Besov classes on finite and infinite dimensional spaces

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Abstract. We give an equivalent description of Besov spaces in terms of a new modulus of continuity. Then we use a similar approach to introduce Besov classes on an infinite-dimensional space endowed with a Gaussian measure.

Bibliography: 25 titles.

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§ 1. Introduction

In this work we continue the study of Nikolskii-Besov classes which started in [1] (see also [2] and [3]), where an equivalent description of these classes was presented, in the spirit of the classical definitions of Sobolev classes and the class of functions of bounded variation, in terms of an action on test functions via integration by parts (the main results of this paper were announced in the notes [4] and [5]). To be precise, a function $f \in L^p(\mathbb{R}^n)$ belongs to the Nikolskii-Besov class $B^{\alpha}_{p,\infty}(\mathbb{R}^n)$ with $0 < \alpha < 1$ if and only if there is a constant $C$ such that

$$
\int_{\mathbb{R}^n} \text{div} \Phi(x)f(x)\,dx \leq C\|\Phi\|_q^\alpha \|\text{div} \Phi\|_q^{1-\alpha}
$$

for each vector field $\Phi$ of class $C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, where $q = p/(p-1)$. If we take $\alpha = 1$ and $p = 1$, we obtain the classical definition of a function of bounded variation. This new characterization has already found some applications in the study of the distributions of polynomials on spaces with Gaussian (or general log-concave) measures (see [6], [7] and also [8]).

In this paper, we give a similar equivalent characterization for the general Besov spaces $B^{\alpha}_{p,\theta}(\mathbb{R}^n)$. We recall that the Besov space $B^{\alpha}_{p,\theta}(\mathbb{R}^n)$ with parameters $\alpha \in (0,1)$, $p \in [1,\infty)$ and $\theta \in [1,\infty]$ consists of all functions $f$ in $L^p(\mathbb{R}^n)$ such that

$$
\left(\int_{\mathbb{R}^n} |h|^{-\alpha} \|f - f_h\|_p^\gamma \|h\|^{-n} \,dh\right)^{1/\theta} < \infty,
$$

where $f_h(x) := f(x-h)$ (see [9]–[12]). However, in what follows it is more convenient to use another equivalent definition in terms of the $L^p$-modulus of continuity. Recall
that the $L^p$-modulus of continuity of a function $f$ from $L^p(\mathbb{R}^n)$ is defined by the equality

$$\omega_p(f, \varepsilon) := \sup_{|h| \leq \varepsilon} \|f_h - f\|_p.$$ 

Note that the function $\omega_p(f, \cdot)$ is nondecreasing and subadditive, which means that

$$\omega_p(f, \varepsilon_1 + \varepsilon_2) \leq \omega_p(f, \varepsilon_1) + \omega_p(f, \varepsilon_2), \quad \varepsilon_1, \varepsilon_2 > 0.$$

A function $f \in L^p(\mathbb{R}^n)$ belongs to the class $B^\alpha_{p,\theta}(\mathbb{R}^n)$ if and only if the quantity

$$\|f\|_{\alpha,p,\theta} := \left( \int_0^{+\infty} \left[ s^{-\alpha} \omega_p(f, s) \right]^\theta s^{-1} ds \right)^{1/\theta}$$

is finite. We define the Besov norm of a function $f$ by the equality

$$\|f\|_{B^\alpha_{p,\theta}(\mathbb{R}^n)} := \|f\|_p + \|f\|_{\alpha,p,\theta}.$$ 

Our equivalent characterization of Besov spaces is based on a new modulus of continuity which is equivalent to $\omega_p(f, \cdot)$ and provides the known characterization (1.1) in the case when $\theta = \infty$. For a function $f \in L^p(\mathbb{R}^n)$ set

$$\sigma_p(f, \varepsilon) := \sup\left\{ \int_{\mathbb{R}^n} \operatorname{div} \Phi(x) f(x) \, dx : \right. \\
\Phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n), \|\Phi\|_{p/(p-1)} \leq 1, \|\Phi\|_{p/(p-1)} \leq \varepsilon \left. \right\}.$$ 

The first main result in this paper asserts the equivalence of $\omega_p(f, \cdot)$ and $\sigma_p(f, \cdot)$: for any function $f \in L^p(\mathbb{R}^n)$,

$$2^{-1} \omega_p(f, 2\varepsilon) \leq \sigma_p(f, \varepsilon) \leq 6n \omega_p(f, \varepsilon).$$

Actually, the function $\sigma_p(f, \cdot)$ has already appeared implicitly in the new definition of Nikolskii-Besov spaces formulated above, since condition (1.1) can be reformulated in the following way:

$$\sup_{s \geq 0} s^{-\alpha} \sigma_p(f, s) < \infty.$$ 

The above equivalence also shows that a function $f$ in $L^p(\mathbb{R}^n)$ belongs to the Besov space $B^\alpha_{p,\theta}(\mathbb{R}^n)$ if and only if

$$\left( \int_0^{+\infty} \left[ s^{-\alpha} \sigma_p(f, s) \right]^\theta s^{-1} ds \right)^{1/\theta} < \infty.$$ 

In the second part of the paper we consider Besov classes on a locally convex space endowed with a centred Gaussian measure. In [1], Nikolskii-Besov classes were introduced on a space with Gaussian measure $\gamma$ by means of relation (1.1) as a definition, where the Gaussian divergence operator $\operatorname{div}_\gamma$ was used in place of
the divergence operator on $\mathbb{R}^n$. If we consider the standard Gaussian measure $\gamma_n$ on $\mathbb{R}^n$, which is the measure with density $(2\pi)^{-n/2}\exp(-|x|^2/2)$, then

$$\text{div}_{\gamma_n} \Phi = \sum_{i=1}^n (\partial_i \Phi_i - x_i \Phi_i) = \text{div} \Phi - \langle x, \Phi \rangle.$$ 

In this paper we propose a similar approach (see Definitions 3.1 and 3.2) to the general Besov classes $B_{p,\theta}^\alpha(\gamma)$ with respect to a Gaussian measure $\gamma$.

The first main result in §3 (presented in Theorem 3.1 and Corollary 3.2) provides an equivalent characterization of the Besov classes we have introduced in terms of a ‘shift’ on the Gaussian space; in a sense this is similar to the classical definition of Besov spaces on $\mathbb{R}^n$. Namely, a function $f$ in $L^p(\gamma)$ with $p > 1$ belongs to the Besov class $B_{p,\theta}^\alpha(\gamma)$ if and only if the quantity

$$\int_0^\infty \left( \int f(e^{-t}x + \sqrt{1-e^{-2t}}y) - f(x) \right)^2 \gamma(dx)\gamma(dy) \frac{1}{t^{1-\theta}} \frac{1}{t^{\theta}} dt$$

is finite. This theorem can also be viewed as an analogue of Theorem 3.2 in [13]. The second main result in §3 is the embedding theorem for Gaussian Besov classes. We recall (see, for example, [14] and [15]) that for an arbitrary function $f$ in the Gaussian Sobolev space $W^{2,1}(\gamma)$ the following logarithmic Sobolev inequality holds:

$$\int f^2 \ln(|f| \|f\|^{-1}) \, d\gamma \leq \int |\nabla f|^2 \, d\gamma.$$ 

There is also an embedding theorem of logarithmic type for the Sobolev class $W^{1,1}(\gamma)$: the space $W^{1,1}(\gamma)$ is continuously embedded into the Orlicz space $L \log L^{1/2}$, which is defined by the condition

$$\int |f| \ln(1 + |f|) \, d\gamma < \infty$$

(see [16] and [17] for the case of functions of bounded variation).

Both results mean that the smoothness of a function provides some higher order of integrability. We could ask whether this effect remains in force for the Besov smoothness condition introduced in our paper. Theorem 3.2 aims to answer this question. It asserts that, for any $\alpha \in (0, 1)$, $\beta \in (0, \alpha)$, $p \in [1, \infty)$ and $\theta \in [1, \infty)$, there is a constant $C = C(p, \theta, \alpha, \beta)$ such that for all functions $f \in B_{p,\theta}^\alpha(\gamma)$,

$$\left( \int |f|^p \ln(|f| \|f\|_{p^{-1}})^{p\beta/2} \, d\gamma \right)^{1/p} \leq C \|f\|_{B_{p,\theta}^\alpha(\gamma)}.$$ 

The main idea of the proof of this result is in spirit of the semigroup approach to the isoperimetric inequality on the Gaussian space proposed by Ledoux in [18] and [16]. Similarly to the cited works, we use information on the short time behaviour of the Ornstein-Uhlenbeck semigroup on functions in the Besov class and the hypercontractivity property of the Ornstein-Uhlenbeck semigroup.

We note that another approach to Besov classes on an infinite-dimensional space with a Gaussian measure was introduced in [19]. In the last section, §4 we show
the equivalence of this approach to the definition of Gaussian Besov spaces given in our paper. Moreover, we obtain an equivalent characterization of Gaussian Besov spaces in terms of the Ornstein-Uhlenbeck semigroup which generalizes some results in [1] to the case of general Gaussian Besov spaces. At the end of the paper, in Theorem 4.2, we provide an estimate of the best approximation to a function in $L^2(\gamma)$ by Hermite polynomials in terms of the Gaussian modulus of continuity we have introduced.

Throughout the paper we assume that $\alpha$ is a fixed number in $(0, 1)$. Given $p \in [1, \infty]$, we let $q$ denote the dual number, so that $1/p + 1/q = 1$. The $L^p$-norm of a function $f$ with respect to a measure $\mu$ is defined as usual by

$$\|f\|_{L^p(\mu)} := \left(\int |f|^p \, d\mu\right)^{1/p}, \quad p \in [1, \infty);$$

and the limiting case when $p = \infty$ is also treated as usual. In §2 the measure $\mu$ will be the standard Lebesgue measure on $\mathbb{R}^n$, but in §§3 and 4 the measure $\mu$ will be a centred Gaussian measure on a locally convex space. We denote the space of all infinitely differentiable functions with compact support on $\mathbb{R}^n$ by $C_\infty(\mathbb{R}^n)$, and the space of all bounded infinitely differential functions with bounded derivatives of every order is denoted by $C_\infty^b(\mathbb{R}^n)$.

§2. Besov classes on $\mathbb{R}^n$

This section is devoted to obtaining a new characterization of Besov classes on $\mathbb{R}^n$ in terms of the new moduli of continuity $\sigma_p(f, \cdot)$ and $\bar{\sigma}_p(f, \cdot)$.

Let $|\cdot|$ denote the standard Euclidean norm on $\mathbb{R}^n$ generated by the standard Euclidean inner product $\langle \cdot, \cdot \rangle$. Let $\lambda^n$ be the standard Lebesgue measure on $\mathbb{R}^n$. We also need the heat semigroup $P_t$ on $\mathbb{R}^n$, which is defined by the equality

$$P_t f(x) := (2\pi t)^{-n/2} \int_{\mathbb{R}^n} f(y) \exp\left(-\frac{|x - y|^2}{2t}\right) \, dy, \quad f \in L^1(\mathbb{R}^n).$$

We start with the following key definitions (recall that $q = p/(p - 1)$).

**Definition 2.1.** Let $f \in L^p(\mathbb{R}^n)$. Set

$$\sigma_p(f, \varepsilon) := \sup \left\{ \int_{\mathbb{R}^n} \text{div} \Phi(x) f(x) \, dx : \Phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n), \|\text{div} \Phi\|_q \leq 1, \|\Phi\|_q \leq \varepsilon \right\}.$$  

**Definition 2.2.** Let $f \in L^p(\mathbb{R}^n)$. Set

$$\bar{\sigma}_p(f, \varepsilon) := \sup \left\{ \int_{\mathbb{R}^n} \partial_e \varphi(x) f(x) \, dx : |e| = 1, \varphi \in C_0^\infty(\mathbb{R}^n), \|\partial_e \varphi\|_q \leq 1, \|\varphi\|_q \leq \varepsilon \right\}.$$  

We now give several important properties of the functions we have introduced.

**Lemma 2.1.** For any function $f \in L^p(\gamma)$, the functions $\sigma_p(f, \cdot)$ and $\bar{\sigma}_p(f, \cdot)$ are nondecreasing, subadditive, concave and continuous on $(0, +\infty)$.
Thus, the function $\|\psi\|$ satisfies $e^{\phi}$.

**Proof.** We consider only the function $\sigma_p(f, \cdot)$, since the proof is essentially the same for the second function. It is readily seen that this function is indeed nondecreasing and subadditive. We now check that it is concave. Let $a, b > 0$ and $t \in (0, 1)$. Then for any two vector fields $\Phi_1, \Phi_2 \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ with $\|\text{div} \Phi_1\|_q \leq 1$, $\|\Phi_1\|_q \leq a$ and $\|\text{div} \Phi_2\|_q \leq 1$, $\|\Phi_2\|_q \leq b$, we have

$$t \int_{\mathbb{R}^n} \text{div} \Phi_1(x)f(x) \, dx + (1 - t) \int_{\mathbb{R}^n} \text{div} \Phi_2(x)f(x) \, dx$$

$$= \int_{\mathbb{R}^n} \text{div}[t\Phi_1(x) + (1 - t)\Phi_2(x)]f(x) \, dx$$

and $\|\text{div}[t\Phi_1 + (1 - t)\Phi_2]\|_q \leq 1$, $\|t\Phi_1 + (1 - t)\Phi_2\|_q \leq ta + (1 - t)b$. Thus,

$$t\sigma_p(f, a) + (1 - t)\sigma_p(f, b) \leq \sigma_p(f, ta + (1 - t)b).$$

The concavity implies the continuity. The lemma is proved.

We now proceed to the main result of this section which shows the equivalence of $\sigma_p(f, \cdot)$, $\tilde{\sigma}_p(f, \cdot)$ and $\omega_p(f, \cdot)$. We start with the following technical lemma (the proof is similar to the proof of Theorem 3.4 in [1]).

**Lemma 2.2.** For any function $f \in L^p(\mathbb{R}^n)$

$$2^{-1}\|f_{2h} - f\|_p \leq \tilde{\sigma}_p(f, |h|) \leq \sigma_p(f, |h|)$$

$$\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \|f_{|h|} - f\|_p (1 + |z|) e^{-|z|^2/2} \, dz.$$ 

**Proof.** For every function $\varphi \in C_0^\infty(\mathbb{R}^n)$ and every unit vector $e \in \mathbb{R}^n$ we can take $\Phi = e\varphi$ and conclude that

$$\tilde{\sigma}_p(f, \varepsilon) \leq \sigma_p(f, \varepsilon).$$

Now let $e = |h|^{-1}h$. For an arbitrary function $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\|\varphi\|_q \leq 1$ we have

$$\int_{\mathbb{R}^n} \varphi(x)(f_h(x) - f(x)) \, dx = \int_{\mathbb{R}^n} [\varphi(x + h) - \varphi(x)]f(x) \, dx$$

$$= \int_{\mathbb{R}^n} \int_0^{|h|} \partial_e \varphi(x + sh) \, ds f(x) \, dx.$$ 

The function

$$\psi(x) = \int_0^{|h|} \varphi(x + se) \, ds \in C_0^\infty(\mathbb{R}^n)$$

satisfies $\|\psi\|_q \leq |h| \|\varphi\|_q \leq |h|$ and $\|\partial_e \psi\|_q \leq 2\|\varphi\|_q \leq 2$, since

$$|\partial_e \psi(x)| = \left| \int_0^{|h|} \partial_e \varphi(x + se) \, ds \right| = |\varphi(x + h) - \varphi(x)|.$$ 

Thus,

$$\int_{\mathbb{R}^n} \varphi(x)(f_h(x) - f(x)) \, dx = \int_{\mathbb{R}^n} \partial_e \psi(x) f(x) \, dx \leq 2\tilde{\sigma}_p\left(f, \frac{|h|}{2}\right).$$
Taking the supremum over the functions \( \varphi \) we get the estimate \( \|f_h - f\|_p \leq 2\sigma_p(f, |h|/2) \).

Finally, for every smooth vector field \( \Phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n) \) we have

\[
\int_{\mathbb{R}^n} \operatorname{div} \Phi(x)f(x)\, dx = \int_{\mathbb{R}^n} \operatorname{div} \Phi(x)(f(x) - P_t f(x))\, dx + \int_{\mathbb{R}^n} \operatorname{div} \Phi(x)P_t f(x)\, dx.
\]

We note that

\[
\|f - P_t f\|_p \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \|f\sqrt{t} - f\|_p e^{-|z|^2/2}\, dz.
\]

Thus, the first term in (2.1) satisfies

\[
\int_{\mathbb{R}^n} \operatorname{div} \Phi(x)(f(x) - P_t f(x))\, dx \leq \|\operatorname{div} \Phi\|_q \|f - P_t f\|_p
\]

\[
\leq \|\operatorname{div} \Phi\|_q (2\pi)^{-n/2} \int_{\mathbb{R}^n} \|f\sqrt{t} - f\|_p e^{-|z|^2/2}\, dz.
\]

Integrating the second term in (2.1) by parts, we obtain

\[
\int_{\mathbb{R}^n} \operatorname{div} \Phi(x)P_t f(x)\, dx = -\int_{\mathbb{R}^n} \langle \Phi(x), \nabla P_t f(x) \rangle\, dx
\]

\[
= t^{-1/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle \Phi(x), (x - y)t^{-1/2}\rangle\, f(y)(2\pi t)^{-n/2}e^{-|x-y|^2/(2t)}\, dy\, dx
\]

\[
= t^{-1/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle \Phi(x), z \rangle\, f(x - \sqrt{t}z)(2\pi)^{-n/2}e^{-|z|^2/2}\, dz\, dx.
\]

Since

\[
\int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \langle \Phi(x), z \rangle e^{-|z|^2/2}\, dz\, dx = 0,
\]

the above expression is equal to

\[
t^{-1/2} \int_{\mathbb{R}^n} (2\pi)^{-n/2}e^{-|z|^2/2} \int_{\mathbb{R}^n} \langle \Phi(x), z \rangle\, (f(x - \sqrt{t}z) - f(x))\, dz\, dx
\]

\[
\leq t^{-1/2}\|\Phi\|_q (2\pi)^{-n/2} \int_{\mathbb{R}^n} |z| e^{-|z|^2/2}\|f\sqrt{t} - f\|_p\, dz.
\]

Thus, we have

\[
\int_{\mathbb{R}^n} \operatorname{div} \Phi(x)f(x)\, dx \leq \|\operatorname{div} \Phi\|_q (2\pi)^{-n/2} \int_{\mathbb{R}^n} \|f\sqrt{t} - f\|_p e^{-|z|^2/2}\, dz
\]

\[
+ t^{-1/2}\|\Phi\|_q (2\pi)^{-n/2} \int_{\mathbb{R}^n} |z| \|f\sqrt{t} - f\|_p e^{-|z|^2/2}\, dz.
\]

Hence

\[
\sigma_p(f, \varepsilon) \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \|f\sqrt{t} - f\|_p e^{-|z|^2/2}\, dz
\]

\[
+ t^{-1/2}\varepsilon (2\pi)^{-n/2} \int_{\mathbb{R}^n} |z|\|f\sqrt{t} - f\|_p e^{-|z|^2/2}\, dz.
\]
Taking $\sqrt{t} = \varepsilon$ we conclude that
\[
\sigma_p(f, \varepsilon) \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} (1 + |z|) \|f_{\varepsilon z} - f\|_p e^{-|z|^2/2} \, dz.
\]

Lemma 2.2 is proved.

As we already mentioned in §1, the function $\omega_p(f, \cdot)$ is nondecreasing and sub-additive; in particular,
\[
\omega_p(f, \tau s) \leq 2\tau \omega_p(f, s) \tag{2.2}
\]
for $\tau \geq 1$ and $s > 0$. In fact, let $k \in \mathbb{N}$ be an integer such that $k \leq \tau < k + 1$. Then
\[
\omega_p(f, \tau s) \leq \omega_p(f, (k+1)s) \leq (k+1)\omega_p(f, s) = k \left(1 + \frac{1}{k}\right) \omega_p(f, s) \leq 2\tau \omega_p(f, s).
\]

Now we are ready to prove the equivalence mentioned above.

**Theorem 2.1.** For any function $f \in L^p(\mathbb{R}^n)$,
\[
2^{-1} \omega_p(f, 2\varepsilon) \leq \tilde{\sigma}_p(f, \varepsilon) \leq \sigma_p(f, \varepsilon) \leq 2(1 + \sqrt{n} + n) \omega_p(f, \varepsilon).
\]

**Proof.** The first two inequalities are straightforward corollaries of Lemma 2.2. To prove the last inequality, using the same lemma we have
\[
\sigma_p(f, \varepsilon) \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \|f_{\varepsilon z} - f\|_p (1 + |z|) e^{-|z|^2/2} \, dz
\]
\[
\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \omega_p(f, \varepsilon |z|) (1 + |z|) e^{-|z|^2/2} \, dz
\]
\[
= (2\pi)^{-n/2} \int_{|z| \leq 1} \omega_p(f, \varepsilon |z|) (1 + |z|) e^{-|z|^2/2} \, dz
\]
\[
+ (2\pi)^{-n/2} \int_{|z| > 1} \omega_p(f, \varepsilon |z|) (1 + |z|) e^{-|z|^2/2} \, dz.
\]
The first integral above is estimated by
\[
\omega_p(f, \varepsilon) (2\pi)^{-n/2} \int_{|z| \leq 1} (1 + |z|) e^{-|z|^2/2} \, dz \leq 2\omega_p(f, \varepsilon)
\]
because the function $\omega_p(f, \cdot)$ is monotonic. By estimate (2.2), the second integral is no greater than
\[
\omega_p(f, \varepsilon) (2\pi)^{-n/2} \int_{|z| > 1} 2|z|(1 + |z|) e^{-|z|^2/2} \, dz \leq \omega_p(f, \varepsilon)(2\sqrt{n} + 2n).
\]
Combining these two estimates we get the third inequality as required.

The theorem is proved.

As a corollary of Theorem 2.1 we get an equivalent characterization of Besov classes on $\mathbb{R}^n$. We introduce the following notation.
Definition 2.3. Let $f \in L^p(\mathbb{R}^n)$, $p \in [1, \infty)$, $\theta \in [1, \infty]$ and $\alpha \in (0, 1)$. Set

$$V^{p, \theta, \alpha}(f) = \left( \int_0^\infty \left[ s^{-\alpha} \sigma_p(f, s) \right]^\theta s^{-1} ds \right)^{1/\theta}.$$ 

Definition 2.4. Let $f \in L^p(\mathbb{R}^n)$, $p \in [1, \infty)$, $\theta \in [1, \infty]$ and $\alpha \in (0, 1)$. Set

$$\tilde{V}^{p, \theta, \alpha}(f) = \left( \int_0^\infty \left[ s^{-\alpha} \tilde{\sigma}_p(f, s) \right]^\theta s^{-1} ds \right)^{1/\theta}.$$ 

Corollary 2.1. For any function $f \in L^p(\mathbb{R}^n)$ the following conditions are equivalent:

(i) $f \in B_{p, \theta}^\alpha(\mathbb{R}^n)$;

(ii) $V^{p, \theta, \alpha}(f) < \infty$;

(iii) $\tilde{V}^{p, \theta, \alpha}(f) < \infty$.

Moreover,

$$2^{\alpha - 1} \| f \|_{\alpha, p, \theta} \leq \tilde{V}^{p, \theta, \alpha}(f) \leq V^{p, \theta, \alpha}(f) \leq 2(1 + \sqrt{n} + n) \| f \|_{\alpha, p, \theta}.$$ 

At the end of this section we discuss an application of the modulus of continuity $\sigma_p(f, \cdot)$ we have introduced in proving the inequality

$$\left( \int_A |f(x)|^p \, dx \right)^{1/p} \leq C(n, p) \sigma_p(f, (\lambda^n(A))^{1/n}),$$

which is equivalent to the bound

$$\left( \int_0^s |f^*(t)|^p \, dt \right)^{1/p} \leq c(n, p) \omega_p(f, s^{1/n})$$

given in [20] and [21], where $f^*$ is the nonincreasing equimeasurable rearrangement of $f$, that is,

$$f^*(t) = \inf\{y > 0 : |\{|f| > y\}| < t\}.$$

This inequality is applied in proving Ul’yanov-type embedding theorems (see [22], [23] and [20], [21], [24]). For example, the following simple sufficient condition for the integrability of the function $U(|f|)$ can be deduced from the above inequality.

Let $f \in L^p(\mathbb{R}^n)$, where $p \in [1, n]$ or $n = p = 1$. Let $U : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function such that $U(0) = 0$, $\lim_{t \to \infty} U(t) = \infty$, and there are positive constants $a$ and $r$ such that $U(t) \leq at^p$ whenever $0 < t < r$. Assume that there is a number $N > 0$ such that

$$\int_N^{+\infty} t^{-1-p} U(C(n, p) t \sigma_p(f, t^{-p/n})) \, dt < \infty,$$

where $C(n, p) = 1 + \nu_n^{-1/p} (n/p - 1)^{1/p-1}$, $\nu_n$ is the surface area of the unit sphere in $\mathbb{R}^n$ and $C(1, 1) = 1$. Then

$$\int_{\mathbb{R}^n} U(|f(x)|) \, dx < \infty.$$ 

Note that a necessary and sufficient condition for such an embedding was obtained in [21], Theorem 5.

We now move on to the proof of the bound (2.3).
\textbf{Lemma 2.3.} Let $f \in L^p(\mathbb{R}^n)$, where $p \in [1, n)$ or $n = p = 1$, and let $u \in C_0^\infty(\mathbb{R}^n)$ be a function such that $\|u\|_q \leq 1$ (recall that $q = p/(p-1)$). Then
\[ \int u(x)f(x)\,dx \leq C(n,p)\sigma_p(f,\|u\|_1^{p/n}), \]
where $C(n,p) = 1 + \nu_n^{-1/p}(n/p - 1)^{1/p-1}$, $\nu_n$ is the surface area of the unit sphere in $\mathbb{R}^n$ and $C(1,1) = 1$.

\textbf{Proof.} Using approximation, for an arbitrary vector field $\Phi \in C^\infty(\mathbb{R}^n)$ with $\|\Phi\|_q \leq \varepsilon$ and $\|\text{div}\,\Phi\|_q \leq 1$ we have
\[ \int \text{div}\,\Phi(x)f(x)\,dx \leq \sigma_p(f,\varepsilon). \]
Assume first that $n > 2$. Consider the function
\[ \varphi(x) = -(n-2)^{-1}\nu_n^{-1}\int_{\mathbb{R}^n} |x-y|^{-n+2}u(y)\,dy. \]
It is known that $\text{div}\,\nabla \varphi = \Delta \varphi = u$. Set
\[ K_1(x) = |x|^{-n+1}\text{Ind}_{\{|x|<R\}}(x) \quad \text{and} \quad K_2(x) = |x|^{-n+1}\text{Ind}_{\{|x|\geq R\}}(x). \]
We estimate $\nabla \varphi$:
\[ |\nabla \varphi(x)| \leq \nu_n^{-1}\int_{\mathbb{R}^n} |x-y|^{-n+1}|u(y)|\,dy = \nu_n^{-1}(K_1 * |u|(x) + K_2 * |u|(x)). \]
Thus,
\[ \|\nabla \varphi\|_q \leq \nu_n^{-1}(\|K_1\|_1\|u\|_q + \|K_2\|_1\|u\|_1) \]
\[ \leq \nu_n^{-1}\int_{\{|x|<R\}} |x|^{-n+1}\,dx + \nu_n^{-1}\|u\|_1 \left( \int_{\{|x|\geq R\}} |x|^{-n+q}\,dx \right)^{1/q} \]
\[ = R + \nu_n^{-1/p}\left((n-1)(q-1) - 1\right)^{-1/q}R^{1-n/p}\|u\|_1 \]
\[ = R + \nu_n^{-1/p}\left(\frac{n}{p} - 1\right)^{1/p-1}R^{1-n/p}\|u\|_1 \]
\[ \leq R + \nu_n^{-1/p}\left(\frac{n}{p} - 1\right)^{1/p-1}R^{1-n/p}\|u\|_1. \]
Now setting $R = \|u\|_1^{p/n}$ we obtain the bound
\[ \|\nabla \varphi\|_q \leq \left(1 + \nu_n^{-1/p}\left(\frac{n}{p} - 1\right)^{1/p-1}\right)\|u\|_1^{p/n}. \]
Thus,
\[ \int u(x)f(x)\,dx = \int \text{div}\,\nabla \varphi(x)f(x)\,dx \]
\[ \leq \sigma_p(f,\left(1 + \nu_n^{-1/p}\left(\frac{n}{p} - 1\right)^{1/p-1}\right)\|u\|_1^{p/n}) \]
\[ \leq \left(1 + \nu_n^{-1/p}\left(\frac{n}{p} - 1\right)^{1/p-1}\right)\sigma_p(f,\|u\|_1^{p/n}). \]
where we have used the concavity of $\sigma_p(f, \cdot)$. Therefore, the given estimate holds for $n > 2$. For $n = 2$ we can take

$$\varphi(x) = -(2\pi)^{-1} \int \ln |x - y| u(y) \, dy$$

and argue as above.

Thus, only the case $n = 1$ remains. In this case we consider the function $\varphi(x) = \int_{-\infty}^x u(t) \, dt$. This function satisfies

$$\int u(x) f(x) \, dx = \int \varphi'(x) f(x) \, dx \leq \sigma_1(f, \|\varphi\|_{\infty}) \leq \sigma_1(f, \|u\|_1).$$

Lemma 2.3 is proved.

**Corollary 2.2.** Let $f \in L^p(\mathbb{R}^n)$, where $p \in [1, n)$ or $n = p = 1$. Then for any Borel set $A$ in $\mathbb{R}^n$,

$$\left( \int_A |f(x)|^p \, dx \right)^{1/p} \leq C(n, p) \sigma_p \left( f, \left( \lambda^n(A) \right)^{1/n} \right)$$

with $C(n, p) = 1 + \nu_n^{-1/p} (n/p - 1)^{1/p-1}$, where $\nu_n$ is the surface area of the unit sphere in $\mathbb{R}^n$, and $C(1, 1) = 1$.

**Proof.** Assume first that $A$ is a bounded set, $A \subset B(0, R)$, where $B(0, R)$ is the ball of radius $R$ centred at the origin. Consider a function $u \in L^q(A)$ with $\|u\|_{L^q(A)} \leq 1$. There is a sequence of functions $u_m \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp}(u_m) \subset B(0, 2R)$, $\|u_m\|_q \leq 1$ and $u_m \to u$ in $L^q(B(0, 2R)) \ (\lambda^n$-a.e. in case $p = 1)$, where $u(x) = 0$ if $x \notin A$. For example, such a sequence can be constructed by means of convolution with a smooth compactly-supported probability density. For each function $u_m$, from Lemma 2.3 we have

$$\int u_m(x) f(x) \, dx \leq C(n, p) \sigma_p \left( f, \|u_m\|_{1}^{p/n} \right).$$

Since $\| \cdot \|_1$ is a continuous function on $L^q(B(0, 2R))$ for $p > 1$ (or by Lebesgue’s dominated convergence theorem in the case when $p = 1$), the above estimate is also valid for the function $u$. Thus, for every function $u \in L^q(A)$ with $\|u\|_{L^q(A)} \leq 1$ we have

$$\int_{A} u(x) f(x) \, dx \leq C(n, p) \sigma_p \left( f, \|u\|^p_{1/n} \right) \leq C(n, p) \sigma_p \left( f, \left( \lambda^n(A) \right)^{1/n} \right).$$

Taking the supremum over all functions $u$ with $\|u\|_{L^q(A)} \leq 1$ we obtain the desired estimate for bounded sets $A$. The case of an arbitrary set $A$ can be obtained by passing to the limit.

The corollary is proved.
§ 3. Besov classes on spaces with Gaussian measures

We now proceed to the infinite-dimensional Gaussian case. Let $X$ be a real Hausdorff locally convex space with topological dual space $X^*$. We recall that a nonnegative Borel measure $\gamma$ on $X$ is called a Radon measure if for every Borel set $B \subset X$ and every $\varepsilon > 0$ there is a compact set $K \subset B$ such that $\gamma(B \setminus K) < \varepsilon$. We also recall that a Radon probability measure $\gamma$ on $X$ is a centered Gaussian measure if, for every continuous linear functional $l$ on $X$, the image measure $\gamma \circ l^{-1}$ is either the Dirac measure at zero or with density of the form

$$
(2\pi c^2)^{-1/2} \exp \left( -\frac{t^2}{2c^2} \right). 
$$

From now on we assume $\gamma$ to be a centered Radon Gaussian measure on $X$. For a function $f \in L^p(\gamma)$ we set

$$
\|f\|_p := \|f\|_{L^p(\gamma)} := \left( \int_X |f|^p \, d\gamma \right)^{1/p}.
$$

Recall that the Cameron-Martin norm of a vector $h \in X$ is defined by

$$
|h|_H = \sup \left\{ l(h) : \int_X l^2 \, d\gamma \leq 1, l \in X^* \right\}.
$$

Let $H \subset X$ be the linear subspace of all vectors $h \in X$ whose Cameron-Martin norm is finite, $|h|_H < \infty$. This subspace $H$ is called the Cameron-Martin space of the measure $\gamma$. If $\gamma$ is the standard Gaussian measure on $\mathbb{R}^n$, then its Cameron-Martin space is $\mathbb{R}^n$ itself, and if $\gamma$ is a countable power of the standard Gaussian measure on the real line, then $H$ is the classical Hilbert space $l^2$. For a general centred Radon Gaussian measure, the Cameron-Martin space is also a separable Hilbert space (see Theorem 3.2.7 and Proposition 2.4.6 in [14]) with the inner product $\langle \cdot, \cdot \rangle_H$ generated by the Cameron-Martin norm $|\cdot|_H$.

Let $\{l_i\}_{i=1}^\infty \subset X^*$ be an orthonormal basis in the closure $X_\gamma^*$ of the set $X^*$ in $L^2(\gamma)$. There exists an orthonormal basis $\{e_i\}_{i=1}^\infty$ in $H$ such that $l_i(e_j) = \delta_{i,j}$ (see [14]). Below, we will use the fact that for any orthonormal family $l_1, \ldots, l_n \in X_\gamma^*$, the distribution of the vector $(l_1, \ldots, l_n)$, that is, the image of the measure $\gamma$, is the standard Gaussian measure $\gamma_n$ on $\mathbb{R}^n$ with density $(2\pi)^{-n/2} \exp(-|x|^2/2)$ with respect to the standard Lebesgue measure on $\mathbb{R}^n$.

Let $\mathcal{F}C^\infty(X)$ denote the set of all functions $\varphi$ of the form

$$
\varphi(x) = \psi(l_1(x), \ldots, l_n(x))
$$
on $X$, where $\psi \in C^\infty_b(\mathbb{R}^n)$ and $l_i \in X^*$, and let $\mathcal{F}C^\infty_0(X)$ denote the set of all functions $\varphi$ of the form $\varphi(x) = \psi(l_1(x), \ldots, l_n(x))$ on $X$, where $\psi \in C^\infty_0(\mathbb{R}^n)$ and $l_i \in X^*$. Let $\mathcal{F}C^\infty(X, H)$ be the set of all vector fields $\Phi$ of the form

$$
\Phi(x) = \sum_{i=1}^n \Psi_i(g_1(x), \ldots, g_n(x))h_i,
$$

where $\Psi_i \in C^\infty_b(\mathbb{R}^n)$, $g_i \in X^*$ and $h_i \in H$ and let $\mathcal{F}C^\infty_0(X, H)$ be the subset of this class that consists of the mappings for which the functions $\Psi_i$ can be chosen...
to have compact support. Note that in the above definition we can actually take vectors $h_i$ orthogonal in $H$ and functionals $g_i$ orthogonal in $X^*_\gamma$ so that $g_i(h_j) = \delta_{ij}$. We will call such vectors and functionals biorthogonal.

For every $\varphi \in \mathscr{F}C^\infty(X)$ of the form $\varphi(x) = \psi(l_1(x), \ldots, l_n(x))$ set

$$\nabla \varphi(x) = \sum_{j=1}^n \partial_j \psi(l_1(x), \ldots, l_n(x)) e_j,$$

where $\{l_i\}$ and $\{e_i\}$ are biorthogonal. Let $\text{div}_\gamma$ be the ‘adjoint’ operator to the gradient operator $\nabla$ with respect to the measure $\gamma$, that is,

$$\int_X (\text{div}_\gamma \Phi) \varphi \, d\gamma = -\int_X \langle \Phi, \nabla \varphi \rangle_H \, d\gamma$$

for arbitrary $\Phi \in \mathscr{F}C^\infty(X, H)$ and $\varphi \in \mathscr{F}C^\infty(X)$. It is easy to check that

$$\text{div}_\gamma \Phi(x) = \sum_{j=1}^n \partial_j \psi_j(l_1(x), \ldots, l_n(x)) - l_j(x) \partial_j \psi_j(l_1(x), \ldots, l_n(x))$$

for a vector field $\Phi \in \mathscr{F}C^\infty(X, H)$ of the form

$$\Phi(x) = \sum_{i=1}^n \psi_i(l_1(x), \ldots, l_n(x)) e_i$$

with biorthogonal $\{l_i\}$ and $\{e_i\}$. We note that the divergence $\text{div}_\gamma \Phi$ of a vector field $\Phi$ in $\mathscr{F}C^\infty(X, H)$ is a bounded function.

Recall that the Ornstein-Uhlenbeck semigroup on the space of functions $f \in L^1(\gamma)$ (see [14] and [25]) is defined by the equality

$$T_t f(x) := \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}y) \, \gamma(dy).$$

Let $L$ be the generator of the Ornstein-Uhlenbeck semigroup considered on the space $L^2(\gamma)$. The operator $L$ is called the Ornstein-Uhlenbeck operator and its action on a function $\varphi \in \mathscr{F}C^\infty(X)$ of the form $\varphi(x) = \psi(l_1(x), \ldots, l_n(x))$ is defined by the equality

$$L \varphi(x) = \text{div}_\gamma \nabla \varphi = \Delta \psi(l_1(x), \ldots, l_n(x)) - \sum_{j=1}^n l_j(x) \partial_j \psi(l_1(x), \ldots, l_n(x)).$$

We now fix an orthonormal basis $\{l_n\} \subset X^*$ in $X^*_\gamma$. For any function $f \in L^1(\gamma)$ let $E_n f$ be a function on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \psi E_n f \, d\gamma_n = \int_X \psi(l_1(x), \ldots, l_n(x)) f(x) \gamma(dx) \quad \forall \psi \in C^\infty_0(\mathbb{R}^n),$$

where $\gamma_n$ is the standard Gaussian measure on $\mathbb{R}^n$. This equality actually means that the function $E_n f(l_1, \ldots, l_n)$ is the conditional expectation of $f$ with respect to
the σ-field generated by the functions \(l_1, \ldots, l_n\). By a known property of conditional expectations, for any function \(f \in L^p(\gamma)\) we have

\[
\|f - \mathbb{E}_n f(l_1, \ldots, l_n)\|_p \to 0 \quad \text{as} \quad n \to \infty.
\]

We also introduce the functions \(C(p)\) and \(c_t\), which will be used below:

\[
C(p) := \left(2\pi\right)^{-1/2} \int_{\mathbb{R}} |s|^p e^{-s^2/2} ds \] \(1/p\) \quad \text{and} \quad c_t := \int_0^t \frac{e^{-\tau}}{\sqrt{1 - e^{-2\tau}}} d\tau.
\]

We note that \(c_t \leq (2t)^{1/2}\) and \(\lim_{t \to \infty} c_t = \pi/2\).

In what follows we will omit the region of integration in integrals over the whole measure space.

Now we define the Gaussian modulus of continuity \(\sigma_{\gamma,p}(f, \cdot, \cdot)\), which is an analogue of the function \(\sigma_p(f, \cdot, \cdot)\) introduced above in the case of \(\mathbb{R}^n\).

**Definition 3.1.** Let \(f \in L^p(\gamma)\). Set

\[
\sigma_{\gamma,p}(f, \varepsilon) := \sup \left\{ \int \text{div}_\gamma \Phi f \, d\gamma : \Phi \in \mathcal{F}_0^\infty(X, H), \|\text{div}_\gamma \Phi\|_q \leq 1, \|\Phi\|_q \leq \varepsilon \right\}.
\]

We note that the function \(\sigma_{\gamma,p}(f, \cdot, \cdot)\) is continuous, concave and nondecreasing on \((0, +\infty)\), which can be proved similarly to Lemma 2.1. Thus, by approximation, in the definition of the quantity \(\sigma_{\gamma,p}(f, \varepsilon)\) the supremum can be taken over all vector fields \(\Phi \in \mathcal{F}_0^\infty(X, H)\) such that \(\|\text{div}_\gamma \Phi\|_q \leq 1\) and \(\|\Phi\|_q \leq \varepsilon\).

Using Definition 3.1 we can now introduce Besov classes on a locally convex space endowed with a Gaussian measure.

**Definition 3.2.** Let \(\alpha \in (0, 1], p \in [1, \infty)\) and \(\theta \in [1, \infty]\). We say that a function \(f \in L^p(\gamma)\) belongs to the Gaussian Besov space \(B_{p,\theta,\alpha}^\gamma(\gamma)\) if the quantity

\[
V_{p,\theta,\alpha}^\gamma(f) = \left( \int_0^\infty \left[ s^{-\alpha} \sigma_{\gamma,p}(f, s) \right]^{\theta} s^{-1} ds \right)^{1/\theta}
\]

is finite.

We note that when \(\theta = \infty\) the above definition coincides with the definition of the Gaussian Nikolskii-Besov class introduced in [1].

We will give an equivalent description of these Gaussian Besov classes in terms of the following two characteristics.

**Definition 3.3.** Given a function \(f \in L^p(\gamma), p \in [1, \infty)\), set

\[
a_{\gamma,p}(f, t) := \left( \iint |f(e^{-tx} + \sqrt{1 - e^{-2ty}}) - f(x)|^p \gamma(dx)\gamma(dy) \right)^{1/p}
\]

and

\[
A_{p,\theta,\alpha}^\gamma(f) := \left( \int_0^\infty \left[t^{-\alpha/2}a_{\gamma,p}(f, t)\right]^{\theta} t^{-1} dt \right)^{1/\theta}.
\]
We note that
\[ \|f - T_t f\|_p \leq a_{\gamma, p}(f, t). \]
In a sense, the function \(a_{\gamma, p}(f, \cdot)\) can be regarded as a Gaussian replacement for the finite-dimensional modulus of continuity \(\omega_p(f, \cdot)\), since we cannot use shifts \(f_h\) of the function \(f \in L^p(\gamma)\) directly as these shifts may fail to be in \(L^p(\gamma)\).

We need the following technical lemma.

**Lemma 3.1.** Let \(\gamma_n\) be the standard Gaussian measure on \(\mathbb{R}^n\). Then for any function \(f \in L^p(\gamma_n)\), where \(p \in [1, \infty)\),
\[ a_{\gamma, p}(f, t) \leq 2\sigma_{\gamma_n, p}(f, 2^{-1} C(p) c_t). \]

**Proof.** For every function \(\varphi \in C_0^\infty(\mathbb{R}^{2n})\) we can write
\[
\int \int \varphi(x, y) [f(e^{-t}x + \sqrt{1 - e^{-2t}}y) - f(x)] \gamma_n(dx) \gamma_n(dy)
= \int f(u) \int [\varphi(e^{-t}u - \sqrt{1 - e^{-2t}}v, \sqrt{1 - e^{-2t}}u + e^{-t}v) - \varphi(u, v)] \gamma_n(dv) \gamma_n(du)
= \int f(u) \int_0^t \frac{\partial}{\partial s} g_s(u) \, ds \, \gamma_n(du),
\]
where
\[ g_s(u) := \int \varphi(e^{-s}u - \sqrt{1 - e^{-2s}}v, \sqrt{1 - e^{-2s}}u + e^{-s}v) \, \gamma_n(dv). \]

We now note that for an arbitrary function \(\psi \in C_0^\infty(\mathbb{R}^n)\) we have
\[
\int \psi(u) \frac{\partial}{\partial s} g_s(u) \, \gamma_n(du)
= \frac{\partial}{\partial s} \int \psi(u) \int \varphi(e^{-s}u - \sqrt{1 - e^{-2s}}v, \sqrt{1 - e^{-2s}}u + e^{-s}v) \, \gamma_n(dv) \, \gamma_n(du)
= \frac{\partial}{\partial s} \int \int \varphi(x, y) \psi(e^{-s}x + \sqrt{1 - e^{-2s}}y) \, \gamma_n(dx) \, \gamma_n(dy)
= \frac{e^{-s}}{\sqrt{1 - e^{-2s}}}
\times \int \int \varphi(x, y) \langle \nabla \psi(e^{-s}x + \sqrt{1 - e^{-2s}}y), e^{-s}y - \sqrt{1 - e^{-2s}}x \rangle \, \gamma_n(dx) \, \gamma_n(dy)
= \frac{e^{-s}}{\sqrt{1 - e^{-2s}}}
\times \int \langle \nabla \psi(u), \int v \varphi(e^{-s}u - \sqrt{1 - e^{-2s}}v, \sqrt{1 - e^{-2s}}u + e^{-s}v) \, \gamma_n(dv) \rangle \, \gamma_n(du)
= - \int \psi(u) \, \text{div}_{\gamma_n} G_s(u) \, \gamma_n(du),
\]
where
\[ G_s(u) \]
\[ := \frac{e^{-s}}{\sqrt{1 - e^{-2s}}} \int v \varphi(e^{-s}u - \sqrt{1 - e^{-2s}}v, \sqrt{1 - e^{-2s}}u + e^{-s}v) \, \gamma_n(dv) \in C_b^\infty(\mathbb{R}^n). \]
Thus, \[
\frac{\partial}{\partial s} g_s(u) = \text{div}\gamma(-G_s(u))
\]
and \[
\int f \int_0^t \frac{\partial}{\partial s} g_s \, ds \, d\gamma_n = \int \text{div}_\gamma \left( - \int_0^t G_s \, ds \right) \, f \, d\gamma_n.
\]
We now observe that \[
\text{div}_\gamma \left( - \int_0^t G_s \, ds \right) = \int_0^t \text{div}_\gamma (-G_s) \, ds = \int_0^t \frac{\partial}{\partial s} g_s(u) \, ds
\]
and that \[
\left\| \text{div}_\gamma \left( - \int_0^t G_s \, ds \right) \right\|_q \leq 2\|\varphi\|_{L^q(\gamma_n \otimes \gamma_n)}.
\]
Moreover, \[
\left\| \int_0^t G_s \, ds \right\|_q \leq \int_0^t \|G_s\|_q \, ds,
\]
and it remains to estimate \(\|G_s\|_q\). To do this, we note that for an arbitrary vector field \(\Psi \in C_0^\infty(\mathbb{R}^n)\) we have
\[
\int \langle \Psi, G_s \rangle \, d\gamma_n
\]
\[
= \frac{e^{-s}}{\sqrt{1-e^{-2s}}} \int \int \langle \Psi(u), v \rangle \varphi(e^{-s}u - \sqrt{1-e^{-2s}}v, \sqrt{1-e^{-2s}}u + e^{-s}v) \gamma_n(dv) \gamma_n(du)
\]
\[
\leq \frac{e^{-s}}{\sqrt{1-e^{-2s}}} \|\varphi\|_{L^q(\gamma_n \otimes \gamma_n)} \left( \int \int \|\Psi(u), v\|^p \gamma_n(dv) \gamma_n(du) \right)^{1/p}
\]
\[
\leq \frac{e^{-s}}{\sqrt{1-e^{-2s}}} C(p) \|\varphi\|_{L^q(\gamma_n \otimes \gamma_n)} \|\Psi\|_p.
\]
Thus, \[
\|G_s\|_q \leq C(p) \frac{e^{-s}}{\sqrt{1-e^{-2s}}} \|\varphi\|_{L^q(\gamma_n \otimes \gamma_n)}
\]
and \[
\left\| \int_0^t G_s \, ds \right\|_q \leq \int_0^t \|G_s\|_q \, ds \leq C(p) c_t \|\varphi\|_{L^q(\gamma_n \otimes \gamma_n)}.
\]
Hence, for an arbitrary function \(\varphi \in C_0^\infty(\mathbb{R}^{2n})\) with \(\|\varphi\|_{L^q(\gamma_n \otimes \gamma_n)} = 1\) we have
\[
\int \int \varphi(x,y) \left[ f(e^{-t}x + \sqrt{1-e^{-2t}y}) - f(x) \right] \gamma_n(dx) \gamma_n(dy)
\]
\[
= \int f \int_0^t \frac{\partial}{\partial s} g_s \, ds \, d\gamma_n = \int \text{div}_\gamma \left( - \int_0^t G_s \, ds \right) \, f \, d\gamma_n \leq 2\sigma_{\gamma_n,p}(f, 2^{-1}C(p)c_t).
\]
Taking the supremum over the functions \(\varphi \in C_0^\infty(\mathbb{R}^{2n})\) with \(\|\varphi\|_{L^q(\gamma_n \otimes \gamma_n)} = 1\) we obtain the required bound.

Lemma 3.1 is proved.
The following theorem is a Gaussian analogue of Theorem 2.1.

**Theorem 3.1.** For any function \( f \in L^p(\gamma) \), where \( p \in [1, \infty) \),
\[
a_{\gamma,p}(f, t) \leq 2\sigma_{\gamma,p}(f, 2^{-1}C(p)c_t).
\]

If \( p > 1 \), then a reverse bound holds:
\[
\sigma_{\gamma,p}(f, \varepsilon) \leq \left(1 + C\left(\frac{p}{p-1}\right)\right)a_{\gamma,p}(f, \varepsilon^2).
\]

**Proof.** To prove the first part of the theorem we fix an orthonormal basis \( \{l_n\} \subset X^* \) in \( X^*_\gamma \). By Lemma 3.1 we have
\[
a_{\gamma,n,p}(E_n f, t) \leq 2\sigma_{\gamma,n,p}(E_n f, 2^{-1}C(p)c_t) \leq 2\sigma_{\gamma,p}(f, 2^{-1}C(p)c_t).
\]

We observe that \( a_{\gamma,n,p}(f, t) \to a_{\gamma,p}(f, t) \) as \( n \to \infty \), which completes the proof.

Now let \( f \in L^p(\gamma) \) for some \( p > 1 \). For an arbitrary vector field \( \Phi \in \mathcal{C}_0^\infty(X, H) \) we can write
\[
\int \text{div}_\gamma \Phi T_t f \, d\gamma = e^{-t} \int \text{div}_\gamma \Phi f \, d\gamma
\]
\[
= -\frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \int \int f(u)\langle \Phi(e^{-t}u - \sqrt{1 - e^{-2t}}v), e^{-t}v + \sqrt{1 - e^{-2t}}u \rangle_H \gamma(dv)\gamma(du)
\]
\[
= -\frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \int \int f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\langle \Phi(x), y \rangle_H \gamma(dy)\gamma(dx).
\]

We observe that
\[
\int f(x)\langle \Phi(x), y \rangle_H \gamma(dy) = 0
\]
for an arbitrary fixed point \( x \). Thus, the last expression is equal to
\[
-\frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \int \int \langle \Phi(x), y \rangle_H [f(e^{-t}x + \sqrt{1 - e^{-2t}}y) - f(x)] \gamma(dy)\gamma(dx)
\]
\[
\leq t^{-1/2}a_{\gamma,p}(f, t)\left(\int \|\langle \Phi(x), y \rangle_H\|_q^q \gamma(dy)\gamma(dx)\right)^{1/q} = C(q)t^{-1/2}a_{\gamma,p}(f, t)\|\Phi\|_q.
\]

So, we have established the estimate
\[
\int \text{div}_\gamma \Phi T_t f \, d\gamma \leq C(q)t^{-1/2}a_{\gamma,p}(f, t)\|\Phi\|_q.
\]

Now we note that
\[
\int \text{div}_\gamma \Phi f \, d\gamma = \int \text{div}_\gamma \Phi f - T_t f \, d\gamma + \int \text{div}_\gamma \Phi T_t f \, d\gamma.
\]

The first term in the above expression is bounded by
\[
a_{\gamma,p}(f, t)\|\text{div}_\gamma \Phi\|_q,
\]
and the second term, as we have proved, is not greater than
\[ C(q) t^{-1/2} a_{\gamma,p}(f, t) \| \Phi \|_q. \]

Taking \( t = \varepsilon^2 \) we obtain
\[ \sigma_{\gamma,p}(f, \varepsilon) \leq (1 + C(q)) a_{\gamma,p}(f, \varepsilon^2). \]

Theorem 3.1 is proved.

**Remark 3.1.** We note that the case when \( p = 1 \) has known specific features for the Gaussian measure, and in this case the existence of the reverse bound in Theorem 3.1 is still unknown.

We also note that from the first inequality in Theorem 3.1, as the function \( \sigma_{\gamma,p}(f, \cdot) \) is concave and monotonic, it follows that
\[ a_{\gamma,p}(f, t) \leq \max\{2, C(p) \sqrt{2}\} \sigma_{\gamma,p}(f, \sqrt{t}). \]

**Corollary 3.1.** For any function \( f \in B_{p,\theta}^\gamma(\gamma) \), where \( p \in [1, \infty) \), and any Lipschitz function \( u: \mathbb{R} \to \mathbb{R} \)
\[ a_{\gamma,p}(u(f), t) \leq 2^{1-\alpha}(\alpha \theta)^{1/\theta} \operatorname{Lip}(u) C(p)^\alpha c_t^\alpha V_p^{\theta,\alpha}(f), \]
where \( \operatorname{Lip}(u) \) is the Lipschitz constant of the function \( u \).

**Proof.** By the Lipschitz continuity of \( u \) and Theorem 3.1 we can write
\[ a_{\gamma,p}(u(f), t) = \left( \int \int |u(f(e^{-t} x + \sqrt{1-e^{-2t}} y)) - u(f(x))|^p \gamma(dy) \gamma(dx) \right)^{1/p} \]
\[ \leq \operatorname{Lip}(u) \left( \int \int |f(e^{-t} x + \sqrt{1-e^{-2t}} y) - f(x)|^p \gamma(dy) \gamma(dx) \right)^{1/p} \]
\[ = \operatorname{Lip}(u) a_{\gamma,p}(f, t) \leq 2 \operatorname{Lip}(u) \sigma_{\gamma,p}(f, 2^{-1} C(p) c_t). \]

We now note that
\[ (2^{-1} C(p) c_t)^{-\alpha \theta} [\sigma_{\gamma,p}(f, 2^{-1} C(p) c_t)]^\theta \]
\[ \leq \alpha \theta \int_{2^{-1} C(p) c_t}^{\infty} r^{-\alpha \theta - 1} [\sigma_{\gamma,p}(f, r)]^\theta dr \leq \alpha \theta [V_p^{\theta,\alpha}(f)]^{\theta}. \]

Thus,
\[ a_{\gamma,p}(u(f), t) \leq 2^{1-\alpha}(\alpha \theta)^{1/\theta} \operatorname{Lip}(u) C(p)^\alpha c_t^\alpha V_p^{\theta,\alpha}(f), \]
as required.

As another corollary, we can also see that for \( p > 1 \) the conditions \( V_p^{\theta,\alpha}(f) < \infty \) and \( A_p^{\theta,\alpha}(f) < \infty \) are equivalent.

**Corollary 3.2.** For any function \( f \in B_{p,\theta}^\gamma(\gamma) \), where \( p \in [1, \infty) \),
\[ A_p^{\theta,\alpha}(f) \leq 2^{1-\alpha+1/\theta} C(p)^\alpha V_p^{\theta,\alpha}(f). \]
Moreover, the converse statement holds for \( p \in (1, \infty) \), that is, if the quantity \( A_p^{\theta,\alpha}(f) \) is finite for a function \( f \in L^p(\gamma) \), then \( f \in B_{p,\theta}^\gamma(\gamma) \) and
\[ V_p^{\theta,\alpha}(f) \leq 2^{-1/\theta} \left( 1 + C\left( \frac{p}{p-1} \right) \right) A_p^{\theta,\alpha}(f). \]
Proof. By Theorem 3.1, any function \( f \in B_{p,\theta}^\alpha(\gamma) \) satisfies
\[
a_{\gamma,p}(f,t) \leq 2\sigma_{\gamma,p}(f,2^{-1}C(p)c_t) \leq 2\sigma_{\gamma,p}(f,2^{-1}C(p)t^{1/2}).
\]
Thus,
\[
[A_{\gamma}^{p,\theta,\alpha}(f)]^\theta = \int_0^\infty [t^{-\alpha/2}a_{\gamma,p}(f,t)]^{\theta} t^{-1} dt \leq 2^\theta \int_0^\infty t^{-\alpha\theta/2} [\sigma_{\gamma,p}(f,2^{-1}C(p)t^{1/2})]^{\theta} t^{-1} dt = 2^{1+\theta-\alpha\theta} C(p)^{\alpha\theta} \int_0^\infty r^{-\alpha\theta} [\sigma_{\gamma,p}(f,r)]^{\theta} r^{-1} dr \leq 2^{1+\theta-\alpha\theta} C(p)^{\alpha\theta} [V_{\gamma}^{p,\theta,\alpha}(f)]^\theta,
\]
which is the required bound.

We now prove the converse statement. For any function \( f \in L^p(\gamma) \) with \( p > 1 \), Theorem 3.1 states that
\[
\sigma_{\gamma,p}(f,\varepsilon) \leq (1 + C(q)) a_{\gamma,p}(f,\varepsilon^2),
\]
which yields
\[
[V_{\gamma}^{p,\theta,\alpha}(f)]^\theta = \int_0^\infty [r^{-\alpha}\sigma_{\gamma,p}(f,r)]^{\theta} r^{-1} dr \leq (1 + C(q))^{\theta} \int_0^\infty [r^{-\alpha}a_{\gamma,p}(f,r^2)]^{\theta} r^{-1} dt = 2^{1+\theta-\alpha\theta} C(p)^{\alpha\theta} [A_{\gamma}^{p,\theta,\alpha}(f)]^\theta.
\]
The corollary is proved.

We now proceed to embedding theorems of log-Sobolev-type for Besov classes with respect to a Gaussian measure. As we mentioned in §1, the main idea of the proof is to use the short-time behaviour of the Ornstein-Uhlenbeck semigroup together with its hypercontractivity property, which is similar to the approach in [18] in a certain sense.

**Theorem 3.2.** For any function \( f \in B_{p,\theta}^\alpha(\gamma) \), where \( p \in [1,\infty) \), and any number \( \beta \in (0,\alpha) \) the function \( |f| \ln |f|^{\beta/2} \) belongs to \( L^p(\gamma) \). Moreover, there is a constant \( C = C(p,\theta,\alpha,\beta) \), depending only on the parameters \( p, \theta, \alpha \) and \( \beta \), such that
\[
\left( \int |f|^p \ln(|f|) \|f\|_p^{-\beta/2} d\gamma \right)^{1/p} \leq C (\|f\|_p + V_{\gamma}^{p,\theta,\alpha}(f)).
\]

**Proof.** Firstly, consider the case of \( p > 1 \). We recall (see [14], Theorem 5.5.3) that for any function \( f \in L^p(\gamma) \) we have
\[
\|T_t f\|_{1+(p-1)e^{2t}} \leq \|f\|_p;
\]
this is called the hypercontractivity property of the Ornstein-Uhlenbeck semigroup. For an arbitrary number \( s > 0 \), let \( A_s := \{|f| \geq s\} \) and let \( I_{A_s} \) be the indicator
function of this set. We note that the function $\tau \mapsto \max\{|\tau|, s\}$ is 1-Lipschitz. Thus, for any function $\varphi \in \mathcal{F}_0^{\infty}(X)$ and any number $t > 0$, by Corollary 3.1 and the hypercontractivity property we have

$$
\int \varphi I_{A_s}(|f| - s)\,d\gamma = \int I_{A_s}\varphi(\max\{|f|, s\} - s)\,d\gamma \\
= \int I_{A_s}\varphi[\max\{|f|, s\} - T_\ell(\max\{|f|, s\})]\,d\gamma + \int I_{A_s}\varphi T_\ell(\max\{|f|, s\} - s)\,d\gamma \\
\leq \|\varphi\|_q \|\max\{|f|, s\} - T_\ell(\max\{|f|, s\})\|_p \\
+ \|I_{A_s}\varphi\|_{1+(p-1)e^{2t}} \|T_\ell(\max\{|f|, s\} - s)\|_{1+(p-1)e^{2t}} \\
\leq 2^{1-\alpha}(\alpha\theta)^{1/\theta}C(p)^\alpha V_\gamma^{p,\theta,\alpha}(f)t^{\alpha/2} + \|I_{A_s}\varphi\|_{1+(p-1)e^{2t}} \|I_{A_s}(|f| - s)\|_p.
$$

We note that

$$
\frac{1+(p-1)e^{2t}}{(p-1)e^{2t}} = 1 + \frac{1}{(p-1)e^{2t}} = q\left(\frac{1}{q} + \frac{1}{pe^{2t}}\right) \leq q.
$$

Thus, we can apply Hölder’s inequality to the expression

$$
\|I_{A_s}\varphi\|_{1+(p-1)e^{2t}} \leq \left[\gamma(A_s)\right]^{e^{2t}/q+pe^{2t}} \|\varphi\|_q.
$$

Taking the supremum over the functions $\varphi$ with $\|\varphi\|_q = 1$ we obtain the estimate

$$
\|I_{A_s}(|f| - s)\|_p \leq 2^{1-\alpha}(\alpha\theta)^{1/\theta}C(p)^\alpha V_\gamma^{p,\theta,\alpha}(f)t^{\alpha/2} + \left[\gamma(A_s)\right]^{e^{2t}/q+pe^{2t}} \|I_{A_s}(|f| - s)\|_p.
$$

We now observe that

$$
e^{2t} - 1 \leq p^{-1} \frac{q^{-1}(e^{2t} - 1)}{1 + q^{-1}(e^{2t} - 1)} \geq p^{-1} \frac{2q^{-1}t}{1 + 2q^{-1}t} \geq \frac{t}{pq}
$$

whenever $t \leq 1/2$. Thus, whenever $t \leq 1/2$, we have

$$
\|I_{A_s}(|f| - s)\|_p \leq 2^{1-\alpha}(\alpha\theta)^{1/\theta}C(p)^\alpha V_\gamma^{p,\theta,\alpha}(f)t^{\alpha/2} + \left[\gamma(A_s)\right]^{(pq)-1} \|I_{A_s}(|f| - s)\|_p.
$$

For the sets $A_s$ with $\gamma(A_s) \leq e^{-2pq}$ we can take $t = pq(-\ln \gamma(A_s))^{-1} \leq 1/2$, which yields the bound

$$
\|I_{A_s}(|f| - s)\|_p \leq 2^{1-\alpha}(\alpha\theta)^{1/\theta}C(p)^\alpha(pq)^{\alpha/2}V_\gamma^{p,\theta,\alpha}(f)[-\ln \gamma(A_s)]^{-\alpha/2} + e^{-1} \|I_{A_s}(|f| - s)\|_p
$$

since

$$
\left[\gamma(A_s)\right]^{(pq)-1} = e^{-(pq)^{-1}[-\ln \gamma(A_s)]t} = e^{-1}
$$
for such $t$. The inequality obtained can be rewritten in the form of

$$
\|I_{A_s}(|f| - s)\|_p \leq C(p, \theta, \alpha) V_p^{\gamma, \theta, \alpha}(f) [-\ln \gamma(A_s)]^{-\alpha/2},
$$

where $C(p, \theta, \alpha) = 2^{1-\alpha}(\alpha \theta)^{1/\theta} C(p)\alpha e(e - 1)^{-1}(pq)^{\alpha/2}$. We now observe that $\gamma(A_s) \leq \|f\|_p s^{-p}$ and $I_{A_s}(|f| - s) \geq 2^{-1} I_{A_{2s}}|f|$. Thus, for $t \geq e^{2q}$, taking $s = t\|f\|_p$ we get

$$
\int I_{\{|f| \geq 2t\|f\|_p\}}|f|^p\, d\gamma \leq (2p^{-\alpha/2} C(p, \theta, \alpha))^{p_{\gamma} V_p^{\gamma, \theta, \alpha}(f)} [\ln t]^{-p\alpha/2}.
$$

Multiplying both sides of the inequality by $t^{-1[\ln t]^{-1+\beta/2}}$ and integrating with respect to $t$ from $e^{2q}$ to $+\infty$ we obtain

$$
\int_{e^{2q}}^{\infty} t^{-1[\ln t]^{-1+\beta/2}} \int I_{\{|f| \geq 2t\|f\|_p\}}|f|^p\, d\gamma \, dt
\leq (2p^{-\alpha/2} C(p, \theta, \alpha))^{p_{\gamma} V_p^{\gamma, \theta, \alpha}(f)} [\ln t]^{-1-\alpha+\beta/2} t^{-1} dt
\leq (2p^{-\alpha/2} C(p, \theta, \alpha))^{p_{\gamma} V_p^{\gamma, \theta, \alpha}(f)} [\ln t]^{-\alpha+\beta/2} [V_p^{\gamma, \theta, \alpha}(f)]^p.
$$

The left-hand side of the above estimate is equal to

$$
\int |f|^p I_{\{|f| \geq 2e^{2q} \|f\|_p\}} \int_{e^{2q}}^{\|f\|_p} t^{-1[\ln t]^{-1+\beta/2}} \, dt \, d\gamma
= 2(p\beta)^{-1} \int |f|^p I_{\{|f| \geq 2e^{2q} \|f\|_p\}} \left(\ln(\|f\|_p^{-1})\|f\|_p^{-1}\right)^{\beta/2} \, d\gamma
\geq 2(p\beta)^{-1} \int |f|^p I_{\{|f| \geq 2e^{2q} \|f\|_p\}} \ln(\|f\|_p^{-1})\|f\|_p^{-1} \, d\gamma
- 2(p\beta)^{-1} (2q)^{\beta/2} \|f\|_p
= 2(p\beta)^{-1} \int |f|^p \ln(\|f\|_p^{-1})\|f\|_p^{-1} \, d\gamma
- 2(p\beta)^{-1} (2q)^{\beta/2} \|f\|_p.
$$

Thus, since $a|\ln a|^{\beta/2} \leq 2e^{2q}(2q + 1)^{\beta/2}$ for $a \in [0, 2e^{2q}]$, we have

$$
\int |f|^p \ln(\|f\|_p^{-1})\|f\|_p^{-1} \, d\gamma \leq C_1(p, \alpha, \beta) [V_p^{\gamma, \theta, \alpha}(f)]^p + C_2(p, \alpha, \beta) \|f\|_p,
$$

where

$$
C_1(p, \theta, \alpha, \beta) = (2p^{-\alpha/2} C(p, \theta, \alpha))^p (\alpha - \beta)^{-1}(2q)^{-p(\alpha-\beta)/2} p^{\beta/2}
$$

and

$$
C_2(p, \theta, \alpha, \beta) = (2p e^{2q} p^{\beta/2} + (2q)^{\beta/2} p^{\beta/2}).
$$

The theorem is proved for $p > 1$. 

Now let \( f \in B_{1,\theta}^{\alpha}(\gamma) \). We note that \( a_{\gamma,2}(\sqrt{|f|}, t) \leq \sqrt{a_{\gamma,1}(f, t)} \). Thus, by Corollary 3.2,

\[
V_2^{2,2\theta,\alpha/2}(\sqrt{|f|}) \leq 2^{-1/\theta}(1 + C(2))A_1^{2,2\theta,\alpha/2}(\sqrt{|f|}) \leq 2^{-1/\theta}(1 + C(2))\sqrt{A_1^{1,\theta,\alpha}(f)}.
\]

\[
\leq 2^{-1/\theta}(1 + C(2))2^{1/2-\alpha/2+1/(2\theta)}C(1)^{\alpha/2}\sqrt{V_2^{1,\theta,\alpha}(f)}.
\]

So, \( \sqrt{|f|} \in B_{2,2\theta}^{\alpha/2}(\gamma) \). We have already proved that for any \( \beta \in (0, \alpha/2) \) the function \( \sqrt{|f|}\ln(\sqrt{|f|})^{\beta/2} \) belongs to the space \( L^2(\gamma) \), and

\[
2^{-\beta/2}\left(\int |f|\ln(|f|)\|f\|_1^{-1})^{2\beta/2}d\gamma\right)^{1/2} \leq C(\|\sqrt{|f|}\|_2 + V_2^{2,2\theta,\alpha/2}(\sqrt{|f|})).
\]

We note that \( 2\beta \) is an arbitrary number in the interval \((0, \alpha)\) and the inequality obtained is equivalent to the required bound.

Theorem 3.2 is proved.

Remark 3.2. We note that the limiting case \( \beta = \alpha \) in Theorem 3.2 remains unclear. It would be interesting to know whether, in fact, the Besov class \( B_{p,\theta}^{\alpha}(\gamma) \) is embedded in the space \( L^p(\log L)^{p\alpha/2} \), as in the classical logarithmic Sobolev inequality.

§ 4. Equivalent definitions of Gaussian Besov spaces

In this section we obtain an equivalent characterization of the Gaussian Besov classes, which we introduced above, in terms of the Ornstein-Uhlenbeck and Poisson-Hermite semigroups. In particular, we prove the equivalence of Definition 3.2 and the definition of Gaussian Besov spaces in [19].

Let \( \mathcal{P}_k(\gamma) \) denote the closure in \( L^2(\gamma) \) of the set of all functions \( g \) of the form

\[
g(x) = P(l_1(x), \ldots, l_n(x)), \quad l_1, \ldots, l_n \in X^*, \quad n \in \mathbb{N},
\]

where \( P \) is an arbitrary polynomial on \( \mathbb{R}^n \) of degree not greater than \( k \). Note that \( \mathcal{P}_k(\gamma) \subset L^p(\gamma) \) for any \( p \in [1, \infty) \). We also note that, in the definition of the space \( \mathcal{P}_k(\gamma) \), the closure can be taken in \( L^p(\gamma) \) for any \( p \in [1, \infty) \). In doing this we obtain the same spaces. Let \( \mathcal{H}_0 \) be the space of all constant functions and let \( \mathcal{H}_k \) be the orthogonal complement of the space \( \mathcal{P}_{k-1}(\gamma) \) with respect to the space \( \mathcal{P}_k(\gamma) \). Elements of the space \( \mathcal{H}_k \) are called Hermite polynomials of degree \( k \).

Recall (see [14], §2.9) that the space \( L^2(\gamma) \) can be decomposed into a direct sum of mutually orthogonal subspaces \( \mathcal{H}_k \):

\[
L^2(\gamma) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.
\]

Note that for any \( g \in \mathcal{H}_k \)

\[
Lg = -kg \quad \text{and} \quad T_t g = e^{-kt} g,
\]

where \( L \) and \( T_t \) are the Ornstein-Uhlenbeck operator and semigroup, respectively.
Define the action of the operator $I_k : L^p(\gamma) \to \mathcal{H}_k$ on a function $f \in L^p(\gamma)$ by the equality

$$\int fg \, d\gamma = \int I_k(f)g \, d\gamma \quad \forall g \in \mathcal{H}_k.$$  

If $f \in L^2(\gamma)$, the image $I_k(f)$ is the orthogonal projection in $L^2(\gamma)$ of the function $f$ onto the space $\mathcal{H}_k$. Note that $I_k$ is a bounded operator in $L^p(\gamma)$ for $p \in (1, \infty)$ (see [14], Corollary 5.5.4).

Recall that for a function $f \in L^1(\gamma)$ the Poisson-Hermite semigroup is defined by the equality

$$Q_t f(x) := \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-sT_s f(x)} ds \quad \text{for } t > 0,$$

where $T_t$ is the Ornstein-Uhlenbeck semigroup.

To shorten the notation we will use the following norm:

$$\|\sigma\|_{*, \theta} := \left( \int_0^\infty \left| \sigma(t) t^{\theta - 1} \right| \, dt \right)^{1/\theta} \quad \text{for } \theta \in [1, \infty),$$

$$\|\sigma\|_{*, \infty} := \text{ess sup}_{t > 0} |\sigma(t)|.$$

Note that for $\theta \in [1, \infty]$ we have

$$\|\sigma(t)\|_{*, \theta} = 2^{1/\theta} \|\sigma(s^2)\|_{*, \theta},$$

where $2^{1/\infty} := 1$.

We now recall the definition of the Gaussian Besov classes from [19].

**Definition 4.1** (see [19], Definition 2.1). A function $f \in L^p(\gamma)$ belongs to the Gaussian Besov class with parameters $\theta \in [1, \infty)$, $p \in [1, \infty)$ and $\alpha \in (0, 1)$ if and only if

$$\left\| t^{1-\alpha} \left\| \frac{\partial}{\partial t} Q_t f \right\|_p \right\|_{*, \theta} < \infty.$$

Firstly, we prove the following auxiliary lemma.

**Lemma 4.1.** For any number $p \in (1, \infty)$ there is a constant $c_p$ such that, for every function $f \in L^p(\gamma)$,

$$\|\nabla Q_t f\|_p \leq c_p \left\| \frac{\partial}{\partial t} Q_t f \right\|_p.$$

**Proof.** Recall the following theorem on multipliers (see [14], Theorem 5.6.2).

Let $a_k$ be numbers such that $\sum_{k=0}^\infty |a_k| N^{-k} < \infty$ for some $N \in \mathbb{N}$. Assume that $\varphi(0) = 0$ and $\varphi(n) = \sum_{k=0}^N a_k n^{-k}$ for $n \geq N$. Then

$$\Psi_\varphi f = \sum_{n=0}^\infty \varphi(n) I_n(f)$$

is a bounded operator on the space $L^p(\gamma)$ for every $p \in (1, \infty)$. 
Consider the function $\varphi(n) = \sqrt{1 + 1/n}$ with $\varphi(0) = 0$. Note that
\[
\sqrt{1 + \frac{1}{n}} = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(2k-2)!}{2^{2k-1} k! (k-1)!} n^{-k}.
\]
Thus, for this function $\varphi(n)$ we have
\[
a_0 = 1 \quad \text{and} \quad a_k = \frac{(-1)^{k-1} (2k-2)!}{2^{2k-1} k! (k-1)!}.
\]
Since $|a_k| \sim Ck^{-3/2}$, the operator $\Psi_{\varphi}$ is bounded on the space $L^p(\gamma)$ for this function $\varphi$.

Recall (see [14], Theorem 5.7.1) that there exists a constant $m_p$ such that
\[
\|\nabla Q_t f\|_p \leq m_p \|(I - L)^{1/2} Q_t f\|_p.
\]
For an arbitrary element $f_n \in \mathcal{H}_n$ we have
\[
Q_t f_n = e^{-t \sqrt{n}} f_n, \quad (I - L)^{1/2} Q_t f_n = \sqrt{n + 1} e^{-t \sqrt{n}} f_n
\]
and
\[
\frac{\partial}{\partial t} Q_t f_n = -\sqrt{n} e^{-t \sqrt{n}} f_n.
\]
Without loss of generality we can assume that $\int f \, d\gamma = 0$. Thus,
\[
(I - L)^{1/2} Q_t f = -\sum_{n=1}^{\infty} \varphi(n) I_n \left( \frac{\partial}{\partial t} Q_t f \right).
\]
As the operator $\Psi_{\varphi}$ is bounded, there is a constant $M_p$ such that
\[
\|(I - L)^{1/2} Q_t f\|_p \leq M_p \left\| \frac{\partial}{\partial t} Q_t f \right\|_p.
\]
Therefore,
\[
\|\nabla Q_t f\|_p \leq m_p \|(I - L)^{1/2} Q_t f\|_p \leq m_p M_p \left\| \frac{\partial}{\partial t} Q_t f \right\|_p,
\]
and the lemma is proved.

Below we need the following simple technical result (see [1], Lemma 5.3, for example).

**Proposition 4.1.** For any function $\varphi \in \mathcal{C}^\infty(X)$
\[
\|\nabla T_t \varphi\|_q \leq C \left( \frac{q}{q-1} \right) \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \|\varphi\|_q \quad \forall q \in (1, \infty].
\]
For any mapping $\Phi \in \mathcal{C}^\infty(X, H)$
\[
\|\text{div}_\gamma T_t \Phi\|_p \leq C(p)(1 - e^{-2t})^{-1/2} \|\Phi\|_p \quad \forall p \in [1, \infty),
\]
where
\[
C(p) := \left( \frac{2\pi}{(2\pi)^{1/2}} \int_0^\infty s^{p} e^{-s^2/2} \, ds \right)^{1/p}.
\]
Thus, the first estimate is proved.

**Theorem 4.1.** Let \( f \in L^p(\gamma) \) for some \( p \in (1, \infty) \). Then

\[
\left\| t^{(1-\alpha)/2} \| \nabla T_t f \|_p \right\|_{\ast, \theta} \leq 4C(q)^{1-\alpha} \| t^{-\alpha} \sigma_{\gamma,p}(f, t) \|_{\ast, \theta},
\]

\[
\left\| t^{1-\alpha/2} \| \frac{\partial}{\partial t} T_t f \|_p \right\|_{\ast, \theta} \leq 2C(p) \| t^{(1-\alpha)/2} \| \nabla T_t f \|_p \|_{\ast, \theta},
\]

\[
\left\| t^{1-\alpha} \| \frac{\partial}{\partial t} Q_t f \|_p \right\|_{\ast, \theta} \leq \frac{8}{\sqrt{\pi}(1-\alpha)} \| t^{1-\alpha/2} \| \frac{\partial}{\partial t} T_t f \|_p \|_{\ast, \theta},
\]

and

\[
\| t^{-\alpha} \sigma_{\gamma,p}(f, t) \|_{\ast, \theta} \leq (\alpha^{-1} + c_p) \| t^{1-\alpha} \| \frac{\partial}{\partial t} Q_t f \|_p \|_{\ast, \theta}.
\]

**Proof.** Assume that \( V_{p,\theta}^\gamma(f) < \infty \). Fix a vector field \( \Phi \in \mathcal{F} \mathcal{C}^\infty(E, H) \) with norm \( \| \Phi \|_q \leq 1 \), where \( q = p/(p-1) \). Note that \( T_t \text{div}_{\gamma} \Phi = e^{-t} \text{div}_{\gamma}(T_t \Phi) \). Thus,

\[
\int \langle \Phi, \nabla T_t f \rangle d\gamma = -e^{-t} \int \text{div}_{\gamma}(T_t \Phi) f d\gamma \leq \frac{C(q)}{\sqrt{2t}} \sigma_{\gamma,p} \left( f, \frac{\sqrt{2t}}{C(q)} \right).
\]

Indeed,

\[
\| -e^{-t} T_t \Phi \|_q \leq \| \Phi \|_q \leq 1
\]

and

\[
\| \text{div}_{\gamma}[-e^{-t} T_t \Phi] \|_q \leq C(q) e^{-t}(1 - e^{-2t})^{-1/2} \| \Phi \|_q \leq \frac{C(q)}{\sqrt{2t}}.
\]

Therefore,

\[
\| \nabla T_t f \|_p \leq \frac{C(q)}{\sqrt{2t}} \sigma_{\gamma,p} \left( f, \frac{\sqrt{2t}}{C(q)} \right).
\]

Thus, for \( \theta \in [1, \infty) \) we have

\[
\left\| t^{(1-\alpha)/2} \| \nabla T_t f \|_p \right\|_{\ast, \theta} \leq 2^{-1/2} C(q) \left\| t^{-\alpha/2} \sigma_{\gamma,p} \left( f, \frac{\sqrt{2t}}{C(q)} \right) \right\|_{\ast, \theta} = 2^{(1-\alpha)/2+1/\theta} C(q)^{1-\alpha} \| s^{-\alpha} \sigma_{\gamma,p}(f, s) \|_{\ast, \theta},
\]

where we have made the change of variables \( s = \sqrt{2t}/C(q) \). For \( \theta = \infty \) a similar bound holds:

\[
\left\| t^{(1-\alpha)/2} \| \nabla T_t f \|_p \right\|_{\ast, \infty} \leq 2^{(1-\alpha)/2} C(q)^{1-\alpha} \| s^{-\alpha} \sigma_{\gamma,p}(f, s) \|_{\ast, \infty}.
\]

Thus, the first estimate is proved.

For any function \( \varphi \in \mathcal{F} \mathcal{C}^\infty(\gamma) \) we have

\[
\int \varphi \frac{\partial}{\partial t} T_t f d\gamma = \int \frac{\partial}{\partial t} T_t \varphi f d\gamma = \int L T_t \varphi f d\gamma = \int (L T_{t/2} \varphi)(T_{t/2} f) d\gamma = \int (\text{div}_{\gamma} \nabla T_{t/2} \varphi)(T_{t/2} f) d\gamma.
\]
Using the bound from the Proposition 4.1 we obtain
\[ \| \nabla T_{t/2} \varphi \|_q \leq C(p) \frac{e^{-(t/2)}}{\sqrt{1 - e^{-t}}} \| \varphi \|_q \leq C(p) \frac{1}{\sqrt{t}} \| \varphi \|_q, \]
implying the inequality
\[ \int \varphi \frac{\partial}{\partial t} T_t f \, d\gamma = - \int \langle \nabla T_{t/2} \varphi, \nabla T_{t/2} f \rangle \, d\gamma \leq C(p) \| \varphi \|_q t^{-1/2} \| \nabla T_{t/2} f \|_p. \]
Thus,
\[ t \left\| \frac{\partial}{\partial t} T_t f \right\|_p \leq \sqrt{2} C(p) \sqrt{\frac{t}{2}} \| \nabla T_{t/2} f \|_p, \]
which yields the bound
\[ \left\| t^{1-\alpha/2} \left\| \frac{\partial}{\partial t} T_t f \right\|_p \right\|_{*, \theta} \leq 2^{(1+\alpha)/2} C(p) \left( \frac{t}{2} \right)^{(1-\alpha)/2} \| \nabla T_{t/2} f \|_p \right\|_{*, \theta} = 2^{(1+\alpha)/2} C(p) \| s^{(1-\alpha)/2} \| \nabla T_s f \|_p \right\|_{*, \theta}. \]
The second estimate from the statement of the theorem is proved.
Note now that
\[ \frac{\partial}{\partial t} Q_t f(x) := \frac{t}{2 \sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{s^{3/2}} \left[ \frac{\partial}{\partial \tau} T_\tau f(x) \right] \bigg|_{\tau = t^2/(4s)} ds. \]
So,
\[ \left\| \frac{\partial}{\partial t} Q_t f \right\|_p \leq \frac{t}{2 \sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{s^{3/2}} \left\| \left[ \frac{\partial}{\partial \tau} T_\tau f \right] \bigg|_{\tau = t^2/(4s)} \right\|_p ds, \]
which implies the bound
\[ \left\| t^{1-\alpha} \left\| \frac{\partial}{\partial t} Q_t f \right\|_p \right\|_{*, \theta} \leq \frac{1}{2 \sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{s^{3/2}} \left\| \left[ \frac{\partial}{\partial \tau} T_\tau f \right] \bigg|_{\tau = t^2/(4s)} \right\|_p ds. \]
We now estimate the expression
\[ \left\| t^{2-\alpha} \left[ \frac{\partial}{\partial \tau} T_\tau f \right] \bigg|_{\tau = t^2/(4s)} \right\|_p \right\|_{*, \theta}. \]
First, consider the case \( \theta \in [1, \infty) \). In this case
\[ \left\| t^{2-\alpha} \left[ \frac{\partial}{\partial \tau} T_\tau f \right] \bigg|_{\tau = t^2/(4s)} \right\|_p \right\|_{*, \theta}^\theta = \left\| \int_0^\infty \left[ t^{2-\alpha} \left[ \frac{\partial}{\partial \tau} T_\tau f \right] \bigg|_{\tau = t^2/(4s)} \right\|_p \right\|_{*, \theta}^\theta t^{-1} dt \]
\[ = 2^{1+(2-\alpha)\theta} s^{(1-\alpha)/2} \theta \int_0^\infty \left[ t^{1-\alpha/2} \left[ \frac{\partial}{\partial \tau} T_\tau f \right] \bigg|_{\tau = t^2/(4s)} \right\|_p \right\|_{*, \theta}^\theta t^{-1} d\tau. \]
Thus,
\[
\left\| t^{1-\alpha} \frac{\partial}{\partial t} Q_tf \right\|_{p,*,\theta} \leq \frac{2^{2-\alpha-1/\theta}}{2\sqrt{\pi}} \int_0^\infty e^{-s} s^{1-\alpha/2} ds \left\| t^{1-\alpha/2} \frac{\partial}{\partial \tau} T_t f \right\|_{p,*,\theta}
\]
\[
\leq \frac{8}{\sqrt{\pi}(1-\alpha)} \left\| t^{1-\alpha/2} \frac{\partial}{\partial \tau} T_t f \right\|_{p,*,\theta}.
\]

The case $\theta = \infty$ can be treated similarly. The third estimate is proved.

For the last estimate note that
\[
\int \text{div}_\gamma \Phi f d\gamma = \int \text{div}_\gamma \Phi(f - Q_tf) d\gamma + \int \text{div}_\gamma \Phi Q_tf d\gamma
\]
for any mapping $\Phi \in \mathcal{F}C^\infty(E, H)$. The first term satisfies
\[
\int \text{div}_\gamma \Phi(f - Q_tf) d\gamma \leq \|\text{div}_\gamma \Phi\|_q \|f - Q_tf\|_p \leq \|\text{div}_\gamma \Phi\|_q \int_0^t \left\| \frac{\partial}{\partial s} Q sf \right\|_p ds.
\]
By Lemma 4.1, the second term is estimated as follows:
\[
\int \text{div}_\gamma \Phi Q_tf d\gamma = -\int \langle \Phi, \nabla Q_tf \rangle d\gamma \leq \|\Phi\|_q \|\nabla Q_tf\|_p \leq c_p \left\| \frac{\partial}{\partial t} Q_tf \right\|_p.
\]

Therefore,
\[
\sigma_{\gamma,p}(f, t) \leq \int_0^t \left\| \frac{\partial}{\partial s} Q sf \right\|_p ds + c_p t \left\| \frac{\partial}{\partial t} Q_tf \right\|_p.
\]

For any $\theta \in [1, \infty],
\[
\| t^{1-\alpha} \sigma_{\gamma,p}(f, t) \|_{*,\theta} \leq \| t^{1-\alpha} \int_0^t \left\| \frac{\partial}{\partial s} Q sf \right\|_p ds \|_{*,\theta} + c_p \left\| t^{1-\alpha} \frac{\partial}{\partial t} Q_tf \right\|_{p,*,\theta}.
\]

Let $u(s) := \left\| \frac{\partial}{\partial s} Q sf \right\|_p$. Then
\[
\left\| t^{1-\alpha} \int_0^t \left\| \frac{\partial}{\partial s} Q sf \right\|_p ds \right\|_{*,\theta} = \left\| t^{1-\alpha} \int_0^t u(s) ds \right\|_{*,\theta} = \left\| t^{1-\alpha} \int_0^1 u(t\tau) d\tau \right\|_{*,\theta}
\]
\[
\leq \int_0^1 \left\| t^{1-\alpha} u(t\tau) \right\|_{*,\theta} d\tau = \int_0^1 \tau^{\alpha-1} d\tau \left\| s^{1-\alpha} u(s) \right\|_{*,\theta} = \alpha^{-1} \left\| s^{1-\alpha} u(s) \right\|_{*,\theta}.
\]

Thus,
\[
\| t^{1-\alpha} \sigma_{\gamma,p}(f, t) \|_{*,\theta} \leq \left( \alpha^{-1} + c_p \right) \left\| t^{1-\alpha} \frac{\partial}{\partial t} Q_tf \right\|_{p,*,\theta}.
\]

Theorem 4.1 is proved.

To conclude, we discuss estimates for the best approximation by Hermite polynomials in the $L^2(\gamma)$-norm with respect to a Gaussian measure $\gamma$.

For any function $f \in L^2(\gamma)$ set
\[
E_N(f) := \inf \left\{ \|f - f_N\|_2: f_N \in \bigoplus_{k=0}^{N-1} \mathcal{H}_k \right\}.
\]
The quantity $E_N(f)$ is the error in the best approximation to the function $f$ by linear combinations of Hermite polynomials of given degree. It is clear that

$$E_N(f) = \|f - I_0(f) - \cdots - I_{N-1}(f)\|_2.$$  

We now prove a Jackson-Stechkin-type inequality for the quantity $E_N(f)$ involving the Gaussian modulus of continuity $\sigma_{\gamma,2}(f, \cdot)$.  

**Theorem 4.2.** For any function $f \in L^2(\gamma)$

$$E_N(f) \leq \sigma_{\gamma,2}(f, \sqrt{2\pi N^{-1/2}}).$$

**Proof.** For an arbitrary function $\varphi \in \mathcal{C}^{\infty}(X)$ with $\|\varphi\|_2 \leq 1$ we have

$$\int \varphi(f - I_0(f) - \cdots - I_{N-1}(f)) \, d\gamma = \int (\varphi - I_0(\varphi) - \cdots - I_{N-1}(\varphi)) f \, d\gamma$$

$$= \int \text{div}_\gamma \left( - \int_0^\infty \nabla T_t(\varphi - I_0(\varphi) - \cdots - I_{N-1}(\varphi)) \, dt \right) f \, d\gamma;$$

where we have used the equality

$$\psi(x) = - \int_0^\infty LT_t \psi(x) \, dt = \text{div}_\gamma \left( - \int_0^\infty \nabla T_t \psi(x) \, dt \right)$$

for an arbitrary function $\psi \in \mathcal{C}^{\infty}(X)$ with $\int \psi \, d\gamma = 0$, where $L$ is the Ornstein-Uhlenbeck operator (see §1.4 and Remark 5.8.7 in [14]). Recall (see Proposition 4.1) that

$$\|\nabla T_t \psi\|_2 \leq \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \|\psi\|_2$$

for all functions $\psi \in \mathcal{C}^{\infty}(X)$ and that

$$T_t g = \sum_{k=0}^\infty e^{-kt} I_k(g),$$

which yields the estimate

$$\|T_t (g - I_0(g) - \cdots - I_{N-1}(g))\|_2 \leq e^{-Nt} \|g\|_2$$

for all $g \in L^2(\gamma)$. We now note that

$$\left\| \int_0^\infty \nabla T_t(\varphi - I_0(\varphi) - \cdots - I_{N-1}(\varphi)) \, dt \right\|_2$$

$$\leq \int_0^\infty \|\nabla T_t(\varphi - I_0(\varphi) - \cdots - I_{N-1}(\varphi))\|_2 \, dt$$

$$\leq \int_0^\infty \frac{e^{-t/2}}{\sqrt{1 - e^{-t}}} \|T_{t/2}(\varphi - I_0(\varphi) - \cdots - I_{N-1}(\varphi))\|_2 \, dt$$

$$\leq \|\varphi\|_2 \int_0^\infty \frac{e^{-t/2}}{\sqrt{1 - e^{-t}}} e^{-Nt/2} \, dt = B \left( \frac{N + 1}{2}, \frac{1}{2} \right) \|\varphi\|_2,$$
where $B(x, y)$ is the standard beta function. It can be easily verified that
\[
\frac{\Gamma(x + 1/2)}{\Gamma(x)} \leq \sqrt{x},
\]
where $\Gamma(\cdot)$ is the standard gamma function. Indeed, introducing the probability density
\[
\rho_x(t) := [\Gamma(x)]^{-1} t^{x-1} e^{-t} I_{\{t>0\}}
\]
and applying Jensen’s inequality we obtain
\[
\frac{\Gamma(x + 1/2)}{\Gamma(x)} = \int \sqrt{t} \rho_x(t) \, dt \leq \sqrt{\int t \rho_x(t) \, dt} = \sqrt{\frac{\Gamma(x + 1)}{\Gamma(x)}} = \sqrt{x}.
\]
Thus,
\[
B\left(\frac{N + 1}{2}, \frac{1}{2}\right) = \frac{\Gamma((N + 1)/2)\Gamma(1/2)}{\Gamma(1 + N/2)} = \frac{\sqrt{\pi} \Gamma(N/2 + 1/2)}{N/2\Gamma(N/2)} \leq 2\pi N^{-1/2}.
\]
Therefore,
\[
\left\| \int_0^\infty \nabla T_t(\varphi - I_0(\varphi) - \cdots - I_{N-1}(\varphi)) \, dt \right\|_2 \leq \sqrt{2\pi} N^{-1/2}.
\]
We also note that
\[
\left\| \text{div}_\gamma \left( -\int_0^\infty \nabla T_t(\varphi - I_0(\varphi) - \cdots - I_{N-1}(\varphi)) \, dt \right) \right\|_2 = \left\| \varphi - I_0(\varphi) - \cdots - I_{N-1}(\varphi) \right\|_2 \leq \|\varphi\|_2 \leq 1.
\]
So,
\[
\int \varphi(f - I_0(f) - \cdots - I_{N-1}(f)) \, d\gamma \leq \sigma_{\gamma,2}(f, \sqrt{2\pi} N^{-1/2}),
\]
which completes the proof.

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