Improved bound for Hadwiger’s conjecture

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Abstract

Hadwiger conjectured in 1943 that for every integer $t \geq 1$, every graph with no $K_t$ minor is $(t-1)$-colorable. Kostochka, and independently Thomason, proved every graph with no $K_t$ minor is $O(t \log t^{1/2})$-colorable. Recently, Postle improved it to $O(t (\log \log t)^6)$-colorable. In this paper, we show that every graph with no $K_t$ minor is $O(t (\log \log t)^5)$-colorable.

1 Introduction

Let $G$ be a graph. A minor of $G$ is a graph obtained from $G$ by contracting edges, deleting edges and deleting isolated vertices. Minors play an important role in topological graph theory. In 1937, Wagner [14] showed that a graph is planar if and only if the complete graph on five vertices $K_5$ and the complete bipartite graph with three vertices in each partition $K_{3,3}$ are not minors of $G$. In 1943, Hadwiger [1] conjectured the following.

Conjecture 1.1 (Hadwiger’s conjecture [1]). For every integer $t \geq 1$, every graph with no $K_t$ minor is $(t-1)$-colorable.

Hadwiger’s conjecture is a strengthening of the four-color theorem, is probably the most famous open problem in graph theory. In fact, Hadwiger [1] proved the conjecture for $t \leq 4$ and Wagner [13] established the equivalence of the case when $t = 5$ and the four-color theorem. Robertson, Seymour and Thomas [10] showed Hadwiger’s conjecture when $t = 6$, but it is still open for $t \geq 7$. For a complete survey and background of Hadwiger’s conjecture, we refer the readers to [11].

Consider the following weakening of Hadwiger’s conjecture: What can we show about the chromatic number of graphs with no $K_t$ minor? Kostochka [4, 5] and Thomason [12] showed that every graph with no $K_t$ minor is $O(t \log t^{1/2})$-degenerate, and thus is $O(t \log t^{1/2})$-colorable. Recently, Norin, Song and Postle [6] improved it to $O(t (\log t)^\beta)$ for every $\beta > 1/4$. Subsequently, it is further improved to $O(t (\log \log t)^{18})$ in [8] and $O(t (\log \log t)^6)$ in [7]. It is conjectured in [2, 3, 9] that there exists a constant $C > 0$ such that for every integer $t \geq 1$, every graph with no $K_t$ minor is $Ct$-colorable. In this paper, we show the following.

Theorem 1.2. Every graph with no $K_t$ minor is $O(t (\log \log t)^5)$-colorable.
Our main contribution is the following improvement to the density increment lemma.

Lemma 1.3. There exists a constant \( C = C_{1.3} > 0 \) such that the following holds. Let \( G \) be a graph with \( d(G) \geq C \), and let \( D > 0 \) be a constant. Let \( s = D/d(G) \) and let \( C_{1.3}(s) := C(1 + \log s)^5 \). Then \( G \) contains at least one of the following:

(i) a minor \( J \) with \( d(J) \geq D \), or

(ii) a subgraph \( H \) with \( v(H) \leq C_{1.3}(s) \cdot \frac{D^2}{d(G)} \) and \( d(H) \geq \frac{d(G)}{C_{1.3}(s)} \).

We need the following theorem proved in \([8]\).

Lemma 1.4 (Theorem 2.2 in \([8]\)). Every graph with no \( K_t \) minor has chromatic number at most

\[
O\left(t \cdot \left( C_{1.3}(s) \cdot 3.2 \cdot \sqrt{\log t} + (\log \log t)^2\right)\right).
\]

It is easy to see that Lemmas 1.3 and 1.4 imply Theorem 1.2.

1.1 Notations

Let \( G \) be a graph. Let \( V(G) \) and \( E(G) \) be the vertex set and the edge set of \( G \) respectively. We write the number of vertices \( v(G) = |V(G)| \), the number of edges \( e(G) = |E(G)| \) and density \( d(G) = e(G)/v(G) \). For a vertex \( v \in V(G) \), let \( \deg_G(v) \) be the degree of \( v \) in \( G \) and \( N_G(v) \) be the neighbourhood of \( v \) in \( G \). We use \( \delta(G) \) to denote the minimum degree of \( G \). For \( S \subseteq E(G) \), we denote \( G/S \) to be the graph obtained from \( G \) by contracting all the edges in \( S \). For \( A \subseteq V(G) \), we denote \( G[A] \) to be the induced subgraph of \( G \) on vertex set \( A \).

2 Preliminaries

In this section, we introduce some definitions and lemmas from \([7]\).

Definition 2.1. Let \( G \) be a graph, and let \( K, d \geq 1, \varepsilon \in (0, 1) \) be real. We say that

- a vertex of \( G \) is \((K, d)\)-small in \( G \) if \( \deg_G(v) \leq Kd \) and \((K, d)\)-big otherwise;
- two vertices of \( G \) are \((\varepsilon, d)\)-mates if they have at least \( \varepsilon d \) common neighbours;
- \( G \) is \((K, \varepsilon_1, \varepsilon_2, d)\)-unmated if every \((K, d)\)-small vertex in \( G \) have strictly fewer than \( \varepsilon_1 d \) \((\varepsilon_2, d)\)-mates.

If a graph is not unmated, it must contain a dense subgraph as shown by the following proposition (see Proposition 3.2 in \([7]\)).

Proposition 2.2 (\([7]\)). For all \( K, d \geq 1, \varepsilon_1, \varepsilon_2 \in (0, 1) \) and every graph \( G \) at least one of the following holds:

(i) there exists a subgraph \( H \) of \( G \) with \( v(H) \leq 3Kd \) and \( e(H) \geq \varepsilon_1 \cdot \varepsilon_2 \cdot \frac{d^2}{2} \), or

(ii) \( G \) is \((K, \varepsilon_1, \varepsilon_2, d)\)-unmated.
We need the following definitions about forests and bounded minors.

**Definition 2.3.** Let $F$ be a non-empty forest in a graph $G$. Let $K, k, d, s \geq 1$ be real and let $\varepsilon_2, c \in (0, 1)$. We say $F$ is

- $(K, d)$-small if every vertex in $V(F)$ is $(K, d)$-small in $G$, and
- $(c, d)$-clean if $e(G) - e(G/F) \leq c \cdot d \cdot v(F)$,

**Definition 2.4.** Let $H$ and $G$ be two graphs with $V(H) = [h]$. $(X_1, X_2, \ldots, X_h)$ is a model of $H$ in $G$ if

- $X_1, X_2, \ldots, X_h$ are pairwise disjoint subsets of $V(G)$,
- $G[X_i]$ is connected for every $i \in [h]$, and
- there exists an edge between $X_i$ and $X_j$ in $G$ for every $ij \in E(H)$.

Note that $G$ has an $H$ minor if and only if $G$ contains a model of $H$. For integer $k \geq 1$, we say $G$ has a $k$-bounded $H$ minor if $G$ contains a model of $H$ where $|X_i| \leq k$ for $i \in [h]$.

If a bipartite graph is almost complete on one partition, then it contains either a dense subgraph or a bounded minor with increased density (see Theorem 3.6 in [7]).

**Lemma 2.5 ([7]).** Let $K_0, \ell_0 \geq 2$ be integers with $K_0 \geq \ell_0(\ell_0+1)$, and let $\varepsilon_{1,0} \in \left(0, \frac{1}{\ell_0}\right)$, and $d_0 \geq 1/\varepsilon_{2,0}$ be real. Let $G = (A, B)$ be a bipartite graph such that $|A| \geq \ell_0|B|$ and every vertex in $A$ has at least $d_0$ neighbours in $B$. Then there exists at least one of the following:

(i) a subgraph $H$ of $G$ with $\nu(H) \leq 4K_0d_0$ and $e(H) \geq \varepsilon_{1,0} \cdot \varepsilon_{2,0} \cdot d_0^2/2$.

(ii) a subgraph $H$ of $G$ with $\nu(H) \leq 4\ell_0K_0d_0$ and $e(H) \geq \varepsilon_{2,0}^2 \cdot d_0^2/2$.

(iii) an $(\ell_0+1)$-bounded minor $H$ of $G$ with $d(H) \geq \frac{\ell_0^2}{\ell_0+1} \left(1 - 2\varepsilon_{1,0} - 2\ell_0\varepsilon_{2,0} - \frac{k_0}{K_0}\right) d_0$.

### 3 Dense Subgraphs or Minors with Increased Density

In this section we prove the following lemma which shows every graph must contain a dense subgraph, a bipartite subgraph such that all the vertices on one partition have large degree, or a bounded minor with increased density.

**Lemma 3.1.** Let $K \geq k \geq 100$ be integers with $K \geq 4k^2$. Let $\ell = \left[\frac{k}{\varepsilon_2}\right]$, $\varepsilon_1 \in (0, \frac{1}{K}]$ and $\varepsilon_2 \in (0, \frac{1}{K}]$. Let $G$ be a graph with $d = d(G) \geq \frac{k}{\min\{\varepsilon_1, \varepsilon_2\}}$. Then $G$ contains at least one of the following:

(i) a subgraph $H$ of $G$ with $\nu(H) \leq 3Kd$ and $d(H) \geq \frac{\varepsilon_2^2 d}{6Kk}$, or

(ii) a bipartite subgraph $H = (X, Y)$ of $G$ with $|X| \geq \ell|Y|$ such that every vertex in $X$ has at least $(1 - 6\varepsilon_1)d$ neighbours in $Y$, or
(iii) a $k$-bounded minor $G'$ of $G$ with $d(G') \geq k \left( 1 - \frac{30}{k^2} \right) d$.

**Proof.** For, otherwise, let $G$ be a minimal counterexample, i.e. $G$ satisfies none of (i), (ii) or (iii), and $v(G)$ is minimized. For any proper subgraph $H$ of $G$, since $H$ is not a minimal counterexample, we may assume $d(H) < d(G)$. In particular, for any vertex $v \in V(G)$, $d(G[V(G) \setminus \{v\}]) < d(G)$. This implies $d(v) > d(G)$, so $d(G) > d(G)$.

First we apply Proposition 2.2 to $G$ with $(K, \varepsilon_1, \varepsilon_2, d) \Gamma = (K, \varepsilon_1, \frac{\varepsilon_2}{k}, d)$. If Proposition 2.2 (i) holds, then there exists a subgraph $H$ of $G$ such that $v(H) \leq 3kd$, $e(H) \geq \frac{\varepsilon_1 \varepsilon_2 d}{2k}$. So $d(H) \geq \frac{\varepsilon_1 \varepsilon_2 d}{6k^2}$ and conclusion (i) holds, a contradiction. Hence, Proposition 2.2 (ii) holds, i.e. $G$ is $(K, \varepsilon_1, \frac{\varepsilon_2}{k}, d)$-unmated.

Let $A = \{v : \deg_G(v) \leq Kd\}$ and $B = V(G) \setminus A$. Since $Kd |B| \leq 2e(G) = 2d\nu(G)$, we have $|B| \leq \frac{2}{k} \nu(G) \leq \frac{1}{2k} \nu(G)$. Since $G$ is $(K, \varepsilon_1, \frac{\varepsilon_2}{k}, d)$-unmated, $v$ has fewer than $\varepsilon_1 d$ $(\frac{\varepsilon_2}{k}, d)$-mates in $G$ for every $v \in A$.

Let $c = 4\varepsilon_2$ and $F$ be a maximal $(K, d)$-small, $(c, d)$-clean forest where each component of $F$ is a star of size $k$. Since $F$ is $(K, d)$-small, we have $V(F) \subseteq A$.

**Claim 3.1.1.** If $F_0$ is a star forest in $G$ and $v \in V(G) \setminus V(F_0)$ has at least $2\varepsilon_1 d$ neighbours in $A \setminus V(F_0)$, then there exists a star $T$ of size $k$ in $A \setminus V(F_0)$ with center $v$ such that

$$e(G/E(F_0)) - e(G/(E(F_0) \cup E(T))) \leq 2k\varepsilon_2 d.$$  

**Proof.** For, otherwise, let $S$ be a maximal star in $A \setminus V(F_0)$ with center $v$ such that

$$e(G/E(F_0)) - e(G/(E(F_0) \cup E(S))) \leq 2(\nu(S) - 1)\varepsilon_2 d.$$  

Such $S$ exists as $S$ could be $\{v\}$. By assumption, $\nu(S) \leq k - 1$.

Let $V(S) = \{v_0, v_1, \ldots, v_s\}$ with $v_0 = v$ and $s \leq k - 2$. Let $U = \{u_1, \ldots, u_n\}$ where for each $j \in [n]$, $u_j \in A \setminus (V(F_0) \cup V(S))$ and there exists $i \in \{0, 1, \ldots, s\}$ such that $u_j$ is an $(\frac{\varepsilon_2}{k}, d)$-mate of $v_i$. Since $G$ is $(K, \varepsilon_1, \frac{\varepsilon_2}{k}, d)$-unmated, $v_i$ has fewer than $\varepsilon_1 d$ $(\frac{\varepsilon_2}{k}, d)$-mates in $G$ for $i \in \{0, 1, \ldots, s\}$. So $n < k\varepsilon_1 d$.

Let $G' = G/E(F_0) \cup E(S)$ and $v_S$ be the vertex in $V(G')$ corresponding to $S$. For $u \in A \setminus (V(F_0) \cup V(S) \cup U)$, by definition of $U$, $u$ has fewer than $\frac{\varepsilon_2}{k} d$ common neighbours with $v_i$ in $G$ for all $i \in \{0, 1, \ldots, s\}$. So $u$ has fewer than $(s+1)\varepsilon_2 d$ common neighbours with $v_S$ in $G'$.

Now we show that at most $\varepsilon_1 d$ vertices in $U$ have at least $\varepsilon_2 d$ common neighbours with $v_S$ in $G'$. Consider the following auxiliary graph. Let $W$ be an edge-weighted complete bipartite graph with vertex partition $V(S) \cup U$. For $i \in \{0, 1, \ldots, s\}$ and $j \in [n]$, we define the edge weight $w_{ij}$ to be the number of common neighbours between $v_i$ and $u_j$.

We claim that for $i \in \{0, 1, \ldots, s\}$, $\sum_{j \in [n]} w_{ij} < \frac{\varepsilon_1 \varepsilon_2 d^2}{k}$. For, otherwise, suppose there exists some $i \in \{0, 1, \ldots, s\}$ such that $\sum_{j \in [n]} w_{ij} \geq \frac{\varepsilon_1 \varepsilon_2 d^2}{k}$. Let $H = G'[\{v_i\} \cup N(v_i) \cup U]$. Then $\nu(H) \leq 1 + Kd + n < 2Kd$. By definition of $W$, $e(H) \geq \sum_{j \in [n]} w_{ij} \geq \frac{\varepsilon_1 \varepsilon_2 d^2}{k}$. Hence, $d(H) \geq \frac{\varepsilon_1 \varepsilon_2 d^2}{2K^2}$ and conclusion (i) holds, a contradiction.

Let $\Gamma = \{j \in [n] : \sum_{i \in \{0, 1, \ldots, s\}} w_{ij} \geq \varepsilon_2 d\}$. We have

$$|\Gamma|\varepsilon_2 d \leq \sum_{i \in \{0, 1, \ldots, s\}} \sum_{j \in [n]} w_{ij} \leq (s + 1) \cdot \frac{\varepsilon_1 \varepsilon_2 d^2}{k} < \varepsilon_1 \varepsilon_2 d^2.$$  

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So $|\Gamma| < \varepsilon_1 d$. This implies that fewer than $\varepsilon_1 d$ vertices in $U$ have at least $\varepsilon_2 d$ common neighbours with $v_S$ in $G'$.

Since $v$ has at least $2\varepsilon_1 d$ neighbours in $A\setminus V(F_0)$, there exists a vertex $u \in A\setminus (V(F_0) \cup V(S))$ that is not a $(\varepsilon_2, d)$-mate of $v_S$ in $G'$.

Let $S'$ be the star with $V(S') = V(S) \cup \{u\}$ and $E(S') = E(S) \cup \{uv\}$. Note that $v(S') > v(S)$ and

$$e(G/E(F_0)) - e(G/(E(F_0) \cup E(S'))) = (e(G/E(F_0)) - e(G/(E(F_0) \cup E(S)))) + (e(G/(E(F_0) \cup E(S))) - e(G/(E(F_0) \cup E(S')))) \leq (2(v(S) - 1)\varepsilon_2 d) + (1 + \varepsilon_2 d) \leq 2(v(S') - 1)\varepsilon_2 d.$$ This is a contradiction to the maximality of $S$. \hfill \Box

Next we work on the graph obtained from $G$ by contracting $F$. Let $G_0 = G/E(F)$. Since each component of $F$ is a star of size $k$, $G_0$ is a $k$-bounded minor of $G$. Let $A' = A \setminus V(F)$. By Claim 3.1.1 and the maximality of $F$, we can show that the edges between $F$ and $A'$ are sparse in the following claim.

**Claim 3.1.2.** Every component $T$ of $F$ has at most one vertex with at least $3\varepsilon_1 d$ neighbours in $A'$ in graph $G$.

**Proof.** For, otherwise, suppose there exist distinct $u_1, u_2 \in V(T)$ such that $|N_G(u_i) \cap A'| \geq 3\varepsilon_1 d$. Let $F_0 = F \setminus V(T)$. Note that since $E(F_0) \subseteq E(F)$, we have that $e(G/E(F_0)) \geq e(G/E(F))$.

Since $F$ is $(c, d)$-clean, we have

$$e(G) - e(G/E(F_0)) \leq e(G) - e(G/E(F)) \leq cdv(F).$$

Since $u_1$ has at least $3\varepsilon_1 d$ neighbours in $A'$, by Claim 3.1.1 there exists a star $T_1$ of size $k$ in $A \setminus V(F_0)$ with center $u_1$ such that

$$e(G/E(F_0)) - e(G/(E(F_0) \cup E(T_1))) \leq 2k\varepsilon_2 d.$$

Let $F_1$ be the union of $F_0$ and $T_1$, i.e. $V(F_1) = V(F_0) \cup V(T_1)$ and $E(F_1) = E(F_0) \cup E(T_1)$. Since $u_2$ has at least $3\varepsilon_1 d$ neighbours in $A'$, $u_2$ has at least $3\varepsilon_1 d - k \geq 2\varepsilon_1 d$ neighbours in $A' \setminus V(T_1)$. Again by Claim 3.1.1 there exists a star $T_2$ of size $k$ in $A \setminus V(F_1)$ with center $u_2$ such that

$$e(G/E(F_1)) - e(G/(E(F_1) \cup E(T_2))) \leq 2k\varepsilon_2 d.$$

Let $F_2$ be the union of $F_1$ and $T_2$, i.e. $V(F_2) = V(F_1) \cup V(T_2)$ and $E(F_2) = E(F_1) \cup E(T_2)$. Note that $v(F_2) = v(F_0) + v(T_1) + v(T_2) = v(F_0) + 2k = v(F) + k$. Moreover,

$$e(G) - e(G/E(F_2)) = (e(G) - e(G/E(F_0))) + (e(G/E(F_0)) - e(G/E(F_1))) + (e(G/E(F_1)) - e(G/E(F_2))) \leq cdv(F) + 2k\varepsilon_2 d + 2k\varepsilon_2 d = 4dv(F) + k\varepsilon_2 d.$$
So $F_2$ is $(c, d)$-clean. Note that $F_2$ is a forest where every component is a star of size $k$ and $F_2$ is also $(K, d)$-small. This contradicts the maximality of $F$. 

By similar argument, we can show that edges within $A'$ are also sparse in the following claim.

**Claim 3.1.3.** Every vertex $v$ in $A'$ has at most $2\varepsilon_1d$ neighbours in $A'$ in $G$.

**Proof.** For, otherwise, suppose $v$ has at least $2\varepsilon_1d$ neighbours in $A'$ in $G$. By Claim 3.1.1 there exists a star $T$ of size $k$ in $A \setminus V(F_0)$ with center $v$ such that

$$e(G/E(F)) - e(G/(E(F) \cup E(T))) \leq 2k\varepsilon_2d \leq cdk.$$ 

Let $F'$ be the union of $F$ and $T$, i.e. $V(F') = V(F) \cup V(T)$ and $E(F') = E(F) \cup E(T)$. Then $F'$ is a $(K, d)$-small forest where every component is a star of size $k$. Moreover, $v(F') = v(F) + k$. Since $F$ is $(c, d)$-clean, we have

$$e(G) - e(G/E(F)) \leq cdv(F).$$

Hence,

$$e(G) - e(G/E(F')) \leq cdv(F) + cdk = cdv(F').$$

So $F'$ is also $(c, d)$-clean. This contradicts the maximality of $F$. 

Let $C$ be the set of vertices in $F$ with at least $3\varepsilon_1d$ neighbours in $A'$. By Claim 3.1.2 every component of $F$ has at most one vertex in $C$. Hence $|C| \leq \frac{1}{k}v(G)$. Now we show that $A'$ is small.

**Claim 3.1.4.** $|A'| \leq \frac{2v(G)}{3}$.

**Proof.** For, otherwise, suppose $|A'| > \frac{v(G)}{2}$. First we aim to construct a bipartite graph that satisfies conclusion (ii). Let $A_1 = \{v \in A', |N_G(v) \cap (B \cup C)| \geq (1 - 6\varepsilon_1)d(G)\}$ and let $A_2 = A' \setminus A_1$. Suppose that $|A_1| \geq (\frac{1}{3} + \frac{2}{k})v(G)$. Since $|B \cup C| \leq \frac{2}{k}v(G)$, we have that $|A_1| \geq (\frac{k}{6} + 1)|B \cup C| \geq \ell|B \cup C|$ and hence bipartite graph $(A_1, B \cup C)$ satisfies conclusion (ii), a contradiction.

So we may assume that $|A_1| < (\frac{1}{3} + \frac{2}{k})v(G)$. By Claim 3.1.3 every vertex in $A'$ has at most $2\varepsilon_1d$ neighbors in $A'$. Since $\delta(G) \geq d(G) = d$, it follows that every vertex in $A_2$ has at least $4\varepsilon_1d$ neighbors in $V(F) \setminus C$. Hence

$$e(G(A_2, V(F) \setminus C)) \geq 4\varepsilon_1d \cdot |A_2|.$$ 

By definition of $C$, we have that

$$e(G(A_2, V(F) \setminus C)) \leq 3\varepsilon_1d \cdot |V(F) \setminus C| \leq 3\varepsilon_1d(v(G) - |A_1| - |A_2|).$$

Hence $|A_2| \leq \frac{3}{7}(v(G) - |A_1|)$. Therefore, we have

$$|A'| = |A_1| + |A_2| \leq \frac{3}{7}v(G) + \frac{4}{7}|A_1| \leq \frac{3}{7}v(G) + \frac{4}{7}\left(\frac{1}{3} + \frac{2}{k}\right)v(G) \leq \frac{2v(G)}{3}.$$
Finally let $G' = G_0 \setminus A'$. In the rest of the proof, we show that $G'$ satisfies conclusion (iii). Note that by Claim 3.1.4, we have that $|A' \cup B \cup C| \leq (\frac{3}{k} + \frac{2}{k})\varepsilon(G) < \varepsilon(G)$. Hence $F$ is nonempty. In addition, $G[A' \cup B \cup C]$ is a proper subgraph of $G$. So $d(G[A' \cup B \cup C]) < d$ and

$$e(G[A' \cup B \cup C]) < d \cdot |A' \cup B \cup C| \leq \left(\frac{2}{k} \varepsilon(G) + |A'|\right)d.$$ 

Moreover, by definition of $C$ we have

$$e(G(A', V(F) \setminus C)) \leq 3\varepsilon_1d \cdot |V(F) \setminus C| < 3\varepsilon_1d \cdot \varepsilon(G).$$

Let $a' = \frac{|A'|}{\varepsilon(G)}$. Hence

$$e(G) - e(G \setminus A') \leq \left(\frac{2}{k} + 3\varepsilon_1 + a'\right)d \cdot \varepsilon(G).$$

Since $F$ is $(c, d)$-clean, we have that

$$e(G) - e(G_0) \leq cd \cdot \varepsilon(F) \leq 4\varepsilon_2d \cdot \varepsilon(G).$$

Hence

$$e(G \setminus A') - e(G') \leq 4\varepsilon_2d \cdot \varepsilon(G),$$

and so we can lower bound $e(G')$ by

$$e(G) - e(G') \leq \left(\frac{2}{k} + 3\varepsilon_1 + 4\varepsilon_2 + a'\right)d \cdot \varepsilon(G).$$

Moreover, since $G'$ is $k$-bounded minor, we can upper bound $\varepsilon(G')$ by

$$\varepsilon(G') = |B| + |C| \leq \frac{1}{2k^2} \cdot \varepsilon(G) + \frac{\varepsilon(G) - |A'|}{k} \leq \frac{\varepsilon(G)}{k} \left(\frac{k + 1/2}{k} - a'\right).$$

Thus

$$d(G') \geq \frac{e(G) - \left(\frac{2}{k} + 3\varepsilon_1 + 4\varepsilon_2 + a'\right) \cdot d \cdot \varepsilon(G)}{\varepsilon(G) \left(\frac{k + 1/2}{k} - a'\right)} = kd \cdot \frac{1 - \left(\frac{2}{k} + 3\varepsilon_1 + 4\varepsilon_2\right) - a'}{\frac{k + 1/2}{k} - a'}.$$

Since $a' \leq \frac{2}{3}$, we have

$$d(G') \geq kd \cdot \frac{1 - \left(\frac{2}{k} + 3\varepsilon_1 + 4\varepsilon_2\right) - \frac{2}{3}}{\frac{k + 1/2}{k} - \frac{2}{3}},$$

$$= kd \cdot \frac{1 - 3\left(\frac{2}{k} + 3\varepsilon_1 + 4\varepsilon_2\right)}{1 + \frac{3/2}{k}},$$

$$= kd \cdot \left(1 - 3\left(\frac{2}{k} + 3\varepsilon_1 + 4\varepsilon_2\right)\right) \cdot \left(1 - \frac{1}{k}\right),$$

$$\leq kd \cdot \left(1 - \frac{3\varepsilon_1}{k}\right),$$

since $\varepsilon_1 \leq \frac{1}{k}$ and $\varepsilon_2 \leq \frac{1}{k}$. But now conclusion (iii) holds, a contradiction.
4 Proof of Lemma 1.3

In this section, we prove Lemma 1.3. First, we show the following lemma that is analogous to Theorem 2.1 in [7]. The proof is similar, but we include it for completeness.

**Lemma 4.1.** Let $k \geq 100$ be an integer. Let $G$ be a graph with $d = d(G) \geq k^2$. Then $G$ contains at least one of the following:

(i) a subgraph $H$ with $v(H) \leq 12k^3d$ and $d(H) \geq \frac{d}{24k^3}$, or

(ii) an $m$-bounded minor $G'$ with $d(G') \geq m \cdot (1 - \frac{30}{m}) \cdot d$ for some integer $m \in [\frac{k}{6}, k]$.

**Proof of Theorem 4.1.** We apply Lemma 3.1 to $G$ with $K = k^2$ and $\varepsilon_1 = \frac{1}{k}$ and $\varepsilon_2 = \frac{1}{k}$. If Lemma 3.1(i) holds, $G$ contains a subgraph $H$ with $v(H) \leq 3k^2d$ and $d(H) \geq \frac{\varepsilon_1 \varepsilon_2 d}{6k^2} = \frac{d}{24k^3}$. So (i) holds. If Lemma 3.1(iii) holds, $G$ contains a $k$-bounded minor $G''$ with $d(G'') \geq k \left(1 - \frac{30}{k}\right) \cdot d$. So (ii) holds with $m = k$.

We may assume that Lemma 3.1(ii) holds, i.e., there exists a bipartite subgraph $H = (X, Y)$ with $|X| \geq \frac{k}{6}|Y|$ such that every vertex in $X$ has at least $(1 - 6\varepsilon_1)d$ neighbors in $Y$. Now we apply Lemma 2.5 to $H$ with $(d_0, \ell_0, K_0, \varepsilon_{1,0}, \varepsilon_{2,0}) = ((1 - 6\varepsilon_1)d, \frac{K}{6}, \frac{K}{6} (\frac{k}{6} + 1), \frac{k}{6}, \frac{1}{(k/6)^2})$. Note that $d_0 \geq d/2$ since $k \geq 12$ and hence $d_0 \geq k/2 \geq \frac{1}{\varepsilon_{2,0}}$ as needed.

If Lemma 2.5(i) holds for $H$, $H$ contains a subgraph $H_0$ with $v(H_0) \leq 4K_0d_0 \leq 4k^2d$ and $e(H) \geq \varepsilon_{1,0} \varepsilon_{2,0} \frac{d_0^2}{2}$. Then $d(H_0) \geq \frac{\varepsilon_{1,0} \varepsilon_{2,0} d_0}{8K_0} \geq \frac{d}{16k^3} \geq \frac{d}{24k^3}$, so (i) holds.

If Lemma 2.5(ii) holds for $H$, $H$ contains a subgraph $H_0$ of $H$ with $v(H_0) \leq 4\ell_0 K_0 d_0 \leq 4k^3d$ and $e(H_0) \geq \varepsilon_{1,0} \varepsilon_{2,0} \frac{d_0^2}{2}$. Then $d(H_0) \geq \varepsilon_{1,0} \varepsilon_{2,0} \frac{d_0}{8K_0} \geq \frac{d}{16k^3} \geq \frac{d}{24k^3}$, so (i) holds.

Finally we may assume Lemma 2.5(iii) holds, i.e., $H$ contains an $(\ell_0 + 1)$-bounded minor $H_0$ with

$$d(H_0) \geq \frac{\ell_0^2}{\ell_0 + 1} \left(1 - 2\varepsilon_{1,0} - 2\varepsilon_{2,0} - \frac{\ell_0}{K_0} \right) d_0$$

$$\geq (\ell_0 + 1) \cdot \left(1 - \frac{1}{\ell_0 + 1} \right)^2 \left(1 - \frac{5}{\ell_0} \right) \cdot (1 - 6\varepsilon_1) \cdot d$$

$$\geq (\ell_0 + 1) \cdot \left(1 - \frac{30}{\ell_0 + 1} \right) \cdot d,$$

since $\ell_0 = \frac{k}{6}$. Therefore (ii) holds with $G' = H_0$ and $m = \ell_0 + 1$. □

Now we are ready to derive Lemma 1.3 from Lemma 4.1.

**Proof of Lemma 1.3.** Let $C_{1.3} = 2^{50}$. We proceed by induction on $s$. If $s \leq 1$, then $J = G$ is a minor of $G$ with $d(J) = d(G) \geq sd(G) = D$ and (i) holds as desired. So we may assume that $s > 1$. Hence $D > d(G) \geq C_{1.3}$.

If $C_{1.3}(s) \geq 2d(G)$, then let $H$ be the graph of a single edge $uv$ where $uv \in E(G)$ and (ii) holds since $v(H) = 2 < C_{1.3} < D^2$ and $d(H) = 1/2$. So we may assume that $d(G) > C_{1.3}(s)/2$. 

8
Let $k = \frac{1}{2} \cdot (\frac{\log(3)}{\log(2)})^\frac{1}{2} \cdot (1 + \log s) = 2^9 \cdot (1 + \log s)$. Note that $q_{1.3}(s) = (2k)^5$. Since $\log s \geq 0$, we have that $k \geq 512$. Moreover, $d(G) \geq \frac{q_{1.3}(s)}{2} \geq (2k)^3/2 \geq k^2$.

Now we apply Lemma 4.1 to $G$ and $k$. Note that $d(G) \geq k^2$ as needed. If Lemma 4.1(ii) holds, $G$ contains a subgraph $H$ with $v(H) \leq 12k^3d(G)$ and $d(H) \geq \frac{d(G)}{24k^5}$. Note that

$$v(H) \leq 12k^3d(G) \leq q_{1.3}(s)d(G) \leq q_{1.3}(s)\frac{D^2}{d(G)}$$

and furthermore

$$d(H) \geq \frac{d(G)}{24k^5} \geq \frac{d(G)}{q_{1.3}(s)}.$$ 

Then (ii) holds as desired.

So we may assume that Lemma 4.1(ii) holds, i.e., $G$ contains an $m$-bounded minor $G'$ with $d(G') \geq m \left(1 - \frac{30}{m}\right)d(G)$ for some integer $m \in [\frac{k}{6}, k]$. Let $s' = D/d(G')$. Note that since $k \geq 512$, we have that $m \geq \frac{k}{6} \geq 60$. Hence

$$d(G') \geq m \left(1 - \frac{30}{m}\right)d(G) \geq \frac{m}{2}d(G) > d(G),$$

and

$$s' \leq \frac{s}{m \left(1 - \frac{30}{m}\right)} \leq \frac{2s}{m} < s.$$

Since $s' < s$, we have by induction that at least one of (i) or (ii) holds for $G'$. If (i) holds for $G'$, i.e., $G'$ contains a minor $J$ with $d(J) \geq D$. Then $J$ is also a minor of $G$ and (i) holds for $G$.

So we may assume that (ii) holds for $G'$, i.e., $G'$ contains a subgraph $H'$ with $v(H') \leq q_{1.3}(s')\frac{D^2}{d(G')}^D(G)$ and $d(H') \geq \frac{d(G')}{q_{1.3}(s')}$. Note that $H'$ corresponds to a subgraph $H$ of $G$ with $v(H) \leq mv(H')$ and $e(H) \geq e(H')$. Then

$$v(H) \leq mv(H') \leq mq_{1.3}(s')\frac{D^2}{d(G')} \leq \left(\frac{q_{1.3}(s')}{1 - \frac{30}{m}}\right)\frac{D^2}{d(G)}.$$ 

and

$$d(H) = \frac{e(H)}{v(H)} \geq \frac{e(H')}{mv(H')} = \frac{d(H')}{m} \geq \frac{d(G')}{mq_{1.3}(s')} \geq \left(\frac{1 - \frac{30}{m}}{q_{1.3}(s')}\right)d(G).$$

In the rest of the proof, we show that $H$ satisfies (ii). Since $m \geq \frac{k}{6} \geq 60(1 + \log s)$, we have

$$\frac{1}{1 - \frac{30}{m}} \leq 1 + \frac{60}{m} \leq 1 + \frac{2}{1 + \log s}.$$ 

On the other hand, since $m \geq \frac{k}{6} > e^3$, we have

$$\log s' \leq \log \left(\frac{2s}{m}\right) \leq \log(s) + 1 - \log(m) \leq \log(s) - 2.$$
Thus
\[
\frac{g_{1,3}(s')}{g_{1,3}(s)} \leq \frac{(1 + \log s')^5}{(1 + \log s)^5} \leq \frac{1 + \log s'}{1 + \log s} \leq \frac{1 + \log(s) - 2}{1 + \log s} = 1 - \frac{2}{1 + \log s}.
\]
Now we have
\[
\frac{g_{1,3}(s')}{1 - \frac{30}{m}} \leq \left(1 - \frac{2}{1 + \log s}\right) \left(1 + \frac{2}{1 + \log s}\right) g_{1,3}(s) \leq g_{1,3}(s).
\]
Hence, \(v(H) \leq \frac{g_{1,3}(s)}{d(G)} \frac{d(G)^2}{d(G)}\) and \(d(H) \geq \frac{d(G)}{g_{1,3}(s)}.\) Therefore, (ii) holds.

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