A comparison of Zeroes and Ones of a Boolean Polynomial

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Abstract
In this paper we consider the computational complexity of the following problem. Let \( f \) be a Boolean polynomial. What value of \( f \), 0 or 1, is taken more frequently? The problem is solved in polynomial time for polynomials of degrees 1, 2. The next case of degree 3 appears to be PP-complete under polynomial reductions in the class of promise problems. The proof is based on techniques of quantum computation.

The class PP was defined by J. Gill [4] as probabilistic polynomial time with unbounded error. The following problem represents all computational power of PP: A Boolean function \( f \) is given by a Boolean circuit computing this function. What value of \( f \), 0 or 1, is taken more frequently? In fact, this value comparison problem is PP-complete under polynomial reductions, if promise problems are considered instead of decision problems. A promise problem is a decision problem in which some inputs are excluded. So, a promise problem \( F \) is described by a pair of disjoint sets \((F_{\text{yes}}, F_{\text{no}})\) of strings corresponding to “yes” and “no” instances.

In this paper we restrict the value comparison problem to the case of Boolean polynomials of fixed degree. It is known that the problem of counting of zeroes for polynomials of degree 3 is \#P-complete [2]. An easy corollary of this result is PP-completeness of the value comparison problem for polynomials of degree 4 (see Theorem 1 below).

At other hand, the comparison problem for polynomials of degree 1 is trivial. In the case of degree 2 the problem is solved in polynomial time (by reduction to the canonical form, see [11, 10]).

We address to the remaining case — polynomials of degree 3. It will be shown that it is PP-complete. Surprisingly enough, the proof will use techniques of quantum computation. We will apply the theorem of efficient approximation for unitary operators [5] and the results of [8]. It was shown in [8] that a problem of determination of sign of specific quadratically signed weight enumerators is BQP-complete (again, we mean the completeness in the class of promise problems). It is possible to use the results of [8] directly for the proof of our main theorem. Instead, we prefer to follow the arguments of [8] and present a slightly more restrictive form of the enumerators.

1 Preliminaries

1.1 Another definition of PP

We will also use the definition of the class PP given by Fenner, Fortnow and Kurtz [3]. They introduced the class GapP functions consisting of the closure under subtraction of the set of \#P functions. In other words, for any GapP function \( f : \mathbb{B}^* \rightarrow \mathbb{Z} \) there are predicates \( Q_1(\cdot, \cdot), Q_2(\cdot, \cdot) \in \text{P} \) and a polynomial \( q(\cdot) \) such that for all \( x \)

\[
f(x) = \text{Card}\{y : Q_1(x, y) & |y| = q(|x|)\} - \text{Card}\{y : Q_2(x, y) & |y| = q(|x|)\}.
\]

(1)

The class PP can be defined in these terms as follows:

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The class PP consists of those promise problems $F$ such that for some GapP function $f$ (an indicator function) and all $x$

$$x \in F_{\text{yes}} \implies f(x) > 0, \quad x \in F_{\text{no}} \implies f(x) < 0. \quad (2)$$

1.2 Notations and simple facts about Boolean polynomials

A Boolean polynomial is a polynomial over the field $\mathbb{F}_2$ consisting of two elements.

Let $\#_0 f$ be the number of zeroes for a Boolean polynomial $f$, $\#_1 f$ the number of ones. It is clear that $\#_0 f + \#_1 f = 2^n$. The difference of these numbers will be denoted by $\Delta f$:

$$\Delta f = \#_0 f - \#_1 f = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)}. \quad (3)$$

In the case $\Delta f = 0$ the polynomial will be called balanced.

So, the value comparison problem can be reformulated as the problem of determination of the sign of $\Delta f$.

We shall assume throughout the paper that polynomials are represented in the form of monomial sum.

Now we introduce some simple properties of $\Delta f$.

Let $f|_L$ be the restriction of the polynomial $f \in \mathbb{F}_2[x_1, \ldots, x_n]$ to the subspace $L$ of $\mathbb{F}_2^n$ (hereinafter we will consider affine subspaces of $\mathbb{F}_2^n$).

Lemma 1. $\Delta (1 + f) = - \Delta f$.

Lemma 2. If $\varphi: \mathbb{F}_2^n \to \mathbb{F}_2$ is a non-zero linear functional, then $\Delta f = \Delta f|_{\varphi(x)=0} + \Delta f|_{\varphi(x)=1}$.

By direct use of the second equation in (3), we get the following lemma.

Lemma 3. If $f(x, y) = g(x) + h(y)$, then $\Delta f = \Delta g \cdot \Delta h$.

Lemma 4. If $\ell: \mathbb{F}_2^n \to \mathbb{F}_2$ is a non-zero linear functional on $\mathbb{F}_2^n$, then $\Delta \ell = 0$.

Proof. Changing a basis, we transform $\ell$ into the form $\ell(x) = x_1$. For this functional the statement is obvious. \hfill \qed

Lemma 5. Suppose a subspace $L$ is given by a system of equations $\ell_i(x) = 0, 1 \leq i \leq d$. Define the polynomial $g(x, v)$ of $n + d$ variables by the formula $g = f + \sum_{j=1}^{d} v_j \ell_j(x)$. Then

$$\Delta g = 2^d \Delta f|_L. \quad (4)$$

Proof. Let us consider polynomials $g_x(v) = g(x, v)$. Lemmata 2, 3 imply that for any $x \notin L$ the polynomial $g_x$ is balanced. So, it contributes 0 to the $\Delta g$. For any $x \in L$ the polynomial $g_x$ is not depend on values of $v_j$. Hence, it contributes $2^d(-1)^{f(x)}$ to the $\Delta g$. Summing contributions over all $x \in \mathbb{F}_2^n$, we get (4). \hfill \qed

In the proof of PP-completeness a relation between quadratically signed weight enumerators and $\Delta f$ will be exploited. Namely, consider a quadratically signed weight enumerator

$$S(A, B) = \sum_{A x = 0, x \in \mathbb{F}_2^n} (-1)^{B(x)2^{|x|}4^{n-|x|} \deg B = 2,} \quad (5)$$

where $A$ is a Boolean matrix, $|x|$ is the number of ones in $\{x_j\}_{j=1}^n$. Let $a_{jk}$ are matrix elements of $A$. By $f_{A,B}$ denote the polynomial of $4n$ variables:

$$f_{A,B}(x, y, z, u) = \sum_{j=1}^{n} x_j y_j z_j + B(x) + \sum_{k=1}^{n} a_k \sum_{j=1}^{n} a_{jk} x_j, \quad (6)$$

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Lemma 6. \( S(A, B) = 2^{-n} \Delta f \).

Proof. We calculate \( \Delta f \) for each \( x \) separately. If \( Ax \neq 0 \), then \( f \) is reduced to \( \ell(u) + g(y, z) \), \( \deg \ell = 1 \), and \( \Delta(u) + g(y, z) = 0 \). If \( Ax = 0 \), then \( f \) is reduced to \( f_x = B(x) + \sum_{j: x_j = 1} y_j z_j \) and does not depend on \( u \). In this case we get

\[
\Delta f_x = 2^n \sum_{y, z} (-1)^{f_x(y, z)} = (-1)^B(x) 2^n 4^n - |x| \prod_{j: x_j = 1} (-1)^{y_j z_j} = 2^n (-1)^B(x) 2^{|x|} 4^n - |x|. \tag{7}
\]

Summing \((7)\), we obtain \( \Delta f = 2^n S(A, B) \). \( \square \)

1.3 Some facts about quantum computation

Basics of quantum computation can be found in \([5, 1]\). Here we recall two facts that are used in the following proof.

The main tool will be the theorem on efficient approximation of unitary operators \([5]\).

Theorem. Let elements \( X_1, \ldots, X_l \in \text{SU}(n) \) generate an everywhere dense set in \( \text{SU}(n) \). There is an algorithm which constructs for any matrix \( U \in \text{SU}(n) \) and any precision threshold \( \delta \) an \( \delta \)-approximation \( \tilde{U} \) of \( U \) in the form of product of generators and their inverses \( X_1, \ldots, X_l, X_1^{-1}, \ldots, X_l^{-1} \). The algorithm runs in time \( \exp(O(n) \text{ poly log}(1/\delta)) \).

\( \delta \)-approximation means that \( \| U - \tilde{U} \| < \delta \) in operator norm. An implicit factor in the running time bound may depend on \( X_1, \ldots, X_l \in \text{SU}(n) \). Matrix elements of \( U \) may be any efficiently computable complex numbers.

We will approximate by a set of unitary operators \( \exp(i\varphi \sigma(s)) \), \( \|s\| \leq 2 \), \( \varphi \) is real. In fact, we will need \( \varphi = \arccos(2/\sqrt{5}) \) only. Here the following notations are used

\[
\sigma(s) = \sigma(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n) \overset{\text{def}}{=} \sigma_{a_1, b_1} \otimes \sigma_{a_2, b_2} \otimes \cdots \otimes \sigma_{a_n, b_n}, \quad s \in \mathbb{F}_2^{2n}, \tag{8}
\]

\( \sigma_{a_i, b_i} \) are Pauli matrices:

\[
\sigma_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{01} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z; \quad \sigma_{10} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x; \quad \sigma_{11} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y.
\]

A weight \( \|s\| \) is equal to the number of non-zero pairs \( a_j, b_j \) in \( s \).

The following lemma is due to \([2, 3]\).

Lemma 7. Suppose \( \varphi \) is incommensurable with \( \pi \); then operators \( \exp(i\varphi \sigma(s)) \) generate an everywhere dense set in \( \text{SU}((\mathbb{C}^2)^{\otimes 2}) \).

Sketch of proof. (See \([2, 3]\) for details.) The operators \( \exp(i\varphi \sigma_z) \) and \( \exp(i\varphi \sigma_x) \) do not commute. Hence, the operators \( \exp(i\varphi \sigma_z(1, 0, 0, 0)) \exp(i\varphi \sigma_z(1, 0, 0, 1)) \) and \( \exp(i\varphi \sigma_z(0, 1, 0, 0)) \exp(i\varphi \sigma_z(0, 1, 0, 1)) \) generate an everywhere dense set in \( \text{SU}(\mathbb{C}^{\otimes 2}) \). Similarly, the operators \( \exp(i\varphi \sigma_z(0, 0, 1, 0)) \exp(i\varphi \sigma_z(0, 1, 1, 0)) \) and \( \exp(i\varphi \sigma_z(0, 0, 1, 1)) \exp(i\varphi \sigma_z(0, 1, 0, 1)) \) generate an everywhere dense set in \( \text{SU}(\mathbb{C}^{\otimes 2}) \). Multiplying these sets, we obtain an everywhere dense set in \( \text{SU}(\mathbb{C}^{\otimes 2} \supseteq \mathbb{C}(\{0\}) \supset \mathbb{C}(\{1\}) \supset \mathbb{C}(\{11\})) \). To complete the proof it remains to note that the operator \( \exp(i\varphi(0, 0, 1, 0)) \) does not fix the subspace \( \mathbb{C}(\{11\}) \). \( \square \)

2 The case of polynomials of degree 4

Theorem 1. The value comparison problem for polynomials of degree 4 is PP-complete.

(The problem is considered as a promise problem and the class PP is assumed consisting of promise problems. A reduction is a polynomial reduction in the class of promise problems.)
Proof. Let \( F \in \text{PP} \) be a promise problem, \( f \in \text{GapP} \) its indicator function, and \( Q_1(x, y), Q_2(x, y) \in \text{P} \) predicates from the definition of the class \( \text{GapP} \) applied to \( f \). On inputs \( x \) of length \( n \) the predicates \( Q_j \) are computed by polynomial size Boolean circuits over the basis \{\+, \cdot\}. Adding dummy assignments if necessary, we may assume that both circuit sizes are equal to \( s = \text{poly}(n) \).

By \( z_k^{(j)} \), \( 1 \leq k \leq s \), we denote auxiliary variables of the circuit computing the predicate \( Q_j \). We also assume that the value of the circuit is the value of the variable \( z_k^{(j)} \). Each assignment in a circuit has the form \( z_k^{(j)} := a \cdot b \) where \( * \in \{+, \cdot\} \) and \( a, b \) are either input or auxiliary variables. The equation \( Z_k^{(j)} = z_k^{(j)} + a \cdot b = 0 \) corresponds to this assignment. Note that the values of input variables \( x, y \) determine the values of all auxiliary variables. So, for each \( x \) the number of solutions of the system of equations \( Z_k^{(j)} = 0, 1 \leq k < s, z_k^{(j)} = 1 \) equals \( \text{Card}\{y : Q_j(x, y) \& |y| = q(|x|)\} \). At other hand, by the argument of Lemma \( \text{Lemma 3} \) this number equals \( 2^{-s} \Delta F_x^{(j)} \), where

\[
F_x^{(j)}(y, z, v) = \sum_{k=1}^{s} v_k Z_k^{(j)} + v_0 (z_0^{(j)} + 1).
\]

Therefore, we get \( f(x) = 2^{-s} (\Delta F_x^{(1)} - \Delta F_x^{(2)}) \). Taking into account the relation \( \Delta F_x^{(1)} - \Delta F_x^{(2)} = \Delta((1 + w)F_x^{(1)} + w(1 + F_x^{(2)})) \), we obtain the reduction \( x \mapsto F_x \), where

\[
F_x = (1 + w)F_x^{(1)} + w(1 + F_x^{(2)}).
\]

It is clear that this reduction is polynomial. \( \square \)

3 The case of polynomials of degree 3

Theorem 2. The value comparison problem for polynomials of degree 3 is \( \text{PP-complete} \).

Proof. We will use the Theorem \( \text{Theorem 2} \). So, we will construct for any polynomial \( f \), deg \( f \) = 4, a polynomial \( g \), deg \( g \) = 3, such that the signs of \( \Delta f \) and \( \Delta g \) are equal. The construction should be done in time polynomial of the input size (polynomial of \( n \), where \( n \) is the number of variables of \( f \)).

At first, we define a unitary operator \( U(f) : (\mathbb{C}^2)^{\otimes n} \rightarrow (\mathbb{C}^2)^{\otimes n} \) by the following way. The operator \( S_j = \Lambda^j(-1) \) (controlled phase shift) corresponds to the monomial \( x_J = \prod_{j \in J} x_j \) of \( f \). Controlled phase shift is defined as

\[
\left\{
\begin{array}{ll}
\Lambda^j(-1)\vert x_1 \ldots x_n \rangle = -\vert x_1 \ldots x_n \rangle, & x_J = 1, \\
\Lambda^j(-1)\vert x_1 \ldots x_n \rangle = \vert x_1 \ldots x_n \rangle, & \text{otherwise}.
\end{array}
\right.
\]

Let

\[
U(f) = \prod_{j=1}^{n} H[j] \prod_{J \in \text{M}(f)} S_j \prod_{j=1}^{n} H[j],
\]

where \( H \) is the Hadamard matrix

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]

and \( \text{M}(f) \) is the set of the monomials of \( f \).

We have

\[
\langle 0 \vert U(f) \vert 0 \rangle = \frac{1}{2^n} \sum_{x_1, \ldots, x_n} \langle x_1, \ldots, x_n \vert \prod_{J \in \text{M}(f)} S_j \vert x_1, \ldots, x_n \rangle = \frac{1}{2^n} \sum_{x_1, \ldots, x_n} (-1)^{\sum_{J \in \text{M}(f)} x_J} = \frac{1}{2^n} \sum_{x_1, \ldots, x_n} (-1)^f(x) = 2^{-n} \Delta f.
\]
Note that det $U(f) = \pm 1$. W.l.o.g. we assume that $U(f) \in SU((\mathbb{C}^2)^{\otimes n})$. (The case of det $U(f) = -1$ is essentially the same.)

Now, we approximate $U(f)$ with precision $\delta = 2^{-n-1}$ by an operator $\tilde{U}$ in the form of product of the operators $\exp(\imath \varphi \sigma(s))$, $\|s\| \leq 2$, where $\cos \varphi = 2/\sqrt{5}$. To achieve the precision required we approximate each factor in (11) with greater precision $O(\delta/n^4)$. Each factor in (11) acts on 4 bits at most. So, by the theorem on efficient approximation an operator $\tilde{U}$ can be constructed in poly($n$) time. Assume that

$$\tilde{U} = \prod_{j=1}^{N} \exp(\imath \varphi \sigma(s_j), \quad N = \text{poly}(n), \ s_j = (\alpha_{j1}, \beta_{j1}, \ldots, \alpha_{jN}, \beta_{jN}). \quad (13)$$

From $\|\langle 0 | U(f) - \tilde{U} | 0 \rangle\| = \|U(f) - \tilde{U}\| < \delta$ and (12) we conclude that the sign $\text{Re}\langle 0 | U(f) | 0 \rangle$ equals the sign of $\Delta f$.

The next step is to find out $\langle 0 | \tilde{U} | 0 \rangle$:

$$\langle 0 | \tilde{U} | 0 \rangle = \langle 0 | \prod_{j=1}^{N} \exp(\imath \varphi \sigma(s_j)) | 0 \rangle = \langle 0 | \prod_{j=1}^{N} (\cos \varphi + \imath \sin \varphi \sigma(s_j)) | 0 \rangle = \sum_{s_1, \ldots, s_N} \prod_{j=1}^{N} \cos \varphi \langle 0 | \prod_{j=1}^{N} \imath \sin \varphi \sigma(s_j) | 0 \rangle = \sum_{s_1, \ldots, s_N} \prod_{j=1}^{N} \cos \varphi \prod_{j=1}^{N} \imath \sin \varphi \sigma(s_j) \langle 0 | 0 \rangle. \quad (14)$$

An operator $\sigma(s_j)$ flips a bit $k$ iff $\alpha_{jk} = 1$. Hence, if $Ax \neq 0$, where $A_{jk} = \alpha_{jk}$, then $(x_1, \ldots, x_N)$ contributes a zero to the sum (14).

Let us evaluate a phase factor of $\langle 0 | \prod_{j=1}^{N} \sigma(s_j) | 0 \rangle$. Let $y_{jk}$ be the value of the bit $k$ before application of $\exp(\imath \varphi \sigma(s_j))$. By direct calculation we get $y_{jk} = \sum_{t=1}^{j-1} \alpha_{tk} x_t$. By $\gamma_j$ denote the number of $\sigma_y$ operators in $\sigma(s_j)$. Then $\sigma(s_j)$ multiply a phase by a factor

$$\imath \gamma_j \prod_{k=1}^{n} (-1)^{\beta_{jk}} y_{jk}. \quad (15)$$

We put (15) into (14) and obtain

$$\langle 0 | \tilde{U} | 0 \rangle = \sum_{Ax=0} (\cos \varphi)^{N-|x|} (\sin \varphi)^{|x|} \prod_{j=1}^{N} \sum_{k=1}^{n} (-1)^{\beta_{jk}} y_{jk} = \sum_{Ax=0} (\cos \varphi)^{N-|x|} (\sin \varphi)^{|x|} \prod_{j=1}^{N} \sum_{k=1}^{n} (-1)^{\beta_{jk}} y_{jk} = \frac{1}{5^{N/2}} \sum_{Ax=0} 2^{N-|x|} \prod_{j=1}^{N} (-1)^{\beta_{jk}} \sum_{k=1}^{n} (-1)^{\beta_{jk}} y_{jk}. \quad (16)$$

Let us introduce the notation $\Gamma_t = \{ j : \gamma_j = t \}$. Using the obvious identity

$x_1 \oplus x_2 \oplus \cdots \oplus x_r = x_1 + x_2 + \cdots + x_r - 2s_2(x_1, \ldots, x_r) \pmod{4}$, \quad $x_i \in \{0, 1\}$, \quad $s_2(x_1, \ldots, x_r) = \sum_{j \neq k} x_j x_k$,

we rewrite the power of $i$ in (14) as

$$i^{\sum_{j=1}^{N} (\gamma_j + 1) x_j} = (-1)^{\sum_{j \in \Gamma_1} x_j} \sum_{j=1}^{N} (-1)^{s_2(x_{\Gamma_0}) + s_2(x_{\Gamma_2})} i^{\sum_{j \in \Gamma_0} x_j - \sum_{j \in \Gamma_2} x_j}, \quad (17)$$

where $x_{\Gamma_i}$ is the set of variables $x_j$ whose indexes are in $\Gamma_i$.

Thus, $\text{Re}\langle 0 | \tilde{U} | 0 \rangle$ can be expressed in the form (16). The quadratic weight in this representation is $B(x) = \tilde{B}(x) + s_2(x_{\Gamma_0}) + s_2(x_{\Gamma_2})$ while the subspace is given by equations $Ax = 0$, $gx = 0$ ($g_j = 1$ iff $\gamma_j$ is even):

$$\text{Re}\langle 0 | \tilde{U} | 0 \rangle = \frac{1}{5^{N/2}} \sum_{Ax=0, gx=0} 2^{N-|x|} (-1)^{B(x)} = \frac{1}{20^{N/2}} \sum_{Ax=0, gx=0} 2^{N-|x|} (-1)^{B(x)}. \quad (18)$$

By Lemma 3 up to a positive factor this expression equals $\Delta g$ for some polynomial $g$ of degree 3. It follows from the proof of Lemma 3 and the construction above that the polynomial $g$ can be built in polynomial time from the representation of $\tilde{U}$ in the form (13). □
Remark. The use of Theorem 1 is not necessary. It would be enough to note that the computation of any predicate from the class P on inputs of length $n$ can be done by a reversible circuit of size poly($n$). The rest of proof remains the same.

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