ON STRONGLY \(g(x)\)-CLEAN RINGS

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ABSTRACT. Let \(R\) be an associative ring with identity, \(C(R)\) denote the center of \(R\), and \(g(x)\) be a polynomial in the polynomial ring \(C(R)[x]\). \(R\) is called strongly \(g(x)\)-clean if every element \(r \in R\) can be written as \(r = s + u\) with \(g(s) = 0\), \(u\) a unit of \(R\), and \(su = us\). The relation between strongly \(g(x)\)-clean rings and strongly clean rings is determined, some general properties of strongly \(g(x)\)-clean rings are given, and strongly \(g(x)\)-clean rings generated by units are discussed.

Key Words: strongly \(g(x)\)-clean rings, strongly clean rings, rings generated by units
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1. INTRODUCTION

Let \(R\) be an associative ring with the group of units \(U(R)\). \(R\) is called clean if for every element \(r \in R\), \(r = e + u\) with \(e^2 = e \in R\) and \(u \in U(R)\) [10] and \(R\) is called strongly clean if in addition, \(eu = ue\) [11].

Let \(C(R)\) denote the center of a ring \(R\) and \(g(x)\) be a polynomial in \(C(R)[x]\). Camillo and Simón [2] say \(R\) is \(g(x)\)-clean if for every element \(r \in R\), \(r = s + u\) with \(g(s) = 0\) and \(u \in U(R)\). If \(V\) is a countable dimensional vector space over a division ring \(D\), Camillo and Simón proved that \(End(DV)\) is \(g(x)\)-clean if \(g(x)\) has two distinct roots in \(C(D)\) [2]. Nicholson and Zhou generalized Camillo and Simón’s result by proving that \(End(RM)\) is \(g(x)\)-clean if \(g(x)\) is a semisimple \(R\)-module and \(g(x) \in (x - a)(x - b)C(R)[x]\) where \(a, b \in C(R)\) and \(b, b - a \in U(R)\) [12]. [5] [14] completely determined the relation between clean rings and \(g(x)\)-clean rings independently. What is the relation between strongly clean rings and \(g(x)\)-clean rings?
In this paper, we continue this topic. In Section 2, we define strongly $g(x)$-clean rings and determine the relation between strongly $g(x)$-clean rings and strongly clean rings; in Section 3, some general properties of strongly $g(x)$-clean rings are given; and in Section 4, some classes of strongly $g(x)$-clean rings generated by units are discussed.

Throughout the paper, $T_n(R)$ denotes the upper triangular matrix ring of order $n$ over $R$, $\mathbb{N}$ denotes the set of all positive integers, and $\mathbb{Z}$ represents the ring of integers.

### 2. STRONGLY $g(x)$-CLEAN RINGS VS STRONGLY CLEAN RINGS

**Definition 2.1.** Let $g(x) \in C(R)[x]$ be a fixed polynomial. An element $r \in R$ is strongly $g(x)$-clean if $r = s + u$ with $g(s) = 0$, $u \in U(R)$, and $su = us$. $R$ is strongly $g(x)$-clean if every element of $R$ is strongly $g(x)$-clean.

Strongly clean rings are exactly strongly $(x^2 - x)$-clean rings. However, there are strongly $g(x)$-clean rings which are not strongly clean and vice versa:

- Let $\mathbb{Z}(p) = \{ \frac{m}{n} \in \mathbb{Q} : \gcd(p, n) = 1 \text{ and } p \text{ prime} \}$ be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$ and $C_3$ be the cyclic group of order 3.

**Example 2.2.** Let $R$ be a commutative local or commutative semiperfect ring with $2 \in U(R)$. By the proof of [14, Theorem 2.7], $RC_3$ is strongly $(x^6 - 1)$-clean. In particular, $\mathbb{Z}(7)C_3$ is a strongly $(x^6 - 1)$-clean ring. Furthermore, by [5, Example 2.2], $\mathbb{Z}(7)C_3$ is strongly $(x^4 - x)$-clean. However, $\mathbb{Z}(7)C_3$ is not strongly clean [7, Example 1].

**Example 2.3.** Let $R = \mathbb{Z}(p)$ and $g(x) = (x - a)(x^2 + 1) \in C(R)[x]$. Then $R$ is strongly clean but by a easy verification we know $R$ is not strongly $g(x)$-clean. Let $R$ be a boolean ring with more than two elements with $c \neq 0, 1$. Then $R$ is strongly clean but $R$ is not strongly $g(x) = (x + 1)(x + c)$-clean by [5, Example 2.3].

However, for some type of polynomials, strong cleanness and strong $g(x)$-cleanness are equivalent.

**Theorem 2.4.** Let $R$ be a ring and $g(x) \in (x - a)(x - b)C(R)[x]$ with $a, b \in C(R)$. Then the following hold:

1. $R$ is strongly $(x - a)(x - b)$-clean if and only if $R$ is strongly clean and $(b - a) \in U(R)$. 


(2) If $R$ is strongly clean and $(b-a) \in U(R)$, then $R$ is strongly $g(x)$-clean.

Proof. (1). “⇒”. Let $r \in R$. Since $R$ is strongly clean and $(b-a) \in U(R)$, let $e$ such that $e^2 = e \in R$, $u \in U(R)$, and $eu = ue$. Thus, $r = [e(b-a) + a] + u(b-a)$ where $u(b-a) \in U(R)$, $[e(b-a) + a - a][e(b-a) + a - b] = 0$, and $[e(b-a) + a]u(b-a) = u(b-a)[e(b-a) + a]$. Hence, $R$ is strongly $(x-a)(x-b)$-clean.

“⇐”. Since $a$ is strongly $(x-a)(x-b)$-clean, there exist $u \in U(R)$ and $s \in R$ such that $a = s + u$ with $(s-a)(s-b) = 0$ and $su = us$. Hence, $s = b$. So $(b-a) \in U(R)$. Let $r \in R$. Since $R$ is strongly $(x-a)(x-b)$-clean, $r(b-a) + a = s + u$ where $(s-a)(s-b) = 0$, $u \in U(R)$, and $su = us$. Thus, $r = \frac{s-a}{b-a} + \frac{s-a}{b-a}$ where $\frac{s-a}{b-a} \in U(R)$, $\frac{s-a}{b-a} = \frac{s-a}{b-a}$, and $\frac{s-a}{b-a} = \frac{s-a}{b-a}$ and $\frac{s-a}{b-a} = \frac{s-a}{b-a}$. So $R$ is strongly clean.

(2). By (1). □

Corollary 2.5. For a ring $R$, $R$ is strongly clean if and only if $R$ is strongly $(x^2 + x)$-clean.

Proof. It follows from Theorem 2.4 by letting $a = 0$ and $b = -1$. □

Remark 2.6. The equivalence of strong $(x^2 + x)$-cleanness and strong cleanness is a ring property since it holds for a ring $R$ but it may fail for a single element. For example, $1 + 1 = 2 \in \mathbb{Z}$ is strongly clean but 2 is not strongly $(x^2 + x)$-clean in $\mathbb{Z}$.

Example 2.7. Let $C(X)$ denote the ring of all real-valued continuous functions from a topological space $X$ to the real number field $\mathbb{R}$ and $C^*(X)$ denote the subring of $C(X)$ consisting of all bounded functions in $C(X)$. If $X$ is strongly zero-dimensional, then $C(X)$ and $C^*(X)$ are strongly $(x^2 - nx)$-clean rings for any $n \in \mathbb{N}$ since $C(X)$ and $C^*(X)$ are strongly clean [1] [9] and $n$ is invertible in $C(X)$ and $C^*(X)$. If $X$ is a $P$-space, then $\mathcal{M}_k(C(X))$ is strongly $(x^2 - nx)$-clean for any $n, k \in \mathbb{N}$ because $\mathcal{M}_k(C(X))$ is strongly clean by [6].

3. General properties of strongly $g(x)$-clean rings

Let $R$ and $S$ be rings and $\theta : C(R) \to C(S)$ be a ring homomorphism with $\theta(1) = 1$. For $g(x) = \Sigma a_i x^i \in C(R)[x]$, let $\theta'(g(x)) := \Sigma \theta(a_i) x^i \in C(S)[x]$. Then $\theta$ induces a
map $\theta'$ from $C(R)[x]$ to $C(S)[x]$. If $g(x)$ is a polynomial with coefficients in $\mathbb{Z}$, then $\theta'(g(x)) = g(x)$.

**Proposition 3.1.** Let $\theta : R \rightarrow S$ be a ring epimorphism. If $R$ is strongly $g(x)$-clean, then $S$ is strongly $\theta'(g(x))$-clean.

**Proof.** Let $g(x) = a_0 + a_1 x + \cdots + a_n x^n \in C(R)[x]$. Then $\theta'(g(x)) = \theta(a_0) + \theta(a_1)x + \cdots + \theta(a_n)x^n \in C(S)[x]$. For any $s \in S$, there exists $r \in R$ such that $\theta(r) = s$. Since $R$ is strongly $g(x)$-clean, there exist $t \in R$ and $u \in U(R)$ such that $r = t + u$ with $g(t) = 0$ and $tu = ut$. Then $s = \theta(r) = \theta(t) + \theta(u)$ with $\theta(u) \in U(S)$, $\theta'(g(x))|_{x=\theta(t)} = 0$, and $\theta(t)\theta(u) = \theta(u)\theta(t)$. So $S$ is strongly $\theta'(g(x))$-clean. \hfill $\Box$

**Corollary 3.2.** If $R$ is $g(x)$-clean, then for any ideal $I$ of $R$, $R/I$ is $\overline{g}(x)$-clean with $\overline{g}(x) \in C(R/I)[x]$.

**Corollary 3.3.** Let $R$ be a ring, $g(x) \in C(R)[x]$, and $1 < n \in \mathbb{N}$. If $\mathbb{T}_n(R)$ is strongly $g(x)$-clean, then $R$ is strongly $g(x)$-clean.

**Proof.** Let $A = (a_{ij}) \in \mathbb{T}_n(R)$ with $a_{ij} = 0$ and $1 \leq j < i \leq n$. Note that $\theta : \mathbb{T}_n(R) \rightarrow R$ with $\theta(A) = a_{11}$ is a ring epimorphism. \hfill $\Box$

**Corollary 3.4.** Let $R$ be a ring and $g(x) \in C(R)[x]$. If the formal power series ring $R[[t]]$ is strongly $g(x)$-clean, then $R$ is strongly $g(x)$-clean.

**Proof.** This is because $\theta : R[[t]] \rightarrow R$ with $\theta(f) = a_0$ is a ring epimorphism where $f = \sum_{i \geq 0} a_i t^i \in R[[t]]$. \hfill $\Box$

**Proposition 3.5.** Let $g(x) \in \mathbb{Z}[x]$ and $\{R_i\}_{i \in I}$ be a family of rings. Then $\prod_{i \in I} R_i$ is strongly $g(x)$-clean if and only if $R_i$ is strongly $g(x)$-clean for each $i \in I$.

**Proof.** It is clear by the definition and Proposition 3.1. \hfill $\Box$

For strongly clean rings, the author in [3,4,13] proved that if $R$ is a strongly clean ring and $e^2 = e \in R$, then the corner ring $eRe$ is strongly clean. For strongly $g(x)$-clean rings, we have the following result:

**Theorem 3.6.** Let $R$ be a strongly $(x - a)(x - b)$-clean ring with $a, b \in C(R)$. Then for any $e^2 = e \in R$, $eRe$ is strongly $(x - ea)(x - eb)$-clean. In particular, if $g(x) \in$
(x − ea)(x − eb)C(R)[x] and R is strongly (x − a)(x − b)-clean with a, b ∈ C(R), then eRe is strongly g(x)-clean.

Proof. By Theorem 2.4 R is strongly (x − a)(x − b)-clean if and only if R is strongly clean and b − a ∈ U(R). If R is strongly clean, then eRe is strongly clean by [13]. Again by Theorem 2.4 eRe is strongly (x − ea)(x − eb)-clean.

However, generally, strongly g(x)-clean property is not a Morita invariant: When

g(x) = (x − a)(x − b) where a, b ∈ C(R) with b − a ∈ U(R), the matrix ring over the local ring Z(p) is not strongly clean [3] (hence, not strongly g(x)-clean).

4. STRONGLY g(x)-CLEAN RINGS VS RINGS GENERATED BY UNITS AND ROOTS OF 1

For any n ∈ N, U_n(R) denotes the set of elements of R which can be written as a sum of no more than n units of R [8]. A ring R is called generated by its units if

\[ R = \bigcup_{n=1}^{\infty} U_n(R) \]

We use strong g(x)-cleanness to characterize some rings in which every element can be written as the sum of unit and a root of 1 which commute.

Theorem 4.1. Let R be a ring and n ∈ N. Then the following are equivalent:

1. R is strongly \((x^2 - 2^n x)\)-clean.
2. R is strongly \((x^2 - 1)\)-clean.
3. R is strongly clean and \(2 \in U(R)\).
4. \(R = U_2(R)\) and for any \(a \in R\), a can be expressed as \(a = u + v\) with some \(u, v \in U(R)\), \(uv = vu\), and \(v^2 = 1\).

Proof. (1) ⇒ (3). To prove \(2 \in U(R)\). Suppose \(2 \notin U(R)\), then \(\overline{R} = R/(2^nR) \neq 0\). Let \(2^n = s + u\) with \(s^2 - 2^n s = 0, u \in U(R)\), and \(su = us\). \(\overline{0} = \overline{2^n} = \overline{s} + \overline{u}\) implies that \(\overline{s} = -\overline{u} \in U(\overline{R})\). But \(\overline{s^2} = \overline{s^2} = \overline{2^n s} = \overline{0}\), a contradiction. So \(2 \in U(R)\). Let \(a = 0\) and \(b = 2^n\). Then by (1) of Theorem 2.4 R is strongly clean.

(3) ⇒ (1). By (1) of Theorem 2.4 R is strongly \((x^2 - 2^n x)\)-clean.

(3) ⇒ (4). Let \(a \in R\). By “(1) ⇔ (3)”, let \(n = 1\). Then \(1 - a = s + u\) where \(s^2 = 2s, u \in U(R)\), and \(su = us\). Then \(a = (−u) + (1 − s)\) with \(−u \in U(R)\), \((1 − s)^2 = 1\), and \((-u)(1 − s) = (1 − s)(−u)\).
(4) ⇒ (3). Let \( a \in R \). By (4), \( 1 - a = u + v \) where \( u \in U(R), v^2 = 1 \), and \( uv = vu \).

Thus, \( a = (-u) + (1 - v) \) with \(-u \in U(R), (1 - v)^2 = 2(1 - v), \) and \((-u)(1 - v) = (1 - v)(-u)\). By “(1) \( \iff \) (3)” and \( n = 1 \), we proved that (4) implies (3).

(2) ⇒ (4). If \( R \) is strongly \((x^2 - 1)\)-clean, then for any \( r \in R \), there exist \( v, u \in U(R) \) such that \( r = v + u \) with \( v^2 = 1 \) and \( uv = vu \).

(4) ⇒ (2). Let \( a \in R \). Then \( a \) can be expressed as \( a = u + v \) with \( u, v \in U(R), v^2 = 1 \), and \( uv = vu \). So \( v \) is the root of \( x^2 - 1 \). Hence, \( R \) is strongly \((x^2 - 1)\)-clean.

Example 4.2. Rings in Example 2.7 are strongly \((x^2 - nx)\)-clean. In particular, they are strongly \((x^2 - 2^n x)\)-clean rings in which every element can be written as the sum of a unit and a square root of 1 which commute.

Example 4.3. Let \( F \) be a field and \( V \) be a vector space over \( F \) of infinite dimension, and let \( R \) be the subring of \( E = \text{End}_F(V) \) generated by the identity and the finite rank transformations. Then \( R \) is strongly clean [13] Example 7. In fact, \( E \) is locally Artinian. So the matrix ring \( M_k(E) \) is strongly \((x^2 - nx)\)-clean with \( \text{char} F \mid n \). If \( \text{char} F \neq 2 \), then every element in the matrix ring can be written as the sum of a unit and a square root of 1 which commute.

Example 4.4. Let \( A = F[x_1, x_2, \ldots] \) be the polynomial ring in a countably infinite set of indeterminates \((x_1, x_2, \ldots)\) over a field \( F \), and let \( I = (x_1^{k_1}, x_2^{k_2}, x_3^{k_3}, \ldots) \) with \( k_i > 0 \). Then \( R = A/I \) is a local ring of dimension 0 which is not Noetherian. But \( R \) is locally Artinian. So the matrix ring \( M_k(R) \) is strongly \((x^2 - nx)\)-clean with \( \text{char} F \mid n \). If \( \text{char} F \neq 2 \), then every element in the matrix ring can be written as the sum of a unit and a square root of 1 which commute.

Proposition 4.5. Let \( R \) be a ring with \( c, d \in C(R) \) and \( d \in U(R) \). If \( R \) is strongly \((x^2 + cx + d)\)-clean, then \( R = U_2(R) \). In particular, if \( R \) is strongly \((x^2 + x + 1)\)-clean, then \( R = U_2(R) \) is strongly \((x^4 - x)\)-clean with every element is the sum of a unit and a cubic root of 1 which commute with each other.

Proof. The first statement is clear. Let \( r \in R \). Then \( r = s + u \) with \( u \in U(R), s^2 + s + 1 = 0 \), and \( su = us \). So \( s^4 - s = 0 \). Thus, \( R \) is strongly \((x^4 - x)\)-clean. Moreover,
ever element in strongly \((x^2 + x + 1)\)-clean ring \(R\) can be written as the sum of a unit and a cubic root of 1 which commute with each other.

\[ \]

**Lemma 4.6.** \(\square\) Let \(a \in R\). The following are equivalent for \(n \in \mathbb{N}\):

1. \(a = a(ua)^n\) for some \(u \in U(R)\).
2. \(a = ve\) for some \(e^{n+1} = e\) and some \(v \in U(R)\).
3. \(a = fw\) for some \(f^{n+1} = f\) and some \(w \in U(R)\).

**Proposition 4.7.** Let \(R\) be an strongly \((x^n - x)\)-clean ring where \(n \geq 2\) and \(a \in R\). Then either (i) \(a = u + v\) where \(u \in U(R)\), \(v^{n-1} = 1\), and \(uv = vu\) or (ii) both \(aR\) and \(Ra\) contain non-trivial idempotents.

**Proof.** Since \(R\) is strongly \((x^n - x)\)-clean, \(a = s + u\) with \(u \in U(R)\), \(s^n = s\), and \(su = us\). Then \(s^{n-1}a = s^n - u + s\). So \((1 - s^{n-1})a = (1 - s^n)u\). Since \(1 - s^{n-1}\) is an idempotent, by Lemma 4.6, \((1 - s^n)u = vq\) where \(v \in U(R)\) and \(g^2 = g \in R\). So \(g = v^{-1}(1 - s^n)u \in Ra\). Suppose (i) does not hold, then \(1 - s^{n-1} \neq 0\), this implies \(g \neq 0\). Thus, \(Ra\) contains a non-trivial idempotent. Similarly, \(aR\) contains a non-trivial idempotent. \(\square\)

Finally, we give a property which has nothing to do with rings generated by units but it relates to strongly \((x^n - x)\)-clean rings.

**Proposition 4.8.** Let \(R\) be a ring and \(n \in \mathbb{N}\). Then \(R\) is strongly \((ax^{2n} - bx)\)-clean if and only if \(R\) is strongly \((ax^{2n} + bx)\)-clean.

**Proof.** “\(\Rightarrow\)”. Suppose \(R\) is strongly \((ax^{2n} - bx)\)-clean. Then for any \(r \in R\), \(-r = s + u, as^{2n} - bs = 0, u \in U(R)\), and \(su = us\). So \(r = (-s) + (-u)\) where \((-u) \in U(R)\), \(a(-s)^{2n} + b(-s) = 0\), and \((-s)(-u) = (-u)(-s)\). Hence, \(r\) is strongly \((ax^{2n} + bx)\)-clean. Therefore, \(R\) is strongly \((ax^{2n} + bx)\)-clean.

“\(\Leftarrow\)”. Suppose \(R\) is strongly \((ax^{2n} + bx)\)-clean. Let \(r \in R\). Then there exist \(s\) and \(u\) such that \(-r = s + u, as^{2n} + bs = 0, u \in U(R)\), and \(su = us\). So \(r = (-s) + (-u)\) satisfies \(a(-s)^{2n} - b(-s) = 0\), \(-u \in U(R)\), and \((-s)(-u) = (-u)(-s)\). Hence, \(R\) is strongly \((ax^{2n} - bx)\)-clean. \(\square\)

For \(2n + 1 \in \mathbb{N}\), we do not know if the strong \((x^{2n+1} - x)\)-cleanness of \(R\) is equivalent to the strong \((x^{2n+1} + x)\)-cleanness of \(R\).
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REFERENCES

[1] F. Azarpanah, When is $C(X)$ a clean ring? *Acta Math. Hungar.*, 94 (1-2) (2002): 53-58.
[2] V. P. Camillo and J. J. Simón, The Nicholson-Varadarajan theorem on clean linear transformations, *Glasgow Math. J.*, 44 (2002): 365-369.
[3] J. Chen, X. Yang, and Y. Zhou, On strongly clean matrix and triangular matrix rings, *Comm. Algebra*, 34 (10) (2006): 3659-3674.
[4] W. Chen, A question on strongly clean rings, *Comm. Algebra*, 34 (7) (2006): 2374-2350.
[5] L. Fan and X. Yang, On rings whose elements are the sum of a unit and a root of a fixed polynomial, *Comm. Algebra*, 36 (1) 2008: 269-278.
[6] L. Fan and X. Yang, Strongly clean property and stable range one of some rings, preprint.
[7] J. Han and W. K. Nicholson, Extensions of clean rings, *Comm. Algebra*, 20 (2001): 2589-2596.
[8] M. Henriksen, Two classes of rings generated by their units, *J. Algebra*, 31 (1974): 182-193.
[9] W. W. McGovern, Clean semiprime $f$-rings with bounded inversion, *Comm. Algebra*, 31 (7) (2003): 3295-3304.
[10] W. K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.*, 229 (1977): 269-278.
[11] W. K. Nicholson, Strongly clean rings and Fitting’s lemma, *Comm. Algebra*, 27 (1999): 3583-3592.
[12] W. K. Nicholson and Y. Zhou, Endomorphisms that are the sum of a unit and a root of a fixed polynomial, *Canad. Math. Bull.*, 49 (2006): 265-269.
[13] Sánchez Campos, On strongly clean rings, 2002, unpublished.
[14] Z. Wang and J. Chen, A note on clean rings, *Algebra Colloquium*, 14 (3) (2007): 537-540.