Supplementary Information for

Multidimensional Political Apportionment

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Supporting Information Text

In the \(d\)-dimensional apportionment problem, the input is given by a tuple \((\mathcal{N}, \mathcal{V}, m^-, m^+, H)\) described as follows. For each \(\ell \in \{1, \ldots, d\}\) we have a set \(N_\ell\) such that \(\mathcal{N} = \{N_1, \ldots, N_d\}\) and \(\mathcal{V}\) is a vector with nonnegative integer entries in the Cartesian product \(\prod_{\ell=1}^{d} N_\ell\). For each \(\ell \in \{1, \ldots, d\}\) and each \(v \in N_\ell\) there are integer values \(m^-_v\) and \(m^+_v > 0\) called lower and upper marginals, respectively, and \(H\) is a strictly positive integer value such that \(\sum_{v \in N_\ell} m^-_v \leq \sum_{v \in N_\ell} m^+_v\) for every \(\ell \in \{1, \ldots, d\}\). In the context of a political election, the values \(\mathcal{V}\) represent the number of votes obtained by the tuple \(c, H\) represents the house size, and the value \(m^-\) (resp. \(m^+\)) defines a lower (resp. upper) bound for the amount of seats that each element (party, district, etc.) should get.\(^\dagger\)

For an instance \((\mathcal{N}, \mathcal{V}, m^-, m^+, H)\) we denote by \(E(\mathcal{V})\) the subset of tuples \(c \in \prod_{\ell=1}^{d} N_\ell\) such that \(\mathcal{V}_c > 0\). Given a \(d\)-dimensional instance \((\mathcal{N}, \mathcal{V}, m^-, m^+, H)\) and a signpost sequence \(s\), we say that \(x \in \mathbb{N}^{E(\mathcal{V})}\) is a \(d\)-dimensional proportional apportionment if there exists a strictly positive value \(\mu\), and for every \(\ell \in \{1, \ldots, d\}\) and every \(v \in N_\ell\) there exists a strictly positive value \(\lambda_v\), called multiplier, such that the following holds:

\[
m^-_v \leq \sum_{e \in E(\mathcal{V}), e_\ell = v} x_e \leq m^+_v \quad \text{for every } \ell \in \{1, \ldots, d\} \text{ and every } v \in N_\ell,\]

\[
\sum_{e \in E(\mathcal{V})} x_e = H,\]

\[
s(x_e) \leq \mathcal{V}_c \cdot \mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell} \leq s(x_e + 1) \quad \text{for every } e \in E(\mathcal{V}),\]

and furthermore, we have the following conditions regarding the values of the multipliers for every \(\ell \in \{1, \ldots, d\}\) and every \(v \in N_\ell\),

\[
\text{If } \lambda_v > 1, \text{ then we have } \sum_{e \in E(\mathcal{V}), e_\ell = v} x_e = m^-_v,\]

\[
\text{If } \lambda_v < 1, \text{ then we have } \sum_{e \in E(\mathcal{V}), e_\ell = v} x_e = m^+_v.\]

We denote by \(A_\mu(\mathcal{N}, \mathcal{V}, m^-, m^+, H)\) the set of triplets \((x, \mu, \lambda)\) where \(x\) is integral, \(\mu\) is strictly positive, \(\lambda\) is strictly positive in every entry and the triplet satisfies conditions (1)-(5). Note that for the case \(s(1) = 0\), any \(x\) satisfying (3) must be strictly positive in every entry. Observe that given the strict positivity of the values \(\mu\) and \(\lambda_v\), condition (3) is equivalent to \([\mathcal{V}_c \cdot \mu \cdot \prod_{\ell=1}^{d} \lambda_{e_\ell}]_{s} = x_e\), thus it captures the idea of proportionality. We remark that for \(d = 2\) this corresponds to the proportionality notion of Balinski and Demange, in the sense that their definition of a biproportional apportionment is equivalent to our definition of a 2-dimensional proportional apportionment.

Our integer linear program to study the \(d\)-dimensional apportionment problem is constructed as follows. Consider a \(d\)-dimensional instance \((\mathcal{N}, \mathcal{V}, m^-, m^+, H)\) and a signpost sequence \(s\). For each \(e \in E(\mathcal{V})\) and each \(t \in \{1, \ldots, H\}\) we have a binary variable \(y_{e}^{t}\) and its cost in the objective function is given by \(\log(s(t)/\mathcal{V}_e)\) if \(s(t) > 0\) and zero otherwise.

\[
\begin{align*}
\text{minimize} & \quad \sum_{e \in E(\mathcal{V})} \sum_{t \in \{1, \ldots, H\}} y_{e}^{t} \log \left( \frac{s(t)}{\mathcal{V}_e} \right) \\
\text{subject to} & \quad \sum_{t=1}^{H} y_{e}^{t} = x_e \quad \text{for every } e \in E(\mathcal{V}), \\
& \quad \sum_{e \in E(\mathcal{V})} x_e = H, \\
& \quad \sum_{e \in E(\mathcal{V}), e_\ell = v} x_e \geq m^-_v \quad \text{for every } \ell \in \{1, \ldots, d\} \text{ and every } v \in N_\ell, \\
& \quad \sum_{e \in E(\mathcal{V}), e_\ell = v} x_e \leq m^+_v \quad \text{for every } \ell \in \{1, \ldots, d\} \text{ and every } v \in N_\ell, \\
& \quad y_{e}^{t} \geq \lfloor 1 - s(1) \rfloor \quad \text{for every } e \in E(\mathcal{V}), \\
& \quad y_{e}^{t} \in \{0, 1\} \quad \text{for every } e \in E(\mathcal{V}) \text{ and every } t \in \{1, \ldots, H\}. 
\end{align*}
\]

\(\dagger\) Observe that the definition of the method requires that the votes convey preferences with respect to all dimensions. For instance, it can be directly applied in a standard open list system, whereas closed lists would only be possible for subsets of candidates belonging to the same element in every dimension.

\(\dagger\) Note that, in the two-dimensional example above, we consider single values and not lower and upper bounds for the marginals. This is captured in this general model by simply setting \(m^- = m^+\).
The variable $x_e$ represents the total number of seats to be allocated in the apportionment for the tuple $e$ and constraint (7) takes care of aggregating the seats in these variables. Constraint (8) ensures to respect the house size and constraints (9) and (10) enforces every feasible solution to satisfy the marginals. Finally, constraint (11) ensures $x_e \geq 1$ if $s(1) = 0$. We remark that this integer linear program can be equivalently written by omitting the variables $y$ at the price of having a nonlinear convex objective.

1. Proofs of Section 2

Theorem 1. Let $(N, V, m^-, m^+, H)$ be an instance of the d-dimensional apportionment problem, let $s$ be a signpost sequence and let $x \in \mathbb{N}^{|E(V)|}$. Then, there exist $\mu$ and $\lambda$ such that $(x, \mu, \lambda) \in A_w(N, V, m^-, m^+, H)$ if and only if there exists $y$ such that $(x, y)$ is an optimal solution for the linear relaxation of (6)-(12).

A. Proof of Theorem 1. In order to prove Theorem 1, we first observe that by duality, for any feasible solution $(x, y)$ of the linear relaxation of (6)-(12) we have that $(x, y)$ is optimal if and only if there exists a dual solution $(U, \Lambda^-, \Lambda^+, \beta)$ such that the following conditions hold for every $e \in E(V)$ and every $t \in \{1, \ldots, H\}$:

\[
U + \sum_{t=1}^d (\Lambda_{-e}^t + \Lambda_{+e}^t) + \beta_e^t \leq \log \left( \frac{s(t)}{V_e} \right), \quad \text{if } s(t) > 0, \tag{13}
\]

\[
U + \sum_{t=1}^d (\Lambda_{-e}^t + \Lambda_{+e}^t) + \beta_e^t \leq 0, \quad \text{if } s(t) = 0, \tag{14}
\]

\[
y_e^t \left[ U + \sum_{t=1}^d (\Lambda_{-e}^t + \Lambda_{+e}^t) + \beta_e^t \right] = 0, \quad \text{if } s(t) > 0, \tag{15}
\]

\[
y_e^t \left[ U + \sum_{t=1}^d (\Lambda_{-e}^t + \Lambda_{+e}^t) + \beta_e^t \right] = 0, \quad \text{if } s(t) = 0, \tag{16}
\]

\[
\beta_e^t (y_e^t - 1) = 0, \quad \beta_e^t \leq 0, \quad \text{if } s(t) > 0, \tag{17}
\]

and such that the following conditions hold for every $t \in \{1, \ldots, d\}$ and every $v \in N_t$:

\[
\Lambda_{-e}^t \left( \sum_{e \in E(V) \setminus \{e\}} \sum_{t=1}^H \gamma_e^t - m_v^e \right) = 0, \tag{19}
\]

\[
\Lambda_{+e}^t \left( \sum_{e \in E(V) \setminus \{e\}} \sum_{t=1}^H \gamma_e^t - m_v^e + 1 \right) = 0, \tag{20}
\]

where $U$ is the dual variable associated to the constraint (8), $\Lambda_{-e}^t$ is the dual variable associated to the constraint (9) for every $t \in \{1, \ldots, d\}$ and every $v \in N_t$, and $\beta_e^t$ is the dual variable associated to the upper bound of one on the value of $y_e^t$ for every $e \in E(V)$ and every $t \in \{1, \ldots, H\}$. We refer to a tuple $(x, y, U, \Lambda^-, \Lambda^+, \beta)$ satisfying (13)-(21), with $(x, y)$ feasible for the linear relaxation of (6)-(12), as an optimal primal-dual pair for this linear relaxation. The following lemma states two important properties about the structure of such optimal pair.

Lemma 1. Let $(N, V, m^-, m^+, H)$ be an instance of d-dimensional apportionment problem, let $s$ be a signpost sequence and let $(x, y, U, \Lambda^-, \Lambda^+, \beta)$ be an optimal primal-dual pair for the linear relaxation of (6)-(12). For every $e \in E(V)$, define the value $t_e = \max \{ t \in \{1, \ldots, H\} : y_e^t > 0 \}$ when $x_e > 0$ and $t_e = 0$ when $x_e = 0$. Then, the following holds:

(i) For every $e \in E(V)$, we have that $t_e = \lceil x_e \rceil$. In particular, $y_e^t = 1$ for every $t < t_e$ when $x_e > 1$.

(ii) For every $e \in E(V)$, we have that $s(t_e) \leq V_e \cdot \exp(U) \cdot \prod_{t=1}^H \exp(\Lambda_{-e}^t + \Lambda_{+e}^t) \leq s(t_e + 1)$.

Proof. When $x_e = 0$, part (i) follows directly, so in the following we suppose $x_e > 0$. Clearly, for every $e \in E(V)$ we have that $t_e \geq \lceil x_e \rceil$, otherwise constraint (7) would be violated. Suppose that $e \in E(V)$ is such that $t_e > \lceil x_e \rceil$. Then, there exists $t > \lceil x_e \rceil$ such that $y_e^t$ is strictly positive, and consider the solution $(\bar{x}, \bar{y})$ obtained as follows: We define $\bar{y}_{-e}^t = 1$ for every $t < \lceil x_e \rceil + 1$, $\bar{y}_{-e}^t = x_e - \lceil x_e \rceil$ for $p = \lceil x_e \rceil + 1$ and $\bar{y}_{e}^t = 0$ otherwise; and $\bar{x}_e = \sum_{t=1}^H \bar{y}_{e}^t$ for every $e \in E(V)$. The solution $(\bar{x}, \bar{y})$ is feasible for the linear relaxation of (6)-(12) since $\bar{x}_e = \sum_{t=1}^H \bar{y}_{e}^t = (\lceil x_e \rceil + 1) - 1 + x_e - \lceil x_e \rceil = x_e$. Since the signpost sequence is strictly increasing in $\{1, \ldots, H\}$, for every $e \in E(V)$ the function $\log(s(t)/V_e)$ is strictly increasing as a function.
of $t \in \{1, \ldots, H\}$. Therefore, $\sum_{t=1}^{H} y_{c}^{t} \log(s(t)/\nu_{c}) < \sum_{t=1}^{H} y_{c}^{t} \log(s(t)/\nu_{c})$, but this contradicts the optimality of $(x, y)$. That concludes the proof for (i).

Now we prove (ii). We know that for every $t > t_{c}$ it holds $y_{c}^{t} = 0$, so complementary slackness condition (17) implies that $\beta_{c}^{t} = 0$. Note that $t > t_{c}$ guarantees $s(t) > 0$, and therefore condition (13) implies that $U + \sum_{t=1}^{d} (\Lambda_{c}^{t} + \Lambda_{c}^{+}) \leq \log(s(t)/\nu_{c})$, which is verified in particular for $t = t_{c} + 1$. Therefore, $\nu_{c} \cdot \exp(U) \cdot \prod_{t=1}^{t_{c}+1} \exp(\Lambda_{c}^{t} + \Lambda_{c}^{+}) \leq s(t_{c}+1)$. On the other hand, when $t = t_{c}$ with $s(t_{c}) > 0$, the complementary slackness condition (15) implies that $U + \sum_{t=1}^{d} (\Lambda_{c}^{t} + \Lambda_{c}^{+}) + \beta_{c}^{t} - \log(s(t_{c})/\nu_{c}) = 0$, since $y_{c}^{t_{c}} > 0$. By (18) we have that $\beta_{c}^{t_{c}} \leq 0$ and therefore we conclude that $U + \sum_{t=1}^{d} (\Lambda_{c}^{t} + \Lambda_{c}^{+}) \geq \log(s(t_{c})/\nu_{c})$, implying $s(t_{c}) \leq \nu_{c} \cdot \exp(U) \cdot \prod_{t=1}^{t_{c}} \exp(\Lambda_{c}^{t} + \Lambda_{c}^{+})$. The case $s(t_{c}) = 0$ (which can only happen for $t_{c} \in (0, 1]$) is straightforward, since by definition $\nu_{c} > 0$ for every $e \in E(V)$. That concludes (ii).

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** Let $(x, \mu, \lambda) \in A_{\nu}(N, V, m^{-}, m^{+}, H)$, and recall that $x$ is integral and $\lambda, \mu$ are strictly positive. To prove that there exists a binary vector $y$ such that $(x, y)$ is an optimal solution of (6)-(12) we use duality, and more specifically, we define a dual solution $(U, \Lambda^{-}, \Lambda^{+}, \beta)$ such that $(x, y, U, \Lambda^{-}, \Lambda^{+}, \beta)$ is an optimal primal-dual pair for the linear relaxation of (6)-(12).

Let $U = \log(\mu)$ and for every $t \in \{1, \ldots, d\}$ and every $v \in N_{t}$ we define $\Lambda_{c}^{t}$, $\Lambda_{c}^{+}$ as follows. We let $\Lambda_{c}^{t} = \log(\lambda_{c})$ and $\Lambda_{c}^{+} = 0$ in the following cases: (a) if $\sum_{e \in E(V), e = v_{c}} x_{e} = m_{v} = m_{v}^{+}$ and $\lambda_{c} \geq 1$, or (b) if $\sum_{e \in E(V), e = v_{c}} x_{e} = m_{v}^{+} \neq m_{v}^{+}$. We let $\Lambda_{c}^{t} = 0$ and $\Lambda_{c}^{+} = \log(\lambda_{c})$ in the following cases: (a) if $\sum_{e \in E(V), e = v_{c}} x_{e} = m_{v} = m_{v}^{+}$ and $\lambda_{c} < 1$, or (b) if $\sum_{e \in E(V), e = v_{c}} x_{e} = m_{v}^{+} \neq m_{v}^{+}$.

In any other case, we have $\sum_{e \in E(V), e = v_{c}} x_{e} \notin \{m_{v}^{+}, m_{v}^{+}\}$, so conditions (4)-(5) imply that $\lambda_{c} = 1$, and hence we define $\Lambda_{c}^{t} = \Lambda_{c}^{+} = \log(\lambda_{c}) = 0$. We define $\Lambda_{c}^{-}$ and $\Lambda_{c}^{+}$ in a way such that for every $t \in \{1, \ldots, d\}$ and every $v \in N_{t}$ we have $\Lambda_{c}^{t} + \Lambda_{c}^{+} = \log(\lambda_{c})$. For each $e \in E(V)$ we define $\beta_{c}^{t}$ in the following way: For each $t > x_{c}$ we define $y_{c}^{t} = 0$ and $\beta_{c}^{t} = 0$, and for each $t \leq x_{c}$ we define $y_{c}^{t} = 1$ and $\beta_{c}^{t} = \log(s(t)/\nu_{c}) - U - \sum_{t=1}^{d} (\Lambda_{c}^{t} + \Lambda_{c}^{+})$.

Suppose first that the signpost sequence is such that $(s(1) > 0$. By construction, the solution is feasible for the linear relaxation of (6)-(12) and it satisfies conditions (15), (17), (19) and (20). To check condition (21), note that if $\Lambda_{c}^{t} > 0$ for some $t$, the only possibility from the definition of this variable is that $\lambda_{c} < 1$ and $1 + \sum_{e \in E(V), e = v_{c}} x_{e} = m_{v} = m_{v}^{+} \neq m_{v}^{+}$, but this cannot happen because of condition (5). For $\Lambda_{c}^{+}$ it is analogous. To check condition (18) for $t \leq x_{c}$, observe that

$$\beta_{c}^{t} = \log \left( \frac{s(t)}{\nu_{c}} \right) - U - \sum_{t=1}^{d} (\Lambda_{c}^{t} + \Lambda_{c}^{+}) \leq \log \left( \frac{s(x_{c})}{\nu_{c}} \right) - U - \sum_{t=1}^{d} (\Lambda_{c}^{t} + \Lambda_{c}^{+})$$

$$= \log \left( \frac{s(x_{c})}{\nu_{c} \cdot \mu \cdot \prod_{t=1}^{x_{c}} \lambda_{c}} \right) \leq 0,$$

where the first inequality comes from the monotonicity of the logarithm and the signpost sequence $s$, the equality comes from the definition of $\Lambda^{-}$ and $\Lambda^{+}$ and the last inequality comes from the fact that $(x, \mu, \lambda)$ satisfies (3) and therefore $s(x_{c}) \leq \nu_{c} \cdot \mu \cdot \prod_{t=1}^{x_{c}} \lambda_{c}$. To check that (13) holds, observe that since $\nu_{c} \cdot \mu \cdot \prod_{t=1}^{x_{c}} \lambda_{c} \leq s(x_{c} + 1)$ from condition (3), we have that

$$U + \sum_{t=1}^{d} (\Lambda_{c}^{t} + \Lambda_{c}^{+}) = \log \left( \mu \prod_{t=1}^{x_{c}} \lambda_{c} \right) \leq \log \left( \frac{s(x_{c} + 1)}{\nu_{c}} \right).$$

If $t \geq x_{c} + 1$, since $\beta_{c}^{t} = 0$ we conclude that

$$U + \sum_{t=1}^{d} (\Lambda_{c}^{t} + \Lambda_{c}^{+}) = \log \left( \mu \prod_{t=1}^{x_{c}} \lambda_{c} \right) \leq \log \left( \frac{s(x_{c} + 1)}{\nu_{c}} \right),$$

where the last inequality comes from the monotonicity of the logarithm and the signpost sequence $s$. Otherwise, when $t \leq x_{c}$, by the definition of $\beta_{c}^{t}$ it follows directly that $U + \sum_{t=1}^{d} (\Lambda_{c}^{t} + \Lambda_{c}^{+}) + \beta_{c}^{t} = \log(s(t)/\nu_{c})$. We conclude that $(x, y, U, \Lambda^{-}, \Lambda^{+}, \beta)$ is an optimal primal-dual pair for the linear relaxation of (6)-(12). When $s(1) = 0$, we define $y$ and $\beta$ in the same way as before except when $t = 1$, case for which we consider $\beta_{c}^{t} = -U - \sum_{t=1}^{d} (\Lambda_{c}^{t} + \Lambda_{c}^{+})$ for every $e \in E(V)$. Once again, the solution is feasible for the linear relaxation of (6)-(12) and moreover, as we know that for $t \geq 2$ we have $s(t) > 0$, the previous analysis is still valid for every $e \in E(V)$ and $t \geq 2$. For $t = 1$, we know that $y_{c}^{t} = 1$ for every $e \in E(V)$, so (17) follows directly.

Fixing $\beta_{c}^{t}$ as mentioned, we conclude that $(x, y, U, \Lambda^{-}, \Lambda^{+}, \beta)$ also verifies the remaining conditions (14) and (16). As the proof of conditions (19)-(21) for the case $s(1) > 0$ remain valid, $(x, y, U, \Lambda^{-}, \Lambda^{+}, \beta)$ is an optimal primal-dual pair for the linear relaxation of (6)-(12). By duality, we conclude that $(x, y)$ is an optimal solution for the linear relaxation of (6)-(12).

For the converse, let $(x, y)$ be an integral optimal solution for the linear relaxation of (6)-(12), with a corresponding dual optimal solution $(U, \Lambda^{-}, \Lambda^{+}, \beta)$. Let $\mu = \exp(U)$ and for every $t \in \{1, \ldots, d\}$ and every $v \in N_{t}$ let $\lambda_{c} = \exp(\Lambda_{c}^{t} + \Lambda_{c}^{+})$. We verify in what follows that $(x, \mu, \lambda)$ satisfies (1)-(5). Observe that condition (1) is directly satisfied since $(x, y)$ satisfies constraints (9)-(10) in the program. Condition (2) also follows directly from constraint (8). To check condition (4), let $t \in \{1, \ldots, d\}$ and $v \in N_{t}$ be such that $\lambda_{c} > 1$. From the definition of $\lambda_{c}$, this is equivalent to $\Lambda_{c}^{t} + \Lambda_{c}^{+} > 0$, and since $\Lambda_{c}^{+} \leq 0$ from condition (21),
Theorem 2. There exists an instance \((\mathcal{N}, \mathcal{V}, m^-, m^+, H)\) of the 3-dimensional apportionment problem, with \(\mathcal{V}\) strictly positive in each of its entries, such that for every stationary signpost sequence \(s\) we have \(A_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H) = \emptyset\).

B. Proof of Theorem 2. Recall that \(s\) is a stationary signpost sequence if \(s(q) = q - \Delta\) for some \(\Delta \in [0, 1]\) and every strictly positive integer \(q\). Consider the 3-dimensional instance \((\mathcal{N}, \mathcal{V}, m^-, m^+, H)\) defined as follows: \(N_1 = \{p_1, p_2\}, N_2 = \{r_1, r_2\}\) and \(N_3 = \{g_1, g_2\}\). In order to represent vectors with entries in \(N_1 \times N_2 \times N_3\), we use two matrices, the first one for \(g_1\) and the second one for \(g_2\), which rows represent \(N_1\) (the first row \(p_1\) and the second \(p_2\)) and which columns represent \(N_2\) (the first column \(r_1\) and the second \(r_2\)). The matrix representation of \(\mathcal{V}\) is given by

\[
\begin{pmatrix}
81 & 40 \\
50 & 54 \\
\end{pmatrix}
\text{ and }
\begin{pmatrix}
66 & 53 \\
81 & 48 \\
\end{pmatrix}.
\]

The house size is equal to \(H = 10\) and the marginals \(m^−\) and \(m^+\) are all equal to five. We denote by \(T\) this instance. Consider \((X, Y)\) where the matrix representation of \(X\) is given by

\[
\begin{pmatrix}
1.5 & 1 \\
1 & 1.5 \\
\end{pmatrix}
\text{ and }
\begin{pmatrix}
1 & 1.5 \\
1.5 & 1 \\
\end{pmatrix},
\]

and \(Y\) is defined as in Lemma 1, in particular \(Y^1_e = 1\) for every \(e \in E(\mathcal{V})\), \(Y^2_e = 0.5\) for every \(e \in E(\mathcal{V})\) such that \(X_e = 1.5\) and \(Y^2_e = 0\) in any other case. By construction, \((X, Y)\) is a feasible solution of the linear relaxation of (6)-(12) for every signpost sequence. Consider the solution \((x^*, y^*)\) where the matrix representation of \(x^*\) is given by

\[
\begin{pmatrix}
2 & 1 \\
1 & 1 \\
\end{pmatrix}
\text{ and }
\begin{pmatrix}
1 & 1 \\
1 & 2 \\
\end{pmatrix},
\]

and \(y^*\) is defined as in Lemma 1, meaning \(y^1_e = 1\) for every \(e \in E(\mathcal{V})\), \(y^2_e = 1\) for every \(e \in E(\mathcal{V})\) such that \(x_e^* = 2\) and \(y^1_e = 0\) in any other case.

Claim 1. For every \(\Delta \in [0, 1]\), \((x^*, y^*)\) is the optimal solution of the integer program (6)-(12).

We show how to prove the theorem using Claim 1. For every \(\Delta \in [0, 1]\) we have that the difference of the objective values for the solutions \((x^*, y^*)\) and \((X, Y)\) is equal to

\[
\log \left( \frac{2 - \Delta}{81} \right) + \log \left( \frac{2 - \Delta}{48} \right) - \frac{1}{2} \left[ \log \left( \frac{2 - \Delta}{81} \right) + \log \left( \frac{2 - \Delta}{54} \right) + \log \left( \frac{2 - \Delta}{53} \right) + \log \left( \frac{2 - \Delta}{81} \right) \right]
\]

\[
= \frac{1}{2} \log \left( \frac{53 \cdot 54}{48^2} \right) > 0,
\]

and therefore \((x^*, y^*)\) is not an optimal solution for the linear relaxation of (6)-(12). Since \((x^*, y^*)\) is the optimal integer solution of (6)-(12), any other optimal solution has the same objective value, and therefore we conclude that there is no integral solution that is optimal for the linear relaxation. By Theorem 1 we conclude that \(A_s(\mathcal{N}, \mathcal{V}, m^-, m^+, H) = \emptyset\).

To prove Claim 1 we consider two cases. Suppose first that \(\Delta \in [0, 1)\). In this case, the program (6)-(12) is given by

\[
\begin{align*}
\text{minimize} & \quad \Phi(y, \Delta) \\
\text{subject to} & \quad \sum_{t=1}^{10} y^1_t = x_e \quad \text{for every } e \in N_1 \times N_2 \times N_3, \\
& \quad \sum_{e \in N_1 \times N_2 \times N_3: e \neq v} x_e = 5 \quad \text{for every } \ell \in \{1, 2, 3\} \text{ and every } v \in N_\ell, \\
& \quad y^1_e \in \{0, 1\} \quad \text{for every } e \in N_1 \times N_2 \times N_3 \text{ and } t \in \{1, \ldots, 10\},
\end{align*}
\]

where the function \(\Phi(y, \Delta)\) is given by

\[
\Phi(y, \Delta) = \sum_{e \in N_1 \times N_2 \times N_3} \sum_{t=1}^{10} y^1_t \log \left( \frac{t - \Delta}{V_e} \right).
\]

In particular, \(\Phi\) is strictly decreasing in its second variable \(\Delta \in [0, 1)\), and \(\Phi(y, \Delta) \to -\infty\) when \(\Delta \to 1\) if there exists \(e \in N_1 \times N_2 \times N_3\) such that \(y^1_e > 0\). For the instance \(T\) there are 104 integer vectors \(x\) satisfying (24), hence using
Lemma 1 (i) it can be checked computationally that for $\Delta^* = 0.99995$ and $\varepsilon = 5 \cdot 10^{-7}$ the following holds: For every $\Delta \in \{\varepsilon, 2\varepsilon, \ldots, \Delta^* - \varepsilon, \Delta^*\}$ and every integer solution $(x, y)$ satisfying (23)-(25), we have that $\Phi(y, \Delta) > \Phi(y^*, \Delta - \varepsilon)$ and $\Phi(y, 0) > \Phi(y^*, 0)$. We show in what follows that this implies $\Phi(y, \Delta) > \Phi(y^*, \Delta)$ for every $\Delta \in [0, \Delta^*]$. Let $\Delta \in [0, \Delta^*]$. If $\Delta = 0$, by hypothesis we have $\Phi(y, \Delta) > \Phi(y^*, \Delta)$, so we are done in this case. Otherwise, if $(k - 1)\varepsilon < \Delta \leq k\varepsilon$ for some $k \in \{1, \ldots, \frac{\Delta}{\varepsilon}\}$, we have that $\Phi(y, \Delta) \geq \Phi(y, k\varepsilon) > \Phi(y^*, (k - 1)\varepsilon) > \Phi(y^*, \Delta)$, where the first and last inequalities come from the fact that $\Phi$ is strictly decreasing in its second variable and the second inequality comes from the mentioned property.

It remains to show the inequality for $\Delta \in (\Delta^*, 1)$. We first observe that only four of the integer vectors $x$ satisfying (24) are strictly positive in every entry, including $x^*$. Among integer solutions $(x, y)$ satisfying (23)-(25) with strictly positive $x$, it is easy to check algebraically and using Lemma 1 (i) that for every $\Delta \in [0, 1]$ $(x^*, y^*)$ is the one with smallest objective value. Thus, it only remains to show this for $\Delta \in (\Delta^*, 1)$ and feasible solutions $(x, y)$ where $x$ has at least one component equal to zero, i.e. where $|\text{supp}(x)| = |\{e \in N_1 \times N_2 \times N_3 : x_e \neq 0\}| < 8$. Note that from Lemma 1 (i) and the integrality of $x$ and $x^*$

$$\Phi(y, \Delta) - \Phi(y^*, \Delta) \geq \sum_{e \in N_1 \times N_2 \times N_3} \sum_{x_e > x_e^*} \log \left( \frac{t - \Delta}{V_e} \right) - \sum_{e \in N_1 \times N_2 \times N_3} \sum_{x_e = x_e^*} \log \left( \frac{t - \Delta}{V_e} \right)$$

and denote the first term on the righthand side as $S_1(x, x^*, \Delta)$ and the subtracted term as $S_2(x, x^*, \Delta)$. Suppose first that $|\text{supp}(x)| = 7$. In this case, the fact that $\sum_{e \in N_1 \times N_2 \times N_3} x_e = 10$ and the marginals constraint (24) implies that there are exactly three entries $e \in \text{supp}(x)$ such that $x_e = 2$. As all the entries of $x^*$ are at least one, in this case we can bound $S_1(x, x^*, \Delta)$ as

$$S_1(x, x^*, \Delta) \geq \min_{\Delta \in [0, 1]} 3 \log \left( \frac{2 - \Delta}{V_e} \right) \geq 3 \log \left( \frac{1}{81} \right),$$

where the first inequality follows from the fact that the logarithm is negative since $2 - \Delta < V_e$ for every $\Delta \in [0, 1]$ and every $e \in N_1 \times N_2 \times N_3$ and the second one from replacing the minimizers. Similarly, we can bound $S_2(x, x^*, \Delta)$ as follows, using that one entry of $x$ is 0:

$$S_2(x, x^*, \Delta) \leq \max_{e \in N_1 \times N_2 \times N_3} \log \left( \frac{1 - \Delta}{V_e} \right) \leq \log \left( \frac{1 - \Delta}{40} \right),$$

where once again, the first inequality uses the negativity of logarithms and the second comes from replacing $c$ with the one with the smallest value of $V_e$. Writing all together, we have that for every feasible solution $(x, y)$ it holds that

$$\Phi(y, \Delta) - \Phi(y^*, \Delta) \geq 3 \log \left( \frac{1}{81} \right) - \log \left( \frac{1 - \Delta}{40} \right) = \log \left( \frac{40}{81^3 \cdot (1 - \Delta)} \right).$$

Thus, imposing $81^3 \cdot (1 - \Delta) < 40$ or equivalently $\Delta > 1 - 40/81^3 \approx 0.999925$ we conclude $\Phi(y, \Delta) > \Phi(y^*, \Delta)$.

Now consider the case $|\text{supp}(x)| \leq 6$. In this case, we achieve a lower bound for $S_1(x, x^*, \Delta)$ computing $S_1(x, x^*, 1)$ for each feasible solution with two or more entries equal to zero. The minimum is achieved in the solution $x$ with entries $(p_2, r_2, g_1)$ and $(p_1, r_1, g_2)$ equal to 5 and all the others equal to 0, and the value of $S_1(x, x^*, 1)$ is approximately $-26.36$. For $S_2(x, x^*, \Delta)$, we bound in the same way as before, but using the fact that the entries with zeros are at least two:

$$S_2(x, x^*, \Delta) \leq 2 \max_{e \in N_1 \times N_2 \times N_3} \log \left( \frac{1 - \Delta}{V_e} \right) = 2 \log \left( \frac{1 - \Delta}{40} \right).$$

Using these two bounds, we can bound the difference of the objective values as

$$\Phi(y, \Delta) - \Phi(y^*, \Delta) > -26.5 - 2 \log \left( \frac{1 - \Delta}{40} \right).$$

So a sufficient condition for our result is $2 \log((1 - \Delta)/40) \leq -26.5$, which is equivalent to $\Delta \leq 1 - 40 \exp(-13.25) \approx 0.99993$.

Thus, recalling that $\Delta^* = 0.99995$, in particular we have that $\Phi(y, \Delta) > \Phi(y^*, \Delta)$ for every $\Delta \in (\Delta^*, 1)$. Hence, we have proved the optimality of $(x^*, y^*)$ for every $\Delta < 1$. When $\Delta = 1$ the claim follows by a computational certification of the optimality of $(x^*, y^*)$ for the integer program (6)-(12).

Theorem 3. For every signpost sequence $s$ and every $d \geq 3$, the $(d, s)$-proportional apportionment problem is NP-complete.

C. Proof of Theorem 3. Consider a $d$-partite hypergraph $G = (P, F)$ with vertices partition $P_1, \ldots, P_d$ such that $|P_1| = \cdots = |P_d| = d$. We define two instances for the $(d, s)$-proportional apportionment problem as follows. Let $P = \{P_1, \ldots, P_d\}$ and let $V(G)$ be such that for every $e \in \prod_{i=1}^d P_i$ is defined as follows: $V_e(G) = 1$ when $\{e_1, \ldots, e_d\} \in F$, and $V_e(G) = 0$ otherwise. For every $\ell \in \{1, \ldots, d\}$ and every $v \in P_\ell$, let $m_\ell^1 = 1$ and $m_\ell^2 = |\delta(v)| + 1$, where $\delta(v)$ is the set of hyperedges containing $v$. Consider the apportionment instance $T_\ell(G) = (P, V(G), m_\ell^1, m_\ell^2, c)$ and the instance $T(G) = (P, V(G), m_1^2, m_2^2, |F| + c)$.

The following lemma establishes the correspondence between the $(d, s)$-proportional apportionment problem and the perfect matching problem in $d$-partite hypergraphs.
Lemma 2. Let $G = (P, F)$ be a $d$-partite graph with vertices partition $P_1, \ldots, P_d$ and such that $|P_1| = \cdots = |P_d| = c$. Then, for every signpost sequence $s$, the following holds:

(a) When $s(1) > 0$, we have that $G$ has a perfect matching if and only if $\mathcal{A}_s(T_1(G)) \neq \emptyset$.

(b) When $s(1) = 0$, we have that $G$ has a perfect matching if and only if $\mathcal{A}_s(T_2(G)) \neq \emptyset$.

Proof. Let $G = (P, F)$ be a $d$-partite hypergraph as described in the statement of the lemma. Recall that $E(V(G))$ corresponds to the subset of tuples in the product $\prod_{i=1}^d P_i$ where $V(G)$ is strictly positive, and therefore, $E(V(G)) = \{ e \in \prod_{i=1}^d P_i : \{ e_1, \ldots, e_d \} \in F \}$, that is, exactly the tuples in the product that are in correspondence with the edges of $G$. Within the proof, the following function will be useful. Let $\alpha : E(V(G)) \rightarrow F$ be such that $\alpha(e) = \{ e_1, \ldots, e_d \}$. The function $\alpha$ is a bijection, and it captures the natural representation of the (ordered) tuples in $\prod_{i=1}^d P_i$ as (unordered) hyperedges in $G$.

We first consider the case $s(1) > 0$ and suppose that $G$ has a perfect matching $F' \subseteq F$. Consider the integral vector $x$ with entries in $E(V(G))$ defined as follows: $x_e = 1$ when $\alpha(e) \in F'$ and $x_e = 0$ otherwise. Let $\mu = 1$ and for every $v \in P$, let $\lambda_v = s(1)^{1/d}$. We check now that $(x, \mu, \lambda) \in \mathcal{A}_s(T_1(G))$. Since $F'$ is a perfect matching, for every $\ell \in \{1, \ldots, d\}$ and every $v \in P_\ell$ we have that

$$\sum_{e \in E(V(G)): e \vdash v} x_e = |\delta(v) \cap F'| = 1 = m^1_v,$$

and therefore conditions (1), (2), (4) and (5) are satisfied. For every $e \in E(V(G))$, we have that $\mathcal{V}_v(G) \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell} = 1 \cdot (s(1)^{1/d})^d = s(1)$. When $x_e = 1$, from the monotonicity of the signpost sequence $s$ we therefore have that $s(x_e) = s(1) = \mathcal{V}_v(G) \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell} \leq s(2) = s(x_e + 1)$, and when $x_e = 0$, we have that $s(x_e) = s(0) \leq \mathcal{V}_v(G) \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell} = s(1) = s(x_e + 1)$. We conclude that condition (3) is satisfied and therefore $(x, \mu, \lambda) \in \mathcal{A}_s(T_1(G))$. Conversely, suppose now that there exists a perfect matching. This concludes the proof of (a).

Consider now the case $s(1) = 0$, and suppose that $G$ has a perfect matching $F' \subseteq F$. Consider an integral vector $x$ with entries in $E(V(G))$ defined as follows: $x_e = 2$ when $\alpha(e) \in F'$ and $x_e = 1$ otherwise. Let $\mu = 1$ and for every $v \in P$, let $\lambda_v = s(2)^{1/d}$. We check now that $(x, \mu, \lambda) \in \mathcal{A}_s(T_2(G))$. Since $F'$ is a perfect matching, for every $\ell \in \{1, \ldots, d\}$ and every $v \in P_\ell$ we have that

$$\sum_{e \in E(V(G)): e \vdash v} x_e = |\delta(v) \cap F'| - 1 + 2 = |\delta(v)| + 1 = m^2_v,$$

and therefore conditions (1), (2), (4) and (5) are satisfied. For every $e \in E(V(G))$, we have that $\mathcal{V}_v(G) \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell} = 1 \cdot (s(2)^{1/d})^d = s(2)$. From the monotonicity of the signpost sequence $s$, when $x_e = 2$ we have that $s(x_e) = s(2) = \mathcal{V}_v(G) \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell} \leq s(3) = s(x_e + 1)$, and when $x_e = 1$ we have that $s(x_e) = s(1) \leq \mathcal{V}_v(G) \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell} = s(2) = s(x_e + 1)$. We conclude that condition (3) is satisfied and therefore $(x, \mu, \lambda) \in \mathcal{A}_s(T_2(G))$. Conversely, suppose now that there exists a perfect matching. This concludes the proof of (b).

Proof of Theorem 3. The problem is in NP since for a triplet $(x, \mu, \lambda)$ checking conditions (1)-(5) can be done in polynomial time. Given a signpost sequence $s$, by Lemma 2 we have that the perfect matching problem in $d$-partite hypergraphs where each part is the same size, is reducible to the $(d, s)$-proportional apportionment problem since for every such hypergraph $G$ the instances $T_1(G)$ and $T_2(G)$ can be computed in polynomial time. Since the perfect matching problem in $d$-partite hypergraphs where each part has equal size is NP-complete for every $d \geq 3 \cdot (1, 2)$, we conclude that for every $d \geq 3$ and every signpost sequence $s$ the $(d, s)$-proportional apportionment problem is NP-complete.

2. Proofs of Section 3

Theorem 4. Let $(N, V, m^-, m^+, H)$ be an instance of the $(d, s)$-dimensional apportionment problem and let $s$ be a signpost sequence such that the linear relaxation of (6)-(12) is feasible. Let $u_1, \ldots, u_d$ be nonnegative integer values such that $\sum_{\ell=1}^d 1/(u_\ell + 2) \leq 1$. Then, there exists an integral vector $X \in \mathbb{N}^{E(V)}$ such that the following holds:

(i) $m^-_v - u_\ell \leq \sum_{e \in E(V): e \vdash v} X_e \leq m^+_v + u_\ell$ for every $\ell \in \{1, \ldots, d\}$ and every $v \in N_e$.

(ii) There exists $\mu > 0$ and a vector $\lambda$ with strictly positive entries such that:

1. $s(X_e) \leq \mathcal{V}_v \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{e_\ell} \leq s(X_e + 1)$ for every $e \in E(V)$,

2. For every $\ell \in \{1, \ldots, d\}$ and every $v \in N_{e_\ell}$, if $\lambda_\ell > 1$ then $| \sum_{e \in E(V): e \vdash v} X_e - m^+_v | \leq u_\ell$. 

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(3) For every $\ell \in \{1, \ldots, d\}$ and every $v \in V_\ell$, if $\lambda_v < 1$ then $|\sum_{e \in E(v)} x_e - m_e| \leq u_v$.

Furthermore, $X$ can be found in time polynomial in $|E(V)|$, $\sum_{\ell=1}^d |V_\ell|$ and $H$.

**Theorem 5.** Let $G$ be a $d$-partite hypergraph with vertex partition $\{P_1, \ldots, P_d\}$ and hyperedges $E$. Let $x \in [0,1]^E$ be such that $
exists\ e \in E : y_e = 1$ and let $u_1, \ldots, u_d$ be nonnegative integers such that $\sum_{\ell=1}^d u_\ell / (u_\ell + 2) \leq 1$. Then, there exists $z \in [0,1]^E$ such that for every $\ell \in \{1, \ldots, d\}$ and every $v \in P_\ell$, it holds $|\sum_{e \in \delta(v)} x_e - z_e| \leq u_v$, and $z_e = x_e$ when $x_e$ is integer. Furthermore, $z$ can be computed in time polynomial in $|E|$ and $\sum_{\ell=1}^d |P_\ell|$.

We present an iterative rounding algorithm, inspired by the classical discrepancy minimization result by Beck and Fiala (3), that computes a solution $z$ satisfying the conditions guaranteed by Theorem 5. To present this procedure, we introduce a simple linear program that will be used during its execution. Given a vector $Y \in [0,1]^E$ with $F \subseteq E$, a subset of edges $E \subseteq F$ and a subset of vertices $Q_\ell \subseteq P_\ell$ for each $\ell \in \{1, \ldots, d\}$, we consider the following linear program with variables $y_e$ for each $e \in E$:

$$
\sum_{e \in \delta(v) \cap E} y_e = 1 \quad \text{for every } v \in \bigcup_{\ell=1}^d Q_\ell, \quad 0 \leq y_e \leq 1 \quad \text{for every } e \in E.
$$

We denote by $K(Y,E,Q)$ the polytope of feasible solutions for this linear program.

**Algorithm 1** Iterative Rounding Algorithm

**Require:** A $d$-partite hypergraph $G$ with vertex partition $\{P_1, \ldots, P_d\}$ and hyperedges $E$, a vector $x \in [0,1]^E$ and nonnegative integer values $u_1, \ldots, u_d$.

**Ensure:** Binary vector $z \in \{0,1\}^E$.

1. Initialize $y_e \leftarrow x$ and let $E^0 = \{e \in E : y_e$ is fractional$\}$.
2. For each $\ell \in \{1, \ldots, d\}$, let $Q^\ell = \{v \in P_\ell : |\delta(v) \cap E^\ell| \geq u_\ell + 2\}$.
3. Let $z_e = y_e$ for every $e \notin E^0$ and initialize $t \leftarrow 0$.
4. **while** there exists $\ell \in \{1, \ldots, d\}$ such that $Q^\ell \neq \emptyset$ **do**
5. Compute an extreme point $y^{t+1}$ of $K(y^t, E^t, Q^\ell)$.
6. Let $E^{t+1} = \{e \in E : y_e^{t+1}$ is fractional$\}$.
7. For each $\ell \in \{1, \ldots, d\}$, let $Q^{\ell+1} = \{v \in P_\ell : |\delta(v) \cap E^{\ell+1}| \geq u_\ell + 2\}$.
8. Let $z_e = y_e^{t+1}$ for every $e \in E^t \setminus E^{t+1}$. Update $t \leftarrow t + 1$.
9. Let $T$ be the value of $t$ that did not satisfy the loop condition.
10. Let $z_e \in \{\lfloor y_e^T \rfloor, \lceil y_e^T \rceil\}$ for every $e \in E^T$.
11. Return $z$.

**Algorithm 2** Apportionment Rounding Algorithm

**Require:** A $d$-dimensional instance $(N, V, m^-, m^+, H)$ and $u_1, \ldots, u_d$ with $\sum_{\ell=1}^d 1/(u_\ell + 2) \leq 1$.

**Ensure:** An integral vector $X \in \mathbb{N}^{-}(V)$.

1. Let $(x^*, y^*)$ be an optimal solution of the linear relaxation of (6)-(12) in instance $(N, V, m^-, m^+, H)$.
2. If $x^*$ is integral return $X \leftarrow x^*$.
3. If $x^*$ is fractional, consider the $d$-partite hypergraph $G$ with vertex partition $N_1, \ldots, N_d$ and hyperedges $\alpha(E(V))$. Run Algorithm 1 over the hypergraph $G$, the fractional vector $w$ defined as $w_{e(c)} = x^*_e - \lfloor x^*_e \rfloor$ for every $e \in E(V)$ and the values $u_1, \ldots, u_d$, and let $z \in \{0,1\}^{\alpha(E(V))}$ be its output.
4. Return $X_c = \lfloor x^*_c \rfloor + z_{c(e)}$ for every $e \in E(V)$.

**Proof of Theorem 5.** In the following lemmas, we summarize the key properties of our Algorithm 1.

**Lemma 3.** Let $G$ be a $d$-partite hypergraph with vertex partition $\{P_1, \ldots, P_d\}$ and hyperedges $E$, let $u_1, \ldots, u_d$ be nonnegative integers and let $x \in [0,1]^E$. When $\sum_{\ell=1}^d 1/(u_\ell + 2) \leq 1$, we have that Algorithm 1 terminates. Furthermore, we have $T \leq |E|$, and for every $t \in \{0, \ldots, T-1\}$ we have that $E^{t+1}$ is strictly contained in $E^t$.

**Lemma 4.** Let $G$ be a $d$-partite hypergraph with vertex partition $\{P_1, \ldots, P_d\}$ and hyperedges $E$, let $u_1, \ldots, u_d$ be nonnegative integers and let $x \in [0,1]^E$. Suppose that $T$ as defined in Algorithm 1 is finite and let $z$ be the output of Algorithm 1. Then, for every $\ell \in \{1, \ldots, d\}$ and every $v \in P_\ell$, there exists $t(v) \in \{0, 1, \ldots, T\}$ such that $|\delta(v) \cap E^t| \geq u_\ell + 2$ for every $t < t(v)$, when $t(v) > 0$, and $|\delta(v) \cap E^T| \leq u_\ell + 1$ for every $t \geq t(v)$. Furthermore, for every $t \leq t(v)$ we have that

$$
\sum_{e \in \delta(v)} x_e = \sum_{e \in \delta(v) \cap E^t} y_e^t + \sum_{e \in \delta(v) \setminus E^t} z_e.
$$
Proof of Theorem 5. We show in what follows that Algorithm 1 computes a solution \( z \) that satisfies the conditions guaranteed by Theorem 5. By Lemma 3 we have that the algorithm terminates and \( T \leq |E| \). Furthermore, \( E^{t+1} \) is strictly contained in \( E^t \) for every \( t \in \{0, \ldots, T-1\} \). In the initialization we have that \( z_e = x_e \in \{0, 1\} \) for every \( e \in E \setminus E^0 \), which in particular implies that if \( x_e \) is integer, then \( z_e = x_e \). For every \( t \in \{0, \ldots, T-1\} \) we have \( z_e = y_e^{t+1} \in \{0, 1\} \) for each \( e \in E \setminus E^{t+1} \), and since we have that \( E \setminus E^t \) is equal to the union of \( E \setminus E^0 \) and \( \bigcup_{t=0}^{T-1} (E^t \setminus E^{t+1}) \), where all these unions are disjoint, we conclude that \( z_e \in \{0, 1\} \) for every \( e \not\in E^t \). Since \( z_e \in \{\lfloor y_e^t \rfloor, \lceil y_e^t \rceil\} \) for every \( e \in E^t \), we conclude that \( z \in \{0, 1\}^E \). By Lemma 4, for every \( \ell \in \{1, \ldots, d\} \) and every \( v \in P_t \), there exists \( t(v) \in \{0, 1, \ldots, T\} \) such that \( |\delta(v) \cap E^t| \geq u_\ell + 2 \) for every \( t < t(v) \), when \( t(v) > 0 \), and \( |\delta(v) \cap E^0| \leq u_\ell + 1 \) for every \( t \geq t(v) \). Furthermore

\[
\sum_{e \in \delta(v)} x_e = \sum_{e \in \delta(v) \cap E^t} y_e^{t(v)} + \sum_{e \in \delta(v) \setminus E^t} z_e = \sum_{e \in \delta(v) \cap E^t} y_e^{t(v)} - \sum_{e \in \delta(v) \setminus E^t} z_e + \sum_{e \in \delta(v)} z_e.
\]

We have that \( 0 < y_e^{t(v)} < 1 \) for every \( e \in \delta(v) \cap E^t(v) \), and therefore the above equality implies that

\[
\left| \sum_{e \in \delta(v)} (x_e - z_e) \right| \leq \sum_{e \in \delta(v) \cap E^t(v)} |y_e^{t(v)} - z_e| < |\delta(v) \cap E^t(v)| \leq u_\ell + 1.
\]

Since for every vertex \( v \) of \( G \) we have that \( \sum_{e \in \delta(v)} x_e \) is integer and \( \sum_{e \in \delta(v)} z_e \) is integral as well, we conclude that \( |\sum_{e \in \delta(v)} (x_e - z_e)| \leq u_\ell \) for every \( \ell \in \{1, \ldots, d\} \) and every \( v \in P_t \). Finally, as the algorithm executes at most \( |E| \) iterations, and in each iteration it solves a linear program with at most \( |E| \) variables and at most \( 2|E| + \sum_{\ell=1}^{d} |P_\ell| \) constraints, we conclude that \( z \) can be computed efficiently.

The following two propositions will be used to prove Lemma 3.

**Proposition 1.** Consider a tuple \((Y, E, Q)\) and suppose that \(|\bigcup_{\ell=1}^{d} Q_\ell| < |E|\). Then, for every extreme point \( y \in K(Y, E, Q) \), there exists at least one hyperedge \( e \in E \) such that \( y_e \in \{0, 1\} \).

**Proof.** Observe that the total number of variables in the linear program is \(|E|\) while the number of equality constraints in (26) is equal to \(|\bigcup_{\ell=1}^{d} Q_\ell| \leq |E|\). In particular, since \(|\bigcup_{\ell=1}^{d} Q_\ell| < |E|\) we conclude that the number of variables in the linear program is strictly larger than the number of equality constraints in (26). Therefore, any extreme point \( y \) of \( K(Y, E, Q) \) must satisfy at least one inequality constraint in (27) with equality, so at least one entry of \( y \) is in \( \{0, 1\} \).

**Proposition 2.** Let \( w_1, \ldots, w_d \) be integer numbers such that \( w_\ell \geq 2 \) for every \( \ell \in \{1, \ldots, d\} \). Then, the following holds:

(i) \( q \geq \sum_{\ell=1}^{d} \lfloor q/w_\ell \rfloor \) for every strictly positive \( q \) if and only if \( \sum_{\ell=1}^{d} 1/w_\ell \leq 1 \).

(ii) \( q > \sum_{\ell=1}^{d} \lfloor q/w_\ell \rfloor \) for every strictly positive \( q \) if and only if \( \sum_{\ell=1}^{d} 1/w_\ell < 1 \).

(iii) Suppose that \( q \geq \sum_{\ell=1}^{d} \lfloor q/w_\ell \rfloor \) for every strictly positive \( q \) and let \( \bar{q} \) be a strictly positive integer such that \( \bar{q} = \sum_{\ell=1}^{d} \lfloor q/w_\ell \rfloor \). Then, \( \bar{q} \) is a common multiplier of \( w_1, \ldots, w_d \).

**Proof.** Consider the function \( \phi_d(q, w) = q - \sum_{\ell=1}^{d} \lfloor q/w_\ell \rfloor \). First observe that for any strictly positive \( q \) we have that \( \phi_d(q, w) \geq q - \sum_{\ell=1}^{d} q/w_\ell = q(1 - \sum_{\ell=1}^{d} 1/w_\ell) \). When \( \sum_{\ell=1}^{d} 1/w_\ell \leq 1 \) we have that \( \phi_d(q, w) \geq 0 \). Consider \( \bar{q} = \prod_{\ell=1}^{d} w_\ell \). In particular, we have that \( \lfloor \bar{q}/w_\ell \rfloor = \bar{q}/w_\ell \) for every \( \ell \in \{1, \ldots, d\} \) and therefore \( \phi_d(\bar{q}, w) = \bar{q}(1 - \sum_{\ell=1}^{d} 1/w_\ell) \). Then, when \( \phi_d(\bar{q}, w) \geq 0 \) we have that \( \sum_{\ell=1}^{d} 1/w_\ell \leq 1 \). That proves (i). Property (ii) follows from the same argument by using strict inequalities. For the third property, if \( \bar{q} \) is not a common multiplier of \( w_1, \ldots, w_d \), then we necessarily have that \( \lfloor \bar{q}/w_\ell \rfloor < \bar{q}/w_\ell \) for some \( \ell \in \{1, \ldots, d\} \). Therefore,

\[
\phi_d(\bar{q}, w) = \bar{q} - \sum_{\ell=1}^{d} \frac{\bar{q}}{w_\ell} > \bar{q} - \sum_{\ell=1}^{d} \frac{\bar{q}}{w_\ell} = \bar{q} \left( 1 - \sum_{\ell=1}^{d} \frac{1}{w_\ell} \right).
\]

By property (i) we have that \( \sum_{\ell=1}^{d} 1/w_\ell \leq 1 \) and therefore the last term in the above inequality is nonnegative, which contradicts the fact that \( \phi_d(\bar{q}, w) = 0 \). That concludes (iii).

Now we are ready to prove Lemma 3 and Lemma 4.
Proof of Lemma 3. In what follows we show how to conclude the lemma by using the following claim: Given an integer $t$, if there exists $\ell \in \{1, \ldots, d\}$ for which $Q^\ell \neq \emptyset$ and if $\sum_{\ell=1}^d 1/(u_\ell + 2) \leq 1$, we have that every extreme point of $K(y^\ell, E^\ell, Q^\ell)$ has at least one integral entry. If the claim is true, as $K(y^\ell, E^\ell, Q^\ell)$ is nonempty since $y^\ell$ is always a feasible solution for the linear program (26)-(27), then $E^{\ell+1}$ is strictly contained in $E^{\ell}$, so as long as the loop condition is satisfied, we have that $|E^{\ell}|$ is strictly decreasing in $t$. We conclude that $T$ is finite and furthermore $T \leq |E|$. We now show how to prove the claim. Note that at iteration $t$, for every $\ell \in \{1, \ldots, d\}$ we have $|Q^\ell| \leq |E^\ell|/(u_\ell + 2)$. If this is not the case, it would mean that $\sum_{\ell=1}^d |\delta(v) \cap E^\ell| \geq (u_\ell + 2) \cdot (1 + |E^\ell|/(u_\ell + 2)) > |E^\ell|$, which is a contradiction since each hyperedge in $E^\ell$ contains at most one vertex in $Q^\ell$, hence the term in the left is a sum of cardinalities of disjoint subsets of $E^\ell$. Therefore, we have that

$$\left| \bigcup_{\ell=1}^d Q^\ell \right| \leq \sum_{\ell=1}^d |Q^\ell| \leq \sum_{\ell=1}^d \frac{|E^\ell|}{u_\ell + 2} \leq |E^\ell|,$$

where the last inequality comes from Proposition 2 (i). From Proposition 2 (ii), we know that if $\sum_{\ell=1}^d 1/(u_\ell + 2) < 1$, then we have $\sum_{\ell=1}^d |E^\ell|/(u_\ell + 2) < |E^\ell|$, and therefore we have $|\bigcup_{\ell=1}^d Q^\ell| < |E^\ell|$. By Proposition 1 we conclude that every extreme point of $K(y^\ell, E^\ell, Q^\ell)$ has at least one integral entry, so we are done in this case. In the following suppose that $\sum_{\ell=1}^d 1/(u_\ell + 2) = 1$. From Proposition 2 (i) we know that $\sum_{\ell=1}^d |E^\ell|/(u_\ell + 2) \leq |E^\ell|$, and from Proposition 2 (iii) the equality is only possible when $|E^\ell|$ is a common multiple of the values $u_1, u_2, \ldots, u_d + 2$. In what follows, we suppose the latter holds and we consider two cases.

Case 1. Suppose first that for every $\ell \in \{1, \ldots, d\}$ and every $v \in P_\ell$ we have $|\delta(v) \cap E^\ell| \in \{0, u_\ell + 2\}$. Then, in this case, the polytope $K(y^\ell, E^\ell, Q^\ell)$ is obtained from the linear program

$$\sum_{e \in \delta(v) \cap E^\ell} z_e = \sum_{e \in \delta(v) \cap E^\ell} y^\ell_e \quad \text{for every } \ell \in \{1, \ldots, d\} \text{ and every } v \in P_\ell \text{ s.t. } |\delta(v) \cap E^\ell| = u_\ell + 2,$$

$$0 \leq z_e \leq 1 \quad \text{for every } e \in E^\ell.$$

Since $G$ is a $d$-partite hypergraph, we have that for every $\ell \in \{1, \ldots, d\}$ the set $E^\ell$ is equal to the union of the sets $\delta(v) \cap E^\ell$ over $v \in P_\ell$. Furthermore, for every $\ell \in \{1, \ldots, d\}$ and every $v \in P_\ell \setminus Q^\ell$, we have $|\delta(v) \cap E^\ell| = 0$, and therefore we conclude that $E^\ell$ is equal to the union of the sets $\delta(v) \cap E^\ell$ over $v \in Q^\ell$. Then, taking $\ell \in \{1, \ldots, d\}$ and summing over $v \in Q^\ell$ we have that

$$\sum_{v \in Q^\ell} \sum_{e \in \delta(v) \cap E^\ell} z_e = \sum_{v \in Q^\ell} \sum_{e \in \delta(v) \cap E^\ell} y^\ell_e,$$

which implies $\sum_{e \in E^\ell} z_e = \sum_{e \in E^\ell} y^\ell_e$, and this last equality does not depend on $\ell$. Then, in the linear program defining $K(y^\ell, E^\ell, Q^\ell)$, the constraints given by two different sets $Q^\ell$ and $Q^\ell \setminus \hat{\ell}$, with $\ell \neq \hat{\ell}$, are linearly dependent. Since any extreme point of $K(y^\ell, E^\ell, Q^\ell)$ is obtained by a linear system of $|E^\ell|$ linearly independent constraints, we necessarily have that at least one of them is from the set of constraints (27), so it has at least one integral entry.

Case 2. Suppose there exists a value $\hat{\ell} \in \{1, \ldots, d\}$ and $\hat{v} \in P_{\hat{\ell}}$ such that $|\delta(\hat{v}) \cap E^\ell| \notin \{0, u_\ell + 2\}$. If $1 \leq |\delta(\hat{v}) \cap E^\ell| < u_\ell + 2$, observe that $\hat{v}$ does not induce a constraint in the linear program defining $K(y^\ell, E^\ell, Q^\ell)$. On the other hand, we have $|\bigcup_{v \in P_{\hat{\ell}} \setminus \delta(\hat{v})} (\delta(v) \cap E^\ell)| \leq |E^\ell| - |\delta(\hat{v}) \cap E^\ell|$. Thus, we have that

$$|Q^\ell| \leq \left| \frac{|E^\ell| - |\delta(\hat{v}) \cap E^\ell|}{u_\ell + 2} \right| = \left| \frac{|E^\ell|}{u_\ell + 2} - \frac{|\delta(\hat{v}) \cap E^\ell|}{u_\ell + 2} \right| \leq \frac{|E^\ell|}{u_\ell + 2} - 1,$$

where the last inequality comes from the fact that $|E^\ell|/(u_\ell + 2)$ is an integer and $|\delta(\hat{v}) \cap E^\ell|$ is strictly positive and strictly less than $u_\ell + 2$. Suppose now that $|\delta(\hat{v}) \cap E^\ell| > u_\ell + 2$. We have $|\bigcup_{v \in P_{\hat{\ell}} \setminus \delta(\hat{v})} (\delta(v) \cap E^\ell)| \leq |E^\ell| - |\delta(\hat{v}) \cap E^\ell|$, and therefore

$$|Q^\ell| \leq 1 + \left| \frac{|E^\ell| - |\delta(\hat{v}) \cap E^\ell|}{u_\ell + 2} \right| = 1 + \left| \frac{|E^\ell|}{u_\ell + 2} - \frac{|\delta(\hat{v}) \cap E^\ell|}{u_\ell + 2} \right| \leq 1 + \frac{|E^\ell|}{u_\ell + 2} - 2 = \frac{|E^\ell|}{u_\ell + 2} - 1,$$

where we used that $|E^\ell|/(u_\ell + 2)$ is an integer and that $|\delta(\hat{v}) \cap E^\ell| > u_\ell + 2$. Therefore, in both situations, we have that

$$\left| \bigcup_{\ell=1}^d Q^\ell \right| \leq \sum_{\ell \neq \hat{\ell}} |Q^\ell| + |Q^\ell| \leq \sum_{\ell \neq \hat{\ell}} \frac{|E^\ell|}{u_\ell + 2} - 1 = \sum_{\ell \neq \hat{\ell}} \frac{|E^\ell|}{u_\ell + 2} - 1 = \sum_{\ell=1}^d \frac{|E^\ell|}{u_\ell + 2} - 1 = |E^\ell| - 1,$$

where the first equality comes from the fact that $|Q^\ell|$ is a common multiple of $u_1 + 2, \ldots, u_d + 2$, and the last equality holds since $\sum_{\ell=1}^d 1/(u_\ell + 2) = 1$. Once again, this implies $|\bigcup_{\ell=1}^d Q^\ell| < |E^\ell|$, hence by Proposition 1 we conclude that every extreme point of $K(y^\ell, E^\ell, Q^\ell)$ has at least one integral entry. This concludes the proof of the claim and the proof of the lemma. □
Proof of Lemma 4. We first observe that for every $t \in \{0, \ldots, T - 1\}$ it holds that $\mathcal{E}^{t+1} \subseteq \mathcal{E}^t$, and therefore for every $\ell \in \{1, \ldots, d\}$ and every $v \in P_t$, we have that $\delta(v) \cap \mathcal{E}^{t+1} \subseteq \delta(v) \cap \mathcal{E}^t$. Since Algorithm 1 terminates, we conclude that for every $\ell \in \{1, \ldots, d\}$ and every $v \in P_t$, either $|\delta(v) \cap \mathcal{E}^t| \leq u_t + 1$, and in this case for every $t \geq t(v) = 0$ it holds $|\delta(v) \cap \mathcal{E}^t| \leq u_t + 1$, or there exists $t(v) \in \{1, \ldots, T\}$ such that $|\delta(v) \cap \mathcal{E}^t| \geq u_t + 2$ for every $t < t(v)$ and $|\delta(v) \cap \mathcal{E}^t| \leq u_t + 1$ for every $t \geq t(v)$. Consider now a vertex $v$ in $G$. If $t(v) = 0$, by definition we have that $y_v^0 = x_v$ for every $v \in \delta(v) \cap \mathcal{E}^0$ and $z_v = x_v$ for every $v \in \delta(v) \cap \mathcal{E}^0$, so the identity follows directly. Otherwise, we proceed by induction. For the base case observe that by definition we have that $y_v^0 = x_v$ for every $v \in \delta(v) \cap \mathcal{E}^0$ and $z_v = x_v$ for every $v \in \delta(v) \cap \mathcal{E}^0$. Given a value $t \leq t(v) - 1$, since $\mathcal{E}^{t+1} \subseteq \mathcal{E}^t$ we have that $\delta(v) \cap \mathcal{E}^{t+1} = (\delta(v) \cap \mathcal{E}^t) \setminus (\delta(v) \cap \mathcal{E}^{t+1})$ and $\delta(v) \cap \mathcal{E}^{t+1} = (\delta(v) \cap \mathcal{E}^t) \cup (\delta(v) \cap (\mathcal{E}^t \setminus \mathcal{E}^{t+1}))$. Therefore, we have that
\[
\sum_{e \in \delta(v) \cap \mathcal{E}^{t+1}} y_{e}^{t+1} + \sum_{e \in \delta(v) \cap \mathcal{E}^t} z_{e} = \sum_{e \in \delta(v) \cap \mathcal{E}^{t+1}} y_{e}^{t+1} - \sum_{e \in \delta(v) \cap \mathcal{E}^t \setminus \mathcal{E}^{t+1}} y_{e}^{t+1} + \sum_{e \in \delta(v) \cap \mathcal{E}^t} z_{e} + \sum_{e \in \delta(v) \cap (\mathcal{E}^t \setminus \mathcal{E}^{t+1})} z_{e}.
\]
We have that by construction in Algorithm 1 it holds that $z_{e} = y_{e}^{t+1}$ for every $e \in \mathcal{E}^t \setminus \mathcal{E}^{t+1}$, and therefore
\[
\sum_{e \in \delta(v) \cap \mathcal{E}^{t+1}} y_{e}^{t+1} + \sum_{e \in \delta(v) \cap \mathcal{E}^t} z_{e} = \sum_{e \in \delta(v) \cap \mathcal{E}^{t+1}} y_{e}^{t+1} + \sum_{e \in \delta(v) \cap \mathcal{E}^t} z_{e} = \sum_{e \in \delta(v) \cap \mathcal{E}^{t+1}} y_{e}^{t+1} + \sum_{e \in \delta(v) \cap \mathcal{E}^t} z_{e} = \sum_{e \in \delta(v)} x_{e}.
\]
where the second equality holds since $y_{e}^{t+1} \in \mathcal{K}(y^t, \mathcal{E}^t, Q^t)$ and the third equality holds by the inductive hypothesis. This concludes the proof of the lemma.

We finish by showing the alternative sufficient condition over integers $u_1, \ldots, u_d$ in Theorem 5 pointed at the end of Subsection 3.A. As mentioned, this condition can be replaced by $\sum_{t=1}^d \min\{\lfloor y/(u_t + 2)\rfloor, |P_t|\} < q$ for every strictly positive integer $q$. In fact, given an integer $t$, if there exists $\ell \in \{1, \ldots, d\}$ for which $Q^\ell \neq \emptyset$, this inequality implies that $\sum_{t=1}^d \min\{\lfloor |E^t|/(u_t + 2)\rfloor, |P_t|\} < \sum_{t=1}^d \min\{\lfloor |E^t|/(u_t + 2)\rfloor, |P_t|\} < |E^t|$, where the second inequality comes from the fact that for every $\ell \in \{1, \ldots, d\}$, $|Q^\ell|$ cannot be higher than $|P_t|$. Since this implies $|E^t| > \sum_{t=1}^d |Q_t|$ for every $t$ such that there exists $\ell \in \{1, \ldots, d\}$ for which $Q^\ell \neq \emptyset$, we conclude the same result of Lemma 3, namely that Algorithm 1 terminates after at most $|E|$ steps, and for every $t \in \{0, \ldots, T - 1\}$ we have that $\mathcal{E}^{t+1}$ is strictly contained in $\mathcal{E}^t$. Since Lemma 4 and the analysis on the proof of Theorem 5 under Lemmas 3 and 4 remain valid under this new condition, we conclude the result.

B. Proof of Theorem 4. We first provide one of the main ingredients to analyze Algorithm 2. The next lemma states that the result of any rounding, down or up, of the fractional entries of an optimal solution of the linear relaxation of (6)-(12), satisfies the proportionality condition (3).

Lemma 5. Let $(\mathcal{N}, \mathcal{V}, m^-, m^+, H)$ be an instance of the d-dimensional apportionment problem, let $s$ be a signpost sequence and let $(x, y, U, \Lambda, \beta)$ be an optimal primal-dual pair for the linear relaxation of (6)-(12). Suppose that $\bar{x}$ is an integral vector such that $\bar{x}_e \in \{\lfloor x_e \rfloor, \lceil x_e \rceil\}$ for every $e \in E(\mathcal{V})$. Then, for every $e \in E(\mathcal{V})$ we have that $s(\bar{x}_e) \leq V_e \cdot \exp(U) \cdot \prod_{e \in \delta^+} \exp(\Lambda^+_e + \Lambda^+_e) \cdot \exp(-\beta^e)$. Proof. For every $e \in E(\mathcal{V})$, let $t_e = \max\{t \in \{1, \ldots, H\} : y^e_t > 0\}$ when $x_e > 0$ and $t_e = 0$ when $x_e = 0$. By Lemma 1 (ii) we have that for every $e \in E(\mathcal{V})$, $s(t_e) \leq V_e \cdot \exp(U) \cdot \prod_{e \in \delta^+} \exp(\Lambda^+_e + \Lambda^+_e)$, and by Lemma 1 (i) we have that $t_e = \lfloor x_e \rfloor$. Let $e \in E(\mathcal{V})$. If $x_e$ is integral, then necessarily $\bar{x}_e = x_e = t_e$, so we are done in this case. Now suppose that $x_e$ is fractional. In particular, we have $s(t_e) > 0$, since $x_e > 0$ and if $s(1) = 0$ the fractionality of $x_e$ implies $x_e > 1$. If we apply condition (15) with $t = t_e$, we have that
\[
U + \sum_{e \in \delta^+} (\Lambda^+_e + \Lambda^+_e) + \beta^e - \log\left(\frac{s(t_e)}{V_e}\right) = 0,
\]
since $y_{e}^t$ is strictly positive. By exponentiating and rearranging terms we get that
\[
V_e \cdot \exp(U) \cdot \prod_{e \in \delta^+} \exp(\Lambda^+_e + \Lambda^+_e) = s(t_e) \cdot \exp(-\beta^e).
\]
Since $0 < y_{e}^t < 1$, by the complementary slackness condition (17) we have that $\beta^e = 0$, and therefore we conclude that
\[
V_e \cdot \exp(U) \cdot \prod_{e \in \delta^+} \exp(\Lambda^+_e + \Lambda^+_e) = s(t_e) \cdot \lfloor x_e \rfloor.
\]
When $\bar{x}_e = \lfloor x_e \rfloor$ we have that $s(\bar{x}_e) = s(\lfloor x_e \rfloor) = V_e \cdot \exp(U) \cdot \prod_{e \in \delta^+} \exp(\Lambda^+_e + \Lambda^+_e) \leq s(\lfloor x_e \rfloor + 1) = s(\bar{x}_e + 1)$, where the inequality holds by the monotonicity of $s$. Similarly, when $\bar{x}_e = |x_e|$ we have that
\[
s(\bar{x}_e + 1) = s(\lfloor x_e \rfloor + 1) = V_e \cdot \exp(U) \cdot \prod_{e \in \delta^+} \exp(\Lambda^+_e + \Lambda^+_e) \geq s(\lfloor x_e \rfloor) = s(\bar{x}_e),
\]
and again the inequality comes from the monotonicity of the signpost sequence. Q.E.D.
Proof of Theorem 4. Let \((x^*, y^*, U, \Lambda^-, \Lambda^+\), \(\beta)\) be an optimal primal-dual pair for the linear relaxation of (6)-(12) and define 
\[\mu = \exp(d)\text{ and } \lambda_v = \exp(\Lambda_v^+ + \Lambda_v^-)\] for every \(\ell \in \{1, \ldots, d\}\) and every \(v \in N_\ell\). If the vector \(x^*\) is integral, by Theorem 1 it holds that \((x^*, \mu, \lambda) \in \mathcal{A}_\ell(N, V, m^-, m^+, H)\). That is, the triplet \((x^*, \mu, \lambda)\) satisfies the marginal condition (2) and the proportionality condition (3), as well as conditions (4) and (5), and as such \(x^*\) is integral we are done in this case.

Otherwise, suppose that \(x^*\) is fractional. Consider the \(d\)-partite hypergraph \(G\) with vertex partition \(N_1, \ldots, N_d\) and hyperedges \(\alpha(E(V))\) and let \(z\) be the output of Algorithm 1 over the graph \(G\) and the fractional vector \(w \in [0,1]^{|\alpha(E(V))|}\), defined as \(w_{\alpha(e)} = x^*_e - x^*_e\) for every \(e \in E(V)\). By Theorem 5, we have \(z \in \{0,1\}^{\alpha(E(V))}\) and \(|\sum_{e \in \delta(v)} (z_e - w_e)| \leq \eta_{\ell}\) for every \(\ell \in \{1, \ldots, d\}\) and every \(v \in N_\ell\), where \(\delta(v)\) is the set of hyperedges incident to \(v\) in hypergraph \(G\). This is equivalent to \(|\sum_{e \in E(V)} (z_{\alpha(e)} - x^*_e + [x^*_e])| \leq \eta_{\ell}\) for every \(\ell \in \{1, \ldots, d\}\) and every \(v \in N_\ell\). For every \(e \in E(V)\) let \(z_e = z_{\alpha(e)} + [x^*_e]\).

We therefore have that for every \(\ell \in \{1, \ldots, d\}\) and every \(v \in N_\ell\) it holds that
\[\sum_{e \in E(V); e \notin \ell} X_e - m_v^- \geq \sum_{e \in E(V)} (z_{\alpha(e)} - x^*_e + [x^*_e]) \geq -\eta_{\ell},\]

where the first inequality comes from the definition of \(X\) and since \(x^*\) satisfies the marginals condition (9). If \(\lambda_v > 0\), from condition (21) we must have \(\Lambda_v^+ > 0\), so complementary slackness condition (19) implies \(\sum_{e \notin \ell \in E(V); e \notin \ell} x^*_e = m_v^+\). Therefore, in this case we have \(|\sum_{e \in E(V)} (z_{\alpha(e)} - x^*_e + [x^*_e])| \leq \eta_{\ell}\). Similarly, for every \(\ell \in \{1, \ldots, d\}\) and every \(v \in N_\ell\) we have that
\[\sum_{e \in E(V); e \notin \ell} X_e - m_v^+ \leq \sum_{e \in E(V)} X_e - m_v^+ \leq \sum_{e \in E(V); e \notin \ell} (z_{\alpha(e)} - x^*_e + [x^*_e]) \leq \eta_{\ell},\]

where the first inequality comes from the definition of \(X\) and since \(x^*\) satisfies the marginals condition (10). If \(\lambda_v < 1\), from condition (21) we must have \(\Lambda_v^- < 0\), so complementary slackness condition (20) implies \(\sum_{e \notin \ell \in E(V); e \notin \ell} x^*_e = m_v^-\). Therefore, in this case we have \(|\sum_{e \in E(V)} (z_{\alpha(e)} - x^*_e + [x^*_e])| \leq \eta_{\ell}\). Finally, observe that \(X_e \in \{[x^*_e], [\star]\}\) for every \(e \in E(V)\) and therefore by Lemma 5 we have that \(s(X_e) \leq \nu_e \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{\ell} \leq s(X_e + 1)\) for every \(e \in E(V)\). The complexity follows since the algorithm first solves a linear program with \(O(|H|E(V)|)\) variables and \(O(|H|E(V)| + \sum_{\ell=1}^d |N_\ell|)\) constraints, and Algorithm 1 runs in time polynomial in \(O(|E(V)| + \sum_{\ell=1}^d |N_\ell|)\) from Theorem 5.

C. Tightness of the Bounds for Deviation from the Marginals. For ease of writing, we first define a \(u\)-approximate \(d\)-dimensional proportional apportionment as an apportionment that satisfies the proportionality condition and deviates a maximum fixed amount from the marginals, as described in Theorem 4. Formally, let \((N, V, m^-, m^+, H)\) be an instance of the \(d\)-dimensional apportionment problem, let \(s\) be a signpost sequence, let \(u \in \mathbb{N}^d\) and let \(V \in \mathbb{N}^{|E(V)|}\). We say that \(X\) is a \(u\)-approximate \(d\)-dimensional proportional apportionment for this instance if the following conditions hold:

(i) \(m_v^- - \eta_{\ell} \leq \sum_{e \in E(V); e \notin \ell} X_e \leq m_v^+ + \eta_{\ell}\) for every \(\ell \in \{1, \ldots, d\}\) and every \(v \in N_\ell\).

(ii) There exists \(\mu > 0\) and a vector \(\lambda\) with strictly positive entries such that:

1. \(s(X_e) \leq \nu_e \cdot \mu \cdot \prod_{\ell=1}^d \lambda_{\ell} \leq s(X_e + 1)\) for every \(e \in E(V)\).

2. For every \(\ell \in \{1, \ldots, d\}\) and every \(v \in N_\ell\), if \(\lambda_v > 1\) then \(\sum_{e \in E(V); e \notin \ell} X_e - m_v^- \leq \eta_{\ell}\).

3. For every \(\ell \in \{1, \ldots, d\}\) and every \(v \in N_\ell\), if \(\lambda_v < 1\) then \(\sum_{e \in E(V); e \notin \ell} X_e - m_v^+ \leq \eta_{\ell}\).

The following theorem formally establishes the notion of tightness mentioned in Subsection 3.B.

Theorem 6. Let \(d \in \mathbb{N}\) with \(d \geq 3\) and let \(f_d : \mathbb{N}^d \to \mathbb{R}_+\) be defined as \(f_d(u) = \sum_{i=1}^d \frac{1}{u_i + 1}\). Let \(h_d : \mathbb{N}^d \to \mathbb{R}_+\) be a continuous function such that \(h_d < f_d\) and the limit \(\lim_{u \to \infty} h_d(u)\) exists and is strictly lower than \(\lim_{u \to \infty} f_d(u)\). Then, there exists a vector \(u \in \mathbb{N}^d\) satisfying \(h_d(u) \leq 1\) for which the following holds: There exists an instance \((N, V, m^-, m^+, H)\) of the \(d\)-dimensional apportionment problem and a signpost sequence \(s\) for which the linear relaxation of (6)-(12) is feasible, such that there is no \(u\)-approximate \(d\)-dimensional proportional apportionment for this instance.

In order to prove this result, we first state two lemmas. The first one establishes the nonexistence of \((0,0,c)\)-approximate 3-dimensional proportional apportionments for certain instances and any \(c \in \mathbb{N}\), while the second one allows extending this result to a higher dimension.

Lemma 6. For every nonnegative integer \(c\), there exists an instance \((N, V, m^-, m^+, H)\) of the 3-dimensional apportionment problem, with \(V\) strictly positive in each entry, and a signpost sequence \(s\) for which the linear relaxation of (6)-(12) is feasible, such that there is no \((0,0,c)\)-approximate 3-dimensional proportional apportionment for this instance.
Lemma 7. Let $d \in \mathbb{N}$ with $d \geq 3$, let $u \in \mathbb{N}^d$ and suppose that there exists an instance of the $d$-dimensional apportionment problem and a signpost sequence $s$ for which the linear relaxation of (6)-(12) is feasible, such that there is no $u$-approximate $d$-dimensional proportional apportionment for this instance. Define $u_{\text{max}} = \max(x \in \{1, \ldots, d\} \cup \mathbb{N})$. Then, there exists an instance of the $(d+1)$-dimensional apportionment problem and a signpost sequence $s$ for which the linear relaxation of (6)-(12) is feasible, such that there is no $(u, u_{\text{max}})$-approximate $(d+1)$-dimensional proportional apportionment for this instance.

We now show how to prove Theorem 6 using these lemmas. In the rest of this appendix, we prove Lemma 6 and Lemma 7.

Proof of Theorem 6. We first claim that for every $d \geq 3$ and every $c \in \mathbb{N}$, there exists an instance of the $d$-dimensional apportionment problem and a signpost sequence $s$ for which the linear relaxation of (6)-(12) is feasible, such that there is no $(0, 0, \ldots, c)$-approximate $d$-dimensional proportional apportionment for this instance. This can be seen by induction over $d \geq 3$. The base case is immediate from Lemma 6 and, given that this is true for a certain $d \geq 3$, Lemma 7 allows to conclude the result for the $(d+1)$-dimensional case.

In order to prove the theorem, let $d \geq 3$ and a continuous function $h_d : \mathbb{N}^d \to \mathbb{R}_+$ satisfying $h_d < f_d$ and $\lim_{u \to \infty} h_d(u) < \lim_{u \to \infty} f_d(u)$. In particular, we have that for every $c \in \mathbb{N}$ it holds $h_d(0, 0, \ldots, c) < f_d(0, 0, \ldots, c) = 1 + (d-2)/(c+2)$, where the last expression tends to $1$ as $c$ tends to infinity. Hence, $\lim_{u \to \infty} h_d(0, 0, \ldots, c) < 1$ and since $h_d$ is continuous, there exists $c \in \mathbb{N}$ such that $h_d(0, 0, \ldots, c) \leq 1$. Since the result of the previous paragraph holds in particular for the $d$-dimensional case and taking $c = e$, the theorem follows.

Proof of Lemma 6. Let $c$ be a nonnegative integer and consider the instance $(N, V, m^-, m^+, H)$ of the 3-dimensional apportionment problem defined as follows: $N = \{N_1, N_2, N_3\}$ with $N_1 = \{p_1, \ldots, p_{2(c+1)}\}$, $N_2 = \{r_1, \ldots, r_{2(c+1)}\}$ and $N_3 = \{g_1, g_2\}$.

The vote matrix $V$ is given by

$$V_{p_1, r_{j1}} = \begin{cases} M & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}, \quad V_{p_1, r_{j2}} = \begin{cases} M & \text{if } i = j + 1 \\ 1 & \text{otherwise} \end{cases}$$

for every $i, j \in \{1, \ldots, 2(c+1)\}$, where $M > 1$ and we denote $p_{2(c+1) + 1} = p_{2(c+1) + 2} = r_1$. Note that $E(V) = N_1 \times N_2 \times N_3$. The marginals are $m^+_1 = m^+_2 = 1$ for every $v \in N_1 \cup N_2$ and $m^+_{g_1} = m^+_{g_2} = m^+_c = c + 1$, and the house size is $H = 2(c+1)$. We also consider any signpost sequence $s$ with $s(1) > 0$.

The linear relaxation of (6)-(12) is feasible for this instance. In fact, defining

$$E' = \{e \in E(V) : V_e = M\} = \bigcup_{i=1}^{2(c+1)} \{(p_i, r_i, g_1), (p_{i+1}, r_i, g_2)\},$$

the pair $(x, y)$ given by $x_{e} = 1/2$ for every $e \in E'$, $x_e = 0$ for every $e \in E(V) \setminus E'$, $y_{e} = 1/2$ for every $e \in E'$, $y_{e} = 0$ for every $e \notin E(V)$ and $t \notin \{2, \ldots, H\}$ verifies constraints (7)-(11) and $y_{e} \in [0, 1]$ for every $e \in E(V)$ and every $t \in \{1, \ldots, H\}$. However, we will show that there is no vector satisfying the definition of $(0, 0, c)$-approximate 3-dimensional proportional apportionment for this instance.

Suppose that $X \in \mathbb{N}(E(V))$ is a $(0, 0, c)$-approximate 3-dimensional proportional apportionment for this instance. This means that $\sum_{e \in E(V)} x_{e} = X_\ell = 1$ for every $\ell \in \{1, 2\}$ and every $v \in N_\ell$, $\sum_{e \in E(V)} y_{e} = X_e = (c+1) \leq c$ for every $v \in N_3$ and there exist values $\mu > 0$, $\nu > 0$ for every $v \in N_1 \cup N_2 \cup N_3$ such that $s(X_v) \leq V_e \cdot \mu \cdot \prod_{t=1}^{3} \lambda_{e_t} \leq s(X_e) + 1$ for every $e \in E(V)$. In the following we use these conditions to obtain a contradiction and conclude that such vector $X$ cannot exist, distinguishing the case $\gamma_{g_1} \neq \gamma_{g_2}$ and the case $\gamma_{g_1} = \gamma_{g_2}$.

If $\gamma_{g_1} \neq \gamma_{g_2}$, we must somehow allow $s$ will not lose generality that $\gamma_{g_1} > \gamma_{g_2}$ (if $\gamma_{g_1} < \gamma_{g_2}$ the proof is analogous). Let $j_1 \in \arg \max_{j \in \{1, \ldots, 2(c+1)\} \setminus \{0\}} \gamma_{r_j}$. Note that $V_{p_1, r_{j1}} = M \geq V_{p_1, r_{j2}}$ for every $j \in \{1, \ldots, 2(c+1)\}$ and every $k \in \{1, 2\}$, with strict inequality if $j \neq j_1$ and $k = 1$. Using that $\gamma_{g_1} > \gamma_{g_2}$, we conclude that

$$\nu_{p_1, r_{j1}} : \nu_{p_1, r_{j2}} \cdot \nu_{r_{j1}} \cdot \nu_{g_1} > \nu_{p_1, r_{j2}} : \nu_{p_1, r_{j2}} \cdot \nu_{r_{j1}} \cdot \nu_{g_2}$$

for every $(j, k) \in \{1, \ldots, 2(c+1)\} \times \{1, 2\}$ with $(j, k) \notin \{(j_1, 1)\}$.

Therefore, from the proportionality condition and the monotonicity of $s$ we obtain that $X_{p_1, r_{j1}} \geq X_{p_1, r_{j2}}$ for every $(j, k) \in \{1, \ldots, 2(c+1)\} \times \{1, 2\}$ with $(j, k) \notin \{(j_1, 1)\}$, and since we know that $\sum_{j=1}^{2(c+1)} \sum_{k=1}^{2(c+1)} X_{p_1, r_{j2}} = 1$ from the marginals condition and $X$ is integral, we conclude that $X_{p_1, r_{j1}} \geq 1$. Since the marginals condition also guarantees $\sum_{j=1}^{2(c+1)} \sum_{k=1}^{2(c+1)} X_{p_1, r_{j1}} = 1$, we can delete $p_1$ and $r_{j1}$ an iterate. Specifically, we now let $j_2 \in \arg \max_{j \in \{1, \ldots, 2(c+1)\} \setminus \{j_1\}} \gamma_{r_j}$ and obtain that

$$\nu_{p_2, r_{j2}} : \nu_{p_2, r_{j2}} \cdot \nu_{r_{j2}} \cdot \nu_{g_1} > \nu_{p_2, r_{j2}} : \nu_{p_2, r_{j2}} \cdot \nu_{r_{j2}} \cdot \nu_{g_2}$$

for every $(j, k) \in \{1, \ldots, 2(c+1)\} \times \{1, 2\}$ with $(j, k) \notin \{(j_1, 1), (j_2, 1)\}$.

\footnote{Note that we write $p_1, r_{j1}$ instead of $(p_1, r_{j1})$ for subindices to make the notation easier.}
so $X_{p_{i_2}r_{j_2}g_1} \geq X_{p_{i_2}r_{j_2}g_k}$ for every $(j, k) \in \{1, \ldots, 2(c+1)\} \times \{1, 2\}$ with $(j, k) \not\in \{(j_1, 1), (j_2, 1)\}$. Since $\sum_{i=1}^{2(c+1)} \sum_{k=1}^{2} X_{p_{i_2}r_{j_2}g_k} = 1$, we conclude as before that $X_{p_{i_2}r_{j_2}g_1} = 1$. Repeating this process allows to conclude that $X$ is given by

$$X_{p_{i}r_{j}g_k} = \begin{cases} 1 & \text{if } i = j \text{ and } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

and therefore $\sum_{i=1}^{2(c+1)} \sum_{j=1}^{2} X_{p_{i}r_{j}g_1} = 2(c + 1)$ and $\sum_{i=1}^{2(c+1)} \sum_{j=1}^{2} X_{p_{i}r_{j}g_2} = 0$, so for each $v \in \{g_1, g_2\}$ we have

$$|\sum_{x \in E(V)} V_x - (c + 1)| = c + 1 > c,$n

violating one of the conditions we have imposed over $X$ and therefore obtaining a contradiction.

We now consider the case $\lambda_{g_1} = \lambda_{g_2}$ and denote this term as $\lambda_g$. We first show that there exists $e \in E(V) \setminus E'$ such that $X_e = 1$. To see this, note that if $X_e = 0$ for every $e \in E(V) \setminus E'$, in order to satisfy the marginals condition without deviations for $N_1$ and $N_2$ we must have $x_p_{i_1}r_{j_1}g_1 + x_p_{i_1}r_{j_1}g_2 = 1$ and $x_p_{i_2}r_{j_2}g_1 + x_p_{i_2}r_{j_2}g_2 = 1$ for every $i \in \{1, \ldots, 2(c+1)\}$, implying $x_p_{i_1}r_{j_1}g_1 = x_p_{i_1}r_{j_1}g_2$ for every $i \in \{1, \ldots, 2(c+1)\}$ and therefore either $x_p_{i_1}r_{j_1}g_1 = 1$ for every $i \in \{1, \ldots, 2(c+1)\}$ or $x_p_{i_2}r_{j_2}g_1 = 0$ for every $i \in \{1, \ldots, 2(c+1)\}$. In any case, this implies $|\sum_{x \in E(V)} V_x - (c + 1)| = c + 1 > c$, violating once again a condition we have imposed over $X$. We conclude that there exists $e \in E(V) \setminus E'$ such that $X_e = 1$, and without loss of generality we assume that $e = (p_{i_0}, r_{i_1}, g_2)$ with $i_0, i_1 \in \{1, \ldots, 2(c+1)\}$ and $i_0 \neq i_1 + 1$. In particular, since $V_e$ we have that

$$s(1) \leq \mu \cdot \lambda_{p_{i_0}} \cdot \lambda_{r_{i_1}} \cdot \lambda_g \leq s(2).$$

If $i_0 = i_1$, since $\sum_{i=1}^{2(c+1)} \sum_{k=1}^{2} X_{p_{i}r_{j}g_k} = 1$ from the marginals condition, we must have $X_{p_{i_0}r_{i_0}g_1} = 0$ and therefore $s(0) \leq V_{p_{i_0}r_{i_0}g_1} + \mu \cdot \lambda_{p_{i_0}} \cdot \lambda_{r_{i_0}} \cdot \lambda_g \leq s(1)$. Nevertheless, $V_{p_{i_0}r_{i_0}g_1} = M$, so using (28) with $i_0 = i_1$ we obtain

$$s(1) \leq \mu \cdot \lambda_{p_{i_0}} \cdot \lambda_{r_{i_0}} \cdot \lambda_g < M \cdot \mu \cdot \lambda_{p_{i_0}} \cdot \lambda_{r_{i_0}} \cdot \lambda_g \leq s(1),$$

which is a contradiction.

If $i_0 \neq i_1$, since $\sum_{i=1}^{2(c+1)} \sum_{k=1}^{2} X_{p_{i}r_{j}g_k} = 1$ from the marginals condition, there exists $i_2 \in \{1, \ldots, 2(c+1)\}$ such that $X_{p_{i_1}r_{i_2}g_k} = 1$ for some $k \in \{0, 1\}$. Since $i_2 \neq i_1$ and $2(c+1)$ is a finite value, we can repeat this argument and conclude that there exists a sequence of pairs $(i_0, i_1), (i_1, i_2), \ldots, (i_{q-1}, i_q), (i_q, i_0)$, with $q \geq 1$, such that the following holds:

(i) $i_j \in \{1, \ldots, 2(c+1)\}$ for every $j \in \{0, \ldots, q\}$.

(ii) If $j, j' \in \{0, \ldots, q\}$ with $j \neq j'$, then $i_j \neq i_{j'}$.

(iii) For every $j \in \{0, \ldots, q\}$ there exists $k \in \{1, 2\}$ such that $X_{p_{i_j}r_{i_{j+1}}g_k} = 1$ (where we denote $r_{i_{j+1}} := r_{i_j}$).

From the proportionality condition we have $s(1) \leq V_{p_{i_j}r_{i_{j+1}}g_k} \cdot \mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_{j+1}}} \cdot \lambda_g \leq M \cdot \mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_{j+1}}} \cdot \lambda_g$ for every $j \in \{1, \ldots, q\}$ and some $k \in \{1, 2\}$, and therefore

$$\prod_{j=1}^{q} (\mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_{j+1}}} \cdot \lambda_g) \geq \left(\frac{s(1)}{M}\right)^q.$$  

On the other hand, the marginals condition applied to $p_{i_0}, \ldots, p_{i_q}$ and $r_{i_0}, \ldots, r_{i_q}$ implies $X_{p_{i_j}r_{i_{j+1}}g_k} = 0$ for every $j \in \{0, \ldots, q\}$. Using the proportionality condition, this implies $s(1) \geq V_{p_{i_j}r_{i_{j+1}}g_k} \cdot \mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_{j+1}}} \cdot \lambda_g = M \cdot \mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_{j+1}}} \cdot \lambda_g$ for every $j \in \{0, \ldots, q\}$, and therefore

$$\prod_{j=0}^{q} (\mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_{j+1}}} \cdot \lambda_g) \leq \left(\frac{s(1)}{M}\right)^{q+1}.$$  

Putting (28), (29) and (30) together, we obtain

$$\prod_{j=0}^{q} (\mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_{j+1}}} \cdot \lambda_g) \leq \left(\frac{s(1)}{M}\right)^{q+1} < \frac{(s(1))^{q+1}}{M^q} \leq \prod_{j=0}^{q} (\mu \cdot \lambda_{p_{i_j}} \cdot \lambda_{r_{i_{j+1}}} \cdot \lambda_g).$$

But the first and the last terms are equal, so this is a contradiction. \[\Box\]

**Proof of Lemma 7.** Let $d \in \mathbb{N}$ with $d \geq 3$, let $u \in \mathbb{N}^d$, let $\{\tilde{N}_1, \ldots, \tilde{N}_d\}, \tilde{V}, \tilde{m}^-, \tilde{m}^+$, $\tilde{H}$ be an instance of the $d$-dimensional apportionment problem and let $\tilde{s}$ be a signpost sequence for which the linear relaxation of (6)-(12) is feasible (denote as $\tilde{x}$ any feasible solution for it) and such that there is no $u$-approximate $d$-dimensional apportionment for this instance. Let $\tilde{e} \in \arg\max_{e \in \{1, \ldots, d\}} u_e$ and denote $\tilde{N}_e = \{v_1, \ldots, v_{|\tilde{N}_e|}\}$. Consider the $(d+1)$-dimensional instance $(N, V, m^-, m^+, H)$ defined as follows. $N = \{\tilde{N}_1, \ldots, \tilde{N}_d, \{w_1, \ldots, w_{|\tilde{N}_e|}\}\}, V_e = \tilde{V}_{e_1} \cdots e_d$ if $e_d = v_i$ and $e_{d+1} = w_i$ for some $i \in \{1, \ldots, |\tilde{N}_e|\}$ and
where the first equality comes from the definition of $\bar{X}$ and the second one from the fact that for every $e \in E(V)$ we have that there exists $i \in \{1, \ldots, |N_{\ell}|\}$ for which $e \in v_i$ and $e_{d+1} = w_i$. Similarly, for any $i \in \{1, \ldots, |N_{\ell}|\}$ the same analysis leads to

$$
\sum_{e \in E(V); e_{d+1} = w_i} X_e = \sum_{e \in E(V); e_{d+1} = w_i} X_e = \sum_{e \in E(V); e_{d+1} = w_i} X_e.
$$

Since the marginals $\bar{m}_v^-$ and $\bar{m}_v^+$ are equal to $m_v^-$ and $m_v^+$, respectively, for every $\ell \in \{1, \ldots, d\}$ and every $v \in N_{\ell}$, condition (i) in the definition of a $u$-approximate $d$-dimensional proportional apportionment follows directly, as well as conditions (ii) (2) and (ii) (3) for every $\ell \in \{1, \ldots, d\}$ \{\ell\} and every $v \in N_{\ell}$, because the corresponding multipliers $\bar{\lambda}_v$ are equal to $\hat{\lambda}_v$. To see condition (ii) (2) for $v \in N_{\ell}$, let $i \in \{1, \ldots, |N_{\ell}|\}$ and note that if $\hat{\lambda}_{v_i} = \lambda_{v_i} \lambda_{w_i} > 1$, then necessarily $\lambda_{v_i} > 1$ or $\lambda_{w_i} > 1$. Since $X$ is a $(u, u_2)$-approximate $(d+1)$-dimensional proportional apportionment, we must have either $|\sum_{e \in E(V); e_{d+1} = w_i} X_e - m_{v_i}^-| \leq u_{d+1}$ or $|\sum_{e \in E(V); e_{d+1} = w_i} X_e - m_{v_i}^-| \leq u_{d+1}$, but we know that $\sum_{e \in E(V); e_{d+1} = w_i} X_e = \sum_{e \in E(V); e_{d+1} = w_i} X_e = \sum_{e \in E(V); e_{d+1} = w_i} X_e$, $m_{v_i}^- = m_{v_i}^- = m_{w_i}^-$, and $u_{d+1} = u_{d+1}$, so we conclude that $|\sum_{e \in E(V); e_{d+1} = w_i} X_e - m_{v_i}^-| \leq u_{d+1}$. The proof of condition (ii) (3) for $v \in N_{\ell}$ is completely analogous. Finally, to see condition (ii) (1), let $e \in E(V)$, define $i \in \{1, \ldots, |N_{\ell}|\}$ such that $e_i = v_i$ and note that

$$
\bar{V}_e : \mu \cdot \prod_{\ell = 1}^d \hat{\lambda}_{e_{d+1}} = \sum_{e_{d+1} = w_i} X_e,
$$

so we conclude that $\bar{X}$, with multipliers $\mu$ and $\lambda$, satisfies this condition as well. This concludes the proof of the claim and the proof of the lemma. \hfill \Box

3. Results from the Chilean Constitutional Convention

In this appendix, we present some tables with detailed results of the methods compared in Section 4. Table S1 shows the percentage of votes obtained by each of the lists that obtains at least one seat in one or both of the evaluated methods, as well as the number and percentage of seats obtained by each of these lists according to each system.

Tables S2 and S3 show the same information as the previous table (omitting the percentage of seats), but in this case by district. For each district, the corresponding table displays only the lists that obtained at least one seat in that district in either system.

Using these results, we calculate the standard deviation of each apportionment with respect to the fair share. These values by district and for the whole country are shown in Table S4.
Table S1. Elected candidates by list.

| List | Votes (%) | Seats CCM | Seats CCM (%) | Seats TPM | Seats TPM (%) |
|------|-----------|-----------|---------------|-----------|---------------|
| XP   | 22.79     | 37        | 26.81         | 33        | 23.91         |
| YQ   | 20.77     | 28        | 20.29         | 30        | 21.74         |
| LP   | 18.28     | 27        | 19.57         | 27        | 19.57         |
| YB   | 16.02     | 25        | 18.12         | 23        | 16.67         |
| INN  | 8.73      | 11        | 7.97          | 12        | 8.7           |
| XA   | 3.78      | 0         | 0             | 5         | 3.62          |
| ZR   | 1.02      | 0         | 0             | 1         | 0.72          |
| ZB   | 0.82      | 0         | 0             | 1         | 0.72          |
| YK   | 0.81      | 1         | 0.72          | 1         | 0.72          |
| YU   | 0.81      | 0         | 0             | 1         | 0.72          |
| ZL   | 0.76      | 0         | 0             | 1         | 0.72          |
| T    | 0.74      | 1         | 0.72          | 1         | 0.72          |
| YX   | 0.73      | 0         | 0             | 1         | 0.72          |
| ZW   | 0.70      | 1         | 0.72          | 1         | 0.72          |
| IND9 | 0.68      | 1         | 0.72          | 0         | 0             |
| XI   | 0.65      | 1         | 0.72          | 0         | 0             |
| ZZ   | 0.61      | 1         | 0.72          | 0         | 0             |
| WB   | 0.46      | 1         | 0.72          | 0         | 0             |
| P    | 0.44      | 1         | 0.72          | 0         | 0             |
| A    | 0.22      | 1         | 0.72          | 0         | 0             |
| XM   | 0.20      | 1         | 0.72          | 0         | 0             |
| District 1 (3 seats) | District 2 (3 seats) | District 3 (4 seats) |
|---------------------|---------------------|---------------------|
| **List**            | **Votes (%)**       | **CCM** | **TPM** |
| INN                 | 18.51               | 0       | 1       |
| XP                  | 22.44               | 1       | 0       |
| YB                  | 27.55               | 1       | 1       |
| YQ                  | 25.66               | 1       | 1       |
| **List**            | **Votes (%)**       | **CCM** | **TPM** |
| A                   | 20.73               | 1       | 0       |
| XP                  | 28.61               | 1       | 1       |
| YQ                  | 34.84               | 1       | 2       |

| District 4 (4 seats) | District 5 (6 seats) | District 6 (8 seats) |
|----------------------|----------------------|----------------------|
| **List**             | **Votes (%)**        | **CCM** | **TPM** |
| INN                  | 19.34                | 1       | 1       |
| LP                   | 28.96                | 2       | 2       |
| XP                   | 11.76                | 1       | 0       |
| WB                   | 17.34                | 1       | 1       |
| YB                   | 10.47                | 1       | 0       |
| YQ                   | 35.56                | 2       | 2       |
| YQ                   | 9.63                 | 0       | 1       |

| District 7 (7 seats) | District 8 (7 seats) | District 9 (6 seats) |
|----------------------|----------------------|----------------------|
| **List**             | **Votes (%)**        | **CCM** | **TPM** |
| LP                   | 22.81                | 2       | 2       |
| XP                   | 24.13                | 2       | 2       |
| YB                   | 15.53                | 1       | 1       |
| YQ                   | 26.90                | 2       | 2       |
| **List**             | **Votes (%)**        | **CCM** | **TPM** |
| INN                  | 20.41                | 1       | 2       |
| XP                   | 16.22                | 1       | 1       |
| YB                   | 10.47                | 1       | 0       |
| YQ                   | 35.56                | 3       | 3       |
| YU                   | 9.63                 | 0       | 1       |

| District 10 (7 seats) | District 11 (6 seats) | District 12 (6 seats) |
|-----------------------|-----------------------|-----------------------|
| **List**              | **Votes (%)**         | **CCM** | **TPM** |
| INN                   | 12.55                 | 1       | 1       |
| LP                    | 11.14                 | 1       | 0       |
| XP                    | 23.04                 | 2       | 2       |
| XA                    | 14.32                 | 1       | 1       |
| YB                    | 24.42                 | 2       | 2       |
| ZL                    | 9.79                  | 0       | 1       |
| **List**              | **Votes (%)**         | **CCM** | **TPM** |
| INN                   | 51.16                 | 4       | 4       |
| XP                    | 15.46                 | 1       | 1       |
| YQ                    | 15.03                 | 1       | 1       |
| YU                    | 9.63                  | 0       | 1       |

| District 13 (4 seats) | District 14 (5 seats) | District 15 (5 seats) |
|-----------------------|-----------------------|-----------------------|
| **List**              | **Votes (%)**         | **CCM** | **TPM** |
| LP                    | 30.39                 | 2       | 2       |
| YB                    | 25.31                 | 1       | 1       |
| YQ                    | 18.86                 | 1       | 1       |
| **List**              | **Votes (%)**         | **CCM** | **TPM** |
| XP                    | 20.69                 | 1       | 1       |
| YB                    | 17.06                 | 1       | 0       |
| YQ                    | 9.63                  | 0       | 1       |
| YQ                    | 24.15                 | 1       | 2       |

| District 16 (4 seats) | District 17 (7 seats) | District 18 (4 seats) |
|-----------------------|-----------------------|-----------------------|
| **List**              | **Votes (%)**         | **CCM** | **TPM** |
| LP                    | 18.87                 | 1       | 1       |
| XP                    | 29.99                 | 1       | 1       |
| YB                    | 19.11                 | 1       | 1       |
| YQ                    | 24.78                 | 1       | 1       |
| **List**              | **Votes (%)**         | **CCM** | **TPM** |
| XI                    | 15.50                 | 1       | 0       |
| XP                    | 26.57                 | 2       | 3       |
| YB                    | 14.49                 | 1       | 1       |
| YQ                    | 21.77                 | 2       | 2       |
Table S3. Votes and elected candidates by list in districts 19 to 28.

| District 19 (5 seats) | District 20 (7 seats) | District 21 (4 seats) |
|----------------------|-----------------------|----------------------|
| **List**            | **Votes (%)** | **CCM** | **TPM** | **List**            | **Votes (%)** | **CCM** | **TPM** | **List**            | **Votes (%)** | **CCM** | **TPM** |
| INN                  | 13.90         | 1       | 1       | INN                  | 12.95         | 1       | 1       | INN                  | 19.42         | 1       | 1       |
| LP                   | 22.22         | 1       | 2       | LP                   | 12.53         | 1       | 0       | XP                   | 23.28         | 1       | 1       |
| XP                   | 26.79         | 2       | 1       | XP                   | 20.60         | 2       | 1       | YB                   | 19.41         | 1       | 0       |
| YB                   | 15.06         | 1       | 1       | YB                   | 16.06         | 1       | 1       | XP                   | 19.58         | 1       | 1       |
| **List**            | **Votes (%)** | **CCM** | **TPM** | **List**            | **Votes (%)** | **CCM** | **TPM** | **List**            | **Votes (%)** | **CCM** | **TPM** |
| INN                  | 13.90         | 1       | 1       | INN                  | 12.95         | 1       | 1       | INN                  | 19.42         | 1       | 1       |
| LP                   | 22.22         | 1       | 2       | LP                   | 12.53         | 1       | 0       | XP                   | 23.28         | 1       | 1       |
| XP                   | 26.79         | 2       | 1       | XP                   | 20.60         | 2       | 1       | YB                   | 19.41         | 1       | 0       |
| YB                   | 15.06         | 1       | 1       | YQ                   | 16.06         | 1       | 1       | YQ                   | 19.58         | 1       | 1       |
| INN                  | 13.90         | 1       | 1       | INN                  | 12.95         | 1       | 1       | INN                  | 19.42         | 1       | 1       |
| LP                   | 22.22         | 1       | 2       | LP                   | 12.53         | 1       | 0       | XP                   | 23.28         | 1       | 1       |
| XP                   | 26.79         | 2       | 1       | XP                   | 20.60         | 2       | 1       | YB                   | 19.41         | 1       | 0       |
| YB                   | 15.06         | 1       | 1       | YQ                   | 16.06         | 1       | 1       | YQ                   | 19.58         | 1       | 1       |

| District 22 (3 seats) | District 23 (6 seats) | District 24 (4 seats) |
|-----------------------|-----------------------|-----------------------|
| **List**             | **Votes (%)** | **CCM** | **TPM** | **List**             | **Votes (%)** | **CCM** | **TPM** | **List**             | **Votes (%)** | **CCM** | **TPM** |
| XP                   | 34.74         | 2       | 2       | INN                  | 13.19         | 1       | 1       | XP                   | 14.73         | 0       | 1       |
| YB                   | 25.71         | 1       | 1       | LP                   | 18.76         | 1       | 1       | LP                   | 23.67         | 1       | 1       |
| XP                   | 29.34         | 1       | 1       | XP                   | 25.52         | 2       | 1       | YB                   | 29.78         | 2       | 2       |
| YB                   | 15.18         | 1       | 1       | XP                   | 22.75         | 1       | 1       | XP                   | 22.75         | 1       | 1       |
| YQ                   | 12.70         | 1       | 1       | YQ                   | 14.97         | 1       | 0       | YQ                   | 19.22         | 1       | 0       |

| District 25 (3 seats) | District 26 (4 seats) | District 27 (3 seats) |
|-----------------------|-----------------------|-----------------------|
| **List**             | **Votes (%)** | **CCM** | **TPM** | **List**             | **Votes (%)** | **CCM** | **TPM** | **List**             | **Votes (%)** | **CCM** | **TPM** |
| XP                   | 36.67         | 2       | 1       | INN                  | 14.63         | 1       | 0       | XP                   | 29.51         | 1       | 1       |
| YB                   | 24.20         | 1       | 1       | LP                   | 21.55         | 1       | 1       | YB                   | 34.13         | 1       | 2       |
| YQ                   | 18.03         | 0       | 1       | XP                   | 22.75         | 1       | 1       | YQ                   | 19.22         | 1       | 0       |

| District 28 (3 seats) |
|-----------------------|
| **List**             | **Votes (%)** | **CCM** | **TPM** |
| LP                   | 18.95         | 1       | 1       |
| XM                   | 20.25         | 1       | 0       |
| XP                   | 19.84         | 1       | 1       |
| YQ                   | 18.45         | 0       | 1       |
Table S4. Standard deviation of the list distribution with respect to the fair share.

| District | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|----------|------|------|------|------|------|------|------|------|------|------|
| CCM      | 23.52| 13.54| 14.44| 14.43| 7.04 | 10.89| 7.57 | 15.2 | 16.78| 12.52|
| TPM      | 28.56| 38.28| 32.04| 23.88| 19.45| 15.1 | 7.57 | 15.96| 28.76| 13.97|

| District | 11   | 12   | 13   | 14   | 15   | 16   | 17   | 18   | 19   | 20   |
|----------|------|------|------|------|------|------|------|------|------|------|
| CCM      | 15.65| 14.01| 20.55| 9.34 | 11.04| 9.86 | 7.25 | 19.1 | 15.52| 12.24|
| TPM      | 15.65| 14.01| 20.55| 24.57| 30.45| 9.86 | 23.51| 33.77| 20.59| 24.55|

| District | 21   | 22   | 23   | 24   | 25   | 26   | 27   | 28   | Country |
|----------|------|------|------|------|------|------|------|------|---------|
| CCM      | 13.07| 32.83| 11.07| 26.98| 36.17| 11.17| 14.64| 30   | 6.44    |
| TPM      | 26.55| 32.83| 15.36| 27.20| 18.13| 29.68| 37.99| 31.94| 2.49    |
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