On the congruence \( x^x \equiv \lambda \pmod{p} \)

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Abstract

In the present paper we obtain several new results related to the problem of upper bound estimates for the number of solutions of the congruence

\[ x^x \equiv \lambda \pmod{p}; \quad x \in \mathbb{N}, \quad x \leq p-1, \]

where \( p \) is a large prime number, \( \lambda \) is an integer coprime to \( p \). Our arguments are based on recent estimates of trigonometric sums over subgroups due to Shkredov and Shteinikov.

1 Introduction

For a prime \( p \) and an integer \( \lambda \) let \( J(p; \lambda) \) be the number of solutions of the congruence

\[ x^x \equiv \lambda \pmod{p}; \quad x \in \mathbb{N}, \quad x \leq p-1. \tag{1} \]

Note that the period of the function \( x^x \) modulo \( p \) is \( p(p-1) \), which is larger than the range in congruence (1).

From the works of Crocker [4] and Somer [8] it is known that there are at least \( \lfloor (p-1)/2 \rfloor \) and at most \( 3p/4 + p^{1/2+o(1)} \) incongruent values of \( x^x \) (mod \( p \)) when \( 1 \leq x \leq p-1 \). There are several conjectures in [5] related to this function.

New approaches to study \( J(p; \lambda) \) were given by Balog, Broughan and Shparlinski, see [1] and [2]. In the special case \( \lambda = 1 \) it was shown in [4] that \( J(p; 1) < p^{1/3+o(1)} \). This estimate was slightly improved in our work [3] to the bound \( J(p; 1) \ll p^{1/3-c} \) for some absolute constant \( c > 0 \). Note that the method of [3] applies for a more general exponential congruences, however, the constant \( c \) there becomes too small. In the present paper we use a different approach and prove the following results.
Theorem 1. The number $J(p; 1)$ of solutions of the congruence
\[ x^x \equiv 1 \pmod{p}; \quad x \in \mathbb{N}, \quad x \leq p - 1, \] satisfies $J(p; 1) \lesssim p^{27/82}$.

Here and below we use the notation $A \lesssim B$ to denote that $A < Bp^{o(1)}$; that is, for any $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ such that $A < cBp^\varepsilon$. As usual, $\text{ord}_\lambda$ denotes the multiplicative order of $\lambda$, that is, the smallest positive integer $t$ such that $\lambda^t \equiv 1 \pmod{p}$. We recall that $\text{ord}_\lambda | p - 1$.

Theorem 2. Uniformly over $t | p - 1$, we have, as $p \to \infty$,
\[ \sum_{\substack{1 \leq \lambda \leq p - 1 \\ \text{ord}_\lambda = t}} J(p; \lambda) \lesssim t + p^{1/3} t^{1/2}. \] (3)

In the range $t < p^{1/3}$ our Theorem 2 improves some results of the aforementioned works [1] and [2]. Note that in the case $t = 1$ the estimate of Theorem 1 is stronger. In fact, following the argument that we use in the proof of Theorem 1 it is possible to improve Theorem 2 in specific small ranges of $t$.

Let now $I(p)$ denote the number of solutions of the congruence
\[ x^x \equiv y^y \pmod{p}; \quad x \in \mathbb{N}, \quad y \in \mathbb{N}, \quad x \leq p - 1, \quad y \leq p - 1. \]

There is the following relationship between $I(p)$ and $J(p; \lambda)$:
\[ I(p) = \sum_{\lambda=1}^{p-1} J(p; \lambda)^2. \]

We modify one of the arguments of [1] and obtain the following refinement on [1, Theorem 8].

Theorem 3. We have, as $p \to \infty$,
\[ I(p) \lesssim p^{23/12}. \] (4)

In order to prove our results, we first reduce the problem to estimates of exponential sums over subgroups. In the proof of Theorem 1 we use Shteinikov’s result from [7], while in the proof of Theorem 2 we use Shkredov’s result from [6] (see, Lemma 2 and Lemma 3 below).
In what follows, \(\mathbb{F}_p\) is the field of residue classes modulo \(p\). The elements of \(\mathbb{F}_p\) we associate with their concrete representatives from \(\{0, 1, \ldots, p - 1\}\). For an integer \(m\) coprime to \(p\) by \(m^*\) we denote the smallest positive integer such that \(m^*m \equiv 1 \pmod{p}\). We also use the abbreviation
\[
e_p(z) = e^{2\pi iz/p}.
\]

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2 Lemmas

Lemma 1. Let
\[
\lambda \not\equiv 0 \pmod{p}, \quad n \in \mathbb{N}, \quad 1 \leq M \leq p.
\]
Then for any fixed constant \(k \in \mathbb{N}\) the number \(J\) of solutions of the congruence
\[
x^n \equiv \lambda \pmod{p}, \quad x \in \mathbb{N}, \quad x \leq M,
\]
satisfies
\[
J \lesssim \left(1 + \frac{M}{p^{1/k}}\right)n^{1/k}.
\]
In particular, if \(n = dt < p\) and \(M = p/d\), then we have the bound
\[
J \lesssim \left(d^{1/k} + \left(\frac{p}{d}\right)^{1-1/k}\right)t^{1/k}.
\]

Proof. We have
\[
J^k \lesssim \#\{(x_1, \ldots, x_k) \in \mathbb{N}^k \cap [1, M]^k; \quad (x_1 \ldots x_k)^n \equiv \lambda^k \pmod{p}\}.
\]
Since for a given integer \(\mu\) the congruence
\[
X^n \equiv \mu \pmod{p}, \quad X \in \mathbb{N}, \quad X \leq p,
\]
has at most \(n\) solutions, there exists a positive integer \(\lambda_0 < p\) such that
\[
J^k \lesssim nJ_1,
\]
where $J_1$ is the number of solutions of the congruence

$$x_1 \ldots x_k \equiv \lambda_0 \pmod{p}; \quad (x_1, \ldots, x_k) \in \mathbb{N}^k \cap [1, M]^k.$$

It follows that

$$x_1 \ldots x_k = \lambda_0 + py; \quad (x_1, \ldots, x_k) \in \mathbb{N}^k \cap [1, M]^k, \quad y \in \mathbb{Z}.$$

Since the left hand side of this equation does not exceed $M^k$, we get that $|y| \leq M^k/p$. Hence, for some fixed $y_0$ we have

$$J_1 \lesssim \left(1 + \frac{M^k}{p}\right)J_2,$$

where $J_2$ is the number of solutions of the equation

$$x_1 \ldots x_k = \lambda_0 + py_0; \quad (x_1, \ldots, x_k) \in \mathbb{N}^k \cap [1, M]^k.$$

Hence, from the bound for the divisor function it follows that $J_2 \lesssim 1$. Thus,

$$J^k \lesssim \left(1 + \frac{M^k}{p}\right)n.$$

and the result follows. \qed

Let $H_d$ be the subgroup of $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$ of order $d$. From the classical estimates for exponential sums over subgroups it is known that

$$\left| \sum_{h \in H_d} e_p(ah) \right| \leq p^{1/2}.$$

For a wide range of $d$ this bound has been improved in a serious of works. Here, we need the results due to Shteinikov \[7\] (see Lemma 2 below) and Shkredov \[6\] (see Lemma 3 below). They will be used in the proof of Theorem 1 and Theorem 2, respectively.

**Lemma 2.** Let $H_d$ be the subgroup of $\mathbb{F}_p^*$ of order $d < p^{1/2}$. Then for any integer $a \not\equiv 0 \pmod{p}$ the following bound holds:

$$\left| \sum_{h \in H_d} e_p(ah) \right| \lesssim p^{1/18}d^{101/126}.$$
Lemma 3. Let $H_d$ be the subgroup of $\mathbb{F}_p^*$ of order $d < p^{2/3}$. Then for any integer $a \not\equiv 0 \pmod{p}$ the following bound holds:

$$\left| \sum_{h \in H_d} e_p(ah) \right| \lesssim p^{1/6} d^{1/2}.$$ 

The following two results are due to Balog, Broughan and Shparlinski from [1] and [2].

Lemma 4. Uniformly over $t|p - 1$, we have, as $p \to \infty$,

$$\sum_{\substack{1 \leq \lambda \leq p-1 \\text{ord} \lambda = t}} J(p; \lambda) \lesssim t + p^{1/2}.$$ 

Lemma 5. Uniformly over $t|p - 1$ and all integers $\lambda$ with $\gcd(\lambda, p) = 1$ and $\text{ord} \lambda = t$, we have, as $p \to \infty$,

$$J(p; \lambda) \lesssim pt^{-1/12}.$$ 

We also need the following lemma.

Lemma 6. Let $a, x$ be positive integers and let $d = \gcd(x, p - 1)$. Then $a^d \equiv 1 \pmod{p}$.

This lemma is well-known and the proof is simple. Indeed, if $\text{ind} a$ is indice of $a$ with respect to some primitive root $g$ modulo $p$, then,

$$x \cdot \text{ind} a \equiv 0 \pmod{(p - 1)}.$$ 

Therefore, $d \cdot \text{ind} a \equiv 0 \pmod{(p - 1)}$, whence $a^d \equiv 1 \pmod{p}$.

The following lemma is also well-known; see, for example, exercise and solutions to chapter 3 in Vinogradov’s book [9] for even a more general statement.

Lemma 7. For any integers $U$ and $V > U$ the following bound holds:

$$\sum_{a=1}^{p-1} \left| \sum_{z=U}^{V} e_p(az) \right| \lesssim p.$$ 

5
3 Proof of Theorem \[1\]

We have

\[ J(p; 1) = \sum_{d|p-1} J'_d, \]

where \( J'_d \) is the number of solutions of (2) with \( \gcd(x, p-1) = d \). It then follows by Lemma 6 that

\[ J(p; 1) \leq \sum_{d|p-1} J_d, \]

where \( J_d \) is the number of solutions of the congruence

\[ z^d \equiv (d^d)^* \pmod{p}, \quad z \in \mathbb{N}, \quad z \leq (p-1)/d. \]

We have therefore,

\[ J(p; 1) \leq R_1 + R_2 + R_3 + \sum_{d|p-1 \quad d < p^{3/7}} J_d, \]

where

\[ R_1 = \sum_{d|p-1 \quad d > p^{5/7}} J_d; \quad R_2 = \sum_{d|p-1 \quad p^{3/7} < d < p^{5/7}} J_d; \quad R_3 = \sum_{d|p-1 \quad p^{5/7} < d \leq p^{3/7}} J_d. \]

The trivial estimate \( J_d \leq p/d \) implies that

\[ R_1 \lesssim \sum_{d|p-1 \quad d > p^{5/7}} \frac{p}{d} \lesssim \sum_{d|p-1} p^{2/7} \lesssim p^{2/7}. \]

To estimate \( R_2 \) we use Lemma 11 with \( k = 3 \) and get

\[ R_2 = \sum_{d|p-1 \quad p^{3/7} < d < p^{5/7}} J_d \lesssim \sum_{d|p-1 \quad p^{3/7} < d < p^{5/7}} (d^{1/3} + (p/d)^{2/3}) \lesssim \sum_{d|p-1} p^{2/7} \lesssim p^{2/7}. \]

To estimate \( R_3 \) we use Lemma 11 with \( k = 2 \) and get

\[ R_3 = \sum_{d|p-1 \quad p^{5/7} < d < p^{3/7}} J_d \lesssim \sum_{d|p-1 \quad p^{5/7} < d < p^{3/7}} (d^{1/2} + (p/d)^{1/2}) \lesssim \sum_{d|p-1} p^{2/7} \lesssim p^{2/7}. \]
Thus,
\[ J(p; 1) \lesssim p^{2/7} + \sum_{d | p-1 \atop d < p^{3/7}} J_d. \]

Hence, there exists \( d | p - 1 \) with \( d < p^{3/7} \) such that
\[ J(p; 1) \lesssim p^{2/7} + J_d. \]  \hspace{1cm} (5)

Applying Lemma 1 with \( k = 2 \), we get
\[ J_d \lesssim d^{1/2} + \left( \frac{p}{d} \right)^{1/2} \lesssim \left( \frac{p}{d} \right)^{1/2}. \]  \hspace{1cm} (6)

Let now \( H_d \) be the subgroup of \( \mathbb{F}_p^* \) of order \( d \). We recall that \( J_d \) is the number of solutions of the congruence
\[(dz)^d \equiv 1 \pmod{p}; \quad z \in \mathbb{N}, \quad z \leq (p-1)/d. \]

Therefore,
\[ J_d = \#\{ z \in \mathbb{N}; \quad z \leq (p-1)/d, \quad dz \pmod{p} \in H_d \}. \]

It then follows that
\[ J_d = \frac{1}{p} \sum_{a=0}^{p-1} \sum_{1 \leq z \leq (p-1)/d} \sum_{h \in H_d} e_p(a(dz - h)). \]

Separating the term corresponding to \( a = 0 \) and using Lemma 2 for \( a \neq 0 \), we get
\[ J_d \leq 1 + p^{1/18} d^{101/126} \left( \frac{1}{p} \sum_{a=1}^{p-1} \left| \sum_{1 \leq z \leq (p-1)/d} e_p(adz) \right| \right) \lesssim p^{1/18} d^{101/126}. \]

Using Lemma 7, we get the following bound for the double:
\[ \sum_{a=1}^{p-1} \left| \sum_{1 \leq z \leq (p-1)/d} e_p(adz) \right| = \sum_{b=1}^{p-1} \left| \sum_{1 \leq z \leq (p-1)/d} e_p(bz) \right| \lesssim p. \]

Therefore
\[ J_d \lesssim p^{1/18} d^{101/126}. \]

Comparing this estimate with (6) we obtain
\[ J_d \lesssim p^{27/82}. \]

Incorporating this in (5), we get the desired result.
4 Proof of Theorem 2

In view of Lemma 4, it suffices to deal with the case \( t < p^{1/3} \).

Since \( \lambda' \equiv 1 \pmod{p} \), it follows from (1) that
\[
\sum_{1 \leq \lambda \leq p-1 \atop \ord \lambda = t} J(p; \lambda) \leq \# \{ x \in \mathbb{N}; \ x^{tx} \equiv 1 \pmod{p}, \ x \leq p-1 \}
\]

Hence, denoting \( d = \gcd(x, (p-1)/t) \) and using Lemma 6 we obtain that
\[
\sum_{1 \leq \lambda \leq p-1 \atop \ord \lambda = t} J(p; \lambda) \leq \sum_{d | (p-1)/t} T_d,
\]
where \( T_d \) is the number of solutions of the congruence
\[
z^{dt} \equiv (d^{dt})^* \pmod{p}, \ z \in \mathbb{N}, \ z \leq (p-1)/d.
\]

By the trivial estimate \( T_d \leq p/d \) we have
\[
\sum_{d | (p-1)/t \atop d > p^{1/3}} T_d \leq \sum_{d | (p-1)/t \atop d > p^{1/3}} p^{1/3} \lesssim p^{1/3}.
\]

Furthermore, applying Lemma 4 with \( k = 2 \), we get
\[
\sum_{d | (p-1)/t \atop p^{1/3} < d < p^{2/3}} T_d \leq \sum_{d | (p-1)/t \atop p^{1/3} < d < p^{2/3}} \left( d^{1/2} + (p/d)^{1/2} \right)^t \lesssim p^{1/3} t^{1/2}.
\]

Therefore,
\[
\sum_{1 \leq \lambda \leq p-1 \atop \ord \lambda = t} J(p; \lambda) \leq p^{1/4} t^{1/2} + \sum_{d | (p-1)/t \atop d < p^{1/3}} T_d. \tag{7}
\]

Recall that \( t < p^{1/3} \), thus \( dt | p - 1 \) and \( dt < p^{2/3} \).

Let \( H_{dt} \) be the subgroup of \( \mathbb{F}_p^* \) of order \( dt \). Since \( T_d \) is the number of solutions of the congruence
\[
(dz)^{dt} \equiv 1 \pmod{p}; \ z \in \mathbb{N}, \ z \leq (p-1)/d,
\]

it follows that
\[
T_d = \# \{ z \in \mathbb{N}; \ z \leq (p-1)/d, \ dz \pmod{p} \in H_{dt} \}.
\]
Therefore,
\[ T_d = \frac{1}{p} \sum_{a=0}^{p-1} \sum_{1 \leq z \leq (p-1)/d} \sum_{h \in H_{dt}} e_p(a(dz - h)). \]

Separating the term corresponding to \( a = 0 \) and using Lemma 3 for \( a \neq 0 \) (with \( d \) replaced by \( dt \)), we get
\[ T_d \leq t + p^{1/6} d^{1/2} t^{1/2} \left( \frac{1}{p} \sum_{a=1}^{p-1} \sum_{1 \leq z \leq (p-1)/d} e_p(adz) \right). \]

Applying Lemma 7 to the double sum, as in the proof of Theorem 1, we obtain for \( d < p^{1/3} \) the bound
\[ T_d \lesssim t + p^{1/6} d^{1/2} t^{1/2} \lesssim t + p^{1/3} t^{1/2}. \]

Thus,
\[ \sum_{d | (p-1)/t} T_d \leq \sum_{d | p-1} (t + p^{1/3} t^{1/2}) \lesssim t + p^{1/3} t^{1/2}. \]

Putting this into (7), we conclude the proof.

\section{Proof of Theorem 3}

We follow the arguments of \[ \text{I} \] with some modifications. We have
\[ I(p) = \sum_{\lambda=1}^{p-1} J(p; \lambda)^2 = \sum_{t | p-1} \sum_{1 \leq \lambda \leq p-1 \atop \text{ord} \lambda = t} J(p; \lambda)^2. \]

It then follows that for some fixed order \( t | p - 1 \) we have
\[ I(p) \lesssim \sum_{1 \leq \lambda \leq p-1 \atop \text{ord} \lambda = t} J(p; \lambda)^2. \]

We can split the range of \( J(p; \lambda) \) into \( O(\log p) \) dyadic intervals. Then, for some \( 1 \leq M \leq p \), we have
\[ I(p) \lesssim |A|M^2, \quad \text{(8)} \]
where \(|A|\) is the cardinality of the set
\[
A = \{1 \leq \lambda \leq p - 1; \quad \text{ord} \lambda = t, \quad M \leq J(p; \lambda) < 2M\}.
\]

From Lemma 5 we have
\[
M \lesssim pt^{-1/12}.
\] (9)

On the other hand, by Lemma 4 we also have
\[
|A|M \lesssim \sum_{\lambda \in A} J(p; \lambda) \lesssim \sum_{1 \leq \lambda \leq p-1, \text{ord} \lambda = t} J(p; \lambda) \lesssim t + p^{1/2}.
\]

If \(t < p^{1/2}\), then using (8) we get
\[
I(p) \lesssim |A|M^2 \lesssim (|A|M)^2 \lesssim p,
\]
and the result follows. If \(t > p^{1/2}\), then we get \(|A|M \lesssim t\). Therefore, using (8) and (9) we get
\[
I(p) \lesssim |A|M^2 \lesssim t(pt^{-1/12}) = pt^{11/12} \lesssim p^{23/12}.
\]

This proves Theorem 3.

References

[1] A. Balog, K. A. Broughan and I. E. Shparlinski, ‘On the number of solutions of exponential congruences’, Acta Arith., 148 (2011), 93–103.

[2] A. Balog, K. A. Broughan and I. E. Shparlinski, ‘Some-product estimates with several sets and applications’, Integers, 12 (2012), 895–906.

[3] J. Cilleruelo and M. Z. Garaev, ‘Congruences involving product of intervals and sets with small multiplicative doubling modulo a prime and applications’, Preprint, (2014).

[4] R. Crocker, ‘On residues of \(n^n\)’, Amer. Math. Monthly, 76 (1969), 1028–1029.

[5] J. Holden and P. Moree, ‘Some heuristics and results for small cycles of the discrete logarithm’, Math. Comp., 75 (2006), 419–449.
[6] I. D. Shkredov, ‘On exponential sums over multiplicative subgroups of medium size’, *Finite Fields and Their Applications*, **30** (2014), 72–87.

[7] Yu. N. Shteinokov, ‘Estimates of trigonometric sums modulo a prime’, *Preprint*, 2014.

[8] L. Somer, ‘The residues of $n^n$ modulo $p$', *Fibonacci Quart.*, **19** (1981), 110–117.

[9] I. M. Vinogradov, *Elements of number theory*, Dover Publ., New York 1954.

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