Quantum Secret Sharing by applying Analytic Geometry

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In this paper, we investigate a novel $(2, 2)$-threshold scheme and then generalize this to a $(n, n)$-threshold scheme for quantum secret sharing (QSS) which makes use of the fundamentals of Analytic Geometry. The dealer aptly selects GHZ states related to the coefficients which determine straight lines on a two-dimension plane. Then by computing each two of the lines intercept or not, we obtain a judging matrix whose rank can be used to determine the secret stored in entangled bits. Based on the database technology, authorized participants access to the database to obtain the secret information and hence the secret never appears in the channel. In this way, the eavesdroppers fail to obtain any secret by applying various attack strategies.

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I. INTRODUCTION

The novel discovery that quantum effects could prevent secret information from being eavesdropped in an insecure channel, which is illustrated by Wiesner [1], and then by Bennett et al. [2], attracts all scientists’ eyes to the quantum cryptography. In 1999, Hillery et al. [3] applied three-particle and four-particle Greenberger-Horne-Zeilinger (GHZ) states to implement an initial QSS scheme. From then on, quantum secret sharing plays an increasingly significant role in the secret protection of quantum cryptography.

Specifically, the QSS scheme is known as a threshold scheme [4]. Suppose that we divide a message into $n$ pieces such that any $k$ of the $n$ pieces could recover the message, but any sets of $k - 1$ or fewer pieces failed to, so the scheme is called a $(k, n)$-threshold scheme. In this way, people can forbid some dishonest participants, at most $k - 1$ in this threshold scheme, from knowing the whole message without the help of the honest ones.

Motivated by the unprecedented developing pace of QSS, a myriad of protocols and schemes have been proposed. For instance, cooperated with quantum secure direct communication (QSDC), Zhang presented a novel concept of quantum secret sharing [5], which proved to be known as QSS-SDC, then he made use of swapping quantum entanglement of Bell states to propose a multiparty QSS protocol which was based on the classical-message (QSS-CM) [6]. Moreover, a $(n, n)$-threshold scheme based on GHZ state and teleportation was proposed by Wang et al. [7]; after that Han and Liu et al. [8] illustrated a multiparty QSS-SDC scheme which used photons and random phase shift operations (RP-SOs). Furthermore, a quantum secret sharing experiment based on a single qubit protocol in telecommunication fiber was reported by Bogdanski et al. [9]; then Markham and Sanders [10] illustrated the graph states for quantum secret sharing. Recently, Sarvepalli et al. [11] illustrated the method to share secret by using quantum information, which converted a set of pure $[[n, 1, d]]_q$ Calderbank-Shor-Steane (CSS) codes to perfect secret sharing schemes; after that the first experimental generation and characterization of a six-photon Dicke state was demonstrated by Prevedel et al. [12], then they showed the applications in multiparty quantum networking protocols such as quantum secret sharing, open-destination teleportation, and telecloning.

Hence, with the recent high-speed developments of large-capacity storage technology and its appendant database technology, it is innovative for us to store the GHZ states and their corresponding coefficients of straight lines in a particular table which belongs to a specific database. Also, the security mechanisms of databases protect the information in databases from being known by uncertified users. Therefore, after considering recent discoveries both on theories and experiments, we make use of these characteristics to implement a novel QSS scheme which is based on the fundamentals of Analytic Geometry. In fact, the GHZ states particles just work as the carriers which carry these states to the participants, and the secret never appears in the channel. After the GHZ measurements on received particles, certified participants could access to the database and search for the coefficients of straight lines. The positions of lines are on behalf of the secrets which actually store in entangled bits. Accordingly, the participants judge the lines’ positions together and obtain the secret.

In this paper, we introduce some elementary knowledge of this scheme, and detailedly describe the implementation of the $(2, 2)$-threshold QSS scheme in sec-
tion II. In section III, we generalize the \((2, 2)\)-threshold QSS scheme to a \((n, n)\)-threshold one and depict this implementation in detail. Then in section IV, we consider some different attack strategies and analyze the security of the scheme by calculating the corresponding maximum Von Neumann entropy.

II. TWO-PARTY QUANTUM SECRET
SHARING BY USING THE PRINCIPLES OF
ANALYTIC GEOMETRY

Before we illustrate the scheme, we should review some basic theory of Analytic Geometry. As is known to all, every straight line on a two-dimension plane has an equation of the following form:

\[ Ax + By + C = 0, \]  

(1)

where \(A\) and \(B\) are not both zero. Conversely, if \(A\) and \(B\) are not both zero, then every equation which likes the Eq.(1) determines a straight line. Furthermore, if two straight lines don’t intersect on a two-dimension plane, they must parallel each other on it. (In this paper, we ignore the situation that the two lines coincide each other, which can be understood as a unique case that they are parallel.)

We should prepare some entangled bits as the secrets being shared. In detail, the secrets are actually stored they are parallel.)

We randomly choose two bits which are given by Eq.(2) as the secret “\(M_0\)” and “\(M_1\)”. For instance, we choose the base \(|S_1\rangle = |00\rangle - |11\rangle\) as the secret “\(M_0\)”, and select the \(|S_3\rangle = |01\rangle - |10\rangle\) as the secret “\(M_1\)”. And the coefficients of entangled bits are decided by all participants before every communication. In this way, the entangled bits work as the secret shared in the following scheme. Considering the orthonormal bases of Bell states, we obtain the general form of the entangled bits’ bases:

\[ |S_1\rangle = a|\varphi_1\psi_1\rangle + (-1)^i b(1 - \varphi_1)(1 - \psi_1), \]  

(2)

where

\[ \varphi_1\psi_1 = \begin{cases} 
00 & i \in (0, 1), \\
01 & i \in (2, 3).
\end{cases} \]

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The following four steps: (i) The dealer Alice randomly chooses two entangled bits given by Eq.(2) as the secret \((e.g. \ "M_0\) and \("M_1\)\)” all by herself. Then she negotiates with Bob and Candy to make sure the coefficients of the bits, but she never tells them the bases she had chosen. Besides, she prepares qubits in GHZ states.

In order to make the GHZ states of the particles relate to specific entangled bits, Alice must properly selects the GHZ states.

We can prepare three qubits in a GHZ state of the bases above. Hence, the well-prepared particles can be used to carry GHZ states to participants.

After these preparations talked above, we propose a novel two-party quantum secret sharing scheme. It can be achieved by the following four steps:

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(ii) Alice sends the GHZ state particles to Bob and Candy via a public insecure channel, respectively. After received the triplets, Bob and Candy implement GHZ state measurements on the received particles.

(iii) According to result of the measurement, Bob and Candy look up each separate table in database \(e.g. \) Table I to obtain a set of coefficients of a specific straight line, then they get together to find out whether the two straight lines intersect or not. If the two lines parallel, the secret Alice shared is “\(M_0\)” otherwise the secret is “\(M_1\)”.

(iv) After the judgement, Alice declares the base of the entangled bit she had chosen in the public channel. In this way, both Bob and Candy get to know the secret shared by Alice.

Fig. 1: The figure shows the implementation of \((2, 2)\)-threshold QSS scheme based on Analytic Geometry. The triangle connects the three qubits in GHZ state. The dashed lines which connect two qubits represent the two participants have obtained the shared entangled bits.

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TABLE I: Bob and Candy get a set of coefficients of the specific straight lines as a form $ax + by + c = 0$ after the GHZ state measurement, then together find out whether the two lines intersect or not. To each specific set of the coefficients which satisfies $\frac{a}{a'} = \frac{b}{b'}$ (where $a', b' \neq 0$), the two straight lines are parallel each other, so the secret of Alice is “$M_0$”. If the two lines intersect at a single point, the secret is “$M_1$”.

| Bob’s table | Candy’s table |
|-------------|---------------|
| GHZ\(_0\) | $a_0 \ b_0 \ c_0$ | GHZ\(_1\) | $a'_1 \ b'_1 \ c'_1$ |
| GHZ\(_1\) | $a_1 \ b_1 \ c_1$ | GHZ\(_2\) | $a'_2 \ b'_2 \ c'_2$ |
| GHZ\(_2\) | $a_2 \ b_2 \ c_2$ | GHZ\(_3\) | $a'_3 \ b'_3 \ c'_3$ |
| GHZ\(_3\) | $a_3 \ b_3 \ c_3$ | GHZ\(_4\) | $a'_4 \ b'_4 \ c'_4$ |
| GHZ\(_4\) | $a_4 \ b_4 \ c_4$ | GHZ\(_5\) | $a'_5 \ b'_5 \ c'_5$ |
| GHZ\(_5\) | $a_5 \ b_5 \ c_5$ | GHZ\(_6\) | $a'_6 \ b'_6 \ c'_6$ |
| GHZ\(_6\) | $a_6 \ b_6 \ c_6$ | GHZ\(_7\) | $a'_7 \ b'_7 \ c'_7$ |
| GHZ\(_7\) | $a_7 \ b_7 \ c_7$ |

In fact, Alice should have known the table which Bob and Candy keep in advance. Therefore, in step (i) she can decide which GHZ state bases to choose to prepare the triplets aptly. In Table I, for example, we intend to make the coefficients satisfy the following:

$$\frac{a}{a'} = \frac{b}{b'} \quad (4)$$

or

$$\frac{a}{a'} \neq \frac{b}{b'} \quad (5)$$

Thus if Alice wants to share the secret “$M_0$”, she randomly chooses GHZ states from the table which satisfy Eq.(4); otherwise, she will use the states shown as Eq.(5). Therefore, after the separate measurements, Bob and Candy will find out whether their straight lines parallel or intersect when they get together.

III. EXTENSION OF TWO-PARTY QUANTUM SECRET SHARING

Suppose that N parties would like to take part in this secret sharing, we should generalize the above two-party QSS scheme to a N-party one. If each two of the N lines are parallel each other, the secret is “$M_0$”. On the other hand, if each two are intersecting, the secret proves to be “$M_1$”. Similar to the proposed two-party QSS scheme, we get the following four steps:

(i) The dealer Alice randomly chooses two entangled bits shown as Eq.(2) all by herself. Then she negotiates with all other participants to make sure the coefficients of the bits without telling them the bases she had chosen. What’s more, she prepares the particles by properly choosing the bases given by Eq.(3), which are related to the corresponding coefficients.

(ii) Alice sends the well-prepared GHZ state particles to everyone of the N parties via a public insecure channel. After received the triplets, every member implements a GHZ state measurement on the accepted particle respectively.

(iii) According to result of the measurements, everyone, such as Bob, looks up his individual table in database (e.g. Table I) to obtain a set of coefficients of a specific straight line. Then N members get together to judge whether each two lines are intersecting or parallel. After that Alice declares the bases of entangled bit she had chosen via the public channel.

In order to judge whether each two of the N lines are parallel or intersecting, we first number the N lines as $l_1, l_2, ..., l_n$. Then we create a matrix shown as FIG. 2, the “0” in the matrix means the $l_i$ and $l_j$ ($l_i$ represents the label in the row, and $l_j$ means the label in the column) parallel each other, the number is “0”; if $l_i$ and $l_j$ intersect, the number at corresponding position is “1”. Attention, we regulates the numbers in the diagonal line are all “0”.

![FIG. 2](image)

IV. SECURITY ANALYSIS

As is known to all, we are always challenged by eavesdropping, for the channels used for sending the particles are insecure. What’s worse, one dishonest participant or more may cheat the others to obtain the secret. Hence, in order to analyze the security of the illustrated scheme, we introduce Von Neumann entropy to evalu-
ate the maximum information entropy eavesdroppers can get.

Similar to the Shannon entropy which measures the uncertainty with a classical probability distribution, Von Neumann entropy is described with density operators replacing probability distributions. Hence, we have the density operator \[ \rho = \sum_i p_i |\phi_i\rangle \langle \phi_i| . \] (6)

In the above equation, \(|\phi_i\rangle\) proves to be one of various states in a quantum system, and \(p_i\) is the corresponding probability. Hence Von Neumann defined the entropy of a quantum state by the following equation:

\[ S(\rho) \equiv -tr(\rho \log \rho). \] (7)

In this formula logarithms are taken to base two. If \(\lambda_x\) are the eigenvalues of \(\rho\), then Von Neumann’s definition can be re-expressed as:

\[ S(\rho) = -\sum_x \lambda_x \log \lambda_x. \] (8)

After the introduction, let’s consider the eavesdropping strategies and evaluate the security by computing the maximum Von Neumann entropy.

A. Eavesdropping In the Insecure Channel

If the eavesdropper Eve wants to obtain the information from the channel, he has two methods: (i) Eve intercepts and captures the particles from the sequences which Alice sends to the participants; (ii) Eve intercepts the particles from the channel, then uses the GHZ measurements to measure the particles and resends them to the corresponding parties respectively.

If Eve applies the (i) method to eavesdrop the channel, then \(N\) parties fail to realize that the eavesdroppers have intercepted the particles. In order to obtain the secret, Eve should measure the particles he has captured. Before that, he must access to the separate tables in database which record the coefficients of straight lines from different parties. Otherwise it’s impossible for him to find out the coefficients. However, even though he gets all the tables kept by the \(N\)-party, he may not get the right coefficients selected by Alice. Eve randomly chooses a basis to measure the captured particles, then for each participant, he has a probability of \(\frac{1}{N}\) to get the right GHZ state. Thus the eavesdropper only gets a probability of \((\frac{1}{N})^N\) to acquire all the correct coefficients of the corresponding straight lines. In this way, Eve can hardly obtain the correct judging matrix if \(N\) is large enough. Admittedly, Eve seems to have a rather small probability to obtain the secret. However, since he don’t know the accurate coefficients of the entangled bits, even he get to know the bases of the bits, the eavesdropping would be meaningless.

Then let’s calculate the maximum Von Neumann entropy eavesdroppers can get. According to column matrix form of the four vectors which are obtained from Eq.(2), we get the outer product operator \(|\phi_i\rangle \langle \phi_i|\). Since the probability \(p_i\) of each outer product operator is \((\frac{1}{N})^N\), we obtain the density operator:

\[ \rho = diag \left( \frac{a^2}{3N-2}, \frac{a^2}{3N-2}, \frac{b^2}{3N-2}, \frac{b^2}{3N-2} \right). \] (9)

Therefore, based on the fundamentals of matrix, we obtain the eigenvalues by calculating the \(|\rho - \lambda I| = 0\), where \(I\) is a unit matrix.

Moreover, since \(a\) and \(b\) are the coefficients of basis vectors in the quantum system, we have \(a^2 + b^2 = 1\). According to Eq.(8), we get the equation of maximum Von Neumann entropy:

\[ S(\rho) = \frac{3N-1}{24N-2} - \frac{1}{24N-2}(a^2 \log a + b^2 \log b). \] (10)

Suppose that \(N = 2\), \(a = b = \frac{1}{\sqrt{2}}\), hence we get the maximum Von Neumann entropy:

\[ S(\rho) = \frac{3}{8}(bit). \] (11)

In other words, by applying this attack strategy the maximum Von Neumann entropy Eve can get is only \(\frac{3}{8}\) bit.

In order to avoid this attack strategy, all participants of this communication, including Alice, regulates the length of these sequences in advance. Or Alice just declares the length of the series before she sends these sequences. If the number of particles fails to equal to the declared amount, the corresponding participants inform Alice that eavesdroppers do exist in the channel, then they abort this communication and replace the current channel with another one.

The (ii) attack strategy applied by Eve will surely be detected, for the GHZ measurements will destroy the original state of the particles. After measurements on these received particles, \(N\) parties get together to judge if each two are intersecting or paralleling and create the judgement matrix. If the rank of the matrix is neither 0 nor \(N\), \(N\) participants know that at least one of them has been eavesdropped, hence they terminate the current communication and seek for another channel. Eve only has a probability of \(\frac{1}{N}\) to obtain the correct GHZ state of a straight line, and the probability to get all the coefficients is \((\frac{1}{N})^N\), which approaches to 0 if \(N\) is large enough. Furthermore, even though he gets all the correct GHZ state shared by Alice, he fails to know the coefficients of the entangled bits, he would never get any information in this way. Suppose that \(N = 2\)
and \( a = b = \frac{1}{\sqrt{2}} \), the maximum Von Neumann entropy eavesdropped by Eve also is \( \frac{3}{8} \) bit.

Hence, in face of the eavesdropping in the insecure channel, this scheme will surely avoid giving out any secret shared by Alice, hence it is secure in any means.

**B. Dishonest Participants**

Besides the eavesdropping in the insecure channel, dishonest participants may also reveal the secret. According to the scheme, all participants decide the coefficients of the entangled bits without knowing the bases Alice had chosen. Hence before they together to judge, the dishonest participants could guess the specific entangled bit from a pair, thus each shares a probability of \( \frac{1}{4} \). And a pair of entangled bits is randomly selected from the four entangled bits given by Eq.(2). Hence in the 6 pairs, each specific entangled bit contains a probability of \( \frac{1}{4} \) to be chosen. Accordingly, the final probability to guess the correct basis of entangled bits equals to \( \frac{1}{4} \).

Then let’s compute the maximum Von Neumann entropy one dishonest participant can get. According to the four vectors of the quantum system, and the probability \( p_i \) is \( \frac{1}{4} \), we get the density operator:

\[
\rho = \text{diag} \left( \frac{a^2}{2}, \frac{a^2}{2}, \frac{b^2}{2}, \frac{b^2}{2} \right).
\]

(12)

We calculate the \(|\rho - \lambda I| = 0\) to get the eigenvalues. Hence, we obtain the equation of maximum Von Neumann entropy as follow:

\[
S(\rho) = 1 - 2(a^2 \log a + b^2 \log b).
\]

(13)

Suppose that \( a = b = \frac{1}{\sqrt{2}} \), then

\[
S(\rho) = 2(\text{bit}).
\]

(14)

Hence, in this case, the maximum Von Neumann entropy one dishonest participant can get is 2 bit.

Therefore, it would be difficult for the dishonest participants to obtain just a few pieces of the secret information. In this way, that would be meaningless to apply this eavesdropping strategy.

**V. CONCLUSIONS**

In this paper, we have proposed a novel quantum secret sharing scheme, which makes use of the fundamentals of Analytic Geometry as well as matrix. In fact, we realize this scheme in a completely different method. The GHZ state particles are just worked as carriers of the coefficients of specific straight lines. Hence the secret doesn’t cling to the triplets which are transmitted in the insecure channel. The dealer properly selects the entangled bits and controls corresponding GHZ states. Then the secrets are decided by the positions of specific straight lines which are actually determined by the GHZ states. Since the secrets never appear in the channel, the eavesdroppers fail to obtain any secret by intercepting the particles transmitted in the channel.

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