Gauge-invariance in one-loop quantum cosmology *

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Abstract

We study the problem of gauge-invariance and gauge-dependence in one-loop quantum cosmology. We formulate some requirements which should be satisfied by boundary conditions in order to give gauge-independent path integral. The case of QED is studied in some detail. We outline difficulties in gauge-invariant quantization of gravitational field in a bounded region.

1 Introduction

According to the Hartle–Hawking prescription [1], the wave function of the Universe is given by the Euclidean path integral over all four metrics with a given boundary three-geometry and all configurations of matter fields with some boundary conditions (for a review, see recent monograph [2]). Thus, at least at one-loop approximation the quantum cosmology is equivalent to the theory of path integration in a bounded region of the Euclidean space.

Past years, many contradicting results were reported for the scaling coefficient \( \zeta(0) \) calculated in different approaches. However, during last year most of these contradictions were removed. It was demonstrated [3] that the direct eigenvalue method and analytical expressions in terms of geometric quantities give equivalent results if one takes into account corrections [4] to

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the formulae of [5]. It was shown [6] that disagreement between covariant path integral and that over physical degrees of freedom is due to the use non-covariant measure in the latter one. Complete agreement is restored if one uses covariant path integral measure.

At the same time, some recent computations [7,8,9] indicate that the one-loop amplitudes on the Euclidean four-disk are gauge-dependent even in a framework of the same direct eigenvalue method. This problem is studied in the present work. We demonstrate, that to obtain gauge-independent path integral one should observe some consistency conditions, which are formulated in the next section. In the section 3 we show that these consistency conditions are violated whenever gauge-dependence was observed in QED. We also comment on invariant gauge conditions for gravitational perturbations.

2 Gauge-invariant boundary conditions

A gauge-invariant formulation of the path integral for gauge theories was suggested a long ago by Faddeev and Popov [10]. Consider a standard proof [11] of gauge-independence of path integral paying special attention to specific features of manifolds with boundaries.

Let $\Phi$ be a gauge field. Consider the path integral in some gauge $\chi$ with a gauge-fixing term added to classical action $S(\Phi)$:

$$Z(\alpha, \chi) = \int \mathcal{D}\Phi J(\chi) \exp(-S(\Phi) - \frac{1}{2\alpha} \chi^2)$$

(1)

Integration over space-time coordinates is assumed. $J(\chi)$ is the Faddeev-Popov determinant corresponding the gauge condition $\chi$. It is well known that the path integral (1) can be obtained from another path integral

$$Z(\chi_a) = \int \mathcal{D}\Phi J(\chi) \delta(\chi - a) \exp(-S(\Phi))$$

(2)

after averaging over $a$ with the weight $\exp(-\frac{1}{2\alpha} a^2)$. There is nothing specific at this point with respect to manifolds with boundaries. One can self-consistently define boundary conditions for $a(x)$ through boundary conditions for $\chi(\Phi)$. Hence, it is enough to study gauge-independence of the path integral

$$Z(\chi) = \int \mathcal{D}\Phi J(\chi) \delta(\chi) \exp(-S(\Phi)) .$$

(3)
The equivalence of two path integrals, $Z(\chi_1)$ and $Z(\chi_2)$ can be established by using the Faddeev-Popov trick. One should use twice the following representation of unity

$$1 = \int \mathcal{D}\xi J(\chi)\delta(\chi(\Phi + \delta\xi\Phi)),$$

where $\delta\xi\Phi$ are (linearized) gauge transformations of the field $\Phi$. One should insert (3) with $\chi = \chi_2$ in the integrand of $Z(\chi_1)$, change integration variables to $\Phi - \delta\xi\Phi$, and use again eq. (3) with $\chi = \chi_1$. This procedure can be done successfully if the two gauges $\chi_1$ and $\chi_2$ satisfy the following requirements.

(i) **Gauge-invariance of the boundary conditions.** Let

$$\mathcal{B}\Phi|_{\partial M} = 0$$

be a boundary condition for the fields $\Phi$ with some boundary operator $\mathcal{B}$, which describes Dirichlet, Neumann or mixed boundary conditions. There should exist boundary conditions

$$\mathcal{B}\xi|_{\partial M} = 0$$

for gauge transformation parameters $\xi$ such that

$$\mathcal{B}\delta\xi\Phi|_{\partial M} = 0.$$

The eq. (7) means that gauge transformations map the functional space defined by eq. (5) onto itself for some boundary conditions (6) imposed on gauge parameter $\xi$.

(ii) **Simultaneous admissibility of $\chi_1$ and $\chi_2$.** We call a gauge condition $\chi$ admissible if for given gauge-invariant boundary conditions (5), (6) the equation

$$\chi(\Phi + \delta\xi\Phi) = 0$$

has unique solution $\xi$ for every $\Phi$. The gauges $\chi_1$ and $\chi_2$ should be admissible for the same boundary operators $\mathcal{B}$ and $\mathcal{B}_\xi$.

Using the Faddeev-Popov trick one can easily demonstrate that $Z(\chi_1) = Z(\chi_2)$ with the integration regions defined by (3) with the same boundary operator $\mathcal{B}$ provided the conditions (i) and (ii) are satisfied.

\footnote{Note that we consider only linearized gauge transformations thus avoiding the question of Gribov ambiguities. This restriction is correct at least at one-loop approximation}
3 QED in a bounded region

Consider a simple example of QED on four-dimensional unit disk. The metric has the form
\[ ds^2 = dx_0^2 + x_0^2 d\Omega^2 , \]  
where \( d\Omega^2 \) is the line element on unit three-sphere.

Let us define gauge-invariant boundary conditions for electromagnetic vector potential \( A_\mu \). Gauge transformations of the \( A_\mu \) are
\[ \delta A_\mu = \partial_\mu \omega (x) . \] 
Suppose, that the boundary operator \( B_\omega \) is local and \( SO(3) \)-invariant. This means, that \( \omega \) satisfies either Dirichlet, \( B_\omega = 1 \), or Neumann boundary conditions, \( B_\omega = \partial_0 + C \), with some constant \( C \). It is easy to see, that for the former case the only gauge invariant local boundary conditions for \( A_\mu \) are relative or magnetic boundary conditions:
\[ A_i|_{@M} = 0, \quad (\partial_0 + 3)A_0|_{@M} = 0 \]  
\( i = 1, 2, 3 \). In the latter case it is possible to define gauge-invariant local boundary conditions only for \( C = 0 \). These are the so-called absolute or electric boundary conditions:
\[ \partial_0 A_i|_{@M} = 0, \quad A_0|_{@M} = 0 \]  
The eqs. (11) and (12) give the only local gauge-invariant boundary conditions. Of course, if we abandon the locality requirement, we obtain more boundary conditions satisfying (i).

The conditions (11) and (12) have one extra property, which makes them preferable. The quadratic form of the action is represented by a hermitian second order differential operator.

Let us now analyze gauge conditions satisfying the requirement (ii). As a reference gauge \( \chi_1 \) let us choose the Lorentz gauge
\[ \nabla^\mu A_\mu = \chi_1 = 0 . \]  
This gauge is admissible for both absolute and relative boundary conditions. Let check whether some popular gauges can be chosen as \( \chi_2 \).
The Coulomb gauge. In our coordinate system the Coulomb gauge condition takes the form
\[ (\nabla^i A_i) = \chi_c = 0 \quad (14) \]
Here \((\nabla^i A_i)\) denotes the covariant derivative with respect to three-metric. It is easy to see, that the condition \((14)\) does not fix the gauge freedom completely. The gauge transformations with \(x_i\)-independent parameter, \(\partial_i \omega = 0\), \(i = 1, 2, 3\), remain unfixed. The space of such transformations is spanned by a one-parameter family of eigenfunctions of the scalar Laplace operator, \((x^0)^{-1} J_1(x^0 \lambda)\), where \(J_1\) is the Bessel function; the eigenvalues \(\lambda\) are defined by one of the conditions, \(J_1(\lambda) = 0\) or \((\partial - 1)J_1(\lambda) = 0\), depending on the boundary operator \(\mathcal{B}_\omega\). This means that an additional gauge fixing condition is needed. One can use e.g.
\[ \tilde{\chi} = < A_0 > = 0, \quad (15) \]
where \(< A_0 >\) denotes average value of \(A_0\) on a spatial slices. The condition \((15)\), however, can not be represented in a convenient way as gauge-fixing term in the action. In the view of the above consideration the disagreement between Lorentz and Coulomb gauges on a disk reported recently \[8\] looks quite natural.

Temporal gauge. In our coordinate system the temporal gauge \(A_0 = 0\) in fact coincides with the Fock radial gauge \[12\]. At first glance, this gauge condition does not fix the gauge freedom corresponding to gauge parameter depending on spatial coordinates only. However, looking at scalar harmonics on unit disk, which have the form \((x^0)^{-1} J_{l+1}(\lambda x^0)Y_l(x_i)\), we see, that all harmonics with non-trivial dependence on spatial coordinates, \(l \geq 1\), have zero in the origin of the coordinate system and thus can not be \(x^0\)-independent. This means that on a disk the temporal gauge fixes the gauge freedom completely. It can be seen, that this gauge is admissible in the sense of \(\text{(ii)}\).

For the boundary conditions \((11)\) and \((12)\) the Hodge–de Rham decomposition
\[ A_\mu = A^\perp_\mu + \partial_\mu \omega \quad \nabla^\mu A^\perp_\mu = 0 \quad (16) \]
is orthogonal with respect to ordinary inner product in the space of vector fields without surface terms. The Jacobian factor of the change of variables \(\{A\} \to \{A^\perp, \omega\}\) is just \(J^\frac{1}{2}\), where \(J\) is the ghost determinant in the Lorentz gauge. For any admissible gauge condition \(\chi\) one can express a solution of
equation $\chi(A) = 0$ in the form $A = A^\perp + \partial \omega(A^\perp)$. Thus $A^\perp$ can be used as coordinates on the space of solutions of a gauge condition $\chi$. In this coordinates the equivalence between Lorentz path integral and that in the gauge $\chi$ becomes evident.

**The Esposito gauge.** The Esposito gauge condition $[2, 7]$ reads

$$\chi_{\text{Esp}} = \partial_0 A_0 + (3) \nabla^i A_i = \partial^\mu A_\mu - \frac{3}{x^0} A_0 .$$  \hspace{1cm} (17)$$

Let us decompose spatial components $A_i$ in longitudinal and transversal parts:

$$A_i = A^T_i + \partial_i s , \quad (3) \nabla^i A^T_i = 0$$  \hspace{1cm} (18)$$

The eq. (17) gives

$$\partial_0 A_0 + (3) \Delta s = 0$$  \hspace{1cm} (19)$$

where $\Delta^3$ is the Laplace operator with respect to three-metric. Eq. (19) makes it possible to express $s$ in terms of $A_0$ and thus eliminate one scalar degree of freedom, as any gauge condition should do. Consider now the condition (19) on the boundary. Let $A_\mu$ satisfy the relative boundary conditions (11). We have

$$\chi_{\text{Esp}}(A_\mu)|_{\partial M} = -3A_0|_{\partial M} = 0$$  \hspace{1cm} (20)$$

This means that $A_0$ should in the same time satisfy Dirichlet and Newmann boundary conditions. Thus one more degree of freedom is excluded, and the Esposito gauge is incompatible with relative boundary conditions. The same is also true for absolute boundary conditions. This explains discrepancies $[9]$ between Lorentz and Esposito gauges on a disk.

### 4 Quantum gravity

The problem of formulation of gauge-invariant boundary condition for gravitational perturbations in much more complicated than that for electromagnetic field. One can demonstrate $[13, 14]$, that there are no gauge-invariant boundary conditions for quantum gravity, which are local for both graviton and ghost perturbations if boundary is not totally geodesic. Such boundary conditions can be formulated for quantum gravity with dynamical torsion in two dimensions $[13]$. However, it is not clear, whether this result can be extended to higher dimensions. Though the locality requirement seems to
be technical, local boundary conditions almost automatically lead to self-adjointness of the quadratic form of the action. For example, the Luckock-Moss-Poletti boundary conditions [15] do really lead to self-adjoint Laplace operator. Unfortunately, these boundary conditions are only partially invariant, in the agreement with the above statement.

At present, the non-local boundary conditions suggested by Barvinsky [16] are the best choice. These boundary conditions are gauge invariant. Recently, manifest computations on a disk were performed [14] and a new class of non-local boundary conditions was suggested [17]. However, self-adjointness of the quadratic form of the action has not been proved.

One can formulate most general gauge invariant boundary conditions for graviton fluctuations giving self-adjoint action at least on a disk [18]. Unfortunately, these boundary conditions have a very complicated form and are hardly suitable for actual computations. Probably, a more careful analysis of classical boundary problem is needed in order to formulate basic properties of quantum gravity in a bounded region.

5 Discussion

To the best of our knowledge, in all cases when gauge-dependence of on-shell amplitudes in one-loop quantum cosmology was observed, at least one of the requirement of Sec. 2 is violated. However, it was demonstrated that for QED in a region between two concentric spheres the \( \zeta(0) \) is gauge independent even if (ii) is violated [7-9]. Two explanations to this fact are possible. First, that the \( \zeta(0) \), being odd function of the orientation of normal vector on a boundary, is not sensitive to gauge non-invariant part of the path integral. Second, that since in this region a smooth 3+1 split is possible actual invariance of the path integral is higher than predicted for general case. A simple test which could help to choose between these two explanation may be a computation of more terms of the heat kernel expansion and/or computation of \( \zeta(0) \) for even-dimensional boundary.

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