ON THE MOMENTUM DIFFUSION OF RADIATING ULTRARELATIVISTIC ELECTRONS IN A TURBULENT MAGNETIC FIELD

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Received 2008 March 5; accepted 2008 April 4

ABSTRACT

Here we investigate some aspects of stochastic acceleration of ultrarelativistic electrons by magnetic turbulence. In particular, we discuss the steady state energy spectra of particles undergoing momentum diffusion due to resonant interactions with turbulent MHD modes, taking rigorously into account direct energy losses connected with different radiative cooling processes. For the magnetic turbulence we assume a given power spectrum of the type \(\mathcal{W}(k) \propto k^{-d}\). In contrast to the previous approaches, however, we assume a finite range of turbulent wavevectors \(k\), consider a variety of turbulence spectral indices \(1 \leq q \leq 2\), and concentrate on the case of a very inefficient particle escape from the acceleration site. We find that for different cooling and injection conditions, stochastic acceleration processes tend to establish a modified ultrarelativistic Maxwellian distribution of radiating particles, with the high-energy exponential cutoff shaped by the interplay between cooling and acceleration rates. For example, if the timescale for the dominant radiative process scales with the electron momentum as \(\propto p^{a}\), the resulting electron energy distribution is of the form \(n_e(p) \propto p^{2\exp\left[-(1/\alpha)(p/p_{eq})^{a}\right]}\), where \(a = 2 - q - r\) and \(p_{eq}\) is the equilibrium momentum defined by the balance between the stochastic acceleration and energy loss timescales. We also discuss in more detail the synchrotron and inverse-Compton emission spectra produced by such an electron energy distribution, taking into account Klein-Nishina effects. We point out that the curvature of the high-frequency segments of these spectra, even though they are produced by the same population of electrons, may be substantially different between the synchrotron and inverse-Compton components.

Subject headings: acceleration of particles — radiation mechanisms: nonthermal

1. INTRODUCTION

Stochastic acceleration of ultrarelativistic particles via scatterings by magnetic inhomogeneities was the first process discussed in the context of generation of a power-law energy distribution of cosmic rays (Fermi 1949; Davis 1956). Because the characteristic acceleration timescale for a given velocity of magnetic inhomogeneities, say, Alfvén velocity \(v_{A}\), is \(t_{ac} \propto (v_{A}/c)^{-2}\), the stochastic particle acceleration is often referred as a "second-order Fermi process." For commonly occurring nonrelativistic turbulence, \(v_{A} \ll c\), the turbulent acceleration mechanism is often deemed less efficient when compared to acceleration by shocks where the rate of momentum change \(\delta p/p \sim v_{sh}/c\) (hence, the name first-order Fermi process). However, here one also needs repeated crossings of the shock front by the particles which can come about via scattering by turbulence upstream and downstream of the shock. Thus, again the acceleration rate or timescale is determined by the scattering timescale. For nonrelativistic turbulence \(v_{A} \ll c\), relativistic particles \(p \gg mc^{2}\), and high-\(\beta\) or weakly magnetized plasma, this time is shorter than the stochastic acceleration time, which may not be the case in many astrophysical plasmas. We note that in a relativistic regime, for example, the first-order Fermi process encounters several difficulties in accelerating particles to high energies (e.g., Niemiec & Ostrowski 2006; Niemiec et al. 2006; Lemoine et al. 2006), while at the same time, stochastic particle energization may play a major role, since the velocities of the turbulent modes may be high, \(v_{A} \ll c\). And indeed, second-order Fermi processes have been discussed in the context of different astrophysical sources of high-energy radiation and particles, such as accretion disks (e.g., Liu et al. 2004, 2006), clusters of galaxies (e.g., Petrosian 2001; Brunetti & Lazarian 2007), gamma-ray bursts (e.g., Stern & Poutanen 2004), solar flares (e.g., Petrosian & Donaghy 1999; Petrosian & Liu 2004), blazars (e.g., Katarzyński et al. 2006b; Giebels et al. 2007), or extragalactic large-scale jets (e.g., Stawarz & Ostrowski 2002; Stawarz et al. 2004). We note that although turbulent acceleration is often a process of choice in modeling high-energy emission in different objects and that in fact there may be some other yet much less understood mechanisms responsible for the generation of this (like magnetic reconnection), evidence for the distributed (or in situ) acceleration process taking place in several astrophysical systems is strong (see, e.g., Jester et al. 2001; Kataoka et al. 2006; Hardcastle et al. 2007 in the context of extragalactic jets).

It was pointed out by Schlickeiser (1984, 1985) that continuous (stochastic) acceleration of high-energy electrons undergoing radiative energy losses tends to establish their ultrarelativistic Maxwellian energy distribution, as long as particle escape from the acceleration site is inefficient. This analysis concerned a particular case of the acceleration timescale independent of the electrons’ energy and the dominant synchrotron-type energy losses. Interestingly, very flat (inverted) electron spectra of the ultrarelativistic Maxwellian-type—often approximated as monoenergetic electron distributions—were discussed in the context of flat-spectrum radio emission observed from Sgr A* and several active galactic nuclei (see, e.g., Beckert & Duschl 1997; Birk et al. 2001 and references therein). More recently, it was proposed that such "nonstandard" electron spectra can account for the striking high-energy X-ray emission of large-scale

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jets observed by the Chandra satellite (Stawarz & Ostrowski 2002; Stawarz et al. 2004) or correlated X-ray and γ-ray (TeV) emission from several BL Lac objects detected by the modern ground-based Čerenkov telescopes (Katarzyński et al. 2006a; Giebels et al. 2007). In addition, it was shown that narrow electron spectra, e.g., Maxwellian distributions, can explain properties of extragalactic high brightness temperature radio sources (Tsang & Kirk 2007a, 2007b), alleviating the difficulties associated with the anticipated but not observed inverse-Compton (IC) catastrophe (Ostroero et al. 2006).

Motivated by these most recent observational and theoretical results, in this paper we investigate further some aspects of the stochastic acceleration of ultrarelativistic electrons by magnetic turbulence. In particular, we discuss steady state energy spectra of particles undergoing momentum diffusion due to resonant interactions with turbulent MHD modes, taking rigorously into account direct in situ energy losses connected with different radiative cooling processes. As described in § 2 we use the quasi-linear approximation for the wave-particle interactions, assuming a given power spectrum \( \gamma V(k) \propto k^{-4} \) for magnetic turbulence within some finite range of the turbulent wavevector \( k_1 \leq k \leq k_2 \), and consider turbulence spectral indices in the range \( 1 \leq q \leq 2 \). In § 3 we provide steady state solutions to the momentum diffusion equation corresponding to the case of no particle escape but different cooling and injection conditions. In § 4 some particular solutions are given corresponding to the case of a finite particle escape from the acceleration site. In § 5 we discuss in more detail synchrotron and IC emission spectra of stochastically accelerated electrons, taking into account Klein-Nishina effects. A final discussion and conclusions are presented in § 6.

2. GENERAL DESCRIPTION

Let us denote the phase-space density of ultrarelativistic particles by \( f(x, p, t) \), such that the total number of particles is \( N(t) = \int d^3x \int d^3p f(x, p, t) \). Here, the position coordinate \( x \) and the momentum coordinate \( p \) are not the position and the momentum of some particular particle, but are fixed to the chosen coordinate space and, therefore, are independent variables. In the case of collisionless plasma, the function \( f(x, p, t) \) satisfies the relativistic Vlasov equation with the acceleration term being determined by the Lorentz force due to the average plasma electromagnetic field acting on particles. This averaged field can be found, in principle, through the Maxwell equations, and such an approach would lead to the exact description of the considered system. However, due to the strongly nonlinear character of the resulting equations (and therefore substantial complexity of the problem), in most cases an approximate description is of interest. In the “test particle approach,” for example, one assumes a configuration of the electromagnetic field and solves the particle kinetic equation to determine the particle spectrum. Further simplification can be achieved if one assumes the presence of only a small-amplitude turbulence \((\delta E, \delta B)\) in addition to the large-scale magnetic field \( B_0 \gg \delta B, \) such that the total plasma fields are \( B = B_0 + \delta B \) and \( E = \delta E \).

In order to find the evolution of the particle distribution function in the phase space under the influence of such a fluctuating electromagnetic field, it is convenient to consider an ensemble of the distribution functions (all equal at some initial time), such that the appropriate ensemble averaging gives \( \langle \delta B \rangle = \langle \delta E \rangle = 0 \) and \( f(x, p, t) = (f(x, p, t) + \delta f(x, p, t)) \). It can then be shown via the “quasi-linear approximation” of the Vlasov equation that the ensemble average of the distribution function \( \langle f(x, p, t) \rangle \) satisfies the Fokker-Planck equation (Hall & Sturrock 1967; Melrose 1968). If, in addition, the particle distribution function is only slowly varying in space (“diffusion approximation”) and the scattering time (or mean free path) is shorter than all other relevant times (or mean free paths), the ensemble-averaged particle distribution function can be assumed to be spatially uniform and isotropic in \( p \), namely, \( \langle f(x, p, t) \rangle = \langle f(p, t) \rangle \), and the Fokker-Planck equation can be further reduced to the momentum diffusion equation (see Tsytovich 1977; Melrose 1980; Schlickeiser 2002).

The resulting momentum diffusion equation describing evolution of the particle distribution can be written as

\[
\frac{\partial}{\partial t} \langle f(p, t) \rangle = \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 D(p) \frac{\partial}{\partial p} \langle f(p, t) \rangle \right],
\]

where the momentum diffusion coefficient \( D(p) \) approximates the rate of interaction with the fluctuating electromagnetic field. Several other terms representing physical processes that may influence the evolution of the particle energy spectrum can be added to the diffusion equation (1). In particular, one can include continuous energy gains and losses due to direct acceleration (e.g., by shocks) and radiative cooling. Furthermore, if the diffusion of particles out of the turbulent region is approximated by a catastrophic escape rate (or time \( t_{\text{esc}} \)) and if there is a source term \( \tilde{Q}(p, t) \) representing particle injection into the system, then the spatially integrated (over the turbulent region) one-dimensional particle momentum distribution, \( n(p, t) = \int d^3x \langle f(p, t) \rangle \), is obtained from (see, e.g., Petrosian & Liu 2004)

\[
\frac{\partial n(p, t)}{\partial t} = \frac{\partial}{\partial p} \left[ D(p) \frac{\partial n(p, t)}{\partial p} \right] - \frac{\partial}{\partial p} \left[ \frac{2D(p)}{p} \langle \tilde{Q}(p, t) \rangle n(p, t) \right] - \frac{n(p, t)}{t_{\text{esc}}} + \tilde{Q}(p, t).
\]

Let us further assume the presence of an isotropic Alfvenic turbulence described by the one-dimensional power spectrum \( \gamma V(k) \propto k^{-q} \) with \( 1 \leq q \leq 2 \) in a finite wavevector range \( k_1 \leq k \leq k_2 \), such that the turbulence energy density \( \int k^q dk \gamma V(k) = (\delta B)^2/8\pi \) is small compared with the “unperturbed” magnetic field energy density, \( \zeta \equiv (\delta B)^2/B_0^2 < 1 \). The momentum diffusion coefficient in equations (1) and (2) can then be evaluated (e.g., Melrose 1968; Kulsrud & Pearce 1969; Schlickeiser 1989) as

\[
D(p) \approx \frac{c^2}{q^2} \frac{\zeta}{p^{3-q}} \propto p^q,
\]

4 Because of the expected high conductivity of the plasma, one can neglect the large-scale electric field, \( E_0 = 0 \).

5 The Fokker-Planck equation can be also derived straight from the definition of the function \( f(x, p, t) \), assuming that the interaction of the particles with turbulent waves is a Markov process in which every interaction (collision) changes the particle energy only by a small amount and that the recoil of the turbulent modes during the collision can be neglected (Blandford & Eichler 1987).
where $\lambda_A = 2\pi/k_1$ is the maximum wavelength of the Alfvén modes, $v_A \equiv \beta_c c$ is the Alfvén velocity, and $r_g = pc/eB_0$ is the gyroradius of the ultrarelativistic particles of interest here. Similar formulæ can be derived for the case of fast magnetosonic modes (e.g., Kulsrud & Ferrari 1971; Achterberg 1981; Schlickeiser & Miller 1998). This allows one to find the characteristic acceleration timescale due to stochastic particle-wave interactions, $t_{acc} \equiv p^2/D(p) \propto p^{2-q}/\lambda_A^2$. Similarly, the escape timescale due to particle diffusion from the system of spatial scale $L$ can be evaluated as $t_{esc} = L^2/\kappa_L \propto p^{2-q}$, where the spatial diffusion coefficient $\kappa_L = (1/3)\lambda L$ is given by the appropriate particle mean free path, $\kappa_L \approx (1/3)\kappa^2 r_g (\lambda/2r_g)^{-1} \propto p^{2-q}$, that can be found from the standard relation $D(p) = (1/3)\beta_A^2 p^2c/\lambda$ (for more details see, e.g., Schlickeiser 2002).

For convenience we define the dimensionless momentum variable $\chi \equiv p/p_0$, where $p_0$ is some chosen (e.g., injection) particle momentum. With this, the (stochastic) acceleration and escape timescales can be written as

$$t_{acc} = \tau_{acc} \chi^{2-q}, \quad \text{where} \quad \tau_{acc} = \frac{\lambda_A}{\zeta\beta_A^2 c} \left( \frac{p_0 c}{eB_0 \lambda_A} \right)^{2-q},$$

$$t_{esc} = \tau_{esc} \chi^{q-2}, \quad \text{where} \quad \tau_{esc} = \frac{9L^2}{\lambda \beta c} \left( \frac{p_0 c}{eB_0 \lambda A} \right)^{q-2}.$$ (4)

Hereafter we also consider regular energy changes, strictly energy losses, as being an arbitrary function of the particle energy as given by the appropriate timescale $t_{loss} = t_{loss}(p)$, namely, $(\dot{p}) = -(p/\tau_{loss})$. We define further $\tau = \tau_{acc}/\tau_{esc}$, $N(\chi, \tau) \equiv p_0\varrho(p, t)V$, and $Q(\chi, \tau) \equiv \tau_{esc}p_0\varrho(p, t)V$, where $V = \int d^3x$ is the volume of the system. With such definitions, the momentum diffusion equation (2) reads as

$$\frac{dN}{d\tau} = \frac{\partial}{\partial \chi} \left( \chi^q \frac{\partial N}{\partial \chi} \right) - \frac{\partial}{\partial \chi} \left[ (2\chi^{q-1} - \chi^2)N \right] - \varepsilon \chi^{2-q}N + Q,$$ (5)

or, in its steady state $\left( \partial N/\partial \tau = 0 \right)$ form, as

$$\frac{\partial}{\partial \chi} \left( \chi^q \frac{\partial N}{\partial \chi} \right) - \frac{\partial}{\partial \chi} \left[ (2\chi^{q-1} - \chi^2)N \right] - \varepsilon \chi^{2-q}N = -Q.$$ (6)

In the above, we have introduced

$$\vartheta_{\chi} \equiv \frac{\tau_{acc}}{t_{loss}(\chi)}, \quad \varepsilon \equiv \frac{\tau_{acc}}{\tau_{esc}}.$$ (7)

Some specific solutions to equation (5) were presented in the literature. The majority of investigations concentrated on the “hard-sphere approximation” with $q = 2$, i.e., with the mean free path for the particle-wave interaction independent of particle energy ($\Lambda = \zeta\lambda_A^2/3$; “classical” Fermi II process). It was found that in the absence of regular energy losses ($\vartheta_{\chi} = 0$), the steady state solution of equation (6) with the source term $Q(\chi) \propto \delta(\chi - \chi_{inj})$, where $\delta(\chi)$ is the Dirac $\delta$-function, is of a power-law form $N(\chi > \chi_{inj}) \propto \chi^{-\sigma}$ with $\sigma = -(1/2) + (9/4) + c/12$ (Davis 1956; Achterberg 1979; Park & Petrosian 1995). Note that for $c \ll 1$ this can be approximated by $\sigma \approx 1 - \varepsilon/3$, which is the original result obtained by Fermi (1949). In addition, with the increasing escape timescale, $\varepsilon \rightarrow 0$, the steady state solution approaches $N(\chi > \chi_{inj}) \propto \chi^{-1}$. This agrees with the general finding that for the range $1 \leq q < 2$ and the same injection conditions, the steady state particle energy distribution implied by equation (6) is $N(\chi > \chi_{inj}) \propto \chi^{-1-q}$, as long as the regular energy changes and particle escape can be neglected ($\vartheta_{\chi} = \varepsilon = 0$; Lacombe 1979; Borovskiy & Eilek 1986; Dröge & Schlickeiser 1986; Becker et al. 2006). The whole energy range $0 \leq \chi \leq \infty$ with the appropriate (singular) boundary conditions is considered in Park & Petrosian (1995).

The analytic investigations of the momentum diffusion equation (5) in the $q = 2$ limit including the radiative cooling have concentrated on the synchrotron-type losses $\vartheta_{\chi} \propto \chi$ (see, however, Schlickeiser et al. 1987; Steinacker et al. 1988). The extended discussion on the time-dependent evolution for such a case (eq. [5]) was presented by Kardashev (1962). As for the steady state solution (eq. [6]), it was found that with $Q(\chi) \propto \delta(\chi - \chi_{inj})$ and the range $0 \leq \chi \leq \infty$,

$$N(\chi > \chi_{inj}) \propto \chi^{\sigma+1} e^{-(\chi/\chi_{eq})} U \left( \sigma - 1, 2\sigma + 2, \frac{\chi}{\chi_{eq}} \right).$$ (8)

where $\sigma$ is the energy spectral index introduced above, the equilibrium momentum $\chi_{eq}$ is defined by the $t_{esc} = t_{loss}$ condition (yielding $\vartheta_{\chi} = \chi/(\chi_{eq})$, and $U(a, b, z)$ is a Tricomi confluent hypergeometrical function (Jones 1970; Schlickeiser 1984; Park & Petrosian 1995). For $\chi \ll \chi_{eq}$, i.e., for the particle momenta low enough to neglect radiative losses, the above distribution function has, as expected, a power-law form $N(\chi > \chi_{inj}) \propto \chi^{-\sigma}$. For $\chi \gg \chi_{eq}$ and $\varepsilon \ll 1$, the particle energy spectrum approaches $N(\chi > \chi_{inj}) \propto \chi^2 \exp(-\chi/\chi_{eq})$. That is, as long as particle escape is inefficient, all the two-component stationary energy distribution is formed: a power law $\propto \chi^{-1}$ at low $\chi < \chi_{eq}$ energies and a pile-up bump ("ultrarelativistic Maxwellian distribution") around $\chi \sim \chi_{eq}$. For shorter escape timescales no pile-up form appears, and the resulting particle spectral index depends on the ratio $\varepsilon$ of the escape and the acceleration timescales.

In the case of $q \neq 2$ and synchrotron-type energy losses $\vartheta_{\chi} \propto \chi$, the steady state solution to equation (6) provided by Melrose (1969) was questioned due to the unclear boundary conditions applied (Tademaru et al. 1971; Park & Petrosian 1995). The special case of $q = 1$
with particle escape included (and the infinite energy range $0 \leq \chi \leq \infty$ was considered further by Bogdan & Schlickeiser (1985). It was found that with the injection of the $Q(\chi) \propto \delta(\chi - \chi_{\text{inj}})$ type, the steady state solution of equation (6) is\footnote{The other (nonsynchrotron) radiative loss terms included in the analysis presented by Bogdan & Schlickeiser (1985) were omitted here for clarity.}

$$N(\chi > \chi_{\text{inj}}) \propto \chi^2 e^{-1/(2\chi/\chi_{\text{eq}})} \mathcal{U} \left( \frac{1}{2} \left( \frac{\chi_{\text{eq}}}{\chi_{\text{esc}}} \right)^2, 2; 1 - 2 \left( \frac{\chi}{\chi_{\text{eq}}} \right)^2 \right),$$  

(9)

where the critical escape and equilibrium momenta $\chi_{\text{esc}}$ and $\chi_{\text{eq}}$ are defined by the conditions $t_{\text{esc}} = t_{\text{acc}}$ and $t_{\text{esc}} = t_{\text{osc}}$, respectively, yielding $\varepsilon = 1/\chi_{\text{esc}}^2$ and $\vartheta_1 = \chi/\chi_{\text{eq}}$. This solution implies $N(\chi > \chi_{\text{inj}}) \propto \text{const}$ at low particle momenta for which synchrotron energy losses are negligible ($\chi \ll \chi_{\text{eq}}$), independent of the particular value of the escape timescale. At higher particle energies, an exponential dependence is expected, $N(\chi > \chi_{\text{inj}}) \propto \chi^{2 - (\chi_{\text{esc}}/\chi_{\text{eq}})^2} \exp \left[ -(1/2)(\chi/\chi_{\text{eq}})^2 \right]$. Note that with an increasing escape timescale this approaches $\sim \chi^2 \exp \left[ -(1/2)(\chi/\chi_{\text{eq}})^2 \right]$.

### 3. INEFFICIENT PARTICLE ESCAPE

In this section we are interested in steady state solutions to the momentum diffusion equation (6) in the case of a very inefficient particle escape and a general (i.e., not necessarily synchrotron-type) form of the regular energy changes $\vartheta_\chi$, which is however a continuous function of the particle energy. With $\varepsilon = 0$, the homogeneous form of this equation can therefore be transformed to the self-adjoint form

$$\frac{d}{d\chi} \left[ P(\chi) \frac{d}{d\chi} N(\chi) \right] - G(\chi) N(\chi) = 0,$$

(10)

with

$$P(\chi) = \chi^2 S(\chi), \quad \quad G(\chi) = \left[ 2(q-1)\chi^{-q-2} - \frac{d}{d\chi} (\chi \vartheta_1^2) \right] S(\chi), \quad \quad S(\chi) = \chi^{-2} \exp\int^\chi d\chi' \chi'^{-q} \vartheta_1^2.$$

(11)

We also restrict the analysis to the finite particle energy range $\chi \in [\chi_1, \chi_2]$, where $0 < \chi_1, \chi_2 < \infty$. The justification for this is that for a finite range of the turbulent wavevectors, say, $k \in [k_1, k_2]$, the momentum diffusion coefficient as given in equation (3) is well defined only for a finite range of particle energies (momenta). For example, gyroresonant interactions between the particles and the Alfveén turbulence require the particles’ gyroradii to be comparable to the scale of the interacting modes, or $kr_0 \sim 2\pi$. Hence, the lower and upper limits of the particle energy range could be chosen as $\chi_1 = 2\pi e B_0/k_1cP_0$ and $\chi_2 = 2\pi e B_0/k_2cP_0$, respectively.\footnote{In the case of the magnetoacoustic-type turbulence interacting with particles via transit-time damping satisfying the Cerenkov condition $kr_0 \ll 1$, the low-energy cutoff in the momentum diffusion coefficient could be chosen to be the energy of the particle whose velocity is comparable to the velocity of the fast magnetosonic mode, which is $\sim v_A$ for low-$\beta$ or magnetically dominated plasmas.}

Since all of the functions $P(\chi), P'(\chi), G(\chi)$, and $S(\chi)$ are continuous and $P(\chi)$ and $S(\chi)$ are finite and strictly positive in the considered (closed) energy interval, the appropriate boundary value problem,

$$a_1 N(\chi_1) + a_2 \frac{dN(\chi)}{d\chi} \bigg|_{\chi_1} = 0, \quad \quad b_1 N(\chi_2) + b_2 \frac{dN(\chi)}{d\chi} \bigg|_{\chi_2} = 0,$$

(12)

is regular. If one of these conditions is violated, which is the case for the infinite energy range $0 \leq \chi \leq \infty$, the problem becomes singular, and the extended analysis presented by Park & Petrosian (1995) has to be applied.

The two linearly independent particular solutions to the homogeneous form of equation (10) are

$$y_1(\chi) = S^{-1}(\chi), \quad y_2(\chi) = S^{-1}(\chi) \int^\chi d\chi' \chi'^{-q} S(\chi'),$$

(13)

or any linear combination of these (each involving arbitrary multiplicative constants),

$$u_1(\chi) = y_1(\chi) + \alpha y_2(\chi), \quad u_2(\chi) = y_1(\chi) + \beta y_2(\chi).$$

(14)

By imposing the boundary conditions (12) in the form

$$a_1 u_1(\chi_1) + a_2 \frac{du_1(\chi)}{d\chi} \bigg|_{\chi_1} = 0, \quad b_1 u_2(\chi_2) + b_2 \frac{du_2(\chi)}{d\chi} \bigg|_{\chi_2} = 0,$$

(15)

parameters $\alpha$ and $\beta$ can be determined. With thusly constructed particular solutions to equation (10), one can define the Wronskian $w(\chi) \equiv u_1(\chi)u_2' (\chi) - u_1' (\chi)u_2 (\chi)$ and next construct the Green’s function of the problem,

$$G(\chi, \chi_{\text{inj}}) = \frac{1}{-\chi_{\text{inj}}^2 w(\chi_{\text{inj}})} \times \begin{cases} u_1(\chi) u_2(\chi_{\text{inj}}), \quad \chi_1 \leq \chi < \chi_{\text{inj}}, \\ u_1(\chi_{\text{inj}}) u_2(\chi), \quad \chi_{\text{inj}} < \chi \leq \chi_2, \end{cases}$$

(16)
where $\chi_1 < \chi_{\text{inj}} < \chi_2$. This gives the final solution to equation (6),

$$N(\chi) = \int_{\chi_1}^{\chi_2} d\chi \mathcal{G}(\chi, \chi_{\text{inj}}) Q(\chi_{\text{inj}}).$$

(17)

Steady state solutions exist, however, only for some particular forms of the injection function $Q(\chi, \tau)$. To investigate this issue and to impose correct boundary conditions for the finite energy range $\chi_1 \leq \chi \leq \chi_2$, let us integrate equation (5) over the energies and rewrite it in the form of the continuity equation,

$$\frac{\partial N}{\partial \tau} + F|_{\chi_2} - F|_{\chi_1} = \int_{\chi_1}^{\chi_2} d\chi Q(\chi, \tau).$$

(18)

Here, $N \equiv \int_{\chi_1}^{\chi_2} d\chi N(\chi)$ is the total number of particles, and the particle flux in the momentum space is defined as

$$F(N(\chi)) = (2\chi^{q-1} - \chi \frac{\partial \chi}{\partial \chi}) N - \chi^q \frac{\partial N}{\partial \chi}.$$  

(19)

Note that with the particular solutions $u_1(\chi)$ and $u_2(\chi)$ given in (14), one has

$$F(u_1(\chi)) = -\alpha, \quad F(u_2(\chi)) = -\beta,$$

(20)

independent of the momentum $\chi$ or of the particular form of the direct energy loss function $\vartheta(\chi)$.

Let us comment in this context on the “zero-flux” boundary conditions of type (12), namely, $F|_{\chi_1} = F|_{\chi_2} = 0$. These, with equations (15) and (20), imply $\alpha = \beta = 0$, i.e., $u_1(\chi) = u_2(\chi)$. In other words, one particular solution $y_1(\chi)$ satisfies the “no-flux” boundary condition of the homogeneous form of equation (10) for both $\chi_1$ and $\chi_2$. In such a case, the steady state solution can be constructed using the function $y_1(\chi)$ only if it is orthogonal to the source function, $\int_{\chi_1}^{\chi_2} d\chi y_1(\chi) Q(\chi) = 0$. This condition, for any nonzero particle injection and $y_1(\chi) = S^{-1}(\chi)$ as given in equation (11), cannot be fulfilled (cf. Melrose 1969; Tademaru et al. 1971). “Zero-flux” boundary conditions for nonvanishing $Q(\chi)$ can instead be imposed if the particle injection is balanced by the particle escape from the system (see § 3 below).

In the case of no particle escape, with the stationary injection such that $\int_{\chi_1}^{\chi_2} d\chi Q(\chi) \equiv A$ and with the direct (radiative) energy losses $\vartheta(\chi) \neq 0$, the boundary conditions can be chosen as

$$-F|_{\chi_1} = A, \quad F|_{\chi_2} = 0,$$

(21)

which give $\alpha = A$ and $\beta = 0$, and correspond to the conservation of the total number of particles within the energy range $[\chi_1, \chi_2]$. Let us justify this choice by noting that the radiative loss processes, unlike momentum diffusion strictly related to the turbulence spectrum, is well defined for particle momenta $\chi < \chi_1$ and $\chi > \chi_2$. Hence, with nonvanishing radiative losses (as expected for the ultrarelativistic particles considered here), no flux of particles in the momentum space through the maximum value $\chi_2$ would higher energies is possible (radiative losses in the absence of stochastic acceleration will always prevent the presence of particle flux above $\chi_2$). For the same reason, there is always a possibility for a nonzero particle flux toward lower energies through the $\chi_2$ point, since the stochastic acceleration timescale, even if it is an increasing function of the particle energy, is always finite at $\chi_2 > 0$. Note in this context that the particle flux at $\chi_1$ implied by the chosen boundary conditions (21) must be negative, $F|_{\chi_1} < 0$. That is, there is a continuous flux of particles through the $\chi_1$ point from high to low energies, which—in the absence of particle catastrophic escape from the system—balances particle injection. With these, one can find the Green’s function as

$$\mathcal{G}(\chi, \chi_{\text{inj}})_{\text{loss}} = \begin{cases} S^{-1}(\chi) \left[ A^{-1} + \int_{\chi_1}^{\chi} d\chi' \chi'^{-q} S(\chi') \right], & \chi_1 \leq \chi < \chi_{\text{inj}}, \\ S^{-1}(\chi) \left[ A^{-1} + \int_{\chi_{\text{inj}}}^{\chi} d\chi' \chi'^{-q} S(\chi') \right], & \chi_{\text{inj}} < \chi \leq \chi_2, \end{cases}$$

(22)

where $S(\chi)$, introduced in the equation (11), can be rewritten as

$$S(\chi) = \chi^{-2} \exp \left[ \int_{\chi_{\text{inj}}}^{\chi} d\chi' \frac{t_{\text{esc}}(\chi')}{\chi' t_{\text{esc}}(\chi')} \right].$$

(23)

3.1. Synchrotron Energy Losses

As an example, let us consider synchrotron energy losses of ultrarelativistic electrons, which are characterized by the timescale (e.g., Blumenthal & Gould 1970)

$$t_{\text{syn}} = \tau_{\text{syn}} \chi^{-1}, \quad \text{with} \quad \tau_{\text{syn}} \equiv \frac{6\pi m_e c^2}{\sigma_T p_0 B_0^2},$$

(24)
and which define the equilibrium momentum \( \chi_{eq} = (\tau_{syn}/\tau_{acc})^{1/(3-q)} \) through the condition \( \tau_{acc} = \tau_{syn} \), yielding \( \partial = \chi/\chi_{eq}^{3-q} \). The Green’s function (22) adopts then the form

\[
G(\chi,\chi_{inj})|_{syn} \approx \chi^2 e^{-[1/(3-q)](\chi/\chi_{eq})^{3-q}} \left\{ \frac{1}{1+q} - \frac{\min(\chi_{inj},\chi)}{1+q} \right\} M \left( -1 + q - \frac{2 - 2q}{3 - q} \right)
\]

where \( \Gamma(a, z_1, z_2) \) is a generalized incomplete Gamma function. By expressing the above solution in terms of Kummer confluent hypergeometrical functions \( M(a, b, z) \) using the identity \( \Gamma(a, z_1, z_2) = a^{-1}z_1^2M(a, 1 + a, -z_2) - a^{-1}z_1^2M(a, 1 + a, -z_1) \) (Abramowitz & Stegun 1964), assuming \( \chi_1 \ll \chi_{eq} \), and noting that \( M(a, b, z) \sim 1 \) for \( z \to 0 \), one can rewrite it further as

\[
G(\chi,\chi_{inj})|_{syn} \approx \chi^2 e^{-[1/(3-q)](\chi/\chi_{eq})^{3-q}} \left\{ \frac{1}{1+q} - \frac{\min(\chi_{inj},\chi)}{1+q} \right\} M \left( -1 + q - \frac{2 - 2q}{3 - q} \right)
\]

Finally, noting that \( M[a, b, z] \sim \Gamma(b)e^{2\beta^{-1}b} \Gamma(a) \) for \( z \to \infty \) and neglecting the \( A^{-1} \) term, one finds a rough approximation

\[
G(\chi,\chi_{inj})|_{syn} \sim \left\{ \begin{array}{ll}
\frac{\chi^2 e^{-[1/(3-q)](\chi/\chi_{eq})^{3-q}}}{1+q}, & \min(\chi_{inj},\chi) \ll \chi_{eq}, \\
\chi_{eq}^2 e^{-2(\chi/\chi_{eq})^{3-q}}, & \chi_{eq} \ll \chi \ll \chi_{inj}, \\
\chi_{eq}^2 \chi_{inj}^4 e^{-[1/(3-q)](\chi_{inj}/\chi_{eq})^{3-q}}, & \chi_{inj} \ll \chi_{inj} < \chi.
\end{array} \right.
\]

Hence, as long as \( \min(\chi_{inj},\chi) < \chi_{eq} \), one has \( G(\chi,\chi_{inj})|_{syn} \propto \chi^2 \exp\left(-[1/(3-q)](\chi/\chi_{eq})^{3-q}\right) \). If, however, \( \min(\chi_{inj},\chi) > \chi_{eq} \), the Green’s function retains the spectral shape \( \propto \chi^2 \exp\left(-[1/(3-q)](\chi/\chi_{eq})^{3-q}\right) \) for \( \chi_{inj} < \chi \), while it is of a power-law form \( G(\chi,\chi_{inj})|_{syn} \propto \chi^{-2} \) for \( \chi < \chi_{inj} \).

In Figures 1 and 2 we plot examples of particle spectra obtained from the above solution for the system with fixed plasma parameters \( (B_0, \beta, \chi_{1}, \chi_{2}) \) but different turbulence energy indices, \( q = 2 \) (”hard-sphere” approximation; thick solid lines in the figures), 5/3 (Kolmogorov-type turbulence; thick dashed lines), and 1 (Bohm limit; thick dotted lines). As for the source function, we consider two different forms, namely, \( \dot{Q}(\chi) \propto \delta(\chi - 1) \) in Figure 1 and \( \dot{Q}(\chi) \propto \delta(\chi - 10^5) \) in Figure 2, with the normalizations given in both cases by the same fixed \( \int \dot{Q}(p) \) dp. The emerging particle spectra are compared with the ones expected for the same injection and cooling conditions, but with the momentum diffusion neglected, \( \dot{N}(\chi) \).

Such a steady state electron distribution can be found from the appropriate equation (see eq. [5])

\[
\frac{\partial}{\partial \chi} \left[ \chi \partial_\chi \dot{N}(\chi) \right] + Q(\chi) = 0,
\]
for which one has the straightforward solution (Kardashev 1962; thin solid lines in the bottom panels of Figs. 1 and 2)

\[ \tilde{N}(\chi) = \frac{1}{\chi} \int_{\chi_0}^{\chi_1} Q(\chi_{\text{inj}}) \, d\chi_{\text{inj}} \]  

As shown in the figures and as follows directly from the obtained solution (25)–(27), joint stochastic acceleration and radiative (synchrotron-type) loss processes, in the absence of particle escape, tend to establish \( N(\chi) \propto \chi^2 \exp\left(-\frac{1}{3}q\right)\left(\chi/\chi_{\text{eq}}\right)^{3-q} \) spectra independent of the energy of the injected particles and the form of the source function as long as it has a narrow distribution. Moreover, for \( \chi \ll \chi_{\text{eq}} \), the turbulence energy index \( q \) does not influence the spectral shape of the electron energy distribution. Instead—with fixed normalization of the injection function \( Q(p) \) and fixed plasma parameters (including magnetic field intensities \( B_0 \) and \( \zeta \)—the turbulence power-law slope \( q \) determines (1) the equilibrium momentum \( \chi_{\text{eq}} \), (2) normalization of the electron energy distribution, and (3) the spectral shape of the particle distribution for \( \chi \geq \chi_{\text{eq}} \). In particular, a flatter turbulent spectrum leads to higher values of the equilibrium momentum \( \chi_{\text{eq}} \), lower normalizations of \( N(\chi) \), and steeper exponential cutoffs at \( \chi > \chi_{\text{eq}} \). Note also that if particles with momenta \( \chi_{\text{inj}} \gg \chi_{\text{eq}} \) are being injected into the system, the resulting electron energy distribution may adopt the “standard” form of the synchrotron-cooled source function (29) at the highest momenta \( \chi_{\text{eq}} < \chi < \chi_{\text{inj}} \) [e.g., \( \propto \chi^{-2} \) for the \( Q(\chi) \propto \delta(\chi - 10^7) \) injection in Fig. 2].

3.2. Inverse-Compton Energy Losses and the Klein-Nishina Effects

Let us now investigate the effects of the IC radiative energy losses in the presence of a turbulent particle acceleration. At low energies when the Klein-Nishina (KN) effects are negligible, the IC case is identical to the synchrotron case with the magnetic energy density \( B^2/8\pi \) replaced by the photon energy density \( u_{\text{ph}} \). The two cases differ when KN effects become important at high energies. To include these effects we approximate the radiative loss timescale as

\[ t_{\text{IC}} = \tau_{\text{IC}} \chi^{-1} \left(1 + \frac{\chi}{\chi_{\text{cr}}}\right)^{1.5} \]  

where \( \tau_{\text{IC}} \equiv \frac{3m_e^2c^2}{4\sigma_{T0}u_{\text{ph}}} \) and \( \chi_{\text{cr}} \equiv \frac{m_e c}{4p_0 \epsilon_0} \) (30).

Here, the radiation field involved in the IC scattering was assumed to be monoenergetic, with the total energy density \( u_{\text{ph}} \) and the dimensionless (i.e., expressed in electron mass units) photon energy \( \epsilon_0 \). The above formula properly takes into account the KN effect up to energies \( \chi \leq 10^4 \chi_{\text{cr}} \) (Moderski et al. 2005). Clearly, as long as \( q < 1.5 \), balance between acceleration and cooling timescales takes place at one particular momentum \( \chi_{\text{eq}} = \chi_{\text{T}}^{1.5-q} \), where \( \chi_{\text{cr}} \) is expressed in electron mass units. For \( q > 1.5 \), there may be instead two equilibrium momenta for a given acceleration timescale, \( \chi_{\text{eq}2} = \chi_{\text{cr}} \) and \( \chi_{\text{eq}2} = \chi_{\text{Kan}} \), or no equilibrium momentum at all, if \( t_{\text{acc}} < t_{\text{IC}} \) within the whole considered range \( \chi < 10^4 \chi_{\text{cr}} \). Finally, for \( q = 1.5 \) (that corresponds to the Kraichnan turbulence), the ratio between IC/KN and acceleration timescales is energy independent, since both \( t_{\text{acc}} \propto \chi^{4-q} = \chi^{1.2} \). As given in (30), \( t_{\text{IC}}(\chi > \chi_{\text{cr}}) \propto \chi^{1.2} \).

Assuming hereafter \( q \neq 3/2 \), one can find from equation (23) that

\[ S(\chi) = \chi^{-2} \exp\left[\frac{1}{3} - q\frac{\chi}{\chi_{\text{T}}} F\left(\frac{3}{2}, 3 - q, 4 - q, \frac{\chi}{\chi_{\text{T}}}\right)\right] \]  

(31)
where \( F(a, b, c, z) \) is a Gauss hypergeometric function. This gives the Green’s function

\[
\mathcal{G}(\chi; \chi_{\text{inj}})_{\text{IC}}^{q \neq 1.5} = z^2 \exp \left[ -\frac{1}{3} + q \left( \frac{\chi}{\chi_T} \right)^{3-q} F \left( \frac{3}{2}, 3-q, 4-q, -\frac{\chi}{\chi_{\text{inj}}} \right) \right] 
\]

\[
\times \left\{ \frac{1}{A} + \int_{\chi_1}^{\chi_{\text{inj}}} \frac{d\chi'}{\chi_{\text{Kn}}^{\frac{1}{2}}} \right\} 
\]

\[
\mathcal{G}(\chi; \chi_{\text{inj}})_{\text{IC}}^{q = 1.5} = \chi^{-2} e^{\frac{1}{1-q} \chi^{3-q}(\chi_{\text{inj}}/\chi)^{3-q}}, 
\]

\( \chi < \chi_{\text{inj}} \),

\[
\mathcal{G}(\chi; \chi_{\text{inj}})_{\text{IC}}^{q = 1.5} = \chi^{-2} e^{\frac{1}{1-q} \chi^{3-q}(\chi_{\text{inj}}/\chi)^{3-q}}, 
\]

\( \chi > \chi_{\text{inj}} \).

Below we discuss some properties of the obtained solution by expanding the Gauss hypergeometric functions as \( F(a, b, c, z) \sim 1 \) for \( z \to 0 \) and \( F(a, b, c, z) \sim \Gamma(c) \Gamma(b-a) / \Gamma(b) \Gamma(c-a) \) \( (z)^{-d} + \Gamma(c) \Gamma(a-b) / \Gamma(a) \Gamma(c-b) \) \( (z)^{-d} \) for \( z \to \infty \).

Let us consider first the case of a low-energy injection, such that \( \chi_{\text{inj}} < \min(\chi_T, \chi_{\text{Kn}}) \). The Green’s function (32) can be then approximated roughly by

\[
\mathcal{G}(\chi; \chi_{\text{inj}})_{\text{IC}}^{q \neq 1.5} \sim \left\{ \frac{1}{1+q} \chi^{-1} e^{-\left(1/(3-q)\right) (\chi_{\text{inj}}/\chi)^{3-q}}, \right\}
\]

\( \chi < \chi_{\text{inj}} \),

\[
\mathcal{G}(\chi; \chi_{\text{inj}})_{\text{IC}}^{q = 1.5} \sim \chi^{-2} e^{-\left(1/(1.5-q)\right) (\chi_{\text{inj}}/\chi)^{3-q}}, \chi > \chi_{\text{inj}}.
\]

For \( \chi < \chi_{\text{inj}} \) the Green’s function has the same form as the synchrotron case, which is expected in the Thomson regime. However, for \( \chi > \chi_{\text{inj}} \), the KN effects modify the high-energy segment of the particle energy distribution. In particular, for \( q < 1.5 \) (e.g., \( q = 1 \) in Fig. 3) the spectrum is almost of a “single-bump” form, possessing either a sharp or a smooth exponential cutoff depending on whether we are in the Thomson or KN cooling regime, respectively. On the other hand, with \( q > 1.5 \) (e.g., \( q = 2 \) in Fig. 4) the acceleration and loss timescales can be equal at two different energies, in which case the particle spectra become concave, flattening smoothly from the exponential decrease \( \propto \chi^2 \exp \left(-\left[1/(3-q)\right] (\chi_{\text{Kn}}/\chi)^{3-q}\right) \) at \( \chi < \chi_{\text{Kn}} \) to the asymptotically approached \( \propto \chi^2 \) continuum at \( \chi > \chi_{\text{Kn}} \). Such spectra are shown in Figures 3 and 4 for \( q = 1 \) and \( q = 2 \), respectively, assuming monoenergetic injection with \( \chi_{\text{inj}} = 1, \chi_1 = 10^{-2} \), \( \chi_{\text{Kn}} = 10^4 \), and \( \chi_2 = 10^8 \). In each figure we use two different acceleration timescales (thick solid and dashed lines), but the radiative loss timescale, \( t_{\text{loss}} \), as well as the normalization of the injection function, \( \int dp \rho(p) \), are kept constant. The emerging spectra are compared with the electron energy distribution \( \tilde{N}(\chi) \) corresponding to the same injection and cooling conditions, but with the momentum diffusion neglected (eq. [29]; thin solid lines in the bottom panels of the figures).

In the case of \( q \neq 1.5 \) and high-energy injection \( \chi_{\text{inj}} > \chi_{\text{Kn}} \), the appropriate Green’s function retains again the familiar shape

\[
\mathcal{G}(\chi; \chi_{\text{inj}})_{\text{IC}}^{q \neq 1.5} \sim \chi^{-2} e^{-\left(1/(1.5-q)\right) (\chi_{\text{Kn}}/\chi)^{3-q}}, \chi < \chi_{\text{Kn}} \]

\[
\mathcal{G}(\chi; \chi_{\text{inj}})_{\text{IC}}^{q = 1.5} \sim \chi^{-2} e^{-\left(1/(1.5-q)\right) (\chi_{\text{Kn}}/\chi)^{3-q}}, \chi > \chi_{\text{Kn}}.
\]
for $\chi > \chi_{cr}$. The resulting particle spectra are plotted in Figures 5 and 6, where we consider two limiting cases of $q = 1$ and 2 and assume monoenergetic injection $Q(\chi) \propto \delta(\chi - \chi_{inj})$ with $\chi_{inj} = 10^7$. All the other parameters are fixed as before. As shown, in addition to the spectral features discussed in the previous paragraph for the case of a low-energy injection (Figs. 3 and 4), the radiatively cooled continuum may be observed at high particle energies $\chi < \chi_{inj}$, depending on the efficiency of the acceleration process. The KN effects manifest by means of a characteristic spectral flattening over the "standard" power-law form $\propto \chi^{-2}$, obviously only within the momentum range $\chi_{cr} < \chi < \chi_{inj}$, in agreement with the appropriate $\tilde{N}(\chi)$ distribution (thin solid lines in the bottom panels of Figs. 5 and 6). Such a feature, being a direct result of a dominant IC/KN-regime radiative cooling with the momentum diffusion effects negligible, was discussed previously by, e.g., Kusunose & Takahara (2005), Moderski et al. (2005), and Manolakou et al. (2007).

Finally, for completeness we note that with $q = 1.5$, one can solve equation (23) to obtain

$$S(\chi) = \chi^{-2} \exp \left\{ 2 \left( \frac{\chi_{cr}}{\chi T} \right)^{3/2} \left[ \text{arcsinh} \sqrt{\frac{\chi}{\chi_{cr}}} - \frac{\sqrt{\chi/\chi_{cr}}}{\sqrt{1 + (\chi/\chi_{cr})}} \right] \right\}. \quad (35)$$

This reduces to $S(\chi) \sim \chi^{-2} \exp \left\{ (2/3)(\chi/\chi_{T})^{3/2} \right\}$ for $\chi < \chi_{cr}$ and can be approximated by $S(\chi) \sim 0.54 \chi^{-1} \chi_{cr}^{-1}$ for $\chi > \chi_{cr}$. The resulting particle spectra, shown in Figure 7 for the case of a low-energy injection $Q(\chi) \propto \delta(\chi - 1)$, are therefore $N(\chi < \chi_{cr}) \propto \chi^{-2} \exp \left\{ -(2/3)(\chi/\chi_{T})^{3/2} \right\}$ at low momenta or of the power-law form $N(\chi > \chi_{cr}) \propto \chi^{-\sigma}$ at higher momenta where the KN effects are important. Here, $\sigma \equiv t_{acc}/\tau_{E}(\chi > \chi_{cr}) - 2 = \left( t_{acc}/\tau_{E} \right) \chi_{cr}^{1/5} - 2$.

---

**Fig. 4.**—Same as Fig. 3, but for fixed $q = 2$.

**Fig. 5.**—Same as Fig. 3, but the spectra correspond to the monoenergetic injection $Q(\chi) \propto \delta(\chi - 10^7)$ with fixed $\int dp \tilde{Q}(p)$ and no particle escape.
3.3. Bremsstrahlung and Coulomb Energy Losses

At high densities or low magnetic field (in general, low Alfven velocities), electron-electron and electron-ion interactions become important. These result in an elastic loss due to Coulomb collisions or radiative loss via bremsstrahlung. At low energies the bremsstrahlung loss rate is negligible when compared to the Coulomb loss rate, which is independent of energy for relativistic charged particles (see, e.g., Petrosian 1973, 2001). However, since the bremsstrahlung rate increases nearly linearly with energy, above some critical energy bremsstrahlung becomes dominant. The timescales associated with these processes approximately are

\[
t_{\text{Coul}} = \tau_{\text{Coul}} \chi, \quad \text{where} \quad \tau_{\text{Coul}} = \frac{2}{m_e c^3} \frac{\rho_0}{3 \sigma_T c n_g \ln \Lambda},
\]

\[
t_{\text{brem}} = \tau_{\text{brem}}, \quad \text{where} \quad \tau_{\text{brem}} = \frac{\pi}{3 \alpha_{\text{Fe}} \sigma_T c n_g}.
\]

Here, \( n_g \) is the background plasma density, the Coulomb logarithm \( \ln \Lambda \) varies from 10 to 40 for a variety of astrophysical plasmas, \( \alpha_{\text{Fe}} = 1/137 \) is the fine-structure constant, and the bremsstrahlung rate includes electron-ion and electron-electron bremsstrahlung and assumes a completely unscreened limit with an approximately 10% (fully ionized) helium abundance (Blumenthal & Gould 1970). The timescales are equal at energy \( p_{\text{Coul}} = \pi \ln \Lambda m_e c/(2 \alpha_{\text{Fe}}) \). At higher energies the bremsstrahlung loss becomes unimportant compared to the synchrotron or IC losses. For example, the synchrotron loss becomes equal to and exceeds the bremsstrahlung loss at electron momenta \( p \geq p_{\text{brem}} = (m_e/\rho)(\alpha_{\text{Fe}}/\beta_\Lambda^2)m_e c \) so that for bremsstrahlung to be at all important we need \( 1000 < p/(m_e c) < 10^{-5} \beta_\Lambda^{-2} \), requiring \( \beta_\Lambda < 0.003 \). Below, we investigate in some detail stochastic acceleration for the conditions when the Coulomb and bremsstrahlung processes are the dominant loss processes.

Fig. 6.— Same as Fig. 5, but for fixed \( q = 2 \).

Fig. 7.— Same as Fig. 3, but for fixed \( q = 3/2 \).
At low energies, \( p < p_{\text{Coul}} \), Coulomb collisions dominate. If \( p_0 \gg m_e c \), then in the range \( m_e c \ll p \ll p_{\text{Coul}} \) and for \( q > 1 \), the appropriate Green’s function becomes (see eqs. [22]–[23])

\[
\mathcal{G}(x, \chi_{\text{inj}})_{\text{Coul}}^{q=1} = x^2 e^{-1/[1/(1-q)](\chi/\chi_{\text{eq}})^{1-q}} \left( \frac{1}{A} + \int_{x_1}^{\min(\chi_{\text{inj}}, \chi)} d\chi' \chi^{-(2+q)} e^{1/[1-(1-q)](\chi'/\chi_{\text{eq}})^{1-q}} \right) \\
\approx x^2 e^{-1/[1/(1-q)](\chi/\chi_{\text{eq}})^{1-q}} \chi_{\text{eq}}^{1-q}/(1-q) \left( -1 + q - \frac{\min(\chi_{\text{inj}}, \chi)/\chi_{\text{eq}}}{1-q}, -\frac{\chi(1/\chi_{\text{eq}})^{1-q}}{1-q}, \right),
\]

(38)

where the equilibrium momentum \( \chi_{\text{eq}} = (t_{\text{Coul}}/\tau_{\text{Coul}})^{(1-q)} \) is defined by the condition \( \tau_{\text{acc}} = \tau_{\text{Coul}} \) condition, yielding \( \theta_{\chi} = \chi_{\text{eq}}^{1-q}/\chi \). Note that since \( q > 1 \) are considered, the acceleration timescale is longer than the Coulomb interaction timescale for \( \chi < \chi_{\text{eq}} \). Thus, in the case of a low-energy particle injection with \( \chi_{\text{inj}} < \chi_{\text{eq}} \), the emerging particle spectra are of the “cooled” form \( N(\chi) = \tilde{N}(\chi) \propto \text{const} \) (see eq. [29]) with \( \theta_{\chi} \propto \chi^{-1} \). If, however, higher energy particles are injected into the system, an additional flat-spectrum component \( N(\chi) \propto \chi^{-2} \) is formed at \( \chi > \chi_{\text{eq}} \).

Let us finally note that pure Coulomb energy losses and the Bohm limit \( q = 1 \) correspond to the situation when \( \theta_{\chi} = \text{const} \) and, hence, \( S(\chi) = \chi^{-2}(\tau_{\text{acc}}/\tau_{\text{Coul}}) \). The Green’s function (22) adopts then the form

\[
\mathcal{G}(x, \chi_{\text{inj}})_{\text{Coul}}^{q=1} = x^2 \tau_{\text{acc}}/\tau_{\text{Coul}} \left( \frac{1}{A} + \int_{x_1}^{\min(\chi_{\text{inj}}, \chi)} d\chi' \chi^{1-q+4/\tau_{\text{acc}}/\tau_{\text{Coul}}} \right) \sim \frac{1}{\sigma'_{\chi}} \chi^{-\sigma'_{\chi}} \left( \frac{\chi_{\text{eq}}^{1-q}}{1-q}, \min(\chi_{\text{inj}}, \chi), \tau_{\text{acc}}/\tau_{\text{Coul}} < 2, \right.

(39)

where \( \sigma'_{\chi} \equiv \tau_{\text{acc}}/\tau_{\text{Coul}} - 2 \). Hence, if only \( \tau_{\text{acc}} < 2\tau_{\text{Coul}} \), a power-law particle energy loss is the dominant process and the equilibrium momentum defined by the condition \( \tau_{\text{acc}} = \tau_{\text{Coul}} \) for \( q < 2 \) becomes \( \chi_{\text{eq}} = (\tau_{\text{brem}}/\tau_{\text{acc}})^{(2-q)} \), yielding \( \theta_{\chi} = \chi_{\text{eq}}^{(2-q)} \). Hence, the Green’s function (22) is (eqs. [22]–[23])

\[
\mathcal{G}(x, \chi_{\text{inj}})_{\text{brem}}^{q<2} = x^2 e^{-1/[1/(2-q)](\chi/\chi_{\text{eq}})^{2-q}} \left( \frac{1}{A} + \int_{x_1}^{\min(\chi_{\text{inj}}, \chi)} d\chi' \chi^{-(2+q)} e^{1/[1-(2-q)](\chi'/\chi_{\text{eq}})^{2-q}} \right) \\
\approx x^2 e^{-1/[2-q](\chi/\chi_{\text{eq}})^{2-q}} \chi_{\text{eq}}^{-q}(1-q)^{3/(2-q)} \Gamma \left( \frac{1 + q}{2 - q}, - \frac{\min(\chi_{\text{inj}}, \chi)/\chi_{\text{eq}}}{2-q}, -\frac{\chi(1/\chi_{\text{eq}})^{2-q}}{2-q}, \right),
\]

(40)

In other words, for any injection conditions the expected electron energy distribution is of the form \( N(\chi) \propto \chi^{2-q} \exp \left( -\left[1/(2-q) \right]/(\chi/\chi_{\text{eq}})^{2-q} \right) \) form, except for the case when high-energy particles with \( \chi_{\text{inj}} > \chi_{\text{eq}} \) are injected into the system. Such high-energy particles subjected to the bremsstrahlung energy losses then form an additional “cooled” high-energy power-law tail \( N(\chi) \propto \chi^{-1} \) in the momentum range between \( \chi_{\text{eq}} \) and \( \chi_{\text{inj}} \), in agreement with the appropriate form of \( N(\chi) \) with \( \theta_{\chi} = \text{const} \) (see eq. [29]).

The situation changes for \( q = 2 \), since both the acceleration and cooling timescales are now independent of the electrons’ energy. In this case, \( S(\chi) = \chi^{-2+4/\tau_{\text{acc}}/\tau_{\text{brem}}} \), and the Green’s function (22) adopts the form

\[
\mathcal{G}(x, \chi_{\text{inj}})_{\text{brem}}^{q=2} = x^2 \tau_{\text{acc}}/\tau_{\text{brem}} \left( \frac{1}{A} + \int_{x_1}^{\min(\chi_{\text{inj}}, \chi)} d\chi' \chi^{1-q+4/\tau_{\text{acc}}/\tau_{\text{brem}}} \right) \sim \frac{1}{1-\sigma'_{\chi}} \chi^{1-q'_{\chi}} \left( \frac{\chi_{\text{eq}}^{1-q'-1}}{1-q'-1}, \tau_{\text{acc}}/\tau_{\text{brem}} < 3, \tau_{\text{acc}}/\tau_{\text{brem}} > 3, \right).
\]

(41)

where \( \sigma'_{\chi} \equiv \tau_{\text{acc}}/\tau_{\text{brem}} - 2 \). This is consistent with the appropriate Green’s function found by Schlickeiser et al. (1987), who in the framework of the “hard-core” approximation \( q = 2 \), also considered synchrotron emission and particle escape in addition to the bremsstrahlung radiation. The solution (41) implies that within the whole energy range the expected electron energy distribution is of the power-law form \( N(\chi) \propto \chi^{-\sigma'_{\chi}} \), with the power-law index \( -2 < \sigma'_{\chi} < 1 \). For any longer acceleration timescales, \( \tau_{\text{acc}} > 3\tau_{\text{brem}} \), and monoenergetic injection \( Q(\chi) \propto \delta(\chi - \chi_{\text{inj}}) \), the emerging electron spectra are expected to be of the form \( N(\chi) \propto \chi^{-1} \) for \( \chi < \chi_{\text{inj}} \), while \( N(\chi) \propto \chi^{-2} \) with \( \sigma'_{\chi} > 1 \) for \( \chi > \chi_{\text{inj}} \).

4. Efficient Particle Escape

In this section we investigate steady state solutions to the momentum diffusion equation of radiating ultrarelativistic particles with a finite escape timescale (eq. [6]). Our analytical approach forces us to consider only the limiting cases of turbulent spectral indices \( q = 2 \) or \( 1 \), as well as to restrict the analysis of radiative losses to the synchrotron and/or IC-Thompson regime processes (\( \theta_{\chi} \propto \chi \)). We note that the global approximation to the solution of the momentum diffusion equation that is not necessarily restricted to some particular values of the \( q \) parameter, with the regular energy loss and particle escape terms included, was studied by Gallegos-Cruz & Perez-Peraza
(1995) by using the WKBJ method. Just as before, we consider a finite energy range of particles undergoing momentum diffusion, $0 < \chi_1, \chi_2 < \infty$, strictly related to the finite wavelength range of interacting turbulent modes. We construct the Green’s function according to the procedure outlined in § 3, with the addition of the escape term ($\varepsilon \neq 0$) and with different boundary conditions. Specifically, we change equation (5) to

$$\frac{\partial N}{\partial \tau} + F_{\chi_2} - F_{\chi_1} = \int_{\chi_1}^{\chi_2} d\chi Q(\chi, \tau) - \varepsilon \int_{\chi_1}^{\chi_2} d\chi \chi^{2-q} N(\chi), \quad (42)$$

where the particle flux in the momentum space $F(N(\chi))$ is defined in the same way as above (eq. [19]). As is evident, the no-flux boundary conditions, $F(N(\chi_1)) = F(N(\chi_2)) = 0$, and the conservation of the total number of particles, $\partial N / \partial \tau = 0$ (within the energy range $[\chi_1, \chi_2]$), implies that the particle injection is completely balanced by the particle escape. We assume this to be the case in this section. Physical realization of these would imply the presence of another efficient yet unspecified acceleration process operating at $\chi < \chi_1$, which prevents negative particle momentum flux through the $\chi_1$ boundary. As shown below, the solutions we obtain agree with the ones discussed in the literature for singular boundary conditions for the infinite momentum range (Jones 1970; Schlickeiser 1984; Bogdan & Schlickeiser 1985; Park & Petrosian 1995), as long as we are dealing with particle momenta $\chi \gg \chi_1$ and $\chi \ll \chi_2$.

4.1. “Hard-Sphere” Approximation

The “hard-sphere” approximation for the momentum diffusion of ultrarelativistic electrons undergoing synchrotron energy losses corresponds to fixed $q = 2$ and $\partial \chi = \chi / \chi_{eq}$ (see eqs. [4] and [44]). With these, equation (6) adopts the form

$$\chi^2 N'(\chi) + \chi_{eq}^{-1} \chi^2 N'(\chi) + (2\chi_{eq}^{-1} \chi - 2 - \varepsilon) N(\chi) = -Q(\chi). \quad (43)$$

The two linearly independent particular solutions to the homogeneous form of the above equation are

$$y_1(\chi) = \chi^{\sigma+1} e^{-\chi / \chi_{eq}} U \left( \sigma - 1, 2\sigma + 2, \frac{\chi}{\chi_{eq}} \right), \quad y_2(\chi) = \chi^{\sigma+1} e^{-\chi / \chi_{eq}} M \left( \sigma - 1, 2\sigma + 2, \frac{\chi}{\chi_{eq}} \right), \quad (44)$$

where $U(a, b, z)$ and $M(a, b, z)$ are Tricomi and Kummer confluent hypergeometrical functions, respectively, and $\sigma = -(1/2) + [(9/4) + \varepsilon]^{1/2}$. Introducing next their linear combinations, $u_1(\chi) = y_1(\chi) + \alpha y_2(\chi)$ and $u_2(\chi) = y_1(\chi) + \beta y_2(\chi)$, one may find that the no-flux boundary conditions $F(u_1(\chi_1)) = F(u_2(\chi_2)) = 0$ are fulfilled for

$$\alpha = (2 + \sigma) \frac{U(\sigma, 2\sigma + 2, \chi_1 / \chi_{eq})}{M(\sigma, 2\sigma + 2, \chi_1 / \chi_{eq})}, \quad \beta = (2 + \sigma) \frac{U(\sigma, 2\sigma + 2, \chi_2 / \chi_{eq})}{M(\sigma, 2\sigma + 2, \chi_2 / \chi_{eq})}. \quad (45)$$

This gives the Green’s function of the problem as

$$G(\chi, \chi_{inj})^{\sigma=2}_{esc} = \frac{\Gamma(\sigma - 1)}{\Gamma(2\sigma + 2)} \left( \alpha - \beta \right)^{-1} \chi_{eq}^{-2\sigma - 1} \chi_{inj}^{-2\sigma - 1} e^{\chi_{inj} / \chi_{eq}} \left\{ \begin{array}{ll}
[\chi_1(\chi) + \alpha \chi_2(\chi)] [\chi_1(\chi_{inj}) + \beta \chi_2(\chi_{inj})], & \chi_1 \leq \chi < \chi_{inj}, \\
[\chi_1(\chi_{inj}) + \alpha \chi_2(\chi_{inj})] [\chi_1(\chi) + \beta \chi_2(\chi)], & \chi_{inj} < \chi \leq \chi_2.
\end{array} \right. \quad (46)$$

In order to investigate the above solution, let us consider first the case $\chi_1 \ll \chi_{inj} \ll \chi_{eq} < \chi_2$ and use the standard expansion of the confluent hypergeometrical functions: $U(a, b, z) \sim z^{-a}$ and $M(a, b, z) \sim \Gamma(b) e^z a^{a-b} / \Gamma(a)$ for $z \to \infty$, while $U(a, b, z) \sim \Gamma(b-1) z^{1-b} / \Gamma(a)$ and $M(a, b, z) \sim 1$ for $z \to 0$ (Abramowitz & Stegun 1964). In this limit, one gets

$$G(\chi, \chi_{inj})^{\sigma=2}_{esc, \chi_{inj} < \chi} \sim \begin{cases}
\frac{1}{2\sigma + 1} \chi_{inj}^{-\sigma-2} \chi^{\sigma+1}, & \chi_1 < \chi < \chi_{inj}, \\
\frac{1}{2\sigma + 1} \chi_{inj}^{-\sigma-2} \chi^{-\sigma}, & \chi_{inj} < \chi \ll \chi_{eq}, \\
\Gamma(\sigma - 1) \chi_{inj}^{-\sigma-2} \chi^{\sigma+1} e^{-\chi / \chi_{eq}}, & \chi_{eq} \ll \chi < \chi_2.
\end{cases} \quad (47)$$

Thus, by moving the critical momenta $\chi_1$ and $\chi_2$ toward 0 and $\infty$, respectively, the resultant Green’s function approaches asymptotically—as expected—the corresponding Green’s function for singular boundary conditions obtained by Jones (1970), Schlickeiser (1984), and Park & Petrosian (1995). In particular, one can find that with the monoenergetic injection $Q(\chi) \propto \delta(\chi - \chi_{inj})$, the resulting electron energy distribution is then of the form $N(\chi < \chi_{inj}) \propto \chi^{\sigma+1}$ and $N(\chi > \chi_{inj}) \propto \chi^{-\sigma}$ up to maximum momentum $\chi_{eq}$. Moreover, for the increasing escape timescale $\varepsilon \to 0$, one has $\sigma \approx 1$ and the pile-up bump $N(\chi) \propto \chi^2 \exp(-\chi / \chi_{eq})$ emerging around $\chi \sim \chi_{eq}$ energies. This is shown in Figure 8, where we fixed normalization of the monoenergetic injection $\int dp Q(p)$ and acceleration and
loss timescales, but varied the escape timescale ($\varepsilon = 3, 0.1, \text{ and } 10^{-4}$; dotted, dashed, and solid lines, respectively). For illustration we have selected $\chi_1 = 10^{-2}$, $\chi_{\text{inj}} = 1$, $\chi_{\text{eq}} = 10^6$, and $\chi_2 = 10^8$.

When high-energy particles are injected into the system, such that $\chi_1 < \chi_{\text{eq}} < \chi_{\text{inj}} < \chi_2$, one may find useful the asymptotic expansion of the Green’s function,

$$G(\chi, \chi_{\text{inj}}) \approx \begin{cases} \frac{\Gamma(\sigma - 1)}{\Gamma(2\sigma + 2)} \chi_{\text{eq}}^{-\sigma - 2} \chi^{\sigma + 1} e^{-\chi/\chi_{\text{eq}}}, & \chi_1 < \chi \leq \chi_{\text{eq}}, \\
\chi_{\text{eq}}^{-2} \chi_{\text{eq}} \leq \chi < \chi_{\text{inj}}, \\
\chi_{\text{inj}}^{-4} \chi_{\text{eq}} e^{\chi_{\text{inj}}/\chi_{\text{eq}}} \chi^2 e^{-\chi/\chi_{\text{eq}}}, & \chi_{\text{inj}} < \chi < \chi_2. \end{cases} \quad (48)$$

That is, for the monoenergetic injection $Q(\chi) \propto \delta(\chi - \chi_{\text{inj}})$ with $\chi_{\text{inj}} > \chi_{\text{eq}}$ the resulting electron energy distribution is of the form $N(\chi) \propto \chi^{\sigma+1} \exp(-\chi/\chi_{\text{eq}})$ for $\chi < \chi_{\text{eq}}$. However, within the energy range $\chi_{\text{eq}} < \chi < \chi_{\text{inj}}$ the power-law tail $N(\chi) \propto \chi^{-2}$ emerges, representing radiatively (\$\varepsilon \propto \chi\$) cooled high-energy particles injected into the system, undergoing negligible (when compared to the energy-loss rate) momentum diffusion. At even higher energies, $\chi > \chi_{\text{inj}}$, the particle spectrum cuts off rapidly. This is shown in Figure 9, where as before, we fixed normalization of the monoenergetic injection $\int dp\, Q(p)$ and acceleration and energy loss timescales, but varied the escape timescale ($\varepsilon = 3, 0.1, \text{ and } 10^{-4}$; dotted, dashed, and solid lines, respectively). For illustration we have selected $\chi_1 = 10^{-2}$, $\chi_{\text{eq}} = 10^2$, $\chi_{\text{inj}} = 10^6$, and $\chi_2 = 10^8$. Note that the escape timescale, and hence parameter $\varepsilon$, now influences the slope and normalization of the particle energy distribution only in the “low-energy” regime $\chi < \chi_{\text{eq}}$, such that the spectrum approaches $\propto \chi^{-2}$ for $\varepsilon \to 0$. 

Fig. 8.— “Hard-sphere approximation” ($\sigma = 2$): particle spectra resulting from joint stochastic acceleration, particle escape, and synchrotron energy losses. The spectra correspond to the monoenergetic injection $Q(\chi) \propto \delta(\chi - \chi_{\text{inj}})$ with fixed normalization, fixed acceleration and cooling rates, but different escape timescales (parameter $\varepsilon = 3, 0.1, \text{ and } 10^{-4}$; dotted, dashed, and solid lines, respectively). For illustration, $\chi_1 = 10^{-2}$, $\chi_{\text{inj}} = 1$, $\chi_{\text{eq}} = 10^6$, and $\chi_2 = 10^8$ have been selected.

Fig. 9.— Same as Fig. 8, but for $\chi_{\text{inj}} = 10^6$. 

4.2. Bohm Limit

The Bohm limit for the momentum diffusion of ultrarelativistic electrons undergoing synchrotron energy losses corresponds to $q = 1$ and $\partial \chi = \chi/\chi_{eq}^{2}$ (see eqs. [4] and [24]). The difference with the “hard-sphere” approximation is that the balance between acceleration and escape timescales, $t_{acc} = t_{esc}$, defines now yet another critical energy, $\chi_{esc} = \epsilon^{-1/2}$, and equation (6) takes the form

$$\chi N''(\chi) + \left(\chi_{eq}^{2} \chi^{2} - 1\right)N'(\chi) + \left(2\chi_{eq}^{2} \chi - \chi_{eq}^{2} \chi\right)N(\chi) = -Q(\chi).$$

(49)

The two linearly independent particular solutions to the homogeneous form of the above equation are

$$y_{1}(\chi) = \chi^{2} e^{-\left(1/2\right)(\chi/\chi_{eq})^{2}} U\left(\eta, 2, \frac{1}{2} \left(\frac{\chi}{\chi_{eq}}\right)^{2}\right), \quad y_{2}(\chi) = \chi^{2} e^{-\left(1/2\right)(\chi/\chi_{eq})^{2}} M\left(\eta, 2, \frac{1}{2} \left(\frac{\chi}{\chi_{eq}}\right)^{2}\right),$$

(50)

where $\eta \equiv \frac{1}{2} \left(\chi_{eq}/\chi\right)^{2}$. Defining $u_{1}(\chi) = y_{1}(\chi) + \alpha y_{2}(\chi)$ and $u_{2}(\chi) = y_{1}(\chi) + \beta y_{2}(\chi)$, one finds that the no-flux boundary conditions $\mathcal{F}(u_{1}(1)) = \mathcal{F}(u_{2}(\chi_{2})) = 0$ correspond to

$$\alpha = 2 \frac{U\left(\eta + 1, 3, (1/2)\left(\chi_{1}/\chi_{eq}\right)^{2}\right)}{M\left(\eta + 1, 3, (1/2)\left(\chi_{1}/\chi_{eq}\right)^{2}\right)}, \quad \beta = 2 \frac{U\left(\eta + 1, 3, (1/2)\left(\chi_{2}/\chi_{eq}\right)^{2}\right)}{M\left(\eta + 1, 3, (1/2)\left(\chi_{2}/\chi_{eq}\right)^{2}\right)}.$$  

(51)

This gives the Green’s function of the problem as

$$\mathcal{G}(\chi, \chi_{inj})|_{\chi_{esc}}^{q=1} = \frac{1}{4} \Gamma(\eta)(\alpha - \beta)^{-1} \chi_{inj}^{-2} \chi_{eq}^{2} \left(1/2\right)(\chi_{inj}/\chi_{eq})^{2} \left\{\begin{array}{ll}
y_{1}(\chi) + \alpha y_{2}(\chi) & \text{if } \chi_{1} \leq \chi < \chi_{inj}, \\
y_{1}(\chi_{inj}) + \alpha y_{2}(\chi_{inj}) & \text{if } \chi_{inj} < \chi \leq \chi_{2}.
\end{array}\right.$$  

(52)

Let us consider first the case $\chi_{1} \ll \chi_{inj} \ll \chi_{eq} \ll \chi_{2}$ for which the Green’s function of equation (52) can be approximated as

$$\mathcal{G}(\chi, \chi_{inj})|_{\chi_{esc}, \chi_{inj}}^{q=1} \simeq \left\{\begin{array}{ll}
(1/2)\chi_{inj}^{2} \chi^{2}, & \chi_{1} < \chi < \chi_{inj}, \\
1/2, & \chi_{inj} < \chi < \chi_{eq}, \\
2^{\eta-2} \Gamma(\eta) \chi_{eq}^{-2} (\chi_{eq}^{2} - 2\chi_{eq}^{2} e^{-\left(1/2\right)(\chi/\chi_{eq})^{2}}), & \chi_{eq} \leq \chi < \chi_{2}.
\end{array}\right.$$  

(53)

Note that, as expected, in the limits $\chi_{1} \to 0$ and $\chi_{2} \to \infty$, the Green’s function (52) approaches asymptotically the solution obtained for singular boundary conditions by Bogdan & Schlickeiser (1985). As shown in Figure 10, for a monoenergetic injection $Q(\chi) \propto \delta(\chi - \chi_{inj})$, the resulting electron energy distribution is $N(\chi < \chi_{inj}) \propto \chi^{2}$ and $N(\chi > \chi_{inj}) \propto \text{const}$ up to the maximum momentum $\chi_{eq}$, with the spectral indices independent of the value of the escape timescale. However, for energies near and above $\chi_{eq}$ the spectra depend on the value of $\eta$. For $\eta \to 0$, i.e., when the escape timescale is large, the familiar bump $N(\chi) \propto \chi^{2} \exp\left[-\frac{1}{2}(\chi/\chi_{eq})^{2}\right]$ emerges around $\chi \sim \chi_{eq}$ energies (Fig. 10, solid line). In the opposite case, when $\eta > 1$ (or $\chi_{eq} > \chi_{esc}$), no pile-up bump is present, and the electron
loss timescales, but varied the escape timescale such that $\chi_{\text{esc}} = 10^5, 10^6,$ and $10^7$ (dotted, dashed, and solid lines, respectively). We choose $\chi_1 = 10^{-2}, \chi_{\text{inj}} = 10^6,$ and $\chi_2 = 10^8$. In the case when $\chi_1 \ll \chi_{\text{eq}} \ll \chi_{\text{inj}} \ll \chi_2$. The asymptotic expansion of the Green’s function (52) yields

$$G(\chi, \chi_{\text{inj}})_{\text{esc,eq}, \text{inj}} \approx \begin{cases} \chi^{-2} e^{-\chi^2/\chi_{eq}}, & \chi_1 < \chi \ll \chi_{eq}, \\ \chi_{\text{inj}}^{-2} \chi_{eq}^{-2} \chi^{-2}, & \chi_{eq} \ll \chi < \chi_{\text{inj}}, \\ \chi_{\text{inj}}^{-2} \chi_{eq}^{-2} e^{(1/2)(\chi_{eq}/\chi_{inj})^2}, & \chi_{\text{inj}} \ll \chi < \chi_2. \end{cases}$$

Again, as above, the spectrum is different in the case of a high-energy injection. For example, as shown in Figure 11, for the monoenergetic injection $Q(\chi) \propto h(\chi - \chi_{\text{inj}})$ with $\chi_{\text{inj}} > \chi_{eq}$ the resulting electron energy distribution is of the form $N(\chi) \propto \chi^{-2} e^{-\chi^2/\chi_{eq}}$ for $\chi \ll \chi_{eq}$, while $N(\chi) \propto \chi^{2q-2}$ for $\chi_{eq} \ll \chi < \chi_{\text{inj}}$. It is interesting to note that the Bohm limit case behaves differently from the $q = 2$ case and analogous injection condition. The escape timescale now affects (via the parameter $\chi_{\text{esc}}$ or $q$) the normalization of the low-energy ($\chi < \chi_{eq}$) segment of the particle spectrum but not its power-law slope. It determines, on the other hand, the “radiatively cooled” part of the particle distribution in the range $\chi_{eq} < \chi < \chi_{\text{inj}},$ which is, however, very close to the standard $\chi^{-2}$ for any $\chi_{\text{esc}} \gg \chi_{eq}$ (or $q \ll 1$). Here, as before, we fixed normalization of the monoenergetic injection $\int dp \hat{Q}(p)$ and the acceleration and loss timescales, but varied the escape timescale such that $\chi_{\text{esc}} = 10^5, 10^6,$ and $10^7$ (dotted, dashed, and solid lines, respectively). Also we set $\chi_1 = 10^{-2}, \chi_{\text{inj}} = 1, \chi_{eq} = 10^6,$ and $\chi_2 = 10^8$.

5. EMISSION SPECTRA

In §§3 and 4 we showed that stochastic interactions of radiating ultrarelativistic electrons (Lorentz factors $\gamma \equiv p/m_e c \gg 1$) with turbulence characterized by a power-law spectrum $\nu \propto k^{-q}$ result in the formation of a “universal” high-energy electron energy distribution,

$$n_e(\gamma) = n_0 \gamma^2 \exp \left[-\frac{1}{a} \left(\frac{\gamma}{\gamma_{eq}}\right)^a\right],$$

as long as particle escape from the system is inefficient and the radiative cooling rate scales with some power of electron energy. Here, the equilibrium energy $\gamma_{eq}$ is defined by the balance between the acceleration and the energy loss timescales, while the parameter $a$ depends on the dominant radiative cooling process and the turbulence spectrum. In particular, for either synchrotron or IC/Thomson-regime cooling one has $a = 3 - q$. In the case of dominant IC/KN-regime energy losses (with $q < 1.5$), one has instead $a = 1.5 - q$. Below, we investigate in more detail emission spectra resulting from such an electron distribution.

5.1. Synchrotron Emission

Assuming an isotropic distribution of momenta of radiating electrons with energy spectrum $n_e(\gamma)$, the synchrotron emissivity can be found as

$$j_{\nu, \text{syn}}(\nu) = \frac{3 e^3 B}{4 \pi m_e c^2} \int d\gamma R \left(\frac{\nu}{\nu_{c,\gamma}}\right) n_e(\gamma),$$

![Figure 11](image-url)
where $\nu_c = 3eB/4\pi m_e c$,

$$
\mathcal{R}(x) = \frac{x^2}{2} K_{4/3} \left( \frac{x}{2} \right) K_{1/3} \left( \frac{x}{2} \right) - 0.3 \frac{x^2}{2} \left[ K_{2/3} \left( \frac{x}{2} \right) - K_{1/3} \left( \frac{x}{2} \right) \right],
$$

(57)

and $K_\nu(z)$ is a modified Bessel function of the second order (Crusius & Schlickeiser 1986). The relatively complicated function (57) can instead be conveniently approximated by $\mathcal{R}(x) \approx 1.81(1.33 + x^{-2/3})^{-1}e^{-x}$ (Zirakashvili & Aharonian 2007), allowing for some analytical investigation of the integral (56). In particular, one may find that the synchrotron emissivity in the frequency range $\nu < \nu_{\text{syn}} \equiv \nu_{\text{eq}} \gamma_0^2$ is of the form $j_{\text{e},\text{syn}}(\nu < \nu_{\text{syn}}) \propto \nu^{1/3}$, as expected in the case of a very hard (inverted) electron energy distribution at low energies, $n_e(\gamma < \gamma_{\text{eq}}) \propto \gamma^2$. At higher frequencies, however, the synchrotron spectrum steepens. In order to evaluate such a high-frequency spectral component, we use the introduced approximation for $\mathcal{R}(x)$ and the electron spectrum as given in (55), and with these, we rewrite synchrotron emissivity (56) as

$$
j_{\text{e},\text{syn}}(\nu > \nu_{\text{syn}}) \approx \frac{1.81 \sqrt{3} e^3 B_{\text{eq}}^3 \nu_{\text{eq}}}{4 \pi m_e c^2} \left \{ \frac{2}{\omega h''(\omega, y)} g(\omega, y) \exp\left[-\omega h(\omega, y)\right] \right \},
$$

(58)

where $\omega = \nu/\nu_{\text{syn}}, y = \gamma/\gamma_{\text{eq}}, g(\omega, y) = y^2(1.33 + \omega^{-2/3})^{4/3} \frac{1}{1 + \left( \frac{2\nu}{\nu_{\text{syn}}} \right)^{-2a/(6+3a)} \exp \left[ -\frac{2a}{2a} \left( \frac{2\nu}{\nu_{\text{syn}}} \right)^{a/(2+a)} \right]}$, and $h(\omega, y) \equiv y^2 + y^a(\omega)$. With such a form it can be noted that for large $\omega$, i.e., for $\nu > \nu_{\text{syn}}$, the integral of interest can be performed approximately using the steepest descent method (see Petrosian 1981). This gives

$$
j_{\text{e},\text{syn}}(\nu > \nu_{\text{syn}}) \approx \frac{1.81 \sqrt{3} e^3 B_{\text{eq}}^3 \nu_{\text{eq}}}{4 \pi m_e c^2} \sqrt{\frac{2\pi}{\omega h''(\omega, y)}} \left \{ \frac{2\nu}{\nu_{\text{syn}}} \right \}^{(6-a)/(4+2a)} \left[ 1 + \left( \frac{2\nu}{\nu_{\text{syn}}} \right)^{-2a/(6+3a)} \right]^{-1/2} \exp \left[ -\frac{2a}{2a} \left( \frac{2\nu}{\nu_{\text{syn}}} \right)^{a/(2+a)} \right],
$$

(59)

where $y_\ast = (2\omega)^{1/(2+a)}$ is the global maximum of $h(\omega, y)$, and $h''(\omega, y) = \partial^2 h(\omega, y)/\partial y^2$. Thus, the high-energy synchrotron component drops much less rapidly than suggested by the emissivity of a single electron, $\mathcal{R}(x) \propto e^{-x}$. For example, assuming synchrotron (and/or IC/Thompson-regime) dominance, $a = 3 - q$, the synchrotron emissivity reads very roughly as

$$
j_{\text{e},\text{syn}}(\nu > \nu_{\text{syn}}) \propto \nu^{1/2} \exp\left[-1.4(\nu/\nu_{\text{syn}})^{1/2}\right] \quad \text{for} \quad q = 1,
$$

(60)

or

$$
j_{\text{e},\text{syn}}(\nu > \nu_{\text{syn}}) \propto \nu^{5/6} \exp\left[-1.9(\nu/\nu_{\text{syn}})^{1/3}\right] \quad \text{for} \quad q = 2.
$$

(61)

In the case of the IC/KN-regime dominance, $a = 1.5 - q$, the emerging high-energy exponential cutoff in the synchrotron continuum can be even smoother than this, for example, $j_{\text{e},\text{syn}}(\nu > \nu_{\text{syn}}) \propto \nu^{1.1} \exp\left[-2.9(\nu/\nu_{\text{syn}})^{0.2}\right]$ for $q = 1$. These spectra are shown in Figure 12 for

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As shown by Petrosian (1981), the following spectral form is also true for synchrotron emission by semirelativistic electrons.
fixed parameters $B$, $n_0$, and $\gamma_{eq}$, where both integration of the exact form of $R(s)$ as given in equation (57) was performed (solid lines), and also approximate formulae following from equation (59) were evaluated for comparison (dashed lines). Different cases for the parameter $a$ are considered in the plot, namely, (a) $a = 3 - q$ with $q = 1$, (b) $a = 3 - q$ with $q = 2$, and (c) $a = 1.5 - q$ with $q = 1$. As shown, synchrotron spectra are curved and extend far beyond the equilibrium frequency $\nu_{eq}$. In the case of the dominant IC/KN-regime cooling with $q = 1$, the $\nu_{ic}(\nu) - \nu_{syn}$ synchrotron spectrum peaks around $\sim 10^3 \nu_{syn}$. We emphasize that the approximation (59), although obviously not accurate in the range $\nu \lesssim \nu_{syn}$, works relatively well at higher frequencies, where the standard $\delta$-approximation for the synchrotron emissivity, $e_{\nu, \nu_{syn}}(\nu) \propto \left[ \gamma^3 n_e(\gamma) \right]_{\gamma_{syn}^{1/3}}$, fails.

5.2. Inverse-Compton Emission

Let us consider IC emission of isotropic electrons up-scattering the monoenergetic and isotropic photon field with energy density $u_{ph}$ and dimensionless photon energy $\epsilon_0 \equiv \hbar \nu_0/m_e c^2$. The appropriate emissivity can be then found from

$$j_{\nu, IC}(\nu) = \frac{3c\sigma_T}{16\pi m_e c^2} u_{ph} \int \frac{d\gamma}{(1/2)[1+(\epsilon_0/\gamma)^{1/3}]^2} \frac{\epsilon}{\gamma^2 \epsilon_0^2} J(\epsilon, \epsilon_0, \gamma)n_0(\gamma),$$

(62)

where $\epsilon \equiv \hbar \nu/m_e c^2$ and $J(\epsilon, \epsilon_0, \gamma)$ is the IC kernel (e.g., Blumenthal & Gould 1970),

$$J(\epsilon, \epsilon_0, \gamma) = 2 \frac{\epsilon_0}{(1 - \epsilon_0/\gamma^2)} \frac{L^2 / \gamma^2}{2(1 - L^2)} \quad \text{with} \quad \mathcal{L} \equiv 4 \epsilon_0 \gamma, \quad \mathcal{I} \equiv \frac{\epsilon}{\gamma(\epsilon - \gamma)}.$$  

(63)

Let us discuss first the case when the KN effects are negligible. The IC kernel can then be approximated by $J(\epsilon, \epsilon_0, \gamma) \approx \frac{1}{2}(1 - \omega/\gamma^2)$, with $\omega \equiv \gamma/\gamma_{eq}$, $\omega \equiv \epsilon/\epsilon_{IC/Th}$, and $\epsilon_{IC/Th} \equiv 4\epsilon_0 \gamma^2$, which is the characteristic energy of soft photons IC up-scattered in a Thomson regime by electrons with Lorentz factor $\gamma_{eq}$. Hence, with the electron energy distribution of the form (55), one can find that

$$j_{\gamma, IC/Th}(\epsilon) = \frac{2}{\pi} c\sigma_T u_{ph} n_0 \gamma_{eq}^5 \epsilon \int \frac{dy}{y^2} \left( 1 - \frac{\omega}{y^2} \right) \exp \left( -\frac{1}{a} y^a \right) \approx \frac{2}{\pi} c\sigma_T u_{ph} n_0 \gamma_{eq}^5 \epsilon \left[ a^{1/3} \Gamma \left( a, \frac{1}{3} \right) - a^{-1/3} \omega \Gamma \left( -a^{-1}, \frac{1}{3} \right) \right],$$

(64)

where $\Gamma(a, z)$ is the incomplete Gamma function. With the expansion $\Gamma(a, z) \sim \Gamma(a)$ for $z \to 0$ (Abramowitz & Stegun 1964), one can approximate further

$$j_{\gamma, IC/Th}(\epsilon < \epsilon_{IC/Th}) \sim \frac{2}{\pi} c\sigma_T u_{ph} n_0 \gamma_{eq}^5 \epsilon (1 - a) \epsilon_{IC/Th} \frac{\epsilon}{\epsilon_{IC/Th}}.$$

(65)

In other words, the IC emissivity at low photon energies is of the form $j_{\gamma, IC/Th}(\epsilon < \epsilon_{IC/Th}) \propto \epsilon$. This is the flattest IC/Thomson-regime spectrum, being analogous to the flattest synchrotron one, $j_{\nu, syn}(\nu < \nu_{syn}) \propto \nu^{1/3}$. At higher photon energies, noting that $\Gamma(a, z) \sim z^{a-1} e^{-z}$ for $z \to \infty$, one may find instead

$$j_{\gamma, IC/Th}(\epsilon > \epsilon_{IC/Th}) \sim \frac{2}{\pi} c\sigma_T u_{ph} n_0 \gamma_{eq}^5 \epsilon (1 - a) \frac{\epsilon}{\epsilon_{IC/Th}} \exp \left[ -\frac{1}{a} \left( \frac{\epsilon}{\epsilon_{IC/Th}} \right)^{a/2} \right].$$

(66)
Therefore, the exponential cutoff of the IC/Thomson-regime component is now steeper than the exponential cutoff of the synchrotron component originating from the same particle distribution. In particular, with \( a = 3 - q \) one gets \( j_{\text{IC/Th}}(\epsilon) \propto \epsilon^{1/2} \exp \left[ -\frac{1}{2} (\epsilon/e_{\text{IC/Th}}) \right] \) for \( q = 1 \), while \( j_{\text{IC/Th}}(\epsilon) \propto \epsilon \exp \left[ -\frac{1}{\epsilon/e_{\text{IC/Th}}} \right] \) for \( q = 2 \) (that can be compared with the corresponding synchrotron emissivities provided above). These spectra are shown in Figure 13 for fixed parameters \( B, n_0 \), and \( \gamma_{eq} \). Here, the solid lines correspond to the formulae (64), and dashed lines correspond to the rough approximation (66). Two different parameters \( a = 3 - q \) are considered in the plot, corresponding to the turbulence energy index \( q = 1 \) and \( 2 \) (cases \([a]\) and \([b]\), respectively).

Finally, we comment on the emission spectra produced in a deep KN regime of the IC scattering, i.e., when \( \gamma > \gamma_{eq} \equiv 1/4a_q \) by the highest energy electrons \( \gamma \geq \gamma_{eq} \). In such a case, the emissivity has to be evaluated by performing the integral (62) with the exact IC kernel as given in equation (63). A rather crude approximation for this can be obtained by utilizing the \( \delta \)-approximation for the resulting IC/KN regime photon energy, namely, \( \epsilon = \gamma \). In particular, with the electron energy distribution as given in equation (55) and with all the previous assumptions regarding a monoenergetic and isotropic soft photon field, one finds

\[
j_{\text{IC/KN}}(\epsilon \geq \gamma_{eq}) \simeq \frac{m_e c^2}{4\pi} \frac{\gamma^2 n_0(\gamma)}{\eta_{eq}(\gamma)} \bigg|_{\gamma=\epsilon} \simeq \frac{1}{3\pi} \frac{c e \sigma_T \eta_{eq} n_0(\gamma)}{\gamma_{eq}} \left( \frac{\epsilon}{\gamma_{eq}} \right)^{5/4} \left( 1 + \frac{\epsilon}{\gamma_{eq}} \right)^{-1.5} \exp \left[ -\frac{1}{a} \left( \frac{\epsilon}{\gamma_{eq}} \right)^a \right],
\]

where \( \eta_{eq}(\gamma) \) is the IC cooling timescale as introduced above in equation (30). As shown in Figure 14, the dominant IC/KN cooling results in the fact that the IC spectra cut off sharply above \( \epsilon = \gamma_{eq} \) photon energies, imitating exponential cutoff in the energy distribution of radiating particles. Here, the exact calculations are plotted as solid lines, and rough approximations from equation (67) are plotted as dashed ones. We fix parameters \( B, n_0 \), and \( \gamma_{eq} \), and again, different cases for the parameter \( a \) are considered, \( a = 3 - q \) with \( q = 1 \), \( b = 3 - q \) with \( q = 2 \), and \( c = 1.5 - q \) with \( q = 1 \). We also choose for illustration \( \gamma_{eq}/\gamma_{eq} = 0.01 \).

6. DISCUSSION AND CONCLUSIONS

In this paper we study steady state spectra of ultrarelativistic electrons undergoing momentum diffusion due to resonant interactions with turbulent MHD waves. We assume a given power spectrum \( \eta(k) \propto k^{-q} \) for magnetic turbulence within some finite range of turbulent wavevectors \( k \) and consider a variety of turbulence spectral indices \( 1 \leq q \leq 2 \). For example, \( q = 1 \) corresponds to the “Bohm limit” of the stochastic acceleration processes, \( q = 2 \) represents the “hard-sphere approximation,” while \( q = 5/3 \) and \( q = 3/2 \) correspond to the Kolmogorov or Kreichnan turbulence, respectively. Within the anticipated quasi-linear approximation for particle-wave interactions, such a turbulent spectrum gives the momentum and pitch angle diffusion rates \( \propto p^{q-2} \), or the acceleration and escape timescales \( t_{\text{acc}} \propto p^{2-q} \) and \( t_{\text{esc}} \propto p^{q-2} \). In the analysis, we also include radiative energy losses as an arbitrary function of the electrons’ energy. In most of the cases, however, or at least in some particular energy ranges, the appropriate timescale for the radiative cooling scales simply with some power of the particle momentum, \( t_{\text{loss}} \propto p^{r} \). For example, \( r = -1 \) corresponds to synchrotron or inverse-Compton/Thomson-regime energy losses, \( r = 0 \) (roughly) corresponds to the bremsstrahlung emission, \( r = +1 \) (roughly) corresponds to the Coulomb interactions of ultrarelativistic electrons, while \( r = 1/2 \) may conveniently approximate inverse-Compton cooling in the Klein-Nishina regime on a monoenergetic background soft photon field.

We find that when the particles are confined to the turbulent acceleration region \( (t_{\text{esc}} \rightarrow \infty) \), the resulting steady state particle spectra (for a finite momentum range of interacting electrons) are in general of the modified ultrarelativistic Maxwellian type, \( n_c(p) \propto p^2 \exp \left[ -(1/a)(p/p_{eq})^{q/2} \right] \) with \( a = 2 - q - r \neq 0 \). Here, \( p_{eq} \) is the momentum at which the acceleration and radiative loss timescales are equal, \( t_{\text{esc}}(p_{eq}) = t_{\text{loss}}(p_{eq}) \). This form is independent of the initial energy distribution of the electrons as long as this distribution is not very broad and the bulk of initial particles have \( p < p_{eq} \). However, if high-energy particles with \( p > p_{eq} \) are injected into the system, there will be significant deviations from this simple form. For example, for a \( \delta \)-function initial distribution the spectrum will have a power-law tail \( \propto p^{r-1} \) in addition to the modified Maxwellian bump. Also, if the ratio of acceleration and energy loss timescales is
independent of the electron escape, in other words, if \( 2 - q = r \), then the resulting particle spectra are of the form \( n_e(p) \propto p^{-\sigma'} \), where \( \sigma' \equiv (\alpha_{\text{IC}}/\alpha_{\text{loss}}) - 2 \). Finally, if the particle escape from the acceleration site is finite but still inefficient, a power-law tail \( \propto p^{\alpha_2} \) may be present in the momentum range \( \alpha_{\text{IC}} < p < \alpha_{\text{eq}} \), again in addition to the modified Maxwellian component. When the radiative loss timescale is not a simple power-law function of the electron energy, the emerging spectra may be of a more complex (e.g., concave) form.

We also analyze in more detail synchrotron and inverse-Compton emission spectra of the electrons characterized by the modified ultrarelativistic Maxwellian energy distribution. In order to summarize briefly our findings, let us define the critical synchrotron frequency of the electrons with the equilibrium Lorentz factor \( \gamma_{\text{eq}} \equiv \alpha_{\text{IC}}/\alpha_{\text{loss}} \), namely, \( \nu_{\text{syn}} \equiv (3 e B / 4 \pi m_e c^2)^{1/2} \) and the critical dimensionless energy of the monochromatic \( (h\nu_{\text{eq}} \equiv e_0 m_e c^2) \) soft photon field inverse-Compton up-scattered (in the Thomson regime) by the \( \gamma_{\text{eq}} \) electrons, \( \epsilon_{\text{IC/Th}} = 4 \alpha_{\text{IC}}/\gamma_{\text{eq}} \). With these, one can note that the low-frequency synchrotron emissivity is of the form \( j_{\text{IC}}(\nu < \nu_{\text{syn}}) \propto \nu^{3/2} \), as expected in the case of a very flat (or inverted) electron energy distribution \( n_e(\gamma < \gamma_{\text{eq}}) \propto \gamma^{-2} \). Such flat electron spectra seem to be required to explain several emission properties of relativistic jets in active galactic nuclei (Tsang & Kirk 2007a, 2007b). At higher frequencies, we find a rough approximation \( j_{\text{IC}}(\nu > \nu_{\text{syn}}) \propto \nu^{6/5}(1+4\alpha_2) \exp \left( -(2+\alpha_2)/(2\nu_{\text{syn}}')[(2+\alpha_2)]^{(2+\alpha_2)} \right) \). Thus, the high-energy synchrotron component drops much less rapidly than is suggested by the emissivity of a single electron, and the emerging high-frequency tail of the synchrotron spectrum is of a smoothly curved shape.

It is therefore very interesting to note that almost exactly this kind of curvature is observed at synchrotron X-ray frequencies in several BL Lac objects (Massaro et al. 2004, 2006; Perlman et al. 2005; Tramacere et al. 2007a, 2007b), in particular those detected also at TeV photon energies.

As for the inverse-Compton emission of ultrarelativistic electrons characterized by the modified Maxwellian energy distribution, we find that in the Thomson regime it is of the form \( j_{\text{IC/Th}}(\epsilon < \epsilon_{\text{IC/Th}}) \propto \epsilon \) and \( j_{\text{IC/Th}}(\epsilon > \epsilon_{\text{IC/Th}}) \propto \epsilon^{3-\alpha_2} \exp \left( -(1/\alpha_2)(\epsilon/\epsilon_{\text{IC/Th}})^a \right) \). Both the very flat low-energy part of this component and its curved high-energy segment may contribute to the observed \( \gamma\)-ray emission of some TeV blazars (Katarzyński et al. 2006a; Giebels et al. 2007).

10 The caution here is that the high-energy spectra computed in this paper correspond to the situation of inverse-Comptonization of the monoenergetic seed photon field, which, in addition, is isotropically distributed in the emitting region rest frame. In the case of relativistic blazars, the external radiation (due to the accretion disk, as well as circumnuclear gas and dust) is distributed anisotropically in the jet rest frame, while the isotropic synchrotron emission produced by the jet electrons is not strictly monochromatic (see, e.g., Dermer et al. 1997 and references therein). On the other hand, synchrotron radiation of ultrarelativistic electrons characterized by the Maxwellian-type energy distribution, as analyzed here, is not that far from the monoenergetic approximation, and the relativistic corrections regarding the anisotropic distribution of the soft photons in the emitting region rest frame are not supposed to influence substantially the spectral shape of the IC emission. For these reasons, we believe that the main spectral features of the high-energy emission components computed in this paper are representative for the \( \gamma \)-ray emission of, e.g., TeV blazars.

L. S. was supported by MEiN through the research project 1-P03D-003-29 in years 2005–2008. L. S. acknowledges M. Ostrowski, R. Schlickeiser, and S. Fuerst for helpful comments and discussion.

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