Quantum evolution speed in the finite-temperature bosonic environment

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Abstract
We investigate the quantum evolution speed of a qubit in two kinds of finite-temperature environments. The first environment is a bosonic bath with Ohmic-like spectrum. It is found that the high temperature not only leads to the speed-up but also speed-down processes in the weak-coupling regime, which is different from the strong-coupling case where only exhibits speed-up process, and the effects of Ohmicity parameter of the bath on the quantum evolution speed are also different in the strong-coupling and weak-coupling regimes. Furthermore, we realize the controllable and stationary quantum evolution speed by applying the bang-bang pulse. For the second nonlinear bath, we study the quantum evolution speed of a qubit by resorting to the hierarchical equations of motion method beyond the Born-Markov approximation. It is shown that the performances of quantum evolution speed in weak-coupling and strong-coupling regimes are also different. In particular, the quantum evolution speed can be decelerated by the rise of temperature in the strong-coupling regime which is an anomalous phenomenon and contrary to the common recognition that quantum evolution speed always increases with the temperature.

Keywords: quantum speed limit, quantum coherence, finite-temperature environment, bang-bang pulse

1. Introduction
Quantum speed limit describes the maximal evolution speed of a quantum system from an initial state to a target state, which has been found applications in the fields of quantum computation [1], quantum thermodynamics [2], quantum metrology [3] and quantum control [4]. The minimum evolution time between two distinguishable states of a quantum system is defined as the quantum speed limit time (QSLT) [5–9], and has been widely used to characterize the maximal evolution speed. The QSLT first proposed for a closed quantum system to naturally evolve to an orthogonal state is characterized by unifying the Mandelstam-Tamm (MT) bound [10] and Margolus-Levitin (ML) bound [7]. Since the inevitable interaction of quantum systems with their surrounding environment, the generalizations of QSLT for open quantum systems have attracted much attention, and some valuable works have been done [11–14] in recent years. In Ref. [13], the authors have proposed a unified bound of QSLT including both MT and ML types for non-Markovian dynamics with pure initial states. For a wider range of applications, another unified bound of QSLT applied to both mixed and pure initial states has been derived by introducing relative purity [15] as the distance measure [14]. These results have stimulated the interest of some further research about quantum speed limit.

Recently, some remarkable progresses about the analysis of environmental effects on the quantum speed limit for open quantum systems have been made. For example, in Refs. [13, 14, 16], authors have investigated the quantum speed limit for cavity QED systems and found that the non-Markovianity can speed up the quantum evolution. Some works have provided the quantum speed limit of a central spin trapped in a
2. Quantum evolution speed in the bosonic environment

In this section, we study the quantum evolution speed of a qubit coupled to its own finite-temperature environment. First, we briefly outline the definition of quantum speed limit for an open quantum system. The maximal rate of evolution can be characterized by the QSLT which is defined as the minimal time needed for the quantum system to evolve from an initial state to a final state. In open quantum systems, the dynamical evolutions are governed by the time-dependent master equation (1) which reads

$$
\tau_{QSL} = \max \left\{ \frac{1}{\sum_{i=1}^{n} \sigma_i \rho_i}, \frac{1}{\sum_{i=1}^{n} \sigma_i} \right\} \ast |f_{\tau+\tau_D} - 1| \text{Tr} \left( \rho_f^2 \right)
$$

where $X = \tau_D^{-1} \int^{\tau+\tau_D} X(t')dt'$. $\sigma_i$ and $\rho_i$ are the singular values of $\rho$ and $\rho_f$, respectively. $\tau_D$ denotes the driving time. The relative purity between initial state $\rho_i$ and final state $\rho_{f+\tau_D}$ of the quantum system is defined as $f_{\tau+\tau_D} = \text{Tr} \left[ \rho_i \rho_{f+\tau_D} \right] / \text{Tr} \left( \rho_f^2 \right)$.

The first system under our consideration is a qubit locally coupled to a finite-temperature bosonic environment, also known as the spin-boson model, whose total Hamiltonian is described as ($\hbar = 1$) [35]

$$
\mathcal{H} = \frac{\Omega}{2} \sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k g_k \sigma_z (b_k^\dagger + b_k)
$$
where $\sigma_z$ is the standard Pauli operator in the $z$ direction, $b_k^\dagger$ and $b_k$ denote the creation and annihilation operators of $k$th oscillators with the frequency $\omega_k$, respectively. The $g_k$ represents the coupling strength of qubit to the finite-temperature bath represented by a set of harmonic oscillators. We investigate the dissipative quantum dynamics of the system by making use of the stochastic decoupling approach [22–25], which is previously used in the calculation of partition functions and real-time dynamics for many-body systems. Based on the Hubbard-Stratonovich transformation, the dissipative interaction between the qubit and the heat bath is decoupled via stochastic fields, then the separated system and bath thus evolve in common white noise fields. The reduced density matrix comes out as an ensemble average of its random realizations. By resorting to the Girsanov transformation [22, 24], a stochastic differential equation for the common white noise fields. The reduced density matrix comes out as an ensemble average of its random realizations. By resorting to the Girsanov transformation [22, 24], a stochastic differential equation for the random density matrix is obtained and can be used to derive the desired master equation. Applying to the system of interest, the master equation is given as

$$\frac{d}{dt} \rho_s(t) = -\frac{i}{2} \left[ \sigma_z, \rho_s(t) \right] - D(t) \left[ \sigma_z, \left[ \sigma_z, \rho_s(t) \right] \right],$$

where $D(t) = \int_0^t dt' C_R(t')$ with $C_R(t)$ being the real part of the correlation function $C(t)$. Assuming the bath is in a thermal equilibrium state $\rho_b(0) = e^{-H_bT^{-1}}/Tr_b(e^{-H_bT^{-1}})$ with the Boltzmann constant $k_B = 1$, then the correlation function for this bosonic bath is given by

$$C(t) = \int d\omega J(\omega) \left[ \coth \left( \frac{\omega}{2T} \right) \cos(\omega t) - i \sin(\omega t) \right],$$

in which $J(\omega)$ is the bath spectral density function, and $T$ represents the temperature. As a result, the reduced density matrix of qubit can be obtained by solving the Eq. (3):

$$\rho_z(t) = \begin{pmatrix} \rho_{zz}(0) & \rho_{zg}(0) e^{-i\Omega t - \Gamma(t)} \\ \rho_{gz}(0) e^{i\Omega t - \Gamma(t)} & \rho_{gg}(0) \end{pmatrix},$$

where

$$\Gamma(t) = 4 \int_0^\infty d\omega J(\omega) \frac{1 - \cos(\omega t)}{\omega^2} \coth \left( \frac{\omega}{2T} \right)$$

is the decoherence factor. In this article, we consider the spectral density of the environmental modes is Ohmic-like $J(\omega) = \Lambda(\omega^s/\omega_c^{-s}) e^{-\omega/\omega_c}$ with $\Lambda$ being the dimensionless coupling constant and $\omega_c$ being the cutoff frequency. It is possible to obtain Ohmic reservoirs ($s = 1$) and sub-Ohmic reservoirs ($s < 1$) by changing the Ohmicity parameter $s$.

In the Bloch sphere representation, a generally mixed state $\rho$ of a qubit can be written in terms of Pauli matrices $\rho = \frac{1}{2}(I + v_x \sigma_x + v_y \sigma_y + v_z \sigma_z)$, where the coefficients $v_x$, $v_y$, $v_z$ are the Bloch vector, and $I$ is the identity operator of the qubit. The time evolution of the reduced density matrix $\rho_z$ in the Bloch sphere representation can be derived from Eq. (5)

$$\rho_z(t) = \frac{1}{2} \begin{pmatrix} 1 + v_z & (v_x - iv_y)q_t \\ (v_x + iv_y)q_t^* & 1 - v_z \end{pmatrix},$$

where $q_t = e^{i\Omega t - \Gamma t}$ with $\Gamma_t$ being the decoherence factor, see Eq. (6). It is readily find that the excited state population is unchanged, thus the evolution of qubit is a dephasing process. According to Eq. (1), the QSLT for the qubit to evolve from initial state $\rho_t$ to final state $\rho_{t+\tau_{QSL}}$ in this dephasing model is given by

$$\tau_{QSL} = \frac{1}{\tau_{QSL}} \int_t^{t+\tau_{QSL}} |qt| dt,$$

where

$$\tau_{QSL} = \frac{1}{\sqrt{v_x^2 + v_y^2}} \text{[QSL] = } \frac{1}{\tau_{QSL}} \int_t^{t+\tau_{QSL}} |qt| dt.$$

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The inset shows the QSLT as a function of temperature $T$ for $t = 0.5$ and $t = 3$. Here we set the driving time $\tau_D = 1$. Other parameters are $\Omega = 1$, $\omega_c = 50$ and $s = 1$.

2.1. The evolution of quantum speed limit

In this section, we give the results about the evolution of quantum speed limit for a qubit locally coupled to a finite-temperature environment. For simplicity, we assume the initial state of qubit is $\rho_0 = 1/2(|0\rangle + |1\rangle)\langle 0| + |1\rangle\langle 1|)$ and fix the computational basis \{\ket{0}, \ket{1}\} as the reference basis, in which $|0\rangle$ and $|1\rangle$ are the ground and excited states of Pauli operator $\sigma_z$, respectively. In Fig. 1(a), we plot the QSLT for the Ohmic dephasing model as a function of initial time parameter $t$ with different temperatures $T$ in the strong-coupling regime. The inset shows the variation of QSLT as a function of temperature $T$ for $t = 0.3$. We can find that the QSLT of qubit decays monotonically to zero with the growth of time $t$, and the more the temperature increases, the smaller the QSLT becomes, which means that the high temperature speed up the evolution of qubit. This result can be understood by the fact that a higher bath temperature induces a more severe decoherence which leads to the acceleration of quantum evolution. In the weak-coupling regime, it can be seen from Fig. 1(b) that the temperature shows different effects on the QSLT compared to what it exhibits in the strong-coupling regime. The high temperature initially prolongs the QSLT and suppresses the quantum evolution speed, however, some time later, the high temperature can speed up the quantum evolution, which means that the bath temperature exhibits two sides for the quantum evolution speed in the weak-coupling regime. When the temperature increases, the decay rate of QSLT is enhanced. Moreover, we can observe from the inset of Fig. 1(b) that a asymptotic value of QSLT below the driving time is obtained as the zero temperature is approached. This result suggests that the extreme low bath temperature may freeze the speed of evolution of qubit in this dissipative system.

Here, we would like to provide a possible physical explanation why the QSLT shows different behaviors in the strong-coupling and weak-coupling regimes for changing temperatures in this quantum dissipative system. In the dephasing model, it has been found that the QSLT relates to the quantum coherence of the initial state under a given driving time $\tau_D$ [14], which can also be confirmed from the term $\sqrt{v_x^2 + v_y^2}$ in Eq. (8). If we do a further derivation, Eq. (8) can be rewritten as

$$\tau_{\text{QSL}} = C_t \cdot \frac{|e^{-\Gamma t} - \cos(\Omega \tau_D) e^{-\Gamma t + \tau_D}|}{\frac{1}{\Omega} \int_{t}^{t + \tau_D} e^{-\Gamma \nu} \sqrt{(\Gamma_{\nu}^2 + \Omega^2 \nu^2)} (\Gamma_{\nu}^2 + \Omega^2) d\nu}$$

in which $C_t = \sqrt{v_x^2 + v_y^2} e^{-\Gamma t}$ is the $l_1$-norm quantum coherence of qubit according to the definition in Ref. [36]. Based on Eq. (9), we can explore the relationship between the QSLT and quantum coherence in this
To get more insight on the role of temperature in the quantum evolution speed, we utilize the right of Eq. (9) may lead to the different performances of quantum evolution speed, although which only for different temperatures in the weak-coupling regime. This result suggests that the second term on the gaps enlarge gradually as time goes on, whereas the ratio always have relatively stable gaps from beginning for different temperatures is displayed in Fig. 2(a). We can observe ratio of QSLT to quantum coherence to analysis the different phenomenons observed above. The ratio as a function of the initial time parameter for different temperatures is displayed in Fig. 2(a). We can observe that in the strong-coupling regime, the ratios have little gaps at first for different temperatures, and then the gaps enlarge gradually as time goes on, whereas the ratio always have relatively stable gaps from beginning for different temperatures in the weak-coupling regime. This result suggests that the second term on the right of Eq. (9) may lead to the different performances of quantum evolution speed, although which only gives a superficial interpretation, it provides inspiration for further studying.

Since the expression of QSLT in Eq. (1) is based on the relative purity, we mainly focus on the investigation about relative purity in the following. In Figs. 2(b) and 2(c), we plot the time evolutions of relative purity and \( l_1 \)-norm coherence for different temperatures in the strong-coupling and weak-coupling regimes, respectively. The relative purity gradually increases to the maximum as time goes on in the strong-coupling regime, and the higher temperature leads to faster increase. Instead, the quantum coherence gradually decreases to the minimum in the time evolution, and the higher temperature leads to faster decrease. This is due to the fact that the increase of temperature brings about more intense thermal fluctuation which induces the stronger decoherence. Similar behaviors can be found in the weak-coupling case which is displayed in Fig. 2(c). One difference is that the changing rates of relative purity and quantum coherence are smaller than the ones in the strong-coupling case. Another difference is that the values of relative purity at the initial time \( t = 0 \) are not consistent which leads to the curves for different temperatures cross each.
Comparing to the Fig. 1(b), we can find that the bath temperature plays a role of dual character in affecting the QSLT and this phenomenon may be linked to the performance of relative purity. It is clearly observed from the time evolutions of quantum coherence that the final states $\rho_{0+TD}$ with driving time $\tau_D = 1$ under various temperatures have little difference in the strong-coupling regime, while have obvious gaps in the weak-coupling case, which contributes to the initial values of relative purity are consistent in the strong-coupling regime, however, are different in the weak-coupling case.

Thus, the reason why the quantum evolution speed shows different performances in the strong-coupling and weak-coupling regimes is threefold: first, the quantum evolution speed at time $t$ is related to the quantum coherence at the same moment. Second, the higher temperature brings about more intense thermal fluctuation in the strong-coupling regime than in the weak-coupling case, which leads to the quantum coherence decreasing faster in the strong-coupling regime than in the weak-coupling case. Third, the initial values of relative purity depends on the driving time which may contribute to the different behaviors of relative purity for various temperatures in strong-coupling and weak-coupling regimes. Above discussions may help us understand the effects of bath temperature on the quantum evolution speed of this dephasing model, and realize that the environment-assisted speed-up and speed-down processes are possible.

![Figure 3: Contour plot of variation of the QSLT $\tau_{QSL}/\tau_D$ as a function of temperature $T$ and coupling strength $\Lambda$ for different initial time parameters in Fig. 3. It is quite clear from Fig. 3(a) that the QSLT has a peak in the weak-coupling regime with the initial time parameter $t = 0.1$, which means that the QSLT increases at first, then decreases along with the growth of bath temperature. The bath temperature plays a role of dual character in affecting the quantum evolution speed in the weak-coupling regime. By contrast, in the strong-coupling regime, the QSLT only decreases with the increase of the bath temperature. Furthermore, it is clearly observed from the Fig. 3(b)-(d) that the dual character of temperature exhibited in the weak-coupling regime gradually disappears as the time goes on, and then the quantum evolution speed is accelerated by increasing temperature in both strong-coupling and weak-coupling regimes. Therefore, the bath temperature...](image-url)
Figure 4: The QSLT $\tau_{\text{QSL}}/\tau_D$ of qubit as functions of the initial time parameter $t$ and bath temperature $T$ for the sub-Ohmic spectrum $s = 0.6$ in the (a) strong-coupling regime $\Lambda = 0.2$ and (b) weak coupling regime $\Lambda = 0.001$, respectively. The QSLT $\tau_{\text{QSL}}/\tau_D$ of qubit as functions of the Ohmicity parameter $s$ and initial time parameter $t$ for bath temperature $T = 1$ in the (c) strong-coupling regime $\Lambda = 0.2$ and (d) weak-coupling regime $\Lambda = 0.001$, respectively. Other Parameters are chosen as $\tau_D = 1$, $\omega_c = 50$ and $\Omega = 1$.

plays a more important and broader role in the weak-coupling regime than it does in the strong-coupling one.

In the following, we explore the variations of QSLT of qubit for the sub-Ohmic reservoirs with different relevant parameters. We plot the QSLT as functions of the initial time parameter $t$ and bath temperature $T$ for the sub-Ohmic spectrum $s = 0.6$ in the strong-coupling regime [Fig. 4(a)] and weak coupling regime [Fig. 4(b)], respectively. It is found that the effects of bath temperature on the QSLT of qubit for the sub-Ohmic spectrum are similar to the case for Ohmic spectrum. We can see from Fig. 4(a) that the QSLT decreases monotonically with the initial time parameter $t$ in the strong-coupling regime, and the increase of bath temperature leads to the shorter QSLT, namely, the faster quantum evolution. In the weak-coupling regime, as shown in Fig. 4(b), the bath temperature also plays a role of dual character in influencing the speed of evolution. Initially, the QSLT is a monotonic increasing function of bath temperature $T$, however, some time later, it is changed to be a monotonic decreasing function of $T$. Moreover, a relative steady speed of evolution can be obtained at the zero temperature, which is a unique phenomenon in weak coupling regime.

To get more insight on the role of the sub-Ohmic reservoirs in influencing the QSLT, we display the QSLT as functions of the Ohmicity parameter $s$ and initial time parameter $t$ in the strong-coupling regime [Fig.
4(c)] and weak-coupling regime [Fig. 4(d)], respectively. It is clear to see that the QSLT is indeed extended with the increase of the Ohmicity parameter $s$ in the strong-coupling regime and the QSLT is not a simple monotone function of the Ohmicity parameter $s$ in the weak-coupling regime. In comparison, we can see from Fig. 4(d) that the QSLT decreases to a minimum with the growth of the Ohmicity parameter $s$ in the beginning of the evolution, and after a certain time the QSLT first increases to a maximum with increasing $s$, and then decreases with further increase of $s$, which means that a nonmonotonic behavior of the QSLT is displayed. Furthermore, the quantum evolution speed of qubit for sub-Ohmic spectrum is faster than the one for Ohmic spectrum in the strong-coupling regime, which is inverse in the weak-coupling regime.

2.2. Control of the quantum evolution speed by applying bang-bang pulses

Figure 5: The QSLT $\tau_{QSL}/\tau_D$ of qubit versus the initial time parameter $t$ with different pulse interval : $\Delta t = 0.05$ (green dotted line), $\Delta t = 0.02$ (blue dotted-dashed line), $\Delta t = 0.01$ (black dashed line), $\Delta t = 0.005$ (red solid line) for the Ohmic spectrum in the strong-coupling regime $\Lambda = 0.2$. Other parameters are $\tau_D = 1$, $\omega_c = 20$, $\Omega = 1$ and $T = 1$.

The major obstacle to the development of quantum technologies is the destruction of all quantum properties caused by the inevitable interaction of quantum systems with their surrounding environment. Much effort has been made to minimize the influence of environmental noise or suppress the decoherence induced by environment in the practical realization of the quantum tasks. One of the interesting approaches is the “dynamical decoupling” or “bang-bang” pulses [37–39], which is based on applying the strong and sufficiently fast pulses to restore the quantum coherence of target system.

In this section, we mainly investigate the effect of bang-bang pulses on the quantum evolution speed of qubit. The Hamiltonian of control pulses is given by [38]

$$H_p(t) = \sum_{n=1}^{N} A_n(t)e^{i\Omega t\sigma_z/2}\sigma_x e^{-i\Omega t\sigma_z/2},$$  \hspace{1cm} (10)

where the pulse amplitude $A_n(t) = A$ for $t_n \leq t \leq t_n + \lambda$ and 0 otherwise, lasting for a duration $\lambda \ll \Delta t$, with $t_n = n\Delta t$ being the time at which the n_th pulse is applied. Here, we only consider the $\pi$ pulses for our investigation, which means that the amplitude $A$ and the duration $\lambda$ of a pulse satisfy $2A\lambda = \pm \pi$. It is not difficult to obtain the time evolution operator in the present of dynamical decoupling pulses at time $t = 2N\Delta t + \epsilon$

$$U(t) = \left\{ \begin{array}{ll} U_\epsilon(\epsilon)[U_\epsilon]^{N} & 0 \leq \epsilon < \Delta t \\
U_\epsilon(\epsilon - \Delta t)[U_p(\lambda)]U_\epsilon(\Delta t)[U_\epsilon]^{N} & \Delta t \leq \epsilon < 2\Delta t \end{array} \right. \hspace{1cm} (11)$$

where $N = \lfloor t/(2\Delta t) \rfloor$ with $[\ldots]$ denoting the integer part, and the $\epsilon$ is the residual time after $N$ cycles. $U_\epsilon$ and $U_p$ are the evolution operators corresponding to the original Hamiltonian without and with the
dynamical decoupling pulses, respectively. $\mathcal{U}_e$ represents the time evolution operator for an elementary cycle $2\Delta t$ which is given by $\mathcal{U}_e = \mathcal{U}_p(\lambda)\mathcal{U}_e(\Delta t)\mathcal{U}_p(\lambda)\mathcal{U}_e(\Delta t)$. For simplicity, we only focus on the periodic points $t_{2N} = 2N\Delta t$. It has been found that the decoherence factor $\Gamma(t)$ needs to be replaced by a new function $\Gamma_p(N, \Delta t)$ in the presence of the decoupling pulses \[38\],

$$\Gamma_p(N, \Delta t) = 4 \int d\omega J(\omega) \coth\left(\frac{\omega}{2T}\right) \times \frac{1 - \cos(\omega t_{2N})}{\omega^2} \tan^2\left(\frac{\omega\Delta t}{2}\right). \tag{12}$$

Comparing $\Gamma_p(N, \Delta t)$ with $\Gamma(t)$, we notice that the original bath spectral density has been transformed from $J(\omega)$ to the effective spectral density $J(\omega) \tan^2(\omega\Delta t/2)$ after applying the bang-bang pulses.

In Fig. 5, we illustrate the QSLT of qubit versus the initial time parameter $t$ with different pulse intervals for the Ohmic spectrum in the strong-coupling regime. In the beginning of the evolution, the QSLT of qubit becomes shorter since the smaller pulse interval. On the contrary, the QSLT decreases with the increase of pulse interval in the latter stage. Here, the fast pulse is not only able to accelerate the dynamical evolution, but also able to decelerate the dynamical evolution, which also plays a role of dual character. More interestingly, when $\Delta t$ is small enough (or in the limit $\Delta t \to 0$), the bang-bang pulse enables us to obtain a relative steady quantum evolution speed which almost remains constant. This is due to the fact that bang-bang pulse can effectively suppress the decoherence by averaging out the unwanted effects of environmental interaction\[37–39\].

![Figure 6](image)

**Figure 6**: (a) The QSLT $\tau_{\text{QSL}}/\tau_{\text{D}}$ of qubit as a function of the initial time parameter $t$ and bath temperature $T$ with bang-bang pulse for the sub-Ohmic spectrum $s = 0.6$ in the strong-coupling regime $\Lambda = 0.2$. (b) The QSLT $\tau_{\text{QSL}}/\tau_{\text{D}}$ of qubit as a function of the initial time parameter $t$ and Ohmicity parameter $s$ with bang-bang pulse at bath temperature $T = 1$ in the strong-coupling regime $\Lambda = 0.2$. Other Parameters are chosen as $\tau_D = 1$, $\omega_c = 20$, $\Delta t = 0.05$ and $\Omega = 1$.

Next, we turn to focus on the quantum evolution speed for a sub-Ohmic spectrum in the present of bang-bang pulse. We mainly investigate the effects of bath temperature $T$ [see Fig. 6(a)] and Ohmicity parameter $s$ [see Fig. 6(b)] on the QSLT in the strong-coupling regime. Comparing to Fig. 4(a), we can observe from Fig. 6(a) that the bath temperature plays a role of dual character in influencing the quantum evolution speed in the strong-coupling regime since the applied bang-bang pulse, which can not be found in the case without bang-bang pulse. Figure 6(b) presents the QSLT as a function of the initial time parameter $t$ and Ohmicity parameter $s$ with bang-bang pulse in the strong-coupling regime. In this situation, the Ohmicity parameter $s$ also plays a role of dual character in influencing the evolution of qubit, which is different in the case without bang-bang pulse [Fig. 4(c)]. In the beginning of evolution, the larger $s$ leads to a shorter QSLT which corresponds to a speed-up evolution. After a certain time, instead, the larger $s$ leads to a longer QSLT which corresponds to a speed-down evolution.

In general, on the one hand, the bang-bang pulse can be used to control the quantum evolution speed in this dephasing model. On the other hand, since the presence of bang-bang pulse, the relevant environmental
parameters, such as bath temperature and Ohmicity parameter, play some more complicated and diverse roles in affecting the quantum evolution speed.

3. Quantum evolution speed in the nonlinear environment

Next, we consider another model: a bare qubit (labeled $A$) interacts with the other one (labeled $B$) which is coupled to a thermal bath. The qubit $B$ and thermal bath constitute the well-known spin-boson model which is used as a nonlinear environment of qubit $A$ [26]. The Hamiltonian of total system is given by

$$H = H_s + H_b + H_{int},$$

$$H_s = \frac{\Omega_A}{2} \sigma_z^A + \frac{\Omega_B}{2} \sigma_z^B, \quad H_b = \sum_k \omega_k b_k^\dagger b_k,$$

$$H_{int} = f(s)g(b) + g_0 \sigma_z^A \sigma_z^B$$

where $H_s$ and $H_b$ are the Hamiltonians of two qubits and the bath, respectively. $g_0$ represents the interaction strength between the two qubits. $f(s) = \sigma_z^B$ is the subsystems operator coupled to its surrounding bath. The $g(b) = \sum_k g_k (b_k^\dagger + b_k)$ denotes the bath operator. We only focus on the on-resonance case: $\Omega_A = \Omega_B = \Omega$.

This model has been studied in the precious articles [26, 27], however, there are no exact analytical expressions of the reduced density matrix for qubits, and their results involve the Born-Markov approximation. Fortunately, by resorting to the HEOM method which is beyond the Born-Markov approximation, we can deal with this model numerically. As a nonperturbative numerical method, the HEOM consists of a set of differential equations for the reduced subsystem and enables some rigorous studies in chemical and biophysical systems, such as the optical line shapes of molecular aggregates [32] and the quantum entanglement in photosynthetic light-harvesting complexes [40]. For the finite-temperature case, we consider the Ohmic spectrum with Drude cutoff:

$$J(\omega) = \frac{2\Lambda\omega \zeta}{\pi(\omega^2 + \omega_c^2)},$$

in which $\omega_c$ is the cutoff frequency and $\Lambda$ represents the coupling strength between the qubit and the bath. Then the bath correlation function $C(t)$ can be expressed as [28–31]

$$C(t) = \sum_{k=0}^{\infty} \zeta_k e^{-\nu_k t},$$

where the real and imaginary parts of $\zeta_k$ are respectively given as

$$\zeta_k^R = 4\Lambda\omega_c T \frac{\nu_k}{\nu_k^2 - \omega_c^2}(1 - \delta_{k0}) + \Lambda\omega_c \cot(\frac{\omega_c}{2T})\delta_{k0},$$

$$\zeta_k^I = -i\Lambda\omega_c \delta_{k0},$$

with the $\nu_k = 2k\pi T(1 - \delta_{k0}) + \omega_c \delta_{k0}$ being the $k$-th Matsubara frequency. Since the bath correlation function can be approximately expressed as the sum of the first few terms in the series, the Matsubara frequency has been cut off and the convergence of result has been checked in our numerical calculation.

Following the derivation shown in Ref. [33], the hierarchy equations of reduced quantum subsystem can be obtained as follows:

$$\frac{d}{dt} \rho(t) = -i[H_s + \vec{l} \cdot \vec{\nu}] \rho(t) + \phi \sum_{k=0}^{\epsilon} \rho_{\vec{c}_k} \rho_{\vec{c}_k}^\dagger (t) + \sum_{k=0}^{\epsilon} l_k \psi_k \rho_{\vec{c}_k} \rho_{\vec{c}_k}^\dagger (t),$$

where $\vec{l} = (l_0, l_1, \ldots, l_{\epsilon})$ is an $\epsilon + 1$-dimensional index, $\vec{\nu} = (\nu_0, \nu_1, \ldots, \nu_{\epsilon})$ and $\vec{c}_k = (0, 0, \ldots, 1_k, \ldots, 0)$ are $(\epsilon + 1)$-dimensional vectors with $\epsilon$ being the cutoff number of the Matsubara frequency. Two superoperators $\phi$ and $\psi_k$ are defined as

$$\phi = if(s)^x, \quad \psi_k = i [\zeta_k^R f(s)^x + i\zeta_k^I f(s)^{\dagger}],$$

where $s$ and $t$ are defined as $s = R_{t0} \zeta_k e^{-\nu_k t}$.
Figure 7: The QSLT $\tau_{QSL}/\tau_D$ of qubit $A$ versus the initial time parameter $t$ and the coupling strength $\Lambda$ for the Ohmic spectrum. Other Parameters are chosen as $T = 5$, $\tau_D = 10$, $g_0 = 0.1, \omega_c = 5$ and $\Omega = 1$.

with $X \times Y = [X, Y] = XY - YX$ and $X \circ Y = \{X, Y\} = XY + YX$.

We choose the initial state of two qubits as

$$\rho_{AB}(0) = \frac{1}{2} \left( \begin{array}{ccc} 1 & 1 \\ 1 & 1 \end{array} \right)_A \otimes \frac{1}{2} \left( \begin{array}{ccc} 1 & 1 \\ 1 & 1 \end{array} \right)_B,$$

and display the QSLT of qubit $A$ as function of the initial time parameter and the coupling strength for Ohmic spectrum in Fig. 7. In the strong-coupling regime, the quantum evolution speed of qubit $A$ exhibits a speed-up process since the quantum decoherence effect. In contrast, the QSLT of qubit $A$ in the weak-coupling regime decreases to a minimum in the beginning of the evolution, then revivals and occurs a damped oscillatory behavior. As the coupling strength $\Lambda$ increases, the damped oscillatory behavior fades away. The QSLT behaves different in the strong-coupling and weak-coupling regimes.

In the previous section, it is shown that the quantum evolution speed is related to the dynamics of quantum coherence. Here, we choose the measure of quantum coherence based on the quantum Jensen-Shannon divergence [41] to study the dynamics of quantum coherence of qubit $A$ for getting more insight on the QSLT. The expression of quantum coherence is given by

$$C(\rho) = \sqrt{S \left( \frac{\rho + \rho_{\text{dia}}}{2} \right) - \frac{S(\rho) + S(\rho_{\text{dia}})}{2}}$$

where $S(\rho) = -\text{Tr} \rho \log_2 \rho$ is the von Neumann entropy and $\rho_{\text{dia}}$ is the incoherent state obtained from $\rho$ by deleting all off-diagonal elements[36].

We plot the QSLT and quantum coherence of qubit $A$ as functions of the initial time parameter $t$ and the temperature $T$ in Figs.8. In the weak-coupling regime, as shown in Figs.8(a) and 8(c), the QSLT and quantum coherence exhibit the damped oscillatory behaviors and have similar evolutions. The increase of temperature induces the speed-up evolution since the fact that higher temperature brings more intensive decoherence, which can also be confirmed by the dynamics of quantum coherence in Fig. 8(c). In contrast, there are some rich and anomalous phenomenons in the strong-coupling regime as shown in Figs.8(b) and 8(d). In the low-temperature region, the behaviors of QSLT and quantum coherence are analogous to those in the weak-coupling regime. However, as the temperature increases, the QSLT is extended which means the quantum evolution speed is decelerated and not a monotonic increasing function of the bath temperature any more. Superficially, this result is due to the enhancement of quantum coherence by the high temperature,
Figure 8: The QSLT $\tau_{QSL}/\tau_D$ and quantum coherence $C$ of qubit $A$ versus the initial time parameter $t$ and the temperature $T$ for the (a)(c) weak-coupling $\Lambda = 0.005$ regime and (b)(d) strong-coupling $\Lambda = 0.05$ regime. Other Parameters are chosen as $\tau_D = 10$, $g_0 = 0.1$, $\omega_c = 5$ and $\Omega = 1$.

which can be confirmed in the dynamics of quantum coherence in Fig. 8(d). The underlying reason for this anomalous phenomenon is that the spin-boson model consisting of qubit $B$ and the thermal bath can be seen as a nonlinear environment for qubit $A$, the power spectrum doesn’t necessarily grow with the temperature [26]. Thus, some reversed effects occur, i.e., the increase of bath temperature may give rise to the speed-down evolution and the enhancement of quantum coherence.

4. Conclusion

In conclusion, we have considered two kinds of finite-temperature bosonic baths to investigate the quantum evolution speed of qubit and find that the quantum evolution speed isn’t a monotonic function of temperature. For the spin-boson model, the quantum evolution speed of qubit can be accelerated by the high temperature in the strong-coupling regime. In the weak-coupling regime, the bath temperature plays a role of dual character in affecting the quantum evolution speed, which means that the high temperature not only leads to the speed-up but also speed-down processes. The quantum coherence, relative purity and the driving time are responsible for the different behaviors of quantum evolution speed in the strong-coupling and weak-coupling regimes. Furthermore, we can observe that the quantum evolution speed can be controlled by the bang-bang pulse in the strong-coupling regime, and the relative steady quantum evolution speed can be
obtained by fast pulse. Interestingly, the bath temperature and Ohmicity parameter also play roles of dual character in the strong-coupling regime since the presence of bang-bang pulse, which are not found in the case without pulse. For the nonlinear bath, we study the quantum evolution speed of qubit by applying the hierarchical equations of motion method. It is shown that the performances of quantum evolution speed in weak-coupling and strong-coupling regimes are very different. In the strong-coupling situation, the quantum evolution speed at low-temperature region behaves similarly to that in the weak-coupling situation where the quantum evolution speed is only a monotonic increasing function of temperature. However, the rise of temperature induces the speed-down process, this anomalous phenomenon is on account of the temperature dependence of the spectral profile in nonlinear bath. As a comparison, the dynamics of quantum coherence is also explored in different situations. These results provide the possibilities to control quantum evolution speed by changing the relevant environmental parameters in the finite-temperature bosonic environments. Finally, we expect our studies to be of interest for experimental applications in quantum computation and quantum information processing.

5. Acknowledgments

This project was supported by the National Natural Science Foundation of China (Grant No.11274274) and the Fundamental Research Funds for the Central Universities (Grant No.2017FZA3005 and 2016XZZX002-01).

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