Liouvillian propagators, Riccati equation and
differential Galois theory

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Abstract

In this paper a Galoisian approach to building propagators through Riccati equations is presented. The main result corresponds to the relationship between the Galois integrability of the linear Schrödinger equation and the virtual solvability of the differential Galois group of its associated characteristic equation. As the main application of this approach we solve Ince’s differential equation through the Hamiltonian algebrization procedure and the Kovacic algorithm to find the propagator for a generalized harmonic oscillator. This propagator has applications which describe the process of degenerate parametric amplification in quantum optics and light propagation in a nonlinear anisotropic waveguide. Toy models of propagators inspired by integrable Riccati equations and integrable characteristic equations are also presented.

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1. Introduction

Over the years, the generalized harmonic oscillator has been of consistent scientific interest because of the role it plays in several advanced quantum problems, including Berry’s phase and the quantization of mechanical systems (see [10] and references therein). Quadratic Hamiltonians are particularly interesting in the study of quantum electrodynamics due to the representation of the electromagnetic field as a set of forced harmonic oscillators [7, 8, 11, 14, 36, 40]. A method has been developed to construct explicit propagators for the linear Schrödinger equation with a time-dependent quadratic Hamiltonian based on solutions of the Riccati equation [6, 7]; particular cases such as the propagators for the free particle, the harmonic oscillator, and the Caldirola–Kanai oscillator can be found in a unified manner [6, 7, 19]. Even the Cauchy problem has been studied with this approach in [22, 27]. We will use the main result of this method here.
There has also been increased interest in the study of the Picard–Vessiot theory, also known as the Galois theory for ordinary linear differential equations, where differential equations of such type are analyzed throughout their Galoisian structure. This Galoisian structure depends on the nature of the differential equation’s solutions; one obtains some kind of solvability (virtual) for the Galois group whenever one obtains Liouvillian solutions, and in this case one says that the differential equation is integrable. This means, for example, that when one obtains Airy functions, the differential equation is not integrable, while when one obtains Jacobi elliptic functions, the differential equation is integrable, and one gets virtual solvability of its Galois group.

In this paper we present a Galoisian approach on how to find explicit propagators through Liouvillian solutions for linear second order differential equations associated with Riccati equations. The main application of this Galoisian approach and one of the main results of the paper is the construction of the propagator for the so-called degenerate parametric oscillator:

\[ i\hbar \psi = H(t)\psi \]  

\[ H(t) = \frac{1}{2m} \left( 1 + \frac{\lambda}{\omega} \cos(2\omega t) \right) p^2 + \left( 1 - \frac{\lambda}{\omega} \cos(2\omega t) \right) x^2 \]

\[ + \frac{\lambda}{2} \sin(2\omega t)(px + xp), \quad p = -i\hbar. \]

Another form for \( H(t) \) in terms of annihilation and creation operators \( \hat{a} = \sqrt{1/2\omega}(\omega x + ip), \quad \hat{a}^\dagger = \sqrt{1/2\omega}(\omega x - ip) \) with \( \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \), in the coordinate representation is:

\[ H(t) = \frac{\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) - \frac{\lambda}{2}(e^{2i\omega t}\hat{a}^2 + e^{-2i\omega t}(\hat{a}^\dagger)^2). \]

In quantum optics the first term refers to the self-energy of the oscillator representing the mode of interest, and the second term describes the coupling of the classical pump to that mode giving rise to the parametric amplification process (\( \lambda \) is the phenomenological constant) [10]. It is likely that the oscillator (1)–(2) was first introduced by Takahasi [38] to describe the degenerate parametric amplification process in quantum optics (see also [20, 21, 23, 24, 31, 32, 38]), but the Hamiltonian (2) was also utilized by Angelow and Trifonov [4, 5] in the description of light propagation in a nonlinear anisotropic waveguide. Also, in [16] the authors give a natural description of squeezed photons that can be created as a result of the parametric amplification of quantum fluctuations in the dynamic Casimir effect.

In [10] the authors motivated the investigation of the properties of the degenerate parametric oscillator (a particular case of the generalized harmonic oscillator), including a systematic study of the corresponding non-periodic solutions of Ince’s equation, which seems to be missing in the mathematical literature (see section 8 in [10]). In [10] the authors constructed the propagator for (1)–(2) for the case \( \lambda = \omega = 1 \) using non-periodic solutions of Ince’s equation (compare with the classical results of periodic solutions [25]).

In this paper, using Galoisian theory we show how to find the explicit solution of the associated Ince’s equation presented in [10] for the special case \( \lambda = \omega \). Those authors do not show how they found the explicit solution; see equations (6.1)–(6.3). In fact, they present a non-periodic solution that allows us to write the propagator explicitly; the fact that the solution is non-periodic is fundamental. In this note we present in detail how to find this non-periodic solution by using a combination of the Kovacic algorithm [18] and an algebrization procedure (see [1, 3]), and, further, we find the explicit solution for the general case \( \lambda \neq \omega \) (see (3.4) in [10] and the discussion that follows it). We believe this approach can be extended to the
study of propagators of other generalized harmonic oscillators, but here we restrict ourselves
to (1)–(2) and give some toy examples in section 5.

The aim of this paper is to establish a Galoisian approach to the techniques given by
Suslov et al., see [6–9]. To study Liouvillian solutions for linear second order differential
equations, as well as the integrability of their associated Riccati equations, we use the Kovacic
algorithm (see [18]) and an algebrization procedure (see [1, 3]). These tools were applied to
study differential equations incoming from physics; in particular the integrability analysis of
the one-dimensional linear Schrödinger equation has been studied in [1, 3].

This paper is organized as follows.

• Section 2 contains a brief description of the basic theory concerning the construction of
explicit propagators using the [6–10, 34–37], and a short summary of the Picard–Vessiot
theory is also presented, which was written according to [1, 2, 39].
• Section 3 contains one of the main results of this paper; it corresponds to a Galoisian
approach of propagators. It is devoted to a theoretical Galoisian approach to propagators
starting with Riccati and second order differential equations. The result given here relates
the Galois integrability of the linear Schrödinger equation with the virtual solvability of
the differential Galois group of its associated characteristic equation.
• Section 4 contains another main result of this paper, where we present the Galoisian
analysis of Ince’s differential equations to perform the construction of the propagator
of the degenerate parametric oscillator in more general terms. Also, we compute the
differential Galois group associated with such a propagator, which corresponds to the
differential Galois group of Ince’s characteristic equation.
• Section 5 contains some toy models of new propagators as well as their Green functions
through characteristic equations and Riccati equations.

For suitability, throughout this paper $\frac{\partial}{\partial x}$ denotes $\frac{\partial}{\partial x}$, for higher order derivation $\frac{\partial^n}{\partial x^n}$ denotes
and by $\partial_a(0)$ we mean $\partial_a(t)|_{t=0}$.

2. Theoretical background

2.1. Differential Galois theory

The Galois theory of differential equations, also called differential Galois theory and Picard–
Vessiot theory, has been developed by Picard, Vessiot, Kolchin and many other current
researchers; see [2, 3, 15, 17, 18, 26, 39]. Moreover, recent applications to mathematical
physics can be found in [1, 3, 28, 29, 33]. We consider differential Galois theory in the context
of second order linear differential equations.

A differential field $K$ is a commutative field of characteristic zero with a derivation $\partial_x$, where the field of constants of $K$, denoted by $\mathcal{C}_K$, is algebraically closed and of characteristic
zero. The coefficient field for a differential equation is defined as the smallest differential field
containing all the coefficients of the equation. Let $L$ be a differential field containing $K$: we say
that $L$ is a Picard–Vessiot extension of $K$ if there exist two linearly independent $y_1, y_2 \in L$ such
that $L = K(y_1, y_2)$ and $\mathcal{C}_L = \mathcal{C}_K$. A $K$-automorphism $\sigma$ of the Picard–Vessiot extension $L$ is
called a differential automorphism if $\sigma(\partial_a(x)) = \partial_a(\sigma(x))$ for all $a \in L$ and $\forall a \in K$, $\sigma(a) = a$.
The group of all differential automorphisms of $L$ over $K$ is called the differential Galois group
of $L$ over $K$ and is denoted by $\text{DGal}(L/K)$.

**Theorem 1** ([17]). The differential Galois group $\text{DGal}(L/K)$ is an algebraic subgroup of GL(2, $\mathbb{C}$).
We denote by $G^0$ the connected component of the identity; thus, when $G^0$ satisfies some property, we say that $G$ virtually satisfies such property.

**Theorem 2** (Lie–Kolchin theorem). Let $G \subseteq \text{GL}(2, \mathbb{C})$ be a virtually solvable group. Then $G^0$ is triangularizable.

We say that a linear differential equation $L$ is **integrable** if the Picard–Vessiot extension $L \supset K$ is obtained as a tower of differential fields $K = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_m = L$ such that $L_i = L_{i-1}(\eta)$ for $i = 1, \ldots, m$, where either

1. $\eta$ is algebraic over $L_{i-1}$, that is $\eta$ satisfies a polynomial equation with coefficients in $L_{i-1}$,
2. $\eta$ is primitive over $L_{i-1}$, that is $\partial x \eta \in L_{i-1}$, or
3. $\eta$ is exponential over $L_{i-1}$, that is $\partial x \eta / \eta \in L_{i-1}$.

The solutions obtained through such towers are called **Liouvillian**. The special functions are not always Liouvillian, however. We can see that the Airy equation does not have Liouvillian solutions, while the Bessel equation has Liouvillian solutions for special values of the parameter; see [13, 30]. Thus, by integrable we mean whenever the differential equation has Liouvillian solutions instead of special function solutions.

**Theorem 3** (Kolchin). A linear differential equation is integrable if and only if $\text{DGal}(L/K)$ is virtually solvable.

**Proposition 4** (Riccati transformations, [2]). Let $K$ be a differential field, $a_0(x)$, $a_1(x)$, $a_2(x)$, $r(x)$, $\rho(x)$, $b_0(x)$ and $b_1(x)$ belonging to $K$. Consider now the following forms associated with any second order differential equation (ode) and the Riccati equation.

1. **Second order ode (in general form):**
   \[
   \partial^2_x y + b_1 \partial_x y + b_0 y = 0.
   \]  
   (4)

2. **Second order ode (in reduced form):**
   \[
   \partial^2_x \xi = \rho \xi.
   \]  
   (5)

3. **Riccati equation (in general form):**
   \[
   \partial_x v = a_0 + a_1 v + a_2 v^2,
   \]
   \[
   a_2 \neq 0.
   \]  
   (6)

4. **Riccati equation (in reduced form):**
   \[
   \partial_x w = r - w^2.
   \]  
   (7)

Then, there exist transformations $T$, $B$, $S$ and $R$ leading some of these equations into the other ones, as shown in the following diagram:

\[
\begin{array}{c}
\partial_x v = a_0 + a_1 v + a_2 v^2 \xrightarrow{T} \partial_x w = r - w^2 \\
\partial^2_x y + b_1 \partial_x y + b_0 y = 0 \xrightarrow{S} \partial^2_x \xi = \rho \xi.
\end{array}
\]

The new independent variables are defined by means of

\[
T : v = -\left(\frac{\partial_x a_2}{2a_2^2} + \frac{a_1}{2a_2}\right) - \frac{1}{a_2} w, \\
B : v = -\frac{1}{a_2} \frac{\partial_x y}{y}, \\
S : y = \xi e^{-\frac{1}{2} \int b_1 dx}, \\
R : w = \frac{\partial_x \xi}{\xi}.
\]
and the functions \( r, \rho, b_0 \) and \( b_1 \) are given by
\[
  r = \frac{1}{\beta} (a_0 + a_1 \alpha + a_2 \alpha^2 - \partial_x \alpha),
\]
(8)
\[
  \alpha = -\left( \frac{\partial_x a_2}{2a_2} + \frac{a_1}{2a_2} \right), \quad \beta = -\frac{1}{a_2},
\]
(9)
\[
  b_1 = -\left( a_1 + \frac{\partial_x a_2}{a_2} \right), \quad b_0 = a_0 a_2,
\]
(10)
\[
  \rho = r = \frac{b_1^2}{4} + \frac{\partial_x b_1}{2} - b_0.
\]
(11)

Remark 5. From proposition 4, the well-known result in differential Galois theory is recovered (see for example [39]): the Riccati equation has an algebraic solution over the differential field \( K \) if and only if its associated second order differential equation has two independent Liouvillian solutions (the differential Galois group of the second order differential equation is virtually solvable). Furthermore, the differential Galois group for equation (5) is a subgroup of \( \text{SL}(2, \mathbb{C}) \), such as the case for the stationary Schrödinger equation.

Kovacic (see [18] and improvements in [1, 3]) developed an algorithm to solve the differential equation
\[
  \partial^2 \tau y = ry, \quad r = \frac{f}{g}, \quad f, g \in \mathbb{C}[\tau].
\]

There are four cases in Kovacic’s algorithm. Only for cases 1, 2 and 3 can we solve the differential equation, but for case 4 the differential equation is not integrable. We use the following notations:

\[
  \Gamma' = \{ c \in \mathbb{C} : g(c) = 0 \}, \quad \Gamma = \Gamma' \cup \{ \infty \}.
\]

By \( \circ (r_c) \), we mean the multiplicity of \( c \) as a pole of \( r \), while by \( \circ (r_{\infty}) \) we mean the order of \( \infty \) as a zero of \( r \). Now, we summarize cases 1 and 2 of the Kovacic algorithm that will be used in section 4.

Case 1. Step 1.
(c2) If \( \circ (r_c) = 2 \), and
\[
  r = \cdots + b(\tau - c)^{-2} + \cdots, \quad \text{then}
\]
\[
  [\sqrt{r}]_c = 0, \quad \alpha_c^\pm = \frac{1 \pm \sqrt{1 + 4b}}{2}.
\]
\([\infty_2] \) If \( \circ (r_{\infty}) = 2 \), and \( r = \cdots + \tau^2 + \cdots, \) then
\[
  [\sqrt{r}]_{\infty} = 0, \quad \alpha_{\infty}^\pm = \frac{1 \pm \sqrt{1 + 4b}}{2}.
\]

Step 2. Find \( D \neq \emptyset \) defined by
\[
  D = \left\{ n \in \mathbb{Z}_+ : n = \alpha_{\infty}^{(\infty)} - \sum_{c \in \Gamma'} \alpha_c^{(c)}, \forall (\varepsilon(p))_{p \in \Gamma'} \right\}.
\]
If $D = \emptyset$, then we should start with case 2. Now, if Card($D$) > 0, then for each $n \in D$ we search $\omega \in \mathbb{C}(\tau)$ such that

$$\omega = \varepsilon (\infty) \left[ \sqrt{\tau} \right]_\infty + \sum_{c \in \Gamma'} \left( \varepsilon (c) \left[ \sqrt{\tau} \right]_c + \frac{\alpha^c_2(c)}{(\tau - c)} \right).$$

**Step 3.** For each $n \in D$, search for a monic polynomial $P_n$ of degree $n$ with

$$\partial_r^3 P_n + 2\omega \partial_r P_n + (\partial_r \omega + \omega^2 - r) P_n = 0.$$

If success is achieved, then $y_1 = P_n e^{\int \omega}$ is a solution of the differential equation. Otherwise, case 1 cannot hold.

**Case 2.**

**Step 1.** Search for each $c \in \Gamma'$ and $\infty$ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Gamma'$ and for $\infty$ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows.

(c2) If $\circ (r_c) = 2$, and $r = \cdots + b(\tau - c)^{-2} + \cdots$, then

$$E_c = \{2 + k\sqrt{1 + 4b} : k = 0, \pm 2\} \cap \mathbb{Z}.$$

(\infty 2) If $\circ (r_\infty) = 2$, and $r = \cdots + br^2 + \cdots$, then

$$E_\infty = \{2 + k\sqrt{1 + 4b} : k = 0, \pm 2\} \cap \mathbb{Z}.$$

**Step 2.** Find $D \neq \emptyset$ defined by

$$D = \left\{ n \in \mathbb{Z}_+ : \frac{1}{2} \left( e_\infty - \sum_{c \in \Gamma'} e_c \right), \forall e_p \in E_p, \; p \in \Gamma \right\}.$$

If $D = \emptyset$, then we should start case 3. Now, if Card($D$) > 0, then for each $n \in D$ we search a rational function $\theta$ defined by

$$\theta = \frac{1}{2} \sum_{c \in \Gamma'} e_c \frac{1}{(\tau - c)}.$$

**Step 3.** For each $n \in D$, search a monic polynomial $P_n$ of degree $n$, such that

$$\partial_r^3 P_n + 3\theta \partial_r^2 P_n + (3\partial_r \theta + 3\theta^2 - 4r) \partial_r P_n + (\partial_r^2 \theta + 3\partial_r \theta + \theta^3 - 4r \theta - 2\partial_r \theta) P_n = 0.$$

If $P_n$ does not exist, then case 2 cannot hold. If such a polynomial is found, set $\phi = \theta + \partial_r P_n / P_n$ and let $\omega$ be a solution of

$$\omega^2 + \phi \omega + \frac{1}{2} (\partial_r \phi + \phi^2 - 2r) = 0.$$

Then $y_1 = e^{\int \omega}$ is a solution of the differential equation.

We can see that applying the Kovacic algorithm by hand can be a little difficult, but thankfully it has been implemented in Maple (command kovacicsols) so that we can avoid such calculations. The problem with Maple is that the answers can have complicated expressions that should be transformed into more suitable and readable expressions. On the other hand, the Kovacic algorithm only works with rational coefficients; for instance, when the differential equation does not have rational coefficients, we cannot apply the Kovacic algorithm.

We recall that in most cases the application of the Kovacic algorithm is better done by hand, especially where parameters can appear or the solution is very complicated. For example, the stationary Schrödinger equation $\partial_r^2 \Psi = (\lambda^2 - \lambda) \Psi$ cannot be solved with kovacicsols due to the parameter $\lambda$; thus, the output given by Maple is [\_\_], i.e., we should solve the equation using the Kovacic algorithm by hand to obtain solutions conditioned to some values of $\lambda$. The interested reader can see in \cite{1, 3} the solutions of (stationary) Schrödinger equations using the
Kovacic algorithm done by hand, recalling that it was not possible to obtain such solutions through Kovacic\texttextunderscore sols.

One question concerning the application of the Kovacic algorithm is whether the coefficients are not rational functions. This problem can be solved using Hamiltonian algebrization, a procedure developed in [1, 3]. We present a short summary of the Hamiltonian algebrization process that will be used in section 4. Following [1, 3], we say that \( r = \tau(t) \) is a Hamiltonian change of variable when \((r, \dot{r}, \tau)\) is a solution curve of the Hamiltonian

\[
H = \frac{p^2}{2} + V(\tau), \quad \dot{r}_\tau = \dot{r}_\tau H = p, \quad \dot{r}_p = -\partial_r H = -\partial_r V(\tau), \quad V(\tau) \in \mathbb{C}(\tau).
\]

In this way, we set \( \alpha := \frac{p^2}{2} \), which is dependent of \( \tau \) and leads us to

\[
\alpha = 2H - 2V(\tau) = (\dot{\alpha}_\tau)^2, \quad \dot{\alpha}_\tau = \sqrt{\alpha}.
\]

In particular, because the theory is more general, we can transform differential equations

\[
\partial_t^2 \mu + p\partial_t \mu + q\mu = 0 \Rightarrow \hat{\partial}_t^2 \hat{\mu} + \hat{p}\hat{\mu} + \hat{q}\hat{\mu} = 0,
\]

where \( \hat{\partial}_t = \sqrt{\alpha} \partial_t, \hat{\mu} \circ \tau = \mu, \hat{p} \circ \tau = p, \hat{q} \circ \tau = q \). Moreover, the differential equation

\[
\hat{\partial}_t^2 \hat{\mu} + \hat{p}\hat{\mu} + \hat{q}\hat{\mu} = 0
\]

can be explicitly written as

\[
\hat{\partial}_t^2 \hat{\mu} + \left( \frac{1}{2} \hat{\partial}_t (\ln \alpha) + \hat{\partial}_t \right) \hat{\mu} + \left( \frac{\hat{q}}{\alpha} \right) \hat{\mu} = 0. \tag{12}
\]

In the case that \( \sqrt{\alpha}, \hat{p} \) and \( \hat{q} \) are rational functions in \( \tau \), the equation (12) is the algebraic form of the first one, i.e., the equation \( \hat{\partial}_t^2 \hat{\mu} + \hat{p}\hat{\mu} + \hat{q}\hat{\mu} = 0 \) has been algebrized through a Hamiltonian change of variable. This procedure is called Hamiltonian algebrization, which is an isogaloisian transformation, i.e., the differential Galois group is preserved under the Hamiltonian algebrization procedure. Further details and proofs can be found in [1, 3].

2.2. Propagators and Green functions

In this section, as well as in the rest of the paper, we follow [8] considering the one-dimensional time-dependent Schrödinger equation for a harmonic oscillator

\[
i\partial_t \psi = H \psi, \quad H = a(t)p^2 + b(t)x^2 + d(t)(px + xp), \quad p = -i\partial_x. \tag{13}
\]

The Schrödinger equation with \( 2d(t) = c(t) \) (13) can be written as

\[
i\partial_t \psi = -a(t)\partial_x^2 \psi + b(t)x^2 \psi - id(t)\psi - ic(t)x\partial_x \psi. \tag{14}
\]

We begin considering the Riccati equation

\[
\partial_t \alpha + b(t) + 2c(t)\alpha + 4a(t)\alpha^2 = 0, \tag{15}
\]

where \( a(t), b(t) \) and \( c(t) \) are elements of a differential field \( K \), with the coefficient field \( \mathbb{C} \). By proposition 4 we can transform the Riccati equation (15) through the change of variable

\[
\alpha(t) = \frac{1}{4a(t)} \frac{\partial_t \mu(t)}{\mu(t)} - \frac{d(t)}{2a(t)}, \tag{16}
\]

into the second order differential equation

\[
\hat{\partial}_t^2 \hat{\mu} - \tau(t)\hat{\partial}_t \hat{\mu} + 4\sigma(t)\mu = 0. \tag{17}
\]

Moreover, by remark 5, the differential Galois group of the differential equation (17) is virtually solvable if and only if the Riccati equation (15) has an algebraic solution over the differential field \( K \). Furthermore, by proposition 4, departing from the differential equation (17) we can arrive at the Riccati equation (15) through changes of variables. The following lemmas show how we can construct propagators based on explicit solutions in (15) and (17).
Lemma 6 ([6, 9, 35]). Given \( a(t), b(t) \) and \( c(t) \) are piecewise continuous, there exists an interval \( I \) of time where the following (Riccati-type) system

\[
\begin{align*}
\partial_t \alpha + b(t) + 2c(t) \alpha + 4a(t) \alpha^2 &= 0, \\
\partial_t \beta + (c(t) + 4a(t) \alpha(t)) \beta &= 0, \\
\partial_t \gamma + a(t) \beta^2(t) &= 0
\end{align*}
\]

(18) (19) (20)

has as a fundamental solution in terms of solutions of the following (characteristic) equation

\[
\ddot{\alpha} - \tau(t) \partial_t \alpha + 4 \sigma(t) \alpha = 0
\]

(21)

given by:

\[
\begin{align*}
\alpha_0(t) &= \frac{1}{4a(t)} - \frac{\partial_t \alpha_0(t)}{\mu_0(t)} - \frac{c(t)}{2a(t)}, \\
\beta_0(t) &= -\frac{h(t)}{\mu_0(t)}, \quad h(t) = \exp\left(-\int_0^t c(\tau) - d(\tau) \, d\tau\right), \\
\gamma_0(t) &= \frac{1}{2\mu_1(0) \mu_0(t)} + \frac{c(0)}{2a(0)}
\end{align*}
\]

(23) (24) (25)

provided that \( \mu_0 \) and \( \mu_1 \) are standard solutions of (21) and (22) with \( \mu_0(0) = 0, \partial_t \mu_0(0) = 2a(0) \not= 0, \) and \( \mu_1(0) \not= 0, \partial_t \mu_1(0) = 0. \) Further, the following asymptotics hold [37]:

\[
\begin{align*}
\alpha_0(t) &= \frac{1}{4a(0) t} - \frac{c(0)}{4a(0)} - \frac{\partial_t a(0)}{8a^2(0)} + O(t), \\
\beta_0(t) &= \frac{1}{2a(0) t} + \frac{\partial_t a(0)}{4a^2(0)} + O(t), \\
\gamma_0(t) &= \frac{1}{4a(0) t} + \frac{c(0)}{4a(0)} - \frac{\partial_t a(0)}{8a^2(0)} + O(t)
\end{align*}
\]

(26) (27) (28)

as \( t \to 0 \), with \( a(t), b(t) \) and \( c(t) \) sufficiently smooth.

Lemma 7 ([6, 9, 35, 36]). The Green function, or Feynman’s propagator, corresponding to Schrödinger equation (13) can be obtained as

\[
G(x, y, t) = \frac{1}{\sqrt{2\pi a(0)t}} \exp \left[ -\frac{(x-y)^2}{4a(0)t} \right] \exp \left[ -i \left( \frac{\partial_x a(0)}{8a^2(0)} (x-y)^2 + \frac{c(0)}{4a(0)} (x-y)^2 \right) \right].
\]

(29)

where \( a_0(t), \beta_0(t) \) and \( \gamma_0(t) \) are solutions of the Riccati-type system. Then the superposition principle allows us to solve the corresponding Cauchy initial value problem:

\[
\psi(x, t) = \int_{-\infty}^{\infty} G(x, y, t) \psi(y, 0) \, dy
\]

for suitable data \( \psi(x, 0) = \psi(x) \). Further as \( t \to 0 \),

\[
G(x, y, t) \sim \frac{1}{\sqrt{2\pi a(0)t}} \exp \left[ -\frac{(x-y)^2}{4a(0)t} \right] \exp \left[ -i \left( \frac{\partial_x a(0)}{8a^2(0)} (x-y)^2 + \frac{c(0)}{4a(0)} (x-y)^2 \right) \right].
\]
3. Galoisian approach to propagators

In this section we apply Picard–Vessiot theory in the context of propagators. We are interested in Liouvillian solutions of the linear Schrödinger equation (LSE) (13); that is, the so-called Liouvillian propagator (29) is obtained through Liouvillian functions. In this way, we can give a Galoisian formulation for this kind of integrability.

Definition 8. The linear Schrödinger equation (13) is integrable in the Galoisian sense (Galois integrable) when it has a Liouvillian propagator (29).

Theorem 9 (Galoisian approach to LSE). The linear Schrödinger equation (13) is Galois integrable if and only if the differential Galois group of the characteristic equation (17) is virtually solvable.

Proof. Consider $K$ as the differential field of the characteristic equation and also consider $\mu_0$ and $\mu_1$ as in lemma 6. Let us suppose that $\mu_0$ is a Liouvillian solution of the characteristic equation; then by the D’Alambert reduction method, the second solution $\mu_1$ is Liouvillian too. Thus, in virtue of theorem 3, we see that the differential Galois group of the characteristic equation is virtually solvable and $\alpha$ is an algebraic solution of the Riccati equation over $K$. Now, using $\mu_0$ and $\mu_1$ in $\beta$ and $\gamma$, as in lemma 6, we see that they are Liouvillian functions over $K$ and therefore the propagator is Liouvillian. Thus, by definition 8 we get that the Schrödinger equation is Galois integrable. Conversely, assuming that the Schrödinger equation is Galois integrable we see that it has a Liouvillian propagator that can be obtained through an algebraic solution of the Riccati equation over $K$. For instance, there exists a Liouvillian solution $\mu_0$ of the characteristic equation which by theorem 3 implies that its differential Galois group is virtually solvable. □

Remark 10. By virtue of theorem 9 we can construct many Liouvillian propagators through integrable second order differential equations over a differential field as well as by algebraic solutions of Riccati equations over such a differential field. This is the practical aim of this paper, which will be given in sections 4 and 5.

4. Degenerate parametric oscillator and Ince’s equation by a Galoisian approach

In this section we use the Galoisian approach to LSE (section 3, theorem 9) to study the integrability of Ince’s equation

$$\ddot{\mu} + \frac{2\lambda \omega \sin(2\omega t)}{\omega + \lambda \cos(2\omega t)} \dot{\mu} + \frac{\omega^3 - 3\omega \lambda^2 - (\omega^2 \lambda + \lambda^3) \cos(2\omega t)}{\omega + \lambda \cos(2\omega t)} \mu = 0$$

(30)

that corresponds to the characteristic equation (see (21) and (22) and lemma 6 of section 2) of the parametric oscillator given by the Schrödinger equation (1)–(2). To find the solutions of Ince’s equation (30) we apply two practical tools of differential Galois theory: the Kovacic algorithm developed by J Kovacic in [18] and Hamiltonian algebrization developed by the first author in [1, 3]. Here we follow the version of the Kovacic algorithm given in [1, 3, 18].

4.1. Case $\lambda = \omega$

Ince’s equation (30) becomes

$$\ddot{\mu} + 2\lambda \tan(\lambda t) \dot{\mu} - 2\mu = 0.$$
To solve the characteristic equation (31) through differential Galois theory we apply the Hamiltonian algebrization process and Kovacic’s algorithm to obtain the solutions as well as its differential Galois group. Afterwards, we construct the corresponding propagator associated with this characteristic equation. We consider $K = \mathbb{C}(\tan \lambda t)$ as the differential field of the characteristic equation.

We first consider the Hamiltonian change of variable $\tau = \tan \lambda t$ obtaining $\alpha = \lambda^2 (1 + \tau^2)^2$, and using the Hamiltonian algebrization process we get

$$\frac{d^2}{dt^2} \hat{\mu}(\tau) + \frac{4\tau}{1 + \tau^2} \frac{d}{dt} \hat{\mu}(\tau) - \frac{2}{(1 + \tau^2)^2} \hat{\mu}(\tau) = 0,$$

i.e., the algebraic form of the characteristic equation, $\hat{K} = \mathbb{C}(\tau)$ being its differential field which is isomorphic to $K = \mathbb{C}(\tan \lambda t)$. In order to apply the Kovacic algorithm we should reduce the equation (32) through the change of dependent variable

$$\hat{\mu}(\tau) = y \exp \left( -\frac{1}{2} \int_0^\tau \frac{4s}{1 + s^2} ds \right) = \frac{y}{1 + \tau^2}.$$

Thus we obtain the reduced form

$$\frac{d^2}{dt^2} y = ry, \quad r = \frac{2\tau^2 + 4}{(1 + \tau^2)^2} = \frac{2}{1 + \tau^2} + \frac{2}{(1 + \tau^2)^3},$$

where the Picard–Vessiot extension is the same as in (32), denoted by $\hat{L}$, and the differential Galois group will be the same for both equations due to the fact that their coefficients are rational functions in $\tau$, i.e., $\hat{K} = \mathbb{C}(\tau)$ and their solutions are linked by a rational function in $\tau$ which does not change the Picard–Vessiot extension.

Applying Kovacic’s algorithm we see that $\Gamma = \{i, -i, \infty\}$, $\tilde{r}_1 = \tilde{r}_{-1} = \tilde{r}_\infty = 2$, which implies that equation (33) can fall in cases 1, 2, 3 or 4 of the algorithm. We begin analyzing case 1.

**Case 1.** According to step 1, we check the conditions (c2) and (∞2) to obtain $b_i = b_{-i} = -\frac{1}{2}$ and $b_\infty = 2$. In this way, $\alpha_1^\pm = \alpha_{-1}^\pm = \frac{1}{2}$, $\alpha_\infty^\pm = 1$, $\sqrt{\theta_i} = \sqrt{\theta_{-i}} = \sqrt{\theta_\infty} = 0$. By step 2 we obtain $D = \{1\}$ through two different options: $1 = \alpha_\infty^+ - \alpha_\infty^- - \alpha_{-1}^+ + \alpha_{-1}^-$ and $1 = \alpha_\infty^+ - \alpha_\infty^- - \alpha_1^+ + \alpha_1^-$. We have two possibilities for $\omega$: $\omega_1 = \frac{1}{\tau_{-1}} - \frac{1}{\tau_1}$ and $\omega_2 = \frac{1}{\tau_{-1}} - \frac{1}{\tau_1}$, that is, $\omega_1^2 = \omega_2^2$. Following step 3 we see that there does not exist a polynomial $P_1^\circ(\tau) = \tau + a_0$ corresponding to this part of the algorithm. In this way, we should go to case 2 of the Kovacic algorithm due to $\text{DGal}(\hat{L}/\hat{K})$ not being a subgroup of the triangular group of $\text{SL}(2, \mathbb{C})$.

**Case 2.** According to step 1, we check the conditions (c2) and (∞2) to obtain $b_i = b_{-i} = -\frac{1}{2}$ and $b_\infty = 2$. In this way, $E_i = E_{-i} = \{2\}$, $E_\infty = \{-4, 2, 8\}$. By step 2 we obtain $D = \{2\}$ only for $e_\infty = 8$. In this way, we obtain $\theta = \frac{1}{\tau_{-1}} + \frac{1}{\tau_1}$. Following step 3 we obtain the polynomial $P_2(\tau) = \tau^2 - 1$, which leads us to obtain $\phi = \frac{1}{\tau_{-1}} + \frac{1}{\tau_1} + \frac{2\pi}{\tau_{-1} - \tau_1}$. In this way, solving the algebraic equation

$$\omega^2 + \phi \omega + \frac{1}{4} (\phi, \phi^2 - 2r) = 0,$$

we obtain two solutions for $\omega$:

$$\omega_- = \frac{2}{\tau_{-1} - \tau_1} + \frac{1}{\tau_{-1} - \tau_1} = -1, \quad \omega_+ = \frac{2}{\tau_{-1} + \tau_1} + \frac{1}{\tau_{-1} + \tau_1}.$$

In this way, due to $y = e^{\phi \omega_+ \tau}$, we have the general solution of (33):

$$y = C_1 e^{-\alpha_{\text{arctan}} \tau (\tau - 1) \sqrt{1 + \tau^2}} + C_2 e^{\alpha_{\text{arctan}} \tau (\tau + 1) \sqrt{1 + \tau^2}},$$
\( \mu(\tau) = D_{\infty} \), that is, the infinite dihedral group. Now, the general solution for equation (32) is given by
\[
\hat{\mu}(\tau) = C_1 e^{-\arctan \left( \frac{\tau - 1}{\tau + 1} \right)} + C_2 e^{\arctan \left( \frac{\tau + 1}{\tau - 1} \right)};
\]
the differential Galois group for the algebrized characteristic equation (32) is also the dihedral infinite group \( \mathbb{D}_{\infty} \). Recalling that \( \tau = \tan \lambda t \), we get the general solution of the characteristic equation
\[
\mu(t) = C_1 e^{-\lambda t} (\sin \lambda t - \cos \lambda t) + C_2 e^{\lambda t} (\sin \lambda t + \cos \lambda t),
\]
which can also be written as
\[
\mu(t) = (C_1 + C_2) (\sinh \lambda t \cos \lambda t + \cosh \lambda t \sin \lambda t) + (C_2 - C_1) (\sinh \lambda t \sin \lambda t + \cosh \lambda t \cos \lambda t),
\]
and its differential Galois group is also the dihedral infinite group, i.e., \( \text{DGal}(L/K) = \mathbb{D}_{\infty} \).

Now, we find \( \mu_0(t) \) and \( \mu_1(t) \) satisfying the conditions of lemma 6:
\[
\begin{align*}
\mu_0(t) &= (\sinh \lambda t \cos \lambda t + \cosh \lambda t \sin \lambda t)/\lambda, \quad C_1 = C_2 = \frac{1}{2}, \\
\mu_1(t) &= \sinh \lambda t \sin \lambda t + \cosh \lambda t \cos \lambda t, \quad -C_1 = C_2 = \frac{1}{2},
\end{align*}
\]

which for \( \lambda = 1 \) corresponds to the solutions given in [10]. Therefore, by theorem 9 the Schrödinger equation (1)–(2) is Galois integrable, and its propagator by lemma 7 is given by
\[
\begin{align*}
G(x, y, t) &= \sqrt{2\pi} i (\cos \lambda t \sinh \lambda t + \sin \lambda t \cosh \lambda t) \\
&\times \exp \left[ \frac{\lambda}{2} \left( x^2 - y^2 \right) \sin \lambda t \sinh \lambda t + 2xy - (x^2 + y^2) \cos \lambda t \cosh \lambda t \right],
\end{align*}
\]
and it is Liouvillian.

On the other hand, after the Hamiltonian algebrization process over equation (31), we use the command \texttt{kovacic} over equation (32) to obtain
\[
\begin{bmatrix}
\frac{(i + \tau)^j}{\sqrt{1 + \tau^2}} (1 + \tau) & \frac{-i + \tau}{\sqrt{1 + \tau^2}} (-1 + \tau)^j \\
\end{bmatrix};
\]
therefore we can write the general solution as
\[
\hat{\mu} = C_1 \frac{(i + \tau)^j}{\sqrt{1 + \tau^2}} (1 + \tau) + C_2 \frac{-i + \tau}{\sqrt{1 + \tau^2}} (-1 + \tau)^j.
\]
Recalling that
\[
\cosh z + \sinh z = e^z, \quad \cos (\arctan z) = \frac{1}{\sqrt{1 + z^2}}, \quad \sin (\arctan z) = \frac{z}{\sqrt{1 + z^2}}, \quad \text{arctan} z = \frac{1}{2}(\ln(1 - iz) - \ln(1 + iz)),
\]
equation (35) becomes
\[
\hat{\mu}(\tau) = (C_1 e^{\arctan \tau} - C_2 e^{-\arctan \tau}) \cos(\arctan \tau) + (C_1 e^{\arctan \tau} + C_2 e^{-\arctan \tau}) \sin(\arctan \tau).
\]
Now, recalling \( \tau = \tan \lambda t \), we obtain
\[
\mu(t) = (C_1 e^{\lambda t} - C_2 e^{-\lambda t}) \cos \lambda t + (C_1 e^{\lambda t} + C_2 e^{-\lambda t}) \sin \lambda t.
\]
Finally, from the conditions of \( \mu_0 \) and \( \mu_1 \) stated in lemma 6, \( \mu_0(t) \) and \( \mu_1(t) \) become
\[
\begin{align*}
\mu_0(t) &= \sinh \lambda t \cos \lambda t + \cosh \lambda t \sin \lambda t, \\
\mu_1(t) &= \cosh \lambda t \cos \lambda t + \sinh \lambda t \sin \lambda t,
\end{align*}
\]
which are the same solutions found using the Kovacic algorithm by hand.
4.2. Case \( \lambda \neq \omega \)

For Ince’s equation (30) using the Hamiltonian algebrization procedure and the Kovacic algorithm and by properties of double angle, we can write equation (30) in terms of tan(\(\omega t\)) and we can consider its differential field to \(K = C(\tan \omega t)\). After the Hamiltonian change of variable \(\tau = \tan \omega t\) we obtain \(\omega = \omega^2(1 + \tau^2)^2\), and by the Hamiltonian algebrization procedure we get as an algebraic form of (30)

\[
\frac{\partial^2}{\partial \tau^2} \mu + \phi_1(\tau) \frac{\partial}{\partial \tau} \mu + \phi_0(\tau) \mu = 0, \quad \phi_1(\tau) = \frac{2(\lambda - \omega)\tau^3 - (3\lambda + \omega)\tau}{(1 + \tau^2)((\lambda - \omega)\tau^2 - \lambda - \omega)},
\]

\[
\phi_0(\tau) = -\frac{(\omega^3 - 3\omega \lambda^2 + \omega^2 \lambda + \lambda^3)\tau^4 + \omega^3 - 3\omega \lambda^2 - \omega^2 \lambda - \lambda^3}{(1 + \tau^2)^2((\lambda - \omega)\tau^2 - \lambda - \omega)\omega^2}.
\]

We can eliminate one parameter through the change \(\lambda = \kappa \omega\); thus, our algebraic form becomes

\[
\frac{\partial^2}{\partial \tau^2} \hat{\mu} + \phi_1(\tau) \frac{\partial}{\partial \tau} \hat{\mu} + \phi_0(\tau) \hat{\mu} = 0, \quad \phi_1(\tau) = \frac{2(\kappa - 1)\tau^3 - (3\kappa + 1)\tau}{(1 + \tau^2)((\kappa - 1)\tau^2 - \kappa - 1)},
\]

\[
\phi_0(\tau) = -\frac{(1 - 3\omega^2 + \kappa + \kappa^3)\tau^2 + 1 - 3\kappa^2 - \kappa - \kappa^3}{(1 + \tau^2)^2((\kappa - 1)\tau^2 - \kappa - 1)}, \quad \kappa \neq 1.
\]

Following the same steps for \(\lambda = \omega\) in the Kovacic algorithm, by proposition 4 we transform equation (36) in

\[
\frac{\partial^2}{\partial \tau^2} y = ry, \quad \hat{\mu}(\tau) = y\frac{\sqrt{(\kappa - 1)\tau^2 - 1 - \kappa}}{1 + \tau^2}
\]

\[
r = \frac{((-4\kappa^3 - 4\kappa + 7\kappa^2 + \kappa^4)\tau^4 + (10\kappa^2 - 2\kappa^4)\tau^2 + 4\kappa + 7\kappa^2 + 4\kappa^3 + \kappa^4)}{(1 + \tau^2)^2((-1 + \kappa)\tau^2 - 1 - \kappa)^2}.
\]

We see that \(\Gamma = \{ i, -i, \frac{\kappa + 1}{\kappa + i}, -\frac{\kappa + 1}{\kappa + i}, \infty \}, \forall c \in \Gamma\), which implies that equation (37) could fall in cases 1, 2, 3 or 4 of the algorithm. We discard case 1 for similar reasons when \(\lambda = \omega\).

In the same way as in the case \(\lambda = \omega\), by step 2 and step 3 we obtain the general solution of (37):

\[
y = C_1 e^{-\kappa \arctan \tau (\tau - 1)\sqrt{1 + \tau^2} + \sqrt{(\kappa - 1)\tau^2 - \kappa - 1}} + C_2 e^{\kappa \arctan \tau (\tau + 1)\sqrt{1 + \tau^2} + \sqrt{(\kappa - 1)\tau^2 - \kappa - 1}},
\]

where

\[
\hat{\mu}(\tau) = C_1 e^{-\kappa \arctan \tau (\tau - 1)\sqrt{1 + \tau^2} + \sqrt{(\kappa - 1)\tau^2 - \kappa - 1}} + C_2 e^{\kappa \arctan \tau (\tau + 1)\sqrt{1 + \tau^2} + \sqrt{(\kappa - 1)\tau^2 - \kappa - 1}}.
\]

The differential Galois group for the algebrized characteristic equation (36) is also the dihedral infinite group \(\mathbb{D}_\infty\) for any \(\kappa \neq 0\). Now, the general solution for equation (36) is given by

\[
\hat{\mu}(\tau) = \mu = C_1 e^{-\kappa \tau \sin(\omega t - \cos \omega t)} + C_2 e^{\kappa \tau \sin(\omega t + \cos \omega t)},
\]

which can also be written as

\[
\mu(t) = \mu_0(t) + \mu_1(t),
\]

where

\[
\mu_0(t) = \frac{\sinh \lambda \tau \cos \omega t + \cosh \lambda \tau \sin \omega t}{\omega}, \quad C_1 = C_2 = \frac{1}{2},
\]

\[
\mu_1(t) = \sinh \lambda \tau \sin \omega t + \cosh \lambda \tau \cos \omega t, \quad -C_1 = C_2 = \frac{1}{2}.
\]
which for \( \lambda = 1 \) corresponds to the solutions given in [10]. Therefore, by theorem 9 the Schrödinger equation (1)–(2) is Galois integrable, and its propagator by lemma 7 is given by

\[
G(x, y, t) = \sqrt{\omega} \times \exp \left[ \frac{\omega}{2} \left( x^2 - y^2 \right) \sin \omega t \sinh \lambda t + 2xy - (x^2 + y^2) \cos \omega t \cosh \lambda t \right] \left( \frac{\tau + i}{\tau i} \right)^{2x (1 + \tau)} \frac{\left( \frac{\tau - i}{\tau - i} \right)^{2x} (-1 + \tau)}{\sqrt{\tau + \tau^2}} + C_2 \frac{\left( \frac{\tau - i}{\tau - i} \right)^{2x} (1 - \tau)}{\sqrt{1 + \tau^2}}. \tag{41}\]

Due to \( \kappa \arctan \tau = \kappa \arctan \left( \ln(1 - i\tau) - \ln(1 + i\tau) \right) \), equation (41) becomes

\[
\hat{\mu} = C_1 \frac{\left( \frac{\tau + i}{\tau + i} \right)^{2x} (1 + \tau)}{\sqrt{1 + \tau^2}} + C_2 \frac{\left( \frac{\tau - i}{\tau - i} \right)^{2x} (-1 + \tau)}{\sqrt{1 + \tau^2}}, \quad \mu = C_1 \sin \omega t \cos \omega t \cosh \lambda t + C_2 \sin \omega t - \cos \omega t \sinh \lambda t, \quad \mu_0 = \sin \omega t \cosh \lambda t + \cos \omega t \sinh \lambda t, \quad \mu_1 = \sin \omega t \sinh \lambda t + \cos \omega t \cosh \lambda t.

Thus, we have proven the following result.

**Theorem 11.** The fundamental solution of Ince’s characteristic equation (30) is given by

\[
\mu = C_1 \sin \omega t + \cos \omega t \cosh \lambda t + C_2 \sin \omega t - \cos \omega t \sinh \lambda t \]

and the propagator for the Schrödinger equation (1)–(2) is Galois integrable; its propagator by lemma 7 is given by

\[
G(x, y, t) = \sqrt{\omega} \times \exp \left[ \frac{\omega}{2} \left( x^2 - y^2 \right) \sin \omega t \sinh \lambda t + 2xy - (x^2 + y^2) \cos \omega t \cosh \lambda t \right] \left( \frac{\tau + i}{\tau i} \right)^{2x (1 + \tau)} \frac{\left( \frac{\tau - i}{\tau - i} \right)^{2x} (-1 + \tau)}{\sqrt{1 + \tau^2}} + C_2 \frac{\left( \frac{\tau - i}{\tau - i} \right)^{2x} (1 - \tau)}{\sqrt{1 + \tau^2}}. \tag{42}\]

and it is Liouvillian.

5. Toy examples

In this section we illustrate our Galoisian approach through some elementary examples. We use the Galoisian approach to LSE, theorem 9, to generalize examples introduced in [19] and to introduce toy examples. The starting point is the knowledge of the integrability, in the Picard–Vessiot sense, of the characteristic equation, or equivalently the existence of an algebraic solution over its differential field of its associated Riccati equation. In general, to obtain the solutions of the characteristic equations we can apply the Kovacic algorithm and the Hamiltonian algebrization procedure as in section 4.
5.1. Toy models inspired by integrable Riccati equations

(1) From the Riccati equation
\[ \frac{d\alpha}{dt} + (\cos t)\alpha^2 = 0 \]
we can construct the propagator for the Schrödinger equation
\[ i \frac{\partial \psi}{\partial t} = -\frac{\cos t}{4} \frac{\partial^2 \psi}{\partial x^2}, \]
and the propagator is given by (29) with
\[ \mu_0(t) = 2 \sin t, \quad \alpha_0(t) = \frac{1}{\sin t}, \quad \beta_0(t) = -\frac{1}{2 \sin t}, \quad \gamma_0(t) = \frac{1}{16 \sin t}. \]

(2) From the Riccati equation
\[ \frac{d\alpha}{dt} + 2(t + a_0)\alpha^2 = 0 \]
we can construct the propagator for the Schrödinger equation
\[ i \frac{\partial \psi}{\partial t} = -\left( \frac{t + a_0}{2} \right) \frac{\partial^2 \psi}{\partial x^2}, \]
and the propagator is given by (29) with
\[ \mu_0(t) = t^2 + 2a_0t, \quad \alpha_0(t) = \frac{1}{t^2 + 2a_0t}, \quad \beta_0(t) = -\frac{1}{t^2 + 2a_0t}, \quad \gamma_0(t) = \frac{1}{4(t^2 + 2a_0t)}. \]

(3) From the Riccati equation
\[ \frac{d\alpha}{dt} + 2 \cos t + \frac{1}{\cos t} \alpha^2 = 0 \]
we can construct the propagator for the Schrödinger equation
\[ i \frac{\partial \psi}{\partial t} = -\left( \frac{1}{4 \cos t} \right) \frac{\partial^2 \psi}{\partial x^2} + 2(\cos t)x^2 \psi, \]
and the propagator is given by (29) with
\[ \mu_0(t) = \frac{1}{2} \sin t, \quad \alpha_0(t) = \frac{\cos^2 t}{\sin t}, \quad \beta_0(t) = -\frac{2}{\sin t}, \quad \gamma_0(t) = \frac{1}{\sin t \cos^2 t} - 32 \int_0^t \frac{1}{\cos^3 \tau} d\tau. \] (43)

(4) From the Riccati equation
\[ \frac{d\alpha}{dt} + 2(tan t)\alpha + \frac{1}{4 \cos t} \alpha^2 = 0 \]
we can construct the propagator for the Schrödinger equation
\[ i \frac{\partial \psi}{\partial t} = -\left( \frac{1}{4 \cos t} \right) \frac{\partial^2 \psi}{\partial x^2} - i(tan t)x \frac{\partial \psi}{\partial x}, \]
and the propagator is given by (29) with
\[ \mu_0(t) = \frac{\sin t}{2}, \quad \alpha_0(t) = \frac{\cos^2 t}{\sin t}, \quad \beta_0(t) = -\frac{2 \exp \left( - \int_0^t \tan(\tau) \, d\tau \right)}{\sin t}, \quad \gamma_0(t) = \frac{\exp \left( -2 \int_0^t \tan(\tau) \, d\tau \right)}{\sin t \cos^2 t}. \]
(5) From the Riccati equation
\[ \frac{d\alpha}{dt} = a e^{\lambda t} - a e^{\beta t} - \alpha^2 = 0 \]
we deduce that for the Schrödinger equation
\[ \frac{\partial \psi}{\partial t} = -\frac{1}{4} \frac{\partial^2 \psi}{\partial x^2} - a e^{\lambda t} x^2 \psi + \frac{at e^{\lambda t}}{2} \frac{\partial \psi}{\partial x} - i d(t) \psi, \]
the propagator is given by (29) where
\[ \mu_0(t) = \frac{t}{2}, \quad \alpha_0(t) = -\frac{1}{t}; \]
\[ \beta_0(t) = -\frac{1}{t} \exp\left( \frac{a e^{\lambda t}(\lambda t - 1)}{2\lambda^2} + \frac{a}{2\lambda^2} \right), \]
\[ \gamma_0(t) = -\frac{1}{4t} \exp\left( -\frac{a e^{\lambda t}(\lambda t - 1)}{\lambda^2} - \frac{a}{\lambda^2} \right) + \frac{a}{4\lambda}(e^{\lambda t} - 1). \]

5.2. The characteristic equation \( \partial^2 \mu + t^4 \mu = 0 \)

We consider in this equation a differential field to \( K = \mathbb{C}(t) \). This is a generalization of the case \( n = 1 \), which was presented in [19]. However, the case \( n = 1 \) does not correspond to Galoisian integrability of the characteristic equation: the solutions are not Liouvillian due to the fact that they are Airy functions. Now, we study the integrability, in the Galoisian sense, of this equation through the Kovacic algorithm. As in [33], we obtain three conditions for \( n \) to get virtual solvability of the differential Galois group. If follows the construction of the propagators related to these integrability conditions.

(1) Let us consider the characteristic equation \( \partial^2 \mu + \mu = 0 \) \( (n = 0) \). A basis of solutions is given by \( B = \{ \sin t, \cos t \} \). Thus, the Picard–Vessiot extension is given by \( L = \mathbb{C}(t, e^t) \) and the differential Galois group is \( \text{DGal}(L/K) = G_m \), that is, the diagonal group of \( \text{SL}(2, \mathbb{C}) \). By proposition 4, through the change of variable \( \alpha = \partial_t \mu / \mu \), we obtain the Riccati equation (15), where \( a = 1/4, b = 1 \) and \( c = 0 \). Considering \( \mu_0(t) = \lambda_1 \sin t \) and \( \mu_1(t) = \lambda_2 \cos t \), we see that the conditions of lemma 6 are satisfied when \( \lambda_1 = 1/2 \). Furthermore, \( \lambda_2 \) must be 2 to get \( W(\mu_0, \mu_1) = 1 \). In this way, by theorem 9 the Schrödinger equation is Galois integrable.

(2) Let us consider the characteristic equation \( \partial^2 \mu + \mu/t^4 = 0 \) \( (n = -2) \). A basis of solutions is given by \( B = \{ t^{m+1}, t^{-m} \} \), being \( m = (-1 \pm \sqrt{5})/2 \). Since \( m \notin \mathbb{Q} \), the Picard–Vessiot extension is \( L = \mathbb{C}(t, t^m) \) and therefore \( \text{DGal}(L/K) = G_m \). In [1] there is a complete study of the Galoisian structure of this equation. Thus, by theorem 9 the Schrödinger equation is Galois integrable.

(3) Let us consider the characteristic equation \( \partial^2 \mu + \mu/t^4 = 0 \) \( (n = -4) \). A basis of solutions is given by \( B = \{ t \cos(1/t), t \sin(1/t) \} \). Thus, the Picard–Vessiot extension is \( L = \mathbb{C}(t, e^{t^2}) \) and therefore \( \text{DGal}(L/K) = G_m \). Here, initial conditions for the characteristic equation are satisfied when \( t \to 0^+ \), that is, \( \mu_0(t) \to 0 \) when \( t \to 0^+ \). By theorem 9 the Schrödinger equation is Galois integrable.
Final remarks

This paper is a starting point to study the integrability of partial differential equations in a more general sense through differential Galois theory. With this approach we studied the linear Schrödinger equation corresponding to a generalized (quadratic) harmonic oscillator, where the main result is the obtainment of the general solution of the Ince’s differential equation and the Liouvillian propagator of a degenerate parametric oscillator, which generalizes the particular results obtained in [10, 19]. Although there are plenty of papers relating explicit solutions and harmonic oscillators (see [12]), we recall that differential Galois theory can provide the Liouvillian solutions of characteristic equations, without previous knowledge of such equations, to build new propagators. As an exercise the interested reader can apply this techniques, knowing the algebraic solutions of Riccati equations, to obtain characteristic equations and Liouvillian propagators. This Galoisian approach can be useful for obtaining new propagators in which the characteristic equations are special functions, such as the Heun equation.

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