Hidden Degeneracy in the Brick Wall Model of Black Holes

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Abstract

Quantum field theory in the near-horizon region of a black hole predicts the existence of an infinite number of degenerate modes. Such a degeneracy is regulated in the brick wall model by the introduction of a short distance cutoff. In this Letter we show that states of the brick wall model with non-zero energy admit a further degeneracy for any given finite value of the cutoff. The black hole entropy is calculated within the brick wall model taking this degeneracy into account. Modes with complex frequencies however do not exhibit such a degeneracy.

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1. Introduction

The event horizon plays a central role in various aspects of black hole physics. A simple illustration of this role of the horizon is provided by the brick wall model proposed by ’t Hooft [1, 2]. This model describes the low energy dynamics of a scalar field in the background of a Schwarzschild black hole. The density of states near the black hole horizon is however divergent, due to the presence of an infinite number of degenerate modes in the near-horizon region [1, 2, 3, 4]. Such a divergence is controlled in the brick wall model through the introduction of a short distance cutoff (i.e. a brick wall). For every finite value of the cutoff, this model provides a well defined expression for the density of states and entropy. The entropy of the black hole as determined from this model arises essentially from the near-horizon features of the system.

The above discussion suggests that the ambiguity in determining the density of states in the brick wall model is completely removed by the short distance cutoff. In this Letter we show that even with such a finite cutoff, there is still an additional degeneracy present in the system. This degeneracy is revealed by a study of the brick wall model in the near-horizon region of the black hole. For a given value of the cutoff, the allowed quantum states of this model are labelled by energy and other quantum numbers. We shall show that for any given finite value of the cutoff, every such allowed quantum state with non-zero energy has infinite degeneracy.

The infinite degeneracy found is expected to contribute to the black hole entropy. In order to analyze this contribution, we show that a dimensionless parameter can be introduced to regulate the degeneracy. The resulting expression for the black hole entropy is found to depend on the ratio of this parameter to the brick wall cutoff. The assumption of finiteness of this parameter is found to be consistent within the framework of the brick wall model.

Our analysis can be extended to the case of complex energy values. Such states are analogous to the quasinormal modes of the Schwarzschild black hole [5]. Analysis of the equation satisfied by these modes shows that the degeneracy disappears due to the presence of complex frequencies.

This Letter is organized as follows. In Section 2 we show how the degeneracy appears in the brick wall model. Similar analysis for modes with complex frequencies is performed in Section 3 and it is shown that such modes admit no degeneracy. In Section 4 we discuss how the density of states and black hole entropy are modified in presence of the new degeneracy. Section 5 concludes this Letter with some discussions.

2. Hidden Degeneracy in the Brick Wall Model

The brick wall model [1, 2] describes the low energy quantum dynamics of a scalar field $\phi$ in the background of a massive Schwarzschild black hole of finite mass $M$. For simplicity we shall take the scalar field to be massless. In the spherical polar coordinates coordinates $(r, \theta, \phi)$, the field theory description is assumed to be valid in the region $r > R + h$ where $R = 2M$ is the radius of the horizon and the brick wall cutoff $h(> 0)$ is small compared to $R$. The field $\phi$ is
assumed to satisfy the boundary condition
\[
\phi(r = R + h, \theta, \phi, t) = 0.
\] (2.1)

In addition, the whole system is assumed to be in a box of radius \(L\) which provides an infrared cutoff with the associated boundary condition
\[
\phi(r, \theta, \phi, t) = 0 \quad \text{for } r \geq L.
\] (2.2)

In the background of a Schwarzschild black hole, the field equation for modes with angular momentum \(l\) and energy \(E\) is given by [2]
\[
\left(1 - \frac{2M}{r}\right)^{-1} E^2 \phi + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r(r - 2M) \frac{\partial \phi}{\partial r} \right) - \frac{l(l + 1)}{r^2} \phi = 0.
\] (2.3)

In order to study the near horizon properties of the brick wall model, it is useful to introduce a new coordinate \(x \equiv r - R\). In the near-horizon region, i.e. when \(x \ll R\), Eqn. (3) can be written as
\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{x}{2} \frac{\partial}{\partial x} \left[ \frac{1}{4} + \frac{R^2 E^2}{x^2} \phi - \frac{l(l + 1)}{Rx} \phi \right] = 0.
\] (2.4)

In terms of a new field \(\chi\) defined by
\[
\chi = \sqrt{x} \phi,
\] (2.5)
Eqn. (2.4) can be written as
\[
\frac{\partial^2 \chi}{\partial x^2} + \frac{1}{4} + \frac{R^2 E^2}{x^2} \chi - \frac{l(l + 1)}{Rx} \chi = 0.
\] (2.6)

For small values of \(x\), the angular term in the field Eqn. (2.6) can be ignored as has been noted elsewhere as well [4]. Thus the KG equation in the near-horizon region can be written as
\[
\frac{\partial^2 \chi}{\partial x^2} + \frac{1}{4} + \frac{R^2 E^2}{x^2} \chi = 0.
\] (2.7)

The field equation (2.3) has been analyzed by 't Hooft using the WKB method [1, 2]. Such an analysis with the brick wall boundary conditions is assumed to provide a discrete set of eigenvalues. Let us consider Eqn. (2.7) with such a generic eigenvalue \(E\), the latter being now treated as a parameter already determined by the WKB analysis. We can think of as Eqn. (2.7) as a Schrödinger equation in presence of a singular potential [6, 7] with \(\chi\) now being interpreted as the zero eigenvalue solution of such an equation. The central question that we address here can now be stated as follows: for the allowed values of the parameter \(E\) and in the presence of the brick wall cutoff \(h\), does the singular quantum mechanical problem described by Eqn. (2.7) possess a unique solution or is there any additional degeneracy which is perhaps not manifest? In order to address this issue, it is useful to start by considering a different equation given by
\[
\frac{\partial^2 \chi}{\partial x^2} + \frac{1}{4} + \frac{R^2 E^2}{x^2} \chi = \eta^2 \chi.
\] (2.8)
In what follows, we shall interprete Eqn. (2.7) as the $\eta \to 0$ limit of Eqn. (2.8). We shall show below that in the limit when $\eta \to 0$, Eqn. (2.8) admits an infinite number of solutions even in the presence of the brick wall cutoff. More precisely, we will see that $\eta = 0$ is an accumulation point for the solutions of Eqn. (2.8).

We now proceed to solve Eqn. (2.8) with the brick wall boundary conditions. Eqn. (2.8) can be related to the equation for modified Bessel functions with imaginary order. The two linearly independent solutions $u_1$ and $u_2$ of Eqn. (2.8) can be taken as [7]

\begin{align}
    u_1 &= \sqrt{x} I_{-i\lambda}(\eta x), \\
    u_2 &= \sqrt{x} I_{i\lambda}(\eta x),
\end{align}

where $\lambda = RE$ and $I$ denotes the modified Bessel function [8]. The general solution of Eqn. (2.8) which vanishes at $x = h$ can thus be written as

\begin{equation}
    \chi(x) = A\sqrt{x} [I_{i\lambda}(\eta x)I_{-i\lambda}(\eta h) - I_{-i\lambda}(\eta x)I_{i\lambda}(\eta h)],
\end{equation}

where $A$ is a constant. When $r \to L$, the coordinate $x \to (L - R)$. In this limit and under the condition that $(L - R) \gg h$, the leading divergent behaviour of $\chi(x)$ is given by

\begin{equation}
    \chi(x) = \frac{Ae^{\eta x}}{\sqrt{2\pi\eta}} [I_{-i\lambda}(\eta h) - I_{i\lambda}(\eta h)].
\end{equation}

For $x \geq (L - R)$, Eqn. (2.2) requires that $\chi(x)$ must vanish. From Eqn. (2.12) we see that this is possible only if

\begin{equation}
    I_{i\lambda}(\eta h) = I_{-i\lambda}(\eta h).
\end{equation}

Let us now recall that we are interested in the situation where $h \to 0$, i.e. where the brick wall is very close to the horizon. When $h \to 0$, $I_{i\lambda}(\eta h)$ behaves as [8]

\begin{equation}
    I_{i\lambda}(\eta h) \to \frac{(\eta h)^{i\lambda}}{2^{i\lambda}} \frac{1}{\Gamma(1 + i\lambda)}.
\end{equation}

Using Eqns. (2.13) and (2.14) we see that the eigenvalues of Eqn. (2.8) are given by

\begin{equation}
    \eta_p^2 = \frac{1}{h^2} \exp \left( 2\ln 2 + \frac{2\theta(\lambda)}{\lambda} + 2p\frac{\pi}{\lambda} \right),
\end{equation}

where $p \in \mathbb{Z}$ is an arbitrary integer and the function $\theta(\lambda)$ is the argument of the Gamma function $\Gamma(1 + i\lambda)$. As mentioned before, we are interested only in the zero eigenvalue solutions of Eqn. (2.8). From Eqn. (2.15) we see that $\eta_p \to 0$ when $p \to -\infty$, or equivalently when $p = -n$ where $n$ is an arbitrarily large positive integer. In that case, Eqn. (2.15) can be written as

\begin{equation}
    \eta_n^2 = \frac{1}{h^2} \exp \left( -2n\frac{\pi}{\lambda} \right).
\end{equation}

From Eqn. (2.16) we see that as $n \to \infty$, there are an infinite number of zero modes solutions of Eqn. (2.8). In other words, the zero is an accumulation point for the eigenvalues of Eqn. (2.8).
We shall now point out several implications of the above analysis:

1) The energy eigenvalue $E$ obtained by the WKB analysis of Eqn. (2.3) serves as a parameter in the above analysis. The degeneracy obtained above exists for every real value of $E$ except for $E = 0$. In the latter case we have $\lambda = 0$ and Eqn. (2.13) reduces trivially to an identity.

2) The infinite number of zero eigenvalue states of Eqn. (2.8) exist for every finite value of the brick wall cutoff parameter $h$. This degeneracy is therefore not removed by the brick wall cutoff.

The above analysis thus shows that for any given real value of $E \neq 0$ and for any finite value of the brick wall cutoff parameter $h$, there exist an infinite number of zero eigenvalue solutions of Eqn (2.8), which can be interpreted as solutions of Eqn. (2.7). It is in this sense that we claim about the existence of the additional degeneracy in the near-horizon region as described by the brick wall model. Finally it may be noted that the above result would continue to hold for massive fields as well as the field equations become independent of the mass term in the near-horizon region of the black hole [4].

3. Analysis of Modes with Complex Frequencies

In this Section we shall analyze Eqn. (2.7) for the case of complex frequencies. For the purpose of this Section, we use the notation $\Omega$ instead of $E$. Eqn. (2.7) is therefore replaced by

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{1}{x^2} \left[ \frac{1}{4} + R^2 \Omega^2 \right] \chi = 0,$$

where $\Omega \equiv \omega_1 + i\omega_2$ is taken to be complex. It is tempting to identify $\chi$ in Eqn. (3.1) with the quasinormal mode amplitude for the Schwarzschild black hole. Indeed, the equation for the quasinormal modes for the Schwarzschild case [9] reduces to Eqn. (3.1) in the near-horizon region. There are however certain differences in the boundary conditions obeyed by the quasinormal modes [5, 9] to the ones that are used in the brick wall model. In what follows we shall analyze Eqn. (3.1) with the brick wall boundary conditions.

As described in Section 2, we interprete Eqn. (3.1) as the $\gamma \to 0$ limit of the equation

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{1}{x^2} \left[ \frac{1}{4} + R^2 \Omega^2 \right] \chi = \gamma^2 \chi.$$  (3.2)

As before, we are interested only in solutions of Eqn. (3.2) with $\gamma \to 0$. The general solution of Eqn. (3.2) which vanishes at $x = h$ is given by

$$\chi(x) = B \sqrt{x} \left[ I_{i\mu}(\gamma x) I_{-i\mu}(\gamma h) - I_{-i\mu}(\gamma x) I_{i\mu}(\gamma h) \right],$$  (3.3)

where $\mu = R\Omega$. When $x \geq (L - R)$ and $(L - R) \gg h$, the wavefunction is Eqn. (3.3) can vanish only if

$$I_{i\mu}(\gamma h) = I_{-i\mu}(\gamma h).$$  (3.4)

In the limit when $h \to 0$, Eqn (3.4) gives

$$(\gamma h)^{2i\mu} 2^{2i\mu} = \frac{\Gamma(1 + i\mu)}{\Gamma(1 - i\mu)}.$$  (3.5)
Since $\Omega = \omega_1 + i\omega_2$, the gamma functions in Eqn. (3.5) can be expressed in the polar form as
\[ \Gamma(1 + i\mu) = \Gamma(1 - R\omega_1 + iR\omega_1) = \xi_1 e^{i\theta_1} \]  
\[ \Gamma(1 - \mu) = \Gamma(1 - R\omega_2 - iR\omega_1) = \xi_2 e^{i\theta_2}. \]  
\[ (3.6) \]
\[ (3.7) \]

Using Eqns. (3.6) and (3.7), the imaginary and real parts of Eqn. (3.5) can be written as
\[ 2R\omega_1 (\ln \gamma h + \ln 2) = (\theta_1 - \theta_2) + 2p\pi \]
\[ -2R\omega_2 (\ln \gamma h + \ln 2) = \ln \frac{\xi_1}{\xi_2}, \]
\[ (3.8) \]
\[ (3.9) \]

where $p \in \mathbb{Z}$ is an arbitrary integer. However, the zero eigenvalue solutions of Eqn. (3.2) are obtained only when $p = -n$ where $n$ is an arbitrarily large positive integer. In that case Eqn. (3.8) can be written as
\[ \ln \gamma_n h = -\frac{n\pi}{R\omega_1}. \]  
\[ (3.10) \]

Using Eqns. (3.9) and (3.10) we get
\[ \left( \frac{\omega_1}{2\pi\omega_2} \right) \ln \frac{\xi_1}{\xi_2} = n. \]  
\[ (3.11) \]

Since $n$ is an arbitrarily large, for fixed values of $\omega_1$ and $\omega_2$ Eqn. (3.11) has no solution. We can therefore conclude that there is no additional degeneracy for the case of the modes with complex frequencies.

### 4. Black Hole Entropy

As discussed in Section 2, the solutions of Eqn. (2.8) for any non-zero value of $E$ admit an infinite degeneracy due to the near-horizon effects. Let us also recall that the black hole entropy found in the brick wall model arises essentially from the near-horizon features of the model. It is therefore expected that the degeneracy found in Section 2 would contribute to the black hole entropy, as they are both related to the physics in the near-horizon region. Below we shall discuss the effect of this degeneracy on the density of states and black hole entropy.

We shall first discuss how the degeneracy found in Section 2 affects the density of states. For this purpose, consider two eigenvalues $\eta_{n_1}$ and $\eta_{n_2}$ of the Eqn. (2.8) where the two positive integers $n_1, n_2 \to \infty$. For definiteness let us assume that $n_2 > n_1$. From Eqn. (2.16) we get
\[ \Delta n \equiv n_2 - n_1 = \frac{\lambda}{R} \ln \left( \frac{\eta_{n_1}}{\eta_{n_2}} \right) = \frac{\lambda}{R} \ln \left( \frac{\eta_{n_1}}{\eta_{n_2}} \right) \]  
\[ (\lambda = RE). \]  
\[ (4.1) \]

The quantity $\Delta n$ provides an expression for the degeneracy of the quantum state with energy $E$. The logarithm on the r.h.s. of Eqn. (4.1) contains the ratio of two numbers both of which are tending towards zero. We shall assume that the logarithm of this ratio tends to a positive constant $C$ as $n_1, n_2 \to \infty$. This is an ad hoc assumption which is nevertheless consistent within the framework of this model. Eqn. (4.1) can now be written as
\[ \Delta n = \frac{\lambda}{R} = \frac{cRE}{\pi}. \]  
\[ (4.2) \]
The above expression vanishes for $E = 0$, which is consistent with the observation made at the end of Section 2. The density of states calculated in the brick wall model would now have to be multiplied with this degeneracy factor. The corresponding expression of the free energy $F$ obtained from the brick wall model [1, 2] is now modified to read as

$$\pi \beta F = -2 \frac{(2M)^4}{3} h \int_0^\infty \frac{E^3 dE}{e^{\beta E} - 1}.$$  

(4.3)

The corresponding expression of the black hole entropy is now obtained as

$$S = \frac{80 c (2M)^4}{\pi^3 h \beta^3}.$$  

(4.4)

On the other hand, the application of quantum mechanical scattering theory to this system gives the black hole entropy as [1, 2]

$$S = \frac{4\pi M^2}{G},$$  

(4.5)

where $G$ is the Newton’s constant in units chosen in Ref. [2]. Comparing Eqns. (4.4) with (4.5) we get

$$h = \frac{320 G c}{\pi^5 M}.$$  

(4.6)

We therefore see within the framework presented above, the main effect of the additional degeneracy is to modify the expression of the brick wall cutoff parameter $h$ (compare with Eqn. (8.19) of Ref. [2] with $N = 1$). Finally, the requirement that physical distance of the brick wall from the horizon be of the order of Planck length [2], leads to a finite value of the constant $c$ which is consistent with assumption made above.

5. Conclusion

In this Letter we have shown that there is a hidden degeneracy in the brick wall model in the near-horizon region of the black hole. This degeneracy has been found by the analysis of the zero eigenvalue solutions of Eqn. (2.8) which can be identified with the solutions of Eqn. (2.7). Usually, any reasonable quantum field theory is expected to predict an infinite number of modes in the near-horizon region of a black hole [1, 2, 3, 4]. The divergence from these modes is handled in the brick wall model by the introduction of a short distance cutoff. The degeneracy that we have discussed is different from the usual one in the sense that it is not removed by the brick wall cutoff. We have in fact shown that for any finite value of the brick wall cutoff, every quantum state of the system with non-zero energy still admits an infinite degeneracy.

The additional degeneracy discussed above is essentially a near-horizon feature and is thus expected to affect the black hole entropy. We have estimated the effect of this degeneracy in terms of a parameter $c$, which has been assumed to be finite. The contribution of this degeneracy to the density of states is finite and non-zero except for $E = 0$. The corresponding expression for the free energy and entropy as calculated within the brick wall depends on the ratio $\frac{c}{h}$. The requirement that the physical distance of the brick wall from the horizon is of
the order of Planck length leads to a finite value of \( c \). The assumption of finiteness of \( c \) is thus consistent within the framework of the brick wall model.

We have also analyzed the system with complex frequencies. Such a situation arises in the description of quasinormal modes of Schwarzschild black holes. It has been shown that for finite values of the brick wall cutoff there is no further degeneracy in this case.

It has been argued in Ref. [4] that for a wide variety of quantum fields including massive ones, the field equation in the near horizon region looks essentially the same. The analysis presented in this paper is thus expected to hold for all such fields and also for a variety of black hole metrics with similar near-horizon structures. It is plausible that the degeneracy discussed in this Letter is related to the conformal symmetry which is present in the near-horizon region of a wide class of black holes [10, 11, 12].

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