CONGRUENCES FOR GENERALIZED FROBENIUS PARTITIONS WITH NONZERO ROW DIFFERENCE

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1. Introduction

We wish to extend Andrews’ [1] general principle to include arrays with nonzero row difference and establish some congruences for these arrays. If $f_A(z, q) = f_A(z) = \sum P_A(m, n)z^m q^n$ denotes the generating function for the number of ordinary partitions of $n$ into $m$ parts subject to the set of restrictions $A$, then the coefficient of $z^\alpha$ in

$$f_A(zq)f_B(z^{-1}) = \sum P_A(m_1, n_1)z^{m_1}q^{m_1+n_1} \sum P_B(m_2, n_2)z^{-m_2}q^{n_2}$$

is the generating function

$$\Phi_{A,B,\alpha}(q) = \sum_{n=0}^{\infty} \phi_{A,B,\alpha}(n)q^n$$

where $\phi_{A,B,\alpha}(n)$ is the number of arrays of weight $n = m_1 + n_1 + n_2$ wherein $(a_1, a_2, ..., a_{m_1})$ is a partition of $n_1$ into $m_1$ parts arranged in nonincreasing order and subject to the set of restrictions $A$, and $(b_1, b_2, ..., b_{m_2})$ is a partition of $n_2$ into $m_2$ parts arranged in nonincreasing order and subject to the set of restrictions $B$. We call $\alpha = m_1 - m_2$ the row difference.

2. Generating Functions

It follows directly from the extension of Andrews’ principle that the coefficient of $z^\alpha$ in

$$\prod_{\lambda_i=0}^{\infty} (1 + zq^{\lambda_i+1}) + ... + z^{k}q^{k(\lambda_i+1)})(1 + z^{-1}q^{\lambda_i} + ... + z^{-k}q^{k\lambda_i})$$

is

$$\Phi_{k,\alpha}(q) = \sum_{n=0}^{\infty} \phi_{k,\alpha}(n)q^n$$

where $\phi_{k,\alpha}(n)$ is the number of arrays of weight $n$ and row difference $\alpha$ such that each nonnegative integer is repeated at most $k$ times in each row. Furthermore, the coefficient of $z^\alpha$ in

$$\prod_{i=0}^{\infty} (1 + zq^{\lambda_i+1})^k(1 + z^{-1}q^{\lambda_i})^k$$

is
is

\[ C\Phi_{k,\alpha}(q) = \sum_{n=0}^{\infty} c\phi_{k,\alpha}(n)q^n \]

where \( c\phi_{k,\alpha}(n) \) is the number of arrays of weight \( n \) and row difference \( \alpha \) such that each nonnegative integer is taken from one of \( k \) copies of the integers. We can index which copy of the integers an entry is taken from by thinking of this as a coloring and denoting the color by a subscript from \( \{1, 2, \ldots, k\} \).

Next, we establish the formulas for the single variable generating functions with fixed row difference.

**Theorem 1.** For any positive integer \( k \) and any integer \( \alpha \),

\[
\Phi_{k,\alpha}(q) = \frac{1}{(q; q)_\infty} \sum_{m_1, m_2, \ldots, m_{k-1} = -\infty}^{\infty} (-1)^\alpha \zeta^{m_1(1-k)+m_2(2-k)+\cdots+m_{k-1}(-1)+k\alpha} \times q^{m_1^2+m_2^2+\cdots+m_{k-1}^2-\alpha(m_1+m_2+\cdots+m_{k-1})+a^2+\alpha+\sum_{1 \leq i < j < k-1} m_im_j/2}.
\]

where \( \zeta = e^{2\pi i/k} \).

**Proof.** We have that \( \Phi_{k,\alpha}(q) \) is the coefficient of \( z^\alpha \) in

\[
\varepsilon(z, q) = \prod_{\lambda_i = 0} (1 + zq^{\lambda_i+1} + \cdots + z^kq^{k(\lambda_i+1)}) (1 + z^{-1}q^{\lambda_i} + \cdots + z^{-k}q^{k\lambda_i}).
\]

From Andrews [1],

\[
\varepsilon(z, q) = \frac{1}{(q; q)_\infty} \prod_{j=1}^k \sum_{m_j = -\infty}^{\infty} (-1)^{m_j} q^{(m_j+1)/2} z^{m_j} \zeta^{jm_j},
\]

where \( \zeta = e^{2\pi i/k} \). Then, to get the coefficient of \( z^\alpha \), we set \( m_1 + m_2 + \cdots + m_k = \alpha \), and solving for \( m_k \) we obtain the desired formula. \(\square\)

**Theorem 2.** For any positive integer \( k \) and any integer \( \alpha \),

\[
C\Phi_{k,\alpha}(q) = \sum_{m_1, m_2, \ldots, m_{k-1} = -\infty}^{\infty} qQ(m_1, m_2, \ldots, m_{k-1}) (q; q)_\infty^k
\]

where

\[
Q(m_1, m_2, \ldots, m_{k-1}) = m_1^2 + m_2^2 + \cdots + m_{k-1}^2 - \alpha(m_1 + m_2 + \cdots + m_{k-1}) + \alpha^2 + \alpha + \sum_{1 \leq i < j < k-1} m_im_j/2.
\]

**Proof.** We have that \( C\Phi_{k,\alpha}(n) \) is the coefficient of \( z^\alpha \) in

\[
\varphi^k(z, q) = \prod_{\lambda_i = 0}^{\infty} (1 + zq^{\lambda_i+1})^k (1 + z^{-1}q^{\lambda_i})^k
\]
Then, by the version of the Jacobi Triple Product in [1], we have
\[
\varphi^k(z, q) = \frac{1}{(q; q)_{\infty}^k} \sum_{m_1, m_2, \ldots, m_k = -\infty}^{\infty} z^{m_1 + m_2 + \cdots + m_k} q^{(m_1 + 1) + (m_2 + 1) + \cdots + (m_k + 1)}.
\]

To get the coefficient of \(z^\alpha\), we set \(m_1 + m_2 + \cdots + m_k = \alpha\). Then,
\[
C_{\Phi_{k, \alpha}}(q) = \frac{1}{(q; q)_{\infty}^k} \sum_{m_1, m_2, \ldots, m_k = -\infty}^{\infty} q^{(m_1 + 1) + (m_2 + 1) + (m_{k-1} + 1) + (\alpha - m_k - \ldots - m_2 - m_1 + 1)}
\]
\[
\times q^\frac{\alpha^2 + \alpha \sum_{1 \leq i < j \leq k-1} m_i m_j}{2}.
\]

□

Next, we establish product formulas for \(\Phi_{2, -1}(q)\) and \(C_{\Phi_{2, -1}}(q)\) using the generating functions in Theorems 1 and 2.

**Corollary 1.** For all nonnegative integers \(n\),
\[
\Phi_{2, -1}(q) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})^2(1 - q^{12n-8})(1 - q^{12n-6})(1 - q^{12n-4})(1 - q^{12n})}.
\]

**Proof.** From Theorem 1 we have
\[
\Phi_{2, -1}(q) = \frac{1}{(q; q)_{\infty}^2} \sum_{m = -\infty}^{\infty} (-1)^{-1} \zeta^{-m-2} q^{m^2 + m}
\]
\[
= -\zeta^{-2} \frac{(q; q)_{\infty}^2}{(q; q)_{\infty}^2} \sum_{m = -\infty}^{\infty} \zeta^{-m} q^{m^2 + m}.
\]

By the Jacobi Triple products, we have
\[
\Phi_{2, -1}(q) = -\zeta^{-2} \prod_{i=1}^{\infty} \frac{(1 - q^{2i})(1 + \zeta^{-1} q^{2i})(1 + \zeta q^{2i-2})}{(1 - q^i)^2}
\]
\[
= -\zeta^{-2} (1 + \zeta) \prod_{i=0}^{\infty} \frac{(1 - q^{2i})(1 + \zeta^{-1} q^{2i})(1 + \zeta q^{2i})}{(1 - q^i)^2}
\]
\[
= \prod_{i=0}^{\infty} \frac{(1 - q^{2i})(1 - q^{2i} + q^{4i})}{(1 - q^i)^2}
\]
\[
= \Psi_2(q)
\]

where \(\Psi_2(q)\) is defined, and the remainder of the proof is given by Drake in [2]. □
Corollary 2. For all nonnegative integers \( n \),
\[
C\Phi_{2, -1}(q) = \prod_{i=1}^{\infty} \frac{(1 - q^{2i})(1 + q^{2i})(1 + q^{2i-2})}{(1 - q^i)^2}.
\]

Proof. From Theorem 2 with \( k = 2 \) and \( \alpha = -1 \), we have
\[
C\Phi_{2, -1}(q) = \sum_{m=-\infty}^{\infty} q^{m^2 + m} \prod_{i=1}^{\infty} (1 - q^{2i}).
\]

We apply the Jacobi Triple Product to obtain the desired formula. \( \square \)

3. Congruences

Now, we will use the generating functions established in the previous section to establish some congruences.

Theorem 3. For every nonnegative integer \( n \),
\[
\phi_{2, -1}(5n + 4) \equiv 0 \pmod{5}.
\]

Proof. From Corollary 1,
\[
\Phi_{2, -1}(q) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})^2(1 - q^{12n-8})(1 - q^{12n-6})(1 - q^{12n-4})(1 - q^{12n})}.
\]

Then,
\[
\Phi_{2, -1}(q) = \prod_{n=1}^{\infty} \frac{(1 - q^n)^3(1 - q^{2n})(1 - q^{12n-2})(1 - q^{12n-10})}{(1 - q^{5n})} \pmod{5}
\]
\[
= \sum_{j=0}^{\infty} (-1)^j (2j + 1) q^{\frac{j+1}{2}} \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{12n-2})(1 - q^{12n-10}) \prod_{n=1}^{\infty} (1 - q^{5n})
\]
\[
= \sum_{j=0}^{\infty} (-1)^j (2j + 1) q^{\frac{j+1}{2}} \sum_{m=-\infty}^{\infty} q^{9m^2 - 3m - q^{9m^2 + 9m + 2}} \prod_{n=1}^{\infty} (1 - q^{5n})
\]
\[
= \sum_{j=0}^{\infty} (-1)^j (2j + 1) q^{\frac{j+1}{2}} \sum_{k=0}^{\infty} a_k q^{k^2 + k} \prod_{n=1}^{\infty} (1 - q^{5n})
\]

where \( a_k = 1 \) if \( k \equiv 0, 2 \pmod{3} \) and \( a_k = -2 \) if \( k \equiv 1 \pmod{3} \). So,
\[
\Phi_{2, -1}(q) \equiv \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^j (2j + 1) a_k q^{\frac{j+1}{2} + k^2 + k} \prod_{n=1}^{\infty} (1 - q^{5n}) \pmod{5}
\]
Then, we get a contribution to $q^{5n+4}$ when
\[
\left(\frac{j+1}{2}\right) + k^2 + k \equiv 4 \pmod{5}
\]
or equivalently, when
\[
(2j + 1)^2 + 2(2k + 1)^2 \equiv 0 \pmod{5}.
\]
So $2j + 1 \equiv 2k + 1 \equiv 0 \pmod{5}$. \hfill \Box

**Theorem 4.** For every nonnegative integer $n$,
\[
c\phi_{2,-1}(5n+4) \equiv 0 \pmod{5}.
\]

**Proof.** From Corollary 2,
\[
C\Phi_{2,-1}(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2i})(1+q^{2i})(1+q^{2i-2})}{(1-q^i)^2}.
\]
Then,
\[
C\Phi_{2,-1}(q) = \prod_{n=1}^{\infty} \frac{(1-q^i)^3(1-q^{2i})(1+q^{2i})(1+q^{2i-2})}{(1-q^{5i})} \equiv \prod_{n=1}^{\infty} \frac{(1-q^i)^3(1-q^{2i})(1+q^{2i})(1+q^{2i-2})}{(1-q^{5i})} \pmod{5}
\]
\[
= \sum_{m=0}^{\infty} (-1)^m (2m+1)q^{(m+1)2} \sum_{j=-\infty}^{\infty} q^{j^2+j} \pmod{5}
\]
\[
= \sum_{m=0}^{\infty} \prod_{i=0}^{\infty} (1-q^{5i}) (1-q^{2i})(1+q^{2i})(1+q^{2i-2}) \pmod{5}
\]
So, we get a contribution to $q^{5n+4}$ when
\[
\left(\frac{m+1}{2}\right) + j^2 + j \equiv 4 \pmod{5}
\]
or equivalently,
\[
(2m + 1)^2 + 2(2j + 1)^2 \equiv 0 \pmod{5}
\]
So, $(2m + 1) \equiv 0 \pmod{5}$, and therefore, $c\phi_{2,-1}(5n+4) \equiv 0 \pmod{5}$. \hfill \Box

**References**

[1] G. Andrews, Generalized Frobenius Partitions, Mem. Amer. Math. Soc. 49, 1984, no. 301.

[2] B. Drake, Limits of areas under lattice paths, Discrete Math. 309 (2009) no. 12, 3936–3953.