The fastest possible continued fraction approximations of a class of functions

Xiaodong Cao*, Yoshio Tanigawa and Wenguang Zhai

Abstract

The goal of this paper is to formulate a systematical method for constructing the fastest possible continued fraction approximations of a class of functions. The main tools are the multiple-correction method, the generalized Mortici’s lemma and the Mortici-transformation. As applications, we will present some sharp inequalities, and the continued fraction expansions associated to the volume of the unit ball. In addition, we obtain a new continued fraction expansion of Ramanujan for a ratio of the gamma functions, which is showed to be the fastest possible. Finally, three conjectures are proposed.

1 Introduction

Let \( f(x) \) be a function defined on \((0, +\infty)\) to be approximated. We suppose that there exists a fixed positive integer \( \nu \) and a constant \( c \neq 0 \) such that

\[
\lim_{x \to +\infty} x^\nu f(x) = c.
\]
In this case, we say that the function $f(x)$ is of order $x^{-\nu}$ when $x$ tends to infinity, and denote

$$R(f(x)) := \nu,$$

where $\nu$ is the exponent of $x\nu$. For convenience, $R(0)$ is stipulated to be infinity. Hence, $R(f(x))$ characterizes the rate of convergence for $f(x)$ as $x$ tends to infinity. From [1], there exists a large positive number $X_0$ such that $f(x)/c > 0$ when $x > X_0$.

In analysis, approximation theory, applied mathematics, etc., we often need to investigate the rational function approximation problem. Let $\frac{P_l(x)}{Q_m(x)}$ be an approximation to $f(x)$ as $x$ tends to infinity, where $P_l(x)$ and $Q_m(x)$ are polynomials in $x$. Quite similarly to the rational approximation problem for an irrational number, in order to find a better approximation to $f(x)$, we have to increase the degrees of both $P_l(x)$ and $Q_m(x)$. The main interest in this paper is to try to look for the fastest possible continued fraction approximation or guess its approximation structure for $f(x)$ as $x$ tends to infinity.

The paper is organized as follows. In Sec. 2, we mainly introduce a definition to classify the continued fraction. In Sec. 3, we will prepare two preliminary lemmas for later use. In Sec. 4, we first develop further the previous multiple-correction method. Secondly, we introduce a transformation named as Moritici-transformation to change a kind of continued fraction approximation problem. In addition, we also give its Mathematica program for the reader’s convenience. Thirdly, similarly to Taylor’s formula, we introduce two definitions of the formal Type-I and Type-II continued fraction approximation of order $k$ for a function, and the formal continued fraction expansion, respectively. This section constitutes the main part of this paper. To illustrate our method formulated in Sec. 4, in Sec. 5 we use the volume of the unit ball as an example to present some new inequalities. In Sec. 6, we test the well-known generalized Lord Brouncker’s continued fraction formula, and show that it is the fastest possible. We also give some applications for the continued fraction formula involving the volume of the unit ball. In Sec. 7, we will use a continued fraction formula of Ramanujan to illustrate how to get the fastest possible form of the continued fraction expression. In Sec. 8, we explain how to guess the fastest possible continued fraction expansions, and give three conjectures associated to the special rate of gamma functions. In the last section, we analyze the related perspective of research in this direction.

## 2 Notation and definition

Throughout the paper, we use the notation $\lfloor x \rfloor$ to denote the largest integer not exceeding $x$. The notation $P_k(x)$ (or $Q_k(x)$) means a polynomial of degree $k$ in $x$. We will use the $\Phi(k; x)$ to denote a polynomial of degree $k$ in $x$ with the leading coefficient equals one, which may be different at each occurrence. While, the notation $\Psi(k; x)$ means a polynomial of degree $k$ in $x$ with all coefficients non-negative, which may be different at each occurrence. Let $(a_n)_{n\geq 1}$ and
(\(b_n\))\(_{n \geq 0}\) be two sequences of real numbers with \(a_n \neq 0\) for all \(n \in \mathbb{N}\). The generalized continued fraction

\[
\tau = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \cdots}} = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots = b_0 + \sum_{n=1}^{\infty} \left( \frac{a_n}{b_n} \right)
\]

is defined as the limit of the \(n\)th approximant

\[
\frac{A_n}{B_n} = b_0 + \sum_{k=1}^{n} \left( \frac{a_k}{b_k} \right)
\]
as \(n\) tends to infinity. The canonical numerators \(A_n\) and denominators \(B_n\) of the approximants satisfy the recurrence relations (see [8, p. 105])

\[
A_{n+2} = b_{n+2}A_{n+1} + a_{n+2}A_n, \quad B_{n+2} = b_{n+2}B_{n+1} + a_{n+2}B_n
\]

with the initial values \(A_0 = b_0, B_0 = 1, A_1 = b_0b_1 + a_1\) and \(B_1 = b_1\).

To describe our method clearly, we will introduce two definitions as follows.

**Definition 1.** Let \(c_0 \neq 0\), and \(x\) be a free variable. Let \((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\) and \((c_n)_{n=0}^{\infty}\) be three real sequences. The formal continued fraction

\[
\frac{c_0}{\Phi(\nu; x) + \sum_{n=0}^{\infty} \left( \frac{a_n}{x+b_n} \right)}
\]
is said to be a **Type-I** continued fraction. While,

\[
\frac{c_0}{\Phi(\nu; x) + \sum_{n=0}^{\infty} \left( \frac{a_n}{x^2+b_nx+c_n} \right)}
\]
is said to be a **Type-II** continued fraction.

**Remark 1.** The **Type-I** and **Type-II** are two kinds of fundamental structures we often meet. Certainly, we may define other-type continued fraction. Because of their complexity, in this paper we will not discuss the involved problems.

**Definition 2.** If the sequence \((b_n)_{n=0}^{\infty}\) is a constant sequence \((b)_{n=0}^{\infty}\) in the **Type-I** (or **Type-II**) continued fraction, we call the number \(\omega = b\) (or \(\omega = \frac{c}{2}\)) the MC-point for the corresponding continued fraction. We use \(\hat{x} = x + \omega\) to denote the MC-shift of \(x\).

If there exists the MC-point, we have the following **simplified form**

\[
\frac{c_0}{\Phi_1(\nu; \hat{x}) + \sum_{n=0}^{\infty} \left( \frac{a_n}{x+b_n} \right)} \quad \text{or} \quad \frac{c_0}{\Phi_1(\nu; \hat{x}) + \sum_{n=0}^{\infty} \left( \frac{a_n}{x^2+b_nx+c_n} \right)},
\]

where \(d_n = c_n - \frac{b^2}{4}\).
3 Two preliminary lemmas

Mortici [23] established a very useful tool for measuring the rate of convergence, which claims that a sequence \((x_n)_{n \geq 1}\) converging to zero is the fastest possible when the difference \((x_n - x_{n+1})_{n \geq 1}\) is the fastest possible. Since then, Mortici’s lemma has been effectively applied in many papers such as [10, 11, 13, 14, 24, 25, 26, 27]. The following lemma is a generalization of Mortici’s lemma. For details, readers may refer to [12].

**Lemma 1.** If \(\lim_{x \to +\infty} f(x) = 0\), and there exists the limit

\[
\lim_{x \to +\infty} x^\lambda (f(x) - f(x + 1)) = l \in \mathbb{R},
\]

with \(\lambda > 1\), then

\[
\lim_{x \to +\infty} x^{\lambda - 1} f(x) = \frac{l}{\lambda - 1}.
\]

In this paper, we will use the following simple inequality, which is a consequence of Hermite-Hadamard inequality.

**Lemma 2.** Let \(f\) be twice differentiable with \(f''\) continuous. If \(f''(x) > 0\), then

\[
\int_a^{a+1} f(x)dx > f(a + 1/2).
\]

4 The multiple-correction, the Mortici-transformation and the formal continued fraction expansion

4.1 The multiple-correction method

In this subsection, we will develop further the previous *multiple-correction method* formulated in [11, 12]. For some applications of this method, reader may refer to [10, 12, 13]. In fact, the *multiple-correction method* is a recursive algorithm, and one of its advantages is that by repeating correction-process we always can accelerate the convergence. More precisely, every non-zero coefficient plays an important role in accelerating the convergence. The *multiple-correction method* consists of the following several steps.

**Step 1)** The *initial-correction*. The initial-correction is vital. Determine the initial-correction \(\Phi_0(\nu; x)\) such that

\[
R\left( f(x) - \frac{c}{\Phi_0(\nu; x)} \right) = \max_{\Phi(\nu; x)} R\left(f(x) - \frac{c}{\Phi(\nu; x)} \right).
\]
(Step 2) The first-correction. If there exists a real number $\kappa_0$ such that

\begin{equation}
R \left( f(x) - \frac{c}{\Phi_0(\nu; x) + \frac{\kappa_0}{x}} \right) > R \left( f(x) - \frac{c}{\Phi_0(\nu; x)} \right),
\end{equation}

then we take the first-correction $MC_1(x) = \frac{\kappa_0}{x + \lambda_0}$ with

\begin{equation}
\lambda_0 = \max_{\lambda} R \left( f(x) - \frac{c}{\Phi_0(\nu; x) + \frac{\kappa_0}{x + \lambda}} \right).
\end{equation}

In this case, the first-correction has the form Type-I. Otherwise, we take the first-correction $MC_1(x)$ in the form Type-II, i.e. $MC_1(x) = \frac{\kappa_0}{x^2 + \lambda_0,1 x + \lambda_0,2}$ such that

\begin{equation}
(\kappa_0, \lambda_0,1, \lambda_0,2) = \max_{\kappa,\lambda_1,\lambda_2} R \left( f(x) - \frac{c}{\Phi_0(\nu; x) + \frac{\kappa_0}{x^2 + \lambda_1 x + \lambda_2}} \right).
\end{equation}

If $\kappa_0 = 0$, we stop the correction-process, which means that the rate of convergence can not be further improved only by making use of Type-I or Type-II continued fraction structure.

(Step 3) The second-correction to the $k$th-correction. If $MC_1(x)$ has the form Type-I, we take the second-correction

\begin{equation}
MC_2(x) = \frac{\kappa_0}{x + \lambda_0 + \frac{\kappa_1}{x + \lambda_1}},
\end{equation}

which satisfies

\begin{equation}
(\kappa_1, \lambda_1) = \max_{\kappa,\lambda} R \left( f(x) - \frac{c}{\Phi_0(\nu; x) + \frac{\kappa_0}{x^2 + \lambda_0,1 x + \lambda_0,2}} \right).
\end{equation}

Similarly to the first-correction, if $\kappa_1 = 0$, we stop the correction-process.

If $MC_1(x)$ has the form Type-II, we take the second-correction

\begin{equation}
MC_2(x) = \frac{\kappa_0}{x^2 + \lambda_0,1 x + \lambda_0,2 + \frac{\kappa_1}{x^2 + \lambda_1,1 x + \lambda_1,2}},
\end{equation}

such that

\begin{equation}
(\kappa_1, \lambda_1,1, \lambda_1,2) = \max_{\kappa,\lambda_1,\lambda_2} R \left( f(x) - \frac{c}{\Phi_0(\nu; x) + \frac{\kappa_0}{x^2 + \lambda_0,1 x + \lambda_0,2 + \frac{\kappa_1}{x^2 + \lambda_1,1 x + \lambda_1,2}}} \right).
\end{equation}

If $\kappa_1 = 0$, we also need to stop the correction-process.

If we can continue the above correction-process to determine the $k$th-correction function $MC_k(x)$ until some $k^*$ you want, then one may use a recurrence relation to determine $MC_k(x)$. More precisely, in the case of Type-I we choose

\begin{equation}
MC_k(x) = \frac{\kappa_{k-1}}{x + \lambda_{k-1}} \frac{\kappa_j}{x + \lambda_j}
\end{equation}
such that
\begin{equation}
(\kappa_{k-1}, \lambda_{k-1}) = \max_{\kappa, \lambda} R \left( f(x) - \left( \frac{c}{\Phi_0(\nu; x)} + \frac{\kappa_0}{x + \lambda_0} + \cdots + \frac{\kappa_{k-2}}{x + \lambda_{k-2}} + \frac{\kappa}{x + \lambda} \right) \right).
\end{equation}

While, in the case of Type-II we take
\begin{equation}
MC_k(x) = \prod_{j=0}^{k-1} \left( \frac{\kappa_j}{x^2 + \lambda_{j,1}x + \lambda_{j,2}} \right),
\end{equation}
which satisfies
\begin{equation}
(\kappa_{k-1}, \lambda_{k-1,1}, \lambda_{k-1,2}) = \max_{\kappa, \lambda; \lambda_1, \lambda_2} R \left( f(x) - G(\kappa, \lambda_1, \lambda_2; x) \right),
\end{equation}
where
\begin{equation*}
G(\kappa, \lambda_1, \lambda_2; x) := \frac{c}{\Phi_0(\nu; x)} + \frac{\kappa_0}{x^2 + \lambda_{1,1}x + \lambda_{1,2}} + \cdots + \frac{\kappa_{k-2}}{x^2 + \lambda_{k-2,1}x + \lambda_{k-2,2}} + \frac{\kappa}{x^2 + \lambda_1x + \lambda_2}.
\end{equation*}

Note that in the case of both Type-I and Type-II continued fraction approximation, if \( \kappa_{k-1} = 0 \), we must stop the correction-process. In other words, to improve the rate of convergence, we need to choose some more complex continued fraction structure instead of it.

\textit{Remark 2.} Sometimes, we need to consider its equivalent forms. For example, the Stirling’s formula reads (See, e.g. [1, p. 253])
\begin{equation}
\Gamma(x + 1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x, \quad x \to +\infty,
\end{equation}
which is equivalent to
\begin{equation}
\lim_{x \to \infty} x^3 f(x) = 1,
\end{equation}
where
\begin{equation}
f(x) = 8\pi^3 \left( \frac{x}{e} \right)^6 \Gamma^{-6}(x + 1).
\end{equation}

From the above asymptotic formula, we may study Ramanujan-type continued fraction approximation for the gamma function. For more details, see Cao [12] or next section. Moreover, we note that \( \Gamma \) has many equivalent forms. Hence, it is not difficult to see that the equivalent transformation of a practical problem influences directly the initial-correction and final continued fraction approximation.

\textit{Remark 3.} If \( \nu \) is a negative integer, our method is still efficient, i.e. we may consider the reciprocal of \( f(x) \).

\textit{Remark 4.} For comparison, we use the mathematical notation “\( R \)” and “\( \max \)” in the above definition, which make the method more clearly.
4.2 The Mortici-transformation

In this subsection we will explain how to look for all the related coefficients in $\Phi_0(\nu; x)$ and $MC_k(x)$. If we can expand $f(x)$ into a power series in terms of $1/x$ easily, then it is not difficult to determine $\Phi_0(\nu; x)$ and $MC_k(x)$. Similarly, if we may expand the difference $f(x) - f(x+1)$ into a power series in terms of $1/x$, by the generalized Moritici’s lemma we also can find $\Phi_0(\nu; x)$ and $MC_k(x)$, e.g. the Euler-Mascheroni constant, the constants of Landau, the constants of Lebesgue, etc. (See [11]). However, in many cases the previous two approaches are not very efficient, e.g. gamma function (see, Remark 2) and the ratio of the gamma functions (for example, see Sec. 7 below). Instead, we may employ the following method to achieve it.

First, we introduce the $k$th-correction relative error sequence $(E_k(x))_{k \geq 0}$ as follows

\begin{align}
(4.16) & \quad f(x) = \frac{c}{\Phi_0(\nu; x)} \exp (E_0(x)), \\
(4.17) & \quad f(x) = \frac{c}{\Phi_0(\nu; x) + MC_k(x)} \exp (E_k(x)), \quad k \geq 1,
\end{align}

where $\Phi_0(\nu; x)$ is a polynomial of degree $\nu$ in $x$ with the leading coefficient equals one, to be specified below.

It is easy to verify that

\[
\begin{align*}
& f(x) - \frac{c}{\Phi_0(\nu; x)} = \frac{c}{\Phi_0(\nu; x)} (\exp (E_0(x)) - 1), \\
& f(x) - \frac{c}{\Phi_0(\nu; x) + MC_k(x)} = \frac{c}{\Phi_0(\nu; x) + MC_k(x)} (\exp (E_k(x)) - 1), \quad k \geq 1.
\end{align*}
\]

It is well-known that

\[
\lim_{t \to 0} \frac{\exp(t) - 1}{t} = 1,
\]

by $\lim_{x \to \infty} E_k(x) = 0$ we obtain

\begin{align}
(4.18) & \quad R \left( f(x) - \frac{c}{\Phi_0(\nu; x)} \right) = \nu + R (E_0(x)), \\
(4.19) & \quad R \left( f(x) - \frac{c}{\Phi_0(\nu; x) + MC_k(x)} \right) = \nu + R (E_k(x)), \quad k \geq 1.
\end{align}

In this way, we turn the problem to solve $R (E_k(x))$.

Take the logarithm of (4.16) and (4.17), respectively, we deduce that

\[
\begin{align*}
& \ln \frac{f(x)}{c} = - \ln (\Phi_0(\nu; x)) + E_0(x), \\
& \ln \frac{f(x)}{c} = - \ln (\Phi_0(\nu; x) + MC_k(x)) + E_k(x), \quad k \geq 1.
\end{align*}
\]
Next, let us consider the difference

\begin{align}
E_0(x) - E_0(x + 1) &= \ln \frac{f(x)}{f(x + 1)} + \ln \frac{\Phi_0(\nu; x)}{\Phi_0(\nu; x + 1)}, \\
E_k(x) - E_k(x + 1) &= \ln \frac{f(x)}{f(x + 1)} + \ln \frac{\Phi_0(\nu; x) + MC_k(x)}{\Phi_0(\nu; x + 1) + MC_k(x + 1)}, \quad k \geq 1.
\end{align}

By Lemma 1 (the generalized Moritici’s lemma), we have

\begin{equation}
R \left( E_k(x) \right) = R \left( E_k(x) - E_k(x + 1) \right) - 1.
\end{equation}

Finally, if set \( MC_0(x) \equiv 0 \), then we attain the following useful tool.

**Lemma 3.** Let \( f(x) \) satisfy (1.1). Under the above notation, we have

\begin{equation}
R \left( f(x) - \frac{c}{\Phi_0(\nu; x) + MC_k(x)} \right) = \nu - 1 + R \left( \ln \frac{f(x)}{f(x + 1)} + \ln \frac{\Phi_0(\nu; x) + MC_k(x)}{\Phi_0(\nu; x + 1) + MC_k(x + 1)} \right), \quad k \geq 0.
\end{equation}

The idea of Lemma 3 is first originated from Mortici [23], which will be called a *Mortici-transformation*. We would like to stress that *Mortici-transformation* implies the following assertion

\begin{equation}
\max_{\kappa, \lambda} \left( \text{or } \kappa, \lambda \right) R \left( f(x) - \frac{c}{\Phi_0(\nu; x) + MC_k(x)} \right) = \max_{\kappa, \lambda} \left( \text{or } \kappa, \lambda \right) R \left( \ln \frac{f(x)}{f(x + 1)} + \ln \frac{\Phi_0(\nu; x) + MC_k(x)}{\Phi_0(\nu; x + 1) + MC_k(x + 1)} \right), \quad k \geq 0.
\end{equation}

In the sequel, we will use this relation many times. For the sake of simplicity, we will always assume that the difference

\begin{equation}
\ln \frac{f(1/z)}{c} - \ln \frac{f(1/z + 1)}{c} = \ln \frac{f(1/z)}{f(1/z + 1)}
\end{equation}

is an analytic function in a neighborhood of point \( z = 0 \).

For the reader’s convenience, we would like to give the complete *Mathematica* program for finding all the coefficients in \( \Phi_0(\nu; x) \) and \( MC_k(x) \) by making use of *Mortici-transformation*.

(i). First, let the function \( MT[x] \) be defined by

\[ MT[x] := \ln \frac{f(x)}{f(x + 1)} + \ln \frac{\Phi_0(\nu; x) + MC_k(x)}{\Phi_0(\nu; x + 1) + MC_k(x + 1)}. \]
(ii). Then we manipulate the following Mathematica command to expand $MT[x]$ into a power series in terms of $1/x$:

\[(4.26)\] \[\text{NormalSeries}[MT[x]. \ x \rightarrow 1/u, \{u, 0, l_k]\} / \ x \rightarrow 1/x (// \text{Simplify})\]

We remark that the variable $l_k$ needs to be suitable chosen according to the different function.

(iii). Taking out the first some coefficients in the above power series, then we enforce them to be zero, and finally solve the related coefficients successively.

Remark 5. Actually, once we have found $MC_k(x)$, \[(4.26)\] can be used again to determine the rate of convergence. In addition, we can apply it to check the general term formula for $MC_k(x)$.

### 4.3 The formal continued fraction expansion

Similarly to Taylor’s formula, if the $k$th-correction $MC_k(x)$ for $f(x)$ has the Type-I (or the Type-II) structure, then we may construct the formal Type-I (or Type-II) continued fraction approximation of order $k$ for $f(x)$ as follows:

\[(4.27)\] \[CF_k(f(x)) := \frac{1}{\Phi_0(\nu; x) + MC_k(x)}, \quad k \geq 0.\]

For example, Euler-Mascheroni constant has the formal Type-I continued fraction approximation of order $k$, while both Landau’s constants and Lebesgue’s constants have the formal Type-II continued fraction approximation of order $k$. For details, readers may refer to [11].

**Example 1.** Let $f(x) = \frac{1^4(x+\frac{1}{2})}{1^4(x+1)}$. Then $CF_k(f(x))$ is the Type-I, its MC-point $\omega$ equals $\frac{1}{8}$ (i.e. $\lambda_m \equiv \frac{1}{8}$), and

\[(4.28)\] \[CF_k(f(x)) = \frac{1}{(x + \frac{1}{8})^3 + \frac{7}{128}(x + \frac{1}{8}) + MC_k(x)},\]

where $(\kappa_0, \kappa_1, \kappa_2, \ldots) = (\frac{189}{32768}, \frac{1483}{2688}, \frac{3932523}{87389}, \frac{10136617131375}{25687632567038502}, \ldots).$

**Example 2.** Let $G_\eta(x)$ be defined by \[(8.22)\] below. In the case of $\eta \neq \frac{1}{2}$, $CF_k(G_\eta^2(x))$ is the Type-II, for details see Corollary 2 in Sec. 7. If $\eta \neq \frac{1}{2}$, then $CF_k(G_\eta^2(x))$ is the Type-I, and it has not MC-point. We have

\[(4.29)\] \[CF_2(G_\eta^2(x)) = \frac{1}{x^2 + 2\eta(1-\eta)x + 2\eta^2(\eta - 1)^2 + MC_2(x)},\]

where $\kappa_0 = -\frac{1}{3}\eta^2(1-\eta)^2(2\eta - 1)^2$, $\lambda_0 = \frac{(2\eta - 1)^2}{8} + 4 - \frac{3}{8(2\eta - 1)^2}$, $\kappa_1 = \frac{1}{64} \left( (2\eta - 3)^2(2\eta + 1)^2 + 10 + \frac{45}{(2\eta - 1)^2} \right)$, and

\[\lambda_1 = \frac{(2\eta - 2\eta^2)(2\eta + 1)^2(2\eta - 1)^2}{6(2\eta - 1)^4}.\]

If we rewrite $CF_k(f(x))$ in a rational function of the form $\frac{P_s(x)}{Q_s(x)}$, then $s = k + \nu$ in the case of Type-I, and $s = 2k + \nu$ in the case of Type-II. If we let $R(f(x) - CF_k(f(x))) = K$, then

\[(4.30)\] \[f(x) = CF_k(f(x)) + O(x^{-K}), \quad x \to \infty.\]
Let $\theta_0 = 0$ or 1. A lot of computations reveal that if $CF_k(f(x))$ is the Type-I, then $K = 2k + 2\nu + 1 + \theta_0$, and $K = 4k + 2\nu + 1 + \theta_0$ in the case of Type-II, respectively.

For a suitable “not very large” positive integer $k$, by using of Mortici-transformation and (4.26), we may get the rate of convergence for $f(x) - CF_k(f(x))$ when $x$ tends to infinity. Moreover, by making use of telescoping method, Hermite-Hadamard inequality, etc, sometimes we can prove sharp double inequalities of $f(x) - CF_k(f(x))$ for as smaller $x$ as possible. We will give an example in Sec. 5.

Now let $k$ tend to $\infty$, we get the formal Type-I (or Type-II) continued fraction expansion for $f(x)$, or shortly write

$$(4.31) \quad f(x) \sim CF(f(x)) := CF_\infty(f(x)), \quad x \to \infty.$$ 

In some cases, we can test and guess further the general term of $CF(f(x))$. Here we need to apply some tools in number theory, difference equation, etc. We will show some examples in Sec. 7.

For the formal continued fraction expansion, we are often concerned with the following two main problems.

**Problem 1.** Determine the domains of convergence for the formal continued fraction expansion $CF(f(x))$. We may refer to two very nice books: L. Lorentzen and H. Waadeland [21], and A. Cuyt, V.B. Petersen, B. Verdonk, H. Waadeland, W.B. Jones [17], or some other classical books cited in there.

**Problem 2.** Prove an identity for as the large domains as possible. That is, based on Problem 1, to determine the intervals $I$ such that $f(x) = CF(f(x))$ for all $x \in I$. For example, with the help of continued fraction theory, hypergeometric series, etc., we hope at least to find a interval $(x_0, \infty) \subset I$ for some $x_0 > 0$. Certainly, we may extend it to a complex domain. However, in this paper we will not investigate this topic.

On one hand, to determine all the related coefficients, we often use an appropriate symbolic computation software, which needs a huge of computations. On the other hand, the exact expressions at each occurrence also takes a lot of space. Hence, in this paper we omit some related details for space limitation.

**Remark 6.** From the above discussion, we observe that for a specific function, except a huge of computations, probably only such two kinds of structures can not provide “good continued fraction approximation”. In addition, in the theory of classical continued fraction, even if there is a continued fraction expansion for a given function, we often do not know whether it is the fastest possible or best possible. Generally speaking, for a given continued fraction, finding the rate of convergence for the $k$th approximant is not always easy.
5 The volume of the unit ball

It is well-known that the volume of the unit ball in $\mathbb{R}^n$ is

\[ \Omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)}. \]

(5.1)

Many authors have investigated the inequalities about the $\Omega_n$, e.g. see [2, 3, 4, 5, 6, 9, 13, 20, 22, 25, 26, 30, 32] and references therein.

Chen and Li [16] proved \((a = \frac{e}{2}, b = \frac{1}{3})\):

\[ \frac{1}{\sqrt{\pi (n + a)}} \left(\frac{2\pi e}{n}\right)^{\frac{n}{2}} \leq \Omega_n < \frac{1}{\sqrt{\pi (n + b)}} \left(\frac{2\pi e}{n}\right)^{\frac{n}{2}}. \]

(5.2)

Recently, Mortici [25, Theorem 3] showed that for every integer \(n \geq 3\) in the left-hand side and \(n \geq 1\) in the right-hand side, then we have the following Gospertype inequalities:

\[ \frac{1}{\sqrt{\pi (n + \theta(n))}} \left(\frac{2\pi e}{n}\right)^{\frac{n}{2}} \leq \Omega_n < \frac{1}{\sqrt{\pi (n + \vartheta(n))}} \left(\frac{2\pi e}{n}\right)^{\frac{n}{2}}, \]

(5.3)

where

\[ \theta(n) = \frac{1}{3} + \frac{1}{18n} - \frac{31}{810n^2}, \quad \vartheta(n) = \theta(n) - \frac{139}{9720n^3}. \]

Now we let

\[ V(x) = \frac{\pi^x}{\Gamma(x + 1)}, \quad x > 0. \]

(5.4)

Let us imagine that if \(\frac{1}{\Gamma(x + 1)} \sim H(x)\) when \(x\) tends to infinity, then \(V(x)\) has an asymptotic formula of the form $\pi^n H(x)$. In this sense, by Remark 2 and Remark 3, it suffices to consider the asymptotic formula for the gamma function. In fact, we note that both $f(x)$ and $1/f(x)$ have the same $k$th-correction $MC_k(x)$.

From (4.14), we introduce the relative error sequence \((E_k(x))_{k \geq 0}\) to be defined by

\[ f(x) := 8\pi^3 \left(\frac{x}{e}\right)^6 e^{-6(x + 1)} = \frac{\exp(E_0(x))}{\Phi_0(x)}, \]

(5.5)

\[ f(x) := \frac{\exp(E_k(x))}{\Phi_0(x) + MC_k(x)}, \quad k \geq 1, \]

(5.6)

where $\Phi_0(x) = x^3 + \frac{1}{2} x^2 + \frac{1}{6} x + \frac{1}{240}$, and

\[ MC_k(x) = \sum_{j=0}^{k-1} \frac{k_j}{x + \lambda_j}, \]

(5.7)
κ₀ = -\frac{11}{1920}, \lambda₀ = \frac{79}{1440}, κ₁ = \frac{459733}{11480}, \lambda₁ = -\frac{145925}{709888}. We stress that Φ₀(x) was claimed first by Ramanujan [31], and some more coefficients may be founded in [12]. By employing Lemma 1, (see below) and (1.12), it is not difficult to verify that

\begin{align*}
\lim_{x \to \infty} x^4 E₀(x) &= \frac{11}{1920} := C₀, \\
\lim_{x \to \infty} x^6 E₁(x) &= -\frac{459733}{124185600} := C₁.
\end{align*}

The following theorem tells us how to improve the above results and obtain some sharper estimates for E₀(x) and E₁(x).

**Theorem 1.** Let E₀(x) and E₁(x) be defined as (5.5) and (5.6), respectively.

(i) For every real number x ≥ 6 in the left-hand side and x ≥ 12 in the right-hand side, we have

\begin{align*}
\frac{11}{1920} \frac{1}{(x + 3)^4} < E₀(x) < \frac{11}{1920(x - 5)^4}.
\end{align*}

(ii) For every real number x ≥ 9 in the left-hand side and x ≥ 10 in the right-hand side, then

\begin{align*}
-\frac{459733}{124185600} \frac{1}{(x - 2)^6} < E₁(x) < -\frac{459733}{124185600} \frac{1}{(x + 2)^6}.
\end{align*}

**Proof.** We use the idea of Theorem 2 in [33] or Theorem 1 in [10]. Let Gₖ(x) = Eₖ(x) - Eₖ(x + 1) for k ≥ 0. We will employ the telescoping method. It follows from limₓ→∞ Eₖ(x) = 0 that

\begin{align*}
Eₖ(x) = \sum_{m=0}^{\infty} Gₖ(x + m), \quad (k = 0, 1).
\end{align*}

If g(∞) = g′(∞) = 0, it is not difficult to prove that

\begin{align*}
g(x) = -\int_x^{\infty} g′(s)ds = \int_x^{\infty} \left(\int_s^{\infty} g″(t)dt\right)ds.
\end{align*}

Note that the convenience MC₀(x) = 0. By (5.5) and (5.6), we have

\begin{align*}
Eₖ(x) &= -6 \ln Γ(x + 1) + 2 \ln 2π + 6x(\ln x - 1) + \ln(Φ₀(x) + MCₖ(x)), \\
Gₖ(x) &= Eₖ(x) - Eₖ(x + 1) = 6 \left(1 - x \ln(1 + \frac{1}{x})\right) + \ln \frac{Φ₀(x) + MCₖ(x)}{Φ₀(x + 1) + MCₖ(x + 1)}.
\end{align*}

By using Mathematica software, we can check that if x > 0, then

\begin{align*}
G₀″(x) - \frac{11}{16x^2} + \frac{29}{8x^3} &= \frac{Ψ₁(15; x)}{96x^8(1 + x)^2Ψ₂(12; x)} > 0, \\
G₀″(x) - \frac{11}{16x^2} + \frac{29}{8x^3} - \frac{9031}{800x^9} &= -\frac{Ψ₃(13; x)}{800x^9(1 + x)^2Ψ₂(12; x)} < 0.
\end{align*}
By (5.13), we get that when $x > 0$,

$$\frac{11}{480x^5} - \frac{29}{336x^6} < G_0(x) < \frac{11}{480x^5} - \frac{29}{336x^6} + \frac{9031}{44800x^7}.$$  

Similarly, if $x \geq \frac{1}{16}$, we have

$$G''_1(x) + \frac{459733}{369600x^9} - \frac{39872247}{4743200x^{10}} = -\frac{2845920x^{10}\Psi_5(6; x) (\Psi_6(3; x)(x - \frac{1}{23}) + \frac{7670381}{279841})^2 \Psi_7(8; x)}{1441 - 5872} < 0,$$

$$G''_1(x) + \frac{459733}{369600x^9} - \frac{39872247}{4743200x^{10}} + \frac{32724285440x^{11}}{4743200x^{10}} = \frac{490864281600x^{11}\Psi_5(6; x) (\Psi_6(3; x)(x - \frac{1}{23}) + \frac{7670381}{279841})^2 \Psi_7(8; x)}{2388 - 7275} > 0,$$

and

$$-\frac{459733}{20697600x^7} + \frac{1092949825573}{2945185689600x^9} < G_1(x) = -\frac{459733}{20697600x^7} + \frac{13290749}{113836800x^8}.$$

Now, combining (5.12), (5.18) and (5.21), we attain that

$$0 < E_0(x) - \frac{11}{480} \sum_{m=0}^{\infty} \frac{1}{(x+m)^5} + \frac{29}{336} \sum_{m=0}^{\infty} \frac{1}{(x+m)^6} < \frac{9031}{44800} \sum_{m=0}^{\infty} \frac{1}{(x+m)^7}, \quad (x > 0),$$

$$-\frac{1092949825573}{2945185689600} \sum_{m=0}^{\infty} \frac{1}{(x+m)^9} < E_1(x) + \frac{459733}{20697600} \sum_{m=0}^{\infty} \frac{1}{(x+m)^7} - \frac{13290749}{113836800} \sum_{m=0}^{\infty} \frac{1}{(x+m)^8} < 0, \quad (x > \frac{1}{16}).$$

Let $j \geq 2$ and $x > \frac{1}{2}$. By Lemma 2, we obtain

$$\frac{1}{(j-1)x^{j-1}} = \int_x^{\infty} \frac{dt}{t^j} = \sum_{m=0}^{\infty} \frac{1}{(x+m)^j},$$

$$< \sum_{m=0}^{\infty} \int_{x+m-\frac{1}{2}}^{x+m} \frac{dt}{t^j} = \int_{x-\frac{1}{2}}^{\infty} \frac{dt}{t^j} = \frac{1}{(j-1)(x-\frac{1}{2})^{j-1}}.$$

By applying (5.22) and (5.24), under the condition $x \geq 6$ we have

$$E_0(x) > \frac{11}{480} \frac{1}{4x^4} - \frac{29}{336} \frac{5(x-\frac{1}{2})^5}{5(x-\frac{1}{2})^5} = \frac{11}{1920} \frac{1}{(x+3)^4} + \frac{\Psi_1(7; x)(x-6) + 2164192911}{13440x^4(3+x)^4(-1+2x)^5} > \frac{11}{1920} \frac{1}{(x+3)^4}. $$
Similarly to (5.25), if \( x \geq 12 \), then
\[
E_0(x) < \frac{11}{480} \frac{1}{4(x - \frac{1}{2})^4} - \frac{29}{336} \frac{1}{5x^5} + \frac{9031}{44800} \frac{1}{6(x - \frac{1}{2})^6}
\]
\[
= \frac{11}{1920(x - 5)^4} - \frac{\Psi_2(9; x)(x - 12) + 12561000435989768}{67200(-1 + x)^4x^3(-1 + 2x)^6}
\]
\[
< \frac{11}{1920(x - 5)^4}.
\]
This completes the proof of assertion (i). Finally, it is not difficult to check that if \( x \geq 9 \), then
\[
E_1(x) > -\frac{459733}{20697600} \frac{1}{6(x - 1/2)^6} + \frac{13290749}{113836800} \frac{1}{7x^7} - \frac{1092949825573}{2945185689600} \frac{1}{8(x - 1/2)^8}
\]
\[
= -\frac{459733}{124185600} \frac{1}{(x - 2)^6} + \frac{\Psi_3(13; x)(x - 9) + 6773478135399363858702201}{736296422400(-2 + x)^6x^7(-1 + 2x)^8}
\]
\[
< -\frac{459733}{124185600} \frac{1}{(x - 2)^6},
\]
and if \( x \geq 10 \), we have
\[
E_1(x) < -\frac{459733}{20697600} \frac{1}{6x^6} + \frac{13290749}{113836800} \frac{1}{7(x - 1/2)^7}
\]
\[
= -\frac{459733}{20697600} \frac{1}{6(x + 2)^6} - \frac{\Psi_4(11; x)(x - 10) + 470994290293217661904}{2390572800x^6(2 + x)^6(-1 + 2x)^7}
\]
\[
< -\frac{459733}{124185600} \frac{1}{(x + 2)^6},
\]
This will finish the proof of Theorem 1.

**Theorem 2.** Assume \( n \geq 24 \), we have the following Ramanujan-type inequalities
\[
\frac{1}{\sqrt{\pi}} \left( \frac{2\pi e}{n} \right)^{\frac{3}{2}} \frac{1}{\sqrt[6]{n^3 + n^2 + \frac{n}{2} + \frac{1}{30}}} < \Omega_n < \frac{1}{\sqrt{\pi}} \left( \frac{2\pi e}{n} \right)^{\frac{3}{2}} \exp \left( -\frac{11}{720(n-10)^6} \right).
\]
If \( n \geq 20 \), then
\[
\frac{1}{\sqrt{\pi}} \left( \frac{2\pi e}{n} \right)^{\frac{3}{2}} \frac{1}{\sqrt[6]{n^3 + n^2 + \frac{n}{2} + \frac{1}{30}}} - \frac{459733}{11642400}(-4+n)^6 < \Omega_n
\]
\[
< \frac{1}{\sqrt{\pi}} \left( \frac{2\pi e}{n} \right)^{\frac{3}{2}} \frac{1}{\sqrt[6]{n^3 + n^2 + \frac{n}{2} + \frac{1}{30}}} - \frac{847}{92400 n + 9480}.
\]
Proof. It follows from (5.1), (5.5) and (5.6) that
\[
\Omega_n = \frac{1}{\sqrt{\pi}} \left( \frac{2\pi e}{n} \right)^{\frac{n}{2}} \exp \left( \frac{1}{6} E_0 \left( \frac{n}{2} \right) \right), \quad \Omega_n = \frac{1}{\sqrt{\pi}} \left( \frac{2\pi}{n} \right)^{\frac{n}{2}} \exp \left( \frac{1}{6} E_1 \left( \frac{n}{2} \right) \right)
\]
Now (5.29) follows from (5.10) and (5.31).

We begin to prove (5.30). It is well-known that \(\exp(t) \geq 1 + t\). When \(n \geq 20\), by the inequality of the right-hand side in (5.11), we have the following trivial estimate
\[
\exp \left( \frac{1}{6} E_1 \left( \frac{n}{2} \right) \right) < 1.
\]
In addition, by the lower bound in (5.11), we get
\[
\exp \left( \frac{1}{6} E_1 \left( \frac{n}{2} \right) \right) > 1 + \frac{1}{6} E_1 \left( \frac{n}{2} \right) > 1 - \frac{459733}{11642400(-4 + n)^6}, \quad (n \geq 20).
\]
Combining (5.31), (5.32) and (5.33) completes the proof of (5.30).

Following the same approach as Theorem 2, it is not difficult to prove the following Ramanujan-type inequalities for the gamma function.

**Corollary 1.** Let \(x \geq 12\). Then
\[
\sqrt{\pi} \left( \frac{x}{e} \right)^x \left( 8x^3 + 4x^2 + x + \frac{1}{30} \right)^{\frac{1}{6}} \exp \left( -\frac{11}{11520(x - 5)^4} \right) < \Gamma(x + 1) < \sqrt{\pi} \left( \frac{x}{e} \right)^x \left( 8x^3 + 4x^2 + x + \frac{1}{30} \right)^{\frac{1}{6}}.
\]

**Remark 7.** It should is noted that the method described in Theorem 1 and 2 also can be used to look for \(CF_k(F(x))\), and prove some inequalities involving the ratio of gamma functions.

**Remark 8.** We will give some other results involving \(\Omega_n\) in the subsection 6.2.

6 Lord Brouncker’s continued fraction formula

6.1 Lord Brouncker’s continued fraction formula

The following formula is taken from Corollary 1 of Berndt [8, p. 145], which was first proved by Bauer [7] in 1872.

**Lemma 4.** If \(\text{Re } x > 0\), then
\[
\frac{\Gamma^2 \left( \frac{1}{2}(x + 1) \right)}{\Gamma^2 \left( \frac{1}{2}(x + 3) \right)} = \frac{4}{x + 1} \sum_{m=1}^{\infty} \left( \frac{2m - 1}{2x} \right).
\]

By taking \(x = 4n + 1\) in the above formula, we obtain the so-called Lord Brouncker’s continued fraction formula
\[
q(n) := \frac{\Gamma^2 (n + \frac{1}{2})}{\Gamma^2 (n + 1)} = \frac{4}{4n + 1 + \frac{12}{2(4n+1)+ \frac{4^2}{2(4n+1)+ \cdots}}},
\]

15
For a very interesting history of formula (6.1), see Berndt [8, p. 145]. In addition, Lord Brouncker’s continued fraction formula also plays an important role in Landau’s constants, see [10, 11].

The main aim in this subsection is to illustrate (without proof) that the formula (6.1) is the fastest possible by making use of the method formulated in Sec. 4. Replacing \( x \) by \( 4x + 1 \) in \( (6.1) \) and then making some simple calculation, we obtain its equivalent forms as follows.

**Lemma 5.** Let \( \text{Re} \, x > -\frac{1}{4} \), we have

\[
\frac{\Gamma^2(x + \frac{1}{2})}{\Gamma^2(x + 1)} = \frac{1}{x + \frac{1}{4} + \frac{1}{x + \frac{1}{4} + \sum_{m=1}^\infty \frac{(2m+1)^2}{x + \frac{1}{4}}}}.
\]

(*Proof.*) Now, we are in a position to treat the above formula directly. Let

\[
f(x) = \frac{\Gamma^2(x + \frac{1}{2})}{\Gamma^2(x + 1)}.
\]

By the recurrence relation \( \Gamma(x + 1) = x\Gamma(x) \), we have

\[
\frac{f(x)}{f(x + 1)} = \frac{(x + 1)^2}{(x + \frac{1}{2})^2}.
\]

By the Stirling’s formula, it is not difficult to prove

\[
\lim_{x \to \infty} x^b \Gamma(x + a)\Gamma(x + b) = 1.
\]

Also see [1, p. 257, Eq. 6.1.47] or [8, p. 71, Lemma 2].

It follows readily from (6.6) that

\[
\lim_{x \to +\infty} xf(x) = 1,
\]

i.e, we take \( \nu = 1 \) in (1.1).

**Step 1** The initial-correction. According to (6.7), we take \( \Phi_0(x) = x + a \) for some constant \( a \), to be specified below. From (6.5) and (4.26), it is not difficult to prove that

\[
\ln \frac{f(x)}{f(x + 1)} + \ln \frac{\Phi_0(x)}{\Phi_0(x + 1)} = 2 \ln \frac{x + \frac{1}{2}}{x + 1} + \ln \frac{x + a}{x + 1 + a}
\]

\[
= -\frac{1}{4} + a \frac{1}{x^2} + O \left( \frac{1}{x^3} \right).
\]

Solve the equation \(-1/4 + a = 0\), we get \( a = 1/4 \). By Mortici-transformation, we obtain

\[
\Phi_0(x) = x + \frac{1}{4}, \quad CF_0(f(x)) = \frac{1}{x + \frac{1}{4}}.
\]
As we need to use Mortici-transformation in each correction-process, so will not mention it for the sake of simplicity.

(Step 2) The first-correction. Let us expand the following function into a power series in terms of $1/x$:

$$
\ln \frac{f(x)}{f(x+1)} + \ln \frac{\Phi_0(x) + \frac{\kappa_0}{x}}{\Phi_0(x+1) + \frac{\kappa_0}{x+1}} = 2 \ln \frac{x+1}{x+\frac{1}{2}} + \ln \frac{x+\frac{1}{2} + \frac{\kappa_0}{x}}{x + \frac{3}{4} + \frac{\kappa_0}{x+1}} \\
= -\frac{1}{16} + 2\kappa_0 \frac{1}{x^3} + O\left(\frac{1}{x^4}\right).
$$

(6.10)

We solve the equation $-\frac{1}{16} + 2\kappa_0 = 0$, and obtain $\kappa_0 = 1/32 \neq 0$. Hence we take the first-correction $MC_1(x)$ to be Type-I, i.e.

$$
MC_1(x) = \frac{\kappa_0}{x + \lambda_0}.
$$

(6.11)

Since

$$
\ln \frac{f(x)}{f(x+1)} + \ln \frac{\Phi_0(x) + \frac{\kappa_0}{x+\lambda_0}}{\Phi_0(x+1) + \frac{\kappa_0}{x+1+\lambda_0}} = \frac{3}{128} - \frac{3\lambda_0}{2^5} \frac{1}{x^4} + O\left(\frac{1}{x^5}\right),
$$

we enforce $\frac{3}{128} - \frac{3\lambda_0}{2^5} = 0$, and deduce $\lambda_0 = \frac{1}{4}$. Thus,

$$
MC_1(x) = \frac{\frac{3}{32}}{x + \frac{1}{4}}, \quad CF_1(f(x)) = \frac{1}{x + \frac{1}{4} + \frac{3}{32}}.
$$

(6.12)

(Step 3) The second-correction to the sixth-correction. Now we take $MC_2(x)$ to be Type-I, and let

$$
MC_2(x) = \frac{\kappa_0}{x + \lambda_0 + \frac{\lambda_1}{x+\lambda_1}}.
$$

(6.13)

By using (4.26), we have

$$
\ln \frac{f(x)}{f(x+1)} + \ln \frac{\Phi_0(x) + MC_2(x)}{\Phi_0(x+1) + MC_2(x+1)} \\
= \frac{3}{2^7} - \frac{\kappa_1}{32} \frac{1}{x^5} + 5\left(-27 + 176\kappa_1 + 64\kappa_1\lambda_1\right) \frac{1}{2^6 x^6} + O\left(\frac{1}{x^7}\right),
$$

(6.14)

Solve the equations

$$
\begin{aligned}
&\frac{3}{2^7} - \frac{\kappa_1}{32} = 0, \\
&-27 + 176\kappa_1 + 64\kappa_1\lambda_1 = 0,
\end{aligned}
$$

(6.15)
we attain
\[ \kappa_1 = \frac{9}{64}, \quad \lambda_1 = \frac{1}{4}. \]  
We take the \( k \)-th-correction \( MC_k(x) \) to be \( \text{Type-I} \), then repeat the above approach like the second-correction, and solve successively the coefficients \( \kappa_j \) and \( \lambda_j \) (\( 2 \leq j \leq 6 \)) as follows:

\[
\begin{align*}
\kappa_2 &= \frac{25}{64}, \quad \lambda_2 = \frac{1}{4}; \\
\kappa_3 &= \frac{49}{64}, \quad \lambda_3 = \frac{1}{4}; \\
\kappa_4 &= \frac{81}{64}, \quad \lambda_4 = \frac{1}{4}; \\
\kappa_5 &= \frac{121}{64}, \quad \lambda_5 = \frac{1}{4}; \\
\kappa_6 &= \frac{169}{64}, \quad \lambda_6 = \frac{1}{4}; \\
\kappa_7 &= \frac{225}{64}, \quad \lambda_7 = \frac{1}{4}.
\end{align*}
\]

From these results, it is not difficult to guess that
\[ \kappa_m = \frac{(2m+1)^2}{64}, \quad \lambda_m = \frac{1}{4}. \]

Further, we apply (4.26) to check that the above conjecture holds true for some larger \( m \). In this way, we finally test that the fastest possible formula should be (6.3).

6.2 The continued fraction formulas involving the volume of the unit ball

Let \( \Omega_n \) be defined by (5.1). The main purpose of this subsection is to present the following two theorems.

**Theorem 3.** Let \( n \geq 1 \) be a positive integer. Then
\[ \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = \frac{2(n+1)}{2n+1 + \sum_{m=0}^{\infty} \left( \frac{(2m+1)^2}{2(2n+1)} \right)}. \]  

**Proof.** It follows from (5.1) and the recurrence relation \( \Gamma(x+1) = x\Gamma(x) \) that
\[ \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} = \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)\Gamma\left(\frac{3}{4} + \frac{3}{2}\right)}{\Gamma^2\left(\frac{1}{2} + 1\right)} = \frac{n+1}{2} \frac{\Gamma^2\left(\frac{1}{2} + \frac{1}{2}\right)}{\Gamma^2\left(\frac{3}{4} + 1\right)}. \]

Replacing \( x \) by \( \frac{n}{2} \) in (6.3), then after simplification, we get easily the desired assertion.  

**Theorem 4.** Let \( n \in \mathbb{N} \), then
\[ \frac{\Omega_{n-1}}{\Omega_n} = \frac{1}{2\sqrt{\pi}} \sqrt{2n+1 + \sum_{m=0}^{\infty} \left( \frac{(2m+1)^2}{2(2n+1)} \right)}. \]  

**Proof.** From (5.1), we have
\[ \frac{\Omega_{n-1}}{\Omega_n} = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{3}{4} + \frac{1}{2}\right). \]

Replacing \( x \) by \( \frac{n}{2} \) in (6.3), then taking reciprocals of both sides, finally substituting it into the above formula, this will complete the proof of Theorem 4.
Remark 9. Condition \([1,1]\) is not an essential restriction. Actually, we can extend our method to any negative integer \(\nu\). For example, by taking reciprocals of both sides in \([6,3]\), we have

\[
\frac{\Gamma^2(x + 1)}{\Gamma^2(x + \frac{1}{2})} = x + \frac{1}{4} + \frac{1}{\sqrt{x + \frac{1}{4}}} \sum_{m=1}^{\infty} \frac{(2m+1)^2}{x + \frac{1}{4}}, \quad \Re x > 0.
\]

In this case, we take \(\nu = -1\). It should be remarked that we can discover the above formula directly by using an approach similarly to Lemma 5.

Remark 10. To the best of our knowledge, formula \([6,1]\) and \([6,3]\) were possibly neglected by many mathematicians for about more than twenty years, until 2013 I. Gavrea and M. Ivan mentioned it in their paper \([19]\).

7 A continued fraction formula of Ramanujan

The following lemma is Entry 39 in Berndt [8, p. 159], which is one of three principal formulas involving gamma functions given by Ramanujan. It is very difficult for us to imagine how Ramanujan discovered those beautiful continued fraction formulas. Maybe our method provides a theoretical basis.

**Lemma 6.** Let \(l\) and \(n\) denote arbitrary complex numbers. Suppose that \(x\) is complex with \(\Re x > 0\) or that either \(n\) or \(l\) is an odd integer. Then

\[
P := \frac{\Gamma \left( \frac{1}{4}(x + l + n + 1) \right) \Gamma \left( \frac{1}{4}(x - l - n + 1) \right) \Gamma \left( \frac{1}{4}(x + l - n + 1) \right) \Gamma \left( \frac{1}{4}(x - l + n + 1) \right)}{\Gamma \left( \frac{1}{4}(x + l + n + 3) \right) \Gamma \left( \frac{1}{4}(x - l + n + 3) \right) \Gamma \left( \frac{1}{4}(x + l - n + 3) \right) \Gamma \left( \frac{1}{4}(x - l + n + 3) \right)}
\]

\[
= \frac{8}{(x^2 - l^2 + n^2 - 1)/2} + \frac{12 - n^2}{1} + \frac{12 - l^2}{x^2 - 1} + \frac{3^2 - n^2}{1} + \frac{3^2 - l^2}{x^2 - 1} + \cdots.
\]

By replacing \(x\) by \(4x\), and taking \((l, n) = (0, 0), (l, n) = (1/4, 1/2), (l, n) = (1/3, 1/2), (l, n) = (1/8, 1/2)\), respectively, the authors have checked that Lemma 6 is not optimal continued fraction expansion. Now, by employing these tests, we may refine it in a uniform expression as follows.

**Theorem 5.** Under the same conditions of Lemma 6, we have

\[
P = \frac{8}{\frac{1}{2}(x^2 - l^2 - n^2 + 1)} - \frac{(12 - n^2)(12 - l^2)}{x^2 - l^2 - n^2 + (3^2 + 1^2 - 1)} - \frac{(3^2 - n^2)(3^2 - l^2)}{x^2 - l^2 - n^2 + (5^2 + 3^2 - 1)} - \cdots
\]

\[
= \frac{8}{\frac{1}{2}(x^2 - l^2 - n^2 + 1)} + \sum_{m=1}^{\infty} \left( \frac{-(2m-1)^2 - n^2}{x^2 - l^2 - n^2 + 8m^2 + 1} \right).
\]

**Proof.** We follow the method of Entry 25 in Berndt [8, p. 141]. First, we rewrite Lemma 6 in the form

\[
\frac{8}{P} + \frac{1}{2}(x^2 + l^2 - n^2 - 1) = x^2 - 1 + \frac{12 - n^2}{1} + \frac{12 - l^2}{x^2 - 1} + \frac{3^2 - n^2}{1} + \frac{3^2 - l^2}{x^2 - 1} + \cdots.
\]
Secondly, by Entry 14 of Berndt [8, p. 121] (an infinity form see [8, p. 157]), we have
\[
\frac{1}{8/P + \frac{1}{2}(x^2 + l^2 - n^2 - 1)} = \frac{1}{x^2 - 1} + \frac{1^2 - n^2}{1} + \frac{1^2 - l^2}{x^2 - 1} + \frac{3^2 - n^2}{1} + \frac{3^2 - l^2}{x^2 - 1} + \cdots
\]

Note that \((2m + 1)^2 + (2m - 1)^2 - 1 = 8m^2 + 1\). Now take the reciprocal of both sides above and then solve for \(P\), which again involves taking reciprocals. This will finish the proof of Theorem 5.

The following theorem is the fastest possible form for Entry 26 in Berndt [8, p. 145].

**Theorem 6.** Suppose that either \(n\) is an odd integer and \(x\) is any complex number or that \(n\) is an arbitrary complex number and \(\Re x > 0\). Then
\[
\frac{\Gamma^2 \left( \frac{1}{4} (x + n + 1) \right) \Gamma^2 \left( \frac{1}{4} (x - n + 1) \right)}{\Gamma^2 \left( \frac{1}{4} (x + n + 3) \right) \Gamma^2 \left( \frac{1}{4} (x - n + 3) \right)} = \frac{8}{x^2 - n^2 + 1} + \sum_{m=1}^{\infty} \frac{(- (2m - 1)^2 (2m - 1)^2 - 1)}{x^2 - n^2 + 8m^2 + 1}.
\]

**Proof.** Set \(l = 0\) in Theorem 5, the desired equality follows at once.

Similarly, we give another form of the Corollary in Berndt [8, p. 146].

**Corollary 2.** If \(\Re x > 0\), then
\[
\frac{\Gamma^4 \left( \frac{1}{4} (x + 1) \right)}{\Gamma^4 \left( \frac{1}{4} (x + 3) \right)} = \frac{8}{x^2 + 1} + \sum_{m=1}^{\infty} \frac{(- (2m - 1)^4)}{x^2 + 8m^2 + 1}.
\]

**Proof.** We set \(n = 0\) in Theorem 6, this completes the proof of the corollary readily.

### 8 Some new conjectural continued fraction formulas

In this section, we will give three examples to illustrate how to guess their fastest possible continued fraction expansions. For the recent results involving these functions, see Mortici, Cristea and Lu [27], Cao and Wang [14], and Chen [15].
8.1 For $\frac{\Gamma^3(x+\frac{1}{3})}{\Gamma^3(x+1)}$

In this subsection, we will use the function $\frac{\Gamma^3(x+\frac{1}{3})}{\Gamma^3(x+1)}$ as an example to explain how to guess its fastest possible continued fraction expansion, which consists of the following steps.

(1). Define

$$f(x) := \frac{\Gamma^3(x + \frac{1}{3})}{\Gamma^3(x + 1)}.$$ 

Find the structure of $CF_k(f(x))$ or $MC_k(x)$ by Mortici transformation and (4.26). We may determine that $CF_k(f(x))$ has the form of Type-II, and its MC-point $\omega$ equals to $\frac{1}{6}$. Here we omit the details for finding those coefficients in $CF_k(f(x))$, since the proof is very similar to that of Sec. 5 or Subsection 8.3 below.

(2). We denote $CF_k(f(x))$ in the simplified form like (2.6):

$$CF_k(f(x)) = \frac{1}{(x + \omega)^2 + \lambda - 1 + K_{k-1} \sum_{j=0}^{\infty} \left( \frac{\kappa_j}{(x + \omega)^2 + \lambda_j} \right)}.$$ 

where $\lambda - 1 = \frac{5}{2^{3/4}}$.

(3). We write two sequences $(\kappa_m)_{m \geq 0}$ and $(\lambda_m)_{m \geq 0}$ in the canonical form, then extract their common factors, respectively. For example, one may use Mathematica command “FactorInteger” to do that. In this way, we denote these two sequences in the form

$$(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \ldots) = \frac{2}{3^3} \left( \frac{1^3}{1^3}, \frac{2^3}{3^2}, \frac{3^3}{5^2}, \frac{4^3}{7^2}, \frac{5^3}{9^2}, \ldots \right),$$

$$\lambda_{-1} = \frac{5}{2^{3/4}}, (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots) = \frac{1}{2^{3/4}} \left( \frac{5^2}{1 \cdot 3}, \frac{7^2}{3 \cdot 5}, \frac{17^2}{5 \cdot 7}, \frac{3307}{7 \cdot 9}, \ldots \right).$$

(4). Now we will look for the general terms of the sequences $(\kappa_m)_{m \geq 0}$ and $(\lambda_m)_{m \geq 0}$. We try to decompose them into some more simpler “partial sequences”.

(4-1). We observe easily that the sequence $(a_m)_{m \geq 0} = (1, 3, 5, \ldots)$ has the general term $a_m = 2m + 1$. While, for the sequence $(b_m)_{m \geq 0} = (1 \cdot 3 \cdot 5 \cdot 7 \cdot 9, \ldots)$, its general term is $b_m = (2m + 1)(2m + 3)$.

(4-2). Let us consider the sequence $(a_m)_{m \geq 0} = (1, 2, 4, 5, 7, 8, 10, 11, \ldots)$, which is the sequence generated by deleting the sequence $(3k)_{k \geq 1}$ from the positive integer sequence $(k)_{k \geq 1}$. We can check that the sequence $(a_m)_{m \geq 0}$ satisfies the following difference equation

$$a_m - a_{m-1} = \begin{cases} 1, & \text{if } m \text{ is an odd}, \\ 2, & \text{if } m \text{ is an even}, \end{cases}$$

21
with the initial value $\alpha_0 = 1$. Hence, we deduce that the general term equals to
\[
\alpha_m = m + 1 + \left\lfloor \frac{m}{2} \right\rfloor.
\]

(4-3). Similarly, the sequence $(\beta_m)_{m \geq 0} = (1, 5, 7, 11, 13, 17, 19, \ldots)$ satisfies the following difference equation
\[
\beta_m - \beta_{m-1} = \begin{cases} 
4, & \text{if } m \text{ is an odd}, \\
2, & \text{if } m \text{ is an even},
\end{cases}
\]
with the initial condition $\beta_0 = 1$, and we get
\[
\beta_m = 3m + 1 + \frac{1 - (-1)^m}{2}.
\]

(4-4). The sequence $(\xi_m)_{m \geq 0} = (5^2, 7, 3307, 17167, 5 \cdot 31 - 353, 5 \cdot 13 \cdot 2063, 5 \cdot 7 \cdot 19 \cdot 419, 516847, 7 \cdot 13 \cdot 9697, \ldots)$ is most difficult. Consider a new sequence $(u_m)_{m \geq 0}$ to be defined by
\[
(8.2) \quad u_m := \xi_m \mod (2m + 1)(2m + 3).
\]

By using Mathematica command “mod”, we can verify
\[
(8.3) \quad (u_0, u_1, u_2, \ldots) = (58, 220, 490, 868, 1354, 1948, 2650, 3460, \ldots) = 2(29, 110, 245, 434, 677, 974, 1325, 1730, \ldots) := 2(v_m)_{m \geq 0}.
\]

We may check that the sequence $(v_m)_{m \geq 0}$ satisfies the following difference equation
\[
(8.4) \quad v_m - 2v_{m-1} + v_{m-2} = 108
\]
with the initial conditions $v_0 = 29$ and $v_1 = 110$. Solve this difference equation of order 2, we can deduce that the general term equals to
\[
(8.5) \quad v_m = 27(m + 1)^2 + 2.
\]

Now we rewrite the general term $\xi_m$ in the form
\[
(8.6) \quad \xi_m = 2(2m + 1)(2m + 3)v_m + w_m.
\]

We may check that $(w_0, w_1, w_2, \ldots) = (1, 7, 17, 31, 49, 71, 97, 127, \ldots)$. Quite similarly to the previous sequence $(v_m)_{m \geq 0}$, $(w_m)_{m \geq 0}$ also satisfies a difference equation as follows
\[
(8.7) \quad w_m - 2w_{m-1} + w_{m-2} = 4
\]

22
with the initial conditions \(w_0 = 1\) and \(w_1 = 7\). In this way, we get

\[
(8.8) \quad w_m = 2(m + 1)^2 - 1.
\]

Substituting (8.5) and (8.8) into (8.6), we discover

\[
(8.9) \quad \xi_m = 2(2m + 1)(2m + 3) \left(27(m + 1)^2 + 2\right) + 2(m + 1)^2 - 1.
\]

Combining the above results and after some simplification, we conjecture that the general terms should be

\[
(8.10) \quad \kappa_m = -\frac{2}{729} \frac{(m + 1 + \lfloor \frac{m}{2} \rfloor)^3 \left(3m + 1 + \frac{1 - (-1)^m}{2}\right)^3}{(2m + 1)^2}, \quad (m \geq 0)
\]

\[
(8.11) \quad \lambda_m = \frac{1}{108} \left(2(27(m + 1)^2 + 2) + \frac{2(m + 1)^2 - 1}{(2m + 1)(2m + 3)}\right), \quad (m \geq -1).
\]

Note that we used the fact that the last formula also holds true for \(m = -1\).

\textbf{(5).} Define two sequences \((\kappa_m)_{m \geq 0}\) and \((\lambda_m)_{m \geq -1}\) by (8.10) and (8.11), respectively. By making use of (4.26), we check that the above conjectures are still true for some “larger” \(m\).

\textbf{(6).} Further simplification for the general term \(\kappa_m\). Actually, we have

\[
(8.12) \quad \left(m + 1 + \lfloor \frac{m}{2} \rfloor\right) \left(3m + 1 + \frac{1 - (-1)^m}{2}\right) \frac{9}{8} \left((2m + 1)^2 - \frac{1}{3}\right)^2
\]

\[
= \frac{(3m + 1)(3m + 2)}{2},
\]

which may be proved easily according to \(m\) is an odd and an even, respectively. Hence

\[
(8.13) \quad \kappa_m = -\frac{1}{2916} \frac{(3m + 1)^3(3m + 2)^3}{(2m + 1)^2}, \quad (m \geq 0).
\]

Finally, we propose the following reasonable conjecture.

**Open Problem 1.** Let two sequences \((\kappa_m)_{m \geq 0}\) and \((\lambda_m)_{m \geq -1}\) be define by (8.13) and (8.11), respectively. Let real \(x > -1/6\), then we have

\[
(8.14) \quad \frac{\Gamma^3(x + \frac{1}{3})}{\Gamma^3(x + 1)} = \frac{1}{(x + \frac{1}{6})^2 + \lambda_{-1} + \cdots}\frac{1}{(x + \frac{1}{6})^2 + \lambda_0 + \cdots}\frac{1}{(x + \frac{1}{6})^2 + \lambda_1 + \cdots}\frac{1}{(x + \frac{1}{6})^2 + \lambda_2 + \cdots}\frac{1}{(x + \frac{1}{6})^2 + \lambda_3 + \cdots}.
\]
Remark 11. Open Problem 1 means that if there exists a fastest possible continued fraction expansion for the function \( \frac{\Gamma^3(x+\frac{1}{6})}{\Gamma^3(x+1)} \), then it must be the continued fraction expression of the right side in (8.14).

Replacing \( x \) by \( x - 1/6 \) and then after some simplification, we get the following equivalent forms of Open Problem 1.

**Open Problem 1’.** Let real \( x > 0 \), then

\[
\frac{\Gamma^3(x + \frac{1}{6})}{\Gamma^3(x + \frac{1}{2})} = \frac{1}{x^2 + \frac{3}{108}} + \frac{K}{x^2 + \frac{1}{108}} \left( \frac{-27(3n-2)^2}{2916(2n-1)^2} \right).
\]

(8.15)

8.2 For \( \frac{\Gamma^3(x + \frac{2}{3})}{\Gamma^3(x + 1)} \)

The main purpose of this subsection is to conjecture the fastest possible continued fraction expansion for the function \( f(x) \), which is defined by

\[ f(x) := \frac{\Gamma^3(x + \frac{2}{3})}{\Gamma^3(x + 1)}. \]

We follow the same method described in last subsection. By testing, we observe that \( CF_k(f(x)) \) has the form of Type-I, and its MC-point \( \omega \) is 1/3. Some computation data are listed as follows:

\[
CF_k(f(x)) = \frac{1}{x + \frac{1}{3} + K_{j=0}^{k-1} \left( \frac{\kappa_j}{x + \frac{1}{3}} \right)}.
\]

(8.16)

where

\[
\kappa_0 = \frac{1}{27}, \quad (\kappa_1, \kappa_2, \kappa_3, \ldots) = \left( \frac{2}{1}, \frac{4}{3}, \frac{3}{5}, \frac{7}{5}, \frac{8}{7}, \frac{5}{9}, \frac{10}{9}, \frac{11}{11}, \frac{13}{11}, \frac{14}{11}, \frac{16}{11}, \ldots \right).
\]

(8.17)

Similarly to the sequence \((\alpha_m)_{m\geq0}\) in last subsection, it is not difficult to verify that the general term of the sequence \((\lambda_m)_{m\geq1}\) should be

\[
\lambda_m = \frac{1}{54} \left( m + 1 + \left\lfloor \frac{m}{2} \right\rfloor \right)^3.
\]

(8.18)

**Open Problem 2.** For all real \( x > -\frac{1}{3} \), we have

\[
\frac{\Gamma^3(x + \frac{2}{3})}{\Gamma^3(x + 1)} = \frac{1}{x + \frac{1}{3} + \frac{1}{x + \frac{1}{3} + \frac{1}{x + \frac{1}{3} + \frac{1}{x + \frac{1}{3} + \ldots}}}}.
\]

(8.19)
Replace $x$ by $x - 1/3$, we have the following equivalent forms of Open Problem 2.

**Open Problem 2'.** Let $\kappa_0 = \frac{1}{2^7}$, and the sequence $(\lambda_m)_{m\geq 1}$ be defined as (8.18). Let $x > 0$, then

\[
\frac{\Gamma^3(x + \frac{4}{3})}{\Gamma^3(x + \frac{3}{2})} = \frac{1}{x} + \sum_{m=0}^{\infty} \frac{\kappa_m}{x}.
\]

(8.20)

Since the partial coefficients of the continued fraction of the right side in (8.20) are all positive, we can prove the following consequence easily.

**Corollary 3.** Let $x > 0$. Assume that Open Problem 2' is true, then for all non-negative integer $k$

\[
\frac{1}{x} + \sum_{j=0}^{2k+1} \frac{\kappa_j}{x} < \frac{\Gamma^3(x + \frac{1}{3})}{\Gamma^3(x + \frac{4}{3})} < \frac{1}{x} + \sum_{j=0}^{2k} \frac{\kappa_j}{x}.
\]

(8.21)

**Remark 12.** The authors have checked that Corollary 3 is true for $k \leq 10$.

### 8.3 For $\frac{\Gamma(x+\eta)\Gamma(x+1-\eta)}{\Gamma^2(x+1)}$

Let $\eta$ be a real number with $0 < \eta < 1$. In this subsection, we will discuss the continued fraction approximation for the ratio of the gamma functions

\[
G_\eta(x) := \frac{\Gamma(x + \eta)\Gamma(x + 1 - \eta)}{\Gamma^2(x + 1)}.
\]

(8.22)

It follows from (6.6) that

\[
\lim_{x \to \infty} xG_\eta(x) = 1.
\]

(8.23)

Now let us begin to look for $CF_k(G_\eta(x))$.

**Step 1** The initial-correction. Note that $\nu = 1$ in (8.23). It follows readily from the recurrence formula $\Gamma(x + 1) = x\Gamma(x)$ that

\[
\frac{G_\eta(x)}{G_\eta(x + 1)} = \frac{(x + 1)^2}{(x + \eta)(x + 1 - \eta)}.
\]

Now we apply Mortici-transformation to determine $\Phi_0(x) = x + c_0$. By making use of Mathematica software, one has

\[
\ln \frac{(x + 1)^2}{(x + \eta)(x + 1 - \eta)} + \ln \frac{x + c_0}{x + 1 + c_0} = \frac{c_0 - \eta + \eta^2}{x^2} + O \left( \frac{1}{x^3} \right).
\]

(8.24)

Solve $c_0 - \eta + \eta^2 = 0$, we obtain $c_0 = \eta - \eta^2$. 

25
(Step 2) The first-correction. Let
\[(8.25) \quad MC_1(x) = \frac{\kappa_0}{x + \lambda_0},\]
similarly to the initial-correction, we also have
\[(8.26) \quad \ln \frac{(x + 1)^2}{(x + \eta)(x + 1 - \eta)} + \ln \frac{x + c_0 + MC_1(x)}{x + 1 + c_0 + MC_1(x + 1)} = \frac{2\kappa_0 - \eta^2 + 2\eta^3 - \eta^4}{x^3} + \frac{-3\kappa_0 - 3\kappa_0\lambda_0 - 3\kappa_0\eta + 2\eta^2 + 3\kappa_0\eta^2 - 3\eta^3 - \eta^4 + 3\eta^5 - \eta^6}{x^4} + \frac{g(x)}{x^5} + O \left(\frac{1}{x^6}\right),\]
where
\[g(x) = 4\kappa_0 - 2\kappa_0^2 + 6\kappa_0\lambda_0 + 4\kappa_0\lambda_0^2 + 6\kappa_0\eta + 4\kappa_0\lambda_0\eta - 3\eta^2 - 2\kappa_0\eta^2 - 4\kappa_0\lambda_0\eta^2 + 4\eta^3 - 8\kappa_0\eta^3 + 2\eta^4 + 4\kappa_0\eta^4 - 2\eta^5 - 4\eta^6 + 4\eta^7 - \eta^8.\]
Solve
\[(8.27) \quad \begin{cases} 2\kappa_0 - \eta^2 + 2\eta^3 - \eta^4 = 0 \\ -3\kappa_0 - 3\kappa_0\lambda_0 - 3\kappa_0\eta + 2\eta^2 + 3\kappa_0\eta^2 - 3\eta^3 - \eta^4 + 3\eta^5 - \eta^6 = 0 \\ g(x) = 0, \end{cases}\]
we get
\[(8.28) \quad \kappa_0 = \frac{(-1 + \eta)^2\eta^2}{2}, \quad \lambda_0 = \frac{1 - \eta + \eta^2}{3}.\]

(Step 3) The second-correction to the sixth-correction. Similarly to the first-correction, we use Mathematica software to find that the second-correction to the sixth-correction are the form of Type-I, and then solve all coefficients in these correction functions. It should be remarked that there is a parametric \(\eta\), the related computations will become very huge and complex. So we need to manipulate Mathematica command “Simplify”. Here we list the final computing results as follows:
\[
\begin{align*}
\kappa_1 &= \frac{(-2 - \eta + \eta^2)^2}{36}, & \lambda_1 &= \frac{4 - \eta + \eta^2}{15}; \\
\kappa_2 &= \frac{(-6 - \eta + \eta^2)^2}{100}, & \lambda_2 &= \frac{9 - \eta + \eta^2}{35}; \\
\kappa_3 &= \frac{(-12 - \eta + \eta^2)^2}{196}, & \lambda_3 &= \frac{16 - \eta + \eta^2}{63}; \\
\kappa_4 &= \frac{(-20 - \eta + \eta^2)^2}{324}, & \lambda_4 &= \frac{25 - \eta + \eta^2}{99}; \\
\kappa_5 &= \frac{(-30 - \eta + \eta^2)^2}{484}, & \lambda_5 &= \frac{36 - \eta + \eta^2}{143}.
\end{align*}
\]
Similarly to Open Problem 1, by careful data analysis and further checking, we may propose the following conjecture.

**Open Problem 3.** For \( x > 0 \), then

\[
\frac{\Gamma(x + \eta) \Gamma(x + 1 - \eta)}{\Gamma^2(x + 1)} = \frac{1}{x + \eta - \eta^2 + \sum_{m=0}^{\infty} \left( \frac{x_m}{x + \lambda_m} \right)},
\]

where \( \kappa_0 = \frac{(-\eta + \eta^2)^2}{2} \) and

\[
\kappa_m = \frac{(-m(m + 1) - \eta + \eta^2)^2}{4(2m + 1)^2} = \frac{(m + \eta)^2(m - \eta + 1)^2}{4(2m + 1)^2}, \quad m \geq 1
\]

\[
\lambda_m = \frac{(1 + m)^2 - \eta + \eta^2}{(2m + 1)(2m + 3)}, \quad m \geq 0.
\]

**Remark 13.** If we take \( \eta = \frac{1}{2} \), then the above conjecture implies (6.3) (i.e. the generalized Lord Brouncker’s continued fraction formula). Here we note that

\[
\frac{(m + \eta)^2(m - \eta + 1)^2}{4(2m + 1)^2} = \frac{(m + \frac{1}{2})^4}{4(2m + 1)^2} = \frac{(2m + 1)^2}{2^6},
\]

\[
\frac{(1 + m)^2 - \eta + \eta^2}{(2m + 1)(2m + 3)} = \frac{(1 + m)^2 - \left( \frac{1}{2} \right)^2}{(2m + 1)(2m + 3)} = \frac{(m + \frac{1}{2})(m + \frac{3}{2})}{(2m + 1)(2m + 3)} = \frac{1}{4}.
\]

9 Conclusions

In this paper, we present a systematic way to construct a best possible finite and infinite continued fraction approximations for a class of functions. In particular, the method described in Sec. 4 is suitable for the ratio of the gamma functions, e.g. many examples can be found in the nice survey papers Qi [28] and Qi and Luo [29]. As our method is constructive, so all involving computations may be manipulated by a suitable symbolic computation software, e.g. Mathematica. In some sense, the main advantage of our method is that such formal continued fraction approximation of order \( k \) is the fastest possible when \( x \) tends to infinity. Concerning applications in approximation theory, numerical computation, our method represents a much better approximation formula than the power series approach (e.g. Taylor’s formula) for a kind of “good functions”.

In addition, the multiple-correction method provides a useful tool for testing and guessing the continued fraction expansion involving a specified function. So our method should help advance the approximation theory, the theory of continued fraction, the generalized hypergeometric function, etc. Further, if we can obtain some new continued fraction expansions, probably these formulas could be used to study the irrationality, transcendence of the involved constants.
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