Abstract: We establish a new version of Floer homology for monotone Lagrangian embeddings in symplectic manifolds. As applications, we get assertions for (monotone) Lagrangian submanifolds $L \hookrightarrow M$ which are displaceable through Hamiltonian isotopies (this happens for instance when $M = \mathbb{C}^n$). We show that when $L$ is aspherical, or more generally when the homology of its universal cover vanishes in odd degrees, its Maslov number $N_L$ equals 2. We also give topological characterisations of Lagrangians $L \hookrightarrow M$ with maximal Maslov number: when $N_L = \dim(L) + 1$ then $L$ is homeomorphic to a sphere; when $N_L = n \geq 6$ then $L$ fibers over the circle and the fiber is homeomorphic to a sphere. A consequence is that any exact Lagrangian in $T^*\mathbb{S}^{2k+1}$ whose Maslov class is zero is homeomorphic to $\mathbb{S}^{2k+1}$.

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1 Introduction and main results

1.1 Preliminaries

Let $(M^{2n}, \omega)$ be a symplectic manifold. A submanifold $L^n$ of $M$ is called Lagrangian if the restriction of $\omega$ on $L$ vanishes. Throughout this paper all

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symplectic manifolds are assumed to be either closed or convex at infinity and all Lagrangian submanifolds are assumed to be closed and connected. One of the fundamental questions in symplectic geometry is the following:

What properties has to satisfy a closed manifold \( L \) in order to admit a Lagrangian embedding into a given symplectic manifold \( M \) ?

This question is still widely open even in the case of \( M = \mathbb{C}^n \). The results of the present paper concern symplectic manifolds such as \( M = \mathbb{C}^n \), \( M = \mathbb{C}P^n \), or \( M = T^*K \) for \( K \) is closed, all of them being endowed with their standard symplectic form. We establish new topological constraints on Lagrangian submanifolds \( L \subset M \) which are monotone or exact. These notions are defined using the following two morphisms related to a given Lagrangian \( L \). The morphism \( I_\omega : \pi_2(M, L) \to \mathbb{R} \) is defined by:

\[
I_\omega(A) = \int_A \omega.
\]

In order to define the morphism \( I_\mu : \pi_2(M, L) \to \mathbb{Z} \), pick a smooth map of pairs \( w : (D^2, \partial D^2) \to (M, L) \) in the class \( A \in \pi_2(M, L) \). There is an unique trivialisation (up to homotopy) of the pull-back \( w^*TM \approx D^2 \times \mathbb{C}^n \) as a symplectic vector bundle. This gives a map \( \alpha_w \) from \( S^1 = \partial D^2 \) to \( \Lambda(\mathbb{C}^n) \) - the set of Lagrangian planes in \( \mathbb{C}^n \). On this space there is a well-known Maslov class \( \mu \in H^1(\Lambda(\mathbb{C}^n), \mathbb{Z}) \) (see [2]), so that one can define

\[
I_\mu(A) = \mu(\alpha) \in \mathbb{Z}.
\]

**Definition 1.1** A Lagrangian submanifold \( L \subset M \) is called weakly exact if the morphism \( I_\omega \) vanishes. It is called exact if \( \omega = d\lambda \) and the restriction \( \lambda|_L \) is an exact one-form.

A Lagrangian submanifold is called monotone if there is a constant \( \tau > 0 \) such that

\[
I_\omega = \tau I_\mu.
\]

By definition, only exact symplectic manifolds admit exact Lagrangian submanifolds. It is not obvious, but still true, that monotone Lagrangian submanifolds only exist in monotone symplectic manifolds (i.e. in symplectic manifolds in which the morphism defined on \( \pi_2(M) \) by the symplectic form is a positive multiple of the morphism defined by the first Chern class). Many authors studied monotone and exact Lagrangians and found various obstructions to the existence of such embeddings. A celebrated result of M. Gromov asserts:
Theorem 1.2 [27] There is no weakly exact Lagrangian embedding $L \subset C^n$.

The results on the obstructions to the existence of monotone Lagrangian submanifolds mostly concern their Maslov number. This number, denoted by $N_L$, is defined as the minimal positive integer which is in the image of $I_\mu$. In 1996 Y.-G. Oh established the following inequality [35], improving thus previous results of C. Viterbo [41] and L. Polterovich [37], [38]:

**Theorem 1.3** For any monotone Lagrangian submanifold $L \subset C^n$ we have:

$$1 \leq N_L \leq n.$$

These bounds turn out to be sharp. Indeed, L. Polterovich gave in [38] an example of a monotone Lagrangian $L \subset C^n$ which satisfies $N_L = n$.

Note that both Gromov’s and Oh’s result can be stated in the more general case of symplectic manifolds $M$ which are convex at infinity and have the property that any compact subset is displaceable through a Hamiltonian isotopy. This means that for any compact $K \subset M$ there is a Hamiltonian isotopy $(\phi_t)_{t \in [0,1]}$ such that $\phi_1(K) \cap K = \emptyset$. Symplectic manifolds of the form $C \times W$, or subcritical Stein manifolds [7] satisfy this assumption.

However, in this more general case the statement of Oh’s result is slightly different:

**Theorem 1.4** Let $M$ be a symplectic manifold in which every compact subset is displaceable through a Hamiltonian isotopy. For any monotone Lagrangian submanifold $L \subset M$ we have:

$$1 \leq N_L \leq n + 1$$

and if $N_L = n + 1$, then $L$ is a $\mathbb{Z}/2$-homology sphere.

Actually, more recent results of K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono [23], imply that for $N_L = n + 1$ the Lagrangian $L$ is a $\mathbb{Z}$-homology sphere in the statement above.

### 1.2 Main results

Our results about monotone Lagrangian submanifolds are of two types. First we show that under some topological assumptions on $L$ we have $N_L = 2$. Then, we study the topology of monotone Lagrangian submanifolds with maximal Maslov number $N_L = n + 1$ or $N_L = n$. Here are the statements:
Theorem 1.5 Let $M$ be a monotone symplectic manifold which has the property that any compact subset is displaceable through a Hamiltonian isotopy. Let $L \subset M$ be a monotone Lagrangian submanifold.

(a) If $L$ is aspherical (i.e. $L$ is an Eilenberg-Mac Lane space $K(\pi_1(L),1)$), then $N_L = 2$ if $L$ is orientable and $N_L \in \{1, 2\}$ if $L$ is not orientable.

(b) Denote by $\tilde{L}$ the universal cover of $L$. If $L$ is orientable and has the property

$$H_{2i+1}(\tilde{L}, \mathbb{Z}/2) = 0$$

for any integer $i$ then $N_L = 2$.

(c) Moreover, for any almost complex structure $J$ which is compatible with the symplectic form, a Lagrangian $L$ which satisfies the condition (b) has the property that through every $p \in L$ there is a $J$-holomorphic disk $w : (D, \partial D) \to (M, L)$ such that:

- The Maslov index $\mu(w)$ equals 2.
- $p \in w(\partial D)$.
- $w(\partial D)$ is non zero in $\pi_1(L)$.

Remarks

1. Part (a) of Theorem 1.5 was proved by K. Fukaya for general aspherical Lagrangian submanifolds, but under the additional hypothesis that $L$ is orientable and relatively spin ([22], Th. 12.2). In the case where $L$ is a torus the statement was conjectured by M. Audin [4]. In this particular case, many results were previously obtained by L. Polterovich, C. Viterbo, Y.-G. Oh, Y. Eliashberg, P. Biran, K. Cieliebak and L. Buhovsky.

2. Part (b) of the statement above was proved by K. Fukaya in the case $L = S^1 \times S^{2m}$ without any monotonicity assumption ([22], Th. 13.1). However, our result applies to many more general examples, such as arbitrary products of tori (or other orientable aspherical manifolds) and complex projective spaces, even-dimensional spheres, etc.

3. Many results related to part (c) of the theorem can be found in the paper [10] of P. Biran and O. Cornea. Using their terminology, the Lagrangian $L$ should be called uniruled of type $(0, 1)$ and order 2.

Using the ideas of P. Biran [6], we obtain the following corollary on the monotone Lagrangian submanifolds in the complex projective space:
Theorem 1.6 Let $W$ be a symplectic manifold such that $M = \mathbb{CP}^n \times W$ is monotone (for instance this holds for $\pi_2(W) = 0$ or for $W = \mathbb{CP}^n$). Let $L \subset M$ be a monotone Lagrangian submanifold which is aspherical. Then $N_L = 2$ if $L$ is orientable and $N_L \in \{1, 2\}$ if $L$ is not orientable.

For spin Lagrangian submanifolds and $W = \text{point}$ this result was also proved by K. Fukaya in [22] without any monotonicity assumption. The result is still true in the more general situation where $\mathbb{CP}^n$ is replaced by a symplectic manifold which arises as a hypersurface in a subcritical polarisation. These manifolds were studied in [7] by P. Biran and K. Cieliebak.

Our next result is a topological characterisation of monotone Lagrangian submanifolds with maximal Maslov number.

Theorem 1.7 Let $M$ be a monotone symplectic manifold of dimension $2n$, which has the property that any compact subset is displaceable through a Hamiltonian isotopy. Let $L \subset M$ be a monotone Lagrangian submanifold.

(a) Suppose that $N_L = n + 1$ and $n \geq 2$. Then $n$ is odd and $L$ is homeomorphic to a $n$-sphere.

(b) Suppose that $N_L = n$ and $n \geq 3$.

If $n$ is odd then $\pi_1(L)$ has an infinite cyclic group $G \approx \mathbb{Z}$ of finite index. If moreover $M$ is an exact symplectic manifold then there is an exact sequence of groups

$$0 \rightarrow K \rightarrow \pi_1(L) \rightarrow \mathbb{Z} \rightarrow 0,$$

where $K$ is finite and has odd order.

If $n$ is even then $\pi_1(L) \approx \mathbb{Z}$. If moreover $n \geq 6$, then there is a fibration of $L$ over the circle $S^1$ whose fiber is homeomorphic to the $(n - 1)$-sphere.

Remarks

1. There are examples of monotone Lagrangian submanifolds satisfying the hypothesis on the Maslov number above. Indeed the embedding of $S^{2k+1}$ into $\mathbb{CP}^k \times \mathbb{C}^{k+1}$ given by

$$z \mapsto ([z], \bar{z})$$

is monotone, Lagrangian and its Maslov number is $2k + 2$. This example is due to M. Audin, F. Lalonde and L. Polterovich [3]. An example of a
monotone Lagrangian embedding $S^1 \times S^{2k-1} \subset \mathbb{C}^{2k}$ whose Maslov number equals $2k$ was constructed by L. Polterovich in [38].

2. In [26] A. Gadbled established topological constrains on monotone Lagrangian submanifolds in cotangent bundles which have a large Maslov number (which implies that they are not displaceable through Hamiltonian isotopies).

We prove the following corollaries of this theorem:

**Theorem 1.8**

(a) Let $X$ be a symplectic manifold of dimension $2n + 2$ with $\pi_2(X) = 0$. Let $L \subset \mathbb{C}P^n \times X$ be a Lagrangian submanifold such that $H_1(L, \mathbb{Z}) = 0$. Then $L$ is homeomorphic to $S^{2n+1}$.

(b) Let $L \subset \mathbb{C}P^n \times \mathbb{C}P^n$ be a Lagrangian submanifold such that $H_1(L, \mathbb{Z})$ vanishes. Then $L$ is simply connected and there is a circle fibration $S^{2n+1} \to L$.

(c) Let $L \subset \mathbb{C}P^n$ be a Lagrangian submanifold with $2H^1(L, \mathbb{Z}) = 0$. Then if $n$ is odd we have $\pi_1(L) = \mathbb{Z}/2\mathbb{Z}$ and the universal cover of $L$ is homeomorphic to $S^n$.

In [8] O. Cornea and P. Biran asked whether a Lagrangian as in Theorem 1.8(c) is diffeomorphic (or homeomorphic) to $\mathbb{R}P^n$. Our result goes in this direction but we do not know whether its conclusion implies that $L$ is homeomorphic to the projective space. In the mentioned paper Biran and Cornea proved that under the hypothesis of Theorem 1.8 the cohomology ring (with $\mathbb{Z}/2$-coefficients) of $L$ is isomorphic to the cohomology ring of $\mathbb{R}P^n$. Similar results were previously obtained by P. Biran [6] and P. Seidel [39]. Statement (a) generalizes Th. B of [6] (asserting that $L$ is a homology sphere). Statement (b) generalizes Th. C of [6] (which asserts that $L$ has the homology of $\mathbb{C}P^n$).

**Theorem 1.9**

(a) Let $L \subset T^*S^{2k+1}$ be an exact Lagrangian submanifold with vanishing Maslov class. Then $L$ is homeomorphic to $S^{2k+1}$.

(b) Let $K^{2k+1}$ be a manifold whose universal cover is $S^{2k+1}$. Let $L \subset T^*K$ be an exact Lagrangian submanifold with vanishing Maslov class. Then the universal cover $\tilde{L}$ is homeomorphic to $S^{2k+1}$ (in particular $\pi_1(L)$ is finite). For instance, when $K = \mathbb{R}P^{2k+1}$ then $\pi_1(L) = \mathbb{Z}/2\mathbb{Z}$ and $L$ is double covered by (a manifold homeomorphic to) $S^{2k+1}$. 


This result gives an answer in the case $K = S^{2k+1}$ (under the hypothesis of the vanishing Maslov class) to an open question raised by V.I. Arnold [1):

Is an exact submanifold $L \subset T^*K$ homeomorphic to $K$?

Actually, Arnold asks whether $L$ is Hamiltonian isotopic to the zero section, but this latter question seems out of reach, except for the case $\dim(N) = 2$ (see [28] for related results). In the general case the most striking result was obtained by K. Fukaya, P. Seidel and I. Smith, who proved in [24], [25] that, when $L$ is relatively spin with vanishing Maslov class, its cohomology is isomorphic to the cohomology of $K$. For the case $K = S^m$ similar results were previously obtained by P. Seidel [39] and L. Buhovsky [11].

1.3 Idea of the proof

Recall first the definition of Floer homology. Let $L \subset M$ be a Lagrangian submanifold which is monotone with $N_L \geq 2$, or weakly exact. Consider a Hamiltonian isotopy $(\phi_t)$ defined by a time-dependent Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$ and a time-dependent almost complex structure $J$ which is compatible to the symplectic form $\omega$. For a generic choice of the couple $(H, J)$, A. Floer associated to these data a complex $(C^\cdot(H), \partial_J)$ whose homology does not depend on $(H, J)$ [17], [18], [19]. The Floer complex is free over $\mathbb{Z}/2$, spanned by the intersections $L \cap \phi_1(L)$, which are supposed to be transverse. Its differential $\partial_J$ is defined by counting the isolated holomorphic strips

$$v : \mathbb{R} \times [0, 1] \to M,$$

with boundary in $L \cup \phi_1(L)$ (more precisely $v(\mathbb{R} \times \{i\}) \subset \phi_i(L)$ for $i = 0, 1$) and joining intersection points $x, y \in L \cap \phi_1(L)$, which means that

$$\lim_{s \to -\infty} v(s, t) = x \quad \text{and} \quad \lim_{s \to +\infty} v(s, t) = y.$$

If $n(x, y)$ is the number modulo 2 of such curves then

$$\partial_J(x) = \sum_{y \in L \cap \phi_1(L)} n(x, y)y.$$

Note that if $L$ is orientable and relatively spin, (meaning that the second Stiefel-Whitney class $w_2(L)$ lies in the image of $H^2(M, \mathbb{Z}/2) \to H^2(L, \mathbb{Z}/2)$), then the whole theory works for integer coefficients [23].
A relation between the Floer homology $HF(L)$ and the usual homology can be established. At this end, one should remark that given a Morse function $f : L \to \mathbb{R}$ which is sufficiently $C^2$-small, its graph

$$\{(df_q, q) \mid q \in L\} \subset T^*L$$

can be embedded in $M$ via a Weinstein neighborhood $U(L) \subset M$ and the intersection points $L \cap L_f$ correspond to the critical points of $f$. Moreover, for a good choice of the almost complex structure $J$, the application

$$v(s, t) \mapsto v(s, 0)$$

defines a one-to-one correspondence between the holomorphic strips joining two intersection points $x, y$ which lie in $U(L)$ and the flow lines of a vector field on $M$ which is the gradient of $f$ with respect to some Riemannian metric. So the Morse complex becomes a sub-complex of the Floer complex in this case. If $L$ is weakly exact, and $f$ is chosen sufficiently small, one can prove that no holomorphic strip leaves $U(L)$, so that the two complexes are actually isomorphic and thus Floer homology is isomorphic to usual homology. In the case where $L$ is monotone, Y.-G. Oh shows in [35] that in the case of the particular Hamiltonian isotopy defined by the graph of a small function $f$, the Floer differential decomposes into a sum

$$\partial_J = \partial_0 + \partial_1 + \partial_2 + \cdots,$$

where $\partial_0 : C_k(f) \to C_{k-1}(f)$ is the Morse differential and $\partial_l : C_k(f) \to C_{k-1+lnL}(f)$ for any integer $l$. By comparing the degrees of the $\partial_l$'s in the relation $\partial_2 = 0$ one easily sees that $\partial_1$ defines an application of degree $-1 + nL$ on the usual homology groups of $L$ and moreover that this application is actually a differential. On the resulting homology groups $\partial_2$ defines an application of degree $-1 + 2nL$ which again turns out to satisfy $\partial_2 = 0$, and so on... This feature of the Floer differential can be formalized in the existence of a spectral sequence which converges to the Floer homology $HF(L)$ and whose first page is built using the usual homology groups of $L$ [35], [6].

The main idea of our paper is the following. Fix a covering space $\tilde{L} \to L$. Given a Morse function on $L$ and an associated generic gradient, one can build a free complex, possibly infinite dimensional, by lifting its flow lines to $\tilde{L}$. The homology of this complex is the usual (singular) homology $H_*(\tilde{L})$. 

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Now consider the Floer complex associated to $L$ and to some generic pair $(H, J)$. Look at the collection of paths $s \mapsto v(s, 0) \subset L$ defined by the holomorphic strips $v : \mathbb{R} \times [0, 1] \to M$ which are counted in the Floer differential $\partial_J$. This collection of paths obviously contains all the information about the Floer complex; one can reconstruct it by counting the paths who have the same endpoints $x$ and $y$ and defining thus a differential on the free module spanned by all these endpoints.

On the other hand one can lift these paths to the covering space $\tilde{L}$. The question is:

Do the lifted paths define a complex?

This turns out to be true when $L$ is weakly exact or monotone with $N_L \geq 3$. But it fails for $N_L = 2$, as one can easily see by looking at the example of an embedded circle $L = S^1$ in $\mathbb{C}$ deformed by a small translation $\phi$, so that $L$ and $\phi(L)$ have two intersection points: in this case the lifts of the paths defined on $L$ by the holomorphic strips of the usual Floer complex do not define a differential on the universal cover of the circle.

When the complex on $\tilde{L}$ is defined, we call it lifted Floer complex. In order to compute its homology $FH^{\tilde{L}}(L)$ we consider a Hamiltonian isotopy defined by a graph $L_f$ of a small function on $L$. In this case the differential $\partial_0$ clearly defines the lifted Morse complex described above. Therefore, when it is defined, the lifted Floer homology $FH^{\tilde{L}}(L)$ has analogue features, namely it coincides with the (Morse) homology of $\tilde{L}$ if $L$ is weakly exact and it is the limit of a spectral sequence starting from $H_*(\tilde{L})$ when $L$ is monotone. This allows us to prove the claimed results, using properties of the homology of the covering $\tilde{L}$; for instance the fact that in the case of the universal cover it vanishes in every nonzero degree provided that $L$ is aspherical.

This construction will be formalized in the next section.

2 The lifted Floer complex

Let $L \subset M$ be a Lagrangian submanifold. Let $p : \tilde{L} \to L$ be a covering of $L$. The elements of a fiber of $p$ are indexed by a possibly infinite set $I$. Let $(\phi_t)_{t \in [0, 1]}$ be a Hamiltonian isotopy of $M$ such that $L$ and $\phi_1(L)$ are transverse. For any $x \in L \cap \phi_1(L)$ denote by $(x_i)_{i \in I}$ the elements of $p^{-1}(x)$. We prove the following theorem, which is the main ingredient in the proof of the results that we claimed in the preceding section:
Theorem 2.1 If $L$ is exact or monotone with $N_L \geq 3$, there exists a free $\mathbb{Z}/2$-complex $C_\bullet$ spanned by $\bigcup_{x \in L \cap \phi_1(L)} \{x_i \mid i \in I\}$ such that:

- If $L$ is exact then
  \[ H_\ast(C_\bullet) \approx H_\ast(\bar{L}, \mathbb{Z}/2). \]
- If $L$ is monotone with $N_L \geq 3$ then there exist applications $\delta_1, \delta_2, \ldots, \delta_k, \ldots$ with the following properties:
  (i) $\delta_1 : H_\ast(\bar{L}, \mathbb{Z}/2) \to H_{\ast-1+N_L}(\bar{L}, \mathbb{Z}/2)$ and $\delta_1 \circ \delta_1 = 0$.
  (ii) $\delta_l : H_\ast(\bar{L}, \mathbb{Z}/2) \to H_{\ast-1+lN_L}(\bar{L}, \mathbb{Z}/2)$, $l \geq 2$ is well-defined if $\delta_m = 0$ for $m = 1, \ldots, l-1$ and $\delta_l \circ \delta_l = 0$.
  (iii) If $\delta_l = 0$ for any $l \geq 1$ then
    \[ H_\ast(C_\bullet) \approx H_\ast(\bar{L}, \mathbb{Z}/2). \]

When $L$ is orientable and relatively spin, one can replace $\mathbb{Z}/2$-coefficients with $\mathbb{Z}$-coefficients in the statement of this theorem and get in the monotone case applications $\delta_i$ satisfying (i), (ii) above and moreover

(iii) If $H_\ast(C_\bullet)$ vanishes there is some $\delta_l$ which is not zero and if
    \[ (l+1)N_L > n+1, \]
then $(H_\ast(\bar{L}, \mathbb{Z}), \delta_l)$ is acyclic.

Before giving the proof of the theorem we briefly remind the construction of the of the original Floer complex $FC_\bullet(L, (\phi_1(L)))$ over $\mathbb{Z}/2$.

2.1 The usual Floer complex

The Floer complex $FC_\bullet$ is free, spanned by the intersection points $L \cap \phi_1(L)$. Denote $L_t = \phi_t(L)$. In order to define the differential of $FC_\bullet$, one has to choose a family of almost complex structures $J = (J_t)_{t \in [0,1]}$ on $M$ which are compatible with the symplectic form $\omega$ and to define the space of holomorphic strips with bounded energy $\mathcal{M}(L_0, L_1)$, as follows:

\[
\mathcal{M}(L_0, L_1) = \left\{ v \in C^\infty(\mathbb{R} \times [0,1], M) \left| \begin{array}{c}
\frac{\partial v}{\partial s} + J \frac{\partial v}{\partial t} = 0 \\
v(s, 0) \in L_0, \ v(s, 1) \in L_1 \\
E(v) < +\infty
\end{array} \right. \right\}.
\]
Here, the energy $E(v)$ is defined by the formula:

$$E(v) = \int_{\mathbb{R} \times [0,1]} \left\| \frac{\partial v}{\partial s} \right\|^2 ds dt,$$

the time-dependent norm being defined by the Riemannian metric $\omega(\cdot, J \cdot)$.

Then consider for $x, y \in L_0 \cap L_1$ the space

$$\mathcal{M}(x, y) = \left\{ v \in \mathcal{M}(L_0, L_1) \left| \lim_{s \to -\infty} v(s, \cdot) = x \right. \left. \lim_{s \to +\infty} v(s, \cdot) = y \right\}. $$

The following was proved by A. Floer in [17], [19] and by A. Floer, H. Hofer and D. Salamon in [21]:

**Theorem 2.2** (a) We have:

$$\mathcal{M}(L_0, L_1) = \bigcup_{x,y \in L_0 \cap L_1} \mathcal{M}(x, y).$$

(b) For a generic choice of $J$ the spaces $\mathcal{M}(x, y)$ are finite dimensional manifolds with local dimension at $v \in \mathcal{M}(x, y)$ given by the Maslov-Viterbo index $\mu(v)$ (see [40] for the definition).

Denote by $\mathcal{L}^0(x, y)$ the zero-dimensional component of $\mathcal{L}(x, y) = \mathcal{M}(x, y)/\mathbb{R}$. To define the differential of $FC^\bullet$, we need to prove that $\mathcal{L}^0(x, y)$ is finite. This is a consequence of Gromov’s compactness for holomorphic curves [27] and was proved by A. Floer in [17] and Y.-G. Oh in [33]:

**Theorem 2.3** Suppose that $L$ is exact, or monotone with $N_L \geq 3$. Let $x, y \in L_0 \cap L_1$ and $A > 0$ and let $(v_n) \subset \mathcal{M}(x, y)$ be a sequence of solutions with constant index $\mu(v_n) = \mu_0 \leq 2$. Then there exist a finite collection $(z_i)_{i=0,\ldots,k}$ of points in $L_0 \cap L_1$ with $z_0 = x$ and $z_k = y$, some solutions $v^i \in \mathcal{M}_A(z_{i-1}, z_i)$ for $i = 1, \ldots, k$ and some sequences of real numbers $(\sigma_n^i)_{n}$ for $i = 1, \ldots, k$ such that for all $i = 1, \ldots, k$ a subsequence of $v_n(s + \sigma_n^i, t)$ converges towards $v^i(s, t)$ in $C^\infty_{\text{loc}}$.

Moreover, we have the relation

$$\sum_{i=1}^k \mu(v^i) = \mu_0$$
If \( \mu_0 = 1 \) in the statement above then necessarily \( k = 1 \) due to the latter relation, so we immediately infer that the spaces \( \mathcal{L}^0(x, y) \) are compact, which implies that they are finite. This enables one to define the differential \( \partial : FC_* \to FC_* \) as:

\[
\partial(x) = \sum_{y \in L_0 \cap L_1} n(x, y)y,
\]

where \( n(x, y) = \#\mathcal{L}^0(x, y) \mod(2) \). In order to prove the relation \( \partial^2 = 0 \), one has to study the compactness of the 1-dimensional component \( \mathcal{L}^1(x, y) \) of \( \mathcal{L}(x, y) \). Using again Theorem 2.3 we find:

**Theorem 2.4** Denote by \( \bar{\mathcal{L}}^1(x, y) \) the union

\[
\mathcal{L}^1(x, y) \cup \bigcup_{z \in L_0 \cap L_1} \mathcal{L}^0(x, z) \times \mathcal{L}^0(z, y),
\]

endowed with the topology given by the convergence towards broken orbits which was defined in Theorem 2.3.

Then \( \bar{\mathcal{L}}^1(x, y) \) is a compact 1-dimensional manifold whose boundary is \( \bigcup_{z \in L_0 \cap L_1} \mathcal{L}^0(x, z) \times \mathcal{L}^0(z, y) \).

Note that the proof of the fact that \( \bar{\mathcal{L}}^1(x, y) \) is a manifold with boundary requires a gluing argument as in [18]. Now the fact that a compact 1-dimensional manifold has a boundary of even cardinality immediately implies \( \partial^2 = 0 \), proving thus that \( (FC_*, \partial) \) is a complex.

The following three subsections contain the proof of Theorem 2.1. We start with:

### 2.2 Proof of Theorem 2.1: Construction of the lifted Floer complex.

A construction of a lifted Floer-type complex was already sketched in our previous work [15]. The idea of constructing such a complex was suggested in [3].

Consider the intersection points \( L \cap \phi_1(L) \), viewed as points in \( L \). For two such points \( x, y \), any holomorphic strip \( v \in \mathcal{M}(x, y) \) defines a path \( \gamma : [-\infty, +\infty] \to L \) which joins \( x \) and \( y \):

\[
\gamma(s) = v(s, 0).
\]
We consider the obvious extension of $\gamma$ to $[-\infty, +\infty]$ keeping the same notation for the extended path. Take the collection of intersection points and the paths $\gamma$ as above, defined by the strips $v$ which belong to the one-dimensional components of $M(x, y)$ (which correspond to the zero dimensional components of $L(x, y)$). Denote by $C$ the collection of points and by $\Gamma$ the collection of paths.

Now start with the above collection $(C, \Gamma)$ and fix a covering $p : \tilde{L} \to L$. For any point $x \in C$ denote by $(x_i)_{i \in I}$ the elements of the fiber $p^{-1}(x)$. Consider all the lifts of the paths of $\Gamma$ to the covering space $\tilde{L}$. It is clear that for fixed points $x_i, y_j$ (where $i, j \in I$), the lifted space $L^0(x_i, y_j)$ is finite; its cardinality is obviously less then $\#L^0(p(x_i), p(y_j))$. Let $n(x_i, y_j)$ be the parity of this cardinality. On the free $\mathbb{Z}/2\mathbb{Z}$-complex $C^\bullet_{\tilde{L}}$ spanned by $\bigcup_{x \in L \cap \phi_1(L)} p^{-1}(x)$ one can therefore define an application $\partial^L : C^\bullet_{\tilde{L}} \to C^\bullet_{\tilde{L}}$ by the formula

$$\partial^L(x_i) = \sum_{p(y_j) = y \in C} n(x_i, y_j)y_j.$$ 

The sum above is obviously well-defined since $\Gamma$ is finite and any path $\gamma \in \Gamma$ admits only one lifting starting from $x_i$. We prove

**Proposition 2.5** Under the hypothesis of Theorem 2.1 $(C^\bullet_{\tilde{L}}, \partial^L)$ is a complex.

**Proof**

This is equivalent to the fact that, given $x_i, y_j$, there is an even number of "broken paths" joining them. Broken paths means liftings of concatenations $\gamma_1 \ast \gamma_2$, where $\gamma_i \in \Gamma$ and $\gamma_1$ starts from $x$ and ends into some point $z \in C$, while $\gamma_2$ starts from the same point $z$ and ends in $y$.

Since the paths of $\Gamma$ define a complex (namely the Floer complex $FC^\bullet$), we know that the number of broken paths joining $x$ and $y$ downstairs is even. Moreover, they represent boundary points of a one-dimensional compact manifold, so they can naturally be dispatched in a disjoint union of sets of two elements, corresponding to the boundary points of each component of the mentioned one-dimensional manifold. But in order to get the same property at the level of the covering space $\tilde{L}$ one has to check that the two broken paths in such a pair admit liftings to $\tilde{L}$ which have the same endpoints:

**Proposition 2.6** Let $\{\gamma_1 \ast \gamma_2, \gamma'_1 \ast \gamma'_2\}$ be a set of two broken paths in $L$ as above. Then these broken paths are homotopic in $L$ with fixed endpoints.
Proof
The concatenations $\gamma_1 \ast \gamma_2$ and $\gamma'_1 \ast \gamma'_2$ have the same starting point $x \in \mathcal{C}$ and the same ending point $y \in \mathcal{C}$. We use the following lemma which was proved in [15] (Lemma 3.16).

Lemma 2.7 Let $(v_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{M}(x, y)$, as in Theorem 2.3. Let $\gamma_n : [-\infty, +\infty] \to L_0$ be the path defined by $\gamma_n(s) = v_n(s, 0)$ extended with $x$ in $s = -\infty$ and with $y$ in $s = +\infty$. For $i = 1, \ldots, k$ let $\gamma^i : [-\infty, +\infty] \to L_0$ be the analogous paths defined by the holomorphic strips $v^i$. Then for $n$ large enough $\gamma_n$ and $\gamma = \gamma^1 \ast \gamma^2 \ast \cdots \ast \gamma^k$ are homotopic in $L_0$.

Now the broken holomorphic strip $(v^1, v^2)$ which defines $\gamma_1 \ast \gamma_2$ corresponds to a boundary point in $\mathcal{L}^1(x, y)$, which means that it is the limit of a sequence lying in a 2-dimensional component of $\mathcal{M}(x, y)$. The same is true for the broken path $\gamma'_1 \ast \gamma'_2$. Using the previous lemma we infer that there is some component of $\mathcal{L}^1(x, y)$ such $\gamma_1 \ast \gamma_2$ and $\gamma'_1 \ast \gamma'_2$ are respectively homotopic to paths in $L_0$ defined by some elements in this component. On the other hand all paths defined by the elements of the same component of $\mathcal{L}^1(x, y)$ are obviously homotopic, which finishes the proof of our claim 2.6.

The proof of Proposition 2.5 immediately follows, since the previous proposition implies that the set of paths in $\Gamma$ whose lifts in $\bar{L}$ join some fixed points $x_i, y_j$ is a disjoint union of sets with two elements.

Hamiltonian invariance
The usual way (see [18], [33]) to prove that the homology of the Floer complex $FC_\bullet$ does not depend on the (generic) choice of the Hamiltonian and of the almost complex structure, is to consider a generic homotopy $\Psi_s = (H_s, J_s)$ between two fixed couples $(H, J)$ and $(H', J')$ and to use it to define a chain morphism between $FC_\bullet(H, J) \to FC_\bullet(H', J')$ which induces an isomorphism in homology. More precisely, the homotopy $\Psi$ is used to define moduli spaces $\mathcal{M}_\Psi(x, y)$ for $x \in \phi^1_H(L) \cap L$ and $y \in \phi^1_{H'}(L) \cap L$. The space $\mathcal{M}_\Psi(x, y)$ is defined as follows:

$$\left\{ \begin{array}{l}
v : \mathbb{R} \times [0, 1] \to M \\
v(s, 0) \in L_0, \ v(s, 1) \in \phi^1_H(L_0) \\
\lim_{s \to -\infty} v(s, t) = x, \ \lim_{s \to +\infty} v(s, t) = y \\
\frac{\partial v}{\partial s} + J_s \frac{\partial v}{\partial t} = 0
\end{array} \right\}. $$
For a generic choice of $\Psi$ these spaces are actually finite dimensional manifolds, and the morphism between the two complexes is defined by counting the number of elements (mod 2) of their 0-dimensional components. The latter are proved to be finite by using a compactness result, analogue to Theorem 2.3. The same result shows that the 1-dimensional component of $\mathcal{M}(x, y)$ can be completed to a compact boundary manifold such that the parity of the number of boundary points is equivalent to the fact that the morphism defined by $\Psi$ commutes with the Floer differentials.

The same argument as before can be used to get the invariance of the homology of $\mathcal{C}_L$. Given two collections of points and paths $(\mathcal{C}, \Gamma)$ and $(\mathcal{C}', \Gamma')$, as above, the paths $v(s, 0) \in \Gamma_\Psi$ defined by elements $v$ belonging to 0-dimensional components of $\mathcal{M}_\Psi(x, y)$ define a morphism between the associated lifted complexes $\mathcal{C}_L$ and $(\mathcal{C}'_L)$. Claiming that it is a chain morphism is equivalent to the fact that the number of broken paths in $\Gamma_\Psi$ admitting lifts which join fixed points $x_i$ and $y_j$ is even. As above, this is a consequence of Proposition 2.6, adapted in this new setting.

Finally, the arguments of [17] which show that the morphism defined by $\Psi$ between the two usual Floer complexes induces an isomorphism at the homology level, can be used together with Proposition 2.6 in the same way, in order to show that the homology of the lifted complex $\mathcal{C}_L$ does not depend on $(H_t, J)$ either. We will denote this homology by $FH_L(L)$.

2.3 Proof of Theorem 2.1: Computation of the lifted Floer homology $FH_L(L)$.

When the Lagrangian $L$ is weakly exact its Floer homology is isomorphic to the singular homology of $L$ [13]. To prove this, one has to consider a Morse function $f : L \to \mathbb{R}$ and a particular Hamiltonian isotopy $\phi_t$ which maps $L$ into the graph $L_{\phi_t} \subset U(L) \subset M$, so that the Lagrangian intersections correspond to the critical points of $f$. The notations are those from §1.3, in particular $U(L)$ is a Weinstein tubular neighborhood of $L$. If $f$ is sufficiently $C^2$-small one proves that all the holomorphic strips lie in $U(L)$; the contrary would imply - via Gromov compactness - the existence of a nonconstant holomorphic disk with boundary in $L$ which is impossible for a weakly exact Lagrangian [35]. For a well chosen almost complex structure the canonical
projection \( U(L) \to L \) maps the holomorphic strips onto the flow lines of a gradient vector field of \( f \) with respect to a generic Riemannian metric on \( L \). Moreover, this projection defines a one-to-one correspondence between the isolated holomorphic strips and the gradient lines joining critical points of consecutive indices. This means that for these particular choices (which still satisfy the genericity assumptions required to define Floer homology), the Floer complex is identical to a Morse complex and the result follows.

The latter property of the holomorphic strips above shows that in this particular case the collection \( (C, \Gamma) \) is the one defined by the critical points of \( f \) and the isolated gradient lines which join them. So the lifted complex \( C_L^\bullet \) coincides with the lifted Morse complex on \( \bar{L} \). The homology of the latter is the singular homology of \( \bar{L} \) (recall that the stable manifolds of the gradient vector field associated to a Morse function yield a CW-decomposition of \( L \) whose lift to \( \bar{L} \) computes the homology of this covering space). Therefore we infer:

\[
FH_L^L(L) \approx H_\ast(\bar{L}, \mathbb{Z}/2).
\]

In the monotone case, when one chooses the same particular Hamiltonian isotopy and almost complex structure, it is no longer true that the holomorphic strips lie in a Weinstein neighborhood \( U(L) \). But, as Y.-G. Oh pointed out in [35], the holomorphic strips which lie in \( U(L) \) still project onto the gradient lines and the isolated ones are in bijective correspondence to the gradient lines defining the Morse complex. On the other hand, according to the same paper [35] an isolated holomorphic strip of finite energy which leaves \( U(L) \) connects two critical points \( x, y \) of \( f \) which satisfy

\[
\text{Ind}(x) - \text{Ind}(y) = 1 - lN_L,
\]

for some positive integer \( l \).

In this particular case, the Floer complex can be graded by the Morse index. Therefore, given an integer \( l \), the count (mod 2) of the isolated holomorphic strips satisfying the index relation above defines for each integer \( k \) a map

\[
\partial_l : FC_k \to FC_{k-1+lN_L}.
\]

Of course, \( \partial_l \) vanishes for \( l > \left\lceil \frac{\text{dim}(L)+1}{N_L} \right\rceil \). The Floer differential \( \partial : FC^\bullet \to FC^\bullet \) writes

\[
\partial = \partial_0 + \partial_1 + \cdots + \partial_l + \cdots.
\]

Here \( \partial_0 \) is the Morse differential defined by the (projections on \( L \) of the) holomorphic strips which do not leave \( U(L) \).
From these data Y.-G. Oh \cite{35} and P. Biran \cite{6} inferred the existence of a spectral sequence which converges towards the Floer homology and whose first page is built using the usual (Morse) homology of $L$.

Now, by definition, it is obvious that the differential $\partial^L$ of the lifted complex $C^L_•$ satisfies the same properties, namely:
1. It decomposes into a finite sum $\partial^L = \partial^L_0 + \partial^L_1 + \cdots$
2. For the grading given by the Morse index we have $\partial^L_k : C^L_k \to C^L_{k-1+N_L}$
3. The complex $(C^L_•, \partial^L_0)$ is identical to the lift to $\tilde{L}$ of a Morse complex on $L$.

These properties are sufficient for the existence of a spectral sequence analogous to the one defined by Biran and Oh. This spectral sequence, denoted $E^{p,q}_r$, is associated to an increasing filtration of a complex $\tilde{C}_•$ which is defined as follows:

Denote by $A$ the subring $\mathbb{Z}/2[T^{N_L}, T^{-N_L}]$ of the Laurent polynomials with $\mathbb{Z}/2$-coefficients and by $A^{kN_L} \subset A$ the subgroup $\mathbb{Z}/2 \cdot T^{kN_L}$, for any integer $k$. Define the complex

$$\tilde{C}_l = \bigoplus_{k \in \mathbb{Z}} C^L_{l-kN_L} \otimes A^{kN_L},$$

endowed with the differential

$$\tilde{d} = \partial_0 \otimes 1d + \partial_1 \otimes T^{-N_L}(\cdot) + \partial_2 \otimes T^{-2N_L}(\cdot) + \cdots.$$ 

On this complex define a filtration $\mathcal{F}_p(\tilde{C}_•)$ by:

$$\mathcal{F}_p(\tilde{C}_l) = \bigoplus_{k \leq p} C^L_{l-kN_L} \otimes A^{kN_L},$$

It is easy to prove that the differential $\tilde{d}$ preserves the filtration and that the homology of this complex is canonically isomorphic to the Floer homology $FH^L(L) \cite{6}$. Summarizing, the spectral sequence associated to these data has the following properties:

**Theorem 2.8** The spectral sequence $\{E^{p,q}_r, d_r\}$ associated to the filtration $\mathcal{F}$ converges to the lifted Floer homology $FH^L(L)$ and satisfies the following properties (all the tensor products below are over $\mathbb{Z}/2$):

- $E^{p,q}_0 = C^L_{p+q-N_L} \otimes A^{pN_L}$, $d_0 = [\partial^L_0] \otimes 1d$. 

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\[ E^p_q = H_{p+q-pN_L}(\bar{L}, \mathbb{Z}/2) \otimes A^{pN_L}, \quad d_1 = [\partial_1^{\bar{L}}] \otimes T^{-N_L}(-), \]

where

\[ [\partial_1^{\bar{L}}]: H_{p+q-pN_L}(\bar{L}, \mathbb{Z}/2) \to H_{p+q-1-(p-1)N_L}(\bar{L}, \mathbb{Z}/2) \]

is induced by \( \partial_1^{\bar{L}} \).

- For every \( r \geq 1 \) \( E^p_q \) has the form \( E^p_q = V^p_q \otimes A^{pN_L} \) with \( d_r = \delta_r \otimes T^{-rN_L} \), where \( V^p_q \) are vector spaces over \( \mathbb{Z}/2 \) and \( \delta_r: V^p_q \to V^{p-r,q+r-1}_r \) are homomorphisms defined for every \( p, q \) and satisfying \( \delta_r \circ \delta_r = 0 \). Moreover:

\[ V^{p,q}_{r+1} = \frac{Ker(\delta_r: V^p_q \to V^{p-r,q+r-1}_r)}{Im(\delta_r: V^{p+r,q-r+1}_r \to V^p_q)}, \]

and for \( r = 0, 1 \) we have \( V^0_q = C^L_{p+q-pN_L}, \quad V^p_q = H_{p+q-pN_L}(\bar{L}, \mathbb{Z}/2) \), \( \delta_1 = [\partial_1^{\bar{L}}] \).

- The spectral sequence collapses at page \( \left\lceil \frac{\text{dim}(L)+1}{N_L} \right\rceil + 1 \) and for all \( p \in \mathbb{Z} \), \( \bigoplus_{q \in \mathbb{Z}} E^p_q \otimes F H^L(L) \approx F H^L(L) \).

The proof of Theorem 2.8 is analogous to the proof of Th. 5.2.A in [6]. It is purely algebraic, and it applies to any graded complex whose differential (in our case \( \partial^{\bar{L}} \)) satisfies the conditions 1-3 above.

Remark that the above theorem immediately implies Theorem 2.1 for \( L \) monotone. Indeed, if \( \delta_1 = 0 \), then, according to Theorem 2.8 \( V^2_q = V^p_q = H_{p+q-pN_L}(\bar{L}, \mathbb{Z}/2) \), and

\[ \delta_2: H_{p+q-pN_L}(\bar{L}, \mathbb{Z}/2) \to H_{p+q-1-(p-1)N_L}(\bar{L}, \mathbb{Z}/2). \]

Actually, the proof of [6] shows that \( \delta_2 = [\partial_2^{\bar{L}}] \). Analogously, if \( \delta_1 = \delta_2 = \cdots = \delta_l = 0 \), then \( \delta_i \) is defined on the homology of \( \bar{L} \) (by \( [\partial_i^{\bar{L}}] \)) and its degree is \(-1 + lN_L\). Finally, if all the \( \delta_i \)'s vanish then the spectral sequence \( E^p_q \) collapses at page 1 and therefore, applying again Theorem 2.8 we have

\[ F H^L(L) \approx H_*(\bar{L}, \mathbb{Z}/2). \]

This finishes the proof of Theorem 2.1 for homologies with \( \mathbb{Z}/2 \)-coefficients.
2.4 End of the proof of Theorem 2.1: Change of coefficients

A Lagrangian submanifold $L \subset M$ is called relatively spin if there exists a class $s t \in H^2(M, \mathbb{Z}/2)$ that restricts to the second Stiefel-Whitney class $w_2(L)$ of $L$. For such submanifolds, also supposed to be orientable, it was proved in [23] that the spaces of holomorphic strips $M(x, y)$ can be oriented and that under this hypothesis the Floer complex $FC_\bullet$ can be defined over $\mathbb{Z}$-coefficients.

In our case, when $L$ is orientable and relatively spin, the lifted complex $C^L_\bullet$ is constructed using a collection of oriented paths $\Gamma$, which enables us to define it over $\mathbb{Z}$. To show that it is a $\mathbb{Z}$-complex whose homology only depends on $L \subset M$ and on the chosen covering space $\tilde{L}$ one can use the same proof as above. Analogously we get a spectral sequence $E_{r}^{p,q}$ associated to a filtered complex as above. But since short exact sequences over $\mathbb{Z}$ do not always split, we cannot infer as in Theorem 2.8 that the lifted Floer homology is a direct sum of modules $E_{\infty}^{p,q}$. However, when the former vanishes, we know that $E_{\infty}^{p,q} = 0$ for all $p, q$. On the other hand, the hypothesis of Theorem 2.1 implies that the spectral sequence collapses at page $l + 1$, and since $\delta_m = 0$ for $m = 1, \ldots, m - 1$ (again by the hypothesis of Theorem 2.1), we find that $V_{l}^{p,q} = H_{p+q-pN_{L}(L,\mathbb{Z})}$, whereas $V_{l}^{p,q} = 0$. Therefore, the complex $(H_{*}(\tilde{L},\mathbb{Z}), \delta_{l})$ has to be acyclic.

The proof of Theorem 2.1 is now complete.

Remark 2.9 If we chose the universal cover $\tilde{L}$ as covering space, then $C^L_\bullet$ can be seen as a free, finite dimensional complex over $\mathbb{Z}/2[\pi_1(L)]$ (resp. over $\mathbb{Z}[\pi_1(L)]$ when $L$ is orientable and relatively spin). Of course we can change the coefficients by tensoring it with any $\mathbb{Z}/2[\pi_1(L)]$-module $R$, for instance with the Novikov ring associated to some morphism $u: \pi_1(L) \to \mathbb{Z}$.

In all these situations the homology of the lifted Floer complex $C^L_\bullet$ is related in the same manner as for $\mathbb{Z}/2$-coefficients (resp. for integer coefficients) to the homology of $L$ with coefficients in the new ring. In particular, in the case of the Novikov ring, the latter is the Novikov homology $H_{*}(L, u)$ (for definition and related properties, see for instance [13], [14]).
3 Applications

In this section we prove our main results which we stated in §1 and other applications of Theorem 2.1.

3.1 Aspherical Lagrangian submanifolds. Proof of theorems 1.5 and 1.6

The idea of the proofs is that, under the given hypothesis, the spectral sequence given by Theorem 2.1 collapses at page 1, which means that the lifted Floer homology associated to the universal cover of $L$ is isomorphic to the singular homology of $\tilde{L}$. On the other hand, since $L$ is displaceable through a Hamiltonian isotopy, the lifted Floer homology vanishes. This is contradictory and therefore Theorem 2.1 should not apply here. The only possible reason for that is the fact that the Maslov number $N_L$ is less than 3.

**Proof of Theorem 1.5**

(a) If $N_L \geq 3$ then we get the applications $\delta_i$ provided by Theorem 2.1. But since $L$ is aspherical $H_i(\tilde{L}) = 0$ for $i \neq 0$, which implies that $\delta_i = 0$ for all $i$, therefore, according to Theorem 2.1

$$H(C_{\tilde{L}}) \approx H(\tilde{L}, \mathbb{Z}/2).$$

Since $L$ is displaceable, the left term vanishes, whereas the right term is not zero in degree $i = 0$. This contradiction implies $N_L \leq 2$.

We use the following well-known result [2] (see also [22], Lemma 2.5):

**Proposition 3.1** If $L$ is orientable then $N_L$ is even. The converse is true if $\pi_1(M)$ is trivial.

The conclusion of Theorem 1.5 follows.

(b) Since $L$ is orientable, its Maslov number $N_L$ is even and therefore all the applications $\delta_i$ provided by Theorem 2.1 have an odd degree. Suppose $N_L \geq 3$. The homology of $\tilde{L}$ is zero in odd degrees. This implies $\delta_1 = 0$ and $E_{2,q}^p = E_{1,q}^p$ in the spectral sequence of Theorem 2.8. The same argument shows that all the applications $\delta_i$ vanish, which implies that the singular homology of $\tilde{L}$ is zero. But this is impossible and therefore $N_L = 2$. [20]
(c) Denote by $\mathcal{J}_{reg}$ the generic set of compatible almost complex structures for which the usual Floer complex $(FC_\bullet, \partial)$ is defined. Consider $J \in \mathcal{J}_{reg}$ and denote

$$\mathcal{M}(M, J; 2) = \{ w : (D, \partial D) \to (M, L) | \bar{\partial}_J w = 0, \mu(w) = 2 \}.$$ 

By standard transversality results ([16], see also [9], chap. 3) one gets that for generic $J$, $\mathcal{M}(M, J; 2)$ is a manifold of dimension $n + 2$. It is important to notice here that a crucial point in the proof of the transversality is the fact that all the disks in $\mathcal{M}(M, J; 2)$ are simple. This is a consequence of the monotonicity of $L$ and of a result of L. Lazzarini [31] (see again [9]). The monotonicity of $L$ also implies that $\mathcal{M}(M, J; 2)$ is closed. Indeed all the holomorphic disks of this manifold have the same area, so Gromov’s compactness [27] applies. On the other hand, since $L$ is monotone and the disks have minimal Maslov number, no bubbling can occur.

The unparametrized $J$-holomorphic disks of Maslov number 2 passing through a given point $p \in L$ can be identified with the preimage $ev^{-1}(p)$ of an evaluation map

$$ev : \mathcal{N} \to L,$$

where $\mathcal{N} = (\mathcal{M}(M, J; 2) \times S^1)/PSL(2, \mathbb{R})$ and $PSL(2, \mathbb{R}) = Aut(D)$ acts on $(\mathcal{M}(M, J; 2) \times S^1)$ by

$$h \cdot (w, z) = (w \circ h, h^{-1}(z)).$$

The evaluation map is given by $ev([w, z]) = w(z)$, for $[w, z] \in \mathcal{N}$.

The closed manifold $\mathcal{N}$ is $n$-dimensional and in particular for a generic $p$, the preimage $ev^{-1}(p)$ is finite. Following the notations of [34], denote by $\Phi_L(p)$ the number of elements of $ev^{-1}(p)$, modulo 2. This number does not depend on the choice of the regular value $p$: it is the mod-2 degree of the evaluation map. Y-G. Oh shows in [34] that in order to define the Floer homology $FH(L_0, L_1)$ in the case where $N_{L_i} = 2$ one needs the hypothesis $\Phi_{L_0} + \Phi_{L_1} = 0$. This comes from the fact that the 2-dimensional component of the trajectory spaces $\mathcal{M}(x, y)$ can be compactified by adding the broken trajectories and the holomorphic disks of Maslov index 2 with boundary in one of the $L_i$’s, passing through the intersection points; the latter occur as bubbles of sequences in $\mathcal{M}(x, y)$ (when $x = y$). If their number is even then the relation $\partial^2 = 0$ is still valid.
Note that Oh also shows that when L₁ is a Hamiltonian deformation of L₀ then the above relation is satisfied and therefore the Floer homology FH(L) can be defined.

But for the definition of the lifted Floer homology FH₁(L) this is no longer sufficient as it can be seen in the example of two circles in C intersecting in two points. Recall that the lifted Floer complex was defined (for NL ≥ 3) using the paths w(s,0) defined by the isolated holomorphic strips of M(x,y). To define it for NL = 2 (with Z/2-coefficients) one needs for any x ∈ L ∩ L₁, L₁ = φ₁(L) and for any homotopy class g ∈ π₁(L) an even number of broken isolated trajectories w from x to x whose associated paths w(s,0) ⊂ L define a loop in the class g. This is clearly not true in the case of the two circles, as the two broken paths lie in different homotopy classes.

In the general situation, we claim that the Floer homology FH₁(L) can be defined when for any g ∈ π₁(L) the number of J-holomorphic disks passing through a generic p ∈ L and whose boundary realize g is even. The same arguments as above (fixing the homotopy class of the boundary in the definitions of M, N) show that the parity of this number does not depend on the generic choice of p. Denote it by Φg,L ∈ {0, 1}; clearly

(1) Φ₁ = Φg,L g∈π₁(L) mod 2.

Let us prove our claim. As explained above, it can happen that in order to compactify an one-dimensional connected component of L¹(x,x) (where x ∈ L ∩ φ₁(L)), one has to add a broken trajectory and a holomorphic disk D with boundary in L or φ₁(L). The homotopy class of the boundary ∂D ⊂ L is determined by the paths w(s,0) defined by the holomorphic strips corresponding to the elements in L¹(x,x). When the boundary contains a holomorphic disk (D, ∂D) ⊂ (M, L₁), then the loops defined in L by the trajectories of L₁(x,x) are necessarily contractible in L. Therefore, counting modulo 2, we have Φg,L broken trajectories in the class g when g ≠ 0 and Φ₀ + Φ₁ for g = 0. Using the fact that Φ₁ = Φ₀ and the relation (1) we infer that the lifted Floer complex is defined provided that Φg,L = 0 for any g ∈ π₁(L), g ≠ 0.

Now we are able to finish our proof. If for any non zero g ∈ π₁(L) there is some p ∈ L such that there is no holomorphic disk with boundary in the class g, passing through p, then Φg,L = 0 for any g ≠ 0, so FH₁(L) is defined. But this leads to a contradiction like in the proof of Theorem 1.5.(a). So, the
proof is finished for \( J \in \mathcal{J}_{reg} \). For an arbitrary \( J \), take a sequence \( J_n \in \mathcal{J}_{reg} \) which converges towards \( J \). Fix \( p \in L \) and consider \( J_n \)-holomorphic disks \( w_n : (D, \partial D) \to (M, L) \) such that \( \mu(w_n) = 2 \) and \( p \in w_n(\partial D) \). Using again Gromov’s compactness [27] we find that \( w_n \) converges towards a \( J \)-holomorphic disk whose boundary passes through \( p \). There is no bubbling here because of the monotonicity of \( L \) and of the fact that the Maslov index is minimal. The boundary of the limit is not trivial in \( \pi_1(L) \) by an argument which is similar to the one in the proof of Lemma 2.7 (see [15], Lemma 3.16).

The proof of Theorem 1.5 is now complete.

\[ \square \]

Using a quite similar argument one can prove the following version of Theorem 1.5:

**Theorem 3.2** Let \( M \) be a monotone symplectic manifold which has the property that any compact subset is displaceable through a Hamiltonian isotopy and let \( L \subset M \) be a monotone Lagrangian submanifold.

(a) If for some integer \( k \geq 1 \) we have \( H_i(\tilde{L}, \mathbb{Z}/2) = 0 \) for \( i > k \) then

\[
N_L \in [1, k + 1]
\]

(b) Suppose that \( L \) is orientable. If \( H_*(\tilde{L}, \mathbb{Z}/2) \) is of finite dimension over \( \mathbb{Z}/2 \) and the Euler characteristic

\[
\chi = \sum_{i=0}^{n} (-1)^i \dim(H_i(\tilde{L}, \mathbb{Z}/2))
\]

does not vanish, then \( N_L = 2 \). Moreover, for any almost complex structure \( J \) which is compatible with the symplectic form we have that through every \( p \in L \) passes at least a \( J \)-holomorphic disk \( w : (D, \partial D) \to (M, L) \) whose Maslov index equals 2 and whose boundary is not trivial in \( \pi_1(L) \).

**Proof**

(a) If \( N_L \geq k + 2 \geq 3 \) the lifted Floer homology \( FH^{\tilde{L}} \) is well defined. Since the degree of the applications \( \delta_i \) is greater of equal to \(-1 + N_L \geq k + 1 \), all these application vanish and therefore \( FH^{\tilde{L}} \) is isomorphic to the singular homology of \( \tilde{L} \). On the other hand \( FH^{\tilde{L}} = 0 \), as \( L \) is displaceable, which is absurd.
(b) To prove this statement one has to look at the proof of Th. 5.2.A in [6]. It is shown that the vector spaces $V^{p,q}_r$ in the statement of Theorem 2.8 satisfy $V^{p+1,q}_r = V^{p,q+1-N_L}_r$, and the applications $\delta^{p,q}_r$ have the same property. Therefore, for $p, q$ fixed, $\delta_r$ is a differential on the complex $(V^{p,q+k(1-rN_L)}_r)_{k \in \mathbb{Z}}$. This complex is finite (since this assertion is true for $r = 1$) and its homology is $(V^{p,q+k(1-rN_L)}_{r+1})_{k}$, according to Theorem 2.8. Now fix $p \in \mathbb{Z}$ and consider the Euler characteristic:

$$\chi_r = \sum_{q \in \mathbb{Z}} (-1)^q \dim(V^{p,q}_r).$$

We show that $\chi_r$ does not depend on $r$. Fix a negative odd number $m$. It is quite clear that

$$\chi_r = \sum_{l=m+1}^{0} (-1)^l \sum_{k \in \mathbb{Z}} (-1)^k \dim(V^{p,l+km}_r).$$

Indeed we have just changed the order of the summands in the writing of $\chi_r$. Applying this for $m = 1 - rN_L$ (which is odd, since $L$ is orientable, by Proposition 3.1), we get:

$$\chi_r = \sum_{q=2-rN_L}^{0} (-1)^q \chi^q_r,$$

where $\chi^q_r$ is the Euler characteristic of the complex $(V^{p,q+k(1-rN_L)}_r)_{k}$. The Euler characteristic of the homology is the same, so one can write

$$\chi^q_r = \sum_{k \in \mathbb{Z}} (-1)^k \dim(V^{p,q+k(1-rN_L)}_{r+1}),$$

therefore we get

$$\chi_r = \sum_{q=2-rN_L}^{0} (-1)^q \sum_{k \in \mathbb{Z}} (-1)^k \dim(V^{p,q+k(1-rN_L)}_{r+1}) = \chi_{r+1}.$$  

The latter equality is obtained by applying the property above for $m = 1 - rN_L$ and $r + 1$ instead of $r$.

So $\chi_r$ is independent of $r$. On the other hand, according to Theorem 2.8 we have

$$\chi_1 = \chi(H_*(\tilde{L}, \mathbb{Z}/2)) \neq 0$$

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and since the spectral sequence collapses and its limit is zero we also have 
\( \chi_r = 0 \) for \( r \) sufficiently large. Therefore the lifted Floer complex cannot be 
defined, which means that \( N_L = 2 \).

The proof of the existence of a \( J \)-holomorphic disk of Maslov index 2 
passing through a given \( p \in L \) is similar to the proof of Theorem 1.5(a).

\[ \Diamond \]

### 3.2 Lagrangian submanifolds with maximal Maslov number. Proof of theorems 1.7 and 1.8

**Proof of Theorem 1.7**

(a) As \( N_L = n + 1 \geq 3 \) the lifted Floer complex is well defined. We know 
that its homology \( \FH(L) \) vanishes because \( L \) is displaceable. On the other 
hand, we know by Theorem 2.8 that the spectral sequence \( \{ E_{p,q} \} \) which converges towards this homology, collapses at page 2. So, according to Theorem 2.8 we have

\[
0 = E_2^{p,q} = \frac{\text{Ker}\left( [\partial^L] : E_1^{p,q} \to E_1^{p-1,q} \right)}{\text{Im}\left( [\partial^L] : E_1^{p+1,q} \to E_1^{p,q} \right)},
\]

which for \( q = p - pN_L + i \) gives

\[
0 = E_2^{p,p-pN_L+i} = \frac{\text{Ker}\left( [\partial^L] : H_i(\bar{L},\Z/2) \to H_{i+1}(\bar{L},\Z/2) \right)}{\text{Im}\left( [\partial^L] : H_{i-n}(\bar{L},\Z/2) \to H_i(\bar{L},\Z/2) \right)}.
\]

Applying this equality for \( i = 1, \ldots n - 1 \) we find that \( \bar{L} \) is a \( \Z/2 \)-homology sphere for any covering space \( \bar{L} \). In particular, for \( \bar{L} = L \) we
have $H^i(L, \mathbb{Z}/2) = 0$, for $i = 1, 2$ and therefore we infer that $L$ is spin. So $\dim(L) = N_L - 1$ is odd (by Proposition 3.1 since $L$ is oriented) and the whole theory works for $\mathbb{Z}$-coefficients. The same argument shows then that any covering $\tilde{L}$ is a $\mathbb{Z}$-homology sphere. Let us prove that $L$ is also simply connected. If not, take a non zero element $g \in \pi_1(L)$ and consider the Abelian subgroup $G = \langle g \rangle$ and the associated covering space $\tilde{L}$. Therefore $H_1(\tilde{L}, \mathbb{Z}) = G$ which contradicts the fact that $\tilde{L}$ is a $\mathbb{Z}$-homology sphere. Finally, using the (proofs of the) Poincaré conjecture, (S. Smale, M. Freedman, G. Perelman) we infer that $L$ is homeomorphic to $S^n$, as claimed.

(b) As above, the lifted Floer complex is defined, its homology vanishes, and the spectral sequence converging to it collapses at page 2. Consider an arbitrary covering $\tilde{L} \to L$. We get, as in the proof of a) for $i = 1, \ldots n$:

$$E_2^{p, pN_L+i} = \frac{\text{Ker} \left( \{ \partial^L_1 \} : H_i(\tilde{L}, \mathbb{Z}/2) \to H_{i+n-1}(\tilde{L}, \mathbb{Z}/2) \right)}{\text{Im} \left( \{ \partial^L_1 \} : H_{-n+i+1}(\tilde{L}, \mathbb{Z}/2) \to H_i(\tilde{L}, \mathbb{Z}/2) \right)}.$$  

We infer that

1. $H_i(\tilde{L}, \mathbb{Z}/2) = 0$

for $i = 2, \ldots n - 2$ and

2. $[\partial^L_1] : H_1(\tilde{L}, \mathbb{Z}/2) \approx H_n(\tilde{L}, \mathbb{Z}/2)$.

3. $[\partial^L_1] : H_0(\tilde{L}, \mathbb{Z}/2) \approx H_{n-1}(\tilde{L}, \mathbb{Z}/2)$.

Suppose first that $L$ is not orientable. Choosing $\tilde{L} = \tilde{L}$ we get $H_n(\tilde{L}) = 0$, so $\tilde{L}$ is not compact, and therefore $\pi_1(L)$ is infinite. We know that $H_1(L, \mathbb{Z}/2) \approx H_n(L, \mathbb{Z}/2) = \mathbb{Z}/2\mathbb{Z}$. Take an element $g \in \pi_1(L)$ which is not in the kernel of the Hurewicz morphism $\pi_1(L) \to H_1(L, \mathbb{Z}/2)$ and consider the covering $\tilde{L} \to L$ associated to the Abelian subgroup $G = \langle g \rangle$. If $g$ is of finite order - which has to be even -, the covering $\tilde{L} \to L$ is infinite (since $\pi_1(L)$ is infinite), therefore $H_n(\tilde{L}, \mathbb{Z}/2) = 0$ and $H_1(\tilde{L}, \mathbb{Z}/2) = 0$, by (2). On the other hand $H_1(\tilde{L}, \mathbb{Z}/2) = G \otimes \mathbb{Z}/2 \neq 0$, since $G$ is a cyclic group $\mathbb{Z}/2\mathbb{Z}$ due to the fact that $g$ has an even order. This contradiction implies that $G$ is an infinite cyclic group. In particular $H_1(\tilde{L}, \mathbb{Z}/2) \neq 0$, and by (2) $H_n(\tilde{L}, \mathbb{Z}/2) \neq 0$, which means that $\tilde{L}$ is compact and $G$ has finite index in $\pi_1(L)$, as claimed.

If $M$ is an exact symplectic manifold then Theorem 1.2 applies and $H^1(L, \mathbb{R}) \neq 0$, a non-zero class being given by the restriction to $L$ of a
primitive of the symplectic form. Therefore the first Betti number of $L$ is not zero. Consider a non-vanishing morphism $u : \pi_1(L) \to \mathbb{Z}$ and denote by $K$ its kernel. We show that $K$ is finite, of odd order.

Indeed, take an element $t \in \pi_1(L)$ such that $u(t) = 1$ and consider the covering $\tilde{L} \to L$ corresponding to the infinite cyclic subgroup $G$ spanned by $t$. As above, using the relation (2) we infer that $\tilde{L} \to L$ is a finite covering. It is easy to see that different elements of $K = \text{ker}(u)$ lie in different classes of the quotient $\pi_1(L)/G$ and therefore $K$ is finite. Moreover, if we suppose that an element $g \in K$ has even order, then, taking as $\tilde{L}$ the covering associated to $G = \langle g \rangle$, we find as above that $H_1(\tilde{L}, \mathbb{Z}/2) \neq 0$, whereas $H_n(\tilde{L}, \mathbb{Z}/2) = 0$ since the covering is infinite. This contradicts the relation (2).

Let us consider now the case where $L$ is orientable (so $n$ is even). Since it is also spin (by (1)), the relations (1) and (2) are valid for integer coefficients. Taking $\tilde{L} = L$, we get $H_1(L, \mathbb{Z}) = \mathbb{Z}$, which implies $H^1(L, \mathbb{Z}) = \mathbb{Z}$. Therefore, as above there is an exact sequence of groups

$$0 \to K \to \pi_1(L) \to \mathbb{Z} \to 0,$$

where $K$ is finite. Now, using the same argument as above for the covering $\tilde{L}$ associated to $< g >$, for an arbitrary $g \in K$, we find that $H_1(\tilde{L}, \mathbb{Z}) \neq 0$ unless $g$ is the identity. Using the relation (2), this implies that $K = \{1\}$, so $\pi_1(L) \approx \mathbb{Z}$, as claimed.

For $\dim(L) \geq 6$ F. Latour and A. Pajitnov independently established an algebraic criterion for the existence of a fibration of $L$ over the circle \[30, 36\]. For $\pi_1(L) = \mathbb{Z}$ we get (see for instance \[14\]):

**Theorem 3.4** When $n = \dim(L) \geq 6$ and $\pi_1(L) = \mathbb{Z}$, then there exists a fibration $f : L \to S^1$ if and only if the Novikov homology $H_*(L; u)$ vanishes, where $u = [f^*(d\theta)] \in H^1(L, \mathbb{Z})$.

According to Theorem 2.1 (see Remark 2.9) the relations (1) and (2) are also valid for the Novikov homology with respect to any 1-cohomology class $u$. On the other hand, for any $u \neq 0$ one can show that $H_0(L; u) = 0$ and $H_n(L; u) = 0$ (see for instance \[13\]). Using (1), (2) and (3) we find that for $u = id_\mathbb{Z}$, $H_*(L, u) = 0$ and therefore, $L$ admits a fibration over the circle, by Theorem 3.4. Denote by $F$ a fiber of this fibration. We know that $\pi_1(F) = \text{ker}(u)$, so $F$ is simply connected. We also have that $\tilde{L}$ is
diffeomorphic to $F \times \mathbb{R}$ and in particular

$$H_*(\tilde{L}, \mathbb{Z}) \approx H_*(F, \mathbb{Z}).$$

We infer from (2) that $F$ is a simply connected homology sphere, therefore it is homeomorphic to the standard $(n - 1)$-sphere, using Poincaré.

The proof of Theorem 1.7 is now complete.

Proof of Theorem 1.8

(a) The vanishing of the first homology group of $L$ implies that $L$ is monotone with $N_L = 2(n + 1)$. Applying Proposition 3.3, we get a monotone Lagrangian submanifold $\Gamma_L \subset \mathbb{C}^{n+1} \times X$ which has the same Maslov number $N_{\Gamma_L} = N_L = \text{dim}(\Gamma_L) + 1$. Moreover $\Gamma_L \to L$ is a circle fibration.

The submanifold $\Gamma_L$ satisfies the hypothesis of Th. 1.7(b) and therefore $\pi_1(\Gamma_L) \approx \mathbb{Z}$. Moreover, considering the lifted Floer complex associated to the universal cover $\tilde{\Gamma}_L$, the relations (1), (2) and (3) from the proof of Theorem 1.7 imply that $H_i(\tilde{\Gamma}_L) = 0$ for $i \neq 0, 2n + 1$, and $H_{2n+1}(\tilde{\Gamma}_L) \approx \mathbb{Z}$. Therefore, using Hurewicz’s isomorphism the first $2n + 1$ homotopy groups of $\Gamma_L$ are isomorphic to the corresponding homotopy groups of $S^{2n+1}$. From the long exact sequence of the fibration $\Gamma_L \to L$ we infer that

$$\pi_i(L) \approx \pi_i(S^{2n+1}), \text{ for } i = 2, \ldots, 2n + 1,$$

and, since $\pi_1(\Gamma_L)$ is $\mathbb{Z}$, $\pi_1(L)$ is Abelian. But as $H_1(L, \mathbb{Z}) = 0$, $L$ is simply connected. Therefore $L$ is homeomorphic to $S^{2n+1}$ according to (the proof of) the Poincaré conjecture.

(b) As above $L$ is monotone with $N_L = 2(n + 1)$. Again, we can apply Proposition 3.3 and we get a monotone Lagrangian in $\mathbb{C}^{n+1} \times \mathbb{CP}^n$ which is a circle fibration over $L$ and has the same Maslov number. We can therefore use Th. 1.7(a) and infer that $\Gamma_L$ is homeomorphic to the $(2n + 1)$-sphere, as claimed. In particular $L$ is simply connected.

(c) As pointed out in [6], one can easily see that $L$ is monotone with $N_L = n + 1$. We consider as above the Lagrangian submanifold $\Gamma_L \subset \mathbb{C}^{n+1}$ which has the same Maslov number. We can use Theorem 2.4 with integer coefficients because $\Gamma_L$ is spin (using again Theorem 2.4 with mod-2 coefficients) and orientable (by Proposition 3.1 since $N_{\Gamma_L}$ is even). As in the proof of
(a) above we get that \( \pi_1(\Gamma_L) = \mathbb{Z} \) and the universal cover \( \tilde{\Gamma}_L \) has the same homotopy groups \( \pi_i \) as \( S^n \), for \( i = 1, \ldots n \). Using the long exact sequence of the fibration \( \Gamma_L \to L \) we find \( \pi_n(L) = \mathbb{Z}, \pi_i(L) = 0 \) for \( i = 3, \ldots n - 1 \) and we have an exact sequence:

\[
0 \to \pi_2(L) \to \pi_1(S^1) = \mathbb{Z} \to \pi_1(\Gamma_L) = \mathbb{Z} \to \pi_1(L) \to 0.
\]

It follows that \( \pi_1(L) \) is cyclic Abelian and, since \( H_1(L, \mathbb{Z}) \) is 2-torsioned, it follows that \( \pi_1(L) = \mathbb{Z}/2 \). From the exact sequence we infer then that \( \pi_2(L) = 0 \). So the universal cover of \( L \) is a homotopy sphere and therefore it is homeomorphic to the \( n \)-sphere.

\[\diamondsuit\]

### 3.3 Lagrangian submanifolds in the cotangent bundle.

**Proof of Theorem 1.9**

Let \( K \subset M \) be a Lagrangian submanifold. If \( L \subset T^*K \) is Lagrangian then by Darboux’ Theorem it follows that \( L \) admits a Lagrangian embedding into \( M \). We need the following result:

**Proposition 3.5** Suppose that \( K \subset M \) is monotone and that \( L \subset T^*K \) is exact with vanishing Maslov class. Then \( L \subset M \) is also monotone. Moreover, if the morphism \( \pi_1(L) \to \pi_1(K) \) induced by the projection is surjective, then \( N_L = N_K \).

**Proof**

Since \( L \) is exact it is easy to see that (using the notations of the first section) we have

\[
I^K_{\omega} \to M = I^K_{\omega} \to M \circ p,
\]

where \( p : \pi_2(M, L) \to \pi_2(M, T^*K) \approx \pi_2(M, K) \) is the canonical morphism.

It is also known and not very hard to prove (see [29], Chap.I, Prop. A.3.3) that when \( L \) has a vanishing Maslov class then (again with the notations of §1) we have

\[
I^L_{\mu} \to M = I^K_{\mu} \to M \circ p.
\]

Therefore \( L \subset M \) is monotone. If \( \pi_1(L) \to \pi_1(K) \) is surjective, then \( p \) is also an epimorphism and the conclusion follows.

\[\diamondsuit\]
Proof of Theorem 1.9 (a) Let $L \subset T^* S^{2k+1}$ be exact Lagrangian with vanishing Maslov class. As pointed out in §1.2, the application 
\[ z \mapsto ([z], \bar{z}) \]
defines a monotone Lagrangian embedding of $S^{2k+1}$ into $\mathbb{C}P^k \times \mathbb{C}^{k+1}$. It follows by Proposition 3.5 that $L$ admits a monotone Lagrangian embedding into $\mathbb{C}P^k \times \mathbb{C}^{k+1}$ of Maslov number $N_L = N_{S^{2k+1}} = 2k + 2$. The statement (a) of Theorem 1.7 implies the desired result.

Proof of Theorem 1.9 (b) Let $L \subset T^* K$ be an exact Lagrangian submanifold with vanishing Maslov class. There is a finite cover $\tilde{L}$ of $L$ which admits an exact Lagrangian embedding into $T^* \tilde{K} = T^* S^{2k+1}$ (see [15], Lemma 3.5). Moreover, we have a commutative diagram:

\[
\begin{array}{ccc}
\tilde{L} & \rightarrow & T^* S^{2k+1} \\
\downarrow & & \downarrow \\
L & \rightarrow & T^* K
\end{array}
\]

From this diagram one immediately infers that the Maslov class of $\tilde{L}$ vanishes. Therefore, we can apply the point (a) which asserts that $\tilde{L}$ is homeomorphic to $S^{2k+1}$.

When $K = \mathbb{R}P^{2k+1}$ we show that $\pi_1(L) \rightarrow \pi_1(\mathbb{R}P^{2k+1})$ is an epimorphism. If not, we can lift $L$ to an exact Lagrangian embedding (of $L$) into $T^* S^{2k+1}$, which still has vanishing Maslov class. Using the statement (a) we find that $L$ is homeomorphic to a sphere. Since $\mathbb{R}P^{2k+1}$ admits a Lagrangian embedding into $\mathbb{C}P^{2k+1}$, the same is true for $L$. But this contradicts a well-known result of P. Biran and K. Cieliebak asserting that there is no Lagrangian sphere in $\mathbb{C}P^n$ ([7], Theorem A). So $\pi_1(L) \rightarrow \pi_1(\mathbb{R}P^{2k+1})$ is surjective. In this case, using Lemma 3.5 of [15] as above we get that $\tilde{L} = L$ is a double covering, homeomorphic to $S^{2k+1}$.

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