RIESZ-FEJÉR INEQUALITIES FOR HARMONIC FUNCTIONS

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Abstract. In this article, we prove the Riesz - Fejér inequality for complex-valued harmonic functions in the harmonic Hardy space $h^p$ for all $p > 1$. The result is sharp for $p \in (1, 2]$. Moreover, we prove two variant forms of Riesz-Fejér inequality for harmonic functions, for the special case $p = 2$.

1. Introduction and Main Results

Let $\mathbb{D}$ be the open unit disk in the complex plane, i.e. $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ and let $\mathcal{A}$ denote the class of all analytic functions $f$ defined on $\mathbb{D}$. For $f \in \mathcal{A}$, the integral means $M_p(r, f)$ is defined as

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \quad 0 < p < \infty.$$

The classical Hardy space $H^p$, $0 < p < \infty$, consists of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that $M_p(r, f)$ remains bounded as $r \to 1^-$. The study of $H^p$ spaces attracted the attention of many mathematicians, as it deals with the important problems in function theory such as the existence of the radial limit in almost all directions, growth of the absolute value of functions, bounds on the coefficients and so on. For more details on $H^p$ spaces, one can refer to the books of [3,8]. One of the celebrated results on $H^p$ spaces by Riesz – Fejér is the following inequality:

Theorem A. [3, Theorem 3.13] If $f \in H^p$ $(0 < p < \infty)$, then the integral of $|f(x)|^p$ along the segment $-1 \leq x \leq 1$ converges, and

$$\int_{-1}^{1} |f(x)|^p dx \leq \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta,$$

where $f(e^{i\theta})$ stands for the radial limit of $f$ on the unit circle. The constant $1/2$ is best possible.

The above theorem has a nice geometry: When the unit disk $\mathbb{D}$ is mapped conformally onto the interior of a rectifiable Jordan curve $C$, the image of any diameter has at most half the length of $C$.

Theorem A has been generalized in several settings over the years. For example, Beckenbach [1] proved that the inequality (1) remains valid if $|f|^p$ is replaced by a
non-negative function whose logarithm is subharmonic. Validity of inequality (1) under weak regularity assumptions and generalizations of it may be seen from [1,2,6] and the references therein. However, not much is known about Riesz-Fejér inequality for complex-valued harmonic functions defined on the unit disk $D$. In this article, we prove Riesz-Fejér inequality for complex-valued harmonic functions and the results are sharp for the cases $1 < p \leq 2$.

Let $H$ denote the class of all complex-valued harmonic functions $f = u + iv$, where $u$ and $v$ are real-valued harmonic functions in $D$. It is easy to see that the function $f$ has the unique decomposition $f = h + g$ in the unit disk $D$, where $h$ and $g$ are analytic in $D$ with the normalization $g(0) = 0$. The normalization $g(0) = 0$ does not affect any result pertaining to the integral means as we can always induct $g(0)$ into $h(0)$. For $0 < p \leq \infty$, denote by $h^p$, the space of all complex-valued harmonic functions $f \in H$ satisfying the condition $M_p(r, f)$ is uniformly bounded with respect to $r$. Each function $f \in h^p$ has a radial limit almost everywhere. Throughout this article, we use $f(e^{i\theta})$ to denote the radial limit of $f$ on the unit circle. For $f \in h^p$, define $||f||_p$ as

$$||f||_p := \left( \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$ 

There are many results in the literature that are more or less directly connected with the Riesz-Fejér inequality (1). Riesz and Zygmund proved the harmonic analog of (1) for the special case $p = 1$ which has the same geometric interpretation as that of their analytic counterparts.

**Theorem B** (Riesz-Zygmund inequality; See also [10, Theorem 6.1.7]). If $f \in H$ such that $\partial f(re^{i\theta})/\partial \theta \in h^1$, then

$$\int_{-1}^{1} \left| \frac{\partial f(re^{i\theta})}{\partial r} \right| dr \leq \frac{1}{2} \sup_{0 < r < 1} \int_0^{2\pi} \left| \frac{\partial f(re^{i\theta})}{\partial \theta} \right| d\theta.$$

The constant $1/2$ is sharp.

As a Corollary to Theorem B, one could easily get the following result: When $f$ is a harmonic diffeomorphism from the unit disk $D$ onto a Jordan domain with rectifiable boundary of length $C$, then the image of any diameter under $f$ has at most half the length of $C$.

In this article, we prove the sharp form of Riesz–Fejér inequality for $f \in h^p$ for $p \in (1,2]$ and it is as follows:

**Theorem 1.** Let $p > 1$. If $f \in h^p$ then the following inequality holds:

$$\int_{-1}^{1} |f(xe^{it})|^p dx \leq A_p \int_0^{2\pi} |f(e^{i\theta})|^p d\theta, \text{ for all } t \in \mathbb{R},$$

where

$$A_p := \begin{cases} \frac{1}{2} \sec^p \left( \frac{\pi}{2p} \right) & \text{if } 1 < p \leq 2 \\ 1 & \text{if } p \geq 2. \end{cases}$$

The estimate is sharp for all $p \in (1,2]$. 

Furthermore, we remark here that $A_p \to \infty$ as $p \to 1^+$ and we shall demonstrate this by an example, following the proof of Theorem 1.

**Remark 1.** It would be interesting to obtain an analog of Theorem 1 in multidimensional case. For instance, if $f: \mathbb{C} \to \mathbb{R}^n$ ($n \geq 3$) is harmonic in $\mathbb{D}$, then is it true that

$$\int_{-1}^{1} |f(x)|^p \, dx \leq A_p \int_{0}^{2\pi} |f(e^{i\theta})|^p \, d\theta?$$

Moreover, for the special case $p = 2$, we get two variant forms of Theorem 1 and they are as follow:

**Theorem 2.** Let $f(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k$ be harmonic in $\mathbb{D}$. If $f \in \mathfrak{h}^2$, then the following sharp inequality holds:

$$\int_{-1}^{1} |f(xe^{it})|^2 \, dx \leq \frac{1}{2} \int_{0}^{2\pi} |f(e^{i\theta})|^2 \, d\theta + 2\pi \text{Re} \left( \sum_{k=1}^{\infty} a_k b_k e^{i2kt} \right) \text{ for all } t \in \mathbb{R}.$$

**Theorem 3.** If $f = h + \overline{g} \in \mathfrak{h}^2$, then the following sharp inequalities hold for all $t \in \mathbb{R}$:

$$\int_{-1}^{1} |f(xe^{it})|^2 \, dx \leq \frac{1}{2} \int_{0}^{2\pi} |f(e^{i\theta})|^2 \, d\theta + \|h\|_1 \leq \int_{0}^{2\pi} |f(e^{i\theta})|^2 \, d\theta \text{ for all } t \in \mathbb{R}.$$

Proofs of all these results are presented in Section 2.

2. **Proofs of Main Theorems and related results**

A positive real-valued function $u$ is called log-subharmonic, if $\log u$ is subharmonic. In order to prove our Theorems, we need the following classical inequality of Lozinski [9] which is popularly known as Fejér-Riesz-Lozinski inequality, and also an inequality of Kalaj from [7].

**Lemma C.** [9] Suppose that $\Phi$ is a log-subharmonic function from $\mathbb{D}$ to $\mathbb{R}$, such that $\int_{0}^{2\pi} \Phi^p(r e^{i\theta}) \, d\theta$ are uniformly bounded with respect to $r$ for some $p > 0$. Then the following sharp inequality holds:

$$(4) \quad \int_{-1}^{1} \Phi^p(xe^{it}) \, dx \leq \frac{1}{2} \int_{0}^{2\pi} \Phi^p(e^{i\theta}) \, d\theta \quad \text{for all } t \in \mathbb{R}.$$

If equality is attained for some $t \in \mathbb{R}$, then $\Phi \equiv 0$ in $\mathbb{D}$. The constant $1/2$ is best possible.

As an application to Lemma C, we deduce the following result.

**Lemma 1.** Let $\varphi$ and $\psi$ be a pair of two analytic functions defined on $\mathbb{D}$ such that $\varphi$ and $\psi \in H^p$ for some $p > 1$. Then

$$(5) \quad \int_{-1}^{1} (|\varphi(xe^{it})| + |\psi(xe^{it})|)^p \, dx \leq \frac{1}{2} \int_{0}^{2\pi} (|\varphi(e^{i\theta})| + |\psi(e^{i\theta})|)^p \, d\theta \quad \text{for all } t \in \mathbb{R}.$$

The constant $1/2$ is sharp.
Proof. In order to prove the result, it is enough to show that \( \log(|\varphi(z)| + |\psi(z)|) \) is subharmonic in \( \mathbb{D} \). Then, we can apply Lemma C and obtain the desired conclusion. It is well known that \( \log(|A(z)|^2 + |B(z)|^2) \) is subharmonic in \( \mathbb{D} \) provided \( A(z) \) and \( B(z) \) are analytic in \( \mathbb{D} \). Without loss of generality we can suppose that both the functions \( \varphi \) and \( \psi \) are nonvanishing at each point of the unit disk. Then, there exist two non-vanishing analytic functions \( A(z) \) and \( B(z) \) in \( \mathbb{D} \) such that \( A^2(z) = \varphi(z) \) and \( B^2(z) = \psi(z) \), which clearly implies that \( \log(|\varphi(z)| + |\psi(z)|) \) is subharmonic in \( \mathbb{D} \).

Suppose that \( \varphi(z) \) and \( \psi(z) \) have zero(s) inside \( \mathbb{D} \). Then the zero(s) can be accumulated in the Blaschke product so that

\[
\varphi(z) = B_1(z)A(z) \quad \text{and} \quad \psi(z) = B_2(z)B(z),
\]

where \( B_1(z), B_2(z) \) are Blaschke products, \( A(z) \) and \( B(z) \) are non-vanishing analytic functions in \( \mathbb{D} \). Then, we deduce that

\[
\int_{-1}^{1} (|\varphi(xe^{it})| + |\psi(xe^{it})|)^p dx = \int_{-1}^{1} (|B_1(xe^{it})A(xe^{it})| + |B_2(xe^{it})B(xe^{it})|)^p dx
\]
\[
\leq \int_{-1}^{1} (|A(xe^{it})| + |B(xe^{it})|)^p dx
\]
\[
\leq \frac{1}{2} \int_{0}^{2\pi} (|A(e^{i\theta})| + |B(e^{i\theta})|)^p dx \quad \text{(by Lemma C)}
\]
\[
= \frac{1}{2} \int_{0}^{2\pi} (|\varphi(e^{i\theta})| + |\psi(e^{i\theta})|)^p d\theta.
\]

This completes the proof. \( \square \)

**Theorem D.** Let \( 1 < p < \infty \). Assume that \( f = h + \overline{g} \in h^p \) with \( \text{Re}(h(0)g(0)) = 0 \). Then we have the following sharp inequality

\[
(6) \quad \int_{0}^{2\pi} (|h(e^{i\theta})|^2 + |g(e^{i\theta})|^2)^{p/2} d\theta \leq \frac{1}{(1 - \cos \frac{\pi}{p})^{p/2}} \int_{0}^{2\pi} |f(e^{i\theta})|^p d\theta.
\]

Theorem D has been stated and proved in [7] and the proof uses plurisubharmonic function method initiated by Hollenbeck and Verbitsky [5]. The following result due to Frazer is also useful in the proof of Theorem 1.

**Theorem E.** \([4, \text{Theorem 2.2}]\) If \( U \) is subharmonic, positive, and continuous inside and on a circle \( \Gamma \), and if \( D_0, D_1, \ldots, D_{n-1} \) are \( n \) diameters of \( \Gamma \) such that the angles between the consecutive radii is \( \pi/n \), then there exists a constant \( A > 0 \) such that the following inequality holds:

\[
\sum_{k=0}^{n-1} \int_{D_k} U^p(x) dx \leq B_p \int_{0}^{2\pi} U^p(e^{i\theta}) d\theta,
\]
where

\[ B_p := \begin{cases} \frac{A_n}{p - 1} & \text{if } 1 < p < 2 \\ \csc \left( \frac{\pi}{2n} \right) & \text{if } p \geq 2. \end{cases} \]

We remark here that Theorem E is still valid, if we replace the condition “positive” by “non-negative”.

2.1. Proof of Theorem 1. Let \( f = h + \overline{g} \in \mathcal{H}^p \) for some \( p \in (1, 2] \). Without loss of generality, we can assume that \( g(0) = 0 \). Then we have

\[
\int_{-1}^{1} |f(xe^{it})|^p dx \leq \int_{-1}^{1} (|h(xe^{it})| + |g(xe^{it})|)^p dx \\
\leq \frac{1}{2} \int_{0}^{2\pi} (|h(e^{it})| + |g(e^{it})|)^p d\theta \quad \text{(by Lemma 1)} \\
\leq \frac{2^{p/2}}{2} \int_{0}^{2\pi} (|h(e^{it})|^2 + |g(e^{it})|^2)^{p/2} d\theta \\
\leq \frac{2^{p/2}}{2(1 - |\cos(\pi/p)|)^{p/2}} \int_{0}^{2\pi} |f(e^{it})|^p d\theta \quad \text{(by Theorem D)}.\]

It is a simple exercise to see that

\[
\frac{2^{p/2}}{2(1 - |\cos(\pi/p)|)^{p/2}} = \frac{1}{2} \sec^p \left( \frac{\pi}{2p} \right) \quad \text{for } 1 < p \leq 2.
\]

Now, let us consider the case \( p \geq 2 \). Let \( f \in \mathcal{H}^p \) for some \( p > 2 \). Set \( f_\rho(z) = f(\rho z) \). It is easy to see that \( |f_\rho(z)| \) is subharmonic, non-negative, and continuous inside and on the circle \( |z| = 1 \) for \( 0 < \rho < 1 \). Therefore, we can apply Theorem E with \( n = 1 \) and we get

\[
\int_{-\rho}^{\rho} |f(xe^{it})|^p dx = \int_{-1}^{1} |f_\rho(xe^{it})|^p dx \\
\leq \int_{0}^{2\pi} |f_\rho(e^{it})|^p d\theta \quad \text{(by Theorem E)} \\
\leq \int_{0}^{2\pi} |f(e^{it})|^p d\theta \quad \text{(by the monotonicity of } M_p(r,f))\).
\]

Since the above inequality holds for all \( \rho \) such that \( 0 < \rho < 1 \), the desired conclusion follows.

Now, let us prove the sharpness of the result for \( p \in (1, 2] \) by an example. By \( K(p) \), we mean the optimal constant in the inequality

\[
\int_{-1}^{1} |f(x)|^p dx \leq K(p) \int_{0}^{2\pi} |f(e^{it})|^p d\theta, \quad p \geq 1.
\]

Let us show that

\[
K(p) \geq \frac{1}{2} \cos^{-p} \left( \frac{\pi}{2p} \right), \quad p \geq 1.
\]
In particular $K(1) = +\infty$. For $0 < r < 1$, consider the function

$$f_r(z) = \text{Re}\left(\frac{1 + rz}{1 - rz}\right)^{1/p}, \quad z \in \mathbb{D},$$

where $r$ will be chosen as close to 1. At first, it is easy to compute that

$$\int_{-1}^{1} |f_r(x)|^p dx = \frac{2}{r} \log\left(\frac{1 + r}{1 - r}\right) - 2 = \frac{4}{r} \text{arctanh} r - 2.$$

On the other hand

$$\int_{0}^{2\pi} |f_r(e^{i\theta})|^p d\theta = \int_{0}^{2\pi} \left|\cos^p\left(\frac{1}{p} \arg \frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) \cdot \left|\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right| d\theta\right|$$

(7)

$$= \int_{0}^{2\pi} \cos^p\left(\frac{1}{p} \arctan\frac{2r \sin \theta}{1 - r^2}\right) \left|\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right| d\theta.$$

We are interested in estimating the last integral when $r$ is close to 1 and for that, we need to consider the behaviour of the integrand when $r$ is close to 1. Given any arbitrarily small $\epsilon > 0$ and large $M > 0$, we can find a $\rho \in (0, 1)$ such that

$$\left|\frac{2r \sin \theta}{1 - r^2}\right| > M \quad \text{for all} \quad \theta \in E \quad \text{and} \quad r \in (\rho, 1),$$

where $E = [\epsilon/4, \pi - \epsilon/4] \cup [\pi + \epsilon/4, 2\pi - \epsilon/4]$. It is a routine matter to see that

$$I = \int_{0}^{2\pi} \frac{d\theta}{|1 - re^{i\theta}|} = 4 \int_{0}^{\pi/2} d\theta \sqrt{(1 + r)^2 - 4r \sin^2 \theta} = 4 \int_{0}^{\pi/2} d\theta \sqrt{1 - k^2 \sin^2 \theta} = 4k(\kappa(k)), \quad k = \frac{4r}{(1 + r)^2}, \quad k^2 < 1,$$

where

$$\kappa(k) := \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad (|k| < 1)$$

which is known as the complete elliptic integral of the first kind. Therefore, for $r$ close to 1, the integral in (7) behaves like

$$2 \cos^p\left(\frac{\pi}{2p}\right) \int_{0}^{2\pi} d\theta \left|1 - re^{i\theta}\right| = 8 \cos^p\left(\frac{\pi}{2p}\right) \left[\frac{1}{1 + r} \kappa\left(\frac{2\sqrt{r}}{1 + r}\right)\right].$$

A simple computation gives

$$\lim_{r \to 1} \left(\frac{\int_{-1}^{1} |f_r(x)|^p dx}{\int_{0}^{2\pi} |f_r(e^{i\theta})|^p d\theta}\right) = \frac{1}{2} \cos^{-p}\left(\frac{\pi}{2p}\right),$$
which proves that the bounds in Theorem 1 are sharp for $1 < p \leq 2$. The proof is complete.

The example demonstrated in the proof of Theorem 1 suggests the following.

**Conjecture 1.** If $f \in h^p$ for $p > 2$, then the following sharp inequality holds:

$$\int_{-1}^{1} |f(xe^{it})|^p dx \leq \frac{1}{2} \cos^{-p} \left( \frac{\pi}{2p} \right) \int_{0}^{2\pi} |f(e^{i\theta})|^p d\theta \quad \text{for all} \quad t \in \mathbb{R}. $$

Next, we present the variant forms of Riesz-Fejér theorems for harmonic functions for the special case $p = 2$.

### 2.2. Proof of Theorem 2

It is enough to prove the case $t = 0$ as $F_t(z) = f(z e^{it}) \in h^2$ for every fixed $t \in \mathbb{R}$. Let

$$\Phi(z) = h(z) + g(\overline{z}) = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \overline{z}^k. $$

Then, it is easy to see that

$$\int_{-1}^{1} |f(x)|^2 dx = \int_{-1}^{1} |\Phi(x)|^2 dx \leq \frac{1}{2} \int_{0}^{2\pi} |\Phi(e^{i\theta})|^2 d\theta$$

$$= \frac{1}{2} \int_{0}^{2\pi} (|h(e^{i\theta})|^2 + |g(e^{i\theta})|^2) d\theta + \text{Re} \int_{0}^{2\pi} h(e^{i\theta})g(e^{-i\theta}) d\theta$$

$$= \frac{1}{2} \int_{0}^{2\pi} (|h(e^{i\theta})|^2 + |g(e^{i\theta})|^2) d\theta + 2\pi \text{Re} \sum_{k=0}^{\infty} a_k b_k$$

$$= \frac{1}{2} \int_{0}^{2\pi} |f(e^{i\theta})|^2 d\theta + 2\pi \text{Re} \sum_{k=1}^{\infty} a_k b_k \quad \text{(since} \quad b_0 = 0).$$

This completes the proof. \qed

### 2.3. Proof of Theorem 3

Let $f = h + \overline{g} \in h^2$. Without loss of the generality we may suppose that $g(0) = 0$. Furthermore, it is enough to consider the case $t = 0$ as the other cases follow in a similar way. Accordingly,

$$\int_{-1}^{1} |f(x)|^2 dx = \int_{-1}^{1} |h(x)|^2 dx + \int_{-1}^{1} |g(x)|^2 dx + 2\text{Re} \int_{-1}^{1} h(x)g(x) dx$$

$$\leq \int_{-1}^{1} |h(x)|^2 dx + \int_{-1}^{1} |g(x)|^2 dx + 2 \int_{-1}^{1} |h(x)g(x)| dx.$$
As \( h(z)g(z) \) is analytic in \( \mathbb{D} \), from the Cauchy-Schwarz inequality, we observe that\( hg \in H^1 \), and hence from Riesz-Fejér inequality we get
\[
\int_{-1}^{1} |f(x)|^2 \, dx \leq \frac{1}{2} \int_{0}^{2\pi} |h(e^{i\theta})|^2 \, d\theta + \frac{1}{2} \int_{0}^{2\pi} |g(e^{i\theta})|^2 \, d\theta + \int_{0}^{2\pi} |h(e^{i\theta})g(e^{i\theta})| \, d\theta
\]
\[
= \frac{1}{2} \int_{0}^{2\pi} |f(e^{i\theta})|^2 \, d\theta + ||hg||_1.
\]
Finally, the right hand side inequality in (3) follows from the following inequality
\[
||hg||_1 \leq ||h||_2 ||g||_2 \leq \frac{1}{2} (||h||^2_{L^2} + ||g||^2_{L^2}) = \frac{1}{2} \int_{0}^{2\pi} |f(e^{i\theta})|^2 \, d\theta.
\]
Now let us show that both the inequalities in Theorem 3 are sharp and cannot be improved. For this, we consider
\[
f(z) = h(z) + g(z) = \frac{\sqrt{\varphi'(z)}}{2} + \frac{\sqrt{\varphi'(z)}}{2},
\]
where \( \varphi \) maps the disk \( \mathbb{D} \) onto the rectangle with the vertices \( \pm 1 \pm i\varepsilon \) such that \([-1, 1]\) maps onto itself. Let us remark that \( \varphi'(x) > 0 \) and \( \varphi'(0) \to 0 \) as \( \varepsilon \to 0 \).

From the basic calculs, it is clear that
\[
\int_{-1}^{1} |f(x)|^2 \, dx = \int_{-1}^{1} |\varphi'(x)| \, dx = 2.
\]
On the other hand
\[
\int_{0}^{2\pi} |f(e^{i\theta})|^2 \, d\theta = \int_{0}^{2\pi} |\text{Re} \sqrt{\varphi'(e^{i\theta})}|^2 \, d\theta
\]
\[
= \frac{1}{2} \int_{0}^{2\pi} |\varphi'(e^{i\theta})| \, d\theta + \pi \text{Re}(\varphi'(0))
\]
\[
= \frac{4 + 4\varepsilon}{2} + \pi \text{Re}(\varphi'(0)) \to 2 \text{ as } \varepsilon \to 0.
\]
Furthermore, as \( \varepsilon \to 0 \), we get
\[
||hg||_1 = \int_{0}^{2\pi} |h(e^{i\theta})g(e^{i\theta})| \, d\theta = \frac{1}{4} \int_{0}^{2\pi} |\varphi'(e^{i\theta})| \, d\theta = \frac{1}{2} \int_{0}^{2\pi} |f(e^{i\theta})|^2 \, d\theta.
\]
So, from here we see that the constant on the right hand side of the inequality cannot be less than 1 and both the inequalities are sharp. The proof is complete. □

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