Pullback attractors on time-dependent spaces for fluid-structure interaction systems.

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Abstract

We study the long-time dynamics of a non-autonomous coupled system consisting of the 3D linearized Navier–Stokes equations and nonlinear elasticity equations. We show that this problem generates a process on time-dependent spaces possessing a pullback attractor.

1 Introduction

We consider a coupled non-autonomous system with time-dependent coefficients which describes interaction of a homogeneous viscous fluid which occupies a domain $\Omega$ bounded by the solid walls of the container $S$ and a horizontal boundary $\Omega$ on which a thin nonlinear elastic plate is placed. The motion of the fluid is described by linearized 3D Navier–Stokes equations. To describe deformations of the plate we consider a generalized plate model which accounts only for transversal displacements and covers a general large deflection Karman type model (see, e.g., [12]). However, our results can be also applied in the cases of nonlinear Berger and Kirchhoff plates.

Our mathematical model is formulated as follows.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a sufficiently smooth boundary $\partial \Omega$. We assume that $\partial \Omega = \overline{\Omega} \cup S$, where $\Omega \cap S = \emptyset$ and

$$\Omega \subset \{x = (x_1; x_2; 0) : x' \equiv (x_1; x_2) \in \mathbb{R}^2\}$$

with the smooth contour $\Gamma = \partial \Omega$ and $S$ is a surface which lies in the subspace $\mathbb{R}^3_+ = \{x_3 \leq 0\}$. The exterior normal on $\partial \Omega$ is denoted by $n$. We have that $n = (0; 0; 1)$ on $\Omega$. We consider the following Navier–Stokes equations in $\Omega$ for the fluid velocity field $v = v(x, t) = (v^1(x, t); v^2(x, t); v^3(x, t))$ and for the pressure $p(x, t)$:

$$\mu(t) v_t - \Delta v + \nabla p = f(x, t) \quad \text{in} \quad \Omega \times (\tau, +\infty), \quad \tau \in \mathbb{R}. \quad (1)$$

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\[
\text{div} \, v = 0 \quad \text{in} \quad \emptyset \times (\tau, +\infty), \quad \tau \in \mathbb{R}.
\] (2)

where \( f(x, t) \) is a volume force.

We supplement (1) and (2) with the non-slip boundary conditions imposed on the velocity field \( v = v(x, t) \):

\[
v = 0 \quad \text{on} \quad S; \quad v \equiv (v^1; v^2; v^3) = (0; 0; u_t) \quad \text{on} \quad \Omega.
\] (3)

Here \( u = u(x, t) \) is the transversal displacement of the plate occupying \( \Omega \) and satisfying the following equation:

\[
\rho(t)u_{tt} + \Delta^2 u + F(u) = g(x, t) + p|\Omega, \quad \text{in} \quad \Omega \times (\tau, \infty), \quad \tau \in \mathbb{R}.
\]

where \( g(x, t) \) is a given body force on the plate, \( F(u) \) is a nonlinear feedback force which will be specified later.

We impose clamped boundary conditions on the plate

\[
u|_{\partial\Omega} = \frac{\partial u}{\partial n} = 0
\] (4)

and supply (1)–(4) with initial data of the form

\[
v(x, \tau) = v_\tau, \quad u(x, \tau) = u_\tau^2, \quad u_t(x, \tau) = u_t^1,
\] (5)

We note that (2) and (3) imply the following compatibility condition

\[
\int_{\Omega} u_t(x', t) dx' = 0 \quad \text{for all} \quad t \geq \tau, \quad \tau \in \mathbb{R}.
\] (6)

This condition fulfills when

\[
\int_{\Omega} u(x', t) dx' = \text{const} \quad \text{for all} \quad t \geq \tau, \quad \tau \in \mathbb{R}.
\]

which can be interpreted as preservation of the volume of the fluid.

In this paper our main point of interest is well-posedness and long-time dynamics of solutions to the coupled problem in (1)–(5) for the velocity \( v \) and the displacement \( u \).

We consider this problem under rather general hypotheses concerning nonlinearity. These hypotheses cover the cases of von Karman, Berger and Kirchhoff plates. We show that problem (1)–(5) generates a process on a family of time-dependent energy spaces. Our main result states that under some natural conditions concerning feedback forces system (1)–(5) possesses a pullback attractor. To establish this result we adjust the compensated compactness approach widely used for dynamical systems of autonomous equations (see \[5\], \[6\] and \[7, Chapters 7,8\] and also the references therein) to processes and pullback attractors.

The mathematical studies of the long-time behavior of autonomous problems of fluid-structure interaction in the case of viscous fluids and elastic plates/bodies have a long history (see, e.g. \[3, 4, 8, 10\] and references therein).

In the present work we investigate the existence of a pullback attractor in case of time-dependent coefficients in the main parts of equations. For such problems only a few papers are devoted to the existence of pullback attractors \[1, 2, 9\]. In paper \[1\] a wave equation with time-dependent coefficient before the damping term is considered, consequently the energy space does not depend on the time parameter. Papers \[2, 9\] deal with wave equations with time-dependent coefficients before the second derivatives with respect to time and to
the space variable respectively. In both works the existence of time-dependent pullback attractor in a scale of spaces is established.

In our paper we consider for the first time, to the best of our knowledge, the interaction model for a Newtonian fluid and a plate with time-dependent coefficients before the time derivatives. The peculiarity of the problem considered consists in the absence of any mechanical damping in the plate component and the strong coupling of fluid and plate components.

We prove the well-posedness of the system considered and investigate the long-time dynamics of solutions to the coupled problem in (1)–(5). In order to show the existence of a time-dependent pullback attractor we derive an abstract result on the asymptotic compactness of processes on time-dependent spaces.

The paper is organized as follows. In Section 2 we introduce notations, recall some properties of Sobolev type spaces with non-integer indexes on bounded domains and collect some regularity properties of (stationary) Stokes problem which we use in the further considerations. The main notions from the theory of pullback attractors and new abstract results are presented in Section 3. Our main result in Section 4 is Theorem 2 on well-posedness and existence of time-dependent absorbing set. Our main result in Section 5 states existence of a pullback attractor. The argument is based on the property established in Theorem 1.

2 Spaces and notations.

Now we introduce Sobolev type spaces which are used in what follows (see e.g. [17]).

Let $D$ be a sufficiently smooth domain and $s \in \mathbb{R}$. We denote by $H^s(D)$ the Sobolev space of order $s$ on a set $D$ which we define as restriction (in the sense of distributions) of the space $H^s(\mathbb{R}^d)$ (introduced via Fourier transform). We denote by $\| \cdot \|_{s,D}$ the norm in $H^s(D)$ which we define by the relation

$$\|d\|_{s,D}^2 = \inf \{ \|w\|_{s,\mathbb{R}^d}^2 : w \in H^s(\mathbb{R}^d), \ w = u \text{ on } D \}.$$

We also use the notation $\| \cdot \|_{D} = \| \cdot \|_{0,D}$ for the corresponding $L_2$-norm and, similarly, $(\cdot, \cdot)_D$ for the $L_2$ inner product. We denote by $H^s_0(D)$ the closure of $C_0^\infty(\Omega)$ in $H^s(D)$ (with respect to $\| \cdot \|_{s,D}$) and introduce the spaces

$$H^s_0(D) := \{ f|_D : f \in H^s(\mathbb{R}^d), \ \text{supp } f \subset \overline{D} \}, \ s \in \mathbb{R}.$$

Since the extension by zero of elements from $H^s_0(D)$ gives us an element of $H^s(\mathbb{R}^d)$, these spaces $H^s_0(D)$ can be treated not only as functional spaces defined on $D$ (and contained in $H^s(D)$) but also as (closed) subspaces of $H^s(\mathbb{R}^d)$. Below we need them to describe boundary traces on $\Omega \subset \partial D$. We endow the classes $H^s_0(D)$ with the induced norms $\|f\|_{s,D} = \|f\|_{s,\mathbb{R}^d}$ for $f \in H^s_0(D)$. It is clear that

$$\|f\|_{s,D} \leq \|f\|_{s,D}^*, \ f \in H^s_0(D).$$

It is known (see [17] Theorem 4.3.2/1) that $C_0^\infty(D)$ is dense in $H^s_0(D)$ and

$$H^s_0(D) \subset H^s_0(D) \subset H^s(D), \ s \in \mathbb{R};$$
$$H^s_0(D) = H^s(D), \ -\infty < s \leq 1/2;$$
$$H^s_0(D) = H^s_0(D), \ -1/2 < s < \infty, \ s - 1/2 \notin \{0, 1, 2, \ldots\}.$$
In particular, \( H^s(D) = \mathcal{H}_0^s(D) = H^s(D) \) for \(|s| < 1/2\). By [17] Remark 4.3.2/2 we also have that \( H^s(D) \neq H^s(D) \) for \(|s| > 1/2\). Note that in the notations of [13] the space \( H^{m+1/2}_0(D) \) is the same as \( H^{m+1/2}_0(D) \) for every \( m = 0, 1, 2, \ldots \), and for \( s = m + \sigma \) with \( 0 < \sigma < 1 \) we have

\[
\|u\|_{s,D}^2 = \left\{ \|u\|_{s,D}^2 + \sum_{|n|=m} \int_D \frac{|Du(x)|^2}{d(x, \partial D) - 2} \, dx \right\}^{1/2},
\]

where \( d(x, \partial D) \) is the distance between \( x \) and \( \partial D \). The norm \( \| \cdot \|_{s,D}^2 \) is equivalent to \( \| \cdot \|_{s,D}^2 \) in the case when \( s > 1/2 \) and \( s - 1/2 \notin \{0, 1, 2, \ldots \} \), but not equivalent in general.

Understanding adjoint spaces with respect to duality between \( C_0^\infty(D) \) and \( [C_0^\infty(D)]' \) by Theorems 4.8.1 and 4.8.2 from [17] we also have that

\[
[H^s(D)]' = H^{-s}(D), \quad s \in \mathbb{R}, \quad \text{and} \quad [H^s(D)]' = H^{-s}(D), \quad s \in (-\infty, 1/2).
\]

To describe fluid velocity fields we introduce the following scale of spaces.

Let \( \mathcal{C}(\mathcal{O}) \) be the class of \( C^\infty \) vector-valued solenoidal (i.e., divergence-free) functions \( v = (v^1, v^2, v^3) \) on \( \mathcal{O} \) which vanish in a neighborhood of \( S \) and such that \( v^1 = v^2 = 0 \) on \( \Omega \). We denote by \( X_t \) the closure of \( \mathcal{C}(\mathcal{O}) \) with respect to the following \( L_2 \)-norms

\[
\| \cdot \|_{X_t} = \mu(t) \| \cdot \|_{L_2(\mathcal{O})}
\]

and by \( Y \) the closure with respect to the \( H^1(\mathcal{O}) \)-norm. One can see that

\[
X_t = \left\{ v = (v^1, v^2, v^3) \in [L_2(\mathcal{O})]^3 : \text{div} v = 0; \; \gamma_n v \equiv (v, n) = 0 \text{ on } S, \; t \in \mathbb{R} \right\}
\]

and

\[
Y = \left\{ v = (v^1, v^2, v^3) \in [H^1(\mathcal{O})]' : \begin{array}{l}
\text{div}^\cdot v = 0, \; v = 0 \text{ on } S, \\
v^1 = v^2 = 0 \text{ on } \Omega
\end{array} \right\}.
\]

The space \( Y \) is endowed with the norm \( \| \cdot \|_Y = \| \nabla \cdot \|_{L_2(\mathcal{O})} \). For some details concerning this type spaces we refer to [16], for instance.

We also need the Sobolev spaces consisting of functions with zero average on the domain \( \Omega \), namely we consider the space

\[
\mathcal{L}_2(\Omega) = \left\{ u \in L_2(\Omega) : \int_{\Omega} u(x') \, dx' = 0 \right\}
\]

and also the scale of time-dependent spaces

\[
\mathcal{L}_2^t(\Omega) = \left\{ L^2(\Omega) : \| \cdot \|^2_{L^2(\Omega)} = \rho(t) \| \cdot \|^2_{L^2}, \; t \in \mathbb{R} \right\}.
\]

We use the notation \( \mathcal{H}^s(\Omega) = H^s(\Omega) \cap \mathcal{L}_2(\Omega) \) for \( s > 0 \) with the standard \( H^s(\Omega) \)-norm. The notations \( \mathcal{H}^s_t(\Omega) \) and \( \mathcal{H}^0_t(\Omega) \) have a similar meaning.

## 3 Abstract results on attractors.

We begin with some definitions from the theory of processes.

**Definition 1.** A two parameter family \( U(t, \tau) : H \to H, \; t \geq \tau \in \mathbb{R} \) of operators in a scale of Banach spaces \( H, \; t \in \mathbb{R} \) is called a process if
• $U(\tau, \tau) = I$
• $U(t, s)U(s, \tau) = U(t, \tau), \ t \geq s \geq \tau \in \mathbb{R}$

**Definition 2.** The family of sets $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$, for $B_t \in H_t$ is positively invariant if $U(t, \tau)B_t \subset B_t, \ \forall \tau \in \mathbb{R}$.

**Definition 3.** The family of bounded sets $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$, for $B_t \in H_t$ is uniformly bounded if there exists $R > 0$ such that $B_t \in \mathcal{B}(R) = \{z \in H_t, \|z\|_{H_t} \leq R\}$ for any $t \in \mathbb{R}$.

**Definition 4.** To study the asymptotic behavior of the operators $U(t, \tau)$ we need to define a suitable object which attracts solutions of the system originating sufficiently far in the past. In order to do it we need to introduce the notion of absorption and attraction.

**Definition 5.** The family of uniformly bounded sets $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$, is time-dependent absorbing if for any $R > 0$ there exists $\Theta = \Theta(R)$ such that $U(t, \tau)\mathcal{B}(R) \subset B_t, \ \forall \tau \leq t - \Theta$.

The process $U(t, \tau)$ is called dissipative whenever it admits a pullback absorbing family.

**Definition 6.** A time dependent $\omega$-limit of any pullback absorbing family $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$, for $B_t \in H_t$ is the family $\Omega = \omega_{\mathcal{B}}(\mathbb{R}) = \bigcap_{t \in \mathbb{R}} U(t, \tau)B_t$.\label{7}

**Definition 7.** The family of uniformly bounded sets $\mathcal{K} = \{K_t\}_{t \in \mathbb{R}}$ is pullback attracting if for every uniformly bounded family $\mathcal{B} = \{B_t\}_{t \in \mathbb{R}}$

$$\lim_{\tau \to -\infty} \delta_t(U(t, \tau)B_t, K_t) = 0,$$

where $\delta_t(B, C) = \sup_{x \in B} \inf_{y \in C} \|x - y\|_{H_t}$ denotes the Hausdorff seminorm.

Now we are in position to define the pullback attractor.

**Definition 8.** A process is asymptotically compact if there exists a pullback attracting family of compact sets $\mathcal{K} = \{K_t\}_{t \in \mathbb{R}}, K_t \subset H_t$.

**Definition 9.** Pullback attractor is the smallest element of pullback attracting families $\mathcal{K} = \{\mathcal{K}\}_{t \in \mathbb{R}}$, where $K_t \subset H_t$ are compact in the corresponding spaces.

The classical approach (see, e.g. [A]) to verification of asymptotic compactness of a process consists in finding a decomposition $U(t, \tau) = U_0(t, \tau) + U_1(t, \tau)$ with the properties

$$\|U_0(t, \tau)z\|_{H_t} \leq Ce^{-\delta(t-\tau)}, \ C, \delta > 0, \ z \in H_t$$

and

$$\sup_{t \geq \tau} \|U_1(t, \tau)z\|_{R_t} \leq M,$$

where $R_t$ is a compactly embedded into $H_t$ in Banach space. However, for the system considered it is not obvious how to get such a decomposition due to strong coupling (fluid and plate components cannot be split in terms of construction of Galerkin approximations). Therefore, we need to derive another criterion for asymptotic compactness. In order to do this we adjust to our situation the method of compensated compactness.
Theorem 1. Let $\mathcal{D} = \{D_t \}_{t \in \mathbb{R}}$ be a time-dependent absorbing family of a process $U(t, \tau) : H \to H$, and for any $\varepsilon > 0$ there exists $T_0 = T_0(\varepsilon) > 0$ such that for any $y_1, y_2 \in D_{t-T_0}$

$$
\|U(t, t-T_0)y_1 - U(t, t-T_0)y_2\|_H \leq \varepsilon + \Phi_{T_0,T_0}(y_1, y_2),
$$

where the function $\Phi_{T_0,T_0}(y_1, y_2) : D_{t-T_0} \times D_{t-T_0} \to \mathbb{R}$ possesses the property

$$
\liminf_{n \to \infty} \liminf_{m \to \infty} \Phi_{T_0,T_0}(y_n, y_m) = 0
$$

for any sequence $\{y_n\} \in D_{t-T_0}$.

Proof. We can assume without loss of generality that $\mathcal{D}$ is positively invariant. Otherwise, we can substitute $D_t$ with $\bigcup_{\tau \leq t-T_0} U(t, \tau)D_{t-T_0} \subset D_{t-T_0}$.

We fix $T > 0$. Obviously, we have a representation

$$
\omega_t(D) = \bigcap_{k \in \mathbb{N}} C_k^t,
$$

where

$$
C_k^t = U(t, t-kT)D_{t-kT}.
$$

Now we need to check that

$$
C_{k+1}^t \subset C_k^t.
$$

(10)

Indeed, due to the invariance of the family $D$ we have $U(t-kT, t-(k+1)T)D_{t-(k+1)T} \subset D_{t-kT}$, consequently,

$$
C_{k+1}^t = U(t, t-(k+1)T)D_{t-(k+1)T} = U(t, t-kT)U(t-kT, t-(k+1)T)D_{t-(k+1)T} \subset U(t, t-kT)D_{t-kT} = C_k^t.
$$

Therefore, we have a sequence of nonempty closed sets

$$
C_1 \supset C_2 \supset \ldots \supset C_k \supset C_{k+1} \supset \ldots
$$

To show that $\omega_t(D)$ is nonempty and compact it remains to prove that

$$
\lim_{k \to \infty} \alpha(C_k^t) = 0.
$$

(11)

Due to (10)

$$
\alpha(C_k^t) = \alpha(C_k^t \cup C_{k+1}^t) = \max(\alpha(C_k^t), \alpha(C_{k+1}^t)),
$$

consequently,

$$
\alpha(C_k^t) \geq \alpha(C_{k+1}^t)
$$

(12)

for any $k \in \mathbb{N}$. It follows readily from (12) that to show (11) it is sufficient to prove that for any $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $\alpha(C_{k_0}^t) \leq \varepsilon$.

Now we use the contradiction argument. Let there exists $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$

$$
\alpha(C_k^t) > 6\varepsilon_0.
$$

(13)
For this $\epsilon_0$ we choose $T_0 = T_0(\epsilon_0)$ such that (9), (10) hold. There exist $k_0 \in \mathbb{N}$ and $0 < \delta_0 < T$ such that $T_0 = k_0 T - \delta_0$. We use the notation $\mathcal{L}_0 = U(t, t - T_0)D_{t=T_0} = U(t, t - k_0 T + \delta_0)D_{t=k_0 T+\delta_0}$. Then, 

$$C'_{k_0} = U(t, t - k_0 T)D_{t=k_0 T} = U(t, t - k_0 T + \delta_0)U(t - k_0 T + \delta_0, t - k_0 T)D_{t=k_0 T} \subset U(t, t - T_0)D_{t=T_0} = \mathcal{L}_0. \quad (14)$$

It follows from (12) and (14) that 

$$\alpha(\mathcal{L}_0) \geq \alpha(C'_{k_0}) > 6\epsilon_0.$$ 

This implies that there exists a sequence $\{y_n\}_{n=1}^\infty \in D_{t=T_0}$ such that for any $n, m \in \mathbb{N}$ such that $n \neq m$

$$2\epsilon_0 \leq \|U(t, t - T_0)y_n - U(t, t - T_0)y_m\|_{H_0} \leq \epsilon_0 + \Phi_{T_0,\delta}(y_n, y_m),$$

and, therefore,

$$\Phi_{T_0,\delta}(y_n, y_m) \geq \epsilon_0,$$

which contradicts to (9). This means that $\Omega = \{\omega_t(D)\}_{t \in \mathbb{R}}$ is a pullback attracting family of compact sets.

## 4 Well-posedness and existence of absorbing set.

In this section we prove the existence and uniqueness of solutions to the problem considered, generation of a continuous process, and existence of its time-dependent absorbing set. We introduce the scale of phase spaces

$$H_t = X_t \times H^2_0(\Omega) \times L^2(\Omega)$$

equipped with the norm

$$\|W\|_{H_t} = \mu(t)\|v\|_{L^2(\Omega)}^2 + \rho(t)\|u\|_{L^2(\Omega)}^2 + \|u_t\|_{L^2(\Omega)}^2, \quad W = (v, u, u_t).$$

Now we impose assumptions on the parameters of problem (1)–(5) (cf. 2, 15).

### Assumptions on $\mu$ and $\rho$

(A1) $\mu(t), \rho(t) > 0$.

(A2) $\mu(t), \rho(t) \in C^1(\mathbb{R})$ are decreasing functions.

(A3) There exists $L > 0$ such that

$$\sup_{t \in \mathbb{R}}(\mu(t) + |\mu'(t)| + |\rho(t)| + |\rho'(t)|) \leq L.$$

(A4) $\lim_{t \to +\infty} \mu(t) = 0, \lim_{t \to +\infty} \rho(t) = 0.$

### Assumptions on $F$
(F1) There exists \( \varepsilon > 0 \) such that \( F \) is locally Lipschitz from \( H^2_0(\Omega) \) into \( H^{-1/2}(\Omega) \), i.e.
\[
\|F(u_1) - F(u_2)\|_{-1/2, \Omega} \leq C_R\|u_1 - u_2\|_{2, \Omega},
\]
for any \( u_1, u_2 \in H^2_0(\Omega) \) possessing the property \( \|u_i\|_{2, \Omega} \leq R, i = 1, 2. \)

(F2) There exists a \( C^1 \) functional \( \Pi(u) \) on \( H^2_0(\Omega) \) such that \( F(u) = \Pi'(u) \) and \( \Pi(u) \leq Q(\|u\|_{2, \Omega}) \), where the function \( Q \) is increasing.

(F3) There exist \( 0 < \nu < 1 \) and \( C \geq 0 \) such that
\[
(1 - \nu)\|\Delta u\|_{\Omega}^2 + \Pi(u) + C \geq 0
\]
for any \( u \in H^2_0(\Omega) \).

(F4) There exist \( a_1, a_2 \geq 0 \) and \( 0 < \nu < 1 \) such that
\[
(F(u, u) \geq a_1\Pi(u) - a_2 - (1 - \nu)\|\Delta u\|_{\Omega}^2.
\]

**Assumptions on \( f \) and \( g \).**

(G1) \( f \in L^2_{\text{loc}}(\mathbb{R}; Y') \), \( g \in L^2_{\text{loc}}(\mathbb{R}; H^{-1/2}(\Omega)) \).

(G2) There exist \( \sigma_0, C_{f,g} > 0 \), such that for any \( t \in \mathbb{R} \) and \( \sigma \in [0, \sigma_0] \)
\[
\int_{-\infty}^{t} e^{-\sigma(t-\tau)} \left( \|f(s)\|_{Y'}^2 + \|g(s)\|_{-1/2, \Omega}^2 \right) ds \leq C_{f,g}.
\]

**Remark 1.** We note that assumption (A4) is imposed in order to consider the problem in time-dependent spaces. Otherwise, due to assumption (A2) we obtain the equivalence of the norms
\[
\|W\|^2_{H^1} \leq \|W\|^2_{H_{loc}} \leq \max \left\{ 1, \frac{\mu(\tau)}{\rho(\tau)}, \frac{\rho(\tau)}{\rho(\tau)} \right\} \|W\|_{H^1}^2.
\]

**Remark 2.** The examples of function satisfying assumptions (G1), (G2) are periodic functions or \( e^{-\kappa t}, \kappa > 0 \).

We define the spaces of test functions
\[
\mathcal{L}_T = \{ \psi = (\phi, b) : \phi \in L^2(\tau, T; [H^1(\Omega)]^3), \phi_t \in L^2(\tau, T; [L^2(\Omega)]^3), \text{div}\phi = 0, \phi|_{\Gamma} = 0, \phi|_{\Omega} = (0, 0, b), b \in L^2(\tau, T, H^2_0(\Omega)), b_t \in L^2(\tau, T, L^2(\Omega)) \}
\]
and \( \mathcal{L}^0_T = \{ \psi \in \mathcal{L}_T : \psi(t) = 0 \} \).

In order to make our statements precise we need to introduce the definition of weak solutions to problem (1)–(5).

**Definition 10.** A pair of functions \( (v(t), u(t)) \) is said to be a weak solution to problem (1)–(5) on a time interval \( [\tau, t] \) if

- \( W(t) = (v(t), u(t), u_t(t)) \in L_{\infty}(\tau, T; H_1) \);
for some constant $K$.

Under assumptions (A1)-(A4), (F1)-(F3), (G1)-(G2) problem

Theorem 2. \( >\tau \) holds for every \( t \in [\tau, T] \)

continuous dependence property: for every pair of initial data \( W \)

\[ \parallel \parallel = u_0; \]

For almost all \( t \in [\tau, T] \)

\[ v(t)|_{\Omega} = u(t); \quad (15) \]

For every \( \psi = (\phi, b) \in \mathcal{L}_T^0 \) the following equality holds

\[
- \int_{\tau}^{T} \mu(t)(v, \phi) \, dt - \frac{1}{2} \int_{\tau}^{T} \mu'(t)(v, \phi) \, dt + \int_{\tau}^{T} \mu(t)(\nabla v, \nabla \phi) \, dt
- \int_{\tau}^{T} \rho(t)(u, b) \, dt - \frac{1}{2} \int_{\tau}^{T} \rho'(t)(u, b) \, dt + \int_{\tau}^{T} (\nabla u, \nabla b) \, dt
= \int_{\tau}^{T} (f(t), \phi) \, dt - \int_{\tau}^{T} (g(t), \phi) \, dt + \mu(\tau)(v, \phi) + \rho(\tau)(u, b) = 0. \quad (16)
\]

The following theorem holds true

**Theorem 2.** Under assumptions (A1)-(A4), (F1)-(F3), (G1)-(G2) problem \( \{1\} \)–\( \{5\} \) generates a strongly continuous process \( U(t, \tau) : H_\tau \to H_\tau, t \geq \tau \in \mathbb{R} \), satisfying the following continuous dependence property: for every pair of initial data \( W^1_\tau = (v_1^0, u_1^0, u_1^{i1}) \in H_\tau \) such that \( \|W^1_\tau\|_{H_\tau} \leq R, i = 1, 2, R > 0 \) the difference of the corresponding solutions satisfies

\[ \|U(t, \tau)W^1_\tau - U(t, \tau)W^2_\tau\|_{H_\tau} \leq e^{K(t-\tau)}\|W^1_\tau - W^2_\tau\|_{H_\tau}, t \geq \tau, \quad (17) \]

for some constant \( K = K(R) \geq 0 \).

The energy equality

\[
\mathcal{E}(v(t), u(t), u_i(t)) + \int_{\tau}^{t} \frac{1}{2} \int_{\Omega} \|v\|^2 \, dx \, ds - \frac{1}{2} \int_{\tau}^{t} \mu'(s) \|v\|^2 \, ds - \frac{1}{2} \int_{\tau}^{t} \rho'(s) \|u_i\|^2 \, ds
= \mathcal{E}(v, u_0, u^{i1}) + \int_{\tau}^{t} (f, v) \, dx + \int_{\tau}^{t} (g, u_i) \, dx \quad (18)
\]

holds for every \( t > \tau \), where the energy functional \( \mathcal{E} \) is defined by the relation

\[ \mathcal{E}(v, u, u_i) = \mathcal{E}(v, u, u_i) + \int_{\Omega} \nabla (u) \, dx, \quad (19) \]

here

\[ E(v, u, u_i) = \frac{1}{2} \left[ \mu(t)\|v\|^2 + \rho(t)\|u_i\|^2 + \|\Delta u\|^2 \right]. \quad (20) \]

**Proof.** The proof is quite standard and relies on the method of Galerkin approximations (see e.g.). We place it here for the sake of completeness.

**Step 1. Existence.**
Let \( \{e_i = (e_{i1}, e_{i2})\}_{i \in \mathbb{N}} \) be the orthonormal basis in \( \tilde{X}_i = \{v \in X_i : (v, n)_\Omega = 0\} \) consisting of the eigenvectors of the Stokes problem:

\[
-\Delta e_i + \nabla p_i = \lambda_i e_i \quad \text{in} \ \mathcal{O}, \quad \text{div} e_i = 0, \quad e_i|_{\partial \mathcal{O}} = 0,
\]

where \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \) are the corresponding eigenvalues. The existence of solutions to \((21)\) can be shown in the same way as in \(\[16\]\).

We define an approximate solution as a pair of functions \( (\hat{v}_1, \hat{u}_1) \) which consists of eigenfunctions of the operator \( \tilde{A} \). It is easy to see that \( (21) \) possesses properties \(\[1\]\)

\[
\operatorname{N} u = v \iff \left\{ \begin{array}{ll}
-\Delta v + \nabla p = 0, & \text{div} v = 0 \quad \text{in} \ \mathcal{O}; \\
\nu = 0 & \text{on} \ \partial \mathcal{O} \setminus \Omega; \quad v = u \quad \text{on} \ \Omega.
\end{array} \right.
\]

Operator \( \operatorname{N} \) possesses properties \(\[1\]\)

\[
\operatorname{N} : [\tilde{X}_1(\Omega)]^2 \hookrightarrow [H^{1/2}(\Omega)]^3 \cap X_t
\]

continuously for every \( s \geq -1/2 \) and

\[
\|\operatorname{N} u\|_{2+s, \Omega} \leq C\|u\|_{s, \Omega}^\alpha, \quad u \in [H^s(\Omega)]^2.
\]

We also introduce a positive self-adjoint operator \( \Delta^2 \) with the domain \( \mathcal{D}(\Delta^2) = (H^4 \cap H^2_0)(\Omega) \). It is easy to see that \( \mathcal{D}(\Delta^{1/2}) = H^1_0(\Omega) \). Denote by \( \{g_i\}_{i \in \mathbb{N}} \) the orthonormal basis in \( L^2(\Omega) \) which consists of eigenfunctions of the operator \( \Delta^2 \)

\[
\lambda_i = \kappa_i g_i
\]

with the eigenvalues \( 0 < \kappa_1 \leq \kappa_2 \leq \cdots \).

Let \( \varphi_i = N g_i \), where the operator \( \operatorname{N} \) is defined by \(\[22\]\).

We define an approximate solution as a pair of functions \( (v_{n,m}, u_n) \):

\[
v_{n,m}(t) = \sum_{i=1}^{m} \alpha_i(t) e_i + \sum_{j=1}^{2n} \tilde{\beta}_j(t) \varphi_j, \quad u_n(t) = \sum_{j=1}^{2n} \beta_j(t) g_j
\]

which satisfy the relations

\[
\mu(t) \left( \sum_{i=1}^{m} \dot{\alpha}_i(t)(e_i, \varphi_k)_\Omega + \sum_{j=1}^{2n} \dot{\beta}_j(t)(\varphi_j, \varphi_k)_\Omega \right) + \lambda_i \alpha_i(t) + \sum_{j=1}^{2n} \dot{\beta}_j(t)(\nabla \varphi_j, \nabla e_k)_\Omega = (f, e_k)_\Omega
\]

for \( k = 1, \ldots, m \), and

\[
\mu(t) \left( \sum_{i=1}^{m} \dot{\alpha}_i(t)(e_i, \varphi_k)_\Omega + \sum_{j=1}^{2n} \dot{\beta}_j(t)(\varphi_j, \varphi_k)_\Omega \right) + \rho(t) \dot{\beta}_k(t) + \sum_{i=1}^{m} \alpha_i(t)(\nabla e_i, \nabla \varphi_k)_\Omega + \sum_{j=1}^{2n} \dot{\beta}_j(t)(\nabla \varphi_j, \nabla e_k)_\Omega + \kappa_k \beta_k(t) +
\]

\[
+ (F(u_n(t)), g_k) = (f(t), \varphi_k)_\Omega + (g(t), g_k)_\Omega
\]

for \( k = 1, \ldots, 2n \). This system of ordinary differential equations \(\[27\]–\[28\]\) is endowed with the initial data

\[
v_{n,m}(\tau) = \Pi_m (v_{\tau} - N u^1_{\tau}) + N P_n u^1_{\tau},
\]

\[
u_n(\tau) = P_n u^0_{\tau}, \quad \dot{u}_n(\tau) = P_n u^1_{\tau},
\]
where $\Pi_m$ is an orthoprojector on $\text{Lin}(e_j : j = 1, \ldots, m,)$ in $\tilde{X}_r$. $P_n$ is an orthoprojector on $\text{Lin}(g_i : i = 1, \ldots, n)$ in $L^2_\tau(\Omega)$. Since $\Pi_m$ and $P_n$ are spectral projectors we have that

$$(v_{n,m}(\tau); u_n(\tau); u\epsilon_n(\tau)) \rightarrow (v; u; u\epsilon),$$ strongly in $H_\epsilon$, $m, n \rightarrow \infty$. \hfill (29)

Arguing as in [8] we infer that system (27) and (28) has a unique solution on any time interval $[\tau, T]$. It follows from (26) that

$$v_{n,m}(t) = \sum_{i=1}^m \alpha_i(t) e_i + N[\partial_t u_n(t)],$$

where $N$ is given by (22). This implies the following boundary compatibility condition

$$v_{n,m}(t) = \partial_t u_n(t) \quad \text{on } \Omega. \hfill (30)$$

Multiplying (27) by $\alpha_i(t)$ and (28) by $\beta_i(t)$, after summation we obtain an energy relation of the form (18) for the approximate solutions $(v_{n,m}; u_n)$ (for a similar argument we refer to [8]). Assumptions (A2), (F2), (F3), (G1) together with the trace theorem imply the following a priori estimate:

$$\sup_{t \in [\tau, T]} \left[ \mu(t)\|v_{n,m}(t)\|_{C_0}^2 + \rho(t)\|\partial_t u_n(t)\|_{C_1}^2 + \|\Delta u_n(t)\|_{C_1}^2 \right]$$

$$+ \int_\tau^T \|\nabla v_{n,m}(t)\|_{C_0}^2 dt + \int_\tau^T \|\partial_t u_n(t)\|_{[H^{1/2};(\Omega)]}^2 dt \leq C(T, \|W\|_{H_\epsilon}) \hfill (31)$$

for any existence interval $[\tau, T]$ of approximate solutions, where the constant $C(T, \|W\|_{H_\epsilon})$ does not depend on $n$ and $m$. In particular, this implies that any approximate solution can be extended on any time interval by the standard procedure, i.e., the solution is global.

It also follows from (31) that the sequence $\{(v_{n,m}; u_n; \partial_t u_n)\}$ contains a subsequence such that

$$(v_{n,m}; u_n; \partial_t u_n) \rightharpoonup (v; u; \partial_t u) \quad *\text{-weakly in } L^\infty(\tau, T; H_r), \hfill (32)$$

$$v_{n,m} \rightarrow v \quad \text{weakly in } L^2(\tau, T; H_\epsilon). \hfill (33)$$

Moreover, by the Aubin-Dubinsky theorem (see, e.g., [14, Corollary 4]) we can assert that

$$u_n \rightarrow u \quad \text{strongly in } C(\tau, T; \tilde{H}^{2-\epsilon}_0(\Omega)) \hfill (34)$$

for every $\epsilon > 0$. Besides, the trace theorem yields

$$\partial_t u_n \rightharpoonup \partial_t u \quad \text{weakly in } L^2(\tau, T; [H^{1/2};(\Omega)]^2). \hfill (35)$$

One can see that $(v_{n,m}; u_n; \partial_t u_n(t))$ satisfies (16) with the test function $\phi$ of the form

$$\phi = \phi_{l,q} = \sum_{i=1}^l \gamma_i(t)e_i + \sum_{j=1}^q \delta_j(t)\varphi_j, \hfill (36)$$

where $l \leq m, q \leq n$ and $\gamma_i, \delta_j$ are scalar absolutely continuous functions on $[\tau, T]$ such that $\gamma_i, \delta_j \in L^2(\tau, T)$ and $\gamma(T) = \delta(T) = 0$. Thus using (32)–(33) we can pass to the limit and show that $(v; u; \partial_t u)$ satisfies (27)–(28) with $\phi = \phi_{l,q}$, where $l$ and $q$ are arbitrary. By (29) and (34) we have $W(t) = W_r$. Compatibility condition (15) follows from (30) and (35).

To conclude the proof of the existence of weak solutions we only need to show that any function $\psi$ in $\mathcal{L}^0_T$ can be approximated by a sequence of functions of the form (36). This
can be done in the following way. We first approximate the corresponding boundary value of $b$ by a finite linear combination $h$ of $\xi_j$, then we approximate the difference $\psi - Nh$ (with $N$ define by (22)) by finite linear combination of $\epsilon_i$. Limit transition in nonlinear terms is quite standard, so we omit it here. Thus the existence of weak solutions is proved.

Step 2. Energy equality.

To prove the energy equality for a weak solution we follow the scheme presented in (11). We introduce a finite difference operator $D_h$, depending on a small parameter $h$. Let $g$ be a bounded function on $[\tau, T]$ with values in some Hilbert space. We extend $g(t)$ for all $t \in \mathbb{R}$ by defining $g(t) = g(0)$ for $t < \tau$ and $g(t) = g(T)$ for $t > T$. With this extension we denote

$$
g_h^+(t) = g(t + h) - g(t), \quad g_h^-(t) = g(t) - g(t - h), \quad D_h g(t) = \frac{1}{2h} (g_h^+(t) + g_h^-(t)).
$$

Properties of the operator $D_h$ are collected in Proposition 4.3 (11).

Taking in (16) $\phi(t) = \int_{\tau}^{t} \chi(s)ds \cdot \phi$, where $\chi$ is a smooth scalar function and $\phi$ belongs to the space

$$
\tilde{Y} = \left\{ \phi \in Y \left| \phi|_{\Omega} = b \in \tilde{H}_0^2(\Omega) \right. \right\},
$$

one can see that the weak solution $(v(t); u(t))$ satisfies the relation

$$
\begin{align*}
\mu(t)(v(t), \phi)_{\Omega} + \rho(t)(u(t), b)_{\Omega} = (v, \phi)_{\Omega} + (u^1, b)_{\Omega} + \int_{\tau}^{t} \left[ \frac{1}{2} \mu'(s)(v, \phi)_{\Omega} \\
+ \frac{1}{2} \rho'(s)(u, b)_{\Omega} - (\nabla v, \nabla \phi)_{\Omega} - (\Delta u, \Delta \phi)_{\Omega} + (F(u), b)_{\Omega} + (g, \phi)_{\Omega} + (f, \phi)_{\Omega} \right] ds \quad (38)
\end{align*}
$$

for all $t \in [\tau, T]$ and $\phi \in \tilde{Y}$ with $\phi|_{\Omega} = b$.

The vector $(v(t), u(t), u_\tau(t))$ is weakly continuous in $H$ for any weak solution $(v(t), u(t))$ to problem (11–15). Indeed, it follows from (38) that $(v(t), u(t))$ satisfies the relation

$$
\begin{align*}
\mu(t)(v(t), \phi)_{\Omega} = \mu(\tau)(v, \phi)_{\Omega} + \int_{\tau}^{t} \left[ \frac{1}{2} \mu'(s)(v, \phi)_{\Omega} - (\nabla v, \nabla \phi)_{\Omega} - (f(s), \phi)_{\Omega} \right] ds
\end{align*}
$$

for almost all $t \in [\tau, T]$ and for all $\phi \in Y = \{ \phi \in Y : \phi|_{\Omega} = 0 \} \subset \tilde{Y} \subset Y$, where $\tilde{Y}$ is given by (37). This implies that $v(t)$ is weakly continuous in $Y'_{\theta}$. Since $X_{\theta} \subset Y'_{\theta}$, for any $\tau < t < T$ we can apply Lions lemma (see [13] Lemma 8.1)) and conclude that $v(t)$ is weakly continuous in $X_{\theta}$. The same lemma gives us weak continuity of $u(t)$ in $\tilde{H}_0^2(\Omega)$. Now using (38) again with $\phi \in \tilde{Y}$ we conclude that $t \mapsto (u(t), b)_{\Omega}$ is continuous for every $b \in \tilde{H}_0^2(\Omega)$. This implies that $t \mapsto u_\tau(t)$ is weakly continuous in $[L_2(\Omega)]^2$. Using weak continuity of weak solutions, we can extend the variational relation in (16) on the class of test functions from $L_T$ (instead of $L_T^0$) by an appropriate limit transition. More precisely, one can show that
any weak solution \((v; u)\) satisfies the relation
\[
\begin{align*}
& - \int_\tau^T \mu(t)(v, \phi_t)_{\Omega} dt + \int_\tau^T (\nabla v, \nabla \phi)_{\Omega} dt - \int_\tau^T \rho(t)(u_t, b_t)_{\Omega} dt + \int_\tau^T (F(u), b) dt \\
& + \int_\tau^T (\Delta u, \Delta b)_{\Omega} dt = (v, \phi(\tau))_{\Omega} + (u^1, b(\tau))_{\Omega} - \mu(T)(v(T), \phi(T))_{\Omega} \\
& - \rho(T)(u_T(T), b(T))_{\Omega} + \frac{1}{2} \int_\tau^T \mu'(t)v, \phi dt + \frac{1}{2} \int_\tau^T \rho'(t)u_t, b dt \\
& + \int_\tau^T (f, \phi)_{\Omega} dt + \int_\tau^T (g, b)_{\Omega} dt,
\end{align*}
\]
for every \(\psi = (\phi, b) \in \mathcal{L}_T\).

Let \((v(t), t)\) be a weak solution to problem \((11-5)\). Now we use
\[
\phi = \frac{1}{2h} \int_{t-h}^{t+h} v(s) ds
\]
as a test function in \((39)\). For the shell component we have test function \(b = \phi|_\Omega = D_h u\) – the same one that used in \((11)\) for the full Karman model.

Arguing as in the proof of Proposition 4.3 \((11)\) we can infer
\[
\begin{align*}
& \lim_{h \to 0} \left\{ \int_\tau^T \mu(t)(v(t), D_h v(t))_{\Omega} dt - \frac{1}{2} \int_\tau^T \mu'(t) v(t), \int_{t-h}^{t+h} v(s) ds)_{\Omega} dt \right\} \\
& = \frac{1}{2} \left( \mu(T) ||v(T)||^2_{\Omega} - \mu(T) ||v(T)||^2_{\Omega} \right) \quad (41)
\end{align*}
\]
\[
\begin{align*}
& \lim_{h \to 0} \left\{ \int_\tau^T \rho(t)(u_t(t), D_h u_t(t))_{\Omega} dt - \frac{1}{2} \int_\tau^T \rho'(t) u_t(t), D_h u(t)_{\Omega} dt \right\} \\
& = \frac{1}{2} \left( \rho(T) ||u_T(T)||^2_{\Omega} - \rho(T) ||u_T(T)||^2_{\Omega} \right) \quad (42)
\end{align*}
\]
Then, relying on \((39), (41),\) and \((42)\) we can conclude the proof. All the arguments for the fluid component in our model are the same as in \([7]\), and the arguments for the plate component are analogous to those presented in the proof of Lemma 4.1 \((11)\). This makes it possible to prove the energy equality in \((18)\).

Continuity of weak solutions with respect to \(t\) can be obtained in the standard way from the energy equality and weak continuity (see \([13, \text{Ch. 3]}\) and also \((11)\)).

**Step 3. Continuity with respect to the initial data and uniqueness.**

It follows from energy estimate \((18)\) and (F3) that if \(||W_r||_{\mathcal{H}} \leq R\), then there exists \(C(R) > 0\) such that \(||U(t, \tau)W_r||_{\mathcal{H}} \leq C(R)\). Consequently, the Gronwall lemma and (F1) yield estimate \((17)\). The uniqueness of solutions follows.

Now we are in position to show the existence of a time-dependent absorbing family.
Lemma 1. Let \( t \geq \tau \). Let \( U(t, \tau)W, \) be the solution of (11–15) with initial time \( \tau \) and initial data \( W_0 \in H_\tau \). Then, if (F4) holds, there exist \( \omega > 0, K \geq 0 \) and an increasing positive function \( Q \) such that

\[
\|U(t, \tau)W\|_{\mathcal{H}} \leq Q(\|W\|_{\mathcal{H}}) e^{-(t-\tau)} + K, \quad \tau \leq t.
\]  

(43)

Proof. We construct the Lyapunov functional of the form

\[
L(t) = \mathcal{E}(t) + \delta (\mu(t)(v, Nu)_{\Omega} + \rho(t)(u_i, u)_{\Omega})
\]

(44)

It is easy to see from (F3) and the properties of the operator \( N \) that there exist \( c_i > 0, i = 1, 4 \) such that

\[
-c_1 + c_2 E(t) \leq L(t) \leq c_3 \mathcal{E}(t) + c_4,
\]

where \( E(t) \) is defined by (20). All the calculations below can be performed on Galerkin approximations. It follows from the energy inequality (18) and (F4) that

\[
\frac{d}{dt} L(t) = -\|\nabla v\|_{\Omega}^2 + \rho'(|v|)|u_i|_{H^2}^2 + \mu'(t)\|v\|_{H^2}^2 + (f, v)_{\Omega} + (g, u)_{\Omega} + \delta \mu(t)(v, Nu)_{\Omega}
\]

\[
- \delta \|\Delta u\|_{\Omega}^2 + \delta \rho(t)\|u_i\|_{H^2}^2 + \delta \mu'(t)(u_i, u)_{\Omega} + \delta \mu'(t)(v, Nu)_{\Omega}
\]

\[
- \delta (F(u), u)_{\Omega} + \delta (f, Nu)_{\Omega} + \delta (g, u)_{\Omega} - \delta (\nabla v, \nabla Nu)_{\Omega} \leq -\omega L(t) + C(\|f\|_{\Omega}^2 + \|g\|_{-1/2, \Omega}^2)
\]

for some \( \omega, C > 0 \). Consequently,

\[
\frac{d}{dt} L(t) + \omega L(t) \leq C(\|f\|_{\Omega}^2 + \|g\|_{-1/2, \Omega}^2)
\]

and using the Gronwall lemma, (G2), (F2), and (45) we come to (43). \( \square \)

Lemma 1 yields the existence of a time-dependent absorbing family with the entering time \( \Theta = \max\{0, \frac{1}{\omega} \log \frac{Q(R)}{\tau \mathcal{K}}\} \).

5 Pullback attractor.

In order to establish the existence of a pullback attractor to the system considered, our remaining task is to show estimate (3).

Lemma 2. Let \( W^1(t) = (v^1(t), u^1(t), u_i^1(t)), i = 1, 2 \) be two weak solutions to problem (11–15) with initial conditions \( W^1_0 \in H_{t=T_0}, \|W^1\|_{H_{t=T_0}} \leq R \). Then, for any \( \varepsilon > 0 \) there exists \( T_0 > 0 \) and a positive constant \( C(t, T_0) \) such that

\[
\|W^1(t) - W^2(t)\|_{H_t} \leq \varepsilon + C(T_0, R) \max_{[r-T_0]} (\|u^1(s) - u^2(s)\|_{L^2(\Omega)}^2)
\]

(46)

for any \( \varepsilon > 0 \).

Proof. We use the notations \( W(t) = (v(t), u(t), u_i(t)) = W^1(t) - W^2(t) \). It follows from the energy inequality that

\[
\frac{d}{dt} \mathcal{E}(\xi) \leq -\|\nabla v\|_{\Omega}^2 + \mu'(\xi)\|v\|_{\Omega}^2 + \rho'(\xi)|u_i|_{\Omega}^2 + (F(u^1) - F(u^1), u_i)_{\Omega}.
\]

(47)
Integrating (47) over the interval \([s,t]\) and then \([t - T_0, t]\) we come to

\[
T_0 E(t) \leq \int_{t-T_0}^{t} E(s) ds - \int_{t-T_0}^{t} \int_{s}^{t} \rho \|v\|_{L^2(\Omega)}^2 ds + \int_{t-T_0}^{t} \int_{s}^{t} \mu'(\xi) \|v\|_{L^2(\Omega)}^2 ds
+ \int_{t-T_0}^{t} \int_{s}^{t} \rho'(\xi) \|u\|_{L^2(\Omega)}^2 ds + \int_{t-T_0}^{t} \int_{s}^{t} (F(u^1) - F(u^1), u\xi)_{\Omega} d\xi ds. \tag{48}
\]

It follows from the trace theorem and assumption (F1) that for any \(\sigma > 0\)

\[
\left| \int_{t-T_0}^{t} \int_{s}^{t} (F(u^1) - F(u^1), u\xi)_{\Omega} d\xi ds \right| \leq \sigma \int_{t-T_0}^{t} ||\nabla v||_{L^2(\Omega)}^2 ds + C(T_0, R, \sigma) \max_{\{t - T_0, t\}} ||u||_{L^2(\Omega)}^2. \tag{49}
\]

Integrating (47) over the interval \([t - T_0, t]\) and taking into consideration (A2) and (F1) we obtain

\[
E(t) + \int_{t-T_0}^{t} ||\nabla v||_{L^2(\Omega)}^2 ds \leq C(R) \int_{t-T_0}^{t} \int_{s}^{t} ||u||_{L^2(\Omega)}^2 ds + E(t-T_0) \leq C(T_0, R) \max_{\{t - T_0, t\}} ||u||_{L^2(\Omega)}^2 + C(R). \tag{50}
\]

It is a straightforward consequence of the trace theorem that

\[
\int_{t-T_0}^{t} E(s) ds \leq C \int_{t-T_0}^{t} ||\nabla v||_{L^2(\Omega)}^2 ds + \int_{t-T_0}^{t} ||u||_{L^2(\Omega)}^2 ds. \tag{51}
\]

Now we estimate the last term in (51). Substituting into (16) \(b = u\) and \(\phi = Nu\) and choosing \(\tau = t - T_0\) and \(T = t\) we arrive at

\[
\int_{t-T_0}^{t} ||u||_{L^2(\Omega)}^2 ds \leq \int_{t-T_0}^{t} \rho(s) ||u||_{L^2(\Omega)}^2 ds - \rho(t)(u(t), u(t))_{\Omega} + \rho(t - T_0)(u(t - T_0), u(t - T_0))_{\Omega}
+ \int_{t-T_0}^{t} (F(u^1) - F(u^2), u\xi)_{\Omega} ds + \int_{t-T_0}^{t} \mu(s)(v, Nu)_{\Omega} ds - \int_{t-T_0}^{t} (\nabla v, \nabla Nu)_{\Omega} ds
- \mu(t)(v(t), Nu(t))_{\Omega} + \mu(t - T_0)(v(t - T_0), Nu(t - T_0))_{\Omega} + \int_{t-T_0}^{t} \mu'(s)(v, Nu)_{\Omega} ds.
\]

Relying on the properties of the operator \(N\), the trace theorem, and (A3) we have the estimate

\[
\int_{t-T_0}^{t} ||u||_{L^2(\Omega)}^2 ds \leq C(T_0, R) \max_{\{t - T_0, t\}} ||u||_{L^2(\Omega)}^2 + \int_{t-T_0}^{t} ||\nabla v||_{L^2(\Omega)}^2 ds + C(R). \tag{52}
\]
Combining (48)–(52) and choosing $\sigma$ in (49) we obtain
\[ E(t) \leq \frac{C(R)}{T_0} + C(T_0, R) \max_{[t-T_0, t]} \|u\|_{2-\epsilon, \Omega}^2, \]
which leads immediately to the assertion of the lemma.

Now we formulate our main result. \(\square\)

**Theorem 3.** The process $U(t, \tau)$ generated by problem (1)–(5) possesses a pullback attractor.

**Proof.** There exists a time-dependent absorbing family and we have in hand Lemma 2. Therefore, by using Theorem 1 it remains to show that (9) holds true for $\Phi_{T_0, t}(W_1, W_2) = C(T_0, R) \max_{[t-T_0, t]} \|u^1(s) - u^2(s)\|_{2-\epsilon, \Omega}^2$. Let $W^n(t) = (v^n(t), u^n(t), u^n_\tau(t))$ be a sequence of solutions to problem (1)–(5) corresponding to initial data $W^n$, from $D_{T, T_0}$, i.e. $\|W^n\|_{H_1, T_0} \leq R$. Then, it follows from Lemma 1 that up to a subsequence
\[
\begin{align*}
    u^n_\tau(s) - u^m_\tau(s) &\to 0, \text{ weak-* in } L_\infty(t - T_0, t; \hat{H}_0^2(\Omega)), \\
    u^n(s) - u^m(s) &\to 0, \text{ weakly in } L_2(t - T_0, t; H_0^{1/2}(\Omega)).
\end{align*}
\]
By the Aubin’s compactness lemma [14], we have (9). This together with Theorem 1 completes the proof. \(\square\)

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**References**

[1] T. Caraballo, A. N. Carvalho, J.A. Langa, and F. Rivera. Existence of pullback attractors for asymptotically compact processes. *Cadernos De Matematica*, 10: 179–100, 2006.

[2] M. Conti, V. Pata, and R. Temam. Attractors for processes on time-dependent spaces. Applications to wave equations. *Journal of Differential Equations*, 255:1254–1277, 2012.

[3] I. Chueshov, A global attractor for a fluid-plate interaction model accounting only for longitudinal deformations of the plate, *Math. Methods Appl. Sci.* 34, 1801–1812.

[4] I. Chueshov and T. Fastovska, On interaction of circular cylindrical shells with a Poiseuille type flow, *Evolution Equations and Control Theory*, 5 (2016), 605–629.

[5] I. Chueshov and I. Lasiecka, Attractors for second order evolution equations, *J. Dynam. Diff. Eqs.*, 16 (2004), 469–512.

[6] I. Chueshov and I. Lasiecka, *Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping*, Memoirs of AMS, vol.195, no. 912, AMS, Providence, RI, 2008.
[7] I. Chueshov and I. Lasiecka, *Von Karman Evolution Equations*, Springer, New York, 2010.

[8] I. Chueshov and I. Ryzhkova, A global attractor for a fluid-plate interaction model, *Comm. Pure Appl. Anal.*, 12(2013), 1635–1656.

[9] F. Di Plinio, G.S. Duane, R. Temam Time dependent attractor for the oscillon equation. *Cont. Dyn. Sys.*, 29:141167, 2011.

[10] T. Fastovska, Long-time behaviour of a radially symmetric fluid-shell interaction system, *Discrete and Continuous Dynamical Systems-A*, 38, no.3 (2018), 1315–1348.

[11] H. Koch and I. Lasiecka, Hadamard well-posedness of weak solutions in nonlinear dynamic elasticity-full von Karman systems, *Prog. Nonlinear Differ. Equ. Appl.*, 50 (2002), 197–216.

[12] J. Lagnese, *Boundary Stabilization of Thin Plates*, SIAM, Philadelphia, 1989.

[13] J.-L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. 1, Dunod, Paris, 1968.

[14] J. Simon, *Compact sets in the space L\(^p\)(0, T; B)*, Ann. Mat. Pura Appl., Ser.4 148 (1987), 65–96.

[15] T.F. Ma, R.N. Monteiro, and A.C. Pereira, Pullback Dynamics of Non-autonomous Timoshenko Systems Appl. Math. Optim., (2017), https://doi.org/10.1007/s00245-017-9469-2.

[16] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, Reprint of the 1984 edition, AMS Chelsea Publishing, Providence, RI, 2001.

[17] H. Triebel, *Interpolation Theory, Functional Spaces and Differential Operators*, North Holland, Amsterdam, 1978.

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