Distributed $\mathcal{H}_2$ Control for Interconnected Discrete-Time Systems: A Dissipativity-based Approach

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Abstract—This work is concerned with the analysis, existence and synthesis of distributed output-feedback controllers, that achieve stability and $\mathcal{H}_2$ performance for discrete-time linear interconnected systems. We consider an interconnection structure of local controllers that resembles the plant’s interconnection structure, which may correspond to an arbitrary graph. The dissipativity-based approach to distributed discrete-time $\mathcal{H}_2$ control presented in this paper complements other dissipativity-based approaches in the literature to distributed continuous-time $\mathcal{H}_2$ control and distributed $\mathcal{H}_\infty$ control. Moreover, the developed method yields a convex alternative to state-of-the-art methods for distributed discrete-time $\mathcal{H}_2$ control, which are typically not convex or consider unstructured problems. We provide an overview of related results and show the relation between sufficient conditions for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance, for both discrete- and continuous-time interconnected linear systems. Sufficient conditions are stated for the existence of a distributed controller achieving a pre-specified $\mathcal{H}_2$ performance. A method for subsequent controller reconstruction is provided by an algebraic procedure. We illustrate the controller synthesis for a large-scale oscillator network, for which the central $\mathcal{H}_2$ control problem can be computationally intractable on a modern PC.

I. INTRODUCTION

INTERCONNECTED systems are an indispensable part of society. Typical examples include power networks, networks in systems biology, communication networks, economic systems and chemical plant networks [11–13]. Control of such systems faces various challenges, related to the distribution or dimensionality of the systems. The number of subsystems can be large, increasing the dimensionality of the system. Even when systems are not physically interconnected, the ever-increasing number of system connections through communication networks makes the related control challenges relevant and popular in the literature.

Classical control techniques view the interconnected systems as a single lumped system, which can lead to an impractical communication architecture or computationally intractable problems. As opposed to centralized control of large-scale systems, decentralized control uses only locally available control variables. For feedback control, this means that only local measurements are taken for determining local corrective actions. Because this feature renders the decentralized controller easily implementable, the literature on this topic has been thriving, cf. [2], [4]–[6]. The lack of inter-controller communication, however, can lead to serious issues related to performance or even instability of the system, see, e.g., [1].

In this work we focus on distributed control of interconnected linear systems, where local controllers can be interconnected. Distributed control can overcome limitations induced by decentralized control [7], such as performance limitations or requirements on the information constraints [8].

For continuous-time systems, sufficient conditions for the existence of a controller that admits the same interconnection structure as the plant and achieves unit $\mathcal{H}_\infty$ performance are developed in [7]. The basis for these sufficient conditions is laid by dissipativity theory, introduced by Willems in [9], which is also the cornerstone for this work. Synthesis of the controller in [7] involves solving a single linear matrix inequality for controller existence verification, followed by an algebraic controller reconstruction. Even for a moderate number of subsystems and interconnection variables, the corresponding linear matrix inequality can be of a large size [7], but it is sparse given a sparse interconnected system. The constraint dimension and number of optimization variables grows in fact affinely with respect to the number of subsystems [10]. Investigation of the structure of this inequality led to an algorithm to distribute not only the controller itself, but also the computation of the controller [11]. In [12], a discrete-time analogue of the work in [7] was presented. Additionally, synthesis of the distributed controller in [12] incorporates robust stability and robust $\mathcal{H}_\infty$ performance of the closed-loop system.

The performance criterion of interest in this work is the $\mathcal{H}_2$ norm. The $\mathcal{H}_2$ norm of a system has two interpretations. From a deterministic point-of-view, it coincides with the output energy of the impulse responses, while from a statistical point-of-view, the $\mathcal{H}_2$ norm equals the asymptotic output variance for a white noise excitation [13]. The latter interpretation is particularly interesting in a setting where stochastic assumptions on disturbance signals are key, e.g. in data-driven modeling of interconnected systems [14].

An approach to solve the discrete-time $\mathcal{H}_2$ output-feedback problem for interconnected systems was presented in [15]. The approach therein aims at minimizing a linear combination of the closed-loop system’s $\mathcal{H}_2$ norm and a cost related to the sparsity of the controller matrices. This method, however, unfortunately leads to a non-convex problem in general [15].

This work is supported by the European Research Council (ERC), Advanced Research Grant SYSDYNET, under the European Unions Horizon 2020 research and innovation programme (grant agreement No 694501).

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In [16], the discrete-time $\mathcal{H}_2$ problem is solved for a 'strictly causal' network, via the search for an unstructured controller that is subsequently transformed into a structured one. The continuous-time $\mathcal{H}_2$ counterpart of the $\mathcal{H}_\infty$ problem in [7] is considered in [17]. The authors apply the dissipativity approach to provide sufficient conditions for the continuous-time distributed $\mathcal{H}_2$ problem.

Motivated by the digitalization of controller implementations, in this paper we consider the problem of distributed $\mathcal{H}_2$ control of interconnected discrete-time linear systems. Observe that the considered problem complements existing results on continuous-time $\mathcal{H}_\infty$ [7, 12] and continuous-time $\mathcal{H}_2$ [17] distributed control problems. More specifically, we consider the synthesis of dynamic output-feedback controllers that admit the same interconnection structure as the system of interest and guarantee a given $\mathcal{H}_2$ performance for the controlled discrete-time interconnected system. Although merely sufficient, the derived conditions avoid the existing limitations of current methods regarding the discrete-time distributed $\mathcal{H}_2$ problem, such as unstructured decision variables [16] or non-convex problems [15].

An overview of linear matrix inequalities for discrete-time and continuous-time interconnected systems’ $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance is also provided, accompanied by a detailed controller reconstruction procedure. Such a complete compendium on dissipativity-based $\mathcal{H}_2$ and $\mathcal{H}_\infty$ conditions for interconnected systems is currently missing from the literature [2, 13, 18], while it is relevant due to distributed control problems in, e.g., smart grids [19], buildings [20] and irrigation networks [21]. We illustrate the controller design for a network of 218 oscillators and show that the developed synthesis method is indeed applicable to large-scale systems.

The remainder of this work is organized as follows: In Section II, we provide some preliminaries and define the interconnected system setting. Section III contains the interconnected system analysis. In Section IV, we provide an overview of dissipativity-based results for interconnected systems. Controller existence conditions and controller reconstruction are described in Section V and VI, respectively. In Section VII, we present a numerical example that illustrates the effectiveness of the proposed method and the ability of handling complex networks. Conclusions are summarized in Section VIII.

II. PRELIMINARIES

A. Basic nomenclature

The sets of non-negative integers and non-negative reals are denoted by $\mathbb{N}$ and $\mathbb{R}_{\geq 0}$, respectively. Given $a \in \mathbb{Z}$, $b \in \mathbb{Z}$ such that $a < b$, we denote $\mathbb{Z}_{[a:b]} := \{a, a+1, \ldots, b-1, b\}$. Define $\mathbb{R}^0 := \{\emptyset\}$, i.e., a singleton that contains the empty tuple. Let $I_n \in \mathbb{R}^{n \times n}$ denote the identity matrix. We write $I$ instead of $I_n$ if the matrix size $n$ is clear from the context. The operator $\text{col}()$, respectively $\text{row}()$, stacks its arguments in a column vector, respectively row vector. The block diagonal matrix $\text{diag}(X_1, \ldots, X_m)$ has matrices $X_i$, $i \in \mathbb{N}_{[1:m]}$, in its block diagonal entries. For $S \subseteq \mathbb{Z}$, the block diagonal matrix $\text{diag}_{i \in S} X_i$ has matrices $X_i$, $i \in S$, in its block diagonal entries. For a real symmetric matrix $X$, the inequality $X > 0$, respectively $X \succeq 0$, denotes that $X$ is positive definite, respectively positive semi-definite. The inertia of a real symmetric matrix $X$ is denoted by $\inertia(X) = (n_-, n_0, n_+)$, with $n_- := \text{in}_- X$, $n_0$, and $n_+ := \text{in}_+ X$ the number of negative, zero, and positive eigenvalues of $X$, respectively. For an $x \in \mathbb{R}^n$, let $\|x\|$ denote the Euclidean norm of $x$.

B. Notions for discrete-time systems

Consider a linear discrete-time system $\Sigma$ described by an input/state/output representation

$$\Sigma : \left\{ \begin{array}{l} x(k+1) = Ax(k) + Bd(k), \\ z(k) = Cx(k) + Dd(k), \end{array} \right.$$ 

with state variable $x : \mathbb{Z} \to \mathbb{R}^n$, (disturbance) input variable $d : \mathbb{Z} \to \mathbb{R}^m$ and output variable $z : \mathbb{Z} \to \mathbb{R}^p$.

**Definition II.1.** System $\Sigma$ is called asymptotically stable (AS) if for each $\varepsilon > 0$, there exists $\delta(\varepsilon)$ so that, for $d = 0$, $\|x(0)\| < \delta \Rightarrow \|x(k)\| < \varepsilon$, $\forall k \geq 0$

and

$$x(0) \in \mathbb{R}^n \Rightarrow \|x(k)\| \to 0 \text{ as } k \to \infty.$$ 

**Definition II.2.** The $\mathcal{H}_\infty$ norm of an AS system $\Sigma$ having transfer function $T(z) := C(zI - A)^{-1}B + D$ is defined by

$$\|\Sigma\|_{\mathcal{H}_\infty} := \left( \frac{1}{2\pi} \text{trace} \int_{-\pi}^\pi T^*(e^{j\omega})T(e^{j\omega}) \, d\omega \right)^{\frac{1}{2}}.$$ 

**Lemma II.1.** For an AS system $\Sigma$, $\|\Sigma\|_{\mathcal{H}_\infty}^2 = \text{trace}(B^TMB + D^TD)$ with $M \succeq 0$ satisfying

$$A^TMA - M + C^TC = 0.$$ 

**Proof.** We refer the reader to Appendix A

The following result is a discrete-time version of one of the equivalence results in [13, Proposition 3.13], and will be instrumental for the proof of Proposition III.1.

**Proposition II.1.** Let system $\Sigma$ be AS and let $\gamma \in \mathbb{R}_{>0}$. The following statements are equivalent:

(i) $\|\Sigma\|_{\mathcal{H}_\infty} < \gamma$.

(ii) There exists $X > 0$ so that

$$A^TXA - X + C^TC \prec 0 \text{ and } \text{trace}(B^TXTB + D^TD) < \gamma^2.$$ 

**Proof.** We refer the reader to Appendix B

C. Discrete-time interconnected systems

We represent the structure of the interconnected system by a graph $G = (V, E)$, where $V$ is the vertex set, of cardinality $L \in \mathbb{N}$, and $E \subseteq V \times V$ is the edge set. Each vertex $P_i \in V$, $i \in \mathbb{Z}_{[1:L]}$, corresponds to a discrete-time system, called a subsystem of the interconnected system. An edge $(P_i, P_j) \in E$ exists if subsystems $P_i$ and $P_j$ are interconnected. For every $i \in \mathbb{Z}_{[1:L]}$, subsystem $P_i$ is assumed not to be connected to
itself, i.e., \((\mathcal{P}_i, \mathcal{P}_j) \notin E\). For each \(\mathcal{P}_i \in V\), we define the set of neighboring subsystems \(\mathcal{N}_i := \{\mathcal{P}_j \in V | (\mathcal{P}_i, \mathcal{P}_j) \in E\}\).

Each subsystem \(\mathcal{P}_i \in V\) is assumed to admit a state-space representation

\[
\begin{pmatrix}
i(i(k + 1)) \\
o_i(k) \\
s_i(k) \\
u_i(k)
\end{pmatrix} = \begin{pmatrix}
A^{T_T} & A^{T_S} & B^{T_d} & B^{T_u} \\
A^{T_S} & A^{S_S} & B^{S_d} & B^{S_u} \\
C^{T_T} & C^{T_S} & D^{T_d} & D^{T_u} \\
C^{T_S} & C^{S_S} & D^{S_d} & D^{S_u}
\end{pmatrix} \begin{pmatrix}
x_i(k) \\
s_i(k) \\
d_i(k) \\
u_i(k)
\end{pmatrix},
\]

(1)

where \(x_i : \mathbb{Z} \rightarrow \mathbb{R}^{k_i}\) is the subsystem’s state, \(o_i : \mathbb{Z} \rightarrow \mathbb{R}^{n_i}\) and \(s_i : \mathbb{Z} \rightarrow \mathbb{R}^{n_i}\) are the outgoing and incoming interconnection variables, \(z_i : \mathbb{Z} \rightarrow \mathbb{R}^{n_i}\) and \(d_i : \mathbb{Z} \rightarrow \mathbb{R}^{n_i}\) are the performance output and disturbance input, and \(y_i : \mathbb{Z} \rightarrow \mathbb{R}^{p_i}\) and \(u_i : \mathbb{Z} \rightarrow \mathbb{R}^{p_i}\) are the measured output and control input, respectively.

For each \(i \in \mathbb{Z}_{[1:L]}\), write the in- and outgoing interconnection signals \(s_i\) and \(o_i\) as

\[s_i = \text{col}(s_{i1}, s_{i2}, \ldots, s_{iL})\quad\text{and}\quad o_i = \text{col}(o_{i1}, o_{i2}, \ldots, o_{iL}),\]

so that \(((s_{ij}, o_{ij}))\) denotes the interconnection channel between subsystem \(\mathcal{P}_i\) and subsystem \(\mathcal{P}_j\). For the ease of the interconnection definition, we assume, without loss of generality, that \(o_{ij}, s_{ij}, o_{ji}\) and \(s_{ji}\) are all elements of \(\mathbb{R}^{n_{ij}}\), \(n_{ij} \in \mathbb{Z}_{\geq 0}\), i.e., they have the same size. Indeed, one can always add zero components to signal vectors of smaller size and the corresponding state-space matrices in (1). Note that \((s_{ii}, o_{ii}) \in \mathbb{R}^0\) for all \(i \in \mathbb{Z}_{[1:L]}\) since \((\mathcal{P}_i, \mathcal{P}_i) \notin E\).

The interconnection between subsystem \(i\) and subsystem \(j\) is defined through the interconnection equation

\[
\begin{pmatrix}
o_j(k) \\
s_j(k)
\end{pmatrix} = \begin{pmatrix}
o_i(k) \\
s_i(k)
\end{pmatrix}, \quad \forall k \in \mathbb{Z}.
\]

(2)

System \(i\) and system \(j\) are thus interconnected if \(n_{ij} > 0\), which is equivalent with \((\mathcal{P}_i, \mathcal{P}_j) \in E\), and not interconnected if \(n_{ij} = 0\), which is equivalent with \((\mathcal{P}_i, \mathcal{P}_j) \notin E\).

Consider the case where there are no control inputs and measured outputs are present, i.e., \(u_i, y_i \in \mathbb{R}^0\) for all \(i \in \mathbb{Z}_{[1:L]}\), and let

\[
\begin{align*}
A^{TT} &= \text{diag} A^{TT}_{\mathcal{P}_i}, \quad A^{TS} = \text{diag} A^{TS}_{\mathcal{P}_i}, \\
A^{ST} &= \text{diag} A^{ST}_{\mathcal{P}_i}, \quad A^{SS} = \text{diag} A^{SS}_{\mathcal{P}_i}, \\
C^{T} &= \text{diag} C^{T}_{\mathcal{P}_i}, \quad C^{S} = \text{diag} C^{S}_{\mathcal{P}_i},
\end{align*}
\]

(3)

Using (1), the interconnected system can be compactly represented by

\[
\begin{pmatrix}
x(k + 1) \\
o(k) \\
s(k) \\
u(k)
\end{pmatrix} = \begin{pmatrix}
A^{TT} & A^{TS} & B^{T_d} & B^{T_u} \\
A^{ST} & A^{SS} & B^{S_d} & B^{S_u} \\
C^{T} & C^{S} & D^{T_d} & D^{T_u}
\end{pmatrix} \begin{pmatrix}
x(k) \\
s(k) \\
d(k)
\end{pmatrix},
\]

with \(x(k) := \text{col}_{i \in \mathbb{Z}_{[1:L]}} x_i(k), o(k) := \text{col}_{i \in \mathbb{Z}_{[1:L]}} o_i(k), s(k) := \text{col}_{i \in \mathbb{Z}_{[1:L]}} s_i(k), z(k) := \text{col}_{i \in \mathbb{Z}_{[1:L]}} z_i(k)\) and \(d(k) := \text{col}_{i \in \mathbb{Z}_{[1:L]}} d_i(k)\), with the interconnection equation

\[
o(k) = \Delta s(k), \quad \forall k \in \mathbb{Z},
\]

where \(\Delta \in \mathbb{R}^{n \times n}, n = \sum_{i=1}^{L} n_i\), which is equivalent with interconnection equation (2) for all \((i, j) \in \mathbb{Z}_{[1:L]}\).

Elimination of the interconnection variables yields a state-space representation

\[
\begin{align*}
\mathcal{P}_I : \begin{pmatrix}
x(k + 1) \\
z(k)
\end{pmatrix} &= \begin{pmatrix}
A_Z & B_Z \\
C_Z & D_Z
\end{pmatrix} \begin{pmatrix}
x(k) \\
d(k)
\end{pmatrix}
\end{align*}
\]

(4)

where

\[
\begin{align*}
A_Z & := A^{TT} - \Delta A^{SS} \quad B_Z := A^{TS}, \\
C_Z & := C^{T} - \Delta C^{S}, \quad D_Z := D^{T_d}.
\end{align*}
\]

Introduce the interconnection variable subspace

\[
\mathcal{S}_I = \{(o, s) \in \mathbb{R}^{2n} | o = \Delta s\},
\]

such that (2) is equivalent with \((o(k), s(k)) \in \mathcal{S}_I\) for all \(k \in \mathbb{Z}\), and consider the interconnection variable subspace

\[
\mathcal{S}_I := \{(o, s) \in \mathbb{R}^{2n} | \text{col}(o_i, s_i) \in \text{im} \text{col}(A^{SS}_i, I), \quad i \in \mathbb{Z}_{[1:L]}\}.
\]

Definition II.3. The interconnected system \(\mathcal{P}_I\) is said to be well-posed if \(\mathcal{S}_I \cap \mathcal{S}_I = \{0\}\).

Well-posedness of the interconnected system implies that, given \(x_i(0)\) and \(d_i\), the signals \(x_i, o_i, s_i\) and \(z_i\) are unique, and is equivalent with \(\Delta - A^{SS}\) being non-singular [17, Lemma 1].

D. Distributed controller

The distributed controller that we will consider in the sequel is also an interconnected system, with graph \(G_C = (V_C, E_C)\). Each vertex \(C_i \in V_C\) corresponds to a local controller for subsystem \(\mathcal{P}_i \in V, i \in \mathbb{Z}_{[1:L]}\), so that the cardinality of \(V_C\) is equal to the cardinality of \(V\). Two local controllers are interconnected if and only if the two corresponding subsystems are interconnected, i.e., \((C_i, C_j) \in E_C\) if and only if \((\mathcal{P}_i, \mathcal{P}_j) \in E\). Hence, local controllers are only allowed to communicate directly if their corresponding subsystems are interconnected.

Each local controller \(C_i \in V_C\) is a discrete-time system that admits the state space representation

\[
\begin{align*}
\xi_i(k + 1) & = \begin{pmatrix}
A^{TT}_i & A^{TS}_i & B^{T_d}_i & B^{T_u}_i \\
A^{ST}_i & A^{SS}_i & B^{S_d}_i & B^{S_u}_i \\
C^{T}_i & C^{S}_i & D^{T_d}_i & D^{T_u}_i
\end{pmatrix} \begin{pmatrix}
\xi_i(k) \\
\sigma_i(k) \\
o_i(k) \\
u_i(k)
\end{pmatrix}
\end{align*}
\]

(5)

where \(\xi_i : \mathbb{Z} \rightarrow \mathbb{R}^{k_i}\) is the local controller’s state, and \(\sigma_i^c : \mathbb{Z} \rightarrow \mathbb{R}^{n_i}\) and \(s_i^c : \mathbb{Z} \rightarrow \mathbb{R}^{n_i}\) are the controller’s outgoing and incoming interconnection (communication) variables, respectively.

Similar to the interconnection of subsystems \(\mathcal{P}_i\), the interconnection between local controller \(C_i\) and \(C_j\) is defined through the interconnection variables \(\sigma_{ij}^c, s_{ij}^c \in \mathbb{R}^{n_{ij}}:\)

\[
\begin{pmatrix}
\sigma_{ij}^c(k) \\
s_{ij}^c(k)
\end{pmatrix} = \begin{pmatrix}
\sigma_{ji}^c(k) \\
s_{ji}^c(k)
\end{pmatrix}, \quad \forall k \in \mathbb{Z}.
\]
E. Closed-loop interconnected system

For each subsystem $\mathcal{P}_i$, we assume without loss of generality, see e.g. $[7]$, that there is no direct feed-through from the control input $u_i$ to the measured output $y_i$, i.e., $D_i^{yu} = 0$. In this case, the local closed-loop (controlled) system, $\mathcal{K}_i$, say, can be represented by

$$
\begin{pmatrix}
x_i^C(k+1) \\
o_i^C(k) \\
z_i(k)
\end{pmatrix} =
\begin{pmatrix}
(A_{i}^{TT})_c & (A_{i}^{TS})_c & (B_i^T)_c \\
(A_{i}^{ST})_c & (A_{i}^{SS})_c & (B_i^S)_c \\
(C_i^T)_c & (C_i^S)_c & (D_i)_c
\end{pmatrix}
\begin{pmatrix}
x_i^C(k) \\
o_i^C(k) \\
z_i(k)
\end{pmatrix} +
\begin{pmatrix}
(x_i, \xi_i) \\
(\xi_i, o_i) \\
(s_i, s_i')
\end{pmatrix}.
\tag{6}
$$

where $x_i^C := \text{col}(x_i, \xi_i)$, $o_i^C := \text{col}(o_i, o_i')$ and $s_i^C := \text{col}(s_i, s_i')$. Such a representation is obtained through elimination of the control variables $y_i, u_i$, as depicted in Figure [1]. Moreover, the state-space matrices of the local closed-loop system are affine with respect to the state-space matrices of the local controller:

$$
\Gamma_i = U_i^T \Theta_i V_i + W_i,
$$

with

$$
U_i^T :=
\begin{pmatrix}
0 & 0 & B_i^{Tu} \\
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & D_i^{yu}
\end{pmatrix},
$$

$$
\Theta_i :=
\begin{pmatrix}
(A_i^{TT})_c & (A_i^{TS})_c & (B_i^T)_c \\
(A_i^{ST})_c & (A_i^{SS})_c & (B_i^S)_c \\
(C_i^T)_c & (C_i^S)_c & (D_i)_c
\end{pmatrix},
$$

$$
V_i :=
\begin{pmatrix}
0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I \\
C_i^{Ty} & 0 & C_i^{yS} & 0 & D_i^{yd}
\end{pmatrix},
$$

$$
W_i :=
\begin{pmatrix}
(A_i^{TT})_c & 0 & A_i^{TS} & 0 & B_i^{Te} \\
0 & 0 & 0 & 0 & 0 \\
A_i^{ST} & 0 & A_i^{SS} & 0 & B_i^{Se} \\
C_i^{Te} & 0 & C_i^{yS} & 0 & D_i^{ed}
\end{pmatrix}.
$$

The closed-loop interconnected system $\mathcal{K}_\mathcal{I}$ is obtained analogously to $\mathcal{P}_\mathcal{I}$. An example of a closed-loop interconnected system is depicted in Figure [2].

F. Dissipative interconnected systems

As a basis for the analysis of the interconnected system and synthesis of the distributed controller, we employ the theory of dissipative dynamical systems $[9]$. For the analysis, we consider the uncontrolled subsystems with $u_i, y_i \in \mathbb{R}^0$ for all $i \in \mathbb{Z}_{[1:L]}$, in this subsection.

**Definition II.4.** Subsystem $\mathcal{P}_i$ is said to be dissipative with respect to the supply function $\sigma_i : \mathcal{S}_i \times \mathcal{O}_i \times D_i \times Z_i \to \mathbb{R}$, if there exists a non-negative storage function $V_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$, so that for all $t \in \mathbb{Z}_{\geq 0}$ the inequality

$$
V_i(x_i(t)) - V_i(x_i(0)) \leq \sum_{k=0}^{t-1} \sigma_i(s_i(k), o_i(k), d_i(k), z_i(k))
$$

holds for all solutions $(x_i, s_i, o_i, d_i, z_i)$ for system $[1]$.

In this work, we restrict the class of considered storage functions to quadratic functions

$$
V_i(x_i) := x_i^T X_i x_i, \quad i \in \mathbb{Z}_{[1:L]},
$$

with $X_i > 0$. Supply functions are restricted to be quadratic functions of the form

$$
\sigma_i(s_i, o_i, d_i, z_i) := \sigma_i^{\text{int}}(s_i, o_i) + \sigma_i^{\text{ext}}(d_i, z_i), \quad i \in \mathbb{Z}_{[1:L]},
$$

with ‘internal’ supply functions

$$
\sigma_i^{\text{int}}(s_i, o_i) := \sum_{j=1}^{L} \sigma_{ij}(s_{ij}, o_{ij}),
$$

$$
\sigma_{ij}(s_{ij}, o_{ij}) := \begin{pmatrix} o_{ij} \\ s_{ij} \end{pmatrix}^T X_{ij} \begin{pmatrix} o_{ij} \\ s_{ij} \end{pmatrix},
$$

where $X_{ij}$ is a real symmetric matrix, and ‘external’ supply functions

$$
\sigma_i^{\text{ext}}(d_i, z_i) := \varepsilon_i d_i^T d_i - z_i^T z_i,
$$

where $\varepsilon_i \in \mathbb{R}_{\geq 0}$.\]
For any pair \((i, j) \in \mathbb{Z}_2, i \neq j\), the interconnection between subsystem \(P_i\) and subsystem \(P_j\) is said to be neutral if the internal supply functions satisfy

\[
0 = \sigma_{ij}(s_{ij}, o_{ij}) + \sigma_{ji}(s_{ji}, o_{ji}).
\]

(7)

One can interpret a neutral interconnection as a lossless one; no ‘energy’ is dissipated or supplied through the interconnection channel. For the considered internal supply functions, the neutrality condition \((7)\) is equivalent with

\[
0 = X_{ij} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} X_{ji} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},
\]

where we used the interconnection equations \((2)\).

III. ANALYSIS RESULTS

A. Uncontrolled system analysis

The following result provides sufficient conditions for well-posedness, stability, and the \(H_2\) norm of the interconnected system. The result provides a discrete-time counterpart of the continuous-time result \([17\text{ Theorem 1}]\). Define the matrix

\[
T_i := \begin{pmatrix}
I & 0 & 0 & 0 \\
A_{iT}^T & A_{iS}^T & B_{iT}^d & 0 \\
A_{iS}^T & A_{iS}^S & B_{iS}^d & 0 \\
0 & I & 0 & 0 \\
C_i & C_i & D_i & D_i^d \\
0 & 0 & I & 0
\end{pmatrix}.
\]

Proposition III.1. Consider the interconnected system \(P_i\), with \(u_i, y_i \in \mathbb{R}^0\) for all \(i \in \mathbb{Z}_2\). The interconnected system \(P_i\) is well-posed, asymptotically stable and \(|P_i|_{\mathcal{H}_2} \leq \gamma\), \(\gamma \in \mathbb{R}^\geq\), if \(B_{iS}^d = 0\) for all \(i \in \mathbb{Z}_2\), and there exist \(X_i \in \mathbb{R}^{n_i \times n_i}, X_i > 0, \varepsilon_i \in \mathbb{R}^{n_i}, \) symmetric \(X_{ij} \in \mathbb{R}^{n_j \times n_i}, (i,j) \in \mathbb{Z}_2^2\), and \(X_{ij} \in \mathbb{R}^{n_i \times n_j}, (i,j) \in \mathbb{Z}_2^2\), \(i > j\), with

\[
T_i^T \begin{pmatrix}
-X_i & 0 & 0 & 0 \\
0 & X_i & 0 & 0 \\
0 & 0 & Z_{i1}^{11} & Z_{i1}^{12} \\
0 & 0 & (Z_{i1}^{12})^T & Z_{i1}^{22} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & -\varepsilon_i I
\end{pmatrix} T_i < 0,
\]

(8)

where

\[
Z_{i1}^{11} := - \text{diag} X_{11}^{11}, \quad Z_{i1}^{22} := \text{diag} X_{11}^{11},
\]

\[
Z_{i1}^{12} := \text{diag} \left(- \text{diag} X_{12}^{12}, \text{diag} (X_{12}^{12})^T\right).
\]

Proof. Well-posedness is identically defined for continuous-time systems \([17\text{ Theorem 1}]\) and the proof for well-posedness of \(P_i\) is identical to the first part of the proof of \([7\text{ Theorem 1}]\), since \((8)\) implies the condition used therein. Let \((8)\) be true. We define the candidate local storage functions

\[
V_i(x_i) := x_i^T X_i x_i
\]

and the candidate global storage function

\[
V(x) := \sum_{i=1}^L V_i(x_i).
\]

Multiplication of inequality \((8)\) from the right and from the left with \(\text{col}(x_i(k), s_i(k), d_i(k))\) and its transpose yields

\[
0 > x_i^T (k+1) X_i x_i (k+1) - x_i^T (k) X_i x_i (k)
\]

\[
+ \begin{pmatrix} o_i (k) \\ s_i (k) \end{pmatrix}^T \begin{pmatrix} Z_{i1}^{11} & Z_{i1}^{12} \\ (Z_{i1}^{12})^T & Z_{i1}^{22} \end{pmatrix} \begin{pmatrix} o_i (k) \\ s_i (k) \end{pmatrix}
\]

\[
+ z_i^T (k) z_i (k) - \varepsilon_i d_i (k) d_i (k)
\]

\[
= V_i (x(k+1)) - V_i (x(k)) - \sigma_i^{\text{int}} (s_i (k), o_i (k)) - \sigma_i^{\text{ext}} (d_i (k), z_i (k)).
\]

Thus system \(P_i\) is dissipative with respect to the supply function \(\sigma_i\). Summing the latter inequality over \(i\) yields

\[
V(x(k+1)) - V(x(k)) < \sum_{i=1}^L \sigma_i^{\text{int}} + \sigma_i^{\text{ext}}.
\]

From the neutrality condition \((7)\), we observe that

\[
\sum_{i=1}^L \sigma_i^{\text{int}} = 0, \text{ and thus}
\]

\[
V(x(k+1)) - V(x(k)) < \sum_{i=1}^L \sigma_i^{\text{ext}}.
\]

(10)

To prove stability, consider the case that \(d(k) = 0\). Then

\[
V(x(k+1)) - V(x(k)) < - \sum_{i=1}^L z_i^T (k) z_i (k) \leq 0.
\]

Therefore, \(V\) is a Lyapunov function for the interconnected system \(P_i\) with \(d(k) = 0\), from which we conclude asymptotic stability of the interconnected system \([22\text{ Corollary 1.2}]\).

Next, we prove \(H_2\) performance for \(P_i\). From equation \((3)\) and inequality \((10)\), it follows that for all \((x, d)\)

\[
\begin{pmatrix}
x \\ d
\end{pmatrix}^T \begin{pmatrix} I & 0 \\ A_i & B_i \end{pmatrix} \begin{pmatrix} -X_i & 0 \\ 0 & X_i \end{pmatrix} \begin{pmatrix} I \\ A_i \\ B_i \end{pmatrix} \begin{pmatrix} x \\ d \end{pmatrix} < - \begin{pmatrix} x \\ d \end{pmatrix}^T \begin{pmatrix} C_i & D_i \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -E \\ 0 & I \end{pmatrix} \begin{pmatrix} C_i & D_i \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ d \end{pmatrix},
\]

with \(X_i := \text{diag}_{i \in \mathbb{Z}_2} X_i\) and \(E := \text{diag}_{i \in \mathbb{Z}_2} \varepsilon_i I\). Hence

\[
A_i^T X_i A_i - X_i + C_i^T C_i A_i^T X_i B_i + C_i^T D_i B_i^T X_i A_i + D_i^T C_i 
\]

\[
A_i^T X_i B_i + D_i^T D_i E < 0,
\]

which implies

\[
A_i^T X_i A_i - X_i + C_i^T C_i < 0.
\]

(11)

Since \(B_{iS}^d = 0\) for all \(i \in \mathbb{Z}_2\), we have

\[
\text{trace} \left( B_i^T X_i B_i + D_i^T D_i \right)
\]

\[
= \text{trace} \left( \sum_{i=1}^L (B_i^d)^T X_i B_i^d + (D_i^d)^T D_i^d \right)
\]

\[
= \sum_{i=1}^L \text{trace} \left( (B_i^d)^T X_i B_i^d + (D_i^d)^T D_i^d \right) < \gamma^2.
\]

(14)
Hence, by Proposition II.1 inequalities (11) and (14) imply \( \| P_\infty \|_{\mathcal{H}_2} < \gamma \) and the proof is completed.

We illustrate the analysis conditions in Proposition II.1 by a simple example.

**Example III.1.** Consider two identical scalar subsystems described by

\[
x_i(t+1) = \frac{1}{2}x_i(t) + \frac{1}{10}s_i(t) + d_i(t), \quad i = 1, 2, \quad k \in \mathbb{Z}
\]

and \( z_i(t) = o_i(t) = x_i(t) \), with interconnection constraints \( s_1(t) = o_2(t) \), \( s_2(t) = o_1(t) \). It is easily verified that LMI (2) holds for \( i = 1, 2 \), with \( X_i = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} \), \( \epsilon_i = 20 \), \( X_{11}^{12} = X_{12}^{11} = -\frac{2}{\gamma} \) and \( X_{21}^{12} = 0 \). By Proposition III.2, the interconnected system is well-posed, asymptotically stable and the approximation \( \| P_\infty \|_{\mathcal{H}_2} < \gamma \) holds for all \( \gamma > \sqrt{X_1 + X_2} = \sqrt{2} \approx 1.87 \). The actual \( \mathcal{H}_2 \) norm of the system is \( \| P_\infty \|_{\mathcal{H}_2} = 1.68 \).

**B. Controlled system analysis**

The feasibility test provided by Proposition III.1 directly induces a feasibility test for well-posedness, stability and \( \mathcal{H}_2 \) performance for the closed-loop system, which consists of subsystems (6), as stated in the following corollary. This result is the discrete-time counterpart of [17] Lemma 4. Define the matrix

\[
T_i^K := \begin{pmatrix}
I & 0 & 0 \\
(\mathcal{A}^{TT})_i^K & (\mathcal{A}^{TS})_i^K & (\mathcal{B}^{TT})_i^K \\
(\mathcal{A}^{ST})_i^K & (\mathcal{A}^{SS})_i^K & (\mathcal{B}^{ST})_i^K \\
0 & I & 0 \\
(C_i^T)^K & (C_i^S)^K & (D_i)^K \\
0 & 0 & I
\end{pmatrix}.
\]

**Corollary III.1.** The interconnected system \( K_\infty \) of (6) is well-posed, asymptotically stable and \( \| K_\infty \|_{\mathcal{H}_2} < \gamma \), \( \gamma \in \mathbb{R}_0 \), if (B^K) = 0 for all \( i \in \mathbb{Z}[1:L] \) and there exist \( X_i^K \in \mathbb{R}^{2n_i \times 2n_i} \), \( X_i^K > 0 \), \( \epsilon_i \in \mathbb{R}_0^\infty \), symmetric \( (X_{ij}^{11})_i^K \in \mathbb{R}^{(n_i+n_j)^2 \times (n_i+n_j)^2} \), \( (i,j) \in \mathbb{Z}[1:L]^2 \), and \( (X_{ij}^{12})_i^K \in \mathbb{R}^{(n_i+n_j)^2 \times (n_i+n_j)^2} \), \( (i,j) \in \mathbb{Z}[1:L] \), \( i > j \), with

\[
(T_i^K)^T \begin{pmatrix}
-X_i^K & 0 & 0 \\
0 & X_i^K & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} T_i^K < 0,
\]

(15)

\[
\sum_{i=1}^L \text{trace} (K_i^T X_i^K (B_i^T)^K + (D_i)^K (D_i)^K < \gamma^2, \quad (16)
\]

where

\[
(Z_{1i}^{11})^K := \begin{pmatrix}
(Z_{11}^{11})_i^p & (Z_{11}^{11})_i^p c \\
(Z_{11}^{11})_i^p c & (Z_{11}^{11})_i^c
\end{pmatrix},
\]

\[
(Z_{1i}^{12})^K := \begin{pmatrix}
(Z_{12}^{12})_i^p & (Z_{12}^{12})_i^p c \\
(Z_{12}^{12})_i^p c & (Z_{12}^{12})_i^c
\end{pmatrix},
\]

(17)

and \( (Z_{2i}^{21})^K := \begin{pmatrix}
(Z_{22}^{21})_i^p & (Z_{22}^{21})_i^p c \\
(Z_{22}^{21})_i^p c & (Z_{22}^{21})_i^c
\end{pmatrix} \) with the submatrices defined in Appendix C.

**IV. OVERVIEW OF DISSIPATIVITY-BASED RESULTS FOR INTERCONNECTED SYSTEMS**

The purpose of this section is to provide a systematic overview of dissipativity-based performance results for interconnected systems in the literature. The overview shows how Proposition III.1 is consistent with respect to related results in the literature and the relation among them. Complementing our discrete-time \( \mathcal{H}_2 \) result, we recall the discrete-time \( \mathcal{H}_\infty \) result [12], continuous-time \( \mathcal{H}_\infty \) result [7], and continuous-time \( \mathcal{H}_2 \) result [17]. The focus is on the analysis results, since these form the basis for the controller existence conditions and construction. Furthermore, the step to the synthesis problem is similar for the above results for interconnected systems, as it is true for the centralized \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) problem [13]. We recall the matrices \( T_i, i \in \mathbb{Z}[1:L] \), defined in Section III.A, that will be used throughout this section for both discrete-time and continuous-time results.

**A. Discrete time \( \mathcal{H}_\infty \)**

Sufficient conditions for robust \( \mathcal{H}_\infty \) performance of discrete-time interconnected systems were derived in [12]. To be consistent with the \( \mathcal{H}_2 \) result and for ease of exposition, we recall the robust result from [12] for the nominal case, i.e., for the case that the parametric uncertainty in [12] is zero.

**Theorem IV.1** (Discrete-time \( \mathcal{H}_\infty \) [12]). The interconnected system \( P_\infty \) is well-posed, asymptotically stable and \( \| P_\infty \|_{\mathcal{H}_\infty} < \gamma \), \( \gamma \in \mathbb{R}_0^\infty \), if there exist \( X_i \in \mathbb{R}^{n_i \times n_i} \), \( X_i > 0 \), symmetric \( X_{ij}^{11} \in \mathbb{R}^{n_i \times n_j}, (i,j) \in \mathbb{Z}[1:L]^2 \) and \( X_{ij}^{12} \in \mathbb{R}^{n_j \times n_i}, (i,j) \in \mathbb{Z}[1:L], i > j \), such that

\[
T_i^T P_i T_i < 0,
\]

with

\[
P_i := \text{diag} \left( \begin{pmatrix}
-X_i & 0 \\
0 & X_i
\end{pmatrix}, \begin{pmatrix}
(Z_{11}^{11})_i^p & (Z_{11}^{11})_i^p c \\
(Z_{11}^{11})_i^p c & (Z_{11}^{11})_i^c
\end{pmatrix}, \begin{pmatrix}
1 & 0 \\
0 & -\gamma_i
\end{pmatrix} \right),
\]

(18)

**B. Continuous time \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \)**

We devote a section to related analysis results for continuous-time interconnected systems. The discrete-time \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) results are reminiscent of their continuous-time counterparts. In a continuous-time setting, each subsystem, \( P_i^\mathbb{R} \), say, is described by an input/state/output representation

\[
\begin{pmatrix}
\dot{x}_i(t) \\
o_i(t) \\
z_i(t) \\
y_i(t)
\end{pmatrix} = \begin{pmatrix}
(A_i^T & A_i^S & B_i^T & B_i^s & B_i^T s_i \\
A_i^ST & A_i^{SS} & B_i^{St} & B_i^{sSt} & B_i^{St} s_i \\
C_i^T & C_i^{SS} & D_i^T & D_i^{sSt} & D_i^{sSt} s_i \\
C_i^T & C_i^{SS} & D_i^T & D_i^{sSt} & D_i^{sSt} s_i
\end{pmatrix} \begin{pmatrix}
x_i(t) \\
o_i(t) \\
z_i(t) \\
y_i(t)
\end{pmatrix},
\]

where all signals have a domain \( \mathbb{R} \), instead of \( \mathbb{Z} \) for the discrete-time subsystems (1). The interconnection constraints are identically described by \( o(t) = \Delta s(t) \) for all \( t \in \mathbb{R} \) and well-posedness of the interconnected systems is defined identically. Hence, for \( y_i, u_i \in \mathbb{R}_0^\infty \), a well-posed continuous-time
interconnected system, \( \mathbb{P}^\mathbb{R}_I \), say, admits an input/state/output representation
\[
\mathbb{P}^\mathbb{R}_I : \begin{pmatrix} \dot{x}(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} A_I & B_I \\ C_I & D_I \end{pmatrix} \begin{pmatrix} x(t) \\ d(t) \end{pmatrix}.
\]

The following result was first presented in [23] for unit performance (\( \gamma = 1 \)) and is a nominal version of the robust result presented in [24], which has an application to power systems in [23].

**Theorem IV.2** (Continuous-time \( \mathcal{H}_\infty \) [7]). The interconnected system \( \mathbb{P}^\mathbb{R}_I \) is well-posed, asymptotically stable and \( \| \mathbb{P}^\mathbb{R}_I \|_{\mathcal{H}_\infty} < \gamma \), \( \gamma \in \mathbb{R}_{>0} \), if there exist \( X_i \in \mathbb{R}^{k_i \times k_i}, X_i > 0 \), symmetric \( X_{ij}^{11} \in \mathbb{R}^{n_{ij} \times n_{ij}}, (i,j) \in \mathbb{Z}_{[1:L]}^2 \), and \( X_{ij}^{12} \in \mathbb{R}^{n_{ij} \times n_{ij}}, (i,j) \in \mathbb{Z}_{[1:L]}^2, i > j \), such that
\[
T_i^\top P_i T_i < 0,
\]
with
\[
P_i := \text{diag} \left( \begin{pmatrix} 0 & X_i \\ X_i & 0 \end{pmatrix}, \begin{pmatrix} X_{ij}^{11} & X_{ij}^{12} \\ (X_{ij}^{11})^\top & (X_{ij}^{12})^\top \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -\varepsilon_i I \end{pmatrix} \right).
\]

The following result in [17] provides sufficient conditions for nominal \( \mathcal{H}_2 \) performance of continuous-time interconnected systems.

**Theorem IV.3** (Continuous-time \( \mathcal{H}_2 \) [17]). The interconnected system \( \mathbb{P}^\mathbb{R}_I \) is well-posed, asymptotically stable and \( \| \mathbb{P}^\mathbb{R}_I \|_{\mathcal{H}_2} < \gamma \), \( \gamma \in \mathbb{R}_{>0} \), if 
\[
B_i^{1d} = 0, D_i^{1d} = 0, \quad \text{for all } i \in \mathbb{Z}_{[1:L]}.
\]
and there exist \( X_i \in \mathbb{R}^{k_i \times k_i}, X_i > 0, \varepsilon_i \in \mathbb{R}_{>0} \), symmetric \( X_{ij}^{11} \in \mathbb{R}^{n_{ij} \times n_{ij}}, (i,j) \in \mathbb{Z}_{[1:L]}^2 \), and \( X_{ij}^{12} \in \mathbb{R}^{n_{ij} \times n_{ij}}, (i,j) \in \mathbb{Z}_{[1:L]}^2, i > j \), such that
\[
T_i^\top P_i T_i < 0,
\]
with
\[
P_i := \text{diag} \left( \begin{pmatrix} 0 & X_i \\ X_i & 0 \end{pmatrix}, \begin{pmatrix} Z_{ij}^{11} & Z_{ij}^{12} \\ (Z_{ij}^{11})^\top & (Z_{ij}^{12})^\top \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & -\varepsilon_i I \end{pmatrix} \right),
\]
A discussion of the results can be summarized as follows:

- The continuous-time and discrete-time conditions differ by the upper-left block in \( P_i \) for both the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) results. Additionally, for the continuous-time \( \mathcal{H}_2 \) result, it must hold that \( D_i^{1d} = 0 \) (a necessary condition for the continuous-time system to have a finite \( \mathcal{H}_2 \) norm).
- Hence, inequality [9] has an additional term with respect to inequality [17].
- The \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) conditions differ due to the external supply functions, which is reflected in the lower-right block in \( P_i \), and the additional inequalities [9] and [17].
- The feasibility problems related to the considered \( \mathcal{H}_2 \) analysis problems for interconnected systems have additional decision variables \( \varepsilon_i, i \in \mathbb{Z}_{[1:L]} \), with respect to the distributed \( \mathcal{H}_\infty \) problems, unlike their centralized counterparts [13].

In this section, we have provided a compendium for the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) analysis of interconnected linear systems, governed by either difference equations or differential equations. This compendium complements the recent LMI survey [18] and the book [13], which focus on lumped system analysis and centralized control. The controller synthesis conditions in the next section are directly linked to the analysis conditions. Therefore, in Section V we will focus on the discrete-time distributed \( \mathcal{H}_2 \) control problem only.

**V. Synthesis results**

Given the affine dependence of the closed-loop state-space matrices \( \Gamma_i \) of subsystems \( \mathcal{X}_i \) with respect to the controller parameters \( \Theta_i \), it follows that [15] is not an LMI with respect to the decision variables \( \Theta_i \), \( X_i \), \( (X_{ij}^{11})_K \) and \( (X_{ij}^{12})_K \). We can, however, first eliminate the controller parameters \( \Theta_i \) and arrive at an existence result in terms of LMIs and subsequently construct a controller such that the closed-loop system satisfies the conditions of Corollary [11]. The following lemma is instrumental for eliminating the controller parameters.

**Lemma V.1** ([25]). Consider a symmetric matrix \( P \) with \( \text{in}(P) = (m, 0, n) \). The matrix inequality
\[
\begin{pmatrix} I & 0 \\ U^\top \Theta V + W & P \end{pmatrix} < 0
\]
with \( W \in \mathbb{R}^{n \times m} \) has a solution \( \Theta \) if and only if
\[
V_\perp^\top \begin{pmatrix} I & 0 \\ W^\top & P \end{pmatrix} V_\perp < 0 \quad \text{and} \quad U_\perp^\top \begin{pmatrix} -W_\perp^\top & I \\ 0 & I \end{pmatrix} U_\perp > 0,
\]
with \( U_\perp \) and \( V_\perp \) any matrices whose columns form a basis of \( \ker U \) and \( \ker V \), respectively.

**A. Distributed controller existence**

As a consequence of Lemma [V.1] we have the following result regarding the existence of a distributed controller that guarantees well-posedness, closed-loop stability and \( \mathcal{H}_2 \) performance.

**Proposition V.1.** Let \( B_i^{1d} = 0, D_i^{1d} = 0 \) for all \( i \in \mathbb{Z}_{[1:L]} \). The following statements are equivalent:

- There exist controllers \( C_i \), with \( n_{ij}^C = 3n_{ij} \) for all \( (i,j) \in \mathbb{Z}_{[1:L]}^2 \) so that the controlled interconnected system described by (2), (5) and (6) admits \( \varepsilon_i \in \mathbb{R}_{>0} \), matrices \( X_i^C > 0, i \in \mathbb{Z}_{[1:L]} \), symmetric \( (X_{ij}^{11})_K \), \( (i,j) \in \mathbb{Z}_{[1:L]}^2 \), and \( (X_{ij}^{12})_K \), \( (i,j) \in \mathbb{Z}_{[1:L]}^2, i > j \), that satisfy inequalities [15] and [16].
- There exist \( \varepsilon_i > 0, X_i, Y_i \), symmetric \( (X_{ij}^{11})_P, (Y_{ij}^{11})_P \) for all \( (i,j) \in \mathbb{Z}_{[1:L]}^2 \) and \( (X_{ij}^{12})_P, (Y_{ij}^{12})_P \) for all \( (i,j) \in \mathbb{Z}_{[1:L]}^2, i > j \), that satisfy
\[
\begin{pmatrix} X_i & I \\ I & Y_i \end{pmatrix} > 0,
\]
\[
\sum_{i=1}^L \text{trace} \left( (B_i^{1d})^\top X_i B_i^{1d} + (D_i^{1d})^\top D_i^{1d} \right) < \gamma^2,
\]
We give the non-convex conditions in Proposition [V.1] with variable \( \varepsilon_i \) in (20) and \( \varepsilon_i^{-1} \) in (21), to be consistent with the continuous-time result in [17]. For fixed \( \varepsilon_i \), the conditions (20) and (21) are LMIs. The non-convex existence conditions can be transformed into a bilinear optimization problem subject to LMIs as in [17]. We stress, however, that the existence of \( \varepsilon_i > 0 \) s.t. (20) and (21) hold, is equivalent with the existence of \( \alpha_i > 0 \) and \( \beta_i > 0 \) such that (20) holds with \( \varepsilon_i \) replaced by \( \alpha_i \) and (21) holds with \( \varepsilon_i^{-1} \) replaced by \( \beta_i \). These equivalent conditions are LMIs. The proof follows mutatis mutandis.

**VI. DISTRIBUTED CONTROLLER CONSTRUCTION**

In essence, the controller construction consists of two parts: (i) the extension of the matrices \( X_i, Y_i, (X_{ij}^{11})_p, (Y_{ij}^{11})_p, (X_{ij}^{12})_p \) and \( (Y_{ij}^{12})_p \), obtained through the existence result Proposition [V.1] to the closed-loop matrices \( X_i^K, (Z_i^{11})_k, (Z_i^{12})_k \) and \( (Z_i^{22})_k \) and (ii) the computation of controller matrices \( \Theta_i \) such that the conditions in Corollary [III.1] are satisfied. One procedure to construct the distributed controller is provided in this section.

The controller construction is not limited to the discrete-time \( \mathcal{H}_c \) distributed control problem; it can also be used for the continuous-time \( \mathcal{H}_\infty \) and \( \mathcal{H}_c \) distributed control problem. We emphasize that the controller construction procedure is performed for each controller \( C_i \) individually, while the LMIs (18), (19), (20) and (21) are solved centrally, due to coupling in inequalities (19), (20) and (21).

Let \( X_i, Y_i, (X_{ij}^{11})_p, (Y_{ij}^{11})_p, (X_{ij}^{12})_p \) and \( (Y_{ij}^{12})_p \) satisfy LMIs (19), (19), (20) and (21). Let \( i \in \mathbb{Z}_{[1:L]} \). First, we construct the closed-loop matrices

\[
X_i^K := \begin{pmatrix} X_i & X_i^{PC} \\ Y_i^{PC} & X_i^C \end{pmatrix}, \quad Y_i^K := \begin{pmatrix} Y_i & Y_i^{PC} \\ (Y_i^{PC})^T & Y_i^C \end{pmatrix},
\]

so that \( X_i^K = (Y_i^K)^{-1} > 0 \). The extension of \( X_i \) and \( Y_i \) to their closed-loop counterparts \( X_i^K \in \mathbb{R}^{2k_i \times 2k_i} \) and \( Y_i^K \in \mathbb{R}^{2k_i, \times 2k_i} \) is well-known for the centralized quadratic performance problem (including the \( \mathcal{H}_\infty \) control problem), see e.g. [13, Theorem 4.2], [26], and can be performed as follows. Inequality (18) is equivalent to \( I - X_iY_i < 0 \), hence \( I - X_iY_i \) is of rank \( k_i \). Take non-singular matrices \( M_i, N_i \in \mathbb{R}^{k_i \times k_i} \) so that \( M_iN_i^T = I - X_iY_i \). Now, we find \( Y_i^K \) as the unique solution to the linear equation

\[
\begin{pmatrix} Y_i & I \\ M_i & 0 \end{pmatrix} = Y_i^K \begin{pmatrix} I & X_i \\ 0 & M_i^T \end{pmatrix},
\]

and set \( X_i^K := (Y_i^K)^{-1} \). It is clear that \( X_i^K \) and \( Y_i^K \) are of the form (22). Observe that \( X_i^K > 0 \) and \( (Y_i^K)^{-1} > 0 \) is equivalent to \( I - X_iY_i < 0 \), by application of the Schur complement to the explicit expression of the solution \( Y_i^K \) to (22).

Let \( (i, j) \in \mathbb{Z}_{[1:L]}^2, i > j \) and let \( X_{ij}^{11}, X_{ij}^{12} \in \mathbb{R}^{2n_{ij} \times 2n_{ij}} \) be defined by

\[
X_i^{ij} := \begin{pmatrix} (X_{ij}^{11})_p & (X_{ij}^{12})_p \\ (X_{ij}^{12})^T_p & -(X_{ij}^{11})_p \end{pmatrix},
\]
By [10] Lemma 21, there exist matrices $M_{ij}^{12}, N_{ij}^{12} \in \mathbb{R}^{2n_{ij} \times n_{ij}}$ and $M_{ij}^{22}, N_{ij}^{22} \in \mathbb{R}^{n_{ij} \times n_{ij}}$ so that

$$\begin{pmatrix} X_{ij}^P & M_{ij}^{12} \\ (M_{ij}^{12})^T & M_{ij}^{22} \end{pmatrix} = \begin{pmatrix} Y_{ij}^P & N_{ij}^{12} \\ (N_{ij}^{12})^T & N_{ij}^{22} \end{pmatrix}^{-1},$$

with

$$\ln \left( \begin{pmatrix} X_{ij}^P & M_{ij}^{12} \\ (M_{ij}^{12})^T & M_{ij}^{22} \end{pmatrix} \right) = \left( \epsilon^-_{ij}, 0, t^+_{ij} \right),$$

if and only if

$$\ln^- \left( \begin{pmatrix} X_{ij}^P & I \\ I & M_{ij}^{12} \end{pmatrix} \right) \leq \epsilon^-_{ij} \quad \text{and} \quad \ln^+ \left( \begin{pmatrix} X_{ij}^P & I \\ I & M_{ij}^{12} \end{pmatrix} \right) \leq t^+_{ij}.$$

For $l_{ij} = 6n_{ij}$ and $l_{ij}^+ = t^+_{ij} = 4n_{ij}$, the latter inertia requirements are satisfied [7]. The construction of such $M_{ij}^{12}, N_{ij}^{12}$ and $M_{ij}^{22}, N_{ij}^{22}$ follows from the constructive proof for [10] Lemma 21. Let $M_{ij}^{22} := \text{diag}(I, -I) \in \mathbb{R}^{6n_{ij} \times 6n_{ij}}$ and $M_{ij}^{12} \in \mathbb{R}^{2n_{ij} \times 6n_{ij}}$ so that in $M_{ij}^{22} = (i^-_{ij}, 0, i^+_{ij}) - \text{in} Y_{ij}^P$ and

$$X_{ij}^P - (Y_{ij}^P)^{-1} = M_{ij}^{12} M_{ij}^{22} (M_{ij}^{12})^T = M_{ij}^{12} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} (M_{ij}^{12})^T.$$

Since $X_{ij}^P - (Y_{ij}^P)^{-1}$ is symmetric, it commutes with itself and hence it admits an eigendecomposition [27] Corollary 5.4.4

$$X_{ij}^P - (Y_{ij}^P)^{-1} = V_{ij} \Lambda_{ij} V_{ij}^T,$$

with $\Lambda_{ij} = \text{diag}_{k \in \mathbb{Z}_{[1,2n_{ij}]}} (\lambda_{ij,k})$, $\lambda_{ij,1} \geq \lambda_{ij,2} \geq \cdots \geq (\lambda_{ij,2n_{ij}})$ and $V_{ij}$ a unitary matrix whose unitary columns correspond to eigenvectors. Clearly, if we let $V_{ij} = V_{ij} [\Lambda_{ij}]^T$, then $X_{ij}^P - (Y_{ij}^P)^{-1} = (V_{ij}^T V_{ij}) \text{diag}(I, -I) (V_{ij}^T V_{ij})^T$, with $V_{ij} = (V_{ij}^T V_{ij})$. Thus we take

$$M_{ij}^{12} := \frac{1}{\sqrt{3}} \begin{pmatrix} V_{ij}^T & V_{ij}^T & V_{ij} & V_{ij} & V_{ij} & V_{ij} \\ V_{ij} & V_{ij} & V_{ij}^T & V_{ij} & V_{ij} & V_{ij}^T \\ V_{ij} & V_{ij} & V_{ij} & V_{ij}^T & V_{ij} & V_{ij} \\ V_{ij} & V_{ij} & V_{ij} & V_{ij} & V_{ij} & V_{ij}^T \end{pmatrix},$$

such that (24) holds. Hence, by defining

$$M_{ij}^{12} := \begin{pmatrix} (X_{ij}^{11})_{PC} & (X_{ij}^{12})_{PC} \\ (X_{ij}^{12})_{PC}^T & - (X_{ij}^{11})_{PC} \end{pmatrix},$$

$$M_{ij}^{22} := \begin{pmatrix} (X_{ij}^{11})_{C} & (X_{ij}^{12})_{C} \\ (X_{ij}^{12})_{C}^T & - (X_{ij}^{11})_{C} \end{pmatrix},$$

we can construct the scales

$$Z_{ij}^K := \begin{pmatrix} (Z_{ij}^{11})_K & (Z_{ij}^{12})_K \\ (Z_{ij}^{12})_K^T & (Z_{ij}^{11})_K \end{pmatrix},$$

$$W_{ij}^K := \begin{pmatrix} (W_{ij}^{11})_K & (W_{ij}^{12})_K \\ (W_{ij}^{12})_K^T & (W_{ij}^{11})_K \end{pmatrix},$$

such that $Z_{ij}^K = (W_{ij}^K)^{-1}$, with $(W_{ij}^{11})_K$, $(W_{ij}^{12})_K$ and $(W_{ij}^{22})_K$ analogously defined as $(Z_{ij}^{11})_K$, $(Z_{ij}^{12})_K$ and $(Z_{ij}^{22})_K$ in Appendix C.

For each $i \in \mathbb{Z}_{[1, L]}$, let $P_i := \text{diag}(-X_i^K, X_i^K, Z_i^K, I, -\epsilon, I)$. Permute the rows and columns of $P_i$ to obtain

$$P_i := \begin{pmatrix} -X_i^K & 0 & 0 & 0 & 0 \\ 0 & (Z_i^{22})_K & 0 & 0 & (Z_i^{12})_K^T \\ 0 & 0 & -\epsilon I & 0 & 0 \\ 0 & 0 & 0 & X_i^K & 0 \\ 0 & 0 & 0 & 0 & (Z_i^{11})_K \end{pmatrix},$$

such that

$$\left( V_{ij} \right)^T \begin{pmatrix} I \\ W_i \end{pmatrix} P_i \begin{pmatrix} I \\ W_i \end{pmatrix} \left( V_{ij} \right) \perp < 0 \quad \text{and} \quad \left( U_i \right)^T \begin{pmatrix} -W_i^T \\ I \end{pmatrix} P_i^{-1} \begin{pmatrix} -W_i^T \\ I \end{pmatrix} \left( U_i \right) \perp > 0.$$

By Lemma V.1 there exists a controller matrix $\Theta_i$ so that (15) is satisfied, or, equivalently, so that

$$\left( I \\ U_i^T \Theta_i V_i + W_i \right) P_i \left( I \\ U_i^T \Theta_i V_i + W_i \right) \perp < 0.$$

To construct such a $\Theta_i$, let $H_i$ and $J_i$ be non-singular matrices such that

$$V_i H_i := (V_i 0), \quad U_i J_i := (U_i 0),$$

with $V_i$ and $U_i$ having full column rank. Then with $Q_i := J_i^T W_i H_i$, we can rewrite inequality (30) as (25)

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} H_i^T & 0 \\ 0 & J_i^{-1} \end{pmatrix} P_i \begin{pmatrix} H_i^T & 0 \\ 0 & J_i^{-1} \end{pmatrix} = \text{diag}_{\varepsilon_{ij}},$$

and, hence, as

$$\begin{pmatrix} R_i \quad (I \quad S_i) \end{pmatrix}^T \Pi_i \begin{pmatrix} R_i \quad (I \quad E_i) \quad S_i \end{pmatrix} \perp < 0,$$

with $E_i := \bar{U}_i^T \Theta_i \bar{V}_i + Q_i^{11}$ and

$$R_i := \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \quad S_i := \begin{pmatrix} 0 & Q_i^{12} \\ 0 & 0 & Q_i^{21} \end{pmatrix}.$$

Now, because $E_i$ is an unrestricted unknown in (31), a suitable solution is given by $E_i = (E_2)_i (E_1)_i^{-1}$ [25], with $F_i := \text{col}((E_1)_i, (E_2)_i)$ solving the quadratic inequality

$$\begin{pmatrix} R_i^T \Pi_i R_i - R_i^T \Pi_i S_i (S_i^T \Pi_i S_i)^{-1} S_i^T \Pi_i R_i \end{pmatrix} F_i \perp < 0.$$
Let the columns of $F_i$ be vectors that span the eigenspaces of $\Gamma_i$ that are associated with negative eigenvalues, such that (32) is satisfied. If the resulting $(E_i)_{ii}$ is singular, one can always choose a $\delta_i > 0$ such that $(E_i)_{ii} + \delta_i I$ is non-singular and
\[
\begin{pmatrix}
(E_i)_{ii} + \delta_i I \\
(E_i)_{ij}
\end{pmatrix}^\top \Omega_i \begin{pmatrix}
(E_i)_{ii} + \delta_i I \\
(E_i)_{ij}
\end{pmatrix} < 0.
\]
Finally, a suitable controller matrix $\Theta_i$ can then be constructed by solving the linear equation
\[
\bar{U}_i^\top \Theta_i \bar{V}_i = (E_i)_{ii}((E_i)_{ii} + \delta_i I)^{-1} - Q_i^{11}.
\]

Reconstruction of the distributed controller is summarized in the following algorithm.

**Algorithm VI.1.** For each pair $(i, j) \in \mathbb{Z}^2_{[1:L]}$, let $X_i, Y_i, \xi_i, (X_{ij}^{11}), (Y_{ij}^{11}), p, \text{ and for each pair } (i, j) \in \mathbb{Z}^2_{[1:L]}, i > j, \text{ let } (X_{ij}^{12}), (Y_{ij}^{12}), p, \text{ be computed to satisfy (18), (19), (20), (21)}. \text{ For each } i \in \mathbb{Z}^2_{[1:L]}, \text{ the synthesis of controller } C_i \text{ proceeds as follows:}

- Extend the matrices $X_i, Y_i$ to the closed-loop scales $X_i^k, Y_i^k \in \mathbb{R}^{k_i \times k_i}$ as defined in (22).
- Construct the matrices $(X_{ij}^{11})_{PC}, (X_{ij}^{12})_{PC}, (X_{ij}^{11})_{CP}, \text{ and } (X_{ij}^{12})_{CP} \text{ to obtain the scale } Z_i^k \text{ as defined in (26)}.
- Construct $P_i$ as defined in (28), satisfying (29).
- Solve the linear equation (33) to obtain a controller $\Theta_i$ that satisfies (30).

The steps in Algorithm VI.1 require a matrix decomposition, an eigendecomposition and solving a linear equation, which are standard linear algebra problems.

**VII. Numerical examples**

To illustrate the distributed $\mathcal{H}_2$ controller synthesis method, we consider a (linear) coupled-oscillator network consisting of $L$ oscillators. For each node $i \in \mathbb{Z}^2_{[1:L]}$, the dynamics are described by
\[
m_i \ddot{\theta}_i + b_i \dot{\theta}_i = u_i - \sum_{j \in N_i} k_{ij}(\theta_i - \theta_j) + d_i,
\]
with inertia $m_i$, damping $b_i$ and coupling coefficient $k_{ij} = k_{ji}$. The mechanical analogue of a linear coupled-oscillator network is a network of masses that are interconnected through linear springs and have linear damping. A typical system that is modeled as a linear oscillator network is a (linearized) power network, consisting of generators ($m_i \neq 0$) and loads ($m_i = 0$) [28], [29]. The local measurement is assumed to be $y_i := \dot{\theta}_i$ and the performance output is set equal to the state $z_i := x_i := \col(\theta_i, \dot{\theta}_i)$. We use a zero-order hold discretization with sampling time $T = 0.1$ seconds for each subsystem and an approximation $e^M = I + M$, so that each subsystem $\mathcal{P}_i$ has an input/state/output representation (1) with matrices
\[
A_i^{TT} = \begin{pmatrix}
1 & -\frac{T}{m_i} & 0 \\
\frac{k_{ij}}{m_i} & 1 - \frac{b_i T}{m_i} & 0 \\
-\sum_{j \in N_i} \frac{k_{ij}}{m_i} & 0 & 1
\end{pmatrix},
A_i^{TS} = \begin{pmatrix}
t & 1 - \frac{b_i T}{m_i} \\
\frac{k_{ij}}{m_i} & 1 - \frac{b_i T}{m_i} \\
-\sum_{j \in N_i} \frac{k_{ij}}{m_i} & 0
\end{pmatrix},
A_i^{ST} = C_i^{VT} = \col_j \begin{pmatrix}
1 \\
0
\end{pmatrix},
A_i^{SS} = 0_{n_i \times n_i},
\]
\[
B_i^{Su} = B_i^{Tu} = 0_{n_i \times 1}, B_i^{Td} = B_i^{Tu} = \col(0, \frac{T}{m_i}), C_i^{T} = I_2,
\]
\[
C_i^{TS} = 0_{2 \times n_i}, D_i^{sd} = D_i^{su} = 0_{2 \times 1}, D_i^{sd} = D_i^{su} = 0.
\]

Initially, we consider an oscillator network with a triangular structure, as depicted in Figure 3. The systems’ inertia, damping and coupling coefficients are $m_1 = 3$, $m_2 = 1$, $m_3 = 2$, $b_1 = 2$, $b_2 = 1$, $b_3 = 4$ and $k_{12} = k_{23} = k_{31} = 1$. We aim for the synthesis of a distributed controller that achieves unit $\mathcal{H}_2$ performance for the controlled network, i.e., we verify the feasibility of the LMIs in Proposition VII for $\gamma := 1$. For $\varepsilon = 10$, we find that the LMIs are feasible using MOSEK Optimization Suite [30].
The centralized $H_2$ feasibility problem is computationally intractable on the same computer, due to insufficient memory.

Interconnection of $P_I$ with the computed distributed controller $C_T$ results in the interconnected system $K_T$, which is asymptotically stable and $\|K_T\|_{\mathcal{H}_2} = 0.80$. For simulation of the interconnected closed-loop system, we draw all subsystems’ initial conditions from a normal distribution $\mathcal{N}(0,1)$ and all controllers’ initial conditions set identical to zero, results in the trajectories depicted in Figure 7. We observe that the subsystems’ and controllers’ states asymptotically converge to zero, illustrating asymptotic stability of the closed-loop system.

Next, we consider a large-scale oscillator network consisting of $L = 218$ subsystems, with parameters $m_i$, $b_i$ and $k_{ij}$ random variables drawn from uniform distributions $\mathcal{U}(2,3)$, $\mathcal{U}(2,3)$ and $\mathcal{U}(1,2)$, respectively. The interconnection structure is described by the graph $G$, which is visualized in Figure 5. This graph has 218 vertices and 648 edges. The goal is to synthesize a distributed controller that achieves $\|K_T\|_{\mathcal{H}_2} < \gamma$ for $\gamma = 1$. For each $i \in \mathbb{Z}_{\{1:218\}}$, we select $\varepsilon_i = 10$ and consider the LMIs from Proposition VI.1. The corresponding feasibility problem, a semidefinite programming problem consisting of 873 matrix variables, 2593 scalar variables and 5196 constraints, was solved in 0.73 seconds using MOSEK Optimization Suite 10 on a PC with a 2.3GHz Intel Core i5 processor and 16GB memory. The distributed controller is constructed via the procedure outlined in section VI. For comparison, we consider the feasibility problem for a centralized discrete-time $H_2$ controller via the methodology in [13], for the same oscillator network. The corresponding semidefinite programming problem is considerably larger, consisting of 2 matrix variables (size: $1962 \times 1962$ and $1308 \times 1308$) and 595031 constraints. We found that the centralized $H_2$ feasibility problem is computationally intractable on the same computer, due to insufficient memory.

Interconnection of $P_I$ with the computed distributed controller $C_T$ results in the interconnected system $K_T$, which is asymptotically stable and $\|K_T\|_{\mathcal{H}_2} = 0.80$. For simulation of the interconnected closed-loop system, we draw all subsystems’ initial conditions from a normal distribution $\mathcal{N}(0,1)$ and all controllers’ initial conditions set identical to zero. A plot of the subsystems’ state components $\{x_i\}_1$ and $\{x_i\}_2$, is provided in Figure 6a and 6b, respectively. Figure 7a and 7b show the state components of the distributed controller, for each controller $C_i$, $i \in \mathbb{Z}_{\{1:218\}}$.

For illustration of the controlled network’s ability to reduce output variance in the case of stochastic disturbance signals, we initialize the system with $x(0) = 0$, $\xi(0) = 0$, and apply signals $d_i$, that are mutually uncorrelated Gaussian white-noise processes with unit variance. Asymptotically, the obtained $H_2$ norm for the controlled network is directly related to the output variance through $\lim_{k \to \infty} E z^T(k) z(k) = \|K_T\|_{\mathcal{H}_2}^2$. 

$$X_3 = \begin{pmatrix} 23.2 & 0.394 \\ 0.394 & 15.8 \end{pmatrix}, \quad Y_3 = 10^{-2} \begin{pmatrix} 7.46 & -5.28 \\ -5.28 & 58.6 \end{pmatrix},$$

and $(X_{12})_p = (X_{13})_p = -10.46$, $(X_{13})_p = -10.53$, $(Y_{12})_p = (Y_{13})_p = -11.24$, $(Y_{13})_p = (Y_{12})_p = -11.23$, and $(X_{12})_p = (Y_{12})_p = 0$ for all pairs $(i,j)$ $\in \{(2,1),(3,1),(3,2)\}$. Since the LMIs are feasible, a distributed controller that achieves $\|K_T\|_{\mathcal{H}_2} < 1$ exists. We compute a distributed controller according to Algorithm VI.1, which results in the controller matrices given in Appendix F.
have provided a decentralized procedure for controller reconfiguration. We applied the developed synthesis method to a system that requires computing the solution to two standard dissipativity-based approaches to the continuous-time and the distributed controller proceeds along the same steps for a structured dynamic output-feedback controller that achieves a desired performance of discrete-time linear interconnected systems. Based on the analysis conditions, existence conditions were derived for a structured dynamic output-feedback controller that achieves a desired performance level when interconnected with the plant. Reconfiguration of the distributed controller proceeds on the same steps for the stochastic disturbances by the distributed controller.

VIII. CONCLUSIONS

We have developed sufficient conditions for well-posedness, stability and $\mathcal{H}_2$ performance of discrete-time linear interconnected systems. Based on the analysis conditions, existence conditions were derived for a structured dynamic output-feedback controller that achieves a desired $\mathcal{H}_2$ performance level when interconnected with the plant. Reconfiguration of the distributed controller proceeds on the same steps for the stochastic disturbances by the distributed controller. We applied the developed synthesis method to a large-scale network consisting of 218 coupled oscillators, to illustrate the applicability and computational tractability of our solution to the discrete-time distributed $\mathcal{H}_2$ control problem.

APPENDIX A

PROOF OF LEMMA III.1

By Parseval’s theorem we infer that

$$\|\Sigma\|_{\mathcal{H}_2}^2 = \text{trace} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} T^*(e^{j\omega}) T(e^{j\omega}) \, d\omega \right)$$

$$= \text{trace} \left( D^T D + \sum_{k=0}^{\infty} B^T (A^k)^T C^T C A^k B \right)$$

$$= \text{trace} \left( D^T D + B^T M B \right),$$

with $M := \sum_{k=0}^{\infty} (A^k)^T C^T C A^k \geq 0$ the observability Gramian, that satisfies the matrix equation

$$A^T M A - M + C^T C = 0.$$

APPENDIX B

PROOF OF PROPOSITION III.1

We first show (i) $\Rightarrow$ (ii). Since $\Sigma$ is AS, there exists $P > 0$ so that $A^T P A - P < 0$. Then by Lemma III.1 there exists an $\epsilon \in \mathbb{R}_{>0}$ so that $X := M + \epsilon P$ satisfies

$$\text{trace} \left( B^T X B + D^T D \right)$$

$$= \text{trace} \left( B^T M B + D^T D + \epsilon B^T P B \right) < \gamma^2,$$

with $M \geq 0$ so that $A^T M A - M + C^T C = 0$. Hence, $X > 0$ and

$$A^T X A - X + C^T C$$

$$= A^T M A - M + C^T C + \epsilon (A^T P A - P) < 0.$$

Next, we show (ii) $\Rightarrow$ (i). If (ii) is true, then there exists a matrix $\Gamma$ so that

$$0 = A^T X A - X + C^T C + \Gamma^T \Gamma$$

$$= A^T X A - X + \begin{pmatrix} C & \Gamma \end{pmatrix}^T \begin{pmatrix} C & \Gamma \end{pmatrix}.$$

Hence, with $T_1(z) := \Gamma(z I - A)^{-1} B$, we use Lemma III.1 to conclude that

$$\gamma^2 > \|\text{col}(T, T_1)\|^2_{\mathcal{H}_2}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} T^*(e^{j\omega}) T(e^{j\omega}) + T_1^*(e^{j\omega}) T_1(e^{j\omega}) \, d\omega$$

$$\geq \|T\|^2_{\mathcal{H}_2},$$

which concludes the proof.

APPENDIX C

CLOSED-LOOP MATRICES IN COROLLARY III.1

$$(Z_{11})_{ij} := - \text{diag} \left( (X_{11})_{ij} \right), (Z_{12})_{ij} := \text{diag} \left( (X_{12})_{ij} \right),$$

$$(Z_{21})_{ij} := \text{diag} \left( (X_{21})_{ij} \right), (Z_{22})_{ij} := \text{diag} \left( (X_{22})_{ij} \right),$$

$$(Z_{11})_{ij} := - \text{diag} \left( (X_{11})_{ij} \right), (Z_{12})_{ij} := \text{diag} \left( (X_{12})_{ij} \right),$$

$$(Z_{21})_{ij} := \text{diag} \left( (X_{21})_{ij} \right), (Z_{22})_{ij} := \text{diag} \left( (X_{22})_{ij} \right),$$

$$(Z_{11})_{ij} := - \text{diag} \left( (X_{11})_{ij} \right), (Z_{12})_{ij} := \text{diag} \left( (X_{12})_{ij} \right),$$

$$(Z_{21})_{ij} := \text{diag} \left( (X_{21})_{ij} \right), (Z_{22})_{ij} := \text{diag} \left( (X_{22})_{ij} \right).$$

APPENDIX D

$\mathcal{H}_\infty$ NORM FOR DISCRETE-TIME SYSTEMS

Let $L^2_2$ be the set of all Lebesgue measurable functions $d: \mathbb{N} \to \mathbb{R}^m$ for which

$$\|d\|^2_{L_2} := \sum_{k=0}^{\infty} \|d(k)\|^2 < \infty.$$
**Definition D.1.** The $\mathcal{H}_\infty$ norm of an AS system $\Sigma$ is defined by

$$\|\Sigma\|_{\mathcal{H}_\infty} := \sup_{\theta \neq 0 \in \ell^2_p} \|Td\|_{\ell_2}.$$ 

**APPENDIX E**

**Notions for continuous time systems**

Consider a linear continuous-time system $\Sigma_R$ described by an input/state/output representation

$$\Sigma_R : \begin{cases} \dot{x}(t) = Ax(t) + Bd(t), \\
z(t) = Cx(t) + Dd(t), \end{cases}$$

with state variable $x : \mathbb{R} \to \mathbb{R}^n$, (disturbance) input variable $d : \mathbb{R} \to \mathbb{R}^m$ and output variable $z : \mathbb{R} \to \mathbb{R}^p$.

**Definition E.1.** System $\Sigma_R$ is called asymptotically stable (AS) if for each $\varepsilon > 0$, there exists $\delta(\varepsilon)$ so that, for $d = 0$,

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall k \geq 0$$

and

$$x(0) \in \mathbb{R}^n \Rightarrow \|x(t)\| \to 0 \text{ as } k \to \infty.$$ 

Consider the transfer function $T_R(s) = C(sI - A)^{-1}B + D$ of $\Sigma_R$.

**Definition E.2.** The $\mathcal{H}_2$ norm of an AS system $\Sigma_R$ with $D = 0$ is defined by

$$\|\Sigma_R\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi} \text{ trace} \int_{-\infty}^{\infty} T_R^*(i\omega)T_R(i\omega) d\omega \right)^{\frac{1}{2}}.$$ 

Let $L^2_p$ be the set of all Lebesgue measurable functions $d : \mathbb{R} \to \mathbb{R}^m$ for which $\|d\|^2_{L^2_p} = \int_0^\infty \|d(t)\|^2 dt < \infty$.

**Definition E.3.** The $\mathcal{H}_\infty$ norm of an AS system $\Sigma_R$ is defined by

$$\|\Sigma_R\|_{\mathcal{H}_\infty} := \sup_{\theta \neq 0 \in \ell^2_p} \frac{\|T_Rd\|_{\ell_2}}{\|d\|_{\ell_2}}.$$ 

**APPENDIX F**

**Controller matrices for the triangular oscillator network**

| $\Theta_1$ | $\Theta_2$ | $\Theta_3$ |
|------------|------------|------------|
| -2.34      | -105.0     | -113.0     |
| 2.39       | -68.8      | -1.30      |
| -104.0     | 126.0      | -73.5      |
| 151.0      | -39.6      | 79.7       |
| -47.1      | -11.0      | 125.0      |
| -7.31      | 2.11       | 1.13       |

| $\Theta_1$ | $\Theta_2$ | $\Theta_3$ |
|------------|------------|------------|
| -2.13      | -103.0     | -138.0     |
| 0.81       | 183.0      | 72.5       |
| 1.32       | -103.0     | -138.0     |
| 72.5       | 71.5       | 65.5       |
| -138.0     | 1.94       | -1.30      |

| $\Theta_1$ | $\Theta_2$ | $\Theta_3$ |
|------------|------------|------------|
| -1.30      | 2.20       | -1.42      |
| 1.54       | -1.20      | -1.02      |
| 2.66       | -3.85      | -1.02      |

| $\Theta_1$ | $\Theta_2$ | $\Theta_3$ |
|------------|------------|------------|
| -11.0      | 115.0      | -0.975     |
| 1.55       | -2.52      | 5.55       |
| -0.575     | -113.0     | -56.5      |
| -81.4      | 35.4       | 138.0      |
| -2.50      | 1.63       | 5.05       |
REFERENCES

[1] M. Aoki, “On feedback stabilizability of decentralized dynamic systems,” *Automatica*, vol. 8, no. 2, pp. 163 – 173, 1972.

[2] J. Lunze, *Feedback Control of Large Scale Systems*. Upper Saddle River, NJ, USA: Prentice Hall PTR, 1992.

[3] F. Bullo, *Lectures on Network Systems*, 1st ed. CreateSpace, 2018, with contributions by J. Cortes, F. Dörfler, and S. Martinez.

[4] S.-H. Wang and E. Davison, “On the stabilization of decentralized control systems,” *IEEE Transactions on Automatic Control*, vol. 18, no. 5, pp. 473–478, Oct 1973.

[5] D. Šiljak, *Decentralized Control of Complex Systems*, ser. Mathematics in science and engineering. Academic Press, 1991.

[6] G. Scorletti and G. Duc, “An LMI approach to decentralized $H_\infty$ control,” *International Journal of Control*, vol. 74, no. 3, pp. 211–224, 2001.

[7] C. Langbort, R. S. Chandra, and R. D’Andrea, “Distributed control design for systems interconnected over an arbitrary graph,” *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1502–1519, 2004.

[8] M. Rotkowitz and S. Lall, “A characterization of convex problems in decentralized control,” *IEEE Transactions on Automatic Control*, vol. 51, no. 2, pp. 274–286, 2006.

[9] J. C. Willems, “Dissipative dynamical systems part I: General theory,” *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 321–351, Jan 1972.

[10] G. E. Dullerud and R. D’Andrea, “Distributed control of heterogeneous systems,” *IEEE Transactions on Automatic Control*, vol. 49, no. 12, pp. 2113–2128, Dec 2004.

[11] C. Langbort and R. D’Andrea, “Distributed control of heterogeneous systems interconnected over an arbitrary graph,” in *42nd IEEE International Conference on Decision and Control*, vol. 3, pp. 2835–2840.

[12] T. F. van der Els, “Robust distributed $H_\infty$ control of electrical power systems,” Master’s thesis, Eindhoven University of Technology, 2011. [Online]. Available: https://pure.tue.nl/ws/portalfiles/portal/47025560

[13] C. Scherer and S. Weiland, “Linear matrix inequalities in control,” October 2017, DISC lecture notes.

[14] P. M. J. Van den Hof, A. G. Dankers, P. S. C. Heuberger, and X. Bombois, “Identification of dynamic models in complex networks with prediction error methods – Basic methods for consistent module estimates,” *Automatica*, vol. 49, no. 10, pp. 2994 – 3006, 2013.

[15] J. Eillbrecht, M. Hilg, and O. Stursberg, “Distributed $H_2$-optimized output feedback controller design using the ADMM,” *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 10.389 – 10.394, 2017, 20th IFAC World Congress.

[16] A. S. Mohan Vamsi and N. Elia, “Optimal distributed controllers realizable over arbitrary networks,” *IEEE Transactions on Automatic Control*, vol. 61, no. 1, pp. 129–144, Jan 2016.

[17] X. Chen, H. Xu, and M. Feng, “$H_2$ performance analysis and $H_\infty$ distributed control design for systems interconnected over an arbitrary graph,” *Systems & Control Letters*, vol. 124, pp. 1 – 11, 2019.

[18] R. J. Caverly and J. R. Forbes, “LMI properties and applications in systems, stability, and control theory,” *ArXiv e-prints*, vol. arXiv:1903.08599, 2019.

[19] J. Stoustrup, A. Annaswamy, A. Chakraborty, and Z. Qu, Eds., *Smart Grid Control: Overview and Research Opportunities*, 1st ed. Germany: Springer, 2018.

[20] P. Moroşan, R. Bourdais, D. Dumur, and J. Buisson, “Distributed model predictive control for building temperature regulation,” in *Proceedings of the 2010 American Control Conference*, June 2010, pp. 3174–3179.

[21] M. Cantoni, E. Weyer, Y. Li, S. K. Osi, I. Mareels, and M. Ryan, “Control of large-scale irrigation networks,” *Proceedings of the IEEE*, vol. 95, no. 1, pp. 75–91, Jan 2007.

[22] R. E. Kalman and J. E. Bertram, “Control system analysis and design via the “second method” of Lyapunov; II—discrete-time systems,” *Journal of Basic Engineering*, vol. 82, no. 2, pp. 394–400, 06 1960.

[23] A. Jokic, T. F. van der Els, and S. Weiland, “Robust distributed $H_\infty$ control of electrical power systems,” in *2012 American Control Conference (ACC)*, June 2012, pp. 3637–3642.

[24] C. Scherer, “LPV control and full block multipliers,” *Automatica*, vol. 37, no. 3, pp. 361 – 375, 2001.

[25] P. Gahinet and P. Apkarian, “A linear matrix inequality approach to $H_\infty$ control with application to electrical power systems,” in *Proceedings of the 2010 American Control Conference (ACC)*, June 2010, pp. 3174–3179.

[26] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas*, 2nd ed. Princeton University Press, 2011.

[27] A. R. Bergen and D. J. Hill, “A structure preserving model for power system stability analysis,” *IEEE Transactions on Power Apparatus and Systems*, vol. PAS-100, no. 1, pp. 25–35, Jan 1981.

[28] F. Dörfler, M. Chertkov, and F. Bullo, “Synchronization in complex oscillator networks and smart grids,” *Proceedings of the National Academy of Sciences*, vol. 110, no. 6, pp. 2005–2010, 2013.

[29] MOSEK ApS, *MOSEK Optimization Suite Release 8.1.0.80*, 2019. [Online]. Available: https://docs.mosek.com/8.1/intro.pdf

[30] A. A. Stoorvogel, *The $H_\infty$ control problem: a state space approach*. New York: Prentice Hall, 1992.