Reflected backward stochastic differential equations with optional barriers: monotone approximation

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Abstract In this short note we consider RBSDE with Lipschitz drivers and barrier processes that are optional and right upper semicontinuous. We treat the case when the barrier can be represented as a decreasing limit of cadlag barriers. We combine well-known existence results for cadlag barriers with comparison arguments for the control process to construct solutions. Finally, we highlight the connection of such RBSDEs with usual cadlag BSDEs.

Keywords reflected backward stochastic differential equation, g-expectation, optional barrier, monotone approximation, comparison principle

AMS Subject Classification 60H10; 60G40

1 Introduction

Reflected backward stochastic differential equations (RBSDE) are a well known tool suited to solve the problem of hedging and pricing American options. The control process \( Y \) of the solution triplet \((Y, Z, A)\) guiding the dynamics of RBSDE is reflected at a barrier process \( \xi \), while the increasing process \( A \) is responsible for keeping \( Y \) above \( \xi \). The original continuity assumption of El Karoui et al. [8] on \( \xi \) has been relaxed in a series of papers to various degrees of discontinuity (see [3, 9, 12, 13, 14, 15, 16]). In [19], Peng and Xu dealt with the case of a Brownian filtration and very irregular \( L^2 \)-obstacle, by introducing a new formulation of Skorokhod condition. The fundamental results on RBSDEs when the barrier is not right continuous were obtained in [10]. In [2] Bouhadou and Ouknine treated RBSDEs in the frame of a general filtration and a ladlag predictable barrier. In these references cited above, the construction of the solution was following the classical route of combining a priori inequalities with a recursively given sequence of approximations of the solution via a suitable fixed point argument.

In this short note, we study the existence and uniqueness of the solution when the barrier can be approximated by a decreasing sequence of cadlag Barriers \((\xi_n)_{n \in \mathbb{N}}\) in \( \mathcal{S}^2 \). This allows technically simpler proofs, since we apply classical results on the existence of solution of RBSDEs with RCLL Barriers, and comparison related arguments (see Hamadène, Wang [14]) to show that the associated sequence of control processes \((Y^n)_{n \in \mathbb{N}}\) is also decreasing. Our first main
result, shows that when the driver \( g \) does not depend on \( y \) and \( z \), the limiting process is the solution of the following optimal stopping problem.

\[
Y_S = \underset{\tau \in T_S}{\text{ess sup}} E \left[ \xi_\tau + \int_T^\tau g(u) \, du \mid \mathcal{F}_S \right].
\] (1.1)

In order to prove that \( Y \) provides the good candidate for the solution of RBSDEs with barrier \( \xi \), we use some tools from optimal stopping theory (cf., [7]).

In the second part of this note, we expand our convergence results to the non linear case, by considering the notion of \( g \)-conditional expectations (introduced by Peng [21]) defined through the notion of BSDEs and used to quantify the riskiness of financial positions (see, among many others [4, 1, 21, 23]). We recall that the \( g \)-conditional expectation at a stopping time \( \tau \) such that \( \tau \leq T \) a.s. (where \( T > 0 \) is a fixed final horizon) is the operator which maps a given square integrable terminal condition \( \xi_T \) to the position at \( \tau \) of the first component of the solution to the BSDEs with parameters \((g, \xi_T)\). The operator is denoted by \( \mathcal{E}^g(.) \).

Roughly speaking, if we interpret \( \xi \) as a financial position process and \(-\mathcal{E}^g(.)\) as a dynamic risk measure, \( R^g[\xi](S) \) defined by the following

\[
R^g[\xi](S) := \underset{\tau \in T_S}{\text{ess sup}} \mathcal{E}^g_{S,\tau}(\xi_\tau), \quad S \in T_0,
\] (1.2)

can be seen as the minimal risk at time \( S \). In the present paper, we show that when the barrier can be approximated by RCLL barriers monotonically from above in \( S^2 \), the first component \( Y \) of the solution of RBSDE with barrier \( \xi \) satisfies the following

\[
Y_S = R^g[\xi](S), \quad S \in T_0.
\]

In the last part, we show how optional RBSDEs with non regular barrier, are closely connected with usual RCLL BSDEs, by introducing a new way to approximate the solution \( Y \). The main novelty is to prove that the solution of RBSDE with optional barrier \( \xi \) can be obtained as the limit of the following sequence processes:

\[
\bar{Y}_t^n = \xi_t \vee \left( \bar{\xi}_T + \int_T^t g(u, Y_u^n, Z_u^n) \, du + \int_t^T n(Y_u^n - \bar{\xi}_u^n)^- \, du - \int_t^T Z_u^n \, dW_u - \int_t^T \int_E l_u^n(e) \tilde{N}(du, de) \right).
\]

Where \( \bar{\xi} \) is given by \( \bar{\xi} = R^g(\xi + X) - X \) and \( X \) denotes an optional process satisfying some suitable assumptions.

Let us present briefly our plan. In section 2, we recall the solution concept of optional RBSDE under a suitable version of Skorokhod condition. In section 3, we present our solution by monotone approximation of the barrier. Section 4, is dedicated to give a new approximation of the solution of RBSDE studied in section 2. Section 5, is devoted to make some link between RBSDE with optional Barrier and RBSDE in the sens of Peng-Xu [19], under an additional assumption on \( \xi \).

2 Preliminaries

First we introduce a series of notations that will be used throughout the paper. Let \( T > 0 \) be a fixed positive real number. Let \((E, \mathcal{E})\) be a measurable space equipped with a \( \sigma \)-finite positive
measure \( \mu \). Let \((\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})\) be a probability space. The filtration is assumed to be complete, right continuous and quasi-left continuous. We suppose that \((\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0})\) supports a Brownian motion \( W \) and an independent Poisson random measure \( N \) with intensity \( dt \otimes \mu(\text{d}e) \). We denote \( \hat{N}(dt, \text{d}e) \) its compensated Poisson random measure.

For \( t \in [0, T] \), we denote by \( \mathcal{T}_t \) (resp. \( \mathcal{T}_t^p \)) the set of stopping times (resp. predictable stopping times) \( \tau \) such that \( P(t \leq \tau \leq T) = 1 \). More generally, for a given stopping time \( \nu \in \mathcal{T}_0 \) (resp. \( \nu \in \mathcal{T}_0^p \)), we denote by \( \mathcal{T}_\nu \) (resp. \( \mathcal{T}_\nu^p \)) the set of stopping times (resp. predictable stopping times) \( \tau \) such that \( P(\nu \leq \tau \leq T) = 1 \). We denote by \( \mathcal{P} \) the predictable \( \sigma \)-algebra on \( \Omega \times [0, T] \). We use the following notation:

- \( L^2(\mathcal{F}_T) \) is the set of random variables which are \( \mathcal{F}_T \)-measurable and square-integrable.

- \( L^2(\mathcal{F}_\nu) \) is the set of measurable functions \( \ell : E \to \mathbb{R} \) such that \( ||\ell||_\mu^2 := \int_E |\ell(e)|^2 \mu(\text{d}e) < +\infty \).

- \( \mathcal{H}^2 \) is the set of processes \( \phi \) which are \emph{predictable} such that
  \[
  ||\phi||_{\mathcal{H}^2}^2 := E \left[ \int_0^T \phi_t^2 \, dt \right] < \infty.
  \]

- \( \mathcal{H}^2(\mu) \) is the set of processes \( \phi \) which are \emph{predictable}, that is, measurable \( \phi : (\Omega \times [0, T] \times E, \mathcal{P} \otimes \mathcal{E}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})); \ (\omega, t, e) \mapsto \phi_t(\omega, e) \) such that
  \[
  ||\phi||_{\mathcal{H}^2(\mu)}^2 := E \left[ \int_0^T ||\phi_t||_\mu^2 \, dt \right] < \infty.
  \]

- \( \mathcal{D}^2 \) is the vector space of \( \mathcal{F}_t \)-adapted RCLL processes \( \phi = (\phi_t)_{t \in [0, T]} \) such that
  \[
  ||\phi||_{\mathcal{D}^2}^2 := E \left[ \sup_{t \leq T} |\phi_t|^2 \right] < \infty.
  \]

- \( \mathcal{S}^2 \) is the vector space of real-valued optional processes \( \phi \) such that
  \[
  ||\phi||_{\mathcal{S}^2}^2 := E \left[ \esssup_{\tau \in \mathcal{T}_0} |\phi_\tau|^2 \right] < \infty.
  \]

- \( \mathcal{S}^4 \) is the vector space of real-valued optional processes \( \phi \) such that
  \[
  ||\phi||_{\mathcal{S}^4}^4 := E \left[ \esssup_{\tau \in \mathcal{T}_0} |\phi_\tau|^4 \right] < \infty.
  \]

- \( \mathcal{S}^2_{\mu} \) is the vector space of real-valued predictable, increasing processes \( A \) such that \( A_0 = 0 \), \( E(A_T^2) < \infty \).

We say that an \( \mathcal{F} \)-progressively measurable process \( X \) is of class \((D)\), if the family \( \{X_\tau, \tau \in \mathcal{T}_0\} \) is uniformly integrable.

For a process \( \psi \), we write \( \psi_- \) for the process of left limits \( \psi_{t-} = \lim_{s \uparrow t} \psi_s \), for \( t > 0 \), provided they all exist, and \( \psi_+ \) for the process of right limits \( \psi_{t+} = \lim_{s \downarrow t} \psi_s \) for \( t < T \) in case they all exist.

For a ladlag process \( X \), we denote by \( \Delta^+ X_t := X_{t_+} - X_t \) the size of the right jump of \( X \) at \( t \), and by \( \Delta X_t := X_t - X_{t-} \) the size of the left jump of \( X \) at \( t \).

If \( A \) is an increasing process, then it can be represented in the form \( A = A^r + A^\theta \), with
Let us recall the key section theorem related to indistinguishability of optional processes or predictable processes.

**Theorem 2.1** Let $X = (X_t)$ and $Y = (Y_t)$ be two optional (resp. predictable) processes. If for every bounded stopping time (resp. predictable time) $\tau$, we have $X_{\tau} \leq Y_{\tau}$ a.s. (resp. $X_{\tau} = Y_{\tau}$ a.s.), then $X \leq Y$ (resp. $X$ and $Y$ are indistinguishable).

**Definition 2.1** (Driver, Lipschitz driver) A function $g$ is said to be a driver if

- *(measurability)* $g : \Omega \times [0,T] \times \mathbb{R}^2 \times L^2_{\mu} \to \mathbb{R}$
  
  $g(\omega, t, y, z, k) \mapsto g(\omega, t, y, z, k)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(L^2_{\mu})$ measurable,

- *(integrability)* $g(\cdot, 0, 0, 0) \in \mathbb{H}^2$.

A driver $g$ is called a Lipschitz driver if moreover there exists a constant $K \geq 0$ such that $dP \otimes dt$-a.e., for each $(y_1, z_1, k_1) \in \mathbb{R}^2 \times L^2_{\mu}$, $(y_2, z_2, k_2) \in \mathbb{R}^2 \times L^2_{\mu}$,

$$|g(\omega, t, y_1, z_1, k_1) - g(\omega, t, y_2, z_2, k_2)| \leq K(|y_1 - y_2| + |z_1 - z_2| + \|k_1 - k_2\|_{\mu}).$$

Let $g$ be a Lipschitz driver, and $\xi$ in $L^2(\mathcal{F}_T)$. The BSDE associated with Lipschitz driver $g$, terminal time $T$, and terminal condition $\xi$, is formulated as follows:

$$X_t = \xi + \int_t^T g(s, X_s, Z_s, l_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_{E} l_s(e)\tilde{N}(ds, de) \quad \text{for all } t \in [0, T] \text{ a.s.}$$

We recall that the above BSDE admits a unique solution $(X, Z, l)$ in the space $\mathbb{D}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_{\mu}$ (cf. [9]).

We also recall the definition of the conditional $g$-expectation.

**Definition 2.2** We define for each $t \in [0, T]$, and $\xi \in L^2(\mathcal{F}_T)$

$$\mathcal{E}_{t,T}^g(\cdot) := X_t.$$

We call the (non-linear) operator $\mathcal{E}_{t,T}^g(\cdot) : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_T)$ conditional $g$-expectation at time $t$. As usual, this notion can be extended to the case where the (deterministic) terminal time $T$ is replaced by a (more general) stopping time $\tau \in \mathcal{T}_0$, $t$ is replaced by a stopping time $S$ such that $S \leq \tau$ a.s. and the domain $L^2(\mathcal{F}_T)$ of the operator is replaced by $L^2(\mathcal{F}_\tau)$.

Let $T > 0$ be a fixed terminal time. Let $g$ be a Lipschitz driver. Let $\xi = (\xi_t)_{t \in [0,T]}$ be an optional process in $\mathbb{S}^2$. We suppose moreover that the process $\xi$ is not necessarily left limited. A process $\xi$ satisfying the previous properties will be called a barrier, or an obstacle.

**Definition 2.3** A quadruple $(Y, Z, l, A)$ of $\mathbb{F}$-progressively measurable processes is a solution of the reflected BSDE with Lipschitz driver $g$ and barrier $\xi$ (RBSDE($\xi, g$) for short) if

1. $(Y, Z, l, A) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_{\mu} \times \mathbb{S}^2_{\mu}$.
2. $Y_\tau = \xi_\tau + \int_\tau^T g(s, Y_s, Z_s, l_s)ds + A_T - A_\tau - \int_\tau^T Z_s dW_s - \int_\tau^T \int_{E} l_s(e)\tilde{N}(ds, de)$ a.s. for all $\tau \in \mathcal{T}_0$. 

(iii) \( Y_\tau \geq \xi_\tau \) a.s. for all \( \tau \in \mathcal{T}_0 \),

(iv) \( A \) is non decreasing predictable process with \( A_0 = 0 \) such that

\[
\int_0^T (Y_s - \limsup_{u \uparrow s} \xi_u) dA_s^r = \sum_{s < T} (Y_s - \xi_s) \Delta^+ A_s = 0 \quad \text{a.s.}
\]

Here \( A^r \) denotes the caglad part of the process \( A \).

**Remark 2.1** Since the filtration is quasi-left continuous, martingales have only totally inaccessible jumps. Thus, in this case, \( p Y_\tau = Y_\tau \) for each predictable stopping time \( \tau \in \mathcal{T}_0^p \).

**Remark 2.2** It follows from (ii), and the same arguments as above that \( \Delta Y_\tau = -\Delta A^+_\tau = -\Delta A_\tau \) a.s., for each predictable stopping time \( \tau \). This clearly yields that \( Y_{\tau^-} \geq Y_\tau \) a.s. for each predictable stopping time \( \tau \).

**Definition 2.4** A progressive process \((\xi_t)\) (resp. integrable) is said to be right (resp. left) upper semicontinuous along stopping times (right (left) USC) (resp. along stopping times in expectation (right (left) USCE)) if for all \( \tau \in \mathcal{T}_0 \) and for all sequences of stopping times \((\tau_n)\) such that \( \tau_n \downarrow \tau \) (resp. \( \tau_n \uparrow \tau \)),

\[
\xi_\tau \geq \limsup_{n \to \infty} \xi_{\tau_n} \quad \text{a.s.} \quad (\text{resp. } E[\xi_\tau] \geq \limsup_{n \to \infty} E[\xi_{\tau_n}]). \tag{2.1}
\]

**Remark 2.3** If \((Y, Z, l, A)\) is a solution of RBSDE defined above, then \( \Delta^+ Y_\tau = Y_{\tau^+} - Y_\tau = -\Delta^+ A_\tau \) a.s. for each stopping time \( \tau \in \mathcal{T}_0 \). Roughly speaking, this equality says that the process has only negative right jumps. Note also that \( Y \geq Y^+ \) up to an evanescent set, which means that process \( Y \) is right upper semicontinuous.

**Proposition 2.1** Let \( g \) be a Lipschitz driver and \( \xi \) an obstacle. Let \((Y, Z, l, A)\) be a solution to the RBSDE\((\xi, g)\).

- For each \( \tau \in \mathcal{T}_0 \), we have
  \[
  Y_\tau = \xi_\tau \vee Y^+_{\tau^-} \quad \text{a.s.}
  \]

- For each predictable stopping time \( \tau \in \mathcal{T}_0^p \), we have
  \[
  Y_{\tau^-} = \limsup_{u \uparrow \tau} \xi_u \vee Y_\tau \quad \text{a.s.}
  \]

Proof. Let us show the first assertion. Let \( \tau \in \mathcal{T}_0 \). The inequality \( \xi_\tau \vee Y^+_{\tau^-} \leq Y_\tau \) a.s. follows from the fact that \( \xi_\tau \leq Y_\tau \) a.s. and \( Y^+_{\tau^-} \leq Y_\tau \) a.s. Let us now show the second inequality. Thanks to Remark 2.3, \( \Delta^+ A_\tau = -\Delta^+ Y_\tau \) a.s. Then, from Skorokhod condition (iv), we have \( Y_1 \{ \xi_\tau > Y_\tau \} = Y^+_{\tau^-} 1 \{ \xi_\tau > Y_\tau \} \) a.s. that \( Y_\tau \leq \xi_\tau \vee Y^+_{\tau^-} \) a.s.

The task now is to prove the second assertion. Let \( \tau \in \mathcal{T}_0^p \). We have \( \limsup_{u \uparrow \tau} \xi_u \leq Y_{\tau^-} \) a.s. and \( Y_\tau \leq Y_{\tau^-} \) a.s. Hence, \( \limsup_{u \uparrow \tau} \xi_u \vee Y_\tau \leq Y_{\tau^-} \) a.s. Let us now focus on the first inequality. It follows from Remark 2.2, that \( \Delta Y_\tau = -\Delta A^+_\tau \). Then, using the second inequality of the Skorokhod condition we get, \( Y_1 \{ \xi_\tau > \limsup_{u \uparrow \tau} \xi_u \} = Y_{\tau^-} 1 \{ \xi_\tau > \limsup_{u \uparrow \tau} \xi_u \} \) a.s. Thus we have proved that \( Y_{\tau^-} \leq \limsup_{u \uparrow \tau} \xi_u \vee Y_\tau \) a.s. \( \square \)
Lemma 2.1 If $\xi$ is left USC, then $Y_{\tau} = Y_{\tau^-}$ for each predictable stopping time $\tau \in T_0^p$. On other words, the process $A^\tau$ is continuous.

Proof. Let $\tau$ be a predictable stopping time in $T_0^p$. The second assertion in Proposition 2.1, combined with the fact that $\xi$ is left USC leads to $Y_{\tau^-} \leq \xi_{\tau} \lor Y_{\tau} = Y_{\tau}$ a.s. Otherwise we know that $Y_{\tau} \leq Y_{\tau^-}$ a.s, at last $Y_{\tau} = Y_{\tau^-}$ a.s. The continuity of the process $A^\tau$ follows immediately from Remark 2.2. □

Remark 2.4 If $Y$ is right continuous, then the Skorokhod condition (iv), can be reduced to the following:

\[ \int_0^T (Y_{s^-} - \limsup_{u \uparrow s} \xi_u) dA^\tau_s = 0 \text{ a.s.} \]

Indeed, the right continuity of $Y$ together with the Remark 2.3 implies that $\Delta^+ A_t = 0$ a.s. for all $t \in [0, T]$.

Lemma 2.2 If the obstacle $\xi$ satisfies $\xi \leq \xi^+$ up to an evanescent set, then $Y$ is right-continuous.

Proof. Through the first assertion of Proposition 2.1, we have for any $\tau \in T_0$, $Y_{\tau} = \xi_{\tau} \lor Y_{\tau^+}$ a.s. Under the assumption on $\xi$, we obtain $Y_{\tau} = \xi_{\tau} \lor Y_{\tau^+} \leq \xi_{\tau} \lor Y_{\tau^+} = Y_{\tau^+}$ a.s. Thanks to Proposition 2.3, $Y_{\tau^+} \leq Y_{\tau}$ a.s. This ends the proof. □

Next we introduce the notion of strong supermartingale which extend the classical supermartingales to those connected to the optional $\sigma-$field.

Definition 2.5 An optional process $(Y_t)_{t \in [0,T]}$ such that

- $Y_{\tau}$ is integrable for all $\tau \in T_0$.
- for arbitrary stopping times $\tau \geq \sigma$

\[ Y_{\sigma} \geq E[Y_{\tau} \mid F_{\sigma}] \text{ a.e} \]

is called a strong supermartingale.

Remark 2.5 Every optional strong supermartingale is indistinguishable from a ladlag process, see [5].

3 Monotone approximation of the barrier

Definition 3.1 Let $\xi$ be an optional process. Let

\[ \mathcal{L} = \{X : X \text{ is a cadlag optional process, } X \geq \xi\}, \quad \mathcal{L}_- = \{X_+ : X \in \mathcal{L}\}, \]

and

\[ \bar{\xi} = \text{ess inf } \mathcal{L}, \quad \hat{\xi} = \text{ess inf } \mathcal{L}_-. \]

We call $\bar{\xi}$ upper cadlag envelope of $\xi$, $\hat{\xi}$ left upper cadlag envelope of $\xi$.

Let us provide some properties of the process $\bar{\xi}$.
Lemma 3.1 Let \( \xi \) be an optional process. Then, there exists a non increasing sequence \((X^n)_{n \in \mathbb{N}}\) of cadlag processes in \( \mathcal{L} \) such that
\[
\underline{\xi} = \lim_{n \to \infty} X^n.
\] (3.1)

Proof. The infima of a finite number of processes in \( \mathcal{L} \) belongs to \( \mathcal{L} \). Thus by a result of Neveu [18], \( \text{ess inf } \mathcal{L} \) can be described as the infimum of a sequence of processes in \( \mathcal{L} \). This concludes the proof. \( \square \)

Remark 3.1 It is clear that if the optional process \( \xi \) is RCLL then, its upper cadlag envelope is more closely related to \( \xi \). We give a generalization of this result to the case of right upper semicontinuous process in the following lemma:

Lemma 3.2 Let \( \xi \) be an optional process which is right upper semicontinuous. Then
\[
\overline{\xi}_t = \xi_t, \quad t \in [0, T],
\]
i.e. \( \overline{\xi} \) is a version of \( \xi \). Moreover, the sequence \((X^n)_{n \in \mathbb{N}}\) may be chosen identical to the sequence \((\xi^n)_{n \in \mathbb{N}}\) resulting from Theorem 21 of Dellacherie, Lenglart [6]. Finally, we have
\[
\hat{\xi} = \lim_{n \to \infty} \xi^n.
\]

Proof:
Since for any \( n \in \mathbb{N} \), \( \xi^n \) is RCLL and optional, we have for \( t \in [0, T] \):
\[
\overline{\xi}_t \leq \lim_{n \to \infty} \xi^n_t = \xi_t \leq \underline{\xi}_t.
\]
Thus the equation on \( \hat{\xi} \) follows from the definition of \( \hat{\xi} \). \( \square \)

Remark 3.2 If \( \xi \) is an optional process which is right upper semicontinuous in expectation of class \((D)\), then it is right upper semicontinuous.

Lemma 3.3 Let \( \xi \) be an optional process which is right upper semicontinuous in expectation. Then
\[
\overline{\xi}_t = \xi_t, \quad t \in [0, T],
\]
In particular, we get the following lemma:

Lemma 3.4 Let \( S \) be an optional strong supermartingale of class \((D)\). Then
\[
\overline{S}_t = S_t, \quad t \in [0, T],
\]
Proof. Since \( S \) is an optional strong supermartingale, the application \( \tau \to E(S_{\tau}) \) is non increasing. Thus, \( S \) is clearly right upper semicontinuous in expectation. The result follows from an application of Lemma 3.3. \( \square \)
3.1 Optional RBSDEs from RCLL RBSDEs

We now show how the solution of the RBSDE when optional barrier is approximated by RCLL barriers monotonically from above in $S^2$, can alternatively be constructed along a sequence of RBSDE with RCLL barriers. Let $g$ be a Lipschitz driver.

Assume that $\xi$ is right upper semicontinuous. Through the proof of Proposition 21 in Dellechier-Lenglart [6], there exits a sequence $(\xi^n)_{n \in \mathbb{N}} \in \mathcal{L}$ such that $E[\sup_{t \in [0,T]} (\xi^n)^2] < \infty$ and
\[
\xi^n \downarrow \text{essinf} \mathcal{L} = \xi. \tag{3.2}
\]

Moreover, we assume that $\|\xi^n - \xi\|_{S^2} \to 0$ as $n \to \infty$. By Hamadène and Ouknine [15], there exists $(Y^n, Z^n, l^n, A^n)$ the solution of the following RCLL reflected BSDE:

\[
\begin{cases}
(i) \quad (Y^n, Z^n, l^n, A^n) \in \mathbb{D} \times \mathbb{H}^2 \times \mathbb{H}^2_\mu \times S^{p_2}_\mu, \\
(ii) \quad Y_t^n = \xi^n_T + \int_t^T g(s, Y_s^n, Z_s^n, l_s^n) ds - \int_t^T Z_s^n dW_s - \int_t^T \int_E l_s^n(e) \tilde{N}(ds, de) + A^n_T - A^n_t, \text{ for all } t \in [0, T], \text{ a.s.} \\
(iii) \quad Y^n_t \geq \xi^n_t, \text{ for all } t \in [0, T] \text{ a.s.,} \\
(iv) \quad A^n \text{ is cadlag predictable, increasing with } A^n_0 = 0, E(A^n_T) < \infty, \text{ and satisfies} \\
\int_0^T (Y^n_t - \xi^n_t^-) dA^n_t = 0.
\end{cases}
\]

Note that $dA^n_t$ is the (random) measure on the Borel sets of $[0, T]$ associated with the increasing cadlag process $A^n$. We also remark that the Skorokhod condition in (iv) can be translated into the more detailed condition:
\[
\int_0^T (Y^n_t - \xi^n_t^-) dA^{n,c}_t = 0, \quad \int_0^T (Y^n_t - \xi^n_t^-) dA^{n,d}_t = 0. \tag{3.3}
\]

Here $A^{n,c}$ denotes the continuous part of $A^n$, $A^{n,d}$ its discontinuous part.

**Theorem 3.1** Assume that $\xi$ is in $S^2$. Let a sequence of decreasing RCLL processes $(\xi^n)_{n \in \mathbb{N}}$ be given which satisfies (ii). Let $g$ be a Lipschitz driver. Then, for each $n \in \mathbb{N}$, there exists a quadruple $(Y^n, Z^n, l^n, A^n) \in \mathbb{D} \times \mathbb{H}^2 \times \mathbb{H}^2_\mu \times S^{p_2}_\mu$ of processes which solves the RBSDE (ii), (iii), (iv). Moreover, we have for any $n \geq 0$ and for any $t \in [0, T]$:
\[
Y^n_t \geq Y^{n+1}_t. \tag{3.4}
\]

Proof. This is shown in Hamadène, Wang [14].

Let us now investigate the convergence of the first component of the solution quadruple of Theorem 3.1. First of all, for $n \in \mathbb{N}, t \in [0, T]$, we have
\[
\xi_t \leq Y^n_t \leq Y^1_t.
\]

Hence, by square integrability of $Y^1$ and the fact that $\xi \in S^2$, and by dominated convergence, the sequence $(Y^n)_{n \in \mathbb{N}}$ converges in $L^2(\Omega \times [0, T], \mathcal{F}_T \otimes \mathcal{B}([0, T]), P \otimes dt)$ to a process $Y \in S^2$.

The proof of our main result is based on the following key theorem:

**Theorem 3.2** Suppose that $g$ does not depend on $y, z, l$ that is $g(\omega, t, y, z, l) = g(\omega, t)$, where $g$ is a progressive process with $E(\int_0^T g(t)^2 dt) < +\infty$. Let $\xi$ be an optional process which is right
upper semicontinuous, and let \((ξ^n)_{n ∈ ℕ}\) be given according to (3.2). Let \(Y = \lim_{n → ∞} Y^n\) in \(L^2(Ω × [0, T], \mathcal{F}_T ⊗ \mathcal{B}([0, T]), P ⊗ dt)\), then, for each \(S ∈ \mathcal{T}_0\)

\[
Y_S = \operatorname{ess sup}_{τ ∈ \mathcal{T}_S} E \left[ ξ_τ + \int_0^τ g(u)du \mid \mathcal{F}_S \right].
\]

(3.5)

And the following properties hold:

(i) We have \(Y ≡ ξ ∨ Y^+\).

(ii) We have \(Y_{τ−} = \lim_{u ↑ τ} ξ_u ∨ Y_τ\), for all \(τ ∈ \mathcal{T}_0^n\).

Moreover, the convergence of the sequence \((Y^n)_{n ∈ ℕ}\) to \(Y\) holds in \(\mathcal{S}^2\).

Proof. First, let us show the equality (3.5). We get from Hamadène and Ouknine [15], that for \(n ≥ 0\) and \(S ∈ \mathcal{T}_0^n\):

\[
Y^n_S = \operatorname{ess sup}_{τ ∈ \mathcal{T}_S} E \left[ ξ^n_τ + \int_0^τ g(u)du \mid \mathcal{F}_S \right].
\]

Let \(σ ∈ \mathcal{T}_0\). Let us denote \(\bar{Y}(σ)\) the random variable defined by:

\[
\bar{Y}(σ) := \operatorname{ess sup}_{τ ∈ \mathcal{T}_σ} E \left[ ξ_τ + \int_0^σ g(u)du \mid \mathcal{F}_σ \right].
\]

Therefore,

\[
\bar{Y}(σ) + \int_0^σ g(u)du = \operatorname{ess sup}_{τ ∈ \mathcal{T}_σ} E \left[ ξ_τ + \int_0^τ g(u)du \mid \mathcal{F}_σ \right].
\]

Since the process \((ξ + \int_0^· g(u)du)\) is of class (D), the family \((\bar{Y}(σ), σ ∈ \mathcal{T}_0)\) can be aggregated by a process we denote also \(\bar{Y}\) (cf., [6, Theorem 15]). Therefore

\[
|\bar{Y}_τ − Y^n_σ| ≤ \operatorname{ess sup}_{τ ∈ \mathcal{T}_σ} E \left[ |ξ^n_τ − ξ_τ| \mid \mathcal{F}_σ \right] \quad (3.6)
\]

\[
≤ E \left[ \operatorname{ess sup}_{τ ∈ \mathcal{T}_σ} |ξ^n_τ − ξ_τ| \mid \mathcal{F}_σ \right].
\]

First, let us denote

\[
U^n_σ = E \left[ \operatorname{ess sup}_{τ ∈ \mathcal{T}_σ} |ξ^n_τ − ξ_τ| \mid \mathcal{F}_σ \right].
\]

Note that the process \((U^n_σ)_{σ ∈ [0, T]}\) is right continuous. This together with the definition of the essential supremum give

\[
\operatorname{ess sup}_{σ ∈ \mathcal{T}_0} |U^n_σ|^2 = \sup_{t ∈ [0, T]} |U^n_t|^2 \quad \text{a.s.}
\]

By using (3.6), we obtain

\[
\operatorname{ess sup}_{σ ∈ \mathcal{T}_0} |\bar{Y}_σ − Y^n_σ|^2 ≤ \sup_{t ∈ [0, T]} |U^n_t|^2 \quad \text{a.s.}
\]

We apply Doob’s inequality, to get

\[
\|\bar{Y} − Y^n\|^2_{\mathcal{S}^2} ≤ E \left[ \operatorname{ess sup}_{τ ∈ \mathcal{T}_0} |ξ^n_τ − ξ_τ|^2 \right].
\]
The sequence \((\xi^n)_{n \in \mathbb{N}}\) converges to \(\xi\) in \(S^2\) by hypothesis. Therefore, \(\|Y^n - Y\|_{S^2} \to 0\) as \(n \to \infty\).

Let us recall that the sequence \((Y^n)_{n \in \mathbb{N}}\) converges in \(L^2(\Omega \times [0, T], \mathcal{F}_T \otimes \mathcal{B}([0, T]), P \otimes dt)\) to a process \(Y \in S^2\). Thus, we get

\[
Y(\sigma) = Y_\sigma = Y_\tau \quad \text{a.s. for all } \sigma \in \mathcal{T}_0.
\]

Which establishes that the process \((Y_t + \int_0^t g(u)du)_{t \in [0, T]}\) is indistinguishable from the Snell envelope of the process \((\xi_t + \int_0^t g(u)du)_{t \in [0, T]}\). Assertions (i) and (ii) follow from classical results (cf., for instance [7, Proposition 2.32]).

\[\square\]

**Theorem 3.3** Let \(\xi\) be a right upper semicontinuous barrier which can be approximated by RCLL barriers monotonically from above in \(S^2\). Suppose that \(g\) does not depend on \(y, z, l\) that is \(g(\omega, t, y, z, l) = g(\omega, t)\), where \(g\) is a progressive process with \(E(\int_0^T g(t)^2)dt < +\infty\). The reflected BSDE with one reflecting barrier associated with \((g, \xi)\) has a unique solution \((Y, Z, l, A)\). Where \(Y\) is given according to Theorem 4.2. Moreover, the first component can be characterized as follows:

\[
Y_S = \text{ess sup}_{\tau \in T_S} E \left[ \xi_\tau + \int_0^\tau g(u)du \mid \mathcal{F}_S \right], \quad \text{for all } S \in \mathcal{T}_0, \tag{3.7}
\]

and the following properties hold:

(i) We have \(Y \equiv \xi \vee Y^+\).

(ii) We have \(Y_{\tau^+} = \limsup_{u \uparrow \tau} \xi_u \vee Y_{\tau} \quad \text{a.s., for all } \tau \in \mathcal{T}_0^p.\)

The proof of Theorem 3.3 relies on the Theorem 3.2 and the following lemma:

**Lemma 3.5** (i) The process \((Y_t)_{t \in [0, T]}\) is in \(S^2\) and admits the following optional Mertens decomposition:

\[
Y_\tau = Y_0 - \int_0^\tau g(s, Y_s, Z_s)ds + \int_0^T Z_s dW_s + \int_0^T I_s(e)\tilde{N}(ds, de) - A_\tau, \quad \text{for all } \tau \in \mathcal{T}_0, \tag{3.8}
\]

where \(A\) is a nondecreasing optional process such that \(A_0 = 0\) and \(E(A^2_T) < \infty\).

(ii) \(\int_0^T (Y_{\tau^+} - \limsup_{u \uparrow \tau} \xi_u) dA_s = \sum_{s < T} (Y_s - \xi_s) A_s^+ = 0 \quad \text{a.s.}\)

Proof. Let us prove the first assertion. For each \(S \in \mathcal{T}_0\), we define the random variable \(U(S)\) by

\[
U(S) := Y_S + \int_0^S g(u)du = \text{ess sup}_{\tau \in T_S} E \left[ \xi_\tau + \int_0^\tau g(u)du \mid \mathcal{F}_S \right].
\]

By [17], the process \((Y_t + \int_0^t g(u)du)_{t \in [0, T]}\) is the Snell’s envelope associated to \((\xi_t + \int_0^t g(u)du)\). By this and by using Mertens’ decomposition, we get the equation (3.8).

Now, let us show the assertion \((ii)\).

First let us note that:

\[
\int_0^T (Y_{\tau^+} - \limsup_{u \uparrow \tau} \xi_u) dA_s = 0,
\]
can be written as the following:

\[ \int_0^T (Y_s - \limsup_{u \uparrow s} \xi_u) dA_s^c = 0 \text{ a.s.} \]
\[ \sum_{s \leq T} (Y_s - \limsup_{u \uparrow s} \xi_u) \Delta A_s = 0 \text{ a.s.} \]

The proof of the first inequality is based on the same arguments used in [11].

The second equality is a consequence of (ii) in Theorem 3.2 and Remark 2.4. The following equality

\[ \sum_{s<T} (Y_s - \xi_s) \Delta A_s = 0 \text{ a.s.} \]

follows from (i) in Theorem 3.3 and Remark 2.3.

\[ \square \]

4 Existence and uniqueness in the case of a general driver

Let \( Y^n \) be the first component of the solution of the RBSDE with the RCLL barrier \( \xi^n \) and the driver \( g \). In [22], the authors proved that the value function of the optimal stopping problem coincides with \( Y^n \). Roughly speaking, for each stopping time \( S \in \mathcal{T}_0 \),

\[ Y^n_S = \text{ess sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^g_{S,\tau}(\xi^n_\tau). \]

In this part, we will use this characterisation, to construct the solution of the RBSDE when the optional barrier \( \xi \) can be approximated by RCLL barriers \( \xi^n \) monotonically from above in \( \mathcal{S}_2 \).

But, first let us revisit some properties of the family of random variables \((\mathcal{R}^g[\xi](S), S \in \mathcal{T}_0)\) defined by:

\[ \mathcal{R}^g[\xi](S) := \text{ess sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^g_{S,\tau}(\xi_\tau), \quad S \in \mathcal{T}_0. \quad (4.1) \]

Let us recall the following definition:

**Definition 4.1** We say that a family \( \phi = (\phi(\theta), \theta \in \mathcal{T}_0) \) is admissible if it satisfies the following conditions

1. for all \( \theta \in \mathcal{T}_0 \), \( \phi(\theta) \) is a \( \mathcal{F}_\theta \)-measurable random variable,
2. for all \( \theta, \theta' \in \mathcal{T}_0 \), \( \phi(\theta) = \phi(\theta') \) a.s. on \( \{\theta = \theta'\} \).

**Definition 4.2** An admissible square-integrable family \( U := (U(\theta), \theta \in \mathcal{T}_0) \) is said to be a strong \( \mathcal{E}^g \)-supermartingale family (resp. a strong \( \mathcal{E}^g \)-martingale family), if for any \( \theta, \theta' \in \mathcal{T}_0 \) such that \( \theta' \geq \theta \) a.s.,

\[ \mathcal{E}^g_{\theta,\theta'}(U(\theta')) \leq U(\theta) \quad \text{a.s.} \quad (\text{resp.} \quad \mathcal{E}^g_{\theta,\theta'}(U(\theta')) = U(\theta)). \quad (4.2) \]

The following proposition plays a crucial role to derive some properties of the family \((\mathcal{R}^g[\xi](S), S \in \mathcal{T}_0)\).
Proposition 4.1 Let $S \leq \theta \in T_0$, and let $\alpha$ be a non negative bounded $\mathcal{F}_\theta$-measurable random variable. Then,

$$E_{S,\theta}^{\alpha g}(\alpha R^g[\xi](\theta)) = \text{ess sup}_{\theta \leq \tau \in T_0} E_{S,\theta}^{\alpha g}(\alpha \xi_\tau) \text{ a.s.} \quad (4.3)$$

Proof. Let $\tau \in T_0$. By using the continuity property of $g$-conditional expectations, the fact that $\alpha$ is $\mathcal{F}_\theta$-measurable, $E_{\theta,\tau}^g(\xi_\tau) \leq R^g[\xi](\theta)$ a.s. and the monotonicity property of $g$-conditional expectations, we obtain:

$$E_{S,\theta}^{\alpha g}(\alpha \xi_\tau) = E_{S,\theta}^{\alpha g}(\{ \alpha \xi_\tau \}) = E_{S,\theta}^{\alpha g}(\{ \alpha E_{\theta,\tau}^g(\xi_\tau) \}) \leq E_{S,\theta}^{\alpha g}(\{ \alpha R^g[\xi](\theta) \})$$

By taking the essential supremum over $\tau \in T_0$, the inequality:

$$\text{ess sup}_{\theta \leq \tau \in T_0} E_{S,\theta}^{\alpha g}(\alpha \xi_\tau) \leq E_{S,\theta}^{\alpha g}(\alpha R^g[\xi](\theta)) \quad \text{a.s.}$$

holds. We need to show the reverse inequality. Following [10], there exists a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times in $T_0$ such that the sequence $(E_{\theta,\tau_n}^g(\xi_\tau_n))_{n \in \mathbb{N}}$ is non decreasing and:

$$R^g[\xi](\theta) = \lim_{n \to \infty} \uparrow E_{\theta,\tau_n}^g(\xi_\tau_n) \quad \text{a.s.}$$

By using the fact that $\alpha$ is $\mathcal{F}_\theta$-measurable and a standard property of conditional $g$-expectations, (cf., e.g., Proposition 2.2 in [21]), we obtain:

$$\alpha R^g[\xi](\theta) = \lim_{n \to \infty} \uparrow E_{\theta,\tau_n}^{\alpha g}(\alpha \xi_{\tau_n}) \quad \text{a.s.}$$

Therefore, by applying the property of continuity of BSDEs with respect to terminal condition (cf., Proposition A.6 in [23]) combined with consistency property of $g$-conditional expectations, we get

$$E_{S,\theta}^{\alpha g}(\alpha R^g[\xi](\theta)) = \lim_{n \to \infty} \uparrow E_{S,\theta}^{\alpha g}(\alpha \xi_{\tau_n}) = \lim_{n \to \infty} \uparrow E_{S,\theta}^{\alpha g}(\alpha \xi_{\tau_n}).$$

Hence,

$$E_{S,\theta}^{\alpha g}(\alpha R^g[\xi](\theta)) \leq \text{ess sup}_{\theta \leq \tau \in T_0} E_{S,\theta}^{\alpha g}(\alpha \xi_\tau).$$

Whence the desired result. \qed

Proposition 4.2 The value family $(R^g[\xi](S), S \in T_0)$ is characterized as the strong $E^g$- Snell envelope family associated with $\xi$, that is, the smallest $E^g$-supermartingale family which is greater (a.s.) than or equal to $\xi$.

Proof. Let $\theta \in T_0$. Applying the Proposition 4.1 with $\alpha = 1$, and using that $S \leq \theta$ a.s. we get:

$$E_{S,\theta}^g(R^g[\xi](\theta)) = \text{ess sup}_{\tau \in T_0} E_{S,\theta}^g(\xi_\tau) \leq \text{ess sup}_{S \leq \tau \in T_0} E_{S,\theta}^g(\xi_\tau) = R^g[\xi](S) \quad \text{a.s.}$$

It follows that $R^g[\xi]$ is an $E^g$-supermartingale family. To complete the proof, it remains to show the minimality property. Let $V'$ another $E^g$-supermartingale family, such that $V' \geq \xi$. The monotonicity property of $g$-conditional expectations allows us to write:

$$E_{S,\theta}^g(\xi_\theta) \leq E_{S,\theta}^g(V'(\theta)) \leq V'(S) \quad \text{a.s.},$$
where the last inequality is due to the $\mathcal{S}^g$-supermartingale property of $V'$. By taking the essential supremum over $\theta \in \mathcal{T}_S$, we deduce that

$$R^g(\xi)(S) = \text{ess sup}_{\theta \in \mathcal{T}_S} E^g_{\theta}(\xi_\theta) \leq V'(S) \quad \text{a.s.}$$

This concludes the proof. \qed

**Remark 4.1** If $\xi \leq \hat{\xi}$, then, $R^g(\xi)(S) \leq R^g(\hat{\xi})(S)$ a.s. for all $S \in \mathcal{T}_0$. First let us notice that through the definition of $g$-conditional expectations and comparison theorem for BSDEs (cf. for e.g. [9]), we get for all $\tau \in \mathcal{T}_0$

$$E^g_{\theta,\tau}(\xi_\tau) \leq E^g_{\theta,\tau}(\hat{\xi}_\tau) \quad \text{a.s.}$$

We conclude the inequality by taking the essential supremum over $\tau \in \mathcal{T}_S$.

The purpose of the following theorem, is to show that under suitable type of convergence of a sequence of reward processes $(\xi^n)_{n \in \mathbb{N}}$, the following convergence in terms of BSDEs can be proved, by using some a priori estimates of BSDEs.

**Theorem 4.1** Let $\xi$ be an optional process in $\mathcal{S}^2$. Let $(\xi^n)_{n \in \mathbb{N}}$ be a sequence of optional processes in $\mathcal{S}^2$ such that $\|\xi^n - \xi\|_{\mathcal{S}^2} \to 0$. Then, the sequence $(R^g[\xi^n])_{n \in \mathbb{N}}$ converges in $\mathcal{S}^4$ to $R^g[\xi]$.

**Proof.** We have:

$$\text{ess sup}_{\theta \in \mathcal{T}_0} |R^g[\xi^n](\theta) - R^g[\xi](\theta)|^4 = \text{ess sup}_{\theta \in \mathcal{T}_0} \text{ess sup}_{\tau \in \mathcal{T}_0} |E^g_{\theta,\tau}(\xi^n_\tau) - E^g_{\theta,\tau}(\xi_\tau)|^4 \quad (4.4)$$

On the other hand, we have by a priori estimates of BSDEs (cf., Proposition A.6 in [23]), for each $\tau \in \mathcal{T}_0$

$$|E^g_{\theta,\tau}(\xi^n_\tau) - E^g_{\theta,\tau}(\xi_\tau)|^4 \leq c(E[\|\xi^n_\tau - \xi_\tau\|^2|\mathcal{F}_\theta])^2.$$ 

Here $c$ is a constant which can changes from line to line. Thus,

$$\text{ess sup}_{\theta \in \mathcal{T}_0} \text{ess sup}_{\tau \in \mathcal{T}_0} |E^g_{\theta,\tau}(\xi^n_\tau) - E^g_{\theta,\tau}(\xi_\tau)|^4 \leq \text{ess sup}_{\theta \in \mathcal{T}_0} \text{ess sup}_{\tau \in \mathcal{T}_0} (E[\|\xi^n_\tau - \xi_\tau\|^2|\mathcal{F}_\theta])^2 \leq \text{ess sup}_{\theta \in \mathcal{T}_0} |U^n_\theta|^2,$$

where $U^n$ is given by $U^n_t = E[\text{ess sup}_{\tau \in \mathcal{T}_0} \|\xi^n_\tau - \xi_\tau\|^2|\mathcal{F}_t]$. The process $(U^n_t)_{t \in [0,T]}$ is right continuous. Thus

$$\text{ess sup}_{\theta \in \mathcal{T}_0} |U^n_\theta| = \sup_{t \in [0,T]} |U^n_t|.$$ 

By using this and Doob’s martingale inequality in $L^2$, we obtain:

$$E \left( \text{ess sup}_{\theta \in \mathcal{T}_0} \text{ess sup}_{\tau \in \mathcal{T}_0} |E^g_{\theta,\tau}(\xi^n_\tau) - E^g_{\theta,\tau}(\xi_\tau)|^4 \right) \leq cE(\text{ess sup}_{\tau \in \mathcal{T}_0} |\xi^n_\tau - \xi_\tau|^2). \quad (4.5)$$

By combining the inequalities (4.4) and (4.5) with $\|\xi^n - \xi\|_{\mathcal{S}^2} \to 0$, we derive the desired convergence result. \qed
Theorem 4.2 Let \( g \) be a Lipschitz driver. Let \( \xi \) be an optional process which in \( S^2 \) which is right upper semicontinuous, and let \( (\xi^n)_{n \in \mathbb{N}} \) be given according to (3.2). Let \( Y = \lim_{n \to \infty} Y^n \) in \( L^2(\Omega \times [0, T], \mathcal{F}_T \otimes \mathcal{B}([0, T]), P \otimes dt) \), then, for each \( S \in \mathcal{T}_0 \)

\[
Y_S = \text{ess sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^{g}_{S, \tau}(\xi_{\tau}).
\]

Proof. Let \( (\xi^n)_{n \in \mathbb{N}} \) be given according to 3.2. From a result of [23], for each stopping time \( S \), we have

\[
Y^n_S = \mathcal{R}^g[\xi^n](S).
\]

By letting \( n \) tend to \( \infty \), and using that the sequence \( (Y^n)_{n \in \mathbb{N}} \) converges in \( L^2(\Omega \times [0, T], \mathcal{F}_T \otimes \mathcal{B}([0, T]), P \otimes dt) \) a process \( Y \) together with Theorem 4.1, we obtain that:

\[
Y_S = \text{ess sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^{g}_{S, \tau}(\xi_{\tau}) = \mathcal{R}^g[\xi](S)
\]

\[\Box\]

Theorem 4.3 Let \( g \) be a Lipschitz driver. Let \( \xi \) an optional right upper semicontinuous which can be approximated by RCLL barriers \( \xi^n \) monotonically from above in \( S^2 \). The reflected BSDE with one reflecting barrier associated with \( (g, \xi) \) has a unique solution \( (Y, Z, l, A) \). Where \( Y \) is given according to Theorem 4.2.

\[
Y_S = \text{ess sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^{g}_{S, \tau}(\xi_{\tau}) = \mathcal{R}^g[\xi](S)
\]

Proof. To prove that \( Y \) is the first component of the solution of RBSDE(\( \xi, g \)), we apply Theorem 4.2 combined with Theorem 10.1 in [11]. \[\Box\]

In what follows, the process which aggregates the family \( \{\mathcal{R}^g[\xi](S), S \in \mathcal{T}_0\} \) will also be denoted by \( \mathcal{R}^g[\xi] \).

Proposition 4.3 Let \( X \) be an optional process in \( S^2 \) such that \( \mathcal{R}^g[\xi] + X \) is a strong \( \mathcal{E}^g \)-supermartingale. Let \( \bar{\xi} := \mathcal{R}^g[\xi + X] - X \), then \( \mathcal{R}^g[\bar{\xi}] = \mathcal{R}^g[\xi] \) a.s.

Proof. \( \mathcal{R}^g[\xi + X] \) is the \( \mathcal{E}^g \)-Snell enveloppe of \( \xi + X \). Thus, it is clear that

\[
\xi \leq \mathcal{R}^g[\xi + X] - X \quad \text{a.s.}
\]

By using Remark 4.1, we obtain

\[
\mathcal{R}^g[\xi] \leq \mathcal{R}^g[\mathcal{R}^g[\xi + X] - X] \quad \text{a.s.}
\]

Which yields the first inequality. Now, let us prove the second inequality. We have clearly \( \xi + X \leq \mathcal{R}^g[\xi] + X \) a.s. But \( \mathcal{R}^g[\xi] + X \) is an \( \mathcal{E}^g \)-supermartingale by assumption and \( \mathcal{R}^g[\xi + X] \) is the smallest \( \mathcal{E}^g \)-supermartingale which is greater than or equal to \( \xi + X \). Hence,

\[
\mathcal{R}^g[\xi + X] \leq \mathcal{R}^g[\xi] + X \quad \text{a.s.}
\]

It follows that

\[
\mathcal{R}^g[\xi + X] - X \leq \mathcal{R}^g[\xi] \quad \text{a.s.}
\]

Which yields the desired result:

\[
\mathcal{R}^g(\mathcal{R}^g[\xi + X] - X) \leq \mathcal{R}^g[\xi] \quad \text{a.s.}
\]

\[\Box\]
Remark 4.2 By the property of the Snell envelope, \( \xi \leq \tilde{\xi} \). Moreover, if the process \( X \) is continuous, then, \( \Delta \xi \leq 0 \). This is due to Theorem 3.3 and Remark 2.2.

For optional processes \( Y, Z, l, A \), we set
\[
g_{Y,Z,l}(t) = g(t, Y_t, Z_t, l_t) ; \forall t \in [0, T].
\]

**Proposition 4.4** Let \( X \) and \( \xi \) are as in Proposition 4.3. Suppose that \( X \) is continuous. If \( (Y, Z, l, A) \) is the solution of the reflected BSDE associated with \( (\xi, g) \). Then, \((Y, Z, l, A)\) is the solution of the reflected BSDE associated with \( (\xi, g) \).

**Proof.** Let \((\tilde{Y}, \tilde{Z}, \tilde{l}, \tilde{A})\) be the solution of the reflected BSDE associated with \( (\xi, g_{Y, Z, l}) \).

Let us prove that \((\tilde{Y}, \tilde{Z}, \tilde{l}, \tilde{A}) = (Y, Z, l, A)\). By Theorem 10.1 in [11], \( \tilde{Y} = R^{g_{Y, Z, l}}[\xi] \) and \( Y = R^{g_{Y, Z, l}}[\xi] \). By Proposition 4.3, \( Y = \tilde{Y} \) a.s. Moreover, we get by Remark (4.2)
\[
\int_0^T (\tilde{Y}_t - \limsup_{u \uparrow t} \xi_u) d\tilde{A}_t \leq \int_0^T (\tilde{Y}_t - \limsup_{u \uparrow t} \xi_u) d\tilde{A}_t + \sum_{0 < t \leq T} (\tilde{Y}_t - \limsup_{u \uparrow t} \xi_u) \Delta \tilde{A}_t = 0 \text{ a.s.}
\]

We have also that \( \sum_{s < T} (\tilde{Y}_s - \tilde{\xi}_s) \Delta^+ \tilde{A}_s \leq \sum_{s < T} (\tilde{Y}_s - \xi_s) \Delta^+ \tilde{A}_s \). Therefore, by uniqueness of the solution \((\tilde{Y}, \tilde{Z}, \tilde{l}, \tilde{A}) = (Y, Z, l, A)\). This means that \((Y, Z, l, A)\) is the solution of the reflected BSDE associated with \((\xi, g_{Y, Z, l})\). \( \square \)

In the following theorem, we give the analogous of a result of [16], in the case when the obstacle process \( \xi \) is not necessarily left limited, in the setting where the noise is given by a Brownian motion and an independent Poisson measure.

**Theorem 4.4** Let \( \xi \) be a right upper semicontinuous process in \( S^2 \), such that \( \xi_t < \limsup_{u \uparrow t} \xi_u \), for all \( t \in (0, T] \), and let \((Y, Z, l, A)\) be the solution of the RBSDE(\(\xi, g\)) from Definition 2.3, then, \((Y^+, Z, l, A^+)\) is the solution of the reflected BSDE with parameters \((\xi^+, g)\). Moreover, for each \( S \in \mathcal{T}_0 \)
\[
Y_{S^+} = \text{ess sup}_{t \geq S} E \left( \xi^+_t + \int_S^T g(s, Y^+_s, Z_s) ds | \mathcal{F}_S \right).
\]

**Proof.** Since \( Y \geq \xi \) up to an evanescent set, then, of course \( Y_+ \geq \xi_+ \) up to an evanescent set. Therefore it is sufficient to show that
\[
SK := \int_0^T ((Y^+)_t - \limsup_{u \uparrow t} \xi_u) dA_{t^+} = 0.
\]

First, let us remark that under the hypothesis \( \xi_t < \limsup_{u \uparrow t} \xi_u \), we have \( \limsup_{u \uparrow t} \xi_u \leq \limsup_{u \uparrow t} \xi_u^+ \). Thus,
\[
SK \leq \int_0^T (Y_t - \limsup_{u \uparrow t} \xi_u) dA_{t^+} = \int_0^T (Y_t - \limsup_{u \uparrow t} \xi_u) dA_{t^+} + \sum_{0 < t \leq T} (Y_t - \limsup_{u \uparrow t} \xi_u) \Delta A_{t^+}
\]

The first term on the right-hand side is equal to zero since \((Y, Z, l, A)\) is the solution of RBSDE associated with \((\xi, g)\). Now, let us prove that the second term is null. We have
\[
\sum_{0 < t \leq T} (Y_t - \limsup_{u \uparrow t} \xi_u) \Delta A_{t^+} = \sum_{0 < t < T} (Y_t - \limsup_{u \uparrow t} \xi_u) 1_{\{Y_t = Y_{t^-}\}} \Delta^+ A_t.
\]
Suppose that \( \Delta^+ A_t > 0 \). Then, \( Y_t = \xi_t \) by the Skorokhod condition (iv). This together with assumption, \( \limsup_{u \uparrow t} \xi_u > \xi_t \) yields that Which completes the proof.

\[ \text{Corollary 4.1} \]

Let \( Y \) be the first component of the solution of \( \text{RBSDE}(\xi, g) \) as in precedent theorem. Then, For each \( S \in \mathcal{T}_0 \)

\[
\text{ess sup}_{\tau \geq S} \mathbb{E}\left( \xi^+_{\tau} + \int_S^\tau g(s, Y^+_{s}, Z_s, l_s) \, ds \big| \mathcal{F}_S \right) = \text{ess sup}_{\tau > S} \mathbb{E}\left( \xi^+_{\tau} + \int_S^\tau g(s, Y_s, Z_s, l_s) \, ds \big| \mathcal{F}_S \right).
\]

Proof. Let \( S \in \mathcal{T}_0 \), let us denote \( Y(S) \) the random variable defined by:

\[
Y(S) = \text{ess sup}_{\tau \geq S} \mathbb{E}\left[ \xi^+_{\tau} + \int_S^\tau g(u, Y_u, Z_u, l_u) \, du \big| \mathcal{F}_S \right].
\]

By the same arguments of the proof of Theorem 3.3, the solution \((Y_t)_{t \in [0, T]}\) of \( \text{RBSDE}(\xi, g) \) aggregates the family \((Y(S), S \in \mathcal{T}_0)\). Thus, \((Y_{t^+})_{t \in [0, T]}\) aggregates the family \((Y(S^+), S \in \mathcal{T}_0)\). Now, let

\[
Y^+(S) = \text{ess sup}_{\tau > S} \mathbb{E}\left[ \xi^+_{\tau} + \int_S^\tau g(u, Y_u, Z_u, l_u) \, du \big| \mathcal{F}_S \right].
\]

Moreover, thanks to a result from optimal stopping theory (cf. [17, Proposition 4.14]), \( Y^+(S) = Y(S^+) \) a.s. Thus, the process \((Y_{t^+})_{t \in [0, T]}\) aggregates the family \((Y(S^+), S \in \mathcal{T}_0)\). Hence,

\[
Y_{S^+} = \text{ess sup}_{\tau > S} \mathbb{E}\left( \xi^+_{\tau} + \int_S^\tau g(s, Y^+_s, Z^+_s) \, ds \big| \mathcal{F}_S \right).\]

This yields the desired result.

\[ \text{Remark 4.3} \]

In particular, if \( \xi \) is a right upper semicontinuous optional process satisfying \( \xi_t \leq \limsup_{u \uparrow t} \xi_t \), then

\[
\text{ess sup}_{\tau \geq S} \mathbb{E}\left( \xi^+_{\tau} \big| \mathcal{F}_S \right) = \text{ess sup}_{\tau > S} \mathbb{E}\left( \xi_{\tau} \big| \mathcal{F}_S \right).
\]

\[ \text{Lemma 4.1} \]

Let \( \bar{\xi} \) is given as in Proposition 4.4. Let \((Y^n, Z^n, A^n)\) be the solution of the following BSDE:

\[
Y^n_t = \bar{\xi}_T + \int_t^T g(u, Y^n_u, Z^n_u) \, du + \int_t^T n(Y^n_u - \bar{\xi}^+_u) \, ds - \int_t^T Z^n_u \, dW_u - \int_t^T \int_E l^n_u(e) \tilde{N}(du, de).
\]

Let

\[
\tilde{Y}^n := \xi \vee Y^n_t.
\]

Then,

\[
\tilde{Y}^n_t \uparrow Y_t, \quad t \in [0, T],
\]

where \( Y \) is the first component of the solution \((Y, Z, l, A)\) of \( \text{RBSDE}(\xi, g) \).

Proof. Let \((\tilde{Y}, \tilde{Z}, \tilde{l}, \tilde{A})\) be the solution of \( \text{RBSDE}(\bar{\xi}, g) \). By Proposition 4.4 \((\tilde{Y}, \tilde{Z}, \tilde{l}, \tilde{A}) = (Y, Z, l, A)\). We have by Remark 4.2 that \( \Delta \tilde{\xi} \leq 0 \). Then, by Theorem 4.4, \((Y^+, Z, l, A^+)\) is the solution of \( \text{RBSDE}(\bar{\xi}^+, g) \). On the other hand, by Hamadène and Ouknine [13], \( Y^n \uparrow Y^+ \). Hence, \( \tilde{Y}^n_t \uparrow \xi_t \vee Y^+_t, \quad t \in [0, T] \). The result follows from Proposition 2.1.
5 Generalized Skorohod condition in the sens of Peng-Xu

To show how a solution of BSDE can be reflected by a very irregular $L^2$ obstacle, Peng and Xu [19], found a new formulation of Skorokhod condition.

**Definition 5.1** (RBSDEs in the sens of Peng-Xu) Let $g$ be a driver, $\xi$ an obstacle. We say a triple of processes $(Y, Z, A)$ is a solution of the reflected BSDE with standard parameters $(g, \xi)$ if $(Y, Z, A) \in D^2 \times L^2 \times D^2$,

$$Y_\tau = \xi_T + \int_\tau^T g(\cdot, t, Y_t, Z_t) dt - \int_\tau^T Z_t dW_t + A_T - A_\tau \text{ a.s. for all } \tau \in T.$$   

$Y \geq \xi \text{ } dt \otimes dP$, the following generalized Skorohod condition holds

$$\int_0^T (Y_t - \xi_t^*) dA_t = 0 \text{ a.s. for all } \xi^* \in D^2 \text{ such that } \xi_t \leq \xi_t^* \leq Y_t \text{ a.s., a.e.}$$

In [19], Peng and Xu shows the existence of a solution of a unique solution $(Y, Z, A)$ by penalization method. The aim of this part is to give a new approach which avoids the generalized Skorokhod condition involving $\xi^*$.

**Theorem 5.1** Let $\xi$ be an optional process in $S^2$ such that $\xi \leq \xi^+$ up to an evanescent set, and let $(Y, Z, A)$ be the solution of the reflected BSDE with parameters $(\xi, g)$ from Definition 2.3, then, $(Y, Z, A)$ is the solution of the reflected BSDE with parameters $(\xi, g)$ in the sens of Peng-Xu.

Proof. First, note that the process $Y$ is right continuous. This follows from the assumption $\xi \leq \xi^+$ up to an evanescent set and an application of Lemma 2.2. Let $\xi^*$ be a cadlag process, such that $\xi \leq \xi^* \leq Y \text{ } dt \otimes dP$. We need show that

$$SK := \int_0^T ((Y_{t+})_+ - \xi^*_t) dA_t = 0.$$  

Since $\xi^* \geq \xi \text{ } dt \otimes dP$, we get

$$SK = \int_0^T (Y_{t+} - \xi^*_t) dA_t \leq \int_0^T (Y_{t+} - \limsup_{u \uparrow t} \xi_u) dA_t. \tag{5.1}$$

The term on the right hand side is equal to zero since $(Y, Z, A)$ is the solution of $RBSDE(\xi, g)$. Hence the result. \qed

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