INITIAL VALUE PROBLEM FOR FRACTIONAL VOLterra
INTEGRO-DIFFERENTIAL EQUATIONS
WITH CAPUTO DERIVATIVE

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Dedicated to Tomás Caraballo on his 60th birthday.

ABSTRACT. In this paper, we consider the time-fractional Volterra integro-differential equations with Caputo derivative. For globally Lipschitz source term, we investigate the global existence for a mild solution. The main tool is to apply the Banach fixed point theorem on some new weighted spaces combining some techniques on the Wright functions. For the locally Lipschitz case, we study the existence of local mild solutions to the problem and provide a blow-up alternative for mild solutions. We also establish the problem of continuous dependence with respect to initial data. Finally, we present some examples to illustrate the theoretical results.

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1. Introduction. Let $T$ be a positive number and $D$ be an open, bounded and connected domain in $\mathbb{R}^N$, $N \geq 1$ with a smooth boundary $\partial D$. In this section, we consider the semilinear fractional Volterra integro-differential with Neumann condition as follows

$$
\begin{align*}
\mathcal{D}_t^\alpha u &= \Delta u + F(x, t, u) + \int_0^t g(t - s, u(s)) \, ds \quad \text{in} \ D \times (0, T), \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \ \partial D \times (0, T), \\
u(x, 0) &= u_0(x),
\end{align*}
$$

(1.1)

where $0 < \alpha < 1$, $\mathcal{D}_t^\alpha$ is the Caputo fractional derivative operator of order $\alpha$. When the function $w$ is absolutely continuous in time, the definition in [20] reduces to the traditional form

$$
\mathcal{D}_t^\alpha w(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \frac{dw}{ds} \, ds,
$$

(1.2)

where $\Gamma$ is the Gamma function and $\frac{dw}{ds}$ is the first order integer derivative of function $w(s)$ with respect to independent variable $s$. The function $u_0$ is the initial state, $F, g$ are defined later. The first equation of (1.1) has many physical applications and arises in such problems as heat flow in materials with memory [10, 11, 22].

Our strong motivation for studying Problem (1.1) arises from anomalous diffusion in fractial media. Nowadays fractional PDEs take an important position in describing the phenomena in the fields such as physics, biology and chemistry, where under certain circumstances integer-order PDEs cannot work [25, 19, 30]. So, PDEs with fractional derivatives are a generalization of equations with integer-order partial derivatives and a subject of strong theoretical and practical interest. If $g = 0$, Problem (1.1) with various forms of source function $F$ has been very much studied. Y. Zhou, J. Manimaran, L. Shangerganesh and A. Debbouche [35] introduced a time-fractional Keller-Segel model with Caputo fractional derivative for the time. L. Peng, Y. Zhou and A. Debbouche [24] investigated the problem of solutions to the time-fractional Navier-Stokes equations with Caputo derivative operators. A. Viana [32] studied the local well-posedness for the Cauchy problem of a semilinear fractional diffusion equation. B. Andrade et al [3] established a semilinear fractional differential equations with critical nonlinearities.

Some works on Problem (1.1) with classical derivative of integer order has been considered in recent works [33, 1, 2]. Volterra integro-differential equations arise frequently in the study of natural phenomena where certain memory effects are taken into account. There are many works concerning some partial differential equation with memory. For instance, we can mention:

- Reaction-diffusion problems with memory [14, 21, 31, 1, 33] and references therein.
- Wave equations with memory [15, 16] and references therein.
- Several kinds of plate equations with memory [2, 12, 13] and Navier-Stokes equation [7, 8] with memory and references therein.

As for semilinear Volterra integro-differential equations with integer order derivative, we can list some interesting works. The authors in [18] considered the following Volterra integro-differential equation. The authors studied in [29] the fractional Volterra integro-differential equation with $\psi$-Hilfer fractional derivative. In [4] it
is proved the existence of solutions of certain kinds of nonlinear fractional integro-differential equations in Banach spaces. Some other models on Volterra equations were studied in [26, 27, 17].

To the best of our knowledge, there are not any results for Problem (1.1). The results of Volterra diffusion equations with Caputo derivative is still limited. Our first goal in this paper is to investigate the global solution under the global Lipschitz case of $F$ and $g$. Our second aim is to study the existence, uniqueness, continuous dependence and a blow-up alternative for mild solutions of Problem (1.1). Our study seems to be a continuation of the previous results of B. Andrade and A. Viana [1].

The paper is organized as follows. In Section 2, notations and assumptions are given and we also consider the formulation of a solution to Problem (1.1). In Section 3, we show the global well-posed results in the weighted space under globally Lipschitz source. In Section 4, under locally Lipschitz source term $F$, we obtain a local existence of solutions. The existence of a unique continuation and a blow-up alternative for the mild solution of Problem (1.1) are also given in Section 4. Finally, in Section 5, we show a numerical example to illustrate our theoretical results.

2. Notation and preliminaries. Given two positive quantities $y, z$, we write $y \lesssim z$ if there exists a constant $C > 0$ such that $y \leq Cz$. Let us recall that the spectral problem

$$\begin{align*}
\{ A\phi_j(x) = \lambda_j \phi_j(x), & \quad x \in D, \\
\phi_j(x) = 0, & \quad x \in \partial D,
\end{align*}$$

(2.1)

where $A$ be a positive-definite, unbounded, self-adjoint operator with compact inverse. It admits a family of eigenvalues as follows

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_j \leq \ldots \nearrow \infty.$$ 

For each number $s \geq 0$, we define

$$\mathbb{H}^s(D) = \mathcal{D}(A^s) := \left\{ v = \sum_{j=1}^{\infty} v_j \phi_j \in L^2(D) : \|v\|^2_{\mathbb{H}^s(D)} = \sum_{j=1}^{\infty} v_j^2 \lambda_j^s < \infty \right\},$$

where $v_j = \int_D v(x) \phi_j(x) dx$.

Let $\mathbb{H}^{-s}(D)$ be the dual space of $\mathbb{H}^s$ which corresponds to the dual inner product $(\cdot, \cdot)_{-s,s}$. Then, the operator $A^s : \mathbb{H}^s(D) \to \mathbb{H}^{-s}(D)$ of the fractional power $s$ can be defined by

$$A^s v := \sum_{j=1}^{\infty} \lambda_j^s (v, \phi_j)_{-s,s} \phi_j, \quad \forall v \in \mathbb{H}^s.$$ 

For any $\eta > 0$, we introduce the following space

$$C^\eta([0,T]; L^2(D)) = \left\{ w \in C([0,T]; L^2(D)) : \sup_{0 \leq t < s \leq T} \frac{\|w(\cdot, t) - w(\cdot, s)\|_{L^2(D)}}{|t - s|^{\eta}} < \infty \right\}.$$ 

**Definition 2.1.** We introduce the Mittag-Leffler operators. Let $-\Delta : H_0^1(D) \to L^2(D)$ be the infinitesimal generator of an analytic semigroup $\{ e^{t\Delta}, \ t \geq 0 \}$. Then, for each $\alpha \in (0, 1)$, we define the Mittag-Leffler families $E_{\alpha,1}(t^\alpha \Delta)$ and $E_{\alpha, \alpha}(t^\alpha \Delta)$ as follows

$$E_{\alpha,1}(t^\alpha \Delta) = \int_0^\infty M_{\alpha}(r)e^{t^\alpha \Delta} dr,$$ 

(2.2)
and
\[ E_{\alpha,\alpha}(t^\alpha \Delta) = \int_0^\infty \omega r M_\alpha(r) e^{rt^\alpha \Delta} dr. \] (2.3)

Here \( M_\alpha \) denotes the Wright type function introduced by Mainardi in [23]
\[ M_\alpha(r) = \sum_{n=0}^{\infty} \frac{r^n}{n! \Gamma(1-\alpha(1+n))}, \quad r \in \mathbb{C}. \]
This function is an entire function on \( \mathbb{C} \).

**Proposition 2.1.** (See [23]) For \( \alpha \in (0, 1) \) and \( \theta > -1 \). Then the following properties hold:
\[ M_\alpha(r) \geq 0, \quad \forall r \geq 0, \] and
\[ \int_0^\infty r^\theta M_\alpha(r) dr = \frac{\Gamma(\theta + 1)}{\Gamma(\theta \alpha + 1)}, \quad \forall \theta > -1. \] (2.4)

**Lemma 2.1.** (See [28]) For \( \lambda > 0, \alpha > 0 \) and positive integer \( m \in \mathbb{N} \), we have
\[ \frac{d^m}{dt^n} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \] (2.5)
\[ \frac{d}{dt} \left( t E_{\alpha,2}(-\lambda t^\alpha) \right) = E_{\alpha,1}(-\lambda t^\alpha), \] (2.6)
\[ \frac{d}{dt} \left( t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \right) = -t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha). \] (2.7)

The above settings can be found in [5] (Section 3) and [6] (Section 2). Let us recall the important \( L^p - L^q \) estimates for the Neumann heat semigroup on bounded domains (see e.g., [34]).

**Lemma 2.2.** Suppose \( \{e^{t\Delta}\}_{t \geq 0} \) is the Neumann heat semigroup in \( D \), and \( \lambda_1 > 0 \) denote the first nonzero eigenvalue of \( -\Delta \) in \( D \) under Neumann boundary conditions. Then there exist \( k_1, k_2 > 0 \) which only depend on \( D \) such that the following properties hold:
\[ \|e^{t\Delta}w\|_{L^p(D)} \leq k_1 \|w\|_{L^q(D)} \] for all \( w \in L^q_0(D); \)
\[ \|\nabla e^{t\Delta}w\|_{L^p(D)} \leq k_2 (1 + t^{-\frac{N}{4}}(\frac{4}{\beta} - \frac{1}{2})) e^{-\lambda_1 t} \|w\|_{L^q(D)} \] for all \( t > 0, \] (2.8)
}(2.9)

\[ \text{for each } w \in L^q(D). \]

**Lemma 2.3.** One has
\[ L^p(D) \hookrightarrow D(A^\beta), \quad \text{if} \quad -\frac{N}{4} < \beta \leq 0, \quad p \geq \frac{2N}{N - 4\beta}, \]
\[ D(A^\alpha) \hookrightarrow L^p(D), \quad \text{if} \quad 0 \leq \alpha < \frac{N}{4}, \quad p \leq \frac{2N}{N - 4\alpha}. \] (2.10)

By applying the Duhamel principle, we obtain the following definition for mild solution of Problem (1.1).
Definition 2.2. The function $u$ is called a mild solution of Problem (1.1) if it satisfies that

$$u(t) = E_{\alpha,1}(t^\alpha \Delta)u^0 + \int_0^t (t - r)^{\alpha-1}E_{\alpha,\alpha}((t - r)^\alpha \Delta)F(r, u(r))dr$$

$$+ \int_0^t (t - r)^{\alpha-1}E_{\alpha,\alpha}((t - r)^\alpha \Delta)\int_0^r g(r - \nu, u(\nu))d\nu dr. \quad (2.11)$$

3. Global existence results under a globally Lipschitz case. In this section, we derive the global results under the globally Lipschitz assumption on the nonlinear source function $F$. Let $F$ and $g$ satisfy that

$$\|F(u) - F(v)\|_{L^p(D)} \leq K_F\|u - v\|_{L^p(D)}, \quad (3.1)$$

and

$$\|g(u) - g(v)\|_{L^p(D)} \leq K_g\|u - v\|_{L^p(D)}, \quad (3.2)$$

where $\sigma, p, K_F, K_g$ are postive constants. Our objective in this section is to present the well-posedness of the problem. Here, $H_{b,m}((0, T]; L^p(D))$ denotes the weighted space of all functions $v \in C((0, T]; L^p(D))$ such that

$$\|v\|_{H_{b,m}((0, T]; L^p(D))} := \sup_{t \in [0, T]} t^b e^{-mt} \|v(t, \cdot)\|_{L^p(D)} < \infty,$$

where $m > 0$. First we recall the following lemma which will be useful in our main results (this lemma can be found in [9], Lemma 8, page 9).

Lemma 3.1. Let $a > -1$, $b > -1$ such that $a + b \geq -1$, $h > 0$ and $t \in [0, T]$. For $\mu > 0$, the following limit holds

$$\lim_{\mu \to \infty} \left( \sup_{t \in [0, T]} t^h \int_0^1 s^\alpha (1 - s)^b e^{-\mu t(1-s)}ds \right) = 0.$$

Now, we can introduce the main contributions of this work. Our main results address the existence and regularity of the mild solution.

Theorem 3.1. Let $0 < \alpha < 1$. Let $F$ and $g$ be as in (3.1) and (3.2). Let $u^0 \in \mathcal{D}(\mathcal{A}^d)$ for any $d > \frac{N(p-2)}{4p} - 2$. Let $\alpha$ be

$$\frac{1}{\sigma} - \frac{1}{p} < \frac{2\alpha}{d}.$$

Let $b$ satisfies that

$$\frac{d}{2} \left( \frac{1}{\sigma} - \frac{1}{p} \right) + 1 - \alpha < b < 1. \quad (3.3)$$

Then Problem (1.1) has a unique solution $u$ in $H_{b,m}((0, T]; L^p(D))$ with some $m_0 > 0$.

Remark 3.1. Let us assume $F$ and $g$ are globally Lipschitz in $L^2(D)$, i.e, in (3.1) and (3.2), we take $p = \sigma = 2$. In order to establish the global result, we only use the following norm on $C^m(0, T; L^2(D))$. If $\sigma$ and $q$ are not 2, it is impossible to use the space solution $C^m(0, T; L^2(D))$. Theorem 3.1 provides a new space solution $H_{b,m}((0, T]; L^p(D))$ in order to investigate the global existence result.
In what follows, we shall prove the existence of a unique solution of Problem (1.1). This is based on the Banach fixed-point theorem. Since the Sobolev embedding

$$D(A^m) \hookrightarrow L^p(D), \quad \text{if} \quad 0 \leq m < \frac{N}{4}, \quad p \leq \frac{2N}{N - 4m},$$

and set $\bar{p} = \frac{N(p - 2)}{4p}$, we know that $D(A^{\bar{p}}) \hookrightarrow L^p(D)$. So, we find that

$$\left\| E_{\alpha,1}(-t^\alpha A)u_0 \right\|_{L^p(D)}^2 \leq \left\| E_{\alpha,1}(-t^\alpha A)u_0 \right\|_{D(A^{\bar{p}})}^2 \leq \sum_{j=1}^\infty (u_0^0, \phi_j)^2 (E_{\alpha,1}(-\lambda_j t^\alpha))^2 \tilde{\lambda}_j^p \leq \sum_{j=1}^\infty (u_0^0, \phi_j)^2 \frac{C^2}{(1 + \lambda_j t^\alpha)^{2\gamma}} \tilde{\lambda}_j^p \leq C^2 t^{-2\alpha\gamma} \sum_{j=1}^\infty (u_0^0, \phi_j)^2 \lambda_j^{-2\gamma} = C^2 t^{-2\alpha\gamma} \left\| u_0^0 \right\|_{D(A^{\bar{p}-2\gamma})}^2.$$

It follows from the condition $b > \alpha\gamma$ and $d + 2 > \frac{p}{2}$ that

$$t^b e^{-mt} \left\| E_{\alpha,1}(-t^\alpha A)u_0 \right\|_{L^p(D)} \leq C t^{b - \alpha\gamma} e^{-mt} \left\| u_0^0 \right\|_{D(A^{\bar{p}-2\gamma})} \leq C T^{b - \alpha\gamma} \left\| u_0^0 \right\|_{D(A^{\bar{p}})} \leq T^{b - \alpha\gamma} \left\| u_0^0 \right\|_{D(A^{\bar{p}})}. \quad (3.5)$$

From the latter inequality, we deduce that $u_0^0 \in H_{b,m}(0, T]; L^p(D))$. Now, we need to estimate the term $J_2 = \left\| J_2(t)(-t^\alpha A)E_{\alpha,1}(-(t-r)^\alpha \Delta)\left[F(w_1(r)) - F(w_2(r))\right]dr \right\|_{L^p(D)}$.

Due to (2.3), we deduce that

$$J_2 \leq \alpha \int_0^t (t-r)^{-\alpha - 1} \left\| \int_0^\infty \eta M_\alpha(\eta) e^{\eta(t-r)^\alpha \Delta} \left[F(w_1(r)) - F(w_2(r))\right]d\eta \right\|_{L^p(D)} \, dr \leq \alpha \int_0^t (t-r)^{-\alpha - 1} \left\| \int_0^\infty \eta M_\alpha(\eta) e^{\eta(t-r)^\alpha \Delta} \left[F(w_1(r)) - F(w_2(r))\right] \right\|_{L^p(D)} \, d\eta dr. \quad (3.6)$$

For any $w \in L^p(D)$, we use the following inequality

$$\left\| e^{t^\alpha w} \right\|_{L^p(D)} \leq k_1 e^{-\lambda_1 t} \left(1 + t^{-\frac{N}{2}} \left(\frac{1}{\lambda_1} \right)^{\frac{N}{2}}\right) \left\| w \right\|_{L^p(D)}, \quad \text{for all} \quad t > 0, \quad (3.7)$$
in order to obtain that
\[
\left\|e^{(t-r)^{\alpha} \Delta} [F(w_1(r)) - F(w_2(r))] \right\|_{L^p(D)} \\
\leq k_1 \left(1 + \eta \left(\frac{1}{2} - \frac{1}{q} \right) (t-r)^{-\frac{1}{2} (\frac{1}{p} - \frac{1}{q})} \right) \|F(w_1(r)) - F(w_2(r))\|_{L^q(D)} \\
\leq k_1 K_f (t-r)^{-\frac{1}{2} (\frac{1}{p} - \frac{1}{q})} \|w_1(r) - w_2(r)\|_{L^p(D)} \\
+ k_1 K_f \eta^{-\frac{1}{2} (\frac{1}{p} - \frac{1}{q})} (t-r)^{-\frac{1}{2} (\frac{1}{p} - \frac{1}{q})} \|w_1(r) - w_2(r)\|_{L^p(D)}.
\]
(3.8)

Hence, we derive that
\[
\int_0^\infty \eta \mathcal{M}_\alpha(\eta) \left\|e^{\eta (t-r)^{\alpha} \Delta} [F(w_1(r)) - F(w_2(r))] \right\|_{L^p(D)} d\eta \\
\leq k_1 K_f (t-r)^{-\frac{1}{2} (\frac{1}{p} - \frac{1}{q})} \|w_1(r) - w_2(r)\|_{L^p(D)} \int_0^\infty \eta \mathcal{M}_\alpha(\eta) d\eta \\
+ k_1 K_f (t-r)^{-\frac{1}{2} (\frac{1}{p} - \frac{1}{q})} \|w_1(r) - w_2(r)\|_{L^p(D)} \int_0^\infty \eta^{-\frac{1}{2} (\frac{1}{p} - \frac{1}{q})} \mathcal{M}_\alpha(\eta) d\eta \\
= \frac{k_1 K_f \Gamma(2)}{\Gamma(\alpha + 1)} (t-r)^{-\frac{1}{2} (\frac{1}{p} - \frac{1}{q})} \|w_1(r) - w_2(r)\|_{L^p(D)} \\
+ k_1 K_f \frac{\Gamma(2 - \frac{\alpha}{\Gamma(\alpha + 1))}}{\Gamma(2 - \frac{\alpha}{\Gamma(\alpha + 1))}} (t-r)^{-\frac{1}{2} (\frac{1}{p} - \frac{1}{q})} \|w_1(r) - w_2(r)\|_{L^p(D)}.
\]
(3.9)

where \(\sigma \leq p\). Set \(\mathcal{K}_0 = \frac{k_1 K_f \Gamma(2)}{\Gamma(\alpha + 1)} + k_1 K_f \frac{\Gamma(2 - \frac{\alpha}{\Gamma(\alpha + 1))}}{\Gamma(2 - \frac{\alpha}{\Gamma(\alpha + 1))}}\), we have the following estimate
\[
\mathcal{J}_2 \leq \alpha \mathcal{K}_0 \int_0^t (t-r)^{\alpha-1-\frac{1}{2} (\frac{1}{p} - \frac{1}{q})} \|w_1(r) - w_2(r)\|_{L^p(D)} dr.
\]
(3.10)

Indeed, for \(w_1, w_2 \in H_{0,m}((0, T]; L^p(D))\), we have
\[
\|\mathcal{F}_1 w_1 - \mathcal{F}_1 w_2\|_{H_{0,m}((0, T]; L^p(D))} \\
= \sup_{t \in (0,T]} t^{b} e^{-mt} \left\|\int_0^t (t-r)^{\alpha-1} E_{\alpha, \alpha} (- (t-r)^{\alpha} \Delta) [F(w_1(r)) - F(w_2(r))] dr \right\|_{L^p(D)} \\
\leq \alpha \mathcal{K}_0 \sup_{t \in (0,T]} t^{b} e^{-mt} \int_0^t (t-r)^{\alpha-1} \|w_1(r) - w_2(r)\|_{L^p(D)} dr \\
\leq \alpha \mathcal{K}_0 \|v_1 - v_2\|_{H_{0,m}((0, T]; L^p(D))} \sup_{t \in (0,T]} t^{b} \int_0^t (t-r)^{\alpha-1} \|w_1(r) - w_2(r)\|_{L^p(D)} dr.
\]
(3.11)

So, we find that
\[
\|\mathcal{F}_1 w_1 - \mathcal{F}_1 w_2\|_{H_{0,m}((0, T]; L^p(D))} \leq \mathcal{K}_m \|w_1 - w_2\|_{H_{0,m}((0, T]; L^p(D))},
\]
(3.12)

where
\[
\mathcal{K}_m = \alpha \mathcal{K}_0 \sup_{t \in (0,T]} t^{b} \int_0^t (t-r)^{\alpha-1} \|w_1(r) - w_2(r)\|_{L^p(D)} dr.
\]

By a similar argument as above, we obtain
\[
\|\mathcal{F}_2 w_1 - \mathcal{F}_2 w_2\|_{H_{0,m}((0, T]; L^p(D))} \leq \alpha \mathcal{K}_0 T \mathcal{K}_m \|w_1 - w_2\|_{H_{0,m}((0, T]; L^p(D))}.
\]
(3.13)
This together with (3.12), we imply the following bound
\[
\|F w_1 - F w_2\|_{H_{b,m}((0,T];L^p(D))} \\
\leq \|F_1 w_1 - F_1 w_2\|_{H_{b,m}((0,T];L^p(D))} + \|F_2 w_1 - F_2 w_2\|_{H_{b,m}((0,T];L^p(D))} \\
\leq (\alpha K_0 T + \alpha K_1) \|w_1 - w_2\|_{H_{b,m}((0,T];L^p(D))}.
\] (3.14)

From the conditions of \(\alpha, b, \sigma, p\) as introduced in Theorem 3.1, we find that
\(b + \alpha - \frac{1}{2} \left(\frac{1}{\sigma} - \frac{1}{p}\right) > 0, \alpha - 1 - \frac{d}{2} \left(\frac{1}{\sigma} - \frac{1}{p}\right) > -1, -b > -1, \alpha - 1 - \frac{d}{2} \left(\frac{1}{\sigma} - \frac{1}{p}\right) - b > -1\).

Applying now Lemma 3.1, we obtain that
\[
\lim_{m \to \infty} \mathcal{K}_m := \lim_{m \to \infty} \left( \sup_{t \in (0,T]} \int_0^t (t-r)^{\frac{\alpha-1}{2} \left(\frac{2}{\sigma} - \frac{1}{p}\right)} r^{-b} e^{-m(t-r)} dr \right)
\]
\[
= \lim_{m \to \infty} \left( \sup_{t \in (0,T]} \int_0^t (1-r)^{\frac{\alpha-1}{2} \left(\frac{2}{\sigma} - \frac{1}{p}\right)} r^{-b} e^{-mt(1-r)} dr \right)
\]
\[
= 0.
\]
Hence, there exists a positive \(m_0 > 0\) such that
\[
\alpha K_0 T \mathcal{K}_m < 1.
\]

It then follows from (3.14) that \(F\) is a contraction mapping on \(H_{b,m}((0,T];L^p(D))\). This together with (3.5) lead to \(F w \in H_{b,m_0}((0,T];L^p(D))\) if \(w \in H_{b,m_0}((0,T];L^p(D))\). Hence, we conclude that \(F\) has a fixed point \(u\) in \(H_{b,m_0}((0,T];L^p(D))\), i.e., \(u\) is a unique mild solution to Problem (1.1). \(\square\)

4. Local existence results and blow up continuation. In this section, we state and prove a local well-posedness result for Problem (1.1), that is, existence, uniqueness and continuous dependence upon the initial data.

4.1. Local well-posedness.

**Lemma 4.1.** Let \(0 < \alpha < 1\) and \(\beta > 0\). For any \(v \in H^q(D)\) and \(\sigma \leq q\) then
\[
\|E_{\alpha,\beta}(-r^{\alpha} \Delta) v\|_{H^q(D)} \leq C_{\alpha,\beta} r^{-\alpha(q-\sigma)/2} \|v\|_{H^q(D)},
\] (4.1)

**Proof.** Let us assume that \(v\) has a Fourier expansion in the form \(v = \sum_{j=1}^\infty < v, \phi_j > \phi_j\). Hence, thanks to the spectral expansion of the operator \(E_{\alpha,\beta}\) and using Paseval’s equality, we obtain
\[
\|E_{\alpha,\beta}(-r^{\alpha} \Delta) v\|_{H^q(D)}^2 = \sum_{j=1}^\infty \lambda_j^q |E_{\alpha,\beta}(-r^{\alpha} \lambda_j)|^2 < v, \phi_j >^2
\]
\[
\leq |C_{\alpha,\beta}|^2 \sum_{j=1}^\infty \frac{\lambda_j^q}{(1 + \lambda_j r^{\alpha})^2} < v, \phi_j >^2,
\] (4.2)

where we used the fact that (as in [28])
\[
|E_{\alpha,\beta}(-z)| \leq \frac{C_{\alpha,\beta}}{1 + |z|^s}, z > 0.
\]
Noting that \(0 < \frac{q-\sigma}{r} < 1\), we find that
\[
(1 + \lambda_j r^{\alpha})^2 = (1 + \lambda_j r^{\alpha})^{q-\sigma}(1 + \lambda_j r^{\alpha})^{2-q+\sigma} \geq (1 + \lambda_j r^{\alpha})^{q-\sigma} \geq \lambda_j^{q-\sigma} r^{(q-\sigma)}.
\]
This together with (4.2) yields that
\[
\left\| E_{\alpha,\beta}(-t^{\alpha} \Delta) v \right\|_{H^s(D)}^2 \leq |C_{\alpha,\beta}|^2 t^{-\alpha(q-\sigma)} \sum_{j=1}^{\infty} \lambda_j^q < v, \phi_j >^2 \\
= |C_{\alpha,\beta}|^2 t^{-\alpha(q-\sigma)} \left\| v \right\|_{H^s(D)}^2,
\]
which allows to obtain the desired result.

**Lemma 4.2.** Let \( w_0 \in H^q(D) \) and \( u \in C([0, T_0], H^q(D)) \) such that \( \sup_{0 \leq t \leq T_0} \| u(t) - w_0 \|_{H^q(D)} \leq M \). If \( 0 \leq \sigma \leq q \) then
\[
\left\| \int_0^t (t - r)^{\alpha-1} E_{\alpha,\alpha} ((t - r)^{\alpha} \Delta) F(r, u(r)) dr \right\|_{H^s(D)} \\
\leq \overline{\mathcal{F}}_1 \left( 1 + \left( \| w_0 \|_{H^q(D)} + M \right)^\alpha \right) t^{\alpha - \alpha(q-\sigma)/2},
\]
and
\[
\left\| \int_0^t (t - r)^{\alpha-1} E_{\alpha,\alpha} ((t - r)^{\alpha} \Delta) \int_0^r g(r - \nu, u(\nu)) d\nu dr \right\|_{H^s(D)} \\
\leq \overline{\mathcal{F}}_2 \left( 1 + \left( \| w_0 \|_{H^q(D)} + M \right)^\delta \right) t^{\alpha + 1 - \alpha(q-\sigma)/2},
\]
where \( \overline{\mathcal{F}}_1 = B(\alpha, 1 - \alpha(q-\sigma)/2) C_{\alpha,\alpha} \mathcal{L}_f \) and \( \overline{\mathcal{F}}_2 = B(\alpha, 2 - \alpha(q-\sigma)/2) C_{\alpha,\alpha} \mathcal{L}_g \). Here \( C_{\alpha,\alpha} \) is defined as in the proof of Lemma 4.1.

**Proof.** Using Lemma (4.1), we find that
\[
\left\| \int_0^t (t - r)^{\alpha-1} E_{\alpha,\alpha} ((t - r)^{\alpha} \Delta) F(r, u(r)) dr \right\|_{H^s(D)} \\
\leq \int_0^t (t - r)^{\alpha-1} \left\| E_{\alpha,\alpha} ((t - r)^{\alpha} \Delta) F(r, u(r)) \right\|_{H^s(D)} dr \\
\leq C_{\alpha,\alpha} \int_0^t (t - r)^{\alpha-1} r^{\alpha(q-\sigma)/2} \| F(r, u(r)) \|_{H^s(D)} dr \\
\leq C_{\alpha,\alpha} \mathcal{L}_f \left( 1 + \left( \| w_0 \|_{H^q(D)} + M \right)^\rho \right) \left( \int_0^t (t - r)^{\alpha-1} r^{\alpha(q-\sigma)/2} dr \right),
\]
where we assume that \( \sigma - q < 2 < \frac{2}{\alpha} \). In the last inequality as above, we used the fact that
\[
\| F(r, u(r)) \|_{H^s(D)} \leq \mathcal{L}_f \left( 1 + \| u(., r) \|_{H^q(D)}^\rho \right) \leq \mathcal{L}_f \left( 1 + \left( \| w_0 \|_{H^q(D)} + M \right)^\rho \right),
\]
for any \( r \in [0, T_0] \). The integral quantity on the right hand side of (4.6) is equal to
\[
\int_0^t (t - r)^{\alpha-1} r^{\alpha(q-\sigma)/2} dr = t^{\alpha - \alpha(q-\sigma)/2} B(\alpha, 1 - \alpha(q-\sigma)/2).
\]
Combining (4.6) and (4.7), we obtain the first desired result. Now, we continue to show the second estimate. By a similar argument as above, we find that

\[
\left\| \int_{0}^{t} (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) \int_{0}^{r} g(r-\nu, u(\nu)) d\nu dr \right\|_{L^q(D)} \\
\leq \int_{0}^{t} (t-r)^{\alpha-1} \left\| E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) \int_{0}^{r} g(r-\nu, u(\nu)) d\nu \right\|_{L^q(D)} dr \\
\leq C_{\alpha,\alpha} \int_{0}^{t} (t-r)^{\alpha-1} r^{\alpha(q-\sigma)/2} \left\| \int_{0}^{r} g(r-\nu, u(\nu)) d\nu \right\|_{L^q(D)} dr \\
\leq C_{\alpha,\alpha} \mathcal{L}_g \left( 1 + \left( \|w_0\|_{H^q(D)} + M \right)^{\delta} \right) \left( \int_{0}^{t} (t-r)^{\alpha-1} r^{\alpha(q-\sigma)/2} \int_{0}^{r} dvdr \right) \\
= C_{\alpha,\alpha} \mathcal{L}_g \left( 1 + \left( \|w_0\|_{H^q(D)} + M \right)^{\delta} \right) \left( \int_{0}^{t} (t-r)^{\alpha-1} r^{1+\alpha(q-\sigma)/2} dvdr \right). \tag{4.8}
\]

It follows from

\[
\int_{0}^{t} (t-r)^{\alpha-1} r^{1+\alpha(q-\sigma)/2} dvdr = t^{\alpha+1+\alpha(q-\sigma)/2} B(\alpha, 2 - \alpha(q-\sigma)/2), \tag{4.9}
\]

that the desired results hold. \hfill \square

**Theorem 4.1.** Let \( 0 < \sigma < q, \rho \geq 1 \) and \( \delta \geq 1 \). Assume that \( F \) satisfies

\[
\left\| F(u) - F(v) \right\|_{H^q(D)} \leq \mathcal{C}_f \|u - v\|_{H^q(D)} \left( 1 + \|u\|_{H^q(D)}^{\rho-1} + \|v\|_{H^q(D)}^{\rho-1} \right), \tag{4.10}
\]
\[
\left\| F(u) \right\|_{H^q(D)} \leq \mathcal{C}_f \left( 1 + \|u\|_{H^q(D)}^{\rho} \right). \tag{4.11}
\]

Let \( g \) satisfy that

\[
\left\| g(u) - g(v) \right\|_{H^q(D)} \leq \mathcal{C}_g \|u - v\|_{H^q(D)} \left( 1 + \|u\|_{H^q(D)}^{\delta-1} + \|v\|_{H^q(D)}^{\delta-1} \right), \tag{4.12}
\]
\[
\left\| g(u) \right\|_{H^q(D)} \leq \mathcal{C}_g \left( 1 + \|u\|_{H^q(D)}^{\delta} \right). \tag{4.13}
\]

Assume that \( 0 < q - \sigma < 2 \). Then there exists \( T_0 > 0 \) such that Problem (1.1) has a unique local mild solution \( u \in C\left( [0, T_0]; H^q(D) \right) \).

**Proof.** Let \( 0 < M \leq 1 \). Let us define the following set

\[
\mathcal{B} = \left\{ u \in C\left( [0, T_0]; H^q(D) \right), \sup_{0 \leq t \leq T_0} \|u(t) - w_0\|_{H^q(D)} \leq M \right\}. \tag{4.14}
\]

It is not difficult to see that \( \mathcal{B} \) is a complete space. Let us define the operator \( \mathcal{B} : \mathcal{B} \to \mathcal{B} \) as follows

\[
\mathcal{B} u(t) = E_{\alpha,1}(t^{\alpha} \Delta) u_0 + \int_{0}^{t} (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) F(r, u(r)) dr \\
+ \int_{0}^{t} (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) \int_{0}^{r} g(r-\nu, u(\nu)) d\nu dr. \tag{4.15}
\]

Let us given \( u \in \mathcal{B} \) and \( 0 < t \leq t + h \leq T_0 \). By a similar way as in [28], it is easy to check that \( \lim_{h \to 0} \|u(t+h) - u(t)\|_{H^q(D)} = 0 \). If \( u \in \mathcal{B} \) then from Lemma 4.2,
we find that
\[
\|D u(t) - w_0\|_{\mathcal{H}^v(D)} \\
\leq \|E_{\alpha,1(t^\alpha \Delta)} u^0 - w_0\|_{\mathcal{H}^v(D)} + \left\|\int_0^t (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) F(r, u(r)) dr\right\|_{\mathcal{H}^v(D)} \\
+ \left\|\int_0^t (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) \int_0^r g(r-\nu, u(\nu)) d\nu dr\right\|_{\mathcal{H}^v(D)} \\
\leq \|E_{\alpha,1(t^\alpha \Delta)} u^0 - u^0\|_{\mathcal{H}^v(D)} + \|u^0 - w_0\|_{\mathcal{H}^v(D)} \\
+ \mathcal{F}_1 \left(1 + \left(\|w_0\|_{\mathcal{H}^v(D)} + \mathcal{M}\right)^{\delta}\right) t^{\alpha - \alpha(q-\sigma)/2} \\
+ \mathcal{F}_2 \left(1 + \left(\|w_0\|_{\mathcal{H}^v(D)} + \mathcal{M}\right)^{\delta}\right) t^{\alpha + 1 - \alpha(q-\sigma)/2}. \tag{4.16}
\]

Let us choose $T_0$ such that for any $t \in [0, T_0]$
\[
\|E_{\alpha,1(t^\alpha \Delta)} u^0 - u^0\|_{\mathcal{H}^v(D)} \leq \frac{\mathcal{M}}{4} \mathcal{F}_1 \left(1 + \left(\|w_0\|_{\mathcal{H}^v(D)} + \mathcal{M}\right)^{\delta}\right) t^{\alpha - \alpha(q-\sigma)/2} \\
\leq \frac{\mathcal{M}}{4} \tag{4.17}
\]

\[
\mathcal{F}_2 \left(1 + \left(\|w_0\|_{\mathcal{H}^v(D)} + \mathcal{M}\right)^{\delta}\right) T_0^{\alpha + 1 - \alpha(q-\sigma)/2} \leq \frac{\mathcal{M}}{4} \tag{4.18}
\]

\[
\mathcal{F}_1 \left(1 + \left(\|w_0\|_{\mathcal{H}^v(D)} + \mathcal{M}\right)^{\delta}\right) T_0^{\alpha - \alpha(q-\sigma)/2} \\
+ \mathcal{F}_2 \left(1 + \left(\|w_0\|_{\mathcal{H}^v(D)} + \mathcal{M}\right)^{\delta}\right) T_0^{\alpha + 1 - \alpha(q-\sigma)/2} < 1. \tag{4.19}
\]

By choosing $T_0$ such as above, we deduce that $\|D u(t) - w_0\|_{\mathcal{H}^v(D)} \leq \mathcal{M}$ for any $u \in \mathcal{B}$. Hence, $D u \in \mathcal{B}$ if $u \in \mathcal{B}$. Now let $v_1, v_2 \in \mathcal{B}$. For any $t \in [0, T_0]$, we obtain the following estimate
\[
\|D v_1(t) - D v_2(t)\|_{\mathcal{H}^v(D)} \\
= \left\|\int_0^t (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) \left(F(r, v_1(r)) - F(r, v_2(r))\right) dr\right\|_{\mathcal{H}^v(D)} \\
+ \left\|\int_0^t (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) \left(\int_0^r g(r-\nu, v_1(\nu)) d\nu - \int_0^r g(r-\nu, v_2(\nu)) d\nu\right) dr\right\|_{\mathcal{H}^v(D)} \\
\leq C_{\alpha,\alpha} \int_0^t (t-r)^{\alpha - 1 - \alpha(q-\sigma)/2} \left\|F(r, v_1(r)) - F(r, v_2(r))\right\|_{\mathcal{H}^v(D)} dr \\
+ C_{\alpha,\alpha} \int_0^t (t-r)^{\alpha - 1 - \alpha(q-\sigma)/2} \left\|\int_0^r g(r-\nu, v_1(\nu)) d\nu - \int_0^r g(r-\nu, v_2(\nu)) d\nu\right\|_{\mathcal{H}^v(D)} dr
\]
\[
L_f C_{\alpha,\alpha} \left[ 1 + 2 \left( M + \|w_0\|_{H^\sigma(D)} \right)^{\rho - 1} \right] \\
\times \mathcal{L}_g \left( 1 + 2 \left( \|w_0\|_{H^\sigma(D)} + M \right)^{\delta} \right) \\
\times \sup_{0 \leq r \leq T_0} \|v_1(r) - v_2(r)\|_{H^\sigma(D)} \int_0^t (t - r)^{\alpha - 1 - \alpha(q - \sigma)/2} dr \\
+ \mathcal{L}_g \left( 1 + 2 \left( \|w_0\|_{H^\sigma(D)} + M \right)^{\delta} \right) \\
\times \sup_{0 \leq r \leq T_0} \|v_1(\nu) - v_2(\nu)\|_{H^\sigma(D)} \left( \int_0^t (t - r)^{\alpha - 1 - \alpha(q - \sigma)/2} dr \right),
\]

where for any \( v_1, v_2 \in \mathcal{B} \), we used the following bound
\[
\left\| F(r, v_1(r)) - F(r, v_2(r)) \right\|_{H^\sigma(D)} \\
\leq \mathcal{L}_f \|v_1(r) - v_2(r)\|_{H^\sigma(D)} \left( 1 + \|v_1(r)\|_{H^\sigma(D)}^{\rho - 1} + \|v_2(r)\|_{H^\sigma(D)}^{\rho - 1} \right) \\
\leq \mathcal{L}_f \sup_{0 \leq r \leq T_0} \|v_1(r) - v_2(r)\|_{H^\sigma(D)} \left[ 1 + 2 \left( M + \|w_0\|_{H^\sigma(D)} \right)^{\rho - 1} \right],
\]

and
\[
\left\| \int_0^r (g(r - \nu, v_1(\nu)) d\nu - \int_0^r g(r - \nu, v_2(\nu)) d\nu \right\|_{H^\sigma(D)} \\
\leq \int_0^r \mathcal{L}_g \|v_1(\nu) - v_2(\nu)\|_{H^\sigma(D)} \left( 1 + \|v_1(\nu)\|_{H^\sigma(D)}^{\delta - 1} + \|v_2(\nu)\|_{H^\sigma(D)}^{\delta - 1} \right) d\nu \\
\leq \mathcal{L}_f \sup_{0 \leq r \leq T_0} \|v_1(r) - v_2(r)\|_{H^\sigma(D)} \left( 1 + 2 \left( \|w_0\|_{H^\sigma(D)} + M \right)^{\delta} \right) \int_0^r d\nu \\
= r \mathcal{L}_f \sup_{0 \leq r \leq T_0} \|v_1(r) - v_2(r)\|_{H^\sigma(D)} \left( 1 + 2 \left( \|w_0\|_{H^\sigma(D)} + M \right)^{\delta} \right).
\]

Combining (4.7), (4.9) and (4.20), we have that
\[
\left\| \mathcal{D} v_1(t) - \mathcal{D} v_2(t) \right\|_{H^\sigma(D)} \\
\leq T_1 \left[ 1 + 2 \left( \|w_0\|_{H^\sigma(D)}^{\rho - 1} \right) T_0^{\alpha - \alpha(q - \sigma)/2} \|v_1 - v_2\|_{C([0, T_0]; H^\sigma(D))} \right] \\
+ T_2 \left[ 1 + 2 \left( \|w_0\|_{H^\sigma(D)}^{\rho - 1} \right) T_0^{\alpha + 1 - \alpha(q - \sigma)/2} \|v_1 - v_2\|_{C([0, T_0]; H^\sigma(D))} \right],
\]

for any \( v_1, v_2 \in \mathcal{B} \) and for all \( t \in [0, T_0] \). From (4.19), we derive that \( \mathcal{D} \) is a contraction mapping on the space \( C([0, T_0]; H^\sigma(D)) \). From the Banach fixed point theorem, it turns out that has one single fixed point \( u \in \mathcal{B} \) which is a mild solution of (1.1).

\[\square\]

4.2. Continuation and blow-up alternative.

**Definition 4.1.** Given a mild solution \( u \in C([0, T_0]; H^\sigma(D)) \) of Problem (1.1), we say that \( \bar{u} \) is a continuation of \( u \) in \( [0, T_0] \) if \( u \in C([0, T_0]; H^\sigma(D)) \) is a mild solution for \( T_0 > T_0 \) and \( u(t) = \bar{u}(t) \) for any \( t \in [0, T_0] \).
Now, we infer the existence of a maximal time by the next theorem.

**Theorem 4.2.** Let $u$ be a mild solution of Problem (1.1) on $[0, T_0]$. Then there exists $\bar{T}_0 > T_0$ and a unique continuation $\tilde{u}$ of $u$ in $[0, \bar{T}_0]$.

**Proof.** Fix $0 < M \leq 1$ and take $\bar{T}_0 > T_0$ such that for $t \in [T_0, \bar{T}_0]$, we denote
\[
\mathcal{H}_M = \left\{ w \in C([0, T_0], \mathbb{H}^q(D)) : \|w(t) - u(T_0)\|_{\mathbb{H}^q(D)} \leq M, w(t) = u(t), t \in [0, T_0] \right\}.
\]

(4.24)

It is not difficult to find that $\mathcal{H}_M$ is a complete metric space with the norm of supremum in $\mathbb{H}^q(D)$. Let us define $\mathcal{P} : \mathcal{H}_M \to \mathcal{H}_M$ by
\[
\mathcal{P}w(t) = E_{\alpha,1}(t^\alpha \Delta)u^0 + \int_0^t (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^\alpha \Delta)F(r,w(r))dr
+ \int_0^t (r-t)^{\alpha-1} E_{\alpha,\alpha}((r-t)^\alpha \Delta)\int_r^t g(r-\nu,w(\nu))d\nu dr. \tag{4.25}
\]

If $v \in \mathcal{H}_M$ then it is obvious to obtain that $\mathcal{P}w(t) = u(t)$ for any $t \in [0, T_0]$. Let any $t \in [T_0, \bar{T}_0]$ and let any $w \in \mathcal{H}_M$, by some simple computations, we get
\[
\mathcal{P}w(t) - u(T_0) = \left( E_{\alpha,1}(t^\alpha \Delta) - E_{\alpha,1}(T_0^\alpha \Delta) \right)u^0
+ \int_0^{T_0} \left( (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^\alpha \Delta) - (T_0 - r)^{\alpha-1} E_{\alpha,\alpha}((T_0 - r)^\alpha \Delta) \right)
\times \int_0^r g(r-\nu,u(\nu))d\nu dr
+ \int_0^{T_0} \left( (r-t)^{\alpha-1} E_{\alpha,\alpha}((r-t)^\alpha \Delta) - (T_0 - r)^{\alpha-1} E_{\alpha,\alpha}((T_0 - r)^\alpha \Delta) \right)F(r,u(r))dr
+ \int_0^{T_0} (r-t)^{\alpha-1} E_{\alpha,\alpha}((r-t)^\alpha \Delta)\int_r^t g(r-\nu,w(\nu))d\nu dr
+ \int_0^{T_0} (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^\alpha \Delta)F(r,w(r))dr = (I) + (II) + (III) + (IV) + (V), \tag{4.26}
\]

where we have used the fact that
\[
u(T_0) = E_{\alpha,1}(T_0^\alpha \Delta)u^0 + \int_0^{T_0} (t-r)^{\alpha-1} E_{\alpha,\alpha}((T_0 - r)^\alpha \Delta)F(r,u(r))dr
+ \int_0^{T_0} (T_0 - r)^{\alpha-1} E_{\alpha,\alpha}((T_0 - r)^\alpha \Delta)\int_r^t g(r-\nu,w(\nu))d\nu dr. \tag{4.27}
\]

Now, we estimate the term $(I)$. Indeed, from the formula
\[
\frac{d}{dt} \left( E_{\alpha,1}(-\Delta t^\alpha v) \right) = -t^{\alpha-1} E_{\alpha,\alpha}(-\Delta t^\alpha)\Delta v,
\]
we obtain the following estimate

\[
\left\| (I) \right\|_{H^s(D)} = \left\| \int_t^T \xi^{\alpha-1} E_{\alpha,\alpha} (-\xi^\alpha \Delta) \Delta u^0 \, d\xi \right\|_{H^s(D)} \leq C_{\alpha,\alpha} \int_T^t \xi^{\alpha-1-\alpha(q+1-\sigma)/2} \left\| u^0 \right\|_{H^s(D)} \, d\xi \leq C_{\alpha,\alpha} \left( \mathcal{M} + \|u_0\|_{H^s(D)} \right) \int_T^t \xi^{\frac{\sigma}{2} - \frac{\sigma}{2}(q-\sigma)} \, d\xi \]

This above display together with the inequality \((a + b)^m \leq a^m + b^m, \, 0 \leq m \leq 1,\) leads to

\[
\left\| (I) \right\|_{H^s(D)} \leq \mathcal{M} \left( T_0 - T_0 \right) \frac{1}{2} - \frac{\sigma}{2}(q-\sigma),
\]

where we note that \(\frac{\sigma}{2} - \frac{\sigma}{2}(q-\sigma) > 0\) and \(\mathcal{M} = \frac{C_{\alpha,\alpha} \left( \mathcal{M} + \|u_0\|_{H^s(D)} \right)}{2 - \frac{\sigma}{2}(q-\sigma)}\). We continue to treat the quantity \((II)\). Thanks to the following fact

\[
\frac{d}{d\xi} \left( \xi^{\alpha-1} E_{\alpha,\alpha} (-\lambda^\alpha) \right) = \xi^\alpha E_{\alpha,\alpha-1} (-\lambda^\alpha),
\]

we deduce the following bound

\[
\left\| (II) \right\|_{H^s(D)} \leq \int_0^T \left( \xi^{\alpha-2} E_{\alpha,\alpha-1} (-\xi^\alpha \Delta) \right) \int_0^r g(r - \nu, u(\nu)) \, d\xi \, d\nu \leq \int_t^T \int_0^r \xi^{\alpha-2} E_{\alpha,\alpha-1} (-\xi^\alpha \Delta) \left\| g(r - \nu, u(\nu)) \right\|_{H^s(D)} \, d\xi \, d\nu \leq C_{\alpha,\alpha} \int_0^T \int_t^T \xi^{\alpha-2-\alpha(q-\sigma)/2} \left\| g(r - \nu, u(\nu)) \right\|_{H^s(D)} \, d\xi \, d\nu \leq L_g C_{\alpha,\alpha} \left[ 1 + \left( \mathcal{M} + \|u_0\|_{H^s(D)} \right) \right] \delta \int_0^T \xi^\alpha d\xi \, d\nu \leq L_g C_{\alpha,\alpha} \left[ 1 + \left( \mathcal{M} + \|u_0\|_{H^s(D)} \right) \right] \delta \int_0^T \frac{(T_0 - r)^{\alpha - \frac{\alpha(q-\sigma)}{2}} - (t - r)^{\alpha - 1}}{1 + \frac{\alpha(q-\sigma)}{2} - \alpha} \, dr.
\]

Now, we look at the integral quantity on the above display. By a simple calculation
In order to estimate the terms \((\text{IV})\), we note that \(\|u(t)\|_{\mathbb{H}^r(D)} \leq \|u(T_0)\|_{\mathbb{H}^r(D)} + M\) for any \(t \in [T_0, T_0]\). In the following, we treat the term \((\text{IV})\). Using Lemma 4.1,
we find that
\[
\left\| (IV) \right\|_{\mathbb{H}^{s}(D)} = \left\| \int_{T_{0}}^{t} (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) \int_{0}^{r} g(r-\nu, w(\nu)) d\nu dr \right\|_{\mathbb{H}^{s}(D)} \\
\leq \int_{T_{0}}^{t} (t-r)^{\alpha-1} \left\| E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) \int_{0}^{r} g(r-\nu, w(\nu)) d\nu \right\|_{\mathbb{H}^{s}(D)} dr \\
\leq C_{\alpha,\alpha} \int_{T_{0}}^{t} (t-r)^{\alpha-1} r^{\alpha/2} \left\| \int_{0}^{r} g(r-\nu, w(\nu)) d\nu \right\|_{\mathbb{H}^{s}(D)} dr \\
\leq C_{\alpha,\alpha} \mathcal{L}_{g} \left( 1 + \left( \| u(T_{0}) \|_{\mathbb{H}^{s}(D)} + \mathcal{M} \right)^{\delta} \right) \left( \int_{T_{0}}^{t} (t-r)^{\alpha-1} r^{-\alpha/2} \int_{0}^{r} d\nu dr \right) \\
= C_{\alpha,\alpha} \mathcal{L}_{g} \left( 1 + \left( \| u(T_{0}) \|_{\mathbb{H}^{s}(D)} + \mathcal{M} \right)^{\delta} \right) \left( \int_{T_{0}}^{t} (t-r)^{\alpha-1} r^{-\alpha/2} \int_{0}^{r} d\nu dr \right). 
\] (4.34)

For the integral term on the right hand side of (4.34), we find that
\[
\int_{T_{0}}^{t} (t-r)^{\alpha-1} r^{-\alpha/2} \int_{0}^{r} d\nu dr \leq (t-T_{0})^{1-\alpha/2} \int_{T_{0}}^{t} (t-r)^{\alpha-1} dr \\
= \frac{(t-T_{0})^{1+\alpha-\alpha/2}}{\alpha}. 
\] (4.35)

Combining (4.34) and (4.35), we derive that
\[
\left\| (IV) \right\|_{\mathbb{H}^{s}(D)} \leq \mathcal{M}_{4} \frac{(T_{0}-T_{0})^{1+\alpha-\alpha/2}}{\alpha}, 
\] (4.36)

where
\[
\mathcal{M}_{4} = C_{\alpha,\alpha} \mathcal{L}_{g} \left( 1 + \left( \| u(T_{0}) \|_{\mathbb{H}^{s}(D)} + \mathcal{M} \right)^{\delta} \right). 
\]

Finally, by similar argument for (IV), the term (V) is estimated as follows
\[
\left\| (V) \right\|_{\mathbb{H}^{s}(D)} \leq C_{\alpha,\alpha} \mathcal{L}_{f} \left( 1 + \left( \| u(T_{0}) \|_{\mathbb{H}^{s}(D)} + \mathcal{M} \right)^{\rho} \right) \left( \int_{T_{0}}^{t} (t-r)^{\alpha-1} r^{-\alpha/2} \int_{0}^{r} d\nu dr \right) \\
\leq C_{\alpha,\alpha} \mathcal{L}_{f} \left( 1 + \left( \| u(T_{0}) \|_{\mathbb{H}^{s}(D)} + \mathcal{M} \right)^{\rho} \right) \int_{T_{0}}^{t} (t-r)^{\alpha-1} dr \\
\leq \mathcal{M}_{5} (T_{0}-T_{0})^{\alpha}, 
\] (4.37)

where
\[
\mathcal{M}_{5} = \frac{C_{\alpha,\alpha} \mathcal{L}_{f}}{\alpha T_{0}^{\alpha/2}} \left( 1 + \left( \| u(T_{0}) \|_{\mathbb{H}^{s}(D)} + \mathcal{M} \right)^{\rho} \right). 
\]

Combining (4.26), (4.29), (4.32), (4.33), (4.36), and (4.37), we find that
\[
\left\| \mathcal{P} w(t) - u(T_{0}) \right\|_{\mathbb{H}^{s}(D)} \leq \mathcal{M}_{1} (T_{0}-T_{0})^{\frac{\alpha}{2} - \frac{\alpha}{2}} + \mathcal{M}_{2} \left[ 1 + \left( \mathcal{M} + \| u_{0} \|_{\mathbb{H}^{s}(D)} \right)^{\delta} \right] (T_{0}-T_{0})^{\alpha-\frac{\alpha(q-\alpha)}{2}} \\
+ \mathcal{M}_{3} \left[ 1 + \left( \mathcal{M} + \| u_{0} \|_{\mathbb{H}^{s}(D)} \right)^{\delta} \right] (T_{0}-T_{0})^{\alpha-\frac{\alpha(q-\alpha)}{2}} + \mathcal{M}_{4} \left( \frac{(T_{0}-T_{0})^{1+\alpha-\alpha/2}}{\alpha} \right) + \mathcal{M}_{5} (T_{0}-T_{0})^{\alpha}. 
\] (4.38)
Hence, we obtain the following estimate

\[
\|Dw(t) - Dv(t)\|_{H^s(D)} \\
\leq \left\| \int_{T_0}^{t} (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) \left( F(r, w(r)) - F(r, v(r)) \right) dr \right\|_{H^s(D)} \\
+ \int_{T_0}^{t} (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) \int_{0}^{r} \left( g(r-\nu, w(\nu)) - g(r-\nu, v(\nu)) \right) dvdr.
\]

(4.39)

Hence, we obtain the following estimate

\[
\|Dw(t) - Dv(t)\|_{H^s(D)} \\
\leq \left\| \int_{T_0}^{t} (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) \left( F(r, w(r)) - F(r, v(r)) \right) dr \right\|_{H^s(D)} \\
+ \int_{T_0}^{t} (t-r)^{\alpha-1} E_{\alpha,\alpha}((t-r)^{\alpha} \Delta) \int_{0}^{r} \left( g(r-\nu, w(\nu)) - g(r-\nu, v(\nu)) \right) dvdr \leq (VI) + (VII).
\]

(4.40)

The quantity \((VI)\) is bounded by

\[
\left\| (VI) \right\|_{H^s(D)} = C_{\alpha,\alpha} \mathcal{L}_f \left( 1 + 2 \left( \|u(T_0)\|_{H^s(D)} + \mathcal{M} \right)^{\alpha-1} \right) \\
\times \left( \int_{T_0}^{t} (t-r)^{\alpha-1} r^{-\alpha(q-\sigma)/2} dr \right) \|w - v\|_{C([0,T_1];H^s(D))} \\
\leq \frac{C_{\alpha,\alpha} \mathcal{L}_f}{T_0^{\alpha(q-\sigma)/2}} \|w - v\|_{C([0,T_1];H^s(D))} \left( 1 + 2 \left( \|u(T_0)\|_{H^s(D)} + \mathcal{M} \right)^{\alpha-1} \right) \int_{T_0}^{t} (t-r)^{\alpha-1} dr \\
\leq \frac{C_{\alpha,\alpha} \mathcal{L}_f (T_0 - T_0)^\alpha}{\alpha T_0^{\alpha(q-\sigma)/2}} \left( 1 + 2 \left( \|u(T_0)\|_{H^s(D)} + \mathcal{M} \right)^{\alpha-1} \right) \|w - v\|_{C([0,T_1];H^s(D))}.
\]

(4.41)

The quantity \((VII)\) is bounded by

\[
\left\| (VII) \right\|_{H^s(D)} = C_{\alpha,\alpha} \mathcal{L}_g \left( 1 + 2 \left( \|u(T_0)\|_{H^s(D)} + \mathcal{M} \right)^{\delta-1} \right) \\
\times \left( \int_{T_0}^{t} (t-r)^{\alpha-1} r^{-\alpha(q-\sigma)/2} \int_{0}^{r} dvdr \right) \|w - v\|_{C([0,T_1];H^s(D))} \\
\leq C_{\alpha,\alpha} \mathcal{L}_g \left( 1 + 2 \left( \|u(T_0)\|_{H^s(D)} + \mathcal{M} \right)^{\delta-1} \right) \left( \int_{T_0}^{t} (t-r)^{\alpha-1} r^{-\alpha(q-\sigma)/2} dr \right) \\
\leq C_{\alpha,\alpha} \mathcal{L}_g \left( 1 + 2 \left( \|u(T_0)\|_{H^s(D)} + \mathcal{M} \right)^{\delta-1} \right) \frac{(T_0 - T_0)^{1+\alpha-\alpha(q-\sigma)/\alpha}}{\alpha} \|w - v\|_{C([0,T_1];H^s(D))}.
\]

(4.42)

Combining (4.40), (4.41), and (4.42), we derive that

\[
\|Dw(t) - Dv(t)\|_{H^s(D)} \\
\leq \left( K_1 (T_0 - T_0)^\alpha + K_2 (T_0 - T_0)^{1+\alpha-\alpha(q-\sigma)/2} \right) \|w - v\|_{C([0,T_1];H^s(D))}.
\]

(4.43)
where
\[ K_1 = \frac{C_{\alpha,\alpha}L_f}{\alpha T_0^{\alpha(q-\sigma)/2}} \left( 1 + 2 \left( \|u(T_0)\|_{\mathcal{H}^q(D)} + M \right)^{q-1} \right), \]
and
\[ K_2 = C_{\alpha,\alpha}L_g \left( 1 + 2 \left( \|u(T_0)\|_{\mathcal{H}^q(D)} + M \right)^{q-1} \right) \frac{1}{\alpha}. \]

Let us choose \( T_0 \) such that the right hand side of (4.38) is less than 1 and
\[ K_1(T_0 - T_0)^{\alpha} + K_2(T_0 - T_0)^{1+\frac{\alpha(q-\sigma)}{2}} < 1. \]
Therefore, we apply the Banach fixed point theorem to obtain a unique fixed point \( \tilde{u} \) on \( \mathcal{H}_M \), which is a continuation of \( u \).

**Theorem 4.3.** Assume that \( F, g \) are as in Theorem 4.1. Let \( u \) be the mild solution of Problem (1.1) defined on \([0, T_{\text{max}}]\), where \( T_{\text{max}} \) is the maximal time of existence of \( u \). Then we have \( T_{\text{max}} = +\infty \) or \( \limsup_{t \to T_{\text{max}}} \|u(t)\|_{\mathcal{H}^q(D)} = \infty \).

**Proof.** Suppose that \( T_{\text{max}} < \infty \). Let us pick a sequence of positive numbers \( t_n \to T_{\text{max}} \); we consider the sequence \( \{u(t_n)\} \) in \( \mathcal{H}^q(D) \). We will prove that this sequence is a Cauchy in the space \( \mathcal{H}^q(D) \). For \( t_m, t_n \in [0, T_{\text{max}}] \) such that \( 0 < t_m < t_n < T_{\text{max}} \), we obtain after some simple caculations
\[
\begin{align*}
  u(t_n) - u(t_m) &= \left( E_{\alpha,1}(t_n^\alpha \Delta) - E_{\alpha,1}(t_m^\alpha \Delta) \right) u^0 \\
  &+ \int_{t_m}^{t_n} (t_n - r)^{\alpha-1} E_{\alpha,\alpha}((t_n - r)^\alpha \Delta) F(r, u(r)) dr \\
  &+ \int_{t_m}^{t_n} (t_n - r)^{\alpha-1} E_{\alpha,\alpha}((t_n - r)^\alpha \Delta) \int_r^t g(r - \nu, u(\nu)) d\nu dr \\
  &= Q_1 + Q_2 + Q_3. \tag{4.44}
\end{align*}
\]

Now, we need to show the upper estimation of \( \|u(t_n) - u(t_m)\|_{\mathcal{H}^q(D)} \). We split the proof into some steps.

**Step 1.** Estimate \( Q_1 \). Using from \( \frac{d}{dt} \left( E_{\alpha,1}(-\Delta^\alpha) \right) = -t^{\alpha-1} E_{\alpha,\alpha}(-\Delta^\alpha) \Delta \),
\[
\left\| Q_1 \right\|_{\mathcal{H}^q(D)} = \left\| \int_{t_m}^{t_n} \xi^{\alpha-1} E_{\alpha,\alpha}(-\xi^\alpha \Delta) u^0 d\xi \right\|_{\mathcal{H}^q(D)} = \left\| \int_{t_m}^{t_n} \xi^{\alpha-1} E_{\alpha,\alpha}(-\xi^\alpha \Delta) u^0 d\xi \right\|_{\mathcal{H}^{q+1}(D)} \\
\leq C_{\alpha,\alpha} \int_{t_m}^{t_n} \xi^{\alpha-1-\alpha(q+1-\sigma)/2} \|u^0\|_{\mathcal{H}^q(D)} d\xi \\
\leq C_{\alpha,\alpha} \left( M + \|u_0\|_{\mathcal{H}^q(D)} \right) \int_{t_m}^{t_n} \xi^{\frac{\alpha}{2} - \frac{\alpha}{2} (q-\sigma)} d\xi \\
= \frac{C_{\alpha,\alpha} \left( M + \|u_0\|_{\mathcal{H}^q(D)} \right)}{\frac{\alpha}{2} - \frac{\alpha}{2} (q-\sigma)} \left( t_n^{\frac{\alpha}{2} - \frac{\alpha}{2} (q-\sigma)} - t_m^{\frac{\alpha}{2} - \frac{\alpha}{2} (q-\sigma)} \right). \tag{4.45}
\]
Applying inequality \((a + b)^m \leq a^m + b^m, 0 \leq m \leq 1\) for the right hand side of (4.45), we find that

\[
\|Q_1\|_{\mathcal{H}^q(D)} \leq \frac{C_{\alpha,\beta} \left( \|M\|_{\mathcal{H}^p(D)} + \|u_0\|_{\mathcal{H}^q(D)} \right) \left( t_n - t_m \right)^{-\frac{2}{\alpha} + \frac{q}{2} - \frac{q}{2} (q - \sigma)} }{\frac{\sigma}{2} + \frac{q}{2} (q - \sigma)}. \tag{4.46}
\]

**Step 2.** Estimate \(Q_2\). Using Lemma 4.1,

\[
\|Q_2\|_{\mathcal{H}^q(D)} = \left\| \int_{t_m}^{t_n} (t_n - r)^{\alpha - 1} E_{\alpha,\alpha}((t - r)^{\alpha} \Delta) F(r, u(r)) dr \right\|_{\mathcal{H}^q(D)}
\leq \int_{t_m}^{t_n} (t_n - r)^{\alpha - 1} \left\| E_{\alpha,\alpha}((t - r)^{\alpha} \Delta) F(r, u(r)) \right\|_{\mathcal{H}^q(D)} dr
\leq C_{\alpha,\beta} \int_{t_m}^{t_n} (t_n - r)^{\alpha - 1} r^{\alpha (q - \sigma)/2} \left\| F(r, u(r)) \right\|_{\mathcal{H}^q(D)} dr
\leq C_{\alpha,\beta} \mathcal{L}_f \left( 1 + \left( \|u_0\|_{\mathcal{H}^q(D)} + \|\mathcal{M}\| \right)^{\delta} \right) \left( \int_{t_m}^{t_n} (t_n - r)^{\alpha - 1} r^{1 - \alpha (q - \sigma)/2} dr \right). \tag{4.47}
\]

We observe that

\[
\int_{t_m}^{t_n} (t_n - r)^{\alpha - 1} r^{1 - \alpha (q - \sigma)/2} dr \leq t_n^{\alpha - \alpha (q - \sigma)/2} \int_{t_m}^{t_n} (t_n - r)^{\alpha - 1} dr = t_n^{\alpha - \alpha (q - \sigma)/2} \frac{(t_n - t_m)^{\alpha}}{\alpha}. \tag{4.48}
\]

From the two previous estimates, we deduce that

\[
\|Q_2\|_{\mathcal{H}^q(D)} \leq C_{\alpha,\beta} \mathcal{L}_f \left( 1 + \left( \|u_0\|_{\mathcal{H}^q(D)} + \|\mathcal{M}\| \right)^{\delta} \right) t_n^{\alpha - \alpha (q - \sigma)/2} \frac{(t_n - t_m)^{\alpha}}{\alpha}. \tag{4.49}
\]

**Step 3.** Estimate \(Q_3\). By a similar argument as in Step 2, we obtain

\[
\|Q_3\|_{\mathcal{H}^q(D)} \leq C_{\alpha,\beta} \mathcal{L}_g \left( 1 + \left( \|u_0\|_{\mathcal{H}^q(D)} + \|\mathcal{M}\| \right)^{\delta} \right) \left( \int_{t_m}^{t_n} (t_n - r)^{\alpha - 1} r^{1 - \alpha (q - \sigma)/2} dr \right). \tag{4.50}
\]

For the integral term on the right hand side of (4.50), we find that

\[
\int_{t_m}^{t_n} (t_n - r)^{\alpha - 1} r^{1 - \alpha (q - \sigma)/2} dr \leq (t_n - t_m)^{1 - \alpha (q - \sigma)/2} \int_{t_m}^{t_n} (t_n - r)^{\alpha - 1} dr = (t_n - t_m)^{1 + \alpha - \alpha (q - \sigma)/2} \frac{\alpha}{\alpha}. \tag{4.51}
\]

This together with (4.50) yields that

\[
\|Q_3\|_{\mathcal{H}^q(D)} \leq C_{\alpha,\beta} \mathcal{L}_g \left( 1 + \left( \|u_0\|_{\mathcal{H}^q(D)} + \|\mathcal{M}\| \right)^{\delta} \right) \frac{(t_n - t_m)^{1 + \alpha - \alpha (q - \sigma)/2}}{\alpha}. \tag{4.52}
\]
Combining (4.44), (4.46), (4.49), (4.52), we get
\[
\left\| u(t_n) - u(t_m) \right\|_{H^\sigma(D)} \leq \sum_{j=1}^{3} \left\| Q_j \right\|_{H^\sigma(D)} \\
\leq \frac{C_{\alpha,\alpha} \left( \mathcal{M} + \left\| w_0 \right\|_{H^\sigma(D)} \right) (t_n - t_m)^{\frac{\alpha}{2} - \frac{\sigma}{2}(q - \sigma)}}{ \frac{q}{2} - \frac{q}{2}(q - \sigma)} + C_{\alpha,\alpha} \mathcal{L}_f \left( 1 + \left( \left\| u_0 \right\|_{H^\sigma(D)} + \mathcal{M} \right)^\alpha \right) t_m^{-\alpha(q - \sigma)/2} (t_n - t_m)^{\alpha/2} \\
+ C_{\alpha,\alpha} \mathcal{L}_g \left( 1 + \left( \left\| u_0 \right\|_{H^\sigma(D)} + \mathcal{M} \right)^\delta \right) (t_n - t_m)^{1+\alpha - \alpha(q - \sigma)/2}.
\]

Now, we claim that \(\{u(t_n)\}\) is a Cauchy sequence in the space \(H^\sigma(D)\). Let us consider any \(\epsilon > 0\). Our purpose here is to finding \(N_*(\epsilon) > 0\) such that for any \(n \geq m > N_*(\epsilon)\), we have
\[
\left\| u(t_n) - u(t_m) \right\|_{H^\sigma(D)} \leq \epsilon.
\]
(4.54)

Since \(\lim_{n \to +\infty} t_n = T_{\text{max}}\), there exists \(N_1(\epsilon) > 0\) such that
\[
t_n \geq \frac{T_{\text{max}}}{2}, \text{ for any } n > N_1(\epsilon).
\]
(4.55)

The sequence \(\{t_n\}\) is a Cauchy sequence in \(\mathbb{R}\), there exists \(N_3(\epsilon) > 0\), \(N_4(\epsilon) > 0\), \(N_5(\epsilon) > 0\) such that
\[
\frac{C_{\alpha,\alpha} \left( \mathcal{M} + \left\| w_0 \right\|_{H^\sigma(D)} \right) (t_n - t_m)^{\frac{\alpha}{2} - \frac{\sigma}{2}(q - \sigma)}}{ \frac{q}{2} - \frac{q}{2}(q - \sigma)} \leq \frac{\epsilon}{3}, \text{ for any } n > m > N_3(\epsilon),
\]
(4.56)

\[
C_{\alpha,\alpha} \mathcal{L}_f \left( 1 + \left( \left\| u_0 \right\|_{H^\sigma(D)} + \mathcal{M} \right)^\alpha \right) \left( \frac{T_{\text{max}}}{2} \right)^{-\alpha(q - \sigma)/2} (t_n - t_m)^{\alpha/2} \leq \frac{\epsilon}{3},
\]
(4.57)

for any \(n > m > N_4(\epsilon)\), and
\[
C_{\alpha,\alpha} \mathcal{L}_g \left( 1 + \left( \left\| u_0 \right\|_{H^\sigma(D)} + \mathcal{M} \right)^\delta \right) (t_n - t_m)^{1+\alpha - \alpha(q - \sigma)/2} \leq \frac{\epsilon}{3},
\]
(4.58)

for any \(n > m > N_5(\epsilon)\). By choosing \(N_*(\epsilon) = \max \left( N_3(\epsilon), N_4(\epsilon), N_5(\epsilon) \right)\), for any \(n \geq m > N_*(\epsilon)\), we derive that (4.54) holds. Hence, we conclude that \(\{u(t_n)\}\) is a Cauchy sequence in \(H^\sigma(D)\). Hence, \(\{u(t_n)\}\) converges to \(\bar{u} \in H^\sigma(D)\) as \(n \to +\infty\). Since \(\{t_n\}\) is arbitrary, we deduce that
\[
\lim_{t \to T_{\text{max}}} \left\| u(t) \right\|_{H^\sigma(D)} = \left\| \bar{u} \right\|_{H^\sigma(D)}.
\]

Then, we may extend \(u\) over \([0, T_{\text{max}}]\). Therefore, we obtain a contradiction with the maximality of \(T_{\text{max}}\). \(\square\)
5. **Numerical results.** In this section, we consider some examples for the Volterra integro-differential equations with Caputo derivative. Through these numerical experiments, we provide now some computational examples to illustrate the validity of the proposed method. The examples are involved with the operator $\frac{\partial^2}{\partial x^2}$ on $L^2(0, \pi)$ on the domain $(x, t) \in (0, \pi) \times (0, 1)$. Then, the problem has the following form

$$\begin{cases}
D^\alpha u_t = \frac{\partial^2}{\partial x^2} u + F(x, t, u) + \int_0^t g(t - s, u(s))ds, &(x, t) \in (0, \pi) \times (0, 1), \\
\frac{\partial}{\partial x} u(x, t) = 0, &(x, t) \in \{0, \pi\} \times (0, 1), \\
u(x, 0) = u^0(x), &x \in (0, \pi),
\end{cases}$$

(5.1)

where the functions $F, g$ and $u^0$ will be given in specific cases later.

We implement the model with the domain $D = [0, \pi]$. The Laplacian operator has eigenfunctions, satisfying the Neumann boundary condition

$$\phi_j(x) = \sqrt{2/\pi} \cos(jx),$$

with corresponding eigenvalues

$$\lambda_j = j^2, j \in \mathbb{N},$$

where the sequence $\{\phi_j\}_{j=0}^\infty$ forms an orthonormal basis of $L^2(0, \pi)$.

A uniform grid of mesh-points $(x_n, t_m)$ is used to discretize the space and time intervals

$$x_n = \frac{n\pi}{N + 1}, \quad t_m = \frac{m}{M + 1}, \quad n = 1, \ldots, N + 1, \quad m = 1, \ldots, M + 1,$$

where $M, N > 0$ are two given integer numbers. To calculate the integrals, the Simpson approximation method is given by

$$\int_a^b f(z) dz \approx \frac{\Delta_z}{3} \sum_{i=1}^{N_z+1} \delta_i f(z_i),$$

(5.2)

where $\Delta_z = \frac{b - a}{N_z}$ the interval $[a, b]$ is split up into $N_z$ sub-intervals for $N_z$ is an even number and

$$\delta_i = \begin{cases} 1, &\text{if } i = 1 \text{ or } i = N_z + 1, \\
2, &\text{if } i \text{ is odd}, \\
4, &\text{if } i \text{ is even}.\end{cases}$$

The Matlab code used to calculate this approximation was written by J.C. Medina, Simpson’s Rule Integration, see https://www.mathworks.com/matlabcentral/fileexchange/28726-simpson-s-rule-integration.

Next, we calculate the Mittag-Leffler function by code Matlab which is written by I. Podlubny with accuracy $P$ for each element of input variable:

$$E_{a,b}(x) = mlf(a, b, x, P),$$

see https://www.mathworks.com/matlabcentral/fileexchange/8738-mittag-leffler-function.
According to (2.11) and representing Fourier series, we have the mild solution of Problem (5.1) as follows

\[
\begin{align*}
    u(x,t) &= \sum_{j=1}^{\infty} \left[ E_{\alpha,1}(t^\alpha \lambda_j)u_0^j + \int_0^t (t-r)^{\alpha-1}E_{\alpha,\alpha}((t-r)^\alpha \lambda_j)F_j(r,u(\cdot,r))dr \\
    &\quad + \int_0^t (t-r)^{\alpha-1}E_{\alpha,\alpha}((t-r)^\alpha \lambda_j) \int_0^r g_j(r-\nu,u(\cdot,\nu))d\nu dr \right] \sqrt{2/\pi} \cos(jx),
\end{align*}
\]

(5.3)

where \( u_0^j = \langle u^0(\cdot), \phi_j(\cdot) \rangle_{L^2(0,\pi)} \), \( F_j = \langle F(\cdot,t), \phi_j(\cdot) \rangle_{L^2(0,\pi)} \), \( g_j = \langle g(\cdot), \phi_j(\cdot) \rangle_{L^2(0,\pi)} \).

By using the partition on the domain \((0, \pi) \times (0, 1)\), let \( N(j) \) be a parameter truncated, we have

\[
\begin{align*}
    u(x_n,t_m) &= \sum_{j=1}^{N(j)} U_j(x_n,t_m) \sqrt{2/\pi} \cos(jx_n), \\
    &= \sqrt{2/\pi} \begin{bmatrix} U_0(x_n,t_m) & U_1(x_n,t_m) & U_2(x_n,t_m) & \cdots & U_{N(j)}(x_n,t_m) \end{bmatrix} \cdot \begin{bmatrix} 1 & \cos(x_n) & \cos(2x_n) & \cdots & \cos(N(j)x_n) \end{bmatrix}^T,
\end{align*}
\]

(5.4)

where

\[
\begin{align*}
    U_j(x_n,t_m) &= E_{\alpha,1}(t_m^\alpha j^2)u_0^j + \int_0^{t_m} (t_m-r)^{\alpha-1}E_{\alpha,\alpha}((t_m-r)^\alpha j^2)F_j(r,U_j(\cdot,r))dr \\
    &\quad + \int_0^{t_m} (t_m-r)^{\alpha-1}E_{\alpha,\alpha}((t_m-r)^\alpha j^2) \int_0^r g_j(r-\nu,u(\cdot,\nu))d\nu dr.
\end{align*}
\]

(5.5)

The matrix form of the solution (5.3) is given by

\[
\begin{bmatrix}
    u(x_1,t_1) & u(x_2,t_1) & u(x_3,t_1) & \cdots & u(x_{N+1},t_1) \\
    u(x_1,t_2) & u(x_2,t_2) & u(x_3,t_2) & \cdots & u(x_{N+1},t_2) \\
    u(x_1,t_3) & u(x_2,t_3) & u(x_3,t_3) & \cdots & u(x_{N+1},t_3) \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    u(x_1,t_{M+1}) & u(x_2,t_{M+1}) & u(x_3,t_{M+1}) & \cdots & u(x_{N+1},t_{M+1})
\end{bmatrix}_{(M+1) \times (N+1)}
\]

Next, by fixing the variable \( t \), we have the absolute error estimation \( \text{Error}_{\alpha^*}^{\alpha^*} \) between the solution for some cases of \( \alpha^* \) and the solution for \( \alpha \) as follows

\[
\text{Error}_{\alpha^*}^{\alpha^*}(t) = \sqrt{\frac{\sum_{n=1}^{N+1} \left| u_{\alpha^*}(t,x_n) - u_{\alpha}(t,x_n) \right|^2}{N+1}},
\]

(5.6)

and the corresponding percentage error is given by

\[
\delta_{\alpha^*}^{\alpha^*}(t) = \text{Error}_{\alpha^*}^{\alpha^*}(t) \sqrt{\frac{\sum_{n=1}^{N+1} \left| u_{\alpha}(t,x_n) \right|^2}{N+1}} \times 100,
\]

(5.7)
Below, we present some examples as follows.

**Example 1.** In this example, we consider a simple case:

\[
\begin{aligned}
F &= t^{-\alpha} \cos(x) + u(x, t) \left(1 - \frac{1}{5} t^2\right), \\
g &= (t-s)u(x, s), \\
 u^0 &= 0.
\end{aligned}
\]

Then we have the solution \( u(x, t) = t \cos(x) \).

| \( \epsilon = |\alpha^* - \alpha| \) | \( N(j) = 10 \) | \( P = 10 \) | \( M = N = 50 \) |
|-----------------|----------------|-----------------|----------------|
|                 | Calculative error | Percent error \( \delta^\alpha \) |
| 0.1             | 0.011036875514009 | 15.46 %          |
| Error^\alpha^* (0.1) | 0.004874547920700 | 6.83 %           |
| 0.001           | 0.002176754401323 | 3.05 %           |
| 0.1             | 0.051353021961229 | 14.38 %          |
| Error^\alpha^* (0.5) | 0.027169280169359 | 7.61 %           |
| 0.001           | 0.012084128051511 | 3.38 %           |
| 0.1             | 0.074877070037197 | 11.65 %          |
| Error^\alpha^* (0.9) | 0.042748723217464 | 6.65 %           |
| 0.001           | 0.028729381706881 | 4.47 %           |

**Table 1.** Ex1. The error estimation for \( \alpha = 0.2 \) and \( t \in \{0.1, 0.5, 0.9\} \)

**Example 2.** In this example, for \( x \in (0, \pi), t \in (0, 1), s \in (0, t) \), we consider a simple case:

\[
\begin{aligned}
F &= u - \cos(x) \left[ \int_0^t \sinh(s) \Gamma(1-\alpha)(t-s)^\alpha ds - \frac{1}{24} \left(4t^3 - 6t + 3 \sinh(2t)\right) \right], \\
g &= (t-s)^2u^2(x, s), \\
 u^0 &= \cos(x).
\end{aligned}
\]

Then we have the solution \( u(x, t) = \cosh(t) \cos(x) \).

The results of this section are presented in Tables 1, 2 and Figures 1, 2, 3, 4. In detail, we show the errors between the solution with order \( \alpha \) the solution with order \( \alpha^* \) and the corresponding percentage error in Table 1 (example 1) and Table 2 (example 2). Besides, we also present the solutions on \( (x, t) \in (0, \pi) \times (0, 1) \) in 3D graph for \( \alpha = 0.2, 0.9 \) respectively. From this results, it is clear that the smaller \( \epsilon \), the smaller output error. In other words, the solution \( u^{\alpha^*} \) tends the solution \( u^\alpha \) when \( \alpha^* \) approaches to \( \alpha \).

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Figure 1. Ex1. The solutions $u(x, t)$ at $t \in \{0.1, 0.5, 0.9\}$ for $\alpha = 0.2$ and $\epsilon \in \{0.1, 0.01, 0.001\}$

| $\epsilon$ | $|\alpha^*-\alpha|$ | $N(j) = 10$, $P = 10$, $M = N = 50$ | Calculative error | Percent error $\delta_{\alpha'}$ |
|------------|-----------------|---------------------------------|----------------|-------------------------------|
| Error$_\alpha^*$ (0.1) | 0.1 | 0.601407040658915 | 83.81 % | |
| | 0.01 | 0.143648723675101 | 20.02 % | |
| | 0.001 | 0.027371563705550 | 3.81 % | |
| Error$_\alpha^*$ (0.5) | 0.1 | 0.64317986712674 | 79.89 % | |
| | 0.01 | 0.167904683423887 | 20.85 % | |
| | 0.001 | 0.025589340748129 | 3.18 % | |
| Error$_\alpha^*$ (0.9) | 0.1 | 0.629790962460797 | 61.54 % | |
| | 0.01 | 0.219601251416672 | 21.46 % | |
| | 0.001 | 0.040291675260527 | 3.94 % | |

Table 2. Ex2. The error estimation for $\alpha = 0.9$ and $t \in \{0.1, 0.5, 0.9\}$

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(a) The solution $u^\alpha(x, t)$

(b) The solution $u^{\alpha^*}(x, t)$ with $\epsilon = 0.1$

(c) The solution $u^{\alpha^*}(x, t)$ with $\epsilon = 0.01$

(d) The solution $u^{\alpha^*}(x, t)$ with $\epsilon = 0.001$

Figure 2. Ex1. The solutions $u(x, t)$ on $(x, t) \in (0, \pi) \times (0, 1)$ for $\alpha = 0.2$ and $\epsilon \in \{0.1, 0.01, 0.001\}$

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Figure 3. Ex2. The solutions $u(x,t)$ at $t \in \{0.1, 0.5, 0.9\}$ for $\alpha = 0.9$ and $\epsilon \in \{0.1, 0.01, 0.001\}$
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(a) The solution $u^\alpha(x, t)$

(b) The solution $u^\alpha^*(x, t)$ with $\epsilon = 0.1$

(c) The solution $u^\alpha^*(x, t)$ with $\epsilon = 0.01$

(d) The solution $u^\alpha^*(x, t)$ with $\epsilon = 0.001$

Figure 4. Ex2. The solutions $u(x, t)$ on $(x, t) \in (0, \pi) \times (0, 1)$ for $\alpha = 0.9$ and $\epsilon \in \{0.1, 0.01, 0.001\}$

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