NONCRITICAL HOLOMORPHIC FUNCTIONS ON STEIN SPACES

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Abstract. We prove that every reduced Stein space admits a holomorphic function without critical points. Furthermore, any closed discrete subset of such a space is the critical locus of a holomorphic function. We also show that for every complex analytic stratification with nonsingular strata on a reduced Stein space there exists a holomorphic function whose restriction to every stratum is noncritical. These result also provide some information on critical loci of holomorphic functions on desingularizations of Stein spaces. In particular, every 1-convex manifold admits a holomorphic function that is noncritical outside the exceptional variety.

1. Introduction

Every Stein manifold $X$ admits a holomorphic function $f \in \mathcal{O}(X)$ without critical points \cite{K;K}. In the algebraic category this fails on any compact Riemann surface of genus $g \geq 1$ with a puncture (every algebraic function on such a surface has a critical point as follows from the Riemann-Hurwitz theorem), but it holds for holomorphic functions of finite order \cite{K}. Noncritical holomorphic functions are of interest in particular since they define nonsingular holomorphic hypersurface foliations; many results on this topic can be found in \cite{K;K;K;K;K}

In this paper we prove that, somewhat surprisingly, the same result is true on Stein spaces.

Theorem 1.1. Every reduced Stein space admits a holomorphic function without critical points.

This is a special case of Theorem 1.3 below. However, before proceeding, we pause for a moment to introduce the relevant notions.

All complex spaces in this paper are assumed to be paracompact and reduced. For the theory of Stein spaces we refer to the monograph \cite{G} of Grauert and Remmert.

Let $X$ be a complex space. Denote by $\mathcal{O}_{X,x}$ the ring of germs of holomorphic function at a point $x \in X$ and by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$, so $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong \mathbb{C}$. Given $f \in \mathcal{O}_{X,x}$ we denote by $f - f(x) \in \mathfrak{m}_x$ the germ obtained by subtracting from $f$ its value $f(x) \in \mathbb{C}$ at $x$.

Definition 1.2. Assume that $x$ is nonisolated point of a complex space $X$.

(a) A germ $f \in \mathcal{O}_{X,x}$ at $x$ is said to be critical (and $x$ is a critical point of $f$) if $f - f(x) \in \mathfrak{m}_x^2$ (the square of the maximal ideal $\mathfrak{m}_x$), and is noncritical if $f - f(x) \in \mathfrak{m}_x \setminus \mathfrak{m}_x^2$.

(b) A germ $f \in \mathcal{O}_{X,x}$ is strongly noncritical at $x$ if the germ at $x$ of the restriction $f|_V$ to any local irreducible component $V$ of $X$ is noncritical.

Any function is considered (strongly) noncritical at an isolated point of $X$.
One can characterize these notions by the (non) vanishing of the differential $df_x$ on the Zariski tangent space $T_xX$. Recall that $T_xX$ is isomorphic to $(m_{x}/m_{x}^{2})^{*}$, the dual of $m_{x}/m_{x}^{2}$, the latter being the cotangent space $T_{x}^{\ast}X$ (cf. [10] p. 78 or [28] p. 111). The number $\dim\mathbb{C}T_{x}X$ is the embedding dimension of the germ $X_{x}$ of $X$ at $x$. The differential $df_{x}:T_{x}X\to\mathbb{C}$ of $f\in\mathcal{O}_{X,x}$ is determined by the class $f-f(x)\in m_{x}/m_{x}^{2}=T_{x}X$, so $f$ is critical at $x$ if and only if $df_{x}=0$. If $X_{x}=\bigcup_{j=1}^{k}V_{j}$ is a decomposition into local irreducible components, then $f$ is strongly noncritical at $x$ if $df_{x}:T_{x}V_{j}\to\mathbb{C}$ is nonvanishing for every $j=1,\ldots,k$.

At a regular point $x\in X\text{reg}$ these notions coincide with the usual one: $x$ is a critical point of $f$ if and only if in some (hence in any) local holomorphic coordinates $z=(z_{1},\ldots,z_{n})$ on a neighborhood of $x$, with $z(x)=0$ and $n=\dim_{x}X$, we have $\frac{\partial f}{\partial z_{j}}(0)=0$ for $j=1,\ldots,n$. Hence the set $\text{Crit}(f)$ of all critical points of a holomorphic function on a complex manifold $X$ is a closed complex subvariety of $X$; on a Stein manifold this set is discrete for a generic choice of $f\in\mathcal{O}(X)$.

We can now state our main result.

**Theorem 1.3.** On every reduced Stein space $X$ there exists a holomorphic function which is strongly noncritical at every point. Furthermore, given a closed discrete set $P=\{p_{1},p_{2},\ldots\}$ in $X$, germs $f_{k}\in\mathcal{O}_{X,p_{k}}$ and integers $n_{k}\in\mathbb{N}$, there exists a function $F\in\mathcal{O}(X)$ which is strongly noncritical on $X\setminus P$ and agrees with $f_{k}$ to order $n_{k}$ at each points $p_{k}\in P$ (i.e., $F_{p_{k}}-f_{k}\in m_{p_{k}}^{n_{k}}$).

The hypothesis on the set $P$ in Theorem [1.3] is natural since the critical locus of a generic holomorphic function on a Stein space is discrete (cf. Corollary [2.11]).

We also prove the following (cf. Theorem [5.1] and Corollary [5.2]). Given a closed complex subvariety $X'$ of a reduced Stein space $X$ and a function $f\in\mathcal{O}(X')$, there exists $F\in\mathcal{O}(X)$ such that $F|_{X'}=f$ and $F$ is strongly noncritical on $X\setminus X'$, or it has critical points at a prescribed discrete set in $X\setminus X'$.

The proof of Theorems [1.1] and [1.3] in the case when $X$ is a Stein manifold relies on two main ingredients (see [13] or [15] Chapter 8):

(i) Runge approximation theorem for noncritical holomorphic functions on polynomially convex subset of $\mathbb{C}^{n}$ by entire noncritical functions (cf. [13] Theorem 3.1) or [15] Theorem 8.11.1, p. 381), and

(ii) a splitting lemma for biholomorphic maps close to the identity on a Cartan pair (cf. [13] Theorem 4.1 or [15] Theorem 8.7.2)].

These tools do not apply directly at singular points of $X$. In addition, the following two phenomena make the analysis very delicate. First, the critical locus of a holomorphic function $f\in\mathcal{O}(X)$ need not be a closed complex subvariety of $X$ near a singularity. An example is $X=\{zw=0\}\subset\mathbb{C}^{2}(z,w)$ and $f(z,w)=z$ with $\text{Crit}(f)=\{(0,w):w\neq0\}$; for another example on an irreducible isolated surface singularity see Example [2.2] in §2. However, $\text{Crit}(f|_{X\text{reg}})\cup X_{\text{sing}}$ is a closed complex subvariety of $X$ (cf. Lemma [2.4]). The second problem is that the class of (strongly) noncritical functions is not stable under small perturbations on compact sets which include singular points of $X$ (see Example [2.3]).

The key idea used in this paper stems from the following observation:

(*) If $S\subset X$ is a local complex submanifold of positive dimension at a point $x\in S$ and if the restriction of a function $f\in\mathcal{O}(X)$ to $S$ is noncritical at $x$, then $f$ is noncritical at $x$ (as a
function on $X$). If such $S$ is contained in every local irreducible component of $X$ at $x$, then $f$ is strongly noncritical at $x$.

This naturally leads one to consider complex analytic stratifications of a Stein space and to construct holomorphic functions that are noncritical on every stratum.

Recall that a (complex analytic) stratification $\Sigma = \{S_j\}$ of a complex space $X$ is a subdivision of $X$ into the union $X = \bigcup_j S_j$ of at most countably many pairwise disjoint connected complex manifolds $S_j$, called the strata of $\Sigma$, such that

- every compact set in $X$ intersects at most finitely many strata, and
- $bS = \overline{S} \setminus S$ is a union of lower dimensional strata for every $S \in \Sigma$.

The pair $(X, \Sigma)$ is called a stratified complex space. Every complex analytic space admits a stratification (cf. Whitney [38, 39]). An example is obtained by taking $X = X_0 \supset X_1 \supset \cdots$, where $X_{j+1} = (X_j)_{\text{sing}}$ for every $j$, and decomposing the smooth differences $X_j \setminus X_{j+1}$ into connected components. This chain of subvarieties is stationary on each compact subset of $X$.

**Definition 1.4.** Let $(X, \Sigma)$ be a stratified complex space. A function $f \in \mathcal{O}(X)$ is said to be a stratified noncritical holomorphic function on $(X, \Sigma)$, or a $\Sigma$-noncritical function, if the restriction $f|_S$ to any stratum $S \in \Sigma$ of positive dimension is a noncritical function on $S$.

Clearly the critical locus of a $\Sigma$-noncritical function on $(X, \Sigma)$ is contained in the union $X_0$ of all 0-dimensional strata of $\Sigma$, a discrete subset of $X$.

**Theorem 1.5.** On every stratified Stein space $(X, \Sigma)$ there exists a $\Sigma$-noncritical function $F \in \mathcal{O}(X)$. Furthermore, $F$ can be chosen to agree to order $n_x \in \mathbb{N}$ with a given germ $f_x \in \mathcal{O}_{X,x}$ at any 0-dimensional stratum $\{x\} \in \Sigma$.

Proof of Theorems 1.3 and 1.5. We may assume that $X$ has no isolated points. Choose a complex analytic stratification $\Sigma$ of $X$ containing a given discrete set $P \subset X$ in the union $X_0 = \{p_1, p_2, \ldots\}$ of its zero dimensional strata. For every $i = 1, 2, \ldots$ let $X_i$ denote the union of all strata of dimension at most $i$ (the $i$-skeleton of $\Sigma$). Note that $X_i$ is a closed complex subvariety of $X$ (since the boundary of each stratum is a union of lower dimensional strata), the difference $X_i \setminus X_{i-1}$ is either empty or a complex manifold of dimension $i$, and

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset \bigcup_{i=0}^{\infty} X_i = X. \quad (1.1)$$

Given germs $f_k \in \mathcal{O}_{X,p_k}$ ($p_k \in X_0$) and integers $n_k \in \mathbb{N}$, Theorem 1.5 furnishes a $\Sigma$-noncritical function $F \in \mathcal{O}(X)$ such that $F_{p_k} - f_{p_k} \in m_{p_k}^{n_k}$ for every $p_k \in X_0$ (i.e., $F$ agrees with $f_k$ to order $n_k$ at $p_k$). We claim that $F$ is strongly noncritical on $X \setminus X_0$. Indeed, given a point $x \in X \setminus X_0$, pick the smallest integer $i \in \mathbb{N}$ such that $x \in X_i$, so $x \in X_i \setminus X_{i-1}$ which is a complex manifold of dimension $i \in \mathbb{N}$. Let $S_i \subset X_i \setminus X_{i-1}$ be the connected component containing $x$. Then the germ of $S_i$ at $x$ is contained in every local irreducible component of $X$ at $x$. Since $x$ is a noncritical point of $F|_{S_i}$, it follows by (*) that $F$ is strongly noncritical at $x$, thereby proving the claim. By choosing $f_k$ to be strongly noncritical at $p_k \in X_0$ and taking $n_k \geq 2$ for each $k$, we obtain a function $F \in \mathcal{O}(X)$ that is strongly noncritical on $X$. (To get a strongly noncritical function at a point $p \in X$, we can embed $X_p$ as a local complex subvariety of the Zariski tangent space $T_pX \cong \mathbb{C}^N$ and choose a linear function on $T_pX$ which is nondegenerate on the tangent space to every local irreducible component of $X$.)
The proof of Theorem 1.5 (cf. §3) proceeds by induction on the skeleta \( X_i \) (1.1). The main induction step is furnished by Theorem 4.1 which provides holomorphic functions on a Stein space without critical points in the regular locus. When passing from \( X_{i-1} \) to \( X_i \), we first apply the transversality theorem to show that the critical locus of a generic holomorphic extension of a given function on \( X_{i-1} \) is discrete and does not accumulate on \( X_{i-1} \) (cf. Lemma 2.4). We then extend the function to \( X_i \) without creating any critical points in \( X_i \setminus X_{i-1} \), keeping it fixed to a high order along \( X_{i-1} \). To this end we adjust one of the main tools from \[13\], namely the splitting lemma for biholomorphic maps close to the identity on a Cartan pair \[13\] Theorem 4.1, to the setting of Stein spaces (see Theorem 3.2 in [8] below).

Besides its original use, this splitting lemma from [13] has found a variety of applications since the publication in 2003. In particular, it was used for exposing boundary points of strongly pseudoconvex domains in 1-convex manifolds, a class of complex manifolds that we discuss below.

We mention a couple of immediate corollaries of Theorem 1.5.

**Corollary 1.6.** Let \((X, \Sigma)\) be a stratified Stein space. Given a closed discrete set \( P \) in \( X \), there exists a function \( F \in \mathcal{O}(X) \) such that for any stratum \( S \in \Sigma \) with \( \dim S > 0 \) we have \( \text{Crit}(F|_S) = P \cap S \).

This follows from Theorem 1.5 applied to a substratification \( \Sigma' \) of \( \Sigma \) which contains the given discrete set \( P \) in the zero dimensional skeleton.

By considering the level sets of a function satisfying Theorem 1.5 we obtain

**Corollary 1.7.** Every stratified Stein space \((X, \Sigma)\) admits a holomorphic foliation \( \mathcal{L} = \{L_a\}_{a \in A} \) with closed leaves such that for every stratum \( S \in \Sigma \) the restricted foliation \( F|_S = \{L_a \cap S\}_{a \in A} \) is a nonsingular hypersurface foliation on \( S \).

In the remainder of this introduction we indicate how Theorems 1.1, 1.3 and 1.5 imply results concerning the critical loci of holomorphic functions on complex manifolds obtained by desingularizing Stein spaces.

The simplest such example is obtained by desingularizing a Stein space \( Y \) with isolated singular points \( Y_{\text{sing}} = \{p_1, p_2, \ldots\} \). Let \( \pi : X \to Y \) be a desingularization (cf. [4, 5, 27]). The fiber \( E_j := \pi^{-1}(p_j) \) over any singular point of \( Y \) is a connected subvariety of \( X \) of positive dimension with negative normal bundle in the sense of Grauert [21]. (A local strongly plurisubharmonic function near \( p_j \in Y \) pulls back to a function that is strongly plurisubharmonic on a deleted neighborhood of \( E_j \).) The set \( \mathcal{E} = \pi^{-1}(Y_{\text{sing}}) = \bigcup_j E_j \) is a complex subvariety of \( X \) with compact irreducible components of positive dimension, and \( \mathcal{E} \) contains any complex compact subvariety of \( X \) without 0-dimensional components. Furthermore, we have \( \pi_* \mathcal{O}_X = \mathcal{O}_Y \) and \( \pi : X \to Y \) is the Remmert reduction of \( X \) [21, 33]. If \( Y \) has only finitely many singular points, the manifold \( X \) is 1-convex and \( \mathcal{E} \) is the exceptional variety of \( X \) [22]. By choosing a noncritical function \( g \in \mathcal{O}(Y) \) furnished by Theorem 1.1 the function \( f = g \circ \pi \in \mathcal{O}(X) \) clearly satisfies \( \text{Crit}(f) \subseteq \mathcal{E} \). Similary, if \( A \) is a discrete set in \( X \) then...
\[ \pi(A) \text{ is discrete in } Y, \text{ and by choosing } g \in \mathcal{O}(Y) \text{ with } \text{Crit}(g) = \pi(A) \text{ we get a function } f = g \circ \pi \in \mathcal{O}(X) \text{ with } \text{Crit}(f) \setminus E = A \setminus E. \] If \( A \) intersects every connected component of \( E \), we have \( \text{Crit}(f) = A \cup E \). This gives the following corollary.

**Corollary 1.8.** A 1-convex manifold \( X \) with the exceptional variety \( E \) admits a holomorphic function \( f \in \mathcal{O}(X) \) with \( \text{Crit}(f) \subset E \). Furthermore, given a closed discrete set \( A \) in \( X \), there exists a function \( f \in \mathcal{O}(X) \) with \( \text{Crit}(f) = A \cup E \).

In general we can not find a holomorphic function \( f \in \mathcal{O}(X) \) on a 1-convex manifold that is noncritical at every point of the exceptional variety \( E \). Indeed, assume that \( E \) is a smooth component of \( E \). Since \( E \) is compact, the restriction \( f|_E \) is constant, so the differential of \( f \) vanishes along \( E \) in the directions tangential to \( E \). Hence, if \( df_x \neq 0 \) for all \( x \in E \), the differential defines a nowhere vanishing section of the conormal bundle of \( E \) in \( X \), a nontrivial condition which does not hold always as is seen in the following example.

**Example 1.9.** Fix an integer \( n > 1 \). Let \( X \) be \( \mathbb{C}^n \) blown up at the origin, and let \( \pi: X \to \mathbb{C}^n \) denote the base point projection. The exceptional variety is \( E = \pi^{-1}(0) \cong \mathbb{CP}^{n-1} \). The conormal bundle of \( E \) is \( \mathcal{O}_{\mathbb{CP}^{n-1}}(1) \) which does not admit any nonvanishing section, so \( X \) does not admit any noncritical holomorphic functions. On the other hand, the function \( g(z) = z_1^2 + z_2^2 + \cdots + z_n^2 \) on \( \mathbb{C}^n \), with \( \text{Crit}(g) = \{0\} \), pulls back to a holomorphic function \( f = g \circ \pi: X \to \mathbb{C} \) with \( \text{Crit}(f) = E \). Similarly, the coordinate functions \( z_j \) on \( \mathbb{C}^n \) pull back to holomorphic functions \( z_j \circ \pi = \pi_j \) on \( X \) which are noncritical on \( X \setminus E \cong \mathbb{C}^n \setminus \{0\} \), and
\[
\text{Crit}(\pi_j) = \{[z_1: z_2: \cdots: z_n] \in E: z_j = 0\} \cong \mathbb{CP}^{n-2}.
\]
Hence the critical locus may be a proper subvariety of the exceptional variety.

**Problem 1.10.** Let \( X \) be a 1-convex manifold. Which closed analytic subsets of its exceptional variety \( E \) are critical loci of holomorphic functions on \( X \)?

Going a step further, recall that a complex space \( X \) is said to be holomorphically convex if for any compact set \( K \subset X \) its \( \mathcal{O}(X) \)-convex hull
\[
\bar{K}_{\mathcal{O}(X)} = \{x \in X: |f(x)| \leq \sup_K |f| \ \forall f \in \mathcal{O}(X)\}
\]
is also compact. This class contains all 1-convex spaces, but many more. For example, the total space of any holomorphic fiber bundle \( X \to Y \) with a compact fiber over a Stein space \( Y \) is holomorphically convex. By Remmert [33], every holomorphically convex space \( X \) admits a proper holomorphic surjection \( \pi: X \to Y \) onto a Stein space \( Y \) such that the (compact) fibers of \( \pi \) are connected, \( \pi_*\mathcal{O}_X = \mathcal{O}_Y \), the map \( f \mapsto f \circ \pi \) is an isomorphism of \( \mathcal{O}(Y) \) onto \( \mathcal{O}(X) \), and every holomorphic map \( X \to S \) to a Stein space \( S \) factors through \( \pi \). If \( g \in \mathcal{O}(Y) \) is a noncritical function on \( Y \) furnished by Theorem 1.1, then the function \( f = g \circ \pi \in \mathcal{O}(X) \) is noncritical on the set where \( \pi \) is a submersion. What else could be said?

Another possible line of investigation is the following. In [13] we proved that on any Stein manifold \( X \) of dimension \( n \) there exist \( q = \lceil \frac{n+1}{2} \rceil \) holomorphic functions \( f_1, \ldots, f_q \in \mathcal{O}(X) \) with pointwise independent differentials, i.e., such that \( df_1 \wedge df_2 \wedge \cdots \wedge df_q \) is a nowhere vanishing holomorphic \((q, 0)\)-form on \( X \), and this number \( q \) is maximal in general by topological reasons. Furthermore, we have the h-principle for holomorphic submersions \( X \to \mathbb{C}^q \) any \( q < n = \dim X \), saying that every \( q \)-tuple of pointwise linearly independent continuous \((1,0)\)-forms can be deformed to a \( q \)-tuple of linearly independent holomorphic differentials \( df_1, \ldots, df_q \). What could be said regarding this problem on Stein spaces? For example:
Problem 1.11. Assume that $X$ is a pure $n$-dimensional Stein space and let $q$ be as above. Do there exist functions $f_1, \ldots, f_q \in \mathcal{O}(X)$ such that $df_1 \wedge df_2 \wedge \cdots \wedge df_q$ is nowhere vanishing on $X_{\text{reg}}$? What is the answer if $X$ has only isolated singularities?

Our methods strongly rely on the fact that the critical locus of a generic holomorphic function on a Stein space is discrete (see [2]). If $q > 1$ then the set $df_1 \wedge df_2 \wedge \cdots \wedge df_q = 0$ (if nonempty) is a subvariety of $X$ of complex dimension $\geq q - 1 > 0$, and we do not know how to ensure nonvanishing of this form on a deleted neighborhood of a subvariety of $X$ as in the case $q = 1$. The problem seems nontrivial even for a deleted neighborhood of an isolated singular point.

2. Critical points of a holomorphic function on a complex space

We begin by recalling certain basic facts of complex analytic geometry.

Let $(X, \mathcal{O}_X)$ be a reduced complex space. Following standard practice we shall simply write $X$ in the sequel. We denote by $\mathcal{O}(X) \cong \Gamma(X, \mathcal{O}_X)$ the algebra of all holomorphic functions on $X$. Given a holomorphic function $f$ on an open set $U \subset X$, we denote by $f_p \in \mathcal{O}_{X, p}$ the germ of $f$ at a point $p \in U$. Similarly, $X_p$ stands for the germ of $X$ at a point $p \in X$.

By $m_p = m_{X, p}$ we denote the maximal ideal of the local ring $\mathcal{O}_{X, p}$, so $\mathcal{O}_{X, p}/m_p \cong \mathbb{C}$. We say that $f \in \mathcal{O}_{X, p}$ vanishes to order $k \in \mathbb{N}$ at the point $p$ if $f \in m_p^k$ (the $k$-th power of the maximal ideal). The quotient ring $\mathcal{O}_{X, p}/m_p^k \cong \mathbb{C} \oplus m_p/m_p^{k+1}$ is a finite dimensional complex vector space, called the space of $k$-jets of holomorphic functions on $X$ at $p$. Recall that $m_p/m_p^2 \cong T_p X$ is the cotangent space and its dual $(m_p/m_p^2)^* \cong T_p X$ is the (Zariski) tangent space of $X$ at $p$.

If $X'$ is a complex subvariety of $X$ and $p \in X'$, then the maximal ideal $m_{X', p}$ of the ring $\mathcal{O}_{X', p}$ consists of all germs at $p$ of the restrictions $f|_{X'}$, where $f \in m_{X, p}$.

Lemma 2.1. Let $X'$ be a closed complex subvariety of a complex space $X$ and $p \in X'$. If $f \in \mathcal{O}_{X', p}$ and $h \in \mathcal{O}_{X, p}$ are such that $f - (h|_{X'})_p \in m_{X', p}^k$ for some $k \in \mathbb{N}$ (i.e., the functions $f$ and $h|_{X'}$ on $X'$ have the same $(k - 1)$-jet at $p$), then there exists $\tilde{h} \in \mathcal{O}_{X, p}$ such that $\tilde{h} - h \in m_{X, p}^k$ and $(\tilde{h}|_{X'})_p = f \in \mathcal{O}_{X', p}$.

Proof. The conditions imply that $f = (h|_{X'})_p + \sum_j \xi_{j, 1} \xi_{j, 2} \cdots \xi_{j, k}$ where $\xi_{j, i} \in m_{X', p}$ for all $i$ and $j$. Then $\xi_{j, i} = \xi_{i, j}|_{X'}$ for some $\xi_j \in m_{X, p}^k$, and the germ $\tilde{h} = h + \sum_j \xi_{j, 1} \xi_{j, 2} \cdots \xi_{j, k} \in \mathcal{O}_{X, p}$ satisfies the stated properties. $\square$

Given a function $f \in \mathcal{O}(X)$, the collection of its differentials $df_x : T_x X \to \mathbb{C}$ over all points $x \in X$ defines the tangent map $T f : TX \to X \times \mathbb{C}$ on the tangent space $TX = \bigcup_{x \in X} T_x X$. Recall that $TX$ carries the structure of a not necessarily reduced linear space over $X$ such that the tangent map $T f$ is holomorphic. Here is a local decription of $TX$ (see e.g. [10, Chapter 2]). Assume that $X$ is a complex analytic subvariety of an open set $U \subset \mathbb{C}^N$, defined by holomorphic functions $h_1, \ldots, h_m \in \mathcal{O}(U)$ which generate the sheaf of ideals $\mathcal{J}_X$ of $X$ (hence $\mathcal{O}_X \cong (\mathcal{O}_U / \mathcal{J}_X)|_X$). Let $(z_1, \ldots, z_N, \xi_1, \ldots, \xi_N)$ be complex coordinates on $U \times \mathbb{C}^N$. Then $TX$ is the closed complex subspace of $U \times \mathbb{C}^N$ generated by the functions

$$h_1, \ldots, h_m \quad \text{and} \quad \frac{\partial h_i}{\partial z_1} \xi_1 + \cdots + \frac{\partial h_i}{\partial z_N} \xi_N \quad \text{for} \quad i = 1, \ldots, m. \tag{2.1}$$

The projection $TX \to X$ is the restriction of the projection $U \times \mathbb{C}^N \to U$, $(z, \xi) \mapsto z$. Different local representations of $X$ give isomorphic representations of $TX$. If $X$ is a complex manifold
then $TX$ is the usual tangent bundle of $X$; this holds in particular over the regular locus $X_{\text{reg}}$ of any complex space. If $f: X \to Y$ is a holomorphic map of complex spaces, then there is the induced tangent map $Tf: TX \to TY$ whose description can be found in [10, pp. 81-82].

Since the critical locus $\text{Crit}(f)$ of a holomorphic function $f \in \mathcal{O}(X)$ is the set of points $x \in X$ at which the differential $df_x: T_xX \to \mathbb{C}$ vanishes, one might expect that $\text{Crit}(f)$ is a closed complex subvariety of $X$. This is clearly true if $X$ is a complex manifold (in particular, it holds on the regular locus $X_{\text{reg}}$ of any complex space), but it fails in general near singularities. Furthermore, unlike in the smooth case, the set of (strongly) noncritical holomorphic functions is not stable under small deformations. The following examples illustrate these phenomena in a simple setting of an irreducible quadratic surface singularity in $\mathbb{C}^3$.

**Example 2.2.** Let $A$ be the subvariety of $\mathbb{C}^3$ given by
\[(2.2) \quad A = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : h(z) = z_1^2 + z_2^2 + z_3^2 = 0\}.
\]
(In the theory of minimal surface this is called the **null quadric**, and complex curves in $\mathbb{C}^3$ whose derivative belongs to $A^* = A \setminus \{(0,0,0)\}$ are called **null holomorphic curves**. Such curves are related to conformally immersed minimal surfaces in $\mathbb{R}^3$. See e.g. [32] for a classical survey of this subject and [3] for some recent results.) Clearly $A_{\text{sing}} = \{(0,0,0)\}$, $A$ is locally and globally irreducible, and $T_{(0,0,0)}A = \mathbb{C}^3$. For any $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ the linear function
\[f_\lambda(z_1, z_2, z_3) = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3\]
restricted to $A$ is strongly noncritical at $(0,0,0)$. However, if $\lambda \in A^*$ then $df_\lambda$ is colinear with $dh = 2(z_1dz_1 + z_2dz_2 + z_3dz_3)$ at every point of the complex line $\Lambda = \{t\lambda : t \in \mathbb{C}\} \subset A$ (and $f_\lambda$ vanishes on this line). Since $T_2A = \ker dh_z$, the differential $(df_\lambda)_z$ vanishes on $T_2A$ for every $z \in \Lambda \setminus \{0\}$, so $\text{Crit}(f_\lambda|A) = \Lambda \setminus \{0\}$ which is not closed. An explicit example is obtained by taking
\[\lambda = (1, i, 0) \in A^*, \quad f(z) = z_1 + iz_2.
\]
(Here $i = \sqrt{-1}$.) By adding to $f$ a generic quadratic term we get a function that is noncritical on $A$. \qed

Let us now show on the same example that the set of (strongly) noncritical functions fails to be stable under small deformations.

**Example 2.3.** Let $A$ be the quadric (2.2). Consider the family of functions
\[f_\epsilon(z_1, z_2, z_3) = z_1 + z_1(z_1 - 2\epsilon) + iz_2, \quad \epsilon \in \mathbb{C}.
\]
Note that $f_\epsilon|A$ is (strongly) noncritical at the origin since $(df_\epsilon)_0(z) = (1 - 2\epsilon)z_1 + iz_2$. A calculation shows that for $\epsilon \neq 1/2$ the differentials $df_\epsilon$ and $dh$ (considered on the tangent bundle $T\mathbb{C}^3$) are colinear at points of the complex curve
\[C = \{(z_1, z_2, 0) \in \mathbb{C}^3 : z_2 = iz_1/(2z_1 - 2\epsilon + 1)\}.
\]
This curve intersects the quadric $A$ at the following points:
\[A \cap C = \{(0, 0, 0), (\epsilon, i\epsilon, 0), (-\epsilon - 1, -i(\epsilon - 1), 0)\}.
\]
Hence the second and the third of these points are the critical points of $f_\epsilon|A$ when $\epsilon \notin \{0, 1\}$. For $\epsilon$ close to 0 the second point $(\epsilon, i\epsilon, 0)$ lies close to the origin while the third point is close to $(-1, i, 0)$. Hence the function $f_0|A$ is noncritical on the intersection of $A$ with the ball of radius $1/2$ around the origin in $\mathbb{C}^3$, but $f_\epsilon|A$ for small $\epsilon \neq 0$ is close to $f_0$ and has a critical point $(\epsilon, i\epsilon, 0) \in A$ near the origin. \qed
Although we have seen in Example 2.2 that \( \text{Crit}(f) \) need not be a closed complex subvariety near singular points of a complex space, we still have the following result.

**Lemma 2.4.** Let \( f \) be a holomorphic function on a complex space \( X \). If \( X' \subset X \) is a closed complex subvariety of \( X \) containing the singular locus \( X_{\text{sing}} \), then the set

\[
C_{X'}(f) := \{ x \in X_{\text{reg}} : df_x = 0 \} \cup X'
\]

is a closed complex subvariety of \( X \).

**Proof.** By the desingularization theorem \([4, 5, 27]\) there are a complex manifold \( M \) and a proper holomorphic surjection \( \pi : M \to X \) such that \( \pi : M \setminus \pi^{-1}(X_{\text{sing}}) \to X \setminus X_{\text{sing}} \) is a biholomorphism and \( \pi^{-1}(X_{\text{sing}}) \) is a compact complex hypersurface in \( M \). Given a function \( f \in \mathcal{O}(X) \), consider the function \( F = f \circ \pi \in \mathcal{O}(M) \) and the subvariety \( M' = \pi^{-1}(X') \) of \( M \). Since \( M \) is a complex manifold, the critical locus \( \text{Crit}(F) \subset M \) is a closed complex subvariety of \( M \), and hence so is the set \( C_{M'}(F) = \text{Crit}(F) \cup M' \). As \( \pi \) is proper, \( \pi(C_{M'}(F)) \) is a closed subvariety of \( X \) according to Remmert \([34]\). Since \( \pi \) is biholomorphic over \( X_{\text{reg}} \), we have \( \pi(C_{M'}(F)) = C_{X'}(f) \) which proves the result. \( \square \)

In spite of the lack of stability of noncritical functions, illustrated by Example 2.3, we shall obtain a certain stability result (cf. Lemma 2.7 below) which will be used in the construction of stratified noncritical holomorphic functions on Stein spaces.

Given a compact set \( K \) in a complex space \( X \), we denote by \( \mathcal{O}(K) \) the space of all functions \( f \) that are holomorphic on an open neighborhood \( U_f \subset X \) of \( K \) (depending on the function), identifying two function that agree on some neighborhood of \( K \). By \( \hat{K} \) we denote the topological interior of a set \( K \).

For any coherent analytic sheaf \( \mathcal{F} \) on a complex space \( X \) the \( \mathcal{O}(X) \)-module \( \mathcal{F}(X) = \Gamma(X, \mathcal{F}) \) of all global sections of \( \mathcal{F} \) over \( X \) can be endowed with a Fréchet space topology (the topology of uniform convergence on compacta in \( X \)) such that for every point \( x \in X \) the natural restriction map \( \mathcal{F}(X) \to \mathcal{F}_x \) is continuous (see Theorem 5 in \([21\text{ p. 167}]) \). The topology on the stalks \( \mathcal{F}_x \) is the sequence topology (cf. \([23\text{ p. 86ff}]\)). Thus every set of the second category in \( \mathcal{F}(X) \) (an intersection of at most countably many open dense sets) is dense in \( \mathcal{F}(X) \). The expression generic holomorphic function on \( X \) will always mean a function in a certain set of the second category in \( \mathcal{O}(X) \), and likewise for \( \mathcal{F}(X) \).

If \( \mathcal{S} \) is a coherent subsheaf of a coherent sheaf \( \mathcal{F} \) over \( X \) then \( \mathcal{S}(X) \) is a closed submodule of \( \mathcal{F}(X) \) (the Closedness Theorem, cf. \([21\text{ p. 169}]\)). Since every \( \mathcal{O}_{X,x} \) submodule \( M \) of the module \( \mathcal{F}_x \) is closed in the sequence topology, it follows that \( \{ f \in \mathcal{F}(X) : f_x \in M \} \) is a closed subspace of \( \mathcal{F}(X) \), hence a Fréchet space.

In particular, if \( X' \) is a closed complex subvariety of a complex space \( X \) and \( \mathcal{J}_{X'} \) is the sheaf of ideals of \( X' \) (a coherent subsheaf of \( \mathcal{O}_X \)), then

\[
\mathcal{J}(X') := \Gamma(X, \mathcal{J}_{X'}) = \{ f \in \mathcal{O}(X) : f|_{X'} = 0 \}
\]

is a closed (hence Fréchet) ideal in \( \mathcal{O}(X) \). Given a function \( g \in \mathcal{O}(X') \) on a closed complex subvariety \( X' \subset X \), the set

\[
\mathcal{O}_{X',g}(X) = \{ f \in \mathcal{O}(X) : f|_{X'} = g \}
\]

is a closed affine subspace of \( \mathcal{O}(X) \) and hence a Baire space.

The closedness property shows that for any point \( x \in X \) and \( k \in \mathbb{N} \) the set

\[
\{ f \in \mathcal{O}(X) : f_x - f(x) \in \mathfrak{m}_x^k \}
\]
is closed in $\mathcal{O}(X)$. For $k = 2$ this is the set of functions with a critical point at $x$.

Let $X_x = \bigcup_{j=1}^{m_j} V_j$ be a decomposition into local irreducible components at a point $x \in X$. According to Definition 1.2, a function $f \in \mathcal{O}_{X,x}$ fails to be strongly noncritical at $x$ if there is a $j \in \{1, \ldots, m\}$ such that $(f|_{V_j})(x) - f(x) \in m^2_{V_j,x}$. This defines a closed subset of $\mathcal{O}_{X,x}$, so the set of all strongly noncritical germs is open in $\mathcal{O}_{X,x}$. Since the restrictions maps in the space of sections of a coherent sheaf are continuous, we get the following conclusion.

**Lemma 2.5.** The set of all functions $f \in \mathcal{O}(X)$ which are noncritical (or strongly noncritical) at a certain point $x \in X$ is open in $\mathcal{O}(X)$.

However, Example 2.2 above shows that the set of functions $f \in \mathcal{O}(X)$ that are noncritical (or strongly noncritical) on a certain compact set $K \subset X$ may fail to be open in $\mathcal{O}(X)$, unless $K$ is contained in $X_{\text{reg}}$.

The following result is [18, Lemma 3.1, p. 52] if $X$ is a Stein manifold; we need it also when $X$ is a Stein space. We use this occasion to correct a misprint in the statement of the lemma in the original source.

**Lemma 2.6 (Bounded extension operator).** Let $X$ be a Stein space, $X'$ a closed complex subvariety of $X$ and $\Omega \subset X$ a Stein domain in $X$. For any relatively compact subdomain $D \subset \Omega$ there exists a bounded linear extension operator $T': H^\infty(\Omega \cap X') \to H^\infty(D)$ such that

$$(Tf)(x) = f(x) \quad \forall f \in H^\infty(\Omega \cap X'), \quad \forall x \in D \cap X'.$$

**Proof.** We embed a Stein neighborhood $W \subset X$ of $\overline{\Omega}$ as a closed complex subvariety of some $\mathbb{C}^N$. Pick a Stein domain $\Omega' \subset \mathbb{C}^N$ such that $\Omega = \Omega' \cap W$. Also choose a domain $D'$ in $\mathbb{C}^N$ such that $D \subset D'$ and $\overline{D'} \subset \Omega'$. By [18, Lemma 3.1], applied with the subvariety $X' \cap W$ of the Stein manifold $\mathbb{C}^N$ and domains $D' \subset \Omega' \subset \mathbb{C}^N$, there exists a bounded linear extension operator $T': H^\infty(\Omega' \cap X') \to H^\infty(D')$. Since $\Omega' \cap W = \Omega$, we obtain by restricting the resulting function $T'f$ to $D' \subset W \cap D'$ a bounded extension operator $T$ as in the lemma. □

**Lemma 2.7 (The Stability Lemma).** Let $X$ be a complex space, $X' \subset X$ a closed complex subvariety containing $X_{\text{sing}}$, and $K \subset L$ compact subsets of $X$ with $K \subset L$. Assume that $f \in \mathcal{O}(X)$ is noncritical on $L \setminus X'$. There exist an integer $r \in \mathbb{N}$ and a number $\epsilon > 0$ such that every function $g \in \mathcal{O}(L)$ satisfying

(i) $f - g \in \Gamma(L, J^r_{X'})$, where $J^r_{X'}$ is the $r$-th power of the ideal $J_{X'}$, and

(ii) $||f - g||_L := \sup_{x \in L} |f(x) - g(x)| < \epsilon$

is noncritical on $K \setminus X'$.

**Proof.** The result holds on compact subsets of $X \setminus X' \subset X_{\text{reg}}$ in view of Lemma 2.5 so it suffices to consider the behavior of $g$ near $K \cap X'$.

Fix a point $p \in K \cap X'$ and embed an open neighborhood $U \subset X$ of $p$ as a closed complex subvariety (still denoted $U$) of an open ball $B \subset \mathbb{C}^N$. We choose $U$ small enough such that $U \subset L$. Pick a slightly smaller ball $B' \subset B$ and set $U' := B' \cap U$. Lemma 2.6 (applied with the domain $\Omega = B$ in $X = \mathbb{C}^N$, the subvariety $X' = U$, and the subdomain $D = B' \subset B$) furnishes a bounded linear extension operator $T$ mapping bounded holomorphic functions on $U$ to bounded holomorphic functions on $B'$. In the embedded picture, a point $x \in U' \setminus X' \subset B'$ is a critical point of $f$ if and only if the differential $d\tilde{f}_x: T_x \mathbb{C}^N \to \mathbb{C}$ of the extended function $\tilde{f} = Tf \in \mathcal{O}(B')$ annihilates the Zariski tangent space $T_x U$. The latter condition is expressed by a finite number of holomorphic equations $F_j(f) = 0$ on $B'$ ($j = 1, \ldots, k$) involving the
values and the first order partial derivatives of $\bar{f}$ and of some fixed holomorphic defining functions $h_1, \ldots, h_m$ for the subvariety $U$ in $B$. (These equations express the fact that the holomorphic gradient of $\bar{f}$ is contained in the linear span of the gradients of the functions $h_1, \ldots, h_m$; compare with the local description (2.1) of $TX$.) By the assumption this system of equations has no solutions on $U \setminus X'$. If a bounded function $g \in \mathcal{O}(U)$ agrees with $f$ to order $r$ along the subvariety $U \cap X'$, then by setting $\tilde{g} = Tg \in \mathcal{O}(B')$ the corresponding functions $F_j(\tilde{g})|_{U'}$ agree with the functions $F_j(f)|_{U'}$ to order $r-1$ along $U' \cap X'$. By choosing the integer $r \in \mathbb{N}$ sufficiently big and the number $\epsilon$ bounded from above by some fixed number $\epsilon_0 > 0$ we can ensure that for any $g$ satisfying conditions (i) and (ii) the system of holomorphic equations on $B' \subset \mathbb{C}^N$

$$h_1 = 0, \ldots, h_m = 0, \quad F_1(g) = 0, \ldots, F_k(g) = 0$$

has no solutions in $W \setminus X'$, where $W \subset U'$ is a neighborhood of $p$ whose size depends on $r$ and $\epsilon_0$. The details of this argument are as in the proof of [12, Theorem 1.3]; see in particular pp. 507–509 in [12]. In fact, looking at the common zero set of the system (2.4) as the inverse image of the origin $0 \in \mathbb{C}^{m+k}$ by the holomorphic map $B' \to \mathbb{C}^{m+k}$ whose components are the functions in (2.4), the local aspect of the cited result from [12] applies verbatim.) Since finitely many open sets $U'$ of this kind cover $K \cap X'$, we see that (2.4) has no solutions on a deleted neighborhood of $K \cap X'$ in $K$. By choosing $\epsilon > 0$ small enough we also ensure in view of Lemma 2.5 that there are no solutions on the rest of $K$. \hfill\Box

Example 2.8. The cusp curve $X = \{(z, w) \in \mathbb{C}^2: z^2 = w^3\}$ has a singularity at the origin $(0,0) \in \mathbb{C}^2$ and is smooth elsewhere. It is desingularized by the map $\pi: \mathbb{C} \to X, \pi(t) = (t^3, t^2)$. The function $f(z, w) = zw$ on $X$ pulls back to the function $h(t) = f(\pi(t)) = t^5$ with the only critical point at $t = 0$, so $f|_X$ is stratified noncritical with respect to $\{(0,0)\} \subset X$. The perturbation of $h$ given by

$$h_\epsilon(t) = t^3(t - \epsilon)^2 = t^5 - 2\epsilon t^4 + \epsilon^2 t^3 = zw - 2\epsilon w^2 + \epsilon^2 z$$

induces a holomorphic function $f_\epsilon: X \to \mathbb{C}$ with a critical point at $(\epsilon^3, \epsilon^2)$, so $f_\epsilon$ is not stratified noncritical on $\{(0,0)\} \subset X$ if $\epsilon \neq 0$. \hfill\Box

We now prove some results concerning the critical locus of a generic holomorphic function on a Stein space.

Lemma 2.9 (The Genericity Lemma). Let $X$ be a Stein space.

(i) For a generic $f \in \mathcal{O}(X)$ the set $A(f) := \text{Crit}(f|_{X_{\text{sing}}})$ is discrete in $X$.

(ii) If $X' \subset X$ a closed complex subvariety containing $X_{\text{sing}}$ and $g \in \mathcal{O}(X')$, then a generic $f \in \mathcal{O}_{X' \setminus g}(X)$ \[2.3\] is such that $\text{Crit}(f|_{X' \setminus X'})$ is discrete in $X$. In particular, a generic holomorphic extension of $g$ is noncritical on a small deleted neighborhood of $X'$ in $X$.

(iii) If $g$ is a holomorphic function on an open neighborhood of $X'$ in $X$ and $n \in \mathbb{N}$, then the conclusion of part (ii) holds for a generic extension $f \in \mathcal{O}(X)$ of $g|_{X'}$, which agrees with $g$ to order $n$ along $X'$. 
Proof. We begin by proving part (i). A point $x \in X_{\text{reg}}$ is a critical point of a holomorphic function $f : X \to \mathbb{C}$ if and only if the partial derivatives $\partial f / \partial z_j$ in any system of local holomorphic coordinates $z = (z_1, \ldots, z_n)$ on an open neighborhood $U \subset X_{\text{reg}}$ of $x$ vanish at the point $z(x)$. (Here $n = \dim_x X$.) This gives $n$ independent holomorphic equations on the 1-jet extension $j_1^* f$ of $f$, so the jet transversality theorem for holomorphic maps $X \to \mathbb{C}$ (cf. [11] or [15, §7.8]) implies that every point $x \in A(f)$ is an isolated point of $A(f)$ for a generic $f \in \mathcal{O}(X)$. (The argument goes as follows: write $X_{\text{reg}} = \bigcup_{j=1}^\infty U_j$ where $U_j \subset X_{\text{reg}}$ is a compact connected coordinate neighborhood for every $j$. The set of all functions $f \in \mathcal{O}(X)$ whose 1-jet extension $U_j \ni x \mapsto j_1^* f \in \mathbb{C}^{n_j}$ (with $n_j = \dim U_j$) is transverse to $0 \in \mathbb{C}^{n_j}$ on the compact set $U_j$, is open and dense in $\mathcal{O}(X)$. Taking the countable intersection of these sets over all $j$ gives the statement.) For any such $f$ the set $A(f)$ is discrete in $X_{\text{reg}}$, and we claim that $A(f)$ is then also discrete in $X$. If not, there is a point $x_0 \in X_{\text{sing}}$ and a sequence $x_j \in A(f)$ with $\lim_{j \to \infty} x_j = x_0$. However, since $C(f) = A(f) \cup X_{\text{sing}}$ is a closed complex subvariety of $X$ by Lemma 2.4, this contradicts the fact that a compact subset of a complex space intersects at most finitely many of its irreducible components.

Part (ii) follows similarly by applying the jet transversality theorem in the Baire space $\mathcal{O}(X,g) = \{ f \in \mathcal{O}(X) : f|_{X'} = g \}$.

Finally let $g$ be as in (iii). Consider the short exact sequence of coherent sheaves $0 \to \mathcal{J}^0_{X'} \to \mathcal{O}_{X'} \to \mathcal{O}_X / \mathcal{J}^0_{X'} \to 0$. The sheaf $\mathcal{O}_X / \mathcal{J}^0_{X'}$ is supported on $X'$ and hence $g$ determines a section of it. Since $H^1(X; \mathcal{J}^0_{X'}) = 0$ by Cartan’s Theorem B, the same section is determined by a function $G \in \mathcal{O}(X)$. This means that $G - g$ vanishes to order $n$ along $X'$. It suffices to apply the transversality theorem in the Baire space $G + \mathcal{J}^0_{X'}(X) \subset \mathcal{O}(X)$; the details are similar as in part (i).

\[\text{Proposition 2.10. If } (X, \Sigma) \text{ is a a stratified Stein space, then the set } \bigcup_{S \in \Sigma} \text{Crit}(f|_S) \text{ is discrete in } X \text{ for a generic } f \in \mathcal{O}(X).\]

Proof. Let $\Sigma = \{S_j\}_j$ where $S_j$ are the (smooth) strata. Each stratum $S_j$ of positive dimension $n_j > 0$ is a union $S_j = \bigcup_k U_{j,k}$ of countably many compact coordinate sets $U_{j,k}$. The same argument as in the proof of Proposition 2.9 shows that set $U_{j,k} \subset \mathcal{O}(X)$, consisting of all $f \in \mathcal{O}(X)$ such that the 1-jet extension $U_{j,k} \ni x \mapsto j_1^* f \in \mathbb{C}^{n_j}$ is transverse to $0 \in \mathbb{C}^{n_j}$ on $U_{j,k}$, is open and dense in $\mathcal{O}(X)$. Hence the intersection $\bigcap_{j,k} U_{j,k}$ is a dense subset of $\mathcal{O}(X)$, and every $f$ in this set satisfies the conclusion of the proposition.

Since every complex space admits a stratification, Proposition 2.10 implies

\[\text{Corollary 2.11. A generic holomorphic function on a Stein space has discrete critical locus.}\]

We also have the following result in which $X$ is not necessarily Stein.

\[\text{Corollary 2.12. Let } X \text{ be a complex space and } X' \subset X \text{ a closed Stein subvariety containing } X_{\text{sing}}. \text{ Given a function } g \in \mathcal{O}(X'), \text{ there are an open neighborhood } U \subset X \text{ of } X' \text{ and a function } f \in \mathcal{O}(U) \text{ such that } f|_{X'} = g \text{ and } f \text{ has no critical points in } U \setminus X'.\]

Proof. According to Siu [36] (see also [15, §3.1] and the additional references therein) a Stein subvariety $X'$ in any complex space $X$ admits an open Stein neighborhood $\Omega \subset X$ containing $X'$ as a closed complex subvariety. The conclusion then follows from Lemma 2.9 applied to the Stein space $\Omega$. \[\square\]

In the proof of Theorem 3.2 we shall also need the following result.
Lemma 2.13. Let $X$ be a reduced complex space and $U \subseteq U'$ open relatively compact sets in $X$. Fix a distance function $\text{dist}$ on $X$ inducing the standard topology. There is a constant $\epsilon > 0$ such that for any holomorphic map $f : U' \to X$ satisfying $\sup_{x \in U'} \text{dist}(x, f(x)) < \epsilon$ the restriction $f|_U : U \to f(U) \subseteq X$ is biholomorphic onto its image.

Proof. This result is well known when $X$ is a complex manifold (i.e., without singularities), and we reduce the proof to this particular case.

We first prove the lemma in the case when $U'$ is Stein and its closure $\overline{U}'$ admits a Stein neighborhood $W$ in $X$. Assuming as we may that $W$ is relatively compact, it has bounded embedding dimension and therefore it embeds as a closed complex subvariety of a Euclidean space $\mathbb{C}^N$ (see [15] Theorem 2.2.8 and the references therein). Since $U'$ is Stein, there is a bounded Stein domain $D' \subseteq \mathbb{C}^N$ such that $D' \cap W = U'$ (here we can use Siu’s theorem [36]). Choose a pair of domains $D_0 \subseteq D$ in $\mathbb{C}^N$ such that $\overline{U} \subseteq D_0 \cap W$ and $\overline{D} \subseteq D'$. Let $T$ be a bounded linear extension operator furnished by Lemma 2.6, mapping bounded holomorphic functions on $U'$ to bounded holomorphic functions on $D$ and satisfying

$$Tg|_{D \setminus U'} = g|_{D \setminus U'} \quad \text{and} \quad ||Tg||_D \leq C||g||_{U'}$$

for some constant $C > 0$ independent of $g$.

Consider a holomorphic map $f : U' \to X$ close to the identity. We may assume that $f(U') \subseteq W \subseteq \mathbb{C}^N$. Write $f(x) = x + g(x)$ for $x \in U'$, where $g : U' \to \mathbb{C}^N$ is close to zero. Applying the operator $T$ to each component of $g$ we get a holomorphic map $F = \text{Id} + Tg : D \to \mathbb{C}^N$ which is close to the identity in the sup norm on $D$. Hence $F$ is biholomorphic on the smaller domain $D_0$ provided that $f$ is close enough to the identity on $U'$. Since $U \subseteq D_0$ and $F|_U = \text{Id}_U + g|_U = f|_U$, we infer that $f : U \to f(U) \subseteq W$ is biholomorphic as well. Furthermore, the inverse map $F^{-1} : F(D_0) \to D_0$ restricted to $F(D_0) \cap W$ has range in $W$ as is easily seen by considering the situation on $W_{\text{reg}}$ and applying the identity principle.

The general case follows as in the standard manifold situation; we include it for completeness. By compactness of $\overline{U}$ we can choose finitely many triples of open sets $V_j \subseteq U_j \subseteq U_j'$ in $X$ $(j = 1, \ldots, m)$ such that

(i) $\overline{U} \subseteq \bigcup_{j=1}^m V_j$ and $\bigcup_{j=1}^m U_j' \subseteq U'$,

(ii) $U_j'$ is Stein and $\overline{U}_j$ has a Stein neighborhood in $X$ for every $j = 1, \ldots, m$.

Pick a number $\epsilon_0 > 0$ such that $\text{dist}(V_j, X \setminus U_j) > 2\epsilon_0$ for every $j = 1, \ldots, m$. By the special case proved above, applied to the pair $U_j \subseteq U_j'$, we can find a number $\epsilon \in (0, \epsilon_0)$ such that $f|_{U_j} : U_j \to f(U_j)$ is biholomorphic for every $j$ provided that $\text{dist}(x, f(x)) < \epsilon$ for all $x \in U'$. Since $U \subseteq \bigcup_{j=1}^m U_j$, it follows that $f|_U : U \to f(U)$ is biholomorphic as long as it is injective. Suppose that $f(x) = f(y)$ for a pair of point $x \neq y$ in $U$. Since the sets $V_j$ cover $U$, we have $x \in V_j$ for some $j$. As $f$ is injective on $U_j$, it follows that $y \in U \setminus U_j$ and hence $\text{dist}(x, y) > 2\epsilon_0$. The triangle inequality and the choice of $\epsilon$ then gives

$$\text{dist}(f(x), f(y)) \geq \text{dist}(x, y) - \text{dist}(x, f(x)) - \text{dist}(y, f(y)) > 2\epsilon_0 - 2\epsilon > 0,$$

a contradiction to $f(x) = f(y)$. Thus $f$ is injective on $U$. \hfill \square

3. A splitting lemma for biholomorphic maps on complex spaces

In this section we prove a splitting lemma for biholomorphic maps close to the identity on certain Cartan pairs in complex spaces (see Theorem 3.2 below). This result will be used for
For every number \( \eta > \gamma \) show that the maps \( \alpha \) of class \( A \) sets glueing pairs of holomorphic functions with control of their critical loci. The nonsingular case is given by [13, Theorem 4.1].

Recall that a compact set \( K \) in a complex space \( X \) is said to be a Stein compact if \( K \) admits a basis of open Stein neighborhoods in \( X \).

**Definition 3.1.** [15] p. 209 (I) A pair \((A, B)\) of compact subsets in a complex space \( X \) is a Cartan pair if it satisfies the following conditions:

(i) \( A, B, D = A \cup B \) and \( C = A \cap B \) are Stein compacta, and
(ii) \( A, B \) are separated in the sense that \( A \setminus B \cap B \setminus A = \emptyset \).

(II) A pair \((A, B)\) of open sets in a complex manifold \( X \) is a strongly pseudoconvex Cartan pair of class \( C^{\ell} (\ell \geq 2) \) if \((A, B)\) is a Cartan pair in the sense of (I) and the sets \( A, B, D = A \cup B \) and \( C = A \cap B \) are Stein domains with strongly pseudoconvex boundaries of class \( C^{\ell} \).

We shall use the following properties of Cartan pairs:

(a) Let \((A, B)\) be a Cartan pair in a complex space \( X \). If \( X \) is a complex subspace of another complex space \( \tilde{X} \), then \((A, B)\) is also a Cartan pair in \( \tilde{X} \) (cf. [15, Lemma 5.7.2, p. 210]).

(b) Every Cartan pair \((A, B)\) in a complex manifold \( X \) can be approximated from the outside by smooth strongly pseudoconvex Cartan pairs (cf. [15, Proposition 5.7.3, p. 210]).

(c) One can solve any Cousin-I problem with sup-norm bounds on a strongly pseudoconvex Cartan pair (cf. [15, Lemma 5.8.2, p. 212]).

We denote by \( \text{dist} \) a distance function on \( X \) which induces its standard complex space topology. (The precise choice will not be important.)

Given a compact set \( K \subset X \) and continuous maps \( f, g: K \to X \), we shall write

\[
\text{dist}_K(f, g) = \sup_{x \in K} \text{dist}(f(x), g(x)).
\]

By \( \text{Id} \) we denote the identity map; its domain will always be clear from the context.

**Theorem 3.2** (Splitting Lemma). Assume that \( X \) is a complex space and \( X' \) is a closed complex subvariety of \( X \) containing the singular locus \( X_{\text{sing}} \). Let \((A, B)\) be a Cartan pair in \( X \) such that \( C := A \cap B \subset X \setminus X' \). For any open set \( \tilde{C} \subset X \) containing \( C \) there exist open sets \( A' \supset A, B' \supset B, C' \supset C \) in \( X \), with \( C' \subset A' \cap B' \subset \tilde{C} \), satisfying the following property. For every number \( \eta > 0 \) there exists a number \( \epsilon_\eta > 0 \) such that for each holomorphic map \( \gamma: \tilde{C} \to X \) with \( \text{dist}_C(\gamma, \text{Id}) < \epsilon_\eta \) there exist biholomorphic maps \( \alpha = \alpha_\gamma: A' \to \alpha(A') \subset X \) and \( \beta = \beta_\gamma: B' \to \beta(B') \subset X \) satisfying the following properties:

(a) \( \gamma \circ \alpha = \beta \) on \( C' \),
(b) \( \text{dist}_{X'}(\alpha, \text{Id}) < \eta \) and \( \text{dist}_{B'}(\beta, \text{Id}) < \eta \), and
(c) \( \alpha \) and \( \beta \) are tangent to the identity map to any given finite order along the subvariety \( X' \) intersected with their respective domain.

The crucial property (a) gives a compositional splitting of \( \gamma \). In view of Lemma [2.13] we can assume (after shrinking the set \( \tilde{C} \) if necessary) that \( \gamma \) is biholomorphic onto its image; our construction will then give biholomorphic maps \( \alpha \) and \( \beta \). As in [13], the proof will also show that the maps \( \alpha_\gamma \) and \( \beta_\gamma \) can be chosen to depend continuously on \( \gamma \) such that \( \alpha_{\text{Id}} = \text{Id} \) and \( \beta_{\text{Id}} = \text{Id} \).
The proof follows in spirit that of [13] Theorem 4.1. We embed a Stein neighborhood of the Stein compact $D = A \cup B$ in $X$ as a closed complex subvariety of a complex Euclidean space $\mathbb{C}^N$. We then use a holomorphic retraction onto $X$ over a neighborhood of the Stein compact $C = A \cap B \subset X_{\text{reg}}$ in order to transport each inductive step in the splitting problem to a suitable 1-parameter family of strongly pseudoconvex Cartan pairs in $\mathbb{C}^N$ (see Lemma 3.4). From this point on we perform a Nash-Moser type iteration, similar to the one that was used in [13], in which the domains of the maps shrink by a controlled amount at every step and the error term converges to zero quadratically. Admittedly the proof is technically more complicated than I would have wished, but I was unable to prove it by a direct application of the corresponding result in the smooth case. The main problem is that the proof of [13, Theorem 4.1] does not ensure that the biholomorphic maps $\alpha$ and $\beta$ preserve $X$ as a subvariety of the ambient space. We now turn to the details.

Proof of Theorem 3.2. Replacing $X$ by an open Stein neighborhood of $D$ we may assume that $X$ is a closed complex subvariety of a Euclidean space $\mathbb{C}^N$ [15] Theorem 2.2.8. The pair $(A, B)$ is then also a Cartan pair in $\mathbb{C}^N$ [15] Lemma 5.7.2, p. 210. We shall assume that the distance function on $X$ is induced by the Euclidean distance on $\mathbb{C}^N$.

By Cartan’s Theorem A there exist functions $h_1, \ldots, h_l \in \mathcal{O}(\mathbb{C}^N)$ such that

$$X = \{z \in \mathbb{C}^N : h_i(z) = 0, \ i = 1, \ldots, l\}$$

and $h_1, \ldots, h_l$ generate the ideal sheaf $\mathcal{I}_X$. (We shall only need finite ideal generation on compact subsets of $\mathbb{C}^N$, but in our case it actually holds globally since $X$ is a relatively compact subset of the original Stein space.) Consider the analytic subsheaf $\mathcal{T}_X \subset \mathcal{O}_{\mathbb{C}^N}$ whose stalk $\mathcal{T}_{X,p}$ at any point $p \in \mathbb{C}^N$ consists of all $N$-tuples $(g_1, \ldots, g_N) \in \mathcal{O}_{\mathbb{C}^N,p}^N$ satisfying

$$\sum_{j=1}^N g_j \frac{\partial h_i}{\partial z_j} \in \mathcal{J}_{X,p}, \ i = 1, \ldots, l.$$ 

The condition is void when $p \notin X$, while at points $p \in X$ it means that the vector $V(p) = (g_1(p), \ldots, g_N(p)) \in \mathbb{C}^N \cong T_p \mathbb{C}^N$ is Zariski tangential to $X$. Observe that $\mathcal{T}_X$ is the preimage of the coherent subsheaf $\left(\mathcal{J}_X^l\right)^t \subset \mathcal{O}_{\mathbb{C}^N}$ under the homomorphism $\sigma: \mathcal{O}_{\mathbb{C}^N}^l \to \mathcal{O}_{\mathbb{C}^N}$ whose $i$-th component equals $\sigma_i(g_1, \ldots, g_N) = \sum_{j=1}^N g_j \frac{\partial h_i}{\partial z_j}$. Therefore $\mathcal{T}_X$ is a coherent analytic subsheaf of $\mathcal{O}_{\mathbb{C}^N}^N$. Sections of $\mathcal{T}_X$ are holomorphic vector fields on $\mathbb{C}^N$ which are tangential to $X$. Note that the quotient $\mathcal{T}_X / \mathcal{J}_X \mathcal{T}_X$ restricted to $X$ is the tangent sheaf of $X$ [10].

Denote by $\mathcal{J}_X$ the sheaf of ideals of the subvariety $X' \subset X$. Fix an integer $n_0 \in \mathbb{N}$ and consider the coherent analytic sheaf $\mathcal{E} := \mathcal{J}_X^{n_0} \mathcal{T}_X$ on $\mathbb{C}^N$. By Cartan’s Theorem A there exist sections $V_1, \ldots, V_m$ of $\mathcal{E}$ which generate $\mathcal{E}$ over the compact set $C = A \cap B \subset X \setminus X'$. These sections are holomorphic vector fields on $\mathbb{C}^N$ which are tangential to $X$ and vanish to order $n_0$ on the subvariety $X'$. Furthermore, as $C$ is contained in the regular locus $X_{\text{reg}}$ and $TX_{\text{reg}}$ is the usual tangent bundle, the vectors $V_1(p), \ldots, V_m(p) \in T_p \mathbb{C}^N$ actually span the tangent space $T_p X \subset T_p \mathbb{C}^N$ at every point $p \in X$ in a neighborhood of $C$. (This need not hold at singular points of $X$ where the vectors $V_j(p)$ may actually vanish.)

Denote by $\phi_t^j$ the local holomorphic flow of $V_j$ for a complex value of time $t$. (For each point $z \in \mathbb{C}^N$ the flow $\phi_t^j(z)$ is defined for $t$ in a neighborhood of $0 \in \mathbb{C}$.) Let $t = (t_1, \ldots, t_m)$ be holomorphic coordinates on $\mathbb{C}^m$. The map

$$s(z, t) = s(z, t_1, \ldots, t_m) = \phi_{t_1}^1 \circ \cdots \circ \phi_{t_m}^m(z), \ z \in \mathbb{C}^N,$$ 

(3.1)
is defined and holomorphic on an open neighborhood of $\mathbb{C}^N \times \{0\}^m$ of $\mathbb{C}^N \times \mathbb{C}^m$ and takes values in $\mathbb{C}^N$. Since the vector fields $V_j$ are tangential to $X$, we have $s(z,t) \in X$ whenever $z \in X$. For any point $z \in X$ we denote by

$$Vd(s)_z = \left. \frac{\partial}{\partial t} \right|_{t=0} s(z,t) : \mathbb{C}^m \to T_zX$$

the partial differential of $s$ in the fiber direction at the point $z$. We call $Vd(s)$ the vertical derivative of $s$ over the subvariety $X$. The definition of the flow implies

$$\left. \frac{\partial s(z,t)}{\partial t} \right|_{t=0} = V_j(z), \quad j = 1, \ldots, m.$$

Since the vectors $V_1(z), \ldots, V_m(z)$ span the tangent space $T_zX$ at every point $z \in C$, the vertical derivative $Vd(s)$ is surjective over a neighborhood of $C$. In the language of sprays, $s$ is a local holomorphic spray on $\mathbb{C}^N$, and the restriction of $s$ to a neighborhood of $C$ in $X$ is dominating spray on $X$; cf. [15, p. 203].

Fix an open Stein set $U_0 \Subset X \setminus X' \subset X_{\text{reg}}$ such that $C \subset U_0$ and $Vd(s)$ is surjective over $\overline{U}_0$. It follows that $U_0 \times \mathbb{C}^m = E \oplus E'$ where $E' = \ker Vd(s)|_{U_0}$ and $E$ is some complementary to $E'$ holomorphic vector subbundle of $U_0 \times \mathbb{C}^m$. (Such $E$ exists since $U_0$ is Stein and hence every holomorphic vector subbundle splits the bundle.) Then $Vd(s) : E|_{U_0} \to TX|_{U_0} = TU_0$ is an isomorphism of holomorphic vector bundles. By the inverse mapping theorem the restriction of $s$ to the fiber $E_z$ for any $z \in U_0$ maps an open neighborhood of the origin in $E_z$ biholomorphically onto an open neighborhood of the point $z$ in $X$. Shrinking $U_0$ slightly around $C$ we get an open set $U_1 \supset C$ in $X$ such that the following holds (cf. [13, Lemma 4.4]).

**Lemma 3.3.** There are a neighborhood $U_1 \subset X \setminus X'$ of $C$ and constants $\epsilon_1 > 0$ and $M_1 \geq 1$ such that for every open set $U \subset U_1$ and every holomorphic map $\gamma : U \to \gamma(U) \subset X$ satisfying $\text{dist}_U(\gamma, \text{Id}) < \epsilon_1$ there exists a unique holomorphic section $c : U \to E|U$ satisfying

$$\gamma(z) = s(z, c(z)) \quad \forall z \in U, \quad M_1^{-1} \text{dist}_U(\gamma, \text{Id}) \leq ||c||_U \leq M_1 \text{dist}_U(\gamma, \text{Id}).$$

Since $E$ is a subbundle of the trivial bundle $U \times \mathbb{C}^m$, we may consider any section $c$ in Lemma 3.3 as a holomorphic map $U \to \mathbb{C}^m$, and $||c||_U$ denotes the sup-norm of $c$ on $U$ measured with respect to the Euclidean metric on $\mathbb{C}^m$.

Since $C = A \cap B \subset X_{\text{reg}}$ is a Stein compact, the Docquier-Grauert theorem [8] (see also [15, Theorem 3.3.3, p. 67]) furnishes an open neighborhood $\Omega \Subset \mathbb{C}^N$ of $C$ and a holomorphic retraction

$$\rho : \Omega \to \Omega \cap X \Subset X_{\text{reg}}.$$ (3.3)

The map

$$T : \mathcal{O}(\Omega \cap X) \to \mathcal{O}(\Omega), \quad c \mapsto Tc = c \circ \rho, \quad \text{is then a bounded extension operator satisfying } ||Tc||_\Omega = ||c||_{\Omega \cap X}. \quad \text{By choosing } \Omega \text{ small enough we may assume that } \overline{\Omega} \cap X \subset \overline{C}, \text{ where } \overline{C} \text{ is as in the statement of Theorem 3.2. We fix } \Omega \text{ and the retraction } \rho \text{ for the rest of the proof.}$$

We now construct a special family of strongly pseudoconvex Cartan pairs.

**Lemma 3.4.** Assume that $X$ be a closed complex subvariety of $\mathbb{C}^N$. Let $(A,B)$ be a Cartan pair in $X$ such that $C := A \cap B \subset X_{\text{reg}}$. Let $\rho : \Omega \to \Omega \cap X$ be a holomorphic retraction with $C \subset \Omega \cap X$. Let $U_A, U_B$ be open sets in $\mathbb{C}^N$ such that $A \subset U_A$, $B \subset U_B$, and $U_A \cap U_B \subset \Omega$. There exists a family of smoothly bounded strongly pseudoconvex Cartan pairs
small (cf. [15, Lemma 5.7.4]). This holds in particular for our Car
tan pair ($A, B$) in $\mathbb{C}^N$, depending smoothly on the parameter $t \in [0, t_0]$ for some $t_0 > 0$, satisfying
the following properties:

(i) For any pair of numbers $t, \tau$ such that $0 \leq t < \tau \leq t_0$ we have $A \subset A_t \subset A_{\tau} \subset U_A$, $B \subset B_t \subset B_{\tau} \subset U_B$, and

\[
\left( \bigcup_{t \in [0, t_0]} A_t \setminus B_t \right) \cap \left( \bigcup_{t \in [0, t_0]} B_t \setminus A_t \right) = \emptyset.
\]

(ii) Set $C_t := A_t \cap B_t$. For any $0 \leq t < \tau \leq t_0$ the distances $\text{dist}(A_t, \mathbb{C}^N \setminus A_{\tau})$, $\text{dist}(B_t, \mathbb{C}^N \setminus B_{\tau})$, and $\text{dist}(C_t, \mathbb{C}^N \setminus C_{\tau})$ are $\geq \tau - t > 0$.

(iii) The boundaries $bA_t$, $bB_t$, and $bC_t$ intersect $X$ transversely at any intersection point belonging to $\Omega \cap X$ for every $t \in [0, t_0]$.

(iv) $\rho(C_t) = C_t \cap X$ for every $t \in [0, t_0]$.

Proof. We follow the proof of Proposition 5.7.3 in [15] p. 210, modifying the construction to also obtain property (iv) which will be crucial in our application. (The first three properties can be obtained directly from the cited result.) For a similar result in a geometrically simpler setting see [37] Proposition 4.4.

In the proof we shall use the function $\text{rmax}\{x, y\}$ on $\mathbb{R}^2$, the regularized maximum of $x$ and $y$ (see e.g. [15] p. 61]). It depends on a positive parameter which will be chosen as close to zero as needed at each application. We have $\max\{x, y\} \leq \text{rmax}\{x, y\}$, the two functions equal outside a small conical neighborhood of the diagonal $\{x = y\}$ in $\mathbb{R}^2$, and they can be made arbitrarily close by choosing the parameter small enough. An important point is that $\text{rmax}$ of two smooth strongly plurisubharmonic functions is still smooth strongly plurisubharmonic. The geometric interpretation is that the domain $\{\text{rmax}\{\phi, \psi\} < 0\}$ is obtained by smoothing the corners of the intersection $\{\phi < 0\} \cap \{\psi < 0\}$.

Since the set $C = A \cap B$ is a Stein compact, we can choose a smoothly bounded strongly pseudoconvex domain $V \Subset U_A \cap U_B$ such that $C \subset V$ and $bV$ intersects $X$ transversely. (Note that $\overline{V} \cap X \subset \Omega \cap X \subset X_{\text{reg}}$, so transversality makes sense.) Let $\theta$ be a smooth strongly plurisubharmonic defining function for $V$ defined on an open neighborhood $V_0 \subset U_A \cap U_B$ of $\overline{V}$.

Given a subset $A \subset \mathbb{C}^N$ and a number $r > 0$, we set

\[
A(r) = \{z \in \mathbb{C}^N : |z - p| < r \text{ for some } p \in A\}.
\]

It is elementary that $(A \cup B)(r) = A(r) \cup B(r)$ and $(A \cap B)(r) \subset A(r) \cap B(r)$; furthermore, if $A$ and $B$ are compact and separated (cf. condition (ii) in Def. 3.11), then for all sufficiently small $r > 0$ we also have

\[
(A \cap B)(r) = A(r) \cap B(r), \quad \overline{A(r) \setminus B(r) \cap B(r) \setminus A(r)} = \emptyset
\]

(cf. [15] Lemma 5.7.4)]. This holds in particular for our Cartan pair $(A, B)$. By choosing $r > 0$ small enough we can also ensure that

\[
(3.5) \quad C(r) = A(r) \cap B(r) \Subset V.
\]

Fix such a number $r$. Since $A \cup B$ is a Stein compact, there is a smoothly bounded strongly pseudoconvex Stein domain $\Omega_0 \subset \mathbb{C}^N$ such that

\[
A \cup B \subset \Omega_0 \Subset A(r) \cup B(r).
\]
Pick a smooth strongly plurisubharmonic defining functions \( \phi : \Omega_0' \to \mathbb{R} \) for \( \Omega_0 \) defined on an open set \( \Omega_0' \supset \Omega_0 \), so \( \Omega_0 = \{ z \in \Omega_0' : \phi(z) < 0 \} \) and \( d\phi \neq 0 \) on \( \partial \Omega_0 = \{ \phi = 0 \} \). We may assume that \( \Omega_0' \in A(r) \cup B(r) \). Choose \( \epsilon_0 > 0 \) such that

\[
(\text{3.6}) \quad \Omega_1' := \{ z \in \Omega_0' : \phi(z) < 3\epsilon_0 \} \in \Omega_0'.
\]

By \cite{35} Lemma 2.2 (see also \cite{37} Proposition 4.1) there exists a smooth function \( \psi \geq 0 \) on \( \mathbb{C}^N \) such that \( \{ \psi = 0 \} = X \), \( \psi \) is strongly plurisubharmonic on \( \mathbb{C}^N \setminus X = \{ \psi > 0 \} \), and the Levi form of \( \psi \) at any point \( z \in X \) is positive except perhaps on the tangent space \( T_zX \).

Choose a smooth function \( \chi : \mathbb{C}^N \to [0,1] \) which equals 0 on a neighborhood of \( \overline{A} \cap \overline{B} \subset \Omega \) and equals 1 on a neighborhood of \( \mathbb{C}^N \setminus \Omega \). Since the function \( \phi \) is strongly plurisubharmonic on \( \Omega_0' \), there is a number \( \epsilon \in (0,\epsilon_0) \) such that the functions \( \phi - 2\epsilon \chi \) and \( \phi + \epsilon \chi \) are strongly plurisubharmonic on \( \Omega_1' \). Fix such \( \epsilon \).

Given constants \( M > 0, M' > 0 \) (to be determined later) we consider the following functions:

\[
(\text{3.7}) \quad \Phi_1 = (\phi - 2\epsilon \chi) \circ \rho + M \psi, \quad \Phi_2 = \phi - 2\epsilon + \epsilon \chi + M' \psi, \quad \Phi = \operatorname{rmax}\{ \Phi_1, \Phi_2 \}.
\]

The function \( \Phi_1 \) is defined on \( \Omega \cap \Omega_0' \) (since the retraction \( \rho \) is defined on \( \Omega \)), while \( \Phi_2 \) is defined on \( \Omega_0' \). We shall see that for suitable choices of \( M > 0, M' > 0 \) the function \( \Phi \) is well defined, smooth and strongly plurisubharmonic on the domain \( \Omega_1' = \{ \phi < 3\epsilon_0 \} \), and we have

\[
(\text{3.8}) \quad A \cup B \subset D_t := \{ z \in \Omega_1' : \Phi(z) < t \} \in \Omega_1'
\]

for all \( t \in \mathbb{R} \) sufficiently close to 0. We shall also obtain a decomposition \( D_t = A_t \cup B_t \) where \((A_t, B_t)\) is a Cauntin pair satisfying the conclusions of the lemma, except perhaps (ii), for a suitable range of \( t \) near 0. To obtain also property (ii) it will suffice to rescale the parameter interval. Let us now prove these claims.

Since \( \phi + \epsilon \chi \) is strongly plurisubharmonic on \( \Omega_1' \) and \( \psi \) is plurisubharmonic on \( \mathbb{C}^N \), the function \( \Phi_2 \) is strongly plurisubharmonic on \( \Omega_1' \) for any choice of \( M' > 0 \).

Consider now the function \( \Phi_1 \). By the choice of \( \epsilon \) the function \( \phi - 2\epsilon \chi \) is strongly plurisubharmonic on \( \Omega_1' \), and hence \( \Phi_1 \) is strongly plurisubharmonic on \( \Omega \cap \Omega_1' \). Indeed, the first summand \( (\phi - 2\epsilon \chi) \circ \rho \) is plurisubharmonic, and its Levi form at any point \( z \in \Omega \cap X \) is positive definite in the directions tangential to \( X \) (since \( \rho|_X \) is the identity on \( X \)). The second summand \( M \psi \) is strongly plurisubharmonic on \( \mathbb{C}^N \setminus X \), and its Levi form at points of \( X \) is positive in directions that are not tangential to \( X \). Hence the Levi form of the sum is positive everywhere. Observe that \( \Phi_1 > 0 \) near \( b\Omega_1' \cap \Omega \cap X \) by the definition of \( \Omega_1' \) (3.6) and the fact that \( 2\epsilon \chi \leq 2\epsilon < 2\epsilon_0 \). By choosing the number \( M > 0 \) sufficiently big we can thus ensure that \( \Phi_1 > 0 \) near \( b\Omega_1' \cap \Omega \). We fix such \( M \) for the rest of the proof.

Next we show that the regularized maximum of \( \Phi_1 \) and \( \Phi_2 \) in (3.7) is well defined if the constant \( M' > 0 \) in \( \Phi_2 \) is chosen big enough. To this end, we need to ensure that \( \Phi_1 < \Phi_2 \) holds on the domain of \( \Phi_1 \) near the boundary of \( \Omega \), so \( \Phi_2 \) takes over in \( \operatorname{rmax} \) before we run out of the domain of \( \Phi_1 \). On \( X = \{ \psi = 0 \} \) this is clear since near \( b\Omega \) we have \( \chi = 1 \) and hence \( \Phi_1 = \phi - 2\epsilon < \phi - \epsilon = \Phi_2 \). By choosing the constant \( M' > 0 \) sufficiently big we get the same inequality \( \Phi_1 < \Phi_2 \) on a neighborhood of \( b\Omega \cap \Omega_1' \). Hence \( \operatorname{rmax}\{ \Phi_1, \Phi_2 \} \) is well defined on \( \Omega_1' \). By increasing \( M' \) we can also ensure that \( \Phi > 0 \) holds near the boundary of \( \Omega_1' \), so the domains \( D_t = \{ \Phi < t \} \) for \( t \) close to 0 (say \( |t| \leq t_1 \) for some \( t_1 > 0 \)) satisfy (3.8). By Sard’s theorem and compactness of the level sets of \( \Phi \) we can find a nontrivial interval \( I \subset [-t_1, +t_1] \)
which contains no critical values of \( \Phi \) and of \( \Phi|_{\Omega_0} \). Hence \( D_t \) for \( t \in I \) are smoothly bounded strongly pseudoconvex domains intersecting \( X \) transversely within the set \( \Omega \).

**Remark 3.5.** One of the key features of the function \( \Phi_1 \) is that on the intersection of its domain with the set \( \{ \chi = 0 \} \) (in particular, on \( U_A \cap U_B \cap \Omega'_1 \)) we have \( \Phi_1 = \phi \circ \rho + M \psi \geq \phi \circ \rho \). It follows that on this set the retraction \( \rho \) projects the domain \( \{ \Phi_1 < t \} \) to \( \{ \Phi_1 < t \} \cap X \). Furthermore, on \( X \cap \{ \chi = 0 \} \) we have \( \psi = 0 \), \( \rho = 1 \), and hence \( \Phi_1 = \phi \geq \phi - 2\epsilon = \Phi_2 \). This shows that the domain \( D_t = \{ \Phi < t \} \), which is contained in \( \{ \Phi_1 < t \} \) since \( \Phi_1 \leq \Phi \), actually agrees with \( \{ \Phi_1 < t \} \) near the set \( X \cap U_A \cap U_B \), so it has the same property as \( \{ \Phi_1 < t \} \) with respect to the retraction \( \rho \). This will be important to obtain property (iv). \( \square \)

It remains to find a Cartan pair decomposition \((A_t, B_t)\) of \( D_t \). Let \( V = \{ \theta < 0 \} \) be as in the beginning of the proof. Recall that \( C(r) \subset V \) by (3.3). Replacing \( \theta \) by \( c\theta \) for a suitably chosen constant \( c > 0 \) we may therefore assume that \( \theta < \Phi \) on \( C(r) \cap \Omega_1'. \) Perturbing \( \theta \) and \( V \) slightly if necessary we can ensure that the real hypersurfaces \( bV \) and \( bD_0 = \{ \Phi = 0 \} \) intersect transversely. Consider the smooth strongly plurisubharmonic function

\[
\phi_C = \max\{\phi, \theta\} : \Omega_1' \cap V_0 \rightarrow \mathbb{R}.
\]

(The value of the parameter in \( \max \) is chosen small enough here and at every step in the sequel.) For every \( t \in I \) the set

\[
C_t' := \{ z \in \Omega_1' \cap V_0 \cap X : \phi_C(z) < t \} \subset X_{\text{reg}}
\]

is then a smoothly bounded strongly pseudoconvex domain which contains \( C(r) \cap X \) in view of (3.5), and is contained in the set \( D_t' := D_t \cap X \) since \( \phi_C \geq \phi \). As \( \theta < \phi \) on \( C(r) \cap \Omega_1' \cap X \), we have \( \phi_C = \phi \) there. This means that the boundaries \( bC_t' \) and \( bD_t' \) coincide along their intersection with the compact set \( C(r) \cap X \). Hence \( C_t' \) separates \( D_t' \) in the sense of a Cartan pair, i.e., we have \( D_t' = A_t' \cup B_t' \) and \( A_t' \cap B_t' = C_t' \).

It remains to complete the picture in the ambient space \( \mathbb{C}^N \). As before, we do this by precomposing our functions by the retraction \( \rho : \Omega \rightarrow \Omega \cap X \) (3.3) and adding a term \( M \psi \). Explicitly, we set

\[
\Theta = \max\{\phi_C \circ \rho + M \psi, \Phi \}, \quad C_t = \{ \Theta < t \}.
\]

Here \( \Phi = \max\{\Phi_1, \Phi_2\} \) is defined by (3.7) and \( M \) is the constant in the definition of \( \Phi_1 \). One easily verifies that \( C_t \) is a strongly pseudoconvex domain which separates \( D_t \) into a Cartan pair \((A_t, B_t)\) with \( D_t = A_t \cap B_t \) and \( C_t = A_t \cap B_t \). Note that \( C_t \) satisfies Lemma 3.4-(iv) as explained in Remark 3.5. By suitably decreasing the parameter interval \( I \) we ensure that \( \Theta \) and \( \Theta|_{\Omega} \) have no critical values in \( I \), so the boundaries \( bC_t \) for \( t \in I \) are smooth and intersect \( X \) transversely. The same is then true for the domains \( A_t \) and \( B_t \) since their boundaries are contained in \( bD_t \cup bC_t \).

Reparametrizing the \( t \) variable and changing the functions \( \Phi \) and \( \Theta \) by an additive constant we may assume that \( I = [0, t_0] \) for some \( t_0 > 0 \) and property (ii) holds. The remaining properties of \( A_t, B_t \) and \( C_t \) follow directly from the construction. \( \square \)

Given an open set \( U \subset X \) and a number \( \delta > 0 \), we shall use the notation

\[
U(\delta) = \{ z \in X : \text{dist}(z, U) < \delta \}.
\]

Recall that \( s \) is the spray (3.1) and \( M_1 \) is the constant from Lemma 3.3.

The following lemma is a special case of [13, Lemma 4.5].
Lemma 3.6. Let $U_1 \subset X \setminus X'$ be the open set from Lemma 3.3. There exist constants $\delta_0 > 0$ (small) and $M_2 > 0$ (big) with the following property. Let $0 < \delta < \delta_0$ and $0 < 4\epsilon < \delta$. Let $U$ be an open set in $X$ such that $U(\delta) \subset U_1$. Assume that $\alpha, \beta, \gamma: U(\delta) \to X$ are holomorphic maps which are $\epsilon$-close to the identity on $U(\delta)$. Then $\tilde{\gamma} := \beta^{-1} \circ \gamma \circ \alpha: U \to X$ is a well defined holomorphic map. Write
\begin{align*}
\alpha(z) &= s(z, a(z)), \\
\beta(z) &= s(z, b(z)), \\
\gamma(z) &= s(z, c(z)), \\
\tilde{\gamma}(x) &= s(z, \tilde{c}(z)),
\end{align*}
where $a, b, c$ are sections of the vector bundle $E|_{U(\delta)}$ and $\tilde{c}$ is a section of $E|_{U}$ furnished by Lemma 3.4. If $c = b - a$ holds on $U(\delta)$, then
\begin{equation}
||\tilde{c}||_U \leq M_2\delta^{-1}e^2 \quad \text{and} \quad \text{dist}_U(\tilde{\gamma}, \text{Id}) \leq M_1M_2\delta^{-1}e^2.
\end{equation}

The next lemma provides a solution of the Cousin-I problem with bounds on the family of strongly pseudoconvex domains $D_t := A_t \cup B_t$ from Lemma 3.3 intersected with the subvariety $X$. We denote by $\mathcal{H}^\infty(D)$ the Banach space of all bounded holomorphic functions on a domain $D$ in a complex space.

Lemma 3.7. Let $(A_t, B_t)$ $(t \in [0, t_0])$ be a family of strongly pseudoconvex Cartan pairs satisfying Lemma 3.3. There is constant $M_3 > 0$ with the following property. For every $t \in [0, t_0]$ and every $c \in \mathcal{H}^\infty(C_t \cap X)$ there exist functions $a \in \mathcal{H}^\infty(A_t)$ and $b \in \mathcal{H}^\infty(B_t)$ such that
\begin{equation}
c = b - a \quad \text{on} \quad C_t \cap X, \quad ||a||_{A_t} \leq M_3||c||_{C_t \cap X}, \quad ||b||_{B_t} \leq M_3||c||_{C_t \cap X}.
\end{equation}
Functions $a$ and $b$ are given by bounded linear operators applied to $c$.

Proof. We begin by finding a constant $M_3 > 0$ independent of $t \in [0, t_0]$ and linear operators
\begin{align*}
A_t: \mathcal{H}^\infty(C_t) &\to \mathcal{H}^\infty(A_t), \\
B_t: \mathcal{H}^\infty(C_t) &\to \mathcal{H}^\infty(B_t)
\end{align*}
such that for every $g \in \mathcal{H}^\infty(C_t)$ $(t \in [0, t_0])$ we have
\begin{equation}
g = A_t(g) - B_t(g) \quad \text{on} \quad C_t,
\end{equation}

\begin{equation}
||A_t g||_{A_t} \leq M_3||g||_{C_t}, \quad ||B_t g||_{B_t} \leq M_3||g||_{C_t}.
\end{equation}
The proof is similar to that of [15] Lemma 5.8.2, p. 212 and uses standard techniques; we include it for completeness. In view of Lemma 3.4(i) there is a smooth function $\xi: \mathbb{C}^N \to [0, 1]$ such that $\xi = 0$ on $\bigcup_{t \in [0, 1]} A_t \setminus B_t$ and $\xi = 1$ on $\bigcup_{t \in [0, 1]} B_t \setminus A_t$. For any $g \in \mathcal{H}^\infty(C_t)$ the product $\xi g$ extends to a bounded smooth function on the domain $A_t$ that vanishes on $A_t \setminus B_t$, and $(\xi-1)g$ extends to a bounded smooth function on $B_t$ that vanishes on $B_t \setminus A_t$. Furthermore, $\overline{\partial}((\xi-1)g) = g\overline{\partial}\xi$ is a smooth bounded $(0, 1)$-form on the strongly pseudoconvex domain $D_t = A_t \cup B_t$ with support contained in $C_t = A_t \cap B_t$.

Let $S_t$ be a sup-norm bounded linear solution operator to the $\overline{\partial}$-equation on $D_t$ at the level of $(0,1)$-forms. (Such $S_t$ can be found as a Henkin-Ramírez integral kernel operator; see the monographs by Henkin and Leiterer [26] or Lieb and Michel [29]. The operators $S_t$ can be chosen to depend smoothly on the parameter $t \in [0, 1]$. For small perturbations of a given strongly pseudoconvex domain this is evident from the construction and is stated explicitly in the cited sources; for compact families of strongly pseudoconvex domains the result follows by applying a smooth partition of unity on the parameter space.) Given $g \in \mathcal{H}^\infty(C_t)$, set
\begin{align*}
A_t g = \xi g - S_t(g\overline{\partial}\xi) \in \mathcal{H}^\infty(A_t), \\
B_t g = (\xi-1)g - S_t(g\overline{\partial}\xi) \in \mathcal{H}^\infty(B_t).
\end{align*}
It is immediate that these operators satisfy the stated properties.
By Lemma 3.3 (iv) the map \( c \mapsto T(c) = c \circ \rho \) induces a linear extension operator \( \mathcal{O}(C_t \cap X) \to \mathcal{O}(C_t) \) satisfying \( \|Tc\|_{C_t} = \|c\|_{C_t \cap X} \). The compositions

\[ A_t \circ T : \mathcal{H}^\infty(C_t \cap X) \to \mathcal{H}^\infty(A_t), \quad B_t \circ T : \mathcal{H}^\infty(C_t \cap X) \to \mathcal{H}^\infty(B_t) \]

are then bounded linear operators satisfying the conclusion of Lemma 3.7. □

**Lemma 3.8.** Let \((A_t, B_t) = (A(t), B(t)) (t \in [0, t_0])\) be a family of strongly pseudoconvex Cartan pairs satisfying Lemma 3.4. Set \( C(t) = A(t) \cap B(t) \). There are constants \( M_4, M_5 > 0 \) satisfying the following. Let \( 0 \leq t < t + \delta \leq t_0 \). For every holomorphic \( \gamma : C(t + \delta) \cap X \to X \) satisfying

\[ \text{dist}_{C(t + \delta) \cap X}(\gamma, \text{Id}) < \frac{\delta}{4M_4} \]

there exist holomorphic maps \( \alpha : A(t + \delta) \to \mathbb{C}^N \) and \( \beta : B(t + \delta) \to \mathbb{C}^N \), tangent to the identity map to order \( n_0 \) along the subvariety \( X' \), such that

\[ \tilde{\gamma} = \beta^{-1} \circ \gamma \circ \alpha : C(t) \cap X \to X \]

is a well defined holomorphic map satisfying

\[ \text{dist}_{C(t) \cap X}(\tilde{\gamma}, \text{Id}) < M_5 \delta^{-1} \text{dist}_{C(t + \delta) \cap X}(\gamma, \text{Id})^2. \]

**Proof.** On the Banach space \( \mathcal{H}^\infty(D)^N \) we use the norm \( \|g\| = \sum_{j=1}^N \|g_j\| \), where \( \|g_j\| \) is the usual sup-norm on \( D \). Set \( \epsilon := \text{dist}_{C(t + \delta)}(\gamma, \text{Id}) \). By Lemma 3.3 there is a holomorphic section \( c : C(t + \delta) \cap X \to E \) of the holomorphic vector bundle \( E \to U_1 \) such that \( \gamma(z) = s(z, c(z)) \) for \( z \in C(t + \delta) \cap X \) and \( \|c\|_{C(t + \delta) \cap X} \leq M_1 \epsilon \). (Here \( M_1 \) is the constant from Lemma 3.3.) By Lemma 3.7 we have \( c = b - a \) where \( a \in \mathcal{H}^\infty(A_t)^N \) and \( b \in \mathcal{H}^\infty(B_t)^N \) satisfy

\[ \|a\|_{A_t} \leq N M_1 M_3 \epsilon, \quad \|b\|_{B_t} \leq N M_1 M_3 \epsilon. \]

Let \( s \) be the spray 3.1. Set

\[ \alpha(z) = s(z, a(z)), \quad z \in A(t + \delta), \]

\[ \beta(z) = s(z, b(z)), \quad z \in B(t + \delta). \]

These maps are tangent to the identity to order \( n_0 \) along the subvariety \( X' \) and satisfy \( \alpha(A_t \cap X) \subset X \) and \( \beta(B_t \cap X) \subset X \). By Lemma 3.3 we have

\[ \text{dist}_{A(t + \delta)}(\alpha, \text{Id}) < N M_1^2 M_3 \epsilon, \quad \text{dist}_{B(t + \delta)}(\beta, \text{Id}) < N M_1^2 M_3 \epsilon. \]

Set \( M_4 = N M_1^2 M_3 \). If \( \epsilon = \text{dist}_{C(t + \delta)}(\gamma, \text{Id}) \) satisfies \( 0 < 4M_4 \epsilon < \delta \) then by Lemma 3.6 the composition \( \tilde{\gamma} = \beta^{-1} \circ \gamma \circ \alpha \) is a well defined holomorphic map on \( C(t) \cap X \) satisfying the estimate 3.11 with the constant \( M_5 = M_2 M_1^2 \). □

We now complete the proof of Theorem 3.2 as in [16 §4]. Recall that the initial map \( \gamma \) is defined on the set \( \tilde{C} \supset C(t_0) \cap X \). For each \( k = 0, 1, 2, \ldots \) we set

\[ t_k = t_0 \prod_{j=1}^k (1 - 2^{-j}), \quad \delta_k = t_k - t_{k+1} = t_k 2^{-k-1}. \]

The sequence \( t_k > 0 \) is decreasing, \( t^* = \lim_{k \to \infty} t_k > 0, \delta_k > t^* 2^{-k-1} \) for all \( k \), and \( \sum_{k=0}^\infty \delta_k = t_0 - t^* \). Set \( A_k = A(t_k), B_k = B(t_k), C_k = C(t_k) = A_k \cap B_k. \)
To begin the induction, set $\gamma_0 = \gamma$ and $\epsilon_0 = \text{dist}_{C_0 \cap X}(\gamma_0, \text{Id})$. Assuming that $4M_4\epsilon_0 < \delta_0 = t_0/2$, Lemma 3.8 gives holomorphic maps $\alpha_1: A_1 \to \mathbb{C}^N$ and $\beta_1: B_1 \to \mathbb{C}^N$ such that $\gamma_1 = \beta_1^{-1} \circ \gamma_0 \circ \alpha_1: C_1 \cap X \to X$ is a well defined holomorphic map satisfying
\[ \text{dist}_{C_1 \cap X}(\gamma_1, \text{Id}) < M_5 \epsilon_0^2 \delta_0^{-1} < 2M_5\epsilon_0^2, \quad M = M_5/t^*.
\]
Set $\epsilon_1 = \text{dist}_{C_1 \cap X}(\gamma_1, \text{Id}) < 2M_5\epsilon_0^2$. Assuming that $4M_4\epsilon_1 < \delta_1$, Lemma 3.8 furnishes holomorphic maps $\alpha_2: A_2 \to \mathbb{C}^N$, $\beta_2: B_2 \to \mathbb{C}^N$ such that $\gamma_2 = \beta_2^{-1} \circ \gamma_1 \circ \alpha_2: C_2 \cap X \to X$ is a well defined holomorphic map satisfying
\[ \text{dist}_{C_2 \cap X}(\gamma_2, \text{Id}) < M_5 \epsilon_1^2 \delta_1^{-1} < 2M_5\epsilon_1^2.
\]
Define $\epsilon_2 = \text{dist}_{C_2 \cap X}(\gamma_2, \text{Id})$. Continuing inductively we get sequences of holomorphic maps $\alpha_k: A_k \to \mathbb{C}^N$, $\beta_k: B_k \to \mathbb{C}^N$, $\gamma_k: C_k \cap X \to X$ for $k = 1, 2, \ldots$ such that $\gamma_{k+1} = \beta_{k+1}^{-1} \circ \gamma_k \circ \alpha_{k+1}: C_{k+1} \cap X \to X$ satisfies
\[ \epsilon_{k+1} := \text{dist}_{C_{k+1} \cap X}(\gamma_{k+1}, \text{Id}) < M_5 \epsilon_k^2 \delta_k^{-1} < 2^{k+1}M_5\epsilon_k^2.
\]
The necessary condition for the induction is that $4M_4\epsilon_k < \delta_k$ holds for each $k$. By [13, Lemma 4.8] this condition is fulfilled, and the resulting sequence $\epsilon_k > 0$ converges to zero very rapidly as $k \to \infty$, as long as the initial number $\epsilon_0 = \text{dist}_{C_0 \cap X}(\gamma, \text{Id})$ is small enough. Set
\[ \tilde{\alpha}_k = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k: A_k \to \mathbb{C}^N, \quad \tilde{\beta}_k = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_k: B_k \to \mathbb{C}^N.
\]
Then $\tilde{\beta}_k \circ \gamma_k = \gamma_0 \circ \tilde{\alpha}_k$ on $C_k \cap X$ for $k = 1, 2, \ldots$. As $k \to \infty$, the sequences $\tilde{\alpha}_k$ and $\tilde{\beta}_k$ converge to holomorphic maps $\alpha: A(t^*) \to \mathbb{C}^N$ and $\beta: B(t^*) \to \mathbb{C}^N$, respectively, and the sequence $\gamma_k$ converges uniformly on $C(t^*) \cap X$ to the identity map. (See [13, §4] for the details of this argument.) In the limit we obtain $\gamma \circ \alpha = \beta$ on $C(t^*) \cap X$. The maps $\alpha$, $\beta$ and $\gamma$ are arbitrarily close to the identity on their respective domains as long as $\epsilon_0 := \text{dist}_{C_0 \cap X}(\gamma, \text{Id})$ is small enough, and hence are biholomorphic on slightly smaller domains by Lemma 2.13. Furthermore, all maps $\alpha_k$ and $\beta_k$ in the sequence are tangent to the identity to order $n_0$ along the subvariety $X'$, so the same is true for the limit maps $\alpha$ and $\beta$. \hfill \Box

**Remark 3.9.** If $X$ is a Stein manifold with the *density property* in the sense of Varolin [15, §4.10] and $(A, B)$ is a Cartan pair in $X$ such that the set $C = A \cap B$ is $\mathcal{O}(X)$-convex, then the conclusion of Theorem 3.2 holds for any biholomorphic map $\gamma$ on a neighborhood $U \subset X$ of $C$ which is isotopic to the identity map on $C$ through a smooth 1-parameter family of biholomorphic maps $\gamma_t: U \to \gamma_t(U) \subset X$ ($t \in [0, 1]$) such that $\gamma_0 = \text{Id}$, $\gamma_1 = \gamma$, and $\gamma_t(C)$ is $\mathcal{O}(X)$-convex for every $t \in [0, 1]$. The main result of the Andersén-Lempert theory (cf. [15, Theorem 4.9.2] for $X = \mathbb{C}^n$ and [15, Theorem 4.10.6] for the general case) implies that $\gamma$ can be approximated uniformly on a neighborhood of $C$ by holomorphic automorphisms $\phi \in \text{Aut}(X)$. This allows us to write $\gamma = \phi \circ \tilde{\gamma}$ where $\tilde{\gamma}$ is a biholomorphic map close to the identity on a neighborhood of $C$. Applying Theorem 3.2 gives $\tilde{\gamma} = \tilde{\beta} \circ \alpha^{-1}$ where $\alpha$ and $\tilde{\beta}$ are biholomorphic maps close to the identity near $A$ and $B$, respectively. Setting $\beta = \phi \circ \tilde{\beta}$ gives $\gamma = \beta \circ \alpha^{-1}$.

## 4. Functions without critical points in the open strata

In this section we construct holomorphic functions that have no critical points in the regular locus of a Stein space. The following result is the main inductive step in the proof of Theorem 1.5 given in the following section.
**Theorem 4.1.** Assume that $X$ is a Stein space, $X' \subset X$ is a closed complex subvariety of $X$ containing $X_{\text{sing}}$, $P = \{p_1, p_2, \ldots \}$ is a closed discrete set in $X'$, $K$ is a compact $\mathcal{O}(X)$-convex set in $X$ (possibly empty), and $f$ is a holomorphic function on a neighborhood of $K \cup X'$ such that $\text{Crit}(f|_{U \setminus X'}) = \emptyset$ for some neighborhood $U \subset X$ of $K$. Then for any $\epsilon > 0$ and integers $r \in \mathbb{N}$ and $n_k \in \mathbb{N}$ $(k = 1, 2, \ldots)$ there exists a holomorphic function $F \in \mathcal{O}(X)$ satisfying the following conditions:

(i) $F - f$ vanishes to order $r$ along the subvariety $X'$,
(ii) $F - f$ vanishes to order $n_k$ at the point $p_k \in P$ for every $k = 1, 2, \ldots$,
(iii) $||F - f||_K < \epsilon$, and
(iv) $F$ has no critical points in $X \setminus X'$.

Applying Theorem 4.1 with the subvariety $X' = X_{\text{reg}}$, we find holomorphic functions on $X$ that have no critical points in $X_{\text{reg}}$.

**Remark 4.2.** Theorem 4.1 implies at no cost the following result. Let $A = \{a_j\}$ be a closed discrete set in $X$ contained in $X \setminus (K \cup X')$. Then there exists a function $F \in \mathcal{O}(X)$ satisfying conditions (i)–(iii) and also the condition

(iv') $\text{Crit}(F|_{X \setminus X'}) = A$.

Indeed, choose any germs $g_j \in \mathcal{O}_{X,a_j}$ at points $a_j \in A$ and apply Theorem 4.1 with the subvariety $X'_0 = A \cup X'$, the discrete set $P_0 = A \cup P$ and the function $f$ extended by $g_j$ to a small neighborhood of the point $a_j \in A$. □

We begin with a couple lemmas. The first one shows that we can replace the function $f$ by a function in $\mathcal{O}(X)$.

**Lemma 4.3.** (Assumptions as in Theorem 4.1) Let $L$ be a compact $\mathcal{O}(L)$-convex set in $X$ such that $K \subset L$. There exists a function $\tilde{f} \in \mathcal{O}(X)$ satisfying the following conditions:

(a) $\tilde{f} - f$ vanishes to order $r$ along the subvariety $X'$,
(b) $\tilde{f} - f$ vanishes to order $n_k$ at the point $p_k \in P$ for every $k = 1, 2, \ldots$,
(c) $||\tilde{f} - f||_K < \epsilon$, and
(d) there is a neighborhood $W \subset X$ of the compact set $K \cup (L \cap X')$ such that $\tilde{f}$ has no critical points in the set $W \setminus X'$.

**Proof.** Let $\mathcal{E} \subset \mathcal{O}_X$ be the coherent sheaf of ideals whose stalk at any point $p_k \in P$ equals $m_{p_k}^{n_k+1}$ and $\mathcal{E}_x = \mathcal{O}_{X,x}$ for every $x \in X \setminus P$. The product

\begin{equation}
\mathcal{E} := \mathcal{E} \mathcal{J}_{X'}^r \subset \mathcal{O}_X
\end{equation}

of $\mathcal{E}$ and the $r$-th power of the ideal sheaf $\mathcal{J}_{X'}$ is a coherent sheaf of ideals in $\mathcal{O}_X$. Consider the short exact sequence of sheaf homomorphisms

\[
0 \to \mathcal{E} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{E} \to 0.
\]

Observe that the quotient sheaf $\mathcal{O}_X/\mathcal{E}$ is supported on $X'$, and hence the function $f$ determines a section of $\mathcal{O}_X/\mathcal{E}$. Since $H^1(X; \mathcal{E}) = 0$ by Theorem B, the same section is induced by a function $g \in \mathcal{O}(X)$. Clearly $g$ satisfies conditions (a) and (b) of the lemma (with $g$ in place of $\tilde{f}$).

To get condition (c) we correct $g$ as follows. Cartan’s Theorem A furnishes sections $\xi_1, \ldots, \xi_m \in \Gamma(X, \mathcal{E})$ which generate the sheaf $\mathcal{E}$ over the compact set $K$. By the choice
of $g$ the difference $f - g$ is a section of $\tilde{E}$ over a neighborhood of $K$. Applying Theorem B to the epimorphism of coherent sheaves

$$\mathcal{O}_X^m \rightarrow \tilde{E} \rightarrow 0, \quad (h_1, \ldots, h_m) \mapsto \sum_{i=1}^m h_i \xi_i,$$

we obtain $f = g + \sum_{i=1}^m h_i \xi_i$ on a neighborhood of $K$ for some holomorphic functions $h_i \in \mathcal{O}(K)$. By the Oka-Weil theorem we can approximate the $h_i$’s uniformly on $K$ by functions $\tilde{h}_i \in \mathcal{O}(X)$. Setting $\tilde{f} = g + \sum_{i=1}^m \tilde{h}_i \xi_i \in \mathcal{O}(X)$ we get a function satisfying properties (a)–(c). By the Stability Lemma [2,7] and the Genericity Lemma [2,9] we can also satisfy condition (d) by choosing $\tilde{f}$ generic and taking $\epsilon > 0$ small enough. □

The next lemma is the main step in the proof of Theorem 4.4. It is here that we use the splitting lemma for biholomorphic maps, furnished by Theorem 3.2. Another key ingredient is the Runge approximation theorem for noncritical holomorphic functions on $\mathbb{C}^n$, provided by Theorem 3.1 in [13].

**Lemma 4.4. (Assumptions as in Theorem 4.1)** Let $L \subset X$ be a compact $\mathcal{O}(X)$-convex set such that $K \subset L$. Then there exists a holomorphic function $F \in \mathcal{O}(X)$ which satisfies conditions (i)–(iii) in Theorem 4.1 and also

$$(iv') \text{Crit}(F|_{U \setminus X'}) = \emptyset, \text{where } U' \subset X \text{ is an open neighborhood of } L.$$

**Proof.** To simplify the exposition we replace the number $r$ by the maximum of $r$ and the numbers $n_k \in \mathbb{N}$ over all points $p_k \in P \cap L$ (a finite set). Choosing $F$ to satisfy condition (i) for this $r$, it will also satisfy condition (ii) at the points $p_k \in P \cap L$.

Let $W$ be the set from Lemma [13, (d)]. By [17, Lemma 8.4, p. 662] there exist finitely many compact $\mathcal{O}(X)$-convex sets $A_0 \subset A_1 \subset \cdots \subset A_m = L$ such that $K \cup (L \setminus X') \subset A_0 \subset W$ and for every $j = 0, 1, \ldots, m - 1$ we have $A_{j+1} = A_j \cup B_j$ where $(A_j, B_j)$ is a Cartan pair (cf. Definition 3.1) and $B_j \subset L \setminus X' \subset X_{\text{reg}}$. Furthermore, the construction in [18] gives for every $j = 0, 1, \ldots, m - 1$ an open set $U_j \subset X_{\text{reg}}$ containing $B_j$ and a biholomorphic map $\phi_j : U_j \rightarrow U_j' \subset \mathbb{C}^n$ onto an open subset of $\mathbb{C}^n$ (where $n$ is the dimension of $X$ at the points of $B_j$) such that the set $\phi_j(C_j)$ is polynomially convex in $\mathbb{C}^n$. (Here $C_j = A_j \cap B_j$.) The proof of Lemma 8.4 in [17] is written in the case when $X$ is a Stein manifold (i.e., without singularities), but it also applies in the present situation, for example, by embedding a relatively compact neighborhood of $L \subset X$ as a closed complex subvariety in some Euclidean space $\mathbb{C}^N$.

We first find a function $\tilde{F}$ that is holomorphic on a neighborhood of $L$ and satisfies the conclusion of the lemma there. This is accomplished by a finite induction, starting with $F_0 = f$ which by the assumption satisfies these properties on $W \supset A_0$. We provide an outline and refer to [13] for further details.

By the assumption $F_0$ is noncritical on a neighborhood of the set $C_0 = A_0 \cap B_0$. Since $C_0$ is polynomially convex in a certain holomorphic coordinate system on a neighborhood of $B_0$ in $X \setminus X' \subset X_{\text{reg}}$, Theorem 3.1 in [13, p. 154] furnishes a noncritical holomorphic function $G_0$ on a neighborhood of $B_0$ in $X \setminus X'$ such that $G_0$ approximates $F_0$ as close as desired uniformly on some neighborhood of $C_0$. Assuming that the approximation is close enough, we can use Theorem 3.2 to glue $F_0$ and $G_0$ into a new function $F_1$ that is holomorphic on a neighborhood of $A_0 \cup B_0 = A_1$ and has no critical points, except perhaps on subvariety $X'$. 

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The gluing of $F_0$ and $G_0$ is accomplished by first finding a biholomorphic map $\gamma$ close to the identity on a neighborhood of the attaching set $C_0$ such that

$$F_0 = G_0 \circ \gamma \quad \text{on a neighborhood of } C_0.$$  

Since $C_0$ is a Stein compact in the complex manifold $X \setminus X'$, such $\gamma$ is furnished by Lemma 5.1 in [13, p. 167]. If $\gamma$ is close enough to $\text{Id}$ (which holds if $G_0$ is chosen sufficiently uniformly close to $F_0$ on a fixed neighborhood of $C_0$), then Theorem 3.2 furnishes a decomposition

$$\gamma \circ \alpha = \beta,$$

where $\alpha$ is a biholomorphic map close to the identity on a neighborhood of $A_0$ in $X$ and $\beta$ is a map with the analogous properties on a neighborhood of $B_0$ in $X$. By Theorem 3.2 we can ensure in addition that $\alpha$ is tangent to the identity to order $r$ along the subvariety $X'$ intersected with its domain. (The domain of $\beta$ does not intersect $X'$.) Then

$$F_0 \circ \alpha = G_0 \circ \beta \quad \text{on a neighborhood of } C_0,$$

so the two sides amalgamate into a holomorphic function $F_1$ on a neighborhood of $A_0 \cup B_0 = A_1$. By the construction, $F_1$ approximates $F_0$ on a neighborhood of $A_0$, $F_1 - F_0$ vanishes to order $r$ along $X'$, and $F_1$ is noncritical except perhaps on $X'$. The last property holds because the maps $\alpha$ and $\beta$ are biholomorphic on their respective domains and $\alpha|_{X'}$ is the identity on $X'$.

Repeating the same construction with $F_1$ we get the next function $F_2$ on a neighborhood of $A_2$, etc. In $m$ steps of this kind we find a function $F = F_m$ on a neighborhood of the set $A_m = L$ satisfying the stated properties.

It remains to replace $\tilde{F}$ by a function $F \in \mathcal{O}(X)$ satisfying the same properties. This is done as in Lemma 4.3 above. Let $\tilde{\mathcal{E}}$ be the sheaf $(4.1)$. Pick sections $\xi_1, \ldots, \xi_m \in \Gamma(X, \tilde{\mathcal{E}})$ which generate $\tilde{\mathcal{E}}$ over the compact set $L$. By the construction of $\tilde{F}$, the difference $\tilde{F} - f$ is a section of $\tilde{\mathcal{E}}$ over a neighborhood of $L$. Hence the Cartan Theorem B furnishes holomorphic functions $\tilde{h}_1, \ldots, \tilde{h}_m \in \mathcal{O}(U')$ an open set $U' \supset L$ such that

$$\tilde{F} = f + \sum_{i=1}^m \tilde{h}_i \xi_i \quad \text{holds on } U'.$$

Choose a compact $\mathcal{O}(X)$-convex set $L'$ such that $L \subset L' \subset L' \subset U'$. Approximating each $\tilde{h}_i$ uniformly on $L'$ by a function $h_i \in \mathcal{O}(X)$ and setting

$$F = f + \sum_{i=1}^m h_i \xi_i \in \mathcal{O}(X)$$

we get a function $F$ satisfying properties (i)–(iii). By the Stability Lemma 2.7, the function $F$ also satisfies property (iv’) provided that the differences $||h_i - \tilde{h}_i||_{L'}$ for $i = 1, \ldots, m$ are small enough.

Proof of Theorem 4.1. In view of Lemma 4.3 we may assume that $f \in \mathcal{O}(X)$. Choose an increasing sequence $K = K_0 \subset K_1 \subset \ldots \subset \bigcup_{i=0}^\infty K_i = X$ of compact $\mathcal{O}(X)$-convex sets satisfying $K_i \subset K_{i+1}$ for every $i = 0, 1, \ldots$. Set $F_0 = f$, $\epsilon_0 = \epsilon/2$, and $r_0 = r$. We inductively construct a sequence of functions $F_i \in \mathcal{O}(X)$ and numbers $\epsilon_i > 0, r_i \in \mathbb{N}$ such that the following conditions hold for every $i = 0, 1, 2, \ldots$:

(a) $\text{Crit}(F_i|_{U_i \setminus X'}) = \emptyset$ for an open neighborhood $U_i \supset K_i$,

(b) $||F_i - F_{i-1}||_{K_{i-1}} < \epsilon_{i-1}$. 


(c) $F_i - F_{i-1}$ vanishes to order $r_i - 1$ along the subvariety $X'$,
(d) $F_i - F_{i-1}$ vanishes to order $n_k$ at each of the point $p_k \in P$,
(e) $0 < \epsilon_i < \epsilon_{i-1}/2$ and $r_i \geq r_{i-1}$, and
(f) if $F \in \mathcal{O}(X)$ is such that $\|F - F_i\|_{K_i} < 2\epsilon_i$ and $F - F_i$ vanishes to order $r_i$ along $X'$,
then $\text{Crit}(F|_{U \setminus X'}) = \emptyset$ for an open neighborhood $U \supset K_{i-1}$.

Assume that we have already found these quantities up to index $i - 1$ for some $i \in \mathbb{N}$. (For $i = 0$ the function $F_0$ satisfies condition (a) and the other conditions are void.) Lemma 4.4 furnishes the next map $F_i \in \mathcal{O}(X)$ in the sequence which satisfies conditions (a)–(d). For this $F_i$ we then pick the next pair of numbers $\epsilon_i > 0$ and $n_i \in \mathbb{N}$ such that conditions (e) and (f) hold. In view of the Stability Lemma 2.7 condition (f) holds by as soon as $\epsilon_i > 0$ is chosen small enough and $r_i \in \mathbb{N}$ is chosen big enough. This completes the induction step.

It is straightforward to verify that the sequence $F_i$ converges uniformly on compacta in $X$ and the limit function $F = \lim_{i \to \infty} F_i \in \mathcal{O}(X)$ satisfies the conclusion of Theorem 4.1. □

In the proof of Theorem 4.5 (see [5]) we shall combine Theorem 4.1 with the following lemma which provides extension from a subvariety and jet interpolation on a discrete set.

**Lemma 4.5 (Extension with jet interpolation).** Let $X$ be a Stein space, $X'$ a closed complex subvariety of $X$ and $P = \{p_1, p_2, \ldots\}$ a closed discrete subset of $X'$. Assume that we have a function $f \in \mathcal{O}(X')$ and germs $f_k \in \mathcal{O}_{X,p_k}$ for $p_k \in P$ such that $f_{pk} - (f_{k,X'})_{pk} \in m_{X',p_k}^{n_k+1}$ for every $p_k \in P$ and some $n_k \in \mathbb{Z}_+$. Then there exists $F \in \mathcal{O}(X)$ such that $F|_{X'} = f$ and $F_{pk} - f_k \in m_{X,p_k}^{n_k+1}$ for every $p_k \in P$.

**Proof.** Let $\mathcal{J}_{X'}$ denote the sheaf of ideals of the subvariety $X'$. By Lemma 2.7 there exists for every point $p_k \in P$ a germ $g_k \in \mathcal{O}_{X,p_k}$ such that $f_k - g_k \in m_{X,p_k}^{n_k+1}$ and $(g_k|_{X'})_{pk} = f_{pk} \in \mathcal{O}_{X',p_k}$.

Pick a function $\tilde{f} \in \mathcal{O}(X)$ with $\tilde{f}|_{X'} = f$; then $\tilde{f}_{pk} - g_k \in \mathcal{J}_{X',p_k}$. Let $E \subset \mathcal{O}_{X}$ be the coherent sheaf of ideals whose stalk at any point $p_k \in P$ equals $m_{p_k}^{n_k+1}$, and $E_x = \mathcal{O}_{X,x}$ for every $x \in X \setminus P$. Consider the following short exact sequence of coherent analytic sheaves on $X$:

$$0 \rightarrow E \mathcal{J}_{X'} \rightarrow \mathcal{J}_{X'} \rightarrow \mathcal{J}_{X'}/(E \mathcal{J}_{X'}) \rightarrow 0.$$ \hspace{1cm}

The quotient sheaf $\mathcal{J}_{X'}/(E \mathcal{J}_{X'})$ is supported on the discrete set $P$, and hence the collection of germs $\tilde{f}_{pk} - g_k \in \mathcal{J}_{X',p_k}$ determines a section of this sheaf. Since $H^1(X; E \mathcal{J}_{X'}) = 0$ by Theorem B, this section lifts to a section $h$ of $\mathcal{J}_{X'}$. Thus $h$ is a holomorphic function on $X$ that vanishes on $X'$ and whose jet of order $n_k$ at the point $p_k \in P$ equals the corresponding jet of $\tilde{f} - g_k$ at $p_k$.

Consider the function $F := \tilde{f} - h \in \mathcal{O}(X)$. We have $F|_{X'} = \tilde{f}|_{X'} = f$. Furthermore, for every point $p_k \in P$ the following identities hold in the ring $\mathcal{O}_{X,p_k}/m_{X,p_k}^{n_k+1}$ of $n_k$-jets at $p_k$:

$$F_{pk} = \tilde{f}_{pk} - h_{pk} = \tilde{f}_{pk} - (\tilde{f}_{pk} - g_k) = g_k = f_k \mod m_{X,p_k}^{n_k+1}.$$ \hspace{1cm}

Thus $F$ satisfies the conclusion of the lemma. □

**Remark 4.6.** Lemma 4.5 gives a version of Theorem 4.1 in which the function $f$ is assumed to be defined and holomorphic only on the subvariety $X' \subset X$ and on a neighborhood of a compact $\mathcal{O}(X)$-convex set $K \subset X$. Furthermore, we are given germs $f_k \in \mathcal{O}_{X,p_k}$ at points $p_k \in P$ such that the conditions of Lemma 4.5 hold. Then there exists $F \in \mathcal{O}(X)$ satisfying Theorem 4.1 except that conditions (i) and (ii) are replaced by the following conditions:

(i) $F|_{X'} = f$, and
(ii) $F - f_k$ vanishes to order $n_k$ at each point $p_k \in P$. 

5. Stratified noncritical functions on Stein spaces

In this section we prove Theorem 1.5 on the existence of stratified noncritical holomorphic functions on stratified Stein spaces. As was shown in the Introduction, this will also prove Theorems 1.1 and 1.3.

Proof of Theorem 1.5. Let \((X, \Sigma)\) be a stratified Stein space (see 1.1). For every integer \(i \in \mathbb{Z}_+\) we let \(\Sigma_i\) denote the collection of all strata of dimension at most \(i\) in \(\Sigma\), and let \(X_i\) (the \(i\)-skeleton of \(\Sigma\)) denote the union of all strata in the family \(\Sigma_i\). Since the boundary of any stratum is a union of lower dimensional strata, \(X_i\) is a closed complex subvariety of \(X\) of dimension \(\leq i\) for every \(i \in \mathbb{Z}_+\). Clearly \(\dim X_i = i\) precisely when \(\Sigma\) contains at least one \(i\)-dimensional stratum; otherwise \(X_i = X_{i-1}\). We have \(X_0 \subset X_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} X_i = X\), the sequence \(X_i\) is stationary on any compact subset of \(X\), and \((X_i, \Sigma_i)\) is a stratified Stein subspace of \((X, \Sigma)\) for every \(i\). Note that \(X_0 = \{p_1, p_2, \ldots\}\) is a discrete subset of \(X\).

By the assumption of Theorem 1.5 we are given for each \(p_k \in X_0\) a germ \(f_k \in \mathcal{O}_{X,p_k}\). Our task is to find a \(\Sigma\)-noncritical function \(F \in \mathcal{O}(X)\) which agrees with the germ \(f_k\) at \(p_k \in X_0\) to order \(n_k \in \mathbb{N}\). If the germs \(f_k \in \mathcal{O}_{X,p_k}\) \((p_k \in X_0)\) are chosen to be (strongly) noncritical, then the resulting function \(F \in \mathcal{O}(X)\) will also be (strongly) noncritical on \(X\) in the sense of Definition 1.2.

Let \(F_0\colon X_0 \to \mathbb{C}\) be the function on the zero dimensional skeleton defined by \(F_0(p_k) = f_k(p_k)\) for every \(p_k \in X_0\). We shall inductively construct a sequence of functions \(F_i \in \mathcal{O}(X_i)\) satisfying the following conditions for \(i = 1, 2, \ldots\):

1. \(F_i|_{X_{i-1}} = F_{i-1}\),
2. \(F_i - f_k|_{X_i}\) vanishes to order \(n_k\) at every point \(p_k \in X_0\) (more precisely, its germ at \(p_k\) belongs to the ideal \(m_{X_i,p_k}^{n_k+1}\)), and
3. \(F_i\) is a stratified noncritical function on the stratified Stein space \((X_i, \Sigma_i)\).

Assuming that we have found functions \(F_1, \ldots, F_{i-1}\) with these properties, we explain how to find the next function \(F_i\) in the sequence. If \(X_i = X_{i-1}\) then we can simply take \(F_i = F_{i-1}\). If this is not the case, then \(X_i \setminus X_{i-1}\) is a complex manifold of dimension \(i\). Apply Lemma 4.5 with \(X = X_i\), \(X' = X_{i-1}\) and \(f = F_{i-1}\) to find a function \(G_i \in \mathcal{O}(X)\) (called \(F\) in the lemma) which satisfies

1. \(G_i|_{X_{i-1}} = F_{i-1}\), and
2. \((G_i)_{|p_k} - (f_k|_{X_i})_{|p_k} \in m_{X_i,p_k}^{n_k+1}\) at every point \(p_k \in X_0\).

Now \(G_i\) satisfies the hypotheses of Theorem 4.1 (with \(X = X_i\), \(X' = X_{i-1}\) and \(f = G_i\)), so we get a function \(F_i \in \mathcal{O}(X)\) which agrees with \(G_i\) on \(X_{i-1}\), it agrees with \(G_i\) (and hence with \(f_k\)) to order \(n_k\) at \(p_k \in X_0\) for each \(k\), and is noncritical on \(X_i \setminus X_{i-1}\). Hence \(F_i\) satisfies properties (i)–(iii) and the induction may proceed.

Since the sequence of subvarieties \(X_0 \subset X_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} X_i = X\) is stationary on any compact subset of \(X\), the sequence of functions \(F_i \in \mathcal{O}(X_i)\) obtained in this way determines a holomorphic function \(F \in \mathcal{O}(X)\) by setting \(F = F_i\) on \(X_i\) for any \(i \in \mathbb{N}\) (no convergence process is needed). It is immediate from the construction that \(F\) satisfies the conclusion of Theorem 1.5.

A similar construction yields the following result.
Theorem 5.1. Given a closed complex subvariety $X'$ of a reduced Stein space $X$ and a function $f \in \mathcal{O}(X')$, there exists a function $F \in \mathcal{O}(X)$ such that $F|_{X'} = f$ and $F$ is strongly noncritical on $X \setminus X'$, or it has critical points at a prescribed discrete set $P$ in $X$ which is contained in $X \setminus X'$.

Proof. We can stratify the difference $X \setminus X' = \bigcup_{j} S_j$ into a union of pairwise disjoint connected complex manifolds (strata) such that

- the boundary $bS_j = \overline{S_j} \setminus S_j$ of any stratum is contained in the union of $X'$ and of lower dimensional strata,
- every point of $P$ is a zero dimensional stratum, and
- every compact set in $X$ intersects at most finitely many strata.

Consider the increasing chain of closed complex subvarieties

$$X' \subset X_0 \subset X_1 \subset \cdots \subset \bigcup_{i=1}^{\infty} X_i = X,$$

where $X_i$ is the union of $X'$ and all strata $S_j$ of dimension at most $i$. In particular, we have $P \cup X' \subset X_0$. Then $X_i \setminus X_{i-1}$ is either empty or a disjoint union of $i$-dimensional complex manifolds contained in $X \setminus X'$.

Let $P = \{p_1, p_2, \ldots \}$, and assume that we are given germs $f_k \in \mathcal{O}_{X,p_k}$ and integers $n_k \in \mathbb{N}$. We start with the function $F_0 \in \mathcal{O}(X_0)$ which equals $f$ on $X'$ and equals $f_k(p_k)$ at $p_k \in P$. By Lemma 4.6 we can find a function $G_1 \in \mathcal{O}(X)$ which agrees with $F_0$ on $X_0$ and satisfies $(G_1)_{p_k} - f_k \in m_{X,p_k}^{n_k+1}$ at $p_k \in P$. Then $F_1$ (denoted $f$ in the theorem) furnishes a function $F_1 \in \mathcal{O}(X_1)$ which agrees with $G_1$ (and hence with $F_0$) on $X_0$ and satisfies $(F_1)_{p_k} - (f_k|_{X_1})_{p_k} \in m_{X_1,p_k}^{n_k+1}$ for every $k$. This completes the first step of the induction. By using again Lemma 4.6 and then Theorem 4.4 we find the next function $F_2 \in \mathcal{O}(X_2)$ such that $F_2|_{X_1} = F_1$ and $(F_2)_{p_k} - (f_k|_{X_2})_{p_k} \in m_{X_2,p_k}^{n_k+1}$ at every point $p_k \in P$. Clearly this process can be continued inductively. We obtain a sequence $F_i \in \mathcal{O}(X_i)$ for $i = 1, 2, \ldots$ such that the function $F \in \mathcal{O}(X)$ defined by $F|_{X_i} = F_i$ for every $i = 1, 2, \ldots$ satisfies the conclusion of Theorem 5.1.

Corollary 5.2. Let $X$ be a reduced Stein space, $X'$ a closed complex subvariety of $X$ without isolated points, and $f \in \mathcal{O}(X')$ a noncritical holomorphic function. Then there exists a noncritical function $F \in \mathcal{O}(X)$ such that $F|_{X'} = f$.

Acknowledgements. Research on this paper was supported in part by the program P1-0291 and the grant J1-5432 from ARRS, Republic of Slovenia.

References

1. Abraham, R.: Transversality in manifolds of mappings. Bull. Amer. Math. Soc., 69, 470–474 (1963)
2. Alarcón, A.; Forstnerič, F.: Every bordered Riemann surface is a complete proper curve in a ball. Math. Ann. 357 (2013) 1049–1070
3. Alarcón, A.; Forstnerič, F.: The Calabi-Yau problem, null curves, and Bryant surfaces. arxiv.org/abs/1308.0903
4. Aroca, J.M.; Hironaka, H.; Vicente, J.L.: Desingularization theorems, Mem. Math. Inst. Jorge Juan, no. 30, Madrid (1977)
5. Bierstone, E.; Milman, P.D.: Canonical desingularization in characteristic zero by blowingup the maximum strata of a local invariant, Invent. Math. 128 (1997) 207–302

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6. Deng, F.; Guan, Q.; Zhang, L.: Properties of squeezing functions and global transformations of bounded domains. http://arxiv.org/abs/1302.5307
7. Diederich, K.; Fornaess, J.E.; Wold, E.F.: Exposing points on the boundary of a strictly pseudoconvex or a locally convexifiable domain of finite 1-type. Math. Z., in press. http://arxiv.org/abs/1303.1976
8. Docquier, F.; Grauert, H.: Levi's Problem and Runge's Satz for Teilgebiete Steinscher Mannigfaltigkeiten. Math. Ann. 140 (1960) 94–123
9. Drnovšek, B.; Forstnerič, F.: Holomorphic curves in complex spaces. Duke Math. J., 139, 203–254 (2007)
10. Fischer, G.: Complex Analytic Geometry. Lecture Notes in Math., vol. 538, Springer-Verlag, Berlin (1976)
11. Forster, O.: Plongements des variétés de Stein. Comment. Math. Helv. 45, 170–184 (1970)
12. Forstnerič, F.: On complete intersections. Ann. Inst. Fourier 51 (2001) 497–512
13. Forstnerič, F.: Noncritical holomorphic functions on Stein manifolds. Acta Math. 191 (2003) 143–189
14. Forstnerič, F.: Holomorphic submersions from Stein manifolds. Ann. Inst. Fourier 54 (2004) 1913–1942
15. Forstnerič, F.: Stein Manifolds and Holomorphic Mappings (The Homotopy Principle in Complex Analysis). Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 56. Springer-Verlag, Berlin-Heidelberg (2011)
16. Forstnerič, F.; Ohsawa, T.: Gunning-Narasimhan's theorem with a growth condition J. Geom. Anal. 23 (2013) 1078–1084
17. Forstnerič, F.; Prezelj, J.: Oka’s principle for holomorphic submersions with sprays. Math. Ann. 322 (2002) 633–666
18. Forstnerič, F.; Prezelj, J.: Extending holomorphic sections from complex subvarieties. Math. Z. 236 (2001) 43–68
19. Forstnerič, F.; Wold, E.F.: Bordered Riemann surfaces in \( \mathbb{C}^2 \). J. Math. Pures Appl. 91 (2009) 100–114
20. Forstnerič, F.; Wold, E.F.: Embeddings of infinitely connected planar domains into \( \mathbb{C}^2 \). Anal. PDE 6 (2013) 499-514
21. Grauert, H.: Über Modifikationen und exceptionelle analytische Mengen. Math. Ann. 146 (1962) 331–368
22. Grauert, H.: Theory of \( q \)-convexity and \( q \)-concavity. In: Several complex variables, VII. Encyclopaedia Math. Sci., vol. 74, pp. 259–284. Springer-Verlag, Berlin (1994)
23. Grauert, H.; Remmert, R.: Analytische Stellenalgebren. Die Grundlehren der mathematischen Wissenschaften, 176. Springer-Verlag, Berlin-New York (1971)
24. Grauert, H.; Remmert, R.: Theory of Stein Spaces. Die Grundlehren der mathematischen Wissenschaften, 227. Springer-Verlag, New York (1979)
25. Gunning, R.C.; Narasimhan, R.: Immersion of open Riemann surfaces. Math. Ann. 174 (1967) 103–108
26. Henkin, G.M.; Leiterer, J.: Theory of Functions on Complex Manifolds. Akademie-Verlag, Berlin (1984)
27. Hironaka, H.: Desingularization of complex-analytic varieties. (in French). Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, 627–631, Gauthier-Villars, Paris (1971)
28. Kaup, L.; Kaup, B.: Holomorphic functions of several variables. An introduction to the fundamental theory. De Gruyter Studies in Mathematics, 3. Walter de Gruyter & Co., Berlin (1983)
29. Lieb, I.; Michel, J.: The Cauchy-Riemann complex. Integral formulæ and Neumann problem. Aspects of Mathematics, E34, Friedr. Vieweg & Sohn, Braunschweig (2002)
30. Majcen, I.: Embedding certain infinitely connected subsets of bordered Riemann surfaces properly into \( \mathbb{C}^2 \). J. Geom. Anal. 19 (2009) 695–707
31. Narasimhan, R.: Imbedding of holomorphically complete complex spaces. Amer. J. Math. 82 (1960) 917–934
32. Osserman, R.: A survey of minimal surfaces. Second ed. Dover Publications, Inc., New York (1986)
33. Remmert, R.: Sur les espaces analytiques holomorphiquement séparables et holomorphiquement convexes. C. R. Acad. Sci. Paris 243 (1956) 118-121
34. Remmert, R.: Holomorphe und meromorphe Abbildungen komplexer Räume. Math. Ann., 133 (1957) 328–370
35. Richberg, R.: Stetige streng pseudoconvexe Funktionen. Math. Ann. 175 (1968) 257–286
36. Siu, J.-T.: Every Stein subvariety admits a Stein neighborhood. Invent. Math. 38 (1976) 89–100
37. Stopar, K.: Approximation of holomorphic mappings on 1-convex domains. Internat. J. Math., to appear.
38. Whitney, H.: Local properties of analytic varieties. In: Differentiable and Combinatorial Topology, A Symposium in honor of Marston Morse, S. Cairns, ed., pp. 205–244. Princeton University Press, Princeton (1965)

39. Whitney, H.: Complex Analytic Varieties. Addison-Wesley, Reading (1972)

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