Fine and hyperfine interaction on the light cone

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Abstract. The formalism for the spin interactions in the front form (light-cone) is re-phrased in terms of an instant form formalism. It is shown how to unitarily transform the Brodsky-LePage spinors to Bjorken-Drell spinors and to rephrase the so called spinor matrix in terms of the interactions one is familiar with from atomic and Dirac theory. — One retrieves the (relativistic) kinetic correction, the hyperfine and the Darwin term which acts even when wave function is spherically symmetric. One also retrieves angular momentum dependent terms like the spin-orbit interaction in a relativistically correct way; and one obtains additional terms which thus far have not been reported particularly various $L^2$-dependent terms. Since the approach includes the full retardation, one gets additional, thus far unknown terms. The differ from atomic and Dirac theory, since there only that part of the vector potential is usually included which is generated by the atomic nucleus. Quite on purpose, the paper is kept formal. —

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1 The light-cone integral equation

This paper number 3 in a row of 3 [1,2] on the bound state problem in gauge theory [3] deals with the technical question of how to formulate the fine and hyperfine interaction in the one-body integral ‘master’ equation which has been previously derived [2,3].

I therefore jump immediately to Eq.(16) of [2],

\[
M^2 \psi_{1,2}(x,k_L) = \left[ \frac{m_1^2 + k_L^2}{x} + \frac{m_2^2 + k_L^2}{1-x} \right] \psi_{1,2}(x,k_L) - \frac{1}{4\pi^2} \sum_{h_1',h_2'} \int \frac{d\sigma' d\lambda' \psi_{h_1',h_2'}(x',k_L')}{\sqrt{x(1-x)}x'(1-x')} \frac{\alpha_c(Q)}{Q^2} R(Q) \times \left[ \gamma(1,1')^\mu u(k_1',h_1') \right] \left[ \gamma(2,2')^\mu v(k_2,h_2') \right].
\]

(1)

Here, $M^2$ is the eigenvalue of the invariant-mass squared. The associated eigenfunction $\psi_{1,2}(x,k_L)$ is the probability amplitude $\langle x,k_L,h_1,1-x,-k_L,h_2|\psi_{q\bar{q}} \rangle$ for finding the quark with momentum fraction $x$, transversal momentum $k_L$ and helicity $h_1$, and correspondingly the anti-quark. Their (effective) masses are denoted by $m_1$ and $m_2$, and $u(k_1,h_1)$ and $v(k_2,h_2)$ are their Dirac spinors in Lepage Brodsky convention, as given in [3]. The (effective) coupling function $\alpha_c(Q) = \frac{1}{3} \pi(Q)$ is also given in [3]. The kernel is governed by the mean four-momentum transfer, $Q^2 = \frac{1}{2} (Q^2 + Q_{\bar{q}}^2)$, where

\[
Q^2_q = -(k_1 - k_1')^2 \quad \text{and} \quad Q^2_{\bar{q}} = -(k_2 - k_2')^2,
\]

(2)

are the Feynman four-momentum transfers of quark and anti-quark, respectively. The regulator function $R(Q)$, finally, removes the ultraviolet singularities and regulates the interaction. Note that the equation is fully relativistic and covariant. It coincides literally with Eq.(4.101) of [3].

Krautgartner et al [4] and Trittmann et al [5] have shown how to solve such an equation numerically with high precision. But since the numerical effort is considerable, it is reasonable to work first with simpler models. The aim of the present work is to derive such ones.

The aspects of regularization and renormalization have been emphasized in [1,2], resulting in an explicit construction of the regulator function $R(Q)$. The case was worked out within the so called Singlet-Triplet model. Here, I address to go beyond that, particularly to derive a model for the spin-orbit interaction, which had been suppressed on purpose in [2].

2 Transforming the integral equation

The light-cone integral equation [1] has the unpleasant aspect that the integration variables have a completely different support,

\[
0 < x < 1, \quad -\infty < k_L < +\infty.
\]

Therefore, practically in all of the numerical work particularly in [4] and [5], the variable transform

\[
x(k_z) = \frac{E_1 + k_z}{E_1 + E_2},
\]

(3)

with $E_{1,2} = E_{1,2}(k) \equiv \sqrt{m_{1,2}^2 + k_z^2 + k_{\perp}^2}$, has been used to transform to integration variables

\[-\infty < k_z < +\infty, \quad -\infty < k_{\perp} < +\infty,\]

and
with the same support. While \( k_z \) varies from \(-\infty\) to \(+\infty\), the \( x(k_z) \) varies from 0 to 1. The particles are then described by their front form four-momenta:

\[
\begin{align*}
    k_1^+ &= k_z + E_1(k), & k_2^- &= -k_z + E_2(k), \\
    k_{1\perp} &= k_{1\perp}, & k_{2\perp} &= -k_{2\perp}.
\end{align*}
\]

Or, they are described by the instant form four-momenta:

\[
\begin{align*}
    k_1^0 &= E_1(k), & k_2^0 &= E_2(k), \\
    k_1 &= k, & k_2 &= -k,
\end{align*}
\]

with \( k \equiv (k_{1\perp}, k_z) \). Such a switching between front form and instant form parameterization is possible, since the four-vectors of the constituents refer to free particles. The free invariant mass of the two particles,

\[
M_{\text{free}}^2 = \frac{m_1^2 + k_1^2}{x} + \frac{m_2^2 + k_2^2}{1 - x} = (E_1(k) + E_2(k))^2,
\]

looks like in the rest frame of the instant form (\( P = 0 \)). For vanishing \( k \) it is \((m_1 + m_2)^2\). The discrepancy can be calculated exactly as follows:

\[
(E_1 + E_2)^2 - (m_1 + m_2)^2 = (E_1 + E_2 - m_1 - m_2)(E_1 + E_2 + m_1 + m_2),
\]

and therefore as

\[
(E_1 + E_2)^2 - (m_1 + m_2)^2 = \left( \frac{E_1^2 - m_1^2}{E_1 + m_1} + \frac{E_2^2 - m_2^2}{E_2 + m_2} \right) (E_1 + E_2 + m_1 + m_2).
\]

With the reduced mass,

\[
\frac{1}{m_r} = \frac{1}{m_1} + \frac{1}{m_2}, \quad m_r = \frac{m_1 m_2}{m_1 + m_2},
\]

and the dimensionless \( A(k) \) and \( B(k) \),

\[
\begin{align*}
    A(k) &\equiv m_r \frac{E_1(k) + E_2(k)}{E_1(k) E_2(k)}, \\
    B(k) &\equiv \frac{E_1(k) + m_1 + E_2(k) + m_2}{(m_1 + m_2)} \\
    &\times \left( \frac{m_1^2}{E_1(k) + m_1} + \frac{m_2^2}{E_2(k) + m_2} \right),
\end{align*}
\]

the free invariant mass and the exact Jacobian of the transformation are therefore

\[
M_{\text{free}}^2 = (m_1 + m_2)^2 + (m_1 + m_2) \frac{k^2}{m_r} B(k),
\]

\[
dx = x(1-x) \frac{dk_z}{m_r} \frac{1}{A(k)},
\]

respectively, see also [6].

The transformation from \((x, k_{1\perp})\) to \((k_z, k_{1\perp})\) will be presented in two steps. In the first step, the variables are transformed and Eq. (11) becomes

\[
\begin{align*}
    M^2 - (E_1(k) + E_2(k))^2 &\psi_{h_1 h_2}(k_z, k_{1\perp}) = \\
    &- \sum_{h_1', h_2'} \int \frac{dk'_z d^2 k'_1}{\sqrt{x'(1-x')}} \psi_{h_1' h_2'}(k'_z, k'_1) \\
    &\times \frac{\alpha_e(Q) R(Q)}{Q^2 4\pi^2 m_r} \\
    &\times \left[ \frac{\sigma(k_1, h_1) \gamma^\mu u(k'_1, h'_1)}{[\sigma(k_2, h'_2) \gamma^\mu v(k_2, h_2)]} \right].
\end{align*}
\]

One notes that the kernel is not symmetric under the exchange of primed ('') and unprimed quantities, as opposed to Eq. (11) which is symmetric. This asymmetry can however be removed as usual, by multiplying the equation in a second step with \( \sqrt{x(1-x)}/A(k) \),

\[
\phi_{h_1 h_2}(k_z, k_{1\perp}).
\]

In the numerical work [11], the reduced wave function \( \phi_{h_1 h_2}(k_z, k_{1\perp}) \) is calculated first. It is then converted to \( \psi_{h_1 h_2}(x, k_{1\perp}) \) by

\[
\psi_{h_1 h_2}(x, k_{1\perp}) = \sqrt{\frac{A(k_z(x), k_{1\perp})}{x(1-x)}} \phi_{h_1 h_2}(x, k_{1\perp}),
\]

i.e. by the substitution \( k_z = k_z(x) \) which is inverse to Eq. (13). Krautgärtner [14] and particularly Trittmann [15] have presented beautiful three-dimensional plots of \( \psi_{h_1 h_2} \).

The variable transformation is applied here in a strict mathematical sense. It does not change the physical content, particularly not the eigenvalue spectrum \( M_r^2 \). The transformed integral equation looks like an equation in usual momentum space. But one should emphasize that it continues to be a front form equation with the sole purpose to generate \( \psi_{h_1 h_2}(x, k_{1\perp}) \). The reduced wave function \( \phi_{h_1 h_2}(x, k_{1\perp}) \) has no physical interpretation.

The kernel of Eq. (13) depends on \( |k'|^2 \) and Lorenz-invariants like \( Q^2 \) or \( \gamma^\mu [\gamma^\nu] \). It is therefore invariant under spatial rotations. The occurrence of spin-degenerate multiplets in the numerical solutions of [13] become thus understandable e postriori. It seems as if all light-cone specific troubles with rotations of the coordinate system are absorbed in the \( x(1-x) \)-factor in Eq. (13). The reduced wave function \( \phi_{h_1 h_2}(k_z, k_{1\perp}) \) transforms covariantly under rotations while \( \psi_{h_1 h_2}(x, k_{1\perp}) \) does not.

3 The Melosh rotated integral equation

Much of the difficulty in getting the reduced wave function in practice [14,15], and to understand the structure of the
The Lepage-Brodsky spinors are [3]:

\[\langle h_1, h_2 | S | h_1', h_2' \rangle = \frac{1}{\sqrt{2m(E + m)}} \begin{pmatrix} s : 1 & 0 \\ 0 & s : 1 \end{pmatrix},\]

with \( E \equiv E(k) \) as in Eq. [8]. The four overlap matrix elements of Eq. [10] are then calculated as

\[\langle s | \omega | h \rangle = \frac{1}{2p^+(E + m)} \begin{pmatrix} k^+ + m & -k_l \\ k_r & k^+ + m \end{pmatrix},\]

with the rows labeled by \( s \) and the columns by \( h \). One calculates

\[\sum_h \langle s | \omega | h \rangle \langle h | \omega' | s' \rangle = \frac{1}{2k^+(E + m)} \times \begin{pmatrix} k^+ + m & -k_l \\ k_r & k^+ + m \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\]

since \((p^+ + m)^2 + pp_r = 2p^+(E + m)\), verifying this way that the transformation is unitary, indeed.

Since the spinors appear in bilinear combinations, it is convenient to define the unitary direct product

\[\langle s_1 s_2 | \Omega | h_1 h_2 \rangle = \langle s_1 | \omega | h_1 \rangle \otimes \langle s_2 | \omega | h_2 \rangle.\]

Introducing a second reduced wave function \( \varphi_{s_1 s_2} \) by

\[\varphi_{s_1 s_2}(k) = \sum_{h_1, h_2} \langle s_1 s_2 | \Omega | h_1 h_2 \rangle \psi_{h_1 h_2}(k),\]

Eq. [13] can be transformed unitarily to

\[\begin{pmatrix} M^2 - (E_1(k) + E_2(k))^2 \end{pmatrix} \varphi_{s_1 s_2}(k) = -\begin{pmatrix} m_1 + m_2 \end{pmatrix} \sum_{s_1', s_2'} \int d^2k_1 d^2k_2 \varphi_{s_1' s_2'}(k') \frac{\alpha_c(Q)}{A(k' A(k')} Q^2 R(Q) \times [\overline{u}(k_1, s_1) \gamma^\nu u(k_1', s_1')][\overline{u}(k_2, s_2) \gamma^\mu u(k_2', s_2')].\]

Once one has the wave functions \( \varphi_{s_1 s_2}(k, k'_\perp) \), one can unitarily transform them back to the light-cone wave functions by

\[\psi_{h_1 h_2}(x, k_\perp) = \frac{\sqrt{A(k_\parallel(x), k_\perp)}}{\sqrt{x(1 - x)}} \times \sum_{s_1, s_2} \langle h_1 h_2 | \Omega^\dagger(k_\parallel(x), k_\perp) | s_1 s_2 \rangle \varphi_{s_1 s_2}(k_\parallel(x), k_\perp),\]

in analogy to Eq. [18].

The spinors in Eq. [20] are Bjørken–Drell spinors. On the technical level, they are much more transparent than those of Lepage–Brodsky, as to be seen next. The previous numerical work [4,5] done with Eq. [20] would have been much easier than with Eq. [9]. But at that time, the present physical insight was lacking.
3.1 Definition of the spinor factor

The spin dependence in Eq. (20) resides in the spinor factor

\[ \langle s_1, s_2 | s'_1, s'_2 \rangle = \frac{\rho_1(k_1, s_1) \gamma^\mu u(k'_1, s'_1) \rho_2(k_2, s_2) \gamma^\mu u(k'_2, s'_2)}{\langle k_1, s_1 | k'_1, s'_1 \rangle \langle k_2, s_2 | k'_2, s'_2 \rangle} . \]

With \( \gamma^0(1) \gamma^0(2) = \gamma^0(1) \gamma^0(2) - \gamma^i(1) \gamma^i(2) \), where

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \{ \gamma^i \} = \gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \]

it can be evaluated in closed form with [8]

\[ u(k_1, s_1) = \sqrt{\frac{E_1 + m_1}{2m_1}} \begin{pmatrix} \sigma \cdot k \\ E_1(k) + m_1 \end{pmatrix} \chi_{s_1}, \]

\[ u(k_2, s_2) = \sqrt{\frac{E_2 + m_2}{2m_2}} \begin{pmatrix} \sigma \cdot k \\ E_2(k) + m_2 \end{pmatrix} \chi_{s_2}. \]

The components of the four current \( j^\mu = (\rho, j) \) are then

\[ \rho_1 = \left[ \vec{\tau}(k_1, s_1) \gamma^0 u(k'_1, s'_1) \right] \left[ \begin{pmatrix} \sigma \cdot k \\ E_1(k) + m_1 \end{pmatrix} \chi_{s_1} \right] , \]

\[ j_1 = \left[ \vec{\tau}(k_1, s_1) \gamma^0 u(k'_1, s'_1) \right] \left[ \begin{pmatrix} \sigma \cdot k \\ E_1(k) + m_1 \end{pmatrix} \chi_{s_1} \right] . \]

The spinor factor becomes then most straightforwardly:

\[ \langle s_1, s_2 | s'_1, s'_2 \rangle = \sqrt{\frac{E_1(k_1) + m_1}{2m_1}} \frac{E_2(k_1) + m_2}{2m_2} \chi_{s_1} \chi_{s_2} \]

\[ \times \left[ \frac{1}{2m_1} \left( \begin{pmatrix} \sigma \cdot k \\ E_1(k) + m_1 \end{pmatrix} \chi_{s_1} \right) \right] \left[ \begin{pmatrix} \sigma \cdot k \\ E_2(k) + m_2 \end{pmatrix} \chi_{s_2} \right] \]

\[ \times \left[ \begin{pmatrix} \sigma \cdot k \\ E_1(k) + m_1 \end{pmatrix} \chi_{s_1} \right] \left[ \begin{pmatrix} \sigma \cdot k \\ E_2(k) + m_2 \end{pmatrix} \chi_{s_2} \right] \]

\[ \times \left( \begin{pmatrix} \sigma \cdot k \gamma^0 \gamma^0 \\ (E_1(k) + m_1) (E_2(k) + m_1) \end{pmatrix} \right) \left[ \begin{pmatrix} \sigma \cdot k \\ (E_1(k) + m_1) \chi_{s_1} \right) \right] \left[ \begin{pmatrix} \sigma \cdot k \\ (E_2(k) + m_2) \chi_{s_2} \right) \right] \]

\[ \times \left[ \begin{pmatrix} \sigma \cdot k \gamma^0 \gamma^0 \\ (E_1(k) + m_1) (E_2(k) + m_2) \end{pmatrix} \right] \left[ \begin{pmatrix} \sigma \cdot k \\ (E_1(k) + m_1) \chi_{s_1} \right) \right] \left[ \begin{pmatrix} \sigma \cdot k \\ (E_2(k) + m_2) \chi_{s_2} \right) \right] \]

The expression is very much simpler than the long tables for the Lepage–Brodsky spinors [9], indeed. The first two lines in the square bracket correspond to the product of the charges, \( \rho_1 \rho_2 \), and the next two lines to the scalar product of the currents, \( j_1 j_2 \). Note that the plus sign of \( j_1 j_2 \), as opposed to the minus sign in \( \gamma^0 \gamma^0 \gamma \gamma = \gamma^0 \gamma^0 \gamma - \gamma \gamma \).

This is due to \( k_1 = + k \) and \( k_2 = - k \), see also Eq. (22).

4 Conversion to a conventional Hamiltonian

Thus far, the \( 4 \times 4 \) matrix of the effective light-cone Hamiltonian in light-cone momentum representation, Eq. (4),

\[ \int d^3k' d^3k'' \langle h_1, h_2 | H_{\text{eff}}(x, k_L; x', k_L') | h'_1, h'_2 \rangle \]

has been transformed to the \( 4 \times 4 \) matrix of the effective light-cone Hamiltonian in usual momentum representation, Eq. (20),

\[ \int d^3k' \langle s_1, s_2 | H_{\text{eff}}(k; k') | s'_1, s'_2 \rangle \varphi_{s_1} \varphi_{s_2} = M^2 \varphi_{s_1} \varphi_{s_2} \]

Here and below the explicit summations over the helicities are replaced by the Einstein convention. The spectrum of invariant mass-squared eigenvalues \( M^2 \) is unchanged by the transformation. One can subtract a c-number from an operator, and divide by a scale, and thus define a new Hamiltonian \( H \) with new eigenvalues \( E \) but the same eigenfunctions \( \varphi_{s_1} \).

\[ H_{\text{eff}}(k; k') = (m_1 + m_2) + 2 (m_1 + m_2) H(k; k') , \]

\[ M^2 = (m_1 + m_2)^2 + 2 (m_1 + m_2) E \]

The eigenvalue \( E \) has the dimension of an energy and is not to be confused with the single particle energy \( E(k) \):

\[ \int d^3k' \langle s_1, s_2 | H(k; k') | s'_1, s'_2 \rangle \varphi_{s_1} \varphi_{s_2} = E \varphi_{s_1} \varphi_{s_2} \]

The kernels for kinetic and potential energy are given by

\[ T(k; k') = \frac{k^2}{2m_r} \delta^{(3)}(k - k') \delta s_1 \delta s_2 \]

\[ U(k; k') = -\frac{m_2}{4} \frac{\alpha_s(Q) S(k; k')}{Q^2} R(Q) \frac{1}{\sqrt{A(k; A')}} , \]

respectively. Both \( A(k) \) and \( B(k) \) were defined in Eq. (3).

5 Explicit calculation of the spinor factor

This section is devoted to carry out explicitly the multiplications in Eq. (24). The bilinear expressions with the same \( \sigma \)'s can be simplified by means of the identities

\[ (k \cdot \sigma) \sigma = k + i (\sigma \wedge k), \]

\[ (k' \cdot \sigma) \sigma = k' - i (\sigma \wedge k'), \]

\[ (\sigma \cdot k') \sigma = k \cdot k' + i \sigma \cdot (k \wedge k') \].

One gets thus identically in a first step:

\[ \langle s_1, s_2 | s'_1, s'_2 \rangle = \sqrt{\frac{E_1(k_1) + m_1}{2m_1}} \frac{E_2(k_1) + m_2}{2m_2} \chi_{s_1} \chi_{s_2} \]

\[ \times \left[ \begin{pmatrix} \sigma \cdot k \\ E_1(k) + m_1 \end{pmatrix} \chi_{s_1} \right] \left[ \begin{pmatrix} \sigma \cdot k \\ E_2(k) + m_2 \end{pmatrix} \chi_{s_2} \right] \]

\[ \times \left[ \begin{pmatrix} \sigma \cdot k \\ E_1(k) + m_1 \end{pmatrix} \chi_{s_1} \right] \left[ \begin{pmatrix} \sigma \cdot k \\ E_2(k) + m_2 \end{pmatrix} \chi_{s_2} \right] \]

\[ \times \left[ \begin{pmatrix} \sigma \cdot k \gamma^0 \gamma^0 \\ (E_1(k) + m_1) (E_2(k) + m_1) \end{pmatrix} \right] \left[ \begin{pmatrix} \sigma \cdot k \\ (E_1(k) + m_1) \chi_{s_1} \right) \right] \left[ \begin{pmatrix} \sigma \cdot k \\ (E_2(k) + m_2) \chi_{s_2} \right) \right] \]

\[ \times \left[ \begin{pmatrix} \sigma \cdot k \gamma^0 \gamma^0 \\ (E_1(k) + m_1) (E_2(k) + m_2) \end{pmatrix} \right] \left[ \begin{pmatrix} \sigma \cdot k \\ (E_1(k) + m_1) \chi_{s_1} \right) \right] \left[ \begin{pmatrix} \sigma \cdot k \\ (E_2(k) + m_2) \chi_{s_2} \right) \right] \]
Carrying out the multiplications one gets in a second step:

\[
\langle s_1, s_2 | S | s_1', s_2' \rangle = \sqrt{\frac{E_1(k) + m_1}{2m_1} \frac{E_2(k) + m_2}{2m_2}} \langle s_1, s_2 | 1 + \frac{k \cdot k'}{E_1(k) + m_1} \frac{E_2(k') + m_2}{2m_2} \rangle \left[ 1 + \frac{k \cdot k'}{E_1(k) + m_1} \frac{E_2(k') + m_2}{2m_2} \right] \langle s_1', s_2' | s_1, s_2 \rangle.
\]

With Eqs. (32), (33), (34) below, one gets in a third step:

\[
\langle s_1, s_2 | S | s_1', s_2' \rangle = \sqrt{\frac{E_1(k) + m_1}{2m_1} \frac{E_2(k) + m_2}{2m_2}} \langle s_1, s_2 | 1 + \frac{k \cdot k'}{E_1(k) + m_1} \frac{E_2(k') + m_2}{2m_2} \rangle \left[ 1 + \frac{k \cdot k'}{E_1(k) + m_1} \frac{E_2(k') + m_2}{2m_2} \right] \langle s_1', s_2' | s_1, s_2 \rangle.
\]

Here, the familiar vector identities like

\[
a \cdot b + c = a \cdot b + c,
\]

\[
(a \cdot b) \cdot c = (a \cdot c)b - (b \cdot c)a,
\]

\[
(a \cdot b) \cdot (c \cdot d) = (a \cdot c)b \cdot d - (b \cdot c)a \cdot d,
\]

were used to derive particularly

\[
(k \cdot k')^2 = k^2 k'^2 - (k \wedge k')^2,
\]

\[
(k + i\sigma_1 \wedge k) \cdot (k' - i\sigma_2 \wedge k') = kk' + (\sigma_1 \wedge k) \cdot (\sigma_2 \wedge k') + i(\sigma_1 + \sigma_2) \cdot (k \wedge k').
\]

In deriving Eq. (33), Eq. (34) was used in line 3 of the square bracket of the equation. In lines 5 and 6, Eq. (31) was applied. In line 4, Eq. (34) was applied as well, but considering the fact that the wedge product disappears for \( k = k' \).

Eq. (31) emphasizes the wedge product \( k \wedge k' \). As to be seen below, it is closely related to the orbital angular momentum \( \mathbf{L} \), which likes for example to be dotted into the spin to produce spin orbit coupling \( \mathbf{L} \cdot \sigma \). It is therefore reasonable to divide the spinor factor into two parts,

\[
S = S_0 + S_1,
\]

with \( S_0 \) being independent of the wedge product, and with \( S_1 \) depending only on the wedge product. Thus:

\[
\langle s_1, s_2 | S_0 | s_1', s_2' \rangle = \sqrt{\frac{E_1(k) + m_1}{2m_1} \frac{E_2(k') + m_2}{2m_2}} \langle s_1, s_2 | 1 + \frac{k \cdot k'}{E_1(k) + m_1} \frac{E_2(k') + m_2}{2m_2} \rangle \left[ 1 + \frac{k \cdot k'}{E_1(k) + m_1} \frac{E_2(k') + m_2}{2m_2} \right] \langle s_1', s_2' | S_0 \rangle.
\]

6 The Fourier approximation

Of course, one can diagonalize the Hamiltonian directly in momentum space, see Eq. (27). But momentum space does not appeal to intuition, at least not to my intuition. One would like to Fourier transform to configuration space, in order to interpret and to understand what one is doing. This however turns out to be impossible for the general case, for two reasons: (1) The Fourier transform cannot be taken analytically, and (2) the square root in \( E(k) = \sqrt{m^2 + \mathbf{p}^2} \) induces strong non-localities in the potential energy part of the Hamiltonian, see for example App. D.

In the further analysis, I will apply therefore what I call the Fourier approximation

\[
E(k) \rightarrow m.
\]

But one must be careful. The un-considered application of Eq. (35) for example to the l.h.s. of Eq. (40) gives plain non-sense.

The non-sense is avoided by substituting in Eq. (27) to the i.h.s. of Eq. (40) gives plain non-sense.

\[
A(k) \rightarrow 1,
\]

\[
B(k) \rightarrow 1.
\]

Similarly, one substitutes the four-momentum transfers in Eq. (2) according to

\[
Q_x^2 = (k - k')^2 - (E_1(k) - E_1(k'))^2 \rightarrow (k - k')^2.
\]

I try to avoid the expression “non-relativistic approximation.” Relativistic or non-relativistic motion is usually associated with the kinetic energy. But here the Fourier approximation takes place essentially in the interaction, under the integral, and is applied, as mentioned, for the sole
purpose of making a Fourier transform possible. Whether this is a good approximation or not, whether Eq. [39] is applicable or not, can be answered only in the solution, a posteriori. In any case, this question must remain on the agenda.

In the Fourier approximation, Eqs. [40]-[43] become very much simplified. Suppression explicit reference to the bras and kets \(|s_1, s_2\) one gets simply

\[
S_0 = 1 + \frac{k^2 - (\sigma_1 \cdot \lambda k) \cdot (\sigma_2 \cdot \lambda k)}{4m_1 m_2} + \frac{k'^2 - (\sigma_1 \cdot \lambda k') \cdot (\sigma_2 \cdot \lambda k')}{4m_1 m_2} + \frac{k \cdot k'}{4m_2^2} \quad \text{(41)}
\]

\[
S_0 = 1 + \frac{\sigma_1 \cdot \lambda k \cdot k'}{4m_1 m_2} + \frac{i(\sigma_1 + \sigma_2) \cdot \lambda k \cdot k'}{2m_1 m_2} + \frac{\sigma_2 \cdot \lambda k \cdot k'}{4m_1 m_2} + \frac{\sigma_2 \cdot \lambda k' \cdot k'}{4m_1 m_2} + \frac{1}{10} \frac{k^2 k'^2}{m_1^2 m_2^2}.
\]

respectively.

With the usual hyperfine approximation, see for example [8],

\[
(\sigma_1 \cdot b) \cdot (\sigma_2 \cdot c) \implies \frac{2}{3}(\sigma_1 \sigma_2)(bc),
\]

with \(b\) or \(c\) being any of the vectors \(k, k',\) or \(k \wedge k'\), these equations can be simplified even more. One gets

\[
S_0 = 1 + \frac{(\sigma_1 \sigma_2)(k - k')^2 + 4m_1 m_2}{4m_1 m_2} + \frac{(k + k')^2}{4m_1 m_2}
+ \frac{k k'}{4m_1 m_2} \left[ m_1^2 + m_2^2 + \frac{1}{10} \frac{k^2 k'^2}{m_1^2 m_2^2} \right] = 1 + S_{\text{hy}} + S_{\text{K}} + S_{\text{D}}.
\]

The last term must be suppressed for consistency with Eq. [38]. Obviously, the identity

\[
\frac{1}{m_1^2} + \frac{1}{m_2^2} = \frac{1}{m_1 m_2} \left[ \frac{m_1^2 + m_2^2}{m_1^2} \right]
\]

has been applied here. Correspondingly one gets

\[
S_1 = \frac{\sigma_1 \cdot \lambda k \cdot k'}{4m_1 m_2} + \frac{i(\sigma_1 + \sigma_2) \cdot \lambda k \cdot k'}{2m_1 m_2} + \frac{\sigma_2 \cdot \lambda k \cdot k'}{4m_1 m_2} + \frac{\sigma_2 \cdot \lambda k' \cdot k'}{4m_1 m_2}.
\]

\[
S_1 = \frac{\sigma_1 \cdot \lambda k \cdot k'}{4m_1 m_2} + \frac{\sigma_2 \cdot \lambda k \cdot k'}{4m_1 m_2} + \frac{\sigma_2 \cdot \lambda k \cdot k'}{4m_1 m_2} + \frac{i(\sigma_1 + \sigma_2) \cdot \lambda k \cdot k'}{2m_1 m_2} + \frac{\sigma_2 \cdot \lambda k \cdot k'}{4m_1 m_2} + \frac{\sigma_2 \cdot \lambda k' \cdot k'}{4m_1 m_2}.
\]

The nomenclature is more transparent in configuration space. The kernel of the potential energy is then

\[
U(k, k') = -\frac{\alpha_c(q)}{2\pi^2} \frac{R(q)}{q^2} S(k, k'),
\]

with \(q = k - k'\) and \(S = S_0 + S_1\).

### 6.1 The hyperfine approximation

Let me first investigate

\[
(\sigma \wedge b) \wedge c = (\sigma \cdot c) b - (b \cdot c) \sigma.
\]

The 3-component of Eq. [49] is then

\[
[(\sigma \wedge b) \wedge c]_3 = ((\sigma_1 + \sigma_2 + \sigma_3) c) b_3
- (b_1 c_1 + b_2 c_2 + b_3 c_3) \sigma_3
= (\sigma_1 + \sigma_2 + \sigma_3) c b_3 - (b_1 c_1 + b_2 c_2 + b_3 c_3) \sigma_3.
\]

If one has reasons to believe that the off-diagonal components of \(b_i c_j\) cancel by symmetry considerations, i.e. \(\langle b_i c_j \rangle \approx \frac{1}{2} \delta_{ij} bc\), one gets \(\langle (\sigma \wedge b) \wedge c \rangle_3 \approx -\frac{4}{3} \sigma_3 (bc)\). The other components behave correspondingly.

Since \((\sigma_1 \wedge b) \cdot (\sigma_2 \cdot c) = -\langle (\sigma_1 \wedge b) \cdot (\sigma_2 \cdot c) \rangle\), Eq. [50] is valid approximately. One is accustomed to this substitution from the theory of hyperfine interactions [8], and I will refer to it as the hyperfine approximation. As shown in the appendices, the hyperfine approximation is a rather weak assumption. Sometimes it is even exact.

### 6.2 Deriving the Singlet-Triplet model

The wedge product \((\sigma_1 \wedge k) \cdot (\sigma_2 \wedge k')\) is defined in Eq. [50]. Replacing it by

\[
(\sigma_1 \wedge k) \cdot (\sigma_2 \wedge k') \implies (\sigma_1 \sigma_2)(kk'),
\]

rather than by the hyperfine approximation in Eq. [51], one gets from Eq. [52] in a first step:

\[
S_0 = 1 + \frac{kk'}{4m_1 m_2^2} + \frac{kk'}{4m_2^2} + \frac{k^2(1 - \sigma_1 \sigma_2) + 2kk' \sigma_1 \sigma_2}{4m_1 m_2^2}.
\]

Adding and subtracting \(2kk' (1 - \sigma_1 \sigma_2)\) gives

\[
S_0 = 1 + \frac{(1 - \sigma_1 \sigma_2)(k - k')^2 + 4m_1 m_2}{4m_1 m_2} + \frac{kk'}{4m_1 m_2} + \frac{4m_1 m_2}{4m_1 m_2}.
\]

In previous work [11], the matrix \(S\) was replaced by its diagonal elements,

\[
\langle s_1, s_2 | S | s_1', s_2' \rangle \implies \delta_{s_1 s_1'} \delta_{s_2 s_2'} \langle s_1, s_2 | S | s_1, s_2 \rangle.
\]

Eq. [53] gives then, up to terms proportional to \(kk'\),

\[
\langle s_1, s_2 | S | s_1, s_2 \rangle = \left\{ \begin{array}{ll}
\frac{1}{Q^2} + \frac{2}{4m_1 m_2} & \text{for } s_1 = -s_2, \\
\frac{2}{Q^2} & \text{for } s_1 = s_2.
\end{array} \right.
\]

One has thus derived the Singlet-Triplet model [11] without ad hoc procedures. The model has played an important role in the development of the theory, particularly its renormalization.
7 The Hamiltonian in configuration space

The eigenvalue equation (25) in momentum space,
\[ \int d^3k' \langle s_1, s_2 | H(k; k') | s_1', s_2' \rangle \varphi_{s_1, s_2}(k') = E \varphi_{s_1, s_2}(k), \]
can be Fourier transformed to configuration space,
\[ \int d^3r' \langle s_1, s_2 | H(r; r') | s_1', s_2' \rangle \Psi_{s_1, s_2}(r') = E \Psi_{s_1, s_2}(r). \]
The eigenvalue equation
\[ H \Psi = E \Psi, \]
has then the structure of the familiar Schrödinger equation. Quite in general, working in configuration space has the advantage of being closer to conventional quantum mechanics and to phenomenological models where our intuition comes from.

Fourier transformations need a sign convention,
\[ H(r; r') = \int d^3k e^{ikr} \int d^3k' e^{-ik'r'} H(k; k'), \]
\[ \Psi_{s_1, s_2}(r) = \int d^3k e^{ikr} \varphi_{s_1, s_2}(k). \]

It could be a source of endless confusion that \( \int d^3r' \varphi_{s_1, s_2}(r') \) are denoted by the same symbol \( H \). But this should not be a problem since the meaning is usually clear from the context.

Following the nomenclature of Eqs. (60) and (61), the Hamiltonian is decomposed into
\[ H = T + V_0 + V_1, \]
with
\[ V_0 = V + V_{hf} + V_K + V_D, \]
\[ V_1 = V_{so1} + V_{so2} + V_L + V_{hfL} + V_{soD}. \]
The kinetic and the potential energy are respectively
\[ T = \frac{\mathbf{p}^2}{2m_r}, \]
\[ V = -\int d^3q e^{iqr} \frac{\alpha_s(q)}{2\pi^2} \frac{R(q)}{q^2}. \]
The hyperfine, kinetic, and Darwin potentials are
\[ V_{hf} = \frac{\sigma_1 \sigma_2}{6m_1 m_2} \nabla^2 V, \]
\[ V_K = \frac{\mathbf{V}}{m_r}, \]
\[ V_D = -\left( \frac{\nabla^2 \mathbf{V}}{4m_1 m_2} + \frac{\mathbf{V} \mathbf{p}^2}{16m_0 m_2} \right) \left[ \frac{m_2}{m_r} + \frac{m_4}{m_1} \right], \]
respectively. The spin-orbit, \( L^2 \), hyperfine-\( L^2 \), and spin-orbit-Darwin potentials are
\[ V_{so1} = \frac{\sigma_1 + \sigma_2}{4m_1 m_2} \left[ \frac{1}{r^2} + \frac{1}{m_r} \right] \left( \frac{d}{dr} \right) \left[ \frac{dV}{dr} \right], \]
\[ V_L = \frac{L^2}{4m_1 m_2} \left[ \frac{1}{r^2} \left( \frac{d^2V}{dr^2} \right) \right], \]
\[ V_{hfL} = \frac{\sigma_1 \sigma_2}{6m_1 m_2} \frac{L^2}{m_2} \left[ \frac{1}{r^2} \left( \frac{d^2V}{dr^2} \right) \right], \]
\[ V_{soD} = \frac{\sigma_1 + \sigma_2}{4m_1 m_2} \frac{L^2}{m_2} \left[ \frac{1}{r^2} \left( \frac{d^2V}{dr^2} \right) \right]. \]

respectively. They depend on the angular momentum operator \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \), see App. C, and vanish for spherically symmetric wave functions. The linear momentum operator is denoted by \( \mathbf{p} = -i \nabla \).

Some of the above terms appear also in the analysis of the Dirac equation with an external vector potential, see for example [8]. One should emphasize that the present analysis is more complete since retardation effects are fully included. They generate the remainder of the terms in the above.

8 Spherically symmetric wave functions

It is our freedom to solve Eq. (22) only for wave functions with spherical symmetry, \( \Psi_{s_1, s_2}(r) = \Psi_{s_1, s_2}(r) \). The s-states, in a way, are also the most interesting, since practically all hadrons suitable for beams or targets have this symmetry. And for them one can solve the problem rigorously.

Since there is no angular momentum, one can couple the quark Bjørken-Drell spins into total spin and thus total angular momentum, by means of Wigner-Eckart [12],
\[ |S, S_z \rangle = \sum_{s_1, s_2} \langle \frac{1}{2} s_1 \frac{1}{2} s_2 | S, S_z \rangle |s_1 \rangle |s_2 \rangle, \]
with the total spin (or total angular momentum \( J \)) is either \( S = 0 \) for singlets or \( S = 1 \) for triplets. Using the completeness relations of the Clebsch-Gordan coefficients
\[ \sum_{S, S_z} \langle \frac{1}{2} s_1 \frac{1}{2} s_2 | S, S_z \rangle |S, S_z \rangle |s_1 \rangle |s_2 \rangle = \delta_{s_1, s'} \delta_{s_2, s''}, \]
I can perform (for the last time) a unitary transformation
\[ \tilde{\varphi}_{S, S_z}(k) = \sum_{s_1, s_2} \varphi_{s_1, s_2}(k) \langle \frac{1}{2} s_1 \frac{1}{2} s_2 | S, S_z \rangle, \]
\[ \tilde{\Psi}_{S, S_z}(r) = \sum_{s_1, s_2} \Psi_{s_1, s_2}(r) \langle \frac{1}{2} s_1 \frac{1}{2} s_2 | S, S_z \rangle, \]
and solve the Hamiltonian equations
\[ H \tilde{\Psi}_{S, S_z}(r) = E \tilde{\Psi}_{S, S_z}(r), \]
\[ H = \sum_{S, S_z} \langle \frac{1}{2} s_1 \frac{1}{2} s_2 | S, S_z \rangle |S, S_z \rangle |s_1 \rangle |s_2 \rangle, \]
for fixed and given \( S \) and \( S_z \) rather than carrying out the spin summations in Eq. (22) explicitly. In this representation, the only spin-off-diagonal operator in Eq. (60) is diagonal, i.e.
\[ \sigma_1 \sigma_2 = 2S(S + 1) - 3 = \begin{cases} +1, & \text{for } S = 1, \text{ triplet}, \\ -3, & \text{for } S = 0, \text{ singlet}. \end{cases} \]

It depends on the taste, whether one works first in configuration space and subsequently Fourier transforms to momentum space, or whether one solves directly Eq. (22) in momentum space. At the end one has \( \tilde{\varphi}_{S, S_z}(k) \), which
can be unitarily transformed back to the light-cone wave function according to
\[ \psi_{h_1h_2}(x, k_L) = \frac{\sqrt{A(k_2(x), k_L)}}{\sqrt{x(1-x)}} \times \sum_{s_1s_2} \langle h_1h_2|T \hat{T}(k_2(x), k_L)|s_1s_2 \rangle \times \sum_{S,S_z} \langle S,S_z| \frac{1}{\sqrt{2}} ( s_1 \frac{1}{\sqrt{2}} s_2 ) \varphi_{S,S_z}(k_2(x), k_L) \rangle , \] (74)

see also Eqs. (10) and (21). Note that the light cone wave function is a superposition of singlet and triplet.

9 Perspectives

I have promised a technical paper. Sometimes technicalities are important.

Most of the above work relates to background knowledge on atomic and Dirac theory and the results speak for themselves.

Instead of discussing the results to some detail, I will highlight only some important aspects:

- Most remarkably, the fine and hyperfine interaction depends on the coupling constant only through the potential energy \( V(r) \).
- One faces a seemingly non-relativistic potential energy which is directly related to the light cone wave functions with all their wonderful advantages to calculate cross sections, structure functions, distribution amplitudes, and the like.
- In fact, one has the wonderful advantage of applying phenomenological approaches working with potential energies, and transforming their solutions to light cone wave functions. One can then calculate dynamic quantities like cross sections. Thus far, this was possible only with a lot of hand waving.
- This opens a broad avenue of model tailoring and comparison to experiment.
- It seems one has gotten the cookie and the cake.

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A Fourier transform of the kinetic energy

The kinetic energy is defined in Eq. (26)
\[ T(k; k') = \frac{k^2}{2m_r} \delta^3(k - k') . \]

The Fourier transform according to Eq. (35) is
\[ \int d^3r T(r; r') \psi(r') = \int d^3r' \int d^3k \int d^3k' \frac{e^{i(kr-k'r')}}{(2\pi)^3} T(k; k') \psi(r') = \int d^3r' \int d^3k \frac{k^2}{2m_r} \frac{e^{i(kr-k'r')}}{(2\pi)^3} \Psi(r') = \frac{-\nabla^2}{2m_r} \psi(r) . \]

B Fourier transform of the potential energy

Inspection of Eq. (15) and Eq. (17) reveals the potential energy \( U(k; k') \) to depend on the momenta through the combinations \( q = k - k' \) and \( (k \wedge k') \). It is therefore convenient to introduce sum and difference, i.e.
\[ k = p + \frac{q}{2}, \quad q = k - k', \quad k' = p - \frac{q}{2}, \quad p = \frac{1}{2}(k + k') . \]

The typical combinations are then
\[ k \cdot k' = p^2 - \frac{q^2}{4}, \]
\[ k \wedge k' = q \wedge p, \]
\[ kr - k'r' = p(r - r') + \frac{q}{2}(r + r') . \]

The general Fourier transform Eq. (35) is then replaced by
\[ H(r; r') = \int \frac{d^3p}{(2\pi)^3} e^{ip(r-r')} e^{i\frac{q}{2}(r+r')} H(q; p) , \]

i.e. the kernel must be expressed in terms of \( q \) and \( p \).

B.1 The central potential

According to Eq. (15) and (17) the central potential is
\[ U_c(q; p) \equiv U(q) = -\frac{\alpha_s(q) R(q)}{q^2} . \]
Define the function \( V(r) \),

\[
V(r) = \int d^3 q \ e^{i q r} U(q),
\]

and get

\[
V_c(r; r') = \int d^3 q \ e^{i q (r+ r')} U(q) \int d^3 p \frac{e^{i p (r- r')}}{(2\pi)^3},
\]

\[
= \delta^{(3)}(r - r') V(r).
\]

Folding with the wave function,

\[
V \otimes \Psi = \int d^3 r' V_c(r; r') \Psi(r') = V(r) \Psi(r) = V \Psi,
\]

produces the potential energy \( V(r) \) as a multiplicative factor.

**B.2 General rules**

Consider first the simpler case of \( U_a(q; p) \equiv q U(q) \). According to Eq. (80), its Fourier transform is

\[
V_a(r; r') = \int d^3 q \int d^3 p \ q U(q) \ \frac{e^{i p (r- r')}}{(2\pi)^3}.
\]

Integrating first over \( p \) gives a Dirac delta function

\[
V_a(r; r') = \delta^{(3)}(r - r') \int d^3 q \ q U(q) \ e^{i q r}.
\]

The \( q \) can be substituted by the partial \( q \to -i \nabla_r \), which gives \( V_a(r; r') = \delta^{(3)}(r - r') \ ( -i \nabla_r V(r)) \). Folding gives

\[
V_a \otimes \Psi = -i \nabla_r (V \Psi),
\]

where \( \nabla_r \) acts only on \( V(r) \). It is clear how to generalize this to arbitrary powers \( U_a(q; p) \equiv (q)^n U(q) \).

Next, consider \( U_b(q; p) \equiv p U(q) \). Take the Fourier transform first over \( q \), thus

\[
V_b(r; r') = \int d^3 p \ p \ \frac{e^{i p (r- r')}}{(2\pi)^3} \int d^3 q \ U(q) e^{i q (r+ r')} = V \left( \frac{1}{2} (r + r') \right) \int d^3 p \ p \ \frac{e^{i p (r- r')}}{(2\pi)^3}.
\]

The \( p \) can be substituted by the partial \( p \to -i \nabla_r \), which gives \( V_b(r; r') = V(r) \ ( -i \nabla_r \delta^{(3)}(r - r')) \). In the folding, the derivative of the delta function generates a derivative of the wave function \( \Psi(r) \),

\[
V_b \otimes \Psi = -i \nabla_r (V \Psi),
\]

where \( \nabla_r \) acts only on \( \Psi(r) \). It is clear how to generalize this to arbitrary powers \( U_b(q; p) \equiv (p)^n U(q) \).

These findings can be cast into the general rules:

1. Consider the kernel of a general integral equation:

\[
U_g(k; k') = f(k, k') U(|k - k'|)
\]

2. Introduce \( p \) and \( q \) according to

\[
\begin{align*}
\mathbf{k} &= \mathbf{p} + \frac{q}{i}, \quad \mathbf{q} = \mathbf{k} - \mathbf{k'}, \\
\mathbf{k'} &= \mathbf{p} - \frac{q}{i}, \quad \mathbf{p} = \frac{1}{2} (\mathbf{k} + \mathbf{k'}). 
\end{align*}
\]

3. Express \( U_g \) in terms of \( p \) and \( q \):

\[
U_g(q; p) = F(q, p) U(q)
\]

4. Substitute \( q \) and \( p \) according to:

\[
\begin{align*}
\mathbf{q} &\to -i \nabla_V, \\
\mathbf{p} &\to -i \nabla_\phi.
\end{align*}
\]

5. Get the folded Fourier transform by

\[
V_k \otimes \Psi = F(-i \nabla_V, -i \nabla_\phi) (V \Psi).
\]

This suffices to Fourier transform all functions of interest.

**C The different terms of the interaction**

According to Eqs. (81 - 83), the hyperfine term is

\[
U_{hf}(k; k') = (k - k')^2 U(q) \ \frac{1}{6m_1 m_2},
\]

\[
V_{hf} \otimes \Psi = -\nabla_V^2 (V \Psi) \ \frac{1}{6m_1 m_2}.
\]

The kinetic term is

\[
U_K(k; k') = \frac{(k-k')^2}{4m_1 m_2} U(q),
\]

\[
V_K \otimes \Psi = -\nabla_V^2 (V \Psi) \ \frac{1}{4m_1 m_2}.
\]

The Darwin term is

\[
U_D(k; k') = \frac{k.k'}{4m_1 m_2} U(q),
\]

\[
V_D \otimes \Psi = -\left( \nabla^2_V - \frac{1}{4} \nabla_\phi^2 \right) (V \Psi) \ \frac{1}{4m_1 m_2}.
\]

The typical spin orbit term is

\[
U_{so1}(k; k') = \frac{\mathbf{q} \cdot \mathbf{k} \cdot \mathbf{k'}}{4m_1} U(q) \ \left[ \frac{1}{m_r} + \frac{1}{m_1} \right],
\]

\[
U_{so1}(q; p) = \frac{\mathbf{q} \cdot \mathbf{p}}{4m_1} U(q) \ \left[ \frac{1}{m_r} + \frac{1}{m_1} \right],
\]

\[
V_{so1} \otimes \Psi = -\frac{\mathbf{q} \cdot \nabla_V \cdot \nabla_\phi}{4m_1} (V \Psi) \ \left[ \frac{1}{m_r} + \frac{1}{m_1} \right].
\]

The \( L^2 \)-term is

\[
U_L(k; k') = -\frac{(k \cdot k')^2}{4(4m_1 m_2)} U(q),
\]

\[
U_L(q; p) = -\frac{(q \cdot p)^2}{4(4m_1 m_2)} U(q),
\]

\[
V_L \otimes \Psi = -\frac{\nabla_V \cdot \nabla_\phi}{4(4m_1 m_2)} (V \Psi).
\]
The hyperfine-$L^2$ term is

\[
U_{hfl}(k,k') = -\frac{(k \cdot k')^2}{4m_1 m_2} U(q) \frac{\sigma_1 \sigma_2}{6m_1 m_2}, \tag{91}
\]

\[
U_{hfl}(q,p) = -\frac{(q \cdot p)^2}{4m_1 m_2} U(q) \frac{\sigma_1 \sigma_2}{6m_1 m_2},
\]

\[
V_{hfl} \otimes \Psi = \frac{(\nabla_V \wedge \nabla_{\Psi})^2}{4m_1 m_2} (V \Psi) \frac{\sigma_1 \sigma_2}{6m_1 m_2}.
\]

The spin-orbit Darwin term is

\[
U_{sod}(k,k') = \frac{i(\sigma_1 + \sigma_2)}{4m_1 m_2} (k \cdot k') U(q) \frac{1}{4m_1 m_2}, \tag{92}
\]

\[
U_{sod}(q,p) = \frac{i(\sigma_1 + \sigma_2)}{4m_1 m_2} (q^2 - \frac{1}{2} p^2) U(q) \frac{1}{4m_1 m_2},
\]

\[
V_{sod} \otimes \Psi = \frac{i(\sigma_1 + \sigma_2)}{4m_1 m_2} (V \nabla_V \wedge \nabla_{\Psi}) (\nabla_V^2 - \frac{4}{2} \nabla_{\Psi}^2) \frac{1}{4m_1 m_2},
\]

finally, which completes taking the Fourier transforms.

The potential $V(r)$ is strictly spherically symmetric. With

\[
\nabla V(r) = \frac{r}{r} \frac{dV}{dr}, \tag{93}
\]

and with the usual orbital angular momentum

\[
L = -i r \wedge \nabla, \tag{94}
\]

one can simplify the above equations considerably. With

\[
(\nabla_V \wedge \nabla_{\Psi}) (V \Psi) = (\nabla_V \wedge \nabla_{\Psi}) = \left[ \frac{1}{r} \frac{dV}{dr} \right] \left[ r \wedge \nabla_{\Psi} \right],
\]

one can substitute in Eqs.\ref{93} \text{-} \ref{94}

\[
(\nabla_V \wedge \nabla_{\Psi}) \Rightarrow i \left[ \frac{1}{r} \frac{dV}{dr} \right] V L_{\Psi},
\]

and replace them by

\[
V_{sod} \otimes \Psi = \left[ \frac{1}{m_1} + \frac{1}{m_1} \right] \frac{\sigma_1 L}{4m_1} \left[ \frac{1}{r} \frac{dV}{dr} \right] V \Psi, \tag{95}
\]

\[
V_L \otimes \Psi = \frac{(L \cdot \Psi)}{(4m_1 m_2)^2} \left[ \frac{1}{r} \frac{dV}{dr} \right] V \Psi, \tag{96}
\]

\[
V_{hfl} \otimes \Psi = \frac{\sigma_1 \sigma_2}{6m_1 m_2} \frac{(L \cdot \Psi)}{(4m_1 m_2)^2} \left[ \frac{1}{r} \frac{dV}{dr} \right] V \Psi, \tag{97}
\]

\[
V_{sod} \otimes \Psi = \frac{(\sigma_1 + \sigma_2)}{(4m_1 m_2)^2} \left[ \frac{1}{r} \frac{dV}{dr} \right] V (\frac{4}{2} \nabla_{\Psi}^2 - \nabla_V^2) V \Psi, \tag{98}
\]

a very suggestive form indeed.

### D The non-local central potential

One can define the kernel of a spherically symmetric potential in which the Fourier approximation has not been made. Restricting to the ‘1’ in Eq.\ref{28}, one gets from Eq.\ref{27},

\[
\tilde{U}(k,k') = -\frac{\alpha_c(Q)}{2\pi^2 Q^2} \frac{R(Q)}{4m_1 m_2 m_r} \tag{99}
\]

\[
\times \sqrt{\frac{E_1(k)E_2(k)}{E_1(k) + E_2(k)}} \left( E_1(k) + m_1 \right) \left( E_2(k) + m_2 \right)
\]

\[
\times \sqrt{\frac{E_1(k')E_2(k')}{E_1(k') + E_2(k')}} \left( E_1(k') + m_1 \right) \left( E_2(k') + m_2 \right).
\]

It is plainly impossible to find an analytical Fourier transform of this, except when applying series expansions in $k^2/m^2$, i.e.

\[
E(k) = m \sqrt{1 + \frac{k^2}{m^2}} \approx m \left[ 1 + \frac{1}{2} \frac{k^2}{m^2} - \frac{1}{8} \left( \frac{k^2}{m^2} \right)^2 + \ldots \right], \tag{100}
\]

but then all the beauty of the present approach gets lost. In the Fourier approximation, Eq.\ref{38}, only the first term is included.