Eppstein’s bound on intersecting triangles revisited

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Abstract

Let \( S \) be a set of \( n \) points in the plane, and let \( T \) be a set of \( m \) triangles with vertices in \( S \). Then there exists a point in the plane contained in \( \Omega\left(\frac{m^3}{n^6 \log^2 n}\right) \) triangles of \( T \). Eppstein (1993) gave a proof of this claim, but there is a problem with his proof. Here we provide a correct proof by slightly modifying Eppstein’s argument.

Keywords: Triangle; Simplex; Selection Lemma; \( k \)-Set

1 Introduction

Let \( S \) be a set of \( n \) points in the plane in general position (no three points on a line), and let \( T \) be a set of \( m \leq \binom{n}{3} \) triangles with vertices in \( S \). Aronov et al. [2] showed that there always exists a point in the plane contained in the interior of

\[
\Omega\left(\frac{m^3}{n^6 \log^2 n}\right)
\]

triangles of \( T \). Eppstein [5] subsequently claimed to have improved this bound to

\[
\Omega\left(\frac{m^3}{n^6 \log^2 n}\right).
\]

There is a problem in Eppstein’s proof, however[1] In this note we provide a correct proof of [2], by slightly modifying Eppstein’s argument.

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[1] The very last sentence in the proof of Theorem 4 (Section 4) in [5] reads: “So \( \epsilon = 1/2^{i+1} \), and \( x = m \epsilon / y = O(m/8^{i}) \), from which it follows that \( x/\epsilon^3 = O(n^2) \).” This is patently false, since what actually follows is that \( x/\epsilon^3 = O(m) \), and the entire argument falls through.
1.1 The Second Selection Lemma and $k$-sets

The above result is the special case $d = 2$ of the following lemma (called the Second Selection Lemma in [6]), whose proof was put together by Bárány et al. [3], Alon et al. [1], and Živaljević and Vrecica [8]:

**Lemma 1.** If $S$ is an $n$-point set in $\mathbb{R}^d$ and $T$ is a family of $m \leq \binom{n}{d+1}$ $d$-simplices spanned by $S$, then there exists a point $p \in \mathbb{R}^d$ contained in at least

$$c_d \left( \frac{m}{n^{d+1}} \right)^{s_d} n^{d+1}$$

simplices of $T$, for some constants $c_d$ and $s_d$ that depend only on $d$.

(Note that $m/n^{d+1} = O(1)$, so the smaller the constant $s_d$, the stronger the bound.) Thus, for $d = 2$ the constant $s_2$ in (3) can be taken arbitrarily close to 3. The general proof of Lemma 1 gives very large bounds for $s_d$; roughly $s_d \approx (4d + 1)^{d+1}$.

The main motivation for the Second Selection Lemma is deriving upper bounds for the maximum number of $k$-sets of an $n$-point set in $\mathbb{R}^d$; see [6, ch. 11] for the definition and details.

2 The proof

We assume that $m = \Omega(n^2 \log^{2/3} n)$, since otherwise the bound (2) is trivial. The proof, like the proof of the previous bound (1), relies on the following two one-dimensional selection lemmas [2]:

**Lemma 2** (Unweighted Selection Lemma). Let $V$ be a set of $n$ points on the real line, and let $E$ be a set of $m$ distinct intervals with endpoints in $V$. Then there exists a point $x$ lying in the interior of $\Omega(m^2/n^2)$ intervals of $E$.

**Lemma 3** (Weighted Selection Lemma). Let $V$ be a set of $n$ points on the real line, and let $E$ be a multiset of $m$ intervals with endpoints in $V$. Then there exists a multiset $E' \subseteq E$ of $m'$ intervals, having as endpoints a subset $V' \subseteq V$ of $n'$ points, such that all the intervals of $E'$ contain a common point $x$ in their interior, and such that

$$\frac{m'}{n'} = \Omega \left( \frac{m}{n \log n} \right).$$

The proof of the desired bound (2) proceeds as follows:

Assume without loss of generality that no two points of $S$ have the same $x$-coordinate. For each triangle in $T$ define its *base* to be the edge with the longest $x$-projection. For each pair of points $a, b \in S$, let $T_{ab}$ be the set of triangles in $T$ that have $ab$ as base, and let $m_{ab} = |T_{ab}|$. (Thus, $\sum_{ab} m_{ab} = m$.)

Discard all sets $T_{ab}$ for which $m_{ab} < m/n^2$. We discarded at most $\left( \binom{n}{2} \right) m/n^2 < m/2$ triangles, so we are left with a subset $T'$ of at least $m/2$ triangles, such that either $m_{ab} = 0$ or $m_{ab} \geq m/n^2$ for each base $ab$.

This critical discarding step is missing in [5], and that is why the proof there does not work.
Partition the bases into a logarithmic number of subsets $E_1, E_2, \ldots, E_k$ for $k = \log_4(n^3/m)$, so that each $E_j$ contains all the bases $ab$ for which

$$\frac{4^j - 1}{n^2} m \leq m_{ab} < \frac{4^j m}{n^2}. \tag{4}$$

Let $T_j = \bigcup_{ab \in E_j} T_{ab}$ denote the set of triangles with bases in $E_j$, and $m_j = |T_j|$ denote their number. There must exist an index $j$ for which

$$m_j \geq 2^{-(j+1)} m,$$

since otherwise the total number of triangles in $T'$ would be less than $m/2$. From now on we fix this $j$, and work only with the bases in $E_j$ and the triangles in $T_j$.

For each pair of triangles $abc, abd$ having the same base $ab \in E_j$, project the segment $cd$ into the $x$-axis, obtaining segment $c'd'$. We thus obtain a multiset $M_0$ of horizontal segments, with

$$|M_0| \geq \frac{m_j}{2} \left( \frac{4^j - 1}{n^2} m - 1 \right) = \Omega \left( \frac{2^j m^2}{n^2} \right).$$

(Each of the $m_j$ triangles in $T_j$ is paired with all other triangles sharing the same base, and each such pair is counted twice.)

We now apply the Weighted Selection Lemma (Lemma 3) to $M_0$, obtaining a multiset $M_1$ of segments delimited by $n_1$ distinct endpoints, all segments containing some point $z_0$ in their interior, with

$$\frac{|M_1|}{n_1} = \Omega \left( \frac{|M_0|}{n \log n} \right) = \Omega \left( \frac{2^j m^2}{n^3 \log n} \right).$$

Let $\ell$ be the vertical line passing through $z_0$. For each horizontal segment $c'd' \in M_1$, each of its (possibly multiple) instances in $M_1$ originates from a pair of triangles $abc, abd$, where points $a$ and $c$ lie to the left of $\ell$, and points $b$ and $d$ lie to the right of $\ell$. Let $p$ be the intersection of $\ell$ with $ad$, and let $q$ be the intersection of $\ell$ with $bc$. Then, $pq$ is a vertical segment along $\ell$, contained in the union of the triangles $abc, abd$ (see Figure 1). Let $M_2$ be the set of all these segments $pq$ for all $c'd' \in M_1$.

Note that the vertical segments in $M_2$ are all distinct, since each such segment $pq$ uniquely determines the originating points $a, b, c, d$ (assuming $z_0$ was chosen in general position).
Let \( n_2 \) be the number of endpoints of the segments in \( M_2 \). We have \( n_2 \leq n_1 \), since each endpoint (such as \( p \)) is uniquely determined by one of \( n_1 \) “inner” vertices (such as \( d \)) and one of at most \( n \) “outer” vertices (such as \( a \)).

Next, apply the Unweighted Selection Lemma (Lemma 2) to \( M_2 \), obtaining a point \( x_0 \in \ell \) that is contained in

\[
\Omega\left( \frac{|M_2|^2}{n_2^2} \right) = \Omega\left( \frac{1}{n^2} \left( \frac{|M_1|}{n_1} \right)^2 \right) = \Omega\left( \frac{4^jm^4}{n^8 \log^2 n} \right)
\]

segments in \( M_2 \). Thus, \( x_0 \) is contained in at least these many unions of pairs of triangles of \( T_j \). But by (4), each triangle in \( T_j \) participates in at most \( 4^jm^2/n^2 \) pairs. Therefore, \( x_0 \) is contained in

\[
\Omega\left( \frac{m^3}{n^6 \log^2 n} \right)
\]

triangles of \( T_j \).

3 Discussion

Eppstein [5] also showed that there always exists a point in \( \mathbb{R}^2 \) contained in \( \Omega(m/n) \) triangles of \( T \). This latter bound is stronger than [2] for small \( m \), namely for \( m = O(n^{5/2} \log n) \).

On the other hand, as Eppstein also showed [5], for every \( n \)-point set \( S \) in general position and every \( m = \Omega(n^2) \), \( m \leq \binom{n}{3} \), there exists a set \( T \) of \( m \) triangles with vertices in \( S \), such that no point in the plane is contained in more than \( O(m^2/n^3) \) triangles of \( T \). Thus, with the current lack of any better lower bound, the bound [2] appears to be far from tight. Even achieving a lower bound of \( \Omega(m^3/n^6) \), without any logarithmic factors, is a major challenge still unresolved.

It is known, however, that if \( S \) is a set of \( n \) points in \( \mathbb{R}^3 \) in general position (no four points on a plane), and \( T \) is a set of \( m \) triangles spanned by \( S \), then there exists a line (in fact, a line spanned by two points of \( S \)) that intersects the interior of \( \Omega(m^3/n^6) \) triangles of \( T \); see [4] and [7] for two different proofs of this.

References

[1] N. Alon, I. Bárány, Z. Füredi, and D. Kleitman, Point selections and weak \( \epsilon \)-nets for convex hulls, *Combin., Probab. Comput.*, 1:189–200, 1992.

[2] B. Aronov, B. Chazelle, H. Edelsbrunner, L. J. Guibas, M. Sharir, and R. Wenger, Points and triangles in the plane and halving planes in space, *Discrete Comput. Geom.*, 6:435–442, 1991.

[3] I. Bárány, Z. Füredi, and L. Lovász, On the number of halving planes, *Combinatorica*, 10:175–183, 1990.

[4] T. K. Dey and H. Edelsbrunner, Counting triangle crossings and halving planes, *Discrete Comput. Geom.*, 12:281–289, 1994.
[5] D. Eppstein, Improved bounds for intersecting triangles and halving planes, *J. Combin. Theory Ser. A*, 62:176–182, 1993.

[6] J. Matoušek, *Lectures on Discrete Geometry*, Springer-Verlag, New York, 2002.

[7] S. Smorodinsky, *Combinatorial problems in computational geometry*, Ph.D. Thesis, Tel Aviv University, June 2003. [http://www.cs.bgu.ac.il/~shakhar/my_papers/phd.ps.gz](http://www.cs.bgu.ac.il/~shakhar/my_papers/phd.ps.gz)

[8] R. T. Živaljević and S. T. Vrećica, The colored Tverberg's problem and complexes of injective functions, *J. Combin. Theory Ser. A*, 61:309–318, 1992.