GLOBAL SOLUTIONS OF 2D ISENTROPIC COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH ONE SLOW VARIABLE

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Abstract. Motivated by [2, 3], we prove the global existence of solutions to the two-dimensional isentropic compressible Navier-Stokes equations with smooth initial data which are slowly varying in one direction and with initial density being away from vacuum. In particular, we present examples of initial data which generate unique global smooth solutions to 2D compressible Navier-Stokes equations with constant viscosity and with initial data which are neither small perturbation of some constant equilibrium state nor of small energy.

Keywords: compressible Navier-Stokes equations, strong solution, large data, slow variable.

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1. Introduction

We investigate the global existence of smooth solutions to the following two-dimensional isentropic compressible Navier-Stokes equations describing the motion of viscous barotropic compressible flows:

\[
\begin{aligned}
\partial_t \rho + \text{div} (\rho u) &= 0, \quad (t, x, y) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) - \mu \Delta u - \nabla (\mu' \text{div} u) + \nabla p(\rho) &= 0,
\end{aligned}
\]

where \(\rho(t, x, y) \in \mathbb{R}_+\) and \(u(t, x, y) \in \mathbb{R}^2\) designate the density and the velocity field of the fluid respectively. Here the shear viscosity \(\mu\) and bulk viscosity \(\mu'\) are assumed to be constant with \(\mu > 0\) and \(\mu + \mu' > 0\). The pressure function \(p\) is a smooth non-decreasing function of the density \(\rho\). For simplicity, we just consider power law pressure with

\[
p(\rho) = a\rho^\gamma, \quad a > 0, \quad \gamma \geq 1.
\]

The typical range of the adiabatic constant is \(1 < \gamma \leq \frac{4}{3}\) where the maximum value \(\gamma = \frac{4}{3}\) is related to monatomic gases, the intermedium value \(\gamma = \frac{7}{5}\) is related to diatomic gases including air, and lower values close to 1 is related to polyatomic gases at high temperature. In this paper, we allow any \(\gamma \in [1, 2]\). One may check [19] for more background of the system (1.1).

The global existence of smooth solution to the multi-dimensional compressible Navier-Stokes equations is one of the most important problems in the theory which describes the motion of the viscous compressible fluids. In [20], Matsumura and Nishida proved the global existence of solutions to the full compressible Navier-Stokes equations when the initial data is close to a constant equilibrium state in \(H^3(\mathbb{R}^n)\). Danchin [4] worked out similar result yet with initial data in certain optimal function spaces. Huang, Li and Xin [12] established the global existence and uniqueness of classical solutions to the Cauchy problem of the barotropic compressible Navier-Stokes equations in three space dimensions with smooth initial data which are of small energy but possibly large oscillations. Concerning general smooth data without size restriction and with initial density being away from vacuum, the first well-posedness result is due to Vaigant and Kazhikhov [30] where the authors proved the global unique solution to the two-dimensional compressible Navier-Stokes system (1.1) in \(\mathbb{R}^+ \times \mathbb{T}^2\) with the bulk viscosity \(\mu' = b\rho^\beta\) and \(b > 0, \beta > 3\). Lately this result was improved by Huang and Li [13] for \(\beta > \frac{4}{3}\). For further results in higher dimensional setting, we refer to [31, 1, 9] and the references therein.

On the other hand, Chemin and Gallagher [2] proved the global existence of smooth solution to three-dimensional incompressible Navier-Stokes equations (NS) with initial data which is slowly varying in one direction. This type of result was extended by Chemin and the second author [3] to the three-dimensional incompressible inhomogeneous Navier-Stokes equations and by Liu and the second author [17] for (NS) with unidirectional derivative of the initial velocity being sufficiently small in some critical functional space.

Motivated by [2, 3, 17], we are looking for global solutions of (1.1) that are periodic in \(x\) variable, and have slow variation in \(y\) variable. This means that we implement the system (1.1) with initial data of the form

\[
(1.3) \quad \rho(0, x, y) = [s_0]_\varepsilon(x, y), \quad u(0, x, y) = ([w_0]_\varepsilon(x, y), [\dot{w}_0]_\varepsilon(x, y)),
\]

where \((x, y) \in \mathbb{T} \times \mathbb{R}\) with \(\mathbb{T}\) being the torus \(\mathbb{R}/\mathbb{Z}\). Here and in all that follows, we shall always denote

\[
[f]_\varepsilon(x, y) \overset{\text{def}}{=} f(x, \varepsilon y).
\]
We aim at showing the global well-posedness of (1.1)–(1.3) without assuming any size restriction on \((\bar{\rho}_0, \bar{w}_0, \bar{w}_0)\) provided that \(\varepsilon\) is sufficiently small. We remark that data of the form (1.3) are neither of small perturbation of some constant equilibrium state, say (1, 0), nor of small energy. Hence in particular, our result gives examples indicating that 2D compressible Navier-Stokes equations with constant viscosity might be globally well-posed even for general initial data with initial density being away from vacuum.

Formally with initial data given by (1.3), we may seek solution of (1.1) as follows

\[
\rho(t, x, y) = [\bar{\xi}]_\varepsilon(t, x, y), \quad u(t, x, y) = [v]_\varepsilon(t, x, y),
\]

then it follows from (1.1) that

\[
\begin{cases}
\partial_t \bar{\xi} + \operatorname{div}_\varepsilon (\bar{\xi} v) = 0, \\
\partial_t (\bar{\xi} v) + \operatorname{div}_\varepsilon (\bar{\xi} v \otimes v) - \mu \Delta_v \bar{\xi} v - \mu' \nabla \varepsilon \operatorname{div}_\varepsilon v + \nabla \varepsilon p(\bar{\xi}) = 0,
\end{cases}
\]

with initial data

\[
\bar{\xi}(0, x, y) = \bar{\xi}_0(x, y), \quad v(0, x, y) = (w_0(x, y), \bar{w}_0(x, y)).
\]

Here

\[
\nabla_\varepsilon \overset{\text{def}}{=} \begin{pmatrix} \partial_x \\ \varepsilon \partial_y \end{pmatrix}, \quad \operatorname{div}_\varepsilon \overset{\text{def}}{=} \nabla_\varepsilon \cdot \quad \text{and} \quad \Delta_\varepsilon \overset{\text{def}}{=} \partial_x^2 + \varepsilon^2 \partial_y^2.
\]

Formally passing \(\varepsilon \to 0\) in (1.4) leads to a limit system of the form with a parameter \(y\) :

\[
\begin{cases}
\partial_t \bar{\eta} + \partial_x (\eta w) = 0, & (t, x, y) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}, \\
\eta (\partial_t w + \partial_x w) - \nu \partial_x^2 w - \partial_x p(\eta) = 0 & \text{with } \nu \overset{\text{def}}{=} \mu + \mu',
\end{cases}
\]

and

\[
\eta (\partial_t w + \partial_x w) - \mu \partial_x^2 w = 0, \quad (t, x, y) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}.
\]

Accordingly, the initial data are

\[
\bar{\eta}(0, x, y) = \bar{\xi}_0(x, y), \quad w(0, x, y) = w_0(x, y), \quad \bar{w}(0, x, y) = \bar{w}_0(x, y).
\]

We observe that the equations of \((\bar{\eta}, w)\) and the equation of \(\bar{w}\) are decoupled. Furthermore, \((\eta, w)\) satisfies one dimensional compressible Navier-Stokes equations with a parameter \(y\). We shall study in detail about this system in Section 3.

Before proceeding, we assume that

\[
\mathcal{F}_0 \leq \bar{\rho}_0 \leq \mathcal{F}_0, \quad (\partial_y^j (\bar{\rho}_0 - 1), \partial_y^j \bar{w}_0, \partial_y^j \bar{w}_0) \in (L^2_v \cap L^\infty_v)(H^k_0) \quad \text{with}
\]

\[
A_k(y) \overset{\text{def}}{=} \sum_{j=0}^2 \| \partial_y^j (\bar{\rho}_0 - 1, \bar{w}_0, \bar{w}_0) \|_{H^k_0}^2 \quad \text{and} \quad \bar{A}_k \overset{\text{def}}{=} \sup_{y \in \mathbb{R}} A_k(y), \quad 0 \leq k \leq 5,
\]

for some positive constants \(\mathcal{F}_0\) and \(\mathcal{F}_0\), and for \(j = 0, 1, 2\). Here the subscript \(h\) (resp. \(v\)) denotes the norm on \(\mathbb{T}_x\) (resp. \(\mathbb{R}_y\)). We assume moreover that

\[
\int_{\mathbb{T}} \bar{\rho}_0(x, y) \, dx = 1, \quad \int_{\mathbb{T}} (\bar{\rho}_0 w_0)(x, y) \, dx = \int_{\mathbb{T}} (\bar{\rho}_0 \bar{w}_0)(x, y) \, dx = 0, \quad \forall y \in \mathbb{R}.
\]

Our first result is concerned with the large time exponential decay for the solutions to the equations (1.6) and (1.7) with initial data (1.8).

**Theorem 1.1.** Let \((\bar{\rho}_0, \bar{w}_0, \bar{w}_0)\) satisfy (1.9) and (1.10). Then for each \(y \in \mathbb{R}\), the system (1.6)–(1.8) has a unique global-in-time strong solution \((\eta, w, \bar{w})\) \(\in C([0, \infty); H^2(\mathbb{T}))\) so that

\[
\eta \leq \eta(t, x, y) \leq \bar{\eta}, \quad \text{for some positive constants } \underline{\eta} \text{ and } \bar{\eta}.
\]
Moreover, there exist positive constants $C$ and $\alpha$ solely depending on $(a, \gamma, \nu, \varsigma_0, \varsigma_0, \bar{A}_4)$ such that for all $t \in \mathbb{R}_+$ and $y \in \mathbb{R}$
\[
\left(\|\eta - 1\|_{H^4_h} + \|\eta w\|_{H^5_h} + \|\eta w(t, \cdot, y)\|_{H^6_h} + \sum_{j=1}^2 \left(\|\partial_y^j \eta\|_{H^{3-j}_h} + \|\partial_y^j w(t, \cdot, y)\|_{H^{2-j}_h}\right)\right) \leq CA_5^4(y)e^{-\alpha t}.
\]

We point out that to prove Theorem 1.1, it is crucial to establish the related estimates for $(\eta, w)$ in (1.12). Indeed with thus obtained estimates for $(\eta, w)$, those estimates for $w$ follow immediately. The proof of Theorem 1.1 will be presented from Section 3 to Section 6.

The main result of this paper states as follows:

**Theorem 1.2.** Let $(s_0, w_0, \eta_0)$ satisfy (1.9) and (1.10). Then (1.1)–(1.3) has a unique global-in-time strong solution $(\rho_\varepsilon, u_\varepsilon) \in C([0, \infty); H^2(\mathbb{T} \times \mathbb{R}))$. Furthermore, let $(\eta, w, \eta_\varepsilon)$ be the global solution to (1.6–1.8) obtained in Theorem 1.1, we denote
\[
\begin{align*}
\bar{y}_\varepsilon &= \rho_\varepsilon - [\eta]_\varepsilon \quad \text{and} \quad R_\varepsilon = u_\varepsilon - ([w]_\varepsilon, [\eta]_\varepsilon)^T,
\end{align*}
\]
and the energy functional
\[
E_\varepsilon(t) \defeq \sup_{0 < t < T} \int_{\mathbb{T} \times \mathbb{R}} \left(\|R_\varepsilon\|_2^2 + |\bar{y}_\varepsilon|_2^2 + |\nabla R_\varepsilon|_2^2 + |\mathcal{D}_t R_\varepsilon|_2^2 + |\nabla \omega_\varepsilon|_2^2\right) \, dx \, dy \\
+ \int_0^T \int_{\mathbb{T} \times \mathbb{R}} \left(|\nabla R_\varepsilon|_2^2 + |\mathcal{D}_t R_\varepsilon|_2^2 + |\nabla \omega_\varepsilon|_2^2 + |\nabla \mathcal{D}_t R_\varepsilon|_2^2\right) \, dx \, dy \, dt,
\]
\[
\theta_\varepsilon(T) \defeq \sup_{0 < t < T} \|\bar{y}_\varepsilon(t)\|_{L^\infty(\mathbb{T} \times \mathbb{R})},
\]
where $\omega_\varepsilon = \text{curl} R_\varepsilon = \partial_y R^1_\varepsilon - \partial_x R^2_\varepsilon$, $\mathcal{D}_t R_\varepsilon = \partial_t R_\varepsilon + u_\varepsilon \cdot \nabla R_\varepsilon$. Then there exists a constant $C$ solely depending on $(a, \gamma, \nu, \mu, \mu', \varsigma_0, \varsigma_0, \|\mathcal{A}_5\|_{L^\infty(\mathbb{R})})$ such that
\[
E_\varepsilon(\infty) + \theta_\varepsilon^2(\infty) \leq C\varepsilon.
\]

We remark that the main idea used to prove Theorem 1.2 is to approximate 2D compressible Navier-Stokes equations with a slow variable via 1D compressible Navier-Stokes equations with a parameter. This is inspired by the study in [2, 3] where 3D incompressible Navier-Stokes equations with a slow variable can be well approximated by 2D Navier-Stokes equations which is globally well-posed.

Let us end this section with some notations that we shall use throughout this paper.

**Notations:** In the whole paper, we designate $\Omega \defeq \mathbb{T} \times \mathbb{R}$, $Q_T \defeq (0, T) \times \Omega$ and $Q_\infty \defeq (0, +\infty) \times \Omega$. We shall always denote $C$ to be a uniform constant which may vary from line to line, and $a \lesssim b$ means that $a \leq Cb$, and $a \sim b$ means that both $a \leq Cb$ and $b \leq Ca$ hold. And we use subscript $h$ (resp. $v$) to denote the norm taken on $\mathbb{T}_x$ (resp. $\mathbb{R}_y$).

2. Ideas and structure of the proof

In this section, we shall sketch the main ideas of the proof to Theorems 1.1 and 1.2.

For each fixed $y \in \mathbb{R}$, the global existence and uniqueness of strong solutions to 1D compressible Navier-Stokes equations (1.6) is well known (see for instance [14, 16, 24, 25, 26]). Although the results in the above references focus on domains either being the whole line $\mathbb{R}$ or being a bounded
interval, the well-posedness results can be easily modified to the torus $\mathbb{T}$. More precisely, under the assumptions that
\begin{align}
0 < \bar{s}_0 \leq s_0 \leq \bar{s}_0 < \infty, \quad s_0 - 1 \in H^1(\mathbb{T}), \quad w_0 \in H^1(\mathbb{T}),
\end{align}
(1.6)--(1.8) has a unique global solution $(\eta, w)$ so that for each $T > 0$
\begin{align}
0 < \bar{s}(T) \leq \eta \leq \bar{s}(T) < \infty, \quad \eta - 1 \in C([0, T]; H^1(\mathbb{T})), \quad \eta_t \in C([0, T]; L^2(\mathbb{T})),
\end{align}
\begin{align}
w \in C([0, T]; H^1(\mathbb{T})), \quad w \in L^2(0, T; H^2(\mathbb{T})), \quad \text{and} \quad w_t \in L^2(0, T; L^2(\mathbb{T})).
\end{align}

In general, the related estimates in (2.2) depend on the time interval $[0, T]$. We shall show below that such estimates hold uniformly in time. Actually, we shall show exponential decay-in-time estimates of the solutions. In the case when spatial domain is a bounded interval and with homogeneous boundary condition for $w$, Straškraba and Zlotnik [26] established exponential decay-in-time estimate for the $H^1$ norms of $\eta - 1$ and $w$. It is not obvious that the same exponential decay estimate holds for the case of torus uniformly in $y \in \mathbb{R}$. One difficulty lies in that, unlike the case in bounded domain with homogeneous boundary condition for $w$, we can not have a similar version of Poincaré inequality on torus $\mathbb{T}$. Another difficulty is due to the presence of the parameter $y \in \mathbb{R}$. We will have to show that the corresponding decay estimates are uniform with respect to $y \in \mathbb{R}$ and $(\partial_y^2 \eta, \partial_y^2 w)$, $j = 1, 2$, share the same decay estimate. To achieve this, we need to show the density function $\eta$ admits a uniform finite upper bound and a uniform positive lower bound with respect to $y \in \mathbb{R}$. In particular, in order to derive the uniform strictly positive lower bound, one needs to show that as time goes to infinity, the kinetic energy goes to zero and the integral of the pressure goes to a strictly positive number with a speed independent of $y \in \mathbb{R}$. The result in [26] can not be directly applied here. Instead, we shall first establish the uniform upper bound of the density by using the idea in [26]. Then we find that this is enough to derive the exponential decay for the basic energy, see Proposition 3.2. Since $(\partial_y^2 \eta, \partial_y^2 w)$, $j = 1, 2$, does not fulfill a typical 1D compressible Navier-Stokes system, there arise new difficulties in this procedure of deriving the exponential decay estimates for $(\partial_y^2 \eta, \partial_y^2 w)$, $j = 1, 2$. A further difficulty comes from the Gagliardo-Nirenberg interpolation inequality in $\Omega$. Since $\Omega = \mathbb{T} \times \mathbb{R}$ which is essentially bounded in $x$ variable, thus the classical Gagliardo-Nirenberg interpolation inequality reads ($p > 2$):
\begin{align}
\|f\|_{L^p(\Omega)} \leq C\|f\|_{L^2(\Omega)}^2 \|f\|_{H^1(\Omega)}^{1-\frac{2}{p}} \leq C\|f\|_{L^2(\Omega)} + C\|f\|_{L^2(\Omega)}^2 \|\nabla f\|_{L^2(\Omega)}^{1-\frac{2}{p}}, \quad \forall f \in H^1(\Omega).
\end{align}

However, it happens to us that we can control the integral of $\|\nabla f\|_{L^2(\Omega)}^2$ in time variable over $\mathbb{R}^+$ for some $f$, but not for the quantity $\|f\|_{L^2(\Omega)}^2$, see for instance (2.34) and (2.35) on the effective viscous flux in the proof of Lemma 2.3. So we need a modified version of Gagliardo-Nirenberg interpolation inequality in $\Omega$ which always involves the term $\|\nabla f\|_{L^2(\Omega)}$, which is Lemma 2.1 below.

In Section 3, we shall present the detailed decay-in-time estimates for solutions of (1.6)--(1.8).

**Proposition 2.1.** Under the assumptions of Theorem 1.1, (1.6)--(1.8) has a unique global solution $(\eta, w)$ and there exist positive constants $\overline{\eta}, \bar{s}, C$ and $\alpha$ solely depending on $(a, \gamma, \nu, \bar{s}_0, \bar{\zeta}_0, \bar{A}_4)$ such that (1.11) and
\begin{align}
(2.3) \quad \|\eta - 1(t, \cdot, y)\|_{H^k_{\bar{s}_0}} + \|w(t, \cdot, y)\|_{H^k_{\bar{s}_0}} + \sum_{i=1}^2 \|\partial_y^i \eta, \partial_y^i w(t, \cdot, y)\|_{H^{k-2i}_{\bar{s}_0}} \leq C\bar{A}_{e}^\frac{1}{2}(y)e^{-\alpha t}.
\end{align}

hold for any $t \in \mathbb{R}_+$ and $y \in \mathbb{R}$.

To avoid notational complexity, we shall denote $E_y(y) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ to a universal function, which is determined by the initial conditions. The positive constants $C$ and $\alpha$ are also determined by the initial conditions. From now on, we shall not point out the precise dependence of $E_y(y)$ and the constants $C, \alpha$, and we neglect the dependence on the $y$ variable.
Proposition 2.2. Under the assumptions of Theorem 1.1, there exist positive constants $C$ and $\alpha$ so that the global solution $(\eta, w)$ of (1.6)–(1.8) verifies
\[
\|\eta_y(t)\|_{H^1_0} + \|w_y(t)\|_{H^1_0} + \|\eta_{yy}w_y(t)\|_{H^1_0} \leq CE^{\frac{1}{2}}(y)e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+, \ y \in \mathbb{R}.
\]

Proposition 2.3. Under the assumptions of Theorem 1.1, there exist positive constants $C$ and $\alpha$ so that the global solution $(\eta, w)$ of (1.6)–(1.8) fulfills
\[
\|\eta_{yy}(t)\|_{H^1_0} + \|w_{yy}(t)\|_{H^1_0} + \|\eta_{yy}w_y(t)\|_{L^2_0} \leq E^{\frac{1}{2}}(y)e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+, \ y \in \mathbb{R}.
\]

Proposition 2.4. Under the assumptions of Theorem 1.1, (1.7)–(1.8) has a unique global solution $\mathbf{w}$ and there exists positive constants $C$ and $\alpha$ so that for all $t \in \mathbb{R}^+$ and $y \in \mathbb{R}$
\[
\begin{align*}
\sum_{j=1}^{2} (\|\eta^{\prime j}_y\|_{H^{2-j}} + \|\eta^{\prime j}_w\|_{H^{2-j}} + \|\eta^{\prime j}w_t\|_{H^{2-j}}) & \leq CE^{\frac{1}{2}}(y)e^{-\alpha t} \quad \forall t \in \mathbb{R}^+, \ y \in \mathbb{R}.
\end{align*}
\]

The proof of Proposition 2.2 will be presented in Section 4. While we shall outline the proof of Propositions 2.3 and 2.4 in Sections 5 and 6 respectively. By combing Propositions 2.1–2.4, we conclude the proof of Theorem 1.1.

Let $(\eta, w, \mathbf{w})$ be the unique global solution of (1.6)–(1.8), which has been constructed in Theorem 1.1. We define
\[
\rho_\varepsilon^a \overset{\text{def}}{=} [\eta]_\varepsilon, \quad u_\varepsilon^a \overset{\text{def}}{=} ([w]_\varepsilon, [\mathbf{w}]_\varepsilon)^T.
\]
Then in view of (1.6) and (1.7), we have
\[
\begin{align*}
\partial_t \rho_\varepsilon^a + \text{div} (\rho_\varepsilon^a u_\varepsilon^a) &= \varepsilon[\eta\mathbf{w}]_\varepsilon, \\
\rho_\varepsilon^a(\partial_t u_\varepsilon^a + u_\varepsilon^a \cdot \nabla u_\varepsilon^a) - \mu \Delta u_\varepsilon^a - \mu \varepsilon \nabla \text{div} u_\varepsilon^a + \nabla p(\rho_\varepsilon^a) &= -G_\varepsilon,
\end{align*}
\]
where $G_\varepsilon = (G_{1,\varepsilon}, G_{2,\varepsilon})^T$ is given by
\[
\begin{align*}
G_{1,\varepsilon} & \overset{\text{def}}{=} -\varepsilon[\eta \mathbf{w} w_y]_\varepsilon + \mu \varepsilon^2[w_{yy}]_\varepsilon + \mu \varepsilon[w_{xy}]_\varepsilon, \\
G_{2,\varepsilon} & \overset{\text{def}}{=} -\varepsilon[\eta \mathbf{w} w_y]_\varepsilon + \varepsilon \varepsilon^2[w_{yy}]_\varepsilon + \mu \varepsilon[w_{xy}]_\varepsilon - \varepsilon[p(\eta)]_\varepsilon.
\end{align*}
\]
Then by virtue of (1.12), we deduce that
\[
\|\partial_t G_\varepsilon\|_{L^2(\Omega)} \leq CE^{\frac{1}{2}}e^{-\alpha t}, \quad \|G_\varepsilon\|_{L^2(\Omega)} \leq CE^{\frac{1}{2}}e^{-\alpha t}, \quad \|G_\varepsilon\|_{L^\infty(\Omega)} \leq CE^{-\alpha t}.
\]

On the other hand, with initial data given by (1.3) and $(\zeta_0, w_0, \mathbf{w}_0)$ satisfying (1.9) and (1.10), there exists a positive time $T_\varepsilon^* > 0$ so that (1.1)–(1.3) has a unique solution $(\rho_\varepsilon, u_\varepsilon)$ on $[0, T_\varepsilon^*)$ and for any $T < T_\varepsilon^*$,
\[
\begin{align*}
0 < \rho_\varepsilon - \tilde{\rho}(T) & \leq \rho_\varepsilon < \infty, \quad \rho_\varepsilon \in C([0, T]; H^2(\Omega)), \quad \partial_t \rho_\varepsilon \in C([0, T]; H^1(\Omega)), \\
u_\varepsilon & \in C([0, T]; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)), \quad \partial_t \rho_\varepsilon \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).
\end{align*}
\]
One may check [21, 23] for details.

Let $T_\varepsilon^*$ be the maximal existence time of $(\rho_\varepsilon, u_\varepsilon)$ so that (2.11) holds. We are going to prove that $T_\varepsilon^* = \infty$ for $\varepsilon$ being sufficiently small. In order to do so, we define the remaining term
\[
\begin{align*}
\varrho_\varepsilon \overset{\text{def}}{=} \rho_\varepsilon^a - \rho_\varepsilon^a \quad \text{and} \quad R_\varepsilon \overset{\text{def}}{=} u_\varepsilon^a - u_\varepsilon^a.
\end{align*}
\]
Recall that $D_t \overset{\text{def}}{=} \partial_t + u_\varepsilon \cdot \nabla$. Then it follows from (1.1) and (2.8) that
\[
\begin{align*}
\begin{cases}
\partial_t \varrho_\varepsilon + \text{div} (\varrho_\varepsilon R_\varepsilon) + \text{div} (\varrho_\varepsilon^a u_\varepsilon^a) = -\varepsilon[\eta \mathbf{w}]_\varepsilon, \\
\varrho_\varepsilon D_t R_\varepsilon - \mu \Delta R_\varepsilon - \mu \varepsilon \nabla \text{div} R_\varepsilon + \nabla (p(\rho_\varepsilon) - p(\rho_\varepsilon^a)) + \rho_\varepsilon R_\varepsilon \cdot \nabla u_\varepsilon^a + \varrho_\varepsilon (\partial_t u_\varepsilon^a + u_\varepsilon^a \cdot \nabla u_\varepsilon^a) = G_\varepsilon, \\
\varrho_\varepsilon(0, x, y) = 0, \quad R_\varepsilon(0, x, y) = 0.
\end{cases}
\end{align*}
\]
We define
\begin{equation}
T^*_{\varepsilon} \text{ def } \sup \{ T < T^*_{\varepsilon}; \ \theta_{\varepsilon}(T) = \sup_{0 < t < T} \| \theta_{\varepsilon} \|_{L^\infty(\Omega)} \leq \frac{1}{2} \min \{ 1, \eta \} \}.
\end{equation}

It is easy to observe that for all \( t < T^*_{\varepsilon} \):
\begin{equation}
0 < \eta/2 \leq \rho_{\varepsilon} \leq \bar{\eta} + \eta/2 < \infty.
\end{equation}

We shall first derive the basic energy estimate for \((\rho_{\varepsilon}, R_{\varepsilon})\) for \( t < T^*_{\varepsilon} \). A standard way to perform this estimate is to test the momentum type equation in (2.13) by \( R_{\varepsilon} \) and using integration by parts. A tricky term to handle is the one related to the pressure:
\begin{equation}
\int_{\Omega} (p(\rho_{\varepsilon}) - p(\rho^*_{\varepsilon})) \text{div } R_{\varepsilon} \ dx \ dy.
\end{equation}

Similar term \( p(\rho_{\varepsilon}) \text{div } u_{\varepsilon} \) appears in the renormalized formulation to the continuity equation of the compressible Navier-Stokes equations:
\[
\rho_{\varepsilon}(t) + \text{div} (P(\rho_{\varepsilon})u_{\varepsilon}) + (\rho_{\varepsilon}P'(\rho_{\varepsilon}) - P(\rho_{\varepsilon})) \text{div } u_{\varepsilon} = 0,
\]
where \( P(\rho) \) is called the pressure potential, which is determined by
\[
\rho P'(\rho) - P(\rho) = p(\rho).
\]

As a result, it comes out
\[
\int_{\Omega} p(\rho_{\varepsilon}) \text{div } u_{\varepsilon} \ dx \ dy = - \frac{d}{dt} \int_{\Omega} P(\rho_{\varepsilon}) \ dx \ dy.
\]

In particular, for our case \( p(\rho) = a\rho^\gamma \) with \( \gamma \geq 1 \), the corresponding pressure potential is
\begin{equation}
P(\rho) \text{ def } \frac{1}{\gamma - 1} \rho^\gamma \text{ if } \gamma > 1, \quad P(\rho) \text{ def } a(\rho \log \rho + 1) \text{ if } \gamma = 1.
\end{equation}

However, it is not clear at a first glance how to use such an argument to deal with the term (2.16). Here we shall employ the well-known relative entropy inequality to derive the basic energy estimate for \( \theta_{\varepsilon} \) and \( R_{\varepsilon} \). Relative entropy inequality is widely used in the study of uniqueness and stability for Navier–Stokes equations and some related models, see [7, 10] for compressible Navier–Stokes equations, [8] for compressible Navier–Stokes–Fourier equations, and [18] for a compressible Oldroyd model. The result states as follows:

**Proposition 2.5.** Let \((\rho_{\varepsilon}, u_{\varepsilon})\) be the local-in-time strong solution of (1.1)-(1.3) satisfying (2.11). For each pair \((\bar{\rho}, \bar{u})\) satisfying the same regularity assumption as that of \((\rho_{\varepsilon}, u_{\varepsilon})\) listed in (2.11), we define the relative energy functional
\begin{equation}
E_1((\rho_{\varepsilon}, u_{\varepsilon})(\bar{\rho}, \bar{u}))(t) \text{ def } \int_{\Omega} \left( \frac{1}{2} \rho_{\varepsilon} | u_{\varepsilon} - \bar{u} |^2 + P(\rho_{\varepsilon}) - P(\bar{\rho}) - P'(\bar{\rho})(\rho_{\varepsilon} - \bar{\rho}) \right) \ dx \ dy,
\end{equation}

where the pressure potential \( P \) is defined by (2.17). Then for any \( t \in (0, T^*_{\varepsilon}) \), the following relative entropy equality holds
\begin{equation}
E_1((\rho_{\varepsilon}, u_{\varepsilon})(\bar{\rho}, \bar{u}))(t) + \int_0^t \int_{\Omega} \left( \mu | \nabla (u_{\varepsilon} - \bar{u}) |^2 + \mu' | \text{div} (u_{\varepsilon} - \bar{u}) |^2 \right) \ dx \ dy \ dt' = E_1((\rho_{\varepsilon}, u_{\varepsilon})|_{t=0}(\bar{\rho}, \bar{u})|_{t=0}) + \int_0^t R(\varepsilon)(t') \ dt',
\end{equation}

where
\begin{equation}
R(\varepsilon)(t) \text{ def } \int_{\Omega} \rho_{\varepsilon} D_t \bar{u} \cdot (\bar{u} - u_{\varepsilon}) \ dx \ dy + \int_{\Omega} (\mu \nabla \bar{u} : \nabla (\bar{u} - u_{\varepsilon}) + \mu' \text{div} \bar{u} \text{div} (\bar{u} - u_{\varepsilon})) \ dx \ dy
\end{equation}

\begin{equation}
+ \int_{\Omega} ((\bar{\rho} - \rho_{\varepsilon}) \partial_t P'(\bar{\rho}) + (\bar{\rho} u_{\varepsilon} - \rho_{\varepsilon} \bar{\rho}) \cdot \nabla P'(\bar{\rho})) \ dx \ dy - \int_{\Omega} \text{div} \bar{u}(p(\rho_{\varepsilon}) - p(\bar{\rho})) \ dx \ dy.
\end{equation}
There are also similar relative entropy inequalities related to finite energy weak solutions of different models: for instance, Theorem 2.4 of [7] for the compressible Navier–Stokes equations and Proposition 5.3 of [18] for a compressible Oldroyd-B model. The proof of Proposition 2.5 follows the same line as the argument in [7] or [18], and we skip the details here.

Thanks to Proposition 2.5, we shall prove in Section 7.1 that

**Proposition 2.6.** Let $T^*_\varepsilon$ be given by (2.14). Then for all $t < T^*_\varepsilon$, one has

\[
\int_\Omega (|\rho_\varepsilon|^2 + \rho_\varepsilon^2)\,d\Omega \leq C \varepsilon, \quad \text{where the positive constants C is independent of } T^*_\varepsilon.
\]

In Section 7.4, we shall prove the energy estimate for the derivatives of $R_\varepsilon$.

**Proposition 2.7.** Let the energy functional $E_\varepsilon(T)$ be given by (1.14). Then for all $T \leq T^*_\varepsilon$, one has

\[
E_\varepsilon(T) \leq C \varepsilon + C \int_0^T \int_\Omega (|\nabla R_\varepsilon|^3 + |\nabla R_\varepsilon|^4) \,d\Omega \,dt.
\]

In order to close the energy estimate, (2.22), we need to handle the estimates of the cubic and quadratic terms:

\[
\int_0^T \int_\Omega |\nabla R_\varepsilon|^3 \,d\Omega \,dt \quad \text{and} \quad \int_0^T \int_\Omega |\nabla R_\varepsilon|^4 \,d\Omega \,dt.
\]

Here we introduce the following refined Gagliardo-Nirenberg interpolation inequality in $\Omega$:

**Lemma 2.1.** Let $2 < p < \infty$, there exists a constant $C$ depending solely on $p$ such that for all $f \in H^1(\Omega)$ there holds

\[
\|f\|_{L^p(\Omega)} \leq C \left( \|f\|_{L^2(\Omega)} \|\nabla f\|_{L^2(\Omega)}^{\frac{2}{p}} + \|f\|_{L^2(\Omega)} \|\nabla f\|_{L^2(\Omega)}^{\frac{2}{p}} \right).
\]

The proof of Lemma 2.1 will be presented in Appendix B.

Motivated by [15, 19], we define the effective viscous flux $\mathfrak{F}_\varepsilon$ for the system (2.13) as follows

\[
\mathfrak{F}_\varepsilon \overset{\text{def}}{=} \nu \text{div } R_\varepsilon - (p(\rho_\varepsilon) - p(\rho_\varepsilon^a)).
\]

**Lemma 2.2.** For $T \leq T^*_\varepsilon$, one has

\[
\int_0^T \int_\Omega \rho_\varepsilon^a \text{dy} \,dt \leq C \varepsilon^3 + C \varepsilon^2 E_\varepsilon(T) + C \int_0^T \int_\Omega |\mathfrak{F}_\varepsilon| \,d\Omega \,dt.
\]

**Proof.** In view of the continuity equations of (1.1) and (2.13), we write

\[
\mathcal{D}_t (\log \rho_\varepsilon - \log \rho_\varepsilon^a) + R_\varepsilon \cdot \nabla \log \rho_\varepsilon^a + \text{div } R_\varepsilon = -\varepsilon (\rho_\varepsilon^a)^{-1} [(\eta \mathfrak{F})]_\varepsilon,
\]

which together with (2.24) implies

\[
\mathcal{D}_t (\log \rho_\varepsilon - \log \rho_\varepsilon^a) = -R_\varepsilon \cdot \nabla \log \rho_\varepsilon^a - \nu^{-1} (\mathfrak{F}_\varepsilon + (p(\rho_\varepsilon) - p(\rho_\varepsilon^a))) - \varepsilon (\rho_\varepsilon^a)^{-1} [(\eta \mathfrak{F})]_\varepsilon.
\]

Multiplying the above equation by $6(\log \rho_\varepsilon - \log \rho_\varepsilon^a)^5$ and then applying Young’s inequality gives

\[
\mathcal{D}_t (\log \rho_\varepsilon - \log \rho_\varepsilon^a)^6 + 6 \nu^{-1} (p(\rho_\varepsilon) - p(\rho_\varepsilon^a)) (\log \rho_\varepsilon - \log \rho_\varepsilon^a)^5
\]

\[\leq \delta (|R_\varepsilon \cdot \nabla \log \rho_\varepsilon^a|^2 + |\mathfrak{F}_\varepsilon|^2 + |(\eta \mathfrak{F})]_\varepsilon| + \delta |\log \rho_\varepsilon - \log \rho_\varepsilon^a|^6.
\]

Notice that for $t < T^*_\varepsilon$, there holds (2.15) so that

\[
6 \nu^{-1} (p(\rho_\varepsilon) - p(\rho_\varepsilon^a)) (\log \rho_\varepsilon - \log \rho_\varepsilon^a)^5 \geq 2C\delta |\log \rho_\varepsilon - \log \rho_\varepsilon^a|^6 \geq \delta |\log \rho_\varepsilon - \log \rho_\varepsilon^a|^5 + \delta (\rho_\varepsilon - \rho_\varepsilon^a)^6.
\]
for some small positive constant $\delta$ and some $C \geq 2$. We thus obtain
\begin{equation}
(2.26) \quad \mathcal{D}_t \left( \log \rho_\varepsilon - \log \rho_0^a \right)^6 + \delta \left( \rho_\varepsilon - \rho_0^a \right)^6 \leq C_\delta \left( |R_\varepsilon \cdot \nabla \log \rho_\varepsilon^a|^6 + |\varepsilon \rho_\varepsilon^a|^{-1}[(\eta \varepsilon \omega)_y]|^6 \right).
\end{equation}

Let us introduce the particle trajectory $X_\varepsilon(t, x, y)$ of $u_\varepsilon$ via
\begin{equation}
(2.27) \quad \frac{d}{dt} X_\varepsilon(t, x, y) = u_\varepsilon(t, X_\varepsilon(t, x, y)); \quad X_\varepsilon(0, x, y) = (x, y).
\end{equation}

Then we deduce from (2.26) that
\begin{equation}
(2.28) \quad \frac{d}{dt} \left( \log \rho_\varepsilon - \log \rho_0^a \right)^6(t, X_\varepsilon(t, x, y)) + \delta \left( \rho_\varepsilon - \rho_0^a \right)^6(t, X_\varepsilon(t, x, y)) \leq C_\delta \left( |R_\varepsilon \cdot \nabla \log \rho_\varepsilon^a|^6 + |\varepsilon \rho_\varepsilon^a|^{-1}[(\eta \varepsilon \omega)_y]|^6 \right)(t, X_\varepsilon(t, x, y)).
\end{equation}

Let $T \leq T^*_\varepsilon$. Integrating the above inequality over $(0, T) \times \Omega$ yields
\begin{equation}
(2.29) \quad C^{-1} \int_{\Omega} g(t, x, y) dx dy \leq \int_{\Omega} g(t, X_\varepsilon(t, x, y)) dx dy \leq C \int_{\Omega} g(t, x, y) dx dy.
\end{equation}

We thus deduce from (1.11) and (2.29) that
\begin{equation}
(3.30) \quad C^{-1} \int_{\Omega} \left| \int_{0}^{T} \left( \rho_\varepsilon - \rho_0^a \right)^6(t, x, y) dx dy dt \right| \leq C \int_{\Omega} \left| \int_{0}^{T} |\varepsilon \rho_\varepsilon^a|^{-1}[(\eta \varepsilon \omega)_y]|^6(t, x, y) dx dy dt \right|.
\end{equation}

It follows from Theorem 1.1, Proposition 2.6 and Lemma 2.1 that
\begin{align*}
&\int_{0}^{T} \int_{\Omega} \left| \nabla \log \rho_\varepsilon^a \right|^6(t, x, y) dx dy dt \\
&\leq C \int_{0}^{T} \int_{\Omega} \left| \nabla \log \rho_\varepsilon^a \right|^6 dx dy dt \\
&\leq C \int_{0}^{T} \sup_{0 \leq t \leq T} \left( \int_{\Omega} \left| \varepsilon \rho_\varepsilon^a \right|^6 dx dy dt \right) e^{-\alpha t} \int_{0}^{T} \left| \varepsilon \rho_\varepsilon^a \right|^6 dx dy dt \\
&\leq C \varepsilon^2 (E_\varepsilon(T) + \varepsilon),
\end{align*}

and
\begin{align*}
\varepsilon^6 \int_{0}^{T} \int_{\Omega} \left| (\varepsilon \rho_\varepsilon^a)^{-1}[(\eta \varepsilon \omega)_y]|^6(t, x, y) dx dy dt \right| &\leq \eta^{-1} \varepsilon^6 \int_{0}^{T} \int_{\Omega} \left| \varepsilon^{-1}[(\eta \varepsilon \omega)_y]|^6 \right| dx dy dt \\
&\leq C \varepsilon^5 \int_{0}^{T} e^{-\alpha t} dt \leq C \varepsilon^5.
\end{align*}
By inserting the above estimates into (2.30), we conclude the proof of (2.25).

\[\int_0^T \int_\Omega |\nabla R_\varepsilon|^6 \, dx \, dy \, dt \leq C \varepsilon^3 + C \varepsilon^2 E_\varepsilon(T) + C \varepsilon E_\varepsilon^2(T).\]

**Lemma 2.3.** For $T \leq T^*_{\varepsilon}$, there holds

\[\int_0^T \int_\Omega |\nabla R_\varepsilon|^6 \, dx \, dy \, dt \leq C \varepsilon^3 + C \varepsilon^2 E_\varepsilon(T) + C \varepsilon E_\varepsilon^2(T).\]

**Proof.** Recall that $\omega_\varepsilon = \partial_y R_\varepsilon^1 - \partial_x R_\varepsilon^2$, we observe that

$$\Delta R_\varepsilon = \nabla \text{div} \, R_\varepsilon + \nabla^\perp \omega_\varepsilon, \quad \nabla^\perp \Xi = \left( \frac{\partial_y}{-\partial_x} \right),$$

which together with (2.24) implies

$$\nu \Delta R_\varepsilon = \nabla \tilde{\omega}_\varepsilon + \nabla \left( p(\rho_\varepsilon) - p(\rho_\varepsilon^0) \right) + \nu \nabla^\perp \omega_\varepsilon.$$

Then we get, by using standard elliptic estimates that

\[\|\nabla R_\varepsilon\|_{L^6(\Omega)} \leq C \left( \|\omega_\varepsilon\|_{L^6(\Omega)} + \|\tilde{\omega}_\varepsilon\|_{L^6(\Omega)} + \|p(\rho_\varepsilon) - p(\rho_\varepsilon^0)\|_{L^6(\Omega)} \right).\]

Next let us estimate term by term on the right-hand side of above equation.

By applying Lemma 2.1, we obtain

$$\|\omega_\varepsilon\|_{L^6(\Omega)} \leq C \left( \|\omega_\varepsilon\|_{L^2(\Omega)}^\frac{1}{2} \|\nabla \omega_\varepsilon\|_{L^2(\Omega)}^\frac{1}{2} + \|\omega_\varepsilon\|_{L^2(\Omega)}^\frac{1}{2} \|\nabla \omega_\varepsilon\|_{L^2(\Omega)}^\frac{1}{2} \right),$$

Together with (1.14) and (2.21), we infer

\[\int_0^T \int_\Omega |\omega_\varepsilon|^6 \, dx \, dy \, dt \leq C \int_0^T \left( \|\omega_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla \omega_\varepsilon\|_{L^2(\Omega)}^2 + \|\omega_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla \omega_\varepsilon\|_{L^2(\Omega)}^2 \right) \, dt \leq C \sup_{0 < t < T} \left( \|\nabla \omega_\varepsilon\|_{L^2(\Omega)}^4 + \|\omega_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla \omega_\varepsilon\|_{L^2(\Omega)}^2 \right) \int_0^T \|\nabla R_\varepsilon\|_{L^2(\Omega)}^2 \, dt \leq C \varepsilon E_\varepsilon^2(T).\]

While in view of (2.21) and (2.24), we have

\[\|\omega_\varepsilon\|_{L^2(\Omega)} \leq \nu \|\nabla R_\varepsilon\|_{L^2(\Omega)} + C \|\rho_\varepsilon - \rho_\varepsilon^0\|_{L^2(\Omega)} \leq C \left( \|\nabla R_\varepsilon\|_{L^2(\Omega)} + \varepsilon \frac{1}{2} \right).\]

By (2.24) and the $R_\varepsilon$ equation of (2.13), we infer

$$\Delta \tilde{\omega}_\varepsilon = \text{div} \left( \mu \Delta R_\varepsilon + \mu \nabla \text{div} \, R_\varepsilon - \nabla \left( p(\rho_\varepsilon) - p(\rho_\varepsilon^0) \right) \right)$$

$$= \text{div} \left( \rho_\varepsilon \partial_t R_\varepsilon + \rho_\varepsilon R_\varepsilon \cdot \nabla u_\varepsilon + \rho_\varepsilon (\partial_t u_\varepsilon^a + u_\varepsilon^a \cdot \nabla u_\varepsilon^a) - \rho_\varepsilon G_\varepsilon \right),$$

from which, we infer

$$\|\nabla \tilde{\omega}_\varepsilon\|_{L^2(\Omega)} \leq C \left( \|\partial_t R_\varepsilon\|_{L^2(\Omega)} + \|R_\varepsilon\|_{L^2(\Omega)} \|\nabla u_\varepsilon^a\|_{L^\infty(\Omega)} \right.$$

$$\left. + \|\rho_\varepsilon\|_{L^2(\Omega)} \|\partial_t u_\varepsilon^a + u_\varepsilon^a \cdot \nabla u_\varepsilon^a\|_{L^\infty(\Omega)} + \|G_\varepsilon\|_{L^2(\Omega)} \right)$$

\[\leq C \left( \|\partial_t R_\varepsilon\|_{L^2(\Omega)} + \varepsilon \frac{1}{2} \right) + C e^{-at} \left( \|R_\varepsilon\|_{L^2(\Omega)} + \|\rho_\varepsilon\|_{L^2(\Omega)} + \varepsilon \frac{1}{2} \right) \leq C \left( \|\partial_t R_\varepsilon\|_{L^2(\Omega)} + \varepsilon \frac{1}{2} \right).\]

Thanks to (2.34) and (2.35), we get, by applying Lemma 2.1, that

\[\int_0^T \int_\Omega |\tilde{\omega}_\varepsilon|^6 \, dx \, dy \, dt \leq C \int_0^T \left( \|\tilde{\omega}_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla \tilde{\omega}_\varepsilon\|_{L^2(\Omega)}^2 + \|\tilde{\omega}_\varepsilon\|_{L^2(\Omega)}^2 \|\nabla \tilde{\omega}_\varepsilon\|_{L^2(\Omega)}^2 \right) \, dt \leq C \left( \|\nabla R_\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \right) \left( \|\partial_t R_\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon e^{-at} \right) \times \left( \|\nabla R_\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon + \|\partial_t R_\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon e^{-at} \right) \, dt \leq C \varepsilon^3 + C \varepsilon E_\varepsilon^2(T).\]
By inserting the estimates (2.25), (2.33) and (2.36) into (2.32), we achieve (2.31). This completes the proof of Lemma 2.3.

Now we are ready to give the following lemma in order to close the energy estimates:

**Lemma 2.4.** For $T \leq T^*_\varepsilon$, there holds

\[
\begin{align*}
\int_0^T \int_\Omega |\nabla R_\varepsilon|^4 \, dx \, dy \, dt &\leq C\varepsilon^2 + C\varepsilon E_\varepsilon(T), \\
\int_0^T \int_\Omega |\nabla R_\varepsilon|^3 \, dx \, dy \, dt &\leq C\varepsilon^{3/4} + C\varepsilon^{1/2} E_\varepsilon(T).
\end{align*}
\]

**Proof.** Indeed it follows from Proposition 2.6, Lemma 2.3 and Hölder’s inequality that

\[
\begin{align*}
\|\nabla R_\varepsilon\|_{L^4((0,T) \times \Omega)}^4 &\leq \|\nabla R_\varepsilon\|_{L^2((0,T) \times \Omega)} \|\nabla R_\varepsilon\|_{L^6((0,T) \times \Omega)}^3 \\
&\leq C\varepsilon^{1/4} (\varepsilon^{3/2} + \varepsilon^{1/2} E_\varepsilon(T)), \quad \|\nabla R_\varepsilon\|_{L^3((0,T) \times \Omega)}^3 &\leq \|\nabla R_\varepsilon\|_{L^6((0,T) \times \Omega)}^2 \|\nabla R_\varepsilon\|_{L^6((0,T) \times \Omega)} \\
&\leq C\varepsilon^{3/4} (\varepsilon^{3/2} + \varepsilon^{1/2} E_\varepsilon(T)).
\end{align*}
\]

Then (2.37) follow immediately.

Next let us turn to the estimate of $\theta_\varepsilon(T)$.

**Lemma 2.5.** For $T \leq T^*_\varepsilon$, there holds

\[
\theta_\varepsilon(T) \leq C\varepsilon + CE_\varepsilon(T).
\]

**Proof.** Notice that for all $T \leq T^*_\varepsilon$, $0 < \eta/2 \leq \rho_\varepsilon, \rho_\varepsilon^0 \leq 3\eta/2$, we deduce from (2.28) that

\[
\begin{align*}
\varrho_\varepsilon^6(T, X_\varepsilon(T,x,y)) &\leq C (\log \rho_\varepsilon - \log \rho_\varepsilon^0)^6(T, X_\varepsilon(T,x,y)) \\
&\leq C \int_0^T \left( |R_\varepsilon \cdot \nabla \log \rho_\varepsilon^0 + |\mathfrak{F}_\varepsilon|^6 + |\varepsilon (\eta \omega_\varepsilon y)\varepsilon|^6 \right) (t, X_\varepsilon(t,x,y)) \, dt,
\end{align*}
\]

from which, we infer

\[
\begin{align*}
\|\varrho_\varepsilon(T)\|_{L^\infty(\Omega)} &\leq C \int_0^T (\|R_\varepsilon\|_{L^\infty(\Omega)} (\rho_\varepsilon^0)^{-1} \|\nabla \rho_\varepsilon^0\|_{L^\infty(\Omega)} + \|\mathfrak{F}_\varepsilon\|_{L^\infty(\Omega)} + \varepsilon \|\eta \omega_\varepsilon y\|_{L^\infty(\Omega)}) \, dt \leq C \int_0^T (e^{-\alpha t} (\|R_\varepsilon\|_{L^\infty(\Omega)} + \varepsilon^6) + \|\mathfrak{F}_\varepsilon\|_{L^\infty(\Omega)}) \, dt.
\end{align*}
\]

It follows from Sobolev embedding theorem, Proposition 2.6 and Lemma 2.3 that

\[
\begin{align*}
\int_0^T e^{-\alpha t} \|R_\varepsilon\|_{L^\infty(\Omega)} \, dt &\leq C \int_0^T e^{-\alpha t} (\|R_\varepsilon\|_{L^2(\Omega)} + \|\nabla R_\varepsilon\|_{L^6(\Omega)}) \, dt \leq C\varepsilon^3 + C\varepsilon E_\varepsilon^2(T).
\end{align*}
\]

Concerning the term related to the effective viscous flux, we get, by using Sobolev embedding inequality, that

\[
\|\mathfrak{F}_\varepsilon\|_{L^\infty(\Omega)} \leq C (\|\mathfrak{F}_\varepsilon\|_{L^6(\Omega)} + \|\nabla \mathfrak{F}_\varepsilon\|_{L^1(\Omega)}).
\]

The estimate related to $\|\mathfrak{F}_\varepsilon\|_{L^6(\Omega)}$ is given in (2.36). While for $\|\nabla \mathfrak{F}_\varepsilon\|_{L^1(\Omega)}$, it follows from a similar derivation of (2.35) that

\[
\|\nabla \mathfrak{F}_\varepsilon\|_{L^3(\Omega)} \leq C \left( \|\mathfrak{D}_t R_\varepsilon\|_{L^3(\Omega)} + \|R_\varepsilon\|_{L^3(\Omega)} \|\nabla u_\varepsilon^0\|_{L^\infty(\Omega)} + \|\varrho_\varepsilon\|_{L^3(\Omega)} \|\partial_t u_\varepsilon^0 + u_\varepsilon^0 \cdot \nabla u_\varepsilon^0\|_{L^\infty(\Omega)} + \|G_\varepsilon\|_{L^3(\Omega)} \right) \leq C \|\mathfrak{D}_t R_\varepsilon\|_{L^3(\Omega)} + C e^{-\alpha t} (\|R_\varepsilon\|_{L^3(\Omega)} + \|\varrho_\varepsilon\|_{L^3(\Omega)} + \varepsilon^2).
\]
where we used the estimate of $G_\varepsilon$ in (2.10). By using the Gagliardo-Nirenberg interpolation inequality and (1.14), we find

$$
\int_0^T \| \mathcal{D}_t R_\varepsilon \|^6_{L^3(\Omega)} \, dt \leq C \int_0^T \| \mathcal{D}_t R_\varepsilon \|_{L^2(\Omega)}^4 \| \mathcal{D}_t R_\varepsilon \|_{H^1(\Omega)}^2 \, dt \\
\leq C \sup_{0 < t < T} \| \mathcal{D}_t R_\varepsilon(t) \|_{L^2(\Omega)}^4 \int_0^T \| \mathcal{D}_t R_\varepsilon \|_{H^1(\Omega)}^2 \, dt \leq C E_\varepsilon^3(T).
$$

Similarly, we have

$$
\int_0^T e^{-\alpha t} \| R_\varepsilon \|^6_{L^3(\Omega)} \, dt \leq C \int_0^T e^{-\alpha t} \| R_\varepsilon \|^4_{L^2(\Omega)} \| R_\varepsilon \|^2_{H^1(\Omega)} \, dt \\
\leq C \sup_{0 < t < T} \| R_\varepsilon(t) \|^4_{L^2(\Omega)} \int_0^T e^{-\alpha t} \| R_\varepsilon \|^2_{H^1(\Omega)} \, dt \leq C \varepsilon_3^3.
$$

Finally, in view of (2.14), one has

$$
\| \varrho_\varepsilon(t) \|_{L^3(\Omega)} \leq \| \varrho_\varepsilon(t) \|_{L^2(\Omega)}^2 \| \varrho_\varepsilon(t) \|_{L^\infty(\Omega)}^{\frac{3}{4}} \leq C \varepsilon_3^\frac{1}{4} \vartheta_\varepsilon^\frac{1}{2}(t),
$$

so that

$$
\int_0^T e^{-\alpha t} \| \varrho_\varepsilon(t) \|^6_{L^3(\Omega)} \, dt \leq C e^2 \int_0^T e^{-\alpha t} \vartheta_\varepsilon^2(t) \, dt \leq C e^2 \vartheta_\varepsilon^2(T).
$$

As a result,

$$
\int_0^T \| \nabla \varphi_\varepsilon(t) \|^6_{L^3(\Omega)} \, dt \leq C \left( E_\varepsilon^3(T) + \varepsilon_3^2 \vartheta_\varepsilon^2(T) + \varepsilon_3^3 \right),
$$

which together with (2.36) and (2.41) ensures that

$$
\int_0^T \| \varphi_\varepsilon(t) \|^6_{L^\infty(\Omega)} \, dt \leq C \int_0^T \left( \| \varphi_\varepsilon \|^6_{L^3(\Omega)} + \| \nabla \varphi_\varepsilon \|^6_{L^3(\Omega)} \right) \, dt \\
\leq C \left( E_\varepsilon^3(T) + \varepsilon_3^2 \vartheta_\varepsilon^2(T) + \varepsilon_3^3 \right).
$$

By inserting the estimates (2.40) and (2.42) into (2.39), we arrive at

$$
\vartheta_\varepsilon^6(T) \leq C \left( E_\varepsilon^3(T) + \varepsilon_3^2 \vartheta_\varepsilon^2(T) + \varepsilon_3^3 \right) \leq C \varepsilon_3^3 + C E_\varepsilon^3(T) + \frac{\vartheta_\varepsilon^6(T)}{2},
$$

which leads to (2.38). \(\square\)

Now we are in a position to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** We first deduce from (2.22) and (2.37) that

$$
E_\varepsilon(T) \leq C \varepsilon + C \varepsilon^\frac{1}{2} E_\varepsilon(T) \quad \text{for} \quad T \leq T^*_\varepsilon,
$$

which implies that for $\varepsilon \leq \varepsilon_1$ small such that $C \varepsilon_1^\frac{1}{2} = \frac{1}{2}$, there holds

$$
E_\varepsilon(T) \leq C \varepsilon \quad \text{for} \quad T \leq T^*_\varepsilon.
$$

This together with Lemma 2.5 ensures that

$$
\vartheta_\varepsilon^6(T) \leq C \varepsilon \quad \text{for} \quad T \leq T^*_\varepsilon.
$$

In particular, if we take $\varepsilon \leq \varepsilon_2$ to be small that $C \varepsilon_2 = \frac{1}{6} \min \{1, \eta \}^2$, (2.44) contradicts with the definition of $T^*_\varepsilon$ given by (2.14). As a consequence, we deduce that $T^*_\varepsilon = T^*_\varepsilon$.

It remains to show that the life-span $T^*_\varepsilon = \infty$. Indeed we have shown that the estimate (2.15) holds for all $t < T^*_\varepsilon$ and $\varepsilon \leq \varepsilon_0 \overset{\text{def}}{=} \min \{\varepsilon_1, \varepsilon_2 \}$, then regularity criteria for smooth solutions of compressible Navier-Stokes equations (see for instance [11, 28, 29]) ensures that $T^*_\varepsilon = \infty$. This completes the proof of Theorem 1.2.
3. 1D compressible Navier-Stokes equations with a parameter

In this section, we investigate the 1D compressible Navier-Stokes equations (1.6) with a parameter \( y \). We assume that \( (\eta, w) \) is a global smooth solution of (1.6) determined at the beginning of Section 2. For simplicity, we shall always neglect \( y \) variable and denote \( D_t \overset{\text{def}}{=} \partial_t + w\partial_x \) to be the material derivative in the rest of this section.

3.1. Conservation of mass, momentum, and energy. Integrating (1.6) in \( x \) over \( \mathbb{T} \) leads to the conservations of the mass and of the momentum:

\[
\frac{d}{dt} \int_{\mathbb{T}} \eta(t, x, y) \, dx = 0, \quad \text{and} \quad \frac{d}{dt} \int_{\mathbb{T}} (\eta w)(t, x, y) \, dx = 0,
\]

which together with (1.10) ensures that

\[
\int_{\mathbb{T}} \eta(t, x, y) \, dx = 1, \quad \int_{\mathbb{T}} (\eta w)(t, x, y) \, dx = 0, \quad \forall \ t \in \mathbb{R}^+, \ y \in \mathbb{R}.
\]

Recall the conservation of energy:

**Lemma 3.1.** There holds

\[
\frac{d}{dt} E_0(t) + \nu \int_{\mathbb{T}} w_x^2 \, dx = 0 \quad \text{with} \quad E_0(t) \overset{\text{def}}{=} \int_{\mathbb{T}} \left( \frac{1}{2} \eta w^2 + P(\eta) \right) \, dx, \quad \text{and}
\]

\[
P(\eta) \overset{\text{def}}{=} \frac{a}{\gamma - 1} \eta^\gamma \quad \text{if} \ \gamma > 1 \quad \text{and} \quad P(\eta) \overset{\text{def}}{=} a(\eta \log \eta + 1) \quad \text{if} \ \gamma = 1.
\]

It follows from (1.9) that

\[
\mathcal{E}_{00}(y) \overset{\text{def}}{=} \int_{\mathbb{T}} \left( \frac{1}{2} \eta w_0^2 + (\varsigma_0 - 1)^2 \right)(x, y) \, dx \in (L^1 \cap L^\infty)(\mathbb{R}) \quad \text{with} \quad \bar{\mathcal{E}}_{00} \overset{\text{def}}{=} \sup_{y \in \mathbb{R}} \mathcal{E}_{00}(y),
\]

\[
E_{00}(y) \overset{\text{def}}{=} \int_{\mathbb{T}} \left( \frac{1}{2} \eta w_0^2 + P(\varsigma_0) \right)(x, y) \, dx \in L^\infty(\mathbb{R}) \quad \text{with} \quad \bar{E}_{00} \overset{\text{def}}{=} \sup_{y \in \mathbb{R}} E_{00}(y).
\]

Then for all \( t \in \mathbb{R}^+, \ y \in \mathbb{R} \), there holds

\[
E_0(t) + \nu \int_0^t \int_{\mathbb{T}} w_x^2 \, dx \, dt' \leq E_{00}(y).
\]

3.2. Upper bound for the density function. We observe from (2.2) that the density function \( \eta \) admits an upper bound depending on time. We shall derive here a time uniform upper bound for \( \eta \). We first recall from [26] that for \( u \in L^1(\mathbb{T}) \) and for all \( x \in \mathbb{T} \)

\[
I(u)(x) \overset{\text{def}}{=} \int_0^x u(x') \, dx', \quad \bar{I}(u) \overset{\text{def}}{=} I(u) - \langle I(u) \rangle \quad \text{and} \quad \langle u \rangle \overset{\text{def}}{=} \int_{\mathbb{T}} u(x) \, dx.
\]

It is easy to observe that \( I(u) \in C(\mathbb{T}) \cap W^{1,1}(\mathbb{T}) \) with

\[
\|I(u)\|_{W^{1,1} \cap L^\infty} \leq \|u\|_{L^1}.
\]

Moreover, for each \( u \in W^{1,1}(\mathbb{T}) \cap C(\mathbb{T}) \), standard density argument implies

\[
I(u_x)(x) = \int_0^x u_x \, dx' = u(x) - u(0) \quad \text{and} \quad \bar{I}(u_x)(x) = u(x) - \langle u \rangle.
\]

**Proposition 3.1.** For all \( (t, x, y) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R} \), there holds

\[
\eta(t, x, y) \leq \max \{ \varsigma_0, \bar{\varsigma}_1 \} \exp \left( 4 \bar{E}_{00}^{-\frac{1}{\gamma}} \right) \overset{\text{def}}{=} \bar{\eta},
\]

where \( \varsigma_0 \) is given in (2.1) and \( \bar{\varsigma}_1 \) will be determined by (3.15) below.
Proof. We rewrite the momentum equation in (1.6) as
\[(\eta w)_t + (\eta w^2)_x - \nu w_{xx} + p(\eta)_x = 0.\]

Applying the operator \(\hat{I}\) to the above equation gives
\[(3.10) \quad \hat{I}(\eta w)_t + \eta w^2 - \nu w_x + p(\eta) - \langle \eta w^2 - \nu w_x + p(\eta) \rangle = 0.\]

We first compute
\[(3.11) \quad D_t \hat{I}(\eta w) = D_t \partial_t \hat{I}(\eta w) - w \partial_x \hat{I}(\eta w) = D_t \hat{I}(\eta w) - \eta w^2.\]

While it follows from the transport equation of (1.6) that
\[-w_x = \frac{-\eta w_x}{\eta} = \frac{\eta_x + \nu w_x}{\eta} = D_t \log \eta.\]

By inserting the above equalities into (3.10) and the fact that \(\langle w_x \rangle = 0\), we obtain
\[(3.11) \quad D_t \hat{I}(\eta w) + \nu D_t \log \eta + p(\eta) - \langle \eta w^2 + p(\eta) \rangle = 0.\]

Let \(X(t, x, y)\) be the trajectory of \(w\), which is determined by
\[(3.12) \quad \frac{d}{dt} X_w(t, x, y) = w(t, X(t, x, y)); \quad X_w(0, x, y) = x.\]

Let \(\eta(t, x, y) = \log \eta(t, X_w(t, x, y), y)\). Then in view of (3.11), we write
\[(3.13) \quad \nu \frac{d}{dt} \eta = g(\eta) + \frac{d}{dt} b\]

with
\[g(\eta) = -p(\eta) + \langle \eta w^2 + p(\eta) \rangle, \quad b = -\hat{I}(\eta w).\]

Before proceeding, we recall the following lemma:

**Lemma 3.2** (Lemma 1.3 of [27]). Suppose \(g \in C(\mathbb{R})\) and \(\eta, b \in W^{1,1}(0, T)\) for all \(T > 0\). Suppose that \(\eta\) verifies
\[\frac{d}{dt} \eta = g(\eta) + \frac{d}{dt} b \quad \text{on} \; \mathbb{R}^+, \quad \eta(0) = \eta_0.\]

If \(g(\infty) = -\infty\) and there exist non-negative constants \(N_0, N_1 \geq 0\) so that for any \(0 \leq t_1 < t_2 < \infty\),
\[b(t_2) - b(t_1) \leq N_0 + N_1 (t_2 - t_1).\]

Then one has
\[\eta(t) \leq \bar{\eta} < \infty, \quad \forall t \in \mathbb{R}^+ \quad \text{with} \quad \bar{\eta} \text{ def } = \max\{\eta_0, \eta_1\} + N_0,\]

where \(\eta_1\) is such that \(g(\eta) \leq -N_1\) for all \(\eta \geq \eta_1\).

Now we would like to apply Lemma 3.2 to derive an upper bound of \(\eta\) via (3.13). Firstly, it follows from (3.5) and (3.4) that
\[g(\eta) = -p(e^\eta) + \int_T (\eta w^2 + p(\eta)) \, dx \leq -p(e^\eta) + \gamma E_0 \rightarrow -\infty, \quad \text{as} \; \eta \rightarrow +\infty.\]

While in view of (3.6), we have for all \(t \in \mathbb{R}^+\),
\[(3.14) \quad |\hat{I}(\eta w)|_{L_1^\infty} \leq 2 \|\eta w\|_{L_1^h} \leq 2 \|\eta\|_{L_1^h} \|\eta w^2\|_{L_1^h} \leq 2 E_0^{1/2},\]

which implies
\[|\hat{I}(\eta w)(t_2) - \hat{I}(\eta w)(t_1)| \leq 4 E_0^{1/2}, \quad \forall t_1, t_2 \in [0, \infty).\]

On the other hand, we observe that
\[(3.15) \quad -p(\zeta_1) + \gamma E_0 = 0 \Rightarrow \zeta_1 \text{ def } = \left(\frac{\gamma E_0}{a}\right)^{1/2}.\]
Hence we get, by applying Lemma 3.2, that

\begin{equation}
\eta \leq \max\{\log \zeta_0, \log \zeta_1\} + 4E_0^\frac{1}{2}, \tag{3.16}
\end{equation}

where \( \zeta_0 \) is given in (2.1). This leads to (3.9).

\section{3.3. Decay estimates of \( L^2 \) norms.} With the upper bound of \( \eta \) obtained in (3.9), similarly to Lemma 2.1 of [32], we have

**Lemma 3.3.** Under the assumptions of (1.10), one has

\[
\int_T \eta w^2 \, dx \leq \eta^2 \int_T w_x^2 \, dx.
\]

**Proof.** Under the assumptions of (1.10), one has (3.2) for all \( t > 0 \). Then we deduce from (3.9) and Poincaré’s inequality that

\[
\int_T \eta w^2 \, dx = \int_T \eta w^2 \, dx - \langle \eta w \rangle^2
\]

\[
= \frac{1}{2} \int_T \int_T \eta(x)\eta(x')|w(x) - w(x')|^2 \, dx \, dx'
\]

\[
\leq \bar{\eta}^2 \int_T \int_T |w(x) - w(x')|^2 \, dx \, dx'
\]

\[
= \bar{\eta}^2 \int_T |w - \langle w \rangle|^2 \, dx \leq \eta^2 \int_T w_x^2 \, dx.
\]

By combining Lemma 3.3 with the energy estimate (3.5), we achieve

\begin{equation}
\int_0^\infty \int_T \eta w^2 \, dx \, dt \leq \bar{\eta}^2 \int_0^\infty \int_T w_x^2 \, dx \, dt \leq \eta^2 \nu^{-1} E_0.
\end{equation}

**Proposition 3.2.** There exist positive constants \( \alpha > 0 \) and \( C \) which depend on \( (a, \gamma, \nu, \bar{\zeta}_0, \bar{E}_0) \) such that

\begin{equation}
\|\eta^{\frac{1}{2}} w(t)\|_{L^2} + \|(\eta - 1)(t)\|_{L^2} \leq CE_0^\frac{1}{2}(y)e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+, \ y \in \mathbb{R}.
\end{equation}

**Proof.** By combining (3.1) with (3.3), we obtain

\begin{equation}
\frac{d}{dt} \int_T \left( \frac{1}{2} \eta w^2 + (P(\eta) - P(1) - P'(1)(\eta - 1)) \right) \, dx + \nu \int_T w_x^2 \, dx = 0.
\end{equation}

While we get, by using Taylor’s formula, that

\[
P(\eta) - P(1) - P'(1)(\eta - 1) = P''(\hat{\eta})(\eta - 1)^2 = \frac{p'(\hat{\eta})}{\hat{\eta}}(\eta - 1)^2 = a\gamma\hat{\eta}^{-2}(\eta - 1)^2,
\]

for some \( \hat{\eta} \) between \( \eta \) and 1. Due to the upper bound \( 0 \leq \eta \leq \bar{\eta} < \infty \), we have

\[
0 \leq \hat{\eta} \leq \max\{1, \bar{\eta}\} \leq 1 + \bar{\eta}.
\]

So that if \( \gamma \in [1, 2] \), we have

\begin{equation}
P(\eta) - P(1) - P'(1)(\eta - 1) \geq a\gamma(1 + \bar{\eta})^{-2}(\eta - 1)^2.
\end{equation}

While for \( \gamma \in [1, 2] \), it is easy to observe that

\begin{equation}
P(\eta) - P(1) - P'(1)(\eta - 1) \leq C(\gamma)a(\eta - 1)^2,
\end{equation}

where the constant \( C \) is solely determined by \( \gamma \). It follows from (3.20) and (3.21) that \( P(\eta) - P(1) - P'(1)(\eta - 1) \) behaves like \( (\eta - 1)^2 \).
Notice that \( I(\eta - 1)|_{x=0} = I(\eta - 1)|_{x=1} = 0 \). By multiplying the momentum equation of (1.6) by \( I(\eta - 1) \) and using integration by parts, one has

\[
(3.22) \quad \int_T p(\eta)(\eta - 1) \, dx = \int_T (\eta w)_t I(\eta - 1) \, dx - \int_T (\eta w^2)(\eta - 1) \, dx + \nu \int_T w_x(\eta - 1) \, dx.
\]

We first compute

\[
\int_T (\eta w)_t I(\eta - 1) \, dx = \frac{d}{dt} \int_T (\eta w) I(\eta - 1) \, dx - \int_T (\eta w) \partial_t I(\eta - 1) \, dx.
\]

By virtue of (3.2) and Lemma 3.3, one has

\[
\int_T (\eta w) \partial_t I(\eta - 1) \, dx = | \int_T (\eta w) I((\eta w)_x) \, dx | = | \int_T (\eta w)(\eta w(t, x) - \eta w(t, 0)) \, dx | \leq \int_T (\eta w^2) \, dx \leq \int_T w_x^2 \, dx.
\]

Similarly, we have

\[
| \int_T (\eta w)_t I(\eta - 1) \, dx | \leq (1 + \bar{\eta}) \int_T \eta w^2 \, dx \leq (1 + \bar{\eta}) \bar{\eta}^2 \int_T w_x^2 \, dx.
\]

While applying Young’s inequality yields

\[
\nu | \int_T w_x(\eta - 1) \, dx | \leq \delta \int_T |\eta - 1|^2 \, dx + \delta^{-1} \nu^2 \int_T w_x^2 \, dx.
\]

By inserting the above estimates into (3.23), we find

\[
(3.24) \quad \int_T p(\eta)(\eta - 1) \, dx - \frac{d}{dt} \int_T (\eta w) I(\eta - 1) \, dx \leq \delta \int_T |\eta - 1|^2 \, dx + (\bar{\eta}^2 + \bar{\eta}^3 + \delta^{-1} \nu^2) \int_T w_x^2 \, dx.
\]

Notice that as long as \( \gamma \geq 1 \), one has

\[
(p(r) - p(1))(r - 1) - a(r - 1)^2 = ar(r^{\gamma - 1} - 1)(r - 1) \geq 0, \quad \forall r \geq 0.
\]

As a result, it comes out

\[
\int_T p(\eta)(\eta - 1) \, dx = \int_T (p(\eta) - p(1))(\eta - 1) \, dx \geq a \int_T (\eta - 1)^2 \, dx.
\]

Then choosing \( \delta = \frac{a}{\gamma} \) in (3.24) yields

\[
(3.25) \quad \frac{a}{\gamma} \int_T (\eta - 1)^2 \, dx - \frac{d}{dt} \int_T (\eta w) I(\eta - 1) \, dx \leq (\bar{\eta}^2 + \bar{\eta}^3 + 2a^{-1} \nu^2) \int_T w_x^2 \, dx.
\]

Multiplying (3.19) by a constant \( A_1 \) and summing up the resulting inequality with (3.25), we get

\[
\frac{d}{dt} \int_T \left( \frac{A_1}{2} \eta w^2 + A_1 (P(\eta) - P(1) - P'(1)(\eta - 1)) - (\eta w) I(\eta - 1) \right) \, dx + A_1 \nu \int_T w_x^2 \, dx + \frac{a}{\gamma} \int_T (\eta - 1)^2 \, dx \leq (\bar{\eta}^2 + \bar{\eta}^3 + 2a^{-1} \nu^2) \int_T w_x^2 \, dx.
\]

Taking \( A_1 \) large enough so that

\[
(3.26) \quad A_1 \geq 4, \quad A_1 a \gamma (1 + \bar{\eta})^{\gamma - 2} \geq 2, \quad A_1 \nu \geq \bar{\eta}^2 + \bar{\eta}^3 + 2a^{-1} \nu^2 + 1,
\]
we obtain
\[
\frac{d}{dt} \int_T \left( \frac{A_1}{2} \eta w^2 + A_1 (P(\eta) - P(1) - P'(1)(\eta - 1)) - (\eta w)I(\eta - 1) \right) \, dx + \int_T w_x^2 \, dx + \frac{a}{2} \int_T (\eta - 1)^2 \, dx \leq 0.
\]

Furthermore, due to
\[
\int_T |(\eta w)I(\eta - 1)| \, dx \leq \int_T |\eta w| \, dx \int_T |\eta - 1| \, dx \leq \int_T \eta w^2 \, dx + \int_T |\eta - 1|^2 \, dx,
\]
and taking into account (3.20) and (3.21), we have
\[
\int_T \left( \frac{A_1}{2} \eta w^2 + A_1 (P(\eta) - P(1) - P'(1)(\eta - 1)) - (\eta w)I(\eta - 1) \right) \, dx \geq \int_T \left( \frac{A_1}{4} \eta w^2 + \frac{A_1}{2} a(1 + \eta) \gamma^2 (\eta - 1)^2 \right) \, dx,
\]
and
\[
\int_T \left( \frac{A_1}{2} \eta w^2 + A_1 (P(\eta) - P(1) - P'(1)(\eta - 1)) - (\eta w)I(\eta - 1) \right) \, dx \leq \int_T \left( \frac{A_1}{2} + 1 \right) \eta w^2 + (A_1 a \gamma + 1)(\eta - 1)^2 \right) \, dx.
\]

Then (3.18) follows from (3.27)–(3.29) and Lemma 3.3. \(\square\)

3.4. **Lower bound of the density function.**

**Proposition 3.3.** There exists a positive constant \(\eta\) which depends on \((a, \gamma, \nu, \varsigma_0, \varsigma_0, \tilde{E}_{00})\) so that
\[
\eta(t, x, y) \geq \eta, \quad \forall (t, x, y) \in \mathbb{R}^+ \times T \times \mathbb{R}.
\]

**Proof.** Let \(\eta_1 \overset{\text{def}}{=} \log \frac{1}{\eta} = - \log \eta.\) Then it follows from (3.11) that
\[
D_t (\nu \eta_1 - \tilde{I}(\eta w)) = p(\eta) - \langle \eta w^2 + p(\eta) \rangle,
\]
from which, we deduce that for each \(t \in \mathbb{R}^+ ,\)
\[
\max_{x \in T} (\nu \eta_1 - \tilde{I}(\eta w))(t, \cdot) \leq \max_{x \in T} (\nu \eta_1 - \tilde{I}(\eta w))(0, \cdot) + \int_0^t \max_{x \in T} (p(\eta) + \langle \eta w^2 + p(\eta) \rangle) \, dt'
\]
\[
\leq -\nu \log \varsigma_0 + E_{00}^2 + (a \gamma + \gamma E_{00})t.
\]

This implies
\[
\eta_1(t) \leq -\log \varsigma_0 + 2 \nu^{-1} E_{00}^2 + \nu^{-1}(a \gamma + \gamma E_{00})t,
\]
which is equivalent to
\[
\eta(t) \geq \varsigma_0 \exp \left( -2 \nu^{-1} E_{00}^2 - \nu^{-1}(a \gamma + \gamma E_{00})t \right) \overset{\text{def}}{=} \eta_1(t).
\]

To obtain a lower bound of the density function for large time, we define
\[
\eta_2 = \exp(\lambda(\nu \eta_1 - \tilde{I}(\eta w))), \quad \lambda > 0
\]
which solves
\[
D_t \eta_2 = \lambda \exp(\lambda(\nu \eta_1 - \tilde{I}(\eta w)))(p(\eta) - \langle \eta w^2 + p(\eta) \rangle)
\]
\[
= -\lambda \eta_2 \langle \eta w^2 + p(\eta) \rangle + \lambda \eta^{-\lambda \nu} \exp(-\lambda \tilde{I}(\eta w)).
\]
Taking \(\lambda\) so that \(\lambda \nu = \gamma\) gives rise to
\[
D_t \eta_2 + \nu^{-1} \gamma \eta_2 \langle \eta w^2 + p(\eta) \rangle = a \nu^{-1} \gamma \exp(-\lambda \tilde{I}(\eta w)).
\]
Yet we observe from Proposition 3.2 that
\[
\langle p(\eta) \rangle = \int_T p(\eta) \, dx \geq p(1) - \int_T |p(\eta) - p(1)| \, dx
\]
(3.33)
\[
\geq p(1) - p'(\eta) \int_\eta |\eta - 1| \, dx \geq a - a\gamma\eta^{-1}(Ce^{-\alpha t})^\frac{1}{2}.
\]
Let \( T_1 \) be such that
(3.34)
\[
\gamma\eta^{-1}(Ce^{-\alpha T_1})^\frac{1}{2} = \frac{1}{2}
\]
Then for all \( t \geq T_1 \), there holds
\[
\langle \eta \nu^2 + p(\eta) \rangle \geq \langle p(\eta) \rangle \geq \frac{a}{2},
\]
so that we deduce from (3.32) that for \( t \geq T_1 \),
(3.35)
\[
\eta_2(t) \leq e^{-\tilde{a}t} \eta_2(T_1) + \int_{T_1}^{t} e^{-\tilde{a}(t-t')} a\nu^{-1} \gamma \exp(-\lambda \tilde{I}(\eta \nu))(t') \, dt',
\]
where \( \tilde{a} \equiv \frac{a\nu^{-1}\gamma}{2} > 0 \). Together with (3.14), (3.35) ensures that for \( t \geq T_1 \),
\[
\eta_2(t) \leq e^{-\tilde{a}t} \eta_2(T_1) + a(\tilde{a}\nu)^{-1} \gamma e^{2\lambda E_{00}},
\]
from which, we infer
\[
\eta^{-\lambda y}(t) \leq e^{-\tilde{a}t} \eta^{-\lambda y}(T_1)e^{2\lambda E_{00}} + a(\tilde{a}\nu)^{-1} \gamma e^{4\lambda E_{00}},
\]
that is, for \( t \geq T_1 \) there holds
(3.36)
\[
\eta(t) \geq \eta_2(T_1) \overset{\text{def}}{=} \left( e^{-\tilde{a}T_1} \eta^{-\lambda y}(T_1)e^{2\lambda E_{00}} + a(\tilde{a}\nu)^{-1} \gamma e^{4\lambda E_{00}} \right)^{-\frac{1}{2}}.
\]
Combining (3.31) and (3.36), we deduce that (3.30) holds with
(3.37)
\[
\bar{\eta} \overset{\text{def}}{=} \min \left\{ \eta_1(T_1), \eta_2(T_1) \right\}.
\]
This completes the proof of the proposition. \( \square \)

3.5. Decay estimate of \( \|w_x(t)\|_{L^8} \). It follows from (1.9) that
(3.38)
\[
E_{10}(y) \overset{\text{def}}{=} \|s_0 - 1\|_{H^1_h}^2 + \|w_0\|_{H_h^1}^2 \in (L^1 \cap L^\infty)(\mathbb{R}) \quad \text{and} \quad \bar{E}_{10} \overset{\text{def}}{=} \sup_{y \in \mathbb{R}} E_{10}(y) < \infty.
\]

We start with the following two lemmas:

**Lemma 3.4.** There holds for all \( t \in \mathbb{R}^+ \)
(3.39)
\[
\frac{d}{dt} \int_T \eta w^4 \, dx + 6\nu \int_T |w|^2 |w_x|^2 \, dx \leq 24\alpha\nu^{-1}\eta^{2\gamma - 1} \int_T w_x^2 \, dx.
\]

**Proof.** Multiplying \( 4w^3 \) to the momentum equation of (1.6) and integrating the resulting equation over \( T \) gives
\[
\frac{d}{dt} \int_T \eta w^4 \, dx + 12\nu \int_T |w|^2 |w_x|^2 \, dx = 12 \int_T p(\eta)w^2 w_x \, dx
\]
\[
\leq 6\nu \int_T |w|^2 |w_x|^2 \, dx + 24\alpha\nu^{-1} \int_T \eta^{2\gamma} w^2 \, dx
\]
\[
\leq 6\nu \int_T |w|^2 |w_x|^2 \, dx + 24\alpha\nu^{-1}\eta^{2\gamma - 1} \int_T w_x^2 \, dx,
\]
where we used Lemma 3.3 in the last step. And (3.39) follows. \( \square \)
Lemma 3.5. There exist positive constants $B_1$ and $B_2$ solely depending on $(a, \gamma, \nu, \varsigma_0, \tilde{E}_{00})$, so that for all $t \in \mathbb{R}^+$

\[
\frac{d}{dt} \int_T \left( \frac{\nu}{2} w_x^2 + \frac{a^2}{2\nu} (\eta^2 - 1) - (p(\eta) - p(1)) w_x \right) dx + \frac{1}{2} \int_T \eta w_t^2 dx \\
\leq 5\tilde{\eta} \int_T |w|^2 |w_x|^2 dx + B_1 \int_T w_x^2 dx + B_2 \int_T (\eta - 1)^2 dx.
\]  

(3.40)

Proof. Multiplying $w_t$ to the momentum equation of (1.6) and integrating the resulting equation over $T$ gives

\[
\int_T (\eta w)_t w_t dx + \int_T (\eta w^2)_x w_t dx + \nu \frac{d}{dt} \int_T w_x^2 dx + \int_T p(\eta)_x w_t dx = 0.
\]  

(3.41)

We now handle term by term above. Firstly, it follows from the continuity equation of (1.6) that

\[
\int_T (\eta w)_t w_t dx = \int_T \eta w_t^2 dx - \int_T (\eta w^2)_x w_t dx + \int_T \eta w w_x w_t dx.
\]  

This implies

\[
\int_T (\eta w)_t w_t dx + \int_T (\eta w^2)_x w_t dx = \int_T \eta w_t^2 dx + \int_T \eta w w_x w_t dx \\
\geq \frac{3}{4} \int_T \eta w_t^2 dx - 4\tilde{\eta} \int_T |w|^2 |w_x|^2 dx.
\]  

(3.42)

It is rather complicated to estimate the term related to the pressure in (3.41). We compute

\[
\int_T p(\eta)_x w_t dx = -\frac{d}{dt} \int_T (p(\eta) - p(1)) w_x dx + \int_T p(\eta)_t w_x dx.
\]  

(3.43)

For the last term of (3.43), we decompose it as

\[
\int_T p(\eta)_t w_x dx = \nu^{-1} \int_T p(\eta)_t (\nu w_x - p(\eta)) dx + \nu^{-1} \int_T p(\eta)_t p(\eta) dx \\
= \nu^{-1} \int_T p(\eta) w (\nu w_{xx} - p(\eta)_x) dx - a(\gamma - 1) \nu^{-1} \int_T \eta^\gamma w_x (\nu w_x - p(\eta)) dx \\
+ \frac{a^2}{2\nu} \frac{d}{dt} \int_T (\eta^2 - 1 - 2\gamma (\eta - 1)) dx.
\]

Observing that

\[
\nu^{-1} \int_T p(\eta) w (\nu w_{xx} - p(\eta)_x) dx = \nu^{-1} \int_T p(\eta) w \eta (w_t + w w_x) dx.
\]

Applying Young’s inequality yields

\[
\nu^{-1} |\int_T p(\eta) w \eta w_t dx| \leq \frac{1}{4} \int_T \eta w_t^2 dx + 4\nu^{-2} \tilde{\eta} \gamma + 1 \int_T w_x^2 dx,
\]

\[
\nu^{-1} |\int_T p(\eta) w \eta w w_x dx| \leq a^2 \nu^{-2} \tilde{\eta} \gamma + 2 \int_T w_x^2 dx + \tilde{\eta} \int_T |w|^2 |w_x|^2 dx,
\]

where we used Lemma 3.3 in the second inequality.

We finally compute

\[
a(\gamma - 1) |\int_T \eta^\gamma w_x^2 dx| \leq a(\gamma - 1) \tilde{\eta} \gamma \int_T w_x^2 dx,
\]
and

$$\nu^{-1}a(\gamma - 1) \left| \int_T \eta^\gamma w_x p(\eta) \, dx \right| = \nu^{-1}a(\gamma - 1) \left| \int_T (\eta^{2\gamma} - 1) w_x \, dx \right|$$

$$\leq \nu^{-1}a(\gamma - 1) \left( \left| \int_T (\eta^{2\gamma} - 1)^2 \, dx + \int_T w_x^2 \, dx \right| \right)$$

$$\leq \nu^{-1}a(\gamma - 1) \left( (2\gamma \eta^{2\gamma - 1})^2 \int_T (\eta - 1)^2 \, dx + \int_T w_x^2 \, dx \right).$$

By inserting the above estimates into (3.43), we achieve

$$\int_T p(\eta)_x w_i \, dx \geq \frac{d}{dt} \int_T \left( \frac{a^2}{2\nu} (\eta^{2\gamma} - 1 - 2\gamma(\eta - 1)) - (p(\eta) - p(1)) w_x \right) \, dx$$

$$- \left( \tilde{\eta} \int_T |w|^2 |w|_x^2 \, dx + \frac{1}{4} \int_T \eta w_i^2 \, dx + B_1 \int_T w_x^2 \, dx + B_2 \int_T (\eta - 1)^2 \, dx \right),$$

where

$$B_1 = 4\nu^{-2}a\bar{\eta}^{2\gamma + 1} + \nu^{-2}a^2\bar{\eta}^{2\gamma + 2} + a(\gamma - 1)(\bar{\eta}^\gamma + \nu^{-1})$$

and $$B_2 = 4a\gamma^2(\gamma - 1)\nu^{-1}\eta^{2(\gamma - 1)}.$$

By substituting (3.42) and (3.44) into (3.41), we achieve (3.40). □

By multiplying (3.39) by a large enough positive constant $$A_2$$, which depends on $$(\nu, \tilde{\eta})$$, and summing up the resulting inequality with (3.40), we obtain

**Corollary 3.1.** Let $$A_2$$ and $$B_3$$ be determined by

$$6A_2\nu = 5\tilde{\eta} + 1, \quad B_3 = B_1 + 24A_2\nu^{-1}a\bar{\eta}^{\gamma + 1}.$$

There holds

$$\frac{d}{dt} \int_T \left( A_2 \eta w^4 + \nu w_x^2 + \frac{a^2}{2\nu} (\eta^{2\gamma} - 1 - 2\gamma(\eta - 1)) - (p(\eta) - p(1)) w_x \right) \, dx$$

$$+ \frac{1}{2} \int_T \eta w_i^2 \, dx + \int_T |w|^2 |w|_x^2 \, dx \leq B_3 \int_T w_x^2 \, dx + B_2 \int_T (\eta - 1)^2 \, dx.$$

**Proposition 3.4.** There exist positive constants $$\alpha$$ and $$C$$ solely depending on $$(a, \gamma, \bar{\eta}, E_0, E_0')$$ such that

$$\|w(t)\|_{H_t^l} + \|w(t)\|_{C_{\bar{\eta}, \tilde{\eta}}^{\frac{1}{2}}} \leq C E_{10}^{\frac{1}{2}}(y) e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+, \ y \in \mathbb{R}.$$

**Proof.** Thanks to (3.27) and (3.45), we deduce that there exists a large enough positive constant $$A_3$$ depending only on $$(a, \gamma, \bar{\eta}, E_0)$$ so that

$$\frac{d}{dt} F_2(t) + \int_T \left( w_x^2 + (\eta - 1)^2 + |w|^2 |w|_x^2 + \frac{1}{2} \eta w_i^2 \right) \, dx \leq 0,$$

where

$$F_2(t) \overset{\text{def}}{=} \int_T \left( \frac{A_1 A_3}{2} \eta w^2 + A_1 A_3 \left( P(\eta) - P(1) - P'(1)(\eta - 1) \right) - A_3(\eta w) I(\eta - 1)$$

$$+ A_2 \eta w^4 + \nu w_x^2 + \frac{a^2}{2\nu} (\eta^{2\gamma} - 1 - 2\gamma(\eta - 1)) - (p(\eta) - p(1)) w_x \right) \, dx.$$
By choosing \( A_3 \) sufficiently large, we have
\[
(3.49) \quad \int_T (\eta w^2 + (\eta - 1)^2 + \eta w^4 + w_x^2) \, dx \leq F_2 \leq C \int_T (\eta w^2 + (\eta - 1)^2 + \eta w^4 + w_x^2) \, dx,
\]
where \( C \) solely depends on \((a, \gamma, \nu, \bar{\omega}, \bar{E}_0)\). Then (3.46) follows from (3.47), (3.49) and (3.18). □

We remark that up to now, the strictly lower bound of \( \eta \) in (3.30) is not really needed. Indeed let us recall

**Lemma 3.6** (Lemma 3.2 in [6]). Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain with \( d \geq 1 \). Let \( \rho \) be a non-negative function satisfying
\[
\int_\Omega \rho \, dx \geq M > 0 \quad \text{and} \quad \int_\Omega \rho^q \, dx \leq E_0 < \infty,
\]
for some \( q > 1 \). Then for each \( u \in H^1(\Omega) \), there holds
\[
\|u\|^2_{L^2(\Omega)} \leq C(M, E_0) \left( \|\nabla u\|^2_{L^2(\Omega)} + \left( \int_\Omega \rho |u| \, dx \right)^2 \right).
\]

**Remark 3.2.** By integrating (3.47) over \([0, t]\), we obtain for all \( t \in \mathbb{R}^+ \)
\[
\int_T (\eta w^2 + (\eta - 1)^2 + \eta w^4 + w_x^2)(t) \, dx + \int_0^t \int_T ((\eta - 1)^2 + |w|^2 |w_x|^2 + \frac{1}{2} \eta w_t^2) \, dx \, dt' \\
\quad \leq C \int_T (\eta w_0^2 + (\eta_0 - 1)^2 + \eta_0 w_0^4 + w_{0,x}^2) \, dx < \infty,
\]
from which and Lemma 3.6, we infer
\[
\int_T |w|^2 \, dx \leq \int_T (\eta w^2 + w_x^2) \, dx \in L^\infty(\mathbb{R}_+).
\]
Then along the same lines of proof of (3.46), yet without using the positive lower bound of \( \eta \), we have
\[
\| (\eta^\frac{2}{3} w, \eta - 1, \eta^\frac{2}{3} w_x, w_x) \|_{L^2_{t,x}} \leq \|w\|_{L^\infty_t(L^2_{x})} \leq CE_{10}^\frac{1}{2}(y) e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+, \, y \in \mathbb{R},
\]
where \( \alpha > 0 \) solely depending on \((a, \gamma, \nu, \bar{\omega}, \bar{E}_0)\) and \( C > 0 \) solely depending on \((a, \gamma, \nu, \bar{\omega}, \bar{E}_0)\).

### 3.6. Decay estimate of \( \|\eta_x(t)\|_{L^2_x} \)

From this subsection on, we need to use the lower bound of \( \eta \) obtained in Proposition 3.3.

**Proposition 3.5.** There exist positive constants \( C \) and \( \alpha \) depending solely on \((a, \gamma, \nu, \bar{\omega}, \bar{E}_0, \bar{\nu}_0)\) so that
\[
(3.50) \quad \|\eta_x(t)\|_{L^2_x} \leq CE_{10}^\frac{1}{2} e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+.
\]

**Proof.** Let \( \zeta \overset{\text{def}}{=} \frac{1}{\eta} \). By multiplying \(-\eta^{-2}\) to the density equation of (1.6), we get
\[
\zeta_t + w\zeta_x - \zeta w_x = 0.
\]
Applying \( \partial_x \) to the above equation gives
\[
(3.51) \quad \zeta_{tx} + w\zeta_{xx} - \zeta w_{xx} = 0 \implies \zeta w_{xx} = \zeta_{tx} + w\zeta_{xx}.
\]
While we rewrite the momentum equation of (1.6) as
\[
w_t + w w_x - \nu (\zeta_{tx} + w \zeta_{xx}) - \zeta^{-1} p'(\eta) \zeta_x = 0.
\]
Multiplying the above equation by \( \eta \) yeilds
\[
(3.52) \quad \eta (w - \nu \zeta_x)_t + w (w - \nu \zeta_x)_x - \eta^2 p'(\eta) \zeta_x = 0.
\]
By multiplying \((w - \nu\zeta_x)\) to (3.52) and integrating the resulting equation over \(T\), we find
\[
\frac{1}{2} \frac{d}{dt} \int_T \eta(w - \nu\zeta_x)^2 \, dx - \int_T \eta^2 p'(\eta)\zeta_x(w - \nu\zeta_x) \, dx = 0.
\]
We compute
\[
- \int_T \eta^2 p'(\eta)\zeta_x(w - \nu\zeta_x) \, dx = \nu \alpha \gamma \int_T \eta^{\gamma+1} \zeta_x^2 \, dx + \int_T p(\eta) x \, dx.
\]
In view of Proposition 3.3, we have \(\eta \geq \eta > 0\), so that
\[
\nu \alpha \gamma \int_T \eta^{\gamma+1} \zeta_x^2 \, dx \geq \nu \alpha \gamma \eta^{\gamma+1} \int_T \zeta_x^2 \, dx.
\]
We observe that
\[
\left| \int_T p(\eta) x \, dx \right| = \left| \int_T p(\eta) w_x \, dx \right| = \left| \int_T (p(\eta) - p(1)) w_x \, dx \right|
\leq \int_T (p(\eta) - p(1))^2 \, dx + \int_T w_x^2 \, dx
\leq a^2 \gamma^2 \eta^{2\gamma-2} \int_T (\eta - 1)^2 \, dx + \int_T w_x^2 \, dx.
\]
By substituting the above estimates into (3.53), we achieve
\[
\frac{1}{2} \frac{d}{dt} \int_T \eta(w - \nu\zeta_x)^2 \, dx + \nu \alpha \gamma \eta^{\gamma+1} \int_T \zeta_x^2 \, dx \leq a^2 \gamma^2 \eta^{2\gamma-2} \int_T (\eta - 1)^2 \, dx + \int_T w_x^2 \, dx.
\]
Let \(A_4\) be a large enough positive constant so that
\[
A_4 \geq 4 + \nu \quad \text{and} \quad \frac{a}{2} A_4 \geq a^2 \gamma^2 \eta^{2\gamma-2} + 1.
\]
Then by multiplying (3.27) by \(A_4\) and summing up the resulting inequality with (3.54), we obtain
\[
\frac{d}{dt} F_3(t) + \int_T (w_x^2 + (\eta - 1)^2 + \nu \alpha \gamma \eta^{\gamma+1} \zeta_x^2) \, dx \leq 0 \quad \text{with}
\]
\[
F_3(t) \overset{\text{def}}{=} \int_T \left( A_4 A_1 \eta w^2 + A_1 A_4 (P(\eta) - P(1) - P'(1)(\eta - 1)) - A_4(\eta w) I(\eta - 1) + \frac{1}{2} \eta^2 (w - \nu\zeta_x)^2 \right) \, dx.
\]
Notice that
\[
2a^2 + (a - b)^2 = a^2 + 2(a - \frac{b}{2})^2 + \frac{b^2}{2} \geq a^2 + \frac{b^2}{2},
\]
by choosing \(A_4\) sufficiently large, we find
\[
\int_T (\eta w^2 + (\eta - 1)^2 + \frac{\nu \eta^2}{4} \zeta_x^2) \, dx \leq F_3(t) \leq C \int_T (\eta w^2 + (\eta - 1)^2 + \frac{\nu}{2} \zeta_x^2) \, dx,
\]
where \(C\) depends solely on \((a, \gamma, \nu, \bar{\nu}, E_0)\).

It follows from (3.56)-(3.57) that
\[
\| \zeta_x(t) \|_{L^2_h} \leq C E_0^{\frac{1}{2}} e^{-\alpha t}, \quad \forall \ t \in \mathbb{R}^+, \quad \alpha \text{ and } C \text{ satisfying the assumptions in the proposition.}
\]
Then (3.50) follows from the fact that
\[
\| \eta_x(t) \|_{L^2_h} = \| \eta_x^2 \zeta_x(t) \|_{L^2_h} \leq \eta^2 \| \zeta_x(t) \|_{L^2_h}.
\]
This completes the proof of Proposition 3.5. \(\square\)
3.7. Decay estimates of $H^2$ norms. We first deduce from (1.9) that

\[ E_{20}(y) \overset{\text{def}}{=} \|s_0 - 1\|^2_{H_h^2} + \|w_0\|^2_{H_h^2} \in (L^1 \cap L^\infty)(\mathbb{R}) \text{ and } \tilde{E}_{20} \overset{\text{def}}{=} \sup_{y \in \mathbb{R}} E_{20}(y) < \infty. \]

**Lemma 3.7.** We have

\[ \int_0^\infty \int_T w_x^2 \, dx \, dt \leq CE_{10}, \]

where $C$ solely depends on $(a, \gamma, \nu, \tilde{s}_0, \tilde{E}_{10}, S_0)$.

**Proof.** Indeed by multiplying $\eta^{-1}w_{xx}$ to the momentum equation of (1.6) and integrating the resulting equation over $T$, we find

\[ \frac{d}{dt} \int_T w_x^2 \, dx + \nu \int_T w_x^2 \, dx \leq \frac{4\eta}{\nu} \left( \int_T w^2 w_x^2 \, dx + a^2 \gamma^2 \int_T \eta^{2\gamma-4} w_x^2 \, dx \right). \]

Yet by virtue of (3.46) and (3.50), we have

\[ \int_T w_x^2(t) \, dx \leq \|w(t)\|^2_{L_h^\infty} \|w_x(t)\|^2_{L_h^2} \leq CE_{10}e^{-\alpha t}, \]

\[ \int_T \eta^{2\gamma-4} w_x^2(t) \, dx \leq \eta^{2\gamma-4} \|w_x(t)\|^2_{L_h^2} \leq CE_{10}e^{-\alpha t}, \]

where $C$ solely depends on $(a, \gamma, \nu, \tilde{s}_0, \tilde{E}_{10}, S_0)$. Then integrating (3.60) over $\mathbb{R}^+$ leads to (3.59). \hfill \Box

**Proposition 3.6.** Let $D_t \overset{\text{def}}{=} \partial_t + w \partial_x$ be the material derivative. Then there exist positive constants $C$ and $\alpha$ depending on $(a, \gamma, \nu, \tilde{s}_0, \tilde{E}_{10})$ so that

\[ \int_T \eta |D_t w|^2 \, dx \leq CE_{20}(y)e^{-\alpha t} \text{ and } \int_0^\infty \int_T |(D_t w)_x|^2 \, dx \, dt \leq CE_{20}(y). \]

**Proof.** Applying the material derivative $D_t$ to the momentum equation of (1.6) gives

\[ \eta D_t^2 w + D_t \eta D_t w - \nu D_t w_{xx} + D_t p(\eta)_x = 0. \]

We compute

\[ D_t \eta D_t w = -\eta w_x D_t w = -w_x(\nu w_{xx} - p(\eta)_x) = -\frac{\nu}{2}(w_x^2)_x + w_x p(\eta)_x, \]

and

\[ -\nu D_t w_{xx} = -\nu(w_{txx} + w w_{xxx}) = -\nu(D_t w)_{xx} + \frac{3\nu}{2}(w_x^2)_x. \]

As a result, it comes out

\[ \eta D_t^2 w - \nu(D_t w)_x + \nu(w_x^2)_x + (wp(\eta)_x)_x + p(\eta)_tx = 0. \]

Multiplying (3.62) by $D_t w$ and integrating the resulting equation over $T$ yields

\[ \frac{1}{2} \frac{d}{dt} \int_T \eta |D_t w|^2 \, dx + \nu \int_T |(D_t w)_x|^2 \, dx \]

\[ = \nu \int_T w_x^2 (D_t w)_x \, dx + \int_T wp(\eta)_x (D_t w)_x \, dx + \int_T p(\eta)_t (D_t w)_x \, dx \]

\[ \leq \frac{\nu}{2} \int_T (|D_t w)_x|^2 \, dx + 8\nu^{-1} \int_T w_x^4 \, dx + C \int_T (\eta_x^2 + w_x^2) \, dx, \]

where $C$ solely depend on $(a, \gamma, \nu, \tilde{s}_0, \tilde{E}_{10,0}, \tilde{S}_0)$, and we used the uniform boundedness of $\|\eta\|_{L^\infty}$ and $\|w\|_{L^\infty}$.

While by applying Sobolev embedding theorem and Hölder inequality, one has

\[ \|w_x\|^2_{L^4} \leq \|w_x\|^2_{L^h} \|w_x\|^2_{L^\infty} \leq C \|w_x\|^2_{L^2} \|w_x\|^2_{H_h^2} \leq Ce^{-2\alpha t}(\|w_x\|^2_{L^2} + \|w_{xx}\|^2_{L^2}). \]
Then we deduce from (3.63) that there exists $C$ solely depending on $(a, \gamma, \nu, \varsigma_0, \bar{E}_{00}, \bar{S}_0, \bar{E}_{10})$ so that
\begin{equation}
\frac{d}{dt} \int_\mathbb{T} |\eta| D_t w|^2 \, dx + \nu \int_\mathbb{T} |(D_t w)_x|^2 \, dx \leq C \int_\mathbb{T} (w_x^2 + w_{xx}^2 + \eta_x^2) \, dx \tag{3.64}
\end{equation}

By multiplying (3.60) by a sufficiently large constant $A_5$ and summing up the resulting inequality with (3.64), we get
\begin{equation}
\frac{d}{dt} \int_\mathbb{T} (A_5 w_x^2 + \eta |D_t w|^2) \, dx + \int_\mathbb{T} w_{xx}^2 \, dx + \nu \int_\mathbb{T} |(D_t w)_x|^2 \, dx \leq C \int_\mathbb{T} (w_x^2 + \eta_x^2) \, dx \tag{3.65}
\end{equation}

Observing that
\[ \int_\mathbb{T} \eta |D_t w|^2 \, dx \leq C \int_\mathbb{T} (\eta w_x^2 + \eta w_{xx}^2) \, dx. \]

Then by virtue of (3.47) and (3.56), we can find a large enough constant $A_6$ such that the quantity
\begin{equation}
F_4(t) \overset{\text{def}}{=} A_6 F_3(t) + \int_\mathbb{T} (A_5 w_x^2 + \eta |D_t w|^2) \, dx \tag{3.66}
\end{equation}
satisfies
\begin{equation}
\int_\mathbb{T} (\eta w^2 + (\eta - 1)^2 + \eta_x^2 + \eta w_x^2 + w_x^2 + \eta |D_t w|^2) \, dx \leq F_4(t) \leq C \int_\mathbb{T} (\eta w^2 + (\eta - 1)^2 + \eta_x^2 + \eta w_x^2 + \eta |D_t w|^2) \, dx,
\end{equation}
and
\[ \frac{d}{dt} F_4(t) + \int_\mathbb{T} (w_x^2 + (\eta - 1)^2 + w^2 |w_x|^2 + \eta w_t^2 + \eta_x^2 + w_{xx}^2) \, dx \leq 0. \]

Here $F_3(t)$ is defined in (3.56) and $C$ solely depends on $(a, \gamma, \nu, \varsigma_0, \bar{E}_{00}, \bar{S}_0, \bar{E}_{10})$. Then there exists $\alpha > 0$ solely depending on $(a, \gamma, \nu, \varsigma_0, \bar{E}_{00}, \bar{S}_0, \bar{E}_{10})$ such that
\[ F_4(t) \leq CE_20e^{-\alpha t}. \]

And (3.61) follows. \qed

3.8. **Proof of Proposition 2.1.** With the estimates obtained in the previous sections, we shall prove Proposition 2.1 by induction method and along the same line as that of Propositions 3.1, 3.2, 3.4, 3.5 and 3.6. Since it involves only technicalities, we postpone the proof in Appendix A.

4. DECAY ESTIMATES OF $(\eta_y, w_y)$

In this section, we investigate the decay in time estimates of $(\eta_y, w_y)$. We first get, by applying $\partial_y$ to (1.6), that
\begin{equation}
\begin{cases}
\eta_y + (\eta w)_{yx} = 0, \\
(\eta w)_y + (\eta w^2)_{yx} - \nu w_y x + p(\eta)_{yx} = 0.
\end{cases}
\end{equation}

Integrating (4.1) with respect to $x$ over $\mathbb{T}$ gives
\[ \frac{d}{dt} \int_\mathbb{T} \eta_y \, dx = 0 \quad \text{and} \quad \frac{d}{dt} \int_\mathbb{T} (\eta w)_y \, dx = 0. \]

It follows from (1.10) that
\[ \int_\mathbb{T} \zeta_0 y \, dx = 0 \quad \text{and} \quad \int_\mathbb{T} (\zeta_0 w_0)_y \, dx = 0. \]

This implies
\begin{equation}
\int_\mathbb{T} \eta_y \, dx = 0 \quad \text{and} \quad \int_\mathbb{T} (\eta w)_y \, dx = 0, \quad \forall t \in \mathbb{R}_+.
\end{equation}
4.1. **Decay estimates of L² norms.** Without loss of generality, we may assume that

\[ p'(1) = 1. \]

Note that this assumption (4.3) can always hold after a suitable normalization. In view of (1.9), one has

\[ E^{(1)}_{00} \overset{\text{def}}{=} \| (\partial_y \varsigma_0, \partial_y w_0) \|^2_{L^2} \in (L^1 \cap L^\infty)(\mathbb{R}) \quad \text{and} \quad \tilde{E}^{(1)}_{00} \overset{\text{def}}{=} \sup_{y \in \mathbb{R}} E^{(1)}_{00} < \infty. \]

Throughout this subsection, \( A, \alpha \) and \( C \) are positive numbers solely depending on \( (a, \gamma, \nu, \varsigma_0, \tilde{E}_{110}, \varsigma_0) \), which may differ from line to line.

**Lemma 4.1.** For all \( t \in \mathbb{R}^+ \), one has

\[ \int_T (\eta w_y^2 + \eta_y^2) \, dx + \nu \int_0^t \int_T |w_{yx}|^2 \, dx \leq CE^{(1)}_{00}. \]

**Proof.** Taking \( L^2(\mathbb{T}) \) inner product of (4.1)\(_2\) with \( w_y \) gives

\[ \int_T (\eta w_y)_yw_y \, dx + \int_T (\eta w_y^2)_{yx} w_y \, dx + \nu \int_T |w_{yx}|^2 \, dx + \int_T p(\eta)_{yx} w_y \, dx = 0. \]

Next we handle term by term above. For the first term in (4.6), we have

\[ \int_T (\eta w)_yw_y \, dx = \int_T (\eta w_y + \eta_y w)_yw_y \, dx = \frac{1}{2} \frac{d}{dt} \int_T \eta w_y^2 \, dx + \frac{1}{2} \int_T \eta w_y^2 \, dx + \int_T (\eta w)_y (ww_y)_x \, dx + \int_T w_t \eta_y w_y \, dx. \]

For the second term in (4.6), we have

\[ \int_T (\eta w^2)_{yx} w_y \, dx = - \int_T (\eta w^2)_{yx} w_y \, dx = - \int_T (2\eta w w_y + w^2 \eta_y) w_{yx} \, dx. \]

And for the last term in (4.6), one has

\[ \int_T p(\eta)_{yx} w_y \, dx = - \int_T p(\eta)_{yx} w_y \, dx = - \int_T p'(\eta) \eta_y w_{yx} \, dx. \]

By substituting the above equalities into (4.6), we achieve

\[ \frac{1}{2} \frac{d}{dt} \int_T \eta w_y^2 \, dx + \nu \int_T |w_{yx}|^2 \, dx \]

\[ = - \frac{1}{2} \int_T \eta w_y^2 \, dx - \int_T (ww_x \eta_y w_y + \eta w_x w_y^2) \, dx \]

\[ - \int_T w_t \eta_y w_y \, dx + \int_T \eta w_y w_{yx} \, dx + \int_T p'(\eta) \eta_y w_{yx} \, dx. \]

On the other hand, by taking \( L^2(\mathbb{T}) \) inner product of (4.1)\(_1\) with \( \eta_y \), we find

\[ \frac{1}{2} \frac{d}{dt} \int_T \eta_y^2 \, dx = - \int_T (\eta w)_{yx} \eta_y \, dx \]

\[ = - \int_T w \eta_x \eta_y \, dx - \int_T (w_x \eta_y^2 + \eta_y w \eta_y) \, dx - \int_T \eta w_{yx} \eta_y \, dx. \]
Summing up (4.7) and (4.8) gives rise to
\[
\frac{1}{2} \frac{d}{dt} \int_T (\eta w_y^2 + \eta_y^2) \, dx + \nu \int_T |w_{yx}|^2 \, dx
\]
\[
= - \int_T \left( \frac{1}{2} \eta_t + \eta w_x \right) w_y^2 \, dx - \int_T (ww_x + w_t + \eta_x) \eta_y w_y \, dx
\]
\[
+ \frac{1}{2} \int_T \eta w_x^2 \, dx + \int_T \left( (p'(\eta)) - \eta \eta_y w_x \right) \eta_y w_y \, dx + \int_T \eta w w_y w_{yx} \, dx,
\]
from which, (4.3), (A.1) and (A.2), we deduce
\[
\frac{d}{dt} \int_T (\eta w_y^2 + \eta_y^2) \, dx + \nu \int_T w_{yx}^2 \, dx \leq C e^{-\alpha t} \int_T (\eta w_y^2 + \eta_y^2) \, dx.
\]
Applying Gronwall’s inequality leads to (4.5).

Proposition 4.1. We have
\[
\int_T (\eta w_y^2 + \eta_y^2)(t) \, dx \leq CE_{10}^{(1)} (y) e^{-\alpha t} \quad \text{and} \quad \int_0^\infty \int_T (w_{yx}^2 + \eta_y^2) \, dx \, dt \leq CE_{10}^{(1)} (y).
\]

Proof. Recall (3.8), we get, by multiplying the momentum equation of (4.1) by \(I(\eta_y)\) and integrating the resulting equality over \(T\), that
\[
\int_T p(\eta)_y \eta_y \, dx = \int_T (\eta w)_y I(\eta_y) \, dx - \int_T (\eta w^2)_y \eta_y \, dx + \nu \int_T w_{yx} \eta_y \, dx.
\]
It is easy to observe that
\[
- \int_T (\eta w^2)_y \eta_y \, dx = - \int_T w^2 \eta_y^2 \, dx - \int_T 2 \eta w w_y \eta_y \, dx,
\]
\[
\nu \int_T w_{yx} \eta_y \, dx \leq C \int_T w_{yx}^2 \, dx + \delta \int_T \eta_y^2 \, dx,
\]
and
\[
\int_T (\eta w)_y I(\eta_y) \, dx = \frac{d}{dt} \int_T (\eta w)_y I(\eta_y) \, dx - \int_T (\eta w)_y I(\eta_y) \, dx.
\]
By virtue of (4.2), one has
\[
- \int_T (\eta w)_y I(\eta_y) \, dx = \int_T (\eta w)_y I((\eta w)_{yx}) \, dx
\]
\[
= \int_T (\eta w) ((\eta w)_y - (\eta w)(t, 0, y)) \, dx = \int_T (\eta w)_y^2 \, dx.
\]
By inserting the above estimates into (4.11), we obtain
\[
\int_T p'(\eta) \eta_y^2 \, dx \leq \frac{d}{dt} \int_T (\eta w)_y I(\eta_y) \, dx + \int_T \eta^2 w_y^2 \, dx + C \int_T w_{yx}^2 \, dx + \delta \int_T \eta_y^2 \, dx.
\]
Choosing \(\delta = \frac{\nu'(\eta)}{2}\) in the above inequality gives rise to
\[
(4.12) \quad \frac{\nu'(\eta)}{2} \int_T \eta_y^2 \, dx - \frac{d}{dt} \int_T (\eta w)_y I(\eta_y) \, dx \leq C \int_T w_{yx}^2 \, dx + \int_T \eta^2 w_y^2 \, dx.
\]
Notice from Lemma 3.3 and (4.2) that
\[
\int_T \eta w_y^2 - \langle \eta w_y \rangle^2 \, dx = \int_T \eta (w - \langle \eta w \rangle)^2 \, dx \leq \eta^2 \int_T w_{yx}^2 \, dx,
\]
from which and (4.2), we infer

\[
\int_T \eta w_y^2 \leq \tilde{\eta}^2 \int_T w_{yx}^2 \, dx + \langle \eta_y w \rangle^2 \leq \tilde{\eta}^2 \int_T w_{yx}^2 \, dx + \int_T \eta_y^2 w^2 \, dx.
\]

Hence thanks to (3.46), we deduce from (4.12) that

\[
\frac{p'(\eta)}{2} \int_T \eta_y^2 \, dx - \frac{d}{dt} \int_T (\eta w)_y I(\eta_y) \, dx \leq C \int_T w_{yx}^2 \, dx + Ce^{-\alpha t} \int_T \eta_y^2 \, dx.
\]

Let \( A \) be a sufficiently large constant, we denote

\[
F_1^{(1)} \overset{\text{def}}{=} \int_T (A(\eta w_y^2 + \eta_y^2) - (\eta w)_y I(\eta_y)) \, dx.
\]

Thanks to Lemma 4.1, we get, by multiplying (4.9) by \( A \) and summing up the resulting inequality with (4.14), that

\[
\frac{d}{dt} F_1^{(1)}(t) + \int_T (w_{yx}^2 + \frac{p'(\eta)}{2} \eta_y^2) \, dx \leq CE_00 e^{-\alpha t}.
\]

Due to

\[
\left| \int_T (\eta w)_y I(\eta_y) \, dx \right| \leq \| \eta_y \|_{L^2} (\| \eta w \|_{L^1} + \| \eta w_y \|_{L^1}) \leq (1 + \| w \|_{L^\infty}) \int_T (\eta w_y^2 + \eta_y^2) \, dx,
\]

we deduce from (4.13) that

\[
\int_T (\eta w_y^2 + \eta_y^2) \, dx \leq F_1^{(1)}(t) \leq C \int_T (\eta w_y^2 + \eta_y^2) \, dx \leq C \int_T (w_{yx}^2 + \eta_y^2) \, dx,
\]

which together with (4.16) ensures (4.10).

\[\square\]

4.2. Decay estimates of \( H^1 \) norms. It follows from (1.9) that

\[
E_{10}^{(1)} \overset{\text{def}}{=} \| \partial_y \tilde{u}_0 \|_{H^1}^2 + \| \partial_y w_0 \|_{H^1}^2 \in (L^1 \cap L^\infty)(\mathbb{R}) \quad \text{and} \quad \bar{E}_{10}^{(1)} \overset{\text{def}}{=} \sup_{y \in \mathbb{R}} E_{10}^{(1)} < \infty.
\]

Throughout this subsection, \( A, \alpha, C \) are positive constants solely depending on \( (a, \gamma, \nu, \varsigma_0, \bar{E}_{10}, \varsigma_0, \bar{E}_{00}^{(1)}) \), which may differ from line to line.

**Lemma 4.2.** For all \( t \in \mathbb{R}^+ \), there holds

\[
\frac{d}{dt} \int_T w_{yx}^2 \, dx + \nu \int_T \eta^{-1} w_{yx}^2 \, dx \leq Ce^{-\alpha t} \int_T (w_{yx}^2 + \eta_y^2) \, dx + C \int_T \eta_y^2 \, dx.
\]

**Proof.** We rewrite the equation (4.1.2) as

\[
\eta(w_{yt} + ww_{yx} + w_y w_x) + \eta_y(w_t + w w_x) - \nu w_{yx} + p(\eta)_{yx} = 0.
\]

Multiplying the above equation by \( \eta^{-1} w_{yx} \) and integrating the resulting equation over \( T \) yields

\[
\frac{1}{2} \frac{d}{dt} \int_T w_{yx}^2 \, dx + \nu \int_T \eta^{-1} w_{yx}^2 \, dx
\]

\[
= \int_T (\eta(ww_{yx} + w_y w_x) + \eta_y(w_t + w w_x) + p(\eta)_{yx}) \eta^{-1} w_{yx} \, dx.
\]
Hence, thanks to (A.3), we deduce from (4.23) that

\[ L \]

By taking

\[ \eta = \omega \]

Notice that

\[ \zeta \]

Let

\[ \delta > 0 \]

Choosing \( \delta \) small so that

\[ \int_T |p(\eta_{yx})|^2 \, dx = \int_T |p'(\eta_{yx}) + p''(\eta_{yx})|^2 \, dx. \]

As a consequence, thanks to (4.13), we thus deduce (4.18) from (4.20).

**Lemma 4.3.** For all \( t \in \mathbb{R}^+ \), there holds

\[ \frac{d}{dt} \int_T (w_y - \nu(\eta^{-1}_{yx}))^2 \, dx + \nu \int_T p'(\eta)\eta^{-2}_{yx} \, dx \leq C \int_T (w^2_{yx} + \eta^2_{yx}) \, dx. \]

**Proof.** We rewrite the equation (4.1)_2 as

\[ D_t w_y + w_x w_y + \eta^{-1}_y D_t w - \nu \eta^{-1} w_{yx} + \eta^{-1} p(\eta_{yx}) = 0. \]

Let \( \zeta = \eta^{-1} \). Then we deduce from equation (4.1)_1 that

\[ \zeta w_{yx} = (\zeta_t + w \zeta_{xx})_y - \zeta_y w_{xx} = D_t \zeta_{yx} + \nu \zeta w_{xx}. \]

Therefore, we obtain

\[ D_t (w_y - \nu \zeta_{yx}) + w_x w_y + \eta^{-1}_y D_t w - \nu (w_y \zeta_{xx} - \zeta_y w_{xx}) + \eta^{-1} p(\eta_{yx}) = 0. \]

By taking \( L^2 \) inner product of the above equation with \( \eta(w_y - \nu \zeta_{yx}) \), we find

\[ \frac{1}{2} \frac{d}{dt} \int_T \eta(w_y - \nu \zeta_{yx})^2 \, dx + \int_T p(\eta_{yx})(w_y - \nu \zeta_{yx}) \, dx \]

\[ + \int_T (\eta w_x w_y + \eta_y D_t w - \nu \eta(y \zeta_{xx} - \zeta_y w_{xx}))(w_y - \nu \zeta_{yx}) \, dx = 0. \]

Notice that

\[ \int_T p(\eta_{yx})(w_y - \nu \zeta_{yx}) \, dx = \int_T (p'(\eta)\eta y_y + p''(\eta)\eta_y \eta_x)(w_y - \nu(2\eta^{-3}_y \eta_y - \eta^{-2}_y \eta_{yx})) \, dx \]

\[ = \nu \int_T p'(\eta)\eta^{-2}_y \eta_x^2 \, dx + \int_T p'(\eta)\eta_{yx}(w_y - 2\nu \eta^{-3}_y \eta_y) \, dx \]

\[ + \int_T p''(\eta)\eta_y \eta_x(w_y - 2\nu \eta^{-3}_y \eta_y + \nu \eta^{-2}_y \eta_{yx}) \, dx. \]

Hence thanks to (A.3), we deduce from (4.23) that

\[ \frac{1}{2} \frac{d}{dt} \int_T \eta(w_y - \nu \zeta_{yx})^2 \, dx + \nu \int_T p'(\eta)\eta^{-2}_y \eta_x^2 \, dx \leq \delta \int_T \eta^2_{yx} \, dx + C \int_T (w^2_{yx} + \eta^2_{yx}) \, dx. \]

Choosing \( \delta > 0 \) small so that \( \delta \leq \frac{\nu}{2} p'(\eta) \eta^{-2}_y \), and using (4.13), we conclude the proof of (4.21).

**Proposition 4.2.** For all \( t \in \mathbb{R}^+ \), there holds

\[ \int_T (w^2_{yx} + \eta^2_{yx}) \, dx \leq CE^{(1)}_{10}(y) e^{-\alpha t}, \]

and

\[ \int_0^\infty \int_T \eta w^2_{yt} \, dx \, dt \leq CE^{(1)}_{10}(y). \]
While we get, by taking $L^2(\mathbb{T})$ inner product of (4.19) with $w_{yt}$, that
\[
\int_{\mathbb{T}} \eta w_{yt}^2 \, dx + \nu \frac{d}{dt} \int_{\mathbb{T}} w_{xy}^2 \, dx = - \int_{\mathbb{T}} (w w_{yx} + w_y w_x + \eta_y (w_t + w w_x) + p(\eta)_{yx}) w_{yt} \, dx,
\]
which implies
\[
\nu \frac{d}{dt} \int_{\mathbb{T}} w_{xy}^2 \, dx + \frac{1}{2} \int_{\mathbb{T}} \eta w_{yt}^2 \, dx \leq C \int_{\mathbb{T}} (w_{yx}^2 + w_{y}^2 + \eta_y^2 + \eta_{yx}^2) \, dx.
\]
By integrating (4.27) over $[0, t]$ and using (4.25), we obtain (4.26). \qed

### 4.3. Decay estimate of $\|w_y\|_{H^2}$

In view of (1.9), we have
\[
E_{20}^{(1)} \overset{\text{def}}= \|\partial_y \eta_0\|^2_{H^2} + \|\partial_y w_0\|^2_{H^2} \in (L^1 \cap L^\infty)(\mathbb{R}) \quad \text{and} \quad \bar{E}_{20}^{(1)} \overset{\text{def}}= \sup_{y \in \mathbb{R}} E_{20}^{(1)} < \infty.
\]

Throughout this subsection, $A, \alpha, C$ are positive numbers solely depending on $(a, \gamma, \nu, \bar{E}_{20}, \bar{\zeta}_0, \bar{E}_{10})$, which may change from line to line.

**Proposition 4.3.** Let $D_t \overset{\text{def}}= \partial_t + w \partial_x$ be the material derivative. Then for all $t \in \mathbb{R}^+$, there holds
\[
\int_{\mathbb{T}} \eta|D_t w_y|^2 \, dx \leq C E_{20}^{(1)}(y) e^{-\alpha t} \quad \text{and} \quad \int_0^\infty \int_{\mathbb{T}} |(D_t w_y)_x|^2 \, dx \, dt \leq C E_{20}^{(1)}(y),
\]
and
\[
\int_{\mathbb{T}} (\eta_{yt}^2 + |w_{yt}|^2 + |w_{yxx}|^2) \, dx \leq C E_{20}^{(1)}(y) e^{-\alpha t}.
\]

**Proof.** Applying $D_t$ to $\eta \times (4.22)$ gives
\[
\eta D_t^2 w_y + D_t \eta D_t w_y + D_t (\eta w_x w_y) + D_t (\eta_y D_t w) - \nu D_t w_{yxx} + D_t p(\eta)_{yx} = 0.
\]

It is easy to observe that
\[
\begin{align*}
D_t \eta D_t w_y &= -\eta w_x D_t w_y = w_x (\eta w_{xy} + \eta_y D_t w - \nu w_{yxx} + p(\eta)_{yx}), \\
D_t (\eta w_x w_y) &= D_t (\eta w_{xy}) + \eta w_x D_t w_y, \\
D_t (\eta y D_t w) &= - (\eta_y w_x + \eta x w_y + \eta w_{yx}) D_t w + \eta_y D_t^2 w, \\
-\nu D_t w_{yxx} &= -\nu (D_t w_{yxx}) + \nu (2 w_x w_{yxx} + w_{xx} w_{yx}),
\end{align*}
\]
and
\[
D_t p(\eta)_{yx} = p(\eta)_{tyx} + wp(\eta)_{yxx}.
\]

Then by taking $L^2(\mathbb{T})$ inner product of (4.31) with $D_t w_y$ and using the fact
\[
w_{x} p(\eta)_{yx} + wp(\eta)_{yxx} = (wp(\eta)_{yx})_{x},
\]
we find
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} \eta|D_t w_y|^2 \, dx + \nu \int_{\mathbb{T}} |(D_t w_y)_x|^2 \, dx \\
\leq \frac{\nu}{2} \int_{\mathbb{T}} |(D_t w_y)_x|^2 \, dx + C \int_{\mathbb{T}} (w_{y}^2 + w_{yx}^2 + w_{yxx}^2 + \eta_y^2 + \eta_{yx}^2) \, dx.
\]
This implies
\[
\frac{d}{dt} \int_{\mathbb{T}} \eta|D_t w_y|^2 \, dx + \nu \int_{\mathbb{T}} |(D_t w_y)_x|^2 \, dx \leq C \int_{\mathbb{T}} (w_{y}^2 + w_{yx}^2 + w_{yxx}^2 + \eta_y^2 + \eta_{yx}^2) \, dx,
\]
from which, Propositions 4.1 and 4.2, Lemmas 4.2 and 4.3, we deduce (4.29) by the similar argument used in the proof of Proposition 3.5.
Thanks to (4.29), we conclude (4.30) by using the mass and momentum equations (4.1). □

4.4. Proof of Proposition 2.2.

Proof of Proposition 2.2. By summarizing Propositions 4.1, 4.2 and 4.3, we deduce that there exist two positive constants $C$ and $\alpha$ solely depends on $(a, \gamma, \nu, \zeta_0, \bar{E}_{20}, \omega_0, \bar{E}_{10}^{(1)})$ such that

\begin{equation}
\|\eta_y\|_{H^1_n} + \|w_y\|_{H^1_n} + \|\eta_yt\|_{L^2_t} + \|w_yt\|_{L^2_t} \leq C \left( E_{20}^{(1)}(y) \right)^{\frac{1}{2}} e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+.
\end{equation}

In what follows, we shall follow the same strategy as that of the proof of Proposition 2.1. We first get, by applying $\partial_x$ to (4.1), that

\begin{equation}
\begin{cases}
\eta_{xxt} + (\eta w)_{yxx} = 0, \\
(\eta w)_{yxt} + (\eta w^2)_{yxx} - \nu w_{yxxx} + p(\eta)_{yxx} = 0.
\end{cases}
\end{equation}

We can also rewrite (4.34) as

\begin{equation}
D_t w_{yx} + (w_y w_{xx} + 2 w_x w_{yx}) + \eta^{-1} \eta_y(D_t w_x + w_x^2)
+ \eta^{-1} \eta_x(D_t w_y + w_x w_y) + \eta^{-1} \eta_y x w_t - \nu \eta^{-1} w_{yxxx} + \eta^{-1} p(\eta)_{yxx} = 0.
\end{equation}

We split the proof of the remaining estimates in (2.4) into the following steps:

**Step 1. Decay estimates for $\eta_{yxx}$.**

Recalling that $\zeta \overset{\text{def}}{=} \eta^{-1}$, we deduce from (4.34) that

\begin{equation}
\zeta w_{yxxx} = D_t \zeta_{yxx} - \zeta_y w_{xx} - \zeta_x w_{yxx} - \zeta_{yy} w_{xxx} - \zeta_{yx} w_{yxx} + \zeta_{yxx} w_x + \zeta_{xx} w_y.
\end{equation}

Plugging (4.36) into (4.35) gives rise to

\begin{align*}
D_t (w_{yx} - \nu \zeta_{yxx}) + w_x (w_{yx} - \nu \zeta_{yxx}) + (w_y w_{xx} + w_x w_{yx})
+ \eta^{-1} \eta_y (D_t w_x + w_x^2) + \eta^{-1} \eta_x (D_t w_y + w_x w_y) + \eta^{-1} \eta_y x w_t
- \nu (-\zeta_y w_{xx} - \zeta_x w_{yxx} - \zeta_{yy} w_{xxx} - \zeta_{yx} w_{yxx} + \zeta_{yxx} w_x + \zeta_{xx} w_y + \zeta_{xx} w_y) + \eta^{-1} p(\eta)_{yxx} = 0.
\end{align*}

By taking $L^2(\mathbb{T})$ inner product of the above equation with $\eta (w_{yx} - \nu \zeta_{yxx})$ and using the decay estimates we have derived in the previous sections, we obtain

\begin{equation}
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_\mathbb{T} \eta (w_{yx} - \nu \zeta_{yxx})^2 \, dx + \int_\mathbb{T} (w_y w_{xx} + w_x w_{yx}) p(\eta)_{yxx} \, dx \\
\leq C \left( E_{20}(y) \right)^{\frac{1}{2}} e^{-\alpha t} \int_\mathbb{T} \eta (w_{yx} - \nu \zeta_{yxx})^2 \, dx + \delta \int_\mathbb{T} \eta (w_{yx} - \nu \zeta_{yxx})^2 \, dx \\
+ C \delta^{-1} \left( E_{30}(y) + E_{20}^{(1)}(y) \right) e^{-\alpha t}.
\end{aligned}
\end{equation}

To handle the term related to the pressure, we write

\begin{align}
p(\eta)_{yxx} &= p'(\eta) \eta_{yxx} + p''(\eta) \eta_y \eta_{yxx} + p'''(\eta) 2 \eta_x \eta_y \eta_{yxx} + \eta''''(\eta) \eta_y \eta_{yxx}^2 \\
&= p'(\eta) (-\zeta_{yy} + 2 \zeta_{y} \zeta_{xx} + 4 \zeta_{yy} \zeta_{xx} - 6 \zeta_{yy} \zeta_{x}^2) \\
&+ p''(\eta) \eta_y \eta_{yxx} + p'''(\eta) 2 \eta_x \eta_y \eta_{yxx} + \eta''''(\eta) \eta_y \eta_{yxx}^2 \\
&= p'(\eta) \eta^2 \nu^{-1} (w_{yx} - \nu \zeta_{yxx}) - p'(\eta) \eta^2 \nu^{-1} w_{yx} + p'(\eta) \eta^3 (2 \zeta_{y} \zeta_{xx} + 4 \zeta_{yy} \zeta_{xx} - 6 \zeta_{yy} \zeta_{x}^2) \\
&+ p''(\eta) \eta_y \eta_{yxx} + p'''(\eta) 2 \eta_x \eta_y \eta_{yxx} + \eta''''(\eta) \eta_y \eta_{yxx}^2.
\end{align}

Notice that

\[ p'(\eta) \eta \geq p'(\eta) \beta > 0, \]
by choosing $\delta$ sufficiently small, we deduce from (4.37) and (4.38) that
\[
\frac{d}{dt} \int_T \eta (w_{yx} - \nu \xi_{yxx})^2 \, dx + \nu^{-1} p'(\eta) \eta \int_T \eta (w_{yx} - \nu \xi_{yxx})^2 \, dx
\leq C e^{-\alpha t} \int_T \eta (w_{yx} - \nu \xi_{yxx})^2 \, dx + C \left( E_{30}(y) + E_{20}^{(1)}(y) \right) e^{-\alpha t}.
\]

Applying Gronwall’s inequality gives
\[
\int_T \eta (w_{yx} - \nu \xi_{yxx})^2 \, dx \leq C \left( E_{30}(y) + E_{20}^{(1)}(y) \right) e^{-\alpha t},
\]
from which and (4.33), we infer
\[
(4.39) \quad \int_T \xi_{yxx}^2 \, dx \leq C \left( E_{30}(y) + E_{20}^{(1)}(y) \right) e^{-\alpha t}.
\]

This leads to
\[
(4.40) \quad \int_T \eta_{yxx}^2 \, dx \leq C \left( E_{30}(y) + E_{20}^{(1)}(y) \right) e^{-\alpha t}.
\]

**Step 2.** Decay estimates for $D_t w_{yx}$.

We first rewrite (4.34) as
\[
\eta D_t w_{yx} - \nu w_{yxxx} + g + p(\eta)_{yxx} = 0,
\]
\[
(4.41) \quad g \overset{\text{def}}{=} \eta (w_y w_{xx} + 2 w_x w_y) + \eta_y (D_t w_x + w_x^2) + \eta_x (D_t w_y + w_x w_y) + \eta yx D_t w.
\]

By taking $L^2(\mathbb{T})$ inner product of (4.41) with $\eta^{-1} w_{yxxx}$, we obtain
\[
(4.42) \quad \int_T D_t w_{yx} w_{yxxx} \, dx - \nu \int_T \eta^{-1} |w_{yxxx}|^2 \, dx + \int_T (g + p(\eta)_{yxx}) \eta^{-1} w_{yxxx} \, dx = 0.
\]

It is easy to observe that
\[
\int_T D_t w_{yx} w_{yxxx} \, dx = \int_T \partial_t w_{yx} w_{yxxx} \, dx + \int_T w w_{yxxx} w_{yxxx} \, dx
\]
\[
= -\frac{1}{2} \frac{d}{dt} \int_T |w_{yxx}|^2 - \frac{1}{2} \int_T w_x (w_{yxx})^2.
\]

While it follows from (A.18), (4.33) and (4.40) that
\[
\int_T (g + p(\eta)_{yxx}) \eta^{-1} w_{yxxx} \, dx \leq C \delta^{-1} E(y)e^{-\alpha t} + \delta \int_T |w_{yxxx}|^2 \, dx.
\]

By inserting the above estimates into (4.42) and choosing $\delta$ suitably small, we achieve
\[
(4.43) \quad \frac{d}{dt} \int_T |w_{yxx}|^2 \, dx + \nu \eta^{-1} \int_T |w_{yxxx}|^2 \, dx \leq C E(y)e^{-\alpha t}.
\]

While by taking $L^2(\mathbb{T})$ inner product of (4.41) with $D_t w_{yx}$, we find
\[
(4.44) \quad \int_T \eta |D_t w_{yx}|^2 \, dx - \nu \int_T D_t w_{yxxx} D_t w_{yx} \, dx + \int_T (D_t g + D_t p(\eta)_{yxx}) D_t w_{yx} \, dx = 0.
\]
Observing that
\[-\nu \int_T w_{yxxx} D_t w_{yx} \, dx = -\nu \int_T w_{yxxx} \partial_t w_{yx} \, dx - \nu \int_T w_{yxxx} w_{yxx} \, dx \]
\[= \frac{\nu}{2} \frac{d}{dt} \int_T |w_{yxxx}|^2 \, dx + \nu \int_T (w_{yxxx})^2 w_{x} \, dx \]
\[\geq \frac{\nu}{2} \frac{d}{dt} \int_T |w_{yxxx}|^2 \, dx - CE(y)e^{-\alpha t}.
\]
By using the estimates (A.18), (4.33) and (4.40), we find
\[\int_T (g + p(\eta)_{yxx}) D_t w_{yx} \, dx \leq C \delta^{-1} E(y)e^{-\alpha t} + \delta \int_T |D_t w_{yx}|^2 \, dx.
\]
By substituting the above estimates into (4.44) and taking \(\delta\) to be suitably small, we achieve
\[(4.45) \quad \frac{\nu}{2} \frac{d}{dt} \int_T |w_{yxxx}|^2 \, dx + \int_T |D_t w_{yx}|^2 \, dx \leq CE(y)e^{-\alpha t}.
\]
On the other hand, we get, by applying \(D_t\) to (4.41) and then taking \(L^2(\mathbb{T})\) inner product of the resulting equation with \(D_t w_{yx}\), that
\[(4.46) \quad \int_T (D_t(\eta D_t w_{yx}) - \nu D_t w_{yxxx} + D_t g + D_t p(\eta)_{yxx}) D_t w_{yx} \, dx = 0.
\]
Next we handle term by term above. We first observe that
\[\int_T D_t(\eta D_t w_{yx}) D_t w_{yx} \, dx = \int_T (D_t \eta) D_t w_{yx} D_t w_{yx} \, dx + \int_T \eta(D_t D_t w_{yx}) D_t w_{yx} \, dx \]
\[= \frac{1}{2} \frac{d}{dt} \int_T |D_t w_{yx}|^2 \, dx + \frac{1}{2} \int_T (D_t \eta) |D_t w_{yx}|^2 \, dx \]
\[\geq \frac{1}{2} \frac{d}{dt} \int_T |D_t w_{yx}|^2 \, dx - C \int_T |D_t w_{yx}|^2.
\]
Notice that
\[D_t w_{yxxx} = (D_t w_{yx})_{xx} - w_{xx} w_{yxx} - 2w_{x} w_{yxxx},
\]
we have
\[-\nu \int_T D_t w_{yxxx} D_t w_{yx} \, dx = -\nu \int_T (D_t w_{yx})_{xx} D_t w_{yx} \, dx + \nu \int_T (w_{xx} w_{yxxx} + 2w_{x} w_{yxxx}) D_t w_{yx} \, dx \]
\[\geq \nu \int_T |(D_t w_{yx})_{x}|^2 \, dx - C \int_T |D_t w_{yx}|^2 + |w_{yxxx}|^2 \, dx.
\]
We get, by applying the estimates (A.18), (4.33) and (4.40), that
\[\int_T (D_t(g - \eta_x D_t w_{y}) D_t w_{yx} \, dx \leq CE(y)e^{-\alpha t} + C \int_T |D_t w_{yx}|^2 \, dx.
\]
In view of (4.22), we write
\[D_t(\eta_x D_t w_{y}) = D_t \eta_x D_t w_{y} - \eta_x D_t (w_{x} w_{y} + \eta^{-1} \eta_y D_t w - \nu \eta^{-1} w_{yxx} + \eta^{-1} p(\eta)_{yx}).
\]
It suffices to deal with the highest order derivative term \(\eta_x \eta^{-1} D_t w_{yxx}\) above. Other terms can be estimated similarly, even easier. Observing that
\[D_t w_{yxx} = (D_t w_{yx})_{x} - w_{x} w_{yxx},
\]
we get
\[\int_T \eta_x \eta^{-1} (D_t w_{yxxx}) D_t w_{yx} \, dx \leq CE(y)e^{-\alpha t} + C \int_T |D_t w_{yx}|^2 \, dx + \nu \int_T |(D_t w_{yx})_{x}|^2 \, dx.
\]
For the term related to the pressure in (4.46), we have
\[
\int_{\mathbb{T}} (D_t p(\eta)_{yy}) D_t w_{yx} \, dx = \int_{\mathbb{T}} (\partial_t p(\eta)_{yy} + wp(\eta)_{yxxx}) D_t w_{yx} \, dx
\]
\[
= -\int_{\mathbb{T}}((\partial_t p(\eta)_{yx} + wp(\eta)_{yxxx})(D_t w_{yx})_x - w_x p(\eta)_{yxx} D_t w_{yx}) \, dx
\]
\[
\leq C \int_{\mathbb{T}} |\partial_t p(\eta)_{yx} + wp(\eta)_{yxxx}|^2 \, dx + \nu \int_{\mathbb{T}} |(D_t w_{yx})_x|^2 \, dx + \int_{\mathbb{T}} (|w_x p(\eta)_{yxx}|^2 + |D_t w_{yx}|^2) \, dx
\]
\[
\leq CE(y)e^{-\alpha t} + C \int_{\mathbb{T}} |D_t w_{yx}|^2 \, dx + \nu \int_{\mathbb{T}} |(D_t w_{yx})_x|^2 \, dx.
\]
By substituting the above estimate into (4.46), we arrive at
\[
(4.47) \quad \frac{d}{dt} \int_{\mathbb{T}} |D_t w_{yx}|^2 \, dx + \nu \int_{\mathbb{T}} |(D_t w_{yx})_x|^2 \, dx \leq CE(y)e^{-\alpha t} + C \int_{\mathbb{T}} (|w_{yxxx}|^2 + |D_t w_{yx}|^2) \, dx.
\]
By virtue of (4.43), (4.45) and (4.47), we can find a large enough positive constant \( A > 0 \) so that
\[
(4.48) \quad \frac{d}{dt} \int_{\mathbb{T}} (A(|w_{yxx}|^2 + |w_{xxx}|^2) + \eta |D_t w_{yx}|^2) \, dx
\]
\[
+ \int_{\mathbb{T}} (|w_{yxxx}|^2 + |D_t w_{yx}|^2 + \nu |D_t w_{yx}|_x^2) \, dx \leq CE(y)e^{-\alpha t}.
\]
Together with (4.33), we deduce from (4.48) that
\[
(4.49) \quad \|D_t w_{yx}\|_{L^2_h}^2 \leq CE(y)e^{-\alpha t}.
\]
Together with (4.33), this implies
\[
(4.50) \quad \|w_{yxxx}\|_{L^2_h}^2 \leq CE(y)e^{-\alpha t}.
\]
By summarizing the estimates (4.33), (4.40), (4.49) and (4.50), we conclude the proof of (2.4). \( \square \)

5. Decay estimates of \((\eta_{yy}, w_{yy})\)

In this section, we investigate the decay in time estimates of \((\eta_{y}, w_{y})\). We first get, by applying \(\partial_y^2\) to (1.6), that
\[
\{ \begin{align*}
\eta_{yyt} + (\eta w)_{yyx} &= 0, \\
(\eta w)_{yyt} + (\eta w^2)_{yyx} - \nu w_{yyxx} + p(\eta)_{yyx} &= 0.
\end{align*}
\]
Integrating (5.1) with respect to \(x\) over \(\mathbb{T}\) gives
\[
\frac{d}{dt} \int_{\mathbb{T}} \eta_{yy} \, dx = 0 \quad \text{and} \quad \frac{d}{dt} \int_{\mathbb{T}} (\eta w)_{yy} \, dx = 0.
\]
This implies
\[
(5.2) \quad \int_{\mathbb{T}} \eta_{yy} \, dx = 0 \quad \text{and} \quad \int_{\mathbb{T}} (\eta w)_{yy} \, dx = 0, \quad \forall t \in \mathbb{R}^+.
\]

5.1. Decay estimates of \(L^2\) norms. Introduce
\[
(5.3) \quad E^{(2)}_0(y) := \|(|\partial_y^2 \bar{\eta}_0, \partial_y^2 w_0)(y)|^2\|_{L^2_h}^2 \in (L^1 \cap L^\infty)(\mathbb{R}) \quad \text{and} \quad \bar{E}^{(2)}_0 = \sup_{y \in \mathbb{R}} E^{(2)}_0 < \infty.
\]
Throughout this subsection,\( A, \alpha \) and \( C \) are positive numbers solely depending on \((\bar{a}, \gamma, \nu, \bar{s}_0, \bar{E}_{20}, \bar{E}^{(2)}_{10}, \bar{s}_0)\), which may differ from line to line. We first give the basic energy estimates:
Lemma 5.1. For all \( t \in \mathbb{R}^+ \), one has

\[
\int_T (\eta w_{yy}^2 + n_{yy}^2) \, dx + \nu \int_0^t \int_T |w_{yyx}|^2 \, dx \leq C \left( E_{10}^{(1)}(y) + E_20(y) \right).
\]

Proof. Firstly, we rewrite (5.1) as

\[
\eta D_t w_{yy} + \eta(2w_y w_{yx} + w_{yy} w_{x}) + 2n_y(D_t w_y + w_y w_x) + \eta_{yy} D_tw - \nu w_{yyxx} + p(\eta)_{yyx} = 0.
\]

By using the decay estimates we have derived in the previous sections and testing (5.5) by \( w_{yy} \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_T \eta w_{yy}^2 \, dx + \nu \int_T |w_{yyx}|^2 \, dx
\leq \frac{\nu}{4} \int_T |w_{yyx}|^2 \, dx + C e^{-\alpha t} \int_T (\eta w_{yy}^2 + n_{yy}^2) \, dx + \int_T (p'(\eta) - \eta) n_{yy} w_{yyx} \, dx + CE_{10}^{(1)}(y)e^{-\alpha t}.
\]

On the other hand, by taking \( L^2(\mathbb{T}) \) inner product of (5.1) with \( \eta_{yy} \), we find

\[
\frac{1}{2} \frac{d}{dt} \int_T \eta_{yy}^2 \, dx = -\int_T (\eta w_{yyx} + n_{yy} + w_{yyx} w + 2n_{yy} w_y + 2n_{yy} w_y + \eta_{yy} w_{yyx} + 2n_{yy} w_{yyx} + n_{yy} + w_{yyx} \eta_{yy} \, dx
\]

By summing up (5.6) with (5.7), we achieve

\[
\frac{d}{dt} \int_T \frac{1}{2} (\eta w_{yy}^2 + n_{yy}^2) \, dx + \nu \int_T |w_{yyx}|^2 \, dx
\leq \frac{\nu}{4} \int_T |w_{yyx}|^2 \, dx + C e^{-\alpha t} \int_T (\eta w_{yy}^2 + n_{yy}^2) \, dx + \int_T (p'(\eta) - \eta) n_{yy} w_{yyx} \, dx + CE_{10}^{(1)}(y)e^{-\alpha t},
\]

from which and the decay of \( p'(\eta) - \eta \) obtained in (2.3), we infer

\[
\frac{d}{dt} \int_T (\eta w_{yy}^2 + n_{yy}^2) \, dx + \nu \int_T |w_{yyx}|^2 \, dx
\leq C e^{-\alpha t} \int_T (\eta w_{yy}^2 + n_{yy}^2) \, dx + CE_{10}^{(1)}(y)e^{-\alpha t}.
\]

Applying Gronwall’s inequality leads to (5.4). \( \square \)

Proposition 5.1. For all \( t \in \mathbb{R}^+ \),

\[
\int_T (\eta w_{yy}^2 + n_{yy}^2)(t) \, dx \leq C \left( E_{10}^{(1)}(y) + E_20(y) \right) e^{-\alpha t}.
\]

Proof. We first get, by multiplying the momentum equation of (5.1) by \( I(\eta_{yy}) \) and integrating the resulting equality over \( \mathbb{T} \), that

\[
\int_T p(\eta)_{yy} \eta_{yy} \, dx = \int_T (\eta w_{yy}) I(\eta_{yy}) \, dx - \int_T (\eta w_{yy}^2) \eta_{yy} \, dx + \nu \int_T w_{yyx} \eta_{yy} \, dx.
\]

We now handle term by term above. For the term on the left-hand side of (5.10), we have

\[
\int_T p(\eta)_{yy} \eta_{yy} \, dx = \int_T p'(\eta)(\eta_{yy})^2 \, dx + \int_T p''(\eta)(\eta_{yy})^2 \eta_{yy} \, dx
\geq \int_T p'(\eta)(\eta_{yy})^2 \, dx - C\delta E_{10}^{(1)}(y)e^{-\alpha t} \int_T (\eta_{yy})^2 \, dx - C\delta^{-1} E_{10}^{(1)}(y)e^{-\alpha t}.
\]
We then compute the first term on the right-hand side of (5.10) as follows

\[
\int_T (\eta w)_{yy} I(\eta_{yy}) \, dx = \frac{d}{dt} \int_T (\eta w)_{yy} I(\eta_{yy}) \, dx - \int_T (\eta w)_{yy} I(\eta_{yy}) \, dx
\]

\[
= \frac{d}{dt} \int_T (\eta w)_{yy} I(\eta_{yy}) \, dx + \int_T (\eta w)_{yy} ((\eta w)_{yy} \, dx
\]

\[
= \frac{d}{dt} \int_T (\eta w)_{yy} I(\eta_{yy}) \, dx + \int_T ((\eta w)_{yy})^2 \, dx
\]

\[
= \frac{d}{dt} \int_T (\eta w)_{yy} I(\eta_{yy}) \, dx + \int_T (\eta_{yy}w + 2\eta_y w_y + \eta w_{yy})^2 \, dx
\]

\[
\leq \frac{d}{dt} \int_T (\eta w)_{yy} I(\eta_{yy}) \, dx + C \left( E_{10}^{(1)}(y) + E_{20}(y) \right) e^{-\alpha t} + C \int_T \eta w_{yy}^2 \, dx,
\]

where we used (5.1), (5.2), (5.4), and the decay estimates we have obtained in the previous sections.

Similarly, we have

\[
- \int_T (\eta w^2)_{yy} \eta_{yy} \, dx = - \int_T (w^2 \eta_{yy} + 2\eta_y w w_y + \eta (w w_{yy} + w_y^2)) \eta_{yy} \, dx
\]

\[
\leq C \left( E_{10}^{(1)}(y) + E_{20}(y) \right) e^{-\alpha t} + C \int_T \eta w_{yy}^2 \, dx.
\]

For the last term in (5.10), we find

\[
\nu \int_T w_{yyx} \eta_{yy} \, dx \leq \delta \nu \int_T \eta_{yy}^2 \, dx + \delta^{-1} \nu \int_T w_{yyx}^2 \, dx.
\]

Notice that \( p'(\eta) \geq p'(\eta) > 0 \), by substituting the above estimates into (5.10) and taking \( \delta \) to be suitably small, we achieve

\[
(5.11) \quad \frac{p'(\eta)}{2} \int_T \eta_{yy}^2 \, dx \leq \frac{d}{dt} \int_T (\eta w)_{yy} I(\eta_{yy}) \, dx + C \left( E_{10}^{(1)}(y) + E_{20}(y) \right) e^{-\alpha t} + C \int_T (\eta w_{yy}^2 + w_{yyx}^2) \, dx.
\]

While it follows from a similar proof of Lemma 3.3 that

\[
\int_T (\eta w_{yy}^2 - (\eta w_{yy})^2) \, dx \leq \hat{\eta}^2 \int_T w_{yyx}^2 \, dx.
\]

In view of (5.2), it holds that

\[
\langle \eta w_{yy} \rangle = -\langle w \eta_{yy} \rangle - \langle 2\eta_y w_y \rangle.
\]

As a result, we infer

\[
\int_T \eta w_{yy}^2 \, dx \leq \hat{\eta}^2 \int_T w_{yyx}^2 \, dx + (\eta w_{yy})^2
\]

\[
\leq \hat{\eta}^2 \int_T w_{yyx}^2 \, dx + 8 \int_T (\eta_{yy} w^2 + \eta_y w_y^2) \, dx
\]

\[
\leq C \left( E_{10}^{(1)}(y) + E_{20}(y) \right) e^{-\alpha t} + C \int_T w_{yyx}^2 \, dx.
\]

Thanks to (5.11) and (5.12), we deduce that

\[
(5.13) \quad \frac{p'(\eta)}{2} \int_T \eta_{yy}^2 \, dx - \frac{d}{dt} \int_T (\eta w)_{yy} I(\eta_{yy}) \, dx \leq C \left( E_{10}^{(1)}(y) + E_{20}(y) \right) e^{-\alpha t} + C \int_T \eta w_{yy}^2 \, dx.
\]
Let $A$ be a sufficiently large positive constant, we denote

\begin{equation}
F_1^{(2)} \overset{\text{def}}{=} \int_T (A(\eta w_y^2 + \eta_{yy}^2) - (\eta w)_{yy} I(\eta_{yy})) \, dx \quad \text{and} \quad F_2^{(2)} \overset{\text{def}}{=} F_1^{(1)} + F_1^{(2)},
\end{equation}

where $F_1^{(1)}$ is given in (4.15).

Then by virtue of (5.8) and (5.13), we get

\begin{equation}
\frac{d}{dt} F_1^{(2)}(t) + \int_T (w_{yy}^2 + \frac{p'(\eta)}{2} \eta_{yy}^2) \, dx \leq C \left( E_{10}^{(1)}(y) + E_{20}(y) \right) e^{-\alpha t}.
\end{equation}

Notice that

\begin{equation}
\left| \int_T (\eta w)_{yy} I(\eta_{yy}) \, dx \right| \leq \| (\eta w)_{yy} \|_{L^1_T} \| \eta_{yy} \|_{L^1_T}
\end{equation}

\begin{equation}
\leq \| \eta_{yy} \| + 2\eta_y w_y + \eta w_{yy} \| \eta_{yy} \|_{L^1_T}
\leq C \int_T (\eta w_y^2 + \eta_{yy}^2 + \eta_y^2 + w_y^2) \, dx.
\end{equation}

Then by (4.16) and (5.15), we infer

\begin{equation}
\frac{d}{dt} F_2^{(2)}(t) + \int_T (w_{yy}^2 + \eta_{yy}^2 + \frac{p'(\eta)}{2} (\eta_y^2 + \eta_{yy}^2)) \, dx \leq C \left( E_{10}^{(1)}(y) + E_{20}(y) \right) e^{-\alpha t}.
\end{equation}

On the other hand, in view of (4.13), (5.12) and (5.16), by choosing $A$ suitably large, we deduce that

\begin{equation}
\int_T (\eta w_y^2 + \eta_{yy}^2 + \eta w_{yy}^2 + \eta_{yy}^2) \, dx \leq F_2^{(2)}(t) \leq C \int_T (\eta w_y^2 + \eta_{yy}^2 + \eta w_{yy}^2 + \eta_{yy}^2) \, dx
\end{equation}

\begin{equation}
\leq C \int_T (w_{yy}^2 + \eta_{yy}^2 + \eta_{yy}^2 + \eta_{yy}^2) \, dx.
\end{equation}

Our desired estimate (5.9) follows from (5.17) and (5.18). \hfill \Box

5.2. Proof of Proposition 2.3.

Proof of Proposition 2.3. We divide the proof into the following two steps:

**Step 1.** Decay estimate of $\eta_{yyx}$.

The idea to derive the decay estimate of $\eta_{yyx}$ is similar as that of $\eta_{yxx}$ in Section 4.4. In what follows, we just outline its derivation. In order to do so, first we rewrite (5.1) as

\begin{equation}
D_t w_y + (2w_y w_{yx} + w_{yy} w_x) + \eta^{-1}(2\eta_y(D_t w_y + w_y w_x) + \eta_{yy} D_t w - \nu w_{yyxx} + p(\eta)_{yyx}) = 0.
\end{equation}

While it follows from (5.1) that

\begin{equation}
\zeta w_{yyxx} = D_t \zeta_{yyx} - \zeta_{yyxx} - \zeta_y w_{yyxx} - \zeta_{yx} w_y + \zeta_{xx} w_{yy}.
\end{equation}

Plugging (5.20) into (5.19) gives

\begin{equation}
D_t(w_y - \nu \zeta_{yyx}) + \eta^{-1} p(\eta)_{yyx} + g_1 = 0,
\end{equation}

with

\begin{equation}
g_1 = (2w_y w_{yx} + w_{yy} w_x) + 2\eta^{-1} \eta_y(D_t w_y + w_y w_x) + \eta^{-1} \eta_{yy} D_t w
\end{equation}

\begin{equation}
- \nu (-\zeta_{yx} w_{xx} - \zeta_y w_{yyx} - \zeta_{yx} w_{xx} - \zeta_{yx} w_{yyx} + \zeta_{xx} w_y + \zeta_{xx} w_{yy}).
\end{equation}

Then we deduce from Propositions 2.1, 2.2 and 5.1 that

\begin{equation}
\| g_2 \|_{L^2_T}^2 \leq CE(y)e^{-\alpha t}.
\end{equation}
By taking $L^2(\mathbb{T})$ inner product of (5.21) with $\eta(w_{yy} - \nu\zeta_{yyx})$ and using (5.22), we find
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} \eta(w_{yy} - \nu\zeta_{yyx})^2 \, dx - \int_{\mathbb{T}} p'(\eta)\zeta^{-2}\zeta_{yxx}(w_{yy} - \nu\zeta_{yyx}) \, dx \\
\leq C\delta^{-1} E(y)e^{-\alpha t} + \delta \int_{\mathbb{T}} \eta(w_{yy} - \nu\zeta_{yyx})^2 \, dx,
\]
from which, we infer
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} \eta(w_{yy} - \nu\zeta_{yyx})^2 \, dx + \nu^{-1} \int_{\mathbb{T}} p'(\eta)\eta^2(w_{yy} - \nu\zeta_{yyx})^2 \, dx \\
\leq C\delta^{-1} E(y)e^{-\alpha t} + \delta \int_{\mathbb{T}} \eta(w_{yy} - \nu\zeta_{yyx})^2 \, dx.
\]
Taking $\delta = (2\nu)^{-1}p'(\eta)\eta$ in (5.23) gives
\[
\frac{d}{dt} \int_{\mathbb{T}} \eta(w_{yy} - \nu\zeta_{yyx})^2 \, dx + \nu^{-1} p'(\eta)\eta \int_{\mathbb{T}} \eta(w_{yy} - \nu\zeta_{yyx})^2 \, dx \leq C E(y)e^{-\alpha t},
\]
from which, we infer
\[
\int_{\mathbb{T}} \eta(w_{yy} - \nu\zeta_{yyx})^2 \, dx \leq C E(y)e^{-\alpha t},
\]
and
\[
\int_{\mathbb{T}} |\zeta_{yyx}|^2 \, dx \leq C E(y)e^{-\alpha t} \quad \text{and} \quad \int_{\mathbb{T}} |\eta_{yyx}|^2 \, dx \leq C E(y)e^{-\alpha t}.
\]

**Step 2.** Decay estimates of $D_t w_{yy}$.

The main idea to derive the decay estimates of $D_t w_{yy}$ is analogues to that of $D_t w_{yx}$ in Section sec-decay-y-all. We shall outline its proof below.

Observing that
\[
\int_{\mathbb{T}} D_t w_{yy} w_{yyx} = \int_{\mathbb{T}} \partial_t w_{yy} w_{yyx} \, dx + \int_{\mathbb{T}} w w_{yy} w_{yyxx} \, dx \\
= -\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} |w_{yyx}|^2 \, dx - \frac{1}{2} \int_{\mathbb{T}} w_x (w_{yy})^2 \, dx,
\]
Then by taking $L^2(\mathbb{T})$ inner product of (5.5) with $\eta^{-1} w_{yyxx}$, we deduce that
\[
\frac{d}{dt} \int_{\mathbb{T}} |w_{yyx}|^2 \, dx + \nu \int_{\mathbb{T}} |w_{yyxx}|^2 \, dx \leq C E(y)e^{-\alpha t}.
\]

While due to
\[
-\nu \int_{\mathbb{T}} w_{yyxx} D_t w_{yy} \, dx = -\nu \int_{\mathbb{T}} w_{yyxx} \partial_t w_{yy} \, dx - \nu \int_{\mathbb{T}} w_{yyxx} w w_{yyx} \, dx \\
= \nu \frac{d}{dt} \int_{\mathbb{T}} |w_{yyx}|^2 + \nu \frac{d}{dt} \int_{\mathbb{T}} (w_{yy})^2 w_x \\
\geq \nu \frac{d}{dt} \int_{\mathbb{T}} |w_{yyx}|^2 \, dx - C E(y)e^{-\alpha t} \int_{\mathbb{T}} |w_{yyx}|^2 \, dx,
\]
we get, by taking $L^2(\mathbb{T})$ inner product of (5.5) with $D_t w_{yy}$, that
\[
\nu \frac{d}{dt} \int_{\mathbb{T}} |w_{yyx}|^2 \, dx + \int_{\mathbb{T}} \eta |D_t w_{yy}|^2 \, dx \leq C E(y)e^{-\alpha t} + C E(y)e^{-\alpha t} \int_{\mathbb{T}} |w_{yyx}|^2 \, dx.
\]
Notice that
\[
\int_T D_t(\eta D_t w_{yy}) D_t w_{yy} \, dx = \int_T (D_t \eta) D_t w_{yy} D_t w_{yy} \, dx + \int_T \eta (D^2_t w_{yy}) D_t w_{yy} \, dx
\]
\[
= \frac{1}{2} \frac{d}{dt} \int_T \eta |D_t w_{yy}|^2 + \frac{1}{2} \frac{d}{dt} \int_T (D_t \eta) |D_t w_{yy}|^2 \, dx
\]
\[
\geq \frac{1}{2} \frac{d}{dt} \int_T \eta |D_t w_{yy}|^2 - C \int_T |D_t w_{yy}|^2 \, dx
\]
and
\[
-\nu \int_T (D_t w_{yy}) D_t w_{yy} \, dx = -\nu \int_T (D_t w_{yy}) D_t w_{yy} \, dx + \int_T \nu (w_{xx} w_{yy} + 2 w_x w_{yy} x) D_t w_{yy} \, dx
\]
\[
\geq \nu \int_T |(D_t w_{yy})|^2 \, dx - C \int_T (|D_t w_{yy}|^2 + |w_{yy} x|^2) \, dx,
\]
we get, by applying $D_t$ to (5.5) and then taking $L^2(T)$ inner product of the resulting equation with $D_t w_{yy}$, that
\[(6.28)\]
\[
\frac{d}{dt} \int_T \eta |D_t w_{yy}|^2 \, dx + \nu \int_T |(D_t w_{yy})|^2 \, dx \leq CE(y) e^{-\alpha t} + C \int_T (|w_{yy} x|^2 + |D_t w_{yy}|^2 + |w_{yy} x|^2) \, dx.
\]

By virtue of (5.15)–(5.18), we deduce from (5.26), (5.27) and (5.28) that
\[(6.29)\]
\[
\|D_t w_{yy}\|_{L^2}^2 \leq CE(y) e^{-\alpha t},
\]
from which and (5.1), we infer
\[(6.30)\]
\[
\|w_{yy} x\|_{L^2}^2 \leq CE(y) e^{-\alpha t}.
\]
By summarizing the estimates (5.9), (5.25), (5.29) and (5.30), we conclude the proof of (2.5). \qed

6. DECAY ESTIMATES OF $w$

With the decay estimates for $(\eta, w)$ obtained in the previous sections, we are going to derive the same exponential decay estimates for $w$. Notice that the equation for $w$, (1.7), is of standard parabolic type, we shall only present the main decay in time estimates of $w$ and skip the derivation of the related estimates concerning $(\partial_y w, \partial_y^2 w)$.

The main results state as follows

**Proposition 6.1.** For all $t \in \mathbb{R}^+$, one has

\[(6.1a)\]
\[
\int_T |w|^2 \, dx \leq \eta^{-1} e^{-\mu \eta^{-2} t} \int_T \eta_0 |w_0|^2 \, dx,
\]

\[(6.1b)\]
\[
\int_T (|w|^2 + |w_x|^2) \, dx \leq CE(y) e^{-\alpha t},
\]

\[(6.1c)\]
\[
\int_T (|w|^2 + |w_x|^2 + |D_t w|^2 + |\partial_{xx} w|^2) \, dx \leq CE(y) e^{-\alpha t}.
\]

**Proof.** 1) We first get, by taking $L^2(T)$ inner product of (1.7) with $w$ and using integrating by parts, that

\[(6.2)\]
\[
\frac{1}{2} \frac{d}{dt} \int_T \eta |w|^2 \, dx + \mu \int_T |w_x|^2 \, dx = 0.
\]

While observing from the density equation of (1.6) and (1.7) that
\[
(\eta w)_t + (\eta w)_x - \mu \partial_x^2 w = 0.
\]
Integrating the above equation over $T$ gives
\[ \frac{d}{dt} \int_T (\eta w) \, dx = 0, \]
which together with (1.10) ensures that
\[ \int_T (\eta w) \, dx = \int_T \eta_0 w_0 \, dx = 0. \]

Then we get, by using a similar proof of Lemma 3.3, that
\[ \int_T \eta |w|^2 \, dx \leq \eta^2 \int_T |w_x|^2 \, dx. \]

By virtue of (6.4), we deduce from (6.2) that
\[ \int_T \eta |w|^2 \, dx \leq e^{\mu t} \int_T \eta_0 |w_0|^2 \, dx, \]
which together with (3.30) ensures (6.1a).

2) By taking $L^2$ inner product of (1.7) with $D_t w$ and using integrating by parts, we obtain
\[ \frac{1}{2} \int_T \eta |D_t w|^2 \, dx + \mu \frac{d}{dt} \int_T |w_x|^2 \, dx \leq \frac{\mu}{2} \|w_x\|_{L^\infty(\Omega)} \int_T |w_x|^2 \, dx. \]

By multiplying (6.2) by $A_1 \overset{\text{def}}{=} 1 + \|w_x\|_{L^\infty((0,\infty) \times \Omega)}$ and summing up the resulting inequality with (6.5), we get
\[ \int_T \eta |D_t w|^2 \, dx + \frac{d}{dt} \int_T (A_1 \eta |w|^2 + \mu |w_x|^2) \, dx + A_1 \int_T \mu |w_x|^2 \, dx \leq 0. \]

Due to $D_t w = \mu \partial_2^2 w$, (6.1b) follows.

3) Applying $D_t$ to (1.7) gives
\[ D_t (\eta D_t w) - \mu D_t \partial_2^2 w = 0. \]

We observe that
\[ D_t (\eta D_t w) = (-\eta \partial_x w) D_t w + \eta D_t^2 w, \]
and
\[ D_t \partial_2^2 w = \partial_2^2(D_t w) - \partial_x (\partial_x w \partial_x w). \]

As a result, it comes out
\[ \eta D_t^2 w - \mu \partial_2^2(D_t w) = (\eta \partial_x w) D_t w - \mu \partial_x (\partial_x w \partial_x w). \]

By taking $L^2(T)$ inner product of (6.6) with $D_t w$, we find
\[ \frac{1}{2} \frac{d}{dt} \int_T \eta |D_t w|^2 \, dx + \mu \int_T |\partial_x(D_t w)|^2 \, dx \]
\[ = \int_T (\eta \partial_x w) |D_t w|^2 \, dx + \int_T \mu (\partial_x w \partial_x w) \partial_x D_t w \, dx \]
\[ \leq \|\partial_x w\|_{L^\infty(\Omega)} \int_T \eta |D_t w|^2 \, dx + \frac{\mu}{2} \|\partial_x w\|^2_{L^\infty(\Omega)} \int_T |\partial_x w|^2 \, dx + \frac{\mu}{2} \int_T |\partial_x(D_t w)|^2 \, dx. \]

Multiplying (6.5) by $A_2 \overset{\text{def}}{=} 2 + 2\|\partial_x w\|^2_{L^\infty((0,\infty) \times \Omega)}$ and summing up the resulting inequality with (6.7) yields
\[ \frac{d}{dt} \int_T (A_1 A_2 \eta |w|^2 + \mu A_2 |w_x|^2 + \eta |D_t w|^2) \, dx + \int_T (\mu |w_x|^2 + \eta |D_t w|^2 + \mu |\partial_x(D_t w)|^2) \, dx \leq 0. \]
Then by using a similar proof of Lemma 3.3 and Gronwall’s inequality, we find

\[
(6.9) \quad \int_T \left( |w|^2 + |w_x| + |D_t w|^2 \right) dx \leq CE(y)e^{-\alpha t}.
\]

Observing that \( \mu \partial_{xx} w = \eta D_t w \), we conclude the proof of (6.1c). This completes the proof of Proposition 6.1.

Let us now outline the proof of Proposition 2.4.

Proof of Proposition 2.4. Along the same line to proof of Proposition 6.1 and through the induction method as what we used in the Appendix A, we deduce that

\[
(6.10) \quad \|w(t)\|_{H^8} + \|w(t)\|_{H^1} + \|w_t(t)\|_{H^4} \leq CE(y)e^{-\alpha t}.
\]

The decay estimates related to \( y \)-derivatives of \( w \) in (2.6) can be derived along the same line. We omit the details here. \( \square \)

7. Energy estimates for the perturbed equations

The purpose of this section is to present the proof of Propositions 2.6 and 2.7. For simplicity, we shall neglect the subscript \( \varepsilon \) in the rest of this section.

7.1. Basic energy estimate. In this subsection, we shall derive a basic energy estimate for all \( t < T^* \). We first deduce from Proposition 2.5 that

Lemma 7.1. Let \( (\rho, u) \) and \( (\rho^a, u^a) \) be respectively given by (2.7) and (2.11). Then one has

\[
(7.1) \quad \mathcal{E}_1((\rho, u)|(\rho^a, u^a))(t) + \int_0^t \int_\Omega (\mu |\nabla (u - u^a)|^2 + \mu' |\text{div} (u - u^a)|^2) \, dx \, dy \, dt' = \int_0^t \mathcal{R}(t') \, dt',
\]

where for \( G \) given by (2.9),

\[
\mathcal{R}(t) \overset{\text{def}}{=} \int_\Omega \left( (\rho^a)^{-1}(\rho - \rho^a)(\mu \Delta u^a + \mu' \nabla \text{div} u^a) + \rho(u - u^a) \cdot \nabla u^a 
\right.
\]

\[
+ \rho(\rho^a)^{-1}G \right) \cdot (u^a - u) \, dx \, dy + \varepsilon \int_\Omega (\rho^a - \rho)(\rho^a)^{-1} p^a(\rho^a)[(\eta \nabla w)|_\varepsilon) \, dx \, dy
\]

\[
- \int_\Omega \text{div} u^a(p(\rho) - p(\rho^a) - p'(\rho^a)(\rho - \rho^a)) \, dx \, dy.
\]

Proof. Since \( (\rho, u) \) and \( (\rho^a, u^a) \) have the same initial data, we get, by applying Proposition 7.1, that (7.1) holds with

\[
(7.3) \quad \mathcal{R}_1(t) \overset{\text{def}}{=} \int_\Omega \left( \rho \partial_t u^a \cdot (u^a - u) + \mu \nabla u^a : \nabla (u^a - u) + \mu' \text{div} u^a \text{div} (u^a - u) 
\right.
\]

\[
+ (\rho^a - \rho) \partial_t P'(\eta) + (\rho^a u^a - \rho u) \cdot \nabla P'(\rho^a) - \text{div} u^a(p(\rho) - p(\rho^a)) \Big) \, dx \, dy.
\]

It follows from the \( u^a \) equation of (2.8) that

\[
\mathcal{R}_1(t) = \int_\Omega \rho \left( \rho^a \right)^{-1}(\mu \Delta u^a + \mu' \nabla \text{div} u^a - \nabla p(\rho^a) + G) + (u - u^a) \cdot \nabla u^a \right) \, dx \, dy.
\]
By using integration by parts and the fact that \( P''(s) = s^{-1}p'(s) \), we find
\[
\mathcal{R}_1(t) = \int_\Omega \left( (\rho^a)^{-1}(\rho - \rho^a)(\mu \Delta u^a + \mu' \nabla \div u^a) - \rho \nabla P'(\rho^a) + \rho(\rho^a)^{-1}G \right) \cdot (u^a - u) \, dx \, dy
- \int_\Omega (\rho \nabla u^a : \nabla (u^a - u) + \mu' \div (u^a - u)) \, dx \, dy
+ \int_\Omega \rho(u - u^a) \cdot \nabla u^a \cdot (u^a - u) \, dx \, dy.
\]
Plugging the above equality into (7.3) gives
\[
\mathcal{R}(t) = \int_\Omega \left( (\rho^a)^{-1}(\rho - \rho^a)(\mu \Delta u^a + \mu' \nabla \div u^a) + \rho(\rho^a)^{-1}G + (u - u^a) \cdot \nabla u^a \right)
- \int_\Omega \left( \rho \nabla P'(\rho^a) \right) \cdot (u^a - u) \, dx \, dy
- \int_\Omega \left( \div (u^a (p(\rho) - p(\rho^a))) \right) \, dx \, dy
+ \int_\Omega \left( (\rho^a - \rho) \partial_t P'(\eta) + (\rho^a u^a - \rho u) \cdot \nabla P'(\rho^a) \right) \, dx \, dy.
\]
Notice that \( P''(s) = s^{-1}p'(s) \) and the renormalized equation
\[
\partial_t P'(\rho^a) + \div (u^a P'(\rho^a)) + (P''(\rho^a)\rho^a - P'(\rho^a)) \div u^a = \varepsilon P''(\rho^a)[(\eta w)\eta]_\varepsilon.
\]
We get, by using the continuity equation (2.8)_1, that
\[
- \rho \nabla P'(\rho^a) \cdot (u^a - u) + (\rho^a - \rho) \partial_t P'(\eta) + (\rho^a u^a - \rho u) \cdot \nabla P'(\rho^a)
= (\rho^a - \rho) \left( \partial_t P'(\rho^a) + u^a \cdot \nabla P'(\rho^a) \right)
= (\rho^a - \rho) \left( \left( \partial_t P'(\rho^a) + \div (u^a P'(\rho^a)) + (P''(\rho^a)\rho^a - P'(\rho^a)) \div u^a \right) - P''(\rho^a)\rho^a \div u^a \right)
= \varepsilon (\rho^a - \rho) (\rho^a - 1) p'(\rho^a) [(\eta w)\eta]_\varepsilon - (\rho^a - \rho) p'(\rho^a) \div u^a.
\]
Then (7.2) follows by inserting (7.6) into (7.4). \( \square \)

We now present the proof of Proposition 2.6.

**Proof of Proposition 2.6.** We first get, by applying (2.15) and Taylor’s expansion, that for \( t < T^* \),
\[
\mathcal{E}_1(t) = \mathcal{E}_1((\rho, u)(\rho^a, u^a))(t)
= \int_\Omega \left( \frac{1}{2} \rho |u - u^a|^2 + P(\rho) - P(\rho^a) - P'(\rho^a)(\rho - \rho^a) \right) \, dx \, dy
\geq \int_\Omega \left( \frac{\eta}{4} |R|^2 + \gamma (2\eta)\gamma^{-2} |\varrho|^2 \right) \, dx \, dy.
\]
While it follows from Lemma 7.1 that
\[
\mathcal{E}_1(t) + \int_t^T \int_\Omega \left( \mu |\nabla R|^2 + \mu' |\div R|^2 \right) \, dx \, dy \, dt' \leq \int_0^T \mathcal{R}(t') \, dt'.
\]
According to (7.2), we decompose \( \mathcal{R} \) as \( \mathcal{R}(t) = \sum_{j=1}^{5} \mathcal{R}_j(t) \). We first deduce from Theorem 1.1 that
\[
\mathcal{R}_1(t)
= \int_\Omega \left( (\rho^a)^{-1}(\rho - \rho^a)(\mu \Delta u^a + \mu' \nabla \div u^a) \cdot (u^a - u) \right) \, dx \, dy
\leq \|u^a\|^2_{L^2(\Omega)} + \|\mathcal{R}\|^2_{L^2(\Omega)} \leq C e^{-\alpha t} \left( \|\varrho\|^2_{L^2(\Omega)} + \|R\|^2_{L^2(\Omega)} \right).
\]
For the second term in (7.2), we have
\[
\mathcal{R}_2(t)
= \int_\Omega \rho(\rho^a)^{-1} G \cdot (u^a - u) \, dx \, dy \leq \|G\|_{L^2(\Omega)} (\rho^a)^{-1} \|\rho\|_{L^\infty(\Omega)} \|G\|^2_{L^2(\Omega)} \|R\|^2_{L^2(\Omega)}.
\]
which together with (2.10) and Theorem 1.1 ensures that
\[ \mathcal{R}_2(t) \leq C e^{-\alpha t} \| R \|_{L^2(\Omega)} . \]

For the third term in (7.2), we get, by applying Theorem 1.1, that
\[ \mathcal{R}_3(t) \triangleq \int_{\Omega} \rho(u - u^a) \cdot \nabla u^a \cdot (u^a - u) \, dx \, dy \leq \| \nabla u^a \|_{L^\infty(\Omega)} \int_{\Omega} \rho |R|^2 \, dx \, dy \leq C e^{-\alpha t} \mathcal{E}_1(t) . \]

Along the same line and thanks to (2.17), we have
\[ \mathcal{R}_4(t) \triangleq - \int_{\Omega} \text{div } u^a (p(\rho) - p(\rho^a) - p'(\rho^a)(\rho - \rho^a)) \, dx \, dy \leq (\gamma - 1) \| \text{div } u^a \|_{L^\infty(\Omega)} \int_{\Omega} (P(\rho) - P(\rho^a) - P'(\rho^a)(\rho - \rho^a)) \, dx \, dy \leq C e^{-\alpha t} \mathcal{E}_1(t) , \]

and
\[ \mathcal{R}_5(t) \triangleq \varepsilon \int_{\Omega} (\rho^a - \rho)(\rho^a)^{-1} p'(\rho^a)(\eta \mathbf{w})_y \, dx \, dy \leq \varepsilon \| (\rho^a)^{-1} p'(\rho^a) \|_{L^\infty(\Omega)} \| (\eta \mathbf{w})_y \|_{L^2(\Omega)} \| \mathbf{R} \|_{L^2(\Omega)} \leq C e^{\frac{\varepsilon^2}{2}} e^{-\alpha t} \| \mathbf{R} \|_{L^2(\Omega)} . \]

By inserting the above estimates into (7.8) and using (7.7), we deduce that
\[ \mathcal{E}_1(t) + \int_0^t \int_{\Omega} \mu |\nabla R|^2 \, dx \, dy \, dt' \leq C \int_0^t e^{-\alpha t'} \left( \varepsilon^{\frac{1}{2}} \mathcal{E}_1(t')^{1/2} + \mathcal{E}_1(t') \right) \, dt' . \]

Applying Gornwall’s inequality gives rise to
\[ \mathcal{E}_1(t) + \int_0^t \int_{\Omega} \mu |\nabla R|^2 \, dx \, dy \, dt' \leq C \varepsilon , \]

which together with (7.7) ensures (2.21). This completes the proof of Proposition 2.6. \( \square \)

### 7.2. Estimates of \( \nabla R \)

**Lemma 7.2.** For each \( t < T^* \), there holds
\[ \int_{\Omega} |\nabla R|^2 \, dx \, dy + \int_0^t \int_{\Omega} |\mathcal{D}_t R|^2 \, dx \, dy \leq C \int_0^t \int_{\Omega} |\nabla R|^3 \, dx \, dy + C \varepsilon . \]

**Proof.** We first get, by taking \( L^2 \) inner product of the \( R \) equation of (2.13) with \( \mathcal{D}_t R \), that
\[ \int_{\Omega} \rho |\mathcal{D}_t R|^2 \, dx \, dy - \int_{\Omega} (\mu \Delta R + \mu' \nabla \text{div } R) \cdot \mathcal{D}_t R \, dx \, dy + \int_{\Omega} \nabla (p(\rho) - p(\rho^a)) \cdot \mathcal{D}_t R \, dx \, dy \]
\[ + \int_{\Omega} \left( \rho R \cdot \nabla u^a + \varepsilon (\partial_t u^a + u^a \cdot \nabla u^a) - G \right) \cdot \mathcal{D}_t R \, dx \, dy = 0 . \]

Let us now handle term by term above. By using integration by parts, we find
\[ \int_{\Omega} \Delta R \cdot \mathcal{D}_t R \, dx \, dy = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla R|^2 \, dx \, dy + \int_{\Omega} \nabla R : (\nabla u \nabla R + (u \cdot \nabla) \nabla R) \, dx \, dy . \]

Due to \( u = u^a + R \), one has
\[ \int_{\Omega} \nabla R : (\nabla u \nabla R) \, dx \, dy = \int_{\Omega} \nabla R : (\nabla u^a \nabla R) \, dx \, dy + \int_{\Omega} \nabla R : (\nabla R \nabla R) \, dx \, dy \leq \| \nabla u^a \|_{L^\infty} \int_{\Omega} |\nabla R|^2 \, dx \, dy + \int_{\Omega} |\nabla R|^3 \, dx \, dy , \]
and 
\[ \int_{\Omega} \nabla R : ((u \cdot \nabla) \nabla R) \, dx \, dy = -\frac{1}{2} \int_{\Omega} (\text{div } u^a)|\nabla R|^2 \, dx \, dy - \frac{1}{2} \int_{\Omega} (\text{div } R)|\nabla R|^2 \, dx \, dy. \]

This together with Theorem 1.1 ensures that
\[ (7.11) \quad -\int_{\Omega} \Delta R \cdot \mathcal{D}_t R \, dx \, dy \geq \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla R|^2 \, dx \, dy - Ce^{-at} \int_{\Omega} |\nabla R|^2 \, dx \, dy - \int_{\Omega} |\nabla R|^3 \, dx \, dy. \]

Exactly along the same line, one has
\[ (7.12) \quad -\int_{\Omega} \nabla \text{div } R \cdot \mathcal{D}_t R \, dx \, dy \geq \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\text{div } R|^2 \, dx \, dy - Ce^{-at} \int_{\Omega} |\nabla R|^2 \, dx \, dy - \int_{\Omega} |\nabla R|^3 \, dx \, dy. \]

It is a little trickier to deal with the pressure term in (7.10). Indeed we first observe that
\[ (7.13) \quad \partial_t (p(\rho) - p(\rho^a)) = -\text{div } (p(\rho)u - p(\rho^a)u^a) - a(\gamma - 1)(\rho^\gamma \text{div } u - (\rho^a)^\gamma \text{div } u^a) - \varepsilon p'(\rho^a)[(\eta u)_y]_\varepsilon \]
so that
\[ \int_{\Omega} \partial_t (p(\rho) - p(\rho^a)) \, dx \, dy = \int_{\Omega} (p(\rho)u - p(\rho^a)u^a) \cdot \nabla \text{div } R \, dx \, dy 
- a(\gamma - 1) \int_{\Omega} (\rho^\gamma \text{div } u - (\rho^a)^\gamma \text{div } u^a) \, dx \, dy - \varepsilon \int_{\Omega} p'(\rho^a)[(\eta u)_y]_\varepsilon \, dx \, dy. \]

While we observe that
\[ -\int_{\Omega} (p(\rho) - p(\rho^a)) \, dx \, dy = -\int_{\Omega} (p(\rho)u - p(\rho^a)u^a) \cdot \nabla \text{div } R \, dx \, dy 
+ \int_{\Omega} (p(\rho^a)R \cdot \nabla \text{div } R \, dx \, dy - \int_{\Omega} (p(\rho) - p(\rho^a))(\nabla u^a + \nabla R) : \nabla R \, dx \, dy. \]

As a result, it comes out
\[ (7.14) \quad \int_{\Omega} \nabla (p(\rho) - p(\rho^a)) \cdot \mathcal{D}_t R \, dx \, dy = -\frac{d}{dt} \int_{\Omega} (p(\rho) - p(\rho^a)) \, dx \, dy 
- a(\gamma - 1) \int_{\Omega} (\rho^\gamma \text{div } u - (\rho^a)^\gamma \text{div } u^a) \, dx \, dy - \varepsilon \int_{\Omega} p'(\rho^a)[(\eta u)_y]_\varepsilon \, dx \, dy 
+ \int_{\Omega} (p(\rho^a)R \cdot \nabla \text{div } R \, dx \, dy - \int_{\Omega} (p(\rho) - p(\rho^a))(\nabla u^a + \nabla R) : \nabla R \, dx \, dy. \]

Observe that
\[ \int_{\Omega} (\rho^\gamma \text{div } u - (\rho^a)^\gamma \text{div } u^a) \, dx \, dy 
\leq \rho^\gamma \int_{\Omega} |\text{div } R|^2 \, dx \, dy + \|\text{div } u^a\|_{L^\infty(\Omega)} \int_{\Omega} ((\rho^\gamma - (\rho^a)^\gamma)^2 + |\text{div } R|^2) \, dx \, dy, \]
By substituting the above estimates into (7.14) and using Theorem 1.1, we find

\[ \varepsilon \int \Omega [p'((\rho^a)(\eta \mu_y) + \Delta \rho)] \, d\nu \leq \varepsilon^2 \|p'(\rho^a)\|_{L^\infty(\Omega)} \|((\eta \mu_y) + \Delta \rho)\|_{L^2(\Omega)} \|\nu\|_{L^2(\Omega)}, \]

and

\[ \int \Omega p(\rho^a) R \cdot \nabla \nu \, d\nu \leq \int \Omega (p(\rho) - p(\rho^a)) \nabla R : \nabla R \, d\nu \]

\[ = - \int \Omega \left( (\nabla p(\rho^a)) \cdot R \cdot \nabla R + (p(\rho) - p(\rho^a)) \nabla R + p(\rho) \nabla R : \nabla R \right) \, d\nu \]

\[ \leq \|\nabla p(\rho^a)\|_{L^\infty} \|R\|_{L^2}^2 + \|\nabla R\|_{L^2}^2 \|\rho - \rho^a\|_{L^2}^2 + C \|\nabla R\|_{L^2}^2. \]

By substituting the above estimates into (7.14) and using Theorem 1.1, we find

\[ \int \Omega \left( \nabla (p(\rho) - p(\rho^a)) \cdot \mathbf{D} \nu R \, d\nu \right) \leq - \frac{\partial}{\partial t} \int \Omega (p(\rho) - p(\rho^a)) \, d\nu \]

\[ + Ce^{-\alpha t} \left( \|\nu\|_{L^2}^2 + \|R\|^2_{L^2} + \|\nabla R\|^2_{L^2} \right) + C \|\nabla R\|_{L^2}^2. \]

We further observe that

\[ \int \Omega \rho \nabla u^a \cdot \mathbf{D} \nu R \, d\nu \leq 2 \|\nabla u^a(t)\|_{L^\infty}^2 \int \Omega \rho |\nu| \, d\nu \]

\[ + \frac{1}{12} \int \Omega \rho (\mathbf{D} \nu R)^2 \, d\nu, \]

\[ \int \Omega \rho (\partial_t u^a + u^a \cdot \nabla u^a) \cdot \mathbf{D} \nu R \, d\nu \leq 2 \|\partial_t u^a + u^a \cdot \nabla u^a\|_{L^\infty}^2 \int \Omega \rho \rho_2 \, d\nu \]

\[ + \frac{1}{12} \int \Omega \rho (\mathbf{D} \nu R)^2 \, d\nu, \]

\[ \int \Omega G \cdot \mathbf{D} \nu R \, d\nu \leq 2 \int \Omega \rho \rho_1 |G| \, d\nu \]

\[ + \frac{1}{12} \int \Omega \rho (\mathbf{D} \nu R)^2 \, d\nu. \]

This together with (2.10) and (2.21) that

\[ \int \Omega \left( \rho R \cdot \nabla u^a + \rho (\partial_t u^a + u^a \cdot \nabla u^a) - G \right) \cdot \mathbf{D} \nu R \, d\nu \leq Ce^{-\alpha t} + \frac{1}{4} \int \Omega \rho (\mathbf{D} \nu R)^2 \, d\nu. \]

By inserting the estimates (7.11), (7.12), (7.15) and (7.16) into (7.10) and and integrating the resulting inequality over [0, t], we achieve

\[ \int \Omega \left( \mu |\nabla R|^2 + \mu^2 |\nabla R|^2 \right) \, d\nu \]

\[ \leq C \varepsilon + \int \Omega (p(\rho) - p(\rho^a)) \, d\nu + 4C \int_0^t \int \Omega |\nabla R|^3 \, d\nu \, dt' + C \int_0^t \int \Omega |\nabla R|^2 \, d\nu \, dt'. \]

Notice that

\[ \int \Omega (p(\rho) - p(\rho^a)) \, d\nu \leq \mu^{-1} \int \Omega |p(\rho) - p(\rho^a)|^2 \, d\nu \]

\[ \leq C \mu^{-1} \|\nu\|_{L^2}^2 + \frac{\mu}{4} \|\nabla R\|_{L^2}^2, \]

from which and (2.21), we deduce (7.9) from (7.17). This completes the proof of the lemma. \( \square \)

7.3. Estimates of \( \mathbf{D} \nu R \).

**Lemma 7.3.** For each \( t < T^* \), one has

\[ \int \Omega |\mathbf{D} \nu R|^2 \, d\nu + \int_0^t \int \Omega |\nabla \mathbf{D} \nu R|^2 \, d\nu \, dt' \leq C \varepsilon + C \int_0^t \int \Omega |\nabla R|^4 \, d\nu \, dt'. \]
Proof. By applying $\mathcal{D}_t$ to the $R$ equation of (2.13) and taking $L^2$ inner product of the resulting equation with $\mathcal{D}_t R$, we find
\begin{equation}
\int_\Omega \left( \mathcal{D}_t (\rho \mathcal{D}_t R) - \mu \mathcal{D}_t \Delta R - \mu' \mathcal{D}_t \nabla (p(\rho) - p(\rho^a)) \\
+ \mathcal{D}_t (\rho R \cdot \nabla u^a) + \mathcal{D}_t (p(\partial_t u^a + u^a \cdot \nabla u^a)) \right) \cdot \mathcal{D}_t R \, dx \, dy = \int_\Omega \mathcal{D}_t G \cdot \mathcal{D}_t R \, dx \, dy.
\end{equation}

Let us now handle term by term above. Notice that
$$
\mathcal{D}_t (\rho D_t R) = \rho D_t^2 R + (\mathcal{D}_t \rho) \mathcal{D}_t R.
$$

Firstly,
\begin{equation}
\int_\Omega \rho \mathcal{D}_t^2 R \cdot \mathcal{D}_t R \, dx \, dy = \frac{1}{2} \frac{d}{dt} \int_\Omega \rho |\mathcal{D}_t R|^2 \, dx \, dy.
\end{equation}

By virtue of (1.1), we write
\begin{equation}
\mathcal{D}_t \rho \mathcal{D}_t R = \operatorname{div} u (-\rho \mathcal{D}_t R)
\end{equation}

\begin{equation}
= \operatorname{div} \left( -\mu \Delta R - \mu' \nabla \operatorname{div} R + \nabla (p(\rho) - p(\rho^a)) + \rho R \cdot \nabla u^a + p(\partial_t u^a + u^a \cdot \nabla u^a) - G \right).
\end{equation}

We shall postpone the estimate of the above terms below.

For $i = 1, 2$, we write
\begin{align*}
-\mu \mathcal{D}_t \Delta R_i &= -\mu \Delta \partial_t R_i - \mu \nabla \cdot (u \cdot \nabla R_i) + \mu \nabla u_j \cdot \partial_j \nabla R_i \\
&= -\mu \Delta \mathcal{D}_t R_i + \mu \nabla \cdot (u \cdot \nabla R_i) + \mu \nabla u_j \cdot \partial_j \nabla R_i.
\end{align*}

The last term above can be written as
\begin{align*}
\mu \nabla u_j \cdot \partial_j \nabla R_i &= \mu \partial_k u_j \partial_j \partial_k R_i \\
&= \mu \partial_j (\partial_k u_j \partial_k R_i) - \mu \partial_k \partial_j u_j \partial_k R_i \\
&= \mu \partial_j (\partial_k u_j \partial_k R_i) - \mu \partial_k (\partial_j u_j \partial_k R_i) + \mu \nabla u \Delta R_i.
\end{align*}

We observe that the last term above cancels with the first term on the right-side of (7.21), so that there holds
\begin{align*}
- \int_\Omega (\operatorname{div} u \Delta R + \mathcal{D}_t \Delta R) \cdot \mathcal{D}_t R \, dx \, dy \\
= - \int_\Omega \Delta D_t R \cdot D_t R \, dx \, dy + \int_\Omega \left( \partial_k (\partial_k u_j \partial_j R) + \partial_j (\partial_k u_j \partial_k R) - \partial_k (\partial_j u_j \partial_k R) \right) \cdot D_t R_i \, dx \, dy.
\end{align*}

We get, by using integration by parts, that
\begin{align*}
- \int_\Omega \Delta \mathcal{D}_t R \cdot \mathcal{D}_t R \, dx \, dy &= \int_\Omega |\nabla \mathcal{D}_t R|^2 \, dx \, dy,
\end{align*}

and
\begin{align*}
\mu \int_\Omega \left( \partial_k (\partial_k u_j \partial_j R) + \partial_j (\partial_k u_j \partial_k R) - \partial_k (\partial_j u_j \partial_k R) \right) \cdot \mathcal{D}_t R \, dx \, dy \\
&\leq 12 \mu \int_\Omega |\nabla u^a|^2 |\nabla R|^2 \, dx \, dy + \frac{\mu}{8} \int_\Omega |\nabla \mathcal{D}_t R|^2 \, dx \, dy \\
&\leq 12 \mu \int_\Omega (||\nabla u^a||^2_{L^\infty} + |\nabla R|^2) |\nabla R|^2 \, dx \, dy + \frac{\mu}{8} ||\nabla \mathcal{D}_t R||^2_{L^2}.
\end{align*}

Thus, we get, by applying Theorem 1.1, that
\begin{equation}
-\mu \int_\Omega (\operatorname{div} u \Delta R + \mathcal{D}_t \Delta R) \cdot \mathcal{D}_t R \, dx \, dy
\geq \frac{7}{8} \mu ||\nabla \mathcal{D}_t R||^2_{L^2} - C e^{-\alpha t} ||\nabla R||^2_{L^2} - 12 \mu \int_\Omega |\nabla R|^4 \, dx \, dy.
\end{equation}
Similar argument leads to

$$-\mu' \int_{\Omega} \left( \text{div} u \nabla R + \mathcal{D}_t \nabla R \right) \cdot \mathcal{D}_t R \, dx \, dy$$

(7.23)

$$\geq \frac{7}{8} \mu'' \| \text{div} \mathcal{D}_t R \|_{L^2}^2 - C e^{-\alpha t} \| \nabla R \|_{L^2}^2 - 12 \mu' \int_{\Omega} | \nabla R |^4 \, dx \, dy.$$

To handle the pressure related terms, we write

$$(\text{div} u) \nabla (p(\rho) - p(\rho^a)) + (u \cdot \nabla) \nabla (p(\rho) - p(\rho^a))$$

$$= \partial_j (u_j \nabla (p(\rho) - p(\rho^a))) = \nabla \text{div} (u(p(\rho) - p(\rho^a))) - \partial_j (\nabla u_j (p(\rho) - p(\rho^a))).$$

In view of the continuity equations of (1.1) and (2.8), one has

$$\partial_t (p(\rho) - p(\rho^a)) = - \text{div} (u(p(\rho) - p(\rho^a)) + p(\rho^a) R)$$

$$- a(\gamma - 1) (\rho^\gamma \text{div} u - (\rho^a)^\gamma \text{div} u^a) - \varepsilon \rho' (\rho^a) \eta \mathbf{v}.$$

As a consequence, we deduce that

$$(\text{div} u) \nabla (p(\rho) - p(\rho^a)) + \mathcal{D}_t \nabla (p(\rho) - p(\rho^a)) = - \partial_j (\nabla u_j (p(\rho) - p(\rho^a))) - \nabla \text{div} (p(\rho^a) R)$$

$$- a(\gamma - 1) \nabla (\rho^\gamma \text{div} u - (\rho^a)^\gamma \text{div} u^a) - \varepsilon \nabla (\rho' (\rho^a) \eta \mathbf{v}),$$

from which and Theorem 1.1, we deduce that

$$\int_{\Omega} \left( (\text{div} u) \nabla (p(\rho) - p(\rho^a)) + \mathcal{D}_t \nabla (p(\rho) - p(\rho^a)) \right) \cdot \mathcal{D}_t R \, dx \, dy$$

(7.24)

$$\leq C e^{-\alpha t} \left( \epsilon + \| a \|_{L^2}^2 + \| R \|_{L^2}^2 \right) + C \| \nabla R \|_{L^2}^2 + \frac{H}{2} \| \nabla \mathcal{D}_t R \|_{L^2}^2.$$

On the other hand, thanks to the continuity equation of (1.6), we infer

$$\int_{\Omega} \left( \mathcal{D}_t (\rho R \cdot \nabla u^a) + (\text{div} u) \rho R \cdot \nabla u^a \right) \cdot \mathcal{D}_t R \, dx \, dy$$

$$= \int_{\Omega} \left( \rho (\mathcal{D}_t R) \cdot \nabla u^a + \rho R \cdot \mathcal{D}_t \nabla u^a \right) \cdot \mathcal{D}_t R \, dx \, dy$$

$$\leq \left( \| \nabla u^a \|_{L^\infty} + \| \partial_t \nabla u^a \|_{L^\infty} + \| u^a \nabla^2 u^a \|_{L^\infty} + \| \nabla^2 u^a \|_{L^\infty} \right)$$

$$\times \int_{\Omega} \rho \left( | \mathcal{D}_t R |^2 + | R |^2 \right) \, dx \, dy + \| \nabla^2 u^a \|_{L^\infty} \int_{\Omega} \rho | R |^4 \, dx \, dy.$$

It follows from the Gagliardo-Nirenberg interpolation inequality and (2.21) that

$$\| R \|_{L^4(\Omega)} \leq C \| R \|_{L^2(\Omega)}^2 \| R \|_{H^1(\Omega)}^2 \leq C \varepsilon^2 + C \varepsilon \| \nabla R \|_{L^2(\Omega)}^2.$$

As a result, we get, by applying Theorem 1.1, that

$$\int_{\Omega} \left( \mathcal{D}_t (\rho R \cdot \nabla u^a) + (\text{div} u) \rho R \cdot \nabla u^a \right) \cdot \mathcal{D}_t R \, dx \, dy$$

(7.25)

$$\leq C e^{-\alpha t} \left( \varepsilon (1 + \| \nabla R \|_{L^2}^2) + \int_{\Omega} \rho | D_t R |^2 \, dx \, dy \right).$$
In view of the \( \varrho \) equation of (2.13), we write
\[
\mathcal{D}_t (\varrho (\partial_t u^a + u^a \cdot \nabla u^a)) + \mathrm{div} \, u (\varrho (\partial_t u^a + u^a \cdot \nabla u^a)) \\
= (\mathcal{D}_t \varrho + \varrho \mathrm{div} \, u) (\partial_t u^a + u^a \cdot \nabla u^a) + \varrho \mathcal{D}_t (\partial_t u^a + u^a \cdot \nabla u^a) \\
= - (\mathrm{div} \, (\varrho^a R) + \varepsilon ((\eta \varpi)_y)_\varepsilon) (\partial_t u^a + u^a \cdot \nabla u^a) + \varrho \mathcal{D}_t (\partial_t u^a + u^a \cdot \nabla u^a).
\]

It is easy to observe that
\[
\int_\Omega \left( \mathrm{div} \, (\varrho^a R) + \varepsilon ((\eta \varpi)_y)_\varepsilon \right) (\partial_t u^a + u^a \cdot \nabla u^a) \cdot \mathcal{D}_t R \, dx \, dy \\
\leq \| (\partial_t u^a + u^a \cdot \nabla u^a) \|_{L^\infty} \left\| \varrho \mathrm{div} \, R + R \cdot \nabla \varrho^a + \varepsilon ((\eta \varpi)_y)_\varepsilon \right\|_{L^2} \left\| \mathcal{D}_t R \right\|_{L^2} \\
\leq C e^{-\alpha t} \left( \| R \|_{L^2} + \| \nabla R \|_{L^2} + \varepsilon^{\frac{1}{2}} \right) \left\| \mathcal{D}_t R \right\|_{L^2},
\]
and
\[
\int_\Omega \varrho \partial_t (\partial_t u^a + u^a \cdot \nabla u^a) \cdot \mathcal{D}_t R \, dx \, dy \leq \| \partial_t (\partial_t u^a + u^a \cdot \nabla u^a) \|_{L^\infty} \| \varrho \|_{L^2} \| \mathcal{D}_t R \|_{L^2},
\]
and
\[
\int_\Omega \varrho u \cdot \nabla (\partial_t u^a + u^a \cdot \nabla u^a) \cdot \mathcal{D}_t R \, dx \, dy \\
\leq \| \mathcal{D}_t R \|_{L^2} \left( \| u^a \cdot \nabla (\partial_t u^a + u^a \cdot \nabla u^a) \|_{L^\infty} \| \varrho \|_{L^2} + \| \varrho \|_{L^\infty} \left( \| \partial_t u^a + u^a \cdot \nabla u^a \|_{L^\infty} \| R \|_{L^2} \right) \\
\leq C e^{-\alpha t} \varepsilon^{\frac{1}{2}} \| D_t R \|_{L^2(\Omega)}.
\]
Therefore, we obtain
\[
(7.26) \int_\Omega \left( \mathcal{D}_t (\varrho (\partial_t u^a + u^a \cdot \nabla u^a)) + \mathrm{div} \, u (\varrho (\partial_t u^a + u^a \cdot \nabla u^a)) \right) \cdot \mathcal{D}_t R \, dx \, dy \\
\leq C e^{-\alpha t} \varepsilon + C e^{-\alpha t} \| \mathcal{D}_t R \|_{L^2(\Omega)}^2.
\]
We remark that it is exactly the term \( \| \partial_t (\partial_t u^a + u^a \cdot \nabla u^a) \|_{L^\infty} \) that needs the highest regularity of the approximate solution. It follows from (1.12) that \( \| \partial_t (\partial_t u^a + u^a \cdot \nabla u^a) \|_{L^\infty} \leq C \).

Finally, thanks to (2.10), we compute
\[
(7.27) \int_\Omega \left( \partial_t G + \mathrm{div} \, (u G) \right) \cdot \mathcal{D}_t R \, dx \, dy \\
\leq \| \partial_t G \|_{L^2} \| \mathcal{D}_t R \|_{L^2} + \| G \|_{L^\infty} \left( \| u^a \|_{L^2} + \| R \|_{L^2} \right) \| \nabla \mathcal{D}_t R \|_{L^2} \\
\leq C \varepsilon^{\frac{1}{2}} e^{-\alpha t} \| \mathcal{D}_t R \|_{L^2} + C \varepsilon^{\frac{1}{2}} e^{-\alpha t} \| \nabla \mathcal{D}_t R \|_{L^2} \\
\leq C \varepsilon e^{-\alpha t} + C e^{-\alpha t} \| \mathcal{D}_t R \|_{L^2}^2 + \frac{1}{8} \| \nabla \mathcal{D}_t R \|_{L^2}^2.
\]
By inserting the estimates (7.20) and (7.22–7.27) into (7.19) and then integrating the resulting inequality over \([0, t] \), we arrive at
\[
(7.28) \| \mathcal{D}_t R \|_{L^2}^2 + \int_0^t \| \nabla \mathcal{D}_t R(t') \|_{L^2}^2 \, dt' \leq C \varepsilon + C \int_0^t | \nabla R(t') |^4 \, dx \, dy + C \int_0^t e^{-\alpha t'} \| D_t R(t') \|_{L^2}^2 \, dt'.
\]
Applying Gronwall’s inequality leads to (7.18). This finishes the proof of Lemma 7.3.

\[\square\]

7.4. **Estimates for the vorticity and proof of Proposition 2.7.** For each vector valued function \( v = (v_1, v_2)^T : \mathbb{R}^2 \to \mathbb{R}^2 \), we define its vorticity \( \mathrm{curl} \, v \) as follows:
\[
\mathrm{curl} \, v \stackrel{\text{def}}{=} \nabla \perp \cdot v = \partial_y v_1 - \partial_x v_2, \quad \nabla \perp \stackrel{\text{def}}{=} \left( \frac{\partial_y}{-\partial_x} \right).
\]
In particular, we denote $\omega \defeq \text{curl} R$. Applying curl to (2.13)_2 gives
\begin{equation}
(7.29) \quad \text{curl} (\rho \partial_t R) - \mu \Delta \omega + \text{curl} (\rho R \cdot \nabla u^a) + \text{curl} \left( \varrho (\partial_t u^a + u^a \cdot \nabla u^a) \right) = \text{curl} G.
\end{equation}

**Lemma 7.4.** Let $\omega \defeq \text{curl} R$. Then there holds
\begin{equation}
(7.30) \quad \int_0^t \int_{\Omega} |\nabla \omega|^2 \, dx \, dy \leq C \varepsilon + C \int_0^t \int_{\Omega} |\nabla R|^4 \, dx \, dy,
\end{equation}
and
\begin{equation}
(7.31) \quad \int_0^t \int_{\Omega} |\nabla \omega|^2 \, dx \, dy \, dt' \leq C \varepsilon + C \int_0^t \int_{\Omega} |\nabla R|^3 \, dx \, dy \, dt'.
\end{equation}

**Proof.** We first get, by taking $L^2$ inner product of (7.29) with $\omega$ and using integration by parts, that
\begin{align*}
\mu \int_{\Omega} |\nabla \omega|^2 \, dx \, dy &= \int_{\Omega} \left( - G \cdot \nabla \omega + \rho \partial_t R \cdot \nabla \omega + \rho R \cdot \nabla u^a \cdot \nabla \omega + \varrho (\partial_t u^a + u^a \cdot \nabla u^a) \cdot \nabla \omega \right) \Delta \omega \, dx \, dy.
\end{align*}
Applying Young's inequality gives
\begin{align*}
\mu \int_{\Omega} |\nabla \omega|^2 \, dx \, dy &\leq \frac{4}{\mu} \int_{\Omega} \left( |G|^2 + |\partial_t R|^2 + |\nabla u^a|^2 |\partial_t u^a|^2 \right) \, dx \, dy + \frac{\mu}{2} \int_{\Omega} |\nabla \omega|^2 \, dx \, dy,
\end{align*}
which together with Proposition 2.6, Lemmas 7.2 and 7.3, and Theorem 1.1 ensures (7.30) and (7.31).

By summarizing Proposition 2.6, Lemmas 7.2, 7.3 and 7.4, we conclude the proof of Proposition 2.7.

**Appendix A. Proof of Proposition 2.1**

**Proof of Proposition 2.1.** By summarizing Propositions 3.1, 3.2, 3.4, 3.5 and 3.6, we conclude that (1.11) holds and there exist two positive constants $C$ and $\alpha$ solely depending on $(a, \gamma, \nu, \bar{s}, E_{10})$ such that
\begin{equation}
(1.11) \quad \| (\eta - 1, w)(t) \|_{H^1} + \| (\eta - 1, w)(t) \|_{L^2} + \| D_t w(t) \|_{L^2} \leq C E_{20}^{\frac{1}{2}}(y) e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+, \ y \in \mathbb{R},
\end{equation}
from which and (1.6), we infer
\begin{equation}
(1.12) \quad \| \eta(t) \|_{L^2} + \| (w_{xx}, w_x)(t) \|_{L^2} \leq C E_{20}^{\frac{1}{2}}(y) e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+, \ y \in \mathbb{R}.
\end{equation}
By virtue of Sobolev embedding, we obtain
\begin{equation}
(1.13) \quad \| \eta(t) \|_{C_h^{0, \frac{1}{2}}} + \| w(t) \|_{C_h^{1, \frac{1}{2}}} \leq C E_{20}^{\frac{1}{2}}(y) e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+, \ y \in \mathbb{R}.
\end{equation}

To derive decay estimates for higher order norms, we are going to use induction method. Let $n \in \mathbb{Z}_+, \ 1 \leq n \leq 3$, we denote
\begin{align}
E_{m+2}(t, y) &\defeq \| (\eta - 1)(t) \|_{H^m}^2 + \| w(t) \|_{H^m}^2 + \| (\partial_t \eta, \partial_t w)(t) \|_{L^2}^2, \\
D_{m+2}(t, y) &\defeq \| (\eta - 1)(t) \|_{H^m}^2 + \| w(t) \|_{H^m}^2 + \| D_t w(t) \|_{L^2}^2.
\end{align}
and assume that for $0 \leq m \leq n - 1$, there exists $F_m(t, y)$ so that

(A.5a) \[ F_m(t, y) \sim (\|(\eta - 1, w)(t)\|_{H^{m+1}_h}^2 + \|D_t w(t)\|_{H^m_h}^2), \]

(A.5b) \[ \frac{d}{dt} F_m(t, y) + \delta D_{m+2}(t, y) \leq CE_{m+2,0}(y)e^{-\alpha t}, \]

(A.5c) \[ E_{m+2}^\frac{1}{2}(t, y) \leq CE_{m+2,0}^\frac{1}{2}(y)e^{-\alpha t}, \]

for some $\alpha > 0$, $\delta > 0$ and all $t \in \mathbb{R}^+$, $y \in \mathbb{R}$, where $E_{m+2,0}(y) \overset{\text{def}}{=} E_{m+2}(0, y)$.

As shown in Section 3, the estimates in (A.5b) and (A.5c) hold when $m = 0$. We would like to prove that (A.5b) and (A.5c) hold for $m = n$. In order to it, we denote

(A.6) \[ \eta^{(n)} \overset{\text{def}}{=} \partial^n_x \eta \quad \text{and} \quad w^{(n)} \overset{\text{def}}{=} \partial^n_x w. \]

Then it follows from (1.6) that $(\eta^{(n)}, w^{(n)})$ verifies

(A.7) \[
\begin{cases}
\eta_t^{(n)} + w\eta_x^{(n)} + \eta w_x^{(n)} = -f_n, \\
\eta(w_t^{(n)} + w w_x^{(n)}) - \nu w_x^{(n)} + p'(\eta)\eta_x^{(n)} = -g_n,
\end{cases}
\]

where

\[
f_n \overset{\text{def}}{=} \sum_{m=1}^n w^{(m)}\eta^{(n-m+1)} + \sum_{m=1}^n \eta^{(m)} w^{(n-m+1)},
\]

\[
g_n \overset{\text{def}}{=} \sum_{m=1}^n \eta^{(m)} w_t^{(n-m)} + \sum_{m=1}^n \eta^{(m)} \partial_x^{n-m}(w w_x) + \eta \sum_{m=1}^n w^{(m)} w^{(n-m+1)} + \sum_{m=1}^n \partial_x^{n-m}(p'(\eta))\eta^{(n-m+1)},
\]

from which, the inductive assumption (A.5c) and Sobolev embedding theorem, we infer

(A.8) \[ \|(f_n, g_n)(t)\|_{L^2_h} \leq CE_{n+2,0}^\frac{1}{2}(y)e^{-\alpha t}. \]

In what follows, we shall decompose the proof into the following steps:

**Step 1.** Decay estimates for $\|\eta^{(n+1)}\|_{L^2_h}$.

Recalling $\zeta = \eta^{-1}$ and (3.51), we find

(A.9) \[ \zeta w^{(n)}_{xx} = D_t \zeta^{(n+1)} + \sum_{m=1}^n w^{(m)} \zeta^{(n-m+2)} - \sum_{m=1}^n \zeta^{(m)} w^{(n-m+2)} \quad \text{with} \quad \zeta^{(n)} \overset{\text{def}}{=} \partial_x^n \zeta. \]

While we observe from (A.7)2 that

\[ D_t w^{(n)} - \nu \eta^{-1} w_{xx}^{(n)} + \eta^{-1} p'(\eta)\eta_x^{(n)} = -\eta^{-1} g_n. \]

By plugging (A.9) into the above equality yields

(A.10) \[ D_t (w^{(n)} - \nu \zeta^{(n+1)}) - \nu \left( \sum_{m=1}^n w^{(m)} \zeta^{(n-m+2)} - \sum_{m=1}^n \zeta^{(m)} w^{(n-m+2)} \right) + \eta^{-1} p'(\eta)\eta_x^{(n)} = -\eta^{-1} g_n. \]
By taking $L^2$ inner product of $\eta(w^{(n)} - \nu\zeta^{(n+1)})$ with (A.10), we find
\[
\frac{1}{2} \frac{d}{dt} \int_T \eta(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx + \int_T p'(\eta)\eta_x^{(n)}(w^{(n)} - \nu\zeta^{(n+1)}) \, dx \\
= - \int_T w_x(\eta(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx + \int_T w_x w^{(n)} \eta(w^{(n)} - \nu\zeta^{(n+1)}) \, dx \\
+ \nu \int_T \left( \sum_{m=2}^{n} w^{(m)}(\zeta^{(n-m+2)} - \sum_{m=1}^{n} \zeta^{(m)} w^{(n-m+2)}) \eta(w^{(n)} - \nu\zeta^{(n+1)}) \, dx \\
- \int_T g_n(w^{(n)} - \nu\zeta^{(n+1)}) \, dx,
\]
from which, and the induction assumption (A.5c), we infer
\[
\frac{1}{2} \frac{d}{dt} \int_T \eta(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx + \int_T p'(\eta)\eta_x^{(n)}(w^{(n)} - \nu\zeta^{(n+1)}) \, dx \\
\leq C\|w_x\|_{L^2} \int_T \eta(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx + 2\delta^{-1} \int_T \eta|w_x w^{(n)}|^2 \, dx \\
+ 2\nu^2 \delta^{-1} \int_T \eta \left| \sum_{m=2}^{n} w^{(m)}(\zeta^{(n-m+2)} - \sum_{m=1}^{n} \zeta^{(m)} w^{(n-m+2)}) \right|^2 \, dx \\
+ 2\delta^{-1} \int_T \eta^{-1}|g_n|^2 \, dx + \delta \int_T \eta(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx \\
\leq C E_m^{2}(y)e^{-\alpha t} \int_T \eta(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx + C\delta^{-1} E_{n+2,0}(y)e^{-\alpha t} \\
+ \delta \int_T \eta(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx,
\]
where $\delta$ is a small positive constant to be determined later on.
While we observe that
\[
\eta_x^{(n)} = \partial_x^n \left(-\zeta^{(n+1)}\right) = -\zeta^{-2}\zeta^{(n+1)} - \sum_{m=1}^{n} \partial_x^m (\zeta^{-2}) \zeta^{(n-m+1)}.
\]
Then we have
\[
\int_T p'(\eta)\eta_x^{(n)}(w^{(n)} - \nu\zeta^{(n+1)}) \, dx = \nu^{-1} \int_T p'(\eta)\eta^2(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx \\
- \nu^{-1} \int_T p'(\eta)\eta^2 w^{(n)}(w^{(n)} - \nu\zeta^{(n+1)}) \, dx - \int_T p'(\eta) \sum_{m=1}^{n} \partial_x^m (\zeta^{-2}) \zeta^{(n-m+1)} (w^{(n)} - \nu\zeta^{(n+1)}) \, dx.
\]
By inserting the above equality into (A.11) and using (1.11), we deduce that
\[
\frac{1}{2} \frac{d}{dt} \int_T \eta(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx + \nu^{-1} p'(\eta)\eta \int_T \eta(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx \\
\leq C E_m^{2}(y)e^{-\alpha t} \int_T \eta(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx + C\delta^{-1} E_{n+2,0}(y)e^{-\alpha t} \\
+ \delta \int_T \eta(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx.
\]
Choosing $\delta = \min\{1, \nu^{-1} p'(\eta)\eta/2\}$ in (A.12) and applying Gronwall’s inequality gives
\[
\int_T \eta(w^{(n)} - \nu\zeta^{(n+1)})^2 \, dx \leq C E_{n+2,0}(y)e^{-\alpha t},
\]
which together with (A.5c) ensures that
\[
\int_T |\zeta^{(n+1)}|^2 \, dx \leq C E_{n+2,0}(y) e^{-\alpha t}.
\]
This together together with the inductive assumption (A.5c) leads to
(A.13) \[
\int_T |\eta^{(n+1)}|^2 \leq C E_{n+2,0}(y) e^{-\alpha t}.
\]
And then we deduce from (A.7) that
(A.14) \[
\|\partial_t \eta^{(n)}\|_{L^2_h} = \|\partial_t (\eta w)\|_{L^2_h} \leq C E_{n+2,0}(y) e^{-\alpha t}.
\]

**Step 2.** Decay estimates of \(\|D_t w^{(n)}\|_{L^2_h}^2\).

We first get, by taking \(L^2_h\) inner product of \(\eta^{-1} w^{(n)}_{xx}\) with the \(w^{(n)}\) equation of (A.7), that
\[
\frac{1}{2} \frac{d}{dt} \int_T |w^{(n)}_x|^2 \, dx + \nu \int_T |w^{(n)}_x|^2 \, dx = \int_T (g_n - p'(\eta) n_x^{(n)} + \eta w w^{(n)}_x) \eta^{-1} w^{(n)}_x \, dx.
\]
Thanks to (1.11), (A.5c) and (A.14), we infer
\[
\frac{1}{2} \frac{d}{dt} \int_T |w^{(n)}_x|^2 \, dx + \nu \eta^{-1} \int_T |w^{(n)}_x|^2 \, dx \leq \frac{1}{2} \nu \eta^{-1} \int_T |w^{(n)}_x|^2 \, dx + C E_{n+2,0}(y) e^{-\alpha t}.
\]
This leads to
(A.15) \[
\frac{d}{dt} \int_T |w^{(n)}_x|^2 + \nu \eta^{-1} \int_T |w^{(n)}_x|^2 \leq C E_{n+2,0}(y) e^{-\alpha t}.
\]
While we get, by taking \(L^2_h\) inner product of \(D_t w^{(n)}\) with the \(w^{(n)}\) equation of (A.7), that
\[
\int_T \eta |D_t w^{(n)}|^2 \, dx - \nu \int_T w^{(n)}_x D_t w^{(n)} \, dx = -\int_T \eta^{-1} g_n D_t w^{(n)} \, dx - \int_T p'(\eta) n_x^{(n)} D_t w^{(n)} \, dx.
\]
Observing that
\[
-\nu \int_T w^{(n)}_x D_t w^{(n)} \, dx = -\nu \int_T w^{(n)}_x \partial_t w^{(n)} \, dx - \nu \int_T w^{(n)}_x w w^{(n)}_x \, dx = \nu \frac{d}{dt} \int_T |w^{(n)}_x|^2 \, dx + \frac{\nu}{2} \int_T (w^{(n)}_x)^2 w_x \, dx,
\]
which together with (1.11), (A.5c) and (A.14) ensures that
(A.16) \[
\nu \frac{d}{dt} \int_T |w^{(n)}_x|^2 \, dx + \int_T \eta |D_t w^{(n)}|^2 \, dx \leq C E_{n+2,0}(y) e^{-\alpha t}.
\]
Before proceeding, let us admit the following lemma, the proof of which will be postponed till we finish the proof of Proposition 2.1.

**Lemma A.1.** For all \(t > 0, y \in \mathbb{R}\), there holds
(A.17) \[
\frac{d}{dt} \int_T \eta |D_t w^{(n)}_x|^2 \, dx + \nu \int_T |(D_t w^{(n)})_x|^2 \, dx \leq C E_{n+2,0}(y) e^{-\alpha t} + C \int_T (|D_t w^{(n)}|^2 + |w^{(n)}_{xx}|^2) \, dx.
\]

**Step 3.** End of the induction and summary of decay estimates
By virtue of (A.15), (A.16) and (A.17), we can find a large enough constant \(A_7\) such that
\[
\frac{d}{dt} \int_T (A_7 |w^{(n)}_x|^2 + \eta |D_t w^{(n)}|^2) \, dx + \int_T (|D_t w^{(n)}|^2 + |w^{(n)}|_x^2 + \nu |(D_t w^{(n)})_x|^2) \, dx \leq C E_{n+2,0}(y) e^{-\alpha t},
\]
from which, the induction assumption (A.5b) for all \(0 \leq m \leq n-1\) and (A.12), we deduce that (A.5b) holds with \(m = n\) for some small positive number \(\delta\). This implies (A.5c) with \(m = n\). We thus complete the induction argument and show in particular that

\[
\| (\eta - 1)(t) \|_{H^4_h}^2 + \| w(t) \|_{H^6_h}^2 + \sum_{m=1}^{2} \| \partial_t^m \eta(t) \|_{H^k_{h^{-m}}}^2 + \| \partial_t^m w(t) \|_{H^0_{h^{-2m}}}^2 \leq C E_{5,0}(y) e^{-\alpha t}.
\]

This finishes the proof of Proposition 2.1. \(\square\)

Now let us present the proof of Lemma A.1.

**Proof of Lemma A.1.** We first rewrite (A.7)_2 as

\[
\eta D_t w^{(n)} - \nu w^{(n)} + (p(\eta))^{(n+1)} = -\tilde{g}_n, \quad \text{with}
\]

\[
\tilde{g}_n \overset{\text{def}}{=} \sum_{m=1}^{n} \eta^{(m)} w^{(n-m)} + \sum_{m=1}^{n} \eta^{(m)} \partial_x^{n-m} (w w_x) + \eta \sum_{m=1}^{n} w^{(m)} w^{(n-m+1)}.
\]

By applying \(D_t\) to (A.19) and then taking \(L^2_h\) inner product of the resulting equation with \(D_t w^{(n)}\), we find

\[
\int_T (\eta D^2_t w^{(n)} + D_t \eta D_t w^{(n)} - \nu D_t w^{(n)} + D_t (p(\eta)) w^{(n)}) D_t w^{(n)} \, dx = -\int_T D_t \tilde{g}_n D_t w^{(n)} \, dx.
\]

In what follows, we shall frequently use the estimates (A.5c), (A.13) and (A.14) to handle the terms above.

We first observe that

\[
\int_T \eta D^2_t w^{(n)} D_t w^{(n)} \, dx = \frac{1}{2} \frac{d}{dt} \int_T \eta |D_t w^{(n)}|^2 \, dx.
\]

For the second term in (A.20), direct calculation gives

\[
\int_T D_t \eta D_t w^{(n)} D_t w^{(n)} \, dx \leq \| w_x \|_{L^\infty_h} \int_T \eta |D_t w^{(n)}|^2 \, dx \leq C \int_T \eta |D_t w^{(n)}|^2 \, dx.
\]

Notice that

\[
D_t w^{(n)} = (D_t w^{(n)})_{xx} - (2w_x w^{(n)} + w_{xx} w_x^{(n)}).
\]

Then, by using integration by parts, we get that

\[
-\nu \int_T D_t w^{(n)}_{xx} D_t w^{(n)} \, dx = \nu \int_T |(D_t w^{(n)})_x|^2 \, dx + \int_T \nu (2w_x w^{(n)} + w_{xx} w_x^{(n)}) D_t w^{(n)} \, dx
\]

\[
\geq \nu \int_T |(D_t w^{(n)})_x|^2 - C \int_T (|w_{xx}|^2 + |w_x^{(n)}|^2 + |D_t w^{(n)}|^2) \, dx.
\]

Similarly, we have

\[
\int_T D_t (p(\eta))^{(n+1)} D_t w^{(n)} \, dx = \int_T \partial_t (p(\eta))^{(n+1)} D_t w^{(n)} \, dx + \int_T w(p(\eta))^{(n+2)} D_t w^{(n)} \, dx
\]

\[
= -\int_T \partial_t (p(\eta))^{(n)} (D_t w^{(n)})_x \, dx - \int_T (p(\eta))^{(n+1)} (w_x D_t w^{(n)} + w(D_t w^{(n)})_x) \, dx,
\]
from which, we infer
\begin{equation}
(A.24)\quad \int_T D_t (p(\eta))^{(n+1)} D_t w^{(n)} \, dx \leq C \int_T |\partial_t (p(\eta))^{(n)}|^2 \, dx + \left( \int_T |D_t w^{(n)}|^2 \, dx + \frac{\nu}{4} \int_T |(D_t w^{(n)})_x|^2 \, dx + C \left( \|w_x\|_{L_h^2}^2 + \|w\|_{L_h^2} \right) \int_T |(p(\eta))^{(n+1)}|^2 \, dx \right) \\
\quad \leq CE_{n+2,0}(y) e^{-\alpha t} + \int_T |D_t w^{(n)}|^2 \, dx + \frac{\nu}{4} \int_T |(D_t w^{(n)})_x|^2 \, dx.
\end{equation}

It remains to handle the source term in (A.20). Indeed in view of (A.20), we find that the most trouble terms are the following ones with highest derivatives on \( \eta \) or \( w \) in \( \tilde{g}_n \):
\begin{equation}
(A.25)\quad \tilde{g}_n^{(1)} \stackrel{\text{def}}{=} \eta^{(1)} w_t^{(n-1)} + \eta^{(1)} w w_x^{n-1} + \eta w^{(1)} w^{(n)} = \eta^{(1)} D_t w^{(n-1)} + \eta w^{(1)} w^{(n)},
\end{equation}
\begin{equation}
\quad \tilde{g}_n^{(2)} \stackrel{\text{def}}{=} \eta^{(n)} + \eta^{(n)} + \eta w^{(n)} w^{(1)}.
\end{equation}

Observe that
\[
D_t (\eta^{(1)} D_t w^{(n-1)}) = (D_t \eta_x) D_t w^{(n-1)} + \eta_x D_t^2 w^{(n-1)}
\]
\[
= D_t \eta_x \eta^{-1}(\nu w_x^{(n-1)} - (p(\eta))^{(n)} - \tilde{g}_n - 1) + \eta_x (D_t \eta^{-1})(\nu w_x^{(n-1)} - (p(\eta))^{(n)} - \tilde{g}_n - 1)
\]
\[
+ \eta_x \eta^{-1} D_t (\nu w_{xx}^{(n-1)} - (p(\eta))^{(n)} - \tilde{g}_n - 1).
\]

It follows from (A.5c), (A.13) and (A.14) that
\[
\int_T D_t \eta_x \eta^{-1}(\nu w_x^{(n-1)} - (p(\eta))^{(n)} - \tilde{g}_n - 1) D_t w^{(n)} \, dx
\]
\[
\leq \eta^{-1} \|\eta_x^{-1}(\nu w_x^{(n-1)} - (p(\eta))^{(n)} - \tilde{g}_n - 1)\|_{H^1_h}^2 \int_T |D_t \eta_x| \, dx + \int_T |D_t w^{(n)}|^2 \, dx
\]
\[
\leq CE_{n+2,0}(y) e^{-\alpha t} + \int_T |D_t w^{(n)}|^2 \, dx,
\]
and
\[
\int_T \eta_x (D_t \eta^{-1})(\nu w_x^{(n-1)} - (p(\eta))^{(n)} - \tilde{g}_n - 1) D_t w^{(n)} \, dx
\]
\[
\leq \|\eta_x (D_t \eta^{-1})(\nu w_x^{(n-1)} - (p(\eta))^{(n)} - \tilde{g}_n - 1)\|_{H^1_h}^2 + \int_T |D_t w^{(n)}|^2 \, dx
\]
\[
\leq CE_{n+2,0}(y) e^{-\alpha t} + \int_T |D_t w^{(n)}|^2 \, dx.
\]

Next we deal with the term \( \int_T \eta_x \eta^{-1} D_t (\nu w_{xx}^{(n-1)} - (p(\eta))^{(n)} - \tilde{g}_n - 1) \, dx \). We compute
\[
\eta_x \eta^{-1} D_t w_{xx}^{(n-1)} = \eta_x \eta^{-1} (D_t w_x^{(n)})_x - \eta_x \eta^{-1} w_{xx} w_x^{(n)},
\]
so that
\[
\int_T \eta_x \eta^{-1} D_t w_{xx}^{(n-1)} D_t w^{(n)} \, dx = \int_T \eta_x \eta^{-1} (D_t w_x^{(n)})_x D_t w^{(n)} \, dx - \int_T \eta_x \eta^{-1} w_{xx} w_x^{(n)} D_t w^{(n)} \, dx
\]
\[
\leq CE_{n+2,0}(y) e^{-\alpha t} + C \int_T |D_t w^{(n)}|^2 \, dx + \frac{\nu}{4} \int_T |(D_t w_x^{(n)})_x|^2 \, dx.
\]
Similarly, we have
\[
\int_T \eta_x \eta^{-1} D_t (p(\eta))^{(n-1)} D_t w(n) \, dx \leq \eta^{-2} \| \eta_x \|_{L^\infty}^2 \int_T |D_t (p(\eta))|_n^2 \, dx + \int_T |D_t w(n)|^2 \, dx \leq C E_{n+2,0}(y) e^{-\alpha t} + C \int_T |D_t w(n)|^2 \, dx.
\]

Finally, by virtue of the definition of \( \tilde{g}_n \) given by (A.19), and the estimates (A.5c), (A.13) and (A.14), we deduce that
\[
\int_T \eta_x \eta^{-1} D_t \tilde{g}_n - D_t w(n-1) \, dx \leq \eta^{-2} \| \eta_x \|_{L^\infty}^2 \int_T |D_t \tilde{g}_n|_n^2 \, dx + \int_T |D_t w(n-1)|^2 \, dx \leq C E_{n+2,0}(y) e^{-\alpha t} + C \int_T |D_t w(n-1)|^2 \, dx.
\]

By summarizing the above estimates, we arrive at
\[
(A.26) \quad \int_T D_t (\eta^{(1)} D_t w(n-1)) D_t w(n) \, dx \leq C E_{n+2,0}(y) e^{-\alpha t} + \int_T (C|D_t w(n)|^2 + \frac{\nu}{4}|(D_t w(n))_x|^2) \, dx.
\]

The other terms in (A.25) can be handled along the same line.

By substituting the estimates (A.21–A.24) and (A.26) into (A.20), we conclude the proof of (A.17). This finishes the proof of Lemma A.1. \( \square \)

**Appendix B. Proof of Lemma 2.1**

Let us recall Lemma 2.1: let \( 2 < p < \infty \) and \( \Omega = \mathbb{T} \times \mathbb{R} \), there exists a constant \( C \) depending solely on \( p \) such that for any \( f \in H^1(\Omega) \), there holds
\[
(B.1) \quad \| f \|_{L^p(\Omega)} \leq C \left( \| f \|_{L^2(\Omega)}^2 \| \nabla f \|_{L^2(\Omega)}^{1-\frac{2}{p}} + \| f \|_{L^2(\Omega)}^{\frac{1}{p} + \frac{1}{2}} \| \nabla f \|_{L^2(\Omega)}^{\frac{1}{p} + \frac{1}{2}} \right).
\]

**Proof of Lemma 2.1.** Let \( f \in H^1(\Omega) \), we split it as follows
\[
f(x, y) = \tilde{f}(x, y) + \bar{f}(y), \quad \text{with} \quad \bar{f}(y) \overset{\text{def}}{=} \int_T f(x, y) \, dx, \quad \text{and} \quad \tilde{f} \overset{\text{def}}{=} f - \bar{f}.
\]

We observe that \( \tilde{f} \in H^1(\mathbb{R}) \). Indeed, applying Hölder’s inequality gives
\[
(B.2) \quad \| \tilde{f} \|_{L^2(\mathbb{R})}^2 = \int_\mathbb{R} \int_T |f(x, y)|^2 \, dy \, dx \leq \int_T \int_\mathbb{R} |f(x, y)|^2 \, dx \, dy = \| f \|_{L^2(\Omega)}^2,
\]
\[
\quad \| \partial_y \tilde{f} \|_{L^2(\mathbb{R})}^2 = \| \partial_y f \|_{L^2(\mathbb{R})}^2 \leq \| \partial_y f \|_{L^2(\Omega)}^2.
\]

Furthermore, \( \tilde{f} \in H^1(\Omega) \) and \( \tilde{f} \in H^1(\mathbb{R}) \), and there hold
\[
(B.3) \quad \| \tilde{f} \|_{L^2(\Omega)} = \| \tilde{f} \|_{L^2(\mathbb{R})}, \quad \| \nabla_{x,y} \tilde{f} \|_{L^2(\Omega)} = \| \partial_y \tilde{f} \|_{L^2(\mathbb{R})}.
\]

For \( \tilde{f} \), we get, by applying the classical two dimensional Gagliardo-Nirenberg interpolation inequality, that
\[
(B.4) \quad \| \tilde{f} \|_{L^p(\Omega)} \leq C \| \tilde{f} \|_{L^2(\Omega)}^\frac{2}{p} \| \tilde{f} \|_{H^1(\Omega)}^{1-\frac{2}{p}} \leq C \| \tilde{f} \|_{L^2(\Omega)}^\frac{2}{p} + C \| \tilde{f} \|_{L^2(\Omega)}^\frac{2}{p} \| \nabla \tilde{f} \|_{L^2(\Omega)}^{1-\frac{2}{p}}.
\]

Notice that
\[
\int_T \tilde{f}(x, y) \, dx = 0, \quad \forall y \in \mathbb{R},
\]
by applying Poincaré’s inequality, one has
\[
\int_T |\tilde{f}(x, y)|^2 \, dx \leq \int_T |\partial_x \tilde{f}(x, y)|^2 \, dx, \quad \forall y \in \mathbb{R}.
\]
This leads to
\[(B.5) \quad \|\tilde{f}\|_{L^2(\Omega)}^2 = \int_{\mathbb{R}} \int_{T} |\tilde{f}(x, y)|^2 \, dx \, dy \leq \int_{\mathbb{R}} \int_{T} |\partial_x \tilde{f}(x, y)|^2 \, dx \, dy \leq \|\nabla \tilde{f}\|_{L^2(\Omega)}^2.\]

Plugging (B.5) into (B.4) and using (B.3), we find
\[(B.6) \quad \|\tilde{f}\|_{L^p(\Omega)} \leq C\|\tilde{f}\|_{L^2(\Omega)}^{\frac{2}{p}} \|\nabla \tilde{f}\|_{L^2(\Omega)}^{1-\frac{2}{p}} \leq C\|\tilde{f}\|_{L^2(\Omega)}^{\frac{2}{p}} \|\nabla \tilde{f}\|_{L^2(\Omega)}^{1-\frac{2}{p}}.\]

While for \(\tilde{f}(y)\), we get, by applying the one dimensional Gagliardo-Nirenberg interpolation inequality (see [5, 22]) and (B.2), that
\[(B.7) \quad \|\tilde{f}\|_{L^p(\mathbb{R})} \leq C\|\tilde{f}\|_{L^2(\mathbb{R})}^{\frac{1}{p}} \|\partial_y \tilde{f}\|_{L^2(\mathbb{R})}^{\frac{1}{p}} \leq C\|\tilde{f}\|_{L^2(\Omega)}^{\frac{1}{p}} \|\partial_y f\|_{L^2(\Omega)}^{\frac{1}{p}}.\]

Then (B.1) follows immediately from (B.6) and (B.7). This finishes proof of Lemma 2.1. □

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