Generalized Riemann curvature corrections to type II supergravity

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Abstract

We observe that the replacement of the Riemann curvature with the generalized Riemann curvature into the corrections to the type II supergravity at order $\alpha'^3$ which are in terms of the contractions of four Riemann curvatures $R^4$, is not fully consistent with the S-matrix elements in the superstring theory. In particular, they produce non-zero S-matrix elements for odd number of B-field strengths which are not consistent with the string theory results. Using the consistency of the couplings with the linear T-duality as a guiding principle, we consider all T-duality invariant couplings and fix their coefficients by requiring them to be consistent with the S-matrix elements. The new Lagrangian density is then equivalent to the replacement of the generalized Riemann curvature into the expression $t_8 t_s R^4$.

Keywords: Effective action, Generalized Riemann curvature, T-duality
Many aspects of string theory can be captured at low-energy by the Wilsonian effective action for massless fields. The leading $\alpha'$-order terms of the effective action of type II superstring theory are given by the supergravity which contains the couplings

$$S \supset \frac{1}{2\kappa^2} \int d^{10}x e^{-2\Phi} \sqrt{-G} \left[ R + 4(\partial \Phi)^2 - \frac{1}{12} H^2 \right]$$

in the NS-NS sector. The next to the leading order couplings of the gravity are given by the curvature couplings at order $\alpha'{}^3$ which have been found by analyzing the sphere-level four-graviton scattering amplitude in the superstring theory \[1\]. The result in the eight-dimensional transverse space of the light-cone formalism, is a polynomial in the linearized Riemann curvature tensors

$$Y \sim t_{i_1 \cdots i_8} t_{j_1 \cdots j_8} R_{i_1 i_2 j_1 j_2} \cdots R_{i_8 j_8 j_1 j_2} + \cdots$$

where $t_8$ is a tensor in eight dimensions which includes the eight-dimensional Levi-Civita tensor $\epsilon_8$ and the tensor $t_8$ that was first introduced in \[2\]. The contraction of $t_8$ with four arbitrary antisymmetric matrices $M^1, \cdots M^4$ is defined as

$$t_{hknpqrs} M^1_{hk} M^2_{mn} M^3_{pq} M^4_{rs} = \frac{1}{\sqrt{6}} (\text{tr} M^1 M^2 M^3 M^4 + 4 \text{tr} M^1 M^2 M^4 + 4 \text{tr} M^1 M^3 M^4 + 4 \text{tr} M^2 M^3 M^4)$$

$$- \frac{1}{4\sqrt{6}} (\text{tr} M^1 M^2 \text{tr} M^3 M^4 + 4 \text{tr} M^1 M^3 \text{tr} M^2 M^4 + 4 \text{tr} M^1 M^4 \text{tr} M^2 M^3)$$

The dots in \(2\) represent terms containing the Ricci and scalar curvature tensors which can not be captured by the four-graviton scattering amplitude as they are zero on-shell. These terms can be absorbed into the supergravity \(1\) by appropriate field redefinition \[1\].

The $t_8 t_8 R^4$ part of the Lagrangian \(2\) has been also found in the covariant path integral formalism in \[3\] and in the pure spinor formalism in \[4\]. The Levi-Civita tensors in $t_8 t_8$ give rise to the covariant coupling $\epsilon_{10} \cdot \epsilon_{10} R^4$ \[5, 6\] which has its first non-zero contribution at five graviton level \[7\]. The presence of this term has been dictated by the sigma model beta function approach \[5, 6\]. Using the definitions of $t_8 t_8$ and $\epsilon_{10} \cdot \epsilon_{10}$, one finds \[5, 6, 10\]

$$Y \sim R_{hmnk} R_{pmon} R_{hrsp} R_{qrsk} + \frac{1}{2} R_{hkmn} R_{pqmn} R_{hrsp} R_{qrsk} + \cdots$$

where dots represent the specific form of the off-shell Ricci and scalar curvature couplings which reproduce the sigma model beta function \[5, 6\]. It has been shown in \[8\] that the above Lagrangian is not consistent with the standard form of the T-duality transformations. They should be invariant under a non-standard form of T-duality transformation which receives quantum corrections.

The $t_8 t_8 R^4$ part of the Lagrangian \(2\) is \[9, 10, 4, 11\]:

$$\mathcal{L}_1(R) = R_{hkmn} R_{knmp} R_{rsqm} R_{shpq} + \frac{1}{2} R_{hkmn} R_{knmp} R_{rsqp} R_{shqm}$$

1We use only subscripts indices and the repeated indices are contracted with the inverse of metric.
It has been shown in [10] that, up to field redefinition, the difference between the above two Lagrangians is the couplings $\epsilon_{10} \cdot \epsilon_{10} R^4$. The supersymmetric extension of the above Lagrangians has been studied in [12, 13, 14].

The B-field and dilaton couplings have been added to the Lagrangians (2) and (5) by extending the linearized Riemann curvature to the generalized Riemann curvature [9]

$$\bar{R}_{abcd} = R_{abcd} - \eta[a^{[c} \Phi,b^{d]}] + e^{-\Phi/2} H_{ab[c,d]}$$

where the bracket notation is $H_{ab[c,d]} = \frac{1}{2}(H_{ab}^{c,d} - H_{ab}^{d,c})$, and comma denotes the partial derivative. We will see that while the two Lagrangians (2) and (5) are identical for the linearized Riemann curvature, they are not identical for the generalized Riemann curvature. As a result, one of them should be consistent with S-matrix elements.

The action corresponding to the Lagrangian (5) in the Einstein frame is [9]

$$S \supset \frac{\gamma}{\kappa^2} \int d^{10}x \sqrt{-G} e^{-3\Phi/2} L_1(\bar{R})$$

where $\gamma = \frac{\alpha'}{2\kappa^2} \zeta(3)$. To study the T-duality of the above action, one should go to the string frame in which the linearized $\bar{R}_{abcd}$ becomes [15]

$$\bar{R}_{abcd} = e^{-\Phi/2} R_{abcd}$$

where $R_{abcd}$ is the following expression

$$R_{abcd} = R_{abcd} + H_{ab[c,d]}$$

It has the symmetries $R_{bacd} = -R_{abcd}$ and $R_{abdc} = -R_{abcd}$. The action (7) in the string frame then becomes

$$S \supset \frac{\gamma}{\kappa^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} L_1(\mathcal{R})$$

The dilaton appears only as the overall factor $e^{-2\Phi} \sqrt{-G}$ which is invariant under standard T-duality. It has been shown in [15] that the Lagrangian $L_1(\mathcal{R})$ is also invariant under the standard linear T-duality transformations.

The Lagrangian $L_1(\mathcal{R})$ contains the couplings $R^4$, $H^4$ and $H^2 R^2$ which are exactly reproduced by string theory S-matrix elements [9]. However, it contains also the couplings $R^3 H$ and $RH^3$ which are not reproduced in string theory. One can easily verify that the supergravity does not produce scattering amplitude of odd number of B-field strengths. Therefore, the string theory S-matrix element which reproduces the supergravity results at the leading order of $\alpha'$, is zero for odd number of $H$.  

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To check that the Lagrangian $L_1(R)$ produces the non-zero couplings for odd number of $H$, one should first replace the generalized Riemann curvature (9) in (10). It produces, for example, 24 couplings between three $H$ and one Riemann curvature, i.e.,

$$S_{HHH} = \frac{\gamma}{\kappa^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left[ -\frac{1}{4} H_{sh[p,q]} H_{kv[p,q]} H_{rs[m,n]} R_{hkmm} + \cdots \right]$$

where dots represent the other 23 terms. To verify that the above couplings do not simplify to zero, one may write the linearized Riemann curvature and the field strength $H$

$$R_{\mu\nu\alpha\beta} = \kappa (h_{\mu\beta,\nu\alpha} + h_{\nu\alpha,\mu\beta} - h_{\mu\alpha,\nu\beta} - h_{\nu\beta,\mu\alpha})$$

$$H_{\mu\nu[a,b]} = \kappa (b_{\mu\beta,\nu\alpha} + b_{\nu\alpha,\mu\beta} - b_{\mu\alpha,\nu\beta} - b_{\nu\beta,\mu\alpha})$$

where as usual the comma represents partial differentiation. The graviton $h_{\mu\nu}$ and the antisymmetric tensor $b_{\mu\nu}$ may be written as

$$h_{\mu\nu} = \frac{1}{2} (\psi_{\mu} \zeta_{\nu} + \psi_{\nu} \zeta_{\mu})$$

$$b_{\mu\nu} = \frac{1}{2} (\psi_{\mu} \zeta_{\nu} - \psi_{\nu} \zeta_{\mu})$$

where $\psi$ and $\zeta$ are two vector fields. Then one may transform the couplings to the momentum space. To this end, one should label the antisymmetric fields by 1,2,3 and the graviton by 4. Then one should add the 6 permutations of the antisymmetric fields. Performing all these steps, one finds that the result is not zero. Even if one uses the on-shell relations $k_i \cdot k_i = 0$ and $k_i \cdot \epsilon_i = 0$ where $\epsilon_i$ is the polarization of the $i$-th particle, the couplings still do not vanish. Doing the same steps for the $HR^3$ couplings, one again finds non-zero couplings. In fact the couplings of odd number of $H$ resulting from the terms in the second line of (5) remain non-zero at the linearized level.

Since the Lagrangian $L_1(R)$ is not fully consistent with the string theory S-matrix elements, one expects there must be another Lagrangian with the following properties:

1-It should produce no couplings $H^4$, $R^4$ or $R^2H^2$.

2-It should produce the couplings $R^3H$ and $RH^3$ which cancel the corresponding couplings in $L_1(R)$.

3-It should be consistent with the standard T-duality.

One may consider all possible contractions of four generalized Riemann curvatures, and may choose unknown coefficient for each of them. Then one may find the coefficients by forcing them to satisfy the above constraints.

To impose the T-duality constraint, we note that under linear T-duality the Riemann curvature with two Killing indices transforms as [8]

$$R_{\mu\nu yy} \rightarrow -R_{\mu\nu yy}$$

\[3\]
where $y$ is the killing index. So under the dimensional reduction on a circle, the couplings with structure $RR_{yy}R_{yy}R_{yy}$ where $R_{yy}$ is the Riemann curvature with two Killing indices, are not consistent with the linear T-duality. To avoid such couplings we consider the contractions of the generalized Riemann curvature in which the first two indices of the curvatures contract among themselves, and the second two indices contract among themselves, as the couplings in (5). Using the symmetries of the curvature $R_{abcd}$, one finds there are eight independent such couplings. Considering them with unknown coefficients, and constraining them to satisfy the conditions 1 and 2, one finds the following couplings:

$$L_2(R) = -\frac{1}{8} R_{hkmn} R_{kprs} R_{hqrs} R_{pqmn} + \frac{1}{8} R_{hkmn} R_{hkrs} R_{pqnr} R_{pqms}$$

$$-\frac{1}{4} R_{hkmn} R_{kpmn} R_{hqrs} R_{pqrs} - \frac{1}{4} R_{hkmn} R_{hknr} R_{pqmr} R_{pqns}$$

(15)

In fact $L_2(R) = 0$ for Riemann curvature, however, it is not an identity any more for the generalized Riemann curvature. The Lagrangian $L_1(R) + L_2(R)$ now has only couplings $H^4, R^4$ and $R^2 H^2$. It has been shown in [15] that such couplings at four-field level are consistent with the linear T-duality.

Since there are eight independent couplings in which the first two indices of the Riemann curvatures contract among themselves, two of the above couplings must have the same structure as the terms in (5). The first and the third terms in (15) have the same structure as the terms in the second line of (5) but their coefficients are different. Adding the Lagrangian $L_1$ to $L_2$, one finds that all independent contraction of the four generalized Riemann curvatures which are consistent with the linear T-duality have non-zero coefficients. Therefore, the action in the string frame which is consistent with the linear T-duality and is fully consistent with the four-point functions of string theory, is

$$S \supset \frac{\gamma}{\kappa^2} \int d^{10}x \sqrt{-G} e^{-2\phi} L(R)$$

(16)

where the Lagrangian density has the following eight independent terms:

$$L(R) = R_{hkmn} R_{krnp} R_{rsmn} R_{hspq} + \frac{1}{2} R_{hkmn} R_{krnp} R_{rspq} R_{hsmp}$$

$$-\frac{1}{4} R_{hkmn} R_{hkns} R_{pqmr} R_{pqmr} + \frac{1}{8} R_{hkmn} R_{hkrs} R_{pqnr} R_{pqms}$$

$$+ \frac{1}{4} R_{hkmn} R_{kpmn} R_{rsmn} R_{hspq} + \frac{1}{8} R_{hkmn} R_{rnsp} R_{rsmn} R_{hspq}$$

$$+ \frac{1}{16} R_{hkmn} R_{hkpq} R_{rsmn} R_{rspq} + \frac{1}{32} R_{hkmn} R_{hkmn} R_{rspq} R_{rspq}$$

(17)

Note that the couplings in the first and the last lines above are the same couplings in (5).

Now let us compare the above Lagrangian with the replacement of the generalized Riemann curvature (9) into $t_8 R^4$. Using the definition of the tensor $t_8$ in (3), and using
the fact that the generalized Riemann curvature $\mathcal{R}_{abcd}$ has the same symmetries of the Riemann curvature except the symmetry under $(ab) \leftrightarrow (cd)$, one finds after some algebra

$$t_8 t_8 R^4 = \mathcal{L}(\mathcal{R}) \quad (18)$$

Therefore, the replacement (9) in the Lagrangian $t_8 t_8 R^4$ is consistent with the S-matrix elements of four NS-NS vertex operators and with the linear T-duality.

The Lagrangian (17) may be extended to nonlinear order by replacing the linearized $\mathcal{R}$ with the nonlinear generalized Riemann curvature

$$\mathcal{R}_{abcd} \rightarrow R_{abcd} + H_{ab[c} + \frac{1}{2} H_{ae[c} H_{be]d]} \quad (19)$$

One can easily verify that the last term has the symmetries of the Riemann curvature, so the above replacement in (17) does not produce odd number of $H$, as expected. It would be interesting to compare the couplings $H^2 R^3$ and $H^4 R$ resulting from the above replacement, with the contact terms of the corresponding sphere-level S-matrix element in string theory. At the one-loop level of type IIA theory, it has been shown in [16] that the above replacement in $t_8 t_8 R^4$ and $B_2 \wedge X_8$ are consistent with S-matrix calculation and with the T-duality, however, this replacement in $\epsilon_{10} \cdot \epsilon_{10} R^4$ is not consistent with the S-matrix calculation.

Acknowledgments: This work is supported by Ferdowsi University of Mashhad under grant 2/20625.

References

[1] D. J. Gross and E. Witten, Nucl. Phys. B 277, 1 (1986).
[2] J. H. Schwarz, Phys. Rept. 89, 223 (1982).
[3] Y. Cai and C. A. Nunez, Nucl. Phys. B 287, 279 (1987).
[4] G. Policastro and D. Tsimpis, Class. Quant. Grav. 23, 4753 (2006) [hep-th/0603165].
[5] M. T. Grisaru and D. Zanon, Phys. Lett. B 177, 347 (1986).
[6] M. D. Freeman, C. N. Pope, M. F. Sohnius and K. S. Stelle, Phys. Lett. B 178, 199 (1986).
[7] B. Zumino, Phys. Rept. 137, 109 (1986).
[8] M. R. Garousi, Phys. Lett. B 718, 1481 (2013) [arXiv:1208.4459 [hep-th]].
[9] D. J. Gross and J. H. Sloan, Nucl. Phys. B 291, 41 (1987).
[10] R. C. Myers, Nucl. Phys. B 289, 701 (1987).

[11] G. Policastro and D. Tsimpis, Class. Quant. Grav. 26, 125001 (2009) [arXiv:0812.3138 [hep-th]].

[12] S. Paban, S. Sethi and M. Stern, Nucl. Phys. B 534, 137 (1998) [hep-th/9805018].

[13] M. B. Green and S. Sethi, Phys. Rev. D 59, 046006 (1999) [hep-th/9808061].

[14] K. Peeters, P. Vanhove and A. Westerberg, Class. Quant. Grav. 18, 843 (2001) [hep-th/0010167].

[15] M. R. Garousi, Phys. Rev. D 87, 025006 (2013) [arXiv:1210.4379 [hep-th]].

[16] J. T. Liu and R. Minasian, arXiv:1304.3137 [hep-th].