INVERSE SPECTRAL PROBLEMS FOR NON-SELF-ADJOINT
STURM-LIOUVILLE OPERATORS WITH DISCONTINUOUS
BOUNDARY CONDITIONS

JUN YAN AND GUOLIANG SHI

Abstract. This paper deals with the inverse spectral problem for a non-
self-adjoint Sturm-Liouville operator with discontinuous conditions inside the
interval. We obtain that if the potential \( q \) is known a priori on a subinterval
\([b, \pi]\) with \( b \in (d, \pi) \) or \( b = d \), then \( h, \beta, \gamma \) and \( q \) on \([0, \pi]\) can be uniquely
determined by partial spectral data consisting of a sequence of eigenvalues
and a subsequence of the corresponding generalized normalizing constants or
a subsequence of the pairs of eigenvalues and the corresponding generalized
ratios. For the case \( b \in (0, d) \), a similar statement holds if \( \beta, \gamma \) are also known
a priori. Moreover, if \( q \) satisfies a local smoothness condition, we provide an
alternative approach instead of using the high-energy asymptotic expansion of
the Weyl \( m \)-function to solve the problem of missing eigenvalues and norming
constants.

1. Introduction

In this paper, we consider the non-self-adjoint Sturm-Liouville operator
\( L := L (q, h, H, \beta, \gamma, d) \) defined by
\[
\ell y := -y'' + q (x) y
\]
on the interval \((0, \pi)\) with the boundary conditions
\[
U (y) := y' (0) - h y (0) = 0, V (y) := y' (\pi) + H y (\pi) = 0
\]
and the discontinuous conditions
\[
y (d + 0) = \beta y (d - 0), \quad y' (d + 0) = \beta^{-1} y' (d - 0) + \gamma y (d - 0),
\]
where \( q \in L^1_{c} [0, \pi] \) is complex-valued, \( h, H \in \mathbb{C} \cup \{ \infty \}, \gamma \in \mathbb{C} \) and \( \beta \in \mathbb{R}, \beta > 0 \).
Note that, in an obvious notation, \( h = \infty \) and \( H = \infty \) single out the Dirichlet
boundary conditions
\[
U^\infty (y) := y (0) = 0 \text{ and } V^\infty (y) := y (\pi) = 0,
\]
respectively. One notes that in the special case \( \beta = 1, \gamma = 0 \), the operator \( L \) reduces
to the classical Sturm-Liouville operator without discontinuities.

Sturm-Liouville operators with discontinuities inside the interval arise in math-
ematics, mechanics, geophysics, and other fields of science and technology. The
inverse spectral problems of such operators is of central importance in disciplines
ranging from engineering to the geosciences. For example, discontinuous inverse
problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [1, 2]. In the last decades, inverse spectral problems for Sturm-Liouville operators with different type discontinuities have attracted tremendous interest [3–18]. These start with the fundamental work given by V. Ambarzumian [19] and then by G. Borg [20], B. Levitan [21, 22], and V. Marchenko [23, 24] for the classical Sturm-Liouville operators.

We emphasize that in 1984, O. H. Hald [5] first generalized Hochstadt–Lieberman’s theorem [25] to the Sturm–Liouville operator \( L \), that is, if \( H \) is given, \( q \) is known on \([\frac{\pi}{2}, \pi]\) and \( d \in (0, \frac{\pi}{2}) \), then one spectra can uniquely determine \( h, \beta, \gamma, d \) and \( q \) on \([0, \pi]\). Motivated by this work, increasing attention has been given to the inverse spectral problem of recovering the operator \( L \) in the self-adjoint case with partial information given on the potential [10, 13, 14]. In contrast, such inverse spectral problem for the non-self-adjoint case has in general been studied considerably less, and it is precisely the starting point of this paper. We investigate the uniqueness problem of determining the non-self-adjoint operator \( L \) with only partial information of \( q \), of the eigenvalues, and of the generalized norming constants. What should be noted is that in the non-self-adjoint setting, complex eigenvalues and multiple eigenvalues may appear, and thus many new ideas and additional effort are required. Before describing the content of this paper, let us first give some notations and basic facts.

To avoid too many case distinctions in the proofs of this paper, we assume that \( h \in \mathbb{C} \). Nevertheless, we expect that the method of the paper can be applied in the case \( h = \infty \). For simplicity we use the notations \( B \) and \( B^\infty \) for the boundary value problems corresponding to \( L \) with \( H \in \mathbb{C} \) and \( H = \infty \), respectively. Assume that \( \varphi (x, \lambda), \psi (x, \lambda), \psi^\infty (x, \lambda) \) are solutions of the equation

\[
\ell y = -y'' + q (x) y = \lambda y
\]

satisfying the discontinuous conditions (1.3) and the initial conditions

\[
\begin{align*}
\varphi (0, \lambda) &= 1, \quad \frac{d \varphi (x, \lambda)}{dx} \bigg|_{x=0} = h \in \mathbb{C}, \\
\psi (\pi, \lambda) &= 1, \quad \frac{d \psi (x, \lambda)}{dx} \bigg|_{x=\pi} = -H \in \mathbb{C}, \\
\psi^\infty (\pi, \lambda) &= 0, \quad \frac{d \psi^\infty (x, \lambda)}{dx} \bigg|_{x=\pi} = 1,
\end{align*}
\]

respectively. Then it is easy to see that eigenvalues of \( B \) and \( B^\infty \) are precisely the zeros of

\[
\Delta (\lambda) := \langle \psi (x, \lambda), \varphi (x, \lambda) \rangle = V (\varphi) = -U (\psi)
\]

and

\[
\Delta^\infty (\lambda) := \langle \psi^\infty (x, \lambda), \varphi (x, \lambda) \rangle = -V^\infty (\varphi) = -U (\psi^\infty),
\]

respectively, where \( \langle y(x), z(x) \rangle := y(x)z'(x) - y'(x)z(x) \). Thus \( \Delta (\lambda) \) and \( \Delta^\infty (\lambda) \) are called the characteristic functions of \( B \) and \( B^\infty \), respectively. Throughout this paper, the algebraic multiplicity of an eigenvalue is the order of it as a zero of the corresponding characteristic function.

**Notation 1.** (1) We denote by \( \sigma (B) := \{ \lambda_n \}_{n \in \mathbb{N}_0} \) the sequence of all the eigenvalues of \( B \) and denote by \( \sigma (B^\infty) := \{ \lambda_n^\infty \}_{n \in \mathbb{N}_0} \) the sequence of all the eigenvalues of
\( B^\infty \). The eigenvalues are assumed to be repeated according to their algebraic multiplicities and labeled in order of increasing moduli. In addition, identical eigenvalues are adjacent.

(2) Denote
\[
S_B := \{ n \in \mathbb{N} | \lambda_{n-1} \neq \lambda_n \} \cup \{ 0 \}, S_{B^\infty} := \{ n \in \mathbb{N} | \lambda_{n-1}^\infty \neq \lambda_n^\infty \} \cup \{ 0 \}.
\]

(3) The symbol \( m_n \) denotes the algebraic multiplicity of the eigenvalue \( \lambda_n, n \in S_B \), and \( m_n^\infty \) denotes the algebraic multiplicity of \( \lambda_n^\infty, n \in S_{B^\infty} \). For sufficiently large \( n \) it is well known that \( m_n^\infty = m_n = 1 \) (see Lemma 2.3 in [17]).

Now we turn to give the definition of the generalized norming constants for the problem \( B \). Denote
\[
\kappa_{n+\nu} := \varphi_{n+\nu}(\pi), \quad \alpha_{n+\nu} := \int_0^\pi \psi_{n+\nu}(x) \psi_{n+m_n-1}(x) \, dx,
\]
where \( n \in S_B, \nu = 0, 1, \ldots, m_n - 1 \), and
\[
\varphi_{n+\nu}(x) := \varphi_{\nu}(x, \lambda_n) := \left. \frac{1}{\nu!} \frac{d^\nu}{d\lambda^\nu} \phi(x, \lambda) \right|_{\lambda=\lambda_n},
\]
\[
\psi_{n+\nu}(x) := \psi_{\nu}(x, \lambda_n) := \left. \frac{1}{\nu!} \frac{d^\nu}{d\lambda^\nu} \psi(x, \lambda) \right|_{\lambda=\lambda_n}.
\]
Then \( \kappa_n \) and \( \alpha_n, n \in \mathbb{N}_0 \), are called the generalized norming constants corresponding to \( \lambda_n \). To distinguish \( \kappa_n \) and \( \alpha_n \), in this paper, \( \kappa_n \) is called the generalized ratio, and \( \alpha_n \) is called the generalized normalizing constant. Moreover, it follows from [17, Theorem 4.1] that for \( n \in S_B, \nu = 0, 1, \ldots, m_n - 1 \),
\[
\frac{d^{m_n+\nu} \Delta(\lambda)}{d\lambda^{m_n+\nu}} \bigg|_{\lambda=\lambda_n} = -(m_n + \nu)! \sum_{j=0}^\nu \kappa_{n+j} \alpha_{n+\nu-j}.
\]
Note that when the multiplicity \( m_n = 1 \), the generalized norming constants \( \kappa_n \) and \( \alpha_n \) coincide with the norming constants for the operator \( L \) in the self-adjoint case (see [15]).

Actually, \( \varphi_{\nu}(x, \lambda_n) \) and \( \psi_{\nu}(x, \lambda_n) \) are the generalized eigenfunctions of \( B \) corresponding to the eigenvalue \( \lambda_n, n \in S_B \). In fact, for \( \nu = 1, 2, \ldots, m_n - 1 \), we notice that
\[
\ell \varphi_{\nu}(x, \lambda_n) = \lambda_n \varphi_{\nu}(x, \lambda_n) + \varphi_{\nu-1}(x, \lambda_n),
\]
\[
\varphi_{\nu}(d + 0, \lambda_n) = \beta \varphi_{\nu}(d - 0, \lambda_n),
\]
\[
\varphi'_{\nu}(d + 0, \lambda_n) = \beta^{-1} \varphi'_{\nu}(d - 0, \lambda_n) + \gamma \varphi_{\nu}(d - 0, \lambda_n),
\]
\[
\varphi_{\nu}(0, \lambda_n) = \varphi'_{\nu}(0, \lambda_n) = 0,
\]
\[
\ell \psi_{\nu}(x, \lambda_n) = \lambda_n \psi_{\nu}(x, \lambda_n) + \psi_{\nu-1}(x, \lambda_n),
\]
\[
\psi_{\nu}(d + 0, \lambda_n) = \beta \psi_{\nu}(d - 0, \lambda_n),
\]
\[
\psi'_{\nu}(d + 0, \lambda_n) = \beta^{-1} \psi'_{\nu}(d - 0, \lambda_n) + \gamma \psi_{\nu}(d - 0, \lambda_n),
\]
\[
\psi_{\nu}(\pi, \lambda_n) = \psi'_{\nu}(\pi, \lambda_n) = 0.
\]
and
\[
\frac{1}{\nu!} \Delta^{(\nu)}(\lambda_n) = \varphi'_{\nu}(\pi, \lambda_n) + H \varphi_{\nu}(\pi, \lambda_n) = 0,
\]
\[
\frac{1}{\nu!} \Delta^{(\nu)}(\lambda_n) = -\psi'_{\nu}(0, \lambda_n) + h \psi_{\nu}(0, \lambda_n) = 0.
\]
Remark 1. Now we define the generalized norming constants for the problem $B^\infty$,

\begin{align}
\kappa_{n,\nu}^\infty &:= \frac{d\varphi_{n,\nu}^\infty(x)}{dx}\bigg|_{x=\pi}, \quad \alpha_{n,\nu}^\infty := \int_0^\pi \psi_{n,\nu}^\infty(x) \psi_{n+m_n-1}^\infty(x) \, dx,
\end{align}

where $n \in S_{B^\infty}$, $\nu = 0, 1, \ldots, m_n^\infty - 1$, and

\begin{align}
\varphi_{n,\nu}^\infty(x) &:= \varphi_{\nu}(x, \lambda_n^\infty) := \frac{1}{\nu!} \frac{d^\nu}{d\lambda^\nu} \varphi(x, \lambda) \bigg|_{\lambda=\lambda_n^\infty}, \\
\psi_{n,\nu}^\infty(x) &:= \psi_{\nu}(x, \lambda_n^\infty) := \frac{1}{\nu!} \frac{d^\nu}{d\lambda^\nu} \psi(x, \lambda) \bigg|_{\lambda=\lambda_n^\infty}.
\end{align}

Then one can also deduce that for $n \in S_{B^\infty}$, $\nu = 0, 1, \ldots, m_n^\infty - 1$,

\begin{align}
\frac{d^{m_n^\infty + \nu}(\Delta^\infty)}{d\lambda^{m_n^\infty + \nu}}(\lambda) \bigg|_{\lambda=\lambda_n^\infty} = -(m_n^\infty + \nu)! \sum_{j=0}^{\nu} \kappa_{n+j,\alpha_{n+\nu-j}}^\infty.
\end{align}

In [17], Y. Liu, G. Shi and J. Yan studied the uniqueness spectral problem of recovering the non-self-adjoint operator $L$ from one of the following spectral characteristics: (1) $\Gamma_1 := \{\lambda_n, \alpha_n\}_{n \in \mathbb{N}_0}$; (2) $\Gamma_2 := \{\lambda_n, \lambda_n^\infty\}_{n \in \mathbb{N}_0}$; (3) the Weyl function $M(\lambda) := \frac{\Delta^\infty(\lambda)}{\Delta(\lambda)}$. This motivates us to investigate the inverse spectral problem with partial information given on the potential. More precisely, assume that $q$ is known on $[b, \pi]$ for some constant $b \in (0, \pi)$, then the uniqueness theorems of this paper will be given in three cases: $b \in (d, \pi]$, $b = d$, $b \in (0, d)$, where $d$ is the discontinuous point. In the case of $b \in (d, \pi]$ or $b = d$, we show that $h$, $\beta$, $\gamma$ and $q$ on $[0, \pi]$ can be uniquely determined by partial information of the eigenvalues $\lambda_n$, $\lambda_n^\infty$, and of the generalized normalizing constants $\alpha_n$, $\alpha_n^\infty$; the uniqueness problem is also considered under the same circumstances but with the normalizing constants $\alpha_n$, $\alpha_n^\infty$ replaced by ratios $\kappa_n$, $\kappa_n^\infty$. Moreover, for the case $b \in (0, d)$, similar uniqueness results can be established with the additional condition that $\beta$, $\gamma$ are known a priori.

We mention that in 1999, F. Gesztesy and B. Simon [27] considered the classical self-adjoint Sturm-Liouville operators and presented a generalization of Hochstadt–Lieberman theorem to the case where the potential $q$ is known on a larger interval $[a, \pi]$ with $a \in (0, \pi]$ and the set of common eigenvalues is sufficiently large. Later, G. Wei, H. K. Xu and Z. Wei [28, 29] provided some uniqueness results for classical self-adjoint Sturm-Liouville operators with only partial information on $q$, on the eigenvalues, and on the normalizing constants. While our results are generalizations of the uniqueness theorems established in [27–29], the non-self-adjointness and the presence of discontinuities produce essential qualitative modifications in the investigation of the operator $L$. To the best of our knowledge, the uniqueness theorems obtained in this paper have not yet been developed even for the non-self-adjoint classical Sturm-Liouville operators (i.e., the case of $\beta = 1$, $\gamma = 0$) and the self-adjoint Sturm-Liouville operators with discontinuous conditions inside (i.e., the real-valued case).

In addition, we show that less knowledge of eigenvalues and norming constants can be required if the potential $q$ satisfies a local smoothness condition, which is a generalization of the results in [27–29]. We notice that the key technique in [27–29] relies on the high-energy asymptotic expansion of the Weyl $m$-function [30], however, in our non-self-adjoint situation, an entirely different approach, based on the asymptotic expansion of the fundamental solutions of the equation (1.4), is
developed (see Proposition 1). Now we briefly present some of these uniqueness results (Theorem 1, Theorem 5, Remark 6, Corollary 1–4) as follows.

(S1) We prove that if $q$ is assumed to be $C^m$ near $\pi$, then $h, \beta, \gamma$ and $q$ on $[0, \pi]$ can be uniquely determined by the values of $q^{(j)}(\pi), j = 0, 1, \ldots, m$, $\{\lambda_n\}_{n \in \mathbb{N}_0 \setminus \Lambda_1}$ (a subsequence of $\sigma(B)$), and $\{\lambda_n^\infty\}_{n \in \mathbb{N}_0 \setminus \Lambda^\infty_1}$ (a subsequence of $\sigma(B^\infty)$), where $\#\Lambda_1 + \#\Lambda^\infty_1 = \lfloor \frac{m+2}{2} \rfloor$.

(S2) When $d \in (0, \frac{\pi}{2})$, we prove that if $q$ is $C^m$ near $\frac{\pi}{2}$ and $q$ on $[\frac{\pi}{2}, \pi]$ is known a priori, then $h, \beta, \gamma$ and $q$ on $[0, \pi]$ can be uniquely determined by all the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}_0}$ of $B$ except for $\left(\left[\frac{m+1}{2}\right]\right)$, or all the eigenvalues $\{\lambda_n^\infty\}_{n \in \mathbb{N}_0}$ of $B^\infty$ except for $\left(\left[\frac{m+1}{2}\right]\right)$; when $d = \frac{\pi}{2}$, the same statement holds if $\beta, \gamma$ are additionally assumed to be known a priori.

Here is a sketch of the contents of this paper. In Section 2, we provide some preliminary lemmas which will be used to prove the main results. In Section 3, assume that $q$ is known on $[b, \pi]$ for some constant $b \in (0, \pi)$, then we discuss the uniqueness theorems for three cases: $b \in (d, \pi)$, $b = d$, and $b \in (0, d)$. Finally, the appendix is devoted to present an important proposition (see Proposition 1), which is necessary to prove our principal results.

We conclude this introduction by briefly summarizing some of the notations used in this paper.

**Notation 2.** $\mathbb{C}$ denotes the complex plane. $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ denotes the set of nonnegative integers. Given a set $A$, the symbol $\# A$ will be used to denote the number of elements in $A$. Moreover, given a sequence $X := \{x_n\}_{n=0}^\infty$ of complex numbers, we use the notation $X_1 << X$ to denote that $X_1$ is a subsequence of $X$, and in addition, $\tilde{X} := \bigcup_{n \in \mathbb{N}_0} \{x_n\}$, $N_X(t) := \# \{n \in \mathbb{N}_0 : |x_n| < t\}$ for each $t \geq 0$.

## 2. Preliminaries

In this section, we provide some preliminaries which will be used in Section 3 to prove the main results.

In order to prove the uniqueness theorems, together with $B \,(B^\infty)$, we consider the boundary value problem $\tilde{B} \left(\tilde{B}^\infty\right)$ of the same form but with different coefficients $\tilde{q}, \tilde{h}, \tilde{H}, \tilde{\beta}, \tilde{\gamma}$ and $\tilde{d}$. We agree that if a certain symbol $\xi$ denotes an object related to $B$ or $B^\infty$, then $\tilde{\xi}$ will denote the analogous object related to $\tilde{B}$ or $\tilde{B}^\infty$, and $\tilde{\xi} := \xi - \xi$.

Now we introduce an entire function of $\lambda \in \mathbb{C}$,

$$F(\lambda) := \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle_{x=\pi}. \tag{2.1}$$

From [17, Theorem 5.2 and Remark 1], the following result can be given.

**Lemma 1.** Suppose that $F(\lambda) \equiv 0$, then $q = \tilde{q}$ a.e. on $[0, \pi], h = \tilde{h}, \beta = \tilde{\beta}, \gamma = \tilde{\gamma}, d = \tilde{d}$.

It should be noted that our main results are based on Lemma 1. Next, we give an important lemma, which plays a key role in this paper.
Lemma 2. Suppose that $H = \bar{H} \in \mathbb{C} \cup \{\infty\}$. If $\lambda_n = \tilde{\lambda}_{\tilde{n}}$ for some $n \in S_B$, $\tilde{n} \in S_{\tilde{B}}$, and $m_n = \tilde{m}_{\tilde{n}}$, then

\[
(2.2) \quad \frac{d^k}{d\lambda^k} F(\lambda) \bigg|_{\lambda = \lambda_n} = 0 \text{ for } k = 0, 1, \ldots, m_n - 1;
\]

In addition, if $\alpha_{n+\nu} = \tilde{\alpha}_{\tilde{n}+\nu}$, $\nu = 0, 1, \ldots, k_n - 1$, where $k_n$ is an integer such that $1 \leq k_n \leq m_n$, then we have

\[
\frac{d^{m_n+\nu}}{d\lambda^{m_n+\nu}} F(\lambda) \bigg|_{\lambda = \lambda_n} = 0 \text{ for } \nu = 0, 1, \ldots, k_n - 1,
\]

that is, in this case, the order of $\lambda_n$ (as a zero of $F(\lambda)$) is at least $(m_n + k_n)$. Similar statement also holds for the case $H = \bar{H} = \infty$.

Proof. We first prove the lemma for $H = \bar{H} \in \mathbb{C}$. From (1.5) and the definition (2.1) of $F(\lambda)$, we have

\[
(2.3) \quad F(\lambda) = \begin{vmatrix} \psi(\pi, \lambda) & \tilde{\psi}(\pi, \lambda) \\ \Delta(\lambda) & \tilde{\Delta}(\lambda) \end{vmatrix}.
\]

Since $m_n = \tilde{m}_{\tilde{n}}$, we know that

\[
(2.4) \quad \frac{d^k}{d\lambda^k} \Delta(\lambda) \bigg|_{\lambda = \lambda_n} = 0, \quad \frac{d^k}{d\lambda^k} \tilde{\Delta}(\lambda) \bigg|_{\lambda = \lambda_n} = 0 \text{ for } k = 0, 1, \ldots, m_n - 1.
\]

This directly yields (2.2). Now we turn to prove the second part of this lemma. It follows from (1.7), (1.8), (1.10), (2.3) and (2.4) that for $\nu = 0, 1, \ldots, k_n - 1$,

\[
\frac{d^{m_n+\nu}}{d\lambda^{m_n+\nu}} F(\lambda) \bigg|_{\lambda = \lambda_n} = \sum_{j=0}^{m_n+\nu} C^j_{m_n+\nu} \left[ \left. \frac{d^{m_n+\nu-j}}{d\lambda^{m_n+\nu-j}} \psi(\pi, \lambda) \right|_{\lambda = \lambda_n} \frac{d^j \Delta(\lambda)}{d\lambda^j} \right.
\]

\[
\left. - \left. \frac{d^{m_n+\nu-j}}{d\lambda^{m_n+\nu-j}} \tilde{\psi}(\pi, \lambda) \right|_{\lambda = \lambda_n} \frac{d^j \tilde{\Delta}(\lambda)}{d\lambda^j} \right] = \sum_{j=0}^{m_n+\nu} C^j_{m_n+\nu} \left[ \left. \frac{d^{m_n+\nu-j}}{d\lambda^{m_n+\nu-j}} \psi(\pi, \lambda) \right|_{\lambda = \lambda_n} \frac{d^j \Delta(\lambda)}{d\lambda^j} \right.
\]

\[
\left. - \left. \frac{d^{m_n+\nu-j}}{d\lambda^{m_n+\nu-j}} \tilde{\psi}(\pi, \lambda) \right|_{\lambda = \lambda_n} \frac{d^j \tilde{\Delta}(\lambda)}{d\lambda^j} \right] = 0.
\]

Let $\tilde{l} = m_n + \nu - l$, $\tilde{j} = m_n + \nu - j$. Then

\[
\frac{d^{m_n+\nu}}{d\lambda^{m_n+\nu}} F(\lambda) \bigg|_{\lambda = \lambda_n} = -\sum_{l=m_n+\nu}^{m_n+\nu} C^{\tilde{l}}_{m_n+\nu} \left[ \left. \frac{d^{\tilde{l}} \psi(\pi, \lambda)}{d\lambda^{\tilde{l}}} \right|_{\lambda = \lambda_n} \frac{d^{m_n+\nu-\tilde{l}} \Delta(\lambda)}{d\lambda^{m_n+\nu-\tilde{l}}} \right.
\]

\[
\left. - \left. \frac{d^{\tilde{l}} \tilde{\psi}(\pi, \lambda)}{d\lambda^{\tilde{l}}} \right|_{\lambda = \lambda_n} \frac{d^{m_n+\nu-\tilde{l}} \tilde{\Delta}(\lambda)}{d\lambda^{m_n+\nu-\tilde{l}}} \right] = 0.
\]
This together (2.5) yield that $\frac{dm_{n+r}}{dx}F(\lambda)|_{\lambda=\lambda_n} = -\frac{dm_{n+r}}{dx}F(\lambda)|_{\lambda=\lambda_n}$, and hence $\frac{dm_{n+r}}{dx}F(\lambda)|_{\lambda=\lambda_n} = 0$ for $\nu = 0, 1, \ldots, k_n - 1$. This proves the lemma for the case $H = \tilde{H} \in \mathbb{C}$. In view of Remark 1 and the fact

$$F(\lambda) := \begin{pmatrix} \varphi(\pi, \lambda) & \varphi'(\pi, \lambda) \\ \varphi(\pi, \lambda) & \varphi'(\pi, \lambda) \end{pmatrix} = -\begin{pmatrix} \Delta(\lambda) & \Delta(\lambda) \\ \Delta'(\lambda) & \Delta'(\lambda) \end{pmatrix},$$

the lemma for $H = \tilde{H} = \infty$ can be proved similarly. \hfill \Box

**Lemma 3.** Assume that $d = \tilde{d}$ and $q = \tilde{q}$ a.e. on $[b, \pi]$ for some $b \in (0, \pi]$, then following expressions hold:

$$F(\lambda) = \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=b} \text{ for } b \in (d, \pi],$$

$$F(\lambda) = \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=d+0} \text{ for } b = d,$$

$$F(\lambda) = \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=d-0} \text{ for } b = d.$$

*Proof.* From the definition (2.1) of $F(\lambda)$, one can easily deduce that

$$F(\lambda) = -\int_0^\pi \tilde{q}(x) \varphi(x, \lambda) \tilde{\varphi}(x, \lambda) dx + \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=0}$$

$$+ \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=d+0} - \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=d-0}.$$

Hence by $q = \tilde{q}$ a.e. on $[b, \pi]$ we infer from the above equality that

$$F(\lambda) = -\int_0^b \tilde{q}(x) \varphi(x, \lambda) \tilde{\varphi}(x, \lambda) dx + \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=0}$$

$$+ \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=d+0} - \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=d-0}.$$

Therefore, this lemma can be directly proved by the following facts

$$-\int_0^b \tilde{q}(x) \varphi(x, \lambda) \tilde{\varphi}(x, \lambda) dx$$

$$= \begin{cases} \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=d-0}^b + \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=d-0}^b \text{ for } b \in (d, \pi], \\
\langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=d-0}^b \text{ for } b = d, \\
\langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle |_{x=d-0}^b \text{ for } b \in (0, d). \end{cases}$$

\hfill \Box

**Lemma 4.** As $|\lambda| \to \infty$,

$$\varphi(x, \lambda) = \begin{cases} \cos(\sqrt{\lambda}x) + O\left(\frac{\exp(|\text{Im}\sqrt{\lambda}|x)}{\sqrt{\lambda}}\right), x < d, \\
b_1 \cos(\sqrt{\lambda}x) + b_2 \cos(\sqrt{\lambda}(2d - x)) + O\left(\frac{\exp(|\text{Im}\sqrt{\lambda}|x)}{\sqrt{\lambda}}\right), x > d, \end{cases}$$

$$\varphi'(x, \lambda) = \begin{cases} -\sqrt{\lambda}\sin(\sqrt{\lambda}x) + O\left(\exp\left(|\text{Im}\sqrt{\lambda}|x\right)\right), x < d, \\
\sqrt{\lambda}\left(-b_1 \sin(\sqrt{\lambda}x) + b_2 \sin(\sqrt{\lambda}(2d - x))\right) + O\left(\exp\left(|\text{Im}\sqrt{\lambda}|x\right)\right), x > d, \end{cases}$$

where $b_1 = \frac{\beta + \beta^{-1}}{2}$ and $b_2 = \frac{\beta - \beta^{-1}}{2}$. \hfill \Box

*Proof.* See [15, p.145-146].
Remark 2. If \( \lambda = iy \) with \( y \in \mathbb{R} \), then by Lemma 4, (1.5) and (1.6), one deduces that as \( |y| \to \infty \),

\[
|\Delta (iy)| = \frac{b_1}{2} |y|^\frac{1}{2} \exp \left( \left| \text{Im} \sqrt{iy} \right| \pi \right) (1 + o(1)), \tag{2.8}
\]

\[
|\Delta^\infty (iy)| = \frac{b_1}{2} \exp \left( \left| \text{Im} \sqrt{iy} \right| \pi \right) (1 + o(1)), \tag{2.9}
\]

\[
|\varphi (b, iy)| = \begin{cases} \frac{b_1}{\sqrt{2}} \exp \left( \left| \text{Im} \sqrt{iy} \right| b \right) (1 + o(1)) & \text{for } b > d, \\ \frac{b_1}{\sqrt{2}} \exp \left( \left| \text{Im} \sqrt{iy} \right| b \right) (1 + o(1)) & \text{for } b < d, \end{cases} \tag{2.10}
\]

\[
|\varphi (d + 0, iy)| = \frac{\beta}{2} \exp \left( \left| \text{Im} \sqrt{iy} \right| d \right) (1 + o(1)). \tag{2.11}
\]

We conclude this section with two lemmas (see Lemma 5 and Lemma 6), which will be used in Section 3 to prove our main results. Now we first give some notations and basic facts.

Recall that \( \sigma (B) := \{ \lambda_n \}_{n \in \mathbb{N}_0} \) and \( \sigma (B^\infty) := \{ \lambda_n^\infty \}_{n \in \mathbb{N}_0} \) are the sequences consisting of all the eigenvalues of \( B \) and \( B^\infty \), respectively. By the asymptotics of the eigenvalues \( \lambda_n \) and \( \lambda_n^\infty \) [17], it is easy to see that there exist constants \( r_1 \) and \( r_2 \) such that

\[
\min_{n \in \mathbb{N}_0} \{ \text{Re} \lambda_n \} \geq r_1, \quad \min_{n \in \mathbb{N}_0} \{ \text{Re} \lambda_n^\infty \} \geq r_2.
\]

Hence by adding (if necessary) a sufficiently large constant to the potential coefficient \( q \), throughout this paper we may assume that

\[
N_{\sigma (B)} (t) = N_{\sigma (B^\infty)} (t) = 0 \text{ for } t \leq 1. \tag{2.12}
\]

By Lemma 4 one can easily deduce that \( \Delta (\lambda) \) and \( \Delta^\infty (\lambda) \) are entire in \( \lambda \in \mathbb{C} \) of order \( \frac{1}{2} \), and hence by Hadamard’s Factorization Theorem [32, Ch. I], there exist constants \( C_B \) and \( C_{B^\infty} \) such that

\[
\Delta (\lambda) = C_B \prod_{n=0}^\infty \left( 1 - \frac{\lambda}{\lambda_n} \right), \tag{2.13}
\]

\[
\Delta^\infty (\lambda) = C_{B^\infty} \prod_{n=0}^\infty \left( 1 - \frac{\lambda}{\lambda_n^\infty} \right). \tag{2.14}
\]

Moreover, it follows from [32, Ch. I, Theorem 4] that

\[
N_{\sigma (B)} (t) \leq C |t|^\rho \quad \text{and} \quad N_{\sigma (B^\infty)} (t) \leq C |t|^\rho \text{ for all } \rho > \frac{1}{2}, \tag{2.15}
\]

where \( C \) is some positive constant.

Lemma 5. Let \( X := \{ x_n \}_{n=0}^\infty \) with \( 0 < |x_0| \leq |x_1| \leq |x_2| \leq \cdots \) be a sequence satisfying

\[
\max_{n \in \mathbb{N}_0} |\text{Im} x_n| \leq c_1 \text{ for some } c_1 > 0, \tag{2.16}
\]

and

\[
N_X (t) = 0 \text{ for } t \leq 1, \tag{2.17}
\]

\[
N_X (t) \leq C |t|^\rho \text{ for all } \rho > \rho_0, \tag{2.18}
\]

where \( C \) is some positive constant and \( \rho_0 \in (0, 1) \) is fixed. If there exist real constants \( l_1, l_2, l_3 \) such that for sufficiently large \( t \in \mathbb{R} \),

\[
N_X (t) \geq l_1 N_{\sigma (B)} (t) + l_2 N_{\sigma (B^\infty)} (t) + l_3, \tag{2.19}
\]
then there exists a constant \( M > 0 \) such that for sufficiently large \( |y| \) (\( y \) being real)

\[
|G_X (iy)| \geq M |y|^{\frac{l_1 + l_2}{2}} e^{\pi (l_1 + l_2) |\text{Im} \sqrt{y}|},
\]

where \( G_X (\lambda) := \prod_{n=0}^{\infty} \left( 1 - \frac{\lambda}{\lambda_n} \right) \).

**Proof.** Note that

\[
\frac{d}{dt} \left[ \frac{1}{2} \ln \left( 1 + \frac{y^2}{t^2} \right) \right] = -\frac{y^2}{t^3 + ty^2}.
\]

Then by (2.17), (2.18), (2.20) and integration by parts, we infer that for \( y \in \mathbb{R} \),

\[
\ln|G_X (iy)| = \frac{1}{2} \sum_{n=0}^{\infty} \ln \left( \frac{1 - \frac{iy}{x_n}}{1 + \frac{iy}{x_n}} \right) + \frac{1}{2} \sum_{n=0}^{\infty} \ln \left( 1 + \frac{|y|^2}{|x_n|^2} \right)
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \ln \left( 1 + \frac{2\text{Im} \lambda_n + y^2}{|\lambda_n|^2} \right) + \frac{1}{2} \int_{0}^{\infty} \ln \left( 1 + \frac{y^2}{t^2} \right) dN_X (t)
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \ln \left( 1 + \frac{2\text{Im} \lambda_n + y^2}{|\lambda_n|^2} \right) + \int_{1}^{\infty} \frac{y^2}{t^3 + ty^2} N_X (t) dt.
\]

Similarly, by (2.12), (2.15) and (2.20) we deduce that

\[
l_1 \ln|G_{\sigma(B)} (iy)| = \frac{l_1}{2} \sum_{n=0}^{\infty} \ln \left( 1 + \frac{2\text{Im} \lambda_n + y^2}{|\lambda_n|^2} \right) + l_1 \int_{1}^{\infty} \frac{y^2}{t^3 + ty^2} N_{\sigma(B)} (t) dt,
\]

\[
l_2 \ln|G_{\sigma(B^\infty)} (iy)| = \frac{l_2}{2} \sum_{n=0}^{\infty} \ln \left( 1 + \frac{2\text{Im} \lambda_n + y^2}{|\lambda_n|^2} \right) + l_2 \int_{1}^{\infty} \frac{y^2}{t^3 + ty^2} N_{\sigma(B^\infty)} (t) dt.
\]

where

\[
G_{\sigma(B)} (\lambda) := \prod_{n=0}^{\infty} \left( 1 - \frac{\lambda}{\lambda_n} \right) \text{ and } G_{\sigma(B^\infty)} (\lambda) := \prod_{n=0}^{\infty} \left( 1 - \frac{\lambda}{\lambda_n^\infty} \right).
\]

Therefore,

\[
\ln|G_X (iy)| - l_1 \ln|G_{\sigma(B)} (iy)| - l_2 \ln|G_{\sigma(B^\infty)} (iy)|
\]

\[
= g(y) + \int_{1}^{\infty} \frac{y^2}{t^3 + ty^2} \left( N_X (t) - l_1 N_{\sigma(B)} (t) - l_2 N_{\sigma(B^\infty)} (t) \right) dt,
\]

where

\[
g(y) := \frac{1}{2} \sum_{n=0}^{\infty} \ln \left( 1 + \frac{2\text{Im} \lambda_n + y^2}{|\lambda_n|^2} \right) - \frac{l_1}{2} \sum_{n=0}^{\infty} \ln \left( 1 + \frac{2\text{Im} \lambda_n + y^2}{|\lambda_n|^2} \right)
\]

\[
- \frac{l_2}{2} \sum_{n=0}^{\infty} \ln \left( 1 + \frac{2\text{Im} \lambda_n + y^2}{|\lambda_n|^2} \right).
\]

Next, we aim to show that there exists a constant \( C_g > 0 \) such that

\[
|g(y)| \leq C_g \text{ for all } y \in \mathbb{R}.
\]
In fact, we first note that there exist constants $c_2$ and $c_3$ such that
\begin{equation}
\max_{n \in \mathbb{N}_0} |\text{Im} \lambda_n| \leq c_2 \quad \text{and} \quad \max_{n \in \mathbb{N}_0} |\text{Im} \lambda_n^\infty| \leq c_3,
\end{equation}
which can be obtained from the asymptotics of the eigenvalues $\lambda_n$ and $\lambda_n^\infty$ [17]. In addition,
\begin{equation}
\frac{d}{dt} \left[ \ln \left( 1 + \frac{2|y|c_i}{t^2 + y^2} \right) \right] = -\frac{4|y|c_i t}{(t^2 + y^2)^2}, \quad i = 1, 2, 3,
\end{equation}
where $c_1$ is defined by (2.16) and $c_2, c_3$ are defined by (2.25). Then by (2.15), (2.16), (2.18), (2.23), (2.25), (2.26) and integration by parts, we obtain that
\begin{align*}
|g(y)| & \leq 1 + 2 \sum_{n=0}^{\infty} \ln \left( 1 + \frac{2|y|c_1}{|x_n|^2 + |y|^2} \right) + \frac{l_1}{2} \sum_{n=0}^{\infty} \ln \left( 1 + \frac{2|y|c_2}{|\lambda_n|^2 + |y|^2} \right) \\
& \quad + \frac{l_2}{2} \sum_{n=0}^{\infty} \ln \left( 1 + \frac{2|y|c_3}{|\lambda_n^\infty|^2 + |y|^2} \right) \\
& = \frac{1}{2} \int_1^\infty \ln \left( 1 + \frac{2|y|c_1}{t^2 + y^2} \right) dN_X(t) + \frac{l_1}{2} \int_1^\infty \ln \left( 1 + \frac{2|y|c_2}{t^2 + y^2} \right) dN_{\sigma(B)}(t) \\
& \quad + \frac{l_2}{2} \int_1^\infty \ln \left( 1 + \frac{2|y|c_3}{t^2 + y^2} \right) dN_{\sigma(B^\infty)}(t) \\
& \leq 2c_1 \int_1^\infty N_X(t) \frac{t |y|}{(t^2 + y^2)^2} dt + 2c_2 |l_3| \int_1^\infty N_{\sigma(B)}(t) \frac{t |y|}{(t^2 + y^2)^2} dt \\
& \quad + 2c_3 |l_2| \int_1^\infty N_{\sigma(B^\infty)}(t) \frac{|y| t}{(t^2 + y^2)^2} dt \\
& \leq C_0 \int_1^\infty \frac{t^2 |y|}{(t^2 + y^2)^2} dt \leq C_0 \int_1^\infty \frac{|y|}{t^2 + y^2} dt \\
& = \frac{C_0 \pi}{2} - C_0 \arctan \frac{1}{|y|} \quad \text{if} \ y \neq 0,
\end{align*}
where $C_0$ is some positive constant. This directly yields (2.24). By hypothesis (2.19) we know that there exist constants $t_0 \geq 1$ and $C_1 \geq 0$ such that
\begin{align}
N_X(t) - l_1 N_{\sigma(B)}(t) - l_2 N_{\sigma(B^\infty)}(t) & \geq l_3, \quad t \geq t_0, \\
N_X(t) - l_1 N_{\sigma(B)}(t) - l_2 N_{\sigma(B^\infty)}(t) & \geq -C_1, \quad t \leq t_0.
\end{align}
Therefore, it follows from (2.22), (2.24), (2.27) and (2.28) that
\begin{align}
\ln \left| \frac{|G_X(iy)|}{|G_{\sigma(B)}(iy)|} \right|_{t_0}^{t_1} & \geq -C_g - \int_1^{t_0} \frac{y^2}{t^3 + ty^2} C_1 dt + \int_{t_0}^{\infty} \frac{y^2}{t^3 + ty^2} l_3 dt \\
& \geq -C_g - (C_1 + l_3) \int_1^{t_0} \frac{y^2}{t^3 + ty^2} dt + l_3 \int_1^{\infty} \frac{y^2}{t^3 + ty^2} dt \\
& = -C_g + (C_1 + l_3) \frac{1}{2} \ln \frac{(t_0^3 + y^2)}{t_0^2 (1 + y^2)} + l_3 \frac{1}{2} \ln (1 + y^2).
\end{align}
In addition, by (2.8), (2.9), (2.13) and (2.14), we infer that

\[
(2.30) \quad \left\{ \begin{array}{l}
|G_{\sigma(B)}(iy)| = \frac{b_{\sigma(B)}}{2e|iy|} |y|^{\frac{1}{2}} \exp \left( |\text{Im}\sqrt{iy}| \pi \right) (1 + o(1)), \\
|G_{\sigma(B^\infty)}(iy)| = \frac{b_{\sigma(B^\infty)}}{2e|iy|} \exp \left( |\text{Im}\sqrt{iy}| \pi \right) (1 + o(1)).
\end{array} \right.
\]

Hence it turns out from (2.29) and (2.30) that there exists a constant \( M > 0 \) such that

\[
|G_X(iy)| \geq M |y|^{\frac{1}{2} + \frac{1}{2}(l_1 + l_2)} e^{\pi(l_1 + l_2)|\text{Im}\sqrt{y}|}
\]

for sufficiently large \(|y|\) and \( y \in \mathbb{R} \). This completes the proof. \( \square \)

**Lemma 6.** Assume that \( g(\lambda) \) is an entire function of order less than one. If

\[
\lim_{|y| \to \infty; y \in \mathbb{R}} |g(iy)| = 0,
\]

then \( g(\lambda) \equiv 0 \).

**Proof.** The proof is referred to [27, 32]. \( \square \)

### 3. Main Results and Proofs

Our goal of this section is to give the main results of this paper. Assume that the potential \( q \) is known on \([b, \pi]\), then due to the presence of discontinuous conditions at \( d \in (0, \pi) \), the uniqueness theorems are given for three cases: \( b \in (d, \pi) \), \( b = d \), and \( b \in (0, d) \). In each case, we first study the uniqueness problem (Theorem 1, Theorem 3, Theorem 5) when only partial information on \( q \), on the eigenvalues, and on the generalized normalizing constants is available, and then we investigate the uniqueness problem (Theorem 2, Theorem 4, Theorem 6) under the same circumstances but with the normalizing constants replaced by ratios. Unless explicitly stated otherwise, \( H \) and \( d \) will be fixed in this section. In addition, let us recall Notation 1 and Notation 2 given in the introduction.

#### 3.1. Case I: \( q \) is known on \([b, \pi]\), where \( b \in (d, \pi) \).

##### 3.1.1. Pairs of Eigenvalues and Normalizing Constants.

**Hypothesis 1.** Consider the subsequences \( W, W_1, W^\infty, W_1^\infty \) satisfying

\[
W_1 \ll W \ll \sigma(B), \ W_1 \ll W \ll \sigma(B^\infty), \\
W_1^\infty \ll W^\infty \ll \sigma(B^\infty), W_1^\infty \ll W^\infty \ll \sigma(B^\infty)
\]

and the following conditions:

(1) for any \( \lambda_n = \bar{\lambda}_n \in \bar{W}_1 \) where \( n \in S_B \) and \( \bar{n} \in S_{\bar{B}} \), suppose that

\[
(3.1) \quad m_n = \bar{m}_n, \ \alpha_n + \nu = \bar{\alpha}_{\bar{n} + \nu} \quad \text{for} \ \nu = 0, 1, \ldots, k_{n} - 1,
\]

where \( k_n \) equals the number of occurrences of the eigenvalue \( \lambda_n \) in \( W_1 \);

(2) for any \( \lambda_n^\infty = \bar{\lambda}_n^\infty \in \bar{W}_1^\infty \) where \( n \in S_{B^\infty} \) and \( \bar{n} \in S_{\bar{B}^\infty} \), suppose that

\[
(3.2) \quad m_n^\infty = \bar{m}_n^\infty, \ \alpha_n^\infty + \gamma = \bar{\alpha}_{\bar{n}^\infty + \gamma} \quad \text{for} \ \gamma = 0, 1, \ldots, k_n^\infty - 1,
\]

where \( k_n^\infty \) equals the number of occurrences of the eigenvalue \( \lambda_n^\infty \) in \( W_1^\infty \).
Theorem 1. Assume Hypothesis 1 and suppose that $q, \tilde{q} \in C^m$ near $b \in (d, \pi]$, $m \in \mathbb{N}_0$, $q = \tilde{q}$ a.e. on $[b, \pi]$ (in particular, for $b = \pi : q^{(j)}(\pi) = \tilde{q}^{(j)}(\pi)$ for $j = 0, 1, \ldots, m$), and

\begin{equation}
N_W(t) + N_W(t) + N_{W^\infty}(t) + N_{W^\infty}(t) \geq AN_{\sigma(B)}(t) + \left(\frac{2b}{\pi} - A\right) N_{\sigma(B^\infty)}(t) - A \frac{m + 1}{2}
\end{equation}

for sufficiently large $t \in \mathbb{R}$. Then $h = \tilde{h}$, $\beta = \beta$, $\gamma = \gamma$ and $q = \tilde{q}$ a.e. on $[0, \pi]$.

Remark 3. By Remark 1, we know that if $q$ and $\tilde{q}$ are assumed to be in $L^1_+ [0, \pi]$, then Theorem 1 should be modified by taking $m = -1$. Thus for brevity $C^{-1}$ means $L^1$ throughout this paper unless explicitly stated otherwise.

Corollary 1. If $q$ is assumed to be $C^m$ near $\pi$, then $h, \beta, \gamma$ and $q$ on $[0, \pi]$ can be uniquely determined by the values of $q^{(j)}(\pi), j = 0, 1, \ldots, m, \{\lambda_n\}_{n \in \mathbb{N}_0 \setminus \Lambda_1}$ (a subsequence of $\sigma(B)$), and $\{\lambda_n^\infty\}_{n \in \mathbb{N}_0 \setminus \Lambda_1^\infty}$ (a subsequence of $\sigma(B^\infty)$), where $\#\Lambda_1 + \#\Lambda_1^\infty = \left\lfloor \frac{m+2}{2} \right\rfloor$.

Corollary 2. Assume that $q$ is $C^m$ near $\pi$ and the values of $q^{(j)}(\pi), j = 0, 1, \ldots, m$, are known a priori. Then $h, \beta, \gamma$ and $q$ on $[0, \pi]$ can be uniquely determined by the following information (1) or (2):

1) all the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}_0}$ of $B$ and a subsequence of the normalizing constants $\{\alpha_{n+1}\}_{n \in S_B \setminus \Lambda}$, where $0 \leq k_n \leq m_n - 1, \Lambda \subset S_B$ and $\sum m_n + \sum_{n \in S_B \setminus \Lambda} (m_n - k_n - 1) = \left\lfloor \frac{m+3}{2} \right\rfloor$;

2) all the eigenvalues $\{\lambda_n^\infty\}_{n \in \mathbb{N}_0}$ of $B^\infty$ and a subsequence of the normalizing constants $\{\alpha_{n+1}^\infty\}_{n \in S_B^\infty \setminus \Lambda^\infty}$, where $0 \leq k_n^\infty \leq m_n^\infty - 1, \Lambda^\infty \subset S_B^\infty$ and $\sum m_n^\infty + \sum_{n \in S_B^\infty \setminus \Lambda^\infty} (m_n^\infty - k_n^\infty - 1) = \left\lfloor \frac{m+1}{2} \right\rfloor$.

Remark 4. Suppose that $b_1 = \frac{\beta + \beta^{-1}}{2}$ is known a priori. Then from (2.8), (2.9), (2.13) and (2.14), one deduces that $\hat{\Delta}(\lambda)$ and $\Delta^\infty(\lambda)$ can be uniquely determined by $\sigma(B)$ and $\sigma(B^\infty)$, respectively; thus by (1.10) and (1.18), we know that Corollary 2 remains valid if the conditions on the normalizing constants $\{\alpha_{n+1}\}_{n \in S_B \setminus \Lambda}$ and $\{\alpha_{n+1}^\infty\}_{n \in S_B^\infty \setminus \Lambda^\infty}$ are replaced by the conditions on the ratios $\{\kappa_{n+1}\}_{n \in S_B \setminus \Lambda}$ and $\{\kappa_{n+1}^\infty\}_{n \in S_B^\infty \setminus \Lambda^\infty}$, respectively.

Corollary 3. Let $d \in (0, \frac{\pi}{2})$. Assume that $q$ is $C^m$ near $\frac{\pi}{2}$ and $q$ on $[\frac{\pi}{2}, \pi]$ are known a priori. Then $h, \beta, \gamma$ and $q$ on $[0, \pi]$ can be uniquely determined by all the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}_0}$ of $B$ except for $(\left\lfloor \frac{m+2}{2} \right\rfloor)$, or all the eigenvalues $\{\lambda_n^\infty\}_{n \in \mathbb{N}_0}$ of $B^\infty$ except for $(\left\lfloor \frac{m+1}{2} \right\rfloor)$.

To prove Theorem 1, we first give a lemma on $F(\lambda)$ defined by (2.1).

Lemma 7. Assume that $q, \tilde{q} \in C^m$ near $b \in (d, \pi], q = \tilde{q}$ a.e. on $[b, \pi]$ (in particular, for $b = \pi : q^{(j)}(\pi) = \tilde{q}^{(j)}(\pi)$ for $j = 0, 1, \ldots, m$). Then one observes that

\[ |F(iy)| = o \left( |y|^{\frac{m+1}{2}} \exp \left( 2 \sqrt{m+2} |b| \right) \right) \text{ as } y \text{ (real)} \to \infty. \]
Proof. Recall Definition 1 (in the Appendix) for the functions \( y_{i,d}(x, \lambda) \) and \( \bar{y}_{i,d}(x, \lambda) \), \( i = 1, 2 \). Then from Lemma 3 we know that for \( b \in (d, \pi] \),

\[
F(\lambda) = \varphi(b, \lambda) \bar{\varphi}'(b, \lambda) - \varphi'(b, \lambda) \bar{\varphi}(b, \lambda)
\]

\[
= \left[ \beta \varphi(d - 0, \lambda) y_{1,d}(b, \lambda) + \left( \beta^{-1} \varphi'(d - 0, \lambda) + \gamma \varphi(d - 0, \lambda) \right) y_{2,d}(b, \lambda) \right] \times \left[ \beta \varphi(d - 0, \lambda) \bar{y}_{1,d}(b, \lambda) + \left( \beta^{-1} \varphi'(d - 0, \lambda) + \gamma \bar{\varphi}(d - 0, \lambda) \right) \bar{y}_{2,d}(b, \lambda) \right] - \left[ \beta \varphi(d - 0, \lambda) y_{1,d}(b, \lambda) + \left( \beta^{-1} \varphi'(d - 0, \lambda) + \gamma \varphi(d - 0, \lambda) \right) y_{2,d}(b, \lambda) \right] \times \left[ \beta \varphi(d - 0, \lambda) \bar{y}_{1,d}(b, \lambda) + \left( \beta^{-1} \varphi'(d - 0, \lambda) + \gamma \bar{\varphi}(d - 0, \lambda) \right) \bar{y}_{2,d}(b, \lambda) \right]
\]

\[
= A_1(\lambda) [y_{1,d}(b, \lambda) \bar{y}_{1,d}(b, \lambda) - y_{1,d}(b, \lambda) \bar{y}_{1,d}(b, \lambda)] + A_2(\lambda) [y_{1,d}(b, \lambda) \bar{y}_{2,d}(b, \lambda) - y_{1,d}(b, \lambda) \bar{y}_{2,d}(b, \lambda)] + A_3(\lambda) [\bar{y}_{1,d}(b, \lambda) y_{2,d}(b, \lambda) - \bar{y}_{1,d}(b, \lambda) y_{2,d}(b, \lambda)] + A_4(\lambda) [y_{2,d}(b, \lambda) \bar{y}_{2,d}(b, \lambda) - y_{2,d}(b, \lambda) \bar{y}_{2,d}(b, \lambda)],
\]

where

\[
A_1(\lambda) = \beta \varphi(d - 0, \lambda) \bar{\varphi}(d - 0, \lambda) = O \left( \exp \left( \frac{1}{\text{Im} \sqrt{\lambda}} d \right) \right),
\]

\[
A_2(\lambda) = \beta \varphi(d - 0, \lambda) \left( \beta^{-1} \varphi'(d - 0, \lambda) + \gamma \varphi(d - 0, \lambda) \right) = O \left( \sqrt{\lambda} \exp \left( \frac{1}{\text{Im} \sqrt{\lambda}} d \right) \right),
\]

\[
A_3(\lambda) = \beta \varphi(d - 0, \lambda) \left( \beta^{-1} \varphi'(d - 0, \lambda) + \gamma \varphi(d - 0, \lambda) \right) = O \left( \sqrt{\lambda} \exp \left( \frac{1}{\text{Im} \sqrt{\lambda}} d \right) \right),
\]

\[
A_4(\lambda) = \left( \beta^{-1} \varphi'(d - 0, \lambda) + \gamma \bar{\varphi}(d - 0, \lambda) \right) \left( \beta^{-1} \varphi'(d - 0, \lambda) + \gamma \varphi(d - 0, \lambda) \right) = O \left( \frac{1}{\text{Im} \sqrt{\lambda}} d \right).
\]

as \( |\lambda| \to \infty \). Note that the asymptotics of \( A_1, A_2, A_3 \) and \( A_4 \) can be directly obtained by Lemma 4. Hence from Proposition 1 it follows that as \( y \to \infty \),

\[
|F(\lambda)|
\]

\[
\leq |A_1(iy)| o \left( \exp \left( \frac{2 \text{Im} \sqrt{i|y|}}{\frac{1}{\sqrt{|y|}^{m+1}}} \right) \left( b - d \right) \right) + |A_2(iy)| o \left( \exp \left( \frac{2 \text{Im} \sqrt{i|y|}}{\frac{1}{\sqrt{|y|}^{m+2}}} \right) \left( b - d \right) \right)
\]

\[
+ |A_3(iy)| o \left( \exp \left( \frac{2 \text{Im} \sqrt{i|y|}}{\frac{1}{\sqrt{|y|}^{m+3}}} \right) \left( b - d \right) \right) + |A_4(iy)| o \left( \exp \left( \frac{2 \text{Im} \sqrt{i|y|}}{\frac{1}{\sqrt{|y|}^{m+4}}} \right) \left( b - d \right) \right)
\]

\[
= o \left( |y|^{\frac{3}{2} + m} \exp \left( \frac{2 \text{Im} \sqrt{i|y|}}{\sqrt{|y|}} \left( b - d \right) \right) \right).
\]

This completes the proof. \qed

Now we turn to prove Theorem 1.

Proof of Theorem 1. For any \( \lambda_n \in \hat{W} \) and \( \lambda_n^\infty \in \hat{W}^\infty \) where \( n \in S_B \) and \( n \in S_{B^\infty} \), let \( \gamma_n \) and \( \gamma_n^\infty \) denote the number of occurrences of \( \lambda_n \) in \( W \) and \( \lambda_n^\infty \) in \( W^\infty \), respectively. Denote

\[
H(\lambda) := \frac{F(\lambda)}{G(\lambda)},
\]

(3.4)
where

$$G_{\Xi}(\lambda) := G_W(\lambda) G_{W_1}(\lambda) G_{W_\infty}(\lambda) G_{W_1^\infty}(\lambda),$$

$$G_W(\lambda) := \prod_{\lambda_n \in W, n \in S_B} \left(1 - \frac{\lambda}{\lambda_n}\right)^{\gamma_n}, G_{W_1}(\lambda) := \prod_{\lambda_n \in W_1, n \in S_B} \left(1 - \frac{\lambda}{\lambda_n}\right)^{k_n},$$

$$G_{W_\infty}(\lambda) := \prod_{\lambda_n^\infty \in W_\infty, n \in S_{B \infty}} \left(1 - \frac{\lambda}{\lambda_n^\infty}\right)^{\gamma_n^\infty}, G_{W_1^\infty}(\lambda) := \prod_{\lambda_n^\infty \in W_1^\infty, n \in S_{B \infty}} \left(1 - \frac{\lambda}{\lambda_n^\infty}\right)^{k_n^\infty}.$$

Then it follows from (3.1), (3.2), Lemma 2, and the fact $\sigma(B) \cap \sigma(B^\infty) = \emptyset$ that $H(\lambda)$ is an entire function. From Lemma 3, we know that $F(\lambda)$ is an entire function of order less than $\frac{1}{2}$; $\Delta(\lambda)$ and $\Delta^\infty(\lambda)$ are entire functions of order $\frac{1}{2}$. Moreover, since the order of canonical product of an entire function is equal to its convergence exponent of zeros ([32, P16]), we can obtain that $G_{\Xi}(\lambda)$ is an entire function of order less than $\frac{1}{2}$, and so the order of $H(\lambda)$ is at most $\frac{1}{2}$.

Now we aim to prove that $H(\lambda) \equiv 0$. By Lemma 6, it is sufficient to prove that $|H(iy)| \to 0$ as $y$ (real) $\to \infty$. From Lemma 5 and the assumption (3.3), we know that there exists a constant $M > 0$ such that

$$|G_{\Xi}(iy)| \geq M |y|^{-\frac{m+1}{2}} \exp\left(2 |\text{Im} iy| b\right),$$

and thus according to (3.4) and Lemma 7, one has

$$|H(iy)| \leq \frac{o\left(|y|^{-\frac{m+1}{2}} \exp\left(2 |\text{Im} iy| b\right)\right)}{M |y|^{-\frac{m+1}{2}} \exp\left(2 |\text{Im} iy| b\right)} = o(1) \text{ as } y \text{ (real) } \to \infty.$$

This implies that $H(\lambda) \equiv 0$ and thus $F(\lambda) \equiv 0$. Then we conclude from Lemma 1 that $h = \tilde{h}$, $\beta = \tilde{\beta}$, $\gamma = \tilde{\gamma}$ and $q = \tilde{q}$ a.e. on $[0, \pi]$. \hfill \Box

3.1.2. Pairs of Eigenvalues and Ratios.

**Hypothesis 2.** Consider the subsequences $W$ and $W_\infty$ satisfying

$$W << \sigma(B), W << \sigma(B^\infty), W_\infty << \sigma(B^\infty), W_\infty << \sigma(B^\infty),$$

and the following conditions:

1. for any $\lambda_n = \tilde{\lambda}_n \in W$ where $n \in S_B$ and $\tilde{n} \in S_{B^\infty}$, suppose that

$$k_{n+\nu} = \tilde{k}_{n+\nu} \text{ for } \nu = 0, 1, \ldots, k_n - 1,$$

where $k_n$ equals the number of occurrences of $\lambda_n$ in $W$;

2. for any $\lambda_n^\infty = \tilde{\lambda}_n^\infty \in W_\infty$ where $n \in S_{B^\infty}$ and $\tilde{n} \in S_{B^{\infty \infty}}$, suppose that

$$k_{n+\gamma} = \tilde{k}_{n+\gamma} \text{ for } \gamma = 0, 1, \ldots, k_n^\infty - 1,$$

where $k_n^\infty$ equals the number of occurrences of $\lambda_n^\infty$ in $W_\infty$.

**Theorem 2.** Assume Hypothesis 2 and suppose that $q = \tilde{q}$ a.e. on $[b, \pi]$, then $h = \tilde{h}$, $\beta = \tilde{\beta}$, $\gamma = \tilde{\gamma}$ and $q = \tilde{q}$ a.e. on $[0, \pi]$.

$$N_W(t) + N_{W_\infty}(t) \geq AN_{\sigma(B)}(t) + \left(\frac{b}{\pi} - A\right) N_{\sigma(B^\infty)}(t) - \frac{A}{2} + \epsilon$$

for sufficiently large $t \in \mathbb{R}$, where $\epsilon$ is an arbitrary positive constant. Then $h = \tilde{h}$, $\beta = \tilde{\beta}$, $\gamma = \tilde{\gamma}$ and $q = \tilde{q}$ a.e. on $[0, \pi]$. 
Proof. Denote

\[ H_1 (\lambda) := \frac{F_1 (\lambda)}{G_\Theta (\lambda)}, \quad H_2 (\lambda) = \frac{F_2 (\lambda)}{G_\Theta (\lambda)}, \]

where

\[ G_\Theta (\lambda) := G_W (\lambda) G_{W^\infty} (\lambda), \]

\[ G_W (\lambda) := \prod_{\lambda_n \in W, n \in S_B} \left(1 - \frac{\lambda}{\lambda_n}\right)^{k_n}, \quad G_{W^\infty} (\lambda) := \prod_{\lambda_n \in W^\infty, n \in S_B} \left(1 - \frac{\lambda}{\lambda_n}\right)^{k_n^\infty} \]

and

\[ F_1 (\lambda) := \varphi (b, \lambda) - \tilde{\varphi} (b, \lambda), \quad F_2 (\lambda) := \varphi' (b, \lambda) - \tilde{\varphi}' (b, \lambda). \]

**Step 1:** This step is devoted to show that \( H_1 (\lambda) \) and \( H_2 (\lambda) \) are entire functions of \( \lambda \in \mathbb{C} \). We first prove that \( \frac{F_1 (\lambda)}{G_W (\lambda)} \) and \( \frac{F_2 (\lambda)}{G_W (\lambda)} \) are entire functions of \( \lambda \in \mathbb{C} \). In fact, from (1.7), (1.11), (1.13), (3.6), \( H = \tilde{H} \) and \( q = \tilde{q} \) a.e. on \([b, \pi] \), one can easily deduce that for \( \lambda_n \in \tilde{W}, n \in S_B, \)

\[ \varphi_\nu \left( x, \lambda_n \right) = \tilde{\varphi}_\nu \left( x, \lambda_n \right), \quad x \in [b, \pi], \]

where \( \nu = 0, 1, \ldots, k_n - 1 \). Thus for \( \lambda_n \in \tilde{W}, n \in S_B, \nu = 0, 1, \ldots, k_n - 1 \), one observes that

\[ \varphi_\nu (b, \lambda_n) = \tilde{\varphi}_\nu (b, \lambda_n), \quad \varphi'_\nu (b, \lambda_n) = \tilde{\varphi}'_\nu (b, \lambda_n), \]

and thus

\[ \frac{d^n F_1 (\lambda)}{d\lambda^n} \bigg|_{\lambda = \lambda_n} : = \nu ! (\varphi_\nu (b, \lambda_n) - \tilde{\varphi}_\nu (b, \lambda_n)) = 0, \]

\[ \frac{d^n F_2 (\lambda)}{d\lambda^n} \bigg|_{\lambda = \lambda_n} : = \nu ! (\varphi'_\nu (b, \lambda_n) - \tilde{\varphi}'_\nu (b, \lambda_n)) = 0. \]

Then in view of (3.9) and (3.10), we infer that \( \frac{F_1 (\lambda)}{G_W (\lambda)} \) and \( \frac{F_2 (\lambda)}{G_W (\lambda)} \) are entire functions of \( \lambda \in \mathbb{C} \). Similarly, we can also prove that \( \frac{F_1 (\lambda)}{G_{W^\infty} (\lambda)} \) and \( \frac{F_2 (\lambda)}{G_{W^\infty} (\lambda)} \) are entire functions of \( \lambda \in \mathbb{C} \). Therefore, from the fact \( \sigma (B) \cap \sigma (B^\infty) = \emptyset \) we conclude that \( H_1 (\lambda) \) and \( H_2 (\lambda) \) are entire functions of \( \lambda \in \mathbb{C} \). Furthermore, it is easy to see that the order of \( H_1 (\lambda) \) and \( H_2 (\lambda) \) are less than \( \frac{1}{2} \).

**Step 2:** Now we want to use Lemma 6 to prove \( H_1 (\lambda) \equiv 0 \). From Lemma 5 and the assumption (3.8), it follows that there exists a constant \( M > 0 \) such that

\[ |G_\Theta (iy)| \geq M |y|^\nu \exp \left( \left| \text{Im} \sqrt{iy} \right| b \right). \]

Moreover, from (2.6) we know that

\[ F_1 (\lambda) = \left( \left( b_1 - \tilde{b}_1 \right) \cos \left( \sqrt{\lambda} b \right) + \left( b_2 - \tilde{b}_2 \right) \cos \left( \sqrt{\lambda} (2d - b) \right) \right) + O \left( \exp \left( \left| \text{Im} \sqrt{\lambda} \right| b \right) \right), \]

and thus

\[ |F_1 (iy)| = \exp \left( \left| \text{Im} \sqrt{iy} \right| b \right) \left( \frac{|b_1 - \tilde{b}_1|}{2} + o (1) \right) \text{ as } y \text{ (real)} \to \infty. \]
Therefore, by (3.9), (3.14) and (3.15), one deduces that

\[ |H_1(iy)| \leq \frac{\exp\left(\text{Im}\sqrt{iy}\right) b \left(\frac{|h - b_1|}{2} + o(1)\right)}{M |y|^\gamma \exp\left(\text{Im}\sqrt{iy}\right) b} = O\left(y^{-\epsilon}\right), \]

as \( y \) (real) \( \to \infty \). By Lemma 6, one deduces that \( H_1(\lambda) \equiv 0 \) and therefore \( F_1(\lambda) \equiv 0 \) for all \( \lambda \in \mathbb{C} \), i.e., \( \varphi(b, \lambda) \equiv \tilde{\varphi}(b, \lambda) \).

**Step 3:** From the fact \( \varphi(b, \lambda) \equiv \tilde{\varphi}(b, \lambda) \), we know that

\[ H_2(\lambda) = \frac{[\varphi'(b, \lambda) - \tilde{\varphi}'(b, \lambda)] \varphi(b, \lambda)}{G_\Theta(\lambda) \varphi(b, \lambda)} = \frac{-F(\lambda)}{G_\Theta(\lambda) \varphi(b, \lambda)}. \]

Hence, from (2.10), (3.14) and Lemma 7, we have

\[ |H_2(iy)| \leq \frac{o \left( \exp \left(2 \left| \text{Im}\sqrt{iy}\right| b \right) \right)}{M |y|^\gamma \exp \left(\text{Im}\sqrt{iy}\right) b \exp \left(\text{Im}\sqrt{iy}\right) b \left(1 + o(1)\right)} \]

\[ = o\left(y^{-\epsilon}\right). \]

Then it follows from Lemma 6 that \( H_2(\lambda) = 0 \) and thus \( F(\lambda) \equiv 0 \) for all \( \lambda \in \mathbb{C} \). Now we can conclude from Lemma 1 that \( h = \bar{h}, \beta = \bar{\beta}, \gamma = \bar{\gamma} \) and \( q = \bar{q} \) a.e. on \([0, \pi]\). The proof is thus completed.

**Remark 5.** If \( b_1 = \frac{4 + \beta^{-1}}{2} \) is given, then it is easy to see from (3.15) that

\[ |F_1(iy)| = o \left( \exp \left(\text{Im}\sqrt{iy}\right) b \right) \text{ as } y \text{ (real)} \to \infty. \]

In this case the assumption (3.8) in Theorem 2 can be replaced by

\[ N_W(t) + N_{W'\infty}(t) \geq AN_{\sigma(B)}(t) + \left(\frac{b}{\pi} - A\right) N_{\sigma(B'\infty)}(t) - \frac{A}{2}. \]

### 3.2. Case II: \( q \) is known on \([b, \pi]\), where \( b = d \).

#### 3.2.1. Pairs of Eigenvalues and Normalizing Constants.

**Theorem 3.** Assume Hypothesis 1 and suppose that \( q = \bar{q} \) a.e. on \([b, \pi]\), and

\[ N_W(t) + N_{W_1}(t) + N_{W'\infty}(t) + N_{W'\infty}(t) \geq AN_{\sigma(B)}(t) + \left(\frac{2d}{\pi} - A\right) N_{\sigma(B'\infty)}(t) - \frac{A}{2} + \frac{1}{2} + \epsilon \]

for sufficiently large \( t \in \mathbb{R} \), where \( \epsilon \) is an arbitrary positive constant. Then \( h = \bar{h}, \beta = \bar{\beta}, \gamma = \bar{\gamma} \) and \( q = \bar{q} \) a.e. on \([0, \pi]\).

**Proof.** Let

\[ H(\lambda) := \frac{F(\lambda)}{G_\Xi(\lambda)}, \]

where \( G_\Xi(\lambda) \) is similarly defined as in (3.5) and \( F(\lambda) \) is defined by (2.1). By Lemma 3 we know that if \( b = d \),

\[ F(\lambda) = \langle \varphi(x, \lambda), \tilde{\varphi}(x, \lambda) \rangle_{x=d \pm 0} \]

\[ = \beta\overline{\beta}^{-1}\varphi(d - 0, \lambda)\varphi'(d - 0, \lambda) - \beta^{-1}\beta\varphi'(d - 0, \lambda)\varphi'(d - 0, \lambda) + \gamma\beta\varphi(d - 0, \lambda)\varphi(d - 0, \lambda) - \gamma\overline{\beta}\varphi(d - 0, \lambda)\varphi(d - 0, \lambda). \]
Moreover, from Lemma 4 it is easy to see that
\begin{equation}
|\varphi (d - 0, iy)| = \frac{1}{2} \exp \left( \left| \Im \sqrt{iy} \right| d \right) (1 + o(1)),
\end{equation}
\begin{equation}
|\varphi' (d - 0, iy)| = \frac{1}{2} |y|^{\frac{1}{2}} \exp \left( \left| \Im \sqrt{iy} \right| d \right) (1 + o(1))
\end{equation}
as \( y \) (real) \to \infty, and hence
\begin{equation}
|F(iy)| = O \left( |y|^{\frac{1}{2}} \exp \left( 2 \left| \Im \sqrt{iy} \right| d \right) \right) \quad \text{as } y \text{ (real) } \to \infty.
\end{equation}
By Lemma 5 and (3.16), we infer that there exists a constant \( M > 0 \) such that
\begin{equation}
|G_{\pm}(iy)| \geq M |y|^{\frac{1}{2} + \epsilon} \exp \left( 2 \left| \Im \sqrt{iy} \right| d \right).
\end{equation}
Therefore, from (3.17), (3.20) and (3.21), we have
\begin{equation}
|H(iy)| \leq \frac{O \left( |y|^{\frac{1}{2}} \exp \left( 2 \left| \Im \sqrt{iy} \right| d \right) \right)}{M |y|^{\frac{1}{2} + \epsilon} \exp \left( 2 \left| \Im \sqrt{iy} \right| d \right)} = O \left( |y|^{-\epsilon} \right)
\end{equation}
as \( y \) (real) \to \infty. This implies that \( H(\lambda) \equiv 0 \) and hence \( F(\lambda) \equiv 0 \) for all \( \lambda \in \mathbb{C} \) by the argument of the proof of Theorem 1. Then the statement of this theorem can be concluded from Lemma 1.

**Remark 6.** (1) If \( \beta = \bar{\beta} \), instead of condition (3.16), we only need the following condition:
\[ N_S(t) + N_{S_1}(t) + N_{S_2}(t) + N_S(t) \geq AN_{\sigma(B\sigma)}(t) + \left( \frac{2d}{\pi} - A \right) N_{\sigma(B\sigma)}(t) - \frac{A}{2} + \epsilon; \]
(2) If \( \beta = \bar{\beta}, \gamma = \bar{\gamma}, q, \bar{q} \in C^m \) near \( d \), then, instead of condition (3.16), we only need the following condition:
\[ N_S(t) + N_{S_1}(t) + N_{S_2}(t) + N_S(t) \geq AN_{\sigma(B\sigma)}(t) + \left( \frac{2d}{\pi} - A \right) N_{\sigma(B\sigma)}(t) - \frac{A}{2} - m + 1. \]
In fact, one notes that for \( x \in (0, d),
\begin{align}
\varphi (x, \lambda) \varphi'(x, \lambda) - \bar{\varphi}(x, \lambda) \varphi'(x, \lambda) &= y_{1,0}(x, \lambda)\bar{y}'_{1,0}(x, \lambda) - y'_{1,0}(x, \lambda)\bar{y}_{1,0}(x, \lambda) \\
&+ h \left( y_{2,0}(x, \lambda)\bar{y}'_{2,0}(x, \lambda) - y'_{2,0}(x, \lambda)\bar{y}_{2,0}(x, \lambda) \right) \\
&+ \bar{h} \left( y_{1,0}(x, \lambda)\bar{y}'_{2,0}(x, \lambda) - y'_{2,0}(x, \lambda)\bar{y}_{1,0}(x, \lambda) \right) \\
&+ h\bar{h} \left( y_{2,0}(x, \lambda)\bar{y}'_{2,0}(x, \lambda) - y'_{2,0}(x, \lambda)\bar{y}_{2,0}(x, \lambda) \right).
\end{align}
Therefore, if \( \beta = \bar{\beta} \), it follows from (3.18), (3.19) and Remark 11 that
\begin{equation}
|F(iy)| = O \left( \exp \left( 2 \left| \Im \sqrt{iy} \right| d \right) \right) \quad \text{as } y \text{ (real) } \to \infty.
\end{equation}
Moreover, if \( \beta = \bar{\beta}, \gamma = \bar{\gamma}, q, \bar{q} \in C^m \) near \( d \), it is easy to see from (3.18) and Proposition 1 that
\begin{equation}
|F(iy)| = o \left( |y|^{-\frac{m+1}{2}} \exp \left( 2 \left| \Im \sqrt{iy} \right| d \right) \right) \quad \text{as } y \text{ (real) } \to \infty.
\end{equation}
Thus by the argument of the proof of Theorem 3, Remark 6 can be directly obtained.

**Corollary 4.** Let \( d = \frac{3}{2} \). Assume that \( q \) is \( C^m \) near \( \frac{3}{2} \) and suppose that \( \beta, \gamma, q \) on \( \left[ \frac{3}{2}, \pi \right] \) are known a priori. Then \( h \) and \( \sigma \) on \( [0, \pi] \) can be uniquely determined by all the eigenvalues \( \{ \lambda_n \}_{n \in \mathbb{N}_0} \) of \( B \) except for \( \left( \frac{m+2}{2} \right) \), or all the eigenvalues \( \{ \lambda^\infty_n \}_{n \in \mathbb{N}_0} \) of \( B^\infty \) except for \( \left( \frac{m+1}{2} \right) \).

**Corollary 5.** Let \( d = \frac{3}{2} \). Assume that \( q \) on \( \left[ \frac{3}{2}, \pi \right] \) and \( \beta \) are known a priori, then \( \sigma(B) \) uniquely determines \( h, \gamma \) and \( q \) a.e. on \( [0, \pi] \).

### 3.2.2. Pairs of Eigenvalues and Ratios.

**Theorem 4.** Assume Hypothesis 2 and suppose that \( q = \tilde{q} \) a.e. on \( [d, \pi] \),

\[
N_W(t) + N_W^\infty(t) \geq AN_{\sigma(B)}(t) + \left( \frac{d}{\pi} - A \right) N_{\sigma(B^\infty)}(t) - \frac{A}{2} + \frac{1}{2} + \epsilon
\]

for sufficiently large \( t \in \mathbb{R} \), where \( \epsilon \) is an arbitrary positive constant. Then \( h = \tilde{h}, \beta = \tilde{\beta}, \gamma = \tilde{\gamma} \) and \( q = \tilde{q} \) a.e. on \( [0, \pi] \).

**Proof.** Denote

\[
H_1(\lambda) := \frac{F_1(\lambda)}{G_\Theta(\lambda)}, \quad H_2(\lambda) := \frac{F_2(\lambda)}{G_\Theta(\lambda)},
\]

where \( G_\Theta(\lambda) := G_W(\lambda) G_{W^\infty}(\lambda) \),

\[
G_W(\lambda) := \prod_{\lambda_n \in \mathbb{Z}^\infty, n \in \mathbb{N}_B} \left( 1 - \frac{\lambda}{\lambda_n} \right)^{k_n}, \quad G_{W^\infty}(\lambda) := \prod_{\lambda_n^\infty \in \mathbb{Z}^{\infty}, n \in \mathbb{N}_{B^\infty}} \left( 1 - \frac{\lambda}{\lambda_n^\infty} \right)^{k_n^\infty}
\]

and

\[
F_1(\lambda) := \varphi(d + 0, \lambda) - \tilde{\varphi}(d + 0, \lambda), \quad F_2(\lambda) := \varphi'(d + 0, \lambda) - \tilde{\varphi}'(d + 0, \lambda).
\]

In view of Lemma 5 and (3.25), one has

\[
|G_\Theta(iy)| \geq M |y|^\epsilon \exp \left( |\text{Im} \sqrt{iy}| \right) d.
\]

In addition, from (2.6) it is easy to see that

\[
|F_1(iy)| = \exp \left( |\text{Im} \sqrt{iy}| \right) \left( \frac{\beta - \tilde{\beta}}{2} + o(1) \right) \text{ as } y \text{ (real)} \to \infty.
\]

Thus it follows from (3.27) and (3.28) that

\[
|H_1(iy)| \leq \left| \exp \left( |\text{Im} \sqrt{iy}| \right) \left( \frac{\beta - \tilde{\beta}}{2} + o(1) \right) \right|
\]

\[
= O \left( y^{-\epsilon - \frac{1}{2}} \right) \text{ as } y \text{ (real)} \to \infty.
\]

By a similar proof to that of Theorem 2, we can obtain that \( H_1(\lambda) \equiv 0 \), and thus \( F_1(\lambda) \equiv 0 \), i.e., \( \varphi(d + 0, \lambda) \equiv \tilde{\varphi}(d + 0, \lambda) \) for all \( \lambda \in \mathbb{C} \). Then it follows from (3.18) and (3.26) that

\[
H_2(\lambda) = \frac{F_2(\lambda)}{G_\Theta(\lambda) \varphi(d + 0, \lambda)} = \frac{-E(\lambda)}{G_\Theta(\lambda) \varphi(d + 0, \lambda)}.
\]
Thus by (2.11), (3.20) and (3.27), we infer that as \( y \) (real) \( \to \infty \),

\[
|H_2 (iy)| \leq \frac{O \left( |y|^\frac{3}{2} \exp \left( 2 |\text{Im} iy| \right) \right)}{M |y|^{\epsilon + \frac{3}{2}} \exp \left( |\text{Im} iy| \right) \frac{\beta}{2} \exp \left( \frac{1}{2} |\text{Im} iy| (1 + o (1)) \right)}
\]

\[
= O \left( |y|^{-\epsilon} \right).
\]

Then by the argument of the proof of Theorem 2, we can obtain that \( F (\lambda) \equiv 0 \). Now we conclude from Lemma 1 that \( h = \bar{h}, \beta = \bar{\beta}, \gamma = \bar{\gamma} \) and \( q = \bar{q} \) a.e. on \([0, \pi]\).

**Remark 7.** (1) If \( \beta \) is known a priori, then by (3.23) and (3.28) one has

\[
|F (iy)| = O \left( \exp \left( 2 |\text{Im} iy| \right) \right)
\]

and \( F_1 (iy) = o \left( \exp \left( 2 |\text{Im} iy| \right) \right) \)\) as \( y \) (real) \( \to \infty \). In this case, the assumption (3.25) can be replaced by

\[
N_W (t) + N_{W_\infty} (t) \geq AN_{\sigma (B)} (t) + \left( \frac{d}{\pi} - A \right) N_{\sigma (B_\infty)} (t) - \frac{A}{2} + \epsilon.
\]

(2) If \( \beta \) and \( \gamma \) are known a priori, by (3.24) (for \( m = -1 \)) and (3.28) one has

\[
|F (iy)| = o \left( \exp \left( 2 |\text{Im} iy| \right) \right)
\]

and \( F_1 (iy) = o \left( \exp \left( 2 |\text{Im} iy| \right) \right) \) as \( y \) (real) \( \to \infty \). In this case, the assumption (3.25) can be replaced by

\[
N_W (t) + N_{W_\infty} (t) \geq AN_{\sigma (B)} (t) + \left( \frac{d}{\pi} - A \right) N_{\sigma (B_\infty)} (t) - \frac{A}{2}.
\]

### 3.3. Case III: \( q \) is known on \([b, \pi]\), where \( b \in (0, d)\).

#### 3.3.1. Pairs of Eigenvalues and Normalizing Constants.

**Theorem 5.** Assume Hypothesis 1 and suppose that \( q, \bar{q} \in C^m \) near \( b \in (0, d) \), \( q = \bar{q} \) a.e. on \([b, \pi]\), \( \beta = \bar{\beta}, \gamma = \bar{\gamma} \) and

\[
N_W (t) + N_{W_1} (t) + N_{W_\infty} (t) + N_{W_1} (t)
\]

\[
\geq AN_{\sigma (B)} (t) + \left( \frac{2b}{\pi} - A \right) N_{\sigma (B_\infty)} (t) - \frac{A}{2} - \frac{m + 1}{2}
\]

for sufficiently large \( t \in \mathbb{R} \). Then \( h = \bar{h} \) and \( q = \bar{q} \) a.e. on \([0, \pi]\).

**Proof.** Denote

\[
H (\lambda) := \frac{F (\lambda)}{G_\Xi (\lambda)},
\]

where \( G_\Xi (\lambda) \) is similarly defined as in (3.5) and \( F (\lambda) \) is defined by (2.1). Then it follows from Lemma 3 that if \( \beta = \bar{\beta}, \gamma = \bar{\gamma}, \)

\[
F (\lambda) = \langle \varphi (x, \lambda), \bar{\varphi} (x, \lambda) \rangle_{x=b}.
\]

In addition, if \( q = \bar{q} \) a.e. on \([b, \pi]\), and \( q, \bar{q} \in C^m \) near \( b \in (0, d) \), one observes from (3.22) and Proposition 1 that

\[
|F (iy)| = o \left( |y|^{-\frac{m+1}{2}} \exp \left( 2 |\text{Im} iy| b \right) \right) \text{ as } y \text{ (real) } \to \infty.
\]

By Lemma 5 and (3.29), we have

\[
|G_\Xi (iy)| \geq M |y|^{-\frac{m+1}{2}} \exp \left( 2 |\text{Im} iy| b \right).
\]
Therefore,
\[
|H_2 (iy)| \leq \frac{o \left( |y|^{-\frac{m+1}{2}} \exp \left( 2 |\text{Im}\sqrt{iy}| b \right) \right)}{M |y|^{-\frac{m}{2}} \exp \left( 2 |\text{Im}\sqrt{iy}| b \right)} = o(1) \text{ as } y \text{ (real)} \to \infty.
\]

This implies that \( H (\lambda) \equiv 0 \) and thus \( F (\lambda) \equiv 0 \) for all \( \lambda \in \mathbb{C} \) by the argument of the proof of Theorem 1. Then we conclude the statement of this theorem from Lemma 1. \qed

3.3.2. Pairs of Eigenvalues and Ratios.

**Theorem 6.** Assume Hypothesis 2 and suppose that \( q = \tilde{q} \text{ a.e. on } [d, \pi], \beta = \tilde{\beta}, \gamma = \tilde{\gamma} \) and

\[
N_W(t) + N_{W^\infty}(t) \geq AN_{\sigma(B)}(t) + \left( \frac{b}{\pi} - A \right) N_{\sigma(B^\infty)}(t) - \frac{A}{2}
\]

\label{equation3.32}

for sufficiently large \( t \in \mathbb{R} \), where \( \epsilon \) is an arbitrary positive constant. Then \( h = \tilde{h} \) and \( q = \tilde{q} \text{ a.e. on } [0, \pi] \).

**Proof.** Denote

\[
H_1 (\lambda) := \frac{F_1 (\lambda)}{G_\Theta (\lambda)}, \quad H_2 (\lambda) = \frac{F_2 (\lambda)}{G_\Theta (\lambda)},
\]

where \( G_\Theta (\lambda) := G_W (\lambda) G_{W^\infty} (\lambda) \),

\[
G_W (\lambda) := \prod_{\lambda_n \in \mathbb{W}, n \in S_B} \left( 1 - \frac{\lambda}{\lambda_n} \right)^{k_n}, \quad G_{W^\infty} (\lambda) := \prod_{\lambda_n \in \mathbb{W}^\infty, n \in S_{\beta^\infty}} \left( 1 - \frac{\lambda}{\lambda_n} \right)^{k_n},
\]

and

\[
F_1 (\lambda) := \varphi (b, \lambda) - \tilde{\varphi} (b, \lambda), \quad F_2 (\lambda) := \varphi' (b, \lambda) - \tilde{\varphi'} (b, \lambda).
\]

By a similar method to that of Theorem 2, one can easily deduce that \( H_1 (\lambda) \) and \( H_2 (\lambda) \) are entire functions of order less than \( \frac{1}{2} \) from the facts (1.7), (1.11), (1.13), (3.6), \( H = \tilde{H}, \beta = \tilde{\beta}, \gamma = \tilde{\gamma} \) and \( q = \tilde{q} \text{ a.e. on } [b, \pi] \).

In view of Lemma 5 and (3.32), one has

\[
|G_\Theta (iy)| \geq M \exp \left( |\text{Im}\sqrt{iy}| b \right).
\]

\label{equation3.34}

By (2.6) we also infer that

\[
|F_1 (iy)| = O \left( |y|^{-\frac{1}{2}} \exp \left( |\text{Im}\sqrt{iy}| b \right) \right) \text{ as } y \text{ (real)} \to \infty.
\]

Therefore,

\[
|H_1 (iy)| \leq \frac{O \left( |y|^{-\frac{1}{2}} \exp \left( |\text{Im}\sqrt{iy}| b \right) \right)}{M \exp \left( |\text{Im}\sqrt{iy}| b \right)} = O \left( y^{-\frac{1}{2}} \right)
\]

as \( y \text{ (real)} \to \infty \). Now by Lemma 6, we can obtain that \( H_1 (\lambda) \equiv 0 \), i.e., \( \varphi (b, \lambda) = \tilde{\varphi} (b, \lambda) \) for all \( \lambda \in \mathbb{C} \). Then it follows from (3.30) and (3.33) that

\[
H_2 (\lambda) = \frac{[\varphi' (b, \lambda) - \tilde{\varphi'} (b, \lambda)] \varphi (b, \lambda)}{G_\Theta (\lambda) \varphi (b, \lambda)} = -\frac{F (\lambda)}{G_\Theta (\lambda) \varphi (b, \lambda)}.
\]
Thus by (2.10), (3.31) (for \(m = -1\)) and (3.34), we have
\[
|H_2 (iy)| \leq \frac{o \left( \exp \left(2 \left| \text{Im} \sqrt{iy} \right| \right) \right) }{M \exp \left( \left| \text{Im} \sqrt{iy} \right| b \right) - \frac{1}{2} \exp \left( \left| \text{Im} \sqrt{iy} \right| b \right) (1 + o (1))} = o (1).
\]
Then by Lemma 6 we infer that \(H_2 (\lambda) \equiv 0\) and then \(F (\lambda) \equiv 0\) for all \(\lambda \in \mathbb{C}\). Now we can conclude from Lemma 1 that \(h = \overline{h}\) and \(q = \overline{q}\) a.e. on \([0, \pi]\). The proof is thus completed.

**APPENDIX**

For the self-adjoint classical Sturm-Liouville operators, an interesting uniqueness result is to assume that the potential \(q\) satisfies a local smoothness condition so that some eigenvalues and norming constants can be missing. While in [27–29] the key technique relies on the high-energy asymptotic expansion of the Weyl \(m\)-function [30], in our non-self-adjoint setting, the key to prove the uniqueness problems (Theorem 1, Theorem 5, Remark 6, Corollary 1–4) will be Proposition 1, to be established below.

**Definition 1.** For \(i = 1, 2\), let \(y_{1,i}(x, \lambda)\) and \(\tilde{y}_{1,i}(x, \lambda)\) be solutions of (1.1) corresponding to the potential \(q\) and \(\tilde{q}\), respectively, where \(y_{1,i}(x, \lambda)\) and \(\tilde{y}_{1,i}(x, \lambda)\) satisfy the initial conditions
\[
\begin{align*}
y_{1,1}(r, \lambda) &= y_{1,1}'(r, \lambda) = 1, \quad y_{2,1}(r, \lambda) = y_{1,1}'(r, \lambda) = 0, \\
\tilde{y}_{1,1}(r, \lambda) &= \tilde{y}_{1,1}'(r, \lambda) = 1, \quad \tilde{y}_{2,1}(r, \lambda) = \tilde{y}_{1,1}'(r, \lambda) = 0, \quad r \in [0, \pi).
\end{align*}
\]
For simplicity, denote \(y_1(x, \lambda) := y_{1,1}(x, \lambda), \quad y_2(x, \lambda) := y_{2,1}(x, \lambda), \quad \tilde{y}_1(x, \lambda) := \tilde{y}_{1,1}(x, \lambda), \quad \tilde{y}_2(x, \lambda) := \tilde{y}_{2,1}(x, \lambda).

**Proposition 1.** Let \(x_0 \in (r, \pi]\) where \(r \in [0, \pi)\) and assume that \(q, \tilde{q} \in C_m [x_0 - \delta, x_0]\) for some sufficiently small \(\delta > 0\) and some \(m \in \mathbb{N}_0\). If \(q^{(j)} (x_0) = q^{(j)} (x_0)\) for \(j = 0, 1, \ldots, m, \) then
\begin{align}
y_{1,i}(x_0, \lambda)\tilde{y}_{1,i}(x_0, \lambda) - y_{1,i}'(x_0, \lambda)\tilde{y}_{1,i}(x_0, \lambda) &= o \left( \frac{\exp \left(2 \left| \text{Im} \sqrt{\lambda} \right| (x_0 - r) \right)}{\sqrt{\lambda}^{m+1}} \right), \\
y_{1,i}(x_0, \lambda)\tilde{y}_{2,i}(x_0, \lambda) - y_{1,i}'(x_0, \lambda)\tilde{y}_{2,i}(x_0, \lambda) &= o \left( \frac{\exp \left(2 \left| \text{Im} \sqrt{\lambda} \right| (x_0 - r) \right)}{\sqrt{\lambda}^{m+2}} \right), \\
\tilde{y}_{1,i}(x_0, \lambda)y_{2,i}(x_0, \lambda) - \tilde{y}_{1,i}(x_0, \lambda)y_{2,i}(x_0, \lambda) &= o \left( \frac{\exp \left(2 \left| \text{Im} \sqrt{\lambda} \right| (x_0 - r) \right)}{\sqrt{\lambda}^{m+2}} \right), \\
y_{2,i}(x_0, \lambda)\tilde{y}_{2,i}(x_0, \lambda) - y_{2,i}'(x_0, \lambda)\tilde{y}_{2,i}(x_0, \lambda) &= o \left( \frac{\exp \left(2 \left| \text{Im} \sqrt{\lambda} \right| (x_0 - r) \right)}{\sqrt{\lambda}^{m+3}} \right).
\end{align}
as \(|\lambda| \to \infty\) in \(\Lambda_\zeta := \{ \lambda \in \mathbb{C} : \zeta < \text{Arg}(\lambda) < \pi - \zeta\ \text{for} \ \zeta > 0\}\).

**Remark 8.** For \(f \in C^m [x_0 - \delta, x_0]\), we adopt following notations in this section:

\[
f_{-}^{(0)}(x_0) : = f(x_0), \quad f_{-}^{(1)}(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},
\]

\[
f_{-}^{(j)}(x_0) : = \lim_{x \to x_0} \frac{f_{-}^{(j-1)}(x) - f_{-}^{(j-1)}(x_0)}{x - x_0} \quad \text{for} \ j = 2, 3, \ldots, m.
\]

In addition, \(f \in C^m [x_0 - \delta, x_0]\) implies \(\lim_{x \to x_0} f^{(j)}(x) = f^{(j)}(x_0)\) for \(j = 0, 1, \ldots, m\).

The proof of Proposition 1 will be given at the end of this appendix after the proof of the following lemma.

**Lemma 8.** Let \(x_0 \in (0, \pi]\) and \(q, \bar{q} \in C^m [0, x_0]\) for some \(m \in \mathbb{N}_0\). If

\[
q_{-}^{(j)}(x_0) = \bar{q}_{-}^{(j)}(x_0)
\]

for \(j = 0, 1, \ldots, m\), then

\[
y_2(x_0, \lambda)\bar{y}_2(x_0, \lambda) - y_2(x_0, \lambda)\bar{y}_2(x_0, \lambda) = o \left( \frac{\exp \left( 2 \frac{\text{Im} \sqrt{\lambda}}{\sqrt{\lambda}} x_0 \right)}{\sqrt{\lambda}^{m+3}} \right),
\]

\[
y_1(x_0, \lambda)\bar{y}_1(x_0, \lambda) - y_1(x_0, \lambda)\bar{y}_1(x_0, \lambda) = o \left( \frac{\exp \left( 2 \frac{\text{Im} \sqrt{\lambda}}{\sqrt{\lambda}} x_0 \right)}{\sqrt{\lambda}^{m+1}} \right),
\]

\[
y_1(x_0, \lambda)\bar{y}_2(x_0, \lambda) - y_1(x_0, \lambda)\bar{y}_2(x_0, \lambda) = o \left( \frac{\exp \left( 2 \frac{\text{Im} \sqrt{\lambda}}{\sqrt{\lambda}} x_0 \right)}{\sqrt{\lambda}^{m+2}} \right),
\]

\[
\bar{y}_1(x_0, \lambda)y_2(x_0, \lambda) - \bar{y}_1(x_0, \lambda)y_2(x_0, \lambda) = o \left( \frac{\exp \left( 2 \frac{\text{Im} \sqrt{\lambda}}{\sqrt{\lambda}} x_0 \right)}{\sqrt{\lambda}^{m+2}} \right)
\]

as \(|\lambda| \to \infty\) in the sector \(\Lambda_\zeta\).

We shall prove Lemma 8 by analyzing the asymptotic expansion of the fundamental solutions (see Lemma 9 and Lemma 10). Now we first give some preliminary facts and notations.

Recall the solution \(y_2\) defined by Definition 1, then it follows from [33] that

\[
y_2(x, \lambda) = \sum_{p=0}^{\infty} S_p(x, \lambda), \quad y_2'(x, \lambda) = \sum_{p=0}^{\infty} C_p(x, \lambda),
\]

where \(S_0(x, \lambda) = \frac{\sin(\sqrt{\lambda} x)}{\sqrt{\lambda}},\ C_0(x, \lambda) = \cos \left( \sqrt{\lambda} x \right),\) and for \(p \geq 1,\)

\[
S_p(x, \lambda) = \int_0^x \frac{\sin(\sqrt{\lambda} (x - t))}{\sqrt{\lambda}} q(t) S_{p-1}(t, \lambda) dt,
\]

\[
C_p(x, \lambda) = \int_0^x \cos \left( \sqrt{\lambda} (x - t) \right) q(t) S_{p-1}(t, \lambda) dt.
\]
Moreover and where for \( (3.48) \)

Assume that Lemma 9.

In what follows, we adopt the following notations:

\[
B \quad f \quad f \\
(\begin{pmatrix}
p, p, j \\ p, j \\
p, j \
\end{pmatrix} = 3 \quad (\begin{pmatrix}
-1 & \text{if } j = 4s, 4s + 1, \\
1 & \text{if } j = 4s + 2, 4s + 3, 
\end{pmatrix}
\]

and

\[
\nu_{2s}(x, \lambda) := \frac{\sin(\sqrt{\lambda x})}{(2\sqrt{\lambda})^{2s}}, \quad \nu_{2s+1}(x, \lambda) := \frac{\cos(\sqrt{\lambda x})}{(2\sqrt{\lambda})^{2s+1}}, \quad s \in \mathbb{N}_0.
\]

Then we have the following statement relating to \( S_p \) defined by (3.45).

**Lemma 9.** Assume that \( q \in C^m [0, \delta] \) for some \( \delta > 0 \) and some \( m \in \mathbb{N} \). Denote \( \sigma (x) := \int_0^x q(t)dt \). Then for \( x \in [0, \delta] \), we have

\[
(3.47) \quad S_1(x, \lambda) = \sum_{j=1}^{m+1} \frac{\nu_j(x, \lambda)}{\sqrt{\lambda}} f_{1,j}(x) + (\pm) \int_0^x \nu_{m+1}(x-2t, \lambda) q^{(m)}(t) dt,
\]

\[
(3.48) \quad S_2(x, \lambda) = \sum_{j=1}^{m+2} \frac{\nu_j(x, \lambda)}{\sqrt{\lambda}} f_{2,j}(x) + B_2(x, \lambda),
\]

\[
(3.49) \quad S_p(x, \lambda) = \sum_{j=1}^{m+2} \frac{\nu_j(x, \lambda)}{\sqrt{\lambda}} f_{p,j}(x) + B_p(x, \lambda) \quad \text{for } p = 3, \ldots, m + 2
\]

where

\[
B_2(x, \lambda) = -\frac{(\pm)_{m+3}}{\sqrt{\lambda}} \int_0^x \nu_{m+2}(x-2t, \lambda) \left( \sum_{j=1}^{m+1} (\pm)_j (q(t) f_{1,j}(t))^{(m+1-j)} \right) dt
\]

\[
+ \frac{(\pm)_{m+2}}{\sqrt{\lambda}} \int_0^x \sin(\sqrt{\lambda}(x-t)) q(t) \int_0^t \nu_{m+1}(t-2s, \lambda) q^{(m)}(s) ds dt,
\]

\[
B_p(x, \lambda) = -\frac{(\pm)_{m+3}}{\sqrt{\lambda}} \int_0^x \nu_{m+2}(x-2t, \lambda) \left( \sum_{j=1}^{m+1} (\pm)_j (q(t) f_{p-1,j}(t))^{(m+1-j)} \right) (t) dt
\]

\[
+ \int_0^x \sin(\sqrt{\lambda}(x-t)) q(t) \left[ \frac{\nu_{m+2}(t, \lambda)}{\sqrt{\lambda}} f_{p-1, m+2}(t) + B_{p-1}(t, \lambda) \right] dt
\]

for \( p = 3, \ldots, m + 2 \), and the functions \( f_{p,j}(x) \) are defined by the recurrence relations

\[
f_{1,j}(x) = (\pm)_{j} \left( \sigma^{(j-1)}(x) - (-1)^{j-1} \sigma^{(j-1)}(0) \right),
\]

\[
f_{p,p}(x) = (-1)^p \int_0^x q(t) f_{p-1,p-1}(t) dt \quad \text{for } p = 2, \ldots, m + 2,
\]

\[
f_{p,j}(x) = \sum_{s=1}^{j-2} (\pm)_s (\pm)_j \left( q f_{p-1,s}(j-s-2) \right) (x) - (-1)^{j-1} \left( q f_{p-1,s}(j-s-2) \right) (0)
\]

\[
+ (-1)^j \int_0^x q(t) f_{p-1,j-1}(t) dt \quad \text{for } j > p \quad \text{and } p = 2, \ldots, m + 2,
\]

\[
f_{p,j}(x) = 0 \quad \text{for } j < p.
\]

Moreover, \( f_{p,j} \in C^{m+p-j+1} [0, \delta] \).
Therefore, by virtue of (3.50), we first note that

\begin{equation}
\frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}} \nu_j(t, \lambda) = (-1)^{j+1} \nu_{j+1}(x, \lambda) + \nu_{j+1}(x-2t, \lambda)
\end{equation}

and for $f \in C^1[0, x]$,

\begin{equation}
\int_0^x \nu_j(x-2t, \lambda) f(t) \, dt
\quad = \quad \nu_{j+1}(x, \lambda) \left( f(x) - (-1)^j f(0) \right) + (-1)^{j+1} \int_0^x \nu_{j+1}(x-2t, \lambda) f'(t) \, dt.
\end{equation}

In view of (3.50) and (3.51), one can easily deduce the expression (3.47). Now we turn to deduce the expressions for the other functions $S_j$. Suppose that $f_j \in C^{m+1-j}[0, x]$, then from (3.50), we know that for $j = 1, \ldots, m+1$,

\begin{equation}
\int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}} \nu_j(t, \lambda) f_j(t) \, dt
\quad = \quad (-1)^{j+1} \nu_{j+1}(x, \lambda) \int_0^x f_j(t) \, dt + \int_0^x \nu_{j+1}(x-2t, \lambda) f_j(t) \, dt,
\end{equation}

Moreover, integrating by parts the first summand on the right-hand side of the above equality $m+1-j$ times and using (3.50), it follows that for $j = 1, \ldots, m$,

\begin{equation}
\int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}} \nu_j(t, \lambda) f_j(t) \, dt
\quad = \quad (-1)^{j+1} \nu_{j+1}(x, \lambda) \int_0^x f_j(t) \, dt - \sum_{s=j+2}^{m+2} \nu_s(x, \lambda) (\pm)_j (\pm)_s (f^{(s-j-2)}_s)(x)
\quad - (\pm)_j (\pm)_{m+3} \int_0^x \nu_{m+2}(x-2t, \lambda) f^{(m+1-j)}_j(t) \, dt.
\end{equation}

Therefore, by virtue of (3.52) (for $j = m+1$) and (3.53), for $x \in (0, \delta]$ we have that

\begin{equation}
\sum_{j=1}^{m+1} \int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}} \nu_j(t, \lambda) f_j(t) \, dt
\quad = \quad (-1)^2 \nu_2(x, \lambda) \int_0^x f_1(t) \, dt + \sum_{j=3}^{m+2} \nu_j(x, \lambda) \left( - \sum_{s=1}^{j-2} (\pm)_s (\pm)_j (f^{(j-s-2)}_s)(x) - (-1)^{j-1} f^{(j-s-2)}_s(0) \right)
\quad + \int_0^x f_{j-1}(t) \, dt - (\pm)_{m+3} \int_0^x \nu_{m+2}(x-2t, \lambda) \sum_{j=1}^{m+1} (\pm)_j f^{(m+1-j)}_j(t) \, dt.
\end{equation}
Now in view of (3.45) and (3.47), we obtain that for \( x \in (0, \delta] \),
\[
S_2(x, \lambda) = \frac{1}{\sqrt{\lambda}} \sum_{j=1}^{m+1} \int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}} \nu_j(t, \lambda)q(t)f_{1,j}(t) \, dt \\
+ \left( \pm \right)_{m+2} \int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}} q(t) \int_0^t \nu_{m+1}(t-2s, \lambda)q^{(m)}(s) ds dt.
\]

Making use of (3.54) with \( f_j(t) \) replaced by \( q(t)f_{1,j}(t) \) and in virtue of the fact \( qf_{1,j} \in C^{m+1-j}[0, \delta] \) for \( j = 1, \ldots, m+1 \), we obtain (3.48). Next, from (3.45) and (3.48), it follows that for \( x \in (0, \delta] \),
\[
S_3(x, \lambda) = \frac{1}{\sqrt{\lambda}} \sum_{j=1}^{m+1} \int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}} \nu_j(t, \lambda)q(t)f_{2,j}(t) \, dt \\
+ \int_0^x \frac{\sin(\sqrt{\lambda}(x-t))}{\sqrt{\lambda}} q(t) \left[ \frac{\nu_{m+2}(t, \lambda)}{\sqrt{\lambda}} f_{2,m+2}(t) + B_2(t, \lambda) \right] dt.
\]

Then the expression (3.49) for \( S_3 \) can be proved by using (3.54) and letting \( f_j(t) := q(t)f_{2,j}(t) \). The proof of the relation (3.49) for \( p = 4, \ldots, m+2 \) can be carried out in the same way.

As a consequence of Lemma 9, we have the following assertion relating to \( C_p \) defined by (3.46).

**Lemma 10.** Assume that \( q \in C^m[0, \delta] \) for some \( \delta > 0 \) and some \( m \in \mathbb{N} \). Denote \( \sigma(x) := \int_0^x q(t) dt \). Then for \( x \in [0, \delta] \), we have
\[
C_1(x, \lambda) = \frac{-f_{1,1}(x)\nu_0(x, \lambda)}{2\sqrt{\lambda}} + \sum_{j=1}^{m} \frac{\nu_j(x, \lambda)}{\sqrt{\lambda}} \left[ f_{1,j}^{'}(x) + \frac{(-1)^{j+1}f_{1,j+1}(x)}{2} \right] \\
+ \left( \pm \right)_{m+2} \int_0^x \frac{d\nu_{m+1}(x-2t, \lambda)}{dx} q^{(m)}(t) dt,
\]
\[
C_2(x, \lambda) = \sum_{j=1}^{m+1} \frac{\nu_j(x, \lambda)}{\sqrt{\lambda}} \left[ f_{2,j}^{'}(x) + \frac{(-1)^{j+1}f_{2,j+1}(x)}{2} \right] + D_2(x, \lambda),
\]
\[
C_p(x, \lambda) = \sum_{j=1}^{m+1} \frac{\nu_j(x, \lambda)}{\sqrt{\lambda}} \left[ f_{p,j}^{'}(x) + \frac{(-1)^{j+1}f_{p,j+1}(x)}{2} \right] + D_p(x, \lambda) \text{ for } p = 3, \ldots, m+2
\]

where \( f_{p,j}(x) \) are the functions defined in Lemma 9, and
\[
D_2(x, \lambda) = \left( \pm \right)_{m+3} \int_0^x \frac{d\nu_{m+2}(x-2t, \lambda)}{dx} \left( \sum_{j=1}^{m+1} \frac{\nu_j(x, \lambda)}{\sqrt{\lambda}} q(t)f_{1,j}(t) \right) dt \\
+ \left( \pm \right)_{m+2} \int_0^x \frac{d\nu_{m+2}(x-2t, \lambda)}{dx} \sum_{j=1}^{m+1} \frac{\nu_j(x, \lambda)}{\sqrt{\lambda}} q(t)f_{1,j}(t) \left[ \frac{\nu_{m+2}(t, \lambda)}{\sqrt{\lambda}} f_{2,m+2}(t) + B_2(t, \lambda) \right] dt.
\]

for \( p = 3, \ldots, m+2 \).
Lemma 11. Assume that \( q \in C^m [0, \delta] \) for some \( \delta > 0 \) and some \( m \in \mathbb{N}_0 \). Then for \( x \in [0, \delta] \), \( y_2(x, \lambda) \) and \( y'_2(x, \lambda) \) can be rewritten as the following form:

\[
y_2(x, \lambda) = \sin \left( \frac{\sqrt{\lambda}x}{\sqrt{\lambda}} \right) + \sum_{j=1}^{m+2} a_j (x) \frac{\nu_j(x, \lambda)}{\sqrt{\lambda}} + \frac{(\pm)^{m+2}}{\sqrt{\lambda}} \int_0^x \nu_{m+1}(x-2t, \lambda)q^{(m)}(t)dt + \sum_{p=2}^{m+2} B_p(x, \lambda) + \sum_{p=m+3}^{\infty} S_p(x, \lambda),
\]

and

\[
y'_2(x, \lambda) = \cos \left( \frac{\sqrt{\lambda}x}{\sqrt{\lambda}} \right) + \sum_{j=0}^{m+1} b_j (x) \frac{\nu_j(x, \lambda)}{\sqrt{\lambda}} + \frac{(\pm)^{m+2}}{\sqrt{\lambda}} \int_0^x \frac{d\nu_{m+1}(x-2t, \lambda)}{dx}q^{(m)}(t)dt + \sum_{p=2}^{m+2} D_p(x, \lambda) + \sum_{p=m+3}^{\infty} C_p(x, \lambda),
\]

where

\[
a_j (x) = \sum_{p=1}^{m+2} f_{p,j} (x) \quad \text{for } j = 1, \ldots, m+1, \quad a_{m+2} (x) = \sum_{p=2}^{m+2} f_{p,m+2} (x),
\]

and

\[
b_0 (x) = \frac{-f_{1,1} (x)}{2}, \quad b_{m+1} (x) = \sum_{p=2}^{m+2} \left( f'_{p,m+1} (x) + \frac{(-1)^{m+2}f_{p,m+2} (x)}{2} \right),
\]

\[
b_j (x) = \sum_{p=1}^{m+2} \left( f'_{p,j} (x) + \frac{(-1)^{j+1}f_{p,j+1} (x)}{2} \right), \quad j = 1, 2, \ldots, m.
\]

Proof. For \( m \in \mathbb{N} \), the expressions (3.55) and (3.56) can be directly obtained from (3.44), Lemma 9 and Lemma 10. For \( m = 0 \), the proof can be carried out in the same way even simpler. \( \square \)

Remark 9. For \( g \in L^1 [0, x] \), one notes that the following identities

\[
\int_0^x \sin \left( \sqrt{\lambda}t \right) g(t)dt = o \left( \exp \left( |Im\sqrt{\lambda}| x \right) \right),
\]

\[
\int_0^x \cos \left( \sqrt{\lambda}t \right) g(t)dt = o \left( \exp \left( |Im\sqrt{\lambda}| x \right) \right)
\]

hold [33]. By virtue of (3.57) and (3.58), it is easy to deduce that

\[
B_p(x, \lambda) = o \left( \frac{\exp \left( |Im\sqrt{\lambda}| x \right)}{|\sqrt{\lambda}|^{m+3}} \right), \quad B'_p(x, \lambda) = o \left( \frac{\exp \left( |Im\sqrt{\lambda}| x \right)}{|\sqrt{\lambda}|^{m+2}} \right).
\]
and

\[ D_p(x, \lambda) = o \left( \frac{\exp \left( \frac{|Im\sqrt{\lambda}|}{x} \right)}{|\sqrt{\lambda}|^{m+2}} \right), \quad D'_p(x, \lambda) = o \left( \frac{\exp \left( \frac{|Im\sqrt{\lambda}|}{x} \right)}{|\sqrt{\lambda}|^{m+1}} \right) \]

hold for \( p = 2, 3, \ldots, m + 2 \).

**Remark 10.** Note that 

\[ S_\lambda(x, \lambda) = \int_{t_0 \leq t_1 \leq \ldots \leq t_{p+1} = x} \prod_{i=1}^{p} s_\lambda(t_{i+1} - t_i) s_\lambda(t_1) q(t_i) dt_1 \cdots dt_p, \]

\[ C_\lambda(x, \lambda) = \int_{t_0 \leq t_1 \leq \ldots \leq t_{p+1} = x} c_\lambda(t_{p+1} - t_p) \prod_{i=1}^{p-1} s_\lambda(t_{i+1} - t_i) s_\lambda(t_1) q(t_i) dt_1 \cdots dt_p, \]

where \( s_\lambda(x) := \frac{\sin(\sqrt{\lambda} x)}{\sqrt{\lambda}}, c_\lambda(x) := \cos(\sqrt{\lambda} x), \) and

\[ |\sin(\sqrt{\lambda} x)| \leq \frac{\exp \left( \frac{|Im\sqrt{\lambda}| x}{\sqrt{\lambda}} \right)}{|\sqrt{\lambda}|^{m+4}}, \quad |\cos(\sqrt{\lambda} x)| \leq \exp \left( \frac{|Im\sqrt{\lambda}| x}{\sqrt{\lambda}} \right). \]

Thus for \( \lambda \in \mathbb{C} \) and \(|\lambda|\) being large enough, one has

\[ |S_\lambda(x, \lambda)| \leq \frac{\exp \left( \frac{|Im\sqrt{\lambda}| x}{\sqrt{\lambda}} \right) \left( \int_0^x |q(t)| dt \right)^p}{|\sqrt{\lambda}|^{m+4}}, \quad p \geq m + 3, \]

\[ |C_\lambda(x, \lambda)| \leq \frac{\exp \left( \frac{|Im\sqrt{\lambda}| x}{\sqrt{\lambda}} \right) \left( \int_0^x |q(t)| dt \right)^p}{|\sqrt{\lambda}|^{m+3}}, \quad p \geq m + 3. \]

This directly yields that as \(|\lambda| \to \infty, \)

\[ \sum_{p=m+3}^{\infty} S_\lambda(x, \lambda) = O \left( \frac{\exp \left( \frac{|Im\sqrt{\lambda}| x}{\sqrt{\lambda}} \right)}{|\sqrt{\lambda}|^{m+4}} \right), \quad \sum_{p=m+3}^{\infty} C_\lambda(x, \lambda) = O \left( \frac{\exp \left( \frac{|Im\sqrt{\lambda}| x}{\sqrt{\lambda}} \right)}{|\sqrt{\lambda}|^{m+3}} \right). \]

Similarly, one can also obtain that \( \sum_{p=m+3}^{\infty} C'_\lambda(x, \lambda) = O \left( \frac{\exp \left( |Im\sqrt{\lambda}| x \right)}{|\sqrt{\lambda}|^{m+2}} \right) \) as \(|\lambda| \to \infty.\)

Now we turn to prove Lemma 8.

**Proof of Lemma 8.** We only aim to prove the relation (3.40), since the other statements can be treated similarly. We first denote

\[ g(x) := \begin{cases} q(x), & x \in [0, x_0], \\ s(x), & x \in (x_0, x_0 + \delta], \end{cases} \quad \tilde{g}(x) := \begin{cases} \tilde{q}(x), & x \in [0, x_0], \\ \tilde{s}(x), & x \in (x_0, x_0 + \delta], \end{cases} \]

where \( s(x) = \sum_{j=0}^{m} q_{-j}(x_0) (x - x_0)^j \) and \( \delta \) is some positive constant. Then by (3.39) it is easy to see that \( g, \tilde{g} \in C^m [0, x_0 + \delta] \) and

\[ g^{(j)}(x_0) = q^{(j)}(x_0) = \tilde{g}^{(j)}(x_0) \quad \text{for} \; j = 0, 1, \ldots, m. \]
For \( i = 1, 2 \), let \( w_2(x, \lambda) \) and \( \bar{w}_2(x, \lambda) \) be the fundamental solutions of the equations

\[-y'' + g(x)y = \lambda y \quad \text{and} \quad -y'' + \bar{g}(x)y = \lambda y, \quad x \in (0, x_0 + \delta)\]

respectively, where \( w_2(x, \lambda) \) and \( \bar{w}_2(x, \lambda) \) are determined by the initial conditions

\[w_2(0, \lambda) = \bar{w}_2(0, \lambda) = 0, \quad w_2'(0, \lambda) = \bar{w}_2'(0, \lambda) = 1.\]

By (3.55), (3.56), Lemma 9, Lemma 10, Remark 9 and Remark 10, it is easy to see that there exist functions \( r_k, u_k, z_k \in C^1[0, x_0 + \delta] \) such that for \( x \in [0, x_0 + \delta] \),

\[
\begin{align*}
\quad w_2(x, \lambda)\bar{w}_2'(x, \lambda) - w_2'(x, \lambda)\bar{w}_2(x, \lambda) \\
&= \sum_{k=0}^{m+3} r_k(x) \frac{\sin(\sqrt{\lambda}x)}{(\sqrt{\lambda})^k} \cos(\sqrt{\lambda}x) + \sum_{k=0}^{m+2} u_k(x) \frac{\cos^2(\sqrt{\lambda}x)}{(\sqrt{\lambda})^k} \\
&\quad + \sum_{k=0}^{m+3} z_k(x) \frac{\sin^2(\sqrt{\lambda}x)}{(\sqrt{\lambda})^k} + \frac{(\pm)^{m+2}}{\sqrt{\lambda}} I_1(x, \lambda) + I_2(x, \lambda)
\end{align*}
\]

(3.60)

where

\[
\begin{align*}
I_1(x, \lambda) &= \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \int_0^x \frac{d\nu_{m+1}(x - 2t, \lambda)}{dx} \left( \bar{g}^{(m)}(t) - g^{(m)}(t) \right) dt \\
&\quad + \cos(\sqrt{\lambda}x) \int_0^x \nu_{m+1}(x - 2t, \lambda) \left( g^{(m)}(t) - \bar{g}^{(m)}(t) \right) dt \\
&= \left\{ \begin{array}{ll}
\int_0^x \frac{\cos(2\sqrt{\lambda}t)}{(2\sqrt{\lambda})^{m+1}} \left( g^{(m)}(t) - \bar{g}^{(m)}(t) \right) dt & \text{if } m \text{ is even,} \\
\int_0^x \frac{\sin(2\sqrt{\lambda}t)}{(2\sqrt{\lambda})^{m+1}} \left( \bar{g}^{(m)}(t) - g^{(m)}(t) \right) dt & \text{if } m \text{ is odd,}
\end{array} \right.
\end{align*}
\]

and as \( |\lambda| \to \infty \),

\[
I_2(x, \lambda) = o \left( \frac{\exp \left( 2 \left| \text{Im}\sqrt{\lambda} \right| x \right)}{\sqrt{\lambda}^{m+3}} \right), \quad I_2'(x, \lambda) = o \left( \frac{\exp \left( 2 \left| \text{Im}\sqrt{\lambda} \right| x \right)}{\sqrt{\lambda}^{m+2}} \right).
\]
In view of (3.60) and the fact \( g = \bar{g} \) on \([x_0, x_0 + \delta]\), one deduces that for \( x \in [x_0, x_0 + \delta] \),
\[
(w_2(x, \lambda)\bar{w}_2(x, \lambda) - w'_2(x, \lambda)\bar{w}_2(x, \lambda))' = r_0(x)\sqrt{\lambda}\cos\left(2\sqrt{\lambda}x\right) - (u_0(x) - z_0(x))\sqrt{\lambda}\sin\left(2\sqrt{\lambda}x\right)
\]
\[
+ \sum_{k=0}^{m+2} \frac{u_k'(x) + z_k'(x)}{2(\sqrt{\lambda})^k} + \sum_{k=0}^{m+2} \left( u_{k+1}(x) + \frac{u_k'(x) - z_k'(x)}{2} \right) \frac{\cos\left(2\sqrt{\lambda}x\right)}{(\sqrt{\lambda})^k}
\]
\[
- \sum_{k=0}^{m+2} \left( u_{k+1}(x) - z_{k+1}(x) - \frac{r_k'(x)}{2} \right) \frac{\sin\left(2\sqrt{\lambda}x\right)}{(\sqrt{\lambda})^k} + o\left( \frac{\exp\left(2\left|\Im\sqrt{\lambda}\right|x_0\right)}{\left|\sqrt{\lambda}\right|^{m+2}} \right)
\]
\[= 0.
\]
This forces that for \( x \in [x_0, x_0 + \delta] \), \( u_0(x) - z_0(x) = r_0(x) \equiv 0 \),
\[
r_{k+1}(x) + \frac{u_k'(x) - z_k'(x)}{2} = 0, \quad u_{k+1}(x) - z_{k+1}(x) - \frac{r_k'(x)}{2} = 0 \text{ for } k = 0, \ldots, m + 2,
\]
and thus for \( k = 0, 1, \ldots, m + 3 \) and \( x \in [x_0, x_0 + \delta] \), one has
\[
(3.62) \quad u_k(x) - z_k(x) = r_k(x) \equiv 0.
\]
Therefore, by (3.60) and (3.62) we infer that
\[
(3.63) \quad w_2(x_0, \lambda)\bar{w}_2(x_0, \lambda) - w'_2(x_0, \lambda)\bar{w}_2(x_0, \lambda)
\]
\[= \sum_{k=0}^{m+3} \frac{u_k(x_0) + z_k(x_0)}{2(\sqrt{\lambda})^k} + \frac{(\pm)m+2}{\sqrt{\lambda}} I_1(x_0, \lambda) + o\left( \frac{\exp\left(2\left|\Im\sqrt{\lambda}\right|x_0\right)}{\left|\sqrt{\lambda}\right|^{m+3}} \right).
\]
Next, we aim to show that
\[
(3.64) \quad I_1(x_0, \lambda) = o\left( \frac{\exp\left(2\left|\Im\sqrt{\lambda}\right|x_0\right)}{\left|\sqrt{\lambda}\right|^{m+2}} \right)
\]
as \( |\lambda| \to \infty \) in the sector \( \Lambda_\ell \). Due to the definition (3.61) of \( I_1(x, \lambda) \), it is sufficient to prove
\[
(3.65) \quad \int_0^{x_0} \cos\left(2\sqrt{\lambda}t\right) \left( g^{(m)}(t) - \bar{g}^{(m)}(t) \right) dt = o\left( \frac{\exp\left(2\left|\Im\sqrt{\lambda}\right|x_0\right)}{\left|\sqrt{\lambda}\right|^{m+2}} \right),
\]
\[
(3.66) \quad \int_0^{x_0} \sin\left(2\sqrt{\lambda}t\right) \left( g^{(m)}(t) - \bar{g}^{(m)}(t) \right) dt = o\left( \frac{\exp\left(2\left|\Im\sqrt{\lambda}\right|x_0\right)}{\left|\sqrt{\lambda}\right|^{m+2}} \right).
\]
In fact, by (3.59) (for \( j = m \)) and the fact \( g, \bar{g} \in C^m \left[0, x_0 + \delta \right] \) we infer that given any \( \epsilon > 0 \), there exists a sufficiently small constant \( \delta_0 > 0 \) such that
\[
|g^{(m)}(t) - \bar{g}^{(m)}(t)| < \epsilon \text{ on } [x_0 - \delta_0, x_0],
\]
and thus for \( \lambda \in \Lambda_\zeta \) and \(|\lambda|\) being sufficiently large, we obtain
\[
\left| \int_0^{x_0} \cos \left(2\sqrt{\lambda} t \right) \left( g^{(m)}(t) - \tilde{g}^{(m)}(t) \right) dt \right| \\
\leq \left| \int_0^{x_0-\delta_0} \cos \left(2\sqrt{\lambda} t \right) \left( g^{(m)}(t) - \tilde{g}^{(m)}(t) \right) dt \right| + \left| \int_{x_0-\delta_0}^{x_0} \cos \left(2\sqrt{\lambda} t \right) \left( g^{(m)}(t) - \tilde{g}^{(m)}(t) \right) dt \right|
\]
\[
\leq \max_{t \in [0,x_0]} |g^{(m)}(t) - \tilde{g}^{(m)}(t)| \int_0^{x_0-\delta_0} \left| \cos \left(2\sqrt{\lambda} t \right) \right| dt + \epsilon \int_{x_0-\delta_0}^{x_0} \left| \cos \left(2\sqrt{\lambda} t \right) \right| dt
\]
\[
\leq \max_{t \in [0,x_0]} |g^{(m)}(t) - \tilde{g}^{(m)}(t)| \int_0^{x_0-\delta_0} \exp \left(2 \left| \text{Im} \sqrt{\lambda} \right| t \right) dt + \epsilon \int_{x_0-\delta_0}^{x_0} \exp \left(2 \left| \text{Im} \sqrt{\lambda} \right| t \right) dt
\]
\[
\leq \max_{t \in [0,x_0]} |g^{(m)}(t) - \tilde{g}^{(m)}(t)| \frac{\exp \left(2 \left| \text{Im} \sqrt{\lambda} \right| (x_0 - \delta_0) \right)} {2 \left| \text{Im} \sqrt{\lambda} \right|} + \epsilon \frac{\exp \left(2 \left| \text{Im} \sqrt{\lambda} \right| x_0 \right)} {2 \sin \frac{\pi}{2}} \left| \sqrt{\lambda} \right|
\]
\[
\leq \frac{\epsilon \exp \left(2 \left| \text{Im} \sqrt{\lambda} \right| x_0 \right)} {\sin \frac{\pi}{2}} \left| \sqrt{\lambda} \right|,
\]
where we have used the inequalities
\[
\left| \cos \left(2\sqrt{\lambda} t \right) \right| \leq \exp \left(2 \left| \text{Im} \sqrt{\lambda} \right| |t| \right) \text{ for } \lambda \in \mathbb{C}, t \in \mathbb{R}
\]
and
\[
(3.67) \quad \left| \text{Im} \sqrt{\lambda} \right| \geq \left| \sqrt{\lambda} \right| \sin \frac{\pi}{2} \text{ for } \lambda \in \Lambda_\zeta.
\]
This proves the equality (3.65). Note that (3.66) can be treated similarly, and thus (3.64) is proved.

Now by (3.63) and (3.64) we have that
\[
w_2(x_0, \lambda) \tilde{w}_2'(x_0, \lambda) - w_2'(x_0, \lambda) \tilde{w}_2(x_0, \lambda) = \sum_{k=0}^{m+3} \frac{u_k(x_0) + z_k(x_0)} {2 \left( \sqrt{\lambda} \right)^k} + o \left( \frac{\exp \left(2 \left| \text{Im} \sqrt{\lambda} \right| x_0 \right)} {\left| \sqrt{\lambda} \right|^{m+3}} \right)
\]
as \(|\lambda| \to \infty\) in the sector \(\Lambda_\zeta\). This together with (3.67) directly yields that
\[
w_2(x_0, \lambda) \tilde{w}_2'(x_0, \lambda) - w_2'(x_0, \lambda) \tilde{w}_2(x_0, \lambda) = o \left( \frac{\exp \left(2 \left| \text{Im} \sqrt{\lambda} \right| x_0 \right)} {\left| \sqrt{\lambda} \right|^{m+3}} \right)
\]
as \(|\lambda| \to \infty\) in the sector \(\Lambda_\zeta\). Now (3.40) is proved, since by the definition of \(g\) and \(\tilde{g}\) we can infer that
\[
w_2(x_0, \lambda) = y_2(x_0, \lambda), \quad \tilde{w}_2(x_0, \lambda) = \tilde{y}_2(x_0, \lambda),
\]
\[
w_2'(x_0, \lambda) = y_2'(x_0, \lambda), \quad \tilde{w}_2'(x_0, \lambda) = \tilde{y}_2'(x_0, \lambda).
\]
Now we are in a position to prove Proposition 1.

Proof of Proposition 1. Note that

\[
\begin{align*}
(3.68) & \quad y_{2,r}(x_0,\lambda) - y_{2,r}(x_0,\lambda) \\
& = \left[ y_{2,r}(x_0 - \delta,\lambda) y_{1,x_0-\delta}(x_0,\lambda) \right] \\
& \quad \times \left[ y_{2,r}(x_0 - \delta,\lambda) y_{1,x_0-\delta}(x_0,\lambda) \right] \\
& \quad - y_{2,r}(x_0 - \delta,\lambda) y_{1,x_0-\delta}(x_0,\lambda) \\
& \quad \times \left[ y_{2,r}(x_0 - \delta,\lambda) y_{1,x_0-\delta}(x_0,\lambda) \right]
\end{align*}
\]

where

\[
\begin{align*}
B_1(\lambda) &= y_{2,r}(x_0 - \delta,\lambda) y_{2,r}(x_0 - \delta,\lambda) = O \left( |\lambda^{-1}| \exp \left( 2 |\text{Im} \lambda| (x_0 - \delta - r) \right) \right), \\
B_2(\lambda) &= y_{2,r}(x_0 - \delta,\lambda) y_{2,r}(x_0 - \delta,\lambda) = O \left( |\sqrt{\lambda}|^{-1} \exp \left( 2 |\text{Im} \lambda| (x_0 - \delta - r) \right) \right), \\
B_3(\lambda) &= y_{2,r}(x_0 - \delta,\lambda) y_{2,r}(x_0 - \delta,\lambda) = O \left( |\text{Im} \lambda|^{-1} \exp \left( 2 |\text{Im} \lambda| (x_0 - \delta - r) \right) \right), \\
B_4(\lambda) &= y_{2,r}(x_0 - \delta,\lambda) y_{2,r}(x_0 - \delta,\lambda) = O \left( \exp \left( 2 |\text{Im} \lambda| (x_0 - \delta - r) \right) \right).
\end{align*}
\]

The above asymptotics of $B_1, B_2, B_3, B_4$ can be obtained from (3.69). Therefore, one can easily deduce from Lemma 8 that

\[
\begin{align*}
y_{1,x_0-\delta}(x_0,\lambda) y_{1,x_0-\delta}(x_0,\lambda) - y_{1,x_0-\delta}(x_0,\lambda) y_{2,x_0-\delta}(x_0,\lambda) &= o \left( \frac{\exp \left( 2 |\text{Im} \lambda| \delta \right)}{|\sqrt{\lambda}|^{m+1}} \right), \\
y_{1,x_0-\delta}(x_0,\lambda) y_{2,x_0-\delta}(x_0,\lambda) - y_{1,x_0-\delta}(x_0,\lambda) y_{2,x_0-\delta}(x_0,\lambda) &= o \left( \frac{\exp \left( 2 |\text{Im} \lambda| \delta \right)}{|\sqrt{\lambda}|^{m+2}} \right), \\
\overline{y}_{1,x_0-\delta}(x_0,\lambda) y_{2,x_0-\delta}(x_0,\lambda) - \overline{y}_{1,x_0-\delta}(x_0,\lambda) y_{2,x_0-\delta}(x_0,\lambda) &= o \left( \frac{\exp \left( 2 |\text{Im} \lambda| \delta \right)}{|\sqrt{\lambda}|^{m+2}} \right), \\
y_{2,x_0-\delta}(x_0,\lambda) y_{2,x_0-\delta}(x_0,\lambda) - y_{2,x_0-\delta}(x_0,\lambda) y_{2,x_0-\delta}(x_0,\lambda) &= o \left( \frac{\exp \left( 2 |\text{Im} \lambda| \delta \right)}{|\sqrt{\lambda}|^{m+3}} \right)
\end{align*}
\]

as $|\lambda| \to \infty$ in $\Lambda_{\xi}$. Thus the equality (3.38) can be directly obtained from (3.68). The statements (3.35) – (3.37) can be proved similarly.

\[ \Box \]

Remark 11. If $q$ and $\overline{q}$ are both assumed to be in $L^1_{\xi}[0,\pi]$, then one can easily find that relations (3.35) – (3.38) still hold by taking $m = -1$. In fact, in this case,
$y_2, r(x, \lambda)$ and $y_2', r(x, \lambda)$ have the following asymptotic form [31]:

\begin{equation}
\begin{aligned}
y_2, r(x, \lambda) &= \frac{\sin(\sqrt{\lambda}(x - r))}{\sqrt{\lambda}} - Q(x) \frac{\cos(\sqrt{\lambda}(x - r))}{2\lambda} + o\left(\frac{\exp\left(\frac{\text{Im} \sqrt{\lambda} (x - r)}{|\lambda|}\right)}{|\lambda|}\right), \\
y_2', r(x, \lambda) &= \frac{\cos(\sqrt{\lambda}(x - r))}{2\sqrt{\lambda}} + \frac{\sin(\sqrt{\lambda}(x - r))}{2\lambda} + o\left(\frac{\exp\left(\frac{2 |\text{Im} \sqrt{\lambda}| (x - r)}{|\lambda|}\right)}{|\lambda|}\right),
\end{aligned}
\end{equation}

where $Q(x) = \int_r^x q(t)dt$. Therefore, it is easy to see that

\begin{equation}
\begin{aligned}
y_2, r(x_0, \lambda) &\tilde{y}_2, r(x_0, \lambda) - y_2', r(x_0, \lambda) \tilde{y}_2, r(x_0, \lambda) \\
&= \int_{x_0}^x (\tilde{q}(t) - q(t)) dt + o\left(\frac{\exp\left(2 |\text{Im} \sqrt{\lambda}| (x_0 - r) / |\lambda|\right)}{|\lambda|}\right).
\end{aligned}
\end{equation}

This directly yields (3.38). (3.35) – (3.37) can be treated similarly.

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School of Mathematics, Tianjin University, Tianjin, 300354, People’s Republic of China

E-mail address: jun.yan@tju.edu.cn

School of Mathematics, Tianjin University, Tianjin, 300354, People’s Republic of China

E-mail address: glshi@tju.edu.cn