A new algorithm for irreducible decomposition of representations of finite groups

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Abstract. An algorithm for irreducible decomposition of representations of finite groups over fields of characteristic zero is described. The algorithm uses the fact that the decomposition induces a partition of the invariant inner product into a complete set of mutually orthogonal projectors. By expressing the projectors through the basis elements of the centralizer ring of the representation, the problem is reduced to solving systems of quadratic equations. The current implementation of the algorithm is able to split representations of dimensions up to hundreds of thousands. Examples of calculations are given.

1. Introduction
The decomposition of linear representations of groups into irreducible subrepresentations is one of the central problems of group theory and its applications in physics. Currently, the most effective algorithm for solving this problem is a Las Vegas type probabilistic algorithm, called MeatAxe [1]. This algorithm is based on the calculation of the characteristic polynomial of a randomly generated matrix of the representation. In case of success, factoring this polynomial and processing the factors allow either to construct a decomposition of the representation, or to prove its irreducibility. The MeatAxe algorithm played an important role in solving the problem of classifying finite simple groups, where it was applied to representations of groups in linear spaces over small finite fields, such as GF(2). However, MeatAxe is inefficient in characteristic zero due to the rapid growth of numerical coefficients of characteristic polynomials with the matrix dimension, and due to the fact that in characteristic zero a random matrix with high probability has an irreducible characteristic polynomial.

The quantum formalism is based on Hilbert spaces over fields of characteristic zero. Traditionally, non-constructive fields C or R are used. Our goal was to develop an algorithm suitable for the study of quantum-mechanical models based on unitary representations of finite groups over constructive fields of characteristic zero [2, 3]. The computer implementation of our algorithm, let’s call it IrreducibleProjectors, splits representations of dimensions up to hundreds of thousands, which is not less than the dimensions achievable for MeatAxe in the computationally easier context of finite fields. On the other hand, unlike MeatAxe, IrreducibleProjectors is of little use in finite-field problems, since it uses the notion of scalar product, which is problematic for spaces over finite fields. In fact, IrreducibleProjectors and MeatAxe have different application areas.

The IrreducibleProjectors algorithm requires knowledge of the centralizer ring of the group representation under consideration. In the general case, the computation of the centralizer ring reduces to a simple problem of linear algebra, namely, to solving a system of matrix equations
of the form $AX =XA$. We will consider here only permutation representations, since (a) any linear representation of a finite group is a subrepresentation of some permutation representation and (b) permutation representations underlie the above mentioned constructive quantum mechanical models. In the case of permutation representations, the computation of the centralizer ring is particularly simple: it reduces to constructing the orbits of the group on the Cartesian square of the set on which the group acts by permutations.

### 2. Basic concepts and notation

Let $G$ (or, in more detail, $G(\Omega)$) be a transitive permutation group on the set $\Omega \cong \{1, \ldots, N\}$. The action of $g \in G$ on $i \in \Omega$ will be denoted by $i^g$. A permutation representation $P$ is a representation of $G$ by matrices of the form $P(g)_{ij} = \delta_{g,j}$. Since $P(g)$ is a $(0,1)$-matrix, the permutation representation can be implemented in vector space over any field $F$. We will consider an $N$-dimensional Hilbert space $H_N$ over the field of scalars $F$, which is some constructive splitting field for the group $G$. As $F$, we may take a suitable subfield of the $m$th cyclotomic field, where $m$ is the exponent of the group $G$. Such a field $F$, being an abelian extension of the field of rational numbers $\mathbb{Q}$, is a constructive dense subfield of the real $\mathbb{R}$ or complex $\mathbb{C}$ field. From the point of view of physics, $F$ is indistinguishable from $\mathbb{R}$ or $\mathbb{C}$ and can be freely used in the formalism of quantum mechanics.

An orbit of $G$ on the Cartesian square $\Omega \times \Omega$ is called an orbital $[4]$. The number $R$ of orbitals is called the rank of the permutation group $G(\Omega)$. If the set of orbitals contains some orbital $\Delta$, then it necessarily contains the transposed orbital $\Delta^T$. The set of orbitals of a transitive group contains a single diagonal orbital $\Delta_1 = \{(i,i) \mid i \in \Omega\}$, which we will always fix as the first element in the list of orbitals $\{\Delta_1, \ldots, \Delta_R\}$. For a transitive group, there is a natural one-to-one correspondence between the orbitals and the orbits of the stabilizer of a point $i \in \Omega$, i.e., the subgroup $G_i \leq G$ such that $g \in G_i \Rightarrow i^g = i$. An orbit of the stabilizer is called a suborbit. The correspondence between the orbital $\Delta$ and the suborbit $\Sigma_i$ has the form $\Delta \leftrightarrow \Sigma_i = \{j \in \Omega \mid (i,j) \in \Delta\}$. The sizes of orbitals and suborbits are related by the equality $|\Delta| = N|\Sigma_i|$.

The invariance condition for a bilinear form $A$ in the space $H_N$ is expressed by the equations $A = P(g)AP(g^{-1})$, $g \in G$. In terms of the matrix entries, these equations have the form $(A)_{ij} = (A)_{i^g,j^g}$. This implies that the basis of all invariant bilinear forms is in one-to-one correspondence with the set of orbitals. Namely, to the orbital $\Delta_r \in \{\Delta_1, \ldots, \Delta_R\}$ corresponds the basis matrix $A_r$ of size $N \times N$ with entries $(A_r)_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \Delta_r, \\ 0, & \text{otherwise}. \end{cases}$

To implement the algorithms and arrange the output of the results of calculations, it is necessary to introduce some ordering of the basis matrices:

$$A_1 < A_2 < \ldots < A_R. \quad (1)$$

We use the following conventions:

(i) $A_r < A_s$, if $|\Delta_r| < |\Delta_s|$ (or, equivalently, $|\Sigma_i)_r| < |\Sigma_i)_s|$—comparing suborbit lengths),

(ii) $A_r < A_s$, if $A_r = A_T^T \land A_s \neq A_T^T$ (symmetric matrices precede asymmetric),

(iii) $A_r < A_s$, if $I_{A_r} < I_{A_s}$, where $I_X = \min (i \mid (X)_{i1} = 1)$ (comparing the positions of the first nonzero element in the first columns of matrices),

(iv) if $A_r \neq A_T^r$, then $A_{r+1} = A_T^r$ (paired matrices are always placed adjacently).

Applying rules $[1] - [IV]$ in the specified order uniquely defines the sequence $[1]$. According to these rules, the diagonal orbital matrix is the first element of the list $[1]$; $A_1 = I_N$.

The set of invariant bilinear forms has the structure of a ring, which is called the centralizer ring (or centralizer algebra). The multiplication table for basic elements $[1]$ has the form

$$A_pA_q = \sum_{r=1}^R C_{pq}^{r} A_r, \quad (2)$$
where the coefficients $C_{pq}^r$ are natural numbers lying within $0 \leq C_{pq}^r < N$. The representation P is multiplicity-free if and only if the centralizer ring is commutative.

3. Algorithm description

Let $T$ be a unitary (we can always provide unitarity) transformation matrix splitting the representation $P$ in the Hilbert space $H_N$ into $M$ irreducible components:

$$T^{-1}P(g) = \mathbb{1} \oplus U_{d_2}(g) \oplus \cdots \oplus U_{d_m}(g) \oplus \cdots \oplus U_{d_M}(g),$$

where $U_{d_m}$ is a $d_m$-dimensional irreducible component.

The standard scalar product in the Hilbert space is represented by the matrix $\mathbb{1}_N$ in any orthonormal basis. In the splitting basis, we have the following decomposition

$$\mathbb{1}_N = \mathbb{1}_{d_1} \oplus \cdots \oplus \mathbb{1}_{d_m} \oplus \cdots \oplus \mathbb{1}_{d_M}.$$ (3)

Here $\mathbb{1}_{d_1} \equiv (1)$ is the scalar product in the one-dimensional trivial subrepresentation that is always present in any permutation representation. The preimage of decomposition (3) in the original permutation basis has the form

$$\mathbb{1}_N = B_1 + \cdots + B_m + \cdots + B_M,$$ (4)

where $B_m$ is defined by the relation

$$T^{-1}B_mT = \mathbb{1}_{d_1} \oplus 1_{d_2} \cdots \oplus 1_{d_m} \oplus 0_{d_{m+1}} \cdots \oplus 0_{d_M} = D_m.$$ (5)

It can be seen from this relation that the matrices $B_m$ are idempotent

$$B_m^2 = B_m$$ (6)

and mutually orthogonal

$$B_mB_{m'} = 0_N \text{ if } m \neq m'.$$ (7)

Relations (4) and (7) together with the completeness condition (3) mean that the set $B_1, \ldots, B_M$ is a complete system of mutually orthogonal projectors in the Hilbert space $H_N$.

The set of irreducible invariant projectors $B_1, \ldots, B_M$ contains complete information about the decomposition of the representation $P$ into irreducible components. In particular, the transformation matrix $T$ can be computed by solving the system of linear equations

$$B_1T - TD_1 = \cdots = B_MT - TD_M = 0_N.$$ Any invariant projector is a solution of the equation

$$X^2 - X = 0_N,$$ (8)

where $X = x_1A_1 + \cdots + x_RA_R$ is a generic invariant bilinear form written in basis (1). Using multiplication table (2) and decomposing (3) into components in basis (1), we obtain the system of R quadratic equations for R unknowns $x_1, \ldots, x_R$

$$E(x_1, \ldots, x_R) = 0 \approx \{E_1(x_1, \ldots, x_R) = 0, \ldots, E_R(x_1, \ldots, x_R) = 0\}.$$ (9)

We will call the left hand sides of these equations idempotency polynomials. An irreducible invariant projector $B_m$ in basis has the form

$$B_m = b_{m,1}A_1 + b_{m,2}A_2 + \cdots + b_{m,R}A_R,$$ (10)

where the vector $B_m = [b_{m,1}, \ldots, b_{m,R}]$ is a solution of the system of equations (9). Due to the invariance of the trace of a matrix under the similarity transformation, relation (5) implies the equality $\text{tr}B_m = d_m$. Combining this equality with the fact that in (10) only $A_1$ has nonzero diagonal elements and $\text{tr}A_1 = N$, we can fix the first coefficient in decomposition (10):

$$b_{m,1} = d_m/N.$$ 

Thus, the possible values of $x_1$ that provide solutions of the polynomial system (9) are fractions of the form $d/N$, where natural numbers $d \in [1, \ldots, N - 1]$ are either irreducible dimensions $d_m$ or sums of such dimensions. Orthogonality condition (7) allows us to exclude from consideration dimensions that are not irreducible. For generic $B = b_1A_1 + \cdots + b_RA_R$ and $X$, the orthogonality
condition can be written as
\[ BX = \emptyset_N. \] (11)

This matrix equation is a system of linear with respect to variables \( x_1, \ldots, x_R \) equations with parameters \( b_1, \ldots, b_R \). Using multiplication table \( \mathbb{2} \), the left hand side of (11) can be represented as a system of \( R \) bilinear forms
\[ O(b_1, \ldots, b_R; x_1, \ldots, x_R) = \begin{cases} O_1(b_1, \ldots, b_R; x_1, \ldots, x_R), \\ \vdots \\ O_R(b_1, \ldots, b_R; x_1, \ldots, x_R). \end{cases} \] (12)
which we will call orthogonality polynomials.

The main part of the algorithm is organized as a cycle starting with \( d = 1 \) and ending when the sum of the irreducible dimensions reaches the value \( N \). The current \( d \) is processed as follows:

(i) Substitute \( x_1 = d/N \) into (9) and solve the system of equations
\[
E(d/N, x_2, \ldots, x_R) = 0.
\] (13)
At the same time, without significant additional calculations, the Hilbert dimension \( h \) of the corresponding polynomial ideal is determined. The solution is always realizable algorithmically, since all the roots of the system belong to abelian extensions of \( \mathbb{Q} \). Modern computer algebra systems, in particular Maple, cope well with this task.

(ii) If system (13) is incompatible, then the current value of \( d \) is not an irreducible dimension and we go to the next value of \( d \) in the loop.

(iii) If the Hilbert dimension \( h = 0 \) and system (13) has \( k \) solutions, then we get \( k \) (different if \( k > 1 \)) \( d \)-dimensional irreducible subrepresentations.

(iv) \( h > 0 \) indicates a \( d \)-dimensional irreducible component of the nontrivial multiplicity \( k \). The corresponding component of the centralizer ring has the structure \( A \otimes \mathbb{1}_d \), where \( A \) is an arbitrary matrix of size \( k \times k \). The idempotency condition, \( (A \otimes \mathbb{1}_d)^2 = A \otimes \mathbb{1}_d \), imposes the constraint on \( A \): \( A^2 - A = 0 \). The complete family of solutions of this equation is a manifold of dimension \( h = \lfloor k^2/2 \rfloor \). Hence, for the multiplicity, we have: \( k = \left\lceil \sqrt{2h} \right\rceil \).
Then, using some procedure, \( k \) arbitrary but mutually orthogonal representatives are selected from the family of equivalent \( d \)-dimensional projectors.

(v) Each of the \( k \) irreducible projectors obtained in items (iii) or (iv) is processed as follows. Projector \( B_m \) is added to the list of irreducible projectors. The corresponding invariant subspace is excluded from further consideration by adding the orthogonality polynomials \( B_mX \) to the set of polynomials \( \mathbb{3} \): \( E(x_1, x_2, \ldots, x_R) \leftarrow E(x_1, x_2, \ldots, x_R) \cup \{B_mX\} \).

(vi) After the described in item (v) processing of all \( k \) irreducible projectors, the transition to the next \( d \) is performed.

The IrreducibleProjectors algorithm is implemented in the form of two procedures, called PreparePolynomialData and SplitRepresentation.

(i) The PreparePolynomialData procedure is implemented in \( C \). The input is the set of generators of \( G(\Omega) \). The program computes basis \( \mathbb{1} \), multiplication table \( \mathbb{2} \), constructs polynomials of idempotency \( \mathbb{9} \) and orthogonality \( \mathbb{12} \), and the code for the procedure SplitRepresentation. This code is task-specific: for non-commutative centralizer ring some additional functions to process multiple subrepresentations are generated.

(ii) SplitRepresentation is a Maple code generated by the PreparePolynomialData. This code performs the above-described cycle over dimensions. The polynomial systems are processed by functions from the Groebner package implemented in Maple.
4. Examples of calculations

The input data are taken from the “Sporadic groups” section of the ATLAS. The ATLAS contains representations of simple groups and some of their extensions. Namely, if a group \( G \) has a non-trivial

(i) second homology group \( H_2(G, \mathbb{Z}) \), called the Schur multiplier and denoted by the symbol \( \text{M}(G) \), then there are nontrivial central extensions of \( G \) by subgroups of \( \text{M}(G) \);

(ii) outer automorphisms \( \text{Out}(G) \), then there are extensions with \( G \) as a normal subgroup.

\( A.B \) denotes a generic extension of \( B \) by \( A \). A split extension is denoted by \( A \rtimes B \). Cyclic groups \( C_n \) are represented by their orders \( n \) in the notation for extensions.

We have tried for completeness to choose examples from all generations of the “Happy Family” and from the “Pariahs” family.

Irreducible components are denoted by their dimensions in bold (possibly with additional indices to distinguish between non-equivalent subrepresentations of the same dimension). Permutation representations are denoted by their dimensions in bold with an underscore. \( B_m \) denotes the irreducible projector corresponding to the irreducible subrepresentation \( m \).

The calculations were performed on a PC with a 3.30GHz CPU and 16GB RAM.

4.1. Detailed example

Here is a compact example of the outputs produced by the programs. The Held group \( \text{He} \) has the properties: \( \text{Ord}(\text{He}) = 4030387200 = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17 \), \( \text{M}(\text{He}) \cong 1 \), \( \text{Out}(\text{He}) \cong \mathbb{C}_2 \).

The program \textsf{PreparePolynomialData}, applied to the 8330-dimensional representation of this group, in addition to the code of the program \textsf{SplitRepresentation} and input data for it, produces the following text:

```
__Action of \text{He} on 8330 points
Rank of \text{He_on_8330}: 7
Dimension: 8330
Suborbit lengths: 1, 105, 720, 840, 840', 1344, 4480.
Centralizer ring is commutative
=> permutation representation is multiplicity free
__Total time: 2.93 sec
__Technical information
Orbital matrices space: 57.9 MB
Orbital path space : 35.6 MB
Total orbital space : 93.5 MB
Maximum number of polynomial terms: 217
```

This text contains information about the rank of the representation, the lengths of the suborbits (the pair 840, 840’ refers to the mutually transposed orbitals), the presence or absence of multiple subrepresentations, as well as the time and memory spent to solve the problem.

\textsf{SplitRepresentation} produces the following decomposition and invariant projectors

\[
\begin{align*}
\mathbf{8330} & \cong 1 \oplus 51 \oplus 5\mathbf{1} \oplus 680 \oplus 1275 \oplus 1920 \oplus 4352 \\
B_1 &= \frac{1}{8330} \left( A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 \right) \\
B_{51} &= \frac{3}{490} \left( A_1 + \frac{A_2}{3} - \frac{A_3}{6} - \frac{1-i\sqrt{7}}{12} A_4 - \frac{1+i\sqrt{7}}{12} A_5 + \frac{A_6}{12} \right) \\
B_{680} &= \frac{4}{49} \left( A_1 + \frac{A_2}{5} + \frac{A_3}{120} + \frac{A_4}{20} + \frac{A_5}{20} - \frac{A_7}{40} \right)
\end{align*}
\]
\[ B_{1275} = \frac{15}{98} \left( A_1 + \frac{A_2}{15} + \frac{A_3}{15} - \frac{A_4}{30} - \frac{A_5}{30} \right) \]
\[ B_{1920} = \frac{192}{833} \left( A_1 - \frac{2A_2}{15} + \frac{A_3}{120} + \frac{A_4}{120} + \frac{A_5}{192} - \frac{3A_7}{320} \right) \]
\[ B_{14352} = \frac{128}{245} \left( A_1 - \frac{A_3}{48} - \frac{A_6}{64} + \frac{A_7}{128} \right) \]

Time: 1.4 sec

Here 51 and 51\textsuperscript{1} are two different complex conjugate representations of dimension 51.

4.2. Comparison with the implementation of Meat\textnormal{A}xe in Magma

The Magma implementation of the Meat\textnormal{A}xe algorithm is considered one of the best. The Magma database contains a 3906-dimensional permutation representation of the group \( G_2(5) \) – an exceptional group of Lie type. The decomposition of this representation into irreducible components over the field \( \text{GF}(2) \) is given in [7] to illustrate the possibilities of Meat\textnormal{A}xe.

The application of our programs to this representation gives the following data:

- Rank: 4
- Suborbit lengths: 1, 30, 750, 3125

\[ 3906 \cong 1 \oplus 930 \oplus 1085 \oplus 1890 \]

\[ B_1 = \frac{1}{3906} (A_1 + A_2 + A_3 + A_4) \]
\[ B_{930} = \frac{5}{21} \left( A_1 + \frac{3}{10} A_2 + \frac{1}{50} A_3 - \frac{1}{125} A_4 \right) \]
\[ B_{1085} = \frac{5}{18} \left( A_1 - \frac{1}{5} A_2 + \frac{1}{25} A_3 - \frac{1}{125} A_4 \right) \]
\[ B_{1890} = \frac{15}{31} \left( A_1 - \frac{1}{30} A_2 - \frac{1}{30} A_3 + \frac{1}{125} A_4 \right) \]

Time C: 0.5 sec. Time Maple: 0.8 sec.

We see that in the characteristic zero the representation splits over the field \( \text{Q} \).

Splitting this representation over \( \text{Q} \) using Magma fails due to memory exhaustion. However, it is possible to reproduce the same set of irreducible dimensions as in the case of characteristic zero, if we split the representation over a finite field with a characteristic that does not divide the order of the group. In our case, we have \( \text{Ord}(G_2(5)) = 5859000000 = 2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31 \). Therefore, the smallest field that “mimics” \( \text{Q} \) in the above sense is \( \text{GF}(11) \). We present a session of the corresponding computation using Magma (execution time is given in seconds).

```plaintext
> load "g25";
Loading "/opt/magma.21-1/libs/pergps/g25"
The Lie group G( 2, 5 ) represented as a permutation
group of degree 3906.
Order: 5 859 000 000 = 2^6 \ast 3^3 \ast 5^6 \ast 7 \ast 31.
Group: G
> time Constituents(PermutationModule(G,GF(11)));
[  
  GModule of dimension 1 over GF(11),
  GModule of dimension 930 over GF(11),
  GModule of dimension 1085 over GF(11),
  GModule of dimension 1890 over GF(11)
]
Time: 282.060
```

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4.3. Some calculations for sporadic groups

The data below contain information about ranks, suborbit lengths, structures of irreducible decompositions, and calculation times. For brevity, we omitted explicit expressions for irreducible projectors \( B_m \). The expression \( \ell^m \) in the list of suborbit lengths means that there are \( m \) suborbits of length \( \ell \). Non-equivalent irreducible components of the same dimension differ, either by the symbol of complex conjugation (overbar), or by the Greek indices, or by the indices \( \pm \), meaning that there are two components having the structure \( A \pm B \). Multiple subrepresentations are underbraced. The execution times are given separately for \textit{PreparePolynomialData (Time C)} and \textit{SplitRepresentation (Time Maple)}.

4.3.1. Mathieu groups. The five Mathieu groups \( M_{11}, M_{12}, M_{22}, M_{23}, M_{24} \) are the first sporadic groups that have been discovered. Each group \( M_n \) is isomorphic to a \textit{multiply transitive} permutation group on \( n \) elements. The 5-transitive group \( M_{12} \) and the 3-transitive group \( M_{22} \) are the only Mathieu groups that have non-trivial Schur multipliers and outer automorphism groups. From the point of view of the structure of irreducible decompositions, the most interesting are the \textit{covers} of the Mathieu group \( M_{22} \).

Main properties of \( M_{22} \): \( \text{Ord}(M_{22}) = 4435200 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \), \( M(22) \cong C_{12} \), \( \text{Out}(M_{22}) \cong C_2 \).

(i) \textbf{990-dimensional representation of } \( 3.M_{22} \)

\[
\text{Rank: } 13. \text{ Suborbit lengths: } 1^3, 7^3, 42^3, 168^3, 336.
\]
\[990 \cong 1 \oplus 21_6 \oplus 21_7 \oplus 21_7 \oplus 55 \oplus 99_4 \oplus 99_7 \oplus 105_\pm \oplus 105_\mp \oplus 105_\pm \oplus 154\]

\( \text{Time C: } 1 \text{ sec. Time Maple: } 28 \text{ sec.} \)

(ii) \textbf{2016-dimensional representation of } \( 3.M_{22} \)

\[
\text{Rank: } 16. \text{ Suborbit lengths: } 1^3, 55^3, 66^3, 165^4, 330^3.
\]
\[2016 \cong 1 \oplus 21_4 \oplus 21_7 \oplus 21_7 \oplus 55 \oplus 105_\pm \oplus 105_\mp \oplus 105_\pm \oplus 154 \oplus 210_6 \oplus 210_7 \oplus 270_6 \oplus 231_6 \oplus 231_7 \oplus 231_7 \]

\( \text{Time C: } 2 \text{ sec. Time Maple: } 1 \text{ h } 15 \text{ min } 52 \text{ sec.} \)

(iii) \textbf{1980-dimensional representation of } \( 6.M_{22} \)

\[
\text{Rank: } 17. \text{ Suborbit lengths: } 1^6, 14^3, 36^3, 84^3, 336^3.
\]
\[1980 \cong 1 \oplus 21_4 \oplus 21_7 \oplus 21_7 \oplus 55 \oplus 99_4 \oplus 99_7 \oplus 105_\pm \oplus 105_\pm \oplus 105_\pm \oplus 120 \oplus 154 \oplus 210 \oplus 330 \oplus 330\]

\( \text{Time C: } 1 \text{ sec. Time Maple: } 6 \text{ h } 34 \text{ min } 14 \text{ sec.} \)

4.3.2. Leech lattice groups.

\textit{Higman-Sims group HS}. \( \text{Ord } = 44352000 = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11, \ M \cong C_2 \), \( \text{Out } \cong C_2 \).

(i) \textbf{5600-dimensional representation of HS}

\[
\text{Rank: } 9. \text{ Suborbit lengths: } 1, 55, 132, 165, 495, 660, 792, 1320, 1980.
\]
\[5600 \cong 1 \oplus 22 \oplus 77 \oplus 154 \oplus 175 \oplus 770 \oplus 825 \oplus 1056 \oplus 2520\]

\( \text{Time C: } 2 \text{ sec. Time Maple: } 2 \text{ sec.} \)

(ii) \textbf{11200-dimensional representation of HS}

\[
\text{Rank: } 16. \text{ Suborbit lengths: } 1^2, 110, 132^2, 165^2, 660^2, 792^2, 990, 1320^2, 1980^2.
\]
\[11200 \cong 1 \oplus 22 \oplus 56 \oplus 77 \oplus 154 \oplus 175 \oplus 176 \oplus 616 \oplus 616 \oplus 770 \oplus 825 \oplus 1056 \oplus 1980 \oplus 1980 \oplus 2520\]

\( \text{Time C: } 7 \text{ sec. Time Maple: } 1 \text{ h } 25 \text{ min } 47 \text{ sec.} \)

(iii) \textbf{1100-dimensional representation of HS } \times 2

\[
\text{Rank: } 5. \text{ Suborbit lengths: } 1, 28, 105, 336, 630.
\]
\[1100 \cong 1 \oplus 77 \oplus 154 \oplus 175 \oplus 693\]

\( \text{Time C: } < 1 \text{ sec. Time Maple: } < 1 \text{ sec.} \)

(iv) \textbf{1408-dimensional representation of HS } \times 2

\[
\text{Rank: } 11. \text{ Suborbit lengths: } 1^4, 50^4, 350^2, 504.
\]
\[1408 \cong 1 \oplus 1^4 \oplus 22 \oplus 22 \oplus 175 \oplus 175 \oplus 308 \oplus 352 \oplus 352\]

\( \text{Time C: } < 1 \text{ sec. Time Maple: } 3 \text{ sec.} \)
**Janko group J₂.**  
Ord = 604800 = \(2^7 \cdot 3^3 \cdot 5^2 \cdot 7, M \cong C_2, \text{Out} \cong C_2.\)

1800-dimensional representation of J₂  
Rank: 18. Suborbit lengths: 1, 14², 21, 28, 42², 84³, 168⁶, 336.

1800 \(\cong\) 1 \oplus 36 \oplus 63 \oplus 63 \oplus 126 \oplus 126 \oplus 160 \oplus 175 \oplus 288 \oplus 336 \oplus 336

Time C: 2 sec. Time Maple: 13 min 29 sec.

**Conway group Co₁.**  
Ord = 415776806543360000 = \(2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23, M \cong C_2, \text{Out} \cong 1.\)

98280-dimensional representation of Co₁  
Rank: 4. Suborbit lengths: 1, 4600, 46575, 47104.

98280 \(\cong\) 1 \oplus 299 \oplus 17250 \oplus 80730

Time C: 43 min 12 sec. Time Maple: 6 sec.

**Remark.** The program PreparePolynomialData uses more than 8.8 GB of RAM for this task.

**Conway group Co₂.**  
Ord = 42305421312000 = \(2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, M \cong 1, \text{Out} \cong 1.\)

4600-dimensional representation of Co₂  
Rank: 5. Suborbit lengths: \(1^2, 891^2, 2816.\)

4600 \(\cong\) 1 \oplus 23 \oplus 275 \oplus 2024 \oplus 2277

Time C: < 1 sec. Time Maple: < 1 sec.

**Conway group Co₃.**  
Ord = 4957666556000 = \(2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23, M \cong 1, \text{Out} \cong 1.\)

48600-dimensional representation of Co₃  
Rank: 8. Suborbit lengths: 1, 253, 506, 1771, 7590, 8855, 14168, 15456.

48600 \(\cong\) 1 \oplus 23 \oplus 253 \oplus 275 \oplus 2024 \oplus 5544 \oplus 8855 \oplus 31625

Time C: 2 min 17 sec. Time Maple: 2 sec.

**McLaughlin group McL.**  
Ord = 8981280000 = \(2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11, M \cong C_3, \text{Out} \cong C_2.\)

(i) 22275-dimensional representation (a) of McL  
Rank: 13. Suborbit lengths: 1, 112, 140, 210, 420, 672, 1680², 2240, 3360³, 5040.

22275 \(\cong\) 1 \oplus 22 \oplus 252 \oplus 252 \oplus 1750 \oplus 1750 \oplus 3520 \oplus 5103 \oplus 9625

Time C: 23 sec. Time Maple: 11 sec.

(ii) 66825-dimensional representation of 3.McL  
Rank: 14. Suborbit lengths: \(1^3, 630, 2240^3, 5040^3, 8064^3, 20160.\)

66825 \(\cong\) 1 \oplus 252 \oplus 252 \oplus 1750 \oplus 2772 \oplus 2772 \oplus 5103 \oplus 5103 \oplus 5103 \oplus 5103 \oplus 5103 \oplus 5103 \oplus 5103 \oplus 6336 \oplus 6336 \oplus 8064 \oplus 8064 \oplus 9625

Time C: 8 min 45 sec. Time Maple: 12 min 59 sec.

(iii) 22275-dimensional representation (a) of McL \(\times 2\)  
Rank: 11. Suborbit lengths: 1, 112, 210, 420, 1120, 1260, 2520², 3360, 4032, 6720.

22275 \(\cong\) 1 \oplus 22 \oplus 252 \oplus 252 \oplus 1750 \oplus 1750 \oplus 3520 \oplus 5103 \oplus 9625

Time C: 23 sec. Time Maple: 5 sec.

**Suzuki group Suz.**  
Ord = 448345497600 = \(2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13, M \cong C_6, \text{Out} \cong C_2.\)

(i) 32760-dimensional representation of Suz  
Rank: 6. Suborbit lengths: 1, 891, 1980, 2816, 6336, 20736.

32760 \(\cong\) 1 \oplus 143 \oplus 364 \oplus 5940 \oplus 12012 \oplus 14300

Time C: 54 sec. Time Maple: 2 sec.

(ii) 65520-dimensional representation of 2.Suz  
Rank: 10. Suborbit lengths: \(1^2, 891^2, 2816^2, 3960, 12672, 20736^2.\)

65520 \(\cong\) 1 \oplus 143 \oplus 364 \oplus 364 \oplus 364 \oplus 364 \oplus 364 \oplus 5940 \oplus 12012 \oplus 14300 \oplus 16016 \oplus 16016

Time C: 6 min 9 sec. Time Maple: 11 sec.

(iii) 98280-dimensional representation of 3.Suz  
Rank: 14. Suborbit lengths: \(1^3, 891^3, 2816^3, 5940, 19008, 20736^3.\)

98280 \(\cong\) 1 \oplus 78 \oplus 78 \oplus 143 \oplus 364 \oplus 1365 \oplus 1365 \oplus 4290 \oplus 4290 \oplus 5940 \oplus 12012 \oplus 14300 \oplus 27027 \oplus 27027

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Remark. The \textit{PreparePolynomialData} program uses more than 17.6 GB of memory for this task, which goes beyond the RAM of our PC, slowing down the calculations.

4.3.4. \textit{Pariahs.}

(iv) 1782-dimensional representation of \textbf{Suz} $\times$ 2

Rank: 3. Suborbit lengths: 1, 416, 1365.

\[1782 \cong 1 \oplus 780 \oplus 1001\]

Time C: < 1 sec. Time \textit{Maple}: < 1 sec.

(v) 5346-dimensional representation of \textbf{3.Suz} $\times$ 2

Rank: 5. Suborbit lengths: 1, 2, 416, 832, 4095.

\[5346 \cong 1 \oplus 132 \oplus 780 \oplus 1001 \oplus 3432\]

Time C: 1 sec. Time \textit{Maple}: < 1 sec.

4.3.3. \textit{Monster sections.} The main properties of the \textbf{Held group} \textbf{He} and the results of calculations for its representation of dimension 8330 are given in Section 4.1.

(i) 29155-dimensional representation of \textbf{He}

Rank: 12. Suborbit lengths: 1, 90, 120, 384, 960, 1440, 2160, 2880, 5760, 11520.

\[29155 \cong 1 \oplus 51 \oplus 51 \oplus 680 \oplus 1275 \oplus 1275 \oplus 1920 \oplus 4352 \oplus 7650 \oplus 11900\]

Time C: 42 sec. Time \textit{Maple}: 11 sec.

(ii) 8330-dimensional representation of \textbf{He} $\times$ 2

Rank: 6. Suborbit lengths: 1, 105, 720, 1344, 1680, 4480.

\[8330 \cong 1 \oplus 102 \oplus 680 \oplus 1275 \oplus 1920 \oplus 4352\]

Time C: 3 sec. Time \textit{Maple}: 1 sec.

\textbf{Fischer group} \textbf{Fi}_{22}. \text{ Ord } = 64561751654400 = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13, \ M \cong C_6, \ Out \cong C_2.

(i) 61776-dimensional representation of \textbf{Fi}_{22}

Rank: 4. Suborbit lengths: 1, 1575, 22400, 37800.

\[61776 \cong 1 \oplus 3080 \oplus 13650 \oplus 45045\]

Time C: 10 min 6 sec. Time \textit{Maple}: 3 sec.

(ii) 28160-dimensional representation of \textbf{2.Fi}_{22}

Rank: 5. Suborbit lengths: 1, 3159, 21840.

\[28160 \cong 1 \oplus 352 \oplus 429 \oplus 13650 \oplus 13728\]

Time C: 39 sec. Time \textit{Maple}: 2 sec.

(iii) 56320-dimensional representation of \textbf{2.Fi}_{22} $\times$ 2

Rank: 9. Suborbit lengths: 1, 278, 1080, 3159, 21840, 25272.

\[56320 \cong 1 \oplus 1 \oplus 352 \oplus 352 \oplus 429 \oplus 429 \oplus 13650 \oplus 13650 \oplus 27456\]

Time C: 3 min 20 sec. Time \textit{Maple}: 5 sec.

\textbf{Fischer group} \textbf{Fi}_{23}. \text{ Ord } = 4089470473293004800 = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23, \ M \cong 1, \ Out \cong 1.

31671-dimensional representation of \textbf{Fi}_{23}

Rank: 3. Suborbit lengths: 1, 3510, 28160.

\[31671 \cong 1 \oplus 782 \oplus 30888\]

Time C: 52 sec. Time \textit{Maple}: 1 sec.

4.3.4. \textit{Pariahs.}

\textbf{Janko group} \textbf{J}_1. \text{ Ord } = 175560 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19, \ M \cong 1, \ Out \cong 1.

1045-dimensional representation of \textbf{J}_1

Rank: 11. Suborbit lengths: 1, 8, 28, 56, 168.

\[1045 \cong 1 \oplus 56 \oplus 56 \oplus 76 \oplus 77 \oplus 77 \oplus 120_0 \oplus 120_0 \oplus 120_0 \oplus 133 \oplus 209\]

Time C: < 1 sec. Time \textit{Maple}: 22 sec.
**Janko group** $J_3$. \( \text{Ord}(J_3) = 50232960 = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19, \ M(J_3) \cong C_3, \ \text{Out}(J_3) \cong C_2. \)

(i) **14688**-dimensional representations (a) and (b) of $J_3$

- **Rank**: 14.
- **Suborbit lengths**: 1, 285, 340, 570, 885, 1140, 1710, 3420.

\[
14688 \cong 1 \oplus 85 \oplus 35 \oplus 1140 \oplus 1140 \oplus 1215 \oplus 1215 \oplus 1615 \oplus 1920, \ \alpha \oplus 1920, \ \beta \oplus 1920, \ \gamma \oplus 2432
\]

- **Time** C: 11 sec. **Time Maple**: 1 min 52 sec.

**Remark.** AtLAS [6] contains two non-equivalent 14688-dimensional representations of $J_3$, (a) and (b), which have the same decomposition structure. The differences are manifested in explicit expressions for irreducible projectors (and in the structure of orbitals).

(ii) **6156**-dimensional representation of $J_3 \rtimes 2$

- **Rank**: 7.
- **Suborbit lengths**: 1, 85, 120, 510, 680, 2040, 2720.

\[
6156 \cong 1 \oplus 324 \oplus 646 \oplus 1140 \oplus 1215 \oplus 1215 \oplus 1615
\]

- **Time** C: 1 sec. **Time Maple**: 1 sec.

**Rudvalis group** $Ru$. \( \text{Ord} = 145926144000 = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29, \ M \cong C_2, \ \text{Out} \cong 1. \)

(i) **4060**-dimensional representation of $Ru$

- **Rank**: 3.
- **Suborbit lengths**: 1, 1755, 2304.

\[
4060 \cong 1 \oplus 783 \oplus 3276
\]

- **Time** C: < 1 sec. **Time Maple**: < 1 sec.

(ii) **16240**-dimensional representation of $2.Ru$

- **Rank**: 9.
- **Suborbit lengths**: 1, 2304, 7020.

\[
16240 \cong 1 \oplus 29 \oplus 28 \oplus 806 \oplus 783 \oplus 3276 \oplus 3654 \oplus 4032 \oplus 4032
\]

- **Time** C: 12 sec. **Time Maple**: 2 sec.

5. **Concluding remarks**

For **PreparePolynomialData**, the main limiting parameter is the representation dimension. Our PC with 16 GB of RAM copes with dimensions not exceeding 100,000. We can expect that with enough RAM, the program will cope with dimensions up to several hundred thousand.

The main bottleneck of **SplitRepresentation** is that it is based on the polynomial algebra methods, which are intrinsically algorithmically difficult. The number of polynomial variables is equal to the rank R of the representation to be split. In practice, the program confidently splits representations with $R \leq 17$, although there are some examples with ranks 18 and 19. However, representations of finite groups often have low ranks. In particular, in AtLAS [6], 761 out of 886, or 86%, permutation representations satisfy the condition $R \leq 17$.

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