Quantum correlation is a key to our understanding of quantum physics. In particular, it is essential for the powerful applications to quantum information and quantum computation. There exist quantum correlations beyond entanglement, such as quantum discord (QD) and measurement-induced nonlocality (MiN) [Phys. Rev. Lett. 106, 120401(2011)]. In [Phys. Rev. A 77, 022113(2008)], a subclass of PPT states so-called strong positive partial transposition (SPPT) states was introduced and it was conjectured there that SPPT states are separable. However, it was illustrated with examples in [Phys. Rev. A 81, 064101(2010)] that this conjecture is not true. Viewing the original SPPT as SPPT up to part B, in the present paper, we define SPPT state up to part A and B respectively and present a separable class of SPPT states, that is the super SPPT (SSPPT) states, in terms of local commutativity. In addition, classical-quantum (CQ) states and nullity of MiN are characterized via local commutativity. Based on CQ states, the geometric measure of quantum discord (GMQD) for infinite-dimensional case is proposed. Consequently, we highlight the relation among MiN, QD (GMQD), SSPPT and separability through a unified approach for both finite- and infinite-dimensional systems: zero MiN implies zero QD (GMQD), zero QD (GMQD) signals SSPPT and SSPPT guarantees separability, but the converses are not valid.

**PACS numbers:** 03.65.Ud, 03.65.Db, 03.67.Mn.

---

**I. INTRODUCTION**

Correlations among subsystems of a composite quantum system, with fundamental applications in many typical of the fields of quantum information and quantum computing [1–15], are usually studied in entanglement-verses-separability framework. However, apart from entanglement, quantum states display other quantum correlations, such as quantum discord (QD) [4, 5] and measurement-induced nonlocality (MiN) [14]. Entanglement lies at the heart of quantum information theory [2, 3]. QD can be used in quantum cryptography, general quantum dense coding [16, 17], remote state control [18, 19], etc. In particular, it is of primary importance to test whether a given quantum state has quantum correlation in it.

Consider the two-mode system labeled by A+B which is described by a complex Hilbert space \( H = H_A \otimes H_B \) with \( \text{dim} H_A \otimes H_B \leq +\infty \). We denote by \( S(H_A \otimes H_B) \) the set of all states acting on \( H_A \otimes H_B \), that is, the set of all positive operators with trace one in \( T(H_A \otimes H_B) \), the space of all trace-class operators. By definition, a quantum state \( \rho \in S(H_A \otimes H_B) \) is separable if it can be written as

\[
\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \quad \sum_i p_i = 1, \quad p_i \geq 0
\]

or can be approximated in the trace norm by the states of the above form [21, 23]. Otherwise, \( \rho \) is called entangled. In particular, a separable state with the form as in Eq. (1) is called countably separable [22, 23]. If \( \text{dim} H_A \otimes H_B < +\infty \), then all separable states \( \rho \) acting on \( H_A \otimes H_B \) are countably separable [21]. But, in the infinite-dimensional case, there exists separable states which are not countably separable [22].

One of the most famous criteria for detecting entanglement is the so-called positive partial transpose (PPT) criterion proposed by Peres and Horodecki [24, 25]: if a quantum state \( \rho \) acting on the Hilbert space \( H_A \otimes H_B \) is separable, then the partial transposes of \( \rho \) are positive operators, that is, \( \rho^{T_A/B} \geq 0 \). There exist entangled PPT states (i.e., states with positive partial transpose) except for 2 \( \otimes 2 \) and 2 \( \otimes 3 \) systems [25]. Consequently, it is important to know which PPT states are separable and which are entangled (PPT entangled states are known as bound states which are not distillable). In [26], a subclass of PPT states, called strong positive partial transpose (SPPT) states, is considered according to the Cholesky decomposition of positive semidefinite matrix. These states have “strong PPT property” which is insured by the canonical construction (see below). Based on several examples of SPPT states, it is conjectured in [26] that all SPPT states are separable. However, it is not true since there exist entangled states which are SPPT [27]. SPPT can be used for witnessing quantum discord in 2 \( \otimes n \) systems because a state \( \rho \) in finite-dimensional bipartite system associated with \( H_A \otimes H_B \) is CQ if and only if it has zero QD with respect to part A [4]. In this case,
letter, we propose a special class of SPPT states which we call super SPPT (SSPPT) states. It turns out every SSPPT state is countably separable and any CQ (resp. QC) state is SSPPT up to part B (resp. A). Furthermore, we find that SSPPT up to part A/B can detect QD with respect to part A/B, and there exist zero QD states with nonzero MiN.

The quantum discord can be viewed as a measure of the minimal loss of correlation in the sense of quantum mutual information. Recall that the quantum discord of a state $\rho$ on finite-dimensional Hilbert space $H_A \otimes H_B$ is defined in [4] by

$$D_A(\rho) := \min_{\Pi^A} \{ I(\rho) - I(\rho|\Pi^A) \},$$

where, the minimum is taken over all local von Neumann measurements $\Pi^A$,

$$I(\rho) := S(\rho_A) + S(\rho_B) - S(\rho)$$

is interpreted as the quantum mutual information,

$$S(\rho) := -\text{Tr}(\rho \log \rho)$$

is the von Neumann entropy,

$$I(\rho|\Pi^A) := S(\rho_B) - S(\rho|\Pi^A),$$

$$S(\rho|\Pi^A) := \sum_k p_k S(\rho_k),$$

and

$$\rho_k = \frac{1}{p_k}(\Pi^A_k \otimes I_B)\rho(\Pi^A_k \otimes I_B)$$

with $p_k = \text{Tr}[(\Pi^A_k \otimes I_B)\rho(\Pi^A_k \otimes I_B)]$, $k = 1, 2, \ldots, \text{dim} H_A$. Throughout this paper, all logarithms are taken to base 2. QD of any state is nonnegative [4, 29]. Also recall that a state $\rho$ on $H_A \otimes H_B$ is said to be a CQ state if it has the form of

$$\rho = \sum_i p_i |i\rangle \langle i| \otimes \rho_i^B,$$

for some orthonormal set $\{|i\rangle\}$ of $H_A$, where $\rho_i^B$s are states of the subsystem $B$, $p_i \geq 0$, $\sum_i p_i = 1$. It is known that a state has zero QD if and only if it is a CQ state. The conditions for nullity of quantum discord may be found in [6, 28, 29].

Measurement-induced nonlocality was firstly proposed by Luo and Fu [14], which can be viewed as a kind of quantum correlation from a geometric perspective based on the local von Neumann measurements from which one of the reduced states is left invariant. The MiN of $\rho$, denoted by $N_A(\rho)$, is defined in [14] by

$$N_A(\rho) := \max_{\Pi^A} \| \rho - \Pi^A(\rho) \|_2^2,$$

where $\| \cdot \|_2$ stands for the Hilbert-Schmidt norm (that is $\| A \|_2 = (\text{Tr}(A^2A))^{1/2}$), and the maximum is taken over all local von Neumann measurement $\Pi^A = (\Pi^A_k)$ with

$$\sum_k \Pi^A_k(\rho \Pi^A) = \rho_A,$$

$$\Pi^A(\rho) = \sum_k (\Pi^A_k \otimes I_B)\rho(\Pi^A_k \otimes I_B).$$

MiN is different from, and in some sense dual to, the geometric measure of quantum discord (GMQD) [30]

$$D_G(\rho) := \min_{\Pi^A} \| \rho - \Pi^A(\rho) \|_2^2$$

where $\Pi^A$ runs over all local von Neumann measurements.

Mathematically, quantumness is always associated with non-commutativity while classical mechanics displays commutativity in some sense [31, 32]. With this idea in mind, it is possible to describe these quantum correlations mentioned above in terms of non-commutativity. The aim of this paper is to find a unified mathematical language in these quantum correlations analysis, and hence shine new light on the structure of quantum correlation, from which we can get more understanding of these different kinds of quantum correlations.

The remainder of this paper is organized as follows. In Sec.II, we define SSPPT states for both finite- and infinite-dimensional systems and prove that these states are separable. Then, in Sec.III, CQ states are characterized, from which we show that CQ is equivalent to zero GMQD and then SSPPT can detect QD, GMQD. Sec.IV distributes to give a necessary and sufficient condition for a state to have zero MiN. A summary is given in the last section.

II. SUPER STRONG POSITIVE PARTIAL TRANSPOSE STATES

In this section we first give definitions of SSPPT states for both finite- and infinite-dimensional systems. Then we show that all SSPPT states are separable.

A. Definitions

Finite-dimensional case In a $m \otimes n$ system with $mn < +\infty$, any state $\rho$ may be viewed as a block $m \times m$ matrix with $n \times n$ blocks. Due to the Cholesky decomposition, there exists block upper triangular matrix $X$ ($m \times m$ block matrix with $n \times n$ blocks),

$$X = \begin{pmatrix}
X_{11} & S_{12}X_1 & S_{13}X_1 & \cdots & S_{1m}X_1 \\
0 & X_2 & S_{23}X_2 & \cdots & S_{2m}X_2 \\
0 & 0 & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & S_{m-1m}X_{m-1} & S_{mm} \\
0 & 0 & 0 & \cdots & X_m
\end{pmatrix}$$

where $X_{ij} = 0$ for $i > j$. From this representation, the super-strongly positive partial transpose (SSPPT) states are defined as those states $\rho = \sum_i p_i | i\rangle \langle i| \otimes \rho_i$ with

$$\sum_i p_i | i\rangle \langle i| \otimes \rho_i$$

are separable.

In Sec.II, we define SSPPT states for both finite- and infinite-dimensional systems and prove that these states are separable. Then, in Sec.III, CQ states are characterized, from which we show that CQ is equivalent to zero GMQD and then SSPPT can detect QD, GMQD. Sec.IV distributes to give a necessary and sufficient condition for a state to have zero MiN. A summary is given in the last section.
such that \( \rho = X^\dagger X \) (the choice of \( X \) is not unique). If \( \rho^T = Y^\dagger Y \) with

\[
Y = \begin{pmatrix}
X_1 & S^\dagger_{12} X_1 & S^\dagger_{13} X_1 & \cdots & S^\dagger_{1m} X_1 \\
0 & X_2 & S^\dagger_{23} X_2 & \cdots & S^\dagger_{2m} X_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & S^\dagger_{m-1, m} X_{m-1} \\
0 & 0 & 0 & 0 & X_m
\end{pmatrix},
\]

\( \rho \) is called a SPPT state \( [27] \). Obviously, if \( S_{ij} \)'s satisfy the condition

\[
[S_{ki}, S^\dagger_{kj}] = 0, \quad k < i \leq j,
\]

then \( \rho \) must be SPPT \( [26] \). In such a case, we say that \( \rho \) is a super SPPT (or SSPPT briefly) state.

Let \( \{ |i \rangle \} \) and \( \{ |j \rangle \} \) be the canonical computational bases of \( \mathbb{C}^m \) and \( \mathbb{C}^n \), respectively. Then \( \rho \) can be presented as

\[
\rho = \sum_{i,j} A_{ij} \otimes F_{ij} = \sum_{k,l} E_{kl} \otimes B_{kl},
\]

where \( F_{ij} = |i \rangle \langle j | \) and \( E_{kl} = |k \rangle \langle l | \). That is, a state \( \rho \) acting on \( H_A \otimes H_B \) can be represented as

\[
\rho = [B_{kl}] \text{ or } \rho = [A_{ij}].
\]

Symmetrically, we can define SPPT and SSPPT states up to part A. Namely, if there exist

\[
\tilde{X} = \begin{pmatrix}
\tilde{X}_1 & \tilde{S}^\dagger_{12} \tilde{X}_1 & \tilde{S}^\dagger_{13} \tilde{X}_1 & \cdots & \tilde{S}^\dagger_{1m} \tilde{X}_1 \\
0 & \tilde{X}_2 & \tilde{S}^\dagger_{23} \tilde{X}_2 & \cdots & \tilde{S}^\dagger_{2m} \tilde{X}_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & \tilde{S}^\dagger_{m-1, m} \tilde{X}_{m-1} \\
0 & 0 & 0 & 0 & \tilde{X}_m
\end{pmatrix}
\]

and

\[
\tilde{Y} = \begin{pmatrix}
\tilde{X}_1 & \tilde{S}^\dagger_{12} \tilde{X}_1 & \tilde{S}^\dagger_{13} \tilde{X}_1 & \cdots & \tilde{S}^\dagger_{1m} \tilde{X}_1 \\
0 & \tilde{X}_2 & \tilde{S}^\dagger_{23} \tilde{X}_2 & \cdots & \tilde{S}^\dagger_{2m} \tilde{X}_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & \tilde{S}^\dagger_{m-1, m} \tilde{X}_{m-1} \\
0 & 0 & 0 & 0 & \tilde{X}_m
\end{pmatrix}
\]

such that

\[
\rho = [A_{ij}] = \tilde{X}^\dagger \tilde{X}
\]

and

\[
\rho^T = \tilde{Y}^\dagger \tilde{Y},
\]

then we call \( \rho \) is SPPT up to part A. In particular, if \( \tilde{S}_{ij} \)'s satisfy

\[
[S_{ki}, \tilde{S}^\dagger_{kj}] = 0, \quad k < i \leq j,
\]

we call that \( \rho \) is SSPPT up to part A.

\textit{Infinite-dimensional case} Note that the Cholesky factorization can be generalized to (not necessarily finite) matrices with operator entries, so we can define SPPT and SSPPT states for infinite-dimensional bipartite systems.

Assume that \( \text{dim} \, H_A \otimes H_B = +\infty \), \( \text{dim} \, H_A = +\infty \), \( \{ |i \rangle \} \) and \( \{ |j \rangle \} \) be any orthonormal bases of \( H_A \) and \( H_B \), respectively. Let \( E_{kl} = |k \rangle \langle l | \). Consequently, any state \( \rho \) acting on \( H_A \otimes H_B \) can be represented by

\[
\rho = \sum_{k,l} E_{kl} \otimes B_{kl},
\]

where \( B_{kl} \) are trace class operators on \( H_B \) and the series converges in trace norm \( [34] \).

That is,

\[
\rho = \begin{pmatrix}
B_{11} & B_{12} & B_{13} & \cdots & \cdots \\
B_{21} & B_{22} & B_{23} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
B_{n1} & B_{n2} & B_{n3} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \cdots & \ddots
\end{pmatrix}
\]

under the given bases. With respect to these bases, if there exist some upper triangular (infinite) operator matrices of the form

\[
X = \begin{pmatrix}
X_1 & S^\dagger_{12} X_1 & S^\dagger_{13} X_1 & \cdots & S^\dagger_{1m} X_1 \\
0 & X_2 & S^\dagger_{23} X_2 & \cdots & S^\dagger_{2m} X_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & S^\dagger_{m-1, m} X_{m-1} \\
0 & 0 & 0 & 0 & X_m
\end{pmatrix}
\]

and

\[
Y = \begin{pmatrix}
\tilde{X}_1 & \tilde{S}^\dagger_{12} \tilde{X}_1 & \tilde{S}^\dagger_{13} \tilde{X}_1 & \cdots & \tilde{S}^\dagger_{1m} \tilde{X}_1 \\
0 & \tilde{X}_2 & \tilde{S}^\dagger_{23} \tilde{X}_2 & \cdots & \tilde{S}^\dagger_{2m} \tilde{X}_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & \tilde{S}^\dagger_{m-1, m} \tilde{X}_{m-1} \\
0 & 0 & 0 & 0 & \tilde{X}_m
\end{pmatrix}
\]

with the same size as that of \( \rho = (B_{kl}) \) in Eq.\( [9] \) such that

\[
\rho = X^\dagger X
\]

and

\[
\rho^T = Y^\dagger Y,
\]

then we call that \( \rho \) is a SPPT state up to part B. Moreover, if \( S_{ij} \)'s are diagonalizable normal operators and satisfy

\[
[S_{ki}, S^\dagger_{kj}] = 0, \quad k < i \leq j,
\]

we call that \( \rho \) is a SSPPT state up to part B. Note that, \( X \) is a Hilbert-Schmidt operator, and in the case that \( \rho \) is SSPPT, \( S_{ij} \) is normal for any \( k < i \leq j \).
Symmetrically, any state $\rho$ acting on $H_A \otimes H_B$ can be represented by

$$\rho = \sum_{i,j} A_{ij} \otimes F_{ij}, \quad (11)$$

where $F_{ij} = \langle i' \rangle \langle j' \rangle$, $A_{ij} \Delta$s are trace class operators on $H_A$ and the series converges in trace norm $\|A\|_\rho$. Analogy to the finite-dimensional case, we can define SSPPT state up to part A when regarding $\rho$ as $\rho = [A_{ij}]$. Furthermore, it is worth mentioning that SSPPT up to A is not equivalent to that up to B.

**Remark.** It is easily checked that the SPPT and the SSPPT are invariant under the local unitary operation. So, the definitions of SPPT and SSPPT are independent to the choice of local bases $\{|i\rangle\}$ and $\{|i'\rangle\}$ of $H_A$ and $H_B$ respectively. Namely, if $\rho$ is SPPT (or SSPPT) with respect to the given local bases $\{|i\rangle\}$ and $\{|i'\rangle\}$ of $H_A$ and $H_B$ respectively, then it is also SPPT (or SSPPT) with respect to the other choice of local bases.

### B. SSPPT states are separable

The main result of this section is the following.

**Theorem 1.** Let $\rho \in S(H_A \otimes H_B)$ with $\dim H_A \otimes H_B \leq +\infty$ be a SSPPT state up to part A or B. Then $\rho$ is countably separable.

**Proof.** We only need to check the case of SSPPT up to part B since the proof for the case of SSPPT up to A is similar. We consider the infinite-dimensional case, the finite-dimensional is then obvious.

Let $\rho$ be a SSPPT state up to part B. Then $\rho = X^\dagger X$ and $\rho^{T_B} = Y^\dagger Y$, where $X$ and $Y$ are upper triangular operator matrices of the form mentioned above. Let $C_k$ be the infinite operator matrix with the same size as that of $X$, which is induced from $X$ by replacing all entries by zero except for the $k$th row of $X$, i.e.,

$$\begin{pmatrix}
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & X_k & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}, \quad k = 1, 2, \ldots.$$ 

Then $C_i$ is a Hilbert-Schmidt operator and

$$\rho = \sum_i C_i^\dagger C_i, \quad C_i^\dagger C_i \geq 0. \quad (12)$$

Write $C_i^\dagger C_i = p_i \rho_i$ where $p_i = \text{Tr}(C_i^\dagger C_i)$. We have

$$p_1 \rho_1 = \begin{pmatrix}
X_1^\dagger X_1 & X_1^\dagger S_{12} X_1 & X_1^\dagger S_{13} X_1 & \cdots & X_1^\dagger S_{1m} X_1 \\
X_1^\dagger S_{12}^\dagger X_1 & X_1^\dagger S_{12} S_{12}^\dagger X_1 & X_1^\dagger S_{12} S_{13}^\dagger X_1 & \cdots & X_1^\dagger S_{12} S_{1m}^\dagger X_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
X_1^\dagger S_{1m-1}^\dagger X_1 & X_1^\dagger S_{1m-1} S_{12}^\dagger X_1 & X_1^\dagger S_{1m-1} S_{13}^\dagger X_1 & \cdots & X_1^\dagger S_{1m-1} S_{1m}^\dagger X_1 \\
X_1^\dagger S_{1m}^\dagger X_1 & X_1^\dagger S_{1m} S_{12}^\dagger X_1 & X_1^\dagger S_{1m} S_{13}^\dagger X_1 & \cdots & X_1^\dagger S_{1m} S_{1m}^\dagger X_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}. \quad (13)$$

We claim that $\rho_1$ is a countably separable state. Note that $X_1$ is a Hilbert-Schmidt operator and $S_{ij}$s are mutually commuting diagonalizable normal operators on $H_B$. Thus

$$X_1^\dagger X_1 = \sum_i a_i |\psi_i\rangle \langle \psi_i|$$

and

$$S_{1l} = \sum_j \beta^{(l)}_j |\phi_j\rangle \langle \phi_j|, \quad l = 2, 3, \ldots$$

for some orthonormal bases $\{|\psi_i\rangle\}$ and $\{|\phi_j\rangle\}$ of $H_B$. Denote

$$\beta_{ij} = \langle \psi_i|\phi_j\rangle.$$
\[
A_i = a_i \begin{pmatrix}
1 & \sum_j |b^{(2)}_j|\beta_{ij}|^2 & \sum_j |b^{(3)}_j|\beta_{ij}|^2 & \ldots & \sum_j |b^{(m)}_j|\beta_{ij}|^2 \\
\sum_j |b^{(2)}_j|\beta_{ij}|^2 & \sum_j |b^{(2)}_j|\beta_{ij}|^2 & \ldots & \sum_j |b^{(2)}_j|\beta_{ij}|^2 \\
\vdots & \vdots & \ddots & \vdots \\
\sum_j |b^{(m-1)}_j|\beta_{ij}|^2 & \sum_j |b^{(m-1)}_j|\beta_{ij}|^2 & \ldots & \sum_j |b^{(m-1)}_j|\beta_{ij}|^2 \\
\sum_j |b^{(m)}_j|\beta_{ij}|^2 & \sum_j |b^{(m)}_j|\beta_{ij}|^2 & \ldots & \sum_j |b^{(m)}_j|\beta_{ij}|^2 \\
\vdots & \vdots & \ddots & \vdots \\
\sum_j |b^{(n)}_j|\beta_{ij}|^2 & \sum_j |b^{(n)}_j|\beta_{ij}|^2 & \ldots & \sum_j |b^{(n)}_j|\beta_{ij}|^2 \\
\end{pmatrix}
\]  
(14)

and

\[B_i = |\psi_i\rangle\langle\psi_i|.
\]

Then we have

\[p_1 p_1 = \sum_i A_i \otimes B_i.
\]  
(15)

So \(A_i \geq 0\) is a trace-class operator for each \(i\) and \(\sum_i \text{Tr}(A_i) = p_1 \leq 1\). Now it is clear that \(p_1\) is countably separable.

Similarly, \(p_i\) is countably separable for each \(i, i \geq 1\). Hence, \(\rho\) is a countably separable state.

In some sense, diagonalizability can be regarded as a kind of commutativity since \(A\) is normal implies \([A, A^\dagger] = 0\). Thus Theorem 1 indicates that diagonalizability of \(S_{ij}\) and commutativity between \(S_{ij}\) guarantee the separability of \(\rho\).

Obviously, SSPPT is not a necessary condition of separability. In fact, for SSPPT states \(\rho_1\) and \(\rho_2\), their convex combination \(t \rho_1 + (1-t) \rho_2\) may not be a SSPPT state in general. That is, the set consisting of SSPPT states is not convex, while both the set of PPT states and the set of separable states are convex. However, for pure states, SPPT, SSPPT and separability are equivalent since a pure state is separable if and only if it is PPT [35]. A little more can be said. In fact we have the following conclusion.

**Proposition 1.** Every product state is SSPPT, and a pure state is separable if and only if it is PPT (or SSPPT).

By Theorem 1, we obtain some simple separability criteria for states in 2 \(\otimes n\) (resp. \(n \otimes 2\)) systems.

**Corollary 1.** Assume that \(\dim H_A \otimes H_B < +\infty\) and \(\dim H_A = 2\) (or \(\dim H_B = 2\)), \(\rho \in S(H_A \otimes H_B)\). Write

\[
\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{pmatrix}
\]

as in Eq. (7). Then the following statements are true:

(i) If \(\rho\) is a SPPT state up to part B (or, up to part A) and \(\rho_{11}\) (or \(\rho_{22}\)) is invertible, then \(\rho\) is separable.

(ii) If \(\rho_{11} \geq \rho_{22}\) or \(\rho_{22} \geq \rho_{11}\) (or, \(\rho_{11} \geq \rho_{22}\) or \(\rho_{22} \geq \rho_{11}\)), then \(\rho\) is separable.

**Proof.** We only give a proof of (ii) here. No loss of generality, we assume that \(\dim H_A = 2\), \(\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} \\
\rho_{21} & \rho_{22}
\end{pmatrix}
\) with \(\rho_{ij} \in T(H_B)\) and \(\rho_{22} \leq \rho_{11}\). We shall show that \(\rho\) is SSPPT and hence is separable by Theorem 1.

Since \(\rho \geq 0\) and \(\rho_{22} \leq \rho_{11}\), there are contractive operators \(T, S \in B(H_B)\) with \(\ker T \cap \ker S \subseteq \ker \rho_{11}\) such that \(\rho_{12} = \sqrt{\rho_{11} T} \sqrt{\rho_{22}}\) and \(\sqrt{\rho_{22}} = \sqrt{\rho_{11}} S = S^\dagger \sqrt{\rho_{11}}\). Let \(\rho_{12} = TS^\dagger\). Then we have

\[\sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}} = \sqrt{\rho_{22}} T \sqrt{\rho_{22}} \leq \rho_{22}.
\]

Let

\[X_2 = [\rho_{22} - \sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}}]^{\dagger},
\]

and

\[Y = \begin{pmatrix}
\sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}} \\
0
\end{pmatrix}.
\]

Then

\[
\rho = X^\dagger X
\]

\[= \begin{pmatrix}
\rho_{11} & \sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}} \\
\sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}} & \rho_{11} + X_2^\dagger X_2
\end{pmatrix}
\]

and

\[
\rho^{TA} = Y^\dagger Y
\]

\[= \begin{pmatrix}
\rho_{11} & \sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}} \\
\sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}} & \rho_{11} + X_2^\dagger X_2
\end{pmatrix}.
\]

Thus \(\rho\) is SPPT. Since \([\rho^{TA}]_{TA} = \rho\), we get

\[\sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}} = \sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}}.
\]

This entails that \(S_{12} S_{12} \sqrt{\rho_{11}} = S_{12} \sqrt{\rho_{11}}\), that is, \(\rho\) is SSPPT. 

**Example 1.** Let \(\rho_{11}, D, T \in M_n(C)\) with \(\rho_{11} \geq 0\), \(|D| \leq 1\) and \(|T| \leq 1\). Then the state \(\rho\) of the form
\[
\rho = \frac{1}{\text{Tr}(\rho_{11} + \sqrt{\rho_{11}D^\dagger D} \sqrt{\rho_{11}})}\left(\sqrt{\rho_{11}D^\dagger D \sqrt{\rho_{11}}} \sqrt{\rho_{11}I^\dagger I \sqrt{\rho_{11}}} \right)^{\frac{1}{2}}
\]

in \(M_2 \otimes M_n\) is separable.

III. GEOMETRIC MEASURE OF QUANTUM DISCORD

The previous section showed that SSPPT is a sufficient but not necessary condition of separability. In this section, in some sense dual to it, we will prove that, SSPPT is a necessary but not sufficient condition of zero GMQD according to the very structure of CQ states.

A. Zero geometric measure of quantum discord states

In order to propose a unified work for both finite- and infinite-dimensional cases, we first generalize QD and CQ to infinite-dimensional systems. The quantum discord for infinite-dimensional systems was firstly introduced and discussed in \([29]\). For readers’ convenience, we restate this concept here.

Quantum discord Let \(\rho \in S(H_A \otimes H_B)\) with \(\dim H_A \otimes H_B = +\infty\). Denote by

\[
I(\rho) := S(\rho_A) + S(\rho_B) - S(\rho)
\]

the quantum mutual information of \(\rho\) whenever \(S(\rho) < +\infty\), where

\[
S(\rho) := -\text{Tr}(\rho \log \rho)
\]

is the von Neumann entropy of the state \(\rho\) (remark here that \(S(\rho)\) may be \(+\infty\)). Let \(\Pi^A = \{\Pi_k^A = |k\rangle\langle k|\}\) be a local von Neumann measurement. Performing \(\Pi^A\) on \(\rho\), the outcome

\[
\Pi^A(\rho) = \sum_k p_k \rho_k,
\]

where

\[
\rho_k = \frac{1}{p_k} (\Pi_k^A \otimes I_B) \rho (\Pi_k^A \otimes I_B)
\]

with probability

\[
p_k = \text{Tr}[ (\Pi_k^A \otimes I_B) \rho (\Pi_k^A \otimes I_B) ].
\]

Define

\[
I(\rho|\Pi^A) := S(\rho_B) - S(\rho|\Pi^A)
\]

and

\[
S(\rho|\Pi^A) := \sum_k p_k S(\rho_k).
\]

If \(I(\rho) < +\infty\), the difference

\[
D_A(\rho) := I(\rho) - \sup_{\Pi^A} I(\rho|\Pi^A)
\]

is defined to be the quantum discord of \(\rho\), where the supremum is taken over all local von Neumann measurement.

\[
D_A(\rho) \geq 0\]

holds for any state \(\rho \in S(H_A \otimes H_B)\) with \(I(\rho) < +\infty\) since the von Neumann entropy is strongly subadditive for both finite- and infinite-dimensional cases (see [29] for detail). One can check that QD can also be calculated as

\[
D_A(\rho) = I(\rho) - \sup_{\Pi^A} I(\Pi^A(\rho)).
\]

Namely, QD is defined as the infimum of the difference of mutual information of the pre-state \(\rho\) and that of the post-state \(\Pi^A(\rho)\) with \(\Pi^A\) runs over all local von Neumann measurements.

Symmetrically, one can define quantum discord \(D_B\) with respect to part B, and the counterpart results are also valid. Note that \(D_A\) and \(D_B\) are asymmetric, i.e.,

\[
D_A(\rho) \neq D_B(\rho)
\]

in general.

For finite-dimensional systems, the classical-quantum (CQ) states attracted much attention since they can be used for quantum broadcasting \([36]\). It was point out in \([4]\) that a state is CQ if and only if it has zero quantum discord. Now we extend the concept of the CQ states to infinite-dimensional case via the same scenario.

Classical-quantum state Similar to Eq.(3), for \(\rho \in S(H_A \otimes H_B)\), \(\dim H_A \otimes H_B = +\infty\), if \(\rho\) admits a representation of the following form

\[
\rho = \sum_k p_k |k\rangle\langle k| \otimes \rho_k^B,
\]

where \(\{|k\rangle\}\) is an orthonormal set of \(H_A\), \(\rho_k^B\)s are states of the subsystem B, \(p_k \geq 0\) and \(\sum_k p_k = 1\), then we call \(\rho\) a classical-quantum (CQ) state.

However for infinite-dimensional case we do not know if every CQ state has zero QD up to part A in general because for some CQ state \(\rho\) we may have \(I(\rho) = +\infty\). So the concept of quantum discord is not very suitable to the states in infinite-dimensional systems. To get a more proper concept that can replace the concept of quantum discord, we generalize the concept of the geometric measure of quantum discord \([3]\) to infinite-dimensional case.

Like to the finite-dimensional case, we define the geometric measure of quantum discord up to part A of a state by

\[
D_A^G(\rho) = \inf\{\|\rho - \pi\|_2^2 : \pi \in CQ\},
\]

where \(\{\pi\}\) is a set of states in the CQ class. However it is not clear whether the infimum is attained at some CQ state in the infinite-dimensional case. In order to overcome this difficulty, we introduce a new concept that can provide a lower bound of the geometric measure of quantum discord

\[
D_A^G(\rho) = \inf\{\|\rho - \pi\|_2^2 : \pi \in CQ, \text{\} entropic distance}\}
\]
where CQ is the set of all CQ states on $H_A \otimes H_B$. That is, the geometric Hilbert-Schmidt quantum discord of a state $\rho$ is the square of the Hilbert-Schmidt distance of the state to the set of all CQ states. $D^G_A(\rho)$ makes sense for any state $\rho$ because states are Hilbert-Schmidt operators.

It is known that, for a state $\rho$ in finite-dimensional system, $D^G_A(\rho) = 0$ if and only if $D_A(\rho) = 0$, and in turn, if and only if $\rho$ is CQ.

In the sequel we show that $D^G_A(\rho) = 0$ if and only if $\rho$ is a CQ state, and thus $D^G_A(\rho)$ is a suitable quantity replacing $D_A(\rho)$. Before doing this let us firstly give a structural feature of CQ states.

Write $\rho = \sum_{i,j} A_{ij} \otimes F_{ij}$ as in Eq. (11). For the case of $\dim H_A \otimes H_B < +\infty$, it is proved in [37] that, if $A_{ij}$s are mutually commuting normal matrices, $\rho$ is separable. We prove below that such a state $\rho$ is not only separable but also a CQ state. In fact, we give a characterization of CQ states in terms of commutativity by showing that $\rho$ is a CQ state if and only if $A_{ij}$s are mutually commuting normal matrices. Moreover, this result is valid for infinite-dimensional cases, too.

**Theorem 2.** Let $\rho \in \mathcal{S}(H_A \otimes H_B)$ with $\dim H_A \otimes H_B \leq +\infty$. Write $\rho = \sum_{i,j} A_{ij} \otimes F_{ij}$ with respect to some given bases of $H_A$ and $H_B$. Then $\rho$ is a CQ state if and only if $A_{ij}$s are mutually commuting normal operators acting on $H_A$.

**Proof.** The ‘if’ part. Assume that $A_{ij}$s are mutually commuting normal operators, then $A_{ij}$s are simultaneously diagonalizable since they are trace-class operators. Thus there exist diagonal operators $D_{ij}$s and a unitary operator $U$ acting on $H_A$ such that

$$(U^\dagger \otimes I_B)\rho(U \otimes I_B) = \sum_{i,j} D_{ij} \otimes F_{ij}.$$ 

With no loss of generality, we may assume

$$\rho = \sum_{i,j} D_{ij} \otimes F_{ij}.$$ 

It turns out that $\rho$ can then be rewritten as

$$\rho = \sum_i \tilde{E}_{ii} \otimes B_{ii},$$

where $\tilde{E}_{ii}$s are orthogonal rank-one projections. Now it is obvious that $B_{ii} \geq 0$ since $\rho \geq 0$, $i = 1, 2, \ldots$. Hence, $\rho$ is a classical-quantum state.

The ‘only if’ part. If $\rho$ is a CQ state, then

$$\rho = \sum_{k} p_k |k\rangle \langle k| \otimes \rho_k^B,$$

$p_k \geq 0$, $\sum_k p_k = 1$ for some orthonormal set $\{|k\rangle\}$ of $H_A$. Extend $\{|k\rangle\}$ to an orthonormal basis of $H_A$ and still denoted by $\{|k\rangle\}$. If $\Pi^A$ is a von Neumann measurement induced from $\{|k\rangle \langle k|\}$, then it follows from $\Pi^A(\rho)$

$$\sum_{k} |k\rangle \langle k| \otimes I_B \sum_{k} p_k |k\rangle \langle k| \otimes \rho_k^B |k\rangle \langle k| \otimes I_B = \rho$$

that

$$\sum_{k} |k\rangle \langle k| \otimes I_B \sum_{i,j} A_{ij} \otimes F_{ij} |k\rangle \langle k| \otimes I_B = \sum_{i,j} A_{ij} \otimes F_{ij}.$$ 

This leads to

$$\sum_{k} |k\rangle \langle k| A_{ij} |k\rangle \langle k| = A_{ij}$$

for any pair $(i, j)$, that is, every $A_{ij}$ is a diagonal operator with respect to the same orthonormal basis $\{|k\rangle\}$. Therefore, $A_{ij}$s are mutually commuting normal operators acting on $H_A$.

Theorem 2 implies that CQ stems from noncommutativity but not from entanglement. We can also find this kind of noncommutativity from another perspective: for finite-dimensional case, it is proved in [38] that if $\rho$ is CQ (equivalently $D_A(\rho) = 0$) then

$$[\rho, \rho_A \otimes I_B] = 0.$$ 

It is easy to check that this result is valid for infinite-dimensional systems as well.

**Proposition 2.** Let $\rho \in \mathcal{S}(H_A \otimes H_B)$ with $\dim H_A \otimes H_B \leq +\infty$. Then

$$\rho = \text{CQ} \Rightarrow [\rho, \rho_A \otimes I_B] = 0. \quad (20)$$

Indeed, if $\rho = \sum_{i,j} A_{ij} \otimes F_{ij}$ as in Eq. (11) with respect to some given bases of $H_A$ and $H_B$ and $\rho$ is a CQ state, then $\rho_A = \sum_i A_{ii}$ commutes with $A_{ij}$ for any $i$ and $j$. This ensures the commutativity of $\rho$ and $\rho_A \otimes I_B$. So the noncommutativity signals quantumness of the state. The converse is not true since for any state with maximal marginal we have Eq. (20) holds in the finite-dimensional case [38]. One can check that the converse of Proposition 2 is not true for infinite-dimensional case, either.

Theorem 2 is powerful for exploring the structure of CQ. For instance, to prove the fact that CQ is equivalent to zero GMQD is still valid for infinite-dimensional case, we need a geometric feature of the set of all CQ states, that is, the set of all CQ states is closed. This can be proved by applying Theorem 2.

**Theorem 3.** The set of all CQ states in $\mathcal{S}(H_A \otimes H_B)$ is a closed set under both the trace norm topology and the Hilbert-Schmidt norm topology.

**Proof.** Let $\rho$ be a state and $\{\rho_n\}_{n=1}^\infty$ be a sequence of CQ states on $H_A \otimes H_B$ such that $\lim_{n \to \infty} \rho_n = \rho$ under the trace norm topology. For an arbitrarily chosen product basis $\{|i\rangle |j\rangle\} \rangle$ of $H_A \otimes H_B$, $\rho_n$ and $\rho$ can be written in the form of

$$\rho_n = \sum_{i,j} A_{ij}^{(n)} \otimes F_{ij} \quad \text{and} \quad \rho = \sum_{i,j} A_{ij} \otimes F_{ij},$$
where $F_{ij} = |i⟩⟨j|$ and $A^{(n)}_{ij}, A_{ij} \in T(H_A)$. As $\rho_n$ is CQ, by Theorem 2, $\{A^{(n)}_{ij}\}_{i,j}$ is a commutative set of normal operators for each $n$. Now $\|A^{(n)}_{ij} - A_{ij}\|_{TV} = \|[I_A \otimes i'\rangle\langle j'|\rho_n - \rho\|(I_A \otimes i'\rangle\langle j'|)||_{TV} \leq \|\rho_n - \rho\|_{TV}$ and $\lim_{n \to \infty} \rho_n = \rho$ imply that $\lim_{n \to \infty} A^{(n)}_{ij} = A_{ij}$ for each pair $(i,j)$. It follows that $\{A_{ij}\}_{i,j}$ is a commutative set of normal operators, too, which ensures that $\rho$ is a CQ state by Theorem 2. Hence the set of all CQ states is closed under the trace norm topology.

Since $\rho_n \to \rho$ under the Hilbert-Schmidt norm topology if and only if $\rho_n \to \rho$ under the trace norm topology for states $\rho_n$ and $\rho$, thus the set of all CQ states is also closed under the Hilbert-Schmidt norm topology. $\blacksquare$

Now the following theorem is obvious.

**Theorem 4.** Let $\rho \in S(H_A \otimes H_B)$ with $\dim H_A \otimes H_B \leq \infty$ be a state. Then $D^A_B(\rho) = 0$ if and only if $\rho$ is a CQ state.

By Theorem 4 we get immediately the following known result for finite-dimensional case.

**Corollary 2.** Let $\rho \in S(H_A \otimes H_B)$ with $\dim H_A \otimes H_B < +\infty$ be a state. Then the following statements are equiverient.

1. $D_A(\rho) = 0$.
2. $D^A_B(\rho) = 0$.
3. $\rho$ is a CQ state.

Symmetrically, we can define quantum-classical (QC) states and it is clear that the counterpart results also valid. Namely, we call $\rho$ a QC state if $\rho$ can be decomposed as

$$\rho = \sum_j q_j \rho_j^A \otimes |j'\rangle\langle j'|, \quad (21)$$

where $\{|j'\rangle\}$ is an orthonormal set of $H_B$, $\rho_j^A$s are states of the subsystem $A$, $q_j \geq 0$ and $\sum_j q_j = 1$. We can also define the geometric measure of quantum discord of a state $\rho$ up to part B by

$$D^G_B(\rho) = \inf \{||\rho - \rho'\|_2 : \rho' \in QC\}, \quad (22)$$

where QC is the set of all QC states on $H_A \otimes H_B$.

The following results are obvious.

**Theorem 2'.** Write $\rho = \sum_{k,l} F_{kl} \otimes B_{kl}$ as in Eq. (8). Then $\rho$ is a QC state if and only if $B_{ij}$s are mutually commuting normal operators acting on $H_B$.

**Proposition 2'.** Let $\rho \in S(H_A \otimes H_B)$ with $\dim H_A \otimes H_B \leq +\infty$. Then $\rho$ is QC implies that $[\rho, I_A \otimes \rho_B] = 0$.

**Theorem 3'.** The set of all QC states is closed.

**Theorem 4'.** Let $\rho \in S(H_A \otimes H_B)$ with $\dim H_A \otimes H_B \leq +\infty$. Then $D^G_B(\rho) = 0$ if and only if $\rho$ is a QC state.

### B. Witnessing (geometric measure of) quantum discord

We now begin to discuss the relationship between zero GMQD states and SSPPT states.

28. Theorem 1] claims that any classical-quantum (CQ) state in $2 \otimes n$ system with $n < \infty$ is not only SPPT up to part A but also SPPT up to part B. And an example is given in 28 to illustrate that this conclusion is not valid for $m \otimes n$ system if $m > 2$. However the example is not correctly given. In fact, we remark here that the above conclusion is valid for any state in $m \otimes n$ with $m, n \leq +\infty$. Much more can be achieved.

First observe that every CQ state is SSPPT up to A and every QC state is SSPPT up to B. These can be checked directly by the definitions.

The following main result of this subsection claims that a CQ/QC state is not only SSPPT up to part A/B but also SSPPT up to part B/A.

**Theorem 5.** Let $\rho \in S(H_A \otimes H_B)$ with $\dim H_A \otimes H_B \leq +\infty$ be a state. If $\rho$ is QC (CQ), then $\rho$ is SSPPT up to part B (up to part A). In particular, if $\rho$ is not SSPPT up to part B or up to part A, then both $D^B_A(\rho)$ and $D^A_B(\rho)$ are nonzero.

This result means that zero GMQD is much more stronger than SSPPT, and any state with non-SSPPT has quantum correlations tested by GMQD.

**Proof of Theorem 5.** We only check the former case here, namely, the case that $\rho$ is QC. We want to show that $\rho$ is SSPPT up to part B. Write $\rho$ in the form

$$\rho = \sum_j q_j \rho_j^A \otimes |j'\rangle\langle j'|,$$

as in Eq. (21). Then, by Theorem 2', $\rho$ can be expressed as in Eq. (9), where $B_{ij}$s are mutually commuting normal trace-class operators acting on $H_B$. We have to show that there exists some $X$ of the form

$$X = \begin{pmatrix} X_1 & S_{12}X_1 & \cdots & S_{1m}X_1 & \cdots \\ 0 & X_2 & S_{23}X_2 & \cdots & S_{2m}X_2 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & X_{m-1} & S_{m-1,m}X_{m-1} & \cdots \\ 0 & 0 & 0 & 0 & X_m & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$Y = \begin{pmatrix} Y_1 & S_{12}^tY_1 & \cdots & S_{1m}^tY_1 & \cdots \\ 0 & Y_2 & S_{23}^tY_2 & \cdots & S_{2m}^tY_2 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & Y_{m-1} & S_{m-1,m}^tY_{m-1} & \cdots \\ 0 & 0 & 0 & 0 & Y_m & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $S_{ij}$s being diagonalizable operators satisfying

$$[S_{ki}, S_{kj}^t] = 0, \quad k < i \leq j,$$

such that $\rho = X^\dagger X$ and $\rho^{T_A} = Y^\dagger Y$. 
Let $A_k = q_k \rho_k^A$. Then

\[
\rho = \sum_k A_k \otimes F_{kk} = \begin{pmatrix}
A_1 & 0 & 0 & 0 & \cdots \\
0 & A_2 & 0 & 0 & \cdots \\
0 & 0 & \ddots & 0 & \cdots \\
0 & 0 & 0 & \ddots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix},
\]

where $F_{kk} = |k'\rangle \langle k'|$.

Thus there are upper triangular Hilbert-Schmidt operators $Z_k$ on $H_A$ such that

\[A_k = Z_k^\dagger Z_k, \quad k = 1, 2, \ldots,\]

and then

\[\rho = Z^\dagger Z\]

with

\[Z = \sum_k Z_k \otimes F_{kk}.\]

Write

\[Z_k = (z_{ij}^{(k)})_{i,j}\]

with $z_{ij}^{(k)} = 0$ whenever $i < j$ and let

\[X = (X_{ij})_{i,j},\]

where

\[X_{ij} = \text{diag}(z_{ij}^{(1)}, z_{ij}^{(2)}, \ldots, z_{ij}^{(n)}, \ldots) = \sum_n z_{ij}^{(n)} |n'\rangle \langle n'|.\]

Then $X_{ij} = 0$ if $i < j$ and

\[Z = X = \sum_{i,j} E_{ij} \otimes X_{ij},\]

which is an upper triangular operator matrix and $\rho = X^\dagger X$.

By Lemma 1 below, we can choose $Z_k$s so that $z_{ii}^{(n)} = 0$ implies that $z_{ij}^{(n)} = 0$. It turns out that $X_{ij}$ can be written in $X_{ij} = S_{ij} X_{ii}$ for some diagonal operator $S_{ij} = \sum_n s_{ij}^{(n)} |n'\rangle \langle n'|$ for any $(i,j)$ with $i < j$. Obviously we have $[S_{ij}, S_{ii}^\dagger] = 0$ for any $i < j \leq l$.

Now it is easily checked that $\rho^T = Y^\dagger Y$. Hence $\rho$ is SSPPT up to part B.

Finally, $D_A^G(\rho) = 0$ if and only if $\rho$ is CQ. By what we proved above, $\rho$ then is SSPPT up to A as well as up to B. So, $\rho$ is not SSPPT up to A or up to B will imply that both $D_A^G(\rho) > 0$ and $D_B^G(\rho) > 0$ hold.

Lemma 1 Let $A$ be a positive finite or infinite matrix with $A = X^\dagger X$ for some finite or infinite upper triangular matrix $X = (x_{ij})_{i,j}$, $\|A\|_F < +\infty$. If $x_{kk} = 0$ for some $k$, then there exists a finite or infinite upper triangular matrix $Y = (y_{ij})_{i,j}$ with $y_{kj} = 0$, $j = 1, 2, \ldots$, such that $A = Y^\dagger Y$.

Proof. Let

\[X = \begin{pmatrix}
x_{11} & x_{12} & x_{13} & \cdots & x_{1n} & \cdots \\
x_{22} & x_{23} & x_{24} & \cdots & x_{2n} & \cdots \\
x_{33} & x_{34} & x_{35} & \cdots & x_{3n} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
x_{nn} & \cdots & \cdots & \cdots & \ddots & \ddots
\end{pmatrix},
\]

\[|\eta_1\rangle = \begin{pmatrix}
x_{11} \\
x_{12} \\
\vdots \\
x_{1n} \\
x_{2n} \\
\vdots
\end{pmatrix},
\]

\[|\eta_2\rangle = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
\bar{x}_{kk} \\
\bar{x}_{k,k+1} \\
\bar{x}_{k,k+2} \\
\vdots
\end{pmatrix},
\]

Then

\[A = \sum_i |\eta_i\rangle \langle \eta_i|.\]  \hspace{1cm} (23)

If $x_{kk} = 0$, we let

\[A = \sum_{i=1}^{k-1} |\eta_i\rangle \langle \eta_i| + 0 \oplus A_{k+1},\]  \hspace{1cm} (24)

where 0 is a $k \times k$ zero matrix. Let

\[A_{k+1} = \bar{X}_{k+1}^\dagger \bar{X}_{k+1},\]

with

\[A_{k+1} = \bar{X}_{k+1}^\dagger \bar{X}_{k+1}.\]
it is clear that $A = Y^\dagger Y$ as desired.

Reviewing the discussion above, we also know that SSPPT up to part B/A is only a necessary condition of zero GMQD.

**Example 2.** The so-called circulant state in $2 \otimes 2$ system [39] is given by

$$\rho = \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}. $$

Assume that $a_{11}b_{11} > 0$. It can be derived form [20] that $\rho$ is SSPPT up to part B if and only if $\hat{a} \geq 0$, $\hat{b} \geq 0$ and $|a_{12}| = |b_{12}|$, where

$$\hat{a} = \begin{pmatrix} a_{11} & b_{21} \\ b_{11} & a_{22} \end{pmatrix}, \quad \hat{b} = \begin{pmatrix} b_{11} & a_{21} \\ a_{12} & b_{22} \end{pmatrix}. $$

On the other hand it is easily checked that $\rho$ is a QC state if and only if $a_{11} = b_{11}$, $a_{22} = b_{22}$ and $|a_{12}| = |b_{12}|$, which implies that there exist SSPPT states that are not QC states.

Facts listed above clearly indicate that zero GMQD (equivalently, zero QD for finite-dimensional case) has strong local commutativity than that of SSPPT.

### IV. NULLITY OF MEASUREMENT-INDUCED NONLOCALITY

In Secs.II-III we comparing SSPPT with separability, CQ, QC and zero QD(GMQD) respectively by means of local commutativity. The present section is devoted to the nullity of MiN (Measurement-induced nonlocality) in terms of local commutativity, from which we get a clearer picture of these different quantum correlations.

With the same spirit as that of the finite-dimensional case, we first generalize the concept of MiN to infinite-dimensional bipartite systems.

**Measurement-induced nonlocality** Assume that $\dim H_A \otimes H_B = +\infty$ and $\rho \in \mathcal{S}(H_A \otimes H_B)$. Let $\Pi^A = \{\Pi^A_k = |k\rangle\langle k|\}$ be a set of mutually orthogonal rank-one projections that sum up to the identity of $H_A$. Similar to the finite-dimensional case, we call such $\Pi^A = \{\Pi^A_k\} \otimes \{\Pi^B_\ell\}$ a local von Neumann measurement. Note that $\sum_k (\Pi^A_k \otimes I_B)(\Pi^A_k \otimes I_B) = \sum_k \Pi^A_k \otimes I_B = I_{AB}$, here the series converges under the strongly operator topology [40]. We define the Measurement-induced nonlocality (MiN, briefly) of $\rho$ by

$$N_A(\rho) := \sup_{\Pi^A} \|\rho - \Pi^A(\rho)\|_2^2, $$

where the supremum is taken over all local von Neumann measurements $\Pi^A = \{\Pi^A_k\}$ that satisfying

$$\sum_k \Pi^A_k \rho_A \Pi^A_k = \rho_A \tag{27}$$

The following properties are straightforward.

(i) $N_A(\rho) = 0$ for any product state $\rho = \rho_A \otimes \rho_B$.

(ii) $N_A(\rho)$ is locally unitary invariant, namely, $N_A((U \otimes V)\rho(U^\dagger \otimes V^\dagger)) = N_A(\rho)$ for any unitary operators $U$ and $V$ acting on $H_A$ and $H_B$, respectively.

(iii) $N_A(\rho) > 0$ whenever $\rho$ is entangled since $\Pi^A(\rho)$ is always a classical-quantum state and thus is separable.

(iv) $0 \leq N_A(\rho) < 4$.

The MiN of a pure state can be easily calculated. Let
\[ |\psi\rangle \in H_A \otimes H_B \text{ and} \]
\[ |\psi\rangle = \sum_k \lambda_k |k\rangle |k'\rangle \]
be its Schmidt decomposition. For the finite-dimensional case, Luo and Fu showed in [14] that
\[ N_A(|\psi\rangle) = 1 - \sum_k \lambda_k^2. \tag{28} \]
This is also true for pure states in infinite-dimensional systems. Dually, one can define MiN with respect to the second subsystem B—\(N_B\), and the corresponding properties are valid. It is easily seen that these two MiNs are asymmetric, namely, the MiN with respect to subsystem A is not equal to the one with respect to subsystem B generally.

Let us now begin to discuss the mullity of MiN. The following is the main result of this section (we only discuss the case of \(N_A\) since the case of \(N_B\) can be obtained by interchanging the role of A and B).

**Theorem 6.** Let \(\rho \in S(H_A \otimes H_B)\) be a state with \(\dim H_A \otimes H_B \leq +\infty\). Let \(\{|k\rangle\}\) and \(\{|i'\rangle\}\) be any orthonormal bases of \(H_A\) and \(H_B\), respectively. Write \(\rho = \sum_{i,j} A_{ij} \otimes F_{ij}\) as in Eq. (11) with respect to the given bases. Then \(N_A(\rho) = 0\) if and only if \(A_{ij}\)s are mutually commuting normal operators and each eigenspace of \(\rho\) contained in some eigenspace of \(A_{ij}\) for all \(i\) and \(j\).

**Proof.** By the definition of \(N_A(\rho)\), it is clear that the condition \(N_A(\rho) = 0\) is equivalent to the condition that \(\Pi^A(\rho) = \rho\) holds for any local von Neumann measurement that make \(\rho\) invariant.

The ‘if’ part. If each eigenspace of \(\rho\) is a one-dimensional space, then \(\rho_A = \sum p_i \langle i | i\rangle\) for some orthonormal base \(\{|i\rangle\}\) and \(\{|p_i\rangle\}\) with \(p_i > 0, p_i \neq p_j\) if \(i \neq j\). Obviously, for any local von Neumann measurement \(\Pi^A = \{\Pi_k^A\}\), \(\sum_k \Pi_k^A \rho_A \Pi_k^A = \rho_A\) implies that, for each \(k\), \(|k\rangle = |i\rangle\) for some \(i\). Thus \(\Pi^A\) is introduced in fact by \(\{|i\rangle\}\). Now it is clear that \(\Pi^A(\rho) = \rho\) as every \(A_{ij}\) commutes with \(\rho\).

Denote
\[ E(\lambda^A) = \ker(\lambda^A - \rho_A) \]
(here, ker(X) stands for the kernel of the operator X), and assume that \(\dim \ker(\lambda^A - \rho_A) \geq 2\) for some nonzero eigenvalue \(\lambda^A\) of \(\rho_A\). Then the restricted operator of \(\rho_A\) on \(E(\lambda^A)\), denoted by \(\rho_A|_{E(\lambda^A)}\), satisfying
\[ \rho_A|_{E(\lambda^A)} = \lambda^A I_{E(\lambda^A)} \]
where \(I_{E(\lambda^A)}\) is the identity operator on \(E(\lambda^A)\). As \(A_{ij}\)s are mutually commuting normal operators and each eigenspace of \(\rho\) contained in some eigenspace of \(A_{ij}\) for all \(i\) and \(j\), we see that
\[ C_{ij} = A_{ij}|_{E(\lambda^A)} = \lambda^{(ij)} I_{E(\lambda^A)} \]
for some eigenvalue \(\lambda^{(ij)}\) of \(A_{ij}\) for any \(i\) and \(j\). This leads to
\[ \sum_k \Pi_k^A A_{ij} \Pi_k^A = A_{ij} \]
for any local von Neumann measurement \(\Pi^A = \{\Pi_k^A\}\) that doesn’t disturb \(\rho_A\) locally, so we have \(\Pi^A(\rho) = \rho\).

The ‘only if’ part. If \(\Pi^A(\rho) = \rho\) for any local von Neumann measurement \(\Pi^A\) that leave \(\rho_A\) invariant, then \(\Pi^A\) satisfying
\[ \sum_k \Pi_k^A A_{ij} \Pi_k^A = A_{ij} \]
for any \(i\), \(j\). This forces that \(A_{ij}\)s are mutually commuting normal operators. We show that each eigenspace of \(\rho\) contained in some eigenspace of \(A_{ij}\) for all \(i\) and \(j\). Or else, we may assume with no loss of generality that
\[ \dim \ker(\lambda^{(i_0j_0)} - A_{i_0j_0}) = 1 \]
while
\[ \dim \ker(\lambda^A - \rho_A) = 2 \]
for some nonzero eigenvalue \(\lambda^{(i_0j_0)}\) of \(A_{i_0j_0}\) and nonzero eigenvalue \(\lambda^A\) of \(\rho_A\). It turns out that there must exist an orthonormal basis \(\{|e_1\rangle\}, \{|e_2\rangle\}\), and a local von Neumann measurement \(\Pi^A\) induced from an orthonormal basis containing \(\{|e_1\rangle\}, \{|e_2\rangle\}\) such that \(\sum_k \Pi_k^A(\rho_A)|_{\Pi_k^A} = \rho_A\) while \(\sum_k \Pi_k^A A_{i_0j_0} \Pi_k^A \neq A_{i_0j_0}\), a contradiction.

Symmetrically, we have

**Theorem 6’.** Let \(\rho = \sum_{k,l} F_{kl} \otimes B_{kl}\) as in Eq. (8) with respect to the given bases. Then \(N_B(\rho) = 0\) if and only if \(B_{kl}\)s are mutually commuting normal operators and each eigenspace of \(\rho_B\) contained in some eigenspace of \(B_{kl}\) for all \(k\) and \(l\).

Theorem 6 and 6’ indicate that the phenomenon of MiN is a manifestation of quantum correlations due to noncommutativity rather than due to entanglement as well. And we claim that the commutativity for a state to have zero MiN is ‘stronger’ than that of zero QD(GMQD) state. We illustrate it with the following example.

**Example 3.** We consider a 3 \(\otimes 2\) system. Let
\[ \rho = \begin{pmatrix} a & 0 & c & 0 & 0 \\ 0 & a & 0 & f & 0 \\ 0 & 0 & b & 0 & 0 \\ e & 0 & c & 0 & 0 \\ 0 & f & 0 & c & 0 \\ 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & d & 0 \end{pmatrix}. \]
It is clear that \(\rho\) is a CQ state for any positive numbers \(a, b, c, d\) and complex numbers \(e, f, g\) that make \(\rho\) a state. However, taking \(\Pi^A = \{|\psi_1\rangle\langle\psi_1|\}_{i=1}^3\) with
\[ |\psi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, |\psi_3\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]
it is easy to see that
\[ \sum_k \Pi_k^A \rho_A \Pi_k^A = \rho_A \]
and
\[ \Pi^A(\rho) \neq \rho \text{ whenever } e \neq f. \]
If $a + c = b + d$, it is easily checked that $N_A(\rho) = 0$ if and only if $a = b$, $c = d$ and $e = f = g$. Hence, there are many CQ states with nonzero MiN.

Let

$$S_{N_{A/B}}^0 = \{ \rho \in S(H_A \otimes H_B) : N_{A/B}(\rho) = 0 \},$$

$$S_{D_{A/B}}^0 = \{ \rho \in S(H_A \otimes H_B) : D_{A/B}^G(\rho) = 0 \},$$

$$CQ = \{ \rho \in S(H_A \otimes H_B) : \rho \text{ is CQ} \},$$

$$QC = \{ \rho \in S(H_A \otimes H_B) : \rho \text{ is QC} \}$$

and $S_{sep}$ be the set of all separable states acting on $H_A \otimes H_B$. The above example shows that, $S_{N_{A/B}}^0$ is a proper subset of $CQ/QC$. In addition, for $0 \leq \epsilon \leq 1$, $\rho_1$, $\rho_2 \in S_{N_{A/B}}^\epsilon$ do not imply $\epsilon \rho_1 + (1 - \epsilon) \rho_2 \in S_{N_{A/B}}^0$ in general, so $S_{N_{A/B}}^0$ is not a convex set. Similarly, $S_{D_{A/B}}^0$, $CQ$ and $QC$ are not convex, either.

Furthermore, equivalent to Theorem 6 and 6’, one can check that

**Corollary 3.** Let $\rho \in S(H_A \otimes H_B)$ with $\dim H_A \otimes H_B \leq +\infty$ be a state. Then

(i) $N_A(\rho) = 0$ if and only if

$$\rho = \sum_k p_k |k\rangle \langle k| \otimes \rho_k^B$$

with $\rho_k^B = \rho_k^B$ whenever $p_k = p_l$;

(ii) $N_B(\rho) = 0$ if and only if

$$\rho = \sum_j q_j \rho_j^A \otimes |j\rangle \langle j'|$$

with $\rho_j^A = \rho_j^A$ whenever $q_j = q_l$.

Comparing with Eqs. (18) and (21), we get a more transparent picture of these two different quantum correlations.

Reviewing the proof of Theorem 6 and 6’, the following is clear:

**Proposition 4.** Let $\rho \in S(H_A \otimes H_B)$, $\dim H_A \otimes H_B \leq +\infty$. Suppose that each eigenspace of $\rho_A$ (resp. $\rho_B$) is of one-dimension and $\rho_A = \sum_k p_k |k\rangle \langle k|$ (resp. $\rho_B = \sum_j q_j |l\rangle \langle l'|$) is the spectral decomposition. Then the local von Neumann measurement $\Pi_A$ (resp. $\Pi_B$) that makes $\rho_A$ (resp. $\rho_B$) invariant is uniquely (up to permutation) induced from $\{ |k\rangle \langle k| \}$ (resp. $\{ |l\rangle \langle l'| \}$), and vice versa.

In Ref. [14], for finite-dimensional case, the authors claim that $N_A(\rho) = 0$ for any classical-quantum state $\rho = \sum_k p_k |k\rangle \langle k| \otimes \rho_k^B$ whose marginal state $\rho^A = \sum_k p_k |k\rangle \langle k|$ is nondegenerate (here, a matrix $A$ is said to be nondegenerate provided that each eigenspace of $A$ is of one-dimension). This is also valid for infinite-dimensional case.

**Corollary 4.** Assume that $\rho \in S(H_A \otimes H_B)$ with $\dim H_A \otimes H_B \leq +\infty$. If $\rho \in CQ$ (resp. QC), then $N_A(\rho) = 0$ (resp. $N_B(\rho) = 0$) provided that each eigenspace of $\rho_A$ (resp. $\rho_B$) is of one-dimension.

It is known that, for the finite-dimensional case, $S_{D_{A/B}}^0$ is a zero-measure set [20] (that is, each point of this set can be approximated by a sequence of states that not belong to this set with respect to the trace norm), and, for the infinite-dimensional case, $S_{sep}$ is a zero-measure set [41]. Thus, both $S_{N_{A/B}}^0$ and $S_{D_{A/B}}^0$ are zero-measure set in both finite- and infinite-dimensional cases. This indicates that MiN is ubiquitous: almost all quantum states have nonzero MiN. In other words, as a resource, we get more states valid in tasks of quantum processing based on MiN.

**V. CONCLUSIONS**

In terms of local commutativity, for both finite- and infinite-dimensional systems, we show that (1) SSPT states are countably separable, (2) SSPT can detects QD(GMQD), and furthermore, the zero MiN states and zero GMQD states are characterized. We argue that MiN is the most essential quantum correlation among MiN, QD, GMQD and entanglement. They all originated from the *supposition* of the states (since for a pure state $\rho$, it is separable if and only if $N_{A/B}(\rho) = D_{A/B}^G(\rho) = 0$).

As a result, we obtain the following chain of (proper) inclusions for finite-dimensional case:

$$S_p \subset S_{N_{A/B}}^0 \subset CQ/QC = S_{D_{A/B}}^0 = S_{D_{A/B}}^G$$

$$\subset S_{SSPT}^B \subset S_{SSPT}^B \subset S_{SSPT}^A \subset S_{SSPT}^A \subset S_{SSPT}^B$$

where $S_p$ denotes the set of all product states, $S_{D_{A/B}}^0$ is the set of all zero QD states, i.e., $S_{D_{A/B}}^0 = \{ \rho \in S(H_A \otimes H_B) : D_{A/B}^G(\rho) = 0 \}$, $S_{SSPT}^A$ denotes the set of all SSPT states up to part A/B, $S_{SSPT}$ stands for the set of all countably separable states and $S_{SSPT}^B$ denotes the set consisting of all separable states and $\mathcal{PPT}$ denotes the set of all PPT states.

The above inclusion chains indicate that the weaker quantum correlation is, the stronger commutativity is. Consequently, we may guess that $\rho$ is separable if and only if its local operators $A_{ij}$s or $B_{kl}$s have certain “commutativity” properties of some degree. This is an interesting task and worth to make a further research.

In addition, our results also suggest several questions for further studies such as, (i) comparing $N_{A/B}(\rho)$ with $D_{A/B}^G(\rho)$ and some other entanglement measures (such as concurrence or entanglement of formation), and (ii)
establishing computable formula of $N_{A/B}(\rho)$ for arbitrary state $\rho$ for both finite- and infinite-dimensional cases.

ACKNOWLEDGMENTS

This work is partially supported by Natural Science Foundation of China (11171249, 11101250), Research Fund for the Doctoral Program of Higher Education of China (20101402110012) and Research start-up fund for the Doctors of Shanxi Datong University.

[1] M. A. Nielsen, I. L. Chuang, *Quantum Computation and Quantum Information*, (Cambridge University Press, Cambridge, 2000).
[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Modern Phys. 81, April-June(2009).
[3] O. G"uhne and G. Tóth, Phys. Reports, 474, 1-75(2009).
[4] H. Ollivier and W. H. Zurek, Phys. Rev. Letters, 88, 017901(2001).
[5] L. Henderson and V. Vedral, J. Phys. A 34, 6899(2001).
[6] B. Dakić, V. Vedral, and Č. Brukner, Phys. Rev. Lett. 105, 190502(2010).
[7] E. Knill and R. Laflamme, Phys. Rev. Lett. 81, 5672(1998).
[8] A. Datta, S. T. Flammia, and C. M. Caves, Phys. Rev. A 72, 042316 (2005).
[9] A. Datta and G. Vidal, Phys. Rev. A 75, 042310(2007).
[10] A. Datta, A. Shaji, and C. M. Caves, Phys. Rev. Lett. 100, 050502(2008).
[11] B. P. Lanyon, M. Barbieri, M. P. Almeida, and A. G. White, Phys. Rev. Lett. 101, 200501(2008).
[12] A. A. Qasimi and D. F. V. James, Phys. Rev. A 83, 032101(2011).
[13] J. Batle, A. Plastino, A. R. Plastino, and M. Casas, arXiv:1103.0704v4(2011).
[14] S.-L. Luo and S.-S. Fu, Phys. Rev. Lett. 106, 120401(2011).
[15] Y. Guo and J.-C. Hou, arXiv: 1107.0355v2(2011).
[16] K. Mattle, H. Weinfurter, P. G. Kwiat, and A. Zeilinger, Phys. Rev. Lett. 76, 4656(1996).
[17] X. Li, Q. Pan, J. Jing, J. Zhang, C. Xie, and K. Peng, Phys. Rev. Lett. 88, 047904(2002).
[18] C. H. Bennett, D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters, Phys. Rev. Lett. 87, 077902(2001).
[19] N. A. Peters, J. T. Barreiro, M. E. Goggin, T.-C. Wei, and P. G. Kwiat, Phys. Rev. Lett. 94, 150502(2005).
[20] R. F. Werner, Phys. rev. A 40, 4277(1989).
[21] P. Horodecki and R. Horodecki, Quant. Inf. Comput. 1(1), 45-75(2001).
[22] A. S. Holevo, M. E. Shirokov, and R. F. Werner, Russian Math. Surveys 60, N2,(2005).
[23] Y. Guo and J.-C. Hou, arXiv:1009.0116v1(2010).
[24] A. Peres, Phys. Rev. Lett. 77, 1413(1996).
[25] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1–8(1996).
[26] D. Chruściński, J. Jurkowski and A. Kossakowski, Phys. Rev. A 77, 022113(2008).
[27] K.-C. Ha, Phys. Rev. A 81, 064101(2010).
[28] B. Bylicka and D. Chruściński, Phys. Rev. A 81, 062102(2010).
[29] A. Datta, arXiv:1003.5256v2(2010).
[30] S.-L. Luo and S.-S. Fu, Phys. Rev. A 82, 034302(2010).
[31] C. Bastos, O. Bertolami, N. C. Dias, and J. N. Prata, J. Math. Phys. 49, 072101(2008).
[32] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. 73, 977(2001).
[33] J. Gamboa, M. Loewe, and J.'C. Rojas, Phys. Rev. D 64, 067901(2001).
[34] Y. Guo and J.-C. Hou, Phys. Lett. A, 375, 1160-1162(2011).
[35] Y. Guo, X.-F. Qi, and J.-C. Hou, Chin. Sci. Bull. 56(9), 840-846(2011).
[36] S.-L. Luo, Lett. Math. Phys. 92, 143-153(2010).
[37] K.-C. Ha, Phys. Rev. A 82, 041402(2010).
[38] A. Ferraro et al, Phys. Rev. A 81, 052318(2010).
[39] D. Chruściński and A. Kossakowski, Phys. Rev. A 76, 032308(2007).
[40] J.-C. Hou, J. Phys A: Math. Theor. 43, 385201(2010).
[41] R. Clifton and H. Halvorson, Phys. Rev. A 61, 012108(1999).