The free energy of spherical pure $p$-spin models: computation from the TAP approach

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Abstract
We compute the free energy at all temperatures for the spherical pure $p$-spin models from the generalized Thouless–Anderson–Palmer representation. This is the first example of a mixed $p$-spin model for which the free energy is computed in the whole replica symmetry breaking phase, without appealing to the famous Parisi formula.

Keywords Spherical spin models · Free energy · TAP approach

Mathematics Subject Classification 82D30 · 60G15 · 60G60

1 Introduction
The spherical pure $p$-spin Hamiltonian is the random field on the sphere of radius $\sqrt{N}$ in dimension $N$, $S^{N-1} := \{\sigma \in \mathbb{R}^N : \|\sigma\| = \sqrt{N}\}$, given by

$$H_{N,p}(\sigma) := N^{-\frac{p-1}{2}} \sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p}, \quad (1.1)$$

where $\sigma = (\sigma_1, \ldots, \sigma_N)$ and $J_{i_1,\ldots,i_p}$ are i.i.d. standard normal variables. More generally, given a sequence of non-negative numbers $\gamma_p$ such that $\sum_{p=2}^{\infty} \gamma_p^2 (1 + \epsilon)^p < \infty$ for small enough $\epsilon > 0$, the mixed $p$-spin Hamiltonian corresponding to the mixture $\nu(t) = \sum_{p=2}^{\infty} \gamma_p^2 t^p$ is

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where the pure Hamiltonians are assumed to be independent for different values of $p$. The covariance function of the centered Gaussian field $H_N(\sigma)$ is given by

$$E H_N(\sigma) H_N(\sigma') = N \nu(R(\sigma, \sigma')),$$

where $R(\sigma, \sigma') := \frac{1}{N} \sigma \cdot \sigma' := \frac{1}{N} \sum_{i \leq N} \sigma_i \sigma'_i$ is called the overlap of $\sigma$ and $\sigma'$.

One of the fundamental problems in the study of mean-field spin glass models is computing, for all inverse-temperatures $\beta \geq 0$, the free energy

$$F(\beta) := \lim_{N \to \infty} F_N(\beta) := \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \int_{S^{N-1}} e^{\beta H_N(\sigma)} d\sigma,$$

where $d\sigma$ denotes integration w.r.t. the uniform measure on $S^{N-1}$, and the $\beta \to \infty$ limit of $\frac{1}{\beta} F(\beta)$, the ground-state energy

$$E_* := \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \max_{\sigma \in S^{N-1}} H_N(\sigma).$$

It was recently proved in Ref. [41] that the limit of the free energy as in (1.3) exists. This of course is also a consequence of Parisi’s formula, but the proof of [41], which uses the Guerra-Toninelli interpolation [25], is independent of the Parisi formula. One can easily verify from this that the limit of the ground-state energy as in (1.4) exists as well, using bounds on the Lipschitz constant of $H_N(\sigma)$ (see e.g. [15, Lemma 6.1]) and Borell’s inequality.

The spherical models were originally proposed in physics as a variant of the Ising spins $p$-spin models. For those models, one treats the Hamiltonian $H_N(\sigma)$ as a function on the hypercube $\Sigma_N = \{1, -1\}^N$, and defines the free energy and ground state energy similarly to the above with $S^{N-1}$ replaced by $\Sigma_N$ and integration by summation.

In the late 70s, Parisi discovered his celebrated formula for the free energy $F(\beta)$ [34, 35]. Although it was originally developed for the Sherrington–Kirkpatrick (SK) model—namely, the pure 2-spin model with Ising spins—the formula applies to general mixed models with either Ising or spherical spins. See in particular the formulation by Crisanti and Sommers for the spherical models [20]. The formula was rigorously proved nearly two and a half decades later. The first breakthrough was made by Guerra [24] who showed that the formula is an upper bound for the free energy. Shortly after, Talagrand proved the matching lower bound in the seminal works [45, 46], assuming that $\gamma_p = 0$ for odd $p \geq 3$. Another breakthrough was made several years later by Panchenko who proved in [31] the Parisi ultrametricity conjecture [29, 30]. Using ultrametricity, the Parisi formula was established for mixed models which include odd interactions by Panchenko [33] for the Ising case and by Chen [12] for the spherical case.

As mentioned above, to express the ground state energy one can use the Parisi formula for $\frac{1}{\beta} F(\beta)$ and take the limit as $\beta \to \infty$. More directly, it is expressed by
the zero temperature analogue of the Parisi formula proved by Chen and Sen [17] and Jagannath and Tobasco [26] in the spherical case and Auffinger and Chen [4] in the Ising case.

In this work we focus on the spherical pure $p$-spin models (1.1) and calculate the free energy and ground state energy by an alternative way to the Parisi formula. Instead, we will use the generalized Thouless–Anderson–Palmer (TAP) approach recently developed in Ref. [39] for general spherical models. To the best of knowledge, this is the first example of a pure or mixed $p$-spin model, either with spherical or Ising spins, for which the free energy can be computed in the whole replica symmetry breaking phase without using the Parisi formula.

The Gibbs measure at inverse-temperature $\beta$ is the random measure on $S^{N-1}$,

$$G_{N,\beta}(A) := \frac{\int_A e^{\beta H_N(\sigma)} d\sigma}{\int_{S^{N-1}} e^{\beta H_N(\sigma)} d\sigma}. \quad (1.5)$$

The following notion of multi-samplable overlaps was introduced in [39]. We say that an overlap value $q \in [0, 1)$ is multi-samplable at $\beta$ if for any $k \geq 1$ and $\epsilon > 0$,

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} G_{N,\beta}^{\otimes k} \left\{ \forall i < j \leq k : |R(\sigma^i, \sigma^j) - q| < \epsilon \right\} = 0, \quad (1.6)$$

where $G_{N,\beta}^{\otimes k}$ denotes the $k$-fold product measure of $G_{N,\beta}$ with itself. In words, if the probability that each pair from $k$ i.i.d. samples from $G_{N,\beta}$ have overlap $q$, up to error $\epsilon$, is not exponentially small in $N$. We will denote by $q_{\beta}$ the maximal $q \in [0, 1)$ which is multi-samplable at $\beta$.\(^1\)

It is not difficult to check (see Sect. 1.2) that there exists a critical inverse-temperature $\beta_c > 0$ such that

$$F(\beta) = \frac{1}{2} \beta^2 v(1) \iff \beta \leq \beta_c. \quad (1.7)$$

(And, by Jensen’s inequality, $F(\beta) < \frac{1}{2} \beta^2 v(1)$ otherwise.) For the critical inverse-temperature, we will abbreviate $q_c := q_{\beta_c}$.

The energy level

$$E_\infty = E_\infty(p) := 2 \sqrt{\frac{p - 1}{p}} \quad (1.8)$$

which arises in our analysis was also relevant in several previous works [3, 7, 21, 28, 37, 38, 40, 42].

Our main results are the following two theorems.

\(^1\) Note that the limit of multi-samplable overlaps is also multi-samplable, and that (e.g. from (1.19)) for fixed $\beta$ there are no multi-samplable overlaps in a small neighborhoods of 1. Note that from the characterization (1.19) below, $q = 0$ is multi-samplable for any $\beta > 0$.\[^{\vphantom{1}}\]
Theorem 1 (Ground state energy and critical parameters). For the pure $p$-spin model $H_N(\sigma) = H_{N,p}(\sigma)$ with $p \geq 3$, $q_c$ is the unique solution in $(0,1)$ of

$$p(1-q)\log(1-q) + pq - (p-1)q^2 = 0, \quad (1.9)$$

the critical inverse-temperature is given by

$$\beta_c = \frac{q_c^{\frac{p}{2} + 1}}{\sqrt{p(1-q_c)}}, \quad (1.10)$$

and the ground state energy is given by

$$E_* = \frac{1}{2} E_{\infty} \left( \frac{1}{\sqrt{(p-1)(1-q_c)}} + \sqrt{(p-1)(1-q_c)} \right). \quad (1.11)$$

Theorem 2 (Free energy). For the pure $p$-spin model $v(q) = q^p$ with $p \geq 3$, for any $\beta > \beta_c$, $q_\beta$ is the larger of the two solutions in $(0,1)$ of

$$\beta q^{p-1} - \frac{1}{\sqrt{p(p-1)}} \left( \frac{E_*}{E_{\infty}} - \sqrt{\frac{E_{*}^2}{E_{\infty}^2} - 1} \right), \quad (1.12)$$

and with $q = q_\beta$, the free energy is given by

$$F(\beta) = \beta E_* q^{p} + \frac{1}{2} \log(1-q) + \frac{1}{2} \beta^2 \left( v(1) - v(q) - v'(q)(1-q) \right). \quad (1.13)$$

While in the theorems above we assume that $p \geq 3$, they still hold when $p = 2$, with the modification that for (1.9) and (1.12) one should take the unique solutions in $[0,1]$, $q_c = 0$ and $q_\beta = 1 - 1/\sqrt{2}\beta$. Analyzing the TAP representation in the case $p = 2$ is immediate and will be done separately in the Appendix.

1.1 The generalized TAP approach

In the late 70s, Thouless, Anderson and Palmer [48] introduced their famous approach to analyze the SK model. Their approach was further developed in physics, see e.g. [10, 11, 21–23, 28, 36], with the general idea that for large $N$, $F_N(\beta)$ is approximated by the sum of free energies associated to the ‘physical’ solutions of the TAP equations. In particular, the spherical pure $p$-spin models were analyzed non-rigorously by Kurchan, Parisi and Virasoro in [28] and Crisanti and Sommers in [21], where (1.13) was predicted as the formula for the free energy. The ground state energy $E_*$, which is needed in order to evaluate $F(\beta)$ from the latter formula, was computed in Refs. [21, 28] using the replica method or from mean ‘complexity’ calculations, while in the current paper it is obtained directly from the TAP representation. (We will show in Sect. 2 that $(E_*, \beta_c, q_c)$ solve a certain set of equations related to the TAP equations from which we will compute the value of $E_*$.)
Recently, we developed in Ref. [39] a generalized TAP approach for spherical models, which can be applied to any multi-samplable overlap. This approach was also extended to mixed models with Ising spins by Chen, Panchenko and the author [15, 16], where for example an analogue of the TAP representation below (1.19) was derived (see also [14] for an earlier result by Chen and Panchenko). In this section we describe the TAP representation for the free energy derived in Ref. [39], which we shall use in the proof of our main theorems above. We emphasize that the results we use from [39] are proved there without appealing to the landmark results in mean-field spin glasses like the Parisi formula [12, 45] or ultrametricity property [29–31].

For $m$ with $\|m\| < \sqrt{N}$ inside the sphere $\mathbb{S}^{N-1}$ and width $\delta > 0$, define the spherical band

$$\text{Band}(m, \delta) := \{ \sigma \in \mathbb{S}^{N-1} : |R(\sigma - m, m)| \leq \delta ||m||/\sqrt{N} \}.$$ 

Define the free energy on the band

$$F_{N, \beta}(m, \delta) := \frac{1}{N} \mathbb{E} \log \int_{\text{Band}(m, \delta)} e^{\beta (H_N(\sigma) - H_N(m))} d\sigma$$

and, for $n \geq 1$ and $\rho > 0$, the replicated free energy

$$F_{N, \beta}(m, \delta, n, \rho) := \frac{1}{Nn} \mathbb{E} \log \int_{\text{Band}(m, \delta, n, \rho)} e^{\beta \sum_{i=1}^{n} (H_N(\sigma_i) - H_N(m))} d\sigma_1 \cdots d\sigma_n,$$

where we define the set of $n$-tuples

$$\text{Band}(m, \delta, n, \rho) := \{ (\sigma_1, \ldots, \sigma_n) \in (\text{Band}(m, \delta))^n : |R(\sigma_i, \sigma_j) - R(m, m)| \leq \rho, \forall i \neq j \}.$$ 

Note that by definition,

$$\frac{\beta}{N} H_N(m) + F_{N, \beta}(m, \delta, n, \rho) \leq \frac{\beta}{N} H_N(m) + F_{N, \beta}(m, \delta) \leq F_{N, \beta}.$$ 

Roughly speaking, it was shown in Ref. [39] (see (1.19) and (1.20)) that as we let $\delta, \rho \to 0$ and $n \to \infty$ slowly, for large $N$ with high probability (w.h.p.) the points $m$ such that

$$\frac{\beta}{N} H_N(m) + F_{N, \beta}(m, \delta, n, \rho) \approx \frac{\beta}{N} H_N(m) + F_{N, \beta}(m, \delta) \approx F_{N, \beta},$$

are the points such that

$$\frac{1}{N} H_N(m) \approx E_*(q) := \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \max_{\|m\|^2 = Nq} H_N(m).$$
and $\|m\|^2 \approx Nq$ for some multi-samplable overlap $q \in [0, 1)$.\(^2\)

Moreover, it was shown in [39] (see Propositions 1 and 22) that as we let $\delta, \rho \to 0$ and $n \to \infty$, w.h.p. and uniformly in $m$,

$$F_{N,\beta}(m, \delta, n, \rho) \approx \frac{1}{2} \log(1 - q) + F(\beta, q), \quad (1.16)$$

where

$$F(\beta, q) := \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log \int_{\mathbb{S}^{N-1}} e^{\beta H_N^q(\sigma)} d\sigma \quad (1.17)$$

is the free energy of the Hamiltonian $H_N^q(\sigma)$ with mixture

$$\nu_q(x) := \nu(q + (1 - q)x) - \nu(q) - \nu'(q)(1 - q)x. \quad (1.18)$$

We recall that the limit as in (1.17) exists by [41].

The first term in the right-hand side of (1.16) accounts for the volume of the band

$$\lim_{\delta \to 0} \lim_{N \to \infty} \frac{1}{N} \log \text{Vol(Band}(m, \delta)) = \frac{1}{2} \log(1 - q).$$

To understand where the second term in (1.16) comes from, first note that the restriction of $H_N(\sigma) - H_N(m)$ to the sphere of co-dimension 1 Band$(m, 0)$ is another spherical Hamiltonian, up to scaling of the space. By a short computation, one finds that the mixture of this Hamiltonian is $\nu(q + (1 - q)x) - \nu(q)$. For reasons which we will not explain here, for the computation of the limit of the replicated free energy (1.16) what is relevant is the same Hamiltonian after its 1-spin interaction is removed. This amounts to subtracting $\nu'(q)(1 - q)x$ from the mixture, resulting in (1.18).

For any multi-samplable $q$, if we take any point satisfying (1.15) and $\|m\|^2 \approx Nq$, then by the above characterization of (1.14) and (1.16), we obtain in the $N \to \infty$ limit a formula for $F(\beta)$. More precisely, the following generalized TAP representation was proved in Theorem 4 of [39]: for any spherical model and $\beta > 0$,

$$F(\beta) = \beta E_\star(q) + \frac{1}{2} \log(1 - q) + F(\beta, q) \quad (1.19)$$

if and only if $q \in [0, 1)$ is multi-samplable. Moreover, if it is not multi-samplable, then

$$F(\beta) > \beta E_\star(q) + \frac{1}{2} \log(1 - q) + F(\beta, q). \quad (1.20)$$

\(^2\) Note that the limit $E_\star(q)$ exists since the restriction of $H_N(\sigma)$ to the sphere $\|m\|^2 = Nq$ is, up to scaling of the space, the Hamiltonian with mixture $\nu(q)$.\(^3\)
It was also proved in Corollary 5 of [39] that for \( q = q\beta \), the last term above is equal to the well-known Onsager reaction term

\[
F(\beta, q) = \frac{1}{2N} \beta^2 \mathbb{E} \left\{ H_N^q(\sigma)^2 \right\} = \frac{1}{2} \beta^2 \nu_q(1). \quad (1.21)
\]

By substituting (1.21) in (1.19) and noting that in the pure case \( E_*(q) = q^\beta E_* \), we obtain the formula for the free energy (1.13). To actually be able to compute the free energy from this formula, one needs to compute \( \beta_c, E_* \) and \( q\beta \) for \( \beta > \beta_c \), and this is the content of our main results, Theorems 1 and 2.

### 1.2 Earlier related works

In this section we survey earlier works where the free energy \( F(\beta) \) was computed without using the Parisi formula. For very small \( \beta \) and in the absence of an external field, one can easily obtain \( F(\beta) \) from the following argument. By Jensen’s inequality,

\[
F_N(\beta) \leq \frac{1}{N} \log \mathbb{E} Z_N(\beta) := \frac{1}{N} \log \mathbb{E} \int_{S^{N-1}} e^{\beta H_N(\sigma)} d\sigma = \frac{1}{2} \beta^2 \nu(1). \quad (1.22)
\]

For an arbitrary model, with either spherical or Ising spins, a short calculation yields that for very small \( \beta \),

\[
\lim_{N \to \infty} \frac{1}{N} \log \frac{\mathbb{E}(Z_N(\beta)^2)}{(\mathbb{E} Z_N(\beta))^2} = 0.
\]

Hence, from the Paley–Zygmund inequality and the well-known concentration of the free energy (see e.g., [32, Theorem 1.2]), in fact, (1.22) holds as equality \( F(\beta) = \beta^2 \nu(1)/2 \). It is not difficult to see\(^3\) that if \( \beta \) satisfies the latter equality, then so does any \( \beta' < \beta \). Since \( F(\beta) \) is continuous, this also explains why a critical \( \beta_c \) as in (1.7) exists.

For the SK model, by a more refined (but still short) calculation of the second moment of \( Z_N(\beta) \), Talagrand proved in [43, Section 2] that \( F(\beta) = \beta^2/2 \) for \( \beta \leq 1/\sqrt{2} \). In fact, Aizenman, Lebowitz and Ruelle proved the same earlier in [1], where they studied the fluctuations of \( Z_{N,\beta} \) (see also the work [19] of Comets and Neveu). Comets [18] showed that whenever \( \beta > 1/\sqrt{2} \), \( F(\beta) < \beta^2/2 \), and therefore Talagrand’s simple argument actually works up to the critical inverse-temperature \( \beta_c = 1/\sqrt{2} \). For the SK model in the presence of an external field, by analyzing the TAP solutions, Bolthausen [8, 9] proved that for small enough \( \beta \), \( F(\beta) \) is given by the replica-symmetric solution of the Parisi formula. For the pure \( p \)-spin model \( \nu(t) = t^p \) with Ising spins, Talagrand [44] used a truncated second moment argument to prove that \( F(\beta) = \beta^2/2 \) for any \( \beta \leq \beta_p \), for some \( \beta_p \) strictly smaller than the critical \( \beta_c \)

\(^3\) For any \( \beta' < \beta \), in distribution, \( \beta H_N(\sigma) = \beta' H'_N(\sigma) + \sqrt{\beta^2 - \beta'^2} H''_N(\sigma) \) where \( H'_N(\sigma) \) and \( H''_N(\sigma) \) are independent copies of \( H_N(\sigma) \). By conditioning on \( H'_N(\sigma) \) and applying Jensen’s inequality, one sees that \( F(\beta) \leq F(\beta') + (\beta^2 - \beta'^2) \nu(1)/2 \).
(see [13, Theorem 2] for a characterization of the replica symmetric phase for the pure p-spin models).

Moving to the spherical models, the Hamiltonian of the pure 2-spin model, $H_{N,2}(\sigma) = \sigma^T M \sigma$, depends in a simple way on a GOE matrix $M = \frac{1}{2\sqrt{N}}(J_{ij} + J_{ji})$. Kosterlitz, Thouless and Jones [27] exploited this to derive an integral formula for the free energy at finite dimension $N$ which they analyzed non-rigorously using the steepest descent method to obtain $F(\beta)$. Baik and Lee [5] provided the necessary estimates and rigorously proved the formula for $F(\beta)$ derived in [27] for all $\beta$. Belius and Kistler [6] also treated the spherical 2-spin model in the presence of an external field and computed $F(\beta)$ for all $\beta$, by proving a TAP variational formula. We emphasize, however, that the spherical pure 2-spin model is replica symmetric at all $\beta$. Namely, the measure achieving the minimum in Parisi’s formula is a delta measure. Hence, all the results we mentioned about the 2-spin model do not concern the replica symmetry breaking phase.

For the spherical pure $p$-spin model with $p \geq 3$, building on the study of critical points [3, 37, 42], we calculated in [38] the free energy for very large $\beta$. This line of works begins with the paper of Auffinger, Ben Arous and Černý [3] where they computed the mean ‘complexity’ of critical points, i.e., the expected number of critical values around any given energy. By Markov’s inequality, the threshold energy beyond which the mean complexity is negative, upper bounds the ground state energy $E_*$. Using the Parisi formula, it was proved in Ref. [3] that in fact this threshold is equal to $E_*$, thus characterizing its value as the solution of an explicit equation. In Ref. [37] we showed that the complexity concentrates around its mean by a second moment calculation. In particular, this gave a proof for the fact that $E_*$ and the aforementioned threshold coincide, independent of the Parisi formula. With Zeitouni we studied in Ref. [42] the extremal point process of critical values near $NE_*$ and showed that it converges to a Poisson process. Finally, in [38] we proved that for very large $\beta$, the free energy is given by (1.13) where $q$ is defined through (1.12). A similar formula to (1.13) for the free energy at very large $\beta$, with $q$ solving an analogue of (1.12), was proved by Ben Arous, Zeitouni and the author [7] for mixed models which are close enough to the pure $p$-spin models. The ground state energy was obtained in the same paper [7] by a second moment argument, while the complexity for mixed spherical models was calculated earlier by Auffinger and Ben Arous in [2].

We reiterate that, excluding [7, 38], all the results about the free energy above concern only the replica symmetric phase. While [7, 38] do concern the symmetry breaking phase, they only cover a part of the phase. Moreover, in contrast to the approach in this work, these are quite technically heavy papers, based on non-trivial computations and results from previous works about critical points [2, 3, 37, 42].

## 2 The basic equations for multi-samplable overlaps

In this section we prove the following equations for $(\beta_c, q_c, E_*)$ and for $(\beta, q_\beta, E_*)$ with $\beta > \beta_c$. We will use them in Sects. 3 and 4 to prove Theorems 1 and 2.
For the pure $p$-spin model $v(q) = q^p$ with $p \geq 3$, the triple $(\beta_c, q_c, E_\star)$ solves
\begin{align}
\frac{1}{1 - q} + \beta^2 (1 - q) v''(q) &= \beta p q^{\frac{p}{2} - 1} E, \\
\beta^2 \left( v(q) + (1 - q)v'(q) \right) &= \beta q^{\frac{p}{2}} E = -\log(1 - q).
\end{align}

Moreover, for any $p \geq 2$ and $\beta > \beta_c$, $(\beta, q_\beta, E_\star)$ solves (2.1).

We will prove the proposition in Sect. 2.3. As an intermediate step, we will first prove the following two lemmas in Sects. 2.1 and 2.2. The existence of the derivatives is part of statement.

**Lemma 4** For the pure $p$-spin model with $p \geq 2$, $F'(\beta_c) = \beta_c v(1)$ and for $(\beta, q) = (\beta_c, q_c)$,
\begin{equation}
\frac{d}{d\beta} F(\beta, q) = \frac{d}{d\beta} \frac{1}{2} \beta^2 v_q(1) = \beta \left( v(1) - v(q) - v'(q)(1 - q) \right).
\end{equation}

**Lemma 5** For the pure $p$-spin model with $p \geq 3$ if $\beta \geq \beta_c$, then $q_\beta > 0$ and with $q = q_\beta$,
\begin{equation}
\frac{d}{dq} F(\beta, q) = \frac{d}{dq} \frac{1}{2} \beta^2 v_q(1) = -\frac{1}{2} \beta^2 (1 - q) v''(q).
\end{equation}

For $p = 2$, the same holds for $\beta > \beta_c$.

Equations (2.3) and (2.4) state that the derivatives of $F(\beta, q)$ are given by formally taking the derivative of $\frac{1}{2} \beta^2 v_q(1)$. Indeed, by (1.21) we have that $F(\beta, q) = \frac{1}{2} \beta^2 v_q(1)$ for the maximal multi-samplable overlap $q = q_\beta$. Note, however, that it is not necessarily true that $F(\beta, q) = \frac{1}{2} \beta^2 v_q(1)$ on a neighborhood of $(\beta, q_\beta)$ so a priori the derivatives may be different from the above.

### 2.1 Proof of Lemma 4

Let us first show that $F'(\beta_c) = \beta_c v(1)$. Recall that $F(\beta) = \frac{1}{2} \beta^2 v(1)$ whenever $\beta \leq \beta_c$ and thus the one-sided derivative from the left is $\frac{d}{d\beta}^- F(\beta_c) = \beta_c v(1)$. For any $\beta$, $F(\beta) \leq \frac{1}{2} \beta^2 v(1)$, and therefore we have an upper bound on the derivative from the right $\frac{d}{d\beta}^+ F(\beta_c) \leq \beta_c v(1)$. Lastly, by Hölder’s inequality $F(\beta)$ is convex, and therefore $\frac{d}{d\beta}^- F(\beta_c) \leq \frac{d}{d\beta}^+ F(\beta_c).$ This of course shows that $F'(\beta_c) = \beta_c v(1)$.

Next we prove (2.3). Suppose that $p \geq 2$, $\beta = \beta_c$ and $q = q_c$ is the maximal multi-samplable overlap. Recall that $F(\beta, q)$ is the limiting free of the model with mixture $v_q(t)$. Note that by (1.21), $\beta = \beta_c$ is less than or equal to the critical inverse-temperature of this model. If it strictly smaller, then $F(\beta', q) = \frac{1}{2} \beta^2 v_q(1)$ for any $\beta'$ in some small neighborhood of $\beta$, and therefore (2.3) follows. If $\beta$ is equal to the critical inverse-temperature of $v_q(t)$, then (2.3) follows by the same argument we used above to prove that $F'(\beta_c) = \beta_c v(1)$.

\[ \square \]
2.2 Proof of Lemma 5

Denote by $F_{N, \beta}^q$ the free energy of $H_N^q(\sigma)$ and recall that $F(\beta, q) = \lim_{N \to \infty} \mathbb{E} F_{N, \beta}^q$, see (1.17). As we explain below, for any given $N$, we may express the derivative $\frac{d}{dq} \mathbb{E} F_{N, \beta}^q$ using the overlap distribution under the Gibbs measure associated to $H_N^q(\sigma)$. By Lemmas 32 and 33 in [39], for a maximal multi-samplable overlap $q$, as $N \to \infty$ the Gibbs measure concentrates at zero. By combining those facts, we can compute the limit of the derivative $\lim_{N \to \infty} \frac{d}{dq} \mathbb{E} F_{N, \beta}^q$. But we need to compute the derivative of the limit $\lim_{N \to \infty} \frac{d}{dq} \mathbb{E} F_{N, \beta}^q$.

Griffiths’ lemma states that if $f_N(x)$ are real convex differentiable functions converging pointwise in an interval to a (convex) function $f(x)$, then $\lim_{N \to \infty} f'_N(x) = f'(x)$ at every point $x$ where $f(x)$ is differentiable (see [47, p. 483]). Unfortunately, $\mathbb{E} F_{N, \beta}^q$ is not convex in $q$ so we cannot apply the lemma directly. Our solution to this will be to replace the Hamiltonian $H_N^q(\sigma)$ by a linearized version of it with the same derivative for the $N \to \infty$ limit, (see (2.6)) which does satisfy the conditions of Griffiths’ lemma. This will allow us to make the interchange of limit and differentiation in the final step of the proof (see (2.11)).

Note that if $q_\beta = 0$ then by the generalized TAP representation (1.19) and (1.21), $\beta \leq \beta_c$. Hence, for any $\beta > \beta_c$ we have that $q_\beta > 0$. We will first prove the lemma assuming explicitly that $q_\beta > 0$. This will imply that the lemma holds for any $\beta > \beta_c$.

Let $p \geq 2$ and $\beta \geq \beta_c$ and assume until said otherwise that $q_\beta > 0$. Recall that $H_N^q(\sigma)$ is the Hamiltonian corresponding to the mixture $\nu_q(x)$ and note that $\nu_q(x) = \sum_{k=2}^p \alpha_k^2(q) x^k$, for $\alpha_k^2(q) = \left(\frac{p}{k}\right)(1 - q)^k q^{p-k}$. We may therefore write

$$H_N^q(\sigma) = \sum_{k=2}^p \alpha_k(q) H_{N,k}(\sigma), \quad (2.5)$$

where the pure $k$-spin models $H_{N,k}(\sigma)$ are independent for different $k$.

For $\epsilon \in \mathbb{R}$, define the linearized Hamiltonian we mentioned above by

$$H_N^{q, \epsilon}(\sigma) := \sum_{k=2}^p \left( \alpha_k(q) + \epsilon \frac{d}{dq} \alpha_k(q) \right) H_{N,k}(\sigma) \quad (2.6)$$

and denote its free energy by

$$F_N(\beta, q, \epsilon) := \frac{1}{N} \mathbb{E} \log \int e^{\beta H_N^{q, \epsilon}(\sigma)} d\sigma.$$
Denote

\[ F(\beta, q, \epsilon) := \lim_{N \to \infty} F_N(\beta, q, \epsilon), \]

where the limit exists by [41].

From Hölder’s inequality, for real \( s \) and \( t \) and \( \lambda \in (0, 1) \),

\[
\log \int e^{\beta H_N^{q_\lambda t + (1-\lambda)s}(\sigma)}d\sigma \\
= \log \int \exp \left( \beta \lambda H_N^{q_\lambda t}(\sigma) + \beta (1-\lambda) H_N^{q_s}(\sigma) \right) d\sigma \\
\leq \lambda \log \int \exp \beta H_N^{q_\lambda t}(\sigma)d\sigma + (1-\lambda) \log \int \exp \beta H_N^{q_s}(\sigma)d\sigma.
\]

Hence, \( F_N(\beta, q, \epsilon) \) and \( F(\beta, q, \epsilon) \) are convex functions of \( \epsilon \).

Write \( H_N^{q_+}(\sigma) - H_N^{q_-}(\sigma) = \sum_{k=2}^{p} t_k H_{N,k}(\sigma) \) with

\[
t_k := \alpha_k(q + \epsilon) - \alpha_k(q) - \epsilon \frac{d}{dq} \alpha_k(q).
\]

Since \( t_k = O(\epsilon^2) \) for small \( \epsilon \), e.g. by [15, Lemma 20], for some constant \( C > 0 \),

\[
\frac{1}{N} \mathbb{E} \sup_{\sigma} |H_N^{q_+}(\sigma) - H_N^{q_-}(\sigma)| < C \epsilon^2.
\]

Combined with the Borell-TIS inequality this implies that for small \( \epsilon \),

\[
\left| \frac{1}{N} \mathbb{E} \log \int e^{\beta H_N^{q_+}(\sigma)}d\sigma - \frac{1}{N} \mathbb{E} \log \int e^{\beta H_N^{q_-}(\sigma)}d\sigma \right| < c \epsilon^2, \tag{2.7}
\]

for some constant \( c > 0 \) independent of \( N \). Therefore,

\[
|F(\beta, q + \epsilon) - F(\beta, q, \epsilon)| < c \epsilon^2.
\]

Hence, for \( q \in (0, 1) \),

\[
\frac{d}{dq}^{\pm} F(\beta, q) = \frac{d}{d\epsilon}^{\pm} F(\beta, q, 0), \tag{2.8}
\]

where \( \frac{d}{dq}^{\pm} \) and \( \frac{d}{d\epsilon}^{-} \) denote the one-sided derivatives from the right and left, respectively. Note that the one sided derivatives on the right-hand side exist from convexity, and therefore using (2.7) so do those on the left-hand side.

Also from convexity,

\[
\frac{d}{d\epsilon}^{-} F(\beta, q, 0) \leq \frac{d}{d\epsilon}^{+} F(\beta, q, 0).
\]
On the other hand, all the terms other than \( F(\beta, q) \) in the TAP representation (2.12) are differentiable in \( q \). Hence, since the TAP representation holds for any \( q \) as an inequality and at \( q = q_\beta \) as equality,

\[
\frac{d}{dq} F(\beta, q_\beta) \geq \frac{d}{dq} F(\beta, q_\beta),
\]

where here we used the assumption that \( q_\beta > 0 \). Thus, all the one-sided derivatives above and therefore also the usual derivatives exist and are equal

\[
\frac{d}{dq} F(\beta, q_\beta) = \frac{d}{d\epsilon} F(\beta, q_\beta, 0). \tag{2.9}
\]

Let \( G_{N,\beta,q} \) denote the Gibbs measure associated to \( H_N^q(\sigma) = H_N^q,0(\sigma) \) at inverse-temperature \( \beta \). From [32, Lemma 1.1],

\[
\frac{d}{d\epsilon} F_N(\beta, q, 0) = \frac{\beta}{N} \mathbb{E} \left( \sum_{k=2}^{p} \frac{d}{dq} \alpha_k(q) H_{N,k}(\sigma^1) \right)
\]

\[
= \frac{\beta^2}{N} \mathbb{E} \left( C(\sigma^1, \sigma^1) - C(\sigma^1, \sigma^2) \right),
\]

where \( \langle \cdot \rangle \) denotes integration w.r.t. \( G_{N,\beta,q} \) and

\[
C(\sigma^1, \sigma^2) := \mathbb{E} \left\{ \sum_{k=2}^{p} \frac{d}{dq} \alpha_k(q) H_{N,k}(\sigma^1) \cdot H_N^q(\sigma^2) \right\}
\]

\[
= N \sum_{k=2}^{p} \alpha_k(q) \frac{d}{dq} \alpha_k(q) R(\sigma^1, \sigma^2)^k = N \frac{1}{2} \frac{d}{dq} v_q(R(\sigma^1, \sigma^2)).
\tag{2.10}
\]

For an appropriate constant \( c > 0 \) and any \( \sigma^1 \) and \( \sigma^2 \),

\[
C(\sigma^1, \sigma^1) = N \frac{d}{2} v_q(1),
\]

\[
|C(\sigma^1, \sigma^2)| \leq N c \left| R(\sigma^1, \sigma^2) \right|.
\]

From Lemmas 32 and 33 in [39], at \( q = q_\beta \), for any \( \delta > 0 \),

\[
\lim_{N \to \infty} \mathbb{E} G_{N,\beta,q}^{\otimes 2} \left\{ |R(\sigma^1, \sigma^2)| > \delta \right\} = 0.
\]

Therefore,

\[
\lim_{N \to \infty} \frac{d}{d\epsilon} F_N(\beta, q_\beta, 0) = \frac{d}{dq} \bigg|_{q=q_\beta} \frac{1}{2} \beta^2 v_q(1).
\]
By Griffiths’ lemma stated in the beginning of the proof,

\[
\frac{d}{d\epsilon} F(\beta, q_\beta, 0) = \frac{d}{d\epsilon} \left|_{\epsilon=0} \right. \lim_{N \to \infty} F_N(\beta, q_\beta, \epsilon) = \lim_{N \to \infty} \frac{d}{d\epsilon} F_N(\beta, q_\beta, 0), \tag{2.11}
\]

which proves (2.4).

At this point, we proved Lemma 5 (either for \( \beta > \beta_c \) or \( \beta = \beta_c \)), provided that \( q_\beta > 0 \). As explained in the beginning of the proof, this implies the lemma for any \( p \geq 2 \) and \( \beta > \beta_c \). The case \( p \geq 3 \) and \( \beta = \beta_c \) follows from the following lemma.

**Lemma 6** For the pure \( p \)-spin model with \( p \geq 3 \), \( q_c > 0 \).

**Proof** Recall that the generalized TAP representation (1.19) and (1.20) states that for any \( \beta \) and \( q \in [0, 1) \),

\[
F(\beta) \geq \beta q^{p/2} E_\star + \frac{1}{2} \log(1 - q) + F(\beta, q), \tag{2.12}
\]

and that there is equality for any multi-samplable overlap, and in particular for \( q = q_\beta \). We therefore have that if \( q_\beta > 0 \), then the derivative in \( q \) of the two sides of (2.12) is equal at \( q = q_\beta > 0 \). In the proof above we saw that if \( q = q_\beta > 0 \), then (2.4) holds.

By combining these two facts, for \( p \geq 2 \) and any \( \beta \) we have that if \( q = q_\beta > 0 \), then with \( E = E_\star \),

\[
\beta \frac{p}{2} q^{\frac{p}{2} - 1} E - \frac{1}{2} \log(1 - q) - \frac{1}{2} \beta^2 (1 - q) v''(q) = 0. \tag{2.13}
\]

Now let \( p \geq 3 \). Let \( \beta_k > \beta_c \) be a sequence such that \( \beta_k \searrow \beta_c \). Since \( \beta_k > \beta_c \), as we explained in the proof of Lemma 5 above, \( q_{\beta_k} > 0 \). Hence, \((\beta_k, q_{\beta_k}, E_\star)\) solves (2.13) for each \( k \). Since \( \beta_k \) is a bounded sequence, using the fact that \( p \geq 3 \), we conclude from (2.13) that for small enough \( \epsilon > 0 \), \( q_{\beta_k} > \epsilon \) for all \( k \). Since the logarithmic term in (1.19) goes to \( -\infty \) as \( q \to 1 \), it is easy to see that \( \sup_{k \geq 1} q_{\beta_k} \leq 1 - \epsilon \) for sufficiently small \( \epsilon > 0 \). Since equality in (1.19) characterizes multi-samplable overlaps and \( \beta \mapsto F(\beta) \) and \((\beta, q) \mapsto F(\beta, q)\) are continuous, any subsequential limit \( q \in [\epsilon, 1 - \epsilon] \) of \( q_{\beta_k} \), is multi-samplable at \( \beta = \beta_c \). In particular, \( q = 0 \) is not the maximal multi-samplable overlap at \( \beta_c \).

\( \Box \)

### 2.3 Proof of Proposition 3

Recall that for any \( p \geq 2 \) and \( \beta > \beta_c \) we have that \( q_\beta > 0 \). By Lemma 6, for \( p \geq 3 \) and \( \beta = \beta_c \) we also have that \( q_\beta > 0 \). As we saw in the proof of Lemma 6, for either of those two choices for \( p \) and \( \beta \), (2.13) holds with \((q, E) = (q_\beta, E_\star)\), from which (2.1) immediately follows.

Now let \( p \geq 3 \). Recall that \( \beta_c > 0 \) and fix \( q = q_c \). Since the TAP representation holds for any \( \beta \) as an inequality (2.12) and at \( \beta_c \) as equality, we have that the derivatives in \( \beta \) of the two sides of (2.12) are equal at \( \beta = \beta_c \). Combined with Lemma 4, this gives that \((\beta_c, q_c, E_\star)\) solves

\[
\beta v(1) = q^\frac{p}{2} E + \beta \left( v(1) - v(q) - v'(q)(1 - q) \right),
\]
from which the first equality of (2.2) follows.

For \((\beta, q, E) = (\beta_c, q_c, E_\ast)\), (2.12) holds with equality, \(F(\beta, q)\) is given by (1.21), and \(F(\beta) = \frac{1}{2} \beta^2 v(1)\). Hence,

\[
\frac{1}{2} \beta^2 v(1) = \beta q E + \frac{1}{2} \log(1 - q) + \frac{1}{2} \beta^2 \left( v(1) - v(q) - (1 - q) v'(q) \right).
\]

By canceling \(\frac{1}{2} \beta^2 v(1)\) from both sides and using the first equality of (2.2), we obtain

the second equality of (2.2).

□

3 Proof of Theorem 1

Let \((\beta, q, E)\) be a solution of the equations in Proposition 3. By (2.1), \(q > 0\). Hence, from (2.2),

\[
\beta pq^\frac{p}{q} E = -\frac{p}{q} \log(1 - q),
\]

\[
\beta^2 = -\frac{\log(1 - q)}{v(q) + (1 - q) v'(q)}.
\]

By substituting this in (2.1), we obtain that

\[
\frac{1}{1 - q} - \frac{\log(1 - q)(1 - q) v'\left(q\right)}{v(q) + (1 - q) v'(q)} = -\frac{p}{q} \log(1 - q). \tag{3.1}
\]

Using the fact that \(v(q) = q^p\), after some algebra we obtain that \(q\) solves (1.9) which we recall, for the convenience of the reader,

\[
a(q) := p(1 - q) \log(1 - q) + pq - (p - 1)q^2 = 0.
\]

Thus, since \((\beta_c, q_c, E_\ast)\) solves the equations of Proposition 3, \(q_c\) solves (1.9) and \(a(0) = a(q_c) = 0\). To prove that there are no other solutions \(a(q) = 0\), it will be enough to show that \(a'(q) = 0\) for at most one point in \((0, 1)\) (and therefore exactly one point).

One can check that \(a'(q) = 0\) if and only if

\[
b(q) := -\frac{\log(1 - q)}{q} = \frac{2(p - 1)}{p}.
\]

We note that \(b(q)\) is strictly increasing on \((0, 1)\), since

\[
b'(q) = \frac{q + \log(1 - q)(1 - q)}{q^2(1 - q)}
\]

is a ratio of positive numbers for \(q \in (0, 1)\). Hence, indeed \(a'(q) = 0\) for one point \(q \in (0, 1)\) at most.
To prove (1.10), we first use (2.1) and the first equality of (2.2) to obtain that, for \((q, \beta) = (q_c, \beta_c)\),

\[
\frac{q}{1-q} + \beta^2 q (1-q) v''(q) = p \beta^2 \left( v(q) + (1-q) v'(q) \right).
\]

Substituting \(v(q) = q^p\) and rearranging yields

\[
\beta^2 = \frac{1}{p(1-q)q^{p-2}},
\]

from which (1.10) follows.

Lastly, by substituting (1.10) and \(v(q) = q^p\) in (2.1) we obtain that, with \((q, E) = (q_c, E^\star)\),

\[
\frac{1}{1-q} + (p-1) = \sqrt{\frac{p}{(1-q)}} E,
\]

from which (1.11) follows. \(\square\)

### 4 Proof of Theorem 2

Throughout the proof assume that \(\beta \geq \beta_c\). First, note that the equality (1.13) follows from the TAP representation (1.19) and (1.21). Next, recall that by Proposition 3, \(q = q_\beta\) satisfies

\[
\frac{1}{1-q} + \beta^2 (1-q)p(p-1)q^{p-2} = \beta p q^{\frac{p}{2}-1} E^\star.
\]

As we will see in a moment, this equation has four solutions in \(q\), for large enough \(\beta\). We will show that \(q = q_\beta\) cannot be equal to three of them. The remaining solution, to which \(q_\beta\) has to be equal, is the solution from the statement of Theorem 2.

There are two reasons for the multiplicity of the solutions of (4.1). First, if we define \(t := \beta q^{\frac{p}{2}-1}(1-q)\), then from (4.1) we obtain the equation

\[
p(p-1)t^2 - p E^\star t + 1 = 0,
\]

which has two solutions

\[
t_\pm = \frac{1}{\sqrt{p(p-1)}} \left( \frac{E^\star}{E_\infty} \pm \sqrt{\frac{E^2_\infty}{E_\infty^2} - 1} \right),
\]

where we recall that \(E_\infty := 2 \sqrt{\frac{p-1}{p}}\) (see (1.8)). Second, for \(t = t_\pm\) there are two solutions in \(q\) for \(t = \beta q^{\frac{p}{2}-1}(1-q)\), assuming \(\beta\) is large enough.
To exclude one of the values for \( t \), we will use the fact that since for \( q = q_\beta \) the Hamiltonian \( H^q_N(\beta) \) with mixture \( \nu_\beta(x) \) is in the replica symmetric phase at inverse-temperature \( \beta \), the 2-spin component of \( H^q_N(\beta) \) alone has to be in the replica symmetric phase as well. For the 2-spin model it is well-known that \( \beta_c = 1/\sqrt{2} \), see [5, 45]. We will also prove this fact in the Appendix, using the TAP representation. By combining these two facts, we will prove the following lemma in Sect. 4.1.

**Lemma 7** For the pure \( p \)-spin model with \( p \geq 3 \) and any \( \beta \),

\[
\beta q_\beta^{-1} (1 - q_\beta) \leq \frac{1}{\sqrt{p(p - 1)}}.
\]

Since \( q_\beta \) solves \((4.1)\), in particular, there exists a solution in \((0, 1)\) to \((4.1)\). In light of \((4.3)\), we therefore must have that \( E_* \geq E_\infty \). (For another proof for this inequality, by construction, see [40].) Note that the function \( x \mapsto x - \sqrt{x^2 - 1} \) is decreasing in \( x \geq 1 \). Thus,

\[
t_- \leq \frac{1}{\sqrt{p(p - 1)}} \leq t_+.
\]

Hence, from the lemma above, only the solution \( t_- \) is relevant to \( q_\beta \). I.e., we have that

\[
\beta q_\beta^{-1} (1 - q_\beta) = \frac{1}{\sqrt{p(p - 1)}} \left( \frac{E_*}{E_\infty} - \sqrt{\frac{E_\infty^2}{E_*^2} - 1} \right).
\]

Denote \( f(q) = q_\beta^{-1} (1 - q) \) and \( \ell = \frac{p - 2}{p} \). The function \( f(q) \) satisfies

\[
f(0) = f(1) = 0, \quad \text{sgn}(f'(q)) = \text{sgn}(\ell - q), \quad \text{on } (0, 1).
\]

In particular, \( f(q) \) is maximal on \([0, 1]\) at \( \ell \). For any \( \beta \geq \beta_c \), and specifically for \( \beta = \beta_c \), \( q_\beta \) solves \( f(q) = t_- / \beta \). Therefore, we have that \( \beta_c \geq t_- / f(\ell) \). For any \( \beta > \beta_c \), \( \beta > t_- / f(\ell) \) and there are exactly two solutions in \( q \) to \( f(q) = t_- / \beta \), one in \((0, \ell)\) and the other in \((\ell, 1)\). Denote the smaller of the two by \( q_\beta^- \) and the larger by \( q_\beta^+ \).

Assume towards contradiction that there exist \( \beta_i < \beta_1 < \beta_2 \) such that \( q_{\beta_i} = q_{\beta_2}^- \). Since \( \beta \mapsto q_\beta^- \) is a strictly decreasing function, \( q_{\beta_2^-} > q_{\beta_1}^- \). Recall that by (1.21), for \( \beta = \beta_i \) and \( q = q_{\beta_i}^- \),

\[
F(\beta, q) = \frac{1}{2} \beta^2 v_q(1) = \frac{1}{2} \beta^2 \left( v(1) - v(q) - v'(q)(1 - q) \right).
\]

Note that if we denote by \( \beta_c(q) \) the critical inverse-temperature that corresponds to the mixture \( v_q(x) \), this exactly means that \( \beta_i \leq \beta_c(q_{\beta_i}^-) \). For the smaller of the two inverse temperatures \( \beta_1 \) we have both

\[
\beta_1 < \beta_2 \leq \beta_c(q_{\beta_2}^-) \quad \text{and} \quad \beta_1 \leq \beta_c(q_{\beta_1}^-).
\]
Thus, (4.5) holds at $\beta = \beta_1$ with both $q = q_{\beta_1}^-$ and $q = q_{\beta_2}^-$. Hence, from the TAP representation (1.19)–(1.20),

$$F(\beta_1) = g(\beta_1, q_{\beta_1}^-) \quad \text{and} \quad F(\beta_1) \geq g(\beta_1, q_{\beta_2}^-),$$

(4.6)

where we denote

$$g(\beta, q) = \beta E^q \frac{p}{2} + \frac{1}{2} \log(1 - q) + \frac{1}{2} \beta^2 \left( \nu(1) - \nu(q) - \nu'(q)(1 - q) \right).$$

Now note that (4.1) is the equation for $d/dq g(\beta, q) = 0$. From (4.2), (4.4) and the fact that $t_\minus \leq t_\plus$, it follows that whenever there is a solution to (4.1), the smallest of the solutions (four at most) is $q_{\beta}^-$. Since the derivative from the right of $g(\beta, 0)$ at $q = 0$ is equal to $-\frac{1}{2}$, $q \mapsto g(\beta_1, q)$ is strictly decreasing on $[0, q_{\beta_1}^-]$. Hence,

$$g(\beta_1, q_{\beta_1}^-) < g(\beta_1, q_{\beta_2}^-),$$

in contradiction to (4.6).

Hence, there exists one value $\beta > \beta_c$ at most such that $q_{\beta} = q_{\beta}^-$. Assume towards contradiction that $\beta$ is such. For any $\beta' > \beta$, we have that $q_{\beta'} = q_{\beta'}^+$. Since equality in (1.19) characterizes multi-samplable overlaps, $\lim_{\beta' \searrow \beta} q_{\beta'}^+ = q_{\beta}^-$ is a multi-samplable overlap of $\nu(x)$ at $\beta$, in contradiction to the fact that $q_{\beta}^-$ is the largest multi-samplable overlap. This proves that for there is no $\beta > \beta_c$ such that $q_{\beta} = q_{\beta}^-$, which completes the proof of (1.12). \qed

4.1 Proof of Lemma 7

Let $\beta \geq 0$ and throughout the proof WLOG assume that $q = q_{\beta} > 0$. Writing $v_q(x) = \sum_{k=2}^{p} \alpha_k^2(q) x^k$ for $\alpha_k^2(q) = \frac{p}{k} (1 - q)^k q^{p-k} > 0$, we have the equality in distribution as in (2.5). By Jensen’s inequality,

$$\frac{1}{N} E \log \int e^{\beta H_{N,2}(\sigma)} d\sigma = \frac{1}{N} E \left\{ \log \int e^{\beta \sum_{k=2}^{p} \alpha_k(q) H_{N,2}(\sigma)} d\sigma \bigg| H_{N,2}(\sigma) \right\} \leq \frac{1}{N} E \left\{ \log E \left[ \int e^{\beta \sum_{k=2}^{p} \alpha_k(q) H_{N,2}(\sigma)} d\sigma \bigg| H_{N,2}(\sigma) \right] \right\} = \frac{1}{N} E \left\{ \log \int e^{\beta \alpha_2(q) H_{N,2}(\sigma)} d\sigma \right\} + \frac{1}{2} \beta^2 \sum_{k=3}^{p} \alpha_k^2(q) \leq \frac{1}{2} \beta^2 \sum_{k=2}^{p} \alpha_k^2(q) = \frac{1}{2} \beta^2 v_q(1).$$
Recall that as \( N \to \infty \) the first term above converges to \( F(\beta, q) = \frac{1}{2} \beta^2 v_q(1) \), see (1.21). Hence,

\[
\lim_{N \to \infty} \frac{1}{N} \sum \log \int e^{\beta \alpha_2(q) H_{N,2}(\sigma)} d\sigma = \frac{1}{2} \beta^2 \alpha_2^2(q).
\]

We note that
\[
\alpha_2^2(q) = \frac{1}{2} v_q''(0) = \frac{1}{2} (1 - q)^2 v''(q) = \frac{1}{2} p(p-1)(1-q)^2 q^{-2}.
\]

And recall that for the pure 2-spin, the critical inverse-temperature is \( \beta_c = 1/\sqrt{2} \) (see Appendix or [5, 45]). Therefore,
\[
\beta \alpha_2(q) = \sqrt{\frac{p(p-1)}{2}} \beta (1-q) q^{-1} \leq \frac{1}{\sqrt{2}},
\]
which proves the lemma. \( \Box \)

**Appendix: the case \( p = 2 \)**

In this appendix we treat the spherical pure 2-spin model. Assume that \( \beta > \beta_c \). Recall that by Proposition 3, \((\beta, q_{\beta}, E_* )\) solves (2.1). Also note that \( H_{N,2}(\sigma) = \sigma^T M \sigma \) where \( M = \frac{1}{2\sqrt{N}} (J_{i,j} + J_{j,i}) \) is a GOE matrix, normalized so that the limiting spectrum is supported on \([-\sqrt{2}, \sqrt{2}]\), and thus \( E_* = \sqrt{2} \). Substituting this in (2.1) and rearranging we obtain that for \( q = q_\beta \),

\[
2\beta^2 (1-q)^2 - 2\sqrt{2}\beta (1-q) + 1 = \left( \sqrt{2}\beta (1-q) - 1 \right)^2 = 0.
\]

Hence,
\[
q_\beta = 1 - \frac{1}{\sqrt{2}\beta}.
\] (4.7)

And, for \( \beta \gg \beta_c \), from the TAP representation and (1.21),

\[
F(\beta) = \sqrt{2}\beta q_\beta + \frac{1}{2} \log(1 - q_\beta) + \frac{1}{2} \beta^2 (1-q_\beta)^2 \nonumber \\
= \sqrt{2}\beta - \frac{1}{2} \log \beta - \frac{1}{4} \log 2 - \frac{3}{4}.
\]

From continuity of \( F(\beta) \), the \( \beta \searrow \beta_c \) limit of the formula above has to coincide with the replica symmetric free energy \( \frac{1}{2} \beta_c^2 \). That is,

\[
\frac{1}{2} \beta_c^2 = \sqrt{2}\beta_c - \frac{1}{2} \log \beta_c - \frac{1}{4} \log 2 - \frac{3}{4}.
\] (4.8)
The difference of the two sides of (4.8) is a strictly monotone function of $\beta_c$ on $[0, \infty)$, and therefore there is a unique value $\beta_c$ that satisfies (4.8). By substitution, we see that $\beta_c = \frac{1}{\sqrt{2}}$.

Lastly, assume towards contradiction that $q_c > 0$. Then, (2.13) and therefore (2.1) hold with $(\beta, q, E) = (\beta_c, q_c, E_\star)$. Hence, by the same argument as above we obtain that (4.7) holds with $\beta_c$, in contradiction to the fact that $q_c > 0$ and $\beta_c = \frac{1}{\sqrt{2}}$. We therefore conclude that $q_c = 0$.

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