Massive particle creation in a static 1+1 dimensional spacetime

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Abstract

We show explicitly that there is particle creation in a static spacetime. This is done by studying the field in a coordinate system based on a physical principle which has recently been proposed. There the field is quantized by decomposing it into positive and negative frequency modes on a particular spacelike surface. This decomposition depends explicitly on the surface where the decomposition is performed, so that an observer who travels from one surface to another will observe particle production due to the different vacuum state.

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I. INTRODUCTION

Before one can quantize a free field propagating on a curved background one must have a unique means of splitting the field into positive and negative frequency modes. This procedure requires a unique definition of time with which to perform the decomposition.

There have been many attempts to define the vacuum for a free field propagating in a nontrivial spacetime. It is a common feature of all these attempts that the choice of vacuum is determined by a particular choice of time coordinate. This is true even for such general quantization procedures as Deutsch and Najmi [1] although there the dependence is not explicit. Instead they require a foliation of spacetime by a family of spacelike hypersurfaces which, in essence, defines “instants of time” and the normals to these surfaces define the “direction of time”. The choice of time coordinate, in most computations, has usually been based on calculational convenience and not on a local physical principle. The physical principle which we use here is that suggested by Capri and Roy [2]. The principle is that on the surface of instantaneousity a 1+1 dimensional Poincaré algebra including the Killing equations for the generators should be valid.

In a globally hyperbolic spacetime with one timelike and one spacelike dimension the surface of instantaneousity, in this case a line, for a given observer is given by the particular spacelike geodesic which passes through the point at which the observer is located and is normal to the observer’s timelike worldline. The direction of time on this surface is then defined to be everywhere normal to this spacelike geodesic. It has been shown that this definition of time is the unique one to obey the physical principle given above [2].

The coordinates on the spacelike surface are chosen, for convenience, to be Riemann coordinates based at the observer’s position, although any other coordinatization of the spacelike surface will do. To define the direction of time Gaussian Geodesic Normal coordinates are then constructed on this spacelike surface so that the time coordinate of some point off the surface is just given by the distance from the point to the spacelike surface.

When one expresses the metric in terms of these new coordinates one finds that the
metric has the form,

\[ ds^2 = dt^2 + g_{11} dx^2, \]  

(1.1)

with \( g_{11} < 0 \) and \( g_{11} = -1 + O(t^2) \) near the origin of coordinates.

To now decompose the field into positive and negative frequency modes we impose initial conditions that force the field to have the correct time dependence \( \exp(-i\omega t) \) in the neighbourhood of this spacelike surface. To ensure this correct time dependence we impose the initial conditions,

\[ \phi_n^+(t, x) \big|_{t=0} = A_n(0, x) \quad \text{and} \quad (\partial_t \phi_n^+(t, x)) \big|_{t=0} = -i\omega_n(0)A_n(0, x). \]  

(1.2)

where the \( A_n(0, x) \) are the spatial eigenmodes of the Laplace-Beltrami operator on the surface \( t = 0 \) where the decomposition is to be performed.

In section II we construct the metric in terms of these physically preferred coordinates and write out explicitly the boundary conditions which determine the positive frequency modes at the surface \( t = 0 \). In section III a complete set of modes for the Laplace-Beltrami operator for a massive scalar field is obtained and in section IV the orthogonality relation satisfied by these modes is calculated. In section V we then use these orthogonality relations to impose the physically relevant boundary conditions which were calculated in section II. In section VI the actual particle creation due to the presence of the gravitational field is calculated for a stationary observer with respect to this spacetime, in terms of the original static coordinates.

**II. THE PREFERRED COORDINATES**

The spacetime we are interested in is described by the metric \[ 3 \]

\[ ds^2 = \alpha(X) dT^2 - \frac{dX^2}{\alpha(X)}, \]  

(2.1)

where
\[ \alpha(X) = 1 - \exp(-q(|X| - r)). \]  

(2.2)

To simplify the technical discussion later on we choose \( r \) such that \( \exp(qr) < 2 \), then \(-1 < \alpha(X) < 1\). This spacetime was first investigated by Witten [3] as a 1 + 1 dimensional eternal blackhole spacetime.

To construct the preferred coordinates we must solve the geodesic equations for this spacetime. The first integrals of the geodesic equations yield

\[
\frac{dT}{ds} = \frac{C_0}{\alpha(X)} ; \quad \frac{dX}{ds} = \epsilon_1(C_0^2 - \epsilon \alpha(X))^{\frac{1}{2}}
\]

(2.3)

where \( \epsilon_1 = \pm 1 \) and \( \epsilon = -1 \) for spacelike geodesics and \( \epsilon = 1 \) for timelike geodesics. The spacelike geodesic which is perpendicular to the timelike vector \( \frac{1}{\sqrt{\alpha}(1, 0)}, (\alpha_0 \equiv \alpha(X_0)) \) and which can be treated as the tangent vector to the worldline of the observer at \( P_0(T_0, X_0) \), is given by setting \( C_0 = 0 \) and \( \epsilon = -1 \). We can therefore see on this surface \( S_0 \) that \( T_0 = T_1 \).

\( T_1 \) is the \( T \) coordinate of the point \( P_1(T_1, X_1) \) which is the point at which the geodesic from the general point \( P(T, X) \) intersects this spacelike surface orthogonally. The preferred time coordinate \( t \) is given by the distance along the timelike geodesic connecting \( P_1 \) to the general point \( P(T, X) \) which is normal to the surface \( S_0 \) at \( P_1 \). This timelike geodesic is given by (2.3) with \( C_0^2 = \alpha_1 \equiv \alpha(X_1) \) and \( \epsilon = 1 \) so that

\[
t = \int_{X_1}^{X} dX' \frac{ds}{dX'} = \int_{X_1}^{X} dX' \frac{\epsilon_1}{\sqrt{\alpha(X_1) - \alpha(X')}}.
\]

(2.4)

One can also calculate the change in the coordinate \( T \) along this geodesic,

\[
T - T_1 = T - T_0 = \int_{P_0}^{P} dT = \int_{X_1}^{X} dX' \frac{\epsilon_1 \epsilon_2 \sqrt{\alpha(X_1)}}{\alpha(X') \sqrt{\alpha(X_1) - \alpha(X')}}.
\]

(2.5)

These two equations allow us to express the metric in terms of the coordinates \( t \) and \( X_1 \). The preferred coordinate \( x \) on \( S_0 \) is now constructed using a 2-bein of orthogonal basis vectors at \( P_0, e_0(P_0) \) and \( e_1(P_0) \). With \( e_0(P_0) \) given by \( \frac{1}{\sqrt{\alpha_0}}(1, 0) \) the tangent to the observer’s worldline and \( p^\mu \) given by the tangent vector at \( P_0 \) to the geodesic connecting \( P_0 \) to \( P_1 \), the Riemann normal coordinates \( \eta^\alpha \) of \( P_1 \) are given by
\[ sp^\mu = \eta^\alpha e^\mu_\alpha(P_0) \] (2.6)

where \( s \) is the distance along the geodesic \( P_0 - P_1 \). Using \( e^\mu_\alpha e_{\beta\mu} = \eta_{\alpha\beta} \) (Minkowski metric), and the orthogonality of \( p^\mu \) to \( e_\mu(P_0) \) we have

\[ \eta^0 = sp^\mu e^0_\mu(P_0) \quad \eta^i = -sp^\mu e^i_\mu(P_0). \] (2.7)

The surface \( S_0 \) is just the surface \( \eta^0 = 0 \) and the coordinate \( x \) is

\[ x = \eta^1 = -sp^\mu e^1_\mu(P_0) = \int_{X_0}^{X_1} dX' \frac{1}{\sqrt{\alpha(X')}} \] (2.8)

The preferred coordinates \( (t, x) \) are then given by solving the above integrals for \( X > 0 \) or \( X < 0 \). After choosing \( \epsilon_1 = -1 \) one obtains for \( X > 0 \),

\[ T = -2 \sqrt{1 + e^{q(-r+X_1)}} \tan^{-1}\left(\sqrt{1 + e^{q(-X+X_1)}}\right) \frac{\epsilon_2}{q} \]
\[ + \frac{2 \tanh^{-1}\left(\sqrt{1 + e^{q(-X+X_1)}}\right)}{\sqrt{1 + e^{q(-r+X_1)}}} \frac{\epsilon_2}{q} + T_0 \] (2.9)

\[ x = \frac{2}{q} \left\{ \tanh^{-1}(\sqrt{\alpha_1}) - \tanh^{-1}(\sqrt{\alpha_0}) \right\} \] (2.10)

\[ t = \frac{2 e^{\frac{q(-r+X_1)}{2}}}{q} \tan^{-1}\left(\sqrt{1 + e^{q(X-X_1)}}\right) \] (2.11)

and for \( X < 0 \),

\[ T = \frac{2 \sqrt{1 + e^{-q(r+X_1)}}}{q} \tan^{-1}\left(\sqrt{1 + e^{q(X-X_1)}}\right) \frac{\epsilon_2}{q} \]
\[ - \frac{2 \tanh^{-1}\left(\sqrt{1 + e^{q(X-X_1)}}\right)}{\sqrt{1 + e^{-q(r+X_1)}}} \frac{\epsilon_2}{q} + T_0 \] (2.12)

\[ x = \frac{2}{q} \left\{ \tanh^{-1}(\sqrt{\alpha_1}) + \tanh^{-1}(\sqrt{\alpha_0}) \right\} \] (2.13)

\[ t = \frac{-2 \tan^{-1}\left(\sqrt{1 + e^{q(X-X_1)}}\right)}{e^{\frac{q(r+X_1)}{2}} \frac{q}{q}} \] (2.14)

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where
\[ \epsilon_2 = \pm 1. \]  

(2.15)

From these coordinate transformations we can see that \( x \) and \( t \) both run from \(-\infty\) to \(+\infty\) and cover the region of the original space corresponding to \(|X| > r\), the region outside the horizon. The region inside the horizon is shrunk to a point. Furthermore, the region between the observer and the horizon (\( r < |X| < |X_0| \)) is covered twice.

In terms of the coordinates \((t, x)\) the metric is now
\[ ds^2 = dt^2 - (1 + tp(x) \tan [tp(x)])^2 dx^2 \]  

(2.16)

where
\[ p(x) = \frac{q}{2} \text{sech}[B(x)] \]  

(2.17)

and
\[ B(x) = \tanh^{-1} \left[ \sqrt{\alpha_0} + \frac{xq}{2} \right] \]  

(2.18)

We can see at this point that in this coordinate system, which does have a physical basis, the metric no longer appears static.

In these new coordinates the Klein-Gordon equation for a massive scalar field is
\[ \partial_t^2 \phi + \frac{1}{2} (\partial_t \ln(|g|)) \partial_t \phi + \frac{1}{\sqrt{|g|}} \partial_x \left( \sqrt{|g|} g^{11} \partial_x \right) \phi + m^2 \phi = 0. \]  

(2.19)

We now define instantaneous eigenfunctions \( A_n(t, x) \) of the spatial part of the Laplace-Beltrami operator, such that
\[ \left[ \frac{1}{\sqrt{|g|}} \partial_x \left( \sqrt{|g|} g^{11} \partial_x \right) + m^2 \right] A_k(t, x) = \omega_k^2(t) A_k(t, x) \]  

(2.20)

The positive frequency solutions of (2.19) are then defined as those which satisfy the initial conditions
\[ \phi_k^+(t, x)|_{t=0} = A_k(0, x) \quad \text{and} \quad (\partial_t \phi_k^+(t, x))|_{t=0} = -i \omega_k(0) A_k(0, x). \]  

(2.21)
These initial conditions ensure that the positive frequency part of the field has the desired
time dependence at $t = 0$. These positive frequency solutions form a vector space which is
made into a Hilbert space using the standard Klein-Gordon inner product.

From the simple form of the metric at $t = 0$ we see that

$$A_k(0, x) = \sin\left(2k \frac{x}{q} B(x)\right) \quad \text{or} \quad \cos\left(2k \frac{x}{q} B(x)\right) \quad \text{and} \quad \omega_k^2(0) = (k^2 + m^2)$$  \hspace{1cm} (2.22)

With the positive frequency solutions defined in this way we can then write out the
quantized field as

$$\Psi_1 = \int_0^\infty dk \frac{1}{\sqrt{2\omega_k}} \left\{ \phi_k^+(t, x)a_k + \phi_k^{(+)*}(t, x)a_k^\dagger \right\},$$ \hspace{1cm} (2.23)

where the subscript 1 of the field simply denotes the surface on which the positive frequency
modes have been defined. In this expression we have written $\omega_k(0)$ as $\omega_k$ and we will
continue this practice. Unfortunately (2.16) is too complicated to obtain the general form
of the modes in terms of the coordinates $(t, x)$. This is, however, not really a problem as
the point of this approach is to find out what boundary conditions should be imposed. It is
therefore sufficient to solve the field equations in whatever coordinate system is convenient
and then express these solutions in the preferred coordinate system to impose the boundary
conditions.

III. MODES OF FIELD EQUATION

From the form of (2.1) we can see that in terms of the original coordinates $(T, X)$ the
field equations are separable. For this reason we solve for the modes in these coordinates
and then express the solutions in terms of the preferred coordinates using the coordinate
transformations given above (2.9-2.14). In terms of the coordinates $(T, X)$ the Klein-Gordon
operator has the form

$$\frac{1}{\alpha(X)} \partial_T^2 \phi - (\partial_X \alpha(X)) \partial_X \phi - \alpha(X) \partial_X^2 \phi + m^2 \phi = 0$$ \hspace{1cm} (3.1)
By assuming a $T$ dependence for the field of the form $\exp(-i\omega_p T)$ we obtain the following differential equation,

$$\partial_X(\alpha(X)\partial_X\phi) + \left(\frac{\omega_p^2}{\alpha(X)} - m^2\right)\phi = 0 \quad (3.2)$$

To construct a self-adjoint extension for this operator we are required to construct solutions which vanish at the horizon where $\alpha(X) = 0$.

By making a change of variable to $z = 1 - \exp(-q(|X| - r)) = \alpha(X)$ we obtain the following differential equation in terms of $z$,

$$z(1 - z)^2\Psi''(z) + (1 - z)(1 - 2z)\Psi'(z) + \left(\frac{p^2}{z} - \mu^2\right)\Psi(z) = 0. \quad (3.3)$$

where

$$p^2 = \frac{\omega_p^2}{q^2} \quad \text{and} \quad \mu^2 = \frac{m^2}{q^2} \quad (3.4)$$

We are interested in constructing solutions outside the horizon so that $|X| > r$ and $z > 0$. As mentioned above we also require that the solutions vanish at $z = 0$.

The two independent solutions to (3.2) are

$$\Psi_{1p}(z) = z^n(1 - z)^l F(a, b, c, z) \quad (3.5)$$

where $F(a, b, c, z)$ is an hypergeometric function and

$$n = ip$$

$$l = i\sqrt{p^2 - \mu^2}$$

$$a = n + l$$

$$b = n + l + 1$$

$$c = 1 + 2n \quad (3.6)$$

and

$$\Psi_{2p}(z) = z^n(1 - z)^l F(a, b, c, z) \quad (3.7)$$
where

\[ n = -ip \]
\[ l = -i\sqrt{p^2 - \mu^2} \]
\[ a = n + l \]
\[ b = n + l + 1 \]
\[ c = 1 + 2n. \] (3.8)

We can now finally write out the desired solution to (3.3)

\[ \Psi(p, X) = \left( \Psi_{2p}(0)\Psi_{1p}(z) - \Psi_{1p}(0)\Psi_{2p}(z) \right) \epsilon(X). \] (3.9)

The general solution to (3.1) can then be written,

\[ \Psi(T, X) = \int_{\mu}^{\infty} dp \left\{ (A(p)\Psi(p, X) \exp(-i\omega_p T) + B(p)\Psi(p, X) \exp(i\omega_p T)) \right\} \] (3.10)

We now impose the initial conditions (2.22) which then give some physical meaning to the decomposition of this field into positive and negative frequency parts. To explicitly impose these initial conditions it is first useful to find the orthogonality relation satisfied by the modes \( \Psi(p, X) \).

### IV. ORTHOGONALITY OF MODES

To find the orthogonality relation satisfied by the mode \( \Psi(p, X) \) we follow standard Sturm-Liouville approach and recall that the mode satisfies,

\[ z(1-z)^2F''(p, z) + (1-z)(1-2z)F'(p, z) + \left( \frac{p^2}{z} - \mu^2 \right) F(p, z) = 0. \] (4.1)

We can also write out a similar equation which is satisfied by the modes \( F^*(k, z) \). If one now multiplies the equation for \( F(p, z) \) by \( F^*(k, z) \) and the equation for \( F^*(k, z) \) by \( F(p, z) \) and looks at the difference of the two equations one can see that after integrating over the range \( z = 0 \) to \( z = 1 \) and integrating the two terms by parts once we are left with the relation,
\begin{equation}
\int_0^1 dz \frac{F(p, z) F^*(k, z)}{z(1-z)} = \lim_{z \to 1} \frac{z(1-z)}{(p^2 - k^2)} \left( F'(p, z) F^*(k, z) - F(p, z) F'^*(k, z) \right) - \frac{z(1-z)}{(p^2 - k^2)} \left( F'(p, z) F^*(k, z) - F(p, z) F'^*(k, z) \right) \bigg|_{z=0} \tag{4.2}
\end{equation}

Because of the boundary conditions satisfied by \( F(p, z) \) and \( F^*(k, z) \) the second term in this relation is identically zero. The first term, as we show, is proportional to a delta function. This shows that these modes are orthogonal. To see that this expression is indeed proportional to a delta function we first smear it with a smooth function of \( p \) and show that the result is proportional to that function evaluated at \( p = k \). When one attempts to evaluate the limit in the first term one finds that all the various terms are proportional to a common factor which produces the delta function, this factor is

\begin{equation}
\lim_{z \to 1} \frac{(1-z)^{-i(\sqrt{k^2-\mu^2}-\sqrt{p^2-\mu^2})} - (1-z)^{i(\sqrt{k^2-\mu^2}-\sqrt{p^2-\mu^2})}}{(p-k)} \tag{4.4}
\end{equation}

To proceed we introduce a regularization \((1-z)^\epsilon\) and write,

\begin{equation}
F(p, z) = \lim_{\epsilon \to 0} F(p, z) (1-z)^\epsilon. \tag{4.5}
\end{equation}

The integral we must evaluate is,

\begin{equation}
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dp \frac{f(p)}{p-k} \lim_{z \to 1} (1-z)^{2\epsilon} \left\{ (1-z)^{-i(\sqrt{k^2-\mu^2}-\sqrt{p^2-\mu^2})} - (1-z)^{i(\sqrt{k^2-\mu^2}-\sqrt{p^2-\mu^2})} \right\} \tag{4.6}
\end{equation}

This shows that there is no contribution to the integral from the regions where \(|p-k| > R\). In these regions the pole at \( p = k \) is not realized so one may interchange the order in which the limits are performed. These contributions then go to zero as the limit \( z \to 1 \) is performed. We are then left with the integral

\begin{equation}
\lim_{\epsilon \to 0} \lim_{R \to 0} \int_{-R}^{k} dp \frac{f(p)}{p-k} \lim_{z \to 1} (1-z)^{2\epsilon} \left\{ (1-z)^{-i(\sqrt{k^2-\mu^2}-\sqrt{p^2-\mu^2})} - (1-z)^{i(\sqrt{k^2-\mu^2}-\sqrt{p^2-\mu^2})} \right\} \tag{4.7}
\end{equation}

It is now convenient to make a change of variable to the variable \( x \) where

\begin{equation}
x = \frac{k \ln(1-z)(p-k)}{\sqrt{k^2-\mu^2}} \tag{4.8}
\end{equation}
We next expand the integrand in powers of \((p - k)\) and find that as \(R \to 0\) we are left with the smooth function \(f(p)\) evaluated at the pole multiplied by a function of \(k\).

Using the above analysis for the orthogonality relations in \(z\) one can then write the orthogonality relations in \(X\),

\[
\int_{|X|>r} dX \frac{\Psi(p, X) \Psi^*(k, X)}{\alpha(X)} = \delta(p - k) |(A\Psi_{1k}(0) + B\Psi_{2k}(0))|^2 \\
\equiv \delta(p - k) |F(k)|^2
\]  

(4.9)

where

\[
A = \frac{\sqrt{2}(k - \sqrt{k^2 - \mu^2}) \Gamma(1 - 2ik)}{\pi \sqrt{kq \sinh(2\pi \sqrt{k^2 - \mu^2})}} \Gamma^2(-i(k - \sqrt{k^2 - \mu^2})) \\
\times \sinh(\pi(k - \sqrt{k^2 - \mu^2})) \sinh(\pi(k + \sqrt{k^2 - \mu^2}))
\]  

(4.10)

\[
B = \frac{\sqrt{2}(k + \sqrt{k^2 - \mu^2}) \Gamma(1 + 2ik)}{\pi \sqrt{kq \sinh(2\pi \sqrt{k^2 - \mu^2})}} \Gamma^2(-i(k + \sqrt{k^2 - \mu^2})) \\
\times \sinh(\pi(k - \sqrt{k^2 - \mu^2})) \sinh(\pi(k + \sqrt{k^2 - \mu^2})).
\]  

(4.11)

**V. FREQUENCY DECOMPOSITION AND THE VACUUM**

We now decompose the field into positive and negative frequency parts by picking out the positive frequency part of the field as that which satisfies the initial conditions in the preferred coordinates. This allows us to extract the annihilation operator for this field and thus define the vacuum state for this field on this particular spacelike hypersurface (i.e. the appropriate hypersurface which passes through the point \((T_0, X_0)\)). The physically relevant question is, of course, how this decomposition depends on the point \((T_0, X_0)\) which could represent the position of an observer. If this decomposition depends on the position of the observer then at some different position presumably the observer would observe some sort of particle density due to the change in composition of the vacuum state. To see this we
must impose the initial conditions relevant to the quantization on this surface. Recall the initial conditions

$$\phi_k^\pm(t, x) \big|_{t=0} = A_k(0, x) \quad \text{and} \quad (\partial_t \phi_k^\mp(t, x)) \big|_{t=0} = -i\omega_k(0)A_k(0, x) \quad (5.1)$$

where

$$A_k(0, x) = \sin\left(\frac{k}{q}B(x)\right) \quad \text{and} \quad \omega_k(0) = \left(k^2 + m^2\right)^{\frac{1}{2}} \quad (5.2)$$

We can thus write the general form of the solution which satisfies these initial conditions for this particular mode $k$

$$\Psi_k(T, X) = \int_\mu^\infty dp \left\{ (A_k(p)\Psi(p, X) \exp(-i\omega_p T) + B_k(p)\Psi(p, X) \exp(i\omega_p T)) \right\} \quad (5.3)$$

where we now regard $T, X$ and $z$ as functions of $(t, x)$. This can be easily done given the coordinate transformations of section II. Because the initial conditions are imposed at $t = 0$ we need only be concerned with the form of this field and its derivative normal to $t = 0$ for $z(t = 0, x)$ in order to evaluate the expansion coefficients $A(p, n), A^*(p, n), B(p, n)$ and $B^*(p, n)$.

By using the orthogonality relations calculated in the last section we can write out the initial condition equations,

$$|F(k)|^2 (A_k(k) \exp(-i\omega_k T_0) + B_k(k) \exp(i\omega_k T_0))$$

$$= \int_{|X|>r} dX \frac{\Psi^*(k, X)\sin\left(\frac{2k}{q}B(x)\right)}{\alpha(X)} \quad (5.4)$$

$$|F(k)|^2 (B_k(k) \exp(-i\omega_k T_0) - B_k(k) \exp(i\omega_k T_0))$$

$$= \int_{|X|>r} dX \frac{\Psi^*(k, X)\sin\left(\frac{2k}{q}B(x)\right)}{\alpha(X)(\frac{\partial X}{\partial t})} \big|_{t=0} \quad (5.5)$$

In taking the time derivative of (5.3) one does not pick up a $\frac{dX}{dt}$ because at $t = 0$ this is zero. Again in these expressions it can be seen that we are still regarding $z$ as $z(0, x)$ and $x$ is the inverse of this function in the integral. We have now determined $A(p, n), A^*(p, n), B(p, n)$ and $B^*(p, n)$ and can therefore decompose the field explicitly in terms of positive and negative frequency modes on this surface,
\[ \Psi_1 = \int_0^\infty dk \frac{1}{\sqrt{2\omega_k}} \left\{ \phi_{k1}^+(t, x) a_1(k) + \phi_{k1}^{(+)*}(t, x) a_1(k)^\dagger \right\} \] 

(5.6)

where the extra subscripts denote the surface on which the frequency decomposition has been performed and the modes \( \phi_{k1}^{(+)*}(t, x) \) and \( \phi_{k1}^+(t, x) \) are the ones constructed with the expansion coefficients which satisfy (5.4) and (5.5). One may now define the vacuum relevant to this field on the surface \( (t = 0) \) in the usual way,

\[ a_1(k) |0_1\rangle = 0 \quad \forall \ k \] 

(5.7)

where again the subscript denotes “when” this is the vacuum state for the field. To see whether particles are created by the gravitational field in this spacetime one must look closely at how this state depends on the surface chosen. So at this stage all one needs is the normal derivatives, with respect to the spacelike surface, of the field on the surface. In Section VI the full transformation equations will be required.

**VI. PARTICLE CREATION**

To see whether particles are created by the gravitational field in the spacetime one must look closely at how the field decomposition depends on the surface chosen (i.e. the position of the observer). To obtain the spectrum of particles created one must calculate the Bogolubov transformation between the different annihilation and creation operators and look at the mixing of positive and negative frequency parts. To calculate the Bogolubov transformation we can just match the field from two different quantizations on a common surface. The easiest way to do this is to propagate one field to the surface on which the second is quantized. We can therefore write

\[ \Psi_1(t, x) = \Psi_2(0, x') \quad \text{and} \quad \partial_t \Psi_1(t, x) = \partial_{t'} \Psi_2(t', x') \big|_{t'=0} \] 

(6.1)

where \( t \) is the proper distance between the two quantization surfaces. This distance will, in general, depend on where on the surface one calculates the distance. To make the calculation
simpler we take \( X'_0 = X_0 \) so that the observer is stationary with respect to the original coordinates where the metric is static.

Because of the simple form of the modes at \( t = 0 \) one can calculate the Bogolubov coefficients and write an expression of the form

\[
a_2(k) = \int dp \left( \alpha(p, k)a_1(p) + \beta(p, k)a_1(p)\dagger \right)
\]

(6.2)

The particle density experienced by an observer travelling from surface 1 to surface 2 is then given by

\[|\beta(p, k)|^2\]

(6.3)

In 1 + 1 dimensions \( \beta(p, k) \) in general has the form,

\[
\beta(p, k) = \int_{-\infty}^{\infty} dx' \frac{q}{2} \frac{A_p(0, x')}{q\pi\sqrt{\omega_p\omega_k}} \left\{ \dot{\phi}_{1k}^+(t, x) \frac{\partial}{\partial t'} + \left( \partial_x \phi_{1k}^+(t, x) \right) \frac{\partial}{\partial x'} - i\omega_p \phi_{1k}^+(t, x) \right\} \Big|_{t'=0}.
\]

(6.4)

In this equation the factors \( \frac{\partial}{\partial t'} \) and \( \frac{\partial}{\partial x'} \) are required because we are matching the field’s normal derivative is done with respect to the second surface.

To calculate an approximate form of \( \beta \), valid for short time intervals, we expand the integrand about \( t = 0 \). To \( O(t^2) \) we obtain,

\[
\beta(p, k) = -\int_{-\infty}^{\infty} dy \frac{\sin^2\left(\frac{2p}{q}y\right)}{q\pi\sqrt{\omega_p\omega_k}} \tan^2(y) \sin\left(\frac{2k}{q}y\right)p^2(x')\omega_k(T'_0 - T_0)^2
\]

(6.5)

where \( p(x') \) is given by (2.17) and we have changed variables from \( x' \) to \( y = B(x') \). In getting from (6.4) to (6.5) the second term of (6.4) doesn’t contribute to the integral because it is odd in \( y \). It should be restated that this is particle creation observed by an observer stationary with respect to the original static coordinates. This can now be rewritten as,

\[
\beta(p, k) = -\frac{(T'_0 - T_0)^2\omega_k q}{4\pi} \frac{1}{\sqrt{\omega_k\omega_p}} \int_{-\infty}^{\infty} dy \frac{\sin^2\left(\frac{2p}{q}y\right)}{q} \tan^2(y) \sin\left(\frac{2k}{q}y\right).
\]

(6.6)

Several comments are in order here. Firstly, \( \beta(p, k) \) is clearly non-zero so that particles are produced in this short time interval \( \delta t = T'_0 - T_0 \).
Secondly, our approximation clearly only holds for $\omega_k < q$ since the expansion breaks down for $\omega_k \delta t > 1$. This means that we can only crudely estimate the number of particles produced in the time $\delta t$ since an ultraviolet cutoff of $k = \sqrt{q^2 - m^2}$ is required.

Putting all this together we see that the momentum density of particles labelled by $k$ produced in the time interval $t$ is:

$$n_t(k) = \int_0^\infty dp |\beta(p, k)|^2$$

$$\simeq \frac{q^2 \delta t^2 \sqrt{\omega_k}}{4\pi} \int_0^\infty \frac{dp}{\sqrt{p^2 + m^2 q^2}} \left| \frac{(p + k)\left(\frac{(p+k)^2}{q^2} + 3\right) - (p - k)\left(\frac{(p-k)^2}{q^2} + 3\right)}{\sinh \frac{\pi(p+k)}{q} - \sinh \frac{\pi(p-k)}{q}} \right|^2 \ (6.7)$$

Within the spirit of the approximation, the total number of particles created in the time $\delta t$ with $\omega_k < q$ is:

$$N_t = \int_0^{\sqrt{q^2 - m^2}} n_t(k) dk \ (6.8)$$

This integral is finite, of course. If the upper limit is allowed to go to $\infty$ then the integral diverges linearly. This does not mean that the Bogolubov transformation is not unitarily implementable. Our approximations simply break down and our results are inconclusive. The difficulty arises from the fact that there are two time scales namely $T_1 = \frac{1}{q}$ and $T_2 = \frac{1}{\omega_k}$.

For a fixed $\omega_k$ it is possible to expand in $\frac{\delta t}{T}$ where $T$ is the smaller of $T_1, T_2$. However, if $\omega_k$ is unbounded no such expansion is possible.

VII. CONCLUSIONS

We have shown that although a spacetime may be static this may not preclude particle creation which is a time dependent phenomenon [4] as the gaussian coordinatization may not be static. The only metrics which always lead to static Gaussian coordinates are those which have been dubbed “ultrastatic” by Fulling [5]. We have shown explicitly in this simple 1+1 dimensional case how the choice of which coordinates should be used leads to some interesting results. In particular, the coordinates which are chosen via a physical principle seem to suggest that although the spacetime may be manifestly static in one coordinate
system these may not be the coordinates that one should use to quantize a field propagating in the spacetime.

Unfortunately the analysis to find out whether the Bogolubov transformation is unitarily implementable was inconclusive. This is due to the fact that the approximate form of $\beta$ which was analysed was not valid for large $k$.

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