Existence Results for Double Phase Problem in Sobolev–Orlicz Spaces with Variable Exponents in Complete Manifold

Ahmed Aberqi, Jaouad Bennouna, Omar Benslimane and Maria Alessandra Ragusa

Abstract. In this paper, we study the existence of non-negative non-trivial solutions for a class of double-phase problems where the source term is a Caratheodory function that satisfies the Ambrosetti–Rabinowitz type condition in the framework of Sobolev–Orlicz spaces with variable exponents in complete manifold. Our approach is based on the Nehari manifold and some variational techniques. Furthermore, the Hölder inequality, continuous and compact embedding results are proved.

Mathematics Subject Classification. Primary 35J20; Secondary 35J47, 35J60.

Keywords. Existence solutions, double phase problem, Sobolev–Orlicz Riemannian manifold, Nehari manifold.

1. Introduction

Let \((M, g)\) be a complete non-compact Riemannian manifold. In this paper, we focused on the existence of non-trivial solutions of the following double phase problem

\[
(P) \begin{cases}
-\text{div}( |\nabla u(x)|^{p(x)-2} \nabla u + \mu(x) |\nabla u(x)|^{q(x)-2} \nabla u) \\
= \lambda |u(x)|^{q(x)-2} u(x) - |u(x)|^{p(x)-2} u(x) + f(x, u(x)) & \text{in } M, \\
0 & \text{on } \partial M,
\end{cases}
\]

where \(-\Delta_{p(x)} u(x) = -\text{div}( |\nabla u(x)|^{p(x)-2} \cdot \nabla u(x))\) and \(-\Delta_{q(x)} u(x) = -\text{div}( |\nabla u(x)|^{q(x)-2} \cdot \nabla u(x))\) are the \(p(x)\)-laplacian and \(q(x)\)-laplacian in \((M, g)\) respectively, \(\lambda > 0\) is a parameter specified later, the function \(\mu : \overline{M} \rightarrow [0, +\infty)\) is supposed to be Lipschitz continuous, and the variables exponents \(p, q \in C(\overline{M})\) satisfy the assumption (3.1) in Sect. 3.

The perturbation \(f(x, u)\) is a Caratheodory function which satisfies the Ambrosetti–Rabinowitz type condition:
(f_1): There exists $\beta > q^+$ and some $A > 0$ such as for each $|\alpha| > A$ we have

$$0 < \int_M F(x, \alpha) \, dv_g(x) \leq \int_M f(x, \alpha) \cdot \frac{\alpha}{\beta} \, dv_g(x) \text{ a.e } x \in M,$$

where $F(x, \alpha) = \int_0^\alpha f(x, t) \, dt$ being the primitive of $f(x, \alpha)$.

And

(f_2): $f(x, 0) = 0$.

(f_3): $\lim_{|\alpha| \to 0} \frac{f(x, \alpha)}{|\alpha|^{p(x)-1}} = 0$ uniformly a.e $x \in M$.

Up to this day, several contributions have been devoted to study double phase problems. This kind of operator was introduced, first, by Zhikov in his relevant paper [24] in order to describe models with strongly anisotropic materials by studying the functional

$$u \mapsto -\int_{\Omega} \left( |\nabla u|^p + \mu(x) |\nabla u|^q \right) \, dx,$$

where $1 < p < q < N$ and with a nonnegative weight function $\mu \in L^\infty(\Omega)$, see also the works of Zhikov [25,26] and the monograph of Zhikov et al. [27]. Indeed, we can easily see that the previous function reduces to p-laplacian if $\mu(x) = 0$ or to the weighted laplacian $(p(x), q(x))$ if $\inf_{x \in \Omega} \mu(x) > 0$, respectively.

In the case of single valued equations, Liu and Dai in [16] discussed double phase problems and proved the existence and the multiplicity of the results, with the sign-changing solutions by variational method. A similar treatment has been recently done by Gasiński and Papageorgiou in [13] via the Nehari manifold method. Following this direction, Papageorgiou et al. [18] studied the existence of positive solutions for a class of double phase Dirichlet equations which has the combined effects of a singular term and of a parametric super-linear term. In particular, in [23] the author provides the Hölder continuity up to the boundary of minimizers of so-called double phase functional with variable exponents, under suitable Dirichlet boundary conditions. For more details, we refer the reader to [3,9,12,14,15,17,20,22] and the references therein.

Also, there are many articles on nonstandard growth problems, especially on $p(x)$-growth and double phase problems. About $p(x)$-growth problems, see [1,2,4–8,10,11,21] and the references given there.

Studying this type of problems is both significant and relevant. In the one hand, we have the physical motivation; since the double phase operator has been used to model the steady-state solutions of reaction-diffusion problems, that arise in biophysic, plasma-physic and in the study of chemical reactions. In the other hand, these operators provide a useful paradigm for describing the behaviour of strongly anisotropic materials, whose hardening properties are linked to the exponent governing the growth of the gradient change radically with the point, where the coefficient $\mu(\cdot)$ determines the geometry of a composite made of two different materials.

Motivated by the aforementioned works, the aim of this paper, is to prove the existence of non-negative non-trivial solutions of the problem $(P)$
where the perturbation \( f(x, u) \) is a Carathéodory function, that satisfies the Ambrosetti–Rabinowitz type condition. To the best of our knowledge, the existence result for double-phase problems (\( P \)) in the framework of Sobolev–Orlicz spaces with variable exponents in complete manifold has not been considered in the literature. The present paper is the first study devoted to this type of problem in the setting of Sobolev–Orlicz spaces with variable exponents in a complete manifold.

We would like to draw attention to the fact that the \( p(x) \)-laplacian operator has more complicated non-linearity than the p-laplacian operator. For example, they are non-homogeneous. Thus, we cannot use the Lagrange Multiplier Theorem in many problems involving this operators, which prove that our problem is more difficult than the operators p-Laplacian type.

The paper is organized as follows. In Sect. 2, we recall the most important and relevant properties and notations of Lebesgue spaces with variable exponents and Sobolev–Orlicz spaces with variable exponents in complete manifold. Moreover, we show two new results: the first one, the Hölder inequality and the second one, the embedding result of these spaces into Lebesgue space with variable exponent. In Sect. 3, we introduce the Nehari manifold associated with (\( P \)) and we study three parts, corresponding to local minima, local maxima and the points of inflection. Finally, in Sect. 4, we demonstrate the existence of two non-negative non-trivial solutions of the problem (\( P \)).

2. Mathematical Background and Auxiliary Results

In this section, we recall the most important and relevant properties and notations about Sobolev spaces with variable exponents and Sobolev spaces with variable exponents in complete manifolds, and we prove some properties, that we will need in our analysis of the problem (\( P \)), by that, referring to [3, 6, 12, 15, 19] for more details.

2.1. Sobolev Spaces with Variable Exponents

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) (\( N \geq 2 \)), we define the Lebesgue space with variable exponent \( L^{q(\cdot)}(\Omega) \) as the set of all measurable function \( u : \Omega \rightarrow \mathbb{R} \) for which the convex modular

\[
\rho_{q(\cdot)}(u) = \int_{\Omega} |u(x)|^{q(x)} \, dx,
\]

is finite. If the exponent is bounded, i.e if \( q^+ = \text{ess sup}\{ q(x)/x \in \Omega \} < +\infty \), then the expression

\[
\| u \|_{q(\cdot)} = \inf\{ \lambda > 0 : \rho_{q(\cdot)} \left( \frac{u}{\lambda} \right) \leq 1 \},
\]

defines a norm in \( L^{q(\cdot)}(\Omega) \), called the Luxemburg norm. The space \( (L^{q(\cdot)}(\Omega), \| \cdot \|_{q(\cdot)}) \) is a separable Banach space. Moreover, if \( 1 < q^- \leq q^+ < +\infty \), then \( L^{q(\cdot)}(\Omega) \) is uniformly convex, where \( q^- = \text{ess inf}\{ q(x)/x \in \Omega \} \), hence reflexive, and its dual space is isomorphic to \( L^{q'(\cdot)}(\Omega) \) where \( \frac{1}{q(x)} + \frac{1}{q'(x)} = 1 \).
Finally, we have the Hölder type inequality:
\[
\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{q^*} + \frac{1}{(q')^*} \right) \| u \|_{q(x)} \| v \|_{q'(x)},
\]
for all \( u \in L^q(\Omega) \) and \( v \in L^{q'}(\Omega) \).

**Proposition 2.1.** [10] (Poincaré inequality) If \( q \in C_+(\Omega) \), then there is a constant \( c > 0 \) such that
\[
\| u \|_{q(x)} \leq c \| \nabla u \|_{q(x)}, \quad \forall u \in W^{1,q(x)}(\Omega).
\]
Consequently, \( \| u \| = \| \nabla u \|_{q(x)} \) and \( \| u \|_{1,q(x)} \) are equivalent norms on \( W^{1,q(x)}(\Omega) \).

### 2.2. Sobolev Spaces with Variable Exponents in Complete Manifolds

**Definition 2.2.** Let \((M,g)\) be a smooth Riemannian manifolds and let \( \nabla \) be the Levi-Civita connection. If \( u \) is a smooth function on \( M \), then \( \nabla^k u \) denotes the \( k \)-th covariant derivative of \( u \), and \( |\nabla^k u| \) the norm of \( \nabla^k u \) defined in local coordinates by
\[
|\nabla^k u|^2 = g^{i_1,j_1} \cdots g^{i_k,j_k} (\nabla^k u)_{i_1 \cdots i_k} (\nabla^k u)_{j_1 \cdots j_k},
\]
where Einstein’s convention is used.

**Definition 2.3.** To define variable Sobolev spaces, given a variable exponent \( q \) in \( \mathcal{P}(M) \) (the set of all measurable functions \( p(\cdot) : M \to [1, \infty] \)) and a natural number \( k \), introduce
\[
C^{q(\cdot)}_k(M) = \{ u \in C^\infty(M) \text{ such that } \forall j \ 0 \leq j \leq k \ |\nabla^j u| \in L^q(\cdot)(M) \}.
\]
On \( C^{q(\cdot)}_k(M) \) define the norm
\[
\| u \|_{L^{q(\cdot)}_k(M)} = \sum_{j=0}^k \| \nabla^j u \|_{L^q(\cdot)(M)}.
\]

**Definition 2.4.** The Sobolev spaces \( L^{q(\cdot)}_k(M) \) is the completion of \( C^{q(\cdot)}_k(M) \) with respect to the norm \( \| u \|_{L^{q(\cdot)}_k} \).

**Definition 2.5.** Given \((M,g)\) a smooth Riemannian manifold, and \( \gamma : [a, b] \to M \) is a curve of class \( C^1 \). The length of \( \gamma \) is
\[
l(\gamma) = \int_a^b \sqrt{g \left( \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right)} \, dt,
\]
and for a pair of points \( x, y \in M \), we define the distance \( d_g(x,y) \) between \( x \) and \( y \) by
\[
d_g(x,y) = \inf \{ l(\gamma) : \gamma : [a, b] \to M \text{ such that } \gamma(a) = x \text{ and } \gamma(b) = y \}.
\]

**Definition 2.6.** A function \( s : M \to \mathbb{R} \) is log-Hölder continuous if there exists a constant \( c \) such that for every pair of points \( \{ x, y \} \) in \( M \) we have
\[
| s(x) - s(y) | \leq \frac{c}{\log(e + \frac{1}{d_g(x,y)})}.
\]
We note by $\mathcal{P}^{log}(M)$ the set of log-Hölder continuous variable exponents. The relation between $\mathcal{P}^{log}(M)$ and $\mathcal{P}^{log}(\mathbb{R}^N)$ is the following:

Proposition 2.7. [3,12] Let $q \in \mathcal{P}^{log}(M)$, and let $(\Omega, \phi)$ be a chart such that

$$\frac{1}{2} \delta_{ij} \leq g_{ij} \leq 2 \delta_{ij}$$

as bilinear forms, where $\delta_{ij}$ is the delta Kronecker symbol. Then $q \circ \phi^{-1} \in \mathcal{P}^{log}(\phi(\Omega))$.

Proposition 2.8. (Hölder’s inequality) For all $u \in L^{q(\cdot)}(M)$ and $v \in L^{q'(\cdot)}(M)$ we have

$$\int_M |u(x) \cdot v(x)| \, dv_g(x) \leq r_q \|u\|_{L^{q(\cdot)}(M)} \cdot \|v\|_{L^{q'(\cdot)}(M)}.$$ 

**Proof.** Obviously, we can suppose that $\|u\|_{L^{q(\cdot)}(M)} \neq 0$ and $\|v\|_{L^{q'(\cdot)}(M)} \neq 0$, we have

$$1 < q(x) < \infty, \ |u(x)| < \infty, \ |v(x)| < \infty \text{ a.e } x \in M.$$ 

By young inequality, we have

$$\frac{u(x) \cdot v(x)}{\|u(x)\|_{L^{q(\cdot)}(M)} \cdot \|v(x)\|_{L^{q'(\cdot)}(M)}} \leq \frac{1}{q(x)} \left( \frac{|u(x)|}{\|u(x)\|_{L^{q(\cdot)}(M)}} \right)^{q(x)} + \frac{1}{q'(x)} \left( \frac{|v(x)|}{\|v(x)\|_{L^{q'(\cdot)}(M)}} \right)^{q'(x)}$$

Integrating over $M$, we obtain

$$\int_M \frac{|u(x) \cdot v(x)|}{\|u(x)\|_{L^{q(\cdot)}(M)} \cdot \|v(x)\|_{L^{q'(\cdot)}(M)}} \, dv_g(x) \leq \frac{1}{q^-} \int_M \left( \frac{|u(x)|}{\|u(x)\|_{L^{q(\cdot)}(M)}} \right)^{q(x)} \, dv_g(x)$$

$$+ \left( 1 - \frac{1}{q^+} \right) \int_M \left( \frac{|v(x)|}{\|v(x)\|_{L^{q'(\cdot)}(M)}} \right)^{q'(x)} \, dv_g(x)$$

$$\leq 1 + \frac{1}{q^-} - \frac{1}{q^+},$$

then,

$$\int_M |u(x) \cdot v(x)| \, dv_g(x) \leq \left( 1 + \frac{1}{q^-} + \frac{1}{q^+} \right) \|u(x)\|_{L^{q(\cdot)}(M)} \cdot \|v(x)\|_{L^{q'(\cdot)}(M)}$$

$$\leq r_q \|u(x)\|_{L^{q(\cdot)}(M)} \cdot \|v(x)\|_{L^{q'(\cdot)}(M)},$$

where $|M|$ be the measure of $M$ and $|M| < \infty$.

Which complete the proof.
Remark 2.9. If \( a \) and \( b \) are two positive functions on \( M \), then by H"older’s inequality and [10,12] we have
\[
\int_{q<2} a^{\frac{q}{2}} b^{\frac{2q-q^2}{2}} \leq 2 \| 1_{q<2} a^{\frac{q}{2}} \|_{L^\frac{q}{2}} \| 1_{q<2} b^{\frac{2q-q^2}{2}} \|_{L^\frac{2q-q^2}{2-q}}. \tag{2.1}
\]
where \( 1 \) is the indicator function of \( M \), moreover, since
\[
\| 1_{q<2} a^{\frac{q}{2}} \|_{L^\frac{q}{2}} \leq \max \{ \rho_1(a), \rho_1(a)^{\frac{q}{2}} \}
\]
and
\[
\| 1_{q<2} b^{\frac{2q-q^2}{2}} \|_{L^\frac{2q-q^2}{2-q}} \leq \max \{ \rho_q(b)^{\frac{2-q}{2-q}}, 1 \},
\]
we get,
\[
\int_{q<2} a^{\frac{q}{2}} b^{\frac{2q-q^2}{2}} \leq 2 \max \{ \rho_1(a), \rho_1(a)^{\frac{q}{2}} \} \max \{ \rho_q(b)^{\frac{2-q}{2-q}}, 1 \}. \tag{2.2}
\]

Definition 2.10. We say that the \( n \)-manifold \((M,g)\) has property \( B_{\text{vol}}(\lambda, v) \) if its geometry is bounded in the following sense:

- The Ricci tensor of \( g \) noted by \( \text{Rc}(g) \) verifies, \( \text{Rc}(g) \geq \lambda(n-1)g \) for some \( \lambda \).
- There exists some \( v > 0 \) such that \( |B_1(x)|_g \geq v \forall x \in M \), where \( B_1(x) \) are the balls of radius 1 centered at some point \( x \) in terms of the volume of smaller concentric balls.

Remark 2.11. If \( M = \Omega \subseteq \mathbb{R}^N \) is a bounded open set, then the following inequality is related to the two exponents \( p, q \) (isotropic case)
\[
\frac{q}{p} < 1 + \frac{1}{N}.
\]
This condition is essential, among others, for the embeddings of spaces to be satisfied.

Proposition 2.12 [3,15]. Let \((M,g)\) be a complete Riemannian \( n \)-manifold. Then, if the embedding \( L^1_q(M) \hookrightarrow L^{n-q}_1(M) \) holds, then whenever the real numbers \( q \) and \( p \) satisfy
\[
1 \leq q < n,
\]
and
\[
q \leq p \leq q^* = \frac{Nq}{n-q},
\]
the embedding \( L^1_q(M) \hookrightarrow L^p(M) \) also holds.

Proposition 2.13 [3,15]. Assume that the complete \( n \)-manifold \((M,g)\) has property \( B_{\text{vol}}(\lambda, v) \) for some \((\lambda, v)\). Then there exist positive constants \( \delta_0 = \delta_0(n, \lambda, v) \) and \( A = A(n, \lambda, v) \), we have, if \( R \leq \delta_0 \), if \( x \in M \) if \( 1 \leq q \leq n \), and if \( u \in L^q_1(B_R(x)) \) the estimate
\[
\| u \|_{L^p} \leq A_p \| \nabla u \|_{L^q},
\]
where \( \frac{1}{p} = \frac{1}{q} - \frac{1}{n} \).
Proposition 2.14 [3, 12, 15]. Assume that for some \((\lambda, v)\) the complete \(n\)-manifold \((M, g)\) has property \(B_{vol}(\lambda, v)\). Let \(p \in \mathcal{P}(M)\) be uniformly continuous with \(q^+ < n\). Then \(L^{q}(M) \hookrightarrow L^{p}(M)\), \(\forall q \in \mathcal{P}(M)\) such that \(q \ll p \ll q^* = \frac{nq}{n-q}\). In fact, for \(\|u\|_{L^{q}(M)}\) sufficiently small we have the estimate

\[
\rho_{p,q}(u) \leq G(\rho_{q,p}(u) + \rho_{q}(|\nabla u|)),
\]

where the positive constant \(G\) depend on \(n, \lambda, v, q,\) and \(p\).

Proposition 2.15. Let \(u \in L^{q}(x)(M), \{u_k\} \subset L^{q}(x)(M), k \in \mathbb{N}\), then we have

(i) \(\|u\|_{q(x)} < 1\) (resp. \(= 1, \delta 1\)) \(\iff\) \(\rho_{q}(u) < 1\) (resp. \(= 1, \delta 1\)),

(ii) \(\|u\|_{q(x)} < 1 \Rightarrow \|u\|_{q^+(x)}^{q^+} \leq \rho_{q}(u) \leq \|u\|_{q^-(x)}^{q^-}\),

(iii) \(\|u\|_{q(x)} > 1 \Rightarrow \|u\|_{q^+(x)}^{q^+} \leq \rho_{q}(u) \leq \|u\|_{q^-(x)}^{q^-}\),

(iv) \(\lim_{k \to +\infty} \|u_k - u\|_{q(x)} = 0 \iff \lim_{k \to +\infty} \rho_{q}(u_k - u) = 0\).

Definition 2.16. The Sobolev space \(W^{1,q}(x)(M)\) consists of such functions \(u \in L^{q}(x)(M)\) for which \(\nabla^{k} u \in L^{q}(x)(M)\) \(k = 1, 2, \ldots, n\). The norm is defined by

\[
\| u \|_{W^{1,q}(x)(M)} = \| u \|_{L^{q}(x)(M)} + \sum_{k=1}^{n} \| \nabla^{k} u \|_{L^{q}(x)(M)}.
\]

The space \(W^{1,q}(x)(M)\) is defined as the closure of \(C_c^\infty(M)\) in \(W^{1,q}(x)(M)\), with \(C_c^\infty(M)\) be the vector space of smooth functions with compact on \(M\).

Theorem 2.17. Let \(M\) be a compact Riemannian manifold with a smooth boundary or without boundary and \(q(x), p(x) \in C(M) \cap L^\infty(M)\). Assume that

\[
q(x) < N, \quad p(x) < \frac{Nq(x)}{N-q(x)} \text{ for } x \in \overline{M}.
\]

Then,

\[
W^{1,q(x)}(M) \hookrightarrow L^{p(x)}(M)
\]

is a continuous and compact embedding.

Proof. This proof is based to an idea introduced in [11, 14]. Let \(f : U(\subset M) \longrightarrow \mathbb{R}^N\) be an arbitrary local chart on \(M\), and \(V\) be any open set in \(M\), whose closure is compact and is contained in \(U\). Choosing a finite subcovering \(\{V_\alpha\}_{\alpha = 1, \ldots, k}\) of \(M\) such that \(V_\alpha\) is homeomorphic to the open unit ball \(B_0(1)\) of \(\mathbb{R}^N\) and for any \(\alpha\) the components \(g_{ij}^\alpha\) of \(g\) in \((V_\alpha, f_\alpha)\) satisfy

\[
\frac{1}{\epsilon \delta_{ij}} \leq g_{ij}^\alpha < \epsilon \delta_{ij}
\]

as bilinear forms, where the constant \(\epsilon > 1\) is given. Let \(\{\pi_\alpha\}_{\alpha = 1, \ldots, k}\) be a smooth partition of unity subordinate to the finite covering \(\{V_\alpha\}_{\alpha = 1, \ldots, k}\). It is obvious that if \(u \in W^{1,q(x)}(M)\), then \(\pi_\alpha u \in W^{1,q(x)}(V_\alpha)\) and \((f_\alpha^{-1})^*(\pi_\alpha u) \in\)
According to proposition 2.12 and the Sobolev embeddings Theorem in [11,12], we obtain the continuous and compact embedding
\[ W^{1,q(x)}(V_\alpha) \hookrightarrow L^{p(x)}(V_\alpha) \quad \text{for each } \alpha = 1, \cdots, k. \]
Since \( u = \sum_{\alpha=1}^{k} \pi_\alpha u \), we can conclude that
\[ W^{1,q(x)}(M) \subset L^{p(x)}(M), \]
and the embedding is continuous and compact. \( \square \)

**Proposition 2.18.** [3] If \((M,g)\) is complete, then \( W^{1,q(x)}(M) = W^{1,q(x)}_0(M) \).

### 3. Nehari Manifold Analysis for \((P)\)

In this section, we note by \( D(M) \) the space of \( C^\infty \) functions with compact support in \( M \).

**Definition 3.1.** \( u \in W^{1,q(x)}_0(M) \) is said to be a weak solution of the problem \((P)\) if for every \( \phi \in D(M) \) we have
\[
\int_M \left( |\nabla u(x) |^{p(x)-2} + \mu(x) |\nabla u(x) |^{q(x)-2} \right) \cdot g(\nabla u(x), \nabla \phi(x)) \, dv_g(x)
= \lambda \int_M |u(x)|^{q(x)-2} \cdot u(x) \phi(x) \, dv_g(x) - \int_M |u(x)|^{p(x)-2} \cdot u(x) \cdot \phi(x) \, dv_g(x)
+ \int_M f(x,u(x)) \cdot \phi(x) \, dv_g(x).
\]

The variable exponents \( p, q \in C(\overline{M}) \), are assumed to satisfy the following assumption:
\[
1 < q^- \leq q^+ < p^- \leq p^+ < N, \tag{3.1}
\]
and suppose \( \frac{p^-}{q^+} \leq 1 + \frac{1}{N} \), the function \( \mu : \overline{M} \to [0, +\infty) \) is Lipschitz continuous.

Let us consider the energy functional \( J_\lambda : W^{1,q(x)}_0(M) \to \mathbb{R} \) associated to problem \((P)\) which is defined by
\[
J_\lambda(u) = \frac{1}{p(x)} \int_M |\nabla u(x) |^{p(x)} \, dv_g(x) + \frac{\mu(x)}{q(x)} \int_M |\nabla u(x) |^{q(x)} \, dv_g(x)
- \frac{\lambda}{q(x)} \int_M |u(x)|^{q(x)} \, dv_g(x) + \frac{1}{p(x)} \int_M |u(x)|^{p(x)} \, dv_g(x)
- \int_M F(x,u(x)) \, dv_g(x).
\]
Indeed, for any \( u \in W^{1,q(x)}_0(M) \) with \( \| u \|_{W^{1,q(x)}_0(M)} > 1 \), we have by \((f_1), (f_2)\), propositions 2.13 and 2.15 that

\[
J_\lambda(u) \geq \frac{1}{p^+} \int_{M} |\nabla u(x)|^{p(x)} \, dv_g(x) + \frac{\mu(x)}{q^+} \int_{M} |\nabla u(x)|^{q(x)} \, dv_g(x) \\
- \frac{\lambda}{q^+} \int_{M} |u(x)|^{q(x)} \, dv_g(x) + \frac{1}{p^+} \int_{M} |u(x)|^{p(x)} \, dv_g(x) \\
- \frac{1}{\beta} \int_{M} |u(x)|^{p(x)} \, dv_g(x)
\]

\[
\geq c_1 \rho_{p(\cdot)}(u) + \frac{\mu(x)}{\lambda \rho_{p(\cdot)}(u)} - \frac{\lambda}{q^+} \rho_{q(\cdot)}(u) - \frac{1}{q^+} \rho_{q(\cdot)}(u)
\]

\[
\geq \left( c_1 + \frac{\mu(x)}{A p q^+} + \frac{1}{p^+} \frac{1}{q^+} \right) \rho_{p(\cdot)}(u) - \frac{\lambda}{q^+} \rho_{q(\cdot)}(u)
\]

\[
\geq \left( c_1 + \frac{\mu(x)}{A p q^+} + \frac{1}{p^+} \frac{1}{q^+} \right) \| u \|^{p^+}_{W^{1,q(x)}_0(M)} - \frac{\lambda}{q^+} \| u \|^{q^+}_{W^{1,q(x)}_0(M)}.
\]

From (3.1), we have that \( J_\lambda \) is not bounded below on \( W^{1,q(x)}_0(M) \).

The Nehari manifold associated to \( J_\lambda \) defined by

\[
\mathcal{N}_\lambda = \{ u \in W^{1,q(x)}_0(M) \setminus \{0\} : \langle J_\lambda'(u), u \rangle = 0 \}.
\]

It is clear that the critical points of the functional \( J_\lambda \) must lie on \( \mathcal{N}_\lambda \) and local minimizers on \( \mathcal{N}_\lambda \) are usually critical points of \( J_\lambda \). Thus, \( u \in \mathcal{N}_\lambda \) if and only if

\[
\langle J_\lambda'(u), u \rangle = \int_{M} |\nabla u(x)|^{p(x)} \, dv_g(x) + \mu(x) \int_{M} |\nabla u(x)|^{q(x)} \, dv_g(x) \\
- \lambda \int_{M} |u(x)|^{q(x)} \, dv_g(x) + \int_{M} |u(x)|^{p(x)} \, dv_g(x) \\
- \int_{M} f(x, u(x)) \cdot u(x) \, dv_g(x) = 0.
\]

Hence, \( \mathcal{N}_\lambda \) contains every nontrivial weak solution of problem \((\mathcal{P})\) (see definition 3.1). Moreover, we have the following result

**Lemma 3.2.** Under assumptions \((f_1) - (f_3)\). The energy functional \( J_\lambda \) is coercive and bounded below on \( W^{1,q(x)}_0(M) \).

**Proof.** Let \( u \in \mathcal{N}_\lambda \) and \( \| u \|_{W^{1,q(x)}_0(M)} > 1 \), we have

\[
J_\lambda(u) \geq \frac{1}{p^+} \int_{M} |\nabla u(x)|^{p(x)} \, dv_g(x) + \frac{\mu(x)}{q^+} \int_{M} |\nabla u(x)|^{q(x)} \, dv_g(x) \\
- \frac{\lambda}{q^+} \int_{M} |u(x)|^{q(x)} \, dv_g(x),
\]
by (3.1), (f₁), (f₃), propositions 2.13 and 2.15 we obtain
\[
J_\lambda(u) \geq \mu(x) \left( \frac{1}{q^+} - \frac{1}{p^+} \right) \int_M |\nabla u(x)|^{q(x)} \, dv_g(x) \\
+ \lambda \left( \frac{1}{p^+} - \frac{1}{q^+} \right) \int_M |u(x)|^{q(x)} \, dv_g(x) \\
\geq \frac{\mu(x)}{Ap} \left( \frac{p^+ - q^+}{p^+ q^+} \right) \rho_{p(\cdot)}(u) + \lambda \left( \frac{q^- - p^+}{p^+ q^-} \right) \rho_{q(\cdot)}(u) \\
\geq \frac{\mu(x)}{Ap} \left( \frac{p^+ - q^+}{p^+ q^+} \right) \| u \|^{p^-}_{W^{1,q(x)}_0(M)} + \lambda \left( \frac{q^- - p^+}{p^+ q^-} \right) \| u \|^{q^+}_{W^{1,q(x)}_0(M)}.
\]
As \( p^- > q^+ \), then \( J_\lambda(u) \to +\infty \) as \( \| u \|_{W^{1,q(x)}_0(M)} \to \infty \). It follows that \( J_\lambda \) is coercive and bounded below on \( \mathcal{N}_\lambda \).

Next, we consider the functional \( \psi : \mathcal{N}_\lambda \to \mathbb{R} \) defined by
\[
\psi_\lambda(u) = \langle J_\lambda'(u), u \rangle \text{ for all } u \in \mathcal{N}_\lambda.
\]
Hence, it is natural to split \( \mathcal{N}_\lambda \) into three part: the first set corresponding to local minima, the second set corresponding to local maxima, and the third one corresponding to points of inflection which defined respectively as follows
\[
\mathcal{N}^+_{\lambda} = \{ u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u \rangle > 0 \}, \\
\mathcal{N}^-_{\lambda} = \{ u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u \rangle < 0 \}, \\
\mathcal{N}^{0}_{\lambda} = \{ u \in \mathcal{N}_\lambda : \langle \psi'_\lambda(u), u \rangle = 0 \}.
\]

Lemma 3.3. Under assumptions (f₁) – (f₃). There exists \( \lambda^* > 0 \) such that for any \( \lambda \in (0, \lambda^*) \) we have \( \mathcal{N}^{0}_{\lambda} = \emptyset \).

Proof. Suppose otherwise, that is \( \mathcal{N}^{0}_{\lambda} \neq \emptyset \) for all \( \lambda \in \mathbb{R} \setminus \{0\} \). Let \( u \in \mathcal{N}^{0}_{\lambda} \) such that \( \| u \| > 1 \). Then by (3.1), (f₁) and the definition of \( \mathcal{N}^{0}_{\lambda} \), we have
\[
0 = \langle \psi'_\lambda(u), u \rangle \geq p^- \int_M |\nabla u(x)|^{p(x)} \, dv_g(x) + \mu(x) q^- \int_M |\nabla u(x)|^{q(x)} \, dv_g(x) \\
- q^+ \left[ \int_M |\nabla u(x)|^{p(x)} \, dv_g(x) + \mu(x) \int_M |\nabla u(x)|^{q(x)} \, dv_g(x) \\
+ \int_M |u(x)|^{p(x)} \, dv_g(x) - \int_M f(x, u(x)) \cdot u(x) \, dv_g(x) \right] \\
+ p^- \int_M |u(x)|^{p(x)} \, dv_g(x) - \int_M F(x, u(x)) \, dv_g(x) \\
\geq \mu(x) (q^- - q^+) \int_M |\nabla u(x)|^{q(x)} \, dv_g(x) \\
+ (c_1 + p^- - q^+) \int_M |u(x)|^{p(x)} \, dv_g(x),
\]
with \( c_1 \) is a constant positive. By proposition 2.14 we get
\[
\frac{1}{G} \rho_{p(\cdot)}(u) - \rho_{q(\cdot)}(u) \leq \rho_{q(\cdot)}(|\nabla u|).
\]
Hence, 
\[
\frac{\mu(x)}{G} (q^- - q^+) \rho_{p(\cdot)}(u) - \mu(x) (q^- - q^+) \rho_{q(\cdot)}(u) \geq \mu(x)(q^- - q^+) \rho_{q(\cdot)}(\|\nabla u\|),
\]
which implies 
\[
\frac{\mu(x)}{G} (q^- - q^+) \rho_{p(\cdot)}(u) - \mu(x) (q^- - q^+) \rho_{q(\cdot)}(u) \geq (c_1 + p^- - q^+) \rho_{p(\cdot)}(u).
\]
Then,
\[
\mu(x)(q^+ - q^-) \rho_{q(\cdot)}(u) \geq \left[c_1 + p^- - q^+ + \frac{\mu(x)}{G}(q^+ - q^-)\right] \rho_{p(\cdot)}(u).
\]
Thus,
\[
\|u\|_{W_0^{1,q(x)}(M)} \geq \left(\frac{\mu(x)(q^+ - q^-)}{c_1 + p^- - q^+ + \frac{\mu(x)}{G}(q^+ - q^-)}\right)^{-\frac{1}{p^- - q^+}}. \tag{3.3}
\]
Analogously:
\[
0 = \langle \psi'_\lambda(u), u \rangle \leq p^+ \int_M |\nabla u(x)|^{p(x)} \, dv_g(x) + \mu(x) q^+ \int_M |\nabla u(x)|^{q(x)} \, dv_g(x) \\
- \lambda q^- \int_M |u(x)|^{q(x)} \, dv_g(x) + p^+ \left[ - \int_M |\nabla u(x)|^{p(x)} \, dv_g(x) \right]
- \mu(x) \int_M |\nabla u(x)|^{q(x)} \, dv_g(x) + \lambda \int_M |u(x)|^{q(x)} \, dv_g(x) \\
+ \int_M f(x, u(x)) \cdot u(x) \, dv_g(x) - \int_M F(x, u(x)) \, dv_g(x) \\
\leq \mu(x)(q^+ - p^+) \int_M |\nabla u(x)|^{q(x)} \, dv_g(x) \\
+ \lambda(p^+ - q^-) \int_M |u(x)|^{q(x)} \, dv_g(x),
\]
by proposition 2.13 we deduce that
\[
0 \leq \frac{\mu(x)}{A_p} (q^+ - p^+) \rho_{p(\cdot)}(u) + \lambda (p^+ - q^-) \rho_{q(\cdot)}(u).
\]
Then,
\[
\frac{\mu(x)}{A_p} (p^+ - q^+) \|u\|_{P^-} \leq \lambda (p^+ - q^-) \|u\|_{q^+}.
\]
Thus,
\[
\|u\|_{W_0^{1,q(x)}(M)} \leq \left(\frac{\lambda A_p}{\mu(x)} (p^+ - q^-) \right)^{-\frac{1}{p^- - q^+}}. \tag{3.4}
\]
For \(\lambda\) sufficiently small \((\lambda < \frac{\mu(x)(p^+ - q^+)}{A_p (p^+ - q^+) c_1 + p^- - q^+ + \frac{\mu(x)(q^+ - q^-)}{G}} = \lambda^*\), if we combining (3.3) and (3.4) we find \(\|u\| < 1\), which contradicts our assumption. Consequently, we can conclude that there exists \(\lambda^* > 0\) such that \(N_\lambda^0 = \emptyset\) for any \(\lambda \in (0, \lambda^*)\). \qed
Remark 3.4. As a consequence of Lemma 3.3, for $0 < \lambda < \lambda^*$, we can write $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$, and we define

$$\theta_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u), \quad \theta_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u).$$

Lemma 3.5. Suppose that $(f_1) - (f_3)$ are true. If $0 < \lambda < \lambda^*$, then for all $u \in \mathcal{N}_\lambda^+$ we have $J_\lambda(u) < 0$.

Proof. Suppose $u \in \mathcal{N}_\lambda^+$, from the definition of $J_\lambda$, we have

$$J_\lambda(u) \leq \frac{1}{p^-} \int_M |\nabla u(x)|^{p(x)} \, dv_g(x) + \frac{\mu(x)}{q^-} \int_M |\nabla u(x)|^{q(x)} \, dv_g(x)$$

$$- \frac{\lambda}{q^+} \int_M |u(x)|^{q(x)} \, dv_g(x) + \frac{1}{p^+} \int_M |u(x)|^{p(x)} \, dv_g(x) - \int_M F(x, u(x)) \, dv_g(x),$$

from (3.2) and (3.5) we have

$$J_\lambda(u) \leq \frac{1}{p^-} \int_M |\nabla u(x)|^{p(x)} \, dv_g(x) + \frac{\mu(x)}{q^-} \int_M |\nabla u(x)|^{q(x)} \, dv_g(x)$$

$$- \frac{\lambda}{q^+} \int_M |u(x)|^{q(x)} \, dv_g(x) + \frac{1}{p^+} \int_M |u(x)|^{p(x)} \, dv_g(x) - \int_M F(x, u(x)) \, dv_g(x),$$

using Poincaré inequality and (3.1) we find:

$$J_\lambda(u) \leq k \| u \|_{W^{1, q(x)}(M)}^p \quad \text{with} \quad k = k(p^+, q^-, c) < 0.$$

Finally, we deduce that $\theta_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u) < 0$. \qed

Lemma 3.6. Under assumptions $(f_1) - (f_3)$. If $0 < \lambda < \lambda^{**}$, then for all $u \in \mathcal{N}_\lambda^-$ we have $J_\lambda(u) > 0$.

Proof. Let $u \in \mathcal{N}_\lambda^-$. By (3.1), (f1), (3.2) and the definition of $J_\lambda$, we find that

$$J_\lambda(u) \geq \frac{1}{p^+} \int_M |\nabla u(x)|^{p(x)} \, dv_g(x) + \frac{\mu(x)}{q^+} \int_M |\nabla u(x)|^{q(x)} \, dv_g(x)$$

$$- \frac{\lambda}{q^-} \int_M |u(x)|^{q(x)} \, dv_g(x) + \frac{1}{p^-} \left[ - \int_M |\nabla u(x)|^{p(x)} \, dv_g(x) - \mu(x) \int_M |\nabla u(x)|^{q(x)} \, dv_g(x) + \lambda \int_M |u(x)|^{q(x)} \, dv_g(x)$$

$$+ \int_M f(x, u(x)) \cdot u(x) \, dv_g(x) \right] - \int_M F(x, u(x)) \, dv_g(x)$$

$$\geq \mu(x) \left( \frac{1}{q^+} - \frac{1}{p^+} \right) \int_M |\nabla u(x)|^{q(x)} \, dv_g(x)$$
\[ + \lambda \left( \frac{1}{p^+} - \frac{1}{q^-} \right) \int_M |u(x)|^{q(x)} \, dv_g(x) \]

\[ + \frac{1}{p^+} \int_M f(x, u(x)) \cdot u(x) \, dv_g(x) \]

\[ - \int_M F(x, u(x)) \, dv_g(x) \]

\[ \geq \mu(x) \left( \frac{1}{q^+} - \frac{1}{p^+} \right) \int_M |\nabla u(x)|^{q(x)} \, dv_g(x) \]

\[ + \lambda \left( \frac{1}{p^+} - \frac{1}{q^-} \right) \int_M |u(x)|^{q(x)} \, dv_g(x), \]

according with proposition 2.14 we deduce that

\[ J_\lambda(u) \geq \frac{\mu(x)}{A \rho} \left( \frac{1}{q^+} - \frac{1}{p^+} \right) \|u\|_{W_0^{1,q(x)}(M)}^{p^-} + \lambda \left( \frac{1}{p^+} - \frac{1}{q^-} \right) \|u\|_{W_0^{1,q(x)}(M)}^{q^+}. \]

Since, \( p^- > q^+ \) we have

\[ J_\lambda(u) \geq \left( \frac{\mu(x)}{A \rho}, \frac{p^+ - q^+}{q^+ p^+} + \lambda \frac{q^- - p^+}{p^+ q^-} \right) \|u\|_{W_0^{1,q(x)}(M)}^{p^-}. \]

Thus, if we choose \( \lambda < \frac{\mu(x) q^- (p^+ - q^+)}{A \rho q^+ (p^+ - q^-)} = \lambda^{**} \), we deduce that \( J_\lambda(u) > 0. \)

It follows that \( \theta_\lambda^* = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u) > 0. \)

Hence, \( \mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^- \) and \( \mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset \), by above Lemma, we must have \( u \in \mathcal{N}_\lambda^- \).

4. **Existence of Non-negative Solutions**

In this section, we prove the existence of two non-negative solutions of problem (P). For this, we first show the existence of minimizers in \( \mathcal{N}_\lambda^+ \) and \( \mathcal{N}_\lambda^- \) for all \( \lambda \in (0, \hat{\lambda}) \), where \( \hat{\lambda} = \min \{ \lambda^*, \lambda^{**} \} \).

**Theorem 4.1.** Suppose that (f1) – (f3) are true, then for all \( \lambda \in (0, \lambda^*) \), there exists a minimizer \( u_0^+ \) of \( J_\lambda(u) \) on \( \mathcal{N}_\lambda^+ \) such that \( J_\lambda(u_0^+) = \theta_\lambda^+ \).

**Proof.** From Lemma 3.2, \( J_\lambda \) is bounded below on \( \mathcal{N}_\lambda \), in particular is bounded below on \( \mathcal{N}_\lambda^+ \). Then there exists a minimizing sequence \( \{ u_n^+ \} \subset \mathcal{N}_\lambda^+ \) such that

\[ \lim_{n \to +\infty} J_\lambda(u_n^+) = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u) = \theta_\lambda^+ < 0. \]

Since, \( J_\lambda \) is coercive, \( \{ u_n^+ \} \) is bounded in \( W_0^{1,q(x)}(M) \). Hence we assume that, without loss generality, \( u_n^+ \rightharpoonup u_0^+ \) in \( W_0^{1,q(x)}(M) \) and by the compact embedding (Theorem 2.17) we have

\[ u_n^+ \longrightarrow u_0^+ \text{ in } L^{p(x)}(M). \tag{4.1} \]

Now, we shall prove \( u_n^+ \rightharpoonup u_0^+ \) in \( W_0^{1,q(x)}(M) \). Otherwise, let \( u_n^+ \nrightarrow u_0^+ \) in \( W_0^{1,q(x)}(M) \). Then, we have

\[ \rho_{q(\cdot)}(u_0^+) < \lim_{n \to +\infty} \inf \rho_{q(\cdot)}(u_n^+), \tag{4.2} \]
using (4.1) we obtain
\[ \int_M |u_0^+|^{p(x)} \, dv_g(x) = \lim_{n \to +\infty} \inf \int_M |u_n^+|^{p(x)} \, dv_g(x), \]
since \( \langle J'_{\lambda}(u_n^+), u_n^+ \rangle = 0 \), we get
\[ J_{\lambda}(u_n^+) \geq \frac{\mu(x)}{Ap} \left( \frac{1}{q^+} - \frac{1}{p^+} \right) \rho_{p}(u_n^+) + \lambda \left( \frac{1}{p^+} - \frac{1}{q^-} \right) \rho_{q}(u_n^+). \]
That is
\[ \lim_{n \to +\infty} J_{\lambda}(u_n^+) \geq \frac{\mu(x)}{Ap} \left( \frac{1}{q^+} - \frac{1}{p^+} \right) \lim_{n \to +\infty} \rho_{p}(u_n^+) + \lambda \left( \frac{1}{p^+} - \frac{1}{q^-} \right) \lim_{n \to +\infty} \rho_{q}(u_n^+). \]
By (4.1) and (4.2), we have
\[ \theta^+_\lambda > \frac{\mu(x)}{Ap} \left( \frac{1}{q^+} - \frac{1}{p^+} \right) ||u_0^+||_{p^+} - \lambda \left( \frac{1}{p^+} - \frac{1}{q^-} \right) ||u_0^+||_{q^+}, \]
since \( p^- > q^+ \), for \( ||u_0^+||_{W_0^{1,q(x)}(M)} > 1 \), we deduce
\[ \theta^+_\lambda = \inf_{u \in N^+_\lambda} J_{\lambda}(u) > 0, \]
which is a contradiction with Lemma 3.5. Hence
\[ u_n^+ \longrightarrow u_0^+ \text{ in } W_0^{1,q(x)}(M), \]
and
\[ \lim_{n \to +\infty} J_{\lambda}(u_n^+) = J_{\lambda}(u_0^+) = \theta^+_\lambda. \]
Consequently, \( u_0^+ \) is a minimizer of \( J_{\lambda} \) on \( N^+_\lambda \). \( \square \)

**Theorem 4.2.** Suppose that conditions \((f_1) - (f_3)\) are true, and for all \( \lambda \in (0, \lambda^{**}) \), there exists a minimizer \( u_0^- \) of \( J_{\lambda} \) on \( N^-_\lambda \) such that \( J_{\lambda}(u_0^-) = \theta^-_\lambda \).

**Proof.** Since \( J_{\lambda} \) is bounded below on \( N_\lambda \) and so on \( N^-_\lambda \). Then, there exists a minimizing sequence \( \{u_n^-\} \subseteq N^-_\lambda \) such that
\[ \lim_{n \to +\infty} J_{\lambda}(u_n^-) = \inf_{u \in N^-_\lambda} J_{\lambda}(u) = \theta^-_\lambda > 0. \]
As \( J_{\lambda} \) is coercive, \( \{u_n^-\} \) is bounded in \( W_0^{1,q(x)}(M) \). Thus without loss of generality, we may assume that, \( u_n^- \rightharpoonup u_0^- \) in \( W_0^{1,q(x)}(M) \) and by Theorem 2.17 we have
\[ u_n^- \longrightarrow u_0^- \text{ in } L^{p(x)}(M). \]
(4.3)
On the other hand, if \( u_0^- \in N^-_\lambda \), then there exists a constant \( t > 0 \) such that \( tu_0^- \in N^-_\lambda \) and \( J_{\lambda}(tu_0^-) \geq J_{\lambda}(u_0^-) \). According to \((f_1)\) and the definition of
ψ', we have

\[ \langle \psi', t u_0^- \rangle = \int_M p(x) |\nabla u_0^- (x)|^{p(x)} \, dv_g(x) \]

\[ + \mu(x) q(x) \int_M |\nabla u_0^- (x)|^{q(x)} \, dv_g(x) \]

\[ - \lambda q(x) \int_M |tu_0^- (x)|^{q(x)} \, dv_g(x) \]

\[ + p(x) \int_M |tu_0^- (x)|^{p(x)} \, dv_g(x) - \int_M F(x, tu_0^- (x)) \, dv_g(x) \]

\[ \leq p^+ t^{p^+} \int_M |\nabla u_0^- |^{p(x)} \, dv_g(x) \]

\[ + \mu(x) q^+ t^{q^+} \int_M |\nabla u_0^- |^{q(x)} \, dv_g(x) \]

\[ - \lambda q^- t^{q^-} \int_M |u_0^- |^{q(x)} \, dv_g(x) \]

\[ + p^+ t^{p^+} \int_M |u_0^- |^{p(x)} \, dv_g(x). \]

Since \( q^- \leq q^+ < p^+ \), and by propositions 2.13 and 2.14, it follows that

\[ \langle \psi', t u_0^- \rangle < 0. \] Hence by the definition of \( N^-_\lambda \), \( t u_0^- \in N^-_\lambda \).

Next, we show that \( u_n^- \rightarrow u_0^- \) in \( W^{1,q(x)}_0 (M) \). Otherwise, suppose \( u_n^- \nrightarrow u_0^- \) in \( W^{1,q(x)}_0 (M) \). Then by Fatou’s Lemma we have

\[ \int_M |\nabla u_0^- (x)|^{q(x)} \, dv_g(x) \leq \lim_{n \to +\infty} \int_M |\nabla u_n^- (x)|^{q(x)} \, dv_g(x). \]

By above inequality (4.3) we get

\[ \int_M |u_0^- (x)|^{p(x)} \, dv_g(x) \leq \lim_{n \to +\infty} \int_M |u_n^- (x)|^{p(x)} \, dv_g(x), \]

and

\[ \int_M |\nabla u_0^- (x)|^{p(x)} \, dv_g(x) \leq \lim_{n \to +\infty} \int_M |\nabla u_n^- (x)|^{p(x)} \, dv_g(x). \]

Then by (f_1), we obtain

\[ J_\lambda(t u_0^-) \leq \frac{t^{p^+}}{p^-} \int_M |\nabla u_0^- (x)|^{p(x)} \, dv_g(x) + \mu(x) \frac{t^{q^+}}{q^-} \int_M |\nabla u_0^- (x)|^{q(x)} \, dv_g(x) \]

\[ - \frac{\lambda t^{q^-}}{q^-} \int_M |u_0^- (x)|^{q(x)} \, dv_g(x) + \frac{t^{p^+}}{p^+} \int_M |u_0^- (x)|^{p(x)} \, dv_g(x) \]

\[ - \int_M F(x, t u_0^- (x)) \, dv_g(x) \]

\[ \leq \lim_{n \to +\infty} \left[ \frac{t^{p^+}}{p^-} \int_M |\nabla u_n^- (x)|^{p(x)} \, dv_g(x) \right] \]

\[ + \mu(x) \frac{t^{q^+}}{q^-} \int_M |\nabla u_n^- (x)|^{q(x)} \, dv_g(x) - \frac{\lambda t^{q^-}}{q^-} \int_M |u_n^- (x)|^{q(x)} \, dv_g(x) \]
Under assumptions introduce the truncation function \( h \) of problem \((\text{P})\) quently all \( W \) then, we conclude that \( u \) and consider the functional \( J \) respectively, we deduce that \( \lambda \) and \( \rho \) are non-negative solutions \( \lambda, \rho \). Then, by proposition 2.13 we have for all \( \lambda \in (0, \bar{\lambda}) \), the problem \( (\text{P}) \) has at least two non-negative weak solutions. **Proof.** Form Theorems 4.1 and 4.2, we deduce that for any \( u \in \mathcal{N}_\lambda^+ \) and \( u \in \mathcal{N}_\lambda^- \) such as

\[
J_\lambda(u_0^+) = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u) \quad \text{and} \quad J_\lambda(u_0^-) = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u).
\]

Then, the problem \( (\text{P}) \) has two solutions \( u_0^+ \in \mathcal{N}_\lambda^+ \) and \( u_0^- \in \mathcal{N}_\lambda^- \) in \( W_{0,1,q(x)}^1(M) \). By Lemma 3.3, it follows that \( \mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^+ = \emptyset \). Then, \( u_0^- \neq u_0^+ \). Thus, these two solutions are distinct.

Next, we prove that \( u_0^- \) and \( u_0^+ \) are non-negative in \( M \). For this, we introduce the truncation function \( h_+ : M \times \mathbb{R} \rightarrow \mathbb{R} \) defined by

\[
h_+(x, s) = \begin{cases} 0 & \text{if } s < 0, \\ h(x, s) & \text{if } s \geq 0. \end{cases}
\]

We set \( H_+(x, s) = f_0^s f(x, t) \, dt \) and consider the \( C^1 \)-functional \( J_\lambda^+ : W_{0,1,q(x)}^1(M) \rightarrow \mathbb{R} \) given by

\[
J_\lambda^+(u) = \int_M \frac{1}{p(x)} |\nabla u(x)|^{p(x)} \, dv_g(x) + \mu(x) \int_M \frac{1}{q(x)} |\nabla u(x)|^{q(x)} \, dv_g(x) - \int_M H_+(x, u(x)) \, dv_g(x).
\]

Then, by proposition 2.13 we have for all \( u_- = \min \{ 0, u(x) \} \in W_{0,1,q(x)}^1(M) \) that

\[
0 = \langle (J_\lambda^+(u_-))', u_- \rangle \geq p^- \rho_{p(\cdot)}(\langle \nabla u_- \rangle) + \mu(x) q^- \rho_{q(\cdot)}(\langle \nabla u_- \rangle)
\]

\[
\geq p^- \rho_{p(\cdot)}(\langle \nabla u_- \rangle) + \frac{\mu(x)}{A^p} q^- \rho_{q(\cdot)}(u_-)
\]

\[
\geq \rho_{p(\cdot)}(u_-) \geq \| u_- \|_{W_{0,1,q(x)}^1(M)}^{p^-}.
\]

Hence, \( \| u_- \|_{W_{0,1,q(x)}^1(M)} = 0 \), and thus \( u = u_+ \). Then, by taking \( u = u_0^- \) and \( u = u_0^+ \) respectively, we deduce that \( u_0^- \) and \( u_0^+ \) are non-negative solutions of problem \( (\text{P}) \).
Conclusion: According to the above results, we can then say that $u^{\pm}$ are critical points of $J_\lambda$ and hence are non-negative weak solutions of problem (P).

Acknowledgements
This paper has been supported by the RUDN University Strategic Academic Leadership Program and P.R.I.N. 2019.

Funding Open access funding provided by Università degli Studi di Catania within the CRUI-CARE Agreement.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References
[1] Aberqi, A., Bennouna, J., Elmassoudi, M., Hammoumi, M.: Existence and uniqueness of a renormalized solution of parabolic problems in Orlicz spaces. Monats. Math. 189, 195–219 (2019)
[2] Aberqi, A., Bennouna, J., Mekkour, M., Redwane, H.: Nonlinear parabolic inequalities with lower order terms. Appl. Anal. 96, 2102–2117 (2017)
[3] Aubin, T.H.: Nonlinear Analysis on Manifolds. Monge-Ampere Equations, p. 252. Springer, Berlin (1982)
[4] Benslimane, O., Aberqi, A., Bennouna, J.: The existence and uniqueness of an entropy solution to unilateral Orlicz anisotropic equations in an unbounded domain. Axioms 9, 109 (2020)
[5] Benslimane, O., Aberqi, A., Bennouna, J.: Existence and uniqueness of entropy solution of a nonlinear elliptic equation in anisotropic Sobolev-Orlicz space. Rend. Circ. Mat. Palermo II. Ser. (2020). https://doi.org/10.1007/s12215-020-00577-4
[6] Benslimane, O., Aberqi, A., Bennouna, J.: Existence and uniqueness of weak solution of $p(x)$-laplacian in Sobolev spaces with variable exponents in complete manifolds. arXiv: 2006.04763 (arXiv preprint) (2020)
[7] Boccardo, L., Gallouët, Th., Vazquez, J.L.: Nonlinear elliptic equations in $\mathbb{R}^N$ without growth restrictions on the data. J. Differ. Equ. 105, 334–363 (1993)
[8] Boccardo, L., Gallouët, Th.: Nonlinear elliptic equations with right hand side measures. Commun. Partial Differ. Equ. 17, 189–258 (1992)

[9] Cencelj, M., Rădulescu, V.D., Repovš, D.D.: Double phase problems with variable growth. Nonlinear Anal. 177, 270–287 (2018)

[10] Fan, X., Zhao, Y., Zhao, D.: Compact imbedding theorems with symmetry of Strauss-Lions type for the space $W^{1,p(x)}(\Omega)$. J. Math. Anal. Appl. 255, 333–348 (2001)

[11] Fan, X., Zhao, D.: On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. J. Math. Anal. Appl. 263, 424–446 (2001)

[12] Gaczkowski, M., Górka, P., Pons, D.J.: Sobolev spaces with variable exponents on complete manifolds. J. Funct. Anal. 270, 1379–1415 (2016)

[13] Gasiński, L., Papageorgiou, N.S.: Constant sign and nodal solutions for superlinear double phase problems. Adv. Calc. Var. 1, 25 (2019)

[14] Guo, L.: The Dirichlet problems for nonlinear elliptic equations with variable exponents on Riemannian manifolds. J. Appl. Anal. Comput. 5, 562–569 (2015)

[15] Hebey, E.: Nonlinear analysis on manifolds: Sobolev spaces and inequalities. Am. Math. Soc. 5, 25 (2000)

[16] Liu, W., Dai, G.: Existence and multiplicity results for double phase problem. J. Differ. Equ. 265, 4311–4334 (2018)

[17] Marino, G., Winkert, P.: Existence and uniqueness of elliptic systems with double phase operators and convection terms. J. Math. Anal. Appl. 492, 124423 (2020)

[18] Papageorgiou, N.S., Repovš, D.D., Vetro, C.: Positive solutions for singular double phase problems. J. Math. Anal. Appl. 20, 123896 (2020)

[19] Rădulescu, V.D., Repovš, D.D.: Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis, p. 9. CRC Press, Boca Raton (2015)

[20] Ragusa, M.A., Tachikawa, A.: Regularity for minimizers for functionals of double phase with variable exponents. Adv. Nonlinear Anal. 9, 710–728 (2019)

[21] Ragusa, M.A., Tachikawa, A.: Partial regularity of the minimizers of quadratic functionals with VMO coefficients. J. Lond. Math. Soc. 72, 609–620 (2005)

[22] Shi, X., Rădulescu, V.D., Repovš, D.D., Zhang, Q.: Multiple solutions of double phase variational problems with variable exponent. Adv. Calc. Var. 13, 385–401 (2020)

[23] Tachikawa, A.: Boundary regularity of minimizers of double phase functionals. J. Math. Anal. Appl. 20, 123946 (2020)

[24] Zhikov, V.V.: Averaging of functionals of the calculus of variations and elasticity theory. Math. USSR-Izvestiy. 29, 33 (1987)

[25] Zhikov, V.V.: On Lavrentiev’s phenomenon. Russ. J. Math. Phys. 3, 2 (1995)

[26] Zhikov, V.V.: On some variational problems. Russ. J. Math. Phys. 5, 105–116 (1997)

[27] Zhikov, V.V., Kozlov, S.M., Oleinik, O.A.: Homogenization of Differential Operators and Integral Functionals. Springer, Berlin (2012)
