We consider an optimal shrinkage algorithm that depends on an effective rank estimation and imputation, coined optimal shrinkage with imputation and rank estimation (OSIR), for matrix denoising in the presence of high-dimensional noise with the separable covariance structure (colored and dependent noise). The algorithm does not depend on estimating separable covariance structure of the noise. On the theoretical side, we study the asymptotic behavior of outlier singular values and singular vectors and prove the delocalization of the non-outlier singular vectors of the associated random matrix with a convergence rate, and apply these results to analyze OSIR over different signal strengths and sizes of data matrices. On the application side, we carry out simulations to demonstrate the effectiveness of OSIR, and apply it to study the fetal electrocardiogram signal processing challenge and the two-dimensional random tomography problem.

Keywords: matrix denoising; random matrix; high dimensional noise noise; spike model; separable covariance.

1. Introduction

We are interested in a $p \times n$ signal-plus-noise data matrix
\begin{equation}
\tilde{S} = S + Z \in \mathbb{R}^{p \times n},
\end{equation}
which is formed by aligning $n \in \mathbb{N}$ samples of dimension $p \in \mathbb{N}$, where $Z$ is a noise-only random matrix and $S$ is a low-rank signal matrix admitting the following singular value decomposition (SVD)
\begin{equation}
S = \sum_{i=1}^{r} d_i u_i v_i^\top,
\end{equation}
where $r \geq 1$ is assumed to be small compared with $p$ and $n$, $u_i \in \mathbb{R}^p$ and $v_i \in \mathbb{R}^n$ are left and right singular vectors, and $d_i > 0$ are the associated singular values which may depend on $n$. We are interested in the high dimensional setup that the dimension of the signal $p$ is comparable to the number the observation $n$; that is, $p = p(n)$ and $p/n \to \beta \in (0, \infty)$. The setting is considered as a generalization of the traditional high dimensional spiked model [6, 7, 9]. It has been shown in [13] that under this high dimensional setup, if the empirical spectral distribution (ESD) of singular values from $Z$ converges to a non-random compactly supported probability measure, the extreme singular values and the direction of extreme left and right singular vectors of $\tilde{S}$ are all biased from those of $S$, and these biases converge to a closed form depending on the D-transform [13] and the Stieltjes transform [57], which are integral transforms of the limiting singular distribution of
the noise only matrix $Z$, and exhibit a phase transition when the signal strength $d_i$ exceeds a critical value, which is usually called the BBP (Baik-Ben Arous-Péché) phase transition named after the authors of the paper [8]. To handle the above-mentioned bias, a celebrated optimal weighting approach, which is later called optimal shrinkage (OS) method in [44, 48, 27], for recovering the data matrix from a noisy data matrix via SVD has been actively studied in the past decade. Overall, the OS algorithm needs a properly designed nonlinear function $\varphi : [0, \infty) \to [0, \infty)$, so that the corresponding estimator $\hat{S}_\varphi$ is constructed by

$$\hat{S}_\varphi = \sum_{i=1}^{p^\wedge n} \varphi(\tilde{\lambda}_i) \tilde{\xi}_i \tilde{\zeta}_i^\top,$$

where $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_{p^\wedge n} \geq 0$, are common eigenvalues of $\tilde{S}\tilde{S}^\top$ and $\tilde{S}^\top \tilde{S}$ and $\{\tilde{\xi}_i\}$ and $\{\tilde{\zeta}_i\}$ are respectively the left and right singular vectors of $\tilde{S}$. Such $\varphi$ is called a shrinker. By choosing a proper loss function $L_n$ that depends on $\hat{S}_\varphi$ and $S$, the optimal shrinker $\varphi^\ast$ is defined as $\varphi^\ast := \arg \min_{\varphi} \lim_{n \to \infty} L_n(\hat{S}_\varphi, S$).

Consider the white noise as a special example; that is, $Z = X$, where the entries of $X$ are i.i.d. with zero mean, variance $\sigma^2/n$ for some noise level $\sigma > 0$, and finite fourth moment. Under this setup, the ESD of $ZZ^\top$ asymptotically converges to the celebrated Marchenko-Pastur (MP) law [40]. In [27], the bulk and the optimal shrinkers $\varphi^\ast(\tilde{\lambda}_i)$ under various loss functions are derived, whose value has a closed form determined solely by $\tilde{\lambda}_i$ and $\beta$ (See the paragraph after Proposition 2.3 for more details). When the noise is not white, the close form of the optimal shrinker with respect to the Frobenius norm loss is derived in [41] (See Proposition 2.3), under the assumptions that the ESD of the noise matrix asymptotically converges to a non-random compactly supported probability measure that satisfies some decay properties in the right most edge of the bulk, and that a delocalization conjecture of the involved singular vectors is satisfied. An algorithm named OptShrink is then provided, assuming the knowledge of the effective rank, $\hat{r}$, which is the number of singular values that exceed some critical value determined by the noise structure. The OS method under the white noise assumption (called TRAD hereafter) considered in [27] has been applied to various applications, for example, the fetal electrocardiogram (ECG) extraction problem from the trans-abdominal maternal ECG (ta-mECG) [52], ECG T wave quality evaluation [53], otoacoustic emission signal denoising [38], stimulation artifact removal from the intracranial electroencephalogram (EEG) [1] and cardiogenic artifact removal from the EEG [16]. We shall mention that the OS idea under the high-dimensional white noise assumption could also lead to a better estimation of covariance matrix [51, 21] and precision matrix [28]. See Figure 1 for an application to the fetal ECG (fECG) extraction, where the ta-mECG shown in Figure 1(a) is truncated into pieces indicated by the red boxes, and the truncated pieces are aligned according to the maternal R peaks indicated by the red arrows. The fetal R peaks are labeled as green dots. The associated data matrix $\tilde{S}$ is shown in Figure 1(b), where the mECG is viewed as the signal, saved as the matrix $S$, and the fECG and inevitable recording noise in each piece is jointly viewed as the noise, saved as the matrix $Z$. The results of TRAD, indicated by $\hat{S}$, are shown as red curves in Figure 1(c), which is the recovered mECG after removing the fECG. In 1(d), we show $S - \hat{S}$, which is the recovered fECG. We could see that the maternal and fECGs have been well decomposed, and
Figure 1. An illustration of extracting the fECG from the ta-mECG shown in (a), where the fetal R peaks labeled by experts are marked as black crosses. (b) is the data matrix including the pieces truncated from the ta-mECG and aligned by the maternal R peaks indicated in red arrows shown in (a). The results of the OS under TRAD and our proposed algorithm, OSIR, both with the operator norm, which is viewed as the estimated mECG, are shown as red and blue curves shown in (c) respectively. By subtracting the mECG from the ta-mECG, we obtain the estimated fECG shown in (d). The error of the mECG estimation could be visualized in the estimated fECG indicated by the black box, which is zoomed in in (e). The covariance structure of the fECG is shown in (f).

we can better visualize the fetal R peaks in Figure 1(d), while there are still visible “remaining” parts around maternal R peaks indicated by the black box, which is zoomed in in Figure 1(e). Note that the recovered fECG is still contaminated by the recording noise, so it is in general not clean, and in the literature we call it the rough fetal ECG. We could apply TRAD on the rough fetal ECG to further recover the fetal ECG, which we do not show details and refer readers with interest to [52] for details. Also, note that such an algorithm could be viewed as a nonlinear variation of the traditional template subtraction algorithm commonly applied in the signal processing society when there is only one channel, and we refer readers to
for a more detailed literature review for other relevant algorithms, particularly those when multiple channels are available.

While TRAD has been successfully applied to several problems, in practice the noise should not be white, and the noises among different samples may not be independent. Under this realistic setup, asymptotically the ESD of $ZZ^\top$ no longer follows the MP law, and hence the performance of TRAD may not be guaranteed. See Figure 1(f) for an illustration of the non-white structure of the fetal ECG when it is viewed as noise, where the simultaneously recorded contact fetal ECG is truncated in the same way. Another common resource leading to the non-white structure is the application of various high and low pass filters. Therefore, a new OS approach respecting such noise is needed. In this paper, we focus on the noise satisfies the separable covariance structure [45]; that is, we consider

$$Z = A^{1/2}XB^{1/2},$$

where the entries of $X$ are i.i.d. with zero mean and mild moment conditions that will be specified later, and $A$ and $B$ are respectively $p \times p$ and $n \times n$ deterministic positive-definite matrices that describe the “colorness” and “dependence” of noise, respectively. A random matrix satisfying the separable covariance structure has been applied to many problems, like spatiotemporal analysis and wireless communication and several recent applications [45, 29, 23]. Many recent works have been dedicated to study this general model. When $B = I_n$, it is well-known that the ESD of $ZZ^\top$ converges to the deformed MP law [40]; the distribution of the largest eigenvalue follows the Tracy-Widom law [54, 55] (or commonly referred to as the edge universality) and has been proved for $A = I_p$ in [30, 46] and for general $A$ in [24, 42, 10, 34, 19, 32] under various moment assumptions on the entries for $X$. For the sample singular vectors, the delocalization [32, 43] and ergodicity [15] have been constructed. For general $A$ and $B$, the convergence of the ESD to a limiting law was shown in [45, 56, 39]. The edge universality and delocalization of eigenvectors of the separable covariance matrices have been established in [19, 59]. The local estimates of the resolvents (or the Green’s functions) of $ZZ^\top$ under the separable covariance structure, which is generally referred to as the local law, has been proved in [58, 59].

Many recent efforts have been devoted to develop a denoising algorithm when the noise model satisfying (4). The model $Z = A^{1/2}X$ is considered in [36] and later the model $Z = A^{1/2}XB^{1/2}$ is considered in [37]. The author proposes to apply the whitening technique before applying TRAD; that is, the author estimates $A$ and $B$, denoted as $\hat{A}$ and $\hat{B}$, and then performs TRAD on the whitened matrix $\hat{A}^{-1/2}\hat{S}\hat{B}^{-1/2}$. This approach works well when $A$ and $B$ can be accurately estimated; for example, when both $A$ and $B$ are diagonal. In [22], an optimal hard thresholding method named ScreeNot under the general set, $A^{1/2}XB^{1/2}$, is developed by finding the hard threshold $\vartheta \geq 0$ such that the denoised matrix is defined as $\hat{S}_{SN} = \sum_{i=1}^{p \wedge n} \sqrt{\lambda_i}1(\hat{\lambda}_i > \vartheta)\hat{\xi}_i\hat{\xi}_i^\top$, where $\vartheta$ is computed by a pseudo distribution constructed by $\hat{\lambda}_i$’s only. The merit of ScreeNot is that the optimal hard threshold is estimated without knowing the underlying low-rank model $X$ or $A$ and $B$ of the noise $Z$, if some mild conditions of the asymptotic spectral density hold.

Inspired by the success and limitation of these results, to achieve a high quality denoise of a noisy data matrix with the separable covariance structure, we propose an OS algorithm, coined optimal shrinkage with imputation and rank estimation
(OSIR). OSIR can be viewed as an extension of OptShrink [41] by introducing a new effective rank estimator, considering various loss functions other than the Frobenius norm, and combining the imputation idea used in [22]. To guarantee the performance OSIR, we provide a series of theoretical results, including the delocalization of the non-outlier singular vectors and a convergence rate. More specifically, we have the following contributions in this paper. First, based on the derived asymptotic behavior of the outlier singular values and the associated biased singular vectors of the noisy data matrix \( \hat{S} = S + A^{1/2}XB^{1/2} \), which is an extension of the results in [11], we establish the rate of convergence, the BBP phase transition, and the delocalization of the non-outlier singular vectors with rates, which have not yet been established in the literature to our best knowledge. Based on this result, we also provide the convergence rate of OSIR. Note that the delocalization of the non-outlier singular vectors with rates is conjectured in [11] to guarantee the function of OptShrink. Second, we provide a novel effective rank estimation algorithm based on the theoretical results. Since the accuracy of OptShrink largely depends on the knowledge of effect rank, the proposed rank estimation algorithm can be applied to OptShrink. Moreover, we construct an estimate of the D-transform using the “imputation” method proposed in [22]. We will show numerically that this new approach has a precise and consistent D-transform estimation compared to the original method in [11] when the rank is overestimated. Furthermore, we consider not only the Frobenius norm loss, but also the operator and nucleus norm losses.

In OSIR, no matter which loss function we choose, we only need the eigenvalues of \( \hat{S}S^\top \). The proposed algorithm is tested on numerical simulations, and applied to the real-world trans-abdominal maternal ECG problem and the two-dimensional random tomography problem [51].

NOTATION: for any random variable \( X \), we denote \( \tilde{X} \) as the perturbed \( X \), and \( \hat{X} \) as the estimator of \( X \). We use \( C \) to denote a generic positive constant, whose value may change from one line to another. For two sequences \( \{a_n\} \) and \( \{b_n\} \) indexed by \( n \), the notation \( a_n = O(b_n) \) means that \( |a_n| \leq C|b_n| \) for some constant \( C > 0 \) as \( n \to \infty \) and \( a_n = o(b_n) \) means that \( |a_n| \leq c_n|b_n| \) for some positive sequence \( c_n \downarrow 0 \) as \( n \to \infty \). We also use the notations \( a_n \lesssim b_n \) if \( a_n = O(b_n) \) and \( |a_n| \ll |b_n| \) if \( a_n = o(b_n) \), and \( a_n \asymp b_n \) if \( a_n = O(b_n) \) and \( b_n = O(a_n) \). For a matrix \( M \), we use \( \|M\| \) to denote its operator norm. For a matrix \( M \) and a number \( a > 0 \), we may abuse the notation and write \( M = O(a) \) if \( \|M\| = O(a) \). We often write an identity matrix of any dimension as \( I \). For \( a, b \in \mathbb{R} \), we denote \( a \vee b \) and \( a \wedge b \) to be the maximal and minimal value of \( a \) and \( b \), respectively. We follow the convention and reserve \( E + i\eta \) to indicate a point in the upper half plane of \( \mathbb{C} \); that is, \( E \in \mathbb{R} \) and \( \eta > 0 \). A list of notations could be found in Tables 1 and 2.

2. Backgrounds

2.1. Asymptotic Framework. We start with some quantities. The Stieltjes transform for a probability measure \( \nu \) is \( m_\nu(z) := \int \frac{1}{X-z}d\nu(\lambda), \) where \( z \in \mathbb{C}^+ \). For an \( n \times n \) symmetric matrix \( H \), the ESD of \( H \) is defined as \( \pi_H^{(n)} := \frac{1}{n} \sum_{i=1}^n \delta_{\xi_i} \), where \( \ell_1 \geq \ell_2 \geq \cdots \geq \ell_n \) are the eigenvalues of \( H \) and \( \delta \) means the Dirac delta measure. We assume that the entries \( x_{ij} \) of \( X \) satisfying \( \mathbb{E} x_{ij} = 0, \mathbb{E} |x_{ij}|^2 = n^{-1} \), and has finite fourth moment. Denote the common eigenvalues of \( ZZ^\top \) and \( Z^\top Z \) (respectively \( \hat{S}S^\top \) and \( \hat{S}^\top \hat{S} \)) by \( \{\lambda_i\}_{i=1}^{\nu} \) (respectively \( \{\tilde{\lambda}_i\}_{i=1}^{\nu} \)). For \( z \in \mathbb{C}^+ \), denote
the Green functions of $ZZ^T$ and $Z^TZ$ as

$$G_1(z) := (Z^T - z)^{-1} \quad \text{and} \quad G_2(z) := (Z^T Z - z)^{-1}$$

respectively. The Stieltjes transforms of ESDs of $ZZ^T$ and $Z^TZ$ are formulated respectively by the Green functions $G_1(z)$ and $G_2(z)$ via

$$m_1^{(p)}(z) := \frac{1}{p} \text{Tr} G_1(z) \quad \text{and} \quad m_2^{(n)}(z) := \frac{1}{n} \text{Tr} G_2(z).$$
Similarly, we denote the Green functions of $\tilde{SS}^T$ and $\tilde{S}^T \tilde{S}$ as $\tilde{G}_1(z) := (\tilde{SS}^T - z)^{-1}$ and $\tilde{G}_2(z) := (\tilde{S}^T \tilde{S} - z)^{-1}$ respectively and the Stieltjes transforms of ESDs of $\tilde{S} \tilde{S}$ and $\tilde{S}^T \tilde{S}$ as $\tilde{m}_1^{(p)}(z) := \frac{1}{p} \text{Tr} \tilde{G}_1(z)$ and $\tilde{m}_2^{(n)}(z) := \frac{1}{n} \text{Tr} \tilde{G}_2(z)$ respectively. Denote $\beta_n := \frac{2}{n}$. By a direct calculation, we have the relationship $m_2^{(n)}(z) = -\frac{1}{\beta_n} + \beta_n \tilde{m}_1^{(p)}(z)$. Moreover, from [45], if $\beta_n \rightarrow \beta \in (0, \infty)$ as $n \rightarrow \infty$, $\pi_A^{(p)}$, $\pi_B^{(n)}$ converge to nonrandom probability distributions $\rho_A$, $\rho_B$ with compact supports, and the entries $X$ are independent with mean 0 and the same variance, then almost surely $\pi_A^{(p)}$ and $\pi_B^{(n)}$ converge to deterministic distributions with compact supports.

Besides, we define $(m_1^{(p)}(z), m_2^{(n)}(z)) \in \mathbb{C}^+ \times \mathbb{C}^+$ as the unique solution to the following system of self-consistent equations

$$m_1^{(p)}(z) = \beta_n \int \frac{x}{-z[1 + zm_2^{(n)}(z)]]} \pi_A^{(p)}(dx) \quad \text{and} \quad m_2^{(n)}(z) = \int \frac{x}{-z[1 + zm_1^{(p)}(z)]} \pi_B^{(n)}(dx).$$

From the above equations, if we define the function $f$ on $\mathbb{C}^+ \times \mathbb{C}^+$ by $f(z, m) := -m + \int \frac{x}{-z + x\beta_n \int \frac{x}{\pi_A^{(p)}(dt) \pi_B^{(n)}(dx)}}$ for $z \in \mathbb{C}^+$, then $m_2^{(n)}(z)$ can be characterized as the unique solution so that $f(z, m_2^{(n)}(z)) = 0$ for $z \in \mathbb{C}^+$ [45]. Moreover, $m_1^{(p)}(z)$ and $m_2^{(p)}(z)$ are the Stieltjes transforms of the densities $\tilde{G}_1^{(p)}$ and $\tilde{G}_2^{(p)}$ so that for $z = E + i\eta \in \mathbb{C}^+$,

$$\tilde{G}_1^{(p)}(E) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \text{Im} m_1^{(p)}(E + i\eta) \quad \text{and} \quad \tilde{G}_2^{(p)}(E) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \text{Im} m_2^{(n)}(E + i\eta).$$

Also, we define

$$m_1^{(p)}(z) := \int \frac{1}{-z[1 + zm_2^{(n)}(z)]} \pi_A^{(p)}(dx) \quad \text{and} \quad m_2^{(n)}(z) := \int \frac{1}{-z[1 + zm_1^{(p)}(z)]} \pi_B^{(n)}(dx).$$

It has been known that when $n \rightarrow \infty$, $m_1^{(p)}(z) - m_1^{(p)}(z)$ and $m_2^{(n)}(z) - m_2^{(n)}(z)$ converges to zero uniformly with respect to $z$. See [29] Theorem 3.6 and [20] Theorem S.3.9 for details. Again, by letting $\eta \downarrow 0$, we obtain their corresponding probability measures with the inverse formula

$$\rho_1^{(p)}(E) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \text{Im} m_1^{(p)}(E + i\eta) \quad \text{and} \quad \rho_2^{(n)}(E) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \text{Im} m_2^{(n)}(E + i\eta).$$

### Table 2. List of default notations, part 2.

| Notation | Description |
|----------|-------------|
| $D_1(\tau_1, \eta) := \{z = E + i\eta: \lambda_+ < E < \xi, -\tau_1 < \eta < \tau_1\}$ | |
| $D_2(\tau_2, \eta) := \{z = E + i\eta: 0 < E < 1/\sqrt{T(\xi)}, -\tau_2 < \eta < \tau_2\}$ | |
| $S_0(\xi, \eta, \omega) := S(\xi, \eta, \omega) \cap \{z = E + i\eta: \lambda_+ < E < \xi\}$ | |
| $S(\xi, \eta) := \{z = E + i\eta: \lambda_+ - \xi \leq E < \xi, -\xi < \eta < \xi\}$ | |
| $S_0(\xi, \eta, \omega) := S(\xi, \eta, \omega) \cap \{z = E + i\eta: \lambda_+ + \xi \leq E < \xi\}$ | |
| $S(\xi, \eta) := \{z = E + i\eta: \lambda_+ + \xi \leq E < \xi, -\xi < \eta < \xi\}$ | |
| $\Pi(\eta) := \Pi_1(\eta) + \Pi_2(\eta)$ | |
| $\Pi_1(\eta) := \frac{1}{\eta - z}$ | |
Throughout the rest of this paper, when there is no danger of confusion, we omit the superscripts \((p)\) and \((n)\) from our notations for simplicity.

We recall the following lemma from Section 3 of [17].

**Lemma 2.1.** Let \(A\) and \(B\) be deterministic symmetric matrices with eigenvalues \(\sigma_1^A \geq \sigma_2^A \geq \ldots \geq \sigma_p^A\) and \(\sigma_1^B \geq \sigma_2^B \geq \ldots \geq \sigma_n^B\) respectively and satisfy

\[
\sigma_i^A \vee \sigma_i^B \leq \tau^{-1} \quad \text{and} \quad \pi_A^{(p)}([0, \tau]) \vee \pi_B^{(n)}([0, \tau]) \leq 1 - \tau.
\]

Then the densities \(\rho_1, \rho_2, \rho_{1c}\) and \(\varrho_2\) all have the same support on \((0, \infty)\), which is a union of finite intervals \(\bigcup_{k=1}^L [e_{2k}, e_{2k-1}] \cap (0, \infty)\), where \(L \in \mathbb{N}\) depends only on \(\pi_A\) and \(\pi_B\) and \(e_1 > e_2 > \ldots > e_{2L}\). Moreover, \((x, m) = (e_k, m_{2c}(e_k))\) are the real solutions of the equations \(f(x, m) = 0\) and \(\frac{\partial f}{\partial m}(x, m) = 0\). Finally, we have \(e_1 = O(1), m_{1c}(e_1) \in (-\max_j \sigma_j^b)^{-1},0)\) and \(m_{2c}(e_1) \in (-\max_i \sigma_i^c)^{-1},0)\).

We call \(e_k\) the spectral edges. In this paper, we focus on the rightmost edge \(\lambda_+ := e_1\). For \(z \in \mathbb{C}^+\), we denote the \(D\)-transform of \(\rho_{1c}\) as

\[
\mathcal{T}(z) := zm_{1c}(z)m_{2c}(z).
\]

See the free probability theory, e.g., [13] Section 2.5, for more details. Note that \(m_{1c}\) and \(m_{2c}\) can be extended to \(x > \lambda_+\) so that \(m_{1c}(x) = \int_0^{\lambda_+} \frac{\rho_{1c}(t)}{t-x}dt\) and \(m_{2c}(x) = \int_0^{\lambda_+} \frac{\rho_{2c}(t)}{t-x}dt\). Thus, when \(x > \lambda_+, m_{1c}(x), m_{2c}(x)\) and \(\mathcal{T}(x)\) are well-defined. Moreover, by a direct calculation, both \(m_{1c}(x)\) and \(m_{2c}(x)\) are negative and monotonically increasing and \(\mathcal{T}(x)\) is monotonically decreasing, such that \(m_{1c}(x), m_{2c}(x)\) and \(\mathcal{T}(x)\) are invertible.

2.2. Model.

**Definition 2.1.** (Bounded support condition). We say a random matrix \(X = [x_{ij}]_{i=1,...,p} = 1,...,n\) has a bounded support \(\phi_n > 0\) if \(\max_{i,j} |x_{ij}| \leq \phi_n\), where \(\phi_n\) is a deterministic parameter and satisfies \(\phi_n = n^{2/a-1/2}\) for some constant \(a > 4\).

This bounded support assumption is introduced to simplify the discussion, and it can be easily removed. Recall that for a random matrix \(X\) whose entries are i.i.d. and have at least \(4 + \epsilon\) moments, it can be reduced to a random matrix with bounded support with probability \(1 - o(1)\) using the truncation, centralization, and rescaling [8] Section 4.3.2]. Assuming that \(X\) satisfies the cut-off model, since we do not assume the entries of \(X\) are identically distributed, the means and variances of the truncated entries may be different.

**Assumption 2.2.** Fix a small constant \(0 < \tau < 1\). We need the following assumptions for the model \(\hat{S} = S + A^{1/2}XB^{1/2}\):

(i) **(Assumption on \(x_{ij}\)).** Suppose \(X\) has a bounded support \(\phi_n\),

\[
\max_{i,j} |E x_{ij}| \leq n^{-2-\tau} \quad \text{and} \quad \max_{i,j} \left| E |x_{ij}|^2 - n^{-1} \right| \leq n^{-2-\tau}.
\]

Further, we assume that there exists a constant \(C_4 > 0\) such that \(E|\sqrt{n}x_{ij}|^4 \leq C_4\).

(ii) **(Assumptions on \(p/n\)).** \(\tau < \frac{\epsilon}{n} < \tau^{-1}\).

(iii) **(Assumption on \(A\) and \(B\)).** Assume \(A\) and \(B\) are deterministic symmetric matrices with eigendecompositions

\[
A = Q^a \Sigma^a (Q^a)^\top \quad \text{and} \quad B = Q^b \Sigma^b (Q^b)^\top
\]
respectively, where $\Sigma^a = \text{diag}(\sigma^a_1, \ldots, \sigma^a_n)$, $\Sigma^b = \text{diag}(\sigma^b_1, \ldots, \sigma^b_n)$, $\sigma^a_i \geq \sigma^b_i \geq \ldots \geq \sigma^b_p$, $\sigma^b_1 \geq \sigma^b_2 \geq \ldots \geq \sigma^b_n$. $Q^a = (q^a_1, \ldots, q^a_p) \in O(p)$ and $Q^b = (q^b_1, \ldots, q^b_n) \in O(n)$. For the ESD of $A$ and $B$, $\pi_A$ and $\pi_B$, we have respectively

\begin{align}
1 + m_1(\lambda_+)\sigma^b_1 & \geq \tau \quad \text{and} \quad 1 + m_2(\lambda_+)\sigma^a_1 \geq \tau.
\end{align}

Moreover, for all sufficiently large $n$, we assume $[10]$ holds.

(iv) (Assumption on the signal strength). We assume

\begin{align}
d_1 \geq d_2 \geq \cdots \geq d_r > 0
\end{align}

for some $r \geq 1$. Further, we assume that $d_i < \tau^{-1}$. Denote a fixed value $\alpha > 0$ as

\begin{align}
\alpha := 1/\sqrt{T(\lambda_+)}.
\end{align}

We assume their exist an integer $1 \leq r_{\text{eff}} \leq r$ such that

\begin{align}
d_k - \alpha > 0, \quad 1 \leq k \leq r_{\text{eff}}.
\end{align}

Moreover, we allow the singular values $d_k$ to depend on $n$ under the condition that there exists an integer $1 \leq r^+ \leq r_{\text{eff}}$ such that

\begin{align}
d_k - \alpha > \phi_n + n^{-1/3}, \quad 1 \leq k \leq r^+
\end{align}

(v) (Assumption on distribution of the singular vectors). Let $G^a_n \in \mathbb{R}^{p \times r}$ and $G^b_n \in \mathbb{R}^{n \times r}$ be two independent matrices with i.i.d entries distributed according to a fixed probability measure $\nu$ on $\mathbb{R}$ with mean zero and variance one, and satisfy the log-Sobolev inequality (see below). We assume that the left and right singular vectors, $u_i \in \mathbb{R}^p$ and $v_i \in \mathbb{R}^n$, are the $i$-th columns obtained from the Gram-Schmidt (or QR factorization) of $\frac{1}{\sqrt{n}}G^a_n$ and $\frac{1}{\sqrt{n}}G^b_n$ respectively.

Remark 2.2. In (iii), [14] and [10] guarantees a regular square-root behavior of the spectral densities $p_{1c}$ and $p_{2c}$ near $\lambda_+$ (see Lemma A.6) and rules out the existence of outlier eigenvalues from $ZZ^\top$. Therefore, it also makes sure that $T(\lambda_+)$ is well-defined so as $\alpha$ in (iv). The second condition means that the spectra of $A$ and $B$ cannot concentrate at zero. In (iv), note that $r^+ \to r_{\text{eff}}$ as $n \to \infty$, and the lower bound in (18) is for the BBP phase transition, which means that sufficiently small signal singular values $d_i$ do not produce outliers beyond the threshold $\alpha$. The detail will be described in the following Theorem 3.2. In (v), the log-Sobolev inequality implies that entries of $u_i$ and $v_i$ have sub-Gaussian tails, and hence nice concentration properties [2 Section 2.3.2] that we will describe in Lemma A.5.

2.3. Optimal shrinkers for various loss functions. Here we recall the definition of asymptotic loss and optimal shrinker.

Definition 2.3 (Asymptotic loss). [27] Definition 1 Recall the signal matrix $S$ in [2]. Let $\mathcal{L} := \{L_{p,n}|p,n \in \mathbb{N}\}$ be a family of loss functions, where each $L_{p,n} : M_{p \times n} \times M_{p \times n} \to [0, \infty)$ is a loss function obeying $L_{p,n}(S, S) = 0$. Let $\varphi : [0, \infty) \to [0, \infty)$ be a nonlinear function and consider $S_{\hat{\varphi}}$ to be the singular shrinkage estimate defined in [3]. Let $p_n$ be a growing sequence such that $\lim_{n \to \infty} p_n/n \to \beta$. When $\lim_{n \to \infty} L_{p,n}$ exists, we define the asymptotic loss of the shrinker $\varphi$ (with respect to $L_{p,n}$) with the signal $d = (d_1, \ldots, d_r)$ as $L_\infty(\varphi|d) = \lim_{n \to \infty} L_{p,n}(S, S_{\hat{\varphi}})$.
Definition 2.4 (Optimal shrinker). [27, Definition 2] Let $L_\infty$ and $\varphi$ as defined in Definition 2.3. If a shrinker $\varphi^*$ has an asymptotic loss that satisfies $L_\infty(\varphi^*(d)) \leq L_\infty(\varphi(d))$ for any other shrinker $\varphi$, any $r \geq 1$, and any $d \in \mathbb{R}^r$, then we say that $\varphi^*$ is unique asymptotically admissible (or simply “optimal”) for the loss family $\mathcal{L}$ and that class of shrinkers.

In [27, 35], it was proved that the asymptotic loss exists for a variety of loss functions and a large class of nonlinearities, and the optimal shrinkers under different loss functions were computed. We list here for readers’ convenience. Denote $\varphi_i := \varphi^*(\lambda_i)$.

Proposition 2.3 (Sections IV. A and C of [27, Lemma 5.1 of [35]). When $d_i \geq \alpha$ (equivalent to $1 \leq i \leq r_{\text{eff}}$), the optimal shrinker is $\varphi_i = d_i \sqrt{\varphi_{1,i} \varphi_{2,i}}$, $\varphi_i = d_i \sqrt{\min\{a_{1,i}, a_{2,i}\}}$ and $\varphi_i = d_i \left(\sqrt{a_{1,i} a_{2,i}} - \sqrt{(1 - a_{1,i})(1 - a_{2,i})}\right)$ when the Frobenius norm, operator norm and nuclear norm are considered in the loss function respectively, where $a_{1,i} := \lim_{n \to \infty} (u_i, \xi_i)^2$ and $a_{2,i} := \lim_{n \to \infty} (v_i, \xi_i)^2$. When $d_i < \alpha$ (equivalent to $r_{\text{eff}} + 1 \leq i \leq r$), for any loss function, we have $\varphi_i = 0$.

Remark 2.4. In [27, Section IV.B], when $p = n$ and the operator norm is used as the loss, $\varphi_i = d_i$. Later, it is shown in [35] that when $p \neq n$, $\varphi_i = d_i \sqrt{\min\{a_{1,i}, a_{2,i}\}} / \max\{a_{1,i}, a_{2,i}\}$.

Motivated by Proposition 2.3, if we can estimate $d_i$, $a_{1,i}$ and $a_{2,i}$ using the available noisy data, we could obtain the above optimal shrinkers. For example, as shown in [13], if the ESD of $Z$ converges almost surely weakly to a nonrandom compactly supported probability measure, then when $d_i > \alpha$, we have that $d_i = 1 / \sqrt{T(\lambda_i)}$, $a_{1,i} = m_{1,i}(\lambda_i)$, $a_{2,i} = m_{2,i}(\lambda_i)$, and the optimal shrinker with respect to the Frobenius norm is derived by replacing $d_i \sqrt{a_{1,i} a_{2,i}}$ with the corresponding values [41]. Moreover, as shown in [13, 27], when $A = I_p$, $B = I_n$ and $X$ have independent entries with zero mean, unit variance, and finite fourth moment, when $\tilde{\lambda}_i > 1 + \sqrt{3}$, we have $d_i = \ell_i$, $a_{1,i} = \ell_i^{1/2} / \ell_i^{1/2} a_{2,i}$ and $a_{2,i} = \ell_i^{1/2} / \ell_i^{1/2}$, where $\ell_i := \frac{1}{\sqrt{2}}(\tilde{\lambda}_i - \beta - 1 + (\tilde{\lambda}_i - 1 - 4\beta)^{1/2})^{1/2}$. Thus, when $\tilde{\lambda}_i > 1 + \sqrt{3}$, the optimal shrinker satisfies $\varphi_i = \frac{1}{\sqrt{2}}(\tilde{\lambda}_i - \beta - 1)^{1/2} (\varphi_i = \ell_i)$ and $\varphi_i = \frac{1}{\sqrt{2}}(\tilde{\lambda}_i - \beta - 1)^{1/2}$ respectively when the Frobenius (operator and nuclear respectively) norm is considered, and when $\tilde{\lambda}_i \leq 1 + \sqrt{3}$, $\varphi_i = 0$ for any loss function.

3. Main Results

We now state the limiting behavior and the associated convergence rate of the biased singular values and singular vectors, and hence our proposed algorithm. To simplify the presentation, we apply the notion of stochastic domination, which is used to state results of the form “$\xi$ is bounded by $\zeta$ with high probability up to a small power of $n$” [24].

Definition 3.1 (Stochastic domination). Let $\xi = (\xi^{(n)}(u) : n \in \mathbb{N}, u \in U^{(n)})$ and $\zeta = (\zeta^{(n)}(u) : n \in \mathbb{N}, u \in U^{(n)})$ be two families of nonnegative random variables, where $U^{(n)}$ is a possibly $n$-dependent parameter set. We say $\xi$ is stochastically dominated by $\zeta$, uniformly in $u$, if for any fixed (small) $\epsilon > 0$ and (large) $D > 0$, $\sup_{u \in U^{(n)}} \mathbb{P} \left[ \xi^{(n)}(u) > n^\epsilon \zeta^{(n)}(u) \right] \leq n^{-D}$ for large enough $n \geq n_0(\epsilon, D)$. 

and we shall use the notation $\xi \prec \zeta$ or $\xi = O_\prec(\zeta)$. Throughout this paper, the stochastic domination will always be uniform in all parameters that are not explicitly fixed, such as matrix indices, and $z$ that takes values in some compact set. Note that $n_\xi(\epsilon, D)$ may depend on quantities that are explicitly constant, such as $\tau$ in Assumption 2.2. Moreover, we say an event $\Xi$ holds with high probability if for any constant $D > 0$, $\mathbb{P}(\Xi) \geq 1 - n^{-D}$, when $n$ is sufficiently large.

3.1. Results of biased singular values and singular vectors. Denote $\{\xi_i\}_{i=1}^p$ and $\{\xi_i\}_{i=1}^n$ the left and right singular vectors of the noise matrix $Z$ respectively, and $\{\xi_i\}_{i=1}^p$ and $\{\xi_i\}_{i=1}^n$ as the left and right singular vectors of $\tilde{S}$ respectively. For a real value $x > \alpha$, denote three functions, $\theta(x)$, $a_1(x)$ and $a_2(x)$, by

$$
\theta(x) := T^{-1}(x^{-2}), \quad a_1(x) = \frac{m_1(\theta(x))}{x^2T(\theta(x))}, \quad a_2(x) = \frac{m_2(\theta(x))}{x^2T(\theta(x))}.
$$

These terms are used to estimate the signal strength $d_i$ and inner products of the clean and noisy singular vectors, $\langle u_i, \xi_i \rangle$ and $\langle v_i, \zeta_i \rangle$, which we will detail in the following theorems.

We first state the results of eigenvalues. Define

$$
\mathcal{O}_+ = \{1, \ldots, r^+\} \quad \text{and} \quad \Delta(d_i) := (d_i - \alpha)^{1/2},
$$

where $\Delta^2(d_i)$ quantifies how far $d_i$ is away from $\alpha$. The followings are our main theorems, and their proofs are postponed to the supplementary. Part of the proof of the following theorems is motivated by the original work in [59] and a recent work [20], where the authors focused on the covariance structure. The main difference comes from the fact that in general, the covariance model in [20] cannot be decomposed into the summation of two independent matrices, like $S$ and $Z = A^{1/2} X B^{1/2}$ in our model. Due to this difference, the results in [20] cannot be directly applied.

**Theorem 3.2.** Suppose Assumption 2.2 holds true. Then we have for $1 \leq i \leq r^+$,

$$
\tilde{\lambda}_i - \theta(d_i) < \phi_n \Delta^2(d_i) + n^{-1/2} \Delta(d_i).
$$

Furthermore, for a fixed integer $\varpi > r$, we have for $r^+ + 1 \leq i \leq \varpi$,

$$
|\tilde{\lambda}_i - \lambda_+| < n^{-2/3} + \phi_n^2.
$$

Next we provide the delocalization result about the singular vectors. We show that when $i \notin \mathcal{O}_+$, the non-outlier singular vectors, $\xi_i$ and $\zeta_i$, are distributed roughly uniformly and approximately perpendicular to the space spanned by $\{u_j\}_{j=1}^r$ and $\{v_j\}_{j=1}^r$ respectively.

**Theorem 3.3.** Suppose Assumption 2.2 holds. For $i = 1, \ldots, n$, denote $\eta_i := n^{-3/4} + n^{-5/6} \xi^{1/3} + n^{-1/2} \phi_n$ and $\kappa_i := \eta_i^{2/3} n^{-2/3}$. For any sufficiently small constant $c$ and $r^+ + 1 \leq i \leq cn$, we have for $j = 1, \ldots, r$,

$$
|\langle u_j, \tilde{\xi}_i \rangle|^2 \vee |\langle v_j, \tilde{\zeta}_i \rangle|^2 < \frac{n^{-1} + \phi_n^3 + \eta_i \sqrt{\kappa_i}}{\Delta(d_i)^4 + \phi_n^2 + \kappa_i}.
$$

If either $A$ or $B$ is diagonal, then the right hand side of (23) becomes $\frac{n^{-1} + \phi_n^3}{\Delta(d_i)^4 + \phi_n^2 + \kappa_i}$. Moreover, fix a constant $\bar{\tau} \in (0, 1/9)$ such that $(\phi_n + n^{-1/3} n^{\bar{\tau}}) \to 0$ as $n \to \infty$. If $i \in \mathcal{O}_+$ satisfies

$$
\Delta(d_i)^2 \leq n^{\bar{\tau}} (\phi_n + n^{-1/3}),
$$

...
then we have for $j = 1, \ldots, r$,

\begin{equation}
|\langle u_j, \tilde{\xi}_i \rangle|^2 \vee |\langle v_j, \tilde{\xi}_i \rangle|^2 \prec n^{3\varphi} \left( n^{-1} + \phi_n + \eta_n \sqrt{\lambda_i} \right).
\end{equation}

Next, for any $A \subset \mathbb{Q}_+$, define

\begin{equation}
\nu_i(A) := \begin{cases} 
\min_{j \notin A} |d_j - d_i|, & \text{if } i \in A, \\
\min_{j \in A} |d_j - d_i|, & \text{if } i \notin A 
\end{cases}
\end{equation}

and two projections,

\begin{equation}
P_A := \sum_{k \in A} \xi_k \xi_k^\top \quad \text{and} \quad P'_A := \sum_{k \in A} \tilde{\xi}_k \tilde{\xi}_k^\top.
\end{equation}

**Theorem 3.4.** Suppose Assumption 2.2 holds. Fix any $A \subset \mathbb{Q}_+$, we have

\begin{equation}
Q(i, j, A, n) = 1(i \in A, j \notin A) \Delta(d_i) \left[ \frac{\phi_n}{\nu_j^{1/2}(A)} + \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j(A)} \right]
+ 1(i \notin A, j \in A) \Delta(d_j) \left[ \frac{\phi_n}{\nu_i^{1/2}(A)} + \frac{\psi_1(d_i) \Delta(d_i)}{\nu_i(A)} \right] + R(i, A),
\end{equation}

where

\begin{equation}
R(i, A) := \frac{1}{\Delta(d_i)} \psi_1(d_i) + \frac{1}{\Delta(d_i)} \psi_1(d_i) \Delta^2(d_i) + \frac{1}{\Delta(\xi_i)} \psi_1(d_i) \Delta(d_i) + \frac{1}{\Delta(\xi_i)} \psi_1(d_i).
\end{equation}

3.2. **Toward our optimal shrinkage algorithm.** Recall the estimator $\hat{\varphi}$ given by (3). We now introduce our OSIR algorithm that gives an adaptive estimator $\hat{\varphi}$ for $\varphi$, and provide an asymptotic analysis of $\hat{\varphi}$.

Recall that the top eigenvalues of $SS^\top$ are deviated from the bulk of eigenvalues of $ZZ^\top$ when the associated signal strength is sufficiently strong. We can viewed these deviated eigenvalues as “missing values” of the bulk. To handle this missing value issue, we adopt the “imputation” method described in (22) to recover the eigenvalues of $ZZ^\top$ that are associated with those strong signals. We have the following heuristic. From Lemma A.6, we have $\rho_{1c}(z) \asymp (\lambda_+ - z)^{1/2}$ as $z \to \lambda_+$. Thus, for a fixed large integer $\varpi < n$, when $p$ is sufficiently large, for $1 \leq \ell \leq \varpi$, we have

\begin{equation}
\frac{\ell - 1}{p} = C \int_{\lambda_1}^{\lambda_\ell} \rho_{1c}(z)dz = C' \int_{\lambda_1}^{\lambda_\ell} (\lambda_+ - z)^{1/2} dz = \frac{2C'}{3} (\lambda_+ - \lambda_\ell)^{3/2}
\end{equation}

for $C, C' > 0$. Thus, for $1 \leq k, \ell \leq \varpi$, we have

\begin{equation}
\lambda_\ell - \lambda_k = C'' \left[ \left( \frac{k - 1}{p} \right)^{2/3} - \left( \frac{\ell - 1}{p} \right)^{2/3} \right],
\end{equation}

for some $C'' > 0$. On the other hand, by (22), we know $\lambda_k$ and $\lambda_\ell$ are close for $r_1 + 1 \leq k \leq \varpi$ when $n$ is sufficiently large. Thus, we have the following estimate for the constant $C''$ from the eigenvalues of $SS^\top$ as $C'' = \frac{\lambda_{2k+1} - \lambda_{k+1}}{(2k/p)^{2/3} - (k/p)^{2/3}}$, assuming
2k + 1 < π. Therefore, by taking a proper \( k \geq r_+ + 1 \), we can view the top \( k \) eigenvalues as missing values and they are reconstructed by

\[
\hat{\lambda}_j := \tilde{\lambda}_{k+1,n} + \frac{1 - \left( \frac{j-1}{k} \right)^{2/3}}{2^{2/3} - 1} \left( \tilde{\lambda}_{2k+1,n} - \tilde{\lambda}_{k+1,n} \right).
\]

The cumulative eigenvalue distribution function (CDF) constructed by imputation is then

\[
F_{\text{imp}}(x) := \frac{1}{p} \sum_{j=k+1}^{p} \mathbb{1}(\hat{\lambda}_j \leq x) + \frac{1}{p} \sum_{j=1}^{k} \mathbb{1}(\lambda_j \leq x).
\]

Denote the CDF of \( ZZ^T \) as \( F_1 \). Recall the following Lemma that compares \( F_{\text{imp}} \) and \( F_1 \).

**Lemma 3.1.** \(^{22}\) **Lemma 4.** Suppose \( k \geq r \). Then the Kolmogorov-Smirnov (KS) distance between \( F_{\text{imp}} \) and \( F_1 \) is controlled by \( d_{KS}(F_{\text{imp}}, F_1) := \sup_x |F_{\text{imp}}(x) - F_1(x)| \leq \frac{k}{p} \).

Now we consider the following “discretization” of the associated quantities. For \( 1 \leq i \leq k \), denote the discrete estimators of \( m_{1,c}(\lambda_i) \) and \( m_{2,c}(\lambda_i) \) as

\[
\hat{m}_{1,i} := \int dF_{\text{imp}}(x) x - \lambda_i = \frac{1}{p} \left( \sum_{j=1}^{k} \frac{1}{\lambda_j - \lambda_i} + \sum_{j=k+1}^{p} \frac{1}{\lambda_j - \lambda_i} \right), \quad \hat{m}_{2,i} := \frac{1}{\lambda_i} - \beta_n \hat{m}_{1,i}.
\]

Similarly, denote the discrete estimators of derivatives \( m'_{1,c}(\lambda_i) \) and \( m'_{2,c}(\lambda_i) \) as

\[
\hat{m}'_{1,i} = \frac{1}{p} \left( \sum_{j=1}^{k} \frac{1}{(\lambda_j - \lambda_i)^2} + \sum_{j=k+1}^{p} \frac{1}{(\lambda_j - \lambda_i)^2} \right), \quad \hat{m}'_{2,i} = \frac{1}{\lambda_i^2} + \beta_n \hat{m}'_{1,i}.
\]

Then, for \( 1 \leq i \leq r \), denote the estimators of the D-transform \( T(\lambda_i) \) and its derivative \( T'(\lambda_i) \) as

\[
\hat{T}_i = \tilde{\lambda}_i \hat{m}_{1,i} \hat{m}_{2,i}, \quad \hat{T}'_i = \hat{T}_i \hat{m}'_{1,i} + \tilde{\lambda}_i \hat{m}'_{2,i} \hat{m}_{2,i} + \tilde{\lambda}_i \hat{m}'_{1,i} \hat{m}_{1,i},
\]

and denote estimators of \( d_i, a_{1,i} = \langle u_i, \tilde{\xi}_i \rangle^2 \), and \( a_{2,i} = \langle v_i, \tilde{\xi}_i \rangle^2 \) respectively as

\[
\hat{d}_i = \sqrt{\frac{1}{\hat{T}'_i}}, \quad \hat{a}_{1,i} = \hat{m}_{1,i} \hat{d}_i^2 \hat{T}'_i \quad \text{and} \quad \hat{a}_{2,i} = \hat{m}_{2,i} \hat{d}_i^2 \hat{T}'_i.
\]

With the above preparation and Proposition \(^{23}\) denote the estimator of the optimal shrinker, \( \varphi_i \), where \( 1 \leq i \leq r_+ \), by

\[
\hat{\varphi}_i = \hat{d}_i \sqrt{\hat{a}_{1,i} \hat{a}_{2,i}} \quad \text{(Frobenius norm)}
\]

\[
\hat{\varphi}_i = \hat{d}_i \sqrt{\min\{\hat{a}_{1,i}, \hat{a}_{2,i}\} \max\{\hat{a}_{1,i}, \hat{a}_{2,i}\}} \quad \text{(Operator norm)}
\]

\[
\hat{\varphi}_i = \hat{d}_i \left( \sqrt{\hat{a}_{1,i} \hat{a}_{2,i}} - \sqrt{(1 - \hat{a}_{1,i})(1 - \hat{a}_{2,i})} \right) \quad \text{(Nuclear norm)}
\]

and \( \hat{\varphi}_i = 0 \) when \( i \geq r_+ + 1 \). The following theorem gives the control between the optimal shrinker and its estimator.
The proposed statistic is ratios of consecutive sample singular values to design our rank estimate statistic.  

\[ \hat{r}^+ = \max_{1 \leq i \leq \varpi} \left\{ i : \frac{\tilde{\lambda}_i}{\lambda_{i+1}} - 1 \geq \vartheta, \quad \tilde{\lambda}_i - \lambda_1 > \vartheta \right\}, \]

where \( \varpi > 0 \) is a fixed large integer (see Theorem 3.2) and \( \vartheta > 0 \) is the chosen threshold. Denote the event \( \Xi := \{ \hat{r}^+ = r^+ \} \). We need the following assumption, which is stronger than (18) by a factor \( n^\varepsilon \).

**Assumption 3.2.** We assume \( d_{r+} - \alpha > n^\varepsilon(\phi_n + n^{-1/3}) \), where \( 0 < \varepsilon < 1/3 \land (1/2 - 2/\alpha) \) is a small constant.

The following theorem says that the proposed statistic could be used to accurately estimate the rank under mild additional assumption.

**Theorem 3.6.** Suppose Assumptions 2.3 and 3.2 hold true. Take the threshold \( \vartheta \geq \varpi \) is a small constant.

\[ \hat{r}^+ \overset{a.s.}{\longrightarrow} r_{\text{eff}} \text{ as } n \to \infty. \]

3.4. **Summary of the proposed OSIR algorithm.** Based on the previous results, we summarize OSIR in Algorithm 1.

**Algorithm 1** Algorithm for adaptive optimal shrinker

\begin{enumerate}
  \item Input : \( S \), \( \vartheta \), a fixed integer \( k > r \), and the desired loss function.
  \item (i) For \( 1 \leq i \leq k \), acquire \( \hat{d}_i, \hat{\alpha}_{1,i} \) and \( \hat{\alpha}_{2,i} \) in (34).
  \item (ii) Acquire \( \hat{r}^+ \) as in (36) by setting \( \varpi = k \).
  \item (iii) For \( 1 \leq i \leq \hat{r}^+ \), acquire \( \hat{\varphi}_i \) as in (35) for the associated norm.
  \item (iv) The estimator of the clean data matrix is \( \hat{S}_\varphi = \sum_{i=1}^{\hat{r}^+} \hat{\varphi}_i \tilde{\xi}_i \tilde{\xi}_i^\top. \)
\end{enumerate}

**Remark 3.3.** If we replace the definition of \( r^+ \) in (18) by a different one, that is, for \( 1 \leq k \leq r^+ \) we have \( d_k - \alpha > 0 \), and replace \( F_{\text{imp}}(x) \) by \( F_{\text{trun}}(x) := \frac{1}{p_r \gamma} \sum_{i=p_r+1}^{p} 1(\lambda_i \leq x) \), then OSIR with the Frobenius norm loss is reduced to OptShrink proposed in [11]. Note that if we select \( \vartheta \) to be too small, by (22) from Theorem 3.2 the rank might be overestimated and larger than \( r_{\text{eff}} \). In the next section, we will show that when the rank is overestimated, using \( F_{\text{trun}}(x) \) results in a growing estimation error as \( \hat{r}^+ \) grows. In contrast, using \( F_{\text{imp}}(x) \) leads to a consistently smaller estimation error.

**Corollary 3.7 (Limiting rank regularized weights).** Suppose Assumptions 2.3 and 3.2 hold. We have that \( \hat{\varphi}_i \overset{a.s.}{\longrightarrow} \varphi_i \), for \( i = 1 \leq i \leq \min\{p,n\} \). Consequently, we have that \( L_\infty(\varphi|d) = \lim_{n \to \infty} L_{p,n}(S, \hat{S}_\varphi) = \lim_{n \to \infty} L_{p,n}(S, \hat{S}_\varphi). \)

**Proof.** This is a direct consequence of Theorems 3.2, 3.3, 3.4, 3.5, and 3.6 \( \square \)
4. Simulations

We consider simulations with different types of noises, the single channel fECG extraction problem and the 2-dim random tomography problem. In all results, we provide the interquartile range error bar or mean ± standard deviation and carry out the paired t test, and view \( p < 0.005 \) as significant and \( 0.005 \leq p < 0.05 \) as suggestive [14]. The Bonferroni correction is carried out when we have multiple testing.

4.1. A simulated signal. We consider different types of noises. Suppose \( X \in \mathbb{R}^{p \times n} \) has i.i.d. entries with Student’s t-distribution with 10 degrees of freedom followed by a proper normalization so that the variance is 1/n. Set \( A \in \mathbb{R}^{p \times p} \) to be
\[
A = \frac{1}{2} Q_A D_A Q_A^T,
\]
where \( D_A = \text{diag}\{\ell_1, \ell_2, \ldots, \ell_p\} \) is a diagonal matrix, \( Q_A \) is an orthonormal matrix generated by the QR decomposition of a \( p \times p \) random matrix, and \( L_A = \sum_{i=1}^p \ell_i \) is a normalizing factor. By assigning different diagonal matrix \( D \), we could create different types of covariance. The same method is applied to generate \( B = \frac{1}{n} Q_B D_B Q_B^T \in \mathbb{R}^{n \times n} \), which is assumed to be independent of \( A \) and \( X \). We set up two types of noise covariance and dependence. The first one (called TYPE1 below) is white with \( D_A = I_p \) and \( D_B = I_n \), and the second one (called TYPE2 below) satisfies \( D_A = \text{diag}\left\{ \sqrt{1 + 9 \times \frac{1}{p}}, \sqrt{1 + 9 \times \frac{2}{p}}, \ldots, \sqrt{1 + 9 \times \frac{p-1}{p}}, \sqrt{10} \right\} \) for \( A \) and \( D_B = \text{diag}\left\{ \sqrt{10 + \frac{1}{n}}, \sqrt{10 + \frac{2}{n}}, \ldots, \sqrt{10 + \frac{p-1}{n}}, 1, \ldots, 1 \right\} \) for \( B \). The signal matrix is assumed to be \( S = \sum_{i=1}^2 d_i u_i v_i^\top \), where the left and right singular vectors are generated by the QR decomposition of two independent random matrices. In this simulation, we set \( n \in \{200, 400, 600, 800, 1000\} \) and \( p/n = 0.5 \) or 1. For each value of \( n \), we realize the simulation for 50 times. For the TYPE1 noise, we set the rank \( r^+ = r_{\text{eff}} = r = 2 \), such that \( \{d_1, d_2\} = \{3, 1.3\} \) when \( p/n = 0.5 \) and \( \{d_1, d_2\} = \{4.1, 1.5\} \) when \( p/n = 1 \). For the TYPE2 noise, we set \( \{d_1, d_2\} = \{3, 1.8\} \) when \( p/n = 0.5 \) and \( \{d_1, d_2\} = \{4.2, 2.1\} \) when \( p/n = 1 \). These singular values are chosen such that the first singular value is away from the bulk and the second singular value is close to the bulk.

We first study the difference between estimating \( d_1 \) and \( d_2 \) using \( F_{\text{imp}} \) used in OSIR and ScreeNot and \( F_{\text{trun}} \) used in OptShrink. Note that this tells us the relationship between OSIR and OptShrink when the Frobenius norm is considered in OSIR. In Figure 2 we compare the error ratio of estimating \( d_1 \) and \( d_2 \) with OSIR while using \( F_{\text{imp}} \) or \( F_{\text{trun}} \) over different estimated rank \( \hat{r}^+ \). We set \( k = \hat{r}^+ \) and fix \( n = 200 \). As \( \hat{r}^+ \) grows, under both types of noise, the errors of both singular values using \( F_{\text{trun}} \) grows. However, the errors using \( F_{\text{imp}} \) are consistent and smaller than using \( F_{\text{trun}} \). This shows the benefit of the imputation approach, and thus, we will keep applying \( F_{\text{imp}} \) in OSIR in the following analysis when we compare the performance of OSIR with TRAD [27] and ScreeNot [22]. For TRAD, the noise level is determined by the same algorithm provided in [27] Section 8.1. Since estimating the covariance structure for the whitening idea [36] needs extra estimation steps, we do not compare it below. For ScreeNot, we follow the suggested imputation as described in [22] Section 4.2 with \( k = 2r = 4 \). For OSIR, we also set \( k = 4 \) for a fair comparison, and set \( \phi_0 = n^{-3/10} \) and \( \vartheta = 2n^{-2/3} \sqrt{\phi_0^2} = 2n^{-3/5} \). We now carry out the following comparisons. First, in Figure 3 we compare the estimated rank in OSIR and ScreeNot. Since the second singular value is very close to the bulk, when
\( n \) is small, our rank estimator sometimes underestimates the rank (about 20\% when \( n = 200 \)). However, ScreeNot often underestimates the rank, and it does not seem to improve when \( n \) grows. In Figure 4 and below, all optimal shrinkers are based on the operator norm loss, and we plot the operator norm errors. While the TYPE1 noise is considered, the errors of ScreeNot is higher compared to other methods, which is caused by the rank estimation error and the fact that the singular value is not corrected, and OSIR has the lowest norm with statistical significance when \( n > 800 \). When the TYPE2 noise is considered, overall OSIR outperforms others with statistical significance. Moreover, in Figure 5 we compare the error ratio of estimating \( d_1 \) and \( d_2 \) by different algorithms. Since ScreeNot does not shrink the noisy singular values, its error ratio is larger. On the contrary, OSIR achieves the lowest error ratio for both noises.

**Figure 2.** Interquartile range error bar of error ratio of estimating \( d_i \) by \( \hat{d}_i \) estimated by different methods over different estimating rank \( \hat{r}_{i+} \), where \( i = 1, 2 \). The black and red lines stand for OptShrink using \( F_{\text{imp}} \) and \( F_{\text{trun}} \) respectively. The solid lines are for the error ratio of \( d_1 \), and the dashed lines are for \( d_2 \). \( n = 200 \) is fixed.

**Figure 3.** A comparison of estimated rank out of 50 realizations. The solid (dashed respectively) lines stand for the ratio of estimated rank 2 (1 respectively). The black lines are for OSIR and the green lines are for ScreeNot.

4.2. **Fetal ECG extraction problem.** The second example is extracting the fECG from the single channel ta-mECG recorded from the mother’s abdomen during pregnancy. The ta-mECG is a linear combination of fECG and mECG. In
Figure 4. Interquartile range error bars of operator norm errors of different algorithms. The black (green and magenta respectively) line stands for OSIR (ScreeNot and TRAD respectively).

Figure 5. Interquartile range error bars of error ratios of estimating $d_i$, $i = 1, 2$, by $\hat{d}_i$ estimated by different algorithms. The black, green, and magenta lines stand for OSIR, ScreeNot, and TRAD. The solid lines are for the error ratio of $d_1$, and the dashed lines are for $d_2$. If the corresponding error ratio is too high, the associated curve is not plotted to enhance the visualization. For example, $d_2$ is often truncated in ScreeNot since the error ratio is 100%.

Our previous work \[52\], TRAD is included as one critical step of the algorithm that is designed when there are only one or two channels of ta-mECG available. The algorithm is composed of two main steps. The first step is mainly for two channels, and can be ignored when we only have one channel. The second step is composed of two sub steps. Step 2-1 is designed to extract the maternal R peaks from the single channel ta-mECG, which is not the concern of OS. Step 2-2 is mainly illustrated in Figure 4, where OS is applied to segments of ta-mECG so that fECG and mECG are decomposed from the ta-mECG. Recall that we view fECG as the noise and mECG as the signal. As is mentioned in Introduction, fECG as noise is not white. Thus, it is natural to consider replacing TRAD approach in \[52\] by ScreeNot or OSIR. To compare the performance, we consider a semi-real simulated database and a real-world database from 2013 PhysioNet/Computing in Cardiology Challenge \[49\], abbreviated as CinC2013. The semi-real ta-mECG data is constructed from the Physikalisch-Technische Bundesanstalt (PTB) Database [https://physionet.org/physiobank/database/ptbdb/] abbreviated as PTBDB,
mECG is created by mECG(VECGs from a healthy recording, denoted as Each signal is digitalized with the sampling rate 1000 Hz. Take 57-second Frank lead includes simultaneously recorded conventional 12 lead and the Frank lead ECGs. 17 to 87 with the mean age 57.2. 52 out of 290 subjects are healthy. Each recording contains 549 recordings from 290 subjects (one to five recordings for one subject) aged following the same way detailed in [52] that we brief here. The database contains simultaneously recorded conventional 12 lead and the Frank lead ECGs. Each signal is digitalized with the sampling rate 1000 Hz. Take 57-second Frank lead ECGs from a healthy recording, denoted as \(V_x(t), V_y(t)\) and \(V_z(t)\) at time \(t \in \mathbb{R}\), as the maternal vectocardiogram (VCG). Take \((\theta_{xy}, \theta_z) = (\frac{\pi}{4}, \frac{\pi}{4})\), and the simulated mECG is created by \(\text{mECG}(t) = (V_x(t) \cos \theta_{xy} + V_y(t) \sin \theta_{xy}) \cdot \cos \theta_z + V_z(t) \cdot \sin \theta_z\). We create 40 mECGs. The simulated fECG of healthy fetus are created from another 40 recordings from healthy subjects, where 114-second V2 and V4 recordings are taken. The simulated and simulated fECG come from different subjects. The simulated fECG is then resampled at 500 Hz. As a result, the simulated fECG has about double the heart rate compared with the simulated mECG if we consider both signals sampled at 1000 Hz. The amplitude of the simulated fECG is normalized to the same level of simulated mECG and then multiplied by \(0 < R < 1\) to make the amplitude relationship consistent with the usual situation of real ta-mECG signals. We generate 40 simulated healthy fECGs. The clean simulated ta-mECG is generated by directly summing simulated mECG and fECG. We then create a simulated noise starting with a random vector \(x = (x_1, x_2, x_3, \ldots)\) with i.i.d entries with student t-10 distribution. The noise is then created and denoted as \(z\) with the entries \(z_i = (1 + 0.5 \sin(i \text{ mod } 500)/500)(x_i + x_{i+1} + x_{i+2})\). The final simulated ta-mECG is generated by adding the created noise to the clean simulated ta-mECG according to the desired SNR ratio. As a result, we acquire following the same way detailed in [52] that we brief here. The database contains 549 recordings from 290 subjects (one to five recordings for one subject) aged 17 to 87 with the mean age 57.2. 52 out of 290 subjects are healthy. Each recording includes simultaneously recorded conventional 12 lead and the Frank lead ECGs. Each signal is digitalized with the sampling rate 1000 Hz. Take 57-second Frank lead ECGs from a healthy recording, denoted as \(V_x(t), V_y(t)\) and \(V_z(t)\) at time \(t \in \mathbb{R}\), as the maternal vectocardiogram (VCG). Take \((\theta_{xy}, \theta_z) = (\frac{\pi}{4}, \frac{\pi}{4})\), and the simulated mECG is created by \(\text{mECG}(t) = (V_x(t) \cos \theta_{xy} + V_y(t) \sin \theta_{xy}) \cdot \cos \theta_z + V_z(t) \cdot \sin \theta_z\). We create 40 mECGs. The simulated fECG of healthy fetus are created from another 40 recordings from healthy subjects, where 114-second V2 and V4 recordings are taken. The simulated and simulated fECG come from different subjects. The simulated fECG is then resampled at 500 Hz. As a result, the simulated fECG has about double the heart rate compared with the simulated mECG if we consider both signals sampled at 1000 Hz. The amplitude of the simulated fECG is normalized to the same level of simulated mECG and then multiplied by \(0 < R < 1\) to make the amplitude relationship consistent with the usual situation of real ta-mECG signals. We generate 40 simulated healthy fECGs. The clean simulated ta-mECG is generated by directly summing simulated mECG and fECG. We then create a simulated noise starting with a random vector \(x = (x_1, x_2, x_3, \ldots)\) with i.i.d entries with student t-10 distribution. The noise is then created and denoted as \(z\) with the entries \(z_i = (1 + 0.5 \sin(i \text{ mod } 500)/500)(x_i + x_{i+1} + x_{i+2})\). The final simulated ta-mECG is generated by adding the created noise to the clean simulated ta-mECG according to the desired SNR ratio. As a result, we acquire

| SNR | \(R\) | TRAD | ScreeNot | OSIR |
|-----|------|------|----------|------|
| 1 dB | 1/4  | 4.30 ± 1.51 | 3.10* ± 1.50 | 2.69* ± 1.26 |
|     | 1/6  | 2.91 ± 1.01 | 2.21* ± 1.62 | 1.89* ± 0.86 |
|     | 1/8  | 2.21 ± 0.76 | 1.71* ± 0.72 | 1.51* ± 0.64 |
| 0 dB | 1/4  | 4.47 ± 1.62 | 3.19* ± 1.50 | 2.77* ± 1.24 |
|     | 1/6  | 3.44 ± 1.10 | 2.21* ± 1.01 | 1.96* ± 0.89 |
|     | 1/8  | 2.31 ± 0.83 | 1.75* ± 0.77 | 1.59* ± 0.70 |
| −1 dB | 1/4 | 4.72 ± 1.80 | 3.25* ± 1.58 | 2.93* ± 1.34 |
|     | 1/6  | 3.19 ± 1.22 | 2.20* ± 1.00 | 2.06* ± 0.98 |
|     | 1/8  | 2.44 ± 0.90 | 1.75* ± 0.76 | 1.65* ± 0.69 |

| SNR | \(R\) | TRAD | ScreeNot | OSIR |
|-----|------|------|----------|------|
| 1 dB | 1/4  | 4.48 ± 2.20 | 2.17* ± 1.87 | 1.84* ± 0.92 |
|     | 1/6  | 3.06 ± 1.50 | 1.53* ± 1.34 | 1.24* ± 0.62 |
|     | 1/8  | 2.37 ± 1.25 | 1.23* ± 1.16 | 0.92* ± 0.46 |
| 0 dB | 1/4  | 4.93 ± 2.37 | 2.36* ± 1.96 | 1.93* ± 0.94 |
|     | 1/6  | 3.43 ± 1.72 | 1.70* ± 1.47 | 1.28* ± 0.63 |
|     | 1/8  | 2.60 ± 1.31 | 1.30* ± 1.16 | 0.94* ± 0.44 |
| −1 dB | 1/4 | 5.49 ± 2.69 | 2.72* ± 2.23 | 1.99* ± 0.93 |
|     | 1/6  | 3.57 ± 1.66 | 1.63* ± 1.22 | 1.30* ± 0.60 |
|     | 1/8  | 2.69 ± 1.27 | 1.25* ± 0.96 | 0.90* ± 0.46 |

Table 3. The comparison of root mean square error for mECG (top) and fECG (bottom) of different algorithms applied to the simulated ta-mECG database. All results are presented as mean ± standard deviation. The asteroids stand for the statistical significance when comparing ScreeNot with OSIR using the paired t-test. The stars stand for the statistical significance when comparing ScreeNot with OSIR using the paired t-test. \(R\) is the amplitude ratio of the simulated fECG.
40 recordings of 57 seconds simulated ta-mECG signals with the sampling rate 1000 Hz. For each recording, we run the fECG extraction algorithm with TRAD, ScreeNot and OSIR, in Step 2-2 and report the root mean square error (RMSE) of the recovered mECG and fECG. For the mECG morphology recovery, we apply different OS algorithms in Step 2-2 on the simulated ta-mECG, and compare it with the ground truth mECG to compute the RMSE. For the fECG morphology recovery, since OSIR leads to a better mECG recovery, we use it to recover mECG and then apply different OS algorithms to recover fECG for a comparison in the following way. For each detected true-positive fetal R peak, we crop fECG cycles from the recovered fECG and the ground truth fECG with a 601 window length with the R-peak being in the middle of the window, and evaluate the RMSE. As is shown in Table 3, OSIR leads to the smallest RMSE in all scenarios.

The next one is about the morphological recovery of the fECG signal using the simulated ta-mECG database detailed in the main article. Particularly, we examine how accurate we can recover the essential “landmarks” of the ECG signal, including the P wave, R peak and T wave. A detected R peak (or P and T waves) is beat-by-beat compared with the ground truth (or provided annotations) with a matching window of 50 ms [4]. For the R peak, We report the F \(_1\) score defined as

\[
F_1 := \frac{2 \cdot TP}{2 \cdot TP + FP + FN},
\]

where TP, FP and FN are the true positive rate (correctly detected R peak), false positive rate (falsely detected R peak), and false negative rate (the existing R peak that is not detected) respectively. The same F \(_1\) definition is used for P and T waves. See Figure [4] for the interquartile range error bars and mean ± standard deviation of F \(_1\) scores when estimating the R peaks and P and T waves from the reconstructed fECG from the single channel simulated ta-mECG. As expected, the higher the ratio of the fECG amplitude is and the higher the SNR is, the higher the F \(_1\) will be. Compared with TRAD, the morphology recovery performance is in general better when ScreeNot or OSIR is applied. In general, OSIR performs better than ScreeNot, while in many cases we cannot see statistical significance.

The final part is the real-world database CinC2013. For CinC2013, we focus on the set A, which is composed of 75 recordings with experts’ annotations of the fetal R peaks. Each recording includes four ta-mECG channels obtained from multiple sources and the simultaneously recorded directly contact fECG resampled at the sampling rate 1000 Hz and lasts for 1 minute duration. Case a54 is discarded based on the suggestion in [4], and we focus on the remaining 74 recordings. Since there is no ground truth fECG for comparison, the algorithm performance is evaluated by the accuracy of fetal R peak detection compared with experts’ annotation, where we report the F \(_1\) score as the above. We also compute the mean absolute error (MAE), which is defined by

\[
MAE = \frac{1}{n_{TP}} \sum_{i=1}^{n_{TP}} |t_i^T - \tilde{t}_i^T|,
\]

where \(n_{TP}\) is the number of TP detected R peaks, and \(t_i^T\) and \(\tilde{t}_i^T\) are the timestamps of the i-th TP detected R peak and the associated annotation. Here, we follow the common approach [3] to consider only TP R peaks to avoid the evaluation dependence on the R peak detection accuracy. We report the mean and standard deviation of F \(_1\) and MAE for each channel combination over all recordings. To have a fair comparison, while comparing OSIR and ScreeNot, we use the rank estimate proposed by ScreeNot. Moreover, for both OSIR and ScreeNot, we select \(k\) equal to \(\lfloor p/4 \rfloor\), which empirically leads to the best F \(_1\) score and MAE. Under such setup, the result in Table 4 shows that OSIR leads to the highest F \(_1\) score and the shortest MAE when channel 1 and
Figure 1. The comparison of $F_1$ scores for the locations of P, R, and T waves when different algorithms are applied to the simulated one channel ta-mECG. The red, green and blue boxes are interquartile range error bar of $F_1$ for TRAD, ScreeNot, and OSIR respectively. Top to bottom: SNR of 1, 0 and -1 dB respectively. Left to right: $f$ECG amplitudes of $R = 1/4$, $1/6$ and $1/8$ respectively. Mean ± standard deviation are shown as well. The asteroids stand for the statistical significance when comparing ScreeNot or OSIR with TRAD using the paired t-test.

channel 4 are considered. However, we do not find any statistical significance due to the large standard deviation, but only a statistically suggestive significance when the fourth channel is considered. Last but not the least, we report the result when two ta-mECG channels are available. This is the case considered in [52]. Note that in this case, the first step of the full algorithm proposed in [52] cannot be ignored—in the first step, we take the physiology of ECG into account, and carry out a search for the optimal linear combination of two ta-mECG channels so that the $f$ECG is the strongest. Then, run Step 2-1 and Step 2-2 on the optimal linear combination to get all desired estimations and evaluations. The result is shown in Table 5. In this case, we see that overall OSIR outperforms TRAD and ScreeNot with a higher $F_1$ and a shorter MAE. However, we do not find any statistical significance due to
the large standard deviation, but only a statistically suggestive significance when the 1 + 4 and 2 + 3 combinations are considered. We shall notice that compared with the simulated ta-mECG signals, in the real database the improvement of OSIR over TRAD and ScreeNot is not that dramatic, particularly when only one channel is available. This might be caused by the more delicate nonlinear interaction of mECG and fECG in the real ta-mECG that is not captured by the simulated ta-mECG. No matter how, overall the result is encouraging and showing that OSIR has a potential in the real world problem. A systematic and large scale evaluation will be reported in our future work to further confirm this encouraging result.

| channels | TRAD [52] | ScreeNot | OSIR  |
|----------|-----------|----------|-------|
|          | F1 (%)    |          |       |
| 1        | 67.37 ± 36.43 | 69.16 ± 35.54 | 68.33 ± 35.89 |
| 2        | 74.73 ± 31.87 | 73.57 ± 32.66 | 73.77 ± 33.31 |
| 3        | 66.96 ± 34.57 | 66.72 ± 35.50 | 66.56 ± 35.45 |
| 4        | 75.31 ± 31.21 | 74.79 ± 32.17 | 76.82* ± 30.55 |
|          | MAE (ms)  |          |       |
| 1        | 11.26 ± 9.59  | 11.13 ± 9.18  | 11.45 ± 9.70  |
| 2        | 9.17 ± 7.80   | 9.51 ± 7.61   | 9.76 ± 8.56   |
| 3        | 11.17 ± 8.45  | 11.85 ± 9.04  | 11.93 ± 9.14  |
| 4        | 8.74 ± 8.23   | 9.22 ± 8.34   | 8.81 ± 8.21   |

Table 4. The comparison of F1 and MAE of different algorithms applied to CinC2013 when the single channel ta-mECG is considered. The subject a54 is removed from the dataset. All data are presented as mean ± standard deviation. The operator norm is used in the proposed method. The asteroids (stars respectively) next to the value of mean stand for the statistically suggestive significance when comparing ScreeNot or OSIR with TRAD (ScreeNot with OSIR respectively) using the paired t-test with Bonferroni correction; that is, the p value is larger than 0.005 but smaller than 0.05.

4.3. Two dimensional random tomography. Next, we consider the 2-dim random tomography problem [51] with the Shepp-Logan phantom. Here we quickly summarize the problem and challenge. Denote $S^2$ to be the canonical 1-dim sphere in $\mathbb{R}^2$. The goal is recovering a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is compacted supported in the unit ball in $\mathbb{R}^2$ from its X-ray transform, which is given by the line integral $X_\theta f(s) = \int_{x \cdot \theta = s} f(x)\,dx$, where $\theta \in S^1$ is the projection direction and $s \in \mathbb{R}$. We call $X_\theta f$ the projected image of $f$ associated with the projection direction $\theta$. When both $\theta$ and $X_\theta f$ are available, we can recover $f$ via the inverse X-ray transform [25]. Motivated by solving the cryogenic electron microscopy problem [50], we assume that we only have noisy projection images $D_p X_\theta f + \xi_i$, $i = 1, \ldots, n$, where $D_p$ is the discretization operator that discretizes $[-1, 1]$ into equally spaced points so that $D_p X_\theta f \in \mathbb{R}^p$ and $\xi_i$ is the random noise, while $\theta_i$ is unknown. Therefore, to recover $f$, the mission is recovering $\theta$ from the collected noisy projection images $D_p X_\theta f + \xi_i$. In this example, we consider the standard Shepp-Logan phantom as the function $f$ and the same noises, TYPE1 and TYPE2, used in Section 4.1, and take $n = 1024$ and $p = 512$. Also, we assume that $\theta_i$ is uniformly sampled from $S^1$.
Table 5. The comparison of F1 and MAE of different algorithms applied to CinC2013 when two ta-mECG channels are considered. The subject a54 is removed from the dataset. All data are presented as mean ± standard deviation. The operator norm is used in the proposed method. The asteroids (stars respectively) next to the value of mean stand for the statistically suggestive significance when comparing ScreeNot or OSIR with TRAD (ScreeNot with OSIR respectively) using the paired t-test with Bonferroni correction; that is, the $p$ value is larger than 0.005 but smaller than 0.05.

|      | TRAD [52] | ScreeNot | OSIR    |
|------|-----------|----------|---------|
| **F1 (%)** |           |          |         |
| 1+2  | 85.40 ± 25.58 | 85.38 ± 26.09 | 85.53 ± 26.16 |
| 1+3  | 89.16 ± 22.93 | 92.16 ± 17.41 | 93.91* ± 13.23 |
| 1+4  | 85.68 ± 26.32 | 87.47 ± 24.36 | 88.32 ± 23.11 |
| 2+3  | 80.30 ± 30.51 | 80.75 ± 30.44 | 82.45* ± 28.61 |
| 2+4  | 80.30 ± 30.51 | 80.75 ± 30.44 | 82.45* ± 28.61 |
| 3+4  | 82.51 ± 28.33 | 83.62 ± 26.87 | 84.07 ± 26.91 |

| **MAE (ms)** |           |          |         |
| 1+2  | 7.18 ± 7.24 | 6.70 ± 5.99 | 6.31 ± 5.35 |
| 1+3  | 6.44 ± 6.07 | 6.31 ± 5.59 | 6.32 ± 6.05 |
| 1+4  | 5.54 ± 4.69 | 5.60 ± 4.49 | 5.48 ± 4.12 |
| 2+3  | 8.39 ± 7.01 | 8.82 ± 7.81 | 8.43 ± 7.56 |
| 2+4  | 7.55 ± 7.03 | 7.24 ± 6.89 | 6.86 ± 6.75 |
| 3+4  | 7.09 ± 6.56 | 7.09 ± 6.99 | 6.86 ± 6.28 |

Note that when $p$ is sufficiently large, these $n$ points are non-uniformly distributed on a 1-dim manifold $M^1$ embedded in $\mathbb{R}^p$ that is diffeomorphic to $S^1$ [51]. It has been shown in [51] that we can apply TRAD to denoise the images and utilize the eigenfunctions of the “properly constructed” graph Laplacian to robustly recover $\theta_i$ up to a negligible error and a global phase difference. Note that in [51], the added noise is white, so TRAD works well. However, in the real world problem, particularly the cryo-electron microscopy problem that motivates this two-dimensional random tomography problem, the noise is not white. We refer readers with interest to [51] for more details of the overall algorithm and other technical issues.

Here we rewrite our model as $\bar{S} = S + \sigma \sqrt{n}Z$, where $S \in \mathbb{R}^{512 \times 1024}$ include $D_{512}X_{\theta_i} f_i$, $i = 1, \ldots, 1024$, $\sigma > 0$ describes the noise strength, and $Z$ is the TYPE2 noise described in the first simulation. We consider $\sqrt{n}Z$ so that the noise is on the same scale as that used in [51] for a fair comparison. We define the signal to noise ratio (SNR, with the unit dB) as $10 \log_{10} \left( \frac{\text{var}(S)}{\sigma^2} \right)$. For various values of $\sigma$ so that $\text{SNR} \in \{2, 1, 0, -1, -2, -3\}$, we compare various OS methods for denosing noisy projection images. For each SNR, we realize the simulation 200 times. See Figure 2 for a comparison of different OS algorithms, where we plot two clean projections and their denoised results using different methods and computed the root mean squared error (RMSE). We see that both of OSIR and ScreeNot achieve a lower RMSE compared with TRAD used in [51], and OSIR is slightly better than ScreeNot in the sense of RMSE with a statistical significance.
Next, we consider the projection angle estimation problem, which is the main mission of the two-dimensional random tomography problem discussed in [51]. For a fair comparison, we apply the same angle estimation algorithm detailed in [51, Section 6], while we omit the rank estimation based on [33] since it is only suitable for the white noise. Instead, we apply our rank estimation algorithm to determine the rank. We compute the $\ell_1$ norm of the error of the estimated angle. Due to the global phase and reflection freedom of the angle estimation algorithm, to achieve this comparison, we carry out the following alignment step. Suppose $m$ projections are kept out of 2048 (including the original projections and their reflections) after the step of identifying false matchings of projections by the Jaccard index. Let $\hat{\theta} = [\hat{\theta}_1, \ldots, \hat{\theta}_m]$ be the estimated angles and $\theta = [\theta_1, \ldots, \theta_m]$ be the associated true angles. The optimal aligned shift is acquired by $k^* = \arg\min_{k \in \{0, 1, \ldots, m-1\}} \frac{\sum_{i=1}^{m} |\theta_i - \hat{\theta}_{\text{mod}(i+k,m)}|}{m}$. To handle the reflection uncertainty, we apply the same steps above on $\hat{\theta} = [\hat{\theta}_m, \ldots, \hat{\theta}_1]$ and suppose we obtain $k'$. The final error is $(\sum_{i=1}^{m} |\theta_i - \hat{\theta}_{\text{mod}(i+k*,m)}|)/m)$ ∧ $(\sum_{i=1}^{m} |\theta_i - \hat{\theta}_{\text{mod}(i+k',m)}|)/m)$. See Figure 3 for an errorbar for a comparison. As the SNR decreases from 2dB to
−3dB, the quality of angle estimation becomes worse. Over each SNR, OSIR and ScreeNot perform better than TRAD \((p < 0.005/6)\). However, there is no statistical significance between the errors using OSIR and ScreeNot, except at \(\text{SNR} = -1\) and \(-2\) dB.

**Figure 3.** Error bar plot of angle estimation errors. The first plot is based on TRAD used in [51], the second plot is based on ScreeNot, and the third plot is based on OSIR. All of them are with respect to the operator norm loss. We use the paired t-test to show the statistical significance when we compare two methods. We take \(p < 0.005\) as statistical significance with the Bonferroni correction. The green asteroids stand for the statistical significance when comparing ScreeNot or OSIR with TRAD. The magenta asteroids stand for the statistical significance when comparing ScreeNot and OSIR.
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Appendix A. Background knowledge

A.1. Some linear algebra tools. We recall a perturbation bound for determinants.

Lemma A.1. Let $A$ and $E$ be two $m \times m$ matrices, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ are singular values of $A$. Then

$$|\det(A + E) - \det(A)| \leq \sum_{i=1}^{m} s_{m-i} \|E\|_2^i,$$

where $s_k := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} \sigma_{i_1} \cdots \sigma_{i_k}$ is the $k$-th elementary symmetric functions of singular values of $A$ and $s_0 := 1$.

Lemma A.2. For any $a, b, \alpha_1, \ldots, \alpha_r \in \mathbb{R}$ and $r \in \mathbb{N}$, we have

$$\det\left( \begin{array}{cc} aI_r & \text{diag}\{\alpha_1, \ldots, \alpha_r\} \\ \text{diag}\{\alpha_1, \ldots, \alpha_r\} & bI_r \end{array} \right) = \prod_{i=1}^{r} (ab - \alpha_i^2).$$

Further, we summarize the Weyl’s inequality for the singular values of rectangular matrices.

Lemma A.3. For any $p \times n$ matrices $A$ and $B$, denote $\sigma_i(A)$ as the $i$-th largest singular value of $A$. Then we have

$$\sigma_{i+j-1}(A + B) \leq \sigma_i(A) + \sigma_j(B), \quad 1 \leq i, j, i + j - 1 \leq p \wedge n.$$

Lemma A.4. \cite[Lemma F.5]{12} Let $z = E + i\eta \in \mathbb{C}^+$. For $n \times n$ Hermitian matrices $A$ and $\tilde{A}$, denote $s(z)$ and $\tilde{s}(z)$ to be the Stieltjes transforms of their ESD’s. Then we have

$$|s(z) - \tilde{s}(z)| \leq \frac{\text{rank}(A - \tilde{A})}{n} \left( \frac{2}{\eta} \wedge \frac{\|A - \tilde{A}\|}{\eta^2} \right).$$

Lemma A.5. \cite[Proposition 6.2]{11} Take a measure $\nu$ satisfying the Assumption 2.2(v). Then there exists a constant $c > 0$ so that for any matrix $A := (a_{jk})_{1 \leq j, k \leq n}$ with complex entries, any $\delta > 0$, and any $g = (g_1, \ldots, g_n)^T$ with i.i.d. entries with law $\nu$, we have

$$\mathbb{P}(\langle(g, Ag) - \mathbb{E}(g, Ag)\rangle > \delta) \leq 4e^{-c\left(\frac{\delta^4}{\delta^2}\right)},$$

where $C = \sqrt{\text{Tr}(AA^T)}$.

A.2. Some complex analysis tools. We first provide the following lemma, which is similar to Lemma 2.6 of \cite{10}.

Lemma A.6. Assume Assumption 2.2 holds. Then there exist constants $c_1, c_2 > 0$ such that for $x \in \mathbb{R}$ and $x \downarrow 0$, we have

$$\rho_1(x) = c_1x^{1/2} + O(x) \quad \text{and} \quad \rho_2(x) = c_2x^{1/2} + O(x),$$

where $\rho_1(x) = \rho_1(\lambda_+ - x)$ and $\rho_2(x) = \rho_2(\lambda_+ - x)$. 


and for \( z \in \mathbb{C}^+ \) and \( z \to \lambda_+ \), we have

\[
m_{1c}(z) = m_{1c}(\lambda_+) + \pi c_1(z - \lambda_+)^{1/2} + O(|z - \lambda_|),
\]

(40)

\[
m_{2c}(z) = m_{2c}(\lambda_+) + \pi c_2(z - \lambda_+)^{1/2} + O(|z - \lambda_|),
\]

where \((z - \lambda_+)^{1/2}\) is taken to be the branch cut with positive imaginary part (the same convention holds in the following). These estimates also hold for \( g_{1c}, g_{2c}, m_{1c} \) and \( m_{2c} \) with different constants. Moreover, there exists a constant \( c_T > 0 \) such that for \( z \in \mathbb{C}^+ \) and \( z \to \lambda_+ \), we have

(41)

\[
T(z) = T(\lambda_+) + \pi c_T(z - \lambda_+)^{1/2} + O(|z - \lambda_|).
\]

Proof. Equations (39) and (40) are directly from Lemma 2.6 of [59]. By (40) and the definition of \( T \) in (11), we obtain (41). \( \square \)

For some properly chosen constants \( \varsigma_1 > 0 \) and \( \varsigma_2 > 1 \), we denote the domain of the spectral parameter \( z \) as

(42)

\[
S(\varsigma_1, \varsigma_2) := \{ z = E + i\eta : \lambda_+ - \varsigma_1 \leq E \leq \varsigma_2 \lambda_+, 0 < \eta \leq 1 \}. \]

Moreover, for any \( z = E + i\eta \), denote

(43)

\[
\kappa_z := |E - \lambda_+|.
\]

The following lemma is similar to Lemma 3.4 in [59] and Lemma S.3.5 in [20].

**Lemma A.7.** Suppose Assumption 2.2 holds. Then we have

(i) for \( z \in S(\varsigma_1, \varsigma_2) \),

(44)

\[
|T(z)| \asymp |m_{1c}(z)| \asymp |m_{2c}(z)| \asymp 1,
\]

\[
\text{Im } T(z) \asymp \text{Im } m_{1c}(z) \asymp \text{Im } m_{2c}(z) \asymp \begin{cases} \frac{\eta}{\sqrt{\varsigma_2 + \eta}}, & \text{if } E \geq \lambda_+ \\ \frac{\eta}{\sqrt{\varsigma_z + \eta}}, & \text{if } E \leq \lambda_+ 
\end{cases},
\]

and

(45)

\[
|\text{Re } T(z) - T(\lambda_+)| \asymp |\text{Re } m_{1c}(z) - m_{1c}(\lambda_+)| \asymp |\text{Re } m_{2c}(z) - m_{2c}(\lambda_+)| \asymp \begin{cases} \frac{\eta}{\sqrt{\varsigma_2 + \eta}}, & \text{if } E \geq \lambda_+ \\ \frac{\eta}{\sqrt{\varsigma_z + \eta}} + \kappa_z, & \text{if } E \leq \lambda_+ 
\end{cases}.
\]

The estimates (44)-(45) hold for \( g_{1c}, g_{2c}, m_{1c} \) and \( m_{2c} \).

(ii) there exists constant \( \tau' > 0 \) such that for any \( z \in S(\varsigma_1, \varsigma_2) \),

(46)

\[
\min_{j=1, \ldots, n} |1 + m_{1c}(z) \sigma_j^a| \geq \tau' \quad \text{and} \quad \min_{i=1, \ldots, p} |1 + m_{2c}(z) \sigma_i^b| \geq \tau',
\]

where \( \sigma_j^a \) and \( \sigma_i^b \) are eigenvalues of \( A \) and \( B \) stated in Assumption 2.2.

The above estimates (i)-(ii) hold for \( z \) on the real axis, i.e., \( z \in S(\varsigma_1, \varsigma_2) \).

Proof. The estimates for \( g_{1c}, g_{2c}, m_{1c} \) and \( m_{2c} \) have been proved in Lemma S.3.5 in [20]. Estimates for \( T \) in (44) and (45) can be directly derived from (41). \( \square \)

The following lemma describes the behavior of \( \theta \) and the derivatives of \( m_{1c}, m_{2c} \) and \( T \) on the real line.
Lemma A.8. Suppose Assumption 2.3 holds. For \( d \geq \alpha \), we have

\[
|\theta(d) - \lambda_+| \asymp (d - \alpha)^2.
\]

For \( x > \lambda_+ \), we have

\[
m'_{1c}(x) \asymp \kappa_x^{-1/2}, \quad m'_{2c}(x) \asymp \kappa_x^{-1/2} \quad \text{and} \quad \mathcal{T}'(x) \asymp \kappa_x^{-1/2}.
\]

Proof. With (41), we obtain that

\[
d_i^2 = \mathcal{T}(\theta(d_i)) = \alpha^{-2} + C \sqrt{\theta(d_i) - \lambda_+} + O(|\theta(d_i) - \lambda_+|)
\]

if \( \theta(d_i) - \lambda_+ \leq \varsigma_1 \) for some sufficiently small constant \( 0 < \varsigma_1 < 1 \), and

\[
d_i^2 = \mathcal{T}(\theta(d_i)) \geq \mathcal{T}(\lambda_+ + \varsigma_1) = \alpha^{-2} + C \sqrt{\varsigma_1} + O(\varsigma_1)
\]

if \( \theta(d_i) - \lambda_+ \geq \varsigma_1 \), where in the inequality we use the fact that \( \mathcal{T}(x) \) is monotone increasing when \( x > \lambda_+ \). The above two estimates imply (47). The first two statements of (48) have been proved in Lemma S.3.6 in [20], and the third statement then can be derived by

\[
\mathcal{T}'(x) = m_{1c}(x)m_{2c}(x) + x m'_{1c}(x)m_{2c}(x) + x m_{1c}(x)m'_{2c}(x) \asymp C_1 + C_2 \kappa_x^{-1/2} \asymp \kappa_x^{-1/2},
\]

where we used the first statement of (48) and (44) in the first \( \asymp \) and the fact that \( \kappa_x^{-1/2} \) blows up when \( x \to \lambda_+ \) in the last approximation. \( \square \)

Note that by (47) and (48), for \( d \geq \alpha \), we have

\[
m'_{1c}(\theta(d)) \asymp \frac{1}{d - \alpha}, \quad m'_{2c}(\theta(d)) \asymp \frac{1}{d - \alpha}, \quad \mathcal{T}'(\theta(d)) \asymp \frac{1}{d - \alpha}
\]

and

\[
\theta'(d) \asymp d - \alpha.
\]

Lemma A.9. Suppose Assumption 2.3 holds. Then for any constant \( \varsigma > \lambda_+ \), there exist constants \( \tau_1, \tau_2 > 0 \) such that the following statements hold.

(i) \( \mathcal{T} \) is a holomorphic homeomorphism on the spectral domain

\[
\mathbf{D}_1(\tau_1, \varsigma) := \{ z = E + i\eta : \lambda_+ < E < \varsigma, \quad -\tau_1 < \eta < \tau_1 \}.
\]

As a consequence, the inverse of \( \mathcal{T} \) exists and we denote it by \( \theta \).

(ii) \( \theta \) is holomorphic homeomorphism on \( \mathbf{D}_2(\tau_2, \varsigma) \), where

\[
\mathbf{D}_2(\tau_2, \varsigma) := \{ \zeta = E + i\eta : \alpha < E < 1/\sqrt{\mathcal{T}(\varsigma)}, \quad -\tau_2 < \eta < \tau_2 \},
\]

such that \( \mathbf{D}_2(\tau_2, \varsigma) \subset 1/\sqrt{\mathcal{T}(\mathbf{D}_1(\tau_1, \varsigma))} \).

(iii) For \( z \in \mathbf{D}_1(\tau_1, \varsigma) \), we have

\[
|\mathcal{T}(z) - \mathcal{T}(\lambda_+)| \asymp |z - \lambda_+|^{1/2} \quad \text{and} \quad |\mathcal{T}'(z)| \asymp |z - \lambda_+|^{-1/2}.
\]

(iv) For \( \zeta \in \mathbf{D}_2(\tau_2, \varsigma) \), we have

\[
|\theta(\zeta) - \lambda_+| \asymp |\zeta - \alpha|^2 \quad \text{and} \quad |\theta'(\zeta)| \asymp |\zeta - \alpha|.
\]

(v) For \( z_1, z_2 \in \mathbf{D}_1(\tau_1, \varsigma) \) and \( w_1, w_2 \in \mathbf{D}_2(\tau_2, \varsigma) \), we have

\[
|\mathcal{T}(z_1) - \mathcal{T}(z_2)| \asymp \frac{|z_1 - z_2|}{\max_{i=1,2} |z_i - \lambda_+|^{1/2}}.
\]

and

\[
|\theta(w_1) - \theta(w_2)| \asymp |w_1 - w_2| \cdot \max_{i=1,2} |w_i - \alpha|.
\]
Note that in (19), $\theta$ is only defined on the real line. In Lemma A.9(i), we extend its definition to the complex plane. The relationship between $\theta$ and $T$ is summarized in Figure 4.

Proof. Similar results for $m_{1c}$ and $m_{2c}$ and their proofs can be found in Lemma S.3.7 [20]. We derive this Lemma specifically using the definition of $T(z)$ and $\theta(\zeta)$ with the same approach, and we omit the detail here. □

![Figure 4](image-url)

**Figure 4.** A illustration of the relationship of $T$, $\theta$ and several quantities used in the proof. Note that $\theta(\alpha) = \lambda_+$.  

A.3. Some random matrix tools. We record some results from the random matrix theorem that we will repeatedly apply in the following proofs. Recall that the eigenvalues of $ZZ^T$ are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{p\wedge n}$.

**Definition A.10** (classical locations of the probability density $\rho_{1c}$). *For the probability density $\rho_{1c}$, the $j$-th classical location, where $j = 1, \ldots, n$, is defined as*

\[
\gamma_j := \sup_x \left\{ \int_x^{+\infty} \rho_{1c}(x) dx > \frac{j-1}{n} \right\}.
\]

*In particular, we have $\gamma_1 = \lambda_+$.*

Denote

\[
\Psi(z) := \sqrt{\frac{\text{Im} \ m_{2c}(z)}{n\eta}} + \frac{1}{n\eta}.
\]
Note that for any \( z = E + i\eta \in \mathcal{S}(\varsigma_1, \varsigma_2) \), from (i) of Lemma A.7 and [59], we have [59, (3.19)]

\[
\Psi(z) \gtrsim n^{-1/2}, \quad \Psi^2(z) \lesssim 1/n\eta,
\]

and \( \Psi^2(E + i\eta) + \phi_n/n\eta \) is monotonically decreasing with respect to \( \eta \) [59] below Lemma 3.7. Hence there is a unique \( \eta(E) \) such that

\[
n^{1/2} \left[ \Psi^2(E + i\eta(E)) + \frac{\phi_n}{n\eta(E)} \right] = 1.
\]

Note that by [44] and [59], we have

\[
\eta(E) \asymp n^{-3/4} + n^{-1/2} (\sqrt{\lambda_E} + \phi_n), \quad \text{for } E \leq \lambda_+.
\]

For \( E > \lambda_+ \), we define \( \eta(E) := \eta_1(\lambda_+) = O(n^{-3/4} + n^{-1/2} \phi_n) \). Then we have the following rigidity result.

**Lemma A.11.** [59, Theorem 3.8], Rigidity of eigenvalues). Suppose Assumption 2.2 holds. Then we have the following estimates for any fixed \( 0 < \varsigma < \varsigma_1 \), where \( \varsigma_1 \) comes from the definition of \( \mathcal{S}(\varsigma_1, \varsigma_2) \) in [42]. For any \( j \) such that \( \lambda_+ - \varsigma \leq \gamma_j \leq \lambda_+ \), we have

\[
|\lambda_j - \gamma_j| \asymp n^{-2/3} \left( j^{-1/3} + 1 (j \leq n^{1/4} \phi_n^{3/2}) \right) + \eta_1(\gamma_j) + n^{2/3} j^{-2/3} \eta_1^2(\gamma_j),
\]

where \( \eta_1(\gamma_j) = O(n^{-3/4} + n^{-5/6} j^{1/3} + n^{-1/2} \phi_n) \). Moreover, if either \( A \) or \( B \) is diagonal, the bound (62) is improved to

\[
|\lambda_j - \gamma_j| \asymp n^{-2/3} j^{-1/3}.
\]

Next, we discuss some properties of resolvents and local laws. For any \( z \in \mathbb{C}^+ \), denote

\[
H := H(z) := \begin{pmatrix} 0 & z^{1/2}Z^{1/2} \\ z^{1/2}Z^{1/2} & 0 \end{pmatrix} \quad \text{and} \quad \tilde{H} := \tilde{H}(z) := \begin{pmatrix} 0 & z^{1/2}S^{1/2} \\ z^{1/2}S^{1/2} & 0 \end{pmatrix},
\]

where \( z^{1/2} \) is taken to be the branch cut with positive imaginary part. Note that we have

\[
\tilde{H}(z) = \begin{pmatrix} 0 & z^{1/2}Z^{1/2} \\ z^{1/2}S^{1/2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & z^{1/2}S^{1/2} \\ z^{1/2}S^{1/2} & 0 \end{pmatrix} = H(z) + z^{1/2}UDU^T,
\]

where

\[
D = \begin{pmatrix} 0 & z^{1/2}D^{1/2} \\ z^{1/2}D^{1/2} & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}
\]

such that \( D = \text{diag}\{d_1, \cdots, d_r\} \) is a \( r \times r \) diagonal matrix formed by singular values, and \( U = (u_1, \cdots, u_r) \) and \( V = (v_1, \cdots, v_r) \) are \( p \times r \) and \( n \times r \) matrices formed by left and right singular vectors defined in [2]. The positive eigenvalues of \( \tilde{H} \) coincide with the singular values of \( z^{1/2}S \). Therefore, it suffices to study the Hermitian matrix \( \tilde{H} \). Further, we denote the Green functions of \( H \) and \( \tilde{H} \) respectively as

\[
G := G(z) := (H - z)^{-1} \quad \text{and} \quad \tilde{G} := \tilde{G}(z) := (\tilde{H} - z)^{-1}.
\]
By the Schur’s complement, we have
\begin{equation}
G(z) = \begin{pmatrix}
G_1(z) & z^{-1/2}G_1(z)Z \\
Z^{-1/2}Z^T G_1(z) & G_2(z)
\end{pmatrix}, \quad \bar{G}(z) = \begin{pmatrix}
\bar{G}_1(z) & z^{-1/2}\bar{G}_1(z)Z \\
Z^{-1/2}Z^T \bar{G}_1(z) & \bar{G}_2(z)
\end{pmatrix},
\end{equation}
where $G_1$ and $G_2$ are defined in \[.\] Further, by the spectral decomposition, we have
\begin{equation}
G(z) = \sum_{k=1}^{p+n} \frac{1}{\lambda_k - z} \begin{pmatrix}
\xi_k \xi_k^T & z^{-1/2}\sqrt{\lambda_k}\xi_k \xi_k^T \\
\sqrt{\lambda_k}\xi_k \xi_k^T & \sqrt{\lambda_k}\xi_k \xi_k^T
\end{pmatrix} \in \mathbb{C}^{(p+n)\times(p+n)},
\end{equation}
\begin{equation}
\bar{G}(z) = \sum_{k=1}^{p+n} \frac{1}{\lambda_k - z} \begin{pmatrix}
\bar{\xi}_k \bar{\xi}_k^T & z^{-1/2}\sqrt{\lambda_k}\bar{\xi}_k \bar{\xi}_k^T \\
\sqrt{\lambda_k}\bar{\xi}_k \bar{\xi}_k^T & \sqrt{\lambda_k}\bar{\xi}_k \bar{\xi}_k^T
\end{pmatrix} \in \mathbb{C}^{(p+n)\times(p+n)}.
\end{equation}

The following lemma characterizes the location of the outlier eigenvalue of $\bar{H}$.

**Lemma A.12.** (\[\] Lemma 6.1) Assume $\mu \in \mathbb{R} - \text{Spec}(H)$. Then $\mu \in \text{Spec}(\bar{H})$ if and only if
\begin{equation}
\det(U^T G(\mu) U + D^{-1}) = 0.
\end{equation}

Next lemma provides a link between the Green functions $G(z)$ and $\bar{G}(z)$.

**Lemma A.13.** (\[\] Lemma 4.8) For $z \in \mathbb{C}^+$, we have
\begin{equation}
\bar{G}(z) = G(z) - G(z) U (D^{-1} + U^T G(z) U)^{-1} U^T G(z) U.
\end{equation}

This lemma immediately leads to the following relationship.
\begin{equation}
U^T \bar{G}(z) U = D^{-1} - D^{-1}(D^{-1} + U^T G(z) U)^{-1} D^{-1}.
\end{equation}

Next, for any $z \in \mathbb{C}^+$, we denote
\begin{equation}
\Pi(z) := \begin{pmatrix}
\Pi_1(z) & 0 \\
0 & \Pi_2(z)
\end{pmatrix} \in \mathbb{C}^{(p+n)\times(p+n)},
\end{equation}
where
\begin{equation}
\Pi_1(z) := -z^{-1}(1 + m_{2c}(z) A)^{-1} \quad \text{and} \quad \Pi_2(z) := -z^{-1}(1 + m_{1c}(z) B)^{-1}.
\end{equation}
Denote
\begin{equation}
\bar{\Pi}(z) := \begin{pmatrix}
m_{1c}(z) I_r & 0 \\
0 & m_{2c}(z) I_r
\end{pmatrix} \in \mathbb{C}^{2r \times 2r}.
\end{equation}

Note that $\Pi(z)$ and $\bar{\Pi}(z)$ are related via $m_{1c}(z) = \frac{1}{p} \text{Tr} \Pi_1(z)$ and $m_{2c}(z) = \frac{1}{n} \text{Tr} \Pi_2(z)$, and we will show below how $\Pi(z)$ converges to $\bar{\Pi}(z)$. We denote the difference
\begin{equation}
\Omega(z) := U^T G(z) U - \bar{\Pi}(z),
\end{equation}
which will be used to control the noise part in the analysis. With $\Omega(z)$, we have the following identity
\begin{equation}
(D^{-1} + U^T G(z) U)^{-1} = (D^{-1} + \Pi(z))^{-1} + (D^{-1} + \Pi(z))^{-1} \Omega(z)(D^{-1} + U^T G(z) U)^{-1}.
\end{equation}

By iteration, we have the following $\ell$-th resolvent expansion.
Lemma A.14. (Resolvant identity [12]) For \( \ell \geq 1 \), we have the following \( \ell \)-th order resolvent expansion:

\[
(D^{-1} + U^T G(z) U)^{-1} = \sum_{k=0}^{\ell-1} (D^{-1} + \Pi(z))^{-1}[\Omega(z)(D^{-1} + \Pi(z))]^k
+ [(D^{-1} + \Pi(z))^{-1}\Omega(z)][D^{-1} + U^T G(z) U]^{-1}.
\]

(76)

We define the following spectral regions. Recall the definition of \( S(\varsigma_1, \varsigma_2) \) in [12]. Denote the following domains of the spectral parameter \( z \) as

\[
S_0(\varsigma_1, \varsigma_2, \omega) := \{ z = E + i\eta : \eta \geq \varsigma_1 \},
\]

(77)

\[
\bar{S}_0(\varsigma_1, \varsigma_2, \omega) := \{ z = E + i\eta : \eta \geq \varsigma_1 \}
\]

(78)

and

\[
S_{\text{out}}(\varsigma_2, \omega) := \{ E + i\eta : \lambda_+ + n\eta(n^{-2/3} + n^{-1/3} ) \leq E \leq \varsigma_2 \},
\]

(79)

Inside these domains, we have the following local laws. For any fixed \( \epsilon > 0 \), the following estimates hold.

1. **Anisotropic local law**: For any \( z \in \bar{S}_0(\varsigma_1, \varsigma_2, \omega) \) and deterministic unit vectors \( \mu, \nu \in \mathbb{C}^{p+n} \), we have

\[
|\langle \mu, (G(z) - \Pi(z))\nu \rangle| < \phi_n + \Psi(z).
\]

(80)

2. **Averaged local law**: For any \( z \in \bar{S}_0(\varsigma_1, \varsigma_2, \omega) \), we have

\[
|m_1(z) - m_{1c}(z)| + |m_2(z) - m_{2c}(z)| < (n\eta)^{-1}.
\]

Moreover, when \( z \in \bar{S}_0(\varsigma_1, \varsigma_2, \omega) \), we have the following stronger estimate

\[
|m_1(z) - m_{1c}(z)| + |m_2(z) - m_{2c}(z)| < \frac{n^{-\omega/4}}{n\eta} + \frac{1}{n(\kappa_2 + \eta)} + \frac{1}{(n\eta)^2\kappa_2 + \eta},
\]

which is uniformly on \( z \).

Note that the term “anisotropic” means the resolvent \( G(z) \) is well approximated by a deterministic matrix that is not a multiple of an identity matrix.

Theorem A.15. ([20] Theorem S.3.9], Local laws “near” \( \lambda_+ \)). Suppose that Assumption 2.2 holds. Fix constant \( \varsigma_1 > 0 \) and \( \varsigma_2 > 1 \) as those in Lemma A.7. Then

1. **Anisotropic local law**:

\[
\|(\mu, (G(z) - \Pi(z))\nu)\| \leq \phi_n + \Psi(z).
\]

(81)

2. **Averaged local law**: For any \( z \in S_{\text{out}}(\varsigma_2, \omega) \) and deterministic unit vectors \( \nu, w \in \mathbb{C}^{p+n} \), we have

\[
|\langle \nu, (G(z) - \Pi(z))w \rangle| \leq \phi_n + \sqrt{\text{Im}m_{2c}(z)} - \frac{1}{(n\eta)^{1/2}(\kappa_2 + \eta)^{-1/4}}.
\]

(82)

We derive the following lemma to describe the difference between \( \Pi(z) \) and \( \bar{\Pi}(z) \) near and beyond \( \lambda_+ \).

Lemma A.17. **Under Assumption 2.2**, for any \( z \in S_{\text{out}}(\varsigma_2, \omega) \) and \( i, j = 1, \ldots, r \), we have

\[
|\langle u_i, (z^{-1} + m_{2c}(z)B)^{-1}u_j \rangle - \delta_{ij}m_{1c}(z)\rangle \vee |\langle v_i, (z^{-1} + m_{1c}(z))^{-1}v_j \rangle - \delta_{ij}m_{2c}(z)\rangle < n^{-1}.
\]
that is,
\begin{equation}
\|U^T \Pi(z) U - \bar{\Pi}(z)\| \prec n^{-1}.
\end{equation}

Proof. When \( z \in \tilde{S}_0(\zeta_1, \zeta_2, \omega) \cup S_{out}(\zeta_2, \omega) \), by (8) and Lemma A.5 for any \( \delta > 0 \) we have that
\begin{align*}
P \left( |n(u_i, z^{-1}(1 + m_{2c}(z)A)^{-1}u_j) - \delta_{ij} \operatorname{Tr}(z^{-1}(1 + m_{2c}(z)A)^{-1})| > \delta \right) \\
= P \left( |(u_i, z^{-1}(1 + m_{2c}(z)A)^{-1}u_j) - \delta_{ij}m_{1c}(z)| > n^{-1} \delta \right) \leq 4e^{-\epsilon \left( \frac{\delta^2}{2^2} \right)}
\end{align*}
and some constants \( c, C > 0 \), where \( n \) is used to normalize \( u_i \) and \( u_j \) to fit the conditions of Lemma A.5. Thus, given any small constant \( \epsilon > 0 \) and large constant \( D > 0 \), by letting \( \delta = n^{-\epsilon} \), we have
\begin{align*}
P \left( |(u_i, z^{-1}(1 + m_{2c}(z)A)^{-1}u_j) - \delta_{ij}m_{1c}(z)| > n^{-1+\epsilon} \right) \leq 4n^{-\epsilon} \leq n^{-D}
\end{align*}
when \( n \) is sufficiently large. By a similar approach we have the same control for the other term, and hence the proof. \( \square \)

Appendix B. Proof of Theorem 3.2

With Lemmas A.11, A.13, A.16 and A.17 and triangle inequality, for some given fixed small \( \epsilon > 0 \), we find that there exists an event \( \Xi \) of high probability such that the followings hold conditional on \( \Xi \):

(i). For \( z \in \tilde{S}_0(\zeta_1, \zeta_2, \epsilon) \), we have from Lemmas A.15 and A.17 that for any deterministic unit vectors \( \mu, \nu \in \mathbb{C}^{p+n} \),
\begin{equation}
|\langle \mu, (G(z) - \bar{\Pi}(z))\nu \rangle| \leq n^{\epsilon/2}(\phi_n + \Psi(z)),
\end{equation}
and since \( r \) is bounded,
\begin{equation}
\|U^T G(z) U - \bar{\Pi}(z)\| \leq n^{\epsilon/2}(\phi_n + \Psi(z)).
\end{equation}

(ii). For \( z \in S_{out}(\zeta_2, \epsilon) \), we have from Lemmas A.16 and A.17 for any deterministic unit vectors \( \nu, \omega \in \mathbb{C}^{p+n} \), we have
\begin{equation}
|\langle \nu, (G(z) - \bar{\Pi}(z))\omega \rangle| \leq n^{\epsilon/2}(\phi_n + n^{-1/2}(\kappa_{z} + \eta)^{-1/4}),
\end{equation}
and since \( r \) is bounded,
\begin{equation}
\|U^T G(z) U - \bar{\Pi}(z)\| \leq n^{\epsilon/2}(\phi_n + n^{-1/2}(\kappa_{z} + \eta)^{-1/4}),
\end{equation}

(iii). From Lemma A.11 there exists a large integer \( \varpi \), such that for \( 1 \leq i \leq \varpi \), we have
\begin{equation}
|\lambda_i - \lambda_+| \leq n^{\epsilon}(n^{-1/3} \phi_n^2 + n^{-2/3}),
\end{equation}
where \( \varpi > \varpi \) is a fixed integer. Below, the proof is conditional on \( \Xi \) such that (85), (87) and (88) hold.

Step I. Find asymptotic outlier locations \( \theta(d_i) \) and the threshold \( \alpha \). We start with finding the asymptotic outlier locations for eigenvalues of \( \tilde{S}ST \). As is shown in (87), \( U^T G(z) U \) converges to \( \bar{\Pi}(z) \) in the operator norm for \( z \in S_{out}(\zeta_2, \epsilon) \). Together with Lemma A.12 and the continuity of determinant, \( \mu > \lambda_+ \) is asymptotically an outlier location if and only if
\begin{equation}
\lim_{n \to \infty} \det(U^T G(\mu) U + D^{-1}) = \det(\bar{\Pi}(\mu) + D^{-1}) = 0
\end{equation}
By Lemma A.2, (89) becomes
\[
\det(\Pi(\mu) + D^{-1}) = \det \left( \begin{pmatrix} m_{1c}(\mu)I_r & 0 \\ 0 & m_{2c}(\mu)I_r \end{pmatrix} + D^{-1} \right) \\
(90) \quad = \mu^{-r} \prod_{i=1}^{r} (\mu m_{1c}(\mu)m_{2c}(\mu) - d_i^{-2}) = \mu^{-r} \prod_{i=1}^{r} (\mathcal{T}(\mu) - d_i^{-2}) = 0.
\]
By the definition of \( \theta(\cdot) \), the determinant is zero when \( \mu = \theta(d_i) \) for \( 1 \leq i \leq r \). Note that since \( \mathcal{T}(x) \) is a monotonically decreasing function for \( x > \lambda_+ \), we have \( \theta(d_i) \) as a solution of (90) if and only if \( d_i^{-2} < \lambda_+ m_{1c}(\lambda_+)m_{2c}(\lambda_+) \), which is equivalent to \( d_i > \alpha \). In other words, \( \alpha \) is the desired signal strength threshold.

**Step II. Define the permissible intervals for the spectrum.** Now the strategy is to prove that with high probability there are no eigenvalues outside a neighbourhood of each \( \theta(d_i) \). Define the index set
\[
(91) \quad \mathbb{O}_\epsilon := \{ i : d_i - \alpha \geq n^\epsilon (\phi_n + n^{-1/3}) \} = \{ 1, 2, \ldots, r \} \subset \mathbb{O}_+,\n\]
for some \( r_\epsilon \leq r^+ \). This means the signal strength \( d_i \) is strong enough if \( i \in \mathbb{O}_\epsilon \). For \( 1 \leq i \leq r \), define the interval
\[
(92) \quad I_i := [\theta(d_i) - n^{2\epsilon}(\phi_n + n^{-1/3}), \theta(d_i) + n^{2\epsilon}(\phi_n + n^{-1/3})],
\]
where
\[
(93) \quad \omega(d_i) := \phi_n \Delta(d_i)^2 + n^{-1/2} \Delta(d_i).
\]
Also, define
\[
(94) \quad I_0 := [0, \lambda_+ + n^{3\epsilon}(\phi_n^2 + n^{-2/3})].
\]
Define
\[
(95) \quad I := I_0 \cup \bigcup_{i \in \mathbb{O}_\epsilon} I_i.
\]

**Step III. Show that \( I \) contains all eigenvalues of \( \tilde{H} \).** By (74), we have
\[
(96) \quad \det(\mathbf{U}^T G(z) \mathbf{U} + D^{-1}) = \det(\Pi(\mu) + D^{-1}) + O(n^{\epsilon/2}(\phi_n + n^{-1/2}K_\mu^{-1/4}))
\]
where the first bound comes from Lemma A.1 and the bounded \( 2r \) eigenvalues of \( \Pi(\mu) + D^{-1} \). To prove that \( \text{Spec}(\tilde{H}) \subset I \), by (96) and \( \lambda_+ = O(1) \), it suffices to show that
\[
(97) \quad \min_{1 \leq i \leq r} |\mu m_{1c}(\mu)m_{2c}(\mu) - d_i^{-2}| \gg n^{\epsilon/2}(\phi_n + n^{-1/2}K_\mu^{-1/4})
\]
when \( \mu > \lambda_+ \) and \( \mu \notin I \). Before proving (97), we claim that for any \( \mu \notin I \) and \( 1 \leq i \leq r \),
\[
(98) \quad |\mu - \theta(d_i)| > n^{\epsilon}(\phi_n).
\]
Step IV. Prove Claim \[98\]. To show that the claim is true, we consider two cases. (i) When \(i \in \mathbb{O}_c\), \[98\] is true by the definition of \(I_1\). (ii) When \(i \notin \mathbb{O}_c\), from \[91\] we have
\[
(99) \quad \Delta(d_i)^2 = d_i - \alpha < n^r(\phi_n + n^{-1/3}).
\]
Thus, by \(47\), we have
\[
(100) \quad \theta(d_i) - \lambda_+ \asymp (d_i - \alpha)^2 < 2n^{2r}(\phi_n^2 + n^{-2/3}),
\]
which is controlled by \(n^{3r}(\phi_n^2 + n^{-2/3})\) when \(n\) is sufficiently large. Thus, \(\theta(d_i) \in I_0\) and \(|\mu - \theta(d_i)| > n^{3r}(\phi_n^2 + n^{-2/3})\) when \(\mu \notin I\) by the definition of \(I_0\). Moreover, by the definition of \(\omega(d_i)\) and \(99\), we have
\[
(101) \quad \omega(d_i) \lesssim n^r(\phi_n^2 + n^{-2/3})
\]
for \(i \notin \mathbb{O}_c\), which leads to \(|\mu - \theta(d_i)| > n^r \omega(d_i)\). We thus obtain the claim.

Step V. Prove \[97\]. With the above claim, we are ready to prove \[97\]. Note that
\[
(102) \quad |\mu m_{1,2}(\mu) - d_i| = |\mathcal{T}(\mu) - d_i| = |\mathcal{T}(\mu) - \mathcal{T}(\theta(d_i))|.
\]
We decompose the problem into the following two cases.

Case (a): Suppose \(\theta(d_i) \in [\mu - c\kappa_\mu, \mu + c\kappa_\mu]\) for some constant \(0 < c < 1\), which is chosen sufficiently small so that \(\mu - \lambda_+ = \kappa_x \approx \kappa_\mu\) for \(x \in \mathbb{I}_i\). From \(47\), we have \(\theta(d_i) - \lambda_+ \asymp \Delta(d_i)^4\). Together with \(\mathcal{T}'(x) \asymp \kappa_x^{-1/2}\) from \(48\), for \(x \in \mathbb{I}_i\), we have
\[
(103) \quad |\mathcal{T}'(x)| \asymp |\mathcal{T}'(\theta(d_i))| \asymp |\Delta(d_i)|^{-2}.
\]
From the claim in \(98\), when \(\mu \notin \mathbb{I}_i\), it is either \(\mu < \theta(d_i) - n^r \omega(d_i)\) or \(\mu > \theta(d_i) + n^r \omega(d_i)\). When \(\mu < \theta(d_i) - n^r \omega(d_i)\), we have
\[
\mathcal{T}(\mu) - \mathcal{T}(\theta(d_i)) > \mathcal{T}(\theta(d_i) - n^r \omega(d_i)) - \mathcal{T}(\theta(d_i)) \gtrsim n^r \omega(d_i) \Delta(d_i)^{-2}
\]
\[
= n^r \phi_n + n^{-1/2+\epsilon} \Delta(d_i)^{-1} \asymp n^r \phi_n + n^{-1/2+\epsilon} (\theta(d_i) - \lambda_+)^{-1/4}
\]
\[
\gg n^r/2 \phi_n + n^{-1/2+\epsilon/2} \kappa_\mu^{-1/4} = n^r/2 (\phi_n + n^{-1/2+\epsilon/2} \kappa_\mu^{-1/4}),
\]
where we use the monotonicity of \(\mathcal{T}\) in the first step, the mean value theorem and \(103\) in the second step, the definition of \(\omega(d_i)\) in the third step, and \(47\) in the fourth step. Similarly, when \(\mu \notin \mathbb{I}_i\) such that \(\mu > \theta(d_i) + n^r \omega(d_i)\), the same argument leads to
\[
(105) \quad \mathcal{T}(\theta(d_i)) - \mathcal{T}(\mu) \gg n^r/2 (\phi_n + n^{-1/2+\epsilon/2} \kappa_\mu^{-1/4}).
\]
By \(104\) and \(105\), the relationship \(102\) leads to \(97\) when \(\theta(d_i) \in [\mu - c\kappa_\mu, \mu + c\kappa_\mu]\).

Case (b): Suppose \(\theta(d_i) \notin [\mu - c\kappa_\mu, \mu + c\kappa_\mu]\) for the same constant \(c\) from the previous case. For \(\theta(d_i) > \mu + c\kappa_\mu\), since \(\mathcal{T}\) is monotonically decreasing on \((\lambda_+, +\infty)\), we have that
\[
(106) \quad \mathcal{T}(\mu) - \mathcal{T}(\theta(d_i)) > \mathcal{T}(\mu) - \mathcal{T}(\mu + c\kappa_\mu) \approx \kappa_\mu^{1/2} \gg n^r/2 (\phi_n + n^{-1/2} \kappa_\mu^{-1/4}).
\]
where we used \(48\) and mean value theorem in the second step and \(\kappa_\mu > n^{3r}(\phi_n^2 + n^{-2/3})\) for \(\mu \notin I_0\) by the definition of \(I_0\) in the last step. By a similar argument,
where \( \theta(d_i) < \mu - c\kappa_\mu \), we have
\[
\mathcal{T}(\theta(d_i)) - \mathcal{T}(\mu) \gg n^{\tau/2}(\phi_n + n^{-1/2}\kappa_\mu^{-1/4}).
\]
By (106) and (107), we obtained (97) when \( \theta(d_i) \notin [\mu - c\kappa_\mu, \mu + c\kappa_\mu] \), and hence the proof of (97).

**Step VI.** Show that each \( I_i \) contains exactly the right number of outliers. We apply the continuity argument used in [31] Section 6.5. Set \( S(t) = S(t) + A_1/2BXB^{1/2} \), where the singular values of \( S(t) \) are \( (d_1(t), \ldots, d_r(t)) \) so that \( d_i(t) \) is a continuous function on \([0,1]\) for \( i = 1, \ldots, r \) satisfying some conditions detailed below, \( (d_1(1), \ldots, d_r(1)) = (d_1, \ldots, d_r) \) satisfying Assumption 2.2(iv) and \( (d_1(0), \ldots, d_r(0)) \) are set in the following way.

First, we construct \( (d_1(0), \ldots, d_r(0)) \). Consider \( d_1(0), \ldots, d_r(0) > 0 \) so that Assumption 2.2(iv) is satisfied but independent of \( n \) and denote \( x(0) := (x_1(0), \ldots, x_{r_0}(0)) = (\theta(d_1(0)), \ldots, \theta(d_r(0))) \). Assume \( d_1(0) > d_2(0) > \cdots > d_{r_0}(0) \) and \( d_i(0) - d_{i+1}(0) \geq 1 \) for \( 1 \leq i \leq r_0 - 1 \), such that \( x_1(0) > x_2(0) > \cdots > x_{r_0}(0) \).

We claim that when \( n \) is large enough, each \( I_i(0) := [\theta(d_i(0)) - n^\tau\omega(d_i(0)), \theta(d_i(0)) + n^\tau\omega(d_i(0))] \), \( 1 \leq i \leq r_0 \), contains only the \( i \)-th eigenvalue of \( S(0)S(0)^T \). To show this, fix any \( 1 \leq i \leq r_0 \) and choose a small positively oriented closed contour \( C \subset \mathbb{C}\setminus[0, \lambda_+ ] \) so that \( C \) only enclose \( I_i(0) \) but no other intervals \( I_j(0) \) for \( j \neq i \). Define two functions,
\[
f_0(z) := \det(U^TG(z)U + D(0)^{-1}) \quad \text{and} \quad g_0(z) := \det(\Pi(z) + D(0)^{-1}),
\]
where \( D(0) \) is defined in the same way as (66) with \( d_1(0), \ldots, d_r(0) \). Both \( f(z) \) and \( g(z) \) are holomorphic on and inside \( C \) by the definition. By Lemma A.2 we have \( g_0(z) = z^{-r}\prod_{i=1}^r(zm_{1c}(z)m_{2c}(z) - d_i(0))^{-2} = z^{-r}\prod_{i=1}^r(T(z) - \mathcal{T}(\theta(d_i(0)))) \).

Thus, \( g_0(z) \) has precisely one zero at \( z = \theta(d_i(0)) \) inside \( C \). So, by a proper choice of the contour \( C \), we have
\[
\min_{z \in C} |g_0(z)| = \min_{z \in C} \left| z^{-r}\prod_{i=1}^r(T(z) - \mathcal{T}(\theta(d_i(0)))) \right| \geq n^{\tau\omega(d_i(0)) - \lambda_+^{-1/2}} \prod_{i=1}^r(\theta(d_i(0)) - \lambda_+^{-1/2}) \\
\geq n^{\tau\omega(d_i(0))} \geq n^{3\tau\phi_n(\phi_n + n^{-1/3}2^2 + n^{2}\phi_n - n^{-1/3})},
\]
where first bound comes from the fact that \( \lambda_+ \leq |z| \leq \tau^{-1} \), the control of \( T(z) - \mathcal{T}(\theta(d_j(0))) \) by (56) for any \( 1 \leq j \leq r \), and the control of \( |z - \theta(d_j(0))| \geq \theta(d_j(0)) - \theta(d_j(0))|/2 \geq 1 \) for any \( j \neq i \) by (57). and the assumption \( d_i(0) - d_{i+1}(0) \geq 1 \), the second bound comes from (47) since \( |\theta(d_i(0)) - \lambda_+^{-1/2} \geq |d_i(0) - \alpha|^{-1/2} \leq \tau^{-1} \), and the last bound comes from the definition of \( \omega(d_i(0)) \) and the lower bound assumption of \( d_i(0) \). Also, by (86), we have for any \( z \in C \) that
\[
|f_0(z) - g_0(z)| \leq n^{\tau/2}(\phi_n + n^{-1/2}(\kappa_\tau + \eta)^{-1/4}),
\]
which is clearly dominated by \( \min_{z \in C} |g_0(z)| \). Hence, the claim follows from Rouché’s theorem.

Second, we set the conditions for the paths \( d_i(t) \). Set \( x(t) := (x_1(t), x_2(t), \cdots, x_{r_0}(t)) \) \( (\theta(d_1(t)), \ldots, \theta(d_{r_0}(t))) \). By the construction, \( x(t) \) is a continuous path over \([0, 1]\).
And we require that \( x(t) \) satisfies the following properties:
where we find that eigenvalues of $t$ into two cases.

Case (b): Claim.

I of $B$ $x$ $|$ $\mid$ construction, we have and eigenvalue interlacing from Lemma A.3, we have that for $i$ $\in$ $\cap$(1), and let $I_i$ $\in$ $I(t)$, $1 \leq i \leq r_e$ for all $t \in [0,1]$, and hence the claim.

Case (a): If are $r_e$ intervals are disjoint at $t = 1$, then they are disjoint for all $t \in [0,1]$ by property (ii). Together with the results when $t = 0$ and the continuity of $\lambda_i(t)$, we conclude that $\lambda_i(t) \in I_i(t)$, $1 \leq i \leq r_e$ for all $t \in [0,1]$, and hence the claim.

Case (b): If some of the intervals are not disjoint at $t = 1$, let $B$ denote the finest partition of $\{1, \ldots, r_e\}$ such that $i$ and $j$ belong to the same block of $B$ if $I_i(1) \cap I_j(1) \neq \emptyset$. Denote by $B_i$ the block of $B$ that contains $i$. Note that elements of $B_i$ are sequences of consecutive integers. We now pick any $1 \leq i \leq r_e$ so that $|B_i| > 1$, and let $j \in B_i$ such that it is not the smallest index in $B_i$. Note that

$$x_{j-1}(1) - x_j(1) \leq 2n^j w(d_j)$$

by assumption. Since the number of elements in $B_i$ is bounded by $r_e$, we obtain that

$$\left| \bigcup_{j \in B_i} I_j(1) \right| \leq r_e n^j w(d_{\min\{j, j \in B_i\}}) = r_e n^j w(d_i),$$

where $\left| \bigcup_{j \in B_i} I_j(1) \right|$ stands for the length of $\bigcup_{j \in B_i} I_j(1)$. Thus, by the continuity construction, we have

$$|\lambda_j(1) - \theta(d_j)| \leq r_e n^j w(d_i), \quad j \in B_i,$$

and hence the claim.

Step VII. Locations of weak (non-outlier) signals. First, we fix a configuration $x(0)$ satisfying the same setup in Step VI. In this setup, when $t = 0$, (88) and eigenvalue interlacing from Lemma A.3, we have that for $i \notin O_e$,

$$\tilde{\lambda}_i(0) \in I_0, \quad \text{and} \quad \tilde{\lambda}_i(0) \geq \lambda_+ - n^j(n^{-1/3} \phi_n^2 + n^{-2/3}).$$

Thus, we find that

$$|\tilde{\lambda}_i(0) - \lambda_+| \leq n^j(\phi_n^2 + n^{-2/3}), \quad i \notin O_e.$$

Next we employ a similar continuity argument as that in Step VI. For $t \in [0,1]$, by (A.11) and Lemma A.3, for any $t \in [0,1]$, we always have that

$$\lambda_i(t) \geq \lambda_+ - n^j(n^{-1/3} \phi_n^2 + n^{-2/3}), \quad i \notin O_e.$$
If \( I_0 \) is disjoint from the other \( I_i \)'s, then by the continuity of \( \tilde{\lambda}_i(t) \) and \( \text{Spec}(\tilde{S}^T) \subset I_i \), we can conclude that \( \tilde{\lambda}_i(t) \in I_0(t) \) for all \( t \in [0,1] \). Otherwise, we again consider the partition \( \mathcal{B} \) as in **Step VI**, and let \( B_0 \) be the block of \( \mathcal{B} \) that contains \( i \). With the same arguments, we can prove that

\[
I_0(1) \cup \left( \bigcup_{j \in B_0} I_j(1) \right) \subset [0, \lambda_+ + Cn^{3r}(\phi_n^2 + n^{-2/3})].
\]

Then using (112), (113) and the continuity of the eigenvalues along the path, we obtain that for all \( r_1 < i \leq r \),

\[
|\tilde{\lambda}_i(t) - \lambda_+| \leq Cn^{3r}(\phi_n^2 + n^{-2/3})
\]

for all \( t \in [0,1] \).

**APPENDIX C. PROOF OF THEOREM 3.3**

**Proof.** The proof is composed of several steps.

**Step I. Prepare an event.** Let \( \epsilon > 0 \) be a small positive constant and take \( 0 < \omega < 1 \). By Theorem 3.2 and Lemmas A.11 and A.16, we find that there exists some high probability event \( \Xi \) such that the followings hold when conditional on \( \Xi \).

(i) Recall the definition of \( S_{out} \) in (78). Set \( \zeta = (\lambda_+ - \omega)^{-1} \) and hence

\[
S_{out}(\lambda_+ - \omega)^{-1}, \omega) = \{ E + i\eta : \lambda_+ + n^{r}(n^{-2/3} + n^{-1/3} \phi_n^2) \leq E \leq \omega \eta, \eta \in [0,1] \}.
\]

For all \( z \in S_{out}(\lambda_+ - \omega)^{-1}, \omega) \), we have the anisotropic local law by Theorem A.16

\[
||\Omega|| \leq n^r [\phi_n + n^{-1/2}(\kappa_z + \eta)^{-1/4}].
\]

(ii) For \( i \leq r^+ \), we have

\[
|\tilde{\lambda}_i - \theta(d_i)| \leq n^r [n^{-1/2} \Delta(d_i) + \phi_n \Delta^2(d_i)].
\]

(iii) From Theorem 3.2 and the rigidity of eigenvalues in Lemma A.11, we have

\[
|\tilde{\lambda}_i - \theta(d_i)| \leq n^r (n^{-1/2} \Delta(d_i) + \phi_n \Delta^2(d_i)), \quad 1 \leq i \leq r^+,
\]

\[
|\tilde{\lambda}_i - \lambda_+| \leq n^{r/2} (\phi_n^2 + n^{-2/3}), \quad r^+ + 1 \leq i \leq \omega
\]

for some fixed large integer \( \omega \geq r \), and

\[
|\lambda_i - \gamma_i| \leq n^{r/2} \left[ n^{-2/3} \left( i^{-1/3} + 1 (i \leq n^{1/4} \phi_n^3/2) \right) + \eta_i (\gamma_i) + n^{2/3} i^{-2/3} \eta_i^2 (\gamma_i) \right]
\]

for \( 1 \leq i \leq \omega \).

The rest of the proof is restricted to the event \( \Xi \).

**Step II. Prepare some quantities.** We prepare some notations that will be used in the following proofs. For \( i = 1, \cdots, r \), let \( u_i^e = (u_i^T, 0)^T \) be the embedding of \( u_i \) in \( \mathbb{R}^{p+} \). By (67), it is easy to see that

\[
u_i^T \bar{G}_1(z) u_j = (u_i^e)^T \bar{G}(z) u_j = c_i^T U^T \bar{G}(z) U e_j,
\]
where $e_j \in \mathbb{R}^{2r}$ is a unit vector with the $i$-th entry 1. When $1 \leq i, j \leq r$, by (71), we have

$$u^T \overline{G}_1(z)u_j = e^T \{D^{-1} - D^{-1}(D^{-1} + U^T G(z)U)^{-1}D^{-1}\}e_j.$$ 

To study $D^{-1} + U^T G(z)U)^{-1}$, by the resolvent expansion from (76), we will encounter the term $(D^{-1} + \Pi(z))^{-1}$. By an elementary computation based on the Schur complement, we have

$$[(D^{-1} + \Pi(z))^{-1}]_{ij} = \begin{cases} 
\frac{zm_{2c}(z)}{zm_{1c}(z)m_{2c}(z) - d_i^{-2}}, & 1 \leq i, j \leq r; \\
\frac{zm_{1c}(z)}{zm_{1c}(z)m_{2c}(z) - d_i^{-2}}, & r \leq i, j \leq 2r; \\
-\delta_{ij} \frac{z^{1/2}d_i^{-1}}{zm_{1c}(z)m_{2c}(z) - d_i^{-2}}, & 1 \leq i \leq r, r \leq j \leq 2r; \\
-\delta_{ij} \frac{z^{1/2}d_j^{-1}}{zm_{1c}(z)m_{2c}(z) - d_i^{-2}}, & r \leq i \leq 2r, 1 \leq j \leq r.
\end{cases}$$

Next, by definition of $\lambda_+$ and $\gamma_i$ in Definition A.10 and the approximation of $\rho_{1c}$ around the bulk in (39), when $r^+ + 1 \leq i \leq \omega$, we have

$$|\gamma_i - \lambda_+| \lesssim n^{-2/3}. \quad (122)$$

Together with the triangle inequality and (118), when $r^+ + 1 \leq i \leq r$, we have

$$|\bar{\lambda}_i - \gamma_i| \leq |\bar{\lambda}_i - \lambda_+| + |\lambda_+ - \gamma_i| \leq n^{r/2} \left(n^{-2/3} + \phi_n^2\right). \quad (123)$$

Moreover, when $r + 1 \leq i \leq \omega$, with the interlacing lemma from Lemma A.3, we have $\lambda_i \leq \bar{\lambda}_i \leq \lambda_i - r \leq \lambda_+$. Together with the triangle inequality, (122) and (119), for $r + 1 \leq i \leq \omega$, we have

$$|\bar{\lambda}_i - \gamma_i| \leq |\bar{\lambda}_i - \lambda_+| + |\lambda_+ - \gamma_i| + |\gamma_i - \lambda_i| \lesssim |\gamma_i - \lambda_i| \quad (124)$$

In conclusion, by (119), (123), and (124), for $r^+ + 1 \leq i \leq \omega$ we have

$$|\bar{\lambda}_i - \gamma_i| \leq |\bar{\lambda}_i - \lambda_+| + |\lambda_+ - \gamma_i| + |\gamma_i - \lambda_i| + |\lambda_i - \gamma_i| \leq n^{r/2} \left(i^{-1/3}n^{-2/3} + \eta(\gamma_i) + i^{-2/3}n^{-1/3}\phi_n^2 + \phi_n^2\mathbb{1}_{i \leq r}\right). \quad (125)$$

Next, we claim that for any $1 \leq i \leq \omega$, the equation

$$\eta \operatorname{Im} m_{1c}(\bar{\lambda}_i + i\eta) = n^{2r} \phi_n \eta + n^{-1+6\epsilon} \quad (126)$$

over $\eta \in [0, 1]$ has a unique solution. Indeed, note that $\eta \operatorname{Im} m_{1c}(\bar{\lambda}_i + i\eta)$ is a strictly monotonically increasing function of $\eta$. Since $\lambda_i + i\eta \in S(S_1, S_2)$ when $n$ is sufficiently large, the behavior of $\eta \operatorname{Im} m_{1c}(\lambda_i + i\eta)$ is detailed in (44). First, consider the case $1 \leq i \leq r_+$. Since $\bar{\lambda}_i$ is of order 1 in this case, $\eta \operatorname{Im} m_{1c}(\lambda_i + i\eta)$ grows quadratically when $\eta \leq n^{3\epsilon}\phi_n$. Thus we find a unique solution $\hat{\eta}_i$ over the region $0 \leq \eta \leq n^{3\epsilon}\phi_n$. When $\eta > n^{3\epsilon}\phi_n$, we cannot find a solution since $n^2\phi_n \eta$ dominates $n^{2r}\phi_n$. Second, consider the case $r_+ + 1 \leq i \leq \omega$. In this case, we should consider the case when $\bar{\lambda}_i$ is less than or greater than $\lambda_+$. When $\bar{\lambda}_i \geq \lambda_+$, we have $\operatorname{Im} m_{1c}(\lambda_i + i\eta)$ grows like $\eta^{3/2}$ when $\eta > n^r\phi_n$, and we can find a unique solution $\hat{\eta}_i$ of order $n^{4r}\phi_n^2$. When $0 \leq \eta < n^r\phi_n$, we cannot find a solution since $n^{2r}\phi_n \eta$
dominates $\eta \text{Im } m_{1c}(\bar{\lambda}_i + i\eta)$ in this region. The case $\lambda_i < \lambda_+$ can be argued in the same way. We thus obtain the claim. More precisely, with (44), one can check that

$$
\begin{equation}
\hat{\eta}_i \asymp \begin{cases}
n^4 (\phi_n^2 + n^{-2/3}) & \text{if } |\bar{\lambda}_i - \lambda_+| \leq n^{4\epsilon} (\phi_n^2 + n^{-2/3}) \\
n^{2\epsilon} \phi_n \sqrt{\kappa_{\lambda_i}} + n^{-1/2} + 3\epsilon \kappa_{\lambda_i}^{1/4} & \text{if } \bar{\lambda}_i \geq \lambda_+ + n^{4\epsilon} (\phi_n^2 + n^{-2/3}) \\
n^{-1 + 6\epsilon} \kappa_{\lambda_i}^{-1/2} & \text{if } \bar{\lambda}_i \leq \lambda_+ - n^{4\epsilon} (\phi_n^2 + n^{-2/3})
\end{cases}
\end{equation}
$$

(127)

Note that $i \in \mathbb{O}_+$ when $\hat{\eta}_i \asymp n^{2\epsilon} \phi_n \sqrt{\kappa_{\lambda_i}} + n^{-1/2} + 3\epsilon \kappa_{\lambda_i}^{1/4}$, and $r^+ + 1 \leq i \leq r$ when $\eta_i \asymp n^{4\epsilon} (\phi_n^2 + n^{-2/3})$.

Based on the above claim, in the discussion afterwards, we fix $z_i = \bar{\lambda}_i + i\eta_i \in \bar{S}_0(\xi_1, \xi_2, \epsilon)$, where $\eta_i := \hat{\eta}_i \vee n^4 \eta_i(\gamma_i)$, such that Lemma A.7 can be applied. Next, we claim that for such $z_i$, we have

$$
\|\Omega(z_i)\| = \|U^T G(z_i) U - \Pi(z_i)\| \lesssim n^{-\epsilon} \text{Im } m_{1c}(z_i).
$$

To show this claim, we discuss two cases: (i) $\hat{\eta}_i \geq n^{4\epsilon} \eta_i(\gamma_i)$ and (ii) $\hat{\eta}_i < n^{4\epsilon} \eta_i(\gamma_i)$. In case (i), since $z_i \in \bar{S}(\xi_1, \xi_2, \epsilon)$, (85) can be applied to $z_i$ and we have

$$
\|\Omega(z_i)\| \leq n^{4\epsilon} (\phi_n + \Psi(z_i)) \lesssim n^{-3\epsilon/2} \text{Im } m_{1c}(z_i),
$$

(128)

where the last inequality is by (126) and the definition of $\Psi(z_i)$ in (59). In case (ii), note that by (125) and $\eta_i(\gamma_i) \asymp n^{-3/4} + n^{-1/2} (\sqrt{\kappa_{\lambda_i}} + \phi_n)$ by (61), we have

$$
|\kappa_{\lambda_i} - \kappa_{\gamma_i}| \lesssim n^{7/2} \left( i^{-1/3} n^{-2/3} + n^{-3/4} + n^{-1/2} (\sqrt{\kappa_{\lambda_i}} + \phi_n) + i^{-2/3} n^{-1/3} \phi_n^2 + \phi_n^2 \mathbb{1}_{1 \leq r} \right).
$$

(129)

Since $\kappa_{\gamma_i} = |\lambda_+ - \gamma_i| \leq \lambda_+$ and we consider $1 \leq i \leq \omega$, when $n$ is sufficiently large, we have

$$
\kappa_{\lambda_i} \asymp \kappa_{\gamma_i},
$$

which with (61) immediately leads to

$$
\eta_i(\lambda_i) \asymp \eta_i(\gamma_i).
$$

Recall the condition $\hat{\eta}_i < n^{4\epsilon} \eta_i(\gamma_i)$. By comparing $\hat{\eta}_i$ and $n^{4\epsilon} \eta_i(\gamma_i)$ using (127), we conclude that $\bar{\lambda}_i \leq \lambda_+ - n^{4\epsilon} (\phi_n^2 + n^{-2/3})$, and hence $\kappa_{\lambda_i} \gtrsim n^{4\epsilon} (\phi_n^2 + n^{-2/3})$. Thus, by a direct comparison using (61), we also have

$$
n^{4\epsilon} \eta_i(\gamma_i) \ll \kappa_{\gamma_i}.
$$

(130)

Since $\bar{\lambda}_i \leq \lambda_+ - n^{4\epsilon} (\phi_n^2 + n^{-2/3})$, we have $\kappa_{\lambda_i} \gtrsim n^{4\epsilon} (\phi_n^2 + n^{-2/3})$, which when combined with By (127) we have

$$
n^{5\epsilon} \eta_i(\gamma_i) \lesssim \sqrt{\kappa_{\gamma_i}}.
$$

(131)

Next, from (44) and (131), we derive that

$$
\text{Im } m_{1c}(z_i) \asymp \sqrt{\kappa_{\lambda_i} + n^{4\epsilon} \eta_i(\gamma_i)} \asymp \sqrt{\kappa_{\gamma_i}},
$$

(132)
where the last approximation comes from (131). By putting the above together, we have
\[
\|\Omega(z_i)\| \leq n^{r/2} \left( \phi_n + \Psi(\lambda_i + in^r \eta(\gamma_i)) \right)
\]
\[
\lesssim n^{r/2} \left( \frac{1}{\sqrt{n^r \eta(\gamma_i)}} + \frac{1}{n \eta(\gamma_i)} \right) \leq n^{r/2} \phi_n + n^{-2r} \sqrt{\eta_\gamma} \lesssim n^{-r} \Im m_{1c}(z_i),
\]
where the second bound comes from the definition of \( \Psi \) and (132), the third bound comes from (131), and the last inequality comes from applying (126) and (132).

**Step III. Prove the theorem.** With the above bounds, we start the proof. For \( j = 1, \ldots, r \), set \( u_j^c = (u_j^c, 0)^T \) be the embedding of \( u_j \) in \( \mathbb{R}^{p+n} \). With the spectral decomposition (68), we have that for \( 1 \leq i \leq cn \),
\[
\Im \langle u_j^c, \tilde{G}(z_i) u_j^c \rangle = \sum_{k=1}^{p+n} \frac{\eta_i |(u_j, \xi_k)|^2}{(\lambda_k - \lambda_i)^2 + \eta_i^2} + \sum_{k \neq i} \frac{\eta_i |(u_j, \xi_k)|^2}{(\lambda_k - \lambda_i)^2 + \eta_i^2},
\]
and thus
\[
|\langle u_j, \tilde{\xi}_i \rangle|^2 \leq \eta_i \Im \langle u_j^c, \tilde{G}(z_i) u_j^c \rangle.
\]
By Lemma A.13 we obtain another identity
\[
\langle u_j^c, \tilde{G}(z_i) u_j^c \rangle = -\frac{1}{zd_j^2} [(D^{-1} + U^T G(z_i) U)^{-1}]_{jjj}.
\]
Using the second order resolvent expansion shown in (76) for \((D^{-1} + U^T G(z_i) U)^{-1}\) and (121), we have
\[
\langle u_j^c, \tilde{G}(z_i) u_j^c \rangle = \frac{z_i m_{1c}(z_i)}{zd_j^2} \left[ \frac{1}{T(z_i) - d_j^{-2}} + \frac{z_i f(z_i)}{(T(z_i) - d_j^{-2})^2} \right] + [(D^{-1} + \Pi(z_i))^{-1}\Omega(z_i)]^2 (D^{-1} + U^T G(z_i) U)^{-1} \right]_{jjj},
\]
where \( f(z) = f_1(z) + f_2(z) \) and
\[
f_1(z) := m_{1c}(z) |z m_{1c}(z) \Omega(z)_{jjj} + (-1)^r z^{1/2} d_j^{-1} \Omega(z)_{jjj}|,
\]
\[
f_2(z) := d_j^{-1} [(-1)^r z^{1/2} m_{1c}(z) \Omega(z)_{jjj} + d_j^{-1} \Omega(z)_{jjj}].
\]
To estimate the right-hand side of (136), note that by (14) and the definition of \( f \),
\[
|f(z_i)| \lesssim \|\Omega(z_i)\|
\]
and
\[
\min_j |T(z_i) - d_j^{-2}| \geq \Im T(z_i) = \Im m_{1c}(z_i) \gtrsim \Im m_{1c}(z_i) \gtrsim \|\Omega(z_i)\|.
\]
Jointly by (137) and (138), the second term on the right-hand side of (136) is dominated by the first term. For the third term in (136), we apply the second order resolvent expansion on \((D^{-1} + U^T G(z_i) U)^{-1}\), then we acquire a similar formula as in (136) times \((D^{-1} + \Pi(z_i))^{-1}\Omega(z_i))^2\). By (121) and (138), we have that
\[
\|D^{-1} + U^T G(z_i) U\|^{-1} \lesssim \frac{1}{|T(z_i) - d_j^{-2}|} \lesssim \frac{1}{\Im m_{1c}(z_i)} \ll \|\Omega(z_i)\|^{-1}.
\]
Inserting bounds from (137)-(139) into (136), we obtain that

$$\langle u_j, \hat{G}(z_i) u_j \rangle = \frac{m_{1c}(z_i)}{1 - d_j^2 \mathcal{T}(z_i)} + O \left( \frac{d_j^2 \| \Omega(z_i) \|}{|1 - d_j^2 \mathcal{T}(z_i)|^2} \right).$$

The next lemma provides a lower bound for $|1 - d_j^2 \mathcal{T}(z_i)|$. Its proof is the same as the one for Lemma 5.6 of [18], so we omit it and refer readers with interest to [18].

**Lemma C.1.** Let $z_i = \tilde{\lambda}_i + i \eta_i$ and $1 \leq j \leq r$. For any fixed $\delta \in [0, 1/3 - \omega)$, when $\tilde{\lambda}_i \in [0, \theta(\alpha + (\phi_n + n^{-1/3}) n^\delta + \omega)],$ there exists a constant $c > 0$ such that

$$|1 - d_j^2 \mathcal{T}(z_i)| \geq c d_j^2 (n^{-2\delta} |d_j^2 - \mathcal{T}(\lambda_\pm)| + \text{Im} \mathcal{T}(z_i)).$$

Now we fix the $\delta$ in Lemma C.1. By (135) and (140), we have that

$$\left| \langle u_j, \xi_i \rangle \right|^2 \leq \frac{\eta_i}{|1 - d_j^2 \mathcal{T}(z_i)|^2} \left[ \text{Im} m_{1c}(z_i)(1 - \text{Re}(d_j^2 \mathcal{T}(z_i))) + d_j^2 \text{Re} m_{1c}(z_i) \text{Im} \mathcal{T}(z_i) + C d_j^2 \| \Omega(z_i) \| \right]$$

$$= \frac{\eta_i}{|1 - d_j^2 \mathcal{T}(z_i)|^2} \left[ \text{Im} m_{1c}(z_i)(1 - d_j^2 \mathcal{T}(\lambda_\pm)) + d_j^2 \text{Re} m_{1c}(z_i) \mathcal{T}(\lambda_\pm) + C d_j^2 \| \Omega(z_i) \| \right]$$

$$+ d_j^2 \text{Re} m_{1c}(z_i) \mathcal{T}(z_i) + C d_j^2 \| \Omega(z_i) \||.$$

We next bound the terms in (141) one by one. For the first term, by (45) and (126), we have

$$|\eta_i \text{Im} m_{1c}(z_i)(1 - d_j^2 \mathcal{T}(\lambda_\pm)) + d_j^2 \text{Re}(\mathcal{T}(\lambda_\pm) - \mathcal{T}(z_i))|$$

$$\lesssim |\eta_i \text{Im} m_{1c}(z_i)| \left( |d_j - \alpha| + \sqrt{\kappa_{\lambda_\pm}^2 + \eta_i^2} \right) \sqrt{\frac{\eta_i}{\kappa_{\lambda_\pm}^2 + \eta_i^2}}.$$

For the second item of (141), by Lemma A.7 we have

$$|\eta_i \text{Re} m_{1c}(z_i) \text{Im} \mathcal{T}(z_i)| \asymp |\eta_i \text{Im} m_{1c}(z_i)|.$$

Note that by (126), (131), and (44), we have

$$|\eta_i \text{Im} m_{1c}(z_i)| \lesssim \begin{cases} n^2 \phi_n \eta_i + n^{-1+6\epsilon}, & \text{if } \eta_i \geq n^\epsilon \eta_i(\gamma_i) \\ n^\epsilon \eta_i(\gamma_i) \sqrt{\kappa_{\gamma_i}}, & \text{if } \eta_i < n^\epsilon \eta_i(\gamma_i) \end{cases}.$$

For the third term, by (128), (133) and (144), we have

$$\|\eta_i \Omega(z_i)\| \lesssim \begin{cases} n^\epsilon \phi_n \eta_i + n^{-1+5\epsilon}, & \text{if } \eta_i \geq n^\epsilon \eta_i(\gamma_i) \\ \eta_i(\gamma_i) \sqrt{\kappa_{\gamma_i}}, & \text{if } \eta_i < n^\epsilon \eta_i(\gamma_i) \end{cases}.$$

Inserting the estimates from (127), (142), (143), (144) and (145) into (141) together, we have

$$\left| \langle u_j, \xi_i \rangle \right|^2 \lesssim \frac{n^\epsilon \phi_n \eta_i + n^\epsilon \eta_i(\gamma_i) \sqrt{\kappa_{\gamma_i}} + n^6 n^{-1}}{|1 - d_j^2 \mathcal{T}(z_i)|^2} \lesssim \frac{n^\epsilon \phi_n(\gamma_i) \sqrt{\kappa_{\gamma_i}} + n^{-1}}{|1 - d_j^2 \mathcal{T}(z_i)|^2},$$

where in the last inequality we used that for $\eta_i \geq n^\epsilon \eta_i(\gamma_i)$,

$$\phi_n \eta_i \lesssim n^{3+\epsilon} \phi_n(n^{-2/3}) \lesssim n^{4+\epsilon} \phi_n(\gamma_i).$$

$$\phi_n \eta_i \lesssim n^{3+\epsilon} \phi_n(\gamma_i).$$
We still need to bound the denominator of (146) from below using Lemma C.1, which requires a lower bound on \( \text{Im} \mathcal{T}(z_i) \). When \( i \notin \mathbb{O}^+ \), with (44), (125), and (127), we find that \( \text{Im} \mathcal{T}(z_i) \approx \text{Im} m_{1c}(z_i) \gtrsim \phi_n + \sqrt{\kappa_n} \). Together with (146), this concludes the proof of (23) for \( |\langle u_j, \xi_i \rangle|^2 \). The proof of \( |\langle v_j, \xi_i \rangle|^2 \) is based on the same steps and we omit details, by choosing \( \delta = 0 \) in Lemma C.1. On the other hand, when \( i \in \mathbb{O}^+ \) such that (24) holds, with (117) and (127) we can verify that \( \tilde{\lambda}_i \leq \theta(d_i) + n^{3+\tilde{\tau}}(\phi_n^2 + n^{-3/3}) \) and by (57) we have \( \theta \left( \alpha + n^{3+\tilde{\tau}}(\phi_n + n^{-1/3}) \right) - \theta(d_i) \asymp n^{2+2\tilde{\tau}}(\phi_n^2 + n^{-2/3}) \). By the two inequalities, we have

\[
\tilde{\lambda}_i \lesssim \theta \left( \alpha + n^{3+\tilde{\tau}}(\phi_n + n^{-1/3}) \right),
\]

and thus by (54) we have

\[
(148) \quad |\tilde{\lambda}_i - \lambda_i| \lesssim n^{2\tilde{\tau}+2\epsilon} (n^{-2/3} + \phi_n^2).
\]

Moreover, together with (127) and (44), we conclude that

\[
(149) \quad \text{Im} \mathcal{T}(z_i) \approx \text{Im} m_{1c}(z_i) \gtrsim n^{2\tilde{\tau}+2\epsilon} (\phi_n + n^{-1/3}) \gtrsim n^{2\tilde{\tau}+2\epsilon} (\phi_n + \sqrt{\kappa_n}).
\]

We can therefore conclude the proof of (25) with (146) by letting \( \delta = \tilde{\tau} - \epsilon \) in Lemma C.1. \( \square \)

Appendix D. Proof of Theorem 3.4

We only show the detailed proof for the control of \( |\langle u_i, P_A u_j \rangle - \delta_{ij} \mathbb{1} (i \in A) a_1(d_i) | \) in (28). The control of \( |\langle v_i, P_A^c v_j \rangle - \delta_{ij} \mathbb{1} (i \in A^c) a_2(d_i) | \) is the same and we omit details. The proof is decomposed into three main parts. First, we prove Theorem 3.4 under two stronger assumptions, which is stated in Proposition D.1. Then, we remove these two assumptions. We start with introducing these two assumptions.

**Assumption D.1.** (Non-overlapping condition). For some fixed constant \( \tilde{\tau} > 0 \), we assume that for all \( i \in A \),

\[
(150) \quad \nu_i(A) \gtrsim n^{\tilde{\tau}} \left( \frac{1}{|\Delta(d_i)|}^{-1} n^{-1/2} + \phi_n \right) = n^{\tilde{\tau}} \psi_1(d_i).
\]

By Assumption D.1, an outlier indexed by \( A \) does not overlap with an outlier indexed by \( A^c \). That is, when \( d_i \neq d_j \), they are sufficiently separated if \( i \in A \) and \( j \in A^c \). However, outliers indexed by \( A \) can overlap among themselves.

**Assumption D.2.** For some fixed small constant \( 0 < \tau' < 1/3 \), we assume that for \( i \in A \),

\[
(151) \quad d_i - \alpha \geq n^{\tau'} (\phi_n + n^{-1/3}).
\]

The necessary argument to remove this assumption will be given in Section D.3 after we complete the proof of Theorem 3.3 since we need the delocalization bounds there.

D.1. Proof of Theorem 3.4 Under Stronger Assumptions. We first provide the following proposition, which is needed for Theorem 3.4.
Suppose that Assumptions D.1 and D.2 hold. The set 

\[
\theta
\]

Lemma D.3.

\[
\begin{align*}
\langle \mathbf{u}_i, \mathcal{P}_A \mathbf{u}_j \rangle - \delta_{ij} & = |\mathbb{I}(i \in A)\alpha_1(d_i)| \\
\leq & n^{-1} \left( \frac{1}{\nu_i(A)} + \mathbb{I}(i \in A) \right) \left( \frac{1}{\nu_j(A)} + \mathbb{I}(j \in A) \right) \\
& + \phi_n \left[ \frac{(\Delta^2(d_i)}{\nu_i(A)} + 1 \right] \left( \frac{1}{\nu_j(A)} + \mathbb{I}(j \in A) \right) \right] \\
& + \mathbb{I}(i \in A, j \notin A) \psi_1(d_i) \Delta^2(d_i) / |d_j - d_i| + \mathbb{I}(i \notin A, j \in A) \psi_1(d_j) \Delta^2(d_j) / |d_j - d_i|.
\end{align*}
\]

Proposition D.1. Grant the assumptions and notations in Theorem 3.4. Then under Assumptions D.1 and D.2, we have that for all \(i, j = 1, \ldots, r\),

\[
(152)
\]

\[
\begin{align*}
\langle \mathbf{u}_i, \mathcal{P}_A \mathbf{u}_j \rangle - \delta_{ij} & = |\mathbb{I}(i \in A)\alpha_1(d_i)| \\
\leq & n^{-1} \left( \frac{1}{\nu_i(A)} + \mathbb{I}(i \in A) \right) \left( \frac{1}{\nu_j(A)} + \mathbb{I}(j \in A) \right) \\
& + \phi_n \left[ \frac{(\Delta^2(d_i)}{\nu_i(A)} + 1 \right] \left( \frac{1}{\nu_j(A)} + \mathbb{I}(j \in A) \right) \right] \\
& + \mathbb{I}(i \in A, j \notin A) \psi_1(d_i) \Delta^2(d_i) / |d_j - d_i| + \mathbb{I}(i \notin A, j \in A) \psi_1(d_j) \Delta^2(d_j) / |d_j - d_i|.
\end{align*}
\]

Proof. Consider the event space \(\Xi\) introduced in the proof of Theorem 3.3 in Section C where \(c > 0\) is a small positive constant satisfying \(0 < c < \min\{\tau^*, \tau\}/10\), and let \(\omega < \tau^*/2\). The rest of the proof is restricted to the event \(\Xi\).

For \(i \in A\), we define the contour \(\Gamma_i := \partial B_{\rho_i}(d_i)\), where

\[
(153)
\]

\[
\rho_i := c_i(\nu_i(A) \land (d_i - \alpha)) = c_i(\nu_i(A) \land \Delta^2(d_i))
\]

for some small constant \(0 < c_i < 1\). Define

\[
(154)
\]

\[
\Upsilon := \cup_{i \in A} B_{\rho_i}(d_i) \quad \text{and} \quad \Gamma := \cup_{i \in A} \Gamma_i.
\]

By choosing small enough \(c_i\), we can assume that \(\Upsilon \subset D_2(\tau_2, \varsigma)\), where \(D_2\) is defined in Lemma A.9. The following lemma shows that (i) \(\bar{\theta}(\Upsilon)\) is a subset of \(S_{out}(\omega)\) so that we can use the estimates of \(115\); (ii) \(\partial \theta(\Upsilon) = \theta(\Gamma)\) only encloses the outlier eigenvalues indexed by \(A\). The proof will be provided in Section D.3.

Lemma D.3. Suppose that Assumptions D.1 and D.2 hold. The set \(\bar{\theta}(\Upsilon)\) lies in the spectral set \(S_{out}(\omega)\) as long as the \(c_i\)'s are sufficiently small. Moreover, by selecting proper \(\omega\) and \(\epsilon\), we have \(\{\lambda_{a}\}_{a \in A} \subset \theta(\Upsilon)\) and all the other eigenvalues lie in the complement of \(\theta(\Upsilon)\).

For \(i = 1, \ldots, r\), let \(u_i^T = (u_i^T, 0)^T\) be the embedding of \(u_i\) in \(\mathbb{R}^{p+n}\). By (67), it is easy to see that

\[
(155)
\]

\[
\begin{align*}
\mathbf{u}_i^T \tilde{G}_1(z) \mathbf{u}_j & = (u_i^T)^T \tilde{G}(z) u_j^T.
\end{align*}
\]

By expanding \(\tilde{G}(z)\) by (68). Cauchy’s integral formula over \(\theta(\Gamma)\) and Lemma D.3 we have

\[
(156)
\]

\[
-\frac{1}{2\pi i} \oint_{\theta(\Gamma)} \langle \mathbf{u}_i, \tilde{G}_1(z) \mathbf{u}_j \rangle dz = \langle \mathbf{u}_i, \mathcal{P}_A \mathbf{u}_j \rangle.
\]

We first analyze the condition when \(1 \leq i, j \leq r\), and then extend the result to \(1 \leq i, j \leq p\) afterwards.

Assume \(1 \leq i, j \leq r\). By (155) and (156), since \((u_i^T)^T \tilde{G}(z) u_j^T = e_i^T U^T \tilde{G}(z) U e_j\), where \(e_j \in \mathbb{R}^{2r}\) is a unit vector with the \(i\)-th entry 1, by (71), we obtain that

\[
(157)
\]

\[
\langle \mathbf{u}_i, \mathcal{P}_A \mathbf{u}_j \rangle = \frac{1}{2\pi i d_i d_j} \oint_{\theta(\Gamma)} [(D^{-1} + U^T \tilde{G}(z) U)^{-1}]_{ij} dz.
\]
where $\tilde{i} := i + r$ and $\tilde{j} := j + r$. By the resolvent expansion from (160), we have

\begin{equation}
(u_i, \mathcal{P}_A u_j) = s_0^{ij} + s_1^{ij} + s_2^{ij},
\end{equation}

where $s_0^{ij}, \ell = 0, 1, 2$, are respectively defined as

\begin{align*}
s_0^{ij} &= \frac{1}{2\pi id_i d_j} \oint_{\theta(\Gamma)} [(\mathbf{1} - \bar{\Pi}(z))^{-1}]_{ij} \frac{dz}{z}, \\
s_1^{ij} &= \frac{1}{2\pi id_i d_j} \oint_{\theta(\Gamma)} [(\mathbf{1} - \bar{\Pi}(z))^{-1}(\mathbf{1} - \bar{\Pi}(z))^{-1}]_{ij} \frac{dz}{z}, \\
s_2^{ij} &= \frac{1}{2\pi id_i d_j} \oint_{\theta(\Gamma)} [(\mathbf{1} - \bar{\Pi}(z))^{-1}(\mathbf{1} - \bar{\Pi}(z))^{-1}(\mathbf{1} - \bar{\Pi}(z))^{-1}]_{ij} \frac{dz}{z}.
\end{align*}

From now on, we write $s^{ij}_\ell$ as $s_\ell$ to simplify the notation. By using the expansion in (121), we have

\begin{equation}
s_0 = \delta_{ij} \frac{1}{2\pi i d_i} \oint_{\theta(\Gamma)} \frac{m_{1c}(z)}{T(z) - d_i^2} dz = \delta_{ij} \frac{1}{2\pi i d_i} \oint_{\Gamma} \frac{\phi_1(\theta(\zeta))}{T(z) - \theta'(\zeta) d_i d_i(z)} d\zeta
\end{equation}

\begin{equation}
= -\delta_{ij} \frac{d_i m_{1c}(\theta(d_i)) \theta'(d_i)}{2} = \delta_{ij} \frac{m_{1c}(\theta(d_i))}{d_i^2 T'(\theta(d_i))} = \delta_{ij} a_1(d_i),
\end{equation}

where in the second equality we use the change of variable $z = \theta(\zeta)$, in the third equality of use the residual theorem, and in the fourth equality we simply use $\theta'(d_i) = 2d_i^{-3}(T^{-1})' (d_i^{-2}) = 2d_i^{-3} / T' (\theta(d_i))$. Similarly, for $s_1$ we can write it as

\begin{equation}
s_1 = \frac{1}{2\pi i d_i d_j} \oint_{\theta(\Gamma)} [(\mathbf{1} - \bar{\Pi}(z))^{-1}(\mathbf{1} - \bar{\Pi}(z))^{-1}]_{ij} \frac{dz}{z}
\end{equation}

\begin{equation}
= \frac{d_i d_j}{2\pi i} \oint_{\Gamma} \frac{f(\zeta) \theta'(\zeta) \zeta^4}{(\zeta^2 - d_i^2)(\zeta^2 - d_j^2)} d\zeta,
\end{equation}

where $f(\zeta) = f_1(\zeta) + f_2(\zeta)$ and

\begin{align*}
f_1(\zeta) := m_{1c}(\theta(\zeta)) \sigma(\theta(\zeta)) \sigma(\theta(\zeta) - \theta(d_i))_{ij} + (-1)^{i+j} \theta(\zeta)_{ij} m_{1c}(\theta(\zeta))_{ij}, \\
f_2(\zeta) := d_i^{-1} [(m_{1c}(\theta(\zeta))_{ij} + (1)^{i+j} \theta(\zeta)_{ij} m_{1c}(\theta(\zeta))_{ij} + (-1)^{i+j} \theta(\zeta)_{ij} m_{1c}(\theta(\zeta))_{ij}].
\end{align*}

To continue to bound $s_1$, we prepare a bound. Denote

\begin{equation}
f_{ij}(\zeta) = \frac{f(\zeta) \theta'(\zeta) \zeta^4}{(d_i + \zeta)(d_j + \zeta)}.
\end{equation}

We know $f(\zeta)$ is holomorphic inside the contour $\Gamma$ by the assumption on $\epsilon$ and $\omega$ and $\theta'(\zeta)$ is holomorphic as well by Lemma A.9. So, by Cauchy’s differentiation formula, we have

\begin{equation}
f_{ij}(\zeta) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f_{ij}(\xi)}{\zeta - \xi^2} d\xi,
\end{equation}

where $\mathcal{C}$ is the circle of radius $|\zeta - \alpha|/2$ centered at $\alpha$. For $\zeta = E + i\eta \in \Gamma$,

\begin{align*}
|f(\zeta) \theta'(\zeta) \zeta^4| \lesssim n^s (\phi_n + n^{-1/2} |\kappa_E + \eta|^{-1/4}) |\zeta - T(\lambda_+)| \\
\lesssim n^s (\phi_n + n^{-1/2} |\theta(\zeta) - \lambda_+|^{-1/4}) |\zeta - T(\lambda_+)| \\
\lesssim n^s (\phi_n |\zeta - T(\lambda_+)| + n^{-1/2} |\zeta - T(\lambda_+)|^{1/2})
\end{align*}

\begin{equation}
(163)
\end{equation}

where in the first inequality we use the bound of $|\Omega|$ provided in (115) and $\theta'(\zeta) \approx |\zeta - \alpha|$ given by (55), in the second inequality we use $(\kappa_E + \eta)|z = \theta(\zeta) \lesssim |\theta(\zeta) - \lambda_+|,$
and in the third inequality we use $|\theta(\zeta) - \lambda_+| \approx |\zeta - \alpha|^2$ given in \(55\). As a consequence, we conclude that

\[
|f'_{ij}(\zeta)| \leq Cn(\phi_n + n^{-1/2}|\zeta - \tau(\lambda_+)|^{-1/2}).
\]

Now, we consider three different cases. (i) Suppose that $i \in \mathbf{A}$ and $j \in \mathbf{A}$. If $d_i \neq d_j$, we have

\[
|s_1| \leq \left| \frac{d_i d_j}{2\pi i} \int_{\Gamma} \left( \frac{f_{ij}(\zeta)}{(\zeta - d_i)(\zeta - d_j)} d\zeta \right) \leq C \left| \int_{\Gamma} \frac{1}{d_i - d_j} \left\{ \frac{f_{ij}(\zeta)}{\zeta - d_i} - \frac{f_{ij}(\zeta)}{\zeta - d_j} \right\} d\zeta \right|
\]

\[
= C \left| \frac{f_{ij}(d_i) - f_{ij}(d_j)}{d_i - d_j} \right| \leq \frac{C}{|d_i - d_j|} \left| \int_{d_i}^{d_j} |f'_{ij}(\zeta)| d\zeta \right|
\]

\[
\leq Cn\left[ \phi_n + \frac{n^{-1/2}}{\Delta(d_i) + \Delta(d_j)} \right] \leq Cn\left[ \phi_n + \frac{n^{-1/2}}{\sqrt{\Delta(d_i)\Delta(d_j)}} \right]
\]

for some constant $C > 0$, where we use the arithmetic and geometric means in the last inequality. If $d_i = d_j$, then by the application of the residue’s theorem we get a similar bound

\[
|s_1| \leq \left| \frac{d_i^2}{2\pi i} \int_{\Gamma} \left( \frac{f_{ij}(\zeta)}{(\zeta - d_i)^2} d\zeta \right) \leq C \left| f'_{ij}(d_i) \right| \leq Cn\left[ \phi_n + \frac{n^{-1/2}}{\Delta(d_i)} \right].
\]

(ii) Suppose $i \in \mathbf{A}$ and $j \notin \mathbf{A}$. Then we get from \(163\) that

\[
|s_1| \leq \frac{C\left| f_{ij}(d_i) \right|}{|d_i - d_j|} \leq Cn^{\prime} \frac{n^{-1/2}\Delta(d_i) + \phi_n \Delta^2(d_i)}{|d_i - d_j|} = Cn^{\prime} \frac{\psi_1(d_i)\Delta^2(d_i)}{|d_i - d_j|}.
\]

We have a similar estimate if $i \notin \mathbf{A}$ and $j \in \mathbf{A}$. (iii) Finally, if $i \notin \mathbf{A}$ and $j \notin \mathbf{A}$, we have $s_1 = 0$ by Cauchy’s residue theorem since there is no poles inside the contour. We thus conclude the bound of $s_1$.

It remains to estimate the second order error $s_2$. Recall \(154\) that $\Gamma = \bigcup_{i \in \mathbf{A}} \Gamma_i$ We have the following basic estimates on each of these components, whose proof is given in Section \(D.4\).

**Lemma D.4.** For any $k \in \mathbf{A}$, $1 \leq \ell \leq r$ and $\zeta \in \Gamma_k$, we have

\[
|\zeta - d_\ell| \approx \rho_k + |d_k - d_\ell|.
\]

Now we finish the estimate of $s_2$. By a trivial bound, we have

\[
|s_2| \leq \frac{1}{2\pi i d_\ell} \int_{\partial(\Gamma)} \left| (\mathbf{D}^{-1} + \overline{\mathbf{P}}(z))^{-1}\Omega(z)(\mathbf{D}^{-1} + \overline{\mathbf{P}}(z))^{-1}\Omega(z)(\mathbf{D}^{-1} + \mathbf{U}^T G(z) \mathbf{U})^{-1} \right| |\zeta - d_\ell| dz.
\]

By the bound of $||\Omega||$ in \(115\), $(\kappa_E + n)_{z=\theta(\zeta)} \gtrsim |\theta(\zeta) - \lambda_+|$ and $|\theta(\zeta) - \lambda_+| \approx |\zeta - \alpha|^2$ given by \(55\), the entries of $(\mathbf{D}^{-1} + \overline{\mathbf{P}}(z))^{-1}$ in \(121\) together with \(44\), and a simple change of variable, \(169\) is bounded by

\[
C \int_{\Gamma} n^{2\varepsilon}(\phi_n^2 + n^{-1}|\zeta - \alpha|^{-1}) \frac{n^2(\phi_n^2 + n^{-1}|\zeta - \alpha|^{-1})}{|\zeta - d_\ell|} \times \left| (\mathbf{D}^{-1} + \mathbf{U}^T G(\theta(\zeta)) \mathbf{U})^{-1} \right| |\theta(\zeta)| d\zeta
\]

for some constant $C > 0$, which is further bounded by

\[
C \int_{\Gamma} n^{2\varepsilon}(\phi_n^2|\zeta - \alpha| + n^{-1}) \frac{n^2(\phi_n^2|\zeta - \alpha| + n^{-1})}{|\zeta - d_\ell|} \left| (\mathbf{D}^{-1} + \mathbf{U}^T G(\theta(\zeta)) \mathbf{U})^{-1} \right| d\zeta
\]
by using the fact that $|\theta'(\zeta)| \gtrsim |\zeta - \alpha|$ from (55). To continue to bound (171), we use the same method as that for (139). Assume $\zeta \in \Gamma_k$, we can bound $\|\Omega(\theta(\zeta))\|$ using (115) and obtain

$$(172) \quad \|\Omega(\theta(\zeta))\| \lesssim n^\epsilon \left[ \phi_n + n^{-1/2} \Delta(d_k)^{-1} \right].$$

By (54) and (55), we have for any $1 \leq \ell \leq r$,

$$|T(\theta(\zeta)) - d_\ell^-| = |\zeta - d_\ell^-| \gtrsim |\zeta - d| \geq |\zeta - d_k|$$

(173)

where the last inequality comes from Assumption (D.1). By (172) and (173), we have $|T(\theta(\zeta)) - d_\ell^-| \gg \|\Omega(\theta(\zeta))\|$ since we have assumed $\tau > \epsilon$. Hence, as we derive the bound in (139), by the resolvent expansion and (121), we have

$$(174) \quad \left\| (D^{-1} + U^T G(\theta(\zeta)) U)^{-1} \right\| \lesssim 1/|T(\theta(\zeta)) - d_\ell^-| \lesssim \frac{1}{\rho_k},$$

where the last bound comes from (173). Together with Assumptions (D.1) and (D.2) and the fact that $\Gamma_k$ has length $2\pi \rho_k$, we obtain

$$(175) \quad |s_2| \lesssim \sum_{k \in \Delta_k} \sup_{\zeta \in \Gamma_k} \frac{n^{-1+2\epsilon} + n^2 \phi_n^2 \Delta^2(d_k)}{|\zeta - d_i| |\zeta - d_j|} \lesssim \sum_{k \in \Delta_k} \frac{n^{-1+2\epsilon} + n^2 \phi_n^2 \Delta^2(d_k)}{(\rho_k + |d_k - d_i|)(\rho_k + |d_k - d_j|)},$$

where we apply Lemma (D.4) in the last inequality. Finally, we bound (175). First, by triangle inequality we have

$$\Delta^2(d_k) = |d_k - \alpha| \leq |d_i - \alpha| + |d_k - d_i| = \Delta^2(d_i) + |d_k - d_i|.$$

For $i \notin \Delta_k, k \in \Delta_k$, we have

$$\frac{1}{(\rho_k + |d_k - d_i|)} \leq \frac{1}{|d_k - d_i|} \leq \frac{1}{\nu_i(\Delta_k)}.$$

For $i, k \in \Delta_k$, by triangle inequality, we have

$$\rho_k + |d_k - d_i| \geq \rho_i.$$

Then we have

$$\frac{1}{(\rho_k + |d_k - d_i|)} \leq \frac{1}{\nu_i(\Delta_k)} \lesssim \frac{1}{\nu_i(\Delta_k)} + \frac{1}{\Delta(d_i)^2},$$

where the last inequality is by the definition of $\rho_i$. Plugging the above estimates into (175), we get that

$$|s_2| \lesssim n^{-1+2\epsilon} \left( \frac{1}{\nu_i(\Delta_k)} + \frac{1}{\Delta(d_i)^2} \right) \left( \frac{1}{\nu_j(\Delta_k)} + \frac{1}{\Delta(d_j)^2} \right) + n^2 \phi_n^2 \left[ \left( \frac{\Delta(d_i)^2}{\nu_i(\Delta_k)} + 1 \right) \left( \frac{1}{\nu_j(\Delta_k)} + \frac{1}{\Delta(d_j)^2} \right) \right] \left( \frac{\Delta(d_i)^2}{\nu_j(\Delta_k)} + 1 \right) \left( \frac{1}{\nu_i(\Delta_k)} + \frac{1}{\Delta(d_i)^2} \right) \left( \frac{1}{\nu_j(\Delta_k)} + \frac{1}{\Delta(d_j)^2} \right).$$

Combining (159) for the bound of $s_0$, (165) for the bound of $s_1$, and (167) for the bound of $s_2$, we obtain (162) for $1 \leq i, j \leq r$ since $\epsilon$ can be arbitrarily small.
Finally, we extend the above results to $1 \leq i, j \leq p$. Define $\mathcal{R} := \{1, \ldots, r\} \cup \{i, j\}$. Then we define a perturbed model with SVD as

$$S_\varepsilon := \sum_{i \in \mathcal{R}} \tilde{d}_i u_i v_i^T,$$

where $\varepsilon > 0$ and $\tilde{d}_k = d_k$ when $1 \leq k \leq r$ and $\tilde{d}_k = \varepsilon$ when $k > r$ and $k \in \mathcal{R}$. Then all the previous proof goes through for the perturbed model. Taking $\varepsilon \downarrow 0$ and using continuity, we get (152) for general $i, j \in \{1, \cdots, p\}$.

\[\square\]

Note that Proposition D.1 is essentially Theorem 3.4 with stronger assumptions. Thus, to finish the proof of Theorem 3.4, we need to remove Assumptions D.1 and D.2 and this is done in the following two subsections.

### D.2. Removing Assumption D.1

**Proof.** Recall the constants $\tau'$ in Assumption D.2 and $\bar{\tau}$ in Assumption D.1 and set $\bar{\tau} < \tau'/4$. Recall (91), we write an index set $\mathcal{O}_{\tau'/2} = \{i : d_i - \alpha \geq n^{-\eta/2}(\phi_n + n^{-1/3})\}$. We say that $a, b \in \mathcal{O}_{\tau'/2}$, $a \neq b$, overlap if

$$|d_a - d_b| \leq n^{-\eta} (\psi_1(d_a) \lor \psi_1(d_b)). \tag{177}$$

For $A$ satisfying Assumption D.2, we define sets $L_1(A), L_2(A) \subset \mathcal{O}_{\tau'/2}$, such that $L_1(A) \subset A \subset L_2(A)$. $L_1(A)$ is constructed by successively removing $k \in A$, such that $k$ overlaps with an index of $A^c$. This process is repeated until no such $k$ exists. In other words, $L_1(A)$ is the largest subset of $A$ that do not overlap with $L_1(A)^c$. On the other hand, $L_2(A)$ is constructed by successively adding $k \in \mathcal{O}_{\tau'/2} \setminus A$ into $A$, where $k$ overlaps with an index of $A$. This process is repeated until no such $k$ exits. In other words, $L_2(A)$ is the smallest subset of $\mathcal{O}_{\tau'/2}$ that do not overlap with $L_2(A)^c$. See Figure 5.2 in [152] for an illustration of construction of $L_1(A)$ and $L_2(A)$. It is easy to see that $L_1(A)$ and $L_2(A)$ exist and are unique. The main reason for defining these two sets is that now holds under Assumption D.2 with the parameter sets as ($\tau'/2, L_1(A)$) or ($\tau'/2, L_2(A)$). Now we are ready to prove (28). There are four cases to consider.

**Case (a):** $i, j \notin A$ and $i = j$. If $i \notin L_2(A)$, then using $r$ is bounded, we see that $\nu_i(A) \asymp \nu_i(L_2(A))$. Then using Proposition D.1 and the definition of $\psi_1$, we have

$$\langle u_i, \mathcal{P}_A u_i \rangle \leq \langle u_i, \mathcal{P}_{L_2(A)} u_i \rangle \tag{178}$$

$$\leq \frac{1}{\nu^2_i(L_2(A))} + \phi^2_n \frac{\Delta^2(d_i) + \nu_i(L_2(A))}{\nu_i^2(L_2(A))} \leq \frac{\psi^2_i(d_i) \Delta^2(d_i)}{\nu_i^2(L_2(A))} + \phi^2_n \nu_i(L_2(A)).$$

If $i \in L_2(A)$, which implies that $L_2(A) \setminus A \neq \emptyset$. Since $A$ overlaps with $L_2(A)$, this gives that

$$\nu_i(A) \leq \left[ \Delta(d_i) \right]^{-1} n^{-1/2} + \phi^2_n = n^{\bar{\tau}} \psi_1(d_i) \leq \nu_i(L_2(A)) \tag{179}
\leq \left[ \phi^2_n \right] \phi^2_n \nu_i(L_2(A)).}
Then Proposition [D.1] gives that
\[
|\langle u_i, \mathcal{P}_A u_i \rangle - a_1(d_i)| \leq m_{L_1}(\theta(d_i)) + \phi_n + \frac{1}{n^{1/2} \Delta(d_i)} + \frac{1}{n \nu_i^2(L_2(A))} \\
+ \frac{\phi_n^2 \Delta^2(d_i)}{\nu_i^2(L_2(A))} + \frac{\phi_n^2}{\Delta^2(d_i)} \Delta^2(d_i) \leq \frac{n^{2\gamma} \psi_i^2(d_i) \Delta^2(d_i)}{\nu_i^2(A)} ,
\]
(180)
where in the second step we used [50] such that \( 1/T'(\theta(d_i)) \approx (d_i - \alpha) = \Delta^2(d_i) \), and [179] such that \( \frac{1}{n \nu_i^2(L_2(A))} + \frac{\phi_n}{\Delta^2(d_i)} \leq \Delta^2(d_i) \), and [151] for the rest terms.
From [178] and [180], we conclude
\[
|\langle u_i, \mathcal{P}_A u_i \rangle - a_1(d_i)| \leq \frac{n^{2\gamma} \psi_i^2(d_i) \Delta^2(d_i)}{\nu_i^2(A)}, \quad i \notin A.
\]

**Case (b):** \( i, j \in A \) and \( i = j \). We first consider the case \( i \in L_1(A) \). We can write
\[
|\langle u_i, \mathcal{P}_A u_i \rangle - a_1(d_i)| = |\langle u_i, \mathcal{P}_{L_1(A)} u_i \rangle + \langle u_i, \mathcal{P}_{A \setminus L_1(A)} u_i \rangle|.
\]
Using [152] and the fact that \( \nu_i(A) \approx \nu_i(L_1(A)) \) (because \( i \) do not overlap with either \( A^c \) or \( L_1(A)^c \)), we can estimate the first term as
\[
|\langle u_i, \mathcal{P}_{L_1(A)} u_i \rangle - a_1(d_i)| \leq \psi_i(d_i) + \frac{1}{\nu_i^2(L_1(A))} \Delta^2(d_i) \left( \frac{1}{\nu_i^2(A)} + \frac{1}{\Delta^2(d_i)} \right) \psi_i(d_i) + \frac{1}{\nu_i^2(A)} \Delta^2(d_i) ,
\]
(183)
where we used that \( \nu_i(A) \leq \Delta^2(d_i) \) (by Assumption [D.2]) in the last step. For the second term in [182], it suffices to assume that \( A \setminus L_1(A) \neq \emptyset \) (otherwise it is equal to zero). Then we observe that \( \nu_i(A) \approx \nu_i(A \setminus L_1(A)) \). By [152], similar to [178] with \( A \) replaced by \( A \setminus L_1(A) \), we obtain that
\[
|\langle u_i, \mathcal{P}_{A \setminus L_1(A)} u_i \rangle - a_1(d_i)| \leq \psi_i(d_i) + \frac{1}{\nu_i^2(A)} \Delta^2(d_i) \left( \frac{1}{\nu_i^2(A)} + \frac{1}{\Delta^2(d_i)} \right) \psi_i(d_i) + \frac{1}{\nu_i^2(A)} \Delta^2(d_i) ,
\]
(184)
where the last step comes from the fact that \( i \) does not overlap with its complement since \( i \in L_1(A) \).

Next, for the case \( i \notin L_1(A) \), this implies \( A \setminus L_1(A) \neq \emptyset \), \( A \) overlaps with its complement, and thus \( L_2(A) \setminus A \neq \emptyset \). With these conditions, [179] holds by the same arguments, and with similar steps in deriving [180], we get
\[
|\langle u_i, \mathcal{P}_A u_i \rangle - a_1(d_i)| \leq |\langle u_i, \mathcal{P}_{L_2(A)} u_i \rangle + a_1(d_i) - \nu_i^2(A) \Delta^2(d_i) |.
\]
(185)
Combining [181] and [183]-[185], we conclude that
\[
|\langle u_i, \mathcal{P}_A u_i \rangle - \mathbb{1} \in A \rangle a_1(d_i)| \leq n^{2\gamma} R(i, A).
\]
This concludes [28] for the \( i = j \) case since \( \tau \) can be chosen arbitrarily small.

**Case (c):** \( i \neq j \) and \( i \notin A \) or \( j \notin A \). Using Cauchy-Schwarz inequality such that
\[
|\langle u_i, \mathcal{P}_A u_j \rangle|^2 \leq |\langle u_i, \mathcal{P}_A u_i \rangle| |\langle u_j, \mathcal{P}_A u_j \rangle|.
\]
(187)
and combine with (181) and (186), we find that in this case (152) holds since ##\overline{\tau}## can be chosen arbitrarily small.

**Case (d):** \( i \neq j \) and \( i,j \in A \). Our goal is to prove that

\[
|\langle u_i, P_A u_j \rangle| \sim n^{2\overline{\tau}} \left[ \psi_1^{1/2}(d_i) + \frac{\psi_1(d_j)\Delta(d_i)}{\nu_i(A)} \right] \left[ \psi_1^{1/2}(d_j) + \frac{\psi_1(d_j)\Delta(d_j)}{\nu_j(A)} \right].
\]

We again split \( P_A \) into

\[
\langle u_i, P_A u_j \rangle = \langle u_i, P_{L_1(A)} u_j \rangle + \langle u_i, P_{A \setminus L_1(A)} u_j \rangle.
\]

There are four cases: (i) \( i,j \in L_1(A) \); (ii) \( i \in L_1(A) \) and \( j \notin L_1(A) \); (iii) \( i \notin L_1(A) \) and \( j \in L_1(A) \); (iv) \( i,j \notin L_1(A) \). In case (i), we can bound the first term in (189) using Proposition [D.1] and the estimates that \( \nu_i(A) \approx \nu_i(L_1(A)) \) and \( \nu_j(A) \approx \nu_j(L_1(A)) \). The second term in (189) can be bounded as in case (c) above (with \( A \) replaced by \( A \setminus L_1(A) \)) together with the estimates \( \phi_n \leq \nu_i(A) \leq C\nu_i(A \setminus L_1(A)) \) and \( \phi_n \leq \nu_j(A) \leq C\nu_j(A \setminus L_1(A)) \). In case (ii), we have

\[
\nu_i(A) \approx \nu_i(L_1(A)) \approx \nu_i(A \setminus L_1(A)), \quad \nu_i(A) \leq C|d_i - d_j|,
\]

\[
\nu_j(A) \lesssim \nu_j(A \setminus L_1(A)) \lesssim n^{\overline{\tau}} \psi_1(d_j) \lesssim \nu_j(L_1(A)).
\]

Then with Proposition [D.1] we can bound the first term in (189) as

\[
|\langle u_i, P_{L_1(A)} u_j \rangle| \sim \frac{1}{n
u_i(L_1(A))\nu_j(L_1(A))} + \frac{1}{n\nu_j(L_1(A))\Delta^2(d_j)} + \frac{1}{\nu_i(L_1(A))}|d_i - d_j| \psi_1(d_i)\Delta^2(d_i) + \psi_1(d_i)\Delta^2(d_i)
\]

\[
+ \left[ \psi_1^{1/2}(d_i) + \frac{\psi_1(d_i)\Delta(d_i)}{\nu_i(A)} \right] \left[ \psi_1^{1/2}(d_j) + \frac{\psi_1(d_j)\Delta(d_j)}{\nu_j(A)} \right] + \frac{1}{\nu_j(L_1(A))} \psi_1(d_j)\Delta^2(d_j).
\]

For the last term, we first assume that \( d_j \leq d_i \) and \( d_i - \alpha \leq 2|d_i - d_j| \). Then

\[
\frac{\psi_1(d_i)\Delta^2(d_i)}{|d_i - d_j|} \leq 2\psi_1(d_i) \leq \sqrt{\psi_1(d_i)\psi_1(d_j)}.
\]

On the other hand, if \( d_j \geq d_i \) or \( d_i - \alpha \geq 2|d_i - d_j| \), we have \( \Delta(d_i) \lesssim \Delta(d_j) \). Hence using (190), we get

\[
\frac{\psi_1(d_i)\Delta^2(d_i)}{|d_i - d_j|} \lesssim n^{\overline{\tau}} \psi_1(d_i)\Delta(d_i)\psi_1(d_j)\Delta(d_j).
\]

The above estimates show that \(|\langle u_i, P_{L_1(A)} u_j \rangle|\) can be bounded by the right-hand side of (188). The second term in (189) can be bounded as in case (c) above (with \( A \) replaced by \( A \setminus L_1(A) \)) together with the estimates in (190) such that

\[
\nu_i(A) \approx \nu_i(A \setminus L_1(A)) \gtrsim n^{\overline{\tau}} \phi_n, \quad \nu_j(A) \lesssim \nu_j(A \setminus L_1(A)) \lesssim n^{\overline{\tau}} \psi_1(d_j).
\]
Then we get that
\[
\left| \langle \mathbf{u}_i, \mathcal{P}_{\mathbf{A} \setminus L_1(\mathbf{A})} \mathbf{u}_j \rangle \right| < n^{2\overline{\tau}} \left[ \phi_n \frac{\Delta^2(d_i)}{\nu_i(L_1(\mathbf{A}))} \right] \left[ \psi_1(d_i) \frac{\Delta(d_i)}{\nu_i(\mathbf{A} \setminus L_1(\mathbf{A}))} \right] \left[ \psi_1^{1/2}(d_j) + \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j(\mathbf{A} \setminus L_1(\mathbf{A}))} \right].
\]

This concludes the proof of (188) for case (ii). The case (iii) can be handled in the same way as case (ii) by interchanging \(i\) and \(j\). Finally, we deal with case (iv). Again we again split \(\mathcal{P}_{\mathbf{A}}\) as (189). For the first term in (189), we have
\[
\nu_i(\mathbf{A}) \lesssim \nu_i(L_1(\mathbf{A})), \quad \nu_i(L_1(\mathbf{A})) \gtrsim \psi_1(d_i),
\]
and similar estimates for the \(j\) case. Then using Proposition D.1, we can obtain that
\[
\left| \langle \mathbf{u}_i, \mathcal{P}_{L_1(\mathbf{A})} \mathbf{u}_j \rangle \right| < \frac{1}{n\nu_i(L_1(\mathbf{A}))\nu_j(L_1(\mathbf{A}))} + \frac{1}{\nu_i(L_1(\mathbf{A}))} + \frac{1}{\nu_j(L_1(\mathbf{A}))} \lesssim \frac{\psi_1(d_i) \psi_1(d_j) \Delta(d_i) \Delta(d_j)}{\nu_i(L_1(\mathbf{A}))} \left[ \frac{1}{\nu_i(L_1(\mathbf{A}))} + \frac{1}{\Delta^2(d_i)} \right] \left[ \frac{1}{\nu_j(L_1(\mathbf{A}))} + \frac{1}{\Delta^2(d_j)} \right]^{1/2}.
\]

For the second term in (189), we use the estimate
\[
\nu_i(\mathbf{A}) \lesssim \nu_i(\mathbf{A} \setminus L_1(\mathbf{A})) \lesssim n^{2\overline{\tau}} \psi_1(d_i)
\]
and the same discussion in case (b) to get that
\[
\langle \mathbf{u}_i, \mathcal{P}_{\mathbf{A} \setminus L_1(\mathbf{A})} \mathbf{u}_j \rangle < \Delta^2(d_i) + \psi_1(d_i) + n^{2\overline{\tau}} \left( \frac{\psi_1(d_i) \Delta^2(d_i)}{\nu_i(\mathbf{A} \setminus L_1(\mathbf{A}))} \right) \lesssim n^{2\overline{\tau}} \left( \frac{\psi_1(d_i) \Delta^2(d_i)}{\nu_i(\mathbf{A} \setminus L_1(\mathbf{A}))} \right).
\]

A similar estimate holds for \(\langle \mathbf{u}_j, \mathcal{P}_{\mathbf{A} \setminus L_1(\mathbf{A})} \mathbf{u}_j \rangle\). Then we conclude that
\[
\left| \langle \mathbf{u}_i, \mathcal{P}_{\mathbf{A}} \mathbf{u}_j \rangle \right| \leq \left| \langle \mathbf{u}_i, \mathcal{P}_{\mathbf{A} \setminus L_1(\mathbf{A})} \mathbf{u}_j \rangle \right|^{1/2} \left| \langle \mathbf{u}_j, \mathcal{P}_{\mathbf{A} \setminus L_1(\mathbf{A})} \mathbf{u}_j \rangle \right|^{1/2} < n^{2\overline{\tau}} \left[ \psi_1^{1/2}(d_i) + \frac{\psi_1(d_i) \Delta(d_i)}{\nu_i(\mathbf{A} \setminus L_1(\mathbf{A}))} \right] \left[ \psi_1^{1/2}(d_j) + \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j(\mathbf{A} \setminus L_1(\mathbf{A}))} \right].
\]

This proves (188) for case (iv), and hence concludes the proof for case (d).

Combining cases (c) and (d), we conclude (28) for the \(i \neq j\) case since \(\overline{\tau}\) can be chosen arbitrarily small. This finishes the proof of Theorem 3.4 under Assumption D.2 together with (186). \(\square\)
D.3. Removing Assumption D.2. By (iv) of Assumption D.2, for all $i \in A \subset \mathbb{O}_+$, we have

\[
\Delta^2(d_i) = d_i - \alpha \geq \phi_n + n^{-1/3}.
\]

Recall in Assumption D.2, we had a stronger condition that $d_i - \alpha \geq n^{\tau'}(\phi_n + n^{-1/3})$ for $0 < \tau' < 1/3$, and now we remove it.

Proof. The proof is devoted to showing that (28) holds for $A \subset \mathbb{O}^+$. Fix a small constant $0 < \epsilon < 1/3$. Note that the following gap property can be checked by contradiction; that is, there exists some $x_0 \in [1, r]$ so that for all $k$ such that $d_k > \alpha + x_0 n^{\epsilon} (\phi_n + n^{-1/3})$, we have $d_k \geq \alpha + (x_0 + 1)n^{\epsilon} (\phi_n + n^{-1/3})$. Following the idea in [13, Section 6.2], for such $x_0$, we split $A = S_0 \cup S_1$ such that $d_k \leq \alpha + x_0 n^{\epsilon} (\phi_n + n^{-1/3})$ for $k \in S_0$, and $d_k \geq \alpha + (x_0 + 1)n^{\epsilon} (\phi_n + n^{-1/3})$ for $k \in S_1$. Note that Assumption D.2 fit in $S_1$ by letting $\tau' = \epsilon$, thus Theorem 3.4 is valid when $A = S_1$ as we proved in Section D.2. Therefore, without loss of generality, we assume that $S_0 = \emptyset$. There are totally six cases: (a) $i, j \in S_0$; (b) $i \in S_0$ and $j \notin A$; (c) $i \notin S_0$ and $j \notin S_1$; (d) $i \notin S_0$ and $j \notin A$; (e) $i \in S_1$ and $j \notin A$; (f) $i, j \notin A$.

Case (a): $i, j \in S_0$. We have the splitting

\[
(\mathbf{u}_i, P_A \mathbf{u}_j) = (\mathbf{u}_i, P_{S_0} \mathbf{u}_j) + (\mathbf{u}_i, P_{S_1} \mathbf{u}_j).
\]

Applying Cauchy-Schwarz inequality as in [187] and [25] to the first term, and Theorem 3.4 to the second term, we get that

\[
|\langle \mathbf{u}_i, P_A \mathbf{u}_j \rangle - \delta_{ij} a_1(d_i)| 
\leq n^{3\epsilon}(\phi_n^3 + n^{-1}) \frac{\Delta^2(d_i) \Delta^2(d_j)}{\Delta^2(d_i) \Delta^2(d_j)} + \frac{\phi_n}{\nu_i^{1/2}(S_1)} + \frac{\psi_1(d_i) \Delta(d_i)}{\nu_i^{1/2}(S_1)} \left( \frac{\phi_n}{\nu_j^{1/2}(S_1)} + \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j^{1/2}(S_1)} \right)
\leq n^{4\epsilon} \psi_i^{1/2}(d_i) \psi_j^{1/2}(d_j),
\]

where in the first step we use $\eta(\gamma_i)/\sqrt{\kappa_i} \lesssim n^{-1} + \phi_n n^{-5/6} \lesssim n^{-1} + \phi_n^3$ for $i \in A$ derived by [61] and $\kappa_i \asymp n^{-2/3}$, and

\[
n^{3\epsilon}(\phi_n^3 + n^{-1}) \lesssim n^{3\epsilon}(\phi_n + n^{-1/3}) \lesssim n^{4\epsilon} \psi_i^{1/2}(d_i) \psi_j^{1/2}(d_j),
\]

and

\[
\frac{\phi_n}{\nu_i^{1/2}(S_1)} + \frac{\psi_1(d_i) \Delta(d_i)}{\nu_i^{1/2}(S_1)} \lesssim \frac{\psi_1(d_i)}{\nu_i^{1/2}(S_1)} \lesssim \phi_i^{1/2} + \frac{n^{-1/3}}{\Delta(d_i)} \lesssim (\phi_n + n^{-1/3})^{1/2} \lesssim \psi_i^{1/2}(d_i)
\]

in the second step, since for $d = d_i$ or $d = d_j$, we have $\phi_n + n^{-1/3} \leq \Delta^2(d) \lesssim n^{\epsilon} (\phi_n + n^{-1/3}) \lesssim \nu_i(S_1)$ and $\psi_1(d) = \phi_n + n^{-1/2}/\Delta(d) \gtrsim \phi_n + \phi_n^{-1/2} n^{-1/2} + n^{-1/3-\epsilon/2} \gtrsim n^{-\epsilon/2} (\phi_n + n^{-1/3})$.

Case (b): $i \in S_0$ and $j \in S_1$. Similar to the previous case, by applying Cauchy-Schwarz and Theorem 3.3 to the first term in (192), we get that

\[
|\langle \mathbf{u}_i, P_{S_0} \mathbf{u}_j \rangle| \leq n^{3\epsilon}(n^{-1} + \phi_n^3) \frac{\Delta^2(d_i) \Delta^2(d_j)}{\Delta^2(d_i) \Delta^2(d_j)} \lesssim n^{4\epsilon} \psi_i^{1/2}(d_i) \psi_j^{1/2}(d_j).
\]
For the second term, we first let Assumption [D.1] hold and apply Proposition [D.1]
and get that
\[
|\langle \mathbf{u}_i, \mathcal{P}_{S_i} \mathbf{u}_j \rangle| \lesssim \psi_1(d_i) \Delta^2(d_j) \frac{1}{|d_j - d_i|} + \psi_1^2(d_i) \Delta^2(d_i) \left( \frac{1}{\nu_i(S_1)} + \frac{1}{\Delta^2(d_i)} \right) \left( \frac{1}{\nu_j(S_1)} + \frac{1}{\Delta^2(d_j)} \right)
\]
(194)
\[
\lesssim \left[ \psi_1^{1/2}(d_i) + \frac{\psi_1(d_i) \Delta(d_i)}{\nu_i(A)} \right] \left[ \psi_1^{1/2}(d_j) + \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j(A)} \right],
\]
where we use
\[
\nu_i(S_1) \gtrsim \Delta(d_i), \quad \nu_j(S_1) \gtrsim \Delta(d_j) \wedge \nu_j(A), \quad \psi_1(d_j) \lesssim \psi_1(d_i),
\]
\[
|d_j - d_i| \gtrsim \Delta^2(d_j) \gtrsim \Delta^2(d_i), \quad \psi_1(d_j) \Delta(d_j) \gtrsim \psi_1(d_i) \Delta(d_i).
\]
This concludes the proof of case (b) if the non-overlapping condition hold. Otherwise, we can remove the non-overlapping condition as in Section [D.2].

**Cases (c), (e) and (f):** Note that in all cases \( j \notin A \), and we have \( \nu_j(A) \leq \nu_j(S_1) \). In case (c) with \( i \in S_0 \) and \( j \notin A \), we use the splitting in (192) and apply Cauchy-Schwarz, Theorem [3.3] and Proposition [D.1] to the first term to obtain that
\[
|\langle \mathbf{u}_i, \mathcal{P}_{S_0} \mathbf{u}_j \rangle| \lesssim n^\delta \psi_1^{1/2}(d_i) \left[ \frac{\phi_n}{\nu_j^{1/2}(A)} + \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j(A)} \right],
\]
(195)
and then we use Theorem [3.4] to the second term and obtain that
\[
|\langle \mathbf{u}_i, \mathcal{P}_{S_1} \mathbf{u}_j \rangle| \lesssim \psi_1^{1/2}(d_i) \left[ \frac{\phi_n}{\nu_j^{1/2}(A)} + \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j(A)} \right],
\]
(196)
where in the last step we also used the last inequality in solving case (a). In case (e) with \( i \in S_1 \) and \( j \notin A \), \( |\langle \mathbf{u}_i, \mathcal{P}_{S_0} \mathbf{u}_j \rangle| \) can be bounded in the same way as case (c). On the other hand, by Theorem [3.4] we have
\[
|\langle \mathbf{u}_i, \mathcal{P}_{S_1} \mathbf{u}_j \rangle| \lesssim \Delta(d_i) \left[ \frac{\phi_n}{\nu_j^{1/2}(S_1)} + \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j(S_1)} \right]
\]
\[
\phantom{\lesssim \Delta(d_i)} + \left[ \psi^{1/2}(d_i) + \frac{\psi_1(d_i) \Delta(d_i)}{\nu_i(S_1)} \right] \left[ \frac{\phi_n}{\nu_j^{1/2}(S_1)} + \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j(S_1)} \right]
\]
\[
\lesssim \Delta(d_i) \left[ \frac{\phi_n}{\nu_j^{1/2}(A)} + \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j(A)} \right]
\]
\[
\phantom{\lesssim \Delta(d_i)} + \left[ \psi^{1/2}(d_i) + \frac{\psi_1(d_i) \Delta(d_i)}{\nu_i(A)} \right] \left[ \frac{\phi_n}{\nu_j^{1/2}(A)} + \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j(A)} \right],
\]
where we used \( \nu_i(S_1) \gtrsim \Delta^2(d_i) \wedge \nu_i(A) \) in the second bound. In case (f) with \( i, j \notin A \), by Theorem 3.3 we obtain that
\[
|\langle u_i, P_{S_0} u_j \rangle| \lesssim n^{2\epsilon} n^{-\epsilon} \left( \Delta^2(d_i) + \phi_n + n^{-1/3} \right) \left( \Delta^2(d_j) + \phi_n + n^{-1/3} \right)
\]
where in the second bound we use the fact that for some \( \nu \)
\[
\Delta \left( \frac{\phi_n}{\nu_i^{1/2}(A)} \right) + \psi \left( \frac{\phi_n}{\nu_j^{1/2}(A)} \right)
\]

where we use \( \nu_i(A) \lesssim |d_i - \alpha| + |d_k - \alpha| \lesssim n^{\epsilon}(\Delta^2(d_i) + \phi_n + n^{-1/3}) \),
\[
\nu_j(A) \lesssim |d_j - \alpha| + |d_k - \alpha| \lesssim n^{\epsilon}(\Delta^2(d_j) + \phi_n + n^{-1/3})
\]
and
\[
\Delta(d_i) \psi_1(d_i) + \phi_n \nu_i(A) \lesssim \phi_n + \phi_n n^{-1/6} + n^{-1/2} + \phi_n \nu_i^{1/2}(A) \gtrsim (\phi_n^3 + n^{-1})^{1/2},
\]
\[
\Delta(d_j) \psi_1(d_j) + \phi_n \nu_j(A) \lesssim \phi_n + \phi_n n^{-1/6} + n^{-1/2} + \phi_n \nu_j^{1/2}(A) \gtrsim (\phi_n^3 + n^{-1})^{1/2},
\]
by \( \Delta(d_k^2) = d_k - \alpha \gtrsim \phi_n + n^{-1/3} \), for \( k = 1, \ldots, r \). For the \( P_{S_1} \) term, we have
\[
|\langle u_i, P_{S_1} u_j \rangle| \lesssim \left[ \frac{\phi_n}{\nu_i^{1/2}(S_1)} + \frac{\psi_1(d_i) \Delta(d_i)}{\nu_i(S_1)} \right] \left[ \frac{\phi_n}{\nu_j^{1/2}(S_1)} + \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j(S_1)} \right]
\]
where we use \( \nu_i(A) \leq \nu_i(S_1) \) and \( \nu_j(A) \leq \nu_j(S_1) \) in the second bound.

**Case (d):** \( i, j \in S_1 \). Again, using Theorem 3.3 and similar steps in case (a), we get that
\[
|\langle u_i, P_A u_j \rangle - \delta_{ij} a_1(d_i) | \lesssim n^{4\epsilon} \psi_1^{1/2}(d_i) \psi_1^{1/2}(d_j) + \left[ \frac{\psi_1(d_i) \Delta(d_i)}{\nu_i(S_1)} \right] \left[ \frac{\psi_1(d_j) \Delta(d_j)}{\nu_j(S_1)} \right]
\]

where we used \( \nu_i(S_1) \gtrsim \Delta^2(d_i) \wedge \nu_i(A) \) and \( \nu_j(S_1) \gtrsim \Delta^2(d_j) \wedge \nu_j(A) \) in the second step.

Combining all the above six cases, we conclude that even without Assumption D.2 the estimate (28) still holds with an additional factor \( n^{6\epsilon} \) multiplying on the right hand side. Since \( \epsilon \) can be arbitrarily small, we conclude the proof. \( \square \)

**D.4. Proof of Lemmas D.3 and D.4**

**Proof of Lemma D.3** Let \( \zeta \in \Gamma \). We first show that there exists \( \bar{c}_0 := \bar{c}_0(c_i) \), such that when \( c_i \) is sufficiently small, then \( \zeta \) satisfies (1) \( \text{Re} \zeta \geq \alpha \), (2) \( |\text{Im} \zeta| \leq \bar{c}_0(\text{Re} \zeta - \alpha) \) and (3) \( |\zeta| \leq C \), and then there exists a constant \( \bar{c}_1 := \bar{c}_1(\bar{c}_0, C) \) such that
\[
(197) \quad \text{Re} \theta(\zeta) \geq \lambda_\epsilon + \bar{c}_1(\text{Re} \zeta - \alpha)^2.
\]
By Assumption D.2 and the definition of \( \rho_i \), (1) and (3) are satisfied. For (2), because we have that for all \( \zeta \in \Gamma_i \),

\[
|\text{Im} \zeta| \leq \rho_i \leq c_i(d_i - \alpha)
\]

and

\[
\text{Re} \zeta - \alpha \geq d_i - \rho_i - \alpha \geq d_i - c_i(d_i - \alpha) - \alpha = (1 - c_i)(d_i - \alpha),
\]

which together lead to

\[
|\text{Im} \zeta| \leq \frac{c_i}{1 - c_i}(\text{Re} \zeta - \alpha).
\]

Thus (1), (2) and (3) are fulfilled. To show (197), by (1) and (3), we have

\[
0 \leq \text{Re} \zeta - \alpha \leq c_0
\]

for some constant \( c_0 > 0 \), then (197) follows from (55) that

\[
\text{Re} \theta(\zeta) - \lambda_+ \asymp \text{Re}(\zeta - \alpha)^2 \asymp (\text{Re} \zeta - \alpha)^2,
\]

where the second \( \asymp \) comes from (2) shown above. The claim (197), then follows by first choosing a sufficiently small constant \( c_0 \) and then choosing an appropriate constant \( c_1 \).

Now we can finish the proof of the first statement in the Lemma. By (55), we have \( |\theta(\zeta)| \leq \omega^{-1} \) for all \( \zeta \in \Gamma \) as long as \( \omega \) is sufficiently small. Also, using (197), we can find the lower bound of \( \text{Re} \theta(\zeta) \). Thus we can conclude that \( \theta(\Gamma_i) \subset S_{\text{out}}((\lambda_+ \omega)^{-1}, \omega) \) as long as \( c_i \) is sufficiently small, so as \( \theta(\Gamma) \).

To prove the second statement, it suffices to show that:

(i) \( \lambda_i \in \theta(\Gamma_i) \) for all \( i \in A \);

(ii) \( \lambda_j \notin \theta(\Gamma_i) \) for all \( j \notin A \) and \( i \in A \).

To prove (i), we notice that under Assumptions D.1

\[
(198) \quad \rho_i \geq c_i([\Delta(d_i)]^{-1} n^{-1/2} + \phi_n) n^{\tau}.
\]

Together with (51), by mean value theorem we get that

\[
|\theta(d_i + \rho_i) - \theta(d_i)| \geq (\Delta(d_i)n^{-1/2} + \phi_n \Delta^2(d_i)) n^{\tau}
\]

and

\[
|\theta(d_i - \rho_i) - \theta(d_i)| \geq (\Delta(d_i)n^{-1/2} + \phi_n \Delta^2(d_i)) n^{\tau}
\]

for \( i \in A \). Then we conclude (i) using (116). In order to prove (ii), we consider the two cases: (1) \( j \in \mathbb{O}^+ \setminus A \); (2) \( j \notin \mathbb{O}^+ \). In case (1), if \( d_j > d_i \), we have

\[
\lambda_j - \theta(d_i) > \theta(d_j - \rho_j) - \theta(d_i) \geq \theta'(d_j)(d_j - d_i - \rho_j) \geq (\Delta(d_j)n^{-1/2} + \Delta^2(d_j)\phi_n)n^{\tau},
\]

where we used that \( \theta \) is monotone in the first inequality, mean value theorem in the second inequality, and the definition of \( \rho_i \), (150), and (51) in the third inequality. Similarly, when \( d_j < d_i \), we have

\[
\theta(d_i) - \lambda_j \gtrsim (\Delta(d_j)n^{-1/2} + \Delta^2(d_j)\phi_n)n^{\tau}.
\]

In conclusion, we have

\[
|\lambda_j - \theta(d_i)| \gtrsim (\Delta(d_j)n^{-1/2} + \Delta^2(d_j)\phi_n)n^{\tau}.
\]

Together with (116) and that \( \epsilon < \tau \), case (1) is proved. For case (2), the claim follows from (118) and (55). This concludes the proof.
Proof of Lemma D.4.} The upper bound in (168) follows from the triangle inequality and the definition of $\rho_k$ in (153):

$$|\zeta - d_\ell| \leq \rho_k + |d_k - d_\ell|.$$ 

It remains to prove a lower bound. For $\ell \notin A$, again by the definition of $\rho_k$, we trivially have $|d_k - d_\ell| \geq 2\rho_k$, from which we obtain that

$$|\zeta - d_\ell| \geq |d_k - d_\ell| - \rho_k \geq \rho_k.$$

Next we consider the case $\ell \in A$. Define $\delta := |d_k - d_\ell| - \rho_k - \rho_\ell$, which is the distance between $B_{\rho_k}(d_k)$ and $B_{\rho_\ell}(d_\ell)$. First suppose that $C_0\delta > |d_k - d_\ell|$ for some constant $C_0 > 1$. It then follows that $\rho_k + \rho_\ell \leq \frac{C_0 - 1}{C_0} |d_k - d_\ell|$. As a consequence, we obtain

$$(200)$$

$$|\zeta - d_\ell| \geq |d_k - d_\ell| - \rho_k \geq \frac{1}{C_0} |d_k - d_\ell| + \rho_\ell \geq \frac{1}{C_0} |d_k - d_\ell| \geq \rho_k + |d_k - d_\ell|.$$ 

Suppose now that $C_0\delta \leq |d_k - d_\ell|$. Then we have

$$|d_k - d_\ell| \leq \frac{C_0}{C_0 - 1}(\rho_k + \rho_\ell).$$ 

We claim that for a sufficiently large constant $C_0 > 0$, there exists a constant $\tilde{C}(c_\ell, c_\ell, C_0) > 0$ such that

$$(200)$$

$$\tilde{C}^{-1}\rho_k \leq \rho_\ell \leq \tilde{C}\rho_k.$$ 

If (200) holds, then we have

$$|\zeta - d_\ell| \geq \rho_\ell \geq \rho_k + |d_k - d_\ell|.$$ 

This concludes (168).

It remains to prove (200). Recall the definition of $\rho_k$ in (153). Consider the following two cases. (i) If $\rho_\ell = c_\ell |d_\ell - d_s|$ for some $s$ such that $s \notin A$, we have

$$\rho_k = c_\ell |d_\ell - d_s| \leq |d_k - d_s| + |d_s - d_\ell| \leq \frac{\rho_k}{c_\ell} + \frac{c_\ell}{C_0 - 1}(\rho_k + \rho_\ell).$$

Thus as long as $c_\ell$ and $C_0$ is chosen such that $c_\ell^{-1} > \frac{c_\ell}{C_0 - 1}$, we obtain the upper bound in (200). (ii) If $\rho_\ell = c_\ell(d_\ell - \alpha)$, the proof is the same as that in case (i), and we omit details.

**Appendix E. Proof of Theorems 3.5 and 3.6**

We have the following remarks that guide us toward the proof.

**Remark E.1.** Suppose Assumptions 2.2 holds. Give any constant $0 < \epsilon \leq \epsilon$, we find that there exists an event $\Xi$ of high probability such that the followings hold when conditional on $\Xi$:

1. By Theorem 3.2 and (47), for $1 \leq i \leq r^+$,

$$|\tilde{\lambda}_i - \theta(d_i)| \leq n^\epsilon(\phi_{n}\Delta^2(d_i) + n^{-1/2}\Delta(d_i)).$$

and

$$|\tilde{\lambda}_i - \lambda_+| \leq \Delta(d_i)^4,$$

and for a fixed integer $\varpi > r$, when $r^+ + 1 \leq i \leq \varpi$ we have

$$|\tilde{\lambda}_i - \lambda_+| \leq n^\epsilon(n^{-2/3} + \phi_n^2)$$

We have the following remarks that guide us toward the proof.
(2) By Theorem 3.4, fix any $A \subset \mathbb{C}^+$, we have that for all $i, j = 1, \ldots, r$, 
\[
|\langle u_i, P_A u_j \rangle - \delta_{ij} 1(i \in A) d_1(d_i) + \langle v_i, P_A^* v_j \rangle - \delta_{ij} 1(i \in A) d_2(d_i) | \leq n^r(\phi_n + n^{-1/2}/\Delta(d_i)).
\]

(3) By Theorem 3.3, for $r_+ + 1 \leq i \leq r$, we have for $j = 1, \ldots, r$,
\[
|\langle u_j, \xi_i \rangle|^2 + |\langle v_j, \xi_i \rangle|^2 \leq n^r(\phi_n + n^{-1/3}).
\]

(4) By Lemmas A.11, there exists a large integer $\varpi$, such that for $1 \leq i \leq \varpi$, we have
\[
|\lambda_i - \lambda_+| \leq n^r(\phi_n^2 + n^{-2/3}),
\]
where $\varpi > r$ is a fixed integer.

(5) From Theorem A.13 and that $r$ is bounded, we have that for $z \in S_{out}(\xi_2, \epsilon)$
\[
|m_1(z) - m_{1c}(z)| + |m_2(z) - m_{2c}(z)| \leq n^r(\phi_n + n^{-1/2}(\kappa_2 + \eta)^{-1/4}).
\]

### E.1. Proof of Theorem 3.5

We restrict our discussion in the event space $\Xi$ in Remark E.1. The proof is then divided into several steps for clearance.

**Step 1: Bound $|m_1(\lambda_i) - \tilde{m}_{1,i}|$ and $|m_2(\lambda_i) - \tilde{m}_{2,i}|$:** As the bound derived in [22] page 30, if $F_A$ and $F_b$ are CDFs of probability measures supported on an interval $I$ and $g : I \to \mathbb{R}$ is bounded and continuously differentiable, with an integral by part we have
\[
\int g(t)(dF_A(t) - dF_b(t)) = \int g(t)(F_A(t) - F_b(t))|_I + \int g'(t)(F_A(t) - F_b(t))dt \leq \|g'||_{L^1(I)} d_{KS}(F_A, F_b)
\]

Fix $1 \leq i \leq r^+$. Note that $|\lambda_i - \lambda_+| \leq n^r(\phi_n^2 + n^{-2/3})$ and $|\lambda_i - \lambda_+| \leq n^r(\phi_n^2 + n^{-2/3})$ by Remark E.1 and the definition $\lambda_i$. Together with [203], we have $\lambda_i \lor \tilde{\lambda}_i < \lambda_i$.

Thus, if we take $F_a = F_{imp}$, $F_b = F_1$, $g(t) = 1/(t - \lambda_i)$ and $I = [0, \lambda_i \lor \tilde{\lambda}_i]$ in [209], we have
\[
|m_1(\lambda_i) - \tilde{m}_{1,i}| = |m_1(\lambda_i) - \tilde{m}_{1}(\lambda_i)| \lesssim \frac{1}{n|\lambda_i - \lambda_+ \lor \tilde{\lambda}_i|},
\]

where the last bound comes from Lemma 3.1. By the definition of $\lambda_i$ in [31] and [203], we have
\[
|m_1(\lambda_i) - \tilde{m}_{1,i}| \lesssim \frac{1}{n\Delta^4(d_i)}.
\]

By the same approach, we have
\[
|m_2(\lambda_i) - \tilde{m}_{2,i}| \lesssim \frac{1}{n\Delta^4(d_i)}.
\]

**Step 2: Bound $|\tilde{m}_{1,i} - m_{1c}(\lambda_i)|$ and $|\tilde{m}_{2,i} - m_{2c}(\lambda_i)|$:** By [203] and [208], we have that for $1 \leq i \leq r^+$,
\[
|m_1(\lambda_i) - m_{1c}(\lambda_i)| \leq n^r(\phi_n + n^{-1/2}/\Delta(d_i)),
\]
\[
|m_2(\lambda_i) - m_{2c}(\lambda_i)| \leq n^r(\phi_n + n^{-1/2}/\Delta(d_i)).
\]
By (211), (212) and (213), we conclude that for $1 \leq i \leq r^+$,
\begin{equation}
|\tilde{m}_{1,i} - m_{1c}(\tilde{\lambda}_i)| \leq n^t(\phi_n + n^{-1/2}/\Delta(d_i)), \quad |\tilde{m}_{2,i} - m_{2c}(\tilde{\lambda}_i)| \leq n^t(\phi_n + n^{-1/2}/\Delta(d_i)).
\end{equation}

**Step 3: Finish the claim:** With the above preparation and that $\tilde{\lambda}_i$, $m_{1c}(\tilde{\lambda}_i)$, $m_{2c}(\tilde{\lambda}_i)$, $\tilde{m}_{1,i}$, and $\tilde{m}_{2,i}$ are all $\approx 1$ by (44) and (214), we immediately have
\begin{equation}
|\tilde{T}_i - T(\tilde{\lambda}_i)| = |\tilde{\lambda}_i(\tilde{m}_{1,i}\tilde{m}_{2,i} - m_{1c}(\tilde{\lambda}_i)m_{2c}(\tilde{\lambda}_i))|
= |\tilde{\lambda}_i((\tilde{m}_{1,i} - m_{1c}(\tilde{\lambda}_i))\tilde{m}_{2,i} + m_{1c}(\tilde{\lambda}_i)(\tilde{m}_{2,i} - m_{2c}(\tilde{\lambda}_i)))|
\lesssim n^t(\phi_n + n^{-1/2}/\Delta(d_i)).
\end{equation}

Together with (44) we also have $\tilde{T}_i \approx T(\tilde{\lambda}_i) \approx 1$. Recall the definition of $\tilde{d}_i$ in (34). We have
\begin{equation}
|\tilde{d}_i - 1/\sqrt{T(\tilde{\lambda}_i)}| \approx \sqrt{T(\tilde{\lambda}_i)} - \sqrt{\tilde{T}_i} \approx |T(\tilde{\lambda}_i) - \tilde{T}_i| \lesssim n^t(\phi_n + n^{-1/2}/\Delta(d_i))
\end{equation}

Also, with (202), (51), and the mean value theorem, for some $d$ between $d_i$ and $1/\sqrt{T(\tilde{\lambda}_i)}$ we have that
\begin{equation}
|d_i - 1/\sqrt{T(\tilde{\lambda}_i)}| = |\theta(d_i) - \tilde{\lambda}_i|/\theta'(d) \lesssim n^t(\phi_n + n^{-1/2}/\Delta(d_i)).
\end{equation}

Thus, by combining (216) and (217), and the triangle inequality, we have
\begin{equation}
|\tilde{d}_i - d_i| \lesssim n^t(\phi_n + n^{-1/2}/\Delta(d_i)),
\end{equation}

which concludes the case of the operator norm.

For the case of the Frobenius norm, by triangle inequality, we can show that for $1 \leq i \leq r^+$
\begin{equation}
|a_{1,i} - \tilde{a}_{1,i}| \leq |a_{1,i} - a_1(d_i)| + |a_1(d_i) - \tilde{a}_{1,i}|.
\end{equation}

Now we need to bound the two terms on the right hand side. For the first term, by (205), we have
\begin{equation}
|a_{1,i} - a_1(d_i)| \leq n^t(\phi_n + n^{-1/2}/\Delta(d_i)).
\end{equation}

For the second term, again by triangle inequality, we have
\begin{equation}
|a_1(d_i) - \tilde{a}_{1,i}| = \left| \frac{m_{1c}(\theta(d_i))}{d_1^2 T'(\theta(d_i))} - \frac{\tilde{m}_{1,i}}{d_1^2 T'_i} \right| \leq \left| \frac{m_{1c}(\theta(d_i))}{d_1^2 T'(\theta(d_i))} - \frac{m_{1c}(\tilde{\lambda}_i)}{d_1^2 T'(\tilde{\lambda}_i)} \right| + \left| \frac{m_{1c}(\tilde{\lambda}_i)}{d_1^2 T'(\tilde{\lambda}_i)} - \frac{\tilde{m}_{1,i}}{d_1^2 T'_i} \right|.
\end{equation}

For some $s$ between $\theta(d_i)$ and $\tilde{\lambda}_i$, by mean value theorem and the order of $m_{1c}'$, $T'$ and $T''$ from (50), together with the bound of $|\theta(d_i) - \tilde{\lambda}_i|$ in (202), we have
\begin{equation}
\left| \frac{m_{1c}(\theta(d_i))}{d_1^2 T'(\theta(d_i))} - \frac{m_{1c}(\tilde{\lambda}_i)}{d_1^2 T'(\tilde{\lambda}_i)} \right| = \left| \frac{m_{1c}'(s) T'(s) - m_{1c}'(s) T''(s)}{d_1^2 (T'(s))^2} \right| |\theta(d_i) - \tilde{\lambda}_i| \lesssim n^t(\phi_n T^2(d_i) + n^{-1/2} \Delta(d_i)).
\end{equation}
Moreover, together with (31), we have that for 
\[ r^+ + 1 \leq i \leq \infty, \]  
by (204) and \( \epsilon \leq \varepsilon, \) we have
\begin{align*}
R_i - 1 - \frac{\bar{\lambda}_i - \bar{\lambda}_{i+1}}{\bar{\lambda}_{i+1}} &\leq n^{2\epsilon}(\phi_n^2 + n^{-2/3}) \\
R_{r^+} - 1 &\geq 1 - \mathbb{P}(R_{r^+} - 1 < \theta, \quad \bar{\lambda}_{r^+} - \bar{\lambda}_1 < \theta) \geq n^{2\epsilon}(\phi_n^2 + n^{-2/3}) \tag{230}
\end{align*}
Moreover, together with (31), we have that for 
\[ r^+ + 1 \leq i \leq \infty, \]  
for some constant \( C \)
\begin{align*}
\bar{\lambda}_i &< \bar{\lambda}_1 + C n^{2\epsilon}(\phi_n^2 + n^{-2/3}). 
\end{align*}
By (227) and (229), we have
\[ \mathbb{P}(r^+ = \tilde{r}^+) = \mathbb{P}(\{R_{r^+} - 1 > \theta, \quad \bar{\lambda}_{r^+} - \bar{\lambda}_1 > \theta\} \cap \bigcap_{r^+ + 1 \leq i \leq \infty} \{R_i - 1 \leq \theta, \quad \bar{\lambda}_i - \bar{\lambda}_1 \leq \theta\}) \]
\[ \geq 1 - \mathbb{P}(R_{r^+} - 1 < \theta, \quad \bar{\lambda}_{r^+} - \bar{\lambda}_1 < \theta) - \sum_{j=r^+ + 1}^{\infty} \mathbb{P}(R_{j} - 1 < \theta, \quad \bar{\lambda}_j - \bar{\lambda}_1 > \theta). \]
With the chosen \( \theta, \) we conclude our proof.
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