From small-overlap conditions to automatic semi-groups

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Abstract

We study the connection between small-overlap conditions and automaticity of semi-groups. We restrict the discussion to conditions that imply embeddability and under which each relation decomposes into at least seven pieces. For these hyperbolic-like conditions we show how to construct an automatic structure. Furthermore, we show that the naive approach of considering just geodesics fails in our case.

1 Introduction

Considering semi-groups from the combinatorial and geometric point of view is an active research field in recent years. A major theme in this line of thinking is the transfer of ideas from combinatorial and geometric group theory into the language of semi-group theory. For example, the definitions of hyperbolic groups and automatic groups were extended to semi-groups; see \cite{[1, 2, 8, 9, 10, 16, 20]}.

One source of difficulty comes from the structure of the (right) Cayley graph of the semi-group. In groups the Cayley graph is a homogeneous space and enjoys a natural metric which is known as the word metric. In semi-groups this is no longer true; the Cayley graph is not homogeneous and it is not clear how to define a useful metric on it. We will therefore focus on the case where the semi-group is embeddable. In this case, one may use the metric induced from an embedding of a semi-group Cayley graph into a group Cayley graph. However, as we shall see, this alone doesn’t allow transfer of results from groups to semi-groups.

In \cite{18} the idea of van Kampen diagrams is extended to the case of semi-groups. The author there uses small-overlap conditions to solve the word problem and to prove Adjan’s criterion for embeddability. In groups, small-cancellation conditions imply automaticity \cite{[3, 5, 21]} and certainly hyperbolicity implies automaticity. In this work we will consider hyperbolic-like
small-overlap conditions that imply automaticity (but also embeddability into a group). This will give a partial answer to a question asked in [2].

Before we can state the main theorem we need some terminology. Let $P = \langle X | L_1 = R_1, \ldots, L_n = R_n \rangle$ be a semi-group presentation. The set of relations in $P$ is called the defining relations; the set of words appearing in the defining relations is called the defining words and we denoted it by $R = R(P) = \{ L_1, \ldots, L_n \} \cup \{ R_1, \ldots, R_n \}$. A piece for $P$ is a word $P$ such that there are two defining words $W_1$ and $W_2$ which decomposes as $W_1 = U_1 PU_2$ and $W_2 = V_1 PV_2$, respectively, and either $U_1 \neq V_1$ or $U_2 \neq V_2$. For a word $W \in X^*$ we denote by $\|W\|$ the piece-length of $W$, namely, the minimal $k$ such that $W = P_1 \cdots P_k$ and $P_1, \ldots, P_k$ are pieces (it is zero if $W$ is the empty word and it is $\infty$ if no such decomposition exists). Let $P = \langle X | L_1 = R_1, \ldots, L_n = R_n \rangle$ be a semi-group presentation. We say that $P$ is a $K^2_3$ presentation [13, 6] if the following conditions hold:

(a) Each defining relation $L = R$ has the property that $L$ and $R$ both start (respectively, end) with different generators.

(b) Each defining word $R$ has a piece-length of at least 3 (i.e., $\|R\| \geq 3$).

(c) If $R_1 = L_1$ and $R_2 = L_2$ are two defining relations then all four words $R_1, R_2, L_2,$ and $L_2$ are distinct.

Condition $K^2_3$ implies [6] that the semi-group presented by $P$ is embeddable.

Our main theorem is the following:

**Theorem 1** (Main Theorem). Let $P = \langle X | L_1 = R_1, \ldots, L_n = R_n \rangle$ be a semi-group presentation. Assume that the $K^2_3$ condition holds and also:

(i) Each defining relation $L = R$ has the property that $\|R\| + \|L\| \geq 7$.

Then, the semi-group presented by $P$ is automatic.

Recently, and independently, Mark Kambites [12] has shown that semi-groups for which each defining word has a piece-length at least four, also known as $C(4)$ semi-groups, are asynchronously-automatic. This is a weaker notion then automaticity. However, the $C(4)$ small-overlap assumption is weaker then the assumptions in the main theorem since it does not imply that the semi-group is embeddable. The following is a natural question:

**Question 2.** Can one show that the $C(4)$ small-overlap condition imply that the semi-group is automatic? (or find a counter example.)

$C(7)$ groups (in which each relator decomposes to at least seven pieces) are hyperbolic. Now, in hyperbolic groups one may construct an automatic structure by considering the set of geodesics. We give an example (Section 4) showing that even in our restricted setup one cannot use the set of geodesics as an automatic structure.
Automatic semi-groups (or monoids) are defined using language theoretic notions (see definition \([4]\)). There are ‘geometric characterizations’ \([10, 20]\) which are not as simple (or nice) as in the group case. We show that in the case of embeddable semi-group one can use the same characterization as in the group case (known as fellow-traveller property) if one considers the metric on the Cayley graph which is induced from the embedding of the semi-group.

For semi-groups, the conjugacy problem is the problem of deciding if for two elements \(A\) and \(B\) there are other elements \(U\) and \(W\) such that \(AU = UB\) and \(WA = BW\) (equality of elements in the semi-group). For bi-automatic groups this problem is decidable \([3, \text{Thm. 2.5.7}]\). A straightforward generalization of the proof of the main theorem shows that in fact the semi-groups considered in the main theorem are bi-automatic. Thus, using the same proof as the one in \([3]\) we get that the conjugacy problem for these semi-groups is decidable. Note however that the complexity of the solution is doubly exponential.

The rest of this paper is organized as follows. In section 2 we give the basic definition and notations. In Section 3 we give characterization of automaticity for embeddable semi-groups and we show how to prove automaticity using special order on the elements of the semi-group. In section 4 we give an example of embeddable semi-group for which the conditions of the main theorem hold but the set of geodesics is not an automatic structure. In Section 5 we recall the parts of van Kampen diagram theory we need for the proof. Finally, in Section 6 we prove the main theorem.

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2 Preliminaries

The following notations and definitions are based on \([1]\). Let \(S\) be a semi-group finitely generated by \(X\). We denote by \(X^*\) the set of words with letters in \(X\), i.e., this is the free semi-group over \(X\). We denote by \(\varepsilon\) the empty word. Given a word \(W\) in \(X^*\) we denote by \(\overline{W}\) the element that \(W\) presents and denote by \(\pi_{X,S} : X^* \to S\) the natural map such that \(\pi_{X,S}(W) = \overline{W}\) for all \(W \in X^*\). For the purpose of this work it is enough to consider only semi-groups with 1 (i.e., monoids) such that the empty word in \(X^*\) maps to 1. We will adopt this convention in the sequel. We denote the length of \(W\) by \(|W|\). We say that \(U\) is a sub-word of \(W\) if \(W\) has a decomposition \(W = V_1UV_2\); \(U\) is a prefix of \(W\) if \(V_1 = \varepsilon\) and it is a suffix of \(W\) if \(V_2 = \varepsilon\). If \(W\) and \(V\) in \(X^*\) presents the same element in \(S\) (i.e., \(\pi_{X,S}(W) = \pi_{X,S}(U)\)) then we may emphasis that the equality is in \(S\) by writing \(W =_S U\). If \(S\) and \(X\) are understood from the context we may also simply write \(\pi\) instead of \(\pi_{X,S}\). When needed, we will distinguish between semi-group
presentations and group presentation using the notations \( \text{Sgp} \langle \cdot | \cdot \rangle \) and \( \text{Grp} \langle \cdot | \cdot \rangle \), respectively. Suppose \( \$ \not\in X \). We denote by \( X(2,\$) \) the set \( X \cup \{ \$ \} \times X \cup \{ \$ \} \setminus \{ (\$,\$) \} \).

**Definition 3.** Let the map \( \delta_X : X^* \times X^* \to X(2,\$)^* \) be defined on \((W,U)\), for \( W = x_1 x_2 \cdots x_n \) and \( U = y_1 y_2 \cdots y_m \) as follows:

\[
\delta_X(W,U) = \begin{cases} 
(x_1, y_1) \cdots (x_n, y_n)(\$, y_{n+1}) \cdots (\$, y_m) & n < m \\
(x_1, y_1) \cdots (x_m, y_m)(x_{m+1}, \$) \cdots (x_n, \$) & m < n \\
(x_1, y_1) \cdots (x_n, y_m) & m = n 
\end{cases}
\]

**Definition 4.** A finitely generated semi-group \( S \) is automatic if there is a generating set \( X \) and regular language \( L \subseteq X^* \) such that:

1. \( L \) is onto \( S \) through the natural map.
2. For any \( x \in X \cup \{ \varepsilon \} \) the following set is regular:

\[
L_x = \{ (W,U) \delta_X \mid W, U \in L; \overline{Wx} = \overline{U} \}
\]

A languages \( L \) having these properties is called an automatic structure of \( S \).

We will be using lexicographical ordering in several places. Suppose we are given two vectors \( v = (a_1, \ldots, a_n) \) and \( u = (b_1, \ldots, b_m) \) with entries in some set \( S \). If we wish to compare between \( u \) and \( v \) lexicographically then there will be some (complete) order on \( S \) and, based on that order, \( u \) precedes \( v \) in lexicographic order if:

1. \( n < m \); or,
2. \( n = m \) and there is some index \( 1 \leq k \leq n \) such that \( a_i = b_i \) for all \( 1 \leq i \leq k - 1 \) and \( a_k < b_k \).

In some cases there may be a natural order defined on \( S \) (e.g., if \( S \) consists of natural numbers). If that is the case then the lexicographical ordering (on vectors with entries in \( S \)) will be based on the natural order on \( S \). We may also use the lexicographical ordering to compare between words in \( X^* \). In this case there will be a fix (arbitrary) order on \( X \) and we would consider the elements of \( X^* \) as vectors with entries in \( X \). The lexicographical ordering is denoted by ‘\( \prec_{\text{lex}} \)’.

Suppose we are given a semi-group \( S \) and a (finite) generating set \( X \). The Cayley graph \( \Gamma(S,X) \) of \( S \) (under the generating set \( X \)) is the graph with \( S \) as the vertex set and an edge \( s \xrightarrow{x} sx \) for any \( s \in S \) and \( x \in X \). Each word \( W = x_1 x_2 \cdots x_n \) in \( X^* \) represents a path in \( \Gamma(S,X) \) which has the following vertices:

\[
1, x_1, x_1 x_2, \ldots, x_1 x_2 \cdots x_n
\]
There are several ways to define a metric on $\Gamma(S,X)$. One option is to consider the distance from $s_1$ to $s_2$ as the length of shortest positive path connecting between them. Another option is to consider the path metric on $\Gamma(S,X)$ viewed as non-directed graph. These two options have their advantages and limitations. Here we consider another option which is only available when the semi-group is embeddable. For that end, consider a semi-group $S$ finitely generated by $X$ with a semi-group presentation

\[ \text{Sgp}\{X \mid R_1 = S_1, \ldots, R_n = S_n\} \]

Then, the co-presented group of $S$ is the group $G$ with presentation

\[ \text{Grp}\{X \mid R_1 = S_1, \ldots, R_n = S_n\} \]

If $S$ is embeddable then it is embeddable in the co-presented group as the sub-semi-group of positive words (positive words are words in $X^*$ and negative words are words in $(X^{-1})^*$). Consequently, $\Gamma(S,X)$ is embedded in $\Gamma(G,X)$ and we can define a metric on $\Gamma(S,X)$ which is induced from the word metric on $\Gamma(G,X)$. We term this metric as the *induced metric* on $\Gamma(S,X)$.

We conclude the preliminary section with few important (but easy) consequences of the definition of a piece. Suppose $W$ is a sub-word of a defining word such that $\|W\| \geq 2$. If follows that $W$ fixes two unique elements $U_1$ and $U_2$ such that $U_1WU_2 \in R$ (it is possible that $U_1 = \varepsilon$ or $U_2 = \varepsilon$). The reason is that non-uniqueness would imply, by the definition of pieces, that $W$ is a piece and consequently $\|W\| = 1$. So we have:

**Observation 5.** Let $W$ be sub-word of a defining word such that $\|W\| \geq 2$. Then, there is a unique $R \in R$ such that $W$ is a sub-word of $R$.

Here is another observation that follows from the definition of a piece and condition $K^2_3$.

**Observation 6.** Assume that condition $K^2_3$ holds and suppose $W = W'W''$ is a sub-word of a defining word where $W' \neq \varepsilon$ and $W'' \neq \varepsilon$. If $W''$ is a prefix of some defining word then $\|W''\| = 1$. Similarly, if $W'$ is a suffix of some defining word then $\|W'\| = 1$.

**Proof.** We prove the first case. Take $V_1, V_2$ and $U$ such that $V_1WV_2$ is a defining word and $W''U$ is a defining word. Clearly $V_2 \neq U$ (because, otherwise we would get that $W''U$, a defining word, is a sub-word of $V_1WV_2$, another defining word, and thus $\|W''U\| = 1$ which contradicts the $K^2_3$ condition). Hence, by the definition of a piece we get that $W''$ is a piece and thus $\|W''\| = 1$. \qed
3 Automaticity in Embeddable Semi-Groups

Automatic groups have a so-called geometric characterization through the idea of fellow-travelling paths (see [3, Ch. 2] and Definition 7 below). For semi-groups and monoids such simple geometric characterization does not apply. However, Hoffmann and Thomas [10], and Silva and Steinberg [20] independently gave similar—though less elegant—geometric characterizations for semi-groups and monoids; in their work additional conditions are needed on top of fellow-travelling. For embeddable semi-groups a group-like geometric characterization can be given (Theorem 9). First, here is the definition of fellow-travellers in semi-groups:

**Definition 7 (Fellow-Travellers [3])**. Let $S$ be a semi-group finitely generated by $X$ and let $d(\cdot, \cdot)$ be some metric on $\Gamma(S,X)$. For a word $W \in X^*$ we denote by $W(n)$ the prefix of length $n$ of $W$ (which is $W$ if $n \geq |W|$, the length of $W$). Two words $W$ and $U$ in $X^*$ are called $k$-fellow-travellers (relative to $d$) if for any $n \in \mathbb{N}$:

$$d(W(n), U(n)) \leq k$$

A set of words $L \subseteq X^*$ has the fellow-traveller property if there is some constant $k$ such that for each $W$ and $U$ in $L$ such that $d(W, U) \leq 1$ we have that $W$ and $U$ are $k$-fellow-travellers. In the sequel we will write $d(W, U)$ instead of $d(W, U)$.

Here is a useful feature of the fellow-travelling property.

**Lemma 8.** Suppose $W$ and $U$ are $k$-fellow-travellers and also $U$ and $V$ are $\ell$-fellow-travellers (with respect to some metric $d$) then $W$ and $V$ are $(k + \ell)$-fellow-travellers.

**Proof.** This follows from:

$$d(W(i), V(i)) \leq d(W(i), U(i)) + d(U(i), V(i)) \leq k + \ell$$

Next the characterization of automaticity in embeddable semi-groups. The following theorem seems to be folklore; we give its proof for completeness.

**Theorem 9.** Let $S$ be an embeddable semi-group, finitely generated by $X$. Then, $S$ is automatic if and only if there is a regular language $L \subseteq X^*$ such that $\pi_{X,S}(L) = S$ and $L$ has the fellow-traveller property under the induced metric on $\Gamma(S,X)$.

The “only if” part of the proof of Theorem 9 follows immediately from Lemma 3.12 in [1]. We prove the “if” part. For embeddable semi-groups we have the following lemma:
Lemma 10. Suppose \( S \) is an embeddable semi-group finitely generated by \( X \) and consider the induced metric on \( \Gamma(S, X) \). Let \( k \) be a natural number and let \( x \in X \cup \{ \varepsilon \} \). The following language, denoted by \( FT^k_x(S, X) \), is regular:

\[
FT^k_x(S, X) = \left\{ (W, U) \delta_X \mid W \text{ and } U \text{ are } k\text{-fellow-travellers and } Wx = SU \right\}
\]

Proof. Denote by \( G \) the co-presented group of \( S \) generated as a semi-group by \( X \pm 1 \). Let \( J_x \) denote the following language:

\[
J_x = \left\{ (W, U) \delta_{X \pm 1} \mid W \text{ and } U \text{ are } k\text{-fellow-travellers and } Wx = SU \right\}
\]

It is well known that \( J_x \) is regular (see the proof of Theorem 2.3.4 in [3]). Hence, it follows that \( FT^k_x(S, X) \) is regular since \( FT^k_x(S, X) = J_x \cap (X^* \times X^*)\delta_X \).

Proof of Theorem 9 (if part). Let \( S \) be an embeddable semi-group, finitely generated by \( X \). Suppose \( L \subseteq X^* \) is a regular language which is onto \( S \) through the natural map and which has the fellow-traveller property for some constant \( k \). We show that \( S \) is automatic by showing that for all \( x \in X \cup \{ \varepsilon \} \) the set \( L_x = \{ (W, U) \delta_X \mid W, U \in L; Wx = SU \} \) is regular. First, notice that \( L_x \subseteq FT^k_x(S, X) \) since \( L \) has the fellow-traveller property. Next, since intersection preserves regularity [11, Thm. 4.8] the set

\[
(L \times L)\delta_X \cap FT^k_x(S, X)
\]

is regular. Finally, since the elements in \( L_x \) are elements of \( FT^k_x(S, X) \) which have the form \((W, U)\delta_X\) where \( W \) and \( U \) in \( L \) and thus there is an equality \( L_x \cap FT^k_x(S, X) = (L \times L)\delta_X \cap FT^k_x(S, X) \). This implies that

\[
L_x = (L \times L)\delta_X \cap FT^k_x(S, X)
\]

and thus \( L_x \) is regular.

Next, we show how to generate an automatic structure for embeddable semi-groups through regular partial orders (an order \( \prec \) is regular if the set \{ \((W, U)\delta_X \mid W \prec U \} \) is regular). This technique is called ‘falsification by fellow travellers’ and is based on a work by Davis and Shapiro (see also [17, 21]).

Theorem 11. Let \( S \) be an embeddable semi-group finitely generated by \( X \). Suppose \( \preceq \) is a regular partial order on \( X^* \). Denote by \( M_\preceq \) the following set:

\[
M_\preceq = \{ W \in X^* \mid \text{if } W = SU \text{ and } U \prec W \text{ then } W = U \}
\]

We assume that \( \pi_{X,S}(M_\preceq) = S \). Suppose there is a constant \( k \) such that the following properties of \( \preceq \) holds:
4 EXAMPLE OF NON-GEODESIC STRUCTURE

(R) If $W \not\in M$ then there is $U \in X^*$ such that $W =_S U$, $U \prec W$, and $W, U$ are $k$-fellow-travellers.

(FT) If $W$ and $U$ in $M$ and $Wa =_S U$ for some $a \in X \cup \{ \varepsilon \}$ then $W$ and $U$ are $k$-fellow-travellers.

Then, $S$ is an automatic semi-group.

Proof. By assumption the set $M$ is onto $S$ through the natural map. By Property (FT), the set $M$ has the fellow-traveller property. Hence, to establish automaticity it is enough by Theorem [9] to show that $M$ is regular. We denote the set $FT^k$ by $K$ and the set $\{ (W, U) \delta_X \mid W \prec U \}$ by $P$ (recall that $P$ is regular since we assumed that “$\prec$” is regular). Since intersection preserve regularity, the following set is regular:

$$K \cap P = \{ (W, U) \delta_X \mid W \prec U, W =_S U, \text{ and } W, U \text{ are } k\text{-fellow-travellers} \}$$

Projection also preserves regularity [3, Cor. 1.4.7] and therefore the following set is regular:

$$C = \{ U \mid \exists W : (W, U) \in K \cap P \}$$

By Property (R) an element $W$ is in $C$ if and only if it is not in $M$. Hence, by Property (R) the set $C$ is exactly the complement of $M$. Consequently, $M$ is regular since taking complement preserves regularity [11, Thm. 4.5].

We will call an elements of $M$ an “$\prec$”-minimal element (reads as “order minimal”). The theorem above shows that the set of “$\prec$”-minimal elements is an automatic structure (assuming, of course, that the conditions of the theorem hold).

4 Example of non-geodesic structure

The $K_3^2$ semi-groups considered in the main theorem are embeddable semi-groups. We give in this section an example of a semi-group $S$ which is automatic by the main theorem but for which for a given set of generators the set of geodesics is not an automatic structure. This is in sharp contrast to the situation in $C(7)$ groups. To recall the definition, a word $W$ is geodesic if for every $U$ such that $W = U$ in $S$ we have that $|W| \leq |U|$.

The semi-group we consider is the semi-group with the following presentation:

$$\langle a, b, c \mid abcc = cba \rangle$$

Here, $R = \{ abcc, cba \}$ and $X = \{ a, b, c \}$. There are only three pieces: $a$, $b$, and $c$. Thus, the $K_3^2$ conditions holds by simple inspection and so $S$ is an embeddable semi-group (embeddable in this case in an hyperbolic group
since the co-presented group is a \( C(7) \) group). We give two geodesics, \( V_n \) and \( U_n \) of lengths \( 3n \) and \( 2n + 1 \), respectively, such that \( V_n c = U_n \) in \( S \) (i.e. \( d(U_n, V_n) = 1 \) in the Cayley graph). Hence, if \( k \) is fixed and \( n \) is large enough then \( V_n \) and \( U_n \) are not \( k \)-fellow-travellers (due to the big difference in their lengths). The definitions of \( V_n \) and \( U_n \) follows: let \( n \) be some natural number and let \( V_n = (abc)^n \) and \( U_n = c(ba)^n \). See Figure 1 for an illustration of part of the Cayley graph of \( S \) containing \( V_n \) and \( U_n \).

![Figure 1: Two non-fellow-travelling geodesics](image)

Denote by \( s_n \) and \( t_n \) the elements in \( S \) presented by \( U_n \) and \( V_n \), respectively. Now, since none of the two sides of the relations is a sub-word of \( V_n \) we have that no other element in \( X^* \) presents \( t_n \). Thus, \( V_n \) is a geodesic. For \( U_n \) we do have a sub-word that is one side of the relation. However, by applying the relation one can only increase the length. Thus, \( U_n \) is also a geodesic. Using the relation \( abcc = cba \) we get that \( (abc)^nc = c(ba)^n \) so consequently \( V_n c = U_n \), as claimed.

Another interesting observation regarding the above example is the following: by the above discussion there are no other geodesics presenting \( s_n \) and \( t_n \) in \( X^* \). Therefore, any automatic structure for \( S \) (under the generating set \( X \)) cannot contain just geodesics or it will contain \( U_n \) and \( V_n \) which is impossible.

5 van Kampen Diagrams

We use the theory of van Kampen diagrams, both for semi-groups and groups. See [15, Chapter V, p. 235] for a standard introduction of van Kapmen diagrams of groups and see [18] or [7, p. 73-79] for the van Kampen diagram theory for semi-groups. Here we give a unified treatment for both cases. A \textit{diagram} is a finite planar connected and simply connected 2-complex. We name the 0-cells, 1-cells, and 2-cells by \textit{vertices}, \textit{edges}, and \textit{regions}, respectively. Vertices of valence one or two are allowed. Each edge has an orientation, i.e., a specific choice of initial and terminal vertices. Given an edge \( e \) we denote by \( i(e) \) the initial vertex of \( e \) and by \( t(e) \) the terminal vertex of \( e \). If \( e \) is an oriented edge then \( e^{-1} \) will denote the same edge but with the reverse orientation. A \textit{path} is a series of (oriented) edges \( e_1, e_2, \ldots, e_n \) such that \( t(e_j) = i(e_{j+1}) \) for \( 1 \leq j < n \). The length of a path \( \rho \)
(i.e., the number of edges along $\rho$ is denoted by $|\rho|$. If $\rho$ is the path $e_1 \cdots e_n$ then we denote by $\rho^{-1}$ the path $e_n^{-1} \cdots e_1^{-1}$. If $\rho$ is a path that decomposes as $\rho = \rho_1 \rho_2$ then $\rho_1$ is a prefix of $\rho$ and $\rho_2$ is a suffix of $\rho$.

Given a finite group presentation $\text{Grp} \langle X \mid R \rangle$, a group diagram over this presentation is a diagram where its edges are labelled by elements of $X^{\pm 1}$ and the boundary of every region is labelled by elements of the symmetric closure of $R$. We also require that if an edge $e$ is labelled by $x$ then $e^{-1}$ is labelled by $x^{-1}$. In the context of group diagram we say that an edge $e$ is positive (resp., negative) if its label is in $X$ (resp., in $X^{-1}$). In the same manner, a path is positive (resp., negative) if it consists of positive (resp., negative) edges. A boundary label is the label of some path $\rho$ that coincides with the boundary of the diagram. Next we give the definition of semi-group diagrams; these requires some additional assumptions. Suppose we are given a semi-group presentation $\text{Sgp} \langle X \mid L_1 = R_1, \ldots, L_n = R_n \rangle$. A semi-group diagram $M$ over the given presentation is a group diagram over the co-presented group such that three conditions hold: (1) there is a boundary label $WU^{-1}$ where $W$ and $U$ are positive; (2) any inner vertex is an initial vertex of some positive edge (i.e., there are no inner sink vertices); (3) any inner vertex is a terminal vertex of some positive edge (i.e., there are no inner source vertices). van Kampen theorem state that equality $W = U$ holds in a group (semi-group) if and only if there is a van Kampen group (semi-group) diagram with boundary label $WU^{-1}$ (such diagrams are called equality diagrams). If we don’t explicitly indicate for a given diagram whether it is a group diagram or a semi-group diagram then it may be either one of the two options.

In the sequel, if a word $W$ labels a path on the boundary of $M$ then $W$ would denote both the path and the word; the context would make the distinction clear. The term neighbors, when referred to two regions, means that the intersection of the regions’ boundaries contain an edge; specifically, if the intersection contains only vertices, or is empty, then the two regions are not neighbors. Boundary regions are regions with outer boundary, i.e., the intersection of their boundary and the diagram’s boundary contains at least one edge. If $D$ is a boundary region in $M$ then the outer-boundary of $D$ is $\partial D \cap \partial M$ and the inner boundary of $D$ is the rest of the boundary (i.e., the complement of the outer boundary). Regions which are not boundary regions will be called inner regions. In a similar manner, a boundary edge is an edge in the boundary of the diagram and an inner edge is an edge not on the boundary. A minimal diagram is a diagram with minimal number of regions among the diagram with the same boundary label.

Suppose $D$ and $E$ are neighboring regions in $M$ and let $\delta$ be a connected component of $\partial D \cap \partial E$. It is a well known fact that if $M$ is a minimal group diagram then the label of $\delta$ is a piece (see the introduction for the definition). This may not be the case for general semi-group diagrams. However, as we shall shortly see, in the cases we consider the label of $\delta$ is always a peice.
Definition 12 (Strong s-condition [13]). Let \( P \) be a semi-group presentation of a semi-group \( S \). We say that \( P \) has the strong s-condition if the following hold. Suppose \( W \) and \( U \) are positive words and \( W = U \) is an equality in the co-presented group of \( S \). Suppose further that \( M \) is a (group) van Kampen diagram over the co-presented group of \( S \) with \( WU^{-1} \) as boundary cycle. Then, there is a boundary region \( D \) in \( M \) with boundary cycle \( \rho \delta^{-1} \) such that the labels of \( \rho \) and \( \delta \) are positive and \( \rho \) is the outer boundary of \( D \) and a sub-word of \( W \) or \( U \). See figure 2.

\[
\begin{array}{c}
\text{Figure 2: Illustration of strong s-condition}
\end{array}
\]

The main result of [6] is that \( K_3 \) semi-groups have the strong s-condition (and, hence, are embeddable [13]). Suppose we are given a \( K_3 \) semi-group \( S \) and we consider a group diagram \( M \) with boundary cycle \( WU^{-1} \) over the co-presented group of \( S \) (\( W \) and \( U \) are positive words). By the strong s-condition we have some region \( D \) that we can remove from \( M \) such that the resulting diagram \( \tilde{M} \) has a boundary label \( \tilde{W}\tilde{U}^{-1} \) where \( \tilde{W} \) and \( \tilde{U} \) are positive (and, clearly, \( M \) has less regions comparing to \( M \)). This is the essence of the proof of the following lemma:

Lemma 13. Let \( S \) be semi-group with a \( K_3 \) presentation \( P \) and let \( M \) be a minimal semi-group van Kampen diagram over \( P \). Suppose \( D \) and \( E \) are neighboring regions in \( M \) and that \( \delta \) be a connected component of \( \partial D \cap \partial E \). Then, \( \delta \) is labelled by a piece.

Proof (sketch). Denote by \( G \) the co-presented group of \( S \). It is enough to show that \( M \) is a minimal group diagram over \( G \) (because, in minimal group diagrams every edge is labelled by a pieces). Denote by \( |M| \) the number of regions in \( M \). Let \( N \) be a minimal group diagram over \( G \) with the same boundary label as \( M \). Clearly, \( M \) is a group diagram over \( G \) so it remains to show that it is minimal, or in other words to show that \( |M| = |N| \). It is also clear that \( |N| \leq |M| \). Thus, we need to show that \( |M| \leq |N| \). We will do that by showing that \( N \) is a semi-group diagram. We prove that \( N \), a minimal group diagram, is a semi-group diagram by induction on \( |N| \). If \( |N| = 0 \) (i.e., there are no regions in \( N \)) then clearly there are no inner source or sink vertices in \( N \) so \( N \) is a semi-group diagram. Suppose that
the assertion is true when $|N| < n$ and we have that $|N| = n$. We use the strong s-condition and we denote the region it guarantees by $D$. We remove the region $D$ from $N$ and denote the new diagram by $N'$. Clearly, $|N'| < n$ so by induction hypothesis we get that $N'$ is a semi-group diagram. Finally, by attaching $D$ back to $N'$ (which restores the diagram $N$) we see that $N$ is also a semi-group diagram.

A $(\mu, \sigma)$-thin diagram is a diagram $M$ with boundary cycle $\delta \sigma^{-1}$ where every region $D$ has at most two neighbors and $\partial D$ has non-empty intersection with $\delta$ and $\sigma$. See an illustration of such diagram in Figure 3. The notion of thin diagrams (also known as one layered diagrams) appeared in [17, 21] and in several other earlier works.

![Figure 3: Thin equality diagram](image)

Let $M$ be a diagram and $D$ a region in $M$. Suppose $\omega$ is a sub-path of $\partial D$ with label $W$. We denote by $N^D_\omega$ the number of neighbors of $D$ along $\omega$ counted with multiplicity; see Figure 4 for an illustration of neighbors along a path.

![Figure 4: The neighbors of $D$ along a sub-path $\omega$ of $\partial D$](image)

Suppose next that $M$ is a minimal semi-group van Kampen diagram over a presentation for which the conditions of the main theorem hold. By lemma [13] we have that the neighbors of $D$ along $\omega$ induce a decomposition of $W$ into pieces and thus we get that $N^D_\omega \geq |W|$. An immediate implication of this is that $M$ is a $C(7)$ diagrams (i.e., diagrams where every inner regions
has at least seven neighbors) and we can use the tools of small cancellation theory for these diagrams. The main diagramatical result of this section is the following:

**Proposition 14.** Assume a semi-group $S$ is a semi-group with presentation $\langle X \mid L_1 = R_1, \ldots, L_n = R_n \rangle$ for which the conditions of the main theorem hold. Let $W$ and $U$ be two positive words and let $a \in X \cup \{ \varepsilon \}$. If $M$ is a minimal semi-group diagram over the presentation with boundary cycle $WaU^{-1}$ then there are two options:

1. $M$ is a $(Wa, U)$-thin diagram.
2. There is a boundary region $D$ with $\partial D = \rho\delta^{-1}$ such that:
   (a) $\rho$ is the outer boundary of $D$ and is a sub-path of $W$ or $U$.
   (b) $\delta$ is the inner boundary and $N_D^\rho = 3$.
   (c) $\rho$ and $\delta$ are labelled by positive words.

Before we can give the proof of Proposition 14 we need two diagramatical results. The first is a lemma from [18].

**Lemma 15** (Lemma 4.8(a) of [18]). Let $S$ be a semi-group with presentation $P$ such that any defining word has piece-length at least three. Suppose $W = U$ is an equality in $S$ and $M$ is a minimal van Kampen diagram with boundary label $WU^{-1}$. Let $D$ be a region in $M$ with boundary cycle $\rho\delta^{-1}$ where $\rho$ and $\delta$ are positively labelled. Then, $N_D^\rho \leq 3$ and $N_D^\delta \leq 3$.

The consequence of Lemma 15 is that minimal van Kampen diagrams over presentations having the conditions of the main theorem have no inner regions. Namely, in these diagrams every region has a boundary. Next is a lemma, originally due to Greendlinger [15, Thm. 4.5], which give information on the structure of a $C(7)$ diagrams. A direct proof of (a generalization of) the lemma can be found in [21, Thm. 13].

**Lemma 16** (Greendlinger’s lemma). Let $M$ be a $C(7)$ diagram with boundary label $\mu\rho^{-1}$. Then, one of the following holds:

1. $M$ is a $(\mu, \sigma)$-thin diagram.
2. There is a boundary region $D$ with $\partial D = \rho\delta^{-1}$ such that:
   (a) $\rho$ is the outer boundary of $D$ and is a sub-path of $\mu$ or $\sigma$.
   (b) $\rho$ does not intersect both $\mu$ and $\sigma$.
   (c) $D$ has at most three neighbors in $M$.

Equipped with these two results we can now give the proof of Proposition 14.
Proof of Proposition 14. By Lemma 13 every inner edge in $M$ is labelled by a piece and thus $M$ is a $C(7)$ diagram. Assume that $M$ is not a $(W_a, U)$-thin diagram. By Greendlinger’s Lemma there is a boundary region $D$ in $M$ such that $D$ has at most three neighbors and the outer boundary of $D$ is contained in $W$ or in $V$. Suppose the boundary of $D$ is $\rho^{-1}\delta$ where both $\rho$ and $\delta$ are positively labelled. Since the outer boundary of $D$ is contained in $W$ or in $V$ we get that one of the parts, $\rho$ or $\delta$, of the boundary of $D$ is completely contained in the inner boundary of $D$. Assume w.l.o.g. that $\delta$ is contained in the inner boundary of $D$. Let $V$ be the label of $\delta$ (this is a defining word). Using Lemma 15 and the $K^2_3$ condition we have that $3 \leq |V| \leq N^D_\delta \leq 3$. Consequently, $\delta$ is exactly the inner boundary of $D$ (because it has exactly three neighbors so all the neighbors of $D$ intersect with $\delta$ and not with $\rho$). As a result, $\rho$ does not contain inner edges. This proves the proposition.

We finish the section on diagrams with the next lemma. The lemma characterize the structure of thin equality diagrams over presentations which satisfy the conditions of the main theorem and another technical condition (one which later we can assume).

Lemma 17. Let $P$ be a semi-group presentation which satisfy conditions $K^2_3$ and (†) of the main theorem and let $M$ be a $(\mu \xi, \sigma)$-thin diagram over this presentation. We will assume that $\xi$ is empty or the label of $\xi$ is a generator. We will also assume the following technical condition:

(‡) Suppose $D$ is a region in $M$ with boundary path $\delta\rho^{-1}$ such that $\delta$ and $\rho$ are (positively) labelled by $V_\delta$ and $V_\rho$ and $|V_\delta| > |V_\rho|$. Then, $\delta$ is not a sub-path of $\mu$ and is not a sub-path of $\sigma$.

Then:

1. If $\nu$ is a vertex of $\mu$ of valence at least three that is not a vertex of $\sigma$ then $\nu$ is of valence exactly three. Specifically, if $D$ is a region of $M$ then $\partial D \cap \mu$ and $\partial D \cap \sigma$ both contain an edge.

2. If $D$ is a region in $M$ then the label of $\partial D \cap \mu$ has piece-length at least two.

3. If $D$ is a region in $M$ that has at most one neighbor which its boundary does not contain $\xi$. Then, the piece-length of the label of $\partial D \cap \mu$ is at least three.

4. Suppose that $D_1$ and $D_2$ are two neighboring regions and let $V_1$ and $V_2$ be the labels of $\partial D_2 \cap \mu$ and $\partial D_2 \cap \mu$, respectively. The word $W = V_1 V_2$ has the property that if $U$ is a prefix of $W$ which is a sub-word of defining word then $|U| \leq |V_1|$ (we will later denote such decomposition of $W$ as left-greedy decomposition).
Proof. We prove the different parts one by one:

1. See figure 5. Assume by contradiction that $\nu$ is a vertex of $\mu$ of valance greater than three which is not a vertex of $\sigma$. In this case there is a region $D$ with two inner edges that are adjacent to $\nu$. Thus, if $\partial D = \rho\delta^{-1}$ such that both $\rho$ and $\delta$ are labelled by positive words then one of them would have piece-length at most two (because the diagram is thin so $D$ has at most two neighbors and the inner parts of $\partial D$ are labelled by pieces). This contradicts the $K^2_3$ condition.

![Figure 5: An impossible situation where a vertex has valance greater than three](image)

2. See figure 6. Let $D$ be a region in $M$. We denote by $\omega_u$, $\omega_d$, $\omega_\ell$, and $\omega_r$ the four sides of $D$ such that $\omega_u = \partial D \cap \mu$, $\omega_d = \partial D \cap \sigma$ and the inner boundary of $D$ consists of $\omega_\ell$ and $\omega_r$ (they may be empty and $\omega_r$ may equal $\xi$). Denote by $V_u$, $V_d$, $V_\ell$, and $V_r$ the labels of $\omega_u$, $\omega_d$, $\omega_\ell$, and $\omega_r$, respectively. Clearly, $\|V_\ell\| \leq 1$ and $\|V_r\| \leq 1$ (since they are pieces or empty). We need to show that $\|V_u\| \geq 2$. Assume otherwise by contradiction, namely, that $\|V_u\| \leq 1$. Depending on $V_\ell$ and $V_r$ being positive or negative words, we have that one of the following is a defining relation in $S$:

$$V_uV_r = V_\ell V_d, \quad V_\ell V_u = V_d V_r, \quad V_u = V_\ell V_d V_r, \quad V_\ell V_u V_r = V_d,$$

The first three cannot be a defining relations since the left side decomposes into less than three pieces and thus violate the $K^3_3$ condition. Hence, $V_\ell V_u V_r = V_d$. But, $\|V_\ell V_u V_r\| \leq 3$ and so by condition (†) of the main theorem we have that $\|V_d\| \geq 4$. This contradicts assumption (‡).

3. See Figure 7. Assume the boundary of $D$ does not contain $\xi$ and suppose $D$ is a region that has at most one neighbor. The boundary of $D$ decomposes into three parts: $\omega_\mu = \partial D \cap \mu$, $\omega_\sigma = \partial D \cap \sigma$, and, possibly empty, inner part $\omega_{\text{in}}$ (which, if not empty, is labelled by a piece). We need to show that the piece-length of $\omega_\mu$ is at least three.
Assume otherwise by contradiction. If follows from the $K_3^2$ condition that either $\omega_\mu \omega_{in}$ or $\omega_{in} \omega_\mu$ are positively labelled. Assume w.l.o.g. that $\omega_\mu \omega_{in}$ is positively labelled. By assumption $\omega_\mu \omega_{in}$ has a piece-length at most three. Thus, by condition (†) of the main theorem we get that $\omega_r$ is labelled by a positive label of piece-length at least four. This contradicts assumption (‡).

4. See Figure 8. Let $\delta_1 = \partial D_1 \cap \mu$, $\delta_2 = \partial D_2 \cap \mu$, and $\omega = \partial D_1 \cap \partial D_2$. Assume that $i(\omega)$, the first vertex of $\omega$, is a vertex of $\mu$. Suppose by contradiction that there is a prefix $U$ of $W$ which is a sub-word of a defining word and $|U| > |V_1|$. Thus, we can decompose $\delta_2$ into $\delta_2 = \delta'_2 \delta''_2$ such that the label of $\delta_1 \delta'_2$ is $U$. Since $\|V_1\| \geq 2$ we have Observation 5 that there is a unique defining word $R$ such that $V_1$ is its sub-word. Thus, $U$ is also a sub-word of $R$. Consequently, $U$ is a prefix of the label of $\delta_1 \omega$. We get that $\delta_2$ and $\omega$ are positively labelled and start with the same generator. This is a contradiction to the $K_3^2$ condition.
6 Proof of Main Theorem

In this section we prove Theorem 1. For the rest of this section fix a presentation \( P = \langle X \mid L_1 = R_1, \ldots, L_n = R_n \rangle \) for a semi-group \( S \) for which the conditions of the main theorem hold (conditions \( K_2^3 \) and \((\ddagger)\)). As suggested by Theorem 11, we will prove automaticity by producing a regular order on some generating set. We will assume that any generator \( x \in X \) appears in one of the defining relations. If that doesn’t happen then we can split \( S \) as \( S = S' \ast F \) where \( S' \) has this property, the conditions of the main theorem hold for \( S' \), and \( F \) is a finitely generated free semi-group. Since a free product of automatic semi-groups is automatic [1, Thm. 6.1] it is enough to prove the theorem for \( S' \).

We start by defining the generating set we will be working with. Let \( B \) be the (finite) set of sub-words of the elements in \( R = R(P) = \{ L_1, \ldots, L_n \} \cup \{ R_1, \ldots, R_n \} \) and let \( \Gamma = \{ \gamma_W \mid W \in B \} \). In other words, \( \Gamma \) is a set of symbols which corresponds to the sub-words of elements in \( R \). By our assumption above we have that \( X \subseteq B \) and thus the set \( \pi(\Gamma) \) is a generating set for \( S \). Our automatic structure will be a subset of \( \Gamma^* \) and it will be constructed by defining an order on the words in \( \Gamma^* \).

We write \( d_{\Gamma}(\cdot, \cdot) \) to denote the induced metric on the Cayley graph of \( S \) under the new generating set (see the end of Section 2). The symbols \( A, B \) and \( C \) will denote elements of \( \Gamma^* \) and the symbols \( U, V, W \) will denote elements of \( X^* \). If \( A \in \Gamma^* \) then there are elements \( W_1, \ldots, W_n \) of \( B \) such that \( A = \gamma_{W_1} \cdots \gamma_{W_n} \). In this case we will use the notation \( \eta(A) \) to denote the word \( W_1 \cdots W_n \) in \( X^* \). Thus, \( \pi(A) = \eta(A) \).

Recall from property (c) of the \( K_2^3 \) condition that if \( W \in R \) is a defining word then there is a unique defining word \( U \in R \) such that \( W = U \) or \( U = W \) is a defining relation. In this case we say that \( U \) is the complement of \( W \). We denote the complement of \( W \) by \( c(W) \). Note that \( c(c(W)) = W \).

We next define for each word in \( \Gamma^* \) an auxiliary vector. These vectors will be used to define an order on \( \Gamma^* \).
Definition 18 (Auxiliary vector for $\Gamma^*$). Let $A = \gamma W_1 \cdots \gamma W_n$ be a word in $\Gamma^*$. We define the vector $\kappa_A \in \{0, 1\}^n$ attached to $A$ (i.e., $\kappa_A$ is vector of length $n$ with zero/one entries). The entries of $\kappa_A$ are defined as follows: the $i$-th coordinate of $\kappa_A$ is one if and only if there are decompositions $W_{i-1} = W_{i-1}' W_{i-1}''$ and $W_{i+1} = W_{i+1}' W_{i+1}''$ such that $W_{i-1}' W_i W_{i+1}' \in R$ and $\|W_{i-1}' W_i W_{i+1}'\| > \|c(W_{i-1}' W_i W_{i+1}')\|$. To complete the definition we need to define $W_0$ and $W_{n+1}$ so we set $W_0 = W_{n+1} = \varepsilon$ (where $\varepsilon$ is the empty word).

To give some intuition, the vector $\kappa_A$ marks these points in $\eta(A)$ that are “inefficient” in the number of pieces. We next define an order on $\Gamma^*$ which is based the auxiliary vectors.

Definition 19 (Piefer order “$\prec$”). Let $A$ and $B$ be two elements of $\Gamma^*$. We write $A \prec B$ (read: ‘$A$ precedes $B$ in the Piefer order’) if $\kappa_A = \kappa_B$ or $\kappa_A$ precedes $\kappa_B$ in lexicographical order.

Note that, for example, if $|A| < |B|$ then $A \prec B$. Note also that the order “$\prec$” is regular. An important property of the order “$\prec$” is that for any $s \in S$ there is a “$\prec$”-minimal element $A$ such that $A$ presents $s$ (see the paragraph after Theorem 11 for the definition of “$\prec$”-minimal). This follows from the fact that lexicographical ordering is a well ordering.

The proof of the main theorem will be completed if we establish that conditions (R) and (FT) of Theorem 11 hold for the order “$\prec$”. The proof of these two properties occupies the rest of this section. Consider an element $A \in \Gamma^*$ that is not minimal according to the order “$\prec$”. Suppose another element $B \in \Gamma^*$ has the following three properties:

1. $\pi(A) = \pi(B)$
2. $B \prec A$
3. $A$ and $B$ are $k$-fellow-travellers.

In this case, following the terminology in [17], we will say that “$B$ $k$-refutes $A$”. To show condition (R) we need to show that for any element $A$ that is not minimal according to the order “$\prec$” we have some element $B$ that $k$-refutes $A$.

Definition 20 (Efficient Words in $\Gamma^*$). We say that $A \in \Gamma^*$ is efficient if $\kappa_A$ is a zero vector (i.e., all its entries are zero). Elements of $\Gamma^*$ which are not efficient will be called inefficient.

To distinguish between zero and non-zero vectors so we adopt the notation $\kappa_A = \overline{0}$ to denote that $\kappa_A$ is a zero vector (of some length) and $\kappa_A \neq \overline{0}$ when $\kappa_A$ is not all zeros. Also, to refer to the coordinates of the vector $\kappa_A$ we will use the notations $[\kappa_A]_i$ which will denote the $i$-th coordinate of the vector.
A technical observation is that condition (R) holds for all inefficient elements of $\Gamma^*$. This is stated in the following proposition.

**Proposition 21.** Let $A \in \Gamma^*$. If $A$ is inefficient then $A$ can be 3-refuted.

Here is a small (and easy) part of the proof of Proposition 21. The rest of the proof of the proposition is left to the next sub-section.

**Lemma 22.** Let $A = \gamma_{W_1} \cdots \gamma_{W_n} \in \Gamma^*$. If there is an index $1 \leq i < n$ such that $W_i W_{i+1}$ is an element of $B$ (the set of sub-words of the relations) then there is an element $B$ that 1-refutes $A$. Moreover, we have that $|B| < |A|$ and $\eta(A) = \eta(B)$.

**Proof.** Construct $B$ from $A$ by replacing the two consecutive generators $\gamma_{W_{i-1}} \gamma_{W_i}$ with the generator $\gamma_{W_{i-1} W_i}$. Namely,

$$B = \gamma_{W_1} \cdots \gamma_{W_{i-2}} \gamma_{W_{i-1} W_i} \gamma_{W_{i+1}} \cdots \gamma_{W_n}$$

Then, $|B| < |A|$ implying that $B \prec A$. Clearly we have that $\eta(B) = \eta(A)$ and so $\pi(A) = \pi(B)$. We finish by showing that $A$ and $B$ are 1-fellow-travellers. Recall that $A(j)$ denotes the prefix of $A$ of length $j$, that $\pi(C)$ is the element in $S$ presented by $C$, and that $d_T$ is the induced metric on the Cayley graph. Since $\pi(B(j)) = \pi(A(j))$ for all $1 \leq j \leq \ell - 2$ and $\pi(B(j)) = \pi(A(j) \gamma_{W_{j+1}})$ for all $\ell - 1 \leq j \leq n - 1$ we get that $d_T(A(j), B(j)) \leq 1$ for all $1 \leq j \leq n$ so $B$ and $A$ are 1-fellow-travellers.

The next lemma shows how to check for inefficiency.

**Lemma 23.** Let $A \in \Gamma^*$. Then, $A$ is inefficient if and only if $\eta(A)$ contains a sub-word $W \in \mathcal{R}$ such that $\|W\| > \|c(W)\|$.

**Proof.** The ‘if’ part follows from the definition of $\kappa_A$. We prove the ‘only if’ part. Suppose $A = \gamma_{W_1} \cdots \gamma_{W_n}$. Suppose further that $\eta(A) = V_1 L V_2$ where $L \in \mathcal{R}$ and $L = R$ is a defining relation with the property that $\|L\| > \|R\|$ ($R$ is the complement of $L$). By the (†) condition we have that $\|L\| \geq 4$. Let $k$ be the smallest index such that $V_1$ is a prefix of $W_1 W_2 \cdots W_k$ (which is equal to $\eta(A(k))$). We are done if $V_1 = W_1 W_2 \cdots W_k$ because then $W_{k+1}$ is a prefix of $L$ and thus $[\kappa_A]_{k+1} = 1$. Otherwise, let $W_k = W'_k W''_k$ where $V_1 = W_1 W_2 \cdots W_{k-1} W'_k$ and both $W'_k$ and $W''_k$ are not empty. Decompose $L$ as $L = W''_k T$. If $W_{k+1}$ is a prefix of $T$ then as above we get that $[\kappa_A]_{k+1} = 1$. Thus, we can assume that $T$ is a prefix of $W_{k+1}$. In this case, $\|W''_k\| \leq 1$ (follows from Observation 6) because $W''_k$ is a subword of $W_k$ and a prefix of $L$ and thus $\|T\| \geq 3$. Now, $T$ is a suffix of $L$ so we must have that $T = W_{k+1}$ (follows from Observation 5 since $L$ is the unique element in $\mathcal{R}$ that $T$ is its sub-word). This implies the lemma since we now have that $[\kappa_A]_{k+1} = 1$.

The following corollaries are immediate from Lemma 23.
Corollary 24. Suppose $A, B \in \Gamma^*$ with $\eta(A) = \eta(B)$. Then, $A$ is efficient if and only if $B$ is efficient.

Proof. By Lemma 23 is enough to know $\eta(A)$ to know if $A$ is efficient.

Corollary 25. Suppose $A, B \in \Gamma^*$ are efficient and $M$ is a $(\mu \xi, \sigma)$-thin diagram with $\eta(A)$ the label of $\mu$ and $\eta(B)$ the label of $\sigma$. Suppose further that $\xi$ is empty or is labelled with an element of $X$. Then, condition (†) of Lemma 17 holds for $M$.

Proof. Assume there is a boundary region $D$ in $M$ with boundary $\delta \rho^{-1}$ such that $\delta$ and $\rho$ have positive labels $V_\delta$ and $V_\rho$ and $\|V_\delta\| > \|V_\rho\|$ (recall that $V_\delta$ and $V_\rho$ are defining words and are complements of each other). If $\delta$ is a sub-path of $\mu$ then it follows from Lemma 23 that $A$ is inefficient but that would contradict the assumption. Hence, $\delta$ is not a sub-path of $\mu$. Similarly, $\delta$ is not a sub-path of $\sigma$. This shows that condition (†) holds.

Corollary 26. Let $A \in \Gamma^*$ and let $M$ be a semi-group diagram with a boundary path $\mu$ labelled by $\eta(A)$. Assume there is a boundary region $D$ in $M$ with boundary $\rho \delta^{-1}$ such that: (1) $\rho$ and $\delta$ have positive labels $V_\rho$ and $V_\delta$; (2) $\rho$ is the outer boundary of $D$ and is a sub-path of $\mu$; (3) $\|V_\delta\| \leq 3$. Then, $A$ is inefficient.

Proof. This follows from Lemma 23 since $V_\rho = V_\delta$ is a defining relation and by condition (†) of the main theorem we have that $\|V_\rho\| \geq 4$ so $V_\rho$ is a sub-word of $\eta(A)$ with $\|V_\rho\| > \|V_\delta\| = \|c(V_\rho)\|$.

We continue with a lemma which makes the connection to the diagrams of $S$.

Lemma 27. Let $A, B \in \Gamma^*$ be efficient. Suppose there is an element $x \in X \cup \{\varepsilon\}$ such that $\eta(A)x = \eta(B)$ in $S$. Suppose further that $M$ is a minimal diagram with boundary $\mu \xi \sigma^{-1}$ such that $\mu$ is labelled by $\eta(A)$, $\xi$ is labelled by $x$, and $\sigma$ is labelled by $\eta(B)$. Then, $M$ is a $(\mu \xi, \sigma)$-thin diagram.

Proof. Assume by contradiction that $M$ is not $(\mu \xi, \sigma)$-thin. By Proposition 14 we have, without loss of generality, a region $D$ in $M$ with the following properties:

1. $\partial D = \rho \delta^{-1}$ where $\rho$ and $\delta$ have positive labels.

2. $\rho$ is a sub-path of $\mu$ and $\rho$ does not intersect $\sigma$ (i.e., $\rho$ is not a suffix or prefix of $\mu \xi$).

3. $\delta$ is the inner boundary and $N_\delta^D = 3$ (the number of neighbors of $D$ along $\delta$ is 3).
By the second part we have that $\rho$, the outer boundary of $D$, is a sub-path of $\mu$. Thus, by Corollary 26 we get that $A$ is inefficient which is a contradiction to the assumption on $A$. 

Using the above lemma we can prove the following technical proposition. Similar result in the context of groups is straightforward. It turns out that for semi-groups one must work a little bit harder.

**Proposition 28.** Let $A, B \in \Gamma^*$ be efficient. Suppose that $\pi(A) = \pi(B)$ or there is an element $x \in X$ such that $\pi(A\gamma_x) = \pi(B)$ in $S$. Suppose further that $B$ is a geodesic. Then, one of the following options hold:

1. $A$ and $B$ are 3-fellow-travellers.
2. There is an element $C \in \Gamma^*$ such that $\pi(A) = \pi(C)$, $|C| < |A|$, and $C$ and $A$ are 2-fellow-travellers (namely, $C$ 2-refute $A$).

Before we give the (rather long) proof of the proposition we show that it implies that the conditions of Theorem 11 hold for the order “$\prec$”. In particular, $S$ is an automatic semi-group.

**Corollary 29.** Conditions (R) and (FT) of Theorem 11 hold for the order “$\prec$”. In particular, $S$ is an automatic semi-group.

**Proof.** First we prove that condition (R) holds. Let $A \in \Gamma^*$ be an element that is not “$\prec$”-minimal element. By Proposition 21 we can assume that $A$ is efficient (or otherwise it can be 3-refuted by the proposition). Take $B \in \Gamma^*$ such that $\pi(A) = \pi(B)$ and $B$ is “$\prec$”-minimal. By the same proposition we get that $B$ is efficient. By minimality, $B$ is a geodesic. By Proposition 28 either $A$ and $B$ are 3-fellow-travellers and thus $A$ is 3-refuted by $B$ or there is the element $C$ that 2-refute $A$. This shows that condition (R) holds. Next we prove condition (FT). Let $k$ be a bound on the lengths of the elements in $B$ (recall the $B$ is a finite set). Take two “$\prec$”-minimal elements $A$ and $B$ such that $d_{\Gamma}(A, B) \leq 1$. As above, $A$ and $B$ are efficient and geodesics. If we have that $d_{\Gamma}(A, B) = 0$ then $\pi(A) = \pi(B)$ so by Proposition 28 we have that $A$ and $B$ are 3-fellow-travellers. Assume that $d_{\Gamma}(A, B) = 1$. Then, there is $\gamma_v \in \Gamma$ such that, switching $A$ and $B$ if necessary, $\pi(A\gamma_v) = \pi(B)$. We claim that $A$ and $B$ are 3k-fellow-travellers. Let $V = x_1x_2\cdots x_n$. Take elements $C_0, C_1, C_2, \ldots, C_{n-1}, C_n$ in $\Gamma^*$ such that each $C_j$ is “$\prec$”-minimal, $C_0 = A$ and $\pi(C_j) = \pi(C_{j-1}\gamma_{x_j})$. We have that $\pi(C_n) = \pi(A\gamma_{x_1}\cdots\gamma_{x_n}) = \pi(A\gamma_{x_1}\cdots x_n) = \pi(A\gamma_{v}) = \pi(B)$ and thus we take $C_n$ to be $B$. By Proposition 28 we get that $C_{j-1}$ and $C_j$ are 3-fellow-travellers (by “$\prec$”-minimality and Proposition 21 both are efficient). Consequently, $C_0$ and $C_n$ are 3n-fellow-travellers (follows from Lemma 8). Finally, because $n \leq k$ we get that $A$ and $B$ are 3k-fellow-travellers. 

The rest of the section is devoted to the proof of Proposition 28. An element $A \in \Gamma^*$ is called semi-geodesic if for any other $B \in \Gamma^*$ such that
\[ \eta(A) = \eta(B) \] (equality as elements of \( X^* \)) we have \(|A| \leq |B|\). Obviously, a geodesic is also a semi-geodesic but the converse may not be true.

**Lemma 30.** Let \( A, B \in \Gamma^* \) and suppose \( \eta(A)x = \eta(B) \) for some \( x \in X \cup \{ \varepsilon \} \). If \( B \) is semi-geodesic then either:

1. \( A \) and \( B \) are 1-fellow-travellers; or,
2. there is \( C \in \Gamma^* \) such that \(|C| < |A|\), \( \eta(A) = \eta(C) \), and \( A \) and \( C \) are 2-fellow-travellers.

**Proof.** By Lemma 22 we can assume that there are no two consecutive letters \( \gamma_{W_i}\gamma_{W_{i+1}} \) in \( A \) such that \( W_iW_{i+1} \in \mathcal{B} \) (if this does not hold then the second case hold for \( A \)). For every two indexes \( r \) and \( s \) there is an element \( V_{r,s} \in X^* \) such that either \( \eta(A(r))V_{r,s} = \eta(B(s)) \) or \( \eta(A(r))V_{r,s} \). Let \( D(r,s) \) denote the minimal \( k \) such that \( V_{r,s} \) decomposes as \( V_{r,s} = U_1U_2 \cdots U_k \) and \( U_i \in \mathcal{B} \) for \( 1 \leq i \leq k \) (it is zero if \( V_{r,s} = \varepsilon \)). If \( D(r,s) \leq 1 \) for all \( r \) then clearly \( A \) and \( B \) are 1-fellow-travellers. Otherwise, let \( r \) be the minimal index such \( D(r,r) \geq 2 \). Notice that \(|\eta(A(r))| < |\eta(B(r))|\) by the fact that \( B \) is a semi-geodesic. (If not, take a minimal index \( s \geq r \) such that \(|\eta(B(s))| \geq |\eta(A(r))|\). Then, \( s-r \geq 2 \) and \( D(r,s) \leq 1 \) so we can replace the prefix \( B(s) \) of \( B \) with \( \gamma_{U}A(r) \) for some \( U \in \mathcal{B} \) and thus reduce the length of \( B \) which is impossible if \( B \) is a semi-geodesic.) Next, take a maximal index \( s \) such that \(|\eta(B(s))| \leq |\eta(A(r))|\) so \( r-s \geq 2 \) and \( D(r,s) \leq 1 \). Let \( U = V_{r,s} \). By the fact that \( D(r,s) \leq 1 \) we have that \( U = \varepsilon \) or \( U \in \mathcal{B} \). Suppose \( A = A(r)T \) and let \( C \in \Gamma^* \) such that \( C = B(s)T \) if \( U = \varepsilon \) and \( C = B(s)\gamma_{U}T \) otherwise. Then, \( \eta(C) = \eta(A) \). Also, \(|C| < |A|\) by the following computation:

\[
\begin{align*}
|C| &\leq |B(s)| + 1 + |T| \\
&= s + 1 + |T| \\
&< r + |T| = |A(r)| + |T| = |A|
\end{align*}
\]

We claim that \( A \) and \( C \) are 2-fellow-travellers. For simplicity we will prove this under the assumption that \( U \neq \varepsilon \). By minimality of \( r \) we have that \( A(s) \) and \( C(s) \) are 1-fellow-travellers. By the construction of \( C \) we have that, \( \eta(C(s+1)) = \eta(A(r)) \). Hence, it is enough to show that \( r-s \leq 3 \). Let \( A(r) = A(s)Q \) and suppose \( A = \gamma_{W_1} \cdots \gamma_{W_n} \) then \( \eta(Q) = W_{s+1}W_{s+2} \cdots W_r \). We have that \( D(s, s) \leq 1 \) and \( D(r, s) \leq 1 \) so we have decomposition of \( \eta(Q) \) into \( U_1U_2 \cdots U_k \) where \( U_i \in \mathcal{B} \) for \( 1 \leq i \leq k \) and \( k \leq 2 \). Consequently, by the pigeonhole principle if \( r-s > 3 \) there will be an index \( s+1 \leq i < r \) such that \( W_iW_{i+1} \) is an element of \( \mathcal{B} \). This contradicts our assumption at the beginning of the proof.

We say that \( A = \gamma_{W_1} \cdots \gamma_{W_n} \) in \( \Gamma^* \) is left-greedy if for every \( 1 \leq i < n \) we have that \( W_ix \notin \mathcal{B} \) where \( x \) is the first letter of \( W_{i+1} \). Clearly, for any \( A \in \Gamma^* \) there is only one left-greedy representative \( A' \) such that \( \eta(A) = \eta(A') \).
Lemma 31. For every $A \in \Gamma^*$ there is a unique element $A' \in \Gamma^*$ such that $A'$ is semi-geodesic, left-greedy, and $\eta(A) = \eta(A')$.

Proof. Let $n$ be the length of a semi-geodesic $B$ such that $\eta(A) = \eta(B)$. We prove the lemma by induction on $n$. If $n = 1$ then there is nothing to prove. Suppose the lemma holds for $n - 1$. Let $B = \gamma_{u_1} \cdots \gamma_{u_n}$ be a semi-geodesic as above. Consider the decomposition $U_1U_2 = V_1V_2$ where $V_1$ is the maximal prefix of $U_1U_2$ such that $V_1 \in \mathcal{B}$. Set $C = \gamma_{V_2} \gamma_{U_3} \cdots \gamma_{U_n}$. Then, $C$ is semi-geodesic and $|C| = n - 1$. By induction, there is some $C'$ that is semi-geodesic, left-greedy, and $\eta(C) = \eta(C')$. Thus, $A' = \gamma_{V_1}C'$ is semi-geodesic, left-greedy, and $\eta(A) = \eta(A')$, as needed.

By the uniqueness of the left-greedy representative, it follows from the above lemma that if $A \in \Gamma^*$ is left-greedy then it is necessarily semi-geodesic. This fact is used in next lemma to check that a given element is semi-geodesic. It is useful to remember that $A = \gamma_{W_1} \cdots \gamma_{W_n}$ is left-greedy if and only $\gamma_{W_i} \gamma_{W_{i+1}}$ is left-greedy for all $1 \leq i < n$.

Lemma 32. Let $A_0, \ldots, A_n$ and $B_1, \ldots, B_n$ be left-greedy elements of $\Gamma^*$. Assume the following:

1. If $\gamma_V$ is the first letter of $B_i$ then $||V|| \geq 3$ and $V$ is a prefix of some element in $\mathcal{R}$.

2. If $\gamma_V$ is the last letter of $B_i$ then $||V|| \geq 3$ and $V$ is a suffix of some element in $\mathcal{R}$.

Then, the element $C = A_0B_1A_1 \cdots B_nA_n$ is semi-geodesic.

Proof. The proof is mostly routine. The conditions above say that if $\gamma_{W_j} \gamma_{W_{j+1}}$ (where $1 \leq j < |C|$) are two consecutive letters in $C$ which are not left-greedy then there is an index $i$ such that one of the following holds:

1. $\gamma_{W_j}$ is the last letter of $B_i$ and $\gamma_{W_{j+1}}$ is the first letter of $A_i$.

2. $\gamma_{W_j}$ is the last letter of $A_i$ and $\gamma_{W_{j+1}}$ is the first letter of $B_{i+1}$.

The first case is impossible since we have that $||W_j|| \geq 3$ and $W_j$ is a suffix of some unique element in $\mathcal{R}$ (uniqueness follows from Observation 5). Hence, $W_j$ is not a proper prefix of any element in $\mathcal{B}$ and thus $\gamma_{W_j} \gamma_{W_{j+1}}$ must be left-greedy. So, we are left with the second case. Since only the second case is possible it follows that if $\gamma_{W_j} \gamma_{W_{j+1}}$ is not left greedy and also $\gamma_{W_k} \gamma_{W_{k+1}}$ is not left greedy for some $1 \leq j < k < n$ then $k \geq j + 2$ (i.e., there is no overlap between the two pairs and moreover there is a gap of at least one letter between them). Consequently, if we can show that we can replace each pair of consecutive letters $\gamma_{W_j} \gamma_{W_{j+1}}$ with a left-greedy pair $\gamma_{U_j} \gamma_{U_{j+1}}$ such that $W_jW_{j+1} = U_jU_{j+1}$ and also $U_{j+1} \gamma_{W_{j+2}}$ is left-greedy then we can
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‘fix’ $C$ so it becomes left-greedy and thus we would show that $C$ is semi-
geo
desic. So, suppose we fix $\gamma W_j \gamma W_{j+1}$ into a left-greedy $\gamma U_j \gamma U_{j+1}$
such that $W_j W_{j+1} = U_j U_{j+1}$. This induces a decomposition of $W_{j+1} = W'_j W''_{j+1}$
where $W''_{j+1} = U_{j+1}$. Since $\|W_{j+1}\| \geq 3$ we have by Observation 6 that the
piece-length of $W'_{j+1}$ is at most one and thus $\|U_{j+1}\| \geq 2$. Therefore by
Observation 5 there are unique $V \in B$ and $R \in R$ such that

$U_j V$ is a
suffix of $R$. Because $U_{j+1}$ is a suffix of $W_{j+1}$ we get that also $W_{2} V$ is a
suffix of $R$. Now, $\gamma W_{j+1} \gamma W_{j+2}$ is left-greedy so also $\gamma U_{j+1} \gamma W_{j+2}$ is left greedy. □

Let $M$ be a $(\mu \xi, \sigma)$-thin diagram where $\xi$ is labelled by an element of
$X \cup \{ \varepsilon \}$. A fundamental decomposition of $M$ is a decomposition
$M = \rho_0 \cup M_1 \cup \rho_1 \cup \cdots \cup \rho_{k-1} \cup M_k \cup \rho_k$ such that $M_1, M_2, \ldots, M_k$ are the connected
components of the closure of the interior of $M$ and $\rho_0, \rho_1, \ldots, \rho_k$ are the
paths in the closure of $M \setminus \left( \bigcup_{j=1}^{k} M_k \right)$. The path $\rho_i$ connects $M_i$ to $M_{i+1}$
for $1 \leq i < k$. The paths $\rho_0$ and $\rho_k$ may be empty (and in this case we
think on them as being a single vertex) or otherwise only intersects $M_1$ and
$M_k$, respectively. See Figure 9. The fundamental decomposition induces the
definition of two elements as defined next.

![Figure 9: Illustration of fundamental decomposition](image)

**Definition 33.** Let $M$ be a $(\mu \xi, \sigma)$-thin diagram where $\xi$ is labelled by
an element of $X \cup \{ \varepsilon \}$ and consider a fundamental decomposition
$M = \rho_0 \cup M_1 \cup \rho_1 \cup \cdots \cup \rho_{k-1} \cup M_k \cup \rho_k$. We define two elements, $C_\mu$ and $C_\sigma$, of
$\Gamma^*$. $C_\mu$ is defined to be $C_\mu = C_{\rho_0} C_{M_1}^{\rho_1} C_{\rho_1} \cdots C_{\rho_{k-1}} C_{M_k}^{\rho_k}$ where:

1. $C_{\rho_j}$ is the left-greedy semi-geodesic element such that $\eta(C_{\rho_j})$ is the
label of $\rho_j \cap \mu$

2. Suppose that in $M_j$ the regions $D^j_1, D^j_2, \ldots, D^j_{N_j}$ have the property
that $\partial D^j_i \cap \mu$ contains an edge for $1 \leq i \leq N_j$. We assume that the
indexing of the regions corresponds to the order they intersects with
$\mu$. We define $C_{M_j}^{\rho_j}$ to be the element $\gamma_{V_1} \cdots \gamma_{V_{N_j}}$ where $V_i$ is the label
of $\partial D^j_i \cap \mu$.

$C_\sigma$ is defined similarly by replacing $\mu$ with $\sigma$.

In the next lemma we analyze the properties of the elements $C_\mu$ and $C_\sigma$
from Definition 33. We will use the results of Lemma 17 for the proof.
Lemma 34. Let the notation be as in Definition 33 and let 1 ≤ i ≤ k. Suppose that $C_{\mu}$ and $C_{\sigma}$ are efficient. Then:

1. $|C_{\mu}^{\nu}| = |C_{\mu}^{\sigma}|$. Moreover, suppose that $A$ is a prefix of $C_{\mu}^{\nu}$ of length $n$ and $B$ is a prefix of $C_{\mu}^{\sigma}$ of length $n$. Suppose further that $\delta$ and $\rho$ is a sub-path of $\partial M_i$ which are labelled by $\eta(A)$ and $\eta(B)$, respectively. Then, the terminal vertices of $\delta$ and $\rho$ belong to the boundary of the same region in $M_i$.

2. Suppose $i \neq k$. Then, if $\gamma_V$ is the first (resp., the last) letter of $C_{\mu}^{\nu}$ or $C_{\mu}^{\sigma}$ then $|V| \geq 3$. For $i = k$ the assertion holds for the first letter and hold for the last letter if $\rho_k$ is not empty.

3. $C_{\mu}^{\nu}$ and $C_{\mu}^{\sigma}$ are left-greedy.

Proof. We use the notation of Definition 33. Since $C_{\mu}$ and $C_{\sigma}$ are efficient it follows from Corollary 25 that condition (1) of Lemma 17 holds and thus we can use its conclusions. By Part 1 of Lemma 17 each region $D$ in $M$ has the property that $\partial D \cap \mu$ and $\partial D \cap \sigma$ contain an edge. This shows that if $D_1, \ldots, D_n$ are the regions of $M_i$ then by construction $C_{\mu}^{\nu} = \gamma_{V_1} \cdots \gamma_{V_n}$ and $V_i$ is the label of $\partial D_i \cap \mu$ for $1 \leq j \leq n$. Consequently, $|C_{\mu}^{\nu}| = n$ and similarly $|C_{\mu}^{\sigma}| = n$ so we get the first part of the lemma (the 'moreover' part follows along the same lines). By Part 3 of Lemma 17 we get that $|V_i| \geq 3$ and $|V_n| \geq 3$ when $i \neq k$. For $i = k$ the same holds for $V_1$ and it holds for $V_n$ if $\partial D_n \cap \mu$ does not contain $\xi$ which is the case if $\rho_k$ is not empty. This proves the second part of the lemma. Finally, by Part 4 of Lemma 17 we get that $\gamma_{V_j} \gamma_{V_{j+1}}$ is left greedy for $1 \leq j < k$ so consequently $C_{\mu}^{\nu}$ is left greedy. This proves the last part of the lemma.

Corollary 35. Let the notation be as in Definition 33 and suppose that $C_{\mu}$ and $C_{\sigma}$ are efficient. Then, $C_{\mu}$ and $C_{\sigma}$ are semi-geodesics.

Proof. We prove the corollary for $C_{\mu}$; the proof for $C_{\sigma}$ is similar. First notice that by the definition of $C_{\mu}^{\nu}$ above we get that it starts with a letter $\gamma_{V_1}$ such that $V_1$ is a prefix of some element in $R$. Also, $C_{\mu}^{\nu}$ ends with a letter $\gamma_{V_n}$ such that $V_n$ is a suffix of some element in $R$. It follows from Lemma 34 that the conditions of Lemma 32 holds when we set $A_i = C_{\rho_i}$ and $B_i = C_{\mu}^{\nu}$. Thus, $C_{\mu} = A_0 B_1 A_1 \cdots B_n A_n$ is semi-geodesic.

Lemma 36. Let the notation be as in Definition 33 and suppose that $C_{\mu}$ and $C_{\sigma}$ are efficient. Then, $C_{\mu}$ and $C_{\sigma}$ are 1-fellow-travellers.

Proof. We need to show that $d_1(C_{\mu}(n), C_{\sigma}(n)) \leq 1$ for all $n$. This follows routinely from the construction of $C_{\mu}$ and $C_{\sigma}$. By Part 1 of Lemma 34 we have that $|C_{\mu}^{\nu}| = |C_{\mu}^{\sigma}|$ for all $j$. Thus, again by Part 1 of Lemma 34 the sub-paths of $\mu$ and $\sigma$ labelled by $\eta(C_{\mu}(n))$ and $\eta(C_{\sigma}(n))$, respectively, terminate at the same vertex (if it belong to some $\rho_j$) or at vertices that
belong to the boundary of the same region. Consequently, we have \( V \in \mathcal{B} \cup \{ \varepsilon \} \) such that either \( \eta(C_\mu(n))V = \eta(C_\sigma(n)) \) or \( \eta(C_\mu(n)) = \eta(C_\sigma(n))V \).

So, \( \pi(C_\mu(n)\gamma_V) = \pi(C_\sigma(n)) \) or \( \pi(C_\mu(n)) = \pi(C_\sigma(n)\gamma_V) \). Consequently, \( d_\Gamma(C_\mu(n), C_\sigma(n)) \leq 1 \).

**Proof of Proposition 28.** Suppose we are given two element \( A, B \in \Gamma^* \) such that the conditions of Proposition 28 hold. Let \( M \) be a van Kampen diagram with boundary path \( \mu\xi\sigma^{-1} \) such that \( \mu \) is labelled by \( \eta(A) \), \( \xi \) is empty or is labelled by some \( x \in X \), and \( \sigma \) is labelled by \( \eta(B) \). Then, \( M \) is \( (\mu, \sigma) \)-thin diagram by Lemma 27. Let \( C_\mu \) and \( C_\sigma \) be the elements induced from the fundamental decomposition (Definition 33) of \( M \) and for which \( \eta(A) = \eta(C_\mu) \) and \( \eta(B) = \eta(C_\sigma) \). By Corollary 24 we have that \( C_\mu \) and \( C_\sigma \) are efficient and by Corollary 35 we have that \( C_\mu \) and \( C_\sigma \) are semi-geodesics. Thus, by Lemma 30 we have that \( B \) and \( C_\sigma \) are 1-fellow-travellers (recall that \( B \) is geodesic and \( C_\sigma \) is semi-geodesic). By the same lemma, either \( A \) and \( C_\mu \) are 1-fellow-travellers or we have \( A' \) that 2-refutes \( A \) so the proposition is satisfied. Hence, we can assume that \( A \) and \( C_\mu \) are 1-fellow-travellers. Using Lemma 36 we have that \( C_\mu \) and \( C_\sigma \) are 1-fellow-travellers and thus \( A \) and \( B \) are 3-fellow-travellers (using Lemma 8) which proves the proposition. \( \square \)

### 6.1 Refuting Inefficient Elements

In this sub-section we prove Proposition 21. Recall that by Lemma 22 if for \( A = \gamma_{w_1} \cdots \gamma_{w_n} \in \Gamma^* \) have two consecutive letters \( \gamma_{w_i}\gamma_{w_{i+1}} \) such that \( w_iw_{i+1} \in \mathcal{B} \) then we can 3-refute \( A \). This leads to the following definition:

**Definition 37** (Admissible). We say that \( A = \gamma_{w_1} \cdots \gamma_{w_n} \in \Gamma^* \) is admissible if for all \( 1 \leq i < n \) we have \( w_iw_{i+1} \notin \mathcal{B} \).

So, by this terminology Lemma 22 say that Proposition 21 holds for all non admissible elements. Thus, the main difficulty is to prove Proposition 21 for admissible elements. We introduce the following definition, which allows us to prove Proposition 21 by induction.

**Definition 38** (\( \ell \)-Pacing Pair). Let \( A \) and \( B \) be in \( \Gamma^* \) such that \( \kappa_A \neq \varepsilon \) and let \( 1 \leq \ell \leq |A| \). We say that \( (A, B) \) is an \( \ell \)-pacing pair if the following conditions hold:

\begin{enumerate}
  \item [C1.] \( A \) is admissible and \( |\kappa_A|_\ell = 1 \).
  \item [C2.] \( \pi(A) = \pi(B) \) and \( |A| = |B| \).
  \item [C3.] There is an index \( 1 \leq j \leq \ell \) such that:
    \begin{enumerate}
      \item \( d_\Gamma(A(i), B(i)) = 0 \) for all \( i < j \) and \( i > \ell \).
      \item \( d_\Gamma(A(\ell), B(\ell)) \leq 1 \).
      \item \( d_\Gamma(A(i), B(i)) \leq 2 \) for all \( j \leq i \leq \ell - 1 \).
    \end{enumerate}
\end{enumerate}
Let Lemma 41.

**Proof.** We use conditions (C2) and (C3) of Definition [38]. By (C2) we have that \( \pi(A) = \pi(B) \). By (C3) we have that \( A \) and \( B \) are 2-fellow-travellers. If \( B \) is admissible then by (C4) we have that \( \kappa_B \) precedes \( \kappa_A \) in lexicographical order so \( B < A \). Thus, \( B \) 2-refutes \( A \). On the other hand, if \( B \) is not admissible then by Lemma [22] there is an element \( C \) such that \( |C| < |B| \), \( \pi(C) = \pi(B) \), and \( C \) and \( B \) are 1-fellow-travellers (since \( C \) 1-refutes \( B \)). This implies that \( C \) 3-refutes \( A \) because (i) \( |C| < |B| \) and thus \( C < A \): (ii) \( \pi(C) = \pi(B) = \pi(A) \); (iii) \( C \) and \( A \) are 3-fellow-travellers (this follows from Lemma [8] since \( A \) and \( B \) are 2-fellow-travellers and \( B \) and \( C \) are 1-fellow-travellers).

We complete the proof of Proposition [21] by proving the following proposition:

**Proposition 40.** Let \( A \in \Gamma^* \) be an admissible but not efficient. Then, there is \( B \in \Gamma^* \) such that \( (A, B) \) is a pacing pair.

The proof is broken into several lemmas. We begin by deriving some technical information in the situation where \( [\kappa_A]_\ell = 1 \).

**Lemma 41.** Let \( A = \gamma_{W_1} \cdots \gamma_{W_n} \in \Gamma^* \) be admissible. Suppose there is an index \( 1 \leq \ell \leq n \) where \( W_{\ell-1} \) has a decomposition \( W_{\ell-1} = W_{\ell-1}' W_{\ell-1}'' \) such that \( W_{\ell-1}' W_{\ell-1}'' \) is in \( \mathcal{B} \). Then, \( \|W_{\ell-1}' \| \leq 1 \). Similarly, if \( W_{\ell} \) has a decomposition \( W_{\ell} = W_{\ell}' W_{\ell}'' \) such that \( W_{\ell-1}' W_{\ell}'' \) is in \( \mathcal{B} \) then \( \|W_{\ell}'' \| \leq 1 \).

**Proof.** We prove the first case; the other case is similar. The lemma follows trivially if \( W_{\ell-1}'' \) is the empty word so assume \( W_{\ell-1}'' \neq \varepsilon \). Take \( V_1, V_2 \) and \( U_1, U_2 \) such that \( V_1 W_{\ell-1}' W_{\ell-1}'' V_2 \in \mathcal{R} \) and \( U_1 W_{\ell-1}' W_{\ell-1}'' U_2 \in \mathcal{R} \). Since by admissibility \( W_{\ell-1}' W_{\ell-1}'' W_{\ell} \) is not in \( \mathcal{B} \) we get that \( U_1 \neq V_1 W_{\ell-1}' \). Hence, \( W_{\ell-1}'' \) is a piece and consequently, \( \|W_{\ell-1}'' \| = 1 \). \( \square \)
Lemma 42. Let $A = \gamma_{W_1} \cdots \gamma_{W_n} \in \Gamma^*$ be admissible and suppose that $|\kappa_A|_{\ell} = 1$ for some $1 \leq \ell \leq n$. Let $W_{\ell-1} = W_{\ell-1}''W_{\ell-1}'$ and $W_{\ell+1} = W_{\ell+1}''W_{\ell+1}'$ be the decompositions guaranteed by the definition of $\kappa_A$. Then, $\|W_{\ell-1}''W_{\ell}W_{\ell+1}'\| \geq 4$ and also $\|W_{\ell-1}'\| \cdot \|W_{\ell+1}'\| \leq 1$.

Proof. By the definition of $\kappa_A$ and the $K_3$ condition we have that

$$\|W_{\ell-1}''W_{\ell}W_{\ell+1}'\| > \|c(W_{\ell-1}'W_{\ell}W_{\ell+1}')\| \geq 3$$

Hence, $\|W_{\ell-1}''W_{\ell}W_{\ell+1}'\| \geq 4$. By admissibility and Lemma 41 we have that $\|W_{\ell-1}'\| \leq 1$ and that $\|W_{\ell+1}'\| \leq 1$.

Lemma 43. Let $A = \gamma_{W_1} \cdots \gamma_{W_n} \in \Gamma^*$ be admissible and suppose that $|\kappa_A|_{\ell} = 1$ for some $1 \leq \ell \leq n$. Then there are unique decompositions $W_{\ell-1} = W_{\ell-1}''W_{\ell-1}'$ and $W_{\ell+1} = W_{\ell+1}''W_{\ell+1}'$ such that $W_{\ell-1}'W_{\ell}W_{\ell+1}' \in \mathcal{R}$.

Proof. It follows by Lemma 42 that any possible decomposition as guaranteed in the definition of $\kappa_A$ we have $\|W_{\ell-1}''W_{\ell}W_{\ell+1}'\| \geq 4$, $\|W_{\ell-1}'\| \leq 1$, and $\|W_{\ell+1}'\| \leq 1$. Now, $\|W_{\ell-1}'\| + \|W_{\ell}'\| + \|W_{\ell+1}'\| \geq \|W_{\ell-1}'W_{\ell}W_{\ell+1}'\|$ which show that $\|W_{\ell}'\| \geq 2$. Consequently, we have by Observation 5 that there is a unique element $R \in \mathcal{R}$ that contains $W_{\ell}$ as a sub-word. This shows the uniqueness of the decompositions.

Construction 44 (below) is used in the inductive step of the proof of Proposition 40. After applying the construction to an inefficient element $A$ we get a new element $B$ that its $\kappa_B$ vector is null at some index where $\kappa_A$ is not null.

Construction 44 (Fixing $A$ at location $\ell$). Let $A = \gamma_{W_1} \cdots \gamma_{W_n} \in \Gamma^*$ be admissible. Suppose that for some index $i$, where $1 \leq i \leq n$, we have that $|\kappa_A|_i = 1$. We construct an element $B = \gamma_{U_1} \cdots \gamma_{U_n}$ in the following way which we denote as “fixing $A$ at location $\ell$”. It follows from Lemma 43 that since $|\kappa_A|_i = 1$ there are unique decompositions $W_{\ell-1} = W_{\ell-1}''W_{\ell-1}'$ and $W_{\ell+1} = W_{\ell+1}''W_{\ell+1}'$ such that $W_{\ell-1}''W_{\ell}W_{\ell+1}' \in \mathcal{R}$ and $\|W_{\ell-1}''W_{\ell}W_{\ell+1}'\| > \|c(W_{\ell-1}''W_{\ell}W_{\ell+1}')\|$. Then, $B$ is defined by setting: $U_{\ell-1} = W_{\ell-1}'$, $U_{\ell} = c(W_{\ell-1}'W_{\ell}W_{\ell+1}')$, $U_{\ell+1} = W_{\ell+1}'$, and $U_i = W_i$ for $i \neq \ell, \ell \pm 1$. If $\ell = 1$ or $\ell = n$ then $W_0$ or $W_{n+1}$, respectively, are undefined so we just ignore these indices.

Remark 45. Suppose $B$ is constructed from $A$ by fixing $A$ at location $\ell$ (Construction 44). Here are some immediate consequences of the construction which are relevant to the definition of pacing pairs (the notation of Construction 44 is used).

i) Clearly, $|A| = |B|$ and $\pi(A) = \pi(B)$.\[\]
ii) We have \( \pi(A(j)) = \pi(B(j)) \) for all \( 1 \leq j \leq n \) excluding \( j = \ell - 1 \) and \( j = \ell \) (recall that \( A(j) \) is the prefix of \( A \) of length \( j \)). This shows that \( d_{\ell}(A(j), B(j)) = 0 \) for all \( j < \ell - 1 \) and \( j > \ell \). In addition, if \( W''_{j+1} \) is not empty then the equality \( \pi(B(\ell - 1)\gamma(W''_{\ell - 1})) = \pi(A(\ell - 1)) \) (if \( W''_{\ell - 1} \) is empty then we have the equality \( \pi(B(\ell - 1)) = \pi(A(\ell - 1)) \)). Similarly, if \( W_{\ell + 1} \) is not empty then \( \pi(A(\ell)\gamma(W'_{\ell + 1})) = \pi(B(\ell)) \).

Therefore, \( d_{\ell}(A(j), B(j)) \leq 1 \) for \( j = \ell - 1 \) and \( j = \ell \). Consequently, \( A \) and \( B \) are \( 1 \)-fellow-travellers.

iii) Following the details of the construction we have that \( W_{\ell + 1} \) is equal to \( W'_{\ell + 1}U_{\ell + 1} \) and \( U_j = W_j \) for all \( j \geq \ell + 2 \). So, \( U_{\ell + 1} \) is a suffix of \( W_{\ell + 1} \). Also, \( \|W'_{\ell + 1}\| \leq 1 \) by Lemma 44 so \( \|W_{\ell + 1}\| \leq 1 + \|U_{\ell + 1}\| \) and we get that \( \|U_{\ell + 1}\| \geq \|W_{\ell + 1}\| - 1 \).

iv) We have \( \|U_\ell\| < \|\epsilon(U_\ell)\| \). This follows since \( U_\ell = c(W''_{\ell - 1}W_\ell W'_{\ell + 1}) \) so \( c(U_\ell) = W''_{\ell - 1}W_\ell W'_{\ell + 1} \). Thus, \( \|U_\ell\| < \|W''_{\ell - 1}W_\ell W'_{\ell + 1}\| = \|\epsilon(U_\ell)\| \).

After applying Construction 44 to an inefficient element \( A \in \Gamma^* \) we get an element \( B \in \Gamma^* \) such that the pair \((A, B)\) is almost a pacing pair. Specifically, out of the five conditions in the definition of a pacing pair (Definition 38) the first four conditions always hold. This is the content of the next lemma. Afterward, we give three special situation where the last condition (the fifth one) also hold (Lemma 49).

**Lemma 46.** Suppose that \( A \in \Gamma^* \) is admissible and \( [\kappa_A]_{\ell} = 1 \) for some \( 1 \leq \ell \leq |A| \). Suppose further that we construct \( B \) by fixing \( A \) at location \( \ell \). The first four properties of an \( \ell \)-pacing pair hold for the pair \((A, B)\).

**Proof.** We check the first four conditions one by one (see Definition 38).

Condition C1: By our assumption \( A \) is admissible and \( [\kappa_A]_{\ell} = 1 \).

Condition C2: Follows from Remark 45 - part ii

Condition C3: Follows from Remark 45 - part ii by taking \( j = \ell - 1 \) if \( \ell > 1 \) or \( j = \ell \) if \( \ell = 1 \).

Condition C4: Follows from Remark 45 - part iii

**Lemma 47.** Let \( A \in \Gamma^* \) be admissible and suppose that \( [\kappa_A]_{\ell} = 1 \) for some \( 1 \leq \ell \leq |A| \). If we construct \( B \) by fixing \( A \) at location \( \ell \) (Construction 44) then \( [\kappa_B]_{\ell} = 0 \). In particular, \( [\kappa_B]_{\ell} < [\kappa_A]_{\ell} \).

**Proof.** Suppose \( B = \gamma_{U_1} \cdots \gamma_{U_n} \). Using the notation of Construction 44 we have that \( U_\ell \in \mathcal{R} \). If we assume by contradiction that \( [\kappa_B]_{\ell} = 1 \) then there are unique decompositions \( U_{\ell - 1} = U'_{\ell - 1}U''_{\ell - 1} \) and \( U_{\ell + 1} = U'_{\ell + 1}U''_{\ell + 1} \) such
that $U_{j-1}^\mu U_jU_j' \in \mathcal{R}$. Also, $\|U_{j-1}^\mu U_jU_j'\| > \|c(U_{j-1}^\mu U_jU_j')\|$. Since $U_j \in \mathcal{R}$ we get that $U_{j-1}^\mu U_jU_j' = U_j$ so $\|U_j\| > \|c(U_j)\|$. This leads to a contradiction since $\|U_j\| < \|c(U_j)\|$ by Remark 45 - part IV.

**Lemma 48.** Let $A \in \Gamma^*$ be admissible and suppose that $[\kappa_A]_\ell = 1$ for some $1 \leq \ell \leq |A|$. Suppose further that we construct $B$ by fixing $A$ at location $\ell$ (Construction 44) and $B$ is admissible. Then, $[\kappa_B]_j \leq [\kappa_A]_j$ for all $j < \ell - 1$.

**Proof.** Let $A = \gamma_{W_1} \cdots \gamma_{W_n}$ and $B = \gamma_{U_1} \cdots \gamma_{U_n}$. Since $U_j = W_j$ for all $j < \ell - 1$ we get that $[\kappa_B]_j \leq [\kappa_A]_j$ for all $j < \ell - 2$ (actually there is an equality). This is also true for $j = \ell - 2$ by the following reasons. We need to show that if $[\kappa_B]_{\ell - 2} = 1$ then also $[\kappa_A]_{\ell - 2} = 1$. So assume that $[\kappa_B]_{\ell - 2} = 1$. By the definition of the auxiliary vectors, there are decompositions $U_{\ell - 3} = U_{\ell - 3}^\mu U_{\ell - 3}$ and $U_{\ell - 1} = U_{\ell - 1}^\mu U_{\ell - 1}$ such that $\|U_{\ell - 3}^\mu U_{\ell - 2} U_{\ell - 1}\| > \|c(U_{\ell - 3}^\mu U_{\ell - 2} U_{\ell - 1})\|$. But, $U_{\ell - 1}$ is a prefix of $W_{\ell - 1}$ and so there similar decompositions for $W_{\ell - 3}$ and $W_{\ell - 2}$ and thus $[\kappa_A]_{\ell - 2} = 1$.

**Lemma 49.** Suppose that $A \in \Gamma^*$ is admissible and $[\kappa_A]_\ell = 1$ for some $1 \leq \ell \leq |A|$. Suppose further that we construct $B$ by fixing $A$ at location $\ell$. If we have that one of the following hold then $(A, B)$ is an $\ell$-pacing pair:

(a) $B$ is not admissible.

(b) $\ell = 1$.

(c) $B$ is admissible and $[\kappa_B]_{\ell - 1} \leq [\kappa_A]_{\ell - 1}$.

**Proof.** In any of these instances the pair $(A, B)$ is a pacing pair since the last condition (C5) of Definition 38 hold (the other conditions hold by Lemma 46). This follows trivially for (a), from Lemma 47 for (b), and from Lemma 47 and Lemma 48 for (c).

We are now ready to complete the proof of Proposition 40.

**Proof of Proposition 40.** Suppose we are given some admissible element $A = \gamma_{W_1} \cdots \gamma_{W_n} \in \Gamma^*$ and an index $\ell$ such that $[\kappa_A]_\ell = 1$ (where, $1 \leq \ell \leq |A|$).

We show that there is an element $A = \gamma_{U_1} \cdots \gamma_{U_n} \in \Gamma^*$ such that $(A, B)$ is an $\ell$-pacing pair. We prove by induction on $\ell$. If $\ell = 1$ then we can construct $B$ by fixing $A$ at location 1 (Construction 44) and by Lemma 49 (b) we get that $(A, B)$ is a pacing pair.

Suppose $\ell > 1$ and assume that the lemma holds for $\ell - 1$. We construct $B$ by fixing $A$ at location $\ell$. If $B$ is not admissible we are done by Lemma 49 (a). Hence we can assume that $B$ is admissible. By Lemma 47 we have that $[\kappa_B]_\ell < [\kappa_A]_\ell$ and we are done if $[\kappa_B]_{\ell - 1} \leq [\kappa_A]_{\ell - 1}$ by Lemma 49 (c).

Hence, we may assume that $[\kappa_B]_{\ell - 1} > [\kappa_A]_{\ell - 1}$ so necessarily $[\kappa_B]_{\ell - 1} = 1$ and $[\kappa_A]_{\ell - 1} = 0$. Using induction hypothesis on $B$ (which is admissible and $[\kappa_B]_{\ell - 1} = 1$) we get an element $C = \gamma_{V_1} \cdots \gamma_{V_n}$ such that $(B, C)$ is an
\( (\ell - 1) \)-pacing pair. We claim that \((A, C)\) is an \(\ell\)-pacing pair. We verify this by checking the conditions of the definition of an \(\ell\)-pacing pair (Definition 38), one by one.

**Condition C1.** By assumption, \(A\) is admissible and \([\kappa_A]_{\ell} = 1\).

**Condition C2.** Recall that we constructed \(B\) by fixing \(A\) at location \(\ell\) and that \((B, C)\) is a pacing pair. Thus, we have the following two facts:

(i) By Remark 45 - part [i] we have the equalities \(|A| = |B|\) and \(\pi(A) = \pi(B)\); and,

(ii) By condition C2 of Definition 38 we have the equalities \(|B| = |C|\) and \(\pi(B) = \pi(C)\).

Consequently, \(|A| = |C|\) and \(\pi(A) = \pi(C)\).

**Condition C3.** \((B, C)\) is an \((\ell - 1)\)-pacing pair and thus by condition C3 there is some index \(1 \leq j \leq \ell - 1\) such that:

1. \(d_{\Gamma}(B(i), C(i)) = 0\) for all \(i < j\) and \(i > \ell - 1\).
2. \(d_{\Gamma}(B(\ell - 1), C(\ell - 1)) \leq 1\).
3. \(d_{\Gamma}(B(i), C(i)) \leq 2\) for all \(j \leq i \leq \ell - 2\).

\(B\) was constructed by fixing \(A\) at location \(\ell\) hence \(d_{\Gamma}(A(i), B(i)) = 0\) for all \(i < \ell - 1\) and \(i > \ell\) (Remark 45 - part [ii]). It follows therefore that \(d_{\Gamma}(A(i), C(i)) \leq d_{\Gamma}(A(i), B(i)) + d_{\Gamma}(B(i), C(i)) \leq 0 + 2 = 2\) for all \(j \leq i \leq \ell - 2\). Again by Remark 45 - part [ii] we have \(d_{\Gamma}(A(\ell - 1), B(\ell - 1)) \leq 1\) which implies that \(d_{\Gamma}(A(\ell - 1), C(\ell - 1)) \leq d_{\Gamma}(A(\ell - 1), B(\ell - 1)) + d_{\Gamma}(B(\ell - 1), C(\ell - 1)) \leq 1 + 1 = 2\). Also, using the remark again, we have \(d_{\Gamma}(A(\ell), C(\ell)) \leq d_{\Gamma}(A(\ell), B(\ell)) + d_{\Gamma}(B(\ell), C(\ell)) \leq 1 + 0 = 1\). Finally, by similar consideration, we have that \(d_{\Gamma}(A(i), C(i)) \leq d_{\Gamma}(A(i), B(i)) + d_{\Gamma}(B(i), C(i)) \leq 0 + 0 = 0\) for all \(i < j\) and \(i > \ell\).

**Condition C4.** For \(j \geq \ell + 2\) we have that \(W_j = U_j = V_j\) (using the induction hypothesis and the fact that \(B\) was constructed from \(A\) by fixing it at location \(\ell\)). Moreover, \(V_{\ell+1} = U_{\ell+1}\) and also \(U_{\ell+1}\) is a sub-word of \(W_{\ell+1}\) such that \(\|U_{\ell+1}\| \geq \|W_{\ell+1}\| - 1\) (Remark 45 - part [iii] and thus \(V_{\ell+1}\) is a sub-word of \(W_{\ell+1}\) such that \(\|V_{\ell+1}\| \geq \|W_{\ell+1}\| - 1\).

**Condition C5.** Assume \(C\) is admissible. Since \((B, C)\) is an \((\ell - 1)\)-pacing pair we have that \([\kappa_C]_{\ell-1} < [\kappa_B]_{\ell-1}\) and \([\kappa_C]_j \leq [\kappa_B]_j\) for all \(1 \leq i \leq \ell - 2\). Hence, \([\kappa_C]_{\ell-1} = 0 = [\kappa_A]_{\ell-1}\) (last equality follows from our assumptions above). By Lemma 18 we have that \([\kappa_B]_j \leq [\kappa_A]_j\) for all \(j < \ell - 1\). Thus, \([\kappa_C]_j \leq [\kappa_A]_j\) for all \(j < \ell - 1\). To finish, we need to verify that \([\kappa_C]_{\ell} < [\kappa_A]_{\ell}\). Since \([\kappa_A]_{\ell} = 1\) we need to show that \([\kappa_C]_{\ell} = 0\). \((B, C)\) is an \((\ell - 1)\)-pacing pair hence \(V_{\ell}\) is a sub-word of \(U_{\ell}\).
with $\|V_\ell\| \geq \|U_\ell\| - 1$. Now, $B$ was constructed from $A$ by fixing $A$ at location $\ell$ so $U_\ell$ is an element of $R$ and thus $\|U_\ell\| \geq 3$. This shows that $\|V_\ell\| \geq 2$. Consequently, using Observation we have that $U_\ell$ is the only element in $R$ such that $V_\ell$ is its sub-word. This complete the argument since by part iv of Remark we have that $\|U_\ell\| < \|c(U_\ell)\|$ so necessarily $[\kappa_C]_\ell = 0$.

\[\square\]

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