A Possible Explanation of the Superposition Principle

Kent A. Peacock
Department of Philosophy
University of Lethbridge

ABSTRACT: I tentatively suggest that the superposition principle of quantum mechanics is explicable in a mathematically natural way if it is possible to understand probability amplitudes as complex-valued logarithms. This notion is inspired by the fact that the quantum state may be interpreted as a measure of information.

The object of this note is to sketch what J. A. Wheeler might call an “idea for an idea". The notion I present here points the way toward a possible new interpretation of quantum mechanics, although the development of this approach that I am able to offer in this paper is in itself certainly not yet complete enough to deserve such a lofty description. I hope to produce a more thorough treatment soon; but in the meantime I dare to think that the bare notion I describe here may be of sufficient interest to merit an airing in preprint form, and I commend it to the attention of those who may be more mathematically skilled than I am.

I. The Feynman Problem

Quantum mechanics is beset by a number of interpretational challenges arising from spectacularly non-classical phenomena such as nonlocality and superfluidity, as well as the host of difficulties surrounding the measurement problem. But the deepest mysteries surround our lack of understanding of origin of the basic rules of the theory.

We can frame the problem very clearly by going back to the statement of the basic principles of quantum mechanics given by Richard Feynman and his co-authors in The Feynman Lectures on Physics:

(1) The probability of an event in an ideal experiment is given by the square of the absolute value of a complex number \( \phi \) which is called the probability amplitude:

\[
P = |\phi|^2.
\]

(2) When an event can occur in several alternative ways, the probability amplitude for the event is the sum of the probability amplitudes for each way considered separately. There is

\[
P = \sum |\phi|^2.
\]
interference:

\[ \phi = \phi_1 + \phi_2, \]  
\[ P = |\phi_1 + \phi_2|^2. \]  

(3) If an experiment is performed which is capable of determining whether one or another alternative is actually taken, the probability of the event is the sum of the probabilities for each alternative. The interference in lost:

\[ P = P_1 + P_2. \]  

\[ \text{[6, p. 1–10]} \]  
\[ \text{[I have renumbered the equations.]} \]

Eq. 3 is, of course, also known as the Born Rule. As it shows, the probability amplitude is, so to speak, the “square root” of the probability.

All the multifarious and complex developments of quantum mechanics are applications of these rules, which, so far, must simply be taken for granted. As Feynman et al. say,

One might still like to ask: “How does it work? What is the machinery behind the law?” No one has found any machinery behind the law. No one can “explain” any more than we have just “explained”. No one will give you any deeper representation of the situation. We have no ideas about a more basic mechanism from which these results can be deduced. \[ \text{[6, p. 1–10]} \]  
\[ \text{[See also similar remarks in [4].]} \]

I will call the problem of explaining rules (1) to (3) the “Feynman Problem”. It seems frustrating that something as simple and basic as these rules cannot be explained. This uncomfortable fact implies that we still do not understand the deepest principles of physics, despite the considerable power and sophistication that our physical science has already attained.

There is a clarification that can immediately be added to the above statement of quantum principles. As Feynman et al. go on to explain, what point (2) really says is that if there is no way of telling which route the system takes without measuring a non-commuting observable, then interference of amplitudes may occur. The classic illustration of this (\[ \text{[4]} \]) is the double-slit experiment: if we do something that allows us to determine which slit the electrons go through (thereby finding their positions), we wipe out the interference pattern (which depends upon the particles being in pure momentum states). There would be no interference if all observables commuted. Since all uniquely quantum phenomena are in some way a consequence of interference, all uniquely quantum phenomena are somehow a consequence of non-commutativity. (For a recent and forceful expression of this view, see \[ \text{[4]} \].) Therefore, we could add to Feynman’s list the fundamental problems of understanding the origin of non-commutativity and the intimately related problem of understanding the origin of Planck’s constant of action.
The problem, then, is to answer the following inter-related questions:

1. Why do we have to represent quantum processes and states by complex-valued mathematical objects?

2. Why the superposition principle? — That is, why do we represent quantum states and processes by objects that add up linearly?

3. Where does the Born rule come from? Why is this the right way to calculate probabilities?

4. Why non-commutativity?

5. Why does Planck’s constant of action have the particular magnitude that it has?

(Questions 1 and 2 can be combined into the question, “Why Hilbert space?”)

In the following I will offer a tentative answer to the second question, and make some hesitant suggestions about the others.

II. The Quantum State as a Measure of Information

In recent years, a very fruitful interpretation of the quantum state has begun to emerge — the notion that state vectors are some sort of measure of information[9]. Indeed, one often speaks of state vectors of the form

\[ |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \]  \hspace{1cm} (7)

as “qubits”, in analogy to the bits (“binary information units”) of the classical information theory developed by Shannon and others[11]. (The apt term “qubit” is due to Schumacher[10].) In this formula, \(|0\rangle\) and \(|1\rangle\) are interpreted most naturally as eigenstates of some Hermitian observable (such as the spin of an electron) that is binary in the sense that its spectrum of eigenvalues is the set \(\{0, 1\}\). The complex numbers \(\alpha\) and \(\beta\) are phase factors such that

\begin{align*}
\text{Prob(getting 0)} &= \alpha^2, \\
\text{Prob(getting 1)} &= \beta^2, \quad \text{and} \\
\alpha^2 + \beta^2 &= 1.
\end{align*}

(8) \hspace{1cm} (9) \hspace{1cm} (10)

In the past ten years or so we therefore see the idea emerging that quantum mechanics is a sort of generalized information theory. As Gerard Milburn puts it,

... quantum theory, our best theory of physical reality, is actually a theory, not of physical things, but of physical information (even today not every physicist would accept this point of view). [7, p. 153]
III. Taking the Information-theoretic Interpretation Seriously

If we are to take seriously the notion that quantum theory is a generalization of classical information theory, then we should push the analogy between quantum and classical information theory as far as possible. In the classical theory of Shannon, information is a logarithm — the log of the complexity (or multiplicity) of a system; that is, the log of the number of ways that a process could have gone. We use a logarithmic measure of complexity for the convenient reason that logarithms are additive. The multiplicity of a number of experiments done in succession is the product of their individual complexities; equivalently, the information required to express the result of a concatenated series of experiments is the sum of the information to be had from each individual experiment.

To make the process clear, consider the elementary example of a two-state system such as a coin toss. A single toss can come out in two ways, but its result can be represented by one letter, H or T. Two tosses in succession can come out four different ways, but we only need two letters to represent the outcome; n tosses can come out in $2^n$ ways, but we only need n letters to represent the outcome, and n tosses followed by m tosses can be described by $n + m$ letters. Shannon therefore found it convenient to define the information contained in the outcomes of n binary events by the binary logarithm of the complexity. This can be generalized easily to experiments in which the outcomes are not equiprobable. The key point, again, is that the simplicity comes from the additivity of logarithms.

My suggestion is that the linearity of quantum mechanics might naturally be explained if we could show that probability amplitudes can be treated as logarithms. (Following Feynman’s approach, we focus our attention on the probability amplitude; this involves no loss of generality, since state vectors such as Schumacher’s qubits are simply arrays of amplitudes.) These logs must be complex-valued, and this raises problems of interpretation which I will discuss here but not definitively solve. But suppose that we can intuitively think of amplitudes as representing the logarithms of a complexity, however that complexity may be defined precisely. The superposition of amplitudes would then correspond to the multiplication of the complexities associated with those amplitudes. If this can be made to work in detail, then we would see that quantum mechanics is essentially a calculus of complex-valued logarithms.

Again, here is my conjecture:

Probability amplitudes are complex-valued logarithms of a complex-valued complexity associated (in a way to be determined) with physical transitions of state. The superposition of probability amplitudes (leading to non-classical interference phenomena because the amplitudes are complex) corresponds to the multiplication of complexities associated with the processes associated with the amplitudes for those states.

The idea outlined here is, as emphasized, highly tentative and sketchy. However, it is the first notion that I have encountered that could even count as a candidate for an
explanation for the linearity of quantum mechanics, and it needs to be investigated with some care.

IV. Concluding Observations and Speculations

The notion of seeking a logarithmic interpretation of the elements of a linear vector space is not as odd as it might seem at first glance. Consider, for instance, the set of the first $n$ primes, including 1, and their inverses. Take the logs (to any convenient base) of the members of this set. The log of any composite number that is built up out of multiples of the given $n$ primes (but no others) will be a linear combination of the logs of the $n$ primes. Logs of the reciprocals give additive inverses, and log 1 gives the zero vector. Each such set of $n$ primes together with their inverses therefore defines a vector space of dimension $n - 1$, with the logs of the composite numbers acting as vectors in the space. Whether this has any useful application to number theory I am not sure, but it raises the interesting question of determining the conditions under which any vector space can be usefully interpreted as a space of logarithms. In particular, this raises the disturbing possibility, which I mention only in passing, that ordinary 3-d position space, or space-time itself, could conceivably accept such an interpretation.

The notion that we could explain the superposition principle if probability amplitudes are a kind of logarithm certainly seems mathematically natural. However, to make this workable we still need to understand what it could mean to talk about complex-valued complexities. We also need to see where the Born Rule comes from. Why can we get back to a classical probability by taking the modulus of a complex logarithm?

I have no clear idea at this writing what the answer to the second question would be, although I suspect that it will turn out to be mathematically obvious. The first question is deeper, but one can see a direction that could be worth exploring.

The key may be to follow up Feynman’s pregnant suggestions about negative probabilities [4]. Feynman pointed out that negative probabilities are just as sensible, and possibly just as useful, as ordinary negative numbers, so long as they are used only in intermediate calculations whose final results come out positive: “...conditional probabilities and probabilities of imagined intermediary states may be negative in a calculation of probabilities of physical events or states” [4, p. 238]. And, as Feynman explains, one situation in which we could expect negative probabilities to arise naturally is when a system may be in one of two mutually incompatible conditions, such as a quantum system that may be subjected to non-commuting measurement procedures.

If a physical theory for calculating probabilities yields a negative probability for a given situation under certain assumed conditions, we need not conclude that the theory is incorrect. Two other possibilities of interpretation exist. One is that the conditions (for example, initial conditions) may not be capable of being realized in the physical world. The other possibility is that the situation for which the probability appears to be negative is not one that can be verified directly. A combination of these two, limitation of verifiability and freedom in initial
conditions, may also be a solution to the apparent difficulty. [3, p. 238–9]

Now, it is a very short step from negative frequencies (or probabilities) to complex-valued “roots” of frequencies (or probabilities) — if for no other reason than the fact that if negative probabilities obey algebraic relations higher than first degree, the fundamental theorem of algebra guarantees that these relations may well be satisfied in some cases by complex-valued quantities. My point, therefore, is that there may be a deep but natural connection between the use of complex numbers to represent, as it were, “square roots” of probabilities, and the fact of non-commutativity.

Is the difference between classical and quantum information merely the move from real-valued to complex-valued measures of information? Non-commuting observables $\hat{A}$ and $\hat{B}$ in quantum mechanics obey commutation relations of the general form

$$[\hat{A}, \hat{B}] = i\hbar \hat{C}$$

(11)

where $\hbar$ is Planck’s reduced constant. For all we know, $\hbar$ might have a different numerical value than it happens to have; and therefore there might be a whole class of mathematically possible quantum information theories depending on the value of $\hbar$. On the other hand, it may be that $\hbar$ is somehow mathematically determined, in which case there is only one mathematically possible quantum mechanics. However, I am rapidly approaching the limits of permissible speculation here, and I will conclude by reiterating the point that the need for complex-valued frequencies or complexities is very likely related to the fact of non-commutativity, in a way that remains to be made clear.

If anything like what I suggest here is right, the linearity of quantum theory is a mathematical artifact, stemming from the use of a logarithmic description. It is quite possible, and rather important to note, that the underlying dynamics of quantum phenomena might be highly nonlinear — and indeed this is suggested by the fact of non-commutativity.

It is beyond the scope of this paper to compare the theory sketched here with the numerous other interpretations of state vectors or wave functions that have been advanced in the past. At first glance, it does seem to conflict with realistic interpretations of the wave function such as Everett’s relative state formulation [8] or Bohm’s interpretation [1]. On the other hand, a tsunami can be thought of as merely a probability distribution of water molecules, but it can still pack quite a punch. It would be well to avoid commitment to philosophical preconceptions about the meaning of the quantum state until we are much clearer about the mathematical possibilities—for a likely implication of recent trends in quantum information theory is that there is much more to quantum mechanics that is purely mathematical than first meets the eye.

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