Symmetry breaking in geometry
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Abstract. A geometric mechanism that may, in analogy to similar notions in physics, be considered as “symmetry breaking” in geometry is described, and several instances of this mechanism in differential geometry are discussed: it is shown how spontaneous symmetry breaking may occur, and it is discussed how explicit symmetry breaking may be used to tackle certain geometric problems. A systematic study of symmetry breaking in geometry is proposed, and some preliminary thoughts on further research are formulated.

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Introduction

In physics, the notion of symmetry breaking is well established and has, in some instances, attracted substantial attention. As a consequence, the phenomenon is fairly well understood, and even if the details of symmetry breaking may depend on the specific physical theory it appears in, there seems to be a general agreement about meaning and significance of the notion.

In geometry, on the other hand, a notion of symmetry breaking is mostly unheard of — despite the fact that the notion of symmetry has become central to an understanding of geometry (in fact, geometries), most notably through Klein’s “Erlanger programme” [43], at about the same time when symmetry considerations gained interest in physics, cf [26]. Thus, phenomena that could — in analogy to the notion in physics — be considered as “symmetry breaking” in a geometric context were, at least to our knowledge, never investigated in a systematic way.

The aim of this text is to give a first outline of a more systematic approach to these phenomena, as well as to the related geometric techniques, in order to obtain a better understanding not only of these phenomena but also of the interrelations between different geometries.

Thus we shall present a cabinet of curiosities of what we consider “symmetry breaking” phenomena in (differential) geometry, trying to present a variety of ways resp contexts in which such symmetry...
breaking may occur on the one hand while, on the other hand, attempting to highlight the features of the presented phenomena that in our view make them qualify as “symmetry breaking” in order to obtain a first abstraction for a more systematic study.

Naturally we will draw from our areas of expertise: thus we will focus on symmetry breaking in sphere geometries, where we or our collaborators have encountered, described and “collected” many of these phenomena — in the case of the second author for more than three decades now. Confining to sphere geometries also has the advantage of providing a clear arrangement while offering a context that is rich enough to display a wide variety of symmetry breaking phenomena.

To clarify similarities and differences with symmetry breaking in physics that we see, we will begin with a very brief introduction of the notion from a physics point of view, tailored for the purpose of this text, including an abstract description as well as several examples: the last of those examples, illustrated in Fig 1, beautifully suggests a relationship between symmetry breaking in physics and in geometry. By drawing on analogies with physics in this endeavour we not only hope to justify our terminology but, more importantly, we hope that insight from more than a century of relevant research in physics may help to structure the quest into symmetry breaking in geometry as well as to suggest the “correct” research questions. However, on closer inspection, there are also crucial differences between the notions of “spontaneous symmetry breaking” in physics and geometry; how significant those differences are may become more clear once a definition in geometry is settled.

In the following section we will then set the scene: we make an attempt at formulating a (working) definition of “symmetry breaking” in geometry, as suggested by Klein’s “Erlanger programme” [43]; we give a brief introduction to the sphere geometries that will provide the realm for our curiosity shop to be presented in the main section of the text; and finally we present two examples of symmetry breaking in the context of elementary geometry.

With these preparations we then present a variety of instances of what we consider as “symmetry breaking phenomena” in geometry in the main section of this text, adopting three different points of view at a more technical level. In the first part of this section we describe phenomena from a classical, projective geometry viewpoint, closely following the “mechanics of symmetry breaking” as set out in Sect 2; most of the discussed examples concern classification problems, that show how “spontaneous symmetry breaking” may occur in various flavours. In the second part we switch our viewpoint to a use of homogeneous coordinates that not only allows for a more algebraic treatment of symmetry breaking in the given context but, in several cases, also for a finer control of the symmetry breaking process. In the final third part of the section we demonstrate how the additional structure provided by an integrable systems context provides for a very efficient description of the interaction between symmetry breaking and the geometry of the objects concerned; here we discuss instances of explicit symmetry breaking and how these can be used to tackle, for example, discretization in differential geometry. This last approach also suggests a close relationship between what we consider as “symmetry breaking” in (integrable) geometry and integrable reductions of the underlying differential equations.

Based on this cabinet of curiosities we conclude with a first attempt at formulating some questions or problems that may lead to further insight into symmetry breaking in geometry.

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1. Symmetry breaking in physics

To draw analogies as well as to detect differences between symmetry breaking phenomena in physics and those geometric phenomena sketched in this text later, we give a very brief description of (our understanding of) what symmetry breaking means in physics and illustrate this description by some examples. This in turn may motivate the choice of terminology adopted in our exposition.

The phenomenon of spontaneous symmetry breaking occurs in many different theories of classical and quantum physics. Since quantum theories seem conceptually further away from the differential
geometry of surfaces than classical theories, we focus on the latter. Mathematically, the state or configuration of a classical physical system is described by a map from a domain manifold into a target manifold. Typical examples of domain manifolds are the real time line of Newtonian mechanics or Minkowski spacetime for special relativistic theories. The target manifold describes the degrees of freedom of the system, which may be the position of a particle, the value of some field at a point in spacetime or the angle between components of a kinematic system. The laws of the physical system under consideration are the equations of motion, usually differential equations. Their solutions are the physically realizable states.

Typically, the physical laws are invariant under the action of some Lie group on the configuration space, for example, translations in time or the isometries of the domain or target manifold, when these carry a Riemannian structure. This Lie group is then considered as a symmetry group of the theory.

In general, the space of solutions to the equations of motion does not consist of a single point but rather is a vector space or a manifold, typically of twice the dimension of the target space when the equations of motion are second order differential equations. Therefore, a physical state is not determined solely by the equations of motion, but additional parameters, such as initial conditions, are required to specify a particular state. Even if the laws of motion have certain symmetries, asymmetry of initial conditions leads to asymmetry of the corresponding solution, see also [34, §2.1]. Consequently, the symmetries of the laws of motion are in general not symmetries of states. However, the symmetry group of the laws acts on the configuration space and preserves the subspace of physically realizable states: it maps solutions to solutions, for further details see [58, Chap 3] or [1, Chap 4].

For example, the electromagnetic field can be described by a 2-form field on Minkowski space-time. The homogeneous Maxwell equations confine this 2-form to be closed and the inhomogeneous Maxwell equations relate its divergence to the electric 4-current acting as a source for the electromagnetic field. In the source-free case, the Lorentz group, the isometry group of Minkowski space, is a symmetry group of Maxwell’s equations. As such, it acts on the space of solutions. On the other hand, any particular solution can be specified by prescribing its values on a spacelike hypersurface. The only solution that is invariant under the full Lorentz group is the vacuum solution, where the field vanishes everywhere. This feature, a unique completely symmetric vacuum state, is typical for field theories.

As another example, consider a free particle in 3-dimensional Euclidean space. Its position is described by a map from the real time line to 3-dimensional Euclidean space; Newton’s laws that govern its motion are invariant under the group of Euclidean motions. The space of solutions of Newton’s equations of motion is the space of affine parametrizations of straight lines and of constant curves, that is, points. Any particular solution can be characterized by its position and velocity at an arbitrary instant of time — thus the solution space is a 6-dimensional space, the product of 3-dimensional Euclidean space and a 3-dimensional real vector space. No solution is invariant under the full symmetry group of Euclidean motions, but the constant solutions have higher symmetry than the straight line solutions: any constant solution is invariant under the subgroup of rotations about any axis that contains that point. The remaining group of translations acts transitively on these point states. Moreover, these point states of highest symmetry are all states of lowest energy, that is, ground states.

Although the free particle in Euclidean space is usually not mentioned as an example of spontaneous symmetry breaking, it definitely seems to share some properties with other physical systems, where spontaneous symmetry breaking is said to happen, at least at the purely classical level.

In fact, if the symmetry is broken by means of an asymmetric initial condition, we will speak of induced or explicit symmetry breaking rather than of spontaneous symmetry breaking that occurs “spontaneously”, i.e., without any (recognizable) external cause, cf [22, Sect 4.2]. This distinction will be made in several places throughout the text that follows.

Probably the most famous example of spontaneous symmetry breaking is the Higgs mechanism, see for example [45, Chap 4.4]. The electroweak theory without the Higgs field is a gauge theory with gauge group $SU(2) \times U(1)$. However, this theory predicts massless gauge bosons, which are not observed in experiments. In order to give mass to three of the four bosons, one introduces a new field, the Higgs-field. Its classical field equations do not admit a vacuum solution (state of lowest energy) with the full $SU(2) \times U(1)$ symmetry: thus, no vacuum solution is invariant under
the whole symmetry group of the field equations. Picking one such vacuum solution, describing
the value of the Higgs-field by its difference from the chosen asymmetric vacuum solution and
reformulating the field equations in terms of this difference, hides the symmetry in the field
equations. It appears as if the $SU(2) \times U(1)$ symmetry is broken. In particular, three of the four
gauge bosons appear to acquire mass. It should be mentioned that although no vacuum state
admits the full gauge symmetry, there is a state of higher symmetry that does not have the lowest
energy though.

As a mechanical example for spontaneous symmetry breaking, consider a thin elastic cylindrical
rod of some length and confine its ends to some fixed distance smaller than its length. If no addi-
tional forces are applied, the problem has cylindrical symmetry, but there is no unique, symmetric
state of lowest energy. Instead there is a 1-parameter family of asymmetric states of lowest energy,
one of which can be transformed into any other by a rotation around the axis through the fixed
endpoints. The most symmetric state, where the rod is squeezed to a straight line between its
endpoints still is a solution, but it is not the solution of lowest energy and furthermore highly
unstable.

Finally, we return to the example illustrated in Fig 1, that exposes a physical phenomenon of a
similar flavour as the elastic rod example above, cf Edgerton’s photograph [33] and [55, Chaps 1.3]:
the splash of a drop of fluid (here: coloured and viscosity-altered water) that hits a fountain created
by a pair of preceding drops falling vertically into the same fluid, see [47, Sect 6.3]. Clearly, the
equations of motion as well as the initial state have an $SO(2)$ rotational symmetry about the trajecto-
ry of the drops, as they travel towards the surface of the fluid, as an axis. Upon the
collision of the drop and the fountain the splash at first retains the initial rotational symmetry,
but then forms (roughly equidistant) “tentacles”: hence the smooth rotational symmetry is broken
to yield (approximately) a discrete rotational (dihedral) symmetry, generated by a finite rotation
about the same axis as before.

Note: the notion of spontaneous symmetry breaking in these examples relates to the loss of
symmetry in a “physically optimal” (least energy) solution of the corresponding (symmetric)
equations of motion — while this is a natural notion in physics it is less so in geometry, as it
presupposes a variational description of the geometric configurations under consideration.

2. The mechanics of symmetry breaking in geometry

A mechanism that could be considered as “symmetry breaking” in geometry has been described
beautifully by Klein in his Erlanger programme [43] of 1893, based on a group theoretic approach
to geometry [43, §1]:

Geometry. Given a manifold of geometric elements and a (transformation-)group acting on it;
investigate the configurations belonging to the manifold with respect to those properties that
remain unaltered by the transformations of the group.

This immediately induces a partial order on the geometries on a given manifold, cf [43, §2]:

Subgeometry. A geometry $(M, H)$ is a subgeometry of a geometry $(M, G)$ if the transformation
group $H$ of $M$ is a subgroup of the transformation group $G$ of $M$.

Motivated by the observation, how many classical geometries arise as subgeometries of a suitable
projective geometry, by “adjoining” an “absolute configuration” $C \subset M$, Klein also gives an
alternative characterization of a subgeometry $(M, H)$ of a geometry $(M, G)$ by requiring the
(transformation) subgroup $H \leq G$ to be the stabilizer group of the absolute configuration $C$.
Thinking of a transformation group acting on an manifold as a “symmetry group” of, for example,
a geometric classification problem, a terminology of “symmetry breaking” suggests itself:

Symmetry breaking. We will speak of symmetry breaking when a geometric investigation in a
geometry $(M, G)$ gives rise to a specific subgeometry $(M, H)$ or, more specifically, to an absolute
configuration $C \subset M$ that fixes a subgeometry $(M, H)$ via the stabilizer subgroup $H \leq G$ of $C$.
And, depending on whether $C$ is imposed or appears without apparent cause, we shall refer to the
 corresponding symmetry breaking as explicit resp spontaneous.
Clearly, this description of our use of terminology does not qualify as a mathematical definition, for example, the lack of an “apparent cause” yields anything but a sound mathematical criterion for symmetry breaking to be spontaneous. However, it provides a first abstraction for a series of phenomena that we observe in geometric research and that seem worthy of a more systematic investigation. Thus this description will be used as a working “definition”, until a better understanding of the related phenomena may lead to a “proper” mathematical definition. The aim of what follows is to discuss examples and scope of this notion of symmetry breaking, in order to work towards such a better understanding, and a better definition.

Note: our notion of spontaneous symmetry breaking in geometry does not refer to any particular type of solution of a geometric problem; in fact, we will often exclude trivial or degenerate solutions and observe “spontaneous symmetry breaking” as a loss of symmetry in generic solutions.

2.1. Sphere geometries

In this text, we will focus on “symmetry breaking” in sphere geometries. To this end, we will give a short introduction to the sphere geometries that we shall discuss; for further details the interested reader is referred to [3], [23] or [37].

We start with Möbius (conformal) geometry as probably the simplest and most familiar of the sphere geometries:

**Möbius geometry.** The geometry of the hypersphere-preserving (point-) transformations of a (conformal) sphere.

Thus the elements of Möbius geometry are points and hyperspheres of an n-sphere $S^n$ — that needs to be equipped with a conformal structure if one wishes to give a differential geometric characterization of hyperspheres as subsets of points of $S^n$. In particular, the manifold $M$ of elements of the geometry, that the Möbius group acts on, consists of two components, a set of points and a set of hyperspheres, that the Möbius group acts on separately. Classically, hyperspheres are unoriented in Möbius geometry; however, in various situations it is also useful to consider oriented hyperspheres, e.g., when considering Möbius geometry as a subgeometry of Lie sphere (contact) geometry:

**Lie sphere geometry.** The geometry of transformations of hyperspheres (including points) of a sphere that preserve oriented contact.

Here, the manifold $M$ of elements of the geometry consists of (oriented) hyperspheres and points, as degenerate hyperspheres: the group of Lie sphere transformations acts on hyperspheres, and may turn a hypersphere into a point or vice versa. An example is given by a “parallel transformation” that adds a constant to the radius of every hypersphere. The description of (hyper-)surfaces in Lie sphere geometry relies on the notion of contact elements, 1-parameter families of hyperspheres in oriented contact (at a common touching “point” hypersphere).

Just as Möbius geometry can (by distinguishing “point spheres”) be obtained as a subgeometry of Lie sphere geometry, another subgeometry is obtained by distinguishing hyperplanes among the (oriented) hyperspheres:

**Laguerre geometry.** The geometry of transformations of hyperspheres and -planes, respectively, of a Euclidean space that preserve oriented contact.

Here, we think of a Euclidean space as obtained from a sphere (via stereographic projection) by distinguishing a point (at infinity). In fact, fixing a “point at infinity” in a Möbius geometry yields a similarity subgeometry of Möbius geometry rather than a Euclidean subgeometry, where also a unit length needs to be fixed. Accordingly, a narrow Laguerre geometry is given by the requirement that the tangential distance of hyperspheres be preserved.

2.2. Symmetry breaking: the classical viewpoint

Klein’s Erlanger programme also provides a description of the Möbius and Lie sphere geometries as subgeometries of suitable projective geometries.
Embedding the ambient sphere $S^n$ of Möbius geometry into a projective space $\mathbb{R}P^{n+1}$, hyperspheres appear as hyperplane intersections with $S^n$. In this way, it is readily clear that any projective transformation $\mu$ of $\mathbb{R}P^{n+1}$ that fixes $S^n$, $\mu(S^n) = S^n$, restricts to a Möbius transformation of $S^n$; the converse is true, but the extension of a Möbius transformation to the ambient projective space requires some more detailed work, cf [43, §6] or [37, §1.3.9]. Thus Möbius geometry of $S^n$ is obtained as a subgeometry of the projective geometry of $\mathbb{R}P^{n+1}$ by adjoining $S^n \subset \mathbb{R}P^{n+1}$ as an absolute quadric.

In a similar way, the Lie sphere geometry of $S^n$ is obtained as subgeometry of $\mathbb{R}P^{n+2}$ by adjoining the Lie quadric $L^{n+1} \subset \mathbb{R}P^{n+2}$ as the space of hyperspheres in $S^n$, where $L^{n+1}$ is a quadric that contains lines but no higher dimensional projective subspaces: the projective lines in $L^{n+1}$ yield the contact elements in Lie sphere geometry mentioned above, cf [43, §7] or [23, Sect 1.6].

![Fig 2. Cascade of sphere geometries: the classical viewpoint](image)

Fixing a point sphere complex $S^n = L^{n+1} \cap \mathbb{R}P^{n+1} \subset \mathbb{R}P^{n+2}$ identifies Möbius geometry as a subgeometry of Lie sphere geometry (cf [43, §7]) — restricting projective transformations of $\mathbb{R}P^{n+2}$ to $\mathbb{R}P^{n+1}$ the orientation of hyperspheres is lost as the polar reflection in $\mathbb{R}P^{n+1} \subset \mathbb{R}P^{n+2}$, that reverses orientations of hyperspheres, restricts to the identity.

We already discussed above how (the wider) Laguerre geometry arises as a subgeometry of Lie sphere geometry by distinguishing a “point at infinity”, analogous to the way in which the similarity geometry of a Euclidean space arises as a subgeometry of Möbius geometry from the stabilizer group of a point, that yields the centre of projection for a stereographic projection.

Hyperbolic geometry can be obtained as a subgeometry of Möbius geometry in a similar way, by adjoining a hypersphere (not a point) at infinity or, equivalently, a point “outside” $S^n \subset \mathbb{R}P^{n+1}$; by polarity, this fixes the complex of all hyperspheres that intersect the given hypersphere orthogonally, hence a complex of hyperbolic hyperplanes in a Poincaré half sphere model of hyperbolic space.

Spherical geometry, on the other hand, is obtained by adjoining a great sphere complex, that is, a point “inside” $S^n \subset \mathbb{R}P^{n+1}$, which can be thought of as a centre of $S^n \subset \mathbb{R}P^{n+1}$, or as a centre of an antipodal reflection.

Thus we obtain the cascade of geometries of Fig 2, where arrows indicate a subgeometry relation.

### 2.3. Symmetry breaking: a gauge theoretic approach

Clearly, the mechanisms to pass from a (sphere) geometry to a subgeometry described above in terms of projective geometry are computationally expressed in terms of homogeneous coordinates. However, dropping the homogeneity of coordinate vectors allows for a more detailed geometric description — essentially, working with vector spaces (or bundles) allows to identify curvature instead of just its sign. We shall sketch these ideas below, for details the interested reader is referred to [23] or [37].

As the $n$-sphere $S^n \subset \mathbb{R}P^{n+1}$ and the Lie quadric $L^{n+1} \subset \mathbb{R}P^{n+2}$ are described as projective light cones of $\mathbb{R}^{n+1,1}$ and $\mathbb{R}^{n+1,2}$, respectively, the Lie sphere and Möbius transformation groups are covered (given) by the respective (projective) orthogonal groups $O(n+1, 1)$ resp $O(n+1, 2)$, cf [37, §1.3.14] or [23, Sect 1.7].

Fixing a point sphere complex to pass from Lie sphere geometry to a Möbius subgeometry amounts to fixing the orthogonal complement $\mathbb{R}^{n+1,1} \cong \{p\}^\perp \subset \mathbb{R}^{n+1,2}$ of a timelike line $[p] \subset \mathbb{R}^{n+1,2}$, to obtain the point spheres as elements of the projective light cone in $\{p\}^\perp$. Here $[.]$ denotes
the linear span of vectors. Normalizing \( p \), so that \( (p, p) = -1 \) with the inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^{n+1,2} \), any proper sphere admits homogeneous coordinates \( \sigma = p + s \in [p] \oplus \langle p \rangle^\perp = \mathbb{R}^{n+1,2} \). Now, since \( 0 = \langle \sigma, \sigma \rangle = -1 + \langle s, s \rangle \) we learn that \( s \in \mathbb{S}^n \) and, because reflection in \( \langle p \rangle^\perp \) reverses orientations of spheres, \( \pm s \in \mathbb{S}^n \) represent the two orientations of the Möbius geometric (unoriented) hypersphere \( [s] \in \mathbb{RP}^{n+1} \), cf [23, Sect 1.5] or [37, §1.1.3].

Further, fixing a space form vector \( q \in \mathbb{R}^{n+1,1} \) in the model space \( \mathbb{R}^{n+1,1} \cong \langle p \rangle^\perp \) for Möbius geometry the quadric

\[
Q^n := \{ y \in L^{n+1} \mid \langle y, q \rangle = -1 \},
\]

obtained as a hyperplane intersection with the light cone \( L^{n+1} \subset \mathbb{R}^{n+1,1} \), yields a space of constant sectional curvature \( \kappa = -\langle q, q \rangle \), see [37, §1.4.1]. Now, the (mean) curvature of a hypersphere \( s \in \mathbb{S}^n \) is given by \( -\langle s, q \rangle \), see [37, §1.7.9]; in particular, the hyperplanes of the quadric \( Q^n \) of constant curvature are those spheres orthogonal to \( q \) or, otherwise said, \( \mathbb{S}^n \cap \{ q \}^\perp \subset \mathbb{R}^{n+1,1} \) yields the hyperplane complex of \( Q^n \). Starting from a Lie sphere geometric setting, the symmetry between a quadric \( Q^n \) of constant curvature and its hyperplane complex \( P^n \) becomes more apparent:

\[
\begin{align*}
Q^n &= \{ y \in L^{n+2} \subset \mathbb{R}^{n+1,2} \mid \langle y, p \rangle = 0, \ (y, q) = -1 \}, \\
P^n &= \{ y \in L^{n+2} \subset \mathbb{R}^{n+1,2} \mid \langle y, p \rangle = -1, \ (y, q) = 0 \}.
\end{align*}
\]

This also shows that the passage from Lie sphere geometry to Euclidean geometry via (the narrow) Laguerre geometry is equivalent to that via Möbius geometry, as the two paths only differ by the order in which the point sphere complex and the space form vector are fixed.

Summarizing we obtain the following cascade of geometries, arrows again indicating subgeometry relations:

- projective geometry of \( \mathbb{P}(\mathbb{R}^{n+1,2}) \)
- Lie sphere (contact) geometry of \( \mathbb{S}^n \)
- projective geometry of \( \mathbb{P}(\mathbb{R}^{n+1,1}) \)
- (narrow) Laguerre geometry in \( E^n \)
- Möbius (conformal) geometry of \( \mathbb{S}^n \)
- Euclidean geometry
- hyperbolic geometry
- spherical geometry

*Fig 3.* Cascade of sphere geometries: the gauge theoretic viewpoint

### 2.4. Two examples from elementary geometry

To illustrate the described concepts of symmetry breaking we shall discuss two examples from elementary geometry: the first example is fairly simple, but relevant to certain integrable discretizations of surfaces, while the second displays a fairly rich structure with respect to symmetry breaking.

**Miguel’s theorem.** In discrete differential geometry circular nets are considered as discretizations of curvature line nets on a surface, cf [37, §8.3.16] or [7, Sect 3.1]: a *circular net* is a map from a 2-dimensional quadrilateral cell complex into a (conformal) \( n \)-sphere so that the image of each quadrilateral has a circumcircle. As an analogue of the Ribaucour transformation of a smooth (curvature line parametrized) surface one says that a second circular net, defined on the same domain, is a *Ribaucour transform* of the first if endpoints of corresponding edges of the two nets are concircular, that is, corresponding edges of the two nets form the same kind of quadrilaterals as the faces of each net do.

Now an existence question arises: does a given circular net \( x : Z \to \mathbb{S}^n \) admit Ribaucour transformations for any initial point \( \tilde{x}_0, o \in Z \)? This existence question can be reduced to a single quadrilateral \( (ijkl) \) of the cell complex \( Z \), with edges \((ij), (jk), (kl) \) and \((li)\). Namely, given \( \tilde{x}_i \) the points \( \tilde{x}_j \) and \( \tilde{x}_l \) can be chosen on the circumcircles of \( x_i, \tilde{x}_i \) and \( x_j, \tilde{x}_j \) resp \( x_l, \tilde{x}_l \); then the remaining point \( \tilde{x}_k \) has to lie on three circles, which is an overdetermined system.

However, it is clear that all constructed circles and points lie on the 2-sphere given by the face of the original net and the initial point \( \tilde{x}_i \). Using \( x_i \) as the point at infinity to break the Möbius
geometric symmetry, the circumcircle of the face of the original net and the two circumcircles for edges \((ij)\) and \((il)\) form a Euclidean triangle, with points \(\hat{x}_k, \hat{x}_i\) and \(\hat{x}_l\) marked on its sides. By Miguel’s theorem the circumcircles of each vertex of the triangle and the marked points on the adjacent sides intersect in a common point \(\hat{x}_k\), cf [37, §8.4.10] or [7, Thm 3.2].

Thus Miguel’s theorem from Euclidean geometry provides a solution of a Möbius geometric problem, via suitable symmetry breaking.

**In- and ex-centres of a Euclidean triangle.** Consider the following Möbius geometric configuration [39]: four points and the four circumcircles of any three of the four points. Thus, each pair of points lies on two circles of an elliptic pencil; once all circles are consistently oriented, the (unique) angle bisecting circles in each pencil can be considered: for each point, the three angle bisecting circles through that point belong to an elliptic pencil, hence intersect in a second point.

Breaking symmetry by choosing one of the four points as the point at infinity of a Euclidean plane, its three angle bisecting circles become the angle bisectors of a Euclidean triangle and their second intersection point becomes the in-centre of the triangle. Less obvious is the fact that the other three triplets of angle bisecting circles intersect in the ex-centres of the triangle [39, Lemma 3.3].

Considering the above Möbius geometric construction on an ellipsoid in projective space as a model for 2-dimensional Möbius geometry, it turns out that the two sets of four points are in 4-fold perspective: there are four centres of perspective on suitable lines joining the original points to the constructed points, thus giving rise to a “desmic system”.

In particular, one of the four centres of perspective lies in the inside of the ellipsoid, hence breaking the Möbius geometric symmetry of the construction to a spherical geometry with that centre of symmetry as the centre of a round sphere: now the four original points and the constructed in- and ex-centres become antipodal on this round sphere.

Thus considering a simple geometric configuration in different ambient geometries, via symmetry breaking, reveals a surprisingly rich structure and facilitates a deeper understanding of the geometry of the configuration.

3. Symmetry breaking in differential geometry

In differential geometry spontaneous symmetry breaking seems to occur in a similar way as in physics, often in the context of classification problems: a classification problem is posed in a particular geometric context, but the solution(s) come with additional data or “conserved quantities” attached, that break the symmetry of the original geometric context and naturally place the solution(s) in a subgeometry.

On the other hand, explicit symmetry breaking may be employed to describe and investigate a class of geometric objects in a rather efficient way as a subclass of a (larger) class of objects in a “higher” geometry: knowledge about the properties of objects of the “higher” geometry as well as about the process of symmetry breaking does not only provide information about the objects under investigation but also about their relations to those of the “higher” geometry.

Our aim in this section is to provide evidence for both: we shall provide short descriptions of a variety of instances of spontaneous symmetry breaking, focusing on how symmetry breaking occurs in each case, and we shall give an outline of some examples that show how explicit symmetry breaking can be useful in the study of differential geometric (classes of) objects.

3.1. The classical differential geometry approach

In this section we discuss various examples for our notion of “symmetry breaking” that are either classical or employ a classical point of view: though homogeneous coordinates are used for some arguments, the core results are independent of ambient scalings, hence obtained from symmetry breaking in a projective setting, cf Sect 2.2.

**Thomsen’s theorem.** A typical instance of what we describe as “symmetry breaking” occurs in the classification of isothermic Willmore surfaces, cf [3, §81] or [37, Sect 3.6]: a surface \(x : \Sigma^2 \to \mathbb{R}^3\) is isothermic if it admits conformal curvature line coordinates \((u, v)\),

\[
(dx, dx) = e^{2\varphi}(du^2 + dv^2) \quad \text{and} \quad (dx, dn) = e^{2\varphi}(\kappa_1 du^2 + \kappa_2 dv^2),
\]
with the Gauss map $n : \Sigma^2 \to S^2$ of $x$, and it is a Willmore surface if it is critical for the Willmore functional

$$W(x) = \int_{\Sigma^2} (H^2 - K) \, dA = \frac{1}{4} \int_{\Sigma^2} (\kappa_1 - \kappa_2)^2 \, dA.$$  

A key observation is that both properties, isothermicity and being Willmore, are conformally invariant hence Möbius geometric notions. In particular, the Willmore functional is the area functional of the central sphere congruence of the surface, that can be characterized as the (unique) enveloped sphere congruence with second order contact with the surface in orthogonal directions: hence Willmore surfaces were called “conformally minimal” in the classical literature.

Imposing both conditions for a surface $x : \Sigma^2 \to S^3$, now considered as a surface in the conformal 3-sphere $S^3 \subset \mathbb{R}P^4$, forces its central sphere congruence $c : \Sigma^2 \to \mathbb{R}P^4$ to take values in a fixed sphere complex $k \in \mathbb{R}P^4$, that is, $\gamma \perp q = 0$ for lifts $\gamma$ and $q$ of $c = [\gamma]$ and $k = [q]$, respectively.

Thus symmetry breaking occurs: considering the surface $x$ as a surface in a metric subgeometry determined by $q$, its central sphere congruence becomes its tangent plane congruence. And, as the central sphere congruence can here alternatively be characterized as the mean curvature sphere congruence of the surface — the enveloped congruence of spheres that have the same mean curvature as the surface at every touching point — Thomsen’s theorem is obtained: every isothermic Willmore surface (in Möbius geometry) is a minimal surface in some metric subgeometry.

As every minimal surface is isothermic and a Willmore surface, a classification result for isothermic Willmore surfaces is obtained in this way. It should be noted that minimality of a surface is scale-invariant, hence only the sign of the ambient curvature is relevant, not its magnitude, cf [3, §81].

**Vessiot’s theorem and Willmore channel surfaces.** The classification of isothermic channel surfaces reveals another instance of “symmetry breaking”, caused by a more complex absolute configuration, cf [57, §34] or [37, §3.7.5].

A channel surface is the envelope of a 1-parameter family of spheres; as this is a Lie sphere geometric notion, hence a sensible notion in every subgeometry of Lie sphere geometry, the natural and most general realm of this classification problem is again Möbius geometry. The enveloped 1-parameter family $s : \Sigma^2 \to \mathbb{R}P^4$ of spheres, $rk ds \leq 1$, of a surface $x : \Sigma^2 \to S^3$ is necessarily a (principal) curvature sphere congruence; hence using isothermicity to introduce conformal curvature line parameters $(u, v)$ it only depends on one of the parameters, say, $s = s(u)$. It then follows that the enveloped sphere curve is planar; thus the polar line $a \subset \mathbb{R}P^4$ of this plane yields a sphere pencil as an absolute configuration attached to an isothermic channel surface: it consists of spheres that intersect all spheres of the enveloped sphere curve orthogonally. Depending on the type of the sphere pencil three cases occur, cf [37, §§1.2.3 & 1.8.5]:

- **elliptic sphere pencil**, where the spheres of the pencil intersect in a circle in $S^3$ that can be considered as an axis of rotation of the channel surface $x$ in any metric subgeometry;
- **parabolic sphere pencil**, where the pencil touches $S^3$ in a point $\infty \in a \cap S^3$ at infinity of a similarity subgeometry of Möbius geometry, and the surface $x$ becomes a cylinder;
- **hyperbolic sphere pencil**, where the pencil hits $S^3$ in two point (spheres) $\infty^\pm$, one of which can be used to distinguish a similarity subgeometry where the pencil consists of concentric spheres and the surface $x$ becomes a cone, with the common centre of the spheres as its apex; considering, alternatively, both points as points at infinity of a hyperbolic subgeometry $x$ becomes an equivarient surface, i.e., invariant under a 1-parameter group of isometries.

Vessiot’s theorem [57, §34] provides the Euclidean (similarity) versions of this classification result, and a Lie geometric version has recently been obtained in [41, Thm 4.3].

A less familiar but more direct and also useful interpretation employs the fixed sphere pencil directly as an absolute configuration, which yields symmetry breaking into one of three metric subgeometries with 4-dimensional isometry groups, of $S^3 \times H^2$, $\mathbb{R}^1 \times \mathbb{R}^3$ or $H^1 \times S^2$.

A *Dupin cyclide* is an envelope of two 1-parameter families of (curvature) spheres. As touching is encoded by polarity in Lie sphere geometry, the enveloped sphere curves take values in polar 2-planes in $\mathbb{R}P^5$: a Dupin cyclide can be encoded algebraically as a polar pair of planes in $\mathbb{R}P^5$, or as an orthogonal decomposition of $\mathbb{R}^4 \times = \mathbb{R}^{2,1} \oplus \mathbb{R}^{2,1}$ of the space of homogeneous coordinates into two Minkowski 3-spaces. Thus a Lie sphere geometric classification of Dupin cyclides yields a single surface, cf [51, §6].

A Möbius geometric classification of Dupin cyclides can then be obtained via symmetry breaking,
by choosing a point sphere complex \( S^3 \subset \mathbb{R}P^5 \), i.e., fixing \( p \in \mathbb{R}^{4,2} \) with \( (p, p) = -1 \): now, a 1-parameter family of Dupin cyclides is obtained, characterized by the relative position of the polar pair of planes and the chosen point sphere complex, cf [44], [23, Sect 3.4] or [37, §1.8.8]. However, as both enveloped sphere curves are planar (circles), similar considerations as in Vessiot’s theorem lead to further symmetry breaking (in two ways), and hence the well known rich classification of Dupin cyclides in Euclidean (similarity) geometry is obtained, cf [31, Mem IV §3] or [52].

As any Willmore channel surface is isothermic [37, §3.7.10], the classification of Willmore channel surfaces gives again rise to symmetry breaking in two ways: as isothermic Willmore surfaces and as isothermic channel surfaces. It turns out that these two ways of symmetry breaking are compatible: any Willmore channel surface is an equivariant minimal surface in a space form, where tori and cylinders appear as minimal surfaces in hyperbolic space, whereas Willmore surfaces of revolution appear as minimal surfaces in various ambient curvatures, cf [56], [46] and [37, §3.7.15].

Cyclic Guichard nets. Rather similar instances of “symmetry breaking” as those described above occur in the context of 3-dimensional (locally) conformally flat hypersurfaces in the conformal 4-sphere \( S^4 \) and, closely related, of Guichard nets in the conformal 3-sphere \( S^3 \). The geometry of a generic conformally flat hypersurface, i.e., a hypersurface \( x : \Sigma^3 \to S^4 \) with flat induced conformal structure and with three distinct principal curvatures, is reflected by the intrinsic geometry of its curvature line net \( y : \Sigma^3 \to \mathbb{R}^3 \): this forms a triply orthogonal system of surfaces \( y_i = \text{const} \), satisfying the (conformally invariant) Guichard condition

\[
l_1^2 + l_2^2 = l_3^2,
\]

where \( (dx, dx) = l_1^2 dy_1^2 + l_2^2 dy_2^2 + l_3^2 dy_3^2 \).

In particular, a generic conformally flat hypersurface can be reconstructed uniquely from this canonical principal Guichard net, cf [37, §2.4.6].

A cyclic system is a triply orthogonal system so that the orthogonal trajectories of one family of surfaces consist of circular arcs; or, alternatively, two families of surfaces are (parts of) channel surfaces. Imposing this additional condition of cyclicity it turns out that a partial classification causes symmetry breaking: associated with every cyclic Guichard net in the conformal 3-sphere \( S^3 \subset \mathbb{R}P^4 \) there is a distinguished point \( [q] \in \mathbb{R}P^4 \) — using this \( q \) to fix a metric subgeometry, the circular arcs form normal lines to a family of parallel linear Weingarten surfaces \( y_3 = t \), i.e., surfaces satisfying

\[
0 = a_t K + 2b_t H + c_t = a_1 \kappa_1 \kappa_2 + b_t (\kappa_1 + \kappa_2) + c_t.
\]

On the other hand, most linear Weingarten surfaces in a space of constant sectional curvature can be used to generate a cyclic Guichard net via its parallel family of surfaces, cf [37, Sect 2.6]. Note that a linear Weingarten condition is not scale invariant but the fact that a surface be linear Weingarten is, hence only the sign of the ambient curvature and the type of the linear Weingarten surface (the sign of the discriminant \( b_t^2 - a_1 c_t \) of the equation) is relevant in this context.

This instance of symmetry breaking also sets the scene for a Möbius geometric approach to space form analogues of Bonnet’s classical theorem on parallel constant mean and constant Gauss curvature surfaces, cf [35, Sect II.3] or [37, Sect 2.7] and [41, Prop 2.8], as well as for an approach to discrete flat fronts in hyperbolic space, cf [30] and [40, Sect 4].

A conformally flat hypersurface in \( S^4 \) can be reconstructed from its canonical principal Guichard net, as mentioned above; consequently, the classification of conformally flat hypersurfaces with cyclic Guichard net naturally complements the above classification of cyclic Guichard nets, cf [38]: it turns out that every such hypersurface comes with a sphere pencil \( a \subset \mathbb{R}P^5 \) attached [38, §4.2]. An appropriate choice of \( [q] \in a \) to break the Möbius geometric symmetry then yields a 1-parameter family of parallel hyperspheres in the 4-dimensional space form defined by \( q \). The orthogonal curvature leaves of the cyclic Guichard net of the hypersurface are obtained as intersections of the hypersurface with these hyperspheres, where they are (extrinsically) linear Weingarten surfaces. As in the other cases of classification by symmetry breaking described above, this geometric characterization obtained by symmetry breaking yields a construction method, starting from any suitable linear Weingarten surface in (a hyperplane of) a space form.

3.2. Using homogeneous coordinates

A slight change of viewpoint on the classical approach yields a refinement which admits the treatment of further symmetry breaking phenomena that remain hidden to the classical approach.
Surfaces of constant mean curvature. For example, Thomsen’s theorem above relies on a Möbius geometric characterization of minimal surfaces as those surfaces whose central sphere congruence \( c = [\gamma] \) takes values in a fixed sphere complex \( k = [q] \), the complex of planes in a quadric \( Q^3 \) of constant curvature given by \( q \). A similar characterization of constant mean curvature surfaces in space forms, using the mean curvature of the central spheres, relies on the use of homogeneous coordinates: when \( \gamma \) is normalized, \( (\gamma, \gamma) \equiv 1 \), then \( H = -\langle q, \gamma \rangle \) yields the mean curvature of the central spheres, hence of the surface. Thus a surface \( x : \Sigma^2 \to S^3 \) has constant mean curvature in the quadric \( Q^3 \subset L^4 \) of constant curvature given by \( q \) if and only if its normalized central sphere congruence \( \gamma : \Sigma \to S^{3,1} \) takes values in an affine hyperplane parallel to \( \{ q \} \subset \mathbb{R}^4 \), that is, if \( d\gamma \perp q \), and its mean curvature is then given by \( H = -\langle \gamma, q \rangle \).

This characterization can be employed to show that any surface of constant mean curvature in a space form is isothermic and a constrained Willmore surface, see [10, Sect 3.4]. The converse, that is, an analogue of Thomsen’s theorem, holds for tori [53, Thm 35], but fails for general isothermic constrained Willmore surfaces, cf [9, Sect 4].

Another application of this Möbius geometric characterization of surfaces of constant mean curvature in space forms is a characterization of discrete constant mean curvature nets in space forms that exploits a duality between (a suitable lift of) an isothermic surface \( x : \Sigma^2 \to S^3 \) and its (normalized) central sphere congruence \( \gamma : \Sigma \to S^{3,1} \), see [8, Sect 5]. This yields a description of discrete constant mean curvature nets in space forms via (explicit) symmetry breaking.

Linear Weingarten surfaces. The above characterization of (discrete) surfaces of constant mean curvature in space forms is a particular case of a more general characterization of (discrete) linear Weingarten surfaces in space forms, see [13] and [17, §2]. Here the key observation is that any non-tubular linear Weingarten surface \( x : \Sigma^2 \to Q^3 \) in a quadric of constant curvature envelops a (possibly complex conjugate) pair of isothermic sphere congruences \( [\gamma^\pm] : \Sigma \to L^4 \), thought of as surfaces in the Lie quadric. Hence any (non-tubular) linear Weingarten surface is an \( \Omega \)-surface in the sense of Demoulin [29], and an \( \Omega \)-surface is linear Weingarten in a quadric \( Q^3 \subset L^3 \) of constant curvature if and only if its two enveloped isothermic sphere congruences take values in fixed sphere complexes, \( \gamma^\pm \perp q^\pm \) for some \( q^\pm \in \mathbb{R}^{4,2} \setminus \{0\} \) with \( [q^+, q^-] = [p, q] \). Though this is a Lie-geometric characterization in the classical sense, the recovery of the linear Weingarten equation requires the use of homogeneous coordinates.

Clearly, an \( \Omega \)-surface gives rise to an isothermic surface in the Möbius geometric sense if one of the enveloped isothermic sphere congruences \( [\gamma^\pm] \) consists of point spheres, say, \( \gamma^- \perp p \) for the point sphere complex \( [p] \in \mathbb{R}P^3 \) of a Möbius subgeometry of Lie sphere geometry. In this way, (discrete) constant mean curvature surfaces arise from (discrete) linear Weingarten surfaces via symmetry breaking, cf [17, Expl 4.2].

Weierstrass representations. Besides clarifying (subclass) relations between various classes of surfaces, more specific benefits can be drawn from those characterizations of surface (sub-)classes via symmetry breaking; for example, the rich transformation theories of isothermic and \( \Omega \)-surfaces descend to constant mean curvature resp linear Weingarten surfaces, reflecting the fact that the latter classes are obtained by integrable reductions from the former. This observation, in turn, allows us to put the Weierstrass type representations for minimal, horospherical and, more generally, Bryant or Bianchi type linear Weingarten surfaces into the wider context of a transformation theory, cf [25], [37, §5.6.21], [13, Sect 4] and [48]. As a consequence, analogous representations for discrete or semi-discrete counterparts from said surface classes are obtained in a straightforward manner, cf [5, Sect 7], [37, §5.7.37], [42], [30], [17, Sect 4], [54] and [49].

3.3. A gauge theoretic approach: polynomial conserved quantities

In an integrable systems context the approach via homogeneous coordinates described in the previous section often admits an alternative description, as an integrable reduction of the underlying differential equation(s). Geometrically, corresponding descriptions in terms of “polynomial conserved quantities”, first established in [11], encode our previously described symmetry breaking phenomena in a rather concise and beautiful way, that readily lends itself to a transparent handling of the symmetry breaking and its associated geometric problems.

Polynomial conserved quantities. The key ingredient of this story is a loop of flat connections associated to the geometric objects under investigation: for special surfaces or submanifolds this
may be obtained by injection of a “spectral parameter” into their Weingarten equations, cf [4] and [37, Sect 3.3]; though such families of flat connections may also be described in a more invariant way, without the introduction of frames or other auxiliary data, cf [37, Sects 5.4-5] or [11, Part III]. The existence of these loops of flat connections is intimately related to a corresponding transformation theory and, in particular, non-rigidity of the underlying geometric objects, cf [28], [2] and [21]: for example, considering the (flat) connections of the family associated to an isothermic surface \( x : \Sigma^2 \to \mathcal{Q} \subset V \) in a conformal quadric \( \mathcal{Q} \) as connections \( d t \) on the trivial ambient (vector) space bundle \( \Sigma^2 \times V \), suitable parallel sections \( \dot{x} : \Sigma^2 \to \mathcal{Q} \) yield the Darboux transformations of \( x \), and its conformal (Calapso) deformation is obtained by application \( x_t = g_t x \) of the gauge transformations \( g_t \) that relate the (flat) connections \( d_t \) to the trivial connection.

A polynomial conserved quantity in this context is then a polynomial \( p(t) = \sum_{k=0}^{n} a_k t^k \) with coefficients \( a_k : \Sigma^2 \to V \) that is parallel for the loop of connections, \( d_t p(t) = 0 \); or, otherwise said, \( g_t p(t) \equiv const \), see [11, Parts IV & V] or [14].

**Integrable reductions for \( \Omega \)-surfaces.** We already briefly discussed in the previous section how linear Weingarten surfaces in space forms appear as special \( \Omega \)-surfaces, by means of two (constant) linear sphere complexes that break the Lie sphere geometric symmetry of \( \Omega \)-surfaces to a space form geometry. This instance of symmetry breaking is part of a larger scheme that we shall describe here, mostly summarizing results from [18].

A surface in Lie sphere geometry is conveniently described in terms of two enveloped sphere congruences \( c^\pm : \Sigma^2 \to \mathcal{L}^4 \), that can be chosen to be isothermic (but possibly complex conjugate) surfaces in the Lie quadric \( \mathcal{L}^4 \) in the case of an \( \Omega \)-surface. Thus they come with their isothermic loops of flat connections on the trivial (vector) bundle \( \Sigma^2 \times \mathbb{R}^4 \) (resp its complexification),

\[
d_t^\pm = d + t \eta^\pm, \quad \text{where } \eta^\pm = \gamma^\pm \wedge d \gamma^\mp
\]

for appropriate lifts \( \gamma^\pm : \Sigma^2 \to \mathcal{L}^5 \) of \( c^\pm = [\gamma^\pm] \), cf [18, Sect 2.3]. Note that the two connections are gauge equivalent, \( d_t^\pm = \exp(t \tau) d_t^\mp = d_t^\mp - t \, d \tau \) since \( \tau := \gamma^+ \wedge \gamma^- \) is nilpotent and \( d \tau = t (\eta^+ - \eta^-) \); hence \( \sigma : \Sigma^2 \to \mathbb{R}^4 \) is \( d_t^\pm \)-parallel iff \( \exp(t \tau) \sigma \) is \( d_t^\mp \)-parallel, in particular, polynomial conserved quantities may be “exchanged” between the two connections at the potential cost of changing the polynomial degree by \( 1 \). Similarly, we could also work with the (always real) “middle” connection of [18, (5)], \( d_t^{mid} := \frac{1}{2} (d_t^+ + d_t^-) = \exp(\pm \frac{1}{2} \tau) \cdot d_t^\mp \).

Obviously, if one of the isothermic sphere congruences, say \( c^- \), takes values in a (fixed) point sphere complex \( p \in \mathbb{R}^4 \), \( c^- \perp p \), symmetry is broken and we obtain an isothermic surface \( x = c^- \) in Möbius geometry (with central sphere congruence \( c^+ \)). As \( \gamma^+ \) and \( \gamma^- \) have parallel tangent planes, this configuration can be characterized by the fact that \( p \) is a constant conserved quantity of \( c^- \),

\[
\forall t \in \mathbb{R} : d_t^- p = d p + t (p, \gamma^-) d \gamma^+ - t (p, d \gamma^+) \gamma^- = 0;
\]

for the other connections \( p \) yields linear conserved quantities, e.g., \( p(t) = p + \frac{1}{2} (\gamma^+ \wedge \gamma^-)(p) \) for the middle connection \( d_t^{mid} \); in any case \( p(t), p(t) \equiv -1 \). Similarly, L(agueurre)-isothermic surfaces can be characterized as those \( \Omega \)-surfaces, where one of the enveloped isothermic sphere congruences, say \( c^- \), qualifies as a (Laguerre geometric) tangent plane map, that is, by breaking symmetry with a hyperplane complex \( q \in \mathbb{R}^4 \); for the middle connection this yields again a linear conserved quantity \( p(t) = q + \frac{1}{2} (\gamma^+ \wedge \gamma^-)(q) \), which is null this time, \( p(t), p(t) = 0 \).

A third class of surfaces characterized in a similar vein are Guichard surfaces (in Möbius geometry): following [20] such surfaces \( x : \Sigma^2 \to S^3 \) can be characterized by the existence of curvature line parameters \( (u, v) \) so that (in any space form subgeometry)

\[
\exists c \neq 0 : c \, EG(\kappa_1 - \kappa_2) = E - \varepsilon G, \quad \text{with} \quad \begin{cases}
(dx, dx) = E \, du^2 + G \, dv^2 \\
-(dx, du) = E \kappa_1 \, du^2 + G \kappa_2 \, dv^2,
\end{cases}
\]

where the sign \( \varepsilon \in \{ \pm 1 \} \) distinguishes two possible types; note how isothermic surfaces are obtained as a limiting case \( c = 0 \) of type \( \varepsilon = +1 \) Guichard surfaces. Here the construction of the enveloped isothermic sphere congruences (complex conjugate for type \( \varepsilon = -1 \)) and a linear conserved quantity for the middle connection (with \( \deg(p(t), p(t)) = 1 \)) are somewhat more involved and the reader is referred to [18, Sect 5.2] for details. Summarizing we obtain the following symmetry breaking scheme in terms of linear conserved quantities, where the (constant coefficient)
polynomial \((p(t), p(t)) \in \mathbb{R}[t]\) distinguishes the classes: an \(\Omega\)-surface that admits a linear conserved quantity \(p(t)\) (for its middle connection) with

- \((p(t), p(t)) = -1\) is isothermic in the Möbius subgeometry of \(p(0)\), cf [18, Thm 5.3];
- \((p(t), p(t)) = 0\) is \(L\)-isothermic in the Laguerre subgeometry of \(p(0)\), cf [18, Thm 5.11];
- \((p(t), p(t)) = 2t - 1\) is a Guichard surface in the Möbius subgeometry of \(p(0)\), cf [18, Thm 5.7].

Note how the linear conserved quantity \(p(t)\) and the real polynomial \((p(t), p(t))\) encode the geometry of the symmetry breaking in each case: the (constant) \(t^0\)-coefficient of \(p(t)\) provides the absolute configuration for symmetry breaking, and its \(t^1\)-coefficient yields a particular sphere congruence that is attached to the geometry of the surface; the \(t^0\)-coefficient of \((p(t), p(t))\) allows to read off the type of subgeometry, and its linear coefficient yields information about the interplay between the subgeometry and this special sphere congruence.

Returning to the characterization of (non-tubular) linear Weingarten surfaces as \(\Omega\)-surfaces whose enveloped isothermic sphere congruences \(e^\pm\) take values in (different) fixed sphere complexes \(q^\pm\), we infer from the above discussion that the middle connection of a linear Weingarten surface, thought of as an \(\Omega\)-surface, admits two linearly independent linear conserved quantities, \(p^\pm(t) = a^\pm t + q^\pm\). In particular, as is well known, any linear Weingarten surface in a space form is either isothermic or a Guichard surface. In fact, using a suitable space form projection

\[
(\xi, \nu) : \Sigma^2 \to Q^3 \times P^3 \text{ with } \left\{ \begin{array}{l}
Q^3 = \{ y \in L^5 \mid (y, p) = 0, (y, q) = -1 \} \\
P^3 = \{ y \in L^5 \mid (y, p) = -1, (y, q) = 0 \}
\end{array} \right.
\]

for \(p, q \in [q^+, q^-]\), the linear Weingarten condition, \(0 = aK + 2bH + c\), is encoded in the middle connection as well as in its corresponding linear conserved quantities, see [18, Prop 6.4 & Cor 6.5]:

\[
d_{\text{mid}}^m = d + tc\xi \wedge d\xi - tb(\xi \wedge d\nu + \nu \wedge d\xi) + ta\nu \wedge d\nu \quad \text{and} \quad \left\{ \begin{array}{l}
p(t) = p + t(-b\xi + a\nu), \\
q(t) = q + t(c\xi - b\nu).
\end{array} \right.
\]

Note that the characteristic polynomials, \((p(t), p(t)) = -1 - 2at\) and \((q(t), q(t)) = -\kappa - 2ct\), depend on the choice \((p, q)\) of space form projection data here: polynomial conserved quantities satisfy the usual superposition principle. Thus in the presence of a higher dimensional bundle of polynomial conserved quantities, the individual characteristic polynomials have little significance for the geometry of symmetry breaking but the corresponding Gram matrix provides valuable information, cf [12, Sect 3], in the case at hand

\[
G = \begin{pmatrix}
(p(t), p(t)) & (p(t), q(t)) \\
(q(t), p(t)) & (q(t), q(t))
\end{pmatrix} = \begin{pmatrix}
-1 - 2at & 2b \\
2bt & -\kappa - 2ct
\end{pmatrix},
\]

where \(\det G = \kappa + 2(ac + c)t + 4(ac - b^2)t^2\) returns the ambient space curvature as well as the discriminant of the linear Weingarten condition, which determines its type resp whether the enveloped isothermic sphere congruences are real or complex conjugate.

Higher degree polynomial conserved quantities have been investigated in [14], where isothermic surfaces (in Möbius geometry) with a quadratic conserved quantity have been shown to yield the classical special isothermic surfaces of Darboux [28] and Bianchi [2]: these surfaces generalize constant mean curvature surfaces (which are isothermic with a linear conserved quantity) and exhibit astonishingly rich geometric properties.

**Conserved quantities: spontaneous vs explicit symmetry breaking.** So far, we have discussed how (sub-)classes of surfaces may be characterized by using polynomial conserved quantities to break the symmetry of a higher geometry into a subgeometry, thus we have described instances of explicit rather than of spontaneous symmetry breaking. Obviously though, polynomial conserved quantities can be employed to describe or detect spontaneous symmetry breaking as well — however, as the sketched theory of polynomial conserved quantities seems to be relatively new and not widely adopted yet, we can only reference one non-trivial example at the time of writing [24, Sect 6.1]: if a Lie applicable surface has one family of spherical curvature lines then these spherical curvature lines are Lie sphere transforms of a constrained elastic curve in a space form geometry.

On the other hand, polynomial conserved quantities can, and have been, used very efficiently to describe integrable reductions of transformation theories as well as integrable discretizations of surface classes by (explicit) symmetry breaking.
We first sketch how the isothermic transformations touched upon at the beginning of this section descend to surfaces $\xi: \Sigma^2 \rightarrow Q^3$ of constant mean curvature $H$ in a space form $Q^3$: considering $\xi$ as an isothermic linear Weingarten surface we have

$$d\xi^\text{mid} = d + \frac{t}{2} (2H \xi \wedge d\xi + \xi \wedge H \nu + \nu \wedge d\xi)$$

and

$$\begin{cases} p(t) = p + \frac{t}{2} \xi, \\ q(t) = q + \frac{t}{2} (2H \xi + \nu); \end{cases}$$

thus with $\gamma^- = \xi$ and the central sphere congruence $\gamma^+ = \nu + H \xi$ we obtain an isothermic surface in M"obius geometry with a linear conserved quantity (note that $\tau = \gamma^+ \wedge \gamma^- = \nu \wedge \xi$):

$$d\xi^* = \exp(\frac{t\tau}{2}) \cdot d\xi^\text{mid} = d + t \xi \wedge d\gamma$$

and

$$\begin{cases} p^-(t) = \exp(\frac{t\tau}{2}) p(t) = p, \\ q^-(t) = \exp(\frac{t\tau}{2}) q(t) = q + t\gamma. \end{cases}$$

The key to see that, resp how, the Calapso and Darboux transformations for the isothermic surface $x = [\xi]$ descend to constant mean curvature surfaces is to control how the isothermic loop of flat connections, hence its (linear) conserved quantities change, cf [11] or [14, Sect 3]. The Calapso transformation is readily shown to descend and, in particular, to yield the Lawson correspondence for constant mean curvature surfaces, cf [37, §5.5.29]. The Darboux transformation, on the other hand, generically increments the degree of a polynomial conserved quantity; however, an orthogonality condition (at an initial point) yields a polynomial conserved quantity of the same degree, see [14, Thm 3.1]: consequently, with an appropriate choice of initial value, the Darboux transformation of isothermic surfaces descends to the Bäcklund transformation for surfaces of constant mean curvature. In fact, the presented arguments are independent of the degree of the polynomial conserved quantity used to break the M"obius geometric symmetry, hence apply to the classical special isothermic surfaces of Darboux and Bianchi touched upon above as well, see [14].

The isothermic transformations directly apply to the isothermic sphere congruences $c^\pm$ enveloped by an $\Omega$-surface; to see how this yields transformations for the enveloping $\Omega$-surface requires some thought: the issue is that an $\Omega$-surface comes with a pair of isothermic sphere congruences $c^\pm$, whose Calapso or Darboux transformations need to line up to form a new $\Omega$-surface. However, by the gauge equivalence $d\xi^* = \exp(\tau) \cdot d\xi^+ \wedge \gamma^-$, the Calapso transforms $g^- c^-$ and $g^+ c^+ = g^- c^+$ of $c^-$ and $c^+$ indeed share an enveloping Calapso transform of the original $\Omega$-surface, cf [18, Def 2.14]; and one Darboux transform, say $c^-$, is enough to construct the Ribaucour sphere congruence $\tau c^-$, whose second envelope is the corresponding Darboux transform of the $\Omega$-surface, cf [18, Def 2.17]. At this point, a line of arguments rather similar to the one outlined above shows how the Calapso and Darboux transformations for $\Omega$-surfaces descend to corresponding transformations of subclasses of surfaces that are obtained by breaking the Lie geometric symmetry using a polynomial conserved quantity, cf [18, Sects 5 & 6].

Note how the former instance of symmetry breaking, where constant mean curvature surfaces are obtained from isothermic surfaces in M"obius geometry, is obtained from the latter, where isothermic (and Guichard surfaces) are obtained from $\Omega$-surfaces, by a symmetry breaking process, with a point sphere complex as a constant conserved quantity.

A core observation in integrable discretizations is that iterated Darboux-Bäcklund transformations of a surface generate discrete nets (via permutability properties), as the orbit of a point, that exhibit similar properties as their smooth counterparts: in particular, the smooth transformation theories are faithfully replicated by the discrete nets obtained in this way, see [7, Chap 2]. This discretization scheme can efficiently be implemented, using polynomial conserved quantities, if an integrable discretization of a “higher” surface class is available: this approach was first pursued in [15] and [16], where integrable discretizations for surfaces of constant mean curvature in space forms and for the classical special isothermic surfaces are obtained from the integrable discretization of isothermic surfaces in M"obius geometry of [5], generalizing previous discretizations in Euclidean space, cf [36, Sect 5], [6, Sect 4] or [7, Def 4.47]; a similar strategy can be used to obtain discretizations for linear Weingarten surfaces, cf [17], and for Guichard, isothermic and L-isothermic surfaces by symmetry breaking from an integrable discretization of $\Omega$-surfaces, cf [19, Sect 7.2].

**Conclusions & Questions**

In Sects 3.1 and 3.2 we encountered several classification results, where an absolute configuration appeared without apparent cause — which “spontaneously” led to a symmetry breaking in geom-
etry in the sense described in Sect 2. Subsequently, exploiting the integrable systems context of the situation, we gave a refined description in terms of polynomial conserved quantities and saw how symmetry breaking can be used explicitly to solve certain geometric problems.

Thus it seems fairly clear how what we consider as “symmetry breaking” occurs in geometry: we understand its “mechanics”, at least in a classical setting. However, it remains unclear why such (spontaneous) symmetry breaking phenomena occur: where do these “absolute configurations” come from and why are they of the given types?

Led by some apparent analogies with physics an answer might be sought in the bifurcation theory of differential equations: this may lead to common explanations of (spontaneous) symmetry breaking, in geometry and physics alike — however, symmetry hence symmetry breaking being inherently geometric concepts, explanations of the causes in terms of differential equations alone seem unsatisfactory, and are unlikely to provide an understanding of instances such as those described in Sect 2.4.

In fact, almost all of the instances of spontaneous symmetry breaking that we described have an integrable systems context, hence one may ask whether or not this is coincidental or systemic: can all these instances of symmetry breaking be described in terms of an integrable reduction or of polynomial conserved quantities? And, on the other hand, does symmetry breaking indicate the presence of an integrable system? For example, should we expect an integrable system behind the configuration of in- and ex-centres of a Euclidean triangle?

As annunciated in the introduction, a better understanding of symmetry breaking — why spontaneous symmetry breaking occurs and how it relates to other features of a geometric theory — will require further investigations and a more systematic approach to these phenomena in geometry. A benefit can be expected to be a better understanding of the interrelations between different geometries, hence of “geometry”.

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