Topological Classification 
of Odd-Parity Sphaleron Deformations

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**Abstract**

We discuss topological aspects of the electroweak sphaleron and its odd-parity deformations. We demonstrate that they are uniquely classified in terms of their odd and even-parity pure gauge field behaviour at spatial infinity. Fermion level crossing occurs only for odd-parity configurations which are topologically disconnected from the vacuum. They contribute to the high temperature unsuppressed baryon violating thermal transitions. Deformations with even-parity pure gauge behaviour at spatial infinity are topologically trivial and do not mediate baryon number violation in the early Universe.
1 Introduction

The standard model of electroweak interactions violates baryon number through the chiral anomaly \[^1\]. The anomaly of the chiral current becomes an anomaly in the baryon and lepton currents because the electroweak theory is chiral. At zero temperature \(B\)-violating transitions are exponentially suppressed. At high temperature it has been convincingly argued \[^2\] that the transition rates are dominated by the sphaleron \[^3\]. It is a finite energy saddle point solution to the electroweak equations of motion that has one unstable mode. This is because it is at the highest point of a continuous set of configurations that interpolates between the topologically distinct vacua with Chern-Simons (CS) numbers \(n\), \(n + 1\) (\(n \in \mathbb{Z}\)) and has \(\text{CS} = n + 1/2\).

In thermal equilibrium and at temperature \(T \leq E_{sp}\) the probability of forming a coherent sphaleron configuration in the hot plasma is given by the Boltzman weight of the classical sphaleron energy \(\Gamma \propto B \exp\left(-\frac{E_{sp}}{T}\right)\) \[^4\].

In the presence of high temperature fluctuations of the gauge and Higgs fields transitions across the sphaleron saddle point are accompanied by fermion level crossing. This formally manifests itself with the presence of normalizable zero energy solutions to the three dimensional Dirac equation in the sphaleron background \[^5\]. These transitions are expected to become unsuppressed at \(T \approx E_{sp}\) where perturbative estimates lose their validity. Strictly speaking in an incoherent thermal plasma any of the nearby configurations to the sphaleron with energy \((E \geq E_{sp})\) become likely to be generated spontaneously. This would result from a transition rate estimate in terms of the Gibbs free energy of all such nearby configurations at the top of the barrier instead of just the classical energy of a single sphaleron. These would include deformed sphaleron configurations which are not solutions to the electroweak equations of motion but nevertheless allow for fermion level crossing in their background. We will henceforth generically name such configurations sphaleron deformations.

Fermion level crossing is therefore not restricted only to sphaleron configurations \[^3\] or to deformed sphaleron solutions of the electroweak equations \[^6\]. Sphaleron deformations define the general class of sphaleron-like gauge and Higgs field configurations (non-solutions) in whose background the Dirac equation admits normalizable zero energy solutions or equivalently allows for fermion level crossing. The existence of such configurations, for example, with \(\text{CS} \neq 1/2\) and purely induced by the Yukawa interactions has been recently demonstrated both analytically \[^7\] and numerically \[^8\]. In a first order electroweak phase transition possible departure from thermal equilibrium in the broken phase behind the expanding bubble walls have also been envisioned \[^9\]. This would result from the liberated latent heat of conversion of the false symmetric vacuum into the true vacuum of spontaneously broken symmetry. Sphaleron-like fluctuations are expected to dominate thermal transition rates away from equilibrium.

It is expected therefore, one way or another, that in the hot electroweak plasma sphaleron deformations with equal or higher energy to that of the sphaleron will
also contribute to the rapid baryon violating transition rates. Indeed their presence is especially likely to dominate the symmetric high temperature phase of the electroweak theory \((T \geq T_{sp} \approx M_w/\alpha)\). In fact in this regime scaling arguments indicate unsuppressed baryon violating transition rates \((\Gamma \propto T^4)\) \[10\]. Equivalently there is an infinitude of paths through sphaleron deformations that connect two adjacent vacua. Recent computer simulations of the hot electroweak sphaleron transitions in the symmetric phase corroborate to this physical picture \[8\]. In such a nonperturbative phase and in the presence of large thermal fluctuations of the gauge and Higgs fields sphaleron deformations dominate fermion level crossing. For definiteness in what follows we will focus on the ones with \(\text{CS}=1/2\). They lie on the ridge of the saddle point of the sphaleron and have two unstable modes. The first brings about their rolling down towards the sphaleron and the second towards the vacuum. In the present work we attempt to gain a better understanding of the general properties of such sphaleron deformations. We do it by establishing sufficient conditions in order that a general deformation of the sphaleron is topologically nontrivial and equivalent homotopically to it. We do it for the case of the spherical electroweak sphaleron, i.e. in the limit of zero weak mixing angle.

The paper is organized as follows. In section 2 we give a topological classification for odd-parity gauge fields with pure gauge behaviour at spatial infinity. In section 3 we make explicit constructions of odd-parity sphaleron deformations by using twisted loops of the electroweak Nielsen-Olesen vortex. We use our scheme to classify the latter and identify the ones that are topologically equivalent to the sphaleron. We conclude by discussing the possible cosmological role odd-parity sphaleron deformations might have played in baryogenesis at the electroweak phase transition in the early Universe.

## 2 A Topological Classification

We start with the observation that our familiar static sphaleron configuration has an odd-parity gauge field everywhere in space. By imposing the same property on all possible deformations (they may not be solutions) we find two topologically distinct sectors of configurations that depend on the (even-odd) parity properties of their pure gauge behaviour at spatial infinity.

Let us recall some properties of the sphaleron configuration. Its gauge field reads

\[
W_k = f(r) \frac{\epsilon_{ijk}s^i\tau^j}{r^2} = -if(r)\partial_k U^\dagger_{sph} U_{sph},
\]

where \(k = 1, 2, 3\) and

\[
U_{sph} = \frac{i\sigma_i \tau_i}{r}.
\]

The function \(f(r) \to 1\) at \(r \to \infty\) so that this configuration is purely gauge at infinity.
The Chern-Simons functional is defined to be
\[ CS(W) = \frac{1}{32\pi^2} \int_{D^3} \text{Tr}(WdW - \frac{2i}{3}W^3) \] (2.3)
and for the sphaleron equals 1/2. This fact can be checked almost without calculation due to the observation that this gauge field configuration is odd under parity. One can define correctly the Chern-Simons number for the sphaleron by making a gauge transformation in order to remove the field at infinity. Such a transformation is given by an \( SU(2) \) group element \( U' \) which is smooth everywhere and coincides with \( U_{sph} \) at infinity. Since \( \pi_2(SU(2)) = 0 \) we know that such a field \( U' \) does exist. The Chern-Simons number for the sphaleron is defined as the functional \( CS(W') \), where
\[ W'_k = U'W_k(U')^{-1} + i\partial_k U' (U')^{-1}. \] (2.4)
We observe that
\[ CS(W') = CS(W) + S_{WZW}(U'), \] (2.5)
where \( CS \) stands for the Chern-Simons functional while \( S_{WZW} \) is the Wess-Zumino-Witten functional defined as follows
\[ S_{WZW}(U') = \frac{1}{24\pi^2} \int_{D^3} \text{Tr}(dU' U'^{-1})^3. \] (2.6)
The latter term actually depends on only the behaviour of the field \( U' \) at the boundary \( S^2 \) of the disk \( D^3 \) and is equal to 1/2 for the particular field \( U' \) introduced above. For example we can take
\[ U' = \exp \frac{i\pi}{2} \frac{\tau_i x_i}{\sqrt{x^2 + \rho^2}} \] (2.7)
and check by an explicit calculation that \( S_{WZW}(U') = 1/2 \). In turn \( CS(W) = 0 \) since the field \( W \) is odd under parity, i.e. \( W(-x) = -W(x) \). Thus we conclude that \( CS(W') = 1/2 \). It is now clear that the same value of the Chern-Simons functional corresponds to all odd-parity configurations with this particular behaviour at infinity.

Thus we see that the odd parity of the spherical sphaleron is a very important property. In fact our short computation indicates that it is necessary for the Chern-Simons number of the sphaleron to be exactly 1/2. However it is not sufficient by itself and we have to also make use of the odd parity of \( U' \) at infinity. It is the purpose of the present work to establish a connection between the Chern-Simons number of gauge fields and the parity property of their pure gauge behaviour (i.e. the \( U' \) field itself) at infinity. Actually we shall argue below that a restriction to odd-parity gauge field configurations allows us to introduce a useful topological classification for all of these fields.

We consider the gauge fields which are purely gauge at infinity (i.e. on the boundary of a 3-dimensional ball, which is \( S^2 \))
\[ A_i = -i(\partial_i U) U^{-1}, \] (2.8)
where \( U \) belongs to the \( SU(2) \) group. A restriction of this field \( U \) to the boundary \( S^2 \) of the 3-dimensional ball is a map of \( S^2 \) into \( SU(2) \).

The homotopic group \( \pi_2(SU(2)) \) is trivial and hence all such configurations in the 3-dimensional ball are contractible to unity. We now restrict ourselves to the space of 3-dimensional odd-parity gauge fields. We want to argue that in this space alone there still exists a relevant non-trivial homotopic classification. Indeed let us consider an odd-parity configuration

\[
A_i(-x) = -A_i(x). \tag{2.9}
\]

On the \( S^2 \) boundary this is a pure gauge so that

\[
(\partial_i U)(-x)U^{-1}(-x) = -(\partial_i U)(x)U^{-1}(x). \tag{2.10}
\]

It is easy to show that the field \( U(x) \) can only be either odd or even under parity. Indeed, let us consider the following equation

\[
(\partial_i - iA_i(x))^2 \phi(x) = 0 \tag{2.11}
\]

where \( \phi \) is an \( SU(2) \) doublet. Due to the odd parity of the gauge field \( A_i \) the parity conjugated doublet \( \phi(-x) \) is also a solution to the above equation. On the \( S^2 \) sphere at infinity this solution behaves like

\[
\phi(x) = U(x)\phi_0, \quad \phi(-x) = U(-x)\phi_0 = U(x)\phi'_0, \tag{2.12}
\]

where \( \phi_0 \) and \( \phi'_0 \) are nonvanishing constant doublets. The matrix \( U(-x)^{-1}U(x) \) is a non-degenerate constant \( SU(2) \) matrix since it maps a constant doublet to another one. Thus we get

\[
U(-x) = U(x)V, \tag{2.13}
\]

while \( V \) is a constant matrix of \( SU(2) \). This equation is valid for any point \( x \) on \( S^2 \). Hence changing \( x \to -x \) we get

\[
V^2 = 1. \tag{2.14}
\]

A short inspection now shows that the matrix \( V \) should belong to the centre of \( SU(2) \). In equations

\[
V = \pm 1. \tag{2.15}
\]

Thus we get two different classes of odd-parity gauge fields: those with odd \( U \) and those with even \( U \).

This conclusion reflects a non-triviality of the homotopic group of maps from the projective sphere \( \mathbb{R}P^2 = S^2/\mathbb{Z}_2 \) (where \( \mathbb{Z}_2 \) is a group of parity reflections with respect to some point in 3-dimensional space) to the group \( SO(3) = SU(2)/\mathbb{Z}_2 \) (where \( \mathbb{Z}_2 \) is the centre of \( SU(2) \)). In short \( \pi(\mathbb{R}P^2, SO(3)) = \mathbb{Z}_2 \). The consideration above shows that the odd-parity gauge fields split into two topologically disconnected equivalence classes. In other words it is not possible to get from one to the other continuously through odd-parity gauge field configurations.
Actually the classification of these gauge fields is more complicated. The above homotopic information also implies the following: if we extend an odd-parity field $U(x)$ from the boundary $S^2$ into the 3-dimensional ball then it will encounter a singularity at some point. As we saw above the sphaleron gauge field is (in an appropriate gauge) odd under parity and has an odd $U$ and hence it is disconnected from the vacuum configurations (see eq.(2.2)). In turn configurations with even $U$ fields are continuously connected to the vacua $A^n_i = i(\partial_i U_n)U^{-1}_n$ where the group element $U_n$ is given by the even-parity (at infinity) group elements

$$U_n = \exp(i \pi \frac{\tau_i x_i}{\sqrt{x^2 + \rho^2}}),$$

(2.16)

with $\rho$ being a constant parameter and $n$ an integer. Vacuum configurations are associated with the different Chern-Simons numbers given by $n$. Such a classification is a manifestation of the non-triviality of the homotopic group $\pi_3(SU(2)) = \mathbb{Z}$. The group element $U_n$ is a constant matrix at infinity and hence it corresponds to a compactification of $D_3$ into $S^3$. In turn by taking into consideration the group elements which are odd under parity at infinity we compactify $D^3$ into $\mathbb{R}P^3$. Thus the relevant homotopy group is in this case $\pi(\mathbb{R}P^3, SO(3)) = \mathbb{Z} + \mathbb{Z}_2$. A common feature of all odd-parity, even-$U$ configurations is that they have an integer valued Chern-Simons functional. Indeed similarly with the case of the sphaleron we can make a nonsingular gauge rotation so that we remove the gauge field at infinity. The Wess-Zumino-Witten functional would give us a Chern-Simons number for the gauge field configuration. It is easy to see that the value of $S_{WZW}(U)$ is invariant under even-parity smooth deformations of the $U$ field at infinity. Indeed a variation of the Wess-Zumino-Wess functional reads

$$\delta S_{WZW} = \frac{1}{8\pi^2} \int_{D^3} d\text{Tr}((U^{-1}\delta U)(U^{-1}dU)^2).\tag{2.17}$$

Since the variation of the group element on the surface $S^2$ is odd under parity and its value depends only on the values of the fields at the boundary we immediately conclude that the present variation of the Wess-Zumino-Wess functional equals zero. On the other hand let us consider a product of even-parity (at infinity) group elements $U_1$ and $U_2$ that correspond to any two such gauge fields. We have

$$S_{WZW}(U_1U_2) = S_{WZW}(U_1)+S_{WZW}(U_2)+\frac{1}{8\pi^2} \int_{D^3} \text{Trd}((U_1^{-1}dU_1)(dU_2U_2^{-1})). \tag{2.18}$$

The third term in the left hand side of this equation equals zero due to the odd parity of the integrand at infinity. Hence we see that the Wess-Zumino-Witten functional acts as a homomorphism from the group of maps $U$ to a discrete subgroup of the group of real numbers which is obviously isomorphic to $\mathbb{Z}$. As we argued before the even-$U$ (at infinity) group element is contractible to the vacuum. In turn as it is well known that the vacuum can have any integer value of the Chern-Simons number we conclude that even-parity $U$-fields are indeed classified by $\mathbb{Z}$.

Thus all odd-parity even-$U$ gauge fields split into an infinite set of disconnected equivalence classes which are labeled by integer values of their Chern-Simons numbers.
Let us now consider odd-parity odd-$U$ gauge fields. A similar argument shows that the value of the Chern-Simons functional is a topological invariant while the Wess-Zumino-Witten functional maps the odd-parity $U$ fields to a discrete subgroup of the group of real numbers according to eq.(2.18). On the other hand a product of two odd-parity group elements $U_1$ and $U_2$ is even under parity. By taking also into account that the sphaleron has $S_{WZW}(U) = 1/2$ we conclude that the odd-parity odd-$U$ gauge fields have half-integer values of the Wess-Zumino-Wess functional and hence are classified by $n + 1/2$ ($n \in \mathbb{Z}$) while these equivalence classes are themselves topologically disconnected one from the other for different values of $n$.

Thus we see that the Chern-Simons functional plays the role of a topological charge: it takes values in $\mathbb{Z}$ for even-$U$ and in $\mathbb{Z} + 1/2$ for odd-$U$ fields respectively.

An immediate implication of such a topological index for the fermionic spectrum of a 3-dimensional Dirac operator is the following. Let us consider a Dirac operator $D = \gamma_i (\partial_i - iA_i)$ in an external odd-parity gauge field $A_i$. Its non-zero eigenvalues are paired up $(\lambda, -\lambda)$. Hence when the external field varies continuously the number of zero modes of the Dirac operator is invariant modulo 2. For the sphaleron background this topological invariant is equal to one while for the vacuum its value is zero. This means that it is not possible to get to the vacuum from the sphaleron configuration continuously through odd-parity gauge field configurations.

From the above considerations we conclude that in the presence of an odd-parity external gauge field the number of fermionic zero modes is 0 mod 2 for even-$U$ and 1 mod 2 for odd-$U$ configurations.

In effect only odd-parity gauge fields with odd-$U$ pure gauge behaviour at spatial infinity mediate rapid anomalous baryon violating thermal transitions in the early universe. Our classification certainly does not exhaust the space of all sphaleron deformations. Yet in its restrictiveness and simplicity it provides us with sufficient conditions so that we can construct explicitly genuine “sphaleron-like” configurations. We now proceed to give concrete examples as an illustration of this point. To that end we construct vortex-like deformations of the spherical sphaleron which are simultaneously odd-parity in their gauge field and odd-$U$ in their pure gauge behaviour at infinity. This would be sufficient in order that fermion level crossing is manifest.

### 3 Nielsen-Olesen Sphaleron Deformations

It has recently been speculated that electroweak W and Z vortex solutions [12] might have mediated rapid baryon violation in the electroweak phase transition [13]. These are unstable solutions [14] to the classical electroweak equations of motion of the Nielsen-Olesen type [15]. The possibility that such configurations, if stable, might have contributed substantially to the baryon asymmetry of the universe in a second order electroweak phase transition has also been considered [16]. We will demonstrate the usefulness of our topological classification by constructing configurations
that are topologically equivalent to the sphaleron solution and configurations that are contractible to the trivial vacuum. To that end let us consider a $W$ string loop configuration. A straight segment of a $W$ string corresponds to the following gauge field

$$W_i = -iv(r)\partial_i U U^{-1}, \quad (3.1)$$

where

$$U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (3.2)$$

Here $\theta$ is a cylindrical angle, while $r$ is a radial cylindrical coordinate. The function $v$ is defined as follows

$$v(r) = 1 \quad \text{at} \quad r \rightarrow \infty, \quad (3.3)$$

$$v(r) = v_0 r^2/R^2 \quad \text{at} \quad r \rightarrow 0. \quad (3.3)$$

Let us consider a two-dimensional slice of a segment in the plane perpendicular to its axis. It is easy to show that a two-dimensional Dirac operator has no normalizable zero modes.

Let us consider a $W$-loop with a rotational symmetry with respect to the axis perpendicular to the plane of the loop. The string loop configuration is a loop of $W$ string twisted by an azimuthal angle $\phi$ with respect to the centre of the loop. The gauge field of the $W$-loop is given by eq.(3.1) while

$$U = U_{\text{loop}} = \exp(ik\phi/2 + \tau_3) \frac{1}{\sqrt{(r-x_0)^2 + z^2}} \begin{pmatrix} r-x_0 & z \\ -z & r-x_0 \end{pmatrix} \quad (3.4)$$

and $v = v((r-x_0)^2 + z^2)$. Here $r, \phi$ and $z$ are cylindrical coordinates and $x_0$ is a positive constant that denotes the distance (radius) of the loop from its center $x = y = z = 0$; $k$ is an integer and plays a role of winding number. In the above equation we used the twisting prescription given by [13]. We observe however that this gauge field has a $U(1)$ component. As our topological classification is formulated for purely $SU(2)$ gauge fields we modify the $W$-loop so that we generate an $SU(2)$ gauge field. We do it by considering an alternative (assymmetric) twisting prescription so that the gauge field is given by eq.(3.1) while instead of $U$ we substitute

$$U'_{\text{loop}} = \exp(i\phi/2) \frac{1}{\sqrt{(r-x_0)^2 + z^2}} \begin{pmatrix} r-x_0 & z \\ -z & r-x_0 \end{pmatrix} \exp(-i\phi\tau_3/2). \quad (3.5)$$

Such a configuration is single valued, odd under parity, even-$U$ and hence it is topologically trivial for any profile function $v$. It is therefore disconnected from the sphaleron configuration and has an integer Chern-Simons number. In the presence of this gauge field there is an even number of normalizable fermionic zero modes.

However there exists a different symmetric twisting that gives an odd-parity odd-$U$ gauge field configuration

$$U'_{\text{loop}} = \exp(i\phi/2) \frac{1}{\sqrt{(r-x_0)^2 + z^2}} \begin{pmatrix} r-x_0 & z \\ -z & r-x_0 \end{pmatrix} \exp(i\phi\tau_3/2). \quad (3.6)$$
This configuration is connected to the sphaleron by the topological arguments given above and has $\text{CS} = 1/2$. The number of fermionic zero modes is odd in this case.

It is clear that it is easy to generalize these two constructions to a twisting with any odd winding numbers

$$U'_{\text{loop}} = \exp(i(2n+1)\phi\tau_3/2) \frac{1}{\sqrt{(r-x_0)^2 + z^2}} \begin{pmatrix} r-x_0 & z \\ -z & r-x_0 \end{pmatrix} \exp(i(2m+1)\phi\tau_3/2),$$

where $n$ and $m$ are integers. For topologically trivial configurations ($\text{CS}$ is integer) $n+m$ must be odd while for topologically non-trivial ones ($\text{CS}$ is half integer) $n+m$ must be even.

Physically the assymmetrically twisted loop can be interpreted as a bound state of two sphalerons while the symmetrically twisted one is to be associated to a single deformed sphaleron. This will be clear if we take the limit of a collapsed loop ($z_0 \to 0$) for both cases. For the symmetrically twisted loop we get

$$U'_{\text{loop}}(x, y, z) \to \exp(i\phi\tau_3/2) \frac{1}{\sqrt{r^2 + z^2}} \begin{pmatrix} x + iy & z \\ -z & x - iy \end{pmatrix} \exp(-i\phi\tau_3/2) = (3.8)$$

Thus the collapsed loop coincides (up to a profile function) with the sphaleron configuration.

In the case of a trivially (assymetrically) twisted loop we get

$$U'_{\text{loop}} \to \exp(i\phi\tau_3/2) \frac{1}{\sqrt{r^2 + z^2}} \begin{pmatrix} r & z \\ -z & r \end{pmatrix} \exp(-i\phi\tau_3/2) = (3.9)$$

We may now introduce a continuous parameter $\alpha$. By rescaling $z \to \alpha z$, and by continuously taking $\alpha \to 0$ we see that $U'_{\text{loop}}$ can be continuously deformed to $U = 1$ by preserving its parity properties. This is an illustration of the topological triviality of assymmetrically twisted loops.

We now deform the assymmetrically twisted loop into a configuration which geometrically looks like two loops of equal radii with their centers on the $z$-axis at $z_0$ and $-z_0$ respectively. We assign symmetrical twistings with opposite handedness to both of them. The topological equivalence of each one of them to the sphaleron implies an interpretation for the assymmetrically twisted topologically trivial $W$-loop as a superposition (bound state) of two odd-$U$ sphaleron deformations. This type of deformation can be understood as follows. The topologically trivial $W$-loop corresponds to $U_1 = T U_0 T^+$, where $U_0$ stands for the untwisted $U$ matrix in eq.(3.4) at $k = 0$, and $T$ is a twisting exponential $\exp(i\phi\tau_3/2)$. The matrices $U_+ = T U_0 T$ and
$U_- = T^+ U_0 T^+$ correspond to topologically non-trivial symmetrically twisted loops with opposite handedness. We now consider the trivially twisted loop with $TU_0 T^+$. This matrix can split into a product

$$TU_0 T^+ = TU_0^{1/2} T T U_0^{1/2} T^+. \quad (3.10)$$

The angle $\theta$ which is an angle going around the string from 0 to $2\pi$ splits therefore into two angles $\theta_1$ and $\theta_2$ which take values in $(0, \pi)$ and $(\pi, 2\pi)$ respectively. These two angles correspond to the matrices $U_0^{1/2}$ above. While the $U_0^{1/2}$ is not single valued a product of them can very well be. By changing the coordinate dependence of $U_0^{1/2}$ and that of the profile function one can split one loop of radius $r = x_0$ with a center at $z_0 = 0$ into two loops of equal radii and centers separated by $2z_0$ as follows (see Fig.2)

$$U'_{\text{loop}} \rightarrow \exp(i\phi\tau_3/2) \left[ \frac{1}{\sqrt{(r - x_0)^2 + (z - z_0)^2}} \begin{pmatrix} r - x_0 & z - z_0 \\ -z + z_0 & r - x_0 \end{pmatrix} \right]^{1/2} \exp(i\phi\tau_3/2) \times \exp(-i\phi\tau_3/2). \quad (3.11)$$

The line of zeros of the profile function consequently splits into two lines with $z = z_0$ and $z = -z_0$ respectively. In the limit $z_0 \rightarrow \infty$ the angles $\theta_1, \theta_2$ take effectively values in $(0, 2\pi)$ near the cores of the separate loops. Thus these loops can be viewed as distinct single valued gauge configurations in the limit $z_0 \rightarrow \infty$.

It is clear that under splitting in the $z$ direction the deformation of the asymmetrically twisted loop is not odd under parity. Indeed under parity conjugation we have to interchange the position of the two nontrivial sphaleron deformations. However there is a way to implement an odd-parity preserving deformation. We do it by splitting the trivially twisted loop into two topologically non-trivial and concentric ones ($z_0 = 0$) but with different radii $r$ (see Fig.3). In this case we have

$$U'_{\text{loop}} \rightarrow \exp(i\phi\tau_3/2) \left[ \frac{1}{\sqrt{(r - x_1)^2 + z^2}} \begin{pmatrix} r - x_1 & z \\ -z & r - x_1 \end{pmatrix} \right]^{1/2} \exp(i\phi\tau_3/2) \times \exp(-i\phi\tau_3/2), \quad (3.12)$$

where $x_{1,2}$ are the radii of central lines of strings in the resulting loops. It is important that we maintain the odd parity of the whole gauge field under such a deformation. This is necessary so that we keep the gauge field in the same topological class as before.

Let us now consider possible deformations of a nontrivial symmetrically twisted loop (Fig.1b) similar to those of trivial asymmetrically twisted loop. It is obvious
that a deformation into two assymetrically twisted loops (Fig.1a) is impossible. It can be easily seen however that an odd-parity superposition of one topologically trivial W-loop with a nontrivial one is allowed. In a similar way a symmetrically twisted loop can split into an odd-parity superposition of an assymetrically twisted (i.e. trivial) loop and a symmetrically twisted one. We can now make use of the two elementary deformations presented in order to build non-minimal configurations. By taking into account that \( T^2 = T U_0^{1/2} T + U_0^{-1/2} T \) it is straightforward to show that non-minimal twisted loops with \( m, n \neq 0 \) can be interpreted as multiple sphaleron-antisphaleron bound states. Topologically non-trivial twisted loops are a superposition of an odd number of elementary non-trivial ones while the trivial loops correspond to even numbers of deformed sphalerons and antisphalerons. Notice also that for a general non-minimal twisting it is possible to get an odd-parity splitting of loops in the \( z \) direction too.

The case of a different twisting of the loop associated with a shift \( m, n \rightarrow m + 1/2, n + 1/2 \) does not correspond to any odd-parity gauge field configuration and therefore it does not fit into our classification. We postpone a detailed analysis of a generalization of our classification for future work. Here we only want to indicate that this type of configurations does not have a simple interpretation in terms of deformed (anti)sphalerons.

4 Conclusions

We have argued that any odd-parity sphaleron deformation that contributes to rapid baryon violating transition rates in the early Universe must be topologically connected to the electroweak sphaleron. We demonstrated that a sufficient condition for that is that its pure gauge behaviour at infinity is given by an odd-parity \( U \)-field. Deformations which are not odd under parity such as deformed sphaleron solutions with \( CS \neq 1/2 \) or the ones induced by Yukawa interactions \([7]\) are beyond the reach of our classification.

In closing we discuss the possible role sphaleron deformations might have played in the baryogenesis at the electroweak scale. In the context of a first order electroweak phase transition the expanding bubble wall of the broken symmetry phase drives the baryon and CP violating processes out of thermal equilibrium. Consequently they are the region where the biasing of the baryon number effectively takes place \([8]\). In their absence, such as in a second order transition, it has been argued that a similar biasing effect can arise by the moving edges of the electroweak vortex solutions. If stable they are expected to have been produced via the Kibble mechanism \([16]\). Our present work suggests that sphaleron deformations could also play a similar role. Electroweak vortex solutions constitute a set of measure zero in the large class of such configurations that could contribute to baryogenesis. We would expect, for example, that the evolution of large networks of odd-parity odd-\( U \) \( W \)-loops and their eventual shrinking and contraction to render their phase boundaries with odd-parity even-\( U \) configurations, such as the vacuum, effective regions of
biasing for the rapidly produced baryons. The precise baryon asymmetry produced in such a highly nonperturbative scenario is certainly a challenge to compute and a highly nontrivial dynamical problem.

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Figure Captions

Fig.1. Examples of topologically trivial (Fig.1a) and nontrivial (Fig.1b) sphaleron deformations. They are given by assymetrical-symmetrical twists of the W-loop, hereby depicted by lines with antiparallel-parallel orientation respectively.

Fig.2. Odd-parity violating split (deformation) of a topologically trivial (even-$U$) W-loop.

Fig.3. Odd-parity preserving split (deformation) of a topologically trivial (even-$U$) W-loop.