Solving Unconstrained Global Optimization Problem Using Parameter Free Filled Function in Cooperation with Jameson Gradient Based Method

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Abstract. Global optimization problem still becomes a challenges due to the problem on locating the global optimum of multimodal function. How to reach the better minimizer from the current minimizer and how to decide that the obtained minimizer is the desired one are both major challenges on solving global optimization problem. Filled function method is one of the recent considered deterministic easy applied methods which concerned to the mentioned problems. The basic concept of filled function method is firstly by minimizing the objective function (first phase) then to build such an auxiliary function which to be minimized (second phase) in order to locate a point with lower function value than the current minimizer of the objective function. In the second phase, a local minimization method can be applied. Newton’s method is considered to be fast method on finding the zero of gradient of quadratic function, but may be very expensive or infeasible to determine the Hessian matrix in the case of complex problems. The Jameson gradient based method is the search procedures which avoid the need to store an estimate of the Hessian as well as its inverse and do not require exact line searches. In this paper, an algorithm in cooperation of parameter free filled function method and Jameson gradient method are introduced for solving global optimization problem with two variables. The algorithm is implemented to some benchmark test function. The numerical performance of the method on solving two-dimensional global optimization problems is presented.

1. Introduction
In Mathematical programming, global optimization problems has become a high concern topic, since the application spread widely such as in engineering, biology, medicine, and many more. Generally, the difficulty of designing algorithm method for global optimization is how to determine the obtained minimizer is a global one.
Filled function method firstly introduced by Ge [1], with the two adjustable parameter. A number of filled functions modifications from [1] were proposed. However, there is no efficient criterion to choose the parameter appropriately. Then a question appears to be; Is it possible to build a free-parameter filled function? Ma et al. [2] proposed a parameter free filled function with n variable, however this function discontinuous in particular point. Mohd et al. [3], have proposed a parameter free filled function which is successfully solving single variable multi modal of general global optimization problems. Referring to this idea of filled function, Napitupulu et al [4] proposed a parameter free filled function, combining together with partial radius of curvature, Newton method, steepest descent for solving two variables of twice continuously differentiable objective function. Napitupulu and Mohd [5] proposed the modification of vector direction in [4] for solving two variables of twice continuously differentiable objective function. The filled function is more preferred to be used among others deterministic method since more efficient in computationally and several advantages.

2. Filled Function Method

The filled function considered in this paper is a special function as a bridge for one point to jump from a basin to a lower basin until a global minimizer of a considered multimodal function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^1 \) can be found. The following three assumptions are applied for this process.

Assumption 1 \( f(x) \) is a twice continuously differentiable function

Assumption 2 \( f(x) \) has only a finite number of minimizers

Assumption 3 \( f(x) \rightarrow \infty \) as \( \|x\| \rightarrow \infty \)

The Assumption 3 implies the existence of a closed compact bounded domain \( D \subset \mathbb{R}^n \) such that \( D \) contains all isolated minimizers of \( f(x) \) and the value of \( f(x) \) when \( x \) is on the boundary of \( D \) is greater than any values of \( f(x) \) when \( x \) is in the interior of \( D \).

The special parameter free filled function of \( f(x) \) at the isolated local minimizer \( x^* \) so called IG’s function proposed in [3] is defined by

\[
F_{IG}(x,x^*) = \begin{cases} 
\int_{x}^{x^*} f(s) - f(x^*) ds & (x^* \leq x) \\
\int_{x^*}^{x} f(s) - f(x^*) ds & (x < x^*) 
\end{cases}
\]

The IG’s function is slightly modified as follows. Suppose that a point \( x_0 \) is chosen from the domain of \( f(x) \) to be used as a starting point for minimization of \( f(x) \) to obtain a local isolated minimizer \( x_0^* \).

Consider a function \( f : X \subset \mathbb{R}^2 \Rightarrow \mathbb{R} \) where \( X \) is a box defined by

\[
X = \{ (x_1,x_2) | x_1L \leq x_1 \leq x_1S \land x_2L \leq x_2 \leq x_2S \} = \{ y | y_1 = [x_1L,x_1S] , y_2 = [x_2L,x_2S] \}
\]

where \( x_{1L} \) and \( x_{1S} \) \( (j=1,2) \) are the infimum and supremum of the interval \( x_j = [x_{1L},x_{1S}] \) respectively.

Consider a function \( F:D \subset X \subset \mathbb{R}^2 \Rightarrow \mathbb{R} \) where \( D \) is a box defined by

\[
D = \{ (x_1,x_2) | x_1Z^- \leq x_1 \leq x_1Z^+ \land x_2Z^- \leq x_2 \leq x_2Z^+ \} = \{ y | y_1 = [x_1Z^-,x_1Z^+], y_2 = [x_2Z^-,x_2Z^+] \}
\]

Assume that \( x^* = (x_1^*,x_2^*) \in D \) is a current isolated local minimizer of \( f \). Let \( x_1Z^+, x_1Z^- \) are the first nearest points lied on the right and left hand sides of \( x_1^* \) respectively while \( x_2Z^+ \) and \( x_2Z^- \) are the first nearest points lied on the right and left hand sides of \( x_2^* \) respectively such that

\[
f(x_1^+,x_2^+)=f(x_1^+,x_2^-)=f(x_1^-,x_2^+)=f(x_1^-,x_2^-)=f(x^*)
\]
The domain $D$ can be divided into four sub domains $D_0, D_1, D_2$, and $D_3$ as shown in Fig. 4 such that

$$D = \bigcup_{i=1}^{4} D_i = D_1 \cup D_2 \cup D_3 \cup D_4$$

where

$$D_1 = \left([x_1^*, \overline{x_1}], [x_2^*, \overline{x_2}^+]\right),$$
$$D_2 = \left([x_1^*, \overline{x_1}], [\overline{x_2}^-, x_2^*]\right),$$
$$D_3 = \left([\overline{x_1}^-, x_1^*], [x_2^*, \overline{x_2}^+]\right),$$
$$D_4 = \left([\overline{x_1}^-, x_1^*], [\overline{x_2}^-, x_2^*]\right)$$

In this section we will recall about parameter free filled function in two dimensional case [4], that is the extension from filled function [3], with the form $F_i (i = 0, 1, 2, 3)$, so called IHR’s Function, which defined on sub domain $D_i$ $(i = 0, 1, 2, 3)$, is given by

$$F_i (x, x^*) = \left\{ \begin{array}{ll}
F_{i1} (x, x^*) & = \left[ \int_a^b \left( f(s, x_2) - f(x_1^*, x_2^*) \right) ds \right] \quad (a \leq b) \\
F_{i2} (x, x^*) & = \left[ \int_c^d \left( f(s, x_2) - f(x_1^*, x_2^*) \right) ds \right] \quad (c \leq d)
\end{array} \right.$$ 

where $a, b, c$ and $d$ are shown in Table 1.

| $i$ | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| Boundary of integration | $a = x_1^*, b = x_1$ | $a = x_1^*, b = x_1$ | $a = x_1^*, b = x_1$ | $a = x_1^*, b = x_1$ |
|   | $c = x_2^*, d = x_2$ | $c = x_2^*, d = x_2$ | $c = x_2^*, d = x_2$ | $c = x_2^*, d = x_2$ |

The definition of IHR’s function is given in Definition 1 and its validity as a parameter free filled function is proved by Theorem 1-Theorem 3 as presented in detail in [4].

**Definition 1** $F(x, x^*) = (F_1, F_2, F_3, F_4) (x, x^*)$ defined previously is so called as IHR’s filled function of $f(x)$ at an isolated local minimizer $x^*$ if

(i) $x^*$ is a maximizer of $F_{HH}(x, x^*)$.

(ii) $F_{HH}$ has no stationary point in set $H_1 = \{ x | f(x) > f(x^*), x \in D \}$

(iii) There is a point $x \in H_2 = \{ x | f(x) \leq f(x^*), W(x) = 0, x \in D \backslash \{ x^* \} \}$ such that $x$ is a stationary point of $F_{HH}$

where $a, b, c$ and $d$ are as in Table 1 and

$$W(x) = \left\{ \begin{array}{ll}
W_{i1} (x) & = \frac{\partial}{\partial x_2} \left[ \int_a^b f(s_1, x_2) ds_1 \right] \quad (j = 1) \\
W_{i2} (x) & = \frac{\partial}{\partial x_1} \left[ \int_c^d f(s_1, x_2) ds_2 \right] \quad (j = 2)
\end{array} \right.$$ 

**Theorem 1** If $x^* = (x_1^*, x_2^*)^T \in \mathbb{R}^2$ is an isolated minimizer of $f$ on $D = \bigcup_{i=0}^{3} D_i$, then $x^*$ is an isolated maximizer of $F_{HH}$. □

**Theorem 2** If $(x_1^*, x_2^*)^T$ is a local isolated minimizer of $f$ on $D = \bigcup_{i=0}^{3} D_i$ and
\[ H_1 = \left\{ (x_1, x_2)^T \mid f(x_1, x_2) > f(x_1^*, x_2^*), (x_1, x_2)^T \in D \setminus \{(x_1^*, x_2^*)^T\} \right\} = \left\{ [x_1^Z, x_1^*] \cup (x_1^*, x_1^Z^+), [x_2^Z, x_2^*] \cup (x_2^*, x_2^Z^+] \right\} \]

Then \( F_{HHR} \left( (x_1, x_2)^T, (x_1^*, x_2^*)^T \right) \) has no stationary point in the set \( H_1 \). □

**Theorem 3** If \( H_2 = \left\{ (x_1, x_2)^T \mid f(x_1, x_2) \leq f(x_1^*, x_2^*), W(x_1, x_2) = 0, (x_1, x_2)^T \in D \setminus \{(x_1^*, x_2^*)^T\} \right\} \neq \emptyset \) where

\[
W = \begin{cases} \frac{\partial}{\partial x_2} \left( \int_a^b f(s_1, x_2) ds_1 \right) & (j = 1) \\ \frac{\partial}{\partial x_1} \left( \int_c^d f(x_1, s_2) ds_2 \right) & (j = 2) \end{cases}
\]

and \( a, b, c, d \) are as in Table 1, then there is a point \( x = (x_1, x_2)^T \in H_2 \) such that \( x \) is a stationary point of \( F_{HHR}( (x_1, x_2)^T, (x_1^*, x_2^*)^T ) \). □

### 3. Jameson Gradient Based Optimization Method Modification [6]

In [4] and [5] the process of minimizing filled function is using Newton’s method. However, even Newton’s method is considered to be fast method on finding the zero of gradient of quadratic function, but may be very expensive or infeasible to determine the Hessian matrix in the case of complex problems. The Jameson gradient based method is the search procedures which avoid the need to store an estimate of the Hessian as well as its inverse and do not require exact line searches.

We use the following updates to find the zeros of gradient of filled function \( \nabla F_i(x_j, x_m^*) = 0 \), that is the modification of Jameson gradient method by replace the estimates of error, \( e_i \), by 1,

\[
x_{m+1} := x_m - \frac{\alpha e_m g_n}{\beta + \frac{1}{2} g_n e_m},
\]

with \( \beta \geq 1 \). We define the value of \( \alpha := 0.1 \) and \( \beta := 1 \) to be applied in the algorithm method.

### 4. Algorithm method

The following algorithm method is generally includes of two methods of optimization, that are the steepest descent method in phase one and modification of Jameson gradient method in phase two.

**Data** (initialization): specify \( n, x^0, D, d > 0 \), set \( i = 1, j = 1, m = 1 \)

**step 1** specify initial step size of steepest descent \( \alpha_0 > 0 \), minimize \( f(x) \) starting at \( x_0 \) to obtain \( x_0^* \)

**step 2** construct \( F \) function \( F_i(x, x_m^*) \) at \( x_0^* \)

**step 3** if \( i \leq n \) then choose direction \( e_i \) for respected defined dimension and go to step 4 else **stop**

**step 4** set \( c := 1 \)

**step 5** set initial point \( x_0^c = x_0^* + c d e_i \).

if \( x_0^c \in D \) then go to step 6
else if \( j < n \) then \( j := j + 1 \), go to step 4
else \( j = 1; i := i + 1 \); and go to step 3

**step 6** if \( f(x_0^c) < f(x_0^*) \) then set \( x_0 := x_0^c \), go to step 1
else if one of cond1 true then go to step 7
else \( c := c + 1 \) and go to step 5

**step 7** by Jameson gradient based method, with initial \( x_0^c \) solve \( \nabla F_i(x, x_m^*) = 0 \) to obtain \( x_0^c \)

**step 8** if cond2 and cond1 hold then set \( x_0 := x_0^c, m := m + 1 \), go to step 1
else \( c := c + 1 \) go to step 5
Note:
cond_1 : at least there is exists a \( j \) \((j=1,2)\) such that the following condition holds

\[
\left( \frac{\partial F_{ij}}{\partial x_j} (x = x^*_m + cde_i) \right)^2 < \left( \frac{\partial}{\partial x_j} F_y (x = x^*_m + (c-1)de_i) \right)^2
\]

cond_2 : at least one of following conditions holds for a specified \( j \)

(i). \( \frac{\partial^2}{\partial x_j} F_y (x = x^F) > 0 \) and (ii). \( \left| \frac{\partial^2}{\partial x_j} F_y (x = x^F) \right| < 10^{-4} \)

cond_3 : \( |f(x^F) - f(x_m^*)| < 10^{-2} \)

5. Numerical Results
Following benchmark tests function are used to test the performance of the algorithm method.

- Six Hump Back Camel Function

\[
\min f(x_1,x_2) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 - x_1x_2 - 4x_2^2 + 4x_2^4
\]

\[ s.t. \ -3 \leq x_1,x_2 \leq 3 \]

- Rastrigin Function

\[
\min f(x_1,x_2) = x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2)
\]

\[ s.t. \ -2 \leq x_1,x_2 \leq 2 \]

Numerical experiment is done by visual C++ program, Processor Intel ® Core i3, 2048 MB RAM. The process of minimizing \( f(x) \), steepest descent method with elimination line search step size (see [7]) is used, which its step size is made to be absolute. Stopping criteria for steepest descent method is \( |\nabla f| \leq 10^{-3} \), while for Jameson’s method is \( |\nabla F| \leq 2 \cdot 10^{-4} \).

The numerical results of the given problems are presented in Table 2 and Table 3. The symbols used in the table are as follows:

- \( k \) : number of iteration
- \( F_y \) : the evaluated Three-Dimensional Parameter Free Filled Function
- \( \alpha_0 \) : initial step size of steepest descent method
- \( c \) : integer number to be multiplied to search direction
- \( x_0 \) : initial point for steepest descent method in \( f(x) \)
- \( x^*_k \) : \( k \)-th obtained minimizer
- \( x_0^* \) : initial point for Newton’s method in \( F_y \)
- \( x^F \) : point obtained by Newton’s method or satisfied \( f(x^F) < f(x^*_k) \)
- \( f_0, f_0^*, f_0^F, f^F \) : function value of \( x_0, x^*_k, x_0^*, x^F \)

| \( k \) | \( \alpha_0, F_y, c, d \) | Results |
|---|---|---|
| 0 | \( \alpha_0 = 1 \) | \( x_0 = (2,1)^T, f_0 = 5.7333 \) |
| 0 | \( \alpha_0 = 0.001 \) | \( x^*_1 = (1.60711, -0.568658)^T, f^*_1 = 2.10425 \) |
| 1 | \( F_y, c = 7 \) | \( x_0^* = (1.36211, -0.323658)^T, f_0^* = 2.38716 \) |
| 1 | \( d = 0.1 \) | \( x^F = (0.910266, -0.166634)^T, f^F = 2.1059 \) |
From the numerical results of Table 2, from the starting point \( \mathbf{x}^*_0 \), the point \( \mathbf{x}^F_1 = (0.910266, -0.166634)^T \) is obtained which has the same value to the minimizer \( \mathbf{x}^*_1 \) that is \( f^F = 2.1059 \). Thus this point is used to obtain the next minimizer of the problem by steepest descent routine, that is \( \mathbf{x}^*_2 = (-0.0898664, -0.712651)^T, f^*_2 = -1.03163 \). While from Table 3, for Numerical Results of Rastrigin Function, after the first minimizer is obtained, the point that has function value lower than the current minimizer is obtained, that is \( f^F_2 = -1.76825 \). The numerical results shows that the algorithm success in obtain the global minimizer of the given tests function with the given initial points with value of \( \alpha = 0.1 \) and \( \beta = 1 \).

### Table 3. Numerical Results of Rastrigin Function

| \( k \) | \( a_0, f_{ij}, c, d \) | Results |
|-------|-----------------|---------|
| 0     | \( a_0 = 0.001 \) | \( x^*_0 = (0.5, -0.5)^T, f^*_0 = 2.32226 \) |
| 1     | \( a_0 = 0.001 \) | \( x^*_0 = (0.346924, -0.346924)^T, f^*_0 = -1.7578 \) |
| 0     | \( F_{31}, c = 320, \) | \( x^F_1 = (0.0269238, -0.0269238)^T, f^F_1 = -1.76825 \) |
| 1     | \( d = 0.1 \) | \( x^F_1 = (0.0269238, -0.0269238)^T, f^F_1 = -1.76825 \) |
| 0     | \( a_0 = 0.001 \) | \( x^*_1 = (-1.178 \times 10^{-14}, 1.178 \times 10^{-14})^T, f^*_1 = -2 \) |

6. **Conclusion**

In this paper, we introduce the parameter free filled function method in cooperation with Jameson gradient based method modification for solving global optimization of \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \). The filled function \( F(x, x^*) \) at a current local minimizer \( x^* \) of \( f \) is used as an auxiliary function for finding the zeros of \( \nabla F(x, x^*) \) or to meet the function with lower function value, \( f(x) < f(x^*) \). In the phase of existance of filled function, the Jameson gradient based method modification is used. This method has an advantages compared to Newton’s method, since it avoid the need to store an estimate of the Hessian as well as its inverse and do not require exact line searches. The algorithm method that we proposed is in cooperation of steepest descent method, Jameson gradient based method modification with \( \alpha = 0.1, \beta = 1 \), and the two dimensional parameter free filled function. Based on the numerical performance of the given test functions, we conclude that our proposed method is promising in successfully obtain global minimizer of unconstrained global optimization problem with two variables.

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