SHARP LOG-SOBOLEV INEQUALITIES IN $\text{CD}(0,N)$ SPACES WITH APPLICATIONS

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Abstract. Given $p,N > 1$, we prove the sharp $L^p$-log-Sobolev inequality on noncompact metric measure spaces satisfying the $\text{CD}(0,N)$ condition, where the optimal constant involves the asymptotic volume ratio of the space. This proof is based on a sharp isoperimetric inequality in $\text{CD}(0,N)$ spaces, symmetrisation, and a careful scaling argument. As an application we establish a sharp hypercontractivity estimate for the Hopf–Lax semigroup in $\text{CD}(0,N)$ spaces. The proof of this result uses Hamilton–Jacobi inequality and Sobolev regularity properties of the Hopf–Lax semigroup, which turn out to be essential in the present setting of nonsmooth and noncompact spaces. Moreover, a sharp Gaussian-type $L^2$-log-Sobolev inequality and a hypercontractivity estimate are obtained in $\text{RCD}(0,N)$ spaces. Our results are new, even in the smooth setting of Riemannian/Finsler manifolds. In particular, an extension of the celebrated rigidity result of Ni (J. Geom. Anal., 2004) on Riemannian manifolds will be a simple consequence of our sharp log-Sobolev inequality.

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Date: November 20, 2023.
2020 Mathematics Subject Classification. 28A25, 26D15, 46E35, 49Q22.
Key words and phrases. Log-Sobolev inequality, sharpness, metric spaces, hypercontractivity, $\text{CD}(0,N)$ condition.
Z. M. Balogh and F. Tripaldi were supported by the Swiss National Science Foundation, Grant Nr. 200020_191978.
Research of A. Kristály was done while visiting the Corvinus Institute for Advanced Studies, Corvinus University, Budapest, Hungary; research also supported by the UEFISCEDI/CNCS grant PN-III-P4-ID-PCE2020-1001.
1. Introduction and Main Results

Different forms of the log-Sobolev inequality appear as indispensable tools to describe nonlinear phenomena, such as the solution of Poincaré’s conjecture (see Perelman [61]), quantum field theory (see e.g. Glimm and Jaffe [40]), hypercontractivity estimates for Hopf–Lax semigroups (see e.g. Bobkov, Gentil and Ledoux [19], Gentil [37], Otto and Villani [59, 60]), equilibrium for spin systems (see Guionnet and Zegarlinski [43]), or hydrodynamic scalings for systems of interacting particles (see Yau [67]).

The optimal Euclidean $L^p$-log-Sobolev inequality for $1 < p < n$ in $\mathbb{R}^n$ is due to Del Pino and Dolbeault [32], and states that, for every Sobolev function $u \in W^{1,p}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |u|^p dx = 1$, one has the inequality

$$\int_{\mathbb{R}^n} |u|^p \log |u|^p dx \leq \frac{n}{p} \log \left( \mathcal{L}_{p,n} \int_{\mathbb{R}^n} |\nabla u|^p dx \right). \quad (1.1)$$

In the above inequality, we used the notation $\mathcal{L}_{p,N}$ for any two given numbers $p, N > 1$ to denote

$$\mathcal{L}_{p,N} = \frac{p}{N} \left( \frac{p-1}{\sigma_N} \right)^{p-1} \left( \sigma_N \Gamma \left( \frac{N}{p'} + 1 \right) \right)^{-\frac{1}{p'}},$$

where $\sigma_N = \frac{n^N}{\Gamma \left( \frac{N}{p} + 1 \right)}$, $\Gamma$ being the usual Euler Gamma-function, and $p' = \frac{p}{p-1}$ is the conjugate of $p$. Note that $\mathcal{L}_{p,N}$ is well-defined for any $N > 1$, and not just for integer values. We also mention that the constant $\mathcal{L}_{p,n}$ in the inequality (1.1) is sharp. This inequality appeared first in the paper of Weissler [68] for $p = 2$, which is equivalent to the dimension-free log-Sobolev inequality of Gross [41], and it was stated for the Gaussian probability measure $\gamma_G$ with $d\gamma_G(x) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}} dx$, i.e., for every function $u \in W^{1,2}(\mathbb{R}^n, \gamma_G)$ with $\int_{\mathbb{R}^n} u^2 d\gamma_G = 1$, one has

$$\int_{\mathbb{R}^n} u^2 \log u^2 d\gamma_G \leq 2 \int_{\mathbb{R}^n} |\nabla u|^2 d\gamma_G. \quad (1.2)$$

Soon after the paper [32] was published, Gentil [37] by the Prékopa–Leindler inequality, and Agueh, Ghoussoub and Kang [2] by optimal mass transportation, showed that (1.1) is valid for all $p > 1$.

Moreover, the optimal transport method was applicable even in the case of curved structures, where the influence of the Ricci curvature plays an important role. Indeed, by optimal transport arguments, Cordero-Erausquin, McCann and Schmuckenschläger established in [29] (see also Cordero-Erausquin [28]) a Bakry–Émery-type estimate [13], i.e., if $(M,g)$ is a complete Riemannian manifold, $V : M \to \mathbb{R}$ is a smooth enough function, and $K > 0$ that satisfy the curvature condition

$$\text{Hess}_x V + \text{Ric}_x \geq K \text{Id} \quad \text{for all } x \in M,$$

then, for the weighted Riemannian measure $d\gamma_V = e^{-V} d\text{vol}$, the inequality

$$\int_M u^2 \log u^2 d\gamma_V \leq \frac{2}{K} \int_M |\nabla u|^2 d\gamma_V, \quad (1.3)$$

holds for any $u \in W^{1,2}(M, \gamma_V)$ such that $\int_M u^2 d\gamma_V = 1$. In the Euclidean case where $M = \mathbb{R}^n$, one can obtain the Gaussian log-Sobolev inequality (1.2) from (1.3), by choosing $K = 1$ and $V(x) = \frac{1}{2} |x - x_0|^2$ for some $x_0 \in M$. However, it is not possible to recover by (1.3) the optimal Euclidean $L^2$-log-Sobolev inequality (1.1) by choosing $V \equiv c$. In this respect, note that also in
the work of Barthe and Kolesnikov [18] general versions of log-Sobolev inequalities were proven for Riemannian manifolds for measures with the tail behavior of order \(e^{-|x|^a}\) for some \(a > 0\) whenever \(|x| \gg 1\). Moreover, we observe that the multiplicative constant \(\frac{2}{K}\) on the right side of (1.3) blows up as \(K \to 0\), indicating that it might be difficult (or even impossible) to find an appropriate version of (1.3) for general manifolds with 0 lower bound on their Ricci curvature.

This is precisely the main purpose of our paper; namely, to prove a sharp version of the \(L^p\)-log-Sobolev inequality (1.1) (and eventually a corresponding version of the Gaussian log-Sobolev inequality (1.2)) for a general metric measure space \((X, d, m)\) satisfying the curvature-dimension condition \(\text{CD}(0,N)\) in the sense of Lott–Sturm–Villani (see [51, 65, 66]). This class of metric measure spaces contains as particular examples Riemannian/Finsler manifolds with non-negative Ricci curvature and their Gromov–Hausdorff limits. Metric cones studied by Bacher and Sturm [12] and Ketterer [48] are also included in this class; for more examples, we refer to [17].

In order to formulate our result, we need to introduce an additional notion. Given a metric measure space \((X, d, m)\) that satisfies the curvature-dimension condition \(\text{CD}(0,N)\), according to Bishop–Gromov theorem, see Sturm [66], one can define the asymptotic volume ratio

\[
\text{AVR}_m = \lim_{r \to \infty} \frac{m(B(x,r))}{\sigma_{N} r^{N}},
\]

which is independent of the choice of \(x \in X\). Let \(W^{1,p}(X, d, m)\) be the space of real-valued Sobolev functions over \(X\), and \(|\nabla u| \in L^p(X, m)\) be the minimal \(p\)-weak upper gradient of \(u \in W^{1,p}(X, d, m)\), which exists \(m\text{-a.e. on } X\), see §2; these notions were introduced by Heinonen and Koskela [46] (see also Hajlasz [45] and Shanmugalingam [64]). These concepts were extensively studied by Cheeger [26] in the context of differentiable structures of metric measures spaces. They play an important role in the study of Hamilton–Jacobi equations in the general metric measure setting, see Ambrosio, Gigli and Savaré [8, 9].

Our first main result provides the natural and sharp extension of (1.1) to the class of \(\text{CD}(0,N)\) spaces.

**Theorem 1.1.** Let \(N, p > 1\), and \((X, d, m)\) be a \(\text{CD}(0,N)\) space with \(\text{AVR}_m > 0\). Then, for every \(u \in W^{1,p}(X, d, m)\) with \(\int_X |u|^p dm = 1\), one has that

\[
\int_X |u|^p \log |u|^p dm \leq \frac{N}{p} \log \left( L_{p,N} \text{AVR}_m^{-\frac{p}{N}} \int_X |\nabla u|^p dm \right).
\]

In addition, the constant \(L_{p,N} \text{AVR}_m^{-\frac{p}{N}}\) in (1.4) is sharp: if there exists \(C > 0\) such that for every \(u \in W^{1,p}(X, d, m)\) with \(\int_X |u|^p dm = 1\) one has

\[
\int_X |u|^p \log |u|^p dm \leq \frac{N}{p} \log \left( C \int_X |\nabla u|^p dm \right),
\]

then

\[
C \geq L_{p,N} \text{AVR}_m^{-\frac{p}{N}}.
\]

Let us note that in the case of the Euclidean space we have \(\text{AVR}_m = \text{AVR}_{C^0} = 1\) and thus (1.4) is a direct generalization of (1.1). In the case of a Riemannian manifold \((M, g)\) (endowed with its canonical measure \(dv_g\)), one has \(0 \leq \text{AVR}_{dv_g} \leq 1\), which makes the constant on the right side of (1.4) slightly worse than in the Euclidean case. In particular, Theorem 1.1 implies a generalisation of the rigidity result of Ni [55] in the Riemannian context; see Remark 3.3 for details.
We mention at this point that certain versions of log-Sobolev inequalities proven by Lott and Villani in [51] for CD(K, N) spaces with K > 0 (see also Bakry, Gentil andLedoux [14]) are similar to (1.3), as they also featurea multiplicative factor $\frac{1}{K}$ which goes to $\infty$ as $K \to 0$. Aspreviously observed, this supports the fact that it is not possible to obtain a sharp log-Sobolevinequality by a simple limiting argument, without the additionalassumption that $AVR_m > 0$. Furthermore, onecan see the second statement of Theorem 1.1 as a converse of the first one in the sensethat if $(X, d, m)$ is a CD(0, N) space that supportsan $L^p$-log-Sobolev inequality of the type (1.5), then$AVR_m > 0$, see (1.6). Therefore, the condition $AVR_m > 0$ is a necessary and sufficient conditionfor the validity of an Euclidean-type log-Sobolevinequality in the framework of CD(0, N) spaces with aprecise relationship between the optimal log-Sobolevconstant and the value $AVR_m > 0$; for a moreprecise statement, see Remark 3.1. We notice thatBakry and Ledoux [15, Theorem 3] proved an$L^2$-log-Sobolev inequality for diffusion operators satisfyingacertain CD(0, N) condition in terms ofthe Bakry-Émery carré de champ on Riemannianmanifolds under a special condition on thegenerator of the diffusion operator. However, this resultsdoes not capture the $AVR_m$ constant as theEuclidean constant appears on the right side of thelog-Sobolev inequality.

The proof of (1.4) combines several known results bynow: (a) the recent sharp isoperimetricinequality in CD(0, N) spaces, proved by Balogh andKristály [17]; (b) a sharp weighted log-Sobolevinequality on cones, see Balogh, Don and Kristály [16], and (c) Pólya–Szegö type rearrangementarguments on CD(0, N) spaces, due to Nobili and Violo [56, 57], see also Mondino andSemola [54]. In order to prove the inequality (1.4),a nonnegative function $u \in W^{1,p}(X, d, m)$ is rearrangedwith respect to the 1-dimensional model space $[0, \infty]$ endowed with the measure $\omega = N\sigma_{N,T}^{-1} L^1$,see §2.1.2 for details. Since the entropy-term remainsinvariant under such rearrangement, thePólya–Szegö inequality involving the term $AVR_m$, see [56, 57] (which follows from a suitable co-areaformula and the isoperimetric inequality from [17]), and the sharp weighted log-Sobolevinequality on the model cone $([0, \infty), |\cdot|, \omega)$ imply inequality in (1.4). We noticethat the limit case $p = \infty$ in (1.4) can also beobtained; a similar result is established by Fujita [36] in the Euclidean case.

The proof of the sharpness of the constant $L^p,NAVR_m^{\frac{1}{p}}$ in (1.4) is more complicated, whereacareful transition from the original measure $m$ to the measure $\omega = N\sigma_{N,T}^{-1} L^1$ is performed,combined with a subtle scaling argument which produces the factor $AVR_m$. We noticethat an alternative (but less general) proof of the samesharpness is known from [49], where some robustordinary differential equations/inequalities are compared.

The first nontrivial application of Theorem 1.1 is a sharp hypercontractivity estimate for theHopf–Lax semigroup on metric measure spaces $(X, d, m)$ satisfying the CD(0, N) condition. Let$p > 1$; for $t > 0$ and $u : X \to \mathbb{R}$ we consider the Hopf–Lax formula

$$Q_t u(x) := \inf_{y \in X} \left\{ u(y) + \frac{d(x, y)^p}{p^\alpha t^{p^\alpha - 1}} \right\}, \quad x \in X.$$ 

By convention, $Q_0 u = u$.

Before stating the result, let us notice that when $(X, d, m)$ is a CD(K, N) space with K > 0 (thus, X is compact) and $u$ is a bounded Lipschitz function on X, one has that $Q_t u(x) \in \mathbb{R}$ for every $(t, x) \in (0, \infty) \times X$. Note however that the limit case CD(0, N) requires finer properties onthe function $u : X \to \mathbb{R}$; indeed, if we still keep the above class of functions and $m(X) = +\infty$ (i.e.,m cannot be normalized to a probability measure), then the $L^\alpha(X, m)$-norm of $e^u$ (which appearsin the hypercontractivity estimate, see (1.7) below) would be $+\infty$ for every $\alpha > 0$. Thus, in order
to provide a meaningful hypercontractivity estimate for Hopf–Lax semigroups on CD(0, N) spaces, a suitable class of functions is needed.

Accordingly, for some \( t_0 > 0 \) and \( x_0 \in X \), we shall consider the class of functions \( u : X \to \mathbb{R} \), denoted by \( \mathcal{F}_{t_0, x_0}(X) \), which satisfy the following assumptions:

(A1) \( u \in \text{Lip}_{\text{loc}}(X) \) and the set \( u^{-1}([0, \infty)) \subset X \) is bounded;

(A2) \( Q_{t_0}u(x_0) > -\infty \);

(A3) there exist \( M \in \mathbb{R} \) and \( C_0 > pt_{t_0}^{p-1} \) such that

\[
u(x) \geq M - \frac{d^{p'}(x,x_0)}{C_0}, \quad \forall x \in X.
\]

One can prove that \( Q_t u(x) \in \mathbb{R} \) for every \( (t, x) \in (0, t_0) \times X \), whenever \( u \in \mathcal{F}_{t_0, x_0}(X) \), see Proposition 4.1. Our second result provides the following sharp hypercontractivity estimate on CD(0, N) spaces.

**Theorem 1.2.** Let \( N > 1 \) and \((X, d, m)\) be a CD(0, N) space with AVR \( m > 0 \). Let \( t_0 > 0 \) and \( x_0 \in X \) be fixed. Then for every \( p > 1 \), \( 0 < \alpha \leq \beta \), \( t \in (0, t_0) \) and \( u \in \mathcal{F}_{t_0, x_0}(X) \) with \( (1 + d^{p'}(x_0, \cdot))e^{\alpha u} \in L^1(X, m) \) and \( e^{\frac{\alpha}{ap}} \in W^{1, p}(X, d, m) \), we have that

\[
\|e^{Q_t u}\|_{L^\beta(X,m)} \leq \|e^u\|_{L^\alpha(X,m)} \left( \frac{\beta - \alpha}{t} \right)^{\frac{\alpha}{\beta} + \frac{\alpha}{p}} \left( \frac{\alpha}{\beta} + \frac{\alpha}{p} \right) \left( \frac{N}{p} \right)^{\frac{\beta - \alpha}{\beta} + \frac{\alpha}{p}} \frac{L_{p, N}^{AVR_m} \frac{N e^{p-1}}{p^p}}{\frac{N}{p} \frac{N e^{p-1}}{p^p}}.
\]

In addition, the constant \( L_{p, N}^{AVR_m} \frac{N e^{p-1}}{p^p} \) in (1.7) is sharp whenever \( \alpha < \beta \).

We notice that the equivalence of the log-Sobolev inequality and hypercontractivity of the Hopf–Lax semigroup has already been studied in the Euclidean setting by Gentil [37], and Bobkov, Gentil and Ledoux [19]. The link between these two results is based on the fact that the Hopf–Lax semigroup satisfies the Hamilton–Jacobi equation. For compact metric measure spaces, the same link was established by Lott and Villani [52] and Gozlan, Roberto and Samson [42]. In this case, the Hopf–Lax semigroup satisfies a Hamilton–Jacobi inequality that is enough for the purpose of the equivalence.

In our setting the same general strategy will be applied. However, in the present situation of nonsmooth and noncompact spaces the implementation of this general strategy becomes rather subtle. We need to make sure that we are working in the right function class. For example if we choose \( u \) in the class of bounded functions, then the right side of (1.7) will be infinity and so the claim will be trivially true. This kind of problem motivated us to search for a suitable class of functions with precise regularity and growth properties in order to obtain meaningful estimates. The choice of the function class as described above allows us to establish good properties of integrals involving the Hopf–Lax semigroup \( Q_t u \), see Propositions 4.1 & 4.2, that will enable us to carry out the strategy of the proof. Another result that we needed in the proof was the fact that a Hamilton–Jacobi inequality is satisfied by the Hopf–Lax semigroup in metric measure spaces that was shown by Ambrosio, Gigli and Savaré [8].

The second application of Theorem 1.1 is an \( L^2 \)-Gaussian log-Sobolev inequality; it turns out that our approach requires the framework of metric measure spaces satisfying the Riemannian curvature-dimension condition \( \text{RCD}(0, N) \), studied in great detail by Ambrosio, Gigli and Savaré [7], Ambrosio, Mondino and Savaré [10], Erbar, Kuwada and Sturm [34] and Cavalletti and Milman [24]; for details, see §5, and Ambrosio [4] and Gigli [39] for further insights on the theory of \( \text{RCD}(0, N) \).
spaces. In order to state the result, for a fixed point \( x_0 \in X \), we consider the \( m \)-Gaussian probability measure

\[
dm_{G,x_0}(x) := G^{-1}e^{-\frac{d^2(x_0,x)}{2}} dm(x) \quad \text{with} \quad G = \int_X e^{-\frac{d^2(x_0,x)}{2}} dm(x),
\]

and the \( m \)-density at \( x \in X \), i.e.,

\[
\theta_m(x) = \lim_{r \to 0} \frac{m(B(x,r))}{\sigma_{N,N}^r}.
\]

A simple consequence of the Bishop–Gromov comparison principle is that

\[
\theta_m(x) \geq \AVR_m, \quad \forall x \in X.
\]

The \( L^2 \)-Gaussian log-Sobolev inequality in \( \text{RCD}(0,N) \) spaces reads as follows:

**Theorem 1.3.** Let \((X,d,m)\) be an \( \text{RCD}(0,N) \) space with \( N > 1 \) and assume that \( \AVR_m > 0 \). Given \( x_0 \in X \), for any \( u \in W^{1,2}(X,d,m_{G,x_0}) \) such that \( \int_X u^2 dm_{G,x_0} = 1 \), one has

\[
\int_X u^2 \log u^2 dm_{G,x_0} \leq 2 \int_X |\nabla u|^2 dm_{G,x_0} + \log \frac{\theta_m(x_0)}{\AVR_m}.
\]

In addition, the constant 2 is sharp in \((1.9)\).

In the Euclidean case with the Lebesgue measure, i.e., \((\mathbb{R}^n, | \cdot |, \mathcal{L}^n)\), we have that \( \theta_{\mathcal{L}^n}(x_0) = 1 \) for any \( x_0 \in \mathbb{R}^n \), \( G = (2\pi)^{\frac{n}{2}} \) and \( \AVR_{\mathcal{L}^n} = 1 \), so we recover \((1.2)\) as a particular case of \((1.9)\).

Note also how the statement of the above result might depend qualitatively on the choice of the basepoint \( x_0 \in X \). Indeed, e.g. for the model space \((X,d,m) = ([0,\infty), | \cdot |, \omega)\) with \( \omega = N\sigma_{N,N}^{-1}\mathcal{L}^1 \), we have that \( \theta_m(x_0) = \infty \) for \( x_0 \neq 0 \) and \( \theta_m(x_0) = 1 \) when \( x_0 = 0 \); therefore, the statement of Theorem 1.3 becomes meaningful only by choosing \( x_0 = 0 \).

We pointed out that our argument is based on the infinitesimal Hilbertianity of the space which is guaranteed by \( \text{RCD}(0,N) \) structures. It is known that the constant 2 in \((1.2)\) is sharp, and the equality case has been characterized by Carlen [21, Theorem 5]; here, we provide a new proof – based on scalings – of the sharpness of 2 in the inequality \((1.9)\) on generic \( \text{RCD}(0,N) \) spaces.

We also notice the comparability of \((1.9)\) and \((1.2)\) (resp. \((1.3)\)) as well as the dimension-free character of \((1.9)\), which contains only a quantitative information about the space \((X,d,m)\) in terms of the ‘amplitude’ of the volume ratio \( r \mapsto \frac{m(B(x,r))}{\sigma_{N,N}^r} \), appearing in the form \( \frac{\theta_m(x_0)}{\AVR_m} \geq 1 \). Comparing the inequality \((1.3)\) with the expression of \((1.9)\), we can see that the presence of the additive constant on the right side of \((1.9)\) compensates for the lack of positive lower bound on the Ricci curvature; in this sense we can interpret the \( \AVR_m > 0 \) condition as a kind of “positive curvature at infinity”. Moreover, it turns out that if \( \theta_m(x_0) = \AVR_m \) for some \( x_0 \in X \), then we are in the rigid setting where ‘volume cone implies metric cone’, i.e., the space \((X,d,m)\) is isometric to a certain model metric cone; for details, see De Philippis and Gigli [31]. Due to the defective log-Sobolev inequality \((1.9)\), a hypercontractivity bound for the \( m \)-Gaussian measure is also stated.

The paper is organized as follows. In Section 2, we provide those notions and results that are indispensable to prove our main results; namely, basic properties of \( \text{CD}(0,N) \) spaces (Bishop–Gromov comparison principle and isoperimetric inequality), rearrangement arguments and Pólya–Szegő inequality on \( \text{CD}(0,N) \) spaces, as well as a sharp log-Sobolev inequality on the 1-dimensional weighted model space. Section 3 is devoted to the proof of Theorem 1.1, most of the arguments being focused on the proof of the sharpness of the log-Sobolev inequality \((1.4)\). In Section 4 we prove the sharp hypercontractivity estimate in \( \text{CD}(0,N) \) spaces (i.e., Theorem 1.2). In Section 5
we recall some basic properties of \( \text{RCD}(0, N) \) spaces (e.g. the Laplace comparison for the distance function) which are needed to prove the sharp \( L^2 \)-Gaussian log-Sobolev inequality in \( \text{RCD}(0, N) \) spaces (see Theorem 1.3). Finally, Section 6 is devoted to presenting some problems which are closely related to our results, and represent further open questions.

2. Preliminaries

2.1. Properties of \( \text{CD}(0, N) \) spaces.

2.1.1. \( \text{CD}(0, N) \) spaces, Bishop–Gromov comparison principle and isoperimetric inequality. Let us fix \((X, d, m)\) a metric measure space, i.e., \((X, d)\) is a complete separable metric space and \(m\) is a locally finite measure on \(X\) endowed with its Borel \(\sigma\)-algebra.

For \( p \geq 1 \) we denote by \( L^p(X, m) = \left\{ u: X \to \mathbb{R} : u \text{ is measurable}, \int_X |u|^p \, dm < \infty \right\} \), the set of \(p\)-integrable functions; as usual, functions in the latter definition which are equal \(m\)-a.e. are identified. The notation \( \text{Lip}_c(X) \) stands for the space of real-valued Lipschitz functions with compact support in \( X \) and \( \text{Lip}_{\text{loc}}(X) \) is the space of real-valued locally Lipschitz functions on \( X \).

Let \( P_2(X, d) \) be the \( L^2 \)-Wasserstein space of probability measures on \( X \), while \( P_2(X, d, m) \) is the subspace of \(m\)-absolutely continuous measures on \( X \). For \( N > 1 \), let \( \text{Ent}_N(\cdot|m): P_2(X, d) \to \mathbb{R} \) be the Rényi entropy functional with respect to the measure \( m \), defined by

\[
\text{Ent}_N(\nu|m) = -\int_X \rho^{-\frac{1}{N}} \, d\nu = -\int_X \rho^{1-\frac{1}{N}} \, d\rho,
\]

where \( \rho \) denotes the density function of \( \nu_{\text{ac}} \) in \( \nu = \nu_{\text{ac}} + \nu^s = \rho \, m + \nu^s \), where \( \nu_{\text{ac}} \) and \( \nu^s \) represent the absolutely continuous and singular parts of \( \nu \in P_2(X, d) \), respectively.

The curvature-dimension condition \( \text{CD}(0, N) \) states that for all \( N' \geq N \) the functional \( \text{Ent}_{N'}(\cdot|\cdot u) \) is convex on the \( L^2 \)-Wasserstein space \( P_2(X, d, m) \), i.e., for each \( m_0, m_1 \in P_2(X, d, m) \) there exists a geodesic \( \Gamma: [0, 1] \to P_2(X, d, m) \) joining \( m_0 \) and \( m_1 \) such that for every \( s \in [0, 1] \) one has

\[
\text{Ent}_{N'}(\Gamma(s)|m) \leq (1 - s)\text{Ent}_{N'}(m_0|m) + s\text{Ent}_{N'}(m_1|m),
\]

see Lott and Villani [51] and Sturm [66].

The Minkowski content of \( \Omega \subset X \) is given by

\[
m^+(\Omega) = \liminf_{\varepsilon \to 0^+} \frac{m(\Omega \setminus \varepsilon \Omega)}{\varepsilon},
\]

where \( \Omega_\varepsilon = \{ x \in X : \exists y \in \Omega \text{ such that } d(x, y) < \varepsilon \} \) is the \(\varepsilon\)-neighborhood of \( \Omega \) w.r.t. the metric \( d \).

If \((X, d, m)\) is a metric measure space satisfying the \( \text{CD}(0, N) \) condition for some \( N > 1 \), then the Bishop–Gromov comparison principle states that both functions

\[
r \mapsto \frac{m^+(B(x, r))}{r^{N-1}} \quad \text{and} \quad r \mapsto \frac{m(B(x,r))}{r^N}, \quad r > 0,
\]

are non-increasing on \([0, \infty)\) for every \( x \in X \), see Sturm [66], where \( B(x, r) = \{ y \in X : d(x, y) < r \} \) is the ball with center \( x \in X \) and radius \( r > 0 \). By the latter Bishop–Gromov monotonicity, the asymptotic volume ratio

\[
\text{AVR}_m = \lim_{r \to \infty} \frac{m(B(x, r))}{\sigma_N r^N},
\]

is well-defined, i.e., it is independent of the choice of \( x \in X \).
If $\text{AVR}_m > 0$, we have the *sharp isoperimetric inequality* on $(X, d, m)$, namely, for every bounded Borel subset $\Omega \subset X$ it holds

\[ m^+(\Omega) \geq N \sigma^{\frac{1}{N}}_N \text{AVR}_m^{\frac{1}{N}} m(\Omega)^{\frac{N-1}{N}}, \tag{2.3} \]

and the constant $N \sigma^{\frac{1}{N}}_N \text{AVR}_m^{\frac{1}{N}}$ in (2.3) is sharp, see Balogh and Kristály [17]. We notice that (2.3) has been stated already in the smooth setting of Riemannian manifolds by Agostiniani, Fogagnolo and Mazzieri [1, 35], Brendle [20] and John [47]. By using an approximation argument, see Ambrosio, Di Marino and Gigli [6, Theorem 3.6], or Nobili and Violo [56, Proposition 3.10], relation (2.3) implies that for every open Borel set $\Omega \subset X$ one has

\[ \text{Per}(\Omega) \geq N \sigma^{\frac{1}{N}}_N \text{AVR}_m^{\frac{1}{N}} m(\Omega)^{\frac{N-1}{N}}, \tag{2.4} \]

where

\[ \text{Per}(\Omega) = \inf \left\{ \liminf_{n \to \infty} \int_X |Du_n| dm : u_n \in \text{Lip}_\text{loc}(X), \ u_n \to \chi_\Omega \text{ in } L^1_{\text{loc}}(X) \right\} \]

denotes the perimeter of $\Omega$ in $X$, see Ambrosio and Di Marino [5] and Miranda [53]; here, we used the notion of *local Lipschitz constant* $|Du|(x)$ at $x \in X$ for $u \in \text{Lip}_\text{loc}(X)$, i.e.,

\[ |Du|(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d(x, y)}. \]

2.1.2. **Rearrangement and Pólya–Szegő inequality on CD(0, N) spaces.** Let $(X, d, m)$ be a metric measure space satisfying the CD(0, N) condition for some $N > 1$ and $\text{AVR}_m > 0$. We first introduce the natural Sobolev space over $(X, d, m)$, following e.g. Cheeger [26] and Ambrosio, Gigli and Savaré [8]. Let $p > 1$. The *p-Cheeger energy* $\text{Ch}_p : L^p(X, m) \to [0, \infty]$ is defined as the convex and lower semicontinuous functional

\[ \text{Ch}_p(u) = \inf \left\{ \liminf_{n \to \infty} \int_X |Du_n|^p dm : (u_n) \subset \text{Lip}(X) \cap L^p(X, m), \ u_n \rightarrow u \text{ in } L^p(X, m) \right\}. \]

Then

\[ W^{1,p}(X, d, m) = \{ u \in L^p(X, m) : \text{Ch}_p(u) < \infty \} \]

is the $p$-Sobolev space over $(X, d, m)$, endowed with the norm $\|u\|_{W^{1,p}} = \left(\|u\|_{L^p(X, m)}^p + \text{Ch}_p(u)\right)^{1/p}$. Note that $W^{1,p}(X, d, m)$ is a Banach space, which turns out to be also reflexive (due to the fact that $(X, d, m)$ is doubling), see Ambrosio, Colombo and Di Marino [3]. By the relaxation of the $p$-Cheeger energy, one can define the minimal $m$-a.e. object $|\nabla u| \in L^p(X, m)$, the so-called *minimal $p$-weak upper gradient* of $u \in W^{1,p}(X, d, m)$, such that

\[ \text{Ch}_p(u) = \int_X |\nabla u|^p dm. \]

Since $(X, d, m)$ is a CD(0, N) space, it is doubling (by Bishop–Gromov comparison principle) and supports the $(1, p)$-Poincaré inequality for every $p \geq 1$, see Rajala [63]; thus, for every $u \in \text{Lip}_\text{loc}(X)$ one has

\[ |Du|(x) = |\nabla u|(x), \ m-\text{a.e. } x \in X, \tag{2.5} \]

see Cheeger [26, Theorem 6.1], and

\[ W^{1,p}(X, d, m) = \text{Lip}_c(X)^{\|\cdot\|_{W^{1,p}}} = W^{1,p}(X, d, m)^{\|\cdot\|_{W^{1,p}}}, \tag{2.6} \]
where \( \mathcal{W}^{1,p}(X, d, m) = \{ u \in \text{Lip}_{\text{loc}}(X) \cap L^p(X, m) : |Du| \in L^p(X, m) \} \), see e.g. Gigli [38, Theorem 2.8]. By (2.5), one has for any fixed \( x_0 \in X \) the eikonal equation, i.e.,

\[
|\nabla d_{x_0}| = 1 \quad \text{m-a.e.,}
\]

where \( d_{x_0}(x) := d(x_0, x) \), \( x \in X \); see also Gigli [38, Theorem 5.3].

For a measurable function \( u : X \to [0, \infty) \), we consider its non-increasing rearrangement \( \hat{u} : [0, \infty) \to [0, \infty) \) defined on the 1-dimensional model space \((0, \infty), | \cdot |, \omega = N\sigma_N r^{N-1} L^1 \) so that

\[
m(M_t(u)) = \omega(M_t(\hat{u})), \quad \forall t > 0,
\]

where

\[
M_t(u) = \{ x \in X : u(x) > t \} \quad \text{and} \quad M_t(\hat{u}) = \{ y \in [0, \infty) : \hat{u}(y) > t \},
\]

whenever \( m(M_t(u)) < \infty \) for every \( t > 0 \). In particular, the open set \( \Omega \subset X \) can be rearranged into the interval \( \Omega^* = [0, r] \) with \( m(\Omega^*) = \omega([0, r]) \), i.e., \( m(\Omega) = \sigma_N r^N \), with the convention that \( \Omega^* = [0, \infty) \) if \( m(\Omega) = +\infty \). Using the isoperimetric inequality (2.4) and the co-area formula (see e.g. Miranda [53]), Nobili and Violo proved recently [56, Theorem 3.6] (see also [57]) that the following Pólya–Szegő inequality holds for any \( u \in W^{1,p}(X, d, m) \):

\[
\text{Ch}_p(u) = \int_X |\nabla u|^p dm \geq \text{AVR}_m^{\frac{p}{n}} \int_0^\infty |\hat{u}'(r)|^p d\omega(r).
\]

2.2. Sharp weighted log-Sobolev inequality on the 1-dimensional model space. Let \( E \subset \mathbb{R}^n \) be an open convex cone and \( \omega : E \to (0, \infty) \) be a log-concave homogeneous weight of class \( C^1 \) and degree \( \tau \geq 0 \), i.e., \( \omega(\lambda x) = \lambda^\tau \omega(x) \).

Let \( p > 1 \) and consider \( W^{1,p}(E; \omega) = \{ u \in L^p(E; \omega) : \nabla u \in L^p(E; \omega) \} \). According to a recent result of Balogh, Don and Kristály [16], for every function \( u \in W^{1,p}(E; \omega) \) for which \( \int_E |u|^p \omega dx = 1 \), we have

\[
\mathcal{E}_{\omega,E}(|u|^p) = \int_E |u|^p \log |u|^p \omega dx \leq \frac{n + \tau}{p} \log \left( \mathcal{L}_{\omega,p} \int_E |\nabla u|^p \omega dx \right)
\]

where

\[
\mathcal{L}_{\omega,p} := \frac{p}{n + \tau} \left( \frac{p - 1}{e} \right)^{p-1} \left( \Gamma \left( \frac{n + \tau}{p} + 1 \right) \right) \int_{B \cap E} \omega dx
\]

and \( B \) is the unit ball with center at the origin in \( \mathbb{R}^n \). Moreover, when \( \tau > 0 \), equality holds in (2.10) if and only if the extremal function belongs to the family of Gaussians

\[
u_{\lambda,x_0}(x) = \lambda^{\frac{n+\tau}{p'}}, \quad \mathcal{L}_{\omega,p'} \left( \Gamma \left( \frac{n + \tau}{p'} + 1 \right) \int_{B \cap E} \omega dx \right)^{-\frac{p}{n+\tau}} e^{-\lambda |x+x_0|^2/p'}, \quad x \in E, \quad \lambda > 0,
\]

where \( x_0 \in -\partial E \cap \partial E \) and \( \omega(x+x_0) = \omega(x) \) for every \( x \in E \).

Let \( N > 1 \) and consider the 1-dimensional cone \( E = [0, \infty) \) endowed with the usual Euclidean distance \( | \cdot | \) and measure \( \omega = N\sigma_N r^{N-1} L^1 \). It is clear that \( \omega \) is a log-concave weight of class \( C^1 \) with homogeneity \( \tau = N - 1 > 0 \). In this particular case, (2.10) implies that on the model space \((0, \infty), | \cdot |, \omega \) the following log-Sobolev inequality holds: for every \( v \in W^{1,p}([0, \infty); \omega) \) with \( N\sigma_N \int_0^\infty |v(r)|^p r^{N-1} dr = 1 \), one has

\[
N\sigma_N \int_0^\infty |v(r)|^p \log |v(r)|^p r^{N-1} dr \leq \frac{N}{p} \log \left( \mathcal{L}_{N,p,N\sigma_N} \int_0^\infty |v'(r)|^p r^{N-1} dr \right),
\]
where
\[ \mathcal{L}_{p,N} := \mathcal{L}_{\omega,p} = p \left( \frac{p-1}{e} \right)^{p-1} \left( \sigma_N \Gamma \left( \frac{N}{p'} + 1 \right) \right)^{-\frac{2}{p}}. \]  

(2.13)

3. Sharp log-Sobolev inequality in CD(0, N) spaces: proof of Theorem 1.1

3.1. Proof of the log-Sobolev inequality (1.4). Fix any function \( u \in W^{1,p}(X, d, m) \) with the property \( \int_X |u|^p dm = 1 \). Without loss of generality, we may assume that \( u \geq 0 \), since \( |\nabla u| \leq |\nabla u| \).

On the one hand, one can show that the operation of rearrangement described in §2.1.2 leaves both the \( L^p \)-norm and entropy invariant, i.e., we have that
\[
1 = \int_X u^p dm = N\sigma_N \int_0^\infty \hat{u}(r)^p r^{N-1} dr,
\]
and
\[
\int_X u^p \log u^p dm = N\sigma_N \int_0^\infty \hat{u}(r)^p \log \hat{u}(r)^p r^{N-1} dr.
\]

The proofs are based on the layer cake representation, see e.g. Lieb and Loss [50], combined with (2.8); for the entropy term we use in addition the 'signed measure' described in [50, p. 27] in order to split in a suitable way the function \( s \to s^p \log s^p, s > 0 \).

On the other hand, by the Pólya–Szegö inequality (2.9), we obtain
\[
\int_X |\nabla u|^p dm \geq \text{AVR}_m^N N\sigma_N \int_0^\infty |\hat{u}'(r)|^p r^{N-1} dr.
\]

The required inequality (1.1) follows from the latter relations combined with inequality (2.12).

3.2. Sharpness in the log-Sobolev inequality (1.4). The sharpness of (1.4) follows by the next result.

**Proposition 3.1.** Let \( p, N > 1 \) and \((X, d, m)\) be a CD(0, N) space. Assume that for every \( u \in W^{1,p}(X, d, m) \setminus \{0\} \) one has
\[
\frac{\int_X |u|^p \log |u|^p dm}{\int_X |u|^p dm} - \log \left( \int_X |u|^p dm \right) \leq \frac{N}{p} \log \left( \frac{\int_X |\nabla u|^p dm}{\int_X |u|^p dm} \right)
\]
for some \( C > 0 \). Then
\[
\mathcal{L}_{p,N} \leq \text{AVR}_m^N C.
\]

**Proof.** Fix any point \( x_0 \in X \), and let \( u(x) := \hat{u}(d_{x_0}(x)) \) in (3.1) for any nonnegative \( \hat{u} \in \text{Lip}_c([0, \infty)) \), where \( d_{x_0}(x) = d(x_0, x) \), \( x \in X \). [Note that \( \hat{u}(0) \) need not be 0.] By using the non-smooth chain rule and the eikonal equation (2.7), it turns out that
\[
\frac{\int_X \hat{u}^p(d_{x_0}(x)) \log \hat{u}^p(d_{x_0}(x)) dm(x)}{\int_X \hat{u}^p(d_{x_0}(x)) dm(x)} - \log \left( \int_X \hat{u}^p(d_{x_0}(x)) dm(x) \right) \leq \frac{N}{p} \log \left( \frac{C \int_X |\hat{u}'(d_{x_0}(x))|^p dm(x)}{\int_X \hat{u}^p(d_{x_0}(x)) dm(x)} \right).
\]

(3.3)
If we consider the push-forward measure $\nu = d_{x_0,#}m$ of $m$ on $[0, \infty)$, a change of variable in (3.3) yields that for every nonnegative $\tilde{u} \in \text{Lip}_c([0, \infty)) \setminus \{0\}$, one has

$$
\int_0^\infty \tilde{u}^p(r) \log \tilde{u}^p(r) d\nu(r) - \log \left( \int_0^\infty \tilde{u}^p(r) d\nu(r) \right) \leq \frac{N}{p} \log \left( C \int_0^\infty \tilde{u}^p(r) d\nu(r) \right),
$$

(3.4)

Now, we want to ‘replace’ the measure $\nu$ by $\omega$ in (3.4), where $\omega = N\sigma_N r^{N-1}L^1$, $r > 0$. For simplicity of notations, we consider for every $r > 0$,

$$
\theta^+_N,r = \frac{m^+(B(x_0,r))}{N\sigma_N r^{N-1}},
$$

(3.5)

where $m^+(B(x_0,r))$ is the Minkowski content of $B(x_0,r)$, see (2.2). We are going to prove that

$$
d\nu = \theta^+_N,r \ d\omega \quad \text{for a.e.} \ r > 0,
$$

(3.6)

where $d\nu/d\omega$ stands for the Radon–Nikodym derivative.

We first prove that the push-forward measure $\nu = d_{x_0,#}m$ is absolutely continuous with respect to the measure $\omega = N\sigma_N r^{N-1}L^1$ on $[0, \infty)$. To see this, we are going to prove that for every $A \subset [0, \infty)$ such that $\omega(A) = \mathcal{L}^1(A) = 0$, one has $\nu(A) = 0$, i.e., $m(\hat{A}) = 0$, where $\hat{A} = \cup_{r \in A} \partial B(x_0, r)$. Without loss of generality, we may assume that $\hat{A}$ is compact and $0 \notin A$. Fix $\varepsilon > 0$ arbitrarily small; we show that $m(\hat{A}) < \varepsilon$. Note that by Bishop–Gromov theorem, the function $\phi(r) = \frac{m(B(x_0,r))}{r^N}$ is non-increasing; if $r_0 = \min\{r : r \in A\}$ and $r_1 = \max\{r : r \in A\}$, then $M := \max_{r \in A} \phi(r) = \phi(r_0)$ and $K := N \max_{r \in A} r^{N-1} = Nr_1^{N-1}$. Let $\delta = \frac{\varepsilon}{MK} > 0$. Since $\mathcal{L}^1(A) = 0$, we can find a finite covering of $A$ by disjoint intervals $\{(a_i, b_i)\}_{i=1,...,L}$ such that $\sum_{i=1}^L (b_i - a_i) < \delta$. Due to the fact that $A \subset \cup_{i=1}^L (a_i, b_i)$, one has that $A \subset \cup_{i=1}^L (B(x_0, b_i) \setminus B(x_0, a_i))$. Therefore, by using the above monotonicity properties and the mean-value theorem, it follows that

$$
m(\hat{A}) \leq \sum_{i=1}^L (m(B(x_0, b_i)) - m(B(x_0, a_i))) = \sum_{i=1}^L (\phi(b_i)b_i^N - \phi(a_i)a_i^N)
$$

$$
\leq \sum_{i=1}^L \phi(a_i)(b_i^N - a_i^N) \leq MK \sum_{i=1}^L (b_i - a_i) < MK\delta = \varepsilon.
$$

The arbitrariness of $\varepsilon > 0$ implies that $m(\hat{A}) = 0$, which concludes the proof of the absolutely continuity of $\nu$ with respect to the measure $\omega$.

In order to prove (3.6), we shall use the result of Sturm [66, Theorem 2.3], according to which

$$
m(B(x_0,r)) = \int_0^r m^+(B(x_0,s))ds, \ r > 0,
$$

(3.7)

combined with the fact that $r \mapsto \theta^+_N,r$ is non-increasing on $(0, \infty)$. This implies in particular, that the function $r \mapsto m^+(B(x_0,r))$ is continuous for almost every $r > 0$. At the point of continuity of this function, we have that its integral $r \mapsto m(B(x_0,r))$ is differentiable and its derivative is equal to $m^+(B(x_0,r))$. Moreover, by the definition of the $\nu = d_{x_0,#}m$ we have that

$$
m(B(x_0,r)) = \nu([0,r)), \ r > 0.
$$

(3.8)
Let us fix $r > 0$ such that the function $t \to m(B(x_0, t))$ is differentiable at $r$ and such that 
\[
\frac{dm(B(x_0, r))}{dr}(r) = m^+(B(x_0, r)),
\]
which is of full measure in $(0, \infty)$ due to the above consideration; thus, by (3.7) and (3.8), one has
\[
\frac{d\nu}{d\omega}(r) = \lim_{t \to 0} \frac{\nu((r-t, r+t))}{\omega((r-t, r+t))} = \lim_{t \to 0} \frac{m(B(x_0, r+t)) - m(B(x_0, r-t))}{\sigma_N((r+t)^N - (r-t)^N)} = m^+(B(x_0, r)) = \theta_{N,r}^+,
\]
which is precisely relation (3.6).

By (3.4) and (3.6) one has
\[
\int_0^\infty \tilde{u}^p(r) \log \tilde{u}^p(r) \theta_{N,r}^+ \, d\omega(r) - \log \left( \int_0^\infty \tilde{u}^p(r) \theta_{N,r}^+ \, d\omega(r) \right) \leq \frac{N}{p} \log \left( C \int_0^\infty |\tilde{u}'(r)|^p \theta_{N,r}^+ \, d\omega(r) \right).
\]
If we insert $\tilde{u}(r) = v(\lambda r) \geq 0$ for every $\lambda > 0$ (with $v \in \text{Lip}_c([0, \infty) \setminus \{0\}$) and $0 \leq v \leq 1$) into the latter inequality, by the scaling invariance of the log-Sobolev inequality and a change of variables imply that
\[
\int_0^\infty v^p(s) \log v^p(s) \theta_{N,x}^+ \, d\omega(s) - \log \left( \int_0^\infty v^p(s) \theta_{N,x}^+ \, d\omega(s) \right) \leq \frac{N}{p} \log \left( C \int_0^\infty |v'(s)|^p \theta_{N,x}^+ \, d\omega(s) \right).
\]

Now, we prove that
\[
\text{AVR}_m = \lim_{r \to \infty} \theta_{N,r}^+.
\]
Clearly, the limit exists, due to (3.5) and the Bishop–Gromov monotonicity property. First, for every $R > 0$ one has by (3.7) that
\[
\frac{m(B(x_0, R))}{\sigma_N R^N} = \frac{1}{\sigma_N R^N} \int_0^R m^+(B(x_0, s)) \, ds = \frac{N}{R^N} \int_0^R s^{N-1} \theta_{N,s}^+ \, ds \geq \theta_{N,R}^+.
\]
If $R \to \infty$, it turns out that $\text{AVR}_m \geq \lim_{r \to \infty} \theta_{N,r}^+$. Conversely, if $R > r$, we have that
\[
\frac{m(B(x_0, R)) - m(B(x_0, r))}{\sigma_N R^N} = \frac{1}{\sigma_N R^N} \int_r^R m^+(B(x_0, s)) \, ds = \frac{N}{R^N} \int_r^R s^{N-1} \theta_{N,s}^+ \, ds \leq \frac{R^N - r^N}{R^N} \theta_{N,r}^+.
\]
Letting first $R \to \infty$ and then $r \to \infty$, it follows that $\text{AVR}_m \leq \lim_{r \to \infty} \theta_{N,r}^+$, which concludes the proof of (3.10).

Letting $\lambda \to 0^+$ in (3.9), the limit in (3.10) implies that
\[
\int_0^\infty v^p(s) \log v^p(s) \, d\omega(s) - \log \left( \text{AVR}_m \int_0^\infty v^p(s) \, d\omega(s) \right) \leq \frac{N}{p} \log \left( C \int_0^\infty |v'(s)|^p \, d\omega(s) \right). \tag{3.11}
\]
Since the Gaussian function $v(r) = e^{-\frac{r^2}{2}}$, $r > 0$, can be approximated by Lipschitz functions with compact support in $[0, \infty)$, we can use it as a test function in (3.11); now, for this particular choice
of \( v \) in (3.11), a straightforward computation yields that

\[
-\frac{N}{p'} - \log \left( \text{AVR}_m \sigma_N \Gamma \left( \frac{N}{p'} + 1 \right) \right) \leq \frac{N}{p} \log \left( C \left( \frac{p'}{p} \right)^p N \right).
\]

After a reorganization of the above terms, it follows that \( \mathcal{L}_{p,N} \leq \text{AVR}_m^{\frac{N}{p}} C \), which is precisely the required relation (3.2).

Some comments are in order.

**Remark 3.1.** A closer look at the two statements of Theorem 1.1 gives the following result: Given \( p, N > 1 \) and \((X, d, m)\) a CD\((0, N)\) space, the following two statements are equivalent:

(i) \((X, d, m)\) supports the \( L^p \)-log-Sobolev inequality, i.e., there exists \( C > 0 \) such that for every \( u \in W^{1,p}(X, d, m) \) with \( \int_X |u|^p d m = 1 \) one has that

\[
\int_X |u|^p \log |u|^p d m \leq \frac{N}{p} \log \left( C \int_X |\nabla u|^p d m \right); \tag{3.12}
\]

(ii) \( \text{AVR}_m > 0 \).

We should mention that this consideration might not come as a surprise; indeed, a similar equivalence is well-known in the Riemannian context for Sobolev-type inequalities, see Coulhon and Saloff-Coste [30], do Carmo and Xia [33], and Hebey [44].

Concerning the proof, we observe that the implication (ii) \( \implies \) (i) is precisely the first statement of Theorem 1.1 with \( C = \mathcal{L}_{p,N} \text{AVR}_m^{-\frac{N}{p}} \), while (i) \( \implies \) (ii) follows from its second statement, written equivalently as \( \text{AVR}_m \geq \left( \frac{\mathcal{L}_{p,N}}{C} \right)^{\frac{N}{p}} \). With these observations and by introducing the notation for the optimal log-Sobolev constant

\[
\mathcal{C}_{LS} := \inf \left\{ C > 0 : (3.12) \text{ holds for all } u \in W^{1,p}(X, d, m) \text{ with } \int_X |u|^p d m = 1 \right\},
\]

we can reformulate the statement of Theorem 1.1 simply into \( \text{AVR}_m = \left( \frac{\mathcal{L}_{p,N}}{\mathcal{C}_{LS}} \right)^{\frac{N}{p}} \).

**Remark 3.2.** The proof of Proposition 3.1 is elementary on Riemannian manifolds. Indeed, if \((M, g)\) is an \( n \)-dimensional complete Riemannian manifold endowed with its canonical measure \( m = dv_g \), for every \( \varepsilon > 0 \), there exists a local chart \((\Omega, \phi)\) of \( M \) at the point \( x_0 \in M \) and a number \( \delta > 0 \) such that \( \phi(\Omega) = B(0, \delta) \subset \mathbb{R}^n \), and the components \( g_{ij} \) of the metric \( g \) satisfy

\[
(1 - \varepsilon) \delta_{ij} \leq g_{ij} \leq (1 + \varepsilon) \delta_{ij}, \quad i, j = 1, ..., n, \tag{3.13}
\]

in the sense of bilinear forms; here, \( B(0, \delta) \) is the \( n \)-dimensional Euclidean ball of center 0 and radius \( \delta > 0 \), see e.g. Hebey [44]. Now, by using (3.13), we can ‘replace’ the measure \( m = dv_g \) with the usual Lebesgue measure \( \mathcal{L}^n \) in \( \mathbb{R}^n \), and the use of the Gaussian functions (which are the extremals in the Euclidean log-Sobolev inequality) provides the required estimate. Clearly, such a strategy is not applicable in the nonsmooth setting of CD\((0, N)\) spaces, due to the lack of local charts and the two-sided estimate (3.13).

**Remark 3.3.** Theorem 1.1 provides generalisation with a simple proof of the celebrated rigidity result of Ni [55] for the case \( p = 2 \):
Let \((M, g)\) be a complete \(n\)-dimensional Riemannian manifold with nonnegative Ricci curvature, and \(p > 1\). Then the log-Sobolev inequality
\[
\int_M |u|^p \log |u|^p dv_g \leq \frac{n}{p} \log \left( \mathcal{L}_{p,n} \int_M |\nabla_g u|^p dv_g \right)
\] (3.14)
holds for every \(u \in W^{1,p}(M)\) with \(\int_M |u|^p dv_g = 1\) if and only if \((M, g)\) is isometric to \(\mathbb{R}^n\).

Indeed, if (3.14) holds, by the sharpness of Theorem 1.1 we necessarily have that \(\mathcal{L}_{p,n} \mathrm{AVR}_{dv_g}^{-\frac{1}{p}} \leq \mathcal{L}_{p,n}\), i.e., \(\mathrm{AVR}_{dv_g} \geq 1\). But, by definition, we always have that \(\mathrm{AVR}_{dv_g} \leq 1\), thus \(\mathrm{AVR}_{dv_g} = 1\), which implies that \((M, g)\) is isometric to \(\mathbb{R}^n\), see Petersen [62, Exercise 7.5.10].

4. Sharp hypercontractivity estimates in CD(0,\(N\)) spaces: proof of Theorem 1.2

Before the proof of (1.7), we are going to provide some important properties of the Hopf–Lax semigroup. In the sequel, we assume the assumptions of Theorem 1.2 are satisfied.

4.1. Hamilton–Jacobi inequality of the Hopf–Lax semigroup. The first proposition and its proof are based on ideas developed in [8] and [42].

Proposition 4.1. Let \(p > 1\), \(t_0 > 0\), \(x_0 \in X\) and \(u \in F_{t_0,x_0}(X)\) be fixed. Then the following statements hold:

(i) \(t_*(x) := \sup \{ t \geq 0 : Q_t u(x) > -\infty \} \geq t_0\) for every \(x \in X\);

(ii) \((t, x) \mapsto Q_t u(x)\) is locally Lipschitz on \((0, t_0) \times X\);

(iii) for every \((t, x) \in (0, t_0) \times X\) one has the Hamilton–Jacobi inequality
\[
\frac{d^+}{dt} Q_t u(x) \leq -\frac{|DQ_t u|^p(x)}{p},
\] (4.1)
where \(\frac{d^+}{dt}\) stands for the right derivative;

(iv) for every \(x \in X\) one has
\[
\left. \frac{d^+}{dt} \right|_{t=0} Q_t u(x) \geq -\frac{|Du|^p(x)}{p}. \quad (4.2)
\]

Proof. (i)&(ii) Let us fix arbitrarily \(0 < t_1 < t_2 < t_0\) and the compact set \(K \subset X\). In order to prove the claims, it is enough to show that the function \((t, x) \mapsto Q_t u(x)\) is well-defined and Lipschitz continuous on \([t_1, t_2] \times K\). Indeed, one has for every \((t, x) \in [t_1, t_2] \times K\) that
\[
Q_t u(x) = \inf_{y \in X} \left\{ u(y) + \frac{d(x, y)^p'}{p't_p'^{-1}} \right\}
\]
\[
\geq \inf_{y \in X} \left\{ u(y) + \frac{d(x_0, y)^p'}{p't_0'^{-1}} \right\} + \inf_{y \in X} \left\{ \frac{d(x, y)^p'}{p't_p'^{-1}} - \frac{d(x_0, y)^p'}{p't_0'^{-1}} \right\}
\]
\[
= Q_{t_0} u(x_0) + \inf_{y \in X} \left\{ \frac{d(x, y)^p'}{p't_p'^{-1}} - \frac{d(x_0, y)^p'}{p't_0'^{-1}} \right\} > -\infty,
\]
uniformly in \(x \in K\); here, we used the assumption (A2), i.e., \(Q_{t_0} u(x_0) > -\infty\), combined with the fact that
\[
\lim_{d(x_0, y) \to \infty} \left\{ \frac{d(x, y)^p'}{p't_p'^{-1}} - \frac{d(x_0, y)^p'}{p't_0'^{-1}} \right\} = +\infty,
\]
which follows from $t < x < t_0$ and the triangle inequality. In particular, by the arbitrariness of $t \in [t_1, t_2] \subset (0, t_0)$, we have that $t_* (x) \geq t_0$ for every $x \in X$, which proves (i).

Moreover, we can find $R = R_K > 0$ such that for every $(t, x) \in [t_1, t_2] \times K$, one has

$$Q_t u(x) = \min_{y \in B(x_0, R)} \left\{ u(y) + \frac{d(x, y)^{p'}}{p' \cdot t_n^{p'-1}} \right\}.$$

We observe that the function $[t_1, t_2] \times K \ni (t, x) \mapsto u(y) + \frac{d(x, y)^{p'}}{p' \cdot t_n^{p'-1}}$ is uniformly Lipschitz in $y$, thus $[t_1, t_2] \times K \ni (t, x) \mapsto Q_t u(x)$ is also Lipschitz, as the infimum of a family of uniformly Lipschitz functions.

(iii) The result of Ambrosio, Gigli and Savaré [8, Theorem 3.5] states that the Hamilton–Jacobi inequality (4.1) is valid for $p = 2$, and for every $x \in X$ and $t \in (0, t_*(x))$; in particular, by (i) the inequality (4.1) is obtained for every $(t, x) \in (0, t_0) \times X$. The generic case $p \neq 2$ follows in a similar way as in [8] after an almost trivial adaptation, either from Ambrosio, Gigli and Savaré [9] or Gozlan, Roberto and Samson [42].

(iv) Let $x \in X$ be fixed, and consider a positive sequence $\{t_n\}_{n \in \mathbb{N}}$ with $t_n \to 0^+$ as $n \to \infty$; clearly, we may assume that $t_n < t_\#$ for every $n \in \mathbb{N}$, where $t_\# < t_0$. According to (i) (i.e., $Q_{t_n} u(z) > -\infty$ for every $(t, z) \in (0, t_0) \times X$), there exist $y_n \in X$ ($n \in \mathbb{N}$) and $y_\# \in X$ such that

$$Q_{t_n} u(x) = u(y_n) + \frac{d(x, y_n)^{p'}}{p' \cdot t_n^{p'-1}} \quad \text{and} \quad Q_{t_\#} u(x) = u(y_\#) + \frac{d(x, y_\#)^{p'}}{p' \cdot t_\#^{p'-1}}. \tag{4.3}$$

We are going to prove that the sequence $\{y_n\}$ is bounded. To see this, we have

$$-\infty < Q_{t_\#} u(x) = u(y_\#) + \frac{d(x, y_\#)^{p'}}{p' \cdot t_\#^{p'-1}} \leq u(y_n) + \frac{d(x, y_n)^{p'}}{p' \cdot t_n^{p'-1} \to \infty} \leq u(x) + \frac{1}{p'} d(x, y_n) \left\{ \frac{1}{t_\#^{p'-1}} - \frac{1}{t_n^{p'-1}} \right\}.$$ 

If $d(x, y_n) \to \infty$ as $n \to \infty$, since $t_n < t_\#$ for every $n \in \mathbb{N}$, the latter expression tends to $-\infty$, which contradicts the fact $-\infty < Q_{t_\#} u(x)$. Therefore, there exists $R > 0$ such that $\{y_n\} \subset B(x_0, R)$ for every $n \in \mathbb{N}$.

By (4.3) and $Q_{t_n} u \leq u$, one has

$$\frac{d(x, y_n)^{p'}}{p' \cdot t_n^{p'-1}} \leq u(x) - u(y_n), \quad n \in \mathbb{N}.$$ 

Since $u \in \text{Lip}_{\text{loc}} (X)$, it turns out that $u$ is globally Lipschitz on the compact set $B(x_0, R + d(x_0, x))$ and $x, y_n \in B(x_0, R + d(x_0, x))$, $n \in \mathbb{N}$; in particular, there exists $L > 0$ such that $u(x) - u(y_n) \leq L \cdot d(x, y_n)$ for every $n \in \mathbb{N}$. Rearranging the previous inequality then yields that

$$d(x, y_n) \leq \left( p' \cdot L \right)^{1/p'-1} \cdot t_n, \quad n \in \mathbb{N},$$
and since \( t_n \to 0 \), we also have that \( d(x, y_n) \to 0 \) as \( n \to \infty \). By the definition of the slope \( |Du| \), for a fixed \( \varepsilon > 0 \) there exists \( N_{\varepsilon} \in \mathbb{N} \) such that

\[
\frac{u(y_n) - u(x)}{d(x, y_n)} \geq -|Du|(x) - \varepsilon, \quad \forall n \geq N_{\varepsilon}.
\]

(4.4)

Therefore, by applying (4.3) and (4.4), we have that

\[
\frac{Q_{t_n} u(x) - u(x)}{t_n} = \frac{u(y_n) - u(x)}{t_n} + \frac{d_{p'}(x, y_n)}{t_n} \geq -|Du|(x) \varepsilon + \frac{1}{p'} \left( \frac{d(x, y_n)}{t_n} \right)^{p'} \geq -|Du|(x + \varepsilon)^p,
\]

where the last estimate follows by Young’s inequality, written in the following form:

\[-A s + \frac{1}{p'} \cdot s^{p'} \geq -\frac{A^p}{p'}, \quad \forall A, s > 0.\]

Finally, since \( t_n \to 0 \) as \( n \to \infty \), combined with the arbitrariness of \( \varepsilon > 0 \), we obtain the desired inequality (4.2).

4.2. Sobolev regularity of the Hopf–Lax semigroup. In the sequel, besides the locally Lipschitz property of \( (t, x) \mapsto Q_{t} u(x) \) on \((0, t_0) \times X \) (cf. Proposition 4.1/(ii)), we prove an important regularity property of an integral involving \( e^{Q_{t} u} \). Before doing this, we state an elementary inequality (whose proof is left to the reader), which plays a crucial role in our argument; namely, for every \( x, y, z \in X, t, s > 0 \) and \( r > 1 \), one has

\[
\frac{d'(x, y)}{l^{r-1}} + \frac{d'(y, z)}{s^{r-1}} \geq \frac{d'(x, z)}{(l+s)^{r-1}}.
\]

(4.5)

Proposition 4.2. Let \( p > 1 \), \( t_0 > 0 \), \( x_0 \in X \) and \( 0 < \alpha \leq \beta \) be fixed, and let \( q : [0, \tilde{t}] \to [\alpha, \beta] \) be a \( C^1 \)-function on \((0, \tilde{t})\) for some \( \tilde{t} \in (0, t_0) \). Then for every \( u \in \mathcal{F}_{t_0, x_0}(X) \) with \((1 + d'(x_0, \cdot))e^{\alpha u} \in L^1(X, \mu)\), we have that

(i) \((1 + d'(x_0, \cdot))e^{\beta u} \in L^1(X, \mu)\);

(ii) the function

\[ F(t) = \int_X e^{q(t)Q_{t} u(x)} dm(x) \]

is well-defined and locally Lipschitz in \((0, \tilde{t})\);

(iii) the function \( x \mapsto e^{\frac{\alpha}{p'} Q_{t} u(x)} \) belongs to \( W^{1, p'}(X, d, \mu) \) for every \( t \in (0, \tilde{t}) \).

Proof. (i) Due to assumption (A1), the set \( S := u^{-1}([0, \infty)) \subset X \) is bounded, i.e., there exists \( R > 0 \) such that \( S \subset B(x_0, R) \). In particular, \( m(S) < +\infty \) (see Sturm [66, Theorem 2.3]) and \( U := \sup_X u < +\infty \); thus

\[
\int_S (1 + d'(x_0, x))e^{\beta u(x)} dm(x) \leq (1 + R')e^{\beta U} m(S) < +\infty.
\]
On the other hand, since \( u(x) \leq 0 \) for every \( x \in X \setminus S \) and \( (1 + d^{p'}(x_0, \cdot))e^{\alpha u} \in L^1(X, m) \), for every \( \beta \geq \alpha \) one has that
\[
\int_{X \setminus S} (1 + d^{p'}(x_0, x))e^{\beta u} dm(x) \leq \int_{X \setminus S} (1 + d^{p'}(x_0, x))e^{\alpha u} dm(x) < +\infty.
\]

(ii) We first show that \( e^{q(t)Q_tu} \in L^1(X, m) \) for every \( t \in (0, \tilde{t}) \), which implies the well-definiteness of the function \( F \). Indeed, by definition we have that \( Q_tu \leq u \) for every \( t \in (0, \tilde{t}) \). Therefore, due to the fact that \( q(t) \in [\alpha, \beta] \) for every \( t \in [0, \tilde{t}] \), one has for every \( (t, x) \in (0, \tilde{t}) \times X \) that
\[
e^{q(t)Q_tu(x)} \leq e^{\max\{aQ_tu(x), \beta Q_tu(x)\}} = \max\{e^{\alpha Q_tu(x)}, e^{\beta Q_tu(x)}\} \leq \max\{e^{\alpha u(x)}, e^{\beta u(x)}\}.
\]
Therefore, the assumption \( (1 + d^{p'}(x_0, \cdot))e^{\alpha u} \in L^1(X, m) \), combined with property (i), yields that \( e^{q(t)Q_tu} \in L^1(X, m) \) for every \( t \in (0, \tilde{t}) \).

In order to prove that \( F \) is locally Lipschitz on \( (0, \tilde{t}) \), we are first going to show that there exists \( C_1 > 0 \) such that
\[
|Q_tu(x)| \leq C_1(1 + d^{p'}(x_0, x)), \quad \forall (t, x) \in (0, \tilde{t}) \times X.
\]
On one hand, by assumption (A3) and inequality (4.5) (note that \( C_0 > p' t_0^{p'-1} \) and \( t_0 > \tilde{t} \)), we have for every \( (t, x) \in (0, \tilde{t}) \times X \) that
\[
Q_tu(x) = \inf_{y \in X} \left\{ u(y) + \frac{d(x, y)^{p'}}{p't^{p'-1}} \right\} \geq M + \inf_{y \in X} \left\{ -\frac{d(x_0, y)^{p'}}{C_0} + \frac{d(x, y)^{p'}}{p't^{p'-1}} \right\} \geq M - \frac{d(x_0, x)^{p'}}{b},
\]
where \( b = \left(C_0^{p'-1} - t_0(p')^{p'-1}\right)^{p'-1} > 0 \). One the other hand, by definition we have \( Q_tu \leq u \) for every \( t \in (0, \tilde{t}) \); therefore, combining the latter two estimates, for every \( (t, x) \in (0, \tilde{t}) \times X \) one has
\[
|Q_tu(x)| \leq \max\{|u(x)|, |M| + \frac{d(x_0, x)^{p'}}{b}\}.
\]
Since \( U := \sup_X u < +\infty \) and by assumption (A3), (4.7) follows at once.

Now, let us fix \( t_1, t_2 \in (0, \tilde{t}) \), say \( t_1 < t_2 \). Recall from Proposition 4.1/(ii) that \( \tau \mapsto Q_\tau u(x) \) is locally Lipschitz. We are going to show that there exists \( C_2 > 0 \) such that
\[
\max \{|\xi| : \xi \in \partial^{0}Q_tu(x)\} \leq C_2(1 + d^{p'}(x_0, x)), \quad \forall (t, x) \in (t_1, t_2) \times X,
\]
where \( \partial^{0}Q_tu(x) \) stands for the Clarke generalized gradient of the locally Lipschitz function \( \tau \mapsto Q_\tau u(x) \) at the point \( t \in (t_1, t_2) \), see Clarke [27]. In this particular setting, it turns out that
\[
|\xi| \leq \max\{Q_0^0u(x)(t; -1), Q_t^0u(x)(t; 1)\}, \quad \forall \xi \in \partial^{0}Q_tu(x),
\]
see Clarke [27, pp. 25-27], where
\[
Q_t^p u(x)(t; v) = \limsup_{s \to t, h \to 0^+} \frac{Q_{s+h}u(x) - Q_su(x)}{h}, \quad v \in \mathbb{R},
\]
denotes the Clarke generalized directional derivative of \( \tau \mapsto Q_\tau u(x) \) at \( t \in (t_1, t_2) \) in direction \( v \in \mathbb{R} \). Note that since \( t \mapsto Q_tu \) is non-increasing, one has
\[
Q_0^0u(x)(t; 1) \leq 0.
\]
In order to estimate the term \( Q_t^0u(x)(t; -1) \), for \( s > 0 \) and \( x \in X \) let \( y_{s,x} \in X \) be such that
\[
Q_su(x) = u(y_{s,x}) + \frac{d(x, y_{s,x})^{p'}}{p's^{p'-1}};
\]
denote the set of such elements by $m(x, s) \subset X$. Then, one has

$$Q^0_t u(x)(t; -1) = \limsup_{s \to t; h \to 0^+} Q_{s-h} u(x) - Q_s u(x)$$

$$\leq \limsup_{s \to t; h \to 0^+} \frac{u(y_{s,x}) + \frac{d(x, y_{s,x})^{p'}}{p'(s-h)^{p'-1}} - (u(y_{s,x}) + \frac{d(x, y_{s,x})^{p'}}{p' s^{p'-1}})}{h}$$

$$\leq \frac{1}{p' b} \sup_{s \in [t_1, t_2], y_{s,x} \in m(s, x)} d(x, y_{s,x})^{p'}.$$  (4.11)

It remains to estimate the last term in (4.11). First, since $y_{s,x} \in m(s, x)$ we have for every $(s, x) \in [t_1, t_2] \times X$ that

$$u(y_{s,x}) + \frac{d(x, y_{s,x})^{p'}}{p' s^{p'-1}} = Q_s u(x) \leq u(x).$$

This relation together with assumption (A3) implies that for every $(s, x) \in [t_1, t_2] \times X$ one has

$$\frac{d(x, y_{s,x})^{p'}}{p' t_2^{p'-1}} \leq \frac{d(x, y_{s,x})^{p'}}{p' s^{p'-1}} \leq u(x) - u(y_{s,x}) \leq u(x) - M + \frac{d(x, y_{s,x})^{p'}}{C_0}.$$  

Moreover, as before, by the metric inequality (4.5) one has that

$$\frac{d(x, 0, y_{s,x})^{p'}}{C_0} \leq \frac{d(x, y_{s,x})^{p'}}{p' t_0^{p'-1}} + \frac{d(x, x_0)^{p'}}{b},$$

where $b = (C_0^{p-1} - t_0 (p')^{p-1})^{p-1} > 0$. Combining the latter two inequalities yields

$$\frac{1}{p'} \left( \frac{1}{t_2^{p'-1}} - \frac{1}{t_0^{p'-1}} \right) d(x_0, y_{s,x})^{p'} \leq u(x) - M + \frac{d(x, x_0)^{p'}}{b}.$$  

Since $t_2 < t_0$, the latter estimate with relations (4.9)-(4.11) provide the claimed inequality (4.8).

Now, let $t, s \in [t_1, t_2] \subset (0, t)$ be arbitrarily fixed (say $t < s$). According to Proposition 4.1/(ii), the function $\tau \mapsto e^{q(\tau)}Q_{\theta u(x)}$ is locally Lipschitz on $(0, t_0)$. By using Lebourg’s mean value theorem for the latter function, see Clarke [27, Theorem 2.3.7], there exist $\theta = \theta_x \in (t, s)$ and $\xi = \xi_x \in \partial^0 Q_{\theta u(x)}$ such that

$$e^{q(t)}Q_{\theta u(x)} - e^{q(s)}Q_{\theta u(x)} = e^{q(\theta)}Q_{\theta u(x)}(q'(\theta)Q_{\theta u(x)} + q(\theta)\xi)(t - s).$$

The latter relation together with estimates (4.6)-(4.8) yields the existence of a constant $C_3 > 0$ such that

$$|F(t) - F(s)| \leq \int_X |e^{q(t)}Q_{\theta u(x)} - e^{q(s)}Q_{\theta u(x)}| dm(x)$$

$$\leq C_3 \int_X \max\{e^{\alpha u(x)}, e^{\beta u(x)}\}(1 + d^{p'}(x_0, x)) dm(x) \cdot |t - s|.$$
Due to (i) and assumption $\phi^p(x_0, \cdot))e^{\alpha u} \in L^1(X, m)$, we conclude by the latter estimate that $F$ is locally Lipschitz on $(0, t_0)$.

(iii) The fact that $x \mapsto e^{\frac{q(t)}{p}Q_{\theta}u(x)}$ belongs to $L^p(X, m)$ for every $t \in (0, \tilde{t})$ is equivalent to the well-definiteness of $F$ on $(0, \tilde{t})$, see (ii). It remains to prove that $\left\| \nabla \left( e^{\frac{q(t)}{p}Q_{\theta}u(x)} \right) \right\| \in L^p(X, m)$ for every $t \in (0, \tilde{t})$.

According to (ii), the functions

$$H_{\alpha}(t) = \int_X e^{\alpha Q_{\theta}u(x)} dm(x) \quad \text{and} \quad H_{\beta}(t) = \int_X e^{\beta Q_{\theta}u(x)} dm(x)$$

are locally Lipschitz on $(0, \tilde{t})$ (consider $q(t) = \alpha$ and $q(t) = \beta$, respectively). Since both of these functions are non-increasing, the right derivatives of both functions $H_{\alpha}$ and $H_{\beta}$ exist at every point $t \in (0, \tilde{t})$, i.e., $-\infty < \frac{d^+}{dt}H_{\alpha}(t) < \infty$ and $-\infty < \frac{d^+}{dt}H_{\beta}(t) < \infty$ for every $t \in (0, \tilde{t})$. In addition, by the Hamilton–Jacobi inequality, see Proposition 4.1/(iii), one has for every $t \in (0, \tilde{t})$ that

$$-\infty < \frac{d^+}{dt}H_{\alpha}(t) = \alpha \int_X \frac{d^+}{dt}Q_{\theta}u(x) e^{\alpha Q_{\theta}u(x)} dm(x) \leq -\frac{\alpha}{p} \int_X |DQ_{\theta}u|^p(x) e^{\beta Q_{\theta}u(x)} dm(x).$$

Due to Proposition 4.1/(ii), one has that $Q_{\theta}u \in \text{Lip}_{\text{loc}}(X)$ for every $t \in (0, t_0)$; in particular, by (2.5) we have $|\nabla Q_{\theta}u(x)| = |DQ_{\theta}u(x)|$ for every $t \in (0, t_0)$ and $m$-a.e. $x \in X$. Therefore, the previous estimate implies that

$$\int_X |\nabla Q_{\theta}u|^p(x) e^{\alpha Q_{\theta}u(x)} dm(x) < +\infty, \quad \forall t \in (0, \tilde{t}).$$

In the same way, we also have

$$\int_X |\nabla Q_{\theta}u|^p(x) e^{\beta Q_{\theta}u(x)} dm(x) < +\infty, \quad \forall t \in (0, \tilde{t}).$$

By (4.6) and the above two relations, we obtain that

$$\int_X |\nabla Q_{\theta}u|^p(x) e^{\frac{\alpha\theta(t)}{p}Q_{\theta}u(x)} dm(x) < +\infty, \quad \forall t \in (0, \tilde{t}),$$

which is equivalent, by means of the non-smooth chain rule, to the fact that

$$\int_X \left| \nabla \left( e^{\frac{q(t)}{p}Q_{\theta}u(x)} \right) \right|^p dm(x) < +\infty, \quad \forall t \in (0, \tilde{t}),$$

i.e., $e^{\frac{q(t)}{p}Q_{\theta}u} \in W^{1,p}(X, d, m)$ for every $t \in (0, \tilde{t})$, which concludes the proof. \hfill $\square$

4.3. Proof of the hypercontractivity estimate (1.7). In the case $\alpha = \beta$, relation (1.7) turns out to be trivial; thus, let us fix $\beta > \alpha > 0$ and $u \in F_{t_0, x_0}(X)$ with $(1 + d^p(x_0, \cdot))e^{\alpha u} \in L^1(X, m)$ and $e^{\frac{\alpha\theta}{p}} \in W^{1,p}(X, d, m)$. Fix an arbitrary $\tilde{t} \in (0, t_0)$, and for every $t \in [0, \tilde{t}]$ consider the functions

$$q(t) = \frac{\alpha\beta}{(\alpha - \beta)t/\tilde{t} + \beta} \quad \text{and} \quad \tilde{F}(t) := F(t)^{1/q(t)} = \left( \int_X e^{q(t)Q_{\theta}u(x)} dm \right)^{1/q(t)} = \|e^{Q_{\theta}u}\|_{L^{q(t)}(X, m)}.$$
We observe that \( q(0) = \alpha, q(\tilde{t}) = \beta \), and \( q \) is of class \( C^1 \) in \((0, \tilde{t})\), with \( q'(t) = \frac{\alpha \beta}{(\alpha - \beta)^2 t + \beta} > 0 \); thus, \( t \mapsto q(t) \) is strictly increasing. Finally, we consider the function

\[
x \mapsto w_t(x) := \frac{e^{\int_0^t Q_t u(x)}}{\tilde{F}(t)}.
\]

By Proposition 4.2/(iii), \( w_t \) belongs to \( W^{1,p}(X, d, m) \) for every \( t \in (0, \tilde{t}) \) and

\[
\int_X w_t^p \, dm = 1, \quad \forall t \in (0, \tilde{t}).
\]

According to Theorem 1.1, one has

\[
\int_X w_t^p \log w_t^p \, dm \leq \frac{N}{p} \log \left( L_{p,N} \mathrm{AVR}_m \right) \int_X |\nabla w_t|^p \, dm.
\]  \hspace{1cm} (4.13)

By the non-smooth chain rule, we have

\[
|\nabla w_t(x)| = \frac{e^{\int_0^t Q_t u(x)}}{\tilde{F}(t)} \cdot \frac{q(t)}{p} \cdot |\nabla Q_t u(x)|,
\]  \hspace{1cm} (4.14)

and

\[
\int_X w_t^p \log w_t^p \, dm = \frac{\mathcal{E}(e^{q(t)} Q_t u)}{F(t)} - \log (\tilde{F}(t)^q(t)),
\]  \hspace{1cm} (4.15)

where

\[
\mathcal{E}(e^{q(t)} Q_t u) = \int_X e^{q(t)} Q_t u \log (e^{q(t)} Q_t u) \, dm = \int_X e^{q(t)} Q_t u q(t) Q_t u \, dm.
\]

By Proposition 4.2, it turns out that \( \tilde{F}(t) = F(t)^{1/q(t)} \), and thus \( \tilde{F} \) is a locally Lipschitz function on \((0, \tilde{t})\). In particular, its right derivative exists at every \( t \in (0, \tilde{t}) \) and its expression is given by

\[
\frac{d^+}{dt} \tilde{F}(t) = \frac{d^+}{dt} e^{\log \tilde{F}(t)} = \frac{d^+}{dt} e^{\frac{1}{q(t)} \log \left( \int_X e^{q(t)} Q_t u \, dm \right)}
\]

\[
= \tilde{F}(t) \left( - \frac{q'(t)}{q(t)^2} \log \left( \int_X e^{q(t)} Q_t u \, dm \right) \right) + \frac{q(t)^2}{\tilde{F}(t)^{q(t)}} \int_X e^{q(t)} Q_t u \left( \frac{d^+}{dt} Q_t u + \frac{q'(t)}{q(t)} Q_t u \right) \, dm
\]

\[
= \frac{\tilde{F}(t)}{q(t)^2} \left( - q'(t) \log (\tilde{F}(t)^{q(t)}) \right) + \frac{q(t)^2}{\tilde{F}(t)^{q(t)}} \int_X e^{q(t)} Q_t u \frac{d^+}{dt} Q_t u \, dm + \frac{q'(t)}{\tilde{F}(t)^{q(t)}} \mathcal{E}(e^{q(t)} Q_t u).
\]  \hspace{1cm} (4.16)

Note that all three terms in the last parenthesis are well-defined for every \( t \in (0, \tilde{t}) \). Indeed, the first term is well-defined due to Proposition 4.2/(ii), while the second and third terms are bounded from above (due to the monotonicity of \( t \mapsto Q_t u \) and the log-Sobolev inequality (4.13) combined with the identity (4.15)); if one of them were \(-\infty\), this would contradict the fact that \( \frac{d^+}{dt} \tilde{F}(t) \in \mathbb{R}, \quad t \in (0, \tilde{t}) \).
By using the Hamilton–Jacobi inequality (4.1) together with \(|\nabla Q_t u(x)| = |D Q_t u(x)|\) for every \(t \in (0, \tilde{t})\) and m.a.e. \(x \in X\), see (2.5), we obtain for every \(t \in (0, \tilde{t})\) that

\[
\frac{d}{dt} \tilde{F}(t) \leq \frac{q'(t)}{q(t)^2} \left( \mathcal{E} \left( e^{q(t) Q_t u} \right) - \tilde{F}(t)^{q(t)} \log \left( \tilde{F}(t)^{q(t)} \right) \int_X e^{q(t) Q_t u} |\nabla Q_t u|^p \, dm \right).
\]

Applying the elementary inequality \(\log(e^y) \leq y\) for every \(y > 0\), together with the log-Sobolev inequality (4.13) and relations (4.14) and (4.15), we obtain for every \(t \in (0, \tilde{t})\) that

\[
\frac{d}{dt} \tilde{F}(t) \leq \frac{q'(t)}{q(t)^2} \left[ \frac{N}{p} \log \left( \mathcal{L}_{p,N} \text{AVR}_m^{2 \frac{N}{p}} \int_X |\nabla w_t|^p \, dm \right) - \frac{N}{p} \left( \frac{q(t)}{q'(t)} \int_X e^{q(t) Q_t u} |\nabla Q_t u|^p \, dm \right) \right]
\]

\[
\leq \frac{q'(t)}{q(t)^2} \cdot \frac{N}{p} \left[ \log \left( \mathcal{L}_{p,N} \text{AVR}_m^{2 \frac{N}{p}} \frac{q(t)^p}{q'(t)^{q(t)}} \int_X e^{q(t) Q_t u} |\nabla Q_t u|^p \, dm \right) \right]
\]

\[
- \log \left( \frac{e q(t)^2}{N q'(t) F(t)^{q(t)}} \int_X e^{q(t) Q_t u} |\nabla Q_t u|^p \, dm \right)
\]

\[
= \frac{q'(t)}{q(t)^2} \cdot \frac{N}{p} \log \left( \frac{N \mathcal{L}_{p,N} \text{AVR}_m^{2 \frac{N}{p}}}{e^{q(t)}} \cdot q(t)^{q(t)-2} \right).
\]

By integrating both sides on \((0, \tilde{t})\), and taking into account that \(t \mapsto \tilde{F}(t)\) is absolutely continuous, we obtain the inequality

\[
\tilde{F}(\tilde{t}) \leq \tilde{F}(0) \cdot \left( \frac{\beta - \alpha}{\tilde{t}} \right)^{N \frac{1 - \alpha}{\alpha} \frac{\beta - \alpha}{\alpha}} \cdot C_{\alpha, \beta, N, p, m},
\]

where

\[
C_{\alpha, \beta, p, N, m} = \frac{\alpha^N (p - \frac{2}{p})}{\beta^N (p - \frac{2}{p})} \left( \text{AVR}_m \sigma_N (p') \frac{N}{p'} \Gamma \left( \frac{N}{p'} + 1 \right) \right)^{\frac{\alpha - \beta}{\alpha \beta}},
\]

which concludes the proof of (1.7).

4.4. Sharpness in the hypercontractivity estimate (1.7). We assume by contradiction that one can improve the constant \(\mathcal{L}_{p,N} \text{AVR}_m^{2 \frac{N}{p}} N^{\frac{p-1}{pp}}\) in (1.7), i.e., for convenience of computations, there exists \(\mathcal{C} < \mathcal{L}_{p,N} \text{AVR}_m^{-\frac{2}{pp}}\) such that for every \(\beta > \alpha > 0\), \(t \in (0, t_0)\) and \(u \in F_{t_0, x_0}(X)\) with \((1 + d^p(x_0, \cdot))e^{\alpha u} \in L^1(X, m)\) and \(e^{\frac{\alpha u}{\beta}} \in W^{1,p}(X, d, m)\), one has

\[
\|e^{Q_t u}\|_{L^\beta(X, m)} \leq \|e^u\|_{L^\alpha(X, m)} \left( \frac{\beta - \alpha}{t} \right)^{\frac{N}{p} \frac{\beta - \alpha}{\alpha \beta} \frac{\alpha^N (p - \frac{2}{p})}{\beta^N (p - \frac{2}{p})} \left( \alpha N e^{p-1} \frac{N}{p} \right)^{\frac{\beta - \alpha}{\alpha \beta}}}. \tag{4.19}
\]

Fix a function \(u : X \to \mathbb{R}\) with the above properties, a number \(\alpha > 0\) and the function \(\beta := \beta(t) = \alpha + yt\) with \(t > 0\), where \(y > 0\) will be specified later. With these choices, inequality (4.19) can be equivalently written into the form

\[
\log \|e^{Q_t u}\|_{L^\beta(t)(X, m)} - \log \|e^u\|_{L^\alpha(X, m)} \leq \frac{1}{t} \log \left( \left( \frac{N y t}{p \alpha \beta} \right)^{\frac{\alpha N (p - \frac{2}{p})}{\beta^N (p - \frac{2}{p})}} \cdot \frac{\alpha^N (p - \frac{2}{p})}{\beta^N (p - \frac{2}{p})} \right), \tag{4.20}
\]
for every $t \in (0, t_0)$.  If we introduce
\[
\mathcal{T}(t) := \left( \int_X e^{\beta(t)Q_t u} \, dm \right)^{1/\beta(t)} = \| e^{Q_t u} \|_{L^{\beta(t)}(X, m)} , \quad t \in [0, t_0),
\]
and by taking the limit $t \to 0^+$ in (4.20), we obtain that
\[
\frac{d^+}{dt} \mathcal{T}(t) \bigg|_{t=0} \leq \lim_{t \to 0^+} \log \left[ \left( yC \frac{Ne^{p-1}}{p^p} \right)^{\frac{N_y}{p\alpha \beta(t)}} \right] . \tag{4.21}
\]
A similar computation as in (4.16) and Proposition 4.1/(iv), show that
\[
\frac{d^+}{dt} \mathcal{T}(t) \bigg|_{t=0} = \frac{1}{\beta(t)^2} \left( -\beta'(t) \log \left( \mathcal{T}(t)^{\beta(t)} \right) + \frac{\beta(t)^2}{\mathcal{T}(t)^{\beta(t)}} \int_X e^{\beta(t)Q_t u} \frac{d^+}{dt} Q_t u \, dm \right.
\]
\[
\left. + \frac{\beta'(t)}{\mathcal{T}(t)^{\beta(t)}} \mathcal{E}(e^{\beta(t)Q_t u}) \right) \bigg|_{t=0} \geq \frac{y}{\alpha^2 \| e^u \|_{L^\alpha(X, m)}^\alpha} \left( \mathcal{E}(e^{\alpha u}) - \| e^u \|_{L^\alpha(X, m)}^\alpha \right) \log \left( \| e^u \|_{L^\alpha(X, m)}^\alpha \right) - \frac{\alpha^2}{py} \int_X e^{\alpha u} \| Du \|_{L^1(X, m)}^p \, dm .
\]
The latter estimate and the limit
\[
\lim_{t \to 0^+} \left( \frac{\frac{\alpha^2}{py} \int_X e^{\alpha u} \| Du \|_{L^1(X, m)}^p \, dm}{\| e^u \|_{L^\alpha(X, m)}^\alpha} \right)^{1/\beta(t)} = e^{\frac{Ny(p-2) \log \alpha - p}{\alpha^2 p}}
\]
transform inequality (4.21) into
\[
\mathcal{E}(e^{\alpha u}) - \log(\| e^u \|_{L^\alpha(X, m)}^\alpha) \leq \frac{\alpha^2}{py} \int_X e^{\alpha u} \| Du \|_{L^1(X, m)}^p \, dm + \frac{N}{p} \left( \log \left( yC \frac{Ne^{p-1}}{p^p} \right) + (p-2) \log \alpha - p \right) ;
\]
here we also used relation (2.5), i.e., $\| Du \| = | Du | m$-a.e., based on the fact that $u \in \mathcal{F}_{t_0, x_0}(X) \subset \text{Lip}_{loc}(X)$. By assumption, $e^{\alpha u} \in W^{1,p}(X, d, m)$, thus
\[
w := \frac{e^{\alpha u}}{\| e^u \|_{L^\alpha(X, m)}^\alpha} \in W^{1,p}(X, d, m) \quad \text{and} \quad \int_X w^p \, dm = 1 .
\]
With this notation, the latter inequality can be equivalently written into the form
\[
\int_X w^p \log w^p \leq \frac{\alpha^2}{py} \left( \frac{p}{\alpha} \right)^p \int_X | Du |^p \, dm + \frac{N}{p} \left( \log \left( yC \frac{Ne^{p-1}}{p^p} \right) + (p-2) \log \alpha - p \right) . \tag{4.22}
\]
Minimizing the right hand side of (4.22) in $y > 0$, its optimal value is $y = \frac{\alpha^2}{N} \left( \frac{p}{\alpha} \right)^p \int_X |Du|^p \, dm$. Replacing this value into (4.22), it follows that
\[
\int_X w^p \log w^p \leq \frac{N}{p} \log \left( C \int_X | Du |^p \, dm \right) .
\]
Due to the sharpness in Theorem 1.1, see (1.6), we have that $C \geq \mathcal{L}_{p, N \text{AVR}} m^{\frac{p}{p-1}}$, which contradicts our initial assumption $C < \mathcal{L}_{p, N \text{AVR}} m^{\frac{p}{p-1}}$. \hfill \square
5. Gaussian log-Sobolev inequality in RCD(0, N) spaces: proof of Theorem 1.3

In this section we are going to prove Theorem 1.3; to do this, we need some basic properties of RCD(0, N) spaces that are presented in the subsequent subsection.

5.1. Properties of RCD(0, N) spaces. A metric measure space $(X, d, m)$ satisfies the Riemannian curvature-dimension condition RCD$(0, N)$ for $N > 1$, if it is a CD$(0, N)$ space and it is infinitesimally Hilbertian, i.e., the Banach space $W^{1,2}(X, d, m)$ is Hilbertian. Due to Cavalletti and Milman [24], this definition of RCD$(0, N)$ is equivalent to the previously introduced notions of $\text{RCD}^e(0, N)$ by Erbar, Kuwada and Sturm [34] and of $\text{RCD}^* (0, N)$ by Ambrosio, Mondino and Savaré [10]. The first form of the Riemannian curvature-dimension condition was introduced by Ambrosio, Gigli and Savaré [7], with no upper bound on the dimension. Typical examples of RCD$(0, N)$ spaces include measured Gromov–Hausdorff limit spaces of Riemannian manifolds with non-negative Ricci curvature.

In the sequel, we fix a metric measure space $(X, d, m)$ satisfying RCD$(0, N)$ for some $N > 1$. The infinitesimal Hilbertianity of $(X, d, m)$ can be characterized by its 2-infinitesimal strict convexity and the symmetry of the map

\[(f, g) \mapsto Df(\nabla g) = D^+ f(\nabla g),\]

i.e.,

\[Df(\nabla g) = Dg(\nabla f),\]

where

\[D^+ f(\nabla g) = \inf_{\varepsilon > 0} \frac{|\nabla(g + \varepsilon f)|^2 - |\nabla g|^2}{2\varepsilon}, \quad D^- f(\nabla g) = \sup_{\varepsilon < 0} \frac{|\nabla(g + \varepsilon f)|^2 - |\nabla g|^2}{2\varepsilon},\]

for every $f, g : X \to \mathbb{R}$ with finite 2-Cheeger energy, see Gigli [38, §4.3]. Note that in particular, for every $f : X \to \mathbb{R}$ with finite 2-Cheeger energy, one has

\[Df(\nabla f) = |\nabla f|^2.\]

According to (5.1), we shall use the formal notation $\nabla f \cdot \nabla g$ for the object $Df(\nabla g)$; as a consequence, the linearity of the differential operator $D$, combined with the symmetry property (5.1), implies the bilinearity of $(f, g) \mapsto \nabla f \cdot \nabla g$. In addition, we also have the Leibnitz-type rule

\[\nabla(f_1 f_2) \cdot \nabla g = f_1 \nabla f_2 \cdot \nabla g + f_2 \nabla f_1 \cdot \nabla g \quad \text{m–a.e.,}\]

for every $f_1, f_2 \in L^\infty_{\text{loc}}(X)$ having locally finite 2-Cheeger energies. If $f_1, f_2 : X \to \mathbb{R}$ are two such functions, as a simple consequence of relations (5.1)-(5.3) we have that

\[|\nabla(f_1 f_2)|^2 = f_1^2 |\nabla f_2|^2 + 2f_1 f_2 \nabla f_1 \cdot \nabla f_2 + f_2^2 |\nabla f_1|^2.\]

It turns out that in the framework of RCD$(0, N)$ spaces, a second order calculus related to the Laplacian can be developed. Let us recall that the Laplacian $\Delta : D(\Delta) \to L^2(X, m)$ is a densely defined linear operator whose domain $D(\Delta)$ consists of all functions $f \in W^{1,2}(X, d, m)$ satisfying

\[\int_X hf \, dm = -\int_X \nabla h \cdot \nabla f \, dm \quad \text{for any } h \in W^{1,2}(X, d, m),\]

and the unique $g \in L^2(X, m)$ that satisfies this property is denoted by $\Delta f$. More generally, we say that $f \in W^{1,2}_{\text{loc}}(X, d, m)$ is in the domain of the measure-valued Laplacian, and we write $f \in D(\Delta)$, if there exists a Radon measure $\nu$ on $X$ such that

\[\int_X \psi \, d\nu = -\int_X \nabla f \cdot \nabla \psi \, dm \quad \text{for any } \psi \in \text{Lip}_c(X).\]
In this case, we write $\Delta f := \nu$. If moreover $\Delta f \ll \text{m}$ with $L^2_{\text{loc}}$-density, we denote by $\Delta f$ the unique function in $L^2_{\text{loc}}(M, d, \text{m})$ such that $\Delta f = \Delta f \text{m}$.

Using this notation, one has the following sharp Laplacian comparison estimate for the distance function on an arbitrary $\text{RCD}(0, N)$ space $(X, d, \text{m})$, see e.g. Gigli [38, Corollary 5.15]: if we consider the distance function $d_{x_0} : X \to [0, \infty)$ with $d_{x_0}(x) := d(x_0, x)$ for some $x_0 \in X$, we have that

$$\frac{d_{x_0}^2}{2} \in D(\Delta) \quad \text{and} \quad \Delta \frac{d_{x_0}^2}{2} \leq N \text{m}. \quad (5.5)$$

After this preparation we are ready to present the proof of Theorem 1.3.

5.2. **Proof of the Gaussian log-Sobolev inequality** (1.9). Let $x_0 \in X$ be fixed. We recall from the Introduction the $\text{m}$-Gaussian measure

$$d\text{m}_{G,x_0}(x) := G^{-1}e^{-\frac{d^2_{(x_0,x)}}{2}}d\text{m}(x), \quad (5.6)$$

where

$$G = \int_X e^{-\frac{d^2_{(x_0,x)}}{2}}d\text{m}(x) > 0.$$  

We first notice that $G < +\infty$; indeed, by the layer cake representation one has

$$G = \int_X e^{-\frac{d^2_{(x_0,x)}}{2}}d\text{m}(x) = \int_0^\infty \rho(B(x_0, \rho))\rho e^{-\frac{\rho^2}{2}}d\rho,$$

thus, by the Bishop–Gromov inequality, if $\theta_\text{m}(x_0) = +\infty$, then

$$G \leq \int_0^1 \rho(B(x_0, \rho))\rho e^{-\frac{\rho^2}{2}}d\rho + \text{m}(B(x_0, 1)) \int_1^\infty \rho^N e^{-\frac{\rho^2}{2}}d\rho < +\infty,$$

while if $\theta_\text{m}(x_0) < +\infty$, then

$$G \leq \theta_\text{m}(x_0)\sigma_N \int_0^\infty \rho^N e^{-\frac{\rho^2}{2}}d\rho = \theta_\text{m}(x_0)\sigma_N 2\rho(N)\frac{\Gamma(\frac{N}{2} + 1)}{2} < +\infty. \quad (5.8)$$

If $\theta_\text{m}(x_0) = +\infty$, inequality (1.9) trivially holds; therefore, we may assume that $\theta_\text{m}(x_0) < +\infty$.

Fix $u \in W^{1,2}(X, d, \text{m}_{G,x_0})$ such that

$$\int_X u^2d\text{m}_{G,x_0} = 1,$$

and let us consider the function

$$v(x) := u(x) \cdot \sqrt{\rho_{G,x_0}(x)}, \quad x \in X,$$

where

$$\rho_{G,x_0}(x) = G^{-1}e^{-\frac{d^2_{(x_0,x)}}{2}}$$

is the density of $d\text{m}_{G,x_0}$ with respect to $d\text{m}$. By definition of $v$, we can write

$$\int_X v^2d\text{m} = \int_X u^2\rho_{G,x_0}d\text{m} = \int_X u^2d\text{m}_{G,x_0} = 1. \quad (5.9)$$

We claim that $v \in W^{1,2}(X, d, \text{m})$. Indeed, by relation (5.4), the eikonal equation (2.7), and the non-smooth chain rule, we have m-a.e. that

$$|\nabla v|^2 = |\nabla(u\sqrt{\rho_{G,x_0}})|^2 = \frac{1}{4}u^2\rho_{G,x_0}^2|\nabla d_{x_0}|^2 - \rho_{G,x_0}u\nabla u \cdot \nabla \left(\frac{d^2_{(x_0,x)}}{2}\right) + \rho_{G,x_0}|\nabla u|^2$$
Due to the Laplacian comparison property (5.5), one has that
\[-\int_X \nabla \left( \frac{u^2}{2} \rho_{G,x_0} \right) \cdot \nabla \left( \frac{d_{x_0}^2}{2} \right) \, dm = \int_X \frac{u^2}{2} \rho_{G,x_0} \Delta \left( \frac{d_{x_0}^2}{2} \right) \, dm \leq \frac{N}{2} \int_X u^2 \rho_{G,x_0} \, dm = \frac{N}{2}.\]  
(5.11)
Integrating (5.10) with respect to \( dm \) and given this final estimate, together with the fact that \( u \in W^{1,2}(X, d, m_{G,x_0}) \) (and hence \( \int_X |\nabla u|^2 \, dm_{G,x_0} < \infty \)), we obtain that
\[\int_X |\nabla v|^2 \, dm < \infty.\]
Using this fact and (5.9), we obtain the claim, i.e., \( v \in W^{1,2}(X, d, m) \).
Furthermore, integrating again (5.10) and using the above observations, we also see that
\[\int_X u^2 d_{x_0}^2 \rho_{G,x_0} \, dm = \int_X u^2 d_{x_0}^2 m_{G,x_0} < \infty.\]
Summing up the above estimates, we obtain that
\[\int_X |\nabla v|^2 \, dm \leq \int_X |\nabla u|^2 \, dm_{G,x_0} + \frac{N}{2} - \frac{1}{4} \int_X u^2 d_{x_0}^2 \, dm_{G,x_0}, \]  
(5.12)
and since \( v \in W^{1,2}(X, d, m) \) verifies (5.9), by the \( L^2 \)-log-Sobolev inequality (1.4) one has
\[\int_X v^2 \log v^2 \, dm \leq \frac{N}{2} \log \left( \frac{2}{eN} \left( \sigma_N \Gamma \left( \frac{N}{2} + 1 \right) \text{AVR}_m \right)^{-2} \right) \int_X |\nabla v|^2 \, dm \].  
(5.13)
Writing the left-hand side of (5.13) in terms of \( u \), we obtain
\[\int_X v^2 \log v^2 \, dm = \int_X u^2 \log \left( u^2 \rho_{G,x_0} \right) \rho_{G,x_0} \, dm \]
\[= \int_X u^2 \log u^2 \, dm_{G,x_0} - \log G - \frac{1}{2} \int_X u^2 d_{x_0}^2 \, dm_{G,x_0}. \]  
(5.14)
If we apply the elementary inequality \( \log(e^y) \leq y \), which holds for every \( y > 0 \), the right-hand side of (5.13) can be estimated as
\[\frac{N}{2} \log \left( \frac{1}{2e^2} \left( \sigma_N \Gamma \left( \frac{N}{2} + 1 \right) \text{AVR}_m \right)^{-2} \right) + \frac{N}{2} \log \left( \frac{4e}{N} \int_X |\nabla v|^2 \, dm \right) \leq \frac{N}{2} \log \left( \frac{1}{2e^2} \left( \sigma_N \Gamma \left( \frac{N}{2} + 1 \right) \text{AVR}_m \right)^{-2} \right) + 2 \int_X |\nabla v|^2 \, dm.\]
The latter estimate together with relations (5.12)-(5.14) and (5.8) imply (1.9), i.e.,
\[\int_X u^2 \log u^2 \, dm_{G,x_0} \leq 2 \int_X |\nabla u|^2 \, dm_{G,x_0} + \log \frac{\theta_m(x_0)}{\text{AVR}_m}.\]
Remark 5.1. As we pointed out in the introduction, the Gaussian log-Sobolev inequality (1.3) holds with the measure $dγ = e^{-V}d\text{vol}$ whenever $V$ verifies a strong convexity assumption, see Cordero-Erausquin, McCann and Schmuckenschläger [29]. Note that the Gaussian log-Sobolev inequality (1.9) cannot be obtained via the Prékopa–Leindler inequality, as the function $V(x) = \frac{1}{2}d^2(x_0, x)$ does not provide any reasonable convexity property (even in the setting of Riemannian manifolds with nonnegative Ricci curvature). Such convexity property occurs more naturally on Cartan-Hadamard manifolds, the threshold geometric objects being the Euclidean spaces. However, in the setting of $\text{RCD}(0, N)$ spaces the additive term $\log \frac{θ_m(x_0)}{\text{AVR}_m}$ balances the lack of such convexity.

5.3. Sharpness of the constant 2 in the Gaussian log-Sobolev inequality (1.9). We are going to prove that the constant 2 in (1.9) is sharp (whenever we assume that $θ_m(x_0) < +\infty$). By contradiction, let us assume that there exists $C ∈ (0, 2)$ such that, for every $\int_X u^2 dm_{G, x_0} = 1$, one has

$$\int_X u^2 \log u^2 dm_{G, x_0} \leq C \int_X |∇u|^2 dm_{G, x_0} + \log \frac{θ_m(x_0)}{\text{AVR}_m}. \quad (5.15)$$

For every $λ > 0$, let

$$f(λ) := \int_X e^{-λd(x_0, x)^2} dm(x).$$

Recalling that

$$G = \int_X e^{-\frac{d^2(x_0, x)}{2}} dm(x) > 0,$$

consider the function

$$u_λ(x) = \left(\frac{G}{f(λ)}\right)^{\frac{1}{2}} e^{(\frac{1}{4} - \frac{λ}{2})d(x_0, x)^2}, \quad x ∈ X.$$

One can easily observe that $\int_X u_λ^2 dm_{G, x_0} = 1$. Thus, we can use $u_λ$ in (5.15). By applying the eikonal equation (2.7), a straightforward reorganization of the terms provide the inequality

$$- \log f(λ) \leq \frac{1}{f(λ)} \left(\frac{1}{2} - λ\right) \left(\frac{C}{2} - 1 - Cλ\right) \int_X e^{-λd(x_0, x)^2} dm(x) + \log \frac{θ_m(x_0)}{G \cdot \text{AVR}_m}, \quad ∀λ > 0. \quad (5.16)$$

By the layer cake representation one has that

$$\lim_{λ → 0^+} λ^N f(λ) = 2 \lim_{λ → 0^+} λ^{N+1} \int_0^∞ m(B(x_0, ρ))ρe^{-λρ^2} dρ = \lim_{λ → 0^+} λ^N \int_0^∞ m(B(x_0, \sqrt{t/λ}))e^{-t} dt \quad \text{[change of var. } ρ = \sqrt{t/λ}]$$

$$= \sigma_N \lim_{λ → 0^+} \int_0^∞ m(B(x_0, \sqrt{t/λ}))\frac{1}{σ_N(√t/λ)^N} t^{N/2} e^{-t} dt$$

$$= \sigma_N \text{AVR}_m Γ\left(\frac{N}{2} + 1\right) = π^N \frac{1}{N!} \text{AVR}_m, \quad (5.17)$$

where we also used the monotone convergence theorem (together with the Bishop–Gromov comparison principle) and the definition of $\text{AVR}_m$. 
In a similar way, one has that
\[ \int_X e^{-\lambda d(x_0,x)^2} d(x_0,x)^2 d\mu(x) = 2 \int_0^{\infty} m(B(x_0,\rho)) \rho(\lambda \rho^2 - 1)e^{-\lambda \rho^2} d\rho, \]
and
\[ \lim_{\lambda \to 0^+} \lambda^{\frac{N}{2}+1} \int_X e^{-\lambda d(x_0,x)^2} d(x_0,x)^2 d\mu(x) = \frac{N}{2} \pi^{\frac{N}{2}} \text{AVR}_m. \] (5.18)

Multiplying (5.16) by \( \lambda > 0 \), and letting \( \lambda \to 0^+ \), on account of relations (5.17) and (5.18) we obtain that \( 0 \leq \frac{N}{4} \left( \frac{C}{2} - 1 \right), \) i.e., \( C \geq 2 \), which contradicts our initial assumption \( C \in (0,2) \). \( \square \)

Inspired by Bobkov, Gentil and Ledoux [19, Remark 2.2], we establish a hypercontractivity bound for the \( m \)-Gaussian probability measure \( m_{G,x_0} \), see (5.6). In the sequel, the previous notions are considered for \( p = 2 \).

**Corollary 5.1.** Let \((X,d,m)\) be an \( \text{RCD}(0,N) \) space with \( N > 1 \) and assume that \( \text{AVR}_m > 0 \). Given \( \alpha > 0, x_0 \in X \), for any function \( u : X \to \mathbb{R} \) with \( e^\frac{u}{2} \in W^{1,2}(X,d,m_{G,x_0}) \) and \( t \geq 0 \) one has that
\[ \| e^{\int_0^t u^2} \|_{L^{\alpha+1}(X,m_{G,x_0})} \leq e^{H_{\alpha,t,x_0}} \| e^u \|_{L^\alpha(X,m_{G,x_0})}, \] (5.19)
where \( H_{\alpha,t,x_0} = -\frac{t}{\alpha(a+t)} \log \frac{\theta_m(x_0)}{\text{AVR}_m} \).

**Proof.** For any \( u \in W^{1,2}(X,d,m_{G,x_0}) \) inequality (1.9) can be written into the equivalent form
\[ \int_X u^2 \log u^2 \, d\mu_{G,x_0} - \int_X u^2 \, d\mu_{G,x_0} \log \int_X u^2 \, d\mu_{G,x_0} \leq 2 \int_X |\nabla u|^2 \, d\mu_{G,x_0} + \frac{1}{\text{AVR}_m} \int_X u^2 \, d\mu_{G,x_0}. \] (5.20)

If we fix \( t \geq 0 \) and any function \( u : X \to \mathbb{R} \) with \( e^\frac{u}{2} \in W^{1,2}(X,d,m_{G,x_0}) \), a similar argument as in subsection 4.3 combined with (5.20) yield the required estimate (5.19). \( \square \)

**Remark 5.2.** As expected, (5.19) implies (as \( t \to 0 \)) the validity of the Gaussian log-Sobolev inequality (5.20).

**6. Final remarks**

At the end, we indicate further perspectives related to our results, which can be considered as starting points for forthcoming investigations.

1. **Rigidities.** Since the log-Sobolev inequality in Theorem 1.1 is sharp, a natural question arises: can one characterize the equality case in (1.4)? The proof of Theorem 1.1 shows that if equality holds in (1.4), we necessarily have equality in the Pólya-Szegő inequality (2.9), therefore the rigidity of the sharp isoperimetric inequality (2.3) is expected to hold giving in turn that \( X \) is an Euclidean metric measure cone. However, according to Nobili and Violo [56], some regularity of the extremal function is a priori needed in order to characterize the equality in the Pólya-Szegő inequality (2.9). Therefore, although this approach seems to be promising, technical difficulties prevent to study the rigidity scenario in the sharp log-Sobolev inequality (1.4). We notice that several rigidity results are available in the literature concerning the equality in the sharp isoperimetric inequality (2.3); beside the smooth setting, see Agostiniani, Fogagnolo and Mazzieri [1, 35], Balogh and Kristály [17], Brendle [20] and Johne [47], there are recent achievements also in the nonsmooth setting by Antonelli, Pasqualetto, Pozzetta and Semola [11] for non-collapsed RCD spaces, as well as Cavalletti and Manini [23] for possibly collapsing spaces.
II. MCP(0,N) spaces. In a recent paper, Cavalletti and Manini [22] proved a sharp isoperimetric inequality on MCP(0,N) spaces; namely, if \((X,d,m)\) is a essentially non-branching metric measure space satisfying the MCP(0,N) condition for some \(N > 1\) (see Ohta [58] or Sturm [66] for the definition) and \(\text{AVR}_m > 0\), one has that, for every bounded Borel subset \(\Omega \subset X\),

\[
m^+ (\Omega) \geq \left( N\sigma_N \text{AVR}_m \right)^{\frac{1}{N}} m(\Omega)^{\frac{N-1}{N}},
\]

and the constant \(\left( N\sigma_N \text{AVR}_m \right)^{\frac{1}{N}}\) in (6.1) is sharp. It is worth noticing that this constant in (6.1) is slightly worse than the one from (2.3). The isoperimetric inequality (6.1) together with a suitable co-area formula on MCP(0,N) should yield a Pólya–Szegő inequality. Furthermore, a sharp log-Sobolev inequality on the 1-dimensional MCP model case is needed (see [22, §3.2]), similar to (2.12) as in the CD setting.

III. Gaussian log-Sobolev inequality on CD(0,N) spaces. One of the crucial steps in the proof of Theorem 1.3 is the estimate (5.10), which deeply explores the infinitesimal Hilbertianity of the RCD(0,N) spaces. It is not clear at this moment if the proof can be extended to CD(0,N) (for instance, to non-Riemannian Finsler structures), in spite of the fact that further indispensable ingredients are already known to be valid on such structures, as versions of the Laplace comparison of the distance functions similar to (5.5), see Cavalletti and Mondino [25] and Gigli [38].

Acknowledgment. A. Kristály thanks T. Illés, M. Pintér, M. E.-Nagy, P. Rigó and Z. Szántó, for their kind invitation and hospitality during his visit as a senior research fellow at the Corvinus Centre for Operations Research, Corvinus Institute for Advanced Studies, Corvinus University of Budapest, Budapest, Hungary. The authors thank the reviewers for their careful reading and helpful comments.

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