BALANCED OPERATORS AND OPERATOR DOMAINS

KONRAD SCHMÜDGEN

Abstract. We shall say that a densely defined closed operator $T$ on a Hilbert space is balanced if $\mathcal{D}(T) = \mathcal{D}(T^*)$. Balanced operators are described in terms of their phase operators and their moduli. Examples of balanced operators are developed. A characterization of the domain equality $\mathcal{D}(A) = \mathcal{D}(B)$ for positive self-adjoint operators $A$ and $B$ with bounded inverses is given in terms of their spectral measures.

AMS Subject Classification (2020). 47A05, 47B25.

Key words: unbounded operator, operator domain, polar decomposition, spectral measure

1. Introduction

The aim of this paper is to introduce and study a class of unbounded operators on Hilbert space, called balanced operators.

Suppose $T$ is a densely defined closed operator on a Hilbert space $\mathcal{H}$ with domain $\mathcal{D}(T)$ and let $T^*$ denote its adjoint operator. The crucial definition is the following.

Definition 1. The operator $T$ is called balanced if $\mathcal{D}(T) = \mathcal{D}(T^*)$.

Obviously, any bounded operator with domain $\mathcal{D}(T) = \mathcal{H}$ is balanced, so this notion is only of interest for unbounded operators. Clearly, $T$ is balanced if and only if $T^*$ is balanced.

In unbounded operator theory the bad behavior of an operator $T$ goes often along with a “large” difference between the domains of $T$ and $T^*$. A typical example are the minimal and the maximal operators of a partial differential operator acting in $L^2(\Omega)$ for some domain $\Omega$ of $\mathbb{R}^d$.

For another example, let $T$ be a densely defined closed symmetric operator. Then we have $T \subseteq T^*$ and the difference between the domains of $T$ and $T^*$ is nicely described by the classical von Neumann formula

$$\mathcal{D}(T^*) = \mathcal{D}(T) + \mathcal{N}(T^* - \lambda I) + \mathcal{N}(T^* - \overline{\lambda} I) \quad \text{for } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$ 

Thus, $T$ is balanced if and only if $T$ is self-adjoint and the balanced extensions of $T$ are precisely the self-adjoint extensions of $T$. Obviously, a formally normal operator is balanced if and only if it is normal. Generally speaking, for a number of classes of unbounded operators the balanced operators are precisely the “good” operators of this class. The examples in Section 3 support this statement.

This paper is organized as follows. In Section 2 we develop some basic facts on operator domains and balanced operators. In Section 3 we give a number of interesting weighted shift operators that are balanced.

The main result of this paper (Theorem 19) is proved in Section 5. It contains necessary and sufficient conditions for a densely defined closed operator $T$ to be balanced in terms of the spectral measure of the modulus $|T|$ and the phase operator $U$ appearing in the polar decomposition $T = U|T|$.
The crucial part in the proof of Theorem \[10\] is a criterion for the domain equality \( D(A) = D(B) \) of positive self-adjoint operators \( A \) and \( B \) in terms of their spectral measures (Theorems \[15\] and \[16\]). This result is proved in Section \[4\] and it seems to be of interest in its own.

Let us collect some notations that will be used throughout this paper. The symbol \( H \) refers always to a Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \).

If \( T \) is an operator on \( H \), we denote its domain by \( D(T) \), its kernel by \( \mathcal{N}(T) \) and its the range by \( \mathcal{R}(T) \). For \( \alpha \in \mathbb{R} \), we write \( T + \alpha \) for \( T + \alpha I \). The spectral measure of a self-adjoint operator \( T \) is denoted by \( E_T \).

2. Preliminary facts on operator domains and balanced operators

Suppose that \( T \) is a densely defined closed operator on a Hilbert space \( H \). It is an elementary fact of operator theory that \( |T| := \sqrt{T^* T} \) is a positive self-adjoint operator on \( H \) such that \( D(|T|) = D(T) \). Replacing \( T \) by \( T^* \), we obtain \( D(|T^*|) = D(T^*) \). Therefore, \( T \) is balanced if and only if \( D(|T|) = D(|T^*|) \).

Next we recall some known facts on domains of unbounded operators. For the convenience of the reader, we include proofs of all results.

**Lemma 2.** Suppose that \( B \) is a closed operator and \( A \) is a positive self-adjoint operator with bounded inverse on \( H \). Then we have \( D(A) \subseteq D(B) \) if and only the \( X := BA^{-1} \) is a bounded operator defined on the Hilbert space \( H \). In this case, the adjoint operator \( X^* \) is the closure of the operator \( A^{-1}B \).

**Proof.** Since \( A \) is positive and has a bounded inverse, the domain \( D(A) \), equipped with the scalar product \( \langle \varphi, \psi \rangle' := \langle A \varphi, A \psi \rangle \), is a Hilbert space \( H_A \). Since \( B \) is closed and \( D(A) \subseteq D(B), \) \( B : H_A \rightarrow H \) is a closed mapping. By the closed graph theorem, \( B \) is continuous, that is, there exists a constant \( \gamma > 0 \) such that

\[
\|B \varphi\| \leq \gamma \|A \varphi\|, \quad \varphi \in D(A).
\]

Suppose \( \psi \in H \). Since \( A \) has a bounded inverse, \( \mathcal{R}(A) = H \), so there is a \( \varphi \in D(A) \) such that \( \psi = A \varphi \). Then \( \varphi = A^{-1} \psi \) and therefore \( \|X \psi\| = \|BA^{-1} \psi\| \leq \gamma \|\psi\| \) by \[1\]. This shows that \( X \) is bounded and defined on the whole Hilbert space \( H \).

Let \( \eta \in D(B) \). Then \( \langle \psi, X^* \eta \rangle = \langle X \psi, \eta \rangle = \langle A^{-1} \psi, B \eta \rangle = \langle \psi, A^{-1} B \eta \rangle \) for \( \psi \in H \). Hence \( X^* \supseteq A^{-1} B \) and \( X^* \) is the closure of \( A^{-1} B \). \( \square \)

**Proposition 3.** Suppose \( T \) and \( S \) are densely defined closed operators on a Hilbert space \( H \). Let \( \alpha > 0 \) and \( \beta > 0 \). We define operators \( X \) and \( Y \) by

\[
X := (|S| + \beta)(|T| + \alpha)^{-1} \quad \text{and} \quad Y := (|T| + \alpha)(|S| + \beta)^{-1}.
\]

Then the following statements are equivalent:

(i) \( D(T) = D(S) \).

(ii) The operators \( X \) and \( Y \) are bounded and defined on the whole Hilbert space.

(iii) The exists an operator \( Z \in \mathcal{B}(H) \) with inverse \( Z^{-1} \in \mathcal{B}(H) \) such that

\[
(|T| + \alpha)^{-1} = (|S| + \beta)^{-1} Z.
\]

**Proof.** (i)\((\Rightarrow)\): We apply Lemma \[2\] twice, with \( B = |S| + \beta, A = |T| + \alpha, \) and with \( A = |S| + \beta, B = |T| + \alpha \).

(ii)\((\Rightarrow)\): Set \( Z := X \). Then \( Z \in \mathcal{B}(H) \) by (ii). From \[2\] it follows that

\[
(|S| + \beta)^{-1} Z = (|T| + \alpha)^{-1},
\]

which proves \[3\]. Further, \[2\] implies that \( XY = YX = I \), that is, the bounded operator \( Y \in \mathcal{B}(H) \) is the inverse of \( Z = X \).

(iii)\((\Rightarrow)\): Using \[3\] we derive

\[
D(T) = D(|T| + \alpha) = \mathcal{R}((|T| + \alpha)^{-1}) \subseteq \mathcal{R}((|S| + \beta)^{-1}) = D(|S| + \beta) = D(S).
\]
Applying the adjoint to equation (3) yields

\[ Z^*(|S| + \beta)^{-1} = (|T| + \alpha)^{-1}. \]

Interchanging \(T \) and \( S \) in the preceding reasoning and using (4) instead of (3) we obtain \( D(S) \subseteq D(T) \). Thus \( D(T) = D(S) \), which proves (i).

Suppose \( D(T) = D(S) \). Then the operators \( X \) and \( Y \) defined by (2) have nice properties: They are bounded and inverse to each other, \( X^* \) is a bijection of \( D(S) \) onto \( D(T) \) and \( Y^* \) is a bijection of \( D(T) \) onto \( D(S) \).

In the special case \( S = T^* \), the equivalence (i) \( \leftrightarrow \) (ii) in Proposition \( \mathbf{4} \) gives the following criterion.

**Corollary 4.** Let \( \alpha > 0 \) and \( \beta > 0 \). A densely defined closed operator \( T \) on a Hilbert space is balanced if and only if

\[ (|T| + \beta)(|T| + \alpha)^{-1} \quad \text{and} \quad (|T| + \alpha)(|T^*| + \beta)^{-1}. \]

are bounded operators defined on the whole Hilbert space.

**Remark 5.** In the literature, domains of closed operators are often studied in terms of operator ranges. An operator range is the range \( R(X) = XH \) of some bounded operator \( X \in B(H) \). Since \( X \) and \( \|X^*\| = \sqrt{XX^*} \) have the same range, each operator range is the range of a positive self-adjoint operator. The domain of a densely defined closed operator \( T \) is the operator range \( R((|T| + I)^{-1}) \).

There is an extensive literature on operator ranges and operator domains. Classical papers are \([\mathbf{vN29}], K36a, K36b, D49a, D49b, D66, FW71, F72\). More recent ones include \([\mathbf{vD82}], \mathbf{Sch83}, BN03, K06, ACC13, AZ15, DM20\).

A simple sufficient condition for an operator \( T \) to be balanced is the following.

**Proposition 6.** Suppose \( T \) is a densely defined closed operator. If \( D(T^*T) = D(\sqrt{TT^*}) \), then \( T \) is balanced.

**Proof.** The assertion means that the domain equality \( D(|T|^2) = D(|T^*|^2) \) implies \( D(|T|) = D(|T^*|) \). This follows from the following result: If \( B \) and \( A \) are positive self-adjoint operators on \( H \) such that \( D(A^2) \subseteq D(B^2) \), then \( D(A) \subseteq D(B) \).

We include a proof of this well-known fact. Without loss of generality we can assume that \( A \) and \( B \) have bounded inverses. Since \( D(A^2) \subseteq D(B^2) \), Lemma \( \mathbf{2} \) implies that \( (B^2A^{-2})^\ast \geq A^{-2}B^2 \) is a bounded operator, so there is a constant \( \gamma > 0 \) such that \( \|A^{-2}B^2\| \leq \gamma \|\psi\| \) for \( \psi \in D(B^2) \). Hence \( \|A^{-2}\psi\| \leq \gamma \|B^{-2}\psi\| \) for \( \psi \in H \), so that \( (A^{-2})^2 \leq (\gamma B^{-2})^2 \). Therefore \( A^{-2} \leq \gamma B^{-2} \) by the Heinz inequality (see e.g. \([\mathbf{Sch89}] \) Proposition 10.14). Then \( \|A^{-1}\psi\| \leq \sqrt{\gamma} \|B^{-1}\psi\| \) for \( \psi \in H \) and \( \|A^{-1}B\psi\| \leq \sqrt{\gamma} \|\varphi\| \) for \( \varphi \in D(B) \). The operator \( A^{-1}B \) is bounded, so \( (A^{-1}B)^\ast \geq BA^{-1} \) is bounded. Hence \( D(A) \subseteq D(B) \) again by Lemma \( \mathbf{2} \).

The following example shows that the converse of Proposition \( \mathbf{6} \) does not hold.

**Example 7.** Suppose \( A \) is a positive self-adjoint operator with trivial kernel and \( U \) a unitary operator on \( H \). Then, \( T := UA \) is a densely defined closed operator on \( H \) such that the formula \( T = UA \) is the polar decomposition of \( T \). \n
\[ T^*T = A^2, \quad TT^* = U^2A^2U^*, \quad |T| = A, \quad |T^*| = \sqrt{TT^*} = UAU^*. \]

Therefore, if \( U^*D(A) = D(A) \), then \( D(|T|) = D(|T^*|) \), so \( T \) is balanced. The equality \( D(T^*T) = D(TT^*) \) holds if and only if \( U^*D(A^2) = D(A^2) \).

Now let \( A \) be the differential operator \(-\frac{d}{dx}^2\) on \( L^2(\mathbb{R}) \) and \( U \) the multiplication operator by a function \( \omega(x) \) of modulus one. Hence, if \( \omega \) is in \( C^2(\mathbb{R}) \), but not in \( C^3(\mathbb{R}) \), then \( T = UA \) is balanced, but \( D(T^*T) \neq D(TT^*) \).
3. Examples: Weighted shift operators

Let us consider the Hilbert space $l_2(\mathbb{N}_0)$. Suppose $(\lambda_n)_{n \in \mathbb{N}_0}$ is a complex sequence. There is a densely defined closed linear operator $T$ on $l^2(\mathbb{N}_0)$ defined by

\[(5)\quad T(\varphi_0, \varphi_1, \varphi_2, \ldots) = (\lambda_1 \varphi_1, \lambda_2 \varphi_2, \lambda_3 \varphi_3, \ldots),\]

written shortly as $T(\varphi_n) = (\lambda_{n+1} \varphi_{n+1})$, with domain

\[(6)\quad D(T) = \{ (\varphi_n)_{n \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) : (\lambda_{n+1} \varphi_{n+1})_{n \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \}.\]

The adjoint operators $T^*$ acts by

\[T^*(\varphi_0, \varphi_1, \varphi_2, \ldots) = (0, \lambda_1 \varphi_0, \lambda_2 \varphi_1, \lambda_3 \varphi_2, \ldots),\]

briefly $T^*(\varphi_n) = (\overline{\lambda_n} \varphi_{n-1})$, and has the domain

\[D(T^*) = \{ (\varphi_n)_{n \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) : (\overline{\lambda_n} \varphi_{n-1})_{n \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \},\]

where $\varphi_{-1} := 0$. Then $|T|$ and $|T^*|$ are diagonal operators

\[|T|(\varphi_0, \varphi_1, \varphi_2, \ldots) = (0, |\lambda_1| \varphi_1, |\lambda_2| \varphi_2, |\lambda_3| \varphi_3, \ldots),\]

\[|T^*|(\varphi_0, \varphi_1, \varphi_2, \ldots) = (|\lambda_1| \varphi_0, |\lambda_2| \varphi_1, |\lambda_3| \varphi_2, \ldots).\]

Then, for $\alpha > 0$, $\beta > 0$, $0 \leq (|T^*| + |T|)(|T^*| + |T^*|^{-1})$ and $0 \leq (|T| + |T^*|)(|T^*| + |T^*|^{-1})$ are diagonal operators. For $n \in \mathbb{N}$, the $n$-th diagonal entries are $0$ and $0$ respectively. Therefore, both diagonal operators are bounded if and only if there are constants $c_1 > 0, c_2 > 0, c_3 > 0, c_4 > 0$ such that

\[(7)\quad |\lambda_n| \leq c_1 |\lambda_{n+1}| + c_2, \quad |\lambda_{n+1}| \leq c_3 |\lambda_n| + c_4, \quad n \in \mathbb{N}.\]

Thus, by Corollary \[\text{III}\] we have proved the following.

**Proposition 8.** The operator $T$ defined by \[(3)\] and \[(1)\] is balanced if and only if condition \[(7)\] holds.

Now we mention some interesting examples of balanced weighted shift operators.

**Example 9.** Set $\lambda_n = \sqrt{n}$ for $n \in \mathbb{N}_0$. Then \[(1)\] is obviously true. In this case, $T$ is the creation operator and $T^*$ is the annihilation operator of quantum mechanics. Both operators are balanced. Note that $T$ satisfies the commutation relation

\[(8)\quad TT^* - T^*T = I.\]

**Example 10.** Suppose $q > 0, q \neq 1$, and set $\lambda_n = \sqrt{\frac{1-q^n}{1-q}}$, $n \in \mathbb{N}_0$. Then \[(7)\] is satisfied. The corresponding operator $T$ is the Fock representation (see e.g. \[\text{Sch20}\] Theorem 11.28(i)) for the $q$-oscillator algebra defined by the relation

\[(9)\quad XX^* = qX^*X + I.\]

Proposition \[\text{III}\] remains valid verbatim for two-sided weighted shifts acting on the Hilbert space $l_2(\mathbb{Z})$. We state two examples of balanced two-sided weighted shifts.

**Example 11.** Suppose $q > 0, q \neq 1$. For $\lambda > 0$, we set $\lambda_n = \lambda q^n/2$, $n \in \mathbb{Z}$. Then the operator $T$ is also balanced and it is an irreducible representations (see e.g. \[\text{Sch20}\] Proposition 11.25) of the Hermitean quantum plane given by the relation

\[(10)\quad XX^* = qX^*X.\]

**Example 12.** Suppose $0 < q < 1$. For $\gamma \in [q, 1)$, set $\lambda_n = \sqrt{\frac{1+q^n}{1+q}}$, $n \in \mathbb{Z}$. The corresponding two-sided weighted shift operator $T$ is balanced and it is an irreducible non-Fock representation \[\text{Sch20}\] Theorem 11.28(iii)] of the $q$-oscillator algebra \[\text{IV}\].
4. ON THE EQUALITY OF OPERATOR DOMAINS

Throughout this section we suppose that $A$ and $B$ are positive self-adjoint operators on $\mathcal{H}$. Let
\[ A = \int_0^{+\infty} \lambda dE_A(\lambda) \quad \text{and} \quad B = \int_0^{+\infty} \lambda dE_B(\lambda) \]
be their spectral resolutions.

Our aim is to characterize the domain equality $\mathcal{D}(A) = \mathcal{D}(B)$ in terms of the spectral measures $E_A$ and $E_B$.

**Proposition 13.** Suppose the inverse operator $A^{-1}$ exists and is bounded. If $\mathcal{D}(A) \subseteq \mathcal{D}(B)$, then there is a number $\nu > 0$ such that
\[ E_A((0, b]) \subseteq E_B((0, b\nu]) \quad \text{for all} \quad b > 0. \]
One may take any number $\nu$ such that $\nu > \|BA^{-1}\|$.

**Proof.** Since $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $A^{-1}$ is bounded, it follows from Lemma 2 that
\[ Z_\delta := (B + \delta)A^{-1}, \quad \delta \geq 0, \]
is a bounded operator defined on the whole Hilbert space $\mathcal{H}$. We fix a number $\nu$ such that $\nu > \|BA^{-1}\|$. Clearly, the norm of $Z_\delta$ is the infimum of numbers $C > 0$ satisfying $\|(B + \delta)\xi\| \leq C\|A\xi\|$ for $\xi \in \mathcal{H}$. Therefore, since $A^{-1}$ is bounded, there is a positive number $\delta$ such that $\nu > \|Z_\delta\|$.

Now we take a number $q$, $0 < q < 1$, such that $q\nu \geq \|Z_\delta\|$. Then we have $\|Z_\delta\|b < q(b\nu + \delta)$, so that
\[ \|Z_\delta\|b(b\nu + \delta)^{-1} < q. \]

Fix $b \in \mathbb{R}$, $b > 0$. Suppose that $\varphi \in E_A((0, b])\mathcal{H}$. We can write the vector $\varphi$ as $\varphi = \psi + \eta$ with
\[ \psi \in E_B((0, b\nu])\mathcal{H}, \quad \eta \in E_B((b\nu, +\infty))\mathcal{H}. \]
Clearly, $\varphi \in \mathcal{D}(A)$. By the spectral calculus,
\[ \|A\varphi\|^2 = \int_{(0, b]} \lambda^2 d(E_A(\lambda)\varphi, \varphi) \leq \int_0^{+\infty} b^2 d(E_A(\lambda)\varphi, \varphi) = b^2\|\varphi\|^2, \]
\[ \|(B + \delta)^{-1}\eta\|^2 = \int_{(b\nu, +\infty)} (\lambda + \delta)^{-2} d(E_B(\lambda)\eta, \eta) \leq \int_0^{+\infty} (b\nu + \delta)^{-2} d(E_B(\lambda)\eta, \eta) \]
\[ = (b\nu + \delta)^{-2}\|\eta\|^2. \]

Further, $(Z_\delta)^*\xi = A^{-1}(B + \delta)\xi$ for $\xi \in \mathcal{D}(B)$. Using the preceding facts we derive
\[ \|\varphi - \psi\|^2 = \|\eta\|^2 = |\langle \varphi, \eta \rangle| = |\langle A^{-1}A\varphi, \eta \rangle| \]
\[ = |\langle A\varphi, \eta \rangle| = |\langle A\varphi, Z_\delta^*(B + \delta)^{-1}\eta \rangle| \]
\[ \leq b\|\varphi\|\|Z_\delta\|\|(b\nu + \delta)^{-1}\|\eta\| \]
\[ = \|\varphi\|\|\eta\|\|(Z_\delta b(b\nu + \delta)^{-1}) \]
\[ \leq \|\varphi\|\|\varphi - \psi\| \quad q, \]
where the last inequality follows from (12). Thus,
\[ \|\varphi - \psi\| \leq q\|\varphi\| \quad \text{for} \quad \varphi \in E_A((0, b])\mathcal{H}, \psi = E_B((0, b\nu])E_A((0, b])\varphi. \]
Let us abbreviate $P_A := E_A((0, b])$ and $P_B := E_B((0, b\nu])$. Then (13) means
\[ \|P_A\varphi - P_B P_A \varphi\| \leq q \|P_A \varphi\| \text{ for } \varphi \in \mathcal{H}. \]
Hence $\|(I - P_B) P_A\| \leq q \|P_A\| \leq q$ and therefore $\|P_A ((I - P_B) P_A)^n\| \leq q^n$ for $n \in \mathbb{N}$. Since $0 \leq q < 1$, we conclude that
\[ \lim_{n \to \infty} \|P_A ((I - P_B) P_A)^n\| = 0. \]
Recall that for orthogonal projections $P, Q$ on a Hilbert space there is the well-known formula
\[ P \wedge Q = \lim_{n \to \infty} P [Q P]^n \]
in the strong operator topology (see e.g. [H67, Exercise 96]). Therefore, (13) yields $P_A \wedge (I - P_B) = 0$, so that $P_A \mathcal{H} \cap (I - P_B) \mathcal{H} = \{0\}$ and hence $P_A \mathcal{H} \subseteq P_B \mathcal{H}$. Thus $P_A \leq P_B$, which means that $E_A((0, b]) \leq E_B((0, b\nu])$. □

Interchanging the role of the operators $B$ and $A$ in Proposition 13 gives

**Proposition 14.** Suppose the inverse operator $B^{-1}$ is bounded and $\mathcal{D}(B) \subseteq \mathcal{D}(A)$. Then there is a number $\nu' > 0$ such that
\[ E_B((0, b]) \leq E_A((0, b\nu']) \quad \text{for all } b > 0. \]
Any number $\nu'$ such that $\nu' > \|AB^{-1}\|$ can be taken.

The following theorem is the main result of the section.

**Theorem 15.** Suppose $A$ and $B$ are positive self-adjoint operators on a Hilbert space $\mathcal{H}$ such that inverses $A^{-1}$ and $B^{-1}$ exist and are bounded operators on $\mathcal{H}$. Let $E_A$ and $E_B$ denote their spectral measures. Then the following are equivalent:

(i) $\mathcal{D}(A) = \mathcal{D}(B)$.

(ii) There is a number $\mu > 0$ such that for all $a, b \in \mathbb{R}, 0 < a < b$,
\[ E_A((a, b]) \leq E_B((a\mu^{-1}, b\mu]) \quad \text{and} \quad E_B((a, b]) \leq E_A((a\mu^{-1}, b\mu]). \]

(iii) There is a number $\mu > 0$ such that for all $a, b \in \mathbb{R}, 0 < a < b$,
\[ E_A((a, b]) \leq E_B((a\mu^{-1}, b\mu]) \quad \text{and} \quad E_B((a, b]) \leq E_A((a\mu^{-1}, b\mu]). \]

**Proof.** (i)→(ii): Recall that $E((a, b]) = E((0, b]) - E((0, a])$ for any spectral measure $E$. Let $\mu$ be the maximum of the numbers $\nu$ and $\nu'$ from Propositions 13 and 14. Then $E_A((0, b]) \leq E_B((0, b\nu])$ and $E_B((0, a\mu^{-1}) \leq E_A((0, a])$ by (13) and (14). Hence
\[ E_A((a, b]) = E_A((0, b]) - E_A((0, a]) \leq E_B((0, b\nu]) - E_B((0, a\mu^{-1})) = E_B((a\mu^{-1}, b\mu]), \]
which proves the first relation of (16). The second relation follows by interchanging the role of $A$ and $B$.

(ii)→(i): From spectral theory we recall that a vector $\varphi \in \mathcal{H}$ belongs to $\mathcal{D}(A)$ if and only if $\int_0^\infty \lambda^2 d(E_A(\lambda) \varphi, \varphi) < \infty$; similarly for $B$.

Now we suppose that (16) holds. There is no loss of generality to assume that $\mu > 1$. We abbreviate
\[ f_k := E_B((\mu^k, \mu^{k+1}]), \quad g_k := E_A((\mu^k, \mu^{k+1}]), \quad e_k := E_A((\mu^{k-1}, \mu^{k+2})) \quad \text{for } k \in \mathbb{N}. \]
Then, $f_k \leq e_k$ by (14). Further, since $[\mu^{k-1}, \mu^{k+2}]$ is the union of disjoint intervals $[\mu^{k-1}, \mu^k), [\mu^k, \mu^{k+1}), [\mu^{k+1}, \mu^{k+2}]$, the projection $e_k$ is the sum of the three orthogonal projections $g_{k-1}, g_k, g_{k+1}$. Thus, $f_k \leq e_k = g_{k-1} \oplus g_k \oplus g_{k+1}$.

Our aim is to show that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$. Suppose that $\varphi \in \mathcal{D}(A)$. Let $m \in \mathbb{N}$. Using the preceding facts we derive
\[ \int_{(\mu^a, \mu^{a+1})} \lambda^2 d(E_B(\lambda) \varphi, \varphi) = \sum_{k=1}^m \int_{(\mu^k, \mu^{k+1})} \lambda^2 d(E_B(\lambda) \varphi, \varphi) \]

\begin{align*}
\sum_{k=1}^{m} \mu^{2(k+1)} \| f_k \varphi \|^2 & \leq \sum_{k=1}^{m} \mu^{2(k+1)} \| e_k \varphi \|^2 \\
= \sum_{k=1}^{m} \mu^{2(k+1)} \| (g_{k-1} \oplus g_k \oplus g_{k+1}) \varphi \|^2 \\
= \sum_{k=1}^{m} \mu^{2(k+1)} \| g_{k-1} \varphi \|^2 + \sum_{k=1}^{m} \mu^{2(k+1)} \| g_k \varphi \|^2 + \sum_{k=1}^{m} \mu^{2(k+1)} \| g_{k+1} \varphi \|^2 \\
\leq 3 \mu^3 \sum_{j=0}^{m+1} \| g_j \varphi \|^2 \\
\leq 3 \mu^3 \sum_{j=0}^{m+1} \int_{(\mu^j, \mu^{j+1}]} \lambda^2 d(E_A(\lambda)(\varphi, \varphi)) \\
= 3 \mu^3 \int_{(1, \mu^{m+2}]} \lambda^2 d(E_A(\lambda)(\varphi, \varphi)) \\
\leq 3 \mu^3 \int_{0}^{\infty} \lambda^2 d(E_A(\lambda)(\varphi, \varphi)) = 3 \mu^3 \| A\varphi \|^2.
\end{align*}

Passing to the limit \( m \to \infty \) and remembering that \( \mu > 1 \) we conclude that

\[
\int_{(\mu, +\infty)} \lambda^2 d(E_B(\lambda)(\varphi, \varphi)) \leq 3 \mu^3 \| A\varphi \|^2.
\]

Hence \( \int_{0}^{+\infty} \lambda^2 d(E_B(\lambda)(\varphi, \varphi)) < \infty \) and therefore \( \varphi \in D(B) \). We have shown that \( D(A) \subseteq D(B) \). Since condition (16) is symmetric in \( A \) and \( B \), we also have \( D(A) \subseteq D(B) \). Thus \( D(A) = D(B) \).

(ii)\(\Rightarrow\)(iii): Suppose that the first relation of (16) holds. Then, for \( 0 < a' < a \),

\[ E_A([a, b]) \leq E_A([a', b]) \leq E_B([a' \mu^{-1}, b\mu]). \]

Letting \( a' \uparrow a \), we get \( E_A([a, b]) \leq E_B([a\mu^{-1}, b\mu]), \) which is the first relation of (17).

Conversely, assume the first relation of (17). For \( 0 < a < a' \), we have

\[ E_A([a, b]) \leq E_B([a' \mu^{-1}, b\mu]) \leq E_B([a' \mu^{-1}, b\mu]). \]

Taking \( a' \downarrow a \) yields \( E_A([a, b]) \leq E_B([a\mu^{-1}, b\mu]). \) This is the first relation of (16).

The equivalence of the second relations in (16) and (17) follows by interchanging the role of \( A \) and \( B \).

The preceding proof shows that the implications (iii)\(\Rightarrow\)(ii)\(\Rightarrow\)(i) in Theorem 15 are valid without the assumption that \( A \) and \( B \) have bounded inverses. For the implication (i)\(\Rightarrow\)(ii) this is not true, as we will see in Example 17 below.

From its proof it follows that the assertion of Proposition 13 remains true if we only assume that \( D(A) \subseteq D(B) \) and the inverses \( A^{-1} \) exists and \( BA^{-1} \) is bounded. (Since \( D(A) \subseteq D(B) \), the domain of \( BA^{-1} \) is the range of \( A \), so it is dense if we assume that \( N(A) = \{0\} \).) Similarly, if we assume \( D(A) = D(B) \) together with the existence of \( A^{-1} \) and \( B^{-1} \) and the boundedness of \( BA^{-1} \) and \( AB^{-1} \), the implication (i)\(\Rightarrow\)(ii) of Theorem 15 holds.

If we omit the boundedness assumption for the inverses, then Theorem 16 has the following reformulation.

**Theorem 16.** Suppose \( A \) and \( B \) are positive self-adjoint operators on a Hilbert space with spectral measures \( E_A \) and \( E_B \). Let \( \epsilon > 0 \) and \( \delta > 0 \). Then the following statements are equivalent:

(i) \( D(A) = D(B) \).
(ii) There is a number $\mu > 0$ such that for all $a, b \in \mathbb{R}, 0 < a < b$,
$$E_A((a, b] \cup [b, c]) \leq E_B((a, b\mu, b\mu + [b,c]) \leq E_A((a, b\mu + \varepsilon, b\mu + [b,c])$$.

(iii) There is a number $\mu > 0$ such that for all $a, b \in \mathbb{R}, 0 < a < b$,
$$E_A((a, b] \cup [b, c]) \leq E_B((a, b\mu, b\mu + [b,c]) \leq E_A((a, b\mu + \varepsilon, b\mu + [b,c])$$.

Proof. The self-adjoint operators $A' := A + \varepsilon$ and $B' := B + \delta$ have trivial kernels and bounded inverses and satisfy $\mathcal{D}(A) = \mathcal{D}(A'), \mathcal{D}(B) = \mathcal{D}(B')$, so Theorem 13 applies to these operators. Using the formulas $E_A(M) = E_A(M + \varepsilon)$ and $E_B(M) = E_B(M + \delta)$ this yields the assertion.

We illustrate the preceding by a simple example dealing with multiplication operators.

Example 17. Let $J$ be an interval and $\mathcal{H}$ the Hilbert space $L^2(J)$, with respect to the Lebesgue measure, let $f$ and $g$ be nonnegative Borel functions on $J$ such that the sets $\{t \in J : f(t) = 0\}$ and $\{t \in J : g(t) = 0\}$ have Lebesgue measure zero. Let $A$ and $B$ denote the multiplication operators by $f$ and $g$, respectively. Then $A$ and $B$ are positive self-adjoint operators on $\mathcal{H}$ with trivial kernels.

We have $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ if and only if there are constants $C \geq 0, c \geq 0$ such that
$$g(t) \leq C(f(t) + c) \text{ a.e. on } J.$$

Clearly, $A^{-1}$ is bounded if and only if there is an $\varepsilon > 0$ such that $g(t) \geq \varepsilon$ a.e. on $J$. For a Borel set $M$, the spectral projection $E_A(M)$ is the multiplication operator by the characteristic function of $f^{-1}(M) := \{t \in J : f(t) \in M\}$; similarly for $B$.

Using these facts one can easily construct interesting cases. For instance, set $J = (\alpha, +\infty)$ and $f(x) = x$, $g(x) = x^2$, with $\alpha \geq 0, \beta > 0$.

First we discuss Proposition 13 and condition (11). We note that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ if and only if $\beta \leq 1$, and $A^{-1}$ is bounded if and only if $\alpha > 0$. For $b > 0, \nu > 0$, the spectral projections $E_A([0, b])$ and $E_B([0, b\nu])$ are multiplication operators by the characteristic functions of $\{t \in J : t \leq b\}$ and $\{t \in J : t \leq (\nu b)^{1/\beta}\}$.

Suppose $a > 0$. Then (11) holds if and only if $\beta \leq 1$, or equivalently, $\mathcal{D}(A) \subseteq \mathcal{D}(B)$. In this case, $A^{-1}$ is bounded.

Now let $a = 0$. Then condition (11) is fulfilled if and only if $\beta = 1$, that is, $A = B$. For $\beta < 1$, we have $\mathcal{D}(A) \subseteq \mathcal{D}(B)$, but (11) is not valid. Thus, the assertion of Proposition 13 does not necessarily hold if the operator $A^{-1}$ is not bounded.

Next we consider Theorem 13 and condition (10). Let $f(x) = x$ and $g(x) = x$ if $x \geq 1$, $g(x) = \sqrt{x}$ if $x \leq 1$. Then, obviously, $\mathcal{D}(A) = \mathcal{D}(B)$, but condition (10) is not satisfied. That is, the implication $(i) \rightarrow (ii)$ in Theorem 13 is not true if the assumption that the operators $A^{-1}$ and $B^{-1}$ are bounded is omitted.

5. Polar decomposition and balanced operators

Suppose that $T$ is a densely defined closed operator on $\mathcal{H}$. By the theorem on the polar decomposition (see e.g. [Sch99], Theorem 7.2) or [K00, Chapter VII, §7]) there exists a partial isometry $U$ with initial space $K_1$ and final space $K_2$, where

$$K_1 := \mathcal{N}(T) = \overline{R(T)} = \mathcal{N}(\overline{R(T)}) = U^* \overline{R(T)}$$,
$$K_2 := \overline{R(T)} = \mathcal{N}(T^*) = \mathcal{N}(\overline{R(T^*)}) = \overline{R(T^*)}$$,

such that $T = U|T|$. The formula $T = U|T|$ is called the polar decomposition of $T$, the partial isometry $U$ is the phase operator and $|T|$ is the modulus of $T$. We have

$$U^* U = P_{K_1}, \quad U U^* = P_{K_2}$$.
where $P_{\mathcal{K}_1}$ and $P_{\mathcal{K}_2}$ are the projections onto $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Moreover,
\begin{equation}
T = U[T] = |T|^*U, \quad T^* = U^*[T^*] = |T|U^*
\end{equation}

In fact, $T^* = U^*[T^*]$ is the polar decomposition of the operator $T^*$.

Further, since \( M = 5.15 \) that for any Borel set $\mathcal{M}$, \( M \in \mathcal{M} \),
\begin{equation}
(23) \quad U E_{[T]}(M) = E_{[T^*]}(M) U^*, \quad E_{[T]}(M) U^* = U^* E_{[T^*]}(M).
\end{equation}

Further, since $\mathcal{N}(|T|) = E_{[T]}(0)\mathcal{H}$ and $\mathcal{N}(|T^*|) = E_{[T^*]}(0)\mathcal{H}$, we have
\begin{equation}
(24) \quad \mathcal{N}(U) = E_{[T]}(0)\mathcal{H}, \quad \mathcal{N}(U^*) = E_{[T^*]}(0)\mathcal{H}.
\end{equation}

Now we multiply the first equality of (23) by $U^*$ and the second by $U$ from the right. Using (21) we then obtain for each Borel $M \subseteq (0, +\infty)$,
\begin{equation}
(25) \quad U E_{[T]}(M) U^* = E_{[T^*]}(M), \quad E_{[T]}(M) U^* = U^* E_{[T^*]}(M) U.
\end{equation}

(Note that the second equalities in (23) and (25) follow also from the first ones by applying the adjoint operation.)

Conversely, suppose that $E$ is a spectral measure on $[0, +\infty)$ and $U$ a partial isometry on $\mathcal{H}$ with kernel $E(\{0\})\mathcal{H}$.

The spectral integral
$$A := \int_{[0, +\infty)} \lambda dE(\lambda)$$

is a positive self-adjoint operator on $\mathcal{H}$. Set $T := UA$. Since $\mathcal{D}(T) = \mathcal{D}(A)$, the operator $T$ is densely defined. We show that $T$ is closed. For let $(\varphi_n)$ be a sequence of vectors $\varphi_n \in \mathcal{D}(T)$ such that $\varphi_n \to 0$ and $T \varphi_n \to \psi$ for some vector $\psi \in \mathcal{H}$. Using that $U$ is zero on $\mathcal{N}(A) = E(\{0\})\mathcal{H}$ and isometric on the complement, we derive
\begin{align*}
\|A\varphi_k - A\varphi_n\| &= \|\int_{[0, +\infty)} \lambda dE(\lambda) \varphi_k - \int_{[0, +\infty)} \lambda dE(\lambda) \varphi_n\|
\end{align*}

Therefore, since $A$ is closed, there is a vector $\varphi \in \mathcal{D}(A)$ such that $A\varphi_n \to A\varphi$.

Hence $T \varphi_n = UA\varphi_n \to UA\varphi = T \varphi = \psi$, which proves that $T$ is closed.

From the properties of $U$ it follows that $|T| = A$ and $T = UA$ is the polar decomposition of $T$. By the preceding we have proved the following proposition.

**Proposition 18.** The densely defined closed operators $T$ on $\mathcal{H}$ are in one-to-one correspondence, given by $|T| = \int_{[0, +\infty)} dE(\lambda)$ and $T = U|T|$, with pairs of spectral measures $E$ on $[0, +\infty)$ and partial isometries $U$ on $\mathcal{H}$ with kernels $E(\{0\})\mathcal{H}$.

Now we are ready to characterize a balanced operator $T$ in terms of the spectral measure $E_{[T^*]}$ of its modulus $|T|$ and its phase operator $U$.

**Theorem 19.** Suppose $T$ is a densely defined closed operator on $\mathcal{H}$ with polar decomposition $T = U|T|$. Let $\varepsilon > 0$ and $\delta > 0$ be fixed numbers. Then the following statements are equivalent:

(i) $T$ is balanced.
(ii) There exists a number $\mu > 0$ such that for all $a, b \in \mathbb{R}, 0 < a < b$,

\begin{align}
E_{|T|} & \left( (a + \varepsilon, b + \varepsilon) \right) \leq U E_{|T|} \left( (a \mu^{-1} + \delta, b \mu + \delta) \right) U^*, \\
UE_{|T|} & \left( (a + \delta, b + \delta) \right) U^* \leq E_{|T|} \left( (a \mu^{-1} + \varepsilon, b \mu + \varepsilon) \right).
\end{align}

(iii) There is a number $\mu > 0$ such that for all $a, b \in \mathbb{R}, 0 < a < b$,

\begin{align}
E_{|T|} & \left( (a + \varepsilon, b + \varepsilon) \right) \leq U E_{|T|} \left( (a \mu^{-1} + \delta, b \mu + \delta) \right), \\
UE_{|T|} & \left( (a + \delta, b + \delta) \right) U^* \leq E_{|T|} \left( (a \mu^{-1} + \varepsilon, b \mu + \varepsilon) \right).
\end{align}

Proof. The result is derived from Theorem 16 applied with $B = |T^*|, A = |T|$ and combined with the first equality of (25). Indeed, by (26), the inequality of (26) means that

\[ E_A((a + \varepsilon, b + \varepsilon)) \leq E_B((a \mu^{-1} + \delta, b \mu + \delta)) \]

and the inequality of (27) is

\[ E_B((a + \delta, b + \delta)) \leq E_A((a \mu^{-1} + \varepsilon, b \mu + \varepsilon)). \]

(Since all these intervals are contained in $(0, +\infty)$, (25) applies.) Recall that, as noted above, $T$ is balanced if and only if $\mathcal{D}(|T|) = \mathcal{D}(|T^*|)$. Therefore, the equivalence of (i) and (ii) in Theorem 16 gives the equivalence of assertions (i) and (ii) of Theorem 16. The proof of the equivalence of (i) and (iii) is similar. □

References

[A CG13] Arias, M.L., Corach, G. and M.C. Gonzalez, Additivity properties of operator ranges, Linear Algebra Appl. 439 (2013), 3581–3590.

[AZ15] Arlinski, Yu. and V.A. Zagrebnov, Around the van Daele-Schmüdgen theorem, Integral Equ. Operator Theory 81 (2015), 53–95.

[BN99] Brasche, J.F. and H. Neidhardt, Has every symmetric operator a closed symmetric restriction whose square has a trivial domain?, Acta Sci. Math. (Szeged) 58 (1993), 425–430.

[DM20] Dehimi, S. and M.M. Mortad, Chernoff-like counterexamples related to unbounded operators, Kyushu J. Math. 74 (2020), 105–108.

[D49a] Dixmier, J., Sur les varietes J d’un espace de Hilbert, J. Math. Pures Appl. 28 (1949), 321–358.

[D49b] Dixmier, J., Etude sur les varietes et les operateurs de Julia, Bull. Soc. Math. France 77 (1949), 11–101.

[D66] Douglas, R.G., On majorization, factorization, and range inclusion of operators on Hilbert space, Proc. Amer. Math. Soc. 17 (1966), 413–416.

[FW71] Fillmore, P.A. and J.P. Williams, On operator ranges, Adv. Math. 7 (1971), 254–281.

[F72] Foiaș, C., Invariant para-closed subspaces, Indiana Univ. Math. J. 21 (1972), 887–906.

[H67] Halmos, P.R., A Hilbert Space Problem Book, van Nostrand, Princeton, 1967.

[K66] Kato, T., Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1966.

[K36a] Köthe, G., Das Träheitsgesetz der quadratischen Formen im Hilbertschen Raum, Math. Z. 41 (1936), 137–152.

[K36b] Köthe, G., Die Gleichungstheorie im Hilbertschen Raum, Math. Z. 41 (1936), 153–162.

[K06] Kosaki, H., On intersections of domains of unbounded positive operators, Kyushu J. Math. 60 (2006), 3–25.

[OS14] Ostrovskyi, V.L. and Yu.S. Samoilenko, Introduction to the Theory of Representations of Finitely Presented Algebras I, Cambridge Sci. Publishers, Cambridge, 2014.

[Sch83] Schmüdgen, K., On domains of powers of closed symmetric operators, J. Oper. Theory 9 (1983), 53–75.

[Sch90] Schmüdgen, K., Unbounded Self-adjoint Operators on Hilbert Space, Springer-Verlag, Cham, 2012.

[Sch20] Schmüdgen, K., An Invitation to Unbounded Representations of *-Algebras on Hilbert Space, Springer, New York, 2020.

[vD82] van Daele, A., On pairs of closed operators, Bull. Soc. Math. Belg. Ser B 34 (1982), 25–40.

[vN29] von Neumann, Zur Theorie der unbeschränkten Matrizen, J. Reine Angew. Math. 161 (1929), 208–236.
University of Leipzig, Mathematical Institute, Augustusplatz 10/11, D-04109 Leipzig, Germany

Email address: schmuedgen@math.uni-leipzig.de