DP-colorings of graphs with high chromatic number

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Abstract

DP-coloring is a generalization of list coloring introduced recently by Dvořák and Postle [4]. We prove that for every $n$-vertex graph $G$ whose chromatic number $\chi(G)$ is “close” to $n$, the DP-chromatic number of $G$ equals $\chi(G)$. “Close” here means $\chi(G) \geq n - O(\sqrt{n})$, and we also show that this lower bound is best possible (up to the constant factor in front of $\sqrt{n}$), in contrast to the case of list coloring.

1 Introduction

We use standard notation. In particular, $\mathbb{N}$ denotes the set of all nonnegative integers. For a set $S$, $\text{Pow}(S)$ denotes the power set of $S$, i.e., the set of all subsets of $S$. All graphs considered here are finite, undirected, and simple. For a graph $G$, $V(G)$ and $E(G)$ denote the vertex and the edge sets of $G$, respectively. For a set $U \subseteq V(G)$, $G[U]$ is the subgraph of $G$ induced by $U$. Let $G - U := G[V(G) \setminus U]$, and for $u \in V(G)$, let $G - u := G - \{u\}$. For $U_1, U_2 \subseteq V(G)$, let $E_G(U_1, U_2) \subseteq E(G)$ denote the set of all edges in $G$ with one endpoint in $U_1$ and the other one in $U_2$. For $u \in V(G)$, $N_G(u) \subseteq V(G)$ denotes the set of all neighbors of $u$, and $\deg_G(u) := |N_G(u)|$ is the degree of $u$ in $G$. We use $\delta(G)$ to denote the minimum degree of $G$, i.e., $\delta(G) := \min_{u \in V(G)}\deg_G(u)$.

For $U \subseteq V(G)$, let $N_G(U) := \bigcup_{u \in U} N_G(u)$. To simplify notation, we write $N_G(u_1, \ldots, u_k)$ instead of $N_G(\{u_1, \ldots, u_k\})$. A set $I \subseteq V(G)$ is independent if $I \cap N_G(I) = \emptyset$, i.e., if $uv \notin E(G)$ for all $u, v \in I$. We denote the family of all independent sets in a graph $G$ by $\mathcal{I}(G)$. The complete $k$-vertex graph is denoted by $K_k$.

1.1 The Noel–Reed–Wu Theorem for list coloring

Recall that a proper coloring of a graph $G$ is a function $f : V(G) \to Y$, where $Y$ is a set of colors, such that $f(u) \neq f(v)$ for every edge $uv \in E(G)$. The smallest $k \in \mathbb{N}$ such that there exists a proper coloring $f : V(G) \to Y$ with $|Y| = k$ is called the chromatic number of $G$ and is denoted by $\chi(G)$.

List coloring was introduced independently by Vizing [11] and Erdős, Rubin, and Taylor [6]. A list assignment for a graph $G$ is a function $L : V(G) \to \text{Pow}(Y)$, where $Y$ is a set. For each
Let $u \in V(G)$, the set $L(u)$ is called the list of $u$, and its elements are the colors available for $u$. A proper coloring $f : V(G) \to Y$ is called an $L$-coloring if $f(u) \in L(u)$ for each $u \in V(G)$. The list chromatic number $\chi_L(G)$ of $G$ is the smallest $k \in \mathbb{N}$ such that $G$ is $L$-colorable for each list assignment $L$ with $|L(u)| \geq k$ for all $u \in V(G)$. It is an immediate consequence of the definition that $\chi_L(G) \geq \chi(G)$ for every graph $G$.

It is well-known (see, e.g., [6, 11]) that the list chromatic number of a graph can significantly exceed its ordinary chromatic number. Moreover, there exist 2-colorable graphs with arbitrarily large list chromatic numbers. On the other hand, Noel, Reed, and Wu [7] established the following result, which was conjectured by Ohba [8, Conjecture 1.3]:

**Theorem 1.1** (Noel–Reed–Wu [7]). Let $G$ be an $n$-vertex graph with $\chi(G) \geq (n - 1)/2$. Then $\chi_L(G) = \chi(G)$.

The following construction was first studied by Ohba [8] and Enomoto, Ohba, Ota, and Sakamoto [5]. For a graph $G$ and $s \in \mathbb{N}$, let $J(G, s)$ denote the join of $G$ and a copy of $K_s$, i.e., the graph obtained from $G$ by adding $s$ new vertices that are adjacent to every vertex in $V(G)$ and to each other. It is clear from the definition that for all $G$ and $s$, $\chi(J(G, s)) = \chi(G) + s$. Moreover, we have $\chi_L(J(G, s)) \leq \chi_L(G) + s$; however, this inequality can be strict. Indeed, Theorem 1.1 implies that for every graph $G$ and every $s \geq |V(G)| - 2\chi(G) - 1$,

$$\chi_L(J(G, s)) = \chi(J(G, s)),$$

even if $\chi_L(G)$ is much larger than $\chi(G)$. In view of this observation, it is interesting to consider the following parameter:

$$Z_L(G) := \min \{ s \in \mathbb{N} : \chi_L(J(G, s)) = \chi(J(G, s)) \}, \tag{1.1}$$

i.e., the smallest $s \in \mathbb{N}$ such that the list and the ordinary chromatic numbers of $J(G, s)$ coincide. The parameter $Z_L(G)$ was explicitly defined by Enomoto, Ohba, Ota, and Sakamoto in [5, page 65] (they denoted it $\psi(G)$). Recently, Kim, Park, and Zhu (personal communication, 2016) obtained new lower bounds on $Z_L(K_{2,n})$, $Z_L(K_{n,n})$, and $Z_L(K_{n,n,n})$. One can also consider, for $n \in \mathbb{N}$,

$$Z_L(n) := \max \{ Z_L(G) : |V(G)| = n \}. \tag{1.2}$$

The parameter $Z_L(n)$ is closely related to the Noel–Reed–Wu Theorem, since, by definition, there exists a graph $G$ on $n + Z_L(n) - 1$ vertices whose ordinary chromatic number is at least $Z_L(n)$ and whose list and ordinary chromatic numbers are distinct. The finiteness of $Z_L(n)$ for all $n \in \mathbb{N}$ was first established by Ohba [8, Theorem 1.3]. Theorem 1.1 yields an upper bound $Z_L(n) \leq n - 5$ for all $n \geq 5$; on the other hand, a result of Enomoto, Ohba, Ota, and Sakamoto [5, Proposition 6] implies that $Z_L(n) \geq n - O(\sqrt{n})$.

### 1.2 DP-colorings and the results of this paper

The goal of this note is to study analogs of $Z_L(G)$ and $Z_L(n)$ for the generalization of list coloring that was recently introduced by Dvořák and Postle [4], which we call DP-coloring. Dvořák and Postle invented DP-coloring to attack an open problem on list coloring of planar graphs with no cycles of certain lengths.

**Definition 1.2.** Let $G$ be a graph. A cover of $G$ is a pair $(L, H)$, where $H$ is a graph and $L : V(G) \to \text{Pow}(V(H))$ is a function, with the following properties:
the sets \( L(u), u \in V(G) \), form a partition of \( V(H) \);
- if \( u, v \in V(G) \) and \( L(v) \cap N_H(L(u)) \neq \emptyset \), then \( v \in \{ u \} \cup N_G(u) \);
- each of the graphs \( H[L(u)], u \in V(G) \), is complete;
- if \( uv \in E(G) \), then \( E_H(L(u), L(v)) \) is a matching (not necessarily perfect and possibly empty).

**Definition 1.3.** Let \( G \) be a graph and let \((L, H)\) be a cover of \( G \). An \((L, H)\)-coloring of \( G \) is an independent set \( I \in \mathcal{I}(H) \) of size \(|V(G)|\). Equivalently, \( I \in \mathcal{I}(H) \) is an \((L, H)\)-coloring of \( G \) if \(|I \cap L(u)| = 1\) for all \( u \in V(G) \).

**Remark 1.4.** Suppose that \( G \) is a graph, \((L, H)\) is a cover of \( G \), and \( G' \) is a subgraph of \( G \). In such situations, we will allow a slight abuse of terminology and speak of \((L, H)\)-colorings of \( G' \) (even though, strictly speaking, \((L, H)\) is not a cover of \( G' \)).

The **DP-chromatic number** \( \chi_{DP}(G) \) of a graph \( G \) is the smallest \( k \in \mathbb{N} \) such that \( G \) is \((L, H)\)-colorable for each cover \((L, H)\) with \(|L(u)| \geq k\) for all \( u \in V(G) \).

To show that \( \chi_{DP}(G) \) indeed generalizes \( \chi_{\ell}(G) \), consider a graph \( G \) and let \((\hat{L}, \hat{H})\) be a cover of \( G \) and a list assignment \( \hat{L} \) for \( G \). Define a graph \( H \) as follows: Let \( V(H) := \{(u, c) : u \in V(G) \text{ and } c \in L(u)\} \) and let
\[
(u_1, c_1)(u_2, c_2) \in E(H) \iff (u_1 = u_2 \text{ and } c_1 \neq c_2) \text{ or } (u_1 u_2 \in E(G) \text{ and } c_1 = c_2).
\]

For \( u \in V(G) \), let \( \hat{L}(u) := \{(u, c) : c \in L(u)\} \). Then \((\hat{L}, \hat{H})\) is a cover of \( G \), and there is a one-to-one correspondence between \( L \)-colorings and \((\hat{L}, \hat{H})\)-colorings of \( G \). Indeed, if \( f \) is an \( L \)-coloring of \( G \), then the set \( I_f := \{(u, f(u)) : u \in V(G)\} \) is an \((\hat{L}, \hat{H})\)-coloring of \( G \). Conversely, given an \((\hat{L}, \hat{H})\)-coloring \( I \) of \( G \), we can define an \( L \)-coloring \( f_I \) of \( G \) by the property \((u, f_I(u)) \in I \) for all \( u \in V(G) \). Thus, list colorings form a subclass of DP-colorings. In particular, \( \chi_{DP}(G) \geq \chi_{\ell}(G) \) for each graph \( G \).

Some upper bounds on list-chromatic numbers hold for DP-chromatic numbers as well. For example, for any \( d \)-degenerate graph \( G \), Dvořák and Postle [4] pointed out that Thomassen’s bounds [9, 10] on the list chromatic numbers of planar graphs hold also for their DP-chromatic numbers; in particular, \( \chi_{DP}(G) \leq 5 \) for every planar graph \( G \). On the other hand, there are also some striking differences between DP- and list coloring. For instance, even cycles are 2-list-colorable, while their DP-chromatic number is 3; in particular, the orientation theorems of Alon–Tarsi [2] and the Bondy–Boppana–Siegel Lemma (see [2]) do not extend to DP-coloring (see [3] for further examples of differences between list and DP-coloring).

By analogy with (1.1) and (1.2), we consider the parameters
\[
Z_{DP}(G) := \min \{ s \in \mathbb{N} : \chi_{DP}(J(G, s)) = \chi(J(G, s)) \},
\]
and
\[
Z_{DP}(n) := \max \{ Z_{DP}(G) : |V(G)| = n \}.
\]

Our main result asserts that for all graphs \( G \), \( Z_{DP}(G) \) is finite:

**Theorem 1.5.** Let \( G \) be a graph with \( n \) vertices, \( m \) edges, and chromatic number \( k \). Then \( Z_{DP}(G) \leq 3m \). Moreover, if \( \delta(G) \geq k - 1 \), then
\[
Z_{DP}(G) \leq 3m - \frac{3}{2}(k - 1)n.
\]
Corollary 1.6. For all \( n \in \mathbb{N} \), \( Z_{DP}(n) \leq 3n^2/2 \).

Note that the upper bound on \( Z_{DP}(n) \) given by Corollary 1.6 is quadratic in \( n \), in contrast to the linear upper bound on \( Z_{l}(n) \) implied by Theorem 1.1. Our second result shows that the order of magnitude of \( Z_{DP}(n) \) is indeed quadratic:

Theorem 1.7. For all \( n \in \mathbb{N} \), \( Z_{DP}(n) \geq n^2/4 - O(n) \).

Corollary 1.6 and Theorem 1.7 also yield the following analog of Theorem 1.1 for DP-coloring:

Corollary 1.8. For \( n \in \mathbb{N} \), let \( r(n) \) denote the minimum \( r \in \mathbb{N} \) such that for every \( n \)-vertex graph \( G \) with \( \chi(G) \geq r \), we have \( \chi_{DP}(G) = \chi(G) \). Then
\[
    n - r(n) = \Theta(\sqrt{n}).
\]

We prove Theorem 1.5 in Section 2 and Theorem 1.7 in Section 3. The derivation of Corollary 1.8 from Corollary 1.6 and Theorem 1.7 is straightforward; for completeness, we include it at the end of Section 3.

2 Proof of Theorem 1.5

For a graph \( G \) and a finite set \( A \) disjoint from \( V(G) \), let \( J(G, A) \) denote the graph with vertex set \( V(G) \cup A \) obtained from \( G \) by adding all edges with at least one endpoint in \( A \) (i.e., \( J(G, A) \) is a concrete representative of the isomorphism type of \( J(G, |A|) \)).

First we prove the following more technical version of Theorem 1.5:

Theorem 2.1. Let \( G \) be a \( k \)-colorable graph. Let \( A \) be a finite set disjoint from \( V(G) \) and let \((L, H)\) be a cover of \( J(G, A) \) such that for all \( a \in A \), \( |L(a)| \geq |A| + k \). Suppose that
\[
    |A| \geq \frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) + |A| - |L(v)| + 1, 0\}. \tag{2.1}
\]
Then \( J(G, A) \) is \((L, H)\)-colorable.

Proof. For a graph \( G \), a set \( A \) disjoint from \( V(G) \), a cover \((L, H)\) of \( J(G, A) \), and a vertex \( v \in V(G) \), let
\[
    \sigma(G, A, L, H, v) := \max\{\deg_G(v) + |A| - |L(v)| + 1, 0\}
\]
and
\[
    \sigma(G, A, L, H) := \sum_{v \in V(G)} \sigma(G, A, L, H, v).
\]

Assume, towards a contradiction, that a tuple \((k, G, A, L, H)\) forms a counterexample which minimizes \( k \); then \(|V(G)|\), and then \(|A|\). For brevity, we will use the following shortcuts:
\[
    \sigma(v) := \sigma(G, A, L, H, v); \quad \sigma := \sigma(G, A, L, H).
\]

Thus, (2.1) is equivalent to
\[
    |A| \geq \frac{3\sigma}{2}.
\]

Note that \(|V(G)|\) and \(|A|\) are both positive. Indeed, if \( V(G) = \emptyset \), then \( J(G, A) \) is just a clique with vertex set \( A \), so its DP-chromatic number is \(|A|\). If, on the other hand, \( A = \emptyset \), then (2.1) implies that \(|L(v)| \geq \deg_G(v) + 1\) for all \( v \in V(G) \), so an \((L, H)\)-coloring of \( G \) can be constructed greedily. Furthermore, \( \chi(G) = k \), since otherwise we could have used the same \((G, A, L, H)\) with a smaller value of \( k \).
Claim 2.1.1. For every \( v \in V(G) \), the graph \( J(G-v, A) \) is \((L, H)\)-colorable.

Proof. Consider any \( v_0 \in V(G) \) and let \( G' := G - v_0 \). For all \( v \in V(G') \), \( \deg_G(v') \leq \deg_G(v) \), and thus \( \sigma(G', A, L, H, v) \leq \sigma(v) \). Therefore,

\[
\frac{3}{2} \sigma(G', A, L, H) \leq \frac{3\sigma}{2} \leq |A|.
\]

By the minimality of \(|V(G)|\), the conclusion of Theorem 2.1 holds for \((k, G', A, L, H)\), i.e., \( J(G', A) \) is \((L, H)\)-colorable, as claimed. \( \triangledown \)

Corollary 2.1.2. For every \( v \in V(G) \),

\[
\sigma(v) = \deg_G(v) + |A| - |L(v)| + 1 > 0.
\]

Proof. Suppose that for some \( v_0 \in V(G) \),

\[
\deg_G(v_0) + |A| - |L(v_0)| + 1 \leq 0,
\]

i.e.,

\[
|L(v_0)| \geq \deg_G(v_0) + |A| + 1.
\]

Using Claim 2.1.1, fix any \((L, H)\)-coloring \( I \) of \( J(G-v_0, A) \). Since \( v_0 \) still has at least

\[
|L(v_0)| - (\deg_G(v_0) + |A|) \geq 1
\]

available colors, \( I \) can be extended to an \((L, H)\)-coloring of \( J(G, A) \) greedily; a contradiction. \( \triangledown \)

Claim 2.1.3. For every \( v \in V(G) \) and \( x \in \bigcup_{a \in A} L(a) \), there is \( y \in L(v) \) such that \( xy \in E(H) \).

Proof. Suppose that for some \( a_0 \in A \), \( x_0 \in L(a_0) \), and \( v_0 \in V(G) \), we have \( L(v_0) \cap N_H(x_0) = \emptyset \). Let \( A' := A \setminus \{a_0\} \), and for every \( w \in V(G) \cup A' \), let \( L'(w) := L(w) \setminus N_H(x_0) \). Note that for all \( a \in A' \), \( |L'(a)| \geq |A'| + k \), and for all \( v \in V(G) \), \( \sigma(G, A', L', H, v) \leq \sigma(v) \). Moreover, by the choice of \( x_0 \), \( |L'(v_0)| = |L(v_0)| \), which, due to Corollary 2.1.2, yields \( \sigma(G, A', L', H, v) \leq \sigma(v_0) - 1 \). This implies \( \sigma(G, A', L', H) \leq \sigma - 1 \), and thus

\[
\frac{3}{2} \sigma(G, A', L', H) \leq \frac{3(\sigma - 1)}{2} \leq |A| - \frac{3}{2} < |A'|.
\]

By the minimality of \(|A|\), the conclusion of Theorem 2.1 holds for \((k, G, A', L', H)\), i.e., the graph \( J(G, A') \) is \((L', H)\)-colorable. By the definition of \( L' \), for any \((L', H)\)-coloring \( I \) of \( J(G, A') \), \( I \cup \{x_0\} \) is an \((L, H)\)-coloring of \( J(G, A) \). This is a contradiction. \( \triangledown \)

Corollary 2.1.4. \( k \geq 2 \).

Proof. Let \( v \in V(G) \) and consider any \( a \in A \). Since, by Claim 2.1.3, each \( x \in L(a) \) has a neighbor in \( L(v) \), we have

\[
|L(v)| \geq |L(a)| \geq |A| + k.
\]

Using Corollary 2.1.2, we obtain

\[
0 \leq \deg_G(v) + |A| - |L(v)| \leq \deg_G(v) - k,
\]

i.e., \( \deg_G(v) \geq k \). Since \( V(G) \neq \emptyset \), \( k \geq 1 \), which implies \( \deg_G(v) \geq 1 \). But then \( \chi(G) \geq 2 \), as desired. \( \triangledown \)
Claim 2.1.5. \( H \) does not contain a walk of the form \( x_0 - y_0 - x_1 - y_1 - x_2 \), where

- \( x_0, x_1, x_2 \in \bigcup_{a \in A} L(a) \);
- \( y_0, y_1 \in \bigcup_{v \in V(G)} L(v) \);
- \( x_0 \neq x_1 \neq x_2 \) and \( y_0 \neq y_1 \) (but it is possible that \( x_0 = x_2 \));
- the set \( \{x_0, x_1, x_2\} \) is independent in \( H \).

Proof. Suppose that such a walk exists and let \( a_0, a_1, a_2 \in A \) and \( v_0, v_1 \in V(G) \) be such that \( x_0 \in L(a_0), y_0 \in L(v_0), x_1 \in L(a_1), y_1 \in L(v_1) \), and \( x_2 \in L(a_2) \). Let \( A' := A \setminus \{a_0, a_1, a_2\} \), and for every \( w \in V(G) \cup A' \), let \( L'(w) := L(w) \setminus N_H(x_0, x_1, x_2) \). Since \( \{x_0, x_1, x_2\} \) is an independent set, for all \( a \in A', |L'(a)| \geq |A'| + k \), while for all \( v \in V(G) \), \( \sigma(G, A', L', H, v) \leq \sigma(v) \). Moreover, since for each \( i \in \{0, 1\} \), the set \( \{x_0, x_1, x_2\} \) contains two distinct neighbors of \( y_i \), we have \( \sigma(G, A', L', H, v_i) \leq \sigma(v_i) - 1 \). Therefore, \( \sigma(G, A', L', H) \leq \sigma - 2 \), and thus

\[
\frac{3}{2} \sigma(G, A', L', H) \leq \frac{3(\sigma - 2)}{2} \leq |A| - 3 \leq |A'|.
\]

By the minimality of \( |A| \), the conclusion of Theorem 2.1 holds for \((k, G, A', L', H)\), i.e., the graph \( J(G, A') \) is \((L', H)\)-colorable. By the definition of \( L' \), for any \((L', H)\)-coloring \( I \) of \( J(G, A') \), \( I \cup \{x_0, x_1, x_2\} \) is an \((L, H)\)-coloring of \( J(G, A) \). This is a contradiction.

Due to Corollary 2.1.4, we can choose a pair of disjoint independent sets \( U_0, U_1 \subset V(G) \) such that \( \chi(G - U_0) = \chi(G - U_1) = k - 1 \). Choose arbitrary elements \( a_1 \in A \) and \( x_1 \in L(a_1) \). By Claim 2.1.3, for each \( u \in U_0 \cup U_1 \), there is a unique element \( y(u) \in L(u) \) adjacent to \( x_1 \) in \( H \) (the uniqueness of \( y(u) \)) follows from the definition of a cover). Let

\[
I_0 := \{y(u) : u \in U_0\} \quad \text{and} \quad I_1 := \{y(u) : u \in U_1\}.
\]

Since \( U_0 \) and \( U_1 \) are independent sets in \( G \), \( I_0 \) and \( I_1 \) are independent sets in \( H \).

Claim 2.1.6. There exists an element \( a_0 \in A \setminus \{a_1\} \) such that \( L(a_0) \cap N_H(I_0) \not\subseteq N_H(x_1) \).

Proof. Assume that for all \( a \in A \setminus \{a_1\} \), we have \( L(a) \cap N_H(I_0) \subseteq N_H(x_1) \). Let \( G' := G - U_0 \), and for each \( w \in V(G') \cup A \), let \( L'(w) := L(w) \setminus N_H(I_0) \). By the definition of \( I_0 \), \( L'(a_1) = L(a_1) \setminus \{x_1\} \), so

\[
|L'(a_1)| = |L(a_1)| - 1 \geq |A| + (k - 1).
\]

On the other hand, by our assumption, for each \( a \in A \setminus \{a_1\} \), we have

\[
|L'(a)| = |L(a) \setminus N_H(I_0)| \geq |L(a) \setminus N_H(x_1)| \geq |L(a)| - 1 \geq |A| + (k - 1).
\]

Since for all \( v \in V(G) \), \( \sigma(G', A, L', H, v) \leq \sigma(v) \), the minimality of \( k \) implies the conclusion of Theorem 2.1 for \((k - 1, G, A, L', H)\); in other words, the graph \( J(G', A) \) is \((L', H)\)-colorable. By the definition of \( L' \), for any \((L', H)\)-coloring \( I \) of \( J(G', A) \), \( I \cup I_0 \) is an \((L, H)\)-coloring of \( J(G, A) \); this is a contradiction.

Using Claim 2.1.6, fix some \( a_0 \in A \setminus \{a_1\} \) satisfying \( L(a_0) \cap N_H(I_0) \not\subseteq N_H(x_1) \), and choose any \( x_0 \in (L(a_0) \cap N_H(I_0)) \setminus N_H(x_1) \).

Since \( x_0 \in N_H(I_0) \), we can also choose \( y_0 \in I_0 \) so that \( x_0y_0 \in E(H) \).
Claim 2.1.7. $x_0 \not\in N_H(I_1)$.

Proof. If there is $y_1 \in I_1$ such that $x_0 y_1 \in E(H)$, then $x_0 - y_0 - x_1 - y_1 - x_0$ is a walk in $H$ whose existence is ruled out by Claim 2.1.5.

Claim 2.1.8. There is an element $a_2 \in A \setminus \{a_0, a_1\}$ such that $L(a_2) \cap N_H(I_1) \not\subseteq N_H(x_0, x_1)$.

Proof. The proof is almost identical to the proof of Claim 2.1.6. Assume that for all $a \in A \setminus \{a_0, a_1\}$, we have $L(a) \cap N_H(I_1) \subseteq N_H(x_0, x_1)$. Let $G' := G - U_1$, $A' := A \setminus \{a_0\}$, and for each $w \in V(G') \cup A'$, let $L'(w) := L(w) \setminus N_H(\{x_0 \cup I_1\})$. By the definition of $I_1$, $L(a_1) \cap N_H(I_1) = \{x_1\}$, so

$$|L'(a_1)| \geq |L(a_1)| - 2 \geq |A| + k - 2 = |A'| + (k - 1).$$

On the other hand, by our assumption, for each $a \in A \setminus \{a_0, a_1\}$, we have

$$|L'(a)| \geq |L(a) \setminus N_H(x_0, x_1)| \geq |L(a)| - 2 \geq |A| + k - 2 = |A'| + (k - 1).$$

Since for all $v \in V(G)$, $\sigma(G', A', L', H, v) \leq \sigma(v)$, the minimality of $k$ implies the conclusion of Theorem 2.1 for $(k - 1, G', A', L', H)$; in other words, the graph $J(G', A')$ is $(L, H)$-colorable. By the definition of $L'$, for any $(L', H)$-coloring $I$ of $J(G', A)$, $I \cup \{x_0\} \cup I_1$ is an $(L, H)$-coloring of $J(G, A)$. This is a contradiction.

Now we are ready to finish the proof of Theorem 2.1. Fix some $a_2 \in A \setminus \{a_0, a_1\}$ satisfying $L(a_2) \cap N_H(I_1) \not\subseteq N_H(x_0, x_1)$, and choose any

$$x_2 \in (L(a_2) \cap N_H(I_1)) \setminus N_H(x_0, x_1).$$

Since $x_2 \in N_H(I_1)$, there is $y_1 \in I_1$ such that $x_2 y_1 \in E(H)$. Then $x_0 - y_0 - x_1 - y_1 - x_2$ is a walk in $H$ contradicting the conclusion of Claim 2.1.5. \hfill \square

Now it is easy to derive Theorem 1.5. Indeed, let $G$ be a graph with $n$ vertices, $m$ edges, and chromatic number $k$, let $A$ be a finite set disjoint from $V(G)$, and let $(L, H)$ be a cover of $J(G, A)$ such that for all $v \in V(G)$ and $a \in A$, $|L(v)| = |L(a)| = \chi(J(G, A)) = |A| + k$. Note that

$$\frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) - |L(v)| + |A| + 1, 0\} = \frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) - k + 1, 0\}. $$

If $|A| \geq 3m$, then

$$\frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) - k + 1, 0\} \leq \frac{3}{2} \sum_{v \in V(G)} \deg_G(v) = 3m \leq |A|,$$

so Theorem 2.1 implies that $J(G, A)$ is $(L, H)$-colorable, and hence $Z_{DP}(G) \leq 3m$. Moreover, if $\delta(G) \geq k - 1$, then

$$\frac{3}{2} \sum_{v \in V(G)} \max\{\deg_G(v) - k + 1, 0\} = \frac{3}{2} \sum_{v \in V(G)} (\deg_G(v) - k + 1) = 3m - \frac{3}{2}(k - 1)n,$$

so $Z_{DP}(G) \leq 3m - \frac{3}{2}(k - 1)n$, as desired. Finally, Corollary 1.6 follows from Theorem 1.5 and the fact that an $n$-vertex graph can have at most $\binom{n}{2} \leq n^2/2$ edges.
3 Proof of Theorem 1.7

We will prove the following precise version of Theorem 1.7:

**Theorem 3.1.** For all even \( n \in \mathbb{N} \), \( Z_{DP}(n) \geq n^2/4 - n \).

**Proof.** Let \( n \in \mathbb{N} \) be even and let \( k := n/2 - 1 \). Note that \( n^2/4 - n = k^2 - 1 \). Thus, it is enough to exhibit an \( n \)-vertex bipartite graph \( G \) and a cover \((L, H)\) of \( J(G, k^2 - 2) \) such that \(|L(u)| = k^2\) for all \( u \in V(J(G, k^2 - 2)) \), yet \( J(G, k^2 - 2) \) is not \((L, H)\)-colorable.

Let \( G \cong K_{2n/2,k} \) be an \( n \)-vertex complete bipartite graph with parts \( X = \{x, x_0, \ldots, x_{k-1}\} \) and \( Y = \{y, y_0, \ldots, y_{k-1}\} \), where the indices \( 0, \ldots, k-1 \) are viewed as elements of the additive group \( \mathbb{Z}_k \) of integers modulo \( k \). Let \( A \) be a set of size \( k^2 - 2 \) disjoint from \( X \cup Y \). For each \( u \in X \cup Y \cup A \), let \( L(u) := \{u\} \times \mathbb{Z}_k \times \mathbb{Z}_k \). Let \( H \) be the graph with vertex set \((X \cup Y \cup A) \times \mathbb{Z}_k \times \mathbb{Z}_k \) in which the following pairs of vertices are adjacent:

- \((u, i, j)\) and \((u, i', j')\) for all \( u \in X \cup Y \cup A \) and \( i, j, i', j' \in \mathbb{Z}_k \) such that \((i, j) \neq (i', j')\);
- \((u, i, j)\) and \((v, i, j)\) for all \( u \in \{x, y\} \cup A \), \( v \in N_{J(G,A)}(u) \), and \( i, j \in \mathbb{Z}_k \);
- \((x_s, i, j)\) and \((y_s, i + s, j + t)\) for all \( s, t, i, j \in \mathbb{Z}_k \).

It is easy to see that \((L, H)\) is a cover of \( J(G, A) \). We claim that \( J(G, A) \) is not \((L, H)\)-colorable. Indeed, suppose that \( I \) is an \((L, H)\)-coloring of \( J(G, A) \). For each \( u \in X \cup Y \cup A \), let \( i(u) \) and \( j(u) \) be the unique elements of \( \mathbb{Z}_k \) such that \((u, i(u), j(u)) \in I \). By the construction of \( H \) and since \( I \) is an independent set, we have

\[ (i(u), j(u)) \neq (i(a), j(a)) \]

for all \( u \in X \cup Y \) and \( a \in A \). Since all the \( k^2 - 2 \) pairs \((i(a), j(a))\) for \( a \in A \) are pairwise distinct, \((i(u), j(u))\) can take at most \( 2 \) distinct values as \( u \) is ranging over \( X \cup Y \). One of those \( 2 \) values is \((i(y), j(y))\), and if \( u \in X \), then

\[ (i(u), j(u)) \neq (i(y), j(y)), \]

so the value of \((i(u), j(u))\) must be the same for all \( u \in X \); let us denote it by \((i, j)\). Similarly, the value of \((i(u), j(u))\) is the same for all \( u \in Y \), and we denote it by \((i', j')\).

It remains to notice that the vertices \((x_{r-i}, i, j)\) and \((y_{r-j}, i', j')\) are adjacent in \( H \), so \( I \) is not an independent set. \( \blacksquare \)

Now we can prove Corollary 1.8:

**Proof of Corollary 1.8.** First, suppose that \( G \) is an \( n \)-vertex graph with \( \chi(G) = r \) that maximizes the difference \( \chi_{DP}(G) - \chi(G) \). Adding edges to \( G \) if necessary, we may arrange \( G \) to be a complete \( r \)-partite graph. Assuming \( 2r > n \), at least \( 2r - n \) of the parts must be of size \( 1 \), i.e., \( G \) is of the form \( J(G', 2r - n) \) for some \( 2(n - r) \)-vertex graph \( G' \). By Corollary 1.6, we have \( \chi_{DP}(G) = \chi(G) \) as long as \( 2r - n \geq 6(n - r)^2 \), which holds for all \( r \geq n - (1/\sqrt{6} - o(1))\sqrt{n} \). This establishes the upper bound \( r(n) \leq n - \Omega(\sqrt{n}) \).

On the other hand, due to Theorem 1.7, for each \( n \), we can find a graph \( G \) with \( s \) vertices, where \( s \leq (2 + o(1))\sqrt{n} \), such that \( \chi_{DP}(J(G, n - s)) > \chi(J(G, n - s)) \). Since \( J(G, n - s) \) is an \( n \)-vertex graph, we get

\[ r(n) > \chi(J(G, n - s)) = \chi(G) + n - s \geq n - (2 + o(1))\sqrt{n} = n - O(\sqrt{n}). \] \( \blacksquare \)
Acknowledgements. The authors are grateful to the anonymous referees for their valuable comments and suggestions.

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