RESTRICTION OF SCALARS FOR $L_{\infty}$-MODULES

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Abstract. Let $I : L' \to L$ be a morphism of $L_{\infty}$-algebras. The goal of this paper is to describe restriction of scalars in the setting of $L_{\infty}$-modules and prove that it defines a functor $I^* : L\text{-mod} \to L'\text{-mod}$. A more abstract approach to this problem was recently given by Kraft-Schnitzer. In a subsequent paper, this result is applied to show that there is a well-defined $L_{\infty}$-module structure on the sutured annular Khovanov homology of a link in a thickened annulus.

1. Introduction

The study of $L_{\infty}$-algebras, also known as strong homotopy Lie algebras or sh-Lie algebras, can be traced back to rational homotopy theory and the deformations of algebraic structures, where they first appeared in the form of Lie-Massy operations [1] [15] [16]. Early applications centered around the Quillen spectral sequence and rational Whitehead products, and there has been continued interest in higher order Whitehead products recently; see [3]. There has also been much interest in $L_{\infty}$-algebras in physics, where Lie algebras and their representations play a major role. In particular, $L_{\infty}$-algebra structures have appeared in work on higher spin particles [4], as well as in closed string theory [18] [19]. Stasheff gives a nice overview in a recent survey article [17].

Attention has also been given to modules over $L_{\infty}$-algebras. The notion of an $L_{\infty}$-module was introduced in [11], in which the correspondence between Lie algebra representations and Lie modules was generalized to the $L_{\infty}$ setting. Moreover, homomorphisms between $L_{\infty}$-modules were developed in [2].

While it is possible to give a complex an $L_{\infty}$ structure by writing down explicit formulas, another option is to use homological perturbation theory to transfer an existing $L_{\infty}$ structure from a different complex. Information on how to do so can be found in [8] [9] [7], where this idea is referred to as the homological perturbation lemma, though sometimes it is referred to as the homotopy transfer theorem, as in [12] [13]. An approach using operads was given in [5], where explicit formulas are written down for the $A_{\infty}$ case. Explicit formulas for the $L_{\infty}$ case can be found in [14].

Much of the literature deals with the transfer of $L_{\infty}$-algebra structures; however, given a map between $L_{\infty}$-algebras, it is natural to want to use this map to relate their respective categories of modules. In this paper, we give one explicit formula to do so, giving a proof of the following:

**Theorem 1.** Suppose $L, L'$ are $L_{\infty}$-algebras over $\mathbb{F}_2$ and $I : L' \to L$ is a map of $L_{\infty}$-algebras. Then there is an induced functor $I^* : L\text{-mod} \to L'\text{-mod}$, called restriction of scalars.

Given an $L_{\infty}$-module homomorphism $f : M \to N$, our definition will satisfy $(I^*f)_1 = f_1$. It follows that $I^*$ preserves quasi-isomorphisms; that is, if $M$ and $N$ are quasi-isomorphic, then so too are $I^*M$ and $I^*N$. We also observe that this generalizes the analogous result in the Lie algebra setting:
Corollary 1. If \( L \) and \( L' \) are Lie algebras, and \( \phi : L' \to L \) is a Lie algebra homomorphism, \( \phi^* \) is the usual restriction of scalars for Lie algebra representations.

One reason for interest in this result is a particular application in knot theory. In [6], it was shown that the sutured annular Khovanov homology of a knot can be given the structure of a Lie algebra representation. In subsequent work, we will use the formulas from this paper to upgrade this structure to an \( L_\infty \)-module.

Because \( L_\infty \) modules are defined in the graded setting, keeping track of signs requires a great deal of care. We will ignore signs and work over \( \mathbb{F}_2 \). As mentioned in [2], \( A_\infty \)-modules and maps between them can be reinterpreted in terms of differential comodules. The analogous reformulation in the \( L_\infty \) case is less-understood, but perhaps could facilitate the recording of signs. Moreover, while this paper was in preparation, Kraft-Schnitzer gave a more abstract approach to the restriction of scalars operation in [10]. We present an alternative interpretation, and we emphasize that the explicit formulas developed here are of particular interest for our applications. On the other hand, [10] might serve as a guide for how to deal with signs in the future.

The outline of the paper is as follows. In section 2, we review the definition of an \( L_\infty \)-algebra and explain morphisms between them. In section 3, we provide a similar exposition for \( L_\infty \)-modules, and we also describe how to compose morphisms between \( L_\infty \)-modules. In section 4, we describe \( I^* \), the restriction of scalars functor. We define \( I^* \) on objects and morphisms, and then we prove that it is functorial. The appendix includes supplementary graphics for the proofs presented in the paper, which contain somewhat complicated formulas.

2. \( L_\infty \)-Algebras and \( L_\infty \)-Modules

In this section, we review \( L_\infty \)-algebras and explain morphisms between them. We start by introducing some notation that we will use throughout the paper.

Definition 1. Let \( \sigma \in S^n \) and \( x_i \) be elements of a set \( X \). Define the map \( \sigma^* : X^n \to X^n \) by \( \sigma^*(x_1, x_2, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \). Note that this induces a map on the \( n \)-fold tensor product \( \sigma^* : X^\otimes n \to X^\otimes n \) that sends an element \( x_1 \otimes \cdots \otimes x_n \) to \( x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \).

Definition 2. Fix non-negative integers \( i_1, i_2, \ldots, i_r \), with \( i_1 + i_2 + \cdots + i_r = n \). A permutation \( \sigma \in S_n \) is an \((i_1, i_2, \ldots, i_r)\)-unshuffle if

\[
\sigma(1) < \cdots < \sigma(i_1) \\
\sigma(i_1 + 1) < \cdots < \sigma(i_1 + i_2) \\
\vdots \\
\sigma(i_1 + \cdots + i_{r-1} + 1) < \cdots < \sigma(i_1 + \cdots + i_r)
\]

We will denote the set of \((i_1, i_2, \ldots, i_r)\)-unshuffles in \( S_n \) by \( S(i_1, \ldots, i_r) \). We will also denote by \( S'(i_1, \ldots, i_r) \) the set of \((i_1, i_2, \ldots, i_r)\)-unshuffles \( \sigma \) in \( S_n \) satisfying \( i_1 \leq i_2 \leq \cdots \leq i_r \) and \( \sigma(i_1 + \cdots + i_{r-1} + 1) < \sigma(i_1 + \cdots + i_r + 1) \) if \( i_l = i_{l+1} \). This second condition on \( \sigma \) says that the order is preserved when comparing the first elements of blocks of the same size. Indeed, if \( \sigma \) is a \((1, 2, 2, 3)\)-unshuffle in \( S_7' \), then \( i_2 = i_3 = 2 \), so the order must be preserved when comparing the first element of the \( i_2 \) block to the first element of the \( i_3 \) block.

Example 1. Figure 1 is an example of a \((1, 1, 2, 3)\)-unshuffle in \( S_7 \). That is, \( \sigma = (124653)(7) \), and we have drawn a picture describing \( \sigma^* \). That is, \( x_{\sigma(1)} = x_2, x_{\sigma(2)} = x_4, \) and so on. The picture describes how \( \sigma^* \) permutes \( x_1, \ldots, x_7 \).
In words, a \((1,1,2,3)\)-unshuffle places the numbers 1 through 7 into boxes of size 1,1,2, and 3, where the order is preserved in each box. In this example, the resulting boxes would be (2), (4), (1, 6), and (3, 5, 7).

**Example 2.** A special case of the above definition is if we only have two numbers in our partition of \(n\). In particular, \(\sigma \in S_n\) is a \((p, n-p)\)-unshuffle if \(\sigma(k) < \sigma(k+1)\) whenever \(k \neq p\). In words, this permutation will place the numbers 1 through \(n\) into two boxes, where order is preserved in each. For brevity, we will sometimes refer to a \((p, n-p)\)-unshuffle as a \(p\)-unshuffle if \(n\) is clear.

**Example 3.** In \(S_4\), if we use the notation \(xyzw\) to denote the permutation \((1\ 2\ 3\ 4\ x\ y\ z\ w)\), then we can write down the 1, 2, and 3-unshuffles:

- **1-unshuffles:** 1234, 2134, 3124, 4123
- **2-unshuffles:** 1234, 1324, 1423, 2314, 2413, 3412
- **3-unshuffles:** 1234, 1243, 1342, 2341

We can now state the definition of an \(L_\infty\)-algebra. The general definition involves signs, but we are working over \(\mathbb{F}_2\) throughout this paper.

**Definition 3.** Let \(V\) be a graded vector space. An \(L_\infty\)-algebra structure on \(V\) is a collection of (skew)-symmetric multilinear maps \(\{l_k : V^\otimes k \to V\}\) of degree \(k-2\). That is,

\[
l_k(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}) = l_k(x_1, x_2, \ldots, x_k)
\]

for all \(\sigma \in S_k\) and \(x_i \in V\). Moreover, these maps must satisfy the generalized Jacobi identity:

\[
\sum_{i+j=n+1} \sum_{\sigma \in S(i,n-i)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0
\]

Here, \(i \geq 1\), \(j \geq 1\), \(n \geq 1\), and the inner summation is taken over all \((i, n-i)\)-unshuffles.

**Remark.** We could have also written the (skew)-symmetry condition as \(l_k \circ \sigma^\ast = l_k\) for \(\sigma \in S_k\).

**Remark.** Another way to write the generalized Jacobi identity is by using the notation

\[
\sum_{i+j=n+1} \sum_{\sigma} l_j \circ (l_i \otimes \text{Id}) \circ \sigma^\ast = 0
\]

**Remark.** Figure 2 is a depiction of the generalized Jacobi identity.
Example 4. The morphism relation simplifies to \( f \) when \((\cdot, \cdot)\) to a cochain complex, we require each \( l \) such that permutes the elements in degree \( n - 1 \) such that

\[
\sum_{j+k=n+1} \sum_{\sigma \in S(k,n-k)} f_j \circ (l_k \otimes \text{Id}) \circ \sigma^* = \sum_{\tau \in S'(i_1, \ldots, i_r)} l'_r \circ (f_{i_1} \circ \cdots \circ f_{i_r}) \circ \tau^* 
\]

Remark. This definition assumes that our \( L_\infty \)-algebra is a chain complex. If instead it is a cochain complex, we require each \( l_k \) to have degree \( 2 - k \).

Definition 4. Let \((L, l_i)\) and \((L', l'_i)\) be \( L_\infty \) algebras. An \( L_\infty \)-algebra homomorphism from \( L \) to \( L' \) is a collection \( \{f_n : L^{\otimes n} \to L'\} \) of (skew)-symmetric multilinear maps of degree \( n - 1 \) such that

\[
\sum_{j+k=n+1} \sum_{\sigma \in S(k,n-k)} f_j \circ (l_k \otimes \text{Id}) \circ \sigma^* = \sum_{\tau \in S'(i_1, \ldots, i_r)} l'_r \circ (f_{i_1} \circ \cdots \circ f_{i_r}) \circ \tau^* 
\]

Example 4. The \( n = 2 \) morphism relation says that

\[
f_1(l_2(x_1, x_2)) + f_2(l_1(x_1), x_2) + f_2(l_1(x_2), x_1) = l'_1(f_2(x_1, x_2)) + l'_2(f_1(x_1), f_1(x_2))
\]

When \((L, l_i)\) and \((L', l'_i)\) are \( L_\infty \)-algebras consisting of elements in degree 0 only, the \( n = 2 \) morphism relation simplifies to \( f_1(l_2(x_1, x_2)) + l'_2(f_1(x_1), f_1(x_2)) = 0 \), which is just a Lie algebra homomorphism: \( \phi([x_1, x_2]) = [\phi(x_1), \phi(x_2)] \).

Definition 5. Let \((L, l_k)\) be an \( L_\infty \)-algebra. An \( L_\infty \)-module over \( L \) is a graded vector space \( M \), together with a collection of skew-symmetric multilinear maps \( \{k_n : L^{\otimes n-1} \otimes M \to M \mid 1 \leq n < \infty \} \) of degree \( n - 2 \) such that the following identity holds:

\[
\sum_{p+q=n+1} \sum_{\sigma(n)=n} k_q \circ (l_p \otimes \text{Id}) \circ \sigma^* + \sum_{p+q=n+1} \sum_{\sigma(p)=n} k_q \circ \delta^* \circ (k_p \otimes \text{Id}) \circ \sigma^* = 0
\]

Here, \( \sigma \) is a \( p \)-unshuffle in \( S_n \). Also, because \( k_n : L^{\otimes n-1} \otimes M \to M \), we introduce the permutation \( \delta \) and use skew-symmetry of \( k_n \) in the case \( \sigma(p) = n \). That is, \( \delta^* \) is the map that permutes the \( k_p \) term past the remaining elements:

\[
k_q \left( \prod_{\sigma(n)}^p \frac{x_{\sigma(1)} \cdots x_{\sigma(p)}}{\sum_{\sigma(n)}^p}, \frac{x_{\sigma(p+1)} \cdots x_{\sigma(n)}}{\sum_{\sigma(n)}^p} \right) = k_q \left( \prod_{\sigma(n)}^p \frac{x_{\sigma(1)} \cdots x_{\sigma(p)}}{\sum_{\sigma(n)}^p}, \frac{x_{\sigma(p+1)} \cdots x_{\sigma(n)}}{\sum_{\sigma(n)}^p} \right)
\]

Figure 2. A graphical depiction of the generalized Jacobi identity. This should be interpreted as the sum of all compositions \( l_j \circ (l_i \otimes \text{Id}) \circ \sigma^* \), applied to the input \( x_1 \otimes \cdots \otimes x_n \). That is, this picture represents \( \sum_{i+j=n+1} \sum_{\sigma} l_j \circ (l_i \otimes \text{Id}) \circ \sigma^* (x_1 \otimes \cdots \otimes x_n) = 0 \).
Example 5. The \( n = 1 \) module relation says that \( M \) is a chain complex with differential \( k_1 \):
\[
k_1(k_1(m)) = 0
\]
The \( n = 2 \) module relation says that the action satisfies the Leibniz rule:
\[
k_2(l_1(x_1), m) + k_2(x_1, k_1(m)) + k_1(k_2(x_1, m)) = 0
\]
Using a different notation, we could also write
\[
[\partial x_1, m] + [x_1, \partial m] + \partial [x_1, m] = 0
\]
to remind us of differential graded Lie algebras.

Definition 6. Following [2], let \((L, l_i)\) be an \( L_\infty \)-algebra, and let \((M, k_i)\) and \((M', k'_i)\) be two \( L_\infty \)-modules over \( L \). An \( L_\infty \)-module homomorphism from \( M \) to \( M' \) is a collection \( \{ h_n : L \otimes (n - 1) \otimes M \to M' \} \) of skew-symmetric multilinear maps of degree \( n - 1 \) such that
\[
\sum_{i+j=n+1 \sigma(n)=n} h_j \circ (l_i \otimes \text{Id}) \circ \sigma^* + \sum_{i+j=n+1 \sigma(i)=n} h_j \circ \delta^* \circ (k_i \otimes \text{Id}) \circ \sigma^* = \sum_{r+s=n+1} k'_r \circ (\text{Id} \otimes h_s) \circ (\tau^* \otimes \text{Id})
\]
Here, \( \sigma \) is an \( i \)-shuffle in \( S_n \), and \( \tau \) is an \( (n - s) \)-shuffle in \( S_{n-1} \).

Remark. Figure 3 is a depiction of the \( L_\infty \)-module homomorphism relation.

Example 6. The \( n = 1 \) module homomorphism relation says that \( h_1 \) is a chain map:
\[
h_1k_1(m) = k'_1h_1(m)
\]
The \( n = 2 \) module homomorphism relation says:
\[
h_2(l_1(x_1), m) + h_2(x_1, k_1(m)) + h_1(k_2(x_1, m)) = k'_2(x_1, h_1(m)) + k'_1(h_2(x_1, m))
\]

Definition 7. The identity map, \( \text{Id}_M \), of an \( L_\infty \) module \( M \) is defined as follows. \( (\text{Id}_M)_1 \) is the identity map of the underlying graded vector space \( M \), and \( (\text{Id}_M)_r = 0 \) for \( r \geq 2 \). It is straightforward to check that this satisfies the definition of an \( L_\infty \)-module homomorphism.
Definition 8. Let $L$ be an $L_\infty$-algebra, and let $A, B,$ and $C$ be $L_\infty$-modules over $L$. Given $L_\infty$-module homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$, we define the composition $g \circ f$ by

$$(g \circ f)_n = \sum_{i+j=n+1} \sum_{\sigma(i)=n} g_j \circ \delta^* \circ (f_i \otimes \text{Id}) \circ \sigma^*$$

where $\sigma$ is an $i$-unshuffle in $S_n$, and $\lambda^*$ is the map that permutes the module element to the final input.

![Figure 4](image)

**Figure 4.** A graphical depiction of the composition of two $L_\infty$-module homomorphisms. This should be interpreted as $(g \circ f)_n = \sum g_j \circ \delta^* \circ (f_i \otimes \text{Id}) \circ \sigma^*$.

The following Lemma is perhaps well-known, but we do not know a reference for it. Pictures representing each step in the proof are given in the appendix.

**Lemma 1 (Composition).** Let $(L, l_i)$ be an $L_\infty$-algebra, and let $A, B,$ and $C$ be $L_\infty$-modules over $L$, with module operations denoted by $a_i, b_i, \text{ and } c_i,$ respectively. Given $L_\infty$-module homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$, the composition $g \circ f$ is an $L_\infty$-module homomorphism.

**Proof.** This follows from the fact that both $f$ and $g$ are $L_\infty$-module homomorphisms. Below, we will apply the $L_\infty$-module homomorphism relation for $f$, then we will apply the $L_\infty$-module homomorphism relation for $g$, and then we will conclude the $L_\infty$-module homomorphism relation for $g \circ f$.

**Step 1.** The relation that we need to show is

$$\sum_{i+j=n+1} \sum_{\sigma} (g \circ f)_j \circ (a_i \otimes \text{Id}) \circ \sigma^* = \sum_{r+s=n+1} \sum_{\tau} c_r \circ (\text{Id} \otimes (g \circ f)_s) \circ \tau^*$$

where $\sigma$ is an $(i, n-i)$-unshuffle and $\tau$ is an $(n-s, s-1)$-unshuffle.

**Step 2.** Break the left-hand side into two parts, and replace $(g \circ f)_j$ with its definition

$$\sum_{i+j=n+1} \sum_{\sigma(i)=n} \sum_{p+q=j+1} \delta^* \circ (f_p \otimes \text{Id}) \circ \delta^* \circ (a_i \otimes \text{Id}) \circ \sigma^*$$

$$+ \sum_{i+j=n+1} \sum_{\sigma(n)=n} \sum_{p+q=j+1} \delta^* \circ (f_p \otimes \text{Id}) \circ \delta^* \circ (l_i \otimes \text{Id}) \circ \sigma^*$$
where $\delta^*$ is the map that permutes the module element to the last input.

**Step 3.** In the first sum, applying $\sigma^*$ and $\theta^*$ results in a block of size $i$ being inputted to $a_i$, a block of size $p - 1$ being inputted into $f_p$, together with the output of $a_i$, and then a block of size $j - p$ remaining elements (which will be inputted into $g_q$). An equivalent way to achieve this is to first apply a $(p + i - 1)$-unshuffle $\eta^*$ and then an $i$-unshuffle $\psi^*$. If $\eta(p + i - 1) = n$ and $\psi(i) = p + i - 1$, we again obtain a block of size $i$ being inputted into $a_i$, then a block of size $p - 1$ being inputted into $f_p$, together with the output of $a_i$, with $j - p$ elements remaining.

In the second sum, we do the same thing, except the output of $l_i$ can either go into the first input of $f_p$ or the first input of $g_q$, by the definition of unshuffle. So we decompose the second sum to reflect these two cases.

\[
\begin{align*}
&\sum_{i+j=n+1} \sum_{p+q=j+1} \sum_{\eta \in S(p+i-1,j-p)} \sum_{\psi \in S(i,p-1)} g_q \circ \delta^* \circ (f_p \otimes \text{Id}) \circ \lambda^* \circ (a_i \otimes \text{Id}) \circ (\psi^* \otimes \text{Id}) \circ \eta^* \\
&+ \sum_{i+j=n+1} \sum_{p+q=j+1} \sum_{\eta \in S(p+i-1,j-p)} g_q \circ \delta^* \circ (f_p \otimes \text{Id}) \circ (l_i \otimes \text{Id}) \circ (\psi^* \otimes \text{Id}) \circ \eta^* \\
&+ \sum_{i+j=n+1} \sum_{p+q=j+1} \sum_{\eta \in S(p+i-1,j-p)} g_q \circ \delta^* \circ (l_i \otimes f_p \otimes \text{Id}) \circ (\psi^* \otimes \text{Id}) \circ \eta^* \\
\end{align*}
\]

**Step 4.** Reindex over $\alpha = p + i$.

\[
\begin{align*}
&\sum_{\alpha=2}^{n+1} \sum_{p+i=\alpha} \sum_{\eta \in S(\alpha-1,n+1)} \sum_{\psi \in S(i,\alpha-1)} g_{n+2-\alpha} \circ \delta^* \circ (f_p \otimes \text{Id}) \circ \lambda^* \circ (a_i \otimes \text{Id}) \circ (\psi^* \otimes \text{Id}) \circ \eta^* \\
&+ \sum_{\alpha=2}^{n+1} \sum_{p+i=\alpha} \sum_{\eta \in S(\alpha-1,n+1)} \sum_{\psi \in S(i,\alpha-1)} g_{n+2-\alpha} \circ \delta^* \circ (f_p \otimes \text{Id}) \circ (l_i \otimes \text{Id}) \circ (\psi^* \otimes \text{Id}) \circ \eta^* \\
&+ \sum_{\alpha=2}^{n+1} \sum_{p+i=\alpha} \sum_{\eta \in S(\alpha-1,n+1)} \sum_{\psi \in S(i,\alpha-1)} g_{n+2-\alpha} \circ \delta^* \circ (l_i \otimes f_p \otimes \text{Id}) \circ (\psi^* \otimes \text{Id}) \circ \eta^* \\
\end{align*}
\]

**Step 5.** Apply the module homomorphism relation for $f$ in the first two sums. In the third sum, change notation from $i$ to $t$ and from $p$ to $s$. 

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Step 6. In the first sum, combine $\tau \in S(t-1, s-1)$ and $\eta \in S(\alpha-1, n-\alpha+1)$ into a single $(t-1, s, n-\alpha+1)$-unshuffle, denoted by $\pi$. In the second sum, combine $\psi \in S(t, s-1)$ and $\eta \in S(\alpha, n-\alpha)$ into a single $(t, s, n-\alpha)$-unshuffle, denoted by $\pi$.

$$\sum_{\alpha=2}^{n+1} \sum_{t+s=\alpha} \sum_{\eta \in S(\alpha-1, n-\alpha+1)} \sum_{\tau \in S(t-1, s-1)} g_{n+2-\alpha} \circ (b_t \otimes \text{Id}) \circ (\text{Id} \otimes f_s \otimes \text{Id}) \circ (\tau^* \otimes \text{Id}) \circ \eta^*$$

$$+ \sum_{\alpha=2}^{n+1} \sum_{t+s=\alpha} \sum_{\eta \in S(\alpha-1, n-\alpha)} \sum_{\psi(t)=t} g_{n+2-\alpha} \circ (l_t \otimes f_s \otimes \text{Id}) \circ (\psi^* \otimes \text{Id}) \circ \eta^*$$

Step 7. In the first sum, $\pi$ unshuffles the $n$ elements into a block of size $t-1$, a block of size $s$, and a block of size $n-\alpha+1$. The block of size $s$ is then inputted into $f_s$, and then the output of $f_s$ is then inputted into $b_t$, as the module element, with the block of size $t-1$.

An equivalent way of achieving this is to apply an $(n-s, s-1)$-unshuffle to the $(n-1)$-algebra elements, to form blocks of size $(n-s)$ and $s-1$, and then input the $s-1$ algebra elements into $f_s$, with the module element. Then, apply an $t$-unshuffle $\sigma^*$ to these $n-s+1$ elements. By requiring $\sigma(t) = n-s+1$, we obtain a block of size $t-1$, plus a module element, that we input into $b_t$. We can do an analogous reformulation of the second sum.

$$\sum_{\alpha=2}^{n+1} \sum_{t+s=\alpha} \sum_{\phi \in S(n-s, s-1)} \sum_{\sigma \in S(t, n-s+1)} g_{n+2-\alpha} \circ (b_t \otimes \text{Id}) \circ (\text{Id} \otimes f_s) \circ (\phi^* \otimes \text{Id})$$

$$+ \sum_{\alpha=2}^{n+1} \sum_{t+s=\alpha} \sum_{\phi \in S(n-s, s-1)} \sum_{\sigma(n-s+1)=n-s+1} g_{n+2-\alpha} \circ (l_t \otimes \text{Id}) \circ (\text{Id} \otimes f_s) \circ (\phi^* \otimes \text{Id})$$

Step 8. Reindex, noting that $\sum_{\alpha=2}^{n+1} t+s=\alpha = \sum_{s=1}^{n} \sum_{t=1}^{s+n-1} = \sum_{s=1}^{n} \sum_{x+y=n+2-s}^{s+y=n+2-s}$. 
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$$\sum_{s=1}^{n} \sum_{x+y=n+2-s} \sum_{\phi \in S(n-s,s-1)} \sum_{\sigma(x)=n-s+1} g_y \circ \delta^\bullet \circ (b_x \otimes \text{Id}) \circ \sigma^\bullet \circ (\text{Id} \otimes f_s) \circ (\phi^\bullet \otimes \text{Id})$$

$$+ \sum_{s=1}^{n} \sum_{x+y=n+2-s} \sum_{\phi \in S(n-s,s-1)} \sum_{\sigma(x)=n-s+1} \sum_{\sigma(n-s+1)=n+s+1} g_y \circ (l_x \otimes \text{Id}) \circ \sigma^\bullet \circ (\text{Id} \otimes f_s) \circ (\phi^\bullet \otimes \text{Id})$$

**Step 9.** Apply the morphism relation for $g$.

$$\sum_{s=1}^{n} \sum_{r+q=n-s+2} \sum_{\phi \in S(n-s,s-1)} \sum_{\kappa \in S(r,q-1)} c_r \circ (\text{Id} \otimes g_q) \circ (\kappa^\bullet \otimes \text{Id}) \circ (\text{Id} \otimes f_s) \circ (\phi^\bullet \otimes \text{Id})$$

**Step 10.** Combine $\kappa$ and $\phi$ into a single permutation $\pi$.

$$\sum_{s=1}^{n} \sum_{r+q=n-s+2} \sum_{\pi \in S(r-q-1,-s+1)} c_r \circ (\text{Id} \otimes g_q) \circ (\text{Id} \otimes f_s) \circ (\pi^\bullet \otimes \text{Id})$$

**Step 11.** Split $\pi$ into $\tau$ and $\psi$. The map $\lambda^\bullet$ is needed to permute the module element into the last input of $g_q$.

$$\sum_{s=1}^{n} \sum_{r+q=n-s+2} \sum_{\tau \in S(\tau-1,n-r)} \sum_{\kappa \in S(q,1)} c_r \circ (\text{Id} \otimes [g_q \circ \lambda^\bullet \circ (f_s \otimes \text{Id}) \circ \psi^\bullet]) \circ (\tau^\bullet \otimes \text{Id})$$

**Step 12.** Change how we index over $s, r, q$.

$$\sum_{r=1}^{n} \sum_{s+q=n+2-r} \sum_{\tau \in S(\tau-1,n-r)} \sum_{\psi \in S(s,q-1)} c_r \circ (\text{Id} \otimes [g_q \circ \lambda^\bullet \circ (f_s \otimes \text{Id}) \circ \psi^\bullet]) \circ (\tau^\bullet \otimes \text{Id})$$

**Step 13.** Use the definition of $g \circ f$.

$$\sum_{r=1}^{n} \sum_{\tau \in S(\tau-1,n-r)} c_r \circ (\text{Id} \otimes (g \circ f)_{n+1-r}) \circ (\tau^\bullet \otimes \text{Id})$$

**Step 14.** This is

$$\sum_{r+s=n+1} \sum_{\tau \in S(\tau-1,n-1,s-1)} c_r \circ (\text{Id} \otimes (g \circ f)_s) \circ (\tau^\bullet \otimes \text{Id})$$

□
3. Restriction of Scalars

In this section, we prove the main result. We start by defining the restriction of scalars functor on objects and prove that the result is an $L\infty$-module. We then define the restriction of scalars functor on morphisms, proving that the result is an $L\infty$-module homomorphism. Finally, we complete the proof of functoriality. The end of this section contains a technical lemma that is applied several times throughout the aforementioned proofs.

**Lemma 2 (Objects).** Suppose $I : (L', l') \to (L, l)$ is a map of $L\infty$-algebras. If $(M, k)$ is an $L$-module, then $I^* M := (M, k')$ is an $L'$-module, where $k'_n : L^\otimes n - 1 \otimes M \to M$ is given by

$$k'_n = \sum_{r=1}^{n-1} \sum_{\tau \in S'(i_1, \ldots, i_r)} \sum_{i_1 + \ldots + i_r = n-1} k_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (\tau^* \otimes \text{Id})$$

**Proof.** The idea of the proof is straightforward. We will first make a substitution using the definition of $k'$ (steps 1-2). We will then use the $L\infty$-algebra homomorphism relation for $I$ to exchange any $I$ and $l'$ terms (steps 3-9). The terms that remain will then cancel by applying the $L\infty$-module relation for $k$ (steps 10-19). Pictures representing each step in the proof are given in the appendix.

**Step 1.** The $L\infty$ relation for $k'_n$ that we need to show is zero is:

$$\sum_{p+q=n+1, \sigma(n)=n} \sum_{p<n} k'_q \circ (l'_p \otimes \text{Id}) \circ \sigma^* + \sum_{p+q=n+1, \sigma(p)=n} \sum_{p<n} k'_q \circ \delta^* \circ (k'_p \otimes \text{Id}) \circ \sigma^* = 0$$

**Step 2.** Focusing only on the first double sum for now, we substitute for $k'_q$ using its definition:

$$\sum_{p+q=n+1, \sigma(n)=n} \sum_{p<n} \sum_{\tau \in S'(i_1, \ldots, i_r)} \sum_{1 \leq r \leq q-1} \sum_{i_1 + \ldots + i_r = n-1} k_{r+1} \left( (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (\tau^* \otimes \text{Id}) \circ (l'_p \otimes \text{Id}) \circ \sigma^* \right)$$

**Step 3.** The goal now is to use the morphism relation to commute the $l'_p$ and $I$ terms. To do so, we will break down this sum by the specific morphism relation that we will apply ($k = 1, \ldots, n-1$). In particular, this is determined by the sum of $p$ and the size of the block to which $\tau$ sends $l'_p$. We will denote the block containing $l'_p$ by $i_1$, and we will denote its size by $s$.

$$\sum_{p=1, \sigma(n)=n} \sum_{\tau \in S'(i_1, \ldots, i_r)} \sum_{1 \leq r \leq n-p} \sum_{i_1 + \ldots + i_r = n-p} k_{r+1} \left( (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (\tau^* \otimes \text{Id}) \circ (l'_p \otimes \text{Id}) \circ \sigma^* \right)$$

$$= \sum_{p=1, \sigma(n)=n} \sum_{s=1}^{n-p} \sum_{\tau \in S'(i_1, \ldots, i_r)} \sum_{1 \leq r \leq n-p} \sum_{i_1 + \ldots + i_r = n-p} \sum_{i_l = s} k_{r+1} \left( (I_{i_1} \otimes \cdots \otimes I_{i_l} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (\tau^* \otimes \text{Id}) \circ (l'_p \otimes \text{Id}) \circ \sigma^* \right)$$
We can now reindex over the sum of \( p \) and \( s \) (on the \((p, s)\)-plane, this is summing over the diagonal) to obtain
\[
\sum_{k=1}^{n-1} \sum_{p+s=k+1} \sum_{\sigma(n)=n} \sum_{r\leq n-p} k_{r+1} \left( (I_{i_1} \otimes \cdots \otimes I_{i_t} \otimes \cdots \otimes I_{i_r}) \otimes \text{Id} \right) \circ (\tau^* \otimes \text{Id}) \circ (l'_p \otimes \text{Id}) \circ \sigma^* \right)
\]

**Step 4.** Here, we change \( \tau \) to \( \tau' \) and introduce \( \lambda \). Since \( \tau \) is an unshuffle, we can make two observations. First, \( \tau \) sends \( l'_p \) to the first input of \( I_i \). Second, in the partition \( i_1 + \cdots + i_r = n - p \), the block \( i_t \) is the first of its size (i.e. \( t < l \) implies \( i_t < i_l \)), since the first elements of blocks of the same size are in order. This information allows us to remove \( l'_p \) as an input to \( \tau \), and then put it back in the correct spot after the remaining elements are permuted. That is, \( \tau \) corresponds to an \((i_1, \ldots, i_t - 1, \ldots, i_r)\)-unshuffle \( \tau' \) in \( S_{n-p-1} \), and we will send \( l'_p \) to the first input of \( I_i \) via a permutation \( \lambda \) after we apply \( \tau' \). Special care is needed when \( s = 1 \), in which case \( \tau' \in S(0, i_2, \ldots, i_r) \), and no element will go to the block of size 0. Note: because \( \tau \in S'(i_1, \ldots, \hat{i}_r) \), we had conditions that \( i_1 \leq \cdots \leq i_r \) and that the order of the first elements among these blocks is preserved. In the rest of the proof, we must remember these restrictions inherited from \( \tau \). We obtain,
\[
\sum_{k=1}^{n-1} \sum_{p+s=k+1} \sum_{\sigma(n)=n} \sum_{r\leq n-p} k_{r+1} \left( (I_{i_1} \otimes \cdots \otimes I_{i_t} \otimes \cdots \otimes I_{i_r}) \otimes \text{Id} \right) \circ \lambda^* \circ (\text{Id} \otimes \tau'^* \otimes \text{Id}) \circ (l'_p \otimes \text{Id}) \circ \sigma^* \right)
\]

**Step 5.** Combine \( \sigma \) and \( \tau' \) into \( \psi \). Now we observe that applying a \( p \)-unshuffle and then \( \tau' \) to the remaining inputs is equivalent to doing a \((p, i_1, \ldots, i_r)\)-unshuffle to all of the inputs at once. We obtain
\[
\sum_{k=1}^{n-1} \sum_{p+s=k+1} \sum_{\psi(n)=n} k_{r+1} \left( (I_{i_1} \otimes \cdots \otimes I_{i_t} \otimes \cdots \otimes I_{i_r}) \otimes \text{Id} \right) \circ \lambda^* \circ (l'_p \otimes \text{Id}) \circ \psi^* \right)
\]

**Step 6.** Change from \( \psi \) to \( \mu, \alpha, \omega \). Notice that a \((p, i_1, \ldots, i_l - 1, \ldots, i_r)\)-unshuffle is the same as first doing a \((p + i_l - 1)\)-unshuffle, and then doing a \((p, i_l - 1)\)-unshuffle on the \((p + i_l - 1)\)-block and an \((i_1, \ldots, i_l, \ldots, i_r)\)-unshuffle on the rest. Since we are fixing \( i_l = s \), note that \( p + i_l - 1 = k \).

Afterwards, we need to apply a permutation \( \omega \) to move the strands in the \( i_l \) block back to their original position between the \( i_{l-1} \) and \( i_{l+1} \) blocks. That is, \( \omega \) is the block permutation so that applying \( \omega^* \) to the blocks \( \{1, i_l - 1, i_1, \ldots, \hat{i_l}, \ldots, i_r\} \) yields \( \{1, i_1, \ldots, i_l - 1, \ldots, i_r\} \). We apply \( \lambda^* \) after \( \omega^* \) to move the \( l'_p \) term.
\[
\sum_{k=1}^{n-1} \sum_{p+s=k+1} \sum_{\mu \in S(k, i_1, \ldots, i_t, \ldots, 1)} \sum_{i_1 + \cdots + i_t = n-p \atop \mu(n) = n} \sum_{\alpha \in S(p, k-p) \atop i_1 = s} \mu(n) = n
\]

\[
k_{r+1} \left( (I_{i_1} \otimes \cdots \otimes I_{i_t} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ \lambda^* \circ \omega^* \circ (l_p' \otimes \text{Id}) \circ (\alpha^* \otimes \text{Id}) \circ \mu^* \right)
\]

**Step 7.** Since \( k_{r+1} \) is skew-symmetric, we can move the \( I_{i_1} \) term to the first input.

\[
\sum_{k=1}^{n-1} \sum_{p+s=k+1} \sum_{\mu \in S(k, i_1, \ldots, i_t, \ldots, 1)} \sum_{i_1 + \cdots + i_t = n-p \atop i_1 = s} \sum_{\alpha \in S(p, k-p) \atop \mu(n) = n} \mu(n) = n
\]

\[
k_{r+1} \left( (I_{i_1} \otimes I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (l_p' \otimes \text{Id}) \circ (\alpha^* \otimes \text{Id}) \circ \mu^* \right)
\]

**Step 8.** Rewrite the maps as

\[
\sum_{k=1}^{n-1} \sum_{p+s=k+1} \sum_{\mu \in S(k, i_1, \ldots, i_t, \ldots, 1)} \sum_{i_1 + \cdots + i_t = n-p \atop i_1 = s} \sum_{\alpha \in S(p, k-p) \atop \mu(n) = n} \mu(n) = n
\]

\[
k_{r+1} \left( [I_{i_1} \circ (l_p' \otimes \text{Id}) \circ \alpha^*] \otimes [(I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id})] \circ \mu^* \right)
\]

**Step 9.** Apply the \( L_\infty \)-algebra homomorphism relation to the terms \( I_{i_1} \circ (l_p' \otimes \text{Id}) \circ \alpha^* \).

Since we no longer are keeping track of \( p \), we also use the fact that \( p + s = k + 1 \) to rewrite the conditions for \( \mu \).

\[
\sum_{k=1}^{n-1} \sum_{1 \leq k = a_1 + \cdots + a_t \leq k} \sum_{\gamma \in S'(a_1, \ldots, a_t)} \sum_{\mu \in S(k, i_1, \ldots, i_r, 1) \atop i_1 + \cdots + i_r + i_t = n-1-k \atop \mu(n) = n} \mu(n) = n
\]

\[
k_{r+1} \left( [l_t \circ (I_{a_1} \otimes \cdots \otimes I_{a_t})] \otimes (\gamma^*) \otimes [(I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id})] \circ \mu^* \right)
\]

**Step 10.** Rewrite the maps as
Step 11. We can combine \( \mu \) and \( \gamma \) into one permutation \( \eta \). Indeed, applying \( \mu \) and then an \((a_1, \ldots, a_t)\)-unshuffle on the \( k \)-block is the same as applying an \((a_1, \ldots, a_t, i_1, \ldots, i_r, 1)\)-unshuffle all at once.

\[
\sum_{k=1}^{n-1} \sum_{\substack{1 \leq \ell \leq k \\ a_1 + \ldots + a_t = k \\ a_r \geq 1}} \sum_{\gamma \in S(a_1, \ldots, a_t)} \sum_{\substack{\mu \in S(k, i_t, \ldots, i_r, 1) \\ i_1 + \ldots + i_r + 1 = n-1-k \atop \mu(n) = n}} k_{r+1} \left( (l_t \otimes \text{Id}) \circ (I_{a_1} \otimes \cdots \otimes I_{a_t} \otimes I_{i_1} \otimes \cdots \otimes \widehat{I_{i_t}} \otimes \cdots \otimes I_{i_r}) \circ (\gamma \otimes \text{Id}) \circ \mu^* \right)
\]

Step 12. Since \( k = 1, \ldots, n - 1 \), we can drop the sum over \( k \) from the notation and just require that \( a_1, \ldots, a_t, i_1, \ldots, i_r \) is a partition of \( n - 1 \), with \( a_1 \leq \ldots \leq a_t \), \( i_1 \leq \ldots \leq i_r \), and \( t \geq 1 \) and \( r \geq 1 \). If we fix \( \eta \in S(a_1, \ldots, a_t, i_1, \ldots, i_r) \), we don’t have any relation between the two partitions \( a_1 \leq \ldots \leq a_t \) and \( i_1 \leq \ldots \leq i_r \). That is, the sizes of the blocks are in order as part of their respective partitions, but it might not be the case that \( a_1, \ldots, a_t, i_1, \ldots, i_r \) is in increasing order as a whole. However, from these two partitions, we can use an unshuffle to construct a new partition where the sizes of the boxes are in order. Indeed, define \( \sigma \) so that \((\sigma^{-1})^*\) arranges the \( a_1, \ldots, a_t, i_1, \ldots, i_r \) in increasing order (to get a unique \( \sigma \), require that the order of the \( a \)'s is preserved, the order of the \( i \)'s is preserved, and that, using \( \eta \), the first elements of boxes of same size are in order). Then let \( c_1, \ldots, c_\alpha := (\sigma^*)^{-1}(a_1, \ldots, a_t, i_1, \ldots, i_r) \). To summarize, what we have done is define a new partition \( c_1, \ldots, c_\alpha \) of \( n - 1 \) so that \( c_\sigma(1) = a_1, \ldots, c_\sigma(t) = a_t, c_\sigma(t+1) = i_1, \ldots, c_\sigma(\alpha) = i_r \). Of course, since \( a_1 \leq \ldots \leq a_t \), \( \sigma \) is a \( t \)-unshuffle. Moreover, we define \( \tau \) by requiring that the elements that \( \eta \) puts into the \( a_1, \ldots, a_t \) and \( i_1, \ldots, i_r \)-boxes are precisely those that \( \tau \) puts into the \( c_\sigma(1), \ldots, c_\sigma(t) \) and \( c_\sigma(t+1), \ldots, c_\sigma(\alpha) \)-boxes, respectively. Finally, since \( \alpha = t + r - 1 \), we relabeled \( k_{r+1} \) as \( k_{\alpha+2-t} \). Note that we can reverse this whole construction to obtain an inverse correspondence. This process is similar to Lemma 4.

\[
\sum_{\tau \in S(c_1, \ldots, c_\alpha)} \sum_{\substack{\sigma \in S(t, \alpha+1-t) \\ 1 \leq \alpha \leq n-1 \atop \sigma(\alpha+1) = \alpha + 1 \atop 1 \leq \sigma \leq \alpha+1}} k_{\alpha+2-t} \circ (l_t \otimes \text{Id}) \circ \sigma^* \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \text{Id}) \circ (\tau^* \otimes \text{Id})
\]

Step 13. On the other hand, we now examine the second term in the original sum:

\[
\sum_{p+q=n+1} \sum_{\sigma(p)=n} k_p' \circ \delta^* \circ (k_p' \otimes \text{Id}) \circ \sigma^*
\]
Step 14. Use the definition of $k'$ to substitute for $k'_p$ and $k'_q$. The cases $p = 1$ and $q = 1$ require some care; they correspond to the cases $r = 0$ and $s = 0$, respectively. If $r = 0$, then $\phi = \text{Id}$, and if $s = 0$, then $\psi = \text{Id}$. We also disallow $r$ and $s$ to be zero simultaneously.

$$
\sum_{p+q=n+1} \sum_{\sigma(p)=n} \sum_{1 \leq p \leq n} \sum_{\sigma = 1} \sum_{\sigma = 1} k_{s+1} \circ (I_{j_1} \otimes \cdots \otimes I_{j_s} \otimes \text{Id}) \circ (\psi \otimes \text{Id}) \circ \delta^* \circ (k_{r+1} \otimes \text{Id}) \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (\phi^* \otimes \text{Id} \otimes \psi^*) \circ \sigma^*
$$

Step 15. Commuting composition and tensor product, and replacing $\delta$ with an analogous $\delta'$ that ensures the module element is in the correct spot, we get

$$
\sum_{p+q=n+1} \sum_{\sigma(p)=n} \sum_{1 \leq p \leq n} \sum_{\sigma = 1} \sum_{\sigma = 1} k_{s+1} \circ \delta^* \circ (k_{r+1} \otimes \text{Id}) \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id} \otimes I_{j_1} \otimes \cdots \otimes I_{j_s}) \circ (\phi^* \otimes \text{Id} \otimes \psi^*) \circ \sigma^*
$$

Step 16. Instead of summing over $r$ and $s$ separately, we can sum over the diagonal $\alpha = r + s$.

$$
\sum_{p=1}^n \sum_{\sigma(p)=n} \sum_{1 \leq \sigma \leq n-1} \sum_{\sigma = 1} \sum_{\sigma = 1} k_{s+1} \circ \delta^* \circ (k_{r+1} \otimes \text{Id}) \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id} \otimes I_{j_1} \otimes \cdots \otimes I_{j_s}) \circ (\phi^* \otimes \text{Id} \otimes \psi^*) \circ \sigma^*
$$

Step 17. Apply Lemma 4, where $r + 1$ above corresponds to $t$ below.

$$
\sum_{\tau \in S_{(c_1, \ldots, c_n)}} \sum_{\sigma \in S_{(1, \alpha + 1 - t)}} k_{s+2-t} \circ \delta^* \circ (k_t \otimes \text{Id}) \circ (\sigma^* \otimes \text{Id}) \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\tau^* \otimes \text{Id})
$$

Step 18. Summarizing what we’ve done so far, we’ve shown that the original sum

$$
\sum_{p+q=n+1} \sum_{\sigma(p)=n} k'_q \circ (l'_p \otimes \text{Id}) \circ \sigma^*
$$

$$
+ \sum_{p+q=n+1} \sum_{\sigma(p)=n} k'_q \circ (\text{Id} \otimes k'_p) \circ \sigma^*
$$

can be rewritten as
Step 19. Letting $F = (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\tau \otimes \text{Id})$ and setting $u = \alpha + 2 - t$, this becomes

$$
\sum_{\tau \in S^I_{(c_1, \ldots, c_n)}} \sum_{c_1 + \ldots + c_n = n - 1} k_{\alpha+2-t} \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\tau \otimes \text{Id})
$$

which cancel by the module relation.

Lemma 3 (Morphisms). Suppose $L$ and $L'$ are $L_\infty$-algebras and $M$ and $N$ are $L$-modules. Let $I : L' \to L$ be an $L_\infty$-algebra homomorphism, and let $f : M \to N$ be an $L$-module homomorphism. Set $(I^* f)_1 = f_1$, and for $n \geq 2$, define

$$(I^* f)_n : (L')^{\otimes n-1} \otimes I^* M \to I^* N$$

by

$$(I^* f)_n = \sum_{r=1}^{n-1} \sum_{\tau \in S^I_{(i_1, \ldots, i_r)}} \sum_{i_1 + \ldots + i_r = n-1} f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (\tau \otimes \text{Id})$$

Then $I^* f : I^* M \to I^* N$ is a homomorphism of $L'$-modules.

Proof. We will start by examining the $L_\infty$-module homomorphism relation. After replacing $I^* f$ and $m'$ with their definitions on the left-hand side (steps 1-4), we will rearrange the sum (steps 5-6) and apply the $L_\infty$-algebra relation for $I$ (step 7). We then rewrite the terms (steps 8-9) and apply the module homomorphism relation for $f$ (step 10). We then show that the result is equal to the right-hand side (steps 11-16).
Step 2. Focusing only on the left-hand side, we break this sum up into two parts

\[
\sum_{i+j=n+1} \sum_{\sigma(i)=n} (I^*f)_{ij} \circ \lambda^* \circ (m'_i \otimes \text{Id}) \circ \sigma^*
\]
\[
+ \sum_{i+j=n+1} \sum_{\sigma(n)=n} (I^*f)_{ij} \circ (l'_i \otimes \text{Id}) \circ \sigma^*
\]

where we use skew-symmetry and introduce the permutation \( \lambda \) to insert the module element in the correct spot.

Step 3. Replace \( I^*f \) with its definition. Note that we’ve allowed \( r = 0 \) in the first sum to include the case \( j = 1 \), which corresponds to \( f_1 \circ m'_0 \circ \sigma^* \). If \( j \) is anything but 1, \( r = 0 \) makes no contribution to the sum.

\[
\sum_{i+j=n+1} \sum_{\sigma(n)=n} \sum_{\tau \in S'(i_1, \ldots, i_r)} \sum_{s=0}^{j-1} f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (\tau^* \otimes \text{Id}) \circ \lambda^* \circ (m'_i \otimes \text{Id}) \circ \sigma^*
\]
\[
+ \sum_{i+j=n+1} \sum_{\sigma(n)=n} \sum_{1 \leq i \leq n} \sum_{\tau \in S'(i_1, \ldots, i_r)} \sum_{s=0}^{j-1} f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (\tau^* \otimes \text{Id}) \circ (l'_i \otimes \text{Id}) \circ \sigma^*
\]

Step 4. Now focus on the first sum and replace \( m'_i \) with its definition. Similar to the above, we’ve allowed for the case \( s = 0 \) to include the case \( i = 1 \), which corresponds to \( (I^*f)_n \circ \lambda^* \circ (m'_i \otimes \text{Id}) \circ \sigma^* \). If \( i \) is anything but 1, \( s = 0 \) makes no contribution to the sum.

\[
\sum_{i+j=n+1} \sum_{\sigma(n)=n} \sum_{\tau \in S'(i_1, \ldots, i_r)} \sum_{s=0}^{j-1} \sum_{\psi \in S'(j_1, \ldots, j_s)} f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (\tau^* \otimes \text{Id}) \circ \lambda^* \circ ([m_{s+1} \circ (I_{j_1} \otimes \cdots \otimes I_{j_s} \otimes \text{Id}) \circ \psi^* \otimes \text{Id}] \otimes \text{Id}) \circ \sigma^*
\]

Step 5. Rewrite the sum by commuting composition and tensor product and considering the diagonal \( \alpha = r + s \) instead of \( r \) and \( s \) individually. Observe that one of \( r \) and \( s \) can be 0, but not both at the same time.

\[
\sum_{i+j=n+1} \sum_{\sigma(n)=n} \sum_{1 \leq r \leq \alpha \leq n-1} \sum_{\tau \in S'(i_1, \ldots, i_r)} \sum_{\psi \in S'(j_1, \ldots, j_s)} f_{r+1} \circ \omega^* \circ (m_{s+1} \otimes \text{Id}) \circ (I_{j_1} \otimes \cdots \otimes I_{j_s} \otimes \text{Id} \otimes I_{i_1} \otimes \cdots \otimes I_{i_r}) \circ (\psi^* \otimes \text{Id} \otimes \tau^*) \circ \sigma^*
\]

Step 6. Apply Lemma 4 to obtain
Combine $\tau$ 

Use the map $i$ 

Reindex over the sum of $n$ 

Now, focusing on the $l$ terms (the second sum in Step 3), our goal is to apply the $L_\infty$-algebra relation for $I$. The steps we follow here are essentially the same as in Lemma 2 (steps 3-12), and we direct the reader to them for details and for diagrams. We start with

$$\sum_{\pi \in S'(c_1, \ldots, c_n), c_1 + \ldots + c_n = n-1} \sum_{i \in S(t, \theta + 1 - t)} \sum_{\theta = \sigma + 1} \sum_{t \in S(t', \sigma + 1 - t)} f_{r+1} \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes I_d) \circ (\pi^* \otimes \Id)$$

Denote the block where $l_i'$ goes by $I_{l_i}$. Break down the sum by $i = s$.

$$\sum_{i+j=n+1, \sigma(n) = n} \sum_{r=1}^{j-1} \sum_{\tau \in S'(i_1, \ldots, i_r)} f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes I_d) \circ (\tau^* \otimes \Id) \circ (l_i' \otimes \Id) \circ \sigma^*$$

Remove $j$ from the notation.

$$\sum_{i+1 = n+1, \sigma(n) = n} \sum_{r=1}^{j-1} \sum_{\tau \in S'(i_1, \ldots, i_r)} f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes I_d) \circ (\tau^* \otimes \Id) \circ (l_i' \otimes \Id) \circ \sigma^*$$

Reindex over the sum of $i$ and $s$.

$$\sum_{k=1}^{n-1} \sum_{i+s=k+1, \sigma(n) = n} \sum_{\tau \in S'(i_1, \ldots, i_r)} f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes I_d) \circ (\tau^* \otimes \Id) \circ (l_i' \otimes \Id) \circ \sigma^*$$

Use the map $\lambda^*$ to permute $l_i'$ around $\tau$ and change $\tau$ to $\tau'$.

$$\sum_{k=1}^{n-1} \sum_{i+s=k+1, \sigma(n) = n} \sum_{\tau \in S'(i_1, \ldots, i_r)} f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes I_d) \circ \lambda^* \circ (\Id \otimes \tau'^* \otimes \Id) \circ (l_i' \otimes \Id) \circ \sigma^*$$

Combine $\tau'$ and $\sigma$ into the permutation $\eta$.

$$\sum_{k=1}^{n-1} \sum_{i+s=k+1, \rho \in S(i_1, i_1-1)} \sum_{\eta \in S(i_1, i_1-1, \ldots, i_r-1)} f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \Id) \circ \lambda^* \circ (\Id \otimes \tau'^* \otimes \Id) \circ (l_i' \otimes \Id) \circ \eta^*$$
Use skew-symmetry of $f_{r+1}$ to swap the order of the $I$'s

$$
\sum_{k=1}^{n-1} \sum_{i+s=k+1} \sum_{\rho \in S(i,i-1)} \sum_{\eta \in S(i+i-1,i,...,i_r,1)} f_{r+1} \circ (I_{i_1} \otimes I_{i_2} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (l'_{i-1} \otimes \text{Id}) \circ (\rho^* \otimes \text{Id}) \circ \eta^*
$$

Rewrite suggestively, noting that now the $I_{i_1}$ is omitted from $I_{i_1} \otimes \cdots \otimes I_{i_r}$.

$$
\sum_{k=1}^{n-1} \sum_{i+s=k+1} \sum_{\rho \in S(i,i-1)} \sum_{\eta \in S(i+i-1,i,...,i_r,1)} f_{r+1} \left( (l_i \circ (l'_i \otimes \text{Id}) \circ \rho^*) \otimes (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \right) \circ \eta^*
$$

Apply the morphism relations.

$$
\sum_{k=1}^{n-1} \sum_{\gamma \in S(t_1,...,t_k)} \sum_{\eta \in S(k,i_1,...,i_r,1)} f_{r+1} \left( (l_i \circ (I_{t_1} \otimes \cdots \otimes I_{t_k}) \circ \gamma^*) \otimes (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \right) \circ \eta^*
$$

Combine $\gamma$ and $\eta$ into $\psi$.

$$
\sum_{k=1}^{n-1} \sum_{\psi \in S(t_1,...,t_k,i_1,...,i_r,1)} f_{r+1} \left( (l_i \otimes \text{Id}) \circ (I_{t_1} \otimes \cdots \otimes I_{t_k} \otimes I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \right) \circ \psi^*
$$

This is equivalent to

$$
\sum_{\pi \in S'(c_1,...,c_n)} \sum_{\theta \in S(t,\alpha+1)} f_{\alpha+2-t} \circ (l_i \otimes \text{Id}) \circ \theta^* \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\pi^* \otimes \text{Id})
$$

**Step 8.** In total, combining this with Step 6, we have the sum

$$
\sum_{\pi \in S'(c_1,...,c_n)} \sum_{\theta \in S(t,\alpha+1)} f_{\alpha+2-t} \circ \omega^* \circ (m_t \otimes \text{Id}) \circ \theta^* \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\pi^* \otimes \text{Id})
$$

$$
+ \sum_{\pi \in S'(c_1,...,c_n)} \sum_{\theta \in S(t,\alpha+1)} f_{\alpha+2-t} \circ (l_i \otimes \text{Id}) \circ \theta^* \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\pi^* \otimes \text{Id})
$$

**Step 9.** Change notation; change $t$ to $i$ and $\alpha + 2 - t$ to $j$. 
\[
\sum_{\pi \in S(c_1, \ldots, c_n)} \sum_{i+j=\alpha+2} \sum_{\theta \in S(l, \alpha+1-i)} f_j \circ \omega^* \circ (m_i \otimes \text{Id}) \circ \theta^* \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\pi^* \otimes \text{Id})
\]
\[
+ \sum_{\pi \in S(c_1, \ldots, c_n)} \sum_{i+j=\alpha+2} \sum_{\theta \in S(l, \alpha+1-i)} f_j \circ (l_i \otimes \text{Id}) \circ \theta^* \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\pi^* \otimes \text{Id})
\]

**Step 10.** Applying the module homomorphism relation for \( f \), we obtain
\[
\sum_{\pi \in S(c_1, \ldots, c_n)} \sum_{r+s=\alpha+2} \sum_{\rho \in S(\alpha-s, s)} n_{r} \circ (\text{Id} \otimes f_s) \circ (\rho^* \otimes \text{Id}) \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\pi^* \otimes \text{Id})
\]

**Step 11.** It just remains to show that the sum above is equal to
\[
\sum_{r+s=\alpha+1} n'_r \circ (\text{Id} \otimes (I^* f)_s) \circ (\tau^* \otimes \text{Id})
\]
Therefore, use the definition of \( I^* f \). Like usual, we start indexing at \( x = 0 \) to allow for the \( f_1 \) case.
\[
\sum_{r+s=\alpha+1} n'_r \circ \left[ \text{Id} \otimes \left( (I^* f)_s \circ (\sum_{\gamma \in S(j_1, \ldots, j_y)} (\gamma^* \otimes \text{Id}) \circ (\phi^* \otimes \text{Id})) \right) \right] \circ (\tau^* \otimes \text{Id})
\]

**Step 12.** Now use the definition of \( n' \). Allow for \( y = 0 \) to deal with the \( n_1 \) case.
\[
\sum_{r+s=\alpha+1} \sum_{\tau \in S(\alpha-s, s)} \sum_{x=0}^{s-1} \sum_{\phi \in S(i_1, \ldots, i_x)} \sum_{y=0}^{s-1} \sum_{\gamma \in S(j_1, \ldots, j_y)} n_{y+1} \circ (I_{j_1} \otimes \cdots \otimes I_{j_y} \otimes \text{Id}) \circ (\gamma^* \otimes \text{Id}) \circ \left[ \text{Id} \otimes \left( f_{x+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_x} \otimes \text{Id}) \circ (\phi^* \otimes \text{Id}) \right) \right] \circ (\tau^* \otimes \text{Id})
\]

**Step 13.** Commute composition and tensor product to rewrite as
\[
\sum_{r+s=\alpha+1} \sum_{\tau \in S(\alpha-s, s)} \sum_{x=0}^{s-1} \sum_{\phi \in S(i_1, \ldots, i_x)} \sum_{y=0}^{s-1} \sum_{\gamma \in S(j_1, \ldots, j_y)} n_{y+1} \circ (\text{Id} \otimes f_{x+1}) \circ (I_{j_1} \otimes \cdots \otimes I_{j_y} \otimes I_{i_1} \otimes \cdots \otimes I_{i_x} \otimes \text{Id}) \circ (\gamma^* \otimes \phi^* \otimes \text{Id}) \circ (\tau^* \otimes \text{Id})
\]

**Step 14.** Reindex over the diagonal of \( \alpha = x + y \). Observe that one of \( x \) and \( y \) can be 0, but not both at the same time.
Step 2. Replace Step 1.

Step 15. Apply Lemma 4 to get

$$
\sum_{r+s=n+1} \sum_{\tau \in S(n-s)} \sum_{1 \leq \alpha \leq n-1} \sum_{x+y=\alpha} \sum_{z, y, \geq 0} n_{y+1} \circ (\text{Id} \otimes f_{x+1}) \circ (I_{j_1} \otimes \cdots \otimes I_{j_y} \otimes I_{i_1} \otimes \cdots \otimes I_{i_x} \otimes \text{Id}) \circ (\gamma^* \otimes \phi^* \otimes \text{Id}) \circ (\tau^* \otimes \text{Id})
$$

Step 16. Rewriting this as

$$
\sum_{\pi \in S'(c_1, \ldots, c_n)} \sum_{\theta \in S(\alpha - s, s)} \sum_{c_1 + \cdots + c_n = n - 1} \sum_{1 \leq \alpha \leq n - 1} n_{r} \circ (\text{Id} \otimes f_{s}) \circ \theta^* \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\pi^* \otimes \text{Id})
$$

shows that it is the same as the sum in Step 10, which completes the proof.

\[\square\]

Theorem 3.1 (Functoriality). Suppose $I : (L', l') \to (L, l)$ is a map of $L_\infty$-algebras. Then $I^* : L\text{-mod} \to L'\text{-mod}$ is a functor.

Proof. Suppose we have $L_\infty$-modules $M, N$, and $Q$ over $L$ and $L_\infty$-module homomorphisms $M \xrightarrow{f} N \xrightarrow{g} Q$. We have defined $I^*$ on objects and morphisms, so it remains to show that $I^*(\text{Id}_M) = \text{Id}_{I^*M}$ and that $I^*(g \circ f) = I^*g \circ I^*f$. For the former, observe that $(I^*(\text{Id}_M))_1 = (\text{Id}_M)_1$, and for $n \geq 2$,

$$(I^*(\text{Id}_M))_n = \sum_{r=1}^{n-1} \sum_{\tau \in S(i_1, \ldots, i_r)} (\text{Id}_M)_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (\tau^* \otimes \text{Id})$$

But $(\text{Id}_M)_r = 0$ for $r > 1$, and so we conclude that $(I^*(\text{Id}_M))_n = 0$ for $n \geq 2$. Hence $I^*(\text{Id}_M) = \text{Id}_{I^*M}$.

In remains to show that $I^*(g \circ f) = I^*g \circ I^*f$. We will follow essentially the same procedure as in Lemma 2, steps 13-17.

Step 1. We start with the right-hand side, and replace $[I^*g \circ I^*f]_n$ with its definition

$$
\sum_{i+j=n+1} \sum_{\sigma(i)=n} (I^g)_j \circ \lambda^* \circ ((I^f)_i \otimes \text{Id}) \circ \sigma^*
$$

Step 2. Replace $I^*g$ and $I^*f$ with their definitions.

$$
\sum_{i+j=n+1} \sum_{\sigma(i)=n} \sum_{r=0}^{i-1} \sum_{\phi \in S'(i_1, \ldots, i_r)} \sum_{s=0}^{j-1} \sum_{\psi \in S'(j_1, \ldots, j_s)} \sum_{i_1 + \cdots + i_r = i} \sum_{j_1 + \cdots + j_s = j} [g_{s+1} \circ (I_{j_1} \otimes \cdots \otimes I_{j_s} \otimes \text{Id}) \circ (\psi^* \otimes \text{Id})]
\circ \lambda^* \circ ([f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id}) \circ (\phi^* \otimes \text{Id})] \otimes \text{Id}) \circ \sigma^*
$$
Note that we include the cases \( r = 0 \) and \( s = 0 \) to include the cases \( f_1 \) and \( g_1 \), respectively. In particular, \( r = 0 \) will contribute a nonzero term only when \( i = 1 \), and \( s = 0 \) will only contribute a nonzero term when \( j = 1 \).

**Step 3.** Commute composition and tensor product to rewrite.

\[
\sum_{i+j=n+1} \sum_{\sigma(i)=n} \sum_{r=0}^{i-1} \sum_{\phi \in S'(i_1, \ldots, i_r)} \sum_{s=0}^{j-1} \sum_{\psi \in S'(j_1, \ldots, j_s)} g_{s+1} \circ \lambda^\bullet \circ (f_{r+1} \otimes \text{Id}) \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes I_{j_1} \otimes \cdots \otimes I_{j_s}) \circ (\phi^\bullet \otimes \text{Id} \otimes \psi^\bullet) \circ \sigma^\bullet
\]

Here, \( \lambda^\bullet \) is the map that permutes the module element into the last input of \( g_{s+1} \).

**Step 4.** By Lemma 4, we obtain

\[
\sum_{\tau \in S(c_1, \ldots, c_n)} \sum_{\theta \in S(t+1-\alpha-\ell)} g_{\alpha+1-\ell} \circ \lambda^\bullet \circ (f_{t+1} \otimes \text{Id}) \circ \theta^\bullet \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\tau^\bullet \otimes \text{Id})
\]

**Step 5.** Change notation; let \( p = t + 1 \) and \( q = \alpha + 1 - t \).

\[
\sum_{\tau \in S(c_1, \ldots, c_n)} \sum_{\theta \in S(p, \alpha + 1 - \ell)} g_{q} \circ \lambda^\bullet \circ (f_{p} \otimes \text{Id}) \circ \theta^\bullet \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\tau^\bullet \otimes \text{Id})
\]

**Step 6.** By the definition of \( g \circ f \), this is

\[
\sum_{\tau \in S(c_1, \ldots, c_n)} (g \circ f)_{\alpha+1} \circ (I_{c_1} \otimes \cdots \otimes I_{c_n} \otimes \text{Id}) \circ (\tau^\bullet \otimes \text{Id})
\]

**Step 7.** By the definition of \( I^\bullet \), this is precisely \( [I^\bullet (g \circ f)]_m \), as desired.

\( \square \)

**Corollary 1.** If \( L \) and \( L' \) are Lie algebras, and \( \phi : L' \to L \) is a Lie algebra homomorphism, \( \phi^\bullet \) is the usual restriction of scalars for Lie algebra representations.

**Proof.** Let \( \rho : L \to \mathfrak{gl}(M) \) be a Lie algebra representation. For \( x \in L' \) and \( m \in M \), the usual restriction of scalars for Lie algebra representations is given by \( x \cdot m := \phi(x) \cdot m \). Indeed, \( \rho' : L' \to \mathfrak{gl}(M) \) defined by \( \rho'(y) = \rho(\phi(y)) \) is a homomorphism of Lie algebras. Now, regarding \( \phi \) as an \( L_\infty \)-algebra map with \( \phi_i = 0 \) for \( i \neq 1 \), because there are also no higher operations on \( M \) as an \( L_\infty \) \( L \)-module, the formulas given in Lemma 4 for the induced operation simplify to give the usual restriction of scalars operation described above. \( \square \)

We now prove the technical lemma that was used in the main results above. In particular, this lemma gives two ways to interpret a particular composition of unshuffles.
Lemma 4. For a fixed \( n \),

\[
\sum_{p=1}^{n} \sum_{\sigma(p)=n} \sum_{1 \leq s \leq n-1} \sum_{\phi \in S'(t_1, \ldots, t_r)} \sum_{\psi \in S'(j_1, \ldots, j_s)} \sum_{r+s=n-p} (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \text{Id} \otimes I_{j_1} \otimes \cdots \otimes I_{j_s}) \circ \phi^* \otimes \text{Id} \otimes \psi^* \circ \sigma^*
\]

is the same as

\[
\sum_{\tau \in S'(c_1, \ldots, c_\alpha)} \sum_{\theta \in S(r+1, \alpha-r)} \sum_{c_1 + \cdots + c_\alpha = n-1} \sum_{0 \leq r \leq \alpha} \theta^* \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \text{Id}) \circ (\tau^* \otimes \text{Id}).
\]

**Figure 4.** The left-hand side represents first unshuffling \( n \) elements into two boxes (with the module element by itself) via \( \sigma^* \) and then unshuffling these boxes further into \( r \) boxes and \( s \) boxes via \( \phi^* \) and \( \psi^* \), respectively. The right-hand side represents first unshuffling \( n-1 \) elements into \( \alpha \) boxes via \( \tau^* \) and then unshuffling these \( \alpha \) boxes via \( \sigma^* \).

**Proof.** To see this, it is helpful to examine what the first sum does for a fixed \( p \) and a fixed \( \alpha \). It unshuffles \( n \) elements into a box of size \( p-1 \) and a box of size \( n-p \), with the module element in between. It then unshuffles the box of size \( p-1 \) further via \( \phi \) into \( r \) smaller boxes and the box of size \( n-p \) further via \( \psi \) into \( s \) smaller boxes.

So, if we iterate through \( \alpha = r+s \), this sum describes all possible ways of unshuffling \( n \) elements into \( r \) boxes (which contain a total of \( p-1 \) elements) and \( s \) boxes (which contain a total of \( n-p \) elements), with the module element in between. Then, iterating through all possible \( p \) tells us that the sum describes all ways of unshuffling \( n \) elements into \( r+s \) boxes, with the module element in between. Note that the \( r \) boxes and the \( s \) boxes have to be of increasing size when considered separately, but they need not be in order when considered all together (e.g. some of the \( s \) boxes could be smaller than the last \( r \) box).

On the other hand, the second sum unshuffles the \( n-1 \) algebra elements into \( \alpha \) boxes first (here, the boxes are all of increasing size), and then it picks out \( r \) of these via an \( r \)-unshuffle \( \theta \) in \( S_\alpha \). Since there was a module element between the \( r \) boxes and \( s \) boxes in the first sum, we can view \( \theta \) as an \( (r+1) \)-unshuffle in \( S_\alpha \) where it puts the module element after the \( r \) boxes. So what we have done is the same as before: unshuffle \( n \) elements into a group of \( r \) boxes, a module element, and a group of \( s = \alpha - r \) boxes, where the boxes are of increasing order when considered separately (but not necessarily when considered all together), see Figure 4. An explicit correspondence between the two sums can be written down using formulas. \( \square \)
4. Appendix

4.1. Composition.

Step 1

Step 2
Step 3

Step 5
RESTRICTION OF SCALARS FOR $L_{\infty}$-MODULES

Step 6

Step 8
4.2. Objects.

\[ \begin{align*}
\sigma^* & \quad \cdots \quad \tau^* \\
\cdots & \quad \cdots \\
I_{i_1} & \quad I_{i_2} \quad \cdots \quad I_{i_r} \\
\kappa_{r+1} & \\
\sigma^* & \quad \cdots \\
\cdots & \quad \cdots \\
I_{i_1} & \quad I_{i_2} \quad \cdots \quad I_{i_r} \\
\kappa_{r+1} & \\
\end{align*} \]

Step 1

\[ \begin{align*}
\sigma^* & \quad \cdots \\
\cdots & \quad \cdots \\
I_{i_1} & \quad I_{i_2} \quad \cdots \quad I_{i_r} \\
\kappa_{r+1} & \\
\sigma^* & \quad \cdots \\
\cdots & \quad \cdots \\
I_{i_1} & \quad I_{i_2} \quad \cdots \quad I_{i_r} \\
\kappa_{r+1} & \\
\end{align*} \]

Step 2

\[ \begin{align*}
\sigma^* & \quad \cdots \\
\cdots & \quad \cdots \\
I_{i_1} & \quad I_{i_2} \quad \cdots \quad I_{i_r} \\
\kappa_{r+1} & \\
\sigma^* & \quad \cdots \\
\cdots & \quad \cdots \\
I_{i_1} & \quad I_{i_2} \quad \cdots \quad I_{i_r} \\
\kappa_{r+1} & \\
\end{align*} \]

Step 4
4.3. Morphisms.

Step 1

\[
\begin{align*}
\sigma^* &\quad + \quad (I^*f)_{j} \\
\tau^* &\quad = \quad (I^*f)_{s} \end{align*}
\]

Step 3

\[
\begin{align*}
\sigma^* &\quad + \quad \tau^* \\
\tau^* &\quad + \quad \tau^* \\
\end{align*}
\]
RESTRICTION OF SCALARS FOR $L_\infty$-MODULES

Step 9

Step 10

Step 11
4.4. Functoriality.
Step 6

### References

1. Christopher Allday. Rational Whitehead products and a spectral sequence of Quillen, II. *Houston J. Math.*, 3(3):301–308, 1977.
2. Michael P. Allocca. Homomorphisms of $L_\infty$ modules. *Journal of Homotopy and Related Structures*, 9(2):285–298, Oct 2014.
3. Francisco Belchí, Urtzi Buijs, José M. Moreno-Fernández, and Aniceto Murillo. Higher order Whitehead products and $L_\infty$ structures on the homology of a DGL. *Linear Algebra and its Applications*, 520:16–31, May 2017.
4. Frits A. Berends, G. J. H. Burgers, and H. van Dam. On the theoretical problems in constructing interactions involving higher spin massless particles. *Nucl. Phys. B*, 260:295–322, 1985.
5. Alexander Berglund. Homological perturbation theory for algebras over operads. *Algebraic & Geometric Topology*, 14(5):2511–2548, Nov 2014.
6. J. Eliesenda Grigsby, Anthony M. Licata, and Stephan M. Wehrli. Annular Khovanov homology and knotted Schur–Weyl representations. *Compositio Mathematica*, 154(3):459–502, Nov 2017.
7. V. K. A. M. Gugenheim, L. A. Lambe, and J. D. Stasheff. Perturbation theory in differential homological algebra. *Illinois J. Math.*, 35(3):357–373, 1991.
8. Johannes Huebschmann. The sh-Lie algebra perturbation lemma. 2007.
9. Johannes Huebschmann and Jim Stasheff. Formal solution of the master equation via HPT and deformation theory. 1999.
10. Andreas Kraft and Jonas Schnitzer. An Introduction to $L_\infty$-Algebras and their Homotopy Theory. *arXiv e-prints*, page arXiv:2207.01861, July 2022.
11. Tom Lada and Martin Markl. Strongly homotopy lie algebras. *Communications in Algebra*, 23(6):2147–2161, 1995.
12. Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2012.
13. Marco Manetti. A relative version of the ordinary perturbation lemma. 2010.
14. José Manuel Moreno-Fernández. The Milnor-Moore theorem for $L_\infty$ algebras in rational homotopy theory. 2019.
15. Vladimir Retakh. Massey operations in Lie superalgebras and differentials in the Quillen spectral sequence. *Colloquium mathematicum.*, 50(1):81–94, 1985.
16. Michael Schlessinger and James Stasheff. The Lie algebra structure of tangent cohomology and deformation theory. *Journal of Pure and Applied Algebra*, 38(2):313–322, 1985.
17. Jim Stasheff. $L$-infinity and $A$-infinity structures. 2018.
18. Edward Witten and Barton Zwiebach. Algebraic structures and differential geometry in two-dimensional string theory. *Nuclear Physics B*, 377(1-2):55–112, Jun 1992.
19. Barton Zwiebach. Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation. *Nuclear Physics B*, 390(1):33–152, Jan 1993.