The Uncertainty Principle for relatively dense sets and lacunary spectra

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Abstract. We obtain a new version of the Uncertainty Principle for functions with Fourier transforms supported on a lacunary set of intervals. This is a generalization of Zygmund’s theorem on lacunary trigonometric series to the real line in the spirit of the Logvinenko-Sereda theorem for relatively dense sets.

1. Introduction

We prove a new version of the Uncertainty Principle which is a general statement saying that a function and its Fourier transform cannot be simultaneously concentrated on small sets. Many examples of the Uncertainty Principle can be found in the text book by Havin and Jöricke [4] and a paper by Folland and Sitaram [3]. We restrict ourselves to the following type of the Uncertainty Principle:

\[ \int_E |f|^2 \geq C \|f\|_2^2 \]

for all \( f \) with \( \text{supp} \hat{f} \subset \Sigma \) where \( E^c \) and \( \Sigma \) are small subsets of the real line and \( C = C(E, \Sigma) > 0 \) does not depend on \( f \). Our result is a combination of two versions of this principle. One is Zygmund’s theorem on lacunary trigonometric series ([11], pp. 202-208) and the other is the Logvinenko-Sereda theorem for relatively dense sets ([4], p.113), [8]. As other examples of the inequality (1), we mention the Amrein-Berthier theorem ([1], [4], p. 97, p. 455), [9] (which is a quantitative version of a result due to Benedicks [2]) where \( E^c \) and \( \Sigma \) are sets of finite measure and Wolff’s theorem [10] where \( E^c \) and \( \Sigma \) are so called \( \epsilon \)-thin sets.

A sequence of integers \( \Lambda = \{n_i\}_{i=-\infty}^{\infty} \subset \mathbb{Z} \) is called lacunary with parameter \( R \) if

\[ \sup_{r \neq 0} \text{Card}\{ (i, j) : n_i - n_j = r \} = R < +\infty. \]

For example, a sequence of integers \( \Lambda \) satisfying the Hadamard condition \( \frac{n_{i+1}}{n_i} \geq q > 1 \) is lacunary ([11], p. 203).

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Zygmund’s Theorem: Given any lacunary sequence $\Lambda = \{n_i\}_{-\infty}^{\infty} \subset \mathbb{Z}$ and a measurable set $E \subset [0,1]$ of positive measure then for any $g(x) \in L^2[0,1]$ with $\text{spec} g \subset \Lambda$: 
\[
g(x) = \sum c_n e^{i2\pi n_i x},
\]
we have
\[
\int_E |g|^2 \geq C(E, R) \|g\|^2_2,
\]
where $C(E, \Lambda) > 0$ depends only on $E$ and $\Lambda$.

Moreover, Nazarov showed\(^9\) that actually $C(E, \Lambda)$ can be replaced with $C(|E|, R)$.

A measurable set $E \subset \mathbb{R}$ is called relatively dense if there are $\gamma > 0$ and $a > 0$ such that
\[
|E \cap I| \geq \gamma |I|
\]
for any interval $I$ of length $a$.

Logvinenko-Sereda Theorem: Given any relatively dense set $E \subset \mathbb{R}$ and any function $f \in L^2(\mathbb{R})$ such that $\text{supp} \hat{f} \subset [-b, b]$ then
\[
\int_E |f|^2 \geq C(\gamma, a, b) \|f\|^2_2
\]
where $C(\gamma, a, b) > 0$ depends only on $\gamma$, $a$ and $b$.

In his recent paper\(^5\) the author obtained an optimal estimate of $C(\gamma, a, b) = \left(\frac{\gamma}{\gamma - a}\right)^{C(ab+1)}$ and generalized the theorem to functions whose Fourier transform is supported in a union of finitely many intervals with an estimate on $C$ depending only on the number of intervals but not how they are placed. The results also hold in higher dimensions.

Our main results here are the following two theorems:

**Theorem 1.** If $\Lambda = \{n_i\}_{-\infty}^{\infty} \subset \mathbb{Z}$ is a lacunary sequence with parameter $R$ then there exist $\epsilon(R) < 1$ and an absolute constant $C > 0$ such that for any $f \in L^2$ with $\text{supp} \hat{f} \subset \bigcup \{[n_i - b/4, n_i + b/4] : i \in \mathbb{Z}\}$ and for any relatively dense set $E \subset \mathbb{R}$ which satisfies $|E \cap I| \leq \epsilon(R)|I|$ for any interval $I$ of length $1/b$ provided $b \leq 1$ or $|E \cap I| \leq \frac{\epsilon(R)}{2}|I|$ for any interval $I$ of length $1/b$ provided $b > 1$ we have
\[
\int_E |f|^2 \geq C \|f\|^2_2
\]
where $C > 0$ is an absolute constant.

**Theorem 2.** Let $E$ be a periodic set with period $1$ such that $|E \cap [0,1]| = \gamma > 0$. If $\Lambda = \{n_i\}_{-\infty}^{\infty} \subset \mathbb{Z}$ is a lacunary sequence with parameter $R$ and $f \in L^2$ with $\text{supp} \hat{f} \subset \bigcup \{[n_i - 1/2, n_i + 1/2] \} \subset [0,1]$ then
\[
\int_E |f|^2 \geq C(\gamma, R) \|f\|^2_2
\]
where $C(\gamma, R) > 0$ depends only on $\gamma$ and $R$.

The constant $C$ below is not fixed and might change appropriately from one equality or inequality to another one.
First we will fix some notations. Let \( g(x) \) denote a 1-periodic \( L^2[0,1] \) function with lacunary spectrum: \( \text{spec} \ g \subset \Lambda \), i.e., \( g(x) = \sum c_n e^{i2\pi nx} \) where \( c_n \) stands for the \( n \)-th Fourier coefficient of \( g \). Let \( \phi \) be a fixed \( C_0^\infty \) function with supp \( \phi \subset [-1/2, 1/2] \) and such that \( \phi(x) \equiv 1 \) when \( x \in [-1/4, 1/4] \). It is clear that \( |\hat{\phi}(x)| \leq \frac{C}{1+|x|^2} \). Denote \( g_n(x) = e^{i2\pi nx} \hat{\phi}(x) \). Therefore \( |\hat{g}_n(x)| = |\hat{\phi}(x+n)| \leq \frac{C}{1+(x+n)^2} \).

**Lemma 1.**

\[
\|g\|_4 \leq (1 + R)^4 \|g\|_2
\]

**Proof:** This is true since

\[
\int_0^1 |g|^4 = \int_0^1 (|g|^2)^2 = \sum_n ||g|^2(n)|^2 = \sum_n \sum_{k-l=n} |c_k \bar{c}_l|^2 = \sum_{k \neq 0} |c_k|^2 + \sum_{k-l=n} |c_k \bar{c}_l|^2.
\]

Using Hölder’s inequality we can estimate the second term by \( \sum_{n \neq 0} \sum_{k-l=n} R|c_k|^4|c_l|^2 \leq R \sum_k |c_k|^2 \sum_l |c_l|^2 = R(\sum_k |c_k|^2)^2 \). Thus,

\[
\int_0^1 |g|^4 \leq (\sum_k |c_k|^2 + R(\sum_k |c_k|^2)^2 = (1 + R)\|g\|_2^4.
\]

**Lemma 2.** Let \( E \subset \mathbb{R} \) be a relatively dense set which satisfies \( |E^c \cap I| \leq \epsilon |I| \) for any interval \( I \) of length \( 1/b \) provided \( b \leq 1 \) or \( |E^c \cap I| \leq \frac{\epsilon}{b} |I| \) for any interval \( I \) of length \( 1/b \) provided \( b > 1 \) then

\[
(2) \quad b \int_{E^c} |g(x)|^2 \cdot |\hat{\phi}_k(bx)|^2 dx \leq C(R)\sqrt{|I|} \|g\|_2^2
\]

where \( C(R) \) depends only on \( R \).

**Proof:** We have

\[
b \int_{E^c} |g(x)|^2 \cdot |\hat{\phi}_k(bx)|^2 dx = \sum_n b \int_{E^c \cap [n/b, (n+1)/b]} |g(x)|^2 \cdot |\hat{\phi}_k(bx)|^2 dx
\]

\[
\leq b \sum_n \sqrt{\int_{[n/b, (n+1)/b]} |g(x)|^4 dx} \cdot \sqrt{\int_{[n/b, (n+1)/b]} |\hat{\phi}_k(bx)|^4 dx}
\]

\[
\leq b \sum_n \sqrt{(1+1/b)\|g\|_4^4} \cdot \frac{1/b}{b} \int_{E^c \cap [n/b, (n+1)/b]} |\hat{\phi}_k(x)|^4 dx.
\]

We used Hölder’s inequality to obtain the first inequality. In the second inequality we covered the interval \([n/b, (n+1)/b]\) by no more than \( 1 + 1/b \) intervals of length.
1. Using Lemma 1 and the fact that \( |\hat{\phi}_k(x)| \leq \frac{C}{1 + (x + k)^2} \) we can estimate (3) by

\[
\leq C(R)\sqrt{b + 1} \cdot \|g\|_2^2 \sum_n \frac{1}{1 + (n + k)^2} \cdot \sqrt{\int_{b \cdot E^c \cap [n, n + 1]} \frac{C}{(1 + (x + k)^2)^2} \ dx} \leq C(R)\sqrt{b + 1} \cdot \|g\|_2^2 \sum_n \frac{1}{1 + (n + k)^2} \cdot \sqrt{|b \cdot E^c \cap [n, n + 1]|}.
\]

(4)

Note that \(|b \cdot E^c \cap [n, n + 1]| \leq \epsilon\) if \(b \leq 1\) and \(|b \cdot E^c \cap [n, n + 1]| \leq \epsilon/b\) if \(b > 1\). In both cases we can bound (4) by

\[
\leq C(R)\|g\|_2^2 \sum_n \frac{1}{1 + (n + k)^2} \cdot \sqrt{\epsilon} = C(R)\|g\|_2^2 \cdot \sqrt{\epsilon} = C(R)\|g\|_2^2.
\]

\[\square\]

**Lemma 3.** Let \(E \subset \mathbb{R}\) be a relatively dense set which satisfies \(|E^c \cap I| \leq \epsilon |I|\) for any interval \(I\) of length \(1/b\) provided \(b \leq 1\) or \(|E^c \cap I| \leq \frac{\epsilon}{b} |I|\) for any interval \(I\) of length \(1/b\) provided \(b > 1\) then for \(k \neq l\)

\[
b \int_{E^c} |g(x)|^2 \cdot |\hat{\phi}_k(bx) \cdot \hat{\phi}_l(bx)| dx \leq C(R)\sqrt{\epsilon} \|g\|_2^2 \cdot \frac{1}{|k - l|^2}
\]

where \(C(R)\) depends only on \(R\).

The proof is similar to the one of Lemma 2. Just use the facts that \( |\hat{\phi}_k(x)| \leq \frac{C}{1 + (x + k)^2} \) and

\[
\sum_n \frac{1}{(1 + (n + k)^2)(1 + (n + l)^2)} = \sum_n \frac{1}{(1 + n^2)(1 + (n + l - k)^2)} \leq C \int \frac{1}{(1 + x^2)(1 + (x + l - k)^2)} dx \leq \frac{C}{|k - l|^2}.
\]

Now we are in a position to proceed with the proof of Theorem 1. Since \(\text{supp} \hat{f} \subset \bigcup_{n_i \in \mathbb{N}} [n_i - b/4, n_i + b/4]\) we can choose (not necessarily uniquely)

\[\hat{f}_{n_i} \in L^2 \text{ such that } \text{supp} \hat{f}_{n_i} \subset [n_i - b/4, n_i + b/4], \text{ the supports of } \hat{f}_{n_i} \text{ are disjoint and } \hat{f} = \sum_{n_i} \hat{f}_{n_i}.\]

Note that \(\hat{f}_{n_i} = \hat{f} \cdot \chi_{[n_i - b/4, n_i + b/4]}\) if \(|i|\) is large enough since the intervals \([n_i - b/4, n_i + b/4]\) are disjoint for \(i \neq j\) if \(|i|\) is large enough, which follows from the fact that there are no more than \(R \cdot b\) pairs \((i, j)\) such that \(0 < |n_i - n_j| \leq b/2\). Although \(\hat{f}_{n_i}\) is supported on \([n_i - b/4, n_i + b/4]\), we will define its Fourier series on the larger interval \([n_i - b/2, n_i + b/2]\) converging in \(L^2\) as follows:

\[
\hat{f}_{n_i}(x) = \sum_k c_{n_i}^{(k)} e^{ixk} \cdot \chi_{[n_i - b/2, n_i + b/2]}(x)
\]

where \(c_{n_i}^{(k)} = \frac{1}{b} \int_{n_i - \frac{b}{2}}^{n_i + \frac{b}{2}} \hat{f}_{n_i}(x) e^{-ixk} dx\).

Therefore,
\[ \|f_n\|_2^2 = \|\hat{f}_n\|_2^2 = b \sum_k |c_n^{(k)}|^2 . \]

Since the supports of \( \hat{f}_n \) are disjoint, we have
\[ \|f\|_2^2 = \|\hat{f}\|_2^2 = \sum_{n_i} \|\hat{f}_{n_i}\|_2^2 = \sum_{n_i} b \sum_k |c_{n_i}^{(k)}|^2 = b \sum_k \|g_k\|_2^2 \]

where \( g_k(x) = \sum_{n_i} c_{n_i}^{(k)} e^{2\pi i n_i x} \) are 1-periodic functions with lacunary spectra: \( \text{spec} g_k \subset \Lambda \) and \( \|g_k\|_2^2 = \frac{1}{b} \int_0^1 |g_k(x)|^2 dx. \)

Using the facts that \( \phi \left( \frac{x-n_i}{b} \right) \equiv 1 \) when \( x \in [n_i - b/4, n_i + b/4] \) and \( \text{supp} \hat{f}_{n_i} \subset [n_i - b/4, n_i + b/4] \), we get
\[ \hat{f}_{n_i}(x) = \hat{f}_{n_i}(x) \cdot \phi \left( \frac{x-n_i}{b} \right) = \sum_k c_{n_i}^{(k)} e^{2\pi i n_i x} \cdot \chi_{[n_i - b/2, n_i + b/2]} \cdot \phi \left( \frac{x-n_i}{b} \right). \]

Now use the fact that \( \text{supp} \phi \left( \frac{x-n_i}{b} \right) \subset [n_i - b/2, n_i + b/2] \) to obtain from the previous equality the following:
\[ \hat{f}_{n_i}(x) = \sum_k c_{n_i}^{(k)} e^{2\pi i n_i x} \cdot \phi_k \left( \frac{x-n_i}{b} \right). \]

Taking the inverse Fourier transform, we get
\[ f_{n_i}(x) = b \sum_k c_{n_i}^{(k)} e^{2\pi i n_i x} \cdot \phi_k (bx). \]

Therefore,
\[ f(x) = \sum_{n_i} f_{n_i}(x) = b \sum_{n_i} \sum_k c_{n_i}^{(k)} e^{2\pi i n_i x} \cdot \phi_k (bx) = b \sum_k \phi_k (bx) \left( \sum_{n_i} c_{n_i}^{(k)} e^{2\pi i n_i x} \right) = b \sum_k g_k(x) \phi_k (bx). \]

Now we can estimate \( \int_{E^c} |f|^2 \):
\[
\int_{E^c} |f|^2 = b^2 \int_{E^c} \left| \sum_k g_k(x) \phi_k (bx) \right|^2 dx = b^2 \int_{E^c} \left( \sum_k |g_k(x) \phi_k (bx)|^2 + \sum_{k \neq l} g_k(x) \phi_k (bx) \bar{g}_l(x) \bar{\phi}_l (bx) \right) dx.
\]
Using Lemma 2 we estimate the first term in (6):

$$b^2 \sum_k \int_{E^c} |g_k(x)\hat{\phi}_k(bx)|^2 dx \leq C(R)\sqrt{b} \sum_k \|g_k\|_2^2$$

(7)

$$= C(R)\sqrt{\epsilon} \|f\|_2^2.$$ 

Using Lemma 3 we estimate the second term in (6):

$$b^2 \sum_{k \neq l} \int_{E^c} (|g_k(x)|^2 + |g_l(x)|^2) \cdot |\hat{\phi}_k(bx)\hat{\phi}_l(bx)| dx \leq$$

$$b \cdot C(R)\sqrt{\epsilon} b \sum_{k \neq l} \frac{(\|g_k\|_2^2 + \|g_l\|_2^2)}{|k - l|^2} =$$

$$b \cdot C(R)\sqrt{\epsilon} \sum_{n \neq 0} \sum_k \frac{(\|g_k\|_2^2 + \|g_{k+n}\|_2^2)}{n^2} =$$

$$C(R)\sqrt{\epsilon} b \sum_k \|g_k\|_2^2 \cdot \sum_{n \neq 0} \frac{1}{n^2} \leq$$

$$C(R)\sqrt{\epsilon} \|f\|_2^2.$$ 

(8)

Adding the estimates (7) and (8) we get

$$\int_{E^c} |f|^2 \leq C(R)\sqrt{\epsilon} \|f\|_2^2.$$ 

Now we choose $\epsilon$ such that $C(R)\sqrt{\epsilon} \leq 1/2$. Hence,

$$\int_E |f|^2 \geq \frac{1}{2} \|f\|_2^2.$$ 

□

3. Proof of Theorem 2

A similar case was studied for uniqueness in [7]: if $f$ vanishes on $E$ then $f$ vanishes on the whole real line.

We will start with some results on periodizations. Define a family of periodizations of a function $f \in L^1$:

$$g_t(x) = \sum_{k=-\infty}^{\infty} f(x + k)e^{-2\pi i (x+k)}$$

(9)

where $t \in [-\frac{1}{2}, \frac{1}{2}]$. Then $g_t(x)$ is periodic with period 1, $\|g_t\|_{L^1([0,1])} \leq \|f\|_1$ and its Fourier coefficients are:

$$\hat{g}_t(l) = \hat{f}(l + t).$$
Now we assume that \( f \in L^1 \cap L^2 \). The next argument shows an important relation between the average of the \( L^2 \) norm of periodizations and the \( L^2 \) norm of \( f \):

\[
\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{E \cap [0,1]} |g_t(x)|^2 \, dx \, dt = \sum_{k,l} \int_{E \cap [0,1]} f(x + k) \bar{f}(x + l) e^{-i2\pi(l-k)} \, dx \, dt
\]

\[
= \sum_{k,l} \int_{E \cap [0,1]} f(x + k) \bar{f}(x + l) \, dx
\]

\[
= \sum_{k=-\infty}^{\infty} \int_{E \cap (E - k) \cap [0,1]} |f(x + k)|^2 \, dx
\]

\[
= \int_{E} |f|^2.
\]

We used that \( E = E - k \) since \( E \) is 1-periodic. In particular, it follows that

\[
\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{[0,1]} |g_t(x)|^2 \, dx \, dt = \int_{E} |f|^2.
\]

In the next lemma we extend these results to functions from \( L^2 \).

**Lemma 4.** If \( f \in L^2 \) then there exists a family \( \{g_t(x)\}_{t \in [-\frac{1}{2}, \frac{1}{2}]} \) of periodic functions: \( x \in [0,1] \), a.e. \( t \in [-\frac{1}{2}, \frac{1}{2}] \) with period 1 such that \( g_t(x) \in L^2([0,1] \times [-\frac{1}{2}, \frac{1}{2}]) \),

\[
\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{E \cap [0,1]} |g_t(x)|^2 \, dx \, dt = \int_{E} |f|^2
\]

and

\[
\hat{g}_t(l) = \hat{f}(l + t)
\]

for almost all \( t \) and all \( l \in \mathbb{Z} \).

**Proof:** Consider the cutoff:

\[
f^n(x) = \chi_{[-n,n]}(x).
\]

Since \( f^n \in L^1 \cap L^2 \) and converge to \( f \) in \( L^2 \) we can define corresponding families of periodizations \( g^n_t(x) \) which form a Cauchy sequence in \( L^2([0,1] \times [-\frac{1}{2}, \frac{1}{2}]) \):

\[
\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{[0,1]} |g^n_t(x) - g^m_t(x)|^2 \, dx \, dt = \int_{E} |f^n - f^m|^2.
\]
Let \( g_t(x) \) be the \( L^2([0, 1] \times [-\frac{1}{2}, \frac{1}{2}]) \) limit of \( g^n_t(x) \). Thus, we get the first statement of the lemma

\[
\frac{1}{2} \int_{E} |g_t(x)|^2 \, dx \, dt = \lim_{n \to \infty} \frac{1}{2} \int_{E} |g^n_t(x)|^2 \, dx \, dt
\]

\[
= \lim_{n \to \infty} \int_{E} |f^n|^2 = \int_{E} |f|^2.
\]

To obtain the second statement of Lemma 4 we consider the following sum:

\[
\sum_{l=-\infty}^{\infty} \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{g}_t(l) - \hat{f}(l + t)|^2 \, dt = \sum_{l=-\infty}^{\infty} \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{g}_t(l) - \hat{g}_t^n(l) + \hat{f}_n(l + t) - \hat{f}(l + t)|^2 \, dt
\]

\[
\leq 2 \sum_{l=-\infty}^{\infty} \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{g}_t(l) - \hat{g}_t^n(l)|^2 + |\hat{f}_n(l + t) - \hat{f}(l + t)|^2 \, dt
\]

\[
= 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} |g_t(x) - g^n_t(x)|^2 \, dx \, dt + 2 \int_{E} |f^n - f|^2 \leq \epsilon
\]

where \( \epsilon \) can be arbitrarily small if \( n \) is large enough. \( \square \)

Let \( g_t \) be a family of periodizations of \( f \) as defined in (9). It follows from Lemma 4 that \( \hat{g}_t(l) = \hat{f}(l + t) \) for almost all \( t \in [-\frac{1}{2}, \frac{1}{2}] \). Since \( \text{supp} \hat{f} \subset \bigcup_{n_i - \frac{1}{2}, n_i + \frac{1}{2}}, g_t \) has lacunary spectra for almost all \( t \). Therefore, from Nazarov’s result \( [9] \) for Zygmund’s theorem it follows that

\[
\int_{E \cap [0,1]} |g_t|^2 \geq C(\gamma, R) \int_{E \cap [0,1]} |g_t|^2
\]

for almost all \( t \in [-\frac{1}{2}, \frac{1}{2}] \). Applying Lemma 4 and (10) we obtain

\[
\int_{E} |f|^2 \geq C(\gamma, R) \int_{E} |f|^2.
\]

\( \square \)

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