EXAMPLES OF MINIVERSEL DEFORMATIONS OF INFINITY ALGEBRAS

ALICE FIALOWSKI AND MICHAEL PENKAVA

Abstract. A classical problem in algebraic deformation theory is whether an infinitesimal deformation can be extended to a formal deformation. The answer to this question is usually given in terms of Massey powers. If all Massey powers of the cohomology class determined by the infinitesimal deformation vanish, then the deformation extends to a formal one. We consider another approach to this problem, by constructing a miniversal deformation of the algebra. One advantage of this approach is that it answers not only the question of existence, but gives a construction of an extension as well.

In this paper, we study some examples of miniversal deformations of infinity algebras, and use these examples to illustrate how to use a miniversal deformation to determine when an infinitesimal deformation extends to a formal deformation. Actually, using a miniversal deformation, one can construct such an extension explicitly. Also, the obstruction to an extension can be computed by this method.

An infinitesimal deformation extends to a formal one precisely when the unique morphism from the base of the universal infinitesimal deformation to the base of the given deformation inducing the infinitesimal deformation can be lifted to a morphism from the miniversal deformation to the formal power series ring in the parameter of the deformation. A nice property of this algebraic approach is that it answers more than the question of existence; in fact, it gives a construction of an extension of the infinitesimal deformation to a formal deformation. Moreover, since the question is reduced to studying the morphisms of the base of the miniversal deformation to a formal power series ring, the...

Date: March 29, 2022.

1991 Mathematics Subject Classification. 14D15, 13D10, 14B12, 16S80, 16E40, 17B55, 17B70.

Key words and phrases. Versal Deformations, Infinity Algebras, Cohomology, Infinitesimal Deformation, Extensions.

The research of the authors was supported by grants MTA-OTKA-NSF 38453, OTKA T043641 and T043034 and by grants from the University of Wisconsin-Eau Claire.
general question of which infinitesimal deformations extend to formal deformations is reduced to a simple algebraic question.

Let us point out that the problem of extending a deformation emerges in deformation quantization as well, see [2].

1. Introduction

We work in the framework of the parity reversion $W = \Pi V$ of the usual vector space $V$ on which an $L_\infty$ algebra structure is defined, because in the $W$ framework, an $L_\infty$ structure is simply an odd coderivation $d$ of the symmetric coalgebra $S(W)$, satisfying $d^2 = 0$, in other words, it is an odd codifferential in the $\mathbb{Z}_2$-graded Lie algebra of coderivations of $S(W)$. As a consequence, when studying $\mathbb{Z}_2$-graded Lie algebra structures on $V$, the parity is reversed, so that an $m|n$-dimensional vector space $W$ corresponds to a $n|m$-dimensional $\mathbb{Z}_2$-graded Lie structure on $V$. Moreover, the $\mathbb{Z}_2$-graded anti-symmetry of the Lie bracket on $V$ becomes the $\mathbb{Z}_2$-graded symmetry of the associated coderivation $d$ on $S(W)$.

A formal power series $d = d_1 + \cdots$, with $d_i \in L_i = \text{Hom}(S^i(W), W)$ determines an element in $L = \text{Hom}(S(W), W)$, which is naturally identified with $\text{Coder}(S(W))$, the space of coderivations of the symmetric coalgebra $S(W)$. Thus $L$ is a $\mathbb{Z}_2$-graded Lie algebra. An odd element $d$ in $L$ is a called a codifferential if $[d, d] = 0$. We also say that $d$ is an $L_\infty$ structure on $W$.

A detailed description of $L_\infty$ algebras can be obtained in [15, 16]. The study of examples of $L_\infty$ algebra structures in [9, 11, 10], and especially [1] may be useful to the reader because they contain many examples of $L_\infty$ algebras and their miniversal deformations.

Let us establish some basic notation for the cochains. Suppose $W = \langle w_1, \cdots, w_{m+n} \rangle$ with $w_1, \cdots, w_n$ odd and $w_{n+1}, \cdots, w_{m+n}$ even elements. If $I = \{i_1, \cdots, i_{m+n}\}$ is a multi-index, with $i_k$ either zero or one when $k \leq n$, let $w_I = w_1^{i_1} \cdots w_{m+n}^{i_{m+n}}$. Denote $\deg(I) = i_1 + \cdots + i_n \ (\text{mod} \ 2)$. Then for $n \geq 1$,

$$(S^n(W))^e_\langle = \langle w_I | \text{parity}(I) = 0, \deg(I) = n \rangle$$
$$(S^n(W))^o_\langle = \langle w_I | \text{parity}(I) = 1, \deg(I) = n \rangle$$

For $n = \deg(I)$, for $j = 1, \cdots, m+n$, we define a map $\varphi_j : S^n(W) \to W$ by $\varphi_j(w_I) = I! \delta_{i_j} w_j$, where $I! = i_1! \cdots i_{m+n}!$. Let $L_n := \text{Hom}(S^n(W), W)$, then $L_n = \langle \varphi_j, \deg(I) = n \rangle$. If $\varphi$ is odd, we denote it by the symbol $\psi$ to make it easier to distinguish the even and odd elements.
1.1. Versal Deformations. For a treatment of classical formal deformation theory we refer to [14]. Versal deformation theory was first worked out for the case of Lie algebras in [4, 5, 7] and then extended to $L_\infty$ algebras in [8].

An augmented local ring $A$ with maximal ideal $m$ will be called an infinitesimal base if $m^2 = 0$, and a formal base if $A = \varprojlim_n A/m^n$. A deformation of an $L_\infty$ algebra structure $d$ on $W$ with base given by a local ring $A$ with augmentation $\epsilon : A \rightarrow K$, where $K$ is the field over which $W$ is defined, is an $A$-$L_\infty$ structure $\tilde{d}$ on $W \hat{\otimes} A$ such that the morphism of $A$-$L_\infty$ algebras $\epsilon_* = 1 \otimes \epsilon : L_A \rightarrow L \otimes K = L$ satisfies $\epsilon_* (\tilde{d}) = d$. (Here $W \hat{\otimes} A$ is an appropriate completion of $W \otimes A$.) The deformation is called infinitesimal (formal) if $A$ is an infinitesimal (formal) base.

In general, the cohomology $H(D)$ of $d$ given by the operator $D : L \rightarrow L$ with $D(\varphi) = [\varphi, d]$ may not be finite dimensional. However, $L$ has a natural filtration $L^n = \prod_{i=n}^\infty L_i$, which induces a filtration $H^n$ on the cohomology, because $D$ respects the filtration. We say that $H(D)$ is of finite type if $H^n/H^{n+1}$ is finite dimensional for all $n$. Since this is always true when $W$ is finite dimensional, the examples we study here will always be of finite type. A set $\{\delta_i\}$ will be called a basis of the cohomology, if any element $\delta$ of the cohomology can be expressed uniquely as a formal sum $\delta = \delta_i a^i$. (Here and throughout the paper, we use Einstein's summation convention). If we identify $H(D)$ with a subspace of the space of cocycles $Z(D)$, and we choose a basis $\{\beta_i\}$ of the coboundary space $B(D)$, then any element $\zeta \in Z(D)$ can be expressed uniquely as a sum $\zeta = \delta_i a^i + \beta_i b^i$.

For each $\delta_i$, let $u^i$ be a parameter of opposite parity. Then the infinitesimal deformation $d^1 = d + \delta_i u^i$, with base $A = \mathbb{K}[u^k]/(u^i u^j)$ is universal in the sense that if $d^i$ is any infinitesimal deformation with base $B$, then there is a unique morphism $f : A \rightarrow B$, such that the morphism $f_* = 1 \otimes f : L_A \rightarrow L_B$ satisfies $f_*(d^1) \sim d^i$.

For formal deformations, there is no universal object in the sense above. A versal deformation is a deformation $d^\infty$ with formal base $A$ such that if $d^i$ is any formal deformation with base $B$, then there is some morphism $f : A \rightarrow B$ such that $f_*(d^\infty) \sim d^i$. If $f$ is unique whenever $B$ is infinitesimal, then the versal deformation is called miniversal. In [8], we constructed a miniversal deformation for $L_\infty$ algebras with finite type cohomology.
The method of construction is as follows. Define a coboundary operator $D$ by $D(\varphi) = [\varphi, d]$. First, one constructs the universal infinitesimal deformation $d^1 = d + \delta_i u^i$, where $\delta_i$ is a graded basis of the cohomology $H(D)$ of $d$, or more correctly, a basis of a subspace of the cocycles which projects isomorphically to a basis in cohomology, and $u^i$ is a parameter whose parity is opposite to $\delta_i$. The infinitesimal assumption that the products of parameters are equal to zero gives the property that $[d^1, d^1] = 0$. Actually, we can express $[d^1, d^1] = (-1)^{\delta_i (\delta_i + 1)} [\delta_i, \delta_j] u^i u^j = \delta_k a_{ij}^k u^i u^j + \beta_k b_{ij}^k u^i u^j$, where $\beta_i$ is a basis of the coboundaries, because the bracket of $d^1$ with itself is a cocycle. Note that the right hand side is of degree 2 in the parameters, so it is zero up to order 1 in the parameters.

If we suppose that $D(\gamma_i) = -\frac{1}{2} \beta_i$, then by replacing $d^1$ with $d^2 = d^1 + \gamma_k b_{ij}^k u^i u^j$, one obtains $[d^2, d^2] = \delta_k a_{ij}^k u^i u^j + 2[\delta_i u^i, \gamma_k b_{ij}^k u^i u^j] + [\gamma_k b_{ij}^k u^i u^j, \gamma_l b_{ij}^l u^i u^j]$.

Thus we are able to get rid of terms of degree 2 in the coboundary terms $\beta_i$, but those which involve the cohomology terms $\delta_i$ can not be eliminated. Therefore, $R^k = a_{ij}^k u^i u^j$ must be equal to zero up to order 3, which is accomplished by taking the base of the second order deformation to be the quotient of the ring $\mathbb{K}[u^i]/(u^i u^j u^k)$ by the ideal generated by the second order relations $R^k$. One continues this process, taking the bracket of the $n$-th order deformation $d^n$, adding some higher order terms to cancel coboundaries, obtaining higher order relations, which extend the second order relations.

Either the process continues indefinitely, in which case the miniversal deformation is expressed as a formal power series in the parameters, or after a finite number of steps, the right hand side of the bracket is zero after applying the $n$-th order relations. In this case, the miniversal deformation is simply the $n$-th order deformation. In either case, we obtain a set of relations $R^i$ on the parameters, one for each $\delta_i$, and the algebra $A = \mathbb{C}[[u^i]]/(R^i)$ is called the base of the miniversal deformation. Examples of the construction of miniversal deformations can be found in [6, 7, 13, 9, 11].

1.2. Extensions of Infinitesimal Deformations. Let us put together a general picture of how to use a miniversal deformation to solve the extension problem. Let us suppose that $\psi_k$ and $\phi_k$ are bases of the odd and even parts of a preimage of the cohomology of a codifferential $d$, and $\alpha_k, \beta_k$ are bases of the odd and even parts of a preimage of the...
coboundaries determined by $d$. Then there is a miniversal deformation of the form
\[ d^\infty = d + \psi_k t^k + \phi_k \theta^k + \alpha_k x^k + \beta_k y^k, \]
where the $t^k$ are odd parameters, the $\theta^k$ are even ones, the $x^k$ are odd and the $y^k$ are even formal power series in the parameters, and for each $k$, there are odd relations $r^k_o$ and even relations $r^k_e$, which are formal power series in the parameters. The base of the miniversal deformation is given by $\mathbb{K}[\![t^k, \theta^k]\!]/(r^k_o, r^k_e)$.

Classically, an infinitesimal deformation is given by a single even parameter $u$. It is natural to extend the classical picture by adding an odd parameter $\theta$, so that for our purposes we will state the deformation problem in the following manner. Consider an infinitesimal deformation of the form
\[ d^i = d + \psi u + \varphi \theta, \]
where $\psi$ is an odd and $\varphi$ is an even cocycle. An important question is:

When does this infinitesimal deformation extend to a formal deformation?

Without loss of generality, one can assume that $\psi$ and $\varphi$ are (possibly infinite) linear combinations of the $\psi_k$ and $\phi_k$. This is because one can remove any coboundary term by applying an equivalence. Similarly, any extension of this infinitesimal deformation to an $n$-th order deformation is equivalent to one of the form
\[ d^n = d + \psi_k a^k u^i + \phi_k b^k u^i \theta + \alpha_k g^k u^i + \beta_k h^k u^i \theta, \]
where $a^k = a^k u^i$, $b^k = b^k u^i$, $g^k = g^k u^i$ and $h^k = h^k u^i$ are polynomials of degree less than or equal to $n$ in $u$ without constant term such that $\psi = \psi_1 a^1_1$ and $\varphi = \phi_1 b^1_1$. An important generalization of the first question is:

When does such an $n$-th order deformation extend to a formal deformation?

To answer this question, first note that if we identify $t^k = a^k$ and $\theta^k = b^k \theta$, then

1. The relations on the base are satisfied up to order $n + 2$.
2. $g^k = x^k$ (mod $n + 1$) and $h^k \theta = y^k$ (mod $n + 1$).

The deformation $d^n$ extends to a formal deformation $d^f$ of $d$ precisely when there are extensions of $a^k$ and $b^k$ to formal power series such that the identifications $t^k = a^k$ and $\theta^k = b^k$ satisfy the relations on the base of the formal deformation. Let $f$ be the morphism $f : \mathbb{K}[\![x^k, \theta^k]\!] \rightarrow \mathbb{K}[\![u, \theta]\!]$ induced by the identifications above. Then $f$ descends to a morphism from the base of the miniversal deformation to $\mathbb{K}[\![u, \theta]\!]$, and $d^f = f_*(d^\infty)$ is a formal deformation extending $d^n$. Because there may
be many extensions of \(a^k\) and \(b^k\) to formal power series satisfying the relations, the deformation \(d^f\) is not unique in general.

Given a formal deformation \(d^f\) of the form
\[
d^f = d + \psi_k a^k u^i + \phi_k b^k u^i \theta + \alpha_k g^k u^i + \beta_k h^k u^i \theta,
\]
where now \(a^k\), \(b^k\), \(g^k\) and \(h^k\) are formal power series, there is a unique map \(f\) from the base of the miniversal deformation to \(K[[u,\theta]]\) satisfying \(f^\ast(d^\infty) = d^f\). Thus it may seem that the miniversal deformation is universal. The problem is that we work in the category of equivalence classes of deformations, so that \(d^f\) may be equivalent to other deformations of the form given by equation (1). The necessity of working with equivalence classes is clear from the fact that in general, a formal deformation is not of the form given by equation (1), but merely equivalent to one in such a form, because coboundary terms may appear in \(d^f\). Moreover, an equivalent deformation in the form given by equation (1) is not unique in general. We will give an example later on in the text to illustrate this point.

Our purpose in this article is to construct some nontrivial examples of miniversal deformations and use them to illustrate how to carry out the procedure of determining which infinitesimal deformations extend to a formal one.

2. CODIFFERENTIALS ON A 2|1 DIMENSIONAL SPACE

In this section, we will be studying miniversal deformations of some \(L_\infty\) structures on a 2|1 dimensional space. For this space, our multi-indices \(I\) will be ordered triples. Since \(w_1\) is the only odd basis element, we have
\[
(L_n)_e = (\varphi_1^{1,q,n-q-1}, \varphi_2^{0,p,n-p}, \varphi_3^{0,p,n-p}, 1 \leq q \leq n - 1, 1 \leq p \leq n)
\]
\[
(L_n)_o = (\psi_1^{1,q,n-q-1}, \psi_2^{1,q,n-q-1}, \psi_3^{0,p,n-p}, 1 \leq q \leq n - 1, 1 \leq p \leq n),
\]
so that \(|L_n| = 3n + 2|3n + 1\). In [1], the moduli space of codifferentials of degree two on this space was computed. The degree two codifferentials are divided into two kinds. Codifferentials of the first kind are of the form
\[
d = \psi_2^{1,1,0} x + \psi_3^{1,1,0} a + \psi_2^{1,0,1} b + \psi_3^{1,0,1} c
\]
and those of the second kind are of the form
\[
d = \psi_1^{0,2,0} a + \psi_1^{0,1,1} b + \psi_1^{0,0,2} c.
\]
Miniversal deformations for codifferentials of the first kind were computed in [1]. The codifferentials of degree two of the first kind form a complicated one parameter family, while the codifferentials of degree two of the second kind are all equivalent to one of only two types,
which we called Type (1,0,0) and Type (0,1,0), where the type represents the triple \((a, b, c)\) of coefficients in equation (2). Even though the description of the moduli space of degree two codifferentials of the second kind is simple, the cohomology for both of the degree two codifferentials of the second kind is infinite dimensional. We did not give a complete description of the miniversal deformations of degree two codifferentials of the second kind in [1], so we will give that description here. The miniversal deformations provide some nice examples which illustrate how to use a miniversal deformation to determine whether an infinitesimal deformation extends to a formal deformation.

2.1. Miniversal deformations of Type (1,0,0). Let \(d = \psi_1^{0,2,0}\). We obtain the following table of coboundaries:

\[
\begin{align*}
D(\varphi_1^{1,q,n-q-1}) &= \psi_1^{0,2+q,n-q-1} \\
D(\varphi_2^{0,p,n-p}) &= -2\psi_1^{0,p+1,n-p} \\
D(\varphi_3^{0,p,n-p}) &= 0 \\
D(\psi_0^{0,p,n-p}) &= 0 \\
D(\psi_2^{1,q,n-q-1}) &= 2\varphi_1^{1,q+1,n-q-1} + \varphi_2^{0,q+2,n-q-1} \\
D(\psi_3^{1,q,n-q-1}) &= \varphi_3^{0,q+2,n-q-1}
\end{align*}
\]

The cohomology is given by

\[
\begin{align*}
H^1 &= \langle \psi_1^{0,0,1}, \varphi_1^{0,1,0}, \varphi_3^{0,0,1}, \varphi_3^{0,1,0}, 2\varphi_1^{1,0,0}, \varphi_2^{0,1,0} \rangle \\
H^n &= \langle \psi_1^{0,0,n}, \varphi_3^{0,0,n}, \varphi_3^{0,1,n-1}, 2\varphi_1^{1,0,n-1} + \varphi_2^{0,1,n-1} \rangle, \text{ if } n > 1
\end{align*}
\]

Let us label the cohomology classes as follows

\[
\begin{align*}
\xi &= \psi_1^{0,1,0} \\
\psi_n &= \psi_1^{0,0,n}, \quad \phi_n = \varphi_3^{0,0,n}, n > 0 \\
\sigma_n &= \varphi_3^{0,1,n-1}, \quad \tau_n = 2\varphi_1^{1,0,n-1} + \varphi_2^{0,1,n-1}, \quad n > 0
\end{align*}
\]

In order to construct the miniversal deformation, we choose pre-images of the coboundaries as follows:

\[
\gamma_{k,l} = \frac{1}{2} \varphi_2^{0,k,l}, \quad \alpha_{k,l} = \psi_2^{1,k,l}, \quad \beta_{k,l} = \psi_3^{1,k,l}
\]

Then

\[
D(\gamma_{k,l}) = -\psi_1^{0,k+1,l} \quad D(\alpha_{k,l}) = 2\varphi_1^{1,k+1,l} + \varphi_2^{0,k+2,l} \quad D(\beta_{k,l}) = \varphi_3^{0,k+2,l}
\]

The universal infinitesimal deformation is given by

\[
d^1 = \psi_1^{0,2,0} + \xi s^1 + \psi_n t^n + \phi_n \theta^n + \sigma_n \eta^n + \tau_n \zeta^n,
\]
where \( s^1 \) and \( t^n \) are even parameters and \( \theta^n, \eta^n \) and \( \zeta^n \) are odd parameters.

The brackets we need to compute \([d^1, d^1]\) are

\[
\begin{align*}
[\xi, \phi_k] & = [\xi, \sigma_k] = [\tau_k, \tau_l] = 0 \\
[\xi, \tau_k] & = D(\gamma_{0,k-1}) \\
[\psi_k, \phi_l] & = \psi_{k+l-1} k \\
[\psi_k, \sigma_l] & = -D(\gamma_{0,k+l-2}) k \\
[\psi_k, \tau_l] & = -2\psi_{k+l-1} \\
[\sigma_k, \tau_l] & = \sigma_{k+l-1} + D(\alpha_{0,k+l-3})(1-l) \\
\end{align*}
\]

\[\sum_{k+l=n+1} t^k (k\theta^l - 2\zeta^l) = \frac{1}{2} \sum_{k+l=n+1} (k-l)\theta^k \theta^l = 0\]

\[\sum_{k+l=n+1} \eta^k (\zeta^l + (k-l-1)\theta^l) = \sum_{k+l=n+1} (1-l)\theta^k \zeta^l = 0\]

The second order deformation is easily computed to be

\[d^2 = d^1 + \gamma_{0,1} x^1 + \alpha_{0,1} y^1 + \beta_{0,1} z^1,\]

where

\[
\begin{align*}
x^n & = s^1 \zeta^{n+1} - \sum_{k+l=n+2} kt^k \eta^l \\
y^n & = \sum_{k+l=n+3} (l-1)\eta^k \zeta^l \\
z^n & = \frac{1}{2} \sum_{k+l=n+3} (l-k)\eta^k \eta^l = -\sum_{k+l=n+3} k\eta^k \eta^l \\
\end{align*}
\]

Note that only some of the pre-images of coboundaries actually play any role in the second order deformation. In this example, it turns out that the second order deformation is miniversal, so these are the only cochains which are necessary to add. To see this, let us consider the
EXAMPLES OF MINIVERSAL DEFORMATIONS OF INFINITY ALGEBRAS

brackets which arise in the computation of \([d^2, d^2]\).

\[
\begin{align*}
\left[ \xi, \alpha_{0,n} \right] &= \frac{1}{2} \tau_{n+1} + \gamma_{1,n} \\
\left[ \psi_k, \alpha_{0,l} \right] &= 2 \gamma_{0,k+l} \\
\left[ \alpha_{0,k}, \phi_l \right] &= \alpha_{0,k+l-1} k \\
\left[ \alpha_{0,k}, \sigma_l \right] &= \alpha_{1,k+l-2} k - \beta_{0,k+l-1} \\
\left[ \alpha_{0,k}, \tau_l \right] &= \alpha_{0,k+l-1} \\
\left[ \xi, \beta_{0,n} \right] &= \sigma_{n+1} \\
\left[ \psi_k, \beta_{0,l} \right] &= \phi_{k+l} + \tau_{k+l} - 2 \gamma_{1,k+l-1} k \\
\left[ \beta_{0,k}, \phi_l \right] &= \beta_{0,k+l-1} (k - l) \\
\left[ \beta_{0,k}, \sigma_l \right] &= \beta_{1,k+l-2} (k + 1 - l) \\
\left[ \beta_{0,k}, \tau_l \right] &= \alpha_{1,k+l-2} (1 - l) + 2 \beta_{0,k+l-1} \\
\left[ \beta_{0,k}, \gamma_{0,l} \right] &= - \alpha_{0,k+l-1} \left( \frac{1}{2} \right)
\end{align*}
\]

No coboundaries appear in these brackets, which means that the second order deformation is miniversal. It is also the case that the sum of the terms in the bracket \([d^2, d^2]\) involving the pre-images of a coboundary must vanish. In particular, the sum of all terms involving \(\alpha_{1,n}, \beta_{1,n}\) or \(\gamma_{1,n}\) cochains must vanish. Strictly speaking, it is unnecessary to check this fact, since it is guaranteed by the existence theorem for the miniversal deformation \[8\], but we found it interesting to check the manner in which the terms cancel. In fact, these terms cancel without using the relations on the base, although, as we shall show later, the same is not true for the \(\alpha_{0,n}, \beta_{0,n}\) and \(\gamma_{0,n}\) cochains.

The \(\beta_{1,n}\) terms appear only in the bracket \([\beta_{0,k-1}, \sigma_l]z^{k-1} \eta^l\), so we should have \(\sum_{k+l=n+2} (k + 1 - l) z^{k-1} \eta^l = 0\). We obtain

\[
\sum_{k+l=n+3} (k - l) z^{k-1} \eta^l = \sum_{i+j+l=n+5} -(i + j - 2 - l) i \eta^i j \eta^j l,
\]

which vanishes simply because the \(\eta\) cochains anti-commute.

To see that the \(\alpha_{1,n}\) cochains cancel, note that there are two sources of such terms. From \([\beta_{0,k-1}, \tau_l]z^{k-1} \zeta^l\), we get \(\alpha_{1,k+l-3} (1 - l) z^{k-1} \zeta^l\), while from \([\alpha_{0,k-1}, \sigma_l]y^{k-1} \eta^l\), we get \(\alpha_{1,k+l-3} k y^{k-1} \eta^l\). Substituting for \(y^{k-1}\) and \(z^{k-1}\), and summing, we obtain

\[
\sum_{i+j+l=n+5} -(1-l) i \eta^i \eta^j \zeta^l - (i+j-2)(1-j) i \eta^i j \zeta^j \eta^l = \sum_{i+j+l=n+5} l(2-l) i \eta^i \eta^j \zeta^l = 0
\]
To see that the $\gamma_{1,n}$ terms vanish, we compute
\[
[\sigma_k, \gamma_{0,l}]\eta^k x^l = -\gamma_{1,k+l-2}\eta^k x^l \\
[\psi_k, \beta_{0,l-1}] t^k z^{l-1} = -\gamma_{1,k+l-2}kt^k z^{l-1} \\
[\xi, \alpha_{0,n}] s^1 y^n = \gamma_{1,n} s^1 y^n
\]
Therefore, the sum of all terms involving $\gamma_{1,n}$ has coefficient
\[
s^1 y^n - \sum_{k+l=n+2} l\eta^k s^1 \zeta^{l+1} + \sum_{k+i+j=n+4} (n + 2 - k)\eta^k \eta^i \eta^j + kt\eta^i \eta^j = 0,
\]
since the first sum above is just $s^1 y^n$ and the second sum vanishes by the anticommutativity of $\eta$ cochains.

The relations on the base of the miniversal deformation are
\[
r_1 = -s^1 \zeta^1 + t^1 \eta^1 = 0 \\
r_2^n = \frac{1}{2} s^1 x^n + \sum_{k+l=n+1} t^k (k\theta^l - 2\zeta^l) = 0 \\
r_3^n = \sum_{k+l=n+1} \frac{1}{2}(k-l)\theta^k \theta^l + t^k z^{l-1} + \frac{1}{2}\eta^k x^l = 0 \\
r_4^n = s^1 z^{n-1} + \sum_{k+l=n+1} \eta^k (\zeta^l + (k-l-1)\theta^l) = 0 \\
r_5^n = \frac{1}{2} s^1 y^{n-1} + \sum_{k+l=n+1} (1-l)\theta^k \zeta^l + \frac{k}{2} t^k z^{l-1} = 0
\]
Now let us show that the coefficients of the $\alpha_{0,n}$, $\beta_{0,n}$ and $\gamma_{0,n}$ cochains vanish.

The coefficients of the terms involving $\beta_{0,n}$ are
\[
[\sigma_k, \gamma_{0,l}]\eta^k \eta^{l-1} = -\beta_{0,k+l-2}z^{k-1}\theta^l \\
[\psi_k, \beta_{0,l-1}] \theta^k z^{l-1} = \beta_{0,k+l-2} (k-1-l) z^{k-1}\theta^l \\
[\xi, \alpha_{0,n}] \zeta^{l-1} \zeta^l = \beta_{0,k+l-2} 2z^{k-1}\zeta^l
\]
First, we observe that
\[
\sum_{k+l=n+2} y^{k-1}\eta^l = \sum_{i+j+l=n+4} (j-1)\eta^i \zeta^j \eta^l = 0,
\]
so the coefficients from the $[\alpha_{0,k-1}, \sigma_i]$ terms add up to zero on their own. Next, we have
\[
\sum_{i+j=n+2} (j-i)\eta^i z^{j-1} = -\sum_{i+k+l=n+4} (n+2-2i)\eta^i k\eta^k \eta^l = 0.
\]
Thus we have

\[ 0 = \sum_{i+k=n+3} (k-i)\eta^i r^l_4 = \sum_{i+k+l=n+4} (j+l-1-i)\eta^j \eta^l (\zeta^l + (j-l-1)\theta^l) \]

\[ = \sum_{i+k+l=n+4} -2i\eta^i \eta^j \zeta^l + (-i^2 + (l+3)i)\eta^i \eta^j \theta^l \]

\[ = - \sum_{i+j+l=n+4} i\eta^i \eta^j ((i+j-3-l)\theta^l + 2\zeta^l) \]

\[ = \sum_{k+l=n+2} z^{k-1} ((k-1-l)\theta^l + 2\zeta^l), \]

which shows that the sum of the coefficients of the \( \beta_{0,n} \) terms is zero. We only needed the fourth relation on the base to establish this result.

The coefficients of the terms involving \( \alpha_{0,n} \) are

\[ [\alpha_0, k-1, \phi_l] y^{k-1} \theta^l = \alpha_0, k+l-2 (k-1) y^{k-1} \theta^l \]

\[ [\alpha_0, k-1, \tau_l] y^{k-1} \zeta^l = \alpha_0, k+l-2 y^{k-1} \zeta^l \]

\[ [\beta_0, k-1, \gamma_0, l] z^{k-1} x^l = \alpha_0, k+l-2 \left( -\frac{1}{2} z^{k-1} x^l \right) \]

We will show that the sum of the coefficients here satisfies

\[ \sum_{k+l=n+2} y^{k-1} ((k-1)\theta^l + \zeta^l) - \frac{l}{2} z^{k-1} x^l = \sum_{k+l=n+3} (1-l) r^k_4 \zeta^l + (k-1) r^k_5 \eta^l, \]

and therefore it vanishes. Thus the vanishing of the sum of these coefficients uses only the fourth and fifth relations on the base.
\[ 0 = \sum_{k+l=n+3} (1 - l)r_4^k \zeta^l + (k - 1)r_5^k \eta^l \]
\[ = \sum_{k+l=n+3} (1 - l)(s^1 z^{k-1} + \sum_{i+j=k+1} \eta^i(\zeta^j + (i - j - 1)\theta^j))\zeta^l \]
\[ + \sum_{k+l=n+3} (k - 1)(\frac{1}{2}s^1 y^{k-1} + \sum_{i+j=k+1} (1 - j)\theta^i\zeta^j + \frac{1}{2}t^l z^{j-1})\eta^l \]
\[ = \sum_{i+j+l=n+4} (1 - l)(\eta^i\zeta^j \zeta^l + (i - j - 1)\eta^i \theta^j \zeta^l) + (i + j - 2)(1 - j)\theta^i \zeta^j \eta^l \]
\[ + s^1 \left( \sum_{i+j+l=n+5} -(1 - l)i\eta^i \zeta^l + \frac{i+j-3}{2}(j - 1)\eta^i \zeta^j \eta^l) \right) \]
\[ = \sum_{i+j+l=n+4} (j - 1)(j - 1)(i + j - 3)\eta^i \zeta^j \theta^l + jn^i \zeta^j \zeta^l \]
\[ + \sum_{i+j+l=n+5} \frac{(1-l)j}{2}s^1 \eta^i \zeta^j \zeta^l \]
\[ = \sum_{i+j+l=n+4} (j - 1)(j - 1)(i + j - 3)\eta^i \zeta^j \theta^l + jn^i \zeta^j \zeta^l - \frac{i}{2}s^1 \eta^i \zeta^j \zeta^l+1 \]
\[ = \sum_{i+j+l=n+4} (i + j + l = n + 4(j - 1)\eta^i \zeta^j ((i + j - 3)\theta^l + \zeta^l) - \frac{n}{2}\eta^i \zeta^j x^l \]
\[ = \sum_{k+l=n+2} y^{k-1}((k - 1)\theta^l + \zeta^l) - \frac{1}{2}z^{k-1}x^l \]

The coefficients of the terms involving \( \gamma_0, n \) are

\[
[ \psi_k, \alpha_{0,l-1} ] t^k y^{l-1} = 2\gamma_{k+l-1} t^k y^{l-1} \\
[ \phi_k \gamma_0, l ] \theta^k x^l = -\gamma_{k+l-1} l \theta^k x^l \\
[ \tau_k, \gamma_0 ] \zeta^k x^l = \gamma_{k+l-1} \zeta^k x^l
\]

We claim that

\[
\sum_{k+l=n+1} 2t^k y^{l-1} - l \theta^k x^l + \zeta^k x^l = -r_1 \zeta^{n+1} + s^1 r_5^{n+1} + \sum_{k+l=n+2} kr_5^k \eta^l - tr_4^k t^l
\]
This follows from

\[ 0 = -r_1\zeta^{n+1} + s^1r_5^{n+1} + \sum_{k+l=n+2} k r_2^{k} \eta^l - lr_4^{k} t^l \]

\[ = -r_1\zeta^{n+1} + \frac{1}{2}(s^1)^2 y^n \]

\[ + \sum_{k+l=n+2} s^1(1 - l)\theta^k \zeta^l + \frac{k}{2}s^1 t^k z^{l-1} + \frac{k}{2}s^1 x^k \eta^l - ls^1 z^{k-1} t^l \]

\[ + \sum_{i+j+l=n+3} (i + j - 1)lt^l \theta^j \eta^l - 2(i + j - 1)t^i \zeta^j \eta^l \]

\[ - \sum_{i+j+l=n+3} ln^i \zeta^j t^l - (i - j - 1)ln^i \theta^j t^l \]

\[ = -r_1\zeta^{n+1} + \sum_{k+l=n+2} (1 - l)s^1 \theta^k \zeta^l \]

\[ + \sum_{k+i+j=n+3} (2(k - l) + i)t^l \eta^j \zeta^k - i(i + j - 2)t^i \eta^j \theta^k \]

\[ = s^1 \zeta^1 \zeta^{n+1} - t^1 \eta^1 \zeta^{n+1} - \sum_{k+l=n+1} ls^1 \theta^k \zeta^{l+1} \]

\[ + \sum_{k+i+j=n+3} 2(j - 1)t^k \eta^j \zeta^i + (i + j - 2)i\theta^k t^l \eta^j - i\zeta^k t^l \eta^j \]

\[ = \sum_{k+l=n+1} 2t^k y^{l-1} - \theta^k x^l + \zeta^k x^l \]

The demonstration of the vanishing of the coefficients of the terms appearing in the bracket \([d^\infty, d^\infty]\) is not straightforward. Nevertheless, this demonstration is unnecessary due to the construction of the miniversal deformation which was given in \(\textbf{[8]}\). We included the explicit calculations here as an illustration of the complexity which arises in establishing the vanishing of these coefficients by a direct calculation.

Now let us address the question of when an infinitesimal deformation \(d^k = d + \psi u + \varphi \theta\), where \(\psi\) is an odd and \(\varphi\) an even cocycle, and \(u\) is an even and \(\theta\) an odd parameter, extends to a formal deformation. Without loss of generality, we can assume that \(\psi\) is in the span of the cocycles \(\xi\) and \(\psi_k\), and that \(\varphi\) is in the span of \(\varphi_k, \sigma_k\) and \(\tau_k\), since they differ from elements of this form by coboundaries. Suppose we write

\[ \psi = a_1 \xi + b^k \psi_k, \quad \varphi = c^k \phi_k + g^k \sigma_k + h^k \tau_k \]

A morphism \(f : \mathbb{K}[[s^1, t^k, \theta^k, \eta^k, \zeta^k]] \rightarrow \mathbb{K}[[u, \theta]]\), given by

\[ f(s^1) = a_1 u^n, \quad f(t^k) = b^k u^n \]

\[ f(\theta^k) = c^k u^n \theta, \quad f(\eta^k) = g^k u^n \theta, \quad f(\zeta^k) = h^k u^n \theta \]
descends to one from the base \( \mathcal{A} = \mathbb{K}[[s^k, t^k, \theta^k, \nu^k, \zeta^k]]/(r_m^n) \) to \( \mathbb{K}[[u, \theta]] \) precisely when it vanishes on the relations. Examining the relations carefully, we observe that the third, fourth and fifth ones only have terms involving the product of two odd terms, and since these products are automatically zero in \( \mathbb{K}[[u, \theta]] \), there is nothing to check for these relations. Substituting in the first two relations gives the conditions

\[
0 = \sum_{k+l=n} -a_k h_l^1 + b_k g_l^1
\]

(3)

\[
0 = \sum_{i+j+r=m} \frac{1}{2} a_i a_j h_r^{n+1} - \sum_{k+l=n+2} k a_i b_j^l g_r^l + \sum_{i+j=m} b_k^l (kc_j^l - 2h_j^l),
\]

which must be satisfied for all \( m \) and \( n \). It is much easier to check when the map \( f \) is of degree 1, in which case the second condition breaks up into the two separate conditions

\[
0 = \frac{1}{2} (a_1)^2 h_1^{n+1} - \sum_{k+l=n+2} k a_i b_j^l g_r^l
\]

\[
0 = \sum_{k+l=n+1} b_k^l (kc_1^l - 2h_1^l)
\]

The first of these two conditions coming from the second relation is cubic and the second quadratic in the parameters \( t \) and \( \theta \), while the condition derived from the first relation is also quadratic. Some infinitesimal deformations will not extend to second order, because their coefficients fail the quadratic constraints, while some will extend to second order deformations, but not to third order, because their coefficients fail the cubic constraints. It is also easy to construct examples of infinitesimal deformations which extend to a formal deformation.

For example, \( d^e = d + \xi u + \tau \theta \) fails to extend to a second order deformation, because \( \frac{1}{2}[d^e, d^e] = -\xi u \theta \). On the other hand, \( d^e = d + (\xi + \psi_1)u + \sigma_2 \theta \) extends to the second order deformation \( d^e = d^e - \gamma_{0,1} u \theta \), but this second order deformation fails to extend because \( \frac{1}{2}[d^e, d^e] = -\frac{1}{2} \psi_1 u^2 \theta \). Moreover, in the first case, the (first) obstruction to the extension is given by \(-\xi\), while in the second case, the (second) obstruction to the extension is given by \(-\frac{1}{2} \psi_1\). In fact, it is easy to determine the obstruction to an extension by simply plugging the coefficients into the relations. For example, if \( d^e = d + \psi_1 u + (\sigma_1 + \phi_1 + \tau_2) \theta \),

then \( \frac{1}{2}[d^e, d^e] = (\xi + \psi_1 - 2 \psi_2) u \theta \), so the obstruction is \( \xi + \psi_1 - 2 \psi_2 \). The coefficients of the cocycles in the first obstruction are given by plugging
the coefficients of the infinitesimal extension in the quadratic parts of
the relations. However, it should be pointed out that in general, the
second and higher obstructions are not uniquely defined, because they
depend on the choice of the cochains added at each order.

For example, if \( d_i = d + \xi u \), then of course, since all the relations
vanish, \( d_i \) extends to a formal deformation. (In fact, \( d_i \) is itself a
formal deformation.) On the other hand, the choice of the extension
to a second order deformation can affect the further extendibility. For
example, if \( d_i = d + \xi u + \tau_1 u \theta \), then the deformation cannot be extended
further, but if we let \( d_i = d + \phi_1 u \theta \), then this extension is already a
formal deformation.

What we can say is that the relations determine the maximum ex-
tendibility of our deformation. This is the same property as one ob-
serves with Massey powers. The vanishing of the \( n \)th Massey power
means that the deformation can be extended to order \( n \). The first non-
vanishing Massey power determines the maximal order to which the
deformation can be extended.

We have to be very careful in interpreting how to use the relations,
though. In the example \( d_i = d + (\xi + \psi_1)u + \sigma_2 \theta \), we can adjust the
second order extension we gave before to \( d_i = d^{\prime}(-\gamma_{0,1} + \frac{1}{2}\phi_1)u \theta \), and
now, the bracket \([d_i, d_i] \) vanishes. The introduction of a cohomology
class later in the deformation corresponds to higher order terms in the
polynomial expressions of the parameters. Here, adding the term \( \frac{1}{2}\phi_1 u \theta \)
is the same as choosing \( f(\theta^1) = \frac{1}{2} u \theta \). Thus the computation of the ext-
tendability of an infinitesimal deformation can be cast as follows. Given
the choice of constants \( a_1, b_1, c_1, g_1^1 \) and \( h_1^1 \), do there exist constants
\( a_m, b_m, c_m, g_m, h_m \) so that the relations are satisfied? If so, then
the deformation extends to a formal one. Thus the conditions in
equation (3) need to be solved recursively for the constants. Moreover,
the least \( m \) for which a solution fails to exist determines the maximum
extendibility of the infinitesimal deformation.

Consider the formal deformation \( d^f = d + \psi_1 u \). Since \( \text{ad} \tau_2(\psi_1) = 2\psi_{k+1} \), we have \( \text{ad} \tau_2^k(\psi_1) = 2^k \psi_{k+1} \). Therefore

\[
    d^f = \exp(\text{ad} \tau_2 u)(d^f) = d + \psi_1 u + \sum_{k=1}^{\infty} \frac{\psi_k u}{k!} \psi_{k+1} u^{k+1}.
\]

Both of these formal deformations are expressed as sums of cohomology
classes, so both of them appear as \( f_*(d^{\infty}) \) for obvious morphisms from
the base of the miniversal deformation to \( \mathbb{K}[[u, \theta]] \). This example illus-
trates the nonuniqueness of the morphism from the base of the versal
deformation to the base \( \mathbb{K}[[u, \theta]] \) such that \( f_*(d^{\infty}) \sim d^f \).
2.2. Miniversal Deformations of Type 010. Let \( D(\varphi) = [\varphi, \psi_1^{0,1,1}] \). Then we have the following table of coboundaries.

\[
\begin{align*}
D(\varphi_1^{q,n-q-1}) &= \psi_1^{0,1+q,n-q} & D(\psi_1^{0,p,n-p}) &= 0 \\
D(\varphi_2^{0,p,n-p}) &= -\psi_1^{0,p,n-p+1} & D(\psi_2^{1,q,n-q-1}) &= \varphi_1^{1,q,n-q} + \varphi_2^{0,q+1,n-q} \\
D(\varphi_3^{0,p,n-p}) &= -\psi_1^{0,p+1,n-p} & D(\psi_3^{1,q,n-q-1}) &= \varphi_1^{1,q+1,n-q-1} + \varphi_3^{0,q+1,n-q}
\end{align*}
\]

The cohomology is given by

\[
H^1 = \langle \psi_1^{0,0,1}, \psi_1^{0,1,0}, \varphi_2^{1,0,0} + \varphi_3^{0,0,1}, \varphi_1^{1,0,0} + \varphi_2^{1,0,0} \rangle \\
H^n = \langle \varphi_1^{0,n-1} + \varphi_3^{0,n}, \varphi_1^{1,n-1,0} + \varphi_2^{0,n,0} \rangle \quad n > 1.
\]

Let us label the cohomology classes as follows.

\[
\psi_1 = \psi_1^{0,0,1} \quad \psi_2 = \psi_1^{0,1,0} \\
\phi_n = \varphi_1^{1,0,n} + \varphi_3^{0,0,n+1} \quad \sigma_n = \varphi_1^{1,n,0} + \varphi_2^{0,n+1,0} \quad n \geq 0
\]

The universal infinitesimal deformation is given by

\[
d^1 = \psi_1^{0,1,1} + \psi_1 t^1 + \psi_2 t^2 + \phi_n \theta^n + \sigma_n \eta^n,
\]

where \( t^i \) are even parameters and \( \theta^n \) and \( \eta^n \) are odd parameters. Let

\[
\alpha_{k,l} = \varphi_3^{0,k,l} \quad \beta_k = \varphi_2^{0,k} \quad \tau_{k,l} = \psi_2^{1,k,l} \quad \xi_{k,l} = \psi_3^{1,k,l}
\]

These cochains are preimages of a basis of the coboundaries, so it is possible to express the miniversal deformation in the form

\[
d^\infty = d^1 + \alpha_{k,l} x^{k,l} + \beta_k y^k + \tau_{k,l} u^{k,l} + \xi_{k,l} v^{k,l}.
\]

It turns out that we do not need all of the above cochains to construct the miniversal deformation. Let us denote

\[
\gamma_k = \alpha_{k,1} \quad \alpha_{0,k} = \epsilon_k \quad k > 0
\]

and set \( r^k = x^{k,1} \), \( s^k = x^{0,l} \). Then we will show that the miniversal deformation can be expressed in the form

\[
d^\infty = d^1 + \gamma_k r^k + \epsilon_k s^k + \beta_k y^k + \tau_{k,l} u^{k,l} + \xi_{k,l} v^{k,l},
\]

where \( \gamma_k \), \( \epsilon_k \) and \( \beta_k \) are even cochains defined for \( k \geq 1 \), \( \tau_{k,l} \) and \( \xi_{k,l} \) are odd cochains defined for \( k, l \geq 0 \), and actually, \( v^{k,0} = 0 \).
The brackets we need to compute in order to determine \([d^1, d^1]\) are

\[
\begin{align*}
[\psi_1, \phi_k] &= [\psi_2, \sigma_k] = [\phi_k, \sigma_l] = 0 \\
[\psi_1, \sigma_k] &= \begin{cases} 
-\psi_1 & k = 0 \\
D(\epsilon_1) & k = 1 \\
D(\gamma_{k-1}) & k > 1 
\end{cases} \\
[\psi_2, \phi_k] &= \begin{cases} 
-\psi_2 & k = 0 \\
D(\epsilon_k) & k > 0 
\end{cases} \\
[\phi_k, \phi_l] &= \phi_{k+l}(k-l), \quad [\sigma_k, \sigma_l] = \sigma_{k+l}(k-l)
\end{align*}
\]

The second order relations are

\[
t^1\eta^0 = t^2\theta^0 = \frac{1}{2} \sum_{k+l=n} (k-l) \theta^k \theta^l = \frac{1}{2} \sum_{k+l=n} (k-l) \eta^k \eta^l = 0
\]

The second order deformation is given by

\[
d^2 = d^1 + \epsilon_1 t^1 \eta^1 + \epsilon_k t^2 \theta^k + \gamma_k t^1 \eta^{k+1}.
\]

We next compute the brackets which are necessary to compute \([d^2, d^2]\). These are the brackets of cohomology classes and the \(\gamma\) and \(\epsilon\) terms. Note that since these terms first appeared in the brackets of cohomology classes, their coefficients in the miniversal deformation have order two, so the brackets below have order 3.

\[
\begin{align*}
[\psi_1, \gamma_k] &= \begin{cases} 
-D(\epsilon_1) & k = 1 \\
-D(\gamma_{k-1}) & k > 1 
\end{cases} \\
\psi_1, \epsilon_k] &= \begin{cases} 
\psi_1 & k = 1 \\
-D(\beta_{k-1}) & k > 1 
\end{cases} \\
[\psi_2, \gamma_k] &= 0 \\
[\psi_2, \epsilon_k] &= 0 \\
[\phi_k, \gamma_l] &= D(\xi_{l-1,k}) k \\
[\phi_k, \epsilon_l] &= \phi_{k+l-1} k + \epsilon_l(k-l) \\
[\sigma_k, \gamma_l] &= -\gamma_{k+l} \\
[\sigma_k, \epsilon_l] &= 0
\end{align*}
\]

Let us show that the terms involving \(\gamma\) and \(\epsilon\) cochains cancel, at least up to fourth order. Consider the following terms

\[
\begin{align*}
[\sigma_{k+1}, \gamma_l] \eta^{k+1} t^1 \eta^{l+1} &= -l \gamma_{n+1} t^1 \eta^{k+1} \eta^{l+1}, k + l = n \\
[\gamma_k, \sigma_{l+1}] t^1 \eta^{k+1} \eta^{l+1} &= k \gamma_{n+1} t^1 \eta^{k+1} \eta^{l+1}, k + l = n \\
[\sigma_0, \gamma_{n+1}] t^1 \eta^0 \eta^{n+2} &= -(n+1) \gamma_{n+1} t^1 \eta^0 \eta^{n+2}.
\end{align*}
\]

Summing the first two types and dividing by \(\frac{1}{2}\) and adding the last term gives the second order relation involving \(\eta\) cochains, plus the term \(t^1 \eta^0 \eta^{n+2}\). But this term is zero up to fourth order, using the second order relation \(t^1 \eta^0 = 0\). The terms involving \(\epsilon\) cochains are handled
similarly. The third order relations are
\[ t^1 \eta^0 - t^1 (t^1 \eta^1 + t^2 \theta^1) = t^2 \theta^0 = 0 \]
\[ nt^1 \theta^m \eta^1 + \sum_{k+l=n} \frac{1}{2} (k-l) \theta^k \theta^l + kt^1 \theta^k \theta^{l+1} = \frac{1}{2} \sum_{k+l=n} (k-l) \eta^k \eta^l = 0 \]

Notice that only two of them have been modified from the second order relations. The third order deformation is
\[ d^3 = d^2 - \epsilon_1 (t^1)^2 \eta^2 - \gamma_k (t^1)^2 \eta^{k+2} - \beta_k t^1 t^2 \theta^{k+1} - \xi_{k,l} t^1 t^2 \eta^{k+2}. \]

Next, we compute all brackets whose order is 4. Note that the brackets of \( \gamma \) and \( \epsilon \) terms appear here, because they have order 4, and so play no role in the construction of the third order deformation.

\[ [\gamma_k, \gamma_l] = 0 \]
\[ [\gamma_k, \epsilon_l] = \alpha_{k,l} (1 - l) \]
\[ [\epsilon_k, \epsilon_l] = \epsilon_{k+l-1} (k - l) \]
\[ [\psi_1, \beta_k] = 0 \]
\[ [\psi_2, \beta_k] = \begin{cases} \psi_1 & k = 1 \\ -D(\beta_{k-1}) & \text{otherwise} \end{cases} \]
\[ [\phi_k, \beta_l] = -\beta_{k+l} \]
\[ [\sigma_k, \beta_l] = \begin{cases} \beta_l & k = 0 \\ 2D(\tau_{0,l-1}) - \phi_l + \epsilon_{l+1} & k = 1 \\ D(\tau_{k-1,l-1})(k + 1) - D(\xi_{k-2,l}) + \alpha_{k-1,l+1} & k > 1 \end{cases} \]
\[ [\psi_1, \xi_{k,l}] = \begin{cases} \phi_l & k = 0 \\ D(\xi_{k-1,l}) & \text{otherwise} \end{cases} \]
\[ [\psi_2, \xi_{k,l}] = \begin{cases} \gamma_{k+1} & l = 1 \\ \alpha_{k+1,l} & l > 1 \end{cases} \]
\[ [\phi_n, \xi_{k,l}] = \xi_{k,n+l}(n - l) \]
\[ [\sigma_n, \xi_{k,l}] = -\xi_{k+n,l}(k + 1) \]

Note the appearance of the terms \( \alpha_{n,k} \). Let us show that the coefficient of such terms is zero, up to order 5. We have
\[ [\gamma_k, \epsilon_l] (t^1 t^2 \eta^{k+1} \theta^l) = \alpha_{k,l} (1 - l) t^1 t^2 \eta^{k+1} \theta^l \]
\[ [\sigma_k, \beta_l] (t^1 t^2 \eta^{k+1} \theta^l) = \alpha_{k-1,l+1} (t^1 t^2 \eta^{k+1} \theta^l) \]
\[ [\psi_2, \xi_{k,l}] (t^1 t^2 \theta^l \eta^{k+2}) = \alpha_{k+1,l} (t^1 t^2 \theta^l \eta^{k+2}) \]
Adjusting the indices, and interchanging the odd terms on the third equation, one sees that these terms add up to zero.

In the fourth order deformation, it is necessary to introduce the cochains $\tau_{k,l}$, which therefore will be of order 4. Some modifications to the relations occur, and some additional $\gamma$, $\beta$ and $\xi$ terms will be added. We will not give the fourth order deformation explicitly here, because we will compute the miniversal deformation directly by recursion. In order to do so, we need to compute all the brackets of all of the remaining terms with each other.

\[
\begin{align*}
[\gamma_k, \beta_l] &= \begin{cases} 
-D(\tau_{0,l-1})l + \phi_l + \epsilon_l(1-l) & k = 1 \\
-D(\tau_{k-1,l-1})l + D(\xi_{k-2,l})l - \alpha_{k-1,l+1}(k-l) & k > 1 
\end{cases} \\
[\epsilon_k, \beta_l] &= -\beta_{k+l-1} \\
[\xi_k, \gamma_l] &= -\xi_{k+l-1}(1-l) \\
[\xi_k, \epsilon_l] &= -\xi_{k+l-1}(n-l) \\
[\xi_k, \beta_n] &= -\tau_{k,l+n+1} + \xi_{k-1,l+n} \\
[\psi_1, \tau_{k,l}] &= \begin{cases} 
\beta_{l+1} & k = 0 \\
D(\tau_{0,l}) - \phi_{l+1} + \epsilon_{l+2} & k = 1 \\
D(\tau_{k-1,l}) - D(\xi_{k-2,l}) + \alpha_{k-1,l+2} & \text{otherwise} 
\end{cases} \\
[\psi_2, \tau_{k,l}] &= \begin{cases} 
\sigma_k & l = 0 \\
D(\tau_{0,l}) & \text{otherwise} 
\end{cases} \\
[\tau_{k,l}, \phi_n] &= \tau_{k,n+l}(l+1) \\
[\tau_{k,l}, \sigma_n] &= -\tau_{k+n,l}(n-k) \\
[\tau_{k,l}, \gamma_n] &= \tau_{k+n,l} - \xi_{k+n-1,l+1} \\
[\tau_{k,l}, \epsilon_n] &= \tau_{m,l+n-1} \\
[\tau_{k,l}, \beta_n] &= \tau_{k-1,l+n} \\
[\beta_k, \beta_l] &= [\tau_{k,l}, \tau_{m,n}] = [\tau_{k,l}, \xi_{m,n}] = [\xi_{k,l}, \xi_{m,n}] = 0
\end{align*}
\]

Let us collect the terms involving the coboundaries of the $\gamma$ cochains. Including the coefficients, we obtain

\[
[\psi_1, \sigma_k] t^1 \eta^k = D(\gamma_{k-1}) t^1 \eta^k \\
[\psi_1, \gamma_k] t^1 r^k = - D(\gamma_{k-1}) t^1 r^k \\
[d, \gamma_k] r^k = - D(\gamma_k) r^k
\]

Since the sum of all terms involving the same index in $\gamma$ must vanish, we obtain the recursive relation $r^k = t^1 \eta^{k+1} - t^1 r^{k+1}$, from which it
follows that
\[ r^k = t^1\eta^{k+1} + \sum_{n=1}^{\infty} (-1)^n(t^1)^{n+1}\eta^{k+n+1}. \]

For the ε cochains we have
\[
[\psi_1, \sigma_1]t^1\eta^1 = D(\epsilon_1)t^1\eta^1
\]
\[
[\psi_2, \phi_k]t^2\theta^k = -D(\epsilon_k)t^2\theta^k
\]
\[
[d, \epsilon_k]s^k = -D(\epsilon_k)s^k.
\]
which yields
\[
s^1 = t^2\theta^1 + t^1\eta^1
\]
\[
s^k = t^2\theta^k, k > 1
\]

The terms involving coboundaries of β cochains are
\[
[\psi_1, \epsilon_k]t^1s^kx = -D(\beta_{k-1})t^1s^k
\]
\[
[\psi_2, \beta_k]t^2y^k = -D(\beta_{k-1})t^2y^k
\]
\[
[d, \beta_k]y^k = -D(\beta_k)y^k.
\]
Since we have \(y^k = -t^1s^{k+1} - t^2y^{k+1}\), it follows easily that
\[
y^k = -t^1t^2 \sum_{n=0}^{\infty} (-1)^n(t^2)^n\theta^{n+k+1}.
\]

The terms involving \(\tau_{k,l}\) are more complicated.
\[
[\sigma_k, \beta_l]\eta^k \eta^l = D(\tau_{k-1,l-1})(k+1)\eta^k \eta^l
\]
\[
[\psi_1, \tau_{k,l}]t^1u^{k,l} = D(\tau_{k-1,l})t^1u^{k,l}
\]
\[
[\psi_2, \tau_{k,l}]t^2u^{k,l} = D(\tau_{k,l-1})t^2u^{k,l}
\]
\[
[\gamma_k, \beta_n]r^k y^n = -D(\tau_{k-1,n-1})mr^k y^n
\]
\[
[d, \tau_{k,l}]u^{k,l} = -D(\tau_{k,l})u^{k,l}.
\]
This yields the following recursion relation
\[
u^{k,l} = ((k+2)\eta^{k+1} - (l+1)r^{k+1})y^{l+1} + t^1u^{k+1,l} + t^2u^{k,l+1},
\]
which gives a power series expression for \( x^{k,l} \). Finally, the terms involving \( \xi \)'s are

\[
\begin{align*}
[\varphi_n, \gamma_k] & \theta^n r^k = D(\xi_{k-1,n}) n \theta^n r^k \\
[\sigma_k, \beta_l] & \eta^k y^l = - D(\xi_{k-2,l}) \eta^k y^l \\
[\psi_1, \tau_k,l] & t^1 u^{k,l} = - D(\xi_{k-2,l+1}) t^1 u^{k,l} \\
[\psi_1, \xi_k,l] & t^1 v^{k,l} = D(\xi_{k-1,l}) t^1 v^{k,l} \\
[\gamma_k, \beta_n] & r^k y^n = D(\xi_{k-2,n}) r^k y^n \\
[d, \xi_k,l] & v^{k,l} = - D(\xi_{k,l}) v^{k,l}
\end{align*}
\]

from which we deduce that

\[
v^{k,l} = l \theta^l r^{k+1} - \eta^{k+2} y^l + l t^{k+2} y^l - t^1 u^{k+2,l-1} + t^1 v^{k+1,l}.
\]

Now let us study the relations on the base. These are the coefficients of the cocycles. There are three terms involving \( \psi_1 \). From \([\psi_1, \epsilon_1]\) we obtain \( t^1 s^1 \), from \([\psi_2, \beta_1]\) we obtain \( t^2 y^1 \), and from \([\psi_1, \sigma_0]\) we obtain \( -t^1 \theta^0 \). Thus the coefficient of \( \psi_1 \) is \( -t^1 \theta^0 + t^2 y^1 + t^1 s^1 \). The only term involving \( \psi_2 \) is \([\psi_2, \theta_0]\), giving \( t^2 \theta^0 = 0 \).

For \( \phi_n \) we obtain the terms \( \frac{1}{2} [\phi_k, \phi_l] = \phi_n \frac{1}{2} (k-l) \theta^k \theta^l \), where \( k+l = n \). From the brackets \([\phi_k, \epsilon_l]\), if we re-index in the form \([\phi_k, \epsilon_{l+1}]\), so that we can sum for all \( k, l \), not just \( l > 0 \), and then sum the corresponding terms with \( k \) and \( l \) interchanged, we obtain \( \frac{1}{2} (k-l) \theta^k s^{l+1} \phi_n \), for \( k+l = n \). Similarly, from \([\gamma_1, \beta_n]\), we get \( -\frac{1}{2} (k-l) r^1 y^n \phi_n \). Lastly, there are the terms \([\sigma_1, \beta_n]\), contributing \( -\eta^1 y^n \phi_n \), \([\psi_1, \tau_{1,n-1}]\), giving \( -t^1 u^{1,n-1} \phi_n \), and \([\psi_1, \xi_{0,n}]\), yielding \( t^1 v^{0,n} \phi_n \). These last terms are only defined when \( n \geq 1 \). Putting this altogether, we obtain

\[
-\eta^1 y^n - t^1 u^{1,n-1} + t^1 v^{0,n} - s^1 y^n + \frac{1}{2} \sum_{k+l=n} (k-l)(\theta^k \theta^l + \theta^k s^{l+1})
\]

Finally, let us examine the terms involving the \( \sigma \) cochains. From \( \frac{1}{2} [\sigma_k, \sigma_l] \), we obtain \( \sigma_n \frac{1}{2} (k-l) \eta^k \eta^l \), when \( k+l = n \). From \([\psi_2, \tau_{n,0}]\), we obtain \( t^2 u^{n,0} \). Thus the corresponding relation is

\[
t^2 u^{n,0} + \frac{1}{2} \sum_{k+l=n} (k-l) \eta^k \eta^l.
\]
Putting these all together, the relations on the base of the miniversal deformation are

\[ t^2 \theta^0 = 0 \]
\[ -t^1 \theta^0 + t^1 s^1 + t^2 y^1 = 0 \]
\[ -\eta^1 y^n - t^1 u^{1,n-1} + t^1 v^{0,n} - s^1 y^n + \frac{1}{2} \sum_{k+l=n} (k-l)(\theta^k \theta^l + \theta^k s^{l+1}) = 0 \]
\[ t^2 u^{m,0} + \frac{1}{2} \sum_{k+l=n} (k-l)\eta^k \eta^l = 0 \]

Note that the \( s, y, u \) and \( v \) coefficients above can be expressed in terms of the parameters \( t^1, t^2, \theta \) and \( \eta \). The first two relations are odd, and the last two are even, so only the first two play a role in determining whether an infinitesimal deformation extends to a formal one. Substituting for the \( s^1 \) and \( y^1 \) coefficients in the second relation gives

\[ -t^1 (\theta^0) + t^2 \theta^1 + (t^1)^2 \eta^1 - \sum_{n=0}^{\infty} (t^2)^n + 2 \theta^{n+2} = 0, \]

so that \( t^1 = 0 \) certainly solves this equation. Now suppose that \( d^i = d + \psi u + \varphi \theta \) is an infinitesimal deformation of \( d \). Let

\[ t^i = a_k^i u^k, \quad \theta^i = b_k^i u^k \theta, \quad \eta^i = c_k^i u^k \theta, \]

where \( \psi = a_1^1 \psi_1 + a_1^2 \psi_2 \) and \( \varphi = b_1^i \phi_i + c_1^i \sigma_i \). Our goal is to solve for coefficients \( a_k^i, b_k^i \), and \( c_k^i \) so that the first two relations are satisfied. If \( a_1^1 = a_1^2 = 0 \), then by choosing \( a_k^i = 0 \) for all \( k \), we obtain a solution. The relations transform to

\[ \sum_{i+j=m} a_k^1 b_j^0 = 0 \]
\[- \sum_{i+j=m} a_k^1 b_j^0 + \sum_{i+j+k=m} a_k^1 (a_k^2 b_j^1 + a_k^1 c_j^1) + \sum_{n=0}^{\infty} (-1)^n a_k^1 c_j^{n+2} \prod_{l=0}^{n+2} a_k^2 = 0 \]

Suppose that \( a_k^i \neq 0 \) for some \( k \). Then the second relation transforms into the simpler

\[ -b_j^0 + \sum_{j+k=m} (a_j^2 b_k^1 + a_j^1 c_k^1) + \sum_{n=0}^{\infty} (-1)^n c_j^{n+2} \prod_{l=0}^{n+2} a_k^2 = 0. \]

Otherwise, the second relation is satisfied automatically. Looking at this transformed second relation, we observe that \( b_j^0 = 0 \), since this is the only term in the expression for \( m = 1 \). If \( t^2 = 0 \), then the first
relation is satisfied, and we can always solve for coefficients $b_m^0$ to satisfy the second relation, for arbitrary choices of the other coefficients. On the opposite end of the spectrum, if $a_1^2 \neq 0$, then $b_k^0 = 0$ for all $k$, or we encounter the obstruction $a_1^2 b_m^0 \psi_1$ at the first level $m$ for which $b_m^0 \neq 0$. We may encounter an earlier obstruction coming from the second relation, because, for example, for $m = 2$, we obtain from the second equation the relation $b_2^0 + a_1^2 b_1^1 + a_1^1 c_1^1 = 0$. Actually, in the case that $a_1^1 \neq 0$, this is the only obstruction coming from the second relation, because the term $a_1^1 c_m^1$ appears in the sum for degree $m + 1$, which means that for a certain choice of the coefficient $c_m^1$, one can make the sum equal to zero. Thus, for an infinitesimal deformation, we can resolve the extendibility to a formal deformation as follows.

If $a_1^1 = a_2^1 = 0$, then the deformation extends trivially. In fact, the infinitesimal deformation is already a formal deformation. Otherwise, if $b_1^0 \neq 0$, then the deformation does not extend to second order, and the obstruction is $a_1^1 b_1^0 \psi_1 + -a_2^1 b_1^0 \psi_2$. If $b_1^0 = 0$, then the deformation does extend to second order. If $a_1^1 = 0$, then the second relation vanishes by choosing $a_k^1 = 0$ for all $k$, and the first relation vanishes by choosing $b_k^0 = 0$ for all $k$, so the deformation extends to a formal one. On the other hand, if $a_2^2 = 0$, then by choosing $a_k^2 = 0$ for all $k$, we can make the first relation vanish, and by choosing $b_m^0$ appropriately, we can make the second relation vanish for each order $m$. Thus the deformation extends to a formal one. When neither $a_1^1$ nor $a_2^2$ vanish, we must have $b_k^0 = 0$ for all $k$. If $a_2^2 b_1^1 + a_1^1 c_1^1 \neq 0$, then no second order deformation extends to third order. On the other hand, if it does vanish, then there is always a choice of the coefficients $c_k^1$ such that the deformation extends to a formal one.

3. Conclusions

The notion of miniversal deformations has been around for quite some time. Nevertheless, there were some confusions in the early literature, as was mentioned in [7]. We have felt that some of the confusion arises because of the lack of concrete examples. Our purpose in this article has been to give some explicit constructions of miniversal deformations, and use them to address the classical question of when an $n$-th order deformation extends to a formal one.

In [7], relations on the base of the miniversal deformation were found, without actually constructing a miniversal deformation. Since these relations alone determine the extendibility of a deformation, it is interesting to note that they can sometimes be found without having to carry out the complete construction.
Another important classical question is:

*Given that an extension to a formal deformation exists, how can you determine the equivalence classes of nonequivalent extensions?*

We did not address this problem in this article. In some of our other work [10, 12], we have addressed the problem of how to extend a codifferential of degree $n$ to a more general $L_\infty$ structure, with possibly infinitely many terms. We studied the equivalence classes of these extensions, and the classification problem is quite tricky. Since these extensions can be thought of as specializations of deformations of the $L_\infty$ structure determined by the degree $n$ codifferential, where the even parameters are given fixed values, and the odd parameters have been set equal to zero, there is a close relationship between the problem of classification of extensions and the classification of deformations up to equivalence. Thus, we don’t expect the classification of deformations to be easy.

**References**

[1] A. Bodin, D. Fialowski and M. Penkava, *Classification and versal deformations of $L_\infty$ algebras on a 2|1-dimensional space*, QA/0401025, to appear in Homotopy, Homolgy and Appl.

[2] A. Cattaneo, G. Felder, L. Tamessini, *From local to global deformation quantization of Poisson manifolds*, Duke Math. J. (2002), no. 2, 329–352.

[3] Ch. Doran, S. Wong, *Deformation of Galois Representations*, to appear in AMS-IP Studies in Adv. Math. Series.

[4] A. Fialowski, *Deformations of Lie algebras*, Mathematics of the USSR-Sbornik 55 (1986), no. 2, 467–473.

[5] A. Fialowski, *An example of formal deformations of Lie algebras*, in: "Proceedings of the NATO Conference on Deformation Theory of Algebras and Appl.", Kluwer 1988, 375–401.

[6] A. Fialowski and D. Fuchs, *Singular deformations of Lie algebras on an example*, Topics in Singularity Theory (Providence, RI) (A. Varchenko and V. Vassilie, eds.), A.M.S. Translation Series 2, Vol.180, Amer. Math. Soc., 1997, 77–92, V. I. Arnold 60th Anniversary Collection.

[7] A. Fialowski, *Construction of miniversal deformations of Lie algebras*, Journal of Functional Analysis (1999), no. 161(1), 76–110.

[8] A. Fialowski and M. Penkava, *Deformation theory of infinity algebras*, Journal of Algebra 255 (2002), no. 1, 59–88, [math.RT/0101097](http://arxiv.org/abs/math.RT/0101097)

[9] A. Fialowski and M. Penkava, *Examples of infinity and Lie algebras and their versal deformations*, Banach Center Publications (2002), 27–42, [math.QA/0102140](http://arxiv.org/abs/math.QA/0102140)

[10] A. Fialowski and M. Penkava, *Strongly homotopy Lie algebras of one even and two odd dimension*, preprint QA/0308016.

[11] A. Fialowski and M. Penkava, *Versal deformations of three dimensional Lie algebras as $L_\infty$ algebras*, RT/0303346, to appear in Commun. in Contemp. Math.

[12] A. Fialowski and M. Penkava, *Extensions of $L_\infty$ algebras of two even and one odd dimension*, preprint QA/0403302.
[13] A. Fialowski and G. Post, *Versal Deformation of the Lie Algebra $L_2$*, Journal of Algebra **236** (2001), 93–109.
[14] M. Gerstenhaber, *On the deformations of ringe and algebras I–IV.*, Annals of Mathematics **79** (1964), 59–103, **84** (1966), 1–19, **88** (1968), 1–34, **1974** (1974), 257–276.
[15] M. Penkava, *Infinity algebras and the homology of graph complexes*, Preprint q-alg 9601018, 1996.
[16] ———, *Infinity algebras, cohomology and cyclic cohomology, and infinitesimal deformations*, Preprint math.QA/0111088, 2002.

Eötvös Loránd University, Budapest, Hungary

E-mail address: fialowsk@cs.elte.hu

University of Wisconsin, Eau Claire, WI 54702-4004

E-mail address: penkavmr@uwec.edu