MPHS I: Differential modular forms, Elliptic curves and Ramanujan foliation

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Abstract

In this article we define the algebra of differential modular forms and we prove that it is generated by Eisenstein series of weight 2, 4 and 6. We define Hecke operators on them, find some analytic relations between these Eisenstein series and obtain them in a natural way as coefficients of a family of elliptic curves. Then we describe the relation between the dynamics of a foliation in $\mathbb{C}^3$ induced by the Ramanujan relations, with vanishing of elliptic integrals. The fact that a complex manifold over the Moduli of Polarized Hodge Structures in the case $h^{10} = h^{01} = 1$ has an algebraic structure with an action of an algebraic group plays a basic role in all of the proofs.

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1 Introduction

In [14] Griffiths introduced the Moduli of Polarized Hodge Structures/the period domain $D$ and described a dream to enlarge $D$ to a moduli space of degenerating polarized Hodge structures. Since in general $D$ is not a Hermitian symmetric domain, he asked for the existence of a certain automorphic cohomology theory for $D$, generalizing the usual notion of automorphic forms on symmetric Hermitian domains. Since then there have been many...
efforts in the first part of Griffiths’s dream (see \cite{17} and references there) but the second part still lives in darkness.

I was looking for some analytic spaces over $D$ for which one may state Baily-Borel theorem on the unique algebraic structure of quotients of symmetric Hermitian domains by discrete arithmetic groups (see \cite{22,3}). I realized that even in the simplest case of Hodge structures, namely $h^{01}=h^{10}=1$, such spaces are not well studied. This led me to the definition of a new class of holomorphic functions on the Poincaré upper half plane which generalize the classical modular forms. Since a differential operator acts on them we call them differential modular forms. These new functions are no longer interpreted as holomorphic sections of a positive line bundle on some compactified moduli curve. Nevertheless, they appear in a natural way as coefficients in families of elliptic curves, analogous to Eisenstein series in the Weierstrass Uniformization Theorem. Due to this, I realized that there are very natural holomorphic foliations in the coefficient space of families of varieties whose dynamics is in close relation with the abelian integrals (resp. Hodge structures) of the family. In the case of a three parametric elliptic curve, the mentioned foliation is called Ramanujan foliation because he was the first who obtained a particular leaf of this foliation using Eisenstein series (see \cite{20} Chapter X and \cite{8} bellow). This article will touch some aspects of number theory, holomorphic foliations and Hodge theory which I am going to explain below:

**Number theory:** Recall the Eisenstein series

\begin{equation}
\tag{1}
g_k(z) = a_k \left( 1 + (-1)^k \frac{4k}{B_k} \sum_{n \geq 1} \sigma_{2k-1}(n) e^{2\pi i zn} \right), \quad k = 1, 2, 3, \quad z \in \mathbb{H}
\end{equation}

where $B_k$ is the $k$-th Bernoulli number ($B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, \ldots$), $\sigma_k(n) := \sum_{d \mid n} d^k$,

\begin{equation}
\tag{2}
a_1 = 2 \zeta(2) \frac{-1}{2\pi i}, \quad a_2 = 2 \zeta(4) \frac{60}{(2\pi i)^2}, \quad a_3 = 2 \zeta(6) \frac{-140}{(2\pi i)^3}
\end{equation}

and $\mathbb{H} := \{x + iy \in \mathbb{C} \mid y > 0\}$ is the Poincaré upper half plane. The most well-known differential modular form, which is not a differential of a modular form, is the Eisenstein series $g_1$. The idea of differentiating modular forms and getting new modular forms is old and goes back to Ramanujan. Nesterenko’s method (see \cite{28}) for the proof of transcendency properties of certain numbers is also based on differential equations satisfied by modular forms. However, the precise definition of differential modular forms has been given recently in \cite{7}. In the present article we give another slightly different definition of differential modular forms (see \cite{22,4} over a modular subgroup $\Gamma \subset \text{SL}(2,\mathbb{Z})$). It is based on a canonical behavior of holomorphic functions on the Poincaré upper half plane under the action of $\text{SL}(2,\mathbb{Z})$. This approach has the advantage that it can be generalized to any modular subgroup of $\text{SL}(2,\mathbb{Z})$ but the one in \cite{7} works only in the case of full modular group $\text{SL}(2,\mathbb{Z})$. The set of differential modular forms in the present article is a bigraded algebra $M = \bigoplus_{n \in \mathbb{N}_0, m \in \mathbb{N}} M^m_n$, $M^0_m$ being the set of classical modular forms of weight $m$, in which the differential operator $\frac{d}{dz}$ maps $M^m_n$ to $M^m_{n+1}$. We have $g_1 \in M^1_2$, $g_2 \in M^0_4$, $g_3 \in M^0_6$ and we prove:

**Theorem 1.** The functions $g_1, g_2, g_3$ are algebraically independent and $M$ is freely generated by $g_1, g_2$ and $g_3$ as a $\mathbb{C}$-algebra. In particular each $M^m_n$ is a finite dimensional $\mathbb{C}$-vector space.
This theorem generalizes the first theorem in each modular forms book that the algebra of modular forms is generated freely by the Eisenstein series \( g_2 \) and \( g_3 \). Our proof gives us also the Ramanujan relations between the \( g_i \)'s. We define the action of Hecke operators on \( M_m^n \) and it turns out that this is similar to the case of modular forms:

\[
T_p f(z) = p^{m-n-1} \sum_{d\mid p, 0 \leq b \leq d-1} d^{-m} f \left( \frac{nz + bd}{d} \right), \quad p \in \mathbb{N}, \ f \in M_m^n.
\]

Hecke operators of this type appear in particular in the study of the transfer operator from statistical mechanics which plays an important role in the theory of dynamical zeta functions (see [16]). The differential operator commutes with Hecke operators (§2.2) and so it induces a map from the set of new differential forms (see §2.7) to itself. Another result which we prove in this article and which could be interesting from the number theory point of view is the following: Let

\[
g := (g_1, g_2, g_3) : \mathbb{H} \to \mathbb{C}^3
\]

and

\[
T := \mathbb{C}^3 \setminus \{(t_1, t_2, t_3) \in \mathbb{C}^3 \mid 27t_3^2 - t_2^3 = 0\}.
\]

**Theorem 2.** There are unique analytic functions

\[
B_1, B_2 : T \to \mathbb{R}, \ B_3 : T \to \mathbb{C}
\]

such that \( B_1 \) does not depend on the variable \( t_1 \) and

\[
B_1 \circ g(z) = \text{Im}(z), \ B_1(t_1, t_2 k^{-4}, t_3 k^{-6}) = B_1(t)|k|^2
\]

\[
B_2 \circ g = 0, \ B_2(t_1 k^{-2} + k' k^{-1}, t_2 k^{-4}, t_3 k^{-6}) = B_1(t)|k'|^2 + B_2(t)|k^{-1}|^2 + \text{Im}(B_3(t)k'k^{-1})
\]

\[
B_3 \circ g = 1, \ B_3(t_1 k^{-2} + k' k^{-1}, t_2 k^{-4}, t_3 k^{-6}) = B_3(t)kk'^{-1} + 2\sqrt{-1}kk'B_1(t)
\]

for all \( k \in \mathbb{C}^* \) and \( k' \in \mathbb{C} \). Moreover, \( |B_3| \) restricted to the zero locus of \( B_2 \) is identically one.

**Holomorphic singular foliations:** Nowadays the theory of holomorphic singular foliations is getting a part of Algebraic Geometry. For a general background in this theory see ([11], [10], [6], [25], [9]). In this direction this article has two novelties which together exhibit a connection between the area of holomorphic foliations and Arithmetic Algebraic Geometry. The first one is as follows: After calculating the Gauss-Manin connection of the following family of elliptic curves

\[
E_t : y^2 - 4t_0(x - t_1)^3 + t_2(x - t_1) + t_3, \ t \in \mathbb{C}^4
\]

and considering its relation with the inverse of the period map, we get the following ordinary differential equation:

\[
\text{Ra} : \begin{cases} 
  \dot{t}_1 = t_1^2 - \frac{1}{12}t_2 \\
  \dot{t}_2 = 4t_1t_2 - 6t_3 \\
  \dot{t}_3 = 6t_1t_3 - \frac{1}{3}t_2^2 
\end{cases}
\]
which is called the Ramanujan relations/differential equation/foliation, because he had observed that $g$ is a solution of (8) (one gets the classical relations by changing the coordinates $(t_1, t_2, t_3) \mapsto (\frac{1}{12}t_1, \frac{1}{12}t_2, \frac{1}{12}t_3)$). We denote by $\mathcal{F}(Ra)$ the singular holomorphic foliation induced by (8) in $\mathbb{C}^3$. Its singularities

$$\text{Sing}(Ra) = \{(t_1, 12t_1^2, 8t_1^3) \mid t_1 \in \mathbb{C}\}$$

form a one-dimensional curve in $\mathbb{C}^3$. Consider the family (7) with $t_0 = 1$ and define

$$K := \left\{ t \in T \mid \int_\delta \frac{xdx}{y} = 0, \text{ for some } \delta \in H_1(E_t, \mathbb{Z}) \right\}$$

and

$$M_r := \{ t \in T \mid B_2(t) = r \}, \ r \in \mathbb{R}.$$ 

The last part of Theorem 2 says that $|B_3|$ restricted to $M_0$ is identically 1. We also define

$$N_w := \{ t \in M_0 \mid B_3(t) = w \}, \ |w| = 1, \ w \in \mathbb{C}.$$ 

For $t \in \mathbb{C}^3 \setminus \text{Sing}(\mathcal{F}(Ra))$ we denote by $L_t$ the leaf of $\mathcal{F}(Ra)$ through $t$.

**Theorem 3.** The following is true:

1. The real analytic varieties $M_r, r \in \mathbb{R}, N_w, |w| = 1$, and the set $K$ are $\mathcal{F}(Ra)$-invariant.

2. The set $K$ is a dense subset of $M_0$ with the following property: For all $t \in K$ the leaf $L_t$ intersects $\text{Sing}(Ra)$ transversally at some point $p$.

3. For all $t \in T$ the leaf $L_t$ has an accumulation point at $T$ if and only if $t \in M_0$.

The item 2 says that there is a transverse disk to $\text{Sing}(\mathcal{F}(Ra))$ at some point $p$ such that $D \setminus \{ p \}$ is a part of the leaf $L_t$. For the proof of the above theorem and a precise description of the foliation $\mathcal{F}(Ra)$ see [5]. The proof of this theorem is based on the fact that the foliation $\mathcal{F}(Ra)$ restricted to $T$ is uniformized by the inverse of the period map (see for instance [21] for similar topics).

The second novelty in the area of holomorphic foliations is as follows:

**Theorem 4.** There is no elliptic curve $E$ and a differential form of the second type $\omega$ on $E$, both defined over $\mathbb{Q}$, such that

$$\int_\delta \omega = 0$$

for some non-zero topological cycle $\delta \in H_1(E, \mathbb{Z})$.

This theorem uses Nesterenko’s Theorem (see [28]) on transcendence properties of the values of Eisenstein series. The above theorem for the case in which $\omega$ is of the first kind, is well-known. In this case we can even state it for the field $\mathbb{C}$. However, it is trivially false when $\omega$ is a differential form of the second kind and we allow transcendental coefficients in $\omega$ or the elliptic curve. Generalizations of such theorems to arbitrary curves can be derived from the Abelian Subvariety Theorem (see [5], [27]). Abelian integrals of the type (9) appear in deformations of holomorphic foliations with a first integral in complex
surfaces (see [12, 25]). The above result shows that the zeros of arithmetic abelian integrals are quite different from complex ones.

**Hodge theory:** This article stimulates the hope to realize the second part of Griffiths' dream with a different formulation. Differential modular forms can also be introduced for a complex manifold $\mathcal{P}$ over Griffiths period domain $D$ with an action of an algebraic group $G_0$ from the right. Since they are no longer interpreted as sections of positive line bundles over moduli spaces, the question of the existence of a kind of Baily-Borel Theorem for $\mathcal{P}$ arises. In the case of Hodge structures with $h^{01} = h^{10} = 1$ we have

$$P := \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \text{GL}(2, \mathbb{C}) \mid \text{Im}(x_1 x_3) > 0 \right\}, \quad D = \mathbb{H}$$

and we show in this article that $\text{SL}(2, \mathbb{Z})\backslash \mathcal{P}$ has a canonical structure of an algebraic quasi-affine variety such that the action of $G_0$ from the right is algebraic. More precisely, we prove that $\text{SL}(2, \mathbb{Z})\backslash \mathcal{P}$ is biholomorphic to $\mathbb{C}^4 \{ t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4 \mid 27t_0^2 t_3^2 - t_2^3 = 0 \}$ and under this biholomorphism the action of $G_0$ is given by:

$$t \cdot g := (t_0 k_1^{-1} k_2^{-1} t_1 k_1^{-1} k_2 + k_3 k_1^{-1} t_2 k_1^{-3} k_2, t_3 k_1^{-4} k_2)$$

$$t = (t_0, t_1, t_2, t_3) \in \mathbb{C}^4, g = \begin{pmatrix} k_1 & k_3 \\ 0 & k_2 \end{pmatrix} \in G_0$$

The mentioned biholomorphism is given by the period map (see §4).

**Singularity theory:** The Brieskorn module/lattice and its Gauss-Manin connection in singularity theory play the role of de Rham cohomology of fibered varieties, and it is a useful object when one wants to calculate the Gauss-Manin connection. In §3 we introduce the associated Brieskorn $\mathbb{C}[t, \frac{1}{t_0}]$-module $H$ of the family (7). We prove that $H$ is freely generated and we calculate the Gauss-Manin connection in a canonical basis of $H$. We generalize a classical Weierstrass Theorem and see that for the inverse of the period map, $t_i$ appears as the Eisenstein series $g_i$, $i = 1, 2, 3$. The novelty is the appearance of $t_1$ as the Eisenstein series of weight 2. This makes us think about a generalization of K. Saito’s primitive form theory (see [30]), to the case in which the deformation of a singularity is bigger than the versal deformation.

**Final note:** An elliptic curve (beside $\mathbb{P}^1$) is one of the most well-studied objects in (Arithmetic) Algebraic Geometry. This makes me feel that some parts of this work has connections to the works of many authors that I have not mentioned here. Here I express my sincere apologies. My aim of writing this article was just to show, by some simple examples, how the dynamics of foliations and algebraic geometry of fibered varieties can be connected. One of the points in this article which I enjoyed very much, was the calculation of the Gauss-Manin connection and the Ramanujan relations by **Singular**, [13]. The corresponding algorithms were developed in the articles [24, 26] and the corresponding library in **Singular** is called **brho.lib**. Using this library for families of algebraic varieties one can obtain certain holomorphic foliations, whose dynamics has to do with the Hodge structure of Algebraic varieties and in particular with abelian integrals. This opens a
new connection between (Arithmetic) Algebraic Geometry and the theory of holomorphic foliations.

Let us now explain the structure of this article. §2 is devoted to the definition of differential modular forms and the action of Hecke operators on them. This section is independent of the other sections. We shall only use the property (19) of the Eisenstein series $g_1$. The reader who is interested only in §2 may lose our proof for the main theorem of this section (Theorem 1). §3 is devoted to calculation of the Gauss-Manin connection of the family (7). This section is based on the machinery introduced in [24]. The heart of the present paper is §4 in which we prove that the period map $\text{pm}$ is a biholomorphism and then we take the inverse of $\text{dpm}$ to obtain the Ramanujan relations. In §5 we have described the dynamics of the Ramanujan foliation and in particular its uniformization by the inverse of the period map in some quasi-affine variety. Finally, §6 is devoted to the proofs announced in the Introduction.

Acknowledgement: The main ideas of this paper took place in my mind when I was visiting Prof. Sampei Usui at Osaka University. Here I would like to thank him for encouraging me to study Hodge theory and for his help to understand it. I would like to thank Prof. Karl-Hermann Neeb for his interest and careful reading of the present article.

2 $M_m^n$-functions

In this section we use the notations $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ and

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad g = \begin{pmatrix} k_1 & k_2 \\ 0 & k_3 \end{pmatrix}, \quad x, g \in \text{GL}(2, \mathbb{C}).$$

When there is no confusion we will simply write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We denote by $\mathbb{H}$ the Poincaré upper half plane and

$$j(A, z) := cA z + dA$$

For $A \in \text{SL}(2, \mathbb{R})$ and $m \in \mathbb{Z}$ we use the slash operator

$$f|_m A = (\det A)^{m-1} j(A, z)^{-m} f(Az)$$

For a ring $R$ we denote by $\text{Mat}_p(2, R)$ the set of $2 \times 2$-matrices in $R$ with the determinant $p$.

2.1 Definitions

In this section we define the notion of an $M_m^n$-function. For $n = 0$ an $M_0^0$-function is a classical modular form of weight $m$ on $\mathbb{H}$ (see below). For arbitrary $n$ we define it by induction: A holomorphic function $f$ on $\mathbb{H}$ is called $M_m^n$ if the following three conditions are satisfied:

1.

$$f|_m A - f = \sum_{i=1}^{n} \binom{n}{i} e^i (A, z)^{-i} f_i, \quad \forall A \in \text{SL}(2, \mathbb{Z})$$

where $f_i$ is an $M_{m-2i}^{n-i}$-function.
2. By induction we can write
\[
f_i|_{m-2;\ A} - f_i = \sum_{j=1}^{n-i} \binom{n-i}{j} c_{A}^{j}(A, z)^{-j} f_{ij}
\]
where \( f_{ij} \) is an \( M_{m-2i-2j}^{n-i-j} \)-function. We assume that \( f_{ij} = f_{i-1,j+1} = \ldots = f_{1,i+j-1} = f_{i+j} \).

3. \( f \) has finite growth when \( \text{Im}(z) \) tends to \( +\infty \), i.e.
\[
\lim_{\text{Im}(z) \to +\infty} f(z) = a_{\infty} < \infty, \quad a_{\infty} \in \mathbb{C}
\]
The above definition can be made using a subgroup \( \Gamma \subset \text{SL}(2, \mathbb{Z}) \). In this article we mainly deal with full differential modular forms, i.e. the case \( \Gamma = \text{SL}(2, \mathbb{Z}) \). We will also denote by \( M_{m}^{n} \) the set of \( M_{m}^{n} \)-functions and we set
\[
M := \oplus_{n,m \in \mathbb{Z}, n \geq 0} M_{m}^{n}
\]
A classical modular form satisfies 1 and 2, where instead of the right hand side of (13) we write 0. Note that for an \( M_{m}^{n} \)-function \( f \) the associated functions \( f_{i} \) are unique (consider the right hand side of (13) as a polynomial in \( c_{A}^{j}(A, z)^{-1} \) with coefficients \( \binom{n-i}{j} f_{i} \)). The second condition guarantees that the consequent use of (13) for two matrices \( A, B \in \text{SL}(2, \mathbb{Z}) \) leads to the same result. To see this, it is useful to define
\[
f_{i}||_{m} := (\det A)^{m-n-1} \sum_{i=0}^{n-i} \binom{n-i}{j} c_{A}^{j}(A, z)^{j-i} f_{i}(Az), \quad A \in \text{GL}(2, \mathbb{R}), \quad f \in M_{m}^{n}
\]
The factor \( \det A \) is introduced because of Hecke operators (see § 2.3). This is not an action of \( \text{GL}(2, \mathbb{R}) \) on \( M_{m}^{n} \) from the right. The equalities (13) and (14) are written in the form
\[
f_{i} = f_{i}||_{m-2;\ A} :=
\]
(\det A)^{m-n-i-1} \sum_{j=0}^{n-i} \binom{n-i}{j} c_{A}^{j}(A, z)^{j+2i-m} f_{i+j}(Az), \quad i = 0, 1, 2, \ldots, n, \quad f_{0} := f
\]
for all \( A \in \text{SL}(2, \mathbb{Z}) \) (We have substituted \( A^{-1}z \) for \( z \) and then \( A^{-1} \) for \( A \)).

**Lemma 1.** We have
\[
f||_{m} = f||_{m}(BA), \quad \forall A \in \text{GL}(2, \mathbb{R}), \quad B \in \text{SL}(2, \mathbb{Z})
\]
Proof. The term \((\det A)^{n-m+1}f||_mA(z)\) is equal to:

\[
\begin{align*}
&= \sum_{i=0}^{n} \binom{n}{i} c^i_A \cdot j(A, z)^{i-m} f_i(B^{-1}BAz) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} c^i_A c^j_B \cdot j(j(A, z)^{i-m} j(B^{-1}, BAz)^{m-2i-j} f_{i+j}(BAz) \\
&= \sum_{r=0}^{n} \sum_{s=0}^{r} \binom{n}{r} \binom{n-s}{r-s} c^r_A c^s_B \cdot j(A, z)^{s-m} j(B^{-1}, BAz)^{m-r-s} f_r(BAz) \\
&= \sum_{r=0}^{n} \binom{n}{r} j(BAz, z)^{r-m} f_r(BAz) j(A, z)^{-r} \left( \sum_{s=0}^{r} \binom{r}{s} j(BA, z)^{s} c_A^{-1} c_B^{r-s} \right) \\
&= \sum_{r=0}^{n} \binom{n}{r} j(BA, z)^{r-m} c_B^{r-1} f_r(BAz) = (\det A)^{n-m+1}(f||_mA)(z) \\
\end{align*}
\]

In the second line we have used \((\text{10})\). In the third line we have have changed the counting parameters: \(i + j = r\), \(i = s\), \(0 \leq s \leq r\). In the other lines we have used

\[ j(AB, z) = j(A, Bz)j(B, z) \]

and

\[ j(BA, z)c_A + \det(A)c_B = c_BA\ j(A, z)\quad \forall A, B \in \operatorname{GL}(2, \mathbb{R}) \]

Since \(f||_mT = f\) we can write the Fourier expansion of \(f\) at infinity

\[ f = \sum_{n=-N}^{+\infty} a_n q^n, \quad a_n \in \mathbb{C}, \quad N = 0, 1, 2, \ldots, \infty, \quad q = e^{2\pi iz}. \]

The third condition on \(f\) implies that \(N = 0\).

### 2.2 Algebra of \(M_m^n\)-functions

In this section we use for \(A \in \operatorname{SL}(2, \mathbb{Z})\) the simplification \(c = c_A\). Recall the Eisenstein series \((\text{11})\) and

\[
\begin{align*}
\Delta(z) &:= (27 g_2^3(z) - g_3^3(z)) = -\left(\frac{2\pi i}{12}\right)^3 q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 + \cdots \\
j(z) &:= \frac{g_2^3(z)}{-\Delta(z)} = q^{-1} + 744 + 196884q + \cdots
\end{align*}
\]

Note that \(\zeta(2) = \frac{\pi^2}{6}\), \(\zeta(4) = \frac{\pi^4}{90}\), \(\zeta(6) = \frac{\pi^6}{945}\) and so

\[
\begin{align*}
p_\infty &:=(a_1, a_2, a_3) = \left(\frac{2\pi i}{12}, 12\left(\frac{2\pi i}{12}\right)^2, 8\left(\frac{2\pi i}{12}\right)^3\right)
\end{align*}
\]
where \( a_i \)'s are defined in \([2]\). For \( k \geq 2 \) one can write

\[
g_k(z) = s_k \sum_{0 \neq (m,n) \in \mathbb{Z}^2} \frac{1}{(n + mz)^k} \in M_k^0
\]

where \( s_2 = \frac{60}{(2\pi i)^2} \) and \( s_3 = \frac{-140}{(2\pi i)^3} \). Now \( g_1 \) satisfies

\[
(19) \quad g_1 |_2 A - g_1 = c j(A, z)^{-1}, \ A \in SL(2, \mathbb{Z})
\]

and so \( g_1 \in M_2^1 \) (see for instance \([1]\) p. 69). The following proposition describes the algebraic structure of \( M_m^n \):

**Proposition 1.** The followings are true:

1. For an \( f \in M_m^n \) the function \( z(f(\frac{-1}{z}) - f(z)) \) is in \( M_{m-2}^0 \), i.e. it is a modular form of weight \( m - 2 \).
2. \( M_2^1 \) is a one dimensional \( \mathbb{C} \)-vector space generated by \( g_1 \).
3. If \( n \leq n' \) then \( M_m^n \subset M_{m'}^{n'} \) and

\[
M_m^n M_{m'}^{n'} \subset M_{m+m'}^{n+n'}, \ M_m^n + M_m^{n'} = M_{m+m'}^{n+n'}
\]

4. For a modular form \( f \) of weight \( m \) we have \( f(g_1)^n \in M_{2n+m}^n \).

**Proof.** The first item is a direct consequence of the definition applied to \( A = Q \). The only modular forms of weight \( 0 \) are constant functions. This and 1 imply that \( M_2^1 \) is one dimensional. Item 3 is derived from the definition. Item 4 is a consequence of Item 3. Item 4 was the main idea behind the definition of \( M_m^n \).

The following proposition shows that \( M \) is in fact a differential algebra.

**Proposition 2.** For \( f \in M_m^n \) we have \( \frac{df}{dz} \in M_{m+2}^{n+1} \) and

\[
(20) \quad \frac{d(f|_m A)}{dz} = \frac{df}{dz} |_{m+2} A, \ \forall A \in GL(2, \mathbb{R})
\]

**Proof.** For \( A \in \text{Mat}_p(2, \mathbb{Z}) \) the term \( \frac{d(f|_m A)}{dz} \) is equal to:

\[
= p^{m-n} \left( \sum_{i=0}^{n} \binom{n}{i} c_{A-1}^i ((m - i)c_{A-1} j(A, z)^{i-1} f_i(Az) + j(A, z)^{i-m-2} \frac{df_i}{dz}(A(z)) \right)
= p^{m-n} \left( \sum_{i=1}^{n+1} \binom{n}{i-1} c_{A-1}^i j(A, z)^{i-2} (m - i + 1)f_i-1(Az) \right)
+ \sum_{i=0}^{n} \binom{n}{i} c_{A-1}^i j(A, z)^{i-2} \frac{df_i}{dz}(A(z))
= p^{m-n} \left( \sum_{i=0}^{n+1} \binom{n+1}{i} c_{A-1}^i j(A, z)^{i-2} \frac{df_i}{dz}(A(z)) \right)
\]
where

\[ \tilde{f}_i = \frac{i(m-i+1)}{n+1} f_{i-1} + \frac{n+1-i}{n+1} \int \frac{df_i}{dz}, \quad i = 0, 1, \ldots, n+1, \quad f_{-1} = f_{n+1} := 0 \]

Now \( \frac{df}{dz} \) is a \( M_{m+2}^{n+1} \)-function with the associated \( M_{m-2i}^{n-i} \) function \( \tilde{f}_i \) for \( i = 0, 1, 2, \ldots, n+1 \). The first item of definition has been checked above, using the fact that \( A \in SL(2, \mathbb{Z}) \) and \( f|_m A = f \). We calculate \( \tilde{f}_{ij} \) as above and then we get

\[ \tilde{f}_{ij} = \frac{i(m-i+1)}{n+1} f_{i-1+j} + \frac{n+1-i}{n+1} (j(m-2i-j+1) f_{i+j-1} + \frac{n+1-j}{n+1} \int \frac{df_{i+j}}{dz}) \]

\[ = \frac{(i+j)(m-(i+j)+1)}{n+1} f_{i+j-1} + \frac{n+1-i-j}{n+1} \int \frac{df_{i+j}}{dz} = \tilde{f}_{i+j} \]

This proves Item 2 of the definition of an \( M_{m+2}^{n+1} \)-function. The third item can be checked using

\[ \frac{df}{dz} = 2\pi i q \frac{df}{dq} \]

We consider \( M \) as a graded algebra with \( \deg(f) = m, \ f \in M^m_m \). The nontrivial statement about \( M^m_m \)-functions is Theorem 1 in the Introduction. It can be interpreted also as the equality of graded algebras \( M = \mathbb{C}[g_1, g_2, g_3] \). It implies that each \( f \in M^m_m \) can be written as

\[ f = \sum_{i=0}^{n} f_i g_1^i, \quad f_i \in M_{m-2i}^{0} \]

We prove Theorem 1 in \( \S 4.2 \).

The relations between the \( g_i, i = 1, 2, 3 \) and their derivatives are given by Ramanujan’s equalities:

\[ \frac{dg_1}{dz} = g_1^2 - \frac{1}{12} g_2, \quad \frac{dg_2}{dz} = 4g_1g_2 - 6g_3, \quad \frac{dg_3}{dz} = 6g_1g_3 - \frac{1}{3} g_2^2 \]

(see for instance [20, 28]). The proof of Theorem 1 will contain a new proof of these equalities.

### 2.3 Hecke operators

For \( p \in \mathbb{N} \) let \( SL(2, \mathbb{Z})/\text{Mat}_p(2, \mathbb{Z}) = \{ [A_1], [A_2], \ldots, [A_s] \} \). We define the \( p \)-th Hecke operator in the following way

\[ T_p f = \sum_{k=1}^{s} f|_m A_k, \quad \forall f \in M^m_m \]

Lemma 1 implies that the above definition does not depend on the choice of \( A_k \) in the class \([A_k]\). From Proposition 2 one can deduce that the differential operator \( \frac{d}{dz} \) commutes with the Hecke operator \( T_p \).

**Proposition 3.** \( T_p \) defines a map from \( M^m_m \) to itself.
This will be proved in §2.5. One can take
\[ \tilde{T}_p := \sum_{d|p, 0 \leq b \leq d-1} \begin{pmatrix} p & b \\ d & 0 \end{pmatrix} \in \mathbb{Z}^{\text{Mat}_p(2, \mathbb{Z})} \]
and since for matrices \( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \) the slash operator \( |_m \) is \( p^n \) times \( ||_m \) we have \( T_p f = p^{-n} f|_m \tilde{T}_p \) and we get the expression \( \text{B} \) in the Introduction. In a similar way to the case of modular forms (see [1] §6) one can check that
\[ T_p \circ T_q = \sum_{d|(p,q)} d^{m-n-1} T_{pq} d \]

2.4 The period domain

The group \( \text{SL}(2, \mathbb{Z}) \) acts from left on the period domain \( \mathcal{P} \) defined in (10) and \( G_0 \) in (11) acts from right. We consider holomorphic functions on
\[ \mathcal{L} := \text{SL}(2, \mathbb{Z}) \backslash \mathcal{P} \]
as holomorphic functions
\[ f : \mathcal{P} \to \mathbb{C}, \text{ holomorphic satisfying } f(Az) = f(z), \forall A \in \text{SL}(2, \mathbb{Z}), z \in \mathcal{P} \]
The determinant function is such a function. The Poincaré upper half plane \( \mathbb{H} \) is embedded in \( \mathcal{P} \) in the following way:
\[ z \to \tilde{z} = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \]
We denote by \( \tilde{\mathbb{H}} \) the image of \( \mathbb{H} \) under this map. For a function \( f \) on \( \mathbb{H} \) we denote by \( \tilde{f} \) the corresponding function on \( \tilde{\mathbb{H}} \).

**Proposition 4.** There is a unique map
\[ \phi : M \to \mathcal{O}(\mathcal{P}), \ f \mapsto \phi(f) = F \]
of the algebra of \( M \)-functions into the algebra of holomorphic functions on \( \mathcal{P} \) such that
1. For all \( f \in M \) the restriction of \( F \) to \( \tilde{\mathbb{H}} \) is equal to \( \tilde{f} \).
2. For all \( f \in M \) the holomorphic function \( F \) is \( \text{SL}(2, \mathbb{Z}) \) invariant.
3. We have
\[ F(x \cdot g) = k_2^n k_1^{n-m} \sum_{i=0}^{n} \binom{n}{i} k_3^i k_2^{-i} F_i(x), \forall x \in \mathcal{P}, \ g \in G_0, \]
where \( F_i = \phi(f_i) \).
Conversely, every holomorphic function \( F \) on \( \mathcal{P} \) which satisfies 2,3 and whose restriction to \( \tilde{\mathbb{H}} \) has finite growth at infinity is of the form \( F = \phi(f) \) for some \( f \in M_m^n \).
We denote by $\tilde{M}_{m}^{0}$ the set of holomorphic functions on $\mathcal{P}$ which restricted to $\tilde{H}$ have finite growth at infinity, are $\text{SL}(2,\mathbb{Z})$ invariant and satisfy (22). For a classical modular form $f : \mathbb{H} \to \mathbb{C}$ of weight $m$ the associated $F = \phi(f)$ is

$$F(x) = x_{3}^{n} f\left(\frac{x_{1}}{x_{3}}\right) \in \tilde{M}_{m}^{0}$$

We also have

$$\det \in \tilde{M}_{1}^{0}$$

**Proof.** We have

(23) \[
\begin{pmatrix}
  x_{1} & x_{2} \\
  x_{3} & x_{4}
\end{pmatrix} = \begin{pmatrix} x_{1} & -1 \\
  x_{3} & 0 \end{pmatrix} \begin{pmatrix}
  1 & 0 \\
  0 & \frac{\det(x)}{x_{3}}
\end{pmatrix}
\]

So we expect $F$ to be defined by

$$F(x) = F\left(\begin{pmatrix} x_{1} & -1 \\
  x_{3} & 0 \end{pmatrix} \begin{pmatrix}
  1 & 0 \\
  0 & \frac{\det(x)}{x_{3}}
\end{pmatrix} \begin{pmatrix}
  x_{4} \\
  \frac{\det(x)}{x_{3}}
\end{pmatrix}\right) = x_{3}^{-m} \det(x)^{n} \sum_{i=0}^{n} \binom{n}{i} x_{4}^{i} \det(x)^{-i} f_{i}\left(\frac{x_{1}}{x_{3}}\right)$$

This $F$ restricted to $\mathbb{H}$ is $f$. By definition of $F$ one can rewrite (13) in the form

$$f(\frac{A_{1}x_{1}}{x_{3}}) = (c x_{1} + d x_{3})^{m-n} F\left(\begin{pmatrix} x_{1} & -d \\
  x_{3} & c \end{pmatrix}\right)$$

where $A = \begin{pmatrix} a & b \\
  c & d \end{pmatrix}$. We check item 3: Let

$$g' = \begin{pmatrix} k'_{1} & k'_{3} \\
  0 & k'_{2} \end{pmatrix} = \begin{pmatrix} x_{3} & 0 \\
  0 & \frac{\det(x)}{x_{3}} \end{pmatrix}$$

RHS of (22) = $k_{2}^{n} k_{1}^{n-m} \sum_{i=0}^{n} \binom{n}{i} k_{3}^{i} k_{2}^{-i} F_{1}(x)$

$$= (k_{2} k'_{2})^{n} (k_{1} k'_{1})^{n-m} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{j} \binom{n-i}{j} k_{3}^{i} k_{2}^{-i} k_{1}^{j} k_{2}^{j} k_{3}^{j} f_{ij}\left(\frac{x_{1}}{x_{3}}\right)$$

$$= (k_{2} k'_{2})^{n} (k_{1} k'_{1})^{n-m} \sum_{r=0}^{n} \sum_{s=0}^{r} \binom{n}{s} \binom{n-s}{r-s} k_{3}^{s} k_{2}^{r-s} k_{1}^{r-s} k_{3}^{r-s} f_{rs}\left(\frac{x_{1}}{x_{3}}\right)$$

$$= (k_{2} k'_{2})^{n} (k_{1} k'_{1})^{n-m} \sum_{r=0}^{n} \sum_{s=0}^{r} \binom{n}{r} k_{3}^{r} k_{1}^{r} k_{3}^{r} f_{r}\left(\frac{x_{1}}{x_{3}}\right)$$

$$= F\left(\begin{pmatrix} z & -1 \\
  1 & 0 \end{pmatrix} g' g\right) = F(xg)$$
For all the equalities above we have used the same reasoning as in the proof of Lemma 1. We check that \( F(Ax) = F(x), \ A \in \text{SL}(2, \mathbb{Z}) \). The term \( F(Ax) \) is equal to

\[
= (cx_1 + dx_3)^{-m} \det(x)^n \sum_{i=0}^n \binom{n}{i} (cx_2 + dx_4)^i (cx_1 + dx_3)^i \det(x)^{-i} f_i(A \frac{x_1}{x_3})
\]

\[
= (cx_1 + dx_3)^{-m} \det(x)^n \sum_{i=0}^n \binom{n}{i} (cx_2 + dx_4)^i (cx_1 + dx_3)^i \det(x)^{-i} (cx_1 + dx_3)^{m-2i-(n-i)} F_i \left( \frac{x_1}{x_3} - \frac{c}{d} \right)
\]

\[
= F \left( \frac{x_1}{x_3} - \frac{d}{c} \left( \frac{cx_2 + dx_4}{cx_1 + dx_3} \right) \right) = F(x)
\]

Now let \( F \) satisfy 2,3 and its restriction to \( \mathbb{H} \) has finite growth at infinity put \( f = F |_{\mathbb{H}} \). First we note that

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} = (Az -1) \begin{pmatrix} j(A, z) & -c \\ 0 & j(A, z)^{-1} \det(A) \end{pmatrix}, \ A \in \text{GL}(2, \mathbb{R})
\]

Now

\[
f(Az) = F \left( \begin{pmatrix} Az & -1 \\ 1 & 0 \end{pmatrix} \right) = F \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \right) = j(A, z)^n j(A, z)^{m-n} \sum_{i=0}^n \binom{n}{i} c^i j(A, z)^{-i} f_i(x) = \sum_{i=0}^n \binom{n}{i} c^i j(A, z)^{m-i} f_i(x)
\]

This finishes the proof of our proposition. \( \square \)

### 2.5 Proof of Proposition

We define

\[
T_p : \tilde{M}_m^* \rightarrow \tilde{M}_m, \ T_p F(x) = p^{m-2n-1} \sum_{k=1}^s F(A_k x).
\]

This function has trivially its image in \( \tilde{M}_m^* \). We calculate the corresponding function in \( M_m^n \). The term \( T_p f(z) \) equals

\[
= p^{m-2n-1} \sum_{k=1}^s F(A_k \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix})
\]

\[
= p^{m-2n-1} \sum_{k=1}^s F \left( \begin{pmatrix} A_k z & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} j(A_k, z) & -c \\ 0 & p j(A_k^{-1}, A_k z) \end{pmatrix} \right)
\]

\[
= p^{m-2n-1} \sum_{k=1}^s (p j(A_k^{-1}, A_k z))^{n-m} \sum_{i=0}^n \binom{n}{i} (-c)^i j(A_k^{-1}, A_k z)^{-i} f_i(A_k z)
\]

\[
= \sum_{k=1}^s f|_M A_k
\]

where \( A_k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). This proves Proposition.
2.6 Some non-holomorphic functions on the period domain

On the complex manifold $\mathcal{P}$ we have the following $\text{SL}(2, \mathbb{Z})$ invariant analytic functions:

$$
B_1 := \text{Im}(x_1 x_3), \quad B_2 := \text{Im}(x_2 x_4), \quad B_3 := x_1 x_4 - x_2 x_3
$$

They define analytic functions on $\mathcal{L}$ which we denote them by the same letter. They satisfy

$$
B_1 \mid \tilde{\mathbb{H}} (z) = \text{Im}(z), \quad B_1 (xg) = B_1 (x) |_{k_1}^2 \\
B_2 \mid \tilde{\mathbb{H}} (z) = 0, \quad B_2 (xg) = B_2 (x) |_{k_3}^2 + B_2 (x) |_{k_2}^2 + \text{Im}(B_3 (x) k_3 k_2) \\
B_3 \mid \tilde{\mathbb{H}} (z) = 1, \quad B_3 (xg) = B_3 (x) k_1 k_2 + 2 \sqrt{-1} k_1 k_3 B_1 (x).
$$

By equality (23) one can easily see that every point in $\mathcal{P}$ can be mapped to a point of $\tilde{\mathbb{H}}$ by an action of a unique element of $G_0$. This implies that $\text{SL}(2, \mathbb{Z})$ invariant $B_i$, $i = 1, 2, 3$, with the above properties are unique.

2.7 Other topics

The literature of modular forms and its applications in number theory is huge. The first question which naturally arises at this point is as follows: Which part of the theory of modular forms can be generalized to the context of differential modular forms and which arithmetic properties can one expect to find? Since I am not expert in this area, I just mention some subjects which could fit well into this section.

The first of these, is the Eichler-Manin-Shimuara theory of periods for cusp forms (see [16] and its references). Note that the notion “period” in this theory, as far as I know, has nothing to do with the notion of a period in this article. The notion of period appears there because classical modular forms can be interpreted as sections of a tensor product of the cotangent bundle of a moduli curve and hence a differential multi form, which can be integrated over some path in the moduli curve (see [31]). The differential modular forms are no longer interpreted as sections of line bundles and this makes the situation more difficult. Lewis type equations attached to differential modular forms will be also of interest (see [16]).

Another theory which could be developed for differential modular forms is Atkin-Lehner theory of old and new modular forms (see [2]). This seems to me to be a quite accessible theory. The $L$-functions attached to differential modular forms through their Fourier expansion may also be of interest (see [3]).

3 Families of elliptic curves and the Gauss-Manin connection

First, let us fix some notation. For a ring $R$ we denote by $R[t]$ the polynomial ring with coefficients in $R$ and the variable $t := (t_1, t_2, \ldots, t_s)$. We also define the affine space $\mathbb{A}^s_R = \text{Spec}(R[t])$ defined over $R$. For simplicity we write $\mathbb{A}^s = \mathbb{A}^s_\mathbb{C}$. The set of relative differential $i$-forms in $\mathbb{A}^s_R$ is denoted by $\Omega^i_{\mathbb{A}^s_R/\text{Spec}(R)}$. 

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3.1 The family

We consider the following family of elliptic curves

(24) \[ \mathcal{E}_t : f = 0 \]

\[ f := y^2 - 4T_0(t)x^3 + T_1(t)x^2 + T_2(t)x + T_3(t) + T_4(t)y + T_5(t)xy, \quad t = (t_1, t_2, \ldots, t_s) \in T \]

where \[ T = \mathbb{A}^s \setminus \{ \Delta = 0 \}, \quad T_i \in \mathbb{C}[t], \quad i = 0, 1, \ldots, 5 \]

and

\[ \Delta = (6912T_1^2 - 3457T_1T_2^2 + 432T_3^2)T_4^3 + (1152T_1T_2T_3 - 288T_1T_2T_5^2 - 576T_1T_3T_4T_5 + 144T_1^2T_6^2 + 256T_3^2 - 384T_2T_5 - 288T_2T_3T_5^2 - 120T_2T_4T_5^2 + 144T_3T_4T_5^2 - 4T_3^2T_5^2)T_6^3 + (64T_1^2T_3 - 16T_1^2T_5^2 - 16T_1^2T_2T_5^2 + 16T_1^2T_2T_4T_5 - 48T_2T_3T_5^2 + 8T_1^2T_5^2 - 8T_2T_3T_5^2 - 8T_3^2T_5^2 + 12T_2T_3^2T_5^2 - T_1T_2T_5^2 - T_1T_2^2T_5^2 - T_2T_4T_5^2 + T_2T_4^2T_5^2 - T_3T_5^2)T_6^2 \]

The polynomial \( \Delta \in \mathbb{C}[t] \) is the locus of parameters in which \( \mathcal{E}_t \) is singular. The reader is referred to [24] for the algorithms which calculate \( \Delta \) (see also the proof of Proposition 6). To make our notation simpler, we define

\[ R := \mathbb{C}[t, \frac{1}{T_0}], \quad U_0 := \text{Spec}(R), \quad U_1 := \mathbb{A}^2 \times (\mathbb{A}^s \setminus \{ T_0 = 0 \}) = \text{Spec}(\mathbb{C}[x, y, t, \frac{1}{T_0}]) \]

3.2 De Rham cohomology

Let \( \pi : U_1 \to U_0 \) be the projection on the last \( s \)-coordinates. The following quotient

\[ H = \frac{\Omega^1_{U_1} \cap \Omega^0_{U_1}}{f \Omega^1_{U_1} + df \wedge \Omega^0_{U_1} + \pi^{-1}\Omega^1_{U_0} \wedge \Omega^0_{U_1} + d\Omega^0_{U_1}} \cong \frac{\Omega^1_{U_1/U_0}}{f \Omega^1_{U_1/U_0} + df \wedge \Omega^0_{U_1/U_0} + d\Omega^0_{U_1/U_0}} \]

is an \( R \)-module and will play the role of de Rham cohomology for us. One may call \( H \) the Brieskorn module associated to the family \( \mathcal{E}_t \) in analogy to the local modules introduced by Brieskorn in 1970. The restriction of \( H \) to the elliptic curve \( \mathcal{E}_t \) gives us \( H^1_{\text{dR}}(\mathcal{E}_t) \). Set

(25) \[ \omega_1 := -\frac{2}{5}(2xdy - 3ydx), \quad \text{and} \quad \omega_2 := -\frac{2}{7}x(2xdy - 3ydx). \]

Proposition 5. The \( R \)-module \( H \) is free and \( \omega_1 \) and \( \omega_2 \) form a basis of \( H \).

Proof. We consider the classical Brieskorn module

\[ \tilde{H} = \frac{\Omega^1_{U_1/U_0}}{df \wedge \Omega^0_{U_1/U_0} + d\Omega^0_{U_1/U_0}} \]

It is a \( R[f] \)-module. It is proved in [24] Proposition 1 that \( \tilde{H} \) is freely generated by \( \omega_1, \omega_2 \) as \( R[f] \)-module. Considering the canonical map \( \tilde{H} \to H \) we obtain the assertion of our proposition. \( \square \)
3.3 Gauss-Manin connection

Each element of the \( R \)-module \( H \) can be interpreted as a global section of the first cohomology bundle of the family \( \mathcal{E}_t \) over \( T \). Since the Gauss-Manin connection is a connection in the cohomology bundle, it is natural to find an algebraic definition for it using the \( R \)-module structure of \( H \). In this section we do this. Define

\[
V := \frac{\Omega^2_{\mathfrak{U}_1}}{df \wedge \Omega^1_{\mathfrak{U}_1} + f \Omega^2_{\mathfrak{U}_1} + \pi^{-1} \Omega^1_{\mathfrak{U}_0} \wedge \Omega^1_{\mathfrak{U}_1}} \cong \frac{\Omega^2_{\mathfrak{U}_1/\mathfrak{U}_0}}{df \wedge \Omega^1_{\mathfrak{U}_1/\mathfrak{U}_0} + f \Omega^2_{\mathfrak{U}_1/\mathfrak{U}_0}}
\]

**Proposition 6.** The polynomial \( \Delta \) is a zero divisor of the \( R \)-module \( V \), i.e.

\[
\Delta \cdot V = 0
\]

**Proof.** We define

\[
\tilde{V} := \frac{\Omega^2_{\mathfrak{U}_1/\mathfrak{U}_0}}{df \wedge \Omega^1_{\mathfrak{U}_1/\mathfrak{U}_0}}
\]

and consider it as an \( R[f] \)-module. It is proved in [24] Lemma 4 that \( B = \{dx \wedge dy, xdx \wedge dy\} \) form a basis of \( \tilde{V} \). Let \( A \) be the matrix of multiplication by \( f \) \( R \)-linear map in \( \tilde{V} \) in the basis \( B \). Then \( S(t, f) := \det(A - fI_2) \) has the property \( S(t, 0)V = 0 \). Now \( S(0, t) = 6912.\Delta \) (In fact this is the way we have calculated \( \Delta \)).

We have a well-defined differential map

\[
d : H \rightarrow V
\]

and we define the Gauss-Manin connection \( H \) as follows:

\[
\nabla : H \rightarrow \Omega^1_T \otimes_R H
\]

\[
\nabla \omega = \frac{1}{\Delta} \sum \alpha_i \otimes \beta_i, \quad \text{where} \quad \Delta d\omega = \sum \alpha_i \wedge \beta_i, \quad \alpha_i \in \Omega^1_{\mathfrak{U}_1}, \quad \beta_i \in \Omega^1_{\mathfrak{U}_1}
\]

Let \( U \) be an small open set in \( U \) and \( \{\delta_i\}_{i \in \mathcal{U}}, \delta_i \in H_1(\mathcal{E}_t, \mathbb{Z}) \) be a continuous family of topological one dimensional cycles. The main property of the Gauss-Manin connection is

\[
d(\int_{\delta_t} \omega) = \sum \alpha_i \int_{\delta_t} \beta_i, \quad \nabla \omega = \sum \alpha_i \otimes \beta_i, \quad \alpha_i \in \Omega^1_T, \quad \beta_i \in H.
\]

Let \( \omega_1, \omega_2 \) be a basis of \( H \) and define \( \omega = (\omega_1, \omega_2)^t \). The Gauss-Manin connection in this basis can be written in the following way:

\[
\nabla \omega = A \otimes \omega, \quad A = \frac{1}{\Delta} \left( \sum_{i=1}^s A_i dt_i \right) \in \text{Mat}(2, \Omega^1_T), \quad A_i \in \text{Mat}(2, \mathbb{C}[t]).
\]

**Proposition 7.** Let \( \tilde{\omega} = S \omega \) be another basis of \( H \) and \( \nabla \omega = A \otimes \omega \). Then

\[
\nabla(\tilde{\omega}) = S(S^{-1}dS + A)S^{-1} \otimes \tilde{\omega}
\]

**Proof.** We have

\[
\nabla(\tilde{\omega}) = \nabla(S \omega) = dS \otimes \omega + S \nabla \omega = dS S^{-1} \otimes \tilde{\omega} + SAS^{-1} \otimes \tilde{\omega} = (dS S^{-1} + SAS^{-1}) \otimes \tilde{\omega}
\]

\[\square\]
3.4 Classical differential forms

Recall the canonical basis [25] of $H$. Assume that in the family [24] $T_3$ is of the form $T_3 + t_3$, where $T_3$ and all other $T_i$'s do not depend on $t_3$. Let $A$ be the matrix of the Gauss-Manin connection in the basis $\omega$ as in [27]. We are particularly interested in $A_3$ in this case; it satisfies

$$\det(A_3) = 321052999680 \cdot T_0^4 \cdot \Delta$$

Define

$$\tilde{\omega} = \frac{1}{\Delta} A_3 \omega$$

The classical way of calculating Gauss-Manin connection leads to the equalities

$$\tilde{\omega}_1 = \frac{d\tilde{\omega}_1}{dy} = \frac{dx}{y}, \quad \tilde{\omega}_2 = \frac{d\tilde{\omega}_2}{dy} = \frac{x dy}{y}$$

restricted to the elliptic curves $\mathcal{E}_t$ and up to exact differential forms. Equality (28) implies that $\tilde{\omega}$ form a basis of $H_\Delta$, the localization of $H$ over $\{1, \Delta, \Delta^2, \ldots\}$.

3.5 An example

The following example plays a basic role in the proof of Theorem [11]

$$\mathcal{E}_t : f := y^2 - 4t_0(x - t_1)^3 + t_2(x - t_1) + t_3 = 0$$

In this example

$$\Delta = t_0(27t_0t_3^2 - t_3^3)$$

The calculation of the Gauss-Manin connection with respect to the canonical basis [25] leads to:

$$A_0 = \begin{pmatrix}
21/2t_0t_1t_2t_3 - 9t_0t_2^2 + 3/4t_3^2 & -21/2t_0t_2t_3 \\
21/2t_0t_1t_2t_3 - 9t_0t_1t_2 - 1/2t_1t_2 - 3/8t_3^2 & -21/2t_0t_1t_2t_3 - 18t_0t_3^2 + 5/4t_3^2
\end{pmatrix}$$

$$A_1 = \begin{pmatrix}
0 & 0 \\
t_0t_2^2 & 0
\end{pmatrix}$$

$$A_2 = \begin{pmatrix}
-63/2t_0t_1t_2 - 5/4t_1t_3 & 63/2t_0t_1t_2 \\
-63/2t_0t_1t_2 + 15/8t_0t_2^2 - 63/2t_0t_2^2 + 7/4t_0t_2^2 & -21t_0t_2 + 63/2t_3^2
\end{pmatrix}$$

$$A_3 = \begin{pmatrix}
21t_0t_1t_2 + 45/2t_3^2 & -21t_0t_2 \\
21t_0t_1t_2 - 9t_0t_1t_3 - 5t_0t_2^2 & 21t_0t_1t_2 + 63/2t_3^2
\end{pmatrix}$$

Now, the same with respect to the classical basis $\tilde{\omega}$ of $H_\Delta$ is given by:

$$A_0 = \begin{pmatrix}
3/2t_0t_1t_2t_3 - 9t_0t_2^2 + 1/4t_3^2 & -3/2t_0t_1t_2 \\
3/2t_0t_1t_2t_3 + 9t_0t_1t_2^2 - 1/2t_1t_2 + 1/8t_3^2 & -3/2t_0t_1t_2t_3 - 18t_0t_3^2 + 3/4t_3^2
\end{pmatrix}$$

$$A_1 = \begin{pmatrix}
0 & 0 \\
t_0t_2^2 & 0
\end{pmatrix}$$

$$A_2 = \begin{pmatrix}
-9/2t_0t_1t_2 + 1/4t_1t_3 & 9/2t_0t_1t_2 \\
-9/2t_0t_1t_2 + 1/2t_0t_1t_2^2 - 3/8t_0t_2^2 & 9/2t_0t_1t_3 - 1/4t_0t_2^2
\end{pmatrix}$$

$$A_3 = \begin{pmatrix}
3t_0t_1t_2 - 9t_0t_1t_3 & -3t_0t_2 \\
3t_0t_1t_2 - 9t_0t_1t_3 + 1/4t_0t_2^2 & 3t_0t_1t_2 + 9t_0t_2 + 9/2t_3^2
\end{pmatrix}$$

See [24] [26] for the procedures which calculate all matrices above. In the library $\text{brho.lib}$ the command $\text{gaussmaninp}$ calculate the above matrix. Note that the canonical basis of the Brieskorn module in this library is $\begin{pmatrix} 0 & -12/8 \\ 12 & 0 \end{pmatrix} \omega$. 

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4 The period map

4.1 Derivation of the period map

Let \( \omega = (\omega_1, \omega_2)^t \) be a basis of \( H_\Delta \). The period map associated to the basis \( \omega \) is given by:

\[
\text{pm} : T \rightarrow \text{SL}(2, \mathbb{Z}) \backslash \text{GL}(2, \mathbb{C}), \quad t \mapsto \left[ \frac{1}{\sqrt{2} \pi i} \left( \int_{\delta_1} \omega_1 \int_{\delta_2} \omega_2 \right) \right]
\]

It is well-defined and holomorphic. Here \( \sqrt{i} = e^{2\pi i / 4} \) and \((\delta_1, \delta_2)\) is a basis of the \( \mathbb{Z} \)-module \( H_1(\mathcal{E}_t, \mathbb{Z}) \) such that the intersection matrix in this basis is

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

Let \( \tilde{\omega} = S \omega, S \in \text{Mat}(2, \mathbb{C}[t_0, t_1]) \) be another basis of \( H \) and \( \tilde{\text{pm}} \) be the associated period map. Then it is easy to see that

\[
(32) \quad \tilde{\text{pm}} = \text{pm} \cdot S^t
\]

**Proposition 8.** Let \( \omega \) be a basis of \( H_\Delta \) with

\[
(33) \quad \nabla \omega = A \otimes \omega, \quad A \in \text{Mat}(2, \Omega_1^1)
\]

Then

\[
(34) \quad d(\text{pm})(t) = \text{pm}(t) \cdot A^t, \quad t \in T,
\]

where \( d \) is the differential map.

**Proof.** Let \( \nabla_i := \nabla_{\frac{\partial}{\partial t_i}} \) denote the Gauss-Manin connection with respect to the parameter \( t_i \). Then \( \nabla_i \omega = A_i \omega \) and according to \( (26) \) and \( (32) \)

\[
\left. \frac{\partial}{\partial t_i} \text{pm}(t) = \left( \int_{\delta_1} \nabla_i \tilde{\omega}_1 \int_{\delta_2} \nabla_i \tilde{\omega}_2 \right) \right) = \text{pm}(t) \cdot A_i^t
\]

This proves the desired statement.

For the basis \( \omega = (\omega_1, \omega_2)^t \) if \( \omega_1 \) is such that its restriction to each elliptic curve \( \mathcal{E}_t, t \in T \) is of the first kind then the period map is defined from \( T \) into \( L := \text{SL}(2, \mathbb{Z}) \backslash \mathbb{P} \). For instance, the period map associated to the classical basis \( (29) \) of \( H_\Delta \) has this property.

4.2 The Action of an algebraic group

We consider the family of elliptic curves \( (30) \). It can be checked easily that \( (12) \) is an action of \( G_0 \) on \( \mathbb{A}^4 \). (This can be also verified from the proof of the proposition bellow). It is also easy to verify that \( \mathbb{A}^4/G_0 \) is isomorphic to \( \mathbb{P}^1 \) through the map

\[
(35) \quad s : \mathbb{A}^4/G_0 \rightarrow \mathbb{P}^1, \quad t \rightarrow [t_2 : 27t_0t_3^2 - t_1^3]
\]

and so

\[
(36) \quad j(t) := \frac{t_3^3}{27t_0t_3^2 - t_1^3}
\]

is \( G_0 \)-invariant and gives an isomorphy between \( T/G_0 \) and \( \mathbb{A} \). Recall the basis \( \tilde{\omega} \) of \( H_\Delta \) in \( (29) \).
Proposition 9. The period $\text{pm}$ associated to the basis $\tilde{\omega}$ is a biholomorphism and
\begin{equation}
\text{pm}(t \cdot g) = \text{pm}(t) \cdot g, \ t \in \mathbb{A}^4, \ g \in G_0
\end{equation}

Proof. We first prove (37). Let
\[ \alpha : \mathbb{A}^2 \to \mathbb{A}^2, \ (x, y) \mapsto (k_2^{-1}k_1x - k_3k_2^{-1}, k_2^{-1}k_1^2y) \]

Then
\begin{align*}
    k_2^{-1}k_1^{-4} \alpha^{-1}(f) &= y^2 - 4t_0k_2^2k_1^{-4}(k_2^{-1}k_1x - k_3k_2^{-1} - t_1)^3 + t_2k_2^2k_1^{-4}(k_2^{-1}k_1x - k_3k_2^{-1} - t_1) + t_3k_2^2k_1^{-4} \ \\
    &= y^2 - 4t_0k_1^{-1}k_2^{-1}(x - (t_1k_2k_1^{-1} + k_3k_2^{-1}))^3 + t_2k_1^{-3}k_2(x - (t_1k_2k_1^{-1} + k_3k_2^{-1})) + t_3k_1^{-4}k_2^2
\end{align*}

This implies that $\alpha$ induces an isomorphism of elliptic curves
\[ \alpha : \mathcal{E}_{t \cdot g} \to \mathcal{E}_t \]

Now
\[ \alpha^{-1}\tilde{\omega} = \begin{pmatrix} k_1^{-1} & 0 \\ -k_3k_2^{-1}k_1^{-1} & k_2^{-1} \end{pmatrix} \tilde{\omega} = \begin{pmatrix} k_1 & 0 \\ k_3 & k_2 \end{pmatrix}^{-1} \tilde{\omega} \]

By the equality (32) we have
\[ \text{pm}(t) = \text{pm}(t \cdot g).g^{-1} \]

which proves (37).

The matrix of the Gauss-Manin connection in the basis $\tilde{\omega}$ for the family (30) is calculated in 3.5. Let $B$ be a $4 \times 4$ matrix and the $i$-th row of $B$ constitutes of the first and second rows of $A_i$. Then
\[ \det(B) = \frac{3}{4}t_0\Delta^3 \]

shows that the period map $\text{pm}$ is regular at each point $t \in T$ and hence it is locally a biholomorphism.

The period map $\text{pm}$ induces a local biholomorphic map $\text{pm} : T/G_0 \to \text{SL}(2, \mathbb{Z})\backslash \mathbb{H} \cong \mathbb{C}$ and so we have the local biholomorphism $\text{pm} \circ j^{-1} : \mathbb{A} \to \mathbb{A}$. One can compactify $\text{SL}(2, \mathbb{Z})\backslash \mathbb{H}$ by adding the cusp $\text{SL}(2, \mathbb{Z})/Q = \{c\}$ (see 23) and the map $\text{pm} \circ j^{-1}$ is continuous at $v$ sending $v$ to $c$, where $v$ is the point induced by $t_027t_3^2 - t_3^3 = 0$ in $\mathbb{A}^4/G_0$.

Using Picard's Great Theorem we conclude that $j^{-1} \circ \text{pm}$ is a biholomorphism and so $\text{pm}$ is a biholomorphism. \qed

4.3 The inverse of the period map

We denote by
\[ F = (F_0, F_1, F_2, F_3) : \mathcal{P} \to T \]
the inverse of the period map.

Proposition 10. The following is true:

1. $F_0(x) = \det(x)^{-1}$.

2. For $i = 2, 3$
\[ F_i = \det(x)^{1-i}g_i \in M_{2i}^0 \]

where $g_i$ is the Eisenstein series (7).
3. $F_1 = \tilde{g}_1 \in \tilde{M}_1^1$.

Proof. Taking $F$ of (37) we have

\[ F_0(xg) = F_0(x)k_1^{-1}k_2^{-1}, \]

(38)

\[ F_1(xg) = F_1(x)k_1^{-1}k_2 + k_3k_4^{-1}, \]
\[ F_2(xg) = F_2(x)k_1^{-3}k_2, \quad F_3(xg) = F_3(x)k_1^{-4}k_2^2, \quad \forall x \in \mathcal{L}, \ g \in G_0 \]

By Legendre’s Theorem $\det(x)$ is equal to one on $V := \operatorname{pm}(1 \times 0 \times \mathbb{A} \times \mathbb{A})$ and so the same is true for $F_0\det(x)$. But the last function is invariant under the action of $G_0$ and so it is the constant function 1. This proves the first item. Let $G_i = F_i\det(x)^{i-1}$, $i = 1, 2, 3$. The equalities (38) imply that $G_i, i = 2, 3$ do not depend on $x_2, x_4$. Now the map $(t_2, t_3) \to \pi \circ \operatorname{pm}(1, 0, t_2, t_3)$, where $\pi$ is the projection on the $x_1, x_3$ coordinates, is the classical period map (see for instance see [30] and and the appendix of [19]) and this implies that $G_i = \tilde{g}_i, i = 2, 3$. Note that in our definition of the period map the factor $\frac{1}{\sqrt{2\pi}}$ appears. In particular $F_i, i = 2, 3$ have finite growth at infinity. The fact that $F_1$ has finite growth at infinity follows form the Ramanujan relations (21) and the equality \( \frac{d}{dx} = 2\pi iq \frac{d}{d\eta} \). Since $G_1 \in \tilde{M}_2^1$, $\tilde{M}_2^1$ is a one dimensional space, both $g_1, G_1$ satisfy (38) and $\tilde{M}_2^0 = \emptyset$, we have $G_1 = g_1$. \qed

4.4 Ramanujan relations

We proved in Lemma [23] that the period map $\operatorname{pm}$ associated to $\tilde{\omega}$ is a biholomorphism. According to (34), the inverse $F$ of $\operatorname{pm}$ satisfies the differential equation

\[ x.A(F(x)) = I \]

We consider $\operatorname{pm}$ as a map sending the vector $(t_0, t_1, t_2, t_3)$ to $(x_1, x_2, x_3, x_4)$. Its derivative at $t$ is a $4 \times 4$ matrix whose $i$-th column constitutes of the first and second row of $\frac{1}{\sqrt{2\pi}} x A_i^t$. We use (31) to derive the equality

\[
\det(x)^{-1} \begin{pmatrix}
-F_0 x_4 & -F_0 x_3 & F_0 x_2 & -F_0 x_1 \\
F_0 x_3 & -F_1 x_2 - x_4 & -F_2 x_3 & -F_3 x_2 \\
F_0 x_2 & -F_1 x_2 - x_4 & -F_2 x_3 & -F_3 x_2 \\
F_0 x_1 & -F_1 x_2 - x_4 & -F_2 x_3 & -F_3 x_2
\end{pmatrix} = \frac{d(F)_x}{(d\operatorname{pm})^{-1}} = \begin{pmatrix}
\frac{1}{\sqrt{16}}(12F_0 F_2 x_3 - 12F_0 F_3 x_4 - 4F_2 x_3 - 4F_3 x_2) & -F_1 x_2 + x_4 & \frac{1}{\sqrt{16}}(-12F_0 F_2 x_3 + 12F_0 F_3 x_4 + 6F_2 x_3 + 6F_3 x_2) & -F_0 x_1 \\
\frac{1}{\sqrt{16}}(18F_0 F_1 F_2 x_3 - 12F_0 F_3 x_4 - F_2 x_3) & -F_1 x_2 + x_4 & \frac{1}{\sqrt{16}}(-18F_0 F_1 F_2 x_3 + 12F_0 F_3 x_4 + F_2 x_3) & \frac{1}{\sqrt{16}}(-18F_0 F_1 F_3 x_4 + 12F_0 F_3 x_2 + F_2 x_3)
\end{pmatrix}.
\]

For $g_i := F_i \mid_{\tilde{H}}$ we obtain the Ramanujan relations (21).

4.5 Other topics

We have seen that the period map satisfies the differential equation

\[ d(\operatorname{pm}^t) = A\operatorname{pm}^t \]

This can be interpreted as a multi-variable Fuchsian system/Picard-Fuchs equation. Restricting (39) to a line in $\mathbb{A}^4$, one gets a usual one variable Picard-Fuchs equation. For many of them the solution and hence the associated abelian integral can be given explicitly by hypergeometric functions (see [4]). It seems to me that with the notion of period map in this article it is possible to classify all the algebraic values of $G$-functions obtained by the methods in [4].

Another interesting subject could be the construction of a logarithmic structure on $T$ in the context of algebraic geometry. Such constructions for the moduli of polarized Hodge structures is done in [17] and [18] in the context of analytic geometry.
5 The Ramanujan foliation

The whole theory developed in the previous sections could be done by setting $t_0 = 1$. In this section we assume that $t_0 = 1$ and use the same notations for $p, \mathcal{P}, G_0, T, \Delta$ and so on. For instance redefine

$$\mathcal{P} := \{ x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \text{GL}(2, \mathbb{C}) \mid \text{Im}(x_1x_3) > 0, \det(x) = 1 \}$$

and

$$G_0 = \{ \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \mid k', k \in \mathbb{C} \}$$

The action of $G_0$ on $\mathbb{A}^3$ is given by

$$t \cdot g := (t_1k^{-2} + k'k^{-1}, t_2k^{-4}, t_3k^{-6}), t = (t_1, t_2, t_3) \in \mathbb{A}^3, g = \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \in G_0$$

We also define

$$g = (g_1, g_2, g_3) : \mathbb{H} \to T \subset \mathbb{A}^3$$

5.1 The Ramanujan foliation

We write (8) in the vector field form

$$\text{Ra} := (t_1^2 - \frac{1}{12}t_2^2) \frac{\partial}{\partial t_1} + (4t_1t_2 - 6t_3) \frac{\partial}{\partial t_2} + (6t_1t_3 - \frac{1}{3}t_2^2) \frac{\partial}{\partial t_3}$$

It is also useful to define the differential forms

$$\eta_1 := (t_1^2 - \frac{1}{12}t_2^2)dt_2 - (4t_1t_2 - 6t_3)dt_1$$

$$\eta_2 := (4t_1t_2 - 6t_3)dt_3 - (6t_1t_3 - \frac{1}{3}t_2^2)dt_2,$$

$$\eta_3 := (t_1^2 - \frac{1}{12}t_2^2)dt_3 - (6t_1t_3 - \frac{1}{3}t_2^2)dt_1$$

and say the foliation $\mathcal{F}(dR)$ is induced by $\eta_i, i = 1, 2, 3$. The singularities of $\text{Ra}$ are the points $t \in \mathbb{A}^3$ such that $\text{Ra}(t) = 0$. It turns out that

$$\text{Sing}(\text{Ra}) = \{(t_1, 12t_1^2, 8t_1^3) \mid t_1 \in \mathbb{A} \},$$

which is a one dimensional curve and lies in $\{ \Delta = 0 \}$. We have

$$d\Delta(\text{Ra}) = (2.27t_3dt_3 - 3t_2^2dt_2)(\text{Ra})$$

$$= 2.27t_3(6t_1t_3 - \frac{1}{3}t_2^2) - 3t_2^2(4t_1t_2 - 6t_3)$$

$$= 12t_1\Delta$$

This implies that the variety $\Delta_0 := \{ \Delta = 0 \}$ is invariant by the foliation $\mathcal{F}(\text{Ra})$. Inside $\Delta_0$ we have the algebraic leaf $\{(t_1, 0, 0) \in \mathbb{A}^3 \}$ of $\mathcal{F}(\text{Ra})$. On $\text{Sing}(\text{Ra})$ we have a special point $p_\infty$ given by $[S]$. It is the limit of $g(z)$ when $\text{Im}(z)$ tends to $+\infty$. 

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Proposition 11. The following is a uniformization of the foliation $F(Ra)$ restricted to $T$:

$$u : \mathbb{H} \times (\mathbb{A}^2 \setminus \{(0,0)\}) \to T$$

\begin{equation}
(z, c_2, c_4) \to g(z) \cdot \begin{pmatrix}
(c_4 z - c_2)^{-1} & c_4 \\
0 & c_4 z - c_2
\end{pmatrix}
\end{equation}

$$(g_1(z)(c_4 z - c_2)^2 + (c_4 z - c_2), g_2(z)(c_4 z - c_2)^4, g_3(z)(c_4 z - c_2)^6).$$

Proof. One may check directly that for fixed $c_2, c_4$ the map induced by $u$ is tangent to $\mathbb{H}$. In general it is not a solution of $\mathbb{S}$. This is the main reason for naming “foliation”. We give another proof which uses the period map: From $\mathbb{S}$ we have

$$d(pm)(t) = pm(t) \begin{pmatrix}
\frac{3}{4} \eta_2 \\
\frac{3}{4} (3 t_2^2 - 2 t_2^2 t_3)
\end{pmatrix}^t.$$

Therefore

$$d(pm(t))(Ra(t)) = pm(t) \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}.$$ 

This implies that the $x_2$ and $x_4$ coordinates of the pull forward of the vector field $Ra$ by $pm$ are zero. Therefore, the leaves of $F(Ra)$ in the period domain are of the form

$$\begin{pmatrix}
(c_4 z - c_2)^{-1} & c_2 \\
(c_4 z - c_2)^{-1} & c_4
\end{pmatrix} = \begin{pmatrix}
z & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
(c_4 z - c_2)^{-1} & c_4 \\
0 & c_4 z - c_2
\end{pmatrix}.$$

\qed

Due to Proposition 11 our foliation $F(Ra)$ can be considered as a kind of Hilbert modular foliation (see $\mathbb{S}$).

5.2 The family $y^2 - 4x^3 + t_1 x^2 + t_2 x + t_3$

The family $\mathbb{T}$ can be rewritten in the form

$$y^2 - 4t_0 x^3 + 12t_0 t_1 x^2 + (-12t_0 t_1^2 + t_2)x + (4t_0 t_1^3 - t_2 t_1 + t_3) = 0.$$ 

The mapping

$$\alpha : \mathbb{A}^4 \to \mathbb{A}^4, t \mapsto (t_0, 12t_0 t_1, -12t_0 t_1^2 + t_2, 4t_0 t_1^3 - t_2 t_1 + t_3)$$

is an isomorphism and so we can re state Proposition 9 for the family

$$\mathcal{E}_t : y^2 - 4t_0 x^3 - t_1 x^2 - t_2 x - t_3 = 0.$$ 

The inverse of the period map in this case is given by $G = (G_0, G_1, G_2, G_3)$ with

$$G_0 = F_0 \in M_0^{-1}, \ G_1 = 12 F_0 F_1 \in M_2^0,$$

$$G_2 = -12 F_0 F_1^2 + F_2 \in M_4^1, \ G_3 = 4 F_0 F_1^3 - F_2 F_1 + F_3 \in M_6^2.$$
In this case the singular fibers are parameterized by the zeros of
\[ \Delta := t_0 (432t_0^2 t_3^2 + 72t_0 t_1 t_2 t_3 - 16t_0^3 t_2^2 + 4t_1^3 t_3 - t_1^2 t_2^2). \]
The Ramanujan relations take the simpler form:
\[
\begin{aligned}
\dot{t}_1 &= -t_2 \\
\dot{t}_2 &= -6t_3 \\
\dot{t}_3 &= t_1 t_3 - \frac{1}{4} t_2^2
\end{aligned}
\]
They have the solution
\[ g := (12g_1, -12g_1^2 + g_2, 4g_1^3 - 2g_2 g_1 + g_3) \]

6 Proofs

Now we are in a position to prove the theorems announced in the Introduction.

6.1 Proof of Theorem 1

We prove that \( \tilde{M} \) as a \( \mathbb{C} \)-algebra is freely generated by \( \frac{1}{t_0}, F_i, \ i = 0, 1, 2, 3 \). Let \( \tilde{F} \in \tilde{M}_m^n \) and \( \tilde{F}_i \in M_{m-2i}^n \) be its associated functions. Since the period map \( \mathfrak{p} \mathfrak{m} : T \to \mathcal{L} \) is a biholomorphism, there exist holomorphic functions \( p_i, i = 0, 1, \ldots, n, \ p_0 := p \) defined on \( T \) such that \( \tilde{F}_i = p_i(F_0, F_1, F_2, F_3) \). The property (22) of \( \tilde{F} \) implies that:

\[
\begin{aligned}
p(t \bullet g) &= k_2^m k_1^{-m} \sum_{i=0}^{n} \left( \begin{array}{c}
2
\end{array} \right) \begin{array}{c}
i
\end{array} \begin{array}{c}
k_1
\end{array} \begin{array}{c}
k_2
\end{array} \ p_i(t), \ \forall g \in G_0, \ t \in T.
\end{aligned}
\]

Take \( g = \left( \begin{array}{cc}
1 & t_1 \\
0 & 1
\end{array} \right) \) and \( t = (t_0, 0, t_1, t_3) \). Then
\[ p(t_0, t_1, t_2, t_3) = \sum_{i=0}^{n} \left( \begin{array}{c}
2
\end{array} \right) \begin{array}{c}
i
\end{array} \begin{array}{c}
l_1
\end{array} p_i(t_0, 0, t_2, t_3). \]

This implies that \( p \) is a polynomial of degree at most \( n \) in the variable \( t_1 \). Let us re write \( p(t) = \sum_{i=0}^{n} l_1 q_i \), where \( q_i \)'s are holomorphic functions on \( k^{3} \setminus \{ t \in k^3 | \Delta = 0 \} \). We apply (42) to \( g = \left( \begin{array}{cc}
k & 0 \\
0 & t_0^{-1}k^{-1}
\end{array} \right) \) and consider the coefficients of \( t_1^i, \ i = 1, 2, \ldots, n \). We get
\[
\begin{aligned}
q_i(1, t_2 t_0^{-1} k^{-4}, t_3 t_0^{-1} k^{-6}) &= t_0^{-n} k^{-m+2i} q_i(t), \ i = 1, 2, \ldots, n.
\end{aligned}
\]

Take \( t_0 = 1 \). The growth condition on \( \tilde{F} \) is translated through the period map into the following fact: \( p \) restricted to a transversal disk to \( \Delta = 0 \) at \( p_\infty \) has a holomorphic extension to \( p_\infty \). This will also imply the similar growth conditions for \( q_i \)'s. The classical fact that the set of modular forms is generated by the Eisenstein series \( g_2 \) and \( g_3 \) and (43) with \( t_0 = 1 \) imply that \( q_i(1, t_2, t_3) \) is a homogeneous polynomial of degree \( m-2i \) in the graded ring \( \mathbb{C}[t_2, t_3], \ \deg(t_2) = 4, \deg(t_3) = 6 \). We conclude that \( p \) is of the form
\[ p = t_0^n \sum_{i=0}^{n} t_1^i q_i(1, t_2 t_0^{-1}, t_3 t_0^{-1}). \]

In other words, \( p \) is homogeneous of degree \( m \) in the variables \( t_1, t_2, t_3 \) with \( \deg(t_i) = 2i, \ i = 1, 2, 3. \)
6.2 Proof of Theorem 2

In [26] we described some analytic functions $B_i$, $i = 1, 2, 3$, on $\mathcal{L}$ which had some compatibility properties with the action of $G_0$ on $\mathcal{L}$. We use Proposition [9] and transfer them to the world of coefficients. This will prove the existence and uniqueness of the functions $B_1, B_2, B_3$. For the sake of simplicity we have used the same letters to name these functions.

By the properties that $B_1$ has we can say more about it. In [1] we put $k = 1$ and we conclude that $B_1$ is independent of the variable $t_1$. The function $B_2 \cdot |\Delta|^{1/\beta}$ is $G_0$ invariant and so there is an analytic function $b_2 : \mathbb{A} \to \mathbb{R}$ such that

$$B_2(t) = \frac{b_2(j(t))}{|\Delta(t)|^{1/\beta}}.$$

Translating this to $\mathbb{H}$, we have

$$\text{Im}(z) = \frac{b_2(j(z))}{|\Delta(z)|^{1/\beta}}$$

where the above $j$ and $\Delta$ are the ones on [22].

The proof of the last part of the theorem is as follows: On $M_0$ an $x \in \mathcal{P}$ can be written in the form $(x_1^r x_4^r, x_3^r x_4^r)$, $r \in \mathbb{R}, x_4(x_1 - r x_3) = 1$. Then

$$B_3(x) = \overline{x_4}(x_1 - r x_3) = \frac{x_4}{x_3}.$$

6.3 Proof of Theorem 3

We follow the notations introduced in [5]. In particular we work with the family [11] with $t_0 = 1$. The leaves of the pull-forward of the foliation $\mathcal{F}(R_0)$ by the period map $\mathcal{P}$ have constant $x_2$ and $x_4$ coordinates. By definition of $B_2 := \text{Im}(x_2 x_4)$ in the period domain, we conclude that $M_r$’s are $\mathcal{F}(R_0)$-invariant. The equality (44) implies that $N_w$’s are $\mathcal{F}(R_0)$-invariants.

Let us now prove item 2: Take $t \in K$ and a cycle $\delta \in H_1(\mathcal{E}_r, \mathbb{Z})$ such that $\int_\delta \tilde{\omega} = 0$ and $\delta$ is not of the form $n\delta'$ for some $2 \leq n \in \mathbb{N}$ and $\delta' \in H_1(\mathcal{E}_r, \mathbb{Z})$. We choose another $\delta' \in H_1(\mathcal{E}_r, \mathbb{Z})$ such that $(\delta', \delta)$ is a basis of $H_1(\mathcal{E}_r, \mathbb{Z})$ and the intersection matrix in this basis is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now $\mathcal{P}(t)$ has zero $x_4$-coordinates and so its $B_2$ is zero. This implies that $K \subset M_0$. It is dense because an element $(x_1^r x_3^r x_4^r, x_4^r x_3^r x_4^r) \in M_0 \subset \mathcal{L}$ can be approximated by the elements in $M_0$ with $r \in \mathbb{Q}$.

The image of the map $g$ is the locus of the points $t_0$ in $\mathcal{T}$ such that $\mathcal{P}(t_0)$ is of the form $(z_1, -1, 0)$ in a basis of $H_1(\mathcal{E}_r, \mathbb{Z})$. We look $g$ as a function of $g = e^{2\pi iz}$ and we have

$$g(0) = p_\infty, \quad \frac{\partial g}{\partial \bar{q}}(0) = (-24a_1, 240a_2, -504a_3)$$

where $p_\infty = (a_1, a_2, a_3)$ is the one in [15]. This implies that the image of $g$ intersects $\text{Sing}(R_0)$ transversally. For a $t \in K$ the $x_4$-coordinate of $\mathcal{P}$ is zero and the leaf through $t$, namely $L_t$, has constant $x_2$-coordinate, namely $c_2$. By (44) $L_t$ is uniformized by

$$u(z) = (c_2^2 g_1(z), c_2^4 g_2(z), c_2^6 g_3(z)), \quad z \in \mathbb{H}.$$
This implies that \( L_t \) intersects \( \text{Sing}(Ra) \) transversally at \((c_2^3a_1, c_4^2a_2, c_6^3a_3)\).

We prove item 3: Let \( t \in T \) and the leaf \( L_t \) through \( t \) have an accumulation point at \( t_0 \in T \). We use the period map \( \text{pm} \) and look \( F(Ra) \) in the period domain. For \((c_2, c_4) \in \mathbb{A}^2 \setminus \{0\} \) the set \( S = \{A(c_2, c_4) \mid A \in \text{SL}(2, \mathbb{Z})\} \) has an accumulation point in \( \mathbb{A}^2 \) if and only if \( \frac{c_2}{c_4} \in \mathbb{R} \cup \infty \) or equivalently \( B_2(t) = 0 \).

### 6.4 Proof of Theorem 4

Let \( k \) be an algebraically closed field of characterstic 0, for instance take \( k = \overline{\mathbb{Q}} \).

**Proposition 12.** The quasi affine variety

\[ T = \text{Spec}(k[t, \frac{1}{\Delta}]) \]

is the moduli of \((E, [\omega_1], [\omega_2])\)'s, where \( E \) is an elliptic curve defined over \( k \), \( \omega_1 \) is a differential form of the first kind on \( E \) at \( t \), for some \( g \). This is in contradiction with \( \frac{\omega_1}{\omega_2} \) being a multiple of another \( \delta \) if and only if \( \frac{\omega_1}{\omega_2} \) is a basis of \( H^2_{\text{dR}}(E) \).

**Proof.** For simplicity we do not write more \([\cdot]\) for differential forms. The \( j \) invariant classifies the elliptic curves over \( k \). Therefore, for a given elliptic curve \( E/k \) we can find parameter \( t \in \mathbb{A}^1_k \) such that \( E \cong \mathcal{E}_t \) over \( k \). Under this isomorphism we write

\[ \left( \frac{\omega_1}{\omega_2} \right) = g^*(\frac{dx}{y}, \frac{2dx}{y}) \]

for some \( g \in G_0 \), where \( \omega_1, \omega_2 \) are as in the proposition. Now, the triple \((E, \omega_1, \omega_2)\) is isomorphic to \((\mathcal{E}_t, \frac{dx}{y}, \frac{2dx}{y})\). Since \( j : \mathbb{A}^1/G_0 \to \mathbb{A} \) is an isomorphism, every triple \((E, \omega_1, \omega_2)\) is represented exactly by one parameter \( t \in T \).

By Proposition 12 the hypothesis of Theorem 4 gives us a parameter \( t \in T \) such that \( \int \frac{xdx}{y} = 0 \), for some \( \delta \in H_1(\mathcal{E}_t, \mathbb{Z}) \). We can assume that \( \delta \) is not a multiple of another cycle in \( H_1(\mathcal{E}_t, \mathbb{Z}) \). The corresponding period matrix of \( t \) in a basis \((\delta', \delta)\) of \( H_1(\mathcal{E}_t, \mathbb{Z}) \) has zero \( x_4 \)-coordinate and so the numbers

\[ t_0 = \det(x)^{-1}, \ t_i = \det(x)^{1-i}x_3^{-2i}g_i(x_1, x_3), \ i = 2, 3, \ t_1 = F_1(\left( \begin{array}{cc} x_1 & x_2 \\ x_3 & 0 \end{array} \right)) = \det(x)x_3^{-2}g_1(x_1) \]

all are in \( \overline{\mathbb{Q}} \). This implies that for \( z = \frac{x_1}{x_3} \in \mathbb{H} \) we have

\[ \frac{g_3}{g_1}(z), \frac{g_4}{g_1}(z), \frac{g_2}{g_1}(z) \in \overline{\mathbb{Q}}. \]

This is in contradiction with

**Theorem** (Nesterenko 1996, [28]) For any \( z \in \mathbb{H} \), the set

\[ e^{2\pi iz}, \frac{g_1(z)}{a_1}, \frac{g_2(z)}{a_2}, \frac{g_3(z)}{a_3} \]

contains at least three algebraically independent numbers over \( \mathbb{Q} \).

A direct corollary of Theorem 4 is that the multivalued function

\[ I(t) = \frac{\int_{a_1} \frac{dx}{y}}{\int_{b_1} \frac{dx}{y}} \]

defined in \( T \) never takes algebraic values for algebraic \( t \).
6.5 Other topics

As a person who has started his mathematical career by studying holomorphic foliations in complex manifolds, I would be interested to describe completely the dynamics of the foliation $\mathcal{F}(Ra)$ and in particular the behavior of the leaves near $\Delta = 0$. The leaves of $\mathcal{F}(Ra)$ in $\Delta = 0$ parameterize degenerated elliptic curves. Can one describe their behavior by abelian integrals?

Our proof of Theorem 1 is completely based on the existence of the nice family (7) for which the period map is a biholomorphism. To prove Theorem 1 for certain subgroups of $\text{SL}(2,\mathbb{Z})$ by the methods of this article, one must find a four parameter family of elliptic curves such that the period map is an etale covering, i.e. it is a local biholomorphism of finite degree. The inverse of the period map is a multi valued function whose restriction to the simply connected domain $\tilde{\mathbb{H}}$ gives rise to a finite number of holomorphic functions on $\mathbb{H}$. These new functions are differential modular with respect to a subgroup of $\text{SL}(2,\mathbb{Z})$ which can be calculated by some topological data, such as the homotopy group of $T$, attached to the period map.

The Moduli space $T$ is not a Shimura variety (see [22]). In this point it would be too much speculation to say that spaces like $T$ can be constructed for arbitrary moduli of polarized Hodge structures. Nevertheless, more constructions of such spaces may result in generalizations of Shimura varieties.

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