A simple 2nd order lower bound to the energy of dilute Bose gases

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Abstract

For a dilute system of non-relativistic bosons interacting through a regular, positive potential $v$ with scattering length $a$ we prove that the ground state energy density satisfies the bound $e(\rho) \geq 4\pi a \rho^2 (1 - C\sqrt{\rho a^3})$.

Contents

1 Introduction 2
2 Facts about the scattering solution 4
3 An equivalent problem on Fock space 4
4 Reduction to a small box 5
\quad 4.1 Setup and notation 5
\quad 4.2 Localization of the potential energy 6
\quad 4.3 Localization of the kinetic energy 7
\quad 4.4 The localized Hamiltonian 10
5 Energy in the box 11
\quad 5.1 A priori estimates on particle numbers 11
\quad 5.2 Estimates on non-quadratic terms 12
6 Bogoliubov calculation 17
7 Estimating the energy 22
A Bogoliubov method 24
B Second quantization 24
1 Introduction

We study a system of \( N \) interacting bosons in a large box \( \Lambda \subset \mathbb{R}^3 \) of volume \( |\Lambda| \). For concreteness, we take \( L > 0 \) and define \( \Lambda = [-L/2, L/2]^3 \). We are interested in the thermodynamic limit \( N \to \infty, |\Lambda| \to \infty \) with density \( \tilde{\rho} := N/|\Lambda| \) fixed and small.

The Hamiltonian of the system is
\[
H_N := \sum_{i=1}^{N} -\Delta_i + \sum_{i<j} v(x_i - x_j),
\]
on the symmetric (bosonic) space \( \otimes_s^N L^2(\Lambda) \). We take \( H_N \) with Dirichlet boundary conditions to realize it as a self-adjoint operator and impose the following conditions on the two-particle potential.

**Assumption 1.1.** The potential \( v \neq 0 \) is non-negative and spherically symmetric, i.e. \( v(x) = v(|x|) \geq 0 \), and integrable with compact support. We fix \( R > 0 \) such that \( \text{supp} v \subset B(0, R) \).

We define the ground state energy of the system \( E_0 = E_0(N, \Lambda) \) to be
\[
E_0(N, \Lambda) := \inf \text{Spec} \ H_N
\]
and the ground state energy density as
\[
e(\tilde{\rho}) = \lim_{L \to \infty, N/|\Lambda| = \tilde{\rho}} \frac{E_0(N, \Lambda)}{L^3}.
\]

Our main result, Theorem 1.2, is formulated in terms of the scattering length \( a = a(v) \), where \( v \) is a potential satisfying Assumption 1.1. The definition and useful properties of the scattering length will be given in Section 2.

**Theorem 1.2.** Given a potential \( v \) satisfying Assumption 1.1 there exists a constant \( C > 0 \) (depending on \( v \)) such that for all \( \tilde{\rho} \) sufficiently small,
\[
e(\tilde{\rho}) \geq 4\pi\tilde{\rho}^2 a(1 - C(\tilde{\rho}a^3)^{1/2}) \quad (1.1)
\]

The proof of Theorem 1.2, which is the first general lower bound containing the expected order of the correction term, will be given in Section 3. We will briefly review some previous results below.

The rigorous study of the ground state energy of the interacting boson problem has a long history\(^1\). Bogoliubov’s theory \(^2\) prescribes how to treat the weak coupling limit of interacting bosons. In the context of the present paper, this weak coupling limit corresponds to the dilute gas, i.e. the limit \( \tilde{\rho} \to 0 \) under study. The Bogoliubov theory actually gives much more detailed information, e.g. on the excitation spectrum, but the ground state energy is one of the simplest quantities on which to obtain rigorous information regarding the validity of Bogoliubov’s approach.

It was Lenz \(^3\) who proposed the leading order behavior in (1.1). Dyson \(^5\) proved that the leading order of (1.1) has the correct form. His upper bound provides the sharp constant, while his lower bound only captures the correct leading order and was completed

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\(^1\)When comparing results in the literature, one should notice that some authors study the energy per particle, i.e. \( \lim E_0(N, \Lambda)/N = \tilde{\rho}^{-1} e(\tilde{\rho}) \) instead of the energy density, leading, of course, to slightly different formulae.
Introducing

by the corresponding lower bound 40 years later in \[13\]. To even higher precision, the energy density is expected to behave as

\[
e(\tilde{\rho}) = 4\pi\tilde{\rho}^2 a \left( 1 + \frac{128}{15\sqrt{\pi}} (\tilde{\rho} a^3)^{1/2} + o(\tilde{\rho} a^3)^{1/2} \right),
\]

(1.2)

but this remains unproven. The second term in (1.2) is often referred to as the Lee-Huang-Yang term after \[8\] but is also consistent with Bogoliubov’s treatment. For this and other background information on the Bose gas we refer to \[10\].

In \[6\] an upper bound to \(e(\tilde{\rho})\) is given which correctly reproduces the first term and the order of the second term, however only giving the correct coefficient on the correction term in the additional limit of weak interaction. From \[14\] (see also \[15\] for more information on the Bogoliubov functional) one can actually conclude that to get (1.2) one needs to go beyond states that are quasi-free. Indeed, they prove that for the ground state energy the trial state in \[6\] is essentially optimal among quasi-free states.

The asymptotic result (1.2) has only been proven in cases where the interaction is scaled to become ‘soft’ in a manner depending on \(\tilde{\rho}\) \[4, 7\]. An upper bound consistent with (1.2) has been proved in \[16\] using trial states that are not quasi-free. The recent work that comes closest to establishing (1.2) is probably \[1, 2\], though they address a somewhat different limit. Actually, the result obtained in \[1\] is after scaling very analogous to our analysis of the box Hamiltonian (see Theorem 5.1 below). We have the additional difficulties that for our localized problem, we no longer have translation invariance nor a fixed number of particles. Nevertheless, we believe that our method, at least for the ground state energy, is substantially shorter and simpler than the one of \[1\], which also covers the excitation spectrum.

In the papers \[11, 12\] the Bogoliubov approximation is proved to give the right result in the setting of the ground state energy of a charged gas. In the present paper we use the general strategy laid out in those papers.

Notation. We use the convention that integrals are over all of \(\mathbb{R}^3\) unless the domain of integration is explicitly specified.

Organization of the paper. The paper is organized as follows. We start by recalling basic definitions and results about the scattering length \(a\) and related quantities in Section 2. Then, in Section 3 we reformulate the many-body problem in a Fock space setting—see in particular Theorem 3.1. This will allow us to use a simple version of Bogoliubov’s theory in which the number of particles is not fixed. In Section 4 a localization to boxes of length scale \(\ell \approx 1/\sqrt{\tilde{\rho} a}\) is carried out. This is an important and delicate step since our proof requires the localization to be carried out in such a way as not to lose the Neumann gap. The final result of the section is Theorem 4.8 whereby all we have to study is the ground state energy of one fixed box, which is the purpose of the remainder of the article. The main work is carried out in Section 5. In Lemma 5.4 we estimate the terms in the Hamiltonian that are not quadratic in excitations out of the constant function. The important point here is inspired by the analysis of the Bogoliubov functional and consists of ‘completing a square’ relative to the quartic term in the excitations. The terms remaining are quadratic, thus allowing us to second quantize and use the Bogoliubov method. This we carry out in Section 6. Finally, in Section 7 we put the pieces together to prove Theorem 5.1 which, using the first sections, implies Theorem 1.2.

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2 Facts about the scattering solution

In this short section we establish notation and recall results concerning the scattering length and associated quantities. We suppose that \( v \) satisfies Assumption 1.1 and refer to Appendix C of [10] for details and a more general treatment. The scattering equation reads

\[
(-\Delta + \frac{1}{2}v(x))(1 - \omega(x)) = 0, \quad \text{with } \omega \to 0, \quad \text{as } |x| \to \infty.
\]  
(2.1)

The solution \( \omega \) to this equation satisfies that there exists a constant \( a > 0 \) such that \( \omega(x) = a/|x| \) for \( x \) outside \( \text{supp } v \). This constant \( a \) is the scattering length of the potential \( v \) and we will refer to \( \omega \) as the scattering solution. Furthermore, \( \omega \) is radially symmetric and non-increasing with

\[
0 \leq \omega(x) \leq 1.
\]  
(2.2)

We introduce the functions

\[
f := v\omega, \quad g := v - f = v(1 - \omega).
\]  
(2.3)

The scattering equation can be reformulated as

\[
-\Delta \omega = \frac{1}{2}g.
\]  
(2.4)

From this we deduce, using the divergence theorem, that

\[
a = (8\pi)^{-1} \int g,
\]  
(2.5)

and that the Fourier transform satisfies

\[
\hat{\omega}(k) = \frac{\hat{g}(k)}{2k^2}.
\]  
(2.6)

Noting that \( v \in L^1(\mathbb{R}^3) \) by Assumption 1.1 we define the quantity

\[
a_1 := \frac{1}{8\pi} \int v(x),
\]  
(2.7)

which satisfies the sharp inequality \( a < a_1 \).

3 An equivalent problem on Fock space

For convenience we reformulate the problem on Fock space.

Consider, for given \( \rho_\mu > 0 \), the following operator \( \mathcal{H}_{\rho_\mu} \) on the symmetric Fock space \( \mathcal{F}_s(L^2(\Lambda)) \). The operator \( \mathcal{H}_{\rho_\mu} \) commutes with particle number and satisfies, with \( \mathcal{H}_{\rho_\mu,N} \) denoting the restriction of \( \mathcal{H}_{\rho_\mu} \) to the \( N \)-particle subspace of \( \mathcal{F}_s(L^2(\Lambda)) \),

\[
\mathcal{H}_{\rho_\mu,N} = \sum_{i=1}^{N} \left( -\Delta_i - \rho_\mu \int_{\mathbb{R}^3} v(x_i - y) \, dy \right) + \sum_{i<j} v(x_i - x_j)
\]

\[
= \sum_{i=1}^{N} -\Delta_i + \sum_{i<j} v(x_i - x_j) - 8\pi a_1 \rho_\mu N.
\]  
(3.1)
Notice that the new term in $\mathcal{H}_{\rho_\mu, N}$ plays the role of a chemical potential justifying the notation.

Define the corresponding ground state energy density,

$$
e_0(\rho_\mu) := \lim_{|\Lambda| \to \infty} |\Lambda|^{-1} \inf_{\Psi \in \mathcal{F}_N \setminus \{0\}} \frac{\langle \Psi, \mathcal{H}_{\rho_\mu} \Psi \rangle}{\|\Psi\|^2}.
$$

We formulate the following result, which will be a consequence of Theorems 4.8 and 5.1.

**Theorem 3.1.** Suppose $v$ satisfies Assumption 1.1. Then the ground state energy density of $\mathcal{H}_{\rho_\mu}$ satisfies for all $\rho_\mu$ sufficiently small that there exists a constant $C$ (depending on $v$) such that

$$
e_0(\rho_\mu) \geq 4\pi \rho_\mu^2 a_1 \left(-\frac{a_1}{a} - C(\rho_\mu a^3)^{1/2}\right).
$$

**Proof of Theorem 1.2.** It is easy to deduce Theorem 1.2 from Theorem 3.1. By inserting the ground state of $\mathcal{H}_N$ as a trial state in $\mathcal{H}_{\rho_\mu}$ one gets in the thermodynamic limit for all $\tilde{\rho}, \rho_\mu > 0$ that

$$
e(\tilde{\rho}) \geq e_0(\rho_\mu) + \tilde{\rho} \rho_\mu \int v.
$$

Therefore, by (3.4) and the lower bound from Theorem 3.1 we get

$$
e(\tilde{\rho}) \geq 4\pi a_1 \left[-\frac{a_1}{a} \rho^2_\mu - C \rho_\mu^2 (\rho_\mu a^3)^{1/2} + 2\tilde{\rho} \rho_\mu\right]
= 4\pi a \left[\tilde{\rho}^2 - (\tilde{\rho} - \frac{a_1}{a} \rho_\mu)^2 - C \rho_\mu^2 \frac{a_1}{a} (\rho_\mu a^3)^{1/2}\right].
$$

At this point we can choose $\rho_\mu = \frac{a}{\tilde{\rho} a} \tilde{\rho}$ to get (1.1). \qed

## 4 Reduction to a small box

### 4.1 Setup and notation

The main part of the analysis will be carried out on a small box of size

$$
\ell := K(\rho_\mu a)^{-1/2},
$$

for some $K > 0$ to be chosen sufficiently small but independent of $\rho_\mu$. In this section we will carry out that localization. The main result is given at the end of the section as Theorem 4.8 which states that for a lower bound it suffices to consider a ‘box energy’, i.e. the ground state energy of a Hamiltonian localized to a box of size $\ell$. For convenience, in Theorem 5.1 we state the bound on the box energy that will suffice in order to prove Theorem 3.1.

Let $\chi \in C_c(\mathbb{R}^3)$ be an even localization function, satisfying

$$0 \leq \chi, \quad \int \chi^2 = 1, \quad \text{supp } \chi \subset [-1/2, 1/2]^3.
$$

For given $u \in \mathbb{R}^3$, we define

$$
\chi_u(x) := \chi(\frac{x}{\ell} - u).
$$

Notice that $\chi_u$ localizes to the box $B(u) := \ell u + [-\ell/2, \ell/2]^3$. 

\[5\]
We will also need the sharp localization function $\theta_u$ to the box $B(u)$, i.e.
\[ \theta_u := 1_{B(u)}. \] (4.4)

Define $P_u, Q_u$ to be the orthogonal projections in $L^2(\mathbb{R}^3)$ defined by
\[ P_u \varphi := \ell^{-3}(\theta_u, \varphi) \theta_u, \quad Q_u \varphi := \theta_u \varphi - \ell^{-3}(\theta_u, \varphi) \theta_u. \] (4.5)

Define furthermore
\[ W(x) := \frac{v(x)}{\chi \ast \chi(x/\ell)}, \] (4.6)

That $W$ is well-defined for sufficiently small values of $\rho_\mu$, uses that $v$ has finite range. Man-
ifestly $W$ depends on $\ell$ and thus $\rho_\mu$, but we will not reflect this in our notation. Define the
localized potentials
\[ w_u(x, y) := \chi_u(x)W(x - y)\chi_u(y), \quad w(x, y) := w_{u=0}(x, y). \] (4.7)

Notice the translation invariance,
\[ w_{u+\tau}(x, y) = w_u(x - \ell \tau, y - \ell \tau). \] (4.8)

For $\rho_\mu$ sufficiently small a simple change of variables yields, for all $u \in \mathbb{R}^3$, the identities
\[ \frac{1}{2} \ell^{-6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \chi\left(\frac{x}{\ell}\right) \chi\left(\frac{y}{\ell}\right) W(x - y) \, dx \, dy = \frac{1}{2} \ell^{-6} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} w_u(x, y) \, dx \, dy \]
\[ = \frac{1}{2} \ell^{-3} \int v = 4\pi a_1. \] (4.9)

For some estimates it is convenient to invoke the scattering solution and thus we introduce
the notation, which again is well-defined for $\rho_\mu$ sufficiently small,
\[ W_2(x) := W(x)(1 - \omega(x)) = \frac{g(x)}{\chi \ast \chi(x/\ell)}, \quad w_2(x, y) := w(x, y)(1 - \omega(x - y)). \] (4.10)

**Lemma 4.1.** There exists a constant $C$ (depending on $R$ and $\chi$) such that for all $u$, and all $\ell$ sufficiently large
\[ \max_x \int w_u(x, y) \, dy \leq C a_1. \] (4.11)

**Proof.** By translation invariance, it suffices to consider $u = 0$. By definition,
\[ \int w(x, y) \, dy = \chi(x/\ell) \int \frac{v(x - y)}{\chi \ast \chi((x - y)/\ell)} \chi(y/\ell) \, dy. \] (4.12)

Since supp $v \subset B(0, R)$, $\chi \ast \chi(0) = 1$ and $\chi \ast \chi$ is even, we get
\[ 0 \leq \int w(x, y) \, dy = \int_{|x - y| < R} w(x, y) \, dy \]
\[ \leq (1 - C(R/\ell)^2)^{-1} \chi(x/\ell) \int v(x - y) \chi(y/\ell) \, dy \leq ||\chi||_{L^\infty}^2 (1 - C(R/\ell)^2)^{-1} 8\pi a_1. \]
4.2 Localization of the potential energy

**Lemma 4.2** (Localization of potential energy). If $\rho_\mu$ is sufficiently small, we have for all $x_1, \ldots, x_N \in \Lambda$

$$\sum_{i=1}^{N} -\rho_\mu \int v(x_i - y) \, dy + \sum_{i<j} v(x_i - x_j) = \int_{\ell^{-1}(\Lambda + B(0, \ell/2))} \left[ \sum_{i=1}^{N} -\rho_\mu \int w_u(x_i, y) \, dy + \sum_{i<j} w_u(x_i, x_j) \right] \, du. \quad (4.13)$$

**Proof.** We calculate, using $x_i, x_j \in \Lambda$,

$$\int_{\ell^{-1}(\Lambda + B(0, \ell/2))} w_u(x_i, x_j) \, du = \int_{\ell^{-1}(\Lambda + B(0, \ell/2))} \chi\left(\frac{x_i}{\ell} - u\right)\chi\left(\frac{x_j}{\ell} - u\right) \, du \, W(x_i - x_j) = v(x_i - x_j). \quad (4.14)$$

Here we used that if $\|x_i - x_j\| \leq R$ and $\rho_\mu$ is sufficiently small, then the $u$-integral gives the (non-zero) convolution, which is the denominator in $W$. The other terms are similar. \qed

4.3 Localization of the kinetic energy

In this subsection we prove a localization estimate on the kinetic energy in the box $B(u)$ centered at $\ell u$. The localized kinetic energy operator stems from Lemma 4.7 below and becomes

$$T_u := Q_u \left[ \chi \left( -\Delta - C_{\text{kin}} \ell^2 \right) + \chi + b\ell^{-2} \right] Q_u, \quad (4.15)$$

where $b, C_{\text{kin}} > 0$ are universal constants.

Note that $T_u$ vanishes on constant functions. The last term in $T_u$ will control the gap in the kinetic energy, i.e. on functions orthogonal to constants in the box, $T_u$ is bounded below by at least $b\ell^{-2}$. A key result to obtain (4.15) is the lemma below.

**Lemma 4.3** (Abstract kinetic energy localization). Let $\mathcal{K} : \mathbb{R}^3 \rightarrow [0, \infty)$ be a symmetric, polynomially bounded, continuous function, and define the operator $T$ on $L^2(\mathbb{R}^3)$ by

$$T = \int_{\mathbb{R}^3} Q_u \chi u \mathcal{K}(-i\nabla)\chi u Q_u \, du, \quad (4.16)$$

where $\chi u$ is considered as a multiplication operator in configuration space. This $T$ is translation invariant, i.e. a multiplication operator in Fourier space $T = F(-i\nabla)$, with

$$F(p) = (2\pi)^{-3} \mathcal{K} * |\hat{\chi}|^2(p) - 2(2\pi)^{-3}\hat{\theta}(p)\hat{\chi} * (\mathcal{K}\hat{\chi})(p) + (2\pi)^{-3} \left( \int |\mathcal{K}|^2 \right) \hat{\theta}(p)^2. \quad (4.17)$$

In particular, we have $F(0) = 0$, $F \geq 0$ and $\nabla F(0) = 0$.

**Remark 4.4.** For simplicity, we have chosen to assume that $\chi \in C_c^\infty$ whereby $\hat{\chi}$ has fast decay. The same method works for localization functions with less regularity, the important assumption for Lemma 4.3 being that the integral $\int |\mathcal{K}|^2$ converges. In the accompanying paper [4] it will be important to use this flexibility.
Lemma 4.6. There exist constants $C > 0$ and $s^* > 0$ (depending on the choice of $\chi$) such that for $0 < s \leq s^*$ and any $\ell > 0$ we have the inequality for all $\varphi \in H^1(\mathbb{R}^3)$

$$
\langle \varphi, F_s(\cdot - i\nabla) |\cdot \rangle \varphi \rangle \geq \int \langle \varphi, Q_u \chi_u (\Delta - (s\ell)^{-2})_+ \chi_u Q_u \varphi \rangle \, du,
$$

where

$$
F_s(|p|) = \begin{cases}
(\frac{1}{2} |p|^2 - \frac{1}{2} (s\ell)^{-2}), & \text{if } |p| \geq \frac{2}{5}(s\ell)^{-1}, \\
C_1 s^2 p^2, & \text{if } |p| < \frac{2}{5}(s\ell)^{-1}.
\end{cases}
$$

Proof. By a simple scaling it is enough to consider $\ell = 1$. This is a straightforward calculation. Note that $Q_u$ has the integral kernel $\theta_u(y) \cdot \delta(y - x) - 1 \theta_u(x)$. If we denote by $\tilde{K}$ the inverse Fourier transform of $K$ in the sense of a tempered distribution, then the integral kernel of the operator $Q_u \chi_u K(-i\nabla) \chi_u Q_u$ is given by

$$
\chi_u(x) \tilde{K}(x - y) \chi_u(y) - \chi_u(x) [\tilde{K} * \chi_u](x) \theta_u(y) - \theta_u(x) [\tilde{K} * \chi_u](y) \chi_u(y) + \theta_u(x) [\chi_u K(-i\nabla) \chi_u] \theta_u(y).
$$

Thus the integral kernel of $\int Q_u \chi_u K(-i\nabla) \chi_u Q_u \, du$ is given by

$$
([\chi * \chi] \tilde{K})(x - y) - 2 (\chi [\tilde{K} * \chi]) * \theta(x - y) + (2\pi)^{-3} \left( \int \tilde{K}(p) \tilde{\chi}(p)^2 \, dp \right) \theta * \theta(x - y),
$$

where we used that $\int \tilde{K}(p) \tilde{\chi}(p)^2 \, dp$ is finite by the choice of $K$ and the decay of $\tilde{\chi}$. We arrive at the expression for $F$ by calculating the inverse Fourier transform. The fact that $F(0) = 0$ follows since $\tilde{\theta}(0) = \int \theta = 1$ and

$$
(2\pi)^3 F(0) = 2 \left( \int \tilde{K} \tilde{\chi}^2 \right) (1 - \tilde{\theta}(0))^2 = 0.
$$

That $F \geq 0$ is a direct consequence of (4.16) since $K$ is positive. Because $F$ is differentiable it follows that $\nabla F(0) = 0$. \hfill $\Box$

With $\ell = 1$ this lemma is similar to the generalized IMS localization formula

$$
\int_{\mathbb{R}^3} \chi_u K(-i\nabla) \chi_u \, du = (2\pi)^{-3} K * |\tilde{\chi}|^2,
$$

where $K(p) = p^2$ gives the standard IMS formula since then $(2\pi)^{-3} K * |\tilde{\chi}|^2 = p^2 + \int |\nabla \chi|^2$.

Corollary 4.5. With the same notation as above we have that

$$
\int_{\mathbb{R}^3} Q_u \, du = 1 - \tilde{\theta}(-i\ell\nabla)^2,
$$

i.e. the operator $\int_{\mathbb{R}^3} Q_u \, du$ is the multiplication operator in Fourier space given by $1 - \tilde{\theta}(\ell p)^2$.

Proof. Simply take $K = 1$ and $\chi = \theta$ in the above lemma which is allowed as noticed in Remark 4.3. \hfill $\Box$

We will use Lemma 4.3 for the function $K(p) = [|p|^2 - s^{-2}]_+$, where $s > 0$ is a sufficiently small constant. Here $u_+ = \max\{u, 0\}$ denotes the positive part of $u$.

Lemma 4.6. There exist constants $C > 0$ and $s^* > 0$ (depending on the choice of $\chi$) such that for $0 < s \leq s^*$ and any $\ell > 0$ we have the inequality for all $\varphi \in H^1(\mathbb{R}^3)$

$$
\langle \varphi, F_s(\cdot - i\nabla) |\cdot \rangle \varphi \rangle \geq \int \langle \varphi, Q_u \chi_u (\Delta - (s\ell)^{-2})_+ \chi_u Q_u \varphi \rangle \, du,
$$

where

$$
F_s(|p|) = \begin{cases}
(\frac{1}{2} |p|^2 - \frac{1}{2} (s\ell)^{-2}), & \text{if } |p| \geq \frac{2}{5}(s\ell)^{-1}, \\
C_1 s^2 p^2, & \text{if } |p| < \frac{2}{5}(s\ell)^{-1}.
\end{cases}
$$
Proof. By scaling we may assume $\ell = 1$. We use (4.16) and (4.17) with $K(p) = (|p|^2 - s^{-2})_+$. Since we have chosen $\chi$ to be a Schwartz function, we have $\int |p|^2 |\hat{\chi}|^2$ being finite and that $\|K\hat{\chi}\|_2 \leq CNs^N$. For the first term in (4.17) we find

$$(2\pi)^{-3} K^\times \hat{\chi}^2(p) = (2\pi)^{-3} \left\{ \int (|p - q|^2 - s^{-2}) |\hat{\chi}|^2(q) dq + \int (s^{-2} - p^2 + 2pq - q^2) \hat{\chi}^2(q) dq \right\} 
\leq p^2 - s^{-2} + \frac{6}{5} p^2 + \frac{5}{6} \frac{6}{2} (2\pi)^{-3} \int q^2 \hat{\chi}^2(q) dq
\leq p^2 - s^{-2} + [s^{-2} - \frac{6}{5} p^2],$$

where we used that $t \mapsto [t]_+$ is increasing and $[a + b]_+ \leq [a]_+ + [b]_+$. If $|p| \geq \frac{5}{6} s^{-1}$ we thus find

$$(2\pi)^{-3} K^\times \hat{\chi}^2(p) \leq p^2 - s^{-2} + \frac{6}{5} s^{-2} \leq (p^2 - \frac{1}{2} s^{-2}) - \frac{1}{3} s^{-2}. \quad (4.21)$$

For the second term in (4.17) we find since $\hat{\theta} \leq 1$ that

$$|\hat{\theta}(p)|^2 \int K|\hat{\chi}|^2 \leq \int q^2 |\hat{\chi}|^2 dq \leq C.$$

For $|p| \geq \frac{5}{6} s^{-1}$ we therefore have that the function $F$ in (4.17) satisfies

$$F(p) \leq (|p|^2 - \frac{1}{2} s^{-2}) - \frac{1}{3} s^{-2} + C.$$

With $s^*$ sufficiently small we arrive at the first line in (4.20). We turn to the proof of the second line in (4.20). We know that $F(0) = \nabla F(0) = 0$. The lemma follows from Taylor’s formula if we can show that for $|p| < \frac{5}{6} s^{-1}$, we have

$$|\partial_i \partial_j F(p)| \leq Cs. \quad (4.24)$$

(Actually, the same proof gives $|\partial_i \partial_j F(p)| \leq CNs^N$ for any power $N$, but we do not need this.) For the first term in (4.17) we therefore find for $|p| < \frac{5}{6} s^{-1}$,

$$|\partial_i \partial_j (K^\times \hat{\chi}^2)(p)| = \left\{ \int (|p - q|^2 - s^{-2})_+ \partial_i \partial_j \hat{\chi}^2(q) dq \right\} 
\leq C \int_{|q| \leq (6s)^{-1}} (s^{-2} + |q|^2) |\partial_i \partial_j \hat{\chi}^2(q)| dq
\leq Cs,$$

where we used the fast decay of $\hat{\chi}$ to conclude.

For the second and third term in (4.17) we use the fact that for all $i, j = 1, 2, 3$ the numbers

$$\|\hat{\theta}\|_\infty, \|\hat{\partial_i \hat{\theta}}\|_\infty, \|\partial_i \partial_j \hat{\theta}\|_\infty, \int |\hat{\chi}|^2, \int |\partial_i \hat{\chi}|^2, \int |\partial_i \partial_j \hat{\chi}|^2$$

are bounded by a constant. The same estimates that led to (4.22) and (4.23) then imply (4.24). \qed
Lemma 4.7. There exists a universal constant $b > 0$ such that if $s$ is small enough, then for all $\varphi \in H^1_0(\Lambda)$ and all $\ell > 0$
\[ \langle \varphi, -\Delta \varphi \rangle \geq \int_{\ell^{-1}(\Lambda+B(0,\ell/2))} \langle \varphi, Q_u \left[ \chi_u K(-i\nabla) + b\ell^{-2} \right] Q_u \varphi \rangle \, du, \]
with $K(p) = (|p|^2 - \frac{1}{4}(s\ell)^{-2})_+$. 

Proof. We again consider $\ell = 1$. By Corollary 4.5 and a Taylor expansion at $p = 0$, we have
\[ \int_{\mathbb{R}^3} Q_u \, du \leq \beta^{-1} - \Delta - \Delta + \frac{-\Delta}{-\Delta + \beta}. \] (4.25)
for a universal constant $0 < \beta < 1$. We use Lemma 4.6 with $s$ replaced by $2s$. We then find
\[ \int_{\mathbb{R}^3} Q_u \chi_u (-\Delta - \frac{1}{4}s^{-2} + b) \, du + b \int_{\mathbb{R}^3} Q_u \, du \leq F_{2s}(|-i\nabla|) + b\beta^{-1} \frac{-\Delta}{-\Delta + \beta}. \]

For $|p| < (5/12)s^{-1}$ and $s, b$ sufficiently small we get
\[ F_{2s}(p) + b\beta^{-1} \frac{p^2}{p^2 + \beta} \leq Csp^2 + b\beta^{-1} \frac{p^2}{p^2 + \beta} \leq (Cs + b\beta^{-2})p^2 \leq p^2. \]
For $|p| \geq (5/12)s^{-1}$ and $s, b$ sufficiently small we get
\[ F_{2s}(p) + b\beta^{-1} \frac{p^2}{p^2 + \beta} = (p^2 - \frac{1}{8}s^{-2}) + b\beta^{-1} \frac{p^2}{p^2 + \beta} \leq p^2 - \frac{1}{8}s^{-2} + b\beta^{-1} \leq p^2. \]

4.4 The localized Hamiltonian

Let $T_u$ be the localized kinetic energy operator, as defined in (4.15), and define for $(x_1, \ldots, x_N) \in \mathbb{R}^{3N}$,
\[ W_u(x_1, \ldots, x_N) := \sum_{i=1}^N \mu \int_{\mathbb{R}^3} w_u(x_i, y) \, dy + \sum_{i<j} w_u(x_i, x_j). \] (4.26)
We also abbreviate
\[ T := T_{u=0}, \quad W(x_1, \ldots, x_N) := W_{u=0}(x_1, \ldots, x_N). \] (4.27)
Define the operator $H_{B,u}(\mu)$ on the symmetric Fock space over $L^2(\mathbb{R}^3) \supset L^2(\Lambda)$, to preserve particle number and satisfy that
\[ (H_{B,u}(\mu))_N = \sum_{i=1}^N T_{u,i} + W_u(x_1, \ldots, x_N). \] (4.28)
As above we abbreviate
\[ H_B(\mu) := H_{B,u=0}(\mu). \]
We will also write
\[ \chi_B := \chi_{u=0} = \chi(\cdot/\ell). \]
Define the box energy and box energy density, by
\[ E_B(\mu) := \inf \text{Spec } H_B(\mu), \]
\[ e_B(\mu) := \ell^{-3} \inf \text{Spec } H_B(\mu) = \ell^{-3} E_B(\mu). \] (4.30)
\textbf{Theorem 4.8.} If $\rho_\mu$ is sufficiently small, then we have

\[ e_0(\rho_\mu) \geq e_B(\rho_\mu). \]  

\textit{Proof.} Note that $(\mathcal{H}_{B,u}(\rho_\mu))_N$ and $(\mathcal{H}_{B,u'}(\rho_\mu))_N$ are unitarily equivalent by (4.8).

From Lemma 4.2 and Lemma 4.7 we find that

\[ \mathcal{H}_{\rho_\mu,N}(\rho_\mu) \geq \int_{\ell^{-1}(\Lambda + B(0,\ell/2))} \mathcal{H}_{B,u}(\rho_\mu)_N \, du \geq \ell^{-3} |\Lambda + B(0,\ell/2)| E_B(\rho_\mu). \]  

Now the desired result follows upon using that $|\Lambda + B(0,\ell/2)|/|\Lambda| \to 1$ in the thermodynamic limit. \hfill \qed

\section{Energy in the box}

It is clear, using Theorem 4.8, that Theorem 3.1 is a consequence of the following theorem on the box Hamiltonian.

\textbf{Theorem 5.1.} Suppose $v$ satisfies Assumption 1.1. There exists $K_0 > 0$ such that if the parameter $K$ in the box length scale $\ell$ defined in (4.1) satisfies $K < K_0$, then there exists $C > 0$ (depending on $v$ and $K$) such that the box ground state energy density $e_B(\rho_\mu)$ satisfies the bound

\[ e_B(\rho_\mu) \geq -4\pi \rho_\mu^2 a_1 a - C \rho_\mu^2 a_1 (\rho_\mu a_3)^{3/2}, \]  

for all sufficiently small $\rho_\mu$.

The remainder of this paper will be dedicated to collecting the ingredients that we need for the proof of Theorem 5.1, which will be given on page 23.

\subsection{A priori estimates on particle numbers}

Recall the projections $P_u, Q_u$ defined in (4.5). Since now we are working on a fixed box $B = [-\ell/2, \ell/2]^3$, we will just denote them by $P$ and $Q$. Notice that $P + Q =: I_B$ is the orthogonal projection, in $L^2(\mathbb{R}^3)$ onto the subspace of functions supported in $B$.

Define the operators

\[ n := \sum_{i=1}^N I_{B,i}, \quad n_0 := \sum_{i=1}^N P_i, \quad n_+ := \sum_{i=1}^N Q_i = n - n_0. \]

Because the operator $n$ commutes with $\mathcal{H}_B(\rho_\mu)$, we can restrict to eigenspaces of $n$ and therefore simultaneously treat $n$ as an operator and a parameter.

Recall, that $\rho_\mu$ is the parameter introduced in (3.1). We define

\[ \rho := n\ell^{-3}, \quad \rho_+ := n_+\ell^{-3}, \quad \rho_0 := n_0\ell^{-3}. \]  

We begin with a simple bound, which is consistent with (5.1) since $a_1 a > 1$.

\textbf{Lemma 5.2.} We have

\[ E_B(\rho_\mu) < -4\pi a_1 \rho_\mu^2 \ell^3, \]  

\[ 11 \]
and for all \( \Psi \) in the Fock space \( \mathcal{F}_s(H^1(B)) \),
\[
-C_{\rho_\mu} a_1 \langle \Psi, n \Psi \rangle \leq \langle \Psi, \mathcal{H}_B(\rho_\mu) \Psi \rangle.
\]
(5.4)

In particular, if \( \Psi \) is a normalized \( n \)-particle state satisfying (see (5.1))
\[
\ell^{-3} \langle \Psi, \mathcal{H}_B(\rho_\mu) \Psi \rangle \leq -4\pi \rho_\mu^2 a_1 a,
\]
then
\[
\rho \geq C \frac{a_1}{a} \rho_\mu.
\]
(5.5)

**Proof.** Choose first \( N \) such that \(|N - \rho_\mu \ell^3| < 1\). Then an easy calculation, using (4.9) with the \( N \)-particle condensate function, that is \(|B|^{-\frac{N}{2}} \times \mathbb{1}_B \otimes \cdots \otimes \mathbb{1}_B\), as trial state yields the upper bound in (5.3). The lower bound in (5.4) follows immediately from (4.11) by dropping the positive terms in the Hamiltonian.

**Remark 5.3.** One of the complications that we have to deal with in this paper is that the optimal particle density \( \rho = n \ell^{-3} \) will not precisely match \( \rho_\mu \). Actually, we will rather find the condition (compare with (3.5))
\[
\rho a \approx \rho_\mu a_1.
\]

Therefore, expressions in powers of \( \rho - \rho_\mu \) are not individually small. However, it will be important to obtain information to the effect that \( \rho \rho_\mu^{-1} \) is bounded from above and below. We already obtained the lower bound in (5.5).

The upper bound is more complicated and will not be proved until Lemma 7.2 below. In the following sections we will work under the assumption that \( n \leq M_0 \rho_\mu \ell^3 \), where \( M_0 \) is some fixed constant depending only on \( v \). This assumption is important in order for our methods, in particular the Bogoliubov calculation in Lemma 6.2, to work.

Notice however that if \( n \) is large we can divide the \( n \) particles into groups that each have \( \leq M_0 \rho_\mu \ell^3 \) particles. Upon removing the interaction between particles in different groups we get a lower bound to the total Hamiltonian. The energy of the particles in each group can then be estimated using the methods in the following sections, in particular the Bogoliubov calculation. This is the method in the proof of Lemma 7.2.

### 5.2 Estimates on non-quadratic terms

We can, for each \( j \), write \( I_{B,j} = P_j + Q_j \). Inserting this on both sides of our operator and expanding we will get a number of terms. These we will organize depending on the number of \( Q \)'s involved. For an even finer decomposition of some of the terms we invoke the scattering solution, \( \omega \). The leading order term in (5.1) will be obtained using the Bogoliubov diagonalization carried out in Section 6. This diagonalization involves the terms quadratic in \( Q \) from the localized kinetic energy and most of the terms quadratic in \( Q \) from the localized potential energy. The aim here is to estimate the non-quadratic terms.

**Lemma 5.4.** Define
\[
\tilde{Q}_4 := \sum_{i \neq j} (Q_i Q_j + P_i P_j \omega(x_i - x_j)) \frac{1}{2} (Q_j Q_i + \omega(x_i - x_j)P_j P_i).
\]
(5.6)

Define furthermore,
\[
Q^\text{Bog}_2 := \sum_{i,j} P_i Q_j w(x_i, x_j) P_j Q_i + \frac{1}{2} \sum_{i \neq j} (P_j P_j w_2(x_i, x_j) Q_j Q_i + Q_j Q_j w_2(x_i, x_j) P_j P_i).
\]
(5.7)
Then we have the following lower bound for all $\varepsilon_1, \varepsilon_2 > 0$,

$$
\sum_{i=1}^{N} -\rho_{\mu} \int w(x_i, y) \, dy + \sum_{i<j} w(x_i, x_j) \geq A_1 + A_2 + A_3 + A_4 + Q_2^{\text{Bog}},
$$

(5.8)

with

$$
\begin{align*}
A_1 &:= 4\pi a_1 \ell^3 \left( \rho^2 - 2\rho \rho_{\mu} - 2\rho_+ (\rho - \rho_{\mu}) + \rho_+^2 - \rho_0 \ell^{-3} \right), \\
A_2 &:= -\varepsilon_2 \ell^3 \left( \rho - \rho_{\mu} \frac{a_1}{a} \right)^2 \frac{a^2}{a_1} - C a_1 \rho \ell^3 \left[ \left( \frac{\rho_+ + \ell^{-3}}{\rho} + \frac{R^2}{\ell^2} \right) (\rho + \rho_{\mu}) + \varepsilon_2^{-1} \rho_+ \right], \\
A_3 &:= -Ca_1 \ell^3 (|\rho_{\mu} - \rho| + \rho_+ + \rho_+^2), \\
A_4 &:= (\frac{1}{2} - \varepsilon_1^{-1}) \tilde{Q}_4 - \frac{1}{2} \ell^3 (\rho_0^2 - \rho_0 \ell^{-3}) \int f\omega - \varepsilon_1 Ca_1 \rho \rho_+ \ell^3.
\end{align*}
$$

Lemma 5.4 is more general than we need. In particular, choosing $\varepsilon_1 \geq 2$ allows us to drop the $\tilde{Q}_4$ term, which then is positive. We will use the following simplified version.

**Lemma 5.5.** If $\rho_{\mu}$ is sufficiently small and $\rho \leq M_0 \rho_{\mu}$ for some constant $M_0$, then we have

$$
\ell^{-3} \left\{ \sum_{i=1}^{N} -\rho_{\mu} \int w(x_i, y) \, dy + \sum_{i<j} w(x_i, x_j) \right\} \\
\geq \ell^{-3} Q_2^{\text{Bog}} + 4\pi a_1 (\rho^2 - 2\rho \rho_{\mu}) - \frac{1}{2} \rho^2 \int f\omega \\
- C_1 (\varepsilon_2, M_0) a_1 \rho \rho_+ - \varepsilon_2 \frac{a^2}{a_1} |\rho - \rho_{\mu} a/a|^2 - C_2 (K, M_0) a_1 \rho_{\mu}^2 \sqrt{\rho_{\mu} a^3},
$$

(5.9)

where $C_1$ and $C_2$ are constants depending on $v$ and on the parameters given.

**Proof.** We use that $\rho_+^2 \leq \rho \rho_+$ and $\rho_+ \ell^{-3} = K^{-1} \rho_+^2 \sqrt{\rho_{\mu} a^3}$. Inserting this information, it is easy to simplify (5.8) for sufficiently small $\rho_{\mu}$ and to obtain (5.9). \hfill \Box

**Proof of Lemma 5.4.** We organize the terms according to the number of $Q$’s involved.

**No $Q$ terms.** From a simple calculation, using (4.9), we get

$$
Q_0 := \sum_{i=1}^{N} -\rho_{\mu} P_i \int w_u(x_i, y) \, dy \, P_i + \sum_{i<j} P_i P_j w_u(x_i, x_j) P_j P_i \\
= 4\pi a_1 \ell^3 \left( \rho_0^2 - 2\rho_0 \rho_{\mu} - \rho_0 \ell^{-3} \right).
$$

(5.10)

We would rather compare with the total number of particles, $n = n_0 + n_+$, so we calculate further,

$$
Q_0 = 4\pi a_1 \ell^3 \left( \rho^2 - 2\rho \rho_{\mu} - 2\rho_+ (\rho - \rho_{\mu}) + \rho_+^2 - \rho_0 \ell^{-3} \right).
$$

(5.11)

Thus $Q_0$ contributes with the term $A_1$ in (5.8).
**Terms with 1 Q.** The 1-Q terms are (notice that the restriction \( j < k \) is dropped in the double sum)

\[ Q_1 := \sum_{i=1}^{N} -\rho_\mu P_i \int w(x_i, y) \, dy \, Q_i + \sum_{i,j} P_i P_j w(x_i, x_j) Q_j P_i + h.c. \]  

(5.12)

Again, having a \( P \) on both sides gives an integral, so we get

\[ Q_1 = \sum_{i=1}^{N} P_i \int w(x_i, y) \, dy \, Q_i (\rho_0 - \rho_\mu) + h.c. = Q'_1 + Q''_1, \]

(5.13)

with (where we have used that \( n \) commutes with \( P_i \) and \( Q_i \))

\[ Q'_1 := (\rho - \rho_\mu) \left( \sum_{i=1}^{N} P_i \int w(x_i, y) \, dy \, Q_i \right), \]  

(5.14)

\[ Q''_1 := - \left( \sum_{i=1}^{N} P_i \int w(x_i, y) \, dy \, Q_i \rho_+ + \rho_+ Q_i \int w(x_i, y) \, dy \, P_i \right). \]  

(5.15)

We will also get an effective 1-Q term from a reduction procedure applied to the 3-Q term below. This term is written in terms of the scattering solution \( \omega \) in (2.1) and will also be analyzed here

\[ \tilde{Q}_1 := - \sum_{i,j} P_i P_j w(x_i, x_j) \omega(x_i - x_j) Q_j P_i + h.c. \]  

(5.16)

\[ = \tilde{Q}'_1 + \tilde{Q}''_1, \]

with

\[ \tilde{Q}'_1 := -\rho \left( \sum_{i=1}^{N} P_i \int w(x_i, y) \omega(x_i - y) \, dy \, Q_i \right), \]  

(5.17)

\[ \tilde{Q}''_1 := \sum_{i=1}^{N} P_i \int w(x_i, y) \omega(x_i - y) \, dy \, Q_i \rho_+ + \rho_+ Q_i \int w(x_i, y) \omega(x_i - y) \, dy \, P_i. \]  

(5.18)

**Lemma 5.6** (Estimates on \( Q''_1 \) and \( \tilde{Q}''_1 \)). We can estimate

\[ Q''_1 + \tilde{Q}''_1 \geq -C(\rho + \rho_\mu) a_1(n_+ + 1). \]

**Proof.** We only estimate \( \tilde{Q}''_1 \), the estimate of \( Q''_1 \) being similar. Since \( \sum_i P_i A_i Q_i n_+ = (n_+ + 1) \sum_i P_i A_i Q_i \) (for any bounded, self-adjoint one-particle operator \( A \)), we get

\[ \sum_i P_i A_i Q_i p(n_+) = p(n_+ + 1) \sum_i P_i A_i Q_i, \]

for any polynomial \( p \), and therefore, by a limiting argument,

\[ \sum_i P_i A_i Q_i n_+ = \sum_i P_i A_i Q_i \sqrt{n_+} \sqrt{n_+} = \sqrt{n_+ + 1} \sum_i P_i A_i Q_i \sqrt{n_+}. \]

14
Energy in the box

Now it follows, by Cauchy-Schwarz and \( n_+ \leq n_+ + 1 \), that

\[
\sum_i P_i A_i Q_i n_+ + h.c. \geq -\|A\| n(n_+ + 1).
\]

Therefore, to finish the proof we use \([4.11]\) and that \( 0 \leq \omega \leq 1 \) to estimate

\[
\int |w(x, y)\omega(x - y)| dy \leq \int |w(x, y)| dy \leq CA_1.
\]

\[\square\]

**Lemma 5.7.** If \( \rho_\mu \) is sufficiently small, we have for all \( \varepsilon > 0 \)

\[
Q'_1 + \tilde{Q}'_1 \geq -CA_1 n\ell^{-3}[R^2 \ell^{-2}(n + \rho_\mu \ell^3) + \varepsilon^{-1}n_+] - \varepsilon|na\ell^{-3} - \rho_\mu a_1|^2 \ell^3 a_1^{-1}.
\]

**Proof.** We rewrite

\[
Q'_1 + \tilde{Q}'_1 = (n - \rho_\mu \ell^3)\ell^{-3} \sum_{i=1}^N P_i \chi_B(x_i) W * \chi_B(x_i) Q_i
\]

\[\text{and } -n\ell^{-3} \sum_{i=1}^N P_i \chi_B(x_i)(W\omega) * \chi_B(x_i) Q_i + h.c. \]

Notice, for a given even function \( \Phi(x) \) of compact support contained in \( B(0, R) \) that (by a Taylor expansion to second order and Hölder’s inequality)

\[
\left| \Phi * \chi_B(x) - \chi_B(x) \int \Phi \right| = \left| \int \Phi(y)[\chi((x - y)/\ell) - \chi(x/\ell)] dy \right| \leq CR^2 \ell^{-2} \int |\Phi|,
\]

with \( C = \|D^2 \chi\|_\infty \). This allows to write

\[
Q'_1 + \tilde{Q}'_1 = q_1 + q_2,
\]

with

\[
q_1 := (n - \rho_\mu \ell^3) \int W - n \int W\omega \ell^{-3} \sum_{i=1}^N P_i \chi_B(x_i)^2 Q_i + h.c.,
\]

\[
q_2 := (n - \rho_\mu \ell^3)\ell^{-3} \sum_{i=1}^N P_i \chi_B(x_i) \left[ W * \chi_B(x_i) - \chi_B(x_i) \int W \right] Q_i
\]

\[
- n\ell^{-3} \sum_{i=1}^N P_i \chi_B(x_i) \left[ (W\omega) * \chi_B(x_i) - \chi_B(x_i) \int W\omega \right] Q_i + h.c.
\]

Here \( q_2 \) can be estimated using \([5.21]\), as

\[
q_2 \geq -CR^2 \ell^{-2} a_1 n(n + \rho_\mu \ell^3)\ell^{-3},
\]

which is consistent with \([5.19]\). Using that \( \int v(1 - \omega) = 8\pi a \), and \( 1 \geq \chi * \chi(y/\ell) \geq 1 - CR^2 \ell^{-2} \), for all \( |y| \leq R \), we find

\[
(n - \rho_\mu \ell^3) \int W - n \int W\omega = 8\pi(na - \rho_\mu \ell^3 a_1) + O((n + \rho_\mu \ell^3)a_1 R^2 \ell^{-2}).
\]
In a given state \( \langle \cdot \rangle \) we proceed to estimate

\[
(8\pi (na - \rho \mu \ell^3 a_1) ) \ell^{-3} \sum_{i=1}^{N} P_i \chi_B(x_i)^2 Q_i \geq -4\pi \| \chi_B \|_\infty^2 \left| na \ell^{-3} - \rho \mu a_1 \right| (\bar{\varepsilon} n + \varepsilon^{-1} \langle n_+ \rangle).
\]

We obtain the remaining terms in (5.19) by choosing \( \bar{\varepsilon} = \frac{|n - \rho \mu \ell^3|}{4\pi \| \chi_B \|_\infty n} \).

Combining Lemma 5.6 and 5.7 we conclude Proposition 5.8. Let \( Q_1 \) and \( \tilde{Q}_1 \) be as given in (5.12) and (5.16). If \( \rho \mu \) is sufficiently small, we have for all \( \varepsilon > 0 \)

\[
Q_1 + \tilde{Q}_1 \geq -\varepsilon \left| na \ell^{-3} - \rho \mu a_1 \right| \ell^3 a_1^{-1}
- C a_1 n \ell^{-3} \left[ \left( \frac{n + 1}{n} + \frac{R^2}{\ell^2} \right) (n + \rho \mu \ell^3) + \varepsilon^{-1} n_+ \right].
\]

(5.22)

In other words, \( Q_1 \) and \( \tilde{Q}_1 \) contribute with the term \( \mathcal{A}_2 \) in (5.8).

Terms with 2 \( Q \)'s. The important 2-\( Q \) terms are

\[
Q'_2 := \sum_{i,j} P_i Q_j w(x_i, x_j) P_j Q_i + \sum_{i<j} (P_i P_j w(x_i, x_j) Q_j Q_i + Q_i Q_j w(x_i, x_j) P_j P_i),
\]

(5.23)

which, together with the effective 2-\( Q \) terms appearing in (5.29) form \( Q^\text{Bog}_2 \) defined in (5.7). We will not estimate these terms here, but analyze them using Bogoliubov’s method in Section 6.

The remaining 2-\( Q \) terms are written as

\[
Q''_2 := -\rho \mu \sum_{i=1}^{N} Q_i \int w(x_i, y) dy Q_i + \sum_{i,j} Q_i P_j w(x_i, x_j) P_j Q_i
= (n_0 - \rho \mu \ell^3) \ell^{-3} \sum_{i=1}^{N} Q_i \int w(x_i, y) dy Q_i
= (n - \rho \mu \ell^3 - n_+) \ell^{-3} \sum_{i=1}^{N} Q_i \int w(x_i, y) dy Q_i.
\]

(5.24)

At this point we invoke (4.11) to estimate,

\[
Q''_2 \geq -C (|\rho \mu \ell^3 - n|_+ + n^2_+) a_1 \ell^{-3}.
\]

(5.25)

This is the term \( \mathcal{A}_3 \) in (5.8).
Terms with 3 and 4 Q’s. For the 3-Q and 4-Q terms, we subtract and use a Cauchy inequality to make $\tilde{Q}_4$ from (5.6) appear.

$$Q_3 := \sum_{i \neq j} P_i Q_j w(x_i, x_j) Q_j Q_i + h.c.$$ 

$$= \sum_{i \neq j} P_i Q_j w(x_i, x_j) (Q_j Q_i + \omega(x_i - x_j) P_j P_i) + h.c.$$

$$- \left\{ \sum_{i \neq j} P_i Q_j w(x_i, x_j) \omega(x_i - x_j) P_j P_i + h.c. \right\}$$

$$\geq -\varepsilon \sum_{i \neq j} P_i Q_j w(x_i, x_j) Q_j P_i - \varepsilon^{-1} \tilde{Q}_4$$

$$- \left\{ \sum_{i \neq j} P_i Q_j w(x_i, x_j) \omega(x_i - x_j) P_j P_i + h.c. \right\}$$

$$= -\varepsilon \ell^{-3} \sum_{i=1}^N Q_i \int w(x_i, y) dy Q_i n_0 - \varepsilon^{-1} \tilde{Q}_4 + \tilde{Q}_1, \quad (5.26)$$

where $\tilde{Q}_1$ was introduced in (5.16) and has already been estimated.

The 2-Q term in (5.26) can be estimated similarly to $Q''_2$ as

$$-\varepsilon \ell^{-3} \sum_{i=1}^N Q_i \int w(x_i, y) dy Q_i n_0 \geq -\varepsilon C n a_1 \ell^{-3}. \quad (5.27)$$

Similarly to $Q_3$ we have (using $f = v \omega$)

$$Q_4 := \sum_{i<j} Q_i Q_j w(x_i, x_j) Q_j Q_i$$

$$= \frac{1}{2} \tilde{Q}_4 - \frac{1}{2} \sum_{i \neq j} \left\{ Q_i Q_j w(x_i, x_j) \omega(x_i - x_j) P_j P_i + h.c. \right\} - \frac{1}{2} \ell^{-3} \int f \omega \sum_{i \neq j} P_i P_j$$

$$= \frac{1}{2} \tilde{Q}_4 - \frac{1}{2} \ell^{-3} \int f \omega (n_0^2 - n_0) - \frac{1}{2} \sum_{i \neq j} \left\{ Q_i Q_j w(x_i, x_j) \omega(x_i - x_j) P_j P_i + h.c. \right\}.$$

This yields the bound on $Q_3 + Q_4$,

$$Q_3 + Q_4 \geq \left( \frac{1}{2} - \varepsilon^{-1} \right) \tilde{Q}_4 - \frac{1}{2} \ell^{-3} \int f \omega (n_0^2 - n_0) - \varepsilon C n a_1 \ell^{-3} + \tilde{Q}_1$$

$$- \frac{1}{2} \sum_{i \neq j} \left\{ Q_i Q_j w(x_i, x_j) \omega(x_i - x_j) P_j P_i + h.c. \right\}. \quad (5.28)$$

Here the 2-Q term is the remaining piece of $Q''_2$ (together with $Q'''_2$ from (5.23)) and, as mentioned before, $\tilde{Q}_1$ has already been estimated. The remaining terms in (5.29) are easily seen to correspond to the term $A_4$ in (5.8). This finishes the proof of Lemma 5.4.

6 Bogoliubov calculation

In this section, we will study the ‘effective Bogoliubov’ Hamiltonian, i.e. the remaining quadratic terms in $Q$. We will assume that the number of particles $n$ satisfies $n \leq M_0 \rho \ell^3$, 
where $M_0$ is some fixed constant only depending on $v$. In order to obtain an improved control on the density of exited particles, $\rho_+$, we separate the constant term $b_\ell^2 Q_u$ from (4.15). That is, we define $\mathcal{H}^{\text{Bog}}$ as an operator on the Fock space such that on the $N$-particle sector we have

$$ (\mathcal{H}^{\text{Bog}})_N = \sum_{j=1}^N T_j - b_\ell^2 n_+ + Q^{\text{Bog}}_2 $$

with $Q^{\text{Bog}}_2$ from (5.7) and $T$ from (4.15). We will pass to a second quantized formalism in order to give an effective lower bound to this operator. Some additional background on second quantization is presented in Appendix B. We define $a_0$ as the annihilation operator associated to the condensate function for the box $B$, i.e. for $\Psi \in \mathcal{F}_N^3$ we have

$$ (a_0 \Psi)(x_2, \ldots, x_N) := \sqrt{\frac{N}{\ell^3}} \int \theta(y) \Psi(y, x_2, \ldots, x_N) \, dy $$

Therefore,

$$ \langle \Psi, n_0 \Psi \rangle = \langle \Psi | a_0^* a_0 \Psi \rangle = \frac{N}{\ell^3} \int \int \theta(y) \Psi(y, x_2, \ldots, x_N) \, dy \, dx_2 \cdots dx_N. $$

We define, for $k \in \mathbb{R}^3$,

$$ b_k := a_0^* a(Q(e^{ikx} \chi_B)) \quad \text{and} \quad b_k^* := a(Q(e^{ikx} \chi_B))^* a_0. $$

Then,

$$ [b_k, b_{k'}] = 0, \quad \forall k, k' \in \mathbb{R}^3, $$

and

$$ [b_k, b_{k'}^*] = a_0^* a_0 \langle Q(e^{ikx} \chi_B), Q(e^{ik'x} \chi_B) \rangle - a(Q(e^{ikx} \chi_B))^* a(Q(e^{ik'x} \chi_B)). $$

In particular,

$$ [b_k, b_{k'}^*] \leq a_0^* a_0 \int \chi_B^2 = \ell^3 a_0^* a_0. $$

Furthermore, we introduce the Fourier multiplier corresponding to the localized kinetic energy (after the separation of the constant term), i.e.

$$ \tau(k) := \left( |k|^2 - C_{\text{kin}} \ell^{-2} \right)_+. $$

**Lemma 6.1** (Lower bound by second quantized operator).

For $\rho_\mu$ sufficiently small and with the notation above, in particular (6.1), we have

$$ \mathcal{H}^{\text{Bog}} \geq \mathcal{H}^{\text{Bog}}_1 - C \frac{a_1}{\ell^3} n_+, $$

with

$$ \mathcal{H}^{\text{Bog}}_1 := \frac{1}{2} (2\pi)^{-3} \int_{\mathbb{R}^3} A(k) (b_\ell^* b_k + b_\ell^* b_{-k}) + B(k) (b_\ell^* b_{-k} + b_k b_{-k}) \, dk. $$

Here

$$ A(k) = \frac{\tau(k)}{n} + \frac{\hat{W}(k)}{\ell^3} \quad \text{and} \quad B(k) = \frac{\hat{W}_2(k)}{\ell^3}. $$
Proof. We can write, with $\theta_0 = 1_{B(0)}$,

\[
\langle \varphi | Pe^{ikx} \chi(x/\ell)Q \psi \rangle = \ell^{-3} \langle \varphi | \theta_0 \rangle \langle \theta_0 | e^{ikx} \chi(x/\ell)Q \psi \rangle = \ell^{-3} \langle \varphi | \theta_0 \rangle \langle Q(e^{-ikx} \chi(x/\ell)) | \psi \rangle = \ell^{-3/2} \langle \varphi | b_{-k} \psi \rangle.
\]

(6.11)

Therefore, we see that the second quantization of $Pe^{ikx} \chi(x/\ell)Q$ is $\ell^{-3/2}b_{-k}$.

Similar calculations (see (B.2) in Appendix B for details) yield,

\[
\sum_{j \neq s} P_j Q_s w(x_j, x_s) P_s Q_j = (2\pi)^{-3} \ell^{-3} \int \tilde{W}(k)b_{-k}^* b_{-k} dk + \mathcal{E}_1,
\]

(6.12)

with

\[
\mathcal{E}_1 := -(2\pi)^{-3} \ell^{-3} \int \tilde{W}(k)a^*(Q(e^{ikx} \chi_B))a(Q(e^{ikx} \chi_B)) dk,
\]

(6.13)

and

\[
\sum_{j \neq s} P_j P_s w(x_j, x_s) Q_s Q_j = (2\pi)^{-3} \ell^{-3} \int \tilde{W}(k)b_k b_{-k} dk = \sum_{j \neq s} (2\pi)^{-3} \int \tilde{W}(k)(P_j \chi(x_j/\ell)e^{ikx} Q_j)(P_s \chi(x_s/\ell)e^{-ikx} Q_s) dk.
\]

(6.14)

Therefore, $Q_2^{Bog}$ second quantizes as

\[
Q_2^{Bog} = \frac{1}{2}(2\pi)^{-3} \ell^{-3} \int \left\{ \tilde{W}(k) \left[ b_k^* b_k + b_{-k}^* b_{-k} \right] + \tilde{W}_2(k) \left[ b_k b_{-k} + b_{-k}^* b_k^* \right] \right\} dk + \mathcal{E}_1.
\]

(6.15)

The term $Q \chi(-\Delta) \chi Q$ second quantizes as

\[
Q \chi(-\Delta) \chi Q = (2\pi)^{-3} \int_{\mathbb{R}^3} \tau(k)a(Q(\chi e^{ikx}))^* a(Q(\chi e^{ikx})) dk.
\]

Therefore we estimate $\sum_{j=1}^{N} Q_j \chi(-\Delta) \chi Q_j$ in terms of its second quantization as

\[
\sum_{j=1}^{N} Q_j \chi(-\Delta) \chi Q_j \geq (2\pi)^{-3} \int_{\mathbb{R}^3} \tau(k)a(Q(\chi e^{ikx}))^* \frac{a_0 a_0^*}{n} a(Q(\chi e^{ikx})) dk
\]

\[
= (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{\tau(k)}{n} b_k^* b_k dk
\]

\[
= \frac{1}{2}(2\pi)^{-3} \int_{\mathbb{R}^3} \frac{\tau(k)}{n} \left[ b_k^* b_k + b_{-k}^* b_{-k} \right] dk.
\]

(6.16)

Here we used that $b_k \psi = 0$ if $\psi$ is in the condensate allowing us to assume that $n_+ \geq 1$ such that in fact $a_0 a_0^* \leq n$.

We analyze the error term $\mathcal{E}_1$ a little further. As in the appendix, we can rewrite $\mathcal{E}_1$ (on the $N$-particle sector) as

\[
(\mathcal{E}_1)_N = -\sum_{j=1}^{N} Q_j Z Q_j,
\]

(6.17)
Bogoliubov calculation

where $Z$ is a 1-particle operator with integral kernel

$$Z(x, y) = \ell^{-3} \chi_B(x) W(x - y) \chi_B(y).$$

We use (4.11) to bound $Z$, as an operator on $L^2(\mathbb{R}^3)$, by

$$\|Z\| \leq C \ell^{-3} a_1.$$

So, we see that

$$E_1 \geq -C a_1 \ell^{-3} n_+ + (6.18)$$

This finishes the proof of Lemma 6.1.

**Lemma 6.2 (The Bogoliubov integral).**

Assume that the number of particles $n$ satisfies the bound (5.5) as well as $n \leq M_0 \rho \mu \ell^3$, where $M_0$ is some fixed constant only depending on $v$. Then, for $\rho \mu$ sufficiently small, we have

$$H_{1}^{\text{Bog}} \geq -\frac{1}{2} (2\pi)^{-3} \ell^3 \rho \rho_0 \int \frac{\hat{W}_2(k)^2}{2k^2} \text{dk} - C \ell^3 \rho_0 \rho a (\rho a^3)^{1/2}. \quad (6.19)$$

**Proof.** Recall that in (6.9) we defined

$$H_{1}^{\text{Bog}} := \frac{1}{2} (2\pi)^{-3} \int_{\mathbb{R}^3} \mathcal{A}(k) (b_k^* b_k + b_{-k}^* b_{-k}) + \mathcal{B}(k) (b_k^* b_{-k}^* + b_{k} b_{-k}) \text{dk}, \quad (6.20)$$

with

$$\mathcal{A}(k) = \frac{\tau(k)}{n} + \frac{\hat{W}(k)}{\ell^3} \quad \text{and} \quad \mathcal{B}(k) = \frac{\hat{W}_2(k)}{\ell^3}. \quad (6.21)$$

By assumption $\rho \leq M_0 \rho \mu$. We seek to apply the Bogoliubov method to estimate the quadratic Hamiltonian, see Theorem A.1. In order to do so, we need to verify that the condition $-A < B \leq A$ with the notation from (6.10) is satisfied for all $\rho \mu$ sufficiently small.

For $|k| \geq C \sqrt{\rho \mu a}$, with $C$ sufficiently large (depending on $K$ and on the upper bound on $\rho/\rho \mu$), we have for small $\rho \mu$,

$$\tau(k) \geq \frac{1}{2} |k|^2 > \rho (|\hat{W}_2(k)| + |\hat{W}(k)|).$$

On the other hand, for a potential satisfying Assumption 1.1 we have the strict inequality $a < a_1$. Therefore, using (2.2) and (4.10),

$$0 < \hat{W}_2(0) = 8\pi a < 8\pi a_1 = \hat{W}(0),$$

and by continuity this inequality extends to

$$0 < \hat{W}_2(k) < \hat{W}(k),$$

for $|k| \leq C \sqrt{\rho \mu a}$ for all $\rho \mu$ sufficiently small.

Notice for later use, that we have actually proved the bounds

$$\frac{|B|}{A} \leq \eta < 1 \quad \text{and} \quad A > 0 \quad (6.22)$$

for all $\rho \mu$ sufficiently small and some $\eta$ independent of $\rho \mu$. 

20
By the above we may apply Theorem A.1 to bound \( \mathcal{H}_1^{\text{Bog}} \). We obtain
\[
\mathcal{H}_1^{\text{Bog}} \geq \frac{1}{4} (2\pi)^{-3} \int_{\mathbb{R}^3} \left( \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} - \mathcal{A}(k) \right) \left( |b_k| + |b_{-k}^{\ast}| \right) dk. \tag{6.23}
\]
We insert the bound \( |b_k, b_k^\ast| \leq \ell^3 a_0^\ast a_0 = \ell^3 n_0 \) from (6.6) and get
\[
\mathcal{H}_1^{\text{Bog}} \geq \frac{1}{2} (2\pi)^{-3} \ell^3 n_0 \int_{\mathbb{R}^3} \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} - \mathcal{A}(k) dk. \tag{6.24}
\]
Notice that, using (6.22), there exists \( C > 0 \), depending only on \( \eta \) such that
\[
\left| \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} - \mathcal{A}(k) + \frac{\mathcal{B}(k)^2}{2\mathcal{A}(k)} \right| \leq C \frac{\mathcal{B}(k)^4}{\mathcal{A}(k)^3}. \tag{6.25}
\]
Therefore, (6.19) follows if we can prove that
\[
\ell^3 \rho^{-1} \int \frac{\mathcal{B}(k)^2}{2\mathcal{A}(k)} - \frac{\bar{W}_2(k)^2}{2k^2} \, dk \leq Ca\sqrt{\rho a}, \tag{6.26}
\]
\[
\ell^3 \rho^{-1} \int \frac{\mathcal{B}(k)^4}{2\mathcal{A}^3(k)} \, dk \leq Ca\sqrt{\rho a}, \tag{6.27}
\]
for some constant \( C \) independent of \( \rho, \rho_\mu \).

We first prove the bound in (6.26). Notice that
\[
\int \ell^3 \rho^{-1} \frac{\mathcal{B}(k)^2}{2\mathcal{A}(k)} - \frac{\bar{W}_2(k)^2}{2k^2} \, dk = \int \frac{\bar{W}_2(k)^2}{2(\tau(k) + \rho W(k))} - \frac{\bar{W}_2(k)^2}{2k^2} \, dk. \tag{6.28}
\]
We split this integral in two parts. Choose a constant \( T > 0 \) such that \( T^2 > 3 C_{\text{kin}} K^{-2} \rho a + 3a^{-1} \tilde{W}(0) \) for all \( \rho, \rho_\mu \). This uses the lower bound (5.5) on the number of particles.

For \( |k| \geq T \sqrt{\rho a} \), using the choice of \( T \), we have
\[
\tau(k) = k^2 - C_{\text{kin}} \ell^2, \quad \tau(k) + \rho \tilde{W}(k) \geq \frac{1}{3} k^2. \tag{6.29}
\]
Therefore,
\[
\int_{\{|k| \geq T \sqrt{\rho a}|} \left| \frac{\bar{W}_2(k)^2}{2(\tau(k) + \rho W(k))} - \frac{\bar{W}_2(k)^2}{2k^2} \right| dk \leq \frac{3\tilde{W}_2(0)^2}{2} \int_{\{|k| \geq T \sqrt{\rho a}|} \frac{k^2 - \tau(k) - \rho \tilde{W}(k)}{k^4} \, dk \leq \frac{3\tilde{W}_2(0)^2}{2} \int_{\{|k| \geq T \sqrt{\rho a}|} \frac{C_{\text{kin}} \ell^2 + \rho \tilde{W}(0)}{k^4} \, dk \leq Ca\sqrt{\rho a}, \tag{6.30}
\]
where the constant in particular depends on \( T, K \) and the upper bound on \( \rho_\mu/\rho \).

For \( |k| \leq T \sqrt{\rho a} \) (and \( \rho_\mu \) sufficiently small) we drop \( \tau(k) \) and estimate \( |\bar{W}_2(k)| \leq \tilde{W}_2(0) = 8\pi a \) and \( \tilde{W}(k) \geq \frac{1}{2} \tilde{W}(0) = 4\pi a_1 \). Therefore,
\[
\int_{\{|k| \leq T \sqrt{\rho a}|} \left| \frac{\bar{W}_2(k)^2}{2(\tau(k) + \rho W(k))} - \frac{\bar{W}_2(k)^2}{2k^2} \right| dk \leq Ca\sqrt{\rho a}, \tag{6.31}
\]

with $C$ depending on $T$ and on $a/a_1$. This establishes the bound in (6.26).

The bound in (6.27) is analogous, and we will be a bit less detailed. Notice that

$$\ell^3 \rho^{-1} \frac{B(k)^4}{2A^4(k)} = \rho^2 \frac{\hat{W}_2^4(k)}{(\tau + \rho W(k))^3}. $$

Upon making the same splitting as for the first integral, we see that for $|k| \leq T \sqrt{\rho a}$ the integrand can be bounded by $C \rho^{-1}$ leading to a bound of the right magnitude. For $|k| \geq T \sqrt{\rho a}$ we again use (6.29) and find that the integrand is bounded by $C \rho^2 a^4 k^{-6}$. Upon explicitly integrating this function we again find a bound of the right magnitude.

**Lemma 6.3.** We have

$$(2\pi)^{-3} \int \frac{\hat{W}_2^2(k)}{2k^2} \, dk = \int g(x)\omega(x) + O(a(R/\ell)^2). \quad (6.32)$$

**Proof.** We have $\hat{\omega}(k) = \hat{g}(k) \hat{\omega}(k)$ from (2.8). We calculate the difference between $(2\pi)^{-3} \int \hat{g}(k)\hat{\omega}(k) \, dk$ and $(2\pi)^{-3} \int \frac{\hat{W}_2^2(k)}{2k^2} \, dk$ using the Fourier transformation and the HLS-inequality,

$$\left| (2\pi)^{-3} \int \frac{\hat{W}_2^2(k) - \hat{g}^2(k)}{k^2} \, dk \right| = \left| \int \frac{W_2 - g)(x)(W_2 + g)(y)}{|x - y|} \, dx \, dy \right| \leq C\|W_2 - g\|_{6/5}\|W_2 + g\|_{6/5}. \quad (6.33)$$

Clearly, by (4.10),

$$W_2(0) = g(0) \quad \text{and} \quad |W_2(x) - g(x)| \leq |g(x)|C(R/\ell)^2,$$

so we find

$$(2\pi)^{-3} \int \frac{\hat{W}_2^2(k)}{2k^2} \, dk = (2\pi)^{-3} \int \frac{\hat{g}^2(k)}{2k^2} \, dk + O(a(R/\ell)^2). \quad (6.34)$$

From this the lemma follows from the definition of the scattering solution $\omega$ (see (2.6)) and the Parseval identity. \hfill \Box

## 7 Estimating the energy

**Lemma 7.1.** Suppose $\Psi$ is normalized and an eigenstate for $n$ with $\rho/\rho_\mu \in [\frac{M_0}{T}, M_0]$, where $M_0 = 12 \frac{a_1}{a}$. Suppose furthermore that $K$ is sufficiently small. Then, for all $\rho_\mu$ sufficiently small,

$$\ell^3 \langle \Psi, H_B(\rho_\mu) \Psi \rangle \geq 4\pi a \rho_\mu^2. \quad (7.1)$$

**Proof.** By assumption of the lemma $\rho \leq M_0 \rho_\mu$ as required in Lemma 5.5. Therefore we can combine the estimates (5.9) and (6.1) to find

$$\ell^3 \langle \Psi, H_B(\rho_\mu) \Psi \rangle \geq \ell^3 \langle \Psi, H_B^{\text{Spin}} \Psi \rangle + b\ell^{-2} \rho_\mu + 4\pi a_1 (\rho^2 - 2\rho \rho_\mu) - \frac{1}{2} \rho_\mu^{2} \int f \omega$$

$$- C_1(\varepsilon, M_0) a_1 \rho_\mu \rho_\mu - \varepsilon_2 a_1^2 |\rho - \rho_\mu a_1/a|^2 - C_2(K, M_0) a_1 \rho_\mu^2 \sqrt{\rho_\mu a^3}. \quad (7.2)$$
At this point we use (6.8). We also insert (6.19) and use (6.32). For convenience we drop the indices and parameters in the constants. This yields, after combining the $\omega$-integrals,

$$\ell^{-3}\langle \Psi, \mathcal{H}_B(\rho_\mu)\Psi \rangle$$

$$\geq 4\pi a_1 (\rho^2 - 2\rho \rho_\mu) - 4\pi (a_1 - a) \rho^2 + \frac{b}{2K^2} \rho \rho_\mu \rho +$$

$$- Ca_1 \rho \rho_\mu - \varepsilon_2 \frac{a^2}{a_1} |\rho - \rho_\mu a| - Ca_1 \rho_\mu \sqrt{\rho a^3}$$

$$= -4\pi a_1 \frac{a^2}{a_1} \rho_\mu - Ca_1 \rho_\mu \sqrt{\rho \mu a^3}$$

$$+ (4\pi a - \frac{a^2}{a_1} \varepsilon_2) (\rho - \frac{a_1}{a} \rho_\mu)^2 + \left\{ \frac{b}{K^2} \rho \rho_\mu - Ca_1 \rho_\mu \right\} \rho_+.$$  

(7.3)

With the choice $\varepsilon_2 = \frac{2\pi a_1}{a}$ and the lower bound on $\rho/\rho_\mu$ the quadratic terms become

$$-4\pi a_1 \frac{a^2}{a_1} \rho_\mu + (4\pi a - \frac{a^2}{a_1} \varepsilon_2) (\rho - \frac{a_1}{a} \rho_\mu)^2 \geq \left(-4\pi a_1 \frac{a^2}{a_1} + 2\pi a (3 \frac{a_1}{a} - \frac{a^2}{a^1})\right) \rho_\mu^2 = 4\pi a_1 \frac{a^2}{a_1} \rho_\mu^2,$$  

(7.4)

by our choice of $M_0$.

Furthermore, for $K$ sufficiently small, we see that the coefficient of $\rho_+$ in (7.3) is also positive (notice that this uses the upper bound on $\rho/\rho_\mu$). Therefore, we get with the above choices,

$$\ell^{-3}\langle \Psi, \mathcal{H}_B(\rho_\mu)\Psi \rangle \geq 4\pi a_1 \frac{a^2}{a_1} \rho_\mu^2 - Ca_1 \rho_\mu^2 \sqrt{\rho_\mu a^3} \geq 4\pi a \rho_\mu^2,$$  

(7.5)

for all $\rho_\mu$ sufficiently small.

Using Lemma 7.1 we can now finally give a good estimate on the number $n$ of particles in the box.

**Lemma 7.2** (Upper bound on $n$). Suppose $K$ is chosen sufficiently small. Suppose that $\Psi$ is a normalized eigenvector for $n$ and

$$\langle \Psi, \mathcal{H}_B(\rho_\mu)\Psi \rangle < 0.$$

Then, for $\rho_\mu$ sufficiently small, we have $\rho/\rho_\mu \leq \frac{3a_1}{a}$, with the notation from Lemma 7.1.

**Proof.** Using Lemma 7.1 we may assume that $n \geq M_0 \rho_\mu \ell^3$. We can now split the particles into a number $m$ of groups, each having particle number in the interval $[\frac{M_0}{4} \rho_\mu \ell^3, M_0 \rho_\mu \ell^3]$. Omitting the positive interaction between particles in different groups gives the lower bound

$$\langle \Psi, \mathcal{H}_B(\rho_\mu)\Psi \rangle \geq m \mathcal{G},$$  

(7.6)

where

$$\mathcal{G} = \inf \left\{ \langle \Psi', \mathcal{H}_B(\rho_\mu)\Psi' \rangle \mid \Psi' \text{ has } n' \text{ particles in the box } B, \text{ with } n' \in \left[ \frac{M_0}{4} \rho_\mu \ell^3, M_0 \rho_\mu \ell^3 \right] \right\}.$$

But $\mathcal{G} \geq 0$ by Lemma 7.1. This finishes the proof.  

**Proof of Theorem 5.1.** The beginning of the proof is a repetition of the proof of Lemma 7.1 above, since we may assume, using Lemma 7.2 and Lemma 5.2 that $C \leq \rho/\rho_\mu \leq \frac{M_0}{4}$. Therefore, we find (7.2) and (7.3) exactly as above. For $\varepsilon_2$ sufficiently small, the coefficient of $(\rho - \frac{a_1}{a} \rho_\mu)^2$ is positive. Similarly, given $\varepsilon_2 > 0$, for $K$ sufficiently small the coefficient of $\rho_+$ is positive. This finishes the proof of Theorem 5.1.
Remark 7.3. Notice that the final arguments of the proof imply the bounds
\[
(\rho - \frac{a_1}{a} \rho_\mu)^2 \leq C \rho_\mu^2 \sqrt{\rho_\mu a^3} \quad \text{and} \quad \rho_+ \leq C \rho_\mu \sqrt{\rho_\mu a^3}
\]
(7.7)
for any normalized \(n\)-eigenvector, \(\Psi\), satisfying \(\langle \Psi, \mathcal{H}_B(\rho_\mu) \Psi \rangle \leq -4\pi \frac{a_1^2}{a} \rho_\mu^2 - C a_1 \rho_\mu^2 \sqrt{\rho_\mu a^3} \).

A Bogoliubov method

In this section we recall a simple consequence of the Bogoliubov method. In [4] we use the following version (and allow \(B = -A\) if \(\kappa = 0\))—see also [11, Theorem 6.3].

**Theorem A.1** (Simple case of Bogoliubov’s method).

For arbitrary \(A, B \in \mathbb{R}\) satisfying \(A > 0\), \(-A < B \leq A\) and \(\kappa \in \mathbb{C}\) we have the operator inequality
\[
A(b_+^* b_+ + b_-^* b_-) + B(b_+^* b_- + b_-^* b_+) + \kappa (b_+ + b_-) + \pi (b_+^* + b_-^*) \geq -\frac{1}{2} (A - \sqrt{A^2 - B^2}) ([b_+, b_+] + [b_-, b_-]) - \frac{2|\kappa|^2}{A + B},
\]
where \(b_\pm\) are operators on a Hilbert space satisfying \([b_+, b_-] = 0\).

B Second quantization

In this appendix we include some details on the second quantization in terms of the operators \(b_k = a_0^* a(Q(e^{ikx} \chi_B))\) and \(b_k^* = a^*(Q(e^{ikx} \chi_B)) a_0\), which we introduced in [6,3]. Again, we assume that \(\Psi \in \otimes \mathbb{N} L^2\) and calculate
\[
(2\pi)^{-3} \ell^{-3} \int \hat{W}(k) \langle b_{-k}^* \Psi, b_{-k} \Psi \rangle \, dk
\]
\[
= (2\pi)^{-3} \ell^{-3} \int \hat{W}(k) \left\{ \sum_{j=1}^N \left[ Q_j(e^{-ikx} \chi_B(x_j)) \right] \ell^{-3/2} \int \theta(y) \Psi(\hat{x}_j, y) \, dy \right\} \times \left\{ \sum_{s=1}^N \left[ Q_s(e^{-iks} \chi_B(x_s)) \right] \ell^{-3/2} \int \theta(y) \Psi(\hat{x}_s, y) \, dy \right\} \, dk
\]
\[
= \sum_{j \neq s} \langle P_j Q_s \Psi \mid w(x_j, x_s) \mid P_s Q_j \Psi \rangle
\]
\[
+ (2\pi)^{-3} \ell^{-6} N \int \hat{W}(k) \|Q(e^{-ikx} \chi_B)\|^2 \, dk \times \left( \int \{ \int \theta(y) \Psi(y, x_2, \ldots, x_N) \, dy \}^2 \, dx_2 \cdots dx_N.\right)
\]
Here we used the symmetry of \(\Psi\) repeatedly, that \(Q_j^* = Q_j\) and that factors of \(\theta(x_j), \theta(x_s)\) can be inserted freely into the integrand.

Upon recalling (6.2) we get the formula,
\[
(2\pi)^{-3} \ell^{-3} \int \hat{W}(k) b_{-k} b_{-k}^* \, dk = \sum_{j \neq s} P_j Q_s w(x_j, x_s) P_s Q_j
\]
\[
+ (2\pi)^{-3} \ell^{-3} \int \hat{W}(k) \|Q(e^{-ikx} \chi_B)\|^2 a_0^* a_0 \, dk. \quad (B.1)
\]
From the commutation relation (6.5) we therefore get
\[
(2\pi)^{-3}\ell^{-3} \int \hat{W}(k)b^*_kB_{-k} dk = \sum_{j \neq s} P_jQ_s w(x_j, x_s)P_sQ_j \\
+ (2\pi)^{-3}\ell^{-3} \int \hat{W}(k)a^*(Q(e^{ikx}\chi_B))a(Q(e^{ikx}\chi_B)) dk. \quad (B.2)
\]

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