Regularized exponentially fitted methods for oscillatory problems

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Abstract. The aim of this work is to regularize the expression of the coefficients, expressed in term of trigonometrical or hyperbolic functions, arising from the exponential fitting procedure, by reformulating them in terms of the so-called $\eta_m$ functions. These coefficients are functions of the variable $\nu = \omega h$ where $\omega$ is the frequency and $h$ is the step size. This reformulation eliminates the $0/0$ indeterminate form of the coefficients when $\nu$ tends to 0. This procedure makes the methods more accurate. A numerical evidence is also given.

1. Introduction

The theory of exponential fitting has been introduced in [69] (see also the monograph [74]) for the numerical treatment of problems exhibiting a pronounced oscillatory or hyperbolic behavior. As a matter of fact classical numerical methods may require a very small stepsize in order to accurately reproduce the qualitative behavior of the solution, therefore it is convenient to use special purpose formulae, i.e. numerical methods adapted to the problem, constructed in order to be exact on functions other than polynomials.

Exponentially fitted (ef) numerical methods have been developed in the literature for the numerical solution of a wide range of problems such as interpolation, numerical differentiation and quadrature [33, 34, 36, 38, 70, 72, 76, 75, 88], numerical solution of first order ordinary differential equations [4, 3, 30, 45, 46, 47, 56, 51, 58, 67, 70, 80, 81, 82, 83, 85, 86, 89], second order differential equations [43, 47, 48, 63, 71, 77], integral equations [16, 17, 18, 19, 20, 9], fractional differential equations [1], partial differential equations [15, 53, 54, 59, 59, 61], whereas different estimates for the parameter characterizing the fitting space have been proposed in [45, 44, 56].

A crucial point in the derivation of ef methods is the choice of the fitting space

$$
\mathcal{F} = \{1, t, t^2, \ldots, t^K, e^{\pm \mu t}, t e^{\pm \mu t}, t^2 e^{\pm \mu t}, \ldots, t^P e^{\pm \mu t}\},
$$

where $\mu = \omega$ or $\mu = i \omega$ depending on whether the exact solution belongs to the space spanned by hyperbolic functions or trigonometric functions, respectively.

In all these cases the coefficients of the methods are functions of the product $\nu = \omega h$ where $\omega$ is the frequency and $h$ is the step size of the considered numerical method. An unpleasant feature
with the expressions of the coefficients in the ef-based formulae is that quite often these exhibit an undeterminacy of the form 0/0 when \( \nu \to 0 \). This feature can deteriorate the convergence of the method when \( h \to 0 \), and therefore additional expressions consisting in power expansions in \( \nu \) must be provided for use when \( \nu \) is smaller than some threshold value. In the paper [33] a method and a Mathematica program for the conversion of such coefficients to forms expressed in terms of functions \( \eta_m(Z) \) has been developed, where \( Z = (\mu h)^2 \), in order to eliminate this 0/0 behaviour. It is the purpose of this work to show the effects of this conversion on some relevant numerical methods, by showing as this conversion can restore the convergence of ef numerical methods.

The paper is organized as follows. In Section 2 we recall the procedure for the conversion of coefficients in terms of \( \eta_m(Z) \) functions [33]. In Section 3 we show the reformulation of the coefficients of method derived in [79] for the numerical solution of second order initial value problems having oscillatory solution. In Section 4 some numerical experiments confirming the benefits of this conversion are presented.

2. Procedure for the conversion of coefficients

The set of functions \( \eta_m(Z) \), \( m = -1, 0, 1, \ldots \) is defined as follows

\[
\eta_{-1}(Z) = \begin{cases} \cos(|Z|^{1/2}) & \text{if } Z \leq 0 \\ \cosh(Z^{1/2}) & \text{if } Z > 0 \end{cases}, \quad \eta_0(Z) = \begin{cases} \sin(|Z|^{1/2})/|Z|^{1/2} & \text{if } Z < 0 \\ 1 & \text{if } Z = 0 \\ \sinh(Z^{1/2})/Z^{1/2} & \text{if } Z > 0 \end{cases}
\]

and, for \( Z \neq 0 \),

\[
\eta_m(Z) = [\eta_{m-2}(Z) - (2m - 1)\eta_{m-1}(Z)]/Z, \quad m = 1, 2, 3, ...
\]

while for \( Z = 0 \),

\[
\eta_m(0) = 1/(2m + 1)!!, \quad m = 1, 2, 3, ...
\]

The following theorem is crucial for the description of the procedure [33].

**Theorem 1.** The functions \( \eta_m(Z) \) satisfy the following relations:

\[
\eta_m(Z) = \eta_m(0) + ZD_m(Z), \quad m = -1, 0, 1, 2, \ldots
\]

where

\[
D_m(Z) = \eta_m(0) \left[ \frac{1}{2} \eta_0^2 \left( \frac{Z}{4} \right) - \sum_{i=1}^{m+1} (2i - 3)!! \eta_i(Z) \right]
\]

Let \( \Phi(\nu) \) be a generic linear combination of trigonometrical or hyperbolic functions, with \( \Phi(0) = 0 \)

\[
\Phi(\nu) = \sum_{n=1}^{N} \alpha_n(\nu) \left[ \prod_{i=1}^{l-1,n} \Psi_{-1}(\beta_i^{-1,n} \nu) \right] \left[ \prod_{i=1}^{l_0,n} \Psi_0(\beta_i^{0,n} \nu) \right]
\]

where \( \Psi_{-1}(\nu) = \cos(\nu), \Psi_0(\nu) = \sin(\nu) \) in the trigonometrical case or \( \Psi_{-1}(\nu) = \cosh(\nu), \Psi_0(\nu) = \sinh(\nu) \) in the hyperbolic case, \( \alpha_n(\nu) \) are polynomial coefficients and \( \beta_i^{-1,n}, \beta_i^{0,n} \) are non negative constants.

The aim is to rewrite this coefficient in the form:

\[
\Phi(\nu) = v^r Z^k F(Z)
\]
where $F(Z) \neq 0$, and

$$F(Z) = \sum_{n=1}^{M} a_n(Z) \prod_{j=0}^{k} \left[ \prod_{i=1}^{l_j,n} \eta_j(b_i^{j,n}Z) \right]$$

(9)

where, $M \geq N, b_i^{j,n}$ and $a_n(Z)$ is a polynomial in the variable $Z$.

The first step consist in converting the function $\Phi(\nu)$ in terms of $\eta_{-1}(Z)$ and $\eta_0(Z)$ functions, such that $\Phi(\nu) = v^r f(Z)$, where

$$f(Z) = \sum_{n=1}^{N} a_n(Z) \left[ \prod_{i=1}^{l_{-1,n}} \eta_{-1}(b_i^{0,n}Z) \right] \left[ \prod_{i=1}^{l_{0,n}} \eta_0(b_i^{0,n}Z) \right]$$

(10)

Now, two situations are possible. If $f(0) \neq 0$ the procedure is stopped. If $f(0) = 0$, the procedure continues until

$$f(Z) = Z^k F(Z)$$

(11)

where, $F(Z) \neq 0$.

The procedure starts by placing $f^{(0)}(Z) = f(Z)$ and determining the function $f^{(s+1)}$ such that

$$f^{(s)}(Z) = Z f^{(s+1)}$$

(12)

for $s = 0, \ldots, k - 1$.

If $k = 1$, the expression given by Theorem 1 for $\eta_{-1}(Z), \eta_0(Z)$ is substituted in $f^{(0)}(Z)$, and in this way $f^{(1)}(Z)$ is obtained and $f^{(1)}(0) \neq 0$. If $k > 1$ for each step $s = 0, \ldots, k - 2$, if $f^{(s)}(0) = 0$ then

$$f^{(s+1)}(Z) = \frac{f^{(s)}(Z)}{Z}.$$  

(13)

If $f^{(s)}(0) \neq 0$, then it is possible to write

$$f^{(s)}(Z) = f_0^{(s)}(Z) + Z f_1^{(s)}(Z) + \ldots + Z^M f_M^{(s)}(Z)$$

(14)

where $f_0^{(s)}(Z)$ is a linear combination of products of functions $\eta_j(Z)$ with $j = -1, 0$ for $s = 0$ and $j = 0, \ldots, s$ for $s > 0$ and $f_0^{(s)}(0) \neq 0$.

By substituting in $f_0^{(s)}(Z)$ the expression given by theorem 1, and the expression of $f^{(s+1)}$ is obtained.

At the second last step

$$f^{(k-1)}(Z) = f_0^{(k-1)}(Z) + Z f_1^{(k-1)}(Z) + \ldots + Z^{M_k} f_{M_k}^{(k-1)}(Z)$$

(15)

where $f_0^{(k-1)}(Z)$ is a linear combination of functions $\eta_j$ for $j = 1, \ldots, k-1$ and $\eta_0^{2j}$ for $j = 1, \ldots, k$.

If $f_0^{(k-1)}(0) = 0$, then

$$f^{(k)}(Z) = \frac{f^{(k-1)}(Z)}{Z}.$$  

(16)

Instead, if $f_0^{(k-1)}(0) \neq 0$, then for $s = k-1$, in $f_0^{(k-1)}(Z)$ the expression given by 1 is substituted. In this way, $f^{(k)}(Z)$ with $f^{(k)}(0) \neq 0$ is determined. Finally the reformulated coefficient is $F(Z) = f^{(k)}(Z)$.
3. Reformulation of the coefficients

As an example we consider the method developed in Simos et al. [79] to solve the second order IVP

\[
\begin{align*}
    y'' &= f(x, y(x)) \\
    y(x_0) &= y_0 \\
    y'(x_0) &= y'_0
\end{align*}
\]

(17)

The numerical scheme is of the form

\[
y_{n+1} + d_0y_n + d_1y_{n-1} + d_2y_{n-2} + d_1y_{n-3} + d_0y_{n-4} + y_{n-5} = h^2 \left( \tilde{d}_0y''_{n+1} + \tilde{d}_1y''_{n-1} + d_2y''_{n-2} + \tilde{d}_1y''_{n-3} + \tilde{d}_0y''_{n-4} \right)
\]

(18)

with,

\[
y_{n+1} + c_0y_n + c_1y_{n-1} + c_2y_{n-2} + c_1y_{n-3} + c_0y_{n-4} + y_{n-5} = h^2 \left( \tilde{c}_0y''_{n+1} + \tilde{c}_1y''_{n-1} + \tilde{c}_2y''_{n-2} + \tilde{c}_1y''_{n-3} + \tilde{c}_0y''_{n-4} \right).
\]

(19)

The classical coefficients, derived in [64], assume the form

\[
\tilde{c}_0 = \frac{51484823}{17645880}, \quad \tilde{c}_1 = \frac{23362512}{735245}, \quad \tilde{c}_2 = \frac{723342859}{8822940}
\]

(20)

\[
c_0 = \frac{12519323}{504168}, \quad c_1 = \frac{2712635}{63021}, \quad c_2 = \frac{551}{4},
\]

\[
d_0 = -\frac{23362512}{735245}, \quad d_1 = \frac{84437}{105035}, \quad d_2 = -\frac{9}{5},
\]

\[
\tilde{d}_0 = \frac{1}{15}, \quad \tilde{d}_1 = \frac{209837}{210070}, \quad \tilde{d}_2 = \frac{320221}{315105}, \quad \tilde{d}_3 = \frac{638003}{315105}
\]

(21)

By applying the exponential fitting technique the following coefficients are derived in [79]

\[
\tilde{c}_0 = -\frac{1}{2016762672\nu^6\sin^3(\nu)} \left( 3(4965191\nu^4 - 82890689\nu^2 + 22589400)\sin(\nu) \right.
\]

\[
- 48(639329-2 - 308970)\sin(2\nu) - (4965191\nu^4 - 8059857\nu^2 - 68357160)\sin(3\nu)
\]

\[
+ 575250(3\nu^2 - 10)\sin(4\nu) - (59935259\nu^2 - 95582232)\nu\cos(\nu)
\]

\[
- 32(437993\nu^2 + 2928636)\nu\cos(2\nu) - 3(4965191\nu^2 + 13671432)\nu\cos(3\nu)
\]

\[
+ 7562500\nu\cos(4\nu) + 48(875986\nu^2 - 759933)\nu \right)
\]

(22)

\[
\tilde{c}_1 = \frac{1}{10083366016\nu^5\sin^2(\nu)} \left( -24(875986\nu^4 + 231349\nu^2 - 617940)\sin(\nu) \right.
\]

\[
- 18(14126869\nu^2 - 7562520)\sin(2\nu) + 4(1751972\nu^4 + 9751899\nu^2 - 15198660)\sin(3\nu)
\]

\[
+ 3(4965191\nu^2 + 22785720)\sin(4\nu) + 3781260(3\nu^2 - 20)\sin(5\nu)
\]

\[
+ 64(875986\nu^2 - 2650563)\nu\cos(\nu) - 12(14126869\nu^2 - 4537512)\nu\cos(2\nu)
\]

\[
- 2(4965191\nu^2 + 13671432)\nu\cos(4\nu) + 60500160\nu\cos(5\nu)
\]

\[
+ 6(4965191\nu^2 + 13671432)\nu \right)
\]

(23)
\[
\hat{c}_2 = \frac{1}{2016672\nu^6 \sin^3(\nu)} \left(6(42380607\nu^4 + 125300744\nu^2 - 45276960)\sin(\nu)ight.
- 24(2281313\nu^2 - 679290)\sin(2\nu) - 3(28253738\nu^4 - 74403153\nu^2 + 113732280)\sin(3\nu)
- 48(1821301\nu^2 - 599314)\sin(4\nu) - 3(4965191\nu^2 + 22785720)\sin(5\nu)
- 7562520(\nu^2 - 10)\sin(6\nu) + 160(906559\nu^2 - 2390292)\nu\cos(\nu)
- 16(1751972\nu^2 - 21371481)\nu\cos(2\nu) + 9(33218929\nu^2 + 459640)\nu\cos(3\nu)
- 32(437993\nu^2 + 6709896)\nu\cos(4\nu) - (4965191\nu^2 + 13671432)\nu\cos(5\nu)
- 45375120\nu\cos(6\nu) - 288(437993\nu^2 - 852624)\nu\cos(7\nu) - 6776017920(2 + Z\eta_0(Z/4) - 2*Z\eta_1(Z))^3 \right)
\]

The other coefficients coincide with the ones derived in the classical way (21). We also observe that, for \(\nu \to 0\), the \(\hat{c}\) coefficients tend to classical ones

\[
\begin{align*}
\lim_{\nu \to 0} \hat{c}_0 &= \frac{51484823}{17645880}, \\
\lim_{\nu \to 0} \hat{c}_1 &= \frac{23362512}{735245}, \\
\lim_{\nu \to 0} \hat{c}_2 &= \frac{723342859}{8822940}.
\end{align*}
\]

By applying the procedure described in the previous section, the modified coefficients are:

\[
\begin{align*}
\hat{c}_0 &= \frac{-1071987210\eta_0^4(Z/64) - 535993605\eta_0^2(Z/256)(1 + \eta_0(Z/64)) + \ldots}{13552035840(2 + Z\eta_0^2(Z/4) - 2*Z\eta_1(Z))^3} \\
\hat{c}_1 &= \frac{-2031821820\eta_0^4(Z/64) - 1015910910\eta_0^2(Z/256)(1 + \eta_0(Z/64)) + \ldots}{13552035840(2 + Z\eta_0^2(Z/4) - 2*Z\eta_1(Z))^3} \\
\hat{c}_2 &= \frac{-2146293450\eta_0^4(Z/64) + 1073146725\eta_0^2(Z/256)(1 + \eta_0(Z/64)) + \ldots}{6776017920(2 + Z\eta_0^2(Z/4) - 2*Z\eta_1(Z))^3}
\end{align*}
\]

The full expression of the coefficients can be obtained using the Mathematica modules in [33]. Figures 1, 2, 3 show the behaviour of the three coefficients.

**Figure 1.** Coefficient \(c_0\) in correspondence of \(\omega = 10\)

In particular, we observe that the coefficients \(\hat{c}_0, \hat{c}_1\) and \(\hat{c}_2\), expressed in terms of \(\eta_m(Z)\) functions are converging on the classical value while the coefficient \(\hat{c}_0, \hat{c}_1\) and \(\hat{c}_2\) explode indefinitely.
4. Numerical experiments
In order to show how the coefficients conversion in terms of $\eta_m(Z)$ affects the convergence of the method, we consider the following problem:

$$
\begin{align*}
    y''(t) &= -100y(t) + 99\sin(t) \\
    y(0) &= 1 \\
    y'(0) &= 11
\end{align*}
$$

(29)

with $t \in [0, 20\pi]$, whose analytic solution is $y(t) = \cos(10t) + \sin(10t) + \sin(t)$.

We compare the results obtained by applying

- EF: the ef method (18)-(19) with coefficients (22), (21);
- EF converted: the ef method (18)-(19) with converted coefficients (26)–(28),(21).

Figure 4 shows the error behaviour applied to previous problem. We observe that this reformulation of the coefficients restores the convergence of the method.
5. Conclusions

This paper has provided a regularized formulation of exponentially fitted methods, eliminating the effects in indeterminacy arising in the coefficients when the employed stepsize is small. These methods are here intended for second order differential equations (whose relevant role in a wide range of physical problems is well known, see [68] and references therein) and the numerical experiments confirm the effectiveness of the approach. Future works will be oriented in the regularization of exponentially fitted methods for other kind of problems, such as PDEs [15, 53, 54, 59, 59, 61], as well as to the consequences of the approach in terms of stability properties of the corresponding methods.

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