CARTAN SUBALGEBRAS OF OPERATOR IDEALS

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Abstract. Denote by $U_I(\mathcal{H})$ the group of all unitary operators in $1 + I$ where $\mathcal{H}$ is a separable infinite-dimensional complex Hilbert space and $I$ is any two-sided ideal of $B(\mathcal{H})$. A Cartan subalgebra $C$ of $I$ is defined in this paper as a maximal abelian self-adjoint subalgebra of $I$ and its conjugacy class is defined herein as the set of Cartan subalgebras $\{VCV^* \mid V \in U_I(\mathcal{H})\}$. For nonzero proper ideals $I$ we construct an uncountable family of Cartan subalgebras of $I$ with distinct conjugacy classes. This is in contrast to the by now classical observation of P. de La Harpe who noted that when $I$ is any of the Schatten ideals, there is precisely one conjugacy class under the action of the full group of unitary operators on $\mathcal{H}$. Our perspective is that the action of the full unitary group on Cartan subalgebras of $I$ is transitive, while by shrinking to $U_I(\mathcal{H})$ we obtain an action with uncountably many orbits if $\{0\} \neq I \neq B(\mathcal{H})$.

In the case when $I$ is a symmetrically normed ideal and is the dual of some Banach space, we show how the conjugacy classes of the Cartan subalgebras of $I$ become smooth manifolds modeled on suitable Banach spaces. These manifolds are endowed with groups of smooth transformations given by the action of the group $U_I(\mathcal{H})$ on the orbits, and are equivariantly diffeomorphic to each other. We then find that there exists a unique diffeomorphism class of full flag manifolds of $U_I(\mathcal{H})$ and we give its construction. This resembles the case of compact Lie groups when one has a unique full flag manifold, since all the Cartan subalgebras are conjugated to each other.

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1. Introduction

The correspondence between Lie groups and Lie algebras has been extended far beyond the classical setting of finite-dimensional Lie groups (see for instance [Ne06]). Lie theory consequently impacted various areas of mathematics, in particular representation theory and operator algebras, and this interaction contributed back to further developments in Lie theory itself. We will next illustrate these general remarks by a very few other references in order to explain the motivation for our present investigation on Cartan subalgebras of operator ideals that is a sequel to our introductory survey paper [BPW14b].

Cartan subalgebras play a central role in the structure theory of finite-dimensional Lie algebras, in part by extending the role of diagonal subalgebras to the study of matrix algebras. Consequently there has been a continuing interest in finding the appropriate notion of Cartan subalgebras of various infinite-dimensional Lie algebras. And this has led to progress in various directions of which we will briefly mention only three that are more relevant for this paper:

- Direct limit groups: representation theory based on root-space decompositions associated to Cartan subalgebras [NRW01], [DPW02], [Wo05].
- Classical Banach-Lie groups associated with the Schatten classes: structure and representation theory [dH72], [Boy80], [Ne98], [Ne02], [Ne04], [AV07], [ALR10], [CD13].
- Maximal abelian self-adjoint subalgebras of von Neumann algebras and of $C^*$-algebras [Re08], [SS08].

The relevance of the first direction is obvious from the fact that many direct limit groups are closely related to the classical groups associated with the ideal of finite-rank operators. The relevance of the second direction is seen for instance in our comments following Question 1.2 below (which first appears in [BPW14b]), where we raised the problem of studying the conjugacy classes of Cartan subalgebras of infinite-dimensional classical groups.

There is a huge literature devoted to the third of these research directions, hence we merely cite here a nice survey and a book but we will not discuss these citations further. We next briefly mention a few facts from this deep and very active theory, since in the present paper we will try to find analogs for some of these facts in the theory of operator ideals.

**Cartan subalgebras of proper operator ideals** of $\mathcal{B}(\mathcal{H})$ as defined here are simply the maximal abelian self-adjoint subalgebras of these ideals. This is a topology-free notion since some operator ideals may not have any reasonable complete linear topology.

However, an additional topological regularity property is required for defining the Cartan subalgebras of $\mathcal{B}(\mathcal{H})$ and more generally for Cartan subalgebras of any $C^*$-algebra or of any von Neumann algebra (see [Re08, Def. 2.1, 5.1]). Let $\mathcal{M}$ be any unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ with its unitary group $U_\mathcal{M} := \{u \in \mathcal{M} \mid uu^* = u^*u = 1\}$. For any maximal abelian self-adjoint subalgebra (masa) $\mathcal{C} \subseteq \mathcal{M}$, its normalizer is a subgroup of $U_\mathcal{M}$ that is commonly defined as

$$U_{\mathcal{M},\mathcal{C}} := \{u \in U_\mathcal{M} \mid u\mathcal{C}u^* = \mathcal{C}\}.$$ 

Observe that $U_{\mathcal{M},\mathcal{C}} \supseteq U_\mathcal{M} \cap \mathcal{C}$. If $U_{\mathcal{M},\mathcal{C}} \subseteq \mathcal{C}$, then $U_{\mathcal{M},\mathcal{C}} = U_\mathcal{M} \cap \mathcal{C}$, and we say that $\mathcal{C}$ is singular since its normalizer is minimum possible. To describe the opposite
situation, which should be thought of as a regularity property of \( C \) (see (1.1) below), we single out two particularly important cases:

(i) If \( \mathcal{M} \) is closed in the operator norm topology, that is, \( \mathcal{M} \) is a concrete \( C^* \)-algebra, then \( C \) is also closed in the operator norm topology by its maximality property.

(ii) Similarly, \( \mathcal{M} \) is closed in the weak operator topology, that is, \( \mathcal{M} \) is a von Neumann algebra, then \( C \) is weakly closed again by its maximality property.

Let us denote by \( \tau \) the operator norm topology of \( \mathcal{M} \) in (i) and the weak operator topology in (ii). One calls the masa \( C \) a Cartan subalgebra of \( \mathcal{M} \) if there exists a faithful conditional expectation \( E: \mathcal{M} \to C \), (that is, \( E \) is \( \tau \)-continuous, \( E^2 = E \), \( \|E\| = 1 \), \( E(\mathcal{M}) = C \) and if \( 0 \leq x \in \ker E \) then \( x = 0 \)) and \( C \) has the following regularity property:

\[ \text{The normalizer } U_{\mathcal{M}, C} \text{ spans a } \tau \text{-dense linear subspace of } \mathcal{M}. \quad (1.1) \]

(See [Re08,Defs. 2.1, 5.1] and note that \( 1 \in C \) by the maximality property of \( C \).)

If \( \mathcal{M} = M_n \) is the algebra of complex \( n \times n \) matrices, then every masa of \( M_n \) is a Cartan subalgebra in the above sense. And in particular this is the case for its canonical diagonal subalgebra \( D_n \subset M_n \), since for instance \( U_{M_n, D_n} \) contains all the permutation matrices. However, in the case of the infinite-dimensional von Neumann algebras, it was discovered very early [Di54] that there may exist singular masas, and such singular masas were actually found in every separable \( II_1 \) factor [Po83].

As regards these Cartan subalgebras, their existence has deep implications in the structure of von Neumann algebras [FM77]. However, there exist \( II_1 \) factors that do not have Cartan subalgebras at all, for instance the von Neumann algebras of the free groups with finitely many generators [Vo96]. On the other hand, there exist \( II_1 \) factors, like the finite hyperfinite factor, for which there exist uncountably many conjugacy classes of Cartan subalgebras [FM77], [Par85], where we mean conjugacy by transformations \( x \mapsto u x u^* \) defined by unitaries \( u \). Also \( II_1 \) factors have been exhibited for which all the Cartan subalgebras are conjugated to each other [OP10].

**Description of the present paper.** We investigate here conjugacy properties of Cartan subalgebras of operator ideals of \( B(\mathcal{H}) \) on separable Hilbert space \( \mathcal{H} \). (All ideals herein are considered two-sided.) If we consider the full unitary group \( U(\mathcal{H}) = \{ V \in B(\mathcal{H}) \mid V^* V = V V^* = 1 \} \), then to any nonzero operator ideal \( I \not\subseteq B(\mathcal{H}) \) there corresponds the subgroup of \( U(\mathcal{H}) \):

\[ U_I(\mathcal{H}) := U(\mathcal{H}) \cap (1 + I) \not\subseteq U(\mathcal{H}). \]

Then, as for \( U(\mathcal{H}) = U_2(\mathcal{H}) \) when \( I = B(\mathcal{H}) \), the adjoint action \( \text{Ad}(V)C = VCV^* \), for these unitaries \( V \in U_I(\mathcal{H}) \) in particular, preserves the class of Cartan subalgebras. That is, \( VCV^* \subset I \) is again a Cartan subalgebra for every Cartan subalgebra \( C \subset I \). The group \( U_I(\mathcal{H}) \) thus acts by unitary equivalence on the set of all Cartan subalgebras of \( I \). We prove that there exist uncountably many distinct orbits of this group action (Theorem 4.9). (Recall the obvious fact that distinct orbits are actually disjoint.) This is strikingly different from the well-known situation with \( I = B(\mathcal{H}) \) when one has countably many orbits [BPW14b, Rem. 4.2], or with \( \dim \mathcal{H} < \infty \) when the aforementioned action of group \( U_2(\mathcal{H}) \) on the set of all Cartan subalgebras of \( I \) is transitive (meaning that there exists only one orbit; see
Theorem [1.1] and the discussion preceding it). For a wide class of complete norm ideals, including the Schatten ideals \( \mathcal{S}_p(\mathcal{H}) \) with \( 1 \leq p < \infty \), for each of these orbits we exhibit in Theorem [2.3] a natural structure of a smooth Banach manifold which turns it into a smooth \( U_2(\mathcal{H}) \)-homogeneous space and the group \( U_2(\mathcal{H}) \) is actually a Banach-Lie group. The smooth homogeneous spaces corresponding to the various orbits turn out to be diffeomorphic to each other, and so there again exists an essentially unique full flag manifold just as in the case of finite-dimensional semisimple Lie groups. We also obtain some results on the size and shape of the union of all Cartan subalgebras from a fixed \( U_2(\mathcal{H}) \)-orbit. (See Section 3, the first part.)

By way of explaining the motivation for the present investigation, we recall what happens in the finite-dimensional case where the only nonzero operator ideals are just full matrix algebras \( M_n(\mathbb{C}) \). Any two maximal abelian self-adjoint subalgebras of the matrix algebra \( M_n(\mathbb{C}) \) are mapped to each other by the unitary equivalence \( \mathbb{C} \rightarrow VCV^* \) for a suitably chosen \( V \in U(n) \), as a direct consequence of the spectral theorem for normal matrices. This observation can be thought of as a statement on the compact Lie group \( G = U(n) := \{ V \in M_n(\mathbb{C}) \mid V^*V = I \} \), whose Lie algebra is \( g = u(n) := \{ X \in M_n(\mathbb{C}) \mid X^* = -X \} \). The complexification of the latter algebra is \( \mathfrak{g} = M_n(\mathbb{C}) \) viewed as a Lie algebra with the usual Lie bracket \([X,Y] = XY - YX\) for all \( X,Y \in M_n(\mathbb{C}) \), and the unitary equivalence transform is the adjoint action \( \text{Ad}_G(V)X = V XV^{-1} \) whenever \( V \in G \) and \( X \in \mathfrak{g} \). In this context, the above statement on the compact Lie group \( G = U(n) \) actually holds true for any compact Lie group. This is known as the conjugacy theorem for Cartan subalgebras and can be stated as follows:

**Theorem 1.1.** Let \( G \) be a compact Lie group whose Lie algebra is \( g \). If the complexified Lie algebra \( \mathfrak{g} = g \otimes \mathbb{C} \) is endowed with the involution given by \( (Y+iZ)^* = -Y+iZ \) for \( Y,Z \in \mathfrak{g} \), then any two Cartan subalgebras \( \mathfrak{c}_1 \) and \( \mathfrak{c}_2 \) of \( \mathfrak{g} \) are \( G \)-conjugated to each other. That is, there exists \( g \in G \) such that \( \text{Ad}_G(g)\mathfrak{c}_1 = \mathfrak{c}_2 \), where \( \text{Ad}_G : G \times \mathfrak{g} \to \mathfrak{g} \) stands for the adjoint action of the Lie group \( G \).

**Proof.** See for instance [Kn02, Th. 4.34].

The aim of this paper is to extend this study to operator ideals and their corresponding smaller unitary groups via

**Question 1.2.** To what extent does Theorem [1.1] hold true when the above finite-dimensional Lie groups and Lie algebras are replaced by

\[ \mathfrak{g} = \mathcal{I} \text{ and } G = U_2(\mathcal{H}) := U(\mathcal{H}) \cap (1 + \mathcal{I}) \]

where \( \mathcal{H} \) is a separable infinite-dimensional complex Hilbert space, \( U(\mathcal{H}) \) is the full unitary group on \( \mathcal{H} \), and \( \mathcal{I} \) is an operator ideal in \( B(\mathcal{H}) \)?

The point is that here we instead investigate conjugacy results involving the smaller unitary group \( U_2(\mathcal{H}) \) rather than the full unitary group \( U(\mathcal{H}) \). We recall that the variant of the above question with \( G = U(\mathcal{H}) \) was addressed in [dlH72, page 33] for the ideal of finite-rank operators, and in [dlH72, §II.4, Props. 10 and 12] when \( \mathfrak{g} = \mathcal{I} \) is any of the Schatten ideals. That argument based on the spectral theorem actually carries over directly to fully general proper operator ideals leading to the following infinite-dimensional version of Theorem [1.1].
Remark 1.3. For every operator ideal \( I \subsetneq \mathcal{B}(\mathcal{H}) \) and any two Cartan subalgebras \( C_1 \) and \( C_2 \) of \( I \) one has \( VC_1V^* = C_2 \) for some \( V \in U(\mathcal{H}) \). (See also Remark 1.3 below for a proof of this fact.)

The answer to Question 1.2 is obvious if \( I = \{0\} \) and is also well known in the case \( I = \mathcal{B}(\mathcal{H}) \) (the relevant facts are recalled in [BPW14] Rem. 4.2]). Let us also note that problems similar to Question 1.2 could be raised in connection with the other classical groups associated with operator ideals, and more generally about various infinite-dimensional versions of reductive Lie groups (see [BH72], [Be11], [BPW14], and also [KMRT98] for their finite-dimensional versions). That is, if \( G \) is one of these groups with its corresponding Lie algebra \( \mathfrak{g} \), it would be interesting to investigate the \( G \)-conjugacy classes of the Cartan subalgebras (maximal abelian self-adjoint subalgebras) of \( \mathfrak{g} \). This problem makes sense since \( G \) is a group of invertible operators and it acts on the operator Lie algebra \( \mathfrak{g} \) by the group action \( G \times \mathfrak{g} \to \mathfrak{g}, (V,X) \mapsto VXV^{-1} \).

2. Preliminaries

Operator ideals on Hilbert spaces. Throughout this paper we denote by \( \mathcal{H} \) a separable infinite-dimensional complex Hilbert space with a fixed orthonormal basis \( b = \{b_n\}_{n \geq 1} \), and by \( \mathcal{D} \) the corresponding set of diagonal operators in \( \mathcal{B}(\mathcal{H}) \). If \( \lambda = \langle \lambda_n \rangle_{n = 1}^{\infty} \) is a bounded sequence of complex numbers, then we denote by \( \text{diag} \{\lambda_n\} \) the operator whose sequence of diagonal entries is \( \lambda \).

Let \( \mathcal{K}(\mathcal{H}) \) denote the ideal of compact operators on \( \mathcal{H} \) and \( \mathcal{F}(\mathcal{H}) \) denote the ideal of finite-rank operators on \( \mathcal{H} \). For every \( F \in \mathcal{F}(\mathcal{H}) \) we denote its rank by \( \text{rank} F := \text{dim} F(\mathcal{H}) \), the dimension of the image of \( F \).

We will use the notation for the full unitary group on \( \mathcal{H} \):

\[
U(\mathcal{H}) = \{ V \in \mathcal{B}(\mathcal{H}) \mid V^*V = VV^* = 1 \}.
\]

By operator ideal in \( \mathcal{B}(\mathcal{H}) \), or \( \mathcal{B}(\mathcal{H}) \)-ideal, we will always mean a two-sided ideal of the algebra \( \mathcal{B}(\mathcal{H}) \) of all bounded linear operators on \( \mathcal{H} \), and we say \( I \) is a proper ideal if \( \{0\} \subsetneq I \subsetneq \mathcal{B}(\mathcal{H}) \). Recall \( \mathcal{F}(\mathcal{H}) \subseteq I \subseteq \mathcal{K}(\mathcal{H}) \) for every proper ideal \( I \) of \( \mathcal{B}(\mathcal{H}) \). For an operator ideal \( I \) we denote its real part \( \mathcal{I}^0 := \{ T \in I \mid T = T^* \} \) and its positive cone \( \mathcal{I}^+ := \{ T \in I \mid T \geq 0 \} \). To each ideal \( I \) in \( \mathcal{B}(\mathcal{H}) \) there corresponds the group of unitary operators

\[
U_I(\mathcal{H}) := U(\mathcal{H}) \cap (1 + I).
\]

Thus, if \( I = \mathcal{B}(\mathcal{H}) \) then \( U_I(\mathcal{H}) = U(\mathcal{H}) \), while for \( I = \mathcal{K}(\mathcal{H}) \) we denote \( U_I(\mathcal{H}) =: U_{\mathcal{K}(\mathcal{H})} \). Likewise for \( I = \mathfrak{S}_2(\mathcal{H}) \) we denote \( U_I(\mathcal{H}) =: U_{\mathfrak{S}_2(\mathcal{H})} \).

We have the following spectral characterizations of the operators belonging to the unitary groups \( U(\mathcal{H}) \) and \( U_I(\mathcal{H}) \) for \( I \subsetneq \mathcal{B}(\mathcal{H}) \).

Proposition 2.1. Let \( A \in \mathcal{B}(\mathcal{H}) \) and denote by \( \mathbb{T} \) the unit circle in the complex plane. Then

(i) \( 1 + A \in U(\mathcal{H}) \) if and only if \( A \) is a normal operator with its spectrum contained in the circle \( -1 + \mathbb{T} \).

(ii) \( 1 + A \in U_I(\mathcal{H}) \) if and only if \( A \) is compact normal with spectrum (that is, point spectrum) \( \{ e^{i \theta_k} - 1 \} \) where \( -\pi < \theta_k \leq \pi \), \( \theta_k \downarrow 0 \), and \( \{ e^{i \theta_k} - 1 \} \) is \( \Sigma(I) \), the characteristic set of the ideal \( I \) (see Remark 2.2).
Proof. If $1 + A \in U(H)$, then $\text{spec}(1 + A) \subseteq \mathbb{T}$ hence $\text{spec}(A) \subseteq -1 + \mathbb{T}$, and on the other hand $(1 + A)^* (1 + A) = (1 + A)(1 + A)^* = 1$, which implies $A^* A = AA^*$. (Another way to prove normality is to notice that $A = (1 + A) - 1$ is the difference of two commuting normal operators, and then one could use either the spectral theorem for normal operators or even the Fuglede commutativity theorem.)

We now prove the converse assertion. Since $A$ is normal, $1 + A$ is also normal, so by spectral theorem for normal operators $1 + A$ is unitarily equivalent to a multiplication operator $M_\varphi$ on some $L^2$-space, where $\varphi$ is an $L^\infty$-function. The hypothesis ensures that the spectrum of $1 + A$ is contained in $\mathbb{T}$, hence $|\varphi| = 1$ almost everywhere. Hence $M_\varphi$ is unitary, which further implies that $1 + A$ is a unitary operator.

Remark 2.2. We collect here for later use a few facts on operator ideals; see [DFWW04 Sect. 4] for more details on this terminology. For any integer $n \geq 1$ and any $T \in B(H)$, its $n$-th singular number is defined by $s_n(T) = \inf \{\|T - F\| : F \in B(H), \text{rank } F < n\}$ and we denote $s(T) := (s_n(T))_{n=1}^\infty$. Thus $\|T\| = s_1(T) \geq s_2(T) \geq \cdots$, and $T$ is a compact operator if and only if $\lim_{n \to \infty} s_n(T) = 0$.

Moreover one has the well-known characterization of $B(H)$-ideals in terms of characteristic sets

$$\Sigma(I) := \{s(T) : T \in I\}$$

with map $I \mapsto \Sigma(I)$ an inclusion-preserving lattice isomorphism between the lattice of $B(H)$-ideals and the lattice of characteristic sets (for a modern reference see [DFWW04 Sect. 4]). In particular, $\Sigma(K(H)) = c_0^*$, the cone of decreasing to zero positive sequences, and $\Sigma(F(H))$ are those $c_0$ sequences that also are finitely supported.

An operator ideal $I$ in $B(H)$ is called arithmetic mean closed if it is closed under $s(\cdot)$-majorization, i.e., it satisfies the condition: if $T \in B(H)$ where for some $K \in I$ we have

$$(\forall n \geq 1) \quad s_1(T) + \cdots + s_n(T) \leq s_1(K) + \cdots + s_n(K),$$

then necessarily $T \in I$. Here are some important properties of these ideals:

(i) If $I$ is arithmetic mean closed and $\{P_n\}_{1 \leq n \leq N}$ is a family of mutually orthogonal projections in $H$, where $1 \leq N \leq \infty$, then for every $T \in I$ we have $\sum_{1 \leq n \leq N} P_n T P_n \in I$ as a direct consequence of [GK69 Ch. II, Th. 5.1].

(ii) Conversely, if $\{P_n\}_{1 \leq n \leq N}$ is a family of mutually orthogonal rank-one projections with $\sum_{1 \leq n \leq N} P_n = 1$ and for every $T \in I$ we have $\sum_{1 \leq n \leq N} P_n T P_n \in I$, then the ideal $I$ is arithmetic mean closed by [KW12 Th. 4.5].

(iii) Let $\Phi$ be a symmetric norming function as in [GK69 Ch. III, §3]. In our notation, this equivalently can be defined as a norm on the space of finite-rank diagonal operators $\Phi : D \cap F(H) \to [0, \infty)$ such that $\Phi(P) = 1$ if $P$ is a rank-one orthogonal projection and $\Phi(V_\varphi D V_\varphi^{-1}) = \Phi(D)$ for all $D \in D \cap F(H)$ and $\varphi \in \mathbb{S}_\infty$ (see “Permutation groups” below for definitions of $\mathbb{S}_\infty$ and $V_\varphi$).

Let $\mathcal{S}_\Phi$ be the set of all operators $T \in K(H)$ such that

$$\|T\|_\Phi := \sup_{n \geq 1} \Phi(\text{diag}(s_1(T), \ldots, s_n(T), 0, 0, \ldots)) < \infty.$$ 

Then $\mathcal{S}_\Phi$ is an operator ideal endowed with the complete (symmetric) norm $\| \cdot \|_\Phi$ by [GK69 Ch. III, Th. 4.1], and this ideal is arithmetic mean closed (see
Permutation groups. We denote by $S_\infty$ the group of all permutations of the positive integers $\{1, 2, \ldots\}$ and by $S_{\text{fin}}$ its subgroup consisting of the “finite permutations”, that is, the permutations that leave fixed all but finitely many natural numbers. There is a group homomorphism

$$S_\infty \to U(\mathcal{H}), \quad \sigma \mapsto V_\sigma,$$

which depends on the choice of the orthonormal basis $b$, and is defined by $V_\sigma(b_n) = b_{\sigma^{-1}(n)}$ whenever $n \in \mathbb{N}$ and $\sigma \in S_{\infty}$. Clearly also then $V_\sigma^{-1} = V_{\sigma^{-1}}$.

Remark 2.3. If $D \in \mathcal{D}$, then for every $\sigma \in S_\infty$ we have $V_\sigma D V_\sigma^{-1} \in \mathcal{D}$, and the diagonal entries of the latter diagonal operator are precisely the diagonal entries of $D$ permuted according to $\sigma$.

Proposition 2.4. If $\mathcal{I}$ is a proper ideal of $\mathcal{B}(\mathcal{H})$ and $\sigma \in S_\infty$, then $V_\sigma \in 1 + \mathcal{I}$ if and only if $\sigma \in S_{\text{fin}}$.

Proof. It is clear that if $\sigma \in S_{\text{fin}}$ then $V_\sigma \in 1 + \mathcal{F}(\mathcal{H}) \subseteq 1 + \mathcal{I}$ since, as is well-known, all proper ideals of $\mathcal{B}(\mathcal{H})$ contain the ideal of finite rank operators. Conversely, assume that for $\sigma \in S_\infty$ and $K \in \mathcal{I}$ we have $V_\sigma = 1 + K$. Since $\mathcal{I} \neq \mathcal{B}(\mathcal{H})$, it follows that $K$ is a compact operator, hence $\lim_{n \to \infty} \|K b_n\| = 0$. On the other hand for every $n \in \mathbb{N}$ we have

$$\|K b_n\| = \|(V_\sigma - 1)b_n\| = \|b_{\sigma^{-1}(n)} - b_n\| = \begin{cases} 0 & \text{if } \sigma(n) = n, \\ \sqrt{2} & \text{if } \sigma(n) \neq n. \end{cases}$$

Therefore the condition $\lim_{n \to \infty} \|K b_n\| = 0$ implies $\sigma(n) = n$ eventually, that is, $\sigma \in S_{\text{fin}}$.

3. $U_\mathcal{I}(\mathcal{H})$-diagonalization

The aim of this section is to investigate the $U_\mathcal{I}(\mathcal{H})$-diagonalizable operators in $\mathcal{I}$, that is, investigate the set

$$\mathcal{D}_\mathcal{I} := \{ VDV^* \mid D \in \mathcal{D} \cap \mathcal{I}, V \in U_\mathcal{I}(\mathcal{H}) \} = \bigcup_{V \in U_\mathcal{I}(\mathcal{H})} V(\mathcal{D} \cap \mathcal{I})V^* \subseteq \mathcal{I},$$

for $\mathcal{I}$ any operator ideal in $\mathcal{B}(\mathcal{H})$ and recalling that $\mathcal{D}$ is the set of all diagonal operators with respect to the fixed basis $b = \{b_n\}_{n \geq 1}$. Here we have the union of the sets in the $U_\mathcal{I}(\mathcal{H})$-conjugacy class of the Cartan subalgebra $\mathcal{D} \cap \mathcal{I}$ of $\mathcal{I}$ (see also Proposition 4.3 and Definition 4.7 below). This is a set of normal operators in $\mathcal{I}$ and we will also consider its self-adjoint part

$$\mathcal{D}^a_\mathcal{I} := \mathcal{D}_\mathcal{I} \cap \mathcal{I}^a = \{ VDV^* \mid D = D^* \in \mathcal{D} \cap \mathcal{I}, V \in U_\mathcal{I}(\mathcal{H}) \}. \quad (3.1)$$

It follows by [BPW14b, Prop. 4.3(1)] that

$$\mathcal{I} = \mathcal{F}(\mathcal{H}) \implies \mathcal{D}^a_\mathcal{I} = \mathcal{I}^a, \quad (3.2)$$

but we will prove that this latter equality fails to be true for any other nontrivial ideal (see Corollary 3.9 below).

[DFWW04 subsect. 4.9] or the Dominance Property in [GK69 Ch. III, §4].

In particular, for $\Phi(\cdot) = \| \cdot \|_{\ell^p}$ we obtain that the Schatten ideal $\mathcal{S}_p(\mathcal{H})$ is arithmetic mean closed whenever $1 \leq p \leq \infty$. Note also that for $\Phi(\cdot) = \| \cdot \|_\infty$, $\mathcal{S}_\Phi = \mathcal{S}_\infty(\mathcal{H}) = \mathcal{K}(\mathcal{H})$.

□
We take this opportunity to offer alternative notions of restricted diagonalizability:

\[ \mathcal{D}_{X,J} := \{ VDV^* \mid D \in \mathcal{D} \cap J, \ V \in U_2(\mathcal{H}) \} = \bigcup_{V \in U_2(\mathcal{H})} V(\mathcal{D} \cap J)V^* \subseteq \mathcal{J}. \]

We expect a fair amount of overlap between results in this paper and any investigations into these more general notions. The difference is essentially the focus on which normal pure point spectrum operators each considers.

**General facts** (Proposition 3.1, Corollary 3.9).

We first record the following **uniqueness properties of** \( U_2(\mathcal{H}) \)-diagonalization.

**Proposition 3.1.** If a normal compact operator \( X \in \mathcal{I} \subsetneq \mathcal{B}(\mathcal{H}) \) with spectral multiplicities one is \( U_2(\mathcal{H}) \)-diagonalizable to a diagonal operator \( D \) with respect to the fixed basis \( b = \{ b_n \}_{n \geq 1} \), then \( D \) is unique up to a finite permutation. Furthermore, \( X \) can be \( U_2(\mathcal{H}) \)-diagonalized to every finite permutation of \( D \).

**Proof.** Suppose \( W_1 \) and \( W_2 \) are two unitary operators in \( U_2(\mathcal{H}) \) with \( W_1XW_1^* = D_1 \) and \( W_2XW_2^* = D_2 \). This implies that \( D_1 = \text{diag}(d_n) \) and \( D_2 \) have the same spectrum. Hence \( D_2 = V_\sigma D_1V_\sigma^{-1} = V_\sigma D_1V_{\sigma^{-1}} \) for some permutation \( \sigma : \mathbb{N} \to \mathbb{N} \).

But then \( W_2XW_2^* = V_\sigma D_1V_{\sigma^{-1}} \), hence \( XW_2V_\sigma = W_2^*V_\sigma D_1 \). Applying this to \( b_n \), we obtain \( XW_2V_\sigma b_n = W_2^*V_\sigma D_1 b_n \), so

\[ XW_2^*b_{\sigma^{-1}(n)} = d_n W_2^*b_{\sigma^{-1}(n)}. \]

Since \( X \) has distinct eigenvalues, \( W_2^*b_{\sigma^{-1}(n)} = \alpha_n b_n \) for some \( |\alpha_n| = 1 \) for all \( n \). But \( W_2^* = 1 + K \), so one obtains \( (1 + K)b_{\sigma^{-1}(n)} = \alpha_n b_n \) and then

\[ ||Kb_{\sigma^{-1}(n)}|| = ||\alpha_n b_n - b_{\sigma^{-1}(n)}||. \]

Since \( K \) is a compact operator, \( ||Kb_{\sigma^{-1}(n)}|| \to 0 \) which implies that \( \sigma^{-1}(n) = n \) eventually. Therefore \( \sigma \) must be a finite permutation. This observation together with \( D_2 = V_\sigma D_1V_{\sigma^{-1}} \) proves the claim. Although not used here, it is clear also that \( \alpha_n \to 1 \).

To prove the last assertion of Proposition 3.1 notice that every finite permutation is a unitary of the form \( 1 + F \) with \( F \in \mathcal{F}(\mathcal{H}) \). (This is because for finite permutations \( \sigma, V_\sigma = 1 + (V_{\sigma} - 1) \) and since \( V_{\sigma} b_n = b_n \) eventually, \( V_{\sigma} - 1 = F \) is finite rank.) Therefore, if \( X \) is \( U_2(\mathcal{H}) \)-diagonalizable to a diagonal operator \( D \), that is, \( W_1XW_1^* = D \) for some \( W_1 = 1 + K \in U_2(\mathcal{H}) \), then \( X \) is also \( U_2(\mathcal{H}) \)-diagonalizable to \( D' \), any finite permutation of \( D \). Indeed, \( V_{\sigma^{-1}}W_1XW_1^*V_\sigma = D' \) for some finite permutation \( \sigma \), and so \( V_\sigma = 1 + F \) and

\[ V_{\sigma^{-1}}W_1 = (1 + F^*)(1 + K) = 1 + F^* + K_1 + F^*K_1 \]

is a unitary of the form \( 1 + K \) for \( K \in \mathcal{I} \).

Note \( W_1 \) and \( W_2 \) are related in the above setting. In fact, if \( W_1XW_1^* = D_1 \) and \( W_2XW_2^* = D_2 \) are a permutation of each other (not necessarily finite permutation), without attempting to obtain an explicit form for the unitaries \( W_1 \) and \( W_2 \), we can however obtain an explicit form for \( W_1W_2^* \), that is, information on how \( W_1 \) and \( W_2 \) relate:

Since \( W_2XW_2^* = V_\sigma D_1V_{\sigma^{-1}} \), we have \( X = W_2^*V_\sigma D_1V_{\sigma^{-1}}W_2 = W_2^*D_1W_1 \). Therefore \( W_2W_1^*D_1W_1W_2^* = V_{\sigma^{-1}}D_1V_{\sigma^{-1}} \). If \( U := W_2W_1^* \) then \( UD_1U^* = V_{\sigma}D_1V_{\sigma^{-1}} \), and hence

\[ D_1(U^*V_\sigma) = (U^*V_\sigma)D_1. \]
The multiplicities one condition on $D_1$ then implies that the unitary operator $U^*V_\sigma$ is a diagonal operator, say $D_u \in D$. That is, $W_1W_\sigma^*V_\sigma = D_u$ and hence $W_1W_\sigma^* = V_\sigma^{-1}D_u$. So $W_2$ in terms of $W_1$ has the form $W_2 = D_\sigma^*V_\sigma W_1$ (even if, as in the case of infinite $\sigma$, neither $W_1$ nor $W_2$ are in $U_I(\mathcal{H})$). In the proof of Proposition 3.1 $\sigma$ is in addition a finite permutation.

On the structure of $D_I^a$ and $D_I^{sa}$. The geometric shape of the set $D_I^{sa}$ from (3.1) is not clear in general. For instance Remark 3.2 shows that it need not be a real linear subspace of $I^{sa}$. In the case when $I \neq I^2$, some information on the shape and size of $D_I$ and $D_I^{sa}$ is discussed below in Remark 3.2, Question 3.3, Corollary 3.9 and Remark 3.14.

**Remark 3.2.** When $I = B(\mathcal{H})$ (so $U_I(\mathcal{H}) = U(\mathcal{H})$), then $D_I$ is the set of all normal operators in $B(\mathcal{H})$ with pure point spectrum for which $\mathcal{H}$ decomposes into the orthogonal direct sum of the eigenspaces of such an operator. Hence $D_I^{sa} \subseteq I^{sa}$ because of the existence of self-adjoint operators without eigenvalues.

It is instructive to note that usually (i.e., when $I \neq B(\mathcal{H})$), the basis $b = \{b_n\}_{n \geq 1}$ is tied to $D$ and hence also to $D_I$. However, when $I = B(\mathcal{H})$, all bases yield the same $D_I$. Moreover, $D_I^{sa}$ is not additive. Indeed, the Weyl-von Neumann theorem implies that if $A = A^* \in I = B(\mathcal{H})$, then there exist $A_1 \in D_I^{sa}$ and $A_2 = A_2^* \in K(\mathcal{H})$ for which $A = A_1 + A_2$ (see for instance [Da96, Sect. II.4], or [Kn58] for a generalization involving symmetrically normed ideals). Furthermore, if the self-adjoint operator $A$ does not have pure point spectrum, then we obtain $A_1, A_2 \in D_I^{sa}$ and $A_1 + A_2 = A \notin D_I^{sa}$. This shows that $D_I^{sa}$ fails to be a real linear subspace of $I^{sa}$ if $I = B(\mathcal{H})$. □

Now the following question naturally arises.

**Question 3.3.** Is it true that the final part of Remark 3.2 can be generalized in the sense that for every nonzero ideal $I$ in $B(\mathcal{H})$, the set $D_I^{sa}$ is a proper subset of $I^{sa}$ and beyond Remark 3.2 does it real linearly spans $I^{sa}$? □

We do not know the answer to the second part of this question, that is, whether $I^{sa}$ is the real linear span of $D_I^{sa}$. As regards the part of the above question concerning the proper inclusion $D_I^{sa} \subsetneq I^{sa}$, it is answered in the affirmative in Corollary 3.9 below whose proof is based on the following Proposition 3.5 Corollary 3.8.

**Notation 3.4.** $D(v)$ denotes the set of operators in $B(\mathcal{H})$ that are diagonal with respect to the orthonormal basis of $\mathcal{H}$, $v = \{v_n\}_{n \geq 1}$.

**Proposition 3.5.** Let $v = \{v_n\}_{n \geq 1}$ and $w = \{w_n\}_{n \geq 1}$ be orthonormal bases in $\mathcal{H}$. Suppose the distances between the parts of the 1-dimensional subspaces spanned by $v_m$ and by $w_n$ that lie on the surface of the unit ball are uniformly bounded away from 0. That is, for some $\delta > 0$, $\|\beta w_m - \alpha v_n\| \geq \delta$ for all $n, m \geq 1$ and $|\beta| = |\alpha| = 1$, or equivalently and more simply, $\|w_n - \alpha v_m\| \geq \delta$ for all $n, m \geq 1$ and $|\alpha| = 1$.

Then for every operator $X \in D(v)$ with spectral multiplicities one and every $W \in U_{K(\mathcal{H})}$ one has $WXW^{-1} \notin D(w)$. Consequently, the $U_{K(\mathcal{H})}$-orbits of the multiplicities one operators of $D(v)$ and $D(w)$ are disjoint.

**Proof.** In fact, if $Xv_n = \lambda_n v_n$ for all $n \geq 1$ and if $W \in U_{K(\mathcal{H})}$ and $WXW^{-1}$ were a diagonal operator with respect to the basis $\{w_n\}_{n \geq 1}$, then the 1-dimensional subspace spanned by the vectors of the latter basis are precisely the eigenspaces of
Proof.

(i) For every orthonormal basis \( \{v_n\}_{n \geq 1} \) of \( \mathcal{H} \), one has \( W^{-1}v_n = \lambda'_n v_n \) and \( \lambda'_n = \lambda_{\sigma(n)} \), where \( \lambda_{\sigma(n)} \) is the \( \sigma(n) \)-th eigenvalue of \( W^{-1}v_n \). Thus, every eigenvector of \( W^{-1}v_n \) is an eigenvector of \( \lambda_{\sigma(n)} \) and where by the distinctness of the eigenvalues and all having multiplicity one, we have a bijection \( \sigma \) and scalars \( |\alpha_n| = 1 \) for which \( \lambda'_n = \lambda_{\sigma(n)} \) and \( W^{-1}v_n = \alpha_n v_{\sigma(n)} \) for all \( n \geq 1 \).

Moreover, since \( W^{-1} \in U_K(\mathcal{H}) \), one has \( W^{-1} = 1 + K \) for some \( K \in K(\mathcal{H}) \). Thus for every \( n \geq 1 \), \( Kw_n = \alpha_nv_{\sigma(n)} \), hence \( \|Kw_n\| \geq \delta \). However, since for any \( \alpha, \beta \) such that \( \alpha\beta > 0 \), one has \( \|\alpha\beta - \alpha\beta\| \geq \delta \), and \( \alpha \in \mathbb{C} \) with \( |\alpha| = 1 \), there exists \( \delta > 0 \) such that for arbitrary scalar sequences \( \alpha_n \) and \( \beta_n \) such that \( \alpha_n \beta_n \in \mathbb{C} \) one has \( \|\alpha_n\beta_n - \alpha_n\beta_n\| \geq \delta \) for all \( n \) and \( m \).

Then for every operator \( X \in \mathcal{D}(v) \) with spectral multiplicities one and every \( W \in U_K(\mathcal{H}) \) one has \( WXW^{-1} \notin \mathcal{D}(v) \).

The following corollary is a reformulation of Proposition 3.5 in a slightly more complicated form, but in a more useful form for our approach.

**Corollary 3.6.** Let \( v = \{v_n\}_{n \geq 1} \) and \( w = \{w_n\}_{n \geq 1} \) be orthonormal bases in \( \mathcal{H} \). Assume there exists \( \delta > 0 \) such that for arbitrary scalar sequences \( \alpha_n \) and \( \beta_n \) in the unit circle in \( \mathbb{C} \) one has \( \|\alpha_n\beta_n - \alpha_n\beta_n\| \geq \delta \) for all \( n, m \geq 1 \).

Then for every operator \( X \in \mathcal{D}(v) \) with spectral multiplicities one and every \( W \in U_K(\mathcal{H}) \) one has \( WXW^{-1} \notin \mathcal{D}(w) \).

**Proof.** Use Proposition 3.5.

**Proposition 3.7.** Fix any orthonormal basis \( b = \{b_n\}_{n \geq 1} \) of \( \mathcal{H} \). There exists a family of vectors \( \{v_n^{(t)}\}_{n \geq 1} \) in \( \mathcal{H} \) with \( v^{(0)} = b \) and \( \{v_n^{(t)}\}_{n \geq 1} \) being orthonormal bases in \( \mathcal{H} \) with \( v^{(0)} = b \).

(i) For every \( t \in [0, \pi/2) \) the subfamily \( \{v_n^{(t)}\}_{n \geq 1} \) is an orthonormal basis in \( \mathcal{H} \) with \( v^{(0)} = b \).

(ii) If \( t \not\in [0, \pi/2) \), then there exists \( \delta > 0 \) with \( \|v_n^{(t)} - \alpha v_m^{(s)}\| \geq \delta \) for all \( n, m \geq 1 \) and \( \alpha \in \mathbb{C} \) with \( |\alpha| = 1 \).

**Proof.**

(i) For every \( t \in [0, \pi/2) \) and \( b = \{b_n\}_{n \geq 1} \) we define the orthonormal basis \( v^{(t)} := \{v_n^{(t)}\}_{n \geq 1} \) in \( \mathcal{H} \) by

\[
v_{2r-1}^{(t)} = (\cos t)b_{2r-1} + (\sin t)b_{2r}, \quad v_{2r}^{(t)} = -(\sin t)b_{2r-1} + (\cos t)b_{2r}
\]

for \( r \geq 1 \). Hence we perform a rotation of angle \( t \in [0, \pi/2] \) in the plane \( \mathbb{R} \cdot b_{2r-1} + \mathbb{R} \cdot b_{2r} \) in order to obtain the orthonormal set \( \{v_{2r-1}^{(t)}, v_{2r}^{(t)}\} \) from \( \{b_{2r-1}, b_{2r}\} \). It is then clear that for each \( t \in [0, \pi/2) \) the family \( v^{(t)} = \{v_n^{(t)}\}_{n \geq 1} \) is an orthonormal basis in \( \mathcal{H} \) and that \( v^{(0)} = b \).

(ii) Let \( 0 \leq s < t < \pi/2 \) and \( \alpha \in \mathbb{C} \) be arbitrary with \( |\alpha| = 1 \). If for some \( r \geq 1 \) we have \( n \in \{2r-1, 2r\} \) and \( m \not\in \{2r-1, 2r\} \), then \( v_n^{(t)} \perp v_m^{(s)} \) hence \( \|\alpha v_n^{(t)} - v_m^{(s)}\| = \sqrt{2} \). Moreover, if \( n, m \in \{2r-1, 2r\} \) then

\[
\|\alpha v_n^{(t)} - v_m^{(s)}\|^2 = 2 - 2 \Re (\alpha(v_n^{(t)}, v_m^{(s)})) \geq 2(1 - |\Re (v_n^{(t)}, v_m^{(s)})|).
\]

On the other hand, since \( s < t \), it follows that the orthonormal set \( \{v_{2r-1}^{(t)}, v_{2r}^{(t)}\} \) is obtained from the orthonormal set \( \{v_{2r-1}^{(s)}, v_{2r}^{(s)}\} \) by performing a rotation of angle
t − s in the plane $\mathbb{R} \cdot b_{2r−1} + \mathbb{R} \cdot b_2$, hence we have

$$v_{2r−1}^{(t)} = (\cos(t−s))v_{2r−1}^{(s)} + (\sin(t−s))v_2^{(s)}$$

$$v_2^{(t)} = −(\sin(t−s))v_{2r−1}^{(s)} + (\cos(t−s))v_2^{(s)}.$$ 

It follows by these formulas that if $n, m \in \{2r−1, 2r\}$ then

$$\|(v_n^{(t)}, v_m^{(s)})\| \leq \max\{\cos(t−s), \sin(t−s)\}.$$ 

Therefore

$$\|\alpha_n v_n^{(t)} − v_m^{(s)}\|^2 \geq 2(1 − \max\{\cos(t−s), \sin(t−s)\})$$

and this lower bound is independent of $r \geq 1$ and strictly positive (as $0 < t−s < \pi/2$). Thus we see that condition $\text{[H]}$ is satisfied, completing the proof. \hfill $\square$

The next corollary provides uncountably many orthogonal bases in $\mathcal{H}$ that pairwise satisfy the requirements of Corollary $\text{3.10}$. In the proof of Corollary $\text{3.10}$ we will actually need only one such pair. However the full power of the following corollary will be needed later in order to prove that there exist infinitely many conjugacy classes of Cartan subalgebras of a nonzero operator ideal.

**Corollary 3.8.** Fix any orthonormal basis $b = \{b_n\}_{n \geq 1}$ of $\mathcal{H}$. There exists a family of vectors $\{v_n^{(t)} \mid n \in \mathbb{N}, 0 \leq t < \pi/2\}$ with the following properties:

(i) For every $t \in [0, \pi/2)$ the subfamily $v^{(t)} := \{v_n^{(t)} \mid n \geq 1\}$ is an orthonormal basis in $\mathcal{H}$ with $v^{(0)} = b$.

(ii) If $t \neq s \in [0, \pi/2)$, then there exists $\delta > 0$ with the property that for arbitrary scalar sequences $(\alpha_n)^{\infty}_{n=1}$ and $(\beta_n)^{\infty}_{n=1}$ in the unit circle in $\mathbb{C}$ one has $\|\beta_n v_n^{(t)} − \alpha_m v_m^{(s)}\| \geq \delta$ for all $n, m \geq 1$.

Proof. Use Proposition $\text{3.7}$. \hfill $\square$

**Corollary 3.9.** If $\mathcal{I} \supsetneq \mathcal{F(H)}$ is an operator ideal in $\mathcal{B(H)}$, then $\mathcal{D}_\mathcal{I}^{sa} \subseteq \mathcal{I}^{sa}$.

Proof. The case $\mathcal{I} = \mathcal{B(H)}$ was already settled in Remark $\text{3.2}$ so we may assume $\mathcal{F(H)} \supsetneq \mathcal{I} \subsetneq \mathcal{B(H)}$. Pick any $t_0 \in (0, \pi/2)$ and consider the corresponding orthonormal basis $v := v^{(t_0)} = \{v_n^{(t_0)}\}_{n \geq 1} \neq b$ provided by Corollary $\text{3.11}$. Then choose any self-adjoint operator $X \in \mathcal{D(v)} \cap \mathcal{I}$ and with spectral multiplicities one. Then using Proposition $\text{3.5}$ one obtains $WXW^{-1} \notin \mathcal{D}$ for all $W \in U_{\mathcal{H}}(\mathcal{H})$. So, also for all $W \in U_{\mathcal{I}}(\mathcal{H})$ in particular. The latter means $X \notin \mathcal{D}_\mathcal{I}^{sa}$. \hfill $\square$

Observe choosing $X \geq 0$ in the above proof yields the stronger $\mathcal{D}_\mathcal{I}^{sa} \cap \mathcal{I}^+ \subseteq \mathcal{I}^{sa}$.

Corollary $\text{3.9}$ proves there are self-adjoint operators in $\mathcal{I}$ that are not $U_{\mathcal{I}}(\mathcal{H})$-diagonalizable. Hence the following.

**A necessary condition for $U_{\mathcal{I}}(\mathcal{H})$-diagonalization.**

For the following theorem we recall that for any operator ideal $\mathcal{I}$ in $\mathcal{B(H)}$, the operator ideal $\mathcal{I}^2$ is defined as the linear span of the operator set $\{TS \mid T, S \in \mathcal{I}\}$ and so one has $\mathcal{I}^2 \subseteq \mathcal{I}$.

**Theorem 3.10.** Let $\mathcal{I}$ be any operator ideal in $\mathcal{B(H)}$. If an operator $X \in \mathcal{I}$ is $U_{\mathcal{I}}(\mathcal{H})$-diagonalizable, that is, $WXW^{-1} = D$ for some operators $W \in U_{\mathcal{I}}(\mathcal{H})$ and $D \in \mathcal{D}$, then $X$ is a normal operator and $X − D \in \mathcal{I}^2$. 


More on the size of necessary. More precisely, by using Corollary 3.9 for \( I \)
then \( X \in I \). Proposition 3.13. There exists an injective linear mapping
\( D \) of the set \( \mathbb{R} \) and \( K \in I \), one obtains \( X - D \in I^2 \). \( \Box \)

Remark 3.11. There is a little more to Theorem 3.10 than meets the eye. Re-
framing it creates a context for a possible converse as follows.

Suppose \( X \in I \) is normal so that \( WXW^{-1} = D \) for some \( W \in U(H) \) and \( D \in D \).
Then \( X - D \in I^2 \). The converse question is: Does \( X - D \in I^2 \) imply \( W \) can be
chosen in \( U_7(H) \)? The answer to this is no.

The reason for this is that, fixing an operator \( X \in D(v) \cap I^2 \), \( v \neq b \), using
Proposition 3.5, \( WXW^{-1} \notin D \) for all \( W \in U_K(H) \) (hence also for all \( W \in U_7(H) \)).
But then \( X, D \in I^2 \), so \( X - D \in I^2 \). But \( WXW^{-1} \notin D \) hence \( WXW^{-1} \neq D \) for
any \( W \in U_7(H) \) in particular.

A sufficient condition for \( U_{\Theta_2(H)} \)-diagonalizability.

Remark 3.12. For the sake of completeness, let us mention that in the case when
\( I = \Theta_2(H) \) is the Hilbert-Schmidt ideal, some sufficient conditions for \( U_7(H) \)-
diagonalizability were provided in [Hi85] Th. 1 as follows.

Let \( X \in I^a \) and denote \( x_{ij} = (Xb_{ij}, b_i) \) for all \( i, j \geq 1 \). If there exist \( \rho, s \in \mathbb{R} \)
where \( 0 < \rho < 1 \) and \( 0 < s \leq 3(1 - \rho)/100 \) such that
\[ (\forall j \geq 1) \quad |x_{jj+1}| \leq \rho |x_{jj}| \]
and
\[ (\forall i, j \geq 1, i \neq j) \quad |x_{ij}|^2 \leq \frac{s^2}{(ij)^2} \cdot |x_{ii}x_{jj}| \]
then \( WXW^{-1} \in D \) for some \( W \in U_7(H) \).

The above sufficient conditions for \( U_7(H) \)-diagonalizability are by no means
necessary. More precisely, by using Corollary 3.9 for \( I = \Theta_2(H) \), one can find
\( X \in I^a \setminus D^a \), and for such a self-adjoint operator \( X \) there exist no \( \rho, s \in \mathbb{R} \)
satisfying the above conditions, since \( X \) fails to be \( U_7(H) \)-diagonalizable.

More on the size of \( D_7 \) under a certain homomorphism of triple systems.

In the case when \( I^2 \neq I \), Proposition 3.13 below is related to the part of
Question 3.13 on \( D^a_7 \subset I^a \), in as much as it provides some information on the size
of the set \( D_7 \).

Proposition 3.13. There exists an injective linear mapping \( \Pi: \mathcal{B}(H) \to \mathcal{B}(H) \)
with the properties:

(i) \( \Pi \) is a homomorphism of triple systems, that is, for all \( R, S, T \in \mathcal{B}(H) \) we
have \( \Pi(RST) = \Pi(R)\Pi(S)\Pi(T) \).

(ii) If \( S, T \in \mathcal{B}(H) \), then \( ST = TS \) if and only if \( \Pi(S)\Pi(T) = \Pi(T)\Pi(S) \).

(iii) For every \( T \in \mathcal{B}(H) \), \( \Pi(T^+) = \Pi(T)^+ \).

(iv) \( \mathcal{B}(H)^+ \cap \text{Ran} \Pi = \{0\} \), i.e., \( \text{Ran} \Pi \) contains no nonzero positive operators.

(v) If \( I_1 \) and \( I_2 \) are any operator ideals in \( \mathcal{B}(H) \), then \( I_1 \subseteq I_2 \) if and only if
\( \Pi(I_1) \subseteq \Pi(I_2) \).
(vi) For every operator ideal \( I \) in \( \mathcal{B}(\mathcal{H}) \) we have \( \Pi(I) \subseteq I \), \( \Pi^{-1}(I) \subseteq I \), and \( \Pi^{-1}(\mathcal{D}_X) \subseteq \mathcal{T}^2 \), where for any \( \Pi^{-1}(A) := \{ T \in \mathcal{B}(\mathcal{H}) \mid \Pi(T) \in A \} \) for any \( A \subseteq \mathcal{B}(\mathcal{H}) \).

**Proof.** Express the vectors in \( \widetilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H} \) as two vertically listed column vectors and the operators in \( \mathcal{B}(\widetilde{\mathcal{H}}) \) as \( 2 \times 2 \) operator matrices with entries in \( \mathcal{B}(\mathcal{H}) \). Define in terms of the fixed basis \( b = \{ b_n \}_{n \geq 1} \) of \( \mathcal{H} \) the unitary operator \( \Delta: \mathcal{H} \rightarrow \widetilde{\mathcal{H}} \) as

\[
\Delta(b_{2n-1}) = \begin{pmatrix} b_n \\ 0 \end{pmatrix} \quad \text{and} \quad \Delta(b_{2n}) = \begin{pmatrix} 0 \\ b_n \end{pmatrix} \quad \text{for all } n \geq 1.
\]

Define the operator ideal \( \widetilde{I} := \Delta I \Delta^{-1} \) in \( \mathcal{B}(\widetilde{\mathcal{H}}) \). Recall that unitary operators between Hilbert spaces in this way preserve \( s \)-numbers and hence preserve operator ideals, that is, \( X \mapsto \Delta X \Delta^{-1} \) maps the class of operator ideals in \( \mathcal{B}(\mathcal{H}) \) one-to-one inclusion preserving onto the class of operator ideals in \( \mathcal{B}(\widetilde{\mathcal{H}}) \). We will check that the injective linear mapping

\[
\Pi: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}), \quad T \mapsto \Delta^{-1} \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} \Delta
\]

satisfies the conditions required in the statement. Conditions (i)–(iii) and (v) are straightforward (for (iii) note that \( \Delta^* = \Delta^{-1} \) since \( \Delta \) is unitary).

Condition (vi) is a direct consequence of [BRT07, Lemma 6.3(i)] or can be proved directly by noting that, since \( \Delta \) is unitary, it suffices to check that for every nonzero \( T \in \mathcal{B}(\mathcal{H}) \) the operator \( \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} \) fails to be positive. And this can be obtained by using the special unitary \( U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \) for which \( U \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} U^* = \begin{pmatrix} -T & 0 \\ 0 & T \end{pmatrix} \), and it is clear that the latter \( 2 \times 2 \) block matrix is neither positive nor negative if \( T \neq 0 \).

To check that Condition (vi) is also satisfied, note that \( \Pi(I) \subseteq \mathcal{I} \) and \( \Pi^{-1}(\mathcal{I}) \subseteq \mathcal{I} \) follow directly by the definition of \( \Pi \). Next, for \( A \in \Pi^{-1}(\mathcal{D}_X) \), one needs to show \( A \in \mathcal{T}^2 \). Define

\[
\widetilde{X} := \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \in \mathcal{B}(\widetilde{\mathcal{H}})
\]

and set \( X := \Delta^{-1} \widetilde{X} \Delta \). Since \( X = \Pi(A) \in \mathcal{D}_X \), one has \( W = 1 + K \in \mathcal{U}_X(\mathcal{H}) \) for which \( (I + K)X(I + K^*) = D' \in \mathcal{D} \) and \( K \in \mathcal{I} \) implying

\[
\Delta(1 + K)\Delta^{-1} \Delta \Delta^{-1} \Delta(1 + K)\Delta^{-1} = \Delta D' \Delta^{-1}.
\] (3.3)

Then set \( U_{\mathcal{Z}}(\widetilde{H}) := U(\widetilde{H}) \cap (1 + \widetilde{I}) \) where \( U(\widetilde{H}) \) denotes the group of unitary operators on \( \widetilde{H} \). Notice that

\[
\Delta(1 + K)\Delta^{-1} = 1 + \Delta K \Delta^{-1} \in U_{\mathcal{Z}}(\widetilde{H}),
\]

\[
\widetilde{X} = \Delta X \Delta^{-1} \in \widetilde{I},
\]

\[
\Delta(1 + K^*)\Delta^{-1} = 1 + \Delta K^* \Delta^{-1} \in U_{\mathcal{Z}}(\widetilde{H}).
\]

Also notice that \( \Delta D' \Delta^{-1} \) is a diagonal operator in \( \mathcal{B}(\widetilde{H}) \) relative to the basis \( \{ \begin{pmatrix} b_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b_n \end{pmatrix} \mid n \geq 1 \} \) because of the particular way the operator \( \Delta \) was earlier
defined. So from Equation (3.3) one obtains for \( \widetilde{X} \in \mathcal{I} \), a unitary operator \( \widetilde{W} := 1 + \Delta K \Delta^{-1} \in \mathcal{U}(\mathcal{H}) \) for which \( \widetilde{W} \widetilde{X} \widetilde{W}^{-1} = \Delta D' \Delta^{-1} =: D'' \).

Since \( \Pi(A) \) is a normal compact operator, \( \widetilde{X} \) is a normal compact operator. So applying Theorem 3.10 in the \( B(\mathcal{H}) \) setting to \( \widetilde{W} \widetilde{X} \widetilde{W}^{-1} \) relative to the basis \( \left\{ \left( \begin{smallmatrix} b_n & 0 \\ 0 & b_n \end{smallmatrix} \right) \right\} \), one obtains \( \widetilde{X} - D'' \in \mathcal{I}^2 \). Diagonals in the \( B(\mathcal{H}) \) setting are regarded as direct sums of diagonals in \( B(\mathcal{H}) \). Let us write

\[
D'' = \begin{pmatrix} D''_1 & 0 \\ 0 & D''_2 \end{pmatrix}, \quad \text{hence} \quad \widetilde{X} - D'' = \begin{pmatrix} -D''_1 & A \\ A & -D''_2 \end{pmatrix} \in \mathcal{I}^2.
\]

For \( \tilde{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in U(\mathcal{H}) \) one has

\[
\begin{pmatrix} 0 & 2A \\ 2A & 0 \end{pmatrix} = \begin{pmatrix} -D''_1 & A \\ A & -D''_2 \end{pmatrix} - \tilde{S}^* \begin{pmatrix} D''_1 & A \\ A & -D''_2 \end{pmatrix} \tilde{S} = \widetilde{X} - D'' - \tilde{S}^*(\widetilde{X} - D'') \tilde{S} \in \mathcal{I}^2
\]

and therefore \( A \in \mathcal{I}^2 \), which completes the proof. \( \square \)

**Remark 3.14.** As mentioned immediately prior to Proposition 3.13, in the case when \( \mathcal{I}^2 \neq \mathcal{I} \), Proposition 3.13(vi) is related to the part of Question 3.3 on whether or not \( D^{sa}_I \subseteq I^{sa} \), in as much as it provides some information on the size of the set \( D_I \). We can then begin to develop intuition on how \( D^{sa}_I \) sits inside of the real linear space \( I^{sa} \) beyond this set inequality in Corollary 3.9. As for instance how \( \tilde{S} \), \( \Pi(\tilde{S}) \) sits inside the real linear space \( I^{sa} \).

More precisely as related to \( \Pi \), it follows by Proposition 3.13(vi) and injectivity respectively that

\[
D_I \cap \Pi(I) \subseteq \Pi(I^2) \subseteq \Pi(I).
\]

Moreover,

\[
\{0\} \neq D^{sa}_I \cap \Pi(I^{sa}) \subseteq \Pi(I^{sa}).
\]

Indeed, since the mapping \( \Pi: I \rightarrow I \) is linear and injective and preserves self-adjoints (see Proposition 3.13(iii)), the image of \( I^{sa} \), \( \Pi(I^{sa}) \), is an infinite dimensional real linear subspace of \( I^{sa} \) with the property that the real linear subspace spanned by \( D^{sa}_I \cap \Pi(I^{sa}) \) is a proper real linear subspace of \( \Pi(I^{sa}) \). It is proper because \( D^{sa}_I \cap \Pi(I^{sa}) = \Pi(I^{sa}) \) implies by Proposition 3.13(iii) that \( I^2 \supseteq \Pi^{-1}(D^{sa}_I) \cap I^{sa} = I^{sa} \) contradicting \( I^2 \not\subseteq I^{sa} \). Note also that we have \( D^{sa}_I \cap \Pi(I^{sa}) \neq \{0\} \) since by (3.2) we have \( \{0\} \neq \mathcal{F}(\mathcal{H})^{sa} = D^{sa}_{\mathcal{F}(\mathcal{H})} \subseteq D^{sa}_I \), hence by Proposition 3.13(iii), \( \{0\} \subseteq \Pi(\mathcal{F}(\mathcal{H})^{sa}) \subseteq \mathcal{F}(\mathcal{H})^{sa} \subseteq D^{sa}_I \cap \Pi(I^{sa}) \).

As another geometric feature, it follows by Proposition 3.13(iv) that the aforementioned real linear subspace \( \Pi(I^{sa}) \) meets the positive cone \( I^+ \) only at its vertex 0.

**Example 3.15.** If \( 0 < p < \infty \) and \( I = \mathcal{S}_p(\mathcal{H}) \) is the Schatten \( p \)-ideal, then Proposition 3.13 provides some nontrivial information on the class \( D_I \). In fact, we have \( I^2 = \mathcal{S}_{p/2}(\mathcal{H}) \), hence \( I^2 \neq I \) and so the above Remark 3.14 applies.
4. Conjugacy classes of Cartan subalgebras of operator ideals

Cartan subalgebras of various types of infinite-dimensional Lie algebras have been studied extensively (see for instance [Sch60], [Sch61], [BP66], [dlH72], [St75], [NP03], [Re08], and the references therein). In this section we focus on Cartan subalgebras of operator ideals. More specifically, we first wish to raise the classification problem for the conjugacy classes of these Cartan subalgebras. (See [BPW14b, Rem. 4.2] for the case of \( \mathcal{B}(\mathcal{H}) \).) Our main result, the case of proper ideals of \( \mathcal{B}(\mathcal{H}) \), is recorded as Theorem 4.9 on the uncountability of such conjugacy classes.

The set of conjugacy classes of Cartan subalgebras. The relevant facts concerning this issue in the case of the operator ideal \( I = \mathcal{B}(\mathcal{H}) \) can be found in [BPW14b, Rem. 4.2]. For the main general Theorem 4.9 below for arbitrary proper ideals, it is convenient to introduce the following.

Definition 4.1. For every proper operator ideal \( I \) in \( \mathcal{B}(\mathcal{H}) \) denote by Cartan(\( I \)) the set of all Cartan subalgebras of \( I \), that is, the set of all maximal abelian self-adjoint subalgebras of the associative \( \ast \)-algebra \( I \).

Furthermore, denote by Decomp(\( \mathcal{H} \)) the set of all sets of one-dimensional, mutually orthogonal subspaces of the Hilbert space \( \mathcal{H} \) whose linear span is dense in \( \mathcal{H} \). (I.e., all one-dimensional subspace orthogonal decompositions of \( \mathcal{H} \).) That is, for every \( S \in \text{Decomp}(\mathcal{H}) \) we have the orthogonal direct sum decomposition \( \mathcal{H} = \bigoplus_{V \in S} V \) with one-dimensional summands \( V \).

Remark 4.2. We should point out that this Definition 4.1 terminology agrees with that introduced in [dlH72, Sect. I.3] for the Schatten ideals, the finite-rank operator ideal, and for \( \mathcal{B}(\mathcal{H}) \), where the Cartan subalgebras were actually defined as maximal abelian self-adjoint subalgebras of a complex Lie algebra \( g \) endowed with an antilinear involution. If \( g \) is finite-dimensional, it can be recovered from any of its Cartan subalgebras \( \mathfrak{h} \) as follows. Denote the eigenspaces for the adjoint action of \( \mathfrak{h} \) on \( g \) by

\[
\mathfrak{g}^\alpha := \{ X \in g \mid (\forall H \in \mathfrak{h}) \quad [H, X] = \alpha(H)X \} \quad \text{for} \quad \alpha: \mathfrak{h} \to \mathbb{C} \text{ linear}
\]

and consider the corresponding set of roots \( \Delta(\mathfrak{g}, \mathfrak{h}) := \{ \alpha \in \mathfrak{h}^* \mid \mathfrak{g}^\alpha \neq \{0\} \} \). Then \( \mathfrak{h} = \mathfrak{g}^0 = \{ X \in g \mid [\mathfrak{h}, X] = \{0\} \} \) and there exists the root space decomposition

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \setminus \{0\}} \mathfrak{g}^\alpha \tag{4.1}
\]

by [Kn02, Equations (2.16) and (2.22)]. If however \( g \) is infinite-dimensional, then (4.1) may not hold true, and additional structure on \( g \) is necessary in order to formulate useful versions of that root space decomposition. For instance, if \( g = \mathcal{M} \) is a \( C^\ast \)- or von Neumann algebra, then the regularity property of a masa \( \mathfrak{h} = \mathcal{C} \) as expressed in (1.1) should be viewed as a variant of root space decomposition of \( \mathcal{M} \) with respect to \( \mathcal{C} \), in as much as in both situations the ambient algebra is recovered by using only the Cartan subalgebra under consideration.

In the present situation of the Lie algebra defined by the operator ideal \( I \) with the Lie bracket \( [X, Y] = XY - YX \), a Cartan subalgebra of \( I \) is a linear self-adjoint subspace \( \mathcal{C} \) of \( I \) which is maximal under the condition that for all \( X, Y \in \mathcal{C} \) we have \( [X, Y] = 0 \). The maximality condition requires however that \( \mathcal{C} \) is actually a subalgebra of the associative algebra \( I \). That is, for all \( X, Y \in \mathcal{C} \) one necessarily
has $XY \in \mathcal{C}$, since otherwise one could replace $\mathcal{C}$ by the associative algebra it generates, and one would thus obtain a larger linear self-adjoint subspace consisting of mutually commuting operators. We thus see that the elements of $\text{Cartan}(\mathcal{I})$ are precisely Cartan subalgebras of $\mathcal{I}$ as defined in our Introduction.

Theorem 4.4 below shows that the Cartan subalgebras of complete separable normed ideals enjoy a regularity property which is quite similar to that of the Cartan subalgebras of $C^*$-algebras (see Remark 2.2 and (1.1) in the Introduction). Its proof is based on the following simple fact which in particular generalizes [dH72, Ch. I, Prop. 3A] from the ideal of finite-rank operators to an arbitrary ideal.

**Proposition 4.3.** Let $\mathcal{I}$ be any proper operator ideal in $\mathcal{B}(\mathcal{H})$. The mapping

$$
\Psi: \text{Decomp}(\mathcal{H}) \to \text{Cartan}(\mathcal{I}), \quad S \mapsto \{ T \in \mathcal{I} \mid (\forall V \in S) \quad TV \subseteq V \}
$$

is one-to-one and onto. For every $\mathcal{C} \in \text{Cartan}(\mathcal{I})$, the set $\Psi^{-1}(\mathcal{C})$ is the set of range spaces of minimal projections of the commutative $C^*$-algebra of compact operators generated by $\mathcal{C}$.

**Proof.** We first check that if $S \in \text{Decomp}(\mathcal{H})$ then $\Psi(S) \in \text{Cartan}(\mathcal{I})$. It is clear that $\Psi(S)$ is an abelian self-adjoint subalgebra of $\mathcal{I}$, so we still have to check that it is maximal with these properties. To this end let $T \in \mathcal{I}$ with $[T, \Psi(S)] = \{0\}$, and so it suffices to prove $T \in \Psi(S)$. Indeed, for arbitrary $V \in S$, if we denote by $P_T$ the orthogonal projection onto $V$, then $P_T$ is a rank-one operator. Moreover, since $S$ is a set of mutually orthogonal subspaces of $\mathcal{H}$, it follows that each element of $S$ is invariant under $P_T$, hence $P_T \in \Psi(S)$. The commutator zero assumption on $T$ then implies $TP_T = P_T T$. Since $V \in S$ is arbitrary, we obtain $T \in \Psi(S)$. We thus see that $\Psi(S)$ is a maximal abelian self-adjoint subalgebra of $\mathcal{I}$, that is, $\Psi(S) \in \text{Cartan}(\mathcal{I})$.

Next, for arbitrary $\mathcal{C} \in \text{Cartan}(\mathcal{I})$, denote by $\Phi(\mathcal{C})$ the set of all minimal projections in the commutative $C^*$-algebra of compact operators generated by $\mathcal{C}$. Since every $C^*$-algebra of compact operators is the direct sum of full algebras of compact operators on mutually orthogonal subspaces of $\mathcal{H}$ (see for instance [Da96, Th. I.10.8]), it is then straightforward to prove that the dimensions of all these subspaces are one by using the commutativity of $\mathcal{C}$, and hence that $\Phi(\mathcal{C}) \in \text{Decomp}(\mathcal{H})$.

Moreover, it is not difficult to check that the mappings $\Psi$ and $\Phi$ are inverse to each other.

**Theorem 4.4.** If $\mathcal{I}$ is a complete separable normed ideal and $\mathcal{C} \in \text{Cartan}(\mathcal{I})$ with the normalizer $U_{\mathcal{I},\mathcal{C}}(\mathcal{H}) := \{ W \in U_{\mathcal{I}}(\mathcal{H}) \mid WCW^{-1} = \mathcal{C} \}$, then the linear span of

$$
\{ WD \mid W \in U_{\mathcal{I},\mathcal{C}}(\mathcal{H}), D \in \mathcal{C} \}
$$

is a dense linear subspace of $\mathcal{I}$.

**Proof.** Use Proposition 4.3 to find $S = \{ V_n \mid n \geq 1 \} \in \text{Decomp}(\mathcal{H})$ with $\Psi(S) = \mathcal{C}$. Then for every $n \geq 1$ pick any $v_n \in V_n$ with $\| v_n \| = 1$. In this way we obtain an orthonormal basis $v := \{ v_n \}_{n \geq 1}$ of $\mathcal{H}$ with the property that $\mathcal{C}$ is the set of all operators in $\mathcal{I}$ that are diagonal with respect to the orthonormal basis $v$. For every integer $n \geq 1$ let $P_n$ be the orthogonal projection of $\mathcal{H}$ onto span\{$v_1, \ldots, v_n$\}, hence $P_n \in F(\mathcal{H})$. Also denote

$$
\mathcal{Q} := \{ WD \mid W \in U_{\mathcal{I},\mathcal{C}}(\mathcal{H}), D \in \mathcal{C} \} \subset \mathcal{I}.
$$
By using the classical variant of Proposition \[4.10\] for \( \dim \mathcal{H} < \infty \), it is easily checked and in fact is well known that
\[
P_nTP_n = \text{span}(P_nQP_n) \subseteq \text{span} \ Q.
\]
(For a more general case see for instance \[GW09\] Th. 3.1.1.)

On the other hand, since \( \mathcal{I} \) is a complete separable normed ideal, it follows by \[GK69\] Th. 6.3 in Ch. III that for every \( T \in \mathcal{I} \) we have \( \lim_{n \to \infty} \| T - P_nTP_n \|_{\mathcal{I}} = 0 \). Then the conclusion of the above paragraph shows that \( \text{span} \ Q \) is dense in \( \mathcal{I} \). \( \square \)

**Remark 4.5** \((\mathcal{I})\text{-equivalent bases.}\) Let \( \mathcal{I} \) be an operator ideal in \( \mathcal{B}(\mathcal{H}) \). Proposition \[4.3\] suggests to us an equivalence relation on the set of all orthonormal bases in \( \mathcal{H} \). Namely, if \( e = \{e_n\}_{n \geq 1} \) and \( f = \{f_n\}_{n \geq 1} \) are two orthonormal bases in \( \mathcal{H} \), then we say they are \( \mathcal{I}\text{-equivalent to each other} \) if there exists \( W \in \mathcal{U}(\mathcal{H}) \) with \( We_n = f_n \) for all \( n \geq 1 \). It seems an interesting problem to provide criteria, involving the vectors only, ensuring when two bases are equivalent in this sense, and to study the corresponding equivalence classes of bases for particular choices of operator ideals.

In the case of the Hilbert-Schmidt ideal \( \mathcal{I} = \Theta_2(\mathcal{H}) \), it is easily checked that the bases \( e \) and \( f \) are \( \mathcal{I}\text{-equivalent to each other} \) if and only if
\[
\sum_{n \geq 1} \|e_n - f_n\|^2 < \infty.
\]
Also recall from \[Han82\] Probl. 12] that if \( e = \{e_n\}_{n \geq 1} \) and \( f = \{f_n\}_{n \geq 1} \) are any orthonormal sequences in \( \mathcal{H} \) satisfying \[4.2\], then sequence \( e \) spans \( \mathcal{H} \) (and hence is an orthonormal basis) if and only if so does the sequence \( f \).

The relationship between this \( \mathcal{I}\text{-equivalence} \) and the setting of Proposition \[4.3\] relies on the \( \mathcal{U}(\mathcal{H})\text{-equivariant correspondence} \{e_n\}_{n \geq 1} \mapsto \{Ce_n\}_{n \geq 1} \) from the set of all orthonormal bases in \( \mathcal{H} \) onto the set \( \text{Decomp}(\mathcal{H}) \). (A \( \mathcal{G}\text{-equivariant} \) map \( T \): \( X \to Y \) is a map for which \( T(gx) = gTx \) for all \( g \in \mathcal{G} \) and \( x \in X \), for any group \( \mathcal{G} \) acting on the sets \( X, Y \), with its corresponding group actions denoted by juxtaposition, hence \( (g,x) \mapsto gx \) and \( (g,y) \mapsto gy \), respectively.)

**Remark 4.6.** Let \( \mathcal{I} \) be a proper operator ideal in \( \mathcal{B}(\mathcal{H}) \), \( S \in \text{Decomp}(\mathcal{H}) \), and define
\[
E_S : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \quad E_S(T) = \sum_{V \in S} P_VTP_V.
\]
Since the map \( \Psi \) in Proposition \[4.3\] takes values in \( \text{Cartan}(\mathcal{I}) \) as indicated in its statement, it follows that \( \mathcal{I} \cap \text{Ran} \ E_S \subseteq \text{Cartan}(\mathcal{I}) \).

Moreover one has the characterization \( E_S(\mathcal{I}) = \mathcal{I}^\text{-am} \cap \text{Ran} \ E_S \) where \( \mathcal{I}^\text{-am} \) denotes the \text{arithmetic mean closure} of \( \mathcal{I} \), that is, the set of all operators \( T \in \mathcal{B}(\mathcal{H}) \) for which there exists \( K \in \mathcal{I} \) such that for every integer \( n \geq 1 \) we have \( s_1(T) + \cdots + s_n(T) \leq s_1(K) + \cdots + s_n(K) \). Then \( \mathcal{I}^\text{-am} \cap \text{Ran} \ E_S = E_S(\mathcal{I}) \) by \[KW12\ Cor. 4.4\].

Consequently, if the ideal \( \mathcal{I} \) is arithmetic mean closed, that is, \( \mathcal{I}^\text{-am} = \mathcal{I} \), then for every \( S \in \text{Decomp}(\mathcal{H}) \) we have \( E_S(\mathcal{I}) \subseteq \text{Cartan}(\mathcal{I}) \).

**Definition 4.7.** The natural action of the full unitary group \( U(\mathcal{H}) \) on the set \( \text{Decomp}(\mathcal{H}) \) is
\[
\alpha : U(\mathcal{H}) \times \text{Decomp}(\mathcal{H}) \to \text{Decomp}(\mathcal{H}), \quad (W, S) \mapsto \alpha_W(S) := \{W(V) \mid V \in S\}
\]
and for every $S \in \text{Decomp}(\mathcal{H})$ we define
\[ U(\mathcal{H})_S := \{ W \in U(\mathcal{H}) \mid \alpha_W(S) = S \}. \tag{4.3} \]

For any proper operator ideal $I$ in $\mathcal{B}(\mathcal{H})$ the group action of $U(\mathcal{H})$ on $\text{Cartan}(I)$ is:
\[ \beta : U(\mathcal{H}) \times \text{Cartan}(I) \to \text{Cartan}(I), \quad (W, \mathcal{C}) \mapsto \beta_W(\mathcal{C}) := WCW^{-1}. \]

**Remark 4.8.** It is easily seen that for every proper ideal $I$ in $\mathcal{B}(\mathcal{H})$ the bijective mapping $\Psi$ in Proposition 4.3 is $U(\mathcal{H})$-equivariant, hence the diagram
\[ \text{Decomp}(\mathcal{H}) \xrightarrow{\alpha} \text{Decomp}(\mathcal{H}) \]
\[ \downarrow \Psi \quad \downarrow \Psi \]
\[ U(\mathcal{H}) \times \text{Cartan}(I) \xrightarrow{\beta} \text{Cartan}(I) \]
commutes. This simple fact allows one to derive properties of the action $\beta$ from properties of the action $\alpha$. For instance, one can draw the following direct consequences for any Cartan subalgebra $\mathcal{C}$ of $I$, by merely considering their corresponding properties of $S := \Psi(\mathcal{C}) \in \text{Decomp}(\mathcal{H})$:

(i) One has $\{ \beta_W(\mathcal{C}) \mid W \in U(\mathcal{H}) \} = \text{Cartan}(I)$, since clearly $\{ \alpha_W(S) \mid W \in U(\mathcal{H}) \} = \text{Decomp}(\mathcal{H})$.

(ii) One has $\{ W \in U(\mathcal{H}) \mid \beta_W(\mathcal{C}) = \mathcal{C} \} = U(\mathcal{H})_S$, using (4.3).

Moreover, if one denotes by $[\text{Cartan}(I)]$ the set of all $U_I(\mathcal{H})$-conjugacy classes of Cartan subalgebras of $I$, and for every $\mathcal{C} \in \text{Cartan}(I)$ and $A \subseteq U(\mathcal{H})$ one denotes $\beta_A(\mathcal{C}) := \{ \beta_V(\mathcal{C}) \mid V \in A \}$, then for any fixed $\mathcal{C}_0 \in \text{Cartan}(I)$ it is easily seen that the map
\[ U(\mathcal{H})/U_I(\mathcal{H}) \to [\text{Cartan}(I)], \quad WU_I(\mathcal{H}) \mapsto \beta_W(\beta_{U_I(\mathcal{H})}(\mathcal{C}_0)) \tag{4.4} \]
is well-defined and bijective, hence can be regarded as a parameterization of the set $[\text{Cartan}(I)]$, where $WU_I(\mathcal{H}) := \{ WV \mid V \in U_I(\mathcal{H}) \}$ for all $W \in U(\mathcal{H})$. The above map takes values as indicated since $\beta_{U_I(\mathcal{H})}(\mathcal{C}_0)$ is the $U_I(\mathcal{H})$-conjugacy class of $\mathcal{C}_0$, and for every $W \in U(\mathcal{H})$ and $V \in U_I(\mathcal{H})$ one has $\beta_W(\beta_V(\mathcal{C}_0)) = \beta_{WVW^{-1}}(\beta_W(\mathcal{C}_0))$. But the map $V \mapsto WVW^{-1}$ is a bijection of the set $U_I(\mathcal{H})$ onto itself since $I$ is an ideal of $\mathcal{B}(\mathcal{H})$, hence we obtain $\beta_W(\beta_{U_I(\mathcal{H})}(\mathcal{C}_0)) = \beta_{U_I(\mathcal{H})}(\mathcal{C})$, where $\mathcal{C} := \beta_W(\mathcal{C}_0)$ runs over $\text{Cartan}(I)$ when $W$ runs over $U(\mathcal{H})$ (by Remark 4.3).

**Theorem 4.9.** If $I$ is a proper operator ideal in $\mathcal{B}(\mathcal{H})$, then there exist uncountably many $U_I(\mathcal{H})$-conjugacy classes of Cartan subalgebras of $I$.

**Proof.** Let $\{ v(t) \}_{0 \leq t < \pi/2}$ be the family of orthonormal bases in $\mathcal{H}$ provided by Corollary 3.8. For every $t \in [0, \pi/2)$ we have $\{ \mathcal{C} v_n(t) \}_{n \geq 1} \in \text{Decomp}(\mathcal{H})$. Hence by Proposition 4.3 one obtains $\mathcal{C}(t) := \Psi(\{ \mathcal{C} v_n(t) \}_{n \geq 1}) \in \text{Cartan}(I)$. Equivalently, if we denote by $D(v(t))$ the set of all diagonal operators with respect to the orthonormal basis $v(t)$, then one has $\mathcal{C}(t) = D(v(t)) \cap I$. We will prove that the uncountable family $\{ \mathcal{C}(t) \}_{0 \leq t < \pi/2}$ is a (possibly non-exhaustive) system of representatives for distinct $U_I(\mathcal{H})$-conjugacy classes of Cartan subalgebras of $I$.

To this end let $s, t \in [0, \pi/2)$ be arbitrary with $s \neq t$. We need to show that for every $W \in U_I(\mathcal{H})$ one has $\beta_W(\mathcal{C}(t)) \neq \mathcal{C}(t)$. Two cases may occur:

Case 1. If $\mathcal{F}(\mathcal{H}) \not\subseteq I$, then there exists an operator $X \in D(v(s)) \cap I = \mathcal{C}(s)$ with spectral multiplicities one. It then follows by Corollary 3.8.4 along with
Corollary 3.6 that for every $W \in U_T(H) \subseteq U_{K(H)}$, $WXW^{-1} \not\in D(v^{(t)})$, i.e.,
$
\beta_W(C^{(s)}) \neq C^{(t)}.
$

Case 2. If $T = F(H)$, assume $\beta_W(C^{(s)}) = C^{(t)}$ for some $W \in U_T(H)$, and this will lead to a contradiction. In fact, define $X \in B(H)$ by $Xv_n^{(t)} = (1/n)v_n^{(t)}$ for every $n \geq 1$, and for arbitrary $k \geq 1$ define $X_k \in F(H)$ by $X_kv_n^{(t)} = (1/n)v_n^{(t)}$ if $1 \leq k \leq n$ and $X_kv_n^{(t)} = 0$ if $k > n$. Then for $k \geq 1$ one has $X_k \in C^{(t)}$, hence
$\gamma^{-1}X_kW \in \beta_W^{-1}(C^{(t)}) = C^{(s)} \subseteq D(v^{(s)})$ by our assumption $\beta_W(C^{(s)}) = C^{(t)}$.
Since one knows $\lim_{k \to \infty} ||X_k - X|| = 0$ and $D(v^{(s)})$ is closed in $B(H)$ with respect to the operator norm topology, then it follows that $W^{-1}XW \in D(v^{(s)})$. On the other hand $X \in D(v^{(t)})$ is a self-adjoint operator with spectral multiplicities one and $W \in U_T(H) \subseteq U_{K(H)}$, hence it follows by Corollary 3.5[H] along with Corollary 3.6 that $W^{-1}XW \not\in D(v^{(s)})$, and this achieves the contradiction. □

The following proposition is helpful for computing the normalizers of the Cartan subalgebras of operator ideals. This could be proved by using Proposition 5.1 but we prefer to provide here an alternative method of proof, based on matrix computations. Recall that $D$ stands for the set of all diagonal operators with respect to a fixed orthonormal basis in $H$.

**Proposition 4.10.** If the operator $T \in U(H)$ satisfies $T(D \cap F(H))T^* \subseteq D$, then there exist a unique permutation $\sigma \in S_{\infty}$ and a unique sequence $u = \langle u_n \rangle_{n=1}^{\infty}$ with $u_n \in T$ for every $n \geq 1$, for which $T = V_0 \text{diag} \, u$.

If moreover $T \in U_{K(H)}$, then $\sigma \in S_{\infty}$.

**Proof.** For $T = (t_{jk})_{j,k \geq 1} \in U(H)$ and $D = \text{diag}(d_1, d_2, d_3, \ldots) \in D \cap F(H)$ one has

$$
T(DT^*) = \begin{pmatrix}
    t_{11} & t_{12} & t_{13} & \ldots
    
    t_{21} & t_{22} & t_{23} & \ldots
    
    t_{31} & t_{32} & t_{33} & \ldots
    
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
    d_1 & 0 & 0 & \cdots
    
    0 & d_2 & 0 & \cdots
    
    0 & 0 & d_3 & \cdots
    
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
= \begin{pmatrix}
    \sum d_j t_{1j}^2 & \sum d_j t_{1j} \bar{t}_{2j} & \sum d_j t_{1j} \bar{t}_{3j} & \cdots
    
    \sum d_j t_{2j} \bar{t}_{1j} & \sum d_j t_{2j}^2 & \sum d_j t_{2j} \bar{t}_{3j} & \cdots
    
    \sum d_j t_{3j} \bar{t}_{1j} & \sum d_j t_{3j} t_{2j} & \sum d_j |t_{3j}|^2 & \cdots
    
    \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

The condition that the off-diagonal coefficients of $T(DT^*)$ vanish for every $D \in D \cap F(H)$ is easily verified to be equivalent to

if $j, k, \ell \geq 1$ and $k \neq \ell$, then $t_{kj}t_{\ell j} = 0$.

Now let $j \geq 1$ be fixed for the moment. Since $T \in U(H)$, we have $\text{Ker} \, T = \{0\}$, hence there exists $k \geq 1$ for which $t_{kj} \neq 0$, and then the above condition implies $t_{\ell j} = 0$ whenever $\ell \neq k$. This shows that this subscript $k$ with $t_{kj} \neq 0$ is uniquely determined by $j$, hence we may write $k = \sigma(j)$. We thus obtain a function

$\sigma: \{1, 2, \ldots\} \to \{1, 2, \ldots\}$ with the property that for every $j, k \geq 1$ we have $t_{kj} \neq 0$ if and only if $k = \sigma(j)$. Note that the function $\sigma$ is surjective, since otherwise there exists a row in $T$ consisting only of zeros, and then $\text{Ker} \, T^* \neq \{0\}$. Moreover $\sigma$ is injective since otherwise two columns in the matrix of $T$ would be linearly dependent, which is not possible since $\text{Ker} \, T = \{0\}$. Consequently $\sigma$ is a bijection,
that is, \( \sigma \in \mathbb{S}_\infty \). In summary, the unitary matrix for \( T = (t_{ij})_{i,j \geq 1} \) has \( t_{ij} = 0 \) for all \( i \) except \( i = \sigma(j) \), in which case \( |t_{\sigma(j)j}| = 1 \).

If for \( n \geq 1 \) we choose \( u_n = t_\sigma(n)n \), then \( u_n \in \mathbb{T} \) since \( T \) is unitary and \( u_n \) is the unique nonzero entry on the \( n \)-th column of \( T \). Hence \( T = V_n \text{diag} u \).

Finally, if in addition \( T \in U_\mathcal{K}(\mathcal{H}) \), then \( \mathcal{K}(\mathcal{H}) \ni T - I = V_n \text{diag} u - 1 \), so \( |u_\sigma(n) - u_n| = \|(T - 1)b_n\| \to 0 \), so \( \sigma(n) = n \) for all but finitely many \( n \), i.e., \( \sigma \in \mathbb{S}_\text{fin} \).

The Corollary 4.13 below points to a rather rich hierarchy of conjugacy classes of Cartan subalgebras in operator ideals. This is obtained as a consequence of the interaction between the lattice of operator ideals and the notion of Cartan subalgebras and their conjugacy classes.

**Remark 4.11.** For later use we record a couple of simple remarks on intersections of Cartan subalgebras with smaller ideals.

(i) For all operator ideals \( \mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{B}(\mathcal{H}) \) and every subalgebra \( \mathcal{C} \in \text{Cartan}(\mathcal{J}) \), one has \( \mathcal{C} \cap \mathcal{I} \in \text{Cartan}(\mathcal{I}) \). This follows directly by Proposition 4.3 using the surjectivity of \( \Psi \).

(ii) If in addition \( \mathcal{J} \) is a separable complete normed ideal, \( \mathcal{C} \in \text{Cartan}(\mathcal{J}) \), \( \mathcal{I} \neq \{0\} \), and another subalgebra \( \mathcal{C}_0 \in \text{Cartan}(\mathcal{J}) \) has the property that the subalgebras \( \mathcal{C} \cap \mathcal{I} \) and \( \mathcal{C}_0 \cap \mathcal{I} \) of \( \mathcal{I} \) are \( U_\mathcal{J}(\mathcal{H}) \)-conjugated to each other, then \( \mathcal{C} \) and \( \mathcal{C}_0 \) are \( U_\mathcal{I}(\mathcal{H}) \)-conjugated to each other (hence \( U_\mathcal{J}(\mathcal{H}) \)-conjugated, since \( U_\mathcal{I}(\mathcal{H}) \subseteq U_\mathcal{J}(\mathcal{H}) \)). Indeed, it easily follows that \( \mathcal{C} \cap \mathcal{F}(\mathcal{H}) \) and \( \mathcal{C}_0 \cap \mathcal{F}(\mathcal{H}) \) are \( \mathcal{J} \)-norm-dense in \( \mathcal{C} \) and \( \mathcal{C}_0 \), respectively. Since \( \mathcal{F}(\mathcal{H}) \subseteq \mathcal{I} \), it then follows that also \( \mathcal{C} \cap \mathcal{I} \) and \( \mathcal{C}_0 \cap \mathcal{I} \) are dense in \( \mathcal{C} \) and \( \mathcal{C}_0 \), respectively. Therefore, if \( V \in U_\mathcal{I}(\mathcal{H}) \) for which \( V(\mathcal{C} \cap \mathcal{I})V^{-1} = \mathcal{C}_0 \cap \mathcal{I} \), via \( \| \cdot \|_\mathcal{J} \geq \| \cdot \| \) and \( \mathcal{J} \)-norm limits one has also \( VCV^{-1} = \mathcal{C}_0 \).

The next theorem negates the converse of Remark 4.11(ii). Its proof is based on a construction that sharpens the one from the proof of Proposition 4.7 below. It would be recovered for the constant sequence \( \theta = (1, 1, \ldots) \) in the proof of Theorem 4.12 below. In fact, recalling Proposition 4.3 one can see that Theorem 4.9 is the limit situation for \( \mathcal{J} = \mathcal{B}(\mathcal{H}) \) of Theorem 4.12 below. Although there is a certain overlap of their proofs, we chose to present these theorems separately since the proof of Theorem 4.9 is simpler and moreover for \( \mathcal{J} = \mathcal{B}(\mathcal{H}) \) we did not define \( \text{Cartan}(\mathcal{J}) \) in order to avoid any confusion with the notion of Cartan subalgebras of von Neumann algebras as discussed in the Introduction.

**Theorem 4.12.** For all operator ideals \( \{0\} \subseteq \mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{B}(\mathcal{H}) \) there exists an uncountable family of Cartan subalgebras of \( \mathcal{J} \) that are pairwise \( U_\mathcal{I}(\mathcal{H}) \)-conjugated to each other but whose intersections with \( \mathcal{I} \) are Cartan subalgebras of \( \mathcal{I} \) that pairwise fail to be \( U_\mathcal{J}(\mathcal{H}) \)-conjugated to each other.

It is easily seen in Theorem 4.12 that the above family of Cartan subalgebras of \( \mathcal{J} \) themselves pairwise fail to be \( U_\mathcal{J}(\mathcal{H}) \)-conjugated to each other, since so do their intersections with \( \mathcal{I} \).

**Proof of Theorem 4.12** The proof proceeds in two steps. The first is a construction for a single operator.

**Step 1.** By using the surjectivity of \( \Psi \) in Proposition 4.3 let \( \mathcal{C} := \mathcal{D} \cap \mathcal{J} \in \text{Cartan}(\mathcal{J}) \), where we recall that this means that we fix an orthonormal basis \( b = \{ \ldots, b_1, b_2, \ldots \} \).
\{b_n\}_{n \geq 1}$ in $\mathcal{H}$ and we denoted by $\mathcal{D}$ the corresponding set of diagonal operators in $\mathcal{B}(\mathcal{H})$ with respect to $b$ (see the beginning of Section 2). At this step we prove that if $\theta = \{\theta_n\}_{n=1}^{\infty}$ is any decreasing sequence in $(0, \pi)$ with $\theta \in \Sigma(\mathcal{J}) \setminus \Sigma(\mathcal{I})$ and $A_\theta \in U(\mathcal{H})$ is the operator defined by (4.5)–(4.6) below, then $A \in U_{\mathcal{J}}(\mathcal{H}) \setminus U_{\mathcal{I}}(\mathcal{H})$. Moreover, for the orthonormal basis $v = \{v_n\}_{n \geq 1}$ in $\mathcal{H}$ defined by $v_n := A b_n$ for every $n \geq 1$, if $\mathcal{D}(v)$ is the corresponding set of diagonal operators in $\mathcal{B}(\mathcal{H})$ with respect to $v$, and we define $\mathcal{C}_0 := \mathcal{D}(v) \cap \mathcal{J}$, then $\mathcal{C}, \mathcal{C}_0 \in \text{Cartan}(\mathcal{J})$ are $U_{\mathcal{J}}(\mathcal{H})$-conjugated but fail to be $U_{\mathcal{I}}(\mathcal{H})$-conjugated to each other.

First let $\theta = \{\theta_n\}_{n=1}^{\infty}$ be any decreasing sequence in $(0, \pi)$ and denote $\lambda_n = e^{i \theta_n} \in \mathbb{T}$ for every $n \geq 1$ and $\lambda = \{\lambda_n\}_{n=1}^{\infty}$. For every integer $r \geq 1$ define the unitary operator

$$A_{\theta_r} = \begin{pmatrix}
\cos \theta_r & -\sin \theta_r \\
\sin \theta_r & \cos \theta_r
\end{pmatrix}$$

matricially and canonically on span$\{b_{2r-1}, b_{2r}\}$ (4.5)

and then define the unitary block diagonal operator

$$A_\theta = \begin{pmatrix}
A_{\theta_1} & 0 & 0 & \cdots \\
0 & A_{\theta_2} & 0 & \cdots \\
0 & 0 & A_{\theta_3} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix} \in U(\mathcal{H}).$$

(4.6)

For arbitrary $r \geq 1$ the operator $A_{\theta_r}$ has the eigenvalues $\{\lambda_r, \bar{\lambda}_r\}$ and $|1 - \lambda_r| = 2 \sin \frac{\theta_r}{2}$. Therefore, by using the fact that $1 - A_{\theta_r}$ is a normal operator, it follows that for an arbitrary operator ideal $\mathcal{L}$ with characteristic set $\Sigma(\mathcal{L})$ one has

$$A_\theta \in U_{\mathcal{L}} \iff \left\langle \sin \frac{\theta_r}{2} \right\rangle_{r=1}^{\infty} \in \Sigma(\mathcal{L})$$

and in particular

$$A_\theta \in U_{\mathcal{J}}(\mathcal{H}) \setminus U_{\mathcal{I}}(\mathcal{H}) \iff \left\langle \sin \frac{\theta_r}{2} \right\rangle_{r=1}^{\infty} \in \Sigma(\mathcal{J}) \setminus \Sigma(\mathcal{I}) \iff \theta \in \Sigma(\mathcal{J}) \setminus \Sigma(\mathcal{I})$$

(4.7)

where the last equivalence is based on $\lim_{t \to 0} \sin t = t$. Since $\mathcal{I} \subset \subset \mathcal{J}$, we can choose the sequence $\theta \in \Sigma(\mathcal{J}) \setminus \Sigma(\mathcal{I})$. This sequence will be fixed throughout this step of the proof and we will denote $A := A_\theta \in U_{\mathcal{J}}(\mathcal{H}) \setminus U_{\mathcal{I}}(\mathcal{H})$.

Now consider the orthonormal basis $v = \{v_n\}_{n \geq 1}$ in $\mathcal{H}$ defined by $v_n := A b_n$ for every $n \geq 1$, let $\mathcal{D}(v)$ be the corresponding set of diagonal operators in $\mathcal{B}(\mathcal{H})$, and define $\mathcal{C}_0 := \mathcal{D}(v) \cap \mathcal{J}$.

We now check that the assertions from this step of the proof hold true for $\mathcal{C}$ and $\mathcal{C}_0$. First note that $\mathcal{C}, \mathcal{C}_0 \in \text{Cartan}(\mathcal{J})$ by Proposition 4.3. According to the way the orthonormal basis $v$ was constructed, we have $A \mathcal{D} A^{-1} = \mathcal{D}(v)$. Since $A \mathcal{J} A^{-1} = \mathcal{J}$, we obtain $A \mathcal{C} A^{-1} = A ((\mathcal{D} \cap \mathcal{J})A^{-1} = (A \mathcal{D} A^{-1}) \cap \mathcal{J} = \mathcal{D}(v) \cap \mathcal{J} = \mathcal{C}_0$. This shows that the Cartan subalgebras $\mathcal{C}$ and $\mathcal{C}_0$ of $\mathcal{J}$ are $U_{\mathcal{J}}(\mathcal{H})$-conjugated to each other since $A \in U_{\mathcal{J}}(\mathcal{H})$.

To prove $\mathcal{C} \cap \mathcal{I}$ and $\mathcal{C}_0 \cap \mathcal{I}$ are not $U_{\mathcal{J}}(\mathcal{H})$-conjugated to each other, let us assume otherwise that the Cartan subalgebras $\mathcal{C} \cap \mathcal{I}$ and $\mathcal{C}_0 \cap \mathcal{I}$ of $\mathcal{I}$ were also $U_{\mathcal{J}}(\mathcal{H})$-conjugated to each other. This means that there exists $B \in U_{\mathcal{J}}(\mathcal{H})$ for which $B(\mathcal{C} \cap \mathcal{I})B^{-1} = \mathcal{C}_0 \cap \mathcal{I}$. On the other hand, since we have seen that $A \mathcal{C} A^{-1} = \mathcal{C}_0$, one has also $A(\mathcal{C} \cap \mathcal{I})A^{-1} = \mathcal{C}_0 \cap \mathcal{I}$. It then follows that $B(\mathcal{C} \cap \mathcal{I})B^{-1} = A(\mathcal{C} \cap \mathcal{I})A^{-1}$, hence

$$(A^{-1} B)(\mathcal{C} \cap \mathcal{I})(A^{-1} B)^{-1} = \mathcal{C} \cap \mathcal{I}.$$
By using Lemma 4.10 we now obtain a permutation $\sigma \in S_{\infty}$ and a sequence $u = \langle u_n \rangle_{n=1}^{\infty}$ with $u_n \in \mathbb{I}$ for every $n \geq 1$, for which $A^{-1}B = V_\sigma$ (diag $u$), hence

$$B(\text{diag } u)^{-1} = AV_\sigma.$$  \hfill (4.8)

Recall that $B \in U_2(H)$, hence $B - 1 \in \mathcal{I}$, and this implies implies

$$A - (\text{diag } u)^{-1}V_\sigma^{-1} \in \mathcal{I}$$  \hfill (4.9)

since $A - (\text{diag } u)^{-1}V_\sigma^{-1} = (B - 1)(\text{diag } u)^{-1}V_\sigma^{-1}$ by (4.8).

On the other hand, since $\sigma \in S_{\infty}$, it follows that $V_\sigma$ is identity on all $b_n$ except for finitely many, and so the subtraction of $(\text{diag } u)^{-1}V_\sigma^{-1}$ acts only on the diagonal of $A$, except for finitely many blocks of $A$. In other words, this difference $(\text{diag } u)^{-1}V_\sigma^{-1}$ is a matrix which is block diagonal with off diagonal entries $\sin \frac{\theta}{2}$ in all but finitely many of the blocks. Now each $2 \times 2$ block then has norm $\geq |\sin \frac{\theta}{2}|$, and so also its biggest $s$-number. Using basic $s$-number theory for operator ideals it follows that the $s$-numbers of $A - (\text{diag } u)^{-1}V_\sigma^{-1}$ are bounded below by a constant multiple of the sequence $\theta$ (see [DFWW04 Subsect. 2.7]). Hence the positive operator $|A - (\text{diag } u)^{-1}V_\sigma^{-1}|$ is bounded below by the direct sum of $2 \times 2$ positive matrices with eigenvalues $|\sin \frac{\theta}{2}|$ and $0$. Since $\theta \in \Sigma(\mathcal{J}) \setminus \Sigma(I)$, it then follows that $A - (\text{diag } u)^{-1}V_\sigma^{-1} \in \mathcal{J} \setminus \mathcal{I}$. This contradicts (4.9) and hence completes the proof that $\mathcal{C} \cap \mathcal{I}$ and $\mathcal{C}_0 \cap \mathcal{I}$ fail to be $U_2(H)$-conjugated to each other.

Step 2. Let $\theta = \langle \theta_n \rangle_{n=1}^{\infty}$ is any decreasing sequence in $(0, \pi)$ with $\theta \in \Sigma(\mathcal{J}) \setminus \Sigma(I)$. Then for arbitrary $\alpha \in (0, 1]$ the sequence $\alpha \theta := \langle \alpha \theta_n \rangle_{n=1}^{\infty}$ is again a decreasing sequence in $(0, \pi)$ with $\alpha \theta \in \Sigma(\mathcal{J}) \setminus \Sigma(I)$, hence $A_{\alpha \theta} \in U_2(H) \setminus U_2(H)$ by Step 1 of the present proof. Moreover, since $\alpha \theta + \beta \theta = (\alpha + \beta)\theta$, we obtain $A_{\alpha \theta}A_{\beta \theta} = A_{(\alpha + \beta)\theta}$ if $\alpha + \beta \in (0, 1]$ with $\alpha + \beta \leq 1$, by using (4.5)–(4.6). This shows that for the orthonormal bases $v^{(\alpha)} = \langle v^{(\alpha)}_n \rangle_{n \geq 1}$ in $H$ defined by $v^{(\alpha)}_n := A_{\alpha \theta}b_n$ for every $n \geq 1$, we have $A_{\alpha \theta}v^{(\beta)} = v^{(\alpha + \beta)}$ (vectorwise on these bases) if $\alpha + \beta \leq m$.

Thus any two distinct bases in the set $\{v^{(\alpha)} \mid \alpha \in (0, 1]\}$ are related to each other by a suitable unitary operator of the form $A_{\alpha \theta} \in U_2(H) \setminus U_2(H)$ with $t \in (0, 1]$, just as it was the case with the bases $\theta$ and $v$ in Step 1 of this proof. Hence, if for every basis $v^{(\alpha)}$ one denotes by $D(v^{(\alpha)})$ its corresponding set of diagonal operators in $B(H)$, and we define $\mathcal{C}_\alpha := D(v^{(\alpha)}) \cap \mathcal{J}$, then $\{\mathcal{C}_\alpha \mid \alpha \in (0, 1]\} \subseteq \text{Cartan}(\mathcal{J})$ are pairwise $U_2(H)$-conjugated and their intersections with $\mathcal{I}$ pairwise fail to be $U_2(H)$-conjugated to each other.

**Corollary 4.13.** For all operator ideals $\{0\} \subsetneq \mathcal{I} \subsetneq \mathcal{J} \subsetneq B(H)$ and every $U_2(H)$-conjugacy class $\mathcal{O} \subseteq \text{Cartan}(\mathcal{J})$, the set $\{\mathcal{C} \cap \mathcal{I} \mid \mathcal{C} \in \mathcal{O}\} \subseteq \text{Cartan}(\mathcal{J})$ disjointly partitions into an uncountable family of $U_2(H)$-conjugacy class of Cartan subalgebras of $\mathcal{I}$.

**Proof.** The assertion follows directly from Theorem 4.12. \hfill \square

**Remark 4.14.** Using Remark 4.8 one can translate information provided by Theorems 4.11 and 4.12 from the conjugacy action of unitary groups on Cartan subalgebras to the natural action $\alpha$ of the same groups on Decompo$(H)$ (see Definition 4.7). One thus obtains that for arbitrary operator ideals $\{0\} \subsetneq \mathcal{I} \subsetneq \mathcal{J} \subsetneq B(H)$ one has $U_2(H) \subsetneq U_2(H)$, and for any $\mathcal{S}_0 \in \text{Decomp}(H)$, the $U_2(H)$-orbit $\alpha(U_2(H) \times \mathcal{S}_0)$ partitions into uncountably many $U_2(H)$-orbits.

To see this, first use Proposition 4.3 for the ideal $\mathcal{J}$ to construct $\mathcal{C}_0 := \Psi(S_0) \in \text{Cartan}(\mathcal{J})$ with its $U_2(H)$-conjugacy class $\mathcal{O}$. It follows by Theorems 4.13 and
that there exists an uncountable family $\{C_i\}_{i \in I}$ of elements of $\mathcal{O}$ that pairwise fail to be $U_2(\mathcal{H})$-conjugate to each other. Then $\{\Psi^{-1}(C_i)\}_{i \in I}$ is an uncountable family of elements of $\text{Decomp}(\mathcal{H})$ that (because of the commutative diagram from Remark 4.8) belong to the $U_2(\mathcal{H})$-orbit of $S_0 \in \text{Decomp}(\mathcal{H})$ and yet pairwise fail to be mapped to each other by the action of $U_2(\mathcal{H})$ on $\text{Decomp}(\mathcal{H})$ via $\alpha$. Thus there exist uncountably many distinct $U_2(\mathcal{H})$-orbits contained in the $U_2(\mathcal{H})$-orbit $\alpha(U_2(\mathcal{H}) \times \{S_0\})$.

5. Differentiable structures on conjugacy classes of Cartan subalgebras

In order to state the next proposition, we recall that for every unital $C^*$-algebra $A$, its unitary group $U_A := \{u \in A \mid u^*u = uu^* = 1\}$ has the structure of a Banach-Lie group. Moreover, for every Banach-Lie group $G$ and every automorphism group $S$ of $G$, one can define the semidirect product $S \ltimes G$, and this has a unique Banach-Lie group structure for which $G$ is an open subgroup. See [Be06] for background information on these constructions of Banach-Lie groups.

We now frame this in the specific setting to be used in this section. There is a natural action of the permutation group $S_\infty$ by $*$-automorphisms of the $C^*$-algebra $\ell^\infty(\mathbb{N})$, that is, every $\sigma \in S_\infty$ defines the $*$-automorphism $\alpha_\sigma : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ by $\alpha_\sigma(z) = (z_{\sigma^{-1}(n)})_{n=1}^\infty$ for every $z = (z_n)_{n=1}^\infty \in \ell^\infty(\mathbb{N})$. One has $\alpha_{\sigma \tau} = \alpha_\sigma \alpha_\tau$ for all $\sigma, \tau \in S_\infty$.

By restricting this action of $S_\infty$ from the $C^*$-algebra $\ell^\infty(\mathbb{N})$ to its unitary group $U(\ell^\infty(\mathbb{N})) \simeq \mathbb{T}^\mathbb{N}$, where $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$, we obtain an action of $S_\infty$ by automorphisms of the group $U(\ell^\infty(\mathbb{N}))$. From this we can then construct the semidirect product $S_\infty \ltimes U(\ell^\infty(\mathbb{N}))$, whose underlying set is the Cartesian product $S_\infty \times U(\ell^\infty(\mathbb{N}))$ and whose group operation is

$$(\sigma, z) \cdot (\tau, w) = (\sigma \tau, \alpha_{\tau^{-1}}(z)w)$$

for all $\sigma, \tau \in S_\infty$ and $z, w \in U(\ell^\infty(\mathbb{N}))$. Since $\ell^\infty(\mathbb{N})$ is a unital $C^*$-algebra, its unitary group $U(\ell^\infty(\mathbb{N}))$ is a Banach-Lie group. Moreover, by using the discrete topology of $S_\infty$, we endow the semidirect product $S_\infty \ltimes U(\ell^\infty(\mathbb{N}))$ with the structure of a Banach-Lie group for which the connected component of the identity element is $U(\ell^\infty(\mathbb{N}))$.

**Proposition 5.1.** If $S \in \text{Decomp}(\mathcal{H})$, then $U(\mathcal{H})_S$ and the semidirect product $S_\infty \ltimes U(\ell^\infty(\mathbb{N}))$ are isomorphic Banach-Lie groups.

**Proof.** Recall from Definition 4.7 that $U(\mathcal{H})_S$ is the class of unitaries that fix the orthogonal one dimensional decomposition $\mathcal{S}$ of $\mathcal{H}$. Let us consider any labeling of the elements of $S$ by natural numbers, $S = \{V_n \mid n \in \mathbb{N}\}$, and for every $n \in \mathbb{N}$ pick a unit vector $v_n \in V_n$. If $W \in U(\mathcal{H})_S$, then we obtain a bijection of $\mathcal{S}$ given by $V \mapsto W(V)$, which in turn defines a permutation $\sigma(W) \in S_\infty$. It follows that the unitary operator $\tilde{W} := V_n(W)V_n$ has the property $\tilde{W}(V) = V$ for every $V \in \mathcal{S}$, hence for every $n \in \mathbb{N}$ there exists $z_n(W) \in \mathbb{C}$ such that $|z_n(W)| = 1$ and $\tilde{W}v_n = z_n(W)v_n$. Then $z(W) = \{z_n(W)\}_{n \in \mathbb{N}}$ is a unitary element in the $C^*$-algebra $\ell^\infty(\mathbb{N})$, and it is easily checked that the mapping $W \mapsto (\sigma(W), z(W))$ is a group isomorphism as in the statement of the proposition.

In the following statement, for every Cartan subalgebra $\mathcal{C}$ of a proper operator ideal $\mathcal{I}$, we denote by $\mathcal{D}(\mathcal{C})$ the set of all operators in $\mathcal{B}(\mathcal{H})$ which are diagonal with
In order to prove that $U^u \subset \text{check that the closed real linear subspace}
$its open subgroups. Since $S$
Recall from [Be06, Prop. 9.28] that the Lie algebra of $U$
the other hand, the Lie algebra of $B$
topology inherited from $\Psi$
the following description for the Lie algebra of $U$
Proof. Use Proposition 5.1 along with Proposition 2.4. \hfill \square

**Theorem 5.3.** Assume the proper ideal $I$ in $\mathcal{B}(\mathcal{H})$ is symmetrically normed and its underlying Banach space is the dual of a Banach space. Then for every $C \in \text{Cartan}(I)$ the corresponding isotropy group $U_{I,C}(\mathcal{H})$ is a Banach-Lie subgroup of $U_I(\mathcal{H})$ and the $U_I(\mathcal{H})$-orbit with respect to the group action $\beta$ has a structure of a smooth homogeneous space of the Banach-Lie group $U_I(\mathcal{H})$. Moreover, the Banach manifolds obtained in this way for various choices of $C \in \text{Cartan}(I)$ are diffeomorphic to each other.

Proof. Let us consider the complex associative Banach $*$-algebra $\mathfrak{B} = \mathbb{C}1 + I$ endowed with the norm given by $\|z1 + T\| = |z| + \|T\|_I$, and denote by $\mathfrak{B}^\times$ its group of invertible elements. Define the bounded linear functional $\psi: \mathfrak{B} \to \mathbb{C}$, $\psi(z1 + T) = z$. If we denote by $\{P_n\}_{n \geq 1}$ the family of orthogonal projections corresponding to the 1-dimensional subspaces in $\Psi^{-1}(C) \in \text{Decomp}(\mathcal{H})$, then it is easily seen that

$$U_{I,C}(\mathcal{H}) = \{V \in \mathfrak{B}^\times \mid \psi(V) = 1, V^*V = 1, (\forall n \geq 1) \left[ VP_nV^{-1}, P_n \right] = 0\},$$

the latter condition being equivalent to the $V$-change of basis preserving the 1-dimensional subspaces. This observation shows that $U_{I,C}(\mathcal{H})$ is an algebraic subgroup of $\mathfrak{B}^\times$ of degree $\leq 2$ (see Definition A.3 below) and therefore it follows that $U_{I,C}(\mathcal{H})$ has the structure of a Banach-Lie group whose underlying topology is its topology inherited from $\mathfrak{B}$ (see for instance [Be06] Th. 4.13 and its proof). On the other hand, the Lie algebra of $U_{I,C}(\mathcal{H})$ is equal to the Lie algebra of any of its open subgroups. Since $\mathfrak{S}_{\text{fin}}$ is a discrete group, it follows by Corollary 5.2 that $U(\mathcal{H}) \cap (1 + (\mathcal{D}(C) \cap I))$ is an open subgroup of $U_{I,C}(\mathcal{H})$, hence we obtain at last the following description for the Lie algebra of $U_{I,C}(\mathcal{H})$:

$$u_{I,C}(\mathcal{H}) = \{T \in I \mid T^* = -T, (\forall n \in \mathbb{N}) TP_n = P_nT\}. \quad (5.1)$$

Recall from [Be06] Prop. 9.28 that the Lie algebra of $U_I(\mathcal{H})$ is

$$u_I(\mathcal{H}) = \{T \in I \mid T^* = -T\}.$$
In view of \[BP07\] Th. 3.1(a), it suffices to construct a commutative semigroup of contractive linear maps on \( u_\mathcal{I}(\mathcal{H}) \) whose fixed point set is precisely \( u_\mathcal{I}(\mathcal{H}) \). To this end, note that the condition \( TP_n = P_n T \) from (5.1) is equivalent to the family of fixed point equations \( e^{isaP_n} T e^{-isaP_n} = T \) for all \( s \in \mathbb{R} \). Now for every \( n \geq 1 \) and \( s \in \mathbb{R} \) define

\[ \gamma_{s,n} : \mathcal{I} \to \mathcal{I}, \quad \gamma_{s,n}(T) = e^{isaP_n} T e^{-isaP_n} \]

and set

\[ \Gamma = \{ \gamma_{s_1,n_1} \circ \cdots \circ \gamma_{s_k,n_k} \mid k \geq 1, n_1, \ldots, n_k \geq 1, s_1, \ldots, s_k \in \mathbb{R} \}, \]

which is an abelian group. The above observations show that

\[ u_{\mathcal{I},C}(\mathcal{H}) = \{ T \in u_\mathcal{I}(\mathcal{H}) \mid (\forall \gamma \in \Gamma) \quad \gamma(T) = T \}. \]

Since \( \mathcal{I} \) is a symmetrically normed ideal, it follows that for every \( \gamma \in \Gamma \) we have \( ||\gamma|| = 1 \). Hence we can use \[BP07\] Th. 3.1(a) to obtain a bounded linear operator \( P : u_\mathcal{I}(\mathcal{H}) \to u_\mathcal{I}(\mathcal{H}) \) with \( P^2 = P \) and \( \text{Ran} P = u_{\mathcal{I},C}(\mathcal{H}) \). Then \( u_{\mathcal{I},C}(\mathcal{H}) \cap \text{Ker} P = \{0\} \), that is, \( \text{Ker} P \) is a direct complement for \( u_{\mathcal{I},C}(\mathcal{H}) \).

This completes the proof of the fact that \( u_{\mathcal{I},C}(\mathcal{H}) \) is a Banach-Lie subgroup of \( U_\mathcal{I}(\mathcal{H}) \), and then the quotient \( U_\mathcal{I}(\mathcal{H}) / u_{\mathcal{I},C}(\mathcal{H}) \) has the natural structure of a smooth homogeneous space (see for instance \[Be06\] Th. 4.19). On the other hand, recall that \( u_{\mathcal{I},C}(\mathcal{H}) \) is the isotropy group for the action of \( U_\mathcal{I}(\mathcal{H}) \) on the \( U_\mathcal{I}(\mathcal{H}) \)-conjugacy class of the Cartan subalgebra \( \mathcal{C} \) of \( \mathcal{I} \). Hence we have a canonical \( U_\mathcal{I}(\mathcal{H}) \)-equivariant bijection from that conjugacy class onto \( U_\mathcal{I}(\mathcal{H}) / u_{\mathcal{I},C}(\mathcal{H}) \). This makes the conjugacy class into a smooth homogeneous space.

Finally, if \( \mathcal{C}_1, \mathcal{C}_2 \in \text{Cartan}(\mathcal{I}) \), then by Remark 5.3 there exists a unitary operator \( V \in U(\mathcal{H}) \) such that \( \beta_V(\mathcal{C}_1) = V \mathcal{C}_1 V^{-1} = \mathcal{C}_2 \). The mapping \( X \mapsto XVX^{-1} \) also gives an automorphism of the Banach-Lie group \( U_\mathcal{I}(\mathcal{H}) \) that maps the isotropy group \( U_{\mathcal{I},C_1}(\mathcal{H}) \) onto the isotropy group \( U_{\mathcal{I},C_2}(\mathcal{H}) \), and therefore gives rise to a diffeomorphism \( U_{\mathcal{I},C_1}(\mathcal{H}) / U_{\mathcal{I},C_1}(\mathcal{H}) \simeq U_{\mathcal{I},C_2}(\mathcal{H}) / U_{\mathcal{I},C_2}(\mathcal{H}) \). Therefore the \( U_\mathcal{I}(\mathcal{H}) \)-conjugacy classes of \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are diffeomorphic to each other, and this concludes the proof.

\[ \square \]

\textbf{Remark 5.4.} On the applicability of Theorem 5.3 we note the following. It follows by the classical duality theory of symmetrically normed ideals that there exist a lot of such ideals which are duals of Banach spaces, including for instance the Schatten ideals \( \mathfrak{S}_p(\mathcal{H}) \) with \( 1 \leq p < \infty \); see \[GK69\] Sect. III.12, and also \[Be06\] Sect 9.4. However, Theorem 5.3 cannot be applied for \( \mathcal{I} = \mathcal{K}(\mathcal{H}) = \mathfrak{S}_\infty(\mathcal{H}) \) since it is well known that this ideal has no predual.

\[ \square \]

\textbf{Remark 5.5.} We now summarize some results on conjugacy classes of maximal abelian self-adjoint subalgebras of three types of algebras: matrix algebras, operator ideals, and \( C^* \)-algebras. This summary may be helpful in placing Theorem 5.3 in some perspective.

- As already mentioned in the Introduction, if we replace in Definition 4.7 the ideal \( \mathcal{I} \) by a matrix algebra \( M_n(\mathbb{C}) \), then the action \( \beta \) is transitive, that is, it has only one orbit (see Theorem 1.1 for a more general result).
- It follows by Theorem 4.9 that if \( \mathcal{I} \) is a proper operator ideal in \( \mathcal{B}(\mathcal{H}) \), then the action \( \beta \) restricted to the group \( U_\mathcal{I}(\mathcal{H}) \) has uncountably many orbits.
- If we replace in Definition 4.7 the ideal \( \mathcal{I} \) by a \( C^* \)-algebra of operators, then we still find infinitely many orbits, however there may be only countably many ones; see \[BPW13\] Rem. 4.2 for \( \mathcal{I} = \mathcal{B}(\mathcal{H}) \).
Remark 5.6. The method of proof of Theorem 5.3 can be used for constructing smooth structures on the $U_I(H)$-orbits of other actions as well. For instance, this is the case with the action

$$U_I(H) \times B(H)^{sa} \to B(H)^{sa}, \quad (V,X) \mapsto VXV^{-1}$$

(see [Be06, Th. 4.33] for a related result).

\[ \Box \]

6. Further problems

Problem 6.1. Find a parameterization of the set of all $U_I(H)$-conjugacy classes of Cartan subalgebras of $I$, simpler than the bijective map (4.4) in Remark 4.8. For instance, if $I = B(H)$, then [BPW14b, Rem. 4.2] shows that $\{0, 1, 2, \ldots\} \cup \{-\infty, \infty\}$ is such a set of parameters.

Problem 6.2. A question whose affirmative answer would provide extra motivation for calling the masa’s of an operator ideal $I$ as Cartan subalgebras of $I$: Does there exist any generalization of Theorem 4.4 beyond the class of complete separable norm ideals?

More specifically, let $C \in \text{Cartan}(I)$ be the set of all diagonal operators in $I$ with respect to some fixed orthonormal basis in $H$. Then let $A$ be the set of all operators given by matrices with the entry 1 on some off-diagonal position and 0 elsewhere. Can we make sense of the assertion that every operator in $I$ is an infinite linear combination of elements in $A \cup D$?

The problem is the convergence of such a linear combination. However, the main point would be to make sense of the assertion of Theorem 4.4 for arbitrary ideals, and in this case one should perhaps expect to have “infinite linear combinations” somehow similar to the formal power series, which are also convergent in a suitable sense expressed in purely algebraic terms. Such a suitable sense could be perhaps imagined for arbitrary operator ideals, by using the characteristic sets. Note that we do not mean to sum up anything inside of the diagonal algebra $D$, but rather to make sense of the fact that the farther an entry is from the diagonal, the smaller it should be thought of.

So the point is that main information of a matrix is concentrated on the main diagonal, and when we get far from the main diagonal, the information carried by the off-diagonal entries is less and less significant. This would provide an infinite-dimensional version of the root-space decomposition that occurs in the proof of [BPW14a, Prop. 4.16].

Problem 6.3. On Remark 4.5, where for $e = \{e_n\}_{n \geq 1}$ and $f = \{f_n\}_{n \geq 1}$ orthonormal bases in $H$, we say they are $I$-equivalent to each other if there exists $W \in U_I(H)$ such that $We_n = f_n$ for all $n \geq 1$. It would be useful to study necessary or sufficient conditions of $I$-equivalence, as for instance the following ones for the largest/smallest proper operator ideal:

- In the compact case $I = K(H)$, $e$ and $f$ are $I$-equivalent to each other if and only if $\|e_n - f_n\| \to 0$.
- And for the finite ranks $I = F(H)$, the $I$-equivalence condition is $e_n = f_n$ for all but finitely many $n$.  

\[ \Box \]
In fact, the above condition in the case $\mathcal{I} = \mathcal{F}(\mathcal{H})$ is clearly sufficient, however it is not difficult to see that it is not necessary. To this end define $v := \sum_{n \geq 1} \alpha_n e_n \in \mathcal{H}$ with $\|v\| = 1$, where $(\alpha_n)_{n=1}^\infty \in l^2$ is any sequence with $\alpha_n \neq 0$ for all but finitely many subscripts, for instance $\alpha_n = \frac{1}{2^n}$ for every $n \geq 1$. If $P := (\cdot, v) \in \mathcal{B}(\mathcal{H})$ is the orthogonal projection onto the 1-dimensional subspace spanned by $v$, then $V := 1 - 2P$ is a unitary (self-adjoint) operator, hence $\{f_n := Ve_n \mid n \geq 1\}$ is an orthonormal basis in $\mathcal{H}$. Since $V \in 1 + \mathcal{F}(\mathcal{H})$, one has $V \in U_{\mathcal{F}(\mathcal{H})}$, hence the orthonormal bases $e = \{e_n\}_{n \geq 1}$ and $f = \{f_n\}_{n \geq 1}$ are $\mathcal{F}(\mathcal{H})$-equivalent. However, $e_n - f_n = 2Pe_n = 2\alpha_n v \neq 0$ for infinitely many $n$.

Perhaps using a slightly more complicate example, one could also prove that the above assertion on $K(\mathcal{H})$-equivalent bases also needs extra assumptions in order to hold true.

**Problem 6.4.** Can one obtain sufficient conditions for $U_{\mathcal{I}}(\mathcal{H})$-diagonalizability of normal operators in a $\mathcal{B}(\mathcal{H})$-ideal $\mathcal{I}$? See Remark 3.12 for the case when $\mathcal{I}$ is the Hilbert-Schmidt ideal, and also [BPW14b] Prop. 4.3 for the limit cases when $\mathcal{I}$ is the ideal of finite-rank operators (when no extra condition is needed) or when $\mathcal{I} = \mathcal{B}(\mathcal{H})$. The case when $\mathcal{I}$ is the trace class would be particularly interesting.

**Problem 6.5.** What information does Proposition 3.13 provide for the size of $D_{\mathcal{I}}$, beyond the fact that $\Pi^{-1}D_{\mathcal{I}} \subset \mathcal{I}^2$?

**Problem 6.6.** On the structure of some subsets of $D_{\mathcal{I}}$ and $D_{\mathcal{I}^2}$, namely for operators with spectral multiplicities one, we know from Proposition 3.1 that they are both closed under finite permutations, and in fact the equivalence relation of one diagonal being a finite permutation of the other partitions both $D_{\mathcal{I}}$ and $D_{\mathcal{I}^2}$. It would be perhaps interesting to see in what form the above mentioned proposition generalizes to arbitrary normal compact operators without any multiplicity restriction.

**Appendix A. Lie theory for some infinite-dimensional algebraic groups**

**Infinite-dimensional linear algebraic reductive groups.** The notion of linear algebraic group in infinite dimensions requires the following terminology. If $A$ is a real Banach space, then a vector-valued continuous polynomial function on $A$ of degree $\leq n$ is a function $p: A \to V$, where $V$ is another real Banach space, such that for some continuous multilinear maps

$$\psi_k: \underbrace{A \times \cdots \times A}_{k \text{ times}} \to V$$

(for $k = 0, 1, \ldots, n$) we have $p(a) = \psi_n(a, \ldots, a) + \cdots + \psi_1(a) + \psi_0$ for every $a \in A$, where $\psi_0 \in V$.

Now let $B$ be a real associative unital Banach algebra, hence a real Banach space endowed with a bounded bilinear mapping $B \times B \to B$, $(x, y) \mapsto xy$ which is associative and admits a unit element $1 \in B$. Then the set

$$B^\times := \{x \in B \mid (\forall y \in B) \ xy = yx = 1\}$$

is an open subset of $B$ and has the natural structure of a Banach-Lie group [Up83, Example 6.9]. The Lie algebra of $B^\times$ is again the Banach space $B$, viewed however
as a nonassociative Banach algebra, more precisely as a Banach-Lie algebra whose
Lie bracket is the bounded bilinear mapping \( \mathcal{B} \times \mathcal{B} \to \mathcal{B} \), \( (x, y) \mapsto xy - yx \).

**Definition A.1.** If \( \mathcal{B} \) is a real associative unital Banach algebra and \( G \) is a closed
subgroup of \( \mathcal{B}^{\times} \), then the **Lie algebra of** \( G \) is
\[ g := \{ x \in \mathcal{B} \mid (\forall t \in \mathbb{R}) \exp(tx) \in G \}. \]

**Remark A.2.** In the setting of Definition [A.1] the set \( g \) is a closed Lie subalgebra of \( \mathcal{B} \) \([U85, Corollary 6.8]\).

In fact, since \( G \) is a closed subset of \( \mathcal{B}^{\times} \), it is easily seen that \( g \) is closed in \( \mathcal{B} \). Moreover, by using the well-known formulas \([U85, Proposition 6.7]\)
\[ \exp(t(x + y)) = \lim_{k \to \infty} \left( \exp\left( \frac{t}{k}x \right) \exp\left( \frac{t}{k}y \right) \right)^k, \]
\[ \exp(t^2[x, y]) = \lim_{k \to \infty} \left( \exp\left( \frac{t}{k}x \right) \exp\left( \frac{t}{k}y \right) \exp\left( -\frac{t}{k}x \right) \exp\left( -\frac{t}{k}y \right) \right)^2 \]
which hold true for all \( x, y \in \mathcal{B} \) and \( t \in \mathbb{R} \), it follows that for every \( x, y \in g \) we have \( x + y \in g \) and \( [x, y] \in g \). Then it is easy to check that \( g \) is a real linear subspace of \( \mathcal{B} \).

Moreover, if \( \mathcal{B} \) is endowed with a continuous involution such that for every \( b \in G \) we have \( b^* \in G \), then for every \( x \in g \) we have \( x^* \in g \).

**Definition A.3** \([HK77]\). Let \( \mathcal{B} \) be a real associative unital Banach algebra, \( n \) a positive integer, and \( G \) a subgroup of \( \mathcal{B}^{\times} \). We say that \( G \) is an **algebraic group in** \( \mathcal{B} \) of degree \( \leq n \) if we have
\[ G = \{ b \in \mathcal{B}^{\times} \mid (\forall p \in P) \ p(b, b^{-1}) = 0 \} \]
for some set \( P \) of vector-valued continuous polynomial functions of degree \( \leq n \) on \( \mathcal{B} \times \mathcal{B} \) (see [BPW14b, Sect. 3] for more details and examples). Note that \( G \) is a closed subgroup of \( \mathcal{B}^{\times} \), hence its Lie algebra can be defined as in Definition [A.1].

If moreover \( \mathcal{B} \) is endowed with a continuous involution \( b \mapsto b^* \) and for every \( b \in G \) we have \( b^* \in G \), then we say that the group \( G \) is **reductive**.

**Definition A.4.** Let \( \mathcal{B} \) be a real associative unital Banach algebra, \( n \) a positive integer, and \( G \) an algebraic subgroup of \( \mathcal{B}^{\times} \) of degree \( \leq n \) with the Lie algebra \( g \) \((\subseteq \mathcal{B})\). Then for every one-sided ideal \( \mathfrak{I} \) of \( \mathcal{B} \), the corresponding \( \mathfrak{I} \)-**restricted**
algebraic group is
\[ G_{\mathfrak{I}} := G \cap (1 + \mathfrak{I}) \]
and the **Lie algebra of** \( G \) is \( g_{\mathfrak{I}} := g \cap \mathfrak{I} \).

**Remark A.5.** Here are some simple remarks on algebraic structures that occur in
the preceding definition. Let \( \mathcal{B} \) be a unital ring and \( \mathfrak{I} \) be a one-sided ideal of \( \mathcal{B} \).

(i) The set \( \mathcal{B}^{\times} \cap (1 + \mathfrak{I}) \) is always a subgroup of the group \( \mathcal{B}^{\times} \) of invertible elements in \( \mathcal{B} \).

To see this, let us assume for instance that \( \mathcal{H} \mathcal{B} \subseteq \mathfrak{I} \). Then \( \mathcal{H} \mathcal{I} \subseteq \mathfrak{I} \), hence \((1+\mathcal{I})(1+\mathfrak{I}) \subseteq 1+\mathfrak{I} \), and thus \((1+\mathcal{I})\cap\mathcal{B}^{\times} \) is closed under the product. On the other hand, if \( x \in \mathfrak{I} \), \( b \in \mathcal{B} \) and \( (1+x)b = 1 \), then \( x = 1 - xb \in 1+\mathcal{I} \mathcal{B} \subseteq 1+\mathfrak{I} \), hence \((1+\mathcal{I}) \cap \mathcal{B}^{\times} \) is also closed under the inversion.

(ii) By definition, every one-sided ideal of a real algebra is assumed to be a real linear subspace. Therefore, if the unital ring \( \mathcal{B} \) has the structure of a real algebra, then \( \mathfrak{I} \) is an associative subalgebra of \( \mathcal{B} \) and in particular \( \mathfrak{I} \) has
the natural structure of a real Lie algebra with the Lie bracket defined by
\[ [x, y] := xy - yx \] for all \( x, y \in \mathfrak{g} \).

(iii) If \( \mathfrak{g} \) is a ring endowed with an involution \( b \mapsto b^* \) and \( \mathfrak{g} \) is a self-adjoint
one-sided ideal of \( \mathfrak{g} \), then \( \mathfrak{g} \) is actually a two-sided ideal.

In fact, if we assume for instance \( \mathfrak{g} \mathfrak{B} \subseteq \mathfrak{g} \), then for every \( x \in \mathfrak{g} \) and \( b \in \mathfrak{B} \) we have \( x^* b^* \in \mathfrak{g} \) hence \( b x = (x^* b^*)^* \in \mathfrak{g} \), and thus \( \mathfrak{g} \mathfrak{B} \subseteq \mathfrak{g} \) as well.

In connection with the following observation we emphasize that the ideal \( \mathfrak{g} \) of
the Banach algebra \( \mathfrak{B} \) is not assumed to be closed.

Lemma A.6. Let \( \mathfrak{B} \) be a real associative unital Banach algebra. If \( \mathfrak{g} \) is a one-sided
ideal of \( \mathfrak{B} \) and \( x \in \mathfrak{B} \), then
\[ x \in \mathfrak{g} \iff (\forall t \in \mathbb{R}) \ \exp(t x) \in (1 + \mathfrak{g}) \cap \mathfrak{B}^\times. \]

Proof. Assume that we have for instance \( \mathfrak{g} \mathfrak{B} \subseteq \mathfrak{g} \). If \( x \in \mathfrak{g} \) and \( t \in \mathbb{R} \) then
\[ \exp(t x) \exp(-t x) = \exp(-t x) \exp(t x) = 1, \] hence \( \exp(t x) \in \mathfrak{B}^\times \). Moreover
\[ \exp(t x) = 1 + tx + \frac{t^2}{2} x + \cdots \in 1 + x \mathfrak{B} \subseteq 1 + \mathfrak{g} \mathfrak{B} \subseteq 1 + \mathfrak{g}. \]

Conversely, assume that \( \exp(t x) \in 1 + \mathfrak{g} \) for every \( t \in \mathbb{R} \). Since \( \lim_{t \to 0} \|1 - \exp(t x)\| = 0 \),
there exists \( t_0 \in \mathbb{R} \setminus \{0\} \) such that \( \|1 - \exp(t_0 x)\| < 1 \). Recall that for every \( y \in \mathfrak{B} \)
with \( \|1 - y\| < 1 \), the series
\[ \log(1 - y) := -\sum_{k=1}^{\infty} \frac{1}{k} y^k \in y \mathfrak{B} \]
is uniformly convergent and for \( y = 1 - \exp(t_0 x) \) we have \( \log(1 - y) = t_0 x \) (see [Up85],
Lemma 2.1). Then \( x \in y \mathfrak{B} = 1 - \exp(t_0 x) \mathfrak{B} \) and the hypothesis \( \exp(t_0 x) \in 1 + \mathfrak{g} \)
implies \( x \in \mathfrak{g} \mathfrak{B} = \mathfrak{g} \). \( \square \)

Theorem A.7. Let \( \mathfrak{B} \) be a real associative unital Banach algebra with a one-sided
ideal \( \mathfrak{g} \). If \( G_2 \) is an \( \mathfrak{g} \)-restricted algebraic group in \( \mathfrak{B} \), then its Lie algebra is a Lie
subalgebra of \( \mathfrak{g} \) and can be described as
\[ \mathfrak{g}_3 = \{ x \in \mathfrak{B} \mid (\forall t \in \mathbb{R}) \ \exp(t x) \in G_2 \}. \]

Proof. Recall from Definition A.4 that there is an algebraic group \( G \) in \( \mathfrak{B} \) with the
Lie algebra \( \mathfrak{g} \) such that \( G_2 = G \cap (1 + \mathfrak{g}) \) and \( \mathfrak{g}_3 = \mathfrak{g} \cap \mathfrak{g} \). Since \( \mathfrak{g} \) is a Lie subalgebra
of \( \mathfrak{B} \) by Remark A.2, it then follows that \( \mathfrak{g}_3 \) is a Lie subalgebra of \( \mathfrak{g} \) (see also
Remark A.5(iii)). Moreover, the description of \( \mathfrak{g}_3 \) follows from Lemma A.6. \( \square \)

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