McKay’s $E_7$ observation on the Baby Monster

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Abstract

In this paper, we study McKay’s $E_7$ observation on the Baby Monster. By investigating so called derived $c = 7/10$ Virasoro vectors, we show that there is a natural correspondence between dihedral subgroups of the Baby Monster and certain subalgebras of the Baby Monster vertex operator algebra which are constructed by the nodes of the affine $E_7$ diagram. This allows us to reinterpret McKay’s $E_7$ observation via the theory of vertex operator algebras.

For a class of vertex operator algebras including the Moonshine module, we will show that the product of two Miyamoto involutions associated to derived $c = 7/10$ Virasoro vectors in certain commutant vertex operator algebras is an element of order at most 4. For the case of the Moonshine module, we obtain the Baby monster vertex operator algebra as the commutant and we can identify the group generated by these Miyamoto involutions with the Baby Monster and recover the $\{3, 4\}$-transposition property of the Baby Monster in terms of vertex operator algebras.
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1 Introduction

The purpose of this article is to give a vertex operator algebra (VOA) theoretical interpretation of McKay’s $E_7$ observation on the Baby Monster. The main idea is to relate certain substructures of the Moonshine VOA $V^\natural$, whose full automorphism group is the Monster $\mathbb{M}$ [B, FLM], to some coset (or commutant) subalgebras related to the Baby Monster. In [LYY1, LYY2, LM], McKay’s $E_8$ observation on the Monster has been studied in detail using the correspondence between 2A involutions of the Monster and simple $c = 1/2$ Virasoro vectors in the Moonshine VOA $V^\natural$ [C, Mi2]. It is established that there exists a natural correspondence between the dihedral groups generated by two 2A involutions of the Monster and certain sub-VOAs of $V^\natural$ which are constructed naturally by the nodes of the affine $E_8$ diagram. It turns out that under some general hypotheses [S], the dihedral subalgebras (cf. Section 1.1) associated to the affine $E_8$ diagram exhaust all possible cases. Therefore, the nine algebras associated to the $E_8$ diagram are exactly the nine possible
subalgebras of $V^2$ generated by two simple $c = 1/2$ Virasoro vectors.

In this article, we will study the $E_7$ observation. We first observe that the Baby Monster acts naturally on a certain commutant subalgebra $V^B$ of the Moonshine VOA $V^2$ which is called the Baby Monster VOA (cf. (5.1)). This observation leads us to study the commutant subalgebra of a simple $c = 1/2$ Virasoro vector in some 2A-subalgebras and certain $c = 7/10$ Virasoro vectors, which we call derived Virasoro vectors (cf. Definition 3.5), in $V^B$. We show that there exists a one-to-one correspondence between 2A involutions of the Baby Monster and derived $c = 7/10$ Virasoro vectors in $V^B$ (see Theorems 5.9 and 5.13). The main result is a connection between the dihedral groups of the Baby Monster and certain sub-VOAs constructed by the nodes of the affine $E_7$ diagram in Theorem 5.18. In addition, we show that one can canonically associate involutive automorphisms to derived $c = 7/10$ Virasoro vectors (see Lemma 2.5), which we call $\sigma$-involutions, and they satisfy the \{3, 4\}-transposition property under certain hypotheses satisfied by a class VOAs containing the Baby Monster VOA $V^B$ (see Proposition 3.11 in Section 3.2). In order to study the dihedral groups generated by two 2A-elements in the Baby Monster, one needs to study the VOAs generated by two 2A-subalgebras (cf. Section 3) with some conditions, which corresponds to VOAs generated by three simple $c = 1/2$ Virasoro vectors, while in the $E_8$ case, one only needs to study VOAs generated two simple $c = 1/2$ Virasoro vectors.

In a further paper, we will discuss McKay’s $E_6$ observation on the largest Fischer group [HLY]. Although the general approach to this case is similar, other vertex operator algebras have to be studied and many technical details are different.

To explain our results more precisely, let us review the background of our method and the results established in [LYY1, LYY2, LM]. The main idea is to associate involutions to certain Virasoro vectors of small central charge in $V^2$ and $V^B$.

Let $R$ be a simple root lattice with a simply laced root system $\Phi(R)$. We scale $R$ such that the roots have squared length 2. We will consider the lattice VOA $V_{\sqrt{2}R}$ associated to $\sqrt{2}R$. Here and further we will use the standard notation for lattice VOAs as in [FLM]. In [DLMN] Dong et al. constructed a simple Virasoro vector of $V_{\sqrt{2}R}$ of the form

$$\tilde{\omega}_R := \frac{1}{2h(h+2)} \sum_{\alpha \in \Phi(R)} \alpha(-1)^2 1 + \frac{1}{h+2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}a},$$

(1.1)

where $h$ denotes the Coxeter number of $R$. The central charge of $\tilde{\omega}_R$ is $1/2$, $7/10$ and $6/7$ if $R = E_8$, $E_7$ and $E_6$, respectively.
1.1 The affine $E_8$ diagram and the Monster

We will describe some automorphisms of $V_{\sqrt{2}E_8}$ from McKay’s $E_8$ diagram [C, Mc].

Let $L_{nX}$ be a sublattice of $E_8$ obtained by removing the node labeled $nX$. Then the index of $L_{nX}$ in $E_8$ is $n$ and we have a coset decomposition

$$E_8 = \bigcup_{j=0}^{n-1} (L_{nX} + j\alpha).$$

Correspondingly, we have a decomposition

$$V_{\sqrt{2}E_8} = \bigoplus_{j=0}^{n-1} V_{\sqrt{2}(L_{nX} + j\alpha)}.$$

Define a linear map $\rho_{nX}$ acting on the component $V_{\sqrt{2}(L_{nX} + j\alpha)}$ by $e^{2\pi\sqrt{-1}/n}$. Then $\rho_{nX}$ is an automorphism of $V_{\sqrt{2}E_8}$ of order $n$. Now take the simple $c = 1/2$ Virasoro vector $e = \bar{\omega}_{E_8}$ of $V_{\sqrt{2}E_8}$ defined by (1.1). Denote by $U_{nX}$ the subalgebra of $V_{\sqrt{2}E_8}$ generated by $e$ and $f := \rho_{nX} e$. Note that one can associate involutive automorphisms to these Virasoro vectors via the so-called Miyamoto involutions [Mi1] and therefore the subalgebra $U_{nX}$ represents the symmetry of a dihedral group of order $2n$.

On the other hand, Sakuma [S] showed the following result about subalgebras generated by two simple $c = 1/2$ Virasoro vectors $e$ and $f$: Let $V = \bigoplus_{n \geq 0} V_n$ be a VOA over $\mathbb{R}$ with $V_0 = \mathbb{R}1$ and $V_1 = 0$, and assume that the invariant bilinear form on $V$ is positive definite. Then there are exactly nine possible structures for the Griess subalgebra generated by $e$ and $f$ in the degree two subspace $V_2$ (cf. Theorem 3.8) and they agree with those of the dihedral subalgebras $U_{nX}$ discussed in the previous paragraph. In other words, the dihedral subalgebras $U_{nX}$ exhaust all the possibilities.

In [LYY1, LYY2, LM], the algebra $U_{nX}$ is studied in detail and it is shown that there exists a natural embedding of $U_{nX} \hookrightarrow V^\natural$ for each node $nX$. This together with Sakuma’s theorem explains how the $E_8$ structure is imposed in the Moonshine VOA. In addition, it is shown in [LYY2] that the product $\tau e \tau f$ of the corresponding Miyamoto involutions on $V^\natural$ is exactly in the Monster conjugacy class $nX$. In fact, the subalgebra $U_{nX}$ exactly
corresponds to a dihedral subgroup generated by two 2A-involutions of the Monster via
the one-to-one correspondence between the simple $c = 1/2$ Virasoro vectors and 2A-
involutions. Thus the $E_8$-structure on the Moonshine VOA corresponds naturally to the
$E_8$-structure of the Monster as observed by McKay.

1.2 The affine $E_7$ diagram and the Baby Monster

In this article, we will give a similar correspondence which associates the affine $E_7$ diagram
to the Baby Monster.

The Baby Monster VOA $V\mathbb{B}^\natural$. Let $e$ be a simple $c = 1/2$ Virasoro vector of the
Moonshine VOA. Denote by $\text{Com}_{V^2}(\text{Vir}(e))$ the commutant subalgebra of $\text{Vir}(e)$ in $V^2$.
It follows from the one-to-one correspondence between simple $c = 1/2$ Virasoro vectors in
$V^2$ and 2A-involutions of the Monster (cf. Theorem 5.1) that all simple $c = 1/2$ Virasoro
vectors of $V^2$ are mutually conjugate under the Monster and thus the VOA structure on
$\text{Com}_{V^2}(\text{Vir}(e))$ is independent of $e \in V^2$. Denote by $\tau_e$ the involution corresponding to a
simple $c = 1/2$ Virasoro vector $e$. The centralizer $C_M(\tau_e)$ is a double cover $2.\mathbb{B}$ of the Baby
Monster simple group $\mathbb{B}$ which naturally acts on $\text{Com}_{V^2}(\text{Vir}(e))$ so that we denote it by $V\mathbb{B}^\natural$ and call it the Baby Monster VOA. The Baby Monster VOA as well as its extension
to a superalgebra called the shorter Moonshine module was first constructed by one of
the authors (G.H.) in [Hö1]. It is proved in [Hö2, Y] that the Baby Monster is indeed
the full automorphism group of the Baby Monster VOA and therefore the Baby Monster
VOA $V\mathbb{B}^\natural$ is probably the most natural object to be considered in the study of the Baby
Monster, the second largest of the 26 sporadic groups.

As $\mathbb{B}$ is involved as a double cover $2.\mathbb{B} \subset M$ in the Monster, we have also an embedding

$$L(1/2, 0) \otimes V\mathbb{B}^\natural \simeq \text{Vir}(e) \otimes \text{Com}_{V^2}(\text{Vir}(e)) \hookrightarrow V^2,$$

where $L(c, 0)$ denotes a simple Virasoro VOA with central charge $c$. We will show in
Proposition 5.4 that every 2A-involution $s \in \text{Aut}(\text{Com}_{V^2}(\text{Vir}(e)))$ of the Baby Monster is
covered by a 2A-involution $t \in \text{Aut}(V^2)$ of the Monster in the sense that $s = t|_{\text{Com}_{V^2}(\text{Vir}(e))}$
for some $t \in C_M(\tau_e)$.

We will also show that every 2A-involution of the Baby Monster is induced by a simple
$c = 7/10$ Virasoro vector via Miyamoto involutions in Theorem 5.9 and this correspon-
dence actually is one-to-one if we restrict only to simple $c = 7/10$ Virasoro vectors of

\footnote{In [Hö1] the shorter Moonshine module is denoted by $V\mathbb{B}^\natural$. Our $V\mathbb{B}^\natural$ is the even part of the shorter Moonshine module and corresponds to $V\mathbb{B}^\natural_{(0)}$ in loc. cit.}
\( \sigma \)-type in \( V\mathbb{B}^2 \) as shown in Theorem 5.13. (See Definition 2.4 for the definition of simple \( c = 7/10 \) Virasoro vectors of \( \sigma \)-type.)

**McKay’s observation.** By using the embedding of the \( E_7 \) lattice into the \( E_8 \) lattice and similar ideas as in [LYY1, LYY2], we will construct a certain sub-VOA \( U_{B(nX)} \) of the lattice VOA \( V_{\sqrt{2}E_7} \) associated to each node \( nX \) of the affine \( E_7 \) diagram (cf. Section 4).

We will show that \( U_{B(nX)} \) is contained in the VOA \( V\mathbb{B}^2 \) purely by their VOA structures in Theorem 5.18. These sub-VOAs contain pairs of Virasoro vectors, whose central charge are \( 7/10 \). Then, by identifying the Baby Monster with \( \text{Aut}(V\mathbb{B}^2) \), we also show in Theorem 5.18 that the products of the corresponding \( \sigma \)-involutions belong to the desired conjugacy classes \( nX \) in \( \mathbb{B} \) using the Atlas [ATLAS]. Thus our embeddings of \( U_{B(nX)} \) into \( V\mathbb{B}^2 \), in some sense, encode the \( E_7 \) structure into the Baby Monster VOA \( V\mathbb{B}^2 \) and the Baby Monster which are compatible with the original McKay observation.

**\( N \)-transposition property.** A central point for understanding McKay’s observations is to describe a product of two involutions. In the Monster, the product of two 2A-involutions has order less than or equal to 6. This fact is known as the 6-\textit{transposition property} of the Monster. This fact can be deduced directly from the character table of the Monster. On the other hand, Sakuma [S] showed that the 6-transposition property can also be viewed as a consequence of symmetries of a vertex operator algebras. In this paper, we will use Sakuma’s theorem and deduce that Miyamoto involutions associated to derived \( c = 7/10 \) Virasoro vectors of the commutant subalgebra satisfy the \( \{3, 4\} \)-transposition property under the same assumption as in Sakuma’s theorem (cf. Propositions 3.11). Applying this result, we can recover the \( \{3, 4\} \)-transposition property of the Baby Monster. It is true that one still needs to use the character tables to identify the conjugacy classes of the Baby Monster, but it is worth to emphasize that the bound for the order of products of two involutions is given by the theory of vertex operator algebras.

**1.3 The organization of the paper**

The organization of this article is as follows: In Section 2, we review basic properties about Virasoro VOAs and Virasoro vectors. In Section 3, we study a special vertex operator algebra, which we call the 2A-algebra for the Monster.

In Section 4, we recall the definition of a commutant sub-VOA and define certain commutant subalgebras associated to the root lattice of type \( E_7 \) using the method described in [LYY1, LYY2]. It turns out that these commutant subalgebras contain many Virasoro vectors, which will be used to define some involutions in the last section.
In Section 5, the commutant subalgebra $V^B\natural$ of $V\natural$ is studied. The full automorphism group of $V^B\natural$ is shown to be the Baby Monster simple group in [Hö2, Y]. We show that there is a one-to-one correspondence between 2A-involutions of the Baby Monster and simple $c = 7/10$ Virasoro vectors of $\sigma$-type in $V^B\natural$.

Finally, we discuss the embeddings of the commutant subalgebras constructed in Section 4 into $V^B\natural$ in Section 5.3. We show that the simple $c = 7/10$ Virasoro vectors defined in Section 4.2 can be embedded into $V^B\natural$. Moreover, the product of the corresponding $\sigma$-involutions belongs to the conjugacy class associated to the node.

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**Notation and Terminology.** In this article, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of non-negative integers, integers, reals and complex numbers, respectively. We denote the ring $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}_p$ with a positive integer $p$ and often identify the integers $0, 1, \ldots, p - 1$ with their images in $\mathbb{Z}_p$. We denote the Monster simple group by $\mathbb{M}$, the Baby Monster simple group by $\mathbb{B}$.

Every vertex operator algebra is defined over the field $\mathbb{C}$ of complex numbers unless otherwise stated. A VOA $V$ is called of CFT-type if it is non-negatively graded $V = \bigoplus_{n \geq 0} V_n$ with $V_0 = \mathbb{C}\mathbb{1}$. For a VOA structure $(V, Y(\cdot, z), \mathbb{1}, \omega)$ on $V$, the vector $\omega$ is called the conformal vector of $V$. For simplicity, we often use $(V, \omega)$ to denote the structure $(V, Y(\cdot, z), \mathbb{1}, \omega)$. The vertex operator $Y'(a, z)$ of $a \in V$ is expanded such that $Y'(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$.

For $c, h \in \mathbb{C}$, let $L(c, h)$ be the irreducible highest weight module over the Virasoro algebra with central charge $c$ and highest weight $h$. It is well-known that $L(c, 0)$ has a simple VOA structure [FZ]. An element $e \in V$ is referred to as a Virasoro vector of central charge $c_e \in \mathbb{C}$ if $e \in V_2$ and satisfies $e(1) e = 2e$ and $e(3) e = (c_e/2) \cdot \mathbb{1}$. It is well-known that the associated modes $L^e(n) := e(n+1)$, $n \in \mathbb{Z}$, generate a representation of the Virasoro algebra on $V$ (cf. [Mi1]), i.e., they satisfy the commutator relation

$$[L^e(m), L^e(n)] = (m - n)L^e(m + n) + \delta_{m+n,0} \frac{m^3 - m}{12} c_e.$$ 

Therefore, a Virasoro vector together with the vacuum vector generates a Virasoro VOA inside $V$. We will denote this subalgebra by $\text{Vir}(e)$. 

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In this paper, we define a sub-VOA of $V$ to be a pair $(U, e)$ of a subalgebra $U$ containing the vacuum element $1$ and a conformal vector $e$ for $U$ such that $(U, e)$ inherits the grading of $V$, that is, $U = \bigoplus_{n \geq 0} U_n$ with $U_n = V_n \cap U$, but $e$ may not be the conformal vector of $V$. In the case that $e$ is also the conformal vector of $V$, we will call the sub-VOA $(U, e)$ a full sub-VOA.

For a positive definite even lattice $L$, we will denote the lattice VOA associated to $L$ (cf. [FLM]). We adopt the standard notation for $V_L$ as in [FLM]. In particular, $V_L^+$ denotes the fixed point subalgebra of $V_L$ under a lift of the $(-1)$-isometry on $L$. The letter $\Lambda$ always denotes the Leech lattice, the unique even unimodular lattice of rank 24 without roots.

Given a group $G$ of automorphisms of $V$, we denote by $V^G$ the fixed point subalgebra of $G$ in $V$. The subalgebra $V^G$ is called the $G$-orbifold of $V$ in the literature. For a $V$-module $(M, Y_M(\cdot, z))$ and $\sigma \in \text{Aut}(V)$, we set $\sigma Y_M(a, z) := Y_M(\sigma^{-1}a, z)$ for $a \in V$. Then the $\sigma$-conjugate module $\sigma \circ M$ of $M$ is defined to be the module structure $(M, \sigma Y_M(\cdot, z))$.

### 2 Virasoro vertex operator algebras

Let

$$c_m := 1 - \frac{6}{(m+2)(m+3)}, \quad m = 1, 2, \ldots,$$

$$h_{r,s}^{(m)} := \frac{\{r(m+3) - s(m+2)\}^2 - 1}{4(m+2)(m+3)}, \quad 1 \leq s \leq r \leq m+1.$$  \hfill (2.1)

It is shown in [W] that $L(c_m, 0)$ is rational and $L(c_m, h_{r,s}^{(m)})$, $1 \leq s \leq r \leq m+1$, are all irreducible $L(c_m, 0)$-modules (see also [DMZ]). This is the so-called unitary series of the Virasoro VOAs. The fusion rules among $L(c_m, 0)$-modules are computed in [W] and given by

$$L(c_m, h_{r_1,s_1}^{(m)}) \times L(c_m, h_{r_2,s_2}^{(m)}) = \sum_{i \in I} L(c_m, h_{[r_1-r_2]+2i-1, |s_1-s_2|+2j-1}^{(m)}),$$  \hfill (2.2)

where

$I = \{1, 2, \ldots, \min\{r_1, r_2, m+2-r_1, m+2-r_2\}\},$

$J = \{1, 2, \ldots, \min\{s_1, s_2, m+3-s_1, m+3-s_2\}\}.$

**Definition 2.1.** A Virasoro vector $x$ with central charge $c$ is called simple if $\text{Vir}(x) \simeq L(c, 0)$. A simple $c = 1/2$ Virasoro vector is called an Ising vector.

The fusion rules among $L(c_m, 0)$-modules have a canonical $\mathbb{Z}_2$-symmetry and this symmetry gives rise to an involutive automorphism of a VOA.
**Theorem 2.2** ([Mi1]). Let $V$ be a VOA and $e \in V$ a simple Virasoro vector with central charge $c_m$. Denote by $V_r^{(m)}_r$ the sum of irreducible $\text{Vir}(e) = L(c_m,0)$-submodules isomorphic to $L(c_m,h^{(m)}_{r,s})$, $1 \leq s \leq r \leq m + 1$. Then the linear map

$$\tau_e := \begin{cases} 
(−1)^{r+1} & \text{on } V_e[h^{(m)}_{r,s}] \text{ if } m \text{ is even}, \\
(−1)^{s+1} & \text{on } V_e[h^{(m)}_{r,s}] \text{ if } m \text{ is odd},
\end{cases}$$

defines an automorphism of $V$ called $\tau$-involution associated to $e$.

In this paper, we frequently consider simple Virasoro vectors with central charges $c_1 = 1/2$ and $c_2 = 7/10$. Here we recall the definitions of $\sigma$-type $c = 1/2$ and $c = 7/10$ Virasoro vectors. The corresponding $\sigma$-involutions will also be defined.

**Definition 2.3.** An Ising vector $e$ of a VOA $V$ is said to be of $\sigma$-type on $V$ if $\tau_e = \text{id}$ on $V$, i.e., $V_e[1/16] = 0$.

In this case, one has $V = V_e[0] \oplus V_e[1/2]$ and the map $\sigma_e$ defined by

$$\sigma_e := \begin{cases} 
1 & \text{on } V_e[0], \\
−1 & \text{on } V_e[1/2].
\end{cases}$$

is an automorphism of $V$ (cf. [Mi1]).

**Definition 2.4.** A simple $c = 7/10$ Virasoro vector $u$ of a VOA $V$ is said to be of $\sigma$-type on $V$ if $V_u[7/16] = V_u[3/80] = 0$.

Let $u \in V$ be a simple $c = 7/10$ Virasoro vector of $\sigma$-type. Then one has the isotypical decomposition

$$V = V_u[0] \oplus V_u[3/2] \oplus V_u[1/10] \oplus V_u[3/5].$$

Define a linear automorphism $\sigma_u \in \text{End}(V)$ by

$$\sigma_u := \begin{cases} 
1 & \text{on } V_u[0] \oplus V_u[3/5], \\
−1 & \text{on } V_u[3/2] \oplus V_u[1/10].
\end{cases} \quad (2.3)$$

The fusion rules (2.2) (cf. Theorem 2.2) imply:

**Lemma 2.5.** The linear map $\sigma_u$ is an automorphism of $V$.

We also need the following result:
Lemma 2.6. Let $V$ be a VOA with grading $V = \bigoplus_{n \geq 0} V_n$, $V_0 = \mathbb{C}1$ and $V_1 = 0$, and $u \in V$ a Virasoro vector such that $\text{Vir}(u) \simeq L(c_m,0)$. Then the zero mode $o(u) = u(1)$ acts on the Griess algebra of $V$ semisimply with possible eigenvalues $2$ and $h_{r,s}^{(m)}$, $1 \leq s \leq r \leq m + 1$. Moreover, if $h_{r,s}^{(m)} \neq 2$ for $1 \leq s \leq r \leq m + 1$ then the eigenspace for the eigenvalue $2$ is one-dimensional, namely, it is spanned by the Virasoro vector $u$.

Proof: Since $V$ is a module over a rational VOA $\text{Vir}(u) \simeq L(c_m,0)$, the zero mode $o(u)$ acts on $V$ semisimply. In the following we use the convention as in Theorem 2.2. Let $v \in V_2$ be an eigenvector with eigenvalue $\lambda$. By the linearity, we may assume that $v \in V_u[h_{r,s}^{(m)}]$ with $1 \leq s \leq r \leq m + 1$ and $\lambda = h_{r,s}^{(m)} + n$, $n \in \mathbb{N}$. Suppose $n > 0$. If $h_{r,s}^{(m)} \neq 0$, that is, $(r,s) \neq (1,1)$, then $\text{Vir}(u) \cdot v$ contains a non-zero vector of $u(1)$-weight $\lambda - 1$ which belongs to the weight one subspace of $V$, a contradiction. If $h_{r,s}^{(m)} = 0$, then $\text{Vir}(u)v \simeq L(c_m,0)$ as a $\text{Vir}(u)$-module. That forces $\lambda \leq 2$ and hence $\lambda = 0$ or $2$. If $n = 2$, then there exists $f \in \text{Hom}_{\text{Vir}(u)}(\text{Vir}(u), V)$ such that $v = f(u)$. In this case $u(3)v = u(3)f(u) = f(u(3)u) = (c_m/2) \cdot f(1)$ is a non-zero vector of the weight zero subspace of $V$ and hence $u(3)v$ is a multiple of the vacuum vector of $V$. Write $f(1) = k1$. Then

$$v = f(u) = f(u(-1)1) = u(-1)f(1) = u(-1) \cdot k1 = ku.$$ 

Therefore, the eigenspace for the eigenvalue $2$ is one-dimensional and spanned by $u$. ■

Among $L(c_m,0)$-modules, only $L(c_m,0)$ and $L(c_m,h_{m+1,1}^{(m)})$ are simple currents, and it is shown in [LLY] that $L(c_m,0) \oplus L(c_m,h_{m+1,1}^{(m)})$ forms a simple current extension of $L(c_m,0)$. Note that $h_{m+1,1}^{(m)} = m(m+1)/4$ is an integer if $m \equiv 0, 3 \pmod{4}$ and a half-integer if $m \equiv 1, 2 \pmod{4}$.

Theorem 2.7 ([LLY]). (1) The $\mathbb{Z}_2$-graded simple current extension

$$\mathcal{W}(c_m) := L(c_m,0) \oplus L(c_m,h_{m+1,1}^{(m)})$$

has a unique simple rational vertex operator algebra structure if $m \equiv 0, 3 \pmod{4}$, and a unique simple rational vertex operator superalgebra structure if $m \equiv 1, 2 \pmod{4}$, which extends $L(c_m,0)$.

(2) Let $M$ be an irreducible $L(c_m,0)$-module and $\tilde{M} = L(c_m,h_{m+1,1}^{(m)}) \times M$ be the fusion product. If $\tilde{M}$ is not isomorphic to $M$, then $M$ is uniquely extended to an irreducible either untwisted or $\mathbb{Z}_2$-twisted $\mathcal{W}(c_m)$-module which is given by $M \oplus \tilde{M}$ as an $L(c_m,0)$-module. If $\tilde{M}$ and $M$ are isomorphic $L(c_m,0)$-modules, then $M$ affords a structure of an irreducible either untwisted or $\mathbb{Z}_2$-twisted $\mathcal{W}(c_m)$-module on which there are two inequivalent structures. These structures are $\mathbb{Z}_2$-conjugates of each other and we denote them by $M^\pm$.  

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3 The 2A algebra for the Monster

In this section, we will review and list some properties of a certain VOA called the 2A-algebra for the Monster which is related to a dihedral subgroup of the Monster. As an application of this algebra, we will show that certain commutant algebras of the Virasoro VOA \( L(\frac{1}{2}, 0) \) have a subgroup of automorphisms satisfying the \( \{3, 4\} \)-transposition property. This result will be used in the last section to study the Moonshine VOA and its subalgebra related to the Baby Monster.

By Theorem 2.7, \( W(\frac{1}{2}) = L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \) and \( W(\frac{7}{10}) = L(\frac{7}{10}, 0) \oplus L(\frac{7}{10}, \frac{3}{2}) \) form simple vertex operator superalgebras. As the even part of a tensor product of these SVOAs

\[
U_{2A} := L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}).
\]

forms a simple VOA. We call \( U_{2A} \) the 2A-algebra for the Monster.

The 2A-algebra can be also constructed along with the recipe described in Section 4 via the embedding \( A_1 \oplus E_7 \rightarrow E_8 \).

The structure as well as the representation theory of \( U_{2A} \) is well-studied in [Mi1, LY1]. We first review their results.

Commutant subalgebras. Let \( V \) be a VOA and \((U, e)\) a sub-VOA. Then the commutant subalgebra of \( U \) is defined by

\[
\text{Com}_V(U) := \{ a \in V | a_{(n)}U = 0 \text{ for all } n \geq 0 \}.
\]

(3.1)

It is known (cf. [FZ, Theorem 5.2]) that

\[
\text{Com}_V(U) = \ker_V e(0)
\]

(3.2)

and in particular \( \text{Com}_V(U) = \text{Com}_V(\text{Vir}(e)) \). Namely, the commutant subalgebra of \( U \) is determined only by the conformal vector \( e \) of \( U \). If \( \omega_{(2)}e = 0 \), it is also shown in Theorem 5.1 of [FZ] that \( \omega - e \) is also a Virasoro vector. In that case we have two mutually commuting subalgebras \( \text{Com}_V(\text{Vir}(e)) = \ker_V e(0) \) and \( \text{Com}_V(\text{Vir}(\omega - e)) = \ker_V (\omega - e)(0) \) and the tensor product \( \text{Com}_V(\text{Vir}(\omega - e)) \otimes \text{Com}_V(\text{Vir}(e)) \) forms an extension of \( \text{Vir}(e) \otimes \text{Vir}(\omega - e) \). If \( V_1 = 0 \), we always have \( \omega_{(2)}e = 0 \). More generally, we say a sum \( \omega = e^1 + \cdots + e^n \) is a Virasoro frame if all \( e^i \) are Virasoro vectors and \( [Y(e^i, z_1), Y(e^j, z_2)] = 0 \) for \( i \neq j \).

3.1 Griess algebra and Representation theory

Let \( \omega^1 \) and \( \omega^2 \) be the Virasoro vectors of \( L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \subset U_{2A} \) with central charges \( \frac{1}{2} \) and \( \frac{7}{10} \), respectively, and let \( x \in L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}) \subset U_{2A} \) be a highest weight
vector. It is well-known that the SVOA $W(1/2) = L(1/2, 0) \oplus L(1/2, 1/2)$ has a realization by a single free fermion [FFR, FRW] and the extension $W(7/10) = L(7/10, 0) \oplus L(7/10, 3/2)$ is isomorphic to the Neveu-Schwarz SVOA of central charge $7/10$ [GKO]. Hence, there exists a presentation

$$
\omega^1 = \frac{1}{2} \psi_{-3/2} \psi_{-1/2}, \quad \omega^2 = \frac{1}{2} G(-1/2)G(-3/2), \quad x = \psi_{-1/2} \otimes G(-3/2)
$$

with $Y(x, z) = \psi(z) \otimes G(z), \psi(z) = \sum_{n \in \mathbb{Z}} \psi_{n+1/2} z^{-n-1}, G(z) = \sum_{r \in \mathbb{Z}_{1/2}} G(r) z^{-r-3/2}$ such that

$$
[\psi_r, \psi_s]_+ = \delta_{r+s,0}, \quad [\omega_{(m+1)}, G(r)] = \left( \frac{1}{2} m - r \right) G(m + r),
$$

$$
[G(r), G(s)]_+ = 2 \omega_{(r+s+1)} + \delta_{r+s,0} \frac{7}{30} \left( r^2 - \frac{1}{4} \right).
$$

By the presentation above, we can calculate the structure of the Griess algebra of $U_{2A}$ as follows (cf. [LYY2]):

$$
\begin{array}{c|ccc|c|ccc|c|}
\hline
a_{(1)}b & \omega^1 & \omega^2 & x & \langle a, b \rangle & \omega^1 & \omega^2 & x \\
\hline
\omega^1 & 2 \omega^1 & 0 & \frac{1}{2} x & \omega^1 & \frac{1}{4} & 0 & 0 \\
\omega^2 & 2 \omega^2 & \frac{3}{2} x & \omega^2 & \frac{7}{20} & 0 & \\
x & \frac{14}{15} \omega^1 + 2 \omega^2 & x & \frac{7}{15} & \\
\end{array}
$$

(3.4)

Note that $\{\omega^1, \omega^2, x\}$ forms an orthogonal basis of the Griess algebra. By a direct computation, we can verify the following.

**Lemma 3.1** ([LYY2]). There exist exactly three Ising vectors in $U_{2A}$, given by

$$
\omega^1, \quad e^+ := \frac{1}{8} \omega^1 + \frac{5}{8} \omega^2 + \frac{\sqrt{15}}{8} x, \quad e^- := \frac{1}{8} \omega^1 + \frac{5}{8} \omega^2 - \frac{\sqrt{15}}{8} x.
$$

These Ising vectors are mutually conjugated by the associated $\sigma$-involutions, namely,

$$
\sigma_{\omega^1} e^\pm = e^\mp, \quad \sigma_{e^\pm} \omega^1 = e^\mp, \quad \sigma_{e^\pm} e^\mp = \omega^1.
$$

In particular, $\text{Aut}(U_{2A})$ is isomorphic to $S_3$ which is generated by $\sigma$-type involutions.

By the lemma above, we see that there exist three Virasoro frames inside $U_{2A}$:

$$
\omega = \omega^1 + \omega^2 = e^+ + f^+ = e^- + f^-, \quad f^\pm := \omega - e^\pm,
$$

and these frames are mutually conjugate under the $\sigma$-involutions.

Since $U_{2A}$ is a $\mathbb{Z}_2$-graded simple current extension of a rational VOA $L(1/2, 0) \otimes L(7/10, 0)$, $U_{2A}$ is also rational. The classification of irreducible $U_{2A}$-modules is completed in [LY1].
Theorem 3.2 ([LY2]). The VOA $U_{2A}$ is rational and there are eight isomorphism types of irreducible modules over $U_{2A}$. Their shapes as $\text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \simeq L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0)$-modules are as follows:

$$U(0, 0) = [0, 0] \oplus [\frac{1}{2}, \frac{3}{2}], \quad U(\frac{1}{2}, 0) = [\frac{1}{2}, 0] \oplus [0, \frac{3}{2}], \quad U(0, \frac{1}{10}) = [0, \frac{1}{10}] \oplus [\frac{1}{2}, \frac{3}{5}],$$

$$U(0, \frac{3}{5}) = [0, \frac{3}{5}] \oplus [\frac{1}{2}, \frac{1}{10}], \quad U(\frac{1}{16}, \frac{7}{16})^\pm = [\frac{1}{16}, \frac{7}{16}]^\pm, \quad U(\frac{1}{16}, \frac{3}{80})^\pm = [\frac{1}{16}, \frac{3}{80}]^\pm,$$

where $[h_1, h_2]$ denotes $L(\frac{1}{2}, h_1) \otimes L(\frac{7}{10}, h_2)$ and $M^-$ is the contragradient (or dual) module of $M^+$.

By the list of irreducible modules, we see that $U_{2A}$ is a maximal extension of $L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0)$ as a simple VOA.

We consider the conjugacy of irreducible $U_{2A}$-modules under the action of $\text{Aut}(U_{2A})$. By comparing top weights, we see that $g \circ U(h_1, h_2) \simeq U(h_1, h_2)$ for any $g \in \text{Aut}(U_{2A})$ if $(h_1, h_2) = (0, 0)$ or $(0, \frac{3}{5})$. The remaining modules can be divided into two groups having the same top weights, $1/2$ and $1/10$, each consists of three irreducibles. To determine the isomorphism types of the conjugates of the modules of the form $U(\frac{1}{16}, h)^\pm$ with $h = \frac{7}{16}$ and $\frac{3}{80}$, we first fix the labeling signs.

By Theorem 2.7, there exist two inequivalent $\mathbb{Z}_2$-twisted $\mathcal{W}(\frac{1}{2}) = L(\frac{1}{2}, 0) \oplus L(\frac{1}{2}, \frac{1}{2})$-module structures on $L(\frac{1}{2}, \frac{1}{16})$, which we denote by $L(\frac{1}{2}, \frac{1}{16})^\pm$. On $L(\frac{1}{2}, \frac{1}{16})^\pm$, the vertex operator associated to a highest weight vector $\psi_{-\frac{1}{2}}1$ of $L(\frac{1}{2}, \frac{1}{2})$ acts as a free fermionic field $\phi(z) = \sum_{n \in \mathbb{Z}} \phi_n z^{-n-\frac{1}{2}}$ such that $[\phi_m, \phi_n]_+ = \delta_{m+n, 0}$. The operator $\phi_0$ acts by $\pm \frac{1}{\sqrt{2}}$ on the top levels of $L(\frac{1}{2}, \frac{1}{16})^\pm$. Similarly, the $L(\frac{7}{10}, 0)$-modules $L(\frac{7}{10}, h)$, $h = \frac{7}{16}, \frac{3}{80}$, can be extended to irreducible $\mathbb{Z}_2$-twisted modules $L(\frac{7}{10}, h)^\pm$ over the Neveu-Schwarz SVOA $\mathcal{W}(\frac{7}{10}) = L(\frac{7}{10}, 0) \oplus L(\frac{7}{10}, \frac{3}{2})$, which form the Ramond sectors. Namely, the vertex operator associated to the highest weight vector $G(-\frac{3}{2})1 \in L(\frac{7}{10}, \frac{3}{2})$ has an expression $Y(G(-\frac{3}{2})1, z) = \sum_{r \in \mathbb{Z}} G(r) z^{-r-\frac{3}{2}}$ such that $[G(r), G(s)]_+$ is given as in (3.3). The operator $G(0)$ acts by $\pm \sqrt{h - \frac{1}{24} \cdot \frac{7}{10}}$ on the top levels of $L(\frac{7}{10}, h)^\pm$, $h = \frac{7}{16}, \frac{3}{80}$.

Let us consider the zero mode action $o(x) := x(1)$ of the highest weight vector $x = \psi_{-\frac{1}{2}}1 \otimes G(-\frac{3}{2})1 \in U_{2A}$ on the top level of $L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{1}{10}, h)$, $h = \frac{7}{16}, \frac{3}{80}$. Since $o(x) = \phi_0 \otimes G(0)$ by the representation, it acts by

$$\pm \frac{1}{\sqrt{2}} \cdot \sqrt{h - \frac{1}{24} \cdot \frac{7}{10}}$$

on the top level. We fix the signs such that $o(x)$ acts by $\pm 7/4\sqrt{15}$ on the top levels of $U(\frac{1}{16}, \frac{7}{16})^\pm$. Then we can define the signs of $U(\frac{1}{16}, \frac{3}{80})^\pm$ by the fusion rule

$$U(\frac{1}{16}, \frac{7}{16})^\pm \times U(0, \frac{3}{5}) = U(\frac{1}{16}, \frac{3}{80})^\pm$$

(3.5)
(cf. [LY1, LLY]).

We consider the isomorphism classes of conjugates of irreducible modules with top weight 1/2, namely, \( g \circ U(1/2, 0) \) and \( g \circ U(1/16, 7/16)^\pm \) for \( g \in \{ \sigma_{\omega^1}, \sigma_{e^+}, \sigma_{e^-} \} \). By definition, 
\[ \sigma_{\omega^1} \circ U(1/16, 7/16)^\pm = U(1/16, 7/16)^\mp \] 
since \( \sigma_{\omega^1} \) coincides with the canonical \( \mathbb{Z}_2 \)-symmetry of \( U_{2\mathbb{A}} = [0, 0] \oplus [1/2, 3/2] \). Now consider \( \sigma_{e^\pm} \circ U(1/16, 7/16)^\varepsilon \), \( \varepsilon = \pm \). The zero mode action \( o(x) \) on the top level of \( \sigma_{e^\pm} \circ U(1/16, 7/16)^\varepsilon \) is equivalent to the zero mode of \( \sigma_{e^\pm} x \) on the top level of \( U(1/16, 7/16)^\varepsilon \). We compute \( \sigma_{e^\pm} x \). Since \( \sigma_{e^\pm} \omega^1 = e^\mp \), one has

\[
\omega^1 = \sigma_{e^\pm} e^\mp = \frac{1}{8} \sigma_{e^\pm} \omega^1 + \frac{5}{8} \sigma_{e^\pm} \omega^2 \mp \frac{\sqrt{15}}{8} \sigma_{e^\pm} x = \frac{1}{8} e^\mp + \frac{5}{8} f^\mp \mp \frac{\sqrt{15}}{8} \sigma_{e^\pm} x.
\]

Solving this, we obtain

\[
\sigma_{e^\pm} x = \pm \frac{1}{2\sqrt{15}} (-7 \omega^1 + 5 \omega^2) + \frac{1}{2} x.
\]

Then the zero mode of \( \sigma_{e^\pm} x \) on the top level of \( U(1/16, 7/16)^\varepsilon \) acts by the scalar

\[
\pm \frac{1}{2\sqrt{15}} \left( -7 \cdot \frac{1}{16} + 5 \cdot \frac{7}{16} \right) + \varepsilon \cdot \frac{1}{2} \cdot \frac{7}{4\sqrt{15}}.
\]

In the above, we identify the signs \( \varepsilon = \pm \) with \( \pm 1 \). From this we can determine the isomorphism classes of \( \sigma_{e^\pm} \circ U(1/16, 7/16)^\varepsilon \) and \( \sigma_{e^\pm} \circ U(1/2, 0) \), and the result is summarized in the following table.

\[
\begin{array}{c|ccc}
M & U(1/2, 0) & U(1/16, 7/16)^+ & U(1/16, 7/16)^- \\
\hline
\sigma_{\omega^1} \circ M & U(1/2, 0) & U(1/16, 7/16)^- & U(1/16, 7/16)^+ \\
\sigma_{e^+} \circ M & U(1/16, 7/16)^- & U(1/16, 7/16)^+ & U(1/2, 0) \\
\sigma_{e^-} \circ M & U(1/16, 7/16)^+ & U(1/2, 0) & U(1/16, 7/16)^- \\
\end{array}
\]

Since a conjugation by an automorphism keeps the fusion rules invariant, by the fusion rules (3.5) and \( U(1/2, 0) \times U(0, 3/5) = U(0, 1/10) \) (cf. [LY1, LLY]) we also have the following table.

\[
\begin{array}{c|ccc}
M & U(0, 1/10) & U(1/16, 3/80)^+ & U(1/16, 3/80)^- \\
\hline
\sigma_{\omega^1} \circ M & U(0, 1/10) & U(1/16, 3/80)^- & U(1/16, 3/80)^+ \\
\sigma_{e^+} \circ M & U(1/16, 3/80)^- & U(1/16, 3/80)^+ & U(0, 1/10) \\
\sigma_{e^-} \circ M & U(1/16, 3/80)^+ & U(0, 1/10) & U(1/16, 3/80)^- \\
\end{array}
\]

(3.6)
Remark 3.3. By the table above, we can explicitly compute that the zero mode $o(x)$ acts by $\frac{1}{4}\sqrt{15}$ on the top levels of $U(1/16, \frac{3}{80})^\pm$.

For the $\tau$-involutions associated to $U_{2A}$, one has the following result:

**Theorem 3.4.** Let $V$ be a VOA containing $U_{2A}$ as a sub-VOA. Then as automorphisms on $V$, the $\tau$-involutions associated to Ising vectors of the subalgebra $U_{2A}$ satisfy the relations of a Kleinian 4-group:

$$
\tau_{\omega^1} \tau_{e^\pm} = \tau_{e^\pm} \tau_{\omega^1} = \tau_{e^\mp} \quad \text{and} \quad \tau_{e^+} \tau_{e^-} = \tau_{e^-} \tau_{e^+} = \tau_{\omega^1}.
$$

**Proof:** Since $U_{2A}$ is rational, $V$ is a direct sum of irreducible $U_{2A}$-submodules given in Theorem 3.2. Let $M$ be an irreducible $U_{2A}$-submodule of $V$. It is clear that each $\tau$-involution keeps an irreducible $U_{2A}$-submodule invariant, and the action of $\tau_{\omega^1}$ on $M$ is manifest by its $\text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2)$-module structure. Since $\tau_{e^\pm} = \tau_{\sigma_{e^\pm} \omega^1}$, the action of $\tau_{e^\pm}$ on $M$ is equivalent to that of $\tau_{\omega^1}$ on $\sigma_{e^\pm} o M$. Thus the action of $\tau_{e^\pm}$ on $M$ is determined by the conjugacy relations given in (3.6) and (3.7) and the following table summarizes the result.

| $M$ | $U(0, 0)$ | $U(1/2, 0)$ | $U(1/16, \frac{7}{16})^\pm$ | $U(0, \frac{3}{5})$ | $U(0, \frac{1}{10})$ | $U(1/16, \frac{3}{80})^\pm$ |
|-----|-----------|-------------|-----------------------------|-------------------|---------------------|-----------------------------|
| $\tau_{\omega^1}$ | $+1$ | $+1$ | $-1$ | $+1$ | $+1$ | $-1$ |
| $\tau_{e^+}$ | $+1$ | $-1$ | $\pm1$ | $+1$ | $-1$ | $\pm1$ |
| $\tau_{e^-}$ | $+1$ | $-1$ | $\mp1$ | $+1$ | $-1$ | $\mp1$ |

From the table above one can see the relations as claimed.

**3.2 \{3, 4\}-transposition property of $\sigma$-involutions**

We consider $\sigma$-involutions induced by the 2A-algebra. We refer the reader to Section 2 for the definition of simple $c = \frac{7}{10}$ Virasoro vectors of $\sigma$-type and their corresponding $\sigma$-involutions. See Definition 2.4 and Eq. (2.3) for the details.

Let $V$ be a VOA and let $e \in V$ be an Ising vector.

**Definition 3.5.** A simple $c = \frac{7}{10}$ Virasoro vector $f \in \text{Com}_V(Vir(e))$ is called a **derived Virasoro vector with respect to $e$** if there exists a sub-VOA $U$ of $V$ containing $e$ and $f$ such that $U$ is isomorphic to the 2A-algebra $U_{2A}$ and $e + f$ is the conformal vector of $U$.

**Lemma 3.6.** A derived $c = \frac{7}{10}$ Virasoro vector $f \in \text{Com}_V(Vir(e))$ with respect to $e$ is of $\sigma$-type on the commutant subalgebra $\text{Com}_V(Vir(e))$.  

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Thus \( f \) is of \( \sigma \)-type on \( \text{Com}_V(\text{Vir}(e)) \).

We consider the one-point stabilizer
\[
\text{Stab}_{\text{Aut}(V)}(e) := \{ g \in \text{Aut}(V) \mid ge = e \}. \quad (3.10)
\]
It is clear that \( \text{Stab}_{\text{Aut}(V)}(e) \) forms a subgroup of \( \text{Aut}(V) \). Each \( g \in \text{Stab}_{\text{Aut}(V)}(e) \) keeps the isotypical component \( V_e[h], h \in \{0, \frac{1}{2}, \frac{1}{16}\} \), invariant so that by restriction we obtain a group homomorphism
\[
\varphi_e : \text{Stab}_{\text{Aut}(V)}(e) \longrightarrow \text{Aut}(\text{Com}_V(\text{Vir}(e))),
\]
\[
g \quad \mapsto \quad g|_{\text{Com}_V(\text{Vir}(e))}. \quad (3.11)
\]

Let \( f \in \text{Com}_V(\text{Vir}(e)) \) be a derived Virasoro vector with respect \( e \). By Lemma 3.6 and Lemma 2.5 above, we obtain an involution \( \sigma_f \in \text{Aut}(\text{Com}_V(\text{Vir}(e))) \). Inside \( U \), we have the three Ising vectors \( e, e^+ \) and \( e^- = \sigma_e e^+ \).

**Lemma 3.7.** \( \varphi_e(\tau_{e^+}) = \varphi_e(\tau_{e^-}) = \sigma_f \) in \( \text{Aut}(\text{Com}_V(\text{Vir}(e))) \).

**Proof:** By Theorem 3.4, we see that \( \tau_{e^+} \) and \( \tau_{e^-} \) are in \( \text{Stab}_{\text{Aut}(V)}(e) \). More precisely, we see from (3.8) that both \( \tau_{e^+} \) and \( \tau_{e^-} \) act by \( \pm 1 \) on each isotypical component \( M \otimes H_M \) for \( M \in \text{Irr}(U_{2A}) \). In particular, by (3.8) and (3.9), we see that \( \varphi_e(\tau_{e^+}) \), \( \varphi_e(\tau_{e^-}) \) and \( \sigma_f \) define the same automorphism on the commutant subalgebra \( \text{Com}_V(\text{Vir}(e)) \).

We say a VOA \( W \) over \( \mathbb{R} \) is **compact** if \( W \) has a positive definite invariant bilinear form. We recall the following interesting theorem of Sakuma.
Proof: Suppose otherwise and $2$ is an Ising vector of $\sigma_i$, that is, $\sigma_i$ satisfies the so-called $\{3, 4\}$-transposition property. Based on Sakuma’s theorem, we will prove that the set of involutions $\{\sigma_a \in \text{Aut}(\text{Com}_V(\text{Vir}(e))) \mid u \in I\}$ satisfies the so-called $\{3, 4\}$-transposition property.

Let $I$ be the set of all derived $c = 7/10$ Virasoro vectors of $\text{Com}_V(\text{Vir}(e))$ with respect to $e$. We have seen in Lemma 3.6 that each vector of $I$ is of $\sigma$-type on $\text{Com}_V(\text{Vir}(e))$. Based on Sakuma’s theorem, we will prove that the set of involutions $\{\sigma_a \in \text{Aut}(\text{Com}_V(\text{Vir}(e))) \mid u \in I\}$ satisfies the so-called $\{3, 4\}$-transposition property.

First we will prove a result about regular tetrahedrons of Ising vectors of $\sigma$-type.

**Lemma 3.9.** Suppose that $V$ is a VOA with $V_1 = 0$ and $V$ has a compact real form $V_\mathbb{R}$, that is, $V$ as an $\mathbb{R}$-algebra has a compact $\mathbb{R}$-subalgebra $V_\mathbb{R}$ such that $V = \mathbb{C} \otimes \mathbb{R} V_\mathbb{R}$. Let $a^1$, $a^2$, $a^3$ and $b$ be Ising vectors of $V_\mathbb{R}$ of $\sigma$-type on $V$ such that $a^1$, $a^2$, $a^3$ are the three Ising vectors of a $2A$-algebra. Then it is impossible that $\langle a^i, b \rangle = 1/32$ for all $1 \leq i \leq 3$.

**Proof:** Suppose otherwise and $\langle a^i, b \rangle = 1/32$ for all $1 \leq i \leq 3$. By Lemma 3.1, the $\sigma$-involution $\sigma_{a^1}$, $\sigma_{a^2}$ and $\sigma_{a^3}$ generate a group $H \simeq S_3$ in $\text{Aut}(V_\mathbb{R})$. Using Theorem 3.8, we know that for each $i = 1, 2, 3$, the vectors $a^i$ and $b$ also generate a $2A$-algebra and therefore $\sigma_{a^i}$ and $\sigma_b$ generate a subgroup $H_i \simeq S_3$ in $\text{Aut}(V_\mathbb{R})$. In particular, $[\sigma_b, \sigma_{a^i}] \neq 1$ for $i = 1, 2, 3$. Note that $H \cap H_i \neq 1$.

Let $K$ be the subgroup of $\text{Aut}(V)$ generated by $\{\sigma_{a^1}, \sigma_{a^2}, \sigma_{a^3}, \sigma_b\}$. It is shown in [Ma2] that the group $K$ is a 3-transposition group of symplectic type, which means, the group generated by two (non-commuting) subgroups $H$ and $H_i$, both isomorphic to $S_3$, is isomorphic to either $S_3$ if $H = H_i$ or $S_4$ if $H \neq H_i$.

Assume first that $K = \langle H, H_i \rangle$ is isomorphic to $S_4$. As elements of subgroups isomorphic to $S_3$ the $\sigma$-involutions $\sigma_{a^1}$, $\sigma_{a^2}$, $\sigma_{a^3}$ and $\sigma_b$ would be transpositions. However, given four transpositions in $S_4$, there are at least two which commute. This is a contradiction. Therefore $K$ must be $S_3$, which implies that $\sigma_b$ coincides with one of the $\sigma_{a^i}$, $i = 1, 2, 3$. This contradicts that $\sigma_b$ and $\sigma_{a^i}$ generate a $S_3$ for $i = 1, 2, 3$. Therefore the lemma follows.
Remark 3.10. It is known [ATLAS] that there is no 2A-pure elementary abelian subgroup of order 8 in the Monster. We have essentially shown this fact by the theory of vertex operator algebras in Lemma 3.9.

Proposition 3.11. Suppose that $V$ is a VOA with $V_1 = 0$ and $V$ has a compact real form $V_{\mathbb{R}}$ such that all the Ising vectors of $V$ are in $V_{\mathbb{R}}$. Then for any $u, v \in I$, we have $|\sigma_u\sigma_v| \leq 4$ on $\text{Com}_V(\text{Vir}(e))$.

Proof: First, we note that all derived $c = 7/10$ Virasoro vectors in $I$ are in the real form $V_{\mathbb{R}}$, as it is a $\mathbb{R}$-linear combination of real Ising vectors by Lemma 3.1. Take $u, v \in I$ and consider involutions $\sigma_u$ and $\sigma_v$ defined on $\text{Com}_V(\text{Vir}(e))$. By definition of $\sigma$-involution, both $\sigma_u$ and $\sigma_v$ preserve the real form $V_{\mathbb{R}}$ and therefore the order of $\sigma_u\sigma_v$ on $V$ is the same as that on $V_{\mathbb{R}}$. By definition of derived vectors, there exist subalgebras $U^1, U^2$ of $V$ isomorphic to the 2A-subalgebras such that $e + u$ and $e + v$ are Virasoro frames of $U^1$ and $U^2$, respectively. By Lemma 3.7 there exist Ising vectors $e^1 \in U^1$ and $e^2 \in U^2$ such that $\varphi_e(\tau_{e^1}) = \sigma_u$ and $\varphi_e(\tau_{e^2}) = \sigma_v$. Since $\varphi_e$ is a group homomorphism, the order $|\sigma_u\sigma_v| = |\varphi_e(\tau_{e^1}\tau_{e^2})|$ divides $|\tau_{e^1}\tau_{e^2}|$. By Theorem 3.8, we know that $|\tau_{e^1}\tau_{e^2}| \leq 6$ and we have nothing to prove if $|\tau_{e^1}\tau_{e^2}| \leq 4$. So we assume $|\tau_{e^1}\tau_{e^2}| = 5$ or 6.

Case $|\tau_{e^1}\tau_{e^2}| = 5$: Suppose $|\tau_{e^1}\tau_{e^2}| = 5$. Since $e^1$ is an Ising vector of $U^1$ which is isomorphic to the 2A-algebra, we can take by Theorem 3.4 another Ising vector $e^{1'} \in U^1$ such that $\tau_{e^{1'}} = \tau_{e}\tau_{e^1}$. Since $e$ and $e^2$ are in another 2A-subalgebra $U^2$, we see that $\tau_e$ and $\tau_{e^2}$ commute, from which we obtain $|\tau_{e^{1'}}\tau_{e^2}| = |\tau_{e}\tau_{e^1}\tau_{e^2}| = 10$. This contradicts Sakuma’s theorem. Therefore, $|\tau_{e^1}\tau_{e^2}| \neq 5$ and this case is impossible.

Case $|\tau_{e^1}\tau_{e^2}| = 6$: In this case the Griess algebra structure of the subalgebra generated by $e^1$ and $e^2$ is unique by Sakuma’s theorem and is isomorphic to the 6A-algebra discussed in Appendix A. By the uniqueness, we can identify $e^1$ and $e^2$ with those in Appendix A. Below we will freely use the results there. Set $w := \tau_{e^2}\tau_{e^1}e^2$. Then $w$ is an Ising vector which is denoted by $e^4$ in $U_{6A}$ by (A.2). Since $\langle e^1, w \rangle = 1/32$ by (A.1), the sub-VOA generated by $e^1$ and $w$ is isomorphic to the 2A-algebra $U_{2A}$. The third Ising vector $x$ in this 2A-algebra is given by the equation

$$x := e^1 + w - 4w_{(1)}e^1 = \sigma_{e^1}w$$ (3.12)

and is corresponding to $\omega^1$ in $U_{6A}$ by (A.3). Moreover, since $\langle e, e^1 \rangle = \langle e, e^2 \rangle = 1/32$ and $e \in V(\tau_{e^1}, \tau_{e^2})$, we have

$$\langle e, w \rangle = \langle e, \tau_{e^2}\tau_{e^1}e^2 \rangle = \langle \tau_{e^1}\tau_{e^2}e, e^2 \rangle = \langle e, e^2 \rangle = \frac{1}{32}.$$

Therefore, $e$ and $w$ also generate a 2A-algebra.
For the \( \tau \)-involutions we obtain:

\[
\tau_w = \tau_{e^2} \tau_1 e^2 = \tau_{e^2} \tau_1 \tau e^2 \tau_{e^1} \tau e^2
\]  

(3.13)

as \( \tau_{ge^2} = g \tau_{e^2} g^{-1} \) for \( g \in \text{Aut}(V) \). It follows from \( (\tau_{e^1} \tau e^2)^6 = 1 \) that \( \tau_w \) commutes with \( \tau_{e^1} \). It follows from (3.13) and (A.4) that

\[
\tau_x = \tau_{e^1} \tau_w = (\tau_{e^1} \tau e^2)^3 \quad \text{on} \quad V.
\]  

(3.14)

Thus \( \tau_e \) commutes with both \( \tau_w \) and \( \tau_x \) in \( \text{Aut}(V) \) since \( \tau_e \) commutes with both \( \tau_{e^1} \) and \( \tau_{e^2} \).

Now either \( |\tau_e \tau_{e^1} \tau e^2| = 3 \) or 6 occurs.

**Claim:** \( |\tau_e \tau_{e^1} \tau e^2| = 3 \).

Suppose otherwise and \( |\tau_e \tau_{e^1} \tau e^2| = 6 \). For the third Ising vector \( e'' \) in the 2A-algebra \( U \) generated by \( e \) and \( e^1 \) one has \( \tau_{e''} = \tau_e \tau_{e^1} \) on \( V \). Then \( \tau_{e^1} \tau e^2 = \tau_{e^2} \tau_{e^1} \) is of order 6 and hence \( e'' \) and \( e^2 \) generate a subalgebra isomorphic to the 6A-algebra. Again we have Ising vectors \( w' := \tau_{e^2} \tau_{e^1} e^2 = \tau_{e^2} \tau_{e^1} e^2 = \tau_{e^2} \tau_{e^1} e^2 = w \) and \( x' = \sigma_{e^1} w \) corresponding to \( e' \) and \( \omega^1 \) in \( U_{6A} \), respectively, such that \( e'' \), \( w' = w \) and \( x' \) generate a 2A-algebra.

Note that if we set \( G = \langle \tau_{e}, \tau_{e^1}, \tau e^2 \rangle \) then \( e, e^1, w \) and \( x \) are Ising vectors of \( V^G \) which are of \( \sigma \)-type on it. As \( \langle w', e'' \rangle = 1/32 \) and \( e \) and \( w \) generate a 2A-algebra, the following holds in \( V^G \):

\[
\frac{1}{32} = \langle w', e'' \rangle = \langle w, e'' \rangle = \langle w, \sigma_{e} e^1 \rangle = \langle \sigma_{e} w, e^1 \rangle = \langle \sigma_{w} e, e^1 \rangle = \langle e, \sigma_{w} e^1 \rangle = \langle e, x \rangle.
\]

(3.15)

Therefore, if \( |\tau_e \tau_{e^1} \tau e^2| = 6 \), then we obtain four \( \sigma \)-type Ising vectors \( e, e^1, w \) and \( x \) in \( V^G \) such that \( e', w \) and \( x \) generate a 2A-algebra and

\[
\langle e, e^1 \rangle = \langle e, w \rangle = \langle e, x \rangle = \frac{1}{32}.
\]

(3.16)

By Lemma 3.9 above, such a configuration is impossible. Thus, \( \tau_e \tau_{e^1} \tau e^2 \) cannot have order 6 and the claim follows.

Since \( \tau_e \in \ker \varphi_e \), it follows from the claim above that \( \sigma_{u} \sigma_{v} = \varphi_e(\tau_{e^1} \tau e^2) = \varphi_e(\tau_{e^1} \tau e^2) \) is of order 3 if \( |\tau_{e^1} \tau e^2| = 6 \). This completes the proof of Proposition 3.11.

\section{4 Commutant subalgebras associated to the root lattice \( E_7 \)}

In this section, we will construct sub-VOAs of the lattice VOA \( V_{\sqrt{17}}E_7 \) which will correspond to dihedral subgroups of the Baby Monster. We will use the standard notation for lattice VOAs as in [FLM]. Our construction is similar to the construction in [LYY1] in the case
of the root lattice $E_8$ and works in fact for any root lattice of type $A_n$, $D_n$ or $E_n$. We start by describing our construction in general and then specialize to the case of the root lattice of type $E_7$.

4.1 Definition of the subalgebras

The algebras $U(i)$. Let $R$ be a root lattice of type $A_n$ ($n \geq 1$), $D_n$ ($n \geq 4$) or $E_n$ ($n = 6, 7, 8$) and let $\alpha_1, \ldots, \alpha_n$ be a system of simple roots for $R$. We let $\alpha_0$ be the root such that $-\alpha_0 = \sum_{i=1}^{n} m_i \alpha_i$ is the highest root for the chosen simple roots. Note that all $m_i$ are positive integers. We also set $m_0 = 1$. For any $i = 0, \ldots, n$, we consider the sublattice $L_i$ of $R$ generated by the roots $\alpha_j$, $0 \leq j \leq n$, $j \neq i$. One observes that $L_i$ is also of rank $n$ and the quotient group $R/L_i$ is cyclic of order $m_i$ with generator $\alpha_i + L_i$.

Thus one has

$$R = L_i \sqcup (\alpha_i + L_i) \sqcup (2\alpha_i + L_i) \sqcup \cdots \sqcup ((m_i - 1)\alpha_i + L_i).$$

(4.1)

We denote by $R_1, \ldots, R_\ell$ the indecomposable components of the lattice $L_i$ which are again root lattices of type $A_n$, $D_n$ or $E_n$. Hence $L_i = R_1 \oplus \cdots \oplus R_\ell$ where the direct sum of lattices denotes the orthogonal sum. In fact, the Dynkin diagram of $L_i$ is obtained from the affine Dynkin diagram of $R$ by removing the node $\alpha_i$ and the adjacent edges. We recall here that the affine Dynkin diagram of $R$ is the graph with vertex set $\{\alpha_0, \ldots, \alpha_n\}$ and two nodes $\alpha_i$ and $\alpha_j$, $0 \leq i, j \leq n$, are joined by an edge if $\langle \alpha_i, \alpha_j \rangle = -1$.

The decomposition (4.1) of the lattice $R$ leads to the decomposition

$$V_{V^R} = \bigoplus_{r=0}^{m_i-1} V_{V^R(\alpha_i + L_i)}$$

of the lattice VOA $V_{V^R}$. We define a linear map $\rho_i : V_{V^R} \rightarrow V_{V^R}$ by

$$\rho_i(u) = \zeta_{m_i}^r u \quad \text{for} \quad u \in V_{V^R(\alpha_i + L_i)}, \quad \text{where} \quad \zeta_{m_i} = e^{2\pi\sqrt{-1}/m_i}.$$  

(4.2)

Then $\rho_i$ is an element of $\text{Aut}(V_{V^R})$ of order $m_i$ and the fixed point sub-VOA $V_{V^R(\rho_i)}$ is exactly $V_{V^R(L_i)}$.

For a root lattice $S$ we denote by $\Phi(S)$ its root system. Then by [DLMN] the conformal vector $\omega_R$ of $V_{V^R}$ is given by

$$\omega_R = \frac{1}{4h} \sum_{\alpha \in \Phi(R)} \alpha(-1)^2 \|,$$

where $h$ is the Coxeter number of $R$. Now define

$$\tilde{\omega}_R := \frac{2}{h + 2} \omega_R + \frac{1}{h + 2} \sum_{\alpha \in \Phi(R)} e^{\sqrt{2}\alpha}.$$  

(4.3)
It is shown in [DLMN] that $\tilde{\omega}_R$ is a Virasoro vector of central charge $2n/(n+3)$ if $R$ is of type $A_n$, 1 if $R$ is of type $D_n$ and $6/7$, $7/10$ and $1/2$ if $R$ is of type $E_6$, $E_7$ and $E_8$, respectively. From the irreducible decomposition $L_i = R_1 \oplus \cdots \oplus R_\ell \subset R$ we have sublattices $R_s$ of $R$ and obtain a factorization

$$V_{\sqrt{2}L_i} = V_{\sqrt{2}R_1} \otimes \cdots \otimes V_{\sqrt{2}R_\ell} \subset V_{\sqrt{2}R}.$$  \hspace{1cm} (4.4)

Associated to the root subsystems $\Phi(R_s)$ of $\Phi(R)$, we also have simple Virasoro vectors

$$\omega^s = \tilde{\omega}_{R_s} = \frac{2}{h_s + 2} \omega_{R_s} + \frac{1}{h_s + 2} \sum_{\alpha \in \Phi(R_s)} e^{\sqrt{2} \alpha} \in V_{\sqrt{2}R_s} \subset V_{\sqrt{2}R}, \hspace{1cm} 1 \leq s \leq \ell,$$  \hspace{1cm} (4.5)

where $\omega_{R_s}$ is the conformal vector of $V_{\sqrt{2}R_s}$ and $h_s$ is the Coxeter number of $R_s$. It follows from the definition that the $\omega^s$ are mutually orthogonal simple Virasoro vectors in $V_{\sqrt{2}R}$.

Consider

$$X^r := \sum_{\beta \in \Delta_{\alpha_s + L_i}, \langle \beta, \beta \rangle = 2} e^{\sqrt{2} \beta}, \hspace{1cm} 1 \leq r \leq m_i - 1,$$

in the weight two subspace of $V_{\sqrt{2}R}$. It is shown in Proposition 2.2 of [LYY1] that the vectors $X^r$ are highest weight vectors for $\text{Vir}(\omega^1) \otimes \cdots \otimes \text{Vir}(\omega^\ell)$ with total weight 2.

Since all $\omega^s$, $1 \leq s \leq \ell$, are contained in the fixed point sub-VOA $V_{\sqrt{2}R}^+$, which has a trivial weight one subspace, $\omega_R - (\omega^1 + \cdots + \omega^\ell)$ is a Virasoro vector of $V_{\sqrt{2}R}$ as discussed at the beginning of chapter 3. We are interested in the commutant subalgebras defined by

$$U(i) := \text{Com}_{V_{\sqrt{2}R}}(V_{\sqrt{2}R}^+ (\omega_R - (\omega^1 + \cdots + \omega^\ell)))$$

$$= \ker_{V_{\sqrt{2}R}} (\omega_R - (\omega^1 + \cdots + \omega^\ell))_{(0)}$$  \hspace{1cm} (4.6)

in the case of $R = E_7$. (The case $R = E_8$ is considered in [LYY1, LYY2].) It is clear from the construction that $U(i)$ has a Virasoro frame $\omega^1 + \cdots + \omega^\ell$. We will consider an embedding of $U(i)$ into a larger VOA and then describe the commutant algebra $U(i)$ using the larger VOA.

It is clear that $U(i)$ forms an extension of the tensor product $\text{Vir}(\omega^1) \otimes \cdots \otimes \text{Vir}(\omega^\ell)$ and contains $X^r$, $1 \leq r \leq m_i - 1$. We will see in Section 5 that we can embed $U(i)$ into the Moonshine VOA and therefore $U(i)$ has a trivial weight one subspace. Consequently, the weight two subspace of $U(i)$ carries a structure of a commutative non-associative algebra called the Griess algebra of $U(i)$, even though $V_{\sqrt{2}R}$ has a non-trivial weight one subspace. Below we will explicitly describe the Griess algebra of $U(i)$. Namely, we will see that the Griess algebra of $U(i)$ is given by

$$G(i) := \text{span}_{\mathbb{C}} \{ \omega^s, X^r \mid 1 \leq s \leq \ell, \hspace{0.5cm} 1 \leq r \leq m_i - 1 \}$$
which is of dimension $\ell + m_i - 1$.

By definition, it is clear that $\tilde{\omega}_R$ and $\rho_i\tilde{\omega}_R$, where $\rho_i \in \text{Aut}(V_{\sqrt{2}R})$ is defined as in (4.2), are linear combinations of $\omega^s$ and $X^r$ and hence are contained in $G(i) \subset U(i)$. We will discuss the structure of the subalgebra generated by $\tilde{\omega}_R$ and $\rho_i\tilde{\omega}_R$.

**The algebras $V(i)$**. Let in the following $R = E_7$ and fix an embedding of $R$ into $E_8$. Let

$$Q(R) := \text{Ann}_{E_8}(R) = \{ \alpha \in E_8 \mid \langle \alpha, R \rangle = 0 \}. \quad (4.7)$$

Then $Q(E_7) \simeq A_1$ and $Q(R) \oplus R$ forms a full rank sublattice of $E_8$. Note that such an embedding is unique up to an automorphism of $E_8$.

Recall that $L_i$ is the sublattice of $R$ generated by roots $\alpha_j$, $j \neq i$. Then we have an embedding of $\tilde{L}_i := Q(R) \oplus L_i$ into $E_8$. Since $L_i$ is a full rank sublattice of $R$, $\tilde{L}_i$ is also a full rank sublattice of $E_8$. Thus $E_8/\tilde{L}_i$ is a finite abelian group whose order is $2m_i$. We fix the corresponding embedding $V_{\sqrt{2}\tilde{L}_i} \subset V_{\sqrt{2}E_8}$.

We have the decomposition $\tilde{L}_i = Q(R) \oplus R_1 \oplus \cdots \oplus R_{\ell}$ into a sum of irreducible root lattices, which gives rise to a factorization

$$V_{\sqrt{2}\tilde{L}_i} = V_{\sqrt{2}Q(R)} \otimes V_{\sqrt{2}L_i} = V_{\sqrt{2}Q(R)} \otimes V_{\sqrt{2}R_1} \otimes \cdots \otimes V_{\sqrt{2}R_{\ell}} \subset V_{\sqrt{2}E_8}.$$

Let $\omega_{E_8}$ be the conformal vector of $V_{\sqrt{2}E_8}$ and let $\tilde{\omega}_{Q(R)} \in V_{\sqrt{2}Q(R)}$ and $\omega^s \in V_{\sqrt{2}R_i}$ be the Virasoro vectors defined as in (4.3) and (4.5), respectively. Since $\omega_{E_8} - (\tilde{\omega}_{Q(R)} + \omega^1 + \cdots + \omega^f)$ is by the same argument as for $U(i)$ a Virasoro vector of $V_{\sqrt{2}E_8}$, we can define a commutant subalgebra

$$V(i) := \text{Com}_{V_{\sqrt{2}E_8}}(\text{Vir}(\omega_{E_8} - (\tilde{\omega}_{Q(R)} + \omega^1 + \cdots + \omega^f))). \quad (4.8)$$

**Remark 4.1.** We note that $\text{Com}_{V(i)}(\text{Vir}(\tilde{\omega}_{Q(R)}))$ coincides with $U(i)$ by the definition of commutant subalgebras. For, as we have orthogonal decompositions $\omega_{E_8} = \omega_{Q(R)} + \omega_R$, $\omega_{Q(R)} = (\omega_{Q(R)} - \tilde{\omega}_{Q(R)}) + \tilde{\omega}_{Q(R)}$ and $\omega_R = (\omega_R - (\omega^1 + \cdots + \omega^f)) + (\omega^1 + \cdots + \omega^f)$, we have $\text{Com}_{V(i)}(\text{Vir}(\tilde{\omega}_{Q(R)})) \subset \text{Com}_{V_{\sqrt{2}E_8}}(\text{Vir}(\omega_{Q(R)})) = V_{\sqrt{2}R}$. Since $U(i)$ is the maximal sub-VOA of $V_{\sqrt{2}R}$ having $\omega^1 + \cdots + \omega^f$ as its conformal vector (cf. discussion in Section 3), one has $\text{Com}_{V(i)}(\text{Vir}(\tilde{\omega}_{Q(R)})) \subset U(i)$. On the other hand, it is clear from the definition of $V(i)$ that $\text{Vir}(\tilde{\omega}_{Q(R)} \otimes U(i)) \subset V(i)$ and hence $U(i) \subset \text{Com}_{V(i)}(\text{Vir}(\tilde{\omega}_{Q(R)}))$.

We finally set

$$\mathcal{F}(i) := \{ g \in \text{Aut}(V_{\sqrt{2}E_8}) \mid g = \text{id} \text{ on } V_{\sqrt{2}\tilde{L}_i} \}. \quad (4.9)$$

Then $\mathcal{F}(i)$ is canonically isomorphic to the group of characters of $E_8/\tilde{L}_i$. The subalgebra $V(i)$ of $V_{\sqrt{2}E_8}$ is invariant under the action of $\mathcal{F}(i)$ since all $\tilde{\omega}_{Q(R)}$, $\omega^1$, $\ldots$, $\omega^f$ and the
conformal vector $\omega_{E_8}$ of $V_{\sqrt{2}E_8}$ are clearly fixed by $\mathcal{F}(i)$. Note that the special Ising vector
\begin{equation}
\hat{e} := \tilde{\omega}_{E_8} = \frac{1}{16} \omega_{E_8} + \frac{1}{32} \sum_{\alpha \in \Phi(E_8)} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}E_8}
\end{equation}
is contained in $V(i)$ (cf. [LYY1, LYY2]) and thus
\[ \{ g\hat{e} \mid g \in \mathcal{F}(i) \} \subset V(i). \]

Remark 4.2. Here is a brief explanation of the rôles of the algebras $V(i), U(i)$ and $G(i)$ in this article. Later we will consider the commutant subalgebra $V\mathcal{B}^{\sharp}$ of the Moonshine VOA $V^\sharp$. Then the sub-VOA $V(i)$ corresponds to the one generated by Ising vectors in $V^\sharp$ under a certain configuration, and the sub-VOA $U(i)$ of $V(i)$ describes the subspace of $V(i)$ contained in $V\mathcal{B}^{\sharp}$. The structure of the Griess algebras $G(i)$ would correspond to those in Table 3 of [C] and one can easily relate $G(i)$ with the $E_7$-structure as in the case of $E_8$ [LYY1].

Remark 4.3. Recall that the central charge of the simple Virasoro vector $\tilde{\omega}_R$ is $7/10$. We will show in Section 5 that there is indeed a nice correspondence between 2A-involutions of the Baby Monster and simple $c = 7/10$ Virasoro vectors of $\sigma$-type in the subalgebra $V\mathcal{B}^{\sharp}$ of $V^\sharp$.

### 4.2 Explicit description of the subalgebras

In this subsection we study the case where $R$ is a root lattice of type $E_7$ in detail.

The affine Dynkin diagram of a root lattice of type $E_7$ is the following graph:

\begin{center}
\begin{tikzpicture}
\node (a0) at (0,0) {$\alpha_0$};
\node (a1) at (1,0) {$\alpha_1$};
\node (a2) at (2,0) {$\alpha_2$};
\node (a3) at (3,0) {$\alpha_3$};
\node (a4) at (4,0) {$\alpha_4$};
\node (a5) at (5,0) {$\alpha_5$};
\node (a6) at (6,0) {$\alpha_6$};
\node (a7) at (7,0) {$\alpha_7$};
\draw (a0) -- (a1);
\draw (a1) -- (a2);
\draw (a2) -- (a3);
\draw (a3) -- (a4);
\draw (a4) -- (a5);
\draw (a5) -- (a6);
\end{tikzpicture}
\end{center}

We like to explain McKay’s correspondence [Mc] between the numerical labels $m_i$ of the affine $E_7$ Dynkin diagram and the Baby Monster conjugacy classes 1A, 2B, 2C, 3A and 4B into which the product of two 2A-involutions of the Baby Monster falls as given by the following figure:

\begin{center}
\begin{tikzpicture}
\node (1a) at (0,0) {1A};
\node (2b) at (1,0) {2B};
\node (3a) at (2,0) {3A};
\node (4b) at (3,0) {4B};
\node (3a) at (4,0) {3A};
\node (2b) at (5,0) {2B};
\node (1a) at (6,0) {1A};
\node (2c) at (7.5,0) {2C};
\draw (1a) -- (2b);
\draw (2b) -- (3a);
\draw (3a) -- (4b);
\draw (4b) -- (3a);
\draw (3a) -- (2b);
\draw (2b) -- (1a);
\draw (1a) -- (2c);
\end{tikzpicture}
\end{center}
Note that the correspondence is not one-to-one but only up to diagram automorphism.

To specialize to this situation, we change our notation slightly and denote $L_i$ by $L_{nX}$, $\rho_i$ by $\rho_{nX}$, $\tilde{L}_i$ by $\tilde{L}_{nX}$, $\mathcal{F}(i)$ by $\mathcal{F}_{nX}$, $V(i)$ by $V_{B(nX)}$, $U(i)$ by $U_{B(nX)}$ and $\mathcal{G}(i)$ by $\mathcal{G}_{B(nX)}$, respectively, where $nX \in \{1A, 2B, 2C, 3A, 4B\}$ is the label of the corresponding node in (4.12). Explicitly, we have

$$L_{1A} \simeq E_7, \quad L_{2B} \simeq A_1 \oplus D_6, \quad L_{2C} \simeq A_7, \quad L_{3A} \simeq A_2 \oplus A_5, \quad L_{4B} \simeq A_1 \oplus A_3 \oplus A_3. \quad (4.13)$$

**Remark 4.4.** Apparently, there are two distinct nodes (up to the diagram automorphism) that have the same numerical label 2. Therefore, there is some ambiguity for the assignment of the $2B$ and $2C$ labels. However, the sublattice structure $L_{4B} \subset L_{2B} \subset L_{1A}$ justify our labeling. Note that $L_{2C} \simeq A_7$ does not contain a sublattice isometric to $L_{4B}$. The above inclusions correspond to the power map $(4B)^2 = 2B$ between conjugacy classes of the Baby Monster, and with this rule, the assignment is uniquely determined.

Let $\tilde{a} \in E_8$ be a root so that $Q(E_7) = \mathbb{Z}\tilde{a} \simeq A_1$ and

$$\tilde{\omega}_{Q(E_7)} = \omega^+(\sqrt{2}\tilde{a}) = \frac{1}{8}(\tilde{a}(-1)^21 + \frac{1}{4}(e^{\sqrt{2}\tilde{a}} + e^{-\sqrt{2}\tilde{a}})$$

is an Ising vector in $V_{\sqrt{2}Q(E_7)} = V_{\sqrt{2}\tilde{a}}$.

**Structures of $V_{B(nX)}$ and $U_{B(nX)}$.** We determine the structures of $V_{B(nX)}$ and $U_{B(nX)}$.

1A case. In this case $\tilde{L}_{1A} \simeq A_1 \oplus E_7$ and we know $V_{B(1A)} \simeq U_{2A}$ by [LYY2]. It follows that $U_{B(1A)} \simeq L(7/10, 0)$ (cf. Section 3.1). It is clear that the weight two subspace of $U_{B(1A)}$ is one-dimensional.

2B case. In this case $\tilde{L}_{2B} \simeq A_1 \oplus A_1 \oplus D_6$ and $E_8/\tilde{L}_{2B} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let $\epsilon_1, \ldots, \epsilon_8 \in \mathbb{R}^8$ be such that $(\epsilon_i, \epsilon_j) = 2\delta_{i,j}$ for any $i, j \in \{1, \ldots, 8\}$. Then

$$\sqrt{2}E_8 = \left\{ \sum_{i=1}^{8} a_i\epsilon_i \in \mathbb{R}^8 \mid \text{all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z} + \frac{1}{2} \text{ and } \sum_{i=1}^{8} a_i \equiv 0 \pmod{2} \right\}. $$

It is shown in [LYY2] (see also [DLY2, FLM]) that

$$V_{\sqrt{2}E_8} \simeq V_{X}^+, \quad \mathcal{K} = \left\{ \sum_{i=1}^{8} a_i\epsilon_i \in \mathbb{R}^8 \mid \text{all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z} + \frac{1}{2} \right\}. \quad (4.14)$$

By the same argument as in the monstrous 4A-case in [LYY2], we have $V_{B(2B)} \simeq V_{A_2}^+$, where $A = \text{span}_\mathbb{Z} \{-\epsilon_1 - \epsilon_2, \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_8)\} \simeq \sqrt{2}A_2$. Thus,

$$V_{B(2B)} \simeq V_{\sqrt{2}A_2}^+. \quad (4.15)$$

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There exists a Virasoro frame $L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0) \otimes L(\frac{1}{5}, 0)$ inside $V^+_{\sqrt{2}A_2}$ and the decomposition of $V^+_{\sqrt{2}A_2}$ as a module over the frame is computed in [KMY]. Here we note that $\text{Vir}(\tilde{\omega}_{Q(E_7)}) \simeq L(\frac{1}{2}, 0)$ is a member of the Virasoro frame and therefore from (loc. cit.) we obtain

$$U_{B(2B)} \simeq L(\frac{7}{10}, 0) \otimes L(\frac{1}{5}, 0) \oplus L(\frac{7}{10}, \frac{3}{5}) \otimes L(\frac{1}{5}, \frac{7}{5}).$$

By this decomposition, we see that the weight two subspace of $U_{B(2B)}$ is 3-dimensional and therefore coincides with $G_{B(2B)}$.

**Remark 4.5.** By the same argument as in [DLY2], one can also show that

$$U_{B(2B)} \simeq L(\frac{1}{2}, 0) \otimes V^+_{Z\gamma} \oplus L(\frac{1}{2}, \frac{1}{2}) \otimes V^+_{Z\gamma+\gamma/2},$$

where $\langle \gamma, \gamma \rangle = 12$.

**2C case.** In this case $\tilde{L}_{2C} \simeq A_1 \oplus A_7$ and $E_8/\tilde{L}_{2C} \simeq Z_4$. It is clear that $V_{B(2C)}$ is isomorphic to the monstrous 4B-algebra $U_{4B}$ discussed in [LYY2]. We know (cf. loc. cit.) that

$$U_{4B} \simeq L(\frac{1}{2}, 0) \otimes [L(\frac{7}{10}, 0) \otimes L(\frac{7}{10}, 0) \oplus L(\frac{7}{10}, \frac{3}{2}) \otimes L(\frac{7}{10}, \frac{3}{2})]$$

$$\oplus L(\frac{1}{2}, \frac{1}{2}) \otimes [L(\frac{7}{10}, \frac{3}{2}) \otimes L(\frac{7}{10}, 0) \oplus L(\frac{7}{10}, 0) \otimes L(\frac{7}{10}, \frac{3}{2})]$$

which gives

$$U_{B(2C)} \simeq L(\frac{7}{10}, 0) \otimes L(\frac{7}{10}, 0) \oplus L(\frac{7}{10}, \frac{3}{2}) \otimes L(\frac{7}{10}, \frac{3}{2}).$$

By this decomposition, we see that the weight two subspace of $U_{B(2C)}$ is 2-dimensional and coincides with $G_{B(2C)}$.

**3A case.** In this case $\tilde{L}_{3A} \simeq A_1 \oplus A_2 \oplus A_5$ and $E_8/\tilde{L}_{3A} \simeq Z_6$. It is clear that $V_{B(3A)}$ is isomorphic to the monstrous 6A-algebra $U_{6A}$ discussed in [LYY2] (see also Appendix A) and $\tilde{\omega}_{Q(E_7)}$ corresponds to the Ising vector $\omega^1 \in U_{6A}$ in Appendix A. The commutant subalgebra $U_{B(3A)} \simeq \text{Com}_{U_{6A}}(\text{Vir}(\omega^1))$ does not have a Virasoro frame which consists of rational unitary Virasoro VOAs. Nevertheless, by [LYY2, Appendix B.3], we have the following decomposition:

$$U_{B(3A)} \simeq W(\frac{1}{5}) \otimes W_6(0, 0) \oplus L(\frac{4}{5}, \frac{2}{3})^+ \otimes W_6(0, 4) \oplus L(\frac{4}{5}, \frac{2}{3})^- \otimes W_6(0, 8),$$

where we denoted the $W(\frac{1}{5}) \simeq L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$-modules as in [HLY] and $W_6(0, 0) \simeq \text{Com}_{V(\mathfrak{A}_5)}(\text{Vir}(\omega_{A_5} - \tilde{\omega}_{A_5}))$ is a $W_6$-algebra with central charge $5/4$ and $W_6(0, 4), W_6(0, 8)$
are irreducible $W_6(0, 0)$-modules see [DLe, DLY3, LYY2] for details.\footnote{The labeling of irreducible $W_6$-modules in [LYY2] is different from the one in [DLY3]. Here we adopt the labeling used in [LYY2]. $W_3(0, 0)$ in loc. cit. is called $W(\frac{3}{2})$ in the present paper.} The head characters of $W_6(0, 0)$, $W_6(0, 4)$ and $W_6(0, 8)$ are computed in [LYY2] and it follows that the weight two subspace of $U_{B(3A)}$ is 4-dimensional. Therefore, the Griess algebra of $U_{B(3A)}$ is equal to $G_{B(3A)}$.

**4B case.** In this case, $\tilde{L}_{4B} \simeq A_1 \oplus A_1 \oplus A_3 \oplus A_3$ and $E_8/\tilde{L}_{4B} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4$. With a similar argument as in the monstrous 4A case of the $E_8$-observation in [LYY2] and the Baby-monstrous 2B case, we have $V_{B(4B)} \simeq V_{N}^+$, where $N$ is a rank 3 lattice defined by $N = \text{span}_\mathbb{Z}\{-\epsilon_1 - \epsilon_2, \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_8), \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_5 - \epsilon_6 - \epsilon_7 - \epsilon_8)\}$. Here, the $\epsilon_i$, $i = 1, \ldots, 8$, are defined as in the 2B case and the Gram matrix of $N$ is given by

$$
\begin{pmatrix}
4 & -2 & -2 \\
-2 & 4 & 1 \\
-2 & 1 & 4
\end{pmatrix}
$$

Set

$$
\alpha = -\epsilon_1 - \epsilon_2, \quad \beta = \epsilon_3 + \epsilon_4 + \epsilon_5, \quad \gamma = \epsilon_6 + \epsilon_7 + \epsilon_8.
$$

Then $\langle \alpha, \alpha \rangle = 4$, $\langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = 6$ and $\{\alpha, \beta, \gamma\}$ forms an orthogonal frame for $N$. Moreover, we have

$$
N = K \cup (\delta + K),
$$

where $K = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\gamma$ and $\delta = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_8) = \frac{1}{2}(-\alpha + \beta + \gamma)$. Thus, we have

$$
U_{B(4B)} = \text{Com}_{V_{N}^+}(\omega^+ (\sqrt{2}\alpha))
$$

$$
\simeq L(1/2, 0) \otimes \left[ V_{Z\beta}^+ \otimes V_{Z\gamma}^+ \oplus V_{Z\beta}^- \otimes V_{Z\gamma}^- \right]
$$

$$
\oplus L(1/2, 1/2) \otimes \left[ V_{Z\beta+\beta/2}^+ \otimes V_{Z\gamma+\gamma/2}^- \oplus V_{Z\beta+\beta/2}^- \otimes V_{Z\gamma+\gamma/2}^+ \right]
$$

The commutant sub-VOA $U_{B(4B)}$ does not have a Virasoro frame which consists of rational unitary Virasoro VOAs. Nevertheless, we see from the decomposition above that the weight two subspace of $U_{B(4B)}$ is 6-dimensional and therefore coincides with $G_{B(4B)}$.

**Remark 4.6.** By Lemma 3.3 of [DLY1], we have

$$
\ker_{V_{Y_{A_3}}} (\omega_{A_3} - \tilde{\omega}_{A_3})_{(0)} \simeq V_{\sqrt{6}2}^+.
$$

Since $L_{4B} \simeq A_1 \oplus A_3 \oplus A_3$, we have another proof that $U_{B(4B)}$ contains a full subalgebra of the form

$$
L(1/2, 0) \otimes V_{Z\beta}^+ \otimes V_{Z\gamma}^+,
$$

where $\langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = 6$.\footnotetext{The labeling of irreducible $W_6$-modules in [LYY2] is different from the one in [DLY3]. Here we adopt the labeling used in [LYY2]. $W_3(0, 0)$ in loc. cit. is called $W(\frac{3}{2})$ in the present paper.}
Subalgebras generated by \( \tilde{f} \) and \( \tilde{f}' \). Let
\[
\tilde{f} := \tilde{\omega}_E, \quad \text{and} \quad \tilde{f}' := \rho_{nX} \tilde{\omega}_E. \tag{4.20}
\]
By definition, it is clear that \( \tilde{f} \) and \( \tilde{f}' \) are contained in the Griess subalgebra
\[
\mathcal{G}_{B(n,X)} = \text{span}_\mathbb{C}\{ \omega^s, X^r \mid 1 \leq s \leq \ell, \ 1 \leq r \leq n - 1 \}
\]
of \( U_{B(n,X)} \).

Next, we will discuss \( \mathcal{G}_{B(n,X)} \) and the subalgebra generated by \( \tilde{f} \) and \( \tilde{f}' \).

1A case. In this case \( L_{1A} \simeq E_7 \) and \( \rho_{1A} \) is trivial. Thus \( \tilde{f} = \tilde{f}' \), \( \langle \tilde{f}, \tilde{f}' \rangle = 7/20 \) and \( \tilde{f} \) generates \( \mathcal{G}_{B(1A)} \) and \( U_{B(1A)} \).

2B case. In this case \( L_{2B} \simeq A_1 \oplus D_6 \) and \( \ell = 2 \).

The mutually orthogonal simple Virasoro vectors \( \omega^1 \) and \( \omega^2 \) have the central charges 1/2 and 1, respectively, and \( X = X^1 \) is a highest weight vector for \( \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \) with highest weight \((1/2, 3/2)\). By a direct computation, one finds the following commutative algebra structure on \( \mathcal{G}_{B(2B)} \):

| \( a(1)b \) | \( \omega^1 \) | \( \omega^2 \) | \( X \) | \( \langle a, b \rangle \) | \( \omega^1 \) | \( \omega^2 \) | \( X \) |
|----------|----------|----------|-----|---------|----------|----------|-----|
| \( \omega^1 \) | \( 2\omega^1 \) | 0 | \( \frac{1}{2}X \) | \( \omega^1 \) | \( \frac{1}{4} \) | 0 | 0 |
| \( \omega^2 \) | \( 2\omega^2 \) | \( \frac{3}{4}X \) | \( \omega^2 \) | \( \frac{1}{2} \) | 0 |
| \( X \) | \( 128\omega^1 + 192\omega^2 \) | \( X \) | 64 |

One also finds that
\[
\tilde{f} = \frac{1}{5}\omega^1 + \frac{3}{5}\omega^2 + \frac{1}{20}X, \quad \tilde{f}' = \frac{1}{5}\omega^1 + \frac{3}{5}\omega^2 - \frac{1}{20}X, \quad \langle \tilde{f}, \tilde{f}' \rangle = \frac{3}{100},
\]
and that the algebra \( \mathcal{G}_{B(2B)} \) is generated by \( \tilde{f} \) and \( \tilde{f}' \). Set
\[
u := \frac{4}{5}\omega^1 + \frac{2}{5}\omega^2 - \frac{1}{20}X \quad \text{and} \quad w := -32\omega^1 + 16\omega^2 - X.
\]

Then \( u \) is a simple \( c = 4/5 \) Virasoro vector orthogonal to \( \tilde{f} \) and \( w \) is a highest weight vector for \( \text{Vir}(\tilde{f}) \otimes \text{Vir}(u) \) with highest weight \((3/5, 7/5)\). By the fusion rules
\[
L(\frac{7}{10}, \frac{3}{5}) \times L(\frac{7}{10}, \frac{3}{5}) = L(\frac{7}{10}, 0) + L(\frac{7}{10}, \frac{3}{5}),
\]
\[
L(\frac{4}{5}, \frac{7}{5}) \times L(\frac{4}{5}, \frac{7}{5}) = L(\frac{4}{5}, 0) + L(\frac{4}{5}, \frac{7}{5}),
\]
one sees that \( \tilde{f} \) and \( \tilde{f}' \) generate a subalgebra
\[
L(\frac{7}{10}, 0) \otimes L(\frac{4}{5}, 0) \oplus L(\frac{7}{10}, \frac{3}{5}) \otimes L(\frac{4}{5}, \frac{7}{5})
\]
which is equal to \( U_{B(2B)} \). Note that the decomposition above is not a \( \mathbb{Z}_2 \)-graded extension of \( \text{Vir}(\tilde{f}) \otimes \text{Vir}(u) \) since \( w_{(1)}w = 768\tilde{f} + 1568u - 24w \).
2C case. In this case $L_{2C} \simeq A_7$ and $\ell = 1$. Then $\omega = \omega^1$ is a Virasoro vector of central charge $7/5$ and $X = X^1$ is a highest weight vector for $\text{Vir}(\omega)$ with highest weight $2$. We have the following commutative algebra structure on $G_{B(2C)}$:

$$
\omega(1)X = 2X, \quad X(1)X = 70X, \quad \langle \omega, \omega \rangle = 7/10, \quad \langle X, X \rangle = 70.
$$

One verifies that

$$
\tilde{f} = \frac{1}{2} \omega + \frac{1}{20} X, \quad \tilde{f}' = \frac{1}{2} \omega - \frac{1}{20} X, \quad \langle \tilde{f}, \tilde{f}' \rangle = 0.
$$

Thus, the VOA generated by $\tilde{f}$ and $\tilde{f}'$ is isomorphic to

$$
\text{Vir}(\tilde{f}) \otimes \text{Vir}(\tilde{f}') \simeq L(7/10, 0) \otimes L(7/10, 0).
$$

In this case, $U_{B(2C)} \simeq L(7/10, 0) \otimes L(7/10, 0) \oplus L(7/10, 3/2) \otimes L(7/10, 3/2)$ is not generated by $\tilde{f}$ and $\tilde{f}'$.

3A case. In this case $L_{3A} \simeq A_2 \oplus A_5$ and $\ell = 2$.

Then $\omega^1$ and $\omega^2$ are mutually orthogonal Virasoro vectors with central charges $4/5$ and $5/4$, respectively, and $X^j, j = 1, 2$, are highest weight vectors for $\text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2)$ with highest weight $(2/3, 4/3)$. We have the following commutative algebra structure on $G_{B(3A)}$:

| $a(1)b$ | $\omega^1$ | $\omega^2$ | $X^1$ | $X^2$ | $\langle a, b \rangle$ | $\omega^1$ | $\omega^2$ | $X^1$ | $X^2$ |
|---------|------------|------------|-------|-------|------------------------|------------|------------|-------|-------|
| $\omega^1$ | $2\omega^1$ | $2\omega^1$ | $\frac{3}{2}X^1$ | $\frac{3}{2}X^2$ | $\omega^1$ | $\frac{2}{5}$ | $0$ | $0$ | $0$ |
| $\omega^2$ | $2\omega^2$ | $\frac{4}{3}X^1$ | $\frac{4}{3}X^2$ | $\omega^2$ | $\frac{5}{3}$ | $0$ | $0$ | $0$ |
| $X^1$ | $12X^1$ | $75\omega^1 + 96\omega^2$ | $X^1$ | $0$ | $45$ |
| $X^2$ | $12X^1$ | $X^2$ | $0$ | $0$ | $0$ |

One verifies that

$$
\tilde{f} = \frac{1}{4} \omega^1 + \frac{2}{5} \omega^2 + \frac{1}{20} X^1 + \frac{1}{20} X^2, \quad \tilde{f}' = \frac{1}{4} \omega^1 + \frac{2}{5} \omega^2 + \frac{3}{20} X^1 + \frac{3}{20} X^2, \quad \langle \tilde{f}, \tilde{f}' \rangle = \frac{1}{80}.
$$

and checks again that $\tilde{f}$ and $\tilde{f}'$ generate $G_{B(3A)}$.

4B case. In this case $L_{4B} \simeq A_3 \oplus A_3 \oplus A_1$ and $\ell = 3$.

One gets that $\omega^1$, $\omega^2$ and $\omega^3$ are mutually orthogonal Virasoro vectors with central charges $1$, $1$ and $1/2$, respectively. The vectors $X^1$, $X^2$ and $X^3$ are highest weight vectors for $\text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \otimes \text{Vir}(\omega^3)$ with highest weight $(5/4, 5/4, 1/2)$, $(1, 1, 0)$ and $(3/4, 3/4, 1/2)$, respectively.
We also have the following commutative algebra structure on $G_{B(4B)}$:

| $a_{(1)b}$ | $\omega^1$ | $\omega^2$ | $\omega^3$ | $X^1$ | $X^2$ | $X^3$ |
|-------------|-------------|-------------|-------------|-------|-------|-------|
| $\omega^1$  | $2\omega^1$ | 0           | 0           | $\frac{3}{4}X^1$ | $X^2$ | $\frac{3}{4}X^3$ |
| $\omega^2$  | $2\omega^2$ | 0           | 0           | $\frac{3}{4}X^1$ | $X^2$ | $\frac{3}{4}X^3$ |
| $\omega^3$  | $2\omega^3$ | $\frac{1}{2}X^1$ | 0           | $\frac{1}{2}X^3$ |       |       |
| $X^1$       | 8$X^2$     | 9$X^3$     | 48($\omega^1 + \omega^2$) + 64$\omega^3$ |
| $X^2$       |           | 72($\omega^1 + \omega^2$) | 9$X^1$ |
| $X^3$       |           |           | 8$X^2$ |

$\langle \omega^1, \omega^1 \rangle = \langle \omega^2, \omega^2 \rangle = 1/2$, $\langle \omega^3, \omega^3 \rangle = 1/4$, $\langle X^1, X^3 \rangle = 32$, $\langle X^2, X^2 \rangle = 36$.

One verifies that

$$\tilde{f} = \frac{3}{10}\omega^1 + \frac{3}{10}\omega^2 + \frac{1}{5}\omega^3 + \frac{1}{20}X^1 + \frac{1}{20}X^2 + \frac{1}{20}X^3,$$

$$\tilde{f}' = \frac{3}{10}\omega^1 + \frac{3}{10}\omega^2 + \frac{1}{5}\omega^3 + \frac{54}{20}X^1 + \frac{54}{20}X^2 + \frac{54}{20}X^3, \quad \langle \tilde{f}, \tilde{f}' \rangle = \frac{1}{100}.$$

In this case, the Griess algebra $G_{B(4B)}$ is not generated by $\tilde{f}$ and $\tilde{f}'$. Denote by $\nu$ the nontrivial diagram automorphism of the affine $E_7$ diagram, that is, $\nu$ is defined as: $\alpha_0 \mapsto \alpha_6 \mapsto \alpha_0$, $\alpha_1 \mapsto \alpha_5 \mapsto \alpha_1$, $\alpha_2 \mapsto \alpha_4 \mapsto \alpha_2$, $\alpha_3 \mapsto \alpha_3$ and $\alpha_7 \mapsto \alpha_7$ on the diagram (4.11). Since $\sqrt{2}E_7$ is doubly even, we have a splitting $\text{Aut}(V_{\sqrt{2}E_7}) \simeq \text{Hom}_{\mathbb{Z}}(E_7, \mathbb{C}^*) \rtimes \text{Aut}(E_7)$. Then $\nu$ canonically acts on $G_{B(4B)}$ and we find that $\tilde{f}$ and $\tilde{f}'$ generate the fixed point subalgebra

$$G_{B(4B)}^{(\nu)} = \text{span}_{\mathbb{C}}\{\omega^1 + \omega^2, \omega^3, X^1, X^2, X^3\}.$$

Summarizing the computations above, we have the following table of values of inner products between $\tilde{f}$ and $\tilde{f}'$:

| 7 | 3 | 1 | 1 | 1 | 3 | 7 |
|---|---|---|---|---|---|---|
| 20| 100| 80| 100| 80| 100| 20|

Remark 4.7. By the computations above, we find that the Griess subalgebra generated by $\tilde{f}$ and $\tilde{f}'$ coincides with the fixed point subalgebra $G_{B(nX)}^{(\nu)}$ if $nX = 2C$ and $4B$. These are the cases when the corresponding nodes are the fixed points of the diagram automorphism $\nu$.  

29
5 The Baby Monster

In this section, we will discuss the properties of the commutant vertex operator subalgebra \( V^B \) of the Moonshine VOA \( V^\natural \). It is known that \( \text{Aut}(V^B) = B \). We will show that there exists a one-to-one correspondence between 2A-involutions of \( B \) and simple \( c = 7/10 \) Virasoro vectors of \( \sigma \)-type in \( V^\natural \).

Finally, we will discuss the embedding of \( U_{B(n,X)} \) into \( V^B \). The main idea is to embed the root lattice \( E_7 \) into \( E_8 \) and view \( U_{B(n,X)} \) as a certain commutant subalgebra of the lattice VOA \( V_{\sqrt{2}E_8} \). Then we will show that the product of two \( \sigma \)-involutions generated by simple \( c = 7/10 \) Virasoro vectors in \( U_{B(n,X)} \) exactly belong to the conjugacy class \( nX \) in \( B \). By this procedure, we obtain a VOA description of the \( E_7 \) structure inside \( B \).

The automorphism group of the Moonshine VOA \( V^\natural \) is the Monster \( \mathbb{M} \) [FLM]. Consider the monstrous Griess algebra of dimension 196884 [C, G]. It is known that the monstrous Griess algebra is naturally realized as the subspace of weight 2 of \( V^\natural \) [FLM], which we call the Griess algebra of \( V^\natural \) and denote by \( G^\natural \). We will freely use the character tables in [ATLAS], although no explicit proofs for their correctness have been published up to now.

5.1 The Baby Monster vertex operator algebra \( V^B \)

The following one-to-one correspondence is crucial in the rest of this paper.

**Theorem 5.1** ([C, Mi1, Ma1, Hö2]). The map which associates to an Ising vector of \( V^\natural \) its \( \tau \)-involution given in Theorem 2.2 defines a bijection between the set of Ising vectors of \( V^\natural \) and the 2A-conjugacy class of the Monster \( \mathbb{M} = \text{Aut}(V^\natural) \).

First, Conway [C] showed that every 2A-element determines a so-called axial vector of the Griess algebra which is up to rescaling an Ising vector. Then Miyamoto [Mi1] showed that any Ising vector defines an involutive automorphism of a VOA. In the case of the Moonshine VOA, this recovers the 2A-element [Ma1]. In [Hö2], it was shown that this correspondence is actually one-to-one as remarked in [Mi1].

By Theorem 5.1, we see that the number of Ising vectors of \( V^\natural \) equals the number of 2A-involutions of the Monster. As constructed in [FLM, Mi2], the Moonshine VOA \( V^\natural \) has a compact real form \( V^\natural_\mathbb{R} \). The 196883-dimensional irreducible representation of \( \mathbb{M} \) is real [ATLAS] and it follows that the Griess algebra structure of the weight 2 subspace of \( V^\natural_\mathbb{R} \) is isomorphic to the monstrous Griess algebra over the real numbers considered in [C, G]. It is shown in [C] that the number of Ising vectors in \( V^\natural_\mathbb{R} \) is not less than the number of 2A-involutions of the Monster, and an Ising vector of \( V^\natural_\mathbb{R} \) is still an Ising vector in the complex Moonshine VOA \( V^\natural = \mathbb{C} \otimes_\mathbb{R} V^\natural_\mathbb{R} \). Therefore, there are no “complex” Ising vectors in \( V^\natural \) and we obtain the following.
Proposition 5.2. Let $V_{\mathbb{R}}^z$ be a compact real form of $V^z$ as in [FLM, Mi2]. Then every Ising vector of $V^z$ is contained in $V_{\mathbb{R}}^z$.

Let us recall the Baby Monster VOA discussed in [Hö1, Hö2, Hö3, Y]. We fix an Ising vector $e \in V^z$. The centralizer $C_{BM}(\tau_e)$ is isomorphic to a 2-fold cover $\langle \tau_e \rangle \cdot \mathbb{B}$ of the Baby Monster $\mathbb{B}$ so that the commutant subalgebra

$$V_{\mathbb{B}}^z := \text{Com}_{V^z}(\text{Vir}(e)) = \ker_{V^z} e(0)$$

affords a natural action of the Baby Monster. Since all the Ising vectors of $V^z$ are mutually conjugate, the VOA $V_{\mathbb{B}}^z$ is well-defined up to isomorphism. So we call $V_{\mathbb{B}}^z$ the Baby Monster VOA.\(^3\) The Baby Monster VOA is a framed VOA and its structure as well as its representation are studied in [Hö1, Hö2, Y]. It is proved in [Hö2] that the Baby Monster $\mathbb{B}$ is the full automorphism group of $V_{\mathbb{B}}^z$. (See also [Y] for another proof.)

As obtained in [C, MeN], the $C_{BM}(\tau_e)$-module $G^z$ decomposes as follows:

$$G^z = \mathbf{1} \oplus \mathbf{1} \oplus 4371 \oplus 96255 \oplus 96256$$

(5.2)

The Ising vector $e$ belongs to the first component in the above decomposition, in particular it is contained in $(G^z)^{C_{BM}(\tau_e)}$.

By construction, the Griess algebra of $V_{\mathbb{B}}^z$ is of dimension 96256 and has the decomposition $V_{\mathbb{B}}^z_2 = \mathbf{1} \oplus 96255$ as a $\mathbb{B}$-module, where the principal module is spanned by the conformal vector of $V_{\mathbb{B}}^z$ with central charge $47/2$.

5.2 The $\{3, 4\}$-transposition property

We fix a 2A-involution $t$ of $\mathbb{B} = \text{Aut}(V_{\mathbb{B}}^z)$. Then $C_{\mathbb{B}}(t) = 2 \cdot (2E_6(2)) \cdot 2$ (cf. [ATLAS]) and the Griess algebra of $V_{\mathbb{B}}^z$ has the following decomposition into irreducibles as a $C_{\mathbb{B}}(t)$-module:

$$V_{\mathbb{B}}^z_2 = \mathbf{1} \oplus \mathbf{1} \oplus 1938 \oplus 48620 \oplus 45696.$$  

(5.3)

By the decomposition above, the Griess algebra of the fixed point subalgebra $(V_{\mathbb{B}}^z)^{C_{\mathbb{B}}(t)}$ forms a 2-dimensional semisimple commutative algebra. Below we will argue that the shorter Virasoro vector in $(V_{\mathbb{B}}^z)^{C_{\mathbb{B}}(t)}$ has central charge $c = 7/10$ and is of $\sigma$-type. Then we will prove that the correspondence between 2A-elements of $\mathbb{B}$ and simple $c = 7/10$ Virasoro vectors of $V_{\mathbb{B}}^z$ of $\sigma$-type is one-to-one.

\(^3\)Our $V_{\mathbb{B}}^z$ is actually the even part of the shorter Moonshine module constructed in [Hö1] and it is denoted by $V_{\mathbb{B}}^z_{(0)}$ in [Hö1].
Let \( f \) be a derived \( c = 7/10 \) Virasoro vector in \( \text{Com}_{V^2}(\text{Vir}(e)) \) with respect to \( e \). Let \( U \subset V^2 \) be the corresponding sub-VOA isomorphic to \( U_{2A} \), that is, \( e + f \) is a Virasoro frame of \( U \). For an irreducible \( U \)-module \( M \), we set the space of multiplicities by

\[
H_M := \text{Hom}_{U_{2A}}(M, V^2)
\]

and we have the decomposition

\[
V^2 = \bigoplus_{M \in \text{Irr}(U_{2A})} M \otimes H_M. \tag{5.4}
\]

The space \( H_M \) affords a natural action of the commutant subalgebra \( \text{Com}_{V^2}(U) \). The Virasoro vectors \( e \) and \( f \) are mutually orthogonal and the subalgebra \( \text{Vir}(e) \otimes \text{Vir}(f) \) forms a Virasoro frame of \( U \) isomorphic to \( L(1/2, 0) \otimes L(7/10, 0) \). We adopt the labeling of irreducible \( U_{2A} \)-modules with respect to this frame.

**Lemma 5.3.** The top weight \( h(H_M) \) and the dimension \( d(H_M) \) of the top level of the \( \text{Com}_{V^2}(U) \)-modules \( H_M \) are given by the following table:

| \( M \) | \( U(0, 0) \) | \( U(1/2, 0), U(1/16, 7/16)\) | \( U(0, 3/5) \) | \( U(0, 1/10), U(1/16, 3/80)\) |
|-------|-------------|-----------------|--------------|-----------------|
| \( h(H_M) \) | 0 | 3/2 | 7/5 | 19/10 |
| \( d(H_M) \) | 1 | 2432 | 1938 | 45696 |

Moreover, the dimension of the Griess algebra of the commutant subalgebra \( \text{Com}_{V^2}(U) = H_{U(0,0)} \) is 48621.

**Proof:** For \( h \in \mathbb{C} \) and \( x \in \mathcal{G}^2 \), we set \( \mathcal{G}^2_v[h] := \{ v \in \mathcal{G}^2 \mid x(v) = hv \} \). It is shown in [Hö4] that \( \mathcal{G}^2 \) forms a conformal 11-design\(^4\) and one can use the Matsuo-Norton trace formula obtained in [Ma1] up to five compositions of adjoint actions of elements in \( \mathcal{G}^2 \).

Let \( u \in \mathcal{G}^2 \) be any Ising vector. Applying the formula, the following decomposition is obtained in (loc. cit.):

\[
\mathcal{G}^2 = \mathcal{G}^2_u[2] \oplus \mathcal{G}^2_u[0] \oplus \mathcal{G}^2_u[1/2] \oplus \mathcal{G}^2_u[1/16],
\]

\[
\dim \mathcal{G}^2_u[2] = 1, \quad \dim \mathcal{G}^2_u[0] = \dim \mathcal{G}^2_u[1/16] = 96256, \quad \dim \mathcal{G}^2_u[1/2] = 4371. \tag{5.6}
\]

Let \( v \in \mathcal{G}^2 \) be a simple \( c = 7/10 \) Virasoro vector. It is also calculated in (loc. cit.) by a similar argument that

\[
\mathcal{G}^2 = \mathcal{G}^2_v[2] \oplus \mathcal{G}^2_v[0] \oplus \mathcal{G}^2_v[3/2] \oplus \mathcal{G}^2_v[1/10] \oplus \mathcal{G}^2_v[3/5] \oplus \mathcal{G}^2_v[7/16] \oplus \mathcal{G}^2_v[3/80],
\]

\[
\dim \mathcal{G}^2_v[2] = \dim \mathcal{G}^2_v[3/2] = 1, \quad \dim \mathcal{G}^2_v[0] = 51054, \quad \dim \mathcal{G}^2_v[1/10] = 47634, \tag{5.7}
\]

\[
\dim \mathcal{G}^2_v[3/5] = 1938, \quad \dim \mathcal{G}^2_v[7/16] = 4864, \quad \dim \mathcal{G}^2_v[3/80] = 91392.
\]

\(^4\)Or, one can say the Moonshine VOA is of class \( S^{11} \) as in [Ma1].
This allows to compute the weights and dimensions of the top levels of $H_M$ for $M = U(0,0)$, $U(1/2,0)$, $U(0,3/5)$ and $U(0,1/10)$ as in the Lemma. Since (5.6) and (5.7) hold for arbitrary Ising vectors $u$ and simple $c = 7/10$ Virasoro vectors $v$ of $V^2$, it follows from the conjugate relations (3.6), (3.7) and Lemma 3.1 that $d(H_U(1/2,0)) = d(H_{U(1/16,7/16)}^\pm)$ and $d(H_U(0,1/10)) = d(H_{U(1/16,3/80)}^\pm)$. Then one obtains the table.

Recall the group homomorphism $\varphi_e : \text{Stab}_{\text{Aut}(V^2)}(e) \rightarrow \text{Aut}(\text{Comm}_V(\text{Vir}(e))) \cong \mathbb{B}$ defined in (3.11). By the one-to-one correspondence in Theorem 5.1, we see $\text{Stab}_{\text{Aut}(V^2)}(e) = C_\mathbb{M}(\tau_e) \cong \langle \tau_e \rangle \cdot \mathbb{B}$ and therefore $\varphi_e$ is surjective.

By Lemma 3.1, $U_{2A}$ contains three Ising vectors $e$, $e'$ and $e''$. As a derived Virasoro vector, $f$ is of $\sigma$-type on the commutant $\text{Comm}_V(\text{Vir}(e))$ by Lemma 3.6 and one has $\varphi_e(\tau_{e'}) = \varphi_e(\tau_{e''}) = \sigma_f$ by Lemma 3.7.

**Proposition 5.4.** $\varphi_e(\tau_{e'}) = \varphi_e(\tau_{e''}) = \sigma_f$ defines a $2A$-element of the Baby Monster.

**Proof:** Let us compute the trace of $\sigma_f$ on the Griess algebra of $\text{Comm}_V(\text{Vir}(e))$. As a $\text{Vir}(f) \otimes \text{Comm}_V(U_{2A})$-module, we have the following decomposition:

$$\text{Comm}_V(\text{Vir}(e)) = [0] \otimes H_U(0,0) \oplus [3/2] \otimes H_U(1/2,0) \oplus [1/10] \otimes H_U(0,1/10) \oplus [3/5] \otimes H_U(0,3/5).$$

By (3.8) and (5.5), we have

$$\text{Tr}_{\text{Comm}_V(\text{Vir}(e))} \sigma_f = \dim \text{Vir}(f) + \dim(H_U(0,0)) - d(H_U(0,1/10)) + d(H_U(0,3/5))$$

$$= 1 + 48621 - 45696 + 1938 = 4864.$$

By [ATLAS], we see that $\sigma_f = \varphi_e(\tau_{e'})$ has to be a $2A$-involution of the Baby Monster.

**Remark 5.5.** For the computation of $\text{Tr}_{\text{Comm}_V(\text{Vir}(e))} \sigma_f$, we only need to use the representation theory of $U_{2A}$ and the fact that $V_2^e$ forms a conformal 11-design. No information about $\text{Aut}(V^2)$ nor $\text{Aut}(V\mathbb{B}^3)$ is required for this part.

**Lemma 5.6.** $\varphi_e^{-1} \mathbb{C}_B(\varphi_e(\tau_{e'}))$ stabilizes the subset $\{e', e''\}$ of $V^e$.

**Proof:** Take any $g \in \varphi_e^{-1} \mathbb{C}_B(\varphi_e(\tau_{e'}))$. Since $g \tau_x g^{-1} = \tau_{gx}$ for any Ising vector $x$, it suffices to show $g\{\tau_{e'}, \tau_{e''}\} g^{-1} = \{\tau_{e'}, \tau_{e''}\}$ by the one-to-one correspondence of Theorem 5.1. We have $\varphi_e(g\{\tau_{e'}, \tau_{e''}\} g^{-1}) = \varphi_e(g) \varphi_e(\tau_{e'}) \varphi_e(g^{-1}) = \varphi_e(\tau_{e'})$. By Theorem 3.4 we know $\tau_{e''} = \tau_e \tau_{e'}$. Since $\ker \varphi_e = \langle \tau_e \rangle$, we get $g\{\tau_{e'}, \tau_{e''}\} g^{-1} \subset \varphi_e^{-1}(\{\varphi_e(\tau_{e'})\}) = \{\tau_{e'}, \tau_e \tau_{e'}\} = \{\tau_{e'}, \tau_{e''}\}$.

**Proposition 5.7.** A derived $c = 7/10$ Virasoro vector $f$ of $\text{Comm}_V(\text{Vir}(e))$ with respect to $e$ is fixed by the centralizer of the $2A$-involution $\sigma_f = \varphi_e(\tau_{e'})$ of the Baby Monster.
Proof: By Lemma 3.1, \( f \) has the expression
\[
f = -\frac{1}{5}e + \frac{4}{5}(e' + e'').
\] (5.8)
Therefore, \( f \) is fixed by the centralizer of \( \varphi_e(\tau'') \) by Lemma 5.6.

Remark 5.8. We can define a compact real form \( \mathcal{V}_B \mathbb{R} \) using the compact real form \( \mathcal{V}_2 \mathbb{R} \). It follows from Proposition 5.2 that every derived \( c = 7/10 \) Virasoro vector in \( \mathcal{V}_B \mathbb{R} \) is real, since it is an \( \mathbb{R} \)-linear combination of real Ising vectors by (5.8).

Now we establish the one-to-one correspondence between 2A-involutions of the Baby Monster and derived \( c = 7/10 \) Virasoro vectors of \( \mathcal{V}_B \mathbb{R} \).

Theorem 5.9. The map which associates a derived \( c = 7/10 \) Virasoro vector to its \( \sigma \)-involution defines a bijection between the set of all derived \( c = 7/10 \) Virasoro vectors of \( \text{Com}_{\mathcal{V}_2}(\text{Vir}(e)) \) with respect to \( e \) and the 2A-conjugacy class of the Baby Monster \( \mathcal{B} = \text{Aut}(\text{Com}_{\mathcal{V}_2}(\text{Vir}(e))) \).

Proof: The map of the theorem is equivariant with respect to the natural action of \( \varphi_e(\text{Stab}_{\text{Aut}(\mathcal{V}_2)}(e)) \) on the derived vectors and the conjugation action of \( \mathcal{B} \) on the set of its 2A-involutions, respectively.

The \( U_{2A} \)-algebra can be embedded into \( V^\mathbb{R} \) since \( U_{2A} \subset V_{\sqrt{2}E_8}^+ \). Because the action of \( \text{Aut}(V^\mathbb{R}) \) on the Ising vectors of \( V^\mathbb{R} \) is transitive by Theorem 5.1, we may assume that \( e \) is contained in the chosen embedding of \( U_{2A} \). The vector \( f = \omega_{U_{2A}} - e \) is a derived \( c = 7/10 \) Virasoro vector of \( \text{Com}_{\mathcal{V}_2}(\text{Vir}(e)) \) with respect to \( e \). The transitivity of the conjugation action of \( \mathcal{B} \) on the 2A-involutions shows the surjectivity of the map of the theorem.

For the injectivity, we fix a 2A-involution \( t \) of the Baby Monster. By the arguments above we have determined that the unique shorter Virasoro vector of \( (V^\mathbb{R})_{\mathcal{B}}(t) \) of central charge 7/10.

Let \( t \) be a 2A-involution of the Baby Monster. By the arguments above we have determined that the unique shorter Virasoro vector of \( (V^\mathbb{R})_{\mathcal{B}}(t) \) has central charge 7/10.

Corollary 5.10. Every 2A-involution \( t \) of the Baby Monster \( \mathcal{B} = \text{Aut}(V^\mathbb{R}) \) uniquely defines a simple \( c = 7/10 \) Virasoro vector of the fixed point subalgebra \( (V^\mathbb{R})_{\mathcal{B}}(t) \).

By Proposition 5.2, we can apply Proposition 3.11 to the Moonshine VOA \( V^\mathbb{R} \). It is shown in Proposition 5.4 that the \( \sigma \)-involutions associated to derived \( c = 7/10 \) Virasoro vectors of \( V^\mathbb{R} \) are in the 2A-conjugacy class of the Baby Monster. Therefore, we have recovered the famous \( \{3,4\} \)-transposition property of the Baby Monster from the view point of vertex operator algebras.
Corollary 5.11. The $2A$ involutions of the Baby Monster satisfy the $\{3, 4\}$-transposition property.

Remark 5.12. Although we used the character table to identify $2A$-involutions of the Baby Monster with $\sigma$-involutions associated to derived $c = 7/10$ Virasoro vectors in Proposition 5.4, it is worth to note that the proof of Proposition 3.11 uses only the theory of vertex operator algebras. Hence the result that the product of two $\sigma$-involutions associated to derived $c = 7/10$ Virasoro vectors in $V^\natural$ is bounded by $4$ follows from the theory of vertex operator algebras.

We finally show that every simple $c = 7/10$ Virasoro vector in $V^\natural$ of $\sigma$-type is in fact a derived vector with respect to $e$.

**Theorem 5.13.** The map which associates to a simple $c = 7/10$ Virasoro vector of $V^\natural$ of $\sigma$-type its $\sigma$-involution defines a bijection between the set of simple $c = 7/10$ Virasoro vectors of $V^\natural$ of $\sigma$-type and the $2A$-conjugacy class of the Baby Monster $\mathbb{B} = \text{Aut}(V^\natural)$.

**Proof:** We identify $V^\natural$ with the subalgebra $\text{Com}_{V^\natural}(\text{Vir}(e))$ of $V^\natural$. Let $v$ be a simple $c = 7/10$ Virasoro vector of $V^\natural$ of $\sigma$-type. Then, a possible eigenvalue of $v(1)$ on the Griess algebra of $V^\natural$ is one of $2, 0, 3/5, 1/10$ and $3/2$ by Lemma 2.6. We set $d_v(h) := \dim V^\natural h \cap V^\natural_2$ for $h \in \{0, 3/5, 1/10, 3/2\}$. It is shown in [Hö4] that the Griess algebra of $V^\natural$ forms a conformal $7$-design$^5$ and we can apply the Matsuo-Norton trace formula in [Ma1] to compute the trace of the adjoint action on the Griess algebra of $V^\natural$ for up to three compositions. As a result, we find the following unique solution:

$$d_v(0) = 48621, \quad d_v(3/5) = 1938, \quad d_v(1/10) = 45696, \quad d_v(3/2) = 0.$$  

The trace of the $\sigma$-involution $\sigma_v$ is equal to $1 + 48621 + 1938 - 45696 = 4864$ and we see that $\sigma_v$ must belong to the $2A$-conjugacy class of the Baby Monster by [ATLAS]. Therefore, every simple $c = 7/10$ Virasoro vector of $\sigma$-type defines a $2A$-involution of the Baby Monster.

It is easy to see that $\sigma_g v = g \sigma_v g^{-1}$ for any $g \in \text{Aut}(V^\natural)$, i.e., the assignment is equivariant. Similar as discussed in the proof of Theorem 5.9 it follows that the assignment is surjective.

It remains to show the injectivity. The idea of the following argument is taken from [Hö2]. Let $v$ be a simple $c = 7/10$ Virasoro vector of $V^\natural$ of $\sigma$-type. Then $\sigma_v$ is a $2A$-element of the Baby Monster and there exists by Theorem 5.9 a unique derived $c = 7/10$ Virasoro vector $f \in (V^\natural)^{\mathcal{C}_2(e)}$ such that $\sigma_v = \sigma_f$. We will prove that $v = f$.

$^5$It is shown in [Hö3] that $V^\natural$ is also of class $S^7$ in the sense of Matsuo [Ma1].
Let $V\mathbb{B}_2^k = X^+ \oplus X^-$ be the eigenspace decomposition such that $\sigma_f$ acts by $\pm 1$ on $X^\pm$. Then $X^+ = 1 \oplus 1938 \oplus 48620$ and $X^- = 45696$ as $C_8(\sigma_f)$-modules by (5.3). It is clear that $v \in X^+$ and $\sigma_v$ also acts by $\pm 1$ on $X^\pm$. For $\rho \in \text{End}(X^-)$ and $k \in C_8(\sigma_v)$, we define $k\rho := k\rho k^{-1}$. Then $\text{End}(X^-)$ becomes a left $C_8(\sigma_v)$-module. Define $\mu : X^+ \to \text{End}(X^-)$ by $\mu(a)x = a(1)x$ for $a \in X^+$ and $x \in X^-$. Then for $k \in C_8(\sigma_f)$ we have

$$\mu(ka)x = (ka)(1)x = (ka)(1)(kk^{-1}x) = k(a(1)(k^{-1}x)) = k\mu(a)k^{-1}x = (k\mu)(a)x$$

so that $\mu$ is a $C_8(\sigma_f)$-homomorphism. Let us consider the $C_8(\sigma_f)$-submodule $\mu(X^+)$. As we will see explicitly in Section 5.3, the 1938 and 48620-dimensional components of $X^+$ act non-trivially on $X^-$ via $\mu$. (See Remark 5.19 below.) The $(1 + 1)$-dimensional component is spanned by the simple $c = 7/10$ Virasoro vector $f$ and the conformal vector $\omega$ of $V\mathbb{B}_2^k$, and $\mu(f)$ and $\mu(\omega)$ act on $X^-$ by scalars $1/10$ and $2$, respectively. Therefore, $\ker_{X^+} \mu = 1$ and $\mu(X^+) = 1 \oplus 1938 \oplus 48620$ as $C_8(\sigma_f)$-modules. Since $\mu(v)$ also acts by $1/10$ on $X^-$, we see that $f - v \in \ker_{X^+} \mu$. Hence, $v \in (V\mathbb{B}_2^k)^{C_8(\sigma_f)}$, and it follows $v = f$ since $f$ is the unique simple $c = 7/10$ Virasoro vector in $(V\mathbb{B}_2^k)^{C_8(\sigma_f)}$. This shows that the association $v \mapsto \sigma_v$ is injective and we have established the desired bijection.

**Corollary 5.14.** Every simple $c = 7/10$ Virasoro vector of $\text{Com}_{1/2}(\text{Vir}(e))$ of $\sigma$-type is also a derived Virasoro vector with respect to $e \in V^\natural$.

We have seen in Lemma 3.6 that a derived $c = 7/10$ Virasoro vector is of $\sigma$-type and therefore both notions coincide in the case of $V\mathbb{B}_2^k$.

### 5.3 Embedding of $U_{B(n,X)}$ into $V\mathbb{B}_2^k$

Consider the $E_8$-lattice. We fix an embedding $E_7 \subset E_8$. Then $Q(E_7) = \text{Ann}_{E_8}(E_7) \simeq A_1$ and we have a full rank sublattice $Q(E_7) \oplus E_7 \simeq A_1 \oplus E_7$ of $E_8$. Since the index $[E_8 : A_1 \oplus E_7]$ is 2, we have a coset decomposition

$$E_8 = A_1 \oplus E_7 \uplus (\delta + A_1 \oplus E_7)$$

with some $\delta \in E_8$. Correspondingly, we obtain a decomposition

$$V\sqrt{2}E_8 = V\sqrt{2}(A_1 \oplus E_7) \oplus V\sqrt{2}(\delta + A_1 \oplus E_7).$$

Define $\eta \in \text{Aut}(V\sqrt{2}E_8)$ by

$$\eta = \text{id} \quad \text{on} \quad V\sqrt{2}(A_1 \oplus E_7), \quad \eta = -1 \quad \text{on} \quad V\sqrt{2}(\delta + A_1 \oplus E_7).$$

Then $\eta$ is clearly in $\mathcal{F}_{nX}$, where $\mathcal{F}_{nX}$ is the renamed object $\mathcal{F}(i)$ which is defined in (4.9). Indeed, $\mathcal{F}_{nX}$ is generated by $\eta$ and $\rho_{nX}$. Note that we can write down $\rho_{nX}$ in exponential form

$$\rho_{nX} = \exp(2\pi\sqrt{-1}\gamma_{nX}^X/n) \quad \text{with suitable } \gamma_{nX}^X \in L^*_{nX},$$
which also defines an automorphism of $V_{\sqrt{3}E_8}$ and fixes $V_{\sqrt{3}L_n X}$ pointwisely.

Remark 5.15. Recall that $\tilde{\omega}_{Q(E_7)}$ defined as in (4.3) with $Q(E_7) \simeq A_1$ is an Ising vector and $U_{B(nX)}$ equals the commutant subalgebra $\text{Com}_{\Vir_{B(nX)}}(\Vir(\tilde{\omega}_{Q(E_7)}))$ (cf. Remark 4.1). Note also that $\rho_{nX}$ fixes $\tilde{\omega}_{Q(E_7)}$.

Now let $U_1 = \langle \hat{\epsilon}, \tilde{\omega}_{Q(E_7)} \rangle$ be the subalgebra generated by $\hat{\epsilon}$ (cf. (4.10) and $\tilde{\omega}_{Q(E_7)}$) and $U_2 = \rho_{nX}(U_1)$. Then $U_1 \simeq U_2 \simeq U_{2A}$ and we also have

$$\Vir(\hat{f}) = \text{Com}_{U_1}(\Vir(\tilde{\omega}_{Q(E_7)})),$$

$$\Vir(\rho_{nX}\hat{f}) = \text{Com}_{U_2}(\Vir(\rho_{nX}(\tilde{\omega}_{Q(E_7)}))) = \text{Com}_{U_2}(\Vir(\tilde{\omega}_{Q(E_7)})).$$

Therefore, the simple $c = 7/10$ Virasoro vectors $\tilde{f} = \tilde{\omega}_{E_7}$ and $\tilde{f}' = \rho_{nX}\tilde{f}$ of $U_{B(nX)}$ are derived Virasoro vectors with respect to $\tilde{\omega}_{Q(E_7)}$.

**Proposition 5.16.** For any $nX = 1A, 2B, 2C, 3A$, or $4B$, the VOA $V_{B(nX)}$ can be embedded into the Moonshine VOA $V^2$.

**Proof:** First, we will note that $V_{B(1A)} \simeq U_{2A}$, $V_{B(2C)} \simeq U_{4B}$, $V_{B(3A)} \simeq U_{6A}$. Since it was shown in [LM] that $U_{2A}, U_{4B}$ and $U_{6A}$ can be embedded into $V^2$, we have $V_{B(nX)} \subset V^2$ if $nX = 1A, 2C, 3A$. Moreover, $V_{B(2B)} \simeq V^+_{\sqrt{2}E_8}$. Hence $V_{B(2B)}$ is also contained in $V^+_N \subset V^2$ since $\sqrt{2}A_2$ is clearly contained in the Leech lattice $\Lambda$.

Recall that the Leech lattice $\Lambda$ is generated by elements of the form [CS, Chapter 4] $e_i \pm e_j$ with $i, j \in \Omega$, $\frac{1}{4}e_\Omega - e_1$, and $\frac{1}{2}e_X$, where $e_i = \frac{1}{\sqrt{8}}(0, \ldots, 4, \ldots, 0)$, $e_X = \sum_{i \in X} e_i$ for $X$ a vector in the Golay code $G_{24}$, and $\Omega = \{1, \ldots, 24\}$. Thus, $\Lambda$ contains a sublattice isomorphic to $N$, for example, the sublattice generated by

$$\frac{1}{\sqrt{8}}(4, 0^7, -4, 0^7, 0^8), \quad \frac{1}{\sqrt{8}}(0^8, 4, 0^7, -4, 0^7), \quad \frac{1}{\sqrt{8}}(-2^4, 0^4, 2^4, 0^4, 0^8).$$

Here, we will arrange the coordinates such that $(1^4, 0^4, 1^4, 0^4, 0^4) \in G_{24}$. Therefore, $V_{B(1B)} \simeq V^+_N$ is also contained in $V^2$.

**Remark 5.17.** In [GL1] (see also [GL2]), the possible configurations for a pair of $\sqrt{2}E_8$ sublattices $(M, N)$ in a rootless integral lattice have been determined. There are exactly 11 such configurations and 10 of them can be embedded into the Leech lattice. By using these embeddings, one can also obtain explicit embeddings of $U_{nX}, nX = 1A, 2A, \ldots, 3C$, into $V^+_N \subset V^2$.

**Theorem 5.18.** Let $e$ be an Ising vector in $V^2$. Then for any $nX = 1A, 2B, 2C, 3A$, or $4B$, the VOA $U_{B(nX)}$ can be embedded into $VB^2 = \text{Com}_{V^2}(\Vir(e))$. Moreover, $\sigma \sigma \tilde{f}$ belongs to the class $nX$ of $B = \text{Aut}(VB^2)$.
Proof: By Proposition 5.16, we can embed $V_{B(nX)}$ into $V^2$. Since all Ising vectors of $V^2$ are conjugate under the action of $\text{Aut}(V^2)$, we may identify $\tilde{\omega}_{Q(E^i)}$ with $e$. Thus, we have

$$U_{B(nX)} = \text{Com}_{V_{nX}}(\text{Vir}(\tilde{\omega}_{Q(E^i)})) \subset \text{Com}_{V^2}(\text{Vir}(e)) = VB^2$$

as desired.

Set $h := \sigma_f \sigma_{f'}$. We will show that $h$ belongs to the class $nX$ of $\mathbb{B}$. Recall that there is an exact sequence

$$1 \longrightarrow \langle \tau_e \rangle \longrightarrow C_M(\tau_e) \longrightarrow \text{Aut}(VB^2) \simeq \mathbb{B} \longrightarrow 1$$

with the projection map $\varphi_e : C_M(\tau_e) \to \text{Aut}(VB^2)$. Let $e^1$ and $e^2$ be Ising vectors in $V_{B(nX)}$ such that $\varphi_e(\tau_{e^1}) = \sigma_f$ and $\varphi_e(\tau_{e^2}) = \sigma_{f'}$. Set $g = \tau_{e^1} \tau_{e^2}$. Then $h = \varphi_e(g)$ and the inverse image $\varphi_e^{-1}(h)$ has order $2n$ and is generated by $\tau_e$ and $g$.

1A case: In this case, $\tilde{f} = f^2$ and hence $h = \sigma_f \sigma_{f'}$ belongs to the class 1A.

2B case: In this case, $V_{B(2B)} \simeq V^2_{7 \sqrt{A}_2}$. Recall that $V^2_{7 \sqrt{A}_2}$ has exactly 6 Ising vectors (cf. [KMY, LSY]). The group generated by the corresponding $\tau$-involutions is an elementary abelian group of order 8, which is not 2A-pure in $\mathbb{M}$ since there exists a pair of mutually orthogonal Ising vectors (cf. Lemma 3.9 and Remark 3.10). Therefore, $g = \tau_{e^1} \tau_{e^2}$ has order 2 and the group generated by $\tau_e$ and $g$ is a Klein’s 4-group, which is not 2A-pure in $\mathbb{M}$. By using [ATLAS], it is easy to show that $h = \varphi_e(g)$ belongs to the conjugacy class 2B of $\mathbb{B}$.

3A case: In this case, $V_{B(3A)} \simeq U_{6A}$ and the group generated by $\tau_e, \tau_{e^1}, \tau_{e^2}$ is a dihedral group of order 12 and $\tau_e$ is in the center. Thus, $g = \tau_{e^1} \tau_{e^2}$ has order 3 or 6. The group generated by $\tau_e$ and $g$ is then a cyclic group of order 6 which is generated by a 6A-element of $\mathbb{M}$. By using [ATLAS], one can deduce that $h = \varphi_e(g)$ belongs to the conjugacy class 3A of $\mathbb{B}$.

4B case: In this case, $\tau_e, \tau_{e^1}$ and $\tau_{e^2}$ generate a subgroup isomorphic to a direct product of a cyclic group of order 2 and a dihedral group of order 8 with $\tau_e$ in the center. By the sublattice structure

$$E_8 \supset D_5 \supset A_3 \supset A_1 \oplus A_1 \oplus A_3 \oplus A_3 \simeq \tilde{L}_{4B},$$

it is clear that $U_{4A} \subset V_{B(4B)}$. Thus, $e^1, e^2$ generate a sub-VOA isomorphic to $U_{4A}$ and $g = \tau_{e^1} \tau_{e^2}$ belongs to the conjugacy class 4A of $\mathbb{M}$. Hence, by using [ATLAS], one can deduce that $h = \varphi_e(g)$ belongs to the conjugacy class 4B of $\mathbb{B}$.

2C case: In this case, $V_{B(2C)} \simeq U_{4B}$. The group generated by $\tau_e$ and $g$ is a cyclic group of order 4, which is generated by a 4B-element of $\mathbb{M}$. Now by using [ATLAS], one can deduce that $h = \varphi_e(g)$ belongs to the conjugacy class 2C of $\mathbb{B}$. 

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Remark 5.19. By Theorem 5.18, we can verify that the 1938 and 48620-dimensional components of $X^+$ discussed in the proof of Theorem 5.13 act on $X^-$ non-trivially. Here we recover the convention as in the proof of Theorem 5.13. The simple $c = 7/10$ Virasoro vector $f$ acts on the 1938 and 48620-dimensional components by $3/5$ and 0, respectively, and acts on $X^-$ by $1/10$. Thanks to Theorem 5.18, we can identify $f$ with $\tilde{f} \in U_{B(4B)}$, up to conjugacy. In the Griess algebra $G_{B(4B)}$, $\tilde{f}$ has a 2-dimensional eigenspace for the eigenvalue 0, a 1-dimensional eigenspace for the eigenvalue $3/5$ and a 1-dimensional eigenspace for the eigenvalue $1/10$. One can directly check that these eigenspaces act non-trivially on the $1/10$-eigenspace. Therefore, by the embedding in Theorem 5.18, we see that the corresponding eigensubspaces of $X^+$ act non-trivially on $X^-$.

A Appendix: The 6A-algebra

In this Appendix we describe the 6A-algebra $U_{6A}$ explicitly.

Let $e^0$, $e^1$ be Ising vectors of $V^\natural$ such that $|\tau e^0 \tau e^1| = 6$. Then it is known [ATLAS] that $\tau e^0 \tau e^1$ belongs to the 6A-conjugacy class of the Monster.

Let $U_{6A}$ be the subalgebra generated by $e^0$ and $e^1$. The subalgebra $U_{6A}$ is called the 6A-algebra for the Monster and its structure is well-studied in [LYY2]. The Griess algebra of $U_{6A}$ is 8-dimensional and its structure with respect to a specific linear basis $\{\omega^1, \omega^2, \omega^3, X^1, X^2, X^3, X^4, X^5\}$ is as follows (cf. [LYY2], see also [C]):

$$
\begin{array}{c|cccccccc}
 a_{(1)b} & \omega^1 & \omega^2 & \omega^3 & X^1 & X^2 & X^3 & X^4 & X^5 \\
\hline
\omega^1 & 2\omega^1 & 0 & 0 & \frac{1}{2}X^1 & 0 & \frac{1}{2}X^3 & 0 & \frac{1}{2}X^5 \\
\omega^2 & 2\omega^2 & 0 & \frac{2}{3}X^1 & \frac{2}{3}X^2 & 0 & \frac{2}{3}X^4 & 0 & \frac{2}{3}X^5 \\
\omega^3 & 2\omega^3 & \frac{5}{6}X^1 & \frac{4}{3}X^2 & \frac{3}{2}X^3 & \frac{4}{3}X^4 & \frac{5}{6}X^5 \\
X^1 & 8X^2 & 9X^3 & 8X^4 & 10X^5 & 72\omega^1 + 60\omega^2 + 48\omega^3 \\
X^2 & 12X^4 & 10X^5 & 75\omega^2 + 96\omega^3 & 10X^1 \\
X^3 & & & & 80\omega^1 + 96\omega^3 & 10X^1 & 8X^2 \\
X^4 & & & & 12X^2 & 9X^3 \\
X^5 & & & & & & 8X^4 \\
\end{array}
$$

Thus, $\omega^i$, $i = 1, 2, 3$, are mutually orthogonal Virasoro vectors. The non-trivial linear pairings between these vectors are

$$
\langle \omega^1, \omega^1 \rangle = 1/4, \quad \langle \omega^2, \omega^2 \rangle = 2/5, \quad \langle \omega^3, \omega^3 \rangle = 5/8, \\
\langle X^1, X^5 \rangle = 36, \quad \langle X^2, X^4 \rangle = 45, \quad \langle X^3, X^3 \rangle = 40.
$$

39
The conformal vector of $U_{6A}$ is given by $\omega = \omega^1 + \omega^2 + \omega^3$. We can define a canonical $\mathbb{Z}_6$-symmetry as follows.

\[ \zeta : \omega^i \mapsto \omega^i, \quad 1 \leq i \leq 3, \quad X^j \mapsto e^{\pi j \sqrt{-1/3}} X^j, \quad 1 \leq j \leq 5. \]

**Lemma A.1.** There are exactly seven Ising vectors in $U_{6A}$, namely, $\omega^1$ and $e^j := \zeta^j e^0$, $0 \leq j \leq 5$, where

\[ e^0 = \frac{1}{8} \omega^1 + \frac{5}{32} \omega^2 + \frac{1}{4} \omega^3 + \frac{1}{32} (X^1 + X^2 + X^3 + X^4 + X^5). \]

Moreover, the inner products among these Ising vectors are

\[ \langle \omega^1, e^i \rangle = \frac{1}{32}, \quad \langle e^i, e^j \rangle = \begin{cases} 5/2^{10}, & \text{if } i - j \equiv \pm1 \pmod{6}, \\ 13/2^{10}, & \text{if } i - j \equiv \pm2 \pmod{6}, \\ 1/32, & \text{if } i - j \equiv 3 \pmod{6}. \end{cases} \quad (A.1) \]

By a straightforward computation, we see that $\omega^1 \in U^\tau e^0_{6A}$ and we have the following permutation representation

\[ \tau_{e^0} = (15)(24), \quad \tau_{e^1} = (02)(35) \quad (A.2) \]

on $\{e^j \mid 0 \leq j \leq 5\} \subset U_{6A}$. From this we also find that $\tau_{e^0} \tau_{e^1}$ coincides with $\zeta$ on $U_{6A}$. We also calculate that

\[ \omega^1 = e^i + e^{i+3} - 4e^{(1)i} e^{i+3}, \quad e^i = \omega^1 + e^{i+3} - 4\omega^{(1)} e^{i+3}, \quad 0 \leq i \leq 5. \quad (A.3) \]

From this we also see that the subalgebra generated by $\{\omega^1, e^i, e^{i+3}\}$ is isomorphic to the $2A$-algebra $U_{2A}$ discussed in Section 3.1. Therefore, by Theorem 3.4, we have

\[ \tau_{\omega^1} \tau_{e^i} = \tau_{e^{i+3}}, \quad \tau_{e^i} \tau_{e^{i+3}} = \tau_{\omega^1}. \quad (A.4) \]

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McKay’s $E_6$ observation on the largest Fischer group

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Abstract

In this paper, we study McKay’s $E_6$-observation on the largest Fischer 3-transposition group $Fi_{24}$. We investigate a vertex operator algebra $VF$ of central charge $23\frac{1}{5}$ on which the Fischer group $Fi_{24}$ naturally acts. We show that there is a natural correspondence between dihedral subgroups of $Fi_{24}$ and certain vertex operator subalgebras constructed by the nodes of the affine $E_6$ diagram by investigating so-called derived Virasoro vectors of central charge $6/7$. This allows us to reinterpret McKay’s $E_6$-observation via the theory of vertex operator algebras.

It is also shown that the product of two non-commuting Miyamoto involutions of $\sigma$-type associated to derived $c = 6/7$ Virasoro vectors is an element of order 3, under certain general hypotheses on the vertex operator algebra. For the case of $VF$, we identify these involutions with the 3-transpositions of the Fischer group $Fi_{24}$.

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1 Introduction

This article is a continuation of our previous work [LM, LYY1, HLY] to give a vertex operator algebra (VOA) theoretical interpretation of McKay’s intriguing observations that relate the Monster, the Baby Monster and the largest Fischer 3-transposition group to the affine $E_8$, $E_7$ and $E_6$ Dynkin diagrams. In this article, we will study the $E_6$-observation recalled below. Our approach here is similar to [HLY], in which the $E_7$-observation is studied, but other vertex operator algebras are involved and many technical details are different.

The largest Fischer group. The largest Fischer group $Fi_{24}$ was discovered by B. Fischer [F] as a group of order $2^{22} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$ containing a conjugacy class of 306,936 involutions, which satisfy the 3-transposition property, i.e., any non-commuting pair has product of order 3. The group $Fi_{24}$ contains as subgroup of index 2 the derived
group $F_{24}'$ which is the third largest of the 26 sporadic groups. An extension $3.F_{24}$ of the Fischer group $F_{24}$ by a cyclic group of order 3 is the normalizer of a 3A-element of the Monster, the largest of the sporadic groups. In fact, one can construct $F_{24}$ from the Monster [G1] and derive its 3-transposition property. One of the main motivations of this article is to study the Fischer group $F_{24}$ and to understand the $E_6$ case of McKay's observations by using the theory of vertex operator algebra.

There exists a class of vertex operator algebras which is closely related to 3-transposition groups [G3, KM, Mi1, Ma2]. Let $V$ be vertex operator algebra which has a simple $c = 1/2$ Virasoro vector $e$ of $\sigma$-type, that is $V = V_e[0] \oplus V_e[1/2]$ where $V_e[h]$ denotes the sum of irreducible $\text{Vir}(e)$-submodules of $V$ isomorphic to the irreducible highest weight representation $L(t/2, h)$ of the $c = 1/2$ Virasoro algebra with highest weight $h$ (cf. Section 2). Then one can define an involutive automorphism

$$
\sigma_e = \begin{cases} 
1 & \text{on } V_e[0], \\
-1 & \text{on } V_e[1/2] 
\end{cases}
$$

usually called a $\sigma$-involution if $\sigma_e \neq \text{id}_V$ [Mi1]. It was shown by Miyamoto [Mi1] that if the weight one subspace of a VOA is trivial, then a collection of involutions associated to $c = 1/2$ Virasoro vectors of $\sigma$-type generates a 3-transposition group. Many interesting examples of 3-transposition groups obtained by $\sigma$-type $c = 1/2$ Virasoro vectors have been studied in [G3, KM] and the complete classification is established in [Ma2]. According to [Ma2], all 3-transposition groups realized by $\sigma$-type $c = 1/2$ Virasoro vectors are so-called symplectic type (cf. [CH2]), and as a result, the Fischer 3-transposition group cannot be obtained by $c = 1/2$ Virasoro vectors of $\sigma$-type.

Another result on $c = 1/2$ Virasoro vectors was obtained in [S] where it was shown that the so-called $\tau$-involutions (cf. Theorem 2.2) associated to such vectors generate a 6-transposition group provided the weight one subspace of the VOA is trivial.

In this paper, we introduce a new idea to obtain 3-transposition groups as automorphism groups of vertex operator algebras. We use the so-called 3A-algebra for the Monster (see [Mi3, LYY2, SY] and Section 4) and consider derived $c = 6/7$ Virasoro vectors to define involutive automorphisms of vertex operator algebras. We will show that a collection of involutions associated to derived $c = 6/7$ Virasoro vectors generates a 3-transposition group. The advantage of our method is that we can realize the largest Fischer 3-transposition group as an automorphism subgroup of a special vertex operator algebra $VF_{5}$ explained below. This result enables us to study the Fischer 3-transposition groups via the theory of vertex operator algebras.

**The Fischer group VOA $VF_{5}$.** We will investigate a certain vertex operator algebra $VF_{5}$ of central charge $23\frac{4}{5}$ on which the Fischer group $F_{24}$ naturally acts.
The Monster is the automorphism group of the Moonshine vertex operator algebra $V^\#$ (cf. [FLM, B]). Let $g$ be a 3A-element of the Monster $M$. Then the normalizer $N_M(g)$ is isomorphic to $3\cdot\Fi_{24}$ and acts on $V^\#$. A character theoretical consideration in [C, MeN] indicates that the centralizer $C_M(g) \simeq 3\cdot\Fi_{24}'$ fixes a unique $c = 4/5$ Virasoro vector in $V^\#$. We will show in Theorem 5.1 that $C_M(g)$ actually fixes a unique $c = 4/5$ extended Virasoro vertex operator algebra $W \simeq W(4/5) = L(4/5, 0) \oplus L(4/5, 3)$ in $V^\#$. Let

$$VF^\# = \Com_{V^\#}(W),$$

where $\Com_V(U)$ denotes the commutant subalgebra of $U$ in $V$ (see (3.1) and (5.4) for the precise definition), and we call $VF^\#$ the Fischer group VOA. A simple observation shows that $N_M(g)$ acts naturally on $VF^\# = \Com_{V^\#}(W)$. In fact, we will show that the Fischer group $\Fi_{24}$ can be realized as a subgroup of $\Aut(VF^\#)$.

**Main Theorem 1 (Theorem 5.5).** The automorphism group $\Aut(VF^\#)$ of $VF^\#$ contains $\Fi_{24}$ as a subgroup. Moreover, let $X$ be the full-subalgebra of $VF^\#$ generated by its weight 2 subspace. Then $\Aut(X) \simeq \Fi_{24}$.

Although we only show that $\Fi_{24}$ equals the automorphism group of the Griess algebra of $VF^\#$, we expect that $\Aut(VF^\#)$ is exactly $\Fi_{24}$ and therefore $VF^\#$ would provide a VOA model for studying the Fischer group $\Fi_{24}$.

Since $N_M(g) \simeq 3\cdot\Fi_{24}$ for any 3A-element $g$ of the Monster, the study of the 3-transpositions in $\Fi_{24}$ leads to the study of dihedral subalgebras $U_{3A}$ of type 3A in $V^\#$ (cf. [Mi3, LYY2, SY]). The 3A-algebra $U_{3A}$ contains a unique extended $c = 4/5$ Virasoro sub-VOA $W = W(4/5)$ and the corresponding commutant subalgebra $\Com_{U_{3A}}(W)$ in $U_{3A}$ (cf. [SY]) gives rise to certain $c = 6/7$ Virasoro vectors in $VF^\#$, which we call derived Virasoro vectors (cf. Definition 4.6). Note that $W \subset U_{3A} \subset V^\#$ implies $VF^\# = \Com_{V^\#}(W) \supset \Com_{U_{3A}}(W)$. It is interesting that a natural construction in [DLMN] suggests $c = 6/7$ Virasoro vectors in the case of $E_6$; cf. the discussion below. Motivated by the above observation, we first study subalgebras $U_{3A}$ of any vertex operator algebra $V$ and we show that one can canonically associate involutive automorphisms to derived $c = 6/7$ Virasoro vectors in $\Com_V(W)$, which we call Miyamoto involutions (see Lemma 2.4 and Eq. (2.4)). We will show that the collection of Miyamoto involutions associated to derived $c = 6/7$ Virasoro vectors generates a 3-transposition group.

We say that a VOA $W$ over $\mathbb{R}$ is compact if $W$ has a positive definite invariant bilinear form. A real sub-VOA $W$ is said to be a compact real form of a VOA $V$ over $\mathbb{C}$ if $W$ is compact and $V \simeq \mathbb{C} \otimes_{\mathbb{R}} W$.

**Main Theorem 2 (Theorem 4.10).** Let $V = \bigoplus_{n \geq 0} V_n$ be a VOA. Suppose that $\dim V_0 = 1$, $V_1 = 0$ and $V$ has a compact real form $V_\mathbb{R}$ and every simple $c = 1/2$ Virasoro vector of $V$
is in $V_2$. Then the Miyamoto involutions associated to derived $c = 6/7$ Virasoro vectors in the commutant subalgebra $\text{Com}_V(W(4/5))$ satisfy a 3-transposition property.

The Moonshine vertex operator algebra satisfies the assumption of the theorem above and we recover in a general fashion the 3-transposition property of the largest Fischer group via the commutant subalgebra $\text{VF}^\# = \text{Com}_{\text{VF}}(W(4/5))$ (see Corollary 5.12). Indicated by the fact that 3-transpositions are induced by $c = 6/7$ derived Virasoro vectors, we will also prove the following one-to-one correspondence.

**Main Theorem 3** (Theorem 5.10). There exists a one-to-one correspondence between 3-transpositions of the Fischer group and derived $c = 6/7$ Virasoro vectors in $\text{VF}^\#$ via Miyamoto involutions.

This theorem provides a link for studying the 3-transpositions of $\text{Fi}_{24}$ by using the VOA $\text{VF}^\#$ and it is possible to relate McKay’s $E_6$-observation of $\text{Fi}_{24}$ to the theory of vertex operator algebra as discussed below.

**McKay’s observation.** We are interested in the $E_6$ case of the observation of McKay which relates the Monster group and some sporadic groups involved in the Monster to the affine Dynkin diagrams of types $E_6$, $E_7$ and $E_8$ [Mc].

McKay’s observation in the case of the largest Fischer group says that the orders of the products of any two 3-transpositions of $\text{Fi}_{24}$ belongs to one of the conjugacy classes $1A$, $2A$ or $3A$ of $\text{Fi}_{24}$ such that these conjugacy classes coincide with the numerical labels of the nodes in an affine $E_6$ Dynkin diagram and there is a correspondence as follows:

```
1A

2A

1A 2A 3A 2A 1A
```

This correspondence is not one-to-one but only up to diagram automorphisms.

For the understanding of the $E_8$-case [LM, LYY1, LYY2], the main foothold is the one-to-one correspondence between $2A$-involutions of the Monster and simple $c = 1/2$ Virasoro vectors in $V^\#$ by which one can translate McKay’s $E_8$-observation into a purely vertex operator algebra theoretical problem. For the $E_6$ case, we have a nice correspondence as in Main Theorem 3 and we can also translate the $E_6$-observation into a problem of vertex operator algebras. Based on this correspondence and making use of the 3A-algebra
which is generated by two \( c = 1/2 \) Virasoro vectors, we show that there is a natural connection between dihedral subgroups of \( \text{Fi}_{24} \) and certain sub-VOAs constructed by the nodes of the affine \( E_6 \) diagram, which gives some context of McKay’s observation in terms of vertex operator algebras.

More precisely, let \( S \) be a simple root lattice with a simply laced root system \( \Phi(S) \). We scale \( S \) such that the roots have squared length 2. Let \( V_{\sqrt{2}S} \) be the lattice VOA associated to \( \sqrt{2}S \). Here and further we use the standard notation for lattice VOAs as in [FLM]. In [DLMN] Dong et al. constructed a Virasoro vector of \( V_{\sqrt{2}S} \) of the form

\[
\tilde{\omega}_S := \frac{1}{2h(h+2)} \sum_{\alpha \in \Phi(S)} \alpha (-1)^2 1 + \frac{1}{h+2} \sum_{\alpha \in \Phi(S)} e^{\sqrt{2}\alpha},
\]

where \( h \) denotes the Coxeter number of \( S \). Recall that the central charge of \( \tilde{\omega}_S \) is 6/7 if \( S = E_6 \) [DLMN]. By the expression, it is clear that \( \tilde{\omega}_S \) is invariant under the Weyl group of \( \Phi(S) \).

Our approach to McKay’s observation is to find suitable pairs of derived \( c = 6/7 \) Virasoro vectors in \( V^\natural \) which inherits the \( E_6 \) structure in McKay’s \( E_6 \)-diagram. By using similar ideas as in [LYY1, LYY2], we construct a certain sub-VOA \( U_{\text{Fi}_{24}}(nX) \) of the lattice VOA \( V_{\sqrt{2}E_6} \) associated to each node \( nX \) of the affine \( E_6 \) diagram (cf. Section 3). Utilizing an embedding of the \( E_6 \) lattice into the \( E_8 \) lattice, we show that \( U_{\text{Fi}_{24}}(nX) \) is contained in the VOA \( V^\natural \) purely by their VOA structures (cf. Theorem 5.16 and Appendix A). These sub-VOAs contain pairs of derived \( c = 6/7 \) Virasoro vectors such that the corresponding Miyamoto involutions generate a dihedral group of type \( nX \) in \( \text{Fi}_{24} \subset \text{Aut}(V^\natural) \). Then, using the identification of \( \text{Fi}_{24} \) as a subgroup of \( \text{Aut}(V^\natural) \), we obtain another main result with the help of the Atlas [ATLAS].

**Main Theorem 4** (Theorem 5.16). For any of the cases \( nX = 1A, 2A \) or \( 3A \), the VOA \( U_{\text{Fi}_{24}}(nX) \) can be embedded into \( V^\natural \). Moreover, \( \sigma_{\tilde{v}}\sigma_{\tilde{v}'} \) belongs to the conjugacy class \( nX \) of \( \text{Fi}_{24} = \text{Aut}(X) \), where \( \tilde{v} \) and \( \tilde{v}' \) are defined as in (3.16).

In this way, our embeddings of \( U_{\text{Fi}_{24}}(nX) \) into \( V^\natural \) encode the \( E_6 \) structure into \( V^\natural \) which are compatible with the original McKay observations.

The organization of the paper. The organization of this article is as follows: In Section 2, we review basic properties about Virasoro VOAs and Virasoro vectors.

In Section 3, we recall the definition of commutant sub-VOAs and define certain commutant subalgebras associated to the root lattice of type \( E_6 \) using the method described in [LYY1, LYY2].
In Section 4, we study a vertex operator algebra $U_{3A}$, which we call the 3A-algebra for the Monster, and prove a 3-transposition property for Miyamoto involutions associated to derived $c = 6/7$ Virasoro vectors on commutant subalgebras of VOAs containing $U_{3A}$.

In Section 5, the commutant subalgebra $VF^3$ of $W(4/5)$ in $V^\natural$ is studied. The sub-VOA $VF^3$ has a natural faithful action of the Fischer group $Fi_{24}$. We expect that the full automorphism group of $VF^3$ is $Fi_{24}$ but we cannot give a proof of this prospect. Instead, we will show a partial result that the full automorphism group of the sub-VOA $X$ generated by the weight two subspace of $VF^3$ is isomorphic to $Fi_{24}$. We also establish a one-to-one correspondence between $2C$-involutions of $Fi_{24}$ and derived $c = 6/7$ Virasoro vectors in $VF^3$.

Finally, we discuss the embeddings of the commutant subalgebras constructed in Section 3 into $VF^3$ in Section 5.3. We show that the $c = 6/7$ Virasoro vectors defined in Section 3.1 can be embedded into $VF^3$. Moreover, we verify that the product of the corresponding $\sigma$-involutions belongs to the conjugacy class associated to the node.

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**Notation and Terminology.** In this article, $N$, $Z$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of non-negative integers, integers, real and complex numbers, respectively. We denote the ring $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}_p$ with a positive integer $p$ and often identify the integers $0, 1, \ldots, p - 1$ with their images in $\mathbb{Z}_p$.

Every vertex operator algebra (VOA for short) is defined over the field $\mathbb{C}$ of complex numbers unless otherwise stated. A VOA $V$ is called of CFT-type if it is non-negatively graded $V = \bigoplus_{n \geq 0} V_n$ with $V_0 = \mathbb{C}1$. For a VOA structure $(V, Y(\cdot, z), 1, \omega)$ on $V$, the vector $\omega$ is called the conformal vector of $V$. For simplicity, we often use $(V, \omega)$ to denote the structure $(V, Y(\cdot, z), 1, \omega)$. The vertex operator $Y(a, z)$ of $a \in V$ is expanded as $Y(a, z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}$.

\footnote{The conformal vector of $V$ is often called the Virasoro element of $V$, e.g. [FLM]}
An element \( e \in V \) is referred to as a Virasoro vector of central charge \( c_e \in \mathbb{C} \) if \( e \in V_2 \) and it satisfies \( e(1)e = 2e \) and \( e(3)e = (c_e/2) \cdot 1 \). It is well-known that the associated modes \( L^e(n) := e(n+1), n \in \mathbb{Z} \), generate a representation of the Virasoro algebra on \( V \) (cf. [Mi1]), i.e., they satisfy the commutator relation

\[
[L^e(m), L^e(n)] = (m - n)L^e(m + n) + \delta_{m,n,0} \frac{m^3 - m}{12} c_e.
\]

Therefore, a Virasoro vector together with the vacuum vector generates a Virasoro VOA inside \( V \). We will denote this subalgebra by \( \text{Vir}(e) \).

In this paper, we define a sub-VOA of \( V \) to be a pair \((U, e)\) consisting of a subalgebra \( U \) containing the vacuum element \( 1 \) and a conformal vector \( e \) for \( U \) such that \((U, e)\) inherits the grading of \( V \), that is, \( U = \bigoplus_{n \geq 0} U_n \) with \( U_n = V_n \cap U \), but \( e \) may not be the conformal vector of \( V \). In the case that \( e \) is also the conformal vector of \( V \), we will call the sub-VOA \((U, e)\) a full sub-VOA.

For a positive definite even lattice \( L \), we will denote the lattice VOA associated to \( L \) by \( V_L \) (cf. [FLM]). We adopt the standard notation for \( V_L \) as in [FLM]. In particular, \( V_L^+ \) denotes the fixed point subalgebra of \( V_L \) under a lift of the \((-1)\)-isometry on \( L \). The letter \( \Lambda \) always denotes the Leech lattice, the unique even unimodular lattice of rank 24 without roots.

Given a group \( G \) of automorphisms of \( V \), we denote by \( V^G \) the fixed point subalgebra of \( G \) in \( V \). The subalgebra \( V^G \) is called the \( G \)-orbifold of \( V \) in the literature. For a \( V \)-module \((M, Y_M(\cdot, z))\) and \( \sigma \in \text{Aut}(V) \), we set \( \sigma Y_M(a, z) := Y_M(\sigma^{-1}a, z) \) for \( a \in V \). Then the \( \sigma \)-conjugated module \( \sigma \circ M \) of \( M \) is defined to be the module structure \((M, \sigma Y_M(\cdot, z))\).

## 2 Virasoro vertex operator algebras and their extensions

For complex numbers \( c \) and \( h \), we denote by \( L(c, h) \) the irreducible highest weight representation of the Virasoro algebra with central charge \( c \) and highest weight \( h \). It is shown in [FZ] that \( L(c, 0) \) has a natural structure of a simple VOA.

### 2.1 Unitary Virasoro vertex operator algebras

Let

\[
\begin{align*}
    c_m &:= 1 - \frac{6}{(m + 2)(m + 3)}, \quad m = 1, 2, \ldots, \\
    h_{r,s}^{(m)} &:= \frac{(r(m + 3) - s(m + 2))^2 - 1}{4(m + 2)(m + 3)}, \quad 1 \leq s \leq r \leq m + 1.
\end{align*}
\]
It is shown in [W] that $L(c_m, 0)$ is rational and $L(c_m, h_{r,s}^{(m)})$, $1 \leq s \leq r \leq m + 1$, provide all irreducible $L(c_m, 0)$-modules (see also [DMZ]). This is the so-called unitary series of the Virasoro VOAs. The fusion rules among $L(c_m, 0)$-modules are computed in [W] and given by

$$L(c_m, h_{r_1,s_1}^{(m)}) \times L(c_m, h_{r_2,s_2}^{(m)}) = \sum_{i \in I, j \in J} L(c_m, h_{i,j}^{(m)}),$$

(2.2)

where

$$I = \{1, 2, \ldots, \min\{r_1, r_2, m + 2 - r_1, m + 2 - r_2\}\},$$

$$J = \{1, 2, \ldots, \min\{s_1, s_2, m + 3 - s_1, m + 3 - s_2\}\}.$$

**Definition 2.1.** A Virasoro vector $e$ with central charge $c$ is called simple if $\text{Vir}(e) \simeq L(c, 0)$. A simple $c = 1/2$ Virasoro vector is called an Ising vector.

The fusion rules among $L(c_m, 0)$-modules have a canonical $\mathbb{Z}_2$-symmetry and this symmetry gives rise to an involutive vertex operator algebra automorphism which is known as Miyamoto involution.

**Theorem 2.2 ([Mi1]).** Let $V$ be a VOA and let $e \in V$ be a simple Virasoro vector with central charge $c_m$. Denote by $V_e[h_{r,s}^{(m)}]$ the sum of irreducible $\text{Vir}(e) = L(c_m, 0)$-submodules of $V$ isomorphic to $L(c_m, h_{r,s}^{(m)})$, $1 \leq s \leq r \leq m + 1$. Then the linear map

$$\tau_e := \begin{cases} (-1)^{r+1} \quad &\text{on } V_e[h_{r,s}^{(m)}] \quad \text{if } m \text{ is even}, \\ (-1)^{s+1} \quad &\text{on } V_e[h_{r,s}^{(m)}] \quad \text{if } m \text{ is odd}, \end{cases}$$

defines an automorphism of $V$ called the $\tau$-involution associated to $e$.

Next we introduce a notion of $\sigma$-type Virasoro vectors.

**Definition 2.3.** For $m = 1, 2, \ldots$, let

$$B^{(m)} := \begin{cases} \{h_{1,s}^{(m)} \mid 1 \leq s \leq m + 2\} &\text{if } m \text{ is even}, \\ \{h_{r,1}^{(m)} \mid 1 \leq r \leq m + 1\} &\text{if } m \text{ is odd}. \end{cases}$$

A simple Virasoro vector $e \in V$ with central charge $c_m$ is said to be of $\sigma$-type on $V$ if $V_e[h] = 0$ for all $h \notin B^{(m)}$.

By Eq. (2.2), the fusion rules among irreducible modules $L(c_m, h)$, $h \in B^{(m)}$, are relatively simple. Moreover, $B^{(m)}$ possesses a natural $\mathbb{Z}_2$-symmetry as follows.
**Lemma 2.4.** Let \( e \in V \) be a simple \( c = c_m \) Virasoro vector of \( \sigma \)-type. Then one has the isotypical decomposition

\[
V = \bigoplus_{h \in B^{(m)}} V_e[h]
\]

and the linear map \( \sigma_e \) given by

\[
\sigma_e := \begin{cases} 
(-1)^{s+1} & \text{on } V_e[h_{1,s}] \text{ if } m \text{ is even}, \\
(-1)^{r+1} & \text{on } V_e[h_{r,1}] \text{ if } m \text{ is odd}.
\end{cases}
\]

(2.3)

is an automorphism of \( V \).

We will call the map \( \sigma_e \) above the *Miyamoto involution* (of \( \sigma \)-type) associated to a simple \( c = c_m \) Virasoro vector \( e \) of \( \sigma \)-type.

**Remark 2.5.** (1) By Eq. (2.2), \( B^{(m)} \) is closed under the fusion product, i.e., if \( h, h' \in B^{(m)} \) then \( L(c_m, h) \times L(c_m, h') \) is a sum of irreducible modules with highest weights in \( B^{(m)} \). Therefore, the subspace \( W = \bigoplus_{h \in B^{(m)}} V_e[h] \) forms a sub-VOA of \( V \). Note that \( e \) is of \( \sigma \)-type on \( W \) and one can always define \( \sigma_e \) as an automorphism of \( W \).

(2) By definition, it is clear that \( \tau_e \) acts trivially on \( V_e[h] \) for all \( h \in B^{(m)} \). However, the fixed point sub-VOA of \( \tau_e \) on \( V \) is usually bigger than \( W = \bigoplus_{h \in B^{(m)}} V_e[h] \).

In this article, we will mainly consider the case \( c_4 = 6/7 \). In this case, \( B^{(4)} = \{0, 5, 1/7, 5/7, 12/7, 22/7\} \) and a simple \( c = 6/7 \) Virasoro vector \( e \in V \) is of \( \sigma \)-type on \( V \) if \( V_e[h] = 0 \) for \( h \neq 0, 5, 1/7, 5/7, 12/7, 22/7 \). The corresponding \( \sigma \)-involution is given by

\[
\sigma_e := \begin{cases} 
1 & \text{on } V_e[0] \oplus V_e[5/7] \oplus V_e[22/7], \\
-1 & \text{on } V_e[5] \oplus V_e[12/7] \oplus V_e[1/7].
\end{cases}
\]

(2.4)

We also need the following result:

**Lemma 2.6.** Let \( V \) be a VOA with grading \( V = \bigoplus_{n \geq 0} V_n \), \( V_0 = \mathbb{C}1 \) and \( V_1 = 0 \), and let \( u \in V \) be a Virasoro vector such that \( \text{Vir}(u) \simeq L(c_m, 0) \). Then the zero mode \( \sigma(u) = u_{(1)} \) acts on the Griess algebra of \( V \) semisimply with possible eigenvalues 2 and \( h_{r,s}^{(m)} \), \( 1 \leq s \leq r \leq m+1 \). Moreover, if \( h_{r,s}^{(m)} \neq 2 \) for \( 1 \leq s \leq r \leq m+1 \) then the eigenspace for the eigenvalue 2 is one-dimensional, namely, it is spanned by the Virasoro vector \( u \).

**Proof:** See Lemma 2.6 of [HLY].

### 2.2 Extended Virasoro vertex operator algebras

Among \( L(c_m, 0) \)-modules, only \( L(c_m, 0) \) and \( L(c_m, h_{m+1,1}^{(m)}) \) are simple currents, and it is shown in [LLY] that \( L(c_m, 0) \oplus L(c_m, h_{m+1,1}^{(m)}) \) forms a simple current extension of \( L(c_m, 0) \). Note that \( h_{m+1,1}^{(m)} = h_{1,m+2}^{(m)} = m(m+1)/4 \) is an integer if \( m \equiv 0, 3 \pmod{4} \) and a half-integer if \( m \equiv 1, 2 \pmod{4} \).
**Theorem 2.7 ([LLY]).** The \( \mathbb{Z}_2 \)-graded simple current extension

\[
W(c_m) := L(c_m, 0) \oplus L(c_m, h_{m+1,1}^{(m)})
\]

has a unique simple rational vertex operator algebra structure extending \( L(c_m, 0) \) if \( m \equiv 0, 3 \pmod{4} \), and a unique simple rational vertex operator superalgebra structure extending \( L(c_m, 0) \) if \( m \equiv 1, 2 \pmod{4} \).

By this theorem, we introduce the following notion.

**Definition 2.8.** Let \( m \equiv 0 \) or \( 3 \pmod{4} \). A simple \( c = c_m \) Virasoro vector \( u \) of a VOA \( V \) is called extendable if there exists a non-zero highest weight vector \( w \in V \) of weight \( h_{m+1,1}^{(m)} = m(m+1)/4 \) with respect to \( \text{Vir}(u) \) such that the subalgebra generated by \( u \) and \( w \) is isomorphic to the extended Virasoro VOA \( W(c_m) \). Note that \( w \) is a \( \text{Vir}(u) \)-primary vector, i.e., \( L^u(m)w = 0 \) for all \( m > 0 \), where \( L^u(m) = u_{(m+1)} \). We will call such a \( w \) an \( h_{m+1,1}^{(m)} \)-primary vector associated to \( u \).

**Lemma 2.9.** Let \( m \equiv 0, 3 \pmod{4} \) and \( u \in V \) be a simple extendable \( c = c_m \) Virasoro vector. Then an \( h_{m+1,1}^{(m)} \)-primary vector associated to \( u \) is unique up to scalar multiple.

**Proof:** Let \( w, w' \) be \( h_{m+1,1}^{(m)} \)-primary vectors associated to \( u \) and \( W \) the subalgebra generated by \( u \) and \( w \). Then the \( W \)-submodule \( W' \) generated by \( w' \) is isomorphic to the adjoint module \( W \). Since we have assumed that \( V_0 = \mathbb{C} \mathbb{I} \), we see that \( W = W' \) and the assertion follows.

Next we discuss the irreducible modules for \( W(c_m) \) when \( m \equiv 0 \) or \( 3 \pmod{4} \).

**Lemma 2.10 ([LLY]).** Let \( L(c_m, h_{m+1,1}^{(m)}) \times L(c_m, h_{r,s}^{(m)}) = L(c_m, \tilde{h}_{r,s}) \), where

\[
\tilde{h}_{r,s}^{(m)} = \begin{cases} 
    h_{m+2-r,s}^{(m)} & \text{when } m \equiv 0 \pmod{4}, \\
    h_{r,m+3-s}^{(m)} & \text{when } m \equiv 3 \pmod{4},
\end{cases}
\]

by Eq. (2.2). Then \( \Delta = h_{r,s}^{(m)} - \tilde{h}_{r,s}^{(m)} \) is in either \( \mathbb{Z} \) or \( \frac{1}{2} + \mathbb{Z} \). More precisely, one has:

1. When \( m \equiv 0 \pmod{4} \), \( \Delta \in \mathbb{Z} \) if \( r \) is odd, and \( \Delta \in \frac{1}{2} + \mathbb{Z} \) if \( r \) is even.
2. When \( m \equiv 3 \pmod{4} \), \( \Delta \in \mathbb{Z} \) if \( s \) is odd, and \( \Delta \in \frac{1}{2} + \mathbb{Z} \) if \( s \) is even.
3. \( \Delta = 0 \), i.e., \( h_{r,s}^{(m)} = \tilde{h}_{r,s}^{(m)} \) when the triple \((m, r, s)\) satisfies \( r = (m+2)/2 \) if \( m \equiv 0 \pmod{4} \) or \( s = (m+3)/2 \) if \( m \equiv 3 \pmod{4} \).

**Theorem 2.11 ([LLY]).** Suppose \( m \equiv 0 \) or \( 3 \pmod{4} \). Let \( M = L(c_m, h_{r,s}^{(m)}) \) be an irreducible \( L(c_m, 0) \)-module and \( \tilde{M} = L(c_m, \tilde{h}_{r,s}^{(m)}) := L(c_m, h_{m+1,1}^{(m)}) \times M \).

1. If \( h_{r,s}^{(m)} - \tilde{h}_{r,s}^{(m)} \in \mathbb{Z} \setminus \{0\} \), then \( M \oplus \tilde{M} \) affords a unique structure of an irreducible (untwisted) \( W(c_m) \)-module extending \( M \).
(2) If \( h_{r,s}^{(m)} - \tilde{h}_{r,s}^{(m)} \in \frac{1}{2} + \mathbb{Z} \), then \( M \oplus \tilde{M} \) affords a unique structure of an irreducible \( \mathbb{Z}_2 \)-twisted \( \mathcal{W}(c_m) \)-module extending \( M \).

(3) If \( h_{r,s}^{(m)} = \tilde{h}_{r,s}^{(m)} \), then \( M \oplus \tilde{M} \) is a direct sum of two inequivalent irreducible (untwisted) \( \mathcal{W}(c_m) \)-modules. In this case, there exists two inequivalent structures of an irreducible (untwisted) \( \mathcal{W}(c_m) \)-module on \( M \) and these structures are \( \mathbb{Z}_2 \)-conjugates of each other. We denote them by \( M^\pm \).

Note that Theorem 2.11 together with Lemma 2.10 provides a classification of irreducible \( \mathcal{W}(c_m) \)-modules.

**Remark 2.12.** If \( u \) is a simple extendable \( c = c_m \) Virasoro vector of \( V \), then it follows from Lemma 2.10 and Theorem 2.11 that the automorphism \( \tau_u \) defined in Theorem 2.2 is trivial on \( V \) since \( V \) is an untwisted module over the extended subalgebra \( \mathcal{W}(c_m) \) of \( \text{Vir} \).

In this paper, we will frequently consider simple extendable Virasoro vectors with central charges \( c_3 = 4/5 \) and \( c_4 = 6/7 \). The key feature is that the extended Virasoro VOAs \( \mathcal{W}(4/5) \) and \( \mathcal{W}(6/7) \) have some natural \( \mathbb{Z}_3 \)-symmetries among their irreducible modules.

The extended Virasoro VOA \( \mathcal{W}(4/5) \) is rational and has six inequivalent irreducible modules (cf. [KMY]). They are of the following forms as \( L(4/5,0) \)-modules:

\[
L(4/5,0) \oplus L(4/5,3), \quad L(4/5,2/5) \oplus L(4/5,7/5), \quad L(4/5,3/5)^\pm, \quad L(4/5,1/15)^\pm,
\]

where the ambiguity on choosing signs \( \pm \) is solved by fusion rules (cf. [Mi2]). The extended Virasoro VOA \( \mathcal{W}(6/7) \) is rational and has nine inequivalent irreducible modules (cf. [LLY, LY]). They are of the following forms as \( L(6/7,0) \)-modules:

\[
L(6/7,0) \oplus L(6/7,5), \quad L(6/7,1/7) \oplus L(6/7,22/7), \quad L(6/7,5/7) \oplus L(6/7,12/7),
\]

\[
L(6/7,4/3)^\pm, \quad L(6/7,1/21)^\pm, \quad L(6/7,10/21)^\pm,
\]

where the ambiguity on choosing signs \( \pm \) is again solved by fusion rules (cf. loc. cit.). The fusion rules among irreducible \( \mathcal{W}(4/5) \)-modules and \( \mathcal{W}(6/7) \)-modules are computed in [Mi2, LLY, LY] and they have some natural \( \mathbb{Z}_3 \)-symmetries. We can extend these symmetries to automorphisms of VOAs containing these extended Virasoro VOAs as follows.

**Theorem 2.13** ([Mi2, LLY, LY]). Let \( V \) be a VOA and let \( U \) be a sub-VOA of \( V \).

(1) Suppose \( U \simeq \mathcal{W}(4/5) \). Define a linear automorphism \( \xi_U \) of \( V \) to act on each irreducible \( \mathcal{W}(4/5) \)-submodule \( M \) by

\[
\begin{cases}
1 & \text{if } M \simeq L(4/5,0) \oplus L(4/5,3) \text{ or } L(4/5,2/5) \oplus L(4/5,7/5), \\
1 & \text{if } M \simeq L(4/5,3/5)^\pm \text{ or } L(4/5,1/15)^\pm
\end{cases}
\]

\[e^{\pm 2\pi \sqrt{-1}/3}\]

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as $L(4/5, 0)$-modules. Then $\xi_U$ defines an element in $\text{Aut}(V)$ satisfying $\xi_U^3 = 1$.

(2) Suppose $U \cong W(6/7)$. Define a linear automorphism $\xi_U$ of $V$ to act on each irreducible $W(6/7)$-submodule $M$ by

$$
\begin{cases}
1 & \text{if } M \cong L(6/7, 0) \oplus L(6/7, 5), \ L(6/7, 1/7) \oplus L(6/7, 22/7) \text{ or } L(6/7, 5/7) \oplus L(6/7, 12/7), \\
e^{\pm 2\pi \sqrt{-1/3}} & \text{if } M \cong L(6/7, 4/3)^\pm, \ L(6/7, 1/21)^\pm \text{ or } L(6/7, 10/21)^\pm
\end{cases}
$$

as $L(6/7, 0)$-modules. Then $\xi_U$ defines an element in $\text{Aut}(V)$ satisfying $\xi_U^3 = 1$.

**Remark 2.14.** If a simple $c = 6/7$ Virasoro vector $x \in V$ is extendable, then $x$ is of $\sigma$-type if and only if the automorphism $\xi_U$ defined in (2) of Theorem 2.13 is trivial on $V$, where $U$ is the subalgebra isomorphic to $W(6/7)$ generated by $x$ and its 5-primary vector.

### 3 Commutant subalgebras associated to root lattices

In this section, we will construct sub-VOAs of the lattice VOA $V_{\sqrt{2}E_6}$ which will correspond to dihedral subgroups of the largest Fischer group. Our construction is similar to the construction in [LYY1] and [HLY] in the case of the root lattices $E_8$ and $E_7$.

#### 3.1 The algebras $U_{F(nX)}$ and $V_{F(nX)}$

**Commutant subalgebras.** Let $(V, \omega)$ be a VOA and $(U, e)$ be a sub-VOA. Then the commutant subalgebra of $U$ is defined by

$$\text{Com}_V(U) := \{a \in V \mid a_{(n)}U = 0 \text{ for all } n \geq 0\}. \quad (3.1)$$

It is known (cf. [FZ]) that

$$\text{Com}_V(U) = \ker e_{(0)} \quad (3.2)$$

and in particular $\text{Com}_V(U) = \text{Com}_V(\text{Vir}(e))$. Therefore, the commutant subalgebra of $U$ is determined only by the conformal vector $e$ of $U$. It is also shown in Theorem 5.1 of [FZ] that $\omega - e$ is also a Virasoro vector if $\omega(2)e = 0$. Provided $V_1 = 0$, we always have $\omega(2)e = 0$. In that case, we have two mutually commuting subalgebras $\text{Com}_V(\text{Vir}(e)) = \ker e_{(0)}$ and $\text{Com}_V(\text{Vir}(\omega - e)) = \ker e_{(0)}$ and the tensor product $\text{Com}_V(\text{Vir}(\omega - e)) \otimes \text{Com}_V(\text{Vir}(e))$ forms an extension of $\text{Vir}(e) \otimes \text{Vir}(\omega - e)$. More generally, we say a sum $\omega = e^1 + \cdots + e^n$ is a **Virasoro frame** if all $e^i$ are Virasoro vectors and $[Y(e^i, z_1), Y(e^j, z_2)] = 0$ for $i \neq j$. 

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The algebras $U_{F(n,X)}$. Let $\alpha_1, \ldots, \alpha_6$ be a system of simple roots for $E_6$. We let $\alpha_0$ be the root such that $-\alpha_0 = \sum_{i=1}^{6} m_i \alpha_i$ is the highest root for the chosen simple roots. Note that all $m_i$ are positive integers. We also set $m_0 = 1$. For any $i = 0, \ldots, 6$, we consider the sublattice $L_i$ of $E_6$ generated by the roots $\alpha_j$, $0 \leq j \leq 6$, $j \neq i$. One observes that $L_i$ is also of rank 6 and the quotient group $E_6/L_i$ is cyclic of order $m_i$ with generator $\alpha_i + L_i$. Thus one has

$$E_6 = L_i \sqcup (\alpha_i + L_i) \sqcup (2\alpha_i + L_i) \sqcup \cdots \sqcup ((m_i - 1)\alpha_i + L_i). \quad (3.3)$$

We denote by $R_1, \ldots, R_\ell$ the indecomposable components of the lattice $L_i$ which are root lattices of type $A_n$, $D_n$ or $E_6$. Hence $L_i = R_1 \oplus \cdots \oplus R_\ell$ where the direct sum of lattices denotes the orthogonal sum. In fact, the Dynkin diagram of $L_i$ is obtained from the affine Dynkin diagram of $E_6$ by removing the node $\alpha_i$ and the adjacent edges. We recall here that the affine Dynkin diagram of $E_6$ is the graph with vertex set $\{\alpha_0, \ldots, \alpha_6\}$ and two nodes $\alpha_i$ and $\alpha_j$, $0 \leq i, j \leq 6$, are joined by an edge if $\langle \alpha_i, \alpha_j \rangle = -1$. The diagram has the following form:

![Dynkin Diagram](image)

The decomposition (3.3) of the lattice $E_6$ leads to the decomposition

$$V\sqrt{2}E_6 = \bigoplus_{r=0}^{m_i-1} V\sqrt{2}(r\alpha_i + L_i)$$

of the lattice VOA $V\sqrt{2}E_6$. We define a linear map $\rho_i : V\sqrt{2}E_6 \rightarrow V\sqrt{2}E_6$ by

$$\rho_i(u) = \zeta_{m_i}^r u \quad \text{for } u \in V\sqrt{2}(r\alpha_i + L_i), \quad \text{where } \zeta_{m_i} = e^{2\pi\sqrt{-1}/m_i}. \quad (3.5)$$

Then $\rho_i$ is an element of Aut$(V\sqrt{2}E_6)$ of order $m_i$ and the fixed point sub-VOA $V\sqrt{2}E_6^{(\rho_i)}$ is exactly $V\sqrt{2}L_i$.

For a root lattice $S$, we denote by $\Phi(S)$ its root system. Then, by [DLMN], the conformal vector $\omega_S$ of $V\sqrt{2}S$ is given by

$$\omega_S = \frac{1}{4h} \sum_{\alpha \in \Phi(S)} \alpha(-1)^2 \mathbb{1},$$

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where $h$ is the Coxeter number of $S$. Now define

$$\tilde{\omega}_S := \frac{2}{h+2} \omega_S + \frac{1}{h+2} \sum_{\alpha \in \Phi(S)} e^{\sqrt{2} \alpha}.$$  \hspace{1cm} (3.6)$$

It is shown in [DLMN] that $\tilde{\omega}_S$ is a Virasoro vector of central charge $2n/(n+3)$ if $S$ is of type $A_n$, $1$ if $S$ is of type $D_n$, and $6/7$, $7/10$ and $1/2$ if $S$ is of type $E_6$, $E_7$, $E_8$, respectively.

From the irreducible decomposition $L_i = R_1 \oplus \cdots \oplus R_\ell \subset E_6$, we have sublattices $R_s$ of $E_6$ and obtain a factorization

$$V_{\sqrt{2} L_i} = V_{\sqrt{2} R_1} \otimes \cdots \otimes V_{\sqrt{2} R_\ell} \subset V_{\sqrt{2} E_6}. \hspace{1cm} (3.7)$$

Associated to the root subsystems $\Phi(R_s)$ of $\Phi(E_6)$, we also have Virasoro vectors

$$\omega_s := \tilde{\omega}_{R_s} = \frac{2}{h_s+2} \omega_{R_s} + \frac{1}{h_s+2} \sum_{\alpha \in \Phi(R_s)} e^{\sqrt{2} \alpha} \in V_{\sqrt{2} R_s} \subset V_{\sqrt{2} E_6}, \hspace{1cm} 1 \leq s \leq \ell, \hspace{1cm} (3.8)$$

where $\omega_{R_s}$ is the conformal vector of $V_{\sqrt{2} R_s}$ and $h_s$ is the Coxeter number of $R_s$. It follows from the definition that $\omega^s, 1 \leq s \leq \ell$, are mutually orthogonal simple Virasoro vectors in $V_{\sqrt{2} E_6}$. Consider

$$X^r := \sum_{\beta \in \Phi(R_s), \langle \beta, \beta \rangle = 2} e^{\sqrt{2} \beta}, \hspace{1cm} 1 \leq r \leq m_i - 1,$$

in the weight two subspace of $V_{\sqrt{2} E_6}$. It is shown in Proposition 2.2 of [LYY1] that the vectors $X^r$ are highest weight vectors for $\text{Vir}(\omega^1) \otimes \cdots \otimes \text{Vir}(\omega^\ell)$ with total weight $2$.

Since all $\omega^s, 1 \leq s \leq \ell$, are contained in the fixed point sub-VOA $V_{\sqrt{2} E_6}^+$, which has a trivial weight one subspace, the vector $\omega_{E_6} - (\omega^1 + \cdots + \omega^\ell)$ is a Virasoro vector of $V_{\sqrt{2} E_6}$ as discussed at the beginning of the section. We are interested in the following commutant subalgebras:

**Definition 3.1.** For $i = 0, \ldots, 6$, let $L_i < E_6$ be defined as in (3.3) and $R_1, \ldots, R_\ell$ the indecomposable components of $L_i$. Let $\omega^s = \tilde{\omega}_{R_s}$ be Virasoro vectors defined as in (3.8) for $1 \leq s \leq \ell$. The algebra $U(i)$ is the vertex operator algebra

$$U(i) = \text{Com}_{V_{\sqrt{2} E_6}}(\text{Vir}(\omega_{E_6} - (\omega^1 + \cdots + \omega^\ell))) = \ker_{V_{\sqrt{2} E_6}} (\omega_{E_6} - (\omega^1 + \cdots + \omega^\ell))_{(0)}. \hspace{1cm}$$

It is clear from the construction that $U(i)$ has a Virasoro frame $\omega^1 + \cdots + \omega^\ell$. We will consider an embedding of $U(i)$ into a larger VOA and then describe the commutant algebra $U(i)$ using the larger VOA.
It is clear that $U(i)$ forms an extension of $\text{Vir}(\omega^1) \otimes \cdots \otimes \text{Vir}(\omega^\ell)$ and contains highest weight vectors $X^r$, $1 \leq r < m_i$. We will see in Section 5 that we can embed $U(i)$ into the Moonshine VOA and therefore $U(i)$ has a trivial weight one subspace. Consequently, the weight two subspace of $U(i)$ carries a structure of a commutative non-associative algebra called the Griess algebra of $U(i)$, even though $V_{\sqrt{2}E_6}$ has a non-trivial weight one subspace. In Section 3.2, we will explicitly describe the Griess algebra of $U(i)$. Namely, we will show that the Griess algebra of $U(i)$ is given by

$$G(i) := \text{span}_C\{ \omega^s, X^r \mid 1 \leq s \leq \ell, \ 1 \leq r \leq m_i - 1\},$$

which is of dimension $\ell + m_i - 1$.

Recall $\rho_i \in \text{Aut}(V_{\sqrt{2}E_6})$ defined as in (3.5). By definition, it is clear that $\tilde{\omega}_{E_6}$ and $\rho_i \tilde{\omega}_{E_6}$ are linear combinations of $\omega^s$ and $X^r$, and hence are contained in $G(i) \subset U(i)$. We will also discuss the structure of the subalgebra generated by $\tilde{\omega}_{E_6}$ and $\rho_i \tilde{\omega}_{E_6}$ and compare it with $G(i)$ in Section 3.2.

The algebras $V_{F(n,X)}$. We also consider another class of commutant algebras inside the VOA $V_{\sqrt{2}E_8}$. These commutant algebras will be used in Section 5.3 to show that $U(i)$ defined in Definition 3.1 can be embedded into the Fischer group VOA $VF^3$.

We fix an embedding of $E_6$ into $E_8$. Let

$$Q := \text{Ann}_{E_8}(E_6) = \{ \alpha \in E_8 \mid \langle \alpha, E_6 \rangle = 0 \}. \quad (3.9)$$

Then $Q \simeq A_2$ and $Q \oplus E_6$ forms a full rank sublattice of $E_8$. Note that such an embedding is unique up to an automorphism of $E_8$.

Recall that $L_i$ is the sublattice of $E_6$ generated by roots $\alpha_j$, $j \neq i$. Then we have an embedding of $\tilde{L}_i := Q \oplus L_i$ into $E_8$. Since $L_i$ is a full rank sublattice of $E_6$, $\tilde{L}_i$ is also a full rank sublattice of $E_8$. Thus $E_8/\tilde{L}_i$ is a finite abelian group whose order is $3m_i$. We fix the corresponding embedding $V_{\sqrt{2}L_i} \subset V_{\sqrt{2}E_8}$.

We have the decomposition $\tilde{L}_i = Q \oplus R_1 \oplus \cdots \oplus R_\ell$ into a sum of irreducible root lattices which gives rise to a factorization

$$V_{\sqrt{2}L_i} = V_{\sqrt{2}Q} \otimes V_{\sqrt{2}L_i} = V_{\sqrt{2}Q} \otimes V_{\sqrt{2}R_1} \otimes \cdots \otimes V_{\sqrt{2}R_\ell} \subset V_{\sqrt{2}E_8}.$$

Let $\omega_{E_8}$ be the conformal vector of $V_{\sqrt{2}E_8}$ and let $\tilde{\omega}_Q \in V_{\sqrt{2}Q}$ and $\omega^s \in V_{\sqrt{2}R_s}$ be the Virasoro vectors defined as in (3.6) and (3.8), respectively. By the same argument as for $U(i)$, one sees $\omega_{E_8} - (\tilde{\omega}_Q + \omega^1 + \cdots + \omega^\ell)$ is a Virasoro vector of $V_{\sqrt{2}E_8}$, and we can define a commutant subalgebra:
**Definition 3.2.** The algebra \( V(i) \) is the commutant subalgebra
\[
V(i) := \text{Com}_{\sqrt{2}E_8}(\text{Vir}(\omega_E - (\tilde{\omega}_Q + \omega^1 + \cdots + \omega^\ell))).
\]

**Remark 3.3.** As explained in [HLY], \( \text{Com}_{V(i)}(\text{Vir}(\tilde{\omega}_Q)) \) coincides with \( U(i) \).

We finally set
\[
\mathcal{F}(i) := \{ g \in \text{Aut}(V_{\sqrt{2}E_8}) \mid g = \text{id} \text{ on } V_{\sqrt{2}L_i} \}.
\]
(3.10)

Then \( \mathcal{F}(i) \) is canonically isomorphic to the group of characters of \( E_8/L_i \). The subalgebra \( V(i) \) of \( V_{\sqrt{2}E_8} \) is invariant under the action of \( \mathcal{F}(i) \) since all \( \tilde{\omega}_Q, \omega^1, \ldots, \omega^\ell \) and the conformal vector \( \omega_E \) of \( V_{\sqrt{2}E_8} \) are clearly fixed by \( \mathcal{F}(i) \). Note that the special Ising vector
\[
\hat{e} := \tilde{\omega}_E = \frac{1}{16} \omega_E + \frac{1}{32} \sum_{\alpha \in \Phi(E_8)} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}E_8}
\]
(3.11)
is contained in \( V(i) \) (cf. [LYY1, LYY2]) and thus \( \{ g \hat{e} \in V_{\sqrt{2}E_8} \mid g \in \mathcal{F}(i) \} \subset V(i) \).

**McKay’s \( E_6 \)-correspondence.** We like to explain McKay’s correspondence between the conjugacy classes 1A, 2A and 3A of the Fischer group \( \text{Fi}_{24} \) which are the products of 2C-involutions of the Fischer group and the numerical labels \( m_i \) of the affine \( E_6 \) Dynkin diagram as given by the following figure:

\[
\begin{array}{ccccccc}
\text{1A} & & & & & \text{2A} & \\
& \text{2A} & \text{3A} & \text{2A} & \text{1A} & & \\
\text{1A} & & & & & & \\
\end{array}
\]

Note that the correspondence is not one-to-one but only up to the diagram automorphism.

Because of this correspondence, we change our notation slightly and denote \( L_i \) by \( L_{nX} \), \( \rho_i \) by \( \rho_{nX} \), \( \tilde{L}_i \) by \( \tilde{L}_{nX} \), \( \mathcal{F}(i) \) by \( \mathcal{F}_{nX} \), \( V(i) \) by \( V_{F(nX)} \), \( U(i) \) by \( U_{F(nX)} \) and \( \mathcal{G}(i) \) by \( \mathcal{G}_{F(nX)} \), where \( nX \in \{ 1A, 2A, 3A \} \) is the label of the corresponding node in (3.1). Explicitly, we have:

\[
L_{1A} \simeq E_6, \quad L_{2A} \simeq A_1 \oplus A_5, \quad L_{3A} \simeq A_2 \oplus A_2 \oplus A_2.
\]
(3.12)

We also have that
\[
\tilde{\omega}_Q = \frac{1}{15} (\beta_0(-1)^2 + \beta_1(-1)^2 + \beta_2(-1)^2) \mathbb{1} + \frac{1}{5} \sum_{\alpha \in \Phi(Q)} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}Q}
\]
is a simple \( c = 4/5 \) Virasoro vector in \( V_{\sqrt{2}Q} \), where \( \{ \beta_1, \beta_2 \} \) is a set of simple roots for \( Q \simeq A_2 \) and \( \beta_0 = - (\beta_1 + \beta_2) \).
3.2 Structures of $V_{F(nX)}$ and $U_{F(nX)}$.

We determine the structures of $V_{F(nX)}$ and $U_{F(nX)}$.

1A case. In this case, we have $\tilde{L}_{1A} \simeq A_2 \oplus E_6$ and $V_{F(1A)} \simeq U_{3A}$ by [LYY2]. It is clear that the following holds (see [LY] and Section 4 for details):

Lemma 3.4. $U_{F(1A)} \simeq W(6/7) = L(6/7, 0) \oplus L(6/7, 5)$.

Therefore the weight two subspace of $U_{F(1A)}$ is one-dimensional.

2A case. In this case $\tilde{L}_{2A} \simeq A_2 \oplus A_1 \oplus A_5$ and $E_8/\tilde{L}_{3A} \simeq Z_6$. By construction, the commutant subalgebra $V_{F(2A)}$ is the same as the monstrous 6A-algebra $U_{6A}$ discussed in [LHY] (see also the Appendix of [HLY]). Thus, the following result follows from [LYY2]:

Lemma 3.5. There is the decomposition

$$U_{F(2A)} \simeq W(6/7) \otimes L(25/28, 0) \oplus (L(6/7, 5/7) \oplus L(6/7, 12/7)) \otimes L(25/28, 9/7)$$

$$+ (L(6/7, 1/7) \oplus L(6/7, 22/7)) \otimes L(25/28, 34/7)$$

as a $W(6/7) \otimes L(25/28, 0)$-module.

It follows from the decomposition in the Lemma that the weight two subspace of $U_{F(2A)}$ is 3-dimensional and coincides with $G_{F(2A)}$.

3A case. In this case, $\tilde{L}_{3A} \simeq A_2 \oplus A_2 \oplus A_2 \oplus A_2$ and $E_8/\tilde{L}_{3A} \simeq Z_3 \oplus Z_3$. In fact, the coset structure $E_8/\tilde{L}_{3A}$ can be identified with the ternary tetra code $C_4$, whose generator matrix is given by

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}.$$  

The sub-VOA $V_{F(3A)}$ is indeed the ternary code VOA defined in [KMY]:

$$M_{C_4} = \bigoplus_{\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in C_4} L_{4/5}(\alpha_1) \otimes L_{4/5}(\alpha_2) \otimes L_{4/5}(\alpha_3) \otimes L_{4/5}(\alpha_4),$$  

(3.13)

where $L_{4/5}(0) = L(4/5, 0) \oplus L(4/5, 3) = W(4/5)$ and $L_{4/5}(\pm 1) = L(4/5, 2/3)\pm$ as $W(4/5)$-modules. Note also that the cosets of $L_{3A}$ in $E_6$ can be parameterized by the ternary repetition code of length 3:

$$D = \{(0, 0, 0), (1, 1, 1), (-1, -1, -1)\}.$$  

(3.14)

Thus the commutant subalgebra $U_{F(3A)}$ is the ternary code VOA $M_D$ defined in [KMY]:

$$M_D = (L(4/5, 0) \oplus L(4/5, 3))^{\oplus 3} \oplus (L(4/5, 2/3)^+)^{\oplus 3} \oplus (L(4/5, 2/3)^-)^{\oplus 3}.$$  

(3.15)
It is shown in [KMY] that
\[ \ker_{V^*} (\omega_{A_2} - \tilde{\omega}_{A_2})_{(0)} \cong \mathcal{W}(4/5) = L(4/5, 0) \oplus L(4/5, 3) \]
and we get:

**Lemma 3.6.** One has a decomposition
\[ U_{F(3A)} \cong \mathcal{W}(4/5)^{\otimes 3} \oplus (L(4/5, 2/3)^+)^{\otimes 3} \oplus (L(4/5, 2/3)^-)^{\otimes 3} \]
as a \( \mathcal{W}(4/5)^{\otimes 3} \)-module.

Now we see that the weight two subspace of \( U_{F(3A)} \) is 5-dimensional and coincides with \( \mathcal{G}_{F(3A)} \).

**Remark 3.7.** By the comments after Remark 3.3, we know that \( \mathcal{F}_{3A} \) acts on \( V_{F(3A)} \). In this case, the fixed point space is \( V_{F(3A)}^{\mathcal{F}_{3A}} \cong L_{4/5}(0) \otimes L_{4/5}(0) \otimes L_{4/5}(0) \cong \mathcal{W}(4/5)^{\otimes 4} \) while \( L_{4/5}(\alpha_1) \otimes L_{4/5}(\alpha_2) \otimes L_{4/5}(\alpha_3) \otimes L_{4/5}(\alpha_4) \), \((\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathcal{C}_4\), are character spaces of \( \mathcal{F}_{3A} \) in \( V_{F(3A)} \). The automorphism group of \( V_{F(3A)} \cong M_{4} \) was computed in [KMY, Proposition 5.4]. It is isomorphic to \( 3^2 \cdot (2S_4) \cong 3^2 \cdot \text{Aut}(C_4) \cong AGL_2(3) \). The subgroup \( \mathcal{F}_{3A} : \langle \tau \rangle \cong 3^2 : 2 \) fixes the four \( c = 4/5 \) Virasoro vectors and is the stabilizer of the sub-VOA \( V_{F(3A)}^{\mathcal{F}_{3A}} \cong L_{4/5}(0) \otimes L_{4/5}(0) \otimes L_{4/5}(0) \cong \mathcal{W}(4/5)^{\otimes 4} \).

In the Appendix, we will need a result about generators for \( V_{F(3A)} \).

**Lemma 3.8.** Let \( \rho_1 \) and \( \rho_2 \) be generating \( \mathcal{F}_{3A} \cong 3^2 \). Then \( V_{F(3A)} \) is generated by \( \hat{\varphi}, \rho_1 \hat{\varphi} \) and \( \rho_2 \hat{\varphi} \).

**Proof:** By Remark 3.7, we know that \( \mathcal{F}_{3A} \) acts on \( V_{F(3A)} \) and the fixed point subspace is \( V_{F(3A)}^{\mathcal{F}_{3A}} \cong \mathcal{W}(4/5)^{\otimes 4} \). It is clear that \( e^{i,j} := \rho_1^i \rho_2^j \hat{\varphi}, 0 \leq i, j \leq 2 \), are contained in \( V_{F(3A)} \) since \( \hat{\varphi} \in V_{F(3A)} \).

Let \( W \) be the sub-VOA of \( V_{F(3A)} \) generated by \( \{e^{i,j} \mid 0 \leq i, j \leq 2\} \). For each \( i, j = 0, 1, 2 \) with \( (i, j) \neq (0,0) \), the Ising vectors \( e^{0,0} \) and \( e^{i,j} \) generate a sub-VOA in \( V_{F(3A)} \) isomorphic to \( U_{3A} \) which contains a subalgebra isomorphic to \( \mathcal{W}(4/5) \) fixed by \( \mathcal{F}_{3A} \) (cf. [LYY2]). In fact, the \( W_3 \)-algebra \( \mathcal{W}(4/5) \) is contained in a lattice sub-VOA \( V_{\sqrt{A_2}} \) where \( \sqrt{2}A_2 < \sqrt{2}A_2^4 < \sqrt{2}E_8 \). By varying \( i, j \) (say, \( (i, j) = (1,0), (0,1), (1,1) \) and \( (1,2) \)), one can obtain four mutually orthogonal \( \mathcal{W}(4/5) \) in \( V_{F(3A)}^{\mathcal{F}_{3A}} \). Thus we have \( V_{F(3A)}^{\mathcal{F}_{3A}} \cong \mathcal{W}(4/5)^{\otimes 4} < W \).

Now let \( \zeta = e^{2\pi i/3} \) and let
\[
v_{m,n} = \sum_{i=0}^{2} \sum_{j=0}^{2} e^{mi+nj} e^{-i,j},
\]

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for any \(m, n = 0, 1, 2\). Then by direct calculation, it is easy to verify that
\[
\rho_1^k \rho_2^\ell (v_{m,n}) = \zeta^{km + \ell n} v_{m,n}.
\]

In other words, \(v_{m,n}\) spans a 1-dimensional \(\mathcal{F}_{3A}\)-submodule affording the character \(\chi_{m,n}\), where \(\chi_{m,n}(\rho_1^{i} \rho_2^{j}) = \zeta^{mi + nj}\). It also generates the irreducible \(V_{F(3A)}^{\mathcal{F}_{3A}} \cong W(\ell/5)^\otimes 4\)-submodule affording the character \(\chi_{m,n}\). Hence, \(W\) contains a sub-VOA isomorphic to \(M_c \cong V_{F(3A)}\) and we have \(W = V_{F(3A)}\).

Now note that the 3A-algebra generated by \(e^{0,0}\) and \(e^{1,0}\) contains \(e^{2,0} = \tau_{e^{0,0}} e^{1,0}\). Similarly, we get \(e^{0,2}\). Since \(\rho_1\) and \(\rho_2\) are inverted by \(\tau_\ell\), we have
\[
\tau_{\ell,s} (e^{0,j}) = \tau_{\ell,s}^j (\rho_1^i \rho_2^j) = \rho_1^i \tau_\ell \rho_1^{-i} \rho_2^j \hat{e} = \rho_1^i \rho_2^j \hat{e} = e^{2i,2j}
\]
and thus \(e^{2i,2j}\) is contained in the 3A-algebra generated by \(e^{i,0}\) and \(e^{0,j}\). Hence, \(V_{F(3A)}\) is generated by \(e^{0,0} = \hat{e}, \rho_1 \hat{e} = e^{1,0}\) and \(\rho_2 \hat{e} = e^{0,1}\).

\textbf{Remark 3.9.} The irreducible modules for ternary code VOA \(M_D\) have been studied in [La]. It is known (cf. Theorem 4.8 and 4.10 of [La]) that if \(D\) is a self-dual ternary code, then all irreducible \(M_D\)-modules can be realized (using coset or GKO construction) as submodules of a certain lattice VOA \(V_{T_D}\). We refer to Section 2 of [KMY] or Section 3.2 of [La] for the precise definition of \(\Gamma_D\). When \(D = C_4\) is the tetra code, we have \(\Gamma_D = \Gamma_{C_4} \cong \sqrt{2}E_8\) and all irreducible \(M_{C_4}\)-modules are contained in the lattice VOA \(V_{\sqrt{2}E_8}\). The subgroup \(3^2:2 \cong \mathcal{F}_{3A}:\langle \tau_\ell \rangle < \text{Aut}(V_{\sqrt{2}E_8})\) actually acts faithfully on all irreducible \(M_{C_4}\)-submodules. Thus, if \(V\) is a VOA containing \(V_{F(3A)} \cong M_{C_4}\), then the involutions \(\tau_\ell, \tau_{\rho_1} \hat{e}\) and \(\tau_{\rho_2} \hat{e}\) generate a group of the shape \(3^2:2\) in \(\text{Aut}(V)\).

\textbf{Subalgebras generated by \(\tilde{\nu}\) and \(\tilde{\nu}'\).} Set
\[
\tilde{\nu} := \omega_{E_6} \quad \text{and} \quad \tilde{\nu}' := \rho_{nX} \omega_{E_6}.
\]

By definition, the Virasoro vectors \(\tilde{\nu}\) and \(\tilde{\nu}'\) are contained in \(U_{F(nX)}\). We will discuss \(G_{F(nX)}\) and the subalgebras generated by \(\tilde{\nu}\) and \(\tilde{\nu}'\).

\textbf{1A case.} In this case \(L_{1A} \cong E_6\) and \(\rho_{1A}\) is trivial. Thus \(\tilde{\nu}' = \tilde{\nu}\), \(\langle \tilde{\nu}, \tilde{\nu}' \rangle = 3/7\) and \(\tilde{\nu}\) generates \(G_{F(1A)}\), but not \(U_{F(1A)}\).

\textbf{2A case.} In this case \(L_{2A} \cong A_1 \oplus A_5\) and \(\ell = 2\).

The vectors \(\omega^1\) and \(\omega^2\) are Virasoro vectors with central charges \(1/2\) and \(5/4\), respectively, and \(X = X^1\) is a highest weight vector for \(\text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2)\) with highest weight \((1/2, 3/2)\). One easily obtains:
Lemma 3.10. The Griess algebra \( G_{F(2A)} \) is spanned by \( \omega^1, \omega^2, \) and \( X \) and we have the following commutative algebra structure on \( G_{F(2A)} \):

\[
\begin{array}{c|ccc}
 a(1)b & \omega^1 & \omega^2 & X \\
\hline
\omega^1 & 2\omega^1 & 0 & \frac{1}{2}X \\
\omega^2 & 2\omega^2 & \frac{3}{2}X & \\
X & 80\omega^1 + 96\omega^2 & \\
\end{array}
\quad \begin{array}{c|ccc}
 \langle a, b \rangle & \omega^1 & \omega^2 & X \\
\hline
\omega^1 & \frac{1}{4} & 0 & 0 \\
\omega^2 & \frac{5}{8} & 0 & \\
X & 40 & \\
\end{array}
\]

One verifies that

\[
\tilde{v} = \frac{2}{7}\omega^1 + \frac{4}{7}\omega^2 + \frac{1}{14}X, \quad \tilde{v}' = \frac{2}{7}\omega^1 + \frac{4}{7}\omega^2 - \frac{1}{14}X, \quad \langle \tilde{v}, \tilde{v}' \rangle = \frac{1}{49}.
\]

It is also easily verified that \( G_{F(2A)} \) is generated by \( \tilde{v} \) and \( \tilde{v}' \). Set

\[
u = \frac{5}{7}\omega^1 + \frac{3}{7}\omega^2 - \frac{1}{14}X.
\]

Then \( \tilde{v} \) and \( \nu \) are the mutually orthogonal Virasoro vectors with central charges 6/7 and 25/28, respectively, used in the decomposition of \( U_{F(2A)} \) given before.

3A case. In this case \( L_{3A} \cong A_2 \oplus A_2 \oplus A_2 \) and \( \ell = 3 \).

The three vectors \( \omega^1, \omega^2 \) and \( \omega^3 \) are mutually orthogonal Virasoro vectors with central charge 4/5, and \( X^1, X^2 \) are highest weight vectors for \( \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \otimes \text{Vir}(\omega^3) \) with highest weight \( (\frac{2}{5}, \frac{2}{5}, \frac{2}{5}) \). Again, the following result is easily obtained:

Lemma 3.11. The Griess algebra \( G_{F(3A)} \) is spanned by \( \omega^1, \omega^2, \omega^3, X^1 \) and \( X^2 \). Moreover, we have the following commutative algebra structure on \( G_{F(3A)} \):

\[
\begin{array}{c|cccc}
 a(1)b & \omega^1 & \omega^2 & \omega^3 & X^1 & X^2 \\
\hline
\omega^1 & 2\omega^1 & 0 & 0 & \frac{2}{5}X^1 & \frac{2}{5}X^2 \\
\omega^2 & 2\omega^2 & 0 & \frac{2}{5}X^1 & \frac{2}{5}X^2 & \\
\omega^3 & 2\omega^3 & \frac{2}{5}X^1 & \frac{2}{5}X^2 & \\
X^1 & 8X^1 & 45(\omega^1 + \omega^2 + \omega^3) & \\
X^2 & 8X^1 & \\
\end{array}
\quad \begin{array}{c|cccc}
 \langle a, b \rangle & \omega^1 & \omega^2 & \omega^3 & X^1 & X^2 \\
\hline
\omega^1 & \frac{2}{5} & 0 & 0 & 0 & 0 \\
\omega^2 & \frac{2}{5} & 0 & 0 & 0 & \\
\omega^3 & \frac{2}{5} & 0 & 0 & \\
X^1 & 0 & 27 & \\
X^2 & 0 & \\
\end{array}
\]

One verifies that

\[
\tilde{v} = \frac{5}{14}(\omega^1 + \omega^2 + \omega^3) + \frac{1}{14}X^1 + \frac{1}{14}X^2, \quad \langle \tilde{v}, \tilde{v}' \rangle = \frac{3}{196}, \quad \text{where} \quad \zeta = e^{2\pi\sqrt{-1}/3}.
\]

\[
\tilde{v}' = \frac{5}{14}(\omega^1 + \omega^2 + \omega^3) + \frac{\zeta}{14}X^1 + \frac{\zeta^{-1}}{14}X^2.
\]
In this case, the Griess algebra $G_{F(3A)}$ is not generated by $\tilde{v}$ and $\tilde{v}'$. Let $\nu$ be the diagram automorphism of the affine $E_6$ diagram of order 3 defined as: $\alpha_0 \mapsto \alpha_1 \mapsto \alpha_5 \mapsto \alpha_0$, $\alpha_6 \mapsto \alpha_2 \mapsto \alpha_4 \mapsto \alpha_6$ and $\alpha_3 \mapsto \alpha_3$ on the diagram (3.4). Since $\sqrt{2}E_6$ is doubly even, we have a splitting $\text{Aut}(V_{\sqrt{2}E_6}) \simeq \text{Hom}(E_6, \mathbb{C}) \rtimes \text{Aut}(E_6)$ (see Theorem 2.1 of [DN] and Chapter 5 of [FLM]). Then $\nu$ canonically acts on the Griess subalgebra above and we find that $\tilde{v}$ and $\tilde{v}'$ generate the fixed point subalgebra $G_{F(3A)} = \text{span}_\mathbb{C}\{\omega^1 + \omega^2 + \omega^3, X^1, X^2\}$.

Summarizing, we have obtained the following table of values of inner products between $\tilde{v}$ and $\tilde{v}'$:

\[
\begin{array}{c|c|c|c|c}
\times & 3 & 7 & 1 & 49 \\
\hline
3 & 7 & & & \\
7 & & 1 & 49 & \\
196 & 49 & 1 & 49 & 3 \\
7 & & & & \\
\end{array}
\]

**Remark 3.12.** By the computation above, we see that in the 3A case the Griess subalgebra generated by $\tilde{v}$ and $\tilde{v}'$ coincides with the fixed point subalgebra $G_{F(3A)}^{(\nu)}$. This is the only case where the corresponding node is fixed by the diagram automorphism.

### 4 The 3A-algebra for the Monster

In this section, we will review and list some properties of a VOA called the 3A-algebra for the Monster which is related to certain dihedral groups of order 6 in the Monster (cf. [LYY1, LYY2, S]). By using the VOA structure of the 3A-algebra, we will show in Theorem 4.10 that certain commutant algebras of the Virasoro VOA $L(4/5, 0)$ in an arbitrary VOA, satisfying few mild assumptions, have a subgroup of automorphisms satisfying the 3-transposition property. These results will be used in the last section to study the Moonshine VOA and its subalgebra related to the Fischer group.

We first consider the extended simple Virasoro VOAs $W(4/5) = L(4/5, 0) \oplus L(4/5, 3)$ and $W(6/7) = L(6/7, 0) \oplus L(6/7, 5)$ in Theorem 2.7. It is discussed in [LYY2, Mi3, SY] that the Moonshine VOA contains the following subalgebra, which is a simple current extension of $W(4/5) \otimes W(6/7)$:

$$U_{3A} := \left( L(4/5, 0) \oplus L(4/5, 3) \right) \otimes \left( L(6/7, 0) \oplus L(6/7, 5) \right) \oplus L(4/5, 2/3)^+ \otimes L(6/7, 4/3)^+ \oplus L(4/5, 2/3)^- \otimes L(6/7, 4/3)^-. \tag{4.1}$$
Moreover, a dihedral group of order 6 can be defined using $U_{3A}$ such that all order 3 elements are in the 3A conjugacy class (loc. cit.). Therefore, $U_{3A}$ is closely related to the 3A-element of the Monster and we will call it the 3A-algebra for the Monster.

Remark 4.1. The 3A-algebra can be also constructed along the recipe described in Section 3 via the embedding $A_2 \oplus E_6 \hookrightarrow E_8$, which corresponds to the 3A node of the McKay $E_8$-observation [LYY1].

In Sections 4.1 and 4.2, we will review the results obtained in [LYY2, SY] which we will use in Section 4.3 to prove the 3-transposition property of Miyamoto involutions associated to derived $c = 6/7$ Virasoro vectors (cf. Theorem 4.10).

### 4.1 Griess algebra

In this subsection, we will recall some basic properties of the 3A-algebra $U_{3A}$ from [LYY2, SY]. The Griess algebra of $U_{3A}$ is 4-dimensional and can be described as follows.

Let $\omega^1$ and $\omega^2$ be the Virasoro vectors of the subalgebras $L(4/5, 0)$ and $L(6/7, 0)$ of $U_{3A}$ in (4.1), respectively, and let $X^\pm$ be the highest weight vectors of the components $L(4/5, 2/3)^\pm \otimes L(6/7, 4/3)^\pm$ of $U_{3A}$.

**Lemma 4.2 ([LYY2]).** The commutative algebra structure on the Griess algebra of $U_{3A}$ is given by:

| $a_{(1)}b$ | $\omega^1$ | $\omega^2$ | $X^+$ | $X^-$ | $\langle a, b \rangle$ | $\omega^1$ | $\omega^2$ | $X^+$ | $X^-$ |
|-----------|------------|------------|-------|-------|----------------|------------|------------|-------|-------|
| $\omega^1$ | $2\omega^1$ | $0$ | $2/3 X^+$ | $2/3 X^-$ | $\omega^1$ | $2/5$ | $0$ | $0$ | $0$ |
| $\omega^2$ | $2\omega^2$ | $4/3 X^+$ | $4/3 X^-$ | $\omega^2$ | $3/7$ | $0$ | $0$ | $0$ |
| $X^+$ | $20 X^-$ | $135 \omega^1 + 252 \omega^2$ | $X^+$ | $0$ | $81$ |
| $X^-$ | $20 X^+$ | $X^-$ | $0$ | $0$ |

The Virasoro vectors of $U_{3A}$ are classified in [SY] and there are in total three Ising vectors in $U_{3A}$. Let $\zeta$ be a primitive cubic root of unity. Then

$$e^i = \frac{5}{32} \omega^1 + \frac{7}{16} \omega^2 + \frac{1}{32} \zeta^i X^+ + \frac{1}{32} \zeta^{-i} X^-, \quad i = 0, 1, 2,$$

(4.2)

provide all the Ising vectors of $U_{3A}$. The associated $\tau$-involutions satisfy $|\tau_{e^i} \tau_{e^j}| = 3$ if $i \neq j$ and therefore they generate the symmetric group $S_3$ in Aut($U_{3A}$). Indeed, it is known [LLY, Mi3, SY] that $\tau_{e^i} \tau_{e^j}$ coincides with the order three elements $\xi$ or $\xi^{-1}$ in Theorem 2.13. The VOA $U_{3A}$ is generated by any two of these Ising vectors and correspondingly Aut($U_{3A}$) $\simeq S_3$ is also generated by the associated $\tau$-involutions.
It is shown in [SY] that $U_{3A}$ has exactly four simple Virasoro vectors with central charge $4/5$, namely, $\omega^1$ and the following three vectors:

$$x^i = \frac{1}{16}\omega^1 + \frac{7}{8}\omega^2 - \frac{1}{48}\zeta^i X^+ - \frac{1}{48}\zeta^{-i} X^-, \quad i = 0, 1, 2.$$ (4.3)

Among these four vectors, only $\omega^1$ is characteristic in the sense it is fixed by $\text{Aut}(U_{3A})$, whereas the other three vectors are conjugated by $\tau$-involutions $\tau_{ci}$, $i = 0, 1, 2$. We call $\omega^1 + \omega^2$ the characteristic Virasoro frame of $U_{3A}$. By (4.1), we see that $\omega^1$ and $\omega^2$ are extendable. Here we show that $\omega^1$ is the unique extendable simple $c = 4/5$ Virasoro vector of $U_{3A}$.

**Lemma 4.3 ([LYY2]).** Let $y$ be one of the $x^i$, $i = 0, 1, 2$. Then as a module over $\text{Vir}(y) \simeq L(4/5, 0)$, we have $U_{3A} = (U_{3A})_y[0] \oplus (U_{3A})_y[3] \oplus (U_{3A})_y[2/3] \oplus (U_{3A})_y[1/8] \oplus (U_{3A})_y[13/8]$. Moreover, $(U_{3A})_2 \cap (U_{3A})_{y[13/8]} \neq 0$.

By Theorem 2.11, there is no untwisted $W(\frac{4}{5})$-module which contains an $L(\frac{4}{5}, 0)$-submodule isomorphic to $L(\frac{4}{5}, \frac{13}{8})$, and therefore we see that the $c = 4/5$ Virasoro vectors $x^i$, $i = 0, 1, 2$, are not extendable.

### 4.2 Representation theory

The representation theory of $U_{3A}$ was completed in [SY].

**Theorem 4.4 ([SY]).** The VOA $U_{3A}$ is rational and there are six isomorphism types of irreducible modules over $U_{3A}$ with the following shapes as $\text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2)$-modules:

- $U(0) = U_{3A} = [0, 0] \oplus [3, 0] \oplus [0, 5] \oplus [3, 5] \oplus 2 [\frac{7}{5}, \frac{4}{3}]$,
- $U(\frac{4}{5}) = [0, \frac{1}{5}] \oplus [0, \frac{22}{7}] \oplus [3, \frac{1}{7}] \oplus [3, \frac{22}{7}] \oplus 2 [\frac{5}{5}, \frac{10}{21}]$,
- $U(\frac{5}{7}) = [0, \frac{5}{7}] \oplus [0, \frac{12}{7}] \oplus [3, \frac{5}{7}] \oplus [3, \frac{12}{7}] \oplus 2 [\frac{5}{5}, \frac{21}{7}]$,
- $U(\frac{2}{5}) = [\frac{2}{5}, 0] \oplus [\frac{2}{5}, 5] \oplus [\frac{7}{5}, 0] \oplus [\frac{7}{5}, 5] \oplus 2 [\frac{15}{5}, \frac{4}{3}]$,
- $U(\frac{19}{35}) = [\frac{7}{5}, \frac{1}{7}] \oplus [\frac{7}{5}, \frac{22}{7}] \oplus [\frac{7}{5}, \frac{1}{7}] \oplus [\frac{7}{5}, \frac{22}{7}] \oplus 2 [\frac{15}{5}, \frac{21}{7}]$,
- $U(\frac{4}{35}) = [\frac{2}{5}, \frac{5}{7}] \oplus [\frac{2}{5}, \frac{12}{7}] \oplus [\frac{7}{5}, \frac{5}{7}] \oplus [\frac{7}{5}, \frac{12}{7}] \oplus 2 [\frac{15}{5}, \frac{21}{7}]$,

where $[h_1, h_2]$ denotes an irreducible $\text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \simeq L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0)$-module isomorphic to $L(\frac{4}{5}, h_1) \otimes L(\frac{6}{7}, h_2)$.

By the list of irreducible modules above, we see that $U_{3A}$ is a maximal extension of $L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0)$ as a simple VOA. We remark the following fundamental observation.
Lemma 4.5. Let \( V \) be a VOA and \( e \in V \) be an Ising vector. Then any \( V \)-module \( M \) is \( \tau_e \)-stable, that is, the \( \tau_e \)-conjugated module \( \tau_e \circ M \) is isomorphic to \( M \) itself. In particular, if \( G \) is a subgroup of \( \text{Aut}(V) \) generated by \( \tau \)-involutions associated to Ising vectors of \( V \) then \( M \) is \( G \)-stable, that is, \( g \circ M \simeq M \) for all \( g \in G \).

As we discussed, \( \text{Aut}(U_{3A}) \simeq S_3 \) is generated by \( \tau \)-involutions associated to Ising vectors of \( U_{3A} \) and therefore all irreducible \( U_{3A} \)-modules are \( S_3 \)-invariant. In general, if an irreducible \( V \)-module \( M \) is \( G \)-stable then we have a projective action of \( G \) on \( M \) (cf. [DY]). But in our case, we have an ordinary action of \( S_3 \) on each irreducible \( U_{3A} \)-module. For, we can find all the irreducible \( U_{3A} \)-modules as a submodule of a larger VOA, say \( V_{\sqrt{2}E_8} \), for example, on which we have an ordinary \( S_3 \)-action (cf. [LY, LYY2]; consider \( U_{3A}^{S_3} \subset (V_{\sqrt{2}A_2} \otimes V_{\sqrt{2}E_8})^+ \subset V_{\sqrt{2}E_8}^+ \)). Let \( M_0, M_1 \) and \( M_2 \) be the principal, signature and 2-dimensional irreducible representations of \( S_3 \). As a \( U_{3A}^{S_3} \otimes CS_3 \)-module, one has the following decompositions:

\[
U(0) = ([0, 0] \oplus [3, 5]) \otimes M_0 \oplus ([3, 0] \oplus [0, 5]) \otimes M_1 \oplus \begin{aligned} &\,[2/3, 4/3] \otimes M_2 \\
U(1/7) &\,(=[0, 22/7] \oplus [3, 22/7]) \otimes M_0 \oplus ([0, 1/7] \oplus [3, 1/7]) \otimes M_1 \oplus \begin{aligned} &\,[2/3, 1/7] \otimes M_2, \end{aligned} \\
U(5/7) &\,(=[0, 5/7] \oplus [3, 5/7]) \otimes M_0 \oplus ([0, 12/7] \oplus [3, 12/7]) \otimes M_1 \oplus \begin{aligned} &\,[2/3, 1/7] \otimes M_2, \end{aligned} \\
U(2/5) &\,(=[2/5, 0] \oplus [7/5, 0]) \otimes M_0 \oplus ([2/5, 5] \oplus [7/5, 5]) \otimes M_1 \oplus \begin{aligned} &\,[1/15, 4/3] \otimes M_2, \end{aligned} \\
U(19/35) &\,(=[2/5, 22/7] \oplus [7/5, 22/7]) \otimes M_0 \oplus ([2/5, 1/7] \oplus [7/5, 1/7]) \otimes M_1 \oplus \begin{aligned} &\,[1/15, 1/7] \otimes M_2, \end{aligned} \\
U(4/35) &\,(=[2/5, 5/7] \oplus [7/5, 5/7]) \otimes M_0 \oplus ([2/5, 12/7] \oplus [7/5, 12/7]) \otimes M_1 \oplus \begin{aligned} &\,[1/15, 1/7] \otimes M_2, \end{aligned} \\
\end{aligned} \tag{4.4}
\]

where \([h_1, h_2] \) denotes a \( \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \)-module isomorphic to \( L(4/5, h_1) \otimes L(6/7, h_2) \).

4.3 3-transposition property of \( \sigma \)-involutions

We consider involutions induced by the 3A-algebra. We will again refer to Section 2.1 for the definition of simple \( c = 6/7 \) Virasoro vectors of \( \sigma \)-type and their corresponding \( \sigma \)-involutions. See Lemma 2.4 and Eq. (2.4) for the details.

Let \( V \) be a VOA and let \( u \in V \) be a simple \( c = 4/5 \) Virasoro vector.

Definition 4.6. A simple \( c = 6/7 \) Virasoro vector \( v \in \text{Com}_V(\text{Vir}(u)) \) is called a derived Virasoro vector with respect to \( u \) if there exists a sub-VOA \( U \) of \( V \) isomorphic to the 3A-algebra \( U_{3A} \) such that \( u + v \) is the characteristic Virasoro frame of \( U \).

\footnote{There is an Ising vector \( e \in V_{\sqrt{2}E_8} \) such that \( \tau_e \) acts by \(-1\) on the weight one subspace. Then considering an embedding \( U_{3A}^{S_3} \subset (V_{\sqrt{2}A_2} \otimes V_{\sqrt{2}E_8})^+ \subset V_{\sqrt{2}E_8}^+ \) with \( e \in U_{3A} \) as in [LYY2], one can verify the decomposition.}
Lemma 4.7. A derived $c = 6/7$ Virasoro vector $v \in \text{Com}_V(\text{Vir}(u))$ with respect to $u$ is of $\sigma$-type on the commutant $\text{Com}_V(\text{Vir}(u))$.

Proof: Assume that $V$ contains a subalgebra $U$ isomorphic to the $3A$-algebra $U_{3A}$ as in Definition 4.6. For an irreducible $U$-module $M$, we denote

$$H_M := \text{Hom}_{U_{3A}}(M, V).$$

Then we have the isotypical decomposition

$$V = \bigoplus_{M \in \text{Irr}(U_{3A})} M \otimes H_M.$$ 

By definition, $u + v$ is the characteristic Virasoro frame of $U_{3A}$. Consider $V$ as a module over its subalgebra $\text{Vir}(u) \otimes \text{Vir}(v) \otimes \text{Com}_V(U)$. By Theorem 4.4 we have the following decomposition:

$$V_u[0] = \text{Vir}(u) \otimes \text{Com}_V(\text{Vir}(u)), \quad \text{Com}_V(\text{Vir}(u)) = \left[ 0 \oplus 5 \right] \otimes H_{U(0)} \oplus \left[ 1/7 \oplus 22/7 \right] \otimes H_{U(1/7)} \oplus \left[ 5/7 \oplus 12/7 \right] \otimes H_{U(5/7)}, \quad (4.5)$$

where $[h_1 \oplus h_2]$ denotes an irreducible $W(6/7)$-module isomorphic to $L(6/7, h_1) \oplus L(6/7, h_2)$. By the decomposition above, we see that $v$ is of $\sigma$-type on $\text{Com}_V(\text{Vir}(u))$. 

We consider the one-point stabilizer

$$\text{Stab}_{\text{Aut}(V)}(u) := \{ h \in \text{Aut}(V) \mid h u = u \}. \quad (4.6)$$

Each $h \in \text{Stab}_{\text{Aut}(V)}(u)$ keeps the isotypical component $V_u[h]$ invariant so that by restriction we can define a group homomorphism

$$\psi_u : \text{Stab}_{\text{Aut}(V)}(u) \longrightarrow \text{Aut}(\text{Com}_V(\text{Vir}(u))), \quad (4.7)$$

$$h \quad \longmapsto \quad h|_{\text{Com}_V(\text{Vir}(u)).}$$

Let $v \in \text{Com}_V(\text{Vir}(u))$ be a derived $c = 6/7$ Virasoro vector with respect to $u$. By Lemmas 4.7 and 2.4, we have an involution $\sigma_v \in \text{Aut}(\text{Com}_V(\text{Vir}(u)))$. Now let $e$ be an Ising vector of $U$. By Lemma 4.5 and Eq. (4.4), we see that $\tau_e$ keeps $\text{Com}_V(\text{Vir}(u))$ invariant and, in fact, we have:

Lemma 4.8. $\psi_u(\tau_e) = \sigma_v$ for any Ising vector $e \in U$.

Let $J$ be the set of all derived $c = 6/7$ Virasoro vectors of $\text{Com}_V(\text{Vir}(u))$. We will prove that the set of involutions

$$\{ \sigma_v \in \text{Aut}(\text{Com}_V(\text{Vir}(e))) \mid v \in J \}$$

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Theorem 4.9 ([S]). Let \( W \) be a VOA over \( \mathbb{R} \) with grading \( W = \bigoplus_{n \geq 0} W_n \), \( W_0 = \mathbb{R}1 \) and \( W_1 = 0 \), and assume \( W \) is compact, that is, the normalized invariant bilinear form on \( W \) is positive definite. Let \( x, y \) be Ising vectors in \( W \) and denote by \( U(x, y) \) the subalgebra of \( W \) generated by \( x \) and \( y \). Then:

1. The 6-transposition property \( |\tau_x \tau_y| \leq 6 \) holds on \( W \).
2. There are exactly nine possible inequivalent structures of the Griess algebra on the weight two subspace \( U(x, y)_2 \) of \( U(x, y) \).
3. The Griess algebra structure on \( U(x, y)_2 \) is unique if \( |\tau_x \tau_y| = 6 \) and in this case \( U(x, y) \) is a copy of the 6A-algebra.

The following is the main theorem of this section.

Theorem 4.10. Suppose that \( V_1 = 0 \) and \( V \) has a compact real form \( V_{\mathbb{R}} \) and every Ising vector of \( V \) is in \( V_{\mathbb{R}} \). Then for any \( v^1, v^2 \in J \), we have \( |\sigma_{v^1} \sigma_{v^2}| \leq 3 \) on \( \text{Com}_V(\text{Vir}(u)) \).

Proof: Let \( v^1, v^2 \in J \). Then there exist subalgebras \( U^1 \) and \( U^2 \) of \( V \) isomorphic to the 3A-algebra such that \( u \) is a simple extendable \( c = 4/5 \) Virasoro vector in \( U^1 \cap U^2 \) and \( v^1 \in \text{Com}_{U^1}(\text{Vir}(u)), v^2 \in \text{Com}_{U^2}(\text{Vir}(u)) \). Let \( a, a', a'' \) be the three distinct Ising vectors in \( U^1 \) and \( b, b', b'' \) be the three distinct Ising vectors in \( U^2 \). Set \( g := \tau_a \tau_{a'} \). Then \( g \) is an order three element induced by the extended Virasoro VOA \( W(\frac{4}{5}) \) and we have \( a' = ga \) and \( a'' = g^2a \). By our settings, \( \tau_b \tau_{b'} = g \) or \( g^2 \) so that we may also assume \( g = \tau_b \tau_{b'}, b' = gb \) and \( b'' = g^2b \). Note that \( \psi_u(g) = 1 \).

Now, recall that

\[
\sigma_{v^1} = \psi_u(\tau_a) = \psi_u(\tau_{a'}) = \psi_u(\tau_{a''}) \quad \text{and} \quad \sigma_{v^2} = \psi_u(\tau_b) = \psi_u(\tau_{b'}) = \psi_u(\tau_{b''}).
\]

Thus, the order of \( \sigma_{v^1} \sigma_{v^2} = \psi_u(\tau_a \tau_b) \) must divide that of \( \tau_a \tau_b \).

By Sakuma’s Theorem 4.9, we know that \( |\tau_a \tau_b| \leq 6 \). If \( |\tau_a \tau_b| \leq 3 \), there is nothing to prove. So we assume \( |\tau_a \tau_b| = 4, 5 \) or \( 6 \).

First, we will note that \( g \) commutes with \( \tau_a \tau_b \) since \( \tau_a g = g^{-1} \tau_a \) and \( \tau_b g = g^{-1} \tau_b \).

Case \( |\tau_a \tau_b| = 4 \): In this case, \( \tau_a \tau_b g \) has order 12, which is impossible since \( \tau_a \tau_b g = \tau_a \tau_b (\tau_a \tau_{b'}) = \tau_a \tau_{b'} \) has order \( \leq 6 \).

Case \( |\tau_a \tau_b| = 5 \): In this case, \( \tau_a \tau_b g = \tau_a \tau_{b'} \) has order 15, which is again impossible.
Case $|\tau_a\tau_b| = 6$: In this case, $\tau_a\tau_b g$ and $\tau_a\tau_b g^2$ have order 6 or 2.

Claim: If $|\tau_a\tau_b g| = 6$ then $|\tau_a\tau_b g^2| = 2$.

The reason is as follows. Suppose both of them have order 6. Since $\tau_a\tau_b g = \tau_a\tau_b'$ and $\tau_a\tau_b g^2 = \tau_a\tau_b(\tau_b'\tau_b) = \tau_a\tau_b\tau_b' = \tau_a\tau_b''$, we have $\langle a, b \rangle = \langle a, b' \rangle = \langle a, b'' \rangle = 5/2^{10}$ and hence $\langle g^i a, g^j b \rangle = 5/2^{10}$ for all $i, j = 0, 1, 2$ (cf. the Appendix in [HLY]).

Now by using the structure of the 3A-algebra $U_{3A}$, we may write the Ising vectors $a$ and $b$ as

$$a = \frac{5}{32} u + \frac{7}{16} v^1 + \frac{1}{32} (X^1 + X^2),$$
$$b = \frac{5}{32} u + \frac{7}{16} v^2 + \frac{1}{32} (Y^1 + Y^2),$$

where $X^1, X^2$ (resp. $Y^1, Y^2$) are certain highest weight vectors of weight $(2/3, 4/3)$ with respect to Vir$(u) \otimes$ Vir$(v^1)$ (resp. Vir$(u) \otimes$ Vir$(v^2)$). Using (4.2), we get

$$a + ga + g^2a = a + a' + a'' = \frac{3}{32} (5u + 14v^1),$$
$$b + gb + g^2b = a + a' + a'' = \frac{3}{32} (5u + 14v^2).$$

Thus, we have

$$\langle a + ga + g^2a, b + gb + g^2b \rangle = \frac{9}{2^{10}} \langle 5u + 14v^1, 5u + 14v^2 \rangle.$$

Since $\langle u, u \rangle = 2/5$ and $\langle u, v^i \rangle = 0$ for $i = 1, 2$ by Lemma 4.2, this implies

$$\frac{5}{2^{10}} = \frac{5^2}{2^{10}} \langle u, u \rangle + \frac{7^2}{2^8} \langle v^1, v^1 \rangle = \frac{5}{2^9} + \frac{7^2}{2^8} \langle v^1, v^2 \rangle$$

and thus $\langle v^1, v^2 \rangle = -5/196 < 0$. This is impossible by the Norton inequality

$$\langle v^1 v^1, v^1 v^2 \rangle \geq \langle v^1 v^2, v^1 v^2 \rangle \geq 0$$

(cf. Theorem 6.3 and Lemma 6.5 in [Mi1], see also [B]3) and the claim follows.

Therefore, $\tau_a\tau_b g = \tau_a\tau_b'$ or $\tau_a\tau_b g^2 = \tau_a\tau_b''$ has order 2 and hence

$$\sigma v^i \sigma v^2 = \psi_u(\tau_a\tau_b) = \psi_u(\tau_a\tau_b') = \psi_u(\tau_a\tau_b'')$$

is of order at most 2.

---

3A sketch of proof is given in [B].
5 The Fischer group

In this section, we will discuss the properties of the commutant vertex operator subalgebra \( V_{\mathfrak{f}} \) of the Moonshine VOA \( V^{\natural} \). We will show that the full Fischer 3-transposition group \( \text{Fi}_{24} \) is a subgroup of \( \text{Aut}(V_{\mathfrak{f}}) \). We also show that there exist one-to-one correspondences between 2C-involutions of \( \text{Fi}_{24} \) and derived \( c = 6/7 \) Virasoro vectors in \( V_{\mathfrak{f}}^{\natural} \).

Finally, we will discuss the embeddings of \( U_{F(n,X)} \) into \( V_{\mathfrak{f}}^{\natural} \). The main idea is to embed the root lattice \( E_6 \) into \( E_8 \) and view the \( U_{F(n,X)} \) as certain commutant subalgebras of the lattice VOA \( V_{\sqrt{2}E_8} \). Then we shall show that the product of two \( \sigma \)-involutions generated by \( c = 6/7 \) Virasoro vectors in \( U_{F(n,X)} \) exactly belong to the conjugacy class \( nX \) in \( \text{Fi}_{24} \).

By this procedure, we obtain a VOA description of the \( E_6 \) structure inside \( \text{Fi}_{24} \).

The automorphism group of the Moonshine VOA \( V^{\natural} \) is the Monster simple group \( M^{[FLM]} \). Consider the monstrous Griess algebra of dimension 196884 \([C, G1]\). It is known that the monstrous Griess algebra is naturally realized as the subspace of weight 2 of \( V_{\mathfrak{f}}^{\natural} \), which we call the Griess algebra of \( V^{\natural} \) and denote by \( G^{\natural} \). We will freely use the character tables in \([ATLAS]\).

5.1 The Fischer group vertex operator algebra \( V_{\mathfrak{f}}^{\natural} \)

We denote by \( \text{Fi}_{24} \) the Fischer 3-transposition group and by \( \text{Fi}'_{24} \) its derived subgroup, the 3rd largest sporadic finite simple group. Let \( g \in M \) be a 3A-element. Then \( C_M(g) \simeq 3.\text{Fi}'_{24} \) and it is shown in \([C, MeN]\) that the monstrous Griess algebra \( G^{\natural} \) has an irreducible decomposition

\[
G^{\natural} = \mathbf{1} \oplus \mathbf{1} \oplus 8671 \oplus 57477 \oplus 2 \cdot 783 \oplus 2 \cdot 64584
\]

(5.1)
as a \( C_M(g) \)-module. Therefore, we can take a Virasoro vector \( u \in (G^{\natural})^{C_M(g)} \) such that \( (G^{\natural})^{C_M(g)} = Cu \oplus C(\omega - u) \) is an orthogonal sum. We take \( u \) to be the shorter one, that is, the central charge of \( u \) is smaller than that of \( \omega - u \). It is also shown in loc. cit. that the central charge of the shorter Virasoro vector \( u \) is 4/5 and its spectrum on \( G^{\natural} \) is as follows:

\[
\begin{align*}
G^{\natural} & = \mathbf{1} \oplus \mathbf{1} \oplus 8671 \oplus 57477 \oplus 2 \cdot 783 \oplus 2 \cdot 64584 \\
\quad u(0) & : 0 \quad 2 \quad 2/5 \quad 0 \quad 2/3 \quad 1/15
\end{align*}
\]

(5.2)
The following result about the extendibility seems already known to experts (see for example \([MeN, KMY, Mi2]\)), even though no rigorous proof has been given so far.

**Theorem 5.1.** Let \( g \) be a 3A-element of the Monster. The cyclic group \( \langle g \rangle \) uniquely determines an extendable simple \( c = 4/5 \) Virasoro vector in \( (V^{\natural})^{C_M(g)} \).
Proof: By the decomposition in Eq. (5.1), every cyclic subgroup \( \langle g \rangle \) defines a unique simple \( c = 4/5 \) Virasoro vector in \((V^2)^{C_M(g)}\). We will prove that this Virasoro vector is extendable.

Let \( t \) be a 2A-involution in \( N_M(\langle g \rangle) \) (\( \simeq 3.\text{Fi}_{24} \)) but not in \( C_M(g) \) (\( \simeq 3.\text{Fi}_{24}' \)). Then the subgroup \( H \) generated by \( t \) and \( g \) in \( M \) is isomorphic to \( S_3 \) and \( C_M(H) \simeq \text{Fi}_{23} \) (cf. [ATLAS] and Lemma 13.3 of [G1]). Take an involution \( t' \in H \) such that \( tt' = g \). Then \( t' \) is conjugate to \( t \) and \( H \) is generated by \( t \) and \( t' \). By the one-to-one correspondence between 2A-elements of \( M \) and Ising vectors of \( V^2 \) (cf. [Mi1] and [Ho], Lemma 3; see also [HLY], Theorem 5.1), there exist Ising vectors \( e^0, e^1 \in V^2 \) such that \( \tau_{e^0} = t, \tau_{e^1} = t' \) and \( e^0 \) and \( e^1 \) are fixed by \( C_M(t) \) and \( C_M(t') \), respectively. Since \( C_M(H) \) is a subgroup of both \( C_M(t) \) and \( C_M(t') \), \( e^0 \) and \( e^1 \) are both contained in \((V^2)^{C_M(H)}\). By [ATLAS], one obtains the following decomposition of the Griess algebra \( G^2 \) as a \( C_M(H) \)-module:

\[
G^2 = 5 \cdot 1 \oplus 3 \cdot 782 \oplus 3 \cdot 3588 \oplus 5083 \oplus 25806 \oplus 30888 \oplus 2 \cdot 60996. \tag{5.3}
\]

Let \( U \) be the subalgebra of \( V^2 \) generated by \( e^0 \) and \( e^1 \). Then \( U \) is isomorphic to the 3A-algebra \( U_{3A} \) [Mi3, SY]. Since the Griess algebra of \( U \) is 4-dimensional (cf. Section 4.1), it follows from the decomposition above that the weight two subspace of \((V^2)^{C_M(H)}\) is spanned by that of \( U \) and the conformal vector of \( V^2 \). Now it is clear that all simple \( c = 4/5 \) Virasoro vectors of \((V^2)^{C_M(H)}\) are contained in \( U \). The 3A-algebra has one extendable simple \( c = 4/5 \) Virasoro vector and three non-extendable ones. By Lemma 4.3, the non-extendable ones in the 3A-algebra have eigenvalues 13/8 on the Griess algebra of the 3A-subalgebra and these eigenvalues do not appear in the decomposition (5.2). Therefore, the subalgebra \((V^2)^{C_M(g)}\) contains the unique simple \( c = 4/5 \) Virasoro vector, which is extendable as claimed. 

Let \( e^0 \) and \( e^1 \) be Ising vectors as in the proof above. Without loss, we may assume that the 3A-element \( \xi_u \) associated to \( u \), defined as in Theorem 2.13, coincides with \( \tau_{e^0} \tau_{e^1} \).

It follows from Theorem 2.13 (cf. [KMY, Mi2]) that for each embedding \( W(4/5) \) isomorphic to \( L(4/5, 3) \), one can define an element \( \xi \) of the Monster with \( \xi^3 = 1 \). By computing its trace on \( V_2^2 \), one can show that \( \xi \) belongs to the conjugacy class 3A (cf. Section 4.1 of [Ma1]). We remark here that a single \( L(4/5, 0) \) does not define an order three symmetry, and in order to obtain 3A-elements, we have to extend \( L(4/5, 0) \) to a larger algebra \( W(4/5) \) isomorphic to \( L(4/5, 0) \oplus L(4/5, 3) \) (cf. [Mi2]). It remains a natural question whether the map associating the order three element \( \xi \) defined as in Theorem 2.13 is injective or not. Since the order three element is defined not only by the Griess algebra but also by the 3-primary vector, we would need extra information about the weight three subspace of \( V^2 \) to solve this question.

Because of this problem, we first fix a 3A-element \( g \in M \) and then take the unique
Lemma 5.3. The Fischer group vertex operator algebra is defined as the commutant
\[
\mathcal{V}^F := \text{Com}_{\mathcal{V}^F}(\mathcal{W}(u, w)) = \text{Com}_{\mathcal{V}^F}(\text{Vir}(u)).
\] (5.4)

Below we will show that the VOA \(\mathcal{V}^F\) affords an action of the Fischer 3-transposition group \(\mathcal{F}_{24}\).

Lemma 5.3. \(N_M(\langle \xi_u \rangle) \subset \text{Stab}_M(u)\).

Proof: Since \(\operatorname{Aut}(\langle \xi_u \rangle) \cong \operatorname{Aut}(\mathbb{Z}_3) = \mathbb{Z}_2\), we have either \(N_M(\langle \xi_u \rangle) = C_M(\xi_u)\) or \(C_M(\xi_u)\) is normal of index 2 in \(N_M(\langle \xi_u \rangle)\). We have seen in Theorem 5.1 that \(C_M(\xi_u) \subset \text{Stab}_M(u)\).

For elements \(h \in C_M(\xi_u)\) and \(g \in N_M(\langle \xi_u \rangle)\) we get \(hgu = gh^g u = gu\) where \(h^g = g^{-1}hg \in C_M(\xi_u)\). Thus \(gu\) is fixed by \(C_M(\xi_u)\). By the decomposition (5.1) of \(G^3\) as a \(C_M(\xi_u)\)-module we see that \(gu\) must be equal to the shorter Virasoro element \(u\) in the fixed point subspace \((G^3)^{C_M(\xi_u)}\). Thus \(N_M(\langle \xi_u \rangle) \subset \text{Stab}_M(u)\).

Recall the group homomorphism \(\psi_u\) defined in (4.7). By Lemma 5.3, we have an action of \(N_M(\langle \xi_u \rangle)\) on \(\text{Com}_{\mathcal{V}^F}(\text{Vir}(u))\) via \(\psi_u\).

Proposition 5.4. Consider the homomorphism \(\psi_u : \text{Stab}_M(u) \to \operatorname{Aut}(\text{Com}_{\mathcal{V}^F}(\text{Vir}(u)))\). Then \(\psi_u(N_M(\langle \xi_u \rangle)) \cong \mathcal{F}_{24}\).

Proof: It is clear that \(\langle \xi_u \rangle \subset \ker \psi_u\). By (5.2), it is also clear that \(\psi_u(C_M(\xi_u)) \neq 1\). Therefore, either \(\psi_u(N_M(\langle \xi_u \rangle)) \cong \mathcal{F}_{24}\) or \(\mathcal{F}_{24}\). Since \(\tau, \alpha \in N_M(\langle \xi_u \rangle) \setminus C_M(\xi_u)\) acts non-trivially on \(\text{Com}_{\mathcal{V}^F}(\text{Vir}(u))\) via \(\psi_u\), we see \(\psi_u(N_M(\langle \xi_u \rangle)) \cong \mathcal{F}_{24}\).

By (5.1) and (5.2) the Griess algebra \(\mathcal{V}_2^3\) is of dimension 57478 and has, under \(\psi_u(N_M(\langle \xi_u \rangle)) \cong \mathcal{F}_{24}\), the decomposition
\[
\mathcal{V}_2^3 = 1 \oplus 57477.
\] (5.5)

We have seen in Proposition 5.4 that \(\mathcal{V}_2^3\) naturally affords an action of the full Fischer group \(\mathcal{F}_{24}\). We cannot prove that \(\mathcal{F}_{24}\) is the full automorphism group of \(\mathcal{V}_2^3\) at the moment. Instead, we will prove the following partial result.

Theorem 5.5. Let \(\mathcal{X}\) be the subalgebra of \(\mathcal{V}_2^3\) generated by the weight 2 subspace. Then \(\operatorname{Aut}(\mathcal{X}) \cong N_M(\langle \xi_u \rangle)/\langle \xi_u \rangle \cong \mathcal{F}_{24}\).
Proof: It is shown in [GL, LM] that there exists a 14-dimensional sublattice \( L \) of the Leech lattice \( \Lambda \) such that \( V_L^+ \) contains a sub-VOA isomorphic to \( U_{3A} \). It is also known that the annihilator \( \text{Ann}_L(L) = \{ \alpha \in \Lambda \mid \langle \alpha, \beta \rangle = 0 \text{ for all } \beta \in L \} \) contains a sublattice isomorphic to \( \sqrt{2}A_2^{\oplus 5} \) (cf. [GL]). Therefore we can find a sub-VOA \( U_{3A} \otimes V_{\sqrt{2}A_2}^+ \) of \( V_L^+ \), which is also contained in the Moonshine VOA. This shows that \( V^\mathfrak{c} \) contains a sub-VOA isomorphic to \( V_{\sqrt{2}A_2}^+ \). It is well-known that \( V_{\sqrt{2}A_2}^+ \) contains 6 Ising vectors (cf. [LSY]) and therefore \( V^\mathfrak{c} \) contains at least 6 Ising vectors.

We have already seen that \( \psi_u(N_{M}(\langle \xi_u \rangle)) \simeq \text{Fi}_{24} \) faithfully acts on \( \mathfrak{X} \). Let \( e^0 \) and \( e^1 \) be Ising vectors of \( V^\mathfrak{c} \) such that \( \tau_{e^0} = x_u \). Then \( e^0 \) and \( e^1 \) generate a subalgebra \( U(e^0, e^1) \) isomorphic to the 3A-algebra such that \( W(u, w) \subset U(e^0, e^1) \). We can take an Ising vector \( e^2 \) of \( V^\mathfrak{c} \) orthogonal to both \( e^0 \) and \( e^1 \). Then \( g e^2 \in \mathfrak{X} \) and \( \tau_{g e^2} \in C_M(x_u) \) for any \( g \in \text{Aut}(\mathfrak{X}) \). As we have shown, \( \ker \psi_u = \langle x_u \rangle \) and hence \( \{ \psi_u(\tau_{g e^2}) = g \psi_u(\tau_{e^2}) g^{-1} \mid g \in \text{Aut}(\mathfrak{X}) \} \) define non-trivial involutions on \( \mathfrak{X} \). Thus the subgroup generated by \( \{ \psi_u(\tau_{g e^2}) \mid g \in \text{Aut}(\mathfrak{X}) \} \) is normal in \( \text{Aut}(\mathfrak{X}) \) and isomorphic to \( C_M(\langle x_u \rangle) \). Since \( \text{Out}(\text{Fi}_{24}) = 2 \) by [ATLAS] and \( \psi_u(N_M(\langle x_u \rangle)) \) contains an outer involution defined by a simple \( c = 6/7 \) Virasoro vector \( v \in U(e^0, e^1) \cap \mathfrak{X} \), we see that

\[
\alpha(\text{Aut}(\mathfrak{X})) = \alpha(\psi_u(N_M(\langle x_u \rangle))) = \text{Aut}(\text{Fi}_{24}^\prime) \simeq \text{Fi}_{24}.
\]

This implies \( \text{Aut}(\mathfrak{X}) \simeq \ker \alpha \times \text{Fi}_{24} \). The Griess algebra of \( V^\mathfrak{c} \) is 57478-dimensional and it has a decomposition \( 57478 = \mathbf{1} \oplus 57477 \) as a module over \( \psi_u(C_M(\langle x_u \rangle)) \simeq \text{Fi}_{24}^\prime \). So the Griess algebra of \( V^\mathfrak{c} \) is spanned by its conformal vector \( \omega_{\text{Fi}_{24}} \) and Ising vectors \( \{ g e^2 \mid g \in C_M(\langle x_u \rangle) \} \). Now let \( h \in \ker \alpha \). Then \( h \) acts by a scalar on the 57477-dimensional component, say \( \lambda \). Write \( e^2 = p \omega_{\text{Fi}_{24}} + x \) with \( p \in \mathbb{C} \) and \( x \in 57477 \). Then \( x \neq 0 \) since the central charge of \( \omega_{\text{Fi}_{24}} \) is equal to \( 24 - 4/5 \) and \( h e^2 = p \omega + \lambda x \). Since both \( e^2/2 \) and \( h e^2/2 \) are idempotents in the Griess algebra, it follows that \( \lambda = 1 \) and \( h = 1 \). Therefore \( \ker \alpha = 1 \) and we obtain the desired isomorphism \( \text{Aut}(\mathfrak{X}) = \psi_u(N_M(\langle x_u \rangle)) \simeq \text{Fi}_{24} \).

5.2 The 3-transposition property

In this section, we will establish the correspondence between derived \( c = 6/7 \) Virasoro vectors in \( V^\mathfrak{c} \) and 2C-involutions of the Fischer group \( \text{Fi}_{24} \).

Take a 2C-involution \( t \) of \( \text{Fi}_{24} \) and consider the decomposition of \( V^\mathfrak{c} \) as a module over \( C_{\text{Fi}_{24}}(t) \simeq \text{Fi}_{23} \). In the computation of (5.3) we have already obtained that

\[
V^\mathfrak{c}_2 = \mathbf{1} \oplus \mathbf{1} \oplus 30888 \oplus 25806 \oplus 782 \quad \text{(5.6)}
\]

as a \( C_{\text{Fi}_{24}}(t) \)-module. Therefore, the fixed point subalgebra \( (V^\mathfrak{c}_2)^{C_{\text{Fi}_{24}}(t)} \) is 2-dimensional and forms a commutative associative algebra spanned by two mutually orthogonal Vira-
soro vectors. In order to determine the central charge of the shorter Virasoro vector in $(VF)_{\mathbb{C}P_{24}(t)}$, we use the 3A-algebra for the Monster to obtain suitable decompositions.

Let $v$ be a derived $c = 6/7$ Virasoro vector in $\text{Com}_{VF}(\text{Vir}(u))$ with respect to $u$. Let $U \subset V^2$ be the corresponding sub-VOA isomorphic to $U_{3A}$, that is, $u + v$ is the conformal vector of $U$, and let $e^0$, $e^1$ and $e^2$ be Ising vectors of $U$ such that $\tau_{e_0} \tau_{e^1} = \xi_u$.

For an irreducible $U$-module $M$, we set $H_M := \text{Hom}_U(M, V^\natural)$. Then we have the decomposition

$$V^\natural = \bigoplus_{M \in \text{Irr}(U_{3A})} M \otimes H_M. \quad (5.7)$$

Clearly $H_M$ forms a module over the commutant subalgebra $\text{Com}_{V^\natural}(\text{Vir}(u))$.

**Lemma 5.6.** The top weight $h(H_M)$ and the dimension $d(H_M)$ of the top level of the $\text{Com}_{V^\natural}(U)$-modules $H_M$ are given by the following table

| $M$ | $U(0)$ | $U(1/7)$ | $U(5/7)$ | $U(2/5)$ | $U(19/35)$ | $U(4/35)$ |
|-----|--------|----------|----------|----------|------------|-----------|
| $h(H_M)$ | 0      | 9/7      | 8/5      | 51/35    | 66/35      |
| $d(H_M)$ | 1      | 25806    | 782      | 5083     | 3588       | 60996     |

(5.8)

Moreover, one has $\dim(H_{U(0)})_2 = 30889$.

**Proof:** By Lemma 2.6 and the classification of irreducible $U_{3A}$-modules in Theorem 4.4, we know that the possible eigenvalues of $u_{(1)}$ on $G^2$ are $0, \frac{1}{15}, \frac{2}{5}$ and $\frac{2}{3}$, and those for $v_{(1)}$ are $0, \frac{1}{7}, \frac{5}{7}, \frac{4}{5}, \frac{1}{21}$ and $\frac{10}{21}$. Applying a similar computation as in the proof of Lemma 5.2 of [HLY], we obtain the Lemma.

On $\text{Com}_{V^\natural}(\text{Vir}(u))$, one can define the $\sigma$-involution $\sigma_v$ as in (2.4), which coincides with $\psi_u(\tau_{e^0})$ by Lemma 4.8.

**Proposition 5.7.** The involution $\sigma_v = \psi_u(\tau_{e^0})$ is a 2C-element of $\psi_u(N_M(\langle \xi_u \rangle)) \simeq \text{Fi}_{24}$.

**Proof:** Let us consider the trace of $\sigma_v = \psi_u(\tau_{e^0})$ on the Griess algebra of $VF^\natural = \text{Com}_{V^\natural}(\text{Vir}(u))$. By (5.8) and (4.5), one has

$$\text{Tr}_{VF^\natural} \sigma_v = 1 + \dim(H_{U(0)})_2 + \dim H_{U(5/7)} - \dim H_{U(1/7)} = 5866,$$

which coincides only with the trace of a 2C-involution of $\text{Fi}_{24}$ on $VF^\natural_2 = 1 \oplus 57477$ by [ATLAS].

**Lemma 5.8.** The set $\{e^0, e^1, e^2\}$ consisting of the three Ising vectors of $U$ is stabilized by $\psi_u^{-1}(C_{\text{Fi}_{24}}(\psi_u(\tau_{e^0})))$.  

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Proof: Take any \( g \in \psi_u^{-1}C_{Fi24}(\psi_u(\tau_e)) \subset N_{\mathcal{M}}(\langle \xi_u \rangle) \). Then \([g, \tau_e] \in \langle \xi_u \rangle\) and hence \( \tau_{ge} = g\tau_e g^{-1} \in \{\tau_e, \xi_u \tau_e, \xi_u^2 \tau_e\} \). Since \( \tau_e \tau_e = \xi_u \) we get \( \{\xi_u \tau_e, \xi_u^2 \tau_e\} = \{\tau_e, \tau_e^3\} \). Thus, \( ge^0 \in \{e^0, e^1, e^2\} \) by the one-to-one correspondence [Mi1, Hö]. Similarly, we also have \( ge^1 \in \{e^0, e^1, e^2\} \) and \( ge^2 \in \{e^0, e^1, e^2\} \).

Proposition 5.9. A derived \( c = 6/7 \) Virasoro vector \( v \in \text{Com}_{V\mathcal{F}}(\text{Vir}(u)) \) with respect to \( u \) is fixed by the centralizer of the 2C-involution \( \psi_u(\tau_e) \) of the Fischer group \( Fi_{24} \).

Proof: Let \( \alpha = e^0 + e^1 + e^2 \). Then, \( \alpha \) is fixed by \( \psi_u^{-1}C_{Fi24}(\psi_u(\tau_e)) \) by Lemma 5.8. On the other hand, by Lemma 4.2 and (4.2), we have

\[
\alpha = e^0 + e^1 + e^2 = \frac{15}{32}u + \frac{21}{16}v \quad \text{and} \quad \alpha^2 = 2 \left( \frac{15}{32} \right)^2 u + 2 \left( \frac{21}{16} \right)^2 v,
\]

where \( v \) is the derived \( c = 6/7 \) Virasoro vector in \( U \). Thus,

\[
v = \frac{16}{567}(16\alpha^2 - 15\alpha).
\]

Hence, \( v \) is fixed by the centralizer of the 2C-involution \( \psi_u(\tau_e) \) in \( Fi_{24} \).

Now we establish the one-to-one correspondence between 2C-involutions of \( Fi_{24} \) and derived \( c = 6/7 \) Virasoro vectors of \( VF^2 \), which is one of our main results.

Theorem 5.10. The map which associates a derived \( c = 6/7 \) Virasoro vector to its \( \sigma \)-involution defines a bijection between the set of all derived \( c = 6/7 \) Virasoro vectors in \( \text{Com}_{V\mathcal{F}}(\text{Vir}(u)) \) with respect to \( u \) and the 2C-conjugacy class of \( Fi_{24} = \psi_u(N_{\mathcal{M}}(\langle \xi_u \rangle)) \).

Proof: The map of the theorem is equivariant with respect to the natural action of \( \psi_u(N_{\mathcal{M}}(\langle \xi_u \rangle)) \) on the derived vectors and the conjugation action of \( Fi_{24} \) on the set of its 2C-involutions, respectively.

As seen in the proof of Theorem 5.1, the vector \( u \) is contained in a sub-VOA isomorphic to the 3A-algebra \( U_{3A} \). Thus there exists at least one derived \( c = 6/7 \) Virasoro vector with respect to \( u \) in \( \text{Com}_{V\mathcal{F}}(\text{Vir}(u)) \). The transitivity of the conjugation action on the 2C-involutions shows now the surjectivity of the map.

For the injectivity, fix a 2C-involution \( t \) of \( Fi_{24} = \psi_u(N_{\mathcal{M}}(\langle \xi_u \rangle)) \). By Proposition 5.9, any derived \( c = 6/7 \) Virasoro vector \( v \) in \( \text{Com}_{V\mathcal{F}}(\text{Vir}(u)) \) such that \( \sigma_v = t \) is contained in \( (VF^2)^{C_{Fi_{24}}(t)} \). We have seen in (5.6) that the fixed point subalgebra \( (VF^2)^{C_{Fi_{24}}(t)} \) is spanned by two mutually orthogonal Virasoro vectors. Hence \( v \) must be the unique shorter Virasoro vector of \( (VF^2)^{C_{Fi_{24}}(t)} \) of central charge \( c = 6/7 \).

The proof gives also the following corollary.

Corollary 5.11. Every 2C-involution \( t \) of the Fischer group \( Fi_{24} \) defines an unique derived \( c = 6/7 \) Virasoro vector of the fixed point subalgebra \( (VF^2)^{C_{Fi_{24}}(t)} \).
As a consequence of Theorem 4.10 and Theorem 5.10, we also recover:

**Corollary 5.12.** *The 2C involutions of the Fischer group $\text{Fi}_{24}$ satisfy the 3-transposition property.*

**Remark 5.13.** We expect that the full automorphism group of $VF^2$ is actually $\text{Fi}_{24}$. Because of Theorem 5.5, it suffices to show $VF^2 = \mathcal{X}$, that is, $VF^2$ is generated by its Griess algebra. This is a technically difficult problem since we do not know a nice embedding of $W(\frac{4}{5})$ into $V^2$ to study the commutant subalgebra $VF^2 = \text{Com}_{V^2}(W(\frac{4}{5}))$. It is conjectured in [DM] that one can obtain $V^2$ from $V$ by the $Z_3$-orbifold construction where the $Z_3$-automorphism is induced by an automorphism of the $A_2$ lattice via an embedding $\sqrt{2}A_2^1 \hookrightarrow \Lambda$. If this conjectural $Z_3$-orbifold construction is established, we obtain a natural embedding of $W(\frac{4}{5})$ into $V^2$ and then we can solve the question above immediately by a decomposition of $V^2$ given in [KLY].

### 5.3 Embedding of $U_{F(nX)}$ into $VF^2$

We recall the definition of $V_{F(nX)}$ from Section 3.1. We have a full rank sublattice $Q \oplus E_6 \simeq A_2 \oplus E_6$ of $E_8$. Since the index of $A_2 \oplus E_6$ in $E_8$ is three, we have a coset decomposition

$$E_8 = A_2 \oplus E_6 \sqcup (\delta + A_2 \oplus E_6) \sqcup (2\delta + A_2 \oplus E_6)$$

with some $\delta \in E_8$ and correspondingly we obtain a decomposition

$$V_\sqrt{2}E_8 = V_\sqrt{2}(A_2 \oplus E_6) \oplus V_\sqrt{2}(\delta + A_2 \oplus E_6) \oplus V_\sqrt{2}(2\delta + A_2 \oplus E_6).$$

Define $\eta \in \text{Aut}(V_\sqrt{2}E_8)$ by

$$\eta = \begin{cases} 
1 & \text{on } V_\sqrt{2}(A_2 \oplus E_6), \\
 e^{2\pi \sqrt{-1}/3} & \text{on } V_\sqrt{2}(\delta + A_2 \oplus E_6), \\
 e^{4\pi \sqrt{-1}/3} & \text{on } V_\sqrt{2}(2\delta + A_2 \oplus E_6).
\end{cases}$$

Then $\eta$ is clearly in $\mathcal{F}_{nX}$, see (3.10). Indeed, $\mathcal{F}_{nX}$ is generated by $\eta$ and $\rho_{nX}$. Note that we can write down $\rho_{nX}$ in exponential form

$$\rho_{nX} = \exp(2\pi \sqrt{-1} \gamma_{nX}^{nX}/n) \quad \text{with suitable } \gamma_{nX}^{nX} \in L_{nX},$$

which also defines an automorphism of $V_\sqrt{2}E_8$ and fixes $V_\sqrt{2}L_{nX}$ pointwisely.

**Remark 5.14.** Recall $\tilde{\omega}_Q$ with $Q \simeq A_2$ is a simple extendable $c = 4/5$ Virasoro vector in a lattice VOA $V_\sqrt{2}Q$ and $U_{F(nX)}$ equals the commutant subalgebra $\text{Com}_{V_{F(nX)}}(\text{Vir}(\tilde{\omega}_Q))$ in $V_{F(nX)}$. Moreover, $\rho_{nX}$ fixes $\tilde{\omega}_Q$. Let $U^1$ be the subalgebra generated by $\hat{e}$ and $\eta(\hat{e})$, and
Proposition 5.15. For any $nX = 1A$, $2A$ or $3A$, the VOA $V_{F(nX)}$ can be embedded into the Moonshine VOA $V^\natural$.

Proof: As we have shown in Section 3.2, $V_{F(1A)}$ is isomorphic to the monstrous $3A$-algebra $U_{3A}$ and $V_{F(2A)}$ is isomorphic to the monstrous $6A$-algebra $U_{6A}$ discussed in [LYY2]. It is shown in [LM] that both $U_{3A}$ and $U_{6A}$ are subalgebras of $V^\natural$ and therefore $V_{F(1A)}$ and $V_{F(2A)}$ are also contained in $V^\natural$. That $V_{F(3A)} \simeq M_{4}$ is contained in $V^\natural$ will be shown in Appendix A.

Finally, we will establish our main theorem.

Theorem 5.16. Let $u$ be a simple extendable $c = 4/5$ Virasoro vector in $V^\natural$ such that $u \in (V^\natural)_{C_{6}M}(\xi_{6})$. Then for any $nX = 1A$, $2A$, $3A$, the VOA $U_{F(nX)}$ can be embedded into $VF^\natural = \text{Com}_{V^\natural}(\text{Vir}(u))$. Moreover, $\sigma_{\tilde{v}}\sigma_{\tilde{v}'}$ belongs to the conjugacy class $nX$ of $Fi_{24} \subseteq \text{Aut}(VF^\natural)$.

Proof: First we embed $V_{F(nX)}$ into $V^\natural$ using Proposition 5.15. Let $e$, $e'$ be a pair of Ising vectors of $V_{F(nX)}$ that generate a subalgebra $U$ such that $\tilde{\omega}_{Q} \in U$ and $U \simeq U_{3A}$. Since pairs of Ising vectors in $V^\natural$ generating the $3A$-algebra are mutually conjugate under $\text{Aut}(V^\natural)$, we may identify $\tilde{\omega}_{Q}$ with $u$ by Theorem 5.1. Thus, we have

$$U_{F(nX)} = \text{Com}_{V_{F(nX)}}(\text{Vir}(\tilde{\omega}_{Q}(E_{6}))) \subseteq \text{Com}_{V^\natural}(\text{Vir}(u)) = VF^\natural$$

as desired.

Next we will show that $h := \sigma_{\tilde{v}}\sigma_{\tilde{v}'}$ belongs to the class $nX$ of $Fi_{24} = \text{Aut}(X)$. Note that $\tilde{v}, \tilde{v}' \in VF^\natural \subset X$. Recall that there is an exact sequence

$$1 \rightarrow \langle \xi_{u} \rangle \rightarrow N_{\mathcal{M}}(\langle \xi_{u} \rangle) \rightarrow \text{Aut}(X) \simeq Fi_{24} \rightarrow 1$$

with the projection map $u : N_{\mathcal{M}}(\langle \xi_{u} \rangle) \rightarrow \text{Aut}(X)$. Let $e^{1}$ and $e^{2}$ be Ising vectors in $V_{nX}$ such that $\psi_{u}(\tau_{e^{1}}) = \sigma_{\tilde{v}}$ and $\psi_{u}(\tau_{e^{2}}) = \sigma_{\tilde{v}'}$. Set $g = \tau_{e^{1}}\tau_{e^{2}}$. Then $h = \psi_{u}(g)$ and the inverse image $\psi_{u}^{-1}(\langle h \rangle)$ has order $3n$ and is generated by $\xi_{u}$ and $g$.

1A case: In this case, $\tilde{v} = \tilde{v}'$ and hence $h = \sigma_{\tilde{v}}\sigma_{\tilde{v}'}$ belongs to the class 1A.

2A case: In this case, $V_{F(2A)} \simeq U_{6A}$. Then $g = \tau_{e^{1}}\tau_{e^{2}}$ has order 2 or 6 and the group generated by $\xi_{u}$ and $g$ is a cyclic group of order 6 which is generated by a $6A$-element of $\mathcal{M}$. Let $t$ be the unique involution in $\langle \xi_{u}, g \rangle$. Then by [ATLAS], $t$ belongs to the 2A
conjugacy class of $\mathbb{M}$ and $C_{\mathbb{M}}(t)$ is isomorphic to a double cover of the Baby Monster $\mathbb{B}$. Thus we have an exact sequence

$$1 \longrightarrow \langle t \rangle \longrightarrow C_{\mathbb{M}}(t) \xrightarrow{\varphi_t} \mathbb{B} \longrightarrow 1.$$ 

Since $\langle t \rangle$ and $\langle \xi_u \rangle$ are unique order 2 and order 3 subgroups in $\langle \xi_u, g \rangle$ and $t$ commutes with $\xi_u$, we have

$$N_{\mathbb{M}}(\langle \xi_u, g \rangle) = N_{\mathbb{M}}(\langle \xi_u, t \rangle) = C_{\mathbb{M}}(t) \cap N_{\mathbb{M}}(\langle \xi_u \rangle).$$

Set $G = N_{\mathbb{M}}(\langle \xi_u, g \rangle)$. Then

$$\varphi_t(G) = N_{\mathbb{B}}(\langle \varphi_t(\xi_u) \rangle) \quad \text{and} \quad \psi_u(G) = C_{\text{Fi}_{24}}(\psi_u(t)).$$

Note that $\varphi_t(\xi_u)$ has order 3 and $\psi_u(g) = \psi_u(t)$ has order 2 since $(2, 3) = 1$. By comparing the 3-local subgroups of $\mathbb{B}$ and the 2-local subgroups of $\text{Fi}_{24}$ in [ATLAS], we have

$$\varphi_t(G) \simeq S_3 \times \text{Fi}_{22}:2 \quad \text{and} \quad \psi_u(G) = C_{\text{Fi}_{24}}(\psi_u(g)) \simeq (2 \times 2.\text{Fi}_{22}):2.$$ 

Thus, $h = \psi_u(g)$ belongs to the conjugacy class $2A$ of $\text{Fi}_{24}$ by [ATLAS].

**3A case:** In this case, $V_{F(3A)} \simeq M_{C_4}$, the ternary code VOA associated to the tetra code $C_4$ and $\xi_u$, $\tau_{e_1}$ and $\tau_{e_2}$ generate a subgroup of the shape $3^2:2$, which has exactly 4 distinct subgroups of order 3. Since $V_{F(3A)} \supset W(\frac{1}{2}) \otimes 4$ and each $W(\frac{1}{2})$ defines a non-trivial subgroup of order 3, all order 3 elements of $\langle \xi_u, \tau_{e_1}, \tau_{e_2} \rangle$ belong to the conjugacy class $3A$ of $\mathbb{M}$.

Let $g = \tau_{e_1}\tau_{e_2}$. Then $\xi_u$ and $g$ generate a 3A-pure elementary abelian 3-subgroup of order $3^2$ in $\mathbb{M}$. The normalizer $N_{\mathbb{M}}(\langle \xi_u, g \rangle)$ has the shape $(3^2 \times O^+_8(3)).S_4$ while the centralizer $C_{\mathbb{M}}(\langle \xi_u, g \rangle)$ has the shape $3^2 \times O^+_8(3)$ (cf. [Wi] and page 234 of [ATLAS]). Thus, $N_{\mathbb{M}}(\langle \xi_u, g \rangle)$ acts (by conjugation) on $\langle \xi_u, g \rangle$ as $2S_4(\simeq \text{GL}_2(3))$ and $S_4$ acts as permutations of the 4 distinct subgroups of order 3 in $\langle \xi_u, g \rangle$. Thus, $N_{\mathbb{M}}(\langle \xi_u \rangle) \cap N_{\mathbb{M}}(\langle \xi_u, g \rangle)$ has the shape $(3^2 : 2 \times O^+_8(3)).S_3$ and $\psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle) \cap N_{\mathbb{M}}(\langle \xi_u, g \rangle))$ has the shape $S_3 \times O^+_8(3).S_3$. Since $\psi_u(\langle \xi_u, g \rangle) = \psi_u(g)$, we see that $\psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle) \cap N_{\mathbb{M}}(\langle \xi_u, g \rangle))$ normalizes the subgroup $\langle \psi_u(g) \rangle$ and hence $\psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle) \cap N_{\mathbb{M}}(\langle \xi_u, g \rangle)) < N_{\text{Fi}_{24}}(\langle \psi_u(g) \rangle)$.

By [ATLAS], page 207, $\psi_u(N_{\mathbb{M}}(\langle \xi_u \rangle) \cap N_{\mathbb{M}}(\langle \xi_u, g \rangle))$ is isomorphic to $N_{\text{Fi}_{24}}(3A)$, where $N_{\text{Fi}_{24}}(3A)$ denotes the normalizer of a cyclic subgroup generated by a 3A-element in $\text{Fi}_{24}$, and it is a maximal subgroup of $\text{Fi}_{24}$. Thus, $N_{\text{Fi}_{24}}(\langle \psi_u(g) \rangle) \simeq N_{\text{Fi}_{24}}(3A)$ and $h = \psi_u(g)$ belongs to the conjugacy class $3A$ of $\text{Fi}_{24}$. 

**Remark 5.17.** We note that in the 3A case, the group $N_{\mathbb{M}}(\langle \xi_u, g \rangle)$ acts on $V_{F(3A)} \simeq M_{C_4}$ as $(3^2 : 2)S_4$, which is isomorphic to $\text{Aut}(M_{C_4})$ (cf. Remark 3.7). In fact, by the similar argument as in Remark 3.9, one can show without using the property of the Monster that the whole group $\text{Aut}(M_{C_4})$ can be extended to a subgroup of $\text{Aut}(V^2)$ since all irreducible modules of $M_{C_4}$ can be embedded into $V_{\sqrt{\mathbb{E}_8}}$ and are $\text{Aut}(M_{C_4})$-invariant.
A Appendix: Embedding of $V_{F(3A)}$ into $V^4$

In this Appendix, we give an embedding of $V_{F(3A)}$ into the moonshine VOA which completes the proof of Proposition 5.15. We achieve this by providing an explicit embedding of $V_{F(3A)} \simeq M_{C_4}$ into $V^+_A \subset V^4$, where $M_{C_4}$ refers to the ternary code VOA associated to the tetra code $C_4$ constructed in [KMY] (see Eq. (3.13)). The main idea is essentially given in [GL2] and [GL3] (see also [LM]).

First, we consider some automorphisms of $V_{A_2}$ and $V_{E_8}$. Let $a_i = E_{i,i} - E_{i+1,i+1}$, $x_i^+ = E_{i,i+1}$ and $x_i^- = E_{i+1,i}$ for $i = 1, 2$, where $E_{i,j}$ denotes a $3 \times 3$ matrix whose $(i,j)$-entry is 1 and others are 0. Then $\{a_i, x_i^\pm | i = 1, 2\}$ is a set of Chevalley generators of $\mathfrak{sl}_3(\mathbb{C})$. Let $\alpha_1, \alpha_2$ be simple roots of the root lattice $A_2$. The weight one subspace of $(V_{A_2})_1$ of the lattice VOA $V_{A_2} \simeq M_{C_{A_2}}(1) \otimes \mathbb{C}[A_2]$ forms a Lie algebra isomorphic to $\mathfrak{sl}_3(\mathbb{C})$ by the following correspondence:

$$a_i \mapsto \alpha_{i(-1)} \mathbb{1}, \quad x_i^\pm \mapsto \pm e^{\pm \alpha_i}. \quad \text{(A.1)}$$

The automorphism group of the lattice VOA $V_{A_2}$ is isomorphic to the automorphism group of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$, which is isomorphic to $\text{PSL}_3(\mathbb{C}) \rtimes \mathbb{Z}_2$. Since $V_{A_2}$ is generated by its weight one subspace, $\text{SL}_3(\mathbb{C})$ acts on $V_{A_2}$ via the adjoint map [FLM] under the identification above. Namely, $P \in \text{SL}_3(\mathbb{C})$ acts by $A \mapsto AP = P^{-1}AP$ for $A \in \mathfrak{sl}_3(\mathbb{C}) \simeq (V_{A_2})_1$.

Let $\zeta = e^{2\pi i/3}$ be a cubic root of unity and consider the elements

$$\tau := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad s := \frac{1}{\sqrt{3}} \begin{pmatrix} \zeta & \zeta^2 & 1 \\ \zeta^2 & \zeta & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{(A.2)}$$

in $\text{SU}_3 \subset \text{SL}_3(\mathbb{C})$. Then

$$r := s^{-1} \tau s = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{(A.3)}$$

Let $A_2^*$ be the dual lattice of $A_2$. We can describe the action of $r$ and $\tau$ on $V_{A_2}$ and the $V_{A_2}$-module $V_{A_2}$ using this identification more explicitly. Let $\delta = \alpha_1 + \alpha_2$ be the half sum of positive roots, i.e., the Weyl vector. Then $r(u \otimes e^\alpha) = \zeta^{(\delta,\alpha)} u \otimes e^\alpha$ for all $u \in M(1)$ and $\alpha \in A_2^*$. Since $\tau$ normalizes the canonical Cartan subalgebra of $\mathfrak{sl}_3(\mathbb{C})$, it induces an order 3 automorphism $\bar{\tau}$ of the root lattice $A_2$ explicitly given by

$$\bar{\tau} : \quad \alpha_1 \mapsto \alpha_2 \mapsto - (\alpha_1 + \alpha_2) \mapsto \alpha_1.$$
Now fix a sublattice of type $A_2^4 = A_2 \perp A_2 \perp A_2 \perp A_2$ in $E_8$. Then $E_8/A_2^4 \subset (A_2^4/A_2)^4 \simeq (\mathbb{Z}/3\mathbb{Z})^4$ can be identified with the tetra code $C_4$. The corresponding inclusion $V_{A_2^4}^4 \subset V_{E_8}$ induces an action of $\text{SL}_3(\mathbb{C})^4$ on $V_{E_8}$ by automorphisms where the center $(\mathbb{Z}/3\mathbb{Z})^4$ of $\text{SL}_3(\mathbb{C})^4$ acts via its quotient $\hat{C}_4$, the dual group of $C_4$.

Define
\[
\tilde{h}_1 = 1 \otimes \tau \otimes \tau \otimes \tau, \quad \tilde{h}_2 = \tau \otimes \tau \otimes \tau^{-1} \otimes 1, \\
\rho_1 = 1 \otimes r \otimes r \otimes r, \quad \rho_2 = r \otimes r \otimes r^{-1} \otimes 1, \\
\tilde{s} = s \otimes s \otimes s \otimes s
\]
as automorphisms of $V_{E_8}$.

Let $\mathcal{G}$ be the subgroup generated by $\tilde{h}_1$ and $\tilde{h}_2$ and $\mathcal{F}$ be the subgroup generated by $\rho_1$ and $\rho_2$. Then $\mathcal{F} \simeq \mathcal{G} \simeq 3^2$. Moreover, by (A.3), we have
\[
\mathcal{F} = \tilde{s}^{-1} \mathcal{G} \tilde{s}. \tag{A.4}
\]

Remark A.1. We shall note that $\{x \in E_8 \mid \langle x, (0, 0, 0, 0) \rangle \in 3\mathbb{Z}\} \simeq E_6 \perp A_2$ and $\{x \in E_8 \mid \langle x, (0, 0, 0, 0) \rangle \in 3\mathbb{Z}\} \simeq E_6 \perp A_2$. Moreover, by (A.3), we have
\[
K := \{x \in E_8 \mid \langle x, (0, 0, 0, 0) \rangle \in 3\mathbb{Z} \text{ and } \langle x, (0, 0, 0, 0) \rangle \in 3\mathbb{Z}\} \simeq A_2^4. \tag{A.5}
\]
Thus, we have $V_{E_8}^{(\rho_1)} \simeq V_{E_8}^{(\rho_2)} \simeq V_{E_6} \otimes V_{A_2}$ and $V_{E_8}^{(\rho_1, \rho_2)} \simeq V_{A_2^4}$. However, we shall remark that the sublattice $K \simeq A_2^4$ obtained in (A.5) is not the same $A_2^4$ used to define $h_i$ and $\rho_i$, $i = 1, 2$.

Set
\[
h_1 = \text{id} \oplus \bar{\tau} \oplus \bar{\tau} \oplus \bar{\tau} \quad \text{and} \quad h_2 = \bar{\tau} \oplus \bar{\tau} \oplus \bar{\tau}^{-1} \oplus \text{id}
\]
Then $h_1$ and $h_2$ can be considered as isometries of $E_8 \subset (A_2^4)^4$ induced by $\tilde{h}_1$ and $\tilde{h}_2$ and they also generate a subgroup isomorphic to $3^2$.

Now consider the following sublattices of an orthogonal sum $E_8 \perp E_8$. Let
\[
R = \{(x, x) \in E_8 \perp E_8 \mid x \in E_8\},
\]
\[
R^1 = \{(x, h_1 x) \in E_8 \perp E_8 \mid x \in E_8\},
\]
\[
R^2 = \{(x, h_2 x) \in E_8 \perp E_8 \mid x \in E_8\}.
\]
Then $R \simeq R^1 \simeq R^2 \simeq \sqrt{2}E_8$. Note also that $R^1 = (\text{id} \oplus h_1)R$ and $R^2 = (\text{id} \oplus h_2)R$.

Let
\[
e_R = \frac{1}{16} \omega_R + \frac{1}{32} \sum_{\alpha \in R_4} e_\alpha
\]
be the Ising vector associated to $R$ defined by (3.11), where $R_4 = \{\alpha \in R \mid \langle \alpha, \alpha \rangle = 4\}$.  

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Now let $\tilde{F} = \{ \text{id} \otimes \rho \mid \rho \in \mathcal{F} \}$. Then $\tilde{F}$ stabilizes the lattice VOA $V_{R}$ and by Remark A.1, the fixed point sub-VOA $V_{R}^{\tilde{F}} \simeq V_{\sqrt{3}A_{3}}$. By the definition of $V_{F(3A)}$ (see the 3A case in Sec. 3.2), we can obtain a sub-VOA isomorphic to $V_{F(3A)}$ in $V_{R} \simeq V_{\sqrt{2}E_{8}}$. By Lemma 3.8, this $V_{F(3A)}$ is generated by $e_{R}$, $(\text{id} \otimes \rho_{1})e_{R}$ and $(\text{id} \otimes \rho_{2})e_{R}$.

Consider the automorphism $\hat{\sigma} := \tilde{s} \otimes \tilde{s}^{-1}$ of $V_{E_{8}} \bigoplus V_{E_{8}} \simeq V_{E_{8}} \otimes V_{E_{8}}$. The following lemma is essentially proved in [GL3] with some trivial modification.

**Lemma A.2** (Lemma 2.19 of [GL3]). We have $\hat{\sigma}^{-1}e_{R} \in V_{R}$.

Next, we will show that $\hat{\sigma}^{-1}e$ is in fact in $V_{R}^{+}$. For any even lattice $L$, let $\theta : V_{L} \rightarrow V_{L}$ be the involution defined by

$$
\theta : \alpha_{1}(-n_{1}) \cdots \alpha_{k}(-n_{k}) \otimes e^{\alpha} \mapsto (-1)^{k}\alpha_{1}(-n_{1}) \cdots \alpha_{k}(-n_{k}) \otimes e^{-\alpha} \quad \text{(A.6)}
$$

(cf. [FLM, Mi1]). Note that if $L$ is a root lattice of type $A_{2}$, by identifying $(V_{A_{2}})_{1}$ with $sl_{3}(\mathbb{C})$ as in (A.1) we have

$$
A^{\theta} = -tA, \quad A \in sl_{3}(\mathbb{C}).
$$

If we extend $\theta$ to a map on $V_{A_{2}}$ then we have

$$
\text{Aut}(SL_{3}(\mathbb{C})) \simeq SL_{3}(\mathbb{C}) \rtimes \langle \theta \rangle
$$

where $\theta$ acts by $X \mapsto tX^{-1}$ on $SL_{3}(\mathbb{C})$.

**Lemma A.3.** The Ising vectors $\hat{\sigma}^{-1}e_{R}$, $(\text{id} \otimes \tilde{h}_{1})\hat{\sigma}^{-1}e_{R}$ and $(\text{id} \otimes \tilde{h}_{2})\hat{\sigma}^{-1}e_{R}$ are fixed by $\theta$.

**Proof:** Let $s$ be defined as in (A.2). Then we have $s^{4} = 1$. Thus $s$ and $\theta$ generate a dihedral group of order 8. One has $\theta s \theta s^{-1} = \theta s^{-1} s = s^{2}$ as a direct calculation shows.

Now we consider the action of $\tilde{s}$ and $\theta$ on $V_{E_{8}}$. This action is given by the diagonal embedding of $(s, \theta)$ into $(\text{Aut}(SL_{3}(\mathbb{C})))^{4}$. We get $\theta \tilde{s} \theta \tilde{s}^{-1} = \theta \tilde{s}^{-1} \theta \tilde{s} = \tilde{s}^{2}$.

Since $s^{2}$ is a permutation matrix, $s^{2}$ normalizes the Cartan subalgebra $C_{a_{1}} + C_{a_{2}}$ of $sl_{3}(\mathbb{C})$. Moreover, $\tilde{s}^{2}$ normalizes the standard Cartan subalgebra of $(V_{E_{8}})_{1}$. That means

$$
\tilde{s}^{2}(M_{CE_{8}}(1)) = M_{CE_{8}}(1).
$$

Thus $\tilde{s}^{2}$ induce an isometry $\mu := \tilde{s}^{2}$ of the root lattice $E_{8}$ such that

$$
\tilde{s}^{2}(e^{\alpha}) = \epsilon(\alpha)e^{\mu \alpha} \quad \text{for } \alpha \in E_{8}, \quad \text{(A.7)}
$$

where $\epsilon(\alpha) = \pm 1$. 

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Since \( e^{(\alpha,\alpha)} = e^{(\alpha,0)}(-1)e^{(0,\alpha)} \) in \( V_{E_8 \oplus E_8} \simeq V_{E_8} \otimes V_{E_8} \), one has

\[
\theta \hat{\sigma} \hat{\theta} \hat{\sigma}^{-1}(e^{(\alpha,\alpha)}) = \theta \hat{\sigma} \hat{\theta} \hat{\sigma}^{-1}(e^{(\alpha,0)}(-1)e^{(0,\alpha)}) = (\theta \hat{s} \theta \hat{s}^{-1}(e^{(\alpha,0)}))(-1)(\theta \hat{s}^{-1} \theta \hat{s}(e^{(0,\alpha)})) = (e(\alpha)e^{(\mu,\mu)})(-1)(e(\alpha)e^{(0,\mu)}) = e^{(\mu \alpha, \mu \alpha)}
\]

for any \( \alpha \in E_8 \). Therefore, \( \theta \hat{\sigma} \hat{\theta} \hat{\sigma}^{-1} \) fixes \( e_R \) and so does \( \hat{\sigma} \hat{\sigma}^{-1} = \theta (\theta \hat{\sigma} \hat{\sigma}^{-1}) \) since \( \theta \) also fixes \( e_R \). Thus, \( \hat{\sigma}^{-1}e_R \) is fixed by \( \theta \). Since \( \text{id} \otimes \hat{h}_1 \) and \( \text{id} \otimes \hat{h}_2 \) commute with \( \theta \), \( \text{id} \otimes \hat{h}_1 \hat{\sigma}^{-1} e_R \) and \( \text{id} \otimes \hat{h}_2 \hat{\sigma}^{-1} e_R \) are also fixed by \( \theta \).

Using (A.4) we have that \( \hat{\sigma}^{-1}(V_{F(3A)}) \) is generated by \( \{(\text{id} \otimes \hat{h}) \hat{\sigma}^{-1}e_R \mid \hat{h} \in G\} \) or \( \{\hat{\sigma}^{-1}e_R, (\text{id} \otimes \hat{h}_1) \hat{\sigma}^{-1}e_R, (\text{id} \otimes \hat{h}_2) \hat{\sigma}^{-1}e_R\} \).

Since \( (\text{id} \oplus h_1)R = R^1 \) and \( (\text{id} \oplus h_2)R = R^2 \), we have \( (\text{id} \otimes \hat{h}_1) \hat{\sigma}^{-1}e_R \in V_{R^1}^+ \) and \( (\text{id} \otimes \hat{h}_2) \hat{\sigma}^{-1}e_R \in V_{R^2}^+ \), by Lemma A.3. Hence, \( \hat{\sigma}^{-1}(V_{F(3A)}) \subset V_{R^1+R^2}^+ \). Therefore, it remains to show that \( L = R + R^1 + R^2 \) can be embedded into the Leech lattice \( \Lambda \).

First, we recall the ternary Golay code construction of the Leech lattice \( \Lambda \) [CS].

Let \( S \) be an orthogonal sum of 12 copies of \( A_2 \). Then the discriminant group \( S^* / S \) has a natural identification with \( \mathbb{Z}_3^{12} \). The ternary Golay code \( \mathcal{C}_{12} \subset \mathbb{Z}_3^{12} \) can be defined using the tetracode \( \mathcal{C}_4 \) as the set

\[
\mathcal{C}_{12} = \{(c^0, c^+, c^-) \mid c^0, c^+, c^- \in \mathbb{Z}_3^4, c^0 + c^+ + c^- = -\sum_{i=1}^{4} c_i^0 \cdot (1, 1, 1, 1), c^+ - c^- \in \mathcal{C}_4\}.
\]

For each codeword \( x = (x_1, \ldots, x_{12}) \in \mathcal{C}_{12} \), let \( \gamma_x = (\gamma_{x_1}, \ldots, \gamma_{x_{12}}) \in S^* \) be some vector which modulo \( S \) gives the codeword \( x \). Then

\[
\mathcal{N} := \bigcup_{x \in \mathcal{C}_{12}} (\gamma_x + S)
\]

is isometric to the Niemeier lattice of type \( A_2^{12} \).

Let \( \delta = \alpha_1 + \alpha_2 \) be the half sum of positive roots of \( A_2 \) and let

\[
\hat{\delta} := (\delta, \delta, \delta, -\delta, -\delta, -\delta, \delta, \delta, \delta, \delta, \delta, \delta).
\]

Then

\[
\mathcal{N}^0 = \{\alpha \in \mathcal{N} \mid (\alpha, \hat{\delta}) \in 3\mathbb{Z}\}
\]

is a sublattice of index 3 without roots. Note that \( (\alpha_1, 0, \ldots, 0) + \frac{1}{3} \hat{\delta} \) has norm 4 and the lattice \( \mathcal{N}^0 + \mathbb{Z}((\alpha_1, 0, \ldots, 0) + \frac{1}{3} \hat{\delta}) \) is even unimodular without roots. Hence, it is isometric to the Leech lattice \( \Lambda \) (see Chapter 24 of [CS]).

Next, we construct some \( \sqrt{3}E_8 \) sublattices of \( \mathcal{N}^0 < \Lambda \). Note that

\[
\{(0^4, c, c) \mid c \in \mathcal{C}_4\} < \mathcal{C}_{12}.
\]
Thus
\[ \tilde{R} := \text{span}\{(0, \gamma_c + z, \gamma_c + z) \mid z \in A_2^4, \ c \in C_4\} \subset N. \]

Since \( E := \bigcup_{c \in C_4} (\gamma_c + A_2^4) \simeq E_8 \) it follows that \( \tilde{R} \simeq \sqrt{2}E_8 \).

Let \( E^1 := 0 \oplus E \oplus 0 \) and \( E^2 := 0 \oplus 0 \oplus E \). Then \((E^1, E^2) = 0\) and \( E^1 + E^2 < \frac{1}{3}\Lambda \).

For a codeword \( x \in C_{12} \), let \( h(x) := \tilde{\tau}^{x_1} \oplus \cdots \oplus \tilde{\tau}^{x_{12}} \) where \( \tilde{\tau} \) is the previously defined isometry of \( A_2 \). Then \( h(x) \) is an isometry of \( N \) and \( \Lambda [CS] \).

Consider now the codewords
\[ d^1 = (0, -1, -1, -1, 0, 0, 0, 0, 1, 1, 1, 1) \quad \text{and} \quad d^2 = (1, 1, 0, -1, 0, 0, 0, 0, 1, 1, -1, 0) \]
of \( C_{12} \). Define \( \hat{h}_1 := h(d^1) \) and \( \hat{h}_2 := h(d^2) \). Note that \( \hat{h}_1 \) and \( \hat{h}_2 \) act on \( E^1 + E^2 \) as \( \text{id} \oplus h_1 \) and \( \text{id} \oplus h_2 \), where \( h_1 = \text{id} \oplus \tilde{\tau} \oplus \tilde{\tau} \oplus \tilde{\tau} \) and \( h_2 = \tilde{\tau} \oplus \tilde{\tau} \oplus \tilde{\tau}^{-1} \oplus \text{id} \) in \( O(E_8) \) as previously defined. Then
\[
\begin{align*}
\tilde{R}^1 &= \hat{h}_1(\tilde{R}) = \{(0, \alpha, \hat{h}_1\alpha) \mid \alpha \in E^1\}, \\
\tilde{R}^2 &= \hat{h}_2(\tilde{R}) = \{(0, \alpha, \hat{h}_2\alpha) \mid \alpha \in E^1\}
\end{align*}
\]
are contained in \( \Lambda^0 < \Lambda \). It is also clear that \( \tilde{L} = \tilde{R} + \tilde{R}^1 + \tilde{R}^2 \) is isometric to \( L = R + R^1 + R^2 \).

**Remark A.4.** It is known that a 3-transposition group generated by three involutions \( t_1, t_2, t_3 \) such that \( t_3 \notin \langle t_1, t_2 \rangle \) and any two of them generate \( S_3 \) is either isomorphic to \( S_4, 3^{1+2}:2 \) or \( 3^{2}:2 \) (see Lemma 2.5 of [CH1]). By Remark 3.9, the \( \tau \)-involutions associated to \( \{\hat{\tau}^{-1}e_R, (\text{id} \otimes \hat{h}_1)\hat{\tau}^{-1}e_R, (\text{id} \otimes \hat{h}_2)\hat{\tau}^{-1}e_R\} \) generate a group of the shape \( 3^{2}:2 \) in \( \text{Aut}(V_{\Lambda}^+) \) and in \( \text{Aut}(V^+) \). Finally, we note that the centralizer of a 3C-element in \( C_0 \) has the shape \( 3^{1+4}:\text{Sp}_4(3) \times 2 \). It has a natural subgroup \( 3^{1+2}:2 \) which is generated by involutions with trace \(-8\) on the Leech lattice (see (10.35.3) of [G2] or [ATLAS]). Recall that an involution of trace \(-8\) on \( \Lambda \) has the fixed sublattice isometric to \( \sqrt{2}E_8 \) (cf. Theorem (10.15) of [G2] or Chapter 10 of [CS]). Thus, by formula (3.11), one can construct Ising vectors in \( V_{\Lambda}^+ < V^+ \) such that the corresponding Miyamoto involutions generate a group of the shape \( 3^{1+2}:2 \) in the Monster and the product of any two of them is in the conjugacy class \( 3A \). In this case, the sub-VOA generated by the corresponding Ising vectors will not be isomorphic to \( M_{C_4} \) but the authors do not know the exact structure of such a VOA.

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