Centralizer of the elementary subgroup of an isotropic reductive group

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1 Introduction

Let $R$ be a commutative ring with 1, and let $G$ be an isotropic reductive algebraic group over $R$. In [5] Victor Petrov and the second author introduced a notion of an elementary subgroup $E(R)$ of the group of points $G(R)$.

More precisely, assume that $G$ is isotropic in the following strong sense: it possesses a parabolic subgroup that intersects properly any semisimple normal subgroup of $G$. Such a parabolic subgroup $P$ is called strictly proper. Denote by $E_P(R)$ the subgroup of $G(R)$ generated by the $R$-points of the unipotent radicals of $P$ and of an opposite parabolic subgroup $P^-$. The main theorem of [5] states that $E_P(R)$ does not depend on the choice of $P$, as soon as for any maximal ideal $M$ of $R$ all irreducible components of the relative root system of $G_{R_M}$ (see [2, Exp. XXVI, §7] for the definition) are of rank $\geq 2$. Under this assumption, we call $E_P(R)$ the elementary subgroup of $G(R)$ and denote it simply by $E(R)$. In particular, $E(R)$ is normal in $G(R)$. This definition of $E(R)$ generalizes the well-known definition of an elementary subgroup of a Chevalley group (or, more generally, of a split reductive group), as well as several other definitions of an elementary subgroup of isotropic classical groups and simple groups over fields. The group $E(R)$ is also perfect under natural assumptions on $R$ [3]. Here we continue this theme by proving that the centralizer of $E(R)$ in $G(R)$ coincides with the group of $R$-points of the group scheme center $\text{Cent}(G)$ (see [2, Exp. I 2.3] for the definition). Consequendly, both these subgroups also coincide with the abstract group center of $G(R)$. Our result extends the respective theorem of E. Abe and J. Hurly for Chevalley groups [1]; see also [7, Lemma 2] for a slightly more general statement.

**Theorem 1.** Let $G$ be an isotropic reductive algebraic group over a commutative ring $R$ having a strictly proper parabolic subgroup $P$. Assume that for any maximal ideal $M$ of $R$ all irreducible components of the relative root system of $G_{R_M}$ are of rank $\geq 2$. Then $C_{G(R)}(E(R)) = \text{Cent}(G)(R) = C(G(R))$.

Observe that the condition of the theorem ensures that the elementary subgroup $E(R)$ of $G(R)$ is correctly defined. We refer to [3] for its definition and basic properties, as well as for the preliminaries on relative root subschemes.

**Remark.** One may ask if the statement holds for $E_P(R)$ instead of $E(R)$, if we do not assume that the local relative rank is at least 2. This seems to hold always except for several natural exceptions, similar to the exception for $\text{PGL}_2$ described in [1]. We plan to address this case in the near future.
2 Preliminary lemmas

We refer to [2] and [3] for the preliminaries and notation.

We include the following obvious lemma for the sake of completeness.

Lemma 1. Let $X = \text{Spec } A$ be an affine scheme over $Y = \text{Spec } R$, and let $Z$ be a closed subscheme of $X$. Take $g \in X(R)$. Then $g \in Z(R)$ if and only if $g \in Z(R_M)$ for any maximal ideal $M$ of $R$.

Proof. For any $R$-module $V$, the natural map $V \rightarrow \prod V \otimes R_M$, where the product runs over all maximal ideals $M$ of $R$, is injective (e.g. [8], p. 104, Lemma). Since $g \in Z(R)$ is equivalent to an inclusion between the respective ideals of $M$ over all maximal ideals $X$ subscheme of $R$.

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Lemma 2. Let $R$ be any commutative ring, $G$ an isotropic reductive group over $R$, $P$ a strictly proper parabolic subgroup of $G$. Take any maximal ideal $M$ of $R$ and any strictly proper parabolic subgroup $P'$ of $G_{R_M}$ contained in $P_{R_M}$. Then for any $A \in \Phi_P$ there is a system of generators $e_{A_i}$, $1 \leq i \leq n_A$, of the $R_M$-module $V_A$ such that for all $g$ in the image of $\text{Cent}_{G(R)}(E_P(R))$ in $G(R_M)$, one has $[g, X_A(e_{A_i})] = 1$, $1 \leq i \leq n_A$.

Proof. We assume from the very beginning that we have passed to a member of the disjoint union

$$\text{Spec}(R) = \bigcup_{i=1}^{m} \text{Spec}(R_i),$$

so that the parabolic subgroup $P$ is also provided with a relative root system $\Phi_P$ and corresponding relative root subschemes. Since for any $B \in \Phi_P$ elements of $V_B$ generate $V_B \otimes_R R_M$ as an $R_M$-module, the claim of the lemma holds if $P' = P_{R_M}$.

By [5] Lemma 12, for any two strictly proper parabolic subgroups $Q \leq Q'$ of a reductive group scheme, one can find such $k > 0$ depending only on rank $\Phi_Q$, that for any relative root $A \in \Phi_Q$ and any $v \in V_A$ there exist relative roots $B_i, C_{ij} \in \Phi_{Q'}$, elements $v_i \in V_{B_i}$, $u_{ij} \in V_{C_{ij}}$, and integers $k_i, n_i, l_{ij} > 0$ ($1 \leq i \leq m$, $1 \leq j \leq m_j$), which satisfy the equality

$$X_A(\xi^k v) = \prod_{i=1}^{m} X_{B_i}(\xi^{k_i} v_i^{n_i}) \prod_{j=1}^{m_j} X_{C_{ij}}(\eta^{l_{ij}} u_{ij})$$

where $\xi, \eta$ are free variables. Taking $Q = P'$, $Q' = P_{R_M}$, $\xi = 1$, for any element $v_i$ of a generating system of the $R_M$-module $V_A$ we get a decomposition

$$X_A(\eta^k v) = \prod_{i=1}^{m} X_{B_i}(\eta^{n_i} v_i),$$

for some $B_i \in \Phi_P$ and $v_i \in V_{B_i} \otimes R_M$, $n_i > 0$. Clearly, for any $v_i$ there is an element $s_i \in R \setminus M$ such that $s_i v_i$ belongs to $V_{B_i}$ (strictly speaking, to the image of $V_{B_i}$ in $V_{B_i} \otimes R_M$ under the localisation homomorphism; here and below we allow ourselves this freedom of speech). Set $\eta = s_1 \ldots s_m$. Then $X_A(\eta^k v) \in E_P(R)$, and hence $[g, X_A(\eta^k v)] = 1$ for any $g \in \text{Cent}_{G(R)}(E_P(R))$. Thus, multiplying the elements of a generating system of $V_A$ by certain invertible elements of $R_M$, we obtain a new generating system of $V_A$, which is centralised by $\text{Cent}_{G(R)}(E_P(R))$.

Lemma 3. Let $R$ be a local ring (in particular, $R$ can be a field) with the maximal ideal $M$, and let $G$ be a split reductive group over $R$. Let $P$ be a parabolic subgroup of $G$ such that $\text{rank } \Phi_P \geq 2$. Assume that $g \in G(R)$ is such that for any $A \in \Phi_P$ there is a system of generators $e_{A_i}$, $1 \leq i \leq n_A$, of $V_A$ such that $[g, X_A(e_{A_i})] = 1$ for all $i$. Then $g \in U_P(M)L(R)U_{P\pm}(M)$, where $U_{P\pm}(M) = \langle X_A(MV_A), A \in \Phi_P \rangle$. 

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Proof. First let \( R \) be a field. We need to show that \( g \in L(R) \). We can assume that \( R \) is algebraically closed without loss of generality. Let \( B^\pm \) be opposite Borel subgroups of \( G \) contained in \( P^\pm \), \( U^\pm \) be their unipotent radicals, and \( T \) their common maximal torus. Bruhat decomposition implies that \( g = uhvw \), where \( u \in U^+(R) \), \( h \in T(R) \), \( w \) is a representative of the Weyl group, \( v \in U^+_w(R) = \{ x \in U^+(R) \mid w(x) \in U^-(R) \} \), and this decomposition is unique. We have \( w \in L(R) \) if and only if \( w \) is a product of elementary reflections \( w_\alpha \) for some simple roots \( \alpha_i \) belonging to the root system of \( L \).

Assume first that \( w \notin L \). Then there is a simple root \( \alpha \) not belonging to the root system of \( L \) such that \( w(\alpha) < 0 \). Consider \( A = \pi(\alpha) \). Let \( e_A \in V_A \) be a vector from the generating set existing by the hypothesis of the Lemma such that \( x_\alpha(\xi) = 0 \), occurs in the canonical decomposition of \( x = X_A(e_A) \) into a product of elementary root unipotents from \( U^+ \). Since \([g, x]\) = 1, we have \( x(uhv) = (uhvw)x \). The rightmost factor in the Bruhat decomposition of \( x(uhv) = (xu)hwv \) equals \( v \). However, since \( \alpha \) is a positive root of minimal height, it is clear that the rightmost factor in the Bruhat decomposition of \( (uhv)x \) contains \( x_\alpha(\eta + \xi) \) in its canonical decomposition, if \( v \) contains \( x_\alpha(\eta) \). Therefore, this rightmost factor is distinct from \( v \), a contradiction.

Therefore, \( w \in L(R) \). Then for any \( x \in U^+_P(R) \) we have \( w x w^{-1} \in U^+_P(R) \), hence by the definition of the Bruhat decomposition \( v \in L(R) \cap U^+_P(R) \). This means that \( g = uhvw \in U^+_P(R)L(R) = U^+_P(R)(U^+(R) \cap L(R))L(R) = U^+_P(R)L(R) = P(R) \). Since symmetric reasoning implies that \( g \in P^-(R) \), we have \( g \in P(R) \cap P^-(R) = L(R) \).

Now let \( R \) be any local ring. Recall that \( \Omega_p = U_p L U_p \simeq U_p \times L \times U_p \) is a principal open subscheme of \( G \) (e.g. [11] p. 92). Therefore, if the image of \( g \in G(R) \) under the natural homomorphism \( G(R) \to G(R/M) \) is in \( \Omega_p(R/M) \), then \( g \in \Omega_p(R) \). Since by the above the image of \( g \) is in \( L(R/M) \), and \( \ker(U_{P_1}(R) \to U_{P_2}(R/M)) = U_{P_2}(M) \), we have \( g \in U_p(M)L(R)U_{P_2}(M) \).

**Lemma 4.** Let \( G \) be an isotropic reductive group over a local ring \( R \), \( M \) the maximal ideal of \( R \), \( P \) a parabolic subgroup of \( G \), \( P^- \) an opposite parabolic subgroup. For any \( u \in U^-_P(M) \), \( v \in U^+_P(R) \) there exist \( u' \in U^-_P(M) \), \( v' \in U^+_P(R) \), and \( b \in L(R) \) such that \( uv = v'bu' \).

**Proof.** The image of \( x = uv \) under \( p : G(R) \to G(R/M) \) equals \( p(v) \text{and thus belongs to } \Omega_p(R/M) \), where \( \Omega_p = U_p L U_p \). Since \( \Omega_p \) is a principal open subscheme of \( G \), this implies that \( x \in \Omega_p(R) \), that is, \( x = v'bu' \). Since \( p(u') = 1 \), we have \( u' \in U^-_P(M) \).

**Lemma 5.** Let \( G \) be a reductive group over a commutative ring \( R \), \( P \) a parabolic subgroup of \( G \), \( A, B \in \Phi_P \) two non-proportional relative roots such that \( A + B \in \Phi_P \). Assume that \( A - B \notin \Phi_P \), or \( A, B \) belong to the image of a simply laced irreducible component of the absolute root system of \( G \). Take \( 0 \neq u \in V_B \). Any generating system \( e_1, \ldots, e_n \) of the \( R \)-module \( V_A \) contains an element \( e_i \) such that \( N_{AB11}(e_i, u) \neq 0 \).

**Proof.** Assume that \( N_{AB11}(e_i, u) = 0 \) for all \( 1 \leq i \leq n \). Consider an affine fpqc-covering \( \coprod \text{Spec } S_i \to \text{Spec } R \) that splits \( G \). There is a member \( S_\pi = S \) of this covering such that the image of \( X_B(u) \) under \( G(R) \to G(S) \) is non-trivial. Write

\[
X_B(u) = \prod_{\pi(\beta)=B} x_\beta(a_\beta) \cdot \prod_{i \geq 2} \prod_{\pi(\beta_i)=iB} x_\beta(c_{\beta_i}),
\]

where \( \pi : \Phi \to \Phi_P \) is the canonical projection of the absolute root system of \( G \) onto the relative one, \( x_\beta \) are root subgroups of the split group \( G_S \), and \( a_\beta \subseteq S \). Since \( X_B(u) \neq 0 \), the definition of \( X_B \) implies that there exists \( a_\beta \neq 0 \). Let \( \beta_0 \in \pi^{-1}(B) \) be the root of minimal height with this property. By Lemma 4 there exists a root \( \alpha \in \pi^{-1}(A) \) such that \( \alpha + \beta_0 \in \Phi \). Let \( v \in V_A \otimes_R S \) be such that \( X_A(v) = x_\alpha(1) \prod_{i \geq 2} \prod_{\pi(\gamma)=iA} x_\gamma(d) \), for some \( d_\gamma \subseteq S \). Then the (usual) Chevalley commutator formula implies that \([X_A(v), X_B(u)]\)
contains in its decomposition a factor $x_{\alpha+\beta}(\lambda a_{\beta_0})$, where $\lambda \in \{\pm 1, \pm 2, \pm 3\}$. However, since either $\alpha, \beta$ belong to a simply laced root system, we have $\lambda = \pm 1$. Then $N_{AB11}(v, u) \neq 0$, a contradiction.

Recall [5] that any relative root $A \in \Phi_{I,G}$ can be represented as a (unique) linear combination of simple relative roots. The level $\text{lev}(A)$ of a relative root $A$ is the sum of coefficients in this decomposition.

**Lemma 6.** Let $R$ be a local ring with the maximal ideal $M$, and let $G$ be a reductive group over $R$. Let $P$ be a parabolic subgroup of $G$ such that rank $\Phi_P \geq 2$, and the type of $P$ occurs in the type of a minimal parabolic subgroup of some reductive group over a local ring (not necessarily over $R$). Assume that $g \in G(R)$ is such that for any $A \in \Phi_P$ there is a system of generators $e_{A_i}, 1 \leq i \leq n_A$, of $V_A$ such that $|g, X_A(e_{A_i})| = 1$ for all $i$. If $g \in U_P(M)L(R)U_P(M)$, then $g \in L(R)$.

**Proof.** Write $g = xhy$, where $x \in U_P(M), h \in L(R), y \in U_P(M)$. We have $\prod_{A \in \Phi_P^+, X_A(u_A), y} = \prod_{A \in \Phi_P^+, X_A(u_A), y}$, where the product is taken in any fixed order.

Let $A \in \Phi_P$ be such that $u_A \neq 0$, and $|\text{lev}(A)|$ is minimal among the levels of relative roots with this property. We are going to deduce a contradiction, thus showing that $A$ cannot occur in the decomposition of $g$.

Assume that $A \in \Phi_P^+$; the other case is treated symmetrically. Since the type of $P$ coincides with the type of a minimal parabolic subgroup, $\Phi_P^+$ is isomorphic to a root system as a set with two partially defined operations—addition and multiplication by integers. Then the standard properties of a root system imply that one can find a simple root or a minus simple root $B \in \Phi_P^+, \Phi_P$ containing $A$ is not of type $G_2$, we can, and we will, choose $B$ so that $A - B \notin \Phi_P$. If it is of type $G_2$, this may be impossible; then we stipulate that we take $B$ positive. The classification of Tits indices over local rings [6] also implies that in this case the respective irreducible component of the absolute root system of $G$ is either simply laced or itself of type $G_2$. Assume for now that the latter does not take place; we will treat this exceptional case in the very end of this proof. Then by Lemma [5] one can find an element $e$ of a generating system of $V_B$ centralized by $g$ such that $N_{AB11}(u_{AB}, e) \neq 0$.

We have $1 = [X_B(e), g] = [X_B(e), x][X_B(e), hy][x^{-1}]$. This is equivalent to

$$1 = (x^{-1}[X_B(e), x][X_B(e), hy]) = [x^{-1}, X_B(e)][X_B(e), hy].$$

By [5] Th. 2 we can write

$$x^{-1} = X_A(-u_A) \prod_{C \in \Phi_P^+, C \neq A, \text{lev}(C) \geq \text{lev}(A)} X_C(v_C) = X_A(-u_A) \cdot x_1,$$

and thus

$$[x^{-1}, X_B(e)] = [X_A(-u_A)x_1, X_B(e)]$$

$$= [X_A(-u_A), x_1, X_B(e)] \cdot [X_A(-u_A), X_B(e)] = [X_A(-u_A), X_B(e)].$$

**Case 1:** $B$ is positive, that is, $B$ is a simple root. We study the factor $[X_B(e), hy]$ of [1]. Write $[X_B(e), hy] = X_B(e)h(yX_B(e)X_B(e)^{-1}y^{-1})h^{-1}$, and

$$y = \prod_{C \in \Phi_P^+, C \neq B} X_C(v_C) \cdot \prod_{i > 0} X_{-iB}(u_{-iB}) = y_1y_2.$$

Using Lemma [4] we obtain $yX_B(e)^{-1} = y_1(y_2X_B(e)^{-1}) = y_1 \cdot \prod_{i > 0} X_iB(w_{iB}) \cdot b \cdot \prod_{i > 0} X_{-iB}(w_{iB})$, where $b \in L(R)$. Since relative roots proportional to $B$ does not occur in the decompo-
summing up, the only factor of the form $X_A \sqrt{G}$ component of the absolute root system of $A_1$ with $u$ and $x$ of Chevalley commutator formula, implies that $X_B(e)^{-1} \in \left( \prod_{i>0} X_B(w_{iB}) \right) P^{-} (R)$, and also

$$[X_B(e), h y] \in X_B(e) h \left( \prod_{i>0} X_B(w_{iB}) \right) h^{-1} P^{-} (R) = \left( \prod_{i>0} X_B(z_{iB}) \right) P^{-} (R).$$

Now we consider the first factor $[x^{-1}, X_B(e)]$ of the right side of (1). The generalized Chevalley commutator formula, applied to (2), says that

$$[x^{-1}, X_B(e)] = \prod_{D \in \Phi^+_P} X_D(w_D).$$

Moreover, $D = A + B$ is a root of minimal height in the decomposition (2) satisfying $w_D \neq 0$; in fact, $w_{A+B} = N_{AB11}(-u_A, e)$. Hence, the whole product

$$[x^{-1}, X_B(e)], [X_B(e), h y] \in X_{A+B}(N_{AB11}(-u_A, e)) \cdot \left( \prod_{i>0} X_B(z_{iB}) \right) \cdot \prod_{C \in \Phi^+_P, \lev(C) \geq \lev(A+B)} X_C(t_C) \cdot P^{-} (R)$$

does not equal 1, a contradiction.

**Case 2: $B$ is negative, that is $B' = -B$ is a simple root.** In this case the generalized Chevalley commutator formula immediately implies $[X_B(e), h y] \in P^{-} (R)$. We study (2). Note that the decomposition of $x_1$ does not contain $X_B(v_{B'})$, and, if $2B' \in \Phi_P$, also does not contain $X_{2B'}(v_{2B'})$. Indeed, in the first case we would have $\lev(A) = 1$, hence $A$ is a simple relative root, hence $A + B = A - B'$ is not a relative root. In the second case we would have $\lev(A) = 2$, and, since $A + B \in \Phi_P$, $A = A' + B'$ for a simple relative root $A'$. Since in this case we are in the irreducible component of $\Phi_P$ of type $BC_n$, and $B'$ is an extra-short simple root, we also have $A' + 2B' = A - B \in \Phi_P$. But then by our algorithm we would have taken $(-A')$ instead of $B$, since $A - (-A') = 2A' + B' \notin \Phi_P$.

The above, together with the fact that $B' = -B$ is a simple root, and the generalized Chevalley commutator formula, implies that $[x_1, X_B(e)] = \prod_{D \in \Phi^+_P} X_D(w_D)$. Moreover, if $w_D \neq 0$, then $D \neq A+B$, since $A - B$ is not a relative root by our assumptions, and obviously $D$ is not proportional to $B$. Further, we see that for any relative root $D$, occurring in the decomposition of $[X_A(-u_A), [x_1, X_B(e)]]$ or $[X_A(-u_A), X_B(e)]$, the coefficient near any simple root $A_0 \neq B'$ in the decomposition of $D$ is greater or equal to that in the decomposition of $A$. Summing up, the only factor of the form $X_{A-B}(u)$ in the decompositions of the expressions $[X_A(-u_A), [x_1, X_B(e)]]$, $[x_1, X_B(e)]$, $[X_A(-u_A), X_B(e)]$ is the factor $X_{A-B}(N_{AB11}(-u_A, e))$ in the third one, and no commutator of the factors can give a new factor of the form $X_{A-B}(u)$ with $u \neq 0$. Hence, $[x^{-1}, X_B(e)]$ contains $X_{A-B}(N_{AB11}(-u_A, e)) \neq 1$ in its decomposition, and

$$[x^{-1}, X_B(e)][X_B(e), h y] \in X_{A-B}(N_{AB11}(-u_A, e)) \cdot \prod_{F \in \Phi^+_P, F \neq A-B} X_F(t_F) \cdot P^{-} (R)$$

cannot equal 1, a contradiction.

**Case $G_2$.** We are left with the case when $\Phi_P$ is of type $G_2$, and moreover the relevant component of the absolute root system of $G$ is also of type $G_2$. Then we can assume without loss of generality that all components of the absolute root system are of type $G_2$, and consequently $G$ is quasi-split. There exists a canonical étale extension $R'$ of $R$ such
that \( G \) is a Weil restriction of a split group \( G' \) of type \( G_2 \) over \( R' \), see [2, Exp. XXIV Prop. 5.9]. Then \( G_{R'} \) is a direct product of \( k \) split groups \( G_i \) of type \( G_2 \). To show that \( g \in L(R) \), it is enough to show that the image \( g' \) of \( g \) in \( G(R') \) is in \( L(R') \). We know that \( P_{R'} \) is a Borel subgroup of \( G_{R'} \), and, since \( \Phi_P \) has no multiple roots, for any \( A \in \Phi_P \) we can identify the root subscheme \( X_A(V_A \otimes R') \) with the direct product of \( k \) elementary root subgroups \( x_\alpha(\mathfrak{g}) \) of the groups \( G_i \). Considering the relevant projections of \( g \) and the generating systems of \( V_A \), we are reduced to proving the following: if a point \( h \in H(S) \) of a split reductive group \( H \) of type \( G_2 \) centralizes \( x_\alpha(\mathfrak{g}) \) for some \( \alpha \in \Psi \), for any root \( \alpha \in \Psi \), where \( \Psi \) is the root system of \( H \), then \( h \) belongs to the corresponding split maximal torus. By Lemmas [1, 3] we can also assume that the ring \( \mathfrak{g} \) is the root system of \( H \), then \( h \) belongs to the corresponding split maximal torus. By Lemmas [1, 3] we can also assume that the ring \( \mathfrak{g} \) is the root system of \( H \), then \( h \) belongs to the corresponding split maximal torus.

**Lemma 7.** Let \( G \) be an isotropic reductive algebraic group over a commutative ring \( R \), \( P \) a parabolic subgroup of \( G \), \( L \) a Levi subgroup of \( P \). Assume that \( g \in G(R) \) is such that for any \( A \in \Phi_P \) there is a system of generators \( e_{A_i} \), \( 1 \leq i \leq n_A \), of \( V_A \) such that \( [g, X_A(e_{A_i})] = 1 \) for all \( i \). If \( g \in L(R) \), then \( [g, E_P(R)] = 1 \).

**Proof.** We show that \( [g, X_A(V_A)] = 0 \) for any \( A \in \Phi_P \) by descending induction on the height of \( A \); the case \( A \in \Phi_P \) is symmetric. By [3, Th. 2] for any \( g \in L(S) \) and any \( A \in \Phi_P \) there exists a set of homogeneous polynomial maps \( \varphi^i_{g,A} : V_A \rightarrow V_{iA}, i \geq 1 \), such that for any \( v \in V_A \) one has

\[
gX_A(v)g^{-1} = \prod_{i \geq 1} X_{iA}(\varphi^i_{g,A}(v)).
\]

Since \( \varphi^i_{g,A} \) are homogeneous, \( [g, X_A(v)] = 1 \) for \( v \in V_A \) implies \( [g, X_A(\lambda v)] = 1 \) for any \( \lambda \in \mathfrak{g} \). Also by [3, Th. 2], there exist a set of homogeneous polynomial maps \( q^i_A : V_A \times V_A \rightarrow V_{iA}, i \geq 1 \), such that

\[
X_A(v)X_A(w) = X_A(v + w) \prod_{i \geq 1} X_{iA}(q^i_A(v, w))
\]

for all \( v, w \in V_A \). Assume that \( [g, X_A(v)] = [g, X_A(v)] = 1 \). Then

\[
gX_A(v + w)g^{-1} = gX_A(v)X_A(w)g^{-1} \cdot g \left( \prod_{i \geq 1} X_{iA}(q^i_A(v, w)) \right)^{-1} g^{-1} = 1,
\]

since by inductive hypothesis \( g \) centralizes \( X_{iA}(V_{iA}) \) for all \( i > 0 \). \( \square \)

**3 The proof**

**Proof of Theorem [1].** Let \( g \in G(R) \) centralize \( E(R) = E_Q(R) \), where \( Q \) a strictly proper parabolic subgroup of \( G \). We are going to show that \( g \in \text{Cent}(G)(R) \). By Lemma [1] it is enough to show that \( g \in \text{Cent}(G)(R_M) \) for any maximal ideal \( M \) of \( R \). Fix an ideal \( M \), and set \( R' = R_M \). Let \( P \) be a minimal parabolic subgroup of \( G_{R'} \). By Lemma [2] for any \( A \in \Phi_P \) there is a system of generators \( e_{A_i} \), \( 1 \leq i \leq n_A \), of the \( R' \)-module \( V_A \) such that \( [g, X_A(e_{A_i})] = 1 \), \( 1 \leq i \leq n_A \). Note that \( \Phi_P \) is a root system by [2, Exp. XXVI, §7], and by the assumption of the theorem all irreducible components of \( \Phi_P \) are of rank \( \geq 2 \).

Let \( \prod \text{Spec } S_\tau \rightarrow \text{Spec } R' \) be an fpqc-covering such that \( G \) splits over each \( \text{Spec } S_\tau \). It is enough to check that \( g \in \text{Cent}(G)(S_\tau) \) for every \( \tau \) (here we identify \( g \) with its image under \( G(R') \rightarrow G(S_\tau) \)). Fix one \( \tau \), and set \( S = S_\tau \) for short. Again by Lemma [1] it is enough to show that \( g \in \text{Cent}(G)(S_N) \) for any maximal ideal \( N \) of \( S \).

Since a system of generators \( e_{A_i} \), \( 1 \leq i \leq n_A \), of the \( R' \)-module \( V_A \), also generates \( (V_A \otimes_R S) \otimes S_N \) as an \( S_N \)-module, \( g \) satisfies the conditions of Lemmas [3, 4, 5] for the base
ring $S_N$); hence $g \in L(S_N)$, where $L$ is a Levi subgroup of $P$. By Lemma 7 this implies that $g$ centralizes $E(S_N)$. Since $G_{S_N}$ is split, it has a Borel subgroup $B$, and $E(S_N) = E_B(S_N)$.

Applying Lemmas 3 and 6 to $B$ instead of $P$, we get that $g \in T(S_N)$ for a split maximal subtorus $T$ of $G_{S_N}$. Hence $g \in \text{Hom}(\Lambda/\Lambda_r, S_N) \subseteq \text{Hom}(\Lambda, S_N) = T(S_N)$, where $\Lambda$ is the weight lattice of $G$, and $\Lambda_r$ is the root sublattice. Therefore, $g \in \text{Cent}(G)(S_N)$. \hfill \qed

References

[1] E. Abe, J. F. Hurley, Centers of Chevalley groups over commutative rings, Comm. in Algebra 16 (1988), 57–74.

[2] M. Demazure, A. Grothendieck, Schémas en groupes, Lecture Notes in Mathematics, Vol. 151–153, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[3] A. Luzgarev, A. Stavrova, Elementary subgroup of an isotropic reductive group is perfect, http://arxiv.org/abs/1001.1105, to appear in St. Petersburg Mathematical Journal.

[4] H. Matsumoto, Sur les sous-groupes arithmétiques des groupes semi-simples déployés, Ann. Sci. de l’É.N.S. 4e série, tome 2, n. 1 (1969), 1–62.

[5] V. Petrov, A. Stavrova, Elementary subgroups of isotropic reductive groups, St. Petersburg Math. J. 20 (2009), 625–644.

[6] V. Petrov, A. Stavrova, Tits indices over semilocal rings, to appear in Transformation Groups.

[7] A. Stavrova, Normal structure of maximal parabolic subgroups in Chevalley groups over rings, Algebra Colloq. 16 (2009), 631–648.

[8] W. C. Waterhouse, Introduction to affine group schemes, Springer-Verlag, New York, 1979.