A Quantitative Relation between Modulational Instability and the Well-known 
Nonlinear Excitations

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We study on explicit relations between modulational instability and analytical nonlinear excitations in a self-focusing Kerr nonlinear fiber in anomalous group velocity dispersion regime, such as bright soliton, nonlinear continuous wave, Akhmediev breather, Peregrine rogue wave, and Kuznetsov-Ma breather. We present a quantitative correspondence between them based on the dominant frequency and propagation constant of each perturbation on a continuous wave background. Especially, we find rogue wave comes from modulational instability under the “resonance” perturbation with continuous wave background. These results will deepen our realization on rogue wave excitation and could be helpful for controllable nonlinear waves excitations in nonlinear fiber and other nonlinear systems.

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I. INTRODUCTION

Modulation instability (MI) is a fundamental process observed in many nonlinear dispersive systems, associated with the growth of perturbations on a continuous wave background \cite{1}. In the initial evolution phase of MI, the spectral sidebands associated with the instability experience exponential amplification at the expense of the pump, but the subsequent dynamics are more complex and display cyclic energy exchange between multiple spectral modes \cite{2}. It has found important applications in optical amplification of weak signal, material absorption and loss compensation \cite{3}, dispersion management, all-optical switching \cite{4}, frequency comb for metrology \cite{5}, and so on \cite{6,7}. Recent numerical studies suggested that MI yields a series of high-contrast peaks in the evolving wave field seeded from noise on continuous wave background with randomly varying intensity in a self-focusing Kerr nonlinear fiber in anomalous group velocity dispersion regime, and the structure patterns can be described well by related analytic nonlinear excitation solutions \cite{10}, such as Akhmediev breather(AB), Peregrine rogue wave(RW), and Kuznetsov-Ma breather(KM) solutions \cite{11,12}. This comes from that the noise admits many different frequency modes and different modes demonstrate different dynamical evolutions.

Recently, these analytical nonlinear excitations were excited experimentally in nonlinear fiber \cite{14,15,16,17}, and even high-order RWs in a water wave tank \cite{18,19}. The results indicated that MI can be used to understand the dynamics of these nonlinear excitations \cite{15}. However, most comparisons between the properties of spontaneous MI and the analytic nonlinear excitations have been qualitative rather than quantitative. Quantitative relation between MI and these excitations is still unclear \cite{10}. For example, we just know that RW should come from MI mechanism, but which modes correspond to RW excitation has not been known precisely. It is noted that the analytical solutions can be written in the form of nonlinear continuous wave plus a perturbation term, which can be compared with MI analyze conveniently. This provides possibilities to clarify their explicit relation.

In this paper, we demonstrate that bright soliton(BS), nonlinear continuous wave(CW), AB, RW, and KM excitations can be located quantitatively on the MI gain spectrum plane. Moreover, we find that RW comes from modulational instability under the “resonance” conditions that the dominant frequency and propagation constant of perturbation signal are both equal to the continuous wave background’s. The results would be meaningful for controllable nonlinear wave excitations.

II. THE RELATIONS BETWEEN MODULATION INSTABILITY AND ANALYTIC NONLINEAR EXCITATIONS

In a Kerr nonlinear fiber, the propagation of optical field (pulse duration > 5 ps) can be described by the following nonlinear Schrödinger equation (NLSE) under slowly varying envelope approximation

\[ i\Psi_z + \frac{1}{2}\Psi_{tt} + \sigma|\Psi|^2\Psi = 0. \] (1)

The equation is written in dimensionless form \cite{20}. When the nonlinear coefficient $\sigma < 0$ (corresponding self-defocusing nonlinear fiber), it admits no MI regime on CW background for which dark soliton has been found on CW background \cite{21}. When the nonlinear coefficient $\sigma > 0$ (corresponding self-focusing nonlinear fiber), it admits MI and modulation stability(MS) regime on the MI gain spectrum continuous. Moreover, different types of nonlinear excitations mainly including BS, AB, K-M, and
We perform the standard linear instability analysis on CW background. Linear stability analysis can present us a good description on the amplification of each spectral mode. We perform the standard linear instability analysis on CW background. We present a generalized nonlinear wave solution from the seed solution \( \Psi_0 \) by Galilei transformation without losing any nontrivial dynamical characters. Therefore, we present a generalized nonlinear wave solution from the seed solution \( \Psi_0 \) as follows,
The parameter $b$ determines the initial nonlinear localized wave’s shape, and $s$ is the background amplitude for localized waves. In the following, we discuss properties of the nonlinear wave solution according to the two parameters. It should be noted that the solution mainly has two terms. The first term is the CW background, and the other term corresponds to perturbation term $f_{\text{pert}}$ in the above linear instability analyze. The second term can be varied to be many different type nonlinear excitations which can be analyzed exactly by Fourier transformation. The frequency spectrum admits a distribution. But there is always a dominant one among various frequencies, and the dominant one plays essential role in the perturbation evolution (see analyze on AB and K-M). This can be used to specify the relations between MI and analytical solutions.

Obviously, when $b = 0$, the solution will become the nonlinear CW solution. When $|b| > |s|$ and $s \neq 0$, the solution corresponds to K-M solution [11]. It is seen that the dominant frequency is at zero. Compare with the linear instability analyze results, we can know that

\begin{equation}
\Psi = \left[ s + 2b \frac{b \cos(2bz\sqrt{b^2-s^2}) + i\sqrt{b^2-s^2} \sin(2bz\sqrt{b^2-s^2}) - s \cosh(2t\sqrt{b^2-s^2})}{b \cosh(2t\sqrt{b^2-s^2}) - s \cosh(2bz\sqrt{b^2-s^2})} \right] e^{is^2z}.
\end{equation}

The spectrum analyze on the RW demonstrate that the dominant frequency of perturbation is equal to the CW’s. The MI amplified rate is maximum one for certain $s$. Therefore, K-M excitation is on the “resonance line” ($k' = 0$) except the point ($s = 0, k' = 0$) on the MI plane (see white dashed line in Fig. 1(b)).

When $s = 0$ and $b \neq 0$, the K-M solution can be reduced to a generalized bright soliton solution [22] as

\begin{equation}
\Psi = 2b \text{sech}(2bt)e^{2b^2z}.
\end{equation}

It is known that the this correspond to $s = 0$ and $k' = 0$ point on the MI spectrum. It has been found that BS is stable against small perturbations [20]. Moreover, in a practical situation, the solitons can be subjected to many types of perturbations as they propagate inside an optical fiber. Therefore, BS can be located as a line $s = 0$ on the MI spectrum (see black dotted line in Fig. 1(b)).

When $|b| < |s|$, the $\sqrt{b^2-s^2}$ in Eq. (2) will become $i\sqrt{s^2-b^2}$, and the solution will become the well-known AB solution [12] as

\begin{equation}
\Psi = \left[ s + 2b \frac{b \cosh(2bz\sqrt{s^2-b^2}) - i\sqrt{s^2-b^2} \sinh(2bz\sqrt{s^2-b^2}) - s \cos(2t\sqrt{s^2-b^2})}{b \cos(2t\sqrt{s^2-b^2}) - s \cosh(2bz\sqrt{s^2-b^2})} \right] e^{is^2z}.
\end{equation}

It is seen that the perturbation’s frequency can be varied arbitrarily in the certain regime $|b| < |s|$. Spectral analyze of the AB could tell us that the dominant mode is $2\sqrt{s^2-b^2}$, and there are some other much weaker nonzero frequency-multiplication modes of the perturbation. The spatial period of A-B is indeed determined by the difference between dominant frequency and CW background’s. This partly means that our analyze based on dominant frequency is reasonable. Varying the parameter $b$, the perturbation’s dominant mode can be changed. When $b = s$, the mode is on the resonance line, the AB excitation will become RW excitation [14]. Namely, the maximum peak emerge on the resonance line. Therefore, we call it as resonance excitation. The MI amplification rate will become smaller with decreasing the value of $b$. When $b = 0$, the mode will become the maximum frequencies $k' = \pm 2s$, they correspond to the point on the red dash-dotted line in Fig. 1(b). Namely, AB excitation is in the regimes between the white dashed line and red dash-dotted line in the MI continuous. Moreover, the breathing behavior for AB comes from the frequency difference between perturbation signal’s dominant one and CW background’s. It has been demonstrated that the spectral dynamics of MI can be described exactly by AB solution [27]. Based on the this, we can know that if the weak perturbation modes are in the zero MI band, the perturbation will be stable, and there is no significant excitations on CW background. The state of system can be still seen as CW. Therefore, we denote these regimes as CW phase on MI ground. The state of system can be still seen as CW.

When $b = s$, the K-M and AB solution can be both reduced to be RW solution [13] as

\begin{equation}
\Psi = \left[ 1 + \frac{-8s^4z^2 + 8is^2(z + it^2) + 2}{4s^4z^2 + 4s^2t^2 + 1} \right] s \cdot e^{is^2z}.
\end{equation}

The spectrum analyze on the RW demonstrate that the dominant frequency of RW is on the resonant line except
the point \((s = 0, k' = 0)\) on the MI plane (white dashed line in Fig. 1(b)). It is should be noted that K-M is on the line too. Then, how to distinguish RW and K-M?

Observing the analytical expression of K-M, we can know that K-M’s propagation constant is different from RW’s (see \(\cos(2bz \sqrt{b^2 - s^2})\) terms in Eq. (2)). With \(b \neq s\), the perturbation signal admits many propagation modes such as \(2n b \sqrt{b^2 - s^2} + s^2\) \((n = \pm 1, \pm 2, \pm 3, \cdots)\), with writing the K-M solution as \(\Psi = \Psi_0 + f_{\text{pert.}}\) form. The dominant one is \(\pm 2b \sqrt{b^2 - s^2} + s^2\). And the oscillating period is indeed determined by the dominant one. Therefore, the breathing behavior for K-M comes from \(b \rightarrow s\), the dominant perturbation propagation constant will be equal to the CW’s. Namely, RW also corresponds to the resonant excitation for perturbation propagation constant. This could be used to understand the mathematical process for RW derivation, for which the spectra of Lax-pair should be equal with each other for rational solution. Therefore, the degeneration of Lax-pair spectra corresponds to that the frequency and propagation constant of perturbation signal are both equal to the ones of CW seed. This result also stands for other coupled NLS systems, and NLS with high-order effects, since the RW solutions are all derived under the degenerations of Lax-pair spectra.

### III. CONCLUSION AND DISCUSSION

We present a quantitative correspondence between MI and BS, CW, RW, AB, and K-M solutions in Fig. 1(b), based on the dominant frequency and propagation constant of each perturbation signal. The results provide many possibilities to realize controllable nonlinear excitations, and are helpful to understand further on the numerical simulation results in [10]. Moreover, we find the breathing behavior of AB or K-M comes from the frequency or propagation constant mode difference between perturbation signal and CW background. Especially, RW comes from modulational instability under “resonance” perturbations for which both dominant frequency and propagation constant are equal to continuous wave background’s. It is well known that the degeneration of Lax-pair spectra can be used to derive rational solution which can be used to describe RW in many cases. It could be expected that it should be related with some resonance things. But this has never been known explicitly before. Here we find out that the degenerations correspond to the frequency and propagation constant of the perturbation signal are both equal to the ones of CW seed. This will deepen our realization on RW dynamics in many different physical systems, such as nonlinear fiber, Bose-Einstein condensate, water wave tank, plasma, and even financial systems.

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