Entropy in spin foam models: the statistical calculation

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Abstract
Recently an idea for computing the entropy of black holes in the spin foam formalism has been introduced. Particularly complete calculations for the three-dimensional Euclidean BTZ black hole were performed. The whole calculation is based on observables living at the horizon of the black hole universe. Departing from this idea of observables living at the horizon, we now go further and compute the entropy of the BTZ black hole in the spirit of statistical mechanics. We compare both calculations and show that they are very interrelated and equally valid. This latter behaviour is certainly due to the importance of the observables.

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1. Introduction

A way to compute the entropy of a black hole in the spin foam model description of quantum gravity has been introduced [1, 2]. In particular, complete calculations were performed for the case of the three-dimensional Euclidean BTZ black hole [2]. It is also important to point out two approaches that are in the same spirit. They compute the geometrical entropy associated with microstates of spin networks; in [3], the black hole entropy is studied in terms of quantum gravity and quantum information, that is, in terms of entanglement of states, and in [4, 5], the entropy of a black hole is studied in terms of quantum surface states.

Here in the same context of our previous studies and as a continuation of our research programme, we now compute the entropy of the BTZ black hole [7] in the language of statistical mechanics. Departing from the observables’ idea which was developed in our previous work, we construct the statistical partition function of our black hole universe (system).

We should of course compare both calculations and hope that they give the same or at least equivalent results. We indeed show that they give the same results up to a constant factor.
This led us to conclude that we may be in the correct direction when thinking of a way to study the entropy in spin foam models.

We divide this paper as follows. In section 2 we briefly review the calculation of the entropy of the Euclidean BTZ black hole in the programme of spin foam models of quantum gravity.

In section 3 we continue with our programme of deriving a satisfactory way for computing the entropy of black holes in spin foam models of quantum gravity. We derive a computation in the spirit of statistical mechanics. Moreover, our idea is based on the observables which live at the horizon and which were part of our calculation in the previous work.

Finally in section 4 we conclude with a discussion of our paper and other related ideas.

2. BTZ black hole entropy and spin foams

We briefly review our idea for calculating the entropy of the Euclidean BTZ black hole in the context of spin foams. For a nice introduction to spin foam models based on quantum groups, see [6].

The Euclidean BTZ black hole has a hyperbolic-type metric which after some identifications is topologically a solid torus with its horizon at the core [7]. For a deep study of the BTZ black hole, see [8].

In the three-dimensional case we have that the entropy of the BTZ black hole is given by

$$S \sim 2L,$$

see [8]. After restoring factors $\hbar$ and $G$ it can be rewritten as

$$S \sim \frac{L}{4\hbar G}. \quad (2)$$

The horizon is the core of the torus. We start by triangulating the Euclidean black hole. We consider triangulations of the solid torus containing interior edges, as we want the core of the torus (horizon) to be formed by edges. See figure 1.

The idea is based on the three-dimensional topological spin foam models where the irreducible representations of the quantum group $SU_q(2)$ play a role. They are given by the finite set $\{0, 1, 2, \ldots, r/2\}$ where $r \geq 3$.

A spin foam partition function of any triangulated three-dimensional space-time $M$ is given by

$$Z(M) = \sum_S \prod_{\text{edges}} \dim_q(j) \prod_{\text{tetrahedra}} \{6j\} \quad (3)$$

where the sum is carried over the set of all admissible states $S$ and the amplitude $\{6j\}$ is the $6j$ symbol associated with the six labels of each tetrahedron. The quantum dimension is an amplitude associated with the edges.

![Figure 1. Triangulation of the BTZ Euclidean black hole together with its horizon.](image)
The above partition function can clearly be applied to the BTZ black hole spacetime \( M = T^2 \). But now consider the horizon of this black hole. We now think of the horizon as an observable \( O \) in the following sense. Given the triangulation we consider the spin foam partition function with the difference that now we do not sum over the spins which label the horizon:

\[
Z(T^2, O(j_1, j_2, \ldots, j_k)) = \sum_{S|O} \prod_{\text{edges}} \dim_q(j) \prod_{\text{tetrahedra}} \{6j\}.
\]  

The spin foam approach is analogous to a Feynman path integral. In this case it is the Feynman path integral of spacetime. We therefore define the expectation value or correlation function as

\[
W(T^2, O) = \frac{Z(T^2, O)}{Z(T^2)}.
\]  

This is therefore a function of the labels of the horizon. It is shown in [2] that for a particular triangulation of BTZ we have

\[
W(T^2, O) = \prod_{m} \frac{N_{im,jm}}{\dim_q(j_m)} \dim_q(i_m) \dim_q(j_m)
\]  

where \( i_1, j_1, \ldots, i_n, j_n \) are spins labelling the horizon, and \( \hat{i}_1, \ldots, \hat{j}_n \) are spins labelling edges that do not belong to the horizon; however, each triple \( \{i_m, j_m, \hat{j}_m\} \) forms a triangle. Our particular triangulation has a horizon with an even number of edges.

We propose that the entropy is given by the logarithm of formula (5):

\[
S = \sum_{m=1}^{2n} \log(\dim_q(j_m)) - \sum_{k=1}^{n} \log(\dim_q(\hat{j}_k)).
\]  

It can be seen that the main contribution is given when the spins \( \hat{j}_m \) are zero. This implies that each pair of the edges of the horizon are equal: \( i_1 = j_1, i_2 = j_2, \ldots, i_n = j_n \). The labels of the edges of the horizon by spins \( j \) are interpreted in the spin foam model as giving a discrete length given by \( j + \frac{1}{2} \).

The horizon is discrete and formed by edges with spins \( j_1, \ldots, j_{2n} \).

We have a constraint, since we want the sum of all the discrete lengths of the horizon be \( L \):

\[
(j_1 + \frac{1}{2}) + (j_2 + \frac{1}{2}) + \cdots + (j_{2n} + \frac{1}{2}) = L
\]  

Observe that particularly when all of the spins at the horizon are equal, we have

\[
2nj + \frac{2n}{2} = L,
\]  

which implies that

\[
n = \frac{L}{(2j + 1)}.
\]  

Consider the case in which the number of spins we have goes to infinity, that is, when we go from the quantum group \( SU_q(2) \) to the classical one \( SU(2) \), such that \( \dim_q(j) \rightarrow \dim(j) = (2j + 1) \), then the entropy is given by

\[
S \sim L \frac{2 \log(2j + 1)}{(2j + 1)}.
\]  

\[^1\] We just have the relabel edges \( i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_{2n} \).
Finally, it can be seen that the main contribution is given when \( j = 1 \), that is,
\[
S \sim 2L \frac{\log(3)}{3},
\]
which is proportional to the length of the horizon and to equation (1).

The spin foam model procedure described in this section uses the quantum group \( SU_q(2) \);
at the end we took the limit \( q \to 1 \). Our calculation can be thought as having a regularization procedure; it could be interesting to think if a different regularization procedure could be used as an alternative method to derive the entropy of the black hole in a similar spirit as ours.

3. Entropy in terms of the statistical partition function

In this section, following our previous idea of the calculation of the entropy in spin foam models of quantum gravity, we continue our proposal by deriving the entropy in a statistical spirit. In other words, we now consider the statistical partition function of our model.

The horizon is an observable in our picture. This means that the microstates live at the horizon. And we saw that when considering the main contribution to the entropy in the spin foam formalism, we should only care about what happens at the horizon, that is, only the spins which label them matter.

Since our approach is based on the spin foam models with a cosmological constant\(^2\), we only have a finite number of half-integer spins \( \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{r-2}{2}\} \) where \( r \geq 3 \).

These spins are interpreted as giving a discrete length to any labelled edge which as we saw contribute to the entropy of the black hole. Recall that a spin \( j \) labelling an edge is interpreted as having a discrete length \( j + \frac{1}{2} \). For convenience we consider the length written as \( \frac{2j+1}{2} \), and call the integer \( l_j = 2j + 1 \).

Our horizon is labelled, and must have length \( L \). Therefore formula (8) of section 3 must be satisfied. We rewrite it here, and therefore we have
\[
n_0 l_0 + n_1 l_1/2 + \cdots + n_{r-1} l_{(r-2)/2} = L, \tag{13}
\]
where clearly \( l_0 = 1, l_1/2 = 2, \ldots, l_{(r-2)/2} = (r - 1) \) and \( n_1, n_2, \ldots, n_{r-1} \) are the number of edges labelled with spins \( 0, 1/2, \ldots, (r - 2)/2 \) respectively. Interpret formula (13) as in the case of the harmonic oscillator where now we have energy given by length and occupation numbers given by the finite set of \( \{l_j\} \).

We can consider a Boltzman parameter \( \beta \) in which the half term in the sum (13) is absorbed, that is, \( \beta = \frac{1}{2kT} \). We think of our model as an isolated system, similar to a gas made of an arbitrary number of photons which obeys a Bose–Einstein statistics. In our case the analogy is to consider our horizon to be formed by an arbitrary number of edges and each edge can be in any length state, that is, we can have any number of edges labelled with spin 1/2, some others labelled with spin 1 and so on.

Therefore, the statistical partition function of our model is given by
\[
Z = \sum_{n_1, n_2, \ldots, n_{r-1}} \exp[-\beta(l_0 n_1 + l_1/2 n_2 + \cdots + l_{(r-2)/2} n_{r-1})]
\]
\[
= \sum_{n_1, n_2, \ldots, n_{r-1}} \exp[-\beta(n_1 + 2n_2 + \cdots + (r - 1)n_{r-1})]. \tag{14}
\]
We want to consider the case of a black hole with very large length, that is, when the length goes to infinity. Recall that our set of spins is finite; therefore, we need to consider a horizon\(^2\) Positive cosmological constant in this case although the BTZ black hole has negative cosmological constant, we are not concern with this issue here, but the idea is what matters for us at the moment.
with a very large number of edges which implies that \( n_1, n_2, \ldots, n_{r-1} \) are unrestricted and can go to infinity.

The partition sum can be rewritten as

\[
Z = \sum_{n_1=0}^{\infty} \exp[-\beta(n_1)] \sum_{n_2=0}^{\infty} \exp[-\beta(2n_2)] \cdots \sum_{n_{r-1}=0}^{\infty} \exp[-\beta((r-1)n_{r-1})]
\]

\[
= \prod_{m=1}^{r-1} \sum_{n_{m=0}}^{\infty} \exp[-\beta(nn_m)] = \prod_{m=1}^{r-1} \sum_{n_{m=0}}^{\infty} \exp(-\beta nn_m).
\]  

(15)

If we consider the complex variable \( z = \exp(-\beta) \) in such a way that \(|z| < 1\), we have

\[
\sum_{n_{m=0}}^{\infty} \exp(-\beta nn_m) = \frac{1}{1 - e^{-\beta m}},
\]  

(16)

and the partition function is now given by

\[
Z = \prod_{m=1}^{r-1} \frac{1}{1 - e^{-\beta m}}.
\]  

(17)

Before we continue let us very briefly make the following mathematical observation.

The partition function written as in formula (17) is a generating function of the number of partitions of the positive integers \( Z^+ \) in terms of the finite set \( \{l_0 = 1, l_1/2 = 2, \ldots, l_{(r-2)/2} = (r-1)\} \). We already know that this finite set is related to the set of the discrete lengths associated with the spins from which we are labelling the edges of our horizon\(^3\). This gives a very nice connection to analytic number theory \([9]\). We will come back to this relation later on in the following section.

We can also state that analogous to the gas made of photons the Planck distribution can be derived from formula (16) and it is given by

\[
\tilde{n}_m = \frac{1}{e^{\beta m} - 1}
\]  

(18)

which gives information about the statistical equilibrium. That is, the most probable partition corresponds to these numbers, given by the lowest spins of the finite set \( \{l_0 = 1, l_1/2 = 2, \ldots, l_{(r-2)/2} = (r-1)\} \).

The free energy is given by \( F = -kT \ln Z \). Substituting \( \beta = \frac{1}{2T} \) we have

\[
F = -kT \ln \left[ \prod_{m=1}^{r-1} \frac{1}{1 - e^{-\frac{m}{T}}} \right] = -kT \sum_{m=1}^{r-1} \ln \left( \frac{1}{1 - e^{-\frac{m}{T}}} \right)
\]  

(19)

which gives

\[
F = kT \sum_{m=1}^{r-1} \ln \left( 1 - e^{-\frac{m}{T}} \right).
\]  

(20)

The entropy is then given by

\[
S = -\frac{\partial F}{\partial T} = \left[ k \sum_{m=1}^{r-1} \ln \left( 1 - e^{-\frac{m}{T}} \right) - kT \sum_{m=1}^{r-1} \frac{m e^{-\frac{m}{T}}}{(1 - e^{-\frac{m}{T}})} \right]
\]

\[
= \frac{1}{2T} \sum_{m=1}^{r-1} \frac{m e^{-\frac{m}{T}}}{(1 - e^{-\frac{m}{T}})} - k \sum_{m=1}^{r-1} \ln \left( 1 - e^{-\frac{m}{T}} \right).
\]  

(21)

\(^3 j + \frac{1}{2} = l_{j/2} \).
The set of spins (equivalently the set of $\{l_j\}$) we have depend on the quantum group that we choose, $SU_q(2)$. We can choose to take the limit when $q \to 1$, that is when going to the classical group $SU(2)$; just as we did at the end of our calculation in section 2.

Then the entropy can be considered to be given by

$$S = \frac{1}{2T} \sum_{m=1}^{\infty} \frac{m e^{-\frac{m \pi}{T}}}{1 - e^{-\frac{m \pi}{T}}} - k \sum_{m=1}^{\infty} \ln \left(1 - e^{-\frac{m \pi}{T}}\right).$$

(22)

If we go further and approximate the sums by integrals, we have

$$S = \frac{1}{2T} \int_{1}^{\infty} \frac{me^{-\frac{m \pi}{T}} dm}{1 - e^{-\frac{m \pi}{T}}} - k \int_{1}^{\infty} \ln \left(1 - e^{-\frac{m \pi}{T}}\right) dm.$$  

(23)

The integrals can be computed for example with a mathematical programme and we then obtain that the entropy is given by

$$S = \frac{1}{2T} \left(\frac{1}{2} + \frac{4\pi^2 k^2 T^2}{3} - 2kT \ln \left(1 - e^{\frac{\pi}{T}}\right) - 4k^2 T^2 \text{Li}_2(e^{\frac{\pi}{T}})\right)$$

$$- k \left(-\frac{1}{4kT} + \frac{2\pi^2 k T}{3} \ln \left(1 - e^{\frac{\pi}{T}}\right) - \ln \left(1 - e^{\frac{\pi}{T}}\right) + 2kT \text{Li}_2(e^{\frac{\pi}{T}})\right)$$

(24)

where the terms $\text{Li}_2(e^{\frac{\pi}{T}})$ are called polylogarithmic integrals given by

$$\text{Li}_2(e^{\frac{\pi}{T}}) = \int_{0}^{e^{\frac{\pi}{T}}} \frac{\ln(1-x)}{x} dx.$$  

(25)

Arranging the terms, the entropy is given by

$$S = \frac{4\pi^2 k^2 T}{6} + \frac{1}{2T} - 2k \ln \left(1 - e^{\frac{\pi}{T}}\right) + k \ln \left(1 - e^{\frac{\pi}{T}}\right) - 4k^2 T \text{Li}_2(e^{\frac{\pi}{T}}).$$

(26)

Our spin foam model description of the black hole entropy is a first step towards the microscopic description in terms of this quantum gravity direction. We are also considering the Euclidean black hole and our model is in fact a toy model. Let us go a bit further and suppose for a moment that the relation between temperature and length is fulfilled. For the BTZ black hole the temperature is given by $T = \frac{\pi}{2R}$, where $R$ is the radius of the horizon, see for instance [8].

In the present case, our statistical partition function (14) can be thought as a mathematical partition of the number $2L$, which therefore, in this case, leads us to consider the temperature given by $T = \frac{\pi}{2r}$. With this value of the temperature, the entropy (first term) is therefore given by

$$S \sim \frac{4\pi r}{6} k^2 = 2L \frac{k^2}{6},$$

(27)

which is indeed proportional to the formula obtained in our previous work [2], given here by formula (12). They will be exactly equal if we have units in which we consider $\frac{k^2}{6} = 2 \log(3)$. Moreover, after restoring the constants $\hbar$ and $G$ it can be argued that we are indeed obtaining the correct entropy of $\frac{k}{\hbar \Omega}$ up to a constant factor.

4. Discussion

We have calculated the entropy of the Euclidean three-dimensional black hole in the spin foam formalism [2]. We reviewed it in section 2 and extended the idea to the spin foam statistical calculation based on the observables. The observables played a significant role.
The interpretation is that it only matters what happens at the horizon, when considering the entropy.

Let us mention some interesting ideas that come up here. How can all our calculations be extended to an isolated four-dimensional black hole universe? That is, develop a spin foam approach to the entropy of four-dimensional black holes [1], as it is done in loop quantum gravity [10–15].

We may have naively chosen units such that \( k^2 = 2 \log(3) \) so that our formulae (12) and (26) match. What does it have to do with the fact that the spectrum of a quantum nonrotating black hole such as Schwarzschild is evenly spaced containing a factor \( \log(3) \) as discussed in [16], and later in [17].

Another interesting fact is the relation of counting microstates which account for a fixed area and partitions of integers in the field of number theory, principally analytic [9]. For instance recent developments about these number theory relations for the case of loop quantum gravity were introduced by [15] and continued in [18, 19].

Now particularly for this paper the relation is as follows. Suppose we want to count the number of undistinguishable microstates which account for a fixed-length horizon \( L \). Then we would have to proceed as follows.

The statistical partition function, written as formula (17), is the generating function for the number of partitions of the positive integers in terms of the finite set \( \{ l_0 = 1, l_{1/2} = 2, \ldots, l_{(r-2)/2} = (r - 1) \} \).

Such a formula can be expanded as follows:

\[
Z = \prod_{m=1}^{r-1} \frac{1}{1 - e^{-\beta m}} = \sum_{n=0}^{\infty} p_{r-1}(n) e^{-\beta n}
\] (28)

where \( p_{r-1} \) is the number of partitions of the number \( n \) in terms of integers not exceeding \( r - 1 \), that is in terms of our finite set \( \{ l_0 = 1, l_{1/2} = 2, \ldots, l_{(r-2)/2} = (r - 1) \} \).

If considering a large horizon length \( L \), then counting the number of partitions of such number \( L \) gives us back \( p_{r-1}(L) \). What we would be really counting with \( p_{r-1}(L) \) is the number of undistinguishable microstates of the black hole. This means that we would not be distinguishing between a given microstate which accounts for a fixed length and a permutation of this microstate.

In this case the number of microstates goes asymptotically as follows:

\[
p_{r-1}(L) = \frac{L^{r-2}}{(r-1)! (r-2)!} \] (29)

as can be seen in [9]. Therefore, if we want to calculate the entropy when the microstates are thought to be undistinguishable, it is just mainly given by the logarithm of formula (28).

References

[1] Manuel Garcia Islas J 2008 Towards a spin foam model description of black hole entropy Class. Quantum Grav. 25 238001 (arXiv:0809.0304v1 [gr-qc])
[2] Manuel Garcia Islas J 2008 BTZ black hole entropy: a spin foam model description Class. Quantum Grav. 25 245001 (arXiv:0804.2082v2 [gr-qc])
[3] Livine E R and Terno D R 2006 Quantum black holes: entropy and entanglement on the horizon Nucl. Phys. B 741 131–61 (arXiv:gr-qc/0508085v3)
[4] Ansari M H 2007 Spectroscopy of a canonically quantized horizon Nucl. Phys. B 783 179–212 (arXiv:hep-th/0607081v4)
[5] Ansari M H 2008 Generic degeneracy and entropy in loop quantum gravity Nucl. Phys. B 795 635–44 (arXiv:gr-qc/0603121v5)
[6] Baez J C 2000 An introduction to spin foam models of quantum gravity and BF theory Lect. Notes Phys. 543 25–94 (arXiv:gr-qc/9905087v1)
[7] Bañados M, Teitelboim C and Zanelli J 1992 Black hole in three-dimensional spacetime Phys. Rev. Lett. 69 1849–51
[8] Carlip S 1995 The (2+1)-dimensional black hole Class. Quantum Grav. 12 2853–80 (arXiv:gr-qc/9506079v1)
[9] Newman D J 2000 Analytic Number Theory (Graduate Texts in Mathematics vol 177) (Berlin: Springer)
[10] Rovelli C 1996 Black hole entropy from loop quantum gravity Phys. Rev. Lett. 77 3288–91 (arXiv:gr-qc/9603063v1)
[11] Krasnov K 1997 Geometrical entropy from loop gravity Phys. Rev. D 55 3505
[12] Ashtekar A, Baez J, Corichi A and Krasnov K 1998 Quantum geometry and black hole entropy Phys. Rev. Lett. 80 904–07 (arXiv:gr-qc/9710007)
[13] Domagala M and Lewandowski J 2004 Black hole entropy from quantum geometry Class. Quantum Grav. 21 5233 (arXiv:gr-qc/0407051)
[14] Corichi A, Diaz-Polo J and Fernandez-Borja E 2007 Quantum geometry and microscopic black hole entropy Class. Quantum Grav. 24 243 (arXiv:gr-qc/0605014)
[15] Agullo I, Fernando Barbero J, Diaz-Polo J, Fernandez–Borja E and Villaseñor E J S 2008 Black hole state counting in loop quantum gravity: a number theoretical approach Phys. Rev. Lett. 100 211301 (arXiv:0802.4077v2 [gr-qc])
[16] Hod S 1998 Bohr’s correspondence principle and the area spectrum of quantum black holes Phys. Rev. Lett. 81 4293 (arXiv:gr-qc/9812002v2)
[17] Dreyer O 2003 Ln(3) and black hole entropy Proc. 3rd Int. Symp. on Quantum Theory and Symmetries Cincinnati (arXiv:gr-qc/0404055v2)
[18] Fernando Barbero J and Villaseñor E J S 2008 Generating functions for black hole entropy in loop quantum gravity Phys. Rev. D 77 121502 (arXiv:0804.4784v1 [gr-qc])
[19] Fernando Barbero J and Villaseñor E J S 2009 On the computation of black hole entropy in loop quantum gravity Class. Quantum Grav. 26 035017 (arXiv:0810.1599v1 [gr-qc])