THE ROLE OF INTRINSIC DISTANCES IN THE RELAXATION OF $L^\infty$-FUNCTIONALS

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Abstract. We consider a supremal functional of the form

$$F(u) = \operatorname{ess sup}_{x \in \Omega} f(x, Du(x))$$

where $\Omega \subseteq \mathbb{R}^N$ is a regular bounded open set, $u \in W^{1,\infty}(\Omega)$ and $f$ is a Borel function. Assuming that the intrinsic distances

$$d^\lambda_F(x, y) := \sup \left\{ u(x) - u(y) : F(u) \leq \lambda \right\}$$

are locally equivalent to the euclidean one for every $\lambda > \inf_{W^{1,\infty}(\Omega)} F$, we give a description of the sublevel sets of the weak$^*$-lower semicontinuous envelope of $F$ in terms of the sub-level sets of the difference quotient functionals

$$R^d\lambda_F(u) := \sup_{x \neq y} \frac{u(x) - u(y)}{d^\lambda_F(x, y)}.$$  

As a consequence we prove that the relaxed functional of positive 1-homogeneous supremal functionals coincides with $R^d_1 F$. Moreover, for a more general supremal functional $F$ (a priori non coercive), we prove that the sublevel sets of its relaxed functionals with respect to the weak$^*$ topology, the weak$^*$ convergence and the uniform convergence are convex. The proof of these results relies both on a deep analysis of the intrinsic distances associated to $F$ and on a careful use of variational tools such as $\Gamma$-convergence.

Mathematics Subject Classification (2000): 47J20, 58B20, 49J45.

Keywords: supremal functionals, level convex functions, intrinsic distances, relaxation, Calculus of Variations in $L^\infty$, $\Gamma$-convergence.

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1. Introduction

In this paper we will consider functionals $F: W^{1,\infty}(\Omega) \to \mathbb{R}$ of the form

$$F(u) = \operatorname{ess sup}_{x \in \Omega} f(x, Du(x)),$$

(1.1)
where \( f : \Omega \times \mathbb{R}^N \to \mathbb{R} \) is a Borel function and \( \Omega \subseteq \mathbb{R}^N \) is a regular bounded open set. According to an established notation we will refer to energies in \( W^{1,\infty}(\Omega) \) as supremal or \( L^\infty \)-functionals on \( W^{1,\infty}(\Omega) \) and we will use the term supramand to denote the function \( f \) which represents the functional \( \Gamma \).

A main topic in the study of supremal functionals is to give sufficient and necessary conditions on the topology \( \tau \) of minimizing sequences when \( F \) satisfies a coercivity assumption. Indeed, this topology is natural in order to study minimum problems since we have compactness of minimizing sequences when \( F \) satisfies a coercivity assumption.

A sufficient condition for the sequential lower semicontinuity of a supramand functional with respect to \( \tau \) requires that for a.e. \( x \in \Omega \) the function \( f(x, \cdot) \) is lower semicontinuous and level convex, i.e.,

\[
f(x, \theta \xi_1 + (1-\theta)\xi_2) \leq f(x, \xi_1) \lor f(x, \xi_2) \quad \forall \theta \in (0,1) \quad \forall \xi_1, \xi_2 \in \mathbb{R}^N.
\]

If \( F \) is weak* lower semicontinuous, in general the vice-versa does not hold (see Remark 3.1), due to the non-uniqueness of the supramand \( f \) which represents \( F \). Note that if \( f(x, \cdot) \) is level convex then the supramand functional \( \Gamma \) is lower semicontinuous and level convex, that is, for every \( \lambda \in \mathbb{R} \) the sublevel set of \( F \), denoted by

\[
E_\lambda = \left\{ u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda \right\},
\]

is convex.

In this paper we study the properties of the relaxed functional of \( F \) given by \( \Gamma \), i.e., the greatest lower semicontinuous functional \( \Gamma_\tau(F) \) less than or equal to \( F \) with respect to a fixed topology \( \tau \in W^{1,\infty}(\Omega) \) chosen among the weak*, the sequential weak* and the uniform one (see Section 2). This extensive analysis relies on the fact that, in general, in absence of further coercivity hypotheses, \( \Gamma_\tau(F) \) is affected by the choice of the topology, as shown by Maggi and Gori in [13].

We note that, an explicit representation formula of \( \Gamma_\tau(F) \) in a supramand form has been established when \( f \) is a continuous and coercive function (see [12]), while the supremality of \( \Gamma_\tau(F) \) is still unknown in the general case. Despite the lack of a representation result for \( \Gamma_\tau(F) \), in this paper we give a detailed description of its sublevel sets by means of the intrinsic distance structures induced by \( F \) (see Theorems 3.1 and 3.4). As a result, for any \( \tau \) quoted above, we also show the level convexity of \( \Gamma_\tau(F) \).

One of the main ingredients of our proofs is the introduction of the family of pseudo-distances \( d^\lambda_F \) defined by

\[
d^\lambda_F(x, y) := \sup \left\{ u(x) - u(y) : F(u) \leq \lambda \right\}
\]

for any \( x, y \in \Omega \) and \( \lambda \in (\min \{f \} \cup \{0\}, \infty) \) (see Section 4).

These distances have been introduced by De Cecco and Palmieri in the setting of Finsler metrics, in [12] and [13], where they characterize the class of geodesic distances \( d \) that satisfies the identity \( d = d^1_F \) for a suitable convex 1-homogeneous supramand functional.

In our framework we consider a more general class of distance functions which are not symmetric or finite unless one requires additional hypotheses on \( F \). In order to obtain a description of the sublevel sets of \( \Gamma_\tau(F) \), in Section 4 we introduce the class of the difference quotients \( R_{d^\lambda_F} \) already considered in [14] and defined by

\[
R_{d^\lambda_F}(u) = \sup_{x, y \in \Omega, d^\lambda_F(x, y) \neq 0} \frac{u(x) - u(y)}{d^\lambda_F(x, y)}.
\]

Under the assumption that the pseudo-distances \( d^\lambda_F \) are metrically equivalent to the euclidean metric, that is, for every \( \lambda > \min \{f \} \cup \{0\} \), \( d^\lambda_F \) satisfies

\[
\alpha(\lambda)|x - y| \leq d^\lambda_F(x, y) \leq \beta(\lambda)|x - y|
\]

with \( \alpha(\lambda), \beta(\lambda) \) strictly positive constants, in Theorem 3.1 we show that the sublevel sets of \( \Gamma_\tau(F) \) satisfy the identity

\[
\{ u \in W^{1,\infty}(\Omega) : \Gamma_\tau(F)(u) \leq \lambda \} = \bigcap_{\lambda' > \lambda} \{ u \in W^{1,\infty}(\Omega) : R_{d^{\lambda'}_F}(u) \leq 1 \}.
\]

As a consequence, under the assumption \( (1.3) \), we get that the relaxed functional \( \Gamma_\tau(F) \) is independent of the topology \( \tau \) fixed above and it is level convex.
Afterwards, exploiting Theorem 3.1, we provide a representation result for $\Gamma_{\tau}(F)$ when $F$ is a positively 1-homogeneous supremal functional satisfying (1.5). More in detail, in Theorem 3.2 we represent $\Gamma_{\tau}(F)$ as the different quotient associated to the distance $d^1_{\tau}$, that is

$$\Gamma_{\tau}(F)(u) = R_{d^1_{\tau}}(u) \quad \forall u \in W^{1,\infty}(\Omega)$$

for any $\tau$ topology among those quoted above. We underline that the class of difference quotients strictly contains supremal functionals; under the assumptions of Theorem 3.2, $\Gamma_{\tau}(F)$ can be represented in a supremal form if and only if the distance $d^1_{\tau}$ is intrinsic according to the definition introduced by De Cecco and Palmieri [12, 13] (see Section 2 in [14]).

Actually, the representation result in Theorem 3.2 is a generalization of Theorem 3.5 in [14]. Indeed, here we weaken the growth conditions on the supremand $f$ representing $F$ and we remove its continuity and symmetry hypotheses with respect to the gradient variable $\xi$.

Note that condition (1.5) has a key role in order to get the results quoted above. Hence, in order to clarify the setting of validity of Theorems 3.1 and 3.2 in Theorem 4.6 we characterize the class of supremal functionals satisfying (1.5). Roughly speaking, we prove that condition (1.5) holds if and only if the sublevel sets of $F$ consist of functions with bounded gradients and $0 \in \text{argmin } \Gamma_{\tau}(F)$ is a "continuity" point for $\Gamma_{\tau}(F)$ along the affine functions $u_\xi := \xi \cdot x$.

Finally, in Theorem 3.3, we study the relaxation problem for a more general class of functionals $F$ with respect to those considered in Theorem 3.1. In particular, we drop the boundedness assumption on the sublevel sets of $F$ thanks to an approximation result of $\Gamma_{\tau}(F)$ via $\Gamma$-convergence through a sequence of coercive ($\tau$-lower semicontinuous) functionals (see Proposition 3.1). Moreover, we replace the condition that $0$ is a minimizer for $\Gamma_{\tau}(F)$ (together with the continuity hypothesis above) with a mild hypothesis concerning the behaviour of $F$ along a suitable minimizing sequence. More precisely, we assume the existence of a minimizing sequence for $F$ made up by a sort of "upper semicontinuity" points, that is, $\exists (u_n)_{n \in \mathbb{N}} \subseteq W^{1,\infty}(\Omega)$ such that, set $u_\xi(x) := \xi \cdot x$, it holds

$$\lim_{n \to \infty} F(u_n) = \inf_{W^{1,\infty}(\Omega)} F = \lim_{n \to \infty} \limsup_{\xi \to 0} F(u_n + u_\xi) \quad (1.7)$$

and we show that $\Gamma_{\tau}(F)$ is level convex when $\tau$ is one of the topologies quoted above.

Note that Theorem 3.3 applies to a wide class of supremal functionals $F$: for instance, assumption (1.7) is satisfied by functionals $F$ whose supremand $f(x, \cdot)$ has a uniform modulus of continuity or satisfies the conditions

$$f(x, 0) = 0 = \min_{\xi \in \mathbb{R}^N} f(x, \xi) \text{ for a.e. } x \in \Omega \text{ and } \lim \text{ ess sup}_{\xi \to 0} f(x, \xi) = 0.$$  

In the particular case when $f$ does not depend on $x$, it is sufficient that every sublevel set of $f$ has non empty interior in order to get that $f$ satisfies (1.7) (for more details, see Remark 3.6).

As a byproduct of Theorems 3.1 or 3.3 we obtain that the lower semicontinuous envelope of $F$ with respect to the weak* topology coincides with the sequential weak* envelope. Note that, in general, the relaxed functional with respect to the uniform convergence could be strictly lower than the weak* envelope, (see Example 2.2 in [13]). Moreover, under the assumptions of Theorem 3.1 or 3.3 we get also that if $F$ is $\tau$-lower semicontinuous then $F$ is level convex. Note that with respect to the analogous result in [19] (see Theorem 2.7), we drop the coercivity and symmetry assumptions on $F$ and we do not require that $f$ is a Carathéodory function.

We remark that a different notion of distance functions has been considered by Champion and De Pascale in [2] in case the supremand $f(x, \xi)$ is level convex and coercive in $\xi$ and globally lower semicontinuous on $\Omega \times \mathbb{R}^N$ in order to establish comparison principles for absolute minimizers. In these hypotheses, using control theory formulas, they prove that their class of distances coincides with the family given by (1.3), main tool in our analysis. Moreover, the family of distances (1.3) and the related different quotients (1.4) have been used by Davini and Ponsiglione in [11] in the setting of Finsler metrics, in order to study the closure of two-phase gradient-constraints, in terms of $\Gamma$-convergence.

We finally emphasize that the distance $d_{\tau}^1$ has been also considered in the particular case when $f(x, \xi) = \sqrt{\sum_{i,j} a_{ij}(x) \xi_i \xi_j}$ with $(a_{ij})$ an elliptic matrix in $\Omega$ or, more in general, when $f$ is a Riemannian metric on a Lipschitz manifold $M$. In this case the distance derived from the metric on $M$ has a relevant role in the study of the heat flow associated to Dirichlet forms on $M$ (see [17]), and, for a smooth manifold,
The role of intrinsic distances in the relaxation of $L^\infty$-functionals

The paper is organized as follows: Section 2 is devoted to some preliminary definitions and results concerning lower semicontinuous envelopes, level convex functionals and $\Gamma$-convergence; in Section 3 we introduce the family of the pseudo-distances and their associated difference quotients and we state the main results of the paper. In Section 4 we establish some key results about the family $\{d^\lambda_{\tau}(F)\}_\lambda$ and its connection with the family $\{d^\lambda_{\tau}(F)\}_\lambda$. These results will be instrumental for the proofs of the main results provided in Section 5. Eventually, Section 6 contains some additional results of interest. In particular, we provide a representation result for $\Gamma^\tau(F)$ assuming its level convexity and we also address the problem of representing a weak* lower semicontinuous functional $F$ by means of a level convex supremand. Indeed, such a representation is crucial in problems involving supremal functionals as, for example, for existence of absolute minimizers, in the homogenization problem, in the study of the $L^p$-approximation via $\Gamma$-convergence, in the characterization of the effective strength set in the context of electrical resistivity (see among others [1], [3], [4], [8] and [14]). Finally, in Section 7 we collect some interesting examples showing the optimality of some statements of the paper. In particular, in Example 7.2 we construct a supremal functional $F$ such that the pseudo-distance $d^\lambda_{\tau}(F)$ associated to its relaxed functional $\Gamma^\tau(F)$ is different from $d^\lambda_\tau$, for some values $\lambda > \inf_{W^{1,\infty}(\Omega)} F$; in Examples 7.3 and 7.5 we exhibit discontinuous supremands $f$ such that the associated supremal functional $F$ still satisfies condition (3.8).

2. Notations and preliminary results

Throughout the paper we assume $\Omega$ to be a bounded open set in $\mathbb{R}^N$. We denote by $\mathcal{A}(\Omega)$ the family of all open subsets of $\Omega$, and by $\mathcal{B}_N$ the Borel $\sigma$-algebra of $\mathbb{R}^N$ (when $N = 1$, we simply write $\mathcal{B}$). Moreover, we write $\| \cdot \|$ for the euclidean norm on $\mathbb{R}^N$, $B_r(x)$ for the open ball $\{ y \in \mathbb{R}^N : \| x - y \| < r \}$, and $\mathcal{L}^N$ for the Lebesgue measure in $\mathbb{R}^N$. For any $a, b \in \mathbb{R}$ we will denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. If $\Omega$ is also connected, besides the euclidean one, it is possible to consider on $\Omega$ the so called geodesic distance, that is a distance containing the geometric features of both the open set and its boundary. More precisely, let $\Gamma_{x,y}(\Omega)$ be the set of Lipschitz curves in $\Omega$ with end-points $x$ and $y$. We define the geodesic distance between two points $x, y \in \Omega$ as
\[
|x - y|_\Omega = \inf\{ \mathcal{L}(\gamma) : \gamma \in \Gamma_{x,y}(\Omega) \}
\]
where $\mathcal{L}(\gamma)$ denotes the length of the curve $\gamma$ with respect to the euclidean distance. Note that if $\partial \Omega$ is Lipschitz then there exists a constant $C_\Omega > 0$ such that
\[
|x - y| \leq |x - y|_\Omega \leq C_\Omega |x - y|.
\]

We will use standard notations for Lebesgue and Sobolev spaces $L^p(\Omega), W^{1,p}(\Omega)$. We will also denote by $\text{Lip}(\Omega)$ the space of the Lipschitz continuous functions on $\Omega$ and set
\[
\text{Lip}_\Omega(u) := \sup_{x, y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|}.
\]

When $\Omega$ is bounded, $\text{Lip}(\Omega) \subseteq W^{1,\infty}(\Omega)$ and for any $u \in \text{Lip}(\Omega)$ it holds $\|Du\|_{L^\infty(\Omega)} \leq \text{Lip}_\Omega(u)$. In general, if $\Omega$ is connected, then $\|Du\|_{L^\infty(\Omega)}$ coincides with the Lipschitz constant of $u$ with respect to the geodesic distance as shown in the following Lemma.

Lemma 2.1. Let $\Omega$ be a connected open set in $\mathbb{R}^N$. Then for any $u \in W^{1,\infty}(\Omega)$ it holds
\[
\|Du\|_{L^\infty(\Omega)} = \sup_{x, y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|}.
\]

Moreover, if $\Omega$ is bounded and has Lipschitz continuous boundary then there exists a constant $c_\Omega > 0$ such that
\[
\|Du\|_{L^\infty(\Omega)} \leq \text{Lip}_\Omega(u) \leq c_\Omega \|Du\|_{L^\infty(\Omega)} \quad \forall u \in W^{1,\infty}(\Omega).
\]

In particular, $W^{1,\infty}(\Omega) \equiv \text{Lip}(\Omega)$. In addition, if $\Omega$ is a convex set, then $\|Du\|_{L^\infty(\Omega)} = \text{Lip}_\Omega(u)$. 

f coincides with the metric derivative of the geodesic distance on $M$ (see [12, 13]). More recently, in the context of diffusion problem, Koskela, Shanmugalingam and Zhou establish when the intrinsic differential and the local intrinsic distance structures coincide (see [16]).
Proof. It is well known that
\[ |u(x) - u(y)| \leq ||Du||_{L_\infty(\Omega)}|x - y|, \]
for every \( u \in W^{1,\infty}(\Omega) \) and for every \( x, y \in \Omega \) (see, for example, [5] Remark 7 in Chapter 9). The converse inequality in (2.2) can be established by the fact that any \( u \in W^{1,\infty}(\Omega) \) is differentiable almost everywhere in \( \Omega \) and \( Du \) coincides with the standard gradient of \( u \). Therefore, for every fixed \( u \in W^{1,\infty}(\Omega) \) and for every ball \( B \subseteq \Omega \), we have that
\[
\sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|} \geq \sup_{x,y \in B, x \neq y} \frac{|u(x) - u(y)|}{|x - y|} = \text{Lip}_B(u) \geq ||Du||_{L_\infty(B)}.
\]
By passing to the supremum with respect to \( B \) we obtain (2.2). Inequality (2.3) follows by (2.1). \( \square \)

On \( W^{1,\infty}(\Omega) \) we will consider different topologies: the uniform topology (denoted by \( \tau_\infty \)) induced by natural inclusion of \( W^{1,\infty}(\Omega) \subseteq L_\infty(\Omega) \) and the weak* topology (\( w^* \) for shortly) inherited by \( W^{1,\infty}(\Omega) \) as a (closed) subset of \( L_\infty(\Omega) \times L_\infty(\Omega) \) endowed with the weak* topology. Moreover we will denote by \( w^*_\text{seq} \) the topology on \( W^{1,\infty}(\Omega) \) induced by the \( w^* \)-convergence (see the next subsection).

2.1. Lower semicontinuous envelopes. Throughout the section \((X, \tau)\) is a fixed topological space. For any set \( B \subseteq X \) we denote by \( \overline{B} \) its \( \tau \)-closure and we denote by \( \tau_{\text{seq}} \) the topology on \( X \) whose closed sets are the sequentially \( \tau \)-closed subsets of \( X \). Note that \( \tau_{\text{seq}} \) is in general strictly stronger than \( \tau \).

Definition 2.2. Let \( F : (X, \tau) \to \mathbb{R} \) be a function.
We say that \( F \) is \( \tau \)-lower semicontinuous on \( X \) (shortly \( \tau \)-l.s.c.) when for any \( \lambda \in \mathbb{R} \) the sublevel set \( \{ x \in X | F(x) \leq \lambda \} \) is \( \tau \)-closed.
We say that \( F \) is sequentially \( \tau \)-lower semicontinuous (shortly seq. \( \tau \)-l.s.c.) on \( X \) if for any \( x \in X \) and for any \( (x_n)_n \subseteq X \) \( \tau \)-converging to \( x \) we have
\[ F(x) \leq \liminf_{n \to +\infty} F(x_n). \]

Remark 2.3. Note that
(1) \( F \) is sequentially \( \tau \)-lower semicontinuous if and only if \( F \) is \( \tau_{\text{seq}} \)-lower semicontinuous;
(2) if \( F : (X, \tau) \to \mathbb{R} \) is \( \tau \)-lower semicontinuous then \( F \) is \( \tau_{\text{seq}} \)-lower semicontinuous;
(3) the supremum of a family (also infinite) of \( \tau \)-lower semicontinuous functions is still \( \tau \)-lower semicontinuous.
Moreover, if \( F, G \) are \( \tau \)-lower semicontinuous then \( F + G \) is \( \tau \)-lower semicontinuous.

Definition 2.4. Let \( F : (X, \tau) \to \mathbb{R} \). The lower semicontinuous envelope (or relaxed function) of \( F \) is defined as
\[
\Gamma_\tau(F) := \sup \{ G | (X, \tau) \to \mathbb{R}, G \text{ \( \tau \)-s.c.i. and } G \leq F \text{ on } X \}. \tag{2.4}
\]
By Remark 2.3(2) it follows that \( \Gamma_\tau(F) \leq \Gamma_{\tau_{\text{seq}}}(F) \); if \( (X, \tau) \) satisfies the first axiom of countability then we have the following sequential characterization:
\[
\Gamma_\tau(F)(x) = \Gamma_{\tau_{\text{seq}}}(F)(x) = \min \{ \liminf_{n \to +\infty} F(x_n) | x_n \xrightarrow{\tau} x \},
\]
for any \( x \in X \) (for a proof see [7] Proposition 1.3.3).

The following properties can be easily shown (for more details see [10], Chapters 3 and 6).

Proposition 2.5. Let \( F : (X, \tau) \to \mathbb{R} \), Then the following properties hold.
(1) \( \Gamma_\tau(F) \) is \( \tau \)-lower semicontinuous;
(2) \( \inf_X F = \inf_X \Gamma_\tau(F) \);
(3) for any \( \tau \)-continuous function \( G : X \to \mathbb{R} \), we have
\[
\Gamma_\tau(F + G) = \Gamma_\tau(F) + G;
\]
(4) for any \( c \in \mathbb{R} \), set \( (F \lor c)(x) := F(x) \lor c \), we have
\[
\Gamma_\tau(F \lor c) = \Gamma_\tau(F) \lor c; \tag{2.5}
\]
(5) if \( X \) is a topological vector space and \( x_0 \in X \), set \( G(\cdot) := F(\cdot + x_0) \), we have
\[
\Gamma_\tau(G)(\cdot) = \Gamma_\tau(F)(\cdot + x_0).
In the following remark we clarify the relationship about the properties of lower semicontinuity of a functional \( F : W^{1,\infty}(\Omega) \to \mathbb{R} \) with respect to the topologies \( w^* \), \( w^*_{\text{seq}} \) and \( \tau_\infty \).

**Remark 2.6.** Let \( F : W^{1,\infty}(\Omega) \to \mathbb{R} \) be a functional. Then the following properties hold.

1. If \( \forall \lambda \in \mathbb{R} \) there exists \( r(\lambda) > 0 \) such that
   \[
   E_\lambda = \{ u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda \} \subseteq \{ u \in W^{1,\infty}(\Omega) : \|Du\|_{L^\infty(\Omega)} \leq r(\lambda) \}
   \]  
then
   \[ F \quad \text{\( w^*_{\text{seq}} \)-l.s.c.} \quad \Rightarrow \quad F \quad \text{\( \tau_\infty \)-l.s.c.} \]
   indeed, thanks to \((2.6)\), any uniformly convergent sequence \((u_n)_n\) in \( W^{1,\infty}(\Omega) \) with \( \liminf_{n \to \infty} F(u_n) < +\infty \) is also \( w^* \)-convergent in \( W^{1,\infty}(\Omega) \).

2. If \( \Omega \) has Lipschitz continuous boundary, by using standard immersion argument, it follows that
   \[ F \quad \text{\( \tau_\infty \)-l.s.c.} \quad \Rightarrow \quad F \quad \text{\( w^*_{\text{seq}} \)-l.s.c.} \]

Taking into account (1), if \( \Omega \) has Lipschitz continuous boundary and \( F \) satisfies \((2.6)\) then
\[ F \text{ is \( \tau_\infty \)-l.s.c. } \iff \text{ F is \( w^*_{\text{seq}} \)-l.s.c.} \]
In particular it holds
\[ \Gamma_{\tau_\infty}(F) = \Gamma_{w^*_{\text{seq}}}(F). \]

3. For what the \( w^* \) topology is concerned, the following relation holds true without additional hypotheses on \( F \) and \( \Omega \)
\[ F \text{ is \( w^* \)-l.s.c. } \quad \iff \text{ F is \( w^*_{\text{seq}} \)-l.s.c.} \]
if the sublevel sets \( E_\lambda \) of \( F \) are bounded in \( W^{1,\infty}(\Omega) \), then the implication above can be reversed
\[ F \text{ is \( w^*_{\text{seq}} \)-l.s.c. } \quad \iff \text{ F is \( w^* \)-l.s.c.} \]
Indeed on bounded sets of \( W^{1,\infty}(\Omega) \) the weak* topology induced by \( L^\infty \times L^\infty \) is metrizable.

**Remark 2.7.** If we drop \((2.6)\), then the identity above could fails. Indeed in \[15\] Example 2.2 Maggi and Gori exhibit an example of a convex supremal functional \( F \) that is \( w^*_{\text{seq}} \)-l.s.c. (and \( w^* \)-l.s.c.) but not \( \tau_\infty \)-l.s.c.. However, when \( f(\cdot, \xi) \) is uniformly continuous, they show that the supremal functional \( F \) represented by \( f \) is \( w^* \)-l.s.c. if and only if \( F \) is \( \tau_\infty \)-l.s.c. (see \[15\] Theorem 1.4).

### 2.2. Γ-convergence

In order to introduce the notion of Γ-convergence let \((X, \tau)\) be a topological space and denote by \( \mathcal{U}(x) \) the set of all open neighbourhoods of \( x \) in \( X \).

**Definition 2.8.** Let \( F_n : X \to \mathbb{R} \) be a function for every \( n \in \mathbb{N} \). The \( \Gamma(\tau) \)-lower limit and the \( \Gamma(\tau) \)-upper limit of the sequence \((F_n)_{n \in \mathbb{N}}\) are the functions from \( X \) into \( \mathbb{R} \) defined by
\[
\Gamma(\tau)\text{-lim inf}_{n \to \infty} F_n(x) := \sup_{U \in \mathcal{U}(x)} \liminf_{n \to \infty} \inf_{y \in U} F_n(y)
\]
\[
\Gamma(\tau)\text{-lim sup}_{n \to \infty} F_n(x) := \sup_{U \in \mathcal{U}(x)} \limsup_{n \to \infty} \inf_{y \in U} F_n(y)
\]
If there exists a function \( F : X \to \mathbb{R} \) such that \( \Gamma(\tau)\text{-lim inf}_{n \to \infty} F_n = \Gamma(\tau)\text{-lim sup}_{n \to \infty} F_n \), then we write
\[ F = \Gamma(\tau)\text{-lim}_{n \to \infty} F_n \]
and we say that the sequence \((F_n)_n\), \( \Gamma(\tau) \)-converges to \( F \) or that \( F \) is the \( \Gamma(\tau) \)-limit of \((F_n)_n\).

In the following proposition we summarize some properties of the \( \Gamma \)-convergence useful in the sequel (see \[10\] Proposition 6.8, Proposition 6.11, Proposition 5.7, Remark 5.5).

**Proposition 2.9.** Let \( F_n : X \to \mathbb{R} \) be a function for every \( n \in \mathbb{N} \). Then

1. both the \( \Gamma(\tau)\)-lim inf \( F_n \) and \( \Gamma(\tau)\)-lim sup \( F_n \) are \( \tau \)-lower semicontinuous on \( X \);
2. the sequence \((F_n)_n\) \( \Gamma(\tau) \)-converges to \( F \) if and only if the sequence of the relaxed functions \((\Gamma(\tau)(F_n))_{n \in \mathbb{N}}\) \( \Gamma(\tau) \)-converges to \( F \);
3. if \((F_n)_n\) is a not increasing sequence which pointwise converges to \( F \) then \( \Gamma(\tau)\text{-lim}_{n \to \infty} F_n = \Gamma(\tau)(F) \).

In particular if \( F_n = F \) for every \( n \in \mathbb{N} \) then \( \Gamma(\tau)\text{-lim}_{n \to \infty} F = \Gamma(\tau)(F) \).
(4) if $(F_n)_{n \in \mathbb{N}}$ is an increasing sequence of $\tau$-lower semicontinuous functions which pointwise converges to $F$ then $\Gamma(\tau) \lim_{n \to \infty} F_n = F$.

2.3. Level convex functionals. In the framework of supremal functionals, level convexity plays the same main role as convexity in the setting of integral functionals.

Definition 2.10. Let $(X, \tau)$ be a topological vector space. A function $F : X \to \mathbb{R}$ is level convex if

$$F(\theta x_1 + (1 - \theta) x_2) \leq F(x_1) \lor F(x_2), \quad \forall \theta \in (0, 1), \; \forall x_1, x_2 \in X,$$

that is, for every $\lambda \in \mathbb{R}$ the sublevel set

$$E_\lambda = \{ x \in X : F(x) \leq \lambda \}$$

is convex.

Note that the level convexity is stable under both pointwise and $\Gamma$-convergence.

Proposition 2.11. Let $(X, \tau)$ be a topological vector space and for every $n \in \mathbb{N}$ let $F_n : X \to \mathbb{R}$ be a level convex function. Then

1. the function $F^\#(x) = \limsup_{n \to \infty} F_n(x)$ is level convex;
2. the function $F''(x) = \Gamma(\tau) \limsup_{n \to \infty} F_n$ is level convex.

Proof. (1) Let $x_1, x_2 \in X$ and $\theta \in (0, 1)$. Then

$$F^\#(\theta x_1 + (1 - \theta) x_2) = \limsup_{n \to \infty} F_n(\theta x_1 + (1 - \theta) x_2) \leq \limsup_{n \to \infty} (F_n(x_1) \lor F_n(x_2)).$$

Since for every pair of real sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ we have that

$$\limsup_{n \to \infty} (a_n \lor b_n) \leq \limsup_{n \to \infty} a_n \lor \limsup_{n \to \infty} b_n,$$

we get that

$$F^\#(\theta x_1 + (1 - \theta) x_2) \leq (\limsup_{n \to \infty} F_n(x_1)) \lor (\limsup_{n \to \infty} F_n(x_2)) = F^\#(x_1) \lor F^\#(x_2).$$

(2) Let $x_1, x_2 \in X$, $\theta \in (0, 1)$ and let $x := \theta x_1 + (1 - \theta) x_2$. Without loss of generality, assume that $F''(x_1) \lor F''(x_2) < +\infty$. Since the map $(x,y) \mapsto \theta x + (1 - \theta) y$ is continuous from $X \times X$ into $X$, then for every $U \in \mathcal{U}(x)$ there exist $U_1 \in \mathcal{U}(x_1)$ and $U_2 \in \mathcal{U}(x_2)$ such that $U$ contains the set $V := \{ t x_1 + (1 - t) x_2 \in X : y_1 \in U_1, \; y_2 \in U_2 \}$. Then, by applying the level convexity of the functions $F_n$ we obtain that

$$\inf_{y \in U} F_n(y) \leq \inf_{y \in V} F_n(y) = \inf_{y_1 \in U_1, y_2 \in U_2} F_n(t y_1 + (1 - t) y_2) \leq \inf_{y_1 \in U_1} \inf_{y_2 \in U_2} (F_n(y_1) \lor F_n(y_2)) = \inf_{y_1 \in U_1} F_n(y_1) \lor \inf_{y_2 \in U_2} F_n(y_2).$$

Hence, thanks to (2.8),

$$\limsup_{n \to \infty} \inf_{y \in U} F_n(y) \leq \limsup_{n \to \infty} (\inf_{y_1 \in U_1} F_n(y_1) \lor \inf_{y_2 \in U_2} F_n(y_2)) \leq (\limsup_{n \to \infty} \inf_{y_1 \in U_1} F_n(y_1)) \lor (\limsup_{n \to \infty} \inf_{y_2 \in U_2} F_n(y_2)).$$

In particular

$$\Gamma(\tau) \limsup_{n \to \infty} F_n(x) \leq \Gamma(\tau) \limsup_{n \to \infty} F_n(x_1) \lor \Gamma(\tau) \limsup_{n \to \infty} F_n(x_2).$$

Remark 2.12. In general, given a sequence of level convex functions $(F_n)_{n \in \mathbb{N}}$, the functions $F' = \Gamma(\tau) \liminf_{n \to \infty} F_n$ and $F^\# := \liminf_{n \to \infty} F_n$ are not level convex. It is enough to consider the sequence $F_n(x) = (x - (-1)^n)^2$. In this case

$$F^\#(x) = F''(x) = (x + 1)^2 \lor (x - 1)^2.$$

Eventually we mention a further property that holds for level convex functions defined on $X'$ when $X$ is a separable Banach space.
Proposition 2.13. Let $X$ be a separable Banach space and let $F : X' \to \mathbb{R}$. If the relaxed function $\Gamma_{w^*_{\text{seq}}}(F)$ is level convex (where $w^*$ stands for the weak* topology on $X'$), then

$$\Gamma_{w^*}(F) = \Gamma_{w^*_{\text{seq}}}(F).$$

In particular, if $F$ is a level convex function, then

$F$ is $w^*$-lower semicontinuous $\iff F$ is $w^*_{\text{seq}}$-lower semicontinuous.

Proof. For any $\lambda \in \mathbb{R}$, the set $E_\lambda = \{\Gamma_{w^*_{\text{seq}}}(F) \leq \lambda\}$ is convex and sequentially weak* closed. Hence, by applying Banach-Dieudonné-Krein-Smulian Theorem (see [5, Theorem 3.33]) we get that $E_\lambda$ is weak* closed. Therefore $\Gamma_{w^*_{\text{seq}}}(F)$ is $w^*$-lower semicontinuous, which implies in turn that $\Gamma_{w^*_{\text{seq}}}(F) \leq \Gamma_{w^*}(F)$. The other inequality follows by Remark 2.3(2). \qed 

Remark 2.14. Note that, for a general functional $F : W^{1,\infty}(\Omega) \to \mathbb{R}$, the lower semicontinuity with respect to one of the topologies $\tau_\infty$, $w^*$, $w^*_{\text{seq}}$ does not imply the level convexity of $F$. Indeed it is enough to consider the characteristic function of the complement of any $\tau$-closed set $C \subseteq W^{1,\infty}(\Omega)$ that is not convex. For instance $C$ can be chosen as the union of two closed disjoint balls.

3. Main results

Given a supremal functional $F$ of the form

$$F(u) = \text{ess sup}_{x \in \Omega} f(x, Du(x)),$$

where $\Omega$ is a bounded connected open set with Lipschitz continuous boundary, in this section we provide a description of the sublevel sets of $\Gamma_r(F)$, where $r$ is one of the topologies $\tau_\infty$, $w^*$, $w^*_{\text{seq}}$ defined in Section 2 and we show that $\Gamma_r(F)$ is a level convex functional.

We recall that by Lemma 2.1 $|Du|_{L^1(\Omega)}$ coincides with the Lipschitz constant of $u \in W^{1,\infty}(\Omega)$ with respect to the geodetic distance $d_\Omega(x,y) = |x-y|_\Omega$. This means that, when $f(x,\xi) = |\xi|$, the supremal functional (3.1) can be represented as the “difference quotient functional”

$$F(u) = \sup_{x,y \in \Omega, x \neq y} \frac{u(x) - u(y)}{d_\Omega(x,y)}.$$  

(3.2)

In addition, it can be also proved that the geodetic distance satisfies

$$|x-y|_\Omega = \sup \{u(x) - u(y) : \|Du\|_{\infty} \leq 1\}.$$  

Starting from this observation and exploiting the works [12] [13], a representation formula of type (3.2) has been proved in [14] for the relaxation of 1-homogeneous supremal functionals of type (3.1) with suitable growth conditions of linear type.

Motivated by this result, we introduce a family $(d^\lambda_F)_\lambda$ of pseudo-distances associated to the sublevel sets of a general functional $F : W^{1,\infty}(\Omega) \to \mathbb{R}$. With a slight abuse of notation we refer to this family as intrinsic distances associated to $F$. Hence we define the corresponding difference quotient functionals and we investigate their main properties.

For any $\lambda > \inf_{W^{1,\infty}(\Omega)} F$ let $E_\lambda := \{u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda\}$. For any $(x,y) \in \Omega \times \Omega$ we set

$$d^\lambda_F(x,y) := \sup \left\{u(x) - u(y) : u \in E_\lambda\right\}.$$  

(3.3)

Note that although $d^\lambda_F$ is in general not symmetric, by the definition of $d^\lambda_F$ it follows straightforward that $\forall x, y, z \in \Omega$ it holds

$$d^\lambda_F(x,y) \leq d^\lambda_F(x,z) + d^\lambda_F(z,y).$$

We define the difference quotient functional (associated to $d^\lambda_F$) the functional $R_{d^\lambda_F} : W^{1,\infty}(\Omega) \to [0, +\infty]$ given by

$$R_{d^\lambda_F}(u) := \sup_{x,y \in \Omega, d^\lambda_F(x,y) \neq 0} \frac{u(x) - u(y)}{d^\lambda_F(x,y)}.$$  

(3.4)

For the sake of brevity, in the sequel we refer to $d^\lambda_F$ as ‘distances’ and to $R_{d^\lambda_F}$ as difference quotients.

We are now in position to state the main results of this paper.
Theorem 3.1. Let \( \Omega \) be a bounded connected open set with Lipschitz continuous boundary and let \( F \) be the supremal functional represented by a Borel function \( f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R} \). Assume that for any \( \lambda > \inf_{W^{1,\infty}(\Omega)} F \) there exist positive coefficients \( \alpha(\lambda), \beta(\lambda) > 0 \) such that for every \( x, y \in \Omega \)

\[
\alpha(\lambda)|x - y| \leq d_{\lambda}^F(x, y) \leq \beta(\lambda)|x - y|_{\Omega}.
\]

Then

\[
\Gamma_{\tau_{\infty}}(F) \equiv \Gamma_{w^*}(F) \equiv \Gamma_{w_{seq}^*}(F)
\]

and the following identities hold for any \( \lambda > \inf_{W^{1,\infty}(\Omega)} F \):

\[
\{ u \in W^{1,\infty}(\Omega) : \Gamma_{\tau}(F)(u) \leq \lambda \} = \{ u \in W^{1,\infty}(\Omega) : R_{d_{\lambda}^F}(u) \leq 1 \}
\]

(3.6)

where \( \tau \) is any of the topologies \( \tau_{\infty}, w^*, w_{seq}^* \). In particular, \( \Gamma_{\tau}(F) \) is a level convex functional.

As a consequence of Theorem 3.1 we obtain the following representation result for the \( \tau \)-lower semicontinuous envelope of a supremal functional depending only on the gradient and positively \( 1 \)-homogeneous.

Theorem 3.2. Let \( \Omega \) be a bounded connected open set with Lipschitz continuous boundary and let \( F \) be a supremal functional represented by a Borel function \( f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R} \). Assume that

1. \( F \) is positively \( 1 \)-homogeneous, that is, \( F(\lambda u) = \lambda F(u) \) \( \forall \lambda > 0 \) \( \forall u \in W^{1,\infty}(\Omega) \);
2. \( \exists \alpha, \beta > 0 \) such that \( \forall x, y \in \Omega \) \( \alpha|x - y| \leq d_{\lambda}^F(x, y) \leq \beta|x - y|_{\Omega} \).

Then

\[
\Gamma_{\tau_{\infty}}(F) \equiv R_{d_{\lambda}^F}
\]

(3.7)

where \( \tau \) is any of the topologies \( \tau_{\infty}, w^*, w_{seq}^* \).

Remark 3.3. Note that under the hypotheses of Theorem 3.2 it can be proved that \( \inf_{W^{1,\infty}(\Omega)} F = 0 \) and \( \Gamma_{\tau}(F)(u) = \inf_{W^{1,\infty}(\Omega)} F = 0 \) if and only if \( u \) is constant. Moreover, Theorem 3.2 provides a significant generalization of the analogous result obtained in [14]. Indeed in [14] the function \( f(x, \xi) : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R} \) representing the functional \( F \) is assumed to be a Carathéodory function satisfying the conditions

(i) \( f(x, \lambda \xi) = |\lambda|f(x, \xi) \) \( \forall \lambda \in \mathbb{R}, \forall (x, \xi) \in \Omega \times \mathbb{R}^N \)
(ii) \( \exists C_1, C_2 > 0 \) such that \( C_1|\xi| \leq f(x, \xi) \leq C_2|\xi| \) \( \forall (x, \xi) \in \Omega \times \mathbb{R}^N \).

Here we remove the continuity and symmetry hypotheses on \( f \) with respect to the variable \( \xi \) and we weaken hypothesis (ii). More in detail, the linear growth condition on \( f \) from below is replaced by the request that the sublevel sets \( \{(x, \xi) : f(x, \xi) \leq \lambda \} \) are bounded in \( \Omega \times \mathbb{R}^N \), while the condition from above is fulfilled when \( f \) is bounded on bounded sets of \( \Omega \times \mathbb{R}^N \) (see Theorem 3.4 for a deeper insight).

Finally, we remark that, under the assumptions of Theorem 3.2, it can be proved that \( \Gamma_{\tau}(F) \) can be represented in a supremal form if and only if the distance \( d_{\lambda}^F \) is \( intrinsic \) according to the definition introduced by De Cecco and Palmieri in [12] [13] (see Proposition 2.5 in [14]).

We emphasize that, at price of hypothesis \( 3.3 \), Theorem 3.1 above yields a detailed description of the sublevel sets of \( \Gamma_{\tau}(F) \) that allows us, in Theorem 3.2, to characterize as a difference quotient the relaxed functional of \( 1 \)-homogeneous functionals. In the following theorem we replace assumption \( 3.3 \) with the a less restrictive hypothesis \( 3.8 \) and prove the level convexity of the relaxed functionals \( \Gamma_{\tau}(F) \) for a more general class of supremal functionals.

Theorem 3.4. Let \( \Omega \) be a bounded connected open set with Lipschitz continuous boundary and let \( F \) be the supremal functional represented by a Borel function \( f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R} \). Assume that there exists \( (u_n)_{n \in \mathbb{N}} \subseteq W^{1,\infty}(\Omega) \) such that, set \( u_\xi(x) := \xi \cdot x \), it holds

\[
\lim_{n \rightarrow \infty} F(u_n) = \inf_{W^{1,\infty}(\Omega)} F = \lim_{n \rightarrow \infty} \limsup_{\xi \rightarrow 0} F(u_n + u_\xi).
\]

(3.8)

Then \( \Gamma_{\tau}(F) \) is a level convex functional when \( \tau \) is one of the topologies \( \tau_{\infty}, w^*, w_{seq}^* \). In particular

\[
\Gamma_{w^*}(F) \equiv \Gamma_{w_{seq}^*}(F).
\]

Finally, as a corollary of Theorems 3.1 and 3.2 we provide the following result concerning the \( \Gamma \)-limits of sequence of supremal functionals.
Corollary 3.5. Let $\Omega$ be a bounded connected open set with Lipschitz continuous boundary and let $(F_n)_n$ be a sequence of supremal functionals
\begin{equation}
F_n(u) = \text{ess sup}_{x \in \Omega} f_n(x, Du(x)),
\end{equation}
represented by Borel functions $f_n : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$. For every $n \in \mathbb{N}$ assume that $F_n$ satisfies either \textbf{3.8} or \textbf{3.8}. If $F_n(\tau)$-converges to a functional $F : W^{1,\infty}(\Omega) \rightarrow \mathbb{R}$ when $\tau$ is one of the topologies $\tau_{\infty}, w^*, w^*_\text{seq}$, then $F$ is level convex.

Remark 3.6. Note that hypothesis \textbf{3.8} is satisfied in the following cases:
- when the supremand $f$ has a uniform modulus of continuity in $\xi$, that is, for any $M > 0$ there exists some modulus of continuity $\omega_M$ such that $|f(x, \xi) - f(x, \eta)| \leq \omega_M(\|\xi - \eta\|)$ for a.e. $x \in \Omega, \forall \xi, \eta \in B_M(0)$.

Such hypothesis has been already exploited in literature to prove necessary conditions to the weak* lower semicontinuity of $F$ (see [18]);
- when the supremand $f$ satisfies
\begin{equation}
f(x, 0) = 0 = \min_{\xi \in \mathbb{R}^N} f(x, \xi) \text{ for a.e. } x \in \Omega \text{ and } \lim_{\xi \rightarrow 0} \inf_{x \in \Omega} f(x, \xi) = 0,
\end{equation}
(as in the model case when $0 \leq f(x, \xi) \leq \alpha(\|\xi\|)$ for a.e. $x \in \Omega, \forall \xi \in \mathbb{R}^N$ for some positive $\alpha$). Indeed, it suffices to choose $u_0 = 0$ for $n \in \mathbb{N}$;
- when $f = f(\xi)$ is such that the level set $H_\lambda(f) := \{f(\xi) \leq \lambda\}$ has not empty interior for every $\lambda > \inf_{\mathbb{R}^N} f$. Indeed, let $(\lambda_n)$ be such that $\lambda_n \rightarrow \inf_{\mathbb{R}^N} f$ and choose $\eta_n$ in the interior of $H_{\lambda_n}(f)$. Then the sequence $u_n(x) := \eta_n \cdot x$ is a minimizing sequence and $\limsup_{n \rightarrow 0} F(u_n + u_\xi) = \limsup_{\xi \rightarrow 0} f(\eta_n + \xi) \leq \lambda_n$. Up to passing to the limit as $n \rightarrow +\infty$, we get \textbf{3.8}.

Remark 3.7. Note that Theorems \textbf{3.4} and \textbf{3.4} could be generalized to more general bounded sets $\Omega$ in $\mathbb{R}^d$ by requiring that any connected component $\Omega_i$ of $\Omega$ is Lipschitz regular with the constants $c_{\Omega_i}$ appearing in the inequality \textbf{2.1} and involving the geodetic distances associated to any connected component $\Omega_i$ are equibounded (and for Theorem \textbf{3.4} a local version of hypothesis \textbf{3.8} must be satisfied). Indeed, the proof follows by arguing separately on each connected component.

4. INTRINSIC DISTANCES AND RELATED RESULTS.

In this section we prove some key results instrumental for the proofs of Theorems \textbf{3.4} and \textbf{3.4}. According to the hypothesis \textbf{3.5} of Theorems \textbf{3.4} in the sequel we assume that the pseudo-distances $d^\lambda_F$ are metrically equivalent to the euclidean distance on $\Omega$. More in detail, for every $\lambda > \inf_{W^{1,\infty}(\Omega)} F$ $d^\lambda_F$ satisfies
\begin{equation}
\alpha(\lambda)|x - y| \leq d^\lambda_F(x, y) \leq \beta(\lambda)|x - y|_{\Omega} \quad \forall x, y \in \Omega
\end{equation}
for positive $\alpha(\lambda), \beta(\lambda)$. Note that the validity of the right hand-side of the inequality above ensures that the (non symmetric) distance $d^\lambda_F$ is non-degenerate, that is
\begin{equation}
d^\lambda_F(x, y) = 0 \iff x = y.
\end{equation}
In particular, for every $u \in W^{1,\infty}(\Omega)$ and for every $\lambda > \inf_{W^{1,\infty}(\Omega)} F$, we can rewrite the different quotient as
\begin{equation}
R_{d^\lambda_F}(u) = \sup_{x, y \in \Omega, x \neq y} \frac{u(x) - u(y)}{d^\lambda_F(x, y)}
\end{equation}
and, by using Lemma \textbf{2.1} we also get that
\begin{equation}
\frac{1}{\beta(\lambda)}|Du|_\infty \leq R_{d^\lambda_F}(u) \leq \frac{1}{\alpha(\lambda)C_{\Omega}}|Du|_\infty
\end{equation}
where $C_{\Omega}$ is the constant in \textbf{2.1}. First of all we state some comparison results inherited straightforward by the definition that will be useful in the sequel.
(i) if $F, G : W^{1, \infty}(\Omega) \to \mathbb{R}$ and $F \leq G$, then $\forall \lambda > \inf_{W^{1, \infty}(\Omega)} G$ it holds
\[ d^\lambda_{G}(x, y) \leq d^\lambda_{F}(x, y) \quad \forall x, y \in \Omega, \tag{4.2} \]
and
\[ R_{d^\lambda_{F}}(u) \leq R_{d^\lambda_{G}}(u) \quad \forall u \in W^{1, \infty}(\Omega); \tag{4.3} \]

(ii) if $\lambda \leq \lambda'$, then $\forall \lambda > \inf_{W^{1, \infty}(\Omega)} F$ it holds
\[ d^\lambda_{F}(x, y) \leq d^{\lambda'}_{F}(x, y) \quad \forall x, y \in \Omega, \]
\[ R_{d^\lambda_{F}}(u) \leq R_{d^{\lambda'}_{F}}(u) \quad \forall u \in W^{1, \infty}(\Omega). \]

The following proposition follows the same guidelines of Lemma 3.4 in [14] up to some suitable changes necessary to deal with the lack of symmetry and linear growth of $F$.

**Proposition 4.1.** Let $\Omega$ be a bounded connected open set with Lipschitz continuous boundary and let $F : W^{1, \infty}(\Omega) \to \mathbb{R}$ be a supremal functional of the form $F(u) = \sup_{x \in \Omega} f(x, Du(x))$ represented by a Borel function $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$. Assume that $(d^\lambda_{F})_\lambda$ satisfies (3.3). Then for any $v \in W^{1, \infty}(\Omega)$ such that $R_{d^\lambda_{F}}(v) < 1$, there exists a sequence $\{v_n\} \subset W^{1, \infty}(\Omega)$ converging to $v$ in $L^\infty$ with $F(v_n) \leq \lambda$.

**Proof.** Note that for any $x, y \in \Omega$ it holds
\[ d^\lambda_{F}(x, y) = d^\lambda_{F/m}(x, y) \quad \forall m \in \mathbb{R}, \quad \forall \lambda > \inf_{W^{1, \infty}(\Omega)} F. \tag{4.4} \]
We claim that, without loss of generality, we may confine to prove the thesis for $\lambda$ positive. Indeed, let $\lambda > \inf_{W^{1, \infty}(\Omega)} F$ and $v \in W^{1, \infty}(\Omega)$ such that $R_{d^\lambda_{F}}(v) < 1$ be fixed and select $\lambda' \in \mathbb{R}$ such that $\inf_{W^{1, \infty}(\Omega)} F < \lambda' < \lambda$. Then, thanks to (4.4) with $m = \lambda'$ we have $R_{d^{\lambda'-\lambda}_{F}}(v) = R_{d^\lambda_{F}}(v) < 1$. If we find $\{v_n\} \subset W^{1, \infty}(\Omega)$ converging to $v$ in $L^\infty$ with $(F - \lambda')(v_n) \leq \lambda - \lambda'$ then we also deduce that $F(v_n) \leq \lambda'$. Since $\lambda - \lambda' > 0$, we may then assume directly $\lambda$ to be strictly positive. Up to replace $F$ with $F/\lambda$ it is not restrictive to treat the case $\lambda = 1$ and drop the dependence on $\lambda$ in the notation of the associate distance and of $\alpha(\lambda), \beta(\lambda)$.

Let $w \in W^{1, \infty}(\Omega)$ satisfying $F(w) \leq 1$ then, by definition of $d_F(x, y)$, for $x, y \in \Omega$ we have that
\[ w(x) - w(y) \leq d_F(x, y) \leq \beta|x - y| \]
and, switching the role of $x, y$, we deduce
\[ |w(x) - w(y)| \leq \beta|x - y|. \]
Thanks to Lemma 2.1 this implies
\[ \|Du\| \leq \beta. \tag{4.5} \]

Now let us fix $v \in W^{1, \infty}(\Omega)$ such that $R_{d_F}(v) < 1$ and a positive radius $r > 0$. Let $0 < \theta < 1$ such that $R_{d_F}(v) = 1 - \theta < 1$. Then, thanks to the assumption (3.3), for every $x, y \in \Omega$ with $|x - y| = r$ we have that
\[ v(y) - v(x) \leq (1 - \theta)d_F(y, x) = d_F(y, x) - \theta d_F(x, y) \leq d_F(y, x) - \alpha|\theta|x - y| < d_F(y, x) - r\alpha. \]
In particular
\[ v(y) - v(x) < d_F(y, x) - r\alpha. \tag{4.6} \]

Let us fix $0 < \varepsilon < r \min\{\frac{1}{\theta} \alpha, \beta + \|Du\| \infty\}$. For every $x \in \Omega$ and for every $y \in \partial B_r(x) \cap \Omega$, by the definition of $d_F$ there exists a function $w^r_x \in W^{1, \infty}(\Omega)$ such that
1) $F(w^r_x, y, \Omega) \leq 1$;
2) $w^r_x(y) \geq w^r_{x'}(y) + d_F(x, y) - \varepsilon$;
3) $w^r_x(x) = v(x)$;
the third property being possible thanks to the translation invariance of the first two.

By properties 2), 3) and by (4.6), for every $y \in \partial B_r(x) \cap \Omega$ we have that
\[ w^r_x(y) \geq w^r_{x'}(y) + d_F(y, x) - \varepsilon = v(x) + d_F(y, x) - \varepsilon > v(y) + r\theta|\alpha - \varepsilon \]
that is
\[ w^r_{x'}(y) - v(y) > r\theta|\alpha - \varepsilon. \]
for every $y \in \partial B_r(x) \cap \Omega$. Thanks to (4.5), for $\delta = \frac{\epsilon}{r + \|v\|_{L^\infty}}$, we have that for every $z \in \Omega$ such that $|z - y|_\Omega \leq \delta$ we have that
\[
 w^{x,y}_r(z) - v(z) > w^{x,y}_r(y) - v(y) - (\beta + \|Dv\|_{L^\infty})|z - y|_\Omega \geq r\theta\alpha - 2\epsilon,
\]
that is
\[
 w^{x,y}_r(z) > v(z) + \varepsilon \quad \forall y \in \partial B_r(x) \cap \Omega, \forall z \in \Omega : |z - y|_\Omega \leq \delta.
\] (4.7)
Moreover, since $w^{x,y}_r(x) = v(x)$, we have that for every $z \in \Omega$ such that $|z - x|_\Omega < \delta$
\[
 w^{x,y}_r(z) \leq v(z) + (\beta + \|Dv\|_{L^\infty})|z - x|_\Omega < v(z) + \epsilon.
\] (4.8)
Note that, since $\delta \leq \frac{\epsilon}{r}$, we have that $\forall x, y \in \Omega$ such that $|x - y| = r$ it holds $B_\delta(x) \cap B_\delta(y) \cap \Omega = \emptyset$. Moreover the family
\[
 \{B_\delta(y) : y \in \Omega \cap \partial B_r(x)\}
\]
is an open covering of the pre-compact set $\partial B_r(x) \cap \Omega$. For every $x \in \Omega$, let us fix a finite set of points \( \{y_1, \ldots, y_N\} \) on $\partial B_r(x) \cap \Omega$ such that
\[
 \partial B_r(x) \cap \Omega \subset \bigcup_{i=1}^{N} B_\delta(y_i),
\]
and let us set the function $w^x_r : \Omega \to \mathbb{R}$ defined by
\[
 w^x_r(z) := \max_i w^{x,y_i}_r(z).
\] (4.9)
By construction, the following facts hold:
\begin{enumerate}
\item $\text{ess sup}_{B_r(x) \cap \Omega} f(z, Dw^x_r(z)) \leq 1$;
\item $w^x_r(z) > v(z) + \varepsilon$ for every $z \in \partial B_r(x) \cap \Omega$
    (in fact, if $z \in \partial B_r(x) \cap \Omega$ then there exists $y_i$ such that $z \in B_\delta(y_i)$. Therefore by (4.7)
    $w^x_r(z) \geq w^{x,y_i}_r(z) > v(z) + \varepsilon$);
\item $w^x_r(z) < v(z) + \varepsilon$ for every $z \in B_\delta(x) \cap \Omega$ (it follows by (4.8)).
\end{enumerate}
Now let $Z_r$ be a finite set of points of $\Omega$ such that
\[
 \Omega \subset \bigcup_{z \in Z_r} B_\delta(z),
\]
and consider the function $w_r : \bigcup_{z \in Z_r} B_r(z) \cap \Omega \to \mathbb{R}$ defined by
\[
 w_r(x) := \min_{z \in Z_r \cap \partial B_r(x)} w^z_r(x).
\] (4.10)
By properties 2) and 3) above it follows that $w_r$ is continuous on $\Omega$ (one may proceed by induction on the cardinality of $Z_r$).

Moreover, for almost every $x$ in $\Omega$, $Dw_r(x)$ coincides with $Dw^x_r(x)$ for some $z \in Z_r$ and this implies that $w_r \in W^{1,\infty}(\Omega)$ and $F(w_r) \leq 1$.

Now let us prove that $\|w_r - v\|_{L^\infty} \to 0$. To this aim, let us fix $x \in \Omega$, and let $z \in B_r(x) \cap \Omega$ be such that $w_r(x) = w^z_r(x)$. Recalling that by construction $w^z_r(z) = v(z)$, by using (3.7) and Lemma 2.1 and by the regularity of $\partial \Omega$ we conclude that there exists a constant $C_\Omega > 0$ such that
\[
 |w_r(x) - v(x)| \leq |w^z_r(x) - w^z_r(z)| + |w^z_r(z) - v(x)|
 = |w^z_r(x) - w^z_r(z)| + |v(z) - v(x)|
 \leq \max\{d_F(x,z), d_F(z,x)\} + \max\{d_F(x,z), d_F(z,x)\} \leq 2C_\Omega\beta r.
\]
Therefore, the sequence $v_n := w_{1/n}$ satisfies the thesis. \qed

In the following proposition we provide a characterization of functional $F$ on $W^{1,\infty}(\Omega)$ whose $d_F^\lambda$ satisfies
\[
 d_F^\lambda(x,y) \leq \beta(\lambda)|x - y|_\Omega \quad \forall x, y \in \Omega.
\] (4.11)

**Proposition 4.2.** Let $\Omega \subseteq \mathbb{R}^N$ be a connected open set and let $F : W^{1,\infty}(\Omega) \to \mathbb{R}$ be a functional. Let $\lambda > \inf_{W^{1,\infty}(\Omega)} F$. Then the following facts are equivalent:
(i) there exists $\beta = \beta(\lambda) > 0$ such that (4.11) holds;
(ii) there exists $\beta = \beta(\lambda) > 0$ such that

$$\{u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda\} \subseteq \{u \in W^{1,\infty}(\Omega) : \|Du\|_{\infty} \leq \beta(\lambda)\}.$$  

(4.12)

Proof. (i) $\implies$ (ii) Let $u \in W^{1,\infty}(\Omega)$ be such that $F(u) \leq \lambda$. Then for every $x, y \in \Omega$

$$|u(x) - u(y)| \leq d^F_p(x, y) \vee d^F_p(y, x) \leq \beta(\lambda)|x - y|_{\Omega}$$

and, thanks to Lemma 2.1, this implies $\|Du\|_{\infty} \leq \beta(\lambda)$.

(ii) $\implies$ (i) Thanks to Lemma 2.1, for every $u \in E_\lambda$, we have that

$$u(x) - u(y) \leq \|Du\|_{\infty}|x - y|_{\Omega} \leq \beta(\lambda)|x - y|_{\Omega}$$

hence passing to the supremum on $u \in E_\lambda$ we get

$$d^F_p(x, y) \leq \beta(\lambda)|x - y|_{\Omega}.$$  

Remark 4.3. Note that, under assumption (4.12) or, equivalently, under assumption (4.11), by Remark 2.6(2)-(3) we have that

$$\Gamma_{w^\ast}(F) \leq \Gamma_{w^\ast, seq}(F) = \Gamma_{\tau^\infty}(F)$$  

and, taking into account (4.2), for every $\forall \lambda > \inf_{W^{1,\infty}(\Omega)} F$ it holds

$$d^F_{\Gamma_{w^\ast}(F)}(x, y) \geq d^F_{\Gamma_{w^\ast, seq}(F)}(x, y) = d^F_{\Gamma_{\tau^\infty}(F)}(x, y) \geq d^F_{\lambda}(x, y) \quad \forall x, y \in \Omega.$$  

We emphasize that in general, for any $\tau \in \{\tau^\infty, w^\ast, w^*_{seq}\}$, the distance $d^F_{\tau}(F)$ does not coincide with $d^F_{\lambda}$ (see Example 2.2).

In the following proposition we prove that, under assumption (4.12), the distances $d^F_{\tau}(F)$ can be obtained as infimum of the distances associated to $F$ for any $\tau \in \{\tau^\infty, w^*_{seq}\}$. In particular, when (3.3) holds, we get that the different quotients associated to the relaxed functional $\Gamma_{\tau}(F)$ can be obtained as supremum of the different quotients associated to $F$ (see (4.17) for a precise statement).

Afterwards, in Proposition 4.4, we will exploit (4.10) in order to provide a characterization of functionals $F$ on $W^{1,\infty}(\Omega)$ whose $(d^F_p)_{\lambda}$ satisfies

$$d^F_p(x, y) \geq \alpha(\lambda)|x - y| \quad \forall x, y \in \Omega.$$  

(4.14)

We underline that in all these results a relevant hypothesis is that $F$ is translation invariant, that is

$$F(u + c) = F(u) \quad \forall u \in W^{1,\infty}(\Omega), \forall c \in \mathbb{R}.$$  

(4.15)

while we do not need a priori $F$ to be a supremal functional.

Proposition 4.4. Let $\Omega \subseteq \mathbb{R}^N$ be a connected open set with Lipschitz continuous boundary and let $F : W^{1,\infty}(\Omega) \to \mathbb{R}$ be a translation invariant functional satisfying (4.12) for every $\lambda > \inf_{W^{1,\infty}(\Omega)} F$. Then for every $\lambda > \inf_{W^{1,\infty}(\Omega)} F$ and for any $\tau \in \{\tau^\infty, w^*_{seq}\}$ we have

$$\inf_{\lambda' > \lambda} d^\lambda_{\tau}(F)(x, y) = d^\lambda_{\tau, seq}(F)(x, y) \quad \forall x, y \in \Omega.$$  

(4.16)

In addition, if $(d^\lambda_{\tau})_{\lambda}$ satisfies (4.5) holds, then for every $\lambda > \inf_{W^{1,\infty}(\Omega)} F$ and for any $\tau \in \{\tau^\infty, w^*_{seq}\}$ we have

$$R_{d^\lambda_{\tau, seq}(F)}(u) = \sup_{\lambda' > \lambda} R_{d^\lambda_{\tau}}(u) \quad \forall u \in W^{1,\infty}(\Omega).$$  

(4.17)

Proof. By Remark 4.3 it holds $\Gamma_{\tau^\infty}(F) = \Gamma_{w^*_{seq}}(F)$, hence it is sufficient to show the thesis when $\tau = \tau^\infty$.

In order to prove (4.16), let us fix $\lambda > \inf_{W^{1,\infty}(\Omega)} F$. Let $v \in W^{1,\infty}(\Omega)$ be such that $\Gamma_{\tau^\infty}(F)(v) \leq \lambda$. Then for $\lambda' > \lambda$ there exists $(v_n)_n \subseteq W^{1,\infty}(\Omega)$ uniformly converging to $v$ such that $F(v_n) \leq \lambda'$ for every $n \in \mathbb{N}$. Thus, for all $x, y \in \Omega$, passing to the limit in the inequality

$$d^\lambda_{\tau}(x, y) \geq v_n(x) - v_n(y) \quad \forall n \in \mathbb{N}, \forall x, y \in \Omega$$

we have

$$d^\lambda_{\tau}(x, y) \geq v(x) - v(y) \quad \forall x, y \in \Omega.$$
Hence
\[
d_F^\lambda(x,y) \geq \sup\{v(x) - v(y) : \Gamma_\infty(F)(v) \leq \lambda\} = d_{\Gamma_\infty(F)}^\lambda(x,y)
\]
and passing to the infimum, it follows
\[
\inf_{\lambda' > \lambda} d_F^{\lambda'}(x,y) \geq d_{\Gamma_\infty(F)}^\lambda(x,y) \quad \forall x,y \in \Omega.
\]

Vice versa, let us fix \(x,y \in \Omega\) and let \((\lambda_n)\) be a decreasing sequence converging to \(\lambda\) such that
\[
\lim_{n \to \infty} d_{\Gamma_\infty(F)}^\lambda(x,y) = \inf_{\lambda' > \lambda} d_F^{\lambda'}(x,y).
\]
Let us choose \(u_n \in W^{1,\infty}(\Omega)\) such that \(F(u_n) \leq \lambda_n\) with \(d_F^{\lambda_n}(x,y) \leq u_n(x) - u_n(y) + \frac{1}{n}\). Thanks to \textbf{(4.15)} we can assume that \(\int_{\Omega} u_n dx = 0\) for every \(n \in \mathbb{N}\). Since the sequence \((u_n) \subseteq W^{1,\infty}(\Omega)\) has bounded gradients, then, up to a subsequence, \((u_n)\) uniformly converges to a function \(u_0 \in W^{1,\infty}(\Omega)\). By lower semicontinuity we have \(\Gamma_\infty(F)(u_0) \leq \lambda\) and
\[
\inf_{\lambda' > \lambda} d_F^{\lambda'}(x,y) = \lim_{n \to +\infty} u_n(x) - u_n(y) = u_0(x) - u_0(y) \leq d_{\Gamma_\infty(F)}^\lambda(x,y).
\]

In order to show \textbf{(4.17)} we note that, if \(\lambda > \inf_{W^{1,\infty}(\Omega)} F\), thanks to \textbf{4.2} and \textbf{3.5}, we have that
\[
\alpha(\lambda)|x - y| \leq d_F^\lambda(x,y) \leq d_{\Gamma_\infty(F)}^\lambda(x,y).
\]
Therefore
\[
d_{\Gamma_\infty(F)}^\lambda(x,y) = 0 \iff x = y
\]
and thus, for every \(u \in W^{1,\infty}(\Omega)\), we have that
\[
R_{d^\lambda_{\Gamma_\infty(F)}}(u) = \sup_{x,y \in \Omega, x \neq y} \frac{u(x) - u(y)}{d_F^\lambda(x,y)} = \sup_{x,y \in \Omega, x \neq y} \frac{u(x) - u(y)}{\inf_{\lambda' > \lambda} d_F^{\lambda'}(x,y)} = \sup_{\lambda' > \lambda} \sup_{\lambda' > \lambda, x,y \in \Omega, x \neq y} \frac{u(x) - u(y)}{d_F^{\lambda'}(x,y)} = \sup_{\lambda' > \lambda} R_{d_F^{\lambda'}}(u).
\]

\[\Box\]

**Proposition 4.5.** Let \(\Omega\) be a bounded connected open set with Lipschitz continuous boundary and let \(F : W^{1,\infty}(\Omega) \to \mathbb{R}\) be a translation invariant functional satisfying \textbf{(4.12)} for every \(\lambda > \inf_{W^{1,\infty}(\Omega)} F\). If
\[
\lim_{\xi \to 0} \Gamma_\tau(F)(u_\xi) = \inf_{W^{1,\infty}(\Omega)} F
\]
where \(u_\xi(x) := \xi \cdot x\) and \(\tau \in \{\tau_\infty, w^{*}_{seq}\}\), then for every \(\lambda > \inf_{W^{1,\infty}(\Omega)} F\) there exists \(\alpha(\lambda) > 0\) such that \textbf{(4.14)} holds.

**Proof.** Let \(\lambda > \inf_{W^{1,\infty}(\Omega)} F\) be fixed and choose any value \(\tilde{\lambda} \in \mathbb{R}\) such that \(\inf_{W^{1,\infty}(\Omega)} F < \tilde{\lambda} < \lambda\). By the definition of limit there exists \(\delta > 0\), depending only on \(\tilde{\lambda}\), such that for every \(\xi \in \mathbb{R}^N\) such that \(|\xi| \leq \delta\) it holds \(\Gamma_\infty(F)(u_\xi) \leq \tilde{\lambda}\). Set \(\alpha = \alpha(\tilde{\lambda}) = \delta/2\). We claim that for every \(x,y \in \Omega, x \neq y\)
\[
d_F(x,y) \geq \alpha|x - y|.
\]
Indeed, let \(x,y \in \Omega, x \neq y\) and define \(\eta := \alpha\frac{(x-y)}{|x-y|}\). Since \(|\eta| = \alpha < \delta\) we get
\[
\Gamma_\infty(F)(u_\eta) \leq \tilde{\lambda}.
\]
By definition of \(d_{\Gamma_\infty(F)}^\lambda(x,y)\) and by \textbf{(4.10)}, this implies that
\[
d_F^\lambda(x,y) \geq \inf_{\gamma > \lambda} d_F^\gamma(x,y) = d_{\Gamma_\infty(F)}^\lambda(x,y) \geq u_\eta(x) - u_\eta(y) = \alpha|x - y|.
\]

\[\Box\]

Finally we can give a characterization of supremal functionals \(F\) whose distances \(d_F^\lambda\) satisfy \textbf{3.5}.
Theorem 4.6. Let $\Omega$ be a bounded connected open set with Lipschitz continuous boundary and let $F$ be a supremal functional of the form (3.1). Then the following facts are equivalent

(i) $(d_F^\lambda)_\lambda$ satisfies (4.5);
(ii) $F$ satisfies (4.12) $\forall \lambda > \inf_{W^{1,\infty}(\Omega)} F$ and

$$\lim_{\xi \to 0} \Gamma_\tau(F)(u_\xi) = \inf_{W^{1,\infty}(\Omega)} F = \Gamma_\tau(F)(0) \quad \forall \tau \in \{\tau_\infty, w_{seq}^*\}. \quad (4.18)$$

Proof. Assume that $(d_F^\lambda)_\lambda$ satisfies (4.5). By Proposition 4.2, we get that $F$ satisfies (4.12) $\forall \lambda > \inf_{W^{1,\infty}(\Omega)} F$. Moreover by Remark 4.3, it is sufficient to show (4.18) with $\tau = \tau_\infty$. Fix $\lambda > \inf_{W^{1,\infty}(\Omega)} F$. Since

$$R_{d_F^\lambda}(u_\xi) = \sup_{x,y \in \Omega, x \neq y} \frac{u_\xi(x) - u_\xi(y)}{\frac{1}{\lambda} d_F^\lambda(x,y)} \leq |\xi| \sup_{x,y \in \Omega, x \neq y} \frac{|x-y|}{\frac{1}{\lambda} d_F^\lambda(x,y)} \leq |\xi| \sup_{x,y \in \Omega, x \neq y} \frac{|x-y|}{\alpha(\lambda)|x-y|} = \frac{|\xi|}{\alpha(\lambda)},$$

if $|\xi| < \alpha(\lambda)$ then, by Proposition 3.1, there exists $(u_n^\xi)_n \subseteq W^{1,\infty}(\Omega)$ such that $u_n^\xi \to u_\xi$ uniformly and $F(u_n^\xi) \leq \lambda$ for any $n \in \mathbb{N}$. This implies

$$\Gamma_{\tau_\infty}(F)(u_\xi) \leq \liminf_{n \to \infty} F(u_n^\xi) \leq \lambda$$

for every $|\xi| < \alpha(\lambda)$. Up to passing to the limsup as $\xi \to 0$ we deduce that

$$\limsup_{\xi \to 0} \Gamma_{\tau_\infty}(F)(u_\xi) \leq \lambda.$$

Eventually, we let $\lambda \to \inf_{W^{1,\infty}(\Omega)} F$ and we get that

$$\inf_{W^{1,\infty}(\Omega)} F \leq \Gamma_{\tau_\infty}(F)(0) \leq \liminf_{\xi \to 0} \Gamma_{\tau_\infty}(F)(u_\xi) \leq \limsup_{\xi \to 0} \Gamma_{\tau_\infty}(F)(u_\xi) \leq \inf_{W^{1,\infty}(\Omega)} F.$$

The converse implication follows by Propositions 4.2 and 4.3. \qed

Remark 4.7. We remark that hypotheses (4.11), (4.14) can be easily inferred for functionals $F$ with linear growth from above and below respectively. Indeed, if there exists $\beta > 0$ such that

$$F(u) \geq \beta \|Du\|_{L^\infty(\Omega)} \quad \forall u \in W^{1,\infty}(\Omega)$$

then for every $\lambda > 0$ and for every $x,y \in \Omega$ it holds

$$d_F^\lambda(x,y) \leq \frac{\lambda}{\beta} |x-y|.$$ 

On the other hand if there exists $\alpha > 0$ such that

$$F(u) \leq \alpha \|Du\|_{L^\infty(\Omega)} \quad \forall u \in W^{1,\infty}(\Omega)$$

then, for every $\lambda > 0$, $\alpha(\lambda)$ in (4.14) can be chosen equal to $\frac{\lambda}{\alpha}$. We conclude this section by giving an alternative representation for $d_F^\lambda$. To this aim we introduce for any subset $A$ of a topological vector space $(X, \tau)$

$$\tau\text{-co}(A) := \bigcap\{B \text{ $\tau$-closed and convex subset of } X : B \supseteq A\}. \quad (4.19)$$

Proposition 4.8. Let $\Omega \subseteq \mathbb{R}^N$ be a connected open set with Lipschitz continuous boundary. Let $F : W^{1,\infty}(\Omega) \to \mathbb{R}$ be a functional and let $\tau$ be one of the topologies $\tau_\infty, w^*, w_{seq}^*$. Then $\forall \lambda > \inf_{W^{1,\infty}(\Omega)} F$, \forall $x,y \in \Omega$ it holds

$$d_F^\lambda(x,y) = \sup\{u(x) - u(y) : u \in \tau\text{-co}\{v \in W^{1,\infty}(\Omega) : F(v) \leq \lambda\}\}, \quad (4.20)$$

Moreover, if $F$ is a supremal functional and $(d_F^\lambda)_\lambda$ satisfies (4.5) then

$$\tau\text{-co}\{u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda\} = \{u \in W^{1,\infty}(\Omega) : R_{d_F^\lambda}(u) \leq 1\}. \quad (4.21)$$
Proof. Note that, thanks to the Banach-Dieudonné-Krein-Smulian Theorem (see [5, Theorem 3.33]) and the definition of $\tau$-$co(A)$, $u^*$-$co(A) = u^*_{\text{seq}}$-$co(A)$ for any given set $A \subseteq W^{1,\infty}(\Omega)$. Hence it suffices to show (4.20) and (4.21) for $\tau \in \{\tau_{\infty}, w^*_{\text{seq}}\}$.

Moreover, (4.20) follows straightforwardly once we prove that
\[
\sup\{u(x) - u(y) : u \in A\} = \sup\{u(x) - u(y) : u \in \tau$-$co(A)\}
\] for any given set $A \subseteq W^{1,\infty}(\Omega)$. To this aim let us define
\[
\text{co}(A) := \left\{ \sum_{i=1}^{m} \lambda_i u_i : \lambda_i \geq 0, u_i \in A, \sum_{i=1}^{m} \lambda_i = 1, m \in \mathbb{N} \right\}.
\]

It can be easily verified that $\text{co}(A)$ is a convex set containing $A$ and the $\tau$-closure of $\text{co}(A)$ coincides with $\tau$-$co(A)$ for $\tau \in \{\tau_{\infty}, w^*_{\text{seq}}\}$. For such topologies one may argue by sequences and prove easily that
\[
\sup\{u(x) - u(y) : u \in \text{co}(A)\} = \sup\{u(x) - u(y) : u \in \tau$-$co(A)\}.
\]

Thus it suffices to show that
\[
\sup\{u(x) - u(y) : u \in A\} = \sup\{u(x) - u(y) : u \in \text{co}(A)\}.
\]

One inequality follows by comparison. To get the converse inequality, it is enough to note that for any convex combination $\sum_{i=1}^{m} \lambda_i u_i \in \text{co}(A)$ and for any $x, y \in \Omega$ we have
\[
\sum_{i=1}^{m} \lambda_i u_i(x) - \sum_{i=1}^{m} \lambda_i u_i(y) = \sum_{i=1}^{m} \lambda_i (u_i(x) - u_i(y)) \leq \sum_{i=1}^{m} \lambda_i \sup\{u(x) - u(y) : u \in A\} = \sup\{u(x) - u(y) : u \in A\}.
\]

It remains to show (4.21). To this aim, let $\tau \in \{\tau_{\infty}, w^*_{\text{seq}}\}$ and let $u \in W^{1,\infty}(\Omega)$ be such that $F(u) \leq \lambda$. Then, by definition of $d^*_{\phi}$, it holds $R_{d^*_{\phi}}(u) \leq 1$. This in turn implies that $\{u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda\} \subseteq \{u \in W^{1,\infty}(\Omega) : R_{d^*_{\phi}}(u) \leq 1\}$ and, since $R_{d^*_{\phi}}$ is a convex functional, we get that
\[
\tau$-$co\left(\{u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda\}\right) \subseteq \{u \in W^{1,\infty}(\Omega) : R_{d^*_{\phi}}(u) \leq 1\}.
\]

On the other hand, if $R_{d^*_{\phi}}(v) \leq 1$ then, for $0 < \epsilon < 1$, $R_{d^*_{\phi}}(\epsilon v) = \epsilon R_{d^*_{\phi}}(v) < 1$ and, by Proposition 4.1, we have that there exists a sequence $\{v_n\} \subseteq W^{1,\infty}(\Omega)$ converging to $\epsilon v$ in $L^{\infty}$ with $F(v_n) \leq \lambda$. Then $\epsilon v \in \tau$-$co\left(\{u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda\}\right)$ that implies, when $\epsilon \to 1$, that $v \in \tau$-$co\left(\{u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda\}\right)$.

\[\square\]

5. The proofs

Proof of Theorem 3.1. We first prove (3.6) for $\tau = \tau_{\infty}$. Let $\lambda > \inf_{W^{1,\infty}(\Omega)} F = \inf_{W^{1,\infty}(\Omega)} \Gamma_{\tau}(F)$ be fixed.

We note that, by definition,
\[
\{u \in W^{1,\infty}(\Omega) : \Gamma_{\tau}(F)(u) \leq \lambda\} \subseteq \{u \in W^{1,\infty}(\Omega) : R_{d^*_{\phi}}(u) \leq 1\}.
\]

Moreover, by (4.17) in Proposition 4.4 it easily follows that
\[
\{u \in W^{1,\infty}(\Omega) : R_{d^*_{\phi}}(u) \leq 1\} = \{u \in W^{1,\infty}(\Omega) : \sup_{\lambda' > \lambda} R_{d^*_{\phi}}(u) \leq 1\}.
\]

If we prove that
\[
\{u \in W^{1,\infty}(\Omega) : \sup_{\lambda' > \lambda} R_{d^*_{\phi}}(u) \leq 1\} \subseteq \{u \in W^{1,\infty}(\Omega) : \Gamma_{\tau}(F)(u) \leq \lambda\}
\]

then, by (3.1), (3.2) and (3.3), we get (3.6).

Let $u \in W^{1,\infty}(\Omega)$ be such that $R_{d^*_{\phi}}(u) \leq 1$ for every $\lambda' > \lambda$. Then, for every fixed $0 < \theta < 1$ we have that
\[
R_{d^*_{\phi}}(\theta u) = \theta R_{d^*_{\phi}}(u) \leq \theta < 1.
\]
With fixed $\lambda' > \lambda$ we apply Proposition 4.1 and get that there exists a sequence $(u_n^{\lambda',\theta})_n \subset W^{1,\infty}(\Omega)$ converging to $u$ in $L^\infty(\Omega)$ such that $F(\theta u_n^{\lambda',\theta}) \leq \lambda'$. Then $\Gamma_\tau(F)(\theta u) \leq \lambda'$ for every $\lambda' > \lambda$ and for every $0 < \theta < 1$ which implies $\Gamma_\tau(F)(\theta u) \leq \lambda$. Letting $\theta \to 1^-$, by lower semicontinuity, we get $\Gamma_\tau(F)(u) \leq \lambda$.

Now we claim that the functional $\Gamma_\tau(F)$ is level convex.

Indeed, since $R_{d_{\Gamma}^1(F)}$ is convex, we conclude that the sublevel sets of $\Gamma_\tau(F)$ are convex for $\lambda > \inf_{W^{1,\infty}(\Omega)} F$. For $\lambda = \inf_{W^{1,\infty}(\Omega)} F$ it suffices to note that

$$\left\{ u \in W^{1,\infty}(\Omega) : \Gamma_\tau(F)(u) = \inf_{W^{1,\infty}(\Omega)} F \right\} = \bigcap_{\lambda > \inf_{W^{1,\infty}(\Omega)} F} \left\{ u \in W^{1,\infty}(\Omega) : \Gamma_\tau(F)(u) \leq \lambda \right\}.$$

Finally, thanks to Proposition 4.2 the functional $F$ satisfies (4.12) and, by applying Remark 2.6(2), we have that $\Gamma_{w_{seq}}(\tau \lambda d) \equiv \Gamma_{\tau \lambda}$ for $\lambda > 0$. Therefore $\Gamma_{w_{seq}}(\tau \lambda)$ is level convex and by Proposition 2.13 we can conclude that

$$\Gamma_{\tau \lambda}(F) \equiv \Gamma_{w_{\tau \lambda}}(F) \equiv \Gamma_{w_{\tau \lambda}^*}(F).$$

Hence (3.6) holds for any $\tau \in \{ \tau_{\infty}, w^*, w_{seq}^* \}$.

**Proof of Theorem 3.3** First of all we note that given a functional $G$ positively 1-homogeneous for any strictly positive value $\lambda > \inf_{W^{1,\infty}(\Omega)} G$ and $\forall x, y \in \Omega$ it holds

$$d_G^1(x, y) = \sup \{ u(x) - u(y) : G(u) \leq \lambda \} = \lambda \sup \left\{ \frac{u(x)}{\lambda} - \frac{u(y)}{\lambda} : G\left(\frac{u}{\lambda}\right) \leq 1 \right\} = \lambda d_G^1(x, y).$$

In particular the scaling in (5.3) can be applied for $\lambda > 0$ both to $F$ and $\Gamma_\tau(F)$, as it can be easily verified that $\Gamma_\tau(F)$ is also positively 1-homogeneous.

We now claim that a supremal functional $F$ satisfying hypotheses (1), (2) is non negative. In order to prove this claim we assume in the sequel $\tau = \tau_{\infty}$.

Hence we may apply Proposition 4.4 and obtain by (4.16) the equality

$$d_{\Gamma}^1(F)(x, y) = \inf_{\lambda > 1} d_F^1(x, y) = \inf_{\lambda > 1} \lambda d_F^1(x, y) = d_F^1(x, y) \quad \forall x, y \in \Omega.$$  

By using hypothesis (2) on $d_F^1 = d_{\Gamma}^1(F)$, by Proposition 4.2 we infer that the sublevel set$$\{ u \in W^{1,\infty}(\Omega) : \Gamma_\tau(F)(u) \leq 1 \}$$consists of functions with equibounded gradients. Moreover this property is inherited by inclusion by any sublevel set$$\{ u \in W^{1,\infty}(\Omega) : \Gamma_\tau(F)(u) \leq \lambda \}$$for $\lambda < 1$.

Let us assume by contradiction that $\inf_{W^{1,\infty}(\Omega)} F < 0$. By (4.18) in Proposition 4.6 we get that

$$\lim_{\xi \to 0} \Gamma_\tau(F)(u_\xi) = \Gamma_\tau(F)(0) = \inf_{W^{1,\infty}(\Omega)} F < 0.$$

Hence, there exists $\xi_0 \neq 0$ such that $\Gamma_\tau(F)(u_\xi) = \eta < 0$. By the positively 1-homogeneity we also infer that $\forall \mu > 1 \Gamma_\tau(F)(u_\mu) = \Gamma_\tau(F)(\mu u_\xi) = \mu \eta < \eta$ and this leads to a contradiction since the set$$\{ u_{\eta} \in W^{1,\infty}(\Omega) : \Gamma_\tau(F)(u) \leq \eta \}$$consists of functions with equibounded gradients and thus it cannot be contained in the sublevel set$$\{ u \in W^{1,\infty}(\Omega) : \Gamma_\tau(F)(u) \leq \eta \}.$$

Hence, the claim is proved and by (5.4) we have $d_F^1(x, y) = \lambda d_{\Gamma}^1(F)(x, y) \forall x, y \in \Omega$ and $\lambda > \inf_{W^{1,\infty}(\Omega)} F$ as $\inf_{W^{1,\infty}(\Omega)} F \geq 0$. Therefore $F$ satisfies (5.5) and by the first equality of (3.6) in Theorem 3.1 we get

$$\{ u \in W^{1,\infty}(\Omega) : \Gamma_\tau(F)(u) \leq \lambda \} = \{ u \in W^{1,\infty}(\Omega) : R_{d_F^1}(u) \leq \lambda \}$$  

for $\lambda > \inf_{W^{1,\infty}(\Omega)} F \geq 0$.

Thus the statement $\Gamma_\tau(F) = R_{d_{\Gamma}^1(F)}$ is proved once we show that

$$\inf_{W^{1,\infty}(\Omega)} F = \inf_{W^{1,\infty}(\Omega)} \Gamma_\tau(F) = \inf_{W^{1,\infty}(\Omega)} R_{d_{\Gamma}^1(F)}.$$  

To this aim we actually prove that the infima above are minima and they are achieved on constant functions, so that $\min_{W^{1,\infty}(\Omega)} \Gamma_\tau(F) = 0 = \min_{W^{1,\infty}(\Omega)} R_{d_{\Gamma}^1(F)}$.

In order to get $\inf_{W^{1,\infty}(\Omega)} R_{d_{\Gamma}^1(F)} = 0$ it suffices to notice that, thanks to hypothesis (2), the values $d_F^1(x, y)$ are finite $\forall x, y \in \Omega$, thus, up to exchanging the role of $x$ and $y$, the functional $R_{d_{\Gamma}^1(F)}$ is non negative; moreover $R_{d_{\Gamma}^1(F)}(u) = 0$ on any constant function $u$. It remains to prove that $\inf_{W^{1,\infty}(\Omega)} \Gamma_\tau(F) = 0$.
This can be obtained by using (5.6) with $\lambda = 1$ and noticing that the constant function $\bar{u} = 0$ belongs to the sublevel set $\{ R_{\delta} \leq 1 \}$. Hence $\Gamma_\tau(F)(\bar{u}) \leq 1$ and, by the positively 1-homogeneity, it also holds
\[
\Gamma_\tau(F)(0) = \Gamma_\tau(F)(2\bar{u}) = 2\Gamma_\tau(F)(\bar{u}) = 2\Gamma_\tau(F)(0)
\]
and this implies $\Gamma_\tau(F)(0) = 0$. \hfill \square

In order to give the proof of Theorem 5.3 we need to show some preliminary results. The first one concerns an approximation via $\Gamma$-convergence of the $\tau$-lower semicontinuous envelope of a non negative functional $F$ on $W^{1,\infty}(\Omega)$ through a sequence of coercive functionals.

**Proposition 5.1.** Let $\Omega$ be an open subset of $\mathbb{R}^N$. Let $F : W^{1,\infty}(\Omega) \times \mathbb{R}^N \to [0, +\infty]$ be a functional and for every $n \in \mathbb{N}$ let $F_n : W^{1,\infty}(\Omega) \to [0, +\infty]$ be the functional defined by
\[
F_n(u) := F(u) \vee \frac{1}{n}||Du||_{L^\infty(\Omega)}.
\]
Then
1. the sequence $(F_n)_{n \in \mathbb{N}}$ is decreasing and pointwise converges to $F$;
2. $F$ is a level convex functional if and only if $F_n$ is level convex for every $n \in \mathbb{N}$;
3. for any topology $\tau \in \{ \tau_{\infty}, w^*, w_{\text{seq}}^* \}$ the sequences $(F_n)_{n \in \mathbb{N}}$ and $(\Gamma_\tau(F_n))_{n \in \mathbb{N}}$ $\Gamma(\tau)$-converge to $\Gamma_\tau(F)$ in $W^{1,\infty}(\Omega)$.

**Proof.** (1) derives from the fact that $F \geq 0$. In order to prove (2) we first note that if $F$ is level convex then, for every $n \in \mathbb{N}$, then the sublevel set
\[
\{ u \in W^{1,\infty}(\Omega) : F_n(u) \leq \lambda \} = \{ u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda \} \cap \{ u \in W^{1,\infty}(\Omega) : ||Du||_{L^\infty(\Omega)} \leq \lambda \}
\]
is convex. Proposition 2.11 yields the converse implication. Finally, (3) follows by statements (2) and (3) in Proposition 2.9. \hfill \square

Thanks to the next proposition, if $F$ is a supremal functional then $F_n$ defined by (5.7) is itself a supremal functional. Note that in general the functional $F \land G$ is not a supremal functional, as shown in Section 7.

**Proposition 5.2.** Let $F, G : W^{1,\infty}(\Omega) \to \mathbb{R}$ be supremal functionals represented by the supremands $f, g : \Omega \times \mathbb{R}^N \to \mathbb{R}$, respectively. Then the functional $F \lor G$ defined by $F \lor G(u) := F(u) \lor G(u)$ is still a supremal functional, represented by the supremand $f \lor g$.

**Proof.** Set $H(u) := \text{ess sup}_0(f \lor g)(x, Du(x))$ for every $u \in W^{1,\infty}(\Omega)$, we show that $H = F \lor G$. The inequality $H \geq F \lor G$ follows by definition. In order to prove the converse inequality, let $u \in W^{1,\infty}(\Omega)$ and $\delta > 0$ be fixed and select $B_\delta \subset \Omega$ with $|B_\delta| > 0$ such that $(f \lor g)(x, Du(x)) \geq H(u) - \delta$ for every $x \in B_\delta$. Set $B^f_\delta := \{ x \in B_\delta : (f \lor g)(x, Du(x)) = f(x, Du(x)) \}$ and $B^g_\delta := B_\delta \setminus B^f_\delta$. If $|B^f_\delta| > 0$, then $F(u) \geq H(u) - \delta$ while if $|B^g_\delta| > 0$ then $G(u) \geq H(u) - \delta$. In both cases $(F \lor G)(u) \geq H(u) - \delta$ for every $\delta > 0$ and this entails $H \leq (F \lor G)$. In particular, it follows that if $F$ is a supremal functional, the functional $F_n$ given by (5.7) is still a supremal functional. Moreover if $F(u) \geq 0$ for every $u \in W^{1,\infty}(\Omega)$ then $F(u) = F(u) \lor 0 = \text{ess sup}_0 f^+(x, Du(x))$ for every $u \in W^{1,\infty}(\Omega)$ where $f^+ = f \lor 0$. Therefore if $F$ is not negative we can suppose, without loss of generality, that its supremand $f$ is non negative. \hfill \square

We are in position to show Theorem 5.3

**Proof of Theorem 5.3.** We divide the proof into four steps.

**Step 1.** We assume that $u_n \equiv 0$ for any $n \in \mathbb{N}$, that is $0$ is a minimum point for $F$, and that $F$ is coercive, i.e. $\exists \beta > 0$ such that
\[
F(u) \geq \beta ||Du||_{L^\infty(\Omega)} \text{ for every } u \in W^{1,\infty}(\Omega).
\]
Note that, in this case,
\[
\Gamma_\tau(F)(u) \geq \min_{u \in W^{1,\infty}(\Omega)} F(u) = F(0)
\]
for every \( u \in W^{1,\infty}(\Omega) \) and for any \( \tau \)-topology. Thus (3.8) implies that for any \( \tau \)-topology

\[
\min_{W^{1,\infty}(\Omega)} F = \Gamma_\tau(F)(0) \leq \liminf_{\xi \to 0} \Gamma_\tau(F)(u_\xi) \leq \limsup_{\xi \to 0} \Gamma_\tau(F)(u_\xi) \leq \limsup_{\xi \to 0} F(u_\xi) = \min_{W^{1,\infty}(\Omega)} F. \tag{5.9}
\]

By (5.8) and (5.9), taking into account Theorem 4.6, we deduce that for every \( \lambda > \inf_{W^{1,\infty}(\Omega)} F \) there exist \( \beta(\lambda), \alpha(\lambda) > 0 \) such that

\[
\alpha(\lambda)|x - y| \leq d_F^\lambda(x, y) \leq \beta(\lambda)|x - y|_{\Omega} \quad \forall x, y \in \Omega.
\]

By applying Theorem 3.1 we may conclude that \( \Gamma_{\tau}(F) \) is a level convex functional and

\[
\Gamma_{\tau}(F) \equiv \Gamma_{w^*, \tau}(F) \equiv \Gamma_{w_{seq}^*, \tau}(F).
\]

**Step 2.** Here we assume the same hypotheses of Step 1 dropping the coercivity assumption (5.8), that is, we assume \( u_n \equiv 0 \) for any \( n \in \mathbb{N} \) and \( F(0) = \min_{W^{1,\infty}(\Omega)} F \). Without loss of generality, it is not restrictive to suppose \( F(0) = 0 \). Then we may approximate \( F \) through the sequence \( (F_n) \) given by

\[
F_n(u) = F(u) \vee \frac{1}{n}||Du||_{\infty}.
\]

By Proposition 5.2 \( F_n \) is still a supremal functional (represented by \( f_n(x, \xi) := f(x, \xi) \vee \frac{1}{n}||\xi|| \)) and satisfies (3.8). By Step 1 we get that \( \Gamma_{\tau}(F_n) \) is level convex for every \( n \in \mathbb{N} \) when \( \tau \) is one of the topologies \( \tau_{\infty}, w^*, w_{seq}^* \). By taking into account Proposition 5.1(3), the sequence \( \Gamma_{\tau}(F_n) = \Gamma_{\tau}(F) \) with respect to any topology \( \tau \). Therefore, by Proposition 2.11(2), it follows that \( \Gamma_{\tau}(F) \) is a level convex functional for every \( \tau \in \{\tau_{\infty}, w^*, w_{seq}^* \} \).

**Step 3.** We consider the general case and we assume only the additional hypothesis

\[
\inf_{u \in W^{1,\infty}(\Omega)} F(u) \in \mathbb{R}.
\]

By taking (3.8) into account this implies that \( F(u_n), \limsup_{\xi \to 0} F(u_\xi + u_n) \) are finite for \( n \) large enough. For every \( n \in \mathbb{N} \) set

\[
c_n := F(u_n) \vee \limsup_{\xi \to 0} F(u_\xi + u_n)
\]

and define \( G_n : W^{1,\infty}(\Omega) \to [0, +\infty] \) as

\[
G_n(u) := F(u + u_n) \vee c_n - c_n.
\]

By Proposition 2.5(3),(4),(5) we have that

\[
\Gamma_\tau(G_n)(\cdot) = \Gamma_\tau(F)(\cdot + u_n) \vee c_n - c_n.
\]

Note that for every \( n \in \mathbb{N} \) \( G_n \) is a supremal functional satisfying the assumptions of Step 2 since

\[
G_n(0) = 0 = \min_{W^{1,\infty}(\Omega)} G(u) \quad \text{and} \quad \limsup_{\xi \to 0} G_n(u_\xi) = \limsup_{\xi \to 0} F(u_\xi + u_n) \vee c_n - c_n = 0.
\]

Hence \( \Gamma_\tau(G_n) \) is level convex for any \( \tau = \tau_{\infty}, w^*, w_{seq}^* \). It easily follows that also the functional \( \Gamma_{\tau}(F) \vee c_n \) is level convex for every \( n \in \mathbb{N} \). By passing to the pointwise limit when \( n \to \infty \), by Proposition 2.11(1) we get that \( \Gamma_{\tau}(F) \) is level convex.

**Step 4.** Finally we consider the case when \( \inf_{u \in W^{1,\infty}(\Omega)} F(u) = -\infty \). For every \( m \in \mathbb{N} \) let \( G_m : W^{1,\infty}(\Omega) \to \mathbb{R} \cup \{+\infty\} \) be given by

\[
G_m(u) := F(u) \vee (-m).
\]

Note that \( G_m \neq +\infty \) and \( \inf_{W^{1,\infty}(\Omega)} G_m \in \mathbb{R} \). It is easy to show that \( (u_n) \) is a minimizing sequence for \( G_m \) and \( \lim_{\xi \to 0} G_m(u_n + u_\xi) = G_m(u_n) \). Since \( \inf_{W^{1,\infty}(\Omega)} G_m \in \mathbb{R} \), by Step 3 it follows that \( \Gamma_{\tau}(G_m) \) is level convex when \( \tau \) is one of the topologies \( \tau_{\infty}, w^*, w_{seq}^* \). By Proposition 2.5(4) we have that

\[
\Gamma_\tau(G_m) = \Gamma_\tau(F \vee (-m)) = \Gamma_\tau(F) \vee (-m).
\]

By passing to the limit when \( m \to \infty \) and by applying Proposition 2.11(2) it follows that \( \Gamma_{\tau}(F) \) is a level convex functional when \( \tau = \tau_{\infty}, w^*, w_{seq}^* \).

\[\square\]
Proof of Corollary 6.2. Thanks to the hypotheses, by applying Theorem 3.1 or 3.4 we get that, for any \( n \in \mathbb{N} \) the functional \( \Gamma(\tau)(F_n) \) is level convex. The thesis follows taking into account Propositions 2.11 (2) and 2.11 (2).

\[ \] 6. Further results.

In this section we provide some additional results deriving from Theorems 3.1 and 3.4.

We recall that for any function \( F : X \to \mathbb{R} \) the sublevel sets of the relaxed function \( \Gamma(\tau)(F) \) can be represented by using the following identity that holds for any \( \lambda \in \mathbb{R} \)

\[ \{ x \in X : \Gamma(\tau)(F) \leq \lambda \} = \bigcap_{\lambda' > \lambda} \{ x \in X : F(x) \leq \lambda' \}, \]

(see Proposition 3.5 in [10]). Under the hypothesis that \( \Gamma(\tau)(F) \) is level convex, we provide a refined relationship between sublevel sets of \( \Gamma(\tau)(F) \) and \( F \) together with an abstract representation formula for \( \Gamma(\tau)(F) \).

Proposition 6.1. Assume that \( \Gamma(\tau)(F) \) is level convex, then for any \( x \in X \) it holds

\[ \Gamma(\tau)(F)(x) = \inf \{ \lambda : x \in \tau\text{-co}([x' \in X : F(x') \leq \lambda]) \}, \tag{6.1} \]

where for \( A \subseteq X \) \( \tau\text{-co}(A) \) is defined in [4.19].

In addition, for any \( \lambda \in \mathbb{R} \) it holds

\[ \{ x \in X : \Gamma(\tau)(F)(x) \leq \lambda \} = \bigcap_{\lambda' > \lambda} \tau\text{-co}([x \in X : F(x) \leq \lambda']). \tag{6.2} \]

Proof. Set

\[ G(x) = \inf \{ \lambda : x \in \tau\text{-co}([x' \in X : F(x') \leq \lambda]) \}, \]

it can be easily verified that

\[ \{ x \in X : G(x) \leq \lambda \} = \bigcap_{\lambda' > \lambda} \tau\text{-co}([x \in X : F(x) \leq \lambda']). \]

Hence the thesis follows once we prove that \( \Gamma(\tau)(F) = G \). To this aim we note the following facts:

(1) for every function \( f : X \to \mathbb{R} \) it holds \( f(x) = \min \{ \lambda : x \in \{ f(x') \leq \lambda \} \} \);

(2) for any family of subsets of \( X \), \( (C_\lambda)_{\lambda} \), with \( C_\lambda \subseteq C_{\lambda'} \) for \( \lambda \leq \lambda' \) and \( \bigcup_\lambda C_\lambda = X \) one may define

\[ f_C(x) := \inf \{ \lambda : x \in C_\lambda \}. \]

If \( (D_\lambda)_{\lambda} \) is another family of subsets of \( X \), with the same properties of \( (C_\lambda)_{\lambda} \), then

\[ C_\lambda \subseteq D_\lambda \forall \lambda \in \mathbb{R} \Rightarrow f_D(x) := \inf \{ \lambda : x \in D_\lambda \} \leq f_C(x) \text{ for any } x \in X. \]

In order to apply the arguments above to \( G, F \) and \( \Gamma(\tau) \), we note that \( \tau\text{-co}([F \leq \lambda]) \supseteq \{ F \leq \lambda \} \) for any \( \lambda \in \mathbb{R} \). Thus, by (1) e (2) above, we deduce that \( G(x) \leq F(x) \) for every \( x \in X \). Moreover, by the closedness of \( \tau\text{-co}([x \in X : F(x) \leq \lambda']) \) for any \( \lambda \in \mathbb{R} \) we get that \( G \) is \( \tau \)-lower semicontinuous. Hence \( G \leq \Gamma(\tau)(F) \). On the other hand, since \( \{ x \in X : \Gamma(\tau)(F) \leq \lambda \} \) is a closed convex set by hypothesis and \( \{ x \in X : \Gamma(\tau)(F)(x) \leq \lambda \} \supseteq \{ x \in X : F(x) \leq \lambda \} \), we deduce

\[ \{ x \in X : \Gamma(\tau)(F)(x) \leq \lambda \} \supseteq \tau\text{-co}([x \in X : F(x) \leq \lambda]). \]

Hence, we get \( \Gamma(\tau)(F) \leq G \) and this yields (6.1) and (6.2).

In the supremal case, once the level convexity of \( \Gamma(\tau)(F) \) is established in Theorems 3.1 or 3.4 one can easily prove the following result.

Corollary 6.2. Under the same hypotheses of Theorem 3.1 or Theorem 3.4 for any \( u \in W^{1,\infty}(\Omega) \) it holds

\[ \Gamma(\tau)(F)(u) = \inf \{ \lambda : u \in \tau\text{-co}([v \in W^{1,\infty}(\Omega) : F(v) \leq \lambda]) \} \tag{6.3} \]

and, for any \( \lambda \in \mathbb{R} \), we have

\[ \{ v \in W^{1,\infty}(\Omega) : \Gamma(\tau)(F)(v) \leq \lambda \} = \bigcap_{\lambda' > \lambda} \tau\text{-co}([v \in W^{1,\infty}(\Omega) : F(v) \leq \lambda']), \tag{6.4} \]

when \( \tau \) is one of the topologies \( \tau_\infty, \tau^*, \tau^{\text{seq}} \).
Proof. By the level convexity of \( \Gamma_\tau (F) \), by applying Proposition \ref{prop:level_convexity} we deduce \ref{rem:level_convexity_1} and \ref{rem:level_convexity_2}.

\[ \ast \]

**Remark 6.3.** We note that, under hypothesis \ref{hyp:level_convexity}, taking into account \ref{prop:level_convexity} in Proposition \ref{prop:level_convexity} we can also obtain that

\[ \{v \in W^{1,\infty} (\Omega) : \Gamma_\tau (F) (v) \leq \lambda \} = \bigcap_{\lambda' > \lambda} \tau\text{-co} \left( \{u \in W^{1,\infty} (\Omega) : F (u) \leq \lambda \} \right). \]

We conclude this section with a representation result which improves \cite[Theorem 2.3]{18} since we do not require a continuity assumption on \( f (x, \cdot) \). Note that, as shown in Remark 3.1 of \cite{14}, the level convexity of the supremal functional \( F \) does not imply the level convexity of \( f (x, \cdot) \). However, it is possible to show that any level convex supremal functional \( F \) can be represented by a level convex supremand \( \varphi_\Omega \) (possibly different from the level convex envelope of \( f (x, \cdot) \), see \cite[Example 8.1]{18}).

In order to show the result, we recall that a Borel function \( f : \Omega \times \mathbb{R}^N \to \mathbb{R} \) is a normal supremand if for a.e. \( x \in \Omega \) the function \( f (x, \cdot) \) is lower semicontinuous on \( \mathbb{R}^N \).

**Proposition 6.4.** Let \( \Omega \) be a bounded open set with Lipschitz continuous boundary. Let \( f : \Omega \times \mathbb{R}^N \to \mathbb{R} \), be a normal supremand. Let \( F \) be the supremal functional \ref{def:level_convex_supremal_functional} represented by \( f \) and assume that \( F \) satisfies the condition \ref{cond:level_convexity}. Then the following facts are equivalent:

1. \( F \) is \( w^* \)-lower semicontinuous in \( W^{1,\infty} (\Omega) \);
2. \( F \) is \( w^*_\text{seq} \)-lower semicontinuous in \( W^{1,\infty} (\Omega) \);
3. \( F \) is a level convex supremal functional;
4. there exists a level convex normal supremand \( \varphi_\Omega : \Omega \times \mathbb{R}^N \to \mathbb{R} \) given by

\[ \varphi_\Omega (x, \xi) := \inf \{ F (u) \mid u \in W^{1,\infty} (\Omega) \text{ s.t. } x \in \hat{u}, \text{ with } Du (x) = \xi \} \quad (6.5) \]

where

\[ \hat{u} := \{ x \in \Omega : x \text{ is a Lebesgue point of } Du \text{ and a differentiable point of } u \} \]

such that

\[ F (u) = \text{ess sup}_{x \in \Omega} \varphi_\Omega (x, Du (x)). \]

Moreover there exists a negligible set \( N \subset \Omega \) such that \( \varphi_\Omega (x, \xi) \geq f (x, \xi) \forall x \in \Omega \setminus N, \forall \xi \in \mathbb{R}^N \).

**Proof.** (i) \( \implies \) (ii) follows by Remark \ref{rem:level_convexity} while (ii) \( \implies \) (iii) follows by Theorem \ref{thm:level_convexity}. In order to prove (iii) \( \implies \) (iv) let us consider the auxiliary functional \( G : W^{1,\infty} (\Omega) \to [0, +\infty] \) given by

\[ G (u) := \arctan F (u) + \frac{\pi}{2} = \text{ess sup}_{\Omega} \arctan (f (x, Du (x))) + \frac{\pi}{2}. \]

Note that \( G \) is level convex if and only if \( F \) is level convex. Thanks to the result \cite[Theorem 2.4]{18}, the level convex functional \( G \) can be represented through the level convex normal supremand \( g \) given by

\[ g (x, \xi) := \inf \{ G (u) \mid u \in W^{1,\infty} (\Omega) \text{ s.t. } x \in \hat{u}, \text{ with } Du (x) = \xi \} \]

and satisfying

\[ g (x, \xi) \geq \arctan f (x, \xi) + \frac{\pi}{2} \forall x \in \Omega \setminus N, \forall \xi \in \mathbb{R}^N \]

where \( N \subset \Omega \) is a negligible set. Hence it is sufficient to choose

\[ \varphi_\Omega (x, \xi) := \tan (g (x, \xi) - \frac{\pi}{2}) = \inf \{ F (u) \mid u \in W^{1,\infty} (\Omega) \text{ s.t. } x \in \hat{u}, \text{ with } Du (x) = \xi \}. \]

Eventually, (iv) \( \implies \) (i) follows by \cite[Theorem 3.4]{2}.

\[ \square \]

7. Some interesting examples and counterexamples

In this section we collect some examples pointing out the optimality of some of the statements given in the previous sections.
7.1. An infimum of suprimal functionals that is not suprimal. Given $F, G : W^{1,\infty}(\Omega) \to \mathbb{R}$ suprimal functionals represented by the supremands $f, g : \Omega \times \mathbb{R}^N \to \mathbb{R}$, respectively we have seen in Example 5.2 that the functional $F \vee G$ defined by $F \vee G(u) := F(u) \vee G(u)$ is still a suprimal functional. On the other hand, this property may not hold in general for the functional $F \wedge G(u) := F(u) \wedge G(u)$, as shown in the following example.

**Example 7.1.** Let $\Omega = (-1, 1)$ and let $f, g : (-1, 1) \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} 1 & \text{if } x \in (-1, 0) \\ 3 & \text{if } x \in (0, 1) \end{cases}, \quad g(x) := \begin{cases} 4 & \text{if } x \in (-1, 0) \\ 2 & \text{if } x \in (0, 1) \end{cases}.$$ 

We consider the (localized) suprimal functionals $F, G$ with supremands $f, g$, respectively, given by $F(u, A) = \inf \sup_{A} f(x)$ and $G(u, A) = \inf \sup_{A} g(x)$ for any open set $A \subseteq (-1, 1)$ and $u \in W^{1,\infty}((-1, 1))$. We claim that $F \wedge G$ cannot be represented in a suprimal form since it does not satisfy the necessary condition

$$F \wedge G(u) = \bigcup_{i \in I} F(u, A_i) = \bigcup_{i \in I} (F \wedge G)(u, A_i) \quad \forall u \in W^{1,\infty}(\Omega), \forall A_i \in \mathcal{A}(\Omega).$$

Indeed, set $A = (-1, -\frac{1}{2})$ and $B = (0, 1)$, an easy computation shows that

$$F \wedge G(u, A \cup B) = 3 > 2 = (F \wedge G(u, A)) \vee (F \wedge G(u, B)).$$

7.2. A suprimal functional $F$ with $d^1_F < d^1_{\Gamma_\tau(F)}$ for some $\lambda$. In the following example we show that, given a suprimal functional $F$, the intrinsic distance associated to its relaxed functional $\Gamma_\tau(F)$ is in general different from $d^1_F$, for some values $\lambda > \inf_{W^{1,\infty}(\Omega)} F$.

**Example 7.2.** Let $\Omega = (0, 1)$ and $F(u) = \inf \sup_{\Omega} F(u'(x))$ where $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(z) := \begin{cases} -z & \text{if } z \leq 0 \\ 1 + \frac{1}{z} & \text{if } z > 0. \end{cases}$$

It can be verified that, for any $\tau \in \{\tau_\infty, w^*, w^*_{\text{seq}}\}$, the relaxed functional $\Gamma_\tau(F)$ is still a suprimal functional given by $\Gamma_\tau(F)(u) = \inf \sup_{\Omega} \phi(u'(x))$ where $\phi : \mathbb{R} \to \mathbb{R}$ is given by

$$\phi(z) := \begin{cases} -z & \text{if } z \leq 0 \\ 1 & \text{if } z > 0. \end{cases}$$

Taking aside the computation above we will deduce the value of the intrinsic distance functions by computing the level sets of $F$ and taking advantage of the results established in Corollary 6.2. Indeed, the sequence $u_n$ given by $u_n(x) = \frac{\tau - n}{n}$ is a minimizing sequence along which $F$ is continuous and thus $F$ satisfies (3.8).

A direct computation shows that

$$\{u \in W^{1,\infty}(\Omega) : F(u) \leq 1\} = \{u \in W^{1,\infty}(\Omega) : -1 \leq u'(x) \leq 0 \text{ a.e. in } \Omega\} \quad (7.1)$$

and, for a given $\lambda > 1$, we have also

$$\{u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda\} = \{u \in W^{1,\infty}(\Omega) : u'(x) \in [-\lambda, 0] \cup [1/(\lambda - 1), +\infty) \text{ a.e. in } \Omega\}.$$

By (7.1) we deduce that

$$d^1_F(x, y) = \begin{cases} 0 & \text{if } x \geq y \\ |x - y| & \text{if } x < y. \end{cases}$$

On the other hand, it can be shown that for any $\tau \in \{\tau_\infty, w^*, w^*_{\text{seq}}\}$ it holds

$$\tau-\text{co}(\{u \in W^{1,\infty}(\Omega) : F(u) \leq \lambda\}) = \{u \in W^{1,\infty}(\Omega) : u'(x) \in [-\lambda, +\infty) \text{ a.e. in } \Omega\}.$$
and
\[ d^1_{\Gamma^\tau(F)}(x, y) = \begin{cases} +\infty & \text{if } x \geq y \\ |x - y| & \text{if } x < y. \end{cases} \]

Note that this example also provides a counterexample to the validity of the formula
\[ \{ v \in W^{1, \infty}(\Omega) : \Gamma^\tau(F)(v) \leq \lambda \} = \bigcap_{\lambda' \geq \lambda} \tau\text{-co}(\{ v \in W^{1, \infty}(\Omega) : F(v) \leq \lambda' \}). \]

7.3. Functional \( F \) with discontinuous supremand satisfying (3.8). In the following example we exhibit a supremal functional \( F \) represented by a discontinuous supremand \( f \) whose minimum is not attained. We construct a “minimizing” sequence of \( F \) made up by “continuity” points which satisfies (3.8).

Example 7.3. Let \( F : W^{1, \infty}(0, 1) \to \mathbb{R} \) be a functional whose supremand is given by
\[ f(\xi) := \begin{cases} \xi + 2 & \text{if } \xi \geq 0 \\ -\xi & \text{if } \xi < 0. \end{cases} \]

For such functional (3.8) fails as it can be easily shown that \( d^1_{F}(x, y) = 0 \) if \( x < y \). Despite of this if we set \( u_n(x) := -\frac{x}{n + 1}, n \in \mathbb{N}, \) then it holds
\[ \lim_{\xi \to 0} F(u_n + u_\xi) = F(u_n), \quad \lim_{n \to +\infty} F(u_n) = 0 = \inf_{W^{1, \infty}(\Omega)} F. \]

In particular (3.8) is satisfied.

Here below we give an example of a supremal functional \( F \) represented by a function \( f \) that is discontinuous everywhere in \( \mathbb{R} \) and still satisfying (3.8).

Example 7.4. We consider the following function : \( [0, 1] \to \mathbb{R} \)
\[ g(\xi) := \begin{cases} 0 & \text{if } \xi \in \left[ \frac{1}{n}, \frac{1}{n-1} \right] \cap \mathbb{Q}, n \in \mathbb{N}, n \geq 2 \\ \frac{1}{n} & \text{if } \xi \in \left[ \frac{1}{n}, \frac{1}{n-1} \right] \cap (\mathbb{R} \setminus \mathbb{Q}), n \geq 2 \\ 1 & \text{otherwise.} \end{cases} \]

Moreover for every \( \lambda \in \left[ \frac{1}{n}, \frac{1}{n-1} \right] \) we have that
\[ \{ g(\xi) \leq \lambda \} = (0, \frac{1}{n - 1}) \cup \left( \mathbb{Q} \cap \left( \frac{1}{n - 1}, 1 \right) \right). \]

The function \( g \) is discontinuous everywhere in \( [0, 1] \). In particular, its periodic extension \( f \) on \( \mathbb{R} \) is discontinuous everywhere in \( \mathbb{R} \). However the supremal functional \( F(u) = \text{ess sup}_{x \in \Omega} f(Du(x)) \) satisfies (3.8) since the sublevel set of \( f \) have not empty interiors.

7.4. Failure of the local representation by means of the supremand \( \varphi_\Omega \). In the following example we show that, given a normal supremand \( f \), the function \( \varphi_\Omega \) in the representation formula (6.5) is in general affected by the reference set \( \Omega \). More in details we exhibit a supremal functional \( F \) such that the supremand \( \varphi_\Omega \) does not represent the lower semicontinuous envelope of the localized version of \( F \), \( F(\cdot, A) \) defined by
\[ F(u, A) := \text{ess sup}_{A} f(x, Du(x)), \]
for \( A \in \mathcal{A}(\Omega). \)
Example 7.5. Let $\Omega = (-2, 2)$ and let $f : (-2, 2) \times \mathbb{R} \to \mathbb{R}$ be defined as
\[
f(x, \xi) := \begin{cases}
(1 - |\xi|) \lor 0 & \text{if } x \in [-1, 1] \\
2 + |\xi| & \text{if } x \in (-2, -1) \cup (1, 2).
\end{cases}
\]

Let $F$ be the supremal functional given by $F(u) = \text{ess sup}_A f(x, u'(x))$. Note that $F$ is level convex on $W^{1, \infty}(\Omega)$ since $F(u) = 2 + \text{ess sup}_{[-2, 2] \setminus (1, 2]} |u'(x)|$ for every $u \in W^{1, \infty}(\Omega)$.

A direct computation shows that the function $\varphi$ in Corollary 6.4 (iv) is given by
\[
\varphi_{\Omega}(x, \xi) = \begin{cases}
2 & \text{if } x \in (-1, 1) \\
2 + |\xi| & \text{if } x \in (-2, -1) \cup (1, 2).
\end{cases}
\]

Note that $\varphi_{\Omega}$ is level convex and is strictly greater than $f$ on a set of positive measure.

We underline that the function $\varphi_{\Omega}$ is obtained by a minimization process on the whole set $\Omega$ and it is not suitable to represent the lower semicontinuous envelope of the localized functional $F(\cdot, A)$ as shown by the following computation. For any open set $A \subseteq (-2, 2)$ let $F(u, A) := \text{ess sup}_A f(x, u'(x))$ be the supremal functional defined on $W^{1, \infty}((-2, 2), \tau)$ and let $G(u, A) := \Gamma_{\tau}(F)(u, A)$ where $\tau$ is one of the topologies $\tau_\infty, \tau^*, \tau^{*\text{seq}}$. We have that $F$ is a normal supremand and $f(x, \cdot)$ is level convex in $\mathbb{R}$ if and only if $x \in \Omega \setminus [-1, 1]$. According to this the functional $F(\cdot, A)$ is level convex only on the open sets $A$ satisfying the condition $|A \setminus (-1, 1)| > 0$. Indeed, for such an open set $A$ it holds
\[
F(u, A) = \text{ess sup}_{A \setminus (-1, 1)} f(x, u'(x))
\]
for any $u \in W^{1, \infty}(\Omega)$ and the supremand $f(x, \cdot)$ is level convex for any $x \in \Omega \setminus [-1, 1]$.

We claim that for any open set $A$ the relaxed functional $G(\cdot, A)$ can be represented by the function $g : (-2, 2) \times \mathbb{R} \to \mathbb{R}$ defined as
\[
g(x, \xi) := \begin{cases}
0 & \text{if } x \in (-1, 1) \\
2 + |\xi| & \text{if } x \in (-2, -1) \cup (1, 2).
\end{cases}
\]

Indeed, if $A$ is an open set in $\Omega$ with $|A \setminus (-1, 1)| > 0$ it holds
\[
F(u, A) = F(u, A \setminus [-1, 1]) = \text{ess sup}_{A \setminus [-1, 1]} f(x, u'(x))
\]
for any $u \in W^{1, \infty}(\Omega)$ and $G(u, A) = F(u, A)$ since $F(u, A \setminus [-1, 1])$ is lower semicontinuous in $W^{1, \infty}(\Omega)$ by the level convexity of $f$. We have to prove that for any fixed $A$ with $A \subseteq (-1, 1)$ it holds $G(u, A) = 0$ for any $u \in W^{1, \infty}(\Omega)$. Indeed, let $u \in W^{1, \infty}(\Omega)$ be fixed and set $C = \text{ess sup}_A |u'(x)|$. For $n \in \mathbb{N}$ let us define $\psi_n$ as the odd function piecewise affine such that $\psi_n(0) = 0$ and
\[
\psi_n'(x) := \begin{cases}
C + 1 & x \in \left(\frac{2k}{2n}, \frac{2k+1}{2n}\right) \\
-(C + 1) & x \in \left(\frac{2k+1}{2n}, \frac{2k+2}{2n}\right)
\end{cases}
\]
with $k = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$. We have that the sequence $u_n = u + \phi_n$ converges to $u$ in any topology above and $|u_n'(x)| > 1$ for any $x \in A$. Hence $F(u_n, A) = 0$ and, subsequently, $\Gamma_{\tau}(F)(u, A) = 0$. As a consequence, for every $A \in \mathcal{A}(\Omega)$ we have verified that $G(u, A) = \text{ess sup}_{A} g(x, u'(x))$.

In particular, for $A \subseteq (-1, 1)$, we have that $G(u, A) < \text{ess sup}_A \varphi_{\Omega}(x, u'(x))$ for any $u \in W^{1, \infty}(\Omega)$.

REFERENCES

[1] N. Ansini, F. Prinari. Power law approximation of supremal functional under differential constraint. SIAM J. Math. Anal. 2 46 (2014), 1085-111.
[2] E.N. Barron, R. R. Jensen, C.Y. Wang. Lower Semicontinuity of $L^\infty$-Functionals. Ann. Inst. H. Poincaré Anal. Non Linéaire (4) 18 (2001), 495-517.
[3] E.N. Barron, R. R. Jensen, C.Y. Wang. The Euler Equation and Absolute Minimizers of $L^\infty$-Functionals. Arch. Rational Mech. Anal. 157 (2001), 225-283.
[4] M. Bocea and V. Nesi. Γ-convergence of power-law functionals, variational principles in $L^\infty$ and applications. SIAM J. Math. Anal. 39 (2008), 1550–1576.

[5] H. Brezis: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext, Springer, New York (2011).

[6] A. Briani, F. Prinari, A. Garroni. Homogenization of $L^\infty$-functionals. Math. Models Methods Appl. Sci. 14 (2004), no. 12, pp. 1761–1784.

[7] G. Buttazzo. Semiconvergence, Relaxation and Integral Representation in the Calculus of Variation. Pitman Research Notes in Mathematics Series 207, Harlow (1989).

[8] T. Champion, L. De Pascale, F. Prinari. Semiconvergence and absolute minimizers for supremal functionals. ESAIM Control Optim. Calc. (1) 10, (2004), 14–27.

[9] T. Champion, L. De Pascale. Principles of comparison with distance functions for AML. J. Convex Anal. 14 (2007), no. 3, 515–541.

[10] G. Dal Maso. An Introduction to Γ-Convergence. Progress in Nonlinear Differential Equations and their Applications 8, Birkhauser, Boston (1993).

[11] A. Davini, M. Ponsiglione. Homogenization of two-phase metrics and applications. J. Anal. Math., 103 (2007), 157–196.

[12] G. De Cecco, G. Palmieri Integral distance on a Lipschitz Riemannian manifold. Math. Z., 207 (1991), 223-243.

[13] G. De Cecco, G. Palmieri LIP manifolds: from metric to Finslerian structure. Math. Z., 218 (1995), 223-237.

[14] A. Garroni, M. Ponsiglione, F. Prinari. From 1-homogeneous supremal functionals to difference quotients: relaxation and Γ-convergence. Calc. Var. Part. Diff. Eq, 27 (2006), no. 4, 397–420.

[15] M. Gori, F. Maggi. On the lower semicontinuity of supremal functionals. ESAIM Control Optim. Calc. Var. 9 (2003), 135–143.

[16] P. Koskela, N. Shanmugalingam, Y. Zhou. Intrinsic geometry and analysis of diffusion processes and $L^\infty$-variational problems. Arch. Rational Mech. Anal. 214 (2014), 99–142.

[17] J. R. Norris Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds Acta Math., 179 (1997), 79-103

[18] F. Prinari. Semiconvergence and supremal representation in Calculus of Variations, Appl. Mat. Optim., 58, (2008), 111–145.

[19] F. Prinari. Semiconvergence and relaxation of $L^\infty$-functionals. Adv. Calc. Var. (1) 2 (2009), 43–71.

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