathcal{S}_a\). It follows that \(\Omega = \mathcal{S}\).

Let \(\rho\) be a representation of \(L(P)\) on \(\mathbb{Z}_2\)-graded space \(V\). Then \(\rho\) can be expanded to a representation \(\rho_1\) of \(0 \otimes L(P)\) on the space \(0 \otimes V\), defined by

\[
\rho_1(a \otimes x)(b \otimes v) = (-1)^{d(x)d(b)} ab \otimes \rho(x)(v),
\]

where \(a, b \in 0\), \(x \in L(P)\), \(v \in V\).

**Proposition 3.2** ([24], Proposition 2). Let \(\Omega\) be the \(P\)-expansion of \(L\) into \(W\). Then

\[
\hat{\rho}(A) = \rho_1(A) + A \otimes 1, \quad A \in \Omega
\]

defines a representation \(\hat{\rho}\) of \(\Omega\) on \(\mathbb{Z}_2\)-graded space \(0 \otimes V\).

By Proposition 3.2, \(0 \otimes V\) which will be denoted by \(\tilde{V}\) is a \(\Omega\)-module. In [20] the module \(\tilde{V}\) is called the mixed product of \(0 \otimes V\) and the module \(V\).

A \(\mathbb{Z}\)-graded module \(V\) of \(X\) is called positively graded if \(V = \bigoplus_{i=0}^{\xi} V_i\) and \(L_j \cdot V_i \subseteq V_{i+j}\), where \(X\) is a \(\mathbb{Z}\)-graded Lie superalgebra.

Let \(\tilde{V}_i = \{a \otimes v \mid a \in 0_i, v \in V\}\). Then \(\tilde{V} = \bigoplus_{i=0}^{\xi} \tilde{V}_i\), where \(\xi = \sum_{i=1}^{m} \pi_i + n\). Put \(\tilde{V}_i = 0\) for \(i > \xi\). A direct verification shows that \(\Omega_i \cdot \tilde{V}_j \subseteq \tilde{V}_{i+j}\). Hence \(\tilde{V}\) is a positively graded \(\Omega\)-module. Since the \(P\)-expansion of \(gl(m, n)\) and \(s\) into \(W\) are, respectively, \(W\) and \(\mathcal{S}\), Proposition 3.2 shows the following corollary.

**Corollary 3.3.** The following statements hold:

1. If \(V\) is a \(gl(m, n)\)-module, then the mixed product \(\tilde{V}\) is a \(\mathbb{Z}\)-graded \(W\)-module.
2. If \(V\) is an \(s\) \((m, n)\)-module, then the mixed product \(\tilde{V}\) is a \(\mathbb{Z}\)-graded \(\mathcal{S}\)-module and so is a \(\mathbb{Z}\)-graded \(\mathcal{S}\)-module.

If \(V\) is an irreducible \(L(P)\)-module with a highest weight \(\lambda\), denote \(\tilde{V}\) by \(\tilde{V}(\lambda)\). Furthermore, the weight vector associated with the highest weight \(\lambda\) is denoted by \(v_\lambda\), where \(v_\lambda \in \tilde{V}(\lambda)\). Similar to [21, Theorem 1.2] one may obtain that \(\tilde{V}(\lambda)\) has an unique irreducible submodule \(U(L)(1 \otimes V)\), where \(U(L)\) is the universal enveloping algebra of \(L\). We customarily denote the unique irreducible submodule by \(\tilde{V}\). Then \(x^{(a)} x^{(b)} \otimes v_\lambda \in \tilde{V}\) for all \(a = (a_1, \ldots, a_m) \in A, \quad u = (i_1, \ldots, i_r) \in B\).

Suppose that \(\pi = (\pi_1, \ldots, \pi_m), \quad E = (m + 1, \ldots, s)\), then \(\pi \in A, \quad E \in B\). Since Lie superalgebra \(W_0\) is isomorphic to \(gl(m, n)\), the element \(\sum_{i=1}^{s} a_i E_i\) of \(L(P)\) can be identified as the element \(\sum_{i=1}^{s} a_i x_i D_j\) of \(W\), where \(a_{ij} \in \mathbb{F}\). Hence \(x^{(a)} x^{(b)} \otimes v\) can be regarded as an \(L(P)\)-module. Similarly, by [21, Proposition 2.4], we can prove that \(\tilde{V}\) is an irreducible \(L(P)\)-module if and only if \(x^{(a)} x^{(b)} \otimes v\) is an irreducible \(L(P)\)-module. The following proposition is the analogue of [21, Proposition 2.1].
Proposition 3.4. Suppose that $V$ is an irreducible $L(P)$-module, $v_\lambda$ is the weight vector associated with the highest weight $\lambda$ and $\tilde{V}$ is the unique irreducible submodule of $\tilde{V}(\lambda)$, where $L = gl(m,n)$ or $sl(m,n)$, then

(i) $\tilde{V}$ contains $1 \otimes V$ as a submodule.

(ii) If $x^{(n)} x^E \otimes v_\lambda \in \tilde{V}$, then $\tilde{V} = \tilde{V}(\lambda)$, that is $\tilde{V}(\lambda)$ is an irreducible $\Omega$-module, where $\Omega$ is the $P$-expansion of $L(P)$ into $W$.

Proof. Recall that $\Omega = W$ if $L = gl(m,n)$ and $\Omega = \mathbb{F}$ if $L = sl(m,n)$.

(i) Without loss of generality we suppose that $x^{(a)} x^u \otimes v_\lambda \in \tilde{V}$, then

$$1 \otimes v_\lambda = D_1^{a_1} \cdots D_m^{a_n} D_i_1 \cdots D_i_k \cdot (x^{(a)} x^u \otimes v_\lambda) \in \tilde{V}. $$

Because $V$ is an irreducible $L(P)$-module, $1 \otimes V$ is an irreducible $\Omega_0$-module. Then $1 \otimes V = U(\Omega_0)(1 \otimes v_\lambda) \subseteq \tilde{V}$, where $U(\Omega_0)$ is the universal enveloping algebra of $\Omega_0$.

(ii) Since module $V$ is non-trivial, $\tilde{V}(\lambda)_k$ is non-trivial. Let $x^{(n)} x^E \otimes V'$ be a proper $\Omega_0$-submodule of $\tilde{V}(\lambda)_k = x^{(n)} x^E \otimes V$. Then $1 \otimes V'$ is a proper $\Omega_0$-submodule of $1 \otimes V$. Hence $V'$ is a proper $L(P)$-submodule of $V$, which contradicts that $V$ is irreducible. Since $\tilde{V}(\lambda)_k$ is an irreducible $\Omega_0$-module, we have

$$U(\Omega_0)(x^{(a)} x^E \otimes v_\lambda) = x^{(a)} x^E \otimes V \subseteq \tilde{V}. $$

Let $\alpha \in A, u \in \mathbb{B}$. Assuming that $\{\beta_1, \ldots, \beta_m\} = \pi - \alpha$ and $w = (j_1, \ldots, j_k) \in \mathbb{B}$ such that $\{w\} = \{E\} \setminus \{u\}$, then

$$x^{(a)} x^u \otimes v_\lambda = D_1^{\beta_1} \cdots D_m^{\beta_n} D_j_1 \cdots D_j_k \cdot (x^{(a)} x^E \otimes v_\lambda) \in \tilde{V}. $$

Hence $\tilde{V} = \tilde{V}(\lambda)$. 

\[ \square \]

4 Irreducibility of module $\tilde{V}(\lambda)$ over $W$ and $S$

Let $V$ be an irreducible $sl(m,n)$-module with the highest weight $\lambda$. Proposition 3.2 shows that $\tilde{V}(\lambda)$ is an $\mathbb{F}$-module. Clearly, it is also an $S$-module. Assuming that $\tilde{V}$ is irreducible as an $S$-module and if $x^{(n)} x^E \otimes v_\lambda \in \tilde{V}$, then it follows that $\tilde{V} = \tilde{V}(\lambda)$ from the similar methods used in Proposition 3.4 (ii). Therefore, $\tilde{V}(\lambda)$ is an irreducible $S$-module.

We know that the standard Cartan subalgebra $H$ of $S$ is $\langle h_i \mid i = 1, 2, \ldots, s - 1 \rangle$, where

$$h_i = E_{ij} - (-1)^{j+i} \tau^{(i)+j+1} E_{i+j+1}, \quad i = 1, 2, \ldots, s - 1. $$

Let $A_i$ be the linear function on $\langle E_{11}, \ldots, E_{ss} \rangle$ such that $A_i(E_{ij}) = \delta_{ij}$, where $i, j = 1, 2, \ldots, s$. Set

$$\lambda_j = \sum_{j=1}^{i} A_j, \quad i = 1, 2, \ldots, m - 1, $$

$$\lambda_m = \sum_{j=m+1}^{s} A_j, $$

$$\lambda_i = -\sum_{j=1}^{m} A_j + \sum_{j=m+1}^{i} A_j, \quad i = m + 1, \ldots, s. $$

Then $\lambda_i, i = 1, 2, \ldots, s - 1$, is a fundamental weight of $sl(m,n)$ and $\lambda_i(h_j) = \delta_{ij}$. We know that $\lambda_i - \lambda_j$ is a positive root of $sl(m,n)$ and the corresponding vectors of the positive root are $E_{ij}$, where $1 \leq i < j \leq s$.

If $V$ is a finite-dimensional irreducible $sl(m,n)$-module, then $\lambda = \sum_{i=1}^{s-1} c_i A_i$, where $c_i \in \mathcal{F}$. Let $\lambda_{|m} = \sum_{i=1}^{m-1} c_i A_i$ and $\lambda_{|+m} = \sum_{i=m+1}^{s-1} c_i A_i$. Then $\lambda = \lambda_{|m} + c_m \lambda_m + \lambda_{|+m}$. Also, assuming that $\rho$ and $\tilde{\rho}$ are representations corresponding with the modules $V$ and $\tilde{V}(\lambda)$, respectively.
Lemma 4.1. Assuming that $|\lambda|_m \neq 0$. If $\lambda \neq \lambda_i$, $i = 1, 2, \ldots, m - 1$, then $\overline{V}(\lambda)$ is an irreducible $S$-module.

Proof. Since $|\lambda|_m \neq 0$ and $\lambda \neq \lambda_i$, $i = 1, 2, \ldots, m - 1$, it was observed that $\lambda$ must be one of the two cases: (1) $|\lambda|_m = \lambda_i$ and $c_{m\lambda} + |\lambda|_m \neq 0$, (2) $|\lambda|_m \neq \lambda_i$, $i = 1, 2, \ldots, m - 1$.

(1) If $|\lambda|_m = \lambda_i$ and $c_{m\lambda} + |\lambda|_m \neq 0$, where $1 \leq i \leq m - 1$, then $c_i \neq 0$ for $m \leq i \leq s - 1$. Let $c_k$ be the first non-zero element of $\{c_m, c_{m+1}, \ldots, c_{s-1}\}$. If $k > m$, then by virtue of formulae (1), (2), (8) and (9) and

$$
\rho(E_{ii} - E_{k+1-k+1})v_{\lambda} = \rho\left(\sum_{j=1}^{m} h_j - \sum_{j=m+1}^{k} h_j\right)v_{\lambda} = (1 - c_k)v_{\lambda},
$$

where $v_{\lambda}$ is a weight vector associated with the highest weight $\lambda$,

$$
\tilde{\rho}(D_{ij}^{(m)}(x^{(n)}x^E))(1 \otimes v_{\lambda}) = \tilde{\rho}(x^{(n)}x^E D_{ij}^{(m)} - (-1)^{n+m}x^{(n)}x^{E-(k+1)} D_{ij})(1 \otimes v_{\lambda})
$$

$$
= \rho_{ij}^{(n)}(x)^{E-(k+1)} \otimes E_{ij}^{(m)}
$$

$$
- (-1)^{n-1} \sum_{j=m+1}^{s} x^{(n)}x^{E-(k+1)} \otimes E_{ij}^{(m)}(1 \otimes v_{\lambda})
$$

$$
= (-1)^{n-k-m} \sum_{j=1}^{s} x^{(n)}x^{E-(k+1)} \otimes E_{ij}^{(m)}(1 \otimes v_{\lambda})
$$

$$
= (-1)^{n-k-m} x^{(n)}x^{E-(k+1)} \otimes E_{ij}^{(m)}(1 \otimes v_{\lambda})
$$

Applying the formulae (8), (9) and (10) and

$$
\rho(E_{ii} + E_{ij})v_{\lambda} = \rho(E_{ii} - E_{ij})v_{\lambda} = v_{\lambda},
$$

we get

$$
\tilde{\rho}(D_{ii}^{(m)}(x^{(2n)}x^{E}))\tilde{\rho}(D_{ij}^{(m)}(x^{(n)}x^E))(1 \otimes v_{\lambda}) = (-1)^{n}x^{(n)}x^{E} \otimes v_{\lambda} - (-1)^{n}(1 - c_k)x^{(n)}x^{E} \otimes \rho(E_{ii} - E_{ij})v_{\lambda}
$$

$$
= (-1)^{n}c_k x^{(n)}x^{E} \otimes v_{\lambda}.
$$

By Proposition 3.4 (i), we know that $1 \otimes v_{\lambda} \in \overline{V}$. Since $c_k \neq 0$, it shows $x^{(n)}x^{E} \otimes v_{\lambda} \in \overline{V}$. Hence $\overline{V}(\lambda)$ is an irreducible $S$-module.

(2) If $|\lambda|_m \neq \lambda_i$, $i = 1, 2, \ldots, m - 1$, then

$$
\tilde{\rho}(D_{ij}^{(m)}(x^{(2n+2n+1)}))(1 \otimes v_{\lambda}) = (-c^2_i + c_i)x^{(n)}x^{E} \otimes v_{\lambda}.
$$

If $c_i \neq 0$ or 1, then $x^{(n)}x^{E} \otimes v_{\lambda} \in \overline{V}$ and $\overline{V}(\lambda)$ is an irreducible $S$-module. So it was assumed that $c_k = 0$ or $1$, where $k = 1, 2, \ldots, m - 1$. Since $|\lambda|_m \neq 0$ and $|\lambda|_m \neq \lambda_i$, there exist at least two $k \in \{1, 2, \ldots, m - 1\}$ such
that $c_k = 1$. Without loss of generality we assumed that $c_i$ and $c_j$, $i < j$, are the first and second coefficients which are equal to 1. Then

$$\tilde{\rho}(D_{ij+1}(x^{(2\varepsilon_i+2\varepsilon_j)}))(\tilde{\rho}(D_{ij+1}(x^{(n)}x^E))(1 \otimes \nu_\lambda) = (-(c_i + c_j)^2 + (c_i + c_j)x_i^{(n)}x^E \otimes \nu_\lambda = 2x^{(n)}x^E \otimes \nu_\lambda.$$  

Therefore, $x^{(n)}x^E \otimes \nu_\lambda \in \mathcal{V}$ and $\bar{V}(\lambda)$ is an irreducible $S$-module. 

**Lemma 4.2.** Suppose that $n$ is odd. If $c_m \neq -1$ or 0, then $\bar{V}(\lambda)$ is an irreducible $S$-module.

**Proof.** If $c_{m+1} \neq c_m + 1$, then $c_m(1 + c_m - c_{m+1}) \neq 0$. A direct computation shows that

$$\tilde{\rho}(D_{mm+1}(x^{(2\varepsilon_m)}x^{(m+1)}x^{(m+2)}))(\tilde{\rho}(D_{mm+1}(x^{(n)}x^E))(1 \otimes \nu_\lambda) = c_m(1 - (-1)^m(c_m - c_{m+1}))x^{(n)}x^E \otimes \nu_\lambda = c_m(1 + c_m - c_{m+1})x^{(n)}x^E \otimes \nu_\lambda.$$  

Then $x^{(n)}x^E \otimes \nu_\lambda \in \mathcal{V}$ and $\bar{V}(\lambda)$ is an irreducible $S$-module. If $c_{m+1} = c_m + 1$, then

$$\tilde{\rho}(D_{mm+1}(x^{(2\varepsilon_m)}x^{(m+1)}x^{(m+2)}))(\tilde{\rho}(D_{mm+1}(x^{(n)}x^E))(1 \otimes \nu_\lambda) = (-c_m - c_{m+1}) - (-1)^n c_m(1 + c_m - c_{m+1})x^{(n)}x^E \otimes \nu_\lambda = (2 + 2c_m)x^{(n)}x^E \otimes \nu_\lambda.$$  

Since $c_m \neq -1$, we have $2 + 2c_m \neq 0$. Therefore, $x^{(n)}x^E \otimes \nu_\lambda \in \mathcal{V}$ and $\bar{V}(\lambda)$ is irreducible.

**Lemma 4.3.** Suppose that $n$ is odd and $c_m \neq 0$. If $\lambda \neq n\lambda$, then $\bar{V}(\lambda)$ is an irreducible $S$-module.

**Proof.** If $c_{m+1} \neq -1$, then Lemma 4.2 shows that $\bar{V}(\lambda)$ is irreducible. If $c_m = -1$ and $\lambda \leq c_m \neq 0$, then Lemma 4.1 shows that $\bar{V}(\lambda)$ is irreducible. Suppose that $c_m = -1$, $\lambda \leq m = 0$ and $\lambda \neq m \neq 0$. Let $c_k$ be the first non-zero element of $\{c_{m+1}, c_{m+2}, \ldots, c_{s-1}\}$. Then

$$\tilde{\rho}(D_{mm+2}(x^{(2\varepsilon_m)}x^{(m+1)}x^{(m+2)}))(\tilde{\rho}(D_{mm+1}(x^{(n)}x^E))(1 \otimes \nu_\lambda) = c_m - (-1)^n(c_m - c_k)\lambda x^{(n)}x^E \otimes \nu_\lambda = c_kx^{(n)}x^E \otimes \nu_\lambda.$$  

Hence $x^{(n)}x^E \otimes \nu_\lambda \in \mathcal{V}$. Consequently, the module $\bar{V}(\lambda)$ is an irreducible $S$-module.

**Lemma 4.4.** Suppose that $\lambda \leq m = 0$, $c_m \neq 0$ and $\lambda \neq m \neq 0$. If $\bar{V}(\lambda)$ is a reducible $S$-module, then there exists $k \in \{m + 1, m + 2, \ldots, s - 2\}$ such that $\lambda = c_k \lambda \lambda_k = (c_k + 1)\lambda_{k+1}$.  

**Proof.** Since $\lambda \neq m \neq 0$, the elements $c_{m+1}, \ldots, c_{s-1}$ of $S$ are not all zero. So we may suppose that $c_k$ is the first non-zero element of $\{c_{m+1}, c_{m+2}, \ldots, c_{s-1}\}$. Assuming that there exists a $c_{k+i} \neq 0$, where $i > 1$. A direct computation shows that

$$\tilde{\rho}(D_{k+i+k+i+1}(x_kx_k+1x+kx+k+i+1))(\tilde{\rho}(D_{kk+k+1}(x^{(n)}x^E))(1 \otimes \nu_\lambda) = (-1)^n c_k c_{k+i} x^{(n)}x^E \otimes \nu_\lambda.$$  

As $c_k c_{k+i} \neq 0$, we have $x^{(n)}x^E \otimes \nu_\lambda \in \mathcal{V}$. But this conclusion confutes that $\bar{V}(\lambda)$ is a reducible $S$-module. As $c_{k+i} = 0$, $i > 1$ and $\lambda = c_k \lambda \lambda_k + c_{k+1} \lambda_{k+1}$. Then

$$\tilde{\rho}(D_{k+i+k+1}(x_kx_k+1x+kx+k+i+1))(\tilde{\rho}(D_{kk+k+1}(x^{(n)}x^E))(1 \otimes \nu_\lambda) = \begin{cases} (-1)^n c_k(1 + c_k + c_{k+1})x^{(n)}x^E \otimes \nu_\lambda, & m + 2 \leq k \leq s - 2; \\ (-1)^n c_k(1 + c_k + c_{k+1})x^{(n)}x^E \otimes \nu_\lambda, & k = m + 1. \end{cases}$$  

Since $\bar{V}(\lambda)$ is a reducible $S$-module, we have $1 + c_k + c_{k+1} = 0$. Therefore, $c_{k+1} = -(c_k + 1) \lambda = c_k \lambda_k = (c_k + 1)\lambda_{k+1}$. 

**Theorem 4.5.** Let $V$ be a finite-dimensional irreducible $\mathfrak{sl}(m, n)$-module with a non-zero highest weight $\lambda$. Suppose that $n$ is odd. If $\lambda \neq -\lambda_m$ or $(-1)^i \lambda_i$, $i = 1, \ldots, m - 1, m + 2, \ldots, s - 1$, then $\bar{V}(\lambda)$ is an irreducible $S$-module.
Proof. Assume that $\tilde{V}(\lambda)$ is a reducible $S$-module. It suffices to prove $\lambda = -\lambda_m$ or $(-1)^{r(i)}\lambda_i$, where $i = 1, \ldots, m-1, m+2, \ldots, s-1$.

If $|\lambda|_{m} \neq 0$, by Lemma 4.1, then $\lambda = \lambda_i = (-1)^{r(i)}\lambda_i$, where $i \in \{1, \ldots, m-1\}$.

Suppose that $|\lambda|_{m} = 0$. If $c_m \neq 0$, by Lemma 4.3, then $\lambda = -\lambda_m$.

Suppose that $|\lambda|_{m} = 0$ and $c_m = 0$. Then $|\lambda|_{m-1} \neq 0$. Lemma 4.4 shows that $\lambda = c_i\tilde{\lambda}_i - (c_i + 1)\tilde{\lambda}_{i+1}$, where $i \in \{m + 1, \ldots, s - 2\}$. A direct computation shows that

$$\rho(D_{m+1}(x^{(2\epsilon_m)}x_{i+1}x_{i+2}))\tilde{\rho}(D_{m+2}(x^{(\epsilon_{i})}x^E))(1 \otimes v_\lambda) = -2c_i\chi^{(\epsilon_{i})}x^E \otimes v_\lambda.$$ 

Since $V(\lambda)$ is a reducible $S$-module, we have $\chi^{(\epsilon_{i})}x^E \otimes v_\lambda \notin V$. Therefore, $c_i = 0$ and $\lambda = -\lambda_{i+1} = (-1)^{r(i+1)}\lambda_{i+1}$, where $i \in \{m + 1, \ldots, s - 2\}$, that is $\lambda = (-1)^{r(i)}\lambda_i$, where $i \in \{m + 1, \ldots, s - 2\}$. The proof is completed.

The $W$-module $\tilde{V}(\lambda)$ will be discussed in the following theorem.

**Theorem 4.6.** Let $V$ be a finite-dimensional irreducible $gl(m, n)$-module with the non-zero highest weight $\lambda$. Suppose that $n$ is odd. If $\lambda \neq -\lambda_m$, $(-1)^{r(i)}\lambda_i$, $i = 1, \ldots, m - 1, m + 2, \ldots, s - 1$, then $V(\lambda)$ is an irreducible $W$-module.

**Proof.** We may suppose that $\lambda = \sum_{j=1}^{s}c_j\lambda_j$, where $c_j \in F$, and $\rho$ is the representation corresponding to the module $V$. Assume that $V(\lambda)$ is a reducible $W$-module. It suffices to prove $\lambda = -\lambda_m$ or $(-1)^{r(i)}\lambda_i$, where $i = 1, \ldots, m - 1, m + 2, \ldots, s - 1$. Clearly, $V(\lambda)$ is also a reducible $S$-module. By Theorem 4.5, we know that $|\lambda|_H$, the restriction of $\lambda$ to the Cartan subalgebra $H$ of $s(m, n)$, is equal to $-\lambda_m$ or $(-1)^{r(i)}\lambda_i$, where $i = 1, \ldots, m - 1, m + 2, \ldots, s - 1$.

If $|\lambda|_H = -\lambda_m$, then $|\lambda|_H(h_i) = -\lambda_m(h_i)$, $i = 1, \ldots, s - 1$. Hence

$$\lambda = c\sum_{j=1}^{m}A_j + (1 + c)\sum_{j=m+1}^{s}A_j,$$

where $c \in F$. A direct computation shows that

$$\tilde{\rho}(x^{(2\epsilon_i)}D_1)\tilde{\rho}(x^{(\epsilon_{i})}x^E D_1)(1 \otimes v_\lambda) = (c - c^2)\chi^{(\epsilon_{i})}x^E \otimes v_\lambda.$$ 

Since $V(\lambda)$ is reducible, we have $c - c^2 = 0$. It follows $c = 0$ or $1$. By virtue of

$$\tilde{\rho}(x^{(2\epsilon_i)}D_1)\tilde{\rho}(x^{(\epsilon_{i})}x^E D_{m+1})(1 \otimes v_\lambda) = (-1)^{n-1}(1 + c)\chi^{(\epsilon_{i})}x^E \otimes v_\lambda$$

and $V(\lambda)$ is reducible, we have $(1 + c)c = 0$. Then $c = 0$ and $\lambda = \sum_{j=m+1}^{s}A_j = -\lambda_m$.

If $|\lambda|_H = (-1)^{r(i)}\lambda_i$, where $i = 1, \ldots, m - 1$, then $|\lambda|_H = \lambda_i$. Since $\lambda(h_i) = \lambda_i(h_i)$, $j = 1, 2, \ldots, s - 1$, we have

$$\lambda = c\sum_{j=1}^{i}A_j + (c - 1)\sum_{j=i+1}^{m}A_j + (c - 1)\sum_{j=m+1}^{s}A_j,$$

where $c \in F$. It follows that $c = 0$ or $1$ from

$$\tilde{\rho}(x^{(2\epsilon_i)}D_1)\tilde{\rho}(x^{(\epsilon_{i})}x^E D_m)(1 \otimes v_\lambda) = (c - c^2)\chi^{(\epsilon_{i})}x^E \otimes v_\lambda$$

and $V(\lambda)$ is reducible. But the equation

$$\tilde{\rho}(x^{(2\epsilon_m)}D_m)\tilde{\rho}(x^{(\epsilon_{m})}x^E D_m)(1 \otimes v_\lambda) = (-c - 1)^2 + (c - 1)\chi^{(\epsilon_{m})}x^E \otimes v_\lambda,$$

shows that $c \neq 0$. Hence $c = 1$ and $\lambda = \lambda_i = (-1)^{r(i)}\lambda_i$, where $i = 1, \ldots, m - 1$.

If $|\lambda|_H = (-1)^{r(i)}\lambda_i$, where $i = m + 2, \ldots, s - 1$, then $|\lambda|_H = -\lambda_i$. Since $\lambda(h_i) = -\lambda_i(h_i)$, $j = 1, 2, \ldots, s - 1$, we have

$$\lambda = c\sum_{j=1}^{m}A_j - c\sum_{j=m+1}^{i}A_j + (1 + c)\sum_{j=i+1}^{s}A_j,$$
where \( c \in \mathbb{F} \). It follows that \( c = 0 \) or \( 1 \) from
\[
\tilde{\rho}(x^{(2c+1)}D_{1})\tilde{\rho}(x^{(n)}E_{1})(1 \otimes v_{\lambda}) = (c - c^{2})x^{(n)}E \otimes v_{\lambda}
\]
and \( \tilde{V}(\lambda) \) is reducible. Furthermore, the equation
\[
\tilde{\rho}(x_{1}x_{j+1}D_{1})\tilde{\rho}(x^{(n)}E_{j+1})(1 \otimes v_{\lambda}) = (-1)^{n}(1 - c)^{2}x^{(n)}E \otimes v_{\lambda},
\]
shows that \( c = 1 \). Hence
\[
\lambda = \sum_{j=1}^{m} A_{j} - \sum_{j=m+1}^{i} A_{j} = -\lambda_{i} = (-1)^{t(i)}\lambda_{i},
\]
where \( i = m + 2, \ldots, s - 1 \).

In conclusion, the proof is completed.

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References

[1] Rudakov A., Shafarevich I., Irreducible representations of a simple three-dimensional Lie algebra over a field of finite characteristic, Math. Notes., 1967, 2(5), 760–767.

[2] Rudakov A., Irreducible representations of infinite-dimensional Lie algebras of Cartan type, Math. USSR Izv., 1974, 8(4), 836–866.

[3] Grozman P., Leites D., Shchepochkina I., Invariant operators on supermanifolds and standard models, Multiple facets of quantization and supersymmetry, 2002, 508–555.

[4] Olshanetsky M., Lie algebroids as gauge symmetries in topological field theories, Multiple facets of quantization and supersymmetry, 2002, 205–232.

[5] Krylyuk Ya., Maximum dimension of irreducible representations of simple Lie \( p \)-algebras of Cartan series \( S \) and \( II \), Mat. Sb. (N.S.), 1984, 123(165)(1), 108-119 (Russian)

[6] Krylyuk Ya., On the index of algebras of Cartan type in finite characteristic, Math. USSR Izv., 1987, 28, 381–399.

[7] Shu B., Representations of Cartan type Lie algebras in characteristic \( p \), J. Algebra, 2002, 256(1), 7–27.

[8] Zhang C., Representations of the restricted Lie algebras of Cartan type, J. Algebra, 2005, 290(2), 408–432.

[9] Kac V., Characters of typical representations of classical Lie superalgebras, Comm. Algebra, 1977, 5(8), 889–897.

[10] Kac V., Representations of classical Lie superalgebras, Lec. Notes Math., Springer, Berlin, 1978, 676, 597–626.

[11] Serganova V., Kazhdan-Lusztig polynomials for Lie superalgebra \( gl(m|n) \), Adv. Soviet Math., 1993, 16(2), 151–165.

[12] Su Y., Some results on finite dimensional representations of general linear Lie superalgebras, Adv. Lect. Math., 2008, 6, 374–439.

[13] Van der Jeugt J., Hughes J., King R., Thierry-Mieg J., Character formulas for irreducible modules of the Lie superalgebras \( sl(m|n) \), J. Math. Phys., 1990, 31(9), 2278–2304.

[14] Bernstein J., Leites D., Character formulas for irreducible modules of the Lie superalgebras \( s(l(m|n)) \), J. Math. Phys., 1990, 31(9), 2278–2304.

[15] Bernstein J., Leites D., Irreducible representations of infinite-dimensional Lie superalgebras of series \( W \) and \( S \), C. R. Acad. Bulgare Sci., 1979, 32(3), 277–278.

[16] Shapovalov A., Finite-dimensional irreducible representations of Hamiltonian Lie superalgebras, Mat.Sb.(N.S.), 1978, 107(149)(2), 259–274. (Russian)

[17] Shu B., Yao Y., Character formulas for restricted simple modules of the special superalgebras, Math. Nachr., 2012, 285(8-9), 1107–1116.

[18] Shu B., Zhang C., Restricted representations of the Witt superalgebras, J. Algebra, 2010, 324(4), 652–672.

[19] Yao Y., Shu B., Restricted representations of Lie superalgebras of Hamiltonian type, Algebr. Represent. Theory, 2013, 16(3), 615–632.

[20] Shen G., Graded modules of graded Lie algebras of Cartan type I: Mixed products of modules, Sci. Sinica Ser. A., 1986, 29(6), 570–581.

[21] Shen G., Graded modules of graded Lie algebras of Cartan type II: Positive and negative graded modules, Sci. Sinica Ser. A., 1986, 29(10), 1009–1019.

[22] Shen G., Graded modules of graded Lie algebras of Cartan type III: Irreducible modules, Chin. Ann. Math. Ser. B., 1988, 9(4), 404–417.

[23] Zhang Y., \( \mathbb{Z} \)-graded module of Lie superalgebra \( H(n) \) of Cartan type, Chin. Sci. Bull., 1996, 41(10), 813–817.
[24] Zhang Y., Fu H., Finite-dimensional Hamiltonian Lie superalgebra, Comm. Algebra, 2002, 30(6), 2651–2673.
[25] Strade H., Simple Lie algebras over fields of positive characteristic, I, Structure Theory, Walter de Gruyter, Berlin, New York, 2004.
[26] Scheunert M., Theory of Lie superalgebras, Springer-Verlay, Berlin, Heidelberg and New York, 1979.
[27] Zhang Y., Finite-dimensional Lie superalgebras of Cartan-type over fields of prime characteristic, Chin. Sci. Bull., 1997, 42, 720–724.