Morse inequalities at infinity for a resonant mean field equation

Mohameden Ahmedou & Mohamed Ben Ayed

Abstract

In this paper we study the following mean field type equation

\[(MF) \quad -\Delta_g u = \varrho \left( \frac{Ke^u}{\int_{\Sigma} Ke^u dV_g} - 1 \right) \text{ in } \Sigma,\]

where \((\Sigma, g)\) is a closed oriented surface of unit volume \(\text{Vol}_g(\Sigma) = 1\), \(K\) a positive smooth function and \(\varrho = 8\pi m, \ m \in \mathbb{N}\). Building on the critical points at infinity approach initiated in [1] we develop, under generic condition on the function \(K\) and the metric \(g\), a full Morse theory by proving Morse inequalities relating the Morse indices of the critical points, the indices of the critical points at infinity, and the Betti numbers of the space of formal barycenters \(B_m(\Sigma)\).

We derive from these Morse inequalities at infinity various new existence as well as multiplicity results of the mean field equation in the resonant case, i.e. \(\varrho \in 8\pi \mathbb{N}\).

Key Words: Critical points at infinity, Morse theory, Space of formal barycenters.
AMS subject classification: 35C60, 58J60, 35J91.

1 Introduction and statement of the results

Let \((\Sigma, g)\) be a closed and oriented surface of unit volume \(\text{Vol}_g(\Sigma) = 1\). For \(\varrho \in \mathbb{R}^+\) real number and \(K\) positive smooth function, we consider the following Mean Field equation

\[(MF) \quad -\Delta_g u = \varrho \left( \frac{Ke^u}{\int_{\Sigma} Ke^u dV_g} - 1 \right) \text{ in } \Sigma.\]

Problem \((MF)\) arises as limiting equation in some mean field approximation in the study of limit point vortices of Euler flows, in Onsager vortex theory, and also in the description of self dual condensates in some Chern-Simons-Higgs models. See for example the papers [2],[8], [9], [12], [13], [14], [20], [21], [32], [28], and the monographs of Tarantello [29] and of Yang [31] as well as the references therein. Furthermore its study is also motivated by the prescribed Gauss curvature problem in differential geometry. Indeed, if \(\Sigma\) is the standard 2-sphere \(S^2\) endowed with its standard metric \(g_{\text{stand}}\) and \(\varrho = 8\pi\), then problem \((MF)\) is the so called Nirenberg’s problem in conformal geometry, which consists of finding a metric \(g\) conformally equivalent to \(g_{\text{stand}}\) and whose Gauss curvature is given by the function \(K\). See [10], [19], [11] and the references therein.

Equation \((MF)\) has a variational structure and the associated Euler Lagrange functional \(J_\varrho\) is defined by

\[J_\varrho(u) := \frac{1}{2}||u||^2 - \varrho \ln\left( \int_{\Sigma} Ke^u dV_g \right) \quad \text{for } u \in \dot{H}^1(\Sigma)\]

where

\[||u|| := ||\nabla u||_{L^2}\]
is the norm used to equip

\[ H^1(\Sigma) := \left\{ u \in H^1(\Sigma) : \int_\Sigma u dV_g = 0 \right\}. \]

It turns out that the analytical aspect of this variational problem depends on the range of values taken by the parameter \( \rho \). Indeed

- If \( \rho \not\in 8\pi\mathbb{N} \) then the associated variational problem is compact in the sense that changes of the topology of the sublevel sets of \( J_\rho \) are induced only by critical points. We call it the non-resonant case.

- If \( \rho \in 8\pi\mathbb{N} \) then the associated variational problem is not compact, i.e. changes in the difference of topology between its sublevel sets might be induced by critical points at infinity, these are non compact orbits of the gradient flow whose level sets are bounded (see [3] for a related notion for Yamabe type problem). We call it the resonant case.

In the non-resonant case, Yanyan Li [21] proved that the Leray-Schauder degree of the solutions of \((MF)\) is well defined and is a topological invariant. He also advised a method to compute it by analyzing the degree jump across the critical values \( 8\pi\mathbb{N} \). C.C. Chen and C.S. Lin [12], [13], proved a priori estimates for the solutions, and used the strategy advised by Yanyan Li to compute the Leray-Schauder degree. Then they derived that Problem \((MF)\) is solvable provided that the surface has a positive genus. Later Z. Djadli [15], using the topological argument of [16], proved the existence of a solution of \((MF)\) in this case for surfaces of every genus. Furthermore A. Malchiodi [25] developed a Morse theory in this case and F. De Marchis [17] proved full Morse inequalities as well as multiplicity results for generic data \((K,g)\).

For the resonant case the main difficulty lies in the fact that to find critical points of \( J_\rho \) one has to understand the contributions of the critical points at infinity to the topology of its sublevel sets. This amounts to proving a Morse lemma at infinity in a highly degenerate setting where gradient flow lines may converge only in the sense of measures to a sum of Dirac masses. Based on ideas going back to Lions [23] and Struwe [27] one can often perform a "Lyapunov-Schmidt reduction at infinity", and finds that the Dirac deltas are located at critical points of a finite-dimensional reduced function. Such an approach has been developed by the authors in collaboration with M. Lucia in [1]. There, a Morse reduction in the neighborhood of the critical points has been performed and the induced difference of topology between the levels of \( J_\rho \) has been computed. In this paper the authors want to develop further this approach by proving Morse inequalities between critical points at Infinity and use them to deduce new existence and multiplicity results.

To state our main results we introduce the following notation and assumptions:

For \( m \in \mathbb{N} \), we let

\[ F_m(\Sigma) := \{(a_1, \ldots, a_m) \in \Sigma^m ; a_i \neq a_j \text{ if } i \neq j \} \]

denote the space of configurations of \( \Sigma^m \) and we define the following reduced energy functional

\[ F^K_m : F_m(\Sigma) \longrightarrow \mathbb{R}, \]

by

\[ F^K_1(a) := \ln K(a) + 4\pi H(a, a) \quad (\text{for } m = 1; a \in \Sigma), \]

\[ F^K_m(A) := \sum_{i=1}^{m} \left( \ln K(a_i) + 4\pi H(a_i, a_i) + 4\pi \sum_{j \neq i} G(a_i, a_j) \right) \quad (\text{for } m \geq 2; A = (a_1, \ldots, a_m)), \]

where \( G \) is the Green’s function of \(-\Delta_g\) and \( H \) its regular part. See (12) and (14) for the precise definition. We notice that the function \( F^K_m \) achieves its infimum for each \( m \geq 1 \) and for \( m = 1 \) it also achieves its maximum.

Next for \( A = (a_1, \ldots, a_m) \in F_m \) a critical point of \( F^K_m \) we define a vector \((F^K_1, \ldots, F^K_m)\), where

\[ F^K_i : \Sigma \longrightarrow \mathbb{R}, \]
are real valued functions defined as follows:

\((5)\) \(F_i^0(x) := K(x) \exp(8\pi H(a, x)), \) if \(m = 1,\)
\[(6)\] \(F_i^A(x) := K(x) \exp \left(8\pi H(a_i, x) + 8\pi \sum_{j \neq i} G(x, a_j)\right)\) if \(m \geq 2\) for \(i = 1, \ldots, m.\)

Moreover we associate to every critical point \(A = (a_1, \ldots, a_m)\) of \(F_m^K\) the following quantity

\[(7)\] \(L(A) := \sum_{i=1}^{m} (\Delta_g F_i^A (a_i) - 2K_g (a_i) F_i^A (a_i)),\)

where \(K_g\) denotes the Gauss curvature of \((\Sigma, g).\) Next we define the following set

\[(8)\] \(K^m_- := \{A \in \mathbb{R}_m(\Sigma) : \nabla F_m^K (A) = 0, L(A) < 0\}.

Our first type of results are based on the construction of solutions of sub- and sup-approximation with fixed morse indices, which we show that, under appropriate assumptions of critical points of the related reduced energy, they give rise to solutions of Equation \((MF).\)

**Theorem 1.1** Let \(m \geq 1\) and \(g = 8\pi m\) and assume that the reduced energy \(F_m^K\) has only non degenerate critical points. Then

1) if at every local minimum \(A \in \mathbb{R}_m(\Sigma)\) of \(F_m^K\) we have that \(L(A) < 0.\) Then Equation \((MF)\) has at least one solution whose generalized Morse index is \(3m.\)

2) for \(m \geq 2,\) if at every critical point \(A \in \mathbb{R}_m(\Sigma)\) of \(F_m^K\) with morse index \(2\) we have that \(L(A) > 0.\) Then Equation \((MF)\) has at least one solution whose generalized Morse index is \(3m - 3,\)

where the generalized Morse index of a solution \(\omega\) of \((MF)\) is the sum of its Morse index and the dimension of the Kernel of the linearized operator

\[T_{\omega}(\varphi) := -\Delta_g \varphi - 8\frac{K e^{\omega} \varphi}{\int_{\Sigma} K e^{\omega}}.\]

We observe that, for \(m = 1,\) the functional \(J_{8\pi m}\) is bounded below and therefore if there exists a local maximum point \(A\) of \(F^K\) such that \(L(A) > 0\) then \(J_{8\pi m}\) achieves its minimum (see [1], Theorem 1.1). We point out that there is no such points for the Nirenberg’s problem on \(S^2\) and more generally there are no such points if the surface has a positive Gauss curvature.

Our next and main result of this paper is to prove *Morse inequalities*. To that aim we introduce the following non degeneracy assumption:

We say that the pair \((g, K)\) satisfies the condition \((N_m)\) if the following conditions are satisfied:

(i) The critical points of \(F_m^K\) are non degenerate and for every critical point \(A,\) we have \(L(A) \neq 0,\)

(ii) All the critical points of \(J_{8\pi m}\) are non degenerate.

**Remark 1.1** The non degeneracy condition \((N_m)\) is generic. Indeed, denoting by \(\mathcal{M}\) the set of riemannian metrics on \(\Sigma\) and \(C^0_+ (\Sigma)\) the space of positive Lipschitz functions on \(\Sigma.\) It follows from transversality arguments, there is an open and dense subset \(D \in \mathcal{M} \times C^0_+ (\Sigma)\) such that for \((g, h) \in D,\) the functional \(J_g\) is a Morse function. See for example [17], pp. 2179-80, for an argument based on some generic properties for nonlinear boundary value problems proved by Saut and Temam [26]. Moreover the condition \((i)\) is also a generic one.

Furthermore, under the non degeneracy condition \((i)\) of \((N_m),\) we associate to every \(A \in K^m_-\) an index

\[(9)\] \(t_{\infty} : K^m_- \to \mathbb{N} \text{ defined by } t_{\infty}(A) := 3m - 1 - Morse(F_m^K, A),\)

```
In the next result we provide Morse inequalities relating the Morse indices of the solutions of \((\mathcal{F}_m^K, A)\) at its critical point \(A\). Next for \(m \in \mathbb{N}\), let

\[ B_m(\Sigma) := \left\{ \sum_{i=1}^{m} \gamma_i \delta_{a_i}, a_i \in \Sigma, \gamma_i \geq 0; \sum_{i=1}^{m} \gamma_i = m \right\} \]

denote the set of formal barycenters of order \(m\). For \(i \geq 0\), we set

\[ \beta_i^m := \text{rank } H_i(B_m(\Sigma), \mathbb{Z}_2). \]

Then \(B_m(\Sigma)\) is a stratified set of top dimension \(3m - 1\) and therefore \(\beta_i^m = 0\) for \(i \geq 3m\).

In the next result we provide Morse inequalities relating the Morse indices of the solutions of \((MF)\), the \(\iota_\infty\)-indices of the critical points at infinity and the Betti numbers \(\beta_i^m\) of the set of formal barycenters \(B_m(\Sigma)\). Before stating these inequalities, we recall that it follows from the blow up analysis of solutions of \((MF)\), see [22, 12, 13] that their Dirichlet energy is uniformly bounded. See please statement (ii) in Theorem 3.1, p. 1686 in [13]. Hence it follows from Moser-Trudinger inequality and elliptic theory that solutions of \((MF)\) with zero mean value integral are uniformly bounded and hence their Morse indices are uniformly bounded.

We are now ready to state our Morse inequalities in the case \(q = 8\pi m; m \geq 2:\)

**Theorem 1.2** Let \(q = 8\pi m\), \(m \geq 2\) and assume that the function \(K\) satisfies the non degeneracy condition \((N_m)\) and let \(\overline{m}\) be the highest Morse index of the critical points of \(J_q\). Then the following Morse inequalities hold

(a) For every \(2 \leq k \leq \overline{N} := \max(3m - 1, \overline{m})\) there holds

\[ \nu_k + \nu_k^\infty \geq \beta_k^{m-1}, \]

where \(\nu_i\) denotes the number of critical points of \(J_q\) with Morse index \(i\) and

\[ \nu_i^\infty := \# \{ A \in \mathcal{K}_m^\infty; \iota_\infty(A) = q \}. \]

(b) Let \(\chi(\Sigma)\) be the Euler Characteristic of \(\Sigma\), it holds

\[ \sum_{i=0}^{\overline{m}} (-1)^i \nu_i + \sum_{A \in \mathcal{K}_m^\infty} (-1)^{\iota_\infty(A)} = \left( m - 1 - \frac{\chi(\Sigma)}{m - 1} \right). \]

In particular the number of critical points of \(J_q\) is lower bounded by

\[ \left| \left( m - 1 - \frac{\chi(\Sigma)}{m - 1} \right) - \sum_{A \in \mathcal{K}_m^\infty} (-1)^{\iota_\infty(A)} \right|. \]

As corollary of the above Morse inequalities at infinity we derive the following existence result:

**Theorem 1.3** Let \(m \geq 2\) and \(q = 8\pi m\) and assume that the function \(K\) satisfies the non degeneracy condition \((N_m)\). Suppose there exists \(2 \leq q_0 \leq 3m - 3\) such that

- there is no element in \(\mathcal{K}_m^\infty\) of index \(q_0\)
- \(\beta_{q_0}^{m-1} \neq 0\).

Then Equation \((MF)\) has at least one solution whose Morse index is \(q_0\). In particular, since \(\beta_{3m-4}^{m-1} = H_{3m-4}(B_{m-1}(\Sigma)) \neq 0\), if there is no element in \(\mathcal{K}_m^\infty\) of index \(3m - 3\), then Equation \((MF)\) has at least one solution whose Morse index is \(3m - 3\). Furthermore, the number of solutions is lower bounded by \(\beta_{3m-4}^{m-1}\).
As a complement of the statement made in the above theorem, we prove the following result:

**Theorem 1.4** Let $m \geq 2$ and $\varrho = 8\pi m$ and assume that the function $K$ satisfies the non degeneracy condition $(N_m)$. Suppose there exists $q_0$ such that

1. there exists $Q_0 \in K_m^-$ with $\iota_\infty(Q_0) = q_0$,
2. for each $Q \in K_m^-$, we have $\iota_\infty(Q) \notin \{q_0 - 1, q_0 + 1\}$,
3. $\beta^{m-1}_{q_0-1} = 0$.

Then Equation (MF) has at least one solution.

**Remark 1.2** We notice that the indices of the critical points at infinity are lower bounded by $m - 1$. Hence for $m = 4$ suppose

1. there exists $Q_0 \in K_m^-$ such that $\iota_\infty(Q_0) = 3$ (i.e. $F^K_m(Q_0)$ is a local maximum of $F^K_m$),
2. for all $Q \in K_m^-$, we have $\iota_\infty(Q) \neq 4$,

then it follows from Theorem 1.4 that the Equation (MF) has at least one solution.

More generally any violation of the above Morse inequalities at infinity of Theorem 1.2 induces the existence of a solution to Equation (MF). In particular we have the following existence result:

**Theorem 1.5** Let $m \geq 2$ and $\varrho = 8\pi m$ and assume that the function $K$ satisfies the non degeneracy condition $(N_m)$. If one of the following inequalities is not satisfied

- $\nu_{3m-2}^\infty \geq \nu_{3m-1}^\infty$.
- $\nu_{3m-3}^\infty - \beta^{m-1}_{3m-4} \geq \nu_{3m-2}^\infty - \nu_{3m-1}^\infty$.

Then Equation (MF) has at least one solution (where $\nu_0^\infty$ is defined in (11)).

The remainder of this paper is organized as follows: we set up some notation and definitions in Section 2, introduce an $\varepsilon$–neighborhood of potential critical points at infinity in Section 3 and provide in Section 4 useful expansions of the gradient of the Euler-Lagrange functional in the neighborhood at infinity. Section 5 is devoted to the characterization of such critical points at infinity and we provide the proof of our main results in Section 6. Finally we provide useful estimates of the projected bubble (see (17) for the definition) in the appendix.

## 2 Notation and definitions

To state our results we need to fix some notation.

For $\xi \in \Sigma$, let $y_\xi$ be a local chart defined on a neighborhood of $\xi$ onto $B(0, 2\eta_0)$ such that $y_\xi(\xi) = 0$. In the sequel, we will denote $B_\xi(\eta) := y_\xi^{-1}(B(0, \eta))$. Furthermore, thanks to the isothermal coordinates, we have that for every $a \in \Sigma$, there exists a function $u_a \in C^\infty(\Sigma)$, satisfying $u_a(a) = 0$ and $\nabla u_a(a) = 0$, such that for the conformal metric $g_a := e^{u_a}g$, there hold

$$g_a = \delta_{ij} \in B_a(2\eta_0), \ \Delta_g = e^{u_a}\Delta_{g_a}, \ dV_g = e^{-u_a}dV_{g_a}, \quad \text{and} \quad \Delta_{g_a}u_a = 2K_g(\cdot)e^{-u_a} \in B_a(2\eta_0),$$

where $0 < \eta_0 < \eta_a < \frac{1}{2}\text{inj}_{g_a}(\Sigma)$ where inj stands for the injectivity radius.

Next for $a \in \Sigma$, let $G(a, \cdot)$ be the Green’s function of $\Delta_g$ defined by

$$G(a, x) = \begin{cases} -\Delta_g G(a, x) + 1 & = \delta_a \text{ in } \Sigma, \\ \int_\Sigma G(a, x)dV_g(x) & = 0. \end{cases}$$
For $0 < \eta < \frac{1}{4} \eta_0$ and $a \in \Sigma$ we define the smooth cut-off function

\begin{equation}
\psi_a(x) := \psi(|y_a(x)|) \quad \text{where} \quad \psi(t) :=
\begin{cases}
t, & \text{if } t \in [0, \eta], \\
2\eta, & \text{if } t \geq 2\eta,
\end{cases}
\psi(t) \in [\eta, 2\eta], \psi \in C^\infty \quad \text{otherwise.}
\end{equation}

Next for $\xi \in \Sigma$, as usual we decompose $G(\xi, \cdot)$ as follows

\begin{equation}
G(\xi, x) = -\frac{1}{2\pi} \ln(\psi_\xi(x)) + H(\xi, x).
\end{equation}

From the definition of $G(\xi, \cdot)$ we derive that $H(\xi, \cdot)$ has to satisfy

\begin{equation}
\Delta_g H(\xi, \cdot) - 1 = \Delta_g G(\xi, \cdot) - 1 + \frac{1}{2\pi} \Delta_g \ln(\psi(\cdot)) = \begin{cases}
0 \text{ in } (\Sigma \setminus B_\xi(2\eta)) \cup B_\xi(\eta), \\
\frac{\psi_\xi}{2\pi} \Delta_g \ln(\psi(\cdot)) \text{ in } B_\xi(2\eta) \setminus B_\xi(\eta).
\end{cases}
\end{equation}

In the next section, we will describe the lack of compactness occurring in the variational problem associated to Equation $(MF)$. To do so we need to introduce some highly concentrated functions, the so called bubbles and a neighborhood of such bubbles.

First, we remark that the following functions

\begin{equation}
\tilde{\delta}_{a, \lambda}(x) := \ln(\frac{8\lambda^2}{(1 + \lambda^2|a - x|^2)^2}), \quad \text{for } a \in \mathbb{R}^2, \text{ and } \lambda > 0,
\end{equation}

are the solutions of

\[-\Delta u = e^u \quad \text{in } \mathbb{R}^2 \quad \text{with} \quad \int_{\mathbb{R}^2} e^u < \infty.\]

Next for $a \in \Sigma$ and $\lambda > 0$ we define the standard bubble as

\begin{equation}
\delta_{a, \lambda}(x) := \ln(\frac{8\lambda^2}{(1 + \lambda^2\psi_a(x)^2)^2}),
\end{equation}

where $\psi_a$ is defined in (13). In order to construct a suitable approximated solution of $(MF)$ we follow the strategy introduced by A. Bahri and J.M. Coron in their study of the Yamabe equation [4] and we introduce the projected bubble $\varphi_{a, \lambda}$ to be the unique solution of

\begin{equation}
\left\{ \begin{array}{l}
-\Delta_g \varphi_{a, \lambda} = e^{\delta_{a, \lambda} + u_a} - \int_{\Sigma} e^{\delta_{a, \lambda} + u_a} dV_g \text{ in } \Sigma, \\
\int_{\Sigma} \varphi_{a, \lambda} dV_g = 0.
\end{array} \right.
\end{equation}

We observe that for $A := (a_1, \cdots, a_m) \in \Sigma^m$ and $\Lambda := (\lambda_1, \cdots, \lambda_m) \in (\mathbb{R}^+)^m$ the sequence of functions $(U_{A, \Lambda})_\Lambda$ defined by

\begin{equation}
U_{A, \Lambda} := \sum_{i=1}^{m} \varphi_{a_i, \lambda_i}
\end{equation}

is a Palais-Smale sequence for Equation $(MF)$ for $\rho = 8\pi m$ if $A$ and $\Lambda$ satisfy the following balancing condition

\begin{equation}
\forall i \neq j \quad \lambda_i^2 F_i^A(a_i) = \lambda_j^2 F_j^A(a_j)(1 + o_\lambda(1)),
\end{equation}

where $o_\lambda(1) \to 0$ if $\lambda \to +\infty$. See please Lemma 7.5 for the equation satisfied by $U_{A, \Lambda}$. Furthermore we collected in the appendix some useful estimates on $\varphi_{a, \lambda}$. Such estimates are used in the expansion of the Euler Lagrange equation and its gradient in the neighborhood at infinity $V(m, \varepsilon)$. See please (20) for the definition of this set.
3 Lack of compactness and a neighborhood at infinity

Our approach is variational. Therefore, in order to detect critical points for the functional \( J_\varrho \) with \( \varrho = 8\pi m \), we have to find the obstruction in deforming sub-level sets \( J_\varrho^{a} := \{ u : J_\varrho(u) \leq a \} \). Using a deformation Lemma given in [24], it is known that for the functional \( J_\varrho \), besides critical points, the obstruction to decrease the functional comes only from flow lines entering a set that is a neighborhood of solutions \( u_k \) of Problem \( (P_\varrho) \) with \( \varrho_k \nearrow \varrho \).

Assuming that \( J_\varrho \) does not have any critical point, then we derive that \( (u_k) \) has to blow up and the results of [7] combined with [22] imply that \( \varrho \) has to be \( 8\pi m \) with \( m \in \mathbb{N}^* \). Furthermore, from the works of [12] and [13], the solutions \( (u_k) \) have to belong to some set \( V(m, \varepsilon) \), called in the sequel neighborhood of potential critical points at Infinity, which is defined by:

\[
V(m, \varepsilon) := \left\{ u \in \dot{H}^1(\Sigma) : \| \nabla J_\varrho(u) \| < \varepsilon ; \exists \lambda_1, \ldots , \lambda_m > \varepsilon^{-1} \text{ with } \lambda_i < C_1 \lambda_j \text{ such that } \| u - \sum_{i=1}^{m} \varphi_{\lambda_i} \| < \varepsilon \right\},
\]

where the space \( \dot{H}^1(\Sigma) \) is defined in (2), \( \varepsilon \) is a small positive constant and \( C_1, \eta \) are fixed positive constants.

Hence, we are led to study the obstructions to decrease the functional \( J_\varrho \) in the set \( V(m, \varepsilon) \). A first step consists in finding an appropriate parametrization of this set. To that aim, following the ideas of A. Bahri and J.M. Coron we consider the following minimization problem

\[
(21) \quad \min_{\alpha_i > 0 ; \alpha_i \in \Sigma ; \lambda_i > 0} \| u - \sum_{i=1}^{m} \alpha_i \varphi_{\lambda_i} \|.
\]

We have the following Lemma whose proof is identical to the proof of Proposition 7 in [5] (see also Chen and Lin [13, Lemma 3.2]). Namely we have

Lemma 3.1 For \( \varepsilon \) small, Problem (21) has for any \( u \in V(m, \varepsilon) \) only one solution (up to permutations on the indices). The variables \( \alpha_i \)’s satisfy \( |\alpha_i - 1| = O(\varepsilon) \).

Hence every \( u \in V(m, \varepsilon) \) can be written as

\[
(22) \quad u = \sum_{i=1}^{m} \alpha_i \varphi_{\lambda_i} + w,
\]

where \( \alpha_i, w \) satisfy

\[
(23) \quad \left\{ \begin{array}{l}
|\alpha_i - 1| \leq c\varepsilon \quad \forall i, \\
\| w \| \leq c\varepsilon, \\
\langle w, \varphi_{\lambda_i} \rangle_g = \langle w, \partial \varphi_{\lambda_i} / \partial \lambda_i \rangle_g = 0, \\
\langle w, \partial \varphi_{\lambda_i} / \partial a_i \rangle_g = 0 \quad \forall i.
\end{array} \right.
\]

In the following, for \( A = (a_1, \ldots , a_m) \) and \( \Lambda = (\lambda_1, \ldots , \lambda_m) \), we denote

\[
(24) \quad E^{m}_{A, \Lambda} := \{ w \in \dot{H}^1(\Sigma) : \text{w satisfies (23)} \}.
\]

To keep the notation short we will write \( \varphi_i \) instead of \( \varphi_{\lambda_i} \). The next Proposition shows how to deal with the infinite dimensional part \( w \) in the above representation:

Proposition 3.2 [1] Let \( u := \sum_{i=1}^{m} \alpha_i \varphi_i \in V(m, \varepsilon) \). Then there exists a unique \( \varpi := \varpi(u) \in E^{m}_{A, \Lambda} \) such that:

\[
J_\varrho(u + \varpi) = \min\{ J_\varrho(u + w) : w \in E^{m}_{A, \Lambda} \}.
\]

Furthermore, there exists a constant \( C \) such that

\[
(25) \quad \| \varpi \| \leq C \sum_{i=1}^{m} \left( |\alpha_i - 1| + \frac{1}{\lambda_i} \right).
\]
4 The expansion of the gradient in the neighborhood at infinity

In this section we provide an asymptotic expansion of the gradient of the Euler Lagrange functional $J_\varepsilon$ in the neighborhood at infinity $V(m, \varepsilon)$. For the sake of simplicity of the notation and since the variables $\lambda_i$'s are of the same order, we will write in this section and in the sequel $O(1/\lambda^n)$ instead of $\sum O(1/\lambda^n)$.

In the first proposition we expand the gradient with respect to the concentrations rates. Namely we have

**Proposition 4.1** Let $u := \sum_{i=1}^m \alpha_i \varphi_i + w \in V(m, \varepsilon)$ and $g := 8m\pi(1 + \mu)$ with $\mu \geq 0$. It holds

$$\langle \nabla J_\varepsilon(u), \lambda \frac{\partial \varphi_i}{\partial \lambda_i} \rangle_g = 16\pi\alpha_i(\tau_i - \mu + \mu \tau_i) - 64\pi^2 \sum_{j=1}^m \frac{\ln \lambda_i}{\lambda_j^2} + O\left(\sum |\alpha_k| - 1|^2 + \|w\|^2\right) + o\left(\frac{\ln \lambda}{\lambda^2}\right),$$

with

$$\tau_i := 1 - \frac{m\pi}{2\alpha_i - 1} \frac{\lambda_i^{4\alpha_i - 2} f_i^A(a_i) g_i^A(a_i)}{\int_{\Sigma} Ke^u dV_g},$$

where $f_i^A$ and $g_i^A$ are defined in (47). Furthermore, we have the estimate

$$|\tau_i| = O(\varepsilon + \mu), \quad \forall i \in \{1, \cdots, m\}.$$

**Proof.** Recall that

$$\langle \nabla J_\varepsilon(u), h \rangle_g = \langle u, h \rangle_g - \frac{\partial}{\partial \lambda_i} \int_{\Sigma} Ke^u h.$$

Using Lemma 7.2 we get that

$$\langle u, \lambda \frac{\partial \varphi_i}{\partial \lambda_i} \rangle_g = -64\pi^2 \sum_{j=1}^m \frac{\alpha_j}{\lambda_i^2} + 16\pi\alpha_i + O\left(\frac{1}{\lambda^2}\right) = -64\pi^2 \sum_{j=1}^m \frac{\alpha_j}{\lambda_i^2} + 16\pi\alpha_i + o\left(\frac{\ln \lambda}{\lambda^2}\right).$$

For the other term of (28), using Lemma 7.1, we have

$$\int_{\Sigma} Ke^u \lambda \frac{\partial \varphi_i}{\partial \lambda_i} dV_g = \int_{\Sigma} Ke^u \left(\frac{4}{1 + \lambda_i^2 \psi_i^2} - 8\pi \frac{\ln \lambda_i}{\lambda_i^2} + O\left(\frac{1}{\lambda^2}\right)\right) dV_g$$

$$= -8\pi \frac{\ln \lambda_i}{\lambda_i^2} \int_{\Sigma} Ke^u dV_g + \int_{B_i} Ke^u \frac{4}{1 + \lambda_i^2 \psi_i^2} dV_g + O\left(\frac{1}{\lambda^2}\right) \int_{\Sigma} Ke^u dV_g.$$

Now we need to estimate the second integral. Letting $\bar{u} := u - w$ and using Lemma A.4 in [1], we have

$$\int_{B_i} Ke^u \frac{4}{1 + \lambda_i^2 \psi_i^2} dV_g = \int_{B_i} Ke^\varpi \frac{4}{1 + \lambda_i^2 \psi_i^2} + \int_{B_i} Ke^\varpi w \frac{4}{1 + \lambda_i^2 \psi_i^2} + \int_{B_i} Ke^\varpi (e^w - 1 - w) \frac{4}{1 + \lambda_i^2 \psi_i^2}$$

$$= \int_{B_i} Ke^\varpi \frac{4}{1 + \lambda_i^2 \psi_i^2} dV_g + O\left((\sum_{j=1}^m \lambda_{j}^{4\alpha_j - 2})\|w\| \left(\|\lambda_i^2 + |\alpha_i - 1|\right)\right).$$

Concerning the last integral, using Lemma 7.3 we get

$$\int_{B_i} Ke^\varpi \frac{4}{1 + \lambda_i^2 \psi_i^2} dV_g = \left(1 + 4\pi \sum_{j=1}^m \frac{\ln \lambda_j}{\lambda_j^2}\right) \int_{B_i} Ke^\varpi \frac{4\lambda_i^{4\alpha_i} f_i^A g_i^A}{(1 + \lambda_i^2 \psi_i^2)^2 |\alpha_i|^{2\alpha_i + 1}} dV_{g_{\alpha_i}} + o\left(\frac{\ln \lambda}{\lambda^2}\right) \int_{B_i} Ke^\varpi$$

$$= \left(1 + 4\pi \sum_{j=1}^m \frac{\ln \lambda_j}{\lambda_j^2}\right) \frac{2\pi}{\alpha_i} \lambda_i^{4\alpha_i - 2} f_i^A g_i^A(a_i) + o\left(\frac{\ln \lambda}{\lambda^2}\right) \sum_{j=1}^m \lambda_j^{4\alpha_j - 2},$$

by using

$$\int_0^{\lambda_0} \frac{4r}{(1 + r^2)^{2\alpha_i + 1}} dr = \frac{1}{\alpha} + O\left(\frac{1}{\lambda^{4\alpha_i}}\right)$$

and

$$\int_0^{\lambda_0} \frac{r^3}{(1 + r^2)^{2\alpha_i + 1}} dr = O(1).$$
Thus we get
\[
\frac{\theta}{\int_{\Sigma} Ke^u} \int_{\Sigma} Ke^u \lambda_i \frac{\partial^2}{\partial x_i} K e^{-u} dV_g = \left( 1 + 4 \pi \sum_{j=1}^{m} \ln \lambda_j \lambda^2 \right) 16 \pi (1 + \mu) x_{ij} (1 - \tau_i)
+ o \left( \frac{\ln \lambda}{\lambda^2} \right) + O \left( \|w\|^2 + |\alpha_i - 1|^2 \right).
\]

The result follows by summing the previous estimates. \( \square \)

In the next proposition we expand the gradient \( \nabla J_\varphi \) with respect to the gluing parameters \( \alpha_i \)'s.

**Proposition 4.2** Let \( u := \sum_{i=1}^{m} \alpha_i \varphi_i + w \in V(m, \varepsilon) \) and \( g := 8m \pi (1 + \mu) \) with \( \mu \geq 0 \). It holds
\[
\left\langle \nabla J_\varphi (u), \varphi_i \right\rangle = 32 \pi \ln \lambda_i (|\alpha_i - 1| + (\tau_i - \mu + \mu \tau_i)) + O \left( |\alpha_i - 1| + \|w\|^2 \ln \lambda \right)
+ O \left( \|w\||(1 + |\alpha_i - 1| \ln \lambda_i) + \|w\| \ln \lambda_i \right) \sum_j \lambda_j^{4\alpha_j - 2}.
\]

**Proof.** Using Lemma 7.2 we get that
\[
\left\langle u, \varphi_i \right\rangle = \alpha_i (32 \pi \ln \lambda_i + 64 \pi^2 H(a_i, a_i) - 16 \pi) + 64 \pi^2 \sum_{j \neq i} \alpha_j G(a_i, a_j) + O \left( \frac{\ln \lambda}{\lambda^2} \right),
\]
\[
\int_{\Sigma} Ke^u \varphi_i dV_g = \int_{\Sigma} Ke^u \varphi_i dV_g + \int_{\Sigma} Ke^u w \varphi_i dV_g + \int_{\Sigma} Ke^u (e^w - 1 - w) \varphi_i dV_g
= \int_{\Sigma} Ke^u \varphi_i dV_g + O \left( \|w\|(1 + |\alpha_i - 1| \ln \lambda_i + \|w\| \ln \lambda_i) \sum_j \lambda_j^{4\alpha_j - 2} \right).
\]

For the last integral we notice that it follows from Lemma 7.1 that \( \varphi_i \) and \( e^w \) are bounded outside the balls \( B_k \). Moreover inside each ball \( B_k \), for \( k \neq i \), using Lemmas 7.1 and 7.3, we derive
\[
\int_{B_k} Ke^u \varphi_i dV_g = 8 \pi \int_{B_k} \frac{\lambda_k^{4\alpha_i} F_k g_i^A e^{-u} a_{ik}}{1 + \lambda_k^2 |y_{ak}|^2} G(a_i, a_i) dV_{g_{b_k}} + O \left( \frac{\ln \lambda}{\lambda^2} \right) \int K e^w
= 8 \pi \lambda_k^{4\alpha_i - 2} F_k g_i^A G(a_i, a_i) \frac{\pi}{2 \alpha_k - 1} + O \left( \frac{\ln \lambda}{\lambda^2} + \sum_j \alpha_j \lambda_j^{4\alpha_j - 2} \right).
\]

by using the fact that (using (53))
\[
\int_{0}^{\lambda r} \frac{r^3 dr}{(1 + r^2)^2} = \int_{0}^{\lambda r} \frac{r^3 \pi (r) dr}{(1 + r^2)^2} = \int_{0}^{\lambda r} \frac{r^3 dr}{(1 + r^2)^2} + O \left( |\alpha_i - 1| \int_{0}^{\lambda r} \frac{r^3 \sqrt{\tau} dr}{(1 + r^2)^2} \right) = O \left( \ln \lambda + |\alpha_i - 1| \sqrt{\lambda} \right).
\]

In the ball \( B_i \), it holds
\[
\int_{B_i} Ke^u \varphi_i dV_g = \int_{B_i} \frac{\lambda_i^{4\alpha_i} F_i g_i^A e^{-u} a_{ii}}{1 + \lambda_i^2 |y_{ai}|^2} \left( 4 \ln \lambda_i - 2 \ln(1 + \lambda_i^2 |y_{ai}|^2) + 8 \pi H(a_i, a_i) \right) dV_{g_{a_i}} + O \left( \frac{\ln \lambda}{\lambda^2} \right) \int K e^w
= \left( 4 \ln \lambda_i + 8 \pi H(a_i, a_i) \right) \left( \frac{\pi}{2 \alpha - 1} + O \left( \frac{1}{\lambda^{4\alpha_i - 2}} + \frac{\ln \lambda}{\lambda^2} + \frac{|\alpha_i - 1|}{\lambda^{3/2}} \right) \right) \lambda_i^{4\alpha_i - 2} F_i g_i^A (a_i)
- 2 \int_{B_i} \frac{\lambda_i^{4\alpha_i} F_i g_i^A e^{-u} a_{ii}}{1 + \lambda_i^2 |y_{ai}|^2} \ln(1 + \lambda_i^2 |y_{ai}|^2) dV_{g_{a_i}} + O \left( \frac{\ln \lambda}{\lambda^2} \right) \int K e^w.
\]

Observe that
\[
\int_{0}^{\lambda r} \frac{2r}{(1 + r^2)^2} \ln(1 + r^2) dr = \frac{1}{(2 \alpha - 1)^2} + O \left( \frac{\ln \lambda}{\lambda^{4\alpha - 2}} \right),
\]
\[
\int_{0}^{\lambda r} \frac{r^3 \ln(1 + r^2) dr}{(1 + r^2)^2} \leq c \ln \lambda \int_{0}^{\lambda r} \frac{r^3}{(1 + r^2)^2} dr \leq c \ln \lambda \left( \ln \lambda + |\alpha_i - 1| \sqrt{\lambda} \right).
\]
Thus we obtain
\[
\int_{B_i} Ke^\tau \varphi_i dV_g = \left(4 \ln \lambda_i + 8 \pi H(a_i, a_i) \right) \frac{\pi}{2|\alpha_i - 1|} \lambda_i^{4|\alpha_i - 1|} F^A_i g_i^A(a_i) - \frac{2 \pi}{(2\alpha_i - 1)^2} \lambda_i^{4|\alpha_i - 1|} F^A_i g_i^A(a_i)
\]
\[
+ O\left( \ln \lambda + \left( \frac{\ln \lambda}{\lambda^2} + |\alpha_i - 1| \frac{\ln \lambda}{\lambda^{3/2}} \right) \sum \lambda_j^{4|\alpha_j - 1|} \right).
\]
As in the proof of Proposition 4.1, summing the previous estimates, we derive the result. ■

Combining Propositions 4.1 and 4.2, we obtain

**Corollary 4.3** Let \( u := \sum_{i=1}^m \alpha_i \varphi_i + w \in V(m, \varepsilon) \) and \( g := 8m\pi(1 + \mu) \) with \( \mu \geq 0 \). It holds
\[
\left\langle \nabla J_g(u), \frac{\varphi_i}{\ln \lambda_i} - \frac{2 \alpha_i}{\lambda_i} \frac{\partial \varphi_i}{\partial \lambda_i} \right\rangle_g = 32\pi(a_i - 1) + O\left( \|w\|^2 + \|u\| \frac{1}{\ln \lambda} + \ln \lambda \sum |\alpha_i - 1| \right) + \frac{\ln \lambda}{\lambda^2} + \frac{1}{\lambda^{3|\alpha_i - 1|}}.
\]

**Lemma 4.4** Let \( u := \sum_{i=1}^m \alpha_i \varphi_i + w \in V(m, \varepsilon) \) with \( w \in E^m_{A_k} \) and \( g := 8m\pi(1 + \mu) \) with \( \mu \geq 0 \). Let \( \tau_i \) be defined in (26) and assume that \( \sum |\alpha_i - 1| \frac{\ln \lambda_i}{\lambda_i} \) is small. Then, it holds
\[
\sum_{i=1}^m \lambda_i \frac{\ln \lambda_i}{\lambda_i} = 2 \int K e^\tau dV_g(L(A) + 4\pi m \sum_{j=1}^m \ln \lambda_j \lambda_j^2 + o\left( \frac{\ln \lambda}{\lambda^2} \right) + \sum O\left( |\alpha_k - 1|^2 \right).
\]

**Proof.** In this case, since we assumed that \( \sum |\alpha_i - 1| \frac{\ln \lambda_i}{\lambda_i} \) is small, we derive that \( \lambda_i^{4|\alpha_i - 1|} = 1 + O(|\alpha_i - 1| \frac{\ln \lambda_i}{\lambda_i}) \) for each \( i, m \).

Hence the result follows from the previous estimate and (49). ■

Lastly arguing as above, we derive the following asymptotic expansion of the gradient with respect to the concentration points \((a_1, \ldots, a_m)\). Namely we prove that

**Proposition 4.5** Let \( u := \sum_{i=1}^m \alpha_i \varphi_i + w \in V(m, \varepsilon) \) and \( g := 8m\pi(1 + \mu) \) with \( \mu \geq 0 \). It holds
\[
\left\langle \nabla J_g(u), \frac{1}{\lambda_i} \frac{\partial \varphi_i}{\partial \lambda_i} \right\rangle_g = -8\pi(1 + \mu) \frac{\nabla F^A_i(a_i)}{\lambda_i} + O\left( \sum_{i=1}^m \frac{|\alpha_i|}{\lambda_i} + \frac{\ln \lambda_i}{\lambda_i^2} + \sum_{k=1}^m |\alpha_k - 1|^2 + \|w\|^2 \right).
\]

## 5 Critical points at infinity and their indices

Critical points at infinity of the functional \( J_g \) are accumulation points of some orbits of the negative gradient flow \( -\nabla J_g \) which enter some \( V(m, \varepsilon) \) and remain there indefinitely. In this section we recall for the sake of completeness the full characterization of these critical points proved in [1] and restate their contribution to the difference of topology between the level sets of the functional \( J_g \).

We start with an expansion of \( J_g \) in the neighborhood at infinity \( V(m, \varepsilon) \).

**Proposition 5.1** Let \( u := \sum_{i=1}^m \alpha_i \varphi_i + w \in V(m, \varepsilon) \) with \( w \in E^m_{A_k} \). Assume that \( |\alpha_i - 1| \ln \lambda_i \) is small for each \( i \) and (25) holds. Then
\[
J_{sm}(u) = -8\pi m(1 + \ln(m\pi)) - 8\pi F_m^K(a_1, \ldots, a_m) + 4\pi \sum_{i=1}^m |\tau_i|^2 + 16\pi \sum_{i=1}^m (|\alpha_i - 1|^2 + \frac{1}{\lambda_i^2}) + \sum_{k=1}^m \left( \left( |\alpha_k - 1|^2 + |\tilde{\tau}_k|^2 + \frac{1}{\lambda_k^2} \right) \right),
\]
where $F^A_i$ and $g^A_i$ are defined in (47) and

$$\tilde{\tau}_i = 1 - \frac{\alpha_i}{2\alpha_i-1} \frac{4\alpha_i - 2 F_i^A(a_i) g_i^A(a_i)}{\sum_{k=1}^m \alpha_i \lambda_i - 2 F_k^A(a_k) g_k^A(a_k)}.$$

**Proof.** The proof follows from Lemmas 7.2 and 7.4 and the following computations. Let us denote by

$$\Gamma := \sum_{i=1}^m \frac{\alpha_i A_{ij} - 2 F_i^A(a_i) g_i^A(a_i)}{2\alpha_i - 1}.$$

Since $w \in E_{A,A}^m$, then (29) holds and using Lemma 7.4, we have

$$\ln \left( \int_{\Sigma} Ke^u \right) = \ln (\pi \Gamma) + \ln \left( 1 + \frac{1}{2\pi} \sum_{i=1}^m \left( \Delta F_i^A(a_i) - 2 K_g(a_i) F_i^A(a_i) \right) \ln \alpha_i \right) + O\left( \sum_{k=1}^m \left\{ |\alpha_k - 1|^2 + \frac{1}{\lambda_k} \right\} \right)$$

$$= \ln \pi - \frac{1}{m} \sum_{i=1}^m \left( 1 - \tilde{\tau}_i \right) + \frac{1}{2\pi} \sum_{i=1}^m \left( \Delta F_i^A(a_i) - 2 K_g(a_i) F_i^A(a_i) \right) \ln \alpha_i$$

$$+ \frac{1}{m} \sum_{i=1}^m \ln \left( \frac{m \alpha_1 A_{ij} - 2 F_i^A(a_i) g_i^A(a_i)}{2\alpha_i - 1} \right) + \frac{4\pi}{2} \left( \sum_{j=1}^m \ln \lambda_j \right) \sum_{i=1}^m \lambda_i^{4\alpha_i - 2 F_i^A(a_i)}$$

$$+ O\left( \sum_{k=1}^m \left\{ |\alpha_k - 1|^2 + \frac{1}{\lambda_k} \right\} \right)$$

$$= \ln \pi + \frac{1}{m} \sum_{i=1}^m \left( \tilde{\tau}_i + \frac{\tau^2_i}{2} \right) + \ln m - \frac{1}{m} \sum_{i=1}^m \ln (2\alpha_i - 1) + \frac{1}{m} \sum_{i=1}^m (4\alpha_i - 2) \ln \alpha_i$$

$$+ \frac{1}{m} \sum_{i=1}^m \ln \left( F_i^A(a_i) \right) + \frac{1}{m} \sum_{i=1}^m \ln \left( g_i^A(a_i) \right) + \frac{1}{2\pi} \sum_{i=1}^m \left( \Delta F_i^A(a_i) - 2 K_g(a_i) F_i^A(a_i) \right) \ln \alpha_i$$

$$+ \frac{4\pi}{2} \left( \sum_{j=1}^m \ln \lambda_j \right) \sum_{i=1}^m \lambda_i^{4\alpha_i - 2 F_i^A(a_i)} + O\left( \sum_{k=1}^m \left\{ |\alpha_k - 1|^2 + \frac{1}{\lambda_k} \right\} \right).$$

We remark that $\sum_{i=1}^m \tilde{\tau}_i = 0$ and for each $i$, we have $\tilde{\tau}_i = O(\varepsilon)$ which implies that $\lambda_i^2 F_i^A(a_i) = \lambda_j^2 F_j^A(a_j)(1 + O(\sum |\alpha_k - 1| \ln \lambda_k))$ for each $i, j$ and $\ln \lambda_i = \ln \lambda_1 + O(1)$. Furthermore, we have

$$\ln (2\alpha_i - 1) = \ln \left( 1 + 2(\alpha_i - 1) \right) = 2(\alpha_i - 1) + O(|\alpha_i - 1|^2),$$

$$g_i(a_i) = 1 + O\left( \sum_{k=1}^m |\alpha_k - 1| \right) \quad \text{and} \quad \Gamma = \sum_{i=1}^m \lambda_i^2 F_i^A(a_i) + O\left( \sum_{k=1}^m |\alpha_k - 1| \lambda_k^2 \ln \lambda_k \right).$$

Hence, the result follows. \(\blacksquare\)

In [1] we constructed the following decreasing pseudogradient of the Euler Lagrange functional $J_\varphi$ in the neighborhood at Infinity $V(m, \varepsilon)$:

**Proposition 5.2** [1] Let $\varphi = 8\pi m$ with $m \geq 1$ and assume that the function $K$ satisfies the condition (i) of $(N_m)$. Then there exists a pseudo gradient $W$ defined in $V(m, \varepsilon)$ and satisfying the following properties: There exists a constant $C$ independent of $u = \sum_{i=1}^m \alpha_i \varphi_{\alpha_i} \lambda_i + \tilde{w}$ such that

$$\langle - \nabla J_\varphi(u), W \rangle \geq C \sum_{i=1}^m \left( |\alpha_i - 1| + |\tau_i| + \frac{\lambda_i^2}{|\lambda_i|} \right).$$
(2) \[- \nabla J_\theta(u), W + \frac{\partial w(W)}{\partial (\alpha, \lambda, a)} \geq C \sum_{i=1}^{m} \left( |\alpha_i - 1| + |\tau_i| + \frac{|\nabla F_i^A(a_i)|}{\lambda_i} + \frac{\ln \lambda_i}{\lambda_i^2} \right),\]

(3) \[|W| \text{ is bounded and the only region where the variables } \lambda_i \text{'s increase along the flow lines of } W \text{ is the region where } (a_1, \ldots, a_m) \text{ is very close to a critical point } q := (q_1, \ldots, q_m) \text{ of } F^K_m \text{ with } q \in K^-_m.\]

Following the program developed in [1], we deduce from Proposition 5.2 the following characterization of the critical points at infinity:

**Proposition 5.3** [1] Let \( q = 8\pi m, m \geq 1. \) The critical points at Infinity of \( J_\theta \) are in one to one correspondence with critical points \( Q := (q_1, \ldots, q_m) \) of \( F^K_m \) satisfying \( Q \in K^-_m, \) that will be denoted by \( (Q)_{\infty}. \) Furthermore the energy level of such a critical point at Infinity \( (Q)_{\infty} \) denoted \( C_{\infty}(Q)_{\infty} \) is given by:

\[ C_{\infty}(Q)_{\infty} = -8\pi m(1 + \ln(\pi m)) - 8\pi F^K_m(Q). \]

Moreover the Morse index of such a critical point at Infinity \( (Q)_{\infty} \) is given by:

\[ i_{\infty}(Q) := 3m - 1 - \text{Morse}(F^K_m, Q). \]

Actually around a critical point at infinity we have a Morse type reduction. Indeed denoting by

\[ V(m, Q, \varepsilon) := \{ u = \sum_{i=1}^{m} \alpha_i \varphi_i, \lambda_i + w \in V(m, \varepsilon) : \forall i = 1, \ldots, m, |\alpha_i - q_i| < \varepsilon \}, \]

where \( Q := (q_1, \ldots, q_m) \) is a critical point of \( F^K_m \) with \( L(Q) < 0, \) we have

**Lemma 5.4** Let \( u = \sum_{i=1}^{m} \alpha_i \varphi_i, \lambda_i + w \in V(m, Q, \varepsilon), \) then there exists a change of variables

\[
\begin{align*}
\alpha_i &\to \beta_i, i = 1, \ldots, m, \\
\tau_i &\to (\tilde{\tau}_i, \cdots, \tilde{\tau}_m), \\
\lambda_i &\to x_i, \\
w &\to V,
\end{align*}
\]

such that

\[ J_\theta(\sum_{i=1}^{m} \alpha_i \varphi_i, \lambda_i + w) = C_{\infty}(Q)_{\infty} - |A_+|^2 + |A_-|^2 + \sum_{i=1}^{m} \beta_i^2 + x_i^2 - \sum_{i=2}^{m} x_i^2 + ||V||^2, \]

where \( (A_-, A_+) \) are the Morse coordinates of \( F^K_m \) around its critical point \( Q. \)

**Proof.** The proof is inspired from [6] (pages 415-417).

For \( \varepsilon' > 0 \) small (we will take \( \varepsilon/\varepsilon' \) small), the pseudo gradient \( W \) of \( J_\theta \) constructed in Proposition 5.2 satisfies: for \( \varepsilon := \sum \alpha_i \varphi_i + \varepsilon \in V(m, \varepsilon) \) it holds

\[ \langle \nabla J_\theta(\varepsilon), W \rangle \leq -c \sum_{i=1}^{m} \left( |\alpha_i - 1| + |\tau_i| + \frac{|\nabla F_i^A(a_i)|}{\lambda_i} + \frac{\ln \lambda_i}{\lambda_i^2} \right). \]

Now let \( \sigma > 0 \) be a small constant and let

\[ I(\varepsilon) := -8\pi m(1 + \ln(\pi m)) - 8\pi F^K_m(a_1, \ldots, a_m) \\
- (4\pi - \sigma) \sum_{i=1}^{m} \tilde{\tau}_i^2 + (16\pi + \sigma) \sum_{i=1}^{m} (\alpha_i - 1)^2 \ln \lambda_i - (4\pi - \sigma) \frac{\ln \lambda_1}{\lambda_1^2} F^A_1(a_1) \ln L(Q). \]
It is easy to see that

\begin{equation}
0 < I(\bar{\pi}) - J_\epsilon(\bar{\pi}) \leq 2\sigma \sum \left( (\alpha_i - 1)^2 \ln \lambda_i + \frac{\ln \lambda_i}{\lambda_i^2} + \hat{r}_i^2 \right) \quad \text{for each } \bar{\pi}.
\end{equation}

Furthermore it follows from Proposition 5.2 we have that

\begin{equation}
\langle \nabla I(\bar{\pi}), W \rangle \leq -\epsilon \sum \left( |\alpha_i - 1| + |\tau_i| + \frac{|\nabla J_i^A(a_i)|}{\lambda_i} + \ln \lambda_i \right).
\end{equation}

Now, let $\bar{\pi}_0 \in V(m, \epsilon)$, we define the differential equation

\begin{equation}
\frac{\partial u}{\partial s} = W(u) ; \quad u(0) = \bar{\pi}_0,
\end{equation}

whose solution is $h_s(\bar{\pi}_0)$ where $h_s$ is the 1-parameter group generated by $W$. We claim:

**CLAIM:** There exists $\mathfrak{s} > 0$ such that $I(h_\mathfrak{s}(\bar{\pi}_0)) = J_\epsilon(\bar{\pi}_0)$.

Observe that $I(h_s(\bar{\pi}_0))$ is a decreasing function with respect to $s$. Hence there exists at most a unique solution to the equation $I(h_s(\bar{\pi}_0)) = J_\epsilon(\bar{\pi}_0)$. The only cases where there could be no solution are

• either $h_s(\bar{\pi}_0)$ exists $V(m, \epsilon')$ (outside this set, we loose (32) since $W$ is defined only in $V(m, \epsilon')$) before reaching this level.

• or $h_s(\bar{\pi}_0)$ will create an asymptote above the level $J_\epsilon(\bar{\pi}_0)$.

We will prove that these two cases cannot occur. In fact, for the first one, since $\bar{\pi}_0 \in V(m, \epsilon)$ then, to exit $V(m, \epsilon')$, the flow line has to travel from $V(m, \epsilon'/2)$ to the boundary of $V(m, \epsilon')$. Note that, using (34), the derivative of $I(h_s(\bar{\pi}_0))$ along this path is lower-bounded by a constant $c(\epsilon')$, independent of $\epsilon$, but depending on $\epsilon'$. Also, by (34), the time to travel from $V(m, \epsilon'/2)$ to the boundary of $V(m, \epsilon')$ is lower-bounded by a constant $c'(\epsilon')$, because $|W|$ is bounded and the distance to travel is lower-bounded by a constant $c > 0$. Therefore, $I(h_s(\bar{\pi}_0))$ decreases at least by $c(\epsilon')c'(\epsilon')$ during this trip.

However, using (33), since $\bar{\pi}_0 \in V(m, \epsilon)$, it follows that $J_\epsilon(h_s(\bar{\pi}_0)) < I(h_s(\bar{\pi}_0)) \leq J_\epsilon(\bar{\pi}_0) + c(\epsilon)$. Hence we have to choose $\epsilon$ small with respect to $\epsilon'$ so that $I(h_s(\bar{\pi}_0))$ reaches the level $J_\epsilon(\bar{\pi}_0)$ before leaving the set $V(m, \epsilon')$. Concerning the second case, the flow line $h_s(\bar{\pi}_0)$ will enter $V(m, \epsilon_1)$ for each $\epsilon_1 > 0$. Observe that

\[ J_\epsilon(h_s(\bar{\pi}_0)) < J_\epsilon(\bar{\pi}_0) ; \quad 0 < I(h_s(\bar{\pi}_0)) - J_\epsilon(h_s(\bar{\pi}_0)) \to 0 \quad \text{as } s \to \infty \quad (\text{for } \epsilon_1 \to 0). \]

Hence $I(h_s(\bar{\pi}_0))$ has to reach the level $J_\epsilon(\bar{\pi}_0)$ and therefore this case cannot occur. Our claim is thereby proved.

Conversely, taking $\epsilon'' > 0$ small with respect to $\epsilon$ and given $\bar{\pi}_0 \in V(m, \epsilon'')$, arguing by the same way (and using $-W$ as an increasing pseudo gradient) : there exists $\mathfrak{s} > 0$ such that $J_\epsilon(h_{-\mathfrak{s}}(\bar{\pi}_0)) = I(\bar{\pi}_0)$. Hence, $h_s$ is the required isomorphism.

To complete the proof, observe that, for $u := \sum \alpha_i \varphi_i + w \in V(m, \epsilon)$, using Proposition 3.2, there exists a change of variable $w - \bar{w} \to V$ so that

\[ J_\epsilon(u) = J_\epsilon(\bar{\pi}) + \|V\|^2 \quad \text{where } \bar{\pi} := \sum \alpha_i \varphi_i + \bar{w} \]

\[ = I(h_{-\mathfrak{s}}(\bar{\pi})) + \|V\|^2 \quad \text{where } \bar{\pi} \text{ is defined by the above claim} \]

\[ = -8\pi m (1 + \ln(m\pi)) - 8\pi F^K_m(a_1(\bar{\pi}), ..., a_m(\bar{\pi})) \]

\[ - (4\pi - \sigma) \sum_{i=1}^m \hat{r}_i^2(\bar{\pi}) + (16\pi + \sigma) \sum_{i=1}^m (\alpha_i(\bar{\pi}) - 1)^2 \ln \lambda_i(\bar{\pi}) - (4\pi - \sigma) \frac{\ln \lambda_i(\bar{\pi})\mathcal{L}(Q)}{\lambda_i^2(\bar{\pi}) F_1^A(\bar{\pi})(a_1(\bar{\pi}))}. \]

Thus our change of variables follows from the standard Morse Lemma for the function $F^K_m$ at its critical point $Q$ and using the above formula. In fact, we define

\[ \beta_i := \sqrt{16\pi + \sigma(\alpha_i(\bar{\pi}) - 1)} \sqrt{\ln \lambda_i(\bar{\pi})} ; \quad x_1 := \sqrt{4\pi - \sigma} \frac{\sqrt{\ln \lambda_i(\bar{\pi}) \sqrt{-\mathcal{L}(Q)}}}{\lambda_i(\bar{\pi}) \sqrt{F_1^A(\bar{\pi})(a_1(\bar{\pi}))}} \]

(we recall that $\mathcal{L}(Q) < 0$). Finally we notice that $\sum_{i=1}^m \hat{r}_i = 0$ and therefore we will loose one index. ■
6 Proof of the main results

Proof of Theorem 1.1. We start by proving the statement 1). To this aim we consider the following sup-approximation of \((MF)\)

\[
(MF)_\mu - \Delta y u = 8m\pi(1 + \mu)(\frac{K e^u}{\int_{\Sigma} Ke^u dV_y} - 1) \quad \text{in } \Sigma,
\]

where \(\mu > 0\) is a small positive real number.

Regarding Problem \((MF)_\mu\) we prove the following:
Claim 1: For a sequence of \(\mu_k \to 0\), Problem \((MF)_{\mu_k}\) admits a solution \(u_{\mu_k}\) whose generalized Morse index \(\text{Morse}(u_{\mu_k})\) is \(3m\).

Indeed let \(J_{8\pi^m(1+\mu)}\) be the Euler Lagrange functional associated to \((MF)_\mu\) then it follows from [15], [25], [16] that, for large \(L\) the sublevel set \(J_{8\pi^m(1+\mu)}^{-L}\) has the same homotopy type as the set of formal barycenters \(B_m(\Sigma)\) of order \(m\). Note that (see [18, 25]), we have

\[
H_{3m-1}(J_{8\pi^m(1+\mu)}, \mathbb{Z}_2) = H_{3m-1}(B_m(\Sigma), \mathbb{Z}_2) \neq 0.
\]

Let \(\varrho_\mu := 8\pi^m(1+\mu)\), using the fact that \(J_{\varrho_\mu}^L\) is a contractible set and the exact sequence of the pair \((J_{\varrho_\mu}^L, J_{\varrho_\mu}^{-L})\) we derive that

\[
0 = H_{3m}(J_{\varrho_\mu}^L) \to H_{3m}(J_{\varrho_\mu}^L, J_{\varrho_\mu}^{-L}) \to H_{3m-1}(J_{\varrho_\mu}^{-L}) \to H_{3m-1}(J_{\varrho_\mu}^{L}) = 0.
\]

Hence it follows that

\[
H_{3m}(J_{8\pi^m(1+\mu)}^L, J_{8\pi^m(1+\mu)}^{-L}) \neq 0.
\]

Therefore \(J_{8\pi^m(1+\mu)}\) has a critical point whose Morse index is \(3m\).

To conclude the proof of the theorem we prove the following claim:
Claim 2: \(u_{\mu_k} \to u_\infty\) in \(C^{2,\alpha}(\Sigma)\), where \(u_\infty\) is a solution of Equation \((MF)\).

To prove the claim it is enough to rule out the blow up of \(u_{\mu_k}\). Arguing by contradiction it follows from Proposition 7.6 that \(u_{\mu_k} \in V(m, \varepsilon)\) for \(k\) large. Hence this function has to be written as

\[
u_{\mu_k} = \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + w,
\]

where the function \(w \in E^m_{A, \Lambda}\) and satisfies

\[
\|w\| \leq c\left( \sum |\alpha_k - 1| + \frac{1}{\lambda} \right).
\]

Now, using the fact that \(\nabla J_{8\pi^m(1+\mu)}(u_{\mu_k}) = 0\) and Corollary 4.3, we get

\[
\sum |\alpha_i - 1| \leq c\left( \frac{1}{\lambda \ln \lambda} + \frac{1}{\ln \lambda} \sum |\tau_j - \mu + \mu \tau_j| \right).
\]

Now, using Proposition 4.1 we derive that

\[
16\pi \alpha_i (\tau_i - \mu + \mu \tau_i) - 64\pi^2 \sum \frac{\ln \lambda_j}{\lambda_j^2} = O\left( \sum |\alpha_j - 1|^2 \right) + o\left( \frac{\ln \lambda}{\lambda^2} \right).
\]

Using (38) and (39), we get that

\[
\sum |\tau_i - \mu + \mu \tau_i| = O\left( \frac{\ln \lambda}{\lambda^2} \right).
\]

Hence, using (38), (40) and summing (39) for \(i = 1, \cdots, m\), we derive that

\[
\sum_{i=1}^{m} \tau_i - m\mu + \mu \sum_{i=1}^{m} \tau_i = 4\pi m \sum \frac{\ln \lambda_j}{\lambda_j^2} + o\left( \frac{\ln \lambda}{\lambda^2} \right).
\]
Now using Lemma 4.4 we obtain

\[ (41) \quad \mu = \frac{\pi}{2} \mathcal{L}(A) \frac{\ln \lambda_1}{\int_{\Sigma} K e^u} (1 + o(1)), \]

which implies that \( \mathcal{L}(A) \) has to be positive. Furthermore it follows from Proposition 4.5, (38), (40) and (37) that \( A := (a_1, \ldots, a_m) \) converges to a critical point of \( \mathcal{F}_m^K \).

Next expanding the functional \( J_{8m\pi(1+\mu_k)} \) in \( V(m, \varepsilon) \) (following the proof of Proposition 5.1), we obtain

\[ J_{8m\pi(1+\mu_k)}(u_{\mu_k}) = C(m, \mu_k) - 8\pi(1 + \mu_k)\mathcal{F}_m^K(A) + 16\pi \sum_{i=1}^m (\alpha_i - 1)^2 \ln \lambda_i \]

\[ - 4\pi \sum_{i=1}^m \frac{\mathcal{L}(A) \ln \lambda_1}{\int_{\Sigma} K e^u} + o\left(\frac{\ln \lambda_1}{\lambda^2}\right). \]

Hence we derive from this expansion, arguing as in Corollary 5.3 that

\[ \text{Morse}(u_{\mu_k}) = 3m - \text{Morse}(\mathcal{F}_m^K, A). \]

Since \( \text{Morse}(u_{\varepsilon}) = 3m \), we have that \( A \) is a local minimum of \( \mathcal{F}_m^K \) satisfying that \( \mathcal{L}(A) > 0 \). Hence we reach a contradiction to the assumption of Theorem 1.1.

The proof of statement 2) follows the same argument as above. The only difference is that we use a sub-approximation:

\[ (MF)_\mu - \Delta g u = 8m\pi(1 - \mu) \left( \frac{K e^u}{\int_{\Sigma} K e^u dV_g} - 1 \right) \text{ in } \Sigma, \]

where \( \mu > 0 \) is a small positive real number. Using the fact that \( H_{3m-4}(B_{m-1}(\Sigma), \mathbb{Z}_2) \neq 0 \) (see [18], Lemma 8.7) we have that

\[ H_{3m-4}(J_{8m\pi(1-\mu)}^L, \mathbb{Z}_2) = H_{3m-4}(B_{m-1}(\Sigma), \mathbb{Z}_2) \neq 0. \]

Hence we have that

\[ H_{3m-3}(J_{8m\pi(1-\mu)}^L, J_{8m\pi(1-\mu)}^L) \neq 0. \]

Therefore \( J_{8m\pi(1-\mu)} \) has a critical point whose Morse index is \( 3m - 3 \). Next we claim

**Claim 3:** \( u_{\mu_k} \to u_\infty \text{ in } C^{2,\alpha}(\Sigma) \), where \( u_\infty \) is a solution of Equation \((MF)\).

By elliptic regularity, it is enough to rule out the blow up of \( u_{\mu_k} \). Arguing by contradiction we have by Proposition 7.6 that for \( k \) large \( u_{\mu_k} \in V(m, \varepsilon) \). It follows then from Proposition 5.3 that \( u_{\mu_k} \in V(m, Q, \varepsilon) \), where \( Q \) is a critical point of \( \mathcal{F}_m^K \) with \( \mathcal{L}(Q) < 0 \). Moreover we have that

\[ \text{Morse}(u_{\mu_k}) = 3m - 1 - \text{Morse}(\mathcal{F}_m^K, Q). \]

That is \( Q \) is a critical point of \( \mathcal{F}_m^K \) whose Morse index is 2 and \( \mathcal{L}(Q) < 0 \). A contradiction to the assumption 2) of the theorem. Hence the proof of statement 2) is complete. ■

**Proof of Theorem 1.2** We first observe that since the function \( K \) satisfies the non degeneracy condition \((N_m)\), \( J_q \) is a Morse function. Moreover the Morse indices of its critical points are uniformly bounded, say by \( \bar{m} \) and it follows from Corollary 5.3 that the Morse indices of the critical points at Infinity of \( J_{8m\pi m} \) are bounded by \( 3m - 1 \).

Without loss of generality, we may assume that \( J_q \) separates its critical as well as its critical points at Infinity. That is at any critical value there is only one critical point or one critical point at infinity. This can be arranged by perturbing \( J_q \) slightly in disjoint neighborhoods of its critical points (resp. its critical points at Infinity). Next we choose \( L \) such that all critical values, respectively, critical values at infinity are contained in the open interval \((-L, L)\) and we order these critical values as

\[-L < C_1 < \cdots < C_p < L.\]
Now choose regular values $a_0 < \cdots < a_p$ such that
\[ a_0 = -L, a_p = L \] and $a_{i-1} < C_i < a_i, \forall i = 1, \cdots, p$. Moreover we denote by $N_i$ the Morse index of the critical point $u_i$ such that $J^i_0(u_i) = C_i$, resp. by $N^\infty_i$ the index $\iota^\infty$ of the critical point at infinity $u^\infty_i$ such that $C^\infty_i(u^\infty_i) = C_i$ and set $M_i := J^i_0 := \{ u : J^i_0(u) < a_i \}$. We notice that it follows from the standard Morse Lemma in the case of usual critical points and from Lemma 5.4 in the case of critical points at infinity, that $M_i$ is obtained from $M_{i-1}$ by attaching a $N(i)$—cell (resp. $N^\infty(i)$—cell). We set
\[ \theta(i, j) := \dim H_1(M_j, M_{j-1}) ; \quad \mu(i, j) := \dim H_1(M_j, J^i_0-L) \]
and observe that $\theta(i, j) = \delta_{i,N_i}$ (resp. $\theta(i, j) = \delta_{i,N^\infty_i}$) and $\mu(i, j) = 0$ if $i > N := \max(m, 3m - 1)$. Next, considering the triple $(M_j, M_{j-1}, J^i_0-L)$ we have the exact sequence
\[
\begin{array}{cccc}
0 & \longrightarrow & H_*(M_{j-1}, J^i_0-L) & \longrightarrow H_*(M_j, J^i_0-L) & \longrightarrow H_*(M_j, M_{j-1}) \\
& \longrightarrow & H_{*+1}(M_{j-1}, J^i_0-L) & \longrightarrow & \cdots \\
& \longrightarrow & H_0(M_j, M_{j-1}) & \end{array}
\]
We recall that exactness implies the vanishing of corresponding alternating sum of dimensions of the vector spaces in the sequence.

Denoting by $\mathcal{N}_{q,j}$ the kernel of $H_q(M_{j-1}, J^i_0-L) \rightarrow H_q(M_i, J^i_0-L)$ and $\nu_{q,j} := \dim \mathcal{N}_{q,j}$ and using the exactness of the triple $(M_j, M_{j-1}, J^i_0-L)$ we derive that (by grouping 3 terms at a time):
\[
\nu_{q,j} = \sum_{i=0}^{q} (-1)^{i+q} \left( \mu(i, j) - \mu(i, j) + \theta(i, j) \right).
\]
Summing over $j = 1, \cdots, p$ yields
\[
\sum_{j=1}^{p} \nu_{q,j} = \sum_{i=0}^{q} (-1)^{i+q} \left( \mu(i, 0) - \mu(i, p) + \sum_{j=1}^{p} \theta(i, j) \right).
\]
Note that $\mu(i, 0) = 0$ and $\mu(i, p) = \dim(H_i(J^i_0-L, J^i_0-L))$. Using the exact sequence of the pair $(J^i_0-L, J^i_0-L)$ we derive that
\[
\begin{cases}
H_0(J^i_0-L, J^i_0-L) \simeq H_1(J^i_0-L, J^i_0-L) \simeq 0 & \text{and} \\
H_1(J^i_0-L, J^i_0-L) \simeq H_{i-1}(J^i_0-L) \simeq H_{i-1}(B_{m-1}(\Sigma)) := B_{i-1}^{m-1} \forall i \geq 2.
\end{cases}
\]
Moreover, denoting by $\nu_i$ the number of critical points of Morse index $i$ resp. by $\nu^\infty_i$ the number of critical points at Infinity of index $i$, it follows that
\[
\sum_{j=1}^{p} \theta(i, j) = \nu_i + \nu^\infty_i.
\]
Hence we get
\[
\sum_{j=1}^{p} \nu_{q,j} + \sum_{i=2}^{q} (-1)^{q+i} B_{i-1}^{m-1} = \sum_{(A \in \mathcal{K}_m, \iota^\infty(A) \leq q)} (-1)^{\iota^\infty(A)+q} + \sum_{i=0}^{q} (-1)^{i+q} \nu_i.
\]
Now, for $2 \leq k \leq 3m - 1$, summing (44) for $q = k$ and $q = k - 1$, we obtain
\[
\nu_k + \nu^\infty_k - \nu^m_{k-1} = \sum_{j=1}^{N} \nu_{k,j} + \sum_{j=1}^{N} \nu_{k-1,j} \geq 0,
\]
and the statement (a) is proved.

The second claim follows by taking $q = N$ in (44) and using $\nu_{q,j} = 0$ for each $j$ and

$$\sum (-1)^j \beta_q^{m-1} = \chi(B_{m-1}(\Sigma)) = 1 - \left( \frac{m - 1 - \chi(\Sigma)}{m - 1} \right).$$

The proof of Theorem 1.2 is complete. ■

**Proof of Theorem 1.3** Since there are no critical points at infinity of index $q_0$, it follows from (10) that:

$$\nu_{q_0} \geq \beta_{q_0-1}^{m-1}.$$ 

Since by assumption $\beta_{q_0-1}^{m-1} \neq 0$, we deduce that the equation $(MF)$ has at least $\beta_{q_0-1}^{m-1}$ solutions. In particular since $\beta_{3m-4}^{m-1} = H_{3m-4}(B_{m-1}(\Sigma), \mathbb{Z}_2) \neq 0$, if there are no critical points at infinity of index $3m - 3$, we obtain a solution of $(MF)$ whose Morse index is $3m - 3$. ■

**Proof of Theorem 1.4** By contradiction, assume that $(MF)$ does not have solutions and observe that, as in (45) it follows from assumption 2) that

$$\forall j, \quad \nu_{q_0+1,j} = \nu_{q_0,j} = \nu_{q_0-1,j} = \nu_{q_0-2,j} = 0.$$ 

Hence summing (44) for $k = q_0$ and $k = q_0 - 1$ we obtain $\nu_{q_0}^\infty = \beta_{q_0-1}^{m-1}$. A contradiction to the assumptions 1) and 3). ■

**Proof of Theorem 1.5** First observe that if there is no solution and there is no critical point at infinity of index $q_0$, then there holds:

$$\nu_{q_0,j} = 0 \quad \text{and} \quad \nu_{q_0-1,j} = 0 \quad \forall j.$$ 

Indeed, recall that $\nu_{q,j} := dim Ker(i_{q,j})$ where $i_{q,j}$ designs the map

$$i_{q,j} : H_q(M_{j-1}, J_q^{-L}) \rightarrow H_q(M_j, J_q^{-L}).$$

The claim is immediate for $q_0$ since there is no critical point or critical point at infinity of index $q_0$ which implies that $H_{q_0}(M_{j-1}, J_q^{-L}) = H_{q_0}(M_j, J_q^{-L}) = 0$. For $q_0 - 1$, using the exact sequence of the triple $(M_j, M_{j-1}, J_q^{-L})$, we get (since we assumed that there is no critical point/critical point at infinity of index $q_0$)

$$0 = H_{q_0}(M_j, M_{j-1}) \rightarrow H_{q_0-1}(M_{j-1}, J_q^{-L}) \rightarrow H_{q_0-1}(M_j, J_q^{-L}), \quad \text{for each} \ j,$$

which gives the claim for $q_0 - 1$.

Next to prove Theorem 1.5, we assume that there is no solution and observe that $\beta_i^{m-1} = 0$ for each $i > 3m - 4$. Furthermore, since there is no critical point at infinity of index $3m$, we get from the above claim (45) that $\nu_3^{m-1} = 0$ for each $j$. Hence, summing (44) for $q = 3m - 1$ and $q = 3m - 3$, we get

$$0 \leq \sum_{j=1}^{p} \nu_{3m-3,j} = -(-1)^{3m-3} \sum_{A \in \mathcal{K}_m; i_\infty(A) = 3m-1} (-1)^{i_\infty(A)} = \nu_{3m-2} - \nu_{3m-1}.$$ 

In the same way, summing (44) for $q = 3m - 1$ and $q = 3m - 4$, we get

$$\sum_{j=1}^{p} \nu_{3m-4,j} + \beta_{3m-4}^{m-1} = -(-1)^{3m-4} \sum_{A \in \mathcal{K}_m; 3m-3 \leq i_\infty(A) \leq 3m-1} (-1)^{i_\infty(A)} = \nu_{3m-3} - \nu_{3m-2} + \nu_{3m-1}.$$ 

Hence the result follows. ■
7 Appendix

In this section we collect some technical Lemmas used in this paper.

**Lemma 7.1** Let \( \varphi_{a, \lambda} \) and \( \delta_{a, \lambda} \) be defined in (17) and (16). The following expansions hold pointwise

\[
\varphi_{a, \lambda} = \delta_{a, \lambda} + \ln \left( \frac{\lambda^2}{8} \right) + 8\pi H(a, \cdot) + 4\pi \frac{\ln \lambda}{\lambda^2} + O\left( \frac{1}{\lambda^2} \right) \quad \text{in } \Sigma,
\]

\[
\lambda \frac{\partial \varphi_{a, \lambda}}{\partial a}(x) = \frac{4}{1 + \lambda^2 \psi_a^2(x)} - 8\pi \frac{\ln \lambda}{\lambda^2} + O\left( \frac{1}{\lambda^2} \right) \quad \text{for each } x \in \Sigma,
\]

\[
\frac{1}{\lambda} \frac{\partial \varphi_{a, \lambda}}{\partial a}(x) = \frac{1}{\lambda} \frac{\partial \delta_{a, \lambda}}{\partial a}(x) + 8\pi \frac{1}{\lambda} \frac{\partial H(a, x)}{\partial a}(x) + O\left( \frac{\ln \lambda}{\lambda^2} \right) \quad \text{for each } x \in \Sigma.
\]

In particular, we have

\[
\varphi_{a, \lambda} = 8\pi G(a, \cdot) + 4\pi \frac{\ln \lambda}{\lambda^2} + O\left( \frac{1}{\lambda^2} \right) \quad \text{in } \Sigma \setminus B_\lambda(\eta),
\]

\[
\lambda \frac{\partial \varphi_{a, \lambda}}{\partial a} = -8\pi \frac{\ln \lambda}{\lambda^2} + O\left( \frac{1}{\lambda^2} \right) \quad \text{in } \Sigma \setminus B_\lambda(\eta),
\]

\[
\frac{1}{\lambda} \frac{\partial \varphi_{a, \lambda}}{\partial a} = 8\pi \frac{1}{\lambda} \frac{\partial G(a, \cdot)}{\partial a} + O\left( \frac{\ln \lambda}{\lambda^2} \right) \quad \text{in } \Sigma \setminus B_\lambda(\eta).
\]

**Proof.** We remark that the second part of this lemma follows immediately from the first one. Now, we start by proving the first claim. First, observe that

\[
(46) \quad \int_\Sigma e^{\delta_{a, \lambda} + u_\lambda} dV_g = \int_{B_\lambda(\eta)} e^{\delta_{a, \lambda} + u_\lambda} dV_g + \int_{\Sigma \setminus B_\lambda(\eta)} O\left( \frac{1}{\lambda^2} \right) = 8\pi + O\left( \frac{1}{\lambda^2} \right).
\]

Now, let us define the following function \( k_{a, \lambda} := \varphi_{a, \lambda} - \delta_{a, \lambda} - \ln \left( \frac{\lambda^2}{8} \right) - 8\pi H(a, \cdot) \). We recall that \( \delta_{a, \lambda} \) is a constant function in \( \Sigma \setminus B_{a}(2\eta) \). Thus, using (15), we obtain

\[
-\Delta_g k_{a, \lambda} = \begin{cases} 
\int_{\Sigma} e^{\delta_{a, \lambda} + u_\lambda} dV_g + 8\pi = O\left( \frac{1}{\lambda^2} \right) \quad \text{in } \Sigma \setminus B_{a}(2\eta). \\
e^{\delta_{a, \lambda} + u_\lambda} - \int_{\Sigma} e^{\delta_{a, \lambda} + u_\lambda} dV_g = e^{u_\lambda} \Delta_g \delta_{a, \lambda} + 8\pi = O\left( \frac{1}{\lambda^2} \right) \quad \text{in } B_{a}(\eta). 
\end{cases}
\]

It remains the case of \( x \in B_{a}(2\eta) \setminus B_{a}(\eta) \). Using again (15) we get

\[
-\Delta_g k_{a, \lambda} = O\left( \frac{1}{\lambda^2} \right) - 8\pi - 2e^{u_\lambda} \Delta_g \ln(1 + \lambda^2 \psi_a^2) + (8\pi + 2e^{u_\lambda} \Delta_g \ln(\psi_a^2))
\]

\[
= -2e^{u_\lambda} \Delta_g \ln \left( \lambda^2 + \frac{1}{\psi_a} \right) + O\left( \frac{1}{\lambda^2} \right) = O\left( \frac{1}{\lambda^2} \right) \quad \text{in } B_{a}(2\eta) \setminus B_{a}(\eta).
\]

Hence we obtain that \( \Delta_g k_{a, \lambda} = O(1/\lambda^2) \) in \( \Sigma \) and therefore we get that

\[
k_{a, \lambda}(x) - \int_{\Sigma} k_{a, \lambda}(y) dV_g = O\left( \frac{1}{\lambda^2} \right) \quad \text{in } \Sigma.
\]

It remains to estimate the previous integral. Using (12), (14) and (17), we get

\[
\int_{\Sigma} k_{a, \lambda} dV_g = -\int_{\Sigma} \delta_{a, \lambda} + \ln \left( \frac{\lambda^2}{8} \right) + 8\pi H(a, \cdot) dV_g = 2 \int_{B_{a}(\eta)} \ln \left( 1 + \frac{1}{\lambda^2 \psi_a^2} \right) dV_g
\]

\[
= 2 \int_{B_{a}(\eta)} \ln \left( 1 + \frac{1}{\lambda^2 \psi_a^2} \right) dV_g + O\left( \left( \int_{B_{a}(\eta)} \ln \left( 1 + \frac{1}{\lambda^2 \psi_a^2} \right) e^{-u_\lambda} - 1 \right) dV_g + \frac{1}{\lambda^2} \right)
\]

\[
= \frac{4\pi}{\lambda^2} \int_0^{\lambda \eta} \ln \left( 1 + \frac{1}{r^2} \right) r dr + O\left( \frac{1}{\lambda^2} \right) = 4\pi \frac{\ln \lambda}{\lambda^2} + O\left( \frac{1}{\lambda^2} \right).
\]

Hence the proof of the first claim follows.

The other claims can be proved in the same way. ■

18
Lemma 7.2 Let \( \varphi_{a,\lambda} \) be defined in (17). The following expansions hold:

\[
\|\varphi_{a,\lambda}\|^2 = 32\pi \ln \lambda + 64\pi^2 H(a, a) - 16\pi + 64\pi^2 \frac{\ln \lambda}{\lambda^2} + O\left(\frac{1}{\lambda^2}\right),
\]

\[
\langle \varphi_{a,\lambda}, \lambda \frac{\partial \varphi_{a,\lambda}}{\partial \lambda} \rangle_g = 16\pi - 64\pi^2 \frac{\ln \lambda}{\lambda^2} + O\left(\frac{1}{\lambda^2}\right),
\]

\[
\langle \varphi_{a_j,\lambda_j}, \varphi_{a_i,\lambda_i} \rangle_g = 64\pi^2 G(a_j, a_i) + 32\pi^2 \frac{\ln \lambda_j}{\lambda_j^2} + 32\pi^2 \frac{\ln \lambda_i}{\lambda_i^2} + O\left(\frac{1}{\lambda_j^2} + \frac{1}{\lambda_i^2}\right),
\]

\[
\langle \varphi_{a_i,\lambda_i}, \lambda_i \frac{\partial \varphi_{a_i,\lambda_i}}{\partial \lambda_i} \rangle_g = -64\pi^2 \frac{\ln \lambda_i}{\lambda_i^2} + O\left(\frac{1}{\lambda_j^2} + \frac{1}{\lambda_i^2}\right).
\]

PROOF. Since \( \int_{\Sigma} \varphi_{a,\lambda} dV_g = 0 \), using Lemma 7.1 and (46), we get

\[
\|\varphi_{a,\lambda}\|^2 = \int_{\Sigma} e^{u_a} e^{\delta_{a,\lambda}} \left( \delta_{a,\lambda} + \ln \left(\frac{\lambda^2}{8}\right) + 8\pi H(a,.) + 4\pi \ln \lambda + O\left(\frac{1}{\lambda^2}\right) \right) dV_g
\]

\[
= 32\pi^2 \frac{\ln \lambda}{\lambda^2} + O\left(\frac{1}{\lambda^2}\right) + 2 \int_{\Sigma} e^{u_a} e^{\delta_{a,\lambda}} \ln \left(\frac{\lambda^2 \psi^2_a}{1 + \lambda^2 \psi^2_a}\right) dV_g + 8\pi \int_{\Sigma} e^{u_a} e^{\delta_{a,\lambda}} G(a,.) dV_g
\]

Note that (using the fact that \( \int_{\Sigma} G(a,.) dV_g = 0 \))

\[
8\pi \int_{\Sigma} e^{u_a} e^{\delta_{a,\lambda}} G(a,.) dV_g = 8\pi \varphi_{a,\lambda}(a) = 32\pi \ln \lambda + 64\pi^2 H(a, a) + 32\pi^2 \frac{\ln \lambda}{\lambda^2} + O\left(\frac{1}{\lambda^2}\right).
\]

Furthermore, we have

\[
\int_{\Sigma} e^{u_a} e^{\delta_{a,\lambda}} \ln \left(\frac{\lambda^2 \psi^2_a}{1 + \lambda^2 \psi^2_a}\right) dV_g = \int_{B(a)(\eta)} e^{\delta_{a,\lambda}} \ln \left(\frac{\lambda^2 |x - a|^2}{1 + \lambda^2 |x - a|^2}\right) dV_{g_a} + O\left(\frac{1}{\lambda^2}\right)
\]

\[
= \int_{B(0,\eta)} \frac{8\lambda^2}{(1 + \lambda^2 |x|^2)^2} \ln \left(\frac{\lambda^2 |x|^2}{1 + \lambda^2 |x|^2}\right) dx + O\left(\frac{1}{\lambda^2}\right)
\]

\[
= -8\pi + O\left(\frac{1}{\lambda^2}\right).
\]

Hence the proof of the first claim follows.

Concerning the second one, using the fact that \( \int_{\Sigma} \lambda \frac{\partial \varphi_{a,\lambda}}{\partial \lambda} dV_g = 0 \), we have

\[
\langle \varphi_{a,\lambda}, \lambda \frac{\partial \varphi_{a,\lambda}}{\partial \lambda} \rangle_g = \int_{\Sigma} e^{u_a} e^{\delta_{a,\lambda}} \left( \frac{4}{1 + \lambda^2 \psi^2_a(x)} - 8\pi \frac{\ln \lambda}{\lambda^2} + O\left(\frac{1}{\lambda^2}\right) \right) dV_g
\]

\[
= -64\pi^2 \frac{\ln \lambda}{\lambda^2} + O\left(\frac{1}{\lambda^2}\right) + \int_{\Sigma} e^{u_a} e^{\delta_{a,\lambda}} \frac{4}{1 + \lambda^2 \psi^2_a(x)} dV_g.
\]

Observe that, we have

\[
\int_{\Sigma} e^{u_a} e^{\delta_{a,\lambda}} \frac{4}{1 + \lambda^2 \psi^2_a(x)} dV_g = \int_{B(0,\eta)} \frac{32\lambda^2}{(1 + \lambda^2 |x|^2)^2} dx + O\left(\frac{1}{\lambda^4}\right) = 16\pi + O\left(\frac{1}{\lambda^2}\right).
\]

Thus the proof of the second claim follows. Now, we will focus on the third claim. Using \( \int_{\Sigma} |\varphi_{a,\lambda}| = O(1) \) and Lemma 7.1, it holds

\[
\langle \varphi_{a_j,\lambda_j}, \varphi_{a_i,\lambda_i} \rangle_g = \int_{B(a_j)(\eta)} e^{u_a} e^{\delta_{a_j,\lambda_j}} \left( 8\pi G(a_j,.) + 4\pi \frac{\ln \lambda_j}{\lambda_j^2} + O\left(\frac{1}{\lambda_j^2}\right) \right) dV_g + O\left(\frac{1}{\lambda_j^2}\right)
\]

\[
= 32\pi^2 \frac{\ln \lambda_j}{\lambda_j^2} + O\left(\frac{1}{\lambda_j^2}\right) + 8\pi \int_{B(a_j)(\eta)} e^{u_a} e^{\delta_{a_j,\lambda_j}} G(a_j,.) dV_g.
\]
Observe that (using $\int_\Sigma G(a_i, \cdot) dV_g = 0$)

$$\int_{B_{a_i}(\eta)} e^{u_{a_i}} e^{\gamma_{a_i} G(a_i, \cdot)} dV_g = \int_\Sigma e^{u_{a_i}} e^{\gamma_{a_i} G(a_i, \cdot)} dV_g + O\left(\frac{1}{\lambda_j}\right)$$

$$= \varphi_{a_i, \lambda_j}(a_i) + O\left(\frac{1}{\lambda_j}\right) = 8\pi G(a_j, a_i) + 4\pi \frac{\ln \lambda_i}{\lambda_j} + O\left(\frac{1}{\lambda_j}\right).$$

Hence the proof of this claim follows. Concerning the last one, it holds

$$\langle \varphi_{a_j, \lambda_j}, \lambda_j \frac{\partial \varphi_{a_i, \lambda_i}}{\partial \lambda_i} \rangle_g = \int_{B_{a_j}(\eta)} e^{u_{a_j}} e^{\gamma_{a_j} G(a_j, \cdot)} \left( -8 \pi \frac{\ln \lambda_j}{\lambda_j} + O\left(\frac{1}{\lambda_j}\right) \right) dV_g + O\left(\frac{1}{\lambda_j}\right)$$

$$= -64\pi^2 \frac{\ln \lambda_i}{\lambda_i^2} + O\left(\frac{1}{\lambda_j^2} + \frac{1}{\lambda_j}\right).$$

Thereby the proof of this lemma follows. ■

From Lemma 7.1, it is easy to get the following expansion

**Lemma 7.3** Let $u := \sum_{i=1}^m \alpha_i \varphi_{a_i, \lambda_i}$. In $B_{a_i}(\eta)$, it holds

$$K e^u = \frac{\lambda_i^{4\alpha_i} F^A_i g^A_i}{(1 + \lambda_i^2 |y_{a_i}(|x)|^2)^{2\alpha_i}} \left( 1 + 4\pi \sum_{j=1}^m \frac{\ln \lambda_j}{\lambda_j^2} + O\left(\frac{1}{\lambda_j} + \sum |\alpha_j - 1| \frac{\ln \lambda_j}{\lambda_j^2}\right) \right)$$

$$= \frac{\lambda_i^{4\alpha_i} F^A_i g^A_i}{(1 + \lambda_i^2 |y_{a_i}(|x)|^2)^{2\alpha_i}} \left( 1 + 4\pi \sum_{j=1}^m \frac{\ln \lambda_j}{\lambda_j^2} + O\left(\frac{1}{\lambda_j^2} + \sum |\alpha_j - 1| \frac{\ln \lambda_j}{\lambda_j^2}\right) \right) K e^u,$$

where the functions $F^A_i$ (with $A = (a_1, \ldots, a_m)$) and $g^A_i$ are defined as follows

$$F^A_i(x) := K(x) \exp \left( 8\pi H(a_i, x) + 8\pi \sum_{j \neq i} G(a_j, x) \right),$$

$$g^A_i(x) := \exp \left( 8\pi (\alpha_i - 1) H(a_i, x) + 8\pi \sum_{j \neq i} (\alpha_j - 1) G(a_j, x) \right).$$

**Lemma 7.4** Let $u := \sum_{i=1}^m \alpha_i \varphi_i \in V(m, \varepsilon)$. Then,

$$\int_\Sigma K e^u dV_g = \pi \sum_{i=1}^m \frac{\lambda_{i}^{4\alpha_{i}-2}}{2a_{i}-1} (F^A_i g^A_i)(a_i) + O\left(1 + \sum_{k=1}^m \lambda_k^4 \alpha_k - 2 \left( |\alpha_k - 1|^2 + \frac{\ln \lambda_k}{\lambda_k^2}\right) \right).$$

If $\sum_{i=1}^m |\alpha_i - 1| \ln \lambda_i = o(1)$, the expansion (48) can be improved as follows

$$\int_\Sigma K e^u dV_g = \pi \sum_{i=1}^m \frac{\lambda_{i}^{4\alpha_{i}-2}}{2a_{i}-1} (F^A_i g^A_i)(a_i) + \frac{\pi}{2} \sum_{i=1}^m \left( \Delta F^A_i(a_i) - 2 K_g(a_i) F^A_i(a_i) \right) \ln \lambda_i$$

$$+ 4\pi^2 \sum_{j=1}^m \frac{\ln \lambda_j}{\lambda_j^2} \sum_{i=1}^m \lambda_i^{4\alpha_i-2} F^A_i(a_i) + O\left(1 + \sum_{k=1}^m |\alpha_k - 1| \ln^2 \lambda_k\right).$$

**Proof.** Let $B_i := B_{a_i}(\eta)$. Firstly, note that Lemma 7.1 shows that $e^u$ is bounded on $\Sigma \setminus (\cup B_i)$. Hence the integral in this set is bounded. Now, using Lemma 7.3, we have

$$\int_{B_i} K e^u dV_g$$

$$= \int_{B_i} \frac{\lambda_{i}^{4\alpha_{i}} F^A_i g^A_i}{(1 + \lambda_i^2 |y_{a_i}(|x)|^2)^{2\alpha_i}} \left( 1 + 4\pi \sum_{j=1}^m \frac{\ln \lambda_j}{\lambda_j^2} \right) dV_g + O\left(\frac{1}{\lambda_j^2} + \sum |\alpha_j - 1| \frac{\ln \lambda_j}{\lambda_j^2}\right) \int_{B_i} K e^u dV_g$$

$$= \left( 1 + 4\pi \sum_{j=1}^m \frac{\ln \lambda_j}{\lambda_j^2} \right) \int_{B_i} \frac{\lambda_{i}^{4\alpha_{i}} F^A_i g^A_i e^{-u_{a_i}}}{(1 + \lambda_i^2 |y_{a_i}(|x)|^2)^{2\alpha_i}} dV_{g_{a_i}} + O\left(\frac{1}{\lambda_j^2} + \sum |\alpha_j - 1| \frac{\ln \lambda_j}{\lambda_j^2}\right) \int_{\Sigma} K e^u dV_g.$$
Finally, we have

$$\int_{B_i} \lambda_i^{4\alpha_i} F^A_i g_i A e^{-u_{\alpha_i}} dV_{g_{\alpha_i}} = \lambda_i^{4\alpha_i-2} (F^A_i g_i A e^{-u_{\alpha_i}})(a_i) \int_{B_i} \lambda_i^2 \frac{\lambda_i^2 |y_{a_i}(x)|^2}{(1 + \lambda_i^2 |y_{a_i}(x)|^2)^{2\alpha_i}} dV_{g_{\alpha_i}}$$

$$+ \lambda_i^{4\alpha_i-2} O \left( \int_{B_i} \lambda_i^2 |y_{a_i}(x)|^2 (1 + \lambda_i^2 |y_{a_i}(x)|^2)^{2\alpha_i} dV_{g_{\alpha_i}} \right).$$

Note that $u_{a_i}(a_i) = 0$. Furthermore, easy computations imply

$$\int_{B(0,\eta)} \frac{\lambda_i^2}{(1 + \lambda_i^2 |x|^2)^{2\alpha_i}} dx = \frac{\pi}{2\alpha_i - 1} + O \left( \frac{1}{\lambda_i^{4\alpha_i-2}} \right).$$

To estimate the other integral, we introduce the following function

$$\xi(x) = \frac{1}{(1 + \lambda_i^2 |x|^2)^{2\alpha_i}}.$$ 

If $(\alpha - 1) \ln \lambda$ is small, one obtains that $\xi = 1 + o(1)$ uniformly on $B(0,\eta)$. In general, we have

$$|\xi(x) - 1| = |\xi(x) - \xi(0)| = \left| \int_0^1 \frac{4(1 - \alpha)\lambda^2 t|x|^2}{(1 + \lambda^2 t^2 |x|^2)^{2\alpha_i - 1}} dt \right| \leq c|\alpha - 1|\sqrt{\lambda|x|}.$$ 

Now, using (53), we obtain

$$\int_{B(0,\eta)} \frac{\lambda_i^2 |x|^2}{(1 + \lambda_i^2 |x|^2)^{2\alpha_i}} dx = \int_{B(0,\eta)} \frac{\lambda_i^2 |x|^2}{(1 + \lambda_i^2 |x|^2)^{2\alpha_i}} dx + \int_{B(0,\eta)} \frac{\lambda_i^2 |x|^2}{(1 + \lambda_i^2 |x|^2)^{2\alpha_i}} (\xi_i - 1) dx$$

$$= O \left( \ln \frac{\lambda_i}{\lambda_i^2} \right) + O \left( \frac{\alpha_i - 1}{\lambda_i^{3/2}} \right).$$

Hence, the proof of (48) follows. Now, we will focus on (49). In this case, we have $|\alpha_i - 1| \ln \lambda_i$ is small for each $i$ and we need to improve the estimate of (50). Using (51), we obtain

$$\int_{B_i} \lambda_i^{4\alpha_i} F^A_i g_i A e^{-u_{\alpha_i}} dV_{g_{\alpha_i}} = \frac{\pi}{2\alpha_i - 1} \lambda_i^{4\alpha_i-2} (F^A_i g_i A)(a_i) + O(1)$$

$$+ \Delta_{g_{\alpha_i}} (F^A_i g_i A e^{-u_{\alpha_i}})(a_i) \int_{B(0,\eta)} \frac{\lambda_i^{4\alpha_i} |x|^3}{(1 + \lambda_i^2 |x|^2)^{2\alpha_i}} dx + O \left( \int_{B(0,\eta)} \frac{\lambda_i^{4\alpha_i} |x|^3}{(1 + \lambda_i^2 |x|^2)^{2\alpha_i}} dx \right).$$

Observe that

$$\int_{B(0,\eta)} \frac{\lambda_i^{4\alpha_i} |x|^3}{(1 + \lambda_i^2 |x|^2)^{2\alpha_i}} dx = \lambda_i^{4\alpha_i-5} \pi^{\lambda_i^3} \int_0^r \frac{r^4}{(1 + r^2)^{2\alpha_i}} dr = O(1),$$

$$\int_{B(0,\eta)} \frac{\lambda_i^{4\alpha_i} |x|^3}{(1 + \lambda_i^2 |x|^2)^{2\alpha_i}} dx = 2\pi \ln \lambda_i + O(1 + |\alpha_i - 1| \ln \lambda_i) \quad \text{(by using $|\alpha_i - 1| \ln \lambda_i$ is small)}.$$ 

To conclude, we need the following information. We know that the function $\tilde{u}_a(x) := u_a(y_a(x))$ (for $x \in B(a,\eta) \subset \mathbb{R}^2$) satisfies

$$-\Delta \tilde{u}_a = -2K_g(y_a^{-1}(x)) e^{-\tilde{u}_a} \quad \text{in } B(a,\eta)$$

and therefore we derive that

$$\Delta_{g_{a_i}} (F^A_i g_i A e^{-u_{a_i}})(a_i) = \Delta_{g_{a_i}} F^A_i (a_i) - 2F^A_i (a_i) K_g(a_i) + O \left( \sum |\alpha_j - 1| \right),$$

by using the fact that

$$g^A_i (a_i) = 1 + O \left( \sum |\alpha_j - 1| \right), \quad |\nabla g^A_i (a_i)| = O \left( \sum |\alpha_j - 1| \right), \quad |\Delta g^A_i (a_i)| = O \left( \sum |\alpha_j - 1| \right).$$

Summing the above estimates, the result follows. ■
Lemma 7.5 Let $A$ and $\Lambda$ satisfying balancing condition (19) then the approximate solution $U_{A,\Lambda}$ defined in (18) satisfies the following equation

$$-\Delta g U_{A,\Lambda} + 8\pi m = 8\pi m \frac{K^U_{A,\Lambda}}{\int_{\Sigma} K^U_{A,\Lambda}} + f_{A,\Lambda},$$

where

$$f_{A,\Lambda} \rightharpoonup 0 \text{ weakly in } H^1(\Sigma) \text{ as } \lambda_i \to \infty \forall i = 1, \cdots, m.$$

Proof. We first notice that according to the equation (17) satisfied by $\varphi_{a,\lambda}$, using (46), the function $U_{A,\lambda}$ satisfies the equation

$$(54) \quad -\Delta g U_{A,\Lambda} + 8\pi m + O\left(\sum_{i=1}^{m} \frac{1}{\lambda_i^2}\right) = \sum_{i=1}^{m} e^{\delta_{a_i,\lambda_i} + u_{a_i}}.$$

Moreover it follows from Lemmas 7.1 and 7.4 that in $\Sigma \setminus \bigcup_{i=1}^{m} B_{\eta}(a_i)$ we have that

$$8\pi m \frac{K^U_{A,\Lambda}}{\int_{\Sigma} K^U_{A,\Lambda}} = O\left(\frac{\ln \lambda}{\lambda^2}\right).$$

Furthermore it follows from Lemmas 7.3 and 7.4 that in $B_{\eta}(a_i)$ there hold

$$K^U_{a,\lambda} = \frac{1}{8} \frac{\lambda_i^2 F^A_i(a_i)e^{\delta_{a_i,\lambda_i}}}{\sum_{j=1}^{m} \lambda_j^2 F^A_j(a_j)} (1 + O\left(\frac{\ln \lambda}{\lambda^2}\right)) \quad \text{and} \quad \int_{\Sigma} K^U_{A,\Lambda} = \pi \sum_{i=1}^{m} \lambda_i^2 F^A_i(a_i)(1 + O\left(\frac{\ln \lambda}{\lambda^2}\right)).$$

Therefore in $B_{\eta}(a_i)$ we have that

$$8\pi m \frac{K^U_{A,\Lambda}}{\int_{\Sigma} K^U_{A,\Lambda}} = \frac{m \lambda_i^2 F^A_i(a_i)e^{\delta_{a_i,\lambda_i}}}{\sum_{j=1}^{m} \lambda_j^2 F^A_j(a_j)} (1 + O\left(\frac{\ln \lambda}{\lambda^2}\right)).$$

Hence we derive that

$$\sum_{j=1}^{m} e^{\delta_{a_j,\lambda_j} + u_{a_j}} - 8\pi m \frac{K^U_{A,\Lambda}}{\int_{\Sigma} K^U_{A,\Lambda}} = e^{\delta_{a_i,\lambda_i}} \left(\tilde{\tau}_i + O\left(|x - a_i|^2 \frac{\ln \lambda}{\lambda^2}\right)\right) \quad \text{in } B_{\eta}(a_i),$$

where $\tilde{\tau}_i$ is defined in Proposition 5.1 by taking $\alpha_k = 1$ for each $k$.

We notice that it follows from the balancing condition (19) that $\forall i = 1, \cdots, m$ we have that $|a_i| = a_\lambda(1)$.

Summarizing we have that $U_{A,\Lambda}$ satisfies the equation

$$-\Delta g U_{A,\Lambda} + 8\pi m = 8\pi m \frac{K^U_{A,\Lambda}}{\int_{\Sigma} K^U_{A,\Lambda}} + f_{A,\Lambda}, \quad \text{where}$$

$$f_{A,\Lambda} = \begin{cases} O(\ln \lambda/\lambda^2) \text{ in } \Sigma \setminus \bigcup_{i=1}^{m} B_{\eta}(a_i) \\ e^{\delta_{a_i,\lambda_i}} \left(\tilde{\tau}_i + O\left(|x - a_i|^2 + \ln \lambda_i/\lambda_i^2\right)\right) \text{ in } B_{\eta}(a_i). \end{cases}$$

Hence the Lemma follows. ■

Lastly for the sake of completeness we provide the following characterization of approximate blowing up solutions of Equation (MF). Namely we prove:
Proposition 7.6 Let \((\Sigma, g)\) be a closed surface of unit volume and \(u_k\) be a blowing up solution of 

\[ -\Delta_g u_k = g_k(\frac{Ke^{u_k}}{\int_{\Sigma} Ke^{u_k}} - 1) \text{ in } \Sigma; \quad \int_{\Sigma} u_k dV_g = 0, \]

where \(g_k \to 8\pi m, m \in \mathbb{N}. \) Then for \(\varepsilon\) small and \(k\) large we have that \(u_k \in V(m, \varepsilon).\)

**Proof.** We set \(\tilde{u}_k := u_k - \ln(\int_{\Sigma} Ke^{u_k})\) and observe that \(\tilde{u}_k\) satisfies the equation

\[ -\Delta_g \tilde{u}_k = g_k(Ke^{\tilde{u}_k} - 1) \text{ in } \Sigma. \]

Now it follows from the refined blow up analysis performed in \([22, 12, 13]\) that \(\tilde{u}_k\) blows up at \(m\)-points \(a_1, \ldots, a_m \in \Sigma\) with comparable blow up rates \(\lambda_i := e^{u_k(a_i)}\) such that \(d_g(a_i, a_j) \geq 2\eta\) for some \(\eta > 0.\) Moreover we have that 

\[ e^{\tilde{u}_k} = O(\sum_{i=1}^{m} \frac{1}{\lambda_i^2}) \text{ in } \Sigma \setminus \bigcup_{i=1}^{m} B_\eta(a_i) \]

and inside the balls \(B_i := B_\eta(a_i)\) there holds (See Theorem 1.4 in \([12]\))

\[ \tilde{u}_k + \ln(g_k(K(a_i))) = \delta_{a_i, \lambda_i} + O(d_g(a_i, x)). \]

Next we claim that:

**Claim:** For \(1 < p < 2\) there exists a constant \(C > 0\) such that

\[ ||f_k||_{L^p} \leq C \sum_{i=1}^{m} \frac{1}{\lambda_i^{(2/p) - 1}}. \]

Indeed we have that 

\[ \int_{\Sigma} |f_k|^p dV_g = \sum_{i=1}^{m} \int_{B_i} |f_k|^p dV_g + O(\sum_{i=1}^{m} \frac{1}{\lambda_i^{2p}}). \]

Furthermore in \(B_i\) we have that in geodesic local coordinates around \(a_i\) that 

\[ f_k = e^{\tilde{u}_k + \ln(g_k K)} - \sum_{i=1}^{m} e^{\delta_{a_i, \lambda_i} + u_{a_i}} + O(\frac{1}{\lambda^2}) \]

\[ = e^{\delta_{a_i, \lambda_i} + \ln(g_k K) - \ln(g_k K(a_i) + O(|x| - a_i))} - e^{\delta_{a_i, \lambda_i} + u_{a_i}} + O(\frac{1}{\lambda^2}) \]

\[ = O(|x - a_i|e^{\delta_{a_i, \lambda_i}} + \frac{1}{\lambda^2}) \]

Hence for \(1 < p < 2\) and a constant \(C > 0\) there holds

\[ \int_{\Sigma} |f_k|^p dV_g \leq C \sum_{i=1}^{m} \frac{1}{\lambda_i^{2-p}}. \]

Next setting \(v_k := \tilde{v}_k + \ln(\int_{\Sigma} Ke^{u_k}) = u_k - \sum_{i=1}^{m} \varphi_{a_i, \lambda_i}\) we have that \(v_k\) satisfies the equation
\[-\Delta_g v_k + q_k - 8\pi m = f_k \text{ in } \Sigma; \int_{\Sigma} v_k dV_g = 0.\]

where \( f_k \in L^p(\Sigma) \) for \( p \in (1, 2) \). Hence it follows from the Calderon-Zygmund a priori estimate that

\[ ||v_k||_{W^{2,p}} \leq C ||f_k||_{L^p} \leq C \sum_{i=1}^{m} \frac{1}{\lambda_i^{(2/p)-1}}.\]

Therefore it follows from the Sobolev embedding \( W^{2,p}(\Sigma) \hookrightarrow W^{1,2}(\Sigma) \) that

\[ ||v_k||_{H^1} \leq \sum_{i=1}^{m} \frac{1}{\lambda_i^{(2/p)-1}}.\]

Hence choosing \( p = \frac{4}{3} \) we obtain that

\[ ||u_k - \sum_{i=1}^{m} \phi_{a_i, \lambda_i}|| \leq \sum_{i=1}^{m} \frac{1}{\sqrt{\lambda_i}}.\]

Therefore the proposition is fully proven. \( \blacksquare \)

References

[1] Ahmedou, M.; Ben Ayed, M.; Lucia, M. On a resonant mean field type equation: a "critical point at infinity" approach, Discrete Contin. Dyn. Syst. 37 (2017), no. 4, 1789–1818.

[2] Ahmedou, M.; Pistoia, A. On the supercritical mean field equation on pierced domains. Proc. Amer. Math. Soc. 143 (2015), 3969–3984.

[3] Bahri A., Critical points at infinity in some variational problems, Research Notes in Mathematics, 182, Longman-Pitman, London, 1989.

[4] Bahri, A.; Coron, J.-M. On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, Comm. Pure Appl. Math 41(1988), 253–294.

[5] Bahri A., Coron J.-M., The Scalar-Curvature Problem on the Standard Three-Dimensional Sphere, J. Funct. Anal. 95 (1991), 106–172.

[6] Bahri A., An invariant for Yamabe-type flows with applications to scalar-curvature problems in high dimension, Duke Math. J. 81 (1996), 323–466.

[7] Brézis, H.; Merle, F. Uniform estimates and blow-up behavior for solutions of \( -\Delta u = V(x)e^u \) in two dimensions, Comm. Partial Differential Equations 16 (1991), 1223–1253.

[8] Caglioti, E; Lions, P.-L; Marchioro, C; Pulvirenti, M. A special class of stationary flows for two dimensional Euler equations: a statistical mechanics description, Comm. Math. Phys. 143 (1992), 501–525.

[9] Caglioti, E; Lions, P.-L; Marchioro, C; Pulvirenti, M. A special class of stationary flows for two dimensional Euler equations: a statistical mechanics description, Part II, Comm. Math. Phys. 174 (1995), 229–260.

[10] Chang, A.; Yang, P. A perturbation result in prescribing scalar curvature on \( S^n \). Duke Math. J. 64 (1991), 27–69.

[11] Chang, A.; Gursky, M.; Yang, Paul. The scalar curvature equation on 2- and 3-spheres. Calc. Var. Partial Differential Equations 1 (1993), no. 2, 205–229.

[12] Chen, C.C; Lin, C.S. Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces, Comm. Pure Appl. Math. 55 (2002), 728–771.
[13] Chen, C.C; Lin, C.S. Topological degree for a mean field equation on Riemann surfaces, Comm. Pure Appl. Math. 56 (2003), 1667–1727.

[14] Ding, W.; Jost, J; Li, J.; Wang, G. Existence results for mean field equations, Ann. Inst. Henri Poincaré Anal. Non Linéaire 16 (1999), 653–666.

[15] Djadli, Z. Existence result for the mean field problem on Riemann surfaces of all genus, Commun. Contemp. Math. 10 (2008), 205–220.

[16] Djadli, Z; Malchiodi, A. Existence of conformal metrics with constant Q-curvature, Ann. of Math. (2) 168 (2008), 813–858.

[17] De Marchis, F. Generic multiplicity for a scalar field equation on compact surfaces, J. Funct. Anal. 259 (2010), 2165–2192.

[18] Kallel S., Karoui R., Symmetric joins and weighted barycenters, Adv. Nonlinear. Stud. 11 (2011), no. 1, 117-143.

[19] Han, Z-C. Prescribing Gaussian curvature on $S^2$. Duke Math. J. 61 (1990), 679–703.

[20] Kiessling, M. K. H. Statistical mechanics of classical particles with logarithmic interactions, Comm. Pure Appl. Math. 46 (1993), 27–56.

[21] Li, Y. Y. Harnack type inequality: the method of moving planes, Comm. Math. Phys. 200 (1999), 421–444.

[22] Li, Y. Y; Shafir, I. Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two, Indiana Univ. Math. J. 43 (1994), 1255–1270.

[23] Lions, P. L.: The concentration-compactness principle in the calculus of variations. The limit case. Part I. Rev. Mat. Iberoamericano 1 (1985), 145–201.

[24] Lucia, M. A deformation lemma with an application to a mean field equation. Topol. Methods Nonlinear Anal. 30 (2007), no. 1, 113–138.

[25] Malchiodi, A. Morse theory and a scalar field equation on compact surfaces, Adv. Differential Equations 13 (2008), 1109–1129.

[26] Saut, J.M.; Temam, R. Generic properties of nonlinear boundary value problems, Comm. Partial Differential Equations 4 (1979), 293-319.

[27] Struwe, M.: A global compactness result for elliptic boundary value problems involving limiting nonlinearities. Math. Z. 187 (1984), 511–517.

[28] M. Struwe, G. Tarantello, On multivortex solutions in Chern-Simons gauge theory, Boll. Unione. Math. Ital. Sez. B Artic. Ric. Mat. (8) 1 (1998), 109–121.

[29] Tarantello, G.; Selfdual gauge field vortices. Progress in nonlinear differential equations and their applications. An analytical approach 72(2008), Birkhäuser Boston Inc.

[30] Tarantello, G. Analytical, geometrical and topological aspects of a class of mean field equations on surfaces, Discrete Contin. Dyn. Syst. 28(2010).

[31] Yang, Y.; Solitons in field theory and nonlinear analysis. Springer monograph in Mathematics. Springer Verlag, New York, 2001.

[32] Zhang, L. Blow up solutions of some nonlinear elliptic equation involving exponential nonlinearities, Comm. Math. Phys. 268(2006), 105–133.

[33] Zhang, L. A priori estimates for a family of semi-linear elliptic equation involving exponential nonlinearities, J. Diff. Equations 247(2009), 105–133.
Mohameden Ahmedou
Mathematisches Institut der Justus-Liebig-Universität Giessen
Arndtsrasse 2, D-35392 Giessen
Germany
Mohameden.Ahmedou@math.uni-giessen.de

Mohamed Ben Ayed
Université de Sfax, Faculté des Sciences
Département de Mathématiques
Route de Soukra, Sfax, Tunisia
Mohamed.Benayed@fss.rnu.tn