DARBOUX SOLUTIONS OF NON-ABELIAN QUANTUM PAINLEVÉ II EQUATION IN TERMS OF QUASIDETERMINANTS

IRFAN MAHMOOD

ABSTRACT. In this article non-abelian version of quantum Painlevé II equation is presented with its quasideterminant solutions has been derived by using the Darboux transformations. This non-abelian quantum Painlevé II equation may be considered as a specific case of its purely noncommutative analogue presented by V. Retakh and V. Rubtsov. In these computations the quantum Painlevé II symmetric form with commutation relations presented by H. Nagoya are applied to derive Nonabelian quantum Painlevé II equation and a new commutation relation between variable $z$ and the solution $f(z)$ such as $zf - fz = \frac{1}{2}i\hbar f$ is presented. Finally, the Darboux solutions of that system are generalized to the $N$-th form in terms of quasideterminants.

1. INTRODUCTION

The Painlevé equations have got considerable attention because of their wide applications in several areas of mathematics and physics. For example, an important fact was observed in [1, 2] concerning the application of PI equation in two dimensional quantum gravity where its free energy functional can be expressed in terms of PI solution. Further it was observed by V. E. Zakharov, E. A. Kuznetsov and S. L. Musher [3] that the PII equation is helpful to describe the strong wave collapse in the case of three-dimensional nonlinear Schrödinger equation and one of its interesting applications has been studied by A. S. Fokas and S. Tanveer [3] in the analysis of classical Hele-Shaw problems. More over it has been studied that the solutions of some field theoretical problems can be obtained in terms of Painlevé transcendents for example, these transcendents appear in the description of two-point correlation function for the two dimensional Ising model as presented in [4] and also useful in the analysis of two-point correlation function at zero temperature for the case of one dimensional Bose-gas, details can be found in [5]. The more recent and interesting applications of the modern theory of Painlevé equations include differential geometry of surfaces, orthogonal polynomials and string theory, see [3, 6, 7] for detail. The classical Painlevé equations are also regarded as completely integrable equations as mentioned by [8, 9, 10] that they admit linear representations, possess Hamiltonian structures, and obeyed the Painlevé test.
A very interesting aspect of these equations is that they also arise as ordinary differential equations (ODEs) reductions of some integrable systems, i.e., N. Joshi and M. Mazzocco [11, 12] has been shown the the ODE reduction of the KdV equation is Painlevé II (PII) equation.

The quantum analogue of the classical Painlevé II equation has been studied in [13] where the Painlevé II symmetric form

\[
\begin{cases}
    f'_0 = f_0f_2 + f_2f_0 + \alpha_0 \\
    f'_1 = -f_1f_2 - f_2f_1 + \alpha_1 \\
    f'_2 = f_1 - f_0
\end{cases}
\]  

which possesses the affine Weyl group symmetry of type $A_1^l$ applied for the quantization of classical Painlevé II (PII) equation. Further after having the quantization of the classical Painlevé equations the extension of these equations to the noncommutative (NC) spaces got considerable attention. A very initial achievement in NC direction was obtained by V. Retakh and V. Roubtsov in [16] where they have introduced purely NC version of PII equation

\[
f''_2 = 2f_2^3 - 2[z, f_2]_+ + 4(\beta + \frac{1}{2})
\]  

with the help of its symmetric form in their case the fields $f_0, f_1, f_2$ obey a kind of star product, where these fields are purely non-commutative. Further, the quasideterminant solutions of that NC PII equation have been constructed by M. Irfan [17] through its zero-curvature condition and later on these solutions are calculated in [18] by taking NC Toda solutions as seed in NC PII Darboux transformations.

In this article I have extended the H. Nagoya [13] work on quantum PII equation to derive its non-abelian analogue. Here I have introduced a zero-curvature condition that is equivalent to the non-abelian quantum PII equation

\[
\begin{cases}
    f''_2 = 2f_2^3 - 2[z, f_2]_+ + c \\
    zf_2 - f_2z = \frac{1}{\hbar}f_2
\end{cases}
\]

these derivations are also involved the symmetric form (1) of PII equation and the fields $f_0, f_1, f_2$ obey the quantum commutation relation given in [13]. The basic difference between the quantum PII equation and the non-abelian quantum PII equation (15) is that here the variable $z$ and field $f_2$ appear as noncommuting elements but in case of H. Nagoya [13] these elements treated classically, as commuting variables. Further this can be shown that under the classical limit $\hbar \to 0$ the system (15) reduces to its classical analogue. I have also constructed it Riccati form and further its Darboux solutions have also been derived. Finally these solutions are generalized to the $N$-th form in terms of quasideterminants.
2. Zero curvature representation of non-abelian quantum PII equation

In the theory of Integrable systems, we are familiar with the Lax formalism, first time was introduced in [19] and zero curvature representations of these systems. Both the Lax technique and zero curvature condition are extensively applied to construct the solutions of Integrable systems in classical as well as in NC case. These representations involve two linear operators, these operators may be differential operators or matrices. The application of these formalisms to explore the various integrable aspects of nonlinear systems has been studied in wide spectrum, see, for example [20, 21, 22, 23, 24, 25]. If $A$ and $B$ are the linear operators and subjected to a linear system $A(x, t)\Psi = \lambda \Psi$, $\Psi_t = B(x, t)\Psi$ then Lax equation is given by $A_t = [B, A]$ where $\lambda$ is spectral parameter and $[B, A]$ is commutator. The compatibility condition of inverse scattering problem $\Psi_x = A(x, t)\Psi$ and $\Psi_t = B(x, t)\Psi$ yields $A_t - B_x = [B, A]$ which is called the zero curvature representation of integrable systems, which has been applied to many classical and NC systems, for a brief description see [26, 27, 28, 29]. In this article the commutator and anti-commutator will be written as $[,]_-$ and $[,]_+$ respectively. Now the Lax equation and zero curvature condition can be expressed as $A_t = [B, A]_-$ and $A_t - B_x = [B, A]_-$. 

**Proposition 1.1.**

The compatibility condition of following linear system

$$\Psi_\lambda = A(z; \lambda)\Psi, \quad \Psi_z = B(z; \lambda)\Psi$$

(4)

with Lax matrices

$$\begin{align*}
A &= (8i\lambda^2 + if_2^2 - 2iz)\sigma_3 + f_2'\sigma_2 + (\frac{1}{4}c\lambda^{-1} - 4\lambda f_2)\sigma_1 + i\hbar\sigma_2 \\
B &= -2i\lambda\sigma_3 + f_2\sigma_1 + f_2I
\end{align*}$$

(5)

yields non-abelian quantum PII equation with quantum commutation relations given in [13], here $I$ is $2 \times 2$ identity matrix and $\lambda$ is spectral parameter and $c$ is constant.

**Proof:**

The compatibility condition of system (4) yields zero curvature condition

$$A_z - B_\lambda = [B, A]_-.$$  

(6)

We can easily evaluate the values for $A_z$, $B_\lambda$ and $[B, A]_- = BA - AB$ from the linear system (5) as follow

$$A_z = (if_2'f_2 + if_2f_2' - 2i)\sigma_3 + f_2''\sigma_2 - 4\lambda f_2'\sigma_1$$

(7)

$$B_\lambda = -2i\sigma_3$$

(8)

and

$$[B, A] = \begin{pmatrix}
if_2'f_2 + if_2f_2' & [f_2, z]_- - i\hbar \\
\delta^- & \delta^+
\end{pmatrix}$$

(9)
\[ \delta^+ = -if_2'' + 2if_2^3 - 2iz[f_2]_+ + ic + if_2'[f_2]_- + 4i\lambda \hbar \]

and

\[ \delta^- = if_2'' - 2if_2^3 + 2iz[f_2]_+ - ic + if_2'[f_2]_- - 4i\lambda \hbar. \]

now after substituting these values from (7), (8) and (9) in equation (6) we get

\[
\begin{pmatrix}
[f_2, z]_- - \frac{1}{2}i\hbar & \delta^+ \\
\delta^- & [z, f_2]_- + \frac{1}{2}i\hbar
\end{pmatrix} = 0
\]

(10)

and the above result (10) yields the following expressions

\[ [f_2, z] = \frac{1}{2}i\hbar f_2 \]

(11)

and

\[ if_2'' - 2if_2^3 + 2iz[f_2]_+ - ic + if_2'[f_2]_- - 4i\lambda \hbar = 0 \]

(12)

equation (11) shows quantum relation between the variables \( z \) and \( f_2 \). In equation (12) the term \( if_2[f_2]_- - 2i\lambda \hbar \) can be eliminated by using equation \( f'_2 = f_1 - f_0 \) from (11) and quantum commutation relations in [14]. For this purpose let us replace \( f_2 \) by \( -\frac{1}{2} \lambda^{-1} f_2 \), then commutation relations become

\[ [f_0, f_2]_- = [f_2, f_1]_- = -2\lambda \hbar. \]

(13)

Now let us take the commutator of the both side of the equation \( f'_2 = f_1 - f_0 \) with \( f_2 \) from right side, we get

\[ [f'_2, f_2]_- = [f_1, f_2]_- - [f_0, f_2]_- \]

above equation with the commutation relations (13) can be written as

\[ [f'_2, f_2]_- = -4\lambda \hbar. \]

(14)

Now after substituting the value of \( [f'_2, f_2]_- \) from (14) in (12) we get

\[ if_2'' - 2if_2^3 + 2iz[f_2]_+ - ic = 0. \]

Finally, we can say that the compatibility of condition of linear system (4) yields the following expressions

\[
\begin{cases}
  f_2'' = 2f_2^3 - 2iz[f_2]_+ + c \\
  zf_2 - f_2z = \frac{1}{2}i\hbar f_2
\end{cases}
\]

(15)

in above system (15) the first equation can be considered as nonabelian version of quantum Painlevé II equation that is equipped with a quantum commutation relation and that equation can be reduced to the classical PII equation under the classical limit when \( \hbar \to 0 \).

**Remark 1.2.**

The linear system (14) with eigenvector \( \Psi = \begin{pmatrix} \chi \\ \phi \end{pmatrix} \) and setting \( \Delta = \chi \psi^{-1} \) can
be reduced to the PII Riccati equation associated to the nonabelian quantum PII equation
\[ \Delta' = -4i\Delta + f_2 + [f_2, \Delta]_- - \Delta f_2 \Delta. \]

**Proof:**
In order to derive the Riccati equation associated to the system (15), we apply the method of Konno and Wadati [30]. For this purpose let us substitute the eigenvector \( \Psi = (\chi \phi) \) in linear systems (4) and we get
\[
\begin{cases}
\frac{d\chi}{d\lambda} = (8i\lambda^2 + if_2^2 - 2iz)\chi + (-if_2' + \frac{1}{4}C_0\lambda^{-1} - 4\lambda f_2 + \hbar)\phi \\
\frac{d\phi}{d\lambda} = (if_2' + \frac{1}{4}C_0\lambda^{-1} - 4\lambda f_2 - \hbar)\chi + (-8i\lambda^2 - if_2^2 + 2iz)\phi
\end{cases}
\]
(16)
and
\[
\begin{cases}
\chi' = (-2i\lambda + f_2)\chi + f_2\phi \\
\phi' = f_2\chi + (2i\lambda + f_2)\phi
\end{cases}
\]
(17)
where \( \chi' = \frac{d\chi}{dz} \) and now from system (17) we can derive the following expressions
\[
\chi'\phi^{-1} = (-2i\lambda + f_2)\chi\phi^{-1} + f_2 \\
\phi'\phi^{-1} = -2i\lambda + f_2 + f_2\chi\phi^{-1}.
\]
(18)
(19)
Let consider the following substitution
\[ \Delta = \chi\phi^{-1} \]
(20)
now taking the derivation of above equation with respect to \( z \)
\[ \Delta' = \chi'\phi^{-1} - \chi\phi^{-1}\phi'\phi^{-1} \]
after using the (18), (19) and (20) in above equation we obtain
\[ \Delta' = -4i\Delta + f_2 + [f_2, \Delta]_- - \Delta f_2 \Delta \]
(21)
the above expression (21) can be regarded as quantum Riccati equation in \( \Delta \) related to the nonabelian quantum PII equation which has been derived from its associated linear systems (17). In next section the Darboux solutions to no-abelian quantum PII equation are constructed by using the idea of Darboux transformations presented in [31].

3. DARBOUX TRANSFORMATION FOR NONABELIAN QUANTUM PII EQUATION

**Proposition 2.1.**
The Darboux transformation for the solution \( u \) of non-abelian quantum PII equation (15) with the help of its associated linear system can be constructed in the following form
\[ u[1] = -4\lambda \Phi_1\chi^{-1} + \Phi_1\chi^{-1}\Phi_1\chi^{-1} \]

here $u[1]$ is a new solution of QP-II equation generated by initial solution $u$, here $f_2$ has been replaced by $u$, just for a simple notation.

**Proof:**

For the derivation of non-abelian QP-II Darboux transformation we consider the linear system (4) with a column vector $\psi = \left( \begin{array}{c} \chi \\ \Phi \end{array} \right)$. Now the linear system will become

$$\begin{pmatrix} \chi \\ \Phi \end{pmatrix} = \begin{pmatrix} 8i\lambda^2 + iu^2 - 2iz \\ iu_z + \frac{1}{4}C\lambda^{-1} - 4\lambda u - \hbar \end{pmatrix} \begin{pmatrix} \chi \\ \Phi \end{pmatrix}$$

$$\begin{pmatrix} \chi \\ \Phi \end{pmatrix} = \begin{pmatrix} -2i\lambda + u \\ u \\ 2i\lambda + u \end{pmatrix} \begin{pmatrix} \chi \\ \Phi \end{pmatrix}. \tag{23}$$

The standard transformations on $\chi$ and $\Phi$ are given below

$$\chi \rightarrow \chi[1] = \lambda \Phi - \lambda_1 \Phi_1(\lambda_1)\chi_1^{-1}(\lambda_1)\chi \tag{24}$$

$$\Phi \rightarrow \Phi[1] = \lambda\chi - \lambda_1 \chi_1(\lambda_1)\Phi_1^{-1}(\lambda_1)\Phi \tag{25}$$

where $\chi$, $\Phi$ are arbitrary solutions at $\lambda$ and $\chi_1(\lambda_1)$, $\Phi_1(\lambda_1)$ are the particular solutions at $\lambda = \lambda_1$ of equations (22) and (23), these equations will take the following forms under the transformations (24) and (25)

$$\begin{pmatrix} \chi[1] \\ \Phi[1] \end{pmatrix} = \begin{pmatrix} 8i\lambda^2 + iu[1]^2 - 2iz \\ b_+ \\ b_- \end{pmatrix} \begin{pmatrix} \chi[1] \\ \Phi[1] \end{pmatrix} \tag{26}$$

where

$$b_+ = -iu_z[1] + \frac{1}{4}C\lambda^{-1} - 4\lambda u[1] + \hbar$$

and

$$b_- = iu_z[1] + \frac{1}{4}C\lambda^{-1} - 4\lambda u[1] + \hbar$$

$$\begin{pmatrix} \chi[1] \\ \Phi[1] \end{pmatrix} = \begin{pmatrix} -2i\lambda + u[1] \\ u[1] \\ 2i\lambda + u[1] \end{pmatrix} \begin{pmatrix} \chi[1] \\ \Phi[1] \end{pmatrix}. \tag{27}$$

Now from (23) and equation (27) we have the following expressions

$$\chi_z = (-2i\lambda + u)\chi + u\Phi \tag{28}$$

$$\Phi_z = (i\lambda + u)\Phi + u\chi \tag{29}$$

and

$$\chi_z[1] = -(2i\lambda + u[1])\chi[1] + u[1]\Phi[1] \tag{30}$$

$$\Phi_z[1] = (2i\lambda + u[1])\Phi[1] + u[1]\chi[1]. \tag{31}$$

Now substituting the transformed values $\chi[1]$ and $\Phi[1]$ in equation (30) and then after using the (28) and (29) in resulting equation, we get

$$u[1] = -4\lambda\Phi_1\chi_1^{-1} + \Phi_1\chi_1^{-1}u\Phi_1\chi_1^{-1}. \tag{32}$$

Equation (32) represents the Darboux transformation of QPII equation, where $v[1]$ is a new solution of QPII equation, this shows that how the new solution is related to the seed solution $u$. By applying the DT iteratively we can construct the multi-soliton solution of QPII equation.
A Brief Introduction of Quasideterminants

This section is devoted to a brief review of quasideterminants introduced in [32]. Quasideterminants are the replacement to classical determinant of the matrices with noncommutative entries and these determinants play very important role to construct the multi-soliton solutions of NC integrable systems such as for the case of NC KP equation these determinants are applied by C. R. Gilson, J. J. C. Nimmo [33] to construct its solutions. Quasideterminants are not just a noncommutative generalization of usual commutative determinants but rather related to inverse matrices, quasideterminants for the square matrices are defined as if \( A = a_{ij} \) be a \( n \times n \) matrix and \( B = b_{ij} \) be the inverse matrix of \( A \). Here all matrix elements are supposed to belong to a NC ring with an associative product. Quasideterminants of \( A \) are defined formally as the inverse of the elements of \( B = A^{-1} \) this expression under the limit \( \theta^{\mu
u} \to 0 \), means entries of \( A \) are commuting, will reduce to \( |A|_{ij} = (-1)^{i+j} \frac{\det A}{\det A^{ij}} \) where \( A^{ij} \) is the matrix obtained from \( A \) by eliminating the \( i \)-th row and the \( j \)-th column. We can write down more explicit form of quasideterminants. In order to see it, let us recall the following formula for a square matrix

\[
A = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}
\]

(33)

where \( A \) and \( D \) are square matrices, and all inverses are supposed to exist. We note that any matrix can be decomposed as a \( 2 \times 2 \) matrix by block decomposition where the diagonal parts are square matrices, and the above formula can be applied to the decomposed \( 2 \times 2 \) matrix. So the explicit forms of quasideterminants are given iteratively by the following formula

\[
|A|_{ij} = a_{ij} - \sum_{p \neq i, q \neq j} a_{iq} |A^{ij}|^{-1}_{pq} a_{pj}
\]

the number of quasideterminant of a given matrix will be equal to the numbers of its elements for example a matrix of order 3 has nine quasideterminants. It is sometimes convenient to represent the quasi-determinant as follows

\[
|A|_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{ni} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}.
\]

(34)

Let us consider examples of matrices with order 2 and 3, for \( 2 \times 2 \) matrix

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}
\]
The quasideterminants of this matrix are given below:

\[ |A|_{11} = \left| \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right| = a_{11} - a_{12}Ma_{21} - a_{13}Ma_{31} \]

where \( M = \left| \begin{array}{cc} a_{22} & a_{23} \\ a_{32} & a_{33} \end{array} \right|^{-1} \), similarly we can evaluate the other eight quasideterminants of this matrix.

5. QUASIDETERMINANT REPRESENTATION OF DARBOUX TRANSFORMATION

**Proposition 7.4.**
The \( N \)-fold Darboux transformation for non-abelian QPII solution can be derived by the iteration of (32) in the following form, taking \( \lambda = \gamma \)

\[ u[N + 1] = -4\gamma \Pi^N_{k=1} \Theta_k[k] + \Pi^N_{k=1} \Theta_k[k]u[1] \Pi_{j=N}^1 \Theta_j[j] \quad \text{for} \quad N \geq 1 \quad (35) \]

with

\[ \Theta_N[N] = \gamma^\phi_N[N] \gamma^\chi_N[N]^{-1} \]

where \( u[1] \) is seed solution and \( u[N + 1] \) are the new solutions of QPII equation [17] and \( \gamma^N_N \), \( \gamma^\phi_N \) are the quasideterminants of the particular solutions of QPII linear system [11].

**Proof**
In order to prove this proposition, first we express the transformations (24) and (25) in terms of quasideterminants. Now consider the transformation (24) in the following form

\[ \chi[1] = \gamma_0 \Phi_0 - \gamma_1 \Phi_1(\gamma_1)^{-1}(\gamma_1)\chi_0 \]
\[
\chi[1] = \begin{vmatrix}
\chi_1 \\
\gamma_1 \Phi_1 \\
\gamma_0 \Phi_0
\end{vmatrix} = \delta^e_\chi[1]
\] (36)

similarly we can do for the equation (25)

\[
\Phi[1] = \begin{vmatrix}
\Phi_1 \\
\gamma_1 \chi_1 \\
\gamma_0 \chi_0
\end{vmatrix} = \delta^e_\Phi[1]
\] (37)

we have taken \(\gamma = \gamma_0\), \(\chi = \chi_0\) and \(\Phi = \Phi_0\) in order to generalize the transformations in \(N\)th form. Further, we can represent the transformations \(\chi[2]\) and \(\Phi[2]\) by quasideterminants

\[
\chi[2] = \begin{vmatrix}
\chi_2 \\
\gamma_2 \Phi_2 \\
\gamma_1 \Phi_1 \\
\gamma_0 \Phi_0
\end{vmatrix} = \delta^e_\chi[2]
\]

and

\[
\Phi[2] = \begin{vmatrix}
\Phi_2 \\
\gamma_2 \chi_2 \\
\gamma_1 \chi_1 \\
\gamma_0 \chi_0
\end{vmatrix} = \delta^e_\Phi[2]
\]

here superscripts \(e\) and \(o\) of \(\delta\) represent the even and odd order quasideterminants. The \(N\)th transformations for \(\delta^o_\chi[N]\) and \(\delta^o_\Phi[N]\) in terms of quasideterminants are given below

\[
\delta^o_\chi[N] = \begin{vmatrix}
\chi_N & \chi_{N-1} & \cdots & \chi_1 & \chi_0 \\
\gamma_N \Phi_N & \gamma_{N-1} \Phi_{N-1} & \cdots & \gamma_1 \chi_1 & \gamma_0 \Phi_0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\gamma_N^{-1} \Phi_N & \gamma_{N-1}^{-1} \Phi_{N-1} & \cdots & \gamma_1^{-1} \chi_1 & \gamma_0^{-1} \Phi_0 \\
\gamma_N \chi_N & \gamma_{N-1} \chi_{N-1} & \cdots & \gamma_1 \chi_1 & \gamma_0 \chi_0
\end{vmatrix}
\]

and

\[
\delta^o_\Phi[N] = \begin{vmatrix}
\Phi_N & \Phi_{N-1} & \cdots & \Phi_1 & \Phi_0 \\
\gamma_N \chi_N & \gamma_{N-1} \chi_{N-1} & \cdots & \gamma_1 \chi_1 & \gamma_0 \chi_0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\gamma_N^{-1} \chi_N & \gamma_{N-1}^{-1} \chi_{N-1} & \cdots & \gamma_1^{-1} \chi_1 & \gamma_0^{-1} \chi_0 \\
\gamma_N \Phi_N & \gamma_{N-1} \Phi_{N-1} & \cdots & \gamma_1 \Phi_1 & \gamma_0 \Phi_0
\end{vmatrix}
\]

here \(N\) is to be taken as even. In the same way we can write \(N\)th quasideterminant representations of \(\delta^e_\chi[N]\) and \(\delta^e_\Phi[N]\).

\[
\delta^e_\chi[N] = \begin{vmatrix}
\chi_N & \chi_{N-1} & \cdots & \chi_1 & \chi_0 \\
\gamma_N \Phi_N & \gamma_{N-1} \Phi_{N-1} & \cdots & \gamma_1 \chi_1 & \gamma_0 \Phi_0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\gamma_N^{-1} \chi_N & \gamma_{N-1}^{-1} \chi_{N-1} & \cdots & \gamma_1^{-1} \chi_1 & \gamma_0^{-1} \chi_0 \\
\gamma_N \Phi_N & \gamma_{N-1} \Phi_{N-1} & \cdots & \gamma_1 \Phi_1 & \gamma_0 \Phi_0
\end{vmatrix}
\]
is given by this is one fold Darboux transformation. The two fold Darboux transformation
where
\[ W \]
We may rewrite the equation (36) and equation (37) in the following forms
\[ (32) \]
by applying the Darboux transformation iteratively, now consider
\[ (38) \]
with following settings
\[ (38) \]
Similarly, we can derive the expression for \( N \)th soliton solution from equation [52] by applying the Darboux transformation iteratively, now consider
\[ u[1] = -4\gamma \Omega_1^\phi[1]| \Omega_1^\phi[1]^{-1} \Omega_1^\phi[1]| \Omega_1^\phi[1]^{-1} u \Omega_1^\phi[1]| \Omega_1^\phi[1]^{-1} \]
where
\[ \Omega_1^\phi[1] = \Phi_1 \]
\[ \Omega_1^\phi[1] = \chi_1 \]
this is one fold Darboux transformation. The two fold Darboux transformation is given by
\[ u[2] = -4\gamma \phi[1]| \chi^{-1}[1] + \phi[1]| \chi^{-1}[1] u[1]| \phi[1]| \chi^{-1}[1]. \quad (38) \]
We may rewrite the equation (36) and equation (37) in the following forms
\[ \chi[1] = \left| \begin{array}{cc} \chi_1 & \gamma_0 \Phi_0 \\ \gamma_1 \Phi_1 & \gamma_0 \Phi_0 \end{array} \right| = \Omega_2^\phi[2] \]
\[ \Phi[1] = \left| \begin{array}{cc} \Phi_1 & \Phi_0 \\ \gamma_1 \chi_1 & \gamma_0 \chi_0 \end{array} \right| = \Omega_2^\phi[2]. \]
and equation (38) may be written as
\[ u[2] = \Theta_2^{(-)}(-4\gamma + u) \Theta_2^{(+)} \]
where
\[ \Theta_2^{(-)} = \Omega_2^\phi[2]| \Omega_2^\phi[2]^{-1} \Omega_2^\phi[1]| \Omega_2^\phi[1]^{-1} \]
\[ \Theta_2^{(+)} = \Omega_2^\phi[1]| \Omega_2^\phi[1]^{-1} \Omega_2^\phi[2]| \Omega_2^\phi[2]^{-1} \]
In the same way, we can derive the expression for three fold Darboux transformation
\[ u[3] = \Theta_3^{(-)}(-4\gamma + u) \Theta_3^{(+)} \]
where
\[ \Theta_3^{(-)} = \Omega_3^\phi[3]| \Omega_3^\phi[3]^{-1} \Omega_3^\phi[2]| \Omega_3^\phi[2]^{-1} \Omega_3^\phi[1]| \Omega_3^\phi[1]^{-1} \]
\[ \Theta_3^{(+)} = \Omega_3^\phi[1]| \Omega_3^\phi[1]^{-1} \Omega_3^\phi[2]| \Omega_3^\phi[2]^{-1} \Omega_3^\phi[3]| \Omega_3^\phi[3]^{-1} \]
with following settings
\[ \chi[2] = \left| \begin{array}{ccc} \chi_2 & \chi_1 & \gamma_0 \Phi_0 \\ \gamma_2 \Phi_2 & \gamma_1 \Phi_1 & \gamma_0 \Phi_0 \\ \gamma_2 \chi_2 & \gamma_1 \chi_1 & \gamma_0 \chi_0 \end{array} \right| = \Omega_3^\phi[3] \]
and

\[ \Phi[2] = \begin{vmatrix} 
\Phi_2 & \Phi_1 & \Phi_0 \\
\gamma_2 \chi_2 & \gamma_1 \chi_1 & \gamma_0 \chi_0 \\
\gamma_2^2 \Phi_2 & \gamma_1^2 \Phi_1 & \gamma_0^2 \Phi_0 
\end{vmatrix} = \Omega^\phi_3[3]. \]

Finally, by applying the transformation iteratively we can construct the \( N \)-fold Darboux transformation

\[ u[N] = \Omega^\phi_N[N] \Omega^N\chi[N]^{-1} \Omega^\phi_{N-1}[N-1] \Omega^\phi_{N-2}[N-2] \cdots \Omega^\phi_2[2] \Omega^\phi_1[2]^{-1} \Omega^\phi_0[1] \Omega^\phi_0[1]^{-1} \times \]

\[ \times (-4\gamma + u) \Omega^\phi_1[1] \Omega^N\chi[1]^{-1} \Omega^\phi_2[2] \Omega^N\chi[2]^{-1} \cdots \Omega^\phi_{N-1}[N-1] \Omega^N\chi_{N-1}[N-1]^{-1} \gamma^\phi_N[N] \gamma^N\chi[N]^{-1} \]

by considering the following substitution

\[ \Theta_N[N] = \Omega^\phi_N[N] \Omega^N\chi[N]^{-1} \]

in above expression, we get

\[ u[N] = \Theta_N[N] \Theta_{N-1}[N-1] \cdots \Theta_1[1] (-4\gamma + u) \Theta_1[1] \Theta_2[2] \cdots \Theta_{N-1}[N-1] \Theta_N[N] \]

or

\[ u[N] = \Pi_{k=0}^{N-1} \Theta_{N-k}[N-k] (-4\gamma + u) \Pi_0^{N-1} \Theta_{N-j}[N-j] \quad (39) \]

here we present only the \( N \)th expression for odd order quasideterminants \( \Omega^\phi_N[N] \) and \( \Omega^N\chi[N] \)

\[ \Omega^\phi_{2N+1}[2N + 1] = \begin{vmatrix} 
\Phi_{2N} & \Phi_{2N-1} & \cdots & \Phi_1 & \Phi_0 \\
\gamma_2 N \chi_{2N} & \gamma_2 N -1 \chi_{2N-1} & \cdots & \gamma_1 \chi_1 & \gamma_0 \chi_0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\gamma_2^{2N-1} N \chi_{2N} & \gamma_2^{2N-1} N -1 \chi_{2N-1} & \cdots & \gamma_1^{2N-1} \chi_1 & \gamma_0^{2N-1} \chi_0 \\
\gamma_2^{2N} N \chi_{2N} & \gamma_2^{2N} N -1 \chi_{2N-1} & \cdots & \gamma_1^{2N} \chi_1 & \gamma_0^{2N} \chi_0 
\end{vmatrix} \]

and

\[ \Omega^N\chi_{2N+1}[2N + 1] = \begin{vmatrix} 
\chi_{2N} & \chi_{2N-1} & \cdots & \chi_1 & \chi_0 \\
\gamma_2 N \Phi_{2N} & \gamma_2 N -1 \Phi_{2N-1} & \cdots & \gamma_1 \Phi_1 & \gamma_0 \Phi_0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\gamma_2^{2N-1} N \Phi_{2N} & \gamma_2^{2N-1} N -1 \Phi_{2N-1} & \cdots & \gamma_1^{2N-1} \Phi_1 & \gamma_0^{2N-1} \Phi_0 \\
\gamma_2^{2N} N \chi_{2N} & \gamma_2^{2N} N -1 \chi_{2N-1} & \cdots & \gamma_1^{2N} \chi_1 & \gamma_0^{2N} \chi_0 
\end{vmatrix}. \]

6. CONCLUSION

In this paper, I have presented the zero curvature representation of non-abelian quantum PII equation and also its Darboux transformation which are further applied to construct the multi-solution expression to that equation in terms quasideterminants. I have also derived Riccati equation associated to non-abelian quantum PII equation from its linear system by using the method of Konno and Wadati [30]. Symmetrically it is quite interesting to construct zero curvature representations for quantum Painlevé equations PIV, PV by using the procedure presented in this article to derive non-abelian quantum
analogue of these equations involving the symmetric forms of these systems given in [13].

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CHEP, UNIVERSITY OF THE PUNJAB, 54590 LAHORE, PAKISTAN
E-mail address: mahirfan@yahoo.com