An unfitted Eulerian finite element method for the time-dependent Stokes problem on moving domains

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Abstract

We analyse a Eulerian Finite Element method, combining a Eulerian time-stepping scheme applied to the time-dependent Stokes equations using the CutFEM approach with inf-sup stable Taylor-Hood elements for the spatial discretisation. This is based on the method introduced by Lehrenfeld & Olshanskii [ESAIM: M2AN 53(2):585–614] in the context of a scalar convection-diffusion problems on moving domains, and extended to the non-stationary Stokes problem on moving domains by Burman, Frei & Massing [arXiv:1910.03054 [math.NA]] using stabilised equal-order elements. The analysis includes the geometrical error made by integrating over approximated levelset domains in the discrete CutFEM setting. The method is implemented and the theoretical results are illustrated using numerical examples.

1 Introduction

Flow problems on time-dependent domains are important in many different applications in biology, physics and engineering, such as blood-flows [AQR12] or fluid-structure interaction problems [Ric17], e.g., freely moving solids submersed in a fluid or generally in multi-phase flow applications, e.g. coupling a liquid and gas.

The most well established methods for these kind of problems studied in the literature are either based on an at least partially Lagragian or a purely Eulerian description of the domain boundary and its motion. Famous are Arbitrary Lagragian-Eulerian (ALE) methods [DGH82; Don+04] which can be combined with rather standard time stepping schemes or space-time Galerkin formulations [Beh01; KVV06; Beh08; Neu13]. ALE methods rely on geometry-fitted moving meshes or space-time meshes. The motion of corresponding meshes and their necessary adaptations after significant geometry deformations can be an severe burden for those methods, depending on the amount of geometrical change. To circumvent the problem of regular mesh updates or space-time meshing, Eulerian methods can be considered. This is also what we will do in this paper. Here, a static computational background mesh is used to define potential unknowns and the geometry is incorporated seperately resulting. In the context of finite element methods (FEM) these unfitted finite element methods have become popular within the last decade and are known under different names, e.g. XFEM [FB10], CutFEM [Bur+14], Finite Cell Method [PDR07], TraceFEM [ORG09]. Similar concepts have also been used before, e.g. in penalty methods [Bab73; BE86], the fictitious domain method [GPP94a; GPP94b], the immersed boundary method [PM89]. While these methods reached a considerable level of maturity for stationary problems, problems with moving

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geometries – one of the main targets for unfitted finite element methods – are not established that much. This is due to the fact that standard time stepping approaches are not straight-forwardly applicable in an Eulerian framework, where the expression $\partial_t u \approx \Delta t^{-1} (u^n - u^{n-1})$ is not well defined if $u^n$ and $u^{n-1}$ live on different domains.

One approach that has been proven to work despite this inconvenient setting is the class of space–time Galerkin formulations in an Eulerian setting as they have been considered in [LR13; Leh15; Zah18; Pre18] for scalar bulk problems, on moving surfaces in [GOR14a; ORX14; OR14] or for surface-bulk coupled problems in [HLZ16]. Recently, also preliminary steps towards unfitted space–time finite elements for two-phase flows have been addressed in [VR18; AB19]. These methods however have the disadvantage that one has to deal – in one way or another – with a higher dimensional problem. Using an adjusted quadrature rule to reduce the space–time formulation to a classical time stepping scheme as in [FR17] calls for the costly computation of projections between discrete function spaces at times $t_{n-1}$ and $t_n$.

In the following we consider an alternative to space–time methods which allows for a more standard time stepping structure and has been introduced in [LOX18; LO19] and considered for flow problems in [BFM19]. To discretise the time-derivative in the spatially smooth setting, the method applies a standard method-of-lines approach using a backward–differentiation formula in combination with a continuous extension operator in Sobolev spaces. This extension ensures that the previous solution defined on the domain $\Omega(t_{n-1})$ has meaning on the domain $\Omega(t_n)$. In the fully discrete setting the approach uses an unfitted finite element method on a fixed background mesh to discretise the domain and applies additional stabilisation outside the physical domain, such that the discrete solution has meaning on a larger domain $\bar{\Omega}(\Omega_h(t_{n-1})) \supset \Omega_h(t_n)$. The major challenge with this approach, in the context of fluid problems, is that the velocities at different time points, are weakly divergence-free with respect to different pressure spaces. This means that the approximated time-derivative $\Delta t^{-1} (u^n_h - u^{n-1}_h)$ is not weakly divergence-free, which causes stability problems for the pressure, as is known from the fitted case [BW11; BBB13].

The essential idea, to extend the discrete fluid velocity solution onto the active mesh at the next time-step using ghost-penalties, in order to use a method-of-lines approach for the discretisation of the time-derivative, has also been considered in [Sch17, Section 3.6.3]. However, there the extension of the velocity solution has been split into a separate sub-step and has been limited to the extension to a vertex patch of previously active elements, such that the time-step must obey a CFL-condition $\Delta t \leq c h$. Furthermore, the analysis of this split approach currently remains open.

In the recent work [BFM19], the implicit extension technique introduced in [LO19] was also considered for the time-dependent Stokes problem on moving domains. Here, the spatial discretisation consists of equal-order pressure stabilised unfitted finite elements. However, the analysis in this work was restricted to the situation, where the physical domain coincides the the discrete levelset domain.

The remainder of this paper is structured as follows. In Section 2 we formally describe the smooth problem we aim to solve numerically. We then begin with the temporal semi-discretisation of the smooth problem in Section 3, where we also show the stability of the approach in the spatially smooth setting. Section 4 then covers the description of the fully discrete problem, which realises the discrete extension in the method. The main part of this paper is then the fully discrete analysis of the method in Section 5. Here we go into the details of how the geometrical consistency error, inherent in unfitted finite element methods using levelsets, affect the coupling of the velocity and pressure errors. We then present numerical examples for the method in Section 6. Here we show the dominance of the geometrical error in practice, as well as discussing approaches to avoid this issue. Finally, in Section 7 we then discuss the conclusions from this work, and discuss remaining open problems connected to this method.
2 Problem Description

Let us consider a time-dependent domain $\Omega(t) \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ and Lipschitz continuous boundary $\Gamma(t) = \partial \Omega(t)$ over a fixed and bounded time-interval $[0, T]$ and assume that this domain evolves smoothly in time. We then define the space-time domain as $Q := \bigcup_{t \in (0, T)} \Omega(t) \times \{t\}$. In $Q$ we consider the time-dependent Stokes problem: Find the velocity $u$ together with homogeneous Dirichlet boundary conditions on the space-boundary, initial condition $u(0) = u_0$ and a forcing term $f(t)$ with the viscosity $\nu > 0$.

For the well-posedness of this problem, we refer to [BFM19, Section 2.1].

3 Time discretisation

For simplicity, let us consider a uniform time-step $\Delta t = T/N$ for some fixed $N \in \mathbb{N}$. We then denote $t_n = n\Delta t$, $I_n = [t_{n-1}, t_n)$, $\Omega^n = \Omega(t_n)$ and $\Gamma^n = \Gamma(t_n)$. We define the $\delta$-neighbourhood of $\Omega(t)$ as

$$\mathcal{O}_\delta(\Omega(t)) := \{x \in \mathbb{R}^d \mid \text{dist}(x, \Omega(t)) \leq \delta\}.$$  

For our method, we require that the domain $\Omega^n$ lies within the $\delta$-neighbourhood of the previous domain, i.e.,

$$\Omega^n \subset \mathcal{O}_\delta(\Omega^{n-1}), \quad \text{for } n = 1, \ldots, N.$$  

We ensure that this requirement is fulfilled by the choice

$$\delta = c_3 w_{\infty} \Delta t$$  

where $w_{\infty}$ is the maximal normal speed of the domain interface and $c_3 > 1$.

The time discretisation is then based on a method of lines approach, in combination with an extension operator for Sobolev functions to a $\delta$-neighbourhood, which ensures that solutions defined on domains at previous time-steps are well defined on $\Omega^n$. See Section 3.1.3 for details of this extension operator.

3.1 Variational formulation

3.1.1 Notation

We introduce some notation. By $L^2(S)$ we denote the function space of square integrable functions on a domain $S$ while $H^1(S)$ is the usual Sobolev space of functions in $L^2(S)$ which have first order weak derivatives in $L^2(S)$. We define the subspace of $L^2(S)$ of functions with mean value zero $L^2_0(S) := \{v \in L^2(S) \mid \int_S v \, dx = 0\}$ and the subspace of $H^1(S)$ of functions with zero boundary values (in the trace sense) as $H^1_0(S)$. The dual space to $(H^1_0(S), \|\nabla \cdot \|_S)$ is denoted by $H^{-1}(S)$. For vector-valued functions we write those spaces bold. Further, we introduce the Poincaré constant $c_P > 0$ which ensures that for all $v \in H^1_0(\Omega(t))$ and all $t \in (0, T)$ there holds $\|v\|_{\Omega(t)} \leq c_P \|\nabla v\|_{\Omega(t)}$.

3.1.2 Semi-discretization

We discretise the time derivative with the implicit Euler (or BDF1) method in combination with the extension operator. Multiplying (2.1) with a test function, integrating over $\Omega^n$ and using integration by parts to obtain the weak formulations for the diffusion and velocity-pressure coupling terms, the variational formulation of the temporally semi-discrete problem then reads: For $n = 1, \ldots, N$, given

$$\partial_t u - \nu \Delta u + \nabla p = f$$  

$$\nabla \cdot u = 0$$  

together with homogeneous Dirichlet boundary conditions on the space-boundary, initial condition $u(0) = u_0$ and a forcing term $f(t)$ with the viscosity $\nu > 0$. We ensure that this requirement is fulfilled by the choice

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\[ u^{n-1} \in \mathcal{H}_0^1(\Omega^{n-1}) \text{ and } f^n \in \mathcal{H}^{-1}(\Omega^n), \text{ find } (u^n, p^n) \in V^n \times Q^n := \mathcal{H}^1(\Omega^n) \times \mathcal{L}^2(\Omega^n) \text{ such that for all } (v, q) \in V^n \times Q^n \text{ it holds }
\]
\[
\frac{1}{\Delta t} (u^n, v)_{\Omega^n} + a^n(u^n, v) + b^n(p^n, v) + b^n(q, u^n) = (f^n, v)_{(V^n)'}, v^n + \frac{1}{\Delta t} (\mathcal{E} u^{n-1}, v)_{\Omega^n}. \tag{3.1}
\]

Here, \((v, q)_{\Omega^n}\) denotes the \(L^2\)-inner product and the bilinear forms are
\[
a^n(u, v) = \nu \int_{\Omega^n} \nabla u : \nabla v \, dx \quad \text{and} \quad b^n(q, v) = -\int_{\Omega^n} q \nabla \cdot v \, dx
\]
for the diffusion term and the velocity-pressure coupling respectively. \(\mathcal{E} : \mathcal{H}^1(\Omega^{n-1}) \rightarrow \mathcal{H}^1(\mathcal{O}_{\delta}(\Omega^{n-1}))\) is the extension operator – discussed in the next section – that allows us to make sense of the “initial value” \(u^{n-1} \in \mathcal{H}_0^1(\Omega^{n-1})\) in \(\Omega^n \subset \mathcal{O}_{\delta}(\Omega^{n-1})\).

### 3.1.3 Extension operator

For the extension operator, we require the following family of space-time anisotropic spaces
\[
\mathcal{L}^\infty(0, T; \mathcal{H}^k(\Omega(t))) := \left\{ v \in \mathcal{L}^2(Q) \mid v(\cdot, t) \in \mathcal{H}^k(\Omega(t)) \text{ for a.e. } t \in (0, T), \right. \quad \text{ess sup}_{t \in (0, T)} \|v(\cdot, t)\|_{\mathcal{H}^k(\Omega(t))} < \infty \}
\]
for \(k = 0, \ldots, m + 1\). We then denote \(\partial_t v = v_t\) as the weak partial derivative with respect to the time variable, if this exists as an element in the space-time space \(\mathcal{L}^2(Q)\). We also use the standard notation for \(L^2\)-norms of \(\cdot : S = \| \cdot \|_{L^2(S)}\) for some domain \(S\).

We now assume the existence of a spatial extension operator
\[
\mathcal{E} : \mathcal{L}^2(\Omega(t)) \rightarrow \mathcal{L}^2(\mathcal{O}(\Omega(t)))
\]
which fulfils the following properties

**Assumption A1.** Let \(v \in \mathcal{L}^\infty(0, T; \mathcal{H}^{m+1}(\Omega(t))) \cap \mathcal{W}^{2,\infty}(Q)\). There exist positive constants \(c_{A1a}, c_{A1b}\) and \(c_{A1c}\) that are uniform in \(t\), such that
\[
\begin{align*}
\|\mathcal{E} v\|_{\mathcal{H}^k(\mathcal{O}_{\delta}(\Omega(t)))} &\leq c_{A1a} \|v\|_{\mathcal{H}^k(\Omega(t))} \tag{3.2a} \\
\|\nabla (\mathcal{E} v)\|_{\mathcal{O}_{\delta}(\Omega(t))} &\leq c_{A1b} \|\nabla v\|_{\Omega(t)} \tag{3.2b} \\
\|\mathcal{E} v\|_{\mathcal{W}^{2,\infty}(\mathcal{O}_{\delta}(Q))} &\leq c_{A1c} \|v\|_{\mathcal{W}^{2,\infty}(Q)} \tag{3.2c}
\end{align*}
\]
holds. Furthermore, if for \(v \in \mathcal{L}^\infty(0, T; \mathcal{H}^{m+1}(\Omega(t)))\) it holds for the weak partial time-derivative that \(v_t \in \mathcal{L}^\infty(0, T; \mathcal{H}^m(\Omega(t)))\), then
\[
\| (\mathcal{E} v)_t \|_{\mathcal{H}^m(\mathcal{O}_{\delta}(\Omega(t)))} \leq c_{A1d} \left[ \|v\|_{\mathcal{H}^{m+1}(\Omega(t))} + \|v_t\|_{\mathcal{H}^m(\Omega(t))} \right], \tag{3.3}
\]
where the constant \(c_{A1d} > 0\) again only depends on the motion of the spatial domain.

**Remark 3.1.** Such an extension operator can be constructed explicitly from the classical linear and continuous universal extension operator for Sobolev spaces (see e.g. [Ste70, Section VI.3]) when the motion of the domain can be described by a diffeomorphism \(\Psi(t) : \Omega_0 \rightarrow \Omega(t)\) for each \(t \in [0, T]\) from the reference domain \(\Omega_0\) that is smooth in time. See [LO19] for details thereof.

Assuming sufficient regularity of the domain \(\Omega^n\), well-posedness of (3.1) is given for every time-step by the standard theory of the Stokes-Brinkman problem.
3.2 Stability

We now show that the semi-discrete scheme (3.1) gives a stable solution for both the velocity and pressure.

**Lemma 3.2.** Let \( \{u^n\}_{n=1}^N \) be the velocity solution of (3.1) with initial data \( u^0 \in H(\Omega^0) \). Then it holds that

\[
\|u^k\|^2_{\Omega^k} + \Delta t \sum_{n=1}^k \frac{\nu}{2} \|\nabla u^n\|^2_{\Omega^n} \leq \exp(c_{L3.2}t_k) \left[ \|u^0\|^2_{\Omega^0} + \frac{\nu \Delta t}{2} \|\nabla u^0\|^2_{\Omega^0} + \frac{c^2_{L3.2} \Delta t}{\nu} \sum_{n=1}^k \|f^n\|^2_{\Omega^n} \right]
\]

with a constant \( c_{L3.2} \) independent of the time step \( \Delta t \) and the number of steps \( k \).

**Proof.** The proof is analogous to that of [LO19, Lemma 3.6] with the different choice of test-function \( 2\Delta t(u^n, -p^n) \in V^n \times Q^n \) to remove the pressure from the equation.

\[
\frac{\|p^n\|^2_{\Omega^n}}{\beta} \leq \frac{1}{\beta} \left( c_P \|f^n\|_{\Omega^n} + \frac{1}{\Delta t} \|u^n - \mathcal{E} u^{n-1}\|_{\mathcal{H}_{-1}(\Omega^n)} + \nu \|\nabla u^n\|_{\Omega^n} \right).
\]

(3.4)

with a constant \( \beta > 0 \) bounded independent of \( \Delta t \).

**Lemma 3.3.** Let \( \{p^n\}_{n=1}^N \) be the pressure solution of (3.1). Then it holds that

\[
\|p^n\|_{\Omega^n} \leq \frac{1}{\beta} \left( \frac{1}{\Delta t} \|u^n - \mathcal{E} u^{n-1}\|_{\mathcal{H}_{-1}(\Omega^n)} + \frac{c_{L3.2}}{\nu} \sum_{n=1}^k \|f^n\|^2_{\Omega^n} \right)
\]

(3.4)

with a constant \( \beta > 0 \) bounded independent of \( \Delta t \).

**Proof.** Let \( \beta^n \) be the inf-sup constant of the space pair \( V^n \times Q^n = \mathcal{H}^1_{0}(\Omega^n) \times L^2_{0}(\Omega^n) \) and denote \( \beta = \min_{n=0,...,N} \beta^n \). From the inf-sup stability of the velocity and pressure spaces and by rewriting the momentum balance equation, it follows that

\[
\beta^n \|p^n\|_{\Omega^n} \leq \beta^n \|p^n\|_{\Omega^n} \leq \sup_{v \in V^n} \frac{b^n(p^n, v)}{\|\nabla v\|_{\Omega^n}} = \sup_{v \in V^n} \frac{(f^n, v)_{\Omega^n} - \frac{1}{\Delta t}(u^n - \mathcal{E} u^{n-1}, v)_{\Omega^n} - a^n(u^n, v)_{\Omega^n}}{\|\nabla v\|_{\Omega^n}}.
\]

Taking absolute values, applying Cauchy-Schwarz, using the Poincaré inequality and the continuity of the diffusion bilinear form we get the claim.

**Remark 3.4.** At first glance it seems unclear if the estimate in (3.4) yields a pressure bound that is independent of \( \Delta t \). Let us explain why a scaling of \( \|p^n\|_{\Omega^n} \) with \( \Delta t^{-1} \) is not to be expected. The argument is based on an a relation for the discretisation error. The exact solution \( (u(t_n), p(t_n)) \) fulfills for all \( v \in V^n \) and all \( q \in Q^n \) an equation similar to (3.1):

\[
(\partial_t u(t_n), v)_{\Omega^n} + a^n(u(t_n), v) + b^n(p(t_n), v) + b^n(q, u(t_n)) = (f^n, v)_{\Omega^n}.
\]

We find for \( E^k := u(t_k) - u^k \), and \( D^k := p(t_k) - p^k \), \( k = 1, ..., N \) that there holds

\[
\left( -\frac{\nabla E^{n-1}}{\Delta t}, v \right)_{\Omega^n} + a^n(E^n, v) + b^n(D^n, v) + b^n(q, E^n) = \left( -\frac{u(t_n) - \mathcal{E} u(t_{n-1})}{\Delta t}, \partial_t u(t_n), v \right)_{\Omega^n}
\]

for all \( v \in V^n \) and \( q \in Q^n \). Now, assuming sufficient regularity, i.e. \( u \in W^{2,\infty}(Q) \), we easily obtain the bound \( c_{R3.4a} \Delta t \|u\|_{W^{2,\infty}(Q)} \|v\|_{\Omega^n} \) for the r.h.s. with a constant \( c_{R3.4a} \) independent of \( n \), \( u \) and \( \Delta t \). Here, we also made use of (3.2c). As the l.h.s. is the same as in (3.1) we can apply Lemma 3.3 (using \( E^0 = 0 \)) to obtain the bound

\[
\|E^n\|_{\Omega^n} \leq c_{R3.4b} \exp(c_{R3.4c} \Delta t) \|u\|_{W^{2,\infty}(Q)}
\]

Hence, a simple triangle inequality yields

\[
\frac{1}{\Delta t} \left( \|u^n - \mathcal{E} u^{n-1}\|_{\mathcal{H}_{-1}(\Omega^n)} \right) \leq \frac{1}{\Delta t} \|u(t_n) - \mathcal{E} u(t_{n-1})\|_{\mathcal{H}_{-1}(\Omega^n)} + \frac{1}{\Delta t} \|E^n\|_{\mathcal{H}_{-1}(\Omega^n)} + \frac{1}{\Delta t} \|\mathcal{E} E^{n-1}\|_{\mathcal{H}_{-1}(\Omega^n)}
\]

\[
\leq \|\partial_t u(t_n)\|_{\mathcal{H}_{-1}(\Omega^n)} + c_{R3.4d} \Delta t \|u\|_{W^{2,\infty}(Q)}
\]

and hence a bound on the norm of \( p^n \) that is independent on \( \Delta t^{-1} \).
4 The fully discrete method

For the spatial discretisation of the method we use the CutFEM approach [Bur+14]. Within the CutFEM framework, we consider a background domain \( \Omega \subset \mathbb{R}^d \) and assume that \( \Omega(t) \subset \Omega \) for all \( t \in [0, T] \). We then take a simplicial, shape-regular and quasi-uniform mesh \( \mathcal{T}_h \) of \( \Omega \), where \( h > 0 \) is the characteristic size of the simplices. Bad cuts of the mesh with the domain boundary \( \Gamma(t) \) are stabilised using ghost-penality stabilisation [Bur10]. We will discuss the details of this in Section 4.1.1. The fully discrete method realises the necessary extension implicitly, by applying the ghost-penalty stabilisation operation on a larger extension region when solving for the solution in each time step. At each time step, we therefore extend the (discrete) physical domain \( \Omega_h^0 \) by a strip of width

\[ \delta_h = c_\delta_h \omega_\infty^0 \Delta t \]

such that \( \Omega_h^{n+1} \) is a subset of this extended domain with \( c_\delta_h > 1 \), but sufficiently small so that \( O_{\delta_h}(\Omega_h^n) \subset O_{\delta}(\Omega^n) \). We collect all elements which have some part in this extended domain at time \( t = t_n \) in the active velocity mesh, denoted as

\[ \mathcal{T}_{h,\delta_h} := \{ K \in \mathcal{T}_h | \exists x \in K \text{ such that } \text{dist}(x, \Omega_h^n) \leq \delta_h \} \subset \mathcal{T}_h \]

and denote the active domain as

\[ O_{\delta_h}^{n,\mathcal{S}} := \{ x \in K | K \in \mathcal{T}_{h,\delta_h} \} \subset \mathbb{R}^d. \]

We further define the cut mesh as \( \mathcal{T}_h^n = \mathcal{T}_{h,0} \) of all elements which contain some part of the physical domain and the cut domain as \( O_0^{n,\mathcal{S}} = O_{0}^{n,\mathcal{S}} \).

On the active mesh, we consider an inf-sup stable finite element pair \( V_h \times Q_h \) for the Nitsche-CutFEM discretisation of the Stokes problem [BH14; Mas+14]. For an overview of such elements see [GO17]. We shall use the family of Taylor-Hood finite elements for \( k \geq 2 \)

\[ V_h^n := \{ v_h \in C(O_{\delta_h}^{n,\mathcal{S}}) | v_h|_K \in [P^k(K)]^d \text{ for all } K \in \mathcal{T}_{h,\delta_h} \} \]

and

\[ Q_h^n := \{ q_h \in C(O_{\delta}^{n,\mathcal{S}}) | v_h|_K \in \mathbb{P}^{k-1}(K) \text{ for all } K \in \mathcal{T}_h^n \}. \]

4.1 The variational formulation

The fully discrete variational formulation of the method reads as follows: Given an appropriate initial condition \( u_h^n \in V_h^0 \), for \( n = 1, \ldots, N \) find \( (u_h^n, p_h^n) \in V_h^n \times Q_h^n \), such that

\[
\int_{\Omega_h^n} \frac{u_h^n - u_h^{n-1}}{\Delta t} \cdot u_h \, dx + a_h^n(u_h^n, v_h) + b_h^n(p_h^n, v_h) + b_h^n(q_h^n, u_h^n) + s_h^n((u_h^n, p_h^n), (v_h, q_h)) = f_h^n(v_h) \quad (4.1)
\]

for all \((v_h, q_h) \in V_h^n \times Q_h^n\). We impose the homogeneous Dirichlet boundary conditions in a weak sense using the symmetric Nitsches method [Nit71]. For the diffusion term, the bilinear form is then

\[ a_h^n(u_h^n, v_h) := \nu \int_{\Omega_h^n} \nabla u \cdot \nabla v \, dx + \nu N_h^n(u_h, v_h), \]

\[ N_h^n(u_h^n, v_h) := N_{h,c}(u_h^n, v_h) + N_{h,c}(v, u) + N_{h,a}(u_h^n, v_h) \]

where

\[ N_{h,c}(u_h^n, v_h) := \int_{\Gamma_h^n} (\partial_n u_h^n) \cdot v_h \, ds \quad \text{and} \quad N_{h,a}(u_h^n, v_h) := \int_{\Gamma_h^n} \sigma_h \cdot u_h^n \cdot v_h \, ds\]
are the consistency (symmetry) and penalty terms of Nitsche’s method while \( \sigma > 0 \) is the penalty parameter. The velocity-pressure coupling term is given by

\[
b_h^0(v,q) = - \int_{\Omega_h^n} q \nabla \cdot v \, dx + \int_{\Gamma_h^n} pv \cdot n \, ds.
\]

In order to realise the discrete extension of the velocity and to stabilise the method with respect to essentially arbitrary mesh-interface cut positions, we use ghost-penalty stabilisation. This term is given by

\[
s_h^n((u_h^n, p_h^n), (v_h^n, q_h^n)) = \gamma_{s,w} \nu \nu_h^n(u_h^n, v_h^n) + \gamma'_{s,M} \frac{1}{\nu} i_h^n(u_h^n, v_h^n) - \gamma_{s,p} \frac{1}{\nu} j_h^n(p_h^n, q_h^n)
\]

with stabilisation parameters \( \gamma_{s,w}, \gamma'_{s,M}, \gamma_{s,p} > 0 \). A suitable choice for these parameters will be discussed later, cf. Remark 5.7 below. The velocity ghost-penalty operator \( i_h^n(\cdot, \cdot) \) stabilises the velocity w.r.t. arbitrary bad cut configurations and implicitly defines an extension of the velocity field. It will therefore act on a strip of elements both inside and outside the physical domain, in order for us to have control over the velocity on the entire active domain \( \mathcal{H}_{h,\delta_h}^n \). The pressure ghost-penalty operator stabilises the pressure in the \( L^2 \)-norm and is needed to give an inf-sup property for unfitted finite elements [GO17]. This operator will therefore only act in the direct vicinity of the domain boundary and only on elements which have at least some part in the physical domain.

### 4.1.1 The ghost-penalty operator

The stabilisation bilinear form \( s_h^n(\cdot, \cdot) \) has two purposes here. First, it stabilises the discrete problem (4.1) with respect to domain boundary-mesh cut position and it implicitly provides the extension of the velocity field that is needed to allow the method of lines approach. To the best of our knowledge, there are currently three different versions of the ghost-penalty stabilisation operator. An LPS-type version was the first ghost-penalty operator, proposed in [Bur10]. The normal derivative jump version is probably the most widely used variant, see e.g., [Bur+14; BH12; Mas+14; SW14; GM19]. We will use the direct version of the ghost penalty operator introduced in [Pre18]. For details on all three ghost penalty operators, see [LO19]. We will only provide details on the direct version used here. However, the other versions of the ghost-penalty operator could also be used instead.

For the velocity ghost-penalty operator, we define the set of elements in the boundary strip

\[
\mathcal{H}_{h,S \pm}^n := \{ K \in \mathcal{H}_h | \exists x \in K \text{ with } \text{dist}(x, \Gamma_h^n) \leq \delta_h \}
\]

and the set of interior facets in this strip

\[
\mathcal{H}_{h,\delta_h}^n := \{ F = \overline{T}_1 \cap \overline{T}_2 | T_1, T_2 \in \mathcal{H}_{h,\delta_h}^n, T_1 \neq T_2 \text{ and } \text{meas}_{d-1}(F) > 0 \}.
\]

Furthermore, for the pressure ghost-penalty operator we define the set of boundary elements

\[
\mathcal{H}_{h,\Gamma_h^n} := \{ K \in \mathcal{H}_h^n | \exists x \in K \text{ with } x \in \Gamma_h^n \},
\]

and the set of interior facets of these elements

\[
\mathcal{F}_h^n := \{ F = \overline{T}_1 \cap \overline{T}_2 | T_1, T_2 \in \mathcal{H}_h^n, T_1 \neq T_2 \text{ and } \text{meas}_{d-1}(F) > 0 \}.
\]

A sketch of the different sets of element and facets for both the velocity and pressure can be seen in Figure 1.

To define the stabilisation operator, we require some further notation. For a facet in the extension strip \( \mathcal{F}_{h,\delta_h} \ni F = \overline{T}_1 \cap \overline{T}_2 \) let \( \omega_F = T_1 \cup T_2 \) be the corresponding facet patch. We then consider \( \| u \| := u_1 - u_2 \) with \( u_i = E^P u |_{T_i} \), where \( E^P : P^m(K) \to P^m(\mathbb{R}^d) \) is the canonical extension of a polynomial to \( \mathbb{R}^d \).
The velocity ghost-penalty operator is then defined as
\[
i^n_h(u^n_h, v_h) = \sum_{F \in \mathcal{F}^n_{h,\delta h}} \frac{1}{h^2} \int_{\omega_F} \|u_h\| \cdot \|v_h\| \, dx
\]
and the pressure ghost-penalty operator is defined as
\[
j^n_h((p^n_h, q_h)) = \sum_{F \in \mathcal{F}^n_h} \int_{\omega_F} \|p_h\| \cdot \|q_h\| \, dx.
\]

For the analysis, we will also need to insert general \(L^2\)-functions as arguments of the ghost-penalty operators. In this case we take \(u_i = E^p \Pi_T u\big|_{T_i}\), where \(\Pi_T\) is the \(L^2(T_i)\)-projection onto \(P_m(T_i)\).

**Lemma 4.1 (Consistency).** Let \(w \in \mathcal{H}^{m+1}(O_{\delta h,\mathcal{T}})\) and \(r \in \mathcal{H}^m(O^n_{\mathcal{T}})\), \(n = 1, \ldots, N, m \geq 1\). Then it holds that
\[
i^n_h(w, w) \leq ch^{2m} \|w\|^2_{\mathcal{H}^{m+1}(O_{\delta h,\mathcal{T}})}
\]
\[
j^n_h(r, r) \leq ch^{2m} \|r\|^2_{\mathcal{H}^m(O^n_{\mathcal{T}})}
\]
Furthermore, let \(I^*\) be the Lagrangian interpolation operator for the velocity \((I^*: [C^0(O^n_{\delta h,\mathcal{T}})]^d \to V^n_h)\) and for the pressure space \((I^*: [C^0(O^n_{\delta h,\mathcal{T}})] \to Q^n_h)\). Then we also have
\[
i^n_h(w - I^*w, w - I^*w) \leq ch^{2m} \|w\|^2_{\mathcal{H}^{m+1}(O_{\delta h,\mathcal{T}})}
\]
\[
j^n_h(r - I^*r, r - I^*r) \leq ch^{2m} \|r\|^2_{\mathcal{H}^m(O^n_{\delta h,\mathcal{T}})}
\]
**Proof.** We note that for \(d \leq 3\) and \(m \geq 1\) \(\mathcal{H}^{m+1}\) is compactly embedded into \(C^0\) so that \(I^*\) is well-defined. For the estimates with \(i^n_h(\cdot, \cdot)\) we refer to [LO19, Lemma 5.8]. The proof for \(j^n_h(\cdot, \cdot)\) follows analogously with the different \(h\) scaling in the definition of \(j^n_h(\cdot, \cdot)\).

### 5 Analysis of the method

The analysis of the fully discrete method in this section is structured as follows. In Section 5.1 we will introduce further notation, concepts and basic necessary results from the literature needed for the analysis.
Section 5.2 will then cover the existence and uniqueness of the solution to the fully discretised system. The stability of this solution is then discussed in Section 5.3. We then cover some technical details on the geometry approximation made by integrating over discrete approximations $\Omega^n_h$ of the exact domain $\Omega^n$. In Section 5.4, we show the consistency of the method in both time and space in Section 5.5. With this consistency result, we are then able to prove an error estimate for the solution in the energy norm in Section 5.6.

5 Analysis of the method

5.1 Preliminaries

For the analysis, we require some further notation and definitions. We define the extension strip mesh as
\[ \mathcal{T}^n_h := \{ K \in \mathcal{T}_h \mid \exists x \in \widetilde{\Omega} \setminus \Omega^n_h \text{ with } \text{dist}(x, \Gamma^n_h) \leq \delta_h \}. \]

We also define the sharp strips as
\[ S^\pm_h(\Omega^n_h) := \{ x \in \widetilde{\Omega} \mid \text{dist}(x, \Gamma^n_h) \leq \delta_h \} \quad \text{and} \quad S^\pm_\delta(\Omega^n_h) := \{ x \in \widetilde{\Omega} \setminus \Omega^n_h \mid \text{dist}(x, \Gamma^n_h) \leq \delta_h \}. \]

Furthermore, we define the discrete extended domain $\mathcal{O}_{\delta_h}(\Omega^n_h) := S^\pm_h(\Omega^n_h) \cup \Omega^n_h$. In the analysis, we require that $\delta$ is sufficiently large, such that
\[ \mathcal{O}_{\delta_h}(\Omega^n_h) \subset \mathcal{O}_{\delta}(\Omega^n) \quad \text{and} \quad \Omega^n_h \subset \mathcal{O}_{\delta}(\Omega(t)), \quad t \in I_n = [t_{n-1}, t_n), \quad (5.1) \]

for $n = 1, \ldots, N$. Furthermore, for ease and brevity of notation, we write $a \lesssim b$ if it holds that $a \leq cb$ with a constant $c > 0$ independent of the mesh size $h$, the time step $\Delta t$, the time $t$, and the mesh-interface cut position. Similarly, we write $a \simeq b$ if both $a \lesssim b$ and $b \lesssim a$ holds.

For the analysis, we consider the following mesh dependent norms. For the velocity we take
\[ \|v\|^2_n := \|\nabla v\|^2_{\Omega^n_h} + \|h^{-1/2}v\|^2_{\Gamma^n_h} + \|h^{1/2}\partial_n v\|^2_{\Gamma^n_h}, \]
\[ \|v\|^2_{*,n} := \|\nabla v\|^2_{\Omega^n_h,\mathcal{T}^n} + \|h^{-1/2}v\|^2_{\Gamma^n_h}, \]
\[ \|v\|_{-1,n} := \sup_{w \in V^n_h} \frac{(v, w)_{\Omega^n_h}}{\|w\|_{*,n}} \]

for the pressure
\[ \|q\|^2_n := \|q\|^2_{\Omega^n_h} + \|h^{1/2}q\|^2_{\Gamma^n_h}, \]
\[ \|q\|_{*,n} := \|q\|_{\mathcal{T}^n} \]

and for the product space
\[ \|(u, p)\|^2_{*,n} := \|u\|^2_{*,n} + \|p\|^2_{*,n}. \]

Note that $\|\cdot\|_{*,n}$-norms are defined on the physical domain and add control on the normal derivative of the velocity and the trace of the pressure at the boundary. This norm arises naturally to bound the bilinear form $a^n_h(u, v)$ for functions $u, v \in H^1(\Omega^n_h)$. The second type of norms, the $\|\cdot\|_{*,n}$-norms, are defined on the entire active domain and therefore represent proper norms for discrete finite element functions.

5.1.1 Basic estimates

For $v_h \in P^k(K)$, $K \in \mathcal{T}_h$ we have the inverse and trace estimates:
\[ \|\nabla v_h\|_K \lesssim h^{-1}_K \|v_h\|_K, \quad (5.2a) \]
\[ \|h^{1/2}\partial_n v_h\|_F \lesssim \|\nabla v_h\|_K, \quad (5.2b) \]
\[ \|h^{1/2}\partial_n v_h\|_{K \cap \Gamma^n_h} \lesssim \|\nabla v_h\|_K \quad (5.2c) \]
For (5.2a) and (5.2b) see for example [Qua14]. For a proof of (5.2c), see [HH02]. Furthermore, for \( v \in \mathcal{H}^1(K) \), \( K \in \mathcal{T}_{h,\delta_h} \), we have the following trace inequality

\[
\|v\|_{K\cap \Gamma_h^0} \lesssim h_k^{-\frac{1}{2}} \|v\|_K + h_k^{\frac{1}{2}} \|\nabla v\|_K \tag{5.3}
\]

See [HH02] for (5.3). It follows from these estimates that

\[
\|v_h\|_n \lesssim \|v_h\|_{s,n} \quad \text{and} \quad \|q_h\|_n \lesssim \|q_h\|_{s,n} \tag{5.4}
\]

for all \( v_h \in V_h^m \) and \( q_h \in Q_h^m \). Furthermore, we have a discrete version of the Poincaré inequality

\[
\|v_h\|_{\mathcal{O}_{s,n}} \leq c_P h \|v_h\|_{s,n} \tag{5.5}
\]

for all \( v_h \in V_h \), see [Mas+14, Lemma 7.2] for a proof thereof.

### 5.1.2 Interpolation properties

**Lemma 5.1.** Let \( K \in \mathcal{T}_h \) and \( w \in \mathcal{H}^{m+1}(K) \), \( m \geq 1 \). Then it holds for the Lagrange interpolant

\[
\|D^k(w - I^s w)\|_K \leq h^{m+1-k} \|D^{m+1}w\|_K, \quad 0 \leq k \leq m,
\]

and on the boundary of an element

\[
\|w - I^s w\|_{\partial K} \leq h^{m+\frac{1}{2}} \|D^{m+1}w\|_K.
\]

**Proof.** See for example [Ape99].

**Lemma 5.2.** Let \( v \in \mathcal{H}^{m+1}(\Omega^n) \) and \( q \in \mathcal{H}^m(\Omega^n) \), \( m \geq 1 \). Then it holds that

\[
\|\mathcal{E} v - I^s \mathcal{E} v\|_n \lesssim h^m \|v\|_{\mathcal{H}^{m+1}(\Omega^n)} \tag{5.6a}
\]

\[
\|\mathcal{E} q - I^s \mathcal{E} q\|_n \lesssim h^m \|q\|_{\mathcal{H}^m(\Omega^n)}. \tag{5.6b}
\]

**Proof.** See also [Mas+14, Lemma 4.1]. For (5.6a), we consider the volume and boundary contribution to the \( \|\cdot\|_n \)-norm separately. Let \( w = \mathcal{E} v \).

We split the volume terms into element contributions. On interior elements, we simply apply Lemma 5.1. For cut simplices, we extend the norm onto the entire element before applying the interpolation estimate. Summing up over all active elements and applying (3.2a) in combination with (5.1) gives

\[
\|\nabla(w - I^s w)\|_{\Omega^n} \lesssim h^m \|w\|_{\mathcal{H}^{m+1}(\Omega^n)} \lesssim h^m \|v\|_{\mathcal{H}^{m+1}(\Omega^n)}.
\]

Similarly, on each cut element we can estimate the boundary terms using (5.3) and apply the interpolation estimate:

\[
\|h^{\frac{1}{2}} \partial_n (w - I^s w)\|_{K\cap \Gamma_h^0} \lesssim \|\nabla(w - I^s w)\|_K + h \|\nabla^2(w - I^s w)\|_K \lesssim h^m \|w\|_{\mathcal{H}^{m+1}(K)}
\]

and

\[
\|h^{-\frac{1}{2}}(w - I^s w)\|_{K\cap \Gamma_h^0} \lesssim h^{-1} \|w - I^s w\|_K + \|\nabla(w - I^s w)\|_K \lesssim h^m \|w\|_{\mathcal{H}^{m+1}(K)}.
\]

Summing up over all cut elements and again applying (3.2a) in combination with (5.1) proves (5.6a). The proof for (5.6b) follows along the same lines of argument and in the case of \( m = 1 \) we consider the Clément interpolation operator.  \( \square \)
5.1.3 The ghost-penalty mechanism

**Assumption A2.** We assume that for every strip element $K \in \mathcal{T}_{h,\delta_0}^n \setminus \mathcal{T}_{h,\Sigma}^n$ there exists an uncut element $K' \in \mathcal{T}_{h,\delta_0}^n \setminus \mathcal{T}_{h,\Sigma}^n$ which can be reached by a path which crosses a bounded number of facets $F \in \mathcal{T}_h^n$. We assume that the number of facets which have to be crossed to reach $K'$ from $K$ is bounded by a constant $L \lesssim (1 + \frac{\delta_0}{\Delta t})$ and that every uncut element $K' \in \mathcal{T}_{h,\delta_0}^n \setminus \mathcal{T}_{h,\Sigma}^n$ is the end of at most $M$ such paths with $M$ bounded independent of $\Delta t$ and $h$. In other words, each uncut element `supports' at most $M$ strip elements.

See [LO19, Remark 5.4] for a justification as to why the above assumption is reasonable if the mesh resolves the domain boundary sufficiently well.

**Lemma 5.3.** For all $v_h \in V_h^n$ and $q_h \in Q_h^n$ it holds that

\[
\|\nabla v_h\|_{\Omega_h^2}^2 \simeq \|\nabla v_h\|_{\Omega_h^2}^2 + L \cdot i_h^n(v_h, v_h) \tag{5.7a}
\]

\[
\|v_h\|_{\Omega_h^2}^2 \simeq \|v_h\|_{\Omega_h^2}^2 + h^2 L \cdot i_h^n(v_h, v_h) \tag{5.7b}
\]

\[
\|q_h\|_{\Omega_h^2}^2 \simeq \|q_h\|_{\Omega_h^2}^2 + j_h^n(q_h, q_h). \tag{5.7c}
\]

**Proof.** For the first inequality in both (5.7a) and (5.7c) we refer to [LO19, Lemma 5.5]. For the second bound we use the fact that we can bound the direct version of the ghost penalty operator by the normal derivative jump version, see [Pre18, Chapter 3, Remark 6]. The upper bound then follows by an application of the inverse (trace) estimates (5.2a) and (5.2b), see [Mas+14, Proposition 5.1].

5.2 Unique solvability

**Lemma 5.4 (Continuity).** For the diffusion bilinear form we have for all $v, w \in H^1(\Omega_h^n)$ that it holds

\[
a_h^n(v, w) \lesssim ||v||_n ||w||_n \tag{5.8}
\]

and for all $v_h, w_h \in V_h^n$ it holds that

\[
a_h^n(v_h, w_h) + \nu L i_h^n(v_h, w_h) \lesssim ||v_h||_{*,n} ||w_h||_{*,n}. \tag{5.9}
\]

Furthermore, for the velocity-pressure coupling bilinear form we have for all $q \in L^2(\Omega_h^n)$ and $v \in H^1(\Omega_h^n)$ that

\[
b_h^n(q, v) \lesssim ||q||_n ||v||_n
\]

and for all $q_h \in Q_h^n$ and $v_h \in V_h^n$ that

\[
b_h^n(q_h, v_h) \lesssim ||q_h||_{*,n} ||v_h||_{*,n}.
\]

**Proof.** See [BH12] for the diffusion bilinear form and [Mas+14] for the pressure coupling bilinear form. The scaling of the ghost-penalty term in (5.9) is due to the larger extension strip and Lemma 5.3.

**Lemma 5.5 (Coercivity).** There exists a constant $c_{L,5.5} > 0$, independent of $h$ and the mesh-interface cut position, such that for sufficiently large $\sigma > 0$ there holds

\[
a_h^n(u_h, u_h) + \nu L i_h^n(u_h, v_h) \geq \nu c_{L,5.5} ||u_h||_{*,n}^2
\]

for all $u_h \in V_h$.

**Proof.** See for example [BH12, Lemma 6] or [BH14, Lemma 4.2]. The different scaling is again due to Lemma 5.3.
Lemma 5.6. Let $\Omega_{h,i}^\circ := \Omega_h^\circ \setminus \{ x \in K \mid K \in \mathcal{T}_h, 1_h \}$ denote the interior, uncut domain. It then holds for all $q_h \in Q_h^n$ with $q_h|_{\Omega_{h,i}^\circ} \in \mathcal{L}_0^2(\Omega_{h,i})$ that

$$\beta \|q_h\|_{H^1_h} \leq \sup_{v_h \in V_h^n} \frac{b_h(q_h, v_h)}{\|v_h\|_{s,n}} + j_h^n(q_h, q_h)^{1/2}. \quad (5.10)$$

The constant $\beta > 0$ is independent of $h$ and $q_h$.

Proof. See [GO17, Corollary 1] for the proof. \hfill \Box

Remark 5.7 (Choice of ghost-penalty parameter). From Lemma 5.4 and Lemma 5.5 we see that the velocity ghost-penalty parameter should scale with the strip-width $L$. This is necessary, in order for an interior unphysical but active element to obtain support from an uncut interior element for which we have to cross at most $L$ elements to reach it, c.f. Assumption A2. As this first part of the ghost-penalties is related to the stabilization of the viscosity bilinear form $a^\nu_h(\cdot, \cdot)$ only it has a scaling with $\nu$. We require the same mechanism also for the implicit extension of functions. As we will see in the analysis below this requires the same ghost-penalty, however with the different scaling $1/\nu$.

The pressure ghost-penalty operator in Lemma 5.6 does not need a scaling with $L$ as we require these ghost-penalties only for stabilizing the velocity-pressure coupling, but not for the extension of the pressure field into a $\delta$-neighbourhood.

For simplicity of the analysis, we choose a common ghost-penalty stabilisation parameter $\gamma_s$, which we use to set $\gamma_{s,u} = \gamma_{s,u} = L \gamma_s$ and $\gamma_{s,p} = \gamma_s$.

We now collect all fully implicit non-ghost-penalty terms in the bilinear form

$$A^\nu_h((u^n_h, p^n_h), (v_h, q_h)) := \frac{1}{\Delta t} (u^n_h, v_h)_{\Omega_h^n}^\circ + a^\nu_h(u^n_h, v_h) + b_h^n(p^n_h, v_h) + b_h^n(q_h, u^n_h)$$

and the explicit terms in the linear form

$$F^n_h(v_h) := \frac{1}{\Delta t} (u_h^{n-1}, v_h)_{\Omega_h^n}^\circ + f^n_h(v_h).$$

We can then rewrite problem (4.1) as: Find $(u^n_h, p^n_h) \in V^n_h \times Q^n_h$, such that

$$A^\nu_h((u^n_h, p^n_h), (v_h, q_h)) + s_h^n((u^n_h, p^n_h), (v_h, q_h)) = F^n_h(v^n_h) \quad (5.11)$$

for all $(v_h, q_h) \in V^n_h \times Q^n_h$. This problem is indeed well-posed:

Theorem 5.8 (Well posedness). Consider the norm $\| (u^n_h, p^n_h) \|_{b,n}^2 := \frac{1}{\Delta t} \| u^n_h \|_{\Omega_h^n}^2 + \| (u^n_h, p^n_h) \|_{s,n}^2$. There exists a constant $c_{5.8} > 0$ such that for all $(u^n_h, p^n_h) \in V^n_h \times Q^n_h$ it holds that

$$\sup_{(v_h, q_h) \in V^n_h \times Q^n_h} \frac{A^\nu_h((u^n_h, p^n_h), (v_h, q_h)) + s_h^n((u^n_h, p^n_h), (v_h, q_h))}{\|(v_h, q_h)\|_{s,n}} \geq c_{5.8} \| (u^n_h, p^n_h) \|_{b,n} \quad (5.12)$$

The solution $(u^n_h, p^n_h) \in V^n_h \times Q^n_h$ to (5.11) exists and is unique.

Proof. The proof follows the same lines as for the unfitted Stokes problem using CutFEM in [Mas+14; GO17]. Let $(u^n_h, p^n_h) \in V^n_h \times Q^n_h$ be given and let $w_h \in V^n_h$ be the test function, such that (5.10) holds and w.l.o.g that $\|w_h\|_{s,n} = \|p^n_h\|_{\Omega_h^n}$. Testing (5.11) with $(-w_h, 0)$ and using Cauchy-Schwarz, Lemma 5.4,
Lemma 5.6 and the weighted Young’s inequality then gives

\[
(A_h^n + s_h^n)((u_h^n, p_h^n), (-w_h, 0)) = -\frac{1}{\Delta t} (u_h^n, w_h)_{\Omega_h} - (u_h^n + (\nu + 1/\nu)) (u_h^n, w_h) + b_h^n (-p_h^n, w_h)
\]

\[
\geq - \frac{1}{\Delta t} \|u_h^n\|_{\Omega_h} \|w_h\|_{\Omega_h} - c ||u_h^n||_{s,n} \|w_h\|_{s,n} + \beta \|p_h^n\|_{\Omega_h} \|w_h\|_{s,n}
\]

\[
\geq - \frac{1}{2\varepsilon_1 \Delta t^2} \|u_h^n\|_{\Omega_h}^2 - \frac{c_1}{2\varepsilon_2} \|u_h^n\|_{s,n}^2 - \frac{1}{2\varepsilon_3} j_h^n (p_h^n, p_h^n)
\]

Choosing \(\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0\) such that \((\beta - \varepsilon_1/2 - \varepsilon_2/2 - \varepsilon_3/2) > 0\) we then have

\[
(A_h^n + s_h^n)((u_h^n, p_h^n), (-w_h, 0)) \geq - c_1 \|u_h^n\|_{\Omega_h}^2 - c_2 \|u_h^n\|_{s,n}^2 + c_3 \|p_h^n\|_{\Omega_h}^2 - c_4 j_h^n (p_h^n, p_h^n).
\]

Furthermore, with the test function \((v_h, q_h) = (u_h^n, -p_h^n)\) in (5.11) and using Lemma 5.5 we have

\[
(A_h^n + s_h^n)((u_h^n, p_h^n), (u_h^n, -p_h^n)) \geq \frac{1}{\Delta t} \|u_h^n\|_{\Omega_h}^2 + c L_{5.5} \|u_h^n\|_{s,n}^2 + j_h^n (p_h^n, p_h^n).
\]

Combining these two estimates, we have with a suitable choice of \(\Delta t \delta > 0\) that

\[
(A_h^n + s_h^n)((u_h^n, p_h^n), (u_h^n - \Delta t \delta w_h, -p_h^n)) \geq c \|u_h^n, p_h^n\|_{s,n}^2.
\]

Since \(||(u_h^n - \Delta t \delta w_h, -p_h^n)||_{s,n} \leq (1 + \Delta t \delta)(||u_h^n, p_h^n||_{s,n}||w_h, -p_h^n||_{s,n}\), we then have (5.12).

The continuity of \((A_h^n + s_h^n)\) in the \(||(\cdot, \cdot)||_{s,n}\)-norm can be easily shown. The existence and uniqueness of the solution then follows by the Banach-Necas-Babuška Theorem (see e.g. [EG04, Theorem 2.6]).

5.3 Stability

For the fully discrete method, we have a discrete counterpart of Lemma 3.2.

Theorem 5.9. For the velocity solution \(u_h^n \in V_h^n\), \(n = 1, \ldots, N\) of (4.1) we have the stability estimate

\[
\|u_h^n\|_{\Omega_h}^2 + \Delta t \sum_{n=1}^k \left[ \frac{\nu c t L_{5.9a}}{2} \|u_h^n\|_{s,n}^2 + \frac{L}{\nu} \|v_h^n(u_h^n, u_h^n)\right]
\]

\[
\leq \exp(c T_{5.9a} \nu^{-1} t_k) \left[ \|u_h^0\|_{\Omega_h}^2 + \frac{\nu c t L_{5.9a}}{2} \|u_h^0\|_{s,0}^2 + \frac{L}{\nu} \|v_h^0(u_h^0, u_h^0) + \Delta t \sum_{n=1}^k \frac{c t_{L,5.9a}}{2} \|f^n_h\|_{\Omega_h}^2 \right].
\]

Proof. Testing (4.1) with \((v_h, q_h) = 2\Delta t(u_h^n, -p_h^n) \in V_h^n \times Q_h^n\) and using the identity

\[
2(u^n - \mathcal{E} u^{n-1}, u^n)_{\Omega_h} = \|u^n\|_{\Omega_h}^2 + \|u^n - \mathcal{E} u^{n-1}\|_{\Omega_h}^2 - \|\mathcal{E} u^{n-1}\|_{\Omega_h}^2
\]

(5.13)

gives us

\[
\|u_h^n\|_{\Omega_h}^2 + \|u_h^n - u_h^{n-1}\|_{\Omega_h}^2 + 2\Delta t ||u_h^n(u_h^n, u_h^n) + \nu c t (u_h^n, u_h^n) + \frac{\gamma s}{\nu} j_h^n(u_h^n, u_h^n) + \frac{\gamma s}{\nu} j_h^n(p_h^n, p_h^n)\]

\[
= 2\Delta t j_h^n(u_h^n) + \|u_h^{n-1}\|_{\Omega_h}^2.
\]
Applying a discrete Gronwall inequality with the choice $\varepsilon = \nu c_{L5.5}$ then gives

$$
\| u_h^n \|_{V_h}^2 + \Delta t \nu c_{L5.5} \| u_h^n \|_{V_h}^2 \leq \frac{c^2_{P,h} \| f_h^n \|_{V_h}^2}{\nu c_{L5.5}^2} + \| u_h^{n-1} \|_{V_h}^2.
$$

To obtain a bound on $\| u_h^{n-1} \|_{V_h}^2$ we utilise the following result, c.f., [LO19, Lemma 5.7]:

$$
\| u_h \|_{S_h(\Omega_h)}^2 \leq (1 + c_1(\varepsilon) \Delta t) \| u_h \|_{S_h(\Omega_h)}^2 + c_2(\varepsilon) \nu \Delta t \| \nabla u_h \|_{S_h(\Omega_h)}^2 + c_3(\varepsilon, h) \Delta t \| L_i^n(u_h, u_h) \|_{S_h(\Omega_h)}
$$

with $c_1(\varepsilon) = c_c \delta_h u_h^n(1 + \varepsilon^{-1})$, $c_2(\varepsilon) = c_c \delta_h w_h^n \varepsilon / \nu$ and $c_3(\varepsilon) = c_c \delta_h w_h^n (\varepsilon + (1 + \varepsilon^{-1}) \delta_h)$ and $c_c > 0$ independent of $h$ and $\Delta t$. Choosing $\varepsilon = \nu c_{L5.5}/(2 c_c \delta_h w_h^n)$ we then have $c_3 = c_{L5.5}/2$ while we can bound the resulting $c_1(\varepsilon) \leq \bar{\varepsilon} / n u$, with $\bar{\varepsilon}$ independent of $\Delta t$ and $h$ and $c_3 \leq \nu c_{L5.5}/2 + \delta_h^2 / \nu$. This gives the estimate

$$
\| u_h^{n-1} \|_{S_h(\Omega_h)}^2 \leq \| u_h^{n-1} \|_{S_h(\Omega_h)}^2 + \frac{\Delta t c_{L5.5}}{2} \| u_h^{n-1} \|_{S_h(\Omega_h)}^2 + \frac{\varepsilon h^2}{\nu} \| L_i^n(u_h, u_h) \|_{S_h(\Omega_h)}
$$

Inserting this into (5.15) and summing over $n = 1, \ldots, k$ for $k \leq N$ then gives for sufficiently small $h$, such that $\varepsilon h^2 \leq 1$

$$
\| u_h^k \|_{S_h(\Omega_h)}^2 + \Delta t \sum_{n=1}^k \left[ \frac{\nu c_{L5.5}}{2} \| u_h^n \|_{S_h(\Omega_h)}^2 + \frac{L_i^0(u_h^n, u_h^n)}{\nu} \right] \leq \| u_h^k \|_{S_h(\Omega_h)}^2 + \frac{\nu \Delta t c_{L5.5}}{2} \| u_h^k \|_{S_h(\Omega_h)}^2 + \frac{L_i^0(u_h^0, u_h^0)}{\nu} + \Delta t \sum_{n=1}^k \frac{c^2_{P,h}}{\nu c_{L5.5}} \| f_h^n \|_{V_h}^2 + \Delta t \sum_{n=0}^{k-1} \frac{\varepsilon}{\nu} \| u_h^n \|_{V_h}^2.
$$

Applying a discrete Gronwall inequality with $c_{T5.9a} = c_{L5.5}$ and $c_{T5.9b} = \bar{\varepsilon}$ then gives the desired result. \hfill $\square$

**Lemma 5.10.** For the pressure solution $p_h^n \in Q_h^n$ of (4.1) it holds that

$$
\| p_h^n \|_{\Omega_h}^2 \leq c_{L5.10} \left[ \frac{1}{\Delta t} \| (u_h^n - u_h^{n-1}) \|_{-1, n} + \| u_h^n \|_{s,n} + \| f_h^n \|_{\Omega_h}^2 + \| j_h^n(p_h^n, p_h^{n-1}) \|_{\Omega_h}^2 \right].
$$

**Proof.** We have that

$$
b_h(p_h^n, v_h) = -\frac{1}{\Delta t} (u_h^n - u_h^{n-1}, v_h)_{\Omega_h} - (a_h^n + \nu L_h^n)(u_h^n, v_h) - \nu (u_h^n, v_h) + f_h^n(v_h)
$$

$$
\leq \left[ \frac{1}{\Delta t} \| (u_h^n - u_h^{n-1}) \|_{-1, n} + c(\nu + 1/\nu) \| u_h^n \|_{s,n} + c_{P,h} \| f_h^n \|_{\Omega_h} \right] \| v_h \|_{s,n}.
$$

Here we used the continuity of $a_h^n + \nu L_h^n$, the estimate $i_h^n(w, v) \leq i_h^n(w, v)^{1/2} i_h^n(v, v)^{1/2}$ with (5.7a) and the Poincaré inequality. The result then follows from the inf-sup result in Lemma 5.6. \hfill $\square$
Remark 5.11. Lemma 5.10 does not immediately result in a satisfactory $L^2$-stability estimate for the pressure. (5.17) only gives us an estimate for

$$\Delta t^2 \sum_{n=1}^{k} \|p^n_h\|_{H^1}^2.$$

In order to get a proper estimate of the form $\Delta t \sum_{n=1}^{k} \|p^n_h\|_{H^1}^2$ we require an estimate of the term $\frac{1}{\Delta t}\|u^n_h - u_h^{n-1}\|_{-1,n}$ which is independent (of negative powers) of $\Delta t$.

5.4 Geometrical Approximation

We shall assume that we have a higher order approximation of the geometry, i.e.,

$$\text{dist}(\Omega^n, \Omega^n_h) \leq h^{q+1}$$

with the geometry approximation order $q$ and that integrals on $\Omega^n_h$ can be computed sufficiently accurately. We further assume the existence of a mapping which maps the approximated extended domain to the exact extended domain. In other words, we have $\Phi : \Omega_h^b(\Omega^n_h) \to \Omega_h^b(\Omega^n)$ which we assume to be well-defined, continuous, it holds that $\Omega^n = \Phi(\Omega_h^n)$, $\Gamma^n = \Phi(\Gamma_h^n)$ and $\Omega_h^b(\Omega^n) = \Phi(\Omega_h^b(\Omega^n_h))$ and

$$\|\Phi - \text{Id} \|_{L^\infty(\Omega_h^b(\Omega^n_h))} \lesssim h^{q+1},$$

$$\|D\Phi - I\|_{L^\infty(\Omega_h^b(\Omega^n_h))} \lesssim h^q, \quad \|\det(D\Phi) - 1\|_{L^\infty(\Omega_h^b(\Omega^n_h))} \lesssim h^q.$$  \hfill (5.18)

Furthermore, for sufficiently small $h$, the mapping $\Phi$ is invertible. Such a mapping has been constructed, e.g., in [GOR15, Section 7.1]. This mapping $\Phi$ is used here (as in [LO19], and therefore using the same notation) to map from the discrete domain to the exact one. Let $v_h \in V_h^n$ and define $v'_h = v_h \circ \Phi^{-1}$. From the third estimate in (5.18) we have $\det(D\Phi) \approx 1$, hence we then get using integration by substitution

$$\|v_h\|_{\Omega_h^b(\Omega^n)}^2 = \sum_{i=1}^{d} \int_{\Omega_h^b(\Omega^n)} (v_h^i)^2 \, d\vec{x} = \sum_{i=1}^{d} \int_{\Omega_h^b(\Omega^n)} \det(D\Phi)(v_h^i)^2 \, d\vec{x} \approx \|v_h\|_{\Omega_h^b(\Omega^n_h)}^2.$$  \hfill (5.19a)

Using the same argument we also have that

$$\|v_h\|_{\Omega_h^b(\Omega^n)}^2 \approx \|v_h\|_{\Omega_h^b(\Omega^n_h)}^2$$

as well as

$$\|\nabla v_h\|_{\Omega_h^b(\Omega^n)}^2 \approx \|\nabla v_h\|_{\Omega_h^b(\Omega^n_h)}^2 \quad \text{and} \quad \|\nabla v_h\|_{\Omega_h^b(\Omega^n_h)}^2 \approx \|\nabla v_h\|_{\Omega_h^b(\Omega^n)}^2.$$  \hfill (5.19b)

For the extension, we also have the following result, c.f. [GOR15, Lemma 7.3]:

Lemma 5.12. The following estimates

$$\|\mathcal{E}u - u \circ \Phi\|_{\Omega_h^b(\Omega^n)} \lesssim h^{q+1}\|u\|_{H^1(\Omega^n)},$$

$$\|\nabla(\mathcal{E}u) - (\nabla u) \circ \Phi\|_{\Omega_h^b(\Omega^n)} \lesssim h^{q+1}\|u\|_{H^2(\Omega^n)},$$

$$\|\mathcal{E}u - u \circ \Phi\|_{\Gamma_h^n} \lesssim h^{q+1}\|u\|_{H^2(\Omega^n)}.$$  \hfill (5.19c)

hold for all $u \in H^2(\Omega^n)$, $n = 1, \ldots, N$. Furthermore, it also holds that

$$\|\mathcal{E}u - (\mathcal{E}u) \circ \Phi\|_{\Omega_h^b(\Omega^n_h)} \lesssim h^{q+1}\|u\|_{H^1(\Omega^n)},$$

\hfill (5.20)
Proof. For (5.19a) – (5.19c) we refer to [GOR15, Lemma 7.3]. The proof of (5.20) follows the identical lines of (5.19a) but integration over $S_+^3(\Omega_h^n)$ rather than $S_+^1(\Omega_h^n)$. □

Lemma 5.13. For all $v \in \mathcal{H}^{m+1}(\Omega^n)$ and $r \in \mathcal{H}^m(\Omega^n)$ we have the estimates

\begin{align}
&h^{1/2}\|\partial_n v - \partial_n r \circ \Phi\|_{\Gamma_h^n} \lesssim h^q \|v\|_{\mathcal{H}^1(\Omega^n)} + h^{m+1/2}\|v\|_{\mathcal{H}^{m+1}(\Omega^n)} \quad (5.21a) \\
&\|\mathcal{E} v \cdot n - (v \cdot n) \circ \Phi\|_{\Gamma_h^n} \lesssim h^{q+1/2}\|v\|_{\mathcal{H}^1(\Omega^n)} + h^{m+1/2}\|v\|_{\mathcal{H}^{m+1}(\Omega^n)} \quad (5.21b) \\
&h^{1/2}\|\mathcal{E} r - r \circ \Phi\|_{\Gamma_h^n} \lesssim h^q \|r\|_{\mathcal{H}^1(\Omega^n)} + h^m \|r\|_{\mathcal{H}^m(\Omega^n)}. \quad (5.21c)
\end{align}

Proof. To make the proof more readable, we do not write the extension operator explicitly and identify $v$ with its smooth extension $\mathcal{E} v$.

Now let $v \in \mathcal{H}^{m+1}(\Omega^n)$. To prove (5.21a) we insert additive zeros and use the triangle inequality to get

\begin{equation}
\begin{aligned}
&h^{1/2}\|\partial_n v - \partial_n r \circ \Phi\|_{\Gamma_h^n} \lesssim h^{1/2} \left[\|\partial_n v - \partial_n \Phi \circ \Phi\|_{\Gamma_h^n} + \|\partial_n \Phi \circ \Phi - I^* \partial_n v\|_{\Gamma_h^n} \right].
\end{aligned}
\end{equation}

To bound the first term on the right-hand side of (5.22) we divide the norm up into the separate contributions of cut elements, use (5.3), the interpolation estimate in Lemma 5.1 as well as Assumption A1 in combination with (5.1) in order to get

\begin{equation}
\begin{aligned}
\|\partial_n v - \partial_n \Phi \circ \Phi\|_{\Gamma_h^n} &= \|\partial_n v - I^* \partial_n v\|_{K \cap \Gamma_h^n} \\
&= \sum_{K \in \mathcal{T}_h, \Gamma} \|\partial_n v - I^* \partial_n v\|_{K \cap \Gamma_h^n} \\
&\lesssim \sum_{K \in \mathcal{T}_h, \Gamma} h^{-1/2}\|\nabla v - I^* \nabla v\|_K + h^{1/2}\|\nabla (\nabla v - I^* \nabla v)\|_K \\
&\lesssim \sum_{K \in \mathcal{T}_h, \Gamma} h^{-1/2} h^m \|v\|_{\mathcal{H}^{m+1}(K)} + h^{1/2} h^{m-1} \|v\|_{\mathcal{H}^{m+1}(K)} \\
&\lesssim h^{m+1/2}\|v\|_{\mathcal{H}^{m+1}(\Omega_h^n)} \lesssim h^{m+1/2}\|v\|_{\mathcal{H}^{m+1}(\Omega^n)}. \quad (5.23)
\end{aligned}
\end{equation}

The third term in (5.22) can be estimated completely analogously. For the second term we make use of the fact, that the argument of the norm is discrete, therefore permitting the use of the inverse estimate (5.2a). Then with (5.2c) and (5.1) it follows that

\begin{equation}
\begin{aligned}
\|I^* \partial_n v - \partial_n \Phi \circ \Phi\|_{\Gamma_h^n} &= \sum_{K \in \mathcal{T}_h, \Gamma} \|I^* \partial_n v - I^* \partial_n v\|_{K \cap \Gamma_h^n} \\
&\lesssim \sum_{K \in \mathcal{T}_h, \Gamma} h^{-1/2}\|\nabla (I^* v - \nabla v)\|_K \\
&\lesssim \sum_{K \in \mathcal{T}_h, \Gamma} h^{-1/2}\|I^* (v - \Phi - v)\|_K \\
&\lesssim h^{-1/2}\|v - \Phi - v\|_{\mathcal{O}_h(\Omega_h^n)}
\end{aligned}
\end{equation}

We can then apply (5.20) so that we get

\begin{equation}
\begin{aligned}
\|I^* \partial_n v - \partial_n \Phi\|_{\Gamma_h^n} &\lesssim h^{-3/2}\|v - \Phi - v\|_{\mathcal{O}_h(\Omega_h^n)} \\
&\lesssim h^{-1/2}\|v\|_{\mathcal{H}^1(\Omega^n)}. \quad (5.24)
\end{aligned}
\end{equation}
Combining (5.23) and (5.24) then gives us the desired estimate

\[ h^{1/2} \| \partial_n v \circ \Phi - \partial_n v \|_{I_h^0} \lesssim h^{m+1/2} \| v \|_{H^{m+1}(\Omega)} + h^q \| v \|_{H^q(\Omega)} \]

For (5.21c), let \( r \in H^m(\Omega) \). We again introduce two additive zeros as in (5.22) and apply the triangle inequality

\[ \| r - r \circ \Phi \|_{I_h^0} \leq \| r - I^* r \|_{I_h^0} + \| I^* r - I^* (r \circ \Phi) \|_{I_h^0} + \| I^* (r \circ \Phi) - r \circ \Phi \|_{I_h^0} \]

The first term of which we can then be estimated using (5.3), Lemma 5.1 and Assumption A1

\[
\| r - I^* r \|_{I_h^0}^2 = \sum_{K \in \mathcal{T}_h} \| r - I^* r \|_{K \cap I_h^0}^2 \\
\lesssim \sum_{K \in \mathcal{T}_h} h^{-1} \| I^* r \|_{K}^2 + h^1 \| \nabla (r - I^* r) \|_{K}^2 \\
\lesssim h^{-1} \| I^* r \|_{O_h(\Omega_h)}^2 + h^1 \| \nabla (r - I^* r) \|_{O_h(\Omega_h)}^2 \\
\lesssim h^{2m-1} \| r \|_{H^m(O_h(\Omega_h))}^2 + h^{2(m-1)+1} \| r \|_{H^m(O_h(\Omega_h))}^2 \\
\lesssim h^{2m-1} \| r \|_{H^m(\Omega_h)}^2.
\]

The term \( \| I^* (r \circ \Phi) - r \circ \Phi \|_{I_h^0} \) can then be bound along the same lines. For the term \( \| I^* r - I^* (r \circ \Phi) \|_{I_h^0} \) we make use of the fact that the argument of the norm is a discrete function and therefore permitting the inverse estimate (5.2a). Then with (5.20) we get

\[
\| I^* r - I^* r \circ \Phi \|_{I_h^0}^2 = \sum_{K \in \mathcal{T}_h} \| I^* r - I^* r \circ \Phi \|_{K \cap I_h^0}^2 \\
\lesssim \sum_{K \in \mathcal{T}_h} h^{-1} \| I^* r - I^* r \circ \Phi \|_{K}^2 + h^1 \| \nabla (I^* r - I^* r \circ \Phi) \|_{K}^2 \\
\lesssim \sum_{K \in \mathcal{T}_h} h^{-1} \| I^* r - I^* r \circ \Phi \|_{K}^2 \\
\lesssim \| r - r \circ \Phi \|_{O_h(\Omega_h)}^2 \lesssim h^{2q-1} \| r \|_{H^q(\Omega_h)}.
\]

This then gives the desired estimate

\[ h^{1/2} \| r - r \circ \Phi \|_{I_h^0} \lesssim h^q \| r \|_{H^q(\Omega_h)} + h^m \| r \|_{H^m(\Omega_h)}. \]

The proof of (5.21b) follows along the same lines as the other two estimates.

\[ \square \]

### 5.5 Consistency

Using integration by parts, we see that any smooth solution \((u, p)\) to the strong problem (2.1), fulfills

\[
\int_{Q_h^n} \partial_t u(t_n) v_h^t \, d\hat{x} + a^n_1(u(t_n), v_h^t) + b_1^n(p(t_n), v_h^t) + b_1^n(q_h^t, u(t_n)) = f^n(v_h^t) \tag{5.25}
\]

for the test-functions \((v_h^t, q_h^t) = (v_h \circ \Phi^{-1}, q_h \circ \Phi^{-1})\) with \((v_h, q_h) \in V_h^n \times Q_h^n\), the mapping \(\Phi\) from Section 5.4 and the bilinear forms

\[ a^n_1(u, v) := a^n(u, v) + \nu \int_{\Gamma_n} (\partial_n u) \cdot v \, ds \]
and
\[ b^n_i(p, v) := b^n(p, v) + \int_{\Gamma^n} pv \cdot n \, ds. \]

Since \( \Omega_{h,n} \subset \mathcal{O}_h(\Omega(t)) \), \( t \in [t_{n-1}, t_n] \), \( u(t_{n-1}) = u^{n-1} \) is well defined on \( \Omega_{h,n} \). For simplicity of notation we shall identify the smooth extension \( \tilde{E} u \) with \( u \) and denote \( \mathbb{E}^n := u^n - u_{h,n}^n \) and \( \mathbb{D}^n := p^n - \rho_{h,n}^n \). Subtracting (4.1) from (5.25) and adding and subtracting appropriate terms we obtain the error equation
\[
\int_{\Omega_{h,n}} \frac{E^n - E^{n-1}}{\Delta t} \cdot v_h \, dx + a_h^n(E^n, v_h) + b_h^n(\mathbb{D}^n, v_h) + b_h^n(q_h, E^n) + s_h^n((E^n, \mathbb{D}^n), (v_h, q_h)) = f^n(v_h) - f_h^n(v_h) + \int_{\Omega_{h,n}} \frac{u^n - u^{n-1}}{\Delta t} \cdot v_h \, dx - \int_{\Omega_{h,n}} \partial_t u^n \cdot v_h \, dx + a_h^n(u^n, v_h) - a_h^n(u^n, v_h) + b_h^n(p^n, v_h) - b_h^n(p^n, v_h) + b_h^n(q_h, u^n) - b_h^n(q_h, u^n) + s_h^n((u^n, p^n), (v_h, q_h)) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5 + \mathcal{I}_6
\]

These six terms correspond to the forcing, time-derivative, diffusion, pressure, divergence constraint and ghost-penalty contributions respectively.

**Lemma 5.14 (Consistency estimate).** The consistency error has the bound
\[
|\mathcal{E}^n((v_h, q_h))| \lesssim \left( \Delta t + h^3 + \frac{h^m L^{1/2}}{\nu} \right) R_{c,1}(u, p, f)\|v_h\|_{s,n} + (h^q + h^m)R_{c,2}(u, p)\|q_h\|_{s,n}
\]

with
\[
R_{c,1}(u, p, f) = \|u\|_{W^{2,q}(\Omega)} + \|f^n\|_{\mathcal{H}^1(\mathbb{D}^n)} + \sup_{t \in [0,T]} (\|\mathcal{U}(t)\|_{\mathcal{H}^{m+1}(\Omega(t))) + \|p(t)\|_{\mathcal{H}^m(\Omega(t)))}
\]

and
\[
R_{c,2}(u, p) = \sup_{t \in [0,T]} (\|\mathcal{U}(t)\|_{\mathcal{H}^{m+1}(\Omega(t))) + \|p(t)\|_{\mathcal{H}^m(\Omega(t)))}
\]

**Proof.** We estimate each of the five components of the consistency error separately.

\( \mathcal{I}_1 \): See also [GOR14b, Lemma 7.5], the preprint of [GOR15]. Beginning with the forcing contribution \( \mathcal{I}_1 \) and denoting \( J = \det(D\Phi) \), we use integration by substitution to obtain
\[
|\mathcal{I}_1| = |f^n(v_h') - f_h^n(v_h)| = \left| \int_{\Omega_{h,n}} f^n \cdot v_h' \, dx - \int_{\Omega_{h,n}} f^n \cdot v_h \, dx \right| = \left| \int_{\Omega_{h,n}} (J \cdot f^n \circ \Phi - f^n) \cdot v_h \, dx \right|
\]

\[ \lesssim \|f^n \circ \Phi - f^n\|_{L^\infty(\Omega_h^n)} \|v_h\|_{L^\infty} + \|J - 1\|_{L^\infty} \|f^n\|_{L^\infty} \|v_h\|_{L^\infty}
\]

\[ \lesssim h^{q+1} \|f^n\|_{\mathcal{H}^1(\Omega_h^n)} \|v_h\|_{L^\infty} + h^q \|f^n\|_{L^\infty} \|v_h\|_{L^\infty}
\]

\[ \lesssim h^q \|f^n\|_{\mathcal{H}^1(O_h(\Omega_h^n))} \|v_h\|_{L^\infty}.
\]

Here we used (5.18), Lemma 5.12 and Assumption A1.

\( \mathcal{I}_2 \): The proof for the time-derivative component follows the proof of [LO19, Lemma 5.11]. Using integration by parts in time, we have that
\[
\frac{1}{\Delta t} [u(t_n) - u(t_{n-1})] = u(t_n) - \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{\Delta t} \mathcal{U}_t \, dt.
\]
Therefore, we may rewrite $\mathcal{T}_2$ as

$$
\mathcal{T}_2 = \int_{\Omega_h^n} u_t(t_n) \cdot v_h \, dx - \int_{\Omega_h^n} u_t(t_n) \cdot v_h^{t_n} \, dx - \int_{\Omega_h^n} \int_{t_{n-1}}^{t_n} \Delta t \nabla u_{tt} \cdot v_h \, dx.
$$

(5.28)

For the first two terms of (5.28) we have

$$
\int_{\Omega_h^n} u_t(t_n) \cdot v_h \, dx - \int_{\Omega_h^n} u_t(t_n) \cdot v_h^{t_n} \, dx = \int_{\Omega_h^n} u_t(t_n) \cdot v_h \, dx - \int_{\Phi(\Omega_h^n)} u_t(t_n) \cdot (v_h \circ \Phi^{-1}) \, dx
$$

$$
= \int_{\Omega_h^n} u_t(t_n) \cdot v_h \, dx - \int_{\Omega_h^n} (u_t \circ \Phi)(t_n) \cdot v_h \, dx.
$$

We estimate this by

$$
|\mathcal{T}_2| \leq \left| \int_{\Omega_h^n} [u_t(t_n) - (u_t \circ \Phi)(t_n)] \cdot J \cdot v_h \, dx \right| + \left| \int_{\Omega_h^n} u_t(1 - J) \cdot v_h \, dx \right|
$$

$$
\lesssim \|\nabla u_t\|_{L^\infty(\Omega_h^n)} \|\text{Id} - \Phi\|_{\Omega_h^n} \|v_h\|_{\Omega_h^n} + ||J - 1||_{L^\infty(\Omega_h^n)} \|u_t(t_n)\|_{\Omega_h^n} \|v_h\|_{\Omega_h^n}
$$

$$
\lesssim h^{q+1} ||\nabla u_t||_{L^\infty(\Omega_h^n)} \|v_h\|_{\Omega_h^n} + h^{q} ||u_t(t_n)||_{\Omega_h^n} \|v_h\|_{\Omega_h^n}
$$

$$
\lesssim h^{q} \|u_t\|_{W^{2,q}(\Omega)} \|v_h\|_{\Omega_h^n}
$$

where we used (5.18) and

$$
\|u_t(x, t_n) - (u_t \circ \Phi)(x, t_n)\| \leq \|\nabla u_t\|_{L^\infty(\Omega_h^n)} \|x - \Phi(x)\|.
$$

For the third and final term in (5.28) we have

$$
\left| \int_{\Omega_h^n} \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{\Delta t} u_{tt} \cdot v_h \, dx \right| \leq \int_{\Omega_h^n} \max_{s \in [t_{n-1}, t_n]} u_{tt}(s) \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{\Delta t} \Delta t v_h \, dx
$$

$$
\leq \frac{1}{2} \Delta t \|u_{tt}\|_{L^\infty(\Omega_h^n)} \|v_h\|_{L^2(\Omega_h^n)}
$$

$$
\lesssim \Delta t \|u_t\|_{W^{2,\infty}(\Omega)} \|v_h\|_{\Omega_h^n}.
$$

This then gives the bound

$$
|\mathcal{T}_2| \lesssim (\Delta t + h^q) \|u_t\|_{W^{2,\infty}(\Omega)} \|v_h\|_{\Omega_h^n}.
$$

(5.29)

$\mathcal{T}_3$: For the volume integral part in the diffusion consistency error term $\mathcal{T}_3$ we also refer to [GOR15, Lemma 7.4]. We observe that it follows from the chain rule that

$$
\nabla v_h^{t_n}(\hat{x}) = \nabla (v_h \circ \Phi^{-1})(\hat{x}) = \nabla v_h(x) D\Phi(x)^{-1} \quad \text{for } x \in \Omega^n, \ n := \Phi^{-1}(\hat{x}).
$$

(5.30)

With this we can rewrite the volume contribution as

$$
\int_{\Omega_h^n} \nu \nabla u^n : \nabla v_h \, dx - \int_{\Omega_h^n} \nu \nabla u^n : \nabla v_h^{t_n} \, dx
$$

$$
= \int_{\Omega_h^n} \nu \nabla u^n : \nabla v_h \, dx - \int_{\Omega_h^n} \nu \nabla u^n : (\nabla v_h) \circ \Phi^{-1} D\Phi^{-1} 
$$

$$
= \int_{\Omega_h^n} \nu \nabla u^n : \nabla v_h \, dx - \int_{\Omega_h^n} \nu \nabla u^n : \Phi : J\nabla v_h D\Phi^{-1} \, dx := J_2.
$$
First we observe that due to (5.18) we have
\[ \|I - JD\Phi^{-1}\|_{\mathcal{C}^2(\Omega_h^\gamma)} \leq \|(I - JJ)D\Phi^{-1}\|_{\mathcal{C}^2(\Omega_h^\gamma)} + \|I - D\Phi^{-1}\|_{\mathcal{C}^2(\Omega_h^\gamma)} \lesssim h^q. \] (5.31)

With this and Lemma 5.12 we can estimate \([\mathcal{I}_3^3]\) as
\[ |\mathcal{I}_3^3| \leq \int_{\Omega_h^\gamma} \nu(\nabla u^n - \nabla u^\gamma \circ \Phi) : J\nabla v_h D\Phi^{-1} \, dx + \int_{\Omega_h^\gamma} \nu\nabla u^n : \nabla v_h \cdot (I - J D\Phi^{-1}) \, dx \]
\[ \lesssim h^{q+1} \|u^n\|_{\mathcal{H}^2(\Omega_h^\gamma)} \|\nabla v_h\|_{\Omega_h^\gamma} + h^q \|u^n\|_{\mathcal{H}^1(\Omega_h^\gamma)} \|\nabla v_h\|_{\Omega_h^\gamma} \]
\[ \lesssim h^q \|u^n\|_{\mathcal{W}^{2,q}(\Omega_h^\gamma)} \|\nabla v_h\|_{\Omega_h^\gamma}. \]

We split the boundary contributions from \(\mathcal{I}_3 = a_h^n(u^n, v_h) - a_h^n(u^n, v^n)\) into three parts
\[ -\int_{\Gamma_h^\gamma} \nu \partial_n u^n \cdot v_h \, ds + \int_{\Gamma_h^\gamma} \nu \partial_n u^n \cdot v^n \, ds \]
\[ + \int_{\Gamma_h^\gamma} \nu \partial_n u^n \cdot v^n \, ds. \]

For \(\mathcal{I}_3^3\) we use integration by substitution, the third bound in (5.18) and Lemma 5.12, which gives
\[ |\mathcal{I}_3^3| \lesssim \|\partial_t u^n\|_{\Gamma_h^\gamma} \lesssim \|u^n\|_{\mathcal{H}^2(\Omega_h^\gamma)} \lesssim \|u^n\|_{\mathcal{H}^2(\Omega_h^\gamma)}. \]

Using the standard trace inequality we have together with Assumption A1
\[ \|\partial_t u^n\|_{\Gamma_h^\gamma} \lesssim \|u^n\|_{\mathcal{H}^2(\Omega_h^\gamma)} \lesssim \|u^n\|_{\mathcal{H}^2(\Omega_h^\gamma)}. \]

Combining this result with (5.21a) now gives the estimate
\[ |\mathcal{I}_3^3| \lesssim h^q \|u^n\|_{\mathcal{H}^2(\Omega_h^\gamma)} \|h^{-1/2} v_h\|_{\Gamma_h^\gamma} \left[ h^{m+1/2} \|u^n\|_{\mathcal{H}^{m+1}(\Omega_h^\gamma)} + h^q \|u^n\|_{\mathcal{H}^1(\Omega_h^\gamma)} \right] \|h^{-1/2} v_h\|_{\Gamma_h^\gamma}. \]

For the term \(\mathcal{I}_3\) we use the fact that \(u^n\) vanishes on \(\Gamma^m\), so that with (5.30), (5.31) and (5.19c) it follows
\[ |\mathcal{I}_3| \lesssim \|h^{1/2} \partial_t v_h\|_{\Gamma_h^\gamma} \lesssim \sum_{T \in \mathcal{T}_h^m} \|\nabla v_h\|_{T \cap \Gamma_h^\gamma} \lesssim \sum_{T \in \mathcal{T}_h^m} \|\nabla v_h\|_{\nabla v_h^2(\Omega_h^\gamma)} \lesssim \sum_{T \in \mathcal{T}_h^m} \|\nabla v_h\|_{\nabla v_h^2(\Omega_h^\gamma)}. \]

Using the trace estimate (5.2c) we have for the first boundary norm
\[ \|h^{1/2} \partial_t v_h\|_{\Gamma_h^\gamma} \lesssim \sum_{T \in \mathcal{T}_h^m} \|\nabla v_h\|_{T \cap \Gamma_h^\gamma} \lesssim \sum_{T \in \mathcal{T}_h^m} \|\nabla v_h\|_{\nabla v_h^2(\Omega_h^\gamma)}. \]
Furthermore, we can estimate \( \|h^{-1/2}u^n\|_{\Gamma_h^n} \) using the fact that \( u^n \circ \Phi(x) = 0 \) for \( x \in \Gamma_h^n \) and (5.19c) to get

\[
\|h^{-1/2}u^n\|_{\Gamma_h^n} \leq h^{-1/2} \left[ \|u^n - u^n \circ \Phi\|_{\Gamma_h^n} + \|u^n \circ \Phi\|_{\Gamma_h^n} \right] \lesssim h^{q+1/2}\|u^n\|_{\mathcal{H}^q(\Omega^n)}.
\] (5.32)

These two estimates then give the bound

\[
|\mathcal{J}_3^3| \lesssim h^q\|u^n\|_{\mathcal{H}^2(\Omega^n)}\|\nabla v_h\|_{\mathcal{H}^1(\Omega^n)\cap_{v,\tau}}.
\]

For the penalty term \( \mathcal{J}_3^4 \) we proceed as for \( \mathcal{J}_3^3 \) using (5.32):

\[
\mathcal{J}_3^4 = \frac{h}{\sigma} \int_{\Gamma_h^n} \nu u^n \cdot v_h \, ds = \frac{h}{\sigma} \int_{\Gamma_h^n} \nu u^n \cdot v_h \, ds - \frac{h}{\sigma} \int_{\Gamma_h^n} \nu u^n \cdot \Phi \cdot J v_h \, ds
\]

\[
= \frac{h}{\sigma} \int_{\Gamma_h^n} (u^n - u^n \circ \Phi) \cdot J v_h \, ds + \frac{h}{\sigma} \int_{\Gamma_h^n} \nu u^n \cdot (1 - J) v_h \, ds
\]

\[
\lesssim h^q \left[ \|u^n\|_{\mathcal{H}^2(\Omega^n)} + \|h^{-1/2}u^n\|_{\Gamma_h^n} \right] \|h^{-1/2}v_h\|_{\Gamma_h^n}
\]

\[
\lesssim h^q\|u^n\|_{\mathcal{H}^2(\Omega^n)}\|h^{-1/2}v_h\|_{\Gamma_h^n}.
\]

Combining these estimates then gives the bound on \( \mathcal{T}_3 \) as

\[
|\mathcal{T}_3| \leq |\mathcal{J}_3^1| + |\mathcal{J}_3^2| + |\mathcal{J}_3^3| + |\mathcal{J}_3^4|
\]

\[
\lesssim h^q\|u\|_{\mathcal{W}^2,\infty(\Omega)}\|v_h\|_{\mathcal{H}^1,n} + h^{m+1/2}\|u^n\|_{\mathcal{H}^{m+1}(\Omega^n)}\|h^{-1/2}v_h\|_{\Gamma_h^n}. \tag{5.33}
\]

\( \mathcal{T}_4 \): As in (5.30), for the divergence we have

\[
\nabla \cdot v_h(\bar{x}) = \text{tr}(\nabla v_h \circ \Phi^{-1})(\bar{x}) = \text{tr}(\nabla v_h(x)D\Phi(x)^{-1}) \quad \text{for } x \in \Omega^n, \; x := \Phi^{-1}(\bar{x}).
\]

We split the pressure term \( \mathcal{T}_4 = \mathcal{J}_4^1 + \mathcal{J}_4^2 \) into volume and boundary integrals respectively. The volume term can then be rewritten as

\[
\mathcal{J}_4^1 := -\int_{\Omega_h^n} p^n \nabla \cdot v_h \, dx + \int_{\Omega_h^n} p^n \nabla \cdot v_h^\ell \, d\hat{x}
\]

\[
= -\int_{\Omega_h^n} p^n \nabla \cdot v_h \, dx + \int_{\Omega_h^n} p^n \circ \Phi \text{tr}(\nabla v_h D\Phi^{-1})J \, d\hat{x}
\]

\[
= \int_{\Omega_h^n} p^n (J - 1) \nabla \cdot v_h \, dx + \int_{\Omega_h^n} (p^n \circ \Phi - p^n) \text{tr}(\nabla v_h D\Phi^{-1})J \, dx
\]

\[
\quad + \int_{\Omega_h^n} p^n J \text{tr}(\nabla v_h D\Phi^{-1} - \nabla v_h) \, dx.
\]

This can then be bounded by

\[
|\mathcal{J}_4^1| \lesssim h^q\|p^n\|_{\Omega_h^n}\|\nabla \cdot v_h\|_{\Omega_h^n} + \|p^n \circ \Phi - p^n\|_{\Omega_h^n}\|\text{tr}(\nabla v_h D\Phi^{-1})\|_{\Omega_h^n}
\]

\[
\quad + \|p^n\|_{\Omega_h^n}\|\text{tr}(\nabla v_h D\Phi^{-1} - \nabla v_h)\|_{\Omega_h^n}
\]

\[
\lesssim h^q\|p^n\|_{\Omega_h^n}\|\nabla v_h\|_{\Omega_h^n} + h^{q+1}\|p^n\|_{\mathcal{H}^1(\Omega^n)}\|\nabla v_h\|_{\Omega_h^n} + h^q\|p^n\|_{\Omega_h^n}\|\nabla v_h\|_{\Omega_h^n}
\]

\[
\lesssim h^q\|p^n\|_{\mathcal{H}^1(\Omega^n)}\|\nabla v_h\|_{\Omega_h^n}.
\]
For the boundary terms we have

\[ \mathcal{J}_2^b := \int_{\Gamma_h^n} p^n \mathbf{v}_h \cdot n \, ds - \int_{\Gamma_h^n} p^n \mathbf{v}_h' \cdot n \, d\hat{s} \]

\[ = \int_{\Gamma_h^n} p^n \mathbf{v}_h \cdot n \, ds - \int_{\Gamma_h^n} p^n \circ \Phi \mathbf{v}_h \cdot nJ \, ds \]

\[ = \int_{\Gamma_h^n} p^n \mathbf{v}_h \cdot n (1 - J) \, ds + \int_{\Gamma_h^n} (p^n - p^n \circ \Phi) \mathbf{v}_h \cdot nJ \, ds \]

\[ \lesssim \int_{\Gamma_h^n} p^n \| \mathbf{v} \|_{\Gamma_h^n} + \| p^n - p^n \circ \Phi \|_{\Gamma_h^n} \| \mathbf{v} \|_{\Gamma_h^n} \]

(5.34)

For the pressure part in the first of these terms we observe that using (5.3) and Assumption A1 gives us

\[ \| p^n \|_{\Gamma_h^n}^2 = \sum_{K \in \mathcal{K}_S} \int_{\Gamma_h^n} p^n \| p^n \|_{H^1(\Omega_h^n)}^2 \lesssim \sum_{K \in \mathcal{K}_S} \left[ h^{-1} \| p^n \|_{K}^2 + h \| \nabla p^n \|_{K}^2 \right] \]

\[ \lesssim h^{-1} \| p^n \|_{H^1(\Omega_h^n)}^2 \]

\[ \lesssim h^{-1} \| p^n \|_{H^1(\Omega_h^n)} \]

For the pressure part in the second summand of (5.34) we apply (5.21c). Combining these results then gives

\[ |\mathcal{J}_2^b| \lesssim h^q \| p^n \|_{H^1(\Omega_h^n)} \| \nabla u^n \|_{\Gamma_h^n} + h^m \| p^n \|_{H^m(\Omega_h^n)} \| \nabla h^{-1/2} \mathbf{v}_h \|_{\Gamma_h^n} \]

Together, these estimates then give us the bound

\[ |\mathcal{J}_4| \lesssim h^q \| p^n \|_{H^1(\Omega_h^n)} \| \nabla u^n \|_{\Gamma_h^n} + h^m \| p^n \|_{H^m(\Omega_h^n)} \| \nabla h^{-1/2} \mathbf{v}_h \|_{\Gamma_h^n} \]  

(5.35)

\( \Xi_5 \): The volume terms of the divergence constraint can again be rewritten as

\[ \mathcal{J}_b^1 := - \int_{\Omega_h^n} q_h \nabla \cdot u^n \, dx + \int_{\Omega_h^n} q_h \nabla \cdot u^n \circ \Phi \, dx \]

\[ = - \int_{\Omega_h^n} q_h \nabla \cdot u^n \, dx + \int_{\Omega_h^n} q_h \nabla \cdot u^n \circ \Phi \, d\hat{x} \]

\[ = \int_{\Omega_h^n} q_h \nabla \cdot u^n (1 - J - 1) \, dx + \int_{\Omega_h^n} q_h (\nabla \cdot u^n \circ \Phi - \nabla \cdot u^n) J \, dx \]

\( \mathcal{J}_b^1 \) can then be bounded again using (5.18) and Lemma 5.12

\[ |\mathcal{J}_b^1| \lesssim h^q \| q_h \|_{\Omega_h^n} \| \nabla u^n \|_{\Omega_h^n} + \| q_h \|_{\Omega_h^n} \| \nabla u^n \circ \Phi - u^n \|_{\Omega_h^n} \]

\[ \lesssim h^q \| q_h \|_{\Omega_h^n} \| \nabla u^n \|_{\Omega_h^n} + h^{q+1} \| q_h \|_{\Omega_h^n} \| u^n \|_{H^2(\Omega_h^n)} \]

\[ \lesssim h^q \| q_h \|_{\Omega_h^n} \| u^n \|_{H^2(\Omega_h^n)} \]

The boundary contribution in \( \Xi_5 \) are

\[ \mathcal{J}_b^2 := \int_{\Gamma_h^n} q_h u^n \cdot n \, ds - \int_{\Gamma_h^n} q_h u^n' \cdot n \, d\hat{s} \]

\[ = \int_{\Gamma_h^n} q_h u^n \cdot n \, ds - \int_{\Gamma_h^n} q_h (u^n \cdot n) \circ \Phi \, ds \]

\[ = \int_{\Gamma_h^n} q_h (u^n - (u^n \cdot n) \circ \Phi) J \, ds + \int_{\Gamma_h^n} q_h u^n \cdot n (1 - J) \, ds \]

\[ \lesssim \| q_h \|_{\Gamma_h^n} \| u^n \cdot n - (u^n \cdot n) \circ \Phi \|_{\Gamma_h^n} + h^q \| q_h \|_{\Gamma_h^n} \| u^n \cdot n \|_{\Gamma_h^n} \]
Using the trace estimate (5.3) and the inverse estimate (5.2a) gives us
\[ \|q_h \|_{\Gamma_h^1} \lesssim h^{-1/2} \|q_h \|_{\sigma_{h,\nabla}}. \]

For the term \( \| u^n \cdot n - (u^n \cdot n) \circ \Phi \|_{\Gamma_h^2} \) we can apply Lemma 5.13. Furthermore, we have due to the homogeneous Dirichlet conditions that
\[ \| u^n \cdot n \|_{\Gamma_h^2} \leq \| u^n \cdot n - (u^n \cdot n) \circ \Phi \|_{\Gamma_h^2} + \| u^n \cdot n - (u^n \cdot n) \circ \Phi \|_{\Gamma_h^2} \]
\[ \leq \| u^n \cdot n \|_{\Gamma_h^2} + \| u^n \|_{\Gamma_h^2} \]
\[ \lesssim h^{q+1/2} \| u^n \|_{\mathcal{H}^1(\Omega^n)} + h^{m+1/2} \| u^n \|_{\mathcal{H}^{m+1}(\Omega^n)} \]

combining these results then gives us the bound
\[ | \mathcal{E}_8 | \leq (1 + h^q) \left[ h^n \| u^n \|_{\mathcal{H}^1(\Omega^n)} + h^m \| u^n \|_{\mathcal{H}^{m+1}(\Omega^n)} \right] \| q_h \|_{\sigma_{h,\nabla}}. \]

These bound on the volume and boundary contributions of \( \mathcal{E}_5 \) then give
\[ | \mathcal{E}_5 | \lesssim h^q \| u^n \|_{\mathcal{H}^1(\Omega^n)} \| q_h \|_{\sigma_{h,\nabla}} + h^m \| u^n \|_{\mathcal{H}^{m+1}(\Omega^n)} \| q_h \|_{\sigma_{h,\nabla}}. \] (5.36)

\( \mathcal{E}_6 \): To bound \( | \mathcal{E}_6 | \) we use the Cauchy-Schwartz inequality, Lemma 4.1 and Assumption A1 to get
\[ | \mathcal{E}_6 | = \gamma_h |L_\nu i_{h}^n(u^n, v_h) + L_{1/2} i_{h}^n(u^n, v_h) + 1/\nu j_{h}^n(p^n, q_h)| \]
\[ \lesssim (\nu + 1/\nu) h^{m+1/2} i_{h}^n(u^n, v_h) + 1/\nu j_{h}^n(p^n, q_h) \]
\[ \lesssim (\nu + 1/\nu) h^{m+1/2} \| u^n \|_{\mathcal{H}^{m+1}(\sigma_{h,\nabla})} \| v_h \|_{\mathcal{H}^1(\sigma_{h,\nabla})} + h^m \| p^n \|_{\mathcal{H}^1(\sigma_{h,\nabla})} \| q_h \|_{\sigma_{h,\nabla}} \]
\[ \lesssim (\nu + 1/\nu) h^{m+1/2} \| u^n \|_{\mathcal{H}^{m+1}(\Omega^n)} \| v_h \|_{\sigma_{h,\nabla}} + h^m \| p^n \|_{\mathcal{H}^1(\Omega^n)} \| q_h \|_{\sigma_{h,\nabla}}. \] (5.37)

Combining Estimates (5.27), (5.29), (5.33), (5.35), (5.36) and (5.37) then gives the desired result. \( \square \)

### 5.6 Error Estimates

We define \( \mathbf{u}_f^n := \mathcal{T}^* \mathbf{u}^n \) and \( p_f^n := \mathcal{T}^* p^n \). We then split the velocity error \( \mathbf{E}^n \) and the pressure \( \mathcal{D}^n \) each into an interpolation and discretisation error
\[ \mathbf{E}^n = (\mathbf{u}^n - \mathbf{u}_f^n) + (\mathbf{u}_f^n - \mathbf{u}_h^n) = \mathbf{e}^n + \mathbf{e}_h^n \]
\[ \mathcal{D}^n = (p^n - p_f^n) + (p_f^n - p_h^n) = \zeta^n + \mathcal{D}_h^n. \]

Inserting this into the equation relation (5.26) and rearranging terms then yields
\[ \int_{\Omega^n} \frac{\mathbf{e}^n - \mathbf{e}_h^n}{\Delta t} \cdot \mathbf{v}_h \, dx + a_h^n(\mathbf{e}^n, \mathbf{v}_h) + b_h^n(\mathbf{v}_h, \mathbf{e}^n) + s_h^n((\mathbf{e}^n, \mathcal{D}^n), (\mathbf{v}_h, q_h)) \]
\[ = \mathbf{e}_h^n(\mathbf{v}_h, q_h) + \mathbf{e}_f^n(\mathbf{v}_h, q_h) \] (5.38)

with
\[ \mathbf{e}_f^n(\mathbf{v}_h, q_h) = -\int_{\Omega^n} \frac{\eta^n - \eta_h^n}{\Delta t} \cdot \mathbf{v}_h \, dx - a_h^n(\eta^n, \mathbf{v}_h) - b_h^n(\zeta^n, \mathbf{v}_h) - b_h^n(q_h, \eta^n) - s_h^n((\eta^n, \zeta^n), (\mathbf{v}_h, q_h)). \]

For the interpolation error component, we can prove the following estimate.
Lemma 5.15. Assume for the velocity that $u \in L^{\infty}(0,T; H^{m+1}(\Omega(t)))$, $u_t \in L^{\infty}(0,T; H^{m}(\Omega(t)))$ and for the pressure that $p \in L^{\infty}(0,T; H^{m}(\Omega(t)))$. We can then bound the interpolation error term by

$$|\mathcal{E}_{T}^n(u_h, q_h)| \lesssim \frac{h^m L^{1/2}}{\nu} R_{1,1}(u, p)\|v_h\|_{s,n} + h^m R_{I,2}(u, p)\|q_h\|_{s,n}.$$ 

with

$$R_{1,1}(u, p) = \sup_{t \in [0,T]} \left( \|u\|_{H^{m+1}(\Omega(t))} + \|u_t\|_{H^m(\Omega(t))} + \|p\|_{H^m(\Omega(t))} \right)$$

and

$$R_{I,2}(u, p) = \sup_{t \in [0,T]} \left( \|u\|_{H^{m+1}(\Omega(t))} + \|p\|_{H^m(\Omega(t))} \right).$$

Proof. We split the interpolation error terms into five different parts $\mathcal{E}_{T}^n(u_h, q_h) = \mathcal{T}_7 + \mathcal{T}_8 + \mathcal{T}_9 + \mathcal{T}_{10} + \mathcal{T}_{11}$. These are the time-derivative term, the diffusion bilinear form, the pressure coupling term, the divergence constraint and ghost-penalty operator respectively. As in Lemma 5.14, we deal with each constituent term separately.

The proof for the time-derivative approximation $\mathcal{T}_7$ is part of [LO19, Lemma 5.12]. For completeness sake, we also give it here. We extend $u^n_h$ in time as the Lagrange interpolant of $u(t)$ in all nodes of $O_n^{h,n}$, so that $u^n_h(t_{n-1}) = u_{n-1}^h$ on $\Omega_h^n$. As $(u^n_T)$ is the Lagrange interpolant of $u_t$, we then have with (5.1) and (3.2a) that

$$\|\eta^n_t\|_{\Omega_h^n} \lesssim h^m \|u_t\|_{H^{m}(\Omega(t))} \lesssim h^m \|u_t\|_{C^1(\Omega(t))} \lesssim h^m \|u_t\|_{H^{m}(\Omega(t))}, \quad t \in I_n.$$

Using the Cauchy-Schwarz inequality twice as well as (5.39) and (3.3) we then get

$$|\mathcal{T}_7| = \left| \int_{\Omega_h^n} \frac{\eta^n - \eta^{n-1}}{\Delta t} \cdot v_h \, dx \right| \leq |\Delta t|^{-1} \|\eta^n - \eta^{n-1}\|_{\Omega_h^n} \|v_h\|_{\Omega_h^n}$$

$$= |\Delta t|^{-1/2} \left( \int_{t_{n-1}}^{t_n} \|\eta_t(s)\|_{\Omega_h^n} ds \right)^{1/2} \|v_h\|_{\Omega_h^n}$$

$$\leq \sup_{t \in [t_{n-1},t_n]} \|\eta_t(t)\|_{\Omega_h^n} \|v_h\|_{\Omega_h^n} \lesssim h^m \sup_{t \in [t_{n-1},t_n]} \|u_t(t)\|_{H^{m}(\Omega(t))} \|v_h\|_{\Omega_h^n}$$

$$\lesssim h^m \sup_{t \in [0,T]} \left( \|u\|_{H^{m+1}(\Omega(t))} + \|u_t\|_{H^{m}(\Omega(t))} \right) \|v_h\|_{\Omega_h^n}.$$

For the diffusion term $\mathcal{T}_8$ we use the continuity result (5.8) and Lemma 5.2 for the interpolation term and (5.4) for the test function, which gives

$$|\mathcal{T}_8| = | - a_h^n(\eta^n, v_h) | \lesssim \|\eta^n\|_n \|v_h\|_n \lesssim h^m \|u^n\|_{H^{m+1}(\Omega^n)} \|v_h\|_{s,n}.$$ 

using the same technique, we can estimate the pressure and divergence bilinear forms as

$$|\mathcal{T}_9| = | - b_h^n(\eta^n, v_h) | \lesssim h^m \|p^n\|_{H^{m}(\Omega^n)} \|v_h\|_{s,n}$$

$$|\mathcal{T}_{10}| = | - b_h^n(q_h, \eta^n) | \lesssim h^m \|u^n\|_{H^{m+1}(\Omega^n)} \|q_h\|_{s,n}.$$
For the ghost-penalty term $\Sigma_{11}$ (see also [LO19, Lemma 5.12]) we use the Cauchy-Schwarz inequality and Lemma 4.1 with (3.2a)
\[
|\Sigma_{11}| = |x^n_h((\eta^n, \zeta^n), (v_h, q_h))| \lesssim (\nu + 1/\nu) L^{1/2} i^h(\eta^n, \eta^n)^{1/2} L^{1/2} i^h(v_h, v_h)^{1/2} + i^h_n(\zeta^n, \zeta^n)^{1/2} j^h_n(q_h, q_h)^{1/2}
\lesssim (\nu + 1/\nu) h^m L^{1/2} \|u^n\|_{\mathcal{H}^{m+1}(\Omega)} \|v_h\|_{\mathcal{H}^m(\Omega)} + h^m \|p^n\|_{\mathcal{H}^m(\Omega)} \|q_h\|_{\mathcal{H}^m(\Omega)}
\lesssim (\nu + 1/\nu) h^m L^{1/2} \|u^n\|_{\mathcal{H}^{m+1}(\Omega)} \|v_h\|_{\mathcal{H}_s} + h^m \|p^n\|_{\mathcal{H}^m(\Omega)} \|q_h\|_{\mathcal{H}_s}.
\]

\[\square\]

**Theorem 5.16.** For sufficiently small $\Delta t$ and $h$, the velocity error can be bound by
\[
\|\hat{E}^n\|_{\mathcal{H}_h}^2 + \sum_{k=1}^n \left\{ \|\hat{E}^k - \hat{E}^{k-1}\|_{\mathcal{H}_h}^2 + \Delta t \left[ \nu c_{L5.5} \|\hat{E}^k\|^2_{\mathcal{H}_s,k} + \frac{L}{\nu} \hat{E}^k(\hat{E}^k, \hat{E}^k) \right] \right\}
\leq \exp((c_{T5.16A}/\nu) t_n) \left[ \Delta t^2 + \frac{h^{2q} + h^{2m} L}{\nu} + \frac{1}{\Delta t} (h^{2q} + h^{2m}) \right] R(u, p, f)
\]
with $R(u, p, f) = \sup_{t \in [0,T]} \left( \|u^2\|_{\mathcal{H}^{m+1}(\Omega)} + \|u_t^2\|_{\mathcal{H}^m(\Omega)} + \|p\|^2_{\mathcal{H}^m(\Omega)} + \|u^2\|_{L^2(\Omega)} \right)$ and constants $c_{T5.16}$ independent of $\Delta t$, $n$ and $h$.

**Proof.** We prove the result for the discretisation error, since the result then immediately follows by optimal interpolation properties. We start with the velocity estimate. Similar to the stability proof, for $n = k$ we test the error equation (5.38) with the test-function $(v_h, q_h) = 2\Delta t(e_h^k, -d_h^k)$ and use the identity (5.13) to get
\[
\|e_h^k\|^2_{\mathcal{H}_h} + \|e_h - e_h^{k-1}\|_{\mathcal{H}_h}^2 + 2\Delta t(a_h^k + \nu L d_h^k)(e_h^k, e_h^k) + 2L a_{L5.5}(e_h^k, e_h^k) + 2L a_{L5.5}(e_h^k, e_h^k)
= 2\Delta t(e_c^k + e_f^k)(e_h^k, -d_h^k) + \|e_h^{k-1}\|_{\mathcal{H}_h}^2.
\]

Using the coercivity result Lemma 5.5 and (5.16) we get (with the appropriate choice of $\varepsilon$) that
\[
\|e_h^k\|^2_{\mathcal{H}_h} + \|e_h - e_h^{k-1}\|_{\mathcal{H}_h}^2 + 2\Delta t a_{L5.5}(\nu e_h^k, e_h^k) + \Delta t \frac{2L}{\nu} c^k(\nu e_h^k, e_h^k) + \Delta t \frac{2}{\nu} j^h_n(d_h^k, d_h^k)
\leq (1 + \frac{\nu}{2\Delta t}) \|e_h^{k-1}\|_{\mathcal{H}_h}^2 + \Delta t \frac{2L}{\nu} \|e_h^{k-1}\|_{\mathcal{H}_s,k-1}^2 + \Delta t \frac{2}{\nu} \|L d_h^k(e_h^{k-1}, e_h^{k-1})
= 2\Delta t(e_c^k + e_f^k)(e_h^k, -d_h^k) + \|e_h^{k-1}\|_{\mathcal{H}_h}^2.
\]

Applying the weighted Young inequality to Lemma 5.14 and Lemma 5.15 then gives
\[
|e_c^k(e_h^k, d_h^k) + e_f^k(e_h^k, d_h^k)|
\leq \frac{1}{\varepsilon_1} c(\|e^{k-1}\|_{\mathcal{H}_h}^2 + h^{2m} L + h^{2m}) R(u, p, f) + \varepsilon_1 \|e_h^k\|_{\mathcal{H}_s,k}^2 + \frac{1}{\varepsilon_2} c(h^{2q} + h^{2m}) R'(u, p) + \varepsilon_2 \|d_h^k\|_{\mathcal{H}_s,k}^2.
\]
with
\[ R'(u, p) = \sup_{t \in [0, T]} \left( \|u(t)\|_{H^{m+1}(\Omega(t))}^2 + \|p(t)\|_{H^m(\Omega(t))}^2 \right). \]

Now we choose \( \varepsilon_1 = \nu c_{L5.5}/4 \) and \( \varepsilon_2 = \Delta t \beta \sqrt{4c_Y c_Z} \). With the constant \( c_Y > 0 \) to be specified later. Inserting these bound on the consistency and interpolation estimates into the above inequality and summing over \( k = 1, \ldots, n \) and using \( e^0_h = 0 \) gives
\[
\|e^k_h\|_{\Omega_h}^2 \leq \Delta t \sum_{k=1}^n \left[ \nu c_{L5.5} \|e^k_h\|_{\Omega_h}^2 + \frac{L}{\nu} i^k_h(e^k_h, e^k_h) \right] + \Delta t \sum_{k=1}^n \frac{2}{\nu} j^k_h(d^k_h, q^k_h)
\]
\[
+ \Delta t \sum_{k=1}^n \left[ \frac{\beta^2}{2c_Y c_Z} \|q^k_h\|_{\Omega_h}^2 + \Delta t^2 \sum_{k=1}^n \|a_{j, k}^h(q^k_h, q^k_h)\|_{s, k}^2 
\]
\[
+ \Delta t \sum_{k=1}^n \sum_{n=1}^{m} \left[ \Delta t^2 + h^{2q} + \frac{h^{2m} L}{\nu} + \frac{1}{\Delta t} \left( h^{2q} + \frac{h^{2m}}{\nu} \right) \right] R(u, p, f). \quad (5.40)
\]
under the assumption, that \( h \) is sufficiently small, such that \( c' h^2 \leq 1 \). To complete the velocity estimate, we therefore require the pressure estimate.

Rearranging the error equation (5.38) and using the test-function \( q_h = 0 \) gives
\[
(b^k_h, v_h) = -(1/\Delta t(e^k_h - e_h^{k-1}), v_h)_{\Omega_h} - (a^k_h + \nu L d^k_h)(e^k_h, v_h) - \frac{L}{\nu} i^k_h(e^k_h, v_h) \]
\[
+ \frac{\beta^2}{2c_Y c_Z} \|q^k_h\|_{\Omega_h}^2 + \Delta t \sum_{k=1}^n \|a_{j, k}^h(q^k_h, q^k_h)\|_{s, k}^2 
\]
\[
+ \Delta t \sum_{k=1}^n \left[ \Delta t^2 + h^{2q} + \frac{h^{2m} L}{\nu} + \frac{1}{\Delta t} \left( h^{2q} + \frac{h^{2m}}{\nu} \right) \right] R(u, p, f) \]
\[
\]
where \( \hat{c} = c_{L5.14} + c_{L5.15} \). Using the inf-sup result from Lemma 5.6 together with (5.7c), we then have
\[
\beta \|d^k_h\|_{s, k} \leq \frac{C_{P, h}}{\Delta t} \|e^k_h - e_h^{k-1}\|_{\Omega_h}^2 + \nu c_{L5.4} \|e^k_h\|_{s, k} + \frac{L}{\nu} i^k_h(e^k_h, e^k_h)^{1/2} \]
\[
+ \hat{c} \left( \Delta t + h^q + \frac{L^{1/2} h^m}{\nu} \right) (R_{c, 1} + R_{I, 1})(u, p, f) \]

Squaring this, using Young’s inequality to remove the product terms multiplying with \( \Delta t^2 \) and summing over \( k = 1, \ldots, n \), we get
\[
\Delta t^2 \sum_{k=1}^n \frac{\beta^2}{C_{P, h} c_Y} \|d^k_h\|_{s, k}^2 \leq \sum_{k=1}^n \|e^k_h - e_h^{k-1}\|_{\Omega_h}^2 + \Delta t \sum_{k=1}^n \left[ \Delta t (\nu^2 c_{L5.4}) \|e^k_h\|_{s, k}^2 + \frac{\Delta t L}{\nu^2} i^k_h(e^k_h, e^k_h) \right] 
\]
\[
+ \Delta t \sum_{k=1}^n \left[ \Delta t (1 + \beta)^2 r_h^k(d^k_h, d^k_h) + \Delta t \sum_{k=1}^n \Delta t e^2 (\Delta t^2 + h^{2q} + \frac{L h^{2m}}{\nu}) R(u, p, f) \right] \quad (5.41)
\]
where \( c_Y \) stems from the estimate \( \sum_{i=1}^n a_i^2 \leq n \sum_{i=1}^n a_i^2 \). We make the technical assumptions that \( \Delta t \nu^2 c_{L5.4}^2 \leq \nu c_{L5.4} \) and \( \Delta t (1 + \beta)^2 \leq 2/\nu \). Note, that since we are interested in the case of \( \nu \ll 1 \), these assumptions are not problematic and the inequalities are not sharp. For sufficiently small \( \Delta t \), that is \( \Delta t L/\nu^2 \leq 2L/\nu \), we can then bound the right-hand side of (5.41) with (5.40) which proves the error estimate.
The pressure discretisation error on the right hand side of (5.40) can therefore be bound by the other terms on the right-hand side of (5.40). This then gives us the estimate

\[ \|e_h^n\|_{\Omega_h}^2 + \sum_{k=1}^{n} \left\{ \|e_h^k - e_h^{k-1}\|_{\Omega_h}^2 + \Delta t \left[ \nu c_{L^2}(u_k)\|e_h^k\|_{L^2_h}^2 + \frac{L}{\nu} \|e_h^k\|_{H^1_h}^2 \right] \right\} \]

\[ \leq \Delta t \sum_{k=1}^{n-1} \frac{2\pi}{\nu} \|e_h^k\|_{\Omega_h}^2 + \Delta t \sum_{k=1}^{n} 2c \left[ \Delta t^2 + h^{2q} + \frac{h^{2m}L}{\nu} + \frac{1}{\Delta t} \left( h^{2q} + \frac{h^{2m}}{\nu} \right) \right] R(u, p, f). \]  

(5.42)

Applying Gronwall’s Lemma then proves the result. \hfill \square

**Remark 5.17.** To be able to get an optimal estimate for the velocity and pressure error from the above proof, which do not depend on negative powers of \( \Delta t \) we would need an estimate for \( \|e_h^k - e_h^{k-1}\|_{\Omega_h}^2 \) which is independent of negative powers of \( \Delta t \). Standard approaches to estimate this, such as in [BW11], do not work here, since the proof requires \( e_h^k - e_h^{k-1} \) to be weakly divergence free with respect to \( Q_h^k \).

An alternative would be to estimate \( \|\frac{e_h^k - e_h^{k-1}}{\Delta t}\|_{-1} \). However, standard approaches such as in [Fru+15] are again not viable due to lack of weak divergence conformity of the solution at two different time-steps.

If such an estimate was at hand, we would also be able to avoid the negative power of \( \Delta t \) on the right hand side of both the velocity and pressure estimates.

**Remark 5.18.** The choice of \( \Delta t \sim h \) would lead to an overall loss of half an order in the energy norm, provided that the spatial error is dominant. However, since this term is the result of lack of full consistency of the method (due to geometry approximation, ghost-penalties and divergence constraint with respect to different spaces for different time-steps), we do not expect that this is the major, dominating error part.

**Remark 5.19.** It cannot be expected, that the above result is sharp, since we have made some very crude estimates to bound (5.41) with (5.40).

**Remark 5.20 (Extension to higher-order BDF schemes).** As remarked upon in [LO19; BFM19], the extension to higher order BDF methods such as BDF2 is relatively unproblematic.

### 6 Numerical Examples

The method has been implemented using ngsxfem [ngs19], an add-on package to the high-order Finite-Element library NGSolve/Netgen [Sch97; Sch14]. The resulting sparse linear systems are solved using the direct solver Parfiso, as part of the Intel MKL library [Int19].

Full datasets of the computations shown here, as well as scripts to reproduce the results can be found online: DOI: 10.5281/zenodo.3647571.

#### 6.1 General Set-up

As a test case, we consider a basic test case with an analytical solution. On the domain \( \Omega(t) = \{ x \in \mathbb{R}^2 \mid (x_1 - t)^2 + x_2^2 = \frac{1}{2} \} \) we consider the analytical solution

\[ u_{ex}(t) = \begin{pmatrix} 2\pi x_2 \cos((x_1 - t)^2 + x_2^2) \\ -2\pi x_1 \cos((x_1 - t)^2 + x_2^2) \end{pmatrix} \quad \text{and} \quad p_{ex}(t) = \sin(\pi((x_1 - t)^2 + x_2^2)) - 2/\pi. \]

The velocity \( u_{ex}(t) \) then fulfills homogeneous Dirichlet boundary conditions and we have \( p_{ex}(t) \in L^2_{\Omega}(\Omega_n(t)) \).

An illustration of the initial solution on the initial background mesh can be seen in Figure 2. The forcing vector is then chosen accordingly as \( f(t) = \partial_t u_{ex}(t) - \nu \Delta u_{ex}(t) + \nabla p_{ex}(t) \).

We then consider the following set-up: The time interval is \( [0, 1] \) and the background domain is taken as \( \bar{\Omega} = (-1, -2) \times (1, 1) \). The maximum interface speed in our time interval is then \( w_{\text{int}}^\infty = 1 \) and for
the strip-width we choose $c_\delta = 1$. Unless otherwise stated, we take the Nitsche penalty parameter as $\sigma = 40k^2$. On the mesh, we consider the lowest order Taylor-Hood elements, i.e., $k = 2$. However, as the geometry is approximated by a piecewise linear level set function we have $q = 1$, so that we cannot expect to get the full spatial convergence order from the elements.

To quantify the computational results, we will consider the following discrete space-time errors:

$$
\|u_h - u\|_{\ell^2(L^2)}^2 := \Delta t \sum_{k=1}^n \|u_h - u\|_{L^2(\Omega_h^k)}^2 \\
\|\nabla u_h - \nabla u\|_{\ell^2(H^1)}^2 := \Delta t \sum_{k=1}^n \|\nabla u_h - \nabla u\|_{L^2(\Omega_h^k)}^2 \\
\|p_h - p\|_{\ell^2(L^2)}^2 := \Delta t \sum_{k=1}^n \|p_h - p\|_{L^2(\Omega_h^k)}^2.
$$

### 6.2 Parameter dependence

We investigate the robustness of our method with respect to the viscosity $\nu$, the ghost-penalty stabilisation parameter $\gamma_s$ and the extension strip-width, in terms of the number of elements $L$. For this we take the time-step $\Delta t = 0.05$. The viscosity is taken as $\nu \in \{10^0, 10^{-1}, \ldots, 10^{-4}\}$ and the stabilisation is taken as $\gamma_s \in \{0.1, 1, \ldots, 10^3\}$. To increase the strip-width $L = [\delta_h/h_{max}]$ we take $h_{max} = 0.1, 0.025, 0.0125$, resulting in $L = 1, 2, 4$ respectively.

The results for the $\ell^2(L^2)$-velocity error of these computations can be seen in Table 1. Here we observe that the method is robust with respect to over-stabilisation. We also note that within the considered range, the method is also robust with respect to number of elements in the extension strip. The method seems to be particularly robust over a wide range of viscosities. With a decrease of the viscosity by a factor $10^4$, the velocity-error only increased by a factor of 50 on the coarsest mesh, while on the finer meshes, this increase was even smaller. Furthermore, we see that we have a stable solution for very small $\nu$, where the assumption $\Delta t \sim \nu$ made in Theorem 5.16, does not hold anymore.

**Remark 6.1.** In computations, where we extended the pressure into the same exterior domain as the velocity, i.e. $\mathcal{F}_{h,S+}^n$, by using ghost penalties based on the same facets $\mathcal{F}_{h,\delta_s}^n$ and using the same ghost-penalty parameter (up to $h$ and $\nu$-scaling) $\gamma_{s,u} = \gamma_{s,u} = \gamma_{s,p} = L\gamma_s$ lead to qualitatively the same results as the results presented here and in the subsequent sections, only with slightly larger error constants.

### 6.3 Convergence study

To investigate the asymptotic convergence behaviour of the method in both time and space, we compute the experimental order of convergence in space (eoc$_x$) and time (eoc$_t$), based on the errors of two successive
6 Numerical Examples

Consider the following set-up. The viscosity is chosen as
rates are also higher than the expected rates if the geometry error was dominant. We attribute this to an
error while for the
interplay between the geometry error and the
eoc for one refinement in both space and time (eoc

to check whether the factor

\[ \nu \downarrow L_2 \] for which the theory predicts a loss half an order of convergence. However, this

\[ \ell_2(L^2) \] velocity error for the BDF1 method over a range of viscosities and ghost-penalty parameters.

With respect to space (eoc_x) we observe at least second order convergence in all norms. This is as good as

The results for all three discrete space-time norms can be seen in Table 2. With respect to time, we see

To check whether the factor \((h^{2q} + h^{2m})/\Delta t\) is observable, we consider joint refinement of both time

For the \(\ell_2(L^2)\)-pressure error we observe that the experimental order of convergence in space is higher

\[ \ell_2(L^2) \] velocity error for the BDF1 method over a range of viscosities and ghost-penalty parameters.

Table 1: \(\ell_2(0, T; L^2(\Omega(t)))\) velocity error for the BDF1 method over a range of viscosities and ghost-penalty parameters.

\[
\begin{array}{c|ccc|c|c|c}
\nu \downarrow L_2 & 0.1 & 1 & 10 & 100 & 1000 \\
\hline
1 & 4.0 \times 10^{-2} & 4.1 \times 10^{-2} & 4.5 \times 10^{-2} & 6.3 \times 10^{-2} & 1.6 \times 10^{-1} \\
0.1 & 1.8 \times 10^{-1} & 1.9 \times 10^{-1} & 2.3 \times 10^{-1} & 4.5 \times 10^{-1} & 1.2 \times 10^{0} \\
0.01 & 3.7 \times 10^{-1} & 4.5 \times 10^{-1} & 7.8 \times 10^{-1} & 1.4 \times 10^{0} & 1.6 \times 10^{0} \\
0.001 & 4.2 \times 10^{-1} & 6.7 \times 10^{-1} & 1.1 \times 10^{0} & 1.3 \times 10^{0} & 1.3 \times 10^{0} \\
0.0001 & 2.1 \times 10^{0} & 3.6 \times 10^{0} & 4.1 \times 10^{0} & 4.3 \times 10^{0} & 4.3 \times 10^{0} \\
\end{array}
\]

(a) \(h_{\text{max}} = 0.1\) and \(\Delta t = 0.05\) and a resulting strip-width of \(L = 1\)

\[
\begin{array}{c|ccc|c|c|c}
\nu \downarrow L_2 & 0.1 & 1 & 10 & 100 & 1000 \\
\hline
1 & 2.7 \times 10^{-2} & 2.7 \times 10^{-2} & 2.7 \times 10^{-2} & 2.7 \times 10^{-2} & 2.7 \times 10^{-2} \\
0.1 & 1.7 \times 10^{-1} & 1.7 \times 10^{-1} & 1.7 \times 10^{-1} & 1.7 \times 10^{-1} & 1.9 \times 10^{-1} \\
0.01 & 3.3 \times 10^{-1} & 3.3 \times 10^{-1} & 3.4 \times 10^{-1} & 3.8 \times 10^{-1} & 6.5 \times 10^{-1} \\
0.001 & 3.7 \times 10^{-1} & 3.7 \times 10^{-1} & 4.2 \times 10^{-1} & 6.7 \times 10^{-1} & 8.9 \times 10^{-1} \\
0.0001 & 3.5 \times 10^{-1} & 3.5 \times 10^{-1} & 4.4 \times 10^{-1} & 5.6 \times 10^{-1} & 5.7 \times 10^{-1} \\
\end{array}
\]

(b) \(h_{\text{max}} = 0.025\) and \(\Delta t = 0.05\) and a resulting strip-width of \(L = 2\)

\[
\begin{array}{c|ccc|c|c|c}
\nu \downarrow L_2 & 0.1 & 1 & 10 & 100 & 1000 \\
\hline
1 & 2.6 \times 10^{-2} & 2.6 \times 10^{-2} & 2.6 \times 10^{-2} & 2.6 \times 10^{-2} & 2.6 \times 10^{-2} \\
0.1 & 1.7 \times 10^{-1} & 1.7 \times 10^{-1} & 1.7 \times 10^{-1} & 1.7 \times 10^{-1} & 1.7 \times 10^{-1} \\
0.01 & 3.3 \times 10^{-1} & 3.3 \times 10^{-1} & 3.3 \times 10^{-1} & 3.4 \times 10^{-1} & 4.0 \times 10^{-1} \\
0.001 & 3.7 \times 10^{-1} & 3.7 \times 10^{-1} & 3.7 \times 10^{-1} & 4.5 \times 10^{-1} & 7.0 \times 10^{-1} \\
0.0001 & 3.6 \times 10^{-1} & 3.7 \times 10^{-1} & 3.9 \times 10^{-1} & 4.8 \times 10^{-1} & 5.4 \times 10^{-1} \\
\end{array}
\]

(c) \(h_{\text{max}} = 0.0125\) and \(\Delta t = 0.05\) and a resulting strip-width of \(L = 4\)
dominating term here.

6.4 Extension to higher order

6.4.1 BDF2 time discretisation

As an extension to the presented method, we consider the BDF2 formula to discretise the time-derivative. To ensure that the appropriate solution history is available on the necessary elements, we increase the extension strip with the choice \( \delta_h = 2c_\delta w^n_\infty \Delta t \).

We investigate the convergence properties in both time and space. To this end we take the same basic set up as in Section 6.3. However we take \( L_x = 0, \ldots, 5 \) and \( L_t = 0, \ldots, 7 \). The results from these computations can be seen in Table 3.

We observe that with \( L_t = 7 \), the spatial error is dominant on all meshes in all three norms and that \( eoc_x \) similar to the BDF1 case.

With respect to time, we see second order of convergence (\( eoc_t \approx 2 \)), while the temporal error is dominant. There are also some stability issues for large time steps and large \( L = 16, 32 \). However these results are outside the time-step/viscosity range covered by our theory.

6.4.2 Higher-order spatial convergence

As we have seen in the theory and some of the numerical results of the previous section, the piecewise linear level set approach leads to a loss in the maximal spatial convergence rate of the velocity. A simple and effective – though not very efficient – way to hide the geometrical error and reveal the underlying discretisation error, is the approximation of the the geometry based on a piecewise linear level set after \( s \) subdivisions of cut elements. As the resulting geometry approximation order is then of order \( O((\frac{h}{s})^2) \).

The drawback of this approach is a non optimal scaling for \( h \to 0 \), since \( s \) must increase to balance the discretisation error on fine meshes.

To balance the geometry error with the discretisation error, we take \( s = \mathcal{O}(\log_2(1/h)) \) yielding an effective geometry approximation of \( O(h^{q+1}) \) with \( q = 3 \). Let us stress that the purpose of this “trick” is to hide the geometry error and reveal the underlying remaining discretisation errors and is not meant as a solution to the problem of approximating unfitted geometries in general.

Using the same set-up as in Section 6.4.1 and the BDF2 formula to discretise the time-derivative, such that the spatial error is dominant for larger time-steps, we compute our test-problem over a series of uniformly refined meshes. The time-step is chosen as \( \Delta t = 0.1 \cdot 2^{-8} \) and we consider Taylor-Hood elements with \( k = 2 \) and \( k = 3 \).

The results of these computations can be seen in Figure 3. Here we can see, that we have recovered optimal order of convergence in both velocity norms. We also note, that the pressure error converges as before, at an order higher than expected.

7 Conclusions and open problems

We have presented, analysed and implemented a fully Eulerian, inf-sup stable, unfitted finite element method for the time dependent Stokes problem on evolving domains. This followed the previous work in [LO19] for convection-diffusion problems and [BFM19] for the time-dependent Stokes problem using equal order, pressure stabilised elements. The method is simple to implement in existing unfitted finite element libraries, since all operators are standard in unfitted finite elements.

In the analysis, we have seen that the geometrical consistency error, introduced by integrating over discrete, approximated domains \( \Omega^n_h \) plays a major role and causes additional coupling between the velocity and pressure errors which is non-standard. Furthermore, since the time derivative approximation term \( \frac{1}{\Delta t}(u^n_h - u^{n-1}_h) \) is not weakly divergence free with respect to the pressure space \( Q^n_h \), we obtained error
7 Conclusions and open problems

| $L_t \downarrow L_x$ | 0     | 1     | 2     | 3     | 4     | eoc$_t$  |
|---------------------|-------|-------|-------|-------|-------|----------|
| 0                   | $1.3 \times 10^0$ | $7.4 \times 10^{-1}$ | $6.6 \times 10^{-1}$ | $6.5 \times 10^{-1}$ | $6.5 \times 10^{-1}$ | –        |
| 1                   | $1.0 \times 10^0$ | $4.4 \times 10^{-1}$ | $3.4 \times 10^{-1}$ | $3.3 \times 10^{-1}$ | $3.3 \times 10^{-1}$ | 0.99     |
| 2                   | $8.2 \times 10^{-1}$ | $2.8 \times 10^{-1}$ | $1.8 \times 10^{-1}$ | $1.7 \times 10^{-1}$ | $1.7 \times 10^{-1}$ | 0.99     |
| 3                   | $7.3 \times 10^{-1}$ | $1.9 \times 10^{-1}$ | $9.9 \times 10^{-2}$ | $8.6 \times 10^{-2}$ | $8.4 \times 10^{-2}$ | 0.99     |
| 4                   | $6.9 \times 10^{-1}$ | $1.6 \times 10^{-1}$ | $5.7 \times 10^{-2}$ | $4.4 \times 10^{-2}$ | $4.2 \times 10^{-2}$ | 0.99     |
| 5                   | $6.7 \times 10^{-1}$ | $1.4 \times 10^{-1}$ | $3.7 \times 10^{-2}$ | $2.3 \times 10^{-2}$ | $2.1 \times 10^{-2}$ | 0.99     |
| 6                   | $6.6 \times 10^{-1}$ | $1.3 \times 10^{-1}$ | $2.8 \times 10^{-2}$ | $1.2 \times 10^{-2}$ | $1.1 \times 10^{-2}$ | 0.98     |
| 7                   | $6.5 \times 10^{-1}$ | $1.3 \times 10^{-1}$ | $2.3 \times 10^{-2}$ | $7.2 \times 10^{-3}$ | $5.5 \times 10^{-3}$ | 0.97     |
| 8                   | $6.5 \times 10^{-1}$ | $1.2 \times 10^{-1}$ | $2.1 \times 10^{-2}$ | $4.7 \times 10^{-3}$ | $2.9 \times 10^{-3}$ | 0.94     |

---

(a) $\ell^2(0, T; L^2(\Omega(t)))$ velocity error.

| $L_t \downarrow L_x$ | 0     | 1     | 2     | 3     | 4     | eoc$_t$  |
|---------------------|-------|-------|-------|-------|-------|----------|
| 0                   | $8.9 \times 10^0$ | $5.8 \times 10^0$ | $5.4 \times 10^0$ | $5.6 \times 10^0$ | $6.3 \times 10^0$ | –        |
| 1                   | $7.9 \times 10^0$ | $4.0 \times 10^0$ | $3.2 \times 10^0$ | $3.2 \times 10^0$ | $3.4 \times 10^0$ | 0.90     |
| 2                   | $6.9 \times 10^0$ | $2.8 \times 10^0$ | $1.9 \times 10^0$ | $1.8 \times 10^0$ | $1.9 \times 10^0$ | 0.83     |
| 3                   | $6.5 \times 10^0$ | $2.3 \times 10^0$ | $1.1 \times 10^0$ | $1.0 \times 10^0$ | $1.0 \times 10^0$ | 0.87     |
| 4                   | $6.3 \times 10^0$ | $2.1 \times 10^0$ | $7.6 \times 10^{-1}$ | $5.5 \times 10^{-1}$ | $5.5 \times 10^{-1}$ | 0.91     |
| 5                   | $6.2 \times 10^0$ | $2.0 \times 10^0$ | $5.9 \times 10^{-1}$ | $3.0 \times 10^{-1}$ | $2.9 \times 10^{-1}$ | 0.94     |
| 6                   | $6.1 \times 10^0$ | $2.0 \times 10^0$ | $5.3 \times 10^{-1}$ | $1.8 \times 10^{-1}$ | $1.5 \times 10^{-1}$ | 0.96     |
| 7                   | $6.1 \times 10^0$ | $1.9 \times 10^0$ | $5.0 \times 10^{-1}$ | $1.2 \times 10^{-1}$ | $7.6 \times 10^{-2}$ | 0.96     |
| 8                   | $6.1 \times 10^0$ | $1.9 \times 10^0$ | $4.9 \times 10^{-1}$ | $1.0 \times 10^{-1}$ | $4.0 \times 10^{-2}$ | 0.93     |

(b) $\ell^2(0, T; H^1(\Omega(t)))$ velocity error.

| $L_t \downarrow L_x$ | 0     | 1     | 2     | 3     | 4     | eoc$_t$  |
|---------------------|-------|-------|-------|-------|-------|----------|
| 0                   | $7.3 \times 10^{-1}$ | $4.4 \times 10^{-1}$ | $4.1 \times 10^{-1}$ | $4.3 \times 10^{-1}$ | $4.2 \times 10^{-1}$ | –        |
| 1                   | $5.9 \times 10^{-1}$ | $2.6 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | $2.0 \times 10^{-1}$ | 1.08     |
| 2                   | $4.9 \times 10^{-1}$ | $1.7 \times 10^{-1}$ | $1.1 \times 10^{-1}$ | $9.8 \times 10^{-2}$ | $9.7 \times 10^{-2}$ | 1.06     |
| 3                   | $4.4 \times 10^{-1}$ | $1.2 \times 10^{-1}$ | $5.9 \times 10^{-2}$ | $5.0 \times 10^{-2}$ | $4.8 \times 10^{-2}$ | 1.01     |
| 4                   | $4.3 \times 10^{-1}$ | $9.7 \times 10^{-2}$ | $3.5 \times 10^{-2}$ | $2.5 \times 10^{-2}$ | $2.4 \times 10^{-2}$ | 1.00     |
| 5                   | $4.3 \times 10^{-1}$ | $8.7 \times 10^{-2}$ | $2.3 \times 10^{-2}$ | $1.3 \times 10^{-2}$ | $1.2 \times 10^{-2}$ | 0.99     |
| 6                   | $4.5 \times 10^{-1}$ | $8.3 \times 10^{-2}$ | $1.8 \times 10^{-2}$ | $7.4 \times 10^{-3}$ | $6.1 \times 10^{-3}$ | 0.98     |
| 7                   | $5.0 \times 10^{-1}$ | $8.3 \times 10^{-2}$ | $1.5 \times 10^{-2}$ | $4.5 \times 10^{-3}$ | $3.2 \times 10^{-3}$ | 0.96     |
| 8                   | $5.8 \times 10^{-1}$ | $8.7 \times 10^{-2}$ | $1.5 \times 10^{-2}$ | $3.1 \times 10^{-3}$ | $1.7 \times 10^{-3}$ | 0.92     |

(c) $\ell^2(0, T; L^2(\Omega(t)))$ pressure error.

Table 2: Mesh-size and time-step convergence for the BDF1 method with $\nu = 10^{-2}$. 

\[\nu \in \{0.9, 0.93, 0.96, 0.97, 0.99\}\]
Conclusions and open problems

Table 3: Mesh-size and time-step convergence for the BDF2 method with $\nu = 10^{-2}$.

| $L_t \downarrow L_x \rightarrow$ | 0 | 1  | 2       | 3  | 4       | 5     | $\text{eoc}_t$ |
|-------------------------------|---|-----|---------|----|---------|-------|----------------|
| 0                             | $1.4 \times 10^0$ | $5.7 \times 10^{-1}$ | $2.4 \times 10^{-1}$ | $10.0 \times 10^{-2}$ | $1.2 \times 10^{-1}$ | $1.7 \times 10^{-1}$ | – |
| 1                             | $1.1 \times 10^0$ | $3.9 \times 10^{-1}$ | $1.1 \times 10^{-1}$ | $3.5 \times 10^{-2}$ | $1.9 \times 10^{-2}$ | $2.4 \times 10^{-2}$ | $2.80$ |
| 2                             | $9.1 \times 10^{-1}$ | $2.1 \times 10^{-1}$ | $4.9 \times 10^{-2}$ | $1.3 \times 10^{-2}$ | $4.6 \times 10^{-3}$ | $4.0 \times 10^{-3}$ | $5.90$ |
| 3                             | $7.6 \times 10^{-1}$ | $1.6 \times 10^{-1}$ | $3.3 \times 10^{-2}$ | $5.8 \times 10^{-3}$ | $1.4 \times 10^{-3}$ | $9.4 \times 10^{-4}$ | $2.10$ |
| 4                             | $7.0 \times 10^{-1}$ | $1.4 \times 10^{-1}$ | $2.6 \times 10^{-2}$ | $4.2 \times 10^{-3}$ | $5.9 \times 10^{-4}$ | $2.4 \times 10^{-4}$ | $1.99$ |
| 5                             | $6.7 \times 10^{-1}$ | $1.3 \times 10^{-1}$ | $2.2 \times 10^{-2}$ | $3.2 \times 10^{-3}$ | $4.5 \times 10^{-4}$ | $8.3 \times 10^{-5}$ | $1.50$ |
| 6                             | $6.6 \times 10^{-1}$ | $1.3 \times 10^{-1}$ | $2.1 \times 10^{-2}$ | $2.8 \times 10^{-3}$ | $3.7 \times 10^{-4}$ | $6.4 \times 10^{-5}$ | $0.37$ |
| 7                             | $6.5 \times 10^{-1}$ | $1.2 \times 10^{-1}$ | $2.0 \times 10^{-2}$ | $2.7 \times 10^{-3}$ | $3.4 \times 10^{-4}$ | $6.0 \times 10^{-5}$ | $0.10$ |

$\text{eoc}_x$ – 2.39 2.61 2.94 2.96 2.51

(a) $\ell^2(0, T; L^2(\Omega(t)))$ velocity error.

| $L_t \downarrow L_x \rightarrow$ | 0 | 1  | 2       | 3  | 4       | 5     | $\text{eoc}_t$ |
|-------------------------------|---|-----|---------|----|---------|-------|----------------|
| 0                             | $9.2 \times 10^0$ | $5.0 \times 10^0$ | $2.7 \times 10^0$ | $1.3 \times 10^0$ | $2.6 \times 10^0$ | $4.3 \times 10^1$ | – |
| 1                             | $8.2 \times 10^0$ | $3.3 \times 10^0$ | $1.6 \times 10^0$ | $6.0 \times 10^{-1}$ | $2.6 \times 10^{-1}$ | $1.0 \times 10^1$ | $2.03$ |
| 2                             | $7.5 \times 10^0$ | $2.7 \times 10^0$ | $9.0 \times 10^{-1}$ | $3.1 \times 10^{-1}$ | $9.5 \times 10^{-2}$ | $5.2 \times 10^{-2}$ | $7.66$ |
| 3                             | $6.7 \times 10^0$ | $2.3 \times 10^0$ | $7.0 \times 10^{-1}$ | $1.6 \times 10^{-1}$ | $4.5 \times 10^{-2}$ | $1.5 \times 10^{-2}$ | $1.79$ |
| 4                             | $6.4 \times 10^0$ | $2.1 \times 10^0$ | $5.9 \times 10^{-1}$ | $1.3 \times 10^{-1}$ | $2.2 \times 10^{-2}$ | $5.9 \times 10^{-3}$ | $1.36$ |
| 5                             | $6.2 \times 10^0$ | $2.0 \times 10^0$ | $5.3 \times 10^{-1}$ | $1.1 \times 10^{-1}$ | $1.9 \times 10^{-2}$ | $2.7 \times 10^{-3}$ | $1.10$ |
| 6                             | $6.1 \times 10^0$ | $2.0 \times 10^0$ | $5.1 \times 10^{-1}$ | $9.7 \times 10^{-2}$ | $1.5 \times 10^{-2}$ | $2.3 \times 10^{-3}$ | $0.24$ |
| 7                             | $6.1 \times 10^0$ | $1.9 \times 10^0$ | $5.0 \times 10^{-1}$ | $9.4 \times 10^{-2}$ | $1.4 \times 10^{-2}$ | $1.9 \times 10^{-3}$ | $0.29$ |

$\text{eoc}_x$ 1.65 1.97 2.41 2.79 2.83

(b) $\ell^2(0, T; H^1(\Omega(t)))$ velocity error.

| $L_t \downarrow L_x \rightarrow$ | 0 | 1  | 2       | 3  | 4       | 5     | $\text{eoc}_t$ |
|-------------------------------|---|-----|---------|----|---------|-------|----------------|
| 0                             | $8.2 \times 10^{-1}$ | $3.8 \times 10^{-1}$ | $2.2 \times 10^{-1}$ | $1.5 \times 10^{-1}$ | $3.5 \times 10^{-1}$ | $4.0 \times 10^{0}$ | – |
| 1                             | $6.6 \times 10^{-1}$ | $1.6 \times 10^{-1}$ | $6.8 \times 10^{-2}$ | $4.5 \times 10^{-2}$ | $4.1 \times 10^{-2}$ | $9.6 \times 10^{-1}$ | $2.06$ |
| 2                             | $5.6 \times 10^{-1}$ | $1.2 \times 10^{-1}$ | $2.3 \times 10^{-2}$ | $1.1 \times 10^{-2}$ | $1.1 \times 10^{-2}$ | $1.1 \times 10^{-2}$ | $1.81$ |
| 3                             | $4.8 \times 10^{-1}$ | $9.7 \times 10^{-2}$ | $1.9 \times 10^{-2}$ | $2.5 \times 10^{-3}$ | $2.7 \times 10^{-3}$ | $2.8 \times 10^{-3}$ | $1.89$ |
| 4                             | $4.6 \times 10^{-1}$ | $8.7 \times 10^{-2}$ | $1.6 \times 10^{-2}$ | $2.3 \times 10^{-3}$ | $5.1 \times 10^{-4}$ | $6.8 \times 10^{-4}$ | $2.05$ |
| 5                             | $4.8 \times 10^{-1}$ | $8.4 \times 10^{-2}$ | $1.4 \times 10^{-2}$ | $2.0 \times 10^{-3}$ | $2.6 \times 10^{-4}$ | $1.5 \times 10^{-4}$ | $2.22$ |
| 6                             | $5.3 \times 10^{-1}$ | $8.5 \times 10^{-2}$ | $1.4 \times 10^{-2}$ | $1.9 \times 10^{-3}$ | $2.5 \times 10^{-4}$ | $4.3 \times 10^{-4}$ | $1.76$ |
| 7                             | $6.3 \times 10^{-1}$ | $9.0 \times 10^{-2}$ | $1.4 \times 10^{-2}$ | $1.8 \times 10^{-3}$ | $2.5 \times 10^{-4}$ | $4.1 \times 10^{-4}$ | $0.07$ |

$\text{eoc}_x$ – 2.80 2.67 2.95 2.89 2.58

(c) $\ell^2(0, T; L^2(\Omega(t)))$ pressure error.

Figure 3: Results with $k = 2, 3$ using subdivisions for better approximation of the domain boundary.
estimates which are dependent on inverse powers of $\Delta t$. Fortunately, this dependence on $\Delta t^{-1}$ was not observable in our numerical results, suggesting that an estimate of $\| \frac{1}{\Delta t} (u_h^n - u_h^{n-1}) \|_{-1}$ independent (of negative powers) of $\Delta t$ should hold.

In our numerical experiments, we have also seen, that the geometrical error—if low order approximations are used—can indeed be a dominating factor in the final error, corrupting the optimal convergence rate of the $P^2/P^1$ finite element pair. We used a simple, but not efficient, approach for avoiding this in the considered test cases for both $P^2/P^1$ and $P^3/P^2$ elements.

Let us now discuss some issues where further refinement of the method and analysis seem to be of some benefit.

As mentioned above, it seems that it should be possible to prove an estimate of $\| \frac{1}{\Delta t} (u_h^n - u_h^{n-1}) \|_{-1}$ independent of $\Delta t^{-1}$ for $u_h^{n-1}$ which we have $b_h^n(q_h, u_h^{n-1}) \neq 0$ for some $q_h \in Q_h^n$.

In Section 6.4.2 we have shown that it is possible to recover the optimal order of convergence for the Taylor-Hood finite element pair if sufficient computational effort is put on the geometry approximation. However, the subdivision strategy used here is limited in its application, due to the large number of subdivisions needed after several global mesh refinements and for higher order elements, which in turn results in a large amount of memory needed for computations.

One approach to efficiently obtain higher order geometry approximations for level set domains has been introduced in [Leh16] for stationary domains. That approach relies on a slight local deformations of the mesh, depending on the level set function. For unsteady problems the corresponding mesh becomes time-dependent for which efficient and accurate transfer operations are needed from one mesh to the other. This will be discussed in a forthcoming paper.

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