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ABSTRACT. A seminal theorem of Tverberg states that any \( T(r, d) = (r - 1)(d + 1) + 1 \) points in \( \mathbb{R}^d \) can be partitioned into \( r \) subsets whose convex hulls have non-empty \( r \)-fold intersection. Almost any set of fewer points in \( \mathbb{R}^d \) cannot be so divided, and in these cases we ask whether the set can nonetheless be \( P(r, d) \)-partitioned, i.e., divided into \( r \) subsets so that there exist \( r \) points, one from each resulting convex hull, which form the vertex set of a prescribed convex \( d \)-polytope \( P(r, d) \). Our main result shows that this is the case for any generic \( T(r, 2) = 2 \)-2 points in the plane and any \( r \geq 3 \) when \( P(r, 2) \) is a regular \( r \)-gon. For \( r = r_1 \cdots r_k, r_i \geq 3 \), this generalizes to generic sets of \( T(r, 2k) = 2k \) points and orthogonal products of regular polygons \( P(r, 2k) = P_{r_1} \times \cdots \times P_{r_k} \) in \( \mathbb{R}^{2k} \), and likewise to \( T(2r, 2k+1) = 2k+1 \) points and the product polytopes \( P(2r, 2k+1) = P_{r_1} \times \cdots \times P_{r_k} \times P_2 \) in \( \mathbb{R}^{2k+1} \). As with Tverberg’s original theorem, these have topological extensions when \( r \) is a prime power, and, using the “constraint method” of Blagojević, Frick, and Ziegler, can be made to satisfy additional conditions such as those of a van Kampen–Flores type.

1. Introduction and Statement of Main Results

Tverberg’s landmark 1966 theorem [24] states that any \( (r - 1)(d + 1) + 1 \) points in \( \mathbb{R}^d \) can be divided into \( r \) pairwise disjoint subsets whose convex hulls have non-empty \( r \)-fold intersection (called a Tverberg \( r \)-partition). The \( r = 2 \) case recovers Radon’s Theorem, and the result also has deep connections with the classical theorems of Helly and Carathéodory. We refer the reader to the recent surveys [5, 9, 10] for a sampling of the many interesting applications and extensions of Tverberg’s Theorem in discrete geometry, combinatorics, topology, and beyond.

For codimension reasons, almost any collection of \( N + 1 \) points fails to admit a Tverberg \( r \)-partition when \( N < T(r, d) := (r - 1)(d + 1) \), and in these cases one considers weaker conditions on the convex hulls arising from partitions by \( r \) subsets. The most studied problem in this direction (see, e.g., [1, 17]) was initiated by Reay [18], and asks for each \( 2 \leq j < r \) whether there exists some \( N < T(r, d) \) such that any generic set of \( N + 1 \) points in \( \mathbb{R}^d \) can be partitioned by \( r \) subsets so that while any \( j \) of the resulting convex hulls have common intersection, all \( r \) of them do not. In fact, Reay’s “relaxed” conjecture claims that no such \( N \) exists, even when \( j = 2 \).

Instead of partial intersection conditions on convex hulls, we propose a polytopal variant with a slight Ramsey–like flavor:

**Question 1.** Let \( d \geq n \). What is the minimum \( N := N_{(P(r,n),d)} \) such that almost any \( N + 1 \) points in \( \mathbb{R}^d \) be partitioned into \( r \) sets \( A_1, \ldots, A_r \) so that there exist \( r \) points \( x_1 \in \text{Conv}(A_1), \ldots, x_r \in \text{Conv}(A_r) \), one from each resulting convex hull, which are the vertices of a scaled isometric copy of a prescribed \( n \)-dimensional convex polytope \( P(r,n) \)?
Any partition as in Question 1 will be called a $P(r,n)$-partition, or simply a polytopal partition if the context is clear. We denote $N_{(P(r)n);d}$ by $N_{P(r,d)}$ in the special case that $n = d$.

Our central result determines $N_{P_r}$ for all regular polygons $P_r$ in terms of the Tverberg number:

**Theorem 1.1.** $N_{P_r} = T(r, 2) - 2 = 3r - 5$ for all regular $r$-gons $P_r$.

The figure below gives the two possible cases of Theorem 1.1 when $r = 3$, in which either one or two of the vertices of the equilateral triangle come from a given set of 5 (blue) points:

![Diagram](image)

For higher dimensional polytopes, we give upper bounds for $N_{P(r,d)}$ in the case of “multiprisms,” i.e., the Cartesian products

$$P(r,2k) := P_{r_1} \times \cdots \times P_{r_k},$$

(1.1)

of orthogonal regular $r_i$-gons in $\mathbb{R}^{2k}$, $r_i \geq 3$, and the orthogonal products

$$P(2r,2k+1) := P_{r_1} \times \cdots \times P_{r_k} \times P_2,$$

(1.2)

in $\mathbb{R}^{2k+1}$. Here $P_2$ denote a line segment, so that for example $P(2r,3)$ is a right regular prism in $\mathbb{R}^3$.

**Theorem 1.2.** Let $k \geq 1$, let $r = r_1 \cdots r_k$, $r_i \geq 3$, and let $P(r,2k) = P_{r_1} \times \cdots \times P_{r_k}$ and $P(2r,2k+1) = P_{r_1} \times \cdots \times P_{r_k} \times P_2$. Then

$$N_{P(r,2k)} \leq T(r, 2k) - 2k \quad \text{and} \quad N_{P(2r,2k+1)} \leq T(2r, 2k + 1) - (2k + 1).$$

(1.3)

Moreover,

(a) Let $S$ be a set of $T(r,2k) - 2k + 1$ generic points in $\mathbb{R}^{2k}$. For any decomposition of $\mathbb{R}^{2k}$ by $k$ pairwise orthogonal coordinate planes $U_1, \ldots, U_k$, there exists a $P(r,2k)$-partition of $S$ such that $P_{r_i}$ is parallel to $U_i$ for all $1 \leq i \leq k$. Likewise,

(b) Let $S$ be a set of $T(2r,2k+1) - 2k$ generic points in $\mathbb{R}^{2k+1}$. For any $k$ pairwise orthogonal coordinate planes $U_1, \ldots, U_k$ in $\mathbb{R}^{2k+1}$, there exists a $P(2r,2k+1)$-partition of $S$ such that $P_{r_i}$ is parallel to $U_i$ for all $1 \leq i \leq k$.

For example, Theorem 1.2(b) guarantees that for any generic set of $8r - 6$ points in $\mathbb{R}^3$, there exist $2r$ points, one form each convex hull determined by a partition by $2r$ subsets, which are the vertices of a right regular prism whose $r$-gon base can be prescribed parallel to any of the three coordinate planes.
Although parts (a) and (b) of Theorem 1.2 are tight under the given coordinate conditions, this should not be the case for the upper bound (1.3) itself. To see this, consider the $2(d-2)$-dimensional Grassmanian manifold $G_2(\mathbb{R}^d)$ of all linear 2-flats in $\mathbb{R}^d$. Allowing non-coordinate multiprisms, there are $\dim \Pi_{i=0}^{k-1} G_2(\mathbb{R}^{2k-2i}) = 2k(k-1)$ remaining degrees of freedom for the existence of any $P(r,2k)$-partition, and subtracting this from $N = T(r,2k) - 2k$ yields an expected value of $N_{P(r,2k)} = T(r,2k) - 2k^2$. Similar remarks lead one to expect $N_{P(2r,2k+1)} = T(2r,2k+1) - 2k(k+1) - 1$. Nonetheless, we do expect our upper bound will hold for all vertex transitive polytopes:

**Conjecture 1.** If $P(r,d)$ is a vertex transitive $d$-dimensional polytope with $r$ vertices, then

$$N_{P(r,d)} \leq T(r,d) - d. \quad (1.4)$$

As with Tverberg’s theorem, topological extensions of Question 1 arise by viewing $N+1$ points in $\mathbb{R}^d$ as the image of the vertices of the $N$-simplex $\Delta_N$ under a map $f : \Delta_N \to \mathbb{R}^d$. Considering affine linear maps, a Tverberg $r$-partition is equivalent to the existence of $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r \subseteq \Delta_N$ so that $\cap_{i=1}^r f(\sigma_i) \neq \emptyset$. Likewise, a $P(r,n)$-partition is equivalent to finding $x_1, \ldots, x_r$ from pairwise disjoint $\sigma_i$ so that the $f(x_i)$ are the vertex set of some $P(r,n)$. It is the content of the Topological Tverberg theorem of [16] (see also [20, 25]) that affine maps may be replaced by arbitrary continuous ones for all prime powers $r$ when $N = T(r,d)$. On the other hand, counterexamples to such an extension for other $r$ with $d$ sufficiently large were recently produced [6, 12] based on fundamental work of [15].

We refer the reader to the reviews of [5, 9, 10] amongst others for the history of this very famous problem initially raised in [4].

In a similar fashion, we have topological $P(r,2k)$-partitions for all odd prime powers $r = p^k$, as well as for squares $P(4,2)$. For even prime powers, one has similar partitions for $k$-orthotopes $P(2^k,k) = P_2^{x_k}$ in $\mathbb{R}^k$.

**Theorem 1.3.**

(a) Let $N = T(p^k,2k) - 2k$, $p$ an odd prime, and let $f : \Delta_N \to \mathbb{R}^{2k}$ be a generic continuous map. For any decomposition of $\mathbb{R}^{2k}$ by $k$ pairwise orthogonal coordinate planes $U_1, \ldots, U_k$, there exist points $x_1 \in \sigma_1, \ldots, x_r \in \sigma_r$ from pairwise disjoint faces so that $f(x_1), \ldots, f(x_r)$ are the vertex set of a multiprism $P(r,2k) = P_p^{x_k}$ whose regular $p$-gons are parallel to the $U_i$. This also holds for $r = 4$ when $k = 1$.

(b) Let $N = T(2^k,k) - k$, and let $f : \Delta_N \to \mathbb{R}^k$ be a generic continuous map. Then there exist points $x_1 \in \sigma_1, \ldots, x_{2k} \in \sigma_{2k}$ from pairwise disjoint faces so that $f(x_1), \ldots, f(x_r)$ are the vertices of a $k$-orthotope $P^{x_2}$ whose edges are parallel to the coordinate axes.

See Definition 1 below for our precise notion of topological (and affine) genericity. As discussed in Remark 1 following, our condition holds for almost every affine map and remains typical in the continuous setting. In particular, it includes all maps $f : \Delta_N \to \mathbb{R}^d$ with $N < T(r,d)$ which do not admit a Tverberg $r$-partition.

The remainder of this paper proceeds as follows. Theorems 1.1–1.3 have equivalent formulations in terms of Fourier analysis on finite abelian groups, so that the existence of the prescribed $P(r,d)$-partition is equivalent to the annihilation of certain Fourier coefficients. This perspective was first introduced to Tverberg-type problems in [22] and is discussed in Section 2. In Section 3, we show that nearly arbitrary collections can be forced to vanish in the affine setting (Theorem 3.1) provided generically tight dimensional considerations are met. This is ultimately a consequence of Sarkaria’s “Linear Borsuk–Ulam Theorem”
[20], which is itself a corollary of Bárány’s colored Carathéodory Theorem [2]. As we show in Section 4, Theorem 1.2 follows immediately, as do two extension of Theorem 1.1 to higher dimensions (Proposition 4.1), including a general upper bound on $N(P;2d)$. In the continuous setting (Section 5), coefficients can be annihilated just as freely provided one considers elementary abelian groups, so that Theorem 1.3 follows from standard equivariant cohomological techniques. In Section 6, we give two examples of how the “constraint” method of [7] (and implicitly [13]) can be applied to our framework. Thus we have regular polygonal partitions (i) of a van Kampen–Flores type for odd primes (Theorem 6.2 and Corollary 6.3), as well as (ii) a colored variant in the mode of Soberón [23] (Theorem 6.5). We conclude in Section 7 with a return to Question 1 when $P(r,r-1) = \Delta_{r-1}$ is a regular $(r-1)$-simplex. While standard methods yield $N(\Delta_2; d) = 4$ for all $d \geq 2$, an observation we owe to Florian Frick, we show that this approach fails for all $r \geq 4$, including when $r$ is a prime power.

2. A Fourier Analytic Approach

Following [22], Fourier techniques can be applied to any map $f = (f_1, \ldots, f_d) : \Delta_N \rightarrow \mathbb{C}^d$ (including when some $f_i$ are real–valued) by indexing each collection of $r$ points $x_1 \in \sigma_1, \ldots, x_r \in \sigma_r$ from pairwise disjoint faces by a fixed group $G$ of order $r$. For each coordinate map $f_i$ and any such $\{x_g\}_{g \in G}$, evaluation of $f_i$ defines a function $F_i : G \rightarrow \mathbb{C}, \ g \mapsto f_i(x_g)$ (2.1)

which has a Fourier decomposition arising from the complex representation theory of $G$. When $G = \oplus_{j=1}^k \mathbb{Z}_{r_j}$ is abelian, this takes a simple form owing to the fact that the irreducible representations are all one-dimensional and indexed by the group itself. Explicitly, each $\chi_h : G \rightarrow \mathbb{C}^\times$ is given by $\chi_h(g) = \Pi_{j=1}^k \zeta_{r_j}^{h_j+g_j}$, where $h = (h_1, \ldots, h_k), g = (g_1, \ldots, g_k) \in G$ and $\zeta_r = e^{2\pi i/r}$ is the standard $r$-th root of unity. The characters $\chi_h$ form an orthonormal basis for the space of all functions $H : G \rightarrow \mathbb{C}$ under the standard inner product $\langle H_1, H_2 \rangle = \frac{1}{|G|} \sum_{g \in G} H_1(g) \overline{H_2(g)}$ for each $H_1, H_2 : G \rightarrow \mathbb{C}$ (see, e.g., [21]), so that each $F_i$ above can be uniquely expressed as

$F_i = \sum_{h \in G} c_{i,h} \chi_h$, (2.2)

where

$c_{i,h} = \langle F_i, \chi_h \rangle = \frac{1}{|G|} \sum_{g \in G} f_i(x_g) \overline{\chi_h^{-1}(g)}$ (2.3)

is the Fourier coefficient corresponding to $\chi_h$.

As $\chi_0 = 1$, it follows immediately from (2.2) that $F_i$ is constant iff $c_{i,h} = 0$ for all $h \neq 0$. In particular, a Tverberg $r$-partition is equivalent to the vanishing of all Fourier coefficients not arising from the trivial representation.

For the multiprism partitions of Theorem 1.2 and 1.3(a), we will see in Section 4 that it suffices to consider the special case where the $U_1, \ldots, U_k$ are coordinate complex planes, the Fourier characterization of which is given below (the situation for Theorem 1.3(b) is analogous, see Section 5). As usual, $\mathbb{Z}_r^\times$ denotes the units of the ring $\mathbb{Z}_r$, while $e_i$ denotes the $i$-th standard basis vector of $G = \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_k}$. 
Lemma 2.1. Let $r = r_1, \ldots, r_k$, $r_i \geq 3$, let $G = \bigoplus_{i=1}^k \mathbb{Z}_{r_i}$, and let $G' = G \oplus \mathbb{Z}_2$.

(a) Let $f : \Delta_N \to \mathbb{C}^k$, and let $\{x_g\}_{g \in G} \subset \Delta_N$ be a collection of points from pairwise disjoint faces. Then $\{f(x_g)\}_{g \in G}$ is the vertex set of a multiprism $P(r, 2k) = P_{r_1} \times \cdots \times P_{r_k}$ with each $P_{r_i}$ parallel to the $i$-th coordinate plane of $\mathbb{C}^k$ iff for each $1 \leq i \leq k$, there exists some $g_{i,0} \in \mathbb{Z}_{r_i}^\times$ such that (i) $c_{i,g_{i,0}e_i} \neq 0$ and (ii) $c_{i,g} = 0$ for all $g \in G - \{0, g_{i,0}e_i\}$ in the Fourier expansion \([2.2]\).

(b) Let $f : \Delta_N \to \mathbb{C}^k \oplus \mathbb{R}$, and let $\{x_g\}_{g \in G} \subset \Delta_N$ be a collection of points from pairwise disjoint faces. Then $\{f(x_g')\}_{g' \in G'}$ is the vertex set of a multiprism $P(2r, 2k+1) = P_{r_1} \times \cdots \times P_{r_k} \times \mathbb{R}$ with each $P_{r_i}$ parallel to the $i$-th coordinate plane of $\mathbb{C}^k$ iff for each $1 \leq i \leq k+1$ there exists some $g_{i,0} \in \mathbb{Z}_{r_i}^\times$ such that (i) $c_{i,g_{i,0}e_i} \neq 0$ and (ii) $c_{i,g'} = 0$ for all $g' \in G' - \{0, g_{i,0}e_i\}$ in the Fourier expansion \([2.2]\).

Proof. For part (a), consider some collection $\{x_g\}_{g \in G}$ from pairwise disjoint faces. The $f(x_g) = (f_1(x_g), \ldots, f_k(x_g))$ are the vertices of a $P(r, 2k)$ with each $P_{r_i}$ parallel to the $i$-th coordinate plane iff each $f_i(x_g)$ is a regular $r_i$-gon in $\mathbb{C}$. For each $1 \leq i \leq k$, the Fourier decomposition \([2.2]\) of each $f_i$ gives $f_i(x_g) = c_{i,0} + \sum_{h \neq 0} c_{h,i} \chi_h(g)$. As $\chi_{g_{i,0}e_i}(g) = \zeta_{g_{i,0}e_i}$, one has $f_i(x_g) = c_{i,0} + c_{i,g_{i,0}e_i} \zeta_{g_{i,0}e_i}$ if $c_{i,g} = 0$ for all $g \in G - \{0, g_{i,0}e_i\}$. If $g_{i,0} \in \mathbb{Z}_{r_i}^\times$, then $\zeta_{g_{i,0}e_i}$ is a primitive $r_i$-th root of unity, and hence $\{f_i(x_g)\}_{g \in G}$ is the vertex set of a regular $r_i$-gon, provided in addition that $c_{g_{i,0}e_i} \neq 0$. Conversely, it follows from the orthogonality of characters that if $\{f_i(x_g)\}_{g \in G}$ is the vertex set of a regular $r_i$-gon, then $f_i(x_g) = c_{i,0} + c_{i,g_{i,0}e_i} \zeta_{g_{i,0}e_i}$ for some $g_{i,0} \in \mathbb{Z}_{r_i}^\times$. Thus all Fourier coefficients other than $c_{i,0}$ and $c_{i,g_{i,0}e_i}$ vanish, and $c_{g_{i,0}e_i} \neq 0$.

The proof for the multiprisms $P(2r, 2k+1) = P(r, 2k) \times \mathbb{R}$ of part (b) is identical, except now $\{f_k(x_g')\}_{g \in G} \subset \mathbb{R}$ are the endpoints of a segment. \(\square\)

3. Annihilating Coefficients in the Affine Setting

Theorem 1.2 follows once it is ensured that the coefficient conditions prescribed in Lemma 2.1 are met. This follows from the fact that which coefficients can be annihilated nearly arbitrarily in the affine setting once generically tight dimension conditions are satisfied.

Theorem 3.1. Let $f : \Delta_N \to \mathbb{C}^d \oplus \mathbb{R}^{d'}$ be an affine map, let $G = \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_k}$, $r_i \geq 2$, and let $r = r_1 \cdots r_k$.

(i) For each $1 \leq i \leq d$, let $h_{i,1}, \ldots, h_{i,m_i} \in G - \{0\}$ be distinct.

(ii) For each $d + 1 \leq i \leq d + d'$,

(a) let $h_{i,1}, \ldots, h_{i,m_i} \in G$ such that $|h_{i,j}| > 2$ for all $j$ and $h_{i,j} \neq h_{i,k}$ for all $j \neq k$.

(b) let $h'_{i,1}, \ldots, h'_{i,m_i'} \in G$ be any distinct elements of order 2.

Let $m = m_1 + \cdots + m_{d+d'}$ and $m' = m'_{d+1} + \cdots + m_{d+d'}$. If $N = 2m + m' + r - 1$, then there exists some $\{x_g\}_{g \in G} \subset \Delta_N$ from pairwise disjoint faces such that all the resulting $c_{i,h_{i,j}}$ and $c_{i,h'_{i,j}}$ vanish in the Fourier expansions \([2.2]\). Moreover, this fails for almost any $f$ if $N < 2m + m' + r - 1$.

It is implicit here that one may take either $d = 0$ or $d' = 0$ above. To see the motivation behind conditions (a) and (b) in (ii), note that (1) $c_{i,h} = c_{i,-h}$ if $f_i$ is real-valued, so that for $|h| > 2$ there is redundancy in specifying that both $c_{i,h} = 0$ and $c_{i,-h} = 0$ are zero, and that (2) $c_{i,h}$ is real-valued for real-valued $f_i$ iff $|h| = 2$. These observations will be
important in the proof of the optimality of $N = 2m + m' + r - 1$ below.

Our proof relies on the join configurations commonly used in Tverberg–type problems. Recall that the $r$-fold join $\Delta_N^r$ consists of all formal convex sums $\lambda_1 x_1 + \cdots + \lambda_r x_r \in \sigma_1 \ast \cdots \ast \sigma_r$ from the join of any $r$ faces of $\Delta_N$, including the possibility of empty faces. The deleted $r$-fold join

$$(\Delta_N)_r^* = \{ \lambda_1 x_1 + \cdots + \lambda_r x_r \in \sigma_1 \ast \cdots \ast \sigma_r \mid x_i \in \sigma_i \text{ and } \sigma_i \cap \sigma_j = \emptyset \text{ for } i \neq j \} \quad (3.1)$$

is the subcomplex consisting of all points from the joins of pairwise disjoint faces. Following Sarkaria [20], we parametrize each $r$-tuple of disjoint faces by $G$ and denote the resulting complex by $(\Delta_N)^*_G$. The group acts freely on $(\Delta_N)^*_G$ by right translations, so that $g' \cdot \sum_g \lambda_g x_g = \sum_g \lambda_{g-g'} x_g - g'$. On the other hand, one can parametrize $r$ disjoint copies of the vertex set $\{ v_1, \ldots, v_{N+1} \}$ of $\Delta_N$ by $G$ as well, with $v^g_j$ denoting the $g$-th copy of $v_j$. One then has an equivalent parametrization of all points from the join of pairwise disjoint faces by the $(N + 1)$-fold join

$$G^{*(N+1)} = \{ \sum_{j=1}^{N+1} t_j v^g_j \mid g_j \in G \text{ and } \sum_{j=1}^{N+1} t_j = 1, t_j \geq 0 \} \quad (3.2)$$

of the group itself. As with $(\Delta_N)^*_G$, $G$ also acts freely on $G^{*(N+1)}$, now by affine extension of the $G$-action on each copy of $G$ in $G^{*(N+1)}$ by addition: $g' \cdot \sum_{j=1}^{N+1} t_j v^g_j = \sum_{j=1}^{N+1} t_j v^{g+g_j}$. Finally, there is an obvious isomorphism $\iota : (\Delta_N)^*_G \cong G^{*(N+1)}$ between these two simplicial complexes given by grouping. Explicitly, for each $v \in G^{*(N+1)}$ and each $g \in G$, let $J_g = \{ j \mid g_j = g \text{ and } t_j > 0 \}$ for each $g \in G$. One then defines $\iota(v) = \sum_{g \in G} \lambda_g x_g$, where

$$\lambda_g = \sum_{j \in J_g} t_j \text{ and } x_g = \sum_{j \in J_g} \frac{t_j}{\lambda_g} v_j \quad (3.3)$$

if $J_g \neq \emptyset$, and (2) $\lambda_g = 0$ otherwise. Moreover, it is easily seen that $\iota$ respects the two $G$-actions.

Theorem 3.1 will follow from an application of Sarkaria’s “Linear Borsuk–Ulam” Theorem [20, Theorem 2.4]:

**Theorem 3.2.** Let $W$ be a real $N$-dimensional linear representation of $G$ which does not contain the trivial subrepresentation. If $L : G^{*(N+1)} \to W$ is an affine linear $G$-equivariant map, then there exists some $v \in G^{*(N+1)}$ such that $L(v) = 0$.

**Proof of Theorem 3.1.** Let $f : \Delta_N \to \mathbb{C}^d \oplus \mathbb{R}^d$, $N = 2m + m' + r - 1$. To apply Theorem 3.2 we construct a linear $G$-equivariant map $L$ whose zeros correspond to the vanishing of the prescribed Fourier coefficients. To that end, let

$$\sigma = \oplus_{i,j} x_{-h_{i,j}} \oplus_{i,j} \chi_{h_{i,j}} : G \to \mathbb{C}^m \oplus \mathbb{R}^{m'} \quad (3.4)$$

where each $c_{i,h_{i,j}}$ is the Fourier coefficient of $\chi_{h_{i,j}}$, and likewise for the $c_{i,h_{i,j}}$. To rule out empty faces, we also consider $\mathbb{R}^\perp[G] = \{ (\lambda_g)_{g \in G} \mid \sum_{g \in G} \lambda_g = 1, \lambda_g \in \mathbb{R} \}$, the orthogonal complement of the trivial subrepresentation of the regular representation $\mathbb{R}[G]$. The action here is again by right translation, so that $\rho(g')$ sends each $(\lambda_g)_{g \in G}$ to $(\lambda_{g-g'})_{g \in G}$. It follows that

$$W := \mathbb{C}^m \oplus \mathbb{R}^{m'} \oplus \mathbb{R}^\perp[G] \quad (3.5)$$
is a real $N$-dimensional representation which does not contain the trivial subrepresentation.

We define $L = \mathcal{L} \circ \iota : G^s(N+1) \to W$, where

$$
\mathcal{L} = \oplus_{i,j} \mathcal{F}_{i,j} \oplus \mathcal{F}_{i,j}' \oplus \mathcal{R} : (\Delta_N)^G \to \mathbb{C}^m \oplus \mathbb{R}^{m'} \oplus \mathbb{R}^+[G] \tag{3.6}
$$

is given by

$$
\mathcal{F}_{i,j}(\sum_{g \in G} \lambda_g x_g) = \sum_{g \in G} \lambda_g f_i(x_g) \chi_{h_{i,j}}^{-1}(g) \in \mathbb{C} \tag{3.7}
$$

for each $\chi_{h_{i,j}}$,

$$
\mathcal{F}_{i,j}'(\sum_{g \in G} \lambda_g x_g) = \sum_{g \in G} \lambda_g f_i(x_g) \chi_{h_{i,j}'}^{-1}(g) \in \mathbb{R} \tag{3.8}
$$

for each $\chi_{h_{i,j}'}$, and

$$
\mathcal{R}(\sum_{g \in G} \lambda_g x_g) = (\lambda_g - \frac{1}{|G|})_{g \in G} \in \mathbb{R}^+[G]. \tag{3.9}
$$

As $\mathcal{R}(\lambda x) = 0$ iff $\lambda_g = \frac{1}{|G|}$ for all $g \in G$, we see that the zeros of $L$ correspond to those $\{x_g\}_{g \in G}$ from non-empty pairwise disjoint faces for which all the $c_{i,h_{i,j}}$ and $c_{i,h_{i,j}'}$ are given by (2.2) vanish. As $\chi_{h_{i,j}}^{-1} = \chi_{-h_{i,j}}$, $L$ is equivariant with respect to the action on $W$ given by $\sigma \oplus \rho$ so that $L$ is equivariant with respect to that action on $W$ as well.

It remains to check that $L$ is affine. For ease of notation, denote $h_{i,j}$ by $h$ and $\mathcal{F}_{i,j}$ by $\mathcal{F}_i$. We have $(\mathcal{F}_i \circ \iota)((v_j^g) = f_i(v_j) \chi_{h}^{-1}(g)$ for each $g \in G$. Let $v = \sum_{j=1}^{N+1} t_j v_j^g$, and let $\iota(v) = \sum_g \lambda_g x_g$ as above. For those $g$ with $J_g \neq \emptyset$, we have $f_i(x_g) = \frac{1}{\lambda_g} \sum_{j \in J_g} t_j f_i(v_j)$ since $f$ is affine. Thus $(\mathcal{F}_i \circ \iota)(v) = \sum_g \lambda_g f_i(x_g) \chi_{h}^{-1}(g) = \sum_g |J_g| t_j f_i(v_j) \chi_{h}^{-1}(g)] = \sum_g \sum_{j \in J_g} t_j f_i(v_j) \chi_{h}^{-1}(g)) = \sum_{j=1}^{N+1} t_j (\mathcal{F}_i \circ \iota)(v_j^g)$, as desired. An identical argument holds for each $\mathcal{F}_{i,j}'$. For $\mathcal{R}$, we have $(\mathcal{R} \circ \iota)((v_j^g) = e_g - \frac{1}{|G|} \mathbf{1}$, where $e_g$ is the standard basis vector in $\mathbb{R}[G]$ and $\mathbf{1} = \sum_{g \in G} e_g$. For given $v \in G^{1(N+1)}$ and each $g$ such that $J_g \neq \emptyset$, we have $\sum_{j \in J_g} t_j (\mathcal{R} \circ \iota)(v_j^g) = \lambda_g e_g - \frac{1}{|G|} \mathbf{1}$. Thus $\sum_{j=1}^{N+1} t_j (\mathcal{R} \circ \iota)(v_j^g) = \sum_g \sum_{j \in J_g} t_j (\mathcal{R} \circ \iota)(v_j^g) = \sum_{g \in G} (\lambda_g e_g - \frac{1}{|G|} \mathbf{1}) = (\mathcal{R} \circ \iota)(v)$.

To prove that $N = 2m + m' + r - 1$ is tight, suppose that $n < N$ and let $f : \Delta_n \to \mathbb{C}^d \oplus \mathbb{R}^d$. For $W$ as above, the vanishing of all desired coefficients corresponds to a zero of $L : G^s(n+1) \to W$. The argument is then along the lines of [20, Theorem 2.4] and [23, Theorem 1]. Since $L$ is affine, $L(v) = 0$ means that $0 \in \text{Conv}(L(v_1^{g_1}), \ldots, L(v_{n+1}^{g_n+1}))$ for some $(g_1, \ldots, g_{n+1}) \in G^{s(n+1)}$, and by assumption each such hull has dimension at most $n < N$. On the other hand, requiring $L(v_1^{g_1}), \ldots, L(v_{n+1}^{g_{n+1}})$ to “capture the origin” forces their convex hull to have dimension at least $N$, provided the images $f(v)$ of the vertices of $\Delta_n$ are generic. This is seen by tallying the number of independent conditions for $L(v) = 0$, $v = \sum t \ell v_\ell$. First, one has $r - 1$ independent linear conditions on the $t_\ell$ themselves because $\sum_{j \in J_g} t_j = \lambda_g = \frac{1}{|G|}$ for all $g \in G$. It is a direct consequence of assumptions (i) and (ii) above that the vanishing of each Fourier coefficient yields $2m + m'$ additional linearly independent conditions on the $f(v_\ell)$ above. For instance, $c_{i,h_{i,j}} = 0$ means that $\sum_{g \in G} \sum_{\ell \in J_g} t_\ell f_i(v_\ell) \chi_{h_{i,j}}^{-1}(g_\ell) = 0$. As the $h_{i,j}$ are distinct for each $1 \leq j \leq m_i$, 7
3.1. Fourier Genericity. As stated in Theorem 3.1, the ability to annihilate coefficients depends on a particular coordinate decomposition of \( f : \Delta_N \to \mathbb{R}^{2d+d'} \). To remove this, we set the following notation for any decomposition \( \mathbb{R}^{2d+d'} \) by pairwise orthogonal coordinate planes \( U_1, \ldots, U_d \) and coordinate lines \( L_1, \ldots, L_{d'} \). Let

\[
\sigma : \oplus_{i=1}^d U_i \oplus_{j=1}^{d'} L_j \to \mathbb{C}^d \oplus \mathbb{R}^{d'}
\]  

be the orthogonal transformation of \( \mathbb{R}^{2d+d'} \) given by coordinate permutation which sends each \( U_i \) to the \( i \)-th coordinate complex plane and each \( L_j \) to the \((2d+j)\)-th coordinate line, respectively, and let

\[
f_\sigma := \sigma \circ f : \Delta_N \to \mathbb{C}^d \oplus \mathbb{R}^{d'}.
\]  

Thus annihilating the Fourier coefficients of the \( f_\sigma \) is equivalent to annihilation those of the compositions of \( f \) onto the \( U_i \) and \( L_j \).

In light of the optimality of Theorem 3.1, we make the following definition:

**Definition 1.** [Fourier Generic] Let \( f : \Delta_N \to \mathbb{R}^{2d+d'} \), let \( G = \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_k} \), and let \( r = r_1 \cdots r_k \). We say that \( f \) is \( G \)-Fourier generic if the following is true for any decomposition of \( \mathbb{R}^{2d+d'} \) as above, as well as for any choice of \( m_i, m_i', h_{ij}, \) and \( h_{ij}' \) as in the statement of Theorem 3.1: if \( \Delta_n \) is any face of \( \Delta_N \) with \( n < 2m + m' + r - 1 \), then there is no \( \{x_g\}_{g \in G} \subset \Delta_n \) with pairwise disjoint support for which the Fourier coefficients \( c_{i,j} \) and \( c_{i,j}' \) of the restriction \( f_g : \Delta_n \to \mathbb{C}^d \oplus \mathbb{R}^{d'} \) of (3.11) all vanish. Finally, we say that \( f \) is is Fourier generic if it is \( G \)-Fourier generic for any abelian group \( G \) of order \( r \) and all \( r \geq 2 \).

Thus Fourier genericity guarantees that one cannot annihilate “too many” coefficients given the dimension of the simplex. In particular, note that if \( f : \Delta_N \to \mathbb{R}^d \) is Fourier generic and \( N < T(r, d) \), then \( f \) does not admit a Tverberg \( r \)-partition.

**Remark 1.** Theorem 3.1 shows that almost every affine map is Fourier generic. In the topological setting, the generic designation remains appropriate, since again the vanishing of prescribed coefficients is equivalent to a zero of the continuous (and \( G \)-equivariant) map \( L : G^{s(n+1)} \to \mathbb{R}^N \). As \( G^{s(n+1)} \) is \( n \)-dimensional, the number of independent conditions exceeds the degrees of freedom when \( n < N \), hence will not vanish for typical \( f \).

4. Proofs in the Affine Setting

Given Lemma 2.1 and Theorem 3.1 Theorem 1.2 follows easily.

**Proof of Theorem 1.2.** For both (a) and (b), suppose that \( f : \Delta_N \to \mathbb{R}^d \) is Fourier generic and affine, with \( d = 2k \) or \( d = 2k + 1 \).

For (a), let \( r = r_1 \cdots r_k \), \( r_i \geq 3 \), and let \( G = \oplus_{i=1}^k \mathbb{Z}_{r_i} \). Given \( k \) pairwise orthogonal planes \( U_1, \ldots, U_k \), let \( \sigma : \oplus_{i=1}^k U_i \to \mathbb{C}^k \) be the resulting orthogonal transformation (3.10) and consider the corresponding \( f_\sigma : \Delta_N \to \mathbb{C}^k \) (3.11). For \( \{x_g\}_{g \in G} \) with pairwise disjoint
support, \( \{ f(x_g) \}_{g \in G} \) will be a \( P(r, 2k) \) with each \( P_{r_i} \) parallel to \( U_i \) iff \( \{ f_\sigma(x_g) \}_{g \in G} \) is one with each \( P_{r_i} \) parallel to \( i \)-th coordinate plane. Following Lemma 2.1(a), we seek some \( \{ x_g \}_{g \in G} \) so that the \( r - 2 \) coefficients \( c_{i,h,i,j}^{\prime} \) given by \( h_{i,j} \in G - \{ 0, \mathbf{e}_i \} \) vanish for each \( 1 \leq i \leq k \) in the Fourier decompositions (2.2) of \( f_\sigma \). This is guaranteed by Theorem 3.1 with \( N = 2k(r - 2) + r - 1 = T(r, 2k) - 2k \), and for this \( N \) no additional coefficients can vanish because \( f \) is Fourier generic. On the other hand, it is immediate from the optimality of Theorem 3.1 and the Fourier characterization of the \( P(r, 2k) \)-partitions from Lemma 2.1(a) that this \( N \) is optimal. In particular, \( N_{P_r} = T(r, 2) - 2 \) for all \( r \geq 3 \).

The proof of (b) is nearly identical, except in dealing with the final coordinate of the analogous \( f_\sigma : \Delta_N \to \mathbb{C}^k \oplus \mathbb{R} \). Considering \( G' = G \oplus \mathbb{Z}_2 \), we seek \( c_{i,h,i,j}^{\prime} = 0 \) for all \( h_{i,j} \in G' - \{ 0, \mathbf{e}_i \} \), \( 1 \leq i \leq k + 1 \). As \( (f_\sigma)_{k+1} \) is real–valued and none of the \( h_{k+1,j} \) considered have order 2, we only need to annihilate half of these \( 2r - 2 \) Fourier coefficients. Thus all of our desired coefficients vanish provided \( N = 2(2k(2r - 2) + 2k) + (2r - 1) = T(2r, 2k+1) - (2k+1) \), Generiticity of \( f \) again ensures that \( c_{i,h,i,e_i}^{\prime} \neq 0 \) for all \( 1 \leq i \leq k + 1 \), and the optimality of \( N \) is again guaranteed by Theorem 3.1 and Lemma 2.1(b). \( \square \)

As a further application of Theorem 3.1 we give two extensions of Theorem 1.1 to higher dimensions.

**Proposition 4.1.**

(a) Let \( S \) be a set of \( T(r, d) - 1 \) generic points in \( \mathbb{R}^d \). Then for any plane \( U \) in \( \mathbb{R}^d \), there exists a \( P_r \)-partition of \( S \) with \( P_r \) parallel to \( U \).

(b) Let \( d \) be even. Then almost any \( T(r, d) - d + 1 \) points in \( \mathbb{R}^d \) can be \( P_r \)-partitioned so that \( P_r \) lies in a complex 1-flat.

**Proof.** Assume that \( f : \Delta_N \to \mathbb{R}^d \) is Fourier generic. Given a linear 2–flat \( U \), let \( f_\Phi = \Phi \circ f : \Delta_N \to \mathbb{C} \oplus \mathbb{R}^{d-2} \), where \( \Phi \) is some orthogonal transformation sending \( U \) to the first complex coordinate plane. Considering the resulting Fourier decompositions for \( f_\Phi \) with \( G = \mathbb{Z}_r \) and letting \( N = T(r, d) - 2 \), Theorem 3.1 guarantees some \( \{ x_g \}_{g \in \mathbb{Z}_r} \) so that \( c_{i,g} = 0 \) for all \( g \in \mathbb{Z}_r - \{ 0, 1 \} \), and moreover that \( c_{i,g} = 0 \) for each \( g \neq 0 \) for all \( i \geq 2 \). If \( c_{1,1} = 0 \) also, then \( \{ x_g \}_{g \in \mathbb{Z}_r} \) would be a Tverberg \( r \)-partition for \( f_\Phi \) and hence for \( f \) as well. This is impossible, however, because \( N < T(r, d) \) and \( f \) is Fourier generic. Thus the \( f_\Phi(x_g) \) are the vertices of a regular \( r \)-gon parallel to the first coordinate plane, and equivalently the \( f(x_g) \) are the vertices of one which is parallel to \( U \).

For (b), let \( N = T(r, 2d) - 2d \) and consider \( f : \Delta_N \to \mathbb{C}^d \). Given \( \{ x_g \}_{g \in \mathbb{Z}_r} \), annihilating \( c_{i,g} \) for all \( g \in \mathbb{Z}_r - \{ 0, 1 \} \) and all \( 1 \leq i \leq d \) yields \( f(x_g) = c_0 + \zeta^d \mathbf{e}_1 \) for each \( g \in \mathbb{Z}_r \), where \( c_j = (c_{1,j}, \ldots, c_{d,j}) \in \mathbb{C}^d \) for \( j = 0, 1 \).

That the respective \( N \) of Proposition 4.1 (a) and (b) is tight again follows from Theorem 3.1 and the partition’s equivalent Fourier characterization. In particular, part (b) gives the upper bound

\[
N_{(P_r,d)} \leq (r - 2)(d + 1) + 1
\]

for even \( d \). As with Theorem 1.2, however, subtraction of the remaining \( 2(d - 2) = \dim G_2(\mathbb{R}^d) \) degrees of freedom from the \( N \) of part (a) yields the expected value \( N_{(P_r,d)} - (r - 3)(d + 1) = 4 \) for all \( d \geq 2 \). This value is confirmed for all \( r = 3 \) in Section 7.

5. Proofs in the Continuous Setting

For general abelian groups, care must be taken to ensure the vanishing of desired Fourier coefficients when \( f \) is continuous (see, e.g., the polynomial criteria [22, Theorem 3.2]).
As with the Topological Tverberg Theorem, however, standard equivariant cohomological techniques ensure that coefficients can be annihilated as freely as in the affine setting if \( G = \mathbb{Z}_p^{\oplus k} \) and \( p \) is prime.

**Proposition 5.1.**

(a) Let \( f : \Delta_N \to \mathbb{C}^d \), let \( G = \mathbb{Z}_p^{\oplus k} \), \( p \) an odd prime, and for each \( 1 \leq i \leq d \) let \( h_{i,1}, \ldots, h_{i,m_i} \in G - \{0\} \). Let \( m = m_1 + \cdots + m_d \). If \( N = 2m + p^k - 1 \), then there exists some \( \{x_g\}_{g \in G} \subset \Delta_N \) from pairwise disjoint faces such that \( c_{i,g} = 0 \) for all \( 1 \leq j \leq m_i \) and all \( 1 \leq i \leq d \) in the Fourier expansions (2.2).

(b) Let \( f : \Delta_N \to \mathbb{R}^d \), let \( G = \mathbb{Z}_2^{\oplus k} \), and for each \( 1 \leq i \leq d \) let \( h_{i,1}, \ldots, h_{i,m_i} \in G - \{0\} \). Let \( m = m_1 + \cdots + m_d \). If \( N = m + 2^k - 1 \), then there exists some \( \{x_g\}_{g \in G} \subset \Delta_N \) from pairwise disjoint faces such that \( c_{i,g} = 0 \) for all \( 1 \leq j \leq m_i \) and all \( 1 \leq i \leq d \) in the Fourier expansions (2.2).

Although Proposition 5.1 can be proven as a consequence of a slightly more general version of [22, Theorem 3.2], we shall derive it instead from the following lemma of Volovikov [25, Lemma 8].

**Lemma 5.2.** Let \( G = \mathbb{Z}_p^{\oplus k} \), \( p \) prime, and let \( X \) and \( Y \) be fixed point free \( G \)-spaces. If \( Y \) is a finite-dimensional cohomology \( n \)-sphere over the field \( \mathbb{F}_p \) and \( \tilde{H}^i(X; \mathbb{F}_p) = 0 \) for all \( 1 \leq i \leq n \), then there is no continuous \( G \)-equivariant map \( h : X \to Y \).

Using Proposition 5.1, one can prove all prime power cases of Theorem 1.3(a) using the same argument as for Theorem 1.2. Namely, for \( f : \Delta_N \to \mathbb{R}^{2d} \) one annihilates all relevant Fourier coefficients of the map \( f_\sigma : \Delta_N \to \mathbb{C}^k \), and as there Fourier genericity is enough to guarantee that no additional coefficients vanish. The argument for Theorem 1.3(b) is entirely analogous:

**Proof of Theorem 1.3(b).** Given \( f : \Delta_N \to \mathbb{R}^k \) and \( N = T(2^k,k) - k \), there exists some \( \{x_g\}_{g \in \mathbb{Z}_2^{\oplus k}} \) with pairwise disjoint support for which \( c_{i,g} = 0 \) all \( g \in \mathbb{Z}_2^{\oplus k} - \{0,e_i\} \) for each \( 1 \leq i \leq k \). On the other hand, no other Fourier coefficients vanish by genericity. Thus each \( \{f_i(x_g)\}_{g \in \mathbb{Z}_2^{\oplus k}} \) form the endpoint of a segment in \( \mathbb{R} \), hence the \( f(x_g) \) are the vertex set of a \( k \)-orthotope with edges parallel to the coordinate axes. \( \square \)

Proposition 5.1 also gives immediate topological extensions of Proposition 1.1 for odd primes, again with the same proofs as in the affine setting.

**Proposition 5.3.** Let \( f : \Delta_N \to \mathbb{R}^d \) be a Fourier generic continuous map and let \( r \) be an odd prime.

(a) Let \( N = T(r,d) - 2 \). Then for any plane \( U \) in \( \mathbb{R}^d \), there exists points \( x_1, \ldots, x_r \in \Delta_N \) with pairwise disjoint support such that \( f(x_1), \ldots, f(x_r) \) are the vertices of a regular \( r \)-gon parallel to \( U \).

(b) If \( d \) is even and \( N = T(r,d) - d \), then there exist \( x_1, \ldots, x_r \in \Delta_N \) with pairwise disjoint support such that \( f(x_1), \ldots, f(x_r) \) are vertices of a regular \( r \)-gon lying in a complex 1-flat.

**Proof of Proposition 5.1.** We shall use the other main configuration space used for Tverberg problems, namely the deleted product

\[
(\Delta_N)_X^G = \{(x_g)_{g \in G} \mid x_g \in \sigma_g \forall g \in G, \text{ and } \sigma_g \cap \sigma_{g'} = \emptyset \forall g \neq g'\}. \tag{5.1}
\]

For consistency of notation of notation with [25], we let \( G = \mathbb{Z}_p^{\oplus k} \) act freely on \( (\Delta_N)_X^G \) by left translations, so that \( g' \cdot (x_g)_{g \in G} = (x_{g'g})_{g \in G} \). It is a crucial fact (first proved in [4])
that $X := (\Delta_N)_N^G$ is $(N - |G| + 1)$-dimensional and $(N - |G|)$-connected, and therefore has vanishing reduced cohomology $\tilde{H}^i(X; \mathbb{Z}_p)$ for all $1 \leq i \leq N - |G|$ by the Hurewicz Theorem. For (a), let $f : \Delta_N \to \mathbb{C}^d$ and $p$ odd. As in the proof of Theorem 3.1 (or [22]), define $\mathcal{F} : X \to \mathbb{C}^m$ be given by the evaluation of Fourier coefficients, $x = (x_g)_{g \in G} \mapsto \sum_{g \in G} \frac{1}{|G|} f(x_g) \chi_i^{-1}(g)$.

As before, this map is $G$-equivariant map with respect to the linear $G$-action on $\mathbb{C}^m$ given by $\oplus_{i,j} \chi_{i,j}$. If no zero for this map exists, then $x \mapsto \mathcal{F}(x)/||\mathcal{F}(x)||$ would define a $G$-equivariant map from $X$ to the $(2m - 1)$-dimensional unit sphere $Y = S(\mathbb{C}^m)$, violating Volovikov’s Lemma. Thus we have the desired $\{x_g\}_{g \in G}$. The proof of (b) is identical, with $f : \Delta_N \to \mathbb{R}^d$ and $G = \mathbb{Z}_2^{\oplus k}$.

For the $k = 1$ and $r = 4$ case of Theorem 1.3, we use [22, Theorem 3.2] with $d = 1$. As applied to $P_r$-partitions, this asserts that if the polynomial $q(y) = (r - 1)! y^{r-2}$ is non-zero in $\mathbb{Z}[y]/(ry)$, then there exists some $\{x_g\}_{g \in \mathbb{Z}_r}$ from pairwise disjoint faces such that $c_{1,i} = 0$ for all $2 \leq i < r$ in the Fourier expansion (2.2) of the given $f_1 : \Delta_{d(r-2)+1} \to \mathbb{C}$. Clearly, $q(y) \neq 0$ for non--prime $r$ iff $r = 4$.

6. Two Constrained Versions

The “constraint” method of Blagojević, Frick, and Ziegler [7] has proven to be a powerful tool for producing a number of interesting variants of the affine and topological Tverberg theorems with surprising ease. In particular, it was crucial in demonstrating counterexamples to the Topological Tverberg Conjecture for non prime powers [6, 12]. We give two example applications of this method to our schema.

6.1. van Kampen–Flores Type Theorems. Given a continuous map $f : \Delta_N \to \mathbb{R}^d$, one may seek a $r$–Tverberg partition for which each pairwise disjoint face lies in the $k$–skeleton $\Delta_N^{(k)}$. Such dimensionally constrained versions of Tverberg’s Theorem were first given in the continuous setting by van Kampen [14] and Flores [11] when $r = 2$, extended to all prime powers $r$ and more general $j$–wise intersection types in [21, 26], and subsequently sharpened in [7]. For our purposes, we only consider the following [7, Theorem 6.3]:

**Theorem 6.1.** Let $rk \geq d(r-1)$ and $N = (r-1)(d+2)$. Then for any continuous map $f : \Delta_r \to \mathbb{R}^d$, there exists $r$ disjoint faces $\sigma_1, \ldots, \sigma_r$ with $\dim \sigma_i \leq k$ for all $1 \leq i \leq r$ such that $\cap_{i=1}^{r}f(\sigma_i) \neq \emptyset$.

Theorem 6.1 follows directly from the Topological Tverberg Theorem by considering an appropriate “constraint” function, here the distance map $d : \Delta_N \to (\Delta_N)^{(k)}$ given by $d(x) = \text{dist}(x, (\Delta_N)^k)$. For $f \oplus d : (\Delta_{r-1}\oplus 2) \to \mathbb{R}^d \oplus \mathbb{R}$, there exists $x_1 \in \sigma_1, \ldots, x_r \in \sigma_r$ with the $\sigma_i$ disjoint so that both $f(x_i)$ and $d(x_i)$ are constant. As $r(k+2) > N + 1$, the pigeon hole principle shows that at least one $\sigma_i$ is from $(\Delta_N)^{(k)}$, and since $d$ is constantly zero so are all the others. The necessity of $rk \geq d(r-1)$ follows from the optimality of Tverberg’s theorem, since one has a Tverberg $r$–partition for $f$ contained in some $\Delta_{(k+1)r-1}$.

In even dimensions, the same method produces dimensionally restricted topological $P_r$–partitions when the dimension of the simplex of Theorem 6.1 is lowered by $d$:

**Theorem 6.2.** Let $r$ be an odd prime, let $d$ be even, and let $N = (r-2)(d+2)+2$. Suppose further that $d(r-2) \leq rk < d(r-1)$. If $f : \Delta_N \to \mathbb{R}^d$ is a Fourier generic continuous map, then there exists $x_1 \in \sigma_1, \ldots, x_r \in \sigma_r$ from pairwise disjoint faces with $\dim \sigma_i \leq k$ for all $1 \leq i \leq r$ such that $f(x_1), \ldots, f(x_r)$ are the vertices of a regular $r$–gon.
In particular, Theorem 6.2 holds for almost every affine map. Letting \( d = r - 1 \) and \( k = r - 2 \), \( N = T(r, r - 1) \) matches that of the Topological Tverberg Theorem:

**Corollary 6.3.** Let \( r \) be an odd prime. For any Fourier generic continuous map \( f : \Delta_{r(r-1)} \to \mathbb{R}^{r-1} \), there exist \( x_1, \ldots, x_r \in \sigma_1, \ldots, \sigma_r \) from pairwise disjoint faces with \( \dim \sigma_i \leq r - 2 \) for all \( 1 \leq i \leq r \) such that \( f(x_1), \ldots, f(x_r) \) are the vertices of a regular \( r \)-gon.

In the affine setting, Corollary 6.3 states that for almost any set of size \( T(r, r - 1) + 1 \) in \( \mathbb{R}^{r-1} \), it is possible to remove a single point so that the remaining \( T(r, r - 1) \) points can be \( P_r \)-partitioned by subsets of \( r - 1 \) points each. A picture of this situation when \( r = 3 \) is given below, with the dashed lines indicating a full Tverberg partition of 7 points in the plane:

![Diagram of Tverberg partition](image)

**Proof of Theorem 6.2.** Let \( f : \Delta_N \to \mathbb{C}^d \) be Fourier generic. Letting \( f_{d+1} : \Delta_N \to \mathbb{R} \) be the distance map \( f_{d+1}(x) = \text{dist}(x, \Delta_N) \) above, again consider \( f \oplus f_{d+1} : \Delta_N \to \mathbb{C}^d \oplus \mathbb{R} \). For \( N = 2d(r-2) + 2 \cdot \frac{r-1}{2} + (r-1) = (r-2)(2d+2) + 2 \), Proposition 5.1 applied to \( G = \mathbb{Z}_r \) guarantees some \( \{x_g\}_{g \in \mathbb{Z}_r} \) from pairwise disjoint \( \sigma_g \) such that (1) \( c_{i,h} = 0 \) for all \( h \in \mathbb{Z}_r - \{0,1\} \) when \( 1 \leq i \leq d \), as well as (2) that \( c_{d+1,h} = 0 \) for all \( h \in \mathbb{Z}_r - \{0\} \). Condition (2) guarantees that \( f_{d+1}(x_g) \) is constant. Again one has \( r(k+2) > N + 1 \) (since \( rk \geq 2d(r-2) \)), so as before each \( \sigma_g \) comes from the \( k \)-skeleton. As in the proof of Proposition 4.1(a), (1) implies that the \( f(x_g) \) are the vertex set of a regular \( r \)-gon (and in fact one lying in a complex 1-flat), provided at least one of the \( c_{i,1} \) is non–zero, \( 1 \leq i \leq d \).

On the other hand, the \( x_g \) reside in some \( \Delta_n, n = r(k+1) - 1 \). As \( rk < 2d(r-1), n < T(r,2d) \), so by Fourier genericity \( f \) does not admit a \( r \)-Tverberg partition. Thus \( c_{i,1} \neq 0 \) for some \( 1 \leq i \leq d \).

### 6.2. Colored versions with equal barycentric coordinates.

A celebrated variant of Tverberg’s Theorem considers a coloring of the vertices of \( \Delta_{rn-1} \) by \( n \) color classes \( C_1, \ldots, C_n \) with \( r \) points each. Given a map \( f : \Delta_{rn-1} \to \mathbb{R}^d \), one seeks a Tverberg \( r \)-partition \( \sigma_1, \ldots, \sigma_r \) so that the vertices of each \( \sigma_i \) consists of a single point from each color class (such \( \sigma_i \) are called “rainbow” faces). The Bárány–Larman conjecture [3] claims that if \( n = d + 1 \), then such “colorful” Tverberg \( r \)-partitions exist for all affine maps, and likewise in the continuous setting [27]. This has been verified in the affine cases for all \( r \) provided \( d = 2 \) [3], and topologically for all \( d \) if \( r + 1 \) is prime [8].

In the affine setting [23], Soberón proved a variant of this conjecture with the additional condition of equal barycentric coordinates [23, Theorem 1]. This was subsequently recovered in [7], where it was extended to the continuous realm for all prime powers [7, Theorem 8.3].
and 8.1, respectively]. We give this as the unified statement Theorem 6.4 below. By equal barycentric coordinates, one means the following: Let $\Delta_{rn-1}$ be partitioned by $n$ color classes $C_1, \ldots, C_n$ of $r$ points each. Given pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$, let $\{v_1^j, \ldots, v_n^j\}$ denote the vertex set of $\sigma_j$, $v_j^i \in C_i$. Each $y_j \in \sigma_j$ is the unique convex sum $y_j = \sum_{i=1}^{n} t_{i,j} v_j^i$ of one vertex from each color class, and these $t_{i,j}$ are called the barycentric coordinates of $y_j$. One says that $x_1 \in \sigma_1, \ldots, x_r \in \sigma_r$ have equal barycentric coordinates provided that $t_{i,j} = t_i$ is independent of $x_j$ for each $1 \leq i \leq n$.

**Theorem 6.4.** Let $r \geq 2$, $n = (r-1)d + 1$, and $N = rm - 1$. Suppose that $\Delta_N$ is partitioned by $n$ color classes $C_1, \ldots, C_n$ with $r$ points each. Then for any affine map $f : \Delta_N \to \mathbb{R}^d$, there exist points $x_1, \ldots, x_r$ with equal barycentric coordinates from pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ such that $f(x_1) = f(x_2) = \cdots = f(x_r)$. This result also holds for continuous maps provided $r + 1$ is a prime power.

Using an argument analogous to that of Sarkaria’s Linear Borsuk–Ulam, it was shown in [23] that partitions as in Theorem 6.4 do not exist for almost any affine map $f : \Delta_{rn-1} \to \mathbb{R}^d$ if $n < (r-1)d + 1$. As with our proof of Fourier genericity, one sees by the same observations as in Remark [4] that the lack of these partitions for continuous maps in these dimensions is also typical. We shall therefore call such continuous maps Soberón generic.

Removing $dr$ color classes in Soberón’s result, we have the following $P_r$-variant in even dimensions:

**Theorem 6.5.** Let $r \geq 3$, $n = (r-2)d + 1$, and $N = rm - 1$, where $d$ is even. Suppose that $\Delta_N$ is partitioned by $n$ color classes $C_1, \ldots, C_n$ with $r$ points each. Then for almost any affine map $f : \Delta_N \to \mathbb{R}^d$, there exist points $x_1, \ldots, x_r$ with equal barycentric coordinates from pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ so that $f(x_1), \ldots, f(x_r)$ are the vertices of a regular $r$-gon. This also holds for Soberón generic continuous maps $f$ when $r$ is an odd prime.

**Proof of Theorem 6.5.** Again, we follow the proof of Theorems 8.1 and 8.3 of [7] using constraints. Let $v_1, \ldots, v_{N+1}$ denote the vertices of $\Delta_N$. For each $1 \leq i \leq n$ and $x = \sum_{j=1}^{N} t_j v_j$, define $a_i : \Delta_N \to \mathbb{R}$ by $a_i(x) = \sum_{v \in C_i} t_j$. For given $\{x_g\}_{g \in \mathbb{Z}_r}$ from pairwise disjoint $\sigma_g$, the argument there shows that if the $a_i(x_g)$ are constant for each $2 \leq i \leq n$, then (1) each $\sigma_g$ is a $n$-rainbow face and (2) the barycentric coordinates of the $x_g$ are all equal (with $t_{i,g} = a_i(x_g)$). Thus one needs to annihilate $r-1$ coefficients for each $a_i$, while for $f : \Delta_N \to \mathbb{C}^d$, we prescribe $c_{i,g} = 0$ for all $g \in \mathbb{Z}_r - \{0,1\}$ and all $1 \leq i \leq d$ as before. This can be guaranteed for all $r \geq 3$ in the affine case when $N = 2d(r-2) + n(r-1) = rm - 1$ by Theorem 3.1 and for odd primes in the continuous case by Proposition 5.1. As $n < 2d(r-1) + 1$, the $f(x_g)$ do not collapse to a single point if they are assumed Soberón generic, hence are the vertices of some $P_r$ which as before lies in some complex 1-flat. □

Note that $n = d + 1$ in Theorem 6.5 when $r = 3$. This matches that of the Bárány–Larman conjecture, which for $r = 3$ remains open for all $d > 2$, and topologically open if $d = 2$. An illustration of our affine planar version is given below:
7. Question 1 for Regular Simplices

We conclude with a consideration of Question 1 when \( P(r, r - 1) = \Delta_{r-1} \) is itself a regular \((r - 1)\)-simplex. As observed by Florian Frick, standard constructions for Tverberg theorems yield an exact value when \( r = 3 \):

Proposition 7.1. \( N(\Delta_{2}; d) = 4 \) for all \( d \geq 2 \).

Unlike the Topological Tverberg Theorem, however, these constructions fail for all \( r > 3 \), including when \( r \) is a prime power. Namely, suppose that \( N \) and \( d \) are arbitrary and that we seek a \( \Delta_{r-1} \)-partition for some \( f : \Delta_N \to \mathbb{R}^d \). The standard approach here would be to define \( D : (\Delta_N^{\times r})_\Delta \to \mathbb{R}^{\binom{r}{2}} \) by \( x = (x_1, \ldots, x_r) \mapsto \| f(x_i) - f(x_j) \| \) for each \( 1 \leq i < j \leq k \). Permutation of the indices produces a free action of the symmetric group \( \mathcal{S}_r \) on \( (\Delta_N^{\times (r-1)})_\Delta \), as well as on \( \mathbb{R}^{\binom{r}{2}} = \{(x_{i,j})_{1 \leq i < j \leq k} \mid x_{i,j} \in \mathbb{R}\} \) (though not free). As usual, one wants the image of \( (\Delta_N^{\times (r-1)})_\Delta \) to intersect the thin diagonal \( \delta \) of \( \mathbb{R}^{\binom{r}{2}} \) consisting of all \( x = (x_{i,j})_{1 \leq i < j \leq k} \) with equal \( x_{i,j} \). Composing \( D \) with the projection of \( \mathbb{R}^{\binom{r}{2}} \) onto the orthogonal complement \( W := \delta^\perp \) produces a \( \mathcal{S}_r \)-equivariant map \( D : (\Delta_N^{\times r})_\Delta \to W \), a zero of which represents some \( x_1, \ldots, x_r \in \Delta_N \) with pairwise disjoint support for which the distances \( \| f(x_i) - f(x_j) \| \) are equal for all \( 1 \leq i < j \leq k \). Thus one has a \( \Delta_{r-1} \)-partition provided this distance is non-zero and a Tverberg \( r \)-partition otherwise. Dimensional considerations yield the following:

Conjecture 2. Let \( r \geq 3 \). Then \( N(\Delta_{r-1}; d) = \binom{r+1}{2} - 2 \) for all \( d \geq r - 1 \).

The map \( D \) must vanish when \( r = 3 \) and \( N = 4 \), since in this case \( W \) is the standard representation of \( \mathcal{S}_3 \) (see, e.g., [16, Corollary 3.4]). The distances \( \| f(x_i) - f(x_j) \| \) are all equal and cannot be zero because \( 4 < T(3, d) \), so the \( f(x_i) \) are the vertices of a regular \( 2 \)-simplex. On the other hand, that \( N_{\Delta_2; d} \geq 4 \) follows by considering 4 coplanar points in \( \mathbb{R}^d \) and applying \( N_{\Delta_2} = 4 \). Thus \( N(\Delta_2; d) = 4 \) as well.

If \( N = \binom{r+1}{2} - 2 \) and \( r \geq 4 \), however, \( \mathcal{S}_r \)-equivariant maps \( (\Delta_N^{\times r})_\Delta \to W \) without zeros always exist. This can be seen directly by considering the above construction as applied to almost any affine map \( f : \Delta_N \to \mathbb{R}^{r-2} \). As \( \binom{r+1}{2} - 2 < T(r, r - 2) \), \( f \) has no \( r \)-Tverberg partition, so the vanishing of the resulting \( D \) would yield a regular \((r - 1)\)-simplex in \( \mathbb{R}^{r-2} \).
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