Sparsified Block Elimination for Directed Laplacians

Richard Peng
University of Waterloo
y5peng@uwaterloo.ca

Zhuoqing Song
Fudan University
zqsong19@fudan.edu.cn

Abstract
We show that the sparsified block elimination algorithm for solving undirected Laplacian linear systems from [Kyng-Lee-Peng-Sachdeva-Spielman STOC’16] directly works for directed Laplacians. Given access to a sparsification algorithm that, on graphs with \( n \) vertices and \( m \) edges, takes time \( T_S(m) \) to output a sparsifier with \( N_S(n) \) edges, our algorithm solves a directed Eulerian system on \( n \) vertices and \( m \) edges to \( \epsilon \) relative accuracy in time

\[
O(T_S(m) + N_S(n) \log n \log(n/\epsilon) + O(T_S(N_S(n)) \log n),
\]

where the \( \tilde{O}(\cdot) \) notation hides \( \log \log(n) \) factors. By previous results, this implies improved run-times for linear systems in strongly connected directed graphs, PageRank matrices, and asymmetric M-matrices. When combined with slower constructions of smaller Eulerian sparsifiers based on short cycle decompositions, it also gives a solver that runs in \( O(n^{5/2} \log n \log(n/\epsilon)) \) time after \( O(n^2 \log^{O(1)} n) \) pre-processing. At the core of our analyses are constructions of augmented matrices whose Schur complements encode error matrices.

1 Introduction
The design of efficient solvers for systems of linear equations in graph Laplacian matrices and their extensions has been a highly fruitful topic in algorithms. Laplacian matrices directly correspond to undirected graphs: off-diagonal entries are negations of edge weights, while the diagonal entries contain weighted degrees. Solvers for Laplacian matrices led to breakthroughs in fundamental problems in combinatorial optimization. Tools developed during such studies have in turn influenced data structures, randomized numerical linear algebra, scientific computing, and network science [Spi10, Ten10].

An important direction in this Laplacian paradigm of designing graph algorithms is extending tools developed for undirected Laplacian matrices to directed graphs. Here a perspective from random walks and Markov chains leads to directed Laplacian matrices [CKP⁺16]. Such matrices have directed edge weights in off-diagonal entries, and weighted out-degrees on diagonals. In contrast to solving linear systems in undirected Laplacians, solving linear systems in directed Laplacians is significantly less well-understood. Almost-linear time [CKP⁺17] and nearly-linear time solvers [CKK⁺18] were developed very recently, and involve many more moving pieces.

In particular, the nearly-linear time algorithm from [CKK⁺18] combined block Gaussian elimination with single variables/vertex elimination, analyzed using matrix Martingales. In contrast, for undirected Laplacians, both block elimination [KLP⁺16] or matrix Martingales [KSI16] can...
give different nearly-linear time solver algorithms, and there also exists more combinatorial approaches \cite{KOSZ13}. In this paper, we simplify this picture for directed Laplacian solvers by providing an analog of the sparsified Cholesky/multi-grid solver from \cite{KLP+16}. This algorithm’s running time is close to the limit of sparsification based algorithms: the running time of invoking a sparsification routine on its own output. Formally, we show:

**Theorem 1.1.** Given a strongly connected Eulerian Laplacian $L \in \mathbb{R}^{n \times n}$ and an error parameter $\epsilon \in (0, 1)$, we can process it in time $O(T_S(m, n, 1)) + \tilde{O}(T_S(N_S(n, 1), n, 1) \log n)$ so that, with high probability, given any vector $b \in \mathbb{R}^n$ with $b \perp 1$, we can compute a vector $x \in \mathbb{R}^n$ in time $O(N_S(n, 1) \log n \log(n/\epsilon))$ such that

$$
\|x - L^1 b\|_{U[L]} \leq \epsilon \|L^1 b\|_{U[L]},
$$

where $U[L] = (L + L^\top)/2$.

This result improves the at least $\Omega(\log^5 n)$ factor overhead upon sparsification of the previous nearly-linear time directed Laplacian solver \cite{CKK+18}, and is analogous to the current best overheads for sparsification based solvers for undirected Laplacians \cite{KLP+16}. From the existence of sparsifiers of size $O(n \log^d n e^{-2})$ \cite{CGP+18}, we also obtain the existence of $O(n \log^5 n \log(n/\epsilon))$ time solver routines that require quadratic time preprocessing to compute. As with other improved solvers for directed Laplacians our improvements directly applies to applications of such solvers, including random walk related quantities \cite{CKP+16}, as well as PageRank / Perron-Frobenius vectors \cite{AJSS19}.

Our result complements recent developments of better sparsifiers of Eulerian Laplacians \cite{CGP+18 LSY19 PY19}. By analyzing a pseudocode that’s entirely analogous to the undirected block-elimination algorithm from \cite{KLP+16}, we narrow the gap between Laplacian solvers for directed and undirected graphs. Our result also emphasizes the need for better directed sparsification routines. While there is a rich literature on undirected sparsification \cite{BSST13}, the current best directed sparsification algorithms rely on expander decompositions, so have rather large logarithmic factor overheads. We discuss such bounds in detail in Appendix C.

Finally, our analysis of this more direct algorithm require better understanding the accumulation errors in Eulerian Laplacians and their partially eliminated states, known as Schur Complements. It was observed in \cite{CKK+18} that these objects are significantly less robust than their undirected analogs. Our analysis of these objects rely on augmentations of matrices: constructing larger matrices whose Schur complements correspond to the final objects we wish to approximate, and bounding errors on these larger matrices instead. This approach has roots in symbolic computation, and can be viewed as a generalization of low-rank perturbation formulas such as Sherman-Morrison-Woodbury \cite{vdB21}. We believe both this algebraic technique, and the additional robustness properties of directed Schur Complements we show, may be of independent interest.

### 1.1 Related Works

Directed Laplacian matrices arise in problems related to directed random walks / non-reversible Markov chains, such as computations of stationary distributions, hitting times and escape probabilities. A formal treatment of applying an Eulerian solver to these problems can be found in \cite{CKP+16} and \cite{AJSS19}. Adaptations of Eulerian Laplacian solvers have also led to improved bounded-space algorithms for estimating random walk probabilities \cite{AKM+20}.
Our algorithm is most closely related to the previous nearly-linear time directed Laplacian solver \cite{CKK+18}. That algorithm is motivated by single variable elimination and a matrix Martin-gale based analysis. However, it invokes both components of block elimination algorithms: finding strongly diagonally dominant subsets, and invoking sparsification as black-boxes. The runtime overhead of this routine over sparsification is at least $\log^5 n$: in \cite{CKK+18}, Lemma 5.1 gives that each phase (for a constant factor reduction) invokes sparsification $O(\log^2 n)$ times, and each call is ran with error at most $\frac{1}{O(\log^3 n)}$ (divided by $\log^2 n$ in Line 2 of Algorithm 2, and also by $\log n$ in Line 5 of Algorithm 3).

While our algorithms are directed analogs of the undirected block elimination routines from in \cite{KLP+16}, our analyses rely on many structures developed in \cite{CKK+18}. Specifically, our cumulative error during elimination steps is bounded via the matrix that’s the sum of undiretifications of the intermediate directed matrices. On the other hand, we believe our algorithm is more natural: our sampling no longer needs to be locally unbiased, the per-step errors do not need to be decreased by polylog factors, and the algorithm is no longer divided into inner/outer phases. This more streamlined algorithm leads to our runtime improvements.

Our Schur Complement sparsification algorithm is based on the partial block elimination routine from \cite{KLP+16}, which is in turn based on a two-term decomposition formula for (pseudo-)inverses from \cite{PS14}. We remark that there is a good sparsification routine in the low space setting \cite{AKM+20}. There is a subsequent algorithm that replaces this decomposition with directly powering via random walks \cite{CCL+15} that’s also applicable for sparsifying undirected Schur Complements. However, that algorithm relies on sparsifying 3-step random walk polynomials, which to our knowledge, is a subroutine that has not been studied in directed settings. As a result, we are unable to utilize this later development directly.

The existence of $O(n \log^4 n)$ sized sparsifiers in \cite{CGP+18} relies on decomposing unit weighted graphs into short cycles and $O(n)$ extra edges. While this decomposition has a simple $O(m^2)$ time algorithm (peel off all vertices with degree < 3, then return the lowest cross-edge in the BFS tree), the current fastest construction of it takes $m^{1+o(1)}$ time \cite{CGP+18, PY19, LSY19}. As a result, we need to instead invoke the more expensive, graph decomposition based, algorithms from \cite{CKP+17} for sparsification. Also, we can only use the naive $O(m^2)$ construction of $O(\log n)$-lengthed cycle decompositions (after an initial sparsification call to make $m = O(n \log^{O(1)} n)$) because the almost-linear time algorithm in \cite{LSY19} produces $O(\log^2 n)$-lengthed cycles.

\section{Preliminary}

\subsection{Notations}

\textbf{General Notations:} The notation $\tilde{O}(\cdot)$ suppresses the polyloglog$(n)$ factors in this paper. We let $[n] = \{1, 2, \cdots, n\}$. For matrix $A$, $\text{nnz}(A)$ denotes its number of nonzero entries. For matrix $A \in \mathbb{R}^{n \times n}$ and subsets $T_1, T_2 \subseteq [n]$, $A_{T_1,T_2} \in \mathbb{R}^{[T_1] \times [T_2]}$ is the submatrix containing the entries with row indexes and column indexes in $(T_1, T_2)$; and $A_{-T_1,-T_2}$ is the submatrix of $A$ by removing the rows indexed by $T_1$ and columns indexed by $T_2$. For vector $v \in \mathbb{R}^n$ and subset $C \subseteq [n]$, $v_C$ is the subvector of $v$ containing the entries indexed by $C$.

\texttt{arXiv version 1 https://arxiv.org/pdf/1811.10722v1.pdf}
Matrix: We use $I_a$, $0_{b \times c}$ to denote the identity matrix of size $a$ and the $b$-by-$c$ zero matrix, and we sometimes omit the subscripts when their sizes can be determined from the context. For any matrix $X \in \mathbb{R}^{n \times b}$ and set $T_1, T_2 \subseteq [n]$ with $|T_1| = a$, $|T_2| = b$, $P(X, T_1, T_2, n)$ denotes an $n$-by-$n$ matrix whose submatrix indexed by $(T_1, T_2)$ equals $X$ and all the other entries equal 0. In other words, $P(X, T_1, T_2, n)$ can be regarded as replacing the submatrix indexed by $(T_1, T_2)$ with $X$ in the zero matrix $0_{n \times n}$.

For symmetric matrix $A \in \mathbb{R}^{n \times n}$, we use $\lambda_i(A)$ to denote its $i$-th smallest eigenvalue. For symmetric matrices $A, B \in \mathbb{R}^{n \times n}$, we use $A \succeq B$ ($A \succ B$) to indicate that for any $x \in \mathbb{R}^n$, $x^T A x \geq x^T B x$ ($x^T A x > x^T B x$). We define $\preceq$, $\prec$ analogously. A square matrix $A \in \mathbb{R}^{n \times n}$ is positive semidefinite (PSD) iff $A$ is symmetric and $A \succ 0$; $A \in \mathbb{R}^{n \times n}$ is positive definite (PD) iff $A$ is symmetric and $A \succ 0$.

For PSD matrix $A$, $A^{1/2}$ is its square root; $A^\dagger$ denotes its Moore-Penrose pseudoinverse; $A^{1/2}$ is the square root of its Moore-Penrose pseudoinverse.

Vector: $1_a$, $0_b$ denote the $a$-dimensional all-ones vector and $b$-dimensional all-zeros vector; when their sizes can be determined from the context, we sometimes omit the subscripts.

For matrix $A \in \mathbb{R}^{n \times n}$, $\text{Diag}(A)$ is an $n$-by-$n$ diagonal matrix with the same diagonal entries as $A$. For vector $x \in \mathbb{R}^n$, $\text{Diag}(x)$ denotes an $n$-by-$n$ diagonal matrix with its $i$-th diagonal entry equaling $x_i$.

For any positive semidefinite matrix $A$, we define the vector norm $\|x\|_A = \sqrt{x^T Ax}$. And $\|\cdot\|_p$ denotes the $\ell^p$ norm.

Matrix norm: $\|\cdot\|_p$ denotes the $\ell^p$ norm. For instance, for $A \in \mathbb{R}^{n \times n}$, $\|A\|_2 = \sqrt{\lambda_n(A^T A)}$; $\|A\|_\infty = \max_{i \in [n]} \sum_{j=1}^n |A_{ij}|$. For matrix $B \in \mathbb{R}^{n \times n}$ and PSD matrix $A \in \mathbb{R}^{n \times n}$, we denote $\|B\|_{A \rightarrow A} = \sup_{\|x\|_A \neq 0} \frac{\|Bx\|_A}{\|x\|_A}$.

Schur complement: For $A \in \mathbb{R}^{n \times n}$ and $F,C$ a partition of $[n]$ such that $A_{FF}$ is nonsingular, the Schur complement of $F$ in $A$ is defined as $\text{Sc}(A,F) = A_{CC} - A_{CF} A_{FF}^{-1} A_{FC}$.

When we need to emphasize the support set of the entries that remain, we also denote $\text{Sc}(A,-C) = \text{Sc}(A,F)$.

2.2 (Directed) Laplacians, Symmetrizations

A matrix $L \in \mathbb{R}^{n \times n}$ is called a directed Laplacian iff $1^T L = 0^T$ and all off-diagonal entries of $L$ are non-positive, i.e., $L_{ij} = -\sum_{j \neq i} L_{ji}$ for all $i \in [n]$ and $L_{ij} \leq 0$ for all $i \neq j$. A (directed) Laplacian $L$ can be associated with a (directed) graph $G[L]$ whose adjacency matrix is $\hat{A} = \text{Diag}(L) - L^T$. The in-degrees/out-degrees of $L$ are defined as the in-degrees/out-degrees of $G[L]$. For directed Laplacians, its out-degrees equal its diagonal entries. If $G[L]$ is strongly connected, we say the (directed) Laplacian $L$ is strongly connected.

In addition, if $L 1 = 0$, we call $L$ an Eulerian Laplacian. These Laplacians have the property that in-degrees of vertices equal to out-degrees. The undirected Laplacian is a special case where $L = L^T$. We often refer to these as symmetric Laplacians, or just Laplacians.

Symmetrization: For square matrix $A \in \mathbb{R}^{n \times n}$, we define its matrix symmetrization as $U[A] = \frac{A + A^T}{2}$. For a directed Laplacian $L \in \mathbb{R}^{n \times n}$, we define its undirectification as
\[ \mathcal{U}^G[L] = \frac{1}{2}(L + L^\top - \text{Diag}((L + L^\top)1)). \] \( \mathcal{U}^G[L] \) is called the undirectification because it is a symmetric Laplacian whose adjacency matrix is \( U[A] \), where \( \hat{A} = \text{Diag}(L) - L^\top \) is the adjacency matrix of \( G[L] \). For an Eulerian Laplacian \( L \), its matrix symmetrization coincides with its undirectification, i.e., \( U[L] = \mathcal{U}^G[L] \). Eulerian Laplacians are critically important in solvers for directed Laplacians because they are the only setting in which the undirectification is positive semidefinite.

**Row Column Diagonal Dominant (RCDD):** A square matrix \( A \in \mathbb{R}^{n \times n} \) is \( \alpha \)-RCDD iff \( \sum_{j \in [n] \setminus \{i\}} |A_{ij}| \leq \frac{1}{1+\alpha} A_{ii} \) and \( \sum_{j \in [n] \setminus \{i\}} |A_{ji}| \leq \frac{1}{1+\alpha} A_{ii} \) for any \( i \in [n] \). We also say \( A \) is RCDD if \( A \) is 0-RCDD.

### 2.3 Sparsification

All almost-linear time or faster solvers for directed Laplacians to date are built around sparsification: the approximation of graphs by ones with fewer edges. As it’s difficult to even approximate reachability of directed graphs, [CKP+17] introduced the key idea of measuring approximations w.r.t. a symmetric PSD matrix. Such approximations are at the core of all subsequent algorithms, including ours.

**Definition 2.1.** (Asymmetrically bounded) Given a matrix \( A \in \mathbb{R}^{n \times n} \) and a PSD matrix \( U \in \mathbb{R}^{n \times n} \), \( A \) is asymmetrically bounded by \( U \) iff \( \ker(U) \subseteq \ker(A^\top) \cap \ker(A) \) and \( \|U^{1/2} A U^{1/2}\|_2 \leq 1 \).

We denote it by \( A \asymp_U \).

By our definition, \( A \asymp_U U \) is equivalent to \( -A \asymp_U U \). The following lemma is changed slightly from Lemma B.2 of [CKP+17].

**Fact 2.2.** For any matrix \( A \in \mathbb{R}^{n \times n} \) and PSD matrix \( U \in \mathbb{R}^{n \times n} \), the following statements are equivalent:

- \( A \asymp_U U \).
- \( 2x^\top Ay \leq x^\top U x + y^\top U y \), \( \forall x, y \in \mathbb{R}^n \).

**Definition 2.3.** (Approximation of directed Laplacians via undirectification) Given matrix \( A \in \mathbb{R}^{n \times n} \) and directed Laplacian \( B \in \mathbb{R}^{n \times n} \), \( A \) is an \( \epsilon \)-asymmetric approximation of \( B \) iff \( A - B \) is asymmetrically bounded by \( \epsilon \cdot \mathcal{U}^G[B] \).

In particular, for strongly connected Eulerian Laplacians \( A \) and \( B \), \( A \) is an \( \epsilon \)-asymmetric approximation of \( B \) iff \( \|U[B]^{1/2}(A - B) U[B]^{1/2}\|_2 \leq \epsilon \).

We will utilize sparsifiers for Eulerian Laplacians [CKP+17] [CGP+18], as well as implicit sparsifiers for products of directed adjacency matrices as black boxes throughout our presentations. The formal statements of these black boxes are below.

**Theorem 2.4.** (Directed Laplacian sparsification oracle) Given a directed Laplacian \( L \in \mathbb{R}^{n \times n} \) with \( \text{nnz}(L) = m \) and error parameter \( \delta \in (0, 1) \), there is an oracle \text{ORASPARSELAPLACIAN} which runs in at most \( T_S(m, n, \delta) \) time, where \( T_S(m, n, \delta) = O((m \log^{O(1)} n + n \log^{O(1)} n)\delta^{-O(1)}) \), to return with high probability a directed Laplacian \( \tilde{L} \) satisfying:
(i) \( \text{nnz}(\bar{L}) \leq N_S(n, \delta) \) where \( N_S(n, \delta) = O(n \log^{O(1)} n \delta^{-O(1)}) \);

(ii) \( \text{Diag}(\bar{L}) = \text{Diag}(L) \);

(iii) \( \bar{L} - L \xrightarrow{\text{asym}} \delta \cdot U^G[L] \).

Remark 2.5. Conditions (ii), (iii) in Theorem 2.4 above are equivalent to \( \bar{L} \) and \( L \) having the same in- and out-degrees, and \( \| U^G[L]^{1/2} (\bar{L} - L) U^G[L]^{1/2}\|_2 \leq \delta \) respectively. By having the same in-degrees and out-degrees, we mean \( \text{Diag}(\bar{L}) = \text{Diag}(L) \) and \( \bar{L}_1 = L_1 \).

Lemma 2.6. (Lemma 3.18 of [CKP+17]) Let \( x, y \in \mathbb{R}^n \) be nonnegative vectors with \( \text{nnz}(x) + \text{nnz}(y) = m \) and let \( \epsilon, \nu \in (0, 1) \). And we denote \( G = (1^\top \cdot x) \text{Diag}(y) - xy^\top \). Then, there is a routine \text{SparseProduct} which computes with probability at least \( 1 - \nu \) a nonnegative matrix \( A \) in \( O\left( me^{-2} \log \frac{m}{\nu} \right) \) time such that \( \text{nnz}(A) = O\left( me^{-2} \log \frac{m}{\nu} \right) \), \( A - xy^\top \xrightarrow{\text{asym}} \epsilon \cdot U^G[G] \).

Given an Eulerian Laplacian \( L \in \mathbb{R}^{n \times n} \) and a partition \( F, C \) of \( [n] \), by invoking \text{OraSparseLaplacian} on subgraphs with edges inside \((F, F)\), \((F, C)\), \((C, F)\), \((C, C)\) respectively, we can get a Laplacian sparsification procedure \text{SparseEulerianFC} so that the sparsified Eulerian Laplacians returned by \text{SparseEulerianFC} not only satisfy all the properties mentioned in Theorem 2.4, but also keep the in-degrees and out-degrees of the subgraph supported by \((F, F)\). Analogously, a routine \text{SparseProductFC} can be constructed by applying \text{SparseProduct} four times. For explicit definitions of \text{SparseEulerianFC} and \text{SparseProductFC}, see Lemma D.1.

2.4 Sufficiency of Solving Eulerian Systems to Constant Error

Previous works on solvers for directed Laplacians and their generalizations (to RCDD and M-matrices) established that it’s sufficient to solve Eulerian systems to constant relative accuracy in their undirectification.

- The iterative refinement procedure shown in [CKP+17] shows that a constant accuracy solver can be amplified to one with \( \epsilon \) relative accuracy in \( O(\log(1/\epsilon)) \) iterations.

- The stationary computation procedure in [CKP+16] showed that arbitrary strongly connected Laplacians with mixing time \( T_{mix} \) can be solved to 2-norm error \( \epsilon \) by solving \( O(\log(T_{mix}/\epsilon)) \) systems in Eulerian Laplacians. This was subsequently simplified and extended to M-matrices and row-column-diagonally-dominant matrices in [AJSS19] (with an extra \( \log n \) factor in running time). A purely random walk (instead of matrix perturbation) based outer loop is also given in the thesis of Peebles [Pee19].

3 Overview

Our algorithm is based on sparse Gaussian elimination. Before we discuss the block version, it is useful to first describe how the single variable version works on Eulerian Laplacians.

Recall that Eulerian Laplacians store the (weighted) degrees on the diagonal, and the negation of the out edge weights from \( j \) in column \( j \).
Suppose we eliminate vertex \( j \). Then we need to add a rescaled version of row \( j \) to each row \( i \) where \( L_{ij} \) is non-zero. Accounting for \( L_{jj} = d_j \), this weight for row \( i \) is given by \( \frac{w_{j \rightarrow i} w_{k \rightarrow j}}{d_j} \), and the corresponding decrease in entry \( j, k \) is then

\[
\frac{w_{j \rightarrow i} w_{k \rightarrow j}}{d_j}.
\]

In other words, when eliminating vertex \( j \), we add an edge \( k \rightarrow i \) for each triple of vertices \( k \rightarrow j \rightarrow i \), with weight given by above.

The effect of this elimination on the vector \( b \) can also be described through this ‘distribution’ of row \( j \) onto its out-neighbors. However, to start with, it’s useful to focus on what happens to the matrix. The key observation for elimination based Laplacian solvers is that this new matrix remains a graph. In fact, it can be checked that this process exactly preserves the in and out degrees of all neighbors of \( i \), so the graph also remains Eulerian.

However, without additional assumptions on the non-zero structures such as separators, directly performing the above process takes \( O(n^3) \) time: the newly added entries quickly increases the density of the matrix until each row has \( \Theta(n) \) entries. So the starting point of elimination based Laplacian solvers is to address the following two problems:

1. Keeping the results of elimination sparse.
2. Find vertices that are easy to eliminate.

3.1 Block Cholesky

One possible solution to the issue above is to directly sample the edges formed after eliminating each vertex. It leads to sparsified/incomplete Cholesky factorization based algorithms \([KS16, CKK+18]\), including the first nearly-linear time solver for directed Laplacians.

Our algorithm is based on eliminating blocks of vertices, and is closest to the algorithm from \([KLP+16]\). It aims to eliminate a block of \( \Omega(n) \) vertices simultaneously. This subset, which we denote using \( F \), is chosen to be almost independent. That is, \( F \) is picked so that each vertex in \( F \) has at least a constant portion of its out-degree going to \( V \setminus F \), which we denote as \( C \).

This property means that any random walk on \( F \) exits it with constant probability. This intuition, when viewed from iterative methods perspective, implies that power method allows rapid simulation of elimination onto \( C = V \setminus F \). From a matrix perspective, it means these matrices are well-approximated by their diagonal. So subproblem \( L_{FF}^{-1} b \) can be solved to high accuracy in \( O(\log n) \) iterations via power method. We formalize the guarantees of such procedures, Pre-Richardson in Lemma \( 6.5 \) in Section \( 6.2 \).

Compared to single-vertex elimination schemes, block elimination has the advantage of having less error accumulation. Single elimination can be viewed as eliminating 1 vertex per step, while we will show that the block method eliminates \( \Omega(n) \) vertices in \( O(\log \log n) \) steps. This smaller number of dependencies in turn provides us the ability to bound errors more directly.

Formally, given the partition \( F, C \subseteq [n] \) with the permutation matrix \( P \) such that \( P L P \top = \begin{bmatrix} L_{FF} & L_{FC} \\ L_{CF} & L_{CC} \end{bmatrix} \), the block Cholesky factorization of \( L \in \mathbb{R}^{n \times n} \) is given as

\[
L = P \top \begin{bmatrix} I_{|F|} & 0_{|F| \times |C|} \\ L_{CF} L_{FF}^{-1} & I_{|C|} \end{bmatrix} \cdot \begin{bmatrix} L_{FF} & 0_{|F| \times |C|} \\ 0_{|C| \times |F|} & SC(L, F) \end{bmatrix} \cdot \begin{bmatrix} I_{|F|} & L_{FF}^{-1} L_{FC} \\ 0_{|C| \times |F|} & I_{|C|} \end{bmatrix} P.
\]
Algorithm 1: Precondition \( \left( \left\{ S^{(i)} \right\}_{i=1}^{d}, \left\{ F_i \right\}_{i=1}^{d} \right), x, N \)

Input: \((\alpha, \beta, \{\delta_i\}_{i=1}^{d})\)-Schur complement chain \( \left\{ S^{(i)} \right\}_{i=1}^{d}, \left\{ F_i \right\}_{i=1}^{d} \); vector \( x \in \mathbb{R}^n \); error parameter \( \epsilon \in (0, 1) \)

Output: vector \( x \in \mathbb{R}^n \)

1. for \( i = 1, \cdots, d-1 \) do
   2. \( x_{F_i} \leftarrow \text{PreRichardson}( S^{(i)}_{F_i, F_i}, x_{F_i}, \text{Diag}( S^{(i)}_{F_i, F_i})^{-1}, \frac{1}{2}, N ) \)
   3. \( x_{C_i} \leftarrow x_{C_i} - S^{(i)}_{C_i, F_i} x_{F_i} \)
   4. end

5. \( x_{F_d} \leftarrow \left(S^{(d)}\right)_{F_d} \)

6. for \( i = d-1, \cdots, 1 \) do
   7. \( x_{F_i} \leftarrow x_{F_i} - \text{PreRichardson}( S^{(i)}_{F_i, F_i}, S^{(i)}_{F_i, C_i}, x_{C_i}, \text{Diag}( S^{(i)}_{F_i, F_i})^{-1}, \frac{1}{2}, N ) \)
   8. end

9. Let \( x \leftarrow x - \frac{1}{n} x \cdot 1 \)
10. Return \( x \)

Using the above factorization iteratively (with sparsification) generates a Schur complement chain \( \left\{ S^{(i)} \right\}_{i=1}^{d}, \left\{ F_i \right\}_{i=1}^{d} \) (Definition 6.1), the solver algorithm loops through these and solves the subproblems via the projection / prolongation maps defined via the random walk on \( F_i \) respectively, using the power-method based elimination procedure described above. Its pseudocode is given in Algorithm 1 for completeness.

We remark that in practice, the iteration numbers of the preconditioned Richardson iterations in Algorithm 1 can differ with each other. Here, we set a uniform \( N \) merely for simplicity. If we have unlimited (or quadratic) precomputation power, the method described above suffices to give us a fast solver. However, due to the exact Schur Complements being dense, the major remaining difficulty is to efficiently compute an approximate Schur complement.

3.2 Schur Complement Sparsification via Partial Block Elimination

Thus, the main bottleneck toward an efficient algorithm is the fast construction of approximate Schur complements. We will give such an algorithm whose running time is close to those of sparsification primitives via a partial block elimination process.

In simple terms, a step of this process squares the \((F, F)\) block. Repeating this gives quadratic convergence. With \( \alpha \) about \( O(1) \), \( O(\log \log n) \) iterations suffice, so the resulting error is easier to control than martingales.

Lemma 3.1 \([\text{PS}14]\). For any diagonal matrix \( D \in \mathbb{R}^{n \times n} \) and a matrix \( A \in \mathbb{R}^{n \times n} \) with \( D - A \) nonsingular, we have

\[
(D - A)^{-1} = \frac{1}{2} \left( D^{-1} + (I + D^{-1} A) (D - AD^{-1} A)^{-1} (I + AD^{-1}) \right).
\] (1)
The main identity \([1]\) gives rise to our definition for partial-block-eliminated Laplacian of \(L\). Let \(D = \text{Diag}(L)\) and \(A = D - L\), let \(P\) be the permutation matrix such that \(P LP^\top = \begin{bmatrix} L_{FF} & \tilde{L}_{FC} \\ L_{CF} & L_{CC} \end{bmatrix}\).

The partial-block-eliminated operator \(\Phi(L|D_{FF}, F)\) is defined as
\[
\Phi(L|D_{FF}, F) = P^\top \begin{bmatrix} -A_{FC} & \tilde{L}_{CC} \\ -A_{CF} & 2L_{CC} \end{bmatrix} P - A_{:,F}D_{FF}^{-1}A_{:,F}^\top.
\]

We define the first exact partial-block-elimination of \(L\) by \(L^{(1)} = \Phi(L|D_{FF}, F)\). Then, \(\frac{1}{2}L^{(1)}\) is an Eulerian Laplacian which has the same Schur complement of \(F\) in \(L\), i.e.,
\[
\text{Sc}(L, F) = \frac{1}{2}\text{Sc}(L^{(1)}, F).
\]

The 2-nd to the \(K\)-th partially-block-eliminated Laplacians are defining iteratively as \(L^{(2)} = \Phi(L^{(1)}|D_{FF}, F), \ldots, L^{(K)} = \Phi(L^{(K-1)}|D_{FF}, F)\). \(L^{(k)}\) can also be regarded as a partially powered matrix of \(L\), which uses the powering to obtain better spectral properties. Specifically, when we focus on the \((F, F)\) block of \(L^{(k)}\), it is easy to see that \(\|D_{FF}^{-1}A_{FF}^{(k)}\|_\infty\) converges at a quadratic rate, where \(A_{FF}^{(k)} = L_{FF}^{(k)} - D_{FF}\). Formal construction of the \(k\)-th partially-block-eliminated Laplacians and their properties are deferred to Appendix [3].

To encounter the increasing density of \(L^{(k)}\), sparsification blackboxes in Section 2.3 are naturally accompanied with the partial block elimination to yield a Schur complement sparsification method (Algorithm 2). Our algorithm is essentially a directed variant of the one from [KLP+16]. A slight difference is that in the last step, we need to fix the degree discrepancies caused by approximating the strongly RCDD matrix by its diagonal.

The running time of Algorithm 2 is shown in Theorem 3.3. The \(k\)-th iterand \(\tilde{L}^{(k)}\) of Algorithm 2 is termed the approximate \(k\)-th partially-block-eliminated Laplacian, while its exact version is just the (exact) \(k\)-th partially-block-eliminated Laplacian \(L^{(k)}\) defined above. To guarantee the performance of Algorithm 2, the most important thing is to provide relatively tight bounds for the difference \(\tilde{L}^{(k)} - L^{(k)}\).

**Remark 3.2.** This permutation matrix \(P\) defined on line 2 of Algorithm 2 is only used to simplify the pseudocodes in Lines 8, 9. We don’t need to construct it in practice. We use the same \(D_{FF}\) in each iteration to simplify our analysis. It is possible to replace \(D_{FF}\) by \(\text{Diag}(L^{(k-1)})_{FF}\) in iterations \(k\) and achieve similar running time.

**Theorem 3.3.** (Schur complement sparsification) For a strongly connected Eulerian Laplacian \(L \in \mathbb{R}^{n \times n}\), let \(F, C\) be a partition of \([n]\) such that \(L_{FF}\) is \(\alpha\)-RCDD (\(\alpha = O(1)\)) and let \(\delta \in (0, 1)\) be an error parameter, the subroutine \(\text{SPARSE-SCHUR}\) (Algorithm 2) runs in time
\[
O(\mathcal{T}_S(m, n, \delta)) + \tilde{O}(\mathcal{T}_S(\mathcal{N}_S(n, \delta)^{-2}, n, \delta) \log n)
\]
to return with high probability a strongly connected Eulerian Laplacian \(S\) satisfying \(\text{nnz}(S) = O(\mathcal{N}_S(|C|, \delta))\) and
\[
S - \text{Sc}(L, F) \stackrel{\text{asym}}{\asymp} \delta \cdot U[\text{Sc}(L, F)].
\]
When considering the approximate partial block elimination, in one update step, not only new sparsification errors are added into \( \tilde{L}^{(k)} \), the errors accumulated from previous steps will multiply significantly more complicated spectral structures. To analyze them, we develop new interpretations of directed Schur complements based on matrix extensions.

### 3.3 Bounding Error Accumulations in Partially-Eliminated Laplacians by Augmented Matrices

When considering the approximate partial block elimination, in one update step, not only new sparsification errors are added into \( \tilde{L}^{(k)} \), the errors accumulated from previous steps will multiply significantly more complicated spectral structures. To analyze them, we develop new interpretations of directed Schur complements based on matrix extensions.

**Algorithm 2:** \texttt{SparseSchur}(\(L, F, \delta\))

\begin{algorithm}
\begin{algorithmic}
\STATE \textbf{Input:} strongly connected Eulerian Laplacian \( L \in \mathbb{R}^{n \times n} \); partition \( F, C \) of \([n]\); error parameter \( \delta \in (0, 1) \)
\STATE \textbf{Output:} Sparse approximate Schur complement \( S \)
\IF{\text{nnz}(L) \geq O(\mathcal{N}_S(n, \delta))}
\STATE \text{call \texttt{ORASPARSELAPLACIAN} to sparsify \( L \) with error parameter \( O(\delta) \)}
\ENDIF
\STATE Find a permutation matrix \( P \) such that \( P L P^\top = \begin{bmatrix} L_{FF} & L_{FC} \\ L_{CF} & L_{CC} \end{bmatrix} \)
\STATE Set \( K \leftarrow O(\log \log \frac{n}{\delta}) \), \( \epsilon \leftarrow O(\frac{\delta}{K}) \), \( \tilde{L}^{(0)} \leftarrow L \), \( D \leftarrow \text{Diag}(\tilde{L}^{(0)}) \), \( \tilde{A}^{(0)} \leftarrow D - \tilde{L}^{(0)} \)
\FOR{\( k = 1, \ldots, K \)}
\FOR{\( i \in F \)}
\STATE \( \tilde{Y}^{(k,i)} \leftarrow \text{SPARSEPRODUCTFC} \left( \tilde{A}^{(k-1)}_{i,:}, (\tilde{A}^{(k-1)}_{i,:})^\top, \epsilon, F \right) \)
\ENDFOR
\STATE Let \( \tilde{Y}^{(k)} \leftarrow \sum_{i \in F} \frac{1}{D_{ii}} \tilde{Y}^{(k,i)} \), \( \tilde{L}^{(k,0)} \leftarrow P^\top \begin{bmatrix} D_{FF} & -\tilde{A}^{(k-1)}_{FC} \\ -\tilde{A}^{(k-1)}_{CF} & \frac{1}{2}L_{CC}^{(k-1)} \end{bmatrix} P - \tilde{Y}^{(k)} \)
\STATE \( \tilde{L}^{(k)} \leftarrow \text{SPARSEEULERIANFC} \left( \tilde{L}^{(k,0)}, \epsilon, F \right) \) and \( \tilde{A}^{(k)} \leftarrow P^\top \begin{bmatrix} D_{FF} \\ \text{Diag}(L_{CC}^{(k)}) \end{bmatrix} P - \tilde{L}^{(k)} \)
\ENDFOR
\FOR{\( i \in F \)}
\STATE \( \tilde{X}^{(i)} \leftarrow \text{SPARSEPRODUCT} \left( \tilde{A}^{(K)}_{C,i}, (\tilde{A}^{(K)}_{i,C})^\top, \epsilon \right) \)
\ENDFOR
\STATE Let \( \tilde{X} \leftarrow \sum_{i \in F} \frac{1}{D_{ii}} \tilde{X}^{(i)} \) and \( \tilde{S}^{(0)} \leftarrow \frac{1}{2\pi} (L_{CC}^{(K)} - \tilde{X}) \)
\STATE Compute a matching matrix \( R \in \mathbb{R}^{|C| \times |C|} \) with \( R_{2:|C|,1} = -\tilde{S}^{(0)}_{2:|C|,1}, R_{1,2:|C|} = -1^\top \tilde{S}^{(0)}_{:,2:|C|}, R_{1,1} = -R_{1,2:|C|} 1 - 1^\top R_{2:|C|,1} - 1^\top \tilde{S}^{(0)}_{:,1}, \) and \( R_{ij} = 0 \) for \( i \neq 1 \) and \( j \neq 1 \)
\STATE Set \( \tilde{S} = \tilde{S}^{(0)} + R \)
\STATE Return \( S = \text{ORASPARSELAPLACIAN} \left( \tilde{S}, \delta/8 \right) \)
\end{algorithmic}
\end{algorithm}

Compared to the undirected analog from [KLP+16], powered directed matrices exhibit significantly more complicated spectral structures. To analyze them, we develop new interpretations of directed Schur complements based on matrix extensions.
with each other and get possibly amplified. In addition, error accumulations in Schur complements of directed Laplacians are not as straightforward as their undirected counterparts. It’s not the case that for two directed Eulerian Laplacians with the same undirectification, the undirectification of their Schur complements are the same. For instance, consider the undirected vs. directed cycle, eliminated till only two originally diametrically opposite vertices remain. The former has a Schur complement that has weight $2/n$, while the latter has a Schur complement that has weight 1.

By the definition of $\epsilon$-asymmetric approximation, we need to essentially show the following inequality in order to obtain the approximations needed for a nearly-linear time algorithm:

\[
\frac{1}{2k} U \left[ L(k) \right] = \frac{1}{2k} U \left[ \Phi \left( L(k-1), D_{FF}, F \right) \right] = \frac{1}{2k} U \left[ \begin{pmatrix} 2L_{CC}^{(k-1)} - A_{CF}^{(k-1)} & D_{FF}^{-1} A_{FC}^{(k-1)} & -A_{CF}^{(k-1)} \left( I + D_{FF}^{-1} A_{FF}^{(k-1)} \right) \\ -\left( I + A_{FF}^{(k-1)} D_{FF}^{-1} A_{FC}^{(k-1)} \right) & D_{FF} - A_{FF}^{(k-1)} D_{FF}^{-1} A_{FF}^{(k-1)} \end{pmatrix} \right] \leq O(1) \cdot U[L], \ \forall 1 \leq k \leq K.
\]

Here significant difficulties arise due to the already complicated formula of $L(k)$. So we instead express the exact and approximate partial block elimination as Schur complements of large augmented matrices introduced below.

In the rest of Section 3, we assume $C = \{1, 2, \ldots, |C|\}$ for simplicity.

### 3.3.1 A Reformulation for Partial Block Elimination

For the exact and approximate $k$-th partially-block-eliminated matrices $L(k), \tilde{L}(k)$, we define augmented matrices $M(0,k), \tilde{M}(0,k)$ of size $2^k |F| + |C|$. We start with the construction of a desirable $M(0,k)$. To this end, we define a sequence of augmented matrices $\{M^{(i,k)}\}^k_{i=0}$, where $M^{(k,k)} = L(k)$ and each $M^{(i,k)}$ is a Schur complement of $M^{(i-1,k)}$. Here we only give an informal explanation of how we construct $M^{(i,k)}$. The formal definitions of these augmented matrices are given in Section 4. To begin with, for some fixed $k \in [K]$, we define $M^{(k,k)} \overset{def}{=} L(k)$. And we write $F_1 = F$ in the remainder of Section 3 and the entire Section 4 and Section 5.

Next, we take $M^{(k-1,k)}$ and $M^{(k-2,k)}$ as examples to show how we define such a sequence of matrices $M^{(k-1,k)}, \ldots, M^{(0,k)}$.

Define $M^{(k-1,k)}$ as follows

\[
M^{(k-1,k)} \overset{def}{=} \begin{bmatrix} C & F_1 & F_2 \\ 2L_{CC}^{(k-1)} - A_{CF}^{(k-1)} & -A_{CF}^{(k-1)} \\ -A_{FF}^{(k-1)} & D_{FF} - A_{FF}^{(k-1)} \end{bmatrix} \left( \begin{array}{c} C \\ F_1 \\ F_2 \end{array} \right) \left( \begin{array}{c} C \\ F_1 \\ F_2 \end{array} \right) \rightarrow \text{column indexes} \left( \begin{array}{c} C \\ F_1 \\ F_2 \end{array} \right) \rightarrow \text{rows indexed by} \ C \left( \begin{array}{c} C \\ F_1 \\ F_2 \end{array} \right) \rightarrow \text{rows indexed by} \ F_1 \left( \begin{array}{c} C \\ F_1 \\ F_2 \end{array} \right) \rightarrow \text{rows indexed by} \ F_2
\]

Then, it follows by direct calculations that

\[
SC\left( M^{(k-1,k)}, F_2 \right) = \begin{bmatrix} 2L_{CC}^{(k-1)} - A_{CF}^{(k-1)} & D_{FF}^{-1} A_{FC}^{(k-1)} & -A_{CF}^{(k-1)} \left( I + D_{FF}^{-1} A_{FF}^{(k-1)} \right) \\ -\left( I + A_{FF}^{(k-1)} D_{FF}^{-1} A_{FC}^{(k-1)} \right) & D_{FF} - A_{FF}^{(k-1)} D_{FF}^{-1} A_{FF}^{(k-1)} \end{bmatrix}
\]

\[
\overset{48a}{=} L(k).
\]
Next, we define $\mathcal{M}^{(k-2,k)}$ as follows

$$
\mathcal{M}^{(k-2,k)} \text{ def } = \begin{bmatrix}
C & F_1 & F_2 & F_3 & F_4 \\
4L_{CC}^{(k-2)} & -A_{CF}^{(k-2)} & -A_{CF}^{(k-2)} & -A_{CF}^{(k-2)} & -A_{CF}^{(k-2)} \\
-A_{FC}^{(k-2)} & D_{FF} & -A_{FC}^{(k-2)} & -A_{FC}^{(k-2)} & -A_{FC}^{(k-2)} \\
-A_{FC}^{(k-2)} & A_{FC}^{(k-2)} & D_{FF} & -A_{FC}^{(k-2)} & -A_{FC}^{(k-2)} \\
-A_{FC}^{(k-2)} & A_{FC}^{(k-2)} & A_{FC}^{(k-2)} & D_{FF} & -A_{FC}^{(k-2)} \\
\end{bmatrix}
$$

It follows by direct calculations that

$$
\text{Sc}\left(\mathcal{M}^{(k-2,k)}, F_3 \cup F_4\right) = \begin{bmatrix}
4L_{CC}^{(k-2)} & -2A_{CF}^{(k-2)} D_{FF}^{-1} A_{CF}^{(k-2)} & -A_{CF}^{(k-2)} \left( I + D_{FF}^{-1} A_{FF}^{(k-2)} \right) & -A_{CF}^{(k-2)} \left( I + D_{FF}^{-1} A_{FF}^{(k-2)} \right) \\
- \left( I + A_{CF}^{(k-2)} D_{FF}^{-1} \right) A_{CF}^{(k-2)} & D_{FF} & -A_{CF}^{(k-2)} D_{FF}^{-1} A_{CF}^{(k-2)} & -A_{CF}^{(k-2)} D_{FF}^{-1} A_{CF}^{(k-2)} \\
- \left( I + A_{CF}^{(k-2)} D_{FF}^{-1} \right) A_{CF}^{(k-2)} & A_{CF}^{(k-2)} & D_{FF} & -A_{CF}^{(k-2)} D_{FF}^{-1} A_{CF}^{(k-2)} \\
2L_{CC}^{(k-1)} & -A_{CF}^{(k-1)} & -A_{CF}^{(k-1)} & D_{FF} \\
-A_{FC}^{(k-1)} & D_{FF} & -A_{FC}^{(k-1)} & D_{FF} \\
-A_{FC}^{(k-1)} & D_{FF} & -A_{FC}^{(k-1)} & D_{FF} \\
\end{bmatrix}
$$

We will show $\frac{1}{2k} U \left[ L^{(k)} \right] \preccurlyeq O(1) \cdot U \left[ L \right]$ later by analyzing the properties of $\mathcal{M}^{(0,k)}$.

We believe this representation may be of independent interest. We also remark that these augmented matrices only arise during analysis, and are not used in the algorithms.

### 3.3.2 Bounding Error Accumulation of Schur complement sparsification

Next, we propose Lemma 5.2 to bound the errors after taking Schur complements. However, in our analysis, iteratively applying Lemma 5.2 to bound $\frac{1}{2k} \left( L^{(k)} - \tilde{L}^{(k)} \right)$ will lead to more log $n$ factors in the running time.

To derive a tighter bound, we introduce another group of augmented matrices $\left\{ \tilde{\mathcal{M}}^{(0,k)} \right\}$ which are defined by attaching sparsification errors to $\mathcal{M}^{(0,k)}$. $\tilde{\mathcal{M}}^{(0,k)}$ can help us disentangle the sparsification errors generated from different iterations and see how these errors accumulate as we do partial block eliminations more clearly.

We use another group of augmented matrices $\left\{ Q^{(k)} \right\}$ to bound the difference between $L^{(k)}$ and $\tilde{L}^{(k)}$. The augmented matrix $Q^{(k)}$ is defined as the weighted sum of a group of “reptition matrix” (Section 4.2). $Q^{(k)}$ adopts many properties similar to $\mathcal{M}^{(0,k)}$, so it is easy to analyze. Then, we can give tighter bound for $\tilde{\mathcal{M}}^{(K)} - \mathcal{M}^{(K)}$ using the robustness of Schur complements in this case (Section 5).
Here, we use some examples to illustrate how we construct the augmented matrices \( \{Q^{(k)}\} \), \( \{\tilde{M}^{(0,k)}\} \). Formal definitions are in Section 4.

Firstly, we define the sparsification error \( E^{(k)} = \tilde{L}^{(k)} - \Phi \left( \tilde{L}^{(k-1)} | D_{FF}, F \right) \). Take \( \tilde{M}^{(0,2)} \) as an example. We define

\[
\begin{align*}
\tilde{M}^{(1,2)} &= \begin{bmatrix}
2\tilde{L}^{(1)}_{CC} & -A^{(1)}_{CF} & -A^{(1)}_{CF} \\
-\tilde{A}^{(1)}_{FC} & D_{FF} & -\tilde{A}^{(1)}_{FF} \\
-\tilde{A}^{(1)}_{FC} & -\tilde{A}^{(1)}_{FF} & D_{FF}
\end{bmatrix} + \begin{bmatrix}
E^{(2)}_{CC} & E^{(2)}_{CF} \\
E^{(2)}_{FC} & E^{(2)}_{FF} \\
0_{|F|\times|F|}
\end{bmatrix},
\end{align*}
\]

and

\[
\tilde{M}^{(0,2)} = \begin{bmatrix}
4L_{CC} & -A_{CF} & -A_{CF} & -A_{CF} \\
-A_{FC} & D_{FF} & -A_{FF} \\
-A_{FC} & D_{FF} & -A_{FF} \\
-A_{FC} & -A_{FF} & D_{FF} \\
2E^{(1)}_{CC} & E^{(1)}_{CF} & E^{(1)}_{CF} & E^{(1)}_{CF} \\
E^{(1)}_{FC} & 0 & E^{(1)}_{FF} & E^{(1)}_{FF} \\
E^{(1)}_{FC} & E^{(1)}_{FF} & 0 & 0_{|2F|\times|2F|}
\end{bmatrix} + \begin{bmatrix}
E^{(2)}_{CC} & E^{(2)}_{CF} \\
E^{(2)}_{FC} & E^{(2)}_{FF} \\
0_{|3F|\times|3F|}
\end{bmatrix}.
\]

Then, it follows by direct calculations that

\[
\text{Sc}\left( \tilde{M}^{(0,2)}, F_3 \cup F_4 \right) = \tilde{M}^{(1,2)}, \quad \text{Sc}\left( \tilde{M}^{(1,2)}, F_2 \right) = \tilde{L}^{(2)}.
\]

Analogously, for larger \( k \), we can also define a sequence of augmented matrices \( \{\tilde{M}^{(i,k)}\} \) such that each \( \tilde{M}^{(i+1,k)} \) is a Schur complement of \( \tilde{M}^{(i,k)} \) and \( \tilde{M}^{(k,k)} = \tilde{L}^{(k)} \). Then, \( \text{Sc}\left( \tilde{M}^{(0,k)}, -[n] \right) = \tilde{L}^{(k)} \).

Next, we take \( Q^{(1)}, Q^{(2)} \) as examples to show how we construct \( Q^{(k)} \) iteratively. We remark that the \( Q^{(1)}, Q^{(2)} \) defined below are a little different with those in Lemma 4.10. In addition, they are not the best choice up to constants. However, what roles they play in the proofs can be seen easily from the following examples.

We naturally start with \( Q^{(1)} = U \left[ M^{(0,1)} \right] \) and it is easy to show that \( M^{(0,1)} - \tilde{M}^{(0,1)} \text{ asym} \lesssim O(\epsilon) \cdot Q^{(1)} \) and \( \tilde{L}^{(1)} - L^{(1)} \text{ asym} \lesssim O(\epsilon) \cdot \text{Sc}\left( U \left[ Q^{(1)} \right], F \right) \) by Theorem 2.4 and Fact A.15. Using (21)
in the proof of Lemma 4.10 we have

\[
\begin{bmatrix}
2E_{CC}^{(1)} & E_{CF}^{(1)} & E_{CF}^{(1)} \\
E_{FC}^{(1)} & 0 & E_{FF}^{(1)} \\
E_{FC}^{(1)} & E_{FF}^{(1)} & 0
\end{bmatrix}
\begin{bmatrix}
\frac{L_{CC}}{2} - \frac{L_{FC}}{2} & \frac{L_{CF}}{2} & \frac{L_{CF}}{2} \\
\frac{L_{FC}}{2} & L_{FF} & 0 \\
\frac{L_{FC}}{2} & 0 & L_{FF}
\end{bmatrix}
\text{asym}
\begin{bmatrix}
O(\epsilon) \\
0_{|2F| \times |2F|}
\end{bmatrix}
\begin{bmatrix}
2\bar{L}_{CC}^{(1)} & \bar{L}_{CF}^{(1)} & \bar{L}_{CF}^{(1)} \\
\bar{L}_{FC}^{(1)} & \bar{L}_{FF}^{(1)} & 0 \\
\bar{L}_{FC}^{(1)} & 0 & \bar{L}_{FF}^{(1)}
\end{bmatrix}
\begin{bmatrix}
0_{|2F| \times |2F|}
\end{bmatrix}
\]

\[
= O(\epsilon) \cdot U
\begin{bmatrix}
2L_{CC}^{(1)} - 2L_{FC}^{(1)} & L_{CF}^{(1)} & L_{CF}^{(1)} \\
L_{FC}^{(1)} & L_{FF}^{(1)} & 0 \\
L_{FC}^{(1)} & 0 & L_{FF}^{(1)}
\end{bmatrix}
0_{|2F| \times |2F|}
\]

\[
+ O(\epsilon) \cdot U
\begin{bmatrix}
2L_{CC}^{(1)} & L_{CF}^{(1)} & L_{CF}^{(1)} \\
L_{FC}^{(1)} & L_{FF}^{(1)} & 0 \\
L_{FC}^{(1)} & 0 & L_{FF}^{(1)}
\end{bmatrix}
0_{|2F| \times |2F|}
\]

Notice that

\[
\begin{bmatrix}
2L_{CC}^{(1)} - 2L_{FC}^{(1)} & L_{CF}^{(1)} & L_{CF}^{(1)} \\
L_{FC}^{(1)} & L_{FF}^{(1)} & 0 \\
L_{FC}^{(1)} & 0 & L_{FF}^{(1)}
\end{bmatrix}
\text{asym}
\begin{bmatrix}
0_{|2F| \times |2F|}
\end{bmatrix}
\begin{bmatrix}
2Q_{CC}^{(1)} & Q_{CF}^{(1)} & Q_{CF}^{(1)} \\
Q_{FC}^{(1)} & Q_{FF}^{(1)} & 0 \\
Q_{F2,C}^{(1)} & Q_{F2,C}^{(1)} & Q_{F2,F2}^{(1)}
\end{bmatrix}
\]

(3)

and

\[
\begin{bmatrix}
2L_{CC}^{(1)} & L_{CF}^{(1)} & L_{CF}^{(1)} \\
L_{FC}^{(1)} & L_{FF}^{(1)} & 0 \\
L_{FC}^{(1)} & 0 & L_{FF}^{(1)}
\end{bmatrix}
\text{asym}
\begin{bmatrix}
2M_{CC}^{(1,2)} & M_{CF}^{(1,2)} & M_{CF}^{(1,2)} \\
M_{FC}^{(1,2)} & M_{FF}^{(1,2)} & M_{FF}^{(1,2)} \\
M_{F2,C}^{(1,2)} & M_{F2,C}^{(1,2)} & M_{F2,F2}^{(1,2)}
\end{bmatrix}
\]

(4)

In addition,

\[
\begin{bmatrix}
E_{CC}^{(2)} & E_{CF}^{(2)} & E_{CF}^{(2)} \\
E_{FC}^{(2)} & E_{FF}^{(2)} & 0_{|3F| \times |3F|}
\end{bmatrix}
\text{asym}
\begin{bmatrix}
\mathcal{M}^{(0,2)}
\end{bmatrix}
\]

(5)
Define $Q^{(2)}$ as a weighted sum of the RHS of (3), (4), (5). Then, we will have $\tilde{M}^{(0,2)} - M^{(0,2)} \preccurlyeq O(\epsilon) \cdot Q^{(2)}$. Using the robustness of Schur complements, we can derive that $\tilde{L}^{(2)} - L^{(2)} \preccurlyeq O(\epsilon) \cdot \text{SC}(Q^{(2)}, [-n])$. 

Now, our approach to bound $L^{(k)} - \tilde{L}^{(k)}$ is summarized in Figure 1. Essentially, we use augmented matrices to separate the sparsification errors generated from different iterations and see their relations more clearly. The properties of the augmented matrices $\{M^{(0,k)}\}, \{\tilde{M}^{(0,k)}\}$ mentioned in this paper are essentially used to guarantee that we can use the Schur complement robustness safely to bound the difference $\frac{1}{2k}(\tilde{L}^{(K)} - L^{(K)}) \preccurlyeq O(\delta) \cdot U[\tilde{L}]$. In this way, we obtain a sparsified Schur complement with only $O(1)$ calls to the sparsification blackboxes rather than $O\left(\log^{O(1)} n\right)$ times.

4 Partial Block Elimination via Augmented Matrices

In this section, we introduce our augmented matrices based view of partial block elimination. As we will show later, after $O(\log \log n)$ steps of partial elimination, the $(F,F)$ block of the approximate partially-block-eliminated Laplacian $\tilde{L}^{(k)}$ can be approximated by its diagonal “safely”. So, what remains is to bound the error accumulations in the difference $\frac{1}{2k}\left( \text{SC}\left(\tilde{L}^{(k)}, F\right) - \text{SC}\left(L^{(k)}, F\right) \right)$,
which we do by bounding differences in \( \frac{1}{2^k}(\tilde{L}^{(k)} - L^{(k)}) \).

In Section 4.1, we represent the exact \( k \)-th partially-block-eliminated Laplacian \( L^{(k)} \) by a Schur complement of an augmented matrix \( \mathcal{M}^{(0,k)} \in \mathbb{R}^{(2^k|F| + |C|) \times (2^k|F| + |C|)} \).

In Section 4.2, we define \( \tilde{\mathcal{M}}^{(0,k)} \) as an inexact version of \( \mathcal{M}^{(0,k)} \) where sparsification errors accumulate. Then, we introduce a special type of augmented matrices, which we term *reptition matrices*, and bound the difference \( \tilde{\mathcal{M}}^{(0,k)} - \mathcal{M}^{(0,k)} \) in norms based on these *reptition matrices*.

### 4.1 A Reformulation of the Exact Partial Block Elimination

To bound the difference \( \frac{1}{2^k}(\tilde{L}^{(k)} - L^{(k)}) \), a direct way is to bound the difference \( \frac{1}{2^k}(\tilde{L}^{(k)} - L^{(k)}) \) recursively. However, our attempts of doing so lead to errors that grow exponentially in \( k \), the number of partial block elimination steps.

In this section, we provide a reformulation of the exact version of partial block elimination which is more friendly to error analysis. To be specific, our strategy is to construct a large matrix \( \mathcal{M}^{(0,k)} \) such that \( L^{(k)} \) is a Schur complement of the large matrix \( \mathcal{M}^{(0,k)} \). And there is a partition of \( \mathcal{M}^{(0,k)} \) such that each block is a zero matrix or equals some submatrix of \( L \). To construct \( \mathcal{M}^{(0,k)} \), we will construct a sequence of augmented matrices \( \{\mathcal{M}^{(i,k)}\}_{i=0}^k \) satisfying Lemma 4.1. Later, by analyzing the large matrix \( \mathcal{M}^{(0,k)} \), we can derive tighter bounds for quantities related to \( L^{(k)} \).

In Section 4 and Section 5, unless otherwise specified, we assume \( F = \{n - |F| + 1, \ldots, n - 1, n\} \) and \( C = \{1, 2, \ldots, |C|\} = [n] \setminus F \) by default.

Now, we give a rigorous way to construct such a sequence of matrices \( \{\mathcal{M}^{(i,k)}\}_{i=0}^k \) \( (0 \leq k \leq K) \). To begin with, we define some sets to be used later. We define

\[
F_a = \{b \in \mathbb{Z} \colon |C| + (a - 1)|F| + 1 \leq b \leq |C| + a|F|\}, \quad \forall 1 \leq a \leq 2^K.
\]

Note that in our notation, \( F = F_1 \).

Then, we construct a sequence of bijections \( \{\psi^{(i)}(\cdot)\}_{i=0}^K \) which indicate the “positions” of the blocks equalling \( -A^{(k-i,F)}_{k,F} \) in the large augmented matrix \( \mathcal{M}^{(k-i,k)} \).

We start with \( \psi^{(0)}(\cdot) \) and will define these \( \psi^{(i)} \) iteratively. The mapping \( \psi^{(0)}(\cdot) \) is defined as a trivial mapping from \( \{1\} \) to \( \{1\} \) with

\[
\psi^{(0)}(1) = 1.
\]

Then, assume we have defined \( \psi^{(i-1)}(\cdot) \). Now, we define \( \psi^{(i)} \) as follows:

\[
\psi^{(i)}(a) = \begin{cases} 
a + 2^{i-1}, & a \in [2^{i-1}] \\
\psi^{(i-1)}(a - 2^{i-1}), & 2^{i-1} + 1 \leq a \leq 2^i \end{cases}
\]

If \( \psi^{(i-1)}(\cdot) \) is a bijection from \( [2^{i-1}] \) to \( [2^{i-1}] \), then \( \psi^{(i)}(\cdot) \) is a bijection from \( [2^{i-1} + 1, \ldots, 2^i] \) to \( [2^{i-1}] \). And by the definition, \( \psi^{(i)}(\cdot) \) is a bijection from \( [2^{i-1}] \) to \( [2^{i-1} + 1, \ldots, 2^i] \). Then, \( \psi^{(i)} \) is a bijection from \( [2^i] \) to \( [2^i] \).
It follows by induction that for any \(k \in [K]\), \(\psi^{(k)}(\cdot)\) is a bijection from \([2^k]\) to \([2^k]\). And by the fact that \(2^{i-1} + 1 \leq \psi^{(i)}(a) \leq 2^i\) for \(a \in [2^{i-1}]\) and \(\psi^{(i)}(a) \in [2^{i-1}]\) for \(2^{i-1} + 1 \leq a \leq 2^i\), we have the following relation

\[
\psi^{(j)}(a) \neq a, \quad \forall 1 \leq j \leq K, \quad a \in [2^i].
\] (6)

With the notations defined above, we define the matrix \(\mathcal{M}^{(i,k)}\) as

\[
\mathcal{M}^{(i,k)} = 2^{k-i} \mathcal{P}(\mathcal{L}^{(i)}_{CC}, C, C, 2^{k-i}|F| + |C|) + \sum_{a=1}^{2^{k-i}} \left( \mathcal{P}(D_{FF}, F_a, F_a, 2^{k-i}|F| + |C|) \right.
\]

\[
+ \mathcal{P}(-A_{FF}^{(i)}, F_a, F_{\psi^{(k-i)}(a)}, 2^{k-i}|F| + |C|) + \mathcal{P}(-A_{CF}^{(i)}, F_a, C, 2^{k-i}|F| + |C|)
\]

\[
+ \mathcal{P}(-A_{CF}^{(i)}, C, F_a, 2^{k-i}|F| + |C|) \), \quad \forall 0 \leq k \leq K, \quad 0 \leq i \leq k.
\] (7)

where the notation \(\mathcal{P}(X, A, B, n)\) has been defined in Section 2, which means putting matrix \(X\) in the submatrix indexed by \((A, B)\) in a zero matrix \(0_{n \times n}\); \(L^{(k)}\) is the exact \(k\)-th partially-block-eliminated Laplacian and formal definitions of \(L^{(k)}, A^{(k)}\) are in Appendix B.

We have the following properties of \(\{\mathcal{M}^{(i,k)}\}\).

**Lemma 4.1.** For any \(0 \leq k \leq K\), \(0 \leq i \leq k\), \(\mathcal{M}^{(i,k)}\) is an Eulerian Laplacian; \(\mathcal{M}^{(i,k)}_{C,-C}\) are \(\alpha\)-RCDD; the Schur complement satisfies

\[
\text{Sc}(\mathcal{M}^{(i,k)}, \cup_{a=2^{k-i}+1}^{2^k} F_a) = \mathcal{M}^{(i+1,k)}.
\] (8)

Further,

\[
\text{Sc}(\mathcal{M}^{(i,k)}, [-n]) = L^{(k)}.
\] (9)

In addition, for any \(x \in \mathbb{R}^n\), let

\[
\hat{x} = \left( x_C^T \quad x_F^T \cdots x_F^T \right)^T \quad \text{2}^{k} \text{ repetitions of } x_F^T
\] (10)

then,

\[
\hat{x}^T \mathcal{M}^{(0,k)} \hat{x} = 2^k x^T L x.
\] (11)

**Proof.** We only need to prove the cases of \(i < k\). Denote \(F_i^+ = \cup_{a=2^i+1}^{2^{i+1}} F_a\) in this proof. Since \(\psi^{(k-i)}(\cdot)\) is a bijection from \([2^{k-i}]\) to \([2^{k-i}]\), then, for any \(a \in [2^{k-i}]\), \(\mathcal{M}^{(i,k)}_{F_a} \) contains 3 nonzero blocks equaling \(D_{FF}, -A_{FF}^{(i)}, -A_{CF}^{(i)}\), respectively; and the other blocks are all zero matrices. Analogously, for any \(a \in [2^{k-i}]\), \(\mathcal{M}^{(i,k)}_{-F_a} \) contains 3 nonzero blocks equaling \(D_{FF}, -A_{FF}^{(i)}, -A_{CF}^{(i)}\), respectively, while the other blocks are all zero matrices. Since \(L^{(i)}\) is Eulerian (Lemma B.2), we have \(\mathcal{M}^{(i,k)}_{F_a} \cdot 1 = D_{FF} 1 - A_{FF}^{(i)} 1 - A_{CF}^{(i)} 1 = L^{(i)}_{F_a} 1 = 0\). Analogously \(1^T \mathcal{M}^{(i,k)}_{-F_a} = 0^T\). Also by the definition (7),
\( \mathcal{M}_{C_i}^{(i,k)} \) 1 = 2 \( k-1 \) \( L(i)_{CC}^i 1 - A_{i CF}^{(i)} 1 \) = 2 \( k-1 \) \( L(i)_{CC}^i \) 1 = 0. Analogously, \( 1^T \mathcal{M}_{C_i}^{(i,k)} = 0^T \). All off-diagonal entries of \( \mathcal{M}^{(i,k)} \) are non-positive by the definition \[7\]. Thus, \( \mathcal{M}^{(i,k)} \) is an Eulerian Laplacian.

Notice that all blocks equaling \( -A_{i CF}^{(i)} \) are on the rows indexed by \( C \); and all blocks equaling \( -A_{FC}^{(i)} \) are on the columns indexed by \( C \), thus, on each block row or block column of \( \mathcal{M}^{(i,k)}_{C_i-C} \), there is exactly one block equaling \( D_{FF} \) and exactly one block equaling \( -A_{FF}^{(i)} \), and the other blocks are all-zeros matrices. Thus, by \[50\], \( \mathcal{M}^{(i,k)}_{C_i-C} \) is \( \alpha \)-RCDD. Then, \( \mathcal{M}^{(i,k)}_{C_i-C} \) is also \( \alpha \)-RCDD as it is a submatrix of \( \mathcal{M}^{(i,k)}_{C_i-C} \).

Also, as there are only 3 nonzero blocks on each row or column of \( \mathcal{M}^{(i,k)} \), when computing the Schur complement of \( F_a \) \( (2^{k-i} + 1 \leq a \leq 2^{k-i}) \), we only need to focus on the submatrix

\[
\mathcal{M}_{C \cup F_{a-2^{k-i}+1} \cup F_a, \cup F_{(k-i)} \cup F_a, F_a}^{(i,k)} = \begin{bmatrix}
2^{k-i} \cdot L(i)_{CC}^i & -A_{i CF}^{(i)} & -A_{i CF}^{(i)} \\
-A_{i FC}^{(i)} & 0 & -A_{i FF}^{(i)} \\
-A_{i FC}^{(i)} & -A_{i FF}^{(i)} & D_{FF}^{(i)}
\end{bmatrix},
\]

where the block \( \mathcal{M}_{F_{a-2^{k-i}+1} \cup F_a, \cup F_{(k-i)} \cup F_a, F_a}^{(i,k)} = 0 \) is by \[6\].

Then, by direct calculations,

\[
\text{SC} \left( \mathcal{M}_{C \cup F_{a-2^{k-i}+1} \cup F_a, \cup F_{(k-i)} \cup F_a, F_a}^{(i,k)} \right) = \begin{bmatrix}
2^{k-i} \cdot L(i)_{CC}^i - A_{i CF}^{(i)} D_{FF}^{(i)} A_{i FC}^{(i)} & -A_{i CF}^{(i)} \left( I + D_{FF}^{(i)} A_{i FF}^{(i)} \right) \\
-A_{i FC}^{(i)} \left( I + A_{i FF}^{(i)} D_{FF}^{(i)} A_{i FC}^{(i)} \right) & -A_{i FF}^{(i)} D_{FF}^{(i)} A_{i FC}^{(i)}
\end{bmatrix},
\]

When computing the Schur complement of \( F_{k_i} \), the term \( A_{i CF}^{(i)} D_{FF}^{(i)} A_{i FC}^{(i)} \) is subtracted from \( \mathcal{M}_{C}^{(i,k)} = 2^{k-i} \cdot L(i)_{CC}^i \) for \( 2^{k-i} \) times. By combining with the equality \( L(i)_{CC}^i + 1 \) \( k+1 \) \( = 2 \cdot L(i)_{CC}^i - A_{i CF}^{(i)} D_{FF}^{(i)} A_{i FC}^{(i)} \) from \[48a\], we have \( \text{SC} \left( \mathcal{M}_{(i,k)}^{(i,k)}, F_{k_i} \right) = \mathcal{M}_{C}^{(i+1,k)} \).

To derive \[8\], what remains is to show that the “positions” of the blocks equaling \( -A_{i FF}^{(i+1,k)} \) are the same in \( \text{SC} \left( \mathcal{M}_{(i,k)}^{(i,k)}, F_{k_i} \right) \) and \( \mathcal{M}_{(i+1,k)}^{(i+1,k)} \). This can be seen easily from the observation that for \( 2^{k-i} - 1 \leq a \leq 2^{k-i} \), \( \text{SC} \left( \mathcal{M}_{C \cup F_{a-2^{k-i}+1} \cup F_a, \cup F_{(k-i)} \cup F_a, F_a}^{(i,k)} \right) \) is supported on the submatrix indexed by

\[
(C \cup F_{a-2^{k-i}+1}, C \cup F_{(k-i)}(a)).
\]

Thus, the “positions” of blocks equaling \( -A_{i FF}^{(i+1)} \) in \( \text{SC} \left( \mathcal{M}_{(i,k)}^{(i,k)}, F_{k-i} \right) \) are

\[
\left\{ \left( F_{a-2^{k-i}+1}, F_{(k-i)}(a) \right) : 2^{k-i} + 1 \leq a \leq 2^{k-i} \right\} = \left\{ \left( F_{a-2^{k-i}+1}, F_{(k-i)}(a-2^{k-i}+1) \right) : 2^{k-i} + 1 \leq a \leq 2^{k-i} \right\} = \left\{ \left( F_0, F_{(k-i)}(a) \right) : a \in [2^{k-i}] \right\},
\]

18
which are the exact “positions” of the blocks equalling $-A_{FF}^{(i+1)}$ in $M^{(i+1,k)}$ by (7). Therefore, we have proved (8).

The relation (9) follows by Fact A.9 and induction:

$$Sc\left(M^{(i,k)}, -[n]\right) = Sc\left(Sc\left(M^{(i,k)}, F_{k-i}^+\right), -[n]\right) = Sc\left(M^{(i+1,k)}, -[n]\right) = \ldots = M^{(k,k)} = L^{(k)}.$$  

The relation (11) follows by the fact that $\psi(\cdot)$ is a bijection from $[2^k]$ to $[2^k]$ and (7).

The following lemma answers a question in Section 3.3. That is, $\frac{1}{2^k} U[L^{(k)}] \preceq O(1) U[L]$.

**Lemma 4.2.** For any $0 \leq k \leq K$, 

$$\frac{1}{2^k} U[L^{(k)}] \preceq \left(3 + \frac{2}{\alpha}\right) U[L].$$  

**Proof.** By Lemma 4.1, $M^{(0,k)}_{-}[n],-[n]$ is $\alpha$-RCDD, and $M^{(0,k)}$ is an Eulerian Laplacian. Thus, using Fact A.15

$$U[Sc\left(M^{(0,k)}, -[n]\right)] \preceq \left(3 + \frac{2}{\alpha}\right) Sc\left(U[M^{(0,k)}], -[n]\right).$$

By (9), $U[L^{(k)}] \preceq \left(3 + \frac{2}{\alpha}\right) Sc\left(U[M^{(0,k)}], -[n]\right)$.

For any $x \in \mathbb{R}^n$, define $\tilde{x}$ as in (10). By Fact A.6 $U[M^{(0,k)}]_{-}[n],-[n]$ is PD. Then, by Fact A.10 and (11), we have

$$x^\top Sc\left(U[M^{(0,k)}], -[n]\right) x \leq \tilde{x}^\top U[M^{(0,k)}] \tilde{x} = 2^k x^\top U[L] x,$$

i.e., $Sc\left(U[M^{(0,k)}], -[n]\right) \preceq 2^k U[L]$.

Combining the above equations yields that

$$U[L^{(k)}] \preceq \left(3 + \frac{2}{\alpha}\right) Sc\left(U[M^{(0,k)}], -[n]\right) \preceq 2^k \left(3 + \frac{2}{\alpha}\right) U[L].$$

\[\square\]

### 4.2 Bounding Error Accumulation Using Repetition Matrices

Before we define $\tilde{M}^{(0,k)}$, we introduce our notations for the errors induced by sparsification.

$$\tilde{E}^{(k,i)} = A_{i,i}^{(k-1)} A_{i,i}^{(k-1)} - \tilde{Y}^{(k,i)}, \quad \tilde{E}^{(k)} = \sum_{i \in F} \frac{1}{D_{ii}} \tilde{E}^{(k,i)} = \tilde{A}_{i,i}^{(k-1)} D_{FF}^{-1} A_{i,i}^{(k-1)} - \tilde{Y},$$

$$E^{(k,0)} = \tilde{L} - \tilde{L}^{(k,0)}, \quad E^{(k)} = \tilde{E}^{(k)} + E^{(k,0)} = \tilde{L} - \Phi \left(\tilde{L}^{(k-1)} | D_{FF}, F\right),$$

and

$$E^{(i)}_X = A_{C,i}^{(K)} A_{i,C}^{(K)} - \tilde{X}^{(i)}, \quad E_X = \sum_{i \in F} \frac{1}{D_{ii}} E^{(i)}_X = A_{C,F}^{(K)} D_{FF}^{-1} A_{F,C}^{(K)} - \tilde{X}.$$
We also denote in the rest of this paper,
\[
\hat{R} = R + \frac{1}{2K} \left( \tilde{A}_{CF}^{(K)} (D_{FF} - \tilde{A}_{FF}^{(K)})^{-1} - \tilde{A}_{CF}^{(K)} D_{FF}^{-1} \tilde{A}_{FC}^{(K)} \right). \tag{13}
\]

Some elementary facts of the results of Algorithm 2 is given by Lemma D.3 in Appendix D.

By Lemma 2.6 and Lemma D.1, we can provide bounds for the one-step errors in the next lemma. Its proof is deferred to Appendix D.

**Lemma 4.3.** The error matrices satisfies
\[
E^{(k)} \asym \leq \epsilon_0 U \left[ \tilde{L}^{(k-1)} \right], \tag{14}
\]
\[
E_X \asym \leq \epsilon U \left[ \text{Sc}(\tilde{L}^{(k)}, F) \right], \tag{15}
\]
where \( \epsilon_0 = 2 \left( 3 + \frac{2}{\alpha} \right) (2\epsilon + \epsilon^2) \).

In the remainder of this paper, we write \( \epsilon_0 = 2 \left( 3 + \frac{2}{\alpha} \right) (2\epsilon + \epsilon^2) \).

Recall that we define an augmented matrix \( \mathcal{M}^{(0,k)} \) such that \( L^{(k)} \) is its Schur complement in Section 4.1. Now, we define \( \tilde{\mathcal{M}}^{(0,k)} \) which is an inexact version of \( \mathcal{M}^{(0,k)} \) to analyze the properties of \( \tilde{L}^{(k)} \). We first define
\[
\mathcal{R} \left( k, a, E^{(i)} \right) = \mathcal{P} \left( E^{(i)}_{FF}, F_a, F_{\psi(k-i)}(a), 2^{k-i}|F| + |C| \right) + \mathcal{P} \left( E^{(i)}_{FC}, F_a, C, 2^{k-i}|F| + |C| \right)
+ \mathcal{P} \left( E^{(i)}_{CF}, C, F_{\psi(k-i)}(a), 2^{k-i}|F| + |C| \right) + \mathcal{P} \left( E^{(i)}_{CC}, C, C, 2^{k-i}|F| + |C| \right).
\]

Then, we define the error matrices
\[
\mathcal{E}^{(i,k)} = \sum_{a=1}^{2^{k-i}} \mathcal{R} \left( k, a, E^{(i)} \right) \tag{16}
\]
and
\[
\mathcal{E}^{(1:k,k)} = \sum_{i=1}^{k} \mathcal{E}^{(i,k)}.
\]

The matrix \( \tilde{\mathcal{M}}^{(0,k)} \) is defined as follows
\[
\tilde{\mathcal{M}}^{(0,k)} = \mathcal{M}^{(0,k)} + \mathcal{E}^{(1:k,k)}.
\]

**Lemma 4.4.** The Schur complement of \([2^k|F| + |C|]|n] \) in \( \tilde{\mathcal{M}}^{(0,k)} \) satisfies:
\[
\text{Sc} \left( \tilde{\mathcal{M}}^{(0,k)}, -|n] \right) = \tilde{L}^{(k)}. \tag{17}
\]
Proof. We define the following auxiliary matrices in this proof
\[
\begin{align*}
\tilde{N}^{(i,k)} &= 2^{k-i}P\left(L_{CC}, C, C, 2^{k-i}|F| + |C|\right) + \sum_{\alpha=1}^{2^{k-i}} \left( P\left(D_{FF}, F_{\alpha}, F_{\alpha}, 2^{k-i}|F| + |C|\right) \\
&+ P\left(-A^{(i)}_{FF}, F_{\alpha}, F_{\psi(k-1)(\alpha)} , 2^{k-i}|F| + |C|\right) + P\left(-A^{(i)}_{FC}, F_{\alpha}, C, 2^{k-i}|F| + |C|\right) \\
&+ P\left(-A^{(i)}_{CF}, C, F_{\alpha}, 2^{k-i}|F| + |C|\right) \right),
\end{align*}
\]
Since \( \tilde{L}^{(0)} = L \), we have \( \tilde{N}^{(0,k)} = \mathcal{M}^{(0,k)} \). Also, from definition, \( \tilde{N}^{(k,k)} = \tilde{L}^{(k)} \). We also denote \( F_i^+ = \bigcup_{\alpha=2^{i-1}+1}^{2^i} F_{\alpha} \) in this proof.

Define
\[
\tilde{\mathcal{M}}^{(i,k)} = \tilde{N}^{(i,k)} + \sum_{j=i+1}^{k} \mathcal{E}^{(j,k)}.
\]

Similar with the arguments in Lemma 4.1, when computing the Schur complement of \( F_{\alpha} \) in \( \tilde{N}^{(i,k)} + \mathcal{E}^{(i+1,k)} \) where \( 2^{k-i-1} + 1 \leq \alpha \leq 2^{k-i} \), we only need to focus on the following submatrix
\[
\begin{align*}
\left( \tilde{N}^{(i,k)} + \mathcal{E}^{(i+1,k)} \right)_{C \cup F_{\alpha - 2^{k-i-1} \cup F_{\psi(k-1)(\alpha)} \cup F_{\alpha}}}
= \begin{bmatrix}
2^{k-i}L_{CC}^{(i)} & -A_{CF}^{(i)} & -A_{CF}^{(i)} \\
-A_{FC}^{(i)} & 0 & -A_{FF}^{(i)} \\
-A_{FC}^{(i)} & -A_{FF}^{(i)} & D_{FF}
\end{bmatrix}
+ \begin{bmatrix}
2^{k-i-1}E^{(i+1)}_{CC} & E^{(i+1)}_{CF} & 0 \\
E^{(i+1)}_{FC} & E^{(i+1)}_{FF} & 0 \\
0 & 0 & 0
\end{bmatrix},
\end{align*}
\]
where the block \( \tilde{N}_{F_{\alpha - 2^{k-i-1} \cup F_{\psi(k-1)(\alpha)} \cup F_{\alpha}}} = 0 \) is also by (6).

By the definition of \( \mathcal{E}^{(i+1)} \), we have
\[
\text{Sc}\left( \tilde{N}^{(i,k)} + \mathcal{E}^{(i+1,k)} \right)_{C \cup F_{\alpha - 2^{k-i-1} \cup F_{\psi(k-1)(\alpha)} \cup F_{\alpha}}, F_{\alpha}}
= \begin{bmatrix}
2^{k-i}L_{CC}^{(i)} - A_{CF}^{(i)} D_{FF}^{-1} A_{FC}^{(i)} + 2^{k-i-1} E^{(i+1)}_{CC} - A_{CF}^{(i)} \left( I + D_{FF}^{-1} A_{FF}^{(i)} \right) + E^{(i+1)}_{CF} \\
- \left( I + A_{FF}^{(i)} D_{FF}^{-1} A_{FC}^{(i)} \right) + E^{(i+1)}_{FC} - A_{CF}^{(i)} D_{FF}^{-1} A_{FF}^{(i)} + E^{(i+1)}_{FF} \\
(2^{k-i} - 2)L_{CC}^{(i)} + L_{CC}^{(i+1)} - A_{CF}^{(i+1)} - A_{FC}^{(i+1)}
\end{bmatrix}.
\]

By similar arguments with Lemma 4.1, the “positions” of the blocks equaling \( -A_{FF}^{(i+1)} \) in \( \text{Sc}\left( \tilde{N}^{(i,k)} + \mathcal{E}^{(i+1,k)}, F_{k-i}^+ \right) \) are exactly the same as those in \( \tilde{N}^{(i+1,k)} \).

Since \( F_{k-i}^+ \) contains \( 2^{k-i-1} \) sets of size \( |F| \), we have
\[
\text{Sc}\left( \tilde{N}^{(i,k)} + \mathcal{E}^{(i+1,k)}, F_{k-i}^+ \right)_{CC} = 2^{k-i-1} \left( 2L_{CC}^{(i)} - A_{CF}^{(i)} D_{FF}^{-1} A_{FC}^{(i)} + E^{(i+1)}_{CC} \right) = 2^{k-i-1} L_{CC}^{(i+1)}.
\]
Thus, we have shown
\[
\text{Sc}\left(\tilde{N}^{(i,k)} + \mathcal{E}^{(i+1,k),F^+_{k-i}}\right) = \tilde{N}^{(i+1,k)}.
\]

Since the support set of \(\sum_{j=i+2}^{k} \mathcal{E}^{(j,k)}\) is \(|2^{k-i-2}|F| + |C|\) which is disjoint with \(F^+_{k-i}\), we have
\[
\text{Sc}\left(\tilde{M}^{(i,k),F^+_{k-i}}\right) = \text{Sc}\left(\tilde{N}^{(i,k)} + \sum_{j=i+1}^{k} \mathcal{E}^{(j,k),F^+_{k-i}}\right) = \tilde{N}^{(i+1,k)} + \sum_{j=i+2}^{k} \mathcal{E}^{(j,k)} = \tilde{M}^{(i+1,k)}.
\]

By induction,
\[
\text{Sc}\left(\tilde{M}^{(0,k),F^+_{0-k}}\right) = \tilde{M}^{(k,k)} = \tilde{N}^{(k,k)} = \tilde{L}^{(k)}.
\]

To help bound \(\mathcal{E}^{(1,k,k)}\), we define some special kinds of matrices termed “repetition matrices”. We will construct the matrices \(\{Q^{(k)}\}\) as linear combinations of “repetition matrices”.

**Definition 4.5.** ( “Repetition matrices”) We will use the following 3 kinds of “repetition matrices”:

(i) the \(k\)-“repetition matrix” of \(A\) is defined as follows:
\[
\text{Rep}(k,C,A) = \begin{bmatrix}
{k}A_{CC} & A_{CE} & A_{CE} & \cdots & A_{CE} \\
A_{EC} & A_{EE} & 0 & \cdots & 0 \\
A_{EC} & 0 & A_{EE} & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
A_{EC} & 0 & \cdots & 0 & A_{EE}
\end{bmatrix} \in \mathbb{R}^{(k|E|+|C|) \times (k|E|+|C|)},
\]

where the repetition numbers of the blocks \(A_{CE}, A_{EC}, A_{EE}\) are \(k\);

(ii) \(\text{Rep}^{+0}(k,C,A,N)\) is a larger matrix by appending all-zeros rows and columns to \(\text{Rep}(k,C,A)\):
\[
\text{Rep}^{+0}(k,C,A,N) = \begin{bmatrix}
\text{Rep}(k,C,A) & 0 \\
0 & 0
\end{bmatrix} \in \mathbb{R}^{N \times N},
\]

where \(N \geq k|E| + |C|\) is used to indicate the size of \(\text{Rep}^{+0}(k,C,A,N)\);

(iii) if \(F,F_+\) is a partition of \(E\), \(\text{Rep}(k,C,F,A)\) is defined as a permutation of the \(k\)-“repetition matrix” of \(A\), which has the following form:
\[
\text{Rep}(k,C,F,A) = \begin{bmatrix}
kA_{CC} & A_{CF} & \cdots & A_{CF} & A_{CF} & \cdots & A_{CF} \\
A_{FC} & A_{FF} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
A_{F+C} & A_{F+F} & \cdots & \cdots & \cdots & \cdots & \cdots \\
A_{F+C} & A_{F+F} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
A_{F+C} & A_{F+F} & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}.
\]
Now, we define the matrices \( \{ Q^{(k)} \}_{0 \leq k \leq K} \) which are used to bound \( \mathcal{M}^{(0,k)} - \tilde{\mathcal{M}}^{(0,k)} \) and then \( L^{(k)} - \tilde{L}^{(k)} \). We define \( \{ Q^{(k)} \}_{0 \leq k \leq K} \) iteratively together with the error quantities \( \{ \gamma_k \}_{0 \leq k \leq K} \).

We start from \( Q^{(0)} = U[L] \) and \( \gamma_0 = 0 \). If we have defined \( \{ Q^{(i)} \}_{0 \leq i < k} \); \( \{ \gamma_i \}_{0 \leq i < k} \); then \( Q^{(k)} \in \mathbb{R}^{(2^k |F| + |C|) \times (2^k |F| + |C|)} \) and \( \gamma_k \in \mathbb{R}_+ \) are defined as follows:

\[
Q^{(k)} \overset{\text{def}}{=} \frac{k}{4k + \frac{2k}{\alpha} + \sum_{i=0}^{k-1} \gamma_i} U \left[ \mathcal{M}^{(0,k)} \right] \nonumber
\]

\[
+ \frac{1}{4k + \frac{2k}{\alpha} + \sum_{i=0}^{k-1} \gamma_i} \sum_{i=0}^{k-1} \gamma_i \text{Rep} \left( 2^{k-i}, C, F, Q^{(i)} \right) \nonumber
\]

\[
+ \frac{3 + 2}{4k + \frac{2k}{\alpha} + \sum_{i=0}^{k-1} \gamma_i} \sum_{i=0}^{k-1} \text{Rep} \left( 2^{k-i}, C, F, U \left[ \mathcal{M}^{(0,i)} \right] \right) \label{eq:Q_def}
\]

\[
\gamma_k \overset{\text{def}}{=} \sup_{x, y \notin \ker(\text{Sc}(Q^{(k)}, -[n]))} \frac{2x^\top (\tilde{L}^{(k)} - L^{(k)}) y}{x^\top \text{Sc}(Q^{(k)}, -[n]) x + y^\top \text{Sc}(Q^{(k)}, -[n]) y}, \label{eq:gamma_k_def}
\]

**Remark 4.6.** The first term \( U \left[ \mathcal{M}^{(0,k)} \right] \) in \eqref{eq:Q_def} is only used to guarantee that \( Q^{(k)} \gtrless \frac{k}{4k + \frac{2k}{\alpha} + \sum_{i=0}^{k-1} \gamma_i} U \left[ \mathcal{M}^{(0,k)} \right] \) (Fact 4.8(i)). Without this term, \eqref{eq:gamma_k_def} still holds with only slight changes in the constants.

**Remark 4.7.** We use the more complicated definition of \( \gamma_k \) as in \eqref{eq:gamma_k_def} because the relations between the kernels of \( \text{Sc}(Q^{(k)}, -[n]) \) and \( \tilde{L}^{(k)} - L^{(k)} \) is only shown in Section 5. In Lemma 5.3, we show \( \gamma_k < +\infty \). Then, by Fact 2.2, we obtain the simplification,

\[
\gamma_k = \left\| \text{Sc}(Q^{(k)}, -[n])^{1/2} (\tilde{L}^{(k)} - L^{(k)}) \text{Sc}(Q^{(k)}, -[n])^{1/2} \right\|^2_2.
\]

The following elementary properties of \( Q^{(k)} \) follows directly by Lemma 4.1, Fact A.17 and the fact that the coefficients on the RHS of \eqref{eq:Q_def} equals 1.

**Fact 4.8.** \( Q^{(k)} \) is a Laplacian satisfying:

(i) \( U \left[ \mathcal{M}^{(0,k)} \right] \preceq \left( 4 + \frac{2}{\alpha} + \sum_{i=0}^{k-1} \gamma_i \right) Q^{(k)} \);

(ii) \( \text{Diag}(Q^{(k)}) = \text{Diag}(\mathcal{M}^{(0,k)}) \);

(iii) \( Q^{(k)}_{C,-C}, Q^{(k)}_{-[n],-[n]} \) are \( \alpha \)-RCDD;

(iv) \( \left( Q^{(k)} \right)^{1/2} \text{Diag}(\mathcal{M}^{(0,k)})^{-1/2} \right) \leq 2; \)

23
Lemma 4.9. \( \text{Sc} \left( Q^{(k)}, -C \right) \leq 2^k U[\text{Sc}(L, F)] \).

The following lemma shows how we bound the sparsification errors attached to \( \tilde{M}^{(0,k)} \) by \( Q^{(k)} \).

Lemma 4.10.

\[
\mathcal{E}^{(1:k,k)}_{\text{asym}} \leq \epsilon_0 \left( 4k + \frac{2k}{\alpha} + \sum_{i=0}^{k-1} \gamma_i \right) Q^{(k)}. \tag{20}
\]

Proof. We only prove the case when all \( \gamma_k < +\infty \), and the proof for the case when some \( \gamma_k = +\infty \) follows trivially.

We define \( Q^{(0)} = U[L] \), and will construct \( Q^{(k)} \) iteratively. That is, we want to show that given \( Q^{(0)}, \ldots, Q^{(k-1)} \) satisfying these conditions, we can find \( Q^{(k)} \).

Firstly, we fix some \( i \in \{0, 1, \ldots, k - 1\} \). By Lemma 4.3 and Fact 2.2, we have

\[
2x^\top E^{(i+1)} y \leq \epsilon_0 \left( x^\top U \left[ \tilde{L}^{(i)} \right] x + y^\top U \left[ \tilde{L}^{(i)} \right] y \right), \quad \forall x, y \in \mathbb{R}^n.
\]

Then, for any \( x, y \in \mathbb{R}^{2^k|F|+|C|} \), by the definition of \( \mathcal{E}^{(i+1,k)} \) in (16), we have

\[
2x^\top \mathcal{E}^{(i+1,k)} y = 2 \sum_{a=1}^{2^{k-i-1}} x^\top R(k, a, E^{(i+1)}) y = 2 \sum_{a=1}^{2^{k-i-1}} \begin{pmatrix} x_C
x_{F_a} \end{pmatrix}^\top U \left[ \tilde{L}^{(i)} \right] \begin{pmatrix} x_C
x_{F_a} \end{pmatrix} + \begin{pmatrix} y_C
y_{F_a} \end{pmatrix}^\top U \left[ \tilde{L}^{(i)} \right] \begin{pmatrix} y_C
y_{F_a} \end{pmatrix}.
\]

By the fact that \( \psi^{(k-i-1)}(\cdot) \) is a bijection from \( [2^{k-i-1}] \) to \( [2^{k-i-1}] \) and \( U\left[ \tilde{L}^{(i)} \right] \) is PSD (because \( \tilde{L}^{(i)} \) is an Eulerian Laplacian from Lemma 4.3), we have

\[
2x^\top \mathcal{E}^{(i+1,k)} y \leq \epsilon_0 \sum_{a=1}^{2^{k-i-1}} \begin{pmatrix} x_C
x_{F_a} \end{pmatrix}^\top U \left[ \tilde{L}^{(i)} \right] \begin{pmatrix} x_C
x_{F_a} \end{pmatrix} + \begin{pmatrix} y_C
y_{F_a} \end{pmatrix}^\top U \left[ \tilde{L}^{(i)} \right] \begin{pmatrix} y_C
y_{F_a} \end{pmatrix}
\]

\[
= \epsilon_0 \left( x^\top \text{Rep}^+(2^{k-i}, C, U \left[ \tilde{L}^{(i)} \right], 2^k|F|+|C|) x + y^\top \text{Rep}^+(2^{k-i}, C, U \left[ \tilde{L}^{(i)} \right], 2^k|F|+|C|) y \right).
\]

To bound the repetition matrix \( \text{Rep}^+(2^{k-i}, C, U \left[ \tilde{L}^{(i)} \right], 2^k|F|+|C|) \), we bound the matrices \( \text{Rep}^+(2^{k-i}, C, U \left[ \tilde{L}^{(i)} - L^{(i)} \right], 2^k|F|+|C|) \) and \( \text{Rep}^+(2^{k-i}, C, U \left[ L^{(i)} \right], 2^k|F|+|C|) \), respectively.

\footnote{Actually, we will show in Lemma 5.3 that all \( \gamma_k < +\infty \).}
By the definition of \( \gamma_i \) in (19),
\[
U[L^{(i)} - L^{(i)}] \preceq \gamma_i \text{Sc}(Q^{(i)}, -[n]).
\]

Denote the set \( E_k = C \cup \bigcup_{a=1}^{2k} F_a \). It is straightforward that
\[
\text{Sc}(\text{Rep}(2k-i, C, F, Q^{(i)}), -E_{k-i}) = \text{Rep}(2k-i, C, \text{Sc}(Q^{(i)}, -[n])).
\]
Then, by Fact A.23 and Fact A.14, we have
\[
\text{Rep}^{+0}(2k-i, C, U[L^{(i)} - L^{(i)}], 2k|F| + |C|) \preceq \gamma_i \text{Rep}^{+0}(2k-i, C, \text{Sc}(Q^{(i)}, -[n]), 2k|F| + |C|). \tag{22}
\]

By Lemma 4.1, \( \mathcal{M}^{(0,i)}_{-[n],-[n]} \) is \( \alpha \)-RCDD and \( \text{Sc}(\mathcal{M}^{(0,i)}, -[n]) = L^{(i)}. \) Then, by Fact A.15 we have
\[
U[L^{(i)}] = U[\text{Sc}(\mathcal{M}^{(0,i)}, -[n])] \preceq \left( 3 + \frac{2}{\alpha} \right) \text{Sc}(U[\mathcal{M}^{(0,i)}], -[n]).
\]

Similar to (22), we also have
\[
\text{Rep}^{+0}(2k-i, C, U[L^{(i)}], 2k|F| + |C|) \preceq \left( 3 + \frac{2}{\alpha} \right) \text{Rep}(2k-i, C, F, U[\mathcal{M}^{(0,i)}]). \tag{23}
\]

Then, by substituting (22), (23) into (21), summing over \( i = 0, 1, \cdots, k-1 \), and combining with (18) and Fact 2.2 we have (20).

5 Robustness of Schur Complements and Full Error Analysis

In this section, we show additional robustness properties of Schur complements suitable for analyzing errors on the augmented matrices. Specifically, we establish conditions on \( A, B, U \) where \( A - B \preceq \epsilon \cdot U \), as well as the set to be eliminated, \( F \), so that \( \text{Sc}(A, F) - \text{Sc}(B, F) \preceq \delta \cdot \text{Sc}(U, F) \).

Using these properties, we bound the norms of errors in Schur complements of the \( Q^{(k)} \) and \( \gamma_k \). Such bounds allow us to complete the proof of Theorem 3.3.

5.1 Schur Complement Robustness

The next lemma is a perturbed version of Fact A.13. It is used to prove Lemma 5.2 below.

**Lemma 5.1.** Suppose that \( L \in \mathbb{R}^{n \times n} \) is an Eulerian Laplacian, \( D = \text{Diag}(L) \), \( W \) is PSD, \( \|W^{1/2}D^{-1/2}\|_2 \leq a \), and the matrix \( E \in \mathbb{R}^{n \times n} \) satisfies \( E \preceq bW \) with \( a^2b < 2 \). Then the matrix \( M = L + E \) satisfies:
\[
M^T D^{-1} M \preceq \frac{1}{2 - a^2b} \left( (4 + 2a^2b) U[L] + 2bW \right),
\]
\[
M D^{-1} M^T \preceq \frac{1}{2 - a^2b} \left( (4 + 2a^2b) U[L] + 2bW \right).
\]
Proof. For any $x \in \mathbb{R}^n$,
\[
x^T M^T D^{-1} M x = x^T L^T D^{-1} L x + x^T L^T D^{-1} E x + x^T E^T D^{-1} M x.
\]

We bound each of these terms on the RHS separately.

- For the first term, by Fact A.13 $x^T L^T D^{-1} L x \leq 2x^T U[L]x$.

- For the second term, by Fact 2.2 the conditions $E \preceq \abb W$ and $\|W^{1/2}D^{-1/2}\|_2 \leq a$ and Fact A.13
\[
2x^T L^T D^{-1} E x \leq b \left( x^T W x + x^T L^T D^{-1} W D^{-1} L x \right)
\]
\[
= b \left( x^T W x + \|W^{1/2}D^{-1}Lx\|_2 \right) \leq b \left( x^T W x + \|W^{1/2}D^{-1/2}\|_2 \|D^{-1/2}Lx\|_2 \right)
\]
\[
\leq b \left( x^T W x + a^2 x^T L^T D^{-1} L x \right) \leq b \left( x^T W x + 2a^2 x^T U[L]x \right).
\]

- For the third term, using the conditions $E \preceq \abb W$ and $\|W^{1/2}D^{-1/2}\|_2 \leq a$ again yields that
\[
2x^T E^T D^{-1} M x \leq b \left( x^T W x + x^T M^T D^{-1} W D^{-1} M x \right)
\]
\[
= b \left( x^T W x + \|W^{1/2}D^{-1}Mx\|_2 \right) \leq b \left( x^T W x + a^2 \|D^{-1/2}Mx\|_2 \right)
\]
\[
= b \left( x^T W x + a^2 x^T M^T D^{-1} M x \right).
\]

By combining the above equations, we have
\[
2x^T M^T D^{-1} M x \leq (4 + 2a^2b) x^T U[L]x + 2bx^T W x + a^2bx^T M^T D^{-1} M x.
\]

Rearranging the above equations yields that
\[
x^T M^T D^{-1} M x \leq \frac{1}{2 - a^2b} \left( (4 + 2a^2b) x^T U[L]x + 2bx^T W x \right).
\]

which gives the result for $M^T D^{-1} M$. The bound for $MD^{-1}M^T$ follows analogously. \qed

The following lemma shows the robustness of the Schur complements. It is used in the proof of Lemma 5.3 to bound $\gamma_k$.

**Lemma 5.2.** Let $N \in \mathbb{R}^{n \times n}$ be an Eulerian Laplacian, let $M$ be an $n$-by-$n$ matrix, let $U \in \mathbb{R}^{n \times n}$ be PSD and $F,C$ a partition of $[n]$. Suppose that $U_{FF}$ is nonsingular, $U\mathbf{1} = \mathbf{0}$, $N_{FF}$ is $\rho$-RCDD ($\rho > 0$), $U[N]_{FF} \gg \frac{1}{\rho} U_{FF}$, $U[N] \preceq \beta U$, $\|U^{1/2} \text{Diag}(N)^{-1/2}\|_2 \leq a$, and the matrix $E = M - N$ satisfies $E \preceq \abb \cdot U$ with $b < \min\left\{\frac{2}{a^2}, \frac{1}{\rho} \right\}$.

Then, $M_{FF}, N_{FF}$ are nonsingular and
\[
\text{Sc}(M, F) - \text{Sc}(N, F) \preceq \abb \left( 1 + \frac{1}{\rho} \right) \mu^2 (\beta(4 + 2a^2b) + 2b) / (1 - \mu b)^2 (2 - a^2b) \cdot \text{Sc}(U, F). \tag{24}
\]
Proof. Without loss of generality, we assume $F = \{1, \ldots, |F|\}$ and $C = [n]\backslash F$ in this proof.

By the condition $M - N \overset{\text{asym}}{\preccurlyeq} b \cdot U$, we have $2x^\top(M - N)y \leq b(x^\top U x + y^\top U y), \forall x \in \mathbb{R}^n$. Then, $x^\top(U[M]_{FF} - U[N]_{FF})x \leq b x^\top U_{FF} x, \forall x \in \mathbb{R}^{|F|}$. Thus, $U[M]_{FF} \succ U[N]_{FF} - b U_{FF}$.

By the condition $U[N]_{FF} \succ \frac{1}{\mu} U_{FF}$, we have

$$U[M]_{FF} \succ U[N]_{FF} - b U_{FF} \succ (1 - b\mu) U[N]_{FF} \tag{25}$$

and

$$U[M]_{FF} \succ (1 - b\mu) U[N]_{FF} \succ \left( \frac{1}{\mu} - b \right) U_{FF} \succ 0. \tag{26}$$

Since $U$ is PSD, $U_{FF}$ is also PSD. Since $U_{FF}$ is also nonsingular, $U_{FF} \succ 0$. Then, by the condition $b < \frac{1}{\mu}$ and (26), we have $U[M]_{FF} \succ 0$. Then, by Fact A.11, $M_{FF}$ and $N_{FF}$ are nonsingular. And by Fact A.12, we have

$$M_{FF}^{-1} U[M]_{FF} (M_{FF}^{-1})^\top \succ U[M]_{FF}^{-1}, \tag{27}$$

$$\left( N_{FF}^{-1} \right)^\top U[N]_{FF} N_{FF}^{-1} \succ U[N_{FF}]^{-1}. \tag{28}$$

For any $x, y \in \mathbb{R}^{|C|}$, define

$$x_N = \begin{pmatrix} -N_{FF}^{-1}N_FC x \\ x \end{pmatrix}, \quad x_U = \begin{pmatrix} -U_{FF}^{-1}U_FC x \end{pmatrix}$$

and

$$y_M = \begin{pmatrix} -(M_{FF}^{-1})^\top M_{CF}^\top y \\ y \end{pmatrix}, \quad y_U = \begin{pmatrix} -U_{FF}^{-1}U_{CF}^\top y \end{pmatrix}.$$ 

Then,

$$N x_N = \begin{pmatrix} 0_{|F|} \\ S_C(N, F)x \end{pmatrix}, \quad y_M^\top M = \begin{pmatrix} 0_{|F|}^\top \\ y^\top S_C(M, F) \end{pmatrix}.$$ 

Thus, we have

$$y_M^\top M x_N = \begin{pmatrix} 0_{|F|}^\top \\ y^\top S_C(M, F) \end{pmatrix} \begin{pmatrix} -N_{FF}^{-1}N_FC x \\ x \end{pmatrix} = y^\top S_C(M, F)x.$$ 

Similarly, $y_M^\top N x_N = y^\top S_C(N, F)x$. Therefore,

$$y^\top (S_C(M, F) - S_C(N, F))x = y_M^\top (M - N)x_N.$$ 

Combining the above equation with $E \overset{\text{asym}}{\preccurlyeq} b \cdot U$ and Fact 2.2 yields that

$$2y^\top (S_C(M, F) - S_C(N, F))x \leq b \left( y_M^\top U y_M + x_N^\top U x_N \right).$$

Denote the projection matrix onto the image space of $N$ by $\Pi$. It follows by direct calculations that $(x_N)_{F} + U_{FF}^{-1} U_{FC} x = -N_{FF}^{-1}(N x_U)_{F} = -N_{FF}^{-1}(\Pi N x_U)_{F}$.
We denote the matrix
\[ P = \Pi \begin{bmatrix} (N_{FF}^{-1})^\top U_{FF} N_{FF}^{-1} & 0_{|F|\times|C|} \\ 0_{|C|\times|F|} & 0_{|C|\times|C|} \end{bmatrix} \Pi \]
in this proof. Then by Fact A.10 we have
\[ x_N^\top U x_N = \|x\|_{Sc(U,F)}^2 + \|N_{FF}^{-1}(\Pi N x_U)\|_{U_{FF}}^2 = \|x\|_{Sc(U,F)}^2 + \|N x_U\|_P^2. \] (29)

By the condition \( U[N_{FF}] \geq \frac{1}{\mu} U_{FF} \), we have
\[ (N_{FF}^{-1})^\top U_{FF} N_{FF}^{-1} \preceq \mu (N_{FF}^{-1})^\top U[N_{FF}] N_{FF}^{-1}. \] (30)

Since \( N_{FF} \) is \( \rho \)-RCDD, we have \( U[N_{FF}] \geq \frac{\rho}{1+\rho} \text{Diag}(N)_{FF} \), i.e.,
\[ U[N_{FF}]^{-1} \preceq \left(1 + \frac{1}{\rho}\right) \text{Diag}(N)_{FF}^{-1}. \] (31)

Then, combining (30), (28), (31) with Fact A.13 yields that
\[ P \preceq \left(1 + \frac{1}{\rho}\right) \mu \Pi \begin{bmatrix} \text{Diag}(N)_{FF}^{-1} & 0_{|F|\times|C|} \\ 0_{|C|\times|F|} & 0_{|C|\times|C|} \end{bmatrix} \Pi \preceq \left(1 + \frac{1}{\rho}\right) \mu \Pi \text{Diag}(N)^{-1} \Pi 
= \left(1 + \frac{1}{\rho}\right) \mu \left( N^\top \right)^\top N^\top \text{Diag}(N)^{-1} N N^\top \preceq 2 \left(1 + \frac{1}{\rho}\right) \mu \left( N^\top \right)^\top U[N] N^\top 
\preceq 2 \left(1 + \frac{1}{\rho}\right) \mu \beta \left( N^\top \right)^\top U N^\top. \] (32)

We also define the matrix
\[ Q = \tilde{\Pi} \begin{bmatrix} M_{FF}^{-1} U_{FF} (M_{FF}^{-1})^\top & 0_{|F|\times|C|} \\ 0_{|C|\times|F|} & 0_{|C|\times|C|} \end{bmatrix} \tilde{\Pi} \]
in this proof, where \( \tilde{\Pi} \) is the projection matrix onto the image space of \( M^\top \). Similar to (29), we have
\[ y_M^\top U y_M = \|y_U\|_{Sc(U,F)}^2 + \|M^\top y_U\|_Q^2. \] (33)

Then, combining (26), (27), (25), (31) yields that
\[ M_{FF}^{-1} U_{FF} (M_{FF}^{-1})^\top \preceq \frac{\mu}{1-\mu b} M_{FF}^{-1} U[M]_{FF} (M_{FF}^{-1})^\top \preceq \frac{\mu}{1-\mu b} U[M]_{FF}^{-1} \]
\[ \preceq \frac{\mu}{(1-\mu b)^2} U[N]_{FF}^{-1} \preceq \frac{\mu}{(1-\mu b)^2} \left(1 + \frac{1}{\rho}\right) \text{Diag}(N)_{FF}^{-1}. \] (34)

By (34), we have
\[ Q \preceq \frac{\mu}{(1-\mu b)^2} \left(1 + \frac{1}{\rho}\right) \tilde{\Pi} \begin{bmatrix} \text{Diag}(N)_{FF}^{-1} & 0_{|F|\times|C|} \\ 0_{|C|\times|F|} & 0_{|C|\times|C|} \end{bmatrix} \tilde{\Pi} \preceq \frac{\mu}{(1-\mu b)^2} \left(1 + \frac{1}{\rho}\right) \tilde{\Pi} \text{Diag}(N)^{-1} \tilde{\Pi} 
= \frac{\mu}{(1-\mu b)^2} \left(1 + \frac{1}{\rho}\right) M^\top M \text{Diag}(N)^{-1} M^\top (M^\top)^\top. \]
Then, using Lemma 5.1 and $U[N] \ll \beta U$, we have

\[
Q \ll \frac{\mu}{(1-\mu b)^2} \left(1 + \frac{1}{\rho}\right) M^\top \left(\frac{1}{2-a^2b} (4 + 2a^2b) U[N] + 2b U\right) M^\top.
\]

Taking limits.

With the above preparations, our proof for (24) has 2 steps. Firstly, we prove (24) under the condition $\ker(N^\top) \cup \ker(M) \subseteq \ker(U)$. Then, we remove this extra condition by taking limits.

To begin with, we prove (24) under the condition $\ker(N^\top) \cup \ker(M) \subseteq \ker(U)$. Since $\Pi$ is the projection matrix onto the image space of $N$ and $\ker(N^\top) \subseteq \ker(U)$, we have $\Pi U \Pi = U$. Then, by (32), we have $\|N x_U\|^2 \leq 2 \left(1 + \frac{1}{\rho}\right) \mu \beta x_U^\top N^\top (N^\top)^T U N^\top U x_U = 2 \left(1 + \frac{1}{\rho}\right) \mu \beta x_U^\top \Pi U \Pi x_U = 2 \left(1 + \frac{1}{\rho}\right) \mu \beta x_U^\top U x_U.$

Then, combining with (29) yields that

\[
x_N^\top U x_N \leq \|x\|^2_{\text{Sc}(U,F)} + 2 \left(1 + \frac{1}{\rho}\right) \mu \beta x_U^\top U x_U = \left(1 + 2 \left(1 + \frac{1}{\rho}\right) \mu \beta\right) x^\top \text{Sc}(U,F) x.
\]

Analogously, we have $\Pi U \Pi = U$. Then, by (35) and (33), we have

\[
y_M^\top U y_M \leq \left(1 + \left(1 + \frac{1}{\rho}\right) \mu \beta \frac{(4 + 2a^2b)}{(1-\mu b)^2(2-a^2b)}\right) y^\top \text{Sc}(U,F) y.
\]

Combining the above equations yields that when $\ker(N^\top) \cup \ker(M) \subseteq \ker(U)$,

\[
2 y^\top (\text{Sc}(M,F) - \text{Sc}(N,F)) x \leq b \left(1 + 2 \left(1 + \frac{1}{\rho}\right) \mu \beta\right) x^\top \text{Sc}(U,F) x + \left(1 + \frac{1}{\rho}\right) \mu \beta (4 + 2a^2b) \frac{(1-\mu b)^2(2-a^2b)}{(1-\mu b)^2(2-a^2b)} y^\top \text{Sc}(U,F) y \leq b \left(1 + \left(1 + \frac{1}{\rho}\right) \mu \beta \frac{(4 + 2a^2b)}{(1-\mu b)^2(2-a^2b)}\right) x^\top \text{Sc}(U,F) x + y^\top \text{Sc}(U,F) y, \forall x, y \in \mathbb{R}^{|C|}.
\]

Next, we remove the condition $\ker(N^\top) \cup \ker(M) \subseteq \ker(U)$. The PSD matrix $U$ has the spectral decomposition as $U = \sum_{i=1}^n \lambda_i(U) z_i z_i^\top$. Since $U 1 = 0$, without loss of generality, we may assume $z_1 = 1$. Denote $d = \text{rank}(U)$. Then, by adding small symmetric perturbations of the form $\sum_{i=2}^{n-d} \delta(i) z_i z_i^\top (\delta(i) > 0)$, we can have a sequence of PSD matrices $\{U^{(j)}\}_{j \geq 0}$ such that for each $j \geq 0$, $\ker(U^{(j)}) = \text{span}(1)$, $U^{(j)} > U$ and $\lim_{j \to +\infty} U^{(j)} = U$.

By adding undirected edges with small weights in $N$, we can find a sequence of Eulerian Laplacians $\{N^{(j)}\}_{j \geq 0}$ such that each $N^{(j)}$ is strongly connected, $N^{(j)}_{FF}$ is nonsingular and $\lim_{j \to +\infty} N^{(j)} = N$. Then, by Fact A.7

\[
\ker(N^{(j)}) = \ker\left(\left(N^{(j)}\right)^\top\right) = \text{span}(1) = \ker(U^{(j)}).
\]
We can let the weights of the new undirected edges in $N$ tend to zero faster than $U^{(j)} - U$. Then,

$$\lim_{j \to +\infty} \left\| \left( U^{(j)} \right)^{1/2} (N - N^{(j)}) \left( U^{(j)} \right)^{1/2} \right\|_2 = 0.$$  

Since $E \asymp bU$ and $U1 = 0$, we have $E1 = E^\top 1 = 0$. Thus, $M1 = M^\top 1 = 0$. By adding small symmetric perturbations of the form $\sum_{i=2}^n \delta_i z_i z_i^\top$ into $M$, we can find a sequence of matrices $\{M^{(j)}\}_{j \geq 0}$ such that $M_{FF}^{(j)}$ is nonsingular, $\lim_{j \to +\infty} M^{(j)} = M$ and

$$\ker(M^{(j)}) = \ker\left( \left( M^{(j)} \right)^\top \right) = \text{span}(1) = \ker(U^{(j)}).$$

Also, by setting the perturbations added into $M$ to be small enough (with respect to the magnitudes of the perturbations in $U$), we can let

$$\lim_{j \to +\infty} \left\| \left( U^{(j)} \right)^{1/2} (M - M^{(j)}) \left( U^{(j)} \right)^{1/2} \right\|_2 = 0.$$

Define $b^{(j)} = b + \left\| \left( U^{(j)} \right)^{1/2} (N - N^{(j)}) \left( U^{(j)} \right)^{1/2} \right\|_2 + \left\| \left( U^{(j)} \right)^{1/2} (M - M^{(j)}) \left( U^{(j)} \right)^{1/2} \right\|_2$. Then, combining the above equations and using the relation $U^{(j)} > U$ and $M - N \asymp bU$, we have $\lim_{j \to +\infty} b^{(j)} = b$ and

$$M^{(j)} - N^{(j)} \asymp b^{(j)} U^{(j)}.$$  

As the perturbations mentioned above tend to zero, we can also define real numbers $\{a^{(j)}, \rho^{(j)}, \mu^{(j)}, \beta^{(j)}\}_{j \geq J}$ easily such that the matrix sequence $\{M^{(j)}, N^{(j)}, U^{(j)}\}_{j \geq 0}$ satisfy

- $N_{FF}^{(j)}$ is $\rho^{(j)}$-RCDD ($\rho^{(j)} > 0$),
- $U \left[ N^{(j)} \right]_{FF} \geq \frac{1}{\mu^{(j)}} U_{FF}^{(j)}$,
- $U \left[ N^{(j)} \right] \preceq \beta^{(j)} U^{(j)}$,
- $\left\| \left( U^{(j)} \right)^{1/2} \text{Diag} \left( N^{(j)} \right)^{-1/2} \right\|_2 \leq a^{(j)}$,

for any $j \geq J$, and $a^{(j)}, \rho^{(j)}, \mu^{(j)}, \beta^{(j)}$ tend to $a, \rho, \mu, \beta$, respectively.

Since $b < \min \left\{ \frac{2}{\alpha^{(j)}}, \frac{1}{\mu^{(j)}} \right\}$ and $b^{(j)}, a^{(j)}$, $\mu^{(j)}$ tend to $b, a, \mu$ respectively, then, there exists a $J' > 0$ such that for any $j \geq J'$, $b^{(j)} < \min \left\{ \frac{2}{(a^{(j)})^2}, \frac{1}{\mu^{(j)}} \right\}$.

Then by (36), we have for any $j \geq \max\{J, J'\}$ and $x, y \in \mathbb{R}^{|C|}$,

$$2y^\top \left( \text{Sc} \left( M^{(j)}, F \right) - \text{Sc} \left( N^{(j)}, F \right) \right) x$$

$$\leq b^{(j)} \left( 1 + \left( \frac{1}{\rho^{(j)}} \right) \frac{\mu^{(j)}(\beta^{(j)}(4 + 2(a^{(j)})^2 b^{(j)}) + 2b^{(j)})}{(1 - \mu^{(j)}b^{(j)})^2 (2 - (a^{(j)})^2 b^{(j)})} \right) \left( x^\top \text{Sc} \left( U^{(j)}, F \right) x + y^\top \text{Sc} \left( U^{(j)}, F \right) y \right).$$

(37)
Since $M_{FF}$ is nonsingular, we have $\lim_{j \to +\infty} \left( M_{FF}^{(j)} \right)^{-1} = M_{FF}^{-1}$. Thus, $\lim_{j \to +\infty} \text{Sc} \left( M^{(j)}, F \right) = \text{Sc}(M, F)$. Analogously, $\lim_{j \to +\infty} \text{Sc} \left( U^{(j)}, F \right) = \text{Sc}(U, F)$, $\lim_{j \to +\infty} \text{Sc} \left( N^{(j)}, F \right) = \text{Sc}(N, F)$. Then, taking limits on both sides of (37) and combining with Fact 2.2 lead to (24). 

5.2 Inductive Accumulation of Errors

We can now obtain a relatively tight bound for $\text{Sc} \left( L^{(K)}, F \right) - \text{Sc} \left( L^{(K)}, F \right)$ by bounding $\gamma_k$ iteratively.

**Lemma 5.3.** For any $\delta_0 \in (0, 1)$, with a small $\epsilon = O \left( \frac{\delta_0}{\rho} \right)$ in Algorithm 2, the exact and approximate $K$-th partially-block-eliminated Laplacians $L^{(K)}$, $\tilde{L}^{(K)}$ satisfies

$$\frac{1}{2K} \left( \text{Sc} \left( \tilde{L}^{(K)}, F \right) - \text{Sc} \left( L^{(K)}, F \right) \right) \approx O(\delta_0) \cdot U[\text{Sc}(L, F)].$$

**Proof.** First, we will prove

$$\gamma_k \approx O(\delta_0) \quad (\forall 0 \leq k \leq K)$$

by induction, where $\{\gamma_k\}$ are defined in (19).

Since $\tilde{L}^{(0)} = L$, we have $\gamma_0 = 0$. Now, assume $\gamma_i \approx O(\delta_0), \quad (\forall 0 \leq i \leq k - 1)$, we will show that $\gamma_k \approx O(\delta_0)$.

By Fact 4.8, $Q^{(k)}$ is Laplacian, thus, $Q^{(k)}$ is PSD and $Q^{(k)}[1] = 0$.

By Fact 4.10, and the condition $\epsilon_0 = O \left( \frac{\delta_0}{\rho} \right)$, we have

$$U \left[ \mathcal{M}^{(0,k)} \right]_{-n,-[n]} \approx \frac{\alpha}{1 + \alpha} \text{Diag} \left( \mathcal{M}^{(0,k)} \right)_{-n,-[n]} = \frac{\alpha}{1 + \alpha} \text{Diag} \left( Q^{(k)} \right)_{-n,-[n]} \approx \frac{2 + 2\alpha}{2} Q^{(k)}_{-n,-[n]}.$$

By the induction hypothesis, $\sum_{i=0}^{k-1} \gamma_i \approx O(k\delta_0)$. Then, by combining with Lemma 4.10, Fact 4.8[iv], and the condition $\epsilon_0 = O \left( \frac{\delta_0}{\rho} \right)$, we have

$$U \left[ \mathcal{M}^{(0,k)} \right] \approx \left( 4 + \frac{2}{\alpha} + O(\delta_0) \right) Q^{(k)}$$

$$\left\| \left( Q^{(k)} \right)^{1/2} \text{Diag} \left( \mathcal{M}^{(0,k)} \right)^{-1/2} \right\|_2 \leq 2$$

$$\mathcal{E}^{(1:k,k)} \approx \epsilon_0 \left( 4k + \frac{2k}{\alpha} + k\delta_0 \right) \cdot Q^{(k)} \approx O(\delta_0) \cdot Q^{(k)}.$$

Now, we invoke Lemma 5.2 with $N := \mathcal{M}^{(0,k)}$, $M := \tilde{\mathcal{M}}^{(0,k)}$, $U := Q^{(k)}$, $E := \epsilon^{(1:k,k)}$, $m := 2kF + |C|$, $F := [2kF] + |C| \setminus [n]$, $C := [n]$, $\rho := \alpha$, $\mu := 2 + 2\alpha$, $\beta := 4 + \frac{2}{\alpha} = O(\delta_0)$, $a^2 := 2$, $b := O(\delta_0)$. By the arguments above, all the conditions of Lemma 5.2 are satisfied. Then, we have

$$\text{Sc} \left( \tilde{\mathcal{M}}^{(0,k)}_{-n}, -[n] \right) - \text{Sc} \left( \mathcal{M}^{(0,k)}_{-n}, -[n] \right) \approx O(\delta_0) \cdot \text{Sc} \left( Q^{(k)}_{-n}, -[n] \right).$$

31
By combining with the fact $L^{(k)} = \text{Sc}(\mathcal{M}^{(0,k)}, -[n])$, $\tilde{L}^{(k)} = \text{Sc}(\tilde{\mathcal{M}}^{(0,k)}, -[n])$, the definition of $\gamma_k$ in (19) and Fact 2.2 we have $\gamma_k \leq O(\delta_0)$. Then, by induction, we have $\gamma_K \leq O(\delta_0)$.

By setting $N := \mathcal{M}^{(0,k)}$, $M := \tilde{\mathcal{M}}^{(0,k)}$, $U := Q^{(k)}$, $E := \mathcal{E}^{(1,k,k)}$, $m := 2^k|F| + |C|$, $F := [2^k|F| + |C|]|C|$, $C := C$, $\rho := \alpha$, $\mu := \frac{2 + 2\alpha}{\alpha}$, $\beta := 4 + \frac{2}{\alpha} + O(\delta_0)$, $a^2 := 2$, $b := O(\delta_0)$ in Lemma 4.9, it is easy to check that all conditions of Lemma 5.2 are satisfied by similar arguments as above. Thus, similar to (39), we have

$$\text{Sc}(\tilde{\mathcal{M}}^{(0,k)}, -C) - \text{Sc}(\mathcal{M}^{(0,k)}, -C) \overset{\text{asym}}{=} O(\delta_0) \cdot \text{Sc}(Q^{(k)}, -C).$$

(40)

By Fact A.9, Lemma 4.1, Lemma 4.4

$$\text{Sc}(\mathcal{M}^{(0,k)}, -C) = \text{Sc}(\text{Sc}(\mathcal{M}^{(0,k)}, -[n]), F) = \text{Sc}(L^{(k)}, F).$$

$$\text{Sc}(\tilde{\mathcal{M}}^{(0,k)}, -C) = \text{Sc}(\text{Sc}(\tilde{\mathcal{M}}^{(0,k)}, -[n]), F) = \text{Sc}(\tilde{L}^{(k)}, F).$$

By Lemma 4.9 we have $\text{Sc}(Q^{(k)}, -C) \overset{\text{asym}}{=} 2^K U[\text{Sc}(L, F)]$.

Substituting the above 3 equations into (40) and combining with Fact A.2 complete this proof.

\[\square\]

**Remark 5.4.** Since $\tilde{S}^{(0)} = \frac{1}{2K} (\tilde{L}^{(k)}_{CC} - \tilde{X})$ (in Algorithm 2), the \(\frac{1}{2K}\) factor on the LHS of (38) doesn’t matter.

Now, we are prepared to prove Theorem 3.3

**Proof of Theorem 3.3** By Lemma 3.1 we have the expansion

$$S - \text{Sc}(L, F) = S - \tilde{S} + \tilde{S}^{(0)} + R - \frac{1}{2K} \text{Sc}(L^{(k)}, F)$$

$$= S - \tilde{S} + \frac{1}{2K} (\tilde{L}^{(k)}_{CC} - \tilde{X} - \text{Sc}(L^{(k)}, F)) + R$$

$$= S - \tilde{S} + \frac{1}{2K} \left(\text{Sc}(\tilde{L}^{(k)}, F) - \text{Sc}(L^{(k)}, F)\right) + \frac{1}{2K} \left(\tilde{A}^{(k)}_{CF} D^{-1}_{FF} \tilde{A}^{(k)}_{FC} - \tilde{X}\right)$$

$$+ \frac{1}{2K} \left(\tilde{A}^{(k)}_{CF} D^{-1}_{FF} - \tilde{A}_{FF} \tilde{A}^{(k)}_{FC} - \tilde{A}^{(k)}_{CF} D^{-1}_{FF} \tilde{A}^{(k)}_{FC}\right) + R$$

$$= S - \tilde{S} + \frac{1}{2K} \left(\text{Sc}(\tilde{L}^{(k)}, F) - \text{Sc}(L^{(k)}, F)\right) + \frac{1}{2K} E \tilde{X} + \tilde{R}.$$

By Lemma 5.3 and choosing a small \(\epsilon = O(\frac{\delta}{\pi})\) in Algorithm 2 we can have

$$\frac{1}{2K} \left(\text{Sc}(\tilde{L}^{(k)}, F) - \text{Sc}(L^{(k)}, F)\right) \overset{\text{asym}}{=} O(\delta) \cdot \frac{\delta}{4} U[\text{Sc}(L, F)].$$

Then, by Lemma 3.1 and Fact A.3

$$U\left[\text{Sc}(\tilde{L}^{(k)}, F)\right] \overset{\text{asym}}{=} U\left[\text{Sc}(L^{(k)}, F)\right] + 2^K \cdot \frac{\delta}{4} U[\text{Sc}(L, F)] = 2^K \left(1 + \frac{\delta}{4}\right) U[\text{Sc}(L, F)].$$

32
Combining the above equation with (15) and Fact A.2 yields that by choosing $\epsilon \leq \frac{\delta}{4 + \delta}$, we have
\[
\frac{1}{2K} E x \text{ asym} \leq \frac{\epsilon}{2K} U[S(L, F)] \leq \left(1 + \frac{\epsilon}{4}\right) U[S(L, F)] \leq \frac{\delta}{4} U[S(L, F)].
\]
By Fact A.18, Fact A.20 and Cheeger’s inequality, we have $\lambda_2(U[S(L, F)]) \geq \lambda_2(S(CU[L], F)) \geq \lambda_2(U[L]) \geq \frac{\min_{i \leq n}|D_{ii}|}{8(\sum_{i \leq n}|D_{ii}|)^2} = \Omega\left(\frac{1}{\lambda(L)}\right)$. Then, by choosing $K = O(\log \log \frac{n}{\delta})$ and using (53), we can let $\|U[S(L, F)]^{1/2} \tilde{R} U[S(L, F)]^{1/2}\|_2 \leq \frac{1}{\lambda_2(U[S(L, F)])} \|\tilde{R}\|_2 \leq \frac{1}{4}$. Since $L$ is strongly connected, we have $S(L, F)$ is strongly connected. So, $\ker(U[S(L, F)]) = \text{span}(1)$. By combining with the fact $\tilde{R} 1 = \tilde{R}^T 1 = 0$ from Lemma D.3, we have
\[
\tilde{R} \text{ asym} \leq \frac{\delta}{4} U[S(L, F)].
\]
Combining the above equations with Fact A.1 yields that
\[
\tilde{S} - S \text{ asym} \leq \frac{3\delta}{4} U[S(L, F)].
\]
Then, $U[\tilde{S}] \leq (1 + \frac{3\delta}{4}) U[S(L, F)] \leq 2 U[S(L, F)]$. Thus,
\[
\tilde{S} - S \text{ asym} \leq \frac{\delta}{8} U[\tilde{S}] \leq \frac{\delta}{4} U[S(L, F)].
\]
Then, (2) follows.

The connectivity of $S$ can be readily checked as follows. If $S$ is not strongly connected, then, there is a vector $x \neq 0$ and $x$ not parallel to $1$ such that $Sx = 0$. Since $L$ is a strongly connected Eulerian Laplacian, we have $S(L, F)$ is strongly connected, thus, $x^T U[S(L, F)]x > 0$. Then, by (2),
\[
x^T U[S(L, F)]x = x^T (U[S(L, F)] - U[S])x \leq \delta x^T U[S(L, F)]x,
\]
which contradicts the condition $\delta \in (0, 1)$. Thus, $S$ is strongly connected.

Since we call \textsc{SparseEulerianFC} in each iteration and $\epsilon = \tilde{O}(\delta)$, we have $\text{nnz}(\hat{L}^{(k)}) = \tilde{O}(N^* (n, \delta))$. Since $\epsilon = \tilde{O}(\delta)$ and $K = \tilde{O}(1)$, the total running time of \textsc{SparseProductFC} and \textsc{SparseProduct} is $\tilde{O}(N^* (n, \delta)\delta^{-2} \log n)$ and $\text{nnz}(\hat{L}^{(k, \delta)}) = \tilde{O}(N^* (n, \delta)\delta^{-2} \log n)$. As $K = \tilde{O}(1)$, $\epsilon = \tilde{O}(\delta)$, by Theorem 2.4 and Lemma D.1, the total running time of \textsc{SparseEulerianFC} is $O(T^*_S (m, n, \delta)) = \tilde{O}(T^*_S (N^* (n, \delta)\delta^{-2} \log n, n, \delta)) = O(T^*_S (m, n, \delta)) = O(T^*_S (N^* (n, \delta)\delta^{-2}, n, \delta) \log n)$ which gives the overall running time bound for Algorithm 2.

6 A Nearly-linear Time Solver

In this section, we complete the Sparsified Schur Complement based algorithm by invoking the nearly-linear time Schur complement sparsification procedure derived above in Sections 4 and 5. We first call this Schur complement sparsification procedure repeatedly to construct a sparse Schur complement chain, in Section 6.1. Then, in Section 6.2, we show that this Schur complement chain gives a preconditioner \textsc{Precondition} for the initial Eulerian Laplacian matrix. The full high accuracy solver then follows from invoking this preconditioner inside Richardson iteration.
We first define Schur complement chains over directed graphs, which is a variant of the Schur complement chain for undirected graphs in [KLP+16].

**Definition 6.1.** (Schur complement chain) Given a strongly connected Eulerian Laplacian \( L \in \mathbb{R}^{n \times n} \), an \( (\alpha, \beta, \{\delta_i\}_{i=1}^d) \)-Schur complement chain of \( L \) is a sequence of strongly connected Eulerian Laplacians and subsets \( \{\{\tilde{S}^{(i)}\}_{i=1}^d, \{F_i\}_{i=1}^d\} \) satisfying

1. \( \{F_i\}_{i=1}^d \) is a partition of \([n]\); each \( \tilde{S}^{(i)} \) is supported on \((C_{i-1}, C_{i-1})\), where \( C_i \overset{\text{def}}{=} [n] \setminus \bigcup_{j=1}^i F_j \) (\( i = 0, 1, \ldots, d - 1 \)); \( |C_i| \leq (1 - \beta)^i n\); \( |F_d| = |C_{d-1}| = O(1) \).
2. For \( 1 \leq i \leq d - 1 \), \( \tilde{S}^{(i)}_{F_{i}, F_{i}} \) is \( \alpha \)-RCDD.
3. \( \tilde{S}^{(1)} \overset{\text{asymp}}{=} \delta_1 \cdot U[L] \) and \( \tilde{S}^{(i+1)} - \text{Sc}(\tilde{S}^{(i)}, F_i) \overset{\text{asymp}}{=} \delta_{i+1} \cdot U\left[ \text{Sc}(\tilde{S}^{(i)}, F_i) \right] \), \( 1 \leq i \leq d - 1 \).
4. \( U\left[ \tilde{S}^{(1)} \right] \overset{\text{asymp}}{=} U[L] \) and \( U\left[ \tilde{S}^{(i+1)} \right] \overset{\text{asymp}}{=} U\left[ \text{Sc}(\tilde{S}^{(i)}, F_i) \right] \), \( 1 \leq i \leq d - 1 \).

We also denote \( F_0 = C_d = \emptyset \), \( C_0 = [n] \) for notational simplicity.

**Remark 6.2.** Compared with the Schur complement chains for undirected graphs from [KLP+16], the only new condition is Condition (iv). It guarantees the positive semi-definiteness of the symmetrization of the sparsified approximate Eulerian Laplacian \( \tilde{L} \) and the error-bounding matrix \( \tilde{B} \) defined in Section 6.2.

To construct a Schur complement chain, we first use the following lemma to find an \( \alpha \)-RCDD subset \( F_1 \), and then apply the Schur complement sparsification method \textsc{SparseSchur} to compute \( \tilde{S}^{(1)} \) which is an approximation for \( \text{Sc}(L, F_1) \). Then, we repeat this process to get a desirable Schur complement chain.
Lemma 6.3. (Theorem A.1 of \cite{CKK+18}) Given an Eulerian Laplacian \( L \in \mathbb{R}^{n \times n} \) with \( \text{nnz}(L) = m \), the routine \textsc{FindRCDDBlock} outputs a subset \( F \subseteq [n] \) such that \( |F| \geq \frac{n}{16(1+\alpha)} \) and \( L_{FF} \) is \( \alpha \)-RCDD in time \( O \left( m \log \frac{1}{p} \right) \) with probability at least \( 1 - p \).

By Lemma 6.3, we can choose for instance \( \alpha = 0.1 \) in practice. So, we assume \( \alpha = O(1) \), when analyzing the complexities below. Our method to construct a Schur complement chain is illustrated in Algorithm 4. It performance is shown in Theorem 6.4 whose proof is deferred to Appendix D.

Algorithm 4: \textsc{SchurChain}(L, \alpha, \delta)

\begin{itemize}
  \item \textbf{Input}: strongly connected Eulerian Laplacian \( L \in \mathbb{R}^{n \times n} \); parameters \( \alpha > 0, \delta \in (0, 1] \)
  \item \textbf{Output}: \( \left( \frac{1}{10(1+\alpha)}, \{ \frac{\delta}{10} \}_{i=1}^{d} \right) \)-Schur complement chain \( \left\{ \left\{ \tilde{S}^{(i)} \right\}_{i=1}^{d}, \{ F_i \}_{i=1}^{d} \right\} \)
\end{itemize}

1. Set \( \delta'_{i} = \frac{\delta}{10(1+\alpha)} \) for \( i \geq 1 \).
2. Compute \( S^{(1)} = \text{OraSparseLaplacian}(\mathbf{L}, \delta'_{1}) \)
3. Let \( \tilde{S}^{(1)} = S^{(1)} + \frac{\delta'_{1}}{1-\delta'_{1}} \mathbf{U}[S^{(1)}] \).
4. Set \( i = 0, C_{0} = [n] \)
5. while \( |C_{i}| > 100 \) do
6. \( i \leftarrow i + 1 \)
7. \( F_{i} \leftarrow \text{FindRCDDBlock}(\tilde{S}^{(i)}, \alpha) \)
8. \( C_{i} \leftarrow C_{i-1} \setminus F_{i} \)
9. \( S^{(i+1)} \leftarrow \text{SparseSchur}(\tilde{S}^{(i)}, F_{i}, \delta_{i+1}) \)
10. \( \tilde{S}^{(i+1)} \leftarrow S^{(i)} + \frac{\delta_{i+1}}{1-\delta_{i+1}} \mathbf{U}[S^{(i)}] \)
11. end
12. Return \( \left\{ \left\{ \tilde{S}^{(i)} \right\}_{i=1}^{d}, \{ F_{i} \}_{i=1}^{d} \right\} \)

Theorem 6.4. Given a strongly connected Eulerian Laplacian \( L \in \mathbb{R}^{n \times n} \) and parameters \( \alpha = O(1), \delta \in (0, 1] \), the routine \textsc{SchurChain} runs in time

\[ O(\mathcal{T}_{S}(m, n, \delta)) + \tilde{O}(\mathcal{T}_{S}(N_{S}(n, \delta)\delta^{-2}, n, \delta) \log n) \]

with high probability to return an \( \left( \frac{1}{10(1+\alpha)}, \{ \frac{\delta}{10} \}_{i=1}^{d} \right) \)-Schur complement chain, where \( d = O(\log n) \).

In addition, \( \sum_{i=1}^{d} \text{nnz}(\tilde{S}^{(i)}) = O(N_{S}(n, \delta)) \).

6.2 Construction of the Preconditioner and the Solver

After constructing a desirable Schur complement chain, we use the Schur complement chain to construct a preconditioner and solve \( Lx = b \) via the preconditioned Richardson iteration.

Consider a linear system \( Ax = b \), where \( b \) is in the image space of \( A \). Given a preconditioner \( Z \), the classical preconditioned Richardson iteration updates as follows:

\[ x^{(k+1)} \leftarrow x^{(k)} + \eta Z \left( b - Ax^{(k)} \right). \]
We initialize $x^{(0)} = 0$ for simplicity. This procedure is denoted by $x^{(N)} = \text{PreRichardson}(A, b, Z, \eta, N)$.

We will use the following fundamental lemma to guarantee the performance of the preconditioned Richardson iteration in our methods.

**Lemma 6.5.** (Lemma 4.2 of (CKP+17)) Let $A, Z, U \in \mathbb{R}^{n \times n}$, where $U$ is PSD and $\ker(U) \subseteq \ker(Z) = \ker(Z^T) = \ker(A) = \ker(A^T)$. Let $b \in \mathbb{R}^n$ be a vector inside the image space of $A$. Denote the projection onto the image space of $A$ by $P_A$. Denote $x^{(N)} = \text{PreRichardson}(A, b, Z, \eta, N)$. Then, $x^{(N)}$ satisfies

$$\left\| x^{(N)} - A^\dagger b \right\|_U \leq \left\| P_A - \eta ZA \right\|_{U \rightarrow U}^N \left\| A^\dagger b \right\|_U.$$

In addition, the preconditioned Richardson iteration is a linear operator with

$$x^{(N)} = \eta \sum_{k=0}^{N-1} (P_A - \eta ZA)^k Zb. \quad (41)$$

Our construction for the preconditioner is illustrated in Algorithm 1. To analyze Algorithm 1, we define the following matrices.

$\Pi = I - \frac{11}{n}$ is the projection matrix onto the image space of $L$.

$\hat{D}$ is an $n$-by-$n$ diagonal matrix with $\hat{D}_{F_iF_i} = \text{Diag}(\hat{S}_F^{(i)})$ for $i \in [d]$.

$M^{(i,N)}$ is the linear operator corresponding to the preconditioned Richardson iterations

$$M^{(i,N)} = \frac{1}{2} \sum_{k=0}^{N-1} \left( I - \frac{1}{2} \hat{D}_{F_iF_i} \hat{S}_F^{(i)} \right)^k \hat{D}_{F_iF_i} = \frac{1}{2} \sum_{k=0}^{N-1} \hat{D}_{F_iF_i} \left( I - \frac{1}{2} \hat{S}_F^{(i)} \hat{D}_{F_iF_i} \right)^k, \quad i \in [d-1].$$

$\tilde{L}^{(i,N)}$ and $\tilde{U}^{(i,N)}$ are block lower triangular and block upper triangular matrices of the block Cholesky factorization with

$$\tilde{L}^{(i,N)} = \begin{bmatrix} I_{\sum_{j=1}^{i-1} |F_i|} & I & \hat{S}_C^{(i)}M^{(i,N)} & I \end{bmatrix}, \quad \tilde{U}^{(i,N)} = \begin{bmatrix} I_{\sum_{j=1}^{i-1} |F_i|} & I & \hat{S}_F^{(i)} & I \end{bmatrix},$$

where $I_k$ denotes the $k$-by-$k$ identity matrix. $\tilde{D}^{(N)}$ is the block diagonal matrix corresponding to the block Cholesky factorization

$$\tilde{D}^{(N)} = \begin{bmatrix} (M^{(1,N)})^{-1} & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \\ & & & \hat{S}_F^{(d)} \end{bmatrix},$$

where the invertibility of $M^{(i,N)}$ is given by Lemma 6.7.

Note that

$$\left( \tilde{L}^{(i)} \right)^{-1} = \begin{bmatrix} I_{\sum_{j=1}^{i-1} |F_i|} & I & \hat{S}_C^{(i)}M^{(i,N)} & I \end{bmatrix}, \quad \left( \tilde{U}^{(i)} \right)^{-1} = \begin{bmatrix} I_{\sum_{j=1}^{i-1} |F_i|} & \hat{S}_F^{(i)}M^{(i,N)} & I \end{bmatrix}.$$
Then, the routine $\text{PreRichardson}$ is a linear operator which is equivalent to multiplying vector $x$ with the matrix $\Pi \tilde{Z}$, where $\tilde{Z} \in \mathbb{R}^{n \times n}$ is defined as follows:

\[
\tilde{Z} = \left( \tilde{U}^{(1,N)} \right)^{-1} \cdots \left( \tilde{U}^{(d-1,N)} \right)^{-1} \left( \tilde{D}^{(N)} \right)^{-1} \tilde{L}^{(d-1,N)} \cdots \left( \tilde{U}^{(1,N)} \right)^{-1}.
\]

We also define the following matrices which are counterparts of $\{ \tilde{L}^{(i,N)} \}$ when $N = +\infty$ in Algorithm 1:

\[
\tilde{L}^{(i,\infty)} = \begin{bmatrix} \sum_{j=1}^{d-1} I_{F_i-j} & I \\ \tilde{S}^{(i)} & S^{-1} \end{bmatrix}
\]

The matrices $\{ \tilde{U}^{(i,\infty)} \}, \{ \tilde{D}^{(\infty)} \}$ are defined similarly by replacing $M^{(i,N)}$ with $\left( \tilde{S}^{(i)} \right)^{-1}$ in $\{ \tilde{U}^{(i,N)} \}, \{ \tilde{D}^{(N)} \}$.

Define $\tilde{L}$ as an approximation for $L$ with the errors induced by the Schur complement sparsification procedure

\[
\tilde{L} = \tilde{L}^{(1)} + \sum_{i=1}^{d-1} \mathcal{P}\left( \tilde{S}^{(i+1)} - \mathcal{S}(\tilde{S}^{(i)}, F_i), C_i, C_i, n \right), \tag{42}
\]

where the notation $\mathcal{P}(\cdot)$ is defined in Section 4.1, which means putting a matrix on the designated position in an all-zeros matrix with designated size.

Then, by direct calculations,

\[
\tilde{L} = \tilde{L}^{(1,\infty)} \cdots \tilde{L}^{(d-1,\infty)} \tilde{D}^{(\infty)} \tilde{U}^{(d-1,\infty)} \cdots \tilde{U}^{(1,\infty)}. \tag{43}
\]

The following matrices $B$ and $\tilde{B}$ are playing the role of $U$ in Lemma 6.5

\[
B = \delta_1 U [L] + \sum_{i=2}^{d} \delta_i \mathcal{P}\left( U \left[ \tilde{S}^{(i)} \right], C_i-1, C_i-1, n \right),
\]

\[
\tilde{B} = \delta_1 U [\tilde{L}] + \sum_{i=1}^{d-1} \delta_{i+1} \mathcal{P}\left( U \left[ \mathcal{S}(\tilde{L}, \cup_{j=1}^{i} F_j) \right], C_i, C_i, n \right).
\]

The proofs of the following lemmas are deferred to Appendix D.

**Lemma 6.6.** If the input $\{ \alpha, \beta, \{ \delta_i \}_{i=1}^{d} \}$-Schur complement chain satisfies $\sum_{i=1}^{d} \delta_i \leq 1$, then $B \preceq \tilde{B} \preceq 2B$.

**Lemma 6.7.** For $N \geq 1$, $M^{(i,N)}$ is nonsingular and

\[
\left\| \left( \tilde{L}^{(i,N)} \right)^{-1} - \left( \tilde{L}^{(i,\infty)} \right)^{-1} \right\|_1 \leq \frac{(1+\alpha)}{\alpha} \left( \frac{2+\alpha}{2(1+\alpha)} \right)^N,
\]

\[
\left\| \tilde{L}^{(i,N)} \right\|_1 \leq \frac{(1+\alpha)}{\alpha} \left( \frac{2+\alpha}{2(1+\alpha)} \right)^N \left\| \tilde{D}^{-1} \right\|_\infty.
\]
From (41), to analyze the quality of the preconditioner $\Pi \tilde{Z}$, we need to provide bounds for $\Pi - \Pi \tilde{Z} L$.

**Lemma 6.8.** Given $\left\{ \alpha, \beta, \{ \delta_i \}_{i=1}^d \right\}$-Schur complement chain with $d = O(\log n)$ and $\sum_{i=1}^d \delta_i \leq \frac{1}{4}$, by setting $N = O(\log n)$ in Algorithm 1, we have $\left\| \Pi - \Pi \tilde{Z} L \right\|_{B \to \tilde{B}} \leq \frac{1}{2}$.

**Proof.** From the fact $\left\{ \tilde{S}^{(i)} \right\}$ are all Eulerian Laplacians and Fact A.4 we have $\tilde{L} 1 = \tilde{L}^\top 1 = 0$. By (43) and the strong connectivity of $\tilde{S}$, we have $\text{rank}(\tilde{L}) = n - 1$. Then, $\ker(\tilde{L}) = \ker(\tilde{L}^\top) = \text{span}(1)$. Thus, $\tilde{LL}^\top = \tilde{L}^\top \tilde{L} = \Pi$.

Now, we expand $\Pi - \Pi \tilde{Z} L$ as follows

\[
\Pi - \Pi \tilde{Z} L = \tilde{L}^\top (\tilde{L} - L) + (\tilde{L}^\top - \Pi \tilde{Z} \Pi) L.
\]

By the fact that $(\Pi - \Pi \tilde{Z} L) 1 = 0$ and $B \succ U L$, we have $\ker(\Pi - \Pi \tilde{Z} L) \supseteq \ker(B)$. Then, by combining with the definition of $\left\| \cdot \right\|_{\tilde{B} \to \tilde{B}}$, we have

\[
\left\| \Pi - \Pi \tilde{Z} L \right\|_{\tilde{B} \to \tilde{B}}^2 = \left\| \tilde{L}^\top (\tilde{L} - L) + (\tilde{L}^\top - \Pi \tilde{Z} \Pi) L \right\|_{\tilde{B} \to \tilde{B}}^2 = \left\| \tilde{B}^{1/2} (\tilde{L}^\top (\tilde{L} - L) + (\tilde{L}^\top - \Pi \tilde{Z} \Pi) L) \tilde{B}^{1/2} \right\|_2^2 \
\leq 2 \left\| \tilde{B}^{1/2} (\tilde{L}^\top (\tilde{L} - L) \tilde{B}^{1/2} \right\|_2^2 + 2 \left\| \tilde{B}^{1/2} (\tilde{L}^\top - \Pi \tilde{Z} \Pi) LB^{1/2} \right\|_2^2.
\]

By the definitions of $\tilde{L}, B$, we have

\[
2 x^\top (\tilde{L} - L) y \\
\leq x^\top \left( \delta_1 U L + \sum_{i=2}^d \delta_i P \left( U \left[ \tilde{S}^{(i)} \right], C_{i-1}, C_{i-1}, n \right) \right) x + y^\top \left( \delta_1 U L + \sum_{i=2}^d \delta_i P \left( U \left[ \tilde{S}^{(i)} \right], C_{i-1}, C_{i-1}, n \right) \right) y = x^\top B x + y^\top B y.
\]

By combining with Lemma 6.6, we have

\[
\left\| \tilde{B}^{1/2} (\tilde{L} - L) \tilde{B}^{1/2} \right\|_2^2 \leq \left\| \tilde{B}^{1/2} (\tilde{L} - L) \tilde{B}^{1/2} \right\|_2^2 \leq 1.
\]

Next, we bound $(\tilde{L}^\top - \Pi \tilde{Z} \Pi) L$. From the definition of $\left\{ \alpha, \beta, \{ \delta_i \}_{i=1}^d \right\}$-Schur complement chain, we have $U \left[ \tilde{S}^{(i+1)} \right] \succ U \left[ \text{Sc}(\tilde{S}^{(i)}, F_i) \right]$. Combining with Fact A.18, we have $U \left[ \tilde{S}^{(i+1)} \right] \succ \text{Sc}(U \left[ \tilde{S}^{(i)} \right], F_i)$. Then, by Fact A.20, $\lambda_2 \left( U \left[ \tilde{S}^{(i+1)} \right] \right) \geq \lambda_2 \left( \text{Sc}(U \left[ \tilde{S}^{(i)} \right], F_i) \right) \geq \lambda_2 (U \left[ \tilde{S}^{(i)} \right])$. By induction, $\lambda_2 (U \left[ \tilde{S}^{(i)} \right]) \geq \lambda_2 (U[L])$. By Cheeger’s inequality, $\lambda_2 (U[L]) = \Omega \left( \frac{1}{\text{poly}(n)} \right)$. Then,
for any $i \in [d]$, $\lambda_2 \left( U \left[ \tilde{S}^{(i)} \right] \right) = \Omega \left( \frac{1}{\text{poly}(n)} \right)$. By Fact A.20, $\| \tilde{D}^{-1}_{F_i} \|_\infty \leq \frac{2}{\lambda_2 \left( U \left[ \tilde{S}^{(i)} \right] \right)} = O(\text{poly}(n))$.

Also, $\lambda_2 \left( \tilde{B} \right) \geq \lambda_2 \left( B \right) = \Omega \left( \frac{1}{\text{poly}(n)} \right)$. It follows by induction easily that $\| \tilde{B} \|_2 = O(\text{poly}(n))$.

By Fact A.19 and (43), we have

$$\bar{L}^\dagger = \pi \left( \bar{U}^{(1,\infty)} \right)^{-1} \cdots \left( \bar{U}^{(d-1,\infty)} \right)^{-1} \left( \bar{D}^{(\infty)} \right)^\dagger \left( \bar{L}^{(d-1,\infty)} \right)^{-1} \cdots \left( \bar{L}^{(1,\infty)} \right)^{-1} \pi.$$ 

Since $\left\| \left( \tilde{S}_{F_i,C_i}^{(i)} \right)^{-1} \tilde{S}_{F_i,C_i}^{(i)} \right\|_\infty = \frac{1}{2} \sum_{k=0}^{\infty} \left( I - \frac{1}{2} \tilde{D}_{F_i,F_i}^{-1} \tilde{S}_{F_i,F_i}^{(i)} \right) \tilde{D}_{F_i,F_i}^{-1} \tilde{S}_{F_i,C_i}^{(i)} \left\|_\infty \leq \frac{1+\alpha}{\alpha}, \right.$ we have $\left\| \left( \bar{L}^{(i,\infty)} \right)^{-1} \right\|_1 \leq \frac{1+2\alpha}{\alpha}$. Analogously, $\left\| \left( \bar{U}^{(i,\infty)} \right)^{-1} \right\|_1 \leq \frac{1+2\alpha}{\alpha}$. Then, by Fact A.21 and Lemma 6.7, the fact that $\| \tilde{D}_{F_i,F_i}^{-1} \|_\infty = O(\text{poly}(n))$ and $(1+2\alpha) O(\text{log} n) = O(\text{poly}(n))$, we have

$$\| \bar{B}^{1/2} \left( \bar{L}^\dagger - \pi \bar{Z} \pi \right) \bar{L}^\dagger \|_2 \leq \left( \frac{2+\alpha}{2(1+\alpha)} \right)^N \cdot O(\text{poly}(n)).$$

Then, by setting $N = O(\text{log} n)$ in Algorithm 1, we can let

$$\| \bar{B}^{1/2} \left( \bar{L}^\dagger - \pi \bar{Z} \pi \right) \bar{L}^\dagger \|_2^2 \leq \frac{1}{16}. \quad \text{(46)}$$

By Fact A.22, we have

$$(\bar{L}^\dagger)^\top \bar{B} \bar{L}^\dagger \prec \left( \sum_{i=1}^{d} \delta_i \right)^2 \bar{B}^\dagger \prec \frac{1}{16} \bar{B}^\dagger. \quad \text{(47)}$$

Continuing with (44),

$$\| \pi - \pi \tilde{Z} \bar{L} \|_{\bar{B} \rightarrow \bar{B}}^2 \leq 2 \| \bar{B}^{1/2} \bar{L}^\dagger \left( \bar{L} - \bar{L} \right) \bar{B}^{1/2} \|_2^2 + 2 \| \bar{B}^{1/2} \left( \bar{L}^\dagger - \pi \tilde{Z} \pi \right) \bar{L}^{1/2} \|_2^2 = 2 \| \bar{B}^{1/2} \left( \bar{L} - \bar{L} \right)^\dagger \bar{B} \bar{L}^{1/2} \|_2^2 + 2 \| \bar{B}^{1/2} \left( \bar{L}^\dagger - \pi \tilde{Z} \pi \right) \bar{L}^{1/2} \|_2^2 \leq \frac{1}{8} \| \bar{B}^{1/2} \left( \bar{L} - \bar{L} \right)^\dagger \bar{B} \bar{L}^{1/2} \|_2^2 + 2 \| \bar{B}^{1/2} \left( \bar{L}^\dagger - \pi \tilde{Z} \pi \right) \bar{L}^{1/2} \|_2^2 \leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4},$$

where the second inequality is by (47); the last inequality is from (45) and (46).

\[ \square \]

**Proof of Theorem 7.1** By Fact A.15 and the definition of $\left\{ \alpha, \beta, \{\delta_i\}_{i=1}^d \right\}$-Schur complement chain, we have

$$U \left[ \tilde{S}^{(i+1)} \right] \preceq (1 + \delta_{i+1}) \left( 3 + \frac{2}{\alpha} \right) U \left[ SC \left( \tilde{S}^{(i)}, F_i \right) \right]. \quad 39$$
It follows by Fact A.8 and induction that \( U \left[ \mathbf{S}^{(i)} \right] \preceq (3 + \frac{2}{\alpha})^{i-1} \prod_{j=2}^{i} (1 + \delta_j) \text{Sc}(U[L], \cup_{j=1}^{i-1} F_j) \preceq (3 + \frac{2}{\alpha})^{i-1} \prod_{j=1}^{i} (1 + \delta_j) \text{Sc}(U[L], \cup_{j=1}^{i-1} F_j). \)

Since \( \sum_{i=1}^{d} \delta_i = O(1), d = O(\log n) \), we have \( (3 + \frac{2}{\alpha})^{d} \prod_{j=1}^{d} (1 + \delta_j) = O(\text{poly}(n)) \), i.e., \( U \left[ \mathbf{S}^{(i)} \right] \preceq O(\text{poly}(n)) \cdot \text{Sc}(U[L], \cup_{j=1}^{i-1} F_j) \) for any \( i \in [d] \). Thus, by Fact A.14 we have \( P \left( U \left[ \mathbf{S}^{(i)} \right], C_{i-1}, C_{i-1}, n \right) \preceq O(\text{poly}(n)) \cdot U[L] \). By combining with Lemma 6.6 we have

\[
\tilde{B} \preceq 2B \preceq 2 \left( U[L] + \sum_{i=2}^{d} P \left( U \left[ \mathbf{S}^{(i)} \right], C_{i-1}, C_{i-1}, n \right) \right) = O(\text{poly}(n)) \cdot U[L].
\]

By Lemma 6.8 after running \( N' \) iterations of the preconditioned Richardson iteration, \( \left\| \mathbf{x}^{(N')} - L^\dagger \mathbf{b} \right\|_{U[L]} \leq \left\| \mathbf{x}^{(N')} - L^\dagger \mathbf{b} \right\|_{\tilde{B}} \leq \left\| I - \Pi \tilde{Z} \right\|_{\tilde{B}^{-1}} \leq \left\| L^\dagger \mathbf{b} \right\|_{\tilde{B}} \leq \left( \frac{1}{2} \right)^{N'} \cdot O(\text{poly}(n)) \cdot \left\| L^\dagger \mathbf{b} \right\|_{U[L]} \). By setting \( N' = O(\log(n/\epsilon)) \), we can let \( \left\| \mathbf{x}^{(N')} - L^\dagger \mathbf{b} \right\|_{U[L]} \leq \epsilon \left\| L^\dagger \mathbf{b} \right\|_{U[L]} \).

The processing time follows by Theorem 6.4 directly. By Theorem 6.4, \( \sum_{i=1}^{d} \text{nnz} \left( \mathbf{S}^{(i)} \right) = \sum_{i=1}^{d} O\left( N_{S} \left( (1 - \beta)^{i-1} n, \frac{\delta_i}{\epsilon} \right) \right) \). As \( \delta = O(1), \beta = \frac{1}{\log(1 + \alpha)} = O(1), \) we have \( \sum_{i=1}^{d} \text{nnz} \left( \mathbf{S}^{(i)} \right) = O(N_{S}(n, 1)) \). Thus, as we set \( N = O(\log n) \) in PRECONDITION, running PRECONDITION for one time takes \( O(N_{S}(n, 1) \log n) \) time. Then, after obtaining a desirable Schur complement chain, an \( \epsilon \)-accurate vector \( \mathbf{x} \) can be computed in \( O(N_{S}(n, 1) \log n \log(n/\epsilon)) \) time.

\[
\square
\]

Using the smaller Eulerian Laplacian sparsifiers based on short cycle decompositions to sparsify the approximate Schur complements returned by Algorithm 2 we get the following solver which has quadratic processing time, but faster solve time. Its proof is deferred to Appendix C.

**Corollary 6.9.** Given a strongly connected Eulerian Laplacian \( L \in \mathbb{R}^{n \times n} \), we can process it time \( O(n^2 \log^{O(1)} n) \). Then, for each query vector \( \mathbf{b} \in \mathbb{R}^{n} \) with \( \mathbf{b} \perp \mathbf{1} \), we can compute a vector \( \mathbf{x} \in \mathbb{R}^{n} \) with \( \left\| \mathbf{x} - L^\dagger \mathbf{b} \right\|_{U[L]} \leq \epsilon \left\| L^\dagger \mathbf{b} \right\|_{U[L]} \) in time \( O(n \log^5 n \log(n/\epsilon)) \).

**Remark 6.10.** Combining Theorem 1.1 or Corollary 6.9 with Appendix D of [CKP+17] yields full solvers for strongly connected directed Laplacians.

**References**

[AJSS19] AmirMahdi Ahmadinejad, Arun Jambulapati, Amin Saberi, and Aaron Sidford. Perron-frobenius theory in nearly linear time: Positive eigenvectors, m-matrices, graph kernels, and other applications. In Timothy M. Chan, editor, *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019*, pages 1387–1404. SIAM, 2019.

[AKM+20] AmirMahdi Ahmadinejad, Jonathan A. Kelner, Jack Murtagh, John Peebles, Aaron Sidford, and Salil P. Vadhan. High-precision estimation of random walks in small space. In *61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020*,
[BSST13] Joshua Batson, Daniel A. Spielman, Nikhil Srivastava, and Shang-Hua Teng. Spectral sparsification of graphs: theory and algorithms. *CACM*, 56(8):87–94, August 2013.

[CCL+15] Dehua Cheng, Yu Cheng, Yan Liu, Richard Peng, and Shang-Hua Teng. Efficient sampling for gaussian graphical models via spectral sparsification. In Peter Grünwald, Elad Hazan, and Satyen Kale, editors, *Proceedings of The 28th Conference on Learning Theory, COLT 2015, Paris, France, July 3-6, 2015*, volume 40 of *JMLR Workshop and Conference Proceedings*, pages 364–390. JMLR.org, 2015.

[CGP+18] Timothy Chu, Yu Gao, Richard Peng, Sushant Sachdeva, Saurabh Sawlani, and Junxing Wang. Graph sparsification, spectral sketches, and faster resistance computation, via short cycle decompositions. In Mikkel Thorup, editor, *59th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2018, Paris, France, October 7-9, 2018*, pages 361–372. IEEE Computer Society, 2018.

[CKK+18] Michael B. Cohen, Jonathan Kelner, Rasmus Kyng, John Peebles, Richard Peng, Anup B Rao, and Aaron Sidford. Solving directed laplacian systems in nearly-linear time through sparse lu factorizations. In *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 898–909. IEEE, 2018.

[CKP+16] Michael B. Cohen, Jonathan A. Kelner, John Peebles, Richard Peng, Aaron Sidford, and Adrian Vladu. Faster algorithms for computing the stationary distribution, simulating random walks, and more. In Irit Dinur, editor, *IEEE 57th Annual Symposium on Foundations of Computer Science, FOCS 2016, 9-11 October 2016, Hyatt Regency, New Brunswick, New Jersey, USA*, pages 583–592. IEEE Computer Society, 2016.

[CKP+17] Michael B Cohen, Jonathan Kelner, John Peebles, Richard Peng, Anup B Rao, Aaron Sidford, and Adrian Vladu. Almost-linear-time algorithms for markov chains and new spectral primitives for directed graphs. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*, pages 410–419, 2017.

[KLP+16] Rasmus Kyng, Yin Tat Lee, Richard Peng, Sushant Sachdeva, and Daniel A Spielman. Sparsified cholesky and multigrid solvers for connection laplacians. In *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*, pages 842–850, 2016.

[KOSZ13] Jonathan A. Kelner, Lorenzo Orecchia, Aaron Sidford, and Zeyuan Allen Zhu. A simple, combinatorial algorithm for solving SDD systems in nearly-linear time. In Dan Boneh, Tim Roughgarden, and Joan Feigenbaum, editors, *Symposium on Theory of Computing Conference, STOC’13, Palo Alto, CA, USA, June 1-4, 2013*, pages 911–920. ACM, 2013.

[KS16] Rasmus Kyng and Sushant Sachdeva. Approximate gaussian elimination for laplacians - fast, sparse, and simple. In Irit Dinur, editor, *IEEE 57th Annual Symposium on Foundations of Computer Science, FOCS 2016, 9-11 October 2016, Hyatt Regency, New Brunswick, New Jersey, USA*, pages 573–582. IEEE Computer Society, 2016.
A Matrix Facts

Some elementary lemmas used frequently in this paper are listed in this section. Unless otherwise specified, we assume $F = \{1, 2, \cdots, |F|\}$ and $C = [n] \setminus F$ by default in this section.

Fact 2.2 For any matrix $A \in \mathbb{R}^{n \times n}$ and PSD matrix $U \in \mathbb{R}^{n \times n}$, the following statements are equivalent:

- $A \preceq U$.
- $2x^\top Ay \leq x^\top Ux + y^\top Uy$, $\forall x, y \in \mathbb{R}^n$.  

42
Proof. Compared with Lemma B.2 of [CKP+17], we need to show that under the condition
\[ 2x^\top Ay \leq x^\top Ux + y^\top Uy, \quad \forall x, y \in \mathbb{R}^n, \]
we have
\[ \ker(U) \subseteq \ker(A) \cap \ker(A^\top). \]
By the assumption, we have for any \( v \in \ker(U) \setminus \{0\}, \)
\[ 2x^\top Av \leq x^\top Ux, \quad 2v^\top Ay \leq y^\top Uy, \quad \forall x, y \in \mathbb{R}^n. \]
By choosing \( x = cAv, \ y = cA^\top v, \) we have
\[ 2c\|Av\|_2^2 \leq c^2\|U\|_2\|A\|_2^2\|v\|_2^2, \quad 2c\|A^\top v\|_2^2 \leq c^2\|U\|_2\|A\|_2^2\|v\|_2^2, \quad \forall c > 0. \]
By letting \( c \to 0^+, \) we have \( \|Av\|_2 = 0 \) and \( \|A^\top v\|_2 = 0, \) i.e., \( v \in \ker(A) \cap \ker(A^\top). \) Thus, \( \ker(U) \subseteq \ker(A) \cap \ker(A^\top). \)

Fact A.1. If \( A \preccurlyeq \alpha \cdot C, \ B \preccurlyeq b \cdot C, \) then \( A + B \preccurlyeq (a + b) \cdot C. \)

Fact A.2. If \( A \preccurlyeq B, \ B \preccurlyeq C, \) then \( A \preccurlyeq C. \)

Fact A.3. If \( A \preccurlyeq B, \) then \( U[A] \preccurlyeq B. \)

Fact A.4. Schur complements of Eulerian Laplacians are Eulerian Laplacians; Schur complements of strongly connected Eulerian Laplacians are strongly connected Eulerian Laplacians.

Fact A.5. For any Eulerian Laplacian \( L, \ U[L] \) is PSD.

Fact A.6. If matrix \( A \in \mathbb{R}^{n \times n} \) is \( \alpha \)-RCDD \( (\alpha \geq 0), \) then \( U[A] \) is PSD. If \( \alpha > 0, \) then \( U[A] \) is PD.

Fact A.7. Any strongly connected Laplacian \( L \in \mathbb{R}^{n \times n} \) has rank \( n - 1. \)

Fact A.8. (Lemma B.1 of [MP13]) Suppose that \( A, B \in \mathbb{R}^{n \times n} \) are PSD, \( F, C \) is a partition of \( [n], \) where \( A_{FF}, B_{FF} \) are nonsingular and \( A \preccurlyeq B. \) Then, \( \text{Sc}(A, F) \preccurlyeq \text{Sc}(B, F). \)

Fact A.9. (Lemma C.2 of [CKK18]) Let \( M \) be an \( n \)-by-\( n \) matrix and \( F, C \) a partition of \( [n] \) such that \( M_{FF} \) is nonsingular. Let \( F_1, F_2 \) be a partition of \( F \) such that \( M_{F_1F_1} \) is nonsingular. Then,
\[ \text{Sc}(\text{Sc}(M, F_1), F_2) = \text{Sc}(M, F). \]

Fact A.10. For any symmetric matrix \( U \in \mathbb{R}^{n \times n} \) and \( F, C \) a partition of \( [n], \) where \( U_{FF} \) is positive definite, for any \( x \in \mathbb{R}^{[C]}, \hat{x} \in \mathbb{R}^n, \) with \( \hat{x}_C = x, \) we have
\[ \|\hat{x}\|_U^2 = \|x\|_{\text{Sc}(U,F)}^2 + \|\hat{x}_F + U_{FF}^{-1} U_{FC} x\|_{U_{FF}}^2 \geq \|x\|_{\text{Sc}(U,F)}^2. \]
Proof. Define
\[
x_U = \left( -U_{FF}^{-1} U_{FC} x \right).
\]

Then,
\[
U x_U = \begin{pmatrix} 0_F \vspace{1em} \text{Sc}(U,F) x \end{pmatrix}.
\]

Since \((\tilde{x} - x_U)_C = 0\) and \((U x_U)_F = 0\),
\[(\tilde{x} - x_U)^\top U x_U = 0.\]

Thus,
\[
\tilde{x}^\top U \tilde{x} = x_U^\top U x_U + (\tilde{x} - x_U)^\top U (\tilde{x} - x_U) + 2(\tilde{x} - x_U)^\top U x_U
\]
\[= x_U^\top U x_U + (\tilde{x} - x_U)^\top U (\tilde{x} - x_U) = \|x\|_{\text{Sc}(U,F)}^2 + \|\tilde{x} F + U_{FF}^{-1} U_{FC} x\|_{U_{FF}}^2.\]

**Fact A.11.** For matrix \(A \in \mathbb{R}^{n \times n}\), if \(U[A]\) is PSD, then we have \(\ker(A) \subseteq \ker(U[A])\).

Proof. Since \(U[A]\) is PSD, we have
\[Ax = 0 \Rightarrow x^\top U[A] x = 0 \Rightarrow U[A]^{1/2} x = 0 \Rightarrow U[A] x = 0,
\]
i.e., \(\ker(A) \subseteq \ker(U[A])\).

**Fact A.12.** (Lemma B.9 of [CKP+17]) For any matrix \(L \in \mathbb{R}^{n \times n}\) with \(U[L] \succeq 0\) and \(\ker(L) = \ker(L^\top) = \ker(U[L])\), we have
\[U[L] \preceq LU[L]^\top L^\top.
\]

**Fact A.13.** (Lemma 4.5 of [CKK+18]) For Eulerian Laplacian \(L\) and \(D = \text{Diag}(L)\), we have
\[L^\top D^{-1} L \preceq 2U[L].
\]

**Fact A.14.** Consider symmetric matrices \(A \in \mathbb{R}^{n \times n}\), \(F, C\) a partition of \([n]\) and \(B \in \mathbb{R}^{|F| \times |F|}\), where \(A_{FF}\) is PD. Then, \(B \preceq \text{Sc}(A,F)\) is equivalent to
\[
\begin{bmatrix} 0_{|F| \times |F|} & 0_{|F| \times |C|} \vspace{1em} 0_{|C| \times |F|} & B \end{bmatrix} \preceq A,
\]
or equivalently,
\[
x^\top \begin{bmatrix} 0_{|F| \times |F|} & 0_{|F| \times |C|} \vspace{1em} 0_{|C| \times |F|} & B \end{bmatrix} x \leq x^\top A x, \ \forall x \in \mathbb{R}^n.
\]
Proof. If \( B \preceq \text{Sc}(A, F) \), then by Fact A.10, for any \( x \in \mathbb{R}^n \),
\[
x^\top \begin{bmatrix} 0_{|F|} & 0_{|C|} \\ 0_{|C|} & B \end{bmatrix} x = x^\top B x C \leq x^\top \text{Sc}(A, F) x C \leq x^\top A x.
\]
This gives
\[
\begin{bmatrix} 0_{|F|} & 0_{|C|} \\ 0_{|C|} & B \end{bmatrix} \preceq A.
\]
If \( y^\top \begin{bmatrix} 0_{|F|} & 0_{|C|} \\ 0_{|C|} & B \end{bmatrix} y \leq y^\top A y \), \( \forall y \in \mathbb{R}^n \), then for any \( x \in \mathbb{R}^{|C|} \), define \( \hat{x} = \left( -A_{1}^{-1} A_{1}^{-1} B x \right) \).
Then, \( x^\top B x = \hat{x}^\top \begin{bmatrix} 0_{|F|} & 0_{|C|} \\ 0_{|C|} & B \end{bmatrix} \hat{x} \leq \hat{x}^\top A \hat{x} = x^\top \text{Sc}(A, F) x \).
\[\square\]

The following fact is from Lemma 4.4 of [CKK+18] and Fact A.14.

**Fact A.15.** For strongly connected Eulerian Laplacian \( L \in \mathbb{R}^{n \times n} \) and \( F, C \) a partition of \([n]\) such that \( L_{FF} \) is \( \alpha \)-RCDD, we have
\[
U[\text{Sc}(L, F)] \preceq \left( 3 + \frac{2}{\alpha} \right) \text{Sc}(U[L], F)
\]

**Fact A.16.** (Lemma B.4 of [CKP+17]) For positive diagonal matrices \( D_1 \in \mathbb{R}^{m \times m}, D_2 \in \mathbb{R}^{n \times n} \) and arbitrary \( M \in \mathbb{R}^{m \times n} \), we have
\[
\|D_1 M D_2\|_2 \leq \max\{\|D_1^2 M\|_\infty, \|M D_2^2\|_1\}.
\]

**Fact A.17.** For any Eulerian Laplacian \( L \in \mathbb{R}^{n \times n} \), let \( D = \text{Diag}(L) \), then, \( \|D^{-1/2} L D^{-1/2}\|_2 \leq 2 \).

**Proof.** Since \( L \) is an Eulerian Laplacian, \( \|D^{-1}\|_\infty \leq 2 \), \( \|LD^{-1}\|_1 \leq 2 \). Then, the result follows by Fact A.16.
\[\square\]

**Fact A.18.** For any matrix \( L \in \mathbb{R}^{n \times n} \), denote \( U = U[L] \). If \( F, C \) is a partition of \([n]\) such that \( U_{FF} \) is PD, then
\[
\text{Sc}(U, F) \preceq U[\text{Sc}(L, F)]
\]

**Proof.** By Fact A.11, \( L_{FF} \) is nonsingular. For any \( x \in \mathbb{R}^{|C|} \), define \( \hat{x} = \left( -L_{1}^{-1} L_{1}^{-1} x \right) \).
Then, by Fact A.10
\[
x^\top \text{Sc}(U, F) x \leq \hat{x}^\top U \hat{x} = \hat{x}^\top L \hat{x} = x^\top \text{Sc}(L, F) x = x^\top U[\text{Sc}(L, F)] x,
\]
i.e., \( \text{Sc}(U, F) \preceq U[\text{Sc}(L, F)] \).
\[\square\]

**Fact A.19.** (Lemma C.3 of [CKK+18]) Consider matrices \( A \in \mathbb{R}^{m \times m} \), \( B \in \mathbb{R}^{m \times n} \) and \( C \in \mathbb{R}^{n \times n} \). Let \( M = ABC \). Let \( P_M, P_M^\top \) denote the orthogonal projection matrix onto the column space of \( M, M^\top \), respectively. If \( A, C \) are nonsingular, then
\[
M^\top = P_M C^{-1} B^\top A^{-1} P_M^\top.
\]
Fact A.20. Let $U \in \mathbb{R}^{n \times n}$ ($n \geq 2$) be a strongly connected symmetric Laplacian, then, for any partition $F, C$ of $[n]$, we have

$$\lambda_2(U) \leq \lambda_2(\text{Sc}(U, F)).$$

And

$$\min_{i \in [n]} U_{ii} \geq \frac{1}{2} \lambda_2(U).$$

Proof. By Lemma C.1 of [CKK+18], $\text{Sc}(U, F) = (P_S U^\dagger P_S)\dagger$, where $P_S$ is the projection matrix onto the image space of $\text{Sc}(U, F)$. Then, $\lambda_2(\text{Sc}(U, F)) = \frac{1}{\lambda_{\min}(P_S U^\dagger P_S)} \geq \frac{1}{\lambda_{\min}(U)} = \lambda_2(U)$, where the inequality uses the fact that $P_S$ is a projection matrix.

Let $\{e_i\}_{i=1}^n$ be standard basis of $\mathbb{R}^n$, then $\lambda_2(U) = \inf_{\|x\|_2 = 1} x^\top U x = \inf_{x \neq \frac{1}{\sqrt{n}}(x^\top 1)\mathbb{1}} \frac{x^\top U x}{\|x - \frac{1}{\sqrt{n}} 1\|_2} \leq \min_{i \in [n]} \left\| e_i^\top U e_i \right\|_2 \leq \min_{i \in [n]} \left\| U_{ii} \right\|_2 \leq 2 \min_{i \in [n]} U_{ii}$, where the last inequality is from $n \geq 2$.

Fact A.21. For 2 sequences of matrices $A_1, A_2, \ldots, A_N \in \mathbb{R}^{n \times n}$ and $B_1, B_2, \ldots, B_N \in \mathbb{R}^{n \times n}$. For any $1 \leq k \leq N$,

$$\|A_1 \cdots A_N - B_1 \cdots B_N\|_\infty \leq \left( n \sum_{i=1}^k \|A_i - B_i\|_\infty + \sum_{i=k+1}^N \|A_i - B_i\|_1 \right) \prod_{i=1}^k \max\{1, \|A_i\|_\infty + \|A_i - B_i\|_\infty\} \cdot \prod_{i=k+1}^N \max\{1, \|A_i\|_1 + \|A_i - B_i\|_1\}.$$

Proof. It follows directly from the expansion

$$A_1 \cdots A_N - B_1 \cdots B_N = \sum_{i=1}^N \prod_{j=1}^{i-1} A_j (A_i - B_i) \prod_{j=i+1}^N B_j$$

and the fact that $\|C\|_1 \leq n\|C\|_\infty$ for any matrix $C \in \mathbb{R}^{n \times n}$.

Fact A.22. (Lemma 6.3 of [CKK+18], paraphrased) Consider matrix $A \in \mathbb{R}^{n \times n}$, where $\ker(A) = \ker(A^\top)$. Let $F_1, F_2, \ldots, F_d$ be a partition of $[n]$. Denote $E_1 = \emptyset$, $E_i = \cup_{j=1}^{i-1} F_i$ ($2 \leq i \leq d$) and $C_0 = [n]$, $C_i = [n] \setminus (E_i \cup F_i)$ ($1 \leq i \leq d$). Suppose that $U[\text{Sc}(A, E_i)]$ is PSD for any $1 \leq i \leq d - 1$ and $U[\text{Sc}(A, E_i)]$ is PSD for any $1 \leq i \leq d$. Let $\{\theta_i\}_{i=1}^d$ be nonnegative numbers such that $\sum_{i=1}^d \theta_i = 1$. Define the matrix $B = \sum_{i=1}^d \theta_i \text{Sc}(A, E_i)$, $C_{i-1}, C_{i-1}, n)$, then,

$$\left(A^\top \right)^\top B A^\top \preceq B^\top.$$

Fact A.23. If $A \preceq B$, then $\text{Rep}(k, C, A) \preceq \text{Rep}(k, C, B)$, $\text{Rep}^+(k, C, A, N) \preceq \text{Rep}^+(k, C, B, N)$.

Fact A.24. $\text{Rep}(b, C, \text{Rep}(a, C, A)) = \text{Rep}(a \cdot b, C, A)$. 

46
B Exact Partial Block Elimination

Let $P$ be a permutation matrix such that $P L P^\top = \begin{bmatrix} L_{FF} & L_{FC} \\ L_{CF} & L_{CC} \end{bmatrix}$ in this Section. We initiate $L^{(0)} = L$, $D = \text{Diag}(L^{(0)})$ and $A^{(0)} = D - L^{(0)}$, and then update for $k = 1, 2, \cdots$.

\[ L^{(k)} = P^\top \begin{bmatrix} D_{FF} - A_{FF}^{(k-1)} & -A_{FC}^{(k-1)} \\ -A_{CF}^{(k-1)} & 2L_{CC}^{(k-1)} \end{bmatrix} P - A_{:.F}^{(k-1)} D_{FF}^{-1} A_{F:.}^{(k-1)} \]
\[ A^{(k)} = P^\top \begin{bmatrix} D_{FF} - A_{FF}^{(k-1)} & \text{Diag}(L^{(k)}) \end{bmatrix} P - L^{(k)}. \]

The following lemmas characterize the performance of the ideal but inefficient scheme \[48\].

**Lemma B.1.** Let $L^{(k)}$ be the output of running $k$ steps of IDEALSCHUR($L, F$). At any step we have:

\[ \text{Sc}(L^{(k)}, F) = 2^k \text{Sc}(L, F). \] (49)

**Proof.** By \[1\],

\[
2\text{Sc}(L^{(k-1)}, F) = 2L_{CC}^{(k-1)} - 2A_{CF}^{(k-1)} \left( L_{FF}^{(k-1)} \right)^{-1} A_{FC}^{(k-1)}
= 2L_{CC}^{(k-1)} - 2A_{CF}^{(k-1)} \left( D_{FF} - A_{FF}^{(k-1)} \right)^{-1} A_{FC}^{(k-1)}
= 2L_{CC}^{(k-1)} - A_{CF}^{(k-1)} D_{FF}^{-1} A_{FC}^{(k-1)}
- A_{CF}^{(k-1)} \left( I + D_{FF}^{-1} A_{FF}^{(k-1)} \right) \left( D_{FF} - A_{FF}^{(k-1)} D_{FF}^{-1} A_{FF}^{(k-1)} \right)^{-1} \left( I + A_{FF}^{(k-1)} D_{FF}^{-1} A_{FF}^{(k-1)} \right) A_{FC}^{(k-1)}
= L_{CC}^{(k)} - A_{CF}^{(k)} L_{FF}^{(k)} A_{FC}^{(k)} = \text{Sc}(L^{(k)}, F).
\]

Then, the relation \[49\] follows by induction. 

**Lemma B.2.** For any $k \geq 0$, $L^{(k)}$ is an Eulerian Laplacian.

**Proof.** We prove it by induction. Firstly, $L^{(0)} = L$ is an Eulerian Laplacian.

Assuming that $L^{(k-1)}$ is an Eulerian Laplacian, we prove that $L^{(k)}$ is also an Eulerian Laplacian. By induction hypothesis, $L^{(k-1)} 1 = 0$, i.e.,

\[ A_{:.F}^{(k-1)} 1 = A_{FF}^{(k-1)} 1 + A_{FC}^{(k-1)} 1 = D_{FF} 1 \]
\[ A_{CF}^{(k-1)} 1 = L_{CC}^{(k-1)} 1. \]

47
Thus,

\[
P L^{(k)} P^T 1 = \begin{bmatrix} D_{FF} & -A_{FC}^{(k-1)} \\ -A_{CF}^{(k-1)} & 2L_{CC}^{(k-1)} \end{bmatrix} 1 - A_{CF}^{(k-1)} D_{FF}^{-1} A_{CF}^{(k-1)} 1
\]

= \begin{bmatrix} D_{FF} 1 - A_{FC}^{(k-1)} 1 \\ -A_{CF}^{(k-1)} 1 + 2L_{CC}^{(k-1)} 1 \end{bmatrix} - \begin{bmatrix} A_{CF}^{(k-1)} 1 \\ A_{CF}^{(k-1)} 1 \end{bmatrix}

= 0.

Since \( P \) is a permutation matrix, we have \( L^{(k)} 1 = 0 \). Similarly, \( 1^T L^{(k)} = 0^T \). Therefore, \( L^{(k)} \) is an Eulerian Laplacian. The result then follows by induction.

\[\square\]

**Lemma B.3.** The result of the update formula \( (48) \) satisfies \( \lim_{k \to +\infty} \frac{1}{2^k} L_{CC}^{(k)} = \text{Sc}(L, F) \).

**Proof.** Since \( L_{FF} \) is \( \alpha \)-RCDD, \( \| D_{FF}^{-1} A_{FF}^{(0)} \|_\infty \leq \frac{1}{1 + \alpha} \). By the equality \( L_{FF}^{(k)} = D_{FF} - A_{FF}^{(k-1)} D_{FF}^{-1} A_{FF}^{(k-1)} \) from \( (48a) \), we have \( D_{FF}^{-1} A^{(k)} = D_{FF}^{-1} A_{FF}^{(k-1)} D_{FF}^{-1} A_{FF}^{(k-1)} \). Thus, by induction,

\[
\| D_{FF}^{-1} A^{(k)} \|_\infty \leq \left( \frac{1}{1 + \alpha} \right)^{2^k}.
\]

By Lemma \[\square\] \( \| A_{CF}^{(k)} \|_\infty \leq n \| A_{CF}^{(k)} \|_\infty \leq n \| D_{FF} \|_2 \| D_{FF}^{-1} A_{CF}^{(k)} \|_\infty \leq 1 \).

Combining the above arguments,

\[
\begin{align*}
\lim_{k \to +\infty} \frac{1}{2^k} \| A_{CF}^{(k)} \left( L_{FF}^{(k)} \right)^{-1} A_{CF}^{(k)} \|_\infty &= \lim_{k \to +\infty} \frac{1}{2^k} \left\| A_{CF}^{(k)} \left( I - D_{FF}^{-1} A_{FF}^{(k)} \right)^{-1} D_{FF}^{-1} A_{CF}^{(k)} \right\|_\infty \\
&\leq \lim\sup_{k \to +\infty} \frac{1}{2^k} \left\| A_{CF}^{(k)} \|_\infty \left\| \left( I - D_{FF}^{-1} A_{FF}^{(k)} \right)^{-1} \right\|_\infty \| D_{FF}^{-1} A_{CF}^{(k)} \|_\infty \\
&\leq n \| D_{FF} \|_2 \cdot \lim\sup_{k \to +\infty} \frac{1}{2^k} \cdot \lim\sup_{k \to +\infty} \left\| \left( I - D_{FF}^{-1} A_{FF}^{(k)} \right)^{-1} \right\|_\infty = 0,
\end{align*}
\]

i.e.,

\[
\lim_{k \to +\infty} \frac{1}{2^k} A_{CF}^{(k)} \left( L_{FF}^{(k)} \right)^{-1} A_{CF}^{(k)} = 0.
\]

Then, using Lemma \[\square\] and the above equation, we have

\[
\lim_{k \to +\infty} \frac{1}{2^k} L_{CC}^{(k)} = \lim_{k \to +\infty} \frac{1}{2^k} \text{Sc}(L^{(k)}, F) + \lim_{k \to +\infty} \frac{1}{2^k} A_{CF}^{(k)} \left( L_{FF}^{(k)} \right)^{-1} A_{CF}^{(k)} = \text{Sc}(L, F).
\]

\[\square\]
C Sparsifying Directed Laplacians

First, we check that the Eulerian Laplacian sparsifier in Section 3 of [CKP+17] meets the requirements of Theorem 2.4. This procedure can be briefly summarized as:

1. decompose $L = \sum_{i=1}^{K} L^{(i)}$ such that each $U[L^{(i)}]$ is an expander;

2. sample the entries in the adjacency matrix of each $L^{(i)}$ and use a patch matrix to keep the row sums and the column sums invariant.

This procedure was analyzed in [CKP+17] by (1) using matrix concentration inequalities to bound the errors in each adjacency matrix with respect to the in-degree and out-degree diagonal matrix; (2) using the property of the expander to bound the errors with respect to $U[L^{(i)}]$ and in turn $U[L]$.

Next, we give a precise bound of the running time of directed Laplacian sparsification by combining the expander decomposition in [SW19] with the degree-fixing on expanders routine from Section 3 of [CKP+17].

By setting $\phi = O(1/\log^3 n)$ in Theorem 4.1 of [SW19] and deleting the edges recursively, we can have a $(s, \phi, 1)$-decomposition (Definition 3.14 of [CKP+17]) of the original directed graph $\mathcal{G}[L]$, denoted by $\{L^{(i)}\}_{i=1}^{K}$, and each $U[L^{(i)}]$ is a $\phi$-expander. Here $s = n \log n$ is the sum of the sizes of the subgraphs $L^{(i)}$. The running time of this step is $O(m \log^8 n)$.

By Cheeger’s inequality, the spectral gap $\delta$ of each $\phi$-expander is $O(\phi^2)$. Then, by Lemma 3.13 of [CKP+17], we can have a $\tilde{L}^{(i)}$ by sampling edges in $L^{(i)}$, such that $\sum_{i=1}^{K} \text{nnz}(\tilde{L}^{(i)}) \leq s \log n / \delta^2 = O(n \log^4 n)$. And $\tilde{L}^{(i)}$ is an $O(1)$-asymmetric approximation of $L^{(i)}$. By summing up $\tilde{L} = \sum_{i=1}^{K} L^{(i)}$, we have a $O(1)$-asymmetric approximation of $L$, with $\text{nnz}(\tilde{L}) = O(n \log^4 n)$.

Putting these costs into Theorem 1.1 specifically setting $T_S(m, n, 1) = O(m \log^8 n)$, $N_S(n, 1) = O(n \log^4 n)$, gives that the overall (construction + solve) running times of our algorithm is $O(m \log^8 n + n \log^4 n \log^{15} n \log^{1/2} n) + \tilde{O}(n \log^{23} n)$.

With a slower processing time, we can use short cycle based Eulerian sparsifiers to get a smaller Schur complement chain, and solve for each query vector faster.

Given an Eulerian Laplacian $L$, the current best bound for the nonzero entries of its sparsifier is $O(n \log^4 n e^{-2})$ edges (Lemma C.1).

**Lemma C.1. (Existence of Eulerian Laplacian sparsifier) [CGP+18]** For any Eulerian Laplacian $L \in \mathbb{R}^{n \times n}$ with $\text{nnz}(L) = m$ and error parameter $\epsilon \in (0, 1)$, there is a Eulerian Laplacian $\bar{L}$ such that $\text{nnz}(\bar{L}) \leq O(n \log^4 n e^{-2})$ and $\bar{L} - L \asym \leq \epsilon \cdot U[L]$. Such an $\bar{L}$ can be constructed in time $O(mn \log O(1) n)$.

Invoking the Eulerian sparsifier routine of Lemma C.1 above within each iteration of Algorithm 4 with error set to $\epsilon = O(\frac{1}{n^2})$ at the $i$-th iteration, we obtain, after $O(n^2 \log O(1) n)$ preprocessing time, an $\alpha = O(1)$, $d = O(\log n)$ and $\sum_{i=1}^{n} \text{nnz}(S^{(i)}) = O(n \log^4 n)$. Then, Corollary 6.9 follows similarly to the proof of Theorem 1.1.
D Supporting Lemmas and Omitted Proofs

Lemma D.1. There is a routine SparseEulerianFC which takes in an Eulerian Laplacian $L \in \mathbb{R}^{n \times n}$, error parameter $\epsilon \in (0,1)$ and a subset $F \subseteq [n]$, where $nnz(L) = m$. And then, SparseEulerianFC runs in $O(T_S(m, n, \delta))$ time to return an Eulerian Laplacian $\tilde{L} \in \mathbb{R}^{n \times n}$ such that $\text{Diag}(\tilde{L}) = \text{Diag}(L)$, $\tilde{L}_{FF} 1 = L_{FF} 1$, $\tilde{L}_{FF}^\top 1 = L_{FF}^\top 1$, $nnz(\tilde{L}) = O(N_S(n, \delta))$ and $\tilde{L} - L \preceq \epsilon \cdot U[L]$ with high probability.

Analogously, under the same conditions as in Lemma 2.6, there is a routine SparseProductFC which takes in vectors $x, y$, error parameter $\epsilon$, probability $p$ and a subset $F \subseteq [n]$, runs in $O(m \epsilon^{-2} \log \frac{m}{p})$ to return with high probability a nonnegative matrix $B$ which possesses all the properties of $A$ in Lemma 2.6. In addition, $B_{FF}1 = (y_F^\top 1)x_F$ and $1^\top B_{FF} = (x_F^\top 1)y_F$.

Proof. We apply OraSparseLaplacian to the directed Laplacians

$$
\begin{bmatrix}
L_{FF} - \text{Diag}(1^\top L_{FF}) & 0_{|F| \times |C|} & 0_{|F| \times |C|} \\
0_{|C| \times |F|} & 0_{|C| \times |C|} & 0_{|F| \times |C|} \\
-\text{Diag}(1^\top L_{CF}) & 0_{|F| \times |C|} & L_{CC} - \text{Diag}(1^\top L_{CC})
\end{bmatrix}
$$

respectively and summing up the resulting sparsified matrices, we have the OraSparseLaplacian in Lemma D.1. Analogously, by setting $(x, y)$ as

$$
\begin{bmatrix}
(0_{|F|}, y_F) \\
(x_F, 0_{|C|})
\end{bmatrix}
$$

in SparseProduct respectively, we get SparseProductFC in Lemma D.1.

The performance of SparseEulerianFC, SparseProductFC follows directly by the fact that OraSparseLaplacian preserves the diagonal entries and the row sums, SparseProductFC preserves the row and column sums.

Remark D.2. By carefully designing a sampling rule, SparseEulerianFC and SparseProductFC can be replaced by a single sparsification procedure. Here, we use OraSparseLaplacian and SparseProduct to construct SparseEulerianFC and SparseProductFC merely for simplicity. When invoking SparseProduct, SparseProductFC in this paper, we set $p = O\left(\frac{1}{\text{poly}(n)}\right)$ and omit the probability parameter $p$ for notational simplicity. For $x = 0$ or $y = 0$, both SparseProduct and SparseProductFC return $0 \in \mathbb{R}^{n \times n}$ naturally.

We have the following properties of Algorithm 2

Lemma D.3. With high probability, the following statements hold:

(i) $\{\tilde{L}^{(k)}\}_{k=0}^K$ are Eulerian Laplacians;

(ii) $\{\tilde{A}^{(k)}\}_{k=0}^K$ are nonnegative matrices satisfying

$$
\left\|D_{FF}^{-1/2} \tilde{A}^{(k)}_{FF}\right\|_\infty \leq \left(\frac{1}{1+\alpha}\right)^{2k}, \forall 0 \leq k \leq K;
$$

(52)
(iii) $S, \tilde{S}$ are Eulerian Laplacians;

(iv) The matrix $\tilde{R}$ satisfies $\tilde{R}1 = \tilde{R}^\top 1 = 0$ and

$$
\|\tilde{R}\|_2 \leq \frac{n^2 \|D_{FF}\|_2}{2^{K-1} \alpha} \left(\frac{1}{1+\alpha}\right)^{2K}.
$$

(53)

Proof. We prove (i) by induction. Firstly, $\tilde{L}^{(0)} = L$ is an Eulerian Laplacian. Suppose that $\tilde{L}^{(k-1)}$ is an Eulerian Laplacian, we will prove the Eulerianness for $k$.

Since SparseProductFC preserves the row sum,

$$
\tilde{Y}^{(k)}1 = \sum_{i \in F} 1_{D_{ii}} \tilde{A}_{ii:i}^{(k-1)} \tilde{A}_{i:i}^{(k-1)} 1 = \tilde{A}_{:,F}^{(k-1)} D_{FF}^{-1} \tilde{A}_{:,F}^{(k-1)} 1.
$$

In this proof, we denote

$$
W^{(k)} \overset{\text{def}}{=} P^\top \begin{bmatrix} D_{FF} & -\tilde{A}_{CF}^{(k-1)} \\
-\tilde{A}_{CF}^{(k-1)} & 2L_{CC}^{(k-1)}
\end{bmatrix} P - \tilde{A}_{:,F}^{(k-1)} D_{FF}^{-1} \tilde{A}_{:,F}^{(k-1)} 1.
$$

(54)

where $P$ is the permutation matrix defined in Section [B]. By similar arguments with Lemma [B.2] $W^{(k)}$ is an Eulerian Laplacian.

Then, we have

$$
\tilde{L}^{(k,0)} = W^{(k)} 1 + \left(\tilde{A}_{:,F}^{(k-1)} D_{FF}^{-1} \tilde{A}_{:,F}^{(k-1)} - \tilde{Y}^{(k)}\right) 1 = 0.
$$

By combining with the fact that $\tilde{Y}^{(k)}$ is a nonnegative matrix (from Lemma [2.6]), $\tilde{L}^{(k,0)}$ is an Eulerian Laplacian. Then, (i) follows by Lemma [D.1] and induction.

The nonnegativity of $\tilde{A}^{(k)}$ follows directly by Lemma [D.1] Since SparseProductFC preserves the row sum and SparseEulerianFC preserves the diagonal and row sum on the submatrix $\tilde{L}^{(k,0)}_{FF}$, we have

$$
\|D_{FF}^{-1} \tilde{A}_{FF}^{(k)}\|_{\infty} = D_{FF}^{-1} \tilde{A}_{FF}^{(k)} 1 = D_{FF}^{-1} \sum_{i \in F} 1_{D_{ii}} \tilde{Y}^{(k,i)} 1 = D_{FF}^{-1} \sum_{i \in F} 1_{D_{ii}} \tilde{A}_{i:F}^{(k-1)} \tilde{A}_{i,F}^{(k-1)} 1
$$

$$= D_{FF}^{-1} \tilde{A}_{FF}^{(k-1)} D_{FF}^{-1} \tilde{A}_{FF}^{(k-1)} 1 = \left\|D_{FF}^{-1} \tilde{A}_{FF}^{(k-1)} D_{FF}^{-1} \tilde{A}_{FF}^{(k-1)}\right\|_{\infty} \leq \left\|D_{FF}^{-1} \tilde{A}_{FF}^{(k-1)}\right\|_{\infty}^2.
$$

Then, (ii) can be shown by induction.

For (iii), it can be shown directly by the nonnegativity of $\tilde{X}$ and [i] that all off-diagonal entries of $\tilde{S}^{(0)}$ is non-positive. By Fact [A.4] $\text{Sc}\left(\tilde{L}^{(K)}, F\right)$ is an Eulerian Laplacian. As SparseProduct
preserves the row sum,

\[
2^K \tilde{S}^{(0)} 1 = \text{Sc}_{\mathbf{L}(K), F} 1 + \tilde{A}^{(K)}_{CF} \left( D_{FF} - \tilde{A}^{(K)}_{FF} \right)^{-1} \tilde{A}^{(K)}_{FC} 1 - \hat{X} 1 \\
= \tilde{A}^{(K)}_{CF} \left( D_{FF} - \tilde{A}^{(K)}_{FF} \right)^{-1} \tilde{A}^{(K)}_{FC} 1 - \sum_{i \in F} \frac{1}{D_{ii}} \tilde{A}^{(K)} \tilde{A}^{(K)}_{i,C} 1 \\
= \tilde{A}^{(K)}_{CF} D_{FF}^{-1} \tilde{A}^{(K)}_{FF} \left( \left( D_{FF} - \tilde{A}^{(K)}_{FF} \right)^{-1} - D_{FF}^{-1} \right) \tilde{A}^{(K)}_{FC} 1 \\
= -\tilde{A}^{(K)}_{CF} D_{FF}^{-1} \tilde{A}^{(K)}_{FF} \sum_{i=0}^{\infty} \left( D_{FF}^{-1} \tilde{A}^{(K)}_{FF} \right)^i D_{FF}^{-1} \tilde{A}^{(K)}_{FC} 1.
\]

(55)

Then, \( \tilde{S}^{(0)} 1 \) is a nonnegative vector. Analogously, \( 1^T \tilde{S}^{(0)} \) is also nonnegative. So, \( \tilde{S}^{(0)} \) is RCDD. Since \( \tilde{S}^{(0)} 1, 1^T \tilde{S}^{(0)} \) are nonnegative, from the way we compute the patching matrix \( \mathbf{R} \), off-diagonal entries of \( \mathbf{R} \) are non-positive. Thus, all off-diagonal entries of \( \tilde{S} \) are non-positive. It follows by the definition of \( \mathbf{R}_{1,1} \) and direct calculations that \( \tilde{S} 1 = \tilde{S}^T 1 = 0 \). Then, we have shown \( \tilde{S} \) is an Eulerian Laplacian. Thus, \( \tilde{S} \) is also an Eulerian Laplacian by the definition of the oracle OraSparseLaplacian.

By (55) and (13), \( \tilde{S}^{(0)} 1 = \left( \tilde{R} - \mathbf{R} \right) 1 \). Then, as we have just shown \( \tilde{S} \) is an Eulerian Laplacian, \( \tilde{R} 1 = \left( \tilde{S}^{(0)} + \mathbf{R} \right) 1 = \tilde{S} 1 = 0 \). Analogously, \( \tilde{R}^T 1 = 0 \). As we have shown \( \mathbf{R} \) is non-positive, we have \( \| \tilde{R} \|_\infty \leq 1^T \mathbf{R} 1 = 1^T \tilde{S}^{(0)} 1 \). As \( \tilde{R} - \mathbf{R} = \tilde{A}^{(K)}_{CF} D^{-1}_{FF} \tilde{A}^{(K)}_{FF} \left( \left( D_{FF} - \tilde{A}^{(K)}_{FF} \right)^{-1} - D_{FF}^{-1} \right) \tilde{A}^{(K)}_{FC} \) is nonnegative, we have \( \| \tilde{R} - \mathbf{R} \|_\infty \leq 1^T \left( \tilde{R} - \mathbf{R} \right) 1 \leq 1^T \tilde{S}^{(0)} 1 \). Thus, \( \| \tilde{R} \|_\infty \leq \| \mathbf{R} \|_\infty + \| \tilde{R} - \mathbf{R} \|_\infty \leq 2 \cdot 1^T \tilde{S}^{(0)} 1 \). As \( \text{SparseProductFC} \) preserves the row sum, by Lemma B.2, \( \| \tilde{A}^{(K)}_{CF} \|_\infty \leq n \| \tilde{A}^{(K)}_{CF} \|_1 \leq n \| D_{FF}^{-1} \tilde{A}^{(K)}_{FC} \|_\infty \). Since \( \mathbf{L}(K) \) is an Eulerian Laplacian, \( \| D_{FF}^{-1} \tilde{A}^{(K)}_{FC} \|_\infty \leq 1 \). Then, by (55),

\[
1^T \tilde{S}^{(0)} 1 \leq \frac{n \| \tilde{S}^{(0)} 1 \|_\infty}{2^K} \leq \frac{n^2 \| D_{FF} \|_2}{2^K} \left( \frac{1}{1 + \alpha} \right)^{2^K} \sum_{i=0}^{\infty} \left( \frac{1}{1 + \alpha} \right)^i \leq \frac{n^2 \| D_{FF} \|_2}{\alpha} \left( \frac{1}{1 + \alpha} \right)^{2^K}.
\]

So, \( \| \tilde{R} \|_\infty \leq \frac{n^2 \| D_{FF} \|_2}{2^{K+1}(1 + \alpha)^{2^K}}. \) Analogously, \( \| \tilde{R} \|_1 \leq \frac{n^2 \| D_{FF} \|_2}{2^{K+1}(1 + \alpha)^{2^K}} \). Then, (55) follows by Fact A.16.

Proof of Lemma 5.3. For any nonnegative vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^n \), we define \( \mathcal{Y}^G(\mathbf{a}, \mathbf{b}) \) as the undirectification of a biclique as follows:

\[
\mathcal{Y}^G(\mathbf{a}, \mathbf{b}) = \mathcal{U}^G \left[ \left( \mathbf{1}^T \mathbf{a} \right) \text{Diag}(\mathbf{b}) - \mathbf{a} \mathbf{b}^T \right] = \frac{1}{2} \left( \mathbf{b} \mathbf{1} \right) \text{Diag}(\mathbf{a}) + \left( \mathbf{a} \mathbf{1} \right) \text{Diag}(\mathbf{b}) - \mathbf{a} \mathbf{b}^T - \mathbf{b} \mathbf{a}^T.
\]

Then, by Lemma 2.6, Lemma D.1 and Lemma 2.2 we have

\[
2 \mathbf{x}^T \mathbf{E}^{(k,i)} \mathbf{y} \leq \epsilon \left( \mathbf{x}^T \mathcal{Y}^G \left( \tilde{A}^{(k-1)}_{a,i}, \tilde{A}^{(k-1)}_{a,i} \right)^T \mathbf{x} + \mathbf{y}^T \mathcal{Y}^G \left( \tilde{A}^{(k-1)}_{a,i}, \tilde{A}^{(k-1)}_{a,i} \right)^T \mathbf{y} \right).
\]

52
Then, summing over $i \in F$ yields that
\[ 2x^\top \hat{E}^{(k)} y \leq \epsilon \left( x^\top U^{(k)} x + y^\top U^{(k)} y \right), \]  
(56)
where $U^{(k)} = \sum_{i \in F} \frac{1}{n} \nu^G \left( \overline{A}^{(k-1)}_{i:}, \overline{A}^{(k-1)}_{i:} \right)^\top$.

Since $U^{(k)}$ is a weighted summation of symmetric Laplacians, $U^{(k)}$ is also a symmetric Laplacian.

We also define $W^{(k)}$ as in (54) in this proof. And we have shown $W^{(k)}$ is an Eulerian Laplacian. By setting $L = \overline{L}^{(k-1)}$ in Lemma 4.1 and Lemma 4.2, we have
\[ U \left[ W^{(k)} \right] \preceq 2 \left( 3 + \frac{2}{\alpha} \right) U \left[ \overline{L}^{(k-1)} \right]. \]  
(57)

By the definition of $W^{(k)}$, we have
\[ U \left[ W^{(k)} \right] = Z^{(k)} - \frac{1}{2} \left( \overline{A}^{(k-1)}_{i:F} D_{FF}^{-1} \overline{A}^{(k-1)}_{F:i} + \left( \overline{A}^{(k-1)}_{F:i} \right)^\top D_{FF}^{-1} \left( \overline{A}^{(k-1)}_{i:F} \right)^\top \right), \]
where $Z^{(k)}$ is a matrix whose off-diagonal entries are all non-positive.

And by the definition of $U^{(k)}$, we have
\[ U^{(k)} = D^{(k)} - \frac{1}{2} \left( \overline{A}^{(k-1)}_{i:F} D_{FF}^{-1} \overline{A}^{(k-1)}_{F:i} + \left( \overline{A}^{(k-1)}_{F:i} \right)^\top D_{FF}^{-1} \left( \overline{A}^{(k-1)}_{i:F} \right)^\top \right), \]
where $D^{(k)}$ is a diagonal matrix. Thus, the off-diagonal entries of $G^{(k)} \defeq U \left[ W^{(k)} \right] - U^{(k)} = Z^{(k)} - D^{(k)}$ is all non-positive. And since $U \left[ W^{(k)} \right]$ and $U^{(k)}$ are both symmetric Laplacians, we have $G^{(k)} 1 = \left( G^{(k)} \right)^\top 1 = 0$. So, $G^{(k)}$ is a symmetric Laplacian. Then, $G^{(k)} \succ 0$. Thus, we have
\[ U \left[ W^{(k)} \right] = U^{(k)} + G^{(k)} \succ U^{(k)}. \]

By combining with (56), we have
\[ 2x^\top \hat{E}^{(k)} y \leq \epsilon \left( x^\top U \left[ W^{(k)} \right] x + y^\top U \left[ W^{(k)} \right] y \right). \]  
(58)

By Fact 2.2 and Fact A.3, $U \left[ \hat{E}^{(k)} \right] \preceq \epsilon U \left[ W^{(k)} \right]$.

It follows by the definitions of $W^{(k)}$ and $\hat{E}^{(k)}$ that $\hat{L}^{(k,0)} = W^{(k)} + \hat{E}^{(k)}$. Thus,
\[ U \left[ \hat{L}^{(k,0)} \right] = U \left[ W^{(k)} \right] + U \left[ \hat{E}^{(k)} \right] \preceq (1 + \epsilon) U \left[ W^{(k)} \right]. \]

By Lemma D.1,
\[ 2x^\top E^{(k,0)} y \leq \epsilon \left( x^\top U \left[ \hat{L}^{(k,0)} \right] x + y^\top U \left[ \hat{L}^{(k,0)} \right] y \right) \leq \epsilon (1 + \epsilon) \left( x^\top U \left[ W^{(k)} \right] x + y^\top U \left[ W^{(k)} \right] y \right). \]  
(59)
By \([57], [58], [59]\), Fact 2.2 and the relation \(E^{(k)} = E^{(k)} + E^{(k,0)}\), we have
\[
E^{(k)} \overset{\text{asym}}{\preceq} (\epsilon + (1 + \epsilon)\epsilon) U \left[ \mathcal{W}^{(k)} \right] \preceq 2 \left( 3 + \frac{2}{\alpha} \right) (2\epsilon + \epsilon^2) U \left[ \mathcal{L}^{(k-1)} \right].
\]

By the definition of \(\epsilon_0\), (14) follows. The inequality (15) follows analogously.

\(\square\)

**Proof of Lemma 4.9.** Denote
\[
\hat{x}^{(i)} = (x^T_C x^T_F \cdots x^T_F)_{\text{2i repetitions of } x^T_F}.
\]

By Lemma 4.1 \((\hat{x}^{(i)})^T U \left[ \mathcal{M}^{(0,i)} \right] \hat{x}^{(i)} = 2^i x^T U [L] x\). Thus,
\[
(\hat{x}^{(k)})^T \text{Rep} \left( 2^{k-i}, F, C, U \left[ \mathcal{M}^{(0,i)} \right] \right) \hat{x}^{(k)} = 2^{k-i} \cdot 2^i x^T U [L] x = 2^k x^T U [L] x.
\]

Since \(Q^{(0)} = U [L]\) and the sum of coefficients of the terms on the RHS of (18) is 1, it follows by induction that
\[
(\hat{x}^{(k)})^T Q^{(k)} \hat{x}^{(k)} = 2^k x^T U [L] x.
\] (60)

By Fact 4.6 \(Q^{(k)}_{-[n],-[n]}\) is PD. Then, by Fact A.10 and (60), we have
\[
x^T \text{Sc} \left( Q^{(k)}, -[n] \right) x \leq (\hat{x}^{(k)})^T Q^{(k)} \hat{x}^{(k)} = 2^k x^T U [L] x,
\]
i.e., \(\text{Sc} \left( Q^{(k)}, -[n] \right) \preceq 2^k U [L]\). Then, using Fact A.8 and Fact A.18, we have \(\text{Sc} \left( Q^{(k)}, -C \right) \preceq 2^k \text{Sc}(U [L], F) \preceq 2^k U [\text{Sc}(L, F)]\).

\(\square\)

**Proof of Theorem 6.4.** By Lemma 3.3 we have
\[
S^{(i+1)} - \text{Sc} \left( \tilde{S}^{(i)}, F_i \right) \overset{\text{asym}}{\preceq} \delta_{i+1}' U \left[ \text{Sc} \left( \tilde{S}^{(i)}, F_i \right) \right] = \frac{\delta}{3 i^2} U \left[ \text{Sc} \left( \tilde{S}^{(i)}, F_i \right) \right].
\]

Then, \((1 - \delta_{i+1}' U \left[ \text{Sc} \left( \tilde{S}^{(i)}, F_i \right) \right] \preceq U \left[ S^{(i+1)} \right] \preceq (1 + \delta_{i+1}') U \left[ \text{Sc} \left( \tilde{S}^{(i)}, F_i \right) \right]\). Thus, we have
\[
U \left[ \tilde{S}^{(i+1)} \right] = \frac{1}{1 - \delta_{i+1}'} U \left[ S^{(i+1)} \right] \preceq U \left[ \text{Sc} \left( \tilde{S}^{(i)}, F_i \right) \right]
\]
and
\[
\tilde{S}^{(i+1)} - \text{Sc} \left( \tilde{S}^{(i)}, F_i \right) = S^{(i+1)} - \text{Sc} \left( \tilde{S}^{(i)}, F_i \right) + \frac{\delta_{i+1}'}{1 - \delta_{i+1}'} U \left[ S^{(i+1)} \right]
\]
\[
\overset{\text{asym}}{\approx} \left( \delta_{i+1}' + \frac{\delta_{i+1}'}{1 - \delta_{i+1}'} \cdot (1 + \delta_{i+1}') \right) U \left[ \text{Sc} \left( \tilde{S}^{(i)}, F_i \right) \right] \overset{\text{asym}}{\approx} \frac{\delta}{i^2} U \left[ \text{Sc} \left( \tilde{S}^{(i)}, F_i \right) \right].
\]

54
Analogously, \( U[L] \lessapprox \begin{bmatrix} S^{(1)} \end{bmatrix} - \tilde{L} \lessapprox \delta \cdot U[L]. \)

By Lemma 6.3, \( \beta = \frac{1}{16(1 + \alpha)}. \) Thus, the loop will terminate in \( d = O(\log n) \) iterations. Since \( T_S(m, n, \delta) \) and \( N_S(n, \delta) \) depends on \( m, n \) nearly linearly and \( \delta^{-1} \) polynomially, the result follows by combining Theorem 3.3 and Lemma 6.3 with the fact \( \sum_{i=1}^{+\infty} (1 - \beta)^{-1} \) poly(i^2) = \( O(1) \).

Proof of Lemma 6.6. By (42),

\[
\mathcal{P}\left( \text{SC}\left( L, F_1 \right), C_1, C_1, n \right) = \mathcal{P}\left( \text{SC}\left( \tilde{S}^{(1)}, F_1 \right), C_1, C_1, n \right) + \mathcal{P}\left( \tilde{S}^{(1)} - \text{SC}\left( \tilde{S}^{(1)}, F_1 \right), C_1, C_1, n \right)
\]

\[
+ \sum_{i=2}^{d-1} \mathcal{P}\left( \tilde{S}^{(i+1)} - \text{SC}\left( \tilde{S}^{(i)}, F_1 \right), C_1, C_1, n \right)
\]

\[
\approx \tilde{S}^{(1)} + \sum_{i=2}^{d-1} \mathcal{P}\left( \tilde{S}^{(i+1)} - \text{SC}\left( \tilde{S}^{(i)}, F_i \right), C_i, C_i, n \right).
\]

Repeating this process gives us that

\[
\mathcal{P}\left( \text{SC}\left( \tilde{L}, \cup_{j=1}^{j-1} F_j \right), C_{i-1}, C_{i-1}, n \right)
\]

\[
= \mathcal{P}\left( \tilde{S}^{(i)}, C_{i-1}, C_{i-1}, n \right) + \sum_{j=i}^{d-1} \mathcal{P}\left( \tilde{S}^{(j+1)} - \text{SC}\left( \tilde{S}^{(j)}, F_j \right), C_j, C_j, n \right), \forall i \in [d].
\]

Thus,

\[
\tilde{B} = B + \delta_1 \left( U \begin{bmatrix} S^{(1)} \end{bmatrix} - U[L] \right) + \sum_{i=1}^{d-1} \left( \sum_{j=1}^{i+1} \delta_j \right) \mathcal{P}\left( U \begin{bmatrix} \tilde{S}^{(i+1)} \end{bmatrix} - U \begin{bmatrix} \text{SC}\left( \tilde{S}^{(i)}, F_i \right) \end{bmatrix}, C_i, C_i, n \right).
\]

By the definition of Schur complement chain, \( U\begin{bmatrix} \tilde{S}^{(1)} \end{bmatrix} \lessapprox U[L], U\begin{bmatrix} \tilde{S}^{(i+1)} \end{bmatrix} \lessapprox U \begin{bmatrix} \text{SC}\left( \tilde{S}^{(i)}, F_i \right) \end{bmatrix} \)

and \( \tilde{S}^{(1)} - L \lessapprox \delta_1 \cdot U[L], \tilde{S}^{(i+1)} - \text{SC}\left( \tilde{S}^{(i)}, F_i \right) \lessapprox \delta_{i+1} \cdot U \begin{bmatrix} \text{SC}\left( \tilde{S}^{(i)}, F_i \right) \end{bmatrix}. \)

Combining with the condition \( \sum_{i=1}^{d} \delta_i \leq 1, \) we have

\[
B \lessapprox \tilde{B} \lessapprox B + \delta_1^2 U[L] + \sum_{i=1}^{d-1} \left( \sum_{j=1}^{i+1} \delta_j \right) U \begin{bmatrix} \text{SC}\left( \tilde{S}^{(i)}, F_i \right) \end{bmatrix} \lessapprox 2B.
\]

Proof of Lemma 6.7. By the definition of \( M^{(i,N)}, \) we have

\[
\left( \tilde{S}^{(i)}_{F,F_i} \right)^{-1} - M^{(i,N)} = \frac{1}{2} \sum_{k=N}^{+\infty} \left( I - \frac{1}{2} \tilde{D}^{-1}_{F,F_i} \tilde{S}^{(i)}_{F,F_i} \right)^k \tilde{D}^{-1}_{F,F_i} = \left( I - \frac{1}{2} \tilde{D}^{-1}_{F,F_i} \tilde{S}^{(i)}_{F,F_i} \right)^N \left( \tilde{S}^{(i)}_{F,F_i} \right)^{-1}.
\]
By the Gershgorin circle theorem, the modulus of the eigenvalues of $I - \frac{1}{2} \hat{D}_{F_iF_i}^{-1} \tilde{S}_{F_iF_i}^{(i)}$ are no greater than $\frac{2 + \alpha}{2(1 + \alpha)}$. Then, the modulus of the eigenvalues of $\left( I - \frac{1}{2} \hat{D}_{F_iF_i}^{-1} \tilde{S}_{F_iF_i}^{(i)} \right)^N$ are no greater than $\left( \frac{2 + \alpha}{2(1 + \alpha)} \right)^N$. Thus, we have $I - \left( I - \frac{1}{2} \hat{D}_{F_iF_i}^{-1} \tilde{S}_{F_iF_i}^{(i)} \right)^N$ is nonsingular. It follows that

$$M^{(i,N)} = \left( I - \left( I - \frac{1}{2} \hat{D}_{F_iF_i}^{-1} \tilde{S}_{F_iF_i}^{(i)} \right)^N \right) \left( \tilde{S}_{F_iF_i}^{(i)} \right)^{-1}$$

is nonsingular.

Then, we have

$$\left\| \left( \tilde{U}^{(i,N)} \right)^{-1} - \left( \tilde{U}^{(i,\infty)} \right)^{-1} \right\|_\infty = \left\| \left( \tilde{S}_{F_iF_i}^{(i)} \right)^{-1} - M^{(i,N)} \tilde{S}_{F_iF_i}^{(i)} \right\|_\infty = \frac{1}{2} \left\| \sum_{k=N}^{+\infty} \left( I - \frac{1}{2} \hat{D}_{F_iF_i}^{-1} \tilde{S}_{F_iF_i}^{(i)} \right)^k \hat{D}_{F_iF_i}^{-1} \tilde{S}_{F_iF_i}^{(i)} \right\|_\infty \leq \frac{1}{2} \sum_{k=N}^{+\infty} \left( 2 + \alpha \right)^k (2(1 + \alpha)) = \frac{(1 + \alpha)}{2(1 + \alpha)} \left( \frac{2 + \alpha}{2(1 + \alpha)} \right)^N.$$

Analogously, we have

$$\left\| \left( \tilde{L}^{(i,N)} \right)^{-1} - \left( \tilde{L}^{(i,\infty)} \right)^{-1} \right\|_1 \leq \frac{(1 + \alpha)}{\alpha} \left( \frac{2 + \alpha}{2(1 + \alpha)} \right)^N$$

and

$$\left\| \left( \tilde{S}_{F_iF_i}^{(i)} \right)^{-1} - M^{(i,N)} \right\|_\infty \leq \frac{(1 + \alpha)}{\alpha} \left( \frac{2 + \alpha}{2(1 + \alpha)} \right)^N \left\| \hat{D}_{F_iF_i}^{-1} \right\|_\infty.$$