ABUNDANCE OF NON ISOMORPHIC INTERMEDIATE RINGS

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Abstract. It is well known that for a non pseudocompact space $X$, the family $\sum(X)$ of all intermediate subrings of $C(X)$ which contain $C^*(X)$ contains at least $2^\omega$ many distinct rings. We show that if in addition $X$ is first countable and real compact, then there are at least $2^\omega$ many rings in $\sum(X)$, no two of which are pairwise isomorphic.

1. Introduction

In what follows $X$ stands for a completely regular Hausdorff topological space and $C(X)$ be the ring of all real valued continuous functions on $X$. $C^*(X)$ is the subring of $C(X)$ containing all those functions which are bounded over $X$. By an intermediate ring we mean a ring that lies between $C^*(X)$ and $C(X)$. Let $\sum(X)$ be the family of all intermediate rings. It was established first by D. Plank [6] and then independently by using an entirely different technique by Redlin and Watson [8] that the structure space of all the intermediate rings a re one and the same, Viz, the Stone-Čech compactification $\beta X$ of $X$. The structure space of a commutative ring $R$ is the set of all maximal ideals of $R$ with hull kernel topology. Intermediate rings have been studied by several authors, viz., [3], [2], [4], [8], [9], [10]. It is perhaps well known, though we shall give a short proof of it of the fact that if $X$ is not pseudocompact meaning that $C^*(X) \subsetneq C(X)$, then $\sum(X)$ contains at least $2^\omega$ many distinct rings. As far as we dig into the literature, we are of the opinion that the problem of determining the non isomorphic rings in the family $\sum(X)$ is yet unaddressed. This is our principal motivation behind writing this article. It is a standard result in the theory of ring of continuous functions that if $X$ is a non pseudocompact space then the Hewitt real compactification $\upsilon X = \{p \in \beta X \mid \text{each } f \in C(X) \text{ has an extension to a function in } C(X \cup \{p\}) \subseteq \beta X \}$. Furthermore in such a case $\beta X \setminus \upsilon X$ contains a copy of $\beta N \setminus N$ and hence $|\beta X \setminus \upsilon X| \geq 2^\omega$ [9D(3), [8]. With each point $p$ in $\beta X \setminus \upsilon X$, we have associated an intermediate ring $C_p$ in such a manner that whenever $p$ and $q$ are distinct points in $\beta X \setminus \upsilon X$, the corresponding ring $C_p$ and $C_q$ are also distinct. [Corolloary 2.5]. This explains why $\sum(X)$ contains at least $2^\omega$ many different members.

To set further insight into the possible relations existing between a pair of rings of the form $C_p$ and $C_q$ as mentioned above, we have imposed additional condition on the topology of $X \upsilon X$ that $X$ is a first countable realcompact space. It turns out that $C_p$ and $C_q$ are isomorphic as rings when and precisely when the points $p$ and $q$ in $\beta X$ could be exchanged by a homeomorphism on $\beta X$ onto itself [Theorem 2.6]. This opens the flood gate of non isomorphic intermediate rings of the form $C_p$ [Theorem 2.7]. We conclude this article after showing that there are indeed at least $2^\omega$ many pairwise non isomorphic intermediate subrings of $C(X)$ [Theorem 2.11].

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2. ISOMORPHISM BETWEEN RINGS $C_p$

**Definition 2.1.** For a point $p$ in $\beta X$ set $C_p = \{ f \in C(X) \mid f^*(p) \in \mathbb{R} \}$, here $f^*: \beta X \to \mathbb{R} \cup \{\infty\}$ is the continuous extension of $f$.

Since for $f,g \in C_p$ one can easily check that $(f+g)^*(p) = f^*(p) + g^*(p)$ and $(fg)^*(p) = f^*(p)g^*(p)$, hence $C_p$ is a subring of $C(X)$. Furthermore as each $f$ in $C^*(X)$ has an extension to a real valued continuous function over the whole of $\beta X$ it follows that $C^*(X) \subseteq C_p$. Thus $C_p \subseteq \Sigma(X)$.

**Notation 2.2.** For each subset $A$ of $C(X)$, set $\nu_A X = \{ p \in \beta X \mid f^*(p) \in \mathbb{R} \text{ for each } f \in A \}$. Then $\nu_A X$ is a real compact space lying between $X$ and $\beta X$ [8, B(3) [5]]. The following proposition is proved in [1] Lemma 2.2.

**Lemma 2.3.** Given any pair of distinct points $p,q$ from $\beta X \setminus \nu X$, there exists an $f \in O^p$ such that $f^*(q) = \infty$, here $O^p = \{ g \in C(X) \mid p \in \text{int}_{\beta X} cl_{\beta X} Z(g) \}$

As a consequence of this lemma we get the following result:

**Theorem 2.4.** If $p \in \beta X \setminus \nu X$, then $\nu C_p X = \nu X \cup \{p\}$.

**Proof.** It is trivial that $p \in \nu C_p X$. But if $q \neq p$ in $\beta X$, then from Lemma 2.3 there exists an $f \in O^p$ such that $q \notin \nu(f) X$. But $f \in O^p$ implies that $f^*(p) = 0$ and so $f \in C_p$. Hence $q \notin \nu C_p X$. \(\square\)

**Corollary 2.5.** If $p$ and $q$ are distinct points in $\beta X \setminus \nu X$, then $C_p \neq C_q$.

We are now ready to establish the first important technical result of the paper.

**Theorem 2.6.** Let $X$ be a first countable noncompact realcompact space and $p,q \in \beta X \setminus \nu X$. Then the ring $C_p$ is isomorphic to the ring $C_q$ if and only if there is a homeomorphism from $\beta X$ onto itself, which exchanges $p$ and $q$.

**Proof.** Each point $x$ in $X$ has a countable local base $\{U_n\}_{n=1}^\infty$ in the space $X$. It follows that $\{cl_{\beta X} U_n \mid n \in \mathbb{N}\}$ is a countable local base about the same point in the space $\beta X$. Thus $\beta X$ is first countable at each point on it. On the other hand no point of $\beta X \setminus X$ is a $G_\delta$-point of $\beta X$ [Theorem 9.6[5]]. Consequently any homeomorphism $\phi: \beta X \to \beta X$ exchanges the points of $X$; i.e., $\phi(X) = X$.

First assume that $C_p$ is isomorphic to $C_q$ under a map $\psi: C_q \to C_p$. Then $\psi$ induces a homeomorphism $\Phi$ between their common structure space $\beta X$ in the most obvious manner. As every isomorphism between two rings of real valued continuous functions takes real maximal ideals to real maximal ideals it follows that the restriction map $\Phi|_{\nu C_p X} : \nu C_p X \to \nu C_q X$ becomes a homeomorphism onto $\nu C_q X$. Since each homeomorphism $\Phi$ on $\beta X$ exchanges the points of $X$ as observed above it follows that $\Phi(\nu C_p X \setminus X) = \nu C_q X \setminus X$. This implies on using Corollary 2.5 that $\Phi((\nu X \setminus X) \cup \{p\}) = (\nu X \setminus X) \cup \{q\}$. But $X$ is realcompact, therefore $\nu X = X$, and hence $\Phi(p) = q$.

Conversely let there exist a homeomorphism $\Phi : \beta X \to \beta X$ with $\Phi(q) = p$. Choose $f \in C_p$. Then $\Psi(f) = (f^* \circ \phi)|_X \in C(X)$ and we notice that $(\Psi(f))^* = f^* \circ \Phi$ because these two continuous functions agree on $X$, dense in $\beta X$. We further check that $(\Psi(f))^*(q) = f^* \circ \Phi(q) = f^*(p) \in \mathbb{R}$. This implies that $\Psi(f) \in C_q$. Thus a map $\Psi : C_p \to C_q$ is defined in the process. On using the denseness of $X$ in $\beta X$, one can easily verify that $\Psi$ is a ring homomorphism and kernel of $\Psi = \{ f \in C_p : (f^* \circ \Phi)|_X = 0 \} = \{ f \in C_p : f \circ \phi|_X = 0 \}$ is $\{ f \in C_p : f = 0 \}$.
0} = \{0\}. Furthermore for \(g \in C_q, (g^* \circ \Phi^{-1})(p) = g^*(q)\) is a real number. We also note that \((g^* \circ \Phi^{-1})(p) = (g \circ \Phi^{-1})(p)\) because of the denseness of \(X\) in \(\beta X\). Thus \((g^* \circ \Phi^{-1})(p) = g^*\) \(\in C_q\). We see that \(\Psi((g^* \circ \Phi^{-1})(p)) = g^* \circ \Phi^{-1} \circ \Phi(p) = g^*(p) = g\). Thus \(\Psi : C_p \to C_q\) turns out to be an isomorphism. 

The following proposition hints at the possibility of abundance of nonisomorphic intermediate rings.

**Theorem 2.7.** Let \(X\) be first countable noncompact and realcompact. If \(p\) is a \(P\)-point of \(\beta X \setminus X\) and \(q\) a non-\(P\)-point of \(\beta X \setminus X\), then \(C_p\) is not isomorphic to \(C_q\). (\(P\)-points of a space \(X\) are those points whose neighbourhood filters are closed under countable intersections).

**Proof.** Follows immediate from Theorem 2.6 because the property that a point in a topological space is a \(P\)-point is a topological invariant. 

To ensure that nonisomorphic intermediate rings do exist in galore, we need the notion of type of a point in \(\beta N \setminus N\) introduced by Frolik [12] and recorded in [11]. We reproduce below a few relevant information about this notion from the monograph [11], 3.1-4.12. Each permutation \(\sigma : N \to N\) extends to a homeomorphism \(\sigma^* : \beta N \to \beta N\), conversely if \(\Phi : \beta N \to \beta N\) is a homeomorphism then \(\Phi|N\) is a permutation of \(N\), because \(\Phi\) takes isolated points to isolated points and the points of \(N\) are the only isolated points of \(\beta N\). Therefore \(\Phi = \sigma^*\) for unique permutation \(\sigma = \Phi|N\) of \(N\).

**Definition 2.8.** For two points \(p, q \in \beta N \setminus N\), we write \(p \sim q\) when there exists a permutation \(\sigma\) on \(N\) such that \(\sigma(p) = q\). \(\sim\) is an equivalence relation on \(\beta N \setminus N\). Each equivalence class of elements of \(\beta N \setminus N\) is called a type of ultrafilters on \(N\).

**Theorem 2.9** (Frolik [11]). There exists \(2^\omega\) many types of ultrafilters on \(N\).

**Theorem 2.10** (Frolik [11]). If \(N\) is \(C\)-embedded in \(X\), then \(cl_{\beta X} N \subseteq \beta X \setminus X\), essentially \(\beta N \setminus N \subseteq \beta X \setminus X\). If now \(h : \beta X \to \beta X\) is a homeomorphism onto \(\beta X\), and \(p, q \in \beta N \setminus N\) are such that \(h(p) = q\), then \(p\) and \(q\) belong to the same type of ultrafilters on \(N\).

We now use these two theorems of Frolik to establish the last main result of the present paper.

**Theorem 2.11.** Let \(X\) be a first countable noncompact realcompact space. Then there exist at least \(2^\omega\) many intermediate subrings of \(C(X)\), no two of which are isomorphic.

**Proof.** Since \(X\) is a noncompact realcompact space it is not pseudocompact. Hence \(X\) contains a copy of \(N\), \(C\)-embedded in \(X\) [1.21, 5]. As every \(C\)-embedded countable subset of a Tychonoff space is a closed subset of it [3, B3, 5], it follows that \(cl_{\beta X} N \subseteq \beta X \setminus X\) essentially \(\beta N \subseteq \beta X \setminus X\). The result of Theorem 2.7 assures that there exist a subset \(S\) of \(\beta N \setminus N\), consisting exactly one member from each type with the property that \(|S| = 2^\omega\). Let \(p\) and \(q\) be two distinct points of the set \(S\). Then it follows from Theorem 2.10 that no homeomorphism from \(\beta X\) to \(\beta X\) can exchange \(p\) and \(q\). We now use Theorem 2.6 to conclude that the rings \(C_p\) and \(C_q\) are not isomorphic.
References

[1] S. K. Acharyya, B. Mitra: Characterizations of nearly pseudocompact spaces and related spaces, Proceedings of the 19th Summer Conference on Topology and its Applications. Topology Proc. 29 (2005), no. 2, 577-594.

[2] S.K.Acharyya and D.De: A-Compactness and Minimal Subalgebras of C(X), Rocky Mountain journal of Mathematics, vol-36, no-4 (2005).

[3] S.K.Acharyya, K.C.Chattopadhyay and D.P.Ghosh: On a class of subalgebras of C(X) and the intersection of their free maximal ideals, Proc. Amer. Math. Soc. 125 (1997), 611-615.

[4] H.L. Byun, S. Watson: Prime and maximal ideals in subrings of C(X), Topology. Appl. 40(1991), 45-62.

[5] L. Gillman and M. Jerison: Rings of Continuous Functions. Springer-Verlag, New york., 1978.

[6] D. Plank: On a class of subalgebras of C(X) with application to βX \ X, Fund. Math. 64(1969), 41-54.

[7] D.Rudd: On isomorphism between Ideals in Rings of Continuous Functions Trans. AMS, vol-159, 1971.

[8] L. Redlin and S. Watson: Maximal ideals in subalgebras of C(X). Proc. Amer. Math. Soc. 100(1987), 763-766.

[9] L. Redlin and S. Watson: Structure spaces for Rings of Continuous Functions with Application to Realcompactifications. Fund. Math. 152(1997), 151-163.

[10] J. Sack, S. Watson: Characterizations of Ideals in Intermediate C-Rings A(X) via the A-Compactifications of X, Int. J. Math. & Math. Sc., vol.-2013, Article Id-635361, 6 pages.

[11] Russel Walker: The Stone Cech Compactification Springer -Verlag, Berlin Heidelberg, New York, 1974.

[12] Frolik Z: Sums of ultrafilters, Bull. Amer. Math. Soc. 73, 87-91.

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