Certain Classes of Univalent Functions With Negative Coefficients Defined By General Linear Operator

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ABSTRACT

In this study, a subclass $S_O^{μ,δ}(μ, β, δ)$ of an univalent function with negative coefficients which is defined by an new general Linear operator $H_m^{σ,μ}(c_m)\) have been introduced. The sharp results for coefficients estimators, distortion and closure bounds, Hadamard product, and Neighborhood, and this paper deals with the utilizing of many of the results for classical hypergeometric function, where there can be generalized to $m$-hypergeometric functions.

A subclasses of univalent functions are presented, and it has involving operator $H_m^{σ,μ}(c_m, b_j)$ which generalizes many well-known. Denote A the class of functions $f$ and we have other results have been studied.

A function $f \in A$ is said to be starlike of complex order if the following condition (see[4]) is satisfied:

$$Re \left( \frac{zf'(z)}{f(z)} - 1 \right) > \beta, \quad 0 \leq \mu < \frac{1}{2}, \quad 0 < \beta \leq 1, \quad \frac{1}{2} \leq \delta \leq 1 \quad (1.2)$$

For complex parameters $c_1, \ldots, c_r$ and $b_1, \ldots, b_j$ ( $b_j \in \mathbb{C}\{0, -1, -2, \ldots\}$), the $m$-hypergeometric

$$\Psi_t = \sum_{n=0}^{\infty} \frac{(c_1, m)_n, \ldots, (c_r, m)_n}{(m, m)_n} z^n \quad (1.3)$$

$t = r + 1$ such that $t, r \in \mathbb{N}_0 = \{0, 1, 2, 3, 4, \ldots\}$; $Z \in \mathbb{Z}$(1.2)

The study suggests that note that and by utilizing ratio test, the series $(1.3)$ converges absolutely in open unit disk $U$; $|z|<1$.

$\Psi_t = \sum_{n=0}^{\infty} \frac{(c_1, m)_n, \ldots, (c_r, m)_n}{(m, m)_n} z^n \quad (m \in \mathbb{N}_0, |z|<1)$

Is the m-Gauss hypergeometric function see [4,5].
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2. Coefficients estimates and Other properties

Theorem 1. Let \( f \) be defined by (1.7). Then \( f \in S^c_\infty(\mu, \beta, \delta) \) if and only if

\[
\sum_{n=2}^\infty \left| \frac{c_{n-1} f(z)}{\nu_{n-1}} \right|^\beta < \frac{3\delta}{1 + \beta},
\]

where \( 0 \leq \mu < \frac{1}{2\delta}, \beta \leq 1, \frac{1}{2} \leq \delta \leq 1, z \in U. \)

Let \( T \) denote the subclass of \( A \) consisting of function of the form

\[
f(z) = z + \sum_{n=2}^\infty a_n z^n, \quad a_n \geq 0.
\]

(1.7)

Now we define the class \( S^c_\infty(\mu, \beta, \delta) \) by

\[
S^c_\infty(\mu, \beta, \delta) = R^c_\infty(\mu, \beta, \delta) \cap T.
\]

The study have the following class and confirms that note that by specializing the parameters \( \mu, \beta, \delta \)

1. The class \( S^c_{m,0}(a, \beta, \delta) \) is the class studied by A. R.S.Juma and S. R. Kulkarni [8].

2. The class \( S^c_{m,1}(0,1,1) \) is precisely the class of starlike function in \( U. \)

3. The class \( S^c_{m,0}(a,1,1) \) is the class of starlike function of order \( a (0 \leq a < 1). \)

4. The class \( S^c_{m,0}(0, \beta, \frac{\mu+\beta}{2}) \) is the class studied by Lakshminar-srinath[9].

5. The class \( S^c_{0,c}(\mu, \beta, \delta) \) is the class studied by S. R. Kulkarni [10].

Recently Mohammed and Darus [1] defined the following:

\[
1(t_c, b_c; m) f : A \rightarrow A
\]

\[ l(t_c; b_c) \rightarrow A \]

\[
(c_m, n) \rightarrow \left( \sum_{n=2}^\infty \frac{a_n z^n}{(n+c)_n} \right) .
\]

The Srivastava-Atiiva operator \( T_{\infty} \) is defined in [6] as:

\[
T_{\infty}(z) = z + \sum_{n=2}^\infty \left( \frac{c_{n-1}}{\nu_{n-1}} \right)^\beta a_n z^n,
\]

where \( z \in \bar{U}, c \in C / \{ 0, 1, 2, ..., \}, s \in C \) and \( f \in A \). This linear operator \( T_{\infty} \) can be written as

\[
T_{\infty}(z) = G(z) \cdot f(z) = (1+c)^\beta (\phi(z, s, c) f(z)),
\]

by using the Hadamard product (convolution).

Here, \( \phi(z, s, c) \sum_{n=0}^\infty z^{n+s} \) is the well-known Hurwitz -Lerch zeta function (see [6],[7]). It is also an important function of Analytic Number Theory such the De Jonquiere function:

\[
H(z) = \sum_{n=0}^\infty \frac{z^n}{(n+c)^\mu}, \quad (Re(s) > 1 \text{ if } |z| = 1).
\]

We can define the linear operator \( H^c_{\infty}(c, b)(f) : A \rightarrow A \) as follows:

\[
H^c_{\infty}(c, b)f(z) = z + \sum_{n=2}^\infty (c_{n-1} f(z)) \cdot \sum_{n=2}^\infty \left( \frac{c_{n-1}}{\nu_{n-1}} \right)^\beta a_n z^n.
\]

(1.5)

It should be noted that the linear operator (1.5)introduced by A. R.S.Juma and M. Darus [3].

Definition 1. \( f \) is a function and \( f \in U \) is said to be in the class \( \mathcal{R}'_{m}(\mu, \beta, \delta) \) if the following condition holds:

\[
\beta \left( \frac{\sum_{n=2}^\infty (c_{n-1}) f(z)}{\sum_{n=2}^\infty (c_{n-1}) f(z)} \right) < \frac{\sum_{n=2}^\infty (c_{n-1}) f(z)}{\sum_{n=2}^\infty (c_{n-1}) f(z)}.
\]

(1.6)

where \( 0 \leq \mu < \frac{1}{2\delta}, 0 < \beta \leq 1, \frac{1}{2} \leq \delta \leq 1, z \in U. \)

Let \( T \) denote the subclass of \( A \) consisting of function of the form

\[
f(z) = z + \sum_{n=2}^\infty a_n z^n, \quad a_n \geq 0 .
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Now we define the class \( S^c_{m}(\mu, \beta, \delta) \) by

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5. The class \( S^c_{0,c}(\mu, \beta, \delta) \) is the class studied by S. R. Kulkarni [10].

130
The above bounds are sharp.

By theorem 1, we have

\[ \sum_{n=2}^{\infty} \left( \frac{(c_{1,m})_{n-1} \cdots (c_{m-1,m})_{n-1}}{(m,m)_{n-1} \cdots (b,m)_{n-1}} \right) \frac{1 + c}{n} \left[ \sum_{n=2}^{\infty} \left( \frac{(c_{1,m})_{n-1} \cdots (c_{m-1,m})_{n-1}}{(m,m)_{n-1} \cdots (b,m)_{n-1}} \right) \frac{1 + c}{n} \right] \leq 2 \beta \delta (1 - \mu). \]

Then we have

\[ \left( \frac{(c_{1,m})_{n-1} \cdots (c_{m-1,m})_{n-1}}{(m,m)_{n-1} \cdots (b,m)_{n-1}} \right) \frac{1 + c}{n} \leq 2 \beta \delta (1 - \mu). \]

\[ \sum_{n=2}^{\infty} a_n \leq \frac{2 \beta \delta (1 - \mu)}{1 + c}. \]

Therefore, the set \( S^{\infty}_m (\mu, \beta, \delta) \) is the convex set.

Proof. Let \( f \in S^{\infty}_m (\mu, \beta, \delta) \). Then for \( |z| \leq r < 1 \), we get

\[ r^2 = \left( \sum_{n=2}^{\infty} \left( \frac{(c_{1,m})_{n-1} \cdots (c_{m-1,m})_{n-1}}{(m,m)_{n-1} \cdots (b,m)_{n-1}} \right) \frac{1 + c}{n} \right) \leq \frac{2 \beta \delta (1 - \mu)}{1 + c}. \]

By utilizing assumption we get

\[ \sum_{n=2}^{\infty} \left( \frac{(c_{1,m})_{n-1} \cdots (c_{m-1,m})_{n-1}}{(m,m)_{n-1} \cdots (b,m)_{n-1}} \right) \frac{1 + c}{n} \left( 1 - \beta \right) \leq \frac{2 \beta \delta (1 - \mu)}{1 + c}. \]

\[ \sum_{n=2}^{\infty} \left( \frac{(c_{1,m})_{n-1} \cdots (c_{m-1,m})_{n-1}}{(m,m)_{n-1} \cdots (b,m)_{n-1}} \right) \frac{1 + c}{n} \left( 1 - \beta \right) \leq \frac{2 \beta \delta (1 - \mu)}{1 + c}. \]

The above bounds are sharp.

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And versa, suppose that \( \mathcal{H}_m \mathcal{S}^{\infty}_c (\mu, \beta, \delta) \) therefore the condition (1.7) gives us

\[ \sum_{n=2}^{\infty} \left( \frac{(c_{1,m})_{n-1} \cdots (c_{m-1,m})_{n-1}}{(m,m)_{n-1} \cdots (b,m)_{n-1}} \right) \frac{1 + c}{n} \leq 2 \beta \delta (1 - \mu). \]

The result is sharp for function \( f(z) \), defined by

\[ f(z) = \frac{2 \beta \delta (1 - \mu)}{1 + c}. \]

Theorem 2. Let \( f \in S^{\infty}_m (\mu, \beta, \delta) \). Then for \( |z| \leq r < 1 \), we get

\[ r^2 = \left( \sum_{n=2}^{\infty} \left( \frac{(c_{1,m})_{n-1} \cdots (c_{m-1,m})_{n-1}}{(m,m)_{n-1} \cdots (b,m)_{n-1}} \right) \frac{1 + c}{n} \right) \leq \frac{2 \beta \delta (1 - \mu)}{1 + c}. \]

Theorem 3. Let \( 0 < \beta \leq 1, 0 < \mu_1 \leq \mu_2 < \frac{1}{2} \) and \( 1 \leq \delta \leq 1 \) the \( S^{\infty}_m (\mu_2, \beta, \delta) \in S^{\infty}_m (\mu_1, \beta, \delta). \)

Proof. By utilizing assumption we get

\[ \sum_{n=2}^{\infty} \left( \frac{(c_{1,m})_{n-1} \cdots (c_{m-1,m})_{n-1}}{(m,m)_{n-1} \cdots (b,m)_{n-1}} \right) \frac{1 + c}{n} \leq \frac{2 \beta \delta (1 - \mu)}{1 + c}. \]

The above bounds are sharp.

Proof. By theorem 1, we have

\[ \sum_{n=2}^{\infty} \left( \frac{(c_{1,m})_{n-1} \cdots (c_{m-1,m})_{n-1}}{(m,m)_{n-1} \cdots (b,m)_{n-1}} \right) \frac{1 + c}{n} \leq 2 \beta \delta (1 - \mu). \]

The result is sharp for function \( f(z) \), defined by

\[ f(z) = \frac{2 \beta \delta (1 - \mu)}{1 + c}. \]
$$\sum_{n=1}^{\infty} \frac{(c_{1m}m^{-1}-c_{2m}m^{-1})^{1+c_{1m}}}{(m(m-1)(1+c_{1m})+1+c_{2m})(1+b_0(n-1)+2\beta(n-1))}$$

The study shall further try to obtain the extreme points in the following theorem.

**Theorem 5.** Let $f(z) = z$ and

$$f_n(z) = \sum_{n=1}^{\infty} \frac{2\beta(n-1)}{[(m(m-1)(1+c_{1m})+1+c_{2m})(1+b_0(n-1)+2\beta(n-1))]} z^n,$$

for all $n = 2, 3, \ldots$; $0 < \beta \leq 1$, $0 < \mu < \frac{1}{2\delta}$, and $\frac{1}{2} \leq \delta \leq 1$.

Then $f(z)$ is in the class $S^{\infty}_{\infty}(\mu, \beta, \delta)$ if and only if it can be expressed in the form

$$f(z) = z + \sum_{n=1}^{\infty} \frac{2\beta(n-1)}{[(m(m-1)(1+c_{1m})+1+c_{2m})(1+b_0(n-1)+2\beta(n-1))]} y_n z^n,$$

and we obtain

$$\sum_{n=1}^{\infty} \frac{2\beta(n-1)}{[(m(m-1)(1+c_{1m})+1+c_{2m})(1+b_0(n-1)+2\beta(n-1))]} y_n = 1 - y_1 \leq 1.$$

In view of theorem 1, this shows that $f(z) \in S^{\infty}_{\infty}(\mu, \beta, \delta)$.

Conversely,

$$\sum_{n=1}^{\infty} \frac{2\beta(n-1)}{[(m(m-1)(1+c_{1m})+1+c_{2m})(1+b_0(n-1)+2\beta(n-1))]} y_n \leq 1$$

and $y_1 = 1 - \sum_{n=1}^{\infty} y_n$. Then we get

$$f(z) = y_1 f_1(z) + \sum_{n=1}^{\infty} y_n f_n(z).$$

\[ \square \]

### 3. Neighbourhood and Hadamard product properties

**Definition 3.1:** Let $\gamma \geq 0$, $f(z) \in T$ on the $(k, \gamma)$-neighborhood of a function $f(z)$ defined by

$$N_{\gamma}(f) = \{g \in T: g(z) = z - \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} |a_n - b_n| \leq \gamma\}.$$  

For the identity function $e(z) = z$, we get

$$N_{\gamma}(e) = \{g \in T: g(z) = z - \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} |b_n| \leq \gamma\}.  \quad (3.2)$$

**Theorem 6.** Let

$$y = \frac{4\beta(1-\mu)}{[(m(m-1)(1+c_{1m})+1+c_{2m})(1+b_0(n-1)+2\beta(n-1))]} \gamma,$$

then $S^{\infty}_{\infty}(\mu, \beta, \delta) \subset N_{\gamma}(e)$.

**Proof.** Let $f \in S^{\infty}_{\infty}(\mu, \beta, \delta)$. Then we get

$$1 - 2\beta(n-1) + 2\beta(n-1) \geq 2\beta(1-\mu),$$

and also we get $|z| < r$

$$|f(z)| \leq 1 + |z| \sum_{n=1}^{\infty} a_n = 1 + r \sum_{n=1}^{\infty} a_n.$$

In view of (3.3) we get

$$|f(z)| \leq 1 + r \frac{2\beta(1-\mu)}{[(m(m-1)(1+c_{1m})+1+c_{2m})(1+b_0(n-1)+2\beta(n-1))]} \gamma = y,$$

thus $f \in N_{\gamma}(e)$.

\[ \square \]

**Definition 3.2:** The function $f(z)$ defined by (1.7) is said to be a member of the subclass $S^{\infty}_{\infty}(\mu, \beta, \delta)$.

If there exists a function $g \in S^{\infty}_{\infty}(\mu, \beta, \delta)$ such that

$$|f(z)| \leq 1 - \zeta, \quad z \in U, \quad 0 \leq \zeta < 1.$$

**Theorem 7.** Let $g \in S^{\infty}_{\infty}(\mu, \beta, \delta)$ and

$$\zeta = \frac{1}{2} d(\mu, \beta, \delta).$$  

Then $N_{\gamma}(g) \subset S^{\infty}_{\infty}(\mu, \beta, \delta)$ when $0 < \beta \leq 1, 0 < \mu < \frac{1}{2\delta^2}$, and $1 < \beta < 1$. Then $d(\mu, \beta, \delta) = \frac{1}{2} d(\mu, \beta, \delta)$.

**Proof.** Let $F \in N_{\gamma}(g)$. Then by (3.3) we get

$$\sum_{n=2}^{\infty} a_n \leq \frac{4\beta(1-\mu)}{[(m(m-1)(1+c_{1m})+1+c_{2m})(1+b_0(n-1)+2\beta(n-1))]} \gamma.$$
Then, the Hadamard product $h(z)$ defined by

$$h(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$$

is in the subclass $S_{m}^{c}(\mu, \beta, \delta)$ when

$$\mu_2 \leq \left[\frac{((n-1)(1-\beta) + 2\beta\delta(n-\mu_3))}{2\beta\delta(1-\mu_3)}\right]^{\frac{1}{n+\mu}}$$

Proof. By theorem 1, we get

$$\sum_{n=2}^{\infty} \frac{((n-1)(1-\beta) + 2\beta\delta(n-\mu_3))}{2\beta\delta(1-\mu_3)} a_n \leq 1$$

From (3.5)

And

$$\sum_{n=2}^{\infty} \frac{((n-1)(1-\beta) + 2\beta\delta(n-\mu_3))}{2\beta(1-\mu_3)} b_n \leq 1$$

We get only to find the largest $\mu_3$ such that,

$$\sum_{n=2}^{\infty} \frac{((n-1)(1-\beta) + 2\beta\delta(n-\mu_3))}{2\beta(1-\mu_3)} a_n b_n \leq 1$$

Now by Cauchy–Schwarz inequality, we get

$$\sum_{n=2}^{\infty} \frac{((n-1)(1-\beta) + 2\beta\delta(n-\mu_3))}{2\beta(1-\mu_3)} a_n b_n \leq \left[\frac{((n-1)(1-\beta) + 2\beta\delta(n-\mu_3))}{2\beta(1-\mu_3)}\right]^{\frac{1}{n+\mu}}$$

We need only to show that

$$\sum_{n=2}^{\infty} \frac{((n-1)(1-\beta) + 2\beta\delta(n-\mu_3))}{2\beta(1-\mu_3)} a_n b_n \leq 1$$

equivalently

$$\sum_{n=2}^{\infty} \frac{((n-1)(1-\beta) + 2\beta\delta(n-\mu_3))}{2\beta(1-\mu_3)} a_n b_n \leq 1$$

But from (3.7) we get

$$\sum_{n=2}^{\infty} \frac{((n-1)(1-\beta) + 2\beta\delta(n-\mu_3))}{2\beta(1-\mu_3)} a_n b_n \leq 1$$

Consequently, we also need to prove that

$$\sum_{n=2}^{\infty} \frac{((n-1)(1-\beta) + 2\beta\delta(n-\mu_3))}{2\beta(1-\mu_3)} a_n b_n \leq 1$$

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في هذا البحث تم دراسة الفئات الفرعية من الدوال الأحادية التكافؤ مع معاملات سالبة والتي هي معرفة بواسطة العامل الخطي العام الذي قد تم الحصول على النتائج في مقدرات المعاملات الشؤوهات وضرب هادمارد ونتائج أخرى تم دراستها وكذلك في هذا البحث تم استخدام العديد من الدوال الكلاسيكية الهندسية العليا بحيث تستطيع أن تكون من نوع $M^\beta S^\gamma_{\mu \nu} (c_i, b_i)$ وكثير من الفئات الفرعية وتشمل تقديم فئات فرعية.