NONNOETHERIAN LORENTZIAN MANIFOLDS II: ASPECTS OF THE STANDARD MODEL

CHARLIE BEIL

ABSTRACT. A nonnoetherian spacetime is a Lorentzian manifold that contains a set of causal curves with no distinct interior points. We show that on such a spacetime, hidden within the free Dirac Lagrangian is the entire standard model (with four massive neutral scalar bosons), with the correct spin, electric charge, color charge, and relative mass orderings of each particle. Furthermore, using two ‘fusion rules’, we are able to reproduce almost all of the standard model trivalent vertices, as well as electroweak parity violation. Finally, we find that on a nonnoetherian spacetime, C, P, and T each sit in a different connected component of the full Lorentz group, and their product is the identity.

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1. INTRODUCTION

The purpose of this article is to investigate structures arising from nonnoetherian Lorentzian manifolds that are similar to aspects of the standard model of particle physics. Such manifolds were introduced in [B3] in order to incorporate the notion that time passes if and only if something changes into general relativity. The geometry of these manifolds is new, and recently arose in the study of nonnoetherian coordinate rings in algebraic geometry [B1, B2, B4].

We briefly recall their construction. Consider an isolated free particle of dust. By the geodesic hypothesis, the particle’s worldline is a causal geodesic \( \beta \). Since the particle is free, it does not detect change. Therefore, \textit{time does not advance for the}...
particle. Consequently, each point \( p \) along the worldline \( \beta \) is the same point, that is, all the points along \( \beta \) are identified. We are thus led to the following definition.

**Definition 1.1.** Let \((\tilde{M}, g)\) be a 3 + 1-dimensional orientable Lorentzian manifold. Let \( B \) be a collection of piecewise causal geodesic curves, called pointal curves. Consider the multivalued map
\[
\pi : \tilde{M} \to M := \{\{p\} | p \in \tilde{M} \setminus \bigcup_{\beta \in B} \beta\} \cup B,
\]
\[p \mapsto q \text{ if } q \ni p.
\]
We call \( M \) a (nonnoetherian) spacetime, and \( \tilde{M} \) a depiction of \( M \).

This construction allows fundamental particles to be defined in terms of the geometry of spacetime alone:

**Definition 1.2.** A pointal particle is a particle whose worldline is a segment of a pointal curve.

The worldline \( \beta \) of a pointal particle \( \beta(t) \) is thus a continuum of distinct 0-dimensional points in \( \tilde{M} \), and a single 1-dimensional point in \( M \); the notion of a ‘positive-dimensional point’ is made precise using ideals and overrings in \([B2]\).

If two pointal particles meet at a point \( p \in \tilde{M} \) and interact, then they would each detect change. Thus, their worldlines in \( \tilde{M} \) would necessarily end at \( p \). Consequently, pointal particles only interact by creation and annihilation in pairs.

Composite bound states of pointal particles are therefore necessary to model more complex interactions. We call such bound states atoms. In this article we aim to construct a model of atoms that approximates features of the standard model (SM).

We now state our main results.

In Section 2 we introduce a spacetime model of color charge. Consider a pointal particle with a timelike worldline \( \beta \subset \tilde{M} \). In \([B3]\) Section 2.5, we obtain a spatial spin vector \( s \) that is parallel transported along \( \beta \), using Hodge duality and a ‘nonnoetherian metric’ on \( M \). If \( s \) transforms under a representation of charge conjugation that breaks Lorentz invariance, then we say the particle has color charge. However, Lorentz invariance is not broken on \( T\tilde{M} \), and therefore a pointal particle with color charge cannot exist in isolation. Denote by \( V \cong \mathbb{R}^3 \subset T_{\beta(t)}\tilde{M} \) the spatial hypersurface in the particle’s inertial frame. We show the following.

**Theorem 1.3.**

(i) There are precisely six color charges \( r^\pm, g^\pm, b^\pm \), since at most three pairwise orthogonal vectors fit in \( V \).

(ii) A bound state of pointal particles with color charge may exist in isolation if and only if they share a common worldline \( \beta \), and together they transform under an \( SO(V) \)-invariant representation of charge conjugation. There are precisely five irreducible \( SO(V) \)-invariant combinations:
\[
\{r^+, g^+, b^+\}, \quad \{r^-, g^-, b^-\}, \quad \{r^+, r^-\}, \quad \{g^+, g^-\}, \quad \{b^+, b^-\}.
\]
In Section 3, we introduce a preon model of the standard model using atoms of pointal particles, as well as two fusion rules which govern how the atoms may be combined. Our preon model, which we call the pointal model, is based on the chiral decomposition of the free Dirac Lagrangian; see Table 1. In this framework, then, the SM Lagrangian is viewed as an effective theory. Our main theorem is the following.

**Theorem 1.4.** The pointal model yields

(i) exactly the SM fermions, electroweak bosons, and six gluons, each with their correct spin, electric charge, and color charge;

(ii) four massive neutral spin-0 bosons, $h_1, \ldots, h_4$, in two distinct mass types.

Furthermore, the fusion rules produce

(iii) exactly the SM trivalent vertices involving fermions and vector bosons;

(iv) electroweak parity violation; and

(v) the vertices $hW^+W^-, hhZ$, and $hh\gamma$.

The $h$ atoms, being massive neutral spin-0 bosons, are ‘Higgs-like’ particles, but admit different interaction vertices. In particular, there is no $hZZ$ vertex; however, we may model the Higgs boson decay channel $H \to ZZ^*$ using two intermediate steps:

$$h \to Zh \to ZW^+W^- \to ZZ^*.$$ 

In Section 3.9, we obtain fifteen independent particle mass orderings from our pointal model, all which agree with experiment. We also obtain the correct relative orders of magnitude for eight particle masses (rounded to the nearest power of 10 in Mev$^2/c^2$), using only two free parameters in a simplistic model of the spin angular momentum of an atom.

Finally, in Section 4 we show that on a nonnoetherian spacetime, charge conjugation is given by the Lorentz transformation $C = -1_4$, and thus acts on spinors by

$$C \psi(t, x^i) = \gamma^5 \psi(-t, -x^i).$$ 

As a consequence, we obtain the following.

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2In QCD, there is a gluon for each generator of the Lie algebra $su(3)$, whence eight gluons, whereas in our model we do not have the Gell-Mann matrices $\frac{1}{\sqrt{2}}(r^+r^--b^+b^-)$ and $\frac{1}{\sqrt{6}}(r^+r^+ + b^+b^--2g^+g^-)$. Furthermore, in QCD the singlet states $r^+r^-, g^+g^-, b^+b^-$ are not allowed since the corresponding matrices are not traceless, and in our model such gluons are photons by [10].
Theorem 1.5. On a nonnoetherian Lorentzian manifold, the quotient of the full Lorentz group $O(1,3)$ by $SO^+(1,3)$ is generated by charge conjugation $C$, parity $P$, and time reversal $T$,

$$O(1,3)/SO^+(1,3) = \langle C, P, T \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Furthermore, each of the four connected components of $O(1,3)$ contains precisely one of the transformations $I$, $C$, $P$, or $T$, and their product CPT is (proportional to) the identity in the spacetime and spinor representations,

$$C^\mu P^\rho T^\lambda = g^\mu_{\rho} = \delta^\mu_{\rho} \in SO^+(1,3), \quad SCSPST = \gamma^5 \gamma^0 (\gamma^1 \gamma^2 \gamma^3) = -i (\gamma^5)^2 = -i.$$

It follows that CPT invariance trivially holds on a nonnoetherian spacetime.

Other interesting preon models that are related to the geometry of spacetime are the Harari-Shupe (or rishon) model [Har, Shu, HarS], its topological realization with braids [Bi, BMS, BHKS], and Schiller’s strand model [S1, S2].

Notation: Tensors labeled with upper and lower indices $a, b, \ldots$ represent covector and vector slots respectively in Penrose’s abstract index notation (so $v^a \in V$ and $v_a \in V^*$), and tensors labeled with indices $\mu, \nu, \ldots$ denote components with respect to a coordinate basis. We denote by $Z(G)$ the center of a group $G$. Given a curve $\beta : I \to \tilde{M}$, we often denote its image $\beta(I)$ also by $\beta$. We use natural units $\hbar = c = G = 1$ and the signature $(-, +, +, +)$, unless stated otherwise.

2. Color charge from nonnoetherian geometry

We briefly recall the definition of electric charge introduced in [B3, Section 2.3]. Let $M$ be a nonnoetherian spacetime depicted by a Lorentzian manifold $(\tilde{M}, g)$. A ‘pointal metric’ on $M$ is defined by requiring local injectivity of the exponential map from the tangent bundle $T\tilde{M}$ to $M$, generalizing a property of Riemannian manifolds:

**Definition 2.1.** [B3, Definition 2.4] The pointal metric of $g$ at $p \in \tilde{M}$ is the symmetric rank-2 tensor

$$h_{ab} = h : T_p\tilde{M} \otimes T_p\tilde{M} \to \mathbb{R}$$

defined by the following:

(i) If $v \in T_p\tilde{M}$ is a timelike tangent 4-vector to some $\beta \in \pi(p)$, then

$$h(v, -) = 0.$$

(ii) If $w \in T_p\tilde{M}$ satisfies $g(w, v) = 0$ for each vector $v$ tangent to some $\beta \in \pi(p)$ at $p$, then

$$h(w, -) = g(w, -).$$

Fix $p \in \tilde{M}$. By [B3, Lemma 2.5], if $\pi(p) = \beta$ and $p$ is a smooth point of $\beta$, then $h$ is the projection of $T_p\tilde{M}$ onto the subspace orthogonal to the tangent 4-vector $v = v^a$ to $\beta$ at $p$,

$$h^a_b = g^a_b + v^a v_b.$$
Table 1. A pointal model of the standard model particles. The atoms are grouped by orbital momentum type and total charge. Subscripts denote spin/polarization states.

| charge | all minimal excitations $[-+,++]$ of the Dirac Lagrangian |
|--------|----------------------------------------------------------|
| $q = 0$ (bosonic) | $\gamma_0 = [\uparrow\downarrow, \uparrow\downarrow]$ $Z_0 = [\uparrow\uparrow, \downarrow\downarrow]$ $h_1 = [0\uparrow, \uparrow\downarrow]$ |
| $q = 0$ (fermionic) | $\nu_e = [00, \downarrow\phi]$ $\nu_\mu = [\uparrow\downarrow, \downarrow\phi]$ $\nu_\tau = [\uparrow\uparrow, \downarrow\phi]$ |
| $q = q^-$ $\mapsto -1$ | $e_\uparrow = [00, \uparrow 0]$ $\mu_\uparrow = [\downarrow\uparrow, \uparrow 0]$ $\tau_\uparrow = [\uparrow\downarrow, \uparrow 0]$ $W_0^- = [0\uparrow, \downarrow\phi]$ |
| $q = q^+$ $\mapsto +1$ | $\bar{e}_\uparrow = [00, 0\uparrow]$ $\bar{\mu}_\uparrow = [\downarrow\uparrow, 0\uparrow]$ $\bar{\tau}_\uparrow = [\uparrow\downarrow, 0\uparrow]$ $W_0^+ = [0\uparrow, \downarrow\phi]$ |
| $q = c^- + q^+$ $\mapsto \frac{1}{3} + 1 = \frac{2}{3}$ | $u_\uparrow = [00, \uparrow\phi]$ $c_\uparrow = [\downarrow\uparrow, \uparrow\phi]$ $t_\uparrow = [\uparrow\downarrow, \uparrow\phi]$ |
| $q = c^+ - q^-$ $\mapsto \frac{1}{3} - 1 = \frac{2}{3}$ | $\bar{u}_\uparrow = [00, \downarrow\phi]$ $c_\downarrow = [\uparrow\downarrow, \downarrow\phi]$ $t_\downarrow = [\uparrow\downarrow, \downarrow\phi]$ |
| $q = \frac{1}{3}$ | $d_\uparrow = [00, \uparrow 0]$ $s_\uparrow = [\uparrow\downarrow, \uparrow 0]$ $b_\uparrow = [\downarrow\uparrow, \uparrow 0]$ |
| $q = \frac{1}{3}$ | $\bar{d}_\uparrow = [00, 0\downarrow]$ $\bar{s}_\downarrow = [\downarrow\uparrow, 0\downarrow]$ $\bar{b}_\downarrow = [\uparrow\downarrow, 0\downarrow]$ |

In particular, if $v$ is null, then $h = g$.

Now let $\beta$ be the (geodesic) worldline a pointal particle with tangent 4-vector $v$, and suppose $\beta$ is timelike. Although $v$ vanishes on $M$, its Hodge dual $*_v = i_v \text{vol}^4$ does not. We thus replace the tangent vector $v$ with its dual 3-form $*_v$. Since $*_v$ is a pseudo-form, an orientation $o_\beta \in \{\pm 1\}$ must be chosen. Similar to the
Figure 1. Examples of fusions. Each diagram represents the worldlines of the pointal particles in the fusions given to the left, with their spins omitted. The orientation of a worldline (future directed or past directed) is the electric charge of its pointal particle. The two orbits in each atom are given by the inner pair and outer pair of worldlines, though this distinction is only to aid in drawing the figures.

Stückelberg-Feynman interpretation of antimatter, we identify this orientation with electric charge.

Let $V = \mathbb{R}^3 \subset T_{\beta(t)}M$ be the spatial hypersurface in the inertial frame of $\beta$, parallel transported along $\beta$. Let $o_V, o_t \in \{\pm 1\}$ be the respective orientations of $V$ and its orthogonal timelike direction in $T_{\beta(t)}V$. These are related to the orientation $o_\beta$ of the volume form $\text{vol}_4$ along $\beta$ by

$$o_\beta = o_V o_t.$$  

In [B3, Section 2.5], we use Hodge duality and the pointal metric to obtain a spin vector $s^a \in V$ that is parallel transported along $\beta$. Furthermore, in [B3, Section 5.1]
we show that the pointal particle $\beta(t)$ may dually be described by a particle with a
circular trajectory about $\beta$ in $V$. Indeed, the contraction of the spin vector $s^a$ with
the Hodge dual of the tangent vector $v^a$ is given by

\[ (\ast v)_{abc} s^c = \ast o_\beta o_V (\hat{r} \times \hat{u})_d = \ast o_t \frac{u}{r} (\hat{r} \times \hat{u})_d, \]

where $\ast$ is the spatial Hodge dual in $V^*$; $u, \omega = ur^{-1} \geq 0$ are the tangential and
angular velocities of the dual particle; and $r > 0$ is its radius.

Charge conjugation $C$ corresponds to flipping the orientation $o_\beta \mapsto -o_\beta$. By (1)
and (2), this is equivalent to flipping the sign of spin vector $s^a$,

\[ o_\beta \mapsto -o_\beta \iff s^a \mapsto -s^a. \]

This equivalence is a geometric realization of the elementary fact that electrons,
subject to a Lorentz force, follow the right-hand rule if and only if positrons follow
the left-hand rule.$^3$

Since $C^2 = id$, charge conjugation generates the group $\langle C \rangle = \mathbb{Z}_2$. Furthermore,
since there is no distinguished direction in $V$, the spin vector $s = s^a$ transforms under
a representation $\rho_V : \langle C \rangle \to O(V)$ that is rotation invariant: $g\rho_V(C)g^{-1} = \rho_V(C)$ for
all $g \in SO(V)$. Therefore, by Schur’s lemma,

\[ \rho_V : \langle C \rangle \to O(V)^{SO(V)} = Z(O(V)) \cong \mathbb{Z}_2 \]

In particular, $\rho_V(C) = -1_3$.

Now fix a line $L \subset V$, and consider a representation that is only rotation invariant
up to fixing $L$:

\[ \rho_L : \langle C \rangle \to O(V)^{SO(L \oplus L^\perp)} = Z(O(L \oplus L^\perp)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \]

(4)

\[ C \mapsto \begin{cases} -s & \text{if } s \in L \\ s & \text{if } s \in L^\perp \end{cases} \]

Note that $Z(O(L \oplus L^\perp)) = \langle \rho_L(C), \rho_V(C) \rangle$.

**Definition 2.2.** If charge conjugation $C$ acts on a pointal particle by the representa-
tion $\rho_L$ for some line $L \subset V$, then we say the pointal particle has *color*.

Since Lorentz invariance is not violated on $T\tilde{M}$, there is no distinguished direction
in $V$, and therefore a pointal particle with color cannot exist in isolation. However,
a bound state of pointal particles with color may exist in isolation if they share a
common worldline $\beta$, and together their transformation under charge conjugation is

$^3$Suppose we flip the electric charge of $\beta(t)$, $q \mapsto -q$. Geometrically, this corresponds to flipping the
orientation of the volume form $\text{vol}^4$, $o_\beta \mapsto -o_\beta$. By (1), this in turn flips the timelike orientation
$o_t \mapsto -o_t$ if we keep $o_V$ fixed, or the spatial orientation $o_V \mapsto -o_V$ if we keep $o_t$ fixed. In either
case, this flips the rotation direction, clockwise or counterclockwise, of the dual particle, and thus
the sign of the spin vector, $s^a \mapsto -s^a$. 

SO(V)-invariant. Indeed, consider a bound state \( \sigma \) of pointal particles \( \beta_i \) with colors \( \rho_i \), electric charges \( o_{\beta_i} \), and total electric charge \( \sum_i o_{\beta_i} \). Denote by \( \rho_0 \) the trivial representation,

\[
\rho_0 : \langle C \rangle \to O(V)^{SO(V)} \quad C \mapsto (s \mapsto s \quad \forall s \in V)
\]

Then \( \sigma \) may exist in isolation if and only if

\[
\prod_i \rho_i = \begin{cases} 
\rho_V & \text{if } \sigma \text{ has nonzero total electric charge} \\
\rho_0 & \text{otherwise}
\end{cases}
\]

where \( (\prod_i \rho_i)(C) := \prod_i (\rho_i(C)) \). Furthermore, \( C \) acts on any neutral subset of \( \sigma \) by the trivial representation \( \rho_0 \).

**Lemma 2.3.** Modulo \( SO(V) \) change-of-bases, there are precisely three irreducible products \([\rho]\) that are \( SO(V) \)-invariant:

\[
\rho_1 \rho_2 \rho_3 = \rho_V, \quad \text{with } o_{\beta_1} = o_{\beta_2} = o_{\beta_3} \in \{-1, +1\};
\]

\[
\rho_L \rho_L = \rho_0, \quad \text{with } o_{\beta_1} = -o_{\beta_2}.
\]

In the top line we set \( \rho_i := \rho_{e_i} \), where \( \{e_1, e_2, e_3\} \) is an orthogonal Fermi basis of \( V \).

**Proof.** In the case \( \rho_1 \rho_2 \rho_3 = \rho_V \), the constraint \( o_{\beta_1} = o_{\beta_2} = o_{\beta_3} \) arises because \( C \) acts on any neutral subset of \( \sigma \) by \( \rho_0(\mathbf{C}) = 1_3 \). Consequently, no two of \( \rho_i, \rho_j \) with \( i \neq j \) satisfy \( \rho_i(\mathbf{C}) \rho_j(\mathbf{C}) = 1_3 \). \( \square \)

With respect to the orthogonal basis \( \{e_1, e_2, e_3\} \), we have

\[
\rho_1(\mathbf{C}) = \text{diag}(-1, 1, 1), \quad \rho_2(\mathbf{C}) = \text{diag}(1, -1, 1), \quad \rho_3(\mathbf{C}) = \text{diag}(1, 1, -1).
\]

By abuse of terminology, we define the charge of a pointal particle to be its orientation \( o_{\beta} \) and the representation \( \rho \) of \( \langle C \rangle \) that its spin vector transforms under. In particular, we call

\[
q^{o_{\beta}} := (o_{\beta}, \rho_V) \cong o_{\beta} \quad \text{and} \quad c^{o_{\beta}} := (o_{\beta}, \rho_L)
\]

an electric charge and color charge respectively. Note that a color charge consists of both an orientation (= electric charge) and a distinguished line \( L \subset V \).

**Proposition 2.4.** There are precisely six color charges with respect to a fixed orthogonal Fermi basis \( \{e_1, e_2, e_3\} \) of \( V \):

\[
r^{o_{\beta}} := (o_{\beta}, \rho_1), \quad g^{o_{\beta}} := (o_{\beta}, \rho_2), \quad b^{o_{\beta}} := (o_{\beta}, \rho_3).
\]

These color charges admit precisely five irreducible bound states:

\[
\{r^+, g^+, b^+\}, \quad \{r^-, g^-, b^-\}, \quad \{r^+, r^-\}, \quad \{g^+, g^-\}, \quad \{b^+, b^-\}.
\]

**Proof.** Follows from Lemma 2.3 \( \square \)
3. A preon model of the standard model

In this section we construct our preon model and prove Theorem 1.4.

3.1. Excitations of the free Dirac Lagrangian. From the geometry of nonnoetherian spacetime, a pointal particle has spin $\frac{1}{2}$ [B3, Section 2.5], and electric charge given by its spinor chirality [B3, Section 6]. In particular, the chiral spinors

$$\psi^- := P^- \psi = \frac{1}{2} (1 - \gamma^5) \psi \quad \text{and} \quad \psi^+ := P^+ \psi = \frac{1}{2} (1 + \gamma^5) \psi$$

have a negative and positive electric charge, respectively. Recall that pointal particles only interact by creation and annihilation in pairs. It follows that pointal particles are described by the chiral decomposition of the free Dirac Lagrangian:

$$L := \bar{\psi} (i \slashed{D} - m) \psi,$$

where $\bar{\psi} := (\psi^\dagger) \gamma^0 \bar{\psi} = \psi \bar{P} \bar{\psi}$. The mass terms, $m \bar{\psi}^- \psi^+$ and $m \bar{\psi}^+ \psi^-$, represent points in $\tilde{M}$ where two pointal particles of opposite charge are created or annihilated; we call these points apices.

Recall that a timelike pointal particle $\beta(t)$ may be dually described by a particle with a circular trajectory about $\beta$ in $V$, by (2). We denote the dual particle’s radius by $r > 0$, tangential velocity by $u \geq 0$, and angular velocity by $\omega = ur^{-1}$.

We define the mass $m$ of the pointal particle to be

$$m := r^{-1} = \frac{\hbar}{rc}.$$  

The rest energy $E_0 = \omega$ of the particle is then

$$E_0 = \omega = \frac{u}{r} = mu = mcu,$$

where units have been restored on the right. We thus derive a variant of Einstein’s relation $E_0 = mc^2$; Einstein’s relation holds if and only if the tangential velocity $u$ equals the speed of light $c = 1$. This allows a model of off-shell particles for which relativity is never violated: a particle is on-shell if $u = 1$, and off-shell otherwise. However, the energy-momentum relation $E_0^2 = m^2 u^2 = p^2$ always holds [B3, Section 3.4].

The relation (8) modifies the Dirac Lagrangian to

$$L = \bar{\psi} (i \slashed{D} - mu) \psi = \bar{\psi} (i \slashed{D} - \phi) \psi,$$

where we have set $\phi := mu$. The inclusion of the tangential velocity $u$ in $L$ allows $m$ to be promoted to the real scalar field $\phi$ on $\tilde{M}$. Furthermore, this scalar field is coupled to the two mass terms, $\bar{\psi}^- \psi^+$ and $\bar{\psi}^+ \psi^-$, which in turn causes the mass terms to be coupled together,

$$\phi (\bar{\psi}^- \psi^+ + \bar{\psi}^+ \psi^-).$$
We thus obtain a bound state of four chiral spinors. In particular, we may consider all possible excitations of the tensor product

\[(ab, cd) := \phi (\psi_a^- \otimes \psi_b^+ + \psi_c^- \otimes \psi_d^+),\]

where \(a, b, c, d \in \{\uparrow, \downarrow, 0\}\) specify the spin, up or down, of each excited spinor. For example, we have

\[\uparrow 0, 00 = \phi (\psi_\uparrow^- \otimes 1 + 1 \otimes 1) \simeq \psi_\uparrow^- \quad \text{and} \quad 0\uparrow, 00 = \phi (1 \otimes \psi_\uparrow^+ + 1 \otimes 1) \simeq \psi_\uparrow^+.\]

Thus \([\uparrow 0, 00]\) and \([0 \uparrow, 00]\) represent an electron \(e\) and positron \(\bar{e}\) respectively, both with spin up. Similarly, the more complicated excitation

\[\uparrow \downarrow, \downarrow 0 = \phi (\psi_\uparrow^- \otimes \psi_\downarrow^+ + \psi_\downarrow^- \otimes 1).\]

should also represent some elementary particle. Just as Dirac predicted the existence of the positron from his equation in 1928, we would like to retrodict all of the standard model particles from his equation.

**Definition 3.1.** We call an excitation \([ab, cd]\) of the Dirac Lagrangian (7) an *atom*. The pairs \(ab\) and \(cd\) (resp. \(ad\) and \(bc\)) are the *orbitals* (resp. *off-orbitals*) of \([ab, cd]\); the orbitals are bound together since they are both coupled to the scalar field \(\phi\).

We note the following:

(i) By (9), the two orbitals may be freely swapped,

\[ab, cd = [cd, ab].\]

(ii) Two spinors with the same charge and spin cannot exist together in an atom by the Pauli exclusion principle. For example, the configuration \([\uparrow \downarrow, \uparrow 0]\) violates the Pauli exclusion principle since it contains two spinors that are identical.

3.2. **Color charge of massive atoms.** We denote a spinor with color charge in an atom by \(\uparrow\uparrow\uparrow\) or \(\downarrow\downarrow\downarrow\). The sign of the charge (that is, the orientation \(o_\beta\) of the tangent form \(*v\)) is specified by the sign of the slot the spinor occupies, \([-+, -+].\)

Let \(a, b \in \{\uparrow, \downarrow\}\) be spinors of opposite sign in an atom. Recall the representations (3) - (5) of charge conjugation \(\langle C \rangle\). From the relation

\[\rho_L \rho_L = \rho_0 = \rho_V \rho_V,\]

we obtain

\[(ab) = ab.\]

By the construction of the spin vector from the pointal metric and the Hodge star operator [B3 Section 2.5], all the spinors in a massive atom have the same spin line \(L \subset V\). Thus, if \(L\) is distinguished under the action of \(C\), then all of the spinors in the atom have the same (unsigned) color \(\rho_L\). However, (10) then implies that the color charge may be omitted from any pair of spinors of opposite sign in the atom. For example,

\[\downarrow \uparrow, \uparrow 0 = [\downarrow \uparrow, \uparrow 0] = [\downarrow \uparrow, \uparrow 0].\]
In particular, the color charge may be assigned to either spinor of the same sign without changing the atom itself. Furthermore, since bosonic atoms consist of an even number of spinors, no massive bosonic atom (of minimal charge) carries a net color charge. For example,

\[
\begin{align*}
[\downarrow\downarrow, \uparrow\uparrow] &= [\downarrow\downarrow, \uparrow\uparrow] = [\downarrow\downarrow, \uparrow\uparrow] = [\downarrow\downarrow, \uparrow\uparrow] = [\downarrow\downarrow, \uparrow\uparrow].
\end{align*}
\]

3.3. Color charge of massless atoms. If an atom is massless, then it has no inertial frame. Thus, color charge is not intrinsically defined for its constituent pointal particles. However, suppose a segment of a pointal curve is a piece-wise null geodesic, and the segment is maximal. Then the null segment has electric or color charges at its endpoints. Furthermore, these charges should coincide since the segment, being null, effectively removes their spacetime separation (similar to the twistor picture of null geodesics). Therefore there is no restriction on the number of color charges a massless atom can ‘carry’.

In Section 3.9 below, we propose that the mass of an atom without \(\phi\) comes from its spin angular momentum. Thus, the only massless atoms (without \(\phi\)) are \([\uparrow\downarrow, 00]\) and \([\downarrow\uparrow, 00]\), with possible color charges on the spinors. To obtain agreement with the masses of the SM particles, we assume that the spin angular momentum of a spinor with color charge is not equal to that of a spinor without color charge (though we do not know why this should be the case). Under this assumption, the atom \(11\) \([\downarrow\uparrow, 00]\) is not massless. Thus, its constituent pointal particles have the same spin line \(L\). Since \(L\) is either distinguished or not distinguished, its pointal particles either both have color charge, or both not. Consequently, the configuration \(11\) is not possible.

3.4. Charged scalar field. We allow the scalar field \(\phi\) to acquire the electric charge of an unexcited spinor if the other spinor in the orbital is excited:

\[
\begin{align*}
[ab, c\phi] &:= \phi^+(\psi_a^- \otimes \psi_b^+ + \psi_c^- \otimes 1) \quad \text{and} \quad [ab, \phi c] := \phi^-(\psi_a^- \otimes \psi_b^+ + 1 \otimes \psi_c^+).
\end{align*}
\]

By abuse of terminology, we say that \(\psi_c\) and \(\phi\) belong to the same orbital. Since \(\phi\) is the energy of the excited spinors, \(\phi = mu = r^{-1}u = \omega\), \(\phi\) can only be ‘turned on’ as a scalar field if a spinor is excited. Thus, \(\phi\) cannot acquire the electric charge of an unexcited spinor if both spinors in the orbital are unexcited.

Naively, an orbital containing \(\phi\) can assume four possible configurations,

\[
\begin{align*}
\downarrow\phi, \quad \uparrow\phi, \quad \phi\downarrow, \quad \phi\uparrow.
\end{align*}
\]

However, such an orbital is neutral of spin \(1/2\), and so has only two possible states. Thus, two of the four configurations in \(12\) are unphysical, and so must be discarded.

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4The bosonic atoms with non-minimal charge can carry color charge, however; see Section 3.6.
In Table 1, we have chosen to discard $\uparrow \phi$ and $\phi \uparrow$. In Section 3.8, we will show that this exclusion is the fundamental reason for electroweak parity violation.

Now consider such an orbital with nonzero charge $q$. Since the two slots of the orbital have opposite sign, and both slots are occupied (one by $\phi$ and one by a spinor), the orbital itself can only be charged if it has color charge. Furthermore, since $\phi$ is a scalar field, it has no spin vector, and thus $\phi$ cannot possess color charge. Therefore the spinor must have color charge. In particular, there are only four possible configurations of the orbital, rather than eight:

$$\downarrow \phi, \uparrow \phi, \phi \downarrow, \phi \uparrow.$$

3.5. Spin of an atom. If an atom $\alpha$ contains an even (resp. odd) number of spinors, then $\alpha$ is bosonic (resp. fermionic).

**Definition 3.2.** The momentum type of an (off-)orbital is specified by the number of spinors (0, 1, or 2), aligned spinors (0 or 2), and color charges (0 or 1) the orbital contains. Thus, there are four momentum types without color charge:

$$\{00\}, \{\uparrow 0, \downarrow 0, \downarrow \phi, 0 \uparrow, 0 \downarrow, \phi \downarrow\}, \{\uparrow \downarrow, \downarrow \uparrow\}, \{\uparrow \uparrow, \downarrow \downarrow\}.$$

We call a maximum set $p$ of atoms with fixed orbital momentum types and total charge a basis set.

We associate a unique SM particle to each basis set $p$ of nonzero charge. For example, using the labeling of atoms given in Table 1 the basis sets

$$e := \{e_\uparrow, e_\downarrow\}, \quad \bar{e} := \{\bar{e}_\uparrow, \bar{e}_\downarrow\}, \quad W^\pm := \{W_0^\pm, W_\uparrow^\pm, W_\downarrow^\pm\},$$

each correspond to a SM particle. The atoms in a basis set $p$ are either all bosons or all fermions, and form a basis of spin states for the particle. The spin of the particle is therefore determined by the number of atoms in $p$: a massive atom has spin $0$ (resp. $\frac{1}{2}, 1$) if $|p| = 1$ (resp. $2, 3$).

In our pointal model, we model the photon and $Z$ boson by the sets

$$\gamma := \{\gamma_0\} \cup \{\gamma_\uparrow, \gamma_\downarrow\} \quad \text{and} \quad Z := \{Z_0\} \cup \{Z_\uparrow, Z_\downarrow\},$$

each of which consists of three neutral bosonic atoms. If a photon is on-shell (i.e., massless), then it can only be an excitation of one of the two transverse polarizations $\gamma_\uparrow$ or $\gamma_\downarrow$, whereas if it is off-shell (massive), then it may also be an excitation of the longitudinal mode $\gamma_0$.

Finally, according to Table 1 each basis set $p$ consists of at most three atoms, with one exception:

$$h := \{h_1, h_2, h_3, h_4\}.$$

Observe that $h$ is partitioned into two sets, $\{h_1, h_2\}$ and $\{h_3, h_4\}$, by flipping the spinors of each atom. Thus, since each $h_j \in h$ is bosonic, and there is no integer spin containing four states, $h$ must correspond to either two massless vector bosons; one massless vector boson and two spin-0 bosons; or four spin-0 bosons. By the mass
relations we consider in Section 3.9, none of the $h_j$ atoms are massless. Therefore each $h_j$ atom must have spin 0. Consequently, these four neutral massive spin-0 atoms are ‘Higgs-like’ particles.

3.6. Non-minimal excitations. There are six excitations of the Dirac Lagrangian $\mathcal{L}$ that do not have minimal charge:

| charge $q$ | non-minimal excitations of $\mathcal{L}$ |
|-----------|------------------------------------------|
| $q = 2q^-$ | $x = [\uparrow 0, \downarrow 0]$ |
| $q = 2q^+$ | $\bar{x} = [0 \uparrow, 0 \downarrow]$ |
| $q = 2c^- + q^+$ | $y_{\uparrow} = [\uparrow 0, \downarrow \phi]$ |
| $q = 2c^+ + q^-$ | $\bar{y}_{\downarrow} = [0 \uparrow, \phi \downarrow]$ |

We assume that these states are unphysical, but our theory needs further development to justify this claim.

3.7. A preon model. A preon model of the standard model particles is given in Table 1. Recall that a pointal particle either has electric charge $q^{o_\beta} = (o_\beta, \rho_V) \cong o_\beta$ or color charge $c^{o_\beta} = (o_\beta, \rho_L)$, where $o_\beta \in \{\pm 1\}$ is the orientation of its tangent form $*v$. The charge of an atom, which we denote $q$, is the formal sum of its constituent charges. The SM electric charge of each atom is then given by the substitutions $q^\pm \mapsto \pm 1$ and $c^\pm \mapsto \pm \frac{1}{3}$.

We say a spinor is lone if it is not in an orbital with a spinor of opposite charge. For example, the orbitals $\uparrow 0$, $\uparrow \phi$, $\uparrow \downarrow$ contain lone spinors, whereas the orbitals $\uparrow \downarrow$, $\uparrow \uparrow$ do not.

**Definition 3.3.** Consider two atoms $\alpha_1$, $\alpha_2$ which contain lone spinors, and whose orbitals have equal momentum types.

(i) Choose an ordering $[ab, cd]$ or $[cd, ab]$ of each atom, such that the outer two slots are occupied by lone spinors if and only if the inner two slots are occupied by lone spinors.

(ii) Allow any number of apices to form between $\alpha_1$ and $\alpha_2$ which occur between slots of opposite charge in orbitals of equal momentum type.

If the component-wise sum $\alpha_3$ of the resulting two atoms is again an atom, then we say the fusion of $\alpha_1$ and $\alpha_2$ is $\alpha_3$, and write

$$\alpha_1 \ast \alpha_2 \rightarrow \alpha_3.$$

In our preon model, a fusion $\alpha_1 \ast \alpha_2 \rightarrow \alpha_3$ corresponds to the interaction vertex $\alpha_1\alpha_2\alpha_3$. As such, fusions may be rotated in spacetime. Examples of fusions are given in Figure 1; lines between slots denote apices (creation or annihilation). Note that our diagrams are similar to ‘t Hooft’s double line formalism for the quark-gluon interactions [tH]. The following is straightforward to verify.
Proposition 3.4. The fusion rules produce exactly the standard model trivalent vertices involving fermions and vector bosons:

\[ f^* f \rightarrow Z, \quad W^+ W^- \rightarrow Z, \quad \ell^* \nu_\ell \rightarrow W, \quad q^* q \rightarrow g, \]

\[ f^c f \rightarrow \gamma, \quad W^+ W^- \rightarrow \gamma, \quad d^* \bar{d} \rightarrow W, \quad g^* g \rightarrow g, \]

where \( f \) is a fermion, \( f^c \) is a charged fermion, \( q \) is a quark, \( d \) is a down-type quark, \( u \) is an up-type quark, \( \ell \) is a charged lepton, and \( g \) is a gluon.

3.8. A derivation of electroweak parity violation. An important feature of the standard model is electroweak parity violation, namely, that \( e_R \) transforms as an SU(2) singlet, whereas \( (\nu_e e_L) \) transforms as an SU(2) doublet. In our model we replace the chiral states \( e_L, e_R \) with the spin states \( e^\uparrow, e^\downarrow \), since chirality is electric charge in the context of nonnoetherian geometry (see [B3, Section 6] and Section 4.2 below).

Proposition 3.5. The pointal model yields leptonic electroweak parity violation.

Proof. The leptons \( e^\uparrow, \mu^\uparrow, \tau^\uparrow \) can fuse with the respective neutrinos \( \nu_e, \nu_\mu, \nu_\tau \) to form \( W^- \) (see Figure 1). However, the leptons with opposite spin, namely \( e^\downarrow, \mu^\downarrow, \tau^\downarrow \), cannot fuse with these neutrinos since their sums violate the Pauli exclusion principle:

\[
\begin{array}{c|c|c}
 e^\uparrow & \nu_e & \nu_\mu \\
 0 \downarrow \phi & [\phi, 0] & [\phi, \downarrow] \\
 \nu_\tau & [\phi, \downarrow] & [\phi, \downarrow] \\
 0 \downarrow \phi & [\phi, 0] & [\phi, \downarrow] \\
\end{array}
\]

Furthermore, \( e^\downarrow, \mu^\downarrow, \tau^\downarrow \) cannot fuse with \( \bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau \) by rule (i) in Definition 3.3. Similarly, \( \bar{e}^\downarrow, \bar{\mu}^\downarrow, \bar{\tau}^\downarrow \) can fuse with \( \bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau \) to form \( W^+ \), but \( \bar{e}^\uparrow, \bar{\mu}^\uparrow, \bar{\tau}^\downarrow \), cannot fuse with any of the neutrinos. \( \Box \)

We thus obtain a derivation of the parity violation of electroweak interactions in the lepton sector. Another derivation of parity violation was recently given in [F], and it would be interesting to understand whether the two approaches are related.

3.9. Mass orderings and orders of magnitude. In the following, we propose a simplified model where the mass of an atom primarily comes from its spin angular momentum.

Consider an atom \( \alpha = [ab, cd] \). If the spin angular momenta of both orbitals is zero, that is, if both orbitals are in \{00, ^\uparrow_\downarrow, ^\downarrow_\uparrow\}, then we define the mass of \( \alpha \) to be zero. Thus, by Table 1 the only atoms (without \( \phi \)) that have zero mass are the photon and gluon atoms.

So suppose that at least one orbital of \( \alpha \) has nonzero spin angular momentum.

\footnote{In particular, our model does not admit sterile neutrinos, that is, right-handed neutrinos and left-handed anti-neutrinos that do not interact with the \( W^\pm \) atoms.}
(i) First suppose that $\alpha$ does not contain $\phi$. Let $\mu_{14}$ and $\mu_{23}$ be the respective spin angular momenta of the off-orbitals $(a,d)$ and $(b,c)$. We propose that the mass $m$ of $\alpha$ is approximately

$$\log m \sim \mu_{14} + \mu_{23} + \delta,$$

where $\delta = \delta_0 > 0$ if $\alpha$ contains precisely two 0 slots, and $\delta = 0$ otherwise. (Note that (13) cannot model a massless atom since it is a logarithm.)

We order the off-orbital spin momenta by their type,

$$\text{off-orbital} = \begin{pmatrix} (0,0) \lessdot (\uparrow,0) \lessdot (\uparrow,\uparrow) \lessdot (\uparrow,\downarrow) \end{pmatrix}$$

$$\mu_{ij} = \begin{pmatrix} 0 & 1 & 2 & 3 \end{pmatrix}$$

In the top row a representative of each off-orbital momentum type is shown, and in the bottom row we have chosen specific numbers to test the validity of (13). The constant $\delta_0$ is then fixed by requiring all three $Z$ atoms have the same mass. Whence, for these values, $\delta_0 = 4$. We thus find

$$\log m \sim 1 < 3 < 4 < 6 < 7$$

The experimental values for the orders of magnitude of the masses are given in the bottom row, where we have rounded each mass to the nearest power of 10. We find that there is perfect agreement between our model (all shifted by $-1$ for units of MeV/c$^2$) and the experimental values. In particular, from only two free parameters, namely $\mu_{ij}$ for $(\uparrow,\uparrow)$ and $(\uparrow,\downarrow)$, we are able to derive the correct relative orders of magnitude for the masses of eight particles. However, we caution that (13) must be modified to account for color charge, since even though the electron mass, 0.511, and the down quark mass, 4.4, both round to 1 in powers of 10, they also nearly differ by a factor of 10.

(ii) Now suppose $\alpha$ contains $\phi$.

We say a representative $[ab,cd]$ of an atom is uniform if each of its spinors has the same unsigned color charge $c \in \{r,g,b\}$; such representatives always exist by (10).

Consider the qualitative rule:

*If a 0 is replaced by $\phi$ in an off-orbital without color (resp. with color) in a uniform representative of an atom, then the atom’s mass decreases (resp. increases).*

From this rule and Table 1 we obtain the following inequalities, each of which agrees with experiment:

$$m(\nu_e) < m(e), \quad m(\nu_\mu) < m(\mu), \quad m(\nu_\tau) < m(\tau), \quad m(W^\pm) < m(h),$$

$$m(d) > m(u), \quad m(s) < m(c), \quad m(b) < m(t).$$
We note that it is nontrivial that our model is able to reverse the inequality (⋆), given that \(d, s\) (resp. \(u, c\)) have the same charges.

To summarize, our model produces fifteen independent mass orderings of the standard model particles that all agree with experiment.

3.10. Decay channels of the Higgs-like atoms. In Section 3.5, we found that the four neutral \(h\) atoms have spin 0. Furthermore, using our mass model (13) in Section 3.9, we found that the two \(h\) atoms

\[
h_1 = [0^\uparrow, 0^\uparrow] \quad \text{and} \quad h_2 = [0^\downarrow, 0^\downarrow]
\]

have masses that are of the same order of magnitude as the Higgs boson \(H\). We thus may model \(H\) by the atoms \(h_1\) and \(h_2\), though the interaction vertices involving these particles differ from those of \(H\):

**Lemma 3.6.** The only fusions involving the \(h_j\) atoms are

\[
W^- \ast W^+ \to h_{1,2,3,4}, \quad h_{1,2,3,4} \ast h_{1,2,4,3} \to Z, \quad h_{1,2,3,4} \ast h_{2,1,3,4} \to \gamma.
\]

**Proof.** We have the fusions

\[
\begin{array}{cccccc}
W_0^- & [\uparrow 0, \downarrow \phi] & W_0^- & [\uparrow 0, \downarrow \phi] & W_+^- & [\uparrow 0, \phi, \downarrow] & h_2 & [\downarrow 0, 0, \downarrow] & h_1 & [\uparrow 0, 0, \uparrow] \\
W_0^+ & [\phi, \downarrow, 0 \uparrow] & W_0^+ & [\phi, \downarrow, 0 \uparrow] & W_+^+ & [\phi, 0 \uparrow, \downarrow/\downarrow] & h_2 & [0 \downarrow, 0 \downarrow] & h_2 & [0 \downarrow, 0 \downarrow] \\
h_1 & [\uparrow 0, 0 \uparrow] & h_2 & [0 \downarrow, 0 \downarrow] & h_{1/4} & [\uparrow 0, 0 \uparrow/\downarrow] & Z & [\downarrow, 0 \downarrow] & \gamma & [\uparrow, 0 \uparrow]
\end{array}
\]

However, leptons and quarks are not able to form \(h_j\) by rule (i) of Definition 3.3. Indeed, in the following, lone spinors occupy both inner slots but no outer slots:

\[
e_\uparrow [0 \uparrow, 0 \uparrow] \quad \mu_\uparrow [0 \downarrow, 0 \uparrow] \quad \tau_\uparrow [0 \downarrow, 0 \uparrow] \\
\bar{e}_\downarrow [0 \uparrow, 0 \uparrow] \quad \bar{\mu}_\downarrow [0 \uparrow, 0 \uparrow] \quad \bar{\tau}_\downarrow [0 \uparrow, 0 \uparrow] \\
h_1 [0 \uparrow, 0 \uparrow] \quad h_1 [0 \uparrow, 0 \uparrow] \quad h_1 [0 \uparrow, 0 \uparrow]
\]

Finally, two \(Z\) atoms cannot form \(h_j\) since fused atoms must contain a lone spinor. \(\square\)

The lemma appears problematic since \(H \to ZZ\) is a common decay channel in the standard model. In particular, the four lepton decay channel,

\[
H \to ZZ^* \to 4\ell,
\]

has been experimentally observed, where \(Z^*\) is off-shell and \(\ell = e, \mu\). However, we may model the decay \(H \to ZZ\) with two intermediate steps,

\[
h \to Zh \to ZW^+W^- \to ZZ.
\]

Further work is needed to determine whether this decay is able to correctly model \(H \to ZZ\).
4. CPT invariance

4.1. Preliminary. To establish notation, we briefly review the standard derivation of the action of a Lorentz transformation \( \Lambda \in SO(1, 3) \) on a Dirac spinor \( \psi \).

To determine the action of \( \Lambda \) on \( \psi \), the Dirac projection \( (i/\partial - m) \) is required to be invariant under \( \Lambda \),

\[
\Lambda (i/\partial - m) \Lambda^{-1} = i/\partial - m.
\]

We denote a spinor representation of \( \Lambda \) by \( S_\Lambda \in \text{End}(\mathbb{C}^4) \), and a spacetime representation by \( \Lambda^\mu_\nu \); whence

\[
\Lambda . \psi(x^\mu) = S_\Lambda \psi(\Lambda^\mu_\nu x^\nu).
\]

Thus, (14) implies

\[
(S_\Lambda \gamma^\nu)(\Lambda^\mu_\nu \partial^\mu) = \gamma^\mu \partial^\mu S_\Lambda = \gamma^\mu S_\Lambda . \psi.
\]

We may therefore obtain \( S_\Lambda \) (unique up to a phase) by imposing the constraint

\[
S_\Lambda \gamma^\nu \Lambda^\mu_\nu = \gamma^\mu S_\Lambda.
\]

For example, the parity transformation

\[
P = (P^\mu_\nu) = \text{diag}(1, -1, -1, -1)
\]

acts on \( \psi \) by

\[
P . \psi(t, x^i) = \gamma^0 \psi(t, -x^i).
\]

4.2. Charge conjugation as a Lorentz transformation. The electron-photon interaction term \( e \bar{\psi} A \psi \) in the SM Lagrangian is of course absent from the free Dirac Lagrangian \( \mathcal{L} \). The only interaction terms of \( \mathcal{L} \) are the mass terms \( m \bar{\psi}^\pm \psi^\mp \). In contrast to \( e \bar{\psi} A \psi \), these terms should not change sign under charge conjugation since their coupling constants are mass \( m \), not electric charge \( e \).

Geometrically, \( \tilde{M} \) is not equipped with a \( U(1) \) gauge bundle. Instead, the electric charge of a pointal particle arises from the choice of orientation \( o_\beta \in \{ \pm 1 \} \) of its tangent form \( \ast v \).

Consider the four positive energy solutions \( (E = +\sqrt{|p|^2 + m^2}) \) of the Dirac equation \( (i\partial - m)\psi = 0 \) in the Dirac representation:

\[
\begin{align*}
u_1 &= N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z - ip_y}{E + m} \\ \frac{p_x + ip_y}{E + m} \end{pmatrix}, \\
u_2 &= N \begin{pmatrix} 0 \\ 1 \\ \frac{-p_z}{E + m} \\ \frac{-p_x}{E + m} \end{pmatrix}, \\
u_1 &= N \begin{pmatrix} \frac{p_z - ip_y}{E + m} \\ \frac{-p_z}{E + m} \\ 0 \\ 1 \end{pmatrix}, \\
u_2 &= N \begin{pmatrix} \frac{p_z + ip_y}{E + m} \\ \frac{-p_z}{E + m} \\ 0 \\ 1 \end{pmatrix},
\end{align*}
\]

where \( N = \sqrt{E + m} \).

In the standard model, charge conjugation \( C \) acts by reversing the sign of the electric charge \( e \),

\[
C (i\partial - eA - m) C^{-1} = i\partial + eA - m.
\]
Thus $C$ acts on $\psi$ by
\begin{equation}
C\psi = i\gamma^2 \psi^*.
\end{equation}

Consequently, $C$ exchanges spinors with anti-spinors,
\begin{align*}
C u_1 e^{ip_\mu x^\mu} &= v_1 e^{-ip_\mu x^\mu}, \\
C u_2 e^{ip_\mu x^\mu} &= v_2 e^{-ip_\mu x^\mu}.
\end{align*}

Of course, the transformation (18) does not correspond to any Lorentz transformation.

In the context of pointal particles, however, the difference between particles and anti-particles lies in the orientation $o_\beta$ of $\ast v$: a particle has positive orientation, and an anti-particle has negative orientation. Recall the $SO(V)$-invariant representation of charge conjugation, $\rho_V : \langle C \rangle \to O(V)^{SO(V)}$, in (3), namely, $\rho_V(C) = -1$. Here, $V$ is the spatial hypersurface of $T_\beta(t) \tilde{M}$ in the inertial frame of $\beta$. To extend this representation to the whole tangent space $T_\beta(t) \tilde{M}$, we allow $V$ to be the spatial hypersurface of $T_\beta(t) \tilde{M}$ in any inertial frame; then the Lorentz transformation corresponding to charge conjugation is
\begin{equation}
C = (C^\mu_\nu) = \text{diag}(-1, -1, -1, -1).
\end{equation}

The following lemma is necessary in order for our model to be consistent.

**Lemma 4.1.** The Lorentz transformation $C$ exchanges spinors with anti-spinors.

**Proof.** Since $C$ is a Lorentz transformation, we have
\[
C(i\partial - m)C^{-1} = i\partial - m.
\]

Thus, from (15) we obtain $S_C \gamma^\nu C^\mu_\nu = \gamma^\mu S_C$. Whence $S_C = \gamma^5$ (times any phase). Therefore
\begin{equation}
C \psi(t, x^i) = \gamma^5 \psi(-t, -x^i) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \psi(-t, -x^i),
\end{equation}

where the $\gamma^5$ matrix is in the Dirac representation. It follows that
\begin{align*}
C u_1 e^{ip_\mu x^\mu} &= (\gamma^5 u_1) e^{iC^\mu_\nu p_\nu x^\mu} = v_1 e^{-ip_\mu x^\mu}, \\
C u_2 e^{ip_\mu x^\mu} &= (\gamma^5 u_2) e^{iC^\mu_\nu p_\nu x^\mu} = v_2 e^{-ip_\mu x^\mu}.
\end{align*}

Consequently, $C$ exchanges spinors with anti-spinors.

**Remark 4.2.** The two spinor transformations $S_C = i\gamma^2$ and $S_C = \gamma^5$ both exchange spinors with anti-spinors, although the spinors that are exchanged are different. Specifically, $i\gamma^2$ exchanges $u_1$ and $u_2$ with $v_1$ and $v_2$ respectively, whereas $\gamma^5$ exchanges $u_1$ and $u_2$ with $v_2$ and $v_1$ respectively.

**Remark 4.3.** The operator $\gamma^5$ cannot act as charge conjugation if the term $e\bar{\psi}A\psi$ is included in the Lagrangian, since (17) does not hold if $S_C = \gamma^5$. 

4.3. **Time reversal without complex conjugation.** Time reversal is the Lorentz transformation

\[ T = (T^\mu_\nu) = \text{diag}(-1, 1, 1, 1), \]

which we take to act on the tangent space at a point of \( \tilde{M} \). It is standard to assume that \( T \) acts on the spatial velocity vector \( v \) of a particle by reversing its sign, since

\[ T \cdot v = \frac{dx}{d(-t)} = -\frac{dx}{dt} = -v. \]

Under this assumption, \( T \) acts on the position and momentum operators by

\[ T \hat{x} T^{-1} = \hat{x} \quad \text{and} \quad T \hat{p} T^{-1} = -\hat{p}. \]

Thus, time reversal acts on complex numbers by complex conjugation:

\[ T i T^{-1} = T \hat{x} T^{-1} = [\hat{x}, \hat{p}] = [\hat{x}, -\hat{p}] = -i. \]

Consequently, \( T \) acts on the Dirac projection by complex conjugation,

\[ T(i \frac{\partial}{\partial t} - m) T^{-1} = (i \frac{\partial}{\partial t} - m)^*. \]

From \([23]\), and the fact that \( \gamma^2 \) is pure imaginary, it follows that \( T \) acts on \( \psi \) by

\[ T \psi = \gamma^1 \gamma^3 \psi^*. \]

However, we claim that time reversal does not actually reverse the sign of the velocity vector \( v \), and so conjugation of the momentum operator \( \hat{p} \) by \( T \) does not reverse the sign of \( \hat{p} \).

Indeed, consider the worldline \( x^\mu = x^\mu(t) \) of a particle. The particle has four-velocity

\[ (\dot{x}^\mu) = \gamma(1, v), \]

where \( \gamma = (1 - |v|^2)^{-1/2} \). Observe that \( T \) leaves \( v \) unchanged:

\[ (T^\mu_\nu \dot{x}^\nu) = \gamma(-1, v). \]

In other words, the **output** of the spatial component of the time derivative \( \dot{x}^\mu \) is the spatial vector \( v \), and so the **spatial component** of the Lorentz transformation \( T \) acts on \( v \), not the time component. Therefore the particle’s spatial momentum, \( p = \gamma m v \), is also left unchanged by \( T \),

\[ (T^\mu_\nu \hat{p}^\nu) = \gamma m(-1, v). \]

Consequently, we have

\[ T \hat{p} T^{-1} = \hat{p}, \]

in contrast to \([21]\). Whence \([23]\) does not hold; instead, the Dirac projection remains invariant under time reversal, as it is a Lorentz transformation:

\[ T(i \frac{\partial}{\partial t} - m) T^{-1} = i \frac{\partial}{\partial t} - m. \]

---

6The relationship between nonnoetherian spacetime and first and second quantization is discussed in \([B3, \text{Sections 3, 7}]\).
Therefore, applying (15) we find
\begin{equation}
T.\psi(t, x^i) = \gamma^1 \gamma^2 \gamma^3 \psi(-t, x^i).
\end{equation}

**Remark 4.4.** The Lorentz transformation that does reverse the direction of \( p \) is charge conjugation \( C \), as we found in Section 4.2. Whence \( C \dot{p} C^{-1} = -\dot{p} \). But \( C \) also reverses the sign of \( x \), and so \( C \hat{x} C^{-1} = -\hat{x} \). Thus \( C \hat{x} C^{-1} = i \) by (22), and therefore \( C \) does not induce complex conjugation either.

4.4. **CPT invariance.** In Section 4.2, we showed that charge conjugation of a pointal curve is given by the Lorentz transformation \( C = -1/4 \). Furthermore, in Section 4.3, we showed that time reversal \( T \) does not induce complex conjugation.

To summarize, charge conjugation (20), parity (16), and time reversal (24) act on Dirac spinors in the pointal model by
\begin{align}
C.\psi(t, x^i) &= \gamma^5 \psi(-t, -x^i), \\
P.\psi(t, x^i) &= \gamma^0 \psi(t, -x^i), \\
T.\psi(t, x^i) &= \gamma^1 \gamma^2 \gamma^3 \psi(-t, x^i).
\end{align}

Each of these transformations act trivially on the Dirac projection \((i\partial - m)\).

There are two important consequences of (19) and (25), stated in Theorem 1.5:
- The Lorentz group \( O(1, 3) \), as a manifold, has four connected components, and so the quotient group \( O(1, 3)/SO^+(1, 3) \) has four elements. Using (19), *this quotient is generated by \( C \), \( P \), and \( T \):
\[
O(1, 3)/SO^+(1, 3) = \langle C, P, T \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.
\]
Furthermore, each of the four connected components of \( O(1, 3) \) contains precisely one of the transformations \( 1, C, P, \) or \( T \).
- The spacetime (resp. spinor) representation of the product \( CPT \) is the identity (resp. proportional to the identity):
\[
C P T = g^\mu_\nu T^\lambda{}_\nu = \delta^\mu_\nu \in SO^+(1, 3), \\
S C S P S T = \gamma^5 \gamma^0 (\gamma^1 \gamma^2 \gamma^3) = -i(\gamma^5)^2 = -i.
\]
(Recall that \( g^\mu_\nu \) may be taken to act on a Dirac spinor by multiplication by an arbitrary phase \( e^{i\theta} \), by (15).) Consequently, CPT invariance of the pointal model trivially holds.

**Remark 4.5.** The relationships between \( C, P, \) and \( T \) that we have obtained cannot occur in the standard model, because in the standard model charge conjugation does not correspond to a Lorentz transformation.

**Remark 4.6.** A rotation by \( 2\pi \) in spacetime corresponds to the Lorentz transformation \( (PT)^2 \). Furthermore, by (25),
\[
(S P S T)^2 = (\gamma^0 \gamma^1 \gamma^2 \gamma^3)^2 = -1.
\]
We therefore obtain the elementary fact that a spacetime rotation of a spinor \( \psi \) in \( \mathbb{C}^4 \) is trivial if and only if the rotation is a multiple of \( 4\pi \).
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Institut für Mathematik und Wissenschaftliches Rechnen, Universität Graz, Heinrichstrasse 36, 8010 Graz, Austria.

Email address: charles.beil@uni-graz.at