A topological scenario for high-temperature superconductivity of copper oxides

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The structure of the joint phase diagram demonstrating high-$T_c$ superconductivity of copper oxides is studied on the basis of a fermion-condensate hypothesis. Prerequisites of an associated topological rearrangement of the Landau state are established. Related non-Fermi-liquid behavior of the normal states of cuprates is investigated, focusing on manifestations of this behavior in the electrical resistivity $\rho(T)$. An explanation is offered for the observed gradual crossover from normal-state $T$-linear behavior of La$_{2-x}$Sr$_x$CuO$_4$ at doping $x$ below the critical value $x_c^h$ for termination of superconductivity, to the $T$-quadratic behavior at $x > x_c^h$, which is contrary to the prediction of the conventional quantum-critical-point scenario. Additional evidence of the failure of the latter scenario in cuprates is presented. Comparison with available experimental data supports the robustness of the proposed topological scenario for superconductivity of copper oxides.

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Introduction. The phenomenon of high-temperature superconductivity (HTSC) was discovered in two-dimensional electron systems of cuprates in 1986 [9]. The joint phase diagram [10] of hole- and electron-doped compounds in the temperature-doping ($T$-$x$) plane, reproduced in Fig. 1, shows two respective superconducting domes, whose external boundaries are situated at $x_c^h \simeq 0.3$ on the hole-doped side and $x_c^e \simeq 0.2$ on the electron-doped side. Their splitting at $x = 0$ is triggered by an intruding antiferromagnetic insulating Mott phase.

The origin of HTSC still remains a central issue of condensed matter physics, unresolved thirty years after its discovery. Apart from the puzzling arrangement of the phase diagram and the superconducting phase per se, attention has focused on the nature of extraordinary non-Fermi-liquid (NFL) behavior that has been well documented in the normal states of cuprates during the last decade. The dominant attempts to understand this challenging behavior (e.g. [11–16]), already promoted in the 1990s, postulated that its source lies in critical antiferromagnetic fluctuations that generate strong interactions between quasiparticles near the Fermi surface. In the course of time, this picture was embodied in a more sophisticated theoretical framework: the quantum critical point (QCP) scenario. The driving QCPs are identified as $T = 0$ end points of lines of second-order phase transitions, as a rule of the Néel type. NFL behavior is ascribed to the divergence of the QCP density of states $N(T = 0)$ through its proportionality to the effective mass $M^*$ [17–19], given by the standard formula $M^* = z^{-1}(1 - (\partial \Sigma(p, \varepsilon)/\partial \varepsilon)_{p_0})^{-1}$ with $p_0^2 = \hbar^2/2M$. The divergence of $M^*$ is attributed, as a rule, to vanishing of the quasiparticle weight $z = (1 - (\partial \Sigma(p_F, \varepsilon)/\partial \varepsilon)_{p_0})^{-1}$ in single-particle states at the Fermi surface, stemming from divergent contributions to the derivative $(\partial \Sigma(p_F, \varepsilon)/\partial \varepsilon)_{p_0} < 0$ induced by critical fluctuations. Conclusion: the Landau quasiparticle picture ceases to apply, “Quasiparticles get heavy and die” [19].

In actuality, however, spin fluctuations having the antiferromagnetic vector $\mathbf{Q} = (\pi, \pi)$ associated with Néel transitions are irrelevant to the divergence of $M^*$ [20]. Additionally, there exist generic theoretical objections [21], supported by experimental findings, against vanishing of the $z$ factor at points of second-order phase transitions. These objections become especially strong on the external boundaries $x_c^h$ and $x_c^e$ of the superconducting domain where, with certainty, the quasiparticle picture continues to hold. Furthermore, it has been stressed in Ref. [22] that quantum criticality develops a $T = 0$ phase transition into a state of broken symmetry. This fact definitely rules out the relevance of the QCP scenario to the cuprates, given the observation that on both sides of the joint phase diagram, the superconducting domes border on conventional FL phases [10, 23].

In contrast to such a QCP scenario, let us consider instead a different one associated with a topological critical point (TCP) where the Landau state loses its stability. By definition the system exclusively undergoes a rearrangement in which all its inherent symmetries are preserved – a circumstance that conforms with available experimental data on normal states of the cuprates. The nature of such a rearrangement was first addressed by I. M. Lifshitz in a seminal article published in 1960 [24]. In his scenario, the density of states $N(0)$ does not diverge at all, and the rearrangement merely involves a change of the number of sheets of the Fermi surface, with the Landau occupation numbers for quasiparticle states remaining intact at 0 and 1. This property implies that in

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Lifshitz transitions, the minimum of the ground-state energy $E$ is still achieved on the borderlines of the domain $D$ of all possible fermionic distribution functions $n(p)$ satisfying $0 \leq n(p) \leq 1$. Landau FL theory should then be applicable on both sides of a Lifshitz transition, in blatant contradiction to prominent experimental results, thus ruling out this species of topological rearrangement in the cuprates.

On the other hand, there exists a more profound topological rearrangement of the Landau state [1, 14] in which the minimum of $E$ occurs at an internal point of the domain $D$. Certainly, traditional Landau theory is no longer applicable for such systems, but it is natural to pursue its replacement by a more robust extension of the same profound strategy (see Ref. [6]). The crux of the problem then lies in the requirement that the optimal quasiparticle momentum distribution, hereafter denoted $n_s(p)$, must be derived from the variational condition [1]

$$\frac{\delta E}{\delta n(p)} - \mu = 0, \quad p \in \Omega. \quad (1)$$

Since the left side of Eq. (1) is nothing but the quasiparticle energy $\epsilon(p)$, this condition now implies the emergence, in the momentum regime $\Omega$, of a dispersionless, compact part of the single-particle spectrum, originally dubbed the fermion condensate (FC). (Other names commonly associated with the dispersionless portion of the spectrum $\epsilon(p)$ are flat band [2, 3] and zero-energy mode [25]).

A comment on the physical rationale for the term fermion condensation is in order, since this term has lately been been introduced in quite different contexts. (In particular, the dramatic experimental demonstration of fermion condensation in cold atomic gases [26] refers to condensation of fermion pairs, rather than individual fermions.) As background, let us recall that quantitative description of the conversion of a vapor of fermions to a quantum liquid is inherent in the Hohenberg-Kohn theorem [27]. This theorem declares that the ground-state energy $E$ is a unique functional of the density $\rho(r)$, whose equilibrium distribution is found through the variational condition

$$\frac{\delta E(\rho)}{\delta \rho(r)} = \mu. \quad (2)$$

In close correspondence, Landau theory [28] asserts that the ground-state energy $E$ of a Fermi liquid is a functional of the quasiparticle momentum distribution $n(p)$ (for a proof, see Ref. [6]). Investigation of nontrivial solutions of the associated variational condition [11] that violate the topological stability of the Landau FL state is the domain of the theory of fermion condensation.

We note further that fermion condensation shares with Bose-Einstein condensation (BEC) the property that the density of states $N(\epsilon)$ possesses a singular term $\propto \delta(\epsilon)$, although phase coherence is present only in the BEC case. Correspondingly, in both situations there is macroscopic occupation of the single-particle state of zero energy relative to the chemical potential, the Pauli principle being preserved in the case of fermion condensation because the FC particles have different momenta within the domain $p \in \Omega$.

This article is devoted to analysis of the NFL behavior of the copper oxides within the framework of FC theory. We first establish the prerequisites for breakdown of the topological stability of the Landau state. Precursors of its topological rearrangement beyond the TCP are then identified and examined, in turn, for the homogeneous electron liquid and for the 2D electron liquid in cuprates. Finally, we address and resolve the observed dichotomy of the resistivity of the normal states of high-$T_c$ cuprates, inexplicable in the QCP scenario.

**Prerequisites for a topological rearrangement of the Landau state.** As is well known [3, 24], violation of the topological stability of the Landau state is signaled by a change in the number of roots of equation

$$\epsilon(p, x_c) = 0 \quad (3)$$

that determines the structure of the Fermi surface. One sees that the topological rearrangement comes into play at a certain point $p_c$ in momentum space, rather than ubiquitously as in the QCP scenario.

With restriction to the nominal Fermi surface of the system, Eq. (3) may be recast in a more convenient form based on the FL formula $\epsilon(p \rightarrow p_c) = v_F(p_c) \Delta p$, where $\Delta p$ is the distance between the momentum $p$ and its critical value $p_c$. Upon inserting this relation into Eq. (3), it takes the form

$$v_F(p_c, x_c) = 0. \quad (4)$$

As an illustration, let us consider a homogeneous electron liquid having $p_c = p_F$ and $v_F = p_F/M^*$, so that the TCP emerges at critical density $n = n_c$ where $M^*(n)$ diverges, a behavior analogous to that envisioned in the QCP scenario [17, 19]. However, in stark contrast to the latter model, in which the divergence of $M^*$ is triggered by vanishing of the quasiparticle weight $z$, the TCP divergence is ensured by nullification of the difference $1 - (\partial \Sigma(p, \epsilon)/\partial p)^0$, which is associated with an enhancement of momentum-dependent components of the Landau amplitude $f$ of the effective interaction between quasiparticles. Important, once the weight $z$ remains finite, the quasiparticle pattern is preserved through the topological transition. The TCP value of $f$ is found with the aid of the FL relation

$$M/M^*(n) = 1 - \frac{1}{3} F^0_1(n) \quad (5)$$

for the effective mass [28, 29] in terms of the dimensionless first Legendre harmonic of the Landau interaction.
function $F_1^0 = f_1 N_0 > 0$. (This function is normalized by the density of states $N_0 = p_F M/\pi^2$ of the perfect Fermi gas, with $p_F = (3\pi^2 n)^{1/3}$, where $n$ is the particle density.) Accordingly, in 3D liquid $^3$He, the TCP emerges at $F_1^0(n_c) = 3$ (note that $F_1^0(n_0) \simeq 2$, with $n_0 = n(P = 0)$). In a 2D homogeneous electron liquid as realized in MOSFETs, the numerical factor in Eq. (5) changes from $1/3$ to $1/2$.

In essence, the interaction $f(n)$ changes smoothly with density ubiquitously, including the vicinity of the TCP $n_c$, to yield:

$$M/M^*(n) \propto n - n_c,$$

in agreement with experimental studies of the Shubnikov-de Haas magnetic oscillations in MOSFETs [30], which confirm the positivity of the interaction $f$.

**Precursors of topological rearrangement of the Landau state in a homogeneous electron liquid.**

In dealing with transport properties of strongly correlated electron systems of cuprates, predictions of the two scenarios – QCP and TCP – clash with each other even on the FL side of the superconducting phase transition occurring in a homogeneous electron liquid. To demonstrate this fact, we focus on the clean-limit behavior of the resistivity

$$\rho(T) = \rho_0 + A_1 T + A_2 T^2,$$

whose nonzero value is associated, as usual, with the presence of Umclapp processes. In this situation, the generic relation for the electric conductivity

$$\sigma \propto (e^2 n \tau)/m^*$$

is applicable, where $m^*$ represents the effective mass of light carriers and the collision time $\tau$ may be expressed in terms of the conventional Boltzmann integral

$$\tau^{-1} \propto \left\{ W[n_1 n_2 (1-n'_1)(1-n'_2) - (1-n_1)(1-n_2)n'_1 n'_2] \right\},$$

in which $n(\epsilon) = (1 + e^{\epsilon/T})^{-1}$. The block $W \propto |\Gamma|^2$ in the collision integral stands for the transition probability, $\Gamma$ being the scattering amplitude. The brackets in relation [9] signify integration/summation over all intermediate momentum and spin variables.

In this section we address the simplest case in which only a single band crosses the Fermi surface, so that $m^* = M^*$. Standard but lengthy algebra [31, 32] then leads to the behavior

$$\tau^{-1} \propto (M^*)^3 T^2 G^2.$$

Importantly, near the TCP the scalar part $\Gamma_0$ of the scattering amplitude $\Gamma$, associated primarily with the zeroth harmonic $F_0^0 = f_0 N_0 > 1$, takes a universal value [33]

$$\Gamma_0 \simeq \frac{f_0}{1 + f_0 N(0)} = \frac{N_0^{-1}}{1/F_0^0 + M^*/M} \simeq N_0^{-1} M/M^*$$

due to the TCP divergence of the ratio $M^*/M$. The same estimate is obtained for a spin-dependent part of the scattering amplitude by applying Mermin’s sum rule [34]. Upon inserting these results into Eq. (10) we arrive at the relation $\tau^{-1} \propto M^*$ and thus find

$$\rho(T, n) \propto 1/\sigma(T) \propto (M^*(n))^2 T^2 \propto C^2(T, n),$$

where $C(T, n) \propto M^*(n) T$ is the specific heat. We see that Eq. (12) is in agreement with the empirical Kadowaki-Woods (KW) relation $\rho(T)/C(T) = \text{const}.$ [32].

**Precursors of topological rearrangement of the Landau state in the 2D electron liquid of cuprates.**

Galilean invariance, originally employed by Landau in deriving the equations of FL theory, does not apply to the electron liquid present in solids. Nevertheless, an analog of the standard FL equations can be formulated for these systems as well, if one asserts gauge invariance of FL theory. This adjustment, introduced by L. P. Pitaevskii, permits the basic FL equation to be recast in the form [36]:

$$v(p) = v_0(p) + \int f(p, p_1)\n(p_1) \frac{2d^3p_1}{(2\pi)^3}.$$  

In this expression, the function $n(p)$ is the $T = 0$ quasi-particle momentum distribution given by the standard FL formula

$$n(p) = \theta(-\epsilon(p)),$$

which in 3D matter obeys the normalization condition

$$n = \int n(p) \frac{2d^3p}{(2\pi)^3}.$$
The quantity $v_0(p)$ playing the role of the bare group velocity is introduced as $v_0(p) = zT^c(p)/M$, where

$$T^c(p) = \lim_{k \to 0} T(p,k;\omega \to 0; kv_F/\omega \ll 1).$$

In homogeneous matter one has $zT^c(p) = p$ [39], and Eq. (13) coincides with the standard Landau equation [28, 29]. However, for an electron system moving in the external field of a crystal lattice, the quantity $v_0(p)$ should be treated phenomenologically, on an equal footing with the interaction function $f$. Finally, in the 2D electron systems of cuprates, a two-dimensional analog of Eq. (13) takes the form:

$$v(p) = v_0(p) - \int_C f(p,p_1) \cos \theta \frac{2p_1(l_1)dl_1}{(2\pi)^2}, \quad (17)$$

with $\cos \theta \propto (p \cdot v(p_1))$.

The integration path $C$, which, according to Eqs. (13) and (14), coincides with the Fermi line itself, is unknown, except for those few 2D compounds for which detailed angle-resolved photo-electron spectroscopy (ARPES) data are available. Among these are La$_{1-x}$Sr$_x$CuO$_4$ compounds (e.g., see Refs. [37, 39]), where the Fermi line looks like a square with rounded corners, as in the left panel of Fig. 2. If the interaction $f$ were weak, the Lifshitz-Volovik condition [3] would be met at doping $x_B \simeq 0.2$, where the LSCO Fermi line reaches boundaries of the Brillouin zone. Since the bare density of states $N(0)$ diverges logarithmically in this situation, we arrive at the so-called van Hove scenario advanced in Refs. [40], which allows the logarithmic divergence to be treated within the framework of a standard renormalization-group (RG) formalism. Most significantly, the results of such RG calculations [40] demonstrate flattening of the single-particle spectrum $\epsilon(p,x \simeq x_B)$, thereby confirming the robustness of the FC scenario of Ref. [1] based on variational condition [1].

However, the RG scenario is valid only in a narrow region of doping $x$ close to $x_B$, given the unrealistic assumption that the $e-e$ interaction $f(n)$ in the particle-hole channel is small. In reality, as attested by experimental values of the effective masses $M^*$ found in measurements of SdH oscillations, the magnitude of $f(n)$ is large. Therefore the true TCP never coincides with $x_B$, preventing a logarithmic divergence of the density of states $N(0)$ and hence rendering the RG formalism inapplicable.

It is instructive to compare results of calculations based on Eq. (17) in two relevant cases: (i) $p = p_a$, where the momentum $p$ lies on the axes, and (ii) $p = p_d$, where it is located on the zone diagonals, evaluating bare Fermi velocities $v_{Fd}^0$ and $v_{Fa}^0$, assuming as usual tight-binding spectra

$$\epsilon^0(p) = \epsilon_0 - 2t(\cos px + \cos py) - 4t' \cos p_x a \cos p_y a \cdots,$$

for which $v_{Fa}^0 < v_{Fd}^0$. After some algebra one finds

$$v_{Fd} = v_{od} - F_d,$$

$$v_{Fa} = v_{oa} - F_a \sqrt{2},$$

where $F > 0$ stands for the corresponding integral on the right side of Eq. (17) containing the repulsive interaction function $f$. Presumably, $F_a \simeq F_d$, so that the topological stability breaks down on the axes, although the opposite situation cannot be excluded.

As seen from Eq. (19), violation of the topological stability of the Landau state triggered by strong repulsive interactions between electrons in the particle-hole channel exhibits itself ubiquitously in a wide region of doping $x$, giving rise to flattening of the single-particle spectrum $\epsilon(p)$ near corner points of the Fermi line. Furthermore, already on the FL side of the TCP, the total electron system of an overdoped LSCO compound is separated into two subsystems, consisting of: (i) light carriers residing in the nodal region, whose properties remain unchanged through the topological transition, and (ii) heavy carriers occupying the domain associated with the saddle points, whose effective mass $M^*$ is somewhat enhanced at $x \to x^h_B$. The light carriers contribute predominantly to the electric current, whereas the heavy carriers serve to enhance the inverse collision time $\tau^{-1}$, producing NFL behavior of the resistivity $\rho(T)$. This behavior is attributed to the divergence of the density of states $N(0)$ determined by the integral

$$N(0,x) \propto \int \frac{d\phi}{v_F(\phi,x)},$$

which replaces the effective mass $M^*$ in anisotropic 2D electron systems of cuprates.

According to Eqs. (19) and (18), the Fermi velocity $v_F(\phi,x)$ obeys the formulas

$$v_F(\phi,x) - v_F(\phi,x^h_c) \propto x - x^h_c,$$
On the opposite side of the joint phase diagram, where
the Fermi line tends toward a circular shape (right panel of Fig. 2), the situation changes: the coefficient \(a_2(x)\) diverges at \(x \to x_c^e\) [43], which conflicts with Lifshitz’s topological scenario [24]. However, the underlying reason for the discrepancy is associated with the implementation of the topological scenario itself in Ref. [24], rather than with a deficiency of the concept. Indeed, in the present case the line integral in Eq. (17) has the same structure as in a homogeneous 2D electron liquid; consequently, instead of Eq. (19) one obtains

\[
v_F(\phi) = v_{0F}(\phi) - F_1/2, \tag{23}
\]

where \(F_1\) is the first harmonic of the Fourier expansion of the interaction function \(f\). The similarity between Eqs. (19) and (23) implies that the behavior of the Fermi velocity \(v_F(\phi)\) at small \(\phi\) and \(x - x_c^e\), given by Eqs. (21), remains intact in going from the LSCO family to the LCCO family. However, the transition angle \(\phi^e\) now becomes large enough (see Fig. 3) that the overwhelming contributions to density of states come from the region \(\phi < \phi^e\). This makes the pivotal difference, since upon inserting Eq. (21) into Eq. (20), simple algebra leads to a result

\[
N_c(0,x) \propto |x - x_c^e|^{-1/2} \tag{24}
\]

divergent at \(x_c^e\), contradicting the result given in Ref. [24]. We conclude that it is the specific shape of the LCCO Fermi line that underlies the crucial difference between the densities of states \(N_c(0)\) and \(N_{h}(0)\).

Notably, on the electron-doped side of the phase diagram, light carriers occupy almost all vicinities of the Fermi line, except for four spots in the antinodal region where the topological stability of the Landau states breaks down. This implies that dominant contributions to the inverse collision time \(\tau^{-1}\) are due to inelastic scattering processes in which one of the light carriers is converted to a heavy carrier, yielding \(\tau^{-1} \propto N_c(0,x)^{\alpha}\) and consequently

\[
A_2(x \to x_c^e) \propto N_c(0,x) \propto (x - x_c^e)^{-1/2}. \tag{25}
\]

It should be noted that this result differs from the behavior \(A_2(x) \propto (x - x_c^e)^{-1}\) proposed by the authors of the experimental report [43]. However, the empirical value of the exponent \(\alpha\) in the divergent part of \(A_2(x)\) behaving as \((x - x_c^e)^{-\alpha}\) should be extracted by multiplying the experimental points nearest \(x_c^e\) (cf. Fig. 2 of Ref. [43]) by the factor \((x - x_c^e)^\alpha\) and varying \(\alpha\) until the result is independent of \(x\). Straightforward calculations then demonstrate that \(\alpha = 1/2\) gives a better fit than \(\alpha = 1\).

**Dichotomy in the resistivity of normal states of high-\(T_c\) cuprates.** We are now ready to analyze the NFL behavior of the resistivity \(\rho(T,x)\) of normal states of cuprates, several experimental findings having attracted broad attention. First, the dominant feature that must be acknowledged is that \(\rho(T)\) exhibits a predominantly linear dependence on temperature (note especially Refs. [8, 10, 41–47]). Second, instead of collapsing to a single critical point, as the conventional QCP scenario prescribes, the coefficient \(A_1(x)\) is found to grow linearly with the difference \(|x - x_c|\). While the linearity of \(\rho(T)\) might be accounted for within the framework of the Hertz-Millis-Moriya approach [48], the observed growth of the coefficient \(A_1(x)\) with the difference \(|x - x_c|\) defies explanation within the fluctuation scenarios.

In contrast, as seen in particular from Figs. 4 and 5, all these anomalies may be properly elucidated within the theoretical framework of the FC concept [1, 2, 4, 6]. Because the pseudogap phase is nonexistent close to boundaries of the joint phase diagram, [38, 45] one can make
of a Thouless criterion, which reveals the trajectory of trend. Right axis: circles show experimental data \[8, 41, 42\], use of a formula from Nozières \[4\], numerous experimental studies (notably Refs. \[8, 43–46\]).

The correspondence between our predictions and experiment involves the residue of the pole of

\[
\tau^{-1}(x) \propto T \eta(x),
\]

where \(\eta(x)\) measures the ratio of the volume in momentum space occupied by the FC to the total Fermi volume. Thus we see that in external magnetic fields strong enough to suppress superconductivity, the FL behavior \(\rho(T) \propto T^2\) of cuprates is replaced by NFL behavior linear in \(T\), with

\[
A_1(x) \propto \eta(x).
\]

The corresponding between our predictions and experiment is seen in Fig. 4.

Let us now turn to elucidation, within the FC scenario, of the enigmatic proportionality between \(A_1(x)\) and the critical temperature \(T_c(x)\), established in numerous experimental studies (notably Refs. \[8, 43–46\]). The theoretical curve for \(T_c(x)\), is found with the aid of a Thouless criterion, which reveals the trajectory of a pole of the scattering amplitude in the Cooper channel at \(T = T_c\). The desired equation for determining \(T_c\)

\[
\epsilon(p \in \Omega) = T \ln \frac{1 - n_s(p)}{n_s(p)},
\]

which implies that the FC momentum distribution found from Eq. \(1\) at \(T = 0\) remains unchanged as \(T\) is elevated. Upon applying this formula in Eq. \(4\), simple manipulations yield

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equation determining $T_c$:

$$1 = -\left[\lambda_D \ln(\Omega_D/T_c) - \eta^+\eta e_k/T_c\right]/2.$$  

(31)

Here a phenomenological parameter $e_k \simeq e_F^0$ is introduced to render the FC term in Eq. (31) dimensionless, and $\lambda_D$ stands for the conventional Cooper pairing constant, while $\eta^+ = \eta_{k,k+1} > 0$ denotes its FC counterpart.

From Eq. (31), we can readily see that the QCP-BCS choice $\lambda_D < 0$ fails to explain the proportionality between $A_1(x)$ and $T_c(x)$ observed in experiment. At doping $x$ where $\eta(x) = 0$, the critical temperature $T_c(x)$ has already a nonzero value, while $A_1(x)$, given by Eq. (28), vanishes. On the other hand, at $\lambda_D > 0$ where BCS superconductivity is hampered, Eq. (31) can be easily recast in the form:

$$T_c(x) = \eta(x) \frac{\eta^+/2}{1 + \lambda_D L/2},$$  

(32)

having defined a constant $L = \ln[1/\eta^+\eta]$.

Thus, given $\lambda_D > 0$, the ratio $A_1(x)/T_c(x)$ becomes doping independent, in agreement with experiment on both electron-doped and hole-doped sides of the joint phase diagram of cuprates. But importantly, despite common opinion otherwise, this independence demonstrates that the dominant contributions to the quantities of most interest stem from the momentum domain occupied by the FC, rather than from the similarity of relevant interactions between quasiparticles.

**Discussion.** First of all, we observe that the principal difference between the quantum critical point (QCP) and topological critical point (TCP) scenarios for HTSC materials is expressed decisively just near the boundaries of superconducting domes where $T_c(x)$ terminates. In fact, the QCP behavior of $T_c(x \rightarrow x_c)$ is supposed to be fully controlled by the Cooper logarithmic term in Eq. (31), with a pairing constant $\lambda_D(x)$ that is negative inside the superconducting dome, but drops to zero at critical doping $x_c$, demonstrating the behaviors

$$T_c^{QCP}\propto e^{-2/\lambda_D(x)}, \quad \frac{dT_c^{QCP}\propto e^{-2/\lambda_D(x)}}{dx}.$$

(33)

The QCP derivative $dT_c^{QCP}(x)/dx$ must then vanish emphatically at $x = x_c$ by virtue of the condition $\lambda_D(x_c) = 0$, in clear contradiction to experiment, which indicates that this derivative remains finite or even diverges. This discrepancy once again rules out the QCP scenario as an explanation of HTSC of cuprates.

In contrast, the TCP behavior $dT_c(x)/dx \propto d\eta/dx$ associated with the doping evolution of the FC parameter $\eta(x)$ is free from this flaw. In numerous solvable models of fermion condensation as well as numerical calculations, the parameter $\eta(x)$ grows linearly with $x - x_c$ or even more rapidly, in unison with available experimental data.

The scenario of high-$T_c$ superconductivity introduced in the present article is distinguished by the occurrence of several FC spots, domains in momentum space where the quasiparticle distribution $n_e(p)$ departs from that of a FL, which is restricted to the values 0 and 1. It is this departure, stable with respect to elevation of $T$, that triggers NFL behavior of strongly correlated Fermi systems, well documented in diverse experimental studies of the last decade. This feature opens the prospect of high-$T_c$ solutions of the BCS gap equation that depend linearly on an attractive effective coupling constant and possess nontrivial symmetry, notably $D$-wave (cf. Eq. (32)), such solutions being nonexistent in homogeneous 2D electron liquid having a single FC spot. Presumably, this situation is in some respects universal; in particular, it is to be expected in a triangular lattice, where, according to strong-coupling calculations of Ref. [40], the number of the FC spots is equal to 6.

Another important result of our analysis bears on explanation of the so-called anomalous criticality uncovered in conductivity studies of overdoped LSCO compounds. Here we refer to the empirical observation of a gradual crossover of the coefficient $A_2(x)$ with passage through critical doping $x_c^h$. This implies the absence of a divergence of the coefficient $A_2(x)$ on the FL side the phase diagram – thus negating a dictum of the QCP scenario. Another salient feature of anomalous criticality is the emergence of a linear-in-$T$ NFL term in the resistivity $\rho(T)$, whose magnitude $A_1(x)$ grows linearly with the difference $x_c^h - x$.

In view of what the existing observations and their study based on the FC concept has revealed, it is worth repeating these words of Cooper et al. nearly a decade ago: “The strange-metal physics of hole-overdoped cuprates is associated not with the presence of a quantum critical point, but instead with a novel extended phase,” which we have identified as the flat-band state.

In summary, we have demonstrated that in copper oxides, the total domain of high-temperature superconductivity is confined between two critical values of the doping level, $x_c^h$, and $x_c^l$, associated with the boundaries of momentum region occupied by a fermion condensate, which emerges due to breakdown of topological stability of the Landau state. Arguments have been presented, both experimental and theoretical, that refute the alternative view that the extraordinary properties of the cuprates are driven by quantum critical points. Our analysis and its results suggest that while doping of CuO$_2$ planes by electrons or holes may convert an antiferromagnetic insulator such as R$_2$CuO$_4$ into a conductor, this observation does not imply that antiferromagnetism and/or antiferromagnetic fluctuations promote superconductivity of copper oxides, but indeed rather the contrary – quite at variance with widespread belief.
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[1] V. A. Khodel and V. R. Shaginyan, JETP Lett. 51, 553 (1990).
[2] G. E. Volovik, JETP Lett. 53, 222 (1991).
[3] G. E. Volovik, Springer Lecture Notes in Physics 718, 31 (2007).
[4] P. Nozières, J. Phys. I France 2, 443 (1992).
[5] V. A. Khodel, J. W. Clark, M. V. Zverev, Phys. Rev. B 78, 075120 (2008).
[6] V. A. Khodel, J. W. Clark, M. V. Zverev, JETP Letters 105, 267 (2017).
[7] R. A. Cooper et al., Science, 323, 603 (2009).
[8] J. G. Bednorz, K. A. Müller, Z. Phys. B 64, 189 (1986).
[9] N. P. Armitage, P. Fournier, R. L. Greene, Rev. Mod. Phys. 82, 2001 (2012).
[10] P. Monthoux, A. V. Balatsky, D. Pines, Phys. Rev. Lett. 67, 3448 (1991).
[11] T. Moriya, R. Ueda, Adv. in Physics 49, 555 (2000).
[12] H. v. Lüchheysen, A. Rosch, M. Vojta, P. Wölfle, Rev. Mod. Phys. 79 1015 (2007).
[13] P. Gegenwart, Q. Si, F. Steglich, Nat. Phys. 4, 186 (2008).
[14] D. J. Scalapino, Rev. Mod. Phys. 84, 1383 (2012).
[15] H. Kontani, Transport Phenomena in Strongly Correlated Fermi Liquids, Springer-Verlag, Berlin, (2013).
[16] J. A. Hertz, Phys. Rev. B 14, 1165 (1976).
[17] P. Coleman et al., J. Phys.: Condens. Matter 13 R723 (2001).
[18] A. V. Chubukov, V. M. Galitski, V. M. Yakovenko, Phys. Rev. Lett. 94, 046404 (2005).
[19] V. A. Khodel, J. W. Clark, M. V. Zverev, JETP Lett. 90, 628 (2010); 94, 73 (2011).
[20] T. Yoshida et al., Science 349, 506 (2015).
[21] B. Keimer et al., Nature 518, 179 (2015).
[22] I. M. Lifshitz, JETP 11, 1130 (1960).
[23] F. Wilczek, arxiv: 1604.05669.
[24] M. Greiner, C. A. Regal, D. S. Jin, Nature 426, 537 (2003).
[25] P. Hohenberg, W. Kohn, Phys. Rev. B864 (1964).
[26] L. D. Landau, JETP 3, 920 (1957); JETP 8, 70 (1958).
[27] A. A. Abrikosov, L. P. Gor’kov, I. E. Dzjaloshinski, Methods of Quantum Field Theory in Statistical Physics (Pergamon Press, Oxford-London-Edinburgh-New York-Paris-Frankfurt, 1965).
[28] A. A. Shashkin, S. V. Kravchenko, in Strongly Correlated Electrons Two Dimensions, edited by S. V. Kravchenko (Pan Stanford Publishing, Singapore, 2016).
[29] D. Pines, P. Nozières, The Theory of Quantum Liquids Vol. I (W. A. Benjamin, New York, 1966).
[30] A. A. Abrikosov, I. M. Khalatnikov, Usp. Fiz. Nauk. 66, 177 (1958).
[31] V. A. Khodel, P. Schuck, Z. Phys. B104 505 (1997).
[32] N. D. Mermin, Phys. Rev. 159, 161 (1967).
[33] K. Kadowaki, S. B. Woods, Solid State Commun. 58, 507 (1986).
[34] L. P. Pitaevskii, JETP 10, 1267 (1960).
[35] T. Yoshida et al., Phys. Rev. B74, 224510 (2006).
[36] T. Yoshida et al., J. Phys. Condens. Matter 19, 125209 (2007).
[37] T. Yoshida et al., Phys. Rev. B 93, 014513 (2016).
[38] V. Yu, Irkhin, A. A. Katanin, M. I. Katsnelson, Phys. Rev. Lett. 89, 076401 (2002).
[39] N. E. Hussey et al., Phys. Trans. R. Soc. A 369, 1626 (2011).
[40] N. E. Hussey et al., J. Phys.: Conference Series 449, 012004 (2013).
[41] K. Jin et al., Nature 476, 73 (2011).
[42] N. Doiron-Leyraud et al., Phys. Rev. B80, 214531 (2009).
[43] N. Doiron-Leyraud et al., Eur. J. Phys. B78, 23 (2010).
[44] L. Taillefer, Annu. Rev. Condens. Matter Phys. 1, 51 (2010).
[45] I. Bozovic et al., Nature, 536, 309 (2016).
[46] T. Moriya, K. Ueda, Adv. Phys. 49, 555 (2000).
[47] D. Yudin et al., Phys. Rev. Lett. 112, 070403 (2014).