Broken R-symmetry and defect conformal manifolds

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Abstract

Just as exactly marginal operators allow to deform a conformal field theory along the space of theories known as the conformal manifold, appropriate operators on conformal defects allow for deformations of the defects. A setup that guarantees exactly marginal defect operators in theories with extended supersymmetry are defects that break the R-symmetry group. In that case the conformal manifold is the symmetry breaking coset and its Zamolodchikov metric is expressed as the two point function of the exactly marginal operator. As the Riemann tensor on the conformal manifold can be expressed as an integrated correlator of the marginal operators, we find an exact relation to the curvature of the coset space. We examine in detail the case of the 1/2 BPS Maldacena-Wilson loop in $\mathcal{N} = 4$ SYM, which breaks $SO(6) \to SO(5)$ and the 1/2 BPS surface operator of the 6d $\mathcal{N} = (2, 0)$ theory with $SO(5) \to SO(4)$ breaking. We verify this identity against known 4-point functions, previously derived from AdS/CFT and the conformal bootstrap.
1 Introduction and summary

Amongst all operators of a conformal field theory, exactly marginal operators hold a special place, as they allow for continuous deformations of the theory, forming a space of CFTs known as the conformal manifold. In a $D$ dimensional CFT, marginal operators $O_i$ have scaling dimension $D$. If the CFT has an action, the deformations can be written as

$$S \rightarrow S + \int \lambda^i O_i d^D x,$$

where the parameters $\lambda^i$ are local coordinates on the conformal manifold. In the absence of the action, correlation functions of any operators $\phi_a$ in the deformed theory are then written simply with the extra insertion of the exponential of the integral in (1.1):

$$\langle \phi_{a_1} \cdots \phi_{a_n} \rangle_{\lambda^i} = \langle e^{-\int \lambda^i O_i d^D x} \phi_{a_1} \cdots \phi_{a_n} \rangle_0,$$

with subscript 0 indicates the undeformed theory.

Theories with boundaries or defects also have local operators restricted to those submanifolds and for a planar or spherical defect, one can define a defect CFT (dCFT) involving operators on and off the defect. A relatively unexplored topic (notable exceptions are [1–4]) are marginal deformations of dCFTs by defect operators. For a defect of dimension $d$, the marginal operators have to be of dimension $d$ and are integrated as above, only restricted to the defect.

Every defect has a distinguished operator known as the displacement operator, which captures the breaking of translation invariance by the defect. It can be seen in the divergence of the energy momentum tensor

$$\partial_\mu T^{\mu\nu}(x) = \mathcal{D}^n(x_\parallel) \delta^{D-d}(x_\perp),$$

where $x_\parallel$ represent the directions along the defect and $x_\perp$ the transverse ones. As $T^{\mu\nu}$ has dimension $D$, the displacement operator has dimension $d+1$. Another important consequence of this equation is that the normalisation of $\mathcal{D}$ is fixed by the normalisation of the $T^{\mu\nu}$ and therefore

$$\left\langle \mathcal{D}^n(x_\parallel) \mathcal{D}^m(0) \right\rangle = \frac{C_D \delta^{nm}}{x_\parallel^{2d+2}}.$$

The double brace notation represents the correlation function in the dCFT normalized by the VEV of the defect without insertions. The factor $C_D$ may of course depend on marginal couplings.

In theories with supersymmetry, some supercharges should be broken by the defect and with R-symmetry, also the R current giving two more operators

$$\partial_\mu J^\mu = Q_\delta \delta^{D-d}(x_\perp),$$
$$\partial_\mu J^{\mu 0} = \mathcal{O}_i \delta^{D-d}(x_\perp).$$

\footnote{For defect multiplets in $\mathcal{N} = 1$ theories, see [5,6].}
Of interest to us are $O_i$, as they are of protected dimension $d$ and are therefore marginal defect operators satisfying
\[ \langle O_i(x_\parallel)O_j(0) \rangle = \frac{C_O \delta_{ij}}{|x_\parallel|^{2d}}. \]  
(1.6)

This leads naturally to consider a defect conformal manifold, and with the usual rescaling of the operator at infinity, it has the Zamolodchikov metric \[^7\]
\[ g_{ij} = \langle O_i(\infty)O_j(0) \rangle = C_O \delta_{ij}, \quad O_i(\infty) \equiv \lim_{|x_\parallel| \to \infty} |x_\parallel|^{2d}O_i(x_\parallel). \]  
(1.7)

While the metric is locally flat, if the theory has R-symmetry group $G_R$, broken by the defect to $G'_R$, the full defect conformal manifold is $G_R/G'_R$. Furthermore, the size of this manifold is set by $C_O$.

It is no surprise that an object that breaks global symmetries transforms nontrivially under the broken symmetries and is parametrised by this coset. Still, this point of view allows us to find non-trivial identities on integrated correlators. The defect analog of (1.2) is
\[ \langle \phi_a \cdots \phi_a \rangle_{\lambda^i} = \langle e^{-\int \lambda^i O_i d^dx \phi_a \cdots \phi_a} \rangle_0. \]  
(1.8)

In particular for a pair of $\phi = O_i$ we have
\[ \langle O_iO_j \rangle_{\lambda^i} = \langle e^{-\int \lambda^i O_i d^dx O_iO_j} \rangle_0. \]  
(1.9)

This is the extension of the local Zamolodchikov metric (1.7) beyond the flat space approximation and the derivatives with respect to $\lambda^i$ give the Riemann tensor. Indeed, as in \[^8\], one finds
\[ R_{ijkl} = \int d^dx_1 d^dx_2 \left[ \langle O_j(x_1)O_k(x_2)O_i(0)O_l(\infty) \rangle_c - \langle O_j(0)O_k(x_2)O_i(x_1)O_l(\infty) \rangle_c \right], \]  
(1.10)

where $\langle \ldots \rangle_c$ implies the connected correlator, as stressed for example in \[^9\]. This integral is $2d$ dimensional, but it can be reduced to an integral over cross-ratios \[^9\]. See equations (2.20), (3.7) below.

Given that the manifold is $G_R/G'_R$, there is no mystery in the metric. Indeed if it is a maximally symmetric space, the Riemann tensor is determined by the Ricci scalar $R$ as
\[ R_{ijkl} = \frac{R}{p(p-1)} (g_{ik}g_{jl} - g_{il}g_{jk}). \]  
(1.11)

where $p$ is the dimension of the conformal manifold. If we know the exact value of $C_O$, then we know the exact form of the curvature and equating the last two equations gives a non-trivial relation on 4-point function, which is one of our main results.

In the remainder of this paper we apply this idea to two defects. In Section 2 we consider the 1d dCFT of 1/2 BPS Wilson loops in $\mathcal{N} = 4$ SYM. Then in Section 3 we look at the case of surface operators in the 6d $\mathcal{N} = (2,0)$ theory. Some details of the calculations are presented in appendices.
2 Maldacena-Wilson loops

We start by looking at the case of the 1/2 BPS Wilson loop in $\mathcal{N} = 4$ SYM in 4d along the Euclidean time direction

$$ W = \text{Tr} \mathcal{P} e^{\int (iA_0 + \Phi_6) dt}. \quad (2.1) $$

Another 1/2 BPS loop is the circle, and there are some subtle differences between the two [10,11], but of our purposes here the differences are immaterial and one finds the same results with either the circle or the line.

2.1 The Wilson loop dCFT

The defect CFT point of view on the Wilson loop was developed in [12–17]. Defect operators are any adjoint valued word inserted along the Wilson loop and their dimensions can be calculated using integrability [12,18–20].

The lowest dimension insertions are the six scalar fields $\Phi_I$. The one already in the Wilson loop, $\Phi_6$, has an anomalous dimension studied in [21,16,22,23]. In fact this scalar is the marginally irrelevant operator at the bottom of the renormalisation group flow from the non-BPS Wilson loop with no scalar coupling [24,25,11].

The remaining five scalars are marginal and in fact are $\mathcal{O}_i$, the superpartners of the displacement operator. Note that deforming the loop by a finite $\lambda_i \Phi_i$ in the exponent gives a non-BPS loop with a non-vanishing beta function. The exactly marginal deformation is a rotation

$$ \Phi_6 \to \Phi_6 \sqrt{1 - |\lambda|^2} + \lambda_i \Phi_i, \quad (2.2) $$

so for finite deformations, the operator $\Phi_i$ includes the appropriate subtraction of $\Phi_6$ to account for that.

The two point function of $\Phi_i$ is indeed as in (1.7) with $C_{\Phi}$ twice the bremsstrahlung function $B$ given in terms of the expectation value of the circular Wilson loop [26,13,27,20]

$$ C_{\Phi} = 2B = \frac{1}{\pi^2} \lambda \partial_\lambda \log \langle W_o \rangle, \quad W_o = \frac{1}{N} L_N^{1/2} \langle -\lambda/4N \rangle e^{\lambda^2/8N}, \quad (2.3) $$

where here $\lambda$ is the Yang-Mills coupling, so the bulk marginal coupling. Explicitly in the planar limit

$$ C_{\Phi} = \begin{cases} \frac{\lambda}{8\pi^2} - \frac{\lambda^2}{192\pi^2} + \frac{\lambda^3}{3072\pi^2} - \frac{\lambda^4}{46080\pi^2} + O(\lambda^5), & \lambda \ll 1, \\ \frac{3}{2\pi^2} - \frac{3}{4\pi^2} + \frac{3}{16\pi^2} + \frac{16\pi^2}{16\pi^2} + O(\lambda^{-3/2}), & \lambda \gg 1. \end{cases} \quad (2.4) $$

Let us present the general form of the four point functions of the scalars. We define

$$ \Phi(x,t) = t^i \Phi_i(x), \quad (2.5) $$
where \( t^i \) are auxiliary five vectors introduced to contract the R-symmetry indices. It is convenient to write the four point functions as

\[
\langle \Phi(x_1, t_1) \Phi(x_2, t_2) \Phi(x_3, t_3) \Phi(x_4, t_4) \rangle = \frac{t_{12} t_{34}}{x_{12} x_{34}} G(\chi; \zeta_1, \zeta_2),
\]

where \( t_{12} = t_1 \cdot t_2 \) and the cross-ratios \( \chi, \zeta_1, \zeta_2 \) are defined via

\[
\chi = \frac{x_{12} x_{34}}{x_{13} x_{24}}, \quad \zeta_1 \zeta_2 = \frac{t_{12} t_{34}}{t_{13} t_{24}}, \quad (1 - \zeta_1)(1 - \zeta_2) = \frac{t_{14} t_{23}}{t_{13} t_{24}}.
\]

The dynamical part of the correlator is encoded in \( G(\chi; \zeta_1, \zeta_2) \) where it is symmetric under \( \zeta_1 \leftrightarrow \zeta_2 \) and it satisfies the Ward identities \cite{17}

\[
\left( \frac{\partial G}{\partial \zeta_1} + \frac{1}{2} \frac{\partial G}{\partial \chi} \right) \bigg|_{\chi=\zeta_1} = \left( \frac{\partial G}{\partial \zeta_2} + \frac{1}{2} \frac{\partial G}{\partial \chi} \right) \bigg|_{\chi=\zeta_2} = 0.
\]

Moreover, the dependence of \( G(\chi; \zeta_1, \zeta_2) \) on \( \zeta_1, \zeta_2 \) is constrained by the fact that it has to be a polynomial of degree four in the \( t^i \). For a detailed discussion on this, see \cite{17}.

The superconformal Ward identities \eqref{2.8} were solved in \cite{17,28} in the elegant expression

\[
G(\chi; \zeta_1, \zeta_2) = C^2_2 \Phi(F X + D f(\chi)),
\]

where \( F \) does not depend on \( \chi, \zeta_1, \zeta_2 \) and can be determined from the topological sector of the correlators, which occurs for the choice \( \chi = \zeta_1 = \zeta_2 \) \cite{29,30,17}.

\( X \) in \eqref{2.9} is one of two superconformal cross-ratios defined in \cite{17} as follows

\[
\bar{X} = \chi, \quad \bar{\chi} = \frac{(1 - \chi)^2}{(1 - \zeta_1)(1 - \zeta_2)}.
\]

\( D \) is a differential operator given by

\[
D = (2 \chi^{-1} - \zeta_1^{-1} - \zeta_2^{-1}) - \chi^2 (\zeta_1^{-1} - \chi^{-1})(\zeta_2^{-1} - \chi^{-1}) \frac{\partial}{\partial \chi}.
\]

Crossing symmetry \( (x_1 \leftrightarrow x_3) \) is manifested on \( G(\chi; \zeta_1, \zeta_2) \) by

\[
\bar{X} G(\chi; \zeta_1, \zeta_2) = X G(1 - \chi; 1 - \zeta_1, 1 - \zeta_2),
\]

and on \( f(\chi) \) as

\[
(1 - \chi)^2 f(\chi) + \chi^2 f(1 - \chi) = 0.
\]

The \( \zeta_{1,2} \) dependence in \( G(\chi; \zeta_1, \zeta_2) \) comes purely from \( X \) \eqref{2.10} and \( D \) \eqref{2.11}. We can therefore also decompose it as

\[
G(\chi; \zeta_1, \zeta_2) = g_2(\chi)(\zeta_1 \zeta_2)^{-1} + g_1(\chi)(\zeta_1^{-1} + \zeta_2^{-1}) + g_0(\chi),
\]

\footnote{The factor of \( C^2_2 \) in \eqref{2.9} is not in \cite{17,28} because they use operators normalised to the identity.}
where
\[ g_2 = C_{\Phi}^2 \left( \chi^2 \delta f - \chi \frac{\partial f}{\partial \chi} \right), \quad g_1 = C_{\Phi}^2 \left( -f + \chi \frac{\partial f}{\partial \chi} \right), \quad g_0 = C_{\Phi}^2 \left( \frac{2f}{\chi} - \frac{\partial f}{\partial \chi} \right). \] (2.15)

From the crossing symmetry equation of \( G(\chi; \zeta_1, \zeta_2) \) (2.12), we can also find the properties for \( g_{0,1,2} \), which are

\[ \chi^2 g_2(1 - \chi) = (1 - \chi)^2 (g_2(\chi) + 2g_1(\chi) + g_0(\chi)), \]
\[ \chi^2 g_1(1 - \chi) = -(1 - \chi)^2 (g_1(\chi) + g_0(\chi)), \]
\[ \chi^2 g_0(1 - \chi) = (1 - \chi)^2 g_0(\chi). \] (2.16)

To get the four point function of scalar insertions, we need to differentiate with respect to the \( t \)'s, c.f. (2.6)

\[ \langle \Phi_i(x_1)\Phi_j(x_2)\Phi_k(x_3)\Phi_l(x_4) \rangle = \frac{\partial}{\partial t_1^i} \frac{\partial}{\partial t_2^j} \frac{\partial}{\partial t_3^k} \frac{\partial}{\partial t_4^l} \left( \frac{t_{12} t_{34}}{x_{12} x_{34}^2} G(\chi; \zeta_1, \zeta_2) \right) \]
\[ = \frac{1}{x_{12}^2 x_{34}^2} \left( g_2(\chi) \delta_{ik} \delta_{jl} + g_1(\chi) (\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk}) + g_0(\chi) \delta_{ij} \delta_{kl} \right). \] (2.17)

### 2.2 Sum rules for Wilson loop insertions

Clearly the dCFT of the 1/2 BPS Wilson loop has a defect conformal manifold with geometry \( S^5 \) of radius \( \sqrt{C_{\Phi}} \) (2.3). The curvature (1.11) is then

\[ R_{ijkl} = C_{\Phi}(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}), \quad R = \frac{20}{C_{\Phi}}. \] (2.18)

We would now like to identify this with the expression for the curvature (1.10). In the case of a 1d defect there is a subtlety, as the order of the insertions is meaningful. Taking the 4-point function \( \langle \Phi_i(1)\Phi_j(\eta)\Phi_k(\infty)\Phi_l(0) \rangle \) for example, it can be regarded as shorthand for

\[ \langle \Phi_i(1)\Phi_j(\eta)\Phi_k(\infty)\Phi_l(0) \rangle_c = \begin{cases} \langle \Phi_j(\eta)\Phi_i(0)\Phi_i(1)\Phi_k(\infty) \rangle, & \text{for } \eta < 0, \\ \langle \Phi_l(0)\Phi_j(\eta)\Phi_i(1)\Phi_k(\infty) \rangle, & \text{for } 0 < \eta < 1, \\ \langle \Phi_l(0)\Phi_i(1)\Phi_j(\eta)\Phi_k(\infty) \rangle, & \text{for } \eta > 1. \end{cases} \] (2.19)

The curvature tensor (1.10) consists of double integrals of the difference between two 4-point functions. However, noticing that the 4-point functions depend at non-coincident points only on the cross ratio \( \chi \), one can perform one of the two integrals explicitly and reduce the formula to just a single integral. This was done in [9] with careful treatment of the integration domain and possible singularities giving the expression

\[ R_{ijkl} = -\text{RV} \int_{-\infty}^{+\infty} d\eta \log |\eta| \left[ \langle \Phi_i(1)\Phi_j(\eta)\Phi_k(\infty)\Phi_l(0) \rangle_c \right. \\
+ \left. \langle \Phi_i(0)\Phi_j(1-\eta)\Phi_k(\infty)\Phi_l(1) \rangle_c \right]. \] (2.20)
Here the letters RV denote a particular prescription for regularizing and subtracting the divergences—a hard-sphere (point-splitting) cutoff followed by minimal subtraction of the divergences [9].

Taking the order of insertions into consideration as shown in (2.19), the curvature tensor (2.20) is actually the sum of six terms. In each term we make use of the expression (2.17) for the 4-point functions, and conformal symmetry such that the cross ration $\chi$ is in the domain $\chi \in (0, 1)$. Eventually we find

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \left( \int_0^1 \frac{d\chi}{\chi^2} \log \chi (g_2(\chi) - 2g_1(\chi) - 2g_0(\chi)) ight. $$

$$+ \left. \int_0^1 \frac{d\chi}{\chi^2} \log(1 - \chi) \left( g_2(\chi) + 4g_1(\chi) + g_0(\chi) \right) \right).$$

(2.21)

We now change the variable in the second integration from $\chi$ to $1 - \chi$ and then apply the crossing relations for $g_{0,1,2}$ (2.16), to find that the second integral is exactly equal to the first one. Therefore, the curvature tensor becomes

$$R_{ijkl} = 2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \int_0^1 \frac{d\chi}{\chi^2} \log \chi (g_2(\chi) - 2g_1(\chi) - 2g_0(\chi)).$$

(2.22)

Comparing with (2.14), the integrand can be written as

$$\log \frac{\chi}{\chi^2} G(\chi; \zeta_1^*, \zeta_2^*), \quad \zeta_{1,2} = -1 \pm \sqrt{3},$$

(2.23)

such that $\zeta_1^* + \zeta_2^* = \zeta_1^* \zeta_2^* = -2$.

To perform the integral, we plug in the expressions for $g_{0,1,2}$ in terms of $F$ and $f$ (2.15) to find

$$R_{ijkl} = 2C_\Phi^2 (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \int_0^1 d\chi \log \chi \left( F + 2 \left( \frac{1}{\chi^2} - \frac{2}{\chi^3} \right) f(\chi) - \left( 1 + \frac{2}{\chi} - \frac{2}{\chi^2} \right) \frac{\partial f(\chi)}{\partial \chi} \right).$$

(2.24)

The tensor structure is as expected for a maximally symmetric space, and after contracting the indices with the inverse of the metric (1.7), the Ricci scalar is

$$R = 40 \int_0^1 d\chi \log \chi \left( F + 2 \left( \frac{1}{\chi^2} - \frac{2}{\chi^3} \right) f(\chi) - \left( 1 + \frac{2}{\chi} - \frac{2}{\chi^2} \right) \frac{\partial f(\chi)}{\partial \chi} \right).$$

(2.25)

In Appendix A we further simplify this to

$$R = 40 \int_0^1 d\chi \left( - \left( 1 + \frac{1}{\chi} \right) F + \left( 1 - \frac{2}{\chi^3} \right) f(\chi) \right).$$

(2.26)

This integral with $R = 20/C_\Phi$ (2.18) can also be deduced from the integral identities in [31], as shown in Appendix B.
2.3 Comparison to explicit 4-point functions

The 4-point function of $\Phi_i$ insertions was calculated at strong coupling by explicit world-sheet Witten diagrams in [16] and extended up to three loop order in [28] based on the formalism in [14,17].

Representing the 4-point function as in (2.9), $F$ is given by the following series at strong coupling [28]

$$F = -\frac{3}{\sqrt{\lambda}} + \frac{45}{8}\frac{1}{\lambda^2} + \frac{45}{4}\frac{1}{\lambda^3} + O(\lambda^{-\frac{5}{2}})$$

Likewise $f(\chi)$ is expanded in a power series

$$f(\chi) = \sum_{n=1}^{\infty} \lambda^{-\frac{n}{2}} f^{(n)}(\chi),$$

where

$$f^{(1)}(\chi) = -(1 - \chi^2) \log(1 - \chi) + \frac{\chi^3(2 - \chi)}{(1 - \chi)^2} \log(\chi) - \frac{\chi(1 - 2\chi)}{1 - \chi}.$$  

$f^{(n)}$ for $n = 2, 3, 4$ are of transcendentality $\leq n$ and are successively more complicated. They can be found in [28] and are not repeated here.

At these orders, the integrand in (2.26) involves terms of the form

$$r(x) \log^a x \log^b (1 - x) \text{Li}_n(y(x)), \quad y(x) \in \left\{ x, 1 - x, \frac{x}{x - 1} \right\},$$

Where $r(x)$ is a rational function with poles at $x = 0$ and $x = 1$. They can all be evaluated recursively by integration by parts. Doing the integrals and combining back into a power series we find

$$2 \int_0^1 d\chi \left( -\left( 1 + \frac{1}{\chi} \right) F + \left( 1 - \frac{2}{\chi^3} \right) f(\chi) \right) = \frac{2\pi^2}{\sqrt{\lambda}} + \frac{3\pi^2}{\lambda} + \frac{15\pi^2}{4\lambda^2} + \frac{15\pi^2}{4\lambda^2} + O(\lambda^{-\frac{5}{2}}).$$

This exactly agrees with the large $\lambda$ expansion of $1/C_\Phi$, whose inverse is in (2.4).

Expressions for the 4-point function at weak coupling where given in [32,33,31]. As [31] checked their integral identities against those results, and (2.26) can be related to those (see Appendix [B]), it is clearly also satisfied.

3 Surface operators in 6d

The 6d $\mathcal{N} = (2,0)$ theory has 1/2 BPS surface operators [34] with the geometry of the plane or the sphere. In the absence of a Lagrangian description, we cannot write an expression like (2.1). Yet many properties of the surface operators are known: they carry a
representation of the $A_{N-1}$ algebra of the theory \cite{35,37} and we focus on the fundamental representation, described by an M2-brane in $AdS_7 \times S^4$ \cite{38}. Their symmetry algebra is $\mathfrak{osp}(4^*|2)^2$, the anomaly coefficients have been evaluated and properties of their defect CFT were also studied.

### 3.1 The surface dCFT

The defect CFT approach to surface operators was developed in \cite{39}. Again the displacement operator is in a multiplet, whose bosonic operators are the displacement itself and the scalar associated to breaking of $SO(5)$ R-symmetry

\[
\partial_\mu T^{\mu n}(x_\parallel, x_\perp) V = V[D^m(x_\parallel)]\delta^{(4)}(x_\perp), \\
\partial_\mu j^{n5}(x_\parallel, x_\perp) V = V[O^5(x_\parallel)]\delta^{(4)}(x_\perp).
\]  
(3.1)

We write here explicitly the surface operator $V$ that leads to the symmetry breaking and on the right hand side the operator with the appropriate insertion.

Their two point functions take the form

\[
\langle \langle D^m(x_\parallel)D^n(0) \rangle \rangle = \frac{C_D\delta^{mn}}{|x_\parallel|^6}, \quad \langle \langle O_i(x_\parallel)O_j(0) \rangle \rangle = \frac{C_\Omega\delta_{ij}}{|x_\parallel|^4}. 
\]  
(3.2)

As shown in \cite{39}, the normalisation constants $C_D$ and $C_\Omega$ are related to each other and to the anomaly coefficients $c$ and $a_2$ \cite{40,41} by

\[
C_\Omega = \frac{1}{16}C_D = \frac{c}{\pi^2} = -\frac{a_2}{\pi^2} = \frac{1}{\pi^2}\left(\frac{N - 1}{2} - \frac{1}{2N}\right). 
\]  
(3.3)

The value of the anomaly coefficients were fully determined in \cite{42,47}.

In order to write the 4-point function of $O_i$, we define as in the case of the Wilson loop dCFT \cite{25}, the operator $O(x; t) = t^iO_i(x)$ where $t^i$ are constant 4-vectors. We then give their 4-point function as

\[
\langle \langle O(x_1; t_1)O(x_2; t_2)O(x_3; t_3)O(x_4; t_4) \rangle \rangle = \frac{t_{12}t_{34}}{x_{12}^2x_{34}^2}\mathcal{G}(\chi, \bar{\chi}; \alpha, \bar{\alpha}). 
\]  
(3.4)

As in Section 2, $t_{ij} \equiv t_i \cdot t_j$ and now the cross-ratios are $\chi$, $\bar{\chi}$, $\alpha$ and $\bar{\alpha}$ are now (c.f. \cite{27})

\[
U = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2} = \chi \bar{\chi}, \quad V = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2} = (1 - \chi)(1 - \bar{\chi}), \\
\sigma = \frac{t_{13}t_{24}}{t_{12}t_{34}} = \alpha \bar{\alpha}, \quad \tau = \frac{t_{14}t_{23}}{t_{12}t_{34}} = (1 - \alpha)(1 - \bar{\alpha}). 
\]  
(3.5)

Since the correlator is not sensitive to the order of the four $O$'s, it should be invariant under the exchanges of any two $O$. The simple crossing equation arises for $\mathcal{G}$, arises from the $1 \leftrightarrow 3$ exchange

\[
\mathcal{G}\left(1 - \chi, 1 - \bar{\chi}; \frac{\alpha}{\alpha - 1}, \frac{\bar{\alpha}}{\bar{\alpha} - 1}\right) = \frac{|1 - \chi|^4}{|\chi|^4(1 - \alpha)(1 - \bar{\alpha})}\mathcal{G}(\chi, \bar{\chi}; \alpha, \bar{\alpha}). 
\]  
(3.6)

This is the analogue of \cite{2,12}.
### 3.2 Sum rules for surface insertions

The expression for the curvature tensor of the Zamolodchikov metric in terms of the 4-point function is as usual \((1.10)\). In this case that is a pair of two dimensional integrals. This expression was reduced in [9] to the expression with a single integral over the complex cross ratio

\[
R_{ijkl} = -2\pi \text{RV} \int d^2\eta \log |\eta| \left\langle \left\langle \mathcal{O}_i(1)\mathcal{O}_j(\eta)\mathcal{O}_k(\infty)\mathcal{O}_l(0) \right\rangle \right\rangle_c
\]

\[
= -2\pi \lim_{\epsilon \to 0} \left[ \int_{\epsilon < |\eta| < 1-\epsilon} d^2\eta \log |\eta| \left\langle \left\langle \mathcal{O}_i(1)\mathcal{O}_j(\eta)\mathcal{O}_k(\infty)\mathcal{O}_l(0) \right\rangle \right\rangle_c + \Delta R_{ijkl}(\epsilon) \right]. \tag{3.7}
\]

The symbol RV represents a particular regularisation prescription of the integral removing the region around \(\eta = 0\), \(\eta = 1\) and \(\eta = \infty\) as expressed in the second line, and \(\Delta R_{ijkl}(\epsilon)\) is a counterterm removing residual power-law divergences. See [9] for the full details.

We further simplify the integral by splitting the integration domain into three parts

\[
R_1 = \{ \eta \mid \epsilon < |\eta| < 1 - \epsilon \},
\]

\[
R_2 = \{ \eta \mid \epsilon < 1 - |\eta| \} \cap \{ \eta \mid 1 - \epsilon < |\eta| < 1 + \epsilon \}, \tag{3.8}
\]

\[
R_3 = \{ \eta \mid 1 + \epsilon < |\eta| < \epsilon^{-1} \}.
\]

In the region \(R_2\) we checked that the explicit expressions \((3.15), (3.16)\) below are maximised along \(|\eta - 1| = \epsilon\) and is bound by

\[
\sup_{\eta \in R_2} \left\langle \left\langle \mathcal{O}_l(0)\mathcal{O}_i(1)\mathcal{O}_j(\eta)\mathcal{O}_k(\infty) \right\rangle \right\rangle_c = O(\log \epsilon). \tag{3.9}
\]

The integral in \((3.7)\) over \(R_2\) then clearly vanishes without any counterterms.

For the region \(R_3\) we consider the conformal transformation

\[
0 \to 0, \quad \eta \to 1, \quad 1 \to \frac{1}{\eta}, \quad \infty \to \infty. \tag{3.10}
\]

Clearly for \(\eta \in R_3\) we have \(1/\eta \in R_1\), and that the conformal transformation of the 4-point function of marginal operators cancels the \(1/|\eta|^4\) Jacobian, thus the integral in \(R_3\) becomes

\[
-\int_{R_3} d^2\eta \log |\eta| \left\langle \left\langle \mathcal{O}_i(1)\mathcal{O}_j(\eta)\mathcal{O}_k(\infty)\mathcal{O}_l(0) \right\rangle \right\rangle_c = \int_{R_1} d^2\eta \log |\eta| \left\langle \left\langle \mathcal{O}_i(\eta)\mathcal{O}_j(1)\mathcal{O}_k(\infty)\mathcal{O}_l(0) \right\rangle \right\rangle_c. \tag{3.11}
\]

Finally we find an expression for the curvature tensor over \(R_1\) alone and \((3.7)\) becomes

\[
R_{ijkl} = -2\pi \lim_{\epsilon \to 0} \left[ \int_{\epsilon < |\eta| < 1-\epsilon} d^2\eta \log |\eta| \left\langle \left\langle \mathcal{O}_l(0)\mathcal{O}_i(1)\mathcal{O}_j(\eta)\mathcal{O}_k(\infty) \right\rangle \right\rangle_c \right.

\[
\left. - \left\langle \left\langle \mathcal{O}_l(0)\mathcal{O}_i(\eta)\mathcal{O}_j(1)\mathcal{O}_k(\infty) \right\rangle \right\rangle_c \right] + \Delta R_{ijkl}(\epsilon). \tag{3.12}
\]
In our case of marginal operators, it turns out that \( \Delta R_{ijkl}(\epsilon) \to 0 \), but one still has to be careful about the exact domain of integration. In this expression \( \eta \) is equal to the cross-ratio \( \chi \) as defined in (3.5).

### 3.3 Comparison to explicit 4-point functions

We now are ready to evaluate (3.12) for large \( N \) and match it to the curvature as deduced from the breaking of R-symmetry by the surface operator, where the analogue of (2.18) is

\[
R_{ijkl} = C_\infty (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}), \quad R = \frac{12}{C_\infty}. \tag{3.13}
\]

The 4-point function was calculated at leading order at large \( N \) from the M2-brane with the geometry of \( AdS_3 \) in \( AdS_7 \) [48]. The expression for the 4-point function (3.4) can be divided into two parts

\[
\mathcal{G}_{\text{tree}} = \mathcal{G}_1 + \mathcal{G}_2 \tag{3.14}
\]

where

\[
\mathcal{G}_1 = -\frac{6N}{\pi^2} U^2 \left( (U - 1 - V) \bar{D}_{3333} - U \bar{D}_{3322} + \bar{D}_{2222} \right)
+ \sigma \left[ (1 - U - V) \bar{D}_{3333} - \bar{D}_{3322} + \bar{D}_{2222} \right] + \tau \left[ (V - 1 - U) \bar{D}_{3333} - \bar{D}_{3223} + \bar{D}_{2222} \right]
\]

and

\[
\mathcal{G}_2 = -\frac{9N}{2\pi^4} U^2 (\chi - \bar{\chi}) (\alpha - \bar{\alpha}) \bar{D}_{3333}.
\tag{3.16}
\]

The definition of the \( \bar{D} \) functions is given in appendix C. Note that the expressions here are 16 times smaller than in [48] because of a factor of 2 difference in the normalisation of the scalar operators \( O_i \) compared to the \( S^4 \) coordinates \( y_i \) in [48].

The 4-point correlator is given by the \( t_i \) derivatives of (3.4)

\[
\left\langle \mathcal{O}_{i} (\vec{x}_1) \mathcal{O}_j (\vec{x}_2) \mathcal{O}_k (\vec{x}_3) \mathcal{O}_l (\vec{x}_4) \rightangle = \frac{\partial}{\partial t_1^i} \frac{\partial}{\partial t_2^j} \frac{\partial}{\partial t_3^k} \frac{\partial}{\partial t_4^l} \left\langle \mathcal{O} (\vec{x}_1; t_1) \mathcal{O} (\vec{x}_2; t_2) \mathcal{O} (\vec{x}_3; t_3) \mathcal{O} (\vec{x}_4; t_4) \rightangle
\tag{3.17}
\]

\( \mathcal{G}_2 \) is parity odd and does not contribute to the curvature tensor. This is easiest to see by changing the integration variables from \( \chi, \bar{\chi} \) to \( U, V \), which gives

\[
d^2 \chi = \frac{1}{|\chi - \bar{\chi}|} dU \, dV.
\tag{3.18}
\]

Identifying \( U = |\chi|^2 \) and combining with the measure, we have

\[
\frac{\log |\chi|}{U^2} \mathcal{G}_2 \, d^2 \chi = -\frac{9N}{2\pi^4} \log |\chi| \text{sign}(\text{Im} \, \chi) (\alpha - \bar{\alpha}) \bar{D}_{3333} dU \, dV.
\tag{3.19}
\]
Differentiation with respect to $t^i$, leaves this expression odd under $\chi \to \bar{\chi}$ and since the integration domain $R_1$ is symmetric, the integral vanishes.

Ignoring the contribution of $G_2$, we write the contribution of $G_1$ to the 4-point function as

$$\langle \langle O_l(0)O_i(\chi)O_j(1)O_k(\infty) \rangle \rangle_c, G_1 = -\frac{6N}{\pi^4} \left( \delta_{ik}\delta_{jl}[(U-1-V)\bar{D}_{3333} - U\bar{D}_{3322} + \bar{D}_{2222}] 
+ \delta_{il}\delta_{jk}[(1-U-V)D_{3333} - D_{3232} + D_{2222}] 
+ \delta_{ij}\delta_{kl}[(V-1-U)\bar{D}_{3333} - \bar{D}_{3232} + \bar{D}_{2222}] \right).$$  \hspace{1cm} (3.20)

The expression of $\langle \langle O_l(0)O_i(\chi)O_j(1)O_k(\infty) \rangle \rangle_c$ is similar just with $i, j$ exchanged. So the final expression of curvature tensor is

$$R_{ijkl} = \frac{12N}{\pi^3} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \int d^2\chi \log |\chi| \left[ (2|\chi|^2 - 2)\bar{D}_{3333} - |\chi|^2\bar{D}_{3322} + \bar{D}_{2222} \right].$$  \hspace{1cm} (3.21)

By numerical integration, we confirm that

$$R_{ijkl} = \frac{N}{\pi^2} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$  \hspace{1cm} (3.22)

in agreement with $(1.11)$ with Ricci scalar $R = 12/C_O$, and the large $N$ limit of $C_O \sim N/\pi^2$ $(3.3)$.

4 Discussion

In this paper we used the breaking of the global R-symmetry by defects to realise defect conformal manifolds. As the defect conformal manifold is the coset arising from symmetry breaking, its geometry is determined up to an overall scale. The marginal operators for these deformations are superpartners of the displacement operator and an appropriate integral of their 4-point function $(1.10)$ gives the Riemann curvature of the conformal manifold \cite{8}.

Fixing the scale requires knowledge of the normalisation of the marginal operator, which is a natural observable in the defect CFT. With that, one finds an exact identity for the integrated 4-point functions. We studied those in the case of the Maldacena-Wilson line dCFT and for surface operators in the 6d $(2,0)$ theory. The exact identities that we wrote were based on a simplification of the integral due to Friedan and Konenchny \cite{9}, and some further simplifications and are given in $(2.26)$ and $(3.21)$.

We verified those identities against the previously computed 4-point functions in \cite{16,17,48,28} and found perfect agreement.

Similar constraints can be found for higher point functions (see e.g. \cite{33}). The fully integrated correlators are again derivatives of the Zamolodchikov metric, and therefore fixed by the geometry of the manifold.
The identity we derived for the Maldacena-Wilson loop is related to the two identities noted in [31] where it was used as part of their numerical bootstrap studies and also compared to new analytic results. It would be interesting to see if these integral identities can be used in analytic bootstrap calculations to derive results at higher loops and in other systems. Further integral identities were identified in [49–51].

In the two cases that we studied, the defect conformal manifolds are symmetric spaces, $S^{5}$ and $S^{4}$, so have just this single scale, giving one integral identity. Defects that break the R-symmetry in more interesting ways will give more interesting metrics, have a variety of marginal operators with different 2-point functions and one could find integral constraints for different components of the Riemann tensor.

The same analysis can be applied to Wilson loops and surface operators in higher dimensional representations, where some results for the bremsstrahlung function and anomaly coefficient $c$ are known [52,42,47]. This can be compared to explicit holographic computations in terms of D3, D5 and M5-branes [53,58]. In the case of the Wilson loop the explicit calculation was carried out in [59], where the result was proportional to the same function of the cross-ratios (2.29) as in the case of the fundamental string.

A natural next avenue would be to examine line operators in 3d supersymmetric theories, which have a much richer spectrum (see [60,61] for an overview and recent results). Expressions for the bremsstrahlung function were found in [62–64], but their interpretation is not clear, as it is not positive definite. Beyond that, there are many other supersymmetric theories with defects for which these techniques can be applied.

It would be interesting to study defect conformal manifolds that do not arise from broken symmetries. Hopefully exactly marginal defect operators are not as rare or hard to find as bulk marginal operators. It is also natural to look at systems with both defect and bulk marginal operators to construct richer structures. Some work in that direction is in [4] and it would be interesting to see if simple theories like mixed dimensional QED [65] and generalisations thereof admit such deformations.

Acknowledgements

We are indebted to S. Giombi, N. Gromov, Z. Komargodski, C. Meneghelli, M. Probst, A. Stergiou, M. Trépanier and G. Watts for invaluable discussions. ND’s research is supported by the Science Technology & Facilities council under the grants ST/T000759/1 and ST/P000258/1. GS is funded by the Science Technology & Facilities council under the grant ST/W507556/1. The work of ZK was supported by China Scholarship Council under CSC No. 201906340174.
A Simplifying the integral of \(f(\chi)\)

To simplify the expression for the Ricci scalar (2.25)

\[
\frac{R}{40} = \int_0^1 d\chi \log \chi \left( F + 2 \left( \frac{1}{\chi^2} - \frac{2}{\chi^3} \right) f(\chi) - \left( 1 + \frac{2}{\chi} - \frac{1}{\chi^2} \right) \frac{\partial f(\chi)}{\partial \chi} \right),
\]

(A.1)

we integrate the first term, giving \(-F\) and integrate the last term by parts, to find

\[
\frac{R}{40} = -F - \left[ \left( 1 + \frac{2}{\chi} - \frac{2}{\chi^3} \right) f(\chi) \log \chi \right]_0^1 + \int_0^1 \frac{d\chi}{\chi} \left( 1 + \frac{2}{\chi} - \frac{1}{\chi^2} \right) f(\chi) .
\]

(A.2)

Noticing the boundary behaviour of \(f(\chi)\)

\[
\chi \to 0, \quad f(\chi) \sim -\frac{F}{2} \chi^2, \\
\chi \to 1, \quad f(\chi) \sim \frac{F}{2},
\]

(A.3)

we can now use (A.3) to evaluate the boundary term in (A.2)

\[
- \left( 1 + \frac{2}{\chi} - \frac{2}{\chi^3} \right) f(\chi) \log \chi \bigg|_{\chi=1-\epsilon}^{\chi=1} = F \log \epsilon.
\]

(A.4)

The crossing symmetry equation (2.13) for \(f(\chi)\) leads to the integral identities

\[
\int_\epsilon^{1-\epsilon} d\chi \frac{f(\chi)}{\chi^2} = 0, \quad \int_\epsilon^{1-\epsilon} d\chi \frac{f(\chi)}{\chi} = \int_\epsilon^{1-\epsilon} d\chi f(\chi).
\]

(A.5)

This allows us to further simplify (A.2) to

\[
R = 40 \lim_{\epsilon \to 0} \left( -F + F \log \epsilon + \int_\epsilon^{1-\epsilon} d\chi f(\chi) \left( 1 - \frac{2}{\chi^3} \right) \right) \\
= 40 \int_0^1 d\chi \left( - \left( 1 + \frac{1}{\chi} \right) F + \left( 1 - \frac{2}{\chi^3} \right) f(\chi) \right).
\]

(A.6)

B Relation to integral identities of [31]

In this appendix we show that the two integral identities in [31] encode the identity (2.25), and thus our results prove one of the identities stated there.

The 4-point function in [31] is expressed in terms of a function \(\delta G\), which in the notations of Section 2 is

\[
\delta G(\chi) = \chi^2 F - \left( 1 - \frac{2}{\chi} \right) f(\chi) - (\chi^2 - \chi + 1) \frac{\partial f(\chi)}{\partial \chi}.
\]

(B.1)
The two integral constraints noticed in [31] are
\[
\int_0^1 d\chi \frac{\delta G(\chi)}{\chi^2} (1 + \log \chi) = \frac{3C - B}{8B^2}, \quad \int_0^1 d\chi \frac{f(\chi)}{\chi} = \frac{C}{4B^2} + F.
\] (B.2)

\(B = C_\phi/2\) is the bremsstrahlung function and \(C\) is a function defined in [31] and since does not appear in this paper, we can take a linear combination to eliminate it
\[
\int_0^1 d\chi \left( 3\frac{f(\chi)}{\chi} - 2\frac{\delta G(\chi)}{\chi^2} (1 + \log \chi) \right) = \frac{1}{4B} + 3F.
\] (B.3)

Noting that \(\delta G(\chi)/\chi^2\) can be written as
\[
\frac{\delta G(\chi)}{\chi^2} = F - \partial_\chi \left( \left( 1 - \frac{1}{\chi} + \frac{1}{\chi^2} \right) f(\chi) \right).
\] (B.4)

Using (A.3), the left hand side of (B.3) is
\[
2F + F \log \epsilon + \int_0^1 d\chi \left( 3\frac{f(\chi)}{\chi} - 2F(1 + \log \chi) - 2 \left( \frac{1}{\chi} - \frac{1}{\chi^2} + \frac{1}{\chi^3} \right) f(\chi) \right).
\] (B.5)

Then using (A.5), this is
\[
\int_0^1 d\chi \left( \left( 2 - \frac{1}{\chi} \right) F + \left( 1 - \frac{2}{\chi^3} \right) f(\chi) \right),
\] (B.6)

and finally using our expression for the Ricci tensor (2.26), this is
\[
3F + \frac{R}{40} = 3F + \frac{1}{2C_\phi} = 3F + \frac{1}{4B},
\] (B.7)

proving (B.3).

C \(\bar{D}\)-functions

We collect here the recursive definition of \(\bar{D}\)-functions in (3.15), (3.16). These expressions can be used to obtain the explicit form of correlators as functions of the cross ratios \(U, V\) or \(\chi, \bar{\chi}\) in (3.5). See [66] for more details.

The simplest \(\bar{D}\)-function is \(\bar{D}_{1111}\), which is just the scalar one-loop box diagram in four dimensions
\[
\bar{D}_{1111} = \frac{1}{\chi - \bar{\chi}} \left[ \log(\chi\bar{\chi}) \log \left( \frac{1 - \chi}{1 - \bar{\chi}} \right) + 2\text{Li}_2(\chi) - 2\text{Li}_2(\bar{\chi}) \right]
\] (C.1)
To obtain $\bar{D}$-functions with higher weights, we can use the following differential operators

\[
\begin{align*}
\bar{D}_{\Delta_1+1,\Delta_2+1,\Delta_3,\Delta_4} &= -\partial_U \bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}, \\
\bar{D}_{\Delta_1,\Delta_2,\Delta_3+1,\Delta_4+1} &= (\Delta_3 + \Delta_4 - \Sigma - U \partial_U) \bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}, \\
\bar{D}_{\Delta_1,\Delta_2+1,\Delta_3+1,\Delta_4} &= -\partial_V \bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}, \\
\bar{D}_{\Delta_1+1,\Delta_2,\Delta_3+1,\Delta_4+1} &= (\Delta_1 + \Delta_4 - \Sigma - V \partial_V) \bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}, \\
\bar{D}_{\Delta_1,\Delta_2+1,\Delta_3+1,\Delta_4+1} &= (\Delta_2 + U \partial_U + V \partial_V) \bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}, \\
\bar{D}_{\Delta_1+1,\Delta_2+1,\Delta_3,\Delta_4+1} &= (\Sigma - \Delta_4 + U \partial_U + V \partial_V) \bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4},
\end{align*}
\]

where $\Sigma = (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)/2$. 

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