Correlators of the Kazakov–Migdal Model

M.I. Dobroliubov\textsuperscript{a,1}, Yu. Makeenko\textsuperscript{b}, G.W. Semenoff\textsuperscript{a,2}

\textsuperscript{a} Department of Physics, University of British Columbia
  Vancouver, British Columbia, V6T 1Z1 Canada

\textsuperscript{b} Institute of Theoretical and Experimental Physics
  117259 Moscow, Russian Federation

Abstract

We derive loop equations for the one-link correlators of gauge and scalar fields in the Kazakov–Migdal model. These equations determine the solution of the model in the large $N$ limit and are similar to analogous equations for the Hermitean two-matrix model. We give an explicit solution of the equations for the case of a Gaussian, quadratic potential. We also show how similar calculations in a non-Gaussian case reduce to purely algebraic equations.

\textsuperscript{1}Permanent address: Institute for Nuclear Research, Academy of Sciences of Russia, Moscow, Russian Federation

\textsuperscript{2}This work was supported in part by the Natural Sciences and Engineering Research Council of Canada
1 Introduction

Solving QCD in the limit of a large number of colors is a classic problem [1]. Recently, Kazakov and Migdal [2] have noted that it may indeed be possible to find an analytic solution of the large $N$ limit of a certain version of induced $SU(N)$ lattice gauge theory. In their model, the Yang-Mills action is absent at lattice distance scales and is induced in the continuum limit by the interaction of the gauge field with a heavy Hermitean matrix-valued scalar field. The eigenvalues of the scalar field behave as the classical “master field” in the large $N$ limit. The possibility of solving this model has attracted wide attention and the task of unraveling its physical content is a subject of ongoing research. Besides being a candidate for QCD, it is an interesting example of a matrix model in dimensions greater than one for which a solution in the large $N$ limit may be attainable.

The Kazakov-Migdal model has action

$$Z = \int \prod_x d\phi_x \prod_{xy} [dU_{xy}] e^{S[\phi, U]},$$

(1.1)

where

$$S[\phi, U] = -\sum_x N \text{tr} V(\phi_x) + \sum_{xy} N \text{tr} \phi_x U_{xy} \phi_y U_{xy}^\dagger,$$

(1.2)

$\phi_x$ and $U_{xy}$ ($= U_{yx}^\dagger$) are $N \times N$ Hermitean and unitary matrices, which live on lattice sites $x$ and links $xy$ between neighboring sites, respectively. Here $[dU]$ denotes the Haar measure for integration over $U(N)$. Using the Itzykson-Zuber integral [3]

$$I[\phi, \chi] \equiv \int [dU] e^{N \text{tr} \phi U \chi U^\dagger} = \frac{\det_{ij} e^{N \phi_i \chi_j}}{\Delta(\phi) \Delta(\chi)},$$

(1.3)

where $\phi^i, \chi^j$ are the eigenvalues of $\phi$ and $\chi$ and $\Delta(\phi) = \prod_{i<j} (\phi^i - \phi^j)$ is the Vandermonde determinant, (1.1) can be presented as the effective theory for a scalar field

$$Z = \int \prod_x d\phi_x e^{S_{\text{eff}}[\phi]}$$

(1.4)

with the action

$$S_{\text{eff}}[\phi] = -N \sum_{i,x} V(\phi^i_x) + \sum_x \ln \Delta^2(\phi_x) + \sum_{xy} \ln I[\phi_x, \phi_y]$$

(1.5)

(The Vandermonde determinants result from converting the integration measure for matrices to one for eigenvalues.)

When $N$ is large this integral can be evaluated in the saddle-point approximation. The saddle-point equation is

$$\frac{2D}{N} \frac{\partial}{\partial \phi^i} \ln I[\phi, \chi]|_{\chi=\phi} = V'(\phi^i) - \frac{2}{N} \sum_{j \neq i} \frac{1}{\phi^i - \phi^j}$$

(1.6)
This model was formally solved by Migdal [4], who derived a singular nonlinear integral equation for the spectral density of the eigenvalues. This equation is rather complicated and its explicit solution even for the simplest Gaussian case proved to be non-trivial [5]. One of the authors has proposed [6] an approach to this problem which employs the method of loop equations. In this paper we shall generalize these ideas to derive a loop equation for arbitrary one-link correlation functions. Its solution provides the spectral density of eigenvalues of $\phi$ as well as the full set of one-link correlators of the gauge and scalar fields. We shall obtain an explicit solution of these equations in the case of a quadratic potential and reduce the more complicated case of a general potential to the solution of algebraic equations.

The importance of the correlators of a pair of the gauge fields was discussed in [7] where it was argued that, to a large extent, they determine the continuum limit of the theory. More recently, in [8] it was shown that they could also play an essential role in resolving the problem of $Z_N$ gauge invariance of the lattice model [1-5]. Consider the correlator (our normalization differs from the original normalization of Ref. [7] by an extra $1/N$)

$$\frac{1}{N} C_{ij}[x, y] \delta_{il} \delta_{jk} = \left\langle U_{xy}^{ij} U_{yx}^{kl} \right\rangle = \frac{\int d\phi [dU] e^{S_{ij}U_{xy}^{ij} U_{yx}^{kl}}}{\int d\phi [dU] e^{S}}. \quad (1.7)$$

Here, the delta-functions on the right-hand side arise from the gauge invariance of the integral. It also follows from gauge invariance that $C_{ij}$ is symmetric. Unitarity of $U$ implies the sum rule

$$\frac{1}{N} \sum_i C_{ij} = 1. \quad (1.8)$$

In the large $N$ limit, the integration over the scalar field is performed by replacing it with its saddle point value (this should properly be done after $\phi$ is diagonalized). Then, the expectation value of a quantity such as $UU^\dagger$ factorize into one-link expectation values,

$$\frac{1}{N} C_{ij} = \frac{\int [dU] e^{N \text{tr} \phi_i U \phi_j U^\dagger} |U_{ij}|^2}{\int [dU] e^{N \text{tr} \phi_i U \phi_j U^\dagger}}. \quad (1.9)$$

From this equation we can derive the quantity

$$\frac{1}{N} \sum_j C_{ij} \phi_j = \frac{1}{N} \frac{\partial}{\partial \chi_i} \ln I[\phi, \chi]|_{\chi=\phi}, \quad (1.10)$$

which can be obtained from the saddle-point equation (1.6) as

$$\frac{1}{N} \sum_j C_{ij} \phi_j = \frac{1}{2D} \left( V'(\phi_i) - \frac{2}{N} \sum_{j \neq i} \frac{1}{\phi_i - \phi_j} \right). \quad (1.11)$$

Formal expressions for the integrals (1.7) at finite $N$, in terms of the eigenvalues $\phi_i$ have been found in [11]. Those expressions are potentially very useful when $N$ is small
rather than large. Here, to analyze the infinite $N$ limit, we shall take a different approach which requires a simultaneous solution for the correlator and the eigenvalue distribution.

In the large $N$ limit, it is necessary to replace the index of the eigenvalues of $\phi$ with a continuous label. This is done by introducing the eigenvalue density, $\rho(\alpha)$ such that $\rho(\alpha)d\alpha$ is the number of eigenvalues in the interval $[\alpha, \alpha + d\alpha]$. The spectral density typically has support in a finite interval, $[a, b]$, and is normalized so that

$$\int_a^b d\alpha \rho(\alpha) = 1 \quad ,$$

(1.12)

Also, the normalization of the correlator (1.8) becomes

$$\int_a^b d\alpha \rho(\alpha) C(\alpha, \beta) = 1 .$$

(1.13)

The saddle-point equation (1.11) is

$$\int_a^b d\beta \rho(\beta) C(\alpha, \beta) \beta = \frac{1}{2D} \left( V'(\alpha) - 2 \mathcal{R} E_{\alpha} \right) ,$$

(1.14)

where $\mathcal{R} E_{\alpha}$ is the real (and continuous across the support of $\rho$) part of the analytic function (of the complex variable $\lambda$)

$$E_{\lambda} \equiv \left< \frac{\text{tr} \frac{1}{N} \frac{1}{\lambda - \phi_x}}{N} \right> = \int_a^b d\alpha \rho(\alpha) \frac{1}{\lambda - \alpha} .$$

(1.15)

Equations of motion (1.11), (1.14) imply that, anywhere that the single-link operator $U_{xy} \phi_y U_{yx}$ appears with other operators inside a gauge invariant correlation function, and $U_{xy}$ and $\phi_y$ appear no-where else in the other operators in the correlators, in the saddle-point approximation we can replace the diagonal elements of the operator $U_{xy} \phi_y U_{yx}$ by the operator

$$F(\phi_x) \equiv \frac{1}{2D} \left( V'(\phi_x) - \tilde{V}'(\phi_x) \right) ,$$

(1.16)

where $\tilde{V}'(x)$ is the analytic continuation of $2 \mathcal{R} E_{\lambda}$ from the cut on the whole complex plane, so that on the cut

$$\tilde{V}'(\alpha) = 2 \mathcal{R} E_{\alpha} .$$

(1.17)

This looks like the saddle point equation for the eigenvalue distribution of a one-matrix model with potential $\tilde{V}(\phi)$ (see also Eq. (2.16)).

2 Loop equations for one-link correlators
2.1 The two analytic functions

The central quantities of interest in this Section are the two analytic functions: \( E_\lambda \), which is defined by Eq. (1.15), and

\[
G_{\nu \lambda} \equiv \left\langle \frac{1}{N} \nu - \phi_x U_{xy} \frac{1}{\lambda - \phi_y} U_{yx} \right\rangle = \int_a^b d\alpha \rho(\alpha) \int_a^b d\beta \rho(\beta) \frac{C(\alpha, \beta)}{(\nu - \alpha)(\lambda - \beta)}. \tag{2.1}
\]

It is important to note that, due to invariance of the group measure under transformations \( U \to U^\dagger \), \( G_{\nu \lambda} \) is symmetric in \( \nu \) and \( \lambda \).

These two functions are analytic on the whole complex plane excluding cut singularities in the real axis in interval \([a, b]\) (the support of \( \rho \)). At the cut they have following discontinuities

\[
E_{\alpha \pm i0} = \mathcal{P} \int_a^b d\beta \rho(\beta) \frac{1}{\alpha - \beta} \mp i\pi \rho(\alpha), \tag{2.2}
\]

and

\[
G_{\alpha \pm i0, \lambda} = \mathcal{P} \int_a^b d\beta \rho(\beta) \frac{G(\beta)}{\alpha - \beta} \mp i\pi \rho(\alpha) G(\alpha), \tag{2.3}
\]

where

\[
G(\alpha) \equiv \int_a^b d\beta \rho(\beta) \frac{C(\alpha, \beta)}{\lambda - \beta}. \tag{2.4}
\]

Note that this defines a function of the matrix \( \phi_x \), which we can also view as a 1-link expectation value

\[
G(\phi_x) = \left\langle U_{xy} \frac{1}{\lambda - \phi_y} U_{yx} \right\rangle_{1L} \tag{2.5}
\]

(where the bracket denotes integration over \( U_{xy} \) and \( \phi_y \) is evaluated at the saddle point). The function \( G(\phi) \) was introduced by Migdal [4] and was the key component of his approach (see Sect. 3 below).

Since the support of \( \rho \) is finite, the functions \( E_\nu \) and \( G_{\nu \lambda} \) have the following asymptotic expansions

\[
E_\nu = \frac{1}{\nu} + \sum_{k=1}^{\infty} \frac{E_k}{\nu^{k+1}} , \quad E_k = \left\langle \frac{\text{tr} \ N \phi^k}{} \right\rangle \tag{2.6}
\]

and

\[
G_{\nu \lambda} = \frac{E_\lambda}{\nu} + \sum_{n=1}^{\infty} \frac{G_n(\lambda)}{\nu^{n+1}} , \quad G_n(\lambda) = \left\langle \frac{\text{tr} \ N \phi^\nu U \frac{1}{\lambda - \phi} U^\dagger}{} \right\rangle \tag{2.7}
\]

As we see from Eq. (2.7), asymptotics of \( G_{\nu \lambda} \) completely fixes \( E_\lambda \).

2.2 Loop equation for \( G_{\nu \lambda} \)

Our main equation is derived from the identity

\[
\frac{N^{-2}}{Z} \int d\phi [dU] \text{tr} \frac{\partial}{\partial \phi_x^i} e^S \left( \frac{1}{\nu - \phi_x} U_{xy} \frac{1}{\lambda - \phi_y} U_{yx} \right)^{ij} = \left\langle \frac{\text{tr} \ N \phi_x U_{xy} \frac{1}{\nu - \phi_x} U_{yx}}{} \right\rangle + \frac{\text{tr} \ N \phi_x U_{xy} \frac{1}{\nu - \phi_x} U_{yx}}{} + \sum_z \left\langle \frac{\text{tr} \ N \phi_z U_{xy} \frac{1}{\nu - \phi_x} U_{yx}}{} \right\rangle . \tag{2.8}
\]
The first term on the right-hand-side can be factored into two terms. This factorization is valid in the large $N$ limit because of the saddle point evaluation of the integral over $\phi$. The result is a product of two quantities
\[
\left\langle \frac{1}{\nu - \phi_x} \frac{1}{\nu - \phi_y} U_{xy} \frac{1}{\lambda - \phi_y} U_{yx} \right\rangle = E_\nu G_{\nu\lambda}.
\] (2.9)

The third term on the right-hand-side contains link operators which connect $x$ to all neighboring points. One of these connects $x$ to $y$. It contributes
\[
\left\langle \frac{1}{\nu - \phi_x} U_{xy} \phi_y \frac{1}{\lambda - \phi_y} U_{yx} \right\rangle = -E_\nu + \lambda G_{\nu\lambda}
\] (2.10)

There are also $2D - 1$ links which connect to other sites. For these one can use the fact that the quantity $U_{x\phi}F_{\phi U}$ inside the expectation value bracket can be replaced by $F(\phi_x)$ (cf. Eq. (1.16)). With this input we present Eq. (2.8) in the form
\[
0 = E_\nu G_{\nu\lambda} - \left\langle \frac{1}{\nu} L(\phi) \frac{1}{\nu - \phi} U \frac{1}{\lambda - \phi} U^\dagger \right\rangle + \lambda G_{\nu\lambda} - E_\nu
\] (2.11)

Where, for brevity of notation, we have defined
\[
L(\omega) \equiv V'(\omega) - (2D - 1)F(\omega) = \tilde{V}'(\omega) + F(\omega).
\] (2.12)

If we assume that $L(\omega)$ has no singularities in the interval $[a, b]$, this equation can be written in the integral form
\[
\int_{C_1} \frac{d\omega}{2\pi i \nu - \omega} L(\omega) G_{\omega\lambda} = E_\nu G_{\nu\lambda} + \lambda G_{\nu\lambda} - E_\nu
\] (2.13)
where the contour $C_1$ encircles the cut of the function $G_{\nu\lambda}$ (which coincides with the support of the spectral density).

### 2.3 Loop equation for $E_\nu$

Consider the limit $\lambda \to \infty$ in Eq. (2.13). The leading term gives
\[
G_1(\nu) = \int_{C_1} \frac{d\omega}{2\pi i \nu - \omega} L(\omega) E_\omega - E^2_\nu.
\] (2.14)

From the definition (2.1) of $G_{\nu\lambda}$
\[
G_1(\nu) = \left\langle \frac{1}{\nu - \phi} U \phi U^\dagger \right\rangle = \left\langle \frac{1}{\nu - \phi} F(\phi) \right\rangle = \int_{C_1} \frac{d\omega}{2\pi i \nu - \omega} F(\omega) E_\omega
\] (2.15)

---

An equivalent set of equations for an Hermitian two-matrix model was derived by Staudacher [12] and by Ambjorn and Kristjansen [13]. The point is that for the two-matrix model, which is associated with $D = 1/2$ in the formulas, one gets from Eq. (2.12) that $L(\omega) = V'(\omega)$. 

---

3An equivalent set of equations for an Hermitian two-matrix model was derived by Staudacher [12] and by Ambjorn and Kristjansen [13]. The point is that for the two-matrix model, which is associated with $D = 1/2$ in the formulas, one gets from Eq. (2.12) that $L(\omega) = V'(\omega)$.
Plugging this equation back to Eq. (2.14) and taking into account the definition (2.12) one gets the equation for $E_\nu$ first obtained in \[14\]
\[\int_{C_1} d\omega \frac{\tilde{V}'(\omega)}{2\pi i \nu - \omega} E_\omega = E^2_\nu, \tag{2.16}\]
which looks like the large-$N$ loop equation for an Hermitean one-matrix model with the potential $\tilde{V}$. However, in contrast to the one-matrix model where $\tilde{V}$ is usually a polynomial, we shall consider below the case of non-polynomial $\tilde{V}$ which has singularities on the complex plane outside the support of $\rho$.

3 Relation to the Migdal’s approach

The purpose of this section is to demonstrate how, within our approach, one can recover the previous approach to solution of this model due to Migdal \[4\]. For simplicity we restrict ourselves to the case where the density of eigenvalues, $\rho$, has support in a single interval $[a, b]$. It is straightforward to generalize to the situation where the support is in two or more intervals.

Consider our equation (2.13). By construction (cf. Eq. (2.1)), the function $G_{\nu\lambda}$ has singularities only on the support of the spectral density $\rho$, where it has a cut. The contour $C_1$ encircles this cut, and the parameters $\nu$ and $\lambda$ lie outside of it. Recall as well that the function $L(\omega)$ is assumed to be continuous across the cut (cf. Eq. (2.12)).

Let us compress the contour $C_1$ to make it coincide with the cut. Then only the discontinuity of the function $G_{\omega\lambda}$ at this cut (cf. Eq. (2.4)) will contribute to the integral on the l.h.s. of Eq. (2.13). Then Eq. (2.13) can be rewritten in the form
\[\int_a^b d\nu \rho(y) L(y) G_{\lambda}(y) \frac{\nu - y}{\nu - \nu} = E_\nu G_{\nu\lambda} + \lambda G_{\nu\lambda} - E_\nu. \tag{3.1}\]

Consider the limiting case of (3.1) when $\nu$ approaches the cut singularity on the real axis from above:
\[\mathcal{P} \int_a^b d\nu \rho(y) L(y) G_{\nu\lambda}(y) \frac{\nu - y}{\nu - \nu} - i\pi \rho(\nu) L(\nu) G_{\lambda}(\nu) = (\mathcal{P} \int_a^b d\nu \rho(y) G_{\lambda}(y) \frac{\nu - y}{\nu - \nu} - i\pi \rho(\nu) G_{\lambda}(\nu)) = \mathcal{R}E_\nu - i\pi \rho(\nu) + \lambda \mathcal{R}E_\nu + i\pi \rho(\nu). \tag{3.2}\]

Writing a similar equation in the limit when $\nu$ approaches the cut from below and subtracting from it (3.3), we finally get
\[\mathcal{P} \int_a^b d\nu \rho(y) G_{\lambda}(y) \frac{\nu - y}{\nu - \nu} = -1 + (\lambda + \mathcal{R}E_\nu - L(\nu)) G_{\lambda}(\nu) \tag{3.3}\]

Let us now make use of the relation between the real part of the function $E_\nu$ and the function $F(\nu)$ (cf. Eqs. (1.16), (1.17), (2.12))
\[L(\nu) - F(\nu) = \tilde{V}'(\nu) = 2\mathcal{R}E_\nu, \tag{3.4}\]
where $F(\nu)$ is the logarithmic derivative of the Itzykson-Zuber integral which enters the saddle-point equation (1.6). Plugging (3.4) into (3.3), we find exactly the basic equation of the Riemann-Hilbert approach of [4]

$$
\mathcal{P} \int_a^b dy \rho(y) \frac{G_\lambda(y)}{y-\nu} = -1 + \left( \lambda - F(\nu) - \mathcal{RE}_\nu \right) G_\lambda(\nu).
$$

(3.5)

## 4 The exact solutions

### 4.1 The general potential

Let us consider an arbitrary potential $V(\omega)$, which is associated with a general $L(\omega)$ (for simplicity we consider the case when $L(\omega)$ is analytic on the whole complex plane, except, may be, infinity)

$$
L(\omega) = \sum_{m=0}^{\infty} L_m \omega^m.
$$

(4.1)

The equation for $G$ (2.13) in this case is written explicitly

$$(\lambda + E_\nu - L(\nu)) G_{\nu\lambda} = E_\nu - R_\lambda(\nu)
$$

(4.2)

where $R_\lambda(\nu)$ is given by

$$
R_\lambda(\nu) \equiv - \int_{C_2} \frac{d\omega}{2\pi i} \frac{L(\omega)}{\nu - \omega} G_{\omega\lambda} = \sum_{m=1}^{\infty} L_m \sum_{n=0}^{m-1} G_n(\lambda) \nu^{m-n-1}
$$

(4.3)

and the contour $C_2$ encircles both the cut singularity of $G_{\omega\lambda}$ and the pole at $\omega = \nu$. The terms on the right-hand-side result from taking the residue at infinity in the contour integral and the functions $G_n(\lambda)$ are defined by Eq. (2.7).

The formal solution to Eq. (4.2) is

$$
G_{\nu\lambda} = \frac{E_\nu - R_\lambda(\nu)}{\lambda + E_\nu - L(\nu)}
$$

(4.4)

with $R_\lambda(\nu)$ given by (4.3). The functions $G_n(\lambda)$ can be expressed in terms of $E_\lambda$ and $L(\omega)$ using the recurrence relation

$$
G_{n+1}(\lambda) = \int_{C_1} \frac{d\omega}{2\pi i} \frac{L(\omega)}{\lambda - \omega} G_n(\omega) - E_\lambda G_n(\lambda), \quad G_0(\lambda) = E_\lambda
$$

(4.5)

which can easily be obtained by expanding Eq. (2.13) in $1/\lambda$. For $n = 0$ this equation recovers Eq. (2.14).

4It is worth mentioning that (4.4) obviously satisfies Eq. (3.3) as is discussed in the previous section since $R_\lambda(\nu)$ has no discontinuity in $\nu$ at the cut.
It still remains to determine $E_\lambda$. It can be determined from the equation
\begin{equation}
\int_{C_1} \frac{d\omega}{2\pi i} L(\omega) G_{\omega \lambda} = \lambda E_\lambda - 1 \tag{4.6}
\end{equation}
which is the $1/\nu$ term of the asymptotic expansion of Eq. (2.13) in $\nu$. Using the expansion (2.7), one can rewrite this equation as
\begin{equation}
\sum_{m \geq 0} L_m G_m(\lambda) = \lambda E_\lambda - 1 \tag{4.7}
\end{equation}

If $L(\lambda)$ were a polynomial of the highest power $J$, Eq. (4.7) and Eq. (4.5) yield a polynomial equation for $E_\lambda$ containing powers of $E_\lambda$ up to order $J$. Some explicit solutions of this equation are given in the next subsections. Once $E_\lambda$ is known, $\tilde{V}$ can be obtained from Eq. (1.17).

One could consider also the next terms of the $1/\nu$-expansion of Eq. (2.13) which result in the equations
\begin{equation}
\sum_{m \geq 0} L_m G_{m+n}(\lambda) = \lambda G_n(\lambda) + \sum_{k=0}^{n-1} E_k G_{n-1-k}(\lambda) - E_n, \quad n \geq 1 \tag{4.8}
\end{equation}

A question arises as to whether these equations impose new restrictions on $E_\lambda$. The answer is “no” and Eq. (4.8) is automatically satisfied as a consequence of Eqs. (4.5) and (4.7). The proof can be given by induction. Let us calculate the left-hand-side of Eq. (4.8) at some $n = n_0$ assuming that it holds for all lower $n < n_0$. One gets
\begin{equation}
\sum_{m \geq 0} L_m G_{m+n}(\lambda) = \sum_{m \geq 0} L_m \left\{ \int_{C_1} \frac{d\omega}{2\pi i} \frac{L(\omega)}{\lambda - \omega} G_{m+n-1}(\omega) - E_\lambda G_{m+n-1}(\lambda) \right\} = \\
\lambda \left\{ \int_{C_1} \frac{d\omega}{2\pi i} \frac{L(\omega)}{\lambda - \omega} G_{n-1}(\omega) - E_\lambda G_{n-1}(\lambda) \right\} + \sum_{k=0}^{n-1} E_k G_{n-1-k}(\lambda) - \int_{C_1} \frac{d\omega}{2\pi i} L(\omega) G_{n-1}(\omega) = \\
\lambda G_n(\lambda) + \sum_{k=0}^{n-1} E_k G_{n-1-k}(\lambda) - E_n \tag{4.9}
\end{equation}

which completes the proof.

As is discussed above, $G_{\nu \lambda}$ should be symmetric in $\nu$ and $\lambda$. While the right-hand-side of Eq. (4.4) with $R_\lambda(\nu)$ given by Eq. (4.3) looks non-symmetric, it can be shown that the symmetry is restored as a consequence of Eq. (4.7). To prove this, let us multiply the numerator and denominator on the right-hand-side of Eq. (4.4) by $(\nu + E_\lambda - L(\lambda))$. The denominator then becomes explicitly symmetric and the numerator can be transformed
\footnote{The equation which was solved in Ref. [14] by the ansatz $F(\omega) = \omega^2 L^{-1}(\omega)$ is just the $1/\lambda^2$-term of the expansion of Eq. (4.6) in $1/\lambda$ and is, therefore, incomplete. This ansatz is not consistent, generally speaking, with the whole Eq. (1.6).}
as
\[
(E_\nu - R_\lambda(\nu))(\nu + E_\lambda - L(\lambda)) = E_\nu E_\lambda + \nu E_\nu - E_\nu L(\lambda)
\]

\[+ \sum_{m=1}^{\infty} L_m \sum_{n=0}^{m-1} \nu^{m-n-1} \left\{ G_{n+1}(\lambda) - \int_{C_{v_2}} \frac{d\omega}{2\pi i} \frac{L(\omega)}{\lambda - \omega} G_n(\omega) - \nu G_n(\lambda) \right\} =
\]

\[E_\nu E_\lambda + \nu E_\nu - E_\nu L(\lambda) - E_\lambda L(\nu) + \sum_{m=0}^{\infty} L_m G_m(\lambda) + \int_{C_{v_2}} \frac{d\omega}{2\pi i} \frac{L(\omega)}{\lambda - \omega} \int_{C_2} \frac{dz}{2\pi i \nu - z} G_\omega z
\]

\[(4.7)\]

This expression is manifestly symmetric in \(\nu\) and \(\lambda\), what is easy to see taking the imaginary part, which completes the proof.

As is shown below for the quadratic and quartic cases, the symmetry requirement can be used directly to determine \(E_\lambda\) alternatively to Eq. (4.7).

### 4.2 Quadratic potential

Let us show how the algorithm of the previous subsection works for the simplest case of a Gaussian potential. In this case, we begin with the assumption that \(L_m = 0\) for \(m \neq 1\) so that \(L(\lambda)\) is

\[L(\lambda) = L_1 \lambda \equiv \Lambda^{-1} \lambda,
\]

where we have introduced the quantity \(\Lambda\) to establish the connection with the previous studied of the Gaussian model.

In the Gaussian case the solution (4.4) takes the form

\[G^{(0)}(\nu\lambda) = \frac{E_\nu - \Lambda^{-1} E_\lambda}{\lambda + \nu E_\nu - \nu \Lambda^{-1}}
\]

since \(R_\lambda(\nu) = \Lambda^{-1} E_\lambda\). Requiring this expression to be symmetric w.r.t. \(\lambda\) and \(\nu\) one fixes the function \(E_\lambda\) completely. It turns out that \(E_\lambda\) coincides with the function obtained from the semi-circle eigenvalue distribution of the Gaussian matrix model (which is also known to solve the Gaussian Kazakov-Migdal model \[5\]),

\[E^{(0)}_\lambda = \frac{\mu \lambda}{2} - \sqrt{\frac{\mu^2 \lambda^2}{4} - \mu} , \quad \rho^{(0)}(x) = \frac{1}{\pi} \sqrt{\mu - \frac{\mu^2 \lambda^2}{4}}
\]

while \(\mu\) and \(\Lambda\) are related by

\[\mu = \Lambda^{-1} - \Lambda.
\]

Therefore, the symmetry requirement fully determines the Gaussian solution.

This allows to rewrite \(G^{(0)}_{\nu\lambda}\) in the manifestly symmetric form

\[G^{(0)}_{\nu\lambda} = \frac{\Lambda^{-1} E_\lambda E_\nu - \Lambda^{-1}(\lambda E_\lambda + \nu E_\nu) + (\lambda E_\nu + \nu E_\lambda) + \mu}{\lambda^2 - (\Lambda + \Lambda^{-1}) \lambda \nu + \nu^2 + \mu}
\]

(4.15)
It is straightforward to check that this expression is not singular for all \( \nu \) and \( \lambda \).

As for the equation (4.7), in the Gaussian case it involves only \( G_1(\lambda) \) which is fixed by the recurrence relation (4.5) to be

\[
G_1^{(0)}(\lambda) = \frac{1}{\Lambda}(\lambda E_\lambda - 1) - E_\lambda^2. \tag{4.16}
\]

This gives the quadratic equation

\[
\frac{1}{\Lambda} E_\lambda^2 + \left(1 - \frac{1}{\Lambda^2}\right) \lambda E_\lambda = 1 - \frac{1}{\Lambda^2} \tag{4.17}
\]

whose solution, which at large \( \lambda \) behaves like \( 1/\lambda \), is given by Eqs. (4.13), (4.14).

Since (Eq. (2.13))

\[
G_1(\nu) = \int_{C_1} \frac{d\omega}{2\pi i} \frac{F(\omega)}{\nu - \omega} E_\omega = \int_a^b dx \rho(x) \frac{F(x)}{\nu - x} \tag{4.18}
\]

where the last integral is along the cut, \( F(x) \) is easily found from the discontinuity of \( G_1(x) \) across the cut. The result coincides with that of \[5\] \[10\]

\[
F^{(0)}(x) = \Lambda x = \frac{2x}{\mu + \sqrt{\mu^2 + 4}}. \tag{4.19}
\]

It is crucial that \( D \) does not enter explicitly in the relation between \( \mu \) and \( \Lambda \) (cf. Eqs. (4.14), (4.15)). The meaning is that the Kazakov–Migdal model with the quadratic potential reduces to the Hermitean one-matrix model with the same kind of potential.

Taking discontinuities of (4.15) across the cuts according to the definition (2.1) we obtain the following expression for the correlator of the gauge fields

\[
C(\alpha, \beta) = \frac{\Lambda^{-1}}{\alpha^2 - (\Lambda + \Lambda^{-1})\alpha\beta + \beta^2 + \mu}. \tag{4.20}
\]

Its expansion as \( \mu \to \infty \) recovers the leading order of the large mass expansion \[7\]

\[
C_{ij} = 1 + \phi_i \phi_j. \tag{4.21}
\]

On the contrary if \( \mu \to 0 \), one gets from (4.20) the weak coupling result \[8\]

\[
C_{ij} = N\delta_{ij}. \tag{4.22}
\]

While the expression (4.20) involves the quadratic polynomial in the denominator, its zeros, say in the complex \( \lambda \)-plane,

\[
\lambda_{\pm}(\nu) = \frac{1}{2} \left(\Lambda + \frac{1}{\Lambda}\right) \nu \pm \frac{\mu}{2} \sqrt{\nu^2 - \frac{4}{\mu}}. \tag{4.23}
\]
always lies for \( \mu > 0 \) outside the cut.

The situation when \( C(\alpha, \beta) \) were become negative somewhere at the support of \( \rho \), would be associated with a (third order) phase transition in the Itzykson–Zuber integral. The only possibility to have such a phase transition with our Gaussian Eq. (4.20) is when \( \mu = 0 \) so that \( \lambda_\pm = \nu \) belongs to the cut. Because \[ m_0^2 = D\sqrt{\mu^2 + 4 - (D - 1)\mu}, \]

the only possibility to reach the point \( \mu = 0 \) in the strong coupling phase \( m_0^2 \geq 2D \), where the Gaussian model is stable, is at \( D = 1 \). This phase transition at \( D = 1 \) is the standard one which provides the continuum limit.

### 4.3 The quartic case

One of simplest non-Gaussian cases is associated with

\[
L(\lambda) = L_1 \lambda + L_3 \lambda^3.
\]

The meaning of this formula is that we consider Eq. (4.25) as an ansatz for \( L(\lambda) \) and shall solve Eq. (2.13) for \( E_\lambda \) (and, therefore, \( \tilde{V}'(\lambda) = 2\mathcal{RE}_\lambda \) at the cut). The original potential \( V \) of the Kazakov–Migdal model can then be determined due to Eq. (2.12) to be

\[
V'(\lambda) = 2D L(\lambda) - (2D - 1)\tilde{V}'(\lambda).
\]

As was already discussed, \( D = 1/2 \) for the two-matrix model and \( \tilde{V}' \) disappears from this formula. For this reason one might think of \( \int L(\lambda)d\lambda \) as of the potential of the proper two-matrix model which is just quartic for the ansatz (4.25).

For \( L(\lambda) \) given by Eq. (4.25) only the term with \( m = 3 \) survives in (4.3) so that Eq. (4.4) gives

\[
G_{\nu\lambda} = \frac{E_{\nu} - (L_1 + L_3\nu^2)E_\lambda - L_3 \nu G_1(\lambda) - L_3 G_2(\lambda)}{\lambda + E_{\nu} - L(\nu)}
\]

where the recurrence relation (1.3) yields

\[
G_1(\lambda) = (L_1 + L_3\lambda^2)(\lambda E_\lambda - 1) - E_\lambda^2 - L_3 E_2,
\]

\[
G_2(\lambda) = (L(\lambda) - E_\lambda) G_1(\lambda) - \lambda L_3(L_1 E_2 + L_3 E_4 - 1)
\]

with \( E_2 \) and \( E_4 \) defined by Eq. (2.4).

To determine \( E_\lambda \) we use Eq. (4.7) (or, which is equivalent, the requirement that \( G_{\nu\lambda} \) should be symmetric in \( \nu \) and \( \lambda \)) which leads to the following quartic equation

\[
E_\lambda (\lambda - L_1 L(\lambda)) + L_3 G_2(\lambda) (E_\lambda - L(\lambda)) + L_1 E_\lambda^2 + L_3 \lambda^2 (L_1 + L_3 E_2) = \text{const}.
\]
The constant on the right-hand-side of this equation can be found through asymptotic expansion in $\lambda$ of its left-hand-side.

Since the resulting equation (1.29) for $E$ is quartic, its general solution, though explicitly known, is rather obscure and not informative. For our purposes it is more convenient to rewrite it lowering the powers of $E_\lambda$ down to linear using Eq. (2.16) which we rewrite in the form

$$E^2_\lambda = \tilde{V}'(\lambda) E_\lambda - Q(\lambda)$$

(4.30)

where

$$Q(\lambda) \equiv -\int_{Cv_2} \frac{d\omega}{2\pi i} \frac{\tilde{V}'(\omega)}{\lambda - \omega} E_\omega$$

(4.31)

is a polynomial of the highest power $J - 2$ if is $\tilde{V}$ is that of the highest power $J$. It is essential for what follows that nether $Q(\lambda)$ nor $\tilde{V}'(\lambda)$ are singular at the cut of $E_\lambda$.

The resulting equation can be conveniently rewritten in terms of $F(\lambda) = L(\lambda) - \tilde{V}'(\lambda)$. Taking the discontinuity of the resulting equation (i.e. just the factor in front of $E_\lambda$) one gets

$$L_1 F(\lambda) + L_3 \left[ F^3(\lambda) - F(\lambda) Q(\lambda) - \left( Q(\lambda) - (L_1 + L_3 \lambda^2) - L_3 E_2 \right) \left( F(\lambda) + L(\lambda) \right) \right] + \lambda L_3 (L_1 E_2 + L_3 E_4 - 1) = \lambda$$

(4.32)

while the continuous part yields

$$L_1 Q(\lambda) + L_3 \left[ F(\lambda) Q(\lambda) \left( F(\lambda) + L(\lambda) \right) - \left( Q(\lambda) - (L_1 + L_3 \lambda^2) - L_3 E_2 \right) \left( Q(\lambda) - L^2(\lambda) \right) - L_3 \lambda L(\lambda) (L_1 E_2 + L_3 E_4 - 1) \right] = L_3 \lambda (L(\lambda) + \lambda L_2 E_2) + \text{const.}$$

(4.33)

The set of equations (4.32), and (4.33) must have a solution with $F(\lambda)$ and $Q(\lambda)$ nonsingularat the cut. In general, $Q(\lambda)$ should be expressed via $F(\lambda)$, say, from Eq. (1.32). The substitution into Eq. (4.33) then yield a high order algebraic equation for $F(\lambda)$. Presumably, a reasonable way to study the equation for $F(\lambda)$ is to assume some analytic structure and then determine algebraic coefficients from the equation. From a bunch of solutions one should choose the one which recovers the Gaussian solution as $L_3 \to 0$.

While the general solution of Eqs. (4.32), (4.33) looks hopeless, the iterative solution in $L_3$ is straightforward

$$F(\lambda) = \left( \frac{1}{L_1} - \frac{2L_3}{L_4^3} \right) \lambda - \frac{L_3}{L_4^2} \lambda^3$$

(4.34)

and

$$\tilde{V}'(\lambda) = \left( L_1 - \frac{1}{L_1} + \frac{2L_3}{L_4^3} \right) \lambda + L_3 \left( 1 + \frac{1}{L_4} \right) \lambda^3.$$  

(4.35)

Next orders in $L_3$ can easily be calculated which illustrates that the solution exists at least at small enough $L_3$. 

12
Given the solution to Eqs. (4.32), (4.33), $C_{ij}$ can be calculated taking the double discontinuity of (4.27) w.r.t. both $\nu$ and $\lambda$ across the cut which gives

$$C(x, y) = \frac{(L_1 + L_3 x^2) + L_3 x F(y) + L_3 \left[ F^2(y) - Q(y) + (L_1 + L_3 y^2) + L_3 E_2 \right]}{y^2 - y \left( L(x) + F(x) \right) + L(x) F(x) + Q(x)}.$$  \hspace{1cm} (4.36)

As is discussed, this expression will be symmetric in $x$ and $y$ for $F$ and $Q$ being the solutions of Eqs. (4.32), (4.33). This requirement of the symmetry of (4.36) might help to find a solution. Anyway, the ansatz (4.25) was chosen as the simplest one while for other functions $L(\lambda)$ the resulting algebraic equations might simplify.

5 Discussion

In this Paper we have developed a method for calculating the gauge field correlator $C(\alpha, \beta)$. Its knowledge allows one to calculate the extended loop averages

$$G_{\nu\lambda}(\Gamma_{xy}) \equiv \left\langle \frac{\text{tr}}{N} \left( \frac{1}{\nu - \phi_x} U(\Gamma_{xy}) \frac{1}{\lambda - \phi_x} U(\Gamma_{yx}) \right) \right\rangle.$$ \hspace{1cm} (5.1)

where $U(\Gamma_{xy})$ is the ordered product of $U$'s along some path $\Gamma_{xy}$ from $x$ to $y$. If $\Gamma_{xy}$ coincides with one link, $G_{\nu\lambda}(\Gamma_{xy})$ coincides with $G_{\nu\lambda}$ according to the definition (2.1). An explicit formula for $G_{\nu\lambda}(\Gamma_{xy})$ can be obtained substituting the gauge field correlator at each link $l$ by $C(\alpha_l, \alpha_{l+1})$ and integrating over $\alpha_l$'s.

In the case of a matrix chain which is associated with the Kazakov–Migdal model at $D < 1$, this would be a complete set of observables at $N = \infty$. However, a new class of correlators which can be constructed from pure gauge fields appears at $D > 1$. The simplest one is the closed adjoint Wilson loop

$$W_A \equiv \left\langle \left| \frac{\text{tr}}{N} U(\Gamma) \right|^2 \right\rangle.$$ \hspace{1cm} (5.2)

which is well-defined in the ’t Hooft limit and can again be calculated integrating over $U$’s at each link. It is expected \[3, 10\] that this quantity might undergo a first-order phase transition which is associated with a restoration of the area law. For the Gaussian model this phase transition does not happen while more complicated potentials have not yet been studied.

One more interesting problem, for which the approach of this paper might be potentially useful, concerns another kind of pure gauge field averages, the filled Wilson loops which were employed in connection with the Kazakov–Migdal model in \[1\] and whose importance for constructing the $D > 1$-dimensional string theories from the Kazakov–Migdal model was emphasized in \[7\] and, as well as their role in resolving the problem of the local $Z_N$ symmetry, in \[8\].
We hope that the technique developed in this paper could help to solve some of these problems.

Acknowledgements

We are grateful to J. Ambjorn, C. Kristjansen and N. Weiss for useful discussions. Yu. M. thanks the UBC physics department for the hospitality at Vancouver. This work is supported in part by the Natural Sciences and Engineering Research Council of Canada.

References

[1] G. ’t Hooft, *Nucl. Phys.* **B72** (1974) 461.
[2] V.A. Kazakov and A.A. Migdal, *Nucl. Phys.* **B397** (1993) 214.
[3] C. Itzykson and J.B. Zuber, *J. Math. Phys.* **21** (1980) 411; Harish-Chandra, *Amer. J. Math.* **79** (1957) 87.
[4] A.A. Migdal, *Mod. Phys. Lett.* **A8** (1993) 359.
[5] D. Gross, *Phys. Lett.* **293B** (1992) 181.
[6] Yu. Makeenko, *Mod. Phys. Lett.* **A8** (1993) 209.
[7] I.I. Kogan, A. Morozov, G.W. Semenoff and N. Weiss, *Nucl. Phys.* **B395** (1993) 547.
[8] M.I. Dobroliubov, I.I. Kogan, G.W. Semenoff and N. Weiss, *Phys. Lett.* **302B** (1993) 283.
[9] I.I. Kogan, G.W. Semenoff and N. Weiss, *Phys. Rev. Lett.* **69** (1992) 3435.
[10] S. Khokhlachev and Yu. Makeenko, *Phys. Lett.* **297B** (1992) 345.
[11] A. Morozov, *Mod. Phys. Lett.* **A7** (1992) 3503; S. Shatashvili, *Correlation functions in the Itzykson-Zuber model*, preprint IASSNS-HEP-92/61 (September, 1992).
[12] M. Staudacher, *Phys. Lett.* **305B** (1993) 332.
[13] J. Ambjorn and C. Kristjansen, unpublished.
[14] Yu. Makeenko, *An exact solution of induced large-N lattice gauge theory at strong coupling*, preprint ITEP-YM-2-93 (March, 1993).