Ab initio limits of atomic nuclei

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We predict the limits of existence of atomic nuclei, the proton and neutron drip lines, from the light through medium-mass regions. Starting from a chiral two- and three-nucleon interaction with good saturation properties, we use the valence-space in-medium similarity renormalization group to calculate ground-state and separation energies from helium to iron, nearly 700 isotopes in total. From a systematic comparison, we find that the deviation from experiment yields an approximately Gaussian distribution. We use this to provide theoretical uncertainties for our ab initio calculations towards the drip lines. Where the drip lines are known experimentally, our predictions are consistent within the estimated uncertainty. For the neutron-rich fluorine to vanadium isotopes, we provide predictions to be tested at rare-isotope beam facilities.

Atomic nuclei, which form the basis for known matter in the Universe, cannot be made from arbitrary numbers of protons and neutrons. For a given element (i.e., proton number Z) a nucleus can support only so many neutrons, N, and vice versa. The point at which neutrons no longer form a bound system is referred to as the drip line. Specifically, at the drip line one- and two-nucleon separation energies become negative, and nuclei decay via nucleon emission. The proton drip line is known experimentally to the medium-mass region, but to date, the neutron drip line is established only up to oxygen (Z = 8) [1]. Pinning down the neutron drip line to calcium and beyond is a flagship scientific motivation for next-generation rare-isotope beam facilities [2, 3]. Indeed several neutron-rich isotopes, including 46Ca, were recently discovered in this region [4]. Furthermore, knowledge of the neutron drip line is important for r-process simulations modeling the synthesis of heavy elements [5, 6] that occurs in neutron-star mergers [7].

Predicting the location of the drip lines poses a substantial theoretical challenge, particularly because many nuclei far from known data must be calculated systematically. In a pioneering study, the nuclear landscape was predicted from extrapolations of state-of-the-art nuclear density functional theory, and approximately 7000 nuclei were estimated to exist in nature [8]. Since this work, tremendous progress has been made in statistical analyses of nuclear models [9, 10] as well as in ab initio nuclear theory. Developments in chiral effective field theory [11–13] and similarity renormalization group [14, 15] are pushing nuclear forces to new levels of accuracy and ranges of applicability. Though a robust and systematic theoretical framework has not yet been fully achieved, particular nuclear Hamiltonians have been constructed which reproduce ground-state energies up to the tin region [16–18]. Three-nucleon (3N) forces play a key role for understanding the drip lines [19–24]. Moreover, many-body theories [25–30] have advanced to treat medium-mass open-shell systems [22, 31–34], with the primary limitation being computational resources needed to obtain convergence with respect to basis size, laying the groundwork for a new era of ab initio theory.

In this Letter we calculate properties of essentially all nuclei from helium to iron (Z = 2–26), close to 700, to provide a global ab initio survey of ground-state energies and predict the nuclear drip lines. Using two-nucleon (NN) and 3N interactions constrained by only few-body data, we solve the many-body problem with the valence-space formulation of the in-medium similarity renormalization group (VS-IMSRG) [28, 29, 31, 34–36]. Our results yield an overall root-mean-square (rms) deviation of 3.3 MeV from absolute experimental energies and 1.0 MeV from separation energies. In comparison, state-of-the-art energy-density functionals obtain rms deviations in the range 0.6–2.0 MeV for energies and 0.4–1.25 MeV for separation energies [37–40].

While the drip line signature is unambiguous experimentally, from a theoretical perspective an error of a few keV – well beyond current levels of precision – can make the difference between an isotope being bound or unbound. Therefore, an assessment of theoretical uncertainty is mandatory for any meaningful drip line prediction. Ab initio methods present an appealing framework for uncertainty quantification. One begins with the most general Lagrangian compatible with the applicable symmetries, organized by a systematically improvable power counting, then solves the nuclear many-body problem within a controlled and systematically improvable approximation scheme, propagating all uncertainties. Such a prescription has not yet been achieved in practice, so for the present we use a comparison with known data to estimate the reliability of our calculations. A recent analysis of global mass models from Bayesian statistical machine learning algorithms provides one such descrip-
FIG. 1. Calculated probabilities for given isotopes to be bound with respect to one- or two-neutron/proton removal. The gray region indicates nuclei that have been calculated, while the height of the blue boxes corresponds to the estimated probability that a given nucleus is bound with respect to one- or two-neutron (proton) removal in the neutron-rich (deficient) region of the chart. The inset shows the global agreement with experimental ground-state energies.

While the statistical methods [40] are sophisticated (and beyond those used in this work), the results were not confronted with known drip lines for the lower nuclear chart and may be biased towards the existing data to which the models were optimized. The main advantages of our current approach are (i) the predictions should not be biased towards measured data because they were not fit to any data beyond helium and (ii) the predictions can be benchmarked where the proton and neutron drip lines are known experimentally (mass models are typically applied to $Z \gtrsim 8$).

In the VS-IMSRG, a valence-space Hamiltonian of tractable dimension is decoupled from the larger Hilbert space via an approximate unitary transformation. We begin in a harmonic-oscillator basis of 15 major shells (i.e., $\epsilon = 2n + l \leq \epsilon_{\text{max}} = 14$) with an imposed cut of $\epsilon_1 + \epsilon_2 + \epsilon_3 \leq \epsilon_{\text{Max}} = 16$ for 3N matrix elements. The resulting ground-state energies are converged to better than a few hundred keV with respect to these truncations. Transforming to the Hartree-Fock basis, we capture effects of 3N interactions between valence nucleons via the ensemble normal-ordering of Ref. [34]. We then use the Magnus formulation of the IMSRG [28, 41], truncating all operators at the normal-ordered two-body level (the IMSRG(2) approximation), to generate approximate unitary transformations that decouple the core energy and valence-space Hamiltonian for each nucleus to be calculated.

We employ a so-called $0h\omega$ valence space, where valence nucleons occupy the appropriate single major harmonic-oscillator shell (e.g., for $8 < N(Z) < 20$ the $sd$ shell, $20 < N(Z) < 40$ the $pf$ shell, etc.). At $N(Z) = 2, 8, 20, \ldots$, we do not decouple a neutron (proton) valence space, and no explicit neutron (proton) excitations are allowed in the calculation. This shifts the treatment of correlations fully onto the IMSRG transformation, and the IMSRG(2) truncation may be insufficient (we discuss implications below). Finally the resulting valence-space Hamiltonians are diagonalized with the NuShellX@MSU shell-model code [42] (with the exception of $^{69}$Ca, which was computed with the m-scheme code Antoine [43, 44]).

We thus calculate ground (and excited) states of all nuclei from helium to iron, except those for which the shell-model diagonalization is beyond our computational limits. For the input NN+3N interaction, we use the $1.8/2.0(EM)$ potential of Refs. [16, 45], where the 3N
coupings were fit to the $^4$H binding energy and the $^4$He charge radius. This interaction reproduces experimental ground-state energies of light- to medium-mass nuclei remarkably well [18, 46]. Studies of nuclear matter [16, 47, 48] have shown that this interaction saturates at somewhat too high density, leading to too-small radii for finite nuclei [46], and with slightly too much binding. It appears that the extra Coulomb repulsion from the small radii fortuitously counteracts the excess nuclear attraction, leading to remarkable agreement with data. In the Supplemental Material, we provide results for absolute and separation energies for all nuclei calculated. In the inset of Fig. 1, we plot the range of agreement with experiment and find an overall rms deviation of 3.3 MeV. The experimental binding and separation energies are taken from the Atomic Mass Evaluation [49], with additional recent data from Refs. [50–53]. The analogous plot as a function of $Z$ with additional recent data from Refs. [50–53]. The data displayed in the Supplemental Material (Fig. 6), displays a clear trend with $\delta E_{\text{g.s.}}$ decreasing with increasing $Z$. While this might indicate a systematic deficiency in the interaction, it could also be a result of incomplete convergence.

As no experimental input beyond $^4$He is used in the current calculations, our results should not be biased toward known data. Therefore our approach is to use measured data to assess how well separation energies are reproduced in general, then assume they will perform equally well for separation energies which have not yet been measured. Certainly, this neglects effects from new degrees of freedom which may arise in the neutron-rich region, such as halo structures or island-of-inversion physics [54, 55], but our results suggest that these effects tend to lie within our estimated uncertainties.

To characterize the quality of the reproduction of experimental data, we globally assess the residual, $\delta X = X^{(\text{th})} - X^{(\text{exp})}$, for some observable $X$, of one- and two-neutron (proton) separation energies $S_n, S_{2n}$ ($S_p, S_{2p}$). In all cases, away from shell closures, where the valence space is not changed between the nuclei involved in the separation energies, we find an approximately Gaussian distribution for the residual, centered near zero with a rms deviation on the order of 1 MeV. A more careful inspection reveals a systematic trend in the residuals as a function of the corresponding separation energies. To illustrate this, the residual $\delta S_{2n}$ is shown as a function of the theoretical two-neutron separation energy $S_{2n}^{\text{th}}$ in the upper right panel of Fig. 2 (similar trends for $S_n, S_p$, and $S_{2p}$ may be found in the Supplemental Material).

This behavior may be understood as an effect of infrared convergence in the many-body calculation. As mentioned above, our calculations employ a truncated harmonic-oscillator basis. Even if the ground-state energy of a system is not fully converged within our truncation, the bulk of this discrepancy will tend to cancel out in the difference when separation energies are computed. However, near a decay threshold (where the corresponding separation energy approaches zero), the nuclear wave function develops an extended tail which is not captured efficiently in an oscillator basis. As we approach a threshold, the cancellation of infrared errors becomes less complete; the binding of the more weakly bound nucleus is underpredicted, and the corresponding separation energy is shifted to a more negative value. We correct for this artifact of the many-body solution with a linear fit of the residuals, illustrated in Fig. 2.

A second many-body artifact arises for separation energies where the two nuclides entering in the difference are treated in different valence spaces. For example, to compute the $S_{2n}$ of $^{38}$Cl, we require the ground-state energy of $^{38}$Cl ($N=21$), which has a valence neutron in the $pf$ shell, and $^{36}$Cl ($N=19$), which has a valence hole in the $sd$ shell. Because we must decouple different valence spaces using the VS-IMSRG, the errors due to truncating to two-body operators may differ significantly in the two systems and not cancel in the difference. In contrast, when both nuclides are treated in the same valence space, we expect the errors due to truncation to be more similar and thus to cancel more in the difference. To account for this effect, we make a distinction between separation energies calculated within the same valence space, and those calculated with different valence spaces.
Our approach to predicting the drip lines is the following. The result of the VS-IMSRG calculation, corrected with the linear shift to account for the effects of infrared convergence, forms our prediction for the separation energies. A separate linear shift is obtained for cases with and without a valence-space change (indicated with green and brown, respectively, in Fig. 2). The resulting corrected residuals in Fig. 2 have approximately Gaussian distributions. Our central assumption is that the residuals for separation energies which have not been measured will follow the same distribution.

The probability \( P_{1n} \) that an isotope is bound with respect to one-neutron emission is given by the fraction of the Gaussian probability distribution for \( S_n \) that is greater than zero:

\[
P_{1n} = \frac{1}{\sqrt{2\pi}\sigma_{1n}} \int_0^\infty \exp \left[ -\frac{(x - S_n^{\text{th,corr}})^2}{2\sigma_{1n}^2} \right] dx ,
\]

where \( S_n^{\text{th,corr}} \) is the theory separation energy, with the linear correction applied, and \( \sigma_{1n} \) is the corresponding standard deviation indicated in Fig. 2.

We make an analogous analysis for \( S_{2n} \), \( S_p \), and \( S_{2p} \). Not surprisingly, the residuals in \( S_n \) and \( S_{2n} \) are correlated (correlation of \( \approx 0.7 \)), as are the residuals in \( S_p \) and \( S_{2p} \) (see Fig. 2), so the probability that the isotope is bound with respect to one- and two-nucleon emission is given by

\[
P_{\text{bound}} = (P_{1n}P_{2n} + \xi_{1n,2n}) (P_{1p}P_{2p} + \xi_{1p,2p}) ,
\]

where \( \xi_{1n,2n} \) and \( \xi_{1p,2p} \) correct for the correlation (see Supplementary Material). As an illustration, the calculated separation energies, with the 1σ uncertainty band, are shown in Fig. 3, for chlorine isotopes. Analogous figures for all isotopic chains studied are included in the Supplementary Material. In addition, we provide a complete data table as a Supplmentary File.

We translate this large-scale analysis into the main result of this Letter in Fig. 1. For each calculated nuclide from helium to iron, we assign a probability that it is bound with respect to one- or two-nucleon emission. Every nuclide calculated is represented by a box in the plot, where its height and color denotes this probability: a full box is bound with probability 1, and an empty gray box is bound with probability 0. For ease of interpretation, we denote cases which are likely bound (> 66%) in blue, a marginal region (between 33% and 66%) in red, and those likely to be unbound (< 33%) in white. We denote experimentally known drip lines with a closed symbol and the heaviest (lightest) observed isotopes in the neutron-rich (deficient) regions with an open symbol.

Qualitatively, the location of the known drip lines appear to be reproduced well in Fig. 1, both on the proton-rich side, and on the neutron-rich side (where it is known up to \( Z = 8 \)). A more quantitative assessment of the predictions can be obtained by evaluating the Brier score [56], and decomposing the score into reliability, resolution and uncertainty terms [57]. This is done in the Supplementary Material, and indicates that the predictions are indeed "reliable" and "useful". Even the well-known halo systems \(^{11}\text{Li}\) and \(^{22}\text{C}\) are predicted to be either bound or marginal, implying that physics of threshold systems lie within our estimated error bands. Only in closed-shell systems do we see significant deviations from experiment, where we often predict nuclei beyond the drip line to likely be bound (see, e.g., nitrogen and oxygen). This is due to artifacts which arise when changing valence spaces, such as incomplete can-
cellation of many-body errors and potentially using incorrect ground states, contaminating our error estimation. Using instead a cross-shell valence space would likely correct these deficiencies, and such Hamiltonians are currently being developed in the VS-IMSRG.

We now turn to our predictions for the experimentally unknown neutron drip line above oxygen. Beginning in fluorine, which binds at least six more neutrons than oxygen, we predict the current known limit $^{31}$F to be a marginal case. Similarly, in neon, we predict the existence of $^{34}$Ne, but in both chains there is a very small probability that heavier isotopes exist. For all remaining chains through chromium, our calculations indicate the likely existence of at least one isotope beyond the current known limits. In calcium, earlier ab initio calculations have generally agreed that $^{60}$Ca is the heaviest bound isotope [58–60], with a very flat trend in binding energies beyond. Here we find a situation potentially similar to oxygen, where the final bound nucleus could be anomalously close to stability, but there is a reasonable probability that the drip line extends to $^{70}$Ca, as predicted in the statistical analysis of Ref. [9]. We also note the remarkable similarities of the latter results to our ab initio predictions, which thus provides a consistent picture of the neutron drip line up to calcium from independent theoretical approaches.

In summary, we have calculated ground-state energies of essentially all nuclei from helium to iron in the ab initio VS-IMSRG starting from NN and 3N interactions fit to few-body systems only. After benchmarking against known experimental limits, we provide drip line predictions in the neutron-rich region above oxygen to guide ongoing and future efforts at rare-isotope beam facilities worldwide. This work also advances ab initio theory to global calculations, highlighting the rapidly increasing scope of the field, and the potential to provide predictions beyond where data exists with uncertainty estimates. While significant challenges remain in improving the rigor of theoretical error estimates from nuclear forces and many-body methods, the approach presented here indicates a path – once current computational limitations can be overcome – for ab initio input for nucleosynthesis calculations probing the r-process region of extreme neutron-rich nuclei.

We would like to thank K. Fossez, H. Hergert, W. Nazarewicz, and J.K. Smith for enlightening discussions. TRIUMF receives funding via a contribution from Advanced Research Computing (ARC).
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SUPPLEMENTARY MATERIAL

Correction for correlated residuals. As discussed, the residuals $\delta S_n$ and $\delta S_{2n}$ are correlated, and this should be accounted for when estimating the probability that a given isotope is bound with respect to both $1n$ and $2n$ emission. Considering a two-dimensional distribution

$$P(u, v; \rho) = \frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left[ -\frac{u^2 + v^2 - 2uv\rho}{2(1 - \rho^2)} \right] , \quad (3)$$

where $\rho = \langle uv \rangle$ is the correlation of variables $u$ and $v$, we are interested in an integral of the form

$$P(u_0, v_0; \rho) = \int_{u_0}^{\infty} du \int_{0}^{v_0} dv P(u, v; \rho) . \quad (4)$$

This may be evaluated as $P(u_0, v_0; \rho) = P(u_0, v_0; 0) + \xi(u_0, v_0; \rho)$, where the correction for the correlation is

$$\xi(u_0, v_0; \rho) = \frac{1}{\sqrt{8\pi}} \int_{-\infty}^{\infty} du e^{-\frac{u^2}{2}} \left[ \text{Erf} \left( \frac{v_0}{\sqrt{2}} \right) - \text{Erf} \left( \frac{v_0 - \rho u}{\sqrt{2(1 - \rho^2)}} \right) \right] . \quad (5)$$

The uncorrelated probability $P(u_0, v_0; 0)$ is given by the product of two one-parameter Gaussians. We evaluate the integral in Eq. (5) numerically, with

$$u_0 = -(S_{1n}^{(th)} - \mu_{1n})/\sigma_{1n}, \quad v_0 = -(S_{2n}^{(th)} - \mu_{2n})/\sigma_{2n} . \quad (6)$$

The same procedure is used to account for the correlation in $\delta S_p$ and $\delta S_{2p}$. The size of the correction $\xi$ is typically small, but its impact is greatest when $S_{1n}^{(th)} \approx 0$, $S_{2n}^{(th)} \approx 0$, which is the region of interest for the drip lines. The maximum possible correction occurs when $P_{1n} = 0.5$, $P_{2n} = 0.5$, and $\rho = 1$, i.e., the two separation energies are perfectly correlated. In that case, $\xi = 0.25$. However, in the cases encountered in this work, $\xi$ only reaches 0.1 in a few cases and is typically below 0.01.

Residuals and correction for linear dependence. The residuals $\delta S_n$, $\delta S_{2n}$, $\delta S_p$, and $\delta S_{2p}$ are presented in Fig. 4, as a function of the theoretical separation energy. As discussed in the main text, all four residuals display a linear dependence on the separation energy. We remove this dependence by applying a linear correction to the predicted separation energy. This linear shift is an example of what is called the “discrepancy bias” in Ref. [61]. The parameters are given in Table I.

Metric of forecasting quality. A popular way to evaluate the quality of a forecasting model of a binary outcome (e.g., bound or unbound) is the Brier score (BS) [56]:

$$BS = \frac{1}{N} \sum_{i} (p_i - o_i)^2 , \quad (7)$$

where $N$ is the number of samples (nuclides in the present case), $p_i$ is the predicted probability and $o_i$ is the measured outcome (either 1 or 0). The Brier score lies between 0 and 1, with 0 being the best possible forecast. (Note that this definition differs from the original definition [56] by a factor of two.) The Brier score is frequently decomposed into three pieces: reliability (REL), resolution (RES), and uncertainty (UNC) [57]:

$$BS = \frac{1}{N} \sum_{k}^{K} n_k (\bar{p}_k - \bar{o}_k)^2 - \frac{1}{N} \sum_{k}^{K} n_k (\bar{o}_k - \bar{o})^2 + o(1 - \bar{o}) , \quad (8)$$

where one divides the data set into $K$ bins, bin $k$ has $n_k$ entries, the mean predicted probability of bin $k$ is $\bar{p}_k$, and the mean observed outcome in bin $k$ is $\bar{o}_k$. The overall mean outcome is $\bar{o}$. The reliability REL lies between 0 and 1 and characterizes how well the prediction in each bin characterizes the outcomes in that bin, with 0 being the ideal case. The reliability can be adjusted with corrections to the model such as the linear correction to the residuals applied in this work. A common way to display the reliability is the plot shown in Fig. 5, which

![FIG. 4. Residuals (theory minus experiment) as a function of the theoretical separation energies. Separation energies in which the two isotopes entering differ in the employed valence spaces are shown in green and labeled “VS change”.

TABLE I. Slope and offset values of the linear correction made to the theoretical separation energies. Numbers in parentheses indicate uncertainties from the linear fit.

|            | no VS change | offset | VS change | offset |
|------------|--------------|--------|-----------|--------|
| $S_n$      | 0.079(5)     | -0.66(5) | 0.11(3)   | -2.8(3) |
| $S_{2n}$   | 0.083(4)     | -1.34(7) | 0.10(2)   | -3.6(3) |
| $S_p$      | 0.078(5)     | -0.20(7) | 0.07(2)   | -2.0(3) |
| $S_{2p}$   | 0.078(4)     | -0.35(12) | 0.07(1)   | -2.2(2) |
The theoretical probability to be bound is shown on the x-axis, while the fraction bound experimentally is shown on the y-axis. A perfect prediction would lie along the dashed line.

Plots $\bar{o}_k$ vs. $\bar{p}_k$, i.e., the fraction bound experimentally vs. the theoretical probability to be bound for each bin. We choose non-uniform binning (indicated by the solid vertical lines), because the majority of cases have theoretical probability to be bound below 0.01 or above 0.99, and these are not the cases of interest. If the resulting curve lies above the diagonal ($\bar{o}_k = \bar{p}_k$), then our predictions are overly “pessimistic” in that we predict a too low probability to be bound. Likewise, a curve below the diagonal indicates an overly “optimistic” prediction. Figure 5 demonstrates that, within the statistical uncertainty due to the finite number of nuclides in each bin, our predictions are reliable. The vertical error bars represent a 68% confidence interval estimated using a Wilson score with continuity correction (method 4 from Ref. [62]). The horizontal error bars are obtained by dividing the standard deviation of predicted probabilities in a given bin by $\sqrt{n_k}$ to approximate the standard error of the mean $\bar{p}_k$ for that bin.

The resolution RES characterizes the extent to which the separation into different bins captures differences in outcomes. It lies between 0 and 1, with 1 being the ideal case. The uncertainty UNC reflects the variability of measured outcomes in the data set, and so characterizes how difficult the forecast is. The uncertainty lies between 0 and 0.25, with larger values corresponding to higher variability.

A rule of thumb for identifying a useful forecast is that the Brier score should be less than the uncertainty UNC, otherwise one would do better by simply taking the global average $\bar{o}$ as the prediction. This corresponds to the requirement $REL < RES$. The values obtained in the current study are listed in Table II. As mentioned above, the vast majority of cases have predicted probability to be bound below 0.01 or above 0.99. These are in a sense the “easy” forecasts and are of less interest for the drip line. We therefore repeat the analysis, restricted to cases with $0.01 \leq P_{\text{bound}} \leq 0.99$, and we obtain values which indicate that even in the region of interest the model is reliable and useful. The uncertainties presented in Table II are obtained from a simple linearized propagation of uncertainties. A more robust uncertainty estimation for the Brier score is possible [63], but not pursued here.

**Separation energy plots.** We provide here plots of the separation energies, equivalent to Fig. 3 in the main text, for all the elements studied here ($2 \leq Z \leq 26$). In addition, we plot in Fig. 6 the range of agreement with the experimental ground-state energies as a function of $Z$ to complement the information given in the inset to Fig. 1 as a function of $N$.

![FIG. 5. For a given predicted probability to be bound, we plot the corresponding fraction of isotopes that are indeed bound experimentally. A perfect prediction would lie along the dashed line.](image)

**FIG. 6.** Ground-state energy residuals as a function of $Z$. The different lines connect isotones of constant $N$. The mean and standard deviation of the discrepancy are indicated by $\mu$ and $\sigma$, respectively, while the root-mean-square discrepancy is given by rms.

![TABLE II. Brier score and components, both for the full range of probabilities and with the extreme values excluded.](image)
FIG. 7. Separation energies and probabilities to be bound for the He isotopes.

FIG. 8. Separation energies and probabilities to be bound for the Li isotopes.
FIG. 9. Separation energies and probabilities to be bound for the Be isotopes.

FIG. 10. Separation energies and probabilities to be bound for the B isotopes.
FIG. 11. Separation energies and probabilities to be bound for the C isotopes.

FIG. 12. Separation energies and probabilities to be bound for the N isotopes.
FIG. 13. Separation energies and probabilities to be bound for the O isotopes.

FIG. 14. Separation energies and probabilities to be bound for the F isotopes.
FIG. 15. Separation energies and probabilities to be bound for the Ne isotopes.

FIG. 16. Separation energies and probabilities to be bound for the Na isotopes.
FIG. 17. Separation energies and probabilities to be bound for the Mg isotopes.

FIG. 18. Separation energies and probabilities to be bound for the Al isotopes.
FIG. 19. Separation energies and probabilities to be bound for the Si isotopes.

FIG. 20. Separation energies and probabilities to be bound for the P isotopes.
FIG. 21. Separation energies and probabilities to be bound for the S isotopes.

FIG. 22. Separation energies and probabilities to be bound for the Cl isotopes.
FIG. 23. Separation energies and probabilities to be bound for the Ar isotopes.

FIG. 24. Separation energies and probabilities to be bound for the K isotopes.
FIG. 25. Separation energies and probabilities to be bound for the Ca isotopes.

FIG. 26. Separation energies and probabilities to be bound for the Sc isotopes.
FIG. 27. Separation energies and probabilities to be bound for the Ti isotopes.

FIG. 28. Separation energies and probabilities to be bound for the V isotopes.
FIG. 29. Separation energies and probabilities to be bound for the Cr isotopes.

FIG. 30. Separation energies and probabilities to be bound for the Mn isotopes.
FIG. 31. Separation energies and probabilities to be bound for the Fe isotopes.