First eigenvalue of the Laplacian on compact surfaces for large genera

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Abstract
For any Riemannian metric $ds^2$ on a compact surface of genus $g$, Yang and Yau proved that the normalized first eigenvalue of the Laplacian $\lambda_1(ds^2)\frac{\text{Area}(ds^2)}{ds^2}$ is bounded in terms of the genus. In particular, if $\Lambda_1(g)$ is the supremum for each $g$, it follows that the asymptotic growth of the sequence $\Lambda_1(g)$ is no larger than the one of $4\pi g$. Recently Ros, for $g = 3$, and Karpukhin and Vinokurov, for the general case, improve these bounds. In this paper we obtain a sharper result for $\Lambda_1(g)$ and we show that

$$\limsup_{g \to \infty} \frac{1}{g} \Lambda_1(g) \leq 4(3 - \sqrt{5})\pi \approx 3.056\pi.$$ 

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1 Introduction

Let $\Sigma$ a compact orientable surface of genus $g$ and $ds^2$ be a Riemannian metric on it. The Laplacian of the metric is a fundamental operator associated to $ds^2$ and its eigenvalues are given by a divergent sequence

$$0 = \lambda_0(ds^2) < \lambda_1(ds^2) \leq \lambda_2(ds^2) \leq \cdots,$$

where each eigenvalue is counted with multiplicity. The normalized first eigenvalue $\lambda_1(ds^2)\frac{\text{Area}(ds^2)}{ds^2}$ is invariant under homothetic rescaling and we define

$$\Lambda_1(g) = \sup \left\{ \lambda_1(ds^2)\frac{\text{Area}(ds^2)}{ds^2} \mid ds^2 \text{ is a Riemannian metric on } \Sigma \right\}.$$
Yang and Yau [25] gave the upper bound

\[ \Lambda_1(g) \leq \left[ \frac{g + 3}{2} \right] 8\pi, \]  

where \([x]\) denotes the integer part of \(x\). Their argument uses as test functions branched conformal maps between \((\Sigma, ds^2)\) and the unit sphere \(S^2(1) \subset \mathbb{R}^3\). This idea was first used by Hersch [10] who proved that \(\Lambda_1(0) = 8\pi\), the equality holding only for constant curvature metrics, and later by [2, 15] and other authors.

The exact value of \(\Lambda_1(g)\) is only known for a few other cases. Nadirashvili [18] proved that \(\Lambda_1(1) = \frac{48}{3\pi} \approx 14.510\pi\), where the equality holds for the flat equilateral metric, and Nayatani and Shoda [19] show \(\Lambda_1(2) = 16\pi\), the equality holding for the Bolza spherical surface and some other branched spherical metrics.

Regarding the metrics that achieve the supremum, the sequence \(\{\Lambda_1(g)\}_g\) is non decreasing and if \(\Lambda_1(g - 1) < \Lambda_1(g)\) then there exist an extremal metric \(ds^2\) with conical singularities on \(\Sigma\) such that \(\lambda_1(ds^2)Area(ds^2) = \Lambda_1(g)\) and a branched minimal immersion \(f : (\Sigma, ds^2) \rightarrow S^m(1), m \geq 2\), by the first eigenfunctions, Petrides [20]. For other related results see Karpukhin, Nadirashvili, Penskoi and Polterovich [14], Petrides [21], Cianci, Karpukhin and Medvedev [5], El Soufi and Ilias [6], Fraser and Schoen [8], Montiel and Ros [17] and Gomyou and Nayatani [9].

We next briefly explain the results of this paper. Bourguignon, Li and Yau [2] gave a first eigenvalue upper bound for algebraic Kaehler manifolds which extends [25] whose main idea is as follows. There is a natural embedding of the complex projective space \(\mathbb{C}P^n\) with the Fubini-Study metric in the Euclidean space \(HM_1(n+1)\) of square Hermitian matrices of order \(n+1\) and trace 1. A compact Riemannian surface \((\Sigma, ds^2)\) will be seen as a Riemann surface \(\Sigma\) with a conformal metric \(ds^2\) on it. A full holomorphic map in the complex projective space \(A : \Sigma \rightarrow \mathbb{C}P^n \subset HM_1(n+1)\) defines a complex curve and its energy depends only of the degree of the curve. By composing with projective transformations of \(\mathbb{C}P^n\) one can get the center of mass of \(A\) to be proportional to the identity matrix and, thus, they obtained that \(\lambda_1(ds^2)Area(ds^2)\) is bounded in terms of this degree (in fact, for complex curves its result is closely related to [25]).

Lately Ros [23] and Karpukhin and Vinokurov [13] further elaborate on this idea by considering the map \(\phi_a = A + aB : \Sigma \rightarrow HM_1(n+1)\), where \(a \in \mathbb{R}\) and \(B\) is a sphere-valued map depending on the tangent lines of the curve (its Gauss map). The energy of \(\phi_a\) is a function that depends on \(a\), the degree of the curve, the genus of \(\Sigma\) and the number of branch points (if any). Using projective transformations one can control, for certain values of \(a\), the center of mass of \(\phi_a\) and in this way obtain new upper bounds for the normalized first eigenvalue of \(ds^2\). In accordance with this, as any genus-three Riemann surface is either hyperelliptic or a quartic curve in the complex projective plane, in [23] one proves that

\[ \Lambda_1(3) \leq 16(4 - \sqrt{7})\pi \approx 21.668\pi, \]

while the corresponding bound in [25] is \(24\pi\). In [13], using holomorphic maps provided by the Brill-Noether theory and the values of \(a\) between 0 and \(1/\sqrt{2n(n+1)}\), the authors improve the bound of Yang and Yau for all genera \(g \neq 4, 6, 8, 10, 14\).

A fundamental step in the proof of the above results is the ability to properly adjust the center of mass of the test functions. In this paper we obtain a complete solution for the maps \(\phi_a : \Sigma \rightarrow HM_1(n+1)\). In Proposition 7, using the Algebraic Geometry of the curve \(A : \Sigma \rightarrow \mathbb{C}P^n\) and some ideas that originate in Ros [22, 23], we prove that
∀ a, 0 ≤ a < 1/2, after composing with a suitable projective transformation, the center of mass of the map φ_a becomes the point \( \frac{1}{n+1} I \).

Using this property, we prove Theorem 9 that gives us an inequality for the supremum first eigenvalue \( \lambda_1(g) \) which improves [13].

As a main application we focus on the asymptotic behaviour of the sequence \( \{ \lambda_1(g) \} \). From (1) we have that the upper limit of \( \lambda_1(g) \) is \( 4\pi \) and in [13] the authors improve this inequality until \( \leq 3.416\pi \). In the following statement we provide a better upper bound which summarizes our contribution, see Theorem 10:

The sequence \( \{ \lambda_1(g) \} \) of extremal normalized first eigenvalues of the Laplacian on compact surfaces satisfies the following growth properties

\[
\pi \leq \limsup_{g \to \infty} \frac{1}{g} \lambda_1(g) \leq 4(3 - \sqrt{5})\pi \approx 3.056\pi.
\]

The lower bound has been obtained recently by Hide and Magee [11] (they use a family of hyperbolic metrics and improve previous known bounds in that context).

2 Preliminaries

2.1 The complex projective space.

Let \( HM(n+1) = \{ A \in gl(n+1, \mathbb{C}) / \tilde{A} = A^t \} \) the space of \( (n+1) \times (n+1) \)-Hermitian matrices with the Euclidean metric

\[
\langle A, B \rangle = 2 \text{tr} AB \quad \forall A, B \in HM(n+1)
\]

and \( HM_1(n+1) = \{ A \in HM(n+1) / \text{tr} A = 1 \} \simeq \mathbb{R}^{n(n+2)} \) be the hyperplane given by the trace 1 restriction. Along the paper, a clever element of this hyperplane will be the point \( \frac{1}{n+1} I \), \( I \) being the identity matrix.

The submanifold \( \mathbb{C}P^n = \{ A \in HM_1(n+1) / AA = A \} \) with the induced metric is isometric to the complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1, see Ros [22].

The action of the unitary group \( U(n+1) \) on \( \mathbb{C}P^n \) is given by \( (P, A) \mapsto \tilde{P}^t A P \), where \( P \in U(n+1) \) and \( A \in \mathbb{C}P^n \). Hence the embedding of \( \mathbb{C}P^n \) in \( HM_1(n+1) \) is \( U(n+1) \)-equivariant.

For any \( A \in \mathbb{C}P^n \) the tangent space at that point identified with a subspace of \( HM(n+1) \) is given by \( T_A \mathbb{C}P^n = \{ X \in HM(n+1) / XA + AX = X \} \) and the second fundamental form \( \tilde{\sigma} \) of \( \mathbb{C}P^n \subset HM_1(n+1) \) maps the tangent vectors \( X, Y \in T_A \mathbb{C}P^n \) into \( \tilde{\sigma}(X, Y) \in T_A^\perp \mathbb{C}P^n \).

Among properties of the embedding we remark the following ones, [22]:

- Complex projective lines \( \mathbb{C}P^1 \subset \mathbb{C}P^n \) are totally geodesic and, when viewed in \( HM_1(n+1) \), are given by round 2-spheres of radius one.
- If \( J \) is the complex structure in \( \mathbb{C}P^n \), then \( \tilde{\sigma}(JX, JY) = \tilde{\sigma}(X, Y) \), \( X, Y \in T_A \mathbb{C}P^2 \).
- \( \nabla \tilde{\sigma} = 0 \), i.e. the second fundamental form is parallel.
- \( \mathbb{C}P^n \) is a minimal submanifold in the sphere \( S^{n(n+2)-1} \) of \( HM_1(n+1) \) of center \( \frac{1}{n+1} I \) and radius \( \sqrt{\frac{2n}{n+1}} \).

Now we consider the convex hull of the complex projective space in \( HM_1(n+1) \). If \( A \in HM(n+1) \) we will use the notation \( A > 0 \), resp. \( A \geq 0 \), when the Hermitian matrix is...
positive definite, resp. positive semidefinite. Then the convex hull \( \mathcal{H} \) of \( \mathbb{C}P^n \) in \( HM_1(n+1) \) verifies the following properties, see [2, 23]:

- \( \mathcal{H} = \{ A \in HM_1(n+1) / A \geq 0 \} \),
- \( int \mathcal{H} = \{ A \in \mathcal{H} / \text{rank} A = n+1 \} = \{ A \in HM_1(n+1) / A > 0 \} \), where by \( int \mathcal{H} \) we denote the topological interior of \( \mathcal{H} \) in \( HM_1(n+1) \),
- \( \partial \mathcal{H} = \{ A \in \mathcal{H} / \text{rank} A \leq n \} \),
- \( \mathbb{C}P^n = \{ A \in \mathcal{H} / \text{rank} A = 1 \} \),
- \( \{ A \in \mathcal{H} / \text{rank} A \leq 2 \} \) is the union of all the unit 3-balls in \( HM_1(n+1) \) enclosed by complex projective lines \( \mathbb{C}P^1 \subset \mathbb{C}P^n \).

The projection of \( \mathbb{C}^{n+1} - \{0\} \) over \( \mathbb{C}P^n \)

\[ z \mapsto \frac{1}{|z|^2} \bar{z}' z, \]

with \( z = (z_0, z_1, \ldots, z_n) \), defines the identification between \( \mathbb{C}^{n+1} - \{0\} \sim \), the usual projective space with homogeneous coordinates \( [z] = [z_0, z_1, \ldots, z_n] \), and \( \mathbb{C}P^n \) viewed as submanifold of \( HM_1(n+1) \). Along this paper both views of the projective space will be used at our convenience.

Given a regular matrix \( P \in GL(n+1, \mathbb{C}) \), the projective transformation \( f_P : \mathbb{C}P^n \longrightarrow \mathbb{C}P^n \), is given by \( [z] \mapsto [zP] \), i.e. \( [z_0, z_1, \ldots, z_n] \mapsto [(z_0, z_1, \ldots, z_n)P] \). When we describe it in terms of elements of \( \mathbb{C}P^n \) we have

\[ f_P : \frac{1}{|z|^2} \bar{z}' z \mapsto \frac{1}{|zP|^2} \bar{P}' \bar{z}' zP. \quad (2) \]

In this paper, as in [2], we consider projective transformations \( f_P \), where \( P \) is a positive definite matrix in \( HM(n+1) \): Any other projectivity can be decomposed as \( f_P \circ f_Q \), where \( P > 0 \) and \( Q \) is a unitary matrix. Moreover, after multiplying by a positive scalar factor, we assume that \( \text{trace} P = 1 \). Therefore, up to unitary motions, the space of projective transformations is parametrized by the interior of the convex hull of \( \mathbb{C}P^n \),

\[ f_P : \mathbb{C}P^n \longrightarrow \mathbb{C}P^n, \quad P \in int \mathcal{H}. \]

When \( P \in \partial \mathcal{H} \), then \( P \) is not a regular matrix but we will still interested into the associated projection map \( f_P \) defined as follows. Let \( \mathcal{R}, \mathcal{S} \subset \mathbb{C}P^n \) be the subspaces determinate by the linear subspaces \( \text{Kernel}(P), \text{Image}(P) \subset \mathbb{C}^{n+1} \), respectively. Then \( \mathcal{R} \cap \mathcal{S} = \emptyset \), \( \dim \mathcal{R} = n - \text{rank} P \) and \( \dim \mathcal{S} = \text{rank} P - 1 \). The map \( f_P : [z] \mapsto [zP] \) is defined over \( \mathbb{C}P^n \) minus \( \mathcal{R} \), its image is \( \mathcal{S} \),

\[ f_P : \mathbb{C}P^n - \mathcal{R} \longrightarrow \mathcal{S} \quad (3) \]

and \( P \) belongs to the convex hull of \( \mathcal{S} \) in \( HM_1(n+1) \). It can be see as the classic projection map since \( \mathcal{R} \) over \( \mathcal{S} \), see Fig.1, followed by a linear projectivity of the image subspace, \( f_P|_\mathcal{S} : \mathcal{S} \longrightarrow \mathcal{S} \).

Two particular cases of \( f_P \) worth mentioning are \( \text{rank} P = 1 \), where the image is the point \( P \in \mathbb{C}P^n \) and \( \text{rank} P = 2 \), whose image is the unique projective line \( \mathbb{C}P^1 \subset \mathbb{C}P^n \) containing \( P \) in its convex hull.

### 2.2 Geometry of complex curves in \( \mathbb{C}P^n \)

Let \( \Sigma \) a compact Riemann surface of genus \( g \) and \( \varphi : \Sigma \longrightarrow \mathbb{C}P^n \) be a holomorphic map in the complex projective space. If \( \varphi \) is nonconstant then the branch points of the map are
isolated and on the unbranched set $\Sigma^0$ it defines an immersion. In this case we say that $\varphi$ is a complex curve. The curve $\varphi$ is said to be full if its image is not contained in any hyperplane $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

In this paper most of the time we think of $\varphi$ as a map into the space of Hermitian matrices and we represent it as

$$A = A_\varphi : \Sigma \longrightarrow \mathbb{C}P^n \subset HM_1(n + 1).$$

(4)

For every unbranched point $A \in \Sigma^0$, the tangent plane could be identified with the corresponding complex projective line $\mathbb{C}P^1$ which is a unit 2-sphere in the Euclidean space. We define the Gauss map $B : \Sigma^0 \longrightarrow HM(n + 1)$ of the curve as the vector joining $A$ with its antipodal point $A^-$ in this 2-sphere, $B = A^- - A$, see Fig. 2. Note that, as a complex line in $\mathbb{C}P^n$ is determined by two of its points, it follows that the vector $B$ determines the tangent 2-sphere at the point $A$.

At the unbranched points we will consider, unless otherwise stated, the metric $(\cdot, \cdot)$ induced on the surface by the Fubini-Study metric on $\mathbb{C}P^n$ and we denote by $K$, $d\Sigma$ and $\Delta$ the Gauss curvature, the Riemannian measure and Laplacian of the surface.

The complex structure $J$ on $\Sigma$ and the second fundamental form $\sigma$ of $\Sigma^0$ in $\mathbb{C}P^n$ verify $\sigma(JX, Y) = \sigma(X, JY) = J\sigma(X, Y)$ and the Gauss equation says that

$$K = 1 - \frac{1}{2}|\sigma|^2.$$  

(5)

The Hessian operator of the vector valued map $A : \Sigma^0 \longrightarrow HM_1(n + 1)$ will be denoted by $\nabla^2A$. It coincides with the second fundamental form of $\Sigma^0$ in $HM(n + 1)$ and it can decomposed as

$$\nabla^2A(X, Y) = \sigma(X, Y) + \tilde{\sigma}(X, Y) \quad \forall X, Y \in T_A\Sigma.$$
As complex curves are minimal surfaces in the complex projective space, it follows that the Laplacian of the immersion $A : \Sigma^o \longrightarrow HM(n + 1)$ is equal to

$$\Delta A = \sum_i \nabla^2 A(E_i, E_i) = \tilde{\sigma}(E_1, E_1) + \tilde{\sigma}(E_2, E_2) = 2\tilde{\mathcal{H}} = B, \quad (6)$$

$E_1, E_2 \in T_{A}\Sigma^o$, with $E_2 = JE_1$, being an orthonormal basis and $\tilde{\mathcal{H}}$ the mean curvature vector of the immersion.

From Lemma 3.2 in [22], at the points of $\Sigma^o$ we have the first rows below

$$\begin{aligned}
|I|^2 &= 2(n + 1) \quad \langle A, I \rangle = 2 \quad |A|^2 = 2 \\
\langle B, I \rangle &= 0 \quad \langle B, A \rangle = -2 \quad |B|^2 = 4 \\
\langle \Delta B, I \rangle &= 0 \quad \langle \Delta B, A \rangle = 4 \quad \langle \Delta B, B \rangle = -8 - 2|\sigma|^2 \\
|\nabla A|^2 &= 2 \quad \langle \nabla A, \nabla B \rangle = -4 \quad |\nabla B|^2 = 8 + 2|\sigma|^2
\end{aligned} \quad (7)$$

and the last one is a consequence of the above. For instance, the Laplacian of the equation $|B|^2 = 4$ give us $|\nabla B|^2 + \langle \Delta B, B \rangle = 0$ and then we get $|\nabla B|^2 = 8 + 2|\sigma|^2$.

Now we focus at branch points of the curve $A_\varphi$, see also [13]. Given a nonconstant holomorphic map $A : \Sigma \longrightarrow \mathbb{C}P^n \subset HM_1(n + 1)$ and a local complex coordinate $w$, $|w| < \varepsilon$, around a point $p \in \Sigma$, using suitable projective coordinates we have

$$z = z(w) = (1, z_1(w), \ldots, z_n(w)) \in \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{C}P^n,$$

where $z_i(w), i = 1, \ldots, n$, are holomorphic functions. Its relation with the Hermitian matrices is

$$A = A(w) = \frac{1}{|z|^2} \bar{z}'z. \quad (8)$$

For $w = 0$, after a suitable unitary transformation, we have that

$$\begin{aligned}
z(0) &= (1, 0, \ldots, 0) \\
A(0) &= P_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\end{aligned} \quad (9)$$

The tangent vector is

$$z'(w) = (0, z_1'(w), \ldots, z_n'(w)) \in \mathbb{C}^{n+1}.$$

If $z'(w) \neq 0$ the point is unbranched and the affine tangent line $\{z(w) + \lambda z'(w) / \lambda \in \mathbb{C}\}$ determines the projective tangent one $\mathbb{C}P^1 \subset \mathbb{C}P^n$. Moreover, since the Gauss map satisfies $B = \Delta A$ it can be expressed at any point of $\Sigma^o$ as

$$B = \frac{2}{|\partial_{\bar{w}} \partial_w A|} \partial_{\bar{w}} \partial_w A, \quad |w| < \varepsilon, \quad (10)$$

where we have used that the Laplacian $\Delta$ is proportional to the operator $\partial_{\bar{w}} \partial_w$ and the fact that $|B| = 2$.

If $\varphi$ is nonconstant and $z'(0) = 0$, then $p$ is a branch point of the curve. The branch points are isolated and around $w = 0$ we have the product

$$z'(w) = w^m h(w), \quad (11)$$
$h(w)$ being a vectorial holomorphic function with $h(0) \neq 0$, $m \geq 1$ an integer number. We say that $m$ is the branching order at $p$, $b_p(\varphi) = m$. Around this point the induced metric is $|w|^2 m \rho^2 |d\omega|^2$, where $\rho(0) \neq 0$. The total branching number of $\varphi$ is defined as the sum of all branching orders

$$b = b(\varphi) = \sum_p b_p(\varphi).$$

(12)

The measure $K d\Sigma$ is smooth, even in the branched case, and the Gauss-Bonnet formula, e. g. Tribuzy [7], relate the total Gauss curvature with the genus of $\Sigma$ and branching number of the curve

$$\int_\Sigma K d\Sigma = 2\pi(2 - 2g + b).$$

(13)

One other thing worth noting is that the nonzero section $h(w)$, $|w| < \varepsilon$, in (11) allows to extend the tangent projective line of $\varphi$ smoothly through the point $w = 0$ and, therefore, the Gauss map is a differentiable on the whole curve, $B : \Sigma \rightarrow HM(n + 1)$.

**Lemma 1** Let $A : \Sigma \rightarrow \mathbb{C}P^n \subset HM_1(n + 1)$ be a nonconstant holomorphic map and $a \in \mathbb{R}$. Then the map $\phi_a : \Sigma \rightarrow HM(n + 1)$, defined as

$$\phi_a = A + aB,$$

(14)

is smooth everywhere. Furthermore, if $n \geq 2$ then for $0 \leq a \leq 1$ the image of $\phi_a$ in contained in the boundary of the convex hull of $\mathbb{C}P^n$, $\phi_a(\Sigma) \subset \partial H$.

**Proof** The first part follows from the smoothness of the Gauss map $B$. The last assertion is a direct consequence of Fig. 2.

The degree of a nonconstant (possibly branched) complex curve $A : \Sigma \rightarrow \mathbb{C}P^n$ is an algebro-geometric/topological invariant given by a positive integer $d$ satisfying the following:

- The immersed complex curve $A$ is $\mathbb{Z}$-homologous to $d$ times the 2-cycle $\mathbb{C}P^1 \subset \mathbb{C}P^n$.
- With respect to the Fubini-Study metric, the area of the curve $A : \Sigma \rightarrow \mathbb{C}P^n$ is $Area(\Sigma) = 4\pi d$.
- A general complex hyperplane intersects the immersed curve $\Sigma$ at $d$ points (counted with multiplicity).
- If $g$ is the genus of $\Sigma$ and $b$ is the total branching number of $A$, the integral of the Gauss curvature (10) combined with the Gauss–Bonnet theorem for branched metrics (13) gives

$$\int_\Sigma 1 \ d\Sigma = 4\pi d, \quad \int_\Sigma |\sigma|^2 d\Sigma = 8\pi \left(g + d - 1 - \frac{1}{2} b\right).$$

(15)

In particular, the total branching number verifies

$$b/2 \leq g + d - 1.$$

(16)

Now we study the behaviour of the vector functions $\phi_a = A + aB$, see also [13].

**Lemma 2** The image of $\phi_a$ lies in a sphere of $HM_1(n + 1)$,

$$\left|\phi_a - \frac{1}{n + 1} I\right|^2 = (2a - 1)^2 + \frac{n - 1}{n + 1}.$$

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and
\[ \int_\Sigma |\nabla \phi_a|^2 d\Sigma = 8\pi d \left\{ (2a - 1)^2 + 2a^2 \delta \right\}, \]
where \( \delta = 1 + \frac{g-1-b/2}{d} \geq 0. \) Furthermore,

(i) the energy of \( \phi_a \) remains invariant under projective transformations,

(ii) the spherical map \( \phi_a \) is conformal and

(iii) the vector-valued function \( \phi_a \) belongs to the Sobolev space \( W^{1,2}(\Sigma) \).

**Proof** From (7) we obtain the first equation
\[ \left| \phi_a - \frac{1}{n+1} I \right|^2 = a^2 |B|^2 + 2a \left( B, A - \frac{1}{n+1} I \right) + \left| A - \frac{1}{n+1} I \right|^2 \]
\[ = 4a^2 - 4a + \frac{2n}{n+1} = (2a - 1)^2 + \frac{n-1}{n+1}. \]

In the same way, at the unbranched points of \( A \) we have
\[ |\nabla \phi_a|^2 = a^2 |\nabla B|^2 + 2a \langle \nabla A, \nabla B \rangle + |\nabla A|^2 = (8 + 2|\sigma|^2)a^2 - 8a + 2. \]

Integrating with respect to \( d\Sigma \) and using (15)
\[ \int_\Sigma |\nabla \phi_a|^2 d\Sigma = 8\pi d \left( d + g - 1 - \frac{1}{2}b \right) a^2 - 32\pi da + 8\pi d \]
\[ = 8\pi d \left\{ (4 + 2\delta)a^2 - 4a + 1 \right\} = 8\pi d \left\{ (2a - 1)^2 + 2a^2 \delta \right\}. \]

A direct consequence of the above gives (i).

The assertion in (ii) in proved in [23] for \( n = 2 \) and the same computation works for any \( n \).

To prove (iii) first observe that the space \( W^{1,2}(\Sigma) \) is independent of the smooth metric. Then the assertion follows because the Dirichlet integral is conformally invariant and that \( \phi_a \) is a bounded function. \( \Box \)

2.3 The Brill–Noether Theory.

First we recall that the moduli space of closed Riemann surface of genus \( g \geq 2 \), \( \mathcal{M}_g \), is a complex algebraic variety of dimension \( 3g - 3 \) (with nonempty singular set). In this context, to say that a property holds for a closed general curve means that it happens for all \( \Sigma \) in a certain open dense subset of \( \mathcal{M}_g \).

As a consequence of Brill–Noether theory, see Arbarello, Cornalba, Griffiths and Harris [1], Chapter V, we have the following existence result for unbranched full complex curves in term of the genus, degree and dimension of the complex projective space.

**Theorem 3** ([1], p.216) Let \( g, n, d \in \mathbb{N} \) with \( n \geq 2 \) such that the Brill-Noether constant 
\( \rho = \rho(g, d, n) = g - (n + 1)(g - d + n) \) is nonnegative, \( \rho \geq 0 \). Then a closed general Riemann surface \( \Sigma \) of genus \( g \) admits a full holomorphic immersion of degree \( d \) in \( \mathbb{C}P^n \), \( \varphi : \Sigma \rightarrow \mathbb{C}P^n \). In particular, \( \varphi \) is unbranched. If \( n \geq 3 \), then the immersion can be chosen to be an embedding.
The condition that \( \rho \) is nonnegative can be rewritten as \( d \geq n + gn/(n+1) \) or, equivalently, \( d \geq d(g, n) \), where

\[
d(g, n) = \left[ \frac{(g + 1)n}{n + 1} \right] + n
\]

and \([x]\) denotes the integer part of the real number \( x \).

There is another well-known consequence of Brill–Noether theory valid for any compact Riemann surface, although in this case the complex curve is branched

**Theorem 4** If \( \Sigma \) is a closed Riemann surface of genus \( g \) then, for any \( n \geq 1 \), there is a full holomorphic map of \( \Sigma \) in \( \mathbb{C}P^n \) of degree \( d \leq d(n, g) \).

### 3 The center of mass

We consider a compact Riemannian surface \((\Sigma, ds^2)\), \( d\mu \) its Riemannian measure and \( A : \Sigma \to \mathbb{C}P^n \subseteq HM_1(n + 1) \) a holomorphic map of the compact Riemann surface \( \Sigma \) in the complex projective space.

Let’s recall that, in terms of the coordinate \( |w| < \varepsilon \), we have the relations

\[
A = \frac{1}{|z|^2} \bar{z} z P \
B = \frac{2}{|z|^2} \bar{z} \partial_z A, \quad \phi_{P, a} = A_P + aB_P \in \mathcal{H}.
\]

We define the center of mass map \( \Phi_a : int \mathcal{H} \to HM_1(n + 1) \),

\[
\Phi_a(P) = \frac{1}{Area(ds^2)} \int_\Sigma \left( A_P + aB_P \right) d\mu \
\quad \forall P \in int \mathcal{H}.
\]

The map

\[
(a, P) \in [0, 1) \times int \mathcal{H} \longmapsto \Phi_a(P) \in HM_1(n + 1)
\]

is continuous and for any \( P \in int \mathcal{H} \), it follows from (18) that

\[
A_P = \bar{P}^i \{\ldots\} P \quad B_P = \bar{P}^i \{\ldots\} P, \quad \forall P \in int \mathcal{H}.
\]

In the rest of the section we will be mostly interested in the case \( P \in \partial \mathcal{H} \), that is, when \( P \) is a singular matrix. First we examine the situation \( \text{rank} P \geq 2 \).
Lemma 5 Let $\Sigma$ a closed Riemann surface, $A : \Sigma \rightarrow \mathbb{C}P^n$, $n \geq 2$, a full branched complex curve and $P \in \partial \mathcal{H}$ with $2 \leq \text{rank } P = k + 1 \leq n$. Then the map $A_P$ defines a full branched complex curve in a lineal subspace $\mathbb{C}P^k \subset \mathbb{C}P^n$, $A_P : \Sigma \rightarrow \mathbb{C}P^k \subset \text{HM}_1(n + 1)$. In particular, $A_P$ is nonconstant, it has finitely many branch points and the associated Gauss map $B_P : \Sigma \rightarrow \text{HM}(n + 1)$ is well defined at every point.

Proof We known from (3) that the projection map $f_P$ is defined outside of a subspace $\mathcal{R}^{n-k}$ and its image is a $k$-dimensional subspace $\mathbb{C}P^k$.

As $k \geq 1$, the preimage of a hyperplane in $\mathbb{C}P^k$ under $f_P$ is a hyperplane on $\mathbb{C}P^n$. Then, as the initial curve $A$ is full, we get that $A_P : \Sigma \rightarrow \mathbb{C}P^k$ is also full. The rest of the lemma follows directly. $\square$

Note that the set of branch points of $A_P$ contains the ones of $A$ and some others that appear when we project by the matrix $P$. For $k \geq 2$, $A_P : \Sigma \rightarrow \mathbb{C}P^k$ is a projective algebraic complex curve and when $k = 1$ we get a nonconstant meromorphic map $A_P : \Sigma \rightarrow \mathbb{C}P^1$.

Proposition 6 Let $A : \Sigma \rightarrow \mathbb{C}P^n$ be a closed full branched complex curve and $0 \leq a < 1$. Then the behaviour of the center of mass map $\Phi_a$ at the boundary of $\mathcal{H}$ is the following: In the case $a = 0$,

(i) $\Phi_0$ extends to a continuous map $\Phi_0 : \mathcal{H} \rightarrow \text{HM}_1(n + 1)$,
(ii) $\Phi_0(\partial \mathcal{H}) \subset \partial \mathcal{H}$ and the restriction $\Phi_0|_{\partial \mathcal{H}}$ has non-zero degree, and
(iii) $\Phi_0(P) = P, \forall P \in \mathbb{C}P^n$. full stop If $0 < a < 1$ and $n \geq 2$, then
(iv) $\Phi_a$ extends continuously to the complement of $\mathbb{C}P^n$, $\Phi_a : \mathcal{H} - \mathbb{C}P^n \rightarrow \text{HM}_1(n + 1)$,
(v) $\Phi_a(\partial \mathcal{H} - \mathbb{C}P^n) \subset \partial \mathcal{H}$ and
(vi) $\Phi_a$ does not extend, in a continuous way, to the points of $\mathbb{C}P^n$.

Proof (i) and (ii) are proved in [2].

To prove (iii), using a unitary transformation, it is sufficient to check it for $P = P_0$,

$$P_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 \\ 0 & & & 1 \end{pmatrix}.$$  

Except for a finite number of points, for any $[z] = [z_0, \ldots, z_n] \in \Sigma$ we have $z_0 \neq 0$. By direct computation $A_{P_0} : \Sigma \rightarrow \text{HM}_1(n + 1)$ is given by

$$A_{P_0} : [z] \mapsto \frac{1}{|z| P_0^2} P_0 z' P_0 = \frac{|z_0|^2}{|z_0|^2} P_0 = P_0,$$

and therefore

$$\Phi_0(P_0) = \frac{1}{\text{Area}(ds^2)} \int_{\Sigma} A_{P_0} d\mu = P_0.$$

Now we prove (iv). We consider $P \in \partial \mathcal{H}$, $P \notin \mathbb{C}P^n$, $2 \leq \text{rank } P = k + 1 \leq n + 1$. From Lemma 5 we have that $A_P : \Sigma \rightarrow \mathbb{C}P^k$ is a full complex curve and the map $A_P + aB_P : \Sigma \rightarrow \text{HM}_1(n + 1)$ makes sense at every points of $\Sigma$. Therefore the integral

$$\int_{\Sigma} (A_P + aB_P) d\mu.$$
Fig. 3 $\mathcal{H}_r$ is obtained as the intersection of $\mathcal{H}$ with the ball of radius $r$. The distance between the center $\frac{1}{n+1}I$ and $\mathbb{C}P^n$ is greater than 1 and, when $n$ goes to infinity, $\text{dist}(\frac{1}{n+1}I, \partial \mathcal{H}) \rightarrow 0$.

is well-defined. Since the integrand in the definition of the center of mass is uniformly bounded, by the dominated convergence theorem, it follows that the expression (19) has a meaning on $\partial \mathcal{H} - \mathbb{C}P^n$ and this extension is continuous.

The assertion (v) follows directly from (21).

Finally we show (vi). We prove it for the matrix $P_0 \in \mathbb{C}P^1$ used in (3). We have seen that, except in a finite number of points, $A_{P_0} : \Sigma \rightarrow \mathbb{C}P^n$ is constantly equal to $P_0$ and $\Phi_0(P_0) = P_0$.

Let’s consider the point $P_\varepsilon \in \partial \mathcal{H}$, $\varepsilon > 0$ small enough, given by the matrix

$$P_\varepsilon = \begin{pmatrix} 1 \\ \varepsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$ 

The map $A_{P_\varepsilon} : \Sigma \rightarrow \mathbb{C}P^n$ is a nonconstant meromorphic map which consists of projecting $\Sigma$ onto the complex line $\mathbb{C}P^1 = \{z_2 = 0, \ldots, z_n = 0\} \subset \mathbb{C}P^n$ and then contract it towards $P_0$, 

$$(z_0, z_1, 0, \ldots, 0) \mapsto (z_0, \varepsilon z_1, 0, \ldots, 0).$$

When $\varepsilon \rightarrow 0$, outside of finitely many points of $\Sigma$, $A_{P_\varepsilon}$ converges smoothly to the constant map $A_{P_0}$ and so it follows that the Gauss map vector $B_{P_\varepsilon}$ converges in $L^2(d\mu)$ to the normal vector of $\mathbb{C}P^1$ at the point $P_0$. If we denote this normal vector by $B_{P_0}(\mathbb{C}P^1)$ we have that 

$$\lim_{\varepsilon \rightarrow 0} \Phi_a(P_\varepsilon) = P_0 + aB_{P_0}(\mathbb{C}P^1).$$

Thus the limit depends not only on $P_0$ but also on the limit straight line $\mathbb{C}P^1$. Finally, if we repeat the argument by using different projective lines passing through $P_0$, we conclude that $\Phi_a$ does not admit a continuous extension at that point. \hfill $\Box$

**Proposition 7** Let $ds^2$ be a conformal metric on a compact Riemann surface $\Sigma$ and $A : \Sigma \rightarrow \mathbb{C}P^n$, $n \geq 2$, be a full complex curve. Then, for $0 \leq a < 1/2$ the point $\frac{1}{n+1}I$ lies in the image of the center of mass map $\Phi_a : \text{int} \mathcal{H} \rightarrow HM_{1}(n+1)$.

**Proof** The result is known for $a = 0$. Let’s $r$, $1 < r < \sqrt{\frac{2n}{n+1}}$, and consider the convex body

$$\mathcal{H}_r = \mathcal{H} \cap \hat{B} \left( \frac{1}{n+1}I, r \right),$$

see Fig. 3. Note that the points of $\partial \mathcal{H}_r$ lie either in $\partial \mathcal{H} - \mathbb{C}P^n$ or in the sphere of center $\frac{1}{n+1}I$ and radius $r$. 

\[\text{Springer}\]
First we prove that $\Phi_a(\partial\mathcal{H}_r)$ does not contain the point $\frac{1}{n+1} I$. In fact, if we assume that $\Phi_a(P) = \frac{1}{n+1} I$ for some $P \in \partial\mathcal{H}_r$, from Proposition 6(v) we have that $P \notin \partial\mathcal{H} - CP^n$. Thus $P \in \text{int}\,\mathcal{H}$ and $|P - \frac{1}{n+1} I| = r$.

Now, from the identities
$$0 = \Phi_a(P) - \frac{1}{n+1} I = \left(\Phi_0(P) - \frac{1}{n+1} I\right) + \frac{a}{\text{Area}(ds^2)} \int_\Sigma B P \, d\mu$$
and (7) it follows that
$$r = \left|\Phi_0(P) - \frac{1}{n+1} I\right| \leq \frac{a}{\text{Area}(ds^2)} \int_\Sigma |BP| \, d\mu = 2a < 1,$$
which contradicts our choice of $r$. So $\frac{1}{n+1} I \notin \Phi_a(\partial\mathcal{H}_r)$.

We continue with the proof of the result and divide the remaining argument in two steps.

- Suppose $a = 0$. The map $\Phi_0 : \mathcal{H} \rightarrow \mathcal{H}$ is continuous and $\Phi_0 : \partial\mathcal{H} \rightarrow \partial\mathcal{H}$ has non zero topological degree. It follows that, for $r$ a bit smaller than $\sqrt{2n/(n+1)}$, the image hypersurface $\Phi_0(\partial\mathcal{H}_r)$ encloses with nonzero topological degree the center $\frac{1}{n+1} I$ and this point belongs to the image of the domain bounded by the hypersurface: $\frac{1}{n+1} I \in \Phi_0(\text{int}\,\mathcal{H}_r)$.

- Now we leave the previous $r$ fixed and we move the parameter $a$ in the interval $0 \leq a < 1/2$. The immersed hypersurfaces $\Phi_a : \partial\mathcal{H}_r \rightarrow HM_1(n+1) - \frac{1}{n+1} I$ vary continuously. Then, by using a standard topological argument it follows that, $\forall a$, the point $\frac{1}{n+1} I$ belongs to the image by $\Phi_a$ of the domain $\text{int}\,\mathcal{H}_r$. This proves the Proposition. \(\square\)

### 4 Upper bounds for the first eigenvalue of the Laplacian

In this section we estimate the first eigenvalue of the Laplacian of $(\Sigma, ds^2)$, where $\Sigma$ is a compact Riemann surface of genus $g$ and $ds^2$ a conformal metric on it. Given a nonconstant holomorphic map $A : \Sigma \rightarrow CP^n$, at unbranched points, the Riemannian metrics $ds^2$ and $\langle , \rangle$ (the one induced by the Fubini-Study metric) are conformal $ds^2 = e^{2\theta} \langle , \rangle$, $\theta$ being smooth in $\Sigma$, and the Riemannian measures are related in the same way $d\mu = e^{2\theta} d\Sigma$.

The first eigenvalue of the Laplacian of the metric $ds^2$ is the largest positive number $\lambda_1(ds^2)$ characterized by the condition
$$\lambda_1(ds^2) \int_{\Sigma} u^2 \, d\mu \leq \int_{\Sigma} |\nabla ds u|^2 \, d\mu \quad \forall u \in C^1(\Sigma) \text{ s.t. } u \neq 0 \text{ and } \int_{\Sigma} u \, d\mu = 0, \tag{23}$$
where $|\nabla ds u|$ is the length of the $ds$-gradient of $u$.

As test functions we will use the sphere-valued maps $\phi_a - \frac{1}{n+1} I : \Sigma \rightarrow CP^n \subset \mathbb{R}^{n(n+2)}$ associated to a complex curve $A : \Sigma \rightarrow CP^n$. Therefore the left hand side of the inequality in (23) transforms to a certain constant times $\lambda_1(ds^2) \text{Area}(ds^2)$.

The Dirichlet integral is a conformal invariant and the right hand side of (23) could be computed by using the induced metric $\langle , \rangle$. In this way we get the energy of $\phi_a$ can be set from the basic invariants of the complex immersion appearing in Lemma 2.

The following theorems improve, for most of the genera, Karpukhin and Vinokurov, [13]. The tricky point is to be able to get the map $\phi_a - \frac{1}{n+1} I$ to have mean value zero, which we have done in Proposition 7.
Theorem 8  Let $\Sigma$ be closed Riemann surface of genus $g$ and $A : \Sigma \longrightarrow \mathbb{CP}^n$, $n \geq 2$, be a full holomorphic map of degree $d$. Then, for any conformal metric $ds^2$ on $\Sigma$ and $0 \leq a \leq 1/2$,

$$
\lambda_1(ds^2)\text{Area}(ds^2) \leq 8\pi d \left(1 + \frac{2a^2\delta - \frac{n-1}{n+1}}{(2a - 1)^2 + \frac{n-1}{n+1}}\right),
$$

where $\delta = 1 + \frac{g-1-b/2}{d}$, $b$ being the total branching number of $A$.

Proof  Proposition 7 implies that, by applying a certain projectivity to $A$, we can assume that the center of the map of $\phi_a : \Sigma \longrightarrow HM_1(n + 1)$ is equal to $\frac{1}{n+1} I$. Therefore, from Lemma 2 and (23) we get

$$
\lambda_1(ds^2)\text{Area}(ds^2) \leq \int_{\Sigma} |\nabla \phi_a|^2 d\mu = 2\pi d \left(1 + \frac{2a^2\delta - \frac{n-1}{n+1}}{(2a - 1)^2 + \frac{n-1}{n+1}}\right).
$$

Note that (16) implies $0 \leq \delta \leq 1 + (g - 1)/d$. By direct computation we can see that the expression $F(a) = F(n, d, \delta, a), a \in \mathbb{R}$, where

$$
F(n, d, \delta, a) = 8\pi d \left(1 + \frac{2a^2\delta - \frac{n-1}{n+1}}{(2a - 1)^2 + \frac{n-1}{n+1}}\right)
$$
satisfies $F(-\infty) = F(\pm \infty)$, $F'(0) < 0$ and $F'(1/2) > 0$. As the rational function $F(a)$ has just two critical points, it follows that it attains its global minimum for a value $a = a_{\text{min}}$, with $0 < a_{\text{min}} < 1/2$.

Theorem 9  Given integer numbers $g \geq 3$, $n \geq 2$ and $0 \leq a \leq 1/2$, one has the eigenvalue inequality

$$
\Lambda_1(g) \leq 8\pi d \left(1 + \frac{2a^2(1 + \frac{g-1}{d}) - \frac{n-1}{n+1}}{(2a - 1)^2 + \frac{n-1}{n+1}}\right),
$$

for any integer $d$ with

$$
d \geq d(g, n) = \left\lceil \frac{(g + 1)n}{n + 1} \right\rceil + n.
$$

Proof  Brill–Noether’s technique in Theorem 3 says that for a general Riemann surface $\Sigma$ with $g \geq 3$ there is a full holomorphic map $A : \Sigma \longrightarrow \mathbb{CP}^n$ of degree $d \geq d(g, n)$ and branching number $b = 0$. So, for any conformal metric $ds^2$ on $\Sigma$, from Theorem 8 we obtain

$$
\lambda_1(ds^2)\text{Area}(ds^2) \leq 8\pi d \left(1 + \frac{2a^2(1 + \frac{g-1}{d}) - \frac{n-1}{n+1}}{(2a - 1)^2 + \frac{n-1}{n+1}}\right).
$$

To get the result for any metric $ds^2$ on a surface of genus $g$, we observe that, in the space of smooth metrics on a compact surface, a neighborhood of $ds^2$ provides an open family of conformal structures. Therefore, the continuity of the first eigenvalue functional give the inequality (25) not only for metrics on a general Riemann surface, but for arbitrary metrics on the surface of this topology.

Alternatively, we can use Theorem 4. \qed
Finally, we consider the growth of $\Lambda_1(g)/g$. The result improves the bound $3.416\pi$ in [13].

**Theorem 10** The asymptotic growth of the sequence $\{\Lambda_1(g)\}$ verifies the inequality

$$\limsup_{g \to \infty} \frac{1}{g} \Lambda_1(g) \leq 4(3 - \sqrt{5})\pi \approx 3.056\pi.$$

**Proof** For any positive integer $g$ we consider the values

$$d = g + 1, \quad n = \left\lceil \sqrt{g + 1} \right\rceil.$$

The Brill–Noether constant is $\rho = g - (n+1)(g-d+n) = g - (n+1)(n-1) = g + 1 - n^2 \geq 0$ and, so, $d \geq d(g, n)$. From Theorem 9, we have

$$\frac{1}{g} \Lambda_1(g) \leq 8\pi \frac{d}{g} \left(1 + \frac{2a^2(1 + \frac{g-1}{d}) - \frac{n-1}{n+1}}{(2a - 1)^2 + \frac{n-1}{n+1}}\right).$$

Now we take limits

$$\lim_{g \to \infty} \frac{d}{g} = 1, \quad \lim_{g \to \infty} \frac{n-1}{n+1} = 1$$

and we get that, for any $a$ between 0 and 1/2,

$$\limsup_{g \to \infty} \frac{1}{g} \Lambda_1(g) \leq 8\pi \left(1 + \frac{4a^2 - 1}{(2a - 1)^2 + 1}\right).$$

If we call $G(a)$ the expression at the right, by direct calculation we see that its minimum value is attained for

$$a_{min} = \frac{3 - \sqrt{5}}{4} \approx 0.191 \quad \text{and} \quad G(a_{min}) = 4(3 - \sqrt{5})\pi \approx 3.056\pi.$$

This proves the theorem. $\square$

A natural variant is to look at the asymptotic behaviour of the first eigenvalue for some families of Riemannian surfaces. The one that has attracted the most interest is that of hyperbolic metrics, Buser, Burger and Dodziuk [3]. Let $\Lambda_1(g, -1)$ be the supremum of the first eigenvalue of the Laplacian on compact surfaces of genus $g$ and curvature $K \equiv -1$. The limit upper bound of the sequence was known to be $\leq 0.25$ (Cheng [4], Huber [12]) and Hide and Magee [11] prove the equality

$$\lim_{g \to \infty} \Lambda_1(g, -1) = \frac{1}{4}.$$

For others asymptotic properties see e.g. Lipnowski and Wright [16] and Wu and Xue [24].

**Remark 1** As Theorem 9 improves the result of [13], one can ask whether the inequality (24) improve Yang-Yau bound for $g = 4, 6, 8, 10, 14$. The answer is no. For these particular genera the optimal value is given by $n = 1$, see Table 1 in [13].
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