Five Essays on the Geometry of László Fejes Tóth

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Abstract

In this paper we consider the following topics related to results of László Fejes Tóth: (1) The Tammes problem and Fejes Tóth’s bound on circle packings; (2) Fejes Tóth’s problem on maximizing the minimum distance between antipodal pairs of points on the sphere; (3) Fejes Tóth’s problem on the maximum kissing number of packings on the sphere; (4) The Fejes Tóth – Sachs problem on the one–sided kissing numbers; (5) Fejes Tóth’s papers on the isoperimetric problem for polyhedra.

1 Tammes’ problem and Fejes Tóth’s bound on circle packings

1.1 Tammes’ problem

We start with the following classical problem: How should $N$ points be distributed on a unit sphere so that the minimum distance between two points of the set attains its maximum value $d_N$? This problem was first asked by the Dutch botanist Tammes [58] while examining the distribution of openings on the pollen grains of different flowers. This question is also known as the problem of the “inimical dictators” [40], namely “where should $N$ dictators build their palaces on a planet so as to be as far away from each other as possible?” The problem is equivalent with the problem of densest packing of congruent circles on the sphere (see e.g. [11] Section 1.6: Problem 6): How are $N$ congruent, non-overlapping circles distributed on a sphere when the common radius of the circles has to be as large as possible? The higher dimensional analogue of the problem has applications in information theory [60]. This justifies the terminology that a finite subset $X$ of $\mathbb{S}^n$ with

$$\psi(X) := \min_{x, y \in X, x \neq y} \text{dist}(x, y)$$

is called a spherical $\psi(X)$-code.

Tammes’ problem is presently solved only for $N \leq 14$ and $N = 24$. L. Fejes Tóth [18] solved the problem for $N = 3, 4, 6, 12$. Schütte and van der Waerden

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settled the cases \( N = 5, 7, 8, 9 \). The cases \( N = 10 \) and 11 were solved by Danzer [15] (see also the papers by Böröczky [7] for \( N = 11 \) and Hárs [36] for \( N = 10 \)). Robinson [51] solved the problem for \( N = 24 \). In my recent papers with Tarasov [46, 49] we gave a computer–assisted solution for \( N = 13 \) and \( N = 14 \).

Robinson extended Fejes Tóth’s method and and gave a bound valid for all \( N \) that is sharp besides the cases \( N = 3, 4, 6, 12 \) also for \( N = 24 \). The solution of all other cases is based on the investigation of the so called contact graphs associated with a finite set of points. For a finite set \( X \) in \( \mathbb{S}^2 \) the contact graph \( CG(X) \) is the graph with vertices in \( X \) and edges \( (x, y), x, y \in X \), such that \( \text{dist}(x, y) = \psi(X) \). The concept of contact graphs was first used by Schütte and van der Waerden [55]. They used the method also for the solution the thirteen spheres (Newton–Gregory) problem [56].

In Chapter VI of the book [21] the concept of irreducible contact graphs is considered in details. The method of irreducible spherical contact graphs was used also [31, 59, 8, 9, 10] for obtaining bounds for the kissing number and Tammes problem.

1.2 The Fejes Tóth bound

Now we consider a theorem on bounds of equal–circle packing and covering of a sphere proved by László Fejes Tóth in 1943 [18, 21, 25].

**Theorem 1.1 (L. Fejes Tóth [18]).** If \( \Theta \) is the density of a packing of the unit sphere \( \mathbb{S}^2 \) in \( \mathbb{R}^3 \) by \( N \) congruent spherical caps then

\[
\Theta \leq \frac{N}{4}(2 - \csc \omega_N), \text{ where } \omega_N := \frac{N\pi}{6N - 12}
\]

If \( \Omega \) is the density of a covering of \( \mathbb{S}^2 \) by \( N \) congruent spherical caps then

\[
\Omega \geq \frac{N}{2} \left(1 - \frac{1}{\sqrt{3}} \cot \omega_N \right).
\]

Denote by \( A(n, \varphi) \) the maximum cardinality of a \( \varphi \)-code in \( \mathbb{S}^{n-1} \). In other words, \( A(n, \varphi) \) is the maximum cardinality of a packing in \( \mathbb{S}^{n-1} \) by spherical caps of radius \( \varphi/2 \).

The bound in Theorem 1.1 yields

\[
A(3, \varphi) \leq \frac{2\pi}{\Delta(\varphi)} + 2,
\]

where

\[
\Delta(\varphi) = 3 \arccos \left( \frac{\cos \varphi}{1 + \cos \varphi} \right) - \pi
\]

Actually, Danzer’s paper [15] is the English translation of his Habilitationsschrift “Endliche Punktmengen auf der 2-sphäre mit möglichst großem Minimalabstand”, Universität Göttingen, 1963.
is the area of a spherical regular triangle with side length $\varphi$.

The bound is tight for $N = 3, 4, 6$ and 12. So for these $N$ it gives a solution of the Tamme problem. It is also tight asymptotically. However, is not tight for any other cases.

1.3 Coxeter's bound

In 1963 Coxeter [14] proposed an extension of Fejes Tóth’s bound for all dimensions. His bound was based on the conjecture that in $n$-dimensional spherical space equal size balls cannot be packed denser than the density of $n + 1$ mutually touching balls of the same size with respect to the simplex spanned by the centers of the balls. This conjecture has been stated by Fejes Tóth for the 3-dimensional case in [22] and for all dimensions in [23, 24]. Assuming the correctness of the conjecture Coxeter calculated the upper bounds 26, 48, 85, 146, and 244 for the kissing numbers $k(n) = A(n, \pi/3)$ for $n = 4, 5, 6, 7$, and 8, respectively. The conjecture, which was finally confirmed by Böröczky [6] in 1978 also yields that

$$A(4, \pi/5) = 120.$$ 

**Theorem 1.2** (Böröczky [6] and Coxeter [14]).

$$A(n, \varphi) \leq 2F_{n-1}(\alpha)/F_n(\alpha),$$

where

$$\sec 2\alpha = \sec \varphi + n - 2,$$

and the function $F$ is defined recursively by

$$F_{n+1}(\alpha) = \frac{2}{\pi} \int_{\arccos(n)/2}^{\alpha} F_{n-1}(\beta) \, d\beta, \quad \sec 2\beta = \sec 2\theta - 2,$$

with the initial conditions $F_0(\alpha) = F_1(\alpha) = 1$.

2 The problem on maximizing the minimum distance between antipodal pairs of points on the sphere

L. Fejes Tóth [26] considered Tamme’s problem for antipodal sets on $S^2$ i.e. for sets $X$ that are invariant under the antipodal mapping $A : S^d \to S^d$, where $A(x) = -x$. Let

$$a_M := \max_{X = -X \subseteq S^2} \{\psi(X)\}, \quad \text{for } |X| = 2M.$$

For a given $M$, Fejes Tóth’s problem for antipodal sets is to find all configurations

$$X = \{x_1, -x_1, \ldots, x_M, -x_M\}.$$
on $S^2$ such that $\psi(X) = a_M$.

This problem is presently solved only for $M \leq 7$. It is clear that $a_M \leq d_{2M}$. Therefore, if $\psi(X) = d_{2M}$, $|X| = 2M$, and $X$ is antipodal then $a_M = d_{2M}$. Thus, for $M = 3$ and $M = 6$ we have this equality. The following theorem is the main result of [26].

**Theorem 2.1** (L. Fejes Tóth, [26]). Let $P_M \subset S^2$ be a maximal set for the Fejes Tóth problem for antipodal configurations, i.e. $\psi(P_M) = a_M$. Then

1. $P_2$ is the set of vertices of a square on the equator, $a_2 = 90^\circ$;
2. $P_3$ is the set of vertices of a regular octahedron, $a_3 = 90^\circ$;
3. $P_4$ is the set of vertices of a cube, $a_4 = \arccos(1/3)$;
4. $P_5$ consists of five pairs of antipodal vertices of a regular icosahedron, $a_5 = \arccos(1/\sqrt{5})$.
5. $P_6$ is the set of vertices of a regular icosahedron, $a_6 = \arccos(1/\sqrt{5})$.

In our paper with Tarasov [48] we gave an alternative proof of this theorem. In [47] we found the list of all irreducible contact graphs with $N$ vertices on the sphere $S^2$, where $6 \leq N \leq 11$. Since the contact graph of $P_M$ is irreducible the theorem (for $M < 6$) follows from this list.

Fejes Tóth conjectured that the solution of the problem for seven pairs of antipodal points consists of the vertices of a rhombic dodecahedron (see the second edition of [21], page 210). This was proved by Cohn and Wootters [12] as a consequence of a more general theorem.

### 3 Problems on the maximum contact number of packings on the sphere

In [28] (pages 86 and 87) Fejes Tóth raised three problems about the number of touching pairs in a packing of congruent circles on the sphere.

Consider a packing $P$ of $S^2$ by $N$ circles $c_1, \ldots, c_N$ of diameter $d$. In the packing $P$ let $c_i$ be touched by $k_i$ circles. The first problem is to find the maximum number of points of contact:

$$K_N(d) := \max_{P:|P|=N} \frac{k_1 + \ldots + k_N}{2}.$$ 

In other words, $K_N(d)$ is the maximum number of touching pairs in a packing of $N$ spherical caps of diameter $d$.

It is clear that if $d = d_N$, then $K_N(d)$ is realized by the solution of the Tammes problem. It seems that the case $d < d_N$ is not well considered. There is only one paper in this direction [30], where this problem is considered for $N = 12$ and $d = 60^\circ$. There, it is proved that $K_{12}(60^\circ) = 24$. 


Fejes Tóth also proposed the problem of finding the maximum

\[ K(d) := \max_N \frac{K_N(d)}{N} \]

of the average number of points of contacts over all packings of circles of diameter \( d \).

The third problem is: For a given \( N \), find the maximum kissing number \( K_N \) over all packings of equal circles, i.e., find

\[ K_N := \max_{d \leq d_N} K_N(d). \]

Let \( X \) be the set of centers of a packing of congruent circles on \( S^2 \). Denote by \( e(X) \) the number of edges of the contact graph \( CG(X) \). It is easy to see that

\[ K_N = \max_{X \in S^2, |X| = N} e(X). \]

This number is currently known only for \( N \leq 12 \) and \( N = 24, 48, 60, 120 \).

Denote by \( \kappa(d) \) the kissing number of the spherical cap with diameter \( d \) in \( S^2 \), i.e. it is the maximum number of non-overlapping circles of diameter \( d \) that can touch a circle of the same diameter. Note that if \( d \leq \arccos(1/\sqrt{5}) \), then \( \kappa(d) = 5 \).

We say that a packing of \( N \) spherical caps with diameter \( d \) is maximal if

\[ K_N(d) = N\kappa(d)/2. \]

The following theorem has been proved by Robinson [52] and Fejes Tóth [27].

**Theorem 3.1** (Robinson [52], Fejes Tóth [27]). A maximal packing of \( N \) equal spherical caps exists only if \( N = 2, 3, 4, 6, 8, 9, 12, 24, 48, 60 \) or \( 120 \).

This theorem implies

**Corollary 3.1.** \( K_2 = 1, K_3 = 3, K_4 = 6, K_6 = 12, K_8 = 16, K_9 = 18 \) and for \( N = 12, 24, 48, 60 \) or 120 we have \( K_N = 5N/2 \).

In our paper [48] we considered \( K_N \) for \( N < 12 \). In particular, we proved that

**Theorem 3.2** (Musin and Tarasov [48]). \( K_5 = 8, K_7 = 12, K_{10} = 21, \) and \( K_{11} = 25 \).

Note that \( K_5 \) is attained by the set of vertices of a square pyramid. For \( N = 7 \) and \( N = 11 \), \( K_N(d) \) achieves its maximum on optimal configurations for Tammes’ problem. However, the arrangement realizing the optimal value \( K_10 \) is obtained by removing from the set of vertices of a regular icosahedron two adjacent vertices. In this case the contact graph \( CG(X) \) is not irreducible.

Our proof of Theorem 3.2 in [48] is based on two lemmas.
Lemma 3.1. Let $X$ be a finite set on the sphere $\mathbb{S}^2$. If every face of the contact graph $CG(X)$ is either a triangle or a quadrilateral, then this graph is irreducible.

Lemma 3.2. Let $X$ be a finite set on the sphere $\mathbb{S}^2$, where $|X| = N$ and $N > 6$. Suppose that $e(X) \geq 3N - 8$. Then the contact graph $CG(X)$ is irreducible.

Using these lemmas, Theorem 3.2 follows by checking the list of irreducible contact graphs for $N \leq 11$.

4 The Fejes Tóth – Sachs problem on the one–sided kissing numbers

Let $H$ be a closed half-space of $\mathbb{R}^n$. Suppose $S$ is a unit sphere in $H$ that touches the bounding hyperplane of $H$. The one–sided kissing number $B(n)$ is the maximal number of unit non–overlapping spheres in $H$ that can touch $S$.

The problem of finding $B(3)$ was raised by Fejes Tóth and Sachs in 1976 in another context. K. Bezdek and Brass studied the problem in a more general setting and they introduced the term “one-sided Hadwiger number”, which in the case of a ball is the same as the one-sided kissing number. The term “one-sided kissing number” has been introduced by K. Bezdek.

Clearly, $B(2) = 4$. The Fejes Tóth – Sachs problem in three dimensions was solved by G. Fejes Tóth. He proved that $B(3) = 9$ (see also Sachs and A. Bezdek and K. Bezdek for other proofs). Finally, Kertész proved that the maximal one–sided kissing arrangement is unique up to isometry.

The first upper bound for $B(4)$ was given by Szabó. He used the Odlyzko–Sloane bound $k(4) \leq 25$ for the kissing number of the four-dimensional ball to show that $B(4) \leq 20$. Next K. Bezdek, based on the result that $k(4) = 24$, lowered the bound to $B(4) \leq 19$.

In I proved that $B(4) = 18$. This proof relies on the extension of Delsarte’s method that was developed in [44]. However, technically the proof is more complicated than the proof of the fact that $k(4) = 24$. An alternate proof was given in [2] using semidefinite programming. The problem of uniqueness of the maximal one–sided kissing arrangement in four dimensions is still open.

In I conjectured that $B(5) = 32$, $B(8) = 183$ and $B(24) = 144855$. This conjecture for $n = 8$ was proved by Bachoc and Vallentin. In [1] and [45] we proposed several upper bounds on $B(n)$. However, all these bounds were improved in [2].

It is clear that there are some relations between kissing numbers and one–sided kissing numbers. Look at these nice equalities:

\[ n = 2, \quad 4 = B(2) = \frac{k(1) + k(2)}{2} = \frac{2 + 6}{2}; \]
\[ n = 3, \quad 9 = B(3) = \frac{k(2) + k(3)}{2} = \frac{6 + 12}{2}; \]
\[ n = 4, \quad 18 = B(4) = \frac{k(3) + k(4)}{2} = \frac{12 + 24}{2}. \]
We do not know whether the equality
\[ B(n) = \bar{K}(n) := \frac{k(n - 1) + k(n)}{2} \]
holds for all \( n \). However, there are reasons to believe that \( B(n) = \bar{K}(n) \) for \( n = 5, 8 \) and 24. We propose a weaker conjecture, namely, that the equality \( B(n) = \bar{K}(n) \) holds asymptotically:

**Conjecture.** We have
\[
\lim_{n \to \infty} \frac{B(n)}{K(n)} = 1.
\]

## 5 The work of Fejes Tóth on the isoperimetric problem for polyhedra and their extensions

### 5.1 Isoperimetric problem for polyhedra

The isoperimetric problem in space can be formulated as follows: *Find a convex body of given surface area \( F \) which contains the largest volume \( V \).*

The famous isoperimetric inequality states
\[ F^3 \geq 36\pi V^2. \]

For any solid \( P \) consider the *Isoperimetric Quotient*
\[ \text{IQ}(P) = \frac{36\pi V^2}{F^3}, \]
a term introduced by Pólya in [50, Chap. 10, Problem 43]. The isoperimetric inequality implies that \( \text{IQ}(P) \leq 1 \) and the equality holds only if \( P \) is a sphere.

The isoperimetric problem for polyhedra was first considered by Lhuili\'er (1782), see [39], and Steiner (1842), see [54]. Steiner stated the following conjecture.

**Steiner’s conjecture [54].** *Each of the five Platonic solids is the best (i.e. with the highest IQ) among all isomorphic polyhedra.*

This problem is still open for the icosahedron.

Consider the isoperimetric problem for polyhedra with given number of faces \( f \). Actually, this problem is currently solved only for \( f \leq 7 \) and \( f = 12 \). However, the first theorem on this problem was discovered in the 19th century.

**Theorem 5.1** (Lindelöf [38] and Minkowski [41]). *Of all convex polyhedra with the same number of faces, a polyhedron with the highest IQ is circumscribed about a sphere which touches each face in its centroid.*
Note that IQ(tetrahedron) \(\approx 0.302\), IQ(cube) \(\approx 0.524\), IQ(octahedron) \(\approx 0.605\), IQ(dodecahedron) \(\approx 0.755\), and IQ(icosahedron) \(\approx 0.829\) [50, Chap. 10, p. 189].

In fact, there are simple polyhedra \(F_{12}\) and \(C_{36}\) with eight and 20 faces that have greater IQ than, respectively, the regular octahedron and icosahedron. Goldberg [33] computed that IQ(octahedron) < IQ(\(F_{12}\)) \(\approx 0.628\) and IQ(icosahedron) < IQ(\(C_{36}\)) \(\approx 0.848\).

Goldberg proved that the regular dodecahedron is the best polyhedron with 12 facets and stated the following conjecture.

**Goldberg’s conjecture** [33]. If a polyhedron \(P\) with \(f \neq 11, 13\) faces and \(v\) vertices has the greatest IQ, then \(P\) is simple and its faces are \(\left\lfloor 6 - \frac{24}{v + 4} \right\rfloor\)-gons or \(\left\lfloor 7 - \frac{24}{v + 4} \right\rfloor\)-gons.

Note that according to Goldberg’s conjecture if \(v \geq 20\), then the faces of a best polyhedron can be only pentagons and hexagons, in other words \(P\) is a fullerene (see [16]).

Let \(P\) be a convex polyhedron with \(f\) faces. In [33] Goldberg proposed the following inequality:

\[
\frac{F^3}{V^2} \geq 54 (f - 2) \tan \omega_f (4 \sin^2 \omega_f - 1), \quad \omega_f := \frac{\pi f}{6f - 12},
\]

or equivalently

\[
\text{IQ}(P) \leq \frac{2\pi \cot \omega_f}{3(f - 2)(4 \sin^2 \omega_f - 1)},
\]

(5.1)

where the equality holds only if \(f = 4\) (regular tetrahedron), \(f = 6\) (cube) or \(f = 12\) (regular dodecahedron).

This inequality was independently found by Fejes Tóth [19] (see Fejes Tóth’s books [21, 25] and Florian’s survey [32] for references and historical remarks). However, both proofs contained a gap, namely the proof of the convexity of a certain function of two variables. A first rigorous proof of (5.1) was given by Fejes Tóth in the paper [20]. Finally, Florian [31] filled the gap in the previous proof by establishing the convexity of the respective function.

Two conjectures of Fejes Tóth on isoperimetric inequalities are still open.

Let \(P\) be a convex polyhedron with \(v\) vertices. The first conjecture states that

\[
\frac{F^3}{V^2} \geq \frac{27\sqrt{3}}{2} (v - 2)(3 \tan^2 \omega_v - 1).
\]

Let \(P\) be a convex polyhedron with \(f\) faces, \(v\) vertices and \(e\) edges. The second conjecture of Fejes Tóth states that

\[
\frac{F^3}{V^2} \geq 9 \epsilon \sin \frac{2\pi}{p} \left( \tan^2 \frac{\pi}{p} \tan^2 \frac{\pi}{q} - 1 \right),
\]

where \(p := 2e/f\) and \(q := 2e/v\).

Note that the validity of any of these conjectures would yield a proof of the still open conjecture of Steiner concerning the isoperimetric property of the icosahedron.
5.2 The Goldberg – Fejes Tóth inequality

Here we consider a “dual” version of the Goldberg – Fejes Tóth inequality (5.1).

Let $P$ be a convex polyhedron with $f$ faces. Then Euler’s formula implies

$$v \leq 2f - 4,$$

(5.2)

where $v$ is the number of vertices of $P$. The equality holds only if $P$ is a simple polyhedron.

Suppose $P$ is a polyhedron with highest IQ and fixed $f$. Then by the Lindelöf – Minkowski theorem there is a sphere $S$ that touches each face $\Gamma_i$ of $P$ in its centroid $x_i$. Thus, the set $X := \{x_1, \ldots, x_f\}$ is a subset of $S$.

Without loss of generality it can be assumed that $S$ is of radius $r = 1$. Since all faces $\Gamma_i$ touch the unit sphere $S$, we have

$$V = \frac{1}{3}F.$$

Therefore, for a given $f$, $P$ has the highest IQ if and only if $F = \text{area}(P)$ achieves its minimum.

Let $O$ be the center of the sphere $S$. Consider the central projection $g : P \rightarrow S$, i.e. for any point $A \in P$ the projection $g(A)$ is the intersection of the line $OA$ with $S$. It is clear that $g(x_i) = x_i$.

Let $p_1, \ldots, p_v$ be vertices of $P$. Denote by $Q$ the projection of this vertex set, i.e. $Q := \{q_1, \ldots, q_v\}$, where $q_i := g(p_i) \in S$.

The set $Q$ coincides with the set of vertices $\{v_i\}$ of the Voronoi diagram $\text{VD}(X)$ of $X$ in $S$. Equivalently, if $G_{i1}, \ldots, G_{im}$ are faces of $\text{VD}(X)$ with a common vertex $v_{i1}$, then $q_i$ is the circumcenter of the Delaunay cell $D_i$ with vertices $x_{i1}, \ldots, x_{im}$ in $S$. It immediately follows from the fact that $|v_ix_{ij}|$ as well as $|q_ix_{ij}|$ does not depend on $j$. Indeed, we have

$$|v_ix_{ij}|^2 = |Ov_i|^2 - |Ox_{ij}|^2,$$

where $|Ox_{ij}| = 1$ for all $j = 1, \ldots, m$.

Let $G_i := g(\Gamma_i)$, $i = 1, \ldots, f$. Then $G_i$ are Voronoi cells of $\text{VD}(X)$. Let $D_1, \ldots, D_v$ be the cells of the Delaunay tessellation $\text{DT}(X)$ in $S$. Then

$$\text{area}(G_1) + \ldots + \text{area}(G_f) = \text{area}(D_1) + \ldots + \text{area}(D_v) = 4\pi.$$

(5.3)

We have

$$F = \text{area}(\Gamma_1) + \ldots + \text{area}(\Gamma_f) = \text{area}(g^{-1}(G_1)) + \ldots + \text{area}(g^{-1}(G_f)).$$

It is easy to see, that the equality in the Goldberg – Fejes Tóth inequality (see [33] and [21, Sect. V.4]) holds only if $v = 2f - 4$ and all $G_i$ are congruent regular polygons. Equivalently, that means the equality holds only if all $D_k$ are congruent regular triangles.

Denote $\tilde{D}_i = g^{-1}(D_i)$. Then

$$F = \text{area}(\tilde{D}_1) + \ldots + \text{area}(\tilde{D}_v).$$

(5.4)
If \( v = 2f - 4 \) and all \( D_i \) are congruent triangles, then (5.3) yields
\[
\text{area}(D_i) = \frac{2\pi}{f-2}, \quad i = 1, \ldots, v.
\]

Let \( T \) be a regular spherical triangle in \( S \) with area \( (T) = t \). Denote
\[
\rho(t) := \text{area}(\tilde{T}), \quad \tilde{T} := g^{-1}(T) \subseteq P_T.
\]
Thus, we have the following inequality that is equivalent to (5.1).
\[
\text{IQ}(P) \leq \frac{\tau}{\rho(\tau)}, \quad \tau = \frac{2\pi}{f-2}.
\]

(5.5)

5.3 The Goldberg – Fejes Tóth inequality for higher dimensions

Here we consider a \( d \)-dimensional analog of the inequality (5.5).

Let \( P \) be a convex polyhedron in \( \mathbb{R}^d \) with \( v \) vertices and \( n \) facets, i.e. \( v = f_0(P) \) and \( n = f_{d-1}(P) \). Then McMullen’s Upper Bound Theorem \[61, p. 254\] yields the extension
\[
v \leq h_d(n) := \left(n - \left\lceil \frac{d}{2} \right\rceil \right) + \left(n - \left\lfloor \frac{d}{2} \right\rfloor - 1 \right).
\]
(5.6)
of the inequality (5.2) for all dimensions.

Define
\[
\text{IQ}(P) := d^{d-1} \Omega_d \frac{V^{d-1}}{F^d}, \quad \Omega_d := \text{area}(S^{d-1}).
\]

The Lindelöf – Minkowski theorem holds for all dimensions (see \[35, p. 274\]). So we have that \( \text{IQ}(P) \) achieves its maximum only if \( P \) is circumscribed about a sphere \( S \). As above we assume that \( S \) is a unit sphere.

It is easy to see that all definitions from Subsection 5.2 can be extended to all dimensions. Therefore, equality (5.4) holds also for a \( d \)-dimensional polyhedron \( P \). An analog of (5.3) is the following equality:
\[
\text{area}(D_1) + \ldots + \text{area}(D_v) = \Omega_d.
\]
(5.7)

Let \( D \) be a regular spherical simplex in \( S \) with area \( (D) = t \). Denote
\[
\rho_d(t) := \text{area}(\tilde{D}).
\]

Our conjecture is that the following extension of the Goldberg – Fejes Tóth inequality (5.5) holds for all dimensions:
\[
\text{IQ}(P) \leq \frac{\Omega_d}{v \rho_d(\Omega_d/v)}.
\]
(5.8)
Since $v \leq h_d(n)$, in particular we have

$$
\text{IQ}(P) \leq \frac{\tau}{\rho_d(\tau)}, \quad \tau = \frac{\Omega_d}{h_d(n)},
$$

(5.9)

Perhaps, (5.8) can be proved by the same way as László Fejes Tóth proved (5.1) $\equiv$ (5.5) in [20] and [21, Sect. V.4]. Actually, for all dimensions there are extensions of (5.2) – (5.4). We think that the most complicated step here is to prove that $\text{IQ}(P)$ cannot exceed IQ of a such polyhedron with $n$ facets that all its $D_i$ are congruent regular spherical simplices.

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