Exact solution of a Levy walk model for anomalous heat transport

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The Levy walk model is studied in the context of the anomalous heat conduction of one dimensional systems. In this model the heat carriers execute Levy-walks instead of normal diffusion as expected in systems where Fourier’s law holds. Here we calculate exactly the average heat current, the large deviation function of its fluctuations and the temperature profile of the Levy-walk model maintained in a steady state by contact with two heat baths (the open geometry). We find that the current is non-locally connected to the temperature gradient. As observed in recent simulations of mechanical models, all the cumulants of the current fluctuations have the same system-size dependence in the open geometry. For the ring geometry, we argue that a size dependent cut-off time is necessary for the Levy walk model to behave as mechanical models. This modification does not affect the results on transport in the open geometry for large enough system sizes.

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Introduction.— Fourier’s law of heat conduction (in one dimension) states that

\[ J(x, t) = -\kappa \frac{\partial T(x, t)}{\partial x}, \]  

where \( T(x, t) \) and \( J(x, t) \) are the local temperature and heat current density fields and \( \kappa \) is the thermal conductivity. Based on results obtained from a large number of numerical simulations and various analytical approaches it is now believed that Fourier’s law is not valid in one and two dimensional mechanical systems (in particular models where momentum is conserved) and heat conduction is anomalous \[4, 5\]. Anomalous behavior in heat conduction is not only a theoretical issue but recently of experimental relevance in several low-dimensional materials \[4, 5\]. Indicators of the anomaly include — (i) in steady states the dependence of the heat current \( J \) on system size \( L \) shows the scaling behaviour \( J \sim L^{\alpha-1} \) with \( \alpha > 0 \), (ii) the temperature profiles across systems in nonequilibrium steady states are found to be nonlinear, even for very small applied temperature differences and, (iii) the spreading of heat pulses in anharmonic chains is super-diffusive.

The microscopic basis of Fourier’s law lies in the fact that the carriers of heat in a system execute random walks. The simplest “derivation” of Fourier’s law, from kinetic theory, is based on this picture and leads to an expression of the conductivity of a material in terms of the mean free path \( \ell \), mean velocity \( v \) and specific heat capacity \( c \) of the heat carriers: \( \kappa = cv\ell \) (in one dimensions). The breakdown of Fourier’s law thus also implies a breakdown of the random walk picture and of the diffusion equation

\[ \frac{\partial T(x, t)}{\partial t} = \frac{\kappa}{c} \frac{\partial^2 T(x, t)}{\partial x^2}, \]

which describes time-dependent heat transfer (assuming \( \kappa \) has no temperature-dependence). A number of recent studies indicate that a good description of anomalous heat conduction in one dimensional systems is obtained by modeling the motion of the heat carriers as Levy random walks instead of simple random walks \[6–9\]. Numerical studies show that the spreading of heat pulses and the form of steady state temperature profiles can be correctly modeled by means of the Levy walk. In another study of a model with stochastic dynamics, it has been shown, starting from a Boltzmann-equation approach, that the temperature satisfies a fractional diffusion equation corresponding to a Levy stable process \[10\].

While some analytical understanding has been achieved \[5\] it is desirable to further develop the Levy walk theory for anomalous heat conductivity so that (i) one can use it in the way as one uses Eqs. (1,2) for normal diffusion and (ii) one has a clearer idea of the range of applicability of the model. In this Letter we present several exact results for steady state heat transport in the Levy walk model in one dimension. We obtain exact analytic expressions for properties such as the density profiles or the average current, which agree with what was already known numerically. We also obtain new results for properties such as the cumulants of the current fluctuations which had not been considered before. For setting up the steady state we follow the idea in \[9\] of connecting two infinite reservoirs at different temperatures to the system and consider a version where space and time are taken to be continuous.

Our exact results provide several interesting physical perspective on anomalous heat transport. The analytic solution of the particle-density profile exhibits the non-linear (and singular) form typical of temperature profiles in 1D systems \[2, 3\]. The steady state current has the power law dependence on the system-size, characteristic
of anomalous diffusion. Also in contrast to \[1\] for normal heat transport, the current is non-locally connected to the temperature gradient. In addition, we derive the exact cumulant generating function of current for the open geometry. Our results show that all cumulants of the integrated current have the same system-size dependence as the average current. This is consistent with recent numerical results of heat conduction in hard particle systems \[11\] and therefore strongly suggests that the Levy-walk model gives a good description of anomalous heat conduction not only for the average current but also the size dependence of the current fluctuations. For the ring geometry, also, the size dependence of the cumulants obtained in simulations \[11\] of mechanical systems can be recovered by introducing a size dependent cut-off in the distribution of times of the Levy-walk model.

**Levy diffusion on the infinite line.** — In the simplest description we think of energy in the system as being transported by particles performing Levy walks, each particle carrying a single quantum of energy. Therefore the local energy density and energy current at any point are directly proportional to the particle density and current respectively. In this model the temperature is proportional to the energy density and hence to the density of particles. The precise definition of the Levy walk model that we consider here is as follows. For a single particle each step of the walk consists in choosing a time of flight \(\tau\) from a given distribution \(\phi(\tau)\) and then moving it at speed \(v\) over a distance \(x=v\tau\) in either direction, with equal probability. Let us define \(P(x,t)dx\) as the probability that the particle is in the interval \((x, x+dx)\) at time \(t\). Thus \(P(x,t)\) includes events where the particle is crossing the interval \((x, x+dx)\). If a particle starts at the origin at time \(t=0\), the probability \(P(x,t)\) satisfies

\[
P(x,t) = \frac{1}{2} \psi(t) \delta(|x| - vt) + \frac{1}{2} \int_0^t d\tau \phi(\tau) [P(x - vt, t - \tau) + P(x + vt, t - \tau)],
\]

where \(\psi(\tau) = \int_{-\infty}^{\infty} d\tau' \phi(\tau')\) is the probability of choosing a time of flight \(\geq \tau\). The Fourier-Laplace transform of \(P(x,t)\) can be calculated (see supplementary material \[29\]) from \[25\] in terms of the Laplace transform of the distribution of flight times \(\phi(\tau)\) and this gives the time dependence of all the cumulants \(\langle \tau^n \rangle_c\) of the position \(x\) at time \(t\).

If the first moments of the flight times \(\langle \tau^n \rangle\) were finite, the motion would be diffusive. One would then get from \[25\] for the first cumulants of \(x\) at large \(t\)

\[
\frac{\langle x^2 \rangle_c}{v^2 t} \sim \frac{\langle \tau^2 \rangle}{\langle \tau \rangle}; \quad \frac{\langle x^4 \rangle_c}{v^4 t} \sim 3 \frac{\langle \tau^2 \rangle^3}{\langle \tau \rangle^3} - 6 \frac{\langle \tau^2 \rangle \langle \tau^3 \rangle}{\langle \tau \rangle^2} + \frac{\langle \tau^4 \rangle}{\langle \tau \rangle}.
\]

Here we consider Levy walkers with a time-of-flight distribution decaying like a power law at large time

\[
\phi(\tau) \sim A \tau^{-\beta - 1}, \quad 1 < \beta < 2,
\]

(5)

(6)

(7)

(8)

(9)
We next discuss the current. The steady state current $J(x)$ at position $x$ is given by (see supplementary material [29])

$$J(x) = \frac{1}{2} \int_{-\infty}^{\infty} dy \, Q(x-y) \, \text{Sgn}(y) \, \psi(|y|/v) ,$$  \hspace{1cm} (14)

which can be interpreted as the difference between the flow from left to right and from right to left. The contribution to the integral coming from $y > 0$ corresponds to particles crossing the point $x$ from left to right – and is obtained by taking the density of particles at $x - y$ and multiplying by the probability $\psi(y/v)$ that these have a flight time longer than $y/v$. Similarly the other part of the integral (from $y < 0$) corresponds to a right-to-left current. After a partial integration and using the fact that $Q(0) = Q_l$ and $Q(L) = Q_r$, one gets

$$J(x) = -\frac{v}{2} \int_0^L dy \, \chi(|x-y|/v) \, Q'(y) .$$  \hspace{1cm} (15)

We note that $dJ/dx = 0$ gives Eq. (11) and so the current is independent of $x$, as expected. Evaluating the current at $x = 0$ and using Eq. (12), we get for large $L$

$$J \simeq (Q_l - Q_r) \frac{A \, v^\alpha \, \Gamma(\beta) \, \Gamma(1 - \frac{\gamma}{2})}{2 \, \beta(\beta-1) \, \Gamma(\frac{\beta}{2})} \, L^{\alpha-1} , \hspace{1cm} \alpha = 2 - \beta .$$  \hspace{1cm} (16)

From Eq. (6) we then get the relation $\alpha = \gamma - 1$, between the conductivity exponent of anomalous transport and the exponent for Levy-walk diffusion. This relation for Levy diffusion was noted in [6], numerically observed in 1D heat conduction models [8,15] and a derivation based on linear response theory has recently been proposed [10].

In the large $L$ limit by using Eq. (13) in Eq. (15) we obtain

$$J = -\frac{v}{2(\tau)} \int_0^L dy \, \chi(|x-y|/v) \, P(y) .$$  \hspace{1cm} (17)

This is the analogue of Fourier’s Law Eq. (1) in the case of normal heat conduction and can be interpreted as current being non-locally connected to the temperature gradient.

Current fluctuations in the open system.— In the rest of this letter, we discuss current fluctuations.

Since the particles are independent the current fluctuations can be described by a Poissonian process characterized by the rate at which walkers injected (at rate $p_L$) from the left reservoir end up (either after a non stop flight or a non direct flight) into the right reservoir or walkers injected (at rate $p_R$) from the right reservoir end up into the left reservoir (see supplementary material [29]). The current and its fluctuations can then be obtained by considering this process. In the case $Q_l = 1$ and $Q_r = 0$ let $P$ be the rate at which walkers, which will end up into the right reservoir, are injected from the left.

\[ P(x) = \chi(0)Q(x) = \langle \tau \rangle Q(x) . \]

In Fig. 1, we compare numerical results obtained by solving Eqs. (29,30) with the exact results of Eqs. (12,13). The profiles are nonlinear and look similar to those observed for temperature profiles in 1D heat conduction [2,3].
of length $L$. In the steady state, the density of particles is uniform. As the walkers are independent, the cumulants of the integrated current $Q$ are related to those of the displacement $x(t)$ of a single walker on the infinite line (in the steady state)

$$\langle Q^n \rangle_c \sim \frac{N}{L^n}(x(t)^n)_c = \frac{\rho}{L^{n-1}}(x(t)^n)_c$$  \hspace{1cm} (21)

where $\rho$ is the density on the ring. If the walkers perform on the ring the same Levy walks as on the infinite line, the cumulants of $x(t)$ and therefore those of $Q$ grow, as in (17), faster than linearly with time (same exponent but a different prefactor as, on the ring, the walker is in its steady state rather than starting a flight at $t = 0$).

On the other hand suppose one introduces a cut-off time $\tau_L \sim L^\delta$ in the distribution $\phi(\tau)$ (for example by arguing that $\tau_L$ should be of the order of $t^*$, the relaxation time corresponding to the shortest wave number on the ring $k = 2\pi/L$, then using the result $t^* \sim k^{-\beta}$ one gets that $\delta = \beta$; one could alternatively argue that, as for the open geometry, the length of the flights cannot exceed the system size and therefore $\delta = 1$). With such a cut-off $\tau_L$, the cumulants of $Q$ would grow linearly in time (17), with an amplitude which depends on the system size and on the cumulant considered

$$\frac{\langle Q^2 \rangle_c}{L} \sim L^{(2-\beta)\delta-1} \quad ; \quad \frac{\langle Q^4 \rangle_c}{t} \sim L^{(4-\beta)\delta-3}$$  \hspace{1cm} (22)

In one-dimensional mechanical models such as hard-point gas and anharmonic chains, energy transport is mediated by phonons which are weakly scattered. One can then think of these as performing Levy walks and indeed this picture is consistent with simulation data on energy diffusion [3,9]. Here we now see that the cut-off time $\tau_L$ also gives a possible explanation for the behavior seen in simulations on the ring of hard-point alternate gas in figure 3 of [11], where the cumulants grow linearly in time with different system size dependence ($\langle Q^2 \rangle_c/t \sim L^{-0.5}$ and $\langle Q^4 \rangle_c/t \sim L^{0.5}$). Then one gets from (21)

$$\beta \sim 5/3 \quad \text{and} \quad \delta \sim 3/2$$

which leads through (19) to a value $\alpha = 1/3$ for the anomalous Fourier’s law of the hard-point alternate gas in the open geometry consistent with most of the simulations done so far [3,11,18] for this system.

**Discussion.**— In this work we have studied the Levy diffusion model of anomalous heat transport. We have computed the average current (16), the energy profile (12,13) and the large deviation function of the integrated current (18), in the open geometry, i.e. when the system is connected at its two ends to reservoirs. One remarkable result is that all the cumulants of the integrated current have the same anomalous size dependence for the open geometry. We have also proposed a simple possible explanation for the size dependence of the cumulants for

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**Figure 2**: Monte-Carlo results for $\langle Q^2 \rangle_c/\langle Q \rangle$ as a function of measurement time $\tau_m$ for several system sizes with the parameters $\beta = 1.5$, $(Q_l, Q_r) = (1.0)$ for the same model as in Figure 1. The data agree with the results of our theory (19). The inset shows $\langle Q^m \rangle_c/\langle Q \rangle$ versus $\tau_m$ for $m = \infty$, which, as predicted by (19), agrees with the exact value of average current $J$. Numerical errors are smaller than the point-sizes.
the ring geometry. An interesting question would be to see how one could adapt existing theories on anomalous conduction, which usually focus on the Green Kubo formula and on the average current, to predict the higher cumulants of the current both in the open and in the ring geometry. Of course a challenging issue would be to know whether the picture which emerges from the present work, (Levy walkers with a cut-off time in the ring geometry) could be confirmed by these theories.

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[1] F. Bonetto, J.L. Lebowitz, and L. Rey-Bellet, in Mathematical Physics 2000, edited by A. Fokas et. al. (Imperial College Press, London, 2000), p. 128.
[2] S. Lepri, R. Livi, and A. Politi, Phys. Rep. 377, 1 (2003).
[3] A. Dhar, Adv. Phys. 57, 457 (2008).
[4] C. W. Chang et al., Phys. Rev. Lett. 101, 075903 (2008).
[5] D.L. Nika et al., Appl. Phys. Lett. 94, 203103 (2009).
[6] S. Denisov, J. Klafter and M. Urbakh, Phys. Rev. Lett. 91, 194301 (2003).
[7] R. Metzler and I. M. Sokolov, Phys. Rev. Lett. 92, 089401 (2004).
[8] P. Cipriani, S. Denisov, and A. Politi, Phys. Rev. Lett. 94, 244301 (2005); V. Zaburdaev, S. Denisov, and P. Hänggi, Phys. Rev. Lett. 106, 180601 (2011).
[9] S. Lepri and A. Politi, Phys. Rev. E 83, 030107 (2011).
[10] M. Jara, T. Komorowski, and S. Olla, Ann. App. Prob. 19, 2270 (2009).
[11] E. Brunet, B. Derrida, A. Gerschenfeld, Europhys. Lett. 90, 20004 (2010).
[12] G. Zumofen and J. Klafter, Phys. Rev. E 47, 851 (1993).
[13] R. Metzler and A. Compte, Physica A 268, 454 (1999).
[14] S. V. Buldyrev, S. Havlin, A. Ya. Kazakov, M. G. E. da Luz, E. P. Raposo, H. E. Stanley, and G. M. Viswanathan, Phys. Rev. E 64, 041108 (2001).
[15] H. Zhao, Phys. Rev. Lett. 96, 140602 (2006).
[16] S. Liu, N. Li, J. Ren, and B. Li, arXiv:1103.2835.
[17] G. Gallavotti and E. G. D. Cohen, Phys. Rev. Lett. 74, 2694 (1995); J.L. Lebowitz and H. Spohn, J. Stat. Phys., 95, 333 (1999).
[18] P. Grassberger, W. Nadler and L. Yang, Phys. Rev. Lett. 89, 18061 (2002).
[19] H. van Beijeren, Phys. Rev. Lett. 108, 180601 (2012).
[20] O. Narayan and S. Ramaswamy, Phys. Rev. Lett. 89, 200601 (2002).
[21] L. Delfini, S. Lepri, R. Livi, A. Politi, Phys. Rev. E 73, 060201(R) (2006).
[22] J. Lukkarinen and H. Spohn, Commun. Pure Appl. Math. 61, 1753-1786 (2008).
[23] S. Lepri, Phys. Rev. E 58 7165 (1998).
[24] S. Lepri, R. Livi and A. Politi, Phys. Rev. E 68 067102 (2003).
[25] A. Pereverzev, Phys. Rev. E 68 056124 (2003).
[26] B. Li and J. Wang, Phys. Rev. Lett., 91 044301 (2003).
[27] G. Basile, C. Bernardin and S. Olla, Phys. Rev. Lett., 96 204303 (2006).
[28] G. Basile, L. Delfini, S. Lepri, R. Livi, S. Olla and A. Politi, Eur. Phys. J. ST, 151 85 (2007).
[29] See the supplementary material.
The precise definition of the Levy walk model that we consider here is as follows. For a single particle each step of the walk consists in choosing a time of flight $t$ from the distribution $\phi(t)$ and then moving at speed $v$ over a distance $x = vt$ in either direction, with equal probability. At any given time, the particle could either have landed at a point $x$ or could be passing over that point. Accordingly let $Q(x,t) dx dt$ be the probability that a particle has precisely landed in the interval $(x, x + dx)$ during the time interval $(t, t + dt)$, and let $P(x,t) dx$ denote the probability that the particle is in the interval $(x, x + dx)$ at time $t$. Thus $P(x,t)$ includes events where the particle is crossing the interval $dx$.

We also define
\[
\psi(t) = \int_0^\infty d\tau \phi(\tau)
\]
(23)
as the probability of choosing a time of flight $\geq t$ and
\[
\chi(t) = \int_t^\infty d\tau \psi(\tau)
\]
(24)

**Levy diffusion of a single particle on the infinite line:** For a particle starting from the origin $x = 0$ at time $t = 0$, the probability $P(x,t)$ satisfies
\[
P(x,t) = \frac{1}{2} \psi(t) \delta(|x| - vt) + \frac{1}{2} \int_0^t d\tau \phi(\tau) \left[ P(x - v\tau,t - \tau) + P(x + v\tau,t - \tau) \right].
\]

Taking the Fourier Laplace transform $\tilde{P}(k,s) = \int_0^\infty dx \int_0^\infty dt \ P(x,t) e^{ikx-st}$ we get
\[
\tilde{P}(k,s) = \frac{\tilde{\phi}(s - ikv) + \tilde{\phi}(s + ikv)}{2 - \tilde{\phi}(s - ikv) - \tilde{\phi}(s + ikv)},
\]
(25)

where $\tilde{\phi}(s) = \int_0^\infty dt e^{-st}\phi(t)$ and $\tilde{\psi}(s) = \int_0^\infty dt e^{-st}\psi(t) = [1 - \tilde{\phi}(s)]/s$. Analysing the small $k$ and $s$ behavior of $\tilde{P}(k,s)$ allows one to obtain formulae (4), (6) and (7) of the main paper.

**Relaxation of density fluctuations:** From the evolution equation for $P(x,t)$ [Eq.(3) in main text], one can see that a density fluctuation of wave number $k$ relaxes exponentially with a time constant $t^*$ obtained from the solution of $\int_0^{\infty} d\tau \phi(\tau) \cos(kv\tau) \exp[\tau/t^*] = 1$. For $\phi(\tau)$ of the form given by Eq. (5) in main text one gets for small $k$
\[
1/t^* \simeq A\Gamma(-\beta)\cos(\pi(1 - \beta/2)) (kv)^{\beta}/\langle \tau \rangle
\]
(26)

whereas when $\phi(\tau)$ has a finite $\langle \tau^2 \rangle$, the regime is diffusive with $t^* \sim k^2$.

**Levy diffusion in a finite system connected to infinite reservoirs:** In this case we consider our system to be the finite segment between $(0, L)$ and this is connected on the two sides to reservoirs. The left reservoir consists of the region $x \leq 0$ while the right reservoir consists of the region $x \geq L$. We set $Q(x,t) = Q_l$ for points on the left reservoir and $Q(x,t) = Q_r$ for those on the right. In general if we know the distributions $Q(x,\tau)$ and $P(x,\tau)$ for all times $-\infty < \tau < t$ then the distribution at time $t$ is given by:
\[
Q(x,t) = \int_{-\infty}^{\infty} dy \frac{1}{2v} Q(y, t - |x - y|/v) \phi(|x - y|/v),
\]
(27)
\[
P(x,t) = \int_{-\infty}^{\infty} dy \frac{1}{2v} Q(y, t - |x - y|/v) \psi(|x - y|/v).
\]
(28)

In the above expressions $Q(x,t)$ gets contributions from walkers starting from all possible points $y$ and landing precisely at $x$ at time $t$. On the other hand $P(x,t)$ gets contributions from walkers starting at $y$ and being either at or passing $x$ at time $t$. Since the distribution $Q(x,t)$ is constrained to take either of the values $Q_l$ or $Q_r$ in the
where we used the definitions of $\psi$. Taking a time-derivative and using the continuity equation

$$
\frac{\partial Q}{\partial t} + \nabla \cdot J = 0,
$$

the solution of Eq. (29) is given by

$$
Q(x,t) = \frac{Q_l}{2} \int_{-\infty}^{0} dy \frac{1}{2v} \phi((x-y)/v) + \frac{Q_r}{2} \int_{L}^{\infty} dy \frac{1}{2v} \phi((y-x)/v)
$$

$$
+ \int_{0}^{L} dy \frac{1}{2v} Q(y, t - |x-y|/v) \phi(|x-y|/v),
$$

$$
P(x,t) = \frac{Q_l}{2} \int_{-\infty}^{0} dy \frac{1}{2v} \psi((x-y)/v) + \frac{Q_r}{2} \int_{L}^{\infty} dy \frac{1}{2v} \psi((y-x)/v)
$$

$$
+ \int_{0}^{L} dy \frac{1}{2v} Q(y, t - |x-y|/v) \psi(|x-y|/v),
$$

where $\psi(x/y)$ comes from the fact that these particles have a flight time longer than $y/v$. particles crossing the point $x$ corresponds to a right-to-left current. In the steady state, setting $Q(x,t) = Q(x)$ and $P(x,t) = P(x)$, hence we get:

$$
Q(x) - \int_{0}^{L} dy \frac{1}{2v} \phi(|x-y|/v) \frac{Q_l}{2} \psi(x/v) + \frac{Q_r}{2} \psi((L-x)/v),
$$

$$
P(x) = \int_{0}^{L} dy \frac{1}{2v} \psi(|x-y|/v) \frac{Q_l}{2} \chi(x/v) + \frac{Q_r}{2} \chi((L-x)/v).
$$

The solution of Eq. (29) is given by

$$
Q(x) = (Q_l - Q_r) H(x) + Q_r
$$

where $H(x)$ is the probability that a Levy walker starting at position $x$ will first hit the left reservoir before it hits the right reservoir, and satisfies

$$
H(x) - \int_{0}^{L} dy \frac{1}{2v} \phi(|x-y|/v) H(y) = \frac{1}{2} \psi(x/v).
$$

**Steady state current.** We re-write Eq. (28) in the form

$$
P(x,t) = \int_{-\infty}^{t} d\tau [ Q(x - vt + v\tau, \tau) + Q(x + vt - v\tau, \tau) ] \psi(t - \tau)/2.
$$

Taking a time-derivative and using the continuity equation $\partial P(x,t)/\partial t + \partial J(x,t)/\partial x = 0$ we then obtain the following form of the current operator

$$
J(x,t) = \frac{1}{2} \int_{-\infty}^{\infty} dy Q(x - y, t - |y|/v) \text{Sgn}(y) \psi(|y|/v).
$$

This equation is easy to understand physically. The contribution to the integral coming from $y > 0$ corresponds to particles crossing the point $x$ from left to right which started their flight at $x - y$ at time $t - y/v$ (the factor $\psi(y/v)$ comes from the fact that these particles have a flight time longer than $y/v$). Similarly the other part of the integral (from $y < 0$) corresponds to a right-to-left current.

In the steady state, setting $Q(x,t) = Q(x)$ we get the result

$$
J(x) = \frac{1}{2} \int_{-\infty}^{\infty} dy Q(x - y) \text{Sgn}(y) \psi(|y|/v).
$$

Using the values of $Q$ in the reservoirs and the steady state solution given by Eqs. (31) we evaluate the current at $x = 0$ and obtain

$$
J = \frac{(Q_l - Q_r)}{2} \left[ \int_{0}^{\infty} dy \psi(y/v) - \int_{0}^{L} dy H(y) \psi(y/v) \right].
$$

where we used the definitions of $\psi$ and $\chi$ from Eqs. (32) - (34). In the steady state we have $Q(x,t) = Q(x)$ and $P(x,t) = P(x)$, hence we get:

$$
Q(x) - \int_{0}^{L} dy \frac{1}{2v} \phi(|x-y|/v) \frac{Q_l}{2} \psi(x/v) + \frac{Q_r}{2} \psi((L-x)/v),
$$

$$
P(x) = \int_{0}^{L} dy \frac{1}{2v} \psi(|x-y|/v) \frac{Q_l}{2} \chi(x/v) + \frac{Q_r}{2} \chi((L-x)/v).
$$

The solution of Eq. (29) is given by

$$
Q(x) = (Q_l - Q_r) H(x) + Q_r
$$

where $H(x)$ is the probability that a Levy walker starting at position $x$ will first hit the left reservoir before it hits the right reservoir, and satisfies

$$
H(x) - \int_{0}^{L} dy \frac{1}{2v} \phi(|x-y|/v) H(y) = \frac{1}{2} \psi(x/v).
$$

**Steady state current.** We re-write Eq. (28) in the form

$$
P(x,t) = \int_{-\infty}^{t} d\tau [ Q(x - vt + v\tau, \tau) + Q(x + vt - v\tau, \tau) ] \psi(t - \tau)/2.
$$

Taking a time-derivative and using the continuity equation $\partial P(x,t)/\partial t + \partial J(x,t)/\partial x = 0$ we then obtain the following form of the current operator

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This equation is easy to understand physically. The contribution to the integral coming from $y > 0$ corresponds to particles crossing the point $x$ from left to right which started their flight at $x - y$ at time $t - y/v$ (the factor $\psi(y/v)$ comes from the fact that these particles have a flight time longer than $y/v$). Similarly the other part of the integral (from $y < 0$) corresponds to a right-to-left current.

In the steady state, setting $Q(x,t) = Q(x)$ we get the result

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Using the values of $Q$ in the reservoirs and the steady state solution given by Eqs. (31) - (32) we evaluate the current at $x = 0$ and obtain

$$
J = \frac{(Q_l - Q_r)}{2} \left[ \int_{0}^{\infty} dy \psi(y/v) - \int_{0}^{L} dy H(y) \psi(y/v) \right].
$$
Since the system is non-interacting, this result can be obtained directly by noting that the current is due to particles which enter from the left and leave to the right (and to the symmetric contribution). We then simply need to know the rate at which the non-interacting particles enter the system on the left side and leave the system into the right reservoir. This is given by

\[ p_l = \frac{Q_l}{2} \int_{-\infty}^{0} dy \int_{(L-y)/v}^{\infty} d\tau \phi(\tau) + \frac{Q_r}{2} \int_{-\infty}^{0} dy \int_{0}^{L} \frac{dx}{v} [1 - H(x)] \phi[(x-y)/v], \]  

with a similar expression for the right to left rate \( p_r \). The net current given by \( J = p_l - p_r \) is easily seen to be identical to Eq. (35).

**Additivity principle:** An interesting observation is that the generating function of the integrated current \( \mu(\lambda) \) [given by Eq. (19) in main text] matches exactly with the formula obtained from the additivity principle (AP) \[1\] which gives an expression for \( \mu_{AP}(\lambda) \) in terms of the the conductivity \( D \) and equilibrium current fluctuations \( \sigma \) defined respectively as

\[ D(Q) = \lim_{\Delta Q \to 0} L J/\Delta Q, \]
\[ \sigma(Q) = L \lim_{t \to \infty} \langle Q^2 \rangle/t. \]

The expression for \( \mu(\lambda) \) from AP is

\[ \mu_{AP}(\lambda) = -\frac{K}{L} \left[ \int_{Q_l}^{Q_r} dQ \frac{D(Q)}{\sqrt{1 + 2K\sigma(Q)}} \right]^2, \]

with \( \lambda = \int_{Q_l}^{Q_r} dQ \frac{D(Q)}{\sigma(Q)} \left[ \frac{1}{\sqrt{1 + 2K\sigma(Q)}} - 1 \right]. \]  

(38)

(this is a parametric expression: as \( K \) varies, \( \mu \) and \( \lambda \) vary). From our exact results for \( \mu(\lambda) \) we find \( D = Lp \) and \( \sigma = L \mu''(\lambda = 0) = 2DQ \). Using these in Eq. (38) and after explicitly performing the integrals we find \( \mu_{AP}(\lambda) = \mu(\lambda) \). This result is somewhat surprising since the additivity principle is expected normally to hold for diffusive systems (here \( D \) and \( \sigma \) have a \( L \)-dependence, whereas in usual diffusive systems they don’t).

\[1\] T. Bodineau and B. Derrida, Phys. Rev. Lett. 92, 180601 (2004); T. Bodineau and B. Derrida, C. R. Physique 8, 540 (2007).