Clear evasion of the uncertainty relation with very small probability

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Abstract

We entertain the idea that the uncertainty relation is not a principle, but rather it is a consequence of quantum mechanics. The uncertainty relation is then a probabilistic statement and can be clearly evaded in processes which occur with a very small probability in a tiny sector of the phase space. This clear evasion is typically realized when one utilizes indirect measurements, and some examples of the clear evasion appear in the system with entanglement though the entanglement by itself is not essential for the evasion. The standard Kennard’s relation and its interpretation remain intact in our analysis. As an explicit example, we show that the clear evasion of the uncertainty relation for coordinate and momentum in the diffraction process discussed by Ballentine is realized in a tiny sector of the phase space with a very small probability. We also examine the uncertainty relation for a two-spin system with the EPR entanglement and show that no clear evasion takes place in this system with the finite discrete degrees of freedom.

1 Introduction

The Schrödinger amplitude together with the Schrödinger equation describes all the possible quantum states and their time developments. Combined with the Born probability interpretation we have the rules to interpret quantum mechanics [1]. The Heisenberg uncertainty relation [2] (and its reformulation in [3, 4]) and the EPR entanglement [5] represent two characteristic features of quantum mechanics which are foreign to macroscopic phenomena in classical physics. The ultimate purpose of the theory of measurements such as described in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] is to explain how to understand the observed phenomena starting with the basic rules of quantum theory.
It is customary to treat the uncertainty relation as a principle, namely, the *uncertainty principle* which defines the quantum mechanics at the deepest level. In this paper, we entertain the idea that the uncertainty relation is one of the consequences of quantum mechanics. To be more precise, the uncertainty relation is controlled by the probability interpretation of quantum mechanics, and thus the validity of the uncertainty relation is probabilistic one. If one allows a very small probability in a tiny sector of the phase space, a clear evasion of the uncertainty relation is allowed.

We illustrate this idea by using the clear evasion of the uncertainty relation in the diffraction process discussed by Ballentine [9] some time ago. We also discuss two simple gedanken experiments considered by Ozawa [17, 18] as examples of the clear evasion of the uncertainty relation in the present sense. The evasion is typically recognized when one utilizes indirect measurements, and some examples of the clear evasion appear in the system with entanglement though the entanglement by itself is not essential for the evasion. The standard Kennard’s relation [3] is naturally preserved intact in our analysis.

It should be emphasized that the evasion of the uncertainty relation we discuss has no connection with the analysis of the differences between the uncertainty relation in the sense of Kennard [3] and the uncertainty relation appearing in the detailed definition of measurements. The evasion of the uncertainty relation in this latter sense mainly analyzes the model of quantum measurements initiated by von Neumann [6] and its modifications. This class of violation does not associate the violation of the uncertainty relation with a very small probability in a tiny sector of the phase space and often discusses the evasion of the quantum limit of measurements. The notion such as the *evasion of the quantum limit* does not arise in our analysis, since we operate in the framework of quantum mechanics, and the occurrence of the violation takes place simply with a very small probability. Our analysis is thus perfectly consistent with experimental results so far that no clear evasion of the uncertainty relation has been reported.

Our analysis mainly utilizes indirect measurements, as we noted above, and the typical indirect measurement is realized for a system with entanglement. We thus briefly comment on the uncertainty relation for the two-spin system with the EPR entanglement [5]. We show that the clear evasion of the uncertainty relation is not realized for the two-spin system with the finite discrete degrees of freedom, since a tiny sector of the phase space with a very small probability is not simply defined for a finite discrete system.
2 Indirect measurement and uncertainty relation

We here discuss the clear evasion of the uncertainty relation in several simple processes by utilizing indirect measurements. Our discussion heavily relies on the explicit model of Ballentine [9] for a diffraction of a particle by a single slit (or a pin hole), which has been discussed without an emphasis on a very small probability.

2.1 Diffraction and uncertainty relation

Consider the plane wave of a particle with momentum \( p \) moving toward the positive \( x \)-direction and colliding with a plate which is placed perpendicular to the \( x \)-axis at \( x = 0 \) with a pin hole at the origin of the \( y-z \) plane. See Fig. 1. The particle diffracted in the forward direction is measured on the screen which is perpendicular to the \( x \)-axis and placed at \( x = L \).

![Diagram](image)

Fig.1. A schematic arrangement to measure \( y \) and then infer the value of \( p_y \).

Suppose that one detects the diffracted particle by a detector with a size \( \delta y \) which is placed on the \( y \)-axis of the screen at the position \( y \). The \( y \)-component of the momentum of the measured particle is indirectly estimated at

\[
p_y = p \frac{y}{\sqrt{L^2 + y^2}} \tag{2.1}
\]

and the coordinate accuracy is \( \delta y \). The uncertainty in \( p_y \) is estimated by

\[
\delta p_y = p \frac{\delta y}{\sqrt{L^2 + y^2}} \frac{L^2}{\sqrt{L^2 + y^2}} \sim p \frac{\delta y}{L} \tag{2.2}
\]

for fixed \(|y|\) with \( L \geq |y| \). The uncertainty product is then

\[
\delta y \delta p_y \sim p \delta y \frac{\delta y}{L} \rightarrow 0 \tag{2.3}
\]
for $L \to \infty$ for fixed $\delta y$. Thus the conventional uncertainty relation appears to be clearly evaded. See Subsection 3.2 in [9].

In the idealized analysis of Ballentine [9], no variable $\delta l$ corresponding to the size of the opening of the slit appears. A slit of $\delta l$ induces a disturbance

$$\delta p_y \sim \hbar/\delta l$$  (2.4)

in the momentum of the particle immediately after the passage of the slit and it causes the effect of diffraction, but the above conclusion is still valid as long as the scattering by the slit is elastic and thus the energy conservation of the combined system of the particle and the slit holds, namely, the magnitude of the momentum of the particle after the passage of the slit is still given by $p$ to a good accuracy. To be precise, one may choose

$$p \geq \hbar/\delta l$$  (2.5)

or

$$\delta l \geq \hbar/p = \lambda$$  (2.6)

but not much larger than $\lambda$, where $\lambda$ stands for the de Broglie wave length of the particle. Our assumption is that the slit with a large mass absorbs the momentum but no energy is lost due to the possible excitation of the atomic states in the slit, as in the case of the Mössbauer effect.

The setup of the problem in [9] is then analogous to the elastic scattering of a particle by a heavy target located at the center of the slit. To define the geometrical picture in Fig.1 precisely, one needs to set $L \to \infty$. In this case, one may rescale all the length variables by $L$ as in the scattering problem, and thus

$$\tilde{y} = \frac{y}{L}, \quad \delta \tilde{y} = \frac{\delta y}{L}$$  (2.7)

and one may describe the scattering in terms of $\tilde{y}$ and $\delta \tilde{y}$. The uncertainty product (2.3) is then given by

$$\delta y \delta p_y \sim p \delta y \delta y \frac{L}{L} \sim \hbar \frac{L}{\lambda} (\delta \tilde{y})^2 \gg \hbar$$  (2.8)

for fixed finite $\delta \tilde{y}$, where $\lambda$ stands for the de Broglie wave length, and the issue of the uncertainty relation does not arise in this description. This description may be regarded as corresponding to a description in the classical limit.

The clear evasion of the uncertainty relation (2.3) was possible since we restricted the range $\delta y$ to a measure-zero set in the a priori allowed interval $(0, \infty)$, namely,
\( \delta y / \tilde{L} \to 0 \) for \( L \to \) large where \( \tilde{L} \) stands for the size of the spread of the diffracted particle at \( x = L \). A way to understand the special property of (2.3) is to consider the wave function of the diffracted particle which may be represented by

\[
\psi(x, y, z) = \frac{1}{r} \exp \left[ \frac{ipr}{\hbar} \right] f(\theta)
\]

by assuming the symmetry around the x-axis. Here \( r = \sqrt{x^2 + \rho^2} \), \( \rho = \sqrt{y^2 + z^2} \) and \( \sin \theta = \rho/r \); \( \psi(x, y, z) \) may be normalized to unity when integrated over a large hemi-sphere with radius \( r \) since \( x \geq 0 \) in Fig.1. One may then imagine to detect the particle which arrives at a ring (annulus) specified by the radius \( \rho \) and \( \rho + \delta \rho \) on the screen placed at \( x = L \). The probability to detect the particle is then

\[
|\psi(x, y, L)|^2 2\pi \rho \delta \rho = \left| \frac{1}{r} f(\theta) \right|^2 2\pi \rho \delta \rho \sim \frac{\rho \delta \rho}{r^2} \leq \frac{\delta \rho}{r} \to 0
\]

for \( L \to \infty \), and thus the probability is vanishingly small; \( \delta \rho \) here corresponds to \( \delta y \), and this conclusion also holds when one considers the flux instead of the probability. The clear evasion of the uncertainty relation for the set of events with a very small probability does not contradict the principle of quantum mechanics. In the conventional treatment of scattering, one fixes \( \delta \sin \theta = \delta \rho / r \) for \( L \to \infty \) and no evasion of the uncertainty relation occurs. See (2.8).

Immediately after the measurement on the screen, however, the particle is described by a wave packet of the size \( \delta \rho \) in the \( \rho \) direction. Translated into the notation of \( y \), the wave function \( \psi(y) \) (by suppressing the dependence on other variables) does not contain the small factor \( \sim 1/\sqrt{L} \) in front of it (as in the case of approximate plane wave \( \sim (1/\sqrt{L}) \exp[ipy/\hbar] \) before the detection) due to the “reduction” of the wave function. At the same time, we generally expect \( \delta p_y \sim \hbar / \delta y \) after the measurement. The event realized with nearly a unit probability such as the present example after the actual measurement does not lead to a clear evasion of the uncertainty relation. More comments on this problem will be given later.

### 2.2 Other examples of the clear evasion of the uncertainty relation

We next study a two-particle system in one-dimensional space with an equal mass \( m \), and their phase space variables are \((x_1, p_1)\) and \((x_2, p_2)\). We then define

\[
\begin{align*}
P &= p_1 + p_2, \\
Q &= \frac{1}{2}(x_1 + x_2), \\
p &= \frac{1}{2}(p_1 - p_2), \\
q &= x_1 - x_2.
\end{align*}
\]

(2.11)
An example given by Ozawa for which the clear evasion of the uncertainty relation takes place goes as follows (see Section 9 in [17]): Starting with a two-particle system such as (2.11), one assumes the precise measurement of \( x_2 \) and no more measurements of other variables. Thus the measurement is rather incomplete for the system of two degrees of freedom. Apparently the disturbance \( \eta(p_1) \) in the momentum \( p_1 \) of the first particle, which is defined in Appendix, caused by the measurement of \( x_2 \) vanishes

\[
\eta(p_1) = 0. \tag{2.12}
\]

It is shown that the noise (or error) in the coordinate of the first particle \( x_1 \), when the precise measurement of \( x_2 \) is regarded as an indirect measurement of \( x_1 \), is given by

\[
\epsilon(x_1) = \langle (\hat{x}_2 - \hat{x}_1)^2 \rangle^{1/2} < \alpha \tag{2.13}
\]

for any small \( \alpha \). He thus concludes [17] that one can estimate \( x_1 \) with \( \epsilon(x_1) < \alpha \) for any small \( \alpha \) from the measured value of \( x_2 \) without disturbing \( p_1 \), namely, \( \eta(p_1) = 0 \) for all such states, and thus the clear evasion of the uncertainty relation \( \epsilon(x_1)\eta(p_1) \geq \frac{1}{2}\hbar \) in the sense of (A.8) in Appendix.

Without any information about the total system (namely, without any measurement of \( x_1 \)), the *a priori* natural guess is

\[
\langle (\hat{x}_2 - \hat{x}_1)^2 \rangle^{1/2} \sim L \tag{2.14}
\]

where \( L \) stands for the size of the “box” in which the system is contained. In this sense

\[
\Delta(\hat{x}_2 - \hat{x}_1) \leq \langle (\hat{x}_2 - \hat{x}_1)^2 \rangle^{1/2} < \alpha \tag{2.15}
\]

for any small \( \alpha \), where \( \Delta(\hat{x}_2 - \hat{x}_1) \) stands for the standard deviation, implies a significant correlation. Under this assumption of significant correlation, one can estimate the initial value of \( x_1 \) without disturbing \( p_1 \) by using the precisely measured value of \( x_2 \). Stated differently, one can know \( x_1 \) accurately without disturbing \( p_1 \) by measuring \( x_2 \) precisely, if the particle 1 happens to be very close to the particle 2.

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1 The precise measurement of \( x_2 \) implies \( \hat{\mu}_{X_2f} = \hat{x}_2 \) where \( \hat{\mu}_{X_2f} \) stands for the apparatus variable. The noise in \( \hat{x}_1 \) when one regards \( \hat{\mu}_{X_2f} \) as the measuring apparatus of \( \hat{x}_1 \) is then given by \( \epsilon(x_1) = \langle (\hat{\mu}_{X_2f} - \hat{x}_1)^2 \rangle^{1/2} = \langle (\hat{x}_2 - \hat{x}_1)^2 \rangle^{1/2} \). See Section 9 in [17].
The above argument related to (2.13) shows that one can clearly evade the uncertainty relation for any input wave function which is confined in a tiny subspace \(\langle (\hat{x}_2 - \hat{x}_1)^2\rangle^{1/2} < \alpha\) of the allowed range of the variable \(x_1\) with a precisely measured \(x_2\). This subspace is measure-zero compared to the full one-dimensional space, namely, the interval covered by the eigenvalues with \(\langle (\hat{x}_2 - \hat{x}_1)^2\rangle^{1/2} < \alpha\) is negligible compared to the total space of \textit{a priori} allowed eigenvalues \(-L < x_1 - x_2 < L\) for \(L \to \infty\) with a fixed \(x_2\). The probability of observing such an event is very small.

This small probability is understood by considering a generic normalized wave function \(\Psi(x_1, x_2) \simeq \sum_n a_n \psi_n(x_2, q)\) where \(\psi_n(x_2, q)\) stand for normalized non-overlapping wave packets in \(q\) with a width \(\alpha\). The function \(\psi_n(x_2, q)\) may have a peak around the precisely measured value of \(x_2\) and the coefficients are \(|a_n| \sim 1/\sqrt{2L/\alpha}\) without any additional information about \(q\). Here we have a Bloch-type representation of \(\Psi(x_1, x_2)\) in mind. The above specific choice of the initial state in (2.13) corresponds to selectively picking up a specific \(\psi_n(x_2, q)\) which contains \(q = x_1 - x_2 = 0\), but the standard interpretation of the wave function \(\Psi(x_1, x_2)\) in quantum mechanics predicts that such a probability is negligibly small

\[|a_n|^2 \sim \alpha/(2L)\] (2.16)

for \(L \to \infty\). In contrast, after the actual direct measurement of the state \(\psi_n(x_2, q)\) which contains \(q = x_1 - x_2 = 0\), such a state is then realized with a unit probability due to the reduction of the state vector. The momentum disturbance \(\eta(p_1)\) is then generally modified as predicted by quantum mechanics for the state \(\psi_n(x_2, q)\) and thus no clear evasion of the uncertainty relation. (To be precise, the condition \(\eta(p_1) = 0\) may be still preserved even after the actual measurement of the state \(\psi_n(x_2, q)\) which contains \(q = x_1 - x_2 = 0\), but such a probability is negligibly small as is understood by writing \(\psi_n(x_2, q)\) as a superposition of plane waves. The probability of a clear evasion of the uncertainty relation is very small in any case.)

In passing, the average \(\langle (\hat{x}_2 - \hat{x}_1)^2\rangle\) for the state \(\Psi(x_1, x_2)\) is estimated by

\[\langle (\hat{x}_2 - \hat{x}_1)^2\rangle \sim \sum_q q^2 \alpha^2 / 2L \sim \int_{-L}^{L} dq q^2 1/2L \sim L^2 / 3\]

and thus \(\langle (\hat{x}_2 - \hat{x}_1)^2\rangle^{1/2} \sim L\) which is consistent with (2.14) and does not give rise to a clear evasion of the uncertainty relation.

Another example of the clear evasion of the uncertainty relation given by Ozawa [18] goes as follows: When one measures the commuting variables \(\hat{P}\) and \(\hat{q}\) precisely with \(\epsilon(P) = \epsilon(q) = 0\), it is then argued that

\[\epsilon(x_1)\epsilon(P) = 0\] (2.17)
and thus the clear evasion of the uncertainty relation in the sense of (A.7) in Appendix for the pair $[\hat{x}_1, \hat{P}] = i\hbar$. Here $\epsilon(x_1)$ and $\epsilon(P)$ respectively stand for the operationally defined root-mean-square noise in $x_1$ and $P$ which are briefly explained in Appendix.

The essence of the argument for the above relation (2.17) is as follows (see Section 2 in [18]): The precise measurement of $q$ can be interpreted as an approximate indirect measurement of $x_1$, if one takes the output from the measurement of $q$ to be the measured value of $x_1$. The root-mean-square noise

$$\epsilon(x_1) = \langle(\hat{q} - \hat{x}_1)^2\rangle^{1/2} = \langle(\hat{x}_2)^2\rangle^{1/2}$$

(2.18)

is thus a reasonable measure of imprecision for the $x_1$ measurement.

If the mean position of $x_2$ is chosen at the origin, $\langle\hat{x}_2\rangle = 0$, then one has $\epsilon(x_1) = \Delta x_2$. Ozawa then argues that $\epsilon(x_1) = \Delta x_2$ can be made arbitrarily small

$$\epsilon(x_1) = \Delta x_2 < \alpha$$

(2.19)

by choosing input states, and thus one obtains (2.17) when one remembers $\epsilon(P) = 0$ in the present case. To achieve (2.17) it is in fact sufficient to keep $\epsilon(x_1) = \Delta x_2$ bounded by a finite constant.

Note that the proof of the conventional relation

$$\frac{1}{2}\hbar \leq \epsilon(x_1)\epsilon(P)$$

(2.20)

requires that the measurement is unbiased as is explained in Appendix, whereas the definition in (2.19) does not satisfy the unbiased condition in general. However, the correlation between $x_1$ and $x_2$ arising from $\epsilon(q) = \epsilon(P) = 0$ combined with a crucial additional assumption $\langle(\hat{x}_2)^2\rangle^{1/2} < \alpha$ allows one to estimate initial $x_1$ to a good accuracy simultaneously with $\epsilon(P) = 0$. In this sense the above argument implies the clear evasion of the uncertainty relation for $[\hat{x}_1, \hat{P}] = i\hbar$.

In passing, instead of (2.19) one may also consider

$$\epsilon(x_1) = \langle(\hat{x}_2 - c)^2\rangle^{1/2} < \alpha$$

(2.21)

with a suitable constant $c$, namely, the correlation between $x_1 - c$ and $x_2 - c$ arising from $\epsilon(q) = \epsilon(P) = 0$; this notation emphasizes the choice of suitable initial states for a given $c$ such that $\langle(\hat{x}_2 - c)^2\rangle^{1/2} < \alpha$. If one chooses $c = \langle x_2 \rangle$, one recovers (2.19) but one may make a more general choice of $c$.  

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2The precise measurement of $q$ implies $\hat{\mu}_{Qf} = \hat{q}$ where $\hat{\mu}_{Qf}$ stands for the apparatus variable. The noise in $\hat{x}_1$ when one regards $\hat{\mu}_{Qf}$ as the measuring apparatus of $\hat{x}_1$ is then given by $\epsilon(x_1) = \langle(\hat{\mu}_{Qf} - \hat{x}_1)^2\rangle^{1/2} = \langle(\hat{q} - \hat{x}_1)^2\rangle^{1/2}$. 

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The essence of the above analysis is that the quantity \( \epsilon(x_1) = \langle (\hat{x}_2 - c)^2 \rangle^{1/2} \) in (2.19) or (2.21) is kept small by choosing suitable input wave functions, but the range covered by the eigenvalues with \( \epsilon(x_1) = \langle (\hat{x}_2 - c)^2 \rangle^{1/2} < \alpha \) is negligible compared to the total space of \textit{a priori} allowed eigenvalues \(-\infty < x_2 < \infty\). Stated differently, one can guess the position \( x_1 \) accurately simultaneously with \( \epsilon(P) = 0 \) by measuring \( q \) precisely, if the position \( x_2 \) of the particle 2 happens to be well localized around some fixed \( c \). The probability of observing such an event is very small.

This small probability is understood by considering a generic normalized wave function \( \Psi(x_1, x_2) \simeq \sum_n a_n \psi_n(x_2, q) \) where \( \psi_n(x_2, q) \) stand for normalized non-overlapping wave packets in \( x_2 \) with a width \( \alpha \). The function \( \psi_n(x_2, q) \) may have a peak around the precisely measured value of \( q \) and the coefficients are \( |a_n| \sim 1/\sqrt{L/\alpha} \) without any additional information about \( x_2 \). The above specific choice of the initial state in (2.21) corresponds to selectively picking up a specific \( \psi_n(x_2, q) \) which contains \( x_2 = c \), but the standard interpretation of \( \Psi(x_1, x_2) \) in quantum mechanics predicts that such a probability is negligibly small \( |a_n|^2 \sim \alpha/L \) for \( L \to \text{large} \).

These properties related to (2.13) and (2.21) are the general aspects of the clear evasion of the uncertainty relation which is recognized by the help of indirect measurements, and the probability for observing such events which lead to the clear evasion of the uncertainty relation is negligibly small.

We also mention that the essential aspects of the evasion of the uncertainty relation in Subsection 2.2 depend only on the initial quantum variables and initial quantum states, and the analysis depends on the minimal set of definitions summarized in Appendix. Our analysis is insensitive to the specific model of measurements, which in turn suggests that the clear evasion of uncertainty relations discussed here is not an artifact of a specific model of measurements.

No explicit reference to entanglement appears in the above analysis. To discuss the implications of entanglement on the present problem, one needs a quantitative definition of entanglement. The entanglement in the context of standard deviations \( 3 \). We may thus tentatively assume that the precise measurements \( \epsilon(P) = \epsilon(q) = 0 \) have been performed for the initial states with \( \Delta P = \Delta q = 0 \), which is possible though there is no strong reason to assume so. The quantity \( \epsilon(x_1) = \langle (\hat{x}_2)^2 \rangle^{1/2} \) in (2.19) may then be understood as a measure of

\[ 2\hbar > \Delta^2(x_1 - x_2) + \Delta^2(p_1 + p_2) \geq 0 \]

which is also known to be a necessary condition for Gaussian two-particle states[19, 20, 21]. Here \( \Delta(x_1 - x_2) \) and \( \Delta(p_1 + p_2) \) stand for the standard deviations. It is explained in [20] that the conventional EPR-type entanglement is included in this criterion.

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3A sufficient condition of entanglement is written as (with a suitable choice of a dimensional constant)

\[ 2\hbar > \Delta^2(x_1 - x_2) + \Delta^2(p_1 + p_2) \geq 0 \]
correlation between $x_1$ and $x_2$ arising from $\Delta P = \Delta q = 0$ for the initial states. One may also use the relation $\epsilon(P) = 0$ as a resource coming from entanglement in the present sense and choose $\epsilon(p_1) = \langle (\hat{P} - \hat{p}_1)^2 \rangle^{1/2} = \langle (\hat{p}_2)^2 \rangle^{1/2}$. In this case, however, we have $\epsilon(x_1) = \langle (\hat{x}_2)^2 \rangle^{1/2} \geq \frac{1}{2}\hbar$ by noting $\langle (\hat{x}_2)^2 \rangle^{1/2} \geq \Delta x_2$, for example, and no evasion of the uncertainty relation takes place [18]. This argument suggests that the connection of the clear evasion of the uncertainty relation with the entanglement is not simple and the entanglement by itself is not essential for the evasion of the uncertainty relation.

2.3 Simple model

To illustrate our analysis related to (2.17), we consider a simple model of two non-interacting particles contained in a one-dimensional "box" with a length $L$. We impose a periodic boundary condition, for simplicity. The coordinates are then confined in

$$-\frac{1}{2}L \leq x_1, x_2 \leq \frac{1}{2}L$$

and the momenta $p_1, p_2$ are given by $p_1, p_2 = 2n\pi\hbar/L$ with $n = 0, \pm 1, \pm 2, ...$. Without any previous knowledge, the precise measurement of $P$ corresponds to the distinction of $(n_1 + n_2)2\pi\hbar/L$ from $(n_1 + n_2 \pm 1)2\pi\hbar/L$ for a priori quite large $n_1 + n_2$. The natural estimate of the noise (or error) in $P$ may be a fraction of $2\pi\hbar/L$

$$\epsilon(P) \sim 2\pi\hbar/L,$$

and the natural estimate of the noise (or error) in $x_1$, when one knows only $q = x_1 - x_2$ precisely, is a fraction of $L$

$$\epsilon(x_1) = \langle (\hat{x}_2 - c)^2 \rangle^{1/2} \sim L$$

for a fixed $c$. We thus have

$$\epsilon(x_1)\epsilon(P) \sim 2\pi\hbar.$$  

One may of course choose the initial states such that

$$\epsilon(x_1) = \langle (\hat{x}_2 - c)^2 \rangle^{1/2} \leq \alpha$$

for any small $\alpha$ [18] and thus

$$\epsilon(x_1)\epsilon(P) \sim 2\alpha\pi\hbar/L \rightarrow 0$$
for $L \to \infty$. But the probability of hitting $\epsilon(x_1) = \langle (\hat{x}_2 - c)^2 \rangle^{1/2} \leq \alpha$ with a given fixed $c$ in the space of a priori allowed eigenvalues $-\frac{1}{2}L < x_2 < \frac{1}{2}L$ is negligibly small for $L \to \infty$, without any information such as coming from a preceding direct measurement of $x_2$. We emphasize that this vanishingly small probability for $\langle (\hat{x}_2 - c)^2 \rangle^{1/2} \leq \alpha$ is inevitable and not a result of our incompetence in performing experiments; the probability to find the particle 2 in any finite interval is negligibly small if one only knows that two particles with given $q = x_1 - x_2$ exist in the interval $-\infty < x_1, x_2 < \infty$ but without any more information. This is confirmed, as already explained in Subsection 2.2, by considering a generic normalized wave function of the form $\Psi(x_1, x_2) \simeq \sum_n a_n \psi_n(x_2, q)$ where $\psi_n(x_2, q)$ stand for normalized non-overlapping wave packets in $x_2$ with a width $\alpha$ and $|a_n| \sim 1/\sqrt{L/\alpha}$.

The correlation between $x_1$ and $x_2$ (or between $x_1 - c$ and $x_2 - c$) arising from $\epsilon(q) = 0$ combined with a restriction to a measure-zero set of initial states with $\langle (\hat{x}_2 - c)^2 \rangle^{1/2} \leq \alpha$ leads to the clear evasion of the uncertainty relation (2.17) but with a negligibly small probability.

### 2.4 Additional comment on the diffraction

We here comment on a treatment of the diffraction process in Subsection 2.1 in the manner of the relations (2.12) and (2.13).

We consider a combined system of the particle (particle 1 with mass $m_1$) and the idealized center of the slit (particle 2 with mass $m_2$) in Fig.1 as a quantum mechanical object: We define the phase space variables by

\[
\begin{align*}
P_x &= p_{1x} + p_{2x}, & P_y &= p_{1y} + p_{2y}, \\
q_x &= L = x_1 - x_2, & q_y &= y = y_1 - y_2,
\end{align*}
\]

with conjugate variables

\[
\begin{align*}
Q_x &= (m_1 x_1 + m_2 x_2)/(m_1 + m_2), & Q_y &= (m_1 y_1 + m_2 y_2)/(m_1 + m_2), \\
\tilde{p}_x &= \frac{m_2}{m_1 + m_2} p_{1x} - \frac{m_1}{m_1 + m_2} p_{2x}, & \tilde{p}_y &= \frac{m_2}{m_1 + m_2} p_{1y} - \frac{m_1}{m_1 + m_2} p_{2y},
\end{align*}
\]

by treating the problem as a two-dimensional problem for simplicity with $m_2 \gg m_1$.

We start with the particle 1 with $(p_{1x}, p_{1y}) = (p, 0)$ produced by a particle generator placed in the far left in Fig.1 and the slit with $(p_{2x}, p_{2y}) = (0, 0)$. We suppose the steady state with a fixed number of particles passing through the slit per unit time, such as a single particle per second. Assuming the elastic scattering at

---

4Our basic assumption here is that the center of the slit is described by an idealized quantum mechanical object with a very specific scattering power.
the slit, \((P_x, P_y) = (p, 0)\) is preserved, and the magnitude of the relative momentum 
\[ \sqrt{\hat{p}_x^2 + \hat{p}_y^2} \] stays constant since the contribution of the total momentum to the total 
conserved energy is negligible due to the huge mass \(m_2\). This is the initial state to 
analyze the uncertainty relation.

To describe the above quantum state, the position of the slit is crucial and thus 
we assume the precise measurement of \(y_2\) and \(x_2\) with \(\epsilon(y_2) = \epsilon(x_2) = 0\). Then 
apparently \(\eta(p_{1y}) = \eta(p_{1x}) = 0\). This setting may be regarded as corresponding to 
the fact that \(p_y (= p_{1y} \text{ in the present notation})\) in (2.1) and also \(p_{1x}\) are not directly 
measured and \(\delta p_y \to 0\) (and also \(\delta p_x \to 0\)) for \(L \to \infty\) independently of any fixed 
\(\delta y\) and \(\delta L\) in (2.2). One may then regard the precisely measured apparatus variable 
\(\hat{y}_{Y2}\) of \(y_2\) plus a constant \(c\) as an indirect measurement apparatus of \(y_1\) (and also the 
precisely measured apparatus variable \(\hat{y}_{X2}\) of \(x_2\) plus a constant \(L\) as an indirect 
measurement apparatus of \(x_1\) \([17]\), and thus the measurement errors

\[
\epsilon(y_1) = \langle (\hat{y}_2 + c - \hat{y}_1)^2 \rangle^{1/2} = \langle (\hat{q}_y - c)^2 \rangle^{1/2}, \\
\epsilon(x_1) = \langle (\hat{x}_2 + L - \hat{x}_1)^2 \rangle^{1/2} = \langle (\hat{q}_x - L)^2 \rangle^{1/2}. 
\] (2.30)

One can then evade the uncertainty relation for \([y_1, p_{1y}] = i\hbar\) (and also for \([x_1, p_{1x}] = i\hbar\) in the sense

\[
\eta(p_{1y})\epsilon(y_1) = \eta(p_{1y})\langle (\hat{q}_y - c)^2 \rangle^{1/2} = 0, \\
\eta(p_{1x})\epsilon(x_1) = \eta(p_{1x})\langle (\hat{q}_x - L)^2 \rangle^{1/2} = 0, 
\] (2.31)

since \(\eta(p_{1y}) = \eta(p_{1x}) = 0\) and one can choose initial states for any fixed \(c\) and \(L\)

\[
\langle (\hat{q}_y - c)^2 \rangle^{1/2} \leq \alpha, \quad \langle (\hat{q}_x - L)^2 \rangle^{1/2} \leq \alpha 
\] (2.32)

for arbitrarily small \(\alpha\) by following the argument in \([17]\).

One may recognize that we assumed the measurement of \(y\)(or \(q_y\)) within an in-
terval \(\delta y\) in (2.1) and (2.2), whereas in the present case this measurement is replaced 
by the specification of the constants \(c\) and \(L\) contained in the indirect measurements 
and the specification of states which satisfy (2.32). The analysis of the uncertainty 
relation depends on \(\delta p_y = p(\delta y/L)\) and \(\delta y\) in (2.3) and on \(\eta(p_{1y}) = 0\) and \(\alpha\) in the 
present case, and the analysis of the uncertainty relation itself is not sensitive to the 
actual value of \(y\) or \(c\). We consider that the present setting is closer to the actual 
situation described in Fig. 1. In particular, the estimate of the error \(\delta p_y = p(\delta y/L)\) 
in (2.2) is based on an expected error when one specifies \(y\) with an accuracy \(\delta y\) 
without taking into account the possible disturbance in \(p_y\) which is caused by any 
experimental specification of \(y\).
The normalized wave function for the configuration in Fig.1 may be described by a superposition of wave functions at $q_x = L$ (by concentrating on the freedom in the y-direction) such as

$$\Psi(y_1, y_2) \simeq \sum_{n=1}^{N} a_n \psi_n(q_y, y_2)$$

(2.33)

where each $\psi_n(q_y, y_2)$ gives a normalized non-overlapping wave packet in $q_y$ with a size $\alpha (= \delta y)$, and

$$|a_n| \sim \frac{1}{\sqrt{N}} \sim \frac{1}{\sqrt{\tilde{L}/\alpha}}$$

(2.34)

without any additional information about $q_y$. Here $\tilde{L}$ is a length parameter which characterizes the spread of the diffracted particle at $q_x = L$, and $\tilde{L} \to \infty$ for $L \to \infty$. The above special choice of the initial state in (2.32) then corresponds to selectively picking up a special $\psi_n(q_y, y_2)$ which is centered at $q_y \simeq c$, but the standard interpretation of $\Psi(y_1, y_2)$ in quantum mechanics predicts that such a probability is vanishingly small $|a_n|^2 \sim \alpha/\tilde{L} \to 0$ for $L \to \infty$. This is equivalent to choosing a vanishingly narrow phase space. One may also use the outgoing wave in the radial direction such as (2.9) in this analysis.

If one fixes $y$ or $c$ together with $L$ by any means, one can estimate the value of $p_y$ by using the geometrical information (2.1). But we consider that the estimate of the value of $p_y$ itself is not essential in the analysis of the uncertainty relation, since the estimate of $p_y$ depends on the pre-determined magnitude of the relative momentum $\sqrt{\tilde{p}_x^2 + \tilde{p}_y^2}$. Even in the context of the relations (2.12) and (2.13) in Subsection 2.2, one may slightly change the setting of the problem and perform an analogous analysis. One may assume that a steady flux of the particle with momentum $p$ is supplied by a particle generator placed in the far left and the flux is passing through a box with a length $L$ in one-dimensional space. Then the wave function of the particle is expected to be of the form $\psi(x) \sim \frac{1}{\sqrt{L}} \exp[ipx/\hbar]$ inside the box. (The precise measurement of the coordinate of the particle 2 in the example of (2.12) and (2.13) may be regarded as a specification of the origin of the coordinate, since the particle 2 has no interaction with the particle 1 and the determination of the position $x_2$ provides the relative origin to measure $x_1$.) If one assumes the observation of the particle with the well-defined momentum $p$ in a small interval $\delta x$, it leads to a clear evasion of the uncertainty relation but such a probability is very small $|\psi(x)|^2 \delta x \sim \delta x/L$. After an actual observation of the particle location with an accuracy $\delta x$, it is described by a wave packet of the size $\delta x$. A state described
by such a wave packet is then realized with a unit probability due to the reduction of the state; if one assumes the simultaneous observation of the specific well-defined momentum $p$, it then leads to a clear evasion of the uncertainty relation. But this time the probability of observing a state with the well-defined momentum $p$ inside the wave packet is very small, as is confirmed by performing a Fourier analysis. We thus conclude that the probability of the clear evasion of the uncertainty relation is very small in any case. In contrast, if one wants to observe a particle described by a plane wave with nearly unit probability, for example, one needs to cover the range of the order $L$ in $x$ space and then no clear evasion of the uncertainty relation occurs.

2.5 Preparation of initial states

In all the examples of the clear evasion of the uncertainty relation we discussed so far, we have explained that the initial states are assumed to be confined in a subspace of the total $a$ priori allowed phase space, which has a very small measure. This means that the probability of observing such events is very small. On the other hand, if one prepares the initial states by some means and treats the preparation as a part of measurement, it is then shown that the uncertainty relation is not clearly evaded. This feature appears to be quite general [9].

Let us repeat the analysis of the case associated with (2.13). If one does not know anything about the value of $q = x_1 - x_2$ by any previous measurement, the value of $q$ can be very small as well as very large. One may then assume that the value of $q$ is confined in between $q$ and $q + \alpha$, though no reasons to assume so. Combined with the precise measurement of $x_2$, one can then estimate $x_1$ within the accuracy $\alpha$ which one can choose to be arbitrarily small without disturbing $\eta(p_1) = 0$. However, the probability for meeting such a lucky situation for the actual initial states, which specify only the value of $x_2$ precisely, is

$$\text{Probability} = |\psi(q)|^2 \alpha \sim \frac{\alpha}{L}$$

as is explained in (2.16) by using a different notation. See also (2.10) and (2.34). The probability thus becomes vanishingly small for $L \to \infty$, as we discussed already.

On the other hand, one may attempt to prepare the sample, which has a spread in $q$ between $q$ and $q + \alpha$, by some preceding measurements. In this case, the final state of the preceding measurements now becomes the initial state in (2.12) and (2.13). The normalization factor $1/\sqrt{L}$ does not appear in the initial state thus defined due to the “reduction” of the wave function, but one inevitably induces the disturbance of the order of $\hbar/\alpha$ in the conjugate relative momentum $p = (p_1 - p_2)/2$.
and thus in $p_1$ also, namely, $\eta(p_1) \sim \frac{\hbar}{\alpha}$. (Note that the preparation has a meaning only when it takes place with a sizable probability.) One may then perform the precise measurement of $x_2$ as described in (2.12) and (2.13). One can determine the position $x_1$ by the knowledge of $q$, which is localized between $q$ and $q+\alpha$, and the precisely measured $x_2$. In this case we have

$$\eta(p_1)\delta x_1 \sim \frac{\hbar}{\alpha} \times \alpha \sim \hbar$$

(2.36)

where $\delta x_1$ is an error in the estimate of $x_1$, and we have no clear evasion of the uncertainty relation for the combined process of preparation and measurement.

Similarly, in the analysis of (2.17) a preceding measurement of $x_2$ with an accuracy $\delta x_2$ induces the disturbance $\sim \hbar/\delta x_2$ in the variable $p_2$ and thus in $P$ also. (Note again that the preparation has a meaning only when it takes place with a sizable probability.) We thus have

$$\epsilon(x_1)\delta P \sim \Delta x_2\delta P \sim \hbar$$

(2.37)

in (2.17), where we used (2.19) with $\Delta x_2 \sim \delta x_2$ for a system of which $x_2$ is determined with an accuracy $\delta x_2$, and thus no clear evasion of the uncertainty relation. Note that we perform the measurement immediately after the preparation 5, and thus the uncertainty $\delta P$ in $P$, which is induced by the entire measurement process, is a sum of $\sim \hbar/\delta x_2$ and $\epsilon(P) = 0$ in the present case.

3 Uncertainty relation in a two-spin system with EPR entanglement

We here comment on the uncertainty relation for a two-spin system $\vec{s}_1$ and $\vec{s}_2$ with the EPR entanglement. To be precise, our analysis is valid for any two-spin system which is in a general inseparable state since our analysis holds regardless of the spatial separation of two spins as long as the interaction between the spins is ignored. Physically it is most interesting to regard our system as corresponding to a two-spin system spatially far apart as is discussed in [22].

We analyze the uncertainty relation associated with the generic non-Abelian structure of the angular momentum [4]

$$[\hat{J}_k, \hat{J}_l] = i\hbar\epsilon^{klm}\hat{J}_m.$$  (3.1)

5If one does not perform the measurement immediately after the preparation, the system is generally settled to a quite different state as is seen in the case of diffraction process in Fig.1 where the initial fluctuation in $p_y$ immediately after the passage of the slit is eventually absorbed into the effect of diffraction.
One may simultaneously measure spin components $s_{1z}$ and $s_{2y}$ precisely since these variables are commuting. We thus have

$$
\epsilon(s_{1z}) = 0, \quad \epsilon(s_{2y}) = 0.
$$

After the simultaneous precise measurements, we will have, for example,

$$
\psi_{\text{final}} = u(1)v(2)
$$

which is a superposition of the total spin $S = 0$ and $S = 1$. Here we defined $\hat{s}_{1y}v(1) = \pm \frac{1}{2}\bar{h}v(1)$ and $\hat{s}_{2y}v(2) = \pm \frac{1}{2}\bar{h}v(2)$, and also $\hat{s}_{1z}u(1) = \pm \frac{1}{2}\bar{h}u(1)$ and $\hat{s}_{2z}u(2) = \pm \frac{1}{2}\bar{h}u(2)$. This $\psi_{\text{final}}$ is the information we obtain by simultaneous precise measurements $\epsilon(s_{1z}) = 0$ and $\epsilon(s_{2y}) = 0$ without assuming any entanglement in the initial state. The basic question is then if one obtains any extra information about the spin $s_{1y}$ or spin $s_{2z}$ in the initial state by assuming the entanglement specified by $S = 0$, for example.

By assuming the entanglement specified by $S = 0$, for example, we have

$$
\hat{S}_{y}|S = 0, S_{y} = 0\rangle = (\hat{s}_{1y} + \hat{s}_{2y})|S = 0, S_{y} = 0\rangle = 0
$$

and

$$
\langle S = 0, S_{y} = 0|\hat{s}_{1y}\hat{s}_{2y}|S = 0, S_{y} = 0\rangle
$$

$$
= -\frac{1}{4}\bar{h}^{2}
$$

$$
\neq \langle S = 0, S_{y} = 0|\hat{s}_{1y}|S = 0, S_{y} = 0\rangle \langle S = 0, S_{y} = 0|\hat{s}_{2y}|S = 0, S_{y} = 0\rangle = 0
$$

which follows from $\langle S = 0, S_{y} = 0|\hat{S}_{y}^{2}|S = 0, S_{y} = 0\rangle = 0$. It thus appears that one can estimate the precise value of $s_{1y}$ indirectly by means of entanglement from the precisely measured $s_{2y}$ with $\epsilon(s_{2y}) = 0$, simultaneously with the precise measurement of $s_{1z}$ with $\epsilon(s_{1z}) = 0$. This would suggest the simultaneous precise determination of $s_{1y}$ and $s_{1z}$, and thus the uncertainty relation for $s_{1y}$ and $s_{1z}$ appears to be evaded.

This argument however does not quite work if one remembers that the symmetry consideration with respect to $1 \leftrightarrow 2$ implies the precise simultaneous estimate of $s_{2y}$ and $s_{2z}$ also, and thus the precise simultaneous information of $s_{1y}$, $s_{1z}$, $s_{2y}$ and $s_{2z}$. One first recalls the relation for the state with $S = 0$

$$
\psi_{\text{initial}}(0) = |S = 0, S_{y} = 0\rangle
$$

$$
= \frac{i}{\sqrt{2}}[v_{+}(1)v_{-}(2) - v_{-}(1)v_{+}(2)]
$$

$$
= \frac{1}{\sqrt{2}}[u_{+}(1)u_{-}(2) - u_{-}(1)u_{+}(2)]
$$

$$
= |S = 0, S_{z} = 0\rangle
$$

(3.6)
where we used $u_+ = (v_+ + v_-)/\sqrt{2}$ and $u_- = (v_+ - v_-)/(\sqrt{2}i)$.

After the precise measurement of $s_{2y} = \frac{1}{2}\hbar$, for example, one may expect the wave function to be

$$\psi_{\text{final}} = v_-(1)v_+(2)$$

(3.7)

which is the basis why we guess $s_{1y} = -\frac{1}{2}\hbar$ and thus obtain extra information. Similarly, after the precise measurement of $s_{1z} = -\frac{1}{2}\hbar$, for example, one may expect the wave function to be

$$\psi_{\text{final}} = u_- (1) u_+(2)$$

(3.8)

which is the basis why we guess $s_{2z} = \frac{1}{2}\hbar$ and obtain extra information. But

$$v_-(1)v_+(2) \neq u_- (1) u_+(2)$$

(3.9)

since both of $v_-(1)v_+(2)\) and $u_- (1) u_+(2)$ are the superposition of $S = 0$ and $S = 1$, and thus a rotation of 90 degrees around the $x$ axis does not leave the wave function invariant. This relation (3.9) shows that the simultaneous accurate information about $s_{1z}$ and $s_{2z}$ implied by the right-hand side and the simultaneous accurate information about $s_{1y}$ and $s_{2y}$ implied by the left-hand side cannot coexist and thus cannot be realized. Thus we cannot have the precise simultaneous information of $s_{1y}$, $s_{1z}$, $s_{2y}$ and $s_{2z}$, and consequently no clear evasion of the uncertainty relation.

Another possibility one may examine is to precisely measure only $s_{1z}$ and no more measurement. This is analogous to (2.12). One may then naively expect $\eta(s_{2y}) = 0$. Assuming entanglement with $S = 0$ in the initial state, one may guess $s_{2z}$ from the measured value $s_{1z}$. The information about $s_{2z}$ without disturbing $s_{2y}$ with $\eta(s_{2y}) = 0$ would suggest an evasion of the uncertainty relation. This possibility can be examined explicitly by noting that the state after the measurement of $s_{1z}$ is given by

$$\psi_{\text{final}} = u_+(1)u_-(2) = \frac{1}{2i}(v_+(1) + v_- (1))(v_+(2) - v_- (2))$$

(3.10)

which may be compared to the initial state, namely, the first expression in (3.6)

$$\psi_{\text{initial}} = \frac{i}{\sqrt{2}}[v_+(1)v_-(2) - v_- (1)v_+(2)].$$

(3.11)

From this comparison, one first concludes that $s_{1y}$ is influenced by the precise measurement of $s_{1z}$, but from the symmetry under $1 \leftrightarrow 2$ in these expressions one also concludes that $s_{2y}$ cannot stay undisturbed by the precise measurement of $s_{1z}$. To
the extent that $\eta(s_{1y}) \neq 0$ under the precise measurement of $s_{1z}$, one concludes that $\eta(s_{2y}) \neq 0$ under the precise measurement of $s_{1z}$ and thus no clear evasion of the uncertainty relation in the present example.

This example shows that a clear evasion of the uncertainty relation is not realized in the present system with a finite number of discrete degrees of freedom even if one uses the indirect determination on the basis of entanglement. This example also provides a further indication that the entanglement by itself is not responsible for the clear evasion of the uncertainty relation.

4 Discussion and conclusion

Motivated by the examples of the clear evasion of the uncertainty relation in the diffraction process by Ballentine [9] and in the two simple gedanken experiments by Ozawa [17, 18], we analyzed the basic mechanism of the clear evasion of the uncertainty relation. We have shown that the clear evasion of the uncertainty relation is realized in a tiny sector of the phase space with a vanishingly small measure. Namely, the clear evasion of the uncertainty relation is possible but such a probability is very small. This analysis is consistent with the fact that no clear evasion of the uncertainty relation in experiments has been reported. Our emphasis here is that the uncertainty relation including the Kennard’s relation is of probability theoretical nature, as are all the laws in quantum mechanics. We take the uncertainty relation as a consequence of quantum mechanics rather than as a principle.

We have also argued that, if one considers the preparation of the initial states and then the analysis of the uncertainty relation later, the combined total process does not lead to the clear evasion of the uncertainty relation.

In the course of our analysis, we indicated that the entanglement by itself is not essential for the clear evasion of the uncertainty relation. In retrospect, this is quite natural since the entanglement or inseparability is the property of state vectors specified by means of the Kennard’s uncertainty relation [19, 20, 21].

We also think the following fact is worth mentioning: The essential aspects of our analysis of the uncertainty relation depend on only the initial quantum variables and initial quantum states, and thus the analysis is rather insensitive to the detailed model of measurements. If the analysis should be very sensitive to a specific model of measurements, one would naturally ask what will happen if one adopts another model of measurements.

Finally, a remaining more fundamental and difficult issue is an analysis of quantitative conditions for the transitional region where the evasion of the uncertainty relation starts visible, instead of the present analysis which is limited to the case of
the clear evasion of the uncertainty relation, and it is left for a future investigation. If such an analysis is performed, an experimental test of the idea presented here will become possible.

A Summary of some basic quantities

We work in the Heisenberg picture and define the coordinate and momentum observables of a system with one degree of freedom before the measurement

\[
(\hat{x}_i, \hat{p}_i)
\]  

(A.1)

and those observables after the measurement

\[
(\hat{x}_f, \hat{p}_f).
\]

(A.2)

We also define the observables of the measurement apparatus after the measurement

\[
(\hat{\mu}_X f, \hat{\mu}_P f)
\]  

(A.3)

which are mutually commuting \([\hat{\mu}_X f, \hat{\mu}_P f] = 0\) and also with the final state variables \([\hat{\mu}_X f, \hat{x}_f] = [\hat{\mu}_X f, \hat{p}_f] = 0\) and \([\hat{\mu}_P f, \hat{x}_f] = [\hat{\mu}_P f, \hat{p}_f] = 0\). We then define the measurement error operators by \([11, 17, 18]\)

\[
\hat{\epsilon}_x = \hat{\mu}_X f - \hat{x}_i,
\]

\[
\hat{\epsilon}_p = \hat{\mu}_P f - \hat{p}_i
\]

(A.4)

and the disturbance operators as the difference between initial and final operators

\[
\hat{\delta}_x = \hat{x}_f - \hat{x}_i,
\]

\[
\hat{\delta}_p = \hat{p}_f - \hat{p}_i
\]

(A.5)

Finally, we define the root-mean-square quantities

\[
\epsilon(x) = \langle \psi \otimes \phi_{ap} | \hat{\epsilon}_x^2 | \psi \otimes \phi_{ap} \rangle^{1/2}
\]

\[
\epsilon(p) = \langle \psi \otimes \phi_{ap} | \hat{\epsilon}_p^2 | \psi \otimes \phi_{ap} \rangle^{1/2}
\]

\[
\eta(x) = \langle \psi \otimes \phi_{ap} | \hat{\delta}_x^2 | \psi \otimes \phi_{ap} \rangle^{1/2}
\]

\[
\eta(p) = \langle \psi \otimes \phi_{ap} | \hat{\delta}_p^2 | \psi \otimes \phi_{ap} \rangle^{1/2}
\]

(A.6)

with the Heisenberg picture state of the form \(| \psi \otimes \phi_{ap} \rangle\) where \(\psi\) and \(\phi_{ap}\) stand for the initial states of the system and apparatus, respectively.
The uncertainty relations we adopt in our paper are then written in the form

$$\epsilon(x)\epsilon(p) \geq \frac{1}{2} \hbar$$  \hspace{1cm} (A.7)

or in the form

$$\epsilon(x)\eta(p) \geq \frac{1}{2} \hbar, \quad \epsilon(p)\eta(x) \geq \frac{1}{2} \hbar$$  \hspace{1cm} (A.8)

where the relation in (A.7) emphasizes the simultaneous measurements of two conjugate quantities, while the relations in (A.8) emphasize the measurement-disturbance relations [11, 17, 18]. The first relation (A.7) is proved by assuming that the measurement is unbiased in the sense

$$\langle \psi \otimes \phi_{ap} | \hat{e}_x | \psi \otimes \phi_{ap} \rangle = 0, \quad \langle \psi \otimes \phi_{ap} | \hat{e}_p | \psi \otimes \phi_{ap} \rangle = 0$$  \hspace{1cm} (A.9)

for all $\psi$ [11]. The quantities $\epsilon(x)$ and $\epsilon(p)$ may be understood as errors involved in the measurement process. It should however be noted that the essential aspects of the clear evasion of the uncertainty relation in the body of the present paper are understood in the more conventional framework as presented in [9].

The standard deviations are defined for any initial state by

$$\Delta x = \langle \psi | (\hat{x} - \langle \psi | \hat{x} | \psi \rangle)^2 | \psi \rangle^{1/2}$$
$$\Delta p = \langle \psi | (\hat{p} - \langle \psi | \hat{p} | \psi \rangle)^2 | \psi \rangle^{1/2}$$  \hspace{1cm} (A.10)

and the Kennard’s relation

$$\Delta x \Delta p \geq \frac{1}{2} \hbar$$  \hspace{1cm} (A.11)

always holds precisely. $\Delta x$ and $\Delta p$ may be understood as intrinsic uncertainties in the initial state.

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