KUMMER-TYPE CONGRUENCES FOR
MULTI-POLY-BERNOULLI NUMBERS

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Abstract. The multi-poly-Bernoulli numbers are generalizations of the Bernoulli numbers. In this paper, we will prove Kummer-type congruences for multi-poly-Bernoulli numbers via $p$-adic distributions.

1. Introduction

For a non-negative integer $n$, the $(n$-th) Bernoulli number $B_n$ is defined by the generating function

$$\frac{te^t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

as formal power series over $\mathbb{Q}$. It is well known that the following congruence holds (cf. [2, Theorem 11.6]). For positive integers $m, n, N$ and an odd prime $p$, if $m \equiv n \mod (p-1)p^{N-1}$, then we have

$$(1 - p^{m-1}) \frac{B_m}{m} \equiv (1 - p^{n-1}) \frac{B_n}{n} \mod p^N.$$ 

This congruence is called the Kummer congruence.

In [6] and [3], Arakawa and Kaneko introduced the poly-Bernoulli numbers $B_n^{(k)}$ and $C_n^{(k)}$, which are generalizations of the Bernoulli numbers, as follows. Let $k$ be an integer and $n$ be a non-negative integer. Poly-Bernoulli numbers $B_n^{(k)}$ and $C_n^{(k)}$ are defined by

$$\text{Li}_k(1 - e^{-t}) = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!},$$

$$\text{Li}_k(1 - e^{-t}) = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!}$$

respectively, as formal power series over $\mathbb{Q}$. Here,

$$\text{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}$$

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is the $k$-th polylogarithm. Note that $\text{Li}_1(t) = -\log(1-t)$ and $B_n^{(1)} = (-1)^n C_n^{(1)} = B_n$ for $n \geq 0$. Kitahara proved the following congruence for poly-Bernoulli numbers by using $p$-adic distributions.

**Theorem 1.1** ([7, Theorem 12]). Let $k$ be an integer, $p$ be an odd prime, and $m, n$ and $N$ be positive integers with $m, n \geq N$ and $k < p-1$. If $m \equiv n \mod (p-1)p^{N-1}$, then we have

$$p^{2k'}B_m^{(k)} \equiv p^{2k'}B_n^{(k)} \mod p^N,$$

where $k' = \max\{k, 0\}$.

**Remark 1.2.** Sakata gave an elementary proof of Theorem 1.1 in the case $k < 0$ ([9, Theorem 6.1]).

In this paper, we will consider a further generalization of Theorem 1.1.

**Definition 1.3** ([5, Section 1]). For $k = (k_1, \cdots, k_r) \in \mathbb{Z}^r$, define the multiple polylogarithm to be

$$\text{Li}_k(t) = \sum_{0 < m_1 < \cdots < m_r} \frac{t^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}.$$

Multi-poly-Bernoulli numbers $B_n^{(k)}$ and $C_n^{(k)}$ are defined to be the rational numbers satisfying

$$\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!},$$

$$\frac{\text{Li}_k(1 - e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} C_n^{(k)} \frac{t^n}{n!},$$

respectively, as formal power series over $\mathbb{Q}$.

**Remark 1.4.** In [5], some relations between $B_n^{(k)}$ and $C_n^{(k)}$ were proved. For examples, we have relations

$$B_n^{(k)} = \sum_{i=0}^{n} \binom{n}{i} C_n^{(k)},$$

$$C_n^{(k)} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} B_i^{(k)},$$

$$B_n^{(k)} = C_n^{(k)} + C_n^{(k_1-1,k_2,\cdots,k_r)}$$

for any $r \geq 1$, $k = (k_1, k_2, \cdots, k_r) \in \mathbb{Z}^r$ and $n \geq 1$ ([5, Section 2]).

**Remark 1.5.** The multiple polylogarithm was introduced in [3]. It is expected to have relations with the multiple zeta values and the multiple zeta functions. It is also known that the multi-poly-Bernoulli numbers $C_n^{(k)}$ describe the finite multiple zeta values ([5, Theorem 8]).
We call $\mathbf{k} = (k_1, \cdots, k_r) \in \mathbb{Z}^r$ an index. For an index $\mathbf{k}$, we define the weight of $\mathbf{k}$ to be $\text{wt}(\mathbf{k}) = k_1 + \cdots + k_r$ and write $k'_r = \max\{k_i, 0\}$ and $\mathbf{k}' = (k'_1, \cdots, k'_r)$. We will prove the following result in Section 3.

**Theorem 1.6.** Let $\mathbf{k} \in \mathbb{Z}^r$ be an index, $p$ be an odd prime and $m, n$ and $N$ be positive integers with $m, n \geq N$ and $\text{wt}(\mathbf{k}) < p - 1$. If $m \equiv n \mod (p - 1)p^{N - 1}$, then we have

$$p^{2 \text{wt}(\mathbf{k})}B_m^{(\mathbf{k})} \equiv p^{2 \text{wt}(\mathbf{k})}B_n^{(\mathbf{k})} \mod p^N,$$

$$p^{2 \text{wt}(\mathbf{k})}c_m^{(\mathbf{k})} \equiv p^{2 \text{wt}(\mathbf{k})}c_n^{(\mathbf{k})} \mod p^N.$$  

In Section 4, we will consider the multi-poly-Bernoulli-star numbers, which were introduced in [4], and find Kummer-type congruences for the multi-poly-Bernoulli-star numbers which are similar to Theorem 1.6.

**Notation:** In this paper, let $p$ be a prime. For $x \in \mathbb{Q}_p$, we denote the $p$-adic valuation by $\text{ord}_p(x)$. For a real number $x$, $\lfloor x \rfloor$ means the greatest integer less than or equal to $x$.

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2. Preliminaries

In this section, we will recall a theory of $p$-adic distributions.

**Definition 2.1.** Let $h$ be a non-negative integer. Define $LA_h(\mathbb{Z}_p, \mathbb{Q}_p)$ to be the set of functions $f : \mathbb{Z}_p \to \mathbb{Q}_p$ which is locally analytic at each point with radius of convergence $\geq p^{-h}$. For $f \in LA_h(\mathbb{Z}_p, \mathbb{Q}_p)$, the norm of $f$ is given by

$$\|f\|_h = \sup_{n \geq 0, a \in \mathbb{Z}_p} \|p^n a|_p\|
$$

for the expansion $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ on $a + p^h \mathbb{Z}_p$. The set $LA_h(\mathbb{Z}_p, \mathbb{Q}_p)$ is a $\mathbb{Q}_p$-vector space equipped with the topology induced by the norm. Since there exist natural inclusions $LA_h(\mathbb{Z}_p, \mathbb{Q}_p) \to LA_{h+1}(\mathbb{Z}_p, \mathbb{Q}_p)$ for all $h \geq 0$, we may define $LA(\mathbb{Z}_p, \mathbb{Q}_p) = \bigcup_{h \geq 0} LA_h(\mathbb{Z}_p, \mathbb{Q}_p)$ equipped with the inductive limit topology. A continuous $\mathbb{Q}_p$-linear map $\mu : LA(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p$ is called a $p$-adic distribution and we write

$$\int_{\mathbb{Z}_p} f(x)d\mu(x) := \mu(f)$$

for $f \in LA(\mathbb{Z}_p, \mathbb{Q}_p)$. We denote by $D(\mathbb{Z}_p)$ the set of $p$-adic distributions.

It is known that the following theorems hold.
Theorem 2.2 ([8, Lemma 1]). Let \( f: \mathbb{Z}_p \to \mathbb{Q}_p \). The function \( f \) is continuous if and only if there exist \( a_n \in \mathbb{Q}_p \) such that
\[
f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}
\]
and \( a_n \to 0 \) as \( n \to \infty \). Here, we define
\[
\binom{x}{0} = 1, \quad \binom{x}{n} = \frac{x(x-1) \cdots (x-n+1)}{n!} \in \mathbb{Q}[x]
\]
for \( n \geq 1 \).

Theorem 2.3 ([1, Théorème 3]). Let \( h \) be a non-negative integer. For \( f: \mathbb{Z}_p \to \mathbb{Q}_p \),
\( f \in \text{LA}_h(\mathbb{Z}_p, \mathbb{Q}_p) \) if and only if there exist \( a_n \in \mathbb{Q}_p \) such that
\[
f(x) = \sum_{n=0}^{\infty} a_n \left\lfloor \frac{n}{p^h} \right\rfloor \binom{x}{n}
\]
and \( a_n \to 0 \) as \( n \to \infty \). Moreover, \( ||f||_h \leq 1 \) holds if and only if \( a_n \in \mathbb{Z}_p \) for all \( n \geq 0 \).

Theorem 2.4 ([10, Theorem 2.3]). Let \( R \) be the set of formal power series \( f(T) \) over \( \mathbb{Q}_p \) which converges on the open unit disk. Then the map \( D(\mathbb{Z}_p) \to R \) given by
\[
\mu \mapsto \int_{\mathbb{Z}_p} (1 + T)^x d\mu(x) := \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x) T^n
\]
is bijective. The inverse map sends \( \sum_{n=0}^{\infty} c_n T^n \in R \) to the element of \( D(\mathbb{Z}_p) \) given by
\[
(1)
\]
\[
\text{LA}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p; \quad f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \mapsto \sum_{n=0}^{\infty} a_n c_n.
\]

Remark 2.5. Since \( f \in \text{LA}(\mathbb{Z}_p, \mathbb{Q}_p) \) is continuous on \( \mathbb{Z}_p \), it follows from Theorem 2.2 that \( f \) has the expansion as (1) and the infinite sum in (1) is convergent.

Note that, if a formal power series \( f(T) \in R \) corresponds to a \( p \)-adic distribution \( \mu \), we have
\[
\left( (1 + T) \frac{d}{dT} \right)^n f(T) = \int_{\mathbb{Z}_p} x^n (1 + T)^x d\mu(x) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x) T^n
\]
and
\[
(2)
\]
\[
\left. \left( (1 + T) \frac{d}{dT} \right)^n f(T) \right|_{T=0} = \int_{\mathbb{Z}_p} x^n d\mu(x)
\]
for \( n \geq 0 \). Indeed, we can check these by using the property
\[
x \binom{x}{n} = (n + 1) \binom{x}{n+1} + n \binom{x}{n}.
\]
3. Proof of Theorem 1.6

In this section, we will prove Theorem 1.6. Our proof is inspired by the proof of [7, Theorem 12]. In the following, let \( p \) be an odd prime.

For positive integers \( m, n \) and \( N \), by applying Theorem 2.3 to the case \( h = 1 \) and \( p^{-N}(x^m - x^n) \in \mathbb{L} A_1(\mathbb{Z}_{p}, \mathbb{Q}_{p}) \), we obtain \( a_{j} \in \mathbb{Q}_{p} \) satisfying

\[
\frac{x^m - x^n}{p^N} = \sum_{j=0}^{\infty} a_{j} \left\lfloor \frac{j}{p} \right\rfloor (x)
\]

and \( |a_{j}|_{p} \to 0 \) as \( j \to \infty \).

**Lemma 3.1.** If \( m, n \geq N \) and \( m \equiv n \mod (p - 1)p^{N-1} \), then we have \( a_{j} \in \mathbb{Z}_{p} \) for any \( j \geq 0 \).

**Proof.** Put \( P(x) = p^{-N}(x^m - x^n) \). According to Theorem 2.3, we must prove \( ||P(x)||_{1} \leq 1 \) and it suffices to show that \( Q(y) := P(c + py) \in \mathbb{Z}_{p}[y] \) for any \( c = 0, 1, \cdots, p - 1 \). If \( c = 0 \), it is clear.

Suppose that \( c \neq 0 \). We put \( m - n = (p - 1)p^{N-1}d \) with \( d \in \mathbb{Z}_{>0} \) and

\[
Q(y) = p^{-N}(c + py)^{n} ((c + py)^{(p-1)p^{N-1}d} - 1)
\]

\[
= p^{-N}(c + py)^{n} \left\{ \sum_{i=0}^{(p-1)p^{N-1}d} \binom{(p-1)p^{N-1}d}{i} (c + py)^{(p-1)p^{N-1}d-i} p^{dy} - 1 \right\}
\]

\[
= p^{-N}(c + py)^{n} \sum_{i=0}^{(p-1)p^{N-1}d} a_{i} y^{i}.
\]

We will check that \( \text{ord}_{p}(a_{i}) \geq N \) for \( 0 \leq i \leq (p - 1)p^{N-1}d \). Since \( c^{(p-1)p^{N-1}d} \equiv 1 \mod p^{N} \), we find that \( \text{ord}_{p}(a_{0}) \geq N \). If \( i = 1 \), we see that \( \text{ord}_{p}(a_{1}) = \text{ord}_{p}((p - 1)p^{N}d) \geq N \). Suppose that \( i \geq 1 \) and \( \text{ord}_{p}(a_{i}) \geq N \). Since we have

\[
\alpha_{i+1} = \frac{(p - 1)p^{N-1}d - i}{i + 1} pc^{-1} \alpha_{i},
\]

it follows that

\[
\text{ord}_{p}(a_{i+1}) \geq \text{ord}_{p}((p - 1)p^{N-1}d - i) - \text{ord}_{p}(i + 1) + 1 + N \\
\geq \text{ord}_{p}(i) - \text{ord}_{p}(i + 1) + 1 + N \geq N.
\]

\[\square\]

**Proof of Theorem 1.6.** We omit the proof for \( C_{n}^{(k)} \) because it can be checked by the same argument as the following proof for \( B_{n}^{(k)} \). Put

\[
f(x) = \frac{\text{Li}_{k}(1 - e^{x})}{1 - e^{x}}
\]
and \( g(T) = f(\log(1 + T)) \). In other words, we set

\[
\begin{align*}
    f(x) &= \sum_{0 < m_1 < \cdots < m_r} \frac{(1 - e^x)^{m_r - 1}}{m_1^{k_1} \cdots m_r^{k_r}} = \sum_{n=0}^{\infty} (-1)^n B_n^{(k)} \frac{x^n}{n!}, \\
    g(T) &= \sum_{0 < m_1 < \cdots < m_r} \frac{(-1)^{m_r - 1}}{m_1^{k_1} \cdots m_r^{k_r}} T^{-m_r - 1}.
\end{align*}
\]

We can check that \( g(T) \) converges on the open unit disk. Indeed, since we have

\[
\left| \sum_{0 < m_1 < \cdots < m_r} \frac{(-1)^{m_r - 1}}{m_1^{k_1} \cdots m_r^{k_r}} T^{-m_r - 1} \right| \leq m_r^{\omega(k^r)},
\]

it follows that

\[
\limsup_{m_r \to \infty} \left| \sum_{0 < m_1 < \cdots < m_r} \frac{(-1)^{m_r - 1}}{m_1^{k_1} \cdots m_r^{k_r}} T^{-m_r - 1} \right| = 1.
\]

Using Theorem 2.4, we get a \( p \)-adic distribution \( \mu \) corresponding to \( g \). The \( p \)-adic distribution \( \mu : \text{LA}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p \) is given by

\[
\varphi \mapsto \sum_{j=1}^{\infty} (-1)^j a_j \sum_{0 < m_1 < \cdots < m_r < j+1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (j + 1)^{k_r}},
\]

where \( \varphi \) has the expansion \( \varphi(x) = \sum_{j=0}^{\infty} a_j \binom{x}{j} \). According to (2), we obtain that

\[
\int_{\mathbb{Z}_p} x^nd\mu(x) = \left( (1 + T) \frac{d}{dT} \right)^n g(T) \bigg|_{T=0} = \left( \frac{d}{dx} \right)^n f(x) \bigg|_{x=0} = (-1)^n B_n^{(k)}
\]

for \( n \geq 0 \).

For positive integers \( m, n \) and \( N \) with \( m \equiv n \mod (p - 1)p^{N-1} \), Theorem 2.3 implies that there exist \( a_j \in \mathbb{Q}_p \) such that

\[
\frac{x^m - x^n}{p^N} = \sum_{j=0}^{\infty} a_j \left\lfloor \frac{j}{p} \right\rfloor \binom{x}{j}
\]

and \( |a_j|_p \to 0 \) as \( j \to \infty \). Then we have \( a_j \in \mathbb{Z}_p \) for any \( j \geq 0 \) by Lemma 3.1. We see that

\[
\int_{\mathbb{Z}_p} x^m - x^n \frac{d\mu(x)}{p^N} = \sum_{j=0}^{\infty} a_j \left\lfloor \frac{j}{p} \right\rfloor ! \int_{\mathbb{Z}_p} \binom{x}{j} d\mu(x)
\]

\[
= \sum_{j=0}^{\infty} (-1)^j a_j \left\lfloor \frac{j}{p} \right\rfloor ! \sum_{0 < m_1 < \cdots < m_r < j+1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (j + 1)^{k_r}}.
\]
Put
\[
(3) \quad h(j) = \left\lfloor \frac{j}{p} \right\rfloor \sum_{0 < m_1 < \cdots < m_r < j+1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} (j+1)^k}
\]
for \( j \geq r-1 \). Note that the summation in the R.H.S. of (3) is empty for \( 0 \leq j \leq r-2 \) and understood to be 0. We will prove the following lemma soon later.

**Lemma 3.2.** If \( \text{wt}(k^+) < p - 1 \), then we have
\[
\min_{j \geq r-1} \{ \text{ord}_p(h(j)) \} \geq -2 \text{wt}(k^+).
\]

It follows from the above lemma that
\[
\begin{align*}
p^2 \text{wt}(k^+) \int_{\mathbb{Z}_p} \frac{x^m - x^n}{p^N} d\mu &= p^2 \text{wt}(k^+) - N \left\{ (-1)^m B_m^{(k)} - (-1)^n B_n^{(k)} \right\} \in \mathbb{Z}_p.
\end{align*}
\]
It is equivalent to the congruence
\[
p^2 \text{wt}(k^+) B_m^{(k)} \equiv p^2 \text{wt}(k^+) B_n^{(k)} \mod p^N.
\]

We will show Lemma 3.2.

**Proof of Lemma 3.2.** Let \( k = (k_1, \cdots, k_r) \). For \( j \leq p - 1 \), we see that \( \text{ord}_p(h(j)) \geq -k_r \). Set \( j = ap + i \geq p \) with \( a \geq 1 \) and \( 0 \leq i \leq p - 1 \). Then we have
\[
\min_{0 \leq i \leq p-1} \{ \text{ord}_p(h(ap + i)) \} = \min_{0 \leq i \leq p-1} \left\{ \text{ord}_p(a!) - k_r \text{ord}_p(ap + i + 1) + \text{ord}_p \left( \sum_{0 < m_1 < \cdots < m_r < ap+i+1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r-1}} \right) \right\}
\]
\[
\geq \min_{0 \leq i \leq p-1} \left\{ \text{ord}_p(a!) - k'_r \text{ord}_p(ap + i + 1) + \max_{0 < m_1 < \cdots < m_r < ap+i+1} \left\{ \sum_{s=1}^{r-1} k'_s \text{ord}_p(m_s) - k'_r \right\} \right\}
\]
\[
= \text{ord}_p(a!) - k'_r \text{ord}_p(a + 1) - \max_{0 < m_1 < \cdots < m_r < (a+1)p} \left\{ \sum_{s=1}^{r-1} k'_s \text{ord}_p(m_s) \right\} - k'_r
\]
\[
\geq \text{ord}_p(a!) - k'_r \text{ord}_p(a + 1) - \max_{0 < b_1 < \cdots < b_r \leq a} \left\{ \sum_{s=1}^{r-1} k'_s \text{ord}_p(b_s) \right\} - \text{wt}(k^+) =: F(a).
\]

It is enough to prove that \( \min_{a \geq 1} \{ F(a) \} \geq -2 \text{wt}(k^+) \). For \( t \geq 0 \) and \( 0 \leq u \leq p - 1 \), since we see that
\[
\text{ord}_p((tp + u)!) = \text{ord}_p((tp + p - 1)!) = \text{ord}_p((tp + u)!)
\]
and
\[
\max_{0 < b_1 < \cdots < b_r \leq tp + u} \left\{ \sum_{s=1}^{r-1} k'_s \text{ord}_p(b_s) \right\} \leq \max_{0 < b_1 < \cdots < b_r \leq tp + p - 1} \left\{ \sum_{s=1}^{r-1} k'_s \text{ord}_p(b_s) \right\},
\]

it suffices to check the case \( a \equiv p - 1 \mod p \). Putting \( a = qp^l - 1 \) with \( l \geq 1 \), \( q \geq 1 \) and \( p \nmid q \), we have

\[
F(qp^l - 1) = \frac{p^l - 1}{p - 1} + \max_{0 < b_1 < \cdots < b_{r-1} \leq qp^l - 1} \left\{ \sum_{s=1}^{r-1} k'_s \ord_p(b_s) \right\} - \wt(k^+) \]

If \( 1 \leq q \leq p - 1 \), since \( b_s \leq (p - 1)p^l - 1 < p^{l+1} \) and \( \ord_p(b_s) \leq l \) for \( 1 \leq s \leq r - 1 \), we find that

\[
F(qp^l - 1) = \frac{p^l - 1}{p - 1} + (k'_s + 1)l - \max_{0 < b_1 < \cdots < b_{r-1} \leq qp^l - 1} \left\{ \sum_{s=1}^{r-1} k'_s \ord_p(b_s) \right\} - \wt(k^+) \]

Note that we used the assumption \( \wt(k^+) < p - 1 \) in the case \( l \geq 2 \).

If \( q \geq p + 1 \), set \( q = \sum_{i=0}^{d} c_i p^l \) with \( 0 \leq c_i \leq p - 1 \), \( c_0 c_d \neq 0 \) and \( d \geq 1 \). Then it follows that

\[
F(qp^l - 1) \geq \frac{p^l - 1}{p - 1} + \frac{1}{p - 1} \sum_{i=0}^{d} c_i p^l + \frac{1}{p - 1} \sum_{i=1}^{d} c_i (p^l - 1) - (k'_s + 1)l - \left( \sum_{s=1}^{r-1} k'_s \right) (d + l) - \wt(k^+) \]

\[
\geq \frac{p^l - 1}{p - 1} (p^d + 1) + \frac{p^d - 1}{p - 1} - (\wt(k^+) + 1)l - \left( \sum_{s=1}^{r-1} k'_s \right) d - \wt(k^+) \]

\[
= \frac{p^{l+d} + p^l - 2}{p - 1} - (\wt(k^+) + 1)l - \left( \sum_{s=1}^{r-1} k'_s \right) d - \wt(k^+) \]

\[
\geq \frac{p^{d+1} + p - 2}{p - 1} - \left( \sum_{s=1}^{r-1} k'_s \right) d - 2 \wt(k^+) - 1 \]
\[ (1 + \frac{1}{p-1}) p^d \left( \sum_{s=1}^{r-1} k'_s \right) d - 2 \text{wt}(k') - \frac{1}{p-1} \]

\[ \geq (1 + \frac{1}{p-1}) p - \sum_{s=1}^{r-1} k'_s - 2 \text{wt}(k') - \frac{1}{p-1} \]

\[ = \left( p - \sum_{s=1}^{r-1} k'_s \right) + 1 - 2 \text{wt}(k') > -2 \text{wt}(k'). \]

This completes the proof. \( \square \)

**Remark 3.3.** We obtain the explicit formula of \( B^{(k)}_n \) by using the \( p \)-adic distribution \( \mu \) in the proof of Theorem 1.6 as follows. For \( n \geq 0 \), it is known that we have

\[ x^n = \sum_{j=0}^{n} \binom{n}{j} j! \left( x \right)^j, \]

where, for any integers \( a \) and \( b \), \( \{a\}_b \) are called the Stirling numbers of the second kind and defined by the recurrence formula

\[ \begin{cases} a + 1 \\ b \end{cases} = \begin{cases} a \\ b - 1 \end{cases} + b \begin{cases} a \\ b \end{cases} \]

with the conditions \( \{0\}_b = 1 \) and \( \{a\}_b = 0 \) for \( a < b \) ([2, Definition 2.2, Proposition 2.6]). Then we find that

\[ B^{(k)}_n = (-1)^n \int_{\mathbb{Z}_p} x^n d\mu(x) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} j! \int_{\mathbb{Z}_p} \left( x \right)^j d\mu(x) \]

\[ = (-1)^n \sum_{j=0}^{n} \binom{n}{j} j! \sum_{0<m_1<\ldots<m_{n-1}<m_{n+1}} \frac{(-1)^{j}}{m_{1}^{k_{1}} \cdots m_{r-1}^{k_{r}} (j+1)^{k}} \]

\[ = (-1)^n \sum_{0<m_1<\ldots<m_{n-1}<m_{n+1}} \frac{(-1)^{m_{n}-1} (m_{r} - 1) \binom{n}{m_{r}-1}}{m_{1}^{k_{1}} \cdots m_{r-1}^{k_{r}} m_{r+1}^{k_{r}}} \]

By the exactly same way, we get

\[ C^{(k)}_n = (-1)^n \sum_{0<m_1<\ldots<m_{n-1}<m_{n+1}} \frac{(-1)^{m_{n}-1} (m_{r} - 1) \binom{n+1}{m_{r}-1}}{m_{1}^{k_{1}} \cdots m_{r-1}^{k_{r}} m_{r+1}^{k_{r}}} \]

These formulas were proved in [5, Theorem 3] by using the generating functions.

**Remark 3.4.** It was claimed in [7, Theorem 13] that, given an odd prime \( p \) and positive integers \( m, n, k, N \) with \( p \geq \max(k+2, (N+k)/2) \) and \( m \equiv n \mod (p-1)p^N \), one has \( p^r B^{(k)}_m \equiv p^r B^{(k)}_m \mod p^N \). However, there are counterexamples: \( pB^{(1)}_m = p/2 \neq 0 = pB^{(1)}_m \mod p^N \) for \( N \geq 2 \) and \( m = (p-1)p^N + 1 \). (Its proof breaks down at [7, Proposition 11], for which \( j = p^2 + p - 1 \) yields a counterexample.)
4. Multi-poly-Bernoulli-star numbers

At the end of this paper, we will give Kummer-type congruences for other Bernoulli numbers.

Definition 4.1 ([4, Section 1]). For \( k = (k_1, \cdots, k_r) \in \mathbb{Z}^r \), define the non-strict multiple polylogarithm to be

\[
\text{Li}_k^*(t) = \sum_{0 < m_1 \leq \cdots \leq m_r} \frac{t^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}}.
\]

The multi-poly-Bernoulli-star numbers \( B_{n,*}^{(k)} \) and \( C_{n,*}^{(k)} \) are defined to be the rational numbers satisfying

\[
\frac{\text{Li}_k^*(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} \frac{B_{n,*}^{(k)}}{n!} t^n,
\]

\[
\frac{\text{Li}_k^*(1 - e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} \frac{C_{n,*}^{(k)}}{n!} t^n
\]

respectively, as formal power series over \( \mathbb{Q} \).

Remark 4.2. Similar relations to Remark 1.4 were proved in [4, Propositions 2.3, 2.4]. Furthermore, the multi-poly-Bernoulli-star numbers \( B_{n,*}^{(k)} \) and \( C_{n,*}^{(k)} \) verify a duality relation for \( k = (k_1, \cdots, k_r) \in \mathbb{Z}^r > 0 \) ([4, Theorem 3.2]).

Remark 4.3. It is known that the multi-poly-Bernoulli-star numbers \( C_{n,*}^{(k)} \) describe finite multiple zeta-star values ([4, Section 4]).

The following theorem can be shown by the exactly same argument as Theorem 1.6 and hence is omitted.

Theorem 4.4. Let \( k \in \mathbb{Z}^r \) be an index, \( p \) be an odd prime and \( m, n \) and \( N \) be positive integers with \( m, n \geq N \) and \( \text{wt}(k^*) < p - 1 \). If \( m \equiv n \mod (p - 1)p^{N-1} \), then we have

\[
p^2 \text{wt}(k^*) B_{m,*}^{(k)} \equiv p^2 \text{wt}(k^*) B_{n,*}^{(k)} \mod p^N,
\]

\[
p^2 \text{wt}(k^*) C_{m,*}^{(k)} \equiv p^2 \text{wt}(k^*) C_{n,*}^{(k)} \mod p^N.
\]

Remark 4.5. We can check the following formulas

\[
B_{n,*}^{(k)} = (-1)^n \sum_{0 < m_1 \leq \cdots \leq m_r \leq m_{r+1}} \frac{(-1)^{m_r-1}(m_r - 1)! \left\{ \frac{n}{m_{r-1}} \right\}_{m_{r-1} - 1}}{m_1^{k_1} \cdots m_r^{k_r} m_{r+1}^{k_{r+1}}},
\]

\[
C_{n,*}^{(k)} = (-1)^n \sum_{0 < m_1 \leq \cdots \leq m_r \leq m_{r+1}} \frac{(-1)^{m_r-1}(m_r - 1)! \left\{ \frac{n+1}{m_r} \right\}_{m_r - 1}}{m_1^{k_1} \cdots m_{r-1}^{k_{r-1}} m_r^{k_r}}
\]

by the same computation as Remark 3.3. These were obtained in [4, Proposition 2.2] by using the generating functions.
KUMMER-TYPE CONGRUENCES FOR MULTI-POLY-BERNOULLI NUMBERS

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