Generalized quantum measurements and local realism

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Abstract

The structure of a local hidden variable model for experiments involving sequences of measurements is analyzed. Constraints imposed by local realism on the conditional probabilities of the outcomes of such measurement schemes are explicitly derived. The violation of local realism in the case of “hidden nonlocality” is illustrated by an operational example.
I. INTRODUCTION

The question of the possibility of the existence of a local realistic model for quantum predictions for pure entangled states has found a negative answer. Namely, Gisin [1] proved that the only pure states of two-component system which do not violate the Bell-CHSH inequality are the product states (states of such a property are often, slightly misleadingly, called the “local” ones). These results, improved by Gisin and Peres [2], have been generalised by Rohrlich and Popescu [3] to N-component quantum systems.

However in the case of mixed states the problem becomes much more complicated. One might naively think that analogously to the case of pure states, the only mixed states which do not violate Bell’s inequalities are the mixtures of product states (i.e. separable states). In his pioneering paper Werner [4] showed that this conjecture is false. He studied the possibility of a direct construction of a local hidden variable (LHV) model for some families of mixed states, and showed that there is a class of nonseparable mixtures for which the results of performed measurements can be simulated by such a model. However, in a recent development, Popescu [5] noticed that Werner had considered only a restricted class of measurement procedures. Namely, Werner constructed a LHV model for single (i.e., nonsequential) von Neumann measurements. Later, again, Popescu [6] was the first to show that most of the Werner mixtures exhibit violations of local realism if sequences of measurements are taken into account. Such an exposure of the so-called hidden nonlocality [6] involves sequences of two measurements. The initial ensemble of two particle systems is subjected to the first measurement. Afterwards, a subensemble of the pairs which produced some required outcome is selected and tested by measuring the Bell observable. If the subensemble does not satisfy Bell inequalities, then one concludes that the original ensemble violates local realism.

Recently, Gisin [7] has shown that for two-level systems the nonlocality can be revealed by using filters at the first stage of the process (a procedure of this kind can be treated as generalized measurement). Quite recently, Peres [8] considered collective tests of parti-
cles in the Werner state and used consecutive measurements to show the impossibility of constructing a local realistic description for some processes of this kind.

In this paper we aim at a description of “hidden nonlocality” which is parallel to the original Bell approach. Namely, Bell simply showed that some functions cannot be simulated by suitable averages of product ones. It is rather obvious that for better understanding of a “hidden nonlocality” the latter should be presented in complete analogy to Bell’s standard nonlocality.

Rigorous proof that the local hidden variable model for joint probabilities of outcomes of sequences of measurements cannot exist in the case of “hidden nonlocality” will be given in section 2. More precisely, we will explicitly show that, if the conditional probabilities cannot be simulated by suitable averages of product of functions depending upon hidden variables then also the joint probabilities of sequences of outcomes cannot be represented in this way. In section 3 we also provide a proposal of a feasible experiment for which there is no local-realistic description despite no violation of the B-CHSH inequality for standard (non-sequential) experiments.

II. LOCAL REALISTIC DESCRIPTION OF SEQUENCES OF MEASUREMENTS

The local hidden variable model for joint probabilities for obtaining the results $a$ and $b$ upon the performance of the local measurements $A$, $B$, on an ensemble of pairs of particles in a certain quantum-mechanical state, must have the following structure

$$P_{A,B}(a, b) = \int P_A(a; \lambda) \tilde{P}_B(b; \lambda) \varrho(\lambda) d\lambda,$$  

where $\lambda$ is the hidden variable, $\varrho$ is its probability distribution (and is independent of the choice of $A$ and $B$), and $P_A(a; \lambda)$ and $\tilde{P}_B(b; \lambda)$ are the probabilities of obtaining specific

$^1$In fact, it has been proved, that all the inseparable Werner states are nonlocal, by using a recursive protocol in the process of the so-called distillability. More generally, quite recently it is been shown that any inseparable $2 \times 2$ system can be distilled.
results, provided we measure the specified observable \((A\) or \(B)\), and the element of the ensemble is in the hidden variable state \(\lambda\) (for further details, please consult [12]). Usually, to show that statistics produced by quantum mechanical states cannot be simulated by the above formula, \(A\) and \(B\) have been treated as single von Neumann measurements (represented by Hermitian operators). However quantum mechanics allows us to predict statistics of results of much more complicated experiments. Consequently one should put \(A = \{A_1, \ldots A_{n_A}\}\) and \(B = \{B_1, \ldots B_{n_B}\}\) where \(A_i, B_i\) denote single generalized measurements (not only von Neumann ones), with sets of results \(I_{A_i}\) and \(I_{B_i}\) respectively. Then the formula (1) can be rewritten as

\[
P_{A_1^1,\ldots A_i^1, A_{i'},\ldots A_{n_A}, B_1^1,\ldots B_i^1, B_{i'},\ldots B_{n_B}}(a_1^1,\ldots a_i^1, b_1^1,\ldots b_{i^1}, a_{i'}^1,\ldots a_{n_A}^1, b_{i'}^1,\ldots b_{n_B}^1) = \int P_{A_1^1,\ldots A_i^1, A_{i'},\ldots A_{n_A}, B_1^1,\ldots B_i^1, B_{i'},\ldots B_{n_B}}(a_1^1,\ldots a_i^1, b_1^1,\ldots b_{i^1}; \lambda) \tilde{P}_{B_1^1,\ldots B_{i'}^1, B_{i'},\ldots B_{n_B}}(b_1^1,\ldots b_{i'}^1; \lambda) \varrho(\lambda) d\lambda,
\]  

(2)

where \(P_{A_1^1,\ldots A_i^1, B_1^1,\ldots B_{i^1}}(a_1^1,\ldots a_i^1, b_1^1,\ldots b_{i^1})\) stands for the joint probability of obtaining outcomes \(\{a_1^1,\ldots a_i^1, b_1^1,\ldots b_{i^1}\}\) in the measurements \(\{A_1^1,\ldots A_i^1, B_1^1,\ldots B_{i^1}\}\). It should be emphasized here that the measurements are performed on a single (but compound) system which is a member of the ensemble. Now, we can ask whether the statistics predicted by quantum mechanics can be reproduced by the above formula, i.e. whether the statistics can be ascribed to the subsystems in a local-realistic way. Below we will derive constraints implied by the existence of a the local hidden variable model of the form (2). These constraints can be treated as the ones of Bell, albeit generalized to the case of sequences of measurements.

For simplicity we will take into account an experiment consisting of two consecutive measurements on each subsystem. The LHV model must then have the form of

\[
P_{A_1^1, A_2^1, B_1^2, B_2^2}(a_1^1, a_2^1, b_1^2, b_2^2) = \int P_{A_1^1, A_2^1}(a_1^1, a_2^1; \lambda) \tilde{P}_{B_1^2, B_2^2}(b_1^2, b_2^2; \lambda) \varrho(\lambda) d\lambda.
\]  

(3)

Consider the conditional probability

\[
P_{A_1^1, A_2^1, B_1^2, B_2^2}(a_2^2|a_1^1, b_1^2) = \frac{P_{A_1^1, A_2^1, B_1^2, B_2^2}(a_1^1, a_2^1, b_1^2, b_2^2)}{\sum_{a_2^1 \in I_{A_2^1}} P_{A_1^1, A_2^1, B_1^2, B_2^2}(a_1^1, a_2^1, b_1^2, b_2^2)},
\]  

(4)
which is the probability of obtaining outcomes $a_2$ and $b_2$ in the measurement $A_2$, $B_2$ given that the measurement $A_1$, $B_1$ produced outcomes $a_1$, $b_1$, respectively. Now we ask whether the form of the formula (3) implies a similar one for the conditional probabilities (4).

Let us introduce the following shorthand notation

$$P_{A_1,A_2,B_1,B_2}(a_1,b_1) = \sum_{a_2' \in I_{A_2}} \sum_{b_2' \in I_{B_2}} P_{A_1,A_2,B_1,B_2}(a_1,a_2',b_1,b_2'),$$

(5)

and by $P_{A_1,A_2}(a_1;\lambda)$ and $\tilde{P}_{B_1,B_2}(b_1;\lambda)$ let us denote the marginals of $P_{A_1,A_2}(a_1,a_2;\lambda)$ and $\tilde{P}_{B_1,B_2}(b_1,b_2;\lambda)$, respectively. Finally, $P_{A_1,A_2}(a_1|a_2;\lambda)$, $P_{B_1,B_2}(b_1|b_2;\lambda)$ are the appropriate conditional probabilities. Therefore, one can write

$$P_{A_1,A_2,B_1,B_2}(a_2,b_2|a_1,b_1) = \int P_{A_1,A_2}(a_1|a_2;\lambda) \tilde{P}_{B_1,B_2}(b_1|b_2;\lambda) \frac{P_{A_1,A_2}(a_1;\lambda)\tilde{P}_{B_1,B_2}(b_1;\lambda)}{P_{A_1,A_2,B_1,B_2}(a_1,b_1)} \varrho(\lambda) d\lambda.$$

(6)

Now, we observe that the conditional probability is given by the average of the product $P_{A_1,A_2}(a_1|a_2;\lambda)P_{B_1,B_2}(b_1|b_2;\lambda)$ over the new probability distribution $\varrho_{A_1,A_2,B_1,B_2}(a_1,b_1;\lambda)$ defined as

$$\varrho_{A_1,A_2,B_1,B_2}(a_1,b_1;\lambda) = \frac{P_{A_1,A_2}(a_1;\lambda)\tilde{P}_{B_1,B_2}(b_1;\lambda)}{P_{A_1,A_2,B_1,B_2}(a_1,b_1)} \varrho(\lambda).$$

(7)

Note, that what we have done so far is based only upon one assumption of a physical nature contained in (3), while the rest consist of merely mathematical manipulations. Now, we will use an argumentation based upon the principles of local-realism [3]: since the measurement $A_2$, $(B_2)$ is performed after $A_1$, $(B_1)$, therefore the probabilities $P_{A_1,A_2}(a_1;\lambda)$ and $P_{B_1,B_2}(b_1;\lambda)$ cannot depend on $A_2$ and $B_2$, respectively. Further, one can put $P_{A_1,A_2,B_1,B_2}(a_1,b_1) = P_{A_1,B_1}(a_1,b_1)$. Otherwise, we would obtain violation of causality. Thus one can drop the indices $A_2$ and $B_2$ in the distribution (7). Now given arbitrarily chosen measurements $A_1$, $B_1$ and certain outcomes $a_1$, $b_1$, one can denote by $X$ the full set of these conditions (i.e., $X \equiv \{A_1,B_1,a_1,b_1\}$). Thus, the conditional probabilities $P_{A_2,B_2}^X(a_2,b_2) \equiv P_{A_1,A_2,B_1,B_2}(a_2,b_2|a_1,b_1)$ acquire the following form
where $g^X$ is a probability distribution, which is independent of the particular choice of the measurements $A_2$, $B_2$. As $X$ is independent of $A_2$, $B_2$, and therefore also of $a_2$ and $b_2$, the conditional probabilities acquire the typical form for the standard local hidden variable models (8). Obviously, they satisfy Bell’s inequalities. This means that local realism implies that any subensemble selected by local measurements is describable by a LHV model. Thus, if one can show that according to quantum mechanics such a sub-ensemble violates certain Bell’s inequalities, this implies that the original state (for the whole ensemble) does not allow any local-realistic description of sequential measurements. I.e., one can reveal the impossibility of constructing the LHV model for joint probabilities of sequences measurements indirectly, by checking that some conditional probabilities do not admit the model. This task is much easier, as we can use standard Bell’s inequalities, as proposed by Popescu [6], while in general we do not have their counterpart for distributions of rank larger than two. What is very important, the subensemble is selected before the second measurement (of a Bell observable). However, results of the second measurement cannot be described in a local realistic way. Thus, as a subensemble does not admit a LHV description, the full ensemble does not either. In simple words, the existence of a model given by (8) is a necessary condition for the existence of model (3).

The temporal sequence: first the pre-selection, later the actual measurement of the Bell observable, is essential here. Otherwise we would have dealt with a problem equivalent to the one associated with inefficient detection (the so-called detection loophole). Suppose for a while that the measurement $A_1$, $B_1$ is performed after $A_2$, $B_2$ and the outcomes are simply detection or nondetection of the particle. Then the probability (4) is the a posteriori conditional probability under the condition that both of the particles of the pair are detected. Of course, now one cannot drop the indices $A_1$, $B_1$ in the distribution (7). Hence, the condition (3) does not imply similar constraints for the a posteriori probabilities. In other words, if we deal with postselection, the hidden variable can contain the information how
to modify the probability of the outcome of the second measurement according to what observable was measured in the first one. In contrast, if the first measurement is used for a pre-selection, its result is (in accordance with local-realism) independent of what is to be measured in future. Otherwise, causality (locality) does not apply anymore.

With the above reasoning leading to (8), we are now able to broaden the class of known quantum states which have statistical properties which violate the assumptions of local realism. In the quantum case, the general measurement is described by a partition of unity \( \{V_i\}_{i=1}^n \) where \( \sum V_i V_i^\dagger = I \), and each \( V_i \) corresponds to a particular outcome, the probability of which, if the system is in the state \( \rho \), is \( p_i = \text{Tr}(V_i \rho V_i^\dagger) \). If the measurement produces the outcome \( V_i \) the system ends up in the state given by

\[
\tilde{\rho} = \frac{V_i \rho V_i^\dagger}{\text{Tr}V_i \rho V_i^\dagger}.
\]

Thus for any state \( \rho \) acting on Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), if the state \( \tilde{\rho} \) given by

\[
\tilde{\rho} = \frac{V \otimes W \rho V^\dagger \otimes W^\dagger}{\text{Tr}V \otimes W \rho V^\dagger \otimes W^\dagger},
\]

with arbitrary (bounded) operators \( V, W \), does not admit the LHV model for single von Neumann measurements, then the state \( \rho \) violates local realism in sequential measurements. Indeed, one can take as \( A_1 \) and \( B_1 \) the partitions of unity given by \( \{\tilde{V}, \sqrt{I - \tilde{V}\tilde{V}^\dagger}\} \) and \( \{\tilde{W}, \sqrt{I - \tilde{W}\tilde{W}^\dagger}\} \) (where \( \tilde{V} = \frac{V}{||V||} \) with \( ||V|| \) being operator norm of \( V \)) respectively. Here \( \tilde{V} \) and \( \tilde{W} \) should be associated with the outcomes \( a_1 \) and \( b_1 \).

Below we will study an experimental proposal aimed at revealing empirically the non-existence of a local realistic model for a two photon state which does not violate the usual B-CHSH inequality. The scheme is based on the filtering method proposed by Gisin [7].

### III. OPERATIONAL EXAMPLE

In this section we shall present an experimental setup which can be used to demonstrate violation of local realism in the case of sequences of measurements. In other words, we aim at presenting an operational discussion of the thesis presented earlier.
Throughout this section we shall employ solely standard techniques of experimental quantum optics. As the primary sources we shall use laser pumped non-linear crystals in which the phenomenon of parametric down conversion leads to production of pairs of entangled photons.

The exemplary initial state of the two-photon system has been suggested in [11] and is given by

$$\rho = \sum_{i=1}^{2} p_i |\psi_i\rangle\langle \psi_i|,$$

with $p_1 \neq p_2$, and

$$|\psi_1\rangle = \alpha |2\rangle|2\rangle + \beta |1\rangle|1\rangle,$$

$$|\psi_2\rangle = \alpha |2\rangle|1\rangle + \beta |1\rangle|2\rangle,$$

where, we have used the convention that the first ket describes the first subsystem, etc., and we have $\langle 1'|2\rangle = \langle 1|2 \rangle = 0$. The coefficients $\alpha$ and $\beta$ are real and satisfy

$$(p_1 - p_2)^2 \leq (\alpha^2 - \beta^2)^2.$$  (14)

Such states do not violate the CHSH-Bell inequality. Below, it will be shown, in an operational manner, that $\rho$ leads to statistical correlations for sequential local experiments that cannot be reproduced by any local hidden variable theory.

First, let us concentrate on the question of actually producing such states. To this end we propose to use two non-linear crystals, as shown in fig.1. Both are pumped by a single laser. Its coherent beam is split by a beamsplitter of reflectivity $|\alpha|^2$ and transmittivity $|\beta|^2$. At each crystal the spontaneous down conversion process can happen (for a detailed description of this phenomenon, see e.g. [13]). The two photon radiation of the pair of crystals can be described by

$$|\psi\rangle = \alpha |2\rangle|2\rangle + \beta |1\rangle|1\rangle$$

(for details, consult the figure 1). The stable phase relation between the two components of $|\psi\rangle$ can be obtained provided the optical paths linking the crystals with the beamsplitter differ by much less than the coherence length of the laser radiation.
The radiation in the modes 2" and 1" enters a Mach-Zehnder interferometer. If one assumes that the beamsplitters of this device are symmetric 50-50 ones, then when the relative phase shift between the arms is 0 the Mach-Zehnder interferometer acts effectively as a mirror, whereas when the phase shift is \( \pi \) it behaves like a perfectly transparent object. By this we mean that in the first case the state \( |2"\rangle \) is transformed into \( |1'\rangle \) and \( |1"\rangle \) into \( |2'\rangle \) (fig. 1). In the “transparent” mode the state \( |2"\rangle \) changes into \( |2'\rangle \) and \( |1"\rangle \) into \( |1'\rangle \).

The phase shift in the interferometer should change between the two values very rapidly and in a stochastic manner. This can be achieved by various mechanical, acusto-optical, or other methods. In this way the output of the full device of fig. 1 is described by the density matrix \( \rho \).

The usual procedure is to perform a Bell type experiment on the initial two particle system. If one aims at an experiment in which each photon is effectively describable by a two dimensional Hilbert space, one can achieve this by placing on the way of each of the photons a Mach-Zehnder interferometer (fig.2). This device enables one to perform any \( U(2) \) transformation [14]. However, in our case we aim at pre-selecting the ensemble of photon pairs. To this end we place a beamsplitter in the path 2. One of its outputs is fed to a local Mach-Zehnder device, whereas the other one is directed towards a detector \( D \). The full setup is presented in fig.3. To make the measurements sequential in time the length of the optical path, \( l \), between the beamsplitter and the Mach-Zehnder interferometer should satisfy \( l \gg \Delta T_c \), where \( \Delta T \) is the resolution time of the detectors employed.

The beamsplitter \( BS \) of fig.3 has a suitably chosen transmittivity \( |\beta/\alpha|^2 \). Also we assume it to be a symmetric device, which upon reflection adds a phase shift of \( \pi/2 \). Thus the state \( |2\rangle \) can be transformed by BS into

\[
(\beta/\alpha)|2\rangle + i\sqrt{1 - (\beta/\alpha)^2}|D\rangle,
\]

where \( |D\rangle \) denotes the state of a photon on its way to the detector \( D \).

The sub-ensemble of coincident counts behind both Mach-Zehnder interferometers of fig.3 is effectively described by a new density matrix which reads
\begin{equation}
\rho' = \sum_{i=1}^{2} p_i |\psi'_i\rangle\langle \psi'_i|,
\end{equation}

where

\begin{align}
|\psi'_1\rangle &= \frac{1}{\sqrt{2}}(|2\rangle|2'\rangle + |1\rangle|1'\rangle), \\
|\psi'_2\rangle &= \frac{1}{\sqrt{2}}(|2\rangle|1'\rangle + |1\rangle|2'\rangle).
\end{align}

The mixed state described by \( \rho \) violates the CHSH inequalities (as shown by [11]). Thus, via a local selection process we get a subensemble of results which cannot be described by any local hidden variable theory. Therefore, one can infer from the discussion presented in Section 2 that the initial state \( \rho \) (of the full ensemble) gives predictions for sequential measurements which cannot have a local realistic interpretation.

Note added: After this paper was completed, we got the manuscripts of Popescu an Mermin (Conference in Haifa, 1995). The authors present the results contained in the second section (“Local realistic description of sequences of measurements”) of our paper.

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FIGURES

FIG. 1. Generation of the required initial mixed state. Two parametric down converters (PDC) are pumped by a single cw laser. The beamsplitter BS has reflectivity $|\alpha|^2$ and transmittivity $|\beta|^2$. Both optical paths linking this beamsplitter with the PDC’s differ by much less than the coherence length of the laser radiation. A Mach-Zehnder interferometer is placed to the right with respect to the primary PDC sources ($M$ stands for mirror, $\phi$ is the phase shift, the beamsplitters BS are symmetric 50 − 50 ones). The internal phase $\phi$ stochastically jumps between the two values: 0, with probability $p_1$, and $\pi$, with probability $p_2 = 1 - p_1$. The resulting density matrix $\rho$ of the two-photon radiation is given in the main text.

FIG. 2. Mach-Zehner interferometer. The external and internal phase shifters enable one to obtain any $U(2)$ transformation (modulo certain overall phase shifts). Compare the previous figure.

FIG. 3. The overall experimental configuration. The two Mach-Zehnder interferometers (MZ) enable one to measure any dichotomic observables [14] (compare fig.1). We insert a beamsplitter of trasmittivity $|\beta/\alpha|^2$ into the path 2. Only those photons which do not activate the detector $D$ can produce coincidences behind the two spatially separated Mach-Zehnder interferometers. The time required for the light to travel from BS to the left MZ interferometer should be longer than the resolution time of the detectors.