Autonomous quantum rotator

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Abstract – We consider a minimal model of a quantum rotator composed of a single particle confined in a harmonic potential and driven by two temperature-biased heat reservoirs. In the case the particle potential is rendered asymmetric and rotated an angle, a finite angular momentum develops, corresponding to a directed rotary motion. At variance with the classical case, the thermal fluctuations in the baths give rise to a non-vanishing average torque contribution; this is a genuine quantum effect akin to the Casimir effect. In the steady state the heat current flowing between the two baths is systematically converted into particle rotation. We derive exact expressions for the work rate and heat currents in the case in which the system is driven by an external time-periodic mechanical force. We show, in agreement with previous works on classical systems, that for this choice of external manipulation protocol, the rotator cannot work either as a heat pump or as a heat engine. We finally use our exact results to extend an ab initio quantum simulation algorithm to the out-of-equilibrium regime.

There is currently a strong interest in quantum thermal machines [1]. From a theoretical point of view these systems are of interest in elucidating the limits of thermodynamics and the role of fluctuations [2]. Experimentally, it has recently been feasible to manufacture nanodevices acting like reciprocating heat engines [3] or thermoelectric transducers [4].

Small quantum devices are open quantum systems coupled to their environment, possibly characterised by a time-dependent Hamiltonian. Consequently, in the modelling of a quantum heat engine or refrigerator one must include the coupling to the heat reservoirs subject to a proper characterisation [5]. More precisely, and differently from the classical case [6], the heat baths have to be characterised explicitly as larger quantum systems coupled to the small quantum system [5,7–9].

Many efforts have been devoted to the investigation of quantum reciprocating engines, performing, e.g., Carnot, Otto or Stirling cycles. These cycles require an external agent that changes periodically one or more mechanical parameters in the Hamiltonian and the system temperature, see, e.g., [3,10,11] or [2] and references therein. The simplest quantum equivalent to the Carnot cycle is a heat engine proposed by Scovil and Schulz-DuBois [12]: it is a three-level amplifier operating in contact with two heat reservoirs. This model acts like an amplifier when coupled to an external field at resonance [13]. However, it is basically a reversible or quasi-static engine and can therefore operate with efficiency only limited by the Carnot efficiency. Another example of thermal motor is the flywheel introduced in [14] which can store kinetic energy when driven by an external time-periodic field and when monitoring and feedback control are applied.

Another class of thermal machines are the quantum autonomous devices which exhibit the ideal design for engineering purposes since they can operate in steady-state conditions without any external time-dependent drive. While quantum autonomous engines such as refrigerators have been investigated, see, e.g., [15], quantum autonomous motors exhibiting directed transport have not attracted much attention.

An autonomous rotor model has recently been proposed in [16]. In this model the strength of the coupling with the two heat baths depends on the state of the system itself. Another interesting example is the quantum mill [17] whose working fluids are two qubits at different temperatures that can perform work against a dissipative load. Here we show that an autonomous rotor can be obtained with a much simpler design.

A minimal model of a classical autonomous heat motor exhibiting directed rotary motion at a non-vanishing rate was first proposed by Filliger and Reimann [18], and...
later studied in [19]; it has recently been realised experimentally [20], and extended to the underdamped regime in [21]. The original Filliger and Reimann model is based on an overdamped particle moving in a 2D anisotropic and rotated harmonic potential. Driven by two temperature biased heat reservoirs the system enters a non-equilibrium steady state yielding a finite torque acting on the potential. In other words, the heat transmitted to the system is converted into a mechanical torque giving rise to a gyration motion, i.e., an elementary heat engine. This model bears a resemblance to a recent study in [22] where it was found that the minimal requirements for directed transport to emerge in a two-temperature autonomous motor are i) a non-equilibrium thermal state and ii) a broken spatial symmetry.

In the present paper we construct and analyse a microscopic quantum model, fulfilling the requirements i) and ii) above, based on an extension of the harmonic model in [18]. Since a proper quantum treatment requires a Hamiltonian framework, the first step is to extend the model to the underdamped case and introduce a mass. Moreover, the Hamiltonian description of the resulting open quantum system must be extended to the heat reservoirs; here modelled as collections of independent quantum oscillators. Since the combined system is linear, it is most convenient to work in the Heisenberg picture and construct quantum Langevin equations according to the scheme first proposed by Ford, Kac, and Mazur [7–9].

In the Ohmic approximation, corresponding to a particular spectral distribution of the oscillator modes in the heat baths, driving the small system in a stationary non-equilibrium state, the temperature biased heat reservoirs generate a finite quantum angular momentum, corresponding to a rotation. However, since the moment of inertia is fluctuating and strongly correlated with the angular momentum, the quantum gyration or quantum rotor does not act like a rigid body and one cannot meaningfully associate an angular frequency with the motion. At variance with the classical case, the thermal fluctuations, moreover, give rise to an additional contribution to the average torque which is a genuinely quantum effect. Such an effect is akin to the Casimir effect for the rotatory motion, in the sense that the quantum reservoirs give rise to a quantum torque which vanishes in the classical limit.

Quantum rotator model. – Here we set up the quantum model for the rotor; for mathematical details we refer to sect. A of the supplementary material Supplementary material.pdf (SM). We consider a particle of mass $m$ at position $(x_1, x_2)$ moving in a rotated 2D anisotropic harmonic potential. The system is characterised by the quantum Hamiltonian

$$H_0 = -\hbar^2/2m \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + U(x_1, x_2),$$

$$U(x_1, x_2) = \frac{1}{2}Ax_1^2 + \frac{1}{2}Bx_2^2 + Cx_1x_2.$$  

The parameters $A$, $B$, and $C$ are related to the rotated potential $u_1x_1^2 + u_2x_2^2$ by

$$A = u_1 \cos^2 \alpha + u_2 \sin^2 \alpha,$$  

$$B = u_2 \cos^2 \alpha + u_1 \sin^2 \alpha,$$  

$$C = \frac{1}{2}(u_1 - u_2) \sin 2\alpha,$$

where $u_1$ and $u_2$ are the anisotropy parameters (with $u_1$, $u_2 > 0$ for mechanical stability), and $\alpha$ the rotation angle.

The model has the structure of two linear oscillators with frequencies $\omega_1 = (A/m)^{1/2}$ and $\omega_2 = (B/m)^{1/2}$ coupled linearly with strength $C$. Since $U$ originates from a rotated anisotropic potential the parameters are constrained according to $A + B = u_1 + u_2$, $A - B = (u_1 - u_2) \cos 2\alpha$, and $AB - C^2 = u_1u_2$; note that the isotropic case $u_1 = u_2$ implies $A = B$ and $C = 0$, corresponding to rotational invariance and two independent oscillators.

Following the prescription in [23,24] the two independent heat baths including their coupling to the rotator coordinates are given by the quantum Hamiltonians

$$H_n = \sum_k \left( \frac{P_k^{(n)}{(\omega_k^{(n)})^2}}{2m_k^{(n)}} + \frac{1}{2}m_k^{(n)}\omega_k^{(n)}{(X_k^{(n)} - x_n)}^2 \right),$$  

with $n = 1, 2$. The heat baths are characterised by the masses $m_k^{(n)}$ and the frequencies $\omega_k^{(n)}$. The sum runs over the wave number modes $k$. Note that the coupling to the system is absorbed in the definitions of $m_k^{(n)}$ and $\omega_k^{(n)}$. Introducing the density of states $N^{(n)}(\omega) = 2\pi \sum_k m_k^{(n)} \omega_k^{(n)} \delta(\omega - \omega_k^{(n)})$, characterising the spectral distribution of bath nodes, the Ohmic approximation corresponds to choosing a constant density of states, i.e., $N^{(n)} = 2\eta$. Non-Ohmic approximations will in general give rise to memory effects; they will not be considered here.

The quantum Langevin equations for a quantum rotator in the Ohmic approximation thus take the form

$$m\ddot{x}_n = -\partial U/\partial x_n - \eta \dot{x}_n + \xi_n.$$  

Here the quantum aspects are incorporated in the quantum operator $\xi_n$ which plays the role of a “quantum noise”. Performing a statistical quantum average we obtain the quantum relations $\langle \xi_n(t)\xi_m(t') \rangle = \delta_{nm}T_n(t - t')$ where

$$F_n(t) = \eta \int \frac{d\omega}{2\pi} e^{-i\omega t} \hbar \omega \left[ 1 + \coth \left( \frac{\hbar \omega}{2T_n} \right) \right];$$  

here $T_n$ are the temperatures of the respective baths and we have set $k_B = 1$. For clarification we note that the quantum noise $\xi_n$ should not be confused with a classical Gaussian coloured noise. The quantum noise originates from the equilibrium averages over the initial bath variables. Note, however, that in the classical limit $\hbar \to 0$ the
noises commute and we obtain $F_n(t - t') = 2nT_n\delta(t - t')$. As a result eq. (7) reduces to a standard Langevin equation driven by white Gaussian noise. Equations (7) and (8) form the basis for the further analysis in the paper.

**Quantum angular momentum and torque.** – We here present and discuss the results for the angular momentum and torque, for mathematical details see sect. B of the SM. Even in the absence of an external force, the heat reservoirs at temperatures $T_1$ and $T_2$ drive the system into a rotating state, characterised by a non-vanishing mean angular momentum. Using the standard definition for the angular momentum $L = m(x_1\dot{x}_2 - x_2\dot{x}_1)\hat{w}$, where $(AB)\hat{w} = (1/2)(AB + BA)$ denotes the symmetrical Weyl ordering, see, e.g., [25], we find the general expression

$$\langle L \rangle_0 = -4m\eta^2C \int \frac{d\omega}{2\pi} \frac{\omega^2}{Z(\omega)^2} \tilde{G}(\omega, T_1, T_2),$$

$$G(\omega, T_1, T_2) = \frac{\hbar\omega}{2} \left[ \coth \left( \frac{\hbar\omega}{2T_1} \right) - \coth \left( \frac{\hbar\omega}{2T_2} \right) \right],$$

with $Z(\omega) = (A - m\omega^2 - i\omega\eta)(B - m\omega^2 - i\omega\eta) - C^2$. We note that $\langle L \rangle_0 = 0$ for $C = 0$, corresponding to the restoration of rotational invariance. Moreover, $\langle L \rangle_0 = 0$ for $T_1 = T_2$, corresponding to thermal equilibrium. In non-equilibrium we have $\langle L \rangle_0 \propto \text{sign}(C(T_2 - T_1))$.

The integral in eq. (9) cannot be performed analytically by contour integration in general, since the integrand has an infinite (but isolated) number of poles along the complex axis. However, in the classical or high-temperature limit $\hbar\omega \ll T_n$, we have $G(\omega, T_1, T_2) \to T_1 - T_2$, and the integral in eq. (9) yields to a contour integration. In terms of the anisotropy parameters $u_1$ and $u_2$ we obtain

$$\langle L \rangle_0^\alpha = -4m\eta(T_1 - T_2)(u_1 - u_2)\sin 2\alpha \frac{m(u_1 - u_2)^2 + 2\eta^2(u_1 + u_2)}{m(u_1 - u_2)^2 + 2\eta^2(u_1 + u_2)}.$$ 

In fig. 1 we have depicted $\langle L \rangle_0$ and $\langle L \rangle_0^\alpha$ as functions of the temperature scaling factor $\theta$, with $T_n = \tau_n\theta$, and $\tau_n$ constant (top panel) and as a function of the potential rotation angle $\alpha$ (bottom panel).

In ref. [18] the classical overdamped version of the rotor was investigated and the expression for the torque of the friction forces $\langle M_\alpha \rangle_0^\alpha = \eta((\dot{x}_1x_2 - \dot{x}_2x_1))$ was obtained (here we take the two friction coefficients equal, $\eta_1 = \eta_2 = \eta$). While the angular momentum $L = m(x_1\dot{x}_2 - x_2\dot{x}_1)$ is not defined in the overdamped limit $\eta \to 0$, from our result, eq. (11), we can calculate the torque of the friction force as $\langle M_\alpha \rangle_0^\alpha = \eta(m)(\langle L \rangle_0^\alpha)$. By taking the limit $m \to 0$, we obtain from eq. (11)

$$\langle M_\alpha \rangle_0^\alpha = -(T_1 - T_2)(u_1 - u_2)\sin 2\alpha \frac{(u_1 + u_2)}{m(u_1 - u_2)^2 + 2\eta^2(u_1 + u_2)}$$

in accordance with the result obtained in [18].

Since the mean angular momentum is finite and bounded, we conclude that the total torque must vanish on average, i.e., $\langle M \rangle_0 = m((x_1\dot{x}_2 - x_2\dot{x}_1)\hat{w})_0 = 0$. This result follows from $\langle M \rangle_0 = \partial \langle L \rangle_0/\partial t$ and ergodicity and can also be checked by a direct calculation, using the solution of eq. (7), see sect. B of the SM. Consequently, from

$$\langle M \rangle_0 = 0 \quad \text{and from eq. (7) we obtain the identity}$$

$$\eta \langle L \rangle_0 = - \langle (x_1\partial_{x_2}U - x_2\partial_{x_1}U)\hat{w} \rangle_0 + \langle (x_1\xi_2 - x_2\xi_1)\hat{w} \rangle_0,$$

showing that the mean angular momentum is proportional to the torque acting on the potential plus a torque contribution from the quantum noise.

The torque associated with the quantum noise fluctuations, $\langle M \xi_0 \rangle = \langle (x_1\xi_2 - x_2\xi_1)\hat{w} \rangle_0$, is given by

$$\langle M \xi_0 \rangle = 2C\eta \int \frac{d\omega}{2\pi} \frac{\coth}{Z(\omega)^2} \tilde{G}(\omega, T_1, T_2).$$

This torque contribution vanishes when the potential is rotationally symmetric ($C = 0$), at equilibrium $T_1 = T_2$, at equilibrium.
(see the definition of $G(\omega)$ in eq. (10)), and in the classical limit $h \to 0$. The last result can also be found by contour integration of the integral in eq. (13), for details consult sect. B of the SM. Since $(\eta/m)(L)_{0}$ is equal to the dissipative torque $\eta(x_{1}x_{2} - x_{2}x_{1})_{0}$, the result $(M_{2})_{0} = 0$ in the classical limit is consistent with the analysis in the classical overdamped case in [18]. In the insets of fig. 1 the average quantum torque $(M_{2})_{0}$ is plotted as a function of the temperature scaling factor $\theta$ and as a function of the angle $\alpha$.

The observation that the fluctuations in the heat baths give rise to a non-vanishing torque contribution $(M_{2})_{0}$ is a genuine quantum effect. It is akin to the Casimir effect due to quantum vacuum fluctuations, see, e.g., [26,27], or critical fluctuations in classical systems [28]. In fact, the non-vanishing stochastic force torque $(M_{2})_{0}$ is due to the mismatch between the fluctuations in the two quantum baths, a mismatch that arises in the non-equilibrium/non-rotational invariant case. It is worthwhile to note that the quantum noise torque $(M_{2})_{0}$ has always the opposite sign of the mean angular moment $(L)_{0}$, for the broad range of parameter values considered in fig. 1. Consequently, an inspection of eq. (12) indicates that the non-vanishing $(M_{2})_{0}$ decreases the value of the angular momentum thus diminishing the performance of the rotor.

Unlike the case of a rigid body, the moment of inertia $I = m(x_{1}^{2} + x_{2}^{2})$ for the quantum rotator is fluctuating. This follows from the fact that the position of the particle explores the whole potential region, including the origin. Consequently, one cannot meaningfully define an angular velocity $\Omega$ according to $L = I\Omega$. In other words, there are correlations between $I$ and $\Omega$ and the approximation $(L) = (I)\langle \Omega \rangle$ is not a priori justified. Based on a numerical integration of the Langevin equations in the classical limit reported in sect. B.1 of the SM we have demonstrated that the moment of inertia and the angular momentum are in fact strongly correlated.

**Driven quantum rotator.** – Next we consider the case in which an external time-dependent force, $\mathbf{f}(t) = (f_{1}(t), f_{2}(t))$, is applied to the system. In this case the Hamiltonian in eq. (1) takes the form $H_{0} - \mathbf{r} \cdot \mathbf{X}$; details of this section are given in sects. B.2 and C of the SM.

Applying an external periodic force the quantum oscillator is driven into a periodic state, i.e., a limit cycle. Here we drive the quantum rotator with a periodic drive with amplitude $D$ and frequency $\omega_{0}$, setting $\mathbf{f}(t) = D(\cos \omega_{0}t, \sin \omega_{0}t)$. A time-periodic protocol is a standard choice in the implementation of microscopic cyclic devices, both in the classical and in the quantum regime, see, e.g., [29,30]. For the angular momentum we obtain in the steady state the expression

$$
\langle L \rangle = (L)_{0} + D^{2}\omega_{0}K(\omega_{0}),
$$

$$
K(\omega_{0}) = m\frac{(A - m\omega_{0}^{2})(B - m\omega_{0}^{2}) + \omega_{0}^{2}\eta^{2} - C^{2}}{|Z(\omega_{0})|^{2}}.
$$

The angular momentum is composed of two parts. The unperturbed angular momentum $(L)_{0}$ generated by the broken symmetry and the temperature bias plus a contribution $D^{2}\omega_{0}K(\omega_{0})$ due to the drive. We note that the contribution from the drive is odd in the applied frequency $\omega_{0}$. Consequently, choosing a frequency $\omega_{0}$ so that $\omega_{0} = -(L)_{0}/D^{2}K(\omega_{0})$ we can on average arrest the rotary motion.

**Work and heat.** – Finally, we consider the thermodynamic properties of the driven system. The fluctuating rate of work, $r_{w}$, performed on the system by the external drive is

$$
\langle r_{w} \rangle = -\frac{\eta(D\omega_{0})^{2}}{2|Z(\omega_{0})|^{2}} \times [(A - m\omega_{0}^{2})^{2} + (B - m\omega_{0}^{2})^{2} + 2(\omega_{0}\eta)^{2} + 2C^{2}].
$$

We note that the work performed on the system is always positive, indicating that the system cannot perform work on the environment and thus cannot perform as an engine or a motor for this specific manipulation protocol. It follows from eq. (7) that the fluctuating forces associated with the heat reservoirs are given by $(-\eta\dot{\mathbf{x}}_{a}(t) + \xi_{a}(t))$. For the rates of heat acting on the rotator we then have $r_{q_{a}} = \langle \dot{\mathbf{x}}_{a}(t) - \eta\dot{\mathbf{x}}_{a}(t) + \xi_{a}(t) \rangle_{W}$. Inserting the equations of motion we obtain for the fluctuating rates

$$
r_{q_{1}} = \langle \dot{\mathbf{x}}_{1}(m\dot{x}_{1} + Ax_{1} + Cx_{2} - f_{1}) \rangle_{W},
$$

$$
r_{q_{2}} = \langle \dot{\mathbf{x}}_{2}(m\dot{x}_{2} + Bx_{2} + Cx_{1} - f_{2}) \rangle_{W}.
$$

The non-equilibrium heat transfer rate is given by $\langle \Delta r_{q} \rangle = \langle r_{q_{1}} \rangle - \langle r_{q_{2}} \rangle$.

From the definition $\langle L \rangle = m\langle (x_{1}\dot{x}_{2} - x_{2}\dot{x}_{1})_{W} \rangle$ and the definitions in eqs. (18) and (19) we obtain the identity

$$
\langle \Delta r_{q} \rangle = -\frac{C}{m}\langle L \rangle - \langle \dot{\mathbf{x}}_{1} \rangle f_{1} + \langle \dot{\mathbf{x}}_{2} \rangle f_{2}.
$$

In the absence of a drive we obtain in particular

$$
\langle \Delta r_{q} \rangle = 0 = -\frac{C}{m}\langle L \rangle_{0}.
$$

showing that the absorption of heat is completely converted into rotation.

For the individual heat transfers $r_{q_{1}}$ and $r_{q_{2}}$ we find

$$
\langle r_{q_{1}} \rangle = -\frac{C}{2m}\langle L \rangle_{0}
$$

$$
\langle r_{q_{2}} \rangle = +\frac{C}{2m}\langle L \rangle_{0},
$$

$$
\langle r_{q_{1}} \rangle = -\frac{\eta}{2}(D\omega_{0})^{2}\frac{(B - m\omega_{0}^{2})^{2} + (C + \omega_{0}\eta)^{2}}{|Z(\omega_{0})|^{2}},
$$

$$
\langle r_{q_{2}} \rangle = +\frac{\eta}{2}(D\omega_{0})^{2}\frac{(A - m\omega_{0}^{2})^{2} + (C - \omega_{0}\eta)^{2}}{|Z(\omega_{0})|^{2}}.
$$
The mechanical fluctuation-dissipation relation, as expressed by the classical Langevin equation with Gaussian colored noise, leads to the Heisenberg equations of motion for the quantum coordinates in different types of systems at equilibrium with various quantum statistics while using standard molecular dynamics in excellent test beds for checking the validity of the Quantum Molecular Dynamics (QMD) numerical algorithm \cite{32}. It has been tested and provided accurate results, for example, for the average energy and the specific heat in different types of systems at equilibrium with various degrees of anharmonicity \cite{32-37}. The QMD algorithm is analogous to the Casimir effect.

In conclusion, we have studied a minimal model of a quantum thermal motor exhibiting a non-vanishing angular momentum when moving in an asymmetric and rotated potential, while interacting with two quantum heat baths at different temperatures. The rotational motion is sustained by the heat current flowing through the system. There is then a striking difference between the classical and the quantum rotator, in the latter case the bath forces give rise to a systematic torque contribution which is absent in the classical case. This effect is analogous to the Casimir effect.

The rotor cannot perform useful work when manipulated with an external periodic load. This conclusion agrees with previous studies of classical motors where the working fluids are coupled oscillators at different temperatures. However, different designs where the trapping potential is non-linear or where the rotor is connected to a work repository, as in \cite{14}, could change this conclusion and yield a device from which useful work can be extracted.

Since we have exact results for the average values of the systems dynamic quantities, we also checked whether an \textit{ab initio} numerical algorithm for equilibrium quantum simulations could be used in order to investigate the dynamics in the case of multiple baths. We find an excellent agreement with our results. This result thus paves the way to the algorithm application to the study of the dynamics in the out-of-equilibrium regime. In particular, since the equilibrium QMD algorithm has been successfully used on non-linear systems, it is of interest to use the non-equilibrium QMD algorithm in order to investigate non-linear out-of-equilibrium quantum systems for which there is no analytic expression for the dynamic and thermodynamic quantities of interest.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{(Colour online) Work rate \(r_w\), and heat rates \(r_{q_1}\), and \(r_{q_2}\) as functions of the driving frequency \(\omega_0\) as given by eqs. (17), (22), (23), respectively. The curves are obtained with the following set of parameters: \(T_1 = 20\), \(T_2 = 50\), \(u_1 = 1\), \(u_2 = 1/4\), \(\alpha = \pi/4\), \(\hbar = 1\), \(m = \eta = D = 1\).}
\end{figure}

**Numerical algorithm.** Our expressions for the average angular momentum in eq. (9) and the quantum torque in eq. (13) are exact results for two non-linear quantities in an out-of-equilibrium system. They are thus excellent test beds for checking the validity of the Quantum Molecular Dynamics (QMD) numerical algorithm \cite{32} in the non-equilibrium regime. This algorithm accounts for quantum statistics while using standard molecular dynamics. It has been tested and has provided accurate results, for example, for the average energy and the specific heat in different types of systems at equilibrium with various degrees of anharmonicity \cite{32-37}. In the QMD algorithm the Heisenberg equations of motion for the quantum coordinate operator \(x\), as given by eq. (7), is replaced by a classical Langevin equation with Gaussian coloured noise \(\xi(t)\), whose power spectral density is dictated by the quantum-mechanical fluctuation-dissipation relation, as expressed by eq. (8). In the standard Langevin equation for classical systems the total time \(t\) of a single trajectory is divided into \(N_t = t/\delta t\) time steps of length \(\delta t\) and at each time step the Gaussian white noise is generated through a random generator. The QMD exhibits an additional complication with respect to this scheme and is thus more computationally demanding, \textit{i.e.}, at the beginning of each trajectory one has to generate and store an \(N_r\)-long vector of correlated random forces with correlation function given by eq. (8); this is the scheme discussed in \cite{32}.

While the QMD algorithm has previously been used in order to study the interaction of a quantum system with a single bath, we here implement it for the two-bath rotator. The results for the average angular momentum \(\langle L \rangle_q\) and the quantum noise torque \(\langle M_z \rangle\) are reported in fig. 1. The agreement with our exact results is excellent.

**Conclusions.** In conclusion, we have studied a minimal model of a quantum thermal motor exhibiting a non-vanishing angular momentum when moving in an asymmetric and rotated potential, while interacting with two quantum heat baths at different temperatures. The rotational motion is sustained by the heat current flowing through the system. There is then a striking difference between the classical and the quantum rotator, in the latter case the bath forces give rise to a systematic torque contribution which is absent in the classical case. This effect is analogous to the Casimir effect.

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