Some invariant string cosmological models in cylindrically symmetric space-time

Ahmad T Ali\textsuperscript{1,4}, Anil Kumar Yadav\textsuperscript{2}, Farook Rahaman\textsuperscript{3} and Arkopriya Mallick\textsuperscript{3}

\textsuperscript{1}King Abdul Aziz University, Faculty of Science, Department of Mathematics, PO Box 80203, Jeddah, 21589, Saudi Arabia
\textsuperscript{2}KCC Institute of Technology and Management, Greater Noida, 201306, India
\textsuperscript{3}Department of Mathematics, Jadavpur University, Kolkata, 700 032, India

E-mail: atali71@yahoo.com, abanilyadav@yahoo.co.in, rahaman@iucaa.ernet.in and arkopriyamallick@gmail.com

Received 7 March 2014, revised 26 July 2014
Accepted for publication 3 September 2014
Published 23 October 2014

Abstract

In this paper, we derive some new invariant solutions of Einstein-Maxwell’s field equations for string fluid as source of matter in cylindrically symmetric space-time with variable magnetic permeability. We also discuss the physical and geometrical properties of the models derived in the paper. The solutions, at least one of them, are interesting physically as they can explain the accelerating as well as singularity-free Universe.

Keywords: Similarity solutions, magnetized cylindrically symmetric space-time, Inhomogeneous cosmological

(Some figures may appear in colour only in the online journal)

1. Introduction

An important benefit of the general theory of relativity is that it helps us understand the large-scale structure of the Universe. The study of cosmology is devoted to the Universe as a whole. Initial studies were based on Friedmann–Robertson–Walker (FRW) type cosmology, which describes homogeneous and isotropic space. We know that homogeneity and isotropy are symmetries of space. Though current observations indicate that the Universe is almost perfectly homogeneous and isotropic in large scales, the symmetries of space in the very early Universe could have been very different. Today’s observations are unable to provide information about the symmetries of space near initial singularity. Hence, it is very justified to consider inhomogeneous and anisotropic models of the Universe. Cylindrically symmetric space-time is more general than homogeneous and isotropic space-time, and it plays an important role in the study of the universe when anisotropy and inhomogeneity are taken into consideration. Barrow and Kunze [1, 2] investigated the flat and open homogeneous Universe, considering string fluid as a source of matter. In the recent past, Pradhan et al [3] and Yadav et al [4] studied inhomogeneous cosmological models with clouds of strings; the latter were invoked to palliate the problems associated with cylindrically symmetric space-time.

Due to non linearity, general relativity is a very difficult theory. Therefore, to describe the models of the Universe, different group of symmetries are used in the literature. In this study, we apply the so-called symmetry analysis method. The main advantage of this method is that it can be successfully applied to nonlinear differential equations. The similarity solutions are attractive because they result in the reduction of the independent variables of the problem. In our case, the problem under investigation is the system of second-order nonlinear PDEs and the similarity solution that will transform this system of nonlinear partial differential equation (PDE) into a system of ordinary differential equation (ODE). The groups of continuous transformations that leave a given family of equations invariant are known as symmetry groups (isovector fields) [5–9]. In a pioneering work, Ovsianikov [10] mentioned that the usual Lie infinitesimal invariance approach could also be employed to construct symmetry groups [11–13].
In recent years, there has been considerable interest in string cosmology. Cosmic string plays an important role in the study of the early Universe. String cosmology is a relatively new field that tries to apply equations of string theory to solve the questions of early cosmology; a related area of study is brain cosmology. Melvin [14] argued that the presence of a magnetic field is not unrealistic because for a large part of the history of evolution, matter was highly ionized and matter and field were smoothly coupled. Later during the expansion of the Universe, the ions combined to form neutral matter. Therefore, including the magnetic field with variable magnetic permeability is justified for cosmological modeling of the Universe.

In this paper, we attempted to find a new class of exact (similarity) solutions for string cosmological models in a cylindrically symmetric inhomogeneous Universe with electromagnetic perfect fluid distribution with variable magnetic permeability in general relativity.

We organized the paper as follows: In section 2, we introduce a string cosmological model in a cylindrically symmetric inhomogeneous Universe with electromagnetic perfect fluid distribution with variable magnetic permeability in general relativity. In section 3, symmetry analysis and isovector fields for Einstein field equations are obtained. In section 4, we find a new class of exact (similarity) solutions for Einstein field equations. Section 5 is devoted to the study of some physical and geometrical properties of the obtained model. The paper ends with a short discussion.

2. The metric and field equations

We consider the metric in the form
\[
\text{d}s^2 = A^2 (\text{d}x^2 - \text{d}t^2) + B^2 \text{d}y^2 + C^2 \text{d}z^2,
\]
where A, B, and C are functions of x and t. The energy-momentum tensor for the string with an electromagnetic field has the form
\[
T_{ij} = \rho u_i u_j - \lambda x_i x_j + E_{ij},
\]
where \(u_i\) and \(x_i\) satisfy the conditions
\[
u^i \nu_i = -x^1 x_i = -1,
\]
and
\[
u^i x_i = 0.
\]
Here, with \(\rho\) being the rest energy density of the system of strings and \(\lambda\) being the tension density of the strings \(x^i\) is a unit space-like vector representing the direction of strings so that \(x^1 = x^2 = x^4 = 0\) and \(x^3 \neq 0\). \(\nu^i\) is the four velocity vector. \(E_{ij}\) is the electromagnetic field given by Lichnerowicz [15] as
\[
E_{ij} = \rho \left[ h_i h_j (u_i u_j + \frac{1}{2} s_{ij}) - h_i h_j \right],
\]
where \(\rho\) is the magnetic permeability and \(h_i\) is the magnetic flux vector defined by
\[
h_i = \frac{1}{\mu} * F^i_{\mu},
\]
where the dual electro magnetic field tensor, \(* F_{ij}\), is defined by Synge [16] as
\[
* F_{ij} = \frac{\sqrt{-g}}{2} \epsilon_{ijkl} F^{kl}.
\]
Here, \(F_{ij}\) is the electromagnetic field tensor and \(\epsilon_{ijkl}\) is a Levi-Civita tensor density. In the present scenario, the co moving coordinates are taken as
\[
u^i = \left( 0, 0, 0, \frac{1}{A} \right).
\]
We choose the direction of string parallel to the \(x\)-axis, so that
\[
x^i = \left( \frac{1}{A}, 0, 0, 0 \right).
\]
If we consider the current flow along the \(z\)-axis, then \(F_{12}\) is the only non vanishing component of \(F_{ij}\). Then, the Maxwell’s equations
\[
F_{i,jk} + F_{jk,i} + F_{ki,j} = 0
\]
and
\[
\left[ \frac{1}{\mu} F_{ij} \right]_j = J^i
\]
require that \(F_{12}\) be a function of \(x\) alone [17]. We assume that the magnetic permeability is a function of both \(x\) and \(t\). Here, the semicolon represents a covariant differentiation.

The Einstein’s field equation
\[
R_{ij} - \frac{1}{2} g_{ij} R = \chi T_{ij},
\]
for the line-element (1) leads to the following system of equations:
\[
C_{xx} - \frac{B_{xx}}{2} + \frac{B_{tt}}{2} - \frac{A_{xx}}{2} - \frac{A_{tt}}{2} = 0,
\]
\[
\chi A^2 \frac{\lambda}{C} = C_{xx} + \frac{B_{tt}}{B} + \frac{A_{xx}}{A} + \frac{A_{tt}}{A} + \frac{A^2 - A^2}{A^2} - \frac{A_{x} C_{t} - A_{t} C_{x}}{A B} + \frac{A_{x} C_{t} + A_{t} C_{x}}{A B},
\]
\[
\chi A^2 \rho = C_{tt} - 2 C_{xx} - \frac{B_{tt}}{B} - \frac{A_{tt}}{A} + \frac{A_{xx}}{A} + \frac{B_{tt}}{B} + \frac{A_{tt}}{A} + \frac{A_{x} C_{t} + A_{t} C_{x}}{A B} - \frac{A_{x} C_{t} - A_{t} C_{x}}{A B} + \frac{A^2 - A^2}{A^2}.
\]
\[
\frac{\chi F_{i}^{2}}{\mu B^{2}} = \frac{C_{xx} - C_{x}}{C} + \frac{A_{xx} - A_{x}}{A} + \frac{A_{x}^{2} - A_{x}^{2}}{A^{2}},
\]

(17)

The velocity field, \( u_{i}' \), is ir-rotational. The scalar expansion, \( \Theta \), shear scalar, \( \sigma^{2} \), acceleration vector, \( u_{i} \), and proper volume, \( V \), are respectively found to have the following expressions [18, 19]:

\[
\Theta = u_{i}' = \frac{1}{A} \left( \frac{C_{i}}{C} + \frac{B_{t}}{B} + \frac{A_{i}}{A} \right) \quad \text{(18)}
\]

\[
\sigma^{2} = \frac{1}{2} \sigma_{i} \sigma^{i} = \frac{\Theta^{2}}{3} \left( \frac{B_{t} C_{i}}{B C} + \frac{A_{i} C_{t}}{A C} + \frac{A_{i} B_{t}}{A B} \right) \quad \text{(19)}
\]

\[
u_{i} = u_{i} + u_{i}' = \left( \frac{A_{i}}{A}, 0, 0, 0 \right) \quad \text{(20)}
\]

\[
V = \sqrt{-g} = A^{2} B C, \quad \text{(21)}
\]

where \( g \) is the determinant of the metric (1). The shear tensor is

\[
\sigma_{i} \equiv u_{(ij)} + \nu_{(ij)} = - \frac{1}{3} \Theta (g_{ij} + u_{i} u_{j}) \quad \text{(22)}
\]

and the non-vanishing components of the \( \sigma_{i} \) are

\[
\begin{align*}
\sigma_{1}^{2} & = \frac{1}{3} A \left( \frac{2 A_{t}}{A} - \frac{B_{t}}{B} - \frac{C_{t}}{C} \right), \\
\sigma_{2}^{2} & = \frac{1}{3} A \left( \frac{2 B_{t}}{B} - \frac{C_{t}}{C} + \frac{A_{t}}{A} \right), \\
\sigma_{3}^{3} & = \frac{1}{3} A \left( \frac{2 C_{t}}{C} - \frac{B_{t}}{B} + \frac{A_{t}}{A} \right), \\
\sigma_{4}^{4} & = 0.
\end{align*}
\]

Using the field equations and the relations (18) and (19), one obtains the Raychaudhuri’s equation as

\[
\frac{\partial \Theta}{\partial t} = \hat{u}_{i}' - \frac{\Theta^{2}}{3} - 2 \sigma^{2} - \frac{\rho_{p}}{2} \quad \text{(24)}
\]

where

\[
\rho_{p} = 2 R_{ij} u_{i}' u_{j}'.
\]

(25)

The Einstein field equations (13)–(17) constitute a system of five highly non-linear differential equations with six unknowns variables: \( A, B, C, \lambda, \rho \), and \( \frac{\Theta^{2}}{\mu} \). Therefore, one physically reasonable condition among these parameters is required to obtain explicit solutions of the field equations. Let us assume that the expansion scalar, \( \Theta \), in the model (1) is proportional to the eigenvalue, \( \sigma_{1}^{2} \), of the shear tensor, \( \sigma_{i} \). Then from (18) and (23), we get

\[
\frac{2 A_{t}}{A} - \frac{B_{t}}{B} - \frac{C_{t}}{C} = 3 \gamma \left( \frac{A_{t}}{A} + \frac{B_{t}}{B} + \frac{C_{t}}{C} \right) \quad \text{(26)}
\]

where \( \gamma \) is a constant of proportionality. The above equation can be written in the form

\[
\frac{A_{t}}{A} = n \left( \frac{B_{t}}{B} + \frac{C_{t}}{C} \right) \quad \text{(27)}
\]

where \( n = \frac{1 + 3 \gamma}{2 - 3 \gamma} \). If we integrate the above equation with respect to \( t \), we can get the following relation:

\[
\lambda(x, t) = f(x)(B(x, t) C(x, t))^{n}, \quad \text{(28)}
\]

where \( f(x) \) is a constant of integration, which is an arbitrary function of \( x \). If we substitute the metric function \( A \) from (20) in the Einstein field equations, the equations (13)–(14) transform to the nonlinear partial differential equations of the coefficients \( B \) and \( C \) only as the following new form:

\[
E_{1} = \frac{B_{tt}}{B} + \frac{C_{tt}}{C} - 2 n \left( \frac{B_{i} B_{i}}{B} + \frac{B_{t} C_{t}}{B} + \frac{C_{t} C_{t}}{C} \right) - \frac{f'}{f} \left( \frac{B_{t}}{B} + \frac{C_{t}}{C} \right) = 0, \quad \text{(29)}
\]

\[
E_{2} = \left( n + \frac{1}{2} \right) \left[ \frac{B_{xx} - \frac{C_{xx}}{C}}{B} + \frac{C_{xx} - \frac{C_{xx}}{C}}{C} \right] + \frac{n}{f} \left[ \frac{B_{t}^{2} - \frac{B_{t}^{2}}{B^{2}}}{B^{2}} + \frac{C_{t}^{2} - \frac{C_{t}^{2}}{C^{2}}}{C^{2}} \right] - \frac{f'}{f} - \frac{f'^{2}}{f^{2}} = 0, \quad \text{(30)}
\]

where the prime indicates a derivative with respect to the coordinate, \( x \).

### 3. Symmetry analysis method

In order to obtain an exact solution for the system of non-linear partial differential equations (29)–(30), we will use the symmetry analysis method. For this we write

\[
\begin{align*}
x_{i} & = x_{i} + \epsilon \xi_{i}(x, u_{j}), \\
u_{i} & = u_{i} + \epsilon \eta_{i}(x, u_{j}).
\end{align*}
\]

\[
\begin{align*}
i, j, \alpha, \beta & = 1, 2, \\
\end{align*}
\]

as the infinitesimal Lie point transformations. We have assumed that the system (29)–(30) is invariant under the transformations given in equation (31). The corresponding infinitesimal generator of Lie groups (symmetries) is given by

\[
X = \sum_{i=1}^{2} x_{i} \frac{\partial}{\partial x_{i}} + \sum_{\alpha=1}^{2} \eta_{\alpha} \frac{\partial}{\partial u_{\alpha}}, \quad \text{(32)}
\]

where \( x_{1} = x, \ x_{2} = t, \ u_{1} = B, \) and \( u_{2} = C \). The coefficients \( \xi_{t}, \xi_{x}, \eta_{x}, \) and \( \eta_{t} \) are the functions of \( x, \ t, \ B, \) and \( C \), respectively. These coefficients are the components of infinitesimal
symmetries corresponding to \( x, t, B, \) and \( C, \) respectively, to be determined from the invariance conditions,

\[
\text{Pr}^{(2)} X (E_a) \big|_{E_m=0} = 0, \quad (33)
\]

where \( E_m = 0, \ m = 1, 2 \) are the system (29)–(30) under study, and \( \text{Pr}^{(2)} \) is the second prolongation of the symmetry, \( X. \) Since our equations (29)–(30) are at most of order two, we need second-order prolongation of the infinitesimal generator in equation (33). It is worth noting that the 2-th order prolongation is given by

\[
\text{Pr}^{(2)} X = \sum_{j=1}^{2} \eta_j \Phi_{x_j} + \sum_{a=1}^{2} \eta_a \Phi_{t_a} + \sum_{j=1}^{2} \sum_{a=1}^{2} \eta_{a,j} \Phi_{x_a,t_j},
\]

where

\[
\eta_{a,j} = D_j (\eta_a) - \sum_{j=1}^{2} u_{a,j} D_j (\xi_j),
\]

\[
\eta_{a,i,j} = D_j (\eta_{a,i}) - \sum_{k=1}^{2} u_{a,k,i} D_j (\xi_k).
\]

The operator, \( D_x, \) is called the total derivative (Hach operator) and takes the following form:

\[
D_x = \frac{\partial}{\partial x} + \sum_{a=1}^{2} u_{a,i} \frac{\partial}{\partial t_a} + \sum_{j=1}^{2} \sum_{a=1}^{2} u_{a,j} \frac{\partial}{\partial x_a},
\]

where \( u_{a,i,j} = \frac{\partial u_{a,i}}{\partial t_j} \) and \( u_{a,i,j} = \frac{\partial^2 u_{a,i}}{\partial t_j \partial t_k}. \)

Expanding the system of equations (33) along with the original system of equations (29)–(30) to eliminate \( B_{x,x} \) and \( B_{x,t} \) while we set the coefficients involving \( C_x, C_t, C_{xx}, C_{ct}, B_x, B_t, B_{x,t}, \) and various products to zero, gives rise to the essential set of over determined equations. Solving the set of these determining equations, the components of symmetries take the following form:

\[
\xi_1 = c_1 x + c_2, \quad \xi_2 = c_1 t + c_3, \quad \eta_1 = c_4 B, \quad \eta_2 = c_5 C,
\]

such that the function \( A(t) \) must be equal,

\[
\left\{ \begin{array}{ll}
f(x) = c_6 \exp \{c_7 x\}, & \text{if } c_1 = 0, \\
f(x) = c_8 (c_1 x + c_2)^{c_9}, & \text{if } c_1 \neq 0,
\end{array} \right.
\]

where \( c_i, \ i = 1, 2, \ldots, 9 \) are arbitrary constants.

\[\newpage\]

4. Similarity solutions

The characteristic equations corresponding to the symmetries (37) are given by

\[
\frac{dx}{c_1 x + c_2} = \frac{dt}{c_1 t + c_3} = \frac{dB}{c_4 B} = \frac{dC}{c_5 C}. \quad (39)
\]

By solving the above system, we have the following two cases:

Case (1): When \( c_1 = 0, \) the similarity variable and similarity functions can be written as

\[
\xi = a x + b t, \quad B(x, t) = \Psi (\xi) \exp \{c x\}, \quad C(x, t) = \Phi (\xi) \exp \{d x\},
\]

where \( a = c_3, \ b = -c_2, \ c = \frac{c_4}{c_2} \) and \( d = \frac{c_5}{c_3} \) are arbitrary constants. Substituting the transformations (40) in the field equations (20)–(21) leads to the following system of ordinary differential equations:

\[
\alpha \Psi'' + \left[ c_7 - c + 2 n(c + d) \right] \Psi' + a \phi'' \Psi + a \phi' \Psi + \phi' = 2 a \left( \frac{\phi''}{\phi} + \frac{\phi'^2}{\phi^2} \right)
\]

\[\text{(41)}\]

\[
(c^2 + d^2) \left( \frac{\Psi''}{\Psi} + \frac{\phi''}{\phi} \right) - 2 a \left( \frac{\phi'^2}{\phi^2} \right) = c^2 + d^2.
\]

\[\text{(42)}\]

Equations (41) and (42) are non linear ordinary differential equations, which are very difficult to solve. However, it is worth noting that this equation is easy to solve when \( b=0. \) In this case, equation (42) takes the form

\[
\frac{c^2 + d^2}{2 a} - \frac{\phi'^2}{\phi} = \frac{c^2 + d^2}{2 a}.
\]

\[\text{(43)}\]

By integration of the above equation, we get the following:

\[
\Phi (\xi) = q_1 \Psi^{a_1} (\xi) \exp \{a_2 \xi\},
\]

where \( a_1 = \frac{-c}{d} \) and \( a_2 = -\left( \frac{c^2 + d^2}{2 a d} \right), \) while \( q_1 \) is an arbitrary constant of integration. Substitute (44) in (41), and we have the following ordinary differential equation of the function \( \Psi' \) only:

\[
\frac{\Psi''}{\Psi} = \left( a_3 - 1 \right) \frac{\phi'^2}{\phi^2} + \alpha_n (\frac{\Psi'}{\Psi}) + \alpha_s,
\]

where
\[
\begin{cases}
\alpha_3 &= \frac{1 + \alpha_2^2 - 2n(1 + \alpha_1)^2}{1 + \alpha_1}, \\
\alpha_4 &= \frac{2\alpha_2[\alpha_1(1 + \alpha_1) + d\alpha_1]}{(1 + \alpha_1)(1 + \alpha_2)}, \\
\alpha_5 &= \frac{[\alpha_2^2 + d(\alpha_1^2 - 1 - 2n)]}{(\alpha_2^2 + 2\alpha_1 - 1)}.
\end{cases}
\]

If we use the transformation
\[\Psi(\xi) = q_2 \exp \left[ \int \Omega(\xi) d\xi \right].\]
equation (45) becomes:
\[\Omega' = \alpha_3 \Omega^2 + \alpha_4 \Omega + \alpha_5,\]
where \(q_2\) is constant, while \(\Omega(\xi)\) is a new function of \(\xi\). To solve the ordinary differential equation above, we must take the following cases:

**Case (1.1):** When \(\alpha_3 \neq 0, \alpha_4 \neq 0,\) and \(\alpha_5 \neq 0\), the following three cases exist:

**Case (1.1.1):** When \(4\alpha_3\alpha_5 - \alpha_4^2 = \frac{4K_3^2}{a^2}\), the general solution of the equation (48) is
\[\Omega(\xi) = -\frac{\alpha_4}{2\alpha_3} + \frac{K_1}{a\alpha_3} \tan \left[ \frac{K_1}{a} \xi + \xi_0 \right],\]
where \(\xi_0\) is an arbitrary constant of integration.

**Case (1.1.2):** When \(\alpha_4^2 - 4\alpha_3\alpha_5 = \frac{4K_3^2}{a^2}\), the general solution of the equation (48) is
\[\Omega(\xi) = -\frac{\alpha_4}{2\alpha_3} + \frac{K_2}{a\alpha_3} \tanh \left[ \frac{K_2}{a} \xi + \xi_0 \right],\]
where \(\xi_0\) is an arbitrary constant of integration.

**Case (1.1.3):** When \(\alpha_4^2 = 4\alpha_3\alpha_5\), the general solution of the equation (48) is
\[\Omega(\xi) = \frac{\alpha_4}{2\alpha_5} (\xi - \xi_0).\]

**Case (1.2):** When \(\alpha_3 \neq 0, \alpha_4 \neq 0,\) and \(\alpha_5 = 0\), the general solution of the equation (48) is
\[\Omega(\xi) = \frac{\alpha_4}{\exp[\alpha_3 - \alpha_4(\xi + \xi_0)]},\]
where \(\xi_0\) is an arbitrary constant of integration.

**Case (1.3):** When \(\alpha_3 \neq 0, \alpha_4 = 0,\) and \(\alpha_5 \neq 0\), then the following cases exist:

**Case (1.3.1):** When \(\alpha_3 = \pm K_4^2\) and \(\alpha_5 = \pm K_5^2\), the general solution of the equation (48) is
\[\Omega(\xi) = \pm \frac{K_3}{K_4} \tan \left[ K_4 K_3(\xi + \xi_0) \right],\]
where \(\xi_0\) is an arbitrary constant of integration.

**Case (1.3.2):** When \(\alpha_3 = \mp K_6^2\) and \(\alpha_5 = \pm K_7^2\), the general solution of the equation (48) is
\[\Omega(\xi) = \pm \frac{K_4}{K_6} \tanh \left[ K_5 K_6(\xi + \xi_0) \right],\]
where \(\xi_0\) is an arbitrary constant of integration.

**Remark (1):** The two cases (1.3.1) and (1.3.2) are the spacial cases from cases (1.1.1) and (1.1.2), respectively.

**Case (1.4):** When \(\alpha_3 = 0, \alpha_4 \neq 0,\) and \(\alpha_5 \neq 0\), the general solution of the equation (48) is
\[\Omega(\xi) = \exp \left[ \alpha_4(\xi + \xi_0) \right] - \frac{\alpha_5}{\alpha_4}\]
where \(\xi_0\) is an arbitrary constant of integration.

**Case (1.5):** When \(\alpha_3 = \alpha_4 = 0,\) and \(\alpha_5 \neq 0\), the general solution of the equation (48) is
\[\Omega(\xi) = \alpha_5 \xi + \xi_0,\]
where \(\xi_0\) is an arbitrary constant of integration.

**Case (1.6):** When \(\alpha_3 = \alpha_5 = 0\) and \(\alpha_4 \neq 0\), the general solution of the equation (48) is
\[\Omega(\xi) = \exp \left[ \alpha_4(\xi + \xi_0) \right],\]
where \(\xi_0\) is an arbitrary constant of integration.

**Case (1.7):** When \(\alpha_4 = \alpha_5 = 0\) and \(\alpha_3 \neq 0\), the general solution of the equation (48) is
\[\Omega(\xi) = -\frac{1}{\alpha_3(\xi + \xi_0)},\]
where \(\xi_0\) is an arbitrary constant of integration.

**Case (1.8):** When \(\alpha_3 = \alpha_4 = \alpha_5 = 0\), the general solution of the equation (48) is:
\[\Omega(\xi) = \xi_0,\]
where \(\xi_0\) is an arbitrary constant of integration.

**Remark (2):** The three cases (1.6), (1.7), and (1.8) are the spacial cases from cases (1.4), (1.1.3), and (1.5), respectively.

Every solution above gives the solution of the Einstein field equation. Therefore, we will study some of these solutions in the following:

**Solution (1.1.1):** We consider the solution corresponding to the case (1.1.1). Without loss of generality, we can take, in this case, \(d = -c\). Now, by using (49), (47), (44), and (40),
we can find the solution as
\[
\begin{align*}
A(x, t) &= q_1 \exp \left[ c_7 (x + n_1 (x + t)) \right] \times \cos^{2n} n_1 [K_1 (x + t) + \xi_0], \\
B(x, t) &= q_2 \exp \left[ \frac{c}{2} (t - x + n_2 (x + t)) \right] \times \cos^{n_2} n_1 [K_1 (x + t) + \xi_0], \\
C(x, t) &= q_3 \exp \left[ \frac{c}{2} (t - x + n_2 (x + t)) \right] \times \cos^{n_2} [K_1 (x + t) + \xi_0],
\end{align*}
\]
(60)

where \( n = \frac{c^2 + c_7^2 + 4 K_1^2}{4c^2} \), \( n_1 = -\frac{c^2}{c^2 + 4 K_1^2} \), and \( n_2 = -\frac{c c_7}{c^2 + 4 K_1^2} \). We observe from equation (60) that the line element (1) can be written in the following form:
\[
\begin{align*}
ds_{11}^2 &= q_1^2 \exp \left[ 2c_7 (x + n_1 (x + t)) \right] \times \cos^{2n} n_1 [\xi_1 (dx^2 - dt^2)] \\
&+ \exp \left[ c (x - t + n_2 (x + t)) \right] \times \cos^{2n} [\xi_1 (q_2^2 dy^2 + q_3^2 \exp [2c (t - x)]) dz^2],
\end{align*}
\]
(61)

where \( \xi = K_1 (x + t) + \xi_0 \) while \( q_1, q_2, q_3, c, c_7, K_1 \) and \( \xi_0 \) are arbitrary constants.

**Solution (1.1.2):** We consider the solution corresponding to the case (1.1.2). Without loss of generality, we can take, in this case, \( d = -c \). We observe from equation (64) that the line element (1) can be written in the following form:
\[
\begin{align*}
ds_{11}^2 &= q_1^2 (x + t)^{-4} d_1^2 \exp \left[ 4 d_1 d_2 (n t + n_1 x) \left( dx^2 - dt^2 \right) \right] \\
&+ (x + t)^{-2} \exp \left[ 2 (n t + n_1 x) \right] \exp \left[ 2c (t - x) \right] dz^2, \\
&+ q_3^2 \exp \left[ 2 (n t + n_1 x) \right] dz^2.
\end{align*}
\]
(65)

where \( q_1, q_2, q_3, d_1, \) and \( d_2 \) are arbitrary constants.

**Solution (1.2):** We consider the solution corresponding to the case (1.2). Without loss of generality, we can take, in this case, \( d = -c \), \( c = 2 n c \), \( n = -\frac{5}{2} \), and \( a = -5 d_1 \). We observe from equation (64) that the line element (1) can be written in the following form:
\[
\begin{align*}
ds_{12}^2 &= q_1^2 \left( a_1 - e^{\frac{3 a c}{5}} \right)^2 \exp \left[ e^{-\frac{2c (t + 3x)}{5}} \right] \left( dx^2 - dt^2 \right) \\
&+ \left[ a_1 - e^{\frac{3 a c}{5}} \right] \left[ q_2^2 e^{-2 c (t + 3x)} dy^2 \right. \\
&+ q_3^2 e^{-2 c (t + 3x)} dz^2.
\end{align*}
\]
(66)

where \( q_1, q_2, q_3, c, \) and \( a_1 \) are arbitrary constants.

**Solution (1.4):** We consider the solution corresponding to the case (1.4). Without loss of generality, we can take, in this case, \( d = -c \), \( c_7 = b_2 a \), \( c = 2 b_1 \), and \( n = -\frac{1}{4} \). We observe from equation (64) that the line element (1) can be written in the following form:
\[
\begin{align*}
ds_{12}^2 &= q_1^2 \left[ a_1 - e^{\frac{3 a c}{5}} \right]^2 \exp \left[ e^{-\frac{2c (t + 3x)}{5}} \right] \left( dx^2 - dt^2 \right) \\
&+ \left[ a_1 - e^{\frac{3 a c}{5}} \right] \left[ q_2^2 e^{-2 c (t + 3x)} dy^2 \right. \\
&+ q_3^2 e^{-2 c (t + 3x)} dz^2.
\end{align*}
\]
(67)

where \( q_1, q_2, q_3, c, \) and \( a_1 \) are arbitrary constants.
using (55), (47), (44), and (40), we can find the solution as
\[
\begin{align*}
A(x, t) &= q_1 \exp \left\{ \frac{1}{2b_2} \left[ 2b_x^2 x - b_x^2 (x + t) + e^{b_2(x+t)} \right] \right\}, \\
B(x, t) &= q_2 \exp \left\{ \frac{1}{2b_2} \left[ b_1 \left( 2b_x x - (b_1 + b_2)(x + t) \right) + e^{b_2(x+t)} \right] \right\}, \\
C(x, t) &= q_3 \exp \left\{ \frac{1}{2b_2} \left[ b_1 \left( b_2 (t-x) - b_1 (x+t) \right) + e^{b_2(x+t)} \right] \right\}. 
\end{align*}
\] (68)

We observe from equation (68) that the line element (1) can be written in the following form:
\[
ds_{13}^2 = A^2 \left( dx^2 - dt^2 \right) + B^2 dy^2 + C^2 dz^2, \tag{69}
\]
where \( q_1, q_2, q_3, b_1, \) and \( b_2 \) are arbitrary constants.

**Solution (1.5):** We consider the solution corresponding to the case (1.5). Without loss of generality, we can take, in this case, \( d = -c, c_1 = 0, c = 2 \sqrt{2} a_3, \) and \( n = \frac{1}{4}. \) Now, by using (55), (47), (44), and (40), we can find the solution as
\[
\begin{align*}
A(x, t) &= q_1 \exp \left\{ \frac{1}{2b_2} \left[ \left( \frac{x + t}{2} \right) \left( a_2 + \sqrt{2} a_3 + a_3^2 (x + t) \right) \right] \right\}, \\
B(x, t) &= q_2 \exp \left\{ \frac{1}{2b_2} \left[ a_2 (x + t) + a_3^2 (x + t)^2 + 2 \sqrt{2} a_3 x \right] \right\}, \\
C(x, t) &= q_3 \exp \left\{ \frac{1}{2b_2} \left[ a_2 (x + t) + a_3^2 (x + t)^2 + 2 \sqrt{2} a_3 t \right] \right\}. 
\end{align*}
\] (70)

We observe from equation (70) that the line element (1) can be written in the following form:
\[
ds_{15}^2 = q_1^2 \exp \left\{ \left( x + t \right) \left( a_2 + \sqrt{2} a_3 + a_3^2 (x + t) \right) \right\} \left( dx^2 - dt^2 \right) \\
+ \exp \left\{ 2 \left( x + t \right) \left( a_2 + a_3^2 (x + t) \right) \right\} \left( dy^2 + q_3^2 e^{\sqrt{2} a_3} dz^2 \right). \tag{71}
\]

where \( q_1, q_2, q_3, a_2, \) and \( a_3 \) are arbitrary constants.

**Case (2):** When \( c_1 \neq 0, \) the similarity variable and similarity functions can be written as
\[
\tilde{\xi} = \frac{x + a}{t + b}, \quad B(x, t) = (x + a)^c \Psi(\tilde{\xi}), \\
C(x, t) = (x + a)^d \Phi(\tilde{\xi}), \tag{72}
\]
where \( a = \frac{c_2}{c_1}, \quad b = \frac{c_3}{c_1}, \quad c = \frac{c_4}{c_1}, \quad d = \frac{c_5}{c_1}, \) are arbitrary constants. Substituting the transformations (72) in the field equations (20)–(21) leads to the following system of ordinary differential equations:
\[
\begin{align*}
\xi \Psi^\prime + \left[ 1 + c - c_0 - 2n(c + d) \right] \Psi^\prime \\
+ \frac{\xi \Phi^\prime}{\phi} + \left[ 1 + c - c_0 - 2n(c + d) \right] \Phi^\prime \\
= 2n \left( \frac{\Psi^\prime}{\phi} + \frac{\Phi^\prime}{\phi} \right)^2, \tag{73}
\end{align*}
\]
\[
\begin{align*}
\Psi^\prime + \frac{\phi^\prime}{\phi} + \frac{2}{(2n + 1)} \xi \Psi^\prime \\
+ \frac{2}{(2n + 1)} \xi \Phi^\prime \\
- \frac{2n}{2n + 1} \left( \frac{\Psi^\prime}{\phi} + \frac{\Phi^\prime}{\phi} \right) \\
= c(c - 1) + d(d - 1) - 2c_0 - 2n(c + d) \left( \frac{\xi^2 - 1}{\xi^2} \right), \tag{74}
\end{align*}
\]
where \( f(x) = k(x + a)^c, k = c_3 c_1^c. \)

If one solves the system of second-order, non-linear ordinary differential equations (73)–(74), one can obtain the exact solutions of the original Einstein field equations (20)–(21), corresponding to reduction (72). The system (73)–(74) is very difficult to solve in general form. This system may be solved in some special cases in future work.

5. **Physical and geometrical properties of some models**

**For the model (61):**

The expressions for energy density, \( \rho, \) the string tension density, \( \lambda, \) magnetic permeability, \( \mu, \) and the particle density, \( \rho_p, \) for the model (61) are given by
\[
\rho(x, t) = -\lambda(x, t) = n_1 \exp \left\{ 2c_7(n(x + t) - x) \right\} \times \sec^{n_1}[\xi] \left( n_3 \cos[\xi] - n_4 \sin[\xi] \right), \tag{75}
\]
\[
\bar{\mu}(x, t) = \frac{\chi F_{12}^2(x) \exp \left\{ c(t - x - n_1(x + t)) \right\}}{2c n_1 q_2^2 \cos^{n_1}[\xi] \left( 2K_1 \tan[\xi] - c_7 \right)}, \tag{76}
\]
\[
\rho_p(x, t) = n_5 \exp \left\{ 2c_7(n(x + t) - x) \right\} \cos^n[\xi], \tag{77}
\]
where \( n_3 = c_7(c + c_7) + 4K_1^2, \quad n_4 = \frac{2c K_1}{\chi q_1^2}, \quad n_5 = \frac{2c^2}{q_1^2}, \quad \xi = K_1(x + t) + \xi_0, \) and \( F_{12}(x) \) is an arbitrary function of the variable, \( x. \)
The deceleration parameter is given by \[ q = \frac{1}{2} \frac{\partial^2 \Theta}{\partial u^2} + \frac{1}{3} \Theta^2 \]
\[ = -\frac{5}{256} q_1^2 \exp \left[ 4 c_7 (n(x + t) - x) \right] \times (c_7 \cos [\xi] - 2 K_1 \sin [\xi]) \times (c_7^2 - c_7^2 + 20 K_1^2 - (n_1 + 1)) \times \left[ (c_7^2 - 4 K_1^2) \cos [\xi] - 4 c_7 K_1 \sin [\xi] \right]. \] (85)

**Remark (3):** It is worth noting that if we put the transformation \[ K_1 \rightarrow i K_2, \sin \rightarrow \sinh, \cos \rightarrow \cosh, \]
in the model (61), we have obtained the model (63), where \( i = \sqrt{-1} \). Therefore, we can find the physical properties of the model corresponding to case (1.1.2) by putting the above transformation in the model corresponding to case (1.1.1).

For the model (65):

The expressions for energy density, \( \rho \), the string tension density, \( \lambda \), magnetic permeability, \( \mu \), and the particle density, \( \rho_p \), for the model (65) are given by

\[ \rho(x, t) = -\lambda(x, t) = \frac{d_1}{\chi q_1} \exp \left[ -4 d_1 d_2 (n t + n_1 x) \right], \]
\[ \lambda(x, t) = \frac{8 d_1^2}{d_2^2} \exp \left[ -4 d_1 d_2 (n t + n_1 x) \right], \]
\[ \mu(x, t) = \frac{8 d_1^2}{d_2^2} \exp \left[ -4 d_1 d_2 (n t + n_1 x) \right], \]
\[ \rho_p(x, t) = \frac{8 d_1^2}{d_2^2} \exp \left[ -4 d_1 d_2 (n t + n_1 x) \right]. \]

The volume element is
\[ V = q_1^2 q_2 q_3 (x + t)^{1-3} d_3 \]
\[ \times \exp \left[ d_1 \left( 1 + 3 d_2^2 \right) t + 3 \left( d_2^2 - 1 \right) x \right]. \] (89)

The expansion scalar, which determines the volume behavior of the fluid, is given by
\[ \Theta = \left( 5 d_2^2 + 1 \right) \left( d_1 (x + t) - d_2 \right)^{d_2^2 - 1} \]
\[ \times \exp \left[ -2 d_1 d_2 (n t + n_1 x) \right]. \] (90)

The nonvanishing components of the shear tensor, \( \sigma'_i \), are
\[ \sigma'_1 = \frac{2(1 - d_2^2)}{3 \left( 1 + 5 d_2^2 \right)}, \]
\[ \sigma'_2 = \frac{d_2 (1 + d_2) + d_1 (1 - 6 d_2 - d_2^2) (x + t)}{3 \left( 1 + 5 d_2^2 \right) \left[ d_2 - d_1 (x + t) \right]} \Theta, \]
\[ \sigma'_3 = -\left( \sigma'_1 + \sigma'_2 \right). \] (93)

The shear scalar is:
\[ \sigma^2 = \frac{\sigma^2}{3 \left( 1 + 5 d_2^2 \right)^2} \left[ 1 + d_2^2 \left( 10 + d_2^2 \right) \right] \]
\[ + \frac{12 d_1^2 \left( 2 d_1 (x + t) - 2 d_2 + 1 \right)}{d_1 (x + t) - d_2^2}. \] (94)
The nonvanishing components of the shear tensor, $\sigma_i$, are

$$\sigma_1 = \frac{\Theta}{6},$$

$$\sigma_2 = \frac{\Theta}{6} \left[ 5 a_1 - 2 e^{\frac{\Theta}{6}} + e^{\frac{2 \Theta}{6}} \right]^{-1},$$

$$\sigma_3 = -\left( \sigma_1 + \sigma_2 \right).$$

The shear scalar is:

$$\sigma^2 = \frac{\Theta^2}{36} \left[ 25 a_1^2 - 5 a_1 e^{\frac{2 \Theta}{6}} + 7 e^{\frac{2 \Theta}{6}} \right] \left( 5 a_1 + e^{\frac{2 \Theta}{6}} \right)^2.$$
\[ \sigma_2^2 = \frac{\theta}{15} \left( b_1^2 + 6 b_1 b_2 - b_2^2 e^{b_2(x+t)} \right), \]  
(114)

\[ \sigma_3^3 = -\left( \sigma_1^1 + \sigma_2^2 \right). \]  
(115)

The shear scalar is

\[ \sigma_2^2 = \frac{\theta^2}{25} \left[ \frac{1}{3} + 4 b_1^2 b_2^2 \left( b_1^2 - b_2 e^{b_2(x+t)} \right)^2 \right]. \]  
(116)

The acceleration vector is given by

\[ \frac{1}{2d_1} \left[ 2 b_2^2 - b_1^2 + b_2 e^{b_2(x+t)} (1, 0, 0). \right] \]  
(117)

The deceleration parameter is given by

\[ q = -\frac{5 \theta^2}{2 b_2^2 q_1} \left[ b_1^2 + b_2 \left( 3 b_2^2 - 2 b_1^2 \right) e^{b_2(x+t)} + 2 b_2^2 e^{b_2(x+t)} \right] \]  
\[ \times \exp \left[ -\frac{1}{b_2^2} \left( 2 b_2^2 - b_1^2 (x + t) + e^{b_2(x+t)} \right) \right]. \]  
(118)

For the model (71):

The expressions for energy density, \( \rho \), the string tension density, \( \lambda \), magnetic permeability, \( \mu \), and the particle density, \( \rho_p \), for the model (71) are given by

\[ \rho(x, t) = -\chi(x, t) = \frac{4 \sqrt{2}}{x q_1} \left[ a_2^2 + 2 a_3^2 (x + t) \right] \]  
\[ \times \exp \left[ -(x + t) \left( a_2 + \sqrt{2} a_3 + a_3^2 (x + t) \right) \right]. \]  
(119)

\[ \bar{\rho}(x, t) = -\chi F_{\|}(x) \left[ 2 \sqrt{2} a_3 a_2 + 8 a_3^2 + \frac{4 a_3^2 (x + t)^2}{\sqrt{2} a_3 + a_3^2 (x + t)} \right] \]  
\[ \times \exp \left[ -2 \left( 2 \sqrt{2} a_3 + a_3^2 (x + t) + a_3^2 (x + t) \right) \right]. \]  
(120)

\[ \rho_p(x, t) = \frac{16 a_3^2}{q_1^2} \exp \left[ -(x + t) \left( \sqrt{2} a_3 + a_3^2 (x + t) \right) \right], \]  
(121)

where \( F_{\|}(x) \) is an arbitrary function of the variable, \( x \).

The volume element is

\[ V = q_1^2 q_2 q_3 \exp \left[ 3(x + t) \left( \sqrt{2} a_3 + a_2 + a_3^2 (x + t) \right) \right]. \]  
(122)

The expansion scalar, which determines the volume behavior of the fluid, is given by

\[ \Theta = \frac{5}{2 q_1} \left[ \sqrt{2} a_3 + a_2 + 2 a_3^2 (x + t) \right] \]  
\[ \times \exp \left[ -\left( \frac{x + t}{2} \right) \left( \sqrt{2} a_3 + a_2 + a_3^2 (x + t) \right) \right]. \]  
(123)

The nonvanishing components of the shear tensor, \( \sigma_i^j \), are

\[ \sigma_1^1 = -\frac{2 \Theta}{15}. \]  
(124)

\[ \sigma_2^2 = \frac{\Theta}{15} \left[ a_2 - 5 \sqrt{2} a_3 + 2 a_3^2 (x + t) \right]. \]  
(125)

\[ \sigma_3^3 = -\left( \sigma_1^1 + \sigma_2^2 \right). \]  
(126)

The shear scalar is

\[ \sigma_2^2 = \frac{\theta^2}{25} \left[ \frac{1}{3} + 8 a_3^2 + 2 a_3^2 (x + t) \right]. \]  
(127)

The acceleration vector is given by

\[ \ddot{u}_i = \left[ \frac{a_3^2}{\sqrt{2}} + \frac{a_3^2}{2} + a_3^2 (x + t) \right] (1, 0, 0). \]  
(128)

The deceleration parameter is given by

\[ q = -\frac{5 \theta^2}{2 q_1^2} \left[ 2 \sqrt{2} a_3 a_2 + 8 a_3^2 + 4 a_3^2 (x + t)^2 \right] \]  
\[ \times \exp \left[ -(x + t) \left( a_2 + \sqrt{2} a_3 + a_3^2 (x + t) \right) \right] \]  
(129)

6. Conclusion

In this paper, we have derived some new invariant solutions of Einstein–Maxwell’s field equations for string fluid as a source of matter in cylindrically symmetric space-time with variable magnetic permeability. Different sets of solutions are found using different values of the parameters. Note that the cosmological solutions are physically viable for the following reasons:

(i) The energy density is positive and decreasing with an increase in time.

(ii) The volume of the Universe is increasing due the expanding nature of the Universe.

(iii) The deceleration parameter should be negative, as recent observations indicate that our Universe is accelerating.

(iv) \( \frac{\dot{a}}{a} \) will be vanished at large time as the Universe may get isotropized in some later time.

(v) Solutions must be non singular as the existence of Big-Bang singularity is one of the basic failures of the general theory of relativity.

The models (61), (65), (67), and (69) do not meet criterion (i) above. Here, in the models (61) and (65), densities increase with time, whereas in model (67), particle density is
increasing, and in model (69) particle density is negative (figure 1). Therefore, these models are not physically interesting. On the other hand, the model (71) is very much acceptable as it describes a more or less observable Universe. Note that our procedure of solving the field equations (symmetry analysis method) is completely different than the usual methods available in the literature. The derived model starts expanding without Big Bang singularity (figure 2). Also, from a theoretical perspective, the present model can be used to explain the acceleration of the Universe. In other words, the solution presented here has the potential to describe the observed Universe. In our model, it seems that a magnetic field with negative magnetic permeability (figure 3) is responsible for providing an accelerated as well as singularity-free Universe. Nowadays, negative magnetic permeability is not an impossible event, rather, it can be found in a split ring resonator in the visible light region [20]. We also note that $\sigma_{\theta}$ will be vanished at large time (figure 2). This means that the Universe may get isotropized at some later time. Again, one can assume that negative magnetic permeability is responsible for the isotropization. A detailed discussion of all basic cosmological constraints is beyond the scope of the present paper. We hope it will be researched in a future project.

Figure 1. (Left) The variation of energy density with respect to time for the model (61). (First middle) The variation of energy density with respect to time for the model (65). (Second middle) The variation of particle density with respect to time for model (67). (Right) The variation of particle density with respect to time for model (69).

Figure 2. (Left) The variation of the volume of the Universe with respect to time. (Middle) The variation of $\frac{\sigma}{\theta}$ with respect to time. (Right) The deceleration parameter with respect to time.

Figure 3. (Left) The variation of energy density with respect to time. (Middle) The variation of magnetic permeability with respect to time. (Right) The variation of particle density with respect to time.
References

[1] Barrow J D and Kunze K E 1997 Phys. Rev. D 56 741
[2] Barrow J D and Kunze K E 1998 Phys. Rev. D 57 2255
[3] Pradhan A, Yadav A K, Singh R P and Singh V K 2007 Astrophys. Space Science 312 145
[4] Yadav A K, Yadav V K and Yadav L 2009 Int. J. Theor. Phys. 48 568
[5] Attallah S K, El-Sabbagh M F and Ali A T 2007 Commun. Nonlinear Sci. Numer. Simulat. 12 1153
[6] Ali A T 2009 Phys. Scr. 79 035006
[7] Mekheimer K S, Husseny S Z, Ali A T and Abo-Elkhair R E 2011 Phys Scr 83 015017
[8] Ali A T 2013 Phys. Scr. 87 015002
[9] Ali A T, Yadav A K and Mahmoud S R 2014 Astrophys Space Science 349 539
[10] Ovsyannikov L V 1982 Group Analysis of Differential Equations (NY: Academic Press)
[11] Bluman G W and Kumei S 1989 Symmetries and Differential Equations in Applied Sciences (New York: Springer)
[12] Ibragimov N H 1989 Transformation Groups Applied to Mathematical Physics (Dordrecht: Reidel)
[13] Olver P J 1993 Application of Lie Groups to differential equations in graduate texts in mathematics vol 107 2nd edition (New York: Springer)
[14] Melvin M A and Ann N Y 1975 Acad Sci 262 253
[15] Lichnerowicz A 1967 Relativistic Hydrodynamics and Magnetohydrodynamics (New-York: W A Benjamin Inc.) p 93
[16] Synge J L 1960 Relativity: The General Theory (Amesterdam: North-Holland Publ. Co.) p 356
[17] Pradhan A and Mathur P 2009 Fizika B 18 243
[18] Feinstein A and Ibanez J 1993 Class Quantum Grav. 10 L227
[19] Raychaudhuri A K 1979 Theoritical Cosmology (Oxford: Clarendon) p 80
[20] Ishikawa A and Tanaka T 2006 Opt. Commun. 258 300