

**MULTIPARAMETER TWISTED WEYL ALGEBRAS**

VYACHESLAV FUTORNY AND JONAS T. HARTWIG

**Abstract.** We introduce a new family of twisted generalized Weyl algebras, called multiparameter twisted Weyl algebras, for which we parametrize all simple quotients of a certain kind. Both Jordan’s simple localization of the multiparameter quantized Weyl algebra and Hayashi’s $q$-analogue of the Weyl algebra are special cases of this construction. We classify all simple weight modules over any multiparameter twisted Weyl algebra. Extending results by Benkart and Ondrus, we also describe all Whittaker pairs up to isomorphism over a class of twisted generalized Weyl algebras which includes the multiparameter twisted Weyl algebras.

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1. Introduction

Let $R$ be an algebra, $\sigma_1, \ldots, \sigma_n$ commuting algebra automorphisms of $R$, $t_1, \ldots, t_n$ elements from the center of $R$, and $\mu_{ij}$ an $n \times n$ matrix of invertible scalars. To these data one associates a twisted generalized Weyl algebra $A_n(R, \sigma, t)$, an associative $\mathbb{Z}^n$-graded algebra (see Section 2.1 for definition). These algebras were introduced by Mazorchuk and Turowska in the context of quantum group covariant differential calculus. They are examples of twisted generalized Weyl algebras. Contrary to quantized Weyl algebras obtained by Pusz and Woronowicz in the context of quantum group contractions, Jordan found a certain natural simple localization of $A_n$, the multiparameter quantized Weyl algebra, namely the multiparameter twisted Weyl algebra $A_n(R, \sigma, t)$, an associative $\mathbb{Z}^n$-graded algebra, which is in general not simple, even for generic parameters. Jordan’s localization of $A_n$ was introduced by Benkart where $A_n$ were introduced. These algebras are a particular case of the algebras of our class.

Multiparameter quantized Weyl algebras $A_n^\Lambda$ were introduced in as a generalization of the quantized Weyl algebras obtained by Pusz and Woronowicz in the context of quantum group covariant differential calculus. They are examples of twisted generalized Weyl algebras. Contrary to the usual Weyl algebras the algebra, $A_n^\Lambda$ is in general not simple, even for generic parameters.

The first main theorem of the paper parametrizes simple quotients of multiparameter twisted Weyl algebras in terms of maximal ideals of certain Laurent polynomial rings. Jordan’s localization of $A_n^\Lambda$ is an example in this family, as well as Hayashi’s $q$-deformed Weyl algebra.

Theorem A. Let $A = A_n^k(r, s, \Lambda)$ be a multiparameter twisted Weyl algebra.

(a) The assignment

$$n \mapsto A/(n)$$

where $\langle n \rangle$ denotes the ideal in $A$ generated by $n$, is a bijection between the set of maximal ideals in the invariant subring $R^\mathbb{Z}^n$ and the set of simple quotients of $A$ in which all $X_i, Y_i (i = 1, \ldots, n)$ are regular.

(b) For any $n \in \text{Specm}(R^\mathbb{Z}^n)$, the quotient $A/(\langle n \rangle)$ is isomorphic to the twisted generalized Weyl algebra $A_n(R/\mathfrak{m}, \sigma, \hat{t})$, where $\sigma_g(r + \mathfrak{m}) = \sigma_g(r) + \mathfrak{m}$, $\forall g \in \mathbb{Z}^n, r \in R$ and $\hat{t} = t_i + \mathfrak{m}, \forall i$.

(c) $A/\langle n \rangle$ is a domain for all $n \in \text{Specm}(R^\mathbb{Z}^n)$ if and only if $\mathbb{Z}^n/G$ is torsion-free, where $G$ is the gradation group of $R^\mathbb{Z}^n$.

The second main theorem of the paper gives the explicit relation between four twisted generalized Weyl algebras, namely the multiparameter quantized Weyl algebra $A_n^\Lambda$, Jordan’s localization $B_n^{\hat{f}, \Lambda}$, a specific multiparameter twisted Weyl algebra $A_n^k(r, s, \Lambda)$ that we define, and a certain quotient $A_n^\mu(r, s, \Lambda)$ of it which is simple and isomorphic to $B_n^{\hat{f}, \Lambda}$.

Theorem B. We have a commutative diagram in the category of $\mathbb{Z}^n$-graded algebras:

$$\begin{array}{ccc}
A_n^k(r, s, \Lambda) & \cong & B_n^{\hat{f}, \Lambda} \\
\downarrow & & \downarrow \\
A_n^\mu(r, s, \Lambda) & \cong & A_n^\Lambda
\end{array}$$

We end the introduction with an overview of the content of this paper. In Sections 3 and 4 we first consider certain families of twisted generalized Weyl algebras. Section 5 is devoted to
the definition and structural results for multiparameter twisted Weyl algebras, with a proof of Theorem B in Section 5.3. Examples and relations to existing algebras are given in Sections 6 and 7 where Theorem C is proved. Representations of multiparameter twisted Weyl algebras are studied in Sections 8 and 9.

Acknowledgements. This work was carried out during the second author’s postdoc at IME-USP, funded by FAPESP, processo 2008/10688-1. The first author is supported in part by the CNPq grant (301743/2007-0) and by the Fapesp grant (2005/60337-2).

Notation and conventions. By “ring” (“algebra”) we mean unital associative ring (algebra). All ring and algebra morphisms are required to be unital. By “ideal” we mean two-sided ideal unless otherwise stated. An element x of a ring R is said to be regular in R if for all nonzero y ∈ R we have xy ̸= 0 and yx ̸= 0. The set of invertible elements in a ring R will be denoted by R×.

Let R be a ring. Recall that an R-ring is a ring A together with a ring morphism R → A. Let X be a set. Let RXR be the free R-bimodule on X. The free R-ring F_R(X) on X is defined as the tensor algebra of the free R-bimodule on X: F_R(X) = ⊕_{n≥0}(RXR)^{⊗n} where (RXR)^{⊗0} = R by convention and the ring morphism R → F_R(X) is the inclusion into the degree zero component.

2. Twisted generalized Weyl algebras

Throughout this section we fix a commutative ring k.

2.1. Definition. We recall the definition of twisted generalized Weyl algebras [17, 16]. Here we emphasize the initial data more than usual, which will be useful in the next section to express the functoriality of the construction.

Definition 2.1 (TGW datum). Let n a positive integer. A twisted generalized Weyl datum (over k of degree n) is a triple (R, σ, t) where

- R is a unital associative k-algebra,
- σ is a group homomorphism σ : Zn → Autk(R), g → σg,
- t is a function t : {1, ..., n} → Z(R), i → t_i.

A morphism between TGW data over k of degree n,

φ : (R, σ, t) → (R′, σ′, t′)

is a k-algebra morphism φ : R → R′ such that φσ_i = σ′_iφ and φ(t_i) = t′_i for all i ∈ {1, ..., n}.

We let TGW_n(k) denote the category whose objects are the TGW data over k of degree n and morphisms are as above.

For i ∈ {1, ..., n} we put σ_i = σ_e_i, where {e_i}_{i=1}^n is the standard Z-basis for Z^n. A parameter matrix (over k× of size n) is an n × n matrix μ = (μ_ij)_{i,j} without diagonal where μ_ij ∈ k× ∀i ̸= j. The set of all parameter matrices over k× of size n will be denoted by PM_n(k).

Definition 2.2 (TGW construction). Let n ∈ Z_{>0}, (R, σ, t) be an object in TGW_n(k), and μ ∈ PM_n(k). The twisted generalized Weyl construction with parameter matrix μ associated to the TGW datum (R, σ, t) is denoted by C_μ(R, σ, t) and is defined as the free R-ring on the set \{x_i, y_i | i = 1, ..., n\} modulo two-sided ideal generated by the following set of elements:

(2.1a) \( x_i r - σ_i(r) x_i, \quad y_i r - σ_i^{-1}(r) y_i, \quad \forall r ∈ R, i ∈ \{1, ..., n\} \)

(2.1b) \( y_i x_i - t_i, \quad x_i y_i - σ_i(t_i), \quad \forall i ∈ \{1, ..., n\} \)

(2.1c) \( x_i y_j - μ_{ij} y_j x_i, \quad \forall i, j ∈ \{1, ..., n\}, i ̸= j \)

The images in C_μ(R, σ, t) of the elements x_i, y_i will be denoted by \( \hat{X}_i, \hat{Y}_i \) respectively. The ring C_μ(R, σ, t) has a Z^n-gradation given by requiring deg \( \hat{X}_i = e_i, \) deg \( \hat{Y}_i = -e_i, \) deg r = 0 ∀r ∈ R. Let \( I_μ(R, σ, t) \subseteq C_μ(R, σ, t) \) be the sum of all graded ideals J ⊆ C_μ(R, σ, t) having zero intersection with the degree zero component, i.e. such that \( C_μ(R, σ, t)_0 \cap J = \{0\} \). It is easy to see that \( I_μ(R, σ, t) \) is the unique maximal graded ideal having zero intersection with the degree zero component.
Definition 2.3 (TGW algebra). The twisted generalized Weyl algebra with parameter matrix $\mu$ associated to the TGW datum $(R, \sigma, t)$ is denoted $A_\mu(R, \sigma, t)$ and is defined as the quotient $A_\mu(R, \sigma, t) := C_\mu(R, \sigma, t)/I_\mu(R, \sigma, t)$.

Since $I_\mu(R, \sigma, t)$ is graded, $A_\mu(R, \sigma, t)$ inherits a $\mathbb{Z}^n$-gradation from $C_\mu(R, \sigma, t)$. The images in $A_\mu(R, \sigma, t)$ of the elements $\hat{X}_i, \hat{Y}_i$ will be denoted by $X_i, Y_i$. By a monic monomial in a TGW construction $C_\mu(R, \sigma, t)$ (respectively TGW algebra $A_\mu(R, \sigma, t)$) we will mean a product of elements from $\{\hat{X}_i, \hat{Y}_i \mid i = 1, \ldots, n\}$ (respectively $\{X_i, Y_i \mid i = 1, \ldots, n\}$).

The following statements are easy to check.

Lemma 2.4. (a) $A_\mu(R, \sigma, t)$ (respectively $C_\mu(R, \sigma, t)$) is generated as a left and as a right $R$-module by the monic monomials in $X_i, Y_i$ $(i = 1, \ldots, n)$ (respectively $\hat{X}_i, \hat{Y}_i$ $(i = 1, \ldots, n)$).
(b) The degree zero component of $A_\mu(R, \sigma, t)$ is equal to the image of $R$ under the natural map $\rho: R \to A_\mu(R, \sigma, t)$.
(c) Any nonzero graded ideal of $A_\mu(R, \sigma, t)$ has nonzero intersection with the degree zero component.

Definition 2.5 ($\mu$-Consistency). Let $(R, \sigma, t)$ be a TGW datum over $k$ of degree $n$ and $\mu$ be a parameter matrix over $k^\times$ of size $n$. We say that $(R, \sigma, t)$ is $\mu$-consistent if the canonical map $\rho: R \to A_\mu(R, \sigma, t)$ is injective.

Since $I_\mu(R, \sigma, t)$ has zero intersection with the zero-component, $(R, \sigma, t)$ is $\mu$-consistent iff the canonical map $R \to C_\mu(R, \sigma, t)$ is injective. Even in the cases when $\rho$ is not injective, we will often view $A_\mu(R, \sigma, t)$ as a left $R$-module and write for example $rX_i$ instead of $\rho(r)X_i$.

Definition 2.6 (Regularity). A TGW datum $(R, \sigma, t)$ is called regular if $t_i$ is regular in $R$ for all $i$.

The following result was proved in [8, Theorem 6.2].

Theorem 2.7. Let $k$ be a commutative unital ring, $R$ be an associative $k$-algebra, $n$ a positive integer, $t = (t_1, \ldots, t_n)$ be an $n$-tuple of regular central elements of $R$, $\sigma: \mathbb{Z}^n \to \text{Aut}_k(R)$ a group homomorphism, $\mu_{ij}$ $(i, j = 1, \ldots, n, i \neq j)$ invertible elements from $k$, and $A_\mu(R, \sigma, t)$ the corresponding twisted generalized Weyl algebra, equipped with the canonical homomorphism of $R$-rings $\rho: R \to A_\mu(R, \sigma, t)$. Then the following two statements are equivalent:

(a) $\rho$ is injective,
(b) the following two sets of relations are satisfied in $R$:

\begin{align*}
(2.2) \quad & \sigma_i \sigma_j (t_i t_j) = \mu_{ij} \mu_{ji} \sigma_i (t_i) \sigma_j (t_j), \quad \forall i, j = 1, \ldots, n, i \neq j, \\
(2.3) \quad & t_j \sigma_i \sigma_k (t_j) = \sigma_i (t_j) \sigma_k (t_j), \quad \forall i, j, k = 1, \ldots, n, i \neq j \neq k \neq i.
\end{align*}

In particular, if (2.2) and (2.3) are satisfied, then $A_\mu(R, \sigma, t)$ is nontrivial iff $R$ is nontrivial. Moreover, neither of the two conditions (2.2) and (2.3) imply the other.

Lemma 2.8. If $t_i \in R^\times$ for all $i$, then the canonical projection $C_\mu(R, \sigma, t) \to A_\mu(R, \sigma, t)$ is an isomorphism.

Proof. The algebra $C_\mu(R, \sigma, t)$ is a $\mathbb{Z}^n$-crossed product algebra over its degree zero subalgebra, since each homogenous component contains an invertible element. Indeed since $t_i \in R^\times$, each $X_i$ is invertible and thus $X_i^{q_1} \cdots X_i^{q_n}$ has degree $q$ and is invertible. Therefore any nonzero graded ideal in $C_\mu(R, \sigma, t)$ has nonzero intersection with the degree zero component, a property which holds for any strongly graded ring, in particular for crossed product algebras. Thus $I_\mu(R, \sigma, t) = 0$, which proves the claim. \qed

2.2. TGW algebras which are domains. The following condition for a TGW algebra to be a domain will be used.

Proposition 2.9. Let $A = A_\mu(R, \sigma, t)$ be a twisted generalized Weyl algebra where $(R, \sigma, t)$ is $\mu$-consistent. Then $A$ is a domain if and only if $R$ is a domain.
Proof. Clearly $R$ must be a domain if $A$ is a domain. For the converse, assume $R$ is a domain and suppose $a, b \in A$ are nonzero but $ab = 0$. Let $a_k$ and $b_k$ be the leading terms in $a$ and $b$ respectively, with respect to some order (we use here that the group $\mathbb{Z}^n$ is orderable). Then $a_k b_k = 0$. As in the proof of [11, Proposition 3.1] this forces $a_g = 0$ or $b_h = 0$ which is a contradiction. □

3. Finitistic TGW algebras

Throughout the rest of the paper we assume that $k$ is a field.

In [10] the following notion (there called ”locally finite” TGW algebra) was defined.

Definition 3.1. A TGW algebra $A = \mathcal{A}_\mu(R, \sigma, t)$ is called $k$-finitistic if $\dim_k V_{ij} < \infty$ for all $i, j \in \{1, \ldots, n\}$, where

$$V_{ij} = \text{Span}_k \{\sigma^k_i(t_j) \mid k \in \mathbb{Z}\}.$$  

(3.1)

For each $i, j$, we denote by $p_{ij} \in k[x]$ be the minimal polynomial for $\sigma_i$ acting on the finite-dimensional space $V_{ij}$. The following result was proved in [10] (for the case $\mu_{ij} = \mu_{ji}$ and $R$ a commutative domain, but these restrictions are unnecessary).

Theorem 3.2. Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a $k$-finitistic TGW algebra where $(R, \sigma, t)$ is $\mu$-consistent. Let $p_{ij}$ be the minimal polynomials defined above.

(a) Define the matrix $C_A = (a_{ij})$ with integer entries as follows:

$$a_{ij} = \begin{cases} 2, & i = j, \\ 1 - \deg p_{ij}, & i \neq j. \end{cases}$$  

(3.2)

Then $C_A$ is a generalized Cartan matrix.

(b) Assume $t_i$ is regular in $R$ for all $i$. Writing

$$p_{ij}(x) = x^{m_{ij}} + \lambda_{ij}^{(1)} x^{m_{ij} - 1} + \cdots + \lambda_{ij}^{(m_{ij})},$$

where all $\lambda_{ij}^{(k)} \in k$, the following identities hold in $A$, for any $i \neq j$:

$$X_i^{m_{ij}} X_j + \lambda_{ij}^{(1)} \mu_{ij} x^{m_{ij} - 1} X_j X_i + \cdots + \lambda_{ij}^{(m_{ij})} \mu_{ij}^{-m_{ij}} X_j X_i^{m_{ij}} = 0$$

and

$$Y_j Y_i^{m_{ij}} + \lambda_{ij}^{(1)} \mu_{ji}^{-1} Y_j Y_i^{m_{ij} - 1} + \cdots + \lambda_{ij}^{(m_{ij})} \mu_{ji}^{-m_{ij}} Y_j Y_i^{m_{ij}} Y_j = 0.$$  

(3.3) 

(3.4)

Moreover, for any $i \neq j$ and $m < m_{ij}$, the sets $\{X_i^{m-k} Y_j X_i^k\}_{k=0}^m$ and $\{Y_i^{m-k} Y_j X_i^k\}_{k=0}^m$ are linearly independent in $A$ over $k$.

This gives an interpretation of the minimal polynomials $p_{ij}$ for $i \neq j$ in terms of identities in the algebra $A$. If $C_A$ is of type $Z$ (for example $A_n, B_n, C_n, D_n, E_6, E_7, E_8$ etc) we say that $A$ is of Lie type $Z$.

Here we note that the polynomials $p_{ii}$ also give rise to identities in $A$.

Theorem 3.3. Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a $k$-finitistic TGW algebra, where $(R, \sigma, t)$ is regular and $\mu$-consistent. Let $p_{ii} \in k[x]$ be the minimal polynomial for $\sigma_i$ acting on the finite-dimensional spaces $V_{ii}$ defined in (3.1). Writing

$$p_{ii}(x) = x^{m_{ii}} + \lambda_{ii}^{(1)} x^{m_{ii} - 1} + \cdots + \lambda_{ii}^{(m_{ii})},$$

where all $\lambda_{ii}^{(k)} \in k$, the following identities hold in $A$, for any $i$:

$$X_i^{m_{ii}} Y_i + \lambda_{ii}^{(1)} X_i^{m_{ii} - 1} Y_i X_i + \cdots + \lambda_{ii}^{(m_{ii})} Y_i X_i^{m_{ii}} = 0$$

and

$$X_i Y_i^{m_{ii}} + \lambda_{ii}^{(1)} Y_i X_i Y_i^{m_{ii} - 1} + \cdots + \lambda_{ii}^{(m_{ii})} Y_i X_i^{m_{ii}} Y_i = 0.$$  

(3.5) 

(3.6)

Moreover, for any $i$ and $m < m_{ii}$, the sets $\{X_i^{m-k} Y_i X_i^k\}_{k=0}^m$ and $\{Y_i^{m-k} Y_i X_i^k\}_{k=0}^m$ are linearly independent in $A$ over $k$.

Proof. The proof is similar to the proof of Theorem 3.2. □
Example 3.4. Let \( R = k[t_1, \ldots, t_n] \) be the polynomial algebra, \( \sigma_i(t_j) = t_j - \delta_{ij}, \mu_{ij} \in k \setminus \{0\} \) such that \( \mu_{ij} \mu_{ji} = 1 \). Let \( A = A_\mu(R, \sigma, t) \) be the associated TGW algebra. It is easy to see that it is \( k \)-finite. For \( i = j \), the minimal polynomials are \( p_{ii}(x) = (x-1)^2 \). For \( i \neq j \) we have \( p_{ij}(x) = x - 1 \). The matrix \( C_A \), defined in (3.22), is the Cartan matrix of type \((A_1)^n = A_1 \times \cdots \times A_1\) (just a diagonal matrix with 2 on the diagonal). Thus \( A \) is of Lie type \((A_1)^n\). By (3.3) we have \( X_i X_j = \mu_{ij}^{-1} X_j X_i \) for \( i \neq j \). If all \( \mu_{ij} = 1 \) then \( A \) is isomorphic to the \( n \)-th Weyl algebra.

Example 3.5. The following TGW algebra was first mentioned as an example in [17], but a complete presentation by generators and relations was only given in [10]. Let \( n = 2, R = k[H], \sigma_1(H) = H + 1, \sigma_2(H) = H - 1, t_1 = H, t_2 = H + 1, \mu_{12} = \mu_{21} = 1 \) and let \( A = A_\mu(R, \sigma, t) \) be the associated TGW algebra. Clearly \( A \) is locally finite with \( V_{ij} = \mathbb{C}H \oplus \mathbb{C}1 \) for \( i, j = 1, 2 \). Observing that \( \sigma_2(t_1) \) and \( t_1 \) are linearly independent and that

\[
\sigma_2^2(t_1) - 2\sigma_2(t_1) + t_1 = H - 2 - 2(H - 1) + H = 0
\]

we see that the minimal polynomial \( p_{23} \) for \( \sigma_2 \) acting on \( V_{23} \) is given by \( p_{23}(x) = x^2 - 2x + 1 = (x - 1)^2 \). Similarly one checks that in fact \( p_{ij}(x) = (x - 1)^2 \) for all \( i, j = 1, 2 \). Thus \( C_A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \), the Cartan matrix of type \( A_2 \), so \( A \) is of Lie type \( A_2 \). By Theorem (3.2), we have for example \( X_1^2 X_2 - 2X_1 X_2 X_2 + X_2^2 = 0 \) in \( A \), which is precisely one of the Serre relations in the enveloping algebra of \( sl_2(k) \), the simple Lie algebra of type \( A_2 \). It was shown in [10] Example 6.3) that in fact \( A \) is isomorphic to the \( k \)-algebra with generators \( X_1, X_2, Y_1, Y_2, H \) and defining relations

\[
\begin{align*}
X_1 H &= (H + 1)X_1, & X_2 H &= (H - 1)X_2, & X_1^2 X_2 &- 2X_1 X_2 X_1 + X_2^2 = 0, \\
Y_1 H &= (H - 1)Y_1, & Y_2 H &= (H + 1)Y_2, & X_2^2 X_1 &- 2X_2 X_1 X_2 + X_1^2 = 0, \\
Y_1 X_1 &= X_2 Y_2 = H, & Y_2 X_2 &= X_1 Y_1 = H + 1, & Y_1^2 Y_2 &- 2Y_1 Y_2 Y_1 + Y_2^2 = 0, \\
Y_2^2 Y_1 &= Y_2^2 Y_2 + Y_1 Y_2^2 = 0.
\end{align*}
\]

In [10], this TGW algebra was also generalized to arbitrary symmetric generalized Cartan matrices, although explicit presentation was only given in type \( A_2 \).

4. TGW Algebras of Lie type \((A_1)^n\)

4.1. Presentation by generators and relations. Let \( A = A_\mu(R, \sigma, t) \) be a \( k \)-finite TGW algebra of Lie type \((A_1)^n = A_1 \times \cdots \times A_1\), with \( (R, \sigma, t) \) being \( \mu \)-consistent. Thus \( C_A \) has all zeros outside the main diagonal. That is, deg \( p_{ij} \) = 1 for all \( i \neq j \). Equivalently (since \( p_{ij} \) are monic by definition), for \( i \neq j \) we have \( p_{ij}(x) = x - \gamma_{ij} \) for some \( \gamma_{ij} \in k \setminus \{0\} \). By Theorem (3.2) this means that in \( A \) we have

\[
\begin{align*}
(4.1a) & \quad X_i X_j = \gamma_{ij} X_j^{-1} X_j X_i & \forall i \neq j, \\
(4.1b) & \quad Y_i Y_j = \gamma_{ij} Y_j^{-1} Y_i Y_j & \forall i \neq j.
\end{align*}
\]

It also means that

\[
(4.2) \quad \sigma_i(t_j) = \gamma_{ij} t_j \quad \text{for all } i \neq j.
\]

By Theorem (3.2) \((R, \sigma, t)\) is \( \mu \)-consistent if and only if

\[
(4.3) \quad \mu_{ij} \mu_{ji} = \gamma_{ij} \gamma_{ji} \quad \forall i \neq j.
\]

We can now prove that (4.1) generate all relations in the ideal \( I_\mu(R, \sigma, t) \).

Theorem 4.1. Let \( A = A_\mu(R, \sigma, t) \) be a \( k \)-finite TGW algebra of type \((A_1)^n\), where \((R, \sigma, t)\) is regular and \( \mu \)-consistent (i.e. (1.3) holds). Then \( A \) is isomorphic to the \( R \)-ring generated by \( X_1, \ldots, X_n, Y_1, \ldots, Y_n \) modulo the relations

\[
\begin{align*}
(4.4) & \quad X_i r = \sigma_i(r) X_i, \quad Y_i r = \sigma_i^{-1}(r) Y_i \quad \forall r \in R, \forall i, \\
(4.5) & \quad X_i X_j = \mu_{ij} Y_j X_i, \quad X_i Y_j = \sigma_i(t_j) Y_j X_i, \quad Y_j Y_i = \gamma_{ij} \mu_{ij}^{-1} Y_i Y_j, \quad \forall i \neq j.
\end{align*}
\]
Definition. 5.1. Let $R$ be the Laurent polynomial ring over $k$.

Definition 5.2. Theorem 4.2. The following statements are proved.

Part (b) is proved in [11, Th. 7.18] in the more general context of so called $R$-finitistic TGW algebras. The result is a generalization of D. Jordan’s simplicity criterion for generalized Weyl algebras [13 Theorem 6.1].

5. Multiparameter twisted Weyl algebras

Now we define a special class of twisted generalized Weyl algebras. The definition of these algebras was inspired by a class of multiparameter Weyl algebras introduced by Benkart [4].

5.1. Definition. Let $n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}\setminus\{0\}$ and let $\Lambda = (\lambda_{ij})$, $r = (r_{ij})$ and $s = (s_{ij})$ be three $n \times n$-matrix with entries from $k\setminus\{0\}$, such that

\begin{align}
\lambda_{ii} &= 1 \forall i \text{ and } \lambda_{ij} = 1 \forall i \neq j, \\
\frac{r_{ii}}{s_{ii}} &= \text{a nonroot of unity } \forall i, \\
\frac{r_{ij}^k}{s_{ij}^k} &= \text{ a nonroot of unity } \forall i \neq j.
\end{align}

Let

\begin{equation}
R = k[u_1^{\pm 1}, \ldots, u_n^{\pm 1}, v_1^{\pm 1}, \ldots, v_n^{\pm 1}]
\end{equation}

be the Laurent polynomial ring over $k$ in $2n$ indeterminates, define $\sigma_1, \ldots, \sigma_n \in Aut_k(R)$ by

\begin{equation}
\sigma_i(u_j) = r_{ij}^{-1}u_j, \quad \sigma_i(v_j) = s_{ij}^{-1}v_j,
\end{equation}

for all $i, j \in \{1, \ldots, n\}$, and define $t_1, \ldots, t_n \in R$ by

\begin{equation}
t_i = \frac{(r_{ii}u_i)^k - (s_{ii}v_i)^k}{r_{ii}^k - s_{ii}^k}.
\end{equation}

Finally, put

\begin{equation}
\mu_{ij} = r_{ji}^{-k} \lambda_{ji}.
\end{equation}
for all \( i \neq j \). Then one easily checks that the consistency relations (2.2), (3.2) hold. Thus, by Theorem 2.7, the TGW datum \((R, \sigma, t)\) is \(\mu\)-consistent, that is, the natural map \(\rho : R \to A_n(R, \sigma, t)\) is injective. We denote the TGW algebra \(A_n(R, \sigma, t)\) by \(A_n^k(r, s, \Lambda)\) and call it a **multiparameter twisted Weyl algebra**. It is easy to see that it is \(k\)-finite of Lie type \((A_1)^n\) and thus, by Theorem 4.1, \(A_n^k(r, s, \Lambda)\) is isomorphic to the unital associative \(k\)-algebra generated by \(u_i^{\pm 1}, v_i^{\pm 1}, X_i, Y_i (i = 1, \ldots, n)\) modulo the relations

(5.6a) \quad \text{the } u_i^{\pm 1}, v_j^{\pm 1} \text{ all commute and } u_i u_j^{-1} = v_i v_j^{-1} = 1 \forall i, j,

(5.6b) \quad X_i X_j = \left(\frac{r_{ij}}{r_{ij}}\right)^k \lambda_{ij} X_j X_i \quad \forall i, j,

(5.6c) \quad Y_i Y_j = \lambda_{ij} Y_j Y_i \quad \forall i, j,

(5.6d) \quad X_i Y_j = r_{ij}^{-k} \lambda_{ij} Y_j X_i \quad \forall i \neq j,

(5.6e) \quad Y_i X_j = \frac{(r_{ij} u_i)^k - (s_{ij} v_i)^k}{r_{ij}^k - s_{ij}^k} \quad \forall i, j,

(5.6f) \quad X_i u_j = r_{ij}^{-1} u_j X_i, \quad X_i v_j = s_{ij}^{-1} v_j X_i, \quad Y_i u_j = r_{ij} u_j Y_i, \quad Y_i v_j = s_{ij} v_j Y_i \quad \forall i, j.

**Remark 5.1.** One can also consider the larger class of algebras in which \(5.1b\) in the definition of \(A_n^k(r, s, \Lambda)\) is replaced by the weaker condition that \(r_{ii}^k \neq s_{ii}^k\) for all \( i \). However in this paper we will always assume \(5.1b\), which in examples corresponds to that “\( q \) is not a root of unity”.

5.2. **Properties of multiparameter twisted Weyl algebras.** Let \(R^Z = \{ r \in R \mid \sigma(r) = r \forall i = 1, \ldots, n \}\) be the invariant subring of \( R \) under \( Z^n \). For \( d \in Z^n \), put \( u^d = u_1^{d_1} \cdots u_n^{d_n} v_1^{d_{n+1}} \cdots v_n^{d_{2n}} \). Let

(5.7) \quad G = \{ d \in Z^{2n} \mid u^d \in R^Z \}.

We have \(R^Z = \bigoplus_{d \in G} ku^d\).

**Proposition 5.2.** (a) If \( J \) is a proper \( Z^n \)-invariant ideal of \( R \), then the group homomorphism

\[ Z^n \to \text{Aut}_k(R/J), \] induced by the \( Z^n \)-action on \( R \), is injective.

(b) If \( J \) is a proper \( Z^n \)-invariant ideal of \( R \), such that \( R/J \) is \( Z^n \)-simple or a domain, then \( R/J \) is maximal commutative in \( A := A_n(R/J, \sigma, t) \).

**Proof.** (a) Assume \( \gamma = (g_1, \ldots, g_n) \in Z^n \) is such that \( \sigma_\gamma(p + J) = p + J \) for all \( p + J \in R/J \). Suppose that \( g_i \neq 0 \) for some \( i \). Then, taking \( p = u_i \) we have \( u_i + J = \sigma_{kg}(u_i) + J = r_{kg}^k \cdots r_{ni}^k u_i + J \), giving \( (r_{ki}^k \cdots r_{ni}^k - 1) u_i \in J \). Since \( J \) is proper and \( u_i \) invertible we must have \( r_{ki}^k \cdots r_{ni}^k = 1 \). Similarly taking \( p = v_i \) gives that \( s_{ki} \cdots s_{ni} = 1 \). But \( r_{ij}^k = s_{ij}^k \) for \( i \neq j \) and thus we get \( r_{ki}^k \cdots r_{ni}^k = 1 \), contradicting the fact that \( r_{ij}/s_{ij} \) is not a root of unity. Thus \( g_i = 0 \) for all \( i \).

(b) Follows from part (a) and [11, Corollary 5.2].

**Proposition 5.3.** Any ideal of \( A \) is graded.

**Proof.** Let \( J \) be any ideal in \( A \) and let \( a \in J \). Write \( a = \sum_{g \in Z^n} a_g \), where \( a_g \in A_g \) for each \( g \). Pick any \( h \in Z^n \). We will show that \( a_h \in J \). By Proposition 5.2(a), the group morphism \( Z^n \to \text{Aut}_k(R) \) is injective. So if \( g \in Z^n \), \( g \neq h \), then there is a \( d \in Z^{2n} \) such that \( \sigma_\gamma(u^d) \neq \sigma_\gamma(u^d) \). By definition of the automorphisms \( \sigma_\gamma \) we have \( \sigma_\gamma(u^d) = \xi_d u^d \) and \( \sigma_\gamma(u^d) = \xi_d u^d \) for some nonzero \( \xi_d, \xi_d \in k \). Put \( b = \xi_d a - u^{-d} a u^d \). Then \( b \in J \) and writing \( b = \sum_{f \in Z^n} b_f \) where \( b_f \in A_f \) we have \( b_g = \xi_g a_g - u^{-d} a_g u^d = (\xi_g - u^{-d} \sigma(g(u^d))) a_g = 0 \), and \( b_h = \xi_g a_h - u^{-d} a_h u^d = (\xi_g - \xi_h) a_h \). So, replacing \( a \) by \( (\xi_g - \xi_h)^{-1} b \), we have an element in \( J \) with the same degree \( h \) component but with the \( g \) component eliminated. Repeating this we can eliminate all components except \( a_h \) and thus we obtain that \( a_h \in J \).

**Proposition 5.4.** Let \( \mathcal{I}(R^{Z^n}) \) denote the set of ideals of \( R^{Z^n} \) and \( \mathcal{I}(R)^{Z^n} \) denote the set of \( Z^n \)-invariant ideals of \( R \). Consider the maps

\[ \varepsilon : \mathcal{I}(R^{Z^n}) \to \mathcal{I}(R)^{Z^n}, \quad n \mapsto Rn, \]

\[ \rho : \mathcal{I}(R)^{Z^n} \to \mathcal{I}(R^{Z^n}), \quad J \mapsto R^n \cap J. \]
Lemma 5.7. Assume $R/J$ is regular in a nonzero scalar hence the $ξ$ and some $u$ and $i / 0$.

In particular $R$ is semisimple as a module over $k[Z^n]$. Using the strong gradation property we have $J=\oplus_{\chi \in k[Z^n]} R[\chi]$, where $J=\bigoplus_{\chi \in k[Z^n]} R[\chi] \cap J$. Using the strong gradation property we have $J=\bigoplus_{\chi \in k[Z^n]} R[\chi] \cap J$. This proves that $\rho \in$ is the identity. Let $\mathfrak{n}$ be an ideal of $R/Z$. Then $R=\bigoplus_{\chi \in k[Z^n]} R[\chi] \cap J$. Thus $J=\rho$. This proves the claim.

Lemma 5.5. $Rt_i + R\sigma_i^d(t_i) = R$ for all $i = 1, \ldots, n$ and all $d \in Z_{>0}$.

Proof. We have

$$r_{ii}^{-dk} t_i = \sigma_i^d(t_i) \quad \forall i = 1, \ldots, n$$

which is invertible since $r_{ii}/s_{ii}$ is assumed to not be a root of 1 and since $v_i$ is invertible. This proves the claim.

Lemma 5.6. No product of elements of the form $\sigma_i(g)$ ($g \in Z^n \ set i = 1, \ldots, n$) can belong to a $Z^n$-invariant proper ideal of $R$.

Proof. Indeed, such a product can be written $a = \xi \sigma_i^n(t_1) \cdots \sigma_i^n(t_n)$ for some nonzero $\xi \in k$ and some $p_i \in Z$. But then the proper $Z^n$-invariant ideal $L$ containing such element would also contain $\sigma_1(a)$. By Lemma 5.5 $Rt_i + R\sigma_i(t_i) = R$. So for suitable $r_1, r_2 \in R$, $r_1 a + r_2 \sigma_1(a) = \xi \sigma_2^n(t_2) \cdots \sigma_2^n(t_n)$ for some nonzero $\xi \in k$. Continuing this way we would obtain that $L$ contains a nonzero scalar hence the $L = R$ contradicting that $L$ was proper.

Lemma 5.7. Assume $J$ is a maximal $Z^n$-invariant ideal of $R$. Then all $t_i + J$ ($i = 1, \ldots, n$) are regular in $R/J$.

Proof. Let $T$ denote the multiplicative submonoid of $R/J$ generated by $\sigma_i(t_i + J)$ for all $g \in Z^n$ and $i = 1, \ldots, n$. By Lemma 5.6 $0 \notin T$. Observe that the set $L = \{ \bar{t} \in R/J \mid u \bar{t} = 0 \text{ for some } u \in T\}$ is a $Z^n$-invariant ideal in $R/J$. But $L = R/J$ is impossible since the ring $R/J$ is unital and $0 \notin T$. Therefore, since $R/J$ is $Z^n$-simple, we have $L = 0$ which proves that in particular all $t_i + J$ ($i = 1, \ldots, n$) are regular in $R/J$. 

□
5.3. Simple quotients. We come now to the main result on the structure theory of multiparameter twisted Weyl algebras. The following theorem (which is Theorem B from the Introduction) describes all quotients $A/Q$ of $A = A_k^Z(r,s,\Lambda)$ such that $A/Q$ is a simple ring and such that the images of $X_i,Y_i$ in $A/Q$ are regular for all $i$. It also gives a necessary and sufficient condition under which all such quotients are domains.

We would like to emphasize that the subring $R^{Z^n}$ of invariants of $R$ under $Z$ is just a Laurent polynomial ring over the field $k$. Thus there are plenty of explicitly known maximal ideals. Moreover, when $k$ is algebraically closed there is a bijection $\text{Specm}(R^{Z^n}) \to (k(0))^m$ where $m$ is the number of variables in $R^{Z^n}$, i.e. the rank of the subgroup $G \subseteq Z^{2n}$ (see (5.7)). It is in this sense we view the following theorem as a parametrization of the stated family of simple quotients.

**Theorem 5.8.** Let $A = A_k^Z(r,s,\Lambda)$ be a multiparameter twisted Weyl algebra.

(a) The assignment

$$n \mapsto A/(n)$$

where $(n)$ denotes the ideal in $A$ generated by $n$, is a bijection between the set of maximal ideals in $R^{Z^n}$ and the set of simple quotients of $A$ in which all $X_i, Y_i (i = 1, \ldots, n)$ are regular.

(b) For any $n \in \text{Specm}(R^{Z^n})$, the quotient $A/(n)$ is isomorphic to the twisted generalized Weyl algebra $A_n(R/R^n, \bar{\sigma}, \bar{t})$, where $\sigma_n(r + R^n) = \sigma_n(r) + Rn \forall g \in Z^n, r \in R$ and $\bar{t}_i = t_i + Rn \forall i$.

(c) $A/(n)$ is a domain for all $n \in \text{Specm}(R^{Z^n})$ if and only if $Z^{2n}/G$ is torsion-free, where $G$ was defined in (5.7).

**Proof.** We first prove part (b). Let $n \in \text{Specm}(R^{Z^n})$. Put $J = Rn$. Trivially $AJA = (n)$. By Lemma 5.7, $t_i + J$ are regular in $R/J$. For any $g \in Z^n$, $A_g = RZ_{n}^{(g_1)} \cdots Z_{n}^{(g_n)}$, $Z_{n}^{(m)} = X_{i}^{m}$ if $m \geq 0$ and $Z_{n}^{(m)} = Y_{i}^{m}$ if $m < 0$. We know $(R, \sigma, t)$ is $\mu$-consistent (see Section 5.1). Thus by [8, Cor. 6.4], $(R/J, \sigma, t)$ is also $\mu$-consistent. Thus the claim follows from [3, Thm. 4.1] using [8, Rem. 4.2].

Now we prove part (a). Let $n \in \text{Specm}(R^{Z^n})$. By part (b), $A/(n)$ isomorphic to $\bar{A} := A_n(R/J, \bar{\sigma}, \bar{t})$. By Proposition 5.2, $\bar{J}$ is maximal among $Z^n$-invariant ideals of $\bar{R}$, hence $R/J$ is $Z^n$-simple. By Proposition 5.2(b), $R/J$ is maximal commutative in $\bar{A}$. Hence, in particular, $Z(\bar{A}) \subseteq R/J$. Let $\pi : R \to R/J$ be the canonical projection. We have $(R/J)\bar{t}_i + (R/J)\bar{\sigma}_i(\bar{t}_i) = \pi(\bar{t}_i + \bar{\sigma}_i(\bar{t}_i)) + J = R/J$ for any $i \in \{1, \ldots, n\}$ and $d \in Z_{>0}$. Thus the requirements in Theorem 4.2(c) are fulfilled and we conclude that $\bar{A}$ is simple. For each $i \in \{1, \ldots, n\}$, the elements $\bar{t}_i$, hence also $\sigma_i(\bar{t}_i)$, are regular in $R/J$. By the proof of [3, Thm. 5.2(a)], these elements are also regular in $\bar{A}$. Since $\bar{t}_i = Y_iX_i$ and $\bar{\sigma}_i(\bar{t}_i) = X_iY_i$, it follows that $X_i$ and $Y_i$ are regular in $\bar{A}$.

Conversely, assume that $Q$ is any nonzero ideal of $A$ such that $A/Q$ is simple and such that $X_i, Y_i + Q$ are regular in $A/Q$ for all $i$. By Proposition 5.2, $R$ is maximal commutative in $A$, that is $C_A(R) = R$. So by Theorem 4.2(b), $R \cap Q \neq 0$. We claim that $R \cap Q$ is $Z^n$-invariant. Let $i \in \{1, \ldots, n\}$ and $p \in R \cap Q$. Then $X_ip \in Q$ since $Q$ is an ideal. On the other hand, $X_ip = \sigma_i(p)X_i$. Since the image of $X_i$ in $A/Q$ not a zero-divisor we conclude that $\sigma_i(p) \notin Q$. Trivially $\sigma_i(p) \in R$. Thus $\sigma_i(R \cap Q) \subseteq R \cap Q$ for all $i$. Analogously one proves that $\sigma_i^{-1}(R \cap Q) \subseteq R \cap Q$ (or one can use that $R$ is Noetherian). So $R \cap Q$ is indeed $Z^n$-invariant. Next we show that $R \cap Q$ is maximal among $Z^n$-invariant ideals in $R$. Suppose $R \cap Q \subseteq J \subseteq R$ where $J$ is a $Z^n$-invariant ideal of $R$. Since $J$ is $Z^n$-invariant, $AJ$ is a two-sided ideal of $A$. Any element of $AJ + Q$ of degree zero has the form $p + a$ where $p \in J$ and $a$ is the degree zero component of an element of $Q$. But $Q$ is graded by Proposition 5.3 so $a \in Q$. Thus $(AJ + Q) \cap R = J + (Q \cap R) = J$. Thus $AJ + Q$ is an ideal of $A$ which properly contains $Q$. Since $Q$ was maximal, $AJ + Q = A$ and thus $J = (AJ + Q) \cap R = R$. This shows that $R \cap Q$ is maximal among all $Z^n$-invariant ideals of $R$. By Proposition 5.3 we conclude that $R \cap Q$ equals $Rn$ for some maximal ideal $n$ of $R^{Z^n}$. So for this $n$ we have $(n) \subseteq Q$. But we proved above that $A/(n)$ is always simple. Thus $(n)$ is a maximal ideal of $A$ which implies that $(n) = Q$.

Finally, two different ideals $n, n'$ in $R^{Z^n}$ cannot generate the same maximal ideal $L$ in $A$, since then $1 \in n + n' \subseteq L$ which is absurd.
(c) By Proposition 2.4 and part (b) we have that $A/(n)$ is a domain iff $Rn$ is a prime ideal of $R$. Assume $Rn$ is prime for all $n \in \text{Specm}(R^{2n})$. Suppose $d \in \mathbb{Z}^{2n}$, $d \notin G$ but that there is a $p \in \mathbb{Z}_{\geq 0}$ such that $pd \in G$. Without loss of generality we can assume $p$ is prime. Then there is a $j \in \{1, \ldots, n\}$ such that $\sigma_j(u^d) = \zeta u^d$ where $\zeta \in k$, $\zeta \neq 1$, $\zeta^p = 1$. Pick any $\mathbb{Z}$-basis $\{d_1, \ldots, d_N\}$ for $G$ and take $n$ to be the maximal ideal in $R^{2n}$ generated by $u^{d_i} - 1$ for $i = 1, \ldots, N$. Then $u^{pd} - 1 \in n$ also, because $pd$ is a $\mathbb{Z}$-linear combination of the $d_i$. But $u^{pd} - 1 = (u^{d_1} - 1)(u^{d_2} - \zeta)(u^{d_3} - \zeta^2) \cdots (u^{d_N} - \zeta^{p-1})$. Since $Rn$ is prime we conclude that $u^d - \zeta^e \in Rn$ for some $e \in \{0, \ldots, p-1\}$. However $Rn$ is $\mathbb{Z}^n$-invariant and thus $Rn \ni u^{d_1} - \zeta^e \in \zeta^{-1}u^{d_i} - \zeta^{-e}((\zeta^{-1} - 1)\zeta^e$ which is invertible. This contradicts that $Rn$ is a proper ideal of $R$ which we know by Proposition 5.4. Hence $\mathbb{Z}^{2n}/G$ is torsion-free.

Conversely, assume that $\mathbb{Z}^{2n}/G$ is torsion-free. Thus $\mathbb{Z}^{2n} \cong G \oplus G'$ for some subgroup $G'$ of $\mathbb{Z}^{2n}$. Therefore, viewing $R$ as the group algebra $k[\mathbb{Z}^{2n}]$, we have an isomorphism $R = k[\mathbb{Z}^{2n}] \cong k[G] \otimes_k k[G']$. Under this isomorphism, $Rn$ (where $n \in \text{Specm}(R^{2n})$ is arbitrary) is mapped to $n \otimes k[G']$ which is a prime ideal in $k[G] \otimes_k k[G']$ since

$$\frac{k[G] \otimes_k k[G']}{n \otimes k[G']} \cong (R^{2n}/n)[G'],$$

which is a Laurent polynomial algebra over a field. This proves that $Rn$ is a prime ideal of $R$ for any $n \in \text{Specm}(R^{2n})$.

6. Multiparameter Weyl algebras and Hayashi’s $q$-analog of the Weyl algebras

In this section we consider a class of multiparameter Weyl algebras defined in [1], which is a particular case of twisted multiparameter Weyl algebras. For the convenience of the reader we include the definition.

6.1. Definition. Assume $\mathfrak{g} = (r_1, \ldots, r_n)$ and $\mathfrak{s} = (s_1, \ldots, s_n)$ are $n$-tuples of nonzero scalars in a field $k$ such that $(r_is_i^{-1})^2 \neq 1$ for each $i$. Let $A_{\mathfrak{g},\mathfrak{s}}(n)$ be the unital associative algebra over the field $k$ generated by elements $\rho_i, \rho_i^{-1}, \sigma_i, \sigma_i^{-1}, x_i, y_i, i = 1, \ldots, n$, subject to the following relations:

- (R1) The $\rho_i^{\pm 1}, \sigma_i^{\pm 1}$ all commute with one another and $\rho_i\rho_i^{-1} = \sigma_i\sigma_i^{-1} = 1$;
- (R2) $\rho_i x_i = r_i^{\delta_{ij}} x_i \rho_i$ and $\rho_i y_i = r_i^{-\delta_{ij}} y_i \rho_i$ $1 \leq i, j \leq n$;
- (R3) $\sigma_i x_i = s_i^{\delta_{ij}} x_j \sigma_i$ and $\sigma_i y_i = s_i^{-\delta_{ij}} y_j \sigma_i$ $1 \leq i, j \leq n$;
- (R4) $y_i x_i = x_i y_i$ and $y_i y_i = y_i y_i$, $1 \leq i, j \leq n$;
- (R5) $y_i x_i - r_i^2 x_i y_i = \sigma_i^2$ and $y_i x_i - s_i^2 x_i y_i = \rho_i^2$, $1 \leq i \leq n$,

or equivalently

- (R5') $y_i x_i = \frac{r_i^2 \rho_i^2 - s_i^2 \sigma_i^2}{r_i - s_i}$ and $x_i y_i = \frac{\rho_i^2 - \sigma_i^2}{r_i - s_i}$, $1 \leq i \leq n$.

When $r_i = q^{-1}$ and $s_i = q$ for all $i$, we may quotient by the ideal generated by the elements $\sigma_i \rho_i - 1$, $i = 1, \ldots, n$, to obtain Hayashi’s $q$-analog of the Weyl algebras $A^{-}_\mathfrak{g}(n)$ (see [12]).

6.2. Realization as multiparameter twisted Weyl algebras. Take $k = 2$, and for all $i, j$ put $\lambda_{ij} = 1$, $r_{ij} = r_i^{\delta_{ij}}$, $s_{ij} = s_i^{\delta_{ij}}$, where $r_i, s_i \in k\setminus\{0\}$, $i = 1, \ldots, n$. Then $A_{\mathfrak{g},\mathfrak{s}}^*(r, s, \Lambda)$ is isomorphic to $A_{\mathfrak{g},\mathfrak{s}}(n)$.

Let us investigate the ring of invariants $R^{2n}_\mathfrak{g}$. Consider a monomial

$$u^d := u_1^{d_1} \cdots u_n^{d_n} v_1^{d_{n+1}} \cdots v_n^{d_{2n}},$$

where $d \in \mathbb{Z}^{2n}$. We have

$$\sigma_i(u^d) = r_i^{d_i} s_i^{d_{n+i}} u^d.$$

6.3. Generic case. Assuming that for each $i = 1, \ldots, n$, the only pair $(d, d') \in \mathbb{Z}^2$ such that $r_i^d s_i^{d'} = 1$ is the pair $(0, 0)$ we obtain that $R^{2n}_\mathfrak{g} = k$ and thus, by Theorem 5.8 $A_{\mathfrak{g},\mathfrak{s}}(n)$ is a simple ring.
6.4. Hayashi’s $q$-analogs of the Weyl algebras $A_q^n(n)$. Assume instead that for all $i$, $r_i = q^{-1}$ and $s_i = q$, where $q \in \k$ is nonzero and not a root of 1. Then by (6.1), $d$ is fixed by all $\sigma_i$ iff $d_i = d_n - i$ for all $i$. Thus $R^{2n} = k[w_1, \ldots, w_n]$ where $w_i := u_i v_i$. Pick the maximal ideal $n := (w_1 - 1, \ldots, w_n - 1)$ of the invariant subring. Then, by Theorem 5.8, we obtain that the quotient of $A_{-\mathbb{q}}(n)$ by the two-sided ideal generated by $w_1 - 1, \ldots, w_n - 1$ is a twisted generalized Weyl algebra which is simple. It is easy to check that this simple algebra is isomorphic to Hayashi’s $q$-analogs of the Weyl algebras $A_q^n(n)$, see [12].

6.5. Connections with generalized Weyl algebras. Assume now that we are in the generic case as in subsection 6.3. As it was observed in [1], the multiparameter Weyl algebra $A_{\Lambda}(n)$ can be realized as a degree $n$ generalized Weyl algebra. For this construction, let $D_i$ be the subalgebra of $A_{\Lambda}(n)$ generated by the elements $\rho_i, \rho_i^{-1}, \sigma_i, \sigma_i^{-1}$. Thus, $D_i$ is isomorphic to $k[\rho_i^{\pm 1}, \sigma_i^{\pm 1}]$. Set $D = D_1 \otimes D_2 \otimes \cdots \otimes D_n$. Let $\phi_i$ be the automorphism of $D_i$ given by

$$
(6.2) \quad \phi_i(\rho_j) = r_i^{-\delta_{i,j}} \rho_j \quad \phi_i(\sigma_i) = s_i^{-\delta_{i,j}} \sigma_i.
$$

Now set

$$
(6.3) \quad t_i = \frac{r_i^2 - s_i^2 \sigma_i^2}{r_i^2 - s_i^2}, \quad X_i = x_i, \quad Y_i = y_i,
$$

and observe that

$$
Y_i X_i = t_i, \quad \text{and} \quad X_i Y_i = \frac{r_i^2 - \sigma_i^2}{r_i^2 - s_i^2} = \phi_i(t_i)
$$

are just the relations in (R5)’. The relations in (R1) and (R4) are apparent. The identities in (R2) and (R3) are equivalent to the statements $Y_j d = \phi_j^{-1}(d) Y_j, \quad X_j d = \phi_j(d) X_j$ with $d = \rho_i$ and $\sigma_i$. Therefore, there is a surjection $W_n := D(\phi, t) \to A_{\Lambda}(n)$. But since $A_{\Lambda}(n)$ has a presentation by (R1)-(R5), there is a surjection $A_{\Lambda}(n) \to W_n$. Since that map is the inverse of the other one, these algebras are isomorphic. Bavula [3 Prop. 7] has shown that a generalized Weyl algebra $D(\phi, t)$ is left and right Noetherian if $D$ is Noetherian, and it is a domain if $D$ is a domain. Since $D$ is commutative and finitely generated, it is Noetherian, hence so are $W_n$ and $A_{\Lambda}(n)$. Since $D$ is a domain as it can be identified with the Laurent polynomial algebra $k[\rho_i^{\pm 1}, \sigma_i^{\pm 1} \mid i = 1, \ldots, n]$; hence $A_{\Lambda}(n)$ is a domain also. In summary, we have

**Proposition 6.1.** [4] When the parameters $r_i, s_i$ are generic as in Section 6.3, the multiparameter Weyl algebra $A_{\Lambda}(n)$ is isomorphic to the degree $n$ generalized Weyl algebra $W_n = D(\phi, t)$ where $D$ is the $k$-algebra generated by the elements $\rho_i, \rho_i^{-1}, \sigma_i, \sigma_i^{-1}, i = 1, \ldots, n$, subject to the relations in (R1), $\phi_i$ is as in (6.2); and the elements $t_i$ are as in (6.3). Thus, $A_{\Lambda}(n)$ is Noetherian domain.

7. Jordan’s simple localization of the multiparameter quantized Weyl algebra

7.1. Quantized Weyl algebras. Let $\bar{q} = (q_1, \ldots, q_n)$ be an $n$-tuple of elements of $k \setminus \{0\}$. Let $\Lambda = (\lambda_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with $\lambda_{ij} \in k \setminus \{0\}$, multiplicatively skewsymmetric: $\lambda_{ij} \lambda_{ji} = 1$ for all $i, j$. The multiparameter quantized Weyl algebra of degree $n$ over $k$, denoted $A_{\bar{q},\Lambda}^n(k)$, is defined as the unital $k$-algebra generated by $x_i, y_i$, $1 \leq i \leq n$ subject to the following defining relations:

$$
(7.1) \quad y_i y_j = \lambda_{ij} y_j y_i, \quad \forall i, j.
$$

$$
(7.2) \quad x_i x_j = q_i \lambda_{ij} x_j x_i, \quad i < j.
$$

$$
(7.3) \quad x_i y_j = \lambda_{ji} y_j x_i, \quad i < j.
$$

$$
(7.4) \quad x_i y_j = q_j \lambda_{ji} y_j x_i, \quad i > j.
$$

$$
(7.5) \quad x_i y_i - q_i y_i x_i = 1 + \sum_{k=1}^{i-1} (q_k - 1) y_k x_k, \quad \forall i.
$$
This algebra first appeared in [15], and was further studied in [1] and [14] among others. For $k = \mathbb{C}$ and $q_1 = \cdots = q_n = \mu^2$, $\lambda_{ij} = \mu$ for $j < i$, where $\mu \in k\setminus\{0\}$, the algebra $A_n^{q,\Lambda}(k)$ is isomorphic to the quantized Weyl algebra introduced by Pusz and Woronowicz [19].

The quantized Weyl algebra can be realized as a twisted generalized Weyl algebra (first observed in [18]) in the following way. Let $P = k[s_1, \ldots, s_n]$ be the polynomial algebra in $n$ variables and $\tau_i$ the $k$-algebra automorphisms of $P$ defined by

$$
\tau_i(s_j) = \begin{cases} 
  s_j, & j < i, \\
  1 + q_is_i + \sum_{k=1}^{i-1} (q_k - 1)s_k, & j = i, \\
  q_is_j, & j > i.
\end{cases}
$$

One can check that the $\tau_i$ commute. Let $\mu = (\mu_{ij})_{i,j=1}^n$ be defined by

$$
\mu_{ij} = \begin{cases} 
  \lambda_{ij}, & i < j, \\
  q_i\lambda_{ji}, & i > j.
\end{cases}
$$

Put $\tau = (\tau_1, \ldots, \tau_n)$ and $s = (s_1, \ldots, s_n)$. Let $A_\mu(P, \tau, s)$ be the associated twisted generalized Weyl algebra. From (7.6) it is easy to see that $A_\mu(P, \tau, s)$ is $k$-finitistic, and that the minimal polynomials are $p_{ij}(x) = x - 1$ for $i < j$ and $p_{ij}(x) = x - q_i$ for $j > i$, so the algebra is of type $(A_1)^n$. By Theorem 4.1 one checks that $A_\mu(P, \tau, s)$ is isomorphic to $A_n^{q,\Lambda}(k)$ via $X_i \mapsto x_i, Y_i \mapsto y_i$, and $s_i \mapsto y_ix_i$. The representation theory of $A_n^{q,\Lambda}$ has been studied from the point of view of TGW algebras in [18] and [9].

In the following it will be convenient to identify $P$ with its isomorphic image in $A_n^{q,\Lambda}$ via $s_i \mapsto y_ix_i$. Consider the following elements in $A_n^{q,\Lambda}$:

$$
z_i = 1 + \sum_{k \leq i} (q_k - 1)s_k, \quad i = 1, \ldots, n.
$$

It was shown in [14] that the set $Z := \{z_1^{k_1} \cdots z_n^{k_n} \mid k_1, \ldots, k_n \in \mathbb{Z}\}$ is an Ore set in $A_n^{q,\Lambda}$ and that, provided that none of the $q_i$ is a root of unity, the localized algebra $B_n^{q,\Lambda} := Z^{-1}A_n^{q,\Lambda}$ is simple.

The algebra $B_n^{q,\Lambda}$ can also be realized as a twisted generalized Weyl algebra. To see this, consider the following subset of $P$:

$$
S = \{ \alpha z_1^{k_1} \cdots z_n^{k_n} \mid \alpha \in k\setminus\{0\}, k_i \in \mathbb{Z}\}.
$$

where $z_i$ were defined in (7.8). Then $0 \notin S$, $1 \in S$, $a, b \in S \Rightarrow ab \in S$, the elements of $S$ are regular, and moreover $S$ has the virtue of being $\mathbb{Z}^n$-invariant, using the relation

$$
\tau_i(z_j) = \begin{cases} 
  z_j, & j < i, \\
  q_i z_j, & j \geq i,
\end{cases}
$$

which can be proved using (7.8) and (7.6). Thus [3 Thm. 5.2] can be applied to give, together with the isomorphism $A_n^{q,\Lambda} \simeq A_\mu(P, \tau, s)$, that

$$
S^{-1}A_n^{q,\Lambda} \simeq S^{-1}A_\mu(P, \tau, s) \simeq A_\mu(S^{-1}P, \bar{\tau}, s).
$$

But localizing at $S$ is equivalent to localizing at $Z$, and thus

$$
B_n^{q,\Lambda} \simeq A_\mu(S^{-1}P, \bar{\tau}, s).
$$

### 7.2. Relation to multiparameter twisted Weyl algebras

We show here how the algebra $B_n^{q,\Lambda}$ fits into the framework of multiparameter twisted Weyl algebras. We keep all notation from previous section. Let $\bar{q} = (q_1, \ldots, q_n) \in (k\setminus\{0\})^n$ and let $\Lambda = (\lambda_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with $\lambda_{ij} \in k\setminus\{0\}$, $\lambda_{ii} = 1$, $\lambda_{ij}\lambda_{ji} = 1$ for all $i, j$. We assume that none of the $q_i$ is a root of unity. Let $k = 1$ and put

$$
r_{ij} = \begin{cases} 
  1, & j \leq i \\
  q_i^{-1}, & j > i
\end{cases}, \quad s_{ij} = \begin{cases} 
  1, & j < i \\
  q_i^{-1}, & j \geq i
\end{cases}
$$

...
Similarly, while using (7.10) and (7.11) in the last step, (7.14) and (7.15) for all $i, j \in \{1, \ldots, n\}$. Note that the $\mu_{ij}$ in (7.15) coincides with the ones defined in (7.14). The goal now is to explain the following diagram, which proves Theorem C stated in the introduction.

\[ (R, \sigma, t) \xrightarrow{\pi} (R/J, \bar{\sigma}, \bar{t}) \xrightarrow{\psi} (S^{-1}P, \bar{\tau}, s) \]

\[ A_n^k(r, s, \Lambda) = A_\mu(R, \sigma, t) \]

for all $i, j \in \{1, \ldots, n\}$. We claim that $\psi$ is $\mathbb{Z}^n$-equivariant. Indeed,

\[ \psi(u_i) = -q_i^{-1}z_{i-1}, \quad \psi(v_i) = -z_i, \quad i = 1, \ldots, n \]

where $z_0 := 1$. We claim that $\psi$ is $\mathbb{Z}^n$-equivariant. Indeed,

\[ \psi(\sigma_i(u_j)) = \psi(r_{ij}^{-1} u_j) = -r_{ij}^{-1} q_i^{-1} z_{i-1}, \]

while, using (7.10) and (7.11) in the last step,

\[ \bar{\tau}_i(\psi(u_j)) = \bar{\tau}_i(-q_i^{-1} z_{j-1}) = -q_i^{-1} q_i^{-1} z_{i-1}. \]

Similarly $\psi(\sigma_i(v_j)) = \bar{\tau}_i(\psi(v_j))$. This proves that $\psi \sigma_i = \bar{\tau}_i \psi$ for each $i$, so in other words, that $\psi$ is $\mathbb{Z}^n$-equivariant. Also, for any $i \in \{1, \ldots, n\},$

\[ \psi(t_i) = \psi\left(\frac{u_i - q_i^{-1} v_i}{1 - q_i^{-1}}\right) = \frac{-q_i^{-1} z_{i-1} + q_i^{-1} z_i}{1 - q_i^{-1}} = \frac{z_i - z_{i-1}}{q_i - 1} = s_i \]

by (7.8). We have proved that $\psi$ is a morphism in the category $\mathcal{TGW}_n(k)$ between $(R, \sigma, t)$ and $(S^{-1}P, \bar{\tau}, s)$. It is easy to see that $\psi$ is surjective because the image contains both $s_1, \ldots, s_n$ since $\psi(t_i) = s_i$, and the inverses of the $z_i$: $z_i^{-1} = \psi(-v_i^{-1})$. Applying the functor $A$ to $\psi$ gives a surjective k-algebra morphism $A_\mu(\psi): A_\mu(R, \sigma, t) \to A_\mu(S^{-1}P, \tau, s)$. 

\[ A_n^k(r, s, \Lambda) \cong A_\mu(R/J, \bar{\sigma}, \bar{t}) \cong A_\mu(S^{-1}P, \bar{\tau}, s) \cong B_n^{\bar{q}, \Lambda} \]
7.2.2. The map $\pi$. We determine the invariant subring $R^Z$. For any $i \in \{1, \ldots, n\}$ and $d \in Z^n$ we have

$$\sigma_i(u^d) = q_i^{-1} \sum_{j=1}^n d_j - \sum_{j=1}^{n+1} d_{n+j} u^d$$

Thus $u^d \in R^Z$ iff for each $i = 1, \ldots, n$ we have $d_{n+i} + \sum_{j=i+1}^n (d_j + d_{n+j}) = 0$. This system of equations is equivalent to that $d_{2n} = 0$, $d_{2n-1} + d_n = 0$, $d_{2n-2} + d_{n-1} = 0$, $\ldots$, $d_{n+1} + d_2 = 0$. Thus $R^Z = k[w_1, \ldots, w_n]$ where $w_1 = -u_1$, $w_2 = w_2 v_1^{-1}$, $\ldots$, $w_n = w_n v_{n-1}^{-1}$. Pick

$$n := (w_1 - q_1^{-1}, \ldots, w_n - q_n^{-1}) \in \text{Specm}(R^Z).$$

Let $J = R^n$ be the ideal in $R$ generated by $n$. The canonical map $\pi : R \to R/J$ is $Z^n$-equivariant and maps $t_i$ to $t_i = t_i + J$.

7.2.3. The map $\Psi$. We have $\psi(w_i) = q_i^{-1}$ for $i = 1, \ldots, n$ which shows that $J = R^n \subseteq \ker \psi$. Thus $\psi$ induces a map $\Psi : R/J \to S^{-1}P$, also $Z^n$-equivariant and $\Psi(t_i) = s_i$. Since $\psi$ is surjective, so is $\Psi$. Applying the functor from [8, Thm. 3.1] we get a surjective homomorphism $\Psi : \text{Specm}(R^Z) \to \text{Specm}(S^{-1}P, \bar{R}, s)$. However, by Theorem 5.8 the algebra $A_{\mu}(R/J, \sigma, t)$ is simple, and thus $\Psi$ is an isomorphism.

7.2.4. The maps $\varphi, \iota, \Phi$. Similarly one can show that the map $\varphi : P \to R/J$ defined by $\varphi(s_i) = \bar{t}_i$ is $Z^n$-equivariant and that the elements of $S$ are mapped to invertible elements of $R/J$, showing that $\varphi$ factorizes through the canonical map $\iota : P \to S^{-1}P$, inducing a map $\Phi$. Applying the functor $A_{\mu}$ gives corresponding homomorphisms of twisted generalized Weyl algebras.

8. Simple weight modules

In this section we describe the simple weight modules over the simple algebras $A = A^k_r(r, s, \Lambda)/\langle J \rangle$ from Theorem 5.8. We also assume that the ground field $k$ is algebraically closed. We will use notation from Section 5.1.

8.1. Dynamics of orbits and their breaks. The group $Z^n$ acts on $R$ via the automorphisms $\sigma_i$. Explicitly, $g(r) = (\sigma_1^g \cdots \sigma_n^g)(r)$ for $g = (g_1, \ldots, g_n) \in Z^n$ and $r \in R$. Using this action and (2.13) we have $a \cdot r = (\deg a)(r) \cdot a$ for any homogenous $a \in A$ and any $r \in R$. The group $Z^n$ also acts on $\text{Max}(R)$, the set of maximal ideals of $R$. Let $\Omega$ denote the set of orbits of this action. An element $m \in \text{Max}(R)$ is called an i-break if $t_i \in m$. An orbit $O \in \Omega$ is called degenerate if it contains an i-break for some $i$. A break $m$ in an orbit $O$ is called maximal if $m$ is an i-break for all $i$ for which $O$ contains an i-break.

**Proposition 8.1.** Let $m \in \text{Specm}(R)$. Then the stabilizer $\text{Stab}(m)$ is trivial.

**Proof.** Write

$$m = (u_i - \alpha_i, \bar{v}_i - \beta_i \mid i = 1, \ldots, n),$$

where $\bar{u}_i = u_i + J, \bar{v}_i = v_i + J$ and $\alpha_i, \beta_i \in k \setminus \{0\}$. Suppose $g \in \text{Stab}(m)$. Then

$$\sigma_g(m) = (\sigma_g(\bar{u}_i) - \alpha_i, \sigma_g(\bar{v}_i) - \beta_i \mid i = 1, \ldots, n) =
(8.1)$$

$$= ((r_1^{g_1} \cdots r_n^{g_n})^{-1} \bar{u}_i - \alpha_i, (s_1^{g_1} \cdots s_n^{g_n})^{-1} \bar{v}_i - \beta_i \mid i = 1, \ldots, n)$$

Thus

$$r_1^{g_1} \cdots r_n^{g_n} = s_1^{g_1} \cdots s_n^{g_n} = 1$$

Raising all sides to the $k$:th power and using that $r_{ij}^{k} = s_{ij}^{k}$ for all $i \neq j$, we obtain that $r_{ii}^{k^n} = s_{ii}^{k^n} = 1$ for all $i$ which, since $r_{ii}/s_{ii}$ is not a root of unity, implies that $g_i = 0$ for all $i$. □

**Proposition 8.2.** Consider the maximal ideal

$$m = (u_1 - \alpha_1, \ldots, u_n - \alpha_n, v_1 - \beta_1, \ldots, v_n - \beta_n) \in \text{Specm}(R).$$

Then, for all $i \in \{1, \ldots, n\}$ and all $g = (g_1, \ldots, g_n) \in Z^n$,

$$t_i \in \sigma_g(m) \iff (\alpha_i/\beta_i)^k = (s_{ii}/r_{ii})^{(g_i + 1)k}.$$
Proof. By the calculation \( (8.1) \) and the definition \( (5.4) \) of \( t_i \) we have
\[
(t_i \in \sigma_g(m) \iff \left( \frac{r_1^{g_1} \cdots r_n^{g_n}}{s_1^{g_1} \cdots s_n^{g_n}} \right)^k = (s_{ii}/r_{ii})^k
\]
Using that \( r_{ij}^k = s_{ij}^k \) for \( i \neq j \) and simplifying, the claim follows. \( \square \)

**Corollary 8.3.** If \( t_i \in m \), then for \( g = (g_1, \ldots, g_n) \in \mathbb{Z}^n \),
\[
t_i \in \sigma_g(m) \iff g_i = 0.
\]

**Corollary 8.4.** Every degenerate orbit contains a maximal break.

**Remark 8.5.** Corollary 8.3 holds for any TGW algebra of Lie type \((A_1)^n\) using the fact that \( \sigma_j(t_i) = \gamma_j t_i \) for any \( j \neq i \).

### 8.2. General results on simple weight modules with no proper inner breaks

We collect here some notation and results from \([9]\).

Let \( A = A_{\mu}(R, \sigma, t) \) be a twisted generalised Weyl algebra. Let \( V \) be a simple weight module over \( A \).

**Definition 8.6** ([9]). \( V \) has no proper inner breaks if for any \( m \in \text{Supp}(V) \) and any homogenous \( a \) with \( aM_m \neq 0 \) we have \( a' \notin \) for some homogenous \( a' \) with \( \deg(a) = -\deg(a') \).

This definition is slightly different than the one given in \([9]\) but can be proved to be equivalent. Consider the following sets (also equivalent to the definitions in \([9]\)), defined for any \( m \in \text{Specm}(R) \).

\[
(8.3) \quad \widetilde{G}_m := \{ g \in \mathbb{Z}^n \mid A_{-g}A_g \text{ is not contained in } m \},
\]
\[
(8.4) \quad G_m := \widetilde{G}_m \cap \text{Stab}_{\mathbb{Z}^n}(m).
\]

One can show that \( G_m \) is a subgroup of \( \mathbb{Z}^n \) and \( \widetilde{G}_m \) is a union of cosets from \( \mathbb{Z}^n/G_m \).

Fix now \( m \in \text{Supp}(V) \). One checks that the subalgebra \( B(m) := \bigoplus_{g \in \text{Stab}(m)} A_g \) of \( A \) preserves the weight space \( V_m \). For any \( g \in \widetilde{G}_m \) we pick elements \( a_g \in A_g \) and \( a'_g \in A_{-g} \) such that \( a'_g a_g \notin m \). The following theorem describes the simple weight modules with no proper inner breaks up to the structure of \( V_m \) as a \( B(m) \)-module.

**Theorem 8.7** ([9]). Suppose \( V \) has no proper inner breaks. If \( \{v_i\}_{i \in J} \) is a \( k \)-basis for \( V_m \) (\( J \) some index set), then the following is a \( k \)-basis for \( V \):
\[
\{ a_g v_i \mid g \in S, i \in J \}
\]
where \( S \subseteq \widetilde{G}_m \) is a set of representatives for \( \widetilde{G}_m \) modulo \( G_m \). Moreover, for any \( v \in V_m \), any \( i \in \{1, \ldots, n\} \) and \( g \in S \) we have
\[
(8.6) \quad X_i a_g v = \begin{cases} a_h b_{g,i} v, & g + e_i \in \mathbb{G}_m \\ 0, & \text{otherwise,} \end{cases} \quad Y_i a_g v = \begin{cases} a_k c_{g,i} v, & g - e_i \in \mathbb{G}_m \\ 0, & \text{otherwise.} \end{cases}
\]
where \( h, k \in S \) with \( h \in (g + e_i) + G_m \) and \( k \in (g - e_i) + G_m \) and \( b_{g,i}, c_{g,i} \in B(m) \) are given by
\[
(8.7) \quad b_{g,i} = \sigma_h(X_i a_g a'_{g+i}-h a'_h)a_{g+e_i-h}, \quad c_{g,i} = \sigma_k(Y_i a_g a'_{g-e_i-k} a'_k)a_{g-e_i-k}.
\]

### 8.3. The case of trivial stabilizer

We show here a theorem which implies that all simple weight modules over \( A_{\mu}^\infty(r, s, A)/\langle n \rangle \) have no proper inner breaks.

**Theorem 8.8.** If \( V \) is a simple weight module over a twisted generalised Weyl algebra \( A_{\mu}(R, \sigma, t) \) such that the stabilizer \( \text{Stab}(m) \) is trivial for some (hence all) weight \( m \in \text{Supp}(V) \), then \( V \) has no proper inner breaks.

**Proof.** Suppose \( m \in \text{Supp}(V) \) has trivial stabilizer. Let \( g \in \mathbb{Z}^n \) and assume \( a \in A_g \) is such that \( aV_m \neq 0 \). Since \( V \) is simple, \( V_m \cap AaV_m \neq 0 \). But \( V_m \cap AaV_m \subseteq A_{-g}aV_m \) since \( m \) has trivial stabilizer. This shows that there exists an element \( b \in A_{-g} \) such that \( baV_m \neq 0 \). Since \( \deg(ba) = 0 \) we have \( ba \in R \). Then \( baV_m \neq 0 \) implies \( ba \notin m \). \( \square \)
8.4. Abstract description of the simple weight modules in case of trivial stabilizer.

Let \( A = A_\mu(R, \sigma, t) \) be a TGWA where \( \mu \) is symmetric. In [10] a description of all simple weight modules with support in an orbit with trivial stabilizer is given in terms of a Shapovalov type form. The form used in [10] requires the matrix \( \mu \) to be symmetric (due to its formulation in terms of a certain involution on the TGWA). As is observed in [11], there is another way to define a bilinear form which works for general \( \mu \). It is given as follows. Let \( p_0 : A \to A_0 = R \) be the graded projection onto the degree zero component of \( A \) with respect to the standard \( \mathbb{Z}^n \)-gradation on \( A \). Then put

\[
F : A \times A \to R, \quad F(a, b) = p_0(ab).
\]

Such forms have been studied for arbitrary group graded rings [11].

We have the following result.

**Theorem 8.9.** Let \( A = A_\mu(R, \sigma, t) \) be any twisted generalized Weyl algebra. Let \( V \) be any simple weight module over \( A \) such that \( \text{Stab}(m) = \{0\} \) for \( m \in \text{Supp}(V) \). Then \( V \simeq A/N(m) \) where \( A \) is considered as a left module over itself and \( N(m) \) is the left ideal given by

\[
N(m) = \{a \in A | F(b, a) \in m \forall b \in A\}
\]

**Proof.** Similar to the case of symmetric \( \mu \) proved in [13] Lemma 6.1 and Corollary 6.2. \( \square \)

8.5. Bases and explicit action on the simple weight modules over \( A^m_{\mu}(r, s, \Lambda) \).

Let \( n, k, r, s, \Lambda \) be as in Section 5.1. Assume that for each \( i = 1, \ldots, n \), the scalar \( r_{ii}/s_{ii} \) is not a root of unity.

Let \( R, \sigma, t, \mu \) be as in Section 5.1.

Let \( J \) be any \( \mathbb{Z}^n \)-invariant ideal of \( R \). Let \( A = A_\mu(R/J, \sigma, t) \). Thus for \( J = 0 \), \( A \) equals the multiparameter twisted Weyl algebra \( A^m_{\mu}(r, s, \Lambda) \), and for \( J = Rn \) where \( n \in \text{Spec}(R) \), \( A \) equals a simple quotient of the algebra in the former case.

We will describe the simple weight modules over \( A \), using Theorem 8.7.

Let \( V \) be a simple weight module over \( A \). Let \( m \in \text{Supp}(V) \). Since \( k \) is algebraically closed we have

\[
m = (\bar{u}_i - \alpha_i, \bar{v}_i - \beta_i | i = 1, \ldots, n)
\]

where \( \bar{u}_i = u_i + J, \bar{v}_i = v_i + J \) and \( \alpha_i, \beta_i \in k \setminus \{0\} \) for \( i = 1, \ldots, n \).

We determine the set \( \tilde{G}_m \). Let \( g \in \mathbb{Z}^n \). Since \( A_{\bar{g}} = \tilde{RZ}(g) \) (where \( Z(g) = Z^{(g)}_1 \cdots Z^{(g)}_n \) where \( Z^{(g)}_i \) equals \( X_i \) if \( j \geq 0 \) and \( Y_i \) otherwise) and \( \forall i \neq j : \sigma_i(t_j) = \gamma_{ij}t_i \) for some \( \gamma_{ij} \in k \setminus \{0\} \) it is clear that

\[
\tilde{G}_m = \{g \in \mathbb{Z}^n | Z^{(-g)}_1 Z^{(g)}_1 \cdots Z^{(-g)}_n Z^{(g)}_n \notin m \} = \\
\{g \in \mathbb{Z}^n | Z^{(-g)}_1 Z^{(g)}_1 \cdots Z^{(-g)}_n Z^{(g)}_n \notin m \} = \\
\{g \in \mathbb{Z}^n | Z^{(-g)}_1 Z^{(g)}_1 \cdots Z^{(-g)}_n Z^{(g)}_n \notin m \}
\]

where

\[
\tilde{G}_m^{(1)} := \{g \in \mathbb{Z}^n | Z^{(-g)}_1 Z^{(g)}_1 \notin m \}
\]

For \( j > 0 \) we have

\[
Z^{(-j)}_i Z^{(j)}_i = Y_i^j X_i^j = t_i \sigma_i^{-1}(t_i) \cdots \sigma_i^{-j+1}(t_i)
\]

while for \( j < 0 \),

\[
Z^{(-j)}_i Z^{(j)}_i = X_i^{-j} Y_i^{-j} = \sigma_i(t_i) \sigma_i^2(t_i) \cdots \sigma_i^{-j}(t_i).
\]

So, since \( m \) is maximal, hence prime, we see that if \( j > 0 \) and \( g \in \tilde{G}_m^{(1)} \) then \( \{0, 1, \ldots, j\} \subseteq \tilde{G}_m^{(1)} \).

Similarly if \( j < 0 \) and \( g \in \tilde{G}_m^{(1)} \) then \( \{j, j + 1, \ldots, 0\} \subseteq \tilde{G}_m^{(1)} \).

We distinguish between three possibilities. The first case is that \( \tilde{G}_m^{(1)} = \mathbb{Z} \). Then we say that (the support of) \( V \) is generic in the \( j \)-th direction. The second case is \( j \notin \tilde{G}_m^{(1)} \) for some positive integer \( j \). Assuming \( j \) is the smallest such integer, by (8.10) and (8.11) we get \( \sigma_i^{-j+1}(t_i) \in m \). By
Corollary 5.3 it follows that \( \sigma^m_i(t_i) \notin m \) for all integers \( m \neq j \). Thus \( \mathcal{G}_m^j = \{ m \in \mathbb{Z} \mid m \leq j - 1 \} \).

By Theorem 5.7 \( \text{Supp}(V) = \{ \sigma_g(m) \mid g \in G_m \} \) and thus we can replace \( m \) by \( \sigma^k_{i-1}(m) \). Doing this, the new \( j \) just equals 1 and \( \mathcal{G}_m^j = Z_{\mathbb{Z}} \). We say that \( m \) is a highest weight for \( V \) in the \( i \)-th direction. The final case is that \( j \notin \mathcal{G}_m^j \) for some negative integer \( j \). This is analogous to the previous case and leads to that, without loss of generality, \( \mathcal{G}_m^{(1)} = Z_{\mathbb{Z}} \geq 0 \) in which case we say that \( m \) is a lowest weight for \( V \) in the \( i \)-th direction.

In other words, there is an \( m \in \text{Supp}(V) \) such that the shape of the support of \( V \) is characterized by a vector

\[
\tau \in \{-1, 0, 1\}^n
\]

via the relation

\[
\mathcal{G}_m^{(i)} = \{ j \in \mathbb{Z} \mid j \cdot \tau_i \geq 0 \} \quad \forall i \in \{1, \ldots, n\}.
\]

Since the stabilizer of \( m \) is trivial by Proposition 5.3 the subalgebra \( B(m) \) in Theorem 5.7 is just \( R \). From well known results [2] (see [16], Proposition 7.2) for a proof in the TGW algebra case), it follows that \( V_m \) is simple as a \( B(m) \)-module since \( V \) is simple as an \( A \)-module. Thus, since \( R/m = k \), we have \( \dim_k V_m = 1 \). Pick \( v_0 \in V_m \), \( v_0 \neq 0 \). Then Theorem 5.7 implies that the set

\[
C = \{ v_g := Z_1^{(g_1)} \cdots Z_n^{(g_n)} v_0 \mid g = (g_1, \ldots, g_n) \in \mathcal{G}_m \}
\]

is a \( k \)-basis for \( V \), where \( Z_i^{(j)} = Z_i^j \) if \( j \geq 0 \) and \( Y_i^{-j} \) otherwise. Furthermore, the action of \( X_i, Y_i \) on the elements of \( C \) is given by

\[
X_i v_g = \begin{cases} b_{g,i} v_{g+e_i} & \text{if } (g_i + 1) \tau_i \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad Y_i v_g = \begin{cases} c_{g,i} v_{g-e_i} & \text{if } (g_i - 1) \tau_i \geq 0, \\ 0 & \text{otherwise,} \end{cases}
\]

for certain \( b_{g,i}, c_{g,i} \in k \). Although the formulas (5.7) can be used to calculate these scalars, one can also use a more direct approach which is available due to our knowledge of the commutation relations (5.3) among the generators \( X_i, Y_i \) in \( A \). Straightforward calculation gives the following.

\[
b_{g,i} = 1 \text{ if } g_i \geq 0, \quad \frac{1}{r_i^{g_i} x_i^{-(g_i - 1)} k_i^{g_i}} \text{ if } g_i < 0,
\]

\[
c_{g,i} = 1 \text{ if } g_i \geq 0, \quad \frac{k_{i}^{g_i} x_i^{g_i} r_i^{g_i}}{r_i^{g_i} x_i^{g_i}} \text{ if } g_i < 0,
\]

where

\[
\gamma_{ij}^{(l)} = \begin{cases} (r_{ij}^{k_l} \lambda_{ij}^{l})^l & \text{if } l \geq 0, \\ (r_{ij}^{k_l} \lambda_{ij}^{l})^l & \text{if } l < 0. \end{cases} \quad \varepsilon_{ij}^{(l)} = \begin{cases} (r_{ij}^{k_l} \lambda_{ij}^{l})^l & \text{if } l \geq 0, \\ \lambda_{ij}^{l} & \text{if } l < 0. \end{cases}
\]

**9. Whittaker modules**

**Definition 9.1.** Let \( A \) be a twisted generalized Weyl algebra of degree \( n \). A module \( V \) over \( A \) is called a Whittaker module if there exists a vector \( v_0 \in V \) (called Whittaker vector) and nonzero scalars \( \zeta_1, \ldots, \zeta_n \in k \setminus \{0\} \) such that the following conditions hold:

- \( V = Av_0 \),
- \( X_i v_0 = \zeta_i v_0 \) for each \( i = 1, \ldots, n \).

The pair \( (V, v_0) \) is called a Whittaker pair of type \( (\zeta_1, \ldots, \zeta_n) \). A morphism of Whittaker pairs \( (V, v_0) \to (W, w_0) \) is an \( A \)-module morphism \( V \to W \) mapping \( v_0 \) to \( w_0 \).

The following theorem describes Whittaker pairs over a family of TGWAs which properly includes all generalized Weyl algebras in which the \( t_i \) are regular. It is a generalization of [5, Theorem 3.12]. We use the notation from Section 4.4.
Theorem 9.2. Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a $\mathbb{k}$-finitistic TGW algebra of Lie type $(A_1)^n$. Assume that $(R, \sigma, t)$ is $\mu$-consistent and that $R$ is Noetherian.

(a) If $A$ has a Whittaker module, then

$$\gamma_{ij} = \mu_{ij} \text{ for all } i \neq j.$$  

(b) Conversely, if (9.1) holds, then for each $\zeta \in (\mathbb{k}[\{0\}])^n$, there is a bijection

$$\left\{ \text{Isomorphism classes } [(V, v_0)] \text{ of } \right\} \xrightarrow{\Psi} \left\{ \text{Proper } \mathbb{Z}^n\text{-invariant left ideals } Q \text{ of } R \right\}$$

$$[(V, v_0)] \mapsto \text{Ann}_{R}v_0$$

$$[(R/Q, 1 + Q)] \mapsto Q$$

where $R/Q$ is given an $A$-module structure by

$$s \bar{r} = \overline{sr} \quad \forall s \in R,$$

$$X_i \bar{r} = \overline{\zeta_i \sigma_i(r)},$$

$$Y_i \bar{r} = \overline{\zeta_i^{-1} \sigma_i^{-1}(r)t_i},$$

for all $\bar{r} \in R/Q$, where $\bar{r} := r + Q \in R/Q$ for $r \in R$.

c) Furthermore, there is a morphism of Whittaker pairs $(V, v_0) \to (W, w_0)$ iff $\Psi([(V, v_0)]) \subseteq \Psi([(W, w_0)])$.

Proof. (a) Suppose $(V, v_0)$ is a Whittaker pair with respect to $\zeta \in (\mathbb{k}[\{0\}])^n$. Then for $i \neq j$, $X_iX_jv_0 = \zeta_i \zeta_jv_0$. On the other hand, by relation (13.3), $X_iX_j = \gamma_{ij} \mu_{ij}^{-1}X_jX_i$ and thus $X_iX_jv_0 = \gamma_{ij} \mu_{ij}^{-1} \zeta_i \zeta_jv_0$. Since, thus, $v_0$ and all $\zeta_i$ are nonzero by definition, we conclude that (9.1) must hold.

b) Suppose $(V, v_0)$ is a Whittaker pair with respect to $\zeta \in (\mathbb{k}[\{0\}])^n$. Let $Q = \text{Ann}_{R}v_0$. Clearly $Q$ is a proper left ideal of $R$. For any $r \in Q$ we have $0 = X_i rv_0 = \sigma_i(r)X_i v_0 = \zeta_i \sigma_i(r)v_0$ which shows that $\sigma_i(Q) \subseteq Q$ for any $i \in \{1, \ldots, n\}$. Since $R$ is Noetherian, $\sigma_i^{-1}(Q) \subseteq Q$ as well, which proves that $Q$ is $\mathbb{Z}^n$-invariant. In addition, if $(V, v_0)$ and $(W, w_0)$ are two isomorphic Whittaker pairs, then clearly $\text{Ann}_{R}v_0 = \text{Ann}_{R}w_0$. This shows that the map $\Psi$ is well-defined.

To prove that $\Psi$ is surjective, suppose that $Q$ is a proper $\mathbb{Z}^n$-invariant left ideal of $R$. We show that (12.2) extends the natural $R$-module structure on $R/Q$ to an $A$-module structure. We only prove that the following relations are preserved: $X_iY_j = \mu_{ij}Y_jX_i$ (i $\neq$ j) and $Y_jY_i = \gamma_{ij}^{-1}\mu_{ij}^{-1}(\gamma_{ij}^{-1}Y_jX_i)$ (i $\neq$ j). The other cases are identical to the generalized Weyl algebra case considered in [5, Section 3]. We have

$$X_iY_j \bar{r} = X_i \overline{\zeta_j^{-1}\sigma_j^{-1}(r)t_j} = \overline{\zeta_i \zeta_j^{-1}\sigma_i^{-1}(r)\sigma_j^{-1}(t_j)t_j}.$$  

Using that $\sigma_i(t_j) = \gamma_{ij}t_j$ (see (4.2)) and condition (9.1) we see that $X_iY_j \bar{r} = \mu_{ij}Y_jX_i \bar{r}$ for any $\bar{r} \in R/Q$. Similarly $Y_jY_i \bar{r} = \overline{\zeta_j^{-1}\sigma_j^{-1}(r)\sigma_j^{-1}(t_j)t_j}$ so using $\sigma_j^{-1}(t_j)t_j = \gamma_{ji}^{-1}\gamma_{ij}t_i\sigma_i^{-1}(t_j)$ and (9.1) again, we see that $Y_jY_i \bar{r} = \gamma_{ij}^{-1}\mu_{ji}^{-1}Y_jX_i \bar{r}$, $\forall i \neq j$. Thus $R/Q$ becomes an $A$-module which is a Whittaker module of type $\zeta$ with Whittaker vector $1 + Q$.

To prove that $\Psi$ is injective we may, as in [5], construct a universal Whittaker module $V_\mu$ of type $\zeta$ by putting $V_\mu = A \otimes_{A_\mu} k_\zeta$ where $A_\mu$ is the subalgebra of $A$ generated over $k$ by $X_1, \ldots, X_n$, and $k_\zeta$ is the 1-dimensional module over $A_\mu$ given by $X_1 1 := \zeta$. The map $\iota : R \to V_\mu, r \mapsto r \otimes 1$ is an $R$-module isomorphism. Then there is a unique morphism of Whittaker pairs from $(V_\mu, 1 \otimes 1)$ to any other Whittaker pair $(V, v_0)$ of type $\zeta$. And, identifying $V_\mu$ with $R$ via $\iota$, the kernel of the map $V_\mu \to V$, is precisely $\text{Ann}_{R}v_0$. So if $(V, v_0)$ and $(W, w_0)$ are two Whittaker pairs with $\text{Ann}_{R}v_0 = \text{Ann}_{R}w_0$, it means that they are isomorphic to the same quotient of the universal Whittaker pair of type $\zeta$, hence are isomorphic to each other.

c) If $\varphi : (V, v_0) \to (W, w_0)$ is a morphism of Whittaker pairs, then $\varphi(rv_0) = r\varphi(v_0) = rw_0$ so clearly $\text{Ann}_{R}v_0 \subseteq \text{Ann}_{R}w_0$. Conversely, if $Q_1 \subseteq Q_2$ are proper $\mathbb{Z}^n$-invariant left ideals, then there is an $R$-module morphism $\pi : R/Q_1 \to R/Q_2$ mapping $1 + Q_1$ to $1 + Q_2$. Since $\pi$ commutes with the $\mathbb{Z}^n$-action, one verifies that $\pi$ is automatically an $A$-module morphism. \qed
Corollary 9.3. Let $A = \frac{A_k(r, s, \Lambda)}{\langle n \rangle}$ be a simple quotient of a multiparameter twisted Weyl algebra as obtained in Theorem 5.8. Then $A$ has a Whittaker module iff

$$\lambda_{ij} = \left( \frac{r_{ij}}{r_{ji}} \right)^k \quad \forall i, j.$$  \hspace{1cm} (9.3)

Moreover, if (9.3) holds, then for each $\zeta \in (k\setminus\{0\})^n$ there is a unique Whittaker module over $A$ of type $\zeta$, namely the universal one, and it is a simple module.

References

[1] J. Alev, F. Dumas, Sur le corps des fractions de certaines algèbres quantiques, J. Algebra 170 (1994) 229–265.
[2] V.V. Bavula, Generalized Weyl algebras and their representations, Algebra i Analiz 4 No. 1 (1992) 75–97; English transl. in St. Petersburg Math. J. 4 (1993) 71–92.
[3] V.V. Bavula, Generalized Weyl algebras, kernel and tensor-simple algebras, their simple modules, CMS Conf. Proc. 14 (1993) 83–107.
[4] G. Benkart, unpublished manuscript.
[5] G. Benkart, M. Ondrus, Whittaker modules for generalized Weyl algebras, Represent. Theory 13 (2009) 141–164.
[6] M. Cohen, L.H. Rowen, Group graded rings, Comm. Algebra 11 (11) (1983) 1253–1270.
[7] Yu.A. Drozd, V.M. Futorny, S.A. Ovsienko, Harish-Chandra subalgebras and Gelfand-Zetlin modules, in: “Finite-dimensional algebras and related topics (Ottawa, ON, 1992)” NATO Adv. Sci. Inst. Ser. C. Math. Phys. Sci. 424 (1994) 79–93.
[8] V. Futorny, J.T. Hartwig, On the Consistency of Twisted Generalized Weyl Algebras, arXiv:1103.4374v1 [math.RA].
[9] J.T. Hartwig, Locally finite simple weight modules over twisted generalized Weyl algebras, J. Algebra 303 No. 1 (2006) 42–76.
[10] J.T. Hartwig, Twisted generalized Weyl algebras, polynomial Cartan matrices and Serre-type relations, to appear in Comm. Algebra.
[11] J.T. Hartwig, J. Öinert, Simplicity and maximal commutative subalgebras of twisted generalized Weyl algebras, arXiv:1009.4892v2 [math.RA].
[12] T. Hayashi, $q$-analogues of Clifford and Weyl algebras-spinor and oscillator representations of quantum enveloping algebras, Comm. Math. Phys. 127 (1990) 129–144.
[13] D.A. Jordan, Primitivity in skew Laurent polynomial rings and related rings, Math. Z. 213 (1993) 353–371.
[14] D.A. Jordan, A simple localization of the quantized Weyl algebra, J. Algebra 174 (1995) 267–281.
[15] G. Maltsiniotis, Groupes quantiques et structures différentielles, C. R. Acad. Sci. Paris Sér. I Math. 311 No. 12 (1990) 831–834.
[16] V. Mazorchuk, M. Ponomarenko, L. Turowska, Some associative algebras related to $U(g)$ and twisted generalized Weyl algebras, Math. Scand. 92 (2003) 5–30.
[17] V. Mazorchuk, L. Turowska, Simple weight modules over twisted generalized Weyl algebras, Comm. Alg. 27 No. 6 (1999) 2613–2625.
[18] V. Mazorchuk, L. Turowska, *-Representations of twisted generalized Weyl constructions, Algebr. Represent. Theory 5 No. 2 (2002) 163–186.
[19] W. Pusz, S. L. Woronowicz, Twisted second quantization, Reports on Math. Phys. 27 No. 2 (1989) 231–257.
[20] A. L. Rosenberg, Noncommutative algebraic geometry and representations of quantized algebras, Kluwer, Dordrecht (1995).

Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brasil
E-mail address: futorny@ime.usp.br

Department of Mathematics, Stanford University, Stanford, CA, USA
E-mail address: jonas.hartwig@gmail.com