TWISTED AND FOLDED AUSLANDER-REITEN QUIVERs AND APPLICATIONS TO THE REPRESENTATION THEORY OF QUANTUM AFFINE ALGEBRAS

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Abstract. In this paper, we introduce twisted and folded AR-quivers of type $A_{2n+1}$, $D_{n+1}$, $E_6$ and $D_4$ associated to (triply) twisted Coxeter elements. Using the quivers of type $A_{2n+1}$ and $D_{n+1}$, we describe the denominator formulas and Dorey’s rule for quantum affine algebras $U'_q(B_{n+1}^{(1)})$ and $U'_q(C_n^{(1)})$, which are important information of representation theory of quantum affine algebras. More precisely, we can read the denominator formulas for $U'_q(B_{n+1}^{(1)})$ (resp. $U'_q(C_n^{(1)})$) using certain statistics on any folded AR-quiver of type $A_{2n+1}$ (resp. $D_{n+1}$) and Dorey’s rule for $U'_q(B_{n+1}^{(1)})$ (resp. $U'_q(C_n^{(1)})$) applying the notion of minimal pairs in a twisted AR-quiver. By adopting the same arguments, we propose the conjectural denominator formulas and Dorey’s rule for $U'_q(E_6^{(1)})$ and $U'_q(G_2^{(1)})$.

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INTRODUCTION

The Auslander-Reiten (AR) quiver of an Artin algebra is a quiver whose vertices correspond to indecomposable modules of the algebra and whose arrows correspond to irreducible morphisms between the modules. In particular, for a Dynkin quiver $Q$ of finite type $X=ADE$ and the corresponding path algebra $\mathbb{C}Q$, Gabriel [10] showed that the corresponding AR quiver $\Gamma_Q$ consists of the set of vertices identified with the set $\Phi^+$ of positive roots.

On the other hand, for a given Dynkin quiver $Q$, there are the corresponding Coxeter element $\phi_Q$, the convex partial order $\prec_Q$ and the set $[Q]$ of reduced expressions of the longest element $w_0$ in the Weyl group $W$. Interestingly, the AR-quiver $\Gamma_Q$ is closely related to $\phi_Q$, $\prec_Q$ and $[Q]$ as follows (see Section 2.1 and 2.2 for details).

- $\Gamma_Q$ is completely determined by the Coxeter element $\phi_Q$ or the Dynkin quiver $Q$.

$\Gamma_Q$ can be understood as the Hesse diagram of $\prec_Q$ defined on $\Phi^+$. In other words,

$$\text{for } \alpha, \beta \in \Phi^+, \quad \alpha \prec_Q \beta \iff \text{there exists a path from } \beta \text{ to } \alpha \text{ in } \Gamma_Q.$$ 

- A reduced expression $\widetilde{w}_0$ of $w_0$ which is adapted to $Q$ can be obtained by reading the AR-quiver $\Gamma_Q$ properly. The set of reduced expressions adapted to $Q$ forms the commutation class $[Q]$. 

Note that the family of adapted commutation classes \([Q]\) is called the adapted \(r\)-cluster point \([\llbracket Q\rrbracket]\). Here, the notion \(r\)-cluster point implies that these classes are related to each other by so-called reflection functors.

Moreover, in [24, 25, 26], the first named author investigated that certain statistics of AR-quivers provide some significant information in the representation theories of quantum groups \(U_q(X)\), quantum affine algebras \(U'_q(X^{(1)})\) and KLR-algebras. Especially, he read the denominator formulas and Dorey’s rule by constructing a simple algorithm for labeling \(\Gamma_Q\) which depends only on its shape. More precisely, he showed the followings:

(A) Connections between Dorey’s rule and AR-quivers: Dorey [8] described relations between three-point couplings in the simply laced affine Toda field theories (ATFTs) and Lie theories. Afterwards, Chari-Pressley [5] interpreted the phenomenon in terms of finite dimensional integrable \(U'_q(Y^{(1)})\)-modules \((Y = A_n, B_n, C_n, D_n)\), which is now referred to as Dorey’s rule. For type \(A_n\) and \(D_n\), they crucially used Coxeter elements of the corresponding type. Inspired from their work, Dorey’s rule for \(U'_q(A_n^{(1)})\), \(U'_q(D_n^{(1)})\) and \(U'_q(E_n^{(1)})\) can be interpreted in terms of coordinates of AR-quivers [24, 25, 26]. That is, for \(i\)-th fundamental representation \(V(\varpi_i)\), the condition that

\[
\text{Hom}(V(\varpi_k), V(\varpi_i) \otimes V(\varpi_j)) \neq 0
\]

is equivalent to (i) \(x = (-q)^a\), \(y = (-q)^b\), \(z = (-q)^c\) and (ii) \(\alpha, \beta, \gamma \in \Phi^+\) whose coordinates are \((i, a), (j, b), (k, c)\) in an AR-quiver \(\Gamma_Q\) satisfies \(\alpha + \beta = \gamma\). (See also [18] and [23] for the analogous results related to type \(A_n^{(2)}\) and \(D_n^{(2)}\).)

(B) Connections between denominator formulas and AR-quivers: With new statistics introduced in [26], one can read the denominator formulas \(d_{k,l}(z)\) for \(U'_q(A_n^{(t)})\) (resp. \(U'_q(D_n^{(t)})\)) \((t = 1, 2)\) from any \(\Gamma_Q\) of type \(A_n\) (resp. \(D_n\)), which control their representation theory (see Theorem 7.4). In detail, the first named author introduced the distance polynomial \(D_{k,l}(z)\) of any \(\Gamma_Q\) which directly follows from the statistics of \(\Gamma_Q\). Using distance polynomials, the denominator formula \(d_{k,l}(z)\) between fundamental representations \(V(\varpi_k)\) and \(V(\varpi_l)\) over \(U'_q(A_n^{(1)})\) (resp. \(U'_q(D_n^{(1)})\)) can be described as

\[
d_{k,l}(z) = D_{k,l}(z) \times (1 - (-q)^h^\vee)^{\delta_{k,l}}
\]

where \(h^\vee\) is the dual Coxeter number of \(A_n\) (resp. \(D_n\)). Note that the denominator formulas between fundamental representations over classical quantum affine algebras were calculated in [1, 7, 17, 23].

In addition, another interesting application of AR-quivers to the representation theory of \(U'_q(X^{(1)})\) is found by Hernandez-Leclerc in [12]. They introduced the category \(C_Q\) of \(U'_q(X^{(1)})\)-modules whose definition depends on the coordinate system of \(\Gamma_Q\) and proved that each \(C_Q\) provides the categorification of negative part \(U_q^{-}(X)\) of \(U_q(X)\) and the dual PBW-basis of \(U_q^{-}(X)\) associated to the commutation class \([Q]\). In [18], Kang-Kashiwara-Kim-Oh introduced the category \(C_Q^{(2)}\) of \(U'_q(X_n^{(2)})\)-modules \((X = A\) or \(D)\) and proved that \(C_Q^{(2)}\) plays the same role of \(C_Q^{(1)}\) in the sense of categorification, by using Dorey’s rules for \(U'_q(X_n^{(2)})\) and AR-quiver \(\Gamma_Q\) crucially.
The main purpose of this paper is finding analogous results to (A) and (B) for type $BCFG$. In order to do this, we focus on the fact that Chari-Pressley [5] considered a twisted Coxeter element $\hat{\varphi}$ of type $A_{2n+1}$, $D_{n+1}$ to see Dorey’s rule for $U'_q(B_n^{(1)})$ and $U'_q(C_n^{(1)})$. On the other hand, the authors [27] defined combinatorial AR-quiver $\Upsilon_{[i_0]}$ for any commutation class $[i_0]$ of reduced expressions of $w_0$, which is a generalization of $\Gamma_Q$. Indeed, $\Upsilon_{[i_0]}$ reflects the convex partial order $\prec_{[i_0]}$ induced from the commutation class $[i_0]$. Hence combinatorial AR-quivers are good options to substitute AR-quivers in (A) and (B). Indeed, our main results are the followings.

- For each (triply) twisted Coxeter element, we associate a commutation class $[i_0]$ of $w_0$ of types $A_{2n-1}$, $D_{n+1}$ and $E_6$ ($D_4$). Also the commutation classes arising from (triply) twisted Coxeter elements are reflection equivalent.
- For each commutation class $[i_0]$ associated to a (triply) twisted Coxeter element, we fold the combinatorial AR-quiver $\Upsilon_{[i_0]}$ via Dynkin diagram automorphisms (see Section 3). Then we get the folded AR-quiver $\hat{\Upsilon}_{[i_0]}$. Using $\hat{\Upsilon}_{[i_0]}$ instead of $\Gamma_Q$, we can find analogous results to (A) and (B) for type $BC$. Thus one can say that commutation classes associated to twisted Coxeter elements are deeply related to the representation theory of quantum affine algebras of type $B_n^{(1)}$ and $C_n^{(1)}$.
- By the same argument, we can find the conjectural formulas of Dorey’s rule and denominator formulas for $U'_q(F_4^{(1)})$ and $U'_q(G_2^{(1)})$ (see the Appendix A). Indeed, there is no article written about Dorey’s rule or denominator formulas for $U'_q(F_4^{(1)})$ and $U'_q(G_2^{(1)})$ for the best of authors’ knowledge. We expect our conjectural formulas can give reasonable suggestions.
- For each commutation class $[i_0]$ associated to a (triply) twisted Coxeter element, we introduce new subcategories $\mathscr{C}_{[i_0]}$ for quantum affine algebras of untwisted non-simply laced types (such as $B_n^{(1)}$, $C_n^{(1)}$, $F_4^{(1)}$ and $G_2^{(1)}$) which can be understood as twisted analogues of $C_Q$ (see Definition 9.3).

In order to achieve the goals, for a non-trivial Dynkin diagram automorphism of type $A_{2n+1}$, $D_{n+1}$, $E_6$, $D_4$, we study a class $[i_0]$ of reduced expressions associated to a (triply) twisted Coxeter element and the corresponding combinatorial AR-quiver $\Upsilon_{[i_0]}$. More precisely, we mainly discuss about reduced expressions in an $r$-cluster point $[[\mathcal{D}]]$ (resp. $[[\mathcal{Q}]]$), called (triply) twisted adapted cluster point, which contains all reduced expressions related to (triply) twisted Coxeter elements. If we call the combinatorial AR-quiver $\Upsilon_{[i_0]}$ for $[i_0] \in [[\mathcal{D}]]$ or $[[\mathcal{Q}]]$ by a twisted AR-quiver, the crucial part of this paper deals with shapes and labeling of twisted AR-quivers. The followings are main steps to see the relations between twisted AR-quivers and denominator formulas or Dorey’s rule:
- Twisted AR-quivers $\Upsilon_{[i_0]}$ of type $A_{2n+1}$ and $D_{n+1}$ can be obtained from AR-quivers $\Gamma_Q$ of type $A_{2n}$ and $A_n$ by simple surgeries (In Section 4, Algorithm 4.15 and Algorithm 4.26).
- $\Upsilon_{[i_0]}$ is foldable via the corresponding Dynkin diagram automorphism so that we can obtain folded AR-quiver $\hat{\Upsilon}_{[i_0]}$ (Section 5).
- $\Upsilon_{[i_0]}$ and $\hat{\Upsilon}_{[i_0]}$ have natural coordinate systems (Section 5).
- Labels in $\Upsilon_{[i_0]}$ and $\hat{\Upsilon}_{[i_0]}$ of type $A_{2n+1}$ and $D_{n+1}$ are completely determined by the shape of quiver (Section 6).
As consequences, we can read the denominator formulas and Dorey’s rule of type $U_q'(B_{n+1}^{(1)})$ and $U_q'(C_n^{(1)})$ from any $\tilde{\Gamma}_{[i_0]}$ of type $A_{2n+1}$ and $D_{n+1}$ by the algorithm for labeling $\tilde{\Gamma}_{[i_0]}$. Also, Algorithm 4.15 and Algorithm 4.26 explain the similarities of denominator formulas for classical untwisted quantum affine algebras (see (9.1) and (9.3)).

In Section 1, we introduce $r$-cluster points of (classes of) reduced expressions of $w_0$. In Section 2, we review results on reduced expressions in the adapted $r$-cluster point $[[\Delta]]$ and AR-quivers focusing on properties which are useful in the applications (A) and (B). In Section 3, we review twisted Coxeter elements and introduce the twisted adapted $r$-cluster point $[[\mathcal{Q}]]$. For detailed properties of a reduced expression $i_0$ in $[[\mathcal{Q}]]$ and the combinatorial AR-quiver $\mathcal{Y}_{[i_0]}$, we explain in Section 4 to Section 6.

In Section 4, we find the cardinalities of twisted adapted $r$-cluster points of type $A_{2n+1}$ and type $D_{n+1}$ and show twisted AR-quivers of type $A_{2n+1}$ and $D_{n+1}$ can be obtained from AR-quivers of type $A_{2n}$ and $A_n$ by simple surgeries. In Section 5, we consider the folded AR-quiver $\tilde{\mathcal{Y}}_{[i_0]}$ of a twisted AR-quiver $\mathcal{Y}_{[i_0]}$ and give coordinates to twisted AR-quivers and folded AR-quivers. In Section 6, we show how to find labels of twisted and folded AR-quivers by observing their shape only.

From Section 7, we focus on applications of twisted and folded AR-quivers to the representation theory of quantum affine algebras. In Section 7, we review basic notions in quantum affine algebras and their representation theories related to $R$-matrices, denominator formulas and Dorey’s rule. In Section 8, we introduce terms on sequences of positive roots, including Distance polynomials and minimal pairs. Here we used properties of twisted and folded AR-quivers in Section 4 to Section 6. In Section 9, we show denominator formulas and Dorey’s rule can be obtained by statistics of $\mathcal{Y}_{[i_0]}$ stated in the previous sections. In addition, we give conjectural formulas of Dorey’s rule and denominators for $U_q'(E_4^{(1)})$ and $U_q'(G_2^{(1)})$ in Appendix A. In Appendix B, we introduce a (triply) twisted Dynkin quiver and adapted reduced expressions to a (triply) twisted Dynkin quiver. Here we show all reduced expressions adapted to a (triply) twisted Dynkin quiver are in a commutation class in $[[\mathcal{Q}]]$ (resp. $[[\mathcal{Q}]]$). This explains the motivation of the notion “twisted adapted” reduced expressions.

In [21, 30], the first named author and his collaborators proved that $\mathcal{C}_{[i_0]}$ gives a categorification of $U_q^{-}(X)$ ($X = A_{2n+1}, D_{n+1}, E_6$ and $D_4$) and the dual PBW-basis of $U_q^-(X)$ associated to the (triply) twisted adapted class $[i_0]$ by using results in the previous versions of this paper ([28, 29]). Hence we can observe mysterious categorical relations between quantum affine algebras

- $U_q'(A_{2n+1}^{(1)})(t = 1,2)$ and $U_q'(B_{n+1}^{(1)})$,
- $U_q'(D_{n+1}^{(1)})(t = 1,2)$ and $U_q'(C_n^{(1)})$,
- $U_q'(E_6^{(1)})(t = 1,2)$ and $U_q'(F_4^{(1)})$,
- $U_q'(D_4^{(1)})(t = 1,2,3)$, $U_q'(C_3^{(1)})$ and $U_q'(G_2^{(1)})$,

with the results in [17, 18] (see also [14] in the aspect of quantum cluster algebras). Note that such observation was initiated in [9]. In [30], the conjectures in this paper on denominator
formulas and Dorey’s rule for $U'_q(F_4^{(1)})$ and $U'_q(G_2^{(1)})$ are also proved by the first author and Scrimshaw.

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1. Reduced expressions and combinatorial AR-quivers

In this section, we review notions related to reduced expressions of the longest element in a Weyl group and introduce an $r$-cluster point, which consists of commutation classes of reduced expressions. Also, we recall the combinatorial AR-quiver associated to a class of reduced expressions, which is a generalization of the AR-quiver (Section 2) associated to a class of adapted reduced expressions.

1.1. $r$-cluster points. Let us consider the finite type Dynkin diagram $\Delta_n$ of rank $n$, labeled by the index set $I_n$. Let $W_n$ be a Weyl group, generated by the set of simple reflections $\Pi_n := \{s_i \mid i \in I_n\}$ and $w_0$ be the longest element of $W_n$. We usually drop $n$ if there is no danger of confusion. Let us denote by $\Phi^+$ the set of positive roots associated to $\Delta$ and denote $N := \langle s_i \rangle_{i \in I_n}$.\footnote{where $\langle I \rangle$ is the free monoid generated by $I$ and $\langle I \rangle^r$ be the set of reduced words of $\langle I \rangle$ in the sense of Weyl group representation. We usually denote by $i$ for words and by $\bar{i}$ for the corresponding reduced words.}

We say that two reduced words $i$ and $j$ representing $w \in W$ are commutation equivalent, denoted by $i \sim j$, if $i$ is obtained from $j$ by applying the commutation relation $s_a s_b = s_b s_a$, where $a$ and $b$ are non-adjacent vertices in $\Delta$. We denote by $[i]$ the commutation class of $i$ under the equivalence relation $\sim$.

Fix a Dynkin diagram $\Delta$ of finite type. For a commutation class $[i_0]$ representing $w_0$, we say that an index $i$ is a sink (resp. source) of $[i_0]$ if there is a reduced word $j_0 \in [i_0]$ of $w_0$ starting with $i$ (resp. ending with $i$).

Definition 1.1. Let $^*$ be the involution on $I$ induced by $w_0$, that is

$$w_0(\alpha_i) = -\alpha_i^*$$

for each simple root $\alpha_i$.

The following proposition is a well-known to experts (see for example \cite{27}).

Proposition 1.2. For the reduced word $i_0 = i_1 i_2 \cdots i_{N-1} i_N$ of $w_0$, the word $i_0' = i_N^* i_1 i_2 \cdots i_{N-1}$ is again a reduced word of $w_0$ such that $[i_0'] \neq [i_0]$. Similarly, $i_0'' = i_2 \cdots i_{N-1} i_N i_1^*$ is again a reduced word of $w_0$ such that $[i_0''] \neq [i_0]$.

By applying the above proposition, we can obtain new reduced words for $w_0$ by applying the operations so called reflection functors, from a reduced word for $w_0$. The right action of reflection functor $r_i$ on $[i_0]$ is defined by

$$[i_0] r_i = \begin{cases} [i_2 \cdots i_N i_1^*] & \text{if } i \text{ is a sink and } i_0' = i \ i_2 \cdots i_N \in [i_0], \\ [i_0] & \text{if } i \text{ is not a sink of } [i_0]. \end{cases}$$
Similarly, the left action of reflection functor $r_i$ on $[i_0]$ is defined by

$$r_i [i_0] = \begin{cases} 
  i^* i_1 \cdots i_{N-1} & \text{if } i \text{ is a source and } i'_0 = i_1 \cdots i_{N-1} i \in [i_0], \\
  [i_0] & \text{if } i \text{ is not a source of } [i_0].
\end{cases}$$

For the word $w = i_1 \cdots i_k$, the right (resp. left) action of the reflection functor $r_w$ is defined by

$$[i_0] r_w = [i_0] r_{i_1} \cdots r_{i_k} \quad \text{(resp. } r_w [i_0] = r_{i_k} \cdots r_{i_1} [i_0]).$$

**Definition 1.3.** Let $[i_0]$ and $[i'_0]$ be two commutation classes representing $w_0$. We say that $[i_0]$ and $[i'_0]$ are reflection equivalent and write $[i_0] \overset{r}{\sim} [i'_0]$ if $[i'_0]$ can be obtained from $[i_0]$ by a sequence of reflection functors. The family of commutation classes $[[i_0]] := \{ [i'_0] | [i'_0] \overset{r}{\sim} [i_0] \}$ is called an $r$-cluster point (see [27]).

Note that for a reduced expression $i_0 = i_1 i_2 \cdots i_N$ and a word $w = i_N i_{N-1} \cdots i_1 i_2^* \cdots i_N^*$, we have

$$[i_0] r_{i_1} = [i_2 i_3 \cdots i_N i_1^*] = r_w [i_0].$$

In this paper, we deal with two types of $r$-cluster points: adapted $r$-cluster points (Definition 2.2) and twisted adapted $r$-cluster points (Section 3). An adapted $r$-cluster point consists of adapted reduced expressions. This type of $r$-cluster points are well-investigated. On the other hand, twisted adapted $r$-cluster points are newly investigated in this paper for the best of authors’ knowledge.

1.2. Convex orders and combinatorial AR-quivers. Note that, for any reduced expression $i_0$ of $w_0$, we can define a total order $<_{i_0}$ on $\Phi^+$ as follows:

$$\beta^i_{l_k} := s_{i_1} s_{i_2} \cdots s_{i_{l_k}} (\alpha_{i_k}) \quad (1 \leq k \leq N) \quad \text{and} \quad \beta^i_{l_k} <_{i_0} \beta^j_{l_l} \iff k < l.$$ 

Interestingly, the order $<_{i_0}$ is convex in the following sense (see [31, 35]): We say that an order $<$ on $\Phi^+$ is convex if it satisfies the following property: For $\alpha, \beta \in \Phi^+$ satisfying $\alpha + \beta \in \Phi^+$, we have either

$$\alpha < \alpha + \beta < \beta \quad \text{or} \quad \beta < \alpha + \beta < \alpha.$$

By considering $<_{i'_0}$ for all $i'_0 \in [i_0]$, the convex partial order $<_{[i_0]}$ on $\Phi^+$ is defined as follows:

$$\alpha <_{[i_0]} \beta \quad \text{if } \alpha <_{i'_0} \beta \text{ for all } i'_0 \in [i_0].$$

In [27], the authors introduced the combinatorial Auslander-Reiten quiver $\Gamma_{[i_0]}$ for any commutation class $[i_0]$ of any finite type.

**Algorithm 1.4.** The quiver $\Gamma_{i_0} = (\Gamma^0_{i_0}, \Gamma^1_{i_0})$ associated to $i_0$ is constructed by the following algorithm:

(Q1) $\Gamma^0_{i_0}$ consists of $N$ vertices labeled by $\beta^i_{l_1}, \cdots, \beta^i_{l_N}$ as in (1.1).

(Q2) There is an arrow from $\beta^i_{l_k}$ to $\beta^i_{l_j}$ for $1 \leq j < k \leq N$ if the followings hold:

(Ar1) two indices $i_k$ and $i_j$ are distinct and connected in the Dynkin diagram,

(Ar2) for $j'$ such that $j < j' < k$, we have $i_{j'} \neq i_j, i_k$.

(Q3) Assign the color $m_{jk} = -(\alpha_{i_j}, \alpha_{i_k})$ to each arrow $\beta^i_{l_k} \to \beta^i_{l_j}$ in (Q2); that is, $\beta^i_{l_k} \xrightarrow{m_{jk}} \beta^i_{l_j}$.

Replace 1 by $\to$, 2 by $\Rightarrow$ and 3 by $\equiv$. 

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For a given reduced expression $i_0 = (i_1i_2i_3\cdots i_N)$ of the longest element $w_0 \in W$, the residue of the vertex $\beta^{i_0}_k \in \Upsilon_{i_0}$ is $i_k$.

**Theorem 1.5.** [27] Let us choose any commutation class $[i_0]$.

1. If $i_0 \sim i'_0$, then $\Upsilon_{i_0} \simeq \Upsilon_{i'_0}$ as quivers. Hence $\Upsilon_{[i_0]}$ is well-defined.
2. $\alpha \prec_{[i_0]} \beta$ if and only if there exists a path from $\beta$ to $\alpha$ in $\Upsilon_{[i_0]}$.
3. Each $i_0' \in [i_0]$ can be obtained by reading the residue of every vertex in a way compatible with the opposite directions of arrows.

**Remark 1.6.** Conventionally, we assume residues of vertices in $\Upsilon_{[i_0]}$ increase from the north to the south. Also, every arrow in a $\Upsilon_{[i_0]}$ points South-East or North-East direction.

**Example 1.7.** Let $i_0 = (123124123124)$ be a reduced word of $w_0$ of type $D_4$. We can draw $\Upsilon_{[i_0]}$ as follows:

```
1  2  3  4
\alpha_4 \alpha_2\alpha_4 \alpha_1+\alpha_2\alpha_3+\alpha_4 \alpha_1+\alpha_2\alpha_3 \alpha_2 \alpha_1
```

**Remark 1.8.** As one can expect, a combinatorial AR-quiver is a generalization of an AR-quiver, which will be reviewed carefully in Section 2. More precisely, if $[i_0]$ is a class of adapted reduced expressions then the corresponding $\Upsilon_{[i_0]}$ is the same as the corresponding AR-quiver $\Gamma_Q$ (see Section 2).

**Definition 1.9.** [24, Definition 1.6] Fix any class $[j_0]$ of $w_0$ of any finite type.

1. A pair $(\alpha, \beta)$ of positive roots is *sectional* in $\Upsilon_{[j_0]}$ if there exists a path in $\Upsilon_{[j_0]}$ between them consisting of $d(i, j)$-arrows, where $i$ is the residue of $\alpha$, $j$ is the residue of $\beta$ and $d(i, j)$ denotes the number of edges between $i$ and $j$ in Dynkin diagram.
2. A full subquiver $\rho$ of $\Upsilon_{[j_0]}$ is *sectional* if every pair $(\alpha, \beta)$ in $\rho$ is sectional.
3. A connected subquiver $\rho$ in $\Upsilon_{[j_0]}$ is called an $S$-sectional (resp. $N$-sectional) path if it is a concatenation of downward (upward) arrows, and there is no longer connected path consisting of downward arrows (resp. upward arrows) containing $\rho$.

We write $N$-path (resp. $S$-path) instead of $N$-sectional path (resp. $S$-sectional path) for brevity.

## 2. Adapted Reduced Expressions and AR-quivers

The Auslander-Reiten (AR) theory, which is closely related to adapted reduced expressions, have been studied well. In this section, we briefly review properties of adapted reduced expressions, AR-quivers and their applications. For the precise references, we mainly refer [2, 10, 24, 26].

In this section, we assume the set $\Phi^+$ of positive roots and Dynkin diagram $\Delta$ are of type $\tilde{X}_n$, $X = A, D, E$. 
2.1. Coxeter elements and the adapted $r$-cluster point $[[\Delta]]$. Let us consider a Dynkin quiver $Q$, which is obtained by orienting all edges of the Dynkin diagram $\Delta$. For a sink $i$ of $Q$, we denote by $iQ$ the quiver obtained from $Q$ by reversing the orientation of each arrow incidents with $i$ in $Q$. For a reduced word $i$, we say that $i = i_1i_2\cdots i_l$ is adapted to $Q$ if $i_k$ is a sink of $i_{k-1}i_1Q$ for all $1 \leq k \leq l$.

Also, recall that a Coxeter element $\phi = s_{i_1}s_{i_2}\cdots s_{i_n}$ is a product of simple reflections, where $i_1i_2i_3\cdots i_n$ is a rearrangement of $123\cdots n$.

The following theorem shows that Dynkin quivers and Coxeter elements are closely related to classes of reduced expressions.

**Theorem 2.1.**

1. The set of all reduced expressions of $w_0$ adapted to $Q$ forms a commutation class $[Q]$.
2. If $Q \neq Q'$ then we have $[Q] \neq [Q']$.
3. The set of commutation classes $[Q]$ forms an $r$-cluster point $[[\Delta]]$.
4. For a Dynkin quiver $Q$, there exists unique adapted Coxeter element denoted by $\phi_Q$.

By the previous theorem, there are one-to-one and onto correspondences between the set of Dynkin quivers, adapted classes, and Coxeter elements by the maps

$$Q \leftrightarrow [Q] \leftrightarrow \phi_Q.$$ 

Also, by Theorem 2.1 (3), we get the following definition.

**Definition 2.2.** The $r$-cluster point $[[\Delta]]$ consisting of all commutation classes $[Q]$ is called the adapted $r$-cluster point.

By counting the number of Dynkin quivers, we have

the number of commutation classes in $[[\Delta]]$ is $2^{n-1}$.

2.2. AR-quivers and their properties. For each Dynkin quiver $Q$ of type $ADE$, let us denote by $\Gamma_Q$ the AR-quiver which is associated to $Q$ and whose vertices are labeled by $\Phi^+$. In this subsection, we focus on its relations with $\subset_{[Q]}$ and $[Q]$. Recall the Coxeter element $\phi_Q$ of $[Q]$. Then $\Gamma_Q$ can be constructed by using only $\phi_Q$:

(A) For any reduced expression $s_{i_1}s_{i_2}\cdots s_{i_n}$ of $\phi_Q$, the subset

$$\Phi(\phi_Q) := \{\beta_1^{\phi_Q} = a_{i_1}, \beta_2^{\phi_Q} = s_{i_1}(a_{i_2}), \ldots, \beta_n^{\phi_Q} = s_{i_1}\cdots s_{i_{n-1}}(a_{i_n})\},$$

of $\Phi^+$ is well-defined. Furthermore, the index $i_k$ on $\beta_k^{\phi_Q}$ is also well-assigned.

(B) The height function $\xi$ associated to $Q$ is a map on $Q$ satisfying $\xi(j) = \xi(i) + 1$ if there exists an arrow $i \rightarrow j$ in $Q$. Note that the connectedness of $Q$ implies the uniqueness of $\xi$ up to constant.

Thus we can assign $\beta_k^{\phi_Q}$ ($k = 1, \cdots, n$) to $(i_k, \xi(i_k)) \in I \times \mathbb{Z}$ which does depend only on $Q$ and hence $\phi_Q$:...
Algorithm 2.3. The AR-quiver $\Gamma_Q$, whose set of vertices is also identified with a subset of $I \times Z$, can be constructed by the following injective map $\Omega_Q : \Phi^+ \rightarrow I \times Z$ in an inductive way (cf. [12]):

1. $\Omega_Q(\beta_k^\Phi) := (i_k, \xi(i_k))$ for $k = 1, \ldots, n$.
2. If $\Omega_Q(\beta)$ is already assigned as $(i, p)$ and $\phi_Q(\beta) \in \Phi^+$, then
   $$\Omega_Q(\phi_Q(\beta)) = (i, p - 2).$$
3. For $(i, p), (j, q) \in \text{Im}(\Omega_Q)$, there exists an arrow $(i, p) \rightarrow (j, q)$ if and only if $i$ and $j$ are adjacent in $\Delta$ and
   $$p - q = -1.$$

For $\beta$ with $\Omega_Q(\beta) = (i, p)$, we call $i$ the residue of $\beta$ with respect to $[Q]$, and $(i, p)$ the coordinate of $\beta$ in $\Gamma_Q$.

Proposition 2.4. [27] For each $[Q]$, we have
$$\Gamma_Q \simeq \Upsilon_{[Q]}$$

as quivers.

The AR-quiver $\Gamma_Q$ satisfies the additive property with respect to arrows and $\phi_Q$ in the following sense: For $\alpha \in \Phi^+$ with $\beta = \phi_Q(\alpha) \in \Phi^+$, we have

$$\alpha + \beta = \sum_{\eta \in \beta_Q^\alpha} \eta,$$

where

$$\beta_Q^\alpha = \{ \eta \in \Phi^+ \mid \text{there exists a path } \beta \rightarrow \eta \rightarrow \alpha \text{ in } \Gamma_Q \}.$$

Example 2.5. The AR-quiver $\Gamma_Q$ associated to $\xymatrix{1 \ar[r] & 2 \ar[r] & 3 \ar[r] & 4 \ar[r] & 5}$ with the height function $\xi$ such that $\xi(1) = 0$ is given as follows:

\[
\begin{array}{cccccccc}
(i, p) & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
1 & 5 & 4 & 3 & 2 & 1 \\
2 & 4,5 & 4 & 2,3 & 1,3 \\
3 & 2,5 & 1,4 & 3 \\
4 & 2 & 1,5 & 3,4 \\
5 & 1,2 & 3,5 \\
\end{array}
\]

Here $[a, b]$ $(1 \leq a, b \leq 5)$ denotes the positive root $\sum_{k=a}^{b} \alpha_k$ and $[a] := \alpha_a$.

When we want to find $\Gamma_Q$ without labeling, Proposition 2.6 can be an easier alternative method to Algorithm 2.3.

Proposition 2.6. [3, 10, 33] For the dual Coxeter number $h^\vee$ associated to $Q$ and $i \in I$, let

$$r^Q_i = \frac{h^\vee + a^Q_i - b^Q_i}{2}.$$
where \( a_i^Q \) is the number of arrows in \( Q \) between \( i \) and \( i^* \) directed toward \( i \), and \( b_i^Q \) is the number of arrows in \( Q \) between \( i \) and \( i^* \) directed toward \( i^* \). Then the number of vertices in \( \Gamma_Q \cap (\{i\} \times \mathbb{Z}) \) is \( r_i^Q \) and

\[
\Gamma_Q \cap (\{i\} \times \mathbb{Z}) = \{(i, \xi(i) - 2k) | k = 0, \ldots, r_i - 1\}.
\]

**Remark 2.7.** It is known that

(i) the right boundary of \( \Gamma_Q \), the full subquiver consisting of \( \Phi(\phi_Q) \), is isomorphic to \( Q \) as a quiver,

(ii) the left boundary of \( \Gamma_Q \), the full subquiver consisting of \( \{\phi_i^{Q-1} (\beta^Q_i) \} \), is isomorphic to \( Q^* \) as a quiver, where \( Q^* \) is the quiver obtained from \( Q \) by replacing each vertex \( i \) with \( i^* \).

Now, the following theorem shows a combinatorial AR-quiver can be understood as a generalization of an AR-quiver (see Remark 1.8), in the sense that \( \Gamma_Q \) is a visualization of \( <Q := <[Q] \).

**Theorem 2.8.** \([3, 32]\)

1. \( \alpha < Q \beta \) if and only if there exists a path from \( \beta \) to \( \alpha \) inside of \( \Gamma_Q \).
2. By reading residues of vertices in a way compatible with opposite directions of arrows in \( \Gamma_Q \), we can obtain all reduced words \( \phi_i \in [Q] \).

With the above theorem, we can extend the correspondences in (2.1) to the one-to-one and onto correspondences between the set of Dynkin quivers, adapted commutation classes, Coxeter elements, AR-quivers and associated convex orders:

\[
Q \leftrightarrow [Q] \leftrightarrow \phi_Q \leftrightarrow \Gamma_Q \leftrightarrow <Q .
\]

On the other hand, relations between adapted commutation classes can be explained by reflection functors. The reflection functor \( r_i : [Q] \mapsto [Q]r_i \) for a sink \( i \) of \( [Q] \) can be understood by the map from \( \Gamma_Q \) to \( [Q] \Gamma_i \). The functor can be described using coordinates and the dual Coxeter number \( h^\vee \) as follows:

**Algorithm 2.9.** Let \( h^\vee \) be the dual Coxeter number associated to \( Q \) and \( \alpha_i \) (\( i \in I \)) be a sink of \( \Gamma_Q \).

(A1) Remove the vertex \((i,p)\) such that \( \Omega_Q(\alpha_i) = (i,p) \) and the arrows adjacent to \((i,p)\).

(A2) Add the vertex \((i^*, p - h^\vee)\) and the arrows to all \((j, p - h^\vee + 1) \in \Gamma_Q \) for \( j \) adjacent to \( i^* \) in \( \Delta \).

(A3) Label the vertex \((i^*, p - h^\vee)\) with \( \alpha_i \) and change the labels \( \beta \) to \( s_i(\beta) \) for all \( \beta \in \Gamma_Q \setminus \{\alpha_i\} \).

### 2.3. Labeling of (combinatorial) AR-quivers of type \( A_n \).

In this subsection, we briefly review results in \([24]\) and \([27]\) regarding labels of (combinatorial) AR-quivers.

Recall that, for every \( 1 \leq a \leq b \leq n \), \( \beta = \sum_{a \leq b} \alpha_k \) is a positive root in \( \Phi^+_A \) and every positive root in \( \Phi^+_A \) is of the form. Thus we frequently identify \( \beta \in \Phi^+ \) (and hence a vertex in \( \Upsilon_{[j_0]} \)) with the segment \([a,b] \). For \( \beta = [a,b] \), we say \( a \) is the first component of \( \beta \) and \( b \) is the second component of \( \beta \). If \( \beta \) is simple, we write \( \beta = [a] \).

**Proposition 2.10.** \([27, Proposition 4.5]\) Fix any class \([j_0]\) of \( w_0 \) of type \( A_n \). Let \( \rho \) be an \( N \)-path (resp. \( S \)-path) in \( \Upsilon_{[j_0]} \). Then every positive root contained in \( \rho \) has the same first (resp. second) component.
Theorem 2.11. [24, Corollary 1.12] Fix any Dynkin quiver $Q$ of type $A_n$. For $1 \leq i \leq n$, the AR-quiver $\Gamma_Q$ contains an $N$-path with $(n-i)$-arrows exactly once whose vertices share $i$ as the first component. At the same time, $\Gamma_Q$ contains an $S$-path with $(i-1)$-arrows exactly once whose vertices share $i$ as the second component.

With the above theorem, we can label the vertices of $\Gamma_Q$ without computation like (1.1).

3. Twisted Coxeter elements and twisted adapted $r$-cluster point(s)

From Section 3 to Section 6, we shall introduce new $r$-cluster points of type $A_{2n+1}, D_{n+1}, E_6, D_4$, called the (triply) twisted adapted $r$-cluster point(s) and show properties of classes in the (triply) twisted $r$-cluster point(s). In particular, in this section, we define the (triply) twisted adapted $r$-cluster point(s) using the following Dynkin diagram automorphisms $\nu$, which yield Dynkin diagrams of non-simply laced types:

\begin{align*}
\text{(3.1a)} & \quad \begin{array}{c}
\overset{2n}{1} \overset{2n+1}{2} \quad \ldots \quad \overset{n+1}{2n+1} \\
\overset{2n+1}{1} \overset{2n}{2} \quad \ldots \quad \overset{1}{n+1}
\end{array}
\begin{cases}
i^\nu = 2n + 2 - i & \text{if } i \leq n - 1, \\
i^\nu = n + 1 & \text{if } i = n, \\
i^\nu = n & \text{if } i = n + 1.
\end{cases}
\leftrightarrow B_{n+1}

\text{(3.1b)} & \quad \begin{array}{c}
\overset{n+1}{1} \overset{n}{2} \quad \ldots \quad \overset{1}{n+1}
\end{array}
\begin{cases}
i^\nu = i & \text{if } i \leq n - 1, \\
i^\nu = n + 1 & \text{if } i = n, \\
i^\nu = n & \text{if } i = n + 1.
\end{cases}
\leftrightarrow C_n

\text{(3.1c)} & \quad \begin{array}{c}
\overset{6}{1} \overset{3}{2} \overset{4}{3} \overset{5}{4}
\end{array}
\begin{cases}
1^\nu = 5, & 5^\nu = 1, \\
2^\nu = 4, & 4^\nu = 2, \\
3^\nu = 3, & 6^\nu = 6.
\end{cases}
\leftrightarrow F_4

\text{(3.1d)} & \quad \begin{array}{c}
\overset{3}{1} \overset{2}{2} \overset{4}{3}
\end{array}
\begin{cases}
1^\nu = 3, & 4^\nu = 1, \\
2^\nu = 2.
\end{cases}
\leftrightarrow G_2
\end{align*}

3.1. Twisted Coxeter elements. As we showed in Section 2, a Coxeter element has information about the corresponding adapted commutation class. In a similar sense, a twisted Coxeter element introduced in this subsection gives rise to a commutation class, which will be called a twisted adapted class. Note that a twisted Coxeter element is closely related to a twisted Dynkin quiver defined in Appendix B via the correspondence (B.3).

Remark 3.1. We can view the Weyl group $W$ as a subgroup of $GL(\mathbb{C} \Phi)$ generated by the set of simple reflections $\{s_i \mid i \in I\}$. In this case, we use the term “reduced expression” instead of “reduced word”. Moreover, we sometimes abuse the notation $i$ to represent reduced expressions.

Let $\sigma \in GL(\mathbb{C} \Phi)$ be a linear transformation of finite order which preserves a base $\Pi$ of $\Phi$. Then $\sigma$ preserves $\Phi$ itself and normalizes $W$ and so $W$ acts by conjugation on the coset $W\sigma$.

Definition 3.2.

(1) Let $\{\Pi_1, \ldots, \Pi_k\}$ be all orbits of $\Pi$ in $\Phi$ with respect to $\sigma$. For each $r \in \{1, \ldots, k\}$, choose $\alpha_i \in \Pi_r$ arbitrarily, and let $s_i \in W$ denote the corresponding reflection. Let $w$ be the product of $s_{i_1}, \ldots, s_{i_k}$ in any order. The element $w\sigma \in W\sigma$ thus obtained is called a $\sigma$-Coxeter element.

(2) For $\nu$ in (3.1a), (3.1b), (3.1c), $\nu$-Coxeter element is called a twisted Coxeter element.
(3) For \( v \) or \( v^2 \) in (3.1d), \( v \)-Coxeter element is called a \textit{triply twisted Coxeter element}.

**Example 3.3.** Take \( v \) in (3.1a) for \( A_5 \). There are 12 distinct twisted Coxeter elements:

\[
s_1s_2s_3v, s_2s_1s_3v, s_3s_1s_2v, s_3s_2s_1v, s_5s_2s_3v, s_3s_2s_5v, \\
s_1s_4s_3v, s_3s_1s_4v, s_4s_3s_2v, s_5s_3s_4v, s_3s_5s_4v, s_3s_4s_5v.
\]

**Example 3.4.** Take \( v \) in (3.1b) for \( D_4 \): There are 8 distinct twisted Coxeter elements:

\[
s_1s_2s_3v, s_1s_3s_2v, s_2s_1s_3v, s_3s_2s_1v, s_1s_4s_2v, s_2s_1s_4v, s_4s_1s_2v, s_4s_2s_1v.
\]

**Remark 3.5.** If there is no danger of confusion, we simply denote by \( i_1i_2\cdots i_kv \) the twisted Coxeter element \( s_{i_1}s_{i_2}\cdots s_{i_k}v \).

**Proposition 3.6.**

1. The number of twisted Coxeter elements of type \( A_{2n+1} \) associated to (3.1a) is \( 4 \times 3^{n-1} \).
2. The number of twisted Coxeter elements of type \( D_{n+1} \) associated to (3.1b) is \( 2^n \).
3. The number of twisted Coxeter elements of type \( E_6 \) associated to (3.1c) is 24.
4. The number of triply twisted Coxeter elements of type \( D_4 \) associated to (3.1d) is 12.

**Proof.** (1) Suppose the number of twisted Coxeter elements of type \( A_{2n+1} \) is \( 4 \times 3^{n-1} \). Then it is enough to show that a twisted Coxeter element of type \( A_{2n+1} \) induces three distinct twisted Coxeter elements of type \( A_{2n+3} \). Take a twisted Coxeter element \( s_{i_1}s_{i_2}\cdots s_{i_{n+1}}v \) of type \( A_{2n+1} \). If \( 1 \in \{i_1, i_2, \ldots, i_{n+1}\} \) then it induces three twisted Coxeter elements of type \( A_{2n+3} \):

\[
(2n+3)(i_1+1)(i_2+1)\cdots(i_{n+1}+1)v, (i_1+1)(i_2+1)\cdots(i_{n+1}+1)v, (i_1+1)(i_2+1)\cdots(i_{n+1}+1)v.
\]

Note that, since \( 2n+1 \notin \{i_1, i_2, \ldots, i_{n+1}\} \), \( 2n+3 \) commutes with \( (i_1+1), (i_2+1), \ldots, (i_{n+1}+1) \). Hence any twisted Coxeter element of the form \((i_1+1)(i_2+1)\cdots(k+1)(2n+3)(i_{k+1}+1)\cdots(i_{n+1}+1)v\) is the same as the first twisted Coxeter element in (3.2). On the other hand, observe that there is \( i_{k'}, k' = 1, 2, \ldots, n+1 \), such that \( i_{k'} = 1 \). Hence any twisted Coxeter element of the form \((i_1+1)(i_2+1)\cdots(i_{k'}+1)(i_{k'+1}+1)v\) for \( k < k' \) is the same as the second twisted Coxeter element in (3.2) and any twisted Coxeter element of the form \((i_1+1)(i_2+1)\cdots(i_{k-1}+1)(i_{k+1}+1)v\) for \( k > k' \) is the same as the third twisted Coxeter element in (3.2).

Otherwise, \( 2n+3 \in \{i_1, i_2, \ldots, i_{n+1}\} \) and it induces

\[
1(i_1+1)(i_2+1)\cdots(i_{n+1}+1)v, (2n+3)(i_1+1)(i_2+1)\cdots(i_{n+1}+1)v, (i_1+1)(i_2+1)\cdots(i_{n+1}+1)(2n+3)v.
\]

(2) Note that a twisted Coxeter element \( i_1i_2\cdots i_nv \) of type \( D_{n+1} \) has \( k \in \{1, 2, \ldots, n\} \) such that \( i_k = n \) or \( n+1 \). From the twisted Coxeter element, we get the Coxeter element of type \( A_n \)

\[
s_i s_{i_2} \cdots s_{i_n} v,
\]

replacing \( s_{i_k} \) by \( s_n \). Conversely, a Coxeter element \( s_{i_1}s_{i_2}\cdots s_{i_n} \) of type \( A_n \) with \( s_{i_k} = s_n \) induces two distinct twisted Coxeter elements of type \( D_{n+1} \)

\[
(i_1i_2\cdots i_{k-1}i_nv, i_1i_2\cdots(i_{k}+1)\cdots i_nv.
\]

Since we know the number of Coxeter elements of type \( A_n \) is the same as the number of Dynkin quiver, which is \( 2^{n-1} \), we proved (2).

The remaining cases can be checked directly. \( \square \)
3.2. (Triply) Twisted adapted r-cluster point \([\mathcal{O}]\) (resp. \([\mathcal{O}_2]\)). Now, we introduce a special r-cluster point, called the (triply) twisted r-cluster point, associated to a particular (triply) twisted Coxeter element and \(v\).

Note that, for each word \(j = j_1 \cdots j_s\) in \((I)\) and \(k \in \mathbb{Z}_{\geq 0}\), we denote
\[(j_1 \cdots j_s)^v := j_1^v \cdots j_s^v \text{ and } (j_1 \cdots j_s)^k v := ((j_1 \cdots j_s)^v)^{k \text{-times}}.\]

- A reduced expression associated to a twisted Coxeter element, type \(A_{2n+1}\) case: Let us fix the twisted Coxeter element \(1 2 \cdots n + 1\) of type \(A_{2n+1}\) and consider the related word \(i_0^j\) of \(W\) of type \(A_{2n+1}:\)
\[(3.4) \quad i_0^j = \prod_{k=0}^{2n}(1 2 3 \cdots (n + 1))^k v \quad \text{for } v \text{ in } (3.1a).\]

Note that the expression in (3.4) does not correspond to the one-line notation of symmetric group but the word in \((I)\).

**Proposition 3.7.** The word \(i_0^j\) in (3.4) is a reduced expression of \(w_0\) of type \(A_{2n+1}\) which is not adapted to any Dynkin quiver. Hence it can be denoted by \(i_0^j\) instead of \(i_0^j\).

**Proof.** Let us recall that the Weyl group of type \(A_{2n+1}\) is the symmetric group \(S_{2n+2}\) and \(w_0\) satisfies \(w_0(k) = 2n + 3 - k\) for \(k = 1, \ldots, 2n + 2\).

Denote \(t = s_1 s_2 \cdots s_n s_{n+1} s_2 \cdots s_n s_{n+1}\) and \(t = s_1 s_2 \cdots s_n s_{n+1}\). Then one can check that
\[
t = \begin{pmatrix} 1 & 2 & \ldots & n + 1 & n + 2 & n + 3 & \ldots & 2n + 1 & 2n + 2 \\
2 & 3 & \ldots & n + 2 & 1 & n + 3 & \ldots & 2n + 1 & 2n + 2
\end{pmatrix},
\]
\[
t = \begin{pmatrix} 1 & 2 & \ldots & n + 1 & n + 2 & n + 3 & \ldots & 2n + 1 & 2n + 2 \\
2 & 3 & \ldots & 2n + 2 & n + 2 & 1 & \ldots & 2n & 2n + 1
\end{pmatrix},
\]
with the two-line notation of symmetric group. Hence
\[
i_0^j = t^n t = \begin{pmatrix} 1 & 2 & \ldots & 2n + 1 & 2n + 2 \\
2n + 2 & 2n + 1 & \ldots & 2 & 1
\end{pmatrix},
\]
which is the same as \(w_0\). Note that \(i_0^j\) is reduced since the length of \(i_0^j\) is the same as \(|\Phi^+|\). \(\square\)

- A reduced expression associated to a twisted Coxeter element, type \(D_{n+1}\) case: As in the \(A_{2n+1}\) case, let us consider the twisted Coxeter element \(1 2 \cdots n v\) and the related word \(i_0^j\) of \(W\) of type \(D_{n+1}:\)
\[(3.5) \quad i_0^j = \prod_{k=0}^{n}(1 2 \cdots n)^k v \quad \text{for } v \text{ in } (3.1b).\]

**Proposition 3.8.** The word \(i_0^j\) in (3.5) is a reduced expression of \(w_0\) of type \(D_{n+1}\) which is not adapted to any Dynkin quiver. Hence it can be denoted by \(i_0^j\) instead of \(i_0^j\).

**Proof.** Recall that
\[
\Phi_{D_{n+1}} = \{ \varepsilon_{a_1} - \varepsilon_{a_2}, \varepsilon_{b_1} + \varepsilon_{b_2} \mid 1 \leq a_1 < a_2 \leq n + 1, 1 \leq b_1 < b_2 \leq n + 1 \}.
\]
We denote the positive roots by
\[(3.6) \quad \langle a_1, -a_2 \rangle, \langle b_1, b_2 \rangle, \]
respectively. By defining
\[
\beta_{p,q}^0 = \prod_{k=0}^{p-2} (s_1 s_2 \cdots s_n)^{k\nu} (s_1 s_2 \cdots s_{q-1})^{(p-1)\nu} (\alpha_{q(p-1)\nu}) \quad \text{for} \quad p \in \{1, \cdots, n+1\}, \quad q \in \{1, \cdots, n\},
\]
one can check that $\beta_{1,q}^0 = (1, -q-1)$, $\beta_{n+1,q}^0 = (q, n+1)$ and for $2 \leq p \leq n$
\[
\beta_{p,q}^0 = \begin{cases} 
(\langle p, -q-p \rangle) & \text{if } p+q \leq n+1, \\
(\langle p+q-n-1, p \rangle) & \text{if } p+q > n+1.
\end{cases}
\]
Since $\{\beta_{p,q}^0\} = \Phi^+$, the word $i_0^\sharp$ is a reduced expression of $w_0$. \square

- A reduced expression associated to a twisted Coxeter element, type $E_6$ case: Let us consider the twisted Coxeter element $1 2 6 3$ and the related word $i_0^\sharp$ of $W$ of type $E_6$:
\[(3.7) \quad i_0^\sharp = \prod_{k=0}^{8} (1 2 6 3)^{k\nu} \quad \text{for } \nu \text{ given in } (3.1c).\]
Then one can check the following proposition:

**Proposition 3.9.** The word $i_0^\sharp$ in (3.7) is a reduced expression of $w_0$ of type $E_6$ which is not adapted to any Dynkin quiver. Hence it can be denoted by $i_0^\flat$ instead of $i_0^\sharp$.

3.2.1. Twisted adapted $r$-cluster points.

**Definition 3.10.** Let $i_0^\flat$ be the reduced expression in (3.4), (3.5) or (3.7).

1. The $r$-cluster point $[[\mathcal{Q}]] := [[[i_0^\flat]]]$ is called the twisted adapted $r$-cluster point of type $A_{2n+1}$, $D_{n+1}$ or $E_6$.
2. A class $[i_0^\flat] \in [[[\mathcal{Q}]]]$ is called a twisted adapted class of type $A_{2n+1}$, $D_{n+1}$ or $E_6$.

**Remark 3.11.**

1. Regarding the notion of twisted adapted classes, we introduce twisted Dynkin quivers and adapted reduced expressions to a twisted Dynkin quiver, in Appendix B. For type $D_{n+1}$, a class in $[[\mathcal{Q}]]$ is adapted to a twisted Dynkin quiver. However, for type $A_{2n+1}$ and $E_6$ cases, there are classes in $[[\mathcal{Q}]]$ which are not adapted to a twisted Dynkin quiver (see Remark B.13.)
2. Recall that every commutation class associated to a Coxeter element belongs to the unique $r$-cluster point called the adapted $r$-cluster point. In Section 4, we show every commutation class associated to a twisted Coxeter element belongs to $[[\mathcal{Q}]]$. However, there exists a commutation class in $[[\mathcal{Q}]]$ of type $A_{2n+1}$ (resp. $E_6$) which is not related to a twisted Coxeter element.
3.2.2. Triply twisted adapted $r$-cluster points. For type $D_4$, we consider the following two words $i_0$ and $j_0$ of $W$ of type $D_4$:
\begin{equation}
(3.8) \qquad i_0^\dagger = \prod_{k=0}^{5}(2\ 1)^{v_k} \quad \text{and} \quad i_0^\ddagger = \prod_{k=0}^{5}(2\ 1)^{2k_v} \quad \text{for $v$ in (3.1d)}.
\end{equation}

Then one can check the following proposition:

**Proposition 3.12.** The words $i_0^\dagger$ and $i_0^\ddagger$ in (3.8) are reduced expressions of $w_0$ of type $D_4$. Hence we denote them by $i_0^\dagger$ and $i_0^\ddagger$ instead of $i_0^\dagger$ and $i_0^\ddagger$ which are not adapted to any Dynkin quiver.

**Definition 3.13.**

1. The $r$-cluster points $[[\Omega^\dagger]] := [[i_0^\dagger]]$ and $[[\Omega^\ddagger]] := [[i_0^\ddagger]]$ are called the **triply twisted adapted $r$-cluster points**.
2. A class $[i_0] \in [[\Omega]] := [[\Omega^\dagger]] \cup [[\Omega^\ddagger]]$ is called a **triply twisted adapted class**.

**Remark 3.14.** In Appendix B, we show a class $[i_0]$ of type $D_4$ is triply twisted adapted if and only if it is adapted to a triply twisted Dynkin quiver.

### 4. Characterizations of (triply) twisted adapted classes

As we saw in the previous section, we can construct the (triply) twisted adapted $r$-cluster point from a particular (triply) twisted Coxeter element. In this section, we shall show that we can construct $[[\Delta]]$ (resp. $[[\Omega]]$) from any (triply) twisted Coxeter element and count the number of twisted adapted classes in $[[\Delta]]$ (resp. $[[\Omega]]$).

Also, we introduce algorithms of finding the shapes of $\Upsilon_{[i_0]}$ for $[i_0] \in [[\Delta]]$ (cf. Proposition 2.6 for $\Gamma_Q$).

#### 4.1. Type $A_{2n+1}$

Consider the monoid homomorphism
\[ P : \langle I_{2n+1} \rangle \to \langle I_{2n} \rangle \quad \text{such that} \quad P(i) = \begin{cases} i & \text{if } 1 \leq i \leq n, \\ i-1 & \text{if } n+2 \leq i \leq 2n+1, \\ \text{id} & \text{if } i = n+1. \end{cases} \]

The following proposition can be proved by using the argument in the proof of Proposition 3.7.

**Lemma 4.1.** Recall $\hat{i}_0 = \prod_{k=0}^{2n}(1\ 2\ 3\cdots n+1)^{k_v}$ in (3.4).

1. $P(\hat{i}_0) = (1\ 2\ 3\cdots n\ 2n\ 2n-1\cdots n+1)^n(1\ 2\ 3\cdots n)$ is a reduced expression of the longest element $2n w_0$ of $A_{2n}$ and adapted to the Dynkin quiver which has only one source at the vertex $n+1$;
\begin{equation}
(4.1) \quad Q^\ddagger = \begin{array}{cccccccccc} \circ_1 & \circ_2 & \cdots & \circ_n & \circ_{n+1} & \circ_{n+2} & \cdots & \circ_{2n} \end{array}.
\end{equation}

2. For $i_0 \in [\hat{i}_0]$, we have $[P(i_0)] = [P(\hat{i}_0)]$.

For a word $i$ and $J \subset I$, we define a subword $i_{|J}$ of $i = i_1\cdots i_t$ as follows:
\[ i_{|J} := i_{t_1}\cdots i_{t_s} \quad \text{such that} \quad \begin{cases} i_{t_x} \in J \text{ and } 1 \leq t_x < t_y \leq l \text{ for all } 1 \leq x < y \leq s, \\ \text{if } t \notin \{t_1, t_2, \ldots, t_s\} \text{ then } i_t \notin J. \end{cases} \]
Lemma 4.2. For any \([i_0] \in [[\mathcal{Q}]]\), \([i_0]\) satisfies the following properties:

1. There is \(n+1\) between every adjacent \(n\) and \(n+2\) in \(i_0\).
2. Let \(J = \{n, n+1, n+2\} \in I_{2n+1}\). We have \(P(i_0, J) = (n+1)^n n\) or \((n+1)^n n + 1\).

Proof. We know the following facts:

(i) For \(i_0^0\) in (3.4), any reduced expression \(j_0\) in \([i_0]\) satisfies \(j_0, i_0 = (n+1)^n n + 1\).

(ii) \(n^\vee = n+2\), \((n+1)^\vee = n+1\) and \((n+2)^\vee = n\).

Hence \(i_{0, J}\) is one of the followings:

\[
\begin{align*}
(1) & \quad (n+1+n+2+n+1)^n(n+1), \\
(2) & \quad (n+1+n+2+n+1)(n+1+n+2) - (n+1+n+2)(n+1+n+1)^n, \\
(3) & \quad (n+1+n+1)(n+2+n+1+n+1)^n, \\
(4) & \quad (n+1+n+1)(n+2+n+1+n+1)^n.
\end{align*}
\]

We can check that every case in (4.2) satisfies (1) and (2). Hence our assertions follow. \(\square\)

Remark 4.3. In (4.2), one can observe that \(n+1\) is a sink or a source (but not both) for any \([i_0] \in [[\mathcal{Q}]]\).

Lemma 4.4.

1. If \(i_0', i_0'' \in [i_0] \in [[\mathcal{Q}]]\) then \([P(i_0')] = [P(i_0'')]\). Hence we can denote \(P([i_0]) := [P(i_0')]\).

2. For \([i_0] \in [[\mathcal{Q}]]\), we have \([P(i_0)] \in [[\Delta]]\) of type \(A_{2n}\).

Proof. (1) is similar to Lemma 4.1 (2).

(2) By (1), we can see that \(i \in I_{2n+1}\{n+1\}\) is a sink (resp. source) of \(i_0\) if and only if \(P(i) \in I_{2n}\) is a sink (resp. source) of \(P(i_0)\). Also, \(P(i^\vee) = (P(i))^\vee\). Hence If we regard \(r_{id}\) as the identity map then

\[
P([i_0] \cdot r_i) = [P(i_0)] \cdot r_{P(i)}.
\]

In Lemma 4.1, we showed \(P([i_0^b])\) is adapted to the quiver \(Q^b\) of \(A_{2n}\) in (4.1). Since all adapted reduced expressions consist of \([[\Delta]]\), we proved (2). \(\square\)

Example 4.5. For the twisted adapted reduced expression \(i_0^b = 123543123543123\) of type \(A_5\), we have

\[
P([i_0^b]) = 1 \ 2 \ 4 \ 3 \ 1 \ 2 \ 4 \ 3 \ 1 \ 2
\]

which is a reduced expression of \(4w_0\) and adapted to \(Q = \overset{1}{\circ} \overset{2}{\circ} \overset{3}{\circ} \overset{4}{\circ} \).

By Lemma 4.4, if we restrict \(P\) to a map on reduced expressions in \([[\Delta]]\) then the map can be considered as a map between classes in \([[\mathcal{Q}]]\) of \(A_{2n+1}\) and \([[\Delta]]\) of \(A_{2n}\). Hence we use the following notation.

Definition 4.6. The map from \([[\mathcal{Q}]]\) of type \(A_{2n+1}\) to \([[\Delta]]\) of type \(A_{2n}\), induced from \(P\), is denoted by

\[
P_{\mathcal{Q}} : [[\mathcal{Q}]] \rightarrow [[\Delta]], \quad [i_0] \mapsto [P(i_0)] = P_{\mathcal{Q}}([i_0]).
\]
Lemma 4.7. For a given Dynkin quiver $Q$ of $A_{2n}$, there are at least two distinct classes $[i_0'], [i_0''] \in \mathbb{Q}$ such that $P_{\mathbb{Q}}([i_0]) = P_{\mathbb{Q}}([i_0'']) = [Q]$. 

Proof. Consider the monoid homomorphism 

$$R : \langle I_{2n} \rangle \to \langle I_{2n+1} \rangle \text{ such that } i \mapsto \begin{cases} 
  i & \text{if } i = 1, \ldots, n - 1, \\
  i + 1 & \text{if } i = n + 2, \ldots, 2n, \\
  n + 1 & \text{if } i = n, \\
  n + 2 & \text{if } i = n + 1.
\end{cases}$$

Then 

(i) $P \circ R = \text{id}$ and $[R(i)R(j)] = [R(j)R(i)]$ for $i, j$ such that $|i - j| \geq 2$, 

(ii) if $R(i)i$ is a reduced expression of $w_0$ then $[R(i)i] \cdot r_{R(i)} = [iR(i')].$

Let us consider the reduced expression $2n_i0 = (1 \ldots n 2n - 1 \ldots n + 1)\ldots(1 2 \ldots n)$ of the longest element $2n_i0$ of $A_{2n}$. We shall show the following statement. For a class of reduced expression

\begin{equation}
[2n_i0] = [2n_i0] \cdot r_1 = [i_1i_2\cdots i_1], \quad (i : \text{word})
\end{equation}

of $2n_i0$, we can induce a reduced expression $[i_0']$ of $w_0$ such that

\begin{equation}
[i_0'] = [i_0] \cdot r_{R(i)} = [R(2n_i0)] = [R(i_1\cdots i_1)].
\end{equation}

Now, we use an induction on the length of $i$. We know $i_00 = R(2n_i0)$ is a reduced expression of $w_0$. Suppose (4.4) is true for $i$ with length $\ell$ and take the length $\ell + 1$ word $i' = i i$ for a sink $i = i_1$ of $2n_i0$. Then, by (i) and (ii), there is a reduced expression in $[i_0']$ which starts with $R(i)$. Moreover, we have

$[i_0'] \cdot r_{R(i)} = [R(i_1\cdots i_1) R(i_1')] = [R(i_1\cdots i_1 i_1')].$

In other words, (4.4) is true for any word $i$ with length $\ell + 1$. As a conclusion, for any word $i$ consisting of $\{1, \ldots, 2n\}$,

$[i_0] \cdot r_{R(i)} = [R(i_1\cdots i_1)]$ satisfies $P_{\mathbb{Q}}([i_0]) \cdot r_{R(i)} = [i_1\cdots i_1] = [2n_i0] \cdot r_1.$

Since every reduced expression adapted to a quiver consists of one $r$-cluster point, we proved that $P_{\mathbb{Q}}$ is an onto map.

In addition, since a class of reduced expression $[i_0]$ of $w_0$ in the image of $R$ has $n + 1$ as a source, $r_{n+1} \cdot [i_0]$ has $n + 1$ as a sink and these two are distinct classes. The following equation is easy to check:

$P_{\mathbb{Q}}([i_0']) = P_{\mathbb{Q}}(r_{n+1} \cdot [i_0']).$

Hence we proved the lemma.

Lemma 4.8. The map $P_{\mathbb{Q}}$ is a two-to-one and onto map. More precisely, for each Dynkin quiver $Q$ of $A_{2n}$, there is a unique class $[i_0]$ of reduced expressions in $\mathbb{Q}$ satisfying $P_{\mathbb{Q}}([i_0]) = [Q]$ which has $n + 1$ as a source (resp. sink).

Proof. Suppose $i_0'$ and $i_0''$ are two distinct reduced expressions in $\mathbb{Q}$ such that

(i) both $[i_0']$ and $[i_0'']$ have $n + 1$ as a source,

(ii) $P_{\mathbb{Q}}([i_0']) = P_{\mathbb{Q}}([i_0'']) = [Q]$ for a quiver $Q$ of type $A_{2n}$.
As we saw in Lemma 4.2, we have
\[ i'_{0,j} = i''_{0,j} = (R(n)R(n+1))^n R(n) \text{ or } (R(n+1)R(n))^n R(n+1) \]
for \( J = \{n, n+1, n+2\} \). Since \( s_{n+1}s_j = s_js_{n+1} \) for any \( j \in I_{2n+1} \setminus J \), using these commutation relations, we can assume that every \( n+1 \) exists right after \( n \) or \( n+2 \) in \( i'_0 \) and \( i''_0 \), that is \( i'_0 \) and \( i''_0 \) appear as images of \( R \). Also, by letting \( s_{R(n)} = s_n s_{n+1} \) and \( s_{R(n+1)} = s_{n+2} s_{n+1} \), we can check that \( R \) preserves commutation relations. Hence the assumption \( P_{\mathcal{D}}([i'_0]) = P_{\mathcal{D}}([i''_0]) = [Q] \) implies \([i'_0] = [i''_0]\), since \( P \circ R = \text{id} \). In other words, there is a unique class \([i'_0]\) of reduced expressions in \([\mathcal{D}]\) satisfying
\[ P_{\mathcal{D}}([i'_0]) = [Q] \]
which has \( n+1 \) as a source.

Similarly, if \([i'_0]\) and \([i''_0]\) such that \( P_{\mathcal{D}}([i'_0]) = P_{\mathcal{D}}([i''_0]) = [Q] \) do not satisfy (i) but (i') below
(i') both \([i'_0]\) and \([i''_0]\) have \( n+1 \) as a sink,
then we can show \([i'_0] = [i''_0]\). Hence there is a unique class \([i'_0]\) of reduced expressions in \([\mathcal{D}]\) satisfying
\[ P_{\mathcal{D}}([i'_0]) = [Q] \]
which has \( n+1 \) as a sink (resp. source).

Recall that we showed \( P_{\mathcal{D}} \) is an onto map in Lemma 4.7. So we proved the lemma. \( \square \)

**Theorem 4.9.** The number of classes in \([\mathcal{D}]\) is \( 2^{2n} \).

**Proof.** Since there are \((2n-1)\)-many arrows in a Dynkin quiver of \( A_{2n} \) and each arrow has two possible directions, the number of Dynkin quivers is \( 2^{2n-1} \). By Lemma 4.8, the number of classes in \([\mathcal{D}]\) is \( 2 \times 2^{2n-1} = 2^{2n} \). \( \square \)

Now we focus on classes of reduced expressions related to twisted Coxeter elements. Consider \( \vee: i \mapsto 2n+1-i \) for \( i \in I_{2n} \) and let \( i_1 \cdots i_n \vee \) be a twisted Coxeter element of \( W_{2n} \). Then it is well-known that \( i_1 i_2 \cdots i_n \) adapted to \( Q \) is a Coxeter element of \( W_{2n} \) and hence there exist a unique \( Q \) of type \( A_{2n} \) such that \( i_1 i_2 \cdots i_n \) is adapted to \( Q \). Moreover, one can prove the following lemma by using induction on \( n \).

**Lemma 4.10.** For a twisted Coxeter element \( i_1 i_2 \cdots i_n \vee \) of \( W_{2n} \), we have
\[ (i) \quad 2n \prod_{k=0}^{2n} (i_1 i_2 i_3 \cdots i_n)^{k \vee} \]
is a reduced expression of \( 2n \) adapted to a Dynkin quiver \( Q' \) of type \( A_{2n} \).

**Theorem 4.11.** Let \( i_1 i_2 \cdots i_{n+1} \vee \) be a twisted Coxeter element of \( A_{2n+1} \). Then
\[ (i_0) \quad \prod_{k=0}^{2n} (i_1 i_2 \cdots i_{n+1})^{k \vee} \]
is a reduced expression of \( w_0 \) and \([i_0] \in [\mathcal{D}]\).

**Proof.** Any twisted Coxeter element \( i_1 i_2 \cdots i_{n+1} \vee \) satisfies only one of followings:
(i) \( (i_1 i_2 \cdots i_{n+1} (n+1)) \) is not reduced, \( (n+1) (i_1 i_2 \cdots i_{n+1}) \) is not reduced.
Note that, for \( k_1 \) and \( k_2 \) such that \( i_{k_1} = n \) or \( n+2 \) and \( i_{k_2} = n+1 \), if \( k_1 < k_2 \) then the twisted Coxeter element is in the case (i) and if \( k_1 > k_2 \) then it is in the case (ii).
For the case (i), we can assume that $i_{n+1} = n+1$. Then our assertion follows from the facts that
\begin{itemize}
  \item $[P(\prod(i_1 \ i_2 \ \ldots \ i_{n+1}))^{k\nu}] \in \llbracket \Delta \rrbracket$ is a reduced expression of type $A_{2n}$ by Lemma 4.10,
  \item $[i_0] = [R \circ P(\prod(i_1 \ i_2 \ \ldots \ i_{n+1})^{k\nu})] \in \llbracket \mathcal{Q} \rrbracket$ is a reduced expression of type $A_{2n+1}$ by the proof of Lemma 4.7.
\end{itemize}

The case (ii) can be proved in the similar way. $\square$

Remark 4.12. Note that there are only $4 \times 3^{n-1}$-many twisted Coxeter elements of type $A_{2n+1}$ while there are $2^{2n}$-many twisted adapted classes. Thus there are twisted adapted classes which are not associated to any twisted Coxeter element in the sense of Theorem 4.11.

Now, using the properties above, we shall derive an algorithm (Algorithm 4.15) to get the combinatorial AR-quiver associated to a twisted adapted class. Recall that, by Lemma 4.8, we have two distinct twisted adapted classes in $\llbracket \mathcal{Q} \rrbracket$ which are obtained from each $Q$ of type $A_{2n}$. In other words, $P_{[\mathcal{Q}]}^{-1}(Q)$ consists of two commutation classes denoted by

\begin{equation}
[Q^\circ] \quad \text{if } n+1 \text{ is a source of the class},
\end{equation}

\begin{equation}
[Q^\circ] \quad \text{if } n+1 \text{ is a sink of the class}.
\end{equation}

The procedure of finding $i'_0$ in $[Q^\circ]$ or $[Q^\circ]$ is as follows.

(i) Take any $2n i'_0 \in [Q]$.

(ii) Substitute $i \in \{n+1, \ldots, 2n\}$ in $2n i'_0$ by $i^+ = i + 1$.

(iii) Between each adjacent $n$ and $n+2$, insert $n+1$.

(iv) Insert another $n+1$ at the end or at the beginning (not both) of the sequence obtained in (iii).

Example 4.13. Consider the Dynkin quiver $Q$ of $A_6$:

\[
Q = \begin{array}{cccccccc}
    & 1 & \scriptstyle \circ & \scriptstyle \circ & 3 & \scriptstyle \circ & 4 & \scriptstyle \circ & 5 & \scriptstyle \circ & 6 \\
\end{array}
\]

The commutation class adapted to $Q$ is

\[
[Q] = \prod_{k=0}^{6} (5 \ 4 \ 6)^{k\nu}.
\]

Then one can compute that

\[
P_{[\mathcal{Q}]}^{-1}([Q]) = \left\{ [Q^\circ] := \prod_{k=0}^{6} (6 \ 5 \ 7 \ 4)^{k\nu}, \ [Q^\circ] := \prod_{k=0}^{6} (4 \ 6 \ 5 \ 7)^{k\nu} \right\},
\]

where the elements are commutation classes associated to Coxeter elements $4 \ 6 \ 5 \ 7\nu$ and $6 \ 5 \ 4 \ 7\nu$ in the sense of Theorem 4.11. The AR-quiver $\Gamma_Q$ and combinatorial AR-quivers $\Upsilon_{[Q^\circ]}, \Upsilon_{[Q^\circ]}$ can be depicted as follows:
Example 4.14. Consider the Dynkin quiver $Q = \circ_1 \circ_2 \circ_3 \circ_4 \circ_5 \circ_6$ of $A_6$. The commutation class adapted to $Q$ is

$$[Q] = [5 6 4 3 2 1 5 6 4 3 2 1 5 6 4 3 5 6 4 5 6].$$

Then $P_{[Q]}^{-1}([Q])$ is

$$\begin{align*}
\{ [Q^0] & := [6 7 4 5 4 3 2 1 6 7 4 5 4 3 2 1 6 7 4 5 4 3 6 7 4 5 6 7], \\
\{ Q^2 & := [6 7 5 4 3 4 2 1 6 7 5 4 3 4 2 1 6 7 5 4 3 4 6 7 5 4 6 7]\}. \end{align*}$$

These two classes of reduced expressions are not associated to any twisted Coxeter element. The AR-quiver $\Gamma_Q$ and combinatorial AR-quivers related to $P_{[Q]}^{-1}([Q])$ can be depicted as follows:

Algorithm 4.15. As we can see in Example 4.13 and Example 4.14, the combinatorial AR-quiver $\Upsilon_{[i_0]}$ for $[i_0] \in [Q]$ of type $A_{2n+1}$ can be constructed from the AR-quiver $\Gamma_Q$ for $[Q] : = P_{[Q]}^{-1}([i_0])$ of type $A_{2n}$ by the following surgery (see Example 4.13 Example 4.14).

1. we put a vertex on every arrow between a residue $n$ vertex and a residue $n+1$ vertex,
2. since vertices obtained in (1) have residue $n+1$, the original residue $m$ for $m \geq n+1$ should be renamed by the residue $m+1$,
3. break every arrow with a new vertex added in (1) into two arrows: one is from the residue $n$ vertex to the residue $n+1$ vertex and the other one is from the residue $n+1$ vertex to the residue $n+2$ vertex (resp. one is from the residue $n+2$ vertex to the residue $n+1$ vertex and the other one is from the residue $n+1$ vertex to the residue $n$ vertex),
(4) if $\alpha_{n+1}$ is a source (resp. sink) in $\Upsilon_{[\mathfrak{i}_0]}$, add a new vertex at residue $n+1$ and an arrow to make the new vertex as a source (resp. sink) of $\Upsilon_{[\mathfrak{i}_0]}$.

**Definition 4.16.** For $[\mathfrak{i}_0] \in \mathcal{A}$ such that $P([\mathfrak{i}_0]) = [Q]$, we denote by $\Gamma_Q \cap \Upsilon_{[\mathfrak{i}_0]}$ the set of all vertices in $\Upsilon_{[\mathfrak{i}_0]}$ whose residues are contained in $I_{2n+1} \setminus \{ n + 1 \}$ and by $\Upsilon_{[\mathfrak{i}_0]} \setminus \Gamma_Q$ the set of all vertices in $\Upsilon_{[\mathfrak{i}_0]}$ whose residues are $n + 1$.

**Remark 4.17.** By Algorithm 4.15, we sometimes identify a vertex in $\Gamma_Q \cap \Upsilon_{[\mathfrak{i}_0]}$ with a vertex in $\Gamma_Q$ also, and call it *induced*. Also, we call a subquiver $\rho$ in $\Upsilon_{[\mathfrak{i}_0]}$ *induced* if it contains a vertex in $\Gamma_Q \cap \Upsilon_{[\mathfrak{i}_0]}$ and a subquiver $\rho$ in $\Upsilon_{[\mathfrak{i}_0]}$ *totally induced* if all vertices $\rho$ are contained in $\Gamma_Q \cap \Upsilon_{[\mathfrak{i}_0]}$.

### 4.2. Type $D_{n+1}$

For $\mathfrak{v}$ in (3.1b), consider the map

$$p_{A_n}^{D_{n+1}} : \{ \text{twisted Coxeter elements of } D_{n+1} \} \to \{ \text{Coxeter elements of } A_n \}$$

such that $i_1 i_2 \cdots i_n \mathfrak{v} \mapsto \left\{ \begin{array}{ll} i_1 i_2 \cdots i_n & \text{if } i_t = n, \\ (i_1 i_2 \cdots i_n)^{\mathfrak{v}} & \text{if } i_t = n + 1, \end{array} \right.$ for $t$ satisfying $i_t \in \{ n, n + 1 \}$.

**Proposition 4.18.** The map $p_{A_n}^{D_{n+1}}$ is a two-to-one and onto map.

*Proof.* Suppose $i_1 i_2 \cdots i_n$ can be considered as a Coxeter element of $A_n$. Then both $[i_1 i_2 \cdots i_n]\mathfrak{v}$ and $[(i_1 i_2 \cdots i_n)^{\mathfrak{v}}]$ are twisted Coxeter elements of $D_{n+1}$. Thus our assertion follows. □

Recall that there is an involution $\ast$ on the index set $I_{n+1}^D$ such that $w_0(\alpha_i) = -\alpha_i \ast$. If $n + 1$ is odd then $\ast : i \mapsto \left\{ \begin{array}{ll} i & \text{if } i = n, n + 1, \\ i + (-1)^{\delta_{n+1,i}} & \text{if } i = n, n + 1. \end{array} \right.$ On the other hand, if $n + 1$ is even then $i^{\ast} = i$, for $i \in I$.

**Proposition 4.19.** Let $[\mathfrak{i}_0]$ be a twisted adapted class of type $D_{n+1}$. Then there is a twisted Coxeter element $i_1 i_2 \cdots i_n \mathfrak{v}$ such that

$$[\mathfrak{i}_0] = \prod_{k=0}^n (i_1 i_2 \cdots i_n)^{\mathfrak{v}}.$$

*Proof.* Suppose $[\mathfrak{i}_0']$ is a class of reduced expressions of $w_0$ such that $[\mathfrak{i}_0'] = \left[ \prod_{k=0}^n (i'_1 i'_2 \cdots i'_n)^{\mathfrak{v}} \right]$ for a twisted Coxeter element $i'_1 i'_2 \cdots i'_n \mathfrak{v}$. Then

(i) If $i$ is a sink of $[\mathfrak{i}_0']$ then there is a reduced expression $j = j_1 j_2 \cdots j_n$ such that

$$[j_1 j_2 \cdots j_n] = [i'_1 i'_2 \cdots i'_n], \quad j_1 = i \quad \text{and} \quad [\mathfrak{i}_0] = \left[ \prod_{k=0}^n (j_1 j_2 \cdots j_n)^{\mathfrak{v}} \right].$$

(ii) $[\mathfrak{i}_0''] = [\mathfrak{i}_0'] r_{i'_1}$ has the form of

$$[\mathfrak{i}_0''] = \left[ \prod_{k=0}^n (i'_2 i'_3 \cdots i'_n i'_1)^{\mathfrak{v}} \right]$$

and $i'_2 i'_3 \cdots i'_n i'_1^{\mathfrak{v}}$ is a twisted Coxeter element.

Since $[\mathfrak{i}_0] = \left[ \prod_{k=0}^n (1 2 \cdots n)^{\mathfrak{v}} \right] \in \mathcal{A}$, for any $[\mathfrak{i}_0] \in \mathcal{A}$, there is a word $\mathfrak{w}$ such that $[\mathfrak{i}_0] = [\mathfrak{i}_0'] r_{\mathfrak{w}}$. Hence $[\mathfrak{i}_0] = \left[ \prod_{k=0}^n (i_1 i_2 \cdots i_n)^{\mathfrak{v}} \right]$ for a twisted Coxeter element $i_1 i_2 \cdots i_n \mathfrak{v}$. □
Proposition 4.20. For a twisted Coxeter element \(i_1i_2\cdots i_n\) of \(D_{n+1}\), the word
\[
\prod_{k=0}^{n}(i_1i_2\cdots i_n)^{k\nu}
\]
is a reduced expression of \(w_0\).

Proof. Recall that if \(\phi_Q' = i_1'i_2'\cdots i_n'\) is the Coxeter element associated to the Dynkin quiver \(Q'\) of type \(A_n\) then \(\phi_{Q''} = i_2'i_3'\cdots i_1'\) is the Coxeter element associated to the Dynkin quiver \(Q''\) where \(r_{i_1'}Q' = Q''\). Hence if \(n'i_0'\) is the reduced expression of type \(A_n\) associated to the Coxeter elements \(i_1'i_2'\cdots i_n'\), then \(n'i_0'A\) such that \([n'i_0'A] = [n'i_0A]r_{i_1}\) is associated to \(\phi_{Q''}\).

On the other hand, in the proof of Proposition 4.19, we noted that if \([i_0']\) of type \(D_{n+1}\) associated to twisted Coxeter elements \(i_1'i_2'\cdots i_n'\) then \(n'i_0'A\) where \([n'i_0A] = [n'i_0']\) is associated to \(i_1'i_2'\cdots i_n'\) by switching \(n\) and \(n+1\), by induction, we have the following statement.

For \([n'i_0A] \in [\Delta]\) associated to \(\phi_Q = 1\cdots n\), let a word \(w \in \langle I_n \rangle\) have the property
\[
[n'i_0A]w = [n'i_0A]\text{ where } n'i_0A\text{ is associated to } i_1'i_2'\cdots i_n'.
\]
Then we have \(w_{n+1} \in \langle I_{n+1}\rangle\) such that

- We get \(w\) from \(w_{n+1}\) by the map \(\langle I_{n+1}\rangle \to \langle I_n \rangle\) such that \(i \mapsto i\) for \(i \in I_n\) and \(n+1 \mapsto n\).
- For classes in \([\mathcal{D}]\) of type \(D_{n+1}\),
\[
[i_0']w_{n+1} = [\prod_{k=0}^{n} (1\cdots n)^{k\nu}]r_{\mathcal{D}_{n+1}} = [\prod_{k=0}^{n} (j_1'j_2'\cdots j_n')^{k\nu}]
\]
where \(p_{D_{n+1}}^{D_{n+1}}(j_1'j_2'\cdots j_n') = i_1'i_2'\cdots i_n'\) and
\[
[p_{D_{n+1}}^{D_{n+1}}(j_1'j_2'\cdots j_n')]r_{\mathcal{D}_{n+1}} = [\prod_{k=0}^{n} (j_1''j_2''\cdots j_n'')^{k\nu}].
\]

Now, since \(p_{A_n}^{A_n}\) is a surjective two-to-one map such that \((p_{A_n}^{A_n})^{-1}(i_1'i_2'\cdots i_n') = \{j_1''j_2''\cdots j_n''\nu, j_1''j_2''\cdots j_n''\nu\}\), we conclude that
\begin{enumerate}[(a)]
  \item the word \(\prod_{k=0}^{n} (i_1'i_2'\cdots i_n')^{k\nu}\) associated to the twisted Coxeter element \(i_1'i_2'\cdots i_n'\nu\) is obtained by applying a reflection functor to \([i_0']\).
  \item by (i), \(\prod_{k=0}^{n} (i_1'i_2'\cdots i_n')^{k\nu}\) is a reduced expression of \(w_0\).
\end{enumerate}

\[\Box\]

Theorem 4.21. The twisted adapted \(r\)-cluster point \([\mathcal{D}]\) of type \(D_{n+1}\) consists of the classes of the form
\[
[\prod_{k=0}^{n} (i_1i_2\cdots i_n)^{k\nu}],
\]
where \(i_1'i_2'\cdots i_n'\nu\) is a twisted Coxeter element of type \(D_{n+1}\). In addition, the number of classes in the twisted adapted \(r\)-cluster point \([\mathcal{D}]\) of type \(D_{n+1}\) is \(2^n\).

Proof. The first assertion follows from Proposition 4.19 and 4.20. Also, since distinct twisted Coxeter elements of type \(D_{n+1}\) give rise to reduced expressions with distinct combinatorial AR-quivers, the number of classes in \([\mathcal{D}]\) of type \(D_{n+1}\) is the same as the number of twisted Coxeter elements of type \(D_{n+1}\). Hence the number of classes in \([\mathcal{D}]\) of type \(D_{n+1}\) is \(2^n\). \[\Box\]
Remark 4.22. Analogous to the case of adapted commutation classes, there is the natural one-to-one correspondence between the set of twisted Coxeter elements and twisted commutation classes of type $D_{n+1}$. It follows by the fact that the number of elements in both sets is $2^n$.

By Proposition 4.20 and Proposition 4.19, we can consider $p^{D_{n+1}}_{A_n}$ as a two-to-one onto map between twisted adapted classes of type $D_{n+1}$ and adapted classes of type $A_n$, i.e., $[[\mathcal{Q}]] \rightarrow [[\Delta]]$. Thus, from now on, we use the notation

$$
(4.9) \quad p^{D_{n+1}}_{A_n} : [[\mathcal{Q}]] \rightarrow [[\Delta]]
$$

for the map between commutation classes.

Example 4.23. For $[i_0] = [\prod_{k=0}^{12} (2135)^{k'}] \in [[\mathcal{Q}]]$, one can check that $p^{D_{n+1}}_{A_n}([i_0]) = [Q]$ where

$$
Q = \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \quad \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array}
$$

The combinatorial AR-quiver $\Upsilon_{[i_0]}$ can be depicted as follows:

By Remark 2.7 and Proposition 4.19, $\Upsilon_{[i_0]}$ for $[i'_0] \in [[\mathcal{Q}]]$ can be considered as concatenation of $\Gamma_{Q^*}$ and $\Gamma_Q$ for $Q$ of type $A_n$, where $p^{D_{n+1}}_{A_n}([i'_0]) = [Q]$.

Remark 4.24. In the above example, $\Gamma_Q$ is isomorphic to the full subquiver of $\Upsilon_{[i_0]}$ consisting of $\star$'s and $\Gamma_{Q^*}$ is isomorphic to the full subquiver of $\Upsilon_{[i_0]}$ consisting of $\star$'s:

$$
\Gamma_{Q^*} = \begin{array}{c}
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\end{array} \quad \Gamma_Q.
$$

Remark 4.25. As quivers, two AR-quivers $\Gamma_{Q^*}$ and $\Gamma_Q$ are isomorphic to each other. More explicitly, $\Gamma_{Q^*} \cong \Gamma_Q$ by the map which relates a vertex in $\Gamma_{Q^*}$ with residue $i$ to a vertex in $\Gamma_Q$ with residue $i^* = n + 1 - i$. This fact will be used crucially in Section 6.2.

We can derive an algorithm of finding twisted adapted AR-quiver of type $D_{n+1}$ by Remark 4.24.

Algorithm 4.26. We can draw $\Upsilon_{[i_0]}$ for $[i_0] \in [[\mathcal{Q}]]$ from the AR-quivers $\Gamma_{Q^*}$ and $\Gamma_Q$ of type $A_n$ as follows:

1. Draw $\Gamma_{Q^*}$ and $\Gamma_Q$ and juxtapose them.
2. Draw arrows from rightmost vertices in $\Gamma_{Q^*}$ to leftmost vertices in $\Gamma_Q$ if their residues are adjacent to each other in $\Delta_{A_n}$.
3. Change residues of vertices from $n$ to $n + 1$ if they correspond to $n + 1$ in $i_0$. 

By Proposition 4.24 and Proposition 4.19, we can consider $p^{D_{n+1}}_{A_n}$ as a two-to-one onto map between twisted adapted classes of type $D_{n+1}$ and adapted classes of type $A_n$, i.e., $[[\mathcal{Q}]] \rightarrow [[\Delta]]$. Thus, from now on, we use the notation

$$
(4.9) \quad p^{D_{n+1}}_{A_n} : [[\mathcal{Q}]] \rightarrow [[\Delta]]
$$

for the map between commutation classes.

Example 4.23. For $[i_0] = [\prod_{k=0}^{12} (2135)^{k'}] \in [[\mathcal{Q}]]$, one can check that $p^{D_{n+1}}_{A_n}([i_0]) = [Q]$ where

$$
Q = \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \quad \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array}
$$

The combinatorial AR-quiver $\Upsilon_{[i_0]}$ can be depicted as follows:

By Remark 2.7 and Proposition 4.19, $\Upsilon_{[i_0]}$ for $[i'_0] \in [[\mathcal{Q}]]$ can be considered as concatenation of $\Gamma_{Q^*}$ and $\Gamma_Q$ for $Q$ of type $A_n$, where $p^{D_{n+1}}_{A_n}([i'_0]) = [Q]$.

Remark 4.24. In the above example, $\Gamma_Q$ is isomorphic to the full subquiver of $\Upsilon_{[i_0]}$ consisting of $\star$'s and $\Gamma_{Q^*}$ is isomorphic to the full subquiver of $\Upsilon_{[i_0]}$ consisting of $\star$'s:

$$
\Gamma_{Q^*} = \begin{array}{c}
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\end{array} \quad \Gamma_Q.
$$

Remark 4.25. As quivers, two AR-quivers $\Gamma_{Q^*}$ and $\Gamma_Q$ are isomorphic to each other. More explicitly, $\Gamma_{Q^*} \cong \Gamma_Q$ by the map which relates a vertex in $\Gamma_{Q^*}$ with residue $i$ to a vertex in $\Gamma_Q$ with residue $i^* = n + 1 - i$. This fact will be used crucially in Section 6.2.

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Algorithm 4.26. We can draw $\Upsilon_{[i_0]}$ for $[i_0] \in [[\mathcal{Q}]]$ from the AR-quivers $\Gamma_{Q^*}$ and $\Gamma_Q$ of type $A_n$ as follows:

1. Draw $\Gamma_{Q^*}$ and $\Gamma_Q$ and juxtapose them.
2. Draw arrows from rightmost vertices in $\Gamma_{Q^*}$ to leftmost vertices in $\Gamma_Q$ if their residues are adjacent to each other in $\Delta_{A_n}$.
3. Change residues of vertices from $n$ to $n + 1$ if they correspond to $n + 1$ in $i_0$. 

Example 4.27. The two to one and onto map $\mathfrak{p}^{D_{n+1}}_{A_n}$ from $[[\mathcal{Q}]]$ of type $D_{n+1}$ to $[[\Delta]]$ of type $A_n$ can be described as follows: For $Q = \circ \xrightarrow{1} \circ \xrightarrow{2} \circ \xrightarrow{3} \circ \xrightarrow{4}$, we can construct a quiver $\Gamma_Q \cup \Gamma_Q^*$ concatenating $\Gamma_Q^*$ and $\Gamma_Q$ (see Algorithm 4.26):

$$\Gamma_Q \cup \Gamma_Q^* = \begin{array}{cccc}
1 & & & \\
2 & & & \\
3 & & & \\
4 & & & \\
5 & & & \\
\end{array}$$

Then we have two distinct combinatorial AR-quivers in $(\mathfrak{p}^{D_{n+1}}_{A_n})^{-1}([Q])$:

by assigning residues $n$ and $n+1$ (resp. $n+1$ and $n$) to vertices in the last row of $\Gamma_Q \cup \Gamma_Q^*$ alternatingly from the right.

4.3. Type $E_6$. Recall that

$$(4.10) \quad \mathfrak{i}_6 = \prod_{k=0}^{8} (1 \ 2 \ 6 \ 3)^{k \nu}.$$ 

By using the example below and reflection functors, we can check that there are 32 distinct twisted adapted classes in $[[\mathcal{Q}]] = [[\mathfrak{i}_6]]$ while there are only 24 distinct twisted Coxeter elements. Here the number 32 coincides with the number of distinct Dynkin quivers of type $E_6$.

Example 4.28. The combinatorial AR-quiver $\mathcal{Q}_{[\mathfrak{i}_6]}$ can be drawn as follows:

\begin{align*}
(a_1a_2a_3a_4a_5a_6) &= \sum_{i=1}^{6} a_i \alpha_i
\end{align*}
4.4. Triply twisted adapted classes of type $D_4$. Recall that
\begin{equation}
\mathcal{t}^+_0 = \prod_{k=0}^5 (2) \quad \text{and} \quad \mathcal{t}^-_0 = \prod_{k=0}^5 (2)^{2k}.
\end{equation}

Each triply twisted adapted class $[\mathcal{t}_0^+] \in [\Omega]$ consists of a unique reduced expression and there are 6 distinct twisted adapted classes in each triply twisted adapted r-cluster point. Recall 12

is the number of distinct triply twisted Coxeter elements.

Example 4.29. The combinatorial AR-quiver $\Upsilon_{[i_0^+]}$ can be drawn as follows:

```
\begin{align*}
1 & \quad \to \quad 2 \\
2 & \quad \to \quad 3 \\
3 & \quad \to \quad 4 \\
4 & \quad \to \quad 1
\end{align*}
```

5. Twisted AR-quivers and folded AR-quivers

Definition 5.1. A combinatorial AR-quiver $\Upsilon_{[i_0]}$ associated to a (triply) twisted adapted class $[\mathcal{t}_0] \in [\Omega]$ (resp. $[\mathcal{t}_0] \in [\Omega]$) is called a (triply) twisted AR-quiver.

Recall, in Algorithm 2.3, the coordinate system on $\Gamma_Q$ is useful to indicate vertices. Since twisted AR-quivers $\Upsilon_{[i_0]}$ have similar patterns with $\Gamma_Q$ (see Algorithm 4.15 and Algorithm 4.26) it is worth to introduce coordinate systems on twisted AR-quivers. In this section, we introduce coordinate systems on $\Upsilon_{[i_0]}$ and, using the coordinates, define folded AR-quivers.

5.1. Coordinate system on a (triply) twisted AR-quiver. Let us fix an automorphism $\nu$ on the Dynkin diagram of type $\mathfrak{X}$ where $\mathfrak{X} = A_{2n+1}, D_{n+1}, E_6$ and let $I$ be the index set of type $\mathfrak{X}$. Then we can consider its folded type $\hat{\mathfrak{X}}$ and the corresponding orbit index set $\hat{I} = \{ \hat{i} \mid i \in I \}$ of $I$. If we choose $\nu$ as one of (3.1a), (3.1b), (3.1c) and (3.1d), then $\hat{\mathfrak{X}}$ is one of $B_{n+1}, C_n, F_4$ or $G_2$. We denote by $\hat{\Pi} = \{ \alpha_i \mid \hat{i} \in \hat{I} \}$ the set of simple roots of type $\hat{\mathfrak{X}}$.

Now we can give a coordinate system on $\Upsilon_{[i_0]}$ by using root system of type $\hat{\mathfrak{X}}$. To do this,

we fix the length $|\alpha_{i_0}|$ of the longest root as 1.

Definition 5.2. Let $[\mathcal{t}_0] \in [\Omega]$ or $[\Omega]$. For an arrow $\mathcal{a}$ between a vertex of residue $i$ and a vertex of residue $j$ in $\Upsilon_{[i_0]}$, we assign the length $\ell(\mathcal{a})$ that is the minimum of $|\alpha_i|^2$ and $|\alpha_j|^2$:

$\ell(\mathcal{a}) := \min \{|\alpha_i|^2, |\alpha_j|^2\}$

Using the length of an arrow, we can naturally define a coordinate system on $\Upsilon_{[i_0]}$ for $[\mathcal{t}_0] \in [\Omega]$. Precisely, for

$\frac{1}{d} := \min \{|\alpha_i|^2 \mid \hat{i} \in \hat{I}\}$,

we assign a coordinate $(i, p) \in I \times \frac{1}{d} \mathbb{Z}$ to a vertex $v$, where $i$ is the residue of $v$ and $p$ is a number induced from lengths of arrows. For $\beta \in \Phi^+$, we denote by $\Omega_{[i_0]}(\beta) \in I \times \frac{1}{d} \mathbb{Z}$ the coordinate of $\beta$ in $\Upsilon_{[i_0]}$. 

 fidelity: 1
Example 5.3. The coordinate systems for $\Upsilon_{[Q^<]}$ and $\Upsilon_{[Q^>]}$ in Example 4.13 are given as follows:

Note that the coordinate system is unique up to constant. Furthermore, if we choose $v$ as an identity, the coordinate system on $\Upsilon_{[Q]}$ is exactly the same as the original one of $\Gamma_Q$.

For $v$ in Section 3, lengths of arrows in a twisted AR-quiver $\Upsilon_{[i_0]}$ and $d$ in (5.1) are given as follows:

\[
(5.2) \quad \ell(a) = \begin{cases} 
1 \text{ or } 1/2 & \text{if } v \text{ is } (3.1a) \text{ or } (3.1c), \\
1/2 & \text{if } v \text{ is } (3.1b), \\
1/3 & \text{if } v \text{ is } (3.1d), 
\end{cases} \quad \text{and } d = \begin{cases} 
1 & \text{if } v \text{ is id}, \\
2 & \text{if } v \text{ is } (3.1a), (3.1b) \text{ or } (3.1c), \\
3 & \text{if } v \text{ is } (3.1d). 
\end{cases}
\]

5.2. Folded AR-quivers. Now the following lemma tells that a (triply) twisted AR-quiver $\Upsilon_{[i_0]}$ for $[i_0] \in [[2]]$ or $[[Q]]$ is foldable in the following sense:

For distinct vertices $v, w$ in $\Upsilon_{[i_0]}$ whose coordinates are $(i, p)$ and $(j, q)$, $(\widetilde{i}, p)$ and $(\widetilde{j}, q) \in \widetilde{T} \times \frac{1}{2}\mathbb{Z}$ are also distinct.

Lemma 5.4. A (triply) twisted AR-quiver $\Upsilon_{[i_0]}$ is foldable.

Proof. Let $(i, p)$ and $(j, q)$ be coordinates of $v$ and $w$ of $\Upsilon_{[i_0]}$, respectively.

1. Let $i_0$ be of type $A_{2n+1}$ and $v$ be the one in (3.1a). By the surgery in Algorithm 4.15, if $j = 2n + 2 - i$, the parity of $p$ and $q$ are different and hence our assertion follows.

2. For $i_0$ of type $D_{n+1}$ and $v$ in (3.1b), our assertion is obvious from the surgery in Algorithm 4.26.

The remained exceptional cases can be checked directly.

We call the $(\widetilde{i}, p) \in \widetilde{T} \times \frac{1}{2}\mathbb{Z}$ in (5.3) the folded coordinate of $v$. Now we denote by $\widetilde{\Upsilon}_{[i_0]}$ when we assign the folded coordinates system to the twisted AR-quiver $\Upsilon_{[i_0]}$ and call it the folded AR-quiver.

Example 5.5. (1) The folded AR-quiver $\widetilde{\Upsilon}_{[Q^<]}$ of $\Upsilon_{[Q^<]}$ in Example 4.13 can be drawn as follows:
(2) The folded AR-quiver $\widehat{\Upsilon}_{[i_0]}$ of $\Upsilon_{[i_0]}$ in Example 4.23 is given as follows:

```
1  1 1/2 2  2 1/2 3  3 1/2 4  4 1/2 5  5 1/2 6
```

(3) The folded AR-quiver $\widehat{\Upsilon}_{[i_0]}$ of $\Upsilon_{[i_0]}$ in Example 4.28 is given as follows:

```
1  1/2 2  2 1/2 3  3 1/2 4  4 1/2 5  5 1/2 6  6 1/2 7  7 1/2 8  8 1/2 9  9 1/2 10
```

(4) The folded AR-quiver $\widehat{\Upsilon}_{[i_0]}$ of $\Upsilon_{[i_0]}$ in Example 4.29 is given as follows:

```
1  1/2 1 3/2 2  2 1/2 3  3 1/2 4  4 1/2 5  5 1/2 6
```

Now we can describe the algorithm which shows a way of obtaining $\widehat{\Upsilon}_{[i_0]}$, from $\widehat{\Upsilon}_{[i_0]}$ by using the notations on $\widehat{\Delta}$ which is almost same as Algorithm 2.9.

**Algorithm 5.6.** Let $h^\vee$ be a dual Coxeter number of type $X$ and $\alpha_i (i \in I)$ be a sink of $\widehat{\Upsilon}_{[i_0]}$ for $[[2]]$.

(A1) Remove the vertex $(i, p)$ such that $\widehat{\Omega}_{[i_0]}(\alpha_i) = (i, p)$ and the arrows adjacent to $(i, p)$.

(A2) Add the vertex $(i, p - h^\vee)$ and the arrows to all $(j, p - h^\vee + \min(|\alpha_i|^2, |\alpha_j|^2)) \in \widehat{\Upsilon}_{[i_0]}$, for $j$ adjacent to $i$ in $\widehat{\Delta}$.

(A3) Label the vertex $(i, p - h^\vee)$ with $\alpha_i$ and change the labels $\beta$ to $s_i(\beta)$ for all $\beta \in \widehat{\Upsilon}_{[i_0]} \setminus \{\alpha_i\}$.

6. LABELING OF A TWISTED AR-QUIVER

Basically, labels of combinatorial AR-quivers can be obtained by iterative computations, using (1.1). In this section, when $[i_0]$ is a twisted adapted class of type $A_{2n+1}$ or $D_{n+1}$, we shall show that the shape of $\Upsilon_{[i_0]}$ completely determines the labels, without computations.

6.1. Type $A_{2n+1}$. Recall that a twisted AR-quiver $\Upsilon_{[i_0]}$ of type $A_{2n+1}$ can be constructed from some AR-quiver $\Gamma_Q$ with the surgery in Algorithm 4.15. Thus the full subquiver of $\Upsilon_{[i_0]}$ consisting of all vertices whose residues are $\{n, n+1, n+2\}$ can be classified as follows:
Remark 6.1. By the surgery, we know that (1) for each \( k \geq 1 \), the \( N \)-path (resp. \( S \)-path) with \( k \)-arrows in \( \Upsilon_{[\rho]} \) is unique, if it exists, (2) an \( N \)-path (resp. \( S \)-path) consisting of \( k \)-arrows exists only if \( k \) is one of the followings:

\[
\begin{align*}
(6.2) & \quad (a) 1, \ldots, n-1, n+1, n+2, \ldots, 2n \text{ (resp. } 1, \ldots, n-2, n, n+1, \ldots, 2n \text{) in (1) or (4) of (6.1)}, \\
& \quad (b) 1, \ldots, n-2, n, n+1, \ldots, 2n \text{ (resp. } 1, \ldots, n-1, n+1, n+2, \ldots, 2n \text{) in (2) or (3) of (6.1)}. 
\end{align*}
\]

Recall the notions induced and non-induced vertices in Remark 4.17. Note that a sectional path \( \rho \) with \( k(\geq n) \)-arrows contains a non-induced vertex; that is, \( \star \in \rho \). Also, a sectional path \( \rho' \) with \( k(< n) \)-arrows do not contains a non-induced vertex; that is, \( \star \notin \rho' \).

**Proposition 6.2.** Let \( k \geq n \).

1. Every vertex in an \( N \)-path with \( k \)-arrows has \( 2n+1-k \) as the first component.
2. Every vertex in an \( S \)-path with \( k \)-arrows has \( k+1 \) as the second component.

*Proof.* Note that we have a maximal \( N \)-path with \( 2n \)-arrows. By Proposition 2.10, its first component should be 1. Since we exhaust all positive roots of the form \([1, \star] \), we can apply the same argument for \([2, \star], [3, \star], \ldots \), sequentially. The second assertion follows in the same way. \( \square \)

Recall the notions in Definition 4.16 and Remark 4.17 to classify vertices in \( \Upsilon_{[\rho]} \) as follows.

**Definition 6.3.** Fix any class \([\rho] \) in \( [[\varnothing]] \) of type \( A_{2n+1} \) such that \( P_{[[\varnothing]]}([\rho]) = [Q] \).

1. A vertex \( v \) is a central vertex of \( \Upsilon_{[\rho]} \) (i) if it is not induced, that is, \( v = \star \in \Upsilon_{[\rho]} \setminus \Gamma_Q \) or (ii) if it is induced and it is the intersection of two sectional paths with \( \star \)'s.
2. The full subquiver \( \Upsilon_{[\rho]}^C \) of \( \Upsilon_{[\rho]} \) consisting of all central vertices is called the center of \( \Upsilon_{[\rho]} \).
3. The full subquiver \( \Upsilon_{[\rho]}^{\text{NE}} \) (resp. \( \Upsilon_{[\rho]}^{\text{SE}}, \Upsilon_{[\rho]}^{\text{NW}} \) and \( \Upsilon_{[\rho]}^{\text{SW}} \)) of \( \Upsilon_{[\rho]} \) consists of all vertices which are not contained in \( \Upsilon_{[\rho]}^C \) and located in the North-East (resp. South-East, North-West and South-West) part of \( \Upsilon_{[\rho]} \).
Example 6.4. For \([i_0] = [Q^c]\) in Example (5.3), we can decompose \(\Upsilon_{[i_0]}\) into \(\Upsilon_{[i_0]}^{\text{NE}}(\bigtriangledown), \Upsilon_{[i_0]}^{\text{SE}}(\square), \Upsilon_{[i_0]}^{\text{NW}}(\bigtriangleup), \Upsilon_{[i_0]}^{\text{SW}}(\bigtriangleup)\) and \(\Upsilon_{[i_0]}^{\text{C}}(\bullet, \bigstar)\) as follows:

By Theorem 1.5, we can get a reduced word \(i_0' \in [i_0]\) by reading residues of vertices in \(\Upsilon_{[i_0]}\) by the following order

\[
\{\Upsilon_{[i_0]}^{\text{NE}}, \Upsilon_{[i_0]}^{\text{SE}}\}, \{\Upsilon_{[i_0]}^{\text{NW}}, \Upsilon_{[i_0]}^{\text{SW}}\}, \{\Upsilon_{[i_0]}^{\text{C}}\}, \{\Upsilon_{[i_0]}\}\.
\]

Note that

(a) all vertices in \(\Upsilon_{[i_0]}^{\text{NE}}\) and \(\Upsilon_{[i_0]}^{\text{NW}}\) have residues less than or equal to \(n\),
(b) all vertices in \(\Upsilon_{[i_0]}^{\text{SE}}\) and \(\Upsilon_{[i_0]}^{\text{SW}}\) have residues larger than or equal to \(n + 2\),
(c) \(\Upsilon_{[i_0]}^{\text{NE}}, \Upsilon_{[i_0]}^{\text{SW}} \subseteq \Gamma_Q \cap \Upsilon_{[i_0]}\) where \(P_{[\varnothing]}([i_0]) = [Q]\).

By Algorithm 4.15, Theorem 2.11 and (6.4), we have the following lemma:

Lemma 6.5. For \(v \in \Upsilon_{[i_0]}^{\text{NE}}(\bigtriangledown) \cup \Upsilon_{[i_0]}^{\text{SW}}(\bigtriangleup)\) and \(v' \in \Upsilon_{[i_0]}^{\text{SE}}(\square) \cup \Upsilon_{[i_0]}^{\text{NW}}(\bigtriangleup)\), we have

1. \(v\) is labeled by \([a, b]\) \((b \leq n)\) which is the same as the labeling \([a, b]\) of \(v\) in \(\Gamma_Q\),
2. \(v'\) is labeled by \([a + 1, b + 1]\) \((a \geq n + 1)\) where the labeling of \(v\) in \(\Gamma_Q\) is \([a, b]\).

Proof. (1) By reading \(\Upsilon_{[i_0]}^{\text{NE}}\) first in (6.3), the labeling for \(v\) in \(\Upsilon_{[i_0]}\) should be the same as that in \(\Gamma_Q\).
(2) By reading \(\Upsilon_{[i_0]}^{\text{SE}}\) first in (6.3), the labeling for \(v'\) in \(\Upsilon_{[i_0]}\) should be shifted by one.

The remaining assertions follow by considering \(i_0''\) where \(i_0'' = i_0 i_{l-1} \cdots i_1\) for \(i_0 = i_1 i_2 \cdots i_l\).

By Lemma 6.5 and Proposition 6.2, we have the following theorem:

Theorem 6.6. For every vertex in \(\Upsilon_{[i_0]}\), we can label it as \([a, b]\) \(\in \Phi^+\) for some \(1 \leq a \leq b \leq 2n + 1\) without computing like (1.1). As consequences, we have

1. every induced \(N\)-path with \(k\)-arrows shares \(2n + 1 - k\) as the first component,
2. every induced \(S\)-path with \(k\)-arrows shares \(k + 1\) as the second component.

Hence, for every vertex in \(\Upsilon_{[i_0]}\), we can label it as \([a, b]\) \(\in \Phi^+\) for some \(1 \leq a \leq b \leq 2n + 1\).

Proof. Every induced central vertex \(\bullet\) in \(\Upsilon_{[i_0]}^{\text{C}}\) is located at the intersection of two maximal induced (but not totally induced) sectional paths with more that \(n\)-arrows and hence we can label them as \([a, b]\) for some \(1 \leq a \leq b \leq 2n + 1\) by Proposition 6.2. The vertices in \(\Upsilon_{[i_0]} \setminus \Upsilon_{[i_0]}^{\text{C}}\) can be labeled by Lemma 6.5 and Theorem 2.11. Then only vertices \(\bigstar\) are not labeled completely; that is \([a, \bigstar]\) or \([\bigstar, b]\). Due to the system \(\Phi^+\), we can label them \(\bigstar\) completely.
Example 6.7. For a Dynkin quiver \( Q = \circ_1 \rightarrow \circ_2 \rightarrow \circ_3 \rightarrow \circ_4 \rightarrow \circ_5 \rightarrow \circ_6 \), let us consider the combinatorial AR-quiver for \( \Upsilon_{[Q^c]} \)

which is the case (1) in (6.1). By Theorem 6.6, we can complete finding labels for \( \Upsilon_{[Q^c]} \) in three steps as follows:

Corollary 6.8.

(a) For (1) or (4) in (6.1), \( \star \) in \( S \)-path (resp. \( N \)-path) is labeled by \( [n+1, \star] \) (resp. \( [\star, n] \)).

(b) For (2) or (3) in (6.1), \( \star \) in \( S \)-path (resp. \( N \)-path) is labeled by \( [n+2, \star] \) (resp. \( [\star, n+1] \)).

Proof. (a) Note that, by (6.2), \( \Upsilon_{[Q^c]} \) does not contain \( N \)-path (resp. \( S \)-path) with \( n \)-arrows (resp. \( (n-1) \)-arrows). Then our first assertion follows from Theorem 6.6.

(b) The second assertion follows from the same argument.

Definition 6.9. \([15, \S 5.3]\) For \( \alpha = \sum_{i \in I} m_i \alpha_i \in \Phi^+ \), the support of \( \alpha \) is defined by

\[
\text{supp}(\alpha) := \{ \alpha_k \mid m_k \neq 0, k \in I \}.
\]

Also, if \( \alpha_k \in \text{supp}(\alpha) \) then we say \( \alpha_k \) is a support of \( \alpha \).
Since every induced central vertex in \( \Upsilon_{[i_0]} \) is located in a sectional path with more than or equal to \( n \)-arrows, (6.2), Proposition 6.2 and Corollary 6.8 tells the following corollary:

**Corollary 6.10.** For an induced central vertex in \( \Upsilon_{[i_0]} \) with the label \( \beta \in \Phi^+ \),

(a) if \( i_0 \) is in the case of (1) or (4) in (6.1), \( \text{supp}(\beta) \supseteq \{\alpha_n, \alpha_{n+1}\} \);

(b) if \( i_0 \) is in the case of (2) or (3) in (6.1), \( \text{supp}(\beta) \supseteq \{\alpha_{n+1}, \alpha_{n+2}\} \).

For an induced vertex in \( \Upsilon_{[i_0]} \), we can summarize as follows:

**Corollary 6.11.** Consider the map \( \iota^+ : I_{2n} \to I_{2n+1} \) such that \( \iota^+(i) = i \) for \( i = 1, \ldots, n \) and \( \iota^+(i) = i + 1 \) for \( i = n + 1, \ldots, 2n \). Then the labeling for the induced vertex \( v \) in \( \Upsilon_{[i_0]} \) corresponding to \( [a, b] \) in \( \Gamma_Q \) is determined as follows:

\[
\begin{align*}
\text{if } v & \in \Upsilon_{[i_0]}^C, \\
\sum_{i=a}^b \alpha_{\iota^+(i)} + \alpha_{n+1} & \quad \text{otherwise.}
\end{align*}
\]

(6.5)

6.2. **Type \( D_{n+1} \).** In this subsection, we let \([i_0]\) be the twisted adapted class of type \( D_{n+1} \) and \( Q \) be the Dynkin quiver of type \( A_n \) such that \( p_{A_n}^{D_{n+1}}([i_0]) = [Q] \). For a root \( \alpha = (a, b) \) in \( \Phi_{D_{n+1}}^+ \) (see 3.6), we say \( a \) and \( b \) are **components** of \( \alpha \).

As in the previous subsection, \( N \)-paths and \( S \)-paths in \( \Upsilon_{[i_0]} \) do the crucial role. Especially, there are two sectional paths denoted by \( N \) and \( S \) where

- \( N \) is the leftmost \( N \)-path with \( (n - 1) \)-arrows,
- \( S \) is the rightmost \( S \)-path with \( (n - 1) \)-arrows.

Note that

\[(n - 1) \text{ is the largest number of arrows in a sectional path in } \Upsilon_{[i_0]}\]

Also, the followings are useful facts in this section:

\[(6.6)\]

(i) \( N \) and \( S \) do not have intersections.

(ii) Two vertices with residue 1 on \( N \) and \( S \) are adjacent to each other.

**Definition 6.12.** Fix any class \([i_0]\) in \([\mathcal{Z}]\) of type \( D_{n+1} \) such that \( p_{A_n}^{D_{n+1}}([i_0]) = [Q] \) of type \( A_n \).

(a) The full subquiver \( \Gamma_Q^W \) (resp. \( \Gamma_Q^E \)) of \( \Gamma_Q \) is the West part (resp. East part) of \( \Gamma_Q \) whose boundary consists of the \( S \)-path with \( (n - 1) \)-arrows in \( \Gamma_Q \), which is unique.

(b) The full subquiver \( \Upsilon_{[i_0]}^W \) (resp. \( \Upsilon_{[i_0]}^E \)) of \( \Upsilon_{[i_0]} \) is the West part (resp. East part) of \( \Upsilon_{[i_0]} \) whose boundary consists of \( N \) (resp. \( S \)).

(c) The full subquiver \( \Upsilon_{[i_0]}^C \) called the center of \( \Upsilon_{[i_0]} \) is defined by

\[
\Upsilon_{[i_0]}^C = \Upsilon_{[i_0]} \setminus (\Upsilon_{[i_0]}^W \cup \Upsilon_{[i_0]}^E).
\]

Note that, we have quiver isomorphisms

\[(6.7)\]

\[
\iota_E : \Gamma_Q^E \to \Upsilon_{[i_0]}^E, \quad \iota_W : \Gamma_Q^W \to \Upsilon_{[i_0]}^W.
\]
Example 6.13. Recall the quiver $\Upsilon_{[i_0]}$ in Example 4.23.

Then

- $\Upsilon^W_{[i_0]}$ (resp. $\Upsilon^E_{[i_0]}$) is the set of $\star^W$'s (resp. $\star^E$'s),
- $\Upsilon^C_{[i_0]}$ is the set of $\star^C$'s and $\star^{C'}$'s,
- the sectional paths $N$ and $S$ consist of the arrows

Then

On the other hand, consider the quiver $\Gamma_Q$ of type $A_n$:

We can see that

- $\Gamma^W_Q$ (resp. $\Gamma^E_Q$) is the set of $\bullet^W$'s (resp. $\bullet^E$'s) and $\bullet^{W, E}$'s,
- $\Gamma^E_Q \simeq \Upsilon^E_{[i_0]}$ by the canonical map,
- $\Gamma^W_Q \simeq \Upsilon^W_{[i_0]}$ by putting $\Gamma^W_Q$ upside down.

Proposition 6.14. The labeling of $\Gamma^E_Q$ naturally induces the labeling of $\Upsilon^E_{[i_0]}$. More precisely,

(i) if the twisted Coxeter element $i_1 i_2 \cdots i_n \nu$ has the index $n + 1$ then

\[ \iota_E([a, b]) = \begin{cases} 
(a, -b - 1) & \text{if } b \neq n, \\
(a, b + 1) & \text{if } b = n.
\end{cases} \]

(ii) if the twisted Coxeter element has the index $n$ then

\[ \iota_E([a, b]) = (a, -b - 1). \]

Proof. Let us denote the simple root $\alpha_i$ of type $A_n$ by $\alpha_i^A$ and the simple root $\alpha_i$ of type $D_{n+1}$ by $\alpha_i^D$. For $1 \leq a \leq b \leq n$, recall that

\[ [a, b] = \sum_{i=a}^{b} \alpha_i^A, \quad (a, -b - 1) = \sum_{i=a}^{b} \alpha_i^D, \quad (a, n + 1) = \left( \sum_{i=a}^{n-1} \alpha_i^A \right) + \alpha_n^D. \]

For the case (i), if $i_1 i_2 \cdots i_k$ is a compatible reading of $\Upsilon^E_{[i_0]}$ then $(i_1 i_2 \cdots i_k)^\nu$ is a compatible reading of $\Gamma^E_Q$. Hence $\iota_E(\sum_{i=a}^{b} \alpha_i^A) = \sum_{i=a}^{b} \alpha_i^D$. For the case (ii), if $i_1 i_2 \cdots i_k$ is a compatible reading of $\Upsilon^E_{[i_0]}$ then $i_1 i_2 \cdots i_k$ is a compatible reading of $\Gamma^E_Q$. Hence $\iota_E(\sum_{i=a}^{b} \alpha_i^A) = \sum_{i=a}^{b} \alpha_i^D$. \qed
Proposition 6.15. The labeling of $\Gamma^W_Q$ induces the labeling of $\Upsilon^W_{[i_0]}$. More precisely,

(i) if the twisted Coxeter element $i_1i_2\cdots i_nv$ has the index $n+1$ then

$$\iota_W([a,b]) = (a,-b-1).$$

(ii) if the twisted Coxeter element has the index $n$ then

$$\iota_W([a,b]) = \begin{cases} (a,-b-1) & \text{if } b \neq n, \\ (a,b+1) & \text{if } b = n. \end{cases}$$

Proof. Here, we only show the proof when $n+1$ is even and $\Upsilon^W_{[i_0]}$ does not have a vertex with residue $n+1$. Other cases can be proved similarly. In this case, the twisted Coxeter element contains the index $n+1$.

Let $i_1i_2\cdots i_kj_{i-1}\cdots j_2j_1$ be an element in $[i_0]$ such that $j_{i-1}\cdots j_2j_1$ is a compatible reading of $\Upsilon^W_{[i_0]}$. Note that a label $\beta^D$ in $\Upsilon^W_{[i_0]}$ is

$$\beta^D = s_{i_1}\cdots s_{i_k}s_{i_k}s_{j_{i-1}}\cdots s_{j_2}s_{j_1}(\alpha_{j_m}^D) = s_{j_1}^D\cdots s_{j_2}^D(\alpha_{j_m}^D) = s_{j_1}\cdots s_{j_{m-1}}(\alpha_{j_m}^D).$$

Here $D^*$ denotes the involution in Definition 1.1, which is the identity since $n+1$ is even.

On the other hand, $(n+1-j_k)(n+1-j_{i-1})\cdots (n+1-j_2)(n+1-j_1)$ is a compatible reading of $\Gamma^W_Q$ and, for the type $A_n$ involution $A^*$, we have $(n+1-j)^{A^*} = j$. Hence the label $\beta^A := \iota_W^{-1}(\beta^D)$ is

$$\beta^A = s_{j_1}\cdots s_{j_{m-1}}(\alpha_{j_m}^A).$$

By (6.8) and (6.9), we proved the proposition. \qed

Corollary 6.16.

(1) Every vertex in $\mathcal{S}$ shares the second component $\pm(n+1)$ and every vertex in $\mathcal{N}$ shares the second component $\mp(n+1)$.

(2) Let $(a, \pm(n+1))$ be the label of a vertex in $\mathcal{S}$ with folded residue $\hat{i}$. Then $(a, \mp(n+1))$ is the label of a vertex in $\mathcal{N}$ with folded residue $n+1-i$.

Proof. From the above two propositions, our assertions follow by comparing (i) (resp. (ii)'s) of Proposition 6.14 and (i) (resp. (ii)'s) of Proposition 6.15 \qed

Lemma 6.17. Let $\alpha$ and $\beta$ be distinct roots in $\Phi^*$. 

(1) If there are two intersection points $\gamma$ and $\delta$ of sectional paths through $\alpha$ and sectional paths through $\beta$ then (see (II - 1) in (8.8) below)

$$\alpha + \beta = \gamma + \delta.$$ 

(2) Suppose there is only one intersection point $\gamma$ of a sectional path through $\alpha$ and a sectional path through $\beta$. If the folded residues of $\alpha$ and $\beta$ are $\hat{i}$ and $\hat{j}$ and that of $\gamma$ is $i+j$ then (see (II - 3) in (8.8) below)

$$\alpha + \beta = \gamma.$$
Proof. (1) Suppose that the folded coordinates of positive roots $\alpha, \beta, \gamma,$ and $\delta$ are $(\hat{i}, p+1)$, $(\hat{i}, p)$, $(i-1, p+\frac{1}{2})$ and $(i+1, p+\frac{1}{2})$, respectively, for $i \leq n-1$. In other words, let $\alpha, \beta, \gamma, \delta$ consist of size $1 \times 1$ rectangular. Then there is an element $w \in W$ such that

$$\alpha = w(\alpha_i), \beta = ws_i s_{i+1} s_{i-1}(\alpha_i), \gamma = ws_i s_{i+1}(\alpha_{i-1}), \delta = ws_i (\alpha_{i+1}),$$

where $i+1 = n$ or $n + 1$ if $i = n - 1$ and $i + 1 = i + 1$, otherwise. Hence $\alpha + \beta = \gamma + \delta$.

Now, if $\alpha, \beta, \gamma, \delta$ consists of size $m \times n$ rectangular, then by applying the same argument $m \cdot n$ times, we get $\alpha + \beta = \gamma + \delta$.

(2) Suppose that $\alpha, \beta$ and $\gamma$ have the folded coordinates $(\hat{i}, p)$, $(\hat{i}, p + 1)$ and $(\hat{i}, p + \frac{1}{2})$. Then there is an element $w \in W$ such that

$$\alpha = w(\alpha_1), \beta = ws_1 s_2(\alpha_1), \gamma = ws_1 (\alpha_2).$$

Hence $\alpha + \beta = \gamma$. Now, by using (1), we can deduce the lemma \hfill $\Box$

Proposition 6.18. Labeling for vertices in $\Upsilon_{C}^{C}[\tilde{t}_0]$ is completely determined by Lemma 6.17. More precisely, if $\gamma \in \Upsilon_{C}^{C}[\tilde{t}_0]$ is the intersection of

an $N$-path crossing $\alpha = \langle a, \pm(n + 1) \rangle$ and an $S$-path crossing $\beta = \langle b, \pm(n + 1) \rangle$,

then $\gamma = \langle a, b \rangle$.

Proof. Without loss of generality, we can assume that $\alpha$ and $\beta$ are in $S$ and $N$. Since $\alpha, \beta, \gamma$ satisfy assumptions in Lemma 6.17 (2), we have $\gamma = \alpha + \beta$. \hfill $\Box$

Theorem 6.19. We can label $\Upsilon_{[t_0]}$ by only observing its shape.

Proof. (1) By Proposition 6.14, Proposition 6.15, we can label the vertices lying in (i) all sectional paths with less than $(n-1)$-arrows, (ii) $N$ and (iii) $S$ by using Theorem 2.11. Since every vertices in $\Upsilon_{C}^{C}[\tilde{t}_0]$ can be labeled by $N$ and $S$ by Proposition 6.18 and, in addition, Theorem 2.11 and Proposition 6.18 depend only on the shape of $\Upsilon_{[t_0]}$, our assertion follows. \hfill $\Box$

For the rest of this subsection, we shall list up the combinatorial properties of the labeling of $\Upsilon_{[t_0]}$ followed by Theorem 6.19:

Proposition 6.20. A folded AR-quiver $\Upsilon_{[t_0]}$ satisfies the following properties:

1. Every vertex in a sectional path shares a component.
2. Consider the $N$-path and the $S$-path which have the vertices with folded coordinates $(\hat{i}, p)$ and $(\hat{i}, p + 1)$, respectively. If every vertex in the $N$-path shares the component $i$ then every vertex in the $S$-path shares the component $-i$.
3. Consider the $N$-path and the $S$-path which have the vertices with folded coordinates $(\hat{i}, q)$ and $(\hat{i}, q - 1)$, respectively. If every vertex in the $N$-path shares the component $i$ then every vertex in the $S$-path also shares the component $i$.

Remark 6.21. Inspired from Proposition 6.20 (3), we will define a swing in Definition 8.16 below, which plays an important role in later sections.

By Proposition 6.20, we get the following corollary.
Corollary 6.22. If there are two vertices \( \alpha, \beta \) in \( \Upsilon_{[i_0]} \) with folded coordinates \((i, p)\) and \((n+1-i, p + \frac{n+1}{2})\), then there are \( 1 \leq a < b \leq n+1 \) such that \( \{\alpha, \beta\} = \{(a, b), (a, -b)\} \).

Proof. When two vertices \( \alpha \) and \( \beta \) have folded coordinates \((i, p)\) and \((n+1-i, p + \frac{n+1}{2})\), we have

(i) the \( N \)-path passing \( \alpha \) and the \( S \)-path passing \( \beta \) which satisfy the assumptions in Proposition 6.20 (2),

(ii) the \( S \)-path passing \( \alpha \) and the \( N \)-path passing \( \beta \) which satisfy the assumptions in Proposition 6.20 (3).

Hence the corollary follows. \( \square \)

Recall we can identify \( \Upsilon_{[i_0]} \) with \( \Gamma_Q \cup \Gamma_Q^\ast \) (see Algorithm 4.26). Since \( \Gamma_Q^\ast \simeq \Gamma_Q \) as in Remark 4.25, we can consider \( \Upsilon_{[i_0]} \) as a union of two copies of \( \Gamma_Q \). Using this observation, we can find the labeling of \( \Upsilon_{[i_0]} \) in an efficient way:

**Proposition 6.23.** There exists an efficient algorithm for labeling of \( \Upsilon_{[i_0]} \) which is canonically induced from the labeling of \( \Gamma_Q \). Let \( \Gamma_Q \) be the quiver which is obtained by upside down \( \Gamma_Q \) and has the same labeling of \( \Gamma_Q \) and let \( \Gamma_Q^\ast = \Gamma_Q \). Then we define the labeling map

\[
\tau : \Gamma_Q \cup \Gamma_Q^\ast \rightarrow \Upsilon_{[i_0]} \text{ given by } \tau([a, b]) = \begin{cases} \iota_E([a, b]) & \text{for } [a, b] \in \Gamma_Q^E, \\ \omega_W([a, b]) & \text{for } [a, b] \in \Gamma_Q^W, \\ (a, b) & \text{otherwise}. \end{cases}
\]

Here, (i) \( \Gamma_Q \cup \Gamma_Q^\ast \) denotes the quiver obtained by gluing \( \Gamma_Q \) and \( \Gamma_Q^\ast \), (ii) \( \Gamma_Q^W \) is obtained by upside down the quiver \( \Gamma_Q^W \), and (iii) \( \Gamma_Q^E \cup \Gamma_Q^E \) is the same as \( \Gamma_Q^E \).

**Example 6.24.** Let us consider the class \( [i_0] \in \llbracket \mathcal{D} \rrbracket \) of type \( D_{n+1} \) and \( [Q] \in \llbracket \Delta \rrbracket \) of type in Example 6.13. Now we can label \( \Upsilon_{[i_0]} \) by only observing its shape and using the results in this subsection:

1. Draw two copies \( \Gamma_Q \) and \( \Gamma_Q^\ast \) of \( \Gamma_Q \)’s.

\[
\Gamma_Q = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
4 \\
3 \\
2 \\
1
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1^W \\
[3, 4]^W \\
[3]^C \\
[1]^W
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
[1, 4]^W \\
[1, 3]^C \\
[1, 2]^C \\
[2]^C
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
[2, 4]^W \\
[2, 3]^C \\
[2]^C \\
[4]^E
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
[3, 4]^E \\
[1, 4]^E \\
[1, 2]^E \\
[2]^E
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
[3, 4]^E \\
[1, 4]^E \\
[1, 2]^E \\
[2]^E
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}

Recall that we can label \( \Gamma_Q \) by observing its shape only.

2. Glue two quivers. Substitute \( [a_1, a_2] \in \Upsilon_{[i_0]}^E \cup \Upsilon_{[i_0]}^W \) to \( (a_1, -a_2 - 1) \) and substitute \( [a_1, a_2] \in \Upsilon_{[i_0]}^C \) to \( (a_1, a_2 + 1) \).
3. Finally, by considering that the twisted Coxeter element has $5 = n + 1$, we substitute $\langle i, -5 \rangle$ in $S$ to $\langle i, 5 \rangle$ and get $\Upsilon_{i_0}$.

7. Representations of quantum affine algebras

In [24, 25, 26], the first named author interpreted denominator formulas and Dorey’s rule for $U_q'(X(1))$ ($X = A_n, D_n, E_n$), in terms of AR-quivers of type $X$. Similarly, in Section 8, there are analogous results for $U_q' (\tilde{X}(1))$ ($\tilde{X} = B_{n+1}, C_n, F_4, G_2$) using twisted and folded AR-quivers. In this section, we briefly introduce some notions and theorems in the theory of quantum affine algebras including R-matrices, denominator formulas and Dorey’s rule.

7.1. Quantum affine algebras and their representations. Let $I_{\text{aff}} = I \cup \{0\}$ be the set of indices. An affine Cartan datum is a quadruple $(A, P, \Pi, \Pi')$ consisting of

(a) a matrix $A = (a_{ij})_{i,j \in I_{\text{aff}}}$ of corank 1, called the affine Cartan matrix satisfying

(i) $a_{ii} = 2$ ($i \in I_{\text{aff}}$),  
(ii) $a_{ij} \in \mathbb{Z}_{\geq 0}$,  
(iii) $a_{ij} = 0$ if $a_{ji} = 0$

with $D = \text{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I_{\text{aff}})$ making DA symmetric,

(b) a free abelian group $P$ of rank $n + 2$, called the weight lattice,

(c) $\Pi = \{ \alpha_i \mid i \in I_{\text{aff}} \} \subset P$, called the set of simple roots,

(d) $\Pi' = \{ h_i \mid i \in I_{\text{aff}} \} \subset P' := \text{Hom}(P, \mathbb{Z})$, called the set of simple coroots,

which satisfy

(1) $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I_{\text{aff}}$,
(2) $\Pi$ and $\Pi'$ are linearly independent sets,
(3) for each $i \in I_{\text{aff}}$, there exists $\Lambda_i \in P$ such that $\langle h_i, \Lambda_j \rangle = \delta_{ij}$ for all $j \in I_{\text{aff}}$.

We set $Q = \bigoplus_{i \in I_{\text{aff}}} \mathbb{Z} \alpha_i$, $Q_+ = \bigoplus_{i \in I_{\text{aff}}} \mathbb{Z}_{\geq 0} \alpha_i$, $Q' = \bigoplus_{i \in I_{\text{aff}}} \mathbb{Z} h_i$ and $Q'_+ = \bigoplus_{i \in I_{\text{aff}}} \mathbb{Z}_{\geq 0} h_i$. We choose the imaginary root $\delta = \sum_{i \in I_{\text{aff}}} a_i \alpha_i \in Q_+$ and the center $c = \sum_{i \in I_{\text{aff}}} c_i h_i \in Q'_+$ such that ([15, Chapter 4])

$\{ \lambda \in Q \mid \langle h_i, \lambda \rangle = 0 \text{ for every } i \in I_{\text{aff}} \} = \mathbb{Z} \delta$ and $\{ h \in Q' \mid \langle h, \alpha_i \rangle = 0 \text{ for every } i \in I_{\text{aff}} \} = \mathbb{Z} c$.

Set $\mathfrak{h} = Q \otimes_{\mathbb{Z}} P'$. Then there exists a symmetric bilinear form $(\ , \ )$ on $\mathfrak{h}^*$ satisfying

$\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for any $i \in I_{\text{aff}}$ and $\lambda \in \mathfrak{h}^*$.

We normalize the bilinear form by

$\langle c, \lambda \rangle = (\delta, \lambda)$ for any $\lambda \in \mathfrak{h}^*$.

Let us denote by $\mathfrak{g}$ the affine Kac-Moody Lie algebra associated with $(A, P, \Pi, \Pi')$ and by $W_{\text{aff}}$ the Weyl group of $\mathfrak{g}$, generated by $(s_i)_{i \in I_{\text{aff}}}$. We define $\mathfrak{g}_0$ the subalgebra of $\mathfrak{g}$ generated by
the Chevalley generators $e_i, f_i$, and $h_i$ for $i \in I = I_{\text{aff}} \setminus \{0\}$. Then $\mathfrak{g}_0$ becomes a finite dimensional simple Lie algebra.

Let $d$ be the smallest positive integer such that

$$d(\alpha_i, \alpha_i)/2 \in \mathbb{Z} \quad \text{for any } i \in I_{\text{aff}}.$$  

Note that $d$ coincides with $d$ in (5.2).

Let $q$ be an indeterminate. For $m, n \in \mathbb{Z}_{\geq 0}$ and $i \in I_{\text{aff}}$, we define $q_i = q^{(\alpha_i, \alpha_i)/2}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \left[ \frac{m}{n} \right]_i = \frac{[m]_i!}{[m-n]_i! [n]_i!}.$$  

**Definition 7.1.** The quantum affine algebra $U_q(\mathfrak{g})$ associated with $(A, P, \Pi, \Pi')$ is the associative algebra over $\mathbb{Q}(q^{1/d})$ with 1 generated by $e_i, f_i$ ($i \in I_{\text{aff}}$) and $q^h$ ($h \in d^{-1} P'$) satisfying following relations:

1. $q^0 = 1, q^h q^{h'} = q^{h+h'}$ for $h, h' \in d^{-1} P'$,
2. $q^h e_i q^{-h} = q^{(h, \alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h, \alpha_i)} f_i$ for $h \in d^{-1} P'$,
3. $e_i f_j - f_j e_i = \delta_{ij} K_i - K_i^{-1}$, where $K_i = q_i^{h_i},$
4. $\sum_{k=0}^{1-\alpha_{ij}} (-1)^k e_i^{(1-\alpha_{ij}-k)} e_j f_i^{(k)} = \sum_{k=0}^{1-\alpha_{ij}} (-1)^k f_j^{(1-\alpha_{ij}-k)} f_i e_i^{(k)} = 0 \quad \text{for } i \neq j,$

where $e_i^{(k)} = e_i^k/[k]_i!$ and $f_i^{(k)} = f_i^k/[k]_i!$.

We denote by $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i$ (resp. $f_i$) ($i \in I_{\text{aff}}$). Let $U_q(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{-1}$ ($i \in I_{\text{aff}}$) and call it also the quantum affine algebra. We mainly deal with $U_q(\mathfrak{g})$-modules.

For $U_q(\mathfrak{g})$-modules $M$ and $N$, $M \otimes N$ becomes a $U_q(\mathfrak{g})$-module by the coproduct $\Delta$ of $U_q(\mathfrak{g})$:

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i.$$  

A $U_q(\mathfrak{g})$-module $M$ is called **integrable** provided that

a. $M = \bigoplus_{\mu \in P_\delta} M_{\mu}$, where $P_\delta := P/\mathbb{Z} \delta$ and $M_{\mu} := \{ v \in M \mid K_i v = q_i^{(h_i, \mu)} v \}$,

b. $e_i$ and $f_i$ ($i \in I$) act on $M$ nilpotently.

In this paper, we mainly consider

(7.1)

$C_\mathfrak{g} = \text{the abelian tensor category consisting of finite dimensional integrable } U_q(\mathfrak{g})\text{-modules}.$

We are interested in another family of $U_q(\mathfrak{g})$-modules called **good**. Since the whole definition of the good module is not needed, we just refer to [20] for the precise definition. However, the following is one of conditions of a good module, which we want to emphasize: A good module $M$ contains the unique (up to constant) weight vector $v_M$ of weight $\lambda$, such that

$$\text{wt}(M) \subset \lambda + \sum_{i \in I} \mathbb{Z}_{\geq 0} \text{cl}(\alpha_i).$$

We call $v_M$ the **dominant extremal weight vector** and $\lambda$ **dominant extremal weight**.
Let us consider the level 0 fundamental weight \( \varpi_i \), for \( i \in I \), defined by
\[
\varpi_i := \gcd(c_0, c_i)^{-1}(c_0 \lambda_i - c_i \Lambda_0) \in P.
\]
Then \( \{ \varpi_i \mid i \in I \} \), where \( \text{ch}: P \to P_{\text{cl}} \) as the canonical projection, forms a basis for the space of classical integral weight level 0, denoted by \( P_{\text{cl}}^0 \), which is defined as follows:
\[
P_{\text{cl}}^0 = \{ \lambda \in P_{\text{cl}} \mid \langle c, \lambda \rangle = 0 \}.
\]
The Weyl group \( W \) of \( g_0 \), generated by \( (s_i)_{i \in I} \), acts on \( P_{\text{cl}}^0 \) (see [1, §1.2]). We denote by \( w_0 \) the longest element of \( W \).

**Definition 7.2.** [1, §1.3] For \( i \in I \), the \( i \)-th fundamental module is a unique finite dimensional integrable \( U_q'(g) \)-module \( V(\varpi_i) \) satisfying the following properties:

1. The weights of \( V(\varpi_i) \) are contained in the convex hull of \( W_{\text{cl}}(\varpi_i) \).
2. \( V(\varpi_i)_{\text{cl}(\varpi_i)} = \mathbb{C}(q)\varpi_i \). (We call the vector \( \varpi_i \) a dominant integral weight vector.)
3. For any \( \mu \in W_{\text{cl}}(\varpi_i) \), we can associate a non-zero vector \( u_\mu \), called an extremal vector of weight \( \mu \), such that
\[
\text{S} \cdot u_\mu := u_{h_i, \mu} = \begin{cases} f_i^{(h_i, \mu)} u_\mu & \text{if } \langle h_i, \mu \rangle \geq 0, \\ e_i^{(-h_i, \mu)} u_\mu & \text{if } \langle h_i, \mu \rangle \leq 0, \end{cases} \text{ for any } i \in I.
\]
4. \( \varpi_i \) generates \( V(\varpi_i) \) as a \( U_q'(g) \)-module.

For instance, the \( i \)-th fundamental representation is a good and integrable module.

Now, we fix the base field of \( U_q'(g) \)-modules \( k \) as the algebraic closure of \( \mathbb{C}(q) \) in \( \cup_{m>0} \mathbb{C}((q^{1/m})) \). When we deal with \( U_q'(g) \)-modules, we regard the base field as \( k \).

For an indeterminate \( z \) and a \( U_q'(g) \)-module \( M \), let us denote by \( M_z = k[z^{\pm 1}] \otimes M \) the \( U_q'(g) \)-module with the action of \( U_q'(g) \) given by
\[
e_i(u_z) = z^{\delta_i, \sigma}(e_i u)_z, \quad f_i(u_z) = z^{-\delta_i, \sigma}(f_i u)_z, \quad K_i(u_z) = (K_i u)_z.
\]

**Definition 7.3.** ([12]) Let \( Q \) be a Dynkin quiver of type \( X = A_n, D_n \) or \( E_n \). For any positive root \( \beta \) contained in \( \Phi_X^+ \), we set the \( U_q'(X^{(1)}) \)-module \( V_Q(\beta) \) as follows:
\[
V_Q(\beta) := V(\varpi_i)_{(-q)^{\rho}}, \text{ where } \Omega_Q(\beta) = (i, p).
\]

Denote by \( C_Q \) the smallest abelian full subcategory of the category \( C_{X^{(1)}} \) defined in (7.1) such that

- (a) it is stable under subquotient, tensor product and extension,
- (b) it contains \( V_Q(\beta) \) for all \( \beta \in \Phi_X^+ \), and the trivial module \( 1 \).

### 7.2. R-matrices, denominators and Dorey’s rule.
In this subsection, we recall the notion of \( R \)-matrices, denominators and Dorey’s rule for quantum affine algebras. We follow [20, §8]. Let us take a basis \( \{ P_\nu \} \) of \( U_q'(g) \) and a basis \( \{ Q_\nu \} \) of \( U_q'(g) \) which are dual to each other with respect to a suitable coupling on \( U_q'(g) \times U_q'(g) \). Then, for \( U_q'(g) \)-modules \( M \) and \( N \), then there exists the universal \( R \)-matrix (6)
\[
R_{M,N}^{\text{uni}}(u \otimes v) = q^{(\text{wt}(u), \text{wt}(v))} \sum_\nu P_\nu v \otimes Q_\nu u,
\]
so that $R_{M,N}^{\text{univ}}$ gives a $U'_q(\mathfrak{g})$-homomorphism from $M \otimes N$ to $N \otimes M$ provided that an infinite sum has a meaning. For $M, N \in \mathcal{C}_q$, $R_{M,N}^{\text{univ}}$ converges in $(z_N/z_M)$-adic topology. Thus we have a morphism of $k[[z_N/z_M]] \otimes k[z_{\pm 1}]/(z_{\pm 1}^2)$-modules

$$R_{M,N}^{\text{univ}} : k[[z_N/z_M]] \otimes k[z_{\pm 1}]/(z_{\pm 1}^2) \to k[[z_N/z_M]] \otimes k[z_{\pm 1}]/(z_{\pm 1}^2) \otimes R_{M,N}.$$  

We say that $R_{M,N}^{\text{univ}}$ is \textit{rationally renormalizable} if there exist $a \in k[z_N/z_M]$ and a $U'_q(\mathfrak{g})[z_{\pm 1}]/(z_{\pm 1}^2)$-module homomorphism

$$R_{M,N}^{\text{ren}} : M_{z_N} \otimes N_{z_M} \to N_{z_N} \otimes M_{z_M}$$

such that $R_{M,N}^{\text{ren}} = aR_{M,N}^{\text{univ}}$. Then we can choose $R_{M,N}^{\text{ren}}$ so that, for any $c_1, c_2 \in k^\times$, the specialization of $R_{M,N}^{\text{ren}}$ at $z_M = c_1, z_N = c_2$,

$$R_{M,N}^{\text{ren}}|_{z_M = c_1, z_N = c_2} : M_{c_1} \otimes N_{c_2} \to N_{c_2} \otimes M_{c_1}$$

does not vanish under the assumption that $M$ and $N$ are non-zero modules in $\mathcal{C}_q$. Such an $R_{M,N}^{\text{ren}}$ is unique up to $k[(z_M/z_N)^{\pm 1}] = \cup_{a \in k} k[z_N/z_M]^{\pm 1}$ and it is called a \textit{renormalized $R$-matrix}.

We denote by

$$r_{M,N}^{\text{ren}} := R_{M,N}^{\text{ren}}|_{z_M = 1, z_N = 1} : M \otimes N \to N \otimes M$$

and call it the \textit{$R$-matrix}. The $R$-matrix $r_{M,N}$ is well-defined up to a constant multiple when $R_{M,N}^{\text{univ}}$ is rationally renormalizable. By definition, $r_{M,N}$ never vanishes.

For simple $U'_q(\mathfrak{g})$-modules $M$ and $N$ in $\mathcal{C}_q$, the universal $R$-matrix $R_{M,N}^{\text{univ}}$ is rationally renormalizable. Then, for dominant extremal weight vectors $u_M$ and $u_N$ of $M$ and $N$, there exists $a_{M,N}(z_N/z_M) \in k[z_{\pm 1}]/(z_{\pm 1}^2)$ such that

$$R_{M,N}^{\text{univ}}((u_M)_{z_M}(u_N)_{z_N}) = a_{M,N}(z_N/z_M)((u_N)_{z_N}(u_M)_{z_M}).$$

Then

$$R_{M,N}^{\text{norm}} := a_{M,N}(z_N/z_M)^{-1}R_{M,N}^{\text{univ}}$$

is a unique $k[z_M]/(z_M^2)$-module homomorphism sending $(u_M)_{z_M}(u_N)_{z_N}$ to $(u_N)_{z_N}(u_M)_{z_M}$:

$$R_{M,N}^{\text{norm}} : k[z_M]/(z_M^2) \otimes k[z_N]/(z_N^2) \to k[z_M]/(z_M^2) \otimes k[z_N]/(z_N^2).$$

It is known that $k[z_M]/(z_M^2) \otimes k[z_N]/(z_N^2)$ is simple $k[z_M,z_N]/(z_M^2,z_N^2)$-$U'_q(\mathfrak{g})$-module ([20, Proposition 9.5]). We call $R_{M,N}^{\text{norm}}$ the \textit{normalized $R$-matrix}.

Let us denote by $d_{M,N}(u) \in k[u]$ a monic polynomial of the smallest degree such that the image $d_{M,N}(z_N/z_M) R_{M,N}^{\text{norm}}$ is contained in $N_{z_N} \otimes M_{z_M}$. We call $d_{M,N}$ the \textit{denominator} of $R_{M,N}^{\text{norm}}$. Then,

$$R_{M,N}^{\text{ren}} = d_{M,N}(z_N/z_M)R_{M,N}^{\text{norm}}$$

and

$$d_{M,N}(z_N/z_M)R_{M,N}^{\text{norm}}|_{z_M = 1, z_N = 1} = c_{M,N} \cdot r_{M,N}$$

for a constant $c_{M,N}$.

From the following theorem, one can notice that the denominator formulas between fundamental representations provides crucial information of the representation theory on $\mathcal{C}_q$.

\textbf{Theorem 7.4.} [1, 4, 20] (see also [16, Theorem 2.2.1])
Theorem 7.5. [1, 7, 17, 23]

(1) For good modules $M, N$, the zeroes of $d_{M,N}(z)$ belong to $\mathbb{C}[[q^{1/m}]]$ for some $m \in \mathbb{Z}_{>0}$.
(2) Let $M_k$ be a good module with a dominant extremal vector $u_k$ of weight $\lambda_k$, and $a_k \in k^*$ for $k = 1, \ldots, t$. Assume that $a_j/a_i$ is not a zero of $d_{M_i,M_j}(z)$ for any $1 \leq i < j \leq t$. Then the following statements hold.

(i) $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is generated by $u_1 \otimes \cdots \otimes u_t$.
(ii) The head of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is simple.
(iii) Any non-zero submodule of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ contains the vector $u_t \otimes \cdots \otimes u_1$.
(iv) The socle of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is simple.
(v) Let $r: (M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t} \to (M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ be the specialization of $R^n_{M_1,\ldots,M_t}$ at $z_k = a_k$. Then the image of $r$ is simple and it coincides with the head of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ and also with the socle of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$.

(3) For a simple integrable $U_q'(\mathfrak{g})$-module $M$, there exists a finite sequence $((i_1, a_1), \ldots, (i_t, a_t))$ in $I \times k^*$ such that $d_V(\mathfrak{w}_{i_k}) V(\mathfrak{w}_{i_{k'}})(a_{k'}/a_k) \neq 0$ for $1 \leq k < k' \leq t$ and $M$ is isomorphic to the head of $V(\mathfrak{w}_{i_1})_{a_1} \otimes \cdots \otimes V(\mathfrak{w}_{i_t})_{a_t}$. Moreover, such a sequence $((i_1, a_1), \ldots, (i_t, a_t))$ is unique up to permutation.

(4) $d_{k,l}(z) = d_{l,k}(z) = d_{k',l'}(z) = d_{l',k'}(z)$ for $k, l \in I$.

The denominator formulas between fundamental representations over classical quantum affine algebras were calculated in [1, 7, 17, 23]:

Theorem 7.5. [1, 7, 17, 23]
A non-zero $U'_q(\mathfrak{g})$-module homomorphism $\psi$ is called a Dorey's type homomorphism if

$$\psi \in \text{Hom}_{U'_q(\mathfrak{g})}(V(\varpi_k), V(\varpi_i) \otimes V(\varpi_j))$$

for some $i, j, k \in I$ and $x, y, z \in k^*$. By [19, Theorem 3.2], such $\psi$ is unique up to non-zero constant multiple.

The following theorems are referred as Dorey’s rule (see [5]):

**Theorem 7.6.** [5, 24, 25, 26] Let $(i, x)$, $(j, y)$, $(k, z) \in I \times k^*$. Then

$$\text{Hom}_{U'_q(\mathfrak{g}(1))}(V(\varpi_k), V(\varpi_i) \otimes V(\varpi_j)) \neq 0 \quad (X = A_n, D_n \text{ or } E_n)$$

if and only if there exists an adapted class $[Q]$ and $\alpha, \beta, \gamma \in \Phi_X^+$ such that

(i) $(\alpha, \beta)$ is a pair of positive roots such that $\alpha + \beta = \gamma$,

(ii) $V(\varpi_j)_y = V_Q(\beta)_t$, $V(\varpi_i)_x = V_Q(\alpha)_t$, $V(\varpi_k)_z = V_Q(\gamma)_t$ for some $t \in k^*$.

Now we present Dorey’s rule for $U'_q(P_n^{(1)})$ and $U'_q(C_n^{(1)})$ which are interested in this paper:

**Theorem 7.7.** [5, Theorem 8.1] (see also [23])

(a) For $(i, x)$, $(j, y)$, $(k, z) \in I = \{1, 2, \ldots, n, n+1\} \times k^*$,

$$\text{Hom}_{U'_q(\mathfrak{g}(1))}(V(\varpi_k), V(\varpi_i) \otimes V(\varpi_j)) \neq 0$$

if and only if one of the following conditions holds:

\[
\begin{cases}
    \ell := \max(i, j, k) \leq n, \; i + j + k = 2\ell & \text{and} \\
    (y/z, x/z) = \begin{cases}
        ((-1)^{i+k}q^{-i}, (-1)^{i+k}q^{j}), & \text{if } \ell = k, \\
        ((-1)^{i+k}q^{-i}, (-1)^{i+k}q^j), & \text{if } \ell = i, \\
        ((-1)^{i+k}q^{-i}, (-1)^{i+k}q^{2n+1-j}), & \text{if } \ell = j.
    \end{cases}
\end{cases}
\]

(b) For $(i, x)$, $(j, y)$, $(k, z) \in I = \{1, 2, \ldots, n\} \times k^*$,

$$\text{Hom}_{U'_q(\mathfrak{g}(1))}(V(\varpi_k), V(\varpi_i) \otimes V(\varpi_j)) \neq 0$$

if and only if one of the following conditions holds:

\[
\begin{cases}
    \ell := \max(i, j, k) \leq n, \; i + j + k = 2\ell & \text{and} \\
    (y/z, x/z) = \begin{cases}
        ((-q^{1/2})^{-i}, (-q^{1/2})^j), & \text{if } \ell = k, \\
        ((-q^{1/2})^{-i}, (-q^{1/2})^j), & \text{if } \ell = i, \\
        ((-q^{1/2})^{-i}, (-q^{1/2})^{2n+2-j}), & \text{if } \ell = j.
    \end{cases}
\end{cases}
\]
8. Distance and folded distance polynomials

In [26], the first named author described denominator formulas for untwisted affine type ADE using so-called distance polynomials, which are obtained by observing AR-quivers. In this section, we first review the distance polynomials defined on the adapted $r$-cluster point $[[\Delta]]$ of finite type ADE and relations between distance polynomials and denominator formulas. Then we introduce how to generalized the results to the cases of untwisted affine type BCFG by inventing folded distance polynomials on $[[\mathcal{G}]]$. Also, as a generalization of Theorem 7.6, we record the positions of minimal pairs for every $\gamma \in \Phi^+ \setminus \Pi$ in $\Upsilon_{\mathcal{G}}$ to describe Dorey’s rule for $U'_q(B^{(1)}_{n+1})$ and $U'_q(C^{(1)}_n)$, in terms of twisted and folded AR-quivers.

8.1. Notions on sequences of positive roots. In this section, we briefly review the notions on sequences of positive roots which were mainly introduced in [22, 26].

Following (1.1), for a reduced expression $j_0 = i_1 i_2 \cdots i_N$ of $w_0 \in W$, we set

$$j_0^\beta_k := s_{i_1} \cdots s_{i_{k-1}} \alpha_i \in \Phi^+.$$ 

Now, we identify a sequence $m_{j_0} = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}_{\geq 0}^N$ with

$$(m_1 j_0^\beta_1, \ldots, m_N j_0^\beta_N) \in (\mathbb{Z}_{\geq 0} j_0^\beta_k)_{1 \leq k \leq N} \cong \mathbb{Z}_{\geq 0}^N.$$ 

If there is no danger of confusion, we omit the subscript $j_0$.

The weight $\text{wt}(m)$ of a sequence $m$ is defined by

$$\sum_{i=1}^{N} m_i \beta_i.$$ 

Definition 8.1 ([22, 26]). We define the partial orders $\prec^b_{j_0}$ and $\prec^b_{[j_0]}$ on $\mathbb{Z}_{\geq 0}^N$ as follows:

(i) $\prec^b_{j_0}$ is the bi-lexicographical partial order induced by $\prec_{j_0}$. Namely, $m \prec^b_{j_0} m'$ if there exist $j$ and $k (1 \leq j \leq k \leq N)$ such that

- $m_s = m_s'$ for $1 \leq s < j$ and $m_j < m_j'$,
- $m_s = m_s'$ for $k < s \leq N$ and $m_k < m_k'$.

(ii) For sequences $m$ and $m'$, we have $m \prec^b_{[j_0]} m'$ if and only if $\text{wt}_{j_0}(m) = \text{wt}_{j_0}(m')$ and $n \prec^b_{j_0} n'$ for all $j'_0 \in [j_0]$, where $n$ and $n'$ are sequences such that $n_{j'_0} = m_{j_0}$ and $n'_{j'_0} = m'_{j_0}$.

We give the following definitions from [22, 26]. We call a sequence $m$ a pair if $|m| = \sum_{i=1}^{N} m_i = 2$ and $m_i \leq 1$ for $1 \leq i \leq N$. We mainly use the notation $p$ for a pair. Frequently, we write a pair $p$ as $(\alpha, \beta) \in (\Phi^+)^2$.

We say a sequence $m = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}_{\geq 0}^N$ is $[j_0]$-simple if it is minimal with respect to the partial order $\prec^b_{[j_0]}$. For a given $[j_0]$-simple sequence $s = (s_1, \ldots, s_N) \in \mathbb{Z}_{\geq 0}^N$, we call a cover of $s$ under $\prec^b_{[j_0]}$ a $[j_0]$-minimal sequence of $s$. The generalized $[j_0]$-distance $\text{gdist}_{[j_0]}(m)$ of a sequence $m$ is the largest integer $k \geq 0$ such that

$$m^{(0)} \prec^b_{[j_0]} \cdots \prec^b_{[j_0]} m^{(k)} = m.$$ 

\footnote{Recall that a cover of $x$ in a poset $P$ with partial order $<$ is an element $y \in P$ such that $x < y$ and there does not exist $y' \in P$ such that $x < y' < y$.}
and \( m^{(0)} \) is \([j_0]\)-simple.

Consider a pair \( p \) such that there exists a unique \([j_0]\)-simple sequence \( s \) satisfying \( s \preceq_{[j_0]}^b p \); we call \( s \) the \([j_0]\)-socle of \( p \) and denoted it by \( \text{soc}_{[j_0]}(p) \).

For a non-simple positive root \( \gamma \in \Phi^+ \setminus \Pi \), the \([j_0]\)-radius of \( \gamma \), denoted by \( \text{rds}_{[j_0]}(\gamma) \), is the integer defined as follows:

\[
\text{rds}_{[j_0]}(\gamma) = \max(\text{gdist}_{[j_0]}(p) : p \text{ a pair, } \gamma \preceq_{[j_0]}^b p).
\]

For \( \eta = \sum_{i \in I} m_i \alpha_i \in \Phi^+ \), define the multiplicity of \( \eta \) as the integer defined by

\[
m(\eta) = \max\{ m_i \mid i \in I \}.
\]

**Theorem 8.2.** [26, Theorem 4.15, Theorem 4.20][24, Theorem 3.4] Let \( Q \) be any Dynkin quiver of type \( A_n, D_n \) or \( E_n \)

1. For any pair \( p = (\alpha, \beta) \in (\Phi^+)^2 \), we have \( 0 \leq \text{gdist}_Q(\alpha, \beta) \leq \max\{ m(\alpha), m(\beta) \} \).

2. For any \( \gamma \in \Phi^+ \setminus \Pi \), we have \( \text{rds}_Q(\gamma) \leq m(\gamma) \). Equality holds when \( Q \) is of type \( A_n \) or \( D_n \).

3. For any pair \( p = (\alpha, \beta) \in (\Phi^+)^2 \), \( \text{soc}_Q(p) \) is well-defined.

**8.2. Distance polynomials and Dorey’s rule on \([[\Delta]]\).** Let \( Q \) be a Dynkin quiver of type \( ADE \). Following [26], for an AR quiver \( \Gamma_Q \), indices \( k, l \in I \) and an integer \( t \in \mathbb{Z}_{\geq 1} \), we define the subset \( \Omega_Q(k, l)[t] \subset (\Phi^+)^2 \) as the pairs \( (\alpha, \beta) \) such that \( \alpha \) and \( \beta \) are comparable under \( \prec_Q \) and

\[
\{ \Omega_Q(\alpha), \Omega_Q(\beta) \} = \{(k, a), (l, b)\} \quad \text{such that} \quad |a - b| = t.
\]

**Lemma 8.3.** [26, Lemma 6.12] For any \((\alpha^{(1)}, \beta^{(1)})\) and \((\alpha^{(2)}, \beta^{(2)})\) in \( \Omega_Q(k, l)[t] \), we have

\[
o_t^{\text{rev}}(k, l) := \text{gdist}_Q(\alpha^{(1)}, \beta^{(1)}) = \text{gdist}_Q(\alpha^{(2)}, \beta^{(2)}).
\]

We denote by \( Q^{\text{rev}} \) the quiver obtained by reversing all arrows of \( Q \), and by \( Q^* \) the quiver obtained from \( Q \) by replacing vertices of \( Q \) from \( i \) to \( i^* \).

**Proposition 8.4.** [26, Proposition 6.16] The integer, defined by

\[
o_t(k, l) := \max(o_t^Q(k, l), o_t^{\text{rev}}(k, l))
\]

does not depend on the choice of \( Q \); that is,

\[
o_t^Q(k, l) = o_t^{\text{rev}}(k, l)
\]

for any distinct Dynkin quivers \( Q, Q' \) of the same type.

Since \( o_t(k, l) \) does not depend on the choice of \( Q \), we can the define distance polynomials \( D_{k, l}(z) \in k[z] \) on \([[\Delta]]\).

**Definition 8.5.** [26] For \( k, l \in I \), we define the distance polynomial \( D_{k, l}(z) \in k[z] \) on \([[\Delta]]\)

\[
D_{k, l}^X(z) := \prod_{t \in \mathbb{Z}_{\geq 0}} (z - (-1)^t q^t)^{o_t(k, l)}.
\]

Here \( X \) denotes the type of \([[\Delta]]\).
Theorem 8.6. [26, Theorem 6.18] For any Dynkin quiver $Q$ of type $X$, the denominator formulas for the quantum affine algebra $U'_q(X^{(1)})$ can be read as follows ($X = A_n$ or $D_n$):

$$d_{k,l}^{X^{(1)}}(z) = D_{k,l}^X(z) \times (z - (-q)^h)^{h_{k,l}}$$

where $h^r$ is the dual Coxeter number of type $X$.

8.3. Generalized distance and radius on $[\mathcal{Q}]$. For this subsection, we will prove the following theorem:

Theorem 8.7. For a non-trivial automorphism $\nu$, recall $d$ is defined by using $\bar{\Phi}^+$ in (5.2). Take any $[i_0] \in [\mathcal{Q}]$ or $[\mathcal{Q}]$.

1. For any pair $\underline{p} = (\alpha, \beta) \in (\Phi^+)^2$ (not $\bar{\Phi}^+$), we have $0 \leq \text{gdist}_{[i_0]}(\alpha, \beta) \leq d$.
2. For any $\gamma \in \bar{\Phi}^+ \setminus \Pi$, we have $1 \leq \text{rds}_{[i_0]}(\gamma) \leq d$.
3. For any $\underline{p} = (\alpha, \beta) \in (\Phi^+)^2$ and $\nu$ in (3.1a) or (3.1b), $\text{soc}_{[i_0]}(\underline{p})$ is well-defined.

8.3.1. Proof of Theorem 8.7 for type $A_{2n+1}$.

Lemma 8.8. For a non-simple root $\gamma$ corresponding to non-central vertex in $\Upsilon_{[i_0]}$, we have

$$\text{rds}_{[i_0]}(\gamma) = 1.$$

Proof. Let us assume that $\gamma \in \Upsilon_{[i_0]}^{\text{NE}} \cup \Upsilon_{[i_0]}^{\text{SW}}$. Then, by Lemma 6.5 and Theorem 6.6, every nonzero component of a sequence $\underline{m}$ with $\text{wt}(\underline{m}) = \gamma$ should appear in $\Upsilon_{[i_0]}^{\text{NE}} \cup \Upsilon_{[i_0]}^{\text{SW}}$. Hence our assertion immediately follows from Algorithm 4.15, Lemma 6.11 and Theorem 8.2. We can prove for $\gamma \in \Upsilon_{[i_0]}^{\text{SE}} \cup \Upsilon_{[i_0]}^{\text{NW}}$ in the similar way. □

Lemma 8.9. For any $\gamma \in \Phi^+ \setminus \Pi$ corresponding to an induced central vertex,

$$\text{rds}_{[i_0]}(\gamma) \leq d = 2.$$

Proof. By Corollary 6.8 and Corollary 6.10, there exists a unique pair $(\alpha^*, \beta^*)$ lying in the $n + 1$-th layer $\star$ and $\alpha^* + \beta^* = \gamma$. Also, by Algorithm 4.15, Lemma 6.11 and [26, Proposition 4.24], other pairs $(\alpha, \beta)$ such that $\alpha + \beta = \gamma$ correspond to induced vertices. Moreover, the pairs of induced vertices are not $<^{b}_{[i_0]}$-comparable to each other by Theorem 8.2. Thus our assertion follows from the fact that sometimes the exceptional pair $(\alpha^*, \beta^*)$ is comparable to a pair which consisting of induced vertices. In Example 6.7, we can see

$$[3, 5] <^{b}_{[i_0]} (\alpha^*, \beta^*) = ([4, 5], [3]) <^{b}_{[i_0]} ([5], [3, 4]).$$

By [24, Theorem 3.2], the non-induced vertices pairs of weight $\gamma$ are less than other induced vertices pairs of weight $\gamma$ with respect to $<^{b}_{[i_0]}$ whenever they are comparable. □

Lemma 8.10. For any $\gamma \in \Phi^+ \setminus \Pi$ corresponding to a non-induced central vertex,

$$\text{rds}_{[i_0]}(\gamma) = 1.$$

Proof. Let us assume that $[i_0]$ satisfies the case (1) in (6.1). Then $\gamma$ is $[a, n]$ or $[n + 1, b]$ by Corollary 6.8. We assume further that $\gamma = [a, n]$. Then every pair for $\gamma$ is of the form $\{[a, b - 1], [b, n]\}$. Without loss of generality, we assume that there are pairs

$$\{[a, c - 1], [c, n]\} <^{b}_{[i_0]} \{[a, d - 1], [d, n]\}.$$
Then there is a path between \([c, n]\) and \([d, n]\) by Lemma 6.5 and Corollary 6.8.

Suppose that \([d, n] \prec_{[i_0]} [c, n]\). Then there is a path from \([c, n]\) to \([d, n]\). We can take a path going through two vertices: (i) \(V_d = [d, k]\) right before \([d, n]\) lying in the \((n + 2)\)-th layer, (ii) \(V_c = [c, k]\) for some \(k > n\); that is,

\[
[c, n] \xrightarrow{N\text{-sectional}} [c, k] \xrightarrow{S\text{-sectional}} [d, k] \xrightarrow{N\text{-sectional}} [d, n]
\]

Here \(k > n\) by Corollary 6.10 and \([d, k]\) is an induced central vertex. Now we know

- \([d, n] \prec_{[i_0]} [a, c - 1]\) so that \([d, k] \prec_{[i_0]} [a, c - 1], [c, k]\),
- \([a, c - 1], [c, k] \prec_{[i_0]} [a, d - 1]\) by the fact that \([c, k] \prec_{[i_0]} [c, n]\).

Hence

\[
\{[a, c - 1], [c, k]\} \prec_{[i_0]}^b \{[a, d - 1], [d, k]\},
\]

where they are induced. However, it contradicts to Theorem 8.2 (2).

Also, when there is a path from \([d, n]\) to \([c, n]\), we can prove by similar arguments. □

**The first step for Theorem 8.7.** From the above three lemmas, the second assertion of Theorem 8.7 follows. Furthermore, the first and the third assertions for \((\alpha, \beta)\) with \(\alpha + \beta \in \Phi^+\) also hold.

**Proposition 8.11.** [26, Proposition 4.5] For a Dynkin quiver \(Q\) of type \(A_m\) and \((\alpha, \beta)\) with \(\alpha + \beta \notin \Phi^+\) and \(\text{gdist}_Q(\alpha, \beta) = 1\), there exists a unique rectangle in \(\Gamma_Q\) given as follows:

\[
\begin{array}{ccc}
\gamma & \alpha \\
\downarrow & | & \downarrow \\
\beta & | & \gamma \\
\end{array}
\]

(8.2)

where \((\gamma, \eta) \prec_Q^b (\alpha, \beta)\). Furthermore, there is no pair \((\alpha', \beta') \neq (\gamma, \eta)\) such that \((\alpha', \beta') \prec_Q^b (\alpha, \beta)\).

**Proposition 8.12.** For any pair \((\alpha, \beta)\) of type \(A_{2n+1}\) such that \(\alpha + \beta \notin \Phi^+\),

\[
\text{gdist}_{[i_0]}(\alpha, \beta) \leq 2.
\]

**Proof.** For \((\alpha, \beta)\) which satisfies one of the following properties:

- \(\text{supp}(\alpha) \cap \text{supp}(\beta) = \emptyset\),
- their first (resp. second) components are the same,
- they are incomparable with respect to \(\prec_{[i_0]}\),

one can prove easily that \(\text{gdist}_{[i_0]}(\alpha, \beta) = 0\) by using the convexity of \(\prec_{[i_0]}\) and Theorem 6.6.

Thus \(\text{gdist}_{[i_0]}(\alpha, \beta) > 0\) implies that there exists a rectangle like (8.2) or one of \(\alpha\) and \(\beta\) is a non-induced vertex.

(1) Now we assume that \(\alpha, \beta\) are all induced. By Algorithm 4.15 if there is a rectangle like (8.2) then the rectangle can be classified with the followings:

- (i) the rectangle without non-induced vertices on it,
- (ii) the rectangle with two non-induced vertices whose first or second component are the same,
(iii) the rectangle with two non-induced vertices whose sum is contained in $\Phi^+$ by Lemma 6.8.

For (i), the proofs are the same as in [26, Proposition 4.5]. The cases (ii) and (iii) can be depicted as follows.

\[(8.3)\]

where $\star$'s denote non-induced vertices.

Note that if $\underbrace{m}_{\leq [i_0]} (\alpha, \beta)$, then positive roots occurring in $m$ should be contained in or on the rectangle. Also, Theorem 6.6 and Theorem 8.2 tell that $m$ cannot consist of induced vertices except the pair $(\eta, \gamma)$

(ii) By (8.4), $m \neq (\eta, \gamma)$ must contain a vertex $\star$ if it exists, where $\star$'s in (8.3) share second component. However, the convexity $\leq [i_0]$, the system $\Phi^+$ and Corollary 6.8 tell that such an $m$ cannot exist. Thus $\text{gdist}_{[i_0]} (\alpha, \beta) = 1$.

(iii) By Corollary 6.8, $\mu + \nu = \eta$ and hence we have

\[(\eta, \gamma) \leq_{[i_0]} (\nu, \mu, \gamma) \leq_{[i_0]} (\alpha, \beta).\]

As in the case (ii), there is no $m$ in or on the rectangle with wt$(m) = \alpha + \beta$ and different from $(\nu, \mu, \gamma)$ and $(\eta, \gamma)$. Thus $\text{gdist}_{[i_0]} (\alpha, \beta) = 2$.

(2) Now let $\alpha$ be an induced vertex and $\beta$ be a non-induced vertex. As in (1), we can classify as follows:

\[
\text{(i')}\]

where $\beta^-$ is the largest non-induced vertex such that $\beta \leq [i_0] \beta^-$, and $\beta^+ = \beta + \beta^-$.

In order to see $\text{gdist}_{[i_0]} (\alpha, \beta)$, we need to find a set of vertices such that (a) every element is in or on the rectangle determined by $\beta^+$ and $\alpha$ (b) the sum of elements is $\alpha + \beta$. Hence depending on whether $(\alpha, \beta^+)$ is of the case (i) or (ii) in (1), we get $\text{gdist}_{[i_0]} (\alpha, \beta) = 0$ or 1. For the latter case, we have $(\gamma, \nu) <_{[i_0]} (\alpha, \beta)$ in (ii') since

$$\alpha + \beta = \alpha + \beta^+ - \beta^- = \gamma + \eta - \beta^- = \gamma + \nu + \beta^- - \beta^- = \gamma + \nu.$$  

\[\square\]

**The second step for Theorem 8.7.** From the above propositions, the first and the third assertions are completed.  

\[\square\]
Remark 8.13. Note that, for each pair \((\alpha, \beta)\) with \(\text{gdist}_{[i_0]}(\alpha, \beta) = 2\), there exists a non-simple sequence \(m\)

\[ m <^b_{[i_0]} (\alpha, \beta) \]

which tells \(\text{gdist}_{[i_0]}(\alpha, \beta) = 2\). Furthermore,

1. if \(\alpha + \beta \in \Phi^+\), then \(m\) is a pair consisting of non-induced central vertices,

2. if \(\alpha + \beta \notin \Phi^+\), \(m\) is a triple \((\mu, \nu, \eta) \in (\Phi^+)^3\) such that
   (i) \(\mu + \nu \in \Phi^+\), \((\mu, \nu)\) is an \([i_0]\)-minimal pair of \(\mu + \nu\) and \(\alpha - \mu, \beta - \nu \in \Phi^+\),
   (ii) \(\eta\) is not comparable to \(\mu\) and \(\nu\) with respect to \(<_{[i_0]}\),
   (iii) \(\eta = (\alpha - \mu) + (\beta - \nu)\), and \(((\alpha - \mu), (\beta - \nu))\) is an \([i_0]\)-minimal pair for \(\eta\),
   (iv) \((\alpha - \mu, \mu), (\nu, \beta - \nu)\) are \([i_0]\)-minimal pairs for \(\alpha\) and \(\beta\), respectively.

In Example 6.7, we have

\[ m = ([4, 7], [1, 3], [2, 5]) <^b_{[i_0]} ([2, 7], [1, 5]). \]

8.3.2. Proof of Theorem 8.7 for type \(D_{n+1}\). In this subsection, we assume that \([i_0]\) is a twisted adapted class of type \(D_{n+1}\) and \(p_{\Gamma_{[i_0]}^E}([i_0]) = [Q]\). The proof mainly uses the properties of \(\Gamma_Q\), for example Theorem 8.2. We refer to the proof of the theorem (24) for more details.

Proof of Theorem 8.7 (2). It follows by Lemma 8.14 and Lemma 8.15 below.

Lemma 8.14. For a non-simple root \(\gamma\) in \(\Upsilon^W_{[i_0]} \cup \Upsilon^E_{[i_0]}\), we have

\[ \text{rds}_{[i_0]}(\gamma) = 1. \]

Proof. Suppose that \(\gamma = (a, -b - 1) \in \Upsilon^E_{[i_0]}\) and both \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) are pairs with weight \(\gamma\). Then there are three cases:

(i) \(\alpha_1, \alpha_2, \beta_1, \beta_2\) are all in \(\Upsilon^E_{[i_0]}\),

(ii) One of the roots, say \(\beta_2\) is in \(\Upsilon^W_{[i_0]}\) and the others are in \(\Upsilon^E_{[i_0]}\),

(iii) Two of the roots, say \(\beta_1, \beta_2\) are in \(\Upsilon^W_{[i_0]}\) and the others are in \(\Upsilon^E_{[i_0]}\).

Consider the case (i). Since the labeling of \(\Upsilon^E_{[i_0]}\) is naturally induced from the labeling of \(\Gamma_Q^E\), by Theorem 8.2, we can see that \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) are incomparable.

In the case of (ii), without loss of generality, two roots, say \(\alpha_1\) and \(\alpha_2\) are in the same S-path, since two roots should share the component \(-b - 1\). Also, we know that \(\alpha_1, \alpha_2, \beta_1 <_{[i_0]} \beta_2, \alpha_1, \alpha_2, \beta_2 \in \Upsilon^E_{[i_0]}\) and \(\beta_2 \in \Upsilon^W_{[i_0]}\). Hence \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) are comparable if and only if \(\alpha_2 <_{[i_0]} \alpha_1, \beta_1\). On the other hand, properties of \(\Gamma_Q\) and the assumption (ii) implies that \(\alpha_2 <_{[i_0]} \gamma <_{[i_0]} \alpha_1\) and the two roots \(\beta_1\) and \(\alpha_2\) are not comparable. In conclusion, \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) are not comparable.

In the case of (iii), we have \(\beta_1 <_{[i_0]} \beta_2\) if and only if \(\alpha_1 <_{[i_0]} \alpha_2\), by the property of \(\Gamma_Q\). Hence, again, \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) are not comparable.

As a conclusion, we have \(\text{rds}_{[i_0]}(\gamma) = 1\).

Lemma 8.15. For a non-simple root \(\gamma\) in \(\Upsilon^C_{[i_0]}\), we have

\[ \text{rds}_{[i_0]}(\gamma) = 2. \]
Proof. Let us denote $\gamma = (a, b)$ for $1 \leq a, b \leq n$. In order to show that $\mathrm{rd}s_{[i_0]}(\gamma) \geq 2$, consider the pairs $((a, c), (b, -c))$ and $((a, c), (b, c))$ for $b < c \leq n + 1$. Then two pairs are comparable since we have

$$(a, c) \prec_{[i_0]} (b, c) \text{ if and only if } (a, -c) \prec_{[i_0]} (b, -c).$$

Now it is enough to show that $((a, c), (b, -c))$ and $((a, d), (b, -d))$ are not comparable when $d \neq \pm c$. It can be proved using the properties of labeling of $\Gamma_Q$, as we did in Lemma 8.14. Since it is lengthy but straightforward, we omit the detailed proof.

**Proof of Theorem 8.7 (1).** We shall state the theorem more explicitly in Proposition 8.20.

**Definition 8.16.** Take $\alpha \in \Phi^+$.  

1. Suppose that the $N$-path passing $\alpha$ has the vertex with the folded coordinate $(\hat{n}, p)$. Then the union of the $N$-path and the $S$-path with vertex $(\hat{n}, p - 1)$ is called the *$N$-swing associated to $\alpha$.*

2. Suppose that the $S$-path passing $\alpha$ has the vertex with the folded coordinate $(\hat{n}, p)$. Then the union of the $S$-path and the $N$-path with vertex $(\hat{n}, p + 1)$ is called the *$S$-swing associated to $\alpha$.*

Using the new notion, we can state the following lemma from Proposition 6.20.

**Lemma 8.17.**  

1. There are exactly $n$ swings in $\tilde{\Gamma}_{[i_0]}$.
2. If there are two distinct swings $S_{\alpha}$ and $S_{\beta}$ share the components $r_1$ and $r_2$, respectively, then $r_1$ and $r_2$ are distinct elements in $\{1, 2, \ldots, n\}$.
3. Every vertex in a swing shares a component and all the vertices sharing the component consists of a swing.
4. If $S$ is a swing with the shared component $r$ then the only one of the following is true: 
   a. $S$ has the $N$-path passing $(\hat{1}, p)$ and the $S$-path passing $(\hat{1}, p + 1)$ consists of all the roots with the component $-r$.
   b. $S$ has the $S$-path passing $(\hat{1}, p)$ and the $N$-path passing $(\hat{1}, p - 1)$ consists of all the roots with the component $-r$.

**Example 8.18.** The following quiver is folded AR-quiver $\tilde{\Gamma}_{[i_0]}$ corresponding to the twisted Coxeter element $2135 \vee$.

Here, the vertices $\circ$ do not exist in $\tilde{\Gamma}_{[i_0]}$. However, in order to show shapes of swings, we put fake vertices $\circ$ in the quiver. Note that

- The quiver consists of arrows $N_{\text{sw}}$ is the $N$-swing associated to $(2, 3)$. 

![Diagram of the quiver](attachment:quiver.png)
Lemma 8.19. Let \( \gamma \) be a root in \( \Phi^+ \setminus \Pi \). In \( \Upsilon_{[\delta]} \), a pair \((\alpha, \beta)\) with weight \( \gamma \) is one of \((\alpha_0, \beta_0), (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3) \) and \((\alpha_4, \beta_4)\) in the following picture:

![Diagram](image)

In the case of \((\alpha_0, \beta_0)\), we assume only one of \(\alpha_0\) and \(\beta_0\) shares a swing with \(\gamma\). Hence \((\alpha_0, \beta_0)\) and \((\alpha_2, \beta_2)\) indicate different cases.

Proof. Let us denote \( \gamma = \{a, b\} \). Then a pair \((\alpha, \beta)\) has weight \( \gamma \) if and only if (i) there is \( c \neq \pm a, \pm b \) such that \( \alpha \) has \( c \) as a component and \( \beta \) has \(-c\) as a component (ii) \( \alpha \) and \( \gamma \) share a component \( (\text{iii}) \beta \) and \( \gamma \) share the other component of \( \gamma \).

Consider the case when \( 1 \leq a < b \leq n \) and \( c > 0 \). By Lemma 8.17, we know the following facts:

- \( \alpha \) and \( \gamma \) share a swing, namely \( S_\alpha \).
- \( \beta \) is in the other swing, namely \( S_\beta \), which also passes \( \gamma \).
- Consider the other swing \( S'_\alpha \neq S_\alpha \) crossing \( \alpha \) and the sectional path \( P'_\beta \) passing \( \beta \) which is not in \( S_\beta \). Then \( S'_\alpha \) and \( P'_\beta \) have the property in Lemma 8.17 (4).

Similarly, we can deal with the case \( 1 \leq a < b \leq n \) and \( c < 0 \). Hence, when \( 1 \leq a < b \leq n \), we can show \( (\alpha, \beta) \) should be one of \((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3) \) and \((\alpha_4, \beta_4)\) in the picture.

Now, let us consider the case when \( b < 0 \) or \( b = n + 1 \) and suppose \( \alpha \) and \( \gamma \) share the component \( \beta \). Then, by Lemma 8.17, we know the following facts:

- \( \alpha \) and \( \gamma \) share a sectional path, which is not contained in a swing.
- \( \beta \) and \( \gamma \) share a swing, namely \( S_\beta \).
- Consider the swing associated to \( \alpha \) and the sectional path passing \( \beta \) which is not in the swing. Then they have the property in Lemma 8.17 (4).

We can do the similar thing when \( \alpha \) and \( \gamma \) share the component \( \alpha \). Now, we can show \( (\alpha, \beta) \) should be one of \((\alpha_0, \beta_0), (\alpha_3, \beta_3) \) and \((\alpha_4, \beta_4)\) in the picture.

\[ \square \]

Proposition 8.20. Let \( \alpha, \beta \in \Phi^+ \) satisfy \( \beta \preceq_{[\delta]} \alpha \). The following pictures show the sectional paths and swings passing \( \alpha \) or \( \beta \). The value of \( \text{gdist}(\alpha, \beta) \) is determined as follows.
Proof. By Proposition 6.20 and Lemma 6.17, we can check that
\((8.9)\) 
\[
\alpha + \beta = \eta + \xi = \gamma + \delta, \quad \text{and} \quad \alpha + \beta = \eta + \xi = \gamma
\]
in (I-1) and (I-2), respectively. By Lemma 6.17, we have
\((8.10)\) 
\[
\alpha + \beta = \eta + \xi, \quad \eta, \gamma + \delta, \quad \text{and} \quad \gamma
\]
in (I-3), (I-6), (II-1), and (II-2), respectively.

Now, it is enough to show that sequences in (8.9) and (8.10) are all we need to consider. We can check that if a sequence \(\mathbf{m}\) consists only one or two roots and is smaller than \((\alpha, \beta)\) with respect to \(\prec_{\mathbf{m}}\) then \(\mathbf{m}\) is one of sequences we listed in (8.9) or (8.10). (Here we omit the detailed proof but the main idea is the same as the argument in Lemma 8.19.)

The last thing we need to show is that there is no sequence \(\mathbf{m}\) which consists of more than two roots and satisfies \(\mathbf{m} \prec_{\mathbf{m}} (\alpha, \beta)\). Suppose it is not true in the case (I-1). This implies that there is a triple \((\Phi^*)^2 \triangleright (\gamma_1, \gamma_2, \gamma_3) \prec_{\mathbf{m}} (\alpha, \beta)\). Since
\[
\gamma_1 + \gamma_2 + \gamma_3 = \alpha + \beta = \gamma + \delta = \eta + \xi,
\]
the sum of two roots, say \(\gamma_1 + \gamma_2\), in \((\gamma_1, \gamma_2, \gamma_3)\) should be equal to \(\alpha, \beta, \gamma, \delta, \eta\) or \(\xi\). However, by Lemma 8.19, any of \(\alpha, \beta, \gamma, \delta, \eta\) and \(\xi\) cannot be obtained as a sum of two roots which are both
smaller than \( \alpha \) and bigger than \( \beta \) with respect to \( <_{[\mathfrak{i}_0]} \). Hence it contracts to our assumption and there is no such triple. As a conclusion, for the case (I-1), we have \( gdist(\alpha, \beta) = 2 \).

Similarly for other cases, we can prove that there exists no sequence \( \mathbf{m} \) which consists of more than two roots and satisfies \( \mathbf{m} <_{[\mathfrak{i}_0]} (\alpha, \beta) \) by Lemma 8.19. Hence we proved the proposition. \( \square \)

**Proof of Theorem 8.7 (3).** In each case of Proposition 8.20, there is a unique socle \( \text{soc}(\alpha, \beta) \)

**Remark 8.21.** As in Remark 8.13 for type \( A_{2n+1} \), for each pair \((\alpha, \beta)\) with \( gdist_{[\mathfrak{i}_0]}(\alpha, \beta) = 2 \), there exists unique chain of non-simple sequences

\[
\mathbf{m}_1 <_{[\mathfrak{i}_0]} (\alpha, \beta)
\]

which tells \( gdist_{[\mathfrak{i}_0]}(\alpha, \beta) = 2 \). (See the cases of (I-1) and (I-2) of (8.6) in Proposition 8.20.)

8.3.3. **Proof of Theorem 8.7 for exceptional cases.** For \([\mathfrak{i}_0] \in [[\mathcal{Q}]]\) (resp. \([\mathfrak{i}_0] \in [[\Omega]]\)) of type \( E_6 \) with respect \( \nu \) in (3.1c) (resp. \( D_4 \) with respect \( \nu \) in (3.1d)) , we can check that

1. For any pair \((\alpha, \beta) \in (\Phi^*)^2 \) (not \((\Phi^*)^2\)), we have \( 0 \leq gdist_{[\mathfrak{i}_0]}(\alpha, \beta) \leq 2 \leq d \).
2. For any \( \gamma \in (\Phi^* \setminus \Pi) \), we have \( 1 \leq rds_{[\mathfrak{i}_0]}(\gamma) \leq 2 \leq d \),

by observing all twisted AR-quivers.

8.4. **Folded distance polynomials on \([[\mathcal{Q}]]\) and \([[\Omega]]\).** In this subsection, we define folded distance polynomials by considering folded AR-quiver \( \tilde{\mathcal{T}}_{[\mathfrak{i}_0]} \) and prove that the folded distance polynomials are well-defined on \([[\mathcal{Q}]]\) (resp. \([[\Omega]]\)).

**Definition 8.22.** For a folded AR-quiver \( \tilde{\mathcal{T}}_{[\mathfrak{i}_0]} \), indices \( \hat{k}, \hat{l} \in \hat{\mathcal{T}} \) and \( t \in \mathbb{N}/d \), we define the subset \( \Phi_{[\mathfrak{i}_0]}(\hat{k}, \hat{l})[t] \) of \((\Phi^*)^2\) as follows:

A pair \((\alpha, \beta)\) is contained in \( \Phi_{[\mathfrak{i}_0]}(\hat{k}, \hat{l})[t] \) if \( \alpha <_{[\mathfrak{i}_0]} \beta \) or \( \beta <_{[\mathfrak{i}_0]} \alpha \) and

\[
\{ \Omega_{[\mathfrak{i}_0]}(\alpha), \Omega_{[\mathfrak{i}_0]}(\beta) \} = \{ (\hat{k}, \alpha), (\hat{l}, \beta) \} \quad \text{such that} \quad |a - b| = t.
\]

**Lemma 8.23.** For any \((\alpha^{(1)}, \beta^{(1)})\) and \((\alpha^{(2)}, \beta^{(2)})\) in \( \Phi_{[\mathfrak{i}_0]}(\hat{k}, \hat{l})[t] \), we have

\[
\sigma_{t}[^{\mathfrak{i}_0}](\hat{k}, \hat{l}) := gdist_{[\mathfrak{i}_0]}(\alpha^{(1)}, \beta^{(1)}) = gdist_{[\mathfrak{i}_0]}(\alpha^{(2)}, \beta^{(2)}).
\]

**Proposition 8.24.** The integer, defined by

\[
\sigma_{t}[^{\mathfrak{i}_0}](\hat{k}, \hat{l}) := \left\lfloor \frac{\sigma_{t}[^{\mathfrak{i}_0}](\hat{k}, \hat{l})}{d} \right\rfloor,
\]

does not depend on the choice of \([\mathfrak{i}_0] \in [[\mathcal{Q}]]\) that is,

\[
\sigma_{t}[^{\mathfrak{i}_0}](\hat{k}, \hat{l}) = \sigma_{t}[^{\mathfrak{i}_0}](\hat{k}, \hat{l})
\]

for any distinct Dynkin quivers \([\mathfrak{i}_0], [\mathfrak{i}_0'] \in [[\mathcal{Q}]]\) of the same type.

From Proposition 8.24, we can define \( \tilde{D}_{\hat{k}, \hat{l}}(z) \) for the twisted adapted \( r \)-cluster point \([[\mathcal{Q}]]\) as in the below, and call it the folded distance polynomial at \( \hat{k} \) and \( \hat{l} \).
Definition 8.25. For any $\hat{k}, \hat{l} \in \hat{T}$ and folded AR-quiver, we define the folded distance polynomial $D_{\hat{k},\hat{l}}(z) \in k[z]$ on $[[2]]$ as follows:

$$D_{\hat{k},\hat{l}}(z) := \begin{cases} \prod_{t \in \mathbb{N}} (z - (-1)^{k+l}(q^{1/4}t)^{o^{r}[i_{0}]}(k,l)), & \text{if } v \text{ is } (3.1a) \text{ or } (3.1c), \\ \prod_{t \in \mathbb{N}} (z - (-q^{1/4}t)^{o^{r}[i_{0}]}(k,l)), & \text{if } v \text{ is } (3.1b) \text{ or } (3.1d). \end{cases}$$

Here $X$ denotes the type of $[[2]]$ or $[[\Omega]]$.

8.4.1. Type $A_{2n+1}$. Recall that the indices of $\hat{T}$ are given as follows

$$\hat{T} = \{1, 2, \ldots, n, n+1\}.$$  

Proof of Lemma 8.23 for $A_{2n+1}$ case. (1) Assume that $\hat{k}, \hat{l} \in \hat{T} \setminus \{n+1\}$. By Theorem 8.2, the set $\Phi_{(i_{0})}(\hat{k},\hat{l})[t]$ is induced from one of

$$\Phi_Q(k,l)[t] \cup \Phi_Q(k^*,l^*)[t] \text{ and } \Phi_Q(k*,l)[t] \cup \Phi_Q(k,l^*)[t].$$

where $t \in \mathbb{N}$, $P_{[[2]]}(i_{0}) = [Q]$ and $i \not\sim 2n+1-i$. Note that one of the sets in (8.11) must be empty by the parity of $t$.

In each case, if there exists a path from $\beta^{(1)}$ to $\alpha^{(1)}$ passing through two non-induced vertices, then so is $(\alpha^{(2)}, \beta^{(2)})$. Then our assertion for this case follows from Corollary 6.11, Theorem 8.7 and Lemma 8.3.

(2) Assume that $\hat{k} = \hat{l} = \{n+1\}$. By Lemma 6.8 either (i) $\alpha^{(j)} + \beta^{(j)} \notin \Phi^+$ or (ii) $\alpha^{(j)} + \beta^{(j)} \notin \Phi^+$ and they shares one component. Then, for all $j$, Lemma 8.9 tells that we have

$$\text{gdist}_{(i_{0})}(\alpha^{(j)}, \beta^{(j)}) = \begin{cases} 1 & \text{if } (i), \\ 0 & \text{if } (ii). \end{cases}$$

(3) Assume that one of $\hat{k}$ and $\hat{l}$ is $n+1$ and the other is not. Then our assertion follows from Lemma 8.10 and (2) in the proof of Proposition 8.12, since they just consider the local condition determined by the pair $(\alpha, \beta)$. □

Proof of Proposition 8.24. It is enough to consider when $[i_{0}'] = [i_{0}r_{i}]$. Then our assertion is obvious for $i \neq n+1$ by Algorithm 4.15, Proposition 8.4 and Theorem 8.7; that is, $o_{t}[i_{0}](\hat{k},\hat{l}) \neq 0$ implies

(i) $o_{t}[i_{0}](\hat{k},\hat{l}) = 1$ and (ii) $o_{t}[i_{0}](\hat{k},\hat{l}) \neq 0$ if and only if $o_{t}[i_{0}r_{i}](\hat{k},\hat{l}) \neq 0$.

When $i = n+1$ is also obvious from the fact that $[Q^{c}]r_{n+1} = [Q^{c}]$. □

8.4.2. Type $D_{n+1}$.

Proof of Lemma 8.23 and Proposition 8.24 for $D_{n+1}$ case. By Proposition 8.20, gdist$_{(i_{0})}(\alpha, \beta)$ is determined by their relative positions for any $[i_{0}] \in [[2]]$. Hence our assertion follows the fact that gdist$(\alpha, \beta) = 0, 1$ or $2$. □

8.4.3. Remained types. By checking all folded AR-quivers for remained types, one can easily check that Lemma 8.23 and Proposition 8.24 hold for the cases, also.
8.5. **Minimal pairs on** [\([\mathcal{D}]\)]. In this subsection, we shall record the relative positions of \((\alpha, \beta)\) which is an \([i_0]\)-minimal pair of some \(\gamma \in \Phi^+\) for \([i_0] \in [\mathcal{D}]\) of type \(A_{2n+1}\) and \(D_{n+1}\). Due to the well-definedness of folded distance polynomials, the relative positions do not depend on the choice of \([i_0]\).

8.5.1. \(A_{2n+1}\)

**Theorem 8.26.** [24, Theorem 3.2, Theorem 3.4] For a Dynkin quiver \(Q\) of type \(A_{2n}\) and every pair \((\alpha, \beta)\) of \(\alpha + \beta = \gamma \in \Phi^+_A\), we write

\[
\Omega_Q(\alpha) = (i, p), \quad \Omega_Q(\beta) = (j, q) \quad \text{and} \quad \Omega_Q(\gamma) = (k, z).
\]

Then \((\alpha, \beta)\) is \([Q]\)-minimal and

\[
\tag{8.12}
(i) \quad p - z = |i - k| \quad \text{and} \quad q - z = |j - k|,
(ii) \quad i + j = k \quad \text{or} \quad (2n + 1 - i) + (2n + 1 - j) = (2n + 1 - k).
\]

Define

\[
\tilde{i} = \begin{cases} 
    i - 1 & \text{if } i > n + 1, \\
    i & \text{if } i \leq n + 1.
\end{cases}
\]

**Lemma 8.27.** For \(\gamma \in \Phi^+ \setminus \Pi\) with \(\Omega_{[i_0]}(\gamma) = (k, r)\) and \(\text{rds}_{[i_0]}(\gamma) = 2\), an \([i_0]\)-minimal pair \((\alpha, \beta)\) for \(\gamma\) satisfies one of the following conditions: Set \(\Omega_{[i_0]}(\alpha) = (i, p)\) and \(\Omega_{[i_0]}(\beta) = (j, q)\).

(i) \(i = j = n + 1\) such that \(p + q = 2r\),
(ii) \(q = |k^- - i^-| + p, \quad r = p - |k^- - j^-|, \quad i, j \neq n + 1\) and one of the following holds:

\[
\tag{8.13}
\begin{cases} 
    (a) \quad i + j = k \quad \text{and} \quad k \leq n, \\
    (b) \quad (2n + 1 - i^-) + (2n + 1 - j^-) = 2n + 1 - k^-, \quad k \leq n \quad \text{and} \quad \min\{i, j\} \leq n, \\
    (c) \quad i^- + j^- = k^-, \quad k \geq n + 2 \quad \text{and} \quad \max\{i, j\} \geq n + 2, \\
    (d) \quad (2n + 2 - i) + (2n + 2 - j) = 2n + 2 - k \quad \text{and} \quad k \geq n + 2.
\end{cases}
\]

**Proof.** By Lemma 8.9, \(\gamma\) with \(\text{rds}_{[i_0]}(\gamma) = 2\) has a unique pair \((\alpha, \beta)\) which consists of non-induced central vertices and is an \([i_0]\)-minimal pair for \(\gamma\). Then the first assertion follows. The other \([i_0]\)-minimal pairs for \(\gamma\) are induced from \(\Gamma_Q\) and incomparable with each others. Then one can easily check that the other \([i_0]\)-minimal pairs satisfy one of the four conditions in (8.13) by Theorem 8.26. \(\square\)

**Lemma 8.28.** For an induced vertex \(\gamma \in \Phi^+ \setminus \Pi\) with \(\Omega_{[i_0]}(\gamma) = (k, r)\) and \(\text{rds}_{[i_0]}(\gamma) = 1\), an \([i_0]\)-minimal pair \((\alpha, \beta)\) for \(\gamma\) satisfies the following conditions: Set \(\Omega_{[i_0]}(\alpha) = (i, p)\) and \(\Omega_{[i_0]}(\beta) = (j, q)\). Then \(p - r = |k^- - i^-|, \quad q - r = |k^- - j^-|\) and

\[
\tilde{i} + \tilde{j} = k^- \quad \text{or} \quad (2n + 1 - i^-) + (2n + 1 - j^-) = 2n + 1 - k^-.
\]

**Proof.** One can see that all pairs for \(\gamma\) are induced from \(\Gamma_Q\) and they not comparable with each others. Then we can apply the same argument of the previous lemma. \(\square\)

**Lemma 8.29.** For a non-induced central vertex \(\gamma \in \Phi^+ \setminus \Pi\) with \(\Omega_{[i_0]}(\gamma) = (n + 1, r)\), an \([i_0]\)-minimal pair \((\alpha, \beta)\) for \(\gamma\) satisfies one of the following conditions: Set \(\Omega_{[i_0]}(\alpha) = (i, p)\) and \(\Omega_{[i_0]}(\beta) = (j, q)\).
(i) \((i, p) = (\ell, r + \frac{1}{2} + (n - \ell))\) and \((j, q) = (n + 1, r - 2\ell),\)
(ii) \((i, p) = (2n + 2 - \ell, r + \frac{1}{2} + (n - \ell))\) and \((j, q) = (n + 1, r - 2\ell),\)
(iii) \((i, p) = (n + 1, r + 2\ell)\) and \((j, q) = (\ell, r - \frac{1}{2} - (n - \ell)),\)
(iv) \((i, p) = (n + 1, r + 2\ell)\) and \((j, q) = (2n + 2 - \ell, r - \frac{1}{2} - (n - \ell)),\)
for some \(\ell \in \mathbb{Z}_{\geq 1}.\)

**Proof.** Let us assume that \(\gamma = [a, n + 1]\) for some \(a \leq n\) and is contained in the \(N\)-path \(N[a]\) with \((2n + 1 - a)\)-arrows. Note that \(\{\alpha, \beta\} = \{[a, k], [k + 1, n + 1]\}\) for some \(a \leq k < n + 1\). We assume further that \(\beta = [k + 1, n + 1]\) and \(\widehat{\Omega}_{[i_0]}(\beta) = (n + 1, r - 2\ell)\) for some \(\ell \in \mathbb{Z}_{\geq 1}\) (see Corollary 6.8). Note that there exists an \(S\)-path \(S[k]\) with \((k - 1)\)-arrows
- whose vertices shares \(k\) as their second component,
- which intersects with \(N[a]\).

Furthermore, the vertex located at the intersection of \(N[a]\) and \(S[k]\) is \([a, k]\). By the assumption that \([a, k] \subset_{[i_0]} [a, n + 1]\), the \([i_0]\)-residue \(i\) of \([a, k]\) is strictly less than \(n + 1\) by Theorem 6.6. By applying [24, Corollary 1.15] and Theorem 6.6, we have the following in \(\Upsilon_{[i_0]}:\)

![Diagram](image)

Hence we can obtain that \(i = \ell\) which yields our first assertion. For the remained cases, one can prove by applying the similar argument. \(\square\)

Now, we record coordinates of minimal pairs for \(\gamma \in \Phi^+ \setminus \Pi\) in \(\Upsilon_{[i_0]}\). The following proposition is an immediate consequence of Lemma 8.27, Lemma 8.28 and Lemma 8.29:

**Proposition 8.30.** For \(\alpha, \beta, \gamma \in \Phi^+\) with \(\widehat{\Omega}_{[i_0]}(\alpha) = (i, p), \widehat{\Omega}_{[i_0]}(\beta) = (j, q) \widehat{\Omega}_{[i_0]}(\gamma) = (k, r)\) and \(\alpha + \beta = \gamma\) \((i, j, k, r, q \in \overline{\Gamma})\), \((\alpha, \beta)\) is an \([i_0]\)-minimal pair of \(\gamma\) if and only if one of the following conditions holds:

\[
(i) \quad \ell := \max(i, j, k) \leq n, \ i + j + k = 2\ell \quad \text{and} \quad (q - r, p - r) = \begin{cases} (-i, j), & \text{if } \ell = k, \\ (i - (2n + 1), j), & \text{if } \ell = i, \\ (-i, 2n + 1 - j), & \text{if } \ell = j. \end{cases}
\]

\[(8.14)\]

(ii) \(s := \min(i, j, k) \leq n,\) the others are the same as \(n + 1\) and

\[
(q - r, p - r) = \begin{cases} (- (n - k) + 1/2, (n - k) - 1/2), & \text{if } s = k, \\ (- 2i - 2, (n - i) - 1/2), & \text{if } s = i, \\ (- (n - j) + 1/2, 2j + 2), & \text{if } s = j. \end{cases}
\]
8.5.2. $D_{n+1}$. The relative positions for an $[i_0]$-minimal pair $(\alpha, \beta)$ for $\gamma \in \Phi^+$ follow from Lemma 8.19.

**Proposition 8.31.** For $\alpha, \beta, \gamma \in \Phi^+$ with $\Omega_{[i_0]}(\alpha) = (i, p)$, $\Omega_{[i_0]}(\beta) = (j, q)$, $\Omega_{[i_0]}(\gamma) = (k, r)$ ($i, j, k \in \mathbb{I}$) such that $\alpha + \beta = \gamma$, the pair $(\alpha, \beta)$ is an $[i_0]$-minimal pair of $\gamma$ if and only if one of the following conditions holds:

$$
\ell := \max(i, j, k) \leq n, \quad i + j + k = 2\ell \quad \text{and}
$$

$$
(q - r, p - r) = \frac{1}{2} \left\{ \begin{array}{ll}
(i - (2n + 2), j), & \text{if } \ell = k, \\
(-i, 2n + 2 - j), & \text{if } \ell = i,
\end{array} \right.
$$

(8.15)

8.6. **Twisted additive property.** In this subsection, we briefly show that the folded AR-quivers have some property which can be understood as a generalization of the additive property in (2.2), by using the results in previous sections.

**Proposition 8.32.** Let $[i_0] \in [[\mathcal{Z}]]$ or $[[\Omega]]$ be a (triply) twisted adapted class of type $A_{2n+1}$, $D_{n+1}$, $E_6$ or $D_4$. Suppose $\alpha \in \Phi^+$ has residue $i \in \mathbb{I}$ in $Y_{[i_0]}$ and, in the folded AR-quiver, $\Omega_{[i_0]}(\alpha) = (i, p)$. Let $[i]$ be the number of indices in the orbit $i$. If there is $\beta \in \Phi^+$ such that $\Omega_{[i_0]}(\beta) = (i, p - 2\frac{|i|}{d})$ then we have

$$
\alpha + \beta = \sum_{\gamma \in \beta[i_0]} \gamma,
$$

(8.16)

where

$$
\beta[i_0] = \left\{ \gamma \in \Phi^+ \mid \Omega_{[i_0]}(\gamma) = (j, r) \quad \text{such that} \quad \left\{ \begin{array}{l}
(a) \quad |j - \hat{j}| = 1, \\
(b) \quad p - 2\frac{|i|}{d} \leq r \leq p
\end{array} \right. \right\}.
$$

(8.17)

**Proof.** For types $A_{2n+1}$, $D_{n+1}$ and $E_6$, the condition in (8.17) can be re-interpreted as follows:

$$
\beta[i_0] = \left\{ \gamma \in \Phi^+ \mid \text{there exists an arrow } \gamma \to \alpha \text{ or } \beta \to \gamma \text{ in } Y_{[i_0]} \right\}.
$$

For type $A_{2n+1}$, our assertion is a direct consequence of Theorem 6.6. For type $D_{n+1}$, our assertion follows from Lemma 6.17 unless $\hat{i} = n$. If $\hat{i} = n$, then our assertion follows from the property of swing and Lemma 6.17 together. For exceptional cases, one can check by direct computations. \qed

**Remark 8.33.**

1. The set $\beta Q_\alpha$ in (2.3) coincides with $\beta[i_0]$, in (8.17) when $[i_0] = [Q]$ for a Dynkin quiver $Q$.

2. Let us take $[i_0] \in [[\mathcal{Z}]]$ or $[[\Omega]]$ which is associated to a (triply) twisted Coxeter element $\phi_{[i_0]}$. Then, the $\beta$ in (8.16) can be written as follows:

$$
\beta = (\phi_{[i_0]}^\vee)^{\hat{\beta}}(\alpha).
$$

Thus, (8.16) can be said to be the **twisted additive property** of $Y_{[i_0]}$, comparing with (2.2).
9. Applications on denominators and Dorey’s rule for $U_q'(B_{n+1}^{(1)})$ and $U_q'(C_n^{(1)})$

In this section, we shall show that the denominator formulas and Dorey’s rule for $U_q'(B_{n+1}^{(1)})$ and $U_q'(C_n^{(1)})$ are well-reflected onto a folded AR-quiver $\tilde{\Gamma}_{[i_0]}$ for any $[i_0] \in [\mathscr{Q}]$ of type $A_{2n+1}$ and $D_{n+1}$, respectively. More precisely, we shall prove the twisted analogues of Theorem 8.6 and Theorem 7.6, by collecting results in previous sections. We also prove that the additive property of $\tilde{\Gamma}_{[i_0]}$ is related to the T-system of $U_q'(\tilde{X}^{(1)})$.

**Theorem 9.1.** For any $[i_0] \in [\mathscr{Q}]$ of type $X$, the denominator formulas for the quantum affine algebra $U_q'(\tilde{X}^{(1)})$ can be read from $\tilde{\Gamma}_{[i_0]}$ ($X = A_{2n+1}$ or $D_{n+1}$) and ($\tilde{X} = B_{n+1}$ or $C_n$):

$$d_{k,l}^{(i)}(z) = D_{k,l}^{X}(z) \times (z - q^{h'_v})^{b_{i,k}},$$

where $h'_v$ is the dual Coxeter number of type $\tilde{X}$.

**Proof of Theorem 9.1** for $X = A_{2n+1}$ and $\tilde{X} = B_{n+1}$. Fix $Q$ such that $P_{[\mathscr{Q}]}([i_0]) = [Q]$. Recall the denominator formulas for type $A_n^{(1)}$ and $B_{n+1}^{(1)}$ in Theorem 7.5 (a) and (b). By considering the denominator formulas $d_{k,l}^{A_n}(z)$, $d_{k,l}^{B_n}(z)$ and the distance polynomials $D_{k,l}^{A_{2n}}(z)$ for $1 \leq k, l \leq n$, we have an interesting interpretation as follows:

$$\frac{d_{k,l}^{B_n}(z)}{(z - q^{h'_v})^{b_{i,k}}} = D_{k,l}^{A_{2n}}(z) \times D_{k,l}^{A_{2n}}(-z)$$

where $D_{k,l}^{A_{2n}}(z) = D_{k,l}^{B_{n+1}}(z) = D_{l,k}^{A_{n+1}}(z) = D_{l,k}^{A_{n+1}}(z)$ are distance polynomials on $[[\Delta]]$ of type $A_{2n}$, $\ast$ is in Definition 1.1 and $h'_v$ denotes the dual Coxeter number of $B_{n+1}$.

1. Assume that $k, l \in \tilde{T} \setminus \{n+1\}$ where $\tilde{T} = \{1, 2, \ldots, n+1\}$. Then one of $o_{i}^{Q}(k, l)$ and $o_{i}^{Q}(l, k)$ is positive implies that $o_{i}^{[i_0]}(k, l) > 0$ and hence $o_{i}^{[i_0]}(k, l) = 1$ by Proposition 8.12. Thus our assertion for this case follows from (9.1).

2. Assume that $k = l = n+1$, and write $\Omega_{[i_0]}(\alpha) = (n+1, p)$ and $\Omega_{[i_0]}(\beta) = (n+1, q)$. Then our assertion is obvious since either $\Phi^{+} \ni \alpha + \beta \in [i_0]$ or $\Omega_{[i_0]}(\alpha, \beta)$ when $|q - p| = 2s - 1$ for some $s \geq 2$ and $\text{gcd}_{[i_0]}(\alpha, \beta) = 0$ otherwise.

3. In general, it suffices to consider only one folded AR-quiver $\tilde{\Gamma}_{[i_0]}$ by Proposition 8.24. We take the Dynkin quiver

$$Q : \begin{array}{ccccccc}
1 & 2 & \cdots & 2n-1 & 2n
\end{array}$$

of type $A_{2n}$ and $[i_0]$ as $[Q^c]$. Then [26, (6.20)] and Corollary 6.11, we can draw $\tilde{\Gamma}_{[i_0]}$ with its labels. Then one can check that the assertion for $k = n+1$ and $\tilde{l} \neq n+1$ holds by reading $\tilde{\Gamma}_{[i_0]}$. We skip the proof and provide a particular example for $[Q^c]$ of type $A_5$ instead. \qed
Example 9.2. Here are $\Gamma_Q$, $\Upsilon_{[Q^c]}$ and $\Upsilon_{[Q^c]}$ for $Q$:

\[ \begin{array}{c}
1 & [4] & 2 & [3] & 3 & [2] & 4 \\
2 & [2,4] & [1,2] & [1] & 1 & [5] & 3 & [4,5] & [2,4] & [2] & [1,2] & 1\\n3 & [1,4] & 4 & \end{array} \]

\[ \begin{array}{c}
1 & [4] & 2 & [3] & 3 & [2] & 4 \\
2 & [2,4] & [1,2] & [1] & 1 & [5] & 3 & [4,5] & [2,4] & [2] & [1,2] & 1\\n3 & [1,4] & 4 & \end{array} \]

(9.2)

\[ \begin{array}{c}
1 & [5] & 2 & [3,5] & 3 & [4,5] & 4 & [2,4] & [2,3] & [4] & [1,4] & [1,2] \\
\end{array} \]

\[ \begin{array}{c}
1 & [5] & 2 & [3,5] & 3 & [4,5] & 4 & [2,4] & [2,3] & [4] & [1,4] & [1,2] \\
\end{array} \]

Proof of Theorem 9.1 for $X = D_{n+1}$ and $\tilde{X} = C_n$. Recall the denominator formulas for $U'_q(C_n)$:

\[ d_{k,l}^{(1)}(z) = \prod_{s=1}^{\min(k,l,n-k,n-l)} (z - (-q^{1/2})^{k-l}) \prod_{s=1}^{\min(k,l)} (z - (-q^{1/2})^{2n+2-k-l+2s}) \quad 1 \leq k, l \leq n \]

Then, for $1 \leq k, l \leq n$, one can observe that

\[ d_{k,l}^{(1)}(z) = \prod_{s=1}^{\min(k,l,n-k,n-l)} (z - (-q^{1/2})^{k-l}) \prod_{s=1}^{\min(k,l)} (z - (-q^{1/2})^{2n+2-k-l+2s}) \quad 1 \leq k, l \leq n \]

(9.3)

(i) the first factor of $d_{k,l}^{(1)}(z)$ is the same as $d_{k,l}^{(1)}(z)$ $(1 \leq k, l \leq n - 1)$,

(ii) the second factor of $d_{k,l}^{(1)}(z)$ is the same as the second factor of $d_{k,l}^{(1)}(z)$.

Thus we can apply the same argument of [26, Theorem 6.18]. More precisely, (i) is induced from (I-1) and (I-2) in (8.8), and (ii) is induced from (I-1), (I-2), (I-3) and (I-6) in (8.6).

To make a twisted analogue of Theorem 7.6, we first define certain category $\mathcal{C}_{[i_0]}$ of modules over quantum affine algebra $U'_q(\tilde{X})$ associated to a (triply) twisted adapted class $[i_0]$:

Definition 9.3. Let $[i_0]$ be a (triply) twisted adapted class of type $X$. For any positive root $\beta$ contained in $\Phi'_X$, we set the $U'_q(\tilde{X})$-module $V_{[i_0]}(\beta)$ as follows:

\[ V_{[i_0]}(\beta) := \begin{cases} 
V(\varpi_i)(-1)q^{p/2} & \text{if } \nu \text{ is } (3.1a) \text{ or } (3.1c), \\
V(\varpi_i)(-q^{1/4})p & \text{if } \nu \text{ is } (3.1b) \text{ or } (3.1d), 
\end{cases} \]

where $\widehat{\Omega}_{[i_0]}(\beta) = (i, p/d)$.

Denote by $\mathcal{C}_{[i_0]}$ the smallest abelian full subcategory of $\mathcal{C}_{\tilde{X}}$ such that

(a) it is stable under subquotient, tensor product and extension,

(b) it contains $V_{[i_0]}(\beta)$ for all $\beta \in \Phi'_X$ and the trivial module.

Theorem 9.4. Let $(i, x)$, $(j, y)$, $(k, z) \in I \times k^\times$. Then

\[ \text{Hom}_{U'_q(\tilde{X})}(V(\varpi_k)_x, V(\varpi_j)_y \otimes V(\varpi_i)_z) \neq 0 \quad (\tilde{X} = B_{n+1} \text{ or } C_n) \]

if and only if there exists a twisted adapted class $[i_0] \in [[2]]$ of type $X$ ($X = A_{2n+1}$ or $D_{n+1}$) and $\alpha, \beta, \gamma \in \Phi'_X$ such that
can be understood as a generalization of Theorem 13(I.1a)

where (A.1b)

where (A.1c)

denominator formulas and Dorey’s rule for

be an adapted or a (triply) twisted adapted class of type

3.2.1])

we can interpret the twisted additive property described in (8.16) as follows: Let \([i_0]\)

as an AR-quiver

we can read the denominator formulas and Dorey’s rule for

In particular, (9.2) implies that the socle of \(V_{[i_0]}([1,4]) \to V_{[i_0]}([1,2]) \otimes V_{[i_0]}([3,4])\), even

Though

Remark 9.5. By considering the particular case \((k = 1)\) of the T-system (see for example [13, 3.2.1]), we can interpret the twisted additive property described in (8.16) as follows: Let \([i_0]\)

be an adapted or a (triply) twisted adapted class of type \(X\). In the Grothendieck ring of \(C_{[i_0]}\), we have

\[
\begin{align*}
&[V_{[i_0]}(\alpha)][V_{[i_0]}(\beta)] = \text{hd}(V_{[i_0]}(\alpha) \otimes V_{[i_0]}(\beta)) + \prod_{\gamma \in [i_0]} [V_{[i_0]}(\gamma)],
\end{align*}
\]

for \(\alpha, \beta\) in (8.16). Here, \([M]\) denotes the isomorphism class of a \(U_q'\((\text{g})\)\)-module \(M\) in \(C_{U_q'((\text{g}))}\). In particular, (9.5) implies that the socle of \(V_{[i_0]}(\alpha) \otimes V_{[i_0]}(\beta)\) is the same as \(\bigotimes_{i=1} R V_{[i_0]}(\gamma_i)\), when \(\beta[i_0] = \{\gamma_1, \ldots, \gamma_r\}\).

APPENDIX A. CONJECTURES ON TYPES \(F_4^{(1)}\) AND \(G_2^{(1)}\)

As in Section 9, we can read the denominator formulas and Dorey’s rule for \(U'_q(B_7^{(1)})\) (resp. \(U'_q(C_6^{(1)})\)) from any folded AR-quiver \(\tilde{\mathcal{F}}_{[i_0]}\) of type \(A_{2n+1}\) (resp. \(D_{n+1}\)). Thus, we expect the denominator formulas and Dorey’s rule for \(U'_q(F_4^{(1)})\) and \(U'_q(G_2^{(1)})\) from any (triply) folded AR-quiver \(\tilde{\mathcal{F}}_{[i_0]}\) of type \(E_6\) and \(D_4\).

First, we give a conjectural denominator formula for \(U'_q(G_2^{(1)})\).

Conjectural denominator formulas for \(U'_q(G_2^{(1)})\): Set \(q_s\) such that \(q_s^3 = q\). The conjectural \(d_{k,l}(z)\) are given as follows:

\[
\begin{align*}
&d_{k,l}(z) = \tilde{D}_{k,l}[\Omega](z) \times (z - q^{h^*})^{6_{i,k}}.
\end{align*}
\]

(A.1a) \(d_{1,1}(z) = (z - q_s^6)(z - q_s^8)(z - q_s^{10})(z - q_s^{12})\),

(A.1b) \(d_{1,2}(z) = (z + q_s^7)(z + q_s^{11})\),

(A.1c) \(d_{2,2}(z) = (z - q_s^2)(z - q_s^8)(z - q_s^{12})\),

where \([\Omega]\) denotes the triply twisted adapted \(r\)-cluster points in Definition 3.13 and the Dynkin diagram of \(G_2^{(1)}\) is given as follows:

\[
G_2^{(1)} = \begin{array}{c}
0 \rightarrow 1 \rightarrow 2
\end{array}
\]
Similarly, one can guess $U'_q(F_4^{(1)})$ can be obtained from $\widetilde{D}_{k,l}(z) \times (z - q^h)^{\delta_{l,k}}$.

**Conjectural denominator formulas for $U'_q(F_4^{(1)})$**: Set $q_s$ such that $q_s^2 = q$. The conjectural formulas for $d_{k,l}(z)$ are given as follows:

(A.2a) \[d_{1,1}(z) = (z - q_s^4)(z - q_s^{10})(z - q_s^{12})(z - q_s^{18}),\]

(A.2b) \[d_{1,2}(z) = (z + q_s^6)(z + q_s^8)(z + q_s^{10})(z + q_s^{12})(z + q_s^{14})(z + q_s^{16}),\]

(A.2c) \[d_{1,3}(z) = (z - q_s^7)(z - q_s^9)(z - q_s^{13})(z - q_s^{15}),\]

(A.2d) \[d_{1,4}(z) = (z + q_s^8)(z + q_s^{14}),\]

(A.2e) \[d_{2,2}(z) = (z - q_s^4)(z - q_s^6)(z - q_s^{10})(z - q_s^{12})^2(z - q_s^{14})^2(z - q_s^{16})(z - q_s^{18}),\]

(A.2f) \[d_{2,3}(z) = (z + q_s^5)(z + q_s^7)(z + q_s^9)(z + q_s^{11})(z + q_s^{13})(z + q_s^{15})(z + q_s^{17}),\]

(A.2g) \[d_{2,4}(z) = (z - q_s^6)(z - q_s^{10})(z - q_s^{12})(z - q_s^{16}),\]

(A.2h) \[d_{3,3}(z) = (z - q_s^2)(z - q_s^6)(z - q_s^{10})(z - q_s^{12})^2(z - q_s^{16})(z - q_s^{18}),\]

(A.2i) \[d_{3,4}(z) = (z + q_s^3)(z + q_s^7)(z + q_s^{11})(z + q_s^{13})(z + q_s^{17}),\]

(A.2j) \[d_{4,4}(z) = (z - q_s^2)(z - q_s^8)(z - q_s^{12})(z - q_s^{18}).\]

where the Dynkin diagram of $F_4^{(1)}$ is given as follows:

\[
\begin{array}{cccccc}
& 0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

Now we suggest conjectural Dorey’s rule for $U'_q(F_4^{(1)})$ and $U'_q(G_2^{(1)})$ in a canonical way:

**Conjecture A.1.** Let $(i, x), (j, y), (k, z) \in I \times k^\times$. Then

\[
\text{Hom}_{U'_q(\widetilde{X}(1))}(V(\varpi_j)_y \otimes V(\varpi_i)_x, V(\varpi_k)_z) \neq 0 \quad (\widetilde{X} = F_4, G_2)
\]

if and only if there exists an $[i_0] \in [\mathcal{D}]$ (resp. $[i_0] \in [\mathcal{Q}]$) and $\alpha, \beta, \gamma \in \Phi^+_X$ ($X = E_6, D_4$) such that

(i) $(\alpha, \beta)$ is an $[i_0]$-minimal pair of $\gamma$,

(ii) $V(\varpi_j)_y = V([i_0])(\beta)_t$, $V(\varpi_i)_x = V([i_0])(\alpha)_t$, $V(\varpi_k)_z = V([i_0])(\gamma)_t$ for some $t \in k^\times$.

The Dorey’s rule for $U_q(G_2^{(1)})$ can be conjectured as (see (4) in Example 5.5)

(A.3a) \[V(\varpi_2)_{-q_s^1} \otimes V(\varpi_2)_{-q_s} \rightarrow V(\varpi_1),\]

(A.3b) \[V(\varpi_2)_{q_s^1} \otimes V(\varpi_2)_{q_s^1} \rightarrow V(\varpi_2).\]

The Dorey’s rule for $U_q(F_4^{(1)})$ can be conjectured as (see (3) in Example 5.5)

(A.4a) \[V(\varpi_4)_{-q_s^1} \otimes V(\varpi_4)_{-q_s} \rightarrow V(\varpi_3), \quad V(\varpi_1)_{-q_s^2} \otimes V(\varpi_1)_{q_s^1} \rightarrow V(\varpi_2),\]

(A.4b) \[V(\varpi_4)_{q_s^1} \otimes V(\varpi_4)_{q_s^1} \rightarrow V(\varpi_4), \quad V(\varpi_4)_{q_s^1} \otimes V(\varpi_4)_{q_s^1} \rightarrow V(\varpi_1),\]

(A.4c) \[V(\varpi_3)_{q_s^1} \otimes V(\varpi_4)_{q_s^1} \rightarrow V(\varpi_2), \quad V(\varpi_4)_{q_s^2} \otimes V(\varpi_1)_{q_s^1} \rightarrow V(\varpi_4),\]

(A.4d) \[V(\varpi_1)_{q_s^1} \otimes V(\varpi_1)_{q_s^1} \rightarrow V(\varpi_1).\]

We remark that (A.3a) and (A.4a) are given in [11, Page 86].
Remark A.2. After completing this paper, Travis and the first named author proved all conjectures in Appendix A in [30].

APPENDIX B. TWISTED DYNKIN QUIVER

In this appendix, we introduce a twisted Dynkin quiver associated to a twisted Coxeter element.

Definition B.1.

(1) A twisted Dynkin quiver $Q\nu$ for $\nu$ in (3.1a), (3.1b) or (3.1c) has the following properties.
   (i) $Q\nu$ consists of vertices of the form $\left(\frac{k}{ik}\right)$ for $k = 1, 2, \ldots, n + 1$, and $i_k = k$ or $k^\nu$. When $k = k^\nu$, we use $k$ instead of $\left(\frac{k}{ik}\right)$ and $\left(\frac{k^\nu}{k^\nu}\right)$.
   (ii) Two vertices $\left(\frac{k_1}{ik_1}\right)$ and $\left(\frac{k_2}{ik_2}\right)$ are connected by an arrow if $k_1$ and $k_2$ are connected in $\Delta$.
   (iii) Two vertices $\left(\frac{k_1}{ik_1}\right)$ and $\left(\frac{k_2}{ik_2}\right)$ are connected by an edge (without orientation) if $k_i = k_i^\nu$ ($i = 1, 2$), and $k_1$ and $k_2$ are connected in $\Delta$.

(2) A triply twisted Dynkin quiver $Q\nu$ (resp. $Q\nu^2$) of type $D_4$ associated to $\nu$ (resp. $\nu^2$) in (3.1d) has the following properties.
   (i) $Q\nu$ (resp. $Q\nu^2$) consists of two vertices, 2 and $\left(\frac{3}{4}\right)$, $\left(\frac{4}{3}\right)$ (resp. $\left(\frac{4}{3}\right)$, $\left(\frac{3}{4}\right)$ or $\left(\frac{3}{4}\right)$).
   (ii) Two vertices are connected by an arrow.

Note that a type $D_{n+1}$ twisted Dynkin quiver does not have edges, since there are no vertices satisfying (iii) in Definition B.1 (1).

Example B.2.

(1) The following two twisted Dynkin quivers of type $A_5$ are associated to $143\nu$ and $123\nu$:
   (B.1a) \[
   \begin{array}{c}
   \bullet \\
   (\frac{1}{2}) \\
   (\frac{2}{3}) \\
   \bullet \\
   3
   \end{array}
   \]
   (B.1b) \[
   \begin{array}{c}
   \bullet \\
   (\frac{1}{3}) \\
   (\frac{2}{4}) \\
   \bullet \\
   3
   \end{array}
   \]

(2) The following two twisted Dynkin quivers of type $D_6$ are associated to $13245\nu$ and $13246\nu$:
   (B.2a) \[
   \begin{array}{c}
   1 \\
   2 \\
   3 \\
   4 \\
   \bullet \\
   (\frac{6}{5})
   \end{array}
   \]
   (B.2b) \[
   \begin{array}{c}
   1 \\
   2 \\
   3 \\
   4 \\
   \bullet \\
   (\frac{5}{6})
   \end{array}
   \]

Remark B.3. Considering the number of indices for each vertex, the underlying graph of a (triply) twisted Dynkin quiver can be understood as the Dynkin diagram of type $B_n$, $C_n$ and $F_4$ (resp. $G_2$), respectively.
Definition B.4. A vertex $v$ of a twisted Dynkin quiver is called a **sink** if every arrow (oriented edge) connected to $v$ points towards $v$.

Example B.5.

1. In (B.1a), $(\frac{1}{3}, \frac{1}{5})$ and $(\frac{3}{2})$ are sinks, while 3 is not. In (B.1b), $(\frac{1}{3})$ is a sink, while the others are not.
2. In (B.2a), 1 and 3 are sinks, while the others are not. In (B.2b), 1 and 3 are sinks, while the others are not.

Definition B.6. For a twisted Dynkin quiver $Q\vee$ and $i \in I$, we define $r_i(Q\vee)$ as follows:

1. For a twisted Dynkin quiver, if $(\frac{i^\vee}{j^\vee})$ is a sink of $Q\vee$, we define $r_i(Q\vee)$ by the following steps.
   - (i) Reverse all arrows incident to $(\frac{i^\vee}{j^\vee}).$
   - (ii) Replace the $(\frac{i^\vee}{j^\vee})$ by $(\frac{j^\vee}{i^\vee})$.
   - (iii) For $i \neq i^\vee$ and $j \neq j^\vee$, if there exists an arrow between $(\frac{i^\vee}{i^\vee})$ and $(\frac{j^\vee}{j^\vee})$, and $i^\vee$ and $j^\vee$ are connected in $\Delta$, remove the orientation of the arrow.
   - (iv) For $i \neq i^\vee$ and $j \neq j^\vee$, if there exists an edge between $(\frac{i^\vee}{i^\vee})$ and $(\frac{j^\vee}{j^\vee})$, and $i^\vee$ and $j$ are connected in $\Delta$, give an orientation of the edge from $(\frac{i^\vee}{i^\vee})$ to $(\frac{j^\vee}{j^\vee})$.
   If $(\frac{i^\vee}{j^\vee})$ is not a sink, $r_i(Q\vee) := Q\vee.$

2. If $(\frac{i^\vee}{j^\vee})$ is a sink of a triply twisted Dynkin quiver $Q\vee$, reverse all arrows incident with $(\frac{i^\vee}{i^\vee})$ and replace $(\frac{i^\vee}{i^\vee})$ by $(\frac{i^\vee}{j^\vee})$. Otherwise, $r_i(Q\vee) = Q\vee.$

Example B.7. Note that, for a twisted Dynkin quiver

$$
\begin{align*}
\begin{array}{c}
\bullet & \bullet & \bullet \\
\circ & \circ & \circ \\
\frac{1}{3} & \frac{1}{5} & \frac{4}{2}
\end{array}
\end{align*}
$$

$(\frac{1}{3}), (\frac{3}{2})$ are sink, while $(\frac{3}{5}), (\frac{3}{4}), (\frac{7}{2})$ are not. Then we have

1. $r_1\left(\begin{array}{c}
\circ & \circ & \circ \\
\bullet & \bullet & \bullet \\
\frac{1}{3} & \frac{1}{5} & \frac{4}{2}
\end{array}\right) = \left(\begin{array}{c}
\circ & \circ & \circ \\
\bullet & \bullet & \bullet \\
\frac{1}{5} & \frac{4}{2} & \frac{3}
\end{array}\right)$ and $r_4\left(\begin{array}{c}
\circ & \circ & \circ \\
\bullet & \bullet & \bullet \\
\frac{1}{3} & \frac{1}{5} & \frac{4}{2}
\end{array}\right) = \left(\begin{array}{c}
\circ & \circ & \circ \\
\bullet & \bullet & \bullet \\
\frac{1}{5} & \frac{4}{2} & \frac{3}
\end{array}\right)$.
2. $r_2\left(\begin{array}{c}
\circ & \circ & \circ \\
\bullet & \bullet & \bullet \\
\frac{1}{3} & \frac{1}{5} & \frac{4}{2}
\end{array}\right) = \left(\begin{array}{c}
\circ & \circ & \circ \\
\bullet & \bullet & \bullet \\
\frac{1}{5} & \frac{4}{2} & \frac{3}
\end{array}\right)$ and $r_3\left(\begin{array}{c}
\circ & \circ & \circ \\
\bullet & \bullet & \bullet \\
\frac{1}{3} & \frac{1}{5} & \frac{4}{2}
\end{array}\right) = \left(\begin{array}{c}
\circ & \circ & \circ \\
\bullet & \bullet & \bullet \\
\frac{1}{5} & \frac{4}{2} & \frac{3}
\end{array}\right)$.

Definition B.8. A reduced expression $\tilde{w} = i_1 i_2 \cdots i_m$ of type $A_{2n-1}$, $D_{n+1}$ or $E_6$ (resp. $D_4$) is said to be **adapted** to $Q\vee$ if $(\frac{i^\vee_k}{i^\vee_k})$ (resp. $(\frac{i^\vee_k}{i^\vee_k})$) is a sink of the quiver $r_i r_{i-1} \cdots r_{i_2} r_{i_1} Q\vee$

for $k = 1, 2, \ldots, m$. 
Proposition B.9. There is a natural one-to-one correspondence between (triply) twisted Dynkin quivers and (triply) twisted Coxeter elements defined as follows:

\[ Q^∨ \mapsto \phi_{Q^∨}. \]

where \( \phi_{Q^∨} \) is an element in \( W \) all of whose reduced expressions are adapted to \( Q^∨ \).

Proof. By definitions, there exists only one \( \phi_{Q^∨} \) which is adapted to \( Q^∨ \). Conversely, one can find \( Q^∨ \) from \( \phi_{Q^∨} = i_1 i_2 \cdots i_n \) as follows.

- If \( i_{k_1} \neq i_{k_1}^∨ \) \((i = 1, 2)\), and \( i_{k_1} \) and \( i_{k_2}^∨ \) are connected in \( \Delta \), then two vertices \( \left( \frac{i_{k_1}}{i_{k_1}^∨} \right) \left( \frac{i_{k_2}}{i_{k_2}^∨} \right) \) in \( Q^∨ \) are connected by an edge (without direction).
- If \( i_{k_1} \) and \( i_{k_2}^∨ \) are connected in \( \Delta \) and \( k_1 < k_2 \), then there is an arrow in \( Q^∨ \) from \( \left( \frac{i_{k_1}}{i_{k_1}^∨} \right) \) to \( \left( \frac{i_{k_2}}{i_{k_2}^∨} \right) \).

Hence (B.3) is a one-to-one correspondence. The assertion for triply twisted case can be proved directly.

Recall (4.6), (4.8) and (4.10) (resp. (4.11)), a (triply) twisted Coxeter element \( \phi_{Q^∨} \) induces a reduced expression of \( w_0 \). Let us denote the class associated to the (triply) twisted Coxeter element \( \phi_{Q^∨} = i_1 i_2 \cdots i_ℓ \) by \([Q^∨]\):

\[ [Q^∨] := \left[ \prod_{k=0}^{[|Φ^∨|/ℓ]-1} (i_1 i_2 \cdots i_ℓ)^{k∨} \right]. \]

Considering Remark B.3, the following theorem tells that a reduced expression adapted to \( Q^∨ \) can be understood as a reduced expression which is related to a Dynkin diagram of type \( B_n \), \( C_n \) or \( F_4 \) (resp. \( G_2 \)), respectively.

Proposition B.10. If \( i_0 \) is adapted to \( Q^∨ \) then \( i_0 \in [Q^∨] \).

Proof. Here, we only give the proof for twisted cases. Take the adapted reduced expression (B.4) and denote it by \( i_0′ = i_1 i_2 \cdots i_N \). Note that \( i_0′ \) is adapted to \( Q^∨ \).

For a reduced expression \( j_0 = j_1 j_2 \cdots j_N \) adapted to \( Q^∨ \), let us assume

- (i) \( j_t = i_t \) for all \( t = 1, 2, \ldots, k-1 \),
- (ii) \( j_k \neq i_k \),
- (iii) \( k_1 \) is the smallest integer such that \( k_1 > k \) and \( j_{k_1} = i_k \).

By the assumption, both \( \left( \frac{j_k}{j_k} \right) \) and \( \left( \frac{i_k}{i_k} \right) \) are sinks of the twisted quiver \( i_{k-1} \cdots i_1 Q^∨ \). Hence \( j_k \) and \( i_k \) are not connected in \( \Delta \). Moreover, for any \( k_2 \) such that \( k < k_2 < k_1 \), \( j_{k_2} \) is not connected to \( i_k \) in \( \Delta \). Hence there is \( j_0′ = j_1′ j_2′ \cdots j_N′ \in [j_0] \) such that \( j_t′ = i_t \) for all \( t = 1, 2, \ldots, k \). By induction, we can show \( j_0′ \in [i_0′] = [Q^∨] \).

Proposition B.11. Consider a reduced expression \( i_0 \) of type \( D_{n+1} \) (resp. \( D_4 \)) and a twisted (resp. triply twisted) adapted class \([Q^∨]\). If \( i_0 \in [Q^∨] \) then \( i_0 \) is adapted to \( Q^∨ \).

Proof. We shall prove only for twisted cases since the triply twisted cases can be proved directly.

Let us show if \( i_0′ \) is adapted to \( Q^∨ \) then any \( i_0′′ \in [i_0′] \) is adapted to \( Q^∨ \). To see this claim, we aim to show if \( i_0′ = i_1 i_2 \cdots i_m i_m i_{m+1} i_{m+2} \cdots i_N \) is adapted to \( Q^∨ \) and \([i_m i_{m+1}| = [i_{m+1} i_m] \), then
$i''_0 = i_1 i_2 \ldots i_{m-1} i_m i_{m+2} \ldots i_N$ is also adapted to $Q^\vee$. Note that $i_m$ and $i_{m+1}$ are in distinct orbits and $i_m$ and $i_{m+1}$ are not connected in $\Delta$. Moreover, $i_m$ and $i_{m+1}$ (resp. $i'_m$ and $i'_{m+1}$) are not connected in $\Delta$ since we only consider the type $D_{n+1}$ case.

By the observations, $(i_m^\vee, i'_m)$ are both sinks in $r_{i_{m-1}} \cdots r_{i_2} r_{i_1} Q^\vee$ and $r_{i_m}$ (resp. $r_{i_{m+1}}$) does not change any arrows incident to $(i_{m+1}^\vee, i'_m)$ (resp. $(i_m^\vee, i'_m)$). Hence

$$i_{m+1} \ (\text{resp. } i_m) \text{ is a sink in } r_{i_m} r_{i_{m-1}} \cdots r_{i_2} r_{i_1} Q^\vee \ (\text{resp. } r_{i_{m+1}} r_{i_{m-1}} \cdots r_{i_2} r_{i_1} Q^\vee)$$

and

$$r_{i_{m+1}} r_{i_m} r_{i_{m-1}} \cdots r_{i_2} r_{i_1} Q^\vee = r_{i_m} r_{i_{m+1}} r_{i_{m-1}} \cdots r_{i_2} r_{i_1} Q^\vee .$$

Hence $i'_0$ is adapted to $Q^\vee$ if and only if $i''_0$ is adapted to $Q^\vee$.

Now, since $\prod_{k=0}^{(\Phi^+|\ell)_1} (i_1 i_2 \cdots i_\ell)^{k^\vee}$ in (B.4) is adapted to $Q^\vee$, we proved the proposition. \hfill \box

**Remark B.12.** Proposition B.11 does not hold for type $A_{2n+1}$ and $E_6$. For example, consider the twisted Dynkin quiver of type $A_5$:

$$Q^\vee = \begin{array}{c}
\circ \\
(1) \\
(2) \\
\circ \\
3
\end{array}.$$

Then $i_0 = 123543123543123$ is adapted to $Q^\vee$ and $i'_0 = 512343123543123 \in [i_0] = [Q^\vee]$. However, $i''_0$ is not adapted to $Q^\vee$.

The following remark can be understood as the twisted analogue of (2.1).

**Remark B.13.** Every twisted adapted class of type $D_{n+1}$ is induced from a twisted Coxeter element. Hence, in this case, we have the following one-to-one correspondences:

$$\{\text{twisted Dynkin quivers}\} \leftrightarrow \{\text{twisted Coxeter elements}\} \leftrightarrow \{\text{twisted adapted classes}\}.$$ 

Also, since the number of triply adapted classes $[[Q]] := [[Q^\dagger]] \cup [[Q^\dagger]]$ of type $D_4$ is the same as the number of triply twisted Coxeter elements, we have

$$\{\text{triply twisted Dynkin quivers}\} \leftrightarrow \{\text{triply twisted Coxeter elements}\} \leftrightarrow \{\text{triply twisted adapted classes}\}.$$ 

On the other hand, for type $A_{2n+1}$ and $E_6$ twisted adapted classes, we have

$$\{\text{twisted Dynkin quivers}\} \leftrightarrow \{\text{twisted Coxeter elements}\} \leftrightarrow \{\text{twisted adapted classes}\}.$$ 

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