THE ASYMPTOTIC DISTRIBUTION OF CIRCLES IN THE ORBITS OF KLEINIAN GROUPS

HEE OH AND NIMISH SHAH

Abstract. Let \( \mathcal{P} \) be a locally finite circle packing in the plane \( \mathbb{C} \) invariant under a non-elementary Kleinian group \( \Gamma \) and with finitely many \( \Gamma \)-orbits. When \( \Gamma \) is geometrically finite, we construct an explicit Borel measure on \( \mathbb{C} \) which describes the asymptotic distribution of small circles in \( \mathcal{P} \), assuming that either the critical exponent of \( \Gamma \) is strictly bigger than 1 or \( \mathcal{P} \) does not contain an infinite bouquet of tangent circles glued at a parabolic fixed point of \( \Gamma \). Our construction also works for \( \mathcal{P} \) invariant under a geometrically infinite group \( \Gamma \), provided \( \Gamma \) admits a finite Bowen-Margulis-Sullivan measure and the \( \Gamma \)-skinning size of \( \mathcal{P} \) is finite. Some concrete circle packings to which our result applies include Apollonian circle packings, Sierpinski curves, Schottky dances, etc.

1. Introduction

A circle packing in the plane \( \mathbb{C} \) is simply a union of circles (here a line is regarded as a circle of infinite radius). As we allow circles to intersect with each other, our definition of a circle packing is more general than the conventional definition of a circle packing.

For a given circle packing \( \mathcal{P} \) in the plane, we are interested in counting and distribution of small circles in \( \mathcal{P} \). A natural size of a circle is measured by its radius. We will use the curvature of a circle, that is, the reciprocal of its radius, instead.

We suppose that \( \mathcal{P} \) is locally finite in the sense that for any \( T > 1 \), there are only finitely many circles in \( \mathcal{P} \) of curvature at most \( T \) in any fixed bounded subset of \( \mathbb{C} \). Geometrically, \( \mathcal{P} \) is locally finite if there is no infinite sequence of circles in \( \mathcal{P} \) converging to a fixed circle. For instance, if the circles of \( \mathcal{P} \) have disjoint interiors as in Fig. 1, \( \mathcal{P} \) is locally finite.

For a bounded subset \( E \) of \( \mathbb{C} \) and \( T > 1 \), we set

\[
N_T(\mathcal{P}, E) := \# \{ C \in \mathcal{P} : C \cap E \neq \emptyset, \ \text{Curv}(C) < T \}
\]

where \( \text{Curv}(C) \) denotes the curvature of a circle \( C \). The local finiteness assumption on \( \mathcal{P} \) implies that \( N_T(\mathcal{P}, E) < \infty \). Our question is then if there exists a Borel measure \( \omega_\mathcal{P} \) on \( \mathbb{C} \) such that for all nice Borel subsets

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\[ E_1, E_2 \subset \mathbb{C}, \]
\[ \frac{N_T(P, E_1)}{N_T(P, E_2)} \sim_{T \to \infty} \frac{\omega_P(E_1)}{\omega_P(E_2)}, \]
assuming \( N_T(P, E_2) > 0 \) and \( \omega_P(E_2) > 0 \).

Our main theorem applies to a very general packing \( P \), provided \( P \) is invariant under a non-elementary (i.e., non virtually-abelian) Kleinian group satisfying certain finiteness conditions.

Recall that a Kleinian group is a discrete subgroup of \( G := \text{PSL}_2(\mathbb{C}) \) and \( G \) acts on the extended complex plane \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) by Möbius transformations:
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}
\]
where \( a, b, c, d \in \mathbb{C} \) with \( ad - bc = 1 \) and \( z \in \hat{\mathbb{C}} \). A Möbius transformation maps a circle to a circle and by the Poincare extension, \( G \) can be identified with the group of all orientation preserving isometries of \( \mathbb{H}^3 \). Considering the upper-half space model \( \mathbb{H}^3 = \{(z, r) : z \in \mathbb{C}, r > 0\} \), the geometric boundary \( \partial_{\infty}(\mathbb{H}^3) \) is naturally identified with \( \hat{\mathbb{C}} \).

For a Kleinian group \( \Gamma \), we denote by \( \Lambda(\Gamma) = \hat{\mathbb{C}} \) its limit set, that is, the set of accumulation points of an orbit of \( \Gamma \) in \( \hat{\mathbb{C}} \), and by \( 0 \leq \delta_\Gamma \leq 2 \) its critical exponent. For \( \Gamma \) non-elementary, it is known that \( \delta_\Gamma > 0 \). Let \( \{\nu_x : x \in \mathbb{H}^3\} \) be a \( \Gamma \)-invariant conformal density of dimension \( \delta_\Gamma \) on \( \Lambda(\Gamma) \), which exists by the work of Patterson [23] and Sullivan [30].

In order to present our main theorem on the asymptotic of \( N_T(P, E) \) we introduce two invariants associated to \( \Gamma \) and \( P \). The first one is a Borel measure on \( \mathbb{C} \) depending only on \( \Gamma \).
Definition 1.1. Define a Borel measure $\omega_\Gamma$ on $\mathbb{C}$: for $\psi \in C_c(\mathbb{C})$

$$\omega_\Gamma(\psi) = \int_{z \in \mathbb{C}} \psi(z)e^{\delta_\Gamma \beta_s(x,(z,1))} d\nu_s(z)$$

where $x \in \mathbb{H}^3$ and $\beta_s(x_1, x_2)$ is the signed distance between the horospheres based at $z \in \mathbb{C}$ and passing through $x_1, x_2 \in \mathbb{H}^3$.

By the conformal property of $\{\nu_s\}$, $\omega_\Gamma$ is well-defined independent of the choice of $x \in \mathbb{H}^3$.

We have a simple formula: for $j = (0,1) \in \mathbb{H}^3$,

$$d\omega_\Gamma = (|z|^2 + 1)^{\delta_j} d\nu_s.$$

For a vector $u$ in the unit tangent bundle $T^1(\mathbb{H}^3)$, denote by $u^+ \in \hat{\mathbb{C}}$ (resp. $u^- \in \hat{\mathbb{C}}$) the forward (resp. backward) end point of the geodesic determined by $u$. On the contracting horosphere $H_\infty(j) \subset T^1(\mathbb{H}^3)$ consisting of upward unit normal vectors on the horizontal plane $\{(z,1) : z \in \mathbb{C}\}$, the normal vector based at $(z,1)$ is mapped to $z$ via the map $u \mapsto u^-$. Under this correspondence, the measure $\omega_\Gamma$ on $\mathbb{C}$ is equal to the density of the Burger-Roblin measure $m_{\text{BR}}^\Gamma$ (see Def. 2.3) on $H_\infty(j)$.

The second invariant is a number in $[0, \infty]$ measuring a certain size of $\mathcal{P}$.

Definition 1.2 (The $\Gamma$-skinning size of $\mathcal{P}$). For a circle packing $\mathcal{P}$ invariant under $\Gamma$, define $0 \leq \text{sk}_\Gamma(\mathcal{P}) \leq \infty$ as follows:

$$\text{sk}_\Gamma(\mathcal{P}) := \sum_{i \in I} \int_{s \in \text{Stab}_\Gamma(C_i^d)} e^{\delta_\Gamma \beta_s(x,\pi(s))} d\nu_s(s^+)$$

where $x \in \mathbb{H}^3$, $\pi : T^1(\mathbb{H}^3) \to \mathbb{H}^3$ is the canonical projection, $\{C_i : i \in I\}$ is a set of representatives of $\Gamma$-orbits in $\mathcal{P}$, $C_i^d \subset T^1(\mathbb{H}^3)$ is the set of unit normal vectors to the convex hull $\hat{C}_i$ of $C_i$ and $\text{Stab}_\Gamma(C_i^d)$ denotes the set-wise stabilizer of $C_i^d$ in $\Gamma$. Again by the conformal property of $\{\nu_s\}$, the definition of $\text{sk}_\Gamma(\mathcal{P})$ is independent of the choice of $x$ and the choice of representatives $\{C_i\}$.

We remark that the value of $\text{sk}_\Gamma(\mathcal{P})$ can be zero or infinite in general and we do not assume any condition on $\text{Stab}_\Gamma(C_i^d)$’s (they may be trivial).

We denote by $m_{\Gamma}^\text{BMS}$ the Bowen-Margulis-Sullivan measure on the unit tangent bundle $T^1(\Gamma \backslash \mathbb{H}^3)$ associated to the density $\{\nu_s\}$ (Def. 2.2). When $\Gamma$ is geometrically finite, i.e., $\Gamma$ admits a finite sided fundamental domain in $\mathbb{H}^3$, Sullivan showed that $|m_{\Gamma}^\text{BMS}| < \infty$ and that $\delta_\Gamma$ is equal to the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ [30]. A point in $\Lambda(\Gamma)$ is called a parabolic fixed point of $\Gamma$ if it is fixed by a parabolic element of $\Gamma$.

Definition 1.3. By an infinite bouquet of tangent circles glued at a point $\xi \in \mathbb{C}$, we mean a union of two collections, each consisting of infinitely many pairwise internally tangent circles with the common tangent point $\xi$ and their radii tending to 0, such that the circles in each collection are externally tangent to the circles in the other at $\xi$ (see Fig. 2).
Theorem 1.4. Let $\mathcal{P}$ be a locally finite circle packing in $\mathbb{C}$ invariant under a non-elementary geometrically finite group $\Gamma$ and with finitely many $\Gamma$-orbits. If $\delta_\Gamma \leq 1$, we further assume that $\mathcal{P}$ does not contain an infinite bouquet of tangent circles glued at a parabolic fixed point of $\Gamma$. Then $\text{sk}_\Gamma(\mathcal{P}) < \infty$ and for any bounded Borel subset $E$ of $\mathbb{C}$ with $\omega_\Gamma(\partial(E)) = 0$,

$$
\lim_{T \to \infty} \frac{N_T(\mathcal{P}, E)}{T^{\delta_\Gamma}} = \frac{\text{sk}_\Gamma(\mathcal{P})}{\delta_\Gamma \cdot |m_{\text{BMS}}|} \cdot \omega_\Gamma(E).
$$

If $\mathcal{P}$ has infinitely many circles, then $\text{sk}_\Gamma(\mathcal{P}) > 0$.

Remark 1.5. (1) Given a finite collection $\{C_1, \cdots, C_m\}$ of circles in the plane $\mathbb{C}$ and a non-elementary geometrically finite group $\Gamma \leq \text{PSL}_2(\mathbb{C})$, Theorem 1.4 applies to $\mathcal{P} := \bigcup_{i=1}^m \Gamma(C_i)$, provided $\mathcal{P}$ contains neither infinitely many circles converging to a fixed circle nor any infinite bouquet of tangent circles.

(2) In the case when $\delta_\Gamma \leq 1$ and $\mathcal{P}$ contains an infinite bouquet of tangent circles glued at a parabolic fixed point of $\Gamma$, we have $\text{sk}_\Gamma(\mathcal{P}) = \infty$ [19]. In that case if the interior of $E$ intersects $\Lambda(\Gamma)$ non-trivially, the growth order of $N_T(\mathcal{P}, E)$ is $T \log T$ if $\delta_\Gamma = 1$, and it is $T$ if $\delta_\Gamma < 1$ [21].

(3) We note that the asymptotic of $N_T(\mathcal{P}, E)$ depends only on $\Gamma$, except for the $\Gamma$-skinning size of $\mathcal{P}$. This is rather surprising in view of the fact that there are circle packings with completely different configurations but invariant under the same group $\Gamma$.

(4) Theorem 1.4 implies that the asymptotic distribution of small circles in $\mathcal{P}$ is completely determined by the measure $\omega_\Gamma$: for any bounded Borel sets $E_1, E_2$ with $\omega_\Gamma(E_2) > 0$ and $\omega_\Gamma(\partial(E_i)) = 0$, $i = 1, 2$, as $T \to \infty$,

$$
\frac{N_T(\mathcal{P}, E_1)}{N_T(\mathcal{P}, E_2)} \sim \frac{\omega_\Gamma(E_1)}{\omega_\Gamma(E_2)}.
$$

(5) Suppose that all circles in $\mathcal{P}$ can be oriented so that they have disjoint interiors whose union is equal to the domain of discontinuity $\Omega(\Gamma) := \mathbb{C} - \Lambda(\Gamma)$. If either $\mathcal{P}$ is bounded or $\infty$ is a parabolic fixed point for $\Gamma$, then $\delta_\Gamma$ is equal to the circle packing exponent $e_\mathcal{P}$ defined
as:

$$e_\mathcal{P} = \inf\{s : \sum_{C \in \mathcal{P}} \text{Curv}^{-s} < \infty\} = \sup\{s : \sum_{C \in \mathcal{P}} \text{Curv}(C)^{-s} = \infty\}.$$  

This was proved by Parker [22] extending the earlier works of Boyd [3] and Sullivan [31] on bounded Apollonian circle packings.

In the proof of Theorem 1.4, the geometric finiteness assumption on $\Gamma$ is used only to ensure the finiteness of the Bowen-Margulis-Sullivan measure $m^\text{BMS}_\Gamma$. We have the following more general theorem:

**Theorem 1.6.** Let $\mathcal{P}$ be a locally finite circle packing invariant under a non-elementary Kleinian group $\Gamma$ and with finitely many $\Gamma$-orbits. Suppose that

$$|m^\text{BMS}_\Gamma| < \infty \quad \text{and} \quad \text{sk}_\Gamma(\mathcal{P}) < \infty.$$  

Then for any bounded Borel subset $E$ of $\mathbb{C}$ with $\omega_\Gamma(\partial(E)) = 0$,

$$\lim_{T \to \infty} \frac{N_\Gamma(\mathcal{P}, E)}{T^{3r}} = \frac{\text{sk}_\Gamma(\mathcal{P})}{\delta_\Gamma \cdot |m^\text{BMS}_\Gamma|} \cdot \omega_\Gamma(E).$$

If $\mathcal{P}$ is infinite, then $\text{sk}_\Gamma(\mathcal{P}) > 0$.

Since there is a large class of geometrically infinite groups with $|m^\text{BMS}_\Gamma| < \infty$ [24], Theorem 1.6 is not subsumed by Theorem 1.4.

We remark that the condition on the finiteness of $m^\text{BMS}_\Gamma$ implies that the density $\{\nu_x\}$ is determined uniquely up to homothety (see [26, Coro. 1.8]).

**Remark 1.7.** (1) The assumption of $|m^\text{BMS}_\Gamma| < \infty$ implies that $\nu_x$ (and hence $\omega_\Gamma$) is atom-free [26, Sec. 1.5], and hence the above theorem works for any bounded Borel subset $E$ intersecting $\Lambda(\Gamma)$ only at finitely many points.

(2) It is not hard to show that $\Gamma$ is Zariski dense in $\text{PSL}_2(\mathbb{C})$ considered as a real algebraic group if and only if $\Lambda(\Gamma)$ is not contained in a circle in $\hat{\mathbb{C}}$. In such a case, any proper real subvariety of $\hat{\mathbb{C}}$ has zero $\nu_x$-measure. This is shown in [7, Cor.1.4] for $\Gamma$ geometrically finite but its proof works equally well if $\nu_x$ is $\Gamma$-ergodic, which is the case when $|m^\text{BMS}_\Gamma| < \infty$. Hence Theorem 1.6 holds for any Borel subset $E$ whose boundary is contained in a countable union of real algebraic curves.

We now describe some concrete applications of Theorem 1.4.

1.1. **Apollonian gasket.** Three mutually tangent circles in the plane determine a curvilinear triangle, say, $\mathcal{T}$. By a theorem of Apollonius of Perga (c. 200 BC), one can inscribe a unique circle into the triangle $\mathcal{T}$, tangent to all of the three circles. This produces three more curvilinear triangles inside $\mathcal{T}$ and we inscribe a unique circle into each triangle. By continuing to add circles in this way, we obtain an infinite circle packing of $\mathcal{T}$, called the Apollonian gasket for $\mathcal{T}$, say, $\mathcal{A}$ (see Fig. 3).
By adding all the circles tangent to three of the given ones, not only those within $\mathcal{T}$, one obtains an Apollonian circle packing $\mathcal{P} := \mathcal{P}(\mathcal{T})$, which may be bounded or unbounded (cf. [10] [9], [27], [28], [12]).

Fixing four mutually tangent circles in $\mathcal{P}$, consider the four dual circles determined by the six intersection points (see Fig. 4 where the dotted circles are dual circles to the solid ones), and denote by $\Gamma_{P}$ the intersection of $\text{PSL}_2(\mathbb{C})$ and the group generated by the inversions with respect to those dual circles. Then $\Gamma_{P}$ is a geometrically finite Zariski dense subgroup of the real algebraic group $\text{PSL}_2(\mathbb{C})$ preserving $\mathcal{P}$, and its limit set in $\hat{\mathbb{C}}$ coincides with the residual set of $\mathcal{P}$ (cf. [12]).

We denote by $\alpha$ the Hausdorff dimension of the residual set of $\mathcal{P}$, which is known to be $1.3056(8)$ according to McMullen [16].

**Corollary 1.8.** Let $\mathcal{T}$ be a curvilinear triangle determined by three mutually tangent circles and $\mathcal{A}$ the Apollonian gasket for $\mathcal{T}$. Then for any Borel subset $E \subset \mathcal{T}$ whose boundary is contained in a countable union of real algebraic curves,

$$
\lim_{T \to \infty} \frac{N_T(E)}{T^\alpha} = \frac{\text{sk}_{\Gamma_{P}}(\mathcal{P})}{\alpha \cdot |m^{\text{BMS}}_{\Gamma_{P}}|} \cdot \omega_{\Gamma_{P}}(E)
$$

where $N_T(E) := \# \{ C \in \mathcal{A} : C \cap E \neq \emptyset, \ \text{Curv}(C) < T \}$ and $\mathcal{P} = \mathcal{P}(\mathcal{T})$.

Either when $\mathcal{P}$ is bounded and $E$ is the disk enclosed by the largest circle of $\mathcal{P}$, or when $\mathcal{P}$ lies between two parallel lines and $E$ is the whole period,
it was proved in [12] that $N_T(P, E) \sim c \cdot T^\alpha$ for some $c > 0$. This implies that $N_T(T) \asymp T^\alpha$. The approach in [12] was based on the Descartes circle theorem in parameterizing quadruples of circles of curvature at most $T$ as vectors of maximum norm at most $T$ in the cone defined by the Descartes quadratic equation. We remark that the fact that $\alpha$ is strictly bigger than 1 was crucial in the proof of [12] as based on the $L^2$-spectral theory of $\Gamma \backslash \mathbb{H}^3$.

1.2. Counting circles in the limit set $\Lambda(\Gamma)$. If $X$ is a finite volume hyperbolic 3-manifold with totally geodesic boundary, then its fundamental group $\Gamma := \pi_1(X)$ is geometrically finite and $X$ is homeomorphic to $\Gamma \backslash \mathbb{H}^3 \cup \Omega(\Gamma)$ where $\Omega(\Gamma) := \mathcal{C} - \Lambda(\Gamma)$ is the domain of discontinuity [11]. The set $\Omega(\Gamma)$ is a union of countably many disjoint open disks in this case and has finitely many $\Gamma$-orbits by the Ahlfors finiteness theorem [1]. Hence Theorem 1.4 applies to counting these open disks in $\Omega(\Gamma)$ with respect to the curvature.

1$\asymp$ means that the ratio of the two sides is between two uniform constants.
For example, for the group $\Gamma$ generated by reflections in the sides of a unique regular tetrahedron whose convex core is bounded by four $\frac{\pi}{4}$ triangles and four right hexagons, $\Omega(\Gamma)$ is illustrated in the second picture in Fig. 1 (see [15, P.9] for details). This circle packing is called a Sierpinski curve, being homeomorphic to the well-known Sierpinski carpet [1].

Two pictures in Fig. 5 can be found in the beautiful book *Indra's pearls* by Mumford, Series and Wright (see P. 269 and P. 297 of [17]) where one can find many more circle packings to which our theorem applies. The book presents explicit geometrically finite Schottky groups $\Gamma$ whose limit sets are illustrated in Fig. 5. The boundaries of the shaded regions meet $\Lambda(\Gamma)$ only at finitely many points. Hence our theorem applies to counting circles in these shaded regions.

### 1.3. Schottky dance.

Another class of examples is obtained by considering the images of Schottky disks under Schottky groups. Take $k \geq 1$ pairs of mutually disjoint closed disks $\{(D_i, D'_i) : 1 \leq i \leq k\}$ in $\mathbb{C}$ and for each $1 \leq i \leq k$, choose a M"obius transformation $\gamma_i$ which maps the interior of $D_i$ to the exterior of $D'_i$ and the interior of $D'_i$ to the exterior of $D_i$. The group, say, $\Gamma$, generated by $\{\gamma_i : 1 \leq i \leq k\}$ is called a Schottky group of genus $k$ (cf. [13, Sec. 2.7]). The $\Gamma$-orbit of the disks $D_i$ and $D'_i$'s nests down onto the limit set $\Lambda(\Gamma)$ which is totally disconnected. If we denote by $\mathcal{P}$ the
union $\bigcup_{1 \leq i \leq k} \Gamma(C_i) \cup \Gamma(C'_i)$ where $C_i$ and $C'_i$ are the boundaries of $D_i$ and $D'_i$ respectively, then $P$ is locally finite, as the nesting disks become smaller and smaller. The common exterior of hemispheres above the initial disks $D_i$ and $D'_i$ is a fundamental domain for $\Gamma$ in the upper half-space $\mathbb{H}^3$, and hence $\Gamma$ is geometrically finite. Since $P$ contains no infinite bouquet of tangent circles, Theorem 1.4 applies to $P$; for instance, we can count circles in the picture in Fig. 6 ([17, Fig. 4.11]).

On the structure of the proof. In [12], the counting problem for a bounded Apollonian circle packing was related to the equidistribution of expanding closed horospheres on the hyperbolic 3-manifold $\Gamma \backslash \mathbb{H}^3$. For a general circle packing, there is no analogue of the Descartes circle theorem which made such a relation possible. The main idea in our paper is to relate the counting problem for a general circle packing $P$ invariant under $\Gamma$ with the equidistribution of orthogonal translates of a closed totally geodesic surface in $T^1(\Gamma \backslash \mathbb{H}^3)$. Let $C_0$ denote the unit circle centered at the origin and $H$ the stabilizer of $C_0$ in $\text{PSL}_2(\mathbb{C})$. Thus $H \backslash G$ may be considered as the space of totally geodesic planes of $\mathbb{H}^3$. The important starting point is to describe certain subset $B_T(E)$ in $H \backslash G$ so that the number of circles in the packing $P := \Gamma(C_0)$ of curvature at most $T$ intersecting $E$ can be interpreted as the number of points in $B_T(E)$ of a discrete $\Gamma$-orbit on $H \backslash G$. We then describe the weighted limiting distribution of orthogonal translates of an $H$-period $(H \cap \Gamma) \backslash H$ (which corresponds to a properly immersed hyperbolic surface which may be of infinite area) along these sets $B_T(E)$ in terms of the Burger-Roblin measure (Theorem 4.3, using the main result in [19] (see Thm. 2.5). To translate the weighted limiting distribution result into the asymptotic for $N_T(P, E)$, we relate the density of the Burger-Roblin measure of the contracting horosphere $H^\infty(j)$ with the measure $\omega_T$.

A version of Theorem 1.4 in a weaker form, and some of its applications stated above were announced in [18]. We remark that the methods of this paper can be easily generalized to prove a similar result for a sphere packing in the $n$-dimensional Euclidean space invariant under a non-elementary discrete subgroup of $\text{Isom}(\mathbb{H}^{n+1})$.

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2. Expansion of a hyperbolic surface by orthogonal geodesic flow

We use the following coordinates for the upper half space model for $\mathbb{H}^3$:

$$\mathbb{H}^3 = \{ z + rj = (z, r) : z \in \mathbb{C}, r > 0 \}$$
where \( j = (0, 1) \). The isometric action of \( G = \text{PSL}_2(\mathbb{C}) \), via the Poincare extension of the linear fractional transformations, is explicitly given as the following (cf. [6]):

\[
(2.1) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z + rj) = \frac{(az + b)(\bar{c}z + \bar{d}) + a\bar{c}r^2}{|cz + d|^2 + |c|^2r^2} + \frac{r}{|cz + d|^2 + |c|^2r^2} j.
\]

In particular, the stabilizer of \( j \) is the following maximal compact subgroup of \( G \):

\[
K := \text{PSU}(2) = \{ \begin{pmatrix} a & b \\ -b & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \}.
\]

We set
\[
A := \{ a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \}, \quad M := \{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \}
\]

and
\[
N := \{ n_z := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \}, \quad N^- := \{ n_z^- := \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} : z \in \mathbb{C} \}.
\]

We can identify \( \mathbb{H}^3 \) with \( G/K \) via the map \( g(j) \mapsto gK \). Denoting by \( X_0 \in T^1(\mathbb{H}^3) \) the upward unit normal vector based at \( j \), we can also identify the unit tangent bundle \( T^1(\mathbb{H}^3) \) with \( G.X_0 = G/M \): here \( g.X_0 \) is given by \( d\lambda(g)(X_0) \) where \( \lambda(g) : G/K \to G/K \) is the left translation \( \lambda(g)(g'K) = gg'K \) and \( d\lambda(g) \) is its derivative at \( j \).

The geodesic flow \( \{ g^t \} \) on \( T^1(\mathbb{H}^3) \) corresponds to the right translation by \( a_t \) on \( G/M \):

\[
g^t(gM) = ga_tM.
\]

For a circle \( C \) in \( \mathbb{C} \), denote by \( \hat{C} \) its convex hull, which is the northern hemisphere above \( C \).

Set \( C_0 \) to be the unit circle in \( \mathbb{C} \) centered at the origin. The set-wise stabilizer of \( \hat{C}_0 \) in \( G \) is given by

\[
H = \text{PSU}(1, 1) \cup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{PSU}(1, 1)
\]

where
\[
\text{PSU}(1, 1) = \{ \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \}.
\]

Note that \( H \) is equal to the stabilizer of \( C_0 \) in \( G \) and hence \( \hat{C}_0 \) can be identified with \( H/H \cap K \).

We have the following generalized Cartan decomposition (cf. [29]): for \( A^+ = \{ a_t : t \geq 0 \} \),

\[
G = HA^+ K
\]

in the sense that every element of \( g \in G \) can be written as \( g = hak, h \in H, a \in A^+, k \in K \) and \( h_1a_1k_1 = h_2a_2k_2 \) implies that \( a_1 = a_2, h_1 = h_2m \) and \( k_1 = m^{-1}k_2 \) for some \( m \in H \cap K \cap \text{Z}_G(A) = M \).
As $X_0$ is orthogonal to the tangent space $T_x(\hat{C}_0)$, $H.X_0 = H/M$ corresponds to the set of unit normal vectors to $\hat{C}_0$, which we will denote by $C^0$. Note that $C^0_0$ has two connected components, depending on their directions. For $t \in \mathbb{R}$, the set $g^t(C^0_0) = (H/M)at = (Ha_tM)/M$ corresponds to a union of two surfaces consisting of the orthogonal translates of $\hat{C}_0$ by distance $|t|$ in each direction, both having the same boundary $C_0$.

Let $\Gamma < G$ be a non-elementary discrete subgroup. As in the introduction, let $\{\nu_x : x \in \mathbb{H}^3\}$ be a $\Gamma$-invariant conformal density on $\hat{C}$ of dimension $\delta_{\Gamma}$, that is, each $\nu_x$ is a finite measure on $\hat{C}$ satisfying that for any $x, y \in \mathbb{H}^3$, $z \in \hat{C}$ and $\gamma \in \Gamma$,

$$\gamma_\ast \nu_x = \nu_{\gamma x}; \quad \text{and} \quad \frac{d\nu_y}{d\nu_x}(z) = e^{-\delta_{\Gamma} \beta_x(y,x)}.$$ 

Here $\gamma_\ast \nu_x(R) = \nu_x(\gamma^{-1}(R))$ for a Borel subset $R \subset \hat{C}$ and the Busemann function $\beta_x(y_1, y_2)$ is given by $\lim_{t \to \infty} d(y_1, \xi_t) - d(y_2, \xi_t)$ for a geodesic ray $\xi_t$ toward $z$.

For $u \in T^1(\mathbb{H}^3)$, we define $u^+ \in \hat{C}$ (resp. $u^- \in \hat{C}$) to be the forward (resp. backward) end point of the geodesic determined by $u$ and $\pi(u) \in \mathbb{H}^3$ to be the basepoint. Fixing $o \in \mathbb{H}^3$, the map $u \mapsto (u^+, u^-, t := \beta_{u^-}(\pi(u), o))$ is a homeomorphism between $T^1(\mathbb{H}^3)$ and $(\hat{C} \times \hat{C} - \{(\xi, \xi) : \xi \in \hat{C}\}) \times \mathbb{R}$.

**Definition 2.2.** The Bowen-Margulis-Sullivan measure $m_{\text{BMS}}^\Gamma$ associated to $\{\nu_x\}$ ([2, 14, 31]) is the measure on $T^1(\Gamma \backslash \mathbb{H}^3)$ induced by the following $\Gamma$-invariant measure on $T^1(\mathbb{H}^3)$: for $x \in \mathbb{H}^3$,

$$dm_{\text{BMS}}^\Gamma(u) = e^{\delta_{\Gamma} \beta_{u^+}(x, \pi(u))} e^{\delta_{\Gamma} \beta_{u^-}(x, \pi(u))} d\nu_x(u^+)d\nu_x(u^-)dt.$$ 

By the conformal properties of $\{\nu_x\}$, this definition is independent of the choice of $x \in \mathbb{H}^3$. We also denote by $\{m_x : x \in \mathbb{H}^3\}$ a $G$-invariant conformal density of dimension 2, which is unique up to homothety: each $m_x$ a finite measure on $\hat{C}$ which is invariant under $\text{Stab}_G(x)$ and $dm_x(z) = e^{-2\beta_{x}(y,z)}dm_y(z)$ for any $x, y \in \mathbb{H}^3$ and $z \in \hat{C}$.

**Definition 2.3.** The Burger-Roblin measure $m_{\text{BR}}^\Gamma$ associated to $\{\nu_x\}$ and $\{m_x\}$ ([4, 26]) is the measure on $T^1(\Gamma \backslash \mathbb{H}^3)$ induced by the following $\Gamma$-invariant measure on $T^1(\mathbb{H}^3)$:

$$dm_{\text{BR}}^\Gamma(u) = e^{2\beta_{u^+}(x, \pi(u))} e^{2\beta_{u^-}(x, \pi(u))} dm_x(u^+)dm_x(u^-)dt$$ 

for $x \in \mathbb{H}^3$. By the conformal properties of $\{\nu_x\}$ and $\{m_x\}$, this definition is independent of the choice of $x \in \mathbb{H}^3$.

For any circle $C$, let

$$H_C = \{g \in G : gC = C\} = \{g \in G : gC^\dagger = C^\dagger\}.$$
We consider the following two measures on $C^\dagger$: Fix any $x \in \mathbb{H}^3$, and let
\begin{equation}
(2.4) \quad d\mu_{C^\dagger}^{\text{Leb}}(s) := e^{2s_+ + (x,\pi(s))} dm_x(s) \quad \text{and} \quad d\mu_{C^\dagger}^{\text{PS}}(s) := e^{\delta_+ + (x,\pi(s))} d\nu_x(s^+).
\end{equation}
These definitions are independent of the choice of $x$ and $\mu_{C^\dagger}^{\text{Leb}}$ (resp. $\mu_{C^\dagger}^{\text{PS}}$) is left-invariant by $H_C$ (resp. $H_C \cap \Gamma$). Hence we may consider the measures $\mu_{C^\dagger}^{\text{Leb}}$ and $\mu_{C^\dagger}^{\text{PS}}$ on the quotient $(H \cap \Gamma) \backslash C^\dagger$.

We denote by $\text{sk}_\Gamma(C)$ the total mass of $\mu_{C^\dagger}^{\text{PS}}$, that is,
\[ \text{sk}_\Gamma(C) := \int_{s \in (H \cap \Gamma) \backslash C^\dagger} e^{\delta_+ + (x,\pi(s))} d\nu_x(s^+). \]
In general, $\text{sk}_\Gamma(C)$ may be zero or infinite.

**Theorem 2.5** ([19 Theorem 1.9]). Suppose that the natural projection map $\Gamma \cap H_C \backslash \hat{C} \to \Gamma \backslash \mathbb{H}^3$ is proper. Assume that $|m_{\Gamma}^{\text{BMS}}| < \infty$ and $\text{sk}_\Gamma(C) < \infty$. Then for any $\psi \in C_c(\Gamma \backslash G/M)$, as $t \to \infty$,
\[ e^{(2-\delta_\Gamma)t} \int_{s \in (\Gamma \cap H_C) \backslash C^\dagger} \psi(sa_t) d\mu_{C^\dagger}^{\text{Leb}}(s) \sim \frac{\text{sk}_\Gamma(C)}{|m_{\Gamma}^{\text{BMS}}| m_{\Gamma}^{\text{BR}}} \psi. \]
Moreover $\text{sk}_\Gamma(C) > 0$ if $[\Gamma : H_C \cap \Gamma] = \infty$.

Note that if $|m_{\Gamma}^{\text{BMS}}| < \infty$, then $\Gamma$ is of divergence type; that is, the Poincare series of $\Gamma$ diverges at $\delta_\Gamma$. When $\Gamma$ is of divergence type, the $\Gamma$-invariant conformal density $\{\nu_\gamma\}$ of dimension $\delta_\Gamma$ is unique up to homothety (see [26 Remark following Corollary 1.8]): explicitly $\nu_x$ can be taken as the weak-limit as $s \to \delta_\Gamma^+$ of the family of measures
\[ \nu_x(s) := \frac{1}{\sum_{\gamma \in \Gamma} e^{-sd_j(\gamma_\gamma)}} \sum_{\gamma \in \Gamma} e^{-sd_j(\gamma_\gamma)} \delta_\gamma. \]

Recall that $g \in \text{PSL}_2(\mathbb{C})$ is parabolic if and only if $g$ has a unique fixed point in $\hat{C}$.

**Theorem 2.6** ([19 Theorem 5.2]). Let $\Gamma$ be geometrically finite. Suppose that the natural projection map $\Gamma \cap H_C \backslash \hat{C} \to \Gamma \backslash \mathbb{H}^3$ is proper. Then $\text{sk}_\Gamma(C) < \infty$ if and only if either $\delta_\Gamma > 1$ or any parabolic fixed point of $\Gamma$ lying on $C$ is fixed by a parabolic element of $H_C \cap \Gamma$.

**Proof.** Note that in the notation of [19 Theorem 5.2], if we put $E = \hat{C}$, which is a complete totally geodesic submanifold of $\mathbb{H}^3$ of codimension 1, then $\partial(\pi(E)) = C$, $\hat{E} = C^\dagger$, $\Gamma \hat{E} = H_C \cap \Gamma$, and $|\mu_{E}^{\text{PS}}| = \text{sk}_\Gamma(C)$. Hence the conclusion is immediate. \qed

3. **Reformulation into the orbital counting problem on the space of hyperbolic planes**

Let $G = \text{PSL}_2(\mathbb{C})$ and $\Gamma < G$ be a non-elementary discrete subgroup. Let $C$ be a circle in $\hat{C}$ and $H_C$ denote the set-wise stabilizer of $C$ in $G$.

It is clear:
Lemma 3.1. If $\Gamma(C)$ is infinite, then $[\Gamma : H_C \cap \Gamma] = \infty$.

Lemma 3.2. The following are equivalent:

1. A circle packing $\Gamma(C)$ is locally finite;
2. the natural projection map $f : \Gamma \cap H_C \setminus \hat{C} \to \Gamma \setminus \mathbb{H}^3$ is proper;
3. $H_C \backslash H_C \Gamma$ is discrete in $H_C \backslash G$.

Proof. We observe that the properness of $f$ is equivalent to the condition that only finitely many distinct hemispheres in $\Gamma(\hat{C})$ intersects a given compact subset of $\mathbb{H}^3$. Note that any compact subset of $\mathbb{H}^3$ is contained in a compact subset the form $E \times [r_1, r_2] = \{(z, r) : z \in E, r_1 \leq r \leq r_2\}$ for $E \subset \mathbb{C}$ compact and $0 < r_1 < r_2 < \infty$, and that the radius of a circle in $\mathbb{C}$ is same as the height of its convex hull in $\mathbb{H}^3$. Hence the properness of the map $f$ is again equivalent to the condition that for any $r > 0$ and a compact subset $E \subset \mathbb{C}$, there are only finitely many distinct circles in $\Gamma(C)$ intersecting $E$ and of radii at least $r$, that is, $\Gamma(C)$ being locally finite, proving the equivalence of (1) and (2).

It is straightforward to verify that the properness of $f$ and that of the projection map $\Gamma \cap H_C \setminus C^\dagger \to \Gamma \setminus \mathbb{T}^1(\mathbb{H}^3)$ are equivalent. Let $X_C \subset C^\dagger$ such that $X_C^\dagger = \infty \in \hat{C}$. Let $M_C = \{g \in G : gX_C = X_C\}$. Since $\hat{C}$ is the unique totally geodesic submanifold of $\mathbb{H}^3$ orthogonal to $X_C$, $M_C$ is contained in $H_C$. We identify $G \cap M_C$ with $T^1(\mathbb{H}^3)$ via $gM_C \mapsto gX_C$. Since $H/M_C$ identifies with $C^\dagger$, the canonical map $\Gamma \cap H_C \setminus H_C/M_C \to \Gamma \setminus G/M_C$ is proper. Since $M_C$ is compact, it follows that $\Gamma \cap H_C \setminus H_C \to \Gamma \setminus G$ is proper. Equivalently $\Gamma H_C$ is closed in $G$ (see [19] for the equivalence). As $\Gamma$ is countable, this is again equivalent to the discreteness of $H_C \backslash H_C \Gamma$ in $H_C \backslash G$. This proves the equivalence of (2) and (3). \hfill \Box

Remark 3.3. If $\Gamma \cap H_C$ is a lattice in $H_C$, then $\Gamma H_C$ is closed in $G$ ([25, §1]), and hence $\Gamma(C)$ is a locally finite circle packing. In this case, by [19, Theorem 1.11], we have $\mathrm{sk}_\Gamma(C) < \infty$.

Proposition 3.4. Let $\xi \in C$ be a parabolic fixed point of $\Gamma$. Suppose that $\Gamma(C)$ does not contain an infinite bouquet of tangent circles glued at $\xi$. Then $\xi$ is a parabolic fixed point for $H_C \cap \Gamma$.

Proof. Suppose that there exists a parabolic element $\gamma \in \Gamma - H_C$ fixing $\xi \in \hat{C}$. By sending $\xi$ to $\infty \in \hat{C}$ by an element of $G$, we may assume that $\xi = \infty$ and $\gamma$ acts as a translation on $\hat{C}$. Since $\gamma C \neq C$ and $C$ is a circle passing through $\infty$, we have that $\{\gamma^k C : k \in \mathbb{Z}\}$ is an infinite collection of parallel lines. By sending $\infty$ back to the original $\xi$, we see that $\{\gamma^k C : k \in \mathbb{Z}\}$ is an infinite bouquet of tangent circles glued at $\xi$. \hfill \Box

3.1. Deduction of Theorem 1.4 from Theorem 1.6. We only need to ensure that $\mathrm{sk}_\Gamma(P) < \infty$, or equivalently, $\mathrm{sk}_\Gamma(C) < \infty$ for each $C \in P$. By the assumption in Theorem 1.4, if $\xi \in C$ is any parabolic fixed point of $\Gamma$, then by Proposition 3.4, $\xi$ is a parabolic fixed point for $H_C \cap \Gamma$. Therefore by Theorem 2.6, $\mathrm{sk}_\Gamma(C) < \infty$. \hfill \Box
3.2. Relating counting on a single $\Gamma$-orbit to a set $B_T(E) \subset H\backslash G$.

In the rest of this section, let $C_0$ denote the unit circle in $\mathbb{C}$ centered at the origin and let $H := \text{Stab}(C_0)$. We follow notations from Section 2. We assume that $\Gamma(C_0)$ is a locally finite circle packing of $\mathbb{C}$.

Let $E$ be a bounded subset in $\mathbb{C}$ and set

$$N_T(\Gamma(C_0), E) := \#\{C \in \Gamma(C_0) : C \cap E \neq \emptyset, \text{Curv}(C) < T\}.$$ 

For $s > 0$, we set

$$A^+_s := \{a_t : 0 \leq t \leq s\}; \quad A^-_s := \{a_{-t} : 0 \leq t \leq s\}.$$

For a subset $E \subset \mathbb{C}$, we set $N_E := \{n_z : z \in E\}$.

**Definition 3.5** (Definition of $B_T(E)$). For $E \subset \mathbb{C}$ and $T > 1$, we define the subset $B_T(E)$ of $H\backslash G$ to be the image of the set

$$KA^+_0 N^-_E = \{ka_t n_z \in G : k \in K, 0 \leq t < \log T, z \in E\}$$

under the canonical projection $G \rightarrow H\backslash G$.

For a bounded circle $C$ in $\mathbb{C}$, $C^0$ denotes the open disk enclosed by $C$. We will not need this definition for a line since there can be only finitely many lines intersecting a fixed bounded subset in a locally finite circle packing.

**Definition 3.6.** For a given circle packing $\mathcal{P}$, a bounded subset $E \subset \mathbb{C}$ is said to be $\mathcal{P}$-admissible if, for any bounded circle $C \in \mathcal{P}$, $C^0 \cap E \neq \emptyset$ implies $C^0 \subset E$, possibly except for finitely many circles.

The following translation of $N_T(\Gamma(C_0), E)$ as the number of points in $[e] \Gamma \cap B_T(E)$, where $[e] = H \in H\backslash G$, is crucial in our approach:

**Proposition 3.7.** If $E$ is $\Gamma(C_0)$-admissible, there exists $m_0 \in \mathbb{N}$ such that for all $T \gg 1$,

$$\#[e] \Gamma \cap B_T(E) - m_0 \leq N_T(\Gamma(C_0), E) \leq \#[e] \Gamma \cap B_T(E) + m_0.$$ 

**Proof.** Observe that

$$\#[e] \Gamma \cap B_T(E) = \#\{[\gamma] \in \Gamma \cap H\backslash \Gamma : H\gamma \cap KA^+_0 N^-_E \neq \emptyset\}$$

$$= \#\{[\gamma] \in \Gamma / \Gamma \cap H : \gamma HK \cap N_E A^-_{\log T} K \neq \emptyset\}$$

$$= \#\{\gamma(\hat{C}_0) : \gamma HK \cap N_E A^-_{\log T} K \neq \emptyset\}$$

where the second equality is obtained by taking the inverse. Since

$$N_E A^-_{\log T} j = \{(z, r) \in \mathbb{H}^3 : T^{-1} < r \leq 1, z \in E\}$$

and $K$ is the stabilizer of $j$ in $G$, it follows that

$$\#[e] \Gamma \cap B_T(E) = \#\{\gamma(\hat{C}_0) : \gamma(\hat{C}_0) \text{ contains } (z, r) \text{ with } z \in E, \ T^{-1} < r \leq 1\}.$$ 

By the admissibility assumption on $E$, we observe that $\gamma(\hat{C}_0)$ contains $(z, r)$ with $z \in E$ and $T^{-1} < r \leq 1$ if and only if the center of $\gamma(C_0)$ lies in $E$ and the radius of $\gamma(C_0)$ is greater than $T^{-1}$, possibly except for finitely many number (say, $m_0$) of circles. □
4. Uniform distribution along the family $B_T(E)$ and the Burger-Roblin measure

We keep the notations $C_0, H, K, M, A^+, X_0, G, \{m_x : x \in \mathbb{H}^3\}$, etc., from section 2. Denoting by $dm$ the probability invariant measure on $M$,

$$dh = d\mu_{C^+_0}(s)dm$$

is a Haar measure on $H \cong C^+_0 \times M$ as $\mu_{C^+_0}$ is $H$-invariant, and the following defines a Haar measure on $G$: for $g = ha, k \in HA^+K$,

$$dg = 4 \sinh r \cdot \cosh r \, dh \, dm_j(k)$$

where $dm_j(k) := dm_j(k,X^+_0)$.

We denote by $d\lambda$ the unique measure on $H \setminus G$ which is compatible with the choice of $dg$ and $dh$: for $\psi \in C_c(G)$,

$$\int_G \psi \, dg = \int_{[g] \in H \setminus G} \int_{h \in H} \psi(h[g]) \, dh \, d\lambda[g].$$

For a bounded set $E \subset \mathbb{C}$, recall that the set $B_T(E)$ in $H \setminus G$ is the image of the set

$$KA^+_{\log T}N_{-E} = \{ka,t_n \cdot z \in G : k \in K, 0 \leq t < \log T, z \in E\}$$

under the canonical projection $G \to H \setminus G$.

The goal of this section is to deduce the following theorem 4.3 from Theorem 2.5.

**Theorem 4.3.** Let $\Gamma$ be a non-elementary discrete subgroup of $G$. Suppose that $|m_{\Gamma}^{\text{BMS}}| < \infty$ and $sk_{\Gamma}(C_0) < \infty$. Suppose that the natural projection map $\Gamma \cap H \setminus C_0 \to \Gamma \setminus \mathbb{H}^3$ is proper. Then for any bounded Borel subset $E \subset \mathbb{C}$ and for any $\psi \in C_c(\Gamma \setminus G)$, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_{g \in B_T(E)} \int_{h \in \Gamma \setminus H \setminus H} \psi(hg) \, dh \, d\lambda(g) = \frac{sk_{\Gamma}(C_0)}{\delta_{\Gamma} : |m_{\Gamma}^{\text{BMS}}|} \int_{n \in N_{-E}} m_{\Gamma}^{\text{BR}}(\psi_n) \, dn$$

where $\psi_n \in C_c((\Gamma \setminus G)^M)$ is given by $\psi_n(g) = \int_{m \in M} \psi(gmn) \, dm$ and $dn$ is the Lebesgue measure on $N$.

In order to prove this result using Theorem 2.5, it is crucial to understand the shape of the set $B_T(E)$ in the $HA^+K$ decomposition of $G$. This is one of the important technical steps in the proof.

**On the shape of $B_T(E)$:** Fix a left-invariant metric on $G$. For $\epsilon > 0$, let $U_\epsilon$ be the $\epsilon$-ball around $e$ in $G$. For a subset $W$ of $G$, we set $W_\epsilon = W \cap U_\epsilon$.

**Proposition 4.4.** (1) If $a_t \in HKa_sK$ for $s > 0$, then $|t| \leq s$.

(2) Given any $\epsilon > 0$, there exists $T_0 = T_0(\epsilon)$ such that

$$\{k \in K : a_tk \in HKA^+ \text{ for some } t > T_0\} \subset K_\epsilon M.$$
Proof. Suppose $a_t = hk_1a_s k_2$ for $h \in H, k_1, k_2 \in K$. We note that, as $Aj$ is orthogonal to $\hat{C}_0$ and $j \in \hat{C}_0$,

$$|t| = d(\hat{C}_0, a_t j) = d(\hat{C}_0, h k_1 a_s j)$$

$$= d(\hat{C}_0, k_1 a_s j) \leq d(j, k_1 a_s j) = d(j, a_s j) = s,$$

proving the first claim. For the second claim, suppose $a_t k \in HK a_s$ for some $s \geq 0$. Then $ka_{-s} \in a_{-t} HK$. Applying both sides to $j \in \mathbb{H}^3$, $k(e^{-s} j) \in a_{-t} \hat{C}_0$. Now $a_{-t} \hat{C}_0 = e^{-t} \hat{C}_0$ is the northern hemisphere of Euclidean radius $e^{-t}$ about 0 in $\mathbb{H}^3$.

On the other hand $A^- j = (0,1) j$ for $A^- = \{a_{-s} : s \geq 0\}$ and $K_\epsilon \{(0,1) j\}$ consists of geodesic rays in $\mathbb{H}^3$ joining $j$ and points of $K_\epsilon(0) \subset \mathbb{C}$. Now $K_\epsilon(0)$ contains a disk of radius, say $r_\epsilon > 0$, centered at 0 in $\mathbb{C}$, and hence $K_\epsilon \{(0,1) j\}$ contains a Euclidean half ball of radius $r_\epsilon > 0$ centered at 0 in $\mathbb{H}^3$.

Therefore for $t > T_0(\epsilon) := -\log(r_\epsilon)$, $k(e^{-s} j) \in a_{-t} \hat{C}_0$ implies that $k(e^{-s} j) \in K_\epsilon \{(0,1) j\}$, in other words, $ka_{-s} K \subset K_\epsilon A^- K$. By the uniqueness of the left $K$-component, modulo the right multiplication by $M$, in the decomposition $G = K A^- K$, it follows that $k \in K_\epsilon M$, proving the second claim. □

For $t \in \mathbb{R}$ and $T > 1$, set

$$K_T(t) := \{k \in K : a_t k \in HK A^+_{\log T}\}.$$  

As a consequence of Proposition 4.4 we have the following.

**Corollary 4.5.**

1. For all $0 \leq t < \log T$, $e \in K_T(t)$.
2. For all $t > \log T$, $K_T(t) = \emptyset$.
3. For any $\epsilon > 0$, there exists $T_0(\epsilon) \geq 1$ such that we have

$$K_T(t) \subset K_\epsilon M \quad \text{for all } t > T_0(\epsilon).$$

Thus for any $T > 1$,

$$HK A^+_{\log T} = \cup_{0 \leq t < \log T} Ha_t K_T(t).$$

Since $B_T(E) = H \setminus HK A^+_{\log T} N_{-E}$, (4.6) together with Corollary 4.5 shows that $B_T(E)$ is essentially of the form $H \setminus Ha_{\log T} K_\epsilon MN_{-E}$. The following proposition shows that $B_T(E)$ can be basically controlled by the set $H \setminus Ha_{\log T} N_{-E}$.

**Proposition 4.7.** Fix a bounded subset $E$ of $\mathbb{C}$. There exists $\ell = \ell(E) \geq 1$ such that for all sufficiently small $\epsilon > 0$,

$$a_t km_n z \in H\ell ma_t n_z U_{\ell}$$

holds for any $m \in M$, $t > 0$, $z \in E$, and $k \in K_\epsilon$.

Proof. Recalling that $N^-$ denotes the lower triangular subgroup of $G$, we note that the product map $N^- x A x M x N \rightarrow G$ is a diffeomorphism at
a neighborhood of $e$, in particular, bi-Lipschitz. Hence there exists $\ell_1 > 1$ such that for all small $\epsilon > 0$,

\[(4.8) \quad K_\epsilon \subset N_{\ell_1\epsilon}A_{\ell_1\epsilon}M_{\ell_1\epsilon}N_{\ell_1\epsilon}.
\]

Similarly due to the $H \times A \times N$ product decomposition of $G_\epsilon$, there exists $\ell_2 > 1$ such that

\[(4.9) \quad U_\epsilon \subset H_{\ell_2\epsilon}A_{\ell_2\epsilon}N_{\ell_2\epsilon}
\]

for all small $\epsilon > 0$ ([8, Lem 2.4]). We also have $\ell_3 > 1$ such that for all small $\epsilon > 0$,

\[(4.10) \quad A_{(\ell_1+\ell_2)\epsilon}N_{(\ell_1+\ell_2)\epsilon}M_{\ell_1\epsilon} \subset U_{\ell_3\epsilon}.
\]

Now let $t > 0, k \in K_\epsilon, m \in M, n \in N$. Then by (4.8), we may write

\[k = n_1b_1m_1n_1 \in N_{\ell_1\epsilon}A_{\ell_1\epsilon}M_{\ell_1\epsilon}N_{\ell_1\epsilon}.
\]

Since $a_{t\epsilon n_1^-}a_{-t} \in N_\epsilon^-$ for $t > 0$, we have, by (4.9),

\[a_{t\epsilon n_1^-}a_{-t} = h_2b_2m_2n_2 \in H_{\ell_2\epsilon}A_{\ell_2\epsilon}M_{\ell_2\epsilon}N_{\ell_2\epsilon}.
\]

Therefore

\[a_{t\epsilon}mn = (a_{t\epsilon n_1^-}a_{-t})(a_{t\epsilon b_1m_1n_1})mn
= (h_2b_2m_2n_2)a_{t\epsilon b_1m_1n_1}mn
= h_2b_2m_2(a_{t\epsilon b_1b_1^-1}a_{-t})n_2a_1b_1m_1n_1mn
= h_2a_{t\epsilon}(b_2m_2)b_1(b_1^-1a_{-t}n_2a_1b_1)m_1n_1mn
\in h_2a_{t\epsilon}A_{(\ell_1+\ell_2)\epsilon}M_{\ell_2\epsilon}N_{(\ell_1+\ell_2)\epsilon}M_{\ell_1\epsilon}mn \quad \text{by (4.10)}
\subset h_2a_{t\epsilon}U_{\ell_3\epsilon}mn.
\]

As $E$ is bounded, there exists $\ell = \ell(E) > \ell_2$ such that for all small $\epsilon > 0$ and for all $z \in E$,

\[U_{\ell_3\epsilon}mnz \subset mnzU_\epsilon.
\]

Since $a_\ell$ commutes with $m$, we obatin for all $z \in E$ that

\[a_{t\epsilon}mnz \subset H_{\ell_2\epsilon}ma_\ell n_2U_\epsilon.
\]

\[\square
\]

**Proof of Theorem 4.3.** Let $\ell = \ell(E) \geq 1$ be as in Proposition 4.7. For $\psi \in C_c(\Gamma \setminus G)$ and $\epsilon > 0$, we define $\psi_\epsilon^\pm \in C_c(\Gamma \setminus G)$,

\[\psi_\epsilon^+(g) := \sup_{u \in U_\epsilon} \psi(gu) \quad \text{and} \quad \psi_\epsilon^-(g) := \inf_{u \in U_\epsilon} \psi(gu).
\]

For a given $\eta > 0$, there exists $\epsilon = \epsilon(\eta) > 0$ such that for all $g \in \Gamma \setminus G$,

\[|\psi_\epsilon^+(g) - \psi_\epsilon^-(g)| \leq \eta
\]

by the uniform continuity of $\psi$. 

On the other hand, by Theorem 2.5 we have $T_1(\eta) \gg 1$ such that for all $t > T_1(\eta),$

$$
\text{(4.11)}
\int_{h \in \Gamma \cap H \setminus H} \psi^+_{\epsilon}(ha_t n) dh
= \int_{s \in \Gamma \cap H \setminus C_0} \int_{m \in M} \psi^+_{\epsilon}(sa_t mn) dmd\mu_{\text{Leb}}(s)
= (1 + O(\eta)) \frac{\text{sk}_\Gamma(C_0)}{|m_{\text{BMS}}|} m_{\text{BR}}(\psi^+_{\epsilon,n}) e^{(\delta_\epsilon - 2)t}
$$

where $\psi^+_{\epsilon,n}(g) = \int_{m \in M} \psi^+_{\epsilon}(g mn) dm.$

As $N_{-E}$ is relatively compact, the implied constant can be taken uniformly over all $n \in N_{-E}.$ Let $T_0(\epsilon) > T_1(\eta)$ be as in Proposition 4.4. For $[e] = H \in H \setminus G$ and $s > 0,$ set

$$V_T(s) := \bigcup_{s \leq t < \log T} [e] a_t K_T(t) N_{-E}$$

so that

$$B_T(E) = V_T(s) \cup (B_T(E) - V_T(s)).$$

Setting

$$\psi^H(g) := \int_{h \in \Gamma \cap H \setminus H} \psi(hg) dh,$$

note that $\psi^H$ is left $H$-invariant as $dh$ is a Haar measure. We will show that

$$\limsup_{T \to \infty} \frac{1}{T^3} \int_{[g] \in V_T(T_0(\epsilon))} \psi^H(g) d\lambda(g) = \frac{\text{sk}_\Gamma(C_0)}{\delta_\epsilon \cdot |m_{\text{BMS}}|} \int_{n \in N_{-E}} m_{\text{BR}}(\psi^+_n) dn.$$

By Corollary 4.5 we have

$$V_T([e] a_t K_T(t) N_{-E}) = \bigcup_{T_0(\epsilon) \leq t < \log T} [e] a_t K_{t_k} M N_{-E}.$$

Let $[g] \in V_T(T_0(\epsilon)),$ so $[g] = [e] a_t k m n.$ With $T_0(\epsilon) \leq t < \log T,$ $k \in K_{\epsilon},$ $m \in M$ and $n \in N_{-E}.$ By Proposition 4.7 there exist $h_0 \in H$ and $u \in U_{\epsilon}$ such that

$$a_t k m n = h_0 m a_t n u$$

so that $[g] = [e] a_t n u,$ since $M \subset H.$

We have

$$\psi^H(g) = \int_{h \in \Gamma \cap H \setminus H} \psi(ha_t n u) dh \leq \int_{h \in \Gamma \cap H \setminus H} \psi^+_{\epsilon}(ha_t n) dh.$$

The measure $e^{2t} dtdn$ is a right invariant measure of $AN$ and $[e]AN$ is an open subset in $H \setminus G.$ Hence $d\lambda(a_t n)$ (restricted to $[e]AN$) and $e^{2t} dtdn$ are constant multiples of each other. It follows from the formula of $dg$ that

$$d\lambda(a_t n) = e^{2t} dtdn.$$ Therefore

$$\int_{[g] \in V_T(T_0(\epsilon))} \psi^H(g) d\lambda(g) \leq \int_{n \in N_{-E}} \int_{T_0(\epsilon) < t \leq \log T} \int_{h \in \Gamma \cap H \setminus H} \psi^+_{\epsilon}(ha_t n) dhe^{2t} dtdn.$$
Similarly we can show that
\[ \psi_{\epsilon,n} \rightarrow \psi \]\nwhich appears in the asymptotic expression in Theorem 4.3, converges to
\[ \int_{n \in \mathbb{N}} m^\text{BR}_g(\psi_n) \, dn \Rightarrow \int_{n \in \mathbb{N}} m^\text{BR}_g(\psi) \, dn \]
where the implied constant depends only on \( \psi \). Hence by (4.11),
\[
\int_{n \in \mathbb{N}} \int_{T_0(e) < t < \log T} \int_{h \in \Gamma \cap H \setminus H} \psi_{\epsilon,n}^+(ha_t) \, dh \, dt \, dn = (1 + O(\eta)) \frac{\text{skR}(C_0)}{\delta_T \cdot |m^\text{BMS}|} \cdot \int_{n \in \mathbb{N}} m^\text{BR}_g(\psi_n) \, dn \cdot (T^{\delta_T} - e^{\delta_T T_0(e)}).
\]
Hence
\[
\limsup_T \frac{1}{T^\delta} \int_{[g] \in V_T(T_0(e))} \psi^H(g) \, d\lambda(g) = (1 + O(\eta)) \frac{\text{skR}(C_0)}{\delta_T \cdot |m^\text{BMS}|} \cdot \int_{n \in \mathbb{N}} m^\text{BR}_g(\psi_n) \, dn.
\]
On the other hand, since \( \Gamma \setminus G \) is a closed subset of \( \Gamma \setminus \Gamma \), so is \( \cup_{0 \leq t \leq s} \Gamma \setminus \Gamma G N_+ \) for any fixed \( s > 0 \); in particular, its intersection with a compact subset of \( \Gamma \setminus G \) is compact.

Since
\[
\cup_{[g] \in B_T(E) \setminus V_T(T_0(e))} \Gamma \setminus \Gamma G N_+ \subset \cup_{0 \leq t \leq s} \Gamma \setminus \Gamma G N_+ N_-
\]
and \( \psi \) has compact support, we have, as \( T \to \infty \),
\[
\int_{[g] \in B_T(E) \setminus V_T(T_0(e))} \psi(hg) \, dh \, d\lambda(g) = O(1).
\]
Therefore
\[
\limsup_T \frac{1}{T^\delta} \int_{[g] \in B_T(E)} \psi^H(g) \, d\lambda(g) \leq (1 + O(\eta)) \frac{\text{skR}(C_0)}{\delta_T \cdot |m^\text{BMS}|} \cdot \int_{n \in \mathbb{N}} m^\text{BR}_g(\psi_n) \, dn.
\]
As \( \eta > 0 \) is arbitrary and \( \epsilon(\eta) \to 0 \) as \( \eta \to 0 \), we have
\[
\limsup_T \frac{1}{T^\delta} \int_{[g] \in B_T(E)} \psi^H(g) \, d\lambda(g) \leq \frac{\text{skR}(C_0)}{\delta_T \cdot |m^\text{BMS}|} \cdot \int_{n \in \mathbb{N}} m^\text{BR}_g(\psi_n) \, dn.
\]
Similarly we can show that
\[
\liminf_T \frac{1}{T^\delta} \int_{[g] \in B_T(E)} \psi^H(g) \, d\lambda(g) \geq \frac{\text{skR}(C_0)}{\delta_T \cdot |m^\text{BMS}|} \cdot \int_{n \in \mathbb{N}} m^\text{BR}_g(\psi_n) \, dn.
\]

5. On the measure \( \omega_T \)

In this section we will describe a measure \( \omega_T \) on \( \mathbb{C} \) and show that the term
\[
\int_{n \in \mathbb{N}} m^\text{BR}_g(\Psi_n) \, dn,
\]
which appears in the asymptotic expression in Theorem 4.3, converges to \( \omega_T(E) \) as the support of \( \Psi \) shrinks to \([e]\) with \( \int_{\Gamma \setminus G} \Psi \, dg = 1 \).

We keep the notations \( G, K, M, A^+, N, N^-, a_t, n_z, n_z^- \), etc., from section 2. Throughout this section, we assume that \( \Gamma \) is a non-elementary discrete...
subgroup of $G$. Recall that $\{\nu_x = \nu_{\Gamma,x} : x \in \mathbb{H}^3\}$ denotes a $\Gamma$-invariant conformal density for $\Gamma$ of dimension $\delta_\Gamma > 0$.

**Definition 5.1.** Define a Borel measure $\omega_\Gamma$ on $\mathbb{C}$ as follows: for $\psi \in C_c(\mathbb{C})$,

$$\omega_\Gamma(\psi) = \int_{z \in \mathbb{C}} e^{\delta_\Gamma \beta_z(x,z+j)} \psi(z) d\nu_{\Gamma,x}(z)$$

for $x \in \mathbb{H}^3$ and $z + j := (z,1) \in \mathbb{H}^3$.

In order to see that the definition of $\omega_\Gamma$ is independent of the choice of $x \in \mathbb{H}^3$, we observe that for any $x_1, x_2 \in \mathbb{H}^3$ and $z \in \mathbb{C}$,

$$e^{\delta_\Gamma (\beta_{z_1,z_2} - \beta_{z_2,z_1})} \frac{d\nu_{z_1}}{d\nu_{z_2}}(z) = e^{\delta_\Gamma \beta_{z_1,z_2}} \frac{d\nu_{z_1}}{d\nu_{z_2}}(z) = 1$$

by the conformality of $\{\nu_x\}$.

**Lemma 5.2.** For any $x = p + rj \in \mathbb{H}^3$ and $\psi \in C_c(\mathbb{C})$,

$$\omega_\Gamma(\psi) := \int_{z \in \mathbb{C}} (r^{-1}|z - p|^2 + r)^{\delta_\Gamma} \psi(z) d\nu_x(z).$$

**Proof.** It suffices to show that

$$\beta_z(p + rj, z + j) = \log \frac{|z - p|^2 + r^2}{r}.$$ 

We use the fact that the hyperbolic distance $d$ on the upper half space model of $\mathbb{H}^3$ satisfies

$$\cosh(d(z_1 + r_1j, z_2 + r_2j)) = \frac{|z_1 - z_2|^2 + r_1^2 + r_2^2}{2r_1r_2}$$

for $z_i + r_ij \in \mathbb{H}^3$ (cf. [6]).

Note that

$$\beta_z(z, z + j) = \beta_0(j, -z + p + rj) = \lim_{t \to \infty} t - d(-z + p + rj, e^{-t}j) = \lim_{t \to \infty} t - d(p + rj, z + e^{-t}j).$$

Now

$$\cosh d(p + rj, z + e^{-t}j) = \frac{e^{t(|z - p|^2 + r^2)} + e^{-t}}{2r}$$

and hence

$$e^{d(p + rj, z + e^{-t}j)} + e^{-d(p + rj, z + e^{-t}j)} = \frac{e^{t(|z - p|^2 + r^2)} + e^{-t}}{r}.$$ 

Therefore as $t \to \infty$,

$$d(p + rj, z + e^{-t}j) \sim t + \log \frac{|z - p|^2 + r^2}{r}.$$ 

Hence

$$\beta_z(p + rj, z + j) = \log \frac{|z - p|^2 + r^2}{r}.$$ 

□
**Definition 5.3.** For a function \( \psi \) on \( \mathbb{C} \) with compact support, define a function \( \mathcal{R}_\psi \) on \( MAN^-N \subset G \) by

\[
\mathcal{R}_\psi(ma_t n_x n_z) = e^{-\delta t} \psi(-z)
\]

for \( m \in M, t \in \mathbb{R}, x, z \in \mathbb{C} \). If \( \psi \) is the characteristic function of \( E \subset \mathbb{C} \), we put \( \mathcal{R}_E = \mathcal{R}_{\chi_E} \).

Since the product map \( M \times A \times N^- \times N \rightarrow G \) has a diffeomorphic image, the above function is well-defined.

**Proposition 5.4.** For any \( \psi \in C_c(\mathbb{C}) \),

\[
\omega_{\Gamma}(\psi) = \int_{k \in K/M} \mathcal{R}_\psi(k^{-1}) d\nu_j(k(0)).
\]

**Proof.** If \( k \in K \) with \( k^{-1} = ma_t n_x n_z \in MAN^-N \), since \( MAN \) fixes 0,

\[
k(0) = n_z(0) = -z.
\]

We note that \( \lim_{s \to \infty} a_{-s}(j) = 0 \) and compute

\[
0 = \beta_{-z}(k(j), j) \\
= \beta_{-z}(n_z n_x a_{-t j}, j) \\
= \beta_0(n_x a_{-t j}, n_z(j)) \\
= \lim_{s \to \infty} d(n_{-x} a_{-t j}, a_{-s j}) - d(n_z(j), a_{-s j}) \\
= \lim_{s \to \infty} d((a_s n_{-x} a_{-s}) a_{-t j}, j) - d(n_z(j), a_{-s j}) \\
= \lim_{s \to \infty} d(a_{s-t j}, j) - d(n_z(j), a_{-s j}) \\
= \lim_{s \to \infty} s - t - d(n_z(j), a_{-s j})
\]

and hence

\[
-t = \lim_{s \to \infty} d(n_z(j), a_{-s j}) - s = \beta_0(n_z(j), j) = \beta_{-z}(j, -z + j).
\]

Hence for \( k^{-1} \in K \cap MAN^-N \),

\[
\mathcal{R}_\psi(k^{-1}) = e^{\delta t \beta_0(j, n_{k(0)}(j))} \psi(k(0)).
\]
Since the complement of \( NN^- AM/M \) in \( K/M \) is a single point and \( \nu_j \) is atom-free, we have

\[
\int_{k \in K/M} \mathcal{R}_E(k^{-1}) d\nu_j(k(0)) = \int_{k \in (K \cap NN^- AM)/M} \mathcal{R}_E(k^{-1}) d\nu_j(k(0)) = \int_{z \in \mathbb{C}} e^{\delta \beta_{(0)}(j,k(0)+j)} \psi(k(0)) d\nu_j(k(0)) = \int_{z \in \mathbb{C}} e^{\delta \beta_{-z}(j,-z+j)} \psi(-z) d\nu_j(-z) = \int_{z \in \mathbb{C}} e^{\delta \beta_{z}(j,z+j)} \psi(z) d\nu_j(z) = \omega_T(\psi).
\]

Lemma 5.5. If \((ma_t n_x^- n_z)(m_1 a_t n_x^- n_z) = m_0 a_t_0 n_x^- n_z \) in the MAN^-N coordinates, then

\[
t_0 = t + t_1 + 2 \log(1 + e^{-t_1 x_1 z'})
\]

for some \( z' \in \mathbb{C} \) with \( |z'| = |z'| \).

Proof. Note that if \( m_1 = \text{diag}(e^{i\theta_1}, e^{-i\theta_1}) \), then

\[
a_t n_x^- n_z m_1 = m_1 a_t n_x^- n_z e^{i\theta_1 z}.
\]

Hence we may assume \( m_1 = m = e \) without loss of generality. We use the following simple identity for \( z, x \in \mathbb{C} \):

\[
(5.6) \quad n_z n_x^- = \begin{pmatrix} 1 + xz & 0 \\ 0 & (1 + xz)^{-1} \end{pmatrix} n_x^- (1 + xz) n_z (1 + xz)^{-1}.
\]

Hence we have

\[
(a_t n_x^- n_z)(a_t n_x^- n_z) = (a_{t+t_1})(a_t^{-1} n_x^- a_t)(a_t^{-1} n_z a_t) n_x^- n_z = a_{t+t_1} n_x^- n_z e^{-t_1 z} n_x^- n_z = a_{t+t_1} n_x^- (1 + e^{-t_1 x_1 z}) 0 0 (1 + e^{-t_1 x_1 z})^{-1} n_x^- (1 + e^{-t_1 x_1 z}) n_z (1 + e^{-t_1 x_1 z})^{-1} n_z = m a_{t+t_1} 2 \log(1 + e^{-t_1 x_1 z}) n_x^- n_z
\]

for appropriate \( m \in M \) and \( x_2, z_2 \in \mathbb{C} \).

Let \( E \subset \mathbb{C} \) be a bounded subset and \( U_\epsilon \subset G \) a symmetric \( \epsilon \)-neighborhood of \( e \) in \( G \). For \( \epsilon > 0 \), set

\[
E_\epsilon^+ := U_\epsilon(E) \quad \text{and} \quad E_\epsilon^- := \cap_{u \in U_\epsilon} u(E).
\]
Lemma 5.8. There exists \( \ell > 0 \) such that for all small \( \epsilon > 0 \) and any \( g \in U_{\ell \epsilon} \),

\[
\int_{k \in K/M} \mathcal{R}_E(k^{-1}g) d\nu_j(k(0)) = (1 + O(\epsilon)) \cdot \omega_T(E_{\epsilon}^+) \]

where the implied constant depends only on \( E \).

Proof. Write \( k^{-1} = ma_t n_x n_z \) and \( g = m_1 a_t n_x n_z \in U_{\epsilon} \). By Lemma 5.5 we have \( k^{-1}g = m_0 a_t n_x n_z \), where \( t_0 = t + t_1 + 2 \log(|1 + e^{-t_1}x_1z|) \). Since \( \mathcal{R}_E(k^{-1}g) = e^{-\delta t_0} \chi_E(g^{-1}k(0)) \), we have

\[
\int_{k \in K/M} \mathcal{R}_E(k^{-1}g) d\nu_j(k(0)) \\
= \int_{k(0) \in g(E)} e^{-\delta t_0} d\nu_j(k(0)) \\
= \int_{k(0) \in g(E)} e^{-\delta t_0} e^{-\delta(t_1+2 \log(|1+e^{t_1}x_1z|))} d\nu_j(k(0)) \\
= (1 + O(\epsilon)) \int_{k(0) \in E_{\epsilon}^\pm} e^{-\delta t_0} e^{-\delta(t_1+2 \log(|1+e^{t_1}x_1z|))} d\nu_j(k(0)).
\]

Since \( t_1 = O(\epsilon), x_1 = O(\epsilon) \) and \( z = -k(0) \in -g(E) \subset -E_{\epsilon}^+, \)

\[
t_1 + 2 \log(|1 + e^{-t_1}x_1z|) = O(\epsilon)
\]

where the implied constant depends only on \( E \). Hence

\[
\int_{k \in K/M} \mathcal{R}_E(k^{-1}g) d\nu_j(k(0)) \\
= (1 + O(\epsilon)) \int_{k(0) \in E_{\epsilon}^\pm} e^{-\delta t_0} d\nu_j(k(0)) \\
= (1 + O(\epsilon)) \int_{k \in K} \mathcal{R}_E(k^{-1}g) d\nu_j(k(0)) \\
= (1 + O(\epsilon)) \cdot \omega_T(E_{\epsilon}^+).
\]

For \( \epsilon > 0 \), let \( \psi^\epsilon \) be a non-negative continuous function in \( C(G) \) with support in \( U_{\epsilon} \) with integral one and \( \Psi^\epsilon \in C_c(\Gamma \backslash G) \) be the \( \Gamma \)-average of \( \psi^\epsilon \):

\[
\Psi^\epsilon(\Gamma g) := \sum_{\gamma \in \Gamma} \psi^\epsilon(\gamma g).
\]

We define \( \Psi_E^\epsilon \in C_c(\Gamma \backslash G)^M \) by

\[
\Psi_E^\epsilon(g) := \int_{z \in E} \int_{m \in M} \Psi^\epsilon(gmn_z) dmdz.
\]

Lemma 5.9. For a bounded Borel subset \( E \subset \mathbb{C} \), there exists \( c = c(E) > 1 \) such that for all small \( \epsilon > 0 \),

\[
(1 - c \cdot \epsilon) \cdot \omega_T(E_{\epsilon}^-) \leq m^{BR}_T(\Psi_E^\epsilon) \leq (1 + c \cdot \epsilon) \cdot \omega_T(E_{\epsilon}^+).
\]
Proof. Note that $N^-$ is the expanding horospherical subgroup for the right action of $a_t$, i.e., $N^- = \{ g \in G : a_t g a_t^{-1} \to e \text{ as } t \to \infty \}$. We have for $\psi \in C_c(G)^M$,

$$\tilde{m}^{BR}_I(\psi) = \int_{KAN^-} \psi(ka_t n^-) e^{-\delta t} dt d\nu_j(k(0))$$

(cf. [19, 6.2]). We note that $d(a_t n^- x mn_z) = dt dx dm dz$ is the restriction of the Haar measure $dg$ to $AN^- N \subset G/M$.

We deduce

$$m^{BR}_I(\Psi^E) = \int_{z \in -E} \tilde{m}^{BR}(\psi_{n_z}) dz$$

$$= \int_{z \in -E} \int_{KAN^-} \int_{M} \psi^s(ka_t n^- mn_z) e^{-\delta t} dm dx dt d\nu_j(k(0)) dz$$

$$= \int_{k \in K} \int_{AN^- MN} \psi^s(k(a_t n^- mn_z)) \chi_{-E}(z) e^{-\delta t} dx dt dm dz d\nu_j(k(0))$$

$$= \int_{k \in K} \int_{g \in G} \psi^s(kg) R_E(g) d\nu_j(k(0))$$

$$= \int_{g \in U^I} \psi^s(g) \left( \int_{k \in K} R_E(k^{-1} g) d\nu_j(k(0)) \right) dg.$$  

Hence by Lemma 5.8 and the identity $\int_{U^I} \psi^s dg = 1$, we have

$$m^{BR}_I(\Psi^E) = (1 + O(\epsilon)) \omega_I(E^\pm_\epsilon).$$

**Corollary 5.10.** If $\omega_I(\partial(E)) = 0$, then

$$\omega_I(E) = \lim_{\epsilon \to 0} m^{BR}_I(\Psi^E).$$

**Proof.** For any $\eta > 0$, there exists $\epsilon = \epsilon(\eta)$ such that $\omega_I(E^+_\epsilon - E^-_\epsilon) < \eta$.

Together with Lemma 5.9 it implies that

$$m^{BR}_I(\Psi^E) = (1 + O(\epsilon))(1 + O(\eta)) \omega_I(E)$$

and hence the claim follows. $\square$

6. CONCLUSION: COUNTING CIRCLES

Let $\Gamma < G := \text{PSL}_2(\mathbb{C})$ be a non-elementary discrete group with $|m^{BMS}_I| < \infty$. Suppose that $P := \Gamma(C)$ is a locally finite circle packing.

Recall that

$$\text{sk}_I(P) = \text{sk}_I(C) := \int_{s \in \text{Stab}_I(C^\dagger)} e^{\delta_t \beta(x,s)} d\nu_I(x)(s^+)$$

where $C^\dagger$ is the set of unit normal vectors to $\hat{C}$. It follows from the conformal property of $\{\nu_I(x)\}$ that $\text{sk}_I(C)$ is independent of the choice of $C \in \Gamma(C)$, and hence is an invariant of the packing $\Gamma(C)$.

Theorem 1.6 is an immediate consequence of the following statement.
Theorem 6.1. Suppose that $\text{sk}_T(C) < \infty$. For any bounded Borel subset $E$ of $\mathbb{C}$ with $\omega_T(\partial(E)) = 0$, we have

$$\lim_{T \to \infty} \frac{N_T(\mathcal{P}, E)}{T^{\delta_T}} = \frac{\text{sk}_T(\mathcal{P})}{\delta_T \cdot |n^\mathrm{BMS}_T|} \cdot \omega_T(E).$$

Moreover $\text{sk}_T(C) > 0$ if $\mathcal{P}$ is infinite.

The second claim on the positivity of $\text{sk}_T(C)$ follows from the second claim of Theorem 2.5 and Lemma 3.1.

We will first prove Theorem 6.1 for the case when $C$ is the unit circle $C_0$ centered at the origin and deduce the general case from that.

The case of $C = C_0$. Fix $\eta > 0$. As $\omega_T(\partial(E)) = 0$, there exists $\epsilon = \epsilon(\eta) > 0$ such that

$$\omega_T(E^+_{4\epsilon} - E^-_{4\epsilon}) \leq \eta$$

where $E^\pm_{4\epsilon}$ is defined as in (5.7): $E^+_{4\epsilon} := U_{4\epsilon}(E)$ and $E^-_{4\epsilon} := \cap_{u \in U_{4\epsilon}} u(E)$.

We can find a $\mathcal{P}$-admissible Borel subset $\tilde{E}^+_\epsilon$ such that $E \subset \tilde{E}^+_\epsilon \subset E^+_\epsilon$ by adding all the open disks inside $E^+_\epsilon$ intersecting the boundary of $E$. Similarly we can find a $\mathcal{P}$-admissible Borel subset $\tilde{E}^-_\epsilon$ such that $E^-_\epsilon \subset \tilde{E}^-_\epsilon \subset E$ by adding all the open disks inside $E$ intersecting the boundary of $E^-_\epsilon$. By the local finiteness of $\mathcal{P}$, there are only finitely many circles intersecting $E$ (resp. $\tilde{E}^-_\epsilon$) which are not contained in $\tilde{E}^+_\epsilon$ (resp. $E^-_\epsilon$). Therefore there exists $q_\epsilon \geq 1$ (independent of $T$) such that

$$N_T(\mathcal{P}, E^-_\epsilon) - q_\epsilon \leq N_T(\mathcal{P}, E) \leq N_T(\mathcal{P}, E^+_\epsilon) + q_\epsilon.$$  

Recalling the set $B_T(\tilde{E}^+_\epsilon) = H \setminus H K A^+_{\log T} N^-_{\tilde{E}^+_\epsilon} \subset H \setminus G$, it follows from Proposition 3.7 and (6.4) that for all $T \gg 1$,

$$\#B_T(\tilde{E}^-_\epsilon) - m_0 \leq N_T(\Gamma(C_0), E) \leq \#B_T(\tilde{E}^+_\epsilon) + m_0$$

for some fixed $m_0 = m_0(\epsilon) > 1$.

Lemma 6.6. There exists $\ell > 0$ such that for all $T > 1$ and all small $\epsilon > 0$,

$$KA^+_{\log T} U_{\epsilon} \subset KA^+_{\log T+\epsilon} N_{\ell \epsilon}$$

where $N_{\ell \epsilon}$ is the $\ell \epsilon$-neighborhood of $\epsilon$ in $N$.

Proof. We may write $U_{\epsilon} = M_{\epsilon} N_{\epsilon}^- A_\epsilon N_{\epsilon} = K_{\epsilon} A_\epsilon N_{\epsilon}$ up to uniform Lipschitz constants. For $u = mn^{-a} \in M_{\epsilon} N_{\epsilon}^- A_\epsilon N_{\epsilon}$, $a_t u = m(a_t n^{-a_t} a_t u)$ since $a_t n^{-a_t} \in U_{\epsilon}$ for $t > 0$, we may write it as $k_1 a_1 n_1 \in K_{\epsilon} A_\epsilon N_{\epsilon}$. Hence for $0 < t < \log T$, we have $(a^{-1} a_{-t} n_1 a_1) \in N_{\epsilon}$ and

$$a_t u = (mk_1)(a_1 a_1)(a^{-1} a_{-t} n_1 a_1)n \in KA^+_{\log T+2\epsilon} N_{2\epsilon}.$$  

This proves the claim. \qed
Lemma 6.7 (Stability of KAN-decomposition). There exists $\ell_0 > 0$ (depending on $E$) such that for all $T > 1$ and for all small $\epsilon > 0$,

$$KA_{\log T}^+ N_{-E_\epsilon} U_{\ell_0 \epsilon} \subset KA_{\log T + \epsilon}^+ N_{-E_{2\epsilon}};$$
$$KA_{\log T - \epsilon}^+ N_{-E_{2\epsilon}} \subset KA_{\log T}^+ \cap U_{\ell_0 \epsilon}^+ N_{-E_{2\epsilon}}.$$

Proof. There exists $\ell_0 > 0$ depending on $E$ such that $N_{-E_{2\epsilon}} U_{\ell_0 \epsilon} \subset U_{\ell_0 \epsilon}^+ N_{-E_{2\epsilon}}$. Hence the first claim follows from Lemma 6.6. The second claim can be proved similarly. \hfill \Box

Let $\ell_0 > 0$ be as in Lemma 6.7. Without loss of generality, we may assume that $\ell_0 < \ell$ as in Lemma 5.8.

Lemma 6.8. For all $g \in U_{\ell_0 \epsilon}$ and $T \gg 1$,

$$\langle F_{T}^{\epsilon,+} (g) \rangle_{\Gamma \setminus G} - m_0 \leq N_T(\Gamma(C_0), E) \leq \langle F_{T}^{\epsilon,+} (g) \rangle_{\Gamma \setminus G} + m_0. \tag{6.9}$$

Proof. Note that, since $U_{\ell_0 \epsilon}$ is symmetric, for any $g \in U_{\ell_0 \epsilon}$,

$$\# [e] \Gamma \cap B_T(\bar{E}_\epsilon) \leq \# [e] \Gamma \cap B_T(\bar{E}_\epsilon) U_{\ell_0 \epsilon} g^{-1} \leq \# [e] \Gamma g \cap B_{e T}(N_{-E_{2\epsilon}}),$$

by Lemma 6.7, which proves the second inequality by (6.5). The other inequality can be proved similarly. \hfill \Box

For $\epsilon > 0$, let $\psi^\epsilon$ be a non-negative continuous function in $C(G)$ with support in $U_{\ell_0 \epsilon}$ with integral one and $\Psi^\epsilon \in C_c(\Gamma \setminus G)$ be the $\Gamma$-average of $\psi^\epsilon$:

$$\Psi^\epsilon(\Gamma g) := \sum_{\gamma \in \Gamma} \psi^\epsilon(\gamma g).$$

By integrating (6.9) against $\Psi^\epsilon$, we have

$$\langle F_{T}^{\epsilon,-} (\psi^\epsilon) \rangle_{\Gamma \setminus G} - m_0 \leq N_T(\Gamma(C_0), E) \leq \langle F_{T}^{\epsilon,+} (\psi^\epsilon) \rangle_{\Gamma \setminus G} + m_0.$$

Since

$$\langle F_{T}^{\epsilon,+} (\psi^\epsilon) \rangle = \int_{\Gamma \setminus G} \sum_{\gamma \in \Gamma \cap H \Gamma} \chi_{B_{e T}(N_{-E_{2\epsilon}})}([e] \gamma g) \Psi^\epsilon(g) \, dg$$

$$= \int_{[e] \Gamma \cap H \Gamma} \chi_{B_{e T}(N_{-E_{2\epsilon}})}([e] g) \Psi^\epsilon(g) \, dg$$

$$= \int_{[e] \Gamma \cap H \Gamma} \int_{h \in \Gamma \cap H \Gamma} \Psi^\epsilon(h g) \, dh \, d\lambda(g)$$

we deduce from Theorem 4.3 and Lemma 3.2 that

$$\langle F_{T}^{\epsilon,+} (\psi^\epsilon) \rangle \sim \frac{\kappa T}{\delta T \cdot m_{BMS}^H} \cdot \int_{n \in N_{-E_{2\epsilon}}} m_{T}^B(\Psi^\epsilon_n) \, dn \cdot T^\delta \cdot e^{\delta T} \cdot \gamma^{-1}.$$
where $\Psi'_\nu(g) = \int_{m \in M} \Psi'(gmn) dm$.

Therefore by applying Lemma 5.9 to (6.10) and using (6.3), we deduce

$$\limsup_T \frac{\langle F_{r,T}^+, \Psi \rangle}{T^{\delta_T}} \leq (1 + \epsilon) \frac{\text{skr}(C_0)}{\delta_T \cdot |m_T^{BMS}|} \cdot \int_{n \in N_{-2\epsilon}} m_T^{BR}(\Psi'_n) dm$$

$$\leq (1 + \epsilon)(1 + c\epsilon) \frac{\text{skr}(C_0)}{\delta_T \cdot |m_T^{BMS}|} \cdot \omega_T(E_{4\epsilon}^+)$$

$$\leq (1 + c_1 \eta)(1 + c_2 \epsilon) \frac{\text{skr}(\Gamma(C_0))}{\delta_T \cdot |m_T^{BMS}|} \cdot \omega_T(E)$$

where the constants $c, c_1, c_2$ depend only on $E$.

Similarly, we have

$$\liminf_T \frac{\langle F_{r,T}^+, \Psi \rangle}{T^{\delta_T}} \geq (1 - c_1 \eta)(1 - c_2 \epsilon) \frac{\text{skr}(C_0)}{\delta_T \cdot |m_T^{BMS}|} \cdot \omega_T(E).$$

As $\eta > 0$ is arbitrary and $\epsilon = \epsilon(\eta) \to 0$ as $\eta \to 0$, we have

$$\lim_{T \to \infty} \frac{N_T(\Gamma(C_0), E)}{T^{\delta_T}} = \frac{\text{skr}(C_0)}{\delta_T \cdot |m_T^{BMS}|} \cdot \omega_T(E).$$

This proves Theorem 6.1 for $C = C_0$.

The general case. Let $r > 0$ be the radius of $C$ and $p \in \mathbb{C}$ the center of $C$. Set

$$g_0 = n_p^\alpha \log r = \left( \begin{array}{cc} 1 & p \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \sqrt{r} & 0 \\ 0 & \sqrt{r}^{-1} \end{array} \right).$$

Then $g_0^{-1}(z) = r^{-1}(z - p)$ for $z \in \mathbb{C}$ and $g_0^{-1}(C) = C_0$.

Setting $\Gamma_0 = g_0^{-1} \Gamma g_0$, we have

$$N_T(\Gamma(C), E) = \# \{ C \in \Gamma(g_0(C_0)) : C^0 \cap E \neq \emptyset, \text{Curv}(C) < T \}$$

$$= \# \{ g_0^{-1}(C) \in \Gamma_0(C_0) : C^0 \cap E \neq \emptyset, \text{Curv}(C) < T \}$$

$$= \# \{ C_* \in \Gamma_0(C_0) : C_*^0 \cap g_0^{-1}(E) \neq \emptyset, \text{Curv}(C_*) < r^{-1}T \}$$

$$= N_{r^{-1}T}(\Gamma_0(C_0), g_0^{-1}(E)).$$

We claim that

$$(6.11) \quad \frac{1}{|m_{\Gamma_0}^{BMS}|} \cdot \text{skr}_{\Gamma_0}(\Gamma_0(C_0)) \cdot r^{-\delta_T} \cdot \omega_{\Gamma_0}(g_0^{-1}(E)) = \frac{1}{|m_{\Gamma}^{BMS}|} \cdot \text{skr}(\Gamma(C)) \cdot \omega_T(E).$$

Note that the each side of the above is independent of the choices of conformal densities of $\Gamma_0$ and $\Gamma$ respectively.

Fixing a $\Gamma$-invariant conformal density $\{ \nu_{\Gamma,x} \}$ of dimension $\delta_T$, set

$$\nu_{\Gamma,0,x} := g_0^* \nu_{\Gamma,g_0(x)}$$
Lemma 6.12. For a bounded Borel function $\psi(x) = \nu_{T_0}(x)(g_0(R))$. It is easy to check that $\nu_{T_0}$ is supported on $\Lambda(\Gamma_0) = g_0 \Lambda(\Gamma)$ and satisfies

$$\frac{d\nu_{T_0,x}}{d\nu_{T_0,y}}(z) = e^{-\delta_T \beta_+(x,y)}; \quad \gamma_*\nu_{T_0,x} = \nu_{T_0,\gamma(x)}$$

for all $x, y \in \mathbb{H}^3$, $\gamma \in \Gamma_0$ and $z \in \hat{\mathbb{C}}$.

Hence $\{\nu_{T_0,x} : x \in \mathbb{H}^3\}$ is a $\Gamma_0$-invariant conformal density of dimension $\delta_T = \delta_{T_0}$ and satisfies that for $f \in C_c(\mathbb{C})$

$$\int_{g_0(z) \in E} f(z)d\nu_{T_0,x}(z) = \int_{z \in E} f(g_0^{-1}(z))d
u_{T_0,0}(x)(z).$$

We consider the Bowen-Margulis-Sullivan measures $m_{BMS}^\Gamma$ and $m_{BMS}^\Gamma$ on $\Gamma \setminus \mathbb{T}^1(\mathbb{H}^3)$ and $\Gamma_0 \setminus \mathbb{T}^1(\mathbb{H}^3)$ associated to $\{\nu_{T,x}\}$ and $\{\nu_{T_0,x}\}$, respectively.

**Lemma 6.12.** For a bounded Borel function $\psi$ on $\Gamma \setminus \mathbb{T}^1(\mathbb{H}^3)$, consider a function $\psi_{g_0}$ on $\Gamma_0 \setminus \mathbb{T}^1(\mathbb{H}^3)$ given by $\psi_{g_0}(u) := \psi(g_0(u))$. Then

$$m_{BMS}^\Gamma(\psi_{g_0}) = m_{BMS}^\Gamma(\psi).$$

In particular, $|m_{BMS}^\Gamma(\psi_{g_0})| = |m_{BMS}^\Gamma(\psi)|$.

**Proof.** Note that if $v = g(u)$, then

$$\beta_u(x, \pi(u)) = \beta_v(g(x), \pi(v)).$$

Since $\nu_{T_0,x} = g_0^* \nu_{T_0}(x)$, we have

$$m_{BMS}^\Gamma(\psi_{g_0}) = \int_{u \in \Gamma_0 \setminus \mathbb{T}^1(\mathbb{H}^3)} \psi(g_0(u))e^{\delta_T \beta_u(x, \pi(u))} e^{\delta_T \beta_u(-(x, \pi(u))} d\nu_{T_0,x}(u^+)d\nu_{T_0,x}(u^-)dt$$

$$= \int_{\Gamma \setminus \mathbb{T}^1(\mathbb{H}^3)} \psi(v)e^{\delta_T \beta_u(g_0(x), \pi(v))} e^{\delta_T \beta_u(-g_0(x), \pi(v))} d\nu_{T_0,x}(v^+)d\nu_{T_0,x}(v^-)dt$$

$$= m_{BMS}^\Gamma(\psi).$$

\[\square\]

Similarly, we can verify:

**Lemma 6.13.** For any $x \in \mathbb{H}^3$,

$$\int_{s \in \text{Stab}_0(C^0)} e^{\delta_T \beta_+(x,s)} d\nu_{T_0,x}(s^+) = \int_{s \in \text{Stab}_0(C^0) \setminus C^0} e^{\delta_T \beta_+(g_0(x),s)} d\nu_{T_0,0}(g_0(x)^+);$$

that is, $\text{sk}_\Gamma(\Gamma(C)) = \text{sk}_{\Gamma_0}(\Gamma_0(C))$.

**Lemma 6.14.** For any bounded Borel subset $E \subset \mathbb{C}$,

$$\omega_{T_0}(g_0^{-1}(E)) = r^{\delta_T}\omega_{T}(E).$$
Proof. Since $g_0^{-1}(z) = r^{-1}(z - p)$, $r$ is the linear distortion of the map $g_0^{-1}$ in the Euclidean metric, that is, $r = \lim_{w \to w_0} \frac{|g_0^{-1}(w) - g_0^{-1}(w_0)|}{|w - w_0|}$ for any $w_0 \in \mathbb{C}$. Hence

\[ d\nu_{\Gamma, g_0}(j)(w) = r^\delta \nu_{\Gamma, j}(w). \]

Since $\nu_{\Gamma_0, x} = g_0^*\nu_{\Gamma, g_0(x)}$, we deduce

\[ \omega_{\Gamma_0}(g_0^{-1}(E)) = \int_{z \in g_0^{-1}(E)} (|z|^2 + 1)^{\delta r} d\nu_{\Gamma, j}(z) \]
\[ = \int_{u \in E} (|g_0^{-1}(u)|^2 + 1)^{\delta r} d\nu_{\Gamma_0, g_0}(j)(u) \]
\[ = r^\delta \int_{u \in E} (|u|^2 + 1)^{\delta r} d\nu_{\Gamma, j}(u) \]
\[ = r^\delta \omega_{\Gamma}(E). \]

This concludes a proof of (6.11). Therefore, since $\text{sk}_{\Gamma_0}(C_0) < \infty$ and $|m^{\text{BMS}}_{\Gamma_0}| < \infty$, the previous case of $C = C_0$ yields that

\[ \lim_{T \to \infty} \frac{1}{T^\delta} N_T(\Gamma(C), E) = \lim_{T \to \infty} \frac{1}{T^\delta} N_{r^{-1}T}(\Gamma_0(C_0), g_0^{-1}(E)) \]
\[ = \frac{1}{\delta \Gamma_0 \cdot |m^{\text{BMS}}_{\Gamma_0}|} \cdot \text{sk}_{\Gamma_0}(C_0) \cdot r^{-\delta} \cdot \omega_{\Gamma_0}(g_0^{-1}(E)) \]
\[ = \frac{1}{\delta \Gamma \cdot |m^{\text{BMS}}_{\Gamma}|} \cdot \text{sk}_{\Gamma}(C) \cdot \omega_{\Gamma}(E). \]

This completes the proof of Theorem 6.1. \qed

References

[1] Lars V. Ahlfors. Finitely generated Kleinian groups. Amer. J. Math., 86:413–429, 1964.
[2] Rufus Bowen. Periodic points and measures for Axiom A diffeomorphisms. Trans. Amer. Math. Soc., 154:377–397, 1971.
[3] David W. Boyd. The residual set dimension of the Apollonian packing. Mathematika, 20:170–174, 1973.
[4] Marc Burger. Horocycle flow on geometrically finite surfaces. Duke Math. J., 61(3):779–803, 1990.
[5] Schieffelin Claytor. Topological immersion of Peanian continua in a spherical surface. Ann. of Math. (2), 35(4):809–835, 1934.
[6] J. Elstrodt, F. Grunewald, and J. Mennicke. Groups acting on hyperbolic space. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. Harmonic analysis and number theory.
[7] L. Flaminio and R. J. Spatzier. Geometrically finite groups, Patterson-Sullivan measures and Ratner’s rigidity theorem. Invent. Math., 99(3):601–626, 1990.
[8] Alex Gorodnik, Nimish Shah, and Hee Oh. Strong wavefront lemma and counting lattice points in sectors. Israel J. Math, 176:419–444, 2010.
[9] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian circle packings: number theory. *J. Number Theory*, 100(1):1–45, 2003.

[10] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian circle packings: geometry and group theory. I. The Apollonian group. *Discrete Comput. Geom.*, 34(4):547–585, 2005.

[11] Sadayoshi Kojima. Polyhedral decomposition of hyperbolic 3-manifolds with totally geodesic boundary. In *Aspects of low-dimensional manifolds*, volume 20 of *Adv. Stud. Pure Math.*, pages 93–112. Kinokuniya, Tokyo, 1992.

[12] Alex Kontorovich and Hee Oh. Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds (with an appendix by Oh and Shah). *To appear in Journal of AMS*.

[13] A. Marden. *Outer circles*. Cambridge University Press, Cambridge, 2007. An introduction to hyperbolic 3-manifolds.

[14] Gregory Margulis. *On some aspects of the theory of Anosov systems*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004. With a survey by Richard Sharp: Periodic orbits of hyperbolic flows, Translated from the Russian by Valentina Vladimirovna Szulikowska.

[15] C. T. McMullen. Riemann surfaces, dynamics and geometry. *Course notes for Math 275: available at www.math.harvard.edu/~ctm*.

[16] C. T. McMullen. Hausdorff dimension and conformal dynamics. III. Computation of dimension. *Amer. J. Math.*, 120(4):691–721, 1998.

[17] David Mumford, Caroline Series, and David Wright. *Indra’s pearls*. Cambridge University Press, New York, 2002. The vision of Felix Klein.

[18] Hee Oh. Dynamics on Geometrically finite hyperbolic manifolds with applications to Apollonian circle packings and beyond. *Proc. of ICM (Hyderabad, 2010), Vol III, 1308–1331*.

[19] Hee Oh and Nimish Shah. Equidistribution and counting for orbits of geometrically finite hyperbolic groups. *Preprint, arXiv:1001.2096*.

[20] Hee Oh and Nimish Shah. Counting visible circles on the sphere and Kleinian groups *Preprint, arXiv:1004.2129*.

[21] Hee Oh and Nimish Shah. In preparation.

[22] John R. Parker. Kleinian circle packings. *Topology*, 34(3):489–496, 1995.

[23] S.J. Patterson. The limit set of a Fuchsian group. *Acta Mathematica*, 136:241–273, 1976.

[24] Marc Peigné. On the Patterson-Sullivan measure of some discrete group of isometries. *Israel J. Math.*, 133:77–88, 2003.

[25] M.S. Raghunathan. *Discrete subgroups of Lie groups*. Springer-Verlag, Berlin; New York, 1972.

[26] Thomas Roblin. Ergodicité et équidistribution en courbure négative. *Mém. Soc. Math. Fr. (N.S.)*, (95), 2003.

[27] Peter Sarnak. Integral Apollonian packings. *MAA Lecture, 2009, available at www.math.princeton.edu/~sarnak*.

[28] Peter Sarnak. Letter to J. Lagarias, 2007. available at www.math.princeton.edu/~sarnak.

[29] Henrik Schlichtkrull. *Hyperfunctions and harmonic analysis on symmetric spaces*, volume 49 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1984.

[30] Dennis Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.*, (50):171–202, 1979.

[31] Dennis Sullivan. Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. *Acta Math.*, 153(3-4):259–277, 1984.
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Mathematics department, Brown university, Providence, RI and Korea Institute for Advanced Study, Seoul, Korea
E-mail address: heeoh@math.brown.edu

Department of Mathematics, The Ohio State University, Columbus, OH
E-mail address: shah@math.ohio-state.edu