Stochastic Line-Motion and Stochastic Conservation Laws for Non-Ideal Hydromagnetic Models. I. Incompressible Fluids and Isotropic Transport Coefficients

Gregory L. Eyink

Department of Applied Mathematics \& Statistics
The Johns Hopkins University
Baltimore, Maryland, USA

Abstract

We prove that smooth solutions of non-ideal (viscous and resistive) incompressible magneto-hydrodynamic equations satisfy a stochastic law of flux conservation. This property involves an ensemble of surfaces obtained from a given, fixed surface by advecting it backward in time under the plasma velocity perturbed with a random white-noise. It is shown that the magnetic flux through the fixed surface is equal to the average of the magnetic fluxes through the ensemble of surfaces at earlier times. This result is an analogue of the well-known Alfvén theorem of ideal MHD and is valid for any value of the magnetic Prandtl number. A second stochastic conservation law is shown to hold at unit Prandtl number, a random version of the generalized Kelvin theorem derived by Bekenstein-Oron for ideal MHD. These stochastic conservation laws are not only shown to be consequences of the non-ideal MHD equations, but are proved in fact to be equivalent to those equations. We derive similar results for two more refined hydromagnetic models, Hall magnetohydrodynamics and the two-fluid plasma model, still assuming incompressible velocities and isotropic transport coefficients. Finally, we use these results to discuss briefly the infinite-Reynolds-number limit of hydromagnetic turbulence and to support the conjecture that flux-conservation remains stochastic in that limit.
I Introduction

If a plasma is sufficiently collisional, then it can be well-described by fluid-mechanical equations. There is a hierarchy of such hydromagnetic models, ranging from standard magnetohydrodynamics (MHD), to more refined models such as Hall MHD and the two-fluid model, with separate equations for electron and ion fluids [23, 3]. In all of these fluid models the magnetic field lines (or magneto-vortex lines) at the limit of infinite conductivity are “frozen-in” to the plasma, as first observed by Alfvén [1]. The properties of magnetic fields for MHD-type models are closely analogous to the properties of vorticity fields for the Navier-Stokes equation of neutral fluids [29, 38]. Thus, the Helmholtz-Kelvin theorem on conservation of circulations [18, 39] has an MHD analogue in the Alfvén theorem on flux conservation [1], and the Cauchy formula for vorticity [6] has an analogue in the Lundquist formula for magnetic field [27]. These Lagrangian properties of magnetic fields are central to many physical processes in plasmas, such as magnetic dynamo and magnetic reconnection. It has been claimed, with some justification, that “The most important property of an ideal plasma is flux freezing” [24] (section 3.2).

The fundamental Lagrangian laws of conservation and line-motion hold exactly only for smooth solutions of ideal fluid equations, with zero viscosities and resistivities. Recently, however, it has been shown by Constantin and Iyer [9, 19, 20] that the analogous laws of vorticity under ideal Euler evolution remain for the viscous Navier-Stokes solution as stochastic laws. To formulate these results, the equations for Lagrangian fluid particles advected by the Navier-Stokes velocity field must be perturbed by a Gaussian white-noise, with amplitude depending upon the viscosity. A random ensemble of fluid motions results, depending upon the realization of the white-noise process. To calculate the fluid circulation on a given closed loop in the fluid, the loop is evolved backward in time to obtain a random ensemble of loops. The circulation on the given loop is the average over the circulations of the ensemble of loops at the earlier time. Not only do Navier-Stokes solutions enjoy this remarkable “stochastic Kelvin theorem,” or conservation of circulations in the mean, but Constantin and Iyer [9, 19, 20] have shown that this property also uniquely characterizes the velocity fields which satisfy the Navier-Stokes equation. Furthermore, vortex-lines at any chosen initial time are “frozen-in” to the stochastic fluid flows and thus become themselves stochastic. The resultant (deterministic) vorticity at any point at a later time is the average over the random ensemble of vorticity vectors that are advected to that point, stretched and tilted, by the stochastic flows. These results provide an intuitive way to understand vortex dynamics, both stretching and reconnection, in viscous Navier-Stokes fluids.
In this paper, we demonstrate similar stochastic conservation laws and “frozen-in” properties for non-ideal (resistive and viscous) plasma fluid models. The proofs of Constantin and Iyer [9, 19, 20] exploit the Weber formulation [41] of the incompressible Euler equations, and we shall employ here the similar Weber formulations of incompressible hydromagnetic models developed in the work of Ruban and Kuznetsov [35, 25]. Such Weber formulas are implied by the Hamiltonian structure of the ideal fluid models and thus have considerable generality. In the case of standard MHD, the Weber formula of [25] is mathematically equivalent to the generalized Kelvin theorem of Bekenstein and Oron [2], which provides a second Lagrangian conservation law in addition to the Alfvén theorem on flux conservation. We shall prove that non-ideal MHD at unit magnetic Prandtl number enjoys analogues of both of these results as stochastic conservation laws. For general Prandtl number, we shall show that MHD retains at least the stochastic Alfvén theorem, or flux-conservation in the average sense. This result may be formulated equivalently as a stochastic Lundquist formula, according to which the magnetic field vectors are “frozen-in” to the stochastic flows and then ensemble-averaged to yield the resultant magnetic field. Similar results shall be established also for two more refined hydromagnetic models, given by the Hall MHD equations and the two-fluid model of electrons and ions.

There are several further directions in which these methods and ideas may be developed. It has been shown that the Constantin-Iyer formulation of the incompressible Navier-Stokes equation in fact corresponds to a variational principle, a stochastic version of the Hamilton-Maupertuis principle of least-action [14]. In that work the stochastic Kelvin theorem was shown to arise from a symmetry of the stochastic action under the infinite-dimensional particle-relabelling group. The ideal fluid models of hydromagnetics are also Hamiltonian in form, as discussed in [35, 25, 2] and references therein. The results of the present paper can be derived from stochastic action principles and, in particular, the stochastic version of the Bekenstein-Oron generalized Kelvin theorem for MHD can be shown to arise from invariance of the stochastic MHD action under particle-relabelling. These results shall be given elsewhere. We shall also extend the main results of this paper in following work [15] to hydromagnetic models of compressible fluids with anisotropic transport coefficients, i.e. with differing values of viscosity and resistivity in directions longitudinal and transverse to the magnetic field. Such refinements have importance in applications, but we confine ourselves here for simplicity to incompressible fluids and isotropic coefficients. We shall also make only a few brief remarks, in the conclusion section, about turbulent hydromagnetics at very high (kinetic and magnetic) Reynolds numbers [11, 12, 13, 8].
II Standard Magnetohydrodynamics

We consider in this section the standard incompressible MHD equations \[23, 3\] for the velocity field \(u\) and the magnetic field \(B\) in three space-dimensions, written (in cgs units) as:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= -\nabla p + \frac{1}{\rho c} J \times B + \nu \nabla^2 u \quad (1) \\
\partial_t B &= \nabla \times (u \times B) + \lambda \nabla^2 B \quad (2) \\
\nabla \cdot B &= \nabla \cdot u = 0 \quad (3)
\end{align*}
\]

Here the mass density \(\rho\), the kinematic viscosity \(\nu = \mu/\rho\) and the magnetic diffusivity \(\lambda = \eta c/4\pi\) are all assumed to be space-time constants. The electric current \(J\) is given by the nonrelativistic approximation to Ampere’s law as \(J = \frac{c}{4\pi} (\nabla \times B)\). Since the speed of light \(c\) cancels in eq.(1), it is simplifying to set \(c = 1\) and it is also convenient to assume that density \(\rho = 1\). We do both throughout this section.

As did Constantin-Iyer \[9, 19, 20\], we take the flow domain \(\Omega\) to be either the 3-torus \(T^3\) or else 3-dimensional Euclidean space \(\mathbb{R}^3\). In the latter case, we require that \(u, B\) decay sufficiently at infinity.

The existence of weak solutions to the above MHD system (1)-(3) and the existence, uniqueness, and regularity of local-in-time strong solutions are established, for example, in \[10, 36\]. We shall consider here only sufficiently smooth initial data \(u_0, B_0 \in C^{k,\alpha}(\Omega)\) with \(k \geq 3\), where \(C^{k,\alpha}(\Omega)\) for \(k \geq 0\) and \(\alpha \in (0, 1)\) is the Banach space of functions that are \(k\)-times differentiable with \(k\)th partial-derivatives Hölder continuous of exponent \(\alpha\). We shall use the fact that there exists for such initial data a unique solution of (1)-(3) with \(u, B \in C([t_0, t_f], C^{k,\alpha}(\Omega)), k \geq 2\), for some \(T = t_f - t_0 > 0\). It should be possible as in \[19, 20\] to give a self-contained local-existence result, based upon the fixed-point characterization of MHD solutions in the theorems below. However, here we find it simpler to give a more direct argument that existing smooth solutions possess the stated stochastic conservation laws.

II.1 Unit Magnetic Prandtl Number

Our first result shall be for the case of unit magnetic Prandtl number, when \(\nu = \lambda\). In this case, we can prove that there are two stochastic Lagrangian conservation laws for solutions of (1)-(3), one corresponding to the Alfvén theorem \[1\] and another corresponding to a generalized Kelvin theorem \[25, 2\]. As in \[9, 19, 20\], we shall show that the stochastic conservation laws furthermore uniquely characterize the MHD solutions. A precise statement of our result is as follows:
Proposition II.1. Divergence-free fields \( u, B \in C([t_0, t_f], C^{k,\alpha}(\Omega)) \) satisfy the non-ideal, incompressible MHD equations (1)-(2) with initial data \( u_0, B_0 \in C^{k,\alpha}(\Omega) \) for \( k \geq 3 \) and \( \nu = \lambda \), iff for all closed, rectifiable loops \( C \) and for all \( t \in [t_0, t_f] \) (with \( A = \text{curl}^{-1} B \))

\[
\oint_C A(x, t) \cdot dx = \mathbb{E} \left[ \oint_{\tilde{a}(C,t)} A_0(a) \cdot da \right],
\]

\[
\oint_C u(x, t) \cdot dx = \mathbb{E} \left[ \oint_{\tilde{a}(C,t)} [u_0(a) + B_0(a) \times \tilde{R}_+(a,t)] \cdot da \right].
\]

Here \( \tilde{a}(x, t) \) are "back-to-label maps" for stochastic forward flows \( \tilde{x}(a,t) \) solving

\[
d\tilde{x}(a,t) = u(\tilde{x}(a,t),t)dt + \sqrt{2\nu}dW(t), \quad t > t_0, \quad \tilde{x}(a,t_0) = a,
\]

\( \tilde{R}_+(a,t) \) is the Lagrangian-history charge density (charge per unit area) satisfying

\[
\partial_t \tilde{R}_+(a,t) = -J(\tilde{x}(a,t),t)(\nabla a \tilde{x}(a,t))^{-1}, \quad t > t_0, \quad \tilde{R}_+(a,t_0) = 0,
\]

and \( \mathbb{E} \) in (4)-(6) denotes average over realizations of the Brownian motion \( W(t) \) in the SDE (6).

Before we present the proof of the above proposition, let us make a few explanatory remarks. The first result (4) may be re-expressed in terms of a flux integral through a smooth bounding surface \( S \) of the closed loop \( C \), with \( C = \partial S \). It takes the form of a stochastic Alfvén theorem, expressing conservation on average of magnetic flux:

\[
\int_S B(x, t) \cdot dS(x) = \mathbb{E} \left[ \int_{\tilde{a}(S,t)} B_0(a) \cdot dS(a) \right].
\]

This result is, in turn, equivalent to a stochastic Lundquist formula for the local magnetic field,

\[
B(x, t) = \mathbb{E} \left[ B_0(a) \cdot \nabla a \tilde{x}(a,t) \right]_{\tilde{a}(x,t)}.
\]

The second law (5) is a stochastic Kelvin theorem, expressing conservation on average of generalized circulation, analogous to the deterministic result of [2]. It is equivalent to a stochastic Weber formula, corresponding to the deterministic formula of [35, 25],

\[
u u(x, t) = \mathbb{E} \left[ \nabla a \tilde{x}(a,t) \left( u_0(a) + B_0(a) \times \tilde{R}_+(a,t) \right) \right]_{\tilde{a}(x,t)}.
\]

where \( \mathbb{E} \) denotes the Leray-Hodge projection onto divergence-free vector fields [7]. To explain the physical meaning of this formula, it is useful to transform the final term to Eulerian form:

\[
u u(x, t) = \mathbb{E} \left[ \nabla a \tilde{x}(a,t)u_0(\tilde{x}(x,t)) + \tilde{B}(x,t) \times \tilde{R}(x,t) \right].
\]
Proof of Proposition II.1: We first remark that, given the velocity field \( \bar{v}(a, t) \) and \( \bar{R}(a, t) \), and employing the usual rules of calculus, \( \bar{R}(a, t) \) satisfies the Stratonovich SPDE which follows from (7):

\[
d\bar{R}(a, t) = \nabla \times \left( U(a, t) \cdot \bar{R}(a, t) \right) - J(a, t)dt, \quad t > t_0, \quad \bar{R}(a, t_0) = 0,
\]

with \( U(a, t) = \int_{t_0}^t dt' \ u(a, t') + 2\nu \bar{W}(t) \) the (Ito and Stratonovich) infinitesimal generator of the stochastic Lagrangian flow. Consider the flux integral of \( \bar{R}_* \), through any smooth surface:

\[
\bar{Q}(S, t) = \int_S \bar{R}_*(a, t) \cdot dS(a) = \int_{\bar{a}(S, t)} \bar{R}(a, t) \cdot dS(a).
\]

Then, (12) is equivalent to the following equation, valid for all smooth surfaces \( S \):

\[
d\bar{Q}(S, t) = -dt \cdot \int_{\bar{a}(S, t)} J(a, t) \cdot dS(a), \quad t > t_0, \quad \bar{Q}(S, t_0) = 0.
\]

The above equation implies that \( -\bar{Q}(S, t) \) equals the electric charge which flowed across the material surface \( \bar{a}(S, t) \) between times \( t_0 \) and \( t \). This explains the name “Lagrangian-history charge density” for the field \( R_*(a, t) \) used in the above proposition. For any infinitesimal vector surface element \( dS(a) \) starting at point \( a \), \(-R_*(a, t) \cdot dS(a)\) equals the charge crossing the advected surface from \( t_0 \) to \( t \).

Proof of Proposition II.1: We first remark that, given the velocity field \( u \in C([t_0, t_f], C^{k,\alpha}(\Omega)) \) for \( k \geq 1 \), there exists a stochastic flow \( \bar{x}(a, t) \) of \( C^{k,\alpha} \)-diffeomorphisms solving the SDE (3), so that the inverse map \( \bar{a}(x, t) \) exists and belongs to \( C([t_0, t_f], C^{k,\alpha}(\Omega)) \), at least for \( T \) sufficiently small. This follows by the arguments in [19, 20] or the general methods in the monograph [21, Chapter 4]. If we assume that \( u, B \in C([t_0, t_f], C^{k,\alpha}(\Omega)) \) for \( k \geq 3 \), then \( \nabla_a \bar{x}, J \in C([t_0, t_f], C^{k,\alpha}(\Omega)) \) for \( k \geq 2 \), so that \( \bar{R}_* \in C^1([t_0, t_f], C^{k,\alpha}(\Omega)) \) for \( k \geq 2 \). This is sufficient regularity to justify all of our calculations below. It is particularly important that \( \bar{R}_* \) is bounded variation in time and has no martingale part.

We begin by showing the “if” direction. Therefore, assuming that divergence-free fields \( u, B \in C([t_0, t_f], C^{k,\alpha}(\Omega)) \) with \( k \geq 3 \) solve the fixed-point problem (FPP) specified by the eqs. (11)–(7) we must prove that they also satisfy the incompressible MHD equations (11)–(2). The argument closely follows the proof of Theorem 2.2 in Section 4 of [9]. We make the successive definitions

\[
\bar{w}(x, t) = \left[ u_0(a) + B_0(a) \times \bar{R}_*(a, t) \right]_{\bar{a}(x, t)}
\]

\[
\bar{v}(x, t) = \nabla_a \bar{a}(x, t) \bar{w}(x, t)
\]

\[
\bar{u}(x, t) = \bar{P} \bar{v}(x, t) = \bar{v}(x, t) - \nabla_a \bar{\varphi}(x, t)
\]
We see that \( \tilde{w}, \tilde{v}, \tilde{u} \in C([t_0, t_f], C^{k,\alpha}(\Omega)) \) for \( k \geq 2 \) and that the stochastic Weber formula (10) is restated as \( \mathbf{u} = E(\tilde{u}) \). We now develop a stochastic evolution equation for each of these three variables.

Note first that the “back-to-labels map” \( \tilde{a} \) satisfies

\[
d\tilde{a}(x, t) + [(u \cdot \nabla_x) \tilde{a} - \nu \Delta \tilde{a}] \, dt + \sqrt{2\nu}(dW(t) \cdot \nabla_x) \tilde{a} = 0.
\]  

(18)

This is proved in [9], Proposition 4.2. It can also be derived as a special case of the “first Itô formula” for a backward flow; see [24], Theorem 4.4.5. Then, as a consequence of the generalized Ito rule,

\[
d\tilde{w}(x, t) = -[(u \cdot \nabla_x) \tilde{w} + \nu \triangle \tilde{w}] \, dt + \left[ B_0(a) \times \partial_t \tilde{R}_a(a, t) \right] \tilde{a}(x, t) \, dt - \sqrt{2\nu}(dW(t) \cdot \nabla_x) \tilde{w}
\]

(19)

For example, see [9], Corollary 4.3. The term in (19) involving \( \partial_t \tilde{R}_a \) can be evaluated using

\[
\nabla_x \tilde{a}(x, t) \left[ B_0(a) \times \partial_t \tilde{R}_a(a, t) \right] = J(x, t) \times \tilde{B}(x, t),
\]

(20)

which follows from (17), and from the definition

\[
\tilde{B}(x, t) = B_0(a) \cdot \nabla_a \tilde{x}(a, t) |_{\tilde{a}(x, t)}.
\]

(21)

We next calculate the differential of \( \tilde{v} \) using the Ito product rule,

\[
d\tilde{v}(x, t) = \nabla_x \tilde{a}(x, t) \, d\tilde{w}(x, t) + d(\nabla_x \tilde{a}) \tilde{w}(x, t) + d(\nabla_x \tilde{a}, \tilde{w}),
\]

which, together with (18),(19),(20), gives

\[
d\tilde{v}(x, t) = \left[ -(u \cdot \nabla_x) \tilde{v} - (\nabla_x u) \tilde{v} + J(x, t) \times \tilde{B}(x, t) + \nu \triangle \tilde{v} \right] \, dt - \sqrt{2\nu}(dW(t) \cdot \nabla_x) \tilde{v}.
\]

(22)

The rest of the argument goes exactly as in Section 4 of [9]. As in the proof of Theorem 2.2 of [9], the differential of \( \tilde{u} \), as defined in (17), can be expressed as

\[
d\tilde{u}(x, t) = \left[ -(u \cdot \nabla_x) \tilde{u} - (\nabla_x u) \tilde{u} + J(x, t) \times \tilde{B}(x, t) + \nu \triangle \tilde{u} \right] \, dt - \sqrt{2\nu}(dW(t) \cdot \nabla_x) \tilde{u}
\]

\[
- \nabla_x \left[ d\tilde{\varphi} + ((u \cdot \nabla_x) \tilde{\varphi} - \nu \triangle \tilde{\varphi}) \, dt + \sqrt{2\nu}(dW(t) \cdot \nabla_x) \tilde{\varphi} \right]
\]

(23)

and, using again the generalized Ito rule and definition (21),

\[
d\tilde{B}(x, t) = \left[ -(u \cdot \nabla_x) \tilde{B} + (\tilde{B} \cdot \nabla_x) u + \nu \triangle \tilde{B} \right] \, dt - \sqrt{2\nu}(dW(t) \cdot \nabla_x) \tilde{B}
\]

(24)

just as in the proof of Proposition 2.7 of [9]. Taking the expectation over the Brownian motion in eqs.(23)-(24) yields eqs.(1)-(2) with \( \lambda = \nu \) and kinematic pressure \( p = \frac{1}{2} |\mathbf{u}|^2 + \tilde{\varphi} + (u \cdot \nabla_x) \varphi - \nu \triangle \varphi \).
We finally show the “only if” direction. Therefore, assuming that divergence-free fields \( \mathbf{u}, \mathbf{B} \in C([t_0, t_f], C^{k,\alpha}(\Omega)) \) with \( k \geq 3 \) solve the incompressible MHD equations (11–22) we shall show that they also satisfy the fixed-point problem (FPP) specified by the eqs. (4)–(7). Let us define

\[
\mathbf{u}(x, t) = \mathbb{E} \left[ \nabla_x \tilde{\mathbf{u}}(x, t) \cdot \left( \mathbf{u}_0(a) + \mathbf{B}_0(a) \times \tilde{\mathbf{R}}_a(a, t) \right) \right]_{\tilde{a}(x, t)}
\]  

(25)

and

\[
\mathbf{B}(x, t) = \mathbb{E} \left[ \mathbf{B}_0(a) \cdot \nabla_x \tilde{\mathbf{u}}(a, t) \right]_{\tilde{a}(x, t)},
\]  

(26)

where \( \tilde{\mathbf{u}}(a, t) \) solves (16) and \( \tilde{\mathbf{R}}_a(a, t) \) solves (17), for the given \( \mathbf{u}, \mathbf{B} \). It then follows from our previous work that \( \mathbf{u}, \mathbf{B} \in C([t_0, t_f], C^{k,\alpha}(\Omega)) \) with \( k \geq 2 \), are divergence-free, and solve the linear equations

\[
\partial_t \mathbf{u} = - (\mathbf{u} \cdot \nabla) \mathbf{u} - (\nabla \mathbf{u}) \mathbf{u} - \nabla p + J \times \mathbf{B} + \nu \Delta \mathbf{u},
\]  

(27)

\[
\partial_t \mathbf{B} = - (\mathbf{u} \cdot \nabla) \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u} + \nu \Delta \mathbf{B},
\]  

(28)

with initial conditions \( \mathbf{u}(t_0) = \mathbf{u}_0, \mathbf{B}(t_0) = \mathbf{B}_0 \). At least one solution is the pair \( (\mathbf{u}, \mathbf{B}) \) itself, so that, if solutions of the initial-value problem are unique, it must be the case that \( \mathbf{u}, \mathbf{B} = (\mathbf{u}, \mathbf{B}) \). It thus suffices to prove that the linear system (27), (28) has unique solutions for specified initial data.

This may be shown by a standard argument based on an energy estimate (e.g. see [30]). For the pair \( \mathbf{z}(x) = (\mathbf{u}(x), \mathbf{B}(x)) \) define norms

\[
||\mathbf{z}||_2 = \left( \int_{\Omega} d^3x \left[ ||\mathbf{u}(x)||^2 + ||\mathbf{B}(x)||^2 \right] \right)^{1/2},
\]

\[
||\mathbf{z}||_\infty = \sup_{x \in \Omega} \left[ ||\mathbf{u}(x)|| + ||\mathbf{B}(x)|| \right].
\]

An easy calculation then shows for any solution \( \mathbf{z}(x, t) = (\mathbf{u}(x, t), \mathbf{B}(x, t)) \) of (27), (28) that

\[
\frac{d}{dt} ||\mathbf{z}(t)||_2^2 = 2 \int_{\Omega} d^3x \left[ \partial_j u_i(x, t) \left( \partial_j u_i(x, t) \partial_j \mathbf{B}_j(x, t) - \mathbf{u}_i(x, t) \partial_j \mathbf{B}_j(x, t) \right) \right] + \partial_j B_i(x, t) \left( \partial_j \mathbf{u}(x, t) \partial_j \mathbf{B}_j(x, t) - \mathbf{u}_i(x, t) \partial_j \mathbf{B}_j(x, t) \right) - 2\nu ||\nabla \mathbf{z}(t)||^2_2
\]

\[
\leq 2 ||\nabla \mathbf{z}(t)||_\infty ||\mathbf{z}(t)||^2_2 - 2\nu ||\nabla \mathbf{z}(t)||^2_2.
\]

For some \( \epsilon > 0 \), choose \( \gamma > \sup_{t \in [t_0, t_f]} ||\nabla \mathbf{z}(t)||_\infty + \epsilon \). Then it follows that

\[
\frac{d}{dt} \left[ e^{-2\gamma (t-t_0)} ||\mathbf{z}(t)||^2_2 \right] \leq -2e^{-2\gamma (t-t_0)} \left[ \epsilon ||\mathbf{z}(t)||^2_2 + \nu ||\nabla \mathbf{z}(t)||^2_2 \right].
\]

Integration yields the energy inequality

\[
e^{-2\gamma (t_f-t_0)} ||\mathbf{z}(t_f)||^2_2 + 2 \int_{t_0}^{t_f} dt e^{-2\gamma (t-t_0)} \left[ \epsilon ||\mathbf{z}(t)||^2_2 + \nu ||\nabla \mathbf{z}(t)||^2_2 \right] \leq ||\mathbf{z}_0||^2_2,
\]

(29)

which implies uniqueness of solutions of (27), (28) as a direct consequence. \( \square \)
II.2 General Magnetic Prandtl Number

An examination of the proof in the previous subsection reveals an interesting fact that the equation (27) for $\mathbf{u}$ involves both $\mathbf{u}$ and $\mathbf{B}$, but the equation (28) for $\mathbf{B}$ involves only $\mathbf{B}$ itself. This implies that the stochastic representation previously employed for both $\mathbf{u}$ and $\mathbf{B}$ can be exploited for $\mathbf{B}$ alone and, furthermore, at any magnetic Prandtl number. A precise statement of the result is as follows:

**Proposition II.2.** Divergence-free fields $\mathbf{u}, \mathbf{B} \in C([t_0, t_f], C^{k,\alpha}(\Omega))$ satisfy the non-ideal, incompressible MHD equations (1)-(2) with initial data $\mathbf{u}_0, \mathbf{B}_0 \in C^{k,\alpha}(\Omega)$ for $k \geq 3$ iff the momentum equation (1) holds over that interval and simultaneously the stochastic flux conservation holds

$$\int_S \mathbf{B}(x, t) \cdot d\mathbf{S}(x) = E \left[ \int_{\tilde{a}(S,t)} \mathbf{B}_0(a) \cdot d\mathbf{S}(a) \right],$$

(30)

for all smooth surfaces $S$ and all times $t \in [t_0, t_f]$, where $\tilde{a}(x, t)$ are “back-to-label maps” for stochastic forward flows $\tilde{x}(a, t)$ solving the SDE

$$d\tilde{x}(a, t) = \mathbf{u}(\tilde{x}(a, t), t) dt + \sqrt{2\lambda} dW(t), \quad t > t_0, \quad \tilde{x}(a, t_0) = a.$$  

(31)

**Proof of Proposition II.2.** The argument is nearly the same as that for the previous proposition. The “if” direction is immediate, since the result (30) is equivalent to the stochastic Lundquist formula (9) and the equation (2) for $\mathbf{B}$ follows from (9) using the generalized Ito rule, just as before. For the “only if” direction, we define

$$\mathbf{B}(x, t) = E \left[ \mathbf{B}_0(a) \cdot \nabla_a \tilde{x}(a, t) \big| e^{\tilde{x}(a, t)} \right]$$

where $\tilde{x}(a, t)$ is the stochastic flow defined by the SDE (31) for the velocity $\mathbf{u}$ that satisfies the MHD momentum equation (11). It follows that $\mathbf{B}$ satisfies the kinematic dynamo equation

$$\partial_t \mathbf{B} = - \mathbf{u} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u} + \lambda \Delta \mathbf{B}, \quad t > t_0, \quad \mathbf{B}(t_0) = \mathbf{B}_0$$

one of whose solutions is $\mathbf{B} = \mathbf{B}$. Unicity of this solution again follows from an energy inequality

$$e^{-2\gamma(t_f-t_0)} \|\mathbf{B}(t_f)\|^2_2 + 2 \int_{t_0}^{t_f} dt \, e^{-2\gamma(t-t_0)} \left[ \epsilon \|\mathbf{B}(t)\|^2_2 + \lambda \|\nabla \mathbf{B}(t)\|^2_2 \right] \leq \|\mathbf{B}_0\|^2_2,$$

which is derived by a similar calculation as before [30], with $\gamma > \sup_{t \in [t_0, t_f]} \|\nabla \mathbf{u}(t)\|_\infty + \epsilon.$


III Other Incompressible Plasma Fluid Models

We now establish similar stochastic conservation laws for some other non-ideal plasma fluid models, more refined than standard MHD. Keeping within the stated limitations of this paper, we consider here only the versions of these models assuming incompressible fluid velocities and isotropic transport coefficients. We also give only sketches of the proofs of the stated theorems, emphasizing essential differences from those given previously, since most of the details are very similar.

III.1 Hall Magnetohydrodynamics

The equations of incompressible Hall magnetohydrodynamics (HMHD) have the form:

$$\frac{\partial}{\partial t}u + (u \cdot \nabla)u = -\nabla p + \frac{1}{4\pi \rho} (\nabla \times B) \times B + \nu \nabla^2 u \tag{32}$$

$$\frac{\partial}{\partial t}B = \nabla \times \left( u - \frac{\alpha}{4\pi \rho} \nabla \times B \right) \times B + \lambda \Delta B \tag{33}$$

$$\nabla \cdot B = \nabla \cdot u = 0 \tag{34}$$

The magnetic induction equation (33) contains a “Hall drift term” proportional to $$\alpha = mc/e$$, whose importance was first emphasized by Lighthill [26] and which was subsequently extensively investigated; see [42] and [23, 3]. The limit $$\alpha \to 0$$ formally recovers standard MHD. Mathematical properties of HMHD solutions (existence, regularity, etc.) are studied in [31, 32].

Before stating our new theorems, we must review some standard facts about HMHD eqs. (32)-(34), for which, for example, see [35]. If one introduces a vector potential $$A$$ for the magnetic field in Coloumb gauge, $$\nabla \cdot A = 0$$, then, along with a corresponding scalar potential $$\Phi$$, it satisfies

$$\frac{\partial}{\partial t}A = \left( u - \frac{\alpha}{4\pi \rho} \nabla \times B \right) \times B - \nabla \Phi + \lambda \Delta A. \tag{35}$$

HMHD is a Hamiltonian fluid model with two canonical momenta

$$p_i = u + \alpha^{-1} A, \quad p_e = -\alpha^{-1} A, \tag{36}$$

which are both divergence-free, $$\nabla \cdot p_\sigma = 0, \quad \sigma = i, e$$. When $$\nu = \lambda$$, these satisfy the equations

$$\frac{\partial}{\partial t}p_\sigma = u_\sigma \times (\nabla \times p_\sigma) - \nabla p_\sigma + \nu \Delta p_\sigma, \quad \sigma = i, e \tag{37}$$

with $$\pi_i = p + (1/2)|u|^2 + \alpha^{-1} \Phi, \quad \pi_e = -\alpha^{-1} \Phi$$, and with $$u_i$$ the ion fluid velocity and $$u_e$$ the electron fluid velocity, given by

$$u_i = u, \quad u_e = u - \frac{\alpha}{4\pi \rho} \nabla \times B. \tag{38}$$
Note that, if \( \nu \neq \lambda \), then the equation for \( p_i \) would contain an additional term \((\nu - \lambda)\Delta p_e\). Corresponding to the two canonical momenta there are two generalized vorticities \( \Omega_\sigma = \nabla \times p_\sigma \), \( \sigma = i, e \), or concretely

\[
\Omega_i = \omega + \alpha^{-1}B, \quad \Omega_e = -\alpha^{-1}B. \tag{39}
\]

When \( \nu = \lambda \), these generalized vorticities satisfy

\[
\partial_t \Omega_\sigma = \nabla \times (u_\sigma \times \Omega_\sigma) + \nu \Delta \Omega_\sigma, \quad \sigma = i, e. \tag{40}
\]

These equations imply two “frozen-in” fields for ideal HMHD, one for the ion fluid and one for the electron fluid, and two Cauchy-type formulas for the two generalized vorticities. There are likewise two Kelvin-type theorems and two Weber formulas for the two canonical momenta.

We now state stochastic analogues of these results for non-ideal HMHD, first for unit Prandtl number:

**Proposition III.1.1.** Divergence-free fields \( \mathbf{u}, \mathbf{B} \in C([t_0, t_f], C^{k,\alpha}(\Omega)) \) satisfy the non-ideal, incompressible HMHD equations (32)-(33) with initial data \( \mathbf{u}_0, \mathbf{B}_0 \in C^{k,\alpha}(\Omega) \), with \( k \geq 3 \) and for unit magnetic Prandtl number \( \nu/\lambda = 1 \), iff for all closed, rectifiable loops \( C \) and for all \( t \in [t_0, t_f] \)

\[
\oint_C p_\sigma (x, t) \cdot dx = E \left[ \oint_{\tilde{a}_\sigma(C, t)} p_\sigma(\tilde{a}) \cdot d\tilde{a} \right], \quad \sigma = i, e \tag{41}
\]

Here the canonical momenta \( p_\sigma, \sigma = i, e \) are given by eq.(36) and \( \tilde{a}_\sigma(x, t) \) are “back-to-label maps” for stochastic forward flows \( \tilde{x}_\sigma(a, t) \) solving,

\[
d\tilde{x}_\sigma(a, t) = u_\sigma(\tilde{x}_\sigma(a, t), t)dt + \sqrt{2\nu} d\mathbf{W}(t), \quad t > t_0, \quad \tilde{x}_\sigma(a, t_0) = a, \tag{42}
\]

for \( \sigma = i, e \), with velocities \( u_\sigma \) given in eq.(38) and \( \mathbf{W}(t) \) a standard Brownian motion.

**Remarks:** (i) If the noise terms in eq.(42) were chosen to be instead \( \sqrt{2\nu_\sigma} d\mathbf{W}(t), \quad \sigma = i, e \) with \( \nu_i = \nu \) and \( \nu_e = \lambda \), then one would obtain the correct induction equation (33) but the momentum equation would differ from (32), containing an additional term \( \alpha^{-1}(\nu - \lambda) \Delta A = \frac{4\pi e}{mc^2} (\lambda - \nu)J \).

(ii) As we shall see below, it would be enough to assume \( \mathbf{u} \in C([t_0, t_f], C^{k,\alpha}(\Omega)) \) for \( k \geq 2 \), whereas \( k \geq 3 \) is required for \( \mathbf{B} \) because of the Hall drift term.

(iii) The two stochastic Kelvin theorems in (41) are equivalent to *stochastic Weber formulas*

\[
p_\sigma(x, t) = E \left[ \nabla \tilde{a}_\sigma(x, t) p_\sigma(\tilde{a}_\sigma(x, t)) \right], \quad \sigma = i, e. \tag{43}
\]

There are likewise two *stochastic Cauchy formulas* for the two generalized vorticities:

\[
\Omega_\sigma(x, t) = E \left[ \Omega_\sigma(\tilde{a}_\sigma(x, t)) \cdot \nabla \tilde{a}_\sigma(x, t) \right], \quad \sigma = i, e. \tag{44}
\]
\textit{Sketch of proof of Proposition III.1.1.} : The proof is very similar to those given in the previous section and in [9]. The main step is to derive equations for the stochastic time-differential of the variables

\begin{equation}
\bar{p}_\sigma = \mathbb{P} \left[ \nabla_x \tilde{a}_\sigma (p_{\sigma 0} \circ \tilde{a}_\sigma) \right] = \nabla_x \tilde{a}_\sigma (p_{\sigma 0} \circ \tilde{a}_\sigma) - \nabla_x \tilde{\varphi}_\sigma
\end{equation}

for \( \sigma = i, e \), of the form

\begin{equation}
d\bar{p}_\sigma (x, t) = \left[ -(u_\sigma \cdot \nabla_x) \bar{p}_\sigma - (\nabla_x u_\sigma) \bar{p}_\sigma + \nu \Delta \bar{p}_\sigma \right] dt - \sqrt{2\nu} (dW(t) \cdot \nabla_x) \bar{p}_\sigma
\end{equation}

\begin{equation}
- \nabla_x \left[ d\tilde{\varphi}_\sigma + \left( (u_\sigma \cdot \nabla_x) \tilde{\varphi}_\sigma - \nu \Delta \tilde{\varphi}_\sigma \right) dt + \sqrt{2\nu} (dW(t) \cdot \nabla_x) \tilde{\varphi}_\sigma \right]
\end{equation}

The calculations are essentially identical to those presented before. There is just one technical issue related to regularity of the field \( \bar{p}_e \), which should belong to \( C^{2,\alpha}(\Omega) \) in order to give classical meaning to the Laplacian term \( \nu \Delta \bar{p}_e \). However, if \( u, B \in C^{3,\alpha}(\Omega) \), then \( u_i \in C^{3,\alpha}(\Omega), u_e \in C^{2,\alpha}(\Omega) \), so that \( \tilde{x}_i, \tilde{a}_i \in C^{3,\alpha}(\Omega), \tilde{x}_e, \tilde{a}_e \in C^{2,\alpha}(\Omega) \), and thus \( \nabla_x \tilde{a}_i \in C^{2,\alpha}(\Omega), \nabla_x \tilde{a}_e \in C^{1,\alpha}(\Omega) \). This means that \( \tilde{p}_e \) defined by (45) belongs a priori only to \( C^{1,\alpha}(\Omega) \). However, we may use an integration-by-parts identity

\[ \mathbb{P} \left[ (\nabla_x \phi) \psi \right] = -\mathbb{P} \left[ \phi (\nabla_x \psi) \right], \quad \phi, \psi \in C^{1,\alpha}(\Omega), \]

proved as Lemma 3.1 of [20], in order to rewrite the partial derivative \( \partial t \tilde{p}_e \) in terms of \( \nabla_x \tilde{a}_e \) only and eliminate second derivatives of \( \tilde{a}_e \). We can thus conclude that \( \tilde{p}_e \in C^{2,\alpha}(\Omega) \).

Ensemble-averaging the equations (46) for \( \sigma = i, e \) yields

\begin{equation}
\partial \bar{p}_\sigma = -(u_\sigma \cdot \nabla_x) \bar{p}_\sigma - (\nabla_x u_\sigma) \bar{p}_\sigma - \nabla_x \bar{p}_\sigma + \nu \Delta \bar{p}_\sigma,
\end{equation}

with \( \bar{p}_\sigma = \partial \tilde{\varphi}_\sigma + (u_\sigma \cdot \nabla_x) \tilde{\varphi}_\sigma - \nu \Delta \tilde{\varphi}_\sigma \), for \( \sigma = i, e \), or, in terms of \( \bar{u} \) and \( \bar{B} \) variables,

\begin{equation}
\partial \bar{u} = -(u \cdot \nabla) \bar{u} - (\nabla u) \bar{u} - \nabla \bar{u} + \frac{1}{4\pi \rho} (\nabla \times B) \times \bar{B} + \nu \Delta \bar{u},
\end{equation}

\begin{equation}
\partial \bar{B} = \nabla \times \left( \left( u - \frac{\alpha}{4\pi \rho} \nabla \times B \right) \times \bar{B} \right) + \nu \Delta \bar{B}.
\end{equation}

A fixed point satisfying \( (\bar{u}, \bar{B}) = (u, B) \) must therefore also obey the HMHD equations (32)-(33). The converse statement is obtained from the unicity of solutions to the initial-value problem for the above linear equations —either (47), \( \sigma = i, e \) or (48)-(49)—which is proved using energy estimates as before.

Just as in the case of standard MHD, we see that the equation (49) for \( \bar{B} \) does not depend upon \( \bar{u} \). This makes it possible to derive a stochastic conservation law for magnetic-flux at any Prandtl number:
Proposition III.1.2. Divergence-free fields \( u, B \in C([t_0, t_f], C^{k,\alpha}(\Omega)) \) satisfy the non-ideal, incompressible HMHD equations (32)-(33) with initial data \( u_0, B_0 \in C^{k,\alpha}(\Omega) \) for \( k \geq 4 \) iff the momentum equation (32) holds over that interval and simultaneously the stochastic flux conservation holds

\[
\int_S B(x, t) \cdot dS(x) = E \left[ \int_{\tilde{a}(S,t)} B_0(a) \cdot dS(a) \right],
\]

for all smooth surfaces \( S \) and all times \( t \in [t_0, t_f] \), where \( \tilde{a}(x, t) \) are “back-to-label maps” for stochastic forward flows \( \tilde{x}(a, t) \) solving the SDE

\[
d\tilde{x}(a, t) = u_e(\tilde{x}(a, t), t) dt + \sqrt{2\lambda} dW(t), \quad t > t_0, \quad \tilde{x}(a, t_0) = a
\]

with \( u_e \) the electron fluid velocity given by (38).

The proof is as in Proposition II.2, but greater smoothness of \( B \) is required to guarantee that \( \tilde{B} \in C^{2,\alpha} \).

### III.2 Two-Fluid Plasma Model

The most general hydrodynamical model of a fully ionized plasma consisting of electrons and one species of singly-charged ions is the two-fluid model of Braginsky [4], or the Braginsky equations. See also [23, 3].

The basic variables of the model are the two fluid velocities \( u_\sigma, \sigma = i, e \), with \( \sigma = i \) for the ion fluid and \( \sigma = e \) for the electron fluid. In the simple form considered here the equations take the form:

\[
(\partial_t + u_i \cdot \nabla) u_i = \frac{e}{m_i} \left( E + \frac{1}{c} u_i \times B \right) - \nabla p_i + \nu_i \nabla \cdot u_i - \frac{1}{\tau_i} (u_i - u_e), \quad \nabla \cdot u_i = 0
\]

\[
(\partial_t + u_e \cdot \nabla) u_e = -\frac{e}{m_e} \left( E + \frac{1}{c} u_e \times B \right) - \nabla p_e + \nu_e \nabla \cdot u_e - \frac{1}{\tau_e} (u_e - u_i), \quad \nabla \cdot u_e = 0
\]

\[
- \nabla A = \frac{4\pi}{c} J = \frac{4\pi}{c} ne(u_i - u_e), \quad \nabla \cdot A = 0
\]

\[
E = -\frac{1}{c} \partial_t A, \quad B = \nabla \times A
\]

In addition to internal viscosities \( \nu_\sigma, \sigma = i, e \), there are linear drag terms which represent the exchange of momentum between the two fluids by collisions of the constituent particles and which are proportional to the collision frequencies \( 1/\tau_\sigma, \sigma = i, e \). Conservation of momentum requires \( m_e/\tau_e = m_i/\tau_i \). Note that the vector potential \( A \) is not an independent variable, but is completely determined from \( u_e, u_i \) by means of the elliptic equation (54). Mathematical properties of solutions of these equations (existence, regularity, etc.) are studied in [33], including even a separate equation for neutral molecules.
Neglecting viscosities and drag, the two-fluid model is Hamiltonian with canonical momenta:

\[ p_i = u_i + \frac{e}{m_i c} A, \quad p_e = u_e - \frac{e}{m_e c} A, \]

satisfying \( \nabla \cdot p = 0 \) for \( \sigma = i, e \). E.g. see [35]. Using \( (u_\sigma \cdot \nabla) u_\sigma = \nabla (\frac{1}{2} |u_\sigma|^2) - u_\sigma \times (\nabla \times u_\sigma) \), \( \pi_\sigma = p_\sigma + \frac{1}{2} |u_\sigma|^2 \), the eqs. (52)-(55) for \( u_e, u_i \) can be rewritten for \( p_e, p_i \) as

\[ \partial_t p_i = u_i \times (\nabla \times p_i) - \nabla \pi_i + \nu_i \Delta u_i - \frac{1}{\tau_i} (u_i - u_e), \quad (57) \]
\[ \partial_t p_e = u_e \times (\nabla \times p_e) - \nabla \pi_e + \nu_e \Delta u_e - \frac{1}{\tau_e} (u_e - u_i), \quad (58) \]

Define the magnetic diffusivity \( \lambda = m_i c^2/4\pi n_e \tau_i = m_e c^2/4\pi n_e \tau_e \), so that \( \frac{m_e}{\tau_e} (u_e - u_i) = \frac{m_i}{\tau_i} (u_i - u_e) = \lambda \cdot \hat{\xi} \Delta A \). Hence, choosing \( \nu_e = \nu_i = \lambda \), (57), (58) become

\[ \partial_t p_\sigma = u_\sigma \times (\nabla \times p_\sigma) - \nabla \pi_\sigma + \lambda \Delta p_\sigma, \quad \sigma = i, e. \quad (59) \]

The vector potential \( A \) can be recovered from \( p_i, p_e \) by solving the Helmholtz equation

\[ -\Delta A + \kappa^2 A = \frac{4\pi}{c} n_e (p_i - p_e), \quad \nabla \cdot A = 0 \]

with \( \kappa^2 = 4\pi n e^2 / \mu c^2 \) and \( \mu = m_i^{-1} + m_e^{-1} \) and then \( u_i, u_e \) obtained from (56). For this non-ideal version of the two-fluid model there are two stochastic conservation laws corresponding to the two canonical momenta:

**Proposition III.2.** Divergence-free fields \( u_i, u_e \in C([t_0, t_f], C^{k, \alpha}(\Omega)) \) satisfy the non-ideal, incompressible two-fluid equations (52)- (55) with initial data \( u_{i,0}, u_{e,0} \in C^{k, \alpha}(\Omega) \) for \( k \geq 2 \) and for unit magnetic Prandtl numbers \( \nu_e/\lambda = \nu_i/\lambda = 1 \), iff for all closed, rectifiable loops \( C \) and for all \( t \in [t_0, t_f] \)

\[ \oint_C p_\sigma(x, t) \cdot dx = \mathbb{E} \left[ \oint_{\tilde{a}_\sigma(C, t)} p_\sigma(\tilde{a}) \cdot d\tilde{a} \right], \quad \sigma = i, e \quad (61) \]

Here the canonical momenta \( p_\sigma, \sigma = i, e \) are given by eq. (57) and \( \tilde{a}_\sigma(x, t) \) are “back-to-label maps” for stochastic forward flows \( \tilde{x}_\sigma(a, t) \) solving

\[ d\tilde{x}_\sigma(a, t) = u_\sigma(\tilde{x}_\sigma(a, t), t) dt + \sqrt{2\lambda} dW(t), \quad t > t_0, \quad \tilde{x}_\sigma(a, t_0) = a, \quad (62) \]

for \( \sigma = i, e \), with \( W(t) \) a standard Brownian motion.

**Remarks:** (i) If the noise terms in eq. (52) were chosen to be \( \sqrt{2\lambda_\sigma} dW_\sigma(t), \quad \sigma = i, e \) with \( \lambda_e \neq \lambda_i \) then one would obtain two-fluid model (52)-(55) with \( \nu_\sigma = \lambda_\sigma \) and \( \tau_\sigma = m_\sigma c^2/4\pi n e^2 \lambda_\sigma \) for \( \sigma = i, e \). Although mathematically well-posed, this system is unphysical since it violates conservation of momentum.
(ii) The two stochastic Kelvin theorems (61) are mathematically equivalent to stochastic Weber formulas:

\[ p_\sigma(x, t) = \mathbb{E}[\nabla_x \tilde{a}_\sigma(x, t) p_{\sigma 0}(\tilde{a}_\sigma(x, t))], \quad \sigma = i, e, \tag{63} \]

identical in form to (13) for HMHD.

**Sketch of proof of Proposition III.3**. The proof is very similar to that of Proposition III.1.1 and to the proofs in Section 4 of [10]. Stochastic Weber variables \( \tilde{p}_\sigma, \sigma = i, e \) of the same form as (45) are shown to obey stochastic PDE’s of the same form as (46). It is now enough to assume \( u_e, u_i \in C([t_0, t_f], C^{k, \alpha}(\Omega)) \) for \( k \geq 2 \), because the integration-parts-identity [19, 20] can be employed to show that \( \tilde{p}_e, \tilde{p}_i \in C^{2,\alpha}(\Omega) \), just as for \( \tilde{p}_i \) in the proof of Proposition III.1.1.

\[ \square \]

The curl of the two canonical momenta in (56) give two generalized vorticities, \( \Omega_\sigma = \nabla \times p_\sigma \) for \( \sigma = i, e \):

\[ \Omega_i = \omega_i + \frac{e}{m_i c} B, \quad \Omega_e = \omega_e - \frac{e}{m_e c} B. \tag{64} \]

If we assume sufficient smoothness \( (u_e, u_i) \in C([t_0, t_f], C^{k,\alpha}(\Omega)) \) with \( k \geq 3 \), then these satisfy equations

\[ \partial_t \Omega_\sigma = \nabla \times (u_\sigma \times \Omega_\sigma) + \lambda \triangle \Omega_\sigma, \quad \sigma = i, e, \tag{65} \]

and there are two stochastic Cauchy formulas

\[ \Omega_\sigma(x, t) = \mathbb{E}\left[ \Omega_{\sigma 0}(a) \cdot \nabla_x x_\sigma(a, t) |_{\tilde{a}_\sigma(x, t)} \right], \quad \sigma = i, e, \tag{66} \]

of the same form as (44) for HMHD. Unlike for the previous models, for the two-fluid model there is no separate stochastic “frozen-in” property for the magnetic field \( B \) alone, at general Prandtl number.

**IV Discussion**

The results of the present paper provide new tools with which to investigate and explain resistive phenomena in plasma fluids. The stochastic Lagrangian conservation laws derived here are the analogues for non-ideal hydromagnetic systems of flux-freezing for ideal ones. Especially robust is the stochastic Alfvén theorem and stochastic Lundquist formula, which hold in both MHD and HMHD at any magnetic Prandtl number. These results have important implications for resistive magnetic reconnection and related problems such as magnetic dynamo [23, 3], which will be pursued in detail in future publications.
Here we just note that that the stochastic Lundquist formula describes how the resultant magnetic field \( \mathbf{B}(\mathbf{x}, t) \) at a spacetime point \((\mathbf{x}, t)\) is obtained by advecting all magnetic field lines as “frozen-in” to the stochastic flows, with added white-noise, and then averaging those magnetic field vectors that arrive to the given point. The advection by the stochastic flows produces the nonlinear effects of magnetic stretching and tilting, while the average over the Brownian motions represents the resistive “gluing” of the magnetic field, reconnecting the field-lines and changing their topology. Such resistive effects are recognized as important in the dynamo process by the cycle of “stretch-twist-fold-reconnect” \[17\]. Moffatt has referred to the “oxymoronic role” of resistivity, writing that “the dynamo process may be described as a process of ‘regenerative decay’, or perhaps better ‘reinvigorating dissipation’.” \[28\]

A possible criticism of the physical relevance of our results is that molecular resistivity, represented by a Laplacian term in the induction equation, is a poor model of actual dissipative processes in a plasma. In contrast to the iconic status of the viscosity term in the Navier-Stokes equation for neutral fluids, there is considerably less universality in the form of the dissipation in plasmas or, indeed, in the validity of a hydromagnetic description. The two-fluid equations of Braginsky \[4\] (see also \[23\], \[3\]) are more complicated than those discussed in our section \[III.1\] For example, viscosity and resistivity in the standard Braginsky equations are anisotropic, with magnitudes differing along directions longitudinal and transverse to the local magnetic field. Furthermore, microscopic Spitzer resistivity is not the only form of magnetic dissipation that may occur in plasmas. A wide variety of processes, both collisional and non-collisional, have been proposed to lead to “anomalous resistivity” of different forms \[34\], \[40\]. Furthermore, in a partially ionized plasma the collisions of ions with neutral molecules induces an “ambipolar drift” of magnetic field lines with velocity proportional to the Lorenz force \[37\], \[5\] and this can be the most significant form of magnetic dissipation in some cases, e.g. the interstellar medium. Thus, the fluid models that we have considered are not necessarily the most physically realistic.

There are two responses that we can give to this important set of criticisms.

First, the results presented in this paper are far from the most general possible. We have chosen to restrict discussion here to models with incompressible fluids and isotropic transport coefficients, since these hydromagnetic models are widely employed and the proofs of the main results are simpler for them than for more complete models. However, in a following work \[15\], we establish similar results for much more general plasma fluid models, allowing for compressible fluids, anisotropic pressure and transport coefficients, neutral components, etc. Stochastic conservation laws of the sort demonstrated
here are quite general and should hold for a very large class of non-ideal plasma fluid models, when the ideal version of the model possesses a corresponding “frozen-in” field. It may even be possible to prove similar stochastic laws for kinetic models of plasmas with collisions described by Boltzmann kernels \[10\] or Fokker-Planck operators \[23\] (Section 8.3), since the ideal, collisionless Vlasov dynamics possesses analogues of the frozen-in invariants \[43\].

Second, the precise form of the dissipation in hydromagnetic models may not matter, as long as its effects are confined to sufficiently small length-scales. There is then a large “effective Reynolds number” (both magnetic and kinetic) and the plasma fluid becomes turbulent. We have previously argued \[12\] that the laws of flux conservation and magnetic line-motion in hydromagnetic turbulence are intrinsically stochastic in the limit of infinite Reynolds number. Formally, the random white-noise disappears in the equations for stochastic Lagrangian particles

\[
d\vec{x} = u'(\vec{x}(t), t)dt + \sqrt{2\lambda} d\vec{W}(t), \quad t > t_0, \quad \vec{x}(t_0) = x_0
\]

(67)
as \(\nu, \lambda \to 0\) (cf. also eq.(16)). However, the randomness need not vanish if the advecting velocity \(u'\) solving (1) approaches a rough or singular velocity \(u\) in this limit, as expected for a Kolmogorov-type cascade range. As a consequence of “explosive” separation of particles in Richardson two-particle turbulent diffusion, a pair of solutions of (67) with the same initial condition \(x_0\) may separate at time \(t\) to a mean-square distance \(\sim t^3\) in the limit \(\nu, \lambda \to 0\). These statements have been proved rigorously to hold in the Kazantsev-Kraichnan kinematic dynamo model \[21, 22\]. It has furthermore been proved that the Lagrangian trajectories in the Kazantsev-Kraichnan model remain stochastic as \(\nu, \lambda \to 0\), a result that has been termed “spontaneous stochasticity.” The limiting probability distributions of trajectories are known to be very robust and universal for the case of an incompressible fluid velocity, with the same result being obtained for limits of a wide class of regularizations. See \[11, 12\] for references and more detailed discussion. The rigorous results for the Kazantsev-Kraichnan dynamo model and the new results in the present work give further plausibility to the ideas that the precise form of dissipation does not matter in nonlinear hydromagnetic turbulence and that flux conservation and “frozen-in” line-motion will remain as stochastic laws in the limit of very large Reynolds numbers.

**Acknowledgements.** We acknowledge the warm hospitality of the Isaac Newton Institute for Mathematical Sciences during the programme on “The Nature of High Reynolds Number Turbulence”, when this paper was completed. This work was partially supported by NSF grant AST-0428325 at Johns Hopkins University.
References

[1] Alfvén, H., “On the existence of electromagnetic-hydrodynamic waves,” Arkiv f. Mat., Astron. o. Fys. 29B 1–7 (1942).

[2] Bekenstein, J. D. and A. Oron, “Conservation of circulation in magnetohydrodynamics,” Phys. Rev. E 62 5594–5602 (2000).

[3] Bellan, P. M., Fundamentals of Plasma Physics. (Cambridge University Press, Cambridge, UK, 2006)

[4] Braginsky, S. I., “Transport processes in a plasma,” Rev. Plasma Phys. 1 205–311 (1965).

[5] Brandenburg, A. and K. Subramanian, “Astrophysical magnetic fields and nonlinear dynamo theory,” Phys. Rep. 417 1–209 (2005).

[6] Cauchy, A. L., “Théorie de la propagation des ondes à la surface d’un fluide pesant d’une profondeur indéfinie (1815)”, Mém. Divers Savants (2) 1 3; Oeuvres (1) 1 5.

[7] Chorin, A. and J. Marsden, Mathematical Introduction to Fluid Mechanics. (Berlin-Heidelberg-New York, Springer, 2000).

[8] Constantin, P., “Singular, weak and absent: Solutions of the Euler equations,” Physica D 237 1926–1931 (2008)

[9] Constantin, P. and G. Iyer, “A stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations,” Commun. Pure Appl. Math. LXI 0330–0345 (2008).

[10] Duvaut, G. and J. L. Lions, “Inéquations en thermoélasticité et magnétohydrodynamique,” Arch. Ration. Mech. Anal. 46 241–279 (1972).

[11] Eyink, G. L., “Turbulent cascade of circulations,” C. R. Physique 7 449–455 (2006).

[12] Eyink, G. L., “Turbulent diffusion of lines and circulations,” Phys. Lett. A. 368 486–490 (2007).

[13] Eyink, G. L., “Dissipative anomalies in singular Euler flows,” Physica D 237 1956–1968 (2008)

[14] Eyink, G. L., “Stochastic least-action principle for the incompressible Navier-Stokes equation,” Physica D, accepted (2008)
[15] Eyink, G. L. and A. F. Neto, “Stochastic line-motion and stochastic conservation laws for non-ideal hydromagnetic models. II. Compressible fluids and anisotropic transport coefficients,” in preparation (2009).

[16] Fournier, N. and S. Méléard, “A Markov process associated with a Boltzmann equation without cutoff and for non-Maxwell molecules, J. Stat. Phys. 104 359–385 (2001).

[17] Galloway, D., “Fast dynamos,” in: Advances in Nonlinear Dynamos. eds. A. Ferriz-Mas and M. Núñez (CRC Press, 2003).

[18] Helmholtz, H., “Über Integrale der hydrodynamischen Gleichungen welche den Wirbelbewegungen entsprechen,” Crelles Journal 55 25–55 (1858).

[19] Iyer, G., “A stochastic Lagrangian formulation of the Navier-Stokes and related transport equations.” Doctoral dissertation, University of Chicago, 2006.

[20] Iyer, G., “A stochastic perturbation of inviscid flows,” Comm. Math. Phys. 266 631–645 (2006)

[21] Kazantsev, A. P., “Enhancement of a magnetic field by a conducting fluid,” Sov. Phys. JETP 26 1031–1034 (1968).

[22] Kraichnan, R. H., “Small-scale structure of a scalar field convected by turbulence,” Phys. Fluids 11 945–953 (1968).

[23] Kulsrud, R. M., Plasma Physics for Astrophysics. (Princeton University Press, Princeton, NJ, 2005)

[24] Kunita, H., Stochastic Flows and Stochastic Differential Equations. (Cambridge University Press, Cambridge, 1990).

[25] Kuznetsov, E. A. and V. P. Ruban, “Hamiltonian dynamics of vortex and magnetic lines in hydrodynamic type systems”, Phys. Rev. E. 61 831–841 (2000).

[26] Lighthill, M. J., “Studies on MHD waves and other anisotropic wave motion,” Phil. Trans. Roy. Soc. 252A 397–430 (1960).

[27] Lundquist, S., “On the stability of magneto-hydrostatic fields”, Phys. Rev. 83 307–311 (1951).

[28] Moffatt, H. K., “The oxymoronic role of molecular diffusivity in the dynamo process,” Woods Hole Oceanographic Institution Technical Rep. WHOI-78-67, 145–149 (1978).
[29] Newcomb, W. A., “Motion of magnetic lines of force,” Ann. Phys. N.Y. 3 347–385 (1958).

[30] Núñez, M., “Some rigorous results for the kinematic dynamo problem with general boundary conditions,” J. Math. Phys. 38 1583–1592 (1997).

[31] Núñez, M., “Growth of the magnetic field in Hall magnetohydrodynamics,” J. Phys. A: Math. Gen. 37 9317–9323 (2004).

[32] Núñez, M., “Existence theorems for two-fluid magnetohydrodynamics,” J. Math. Phys. 46 083101 (2005).

[33] Núñez, M., “A theorem of existence for the equations of magnetohydrodynamics of partially ionized plasmas,” Proc. R. Soc. A 464 1571–1586 (2008).

[34] Papadopoulos, K., “A review of anomalous resistivity for the ionosphere,” Rev. Geophys. Sp. Phys. 15 113–127 (1977).

[35] Ruban, V. P., “Motion of magnetic flux lines in magnetohydrodynamics,” JETP 89 299–310 (1999).

[36] Sermange, M. and R. Temam, “Some mathematical questions related to the MHD equations,” Commun. Pure Appl. Math. 36 635–664 (1983).

[37] Spitzer, Jr., L., Physical Processes in the Interstellar Medium. (J. Wiley & Sons, New York, 1978)

[38] Stern, D. P., “The motion of magnetic field lines,” Space Science Reviews 6 147–173 (1966).

[39] Thomson, W. (Lord Kelvin), “On vortex motion”, Trans. Roy. Soc. Edin. 25 217–260 (1869).

[40] Treumann, R. A., “Origin of resistivity in reconnection,” Earth Planets Space (Japan) 53 453–462 (2001).

[41] Weber, W., “Über eine Transformation der hydrodynamischen Gleichungen,” J. Reine Angew.Math. 68 286–292 (1868).

[42] Witalis, E. A., “Hall magnetohydrodynamics and its applications to laboratory and cosmic plasma,” IEEE Trans. Plasma Science PS-14 842–848 (1986).

[43] Yan’kov, V. V., “Attractors and frozen-in invariants in turbulent plasma,” Phys. Uspekhi 40 477–493 (1997).