On Lyapunov Functions for Infinite Dimensional Volterra Quadratic Stochastic Operators

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Abstract. In the present paper, we study the existence of Lyapunov functions for Volterra quadratic stochastic operators (QSO) on infinite dimensional simplex. We provide a construction of Lyapunov functions for such kind of operators, which allows us to estimate their limiting points.

1. Introduction

An original work on quadratic stochastic operators (in short QSOs) was done by Bernstein [1] where such kind of operators appeared from the problems of population genetics (see also [2]). These operator appear to have tremendous applications especially in modelings in many different fields such as biology [3] (population and disease dynamics), physics [4, 5](non-equilibrium statistical mechanics), economics and mathematics [2, 5] (replicator dynamics and games). A quadratic stochastic operator is usually used to present the time evolution of species in biology, which arises as follows. Consider evolution of species in biology as given in the following situation. Let $I = \{1, 2, \ldots, n\}$ be the $n$ type of species (or traits) in a population and we denote $x^{(0)}(i) = (x_1^{(0)}, \ldots, x_n^{(0)})$ to be the probability distribution of the species in an early state of that population. By $P_{ij,k}$ we mean the probability of an individual in the $i$th species and $j$th species to cross-fertilize and produce an individual from $k$th species (trait). Given $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$, we can find the probability distribution of the first generation, $x^{(1)} = (x_1^{(1)}, \ldots, x_n^{(1)})$ by using a total probability, i.e.,

$$x_k^{(1)} = \sum_{i,j=1}^{n} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k \in \{1, \ldots, n\}. $$

This relation defines an operator which is denoted by $V$ and it is called quadratic stochastic operator (QSO). Each QSO can be reinterprete as evolutionary operator that describes the sequence of generations in terms of probabilistic distribution if the values of $P_{ij,k}$ and the distribution of the current generation are given.

Currently, there are only a small number of studies on dynamical phenomena on higher dimensional systems, even though they are very important. Note that, most of the studies in
2. Preliminaries

In this paper we are going to consider the set of absolutely summable sequences i.e.,

\[ \ell^1 = \left\{ x = \{x_k\} : \|x\|_1 = \sum_{k=1}^{\infty} |x_k| < \infty \right\}. \quad (2.1) \]

Denote

\[ S = \{x = \{x_k\} \in \ell^1 : x_k \geq 0 \text{ for all } k \in \mathbb{N}, \|x\|_1 = 1\}. \quad (2.2) \]

It is known that \( S = \text{convh}(\text{Extr}S) \), where \( \text{Extr}(S) \) is the extremal points of \( S \) and \( \text{convh}(A) \) is the convex hall of a set \( A \). Any extremal point of \( S \) has the following form:

\[ e_k = (0, \ldots, 0, 1, 0, \ldots), \quad k \in \mathbb{N} \]

where 1 stands in \( k \)th position. That is, vertices of the simplex are extreme points of the simplex. Here and henceforth we denote

\[ \text{ri}S = \{x = (x_1, x_2, \ldots, x_k, \ldots) \in S : x_k > 0, \text{ for all } k \in \mathbb{N}\}, \quad \partial S = S \text{\( \setminus \text{ri}S \)} \]

Next, let \( V \) be a mapping defined by

\[ V(x)_k = \sum_{i,j=1}^{\infty} P_{ij,k} x_i x_j, \quad k \in \mathbb{N}. \quad (2.3) \]

Here, \( P_{ij,k} \) are hereditary coefficients and satisfy

\[ P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{\infty} P_{ij,k} = 1, \quad i, j, k \in \mathbb{N}. \quad (2.4) \]

It is important to see that the mapping \( V \) is well-defined i.e., \( V(x)_k < \infty \) for any \( k \in \mathbb{N} \). Moreover one can check that \( V \) maps \( S \) into itself. Such kind of mapping \( V \) is called Quadratic Stochastic Operator (QSO).

Recall that a QSO \( V : S \rightarrow S \) is called Volterra if one has

\[ P_{ij,k} = 0 \text{ when } k \notin \{i, j\} \quad (2.5) \]
The condition (2.5) biologically means that each individual can inherit only the species of the parents.

Taking into account (2.3), one easily can check that (2.5) is equivalent to the following canonical form of $V$

$$V(x)_k = x_k \left(1 + \sum_{i=1}^{\infty} a_{ki} x_i \right) \text{ for all } k \in \mathbb{N},$$  

(2.6) where $a_{ki} = 1 - 2P_{ki}$. From (2.6), one can deduct that all faces of the simplex $S$ are invariant with respect to $V$. In particular, all the vertices of $S$ are the fixed points. Several properties of infinite dimensional Volterra operators have been investigated in [6].

Next we recall Young’s inequality:

$$b_1^{p_1} \cdots b_n^{p_n} \leq \sum_{i=1}^{n} b_i^{p_i},$$

for any $b_i > 0$ and $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$.

Here and henceforth, we use $V^{(n)}(x)$ to denote the iterations of the given operator $V$ at $x \in S$ i.e.,

$$V^{(n+1)}(x) = V(V^{(n)}(x)), \ n \in \mathbb{N}.$$  

A continuous function $\varphi : S \to \mathbb{R}$ is called a Lyapunov function for $V$ if the limit

$$\lim_{n \to \infty} \varphi(V^{(n)}(x))$$

exists for any initial point $x \in S$.

Let $\{V^{(n)}(x)\}_{n=1}^{\infty}$ be the trajectory of the point $x \in S$ under QSO $V$. By $\omega_V(x)$ we denote the set of limit points with respect to $\ell^1$-norm of the trajectory. Namely, for $x^* \in \omega_V(x)$ means, there exist a subsequence $\{n_k\}$ such that

$$V^{(n_k)}(x) \to x^*$$

in $\ell^1$-norm as $n_k$ goes to infinity.

Obviously, if $\omega_V(x)$ consists of a single point, then the trajectory converges, to a fixed point of (2.5). However, looking ahead, we remark that convergence of the trajectories is not the typical case for the dynamical systems (2.5). Therefore, it is of particular interest to obtain an upper bound for $x_0 \in S$, i.e., to determine a sufficiently "small" set containing $x_0 \in S$.

Obviously, if $\lim_{n \to \infty} \varphi(V^{(n)}(x_0)) = c$, then $\omega_V(x_0) \subset \varphi^{-1}(c)$. Consequently, to estimate of $\omega_V(x_0)$ we should construct the set of Lyapunov functions that is as large as possible.

3. Finite Dimensional Volterra QSO

Before, considering infinite dimensional Volterra QSO, let us observe some facts on finite dimensional Volterra QSO which turns to be very useful in the investigation one of the classes of infinite dimensional Volterra QSO. Denote

$$S_n^{r-1} = \left\{ x = (x_1, \ldots, x_n) : x_i \geq 0 \ \forall i = 1, n; \ \sum_{i=1}^{n} x_i = r \right\},$$

for any $r \in (0, 1]$. Note that, a huge number of papers on the theory of QSO were done on $S_1^{n-1}$ (or simply we write $S^{n-1}$). Therefore, it is self-interest to examine Volterra QSO defined on $S_r^{n-1}$ for any $r \in (0, 1)$. We need an auxiliary result to further in this topic. Denote

$$P_n^{r-1} = \left\{ x \in S^{n-1} : \sum_{i=1}^{n} a_{ki} x_i \geq 0 \text{ for every } k \in \{1, \ldots, n\} \right\}.$$  

Analogously, 
\[ P = \left\{ x \in S : \sum_{i=1}^{\infty} a_{ki}x_i \geq 0 \text{ for every } k \in \mathbb{N} \right\}. \]

We stress that the set \( P^{m-1} \) is non-empty (see [10]), in contrast to the set \( P \) could be empty [6]. Now, we recall some of known facts on Volterra QSO over a finite dimensional simplex.

**Theorem 3.1** ([10]). Let \( V \) be a Volterra QSO. If \( p = (p_1, \ldots, p_n) \in P^{n-1} \), then
\[ \varphi_p(x) = x_1^{p_1} \cdots x_n^{p_n} \tag{3.1} \]
is a Lyapunov function for \( V \) for \( x \in riS \).

Using the constructed Lyapunov function, it was proved the following result

**Theorem 3.2** ([10]). If \( x \in riS^{n-1} \) is not a fixed point, then \( \omega_V(x) \subset \partial S^{n-1} \) where \( \partial S^{n-1} \) is the boundary of the simplex \( S^{n-1} \).

**Proposition 3.3.** Let \( V : S^{n-1} \rightarrow S^{n-1} \) (for any \( r \in (0, 1] \)) be a Volterra QSO associated with a skew-symmetric matrix \( A = (a_{ki})_{k,i=1}^{n} \), then its dynamics is the same as \( \tilde{V} : S^{n-1} \rightarrow S^{n-1} \) where the associated skew-symmetric matrix for \( \tilde{V} \) is given by \( \tilde{A} = rA \) i.e.,
\[ rA = (ra_{ki})_{k,i=1}^{n} \]

**Proof.** Let us prove the statement by showing \( V \) and \( \tilde{V} \) are conjugate to each other i.e., there exist a homeomorphism function \( T : S^{n-1} \rightarrow S^{n-1} \) such that
\[ T^{-1} \circ V \circ T_r(y) = \tilde{V} \]
Indeed, \( V \) is well-defined on \( S^{n-1} \) i.e., it maps from \( S^{n-1} \) to itself. Next, we define a linear operator by
\[ T_r(y) := r(y) = (ry_1, ry_2, \ldots, ry_n) \]
for any \( y \in S^{n-1} \). One can check that \( T_r(S^{n-1}) = S^{n-1} \) and \( T^{-1}(S^{n-1}) = S^{n-1} \), where \( T^{-1}(x) \) is the pre-image of \( x \) under mapping \( T \). Moreover, for any \( y \in S^{n-1} \) we have
\[ T^{-1} \circ V \circ T_r(y) = T^{-1} \circ V(r(y)) = T^{-1} \circ \left( \left\{ ry_k \left( 1 + r \sum_{i=1}^{n} a_{ki}y_i \right) \right\}_{k=1}^{n} \right) = \tilde{V}(y) \]
The last statement means \( V : S^{n-1} \rightarrow S^{n-1} \) is conjugate to \( \tilde{V} : S^{n-1} \rightarrow S^{n-1} \). Hence it proves the statement.

In [6], it has been proved that every infinite dimensional Volterra QSO can be constructed by means of compatible sequence of finite dimensional Volterra QSO. Therefore, it is interesting to know what is the relationship between the sequence of compatible Volterra QSO and the corresponding Lyapunov functions. Let us consider a sequence \( V_n : S^{n-1} \rightarrow S^{n-1} \) finite dimensional Volterra operators, i.e.
\[ (V_n(x))_k = x_k \left( 1 + \sum_{i=1}^{n} a_{ki}x_i \right) \quad k = 1, \cdots, n, \quad (n \in \mathbb{N}), \tag{3.2} \]
We emphasize that $(a_{ij}^{n})$ is a skew-symmetric matrix. We recall that the sequence $\{V_{n}\}$ of Volterra operators is compatible if

$$V_{n+1} \mid S^{n-1} = V_{n}$$  \hspace{1cm} (3.3)

for every $n \in \mathbb{N}$. Here, $V_{n+1} \mid S^{n-1}$ means restriction of $V_{n+1}$ to the simplex $S^{n-1}$.

Remark 3.4. Let $\{V_{n}(x)\}$ be a sequence of compatible Volterra QSO and let $p = (p_{1}, \ldots, p_{m}) \in \mathbb{P}^{m-1}$ for $m < \infty$. If there exists $l_{0} < m$ such that

$$\sum_{i=1}^{l_{0}} p_{i} = 1$$

then $\varphi(x) = \prod_{i=1}^{l_{0}} x_{i}^{p_{i}}$ is a Lyapunov function for $V_{l_{0}}(x)$.

4. Infinite Dimensional Volterra QSO and Lyapunov Function

From Proposition 3.3 one can deduce the following remark.

Remark 4.1. Let $V$ be a Volterra QSO defined on $S$ associated with a skew-symmetric matrix $A = (a_{ij})$ such that we have finitely many $a_{ij} > 0$ i.e., there exist $n$ such that

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & 0 & \cdots \\
a_{21} & a_{22} & \cdots & a_{2n} & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\end{bmatrix}$$

We emphasize that $|a_{ij}| \leq 1$ for any $i, j \in \{1, \ldots, n\}$. Then, for any $x \in S$, the operator $V$ can be written as follows

$$V(x)_{k} = x_{k} \left(1 + \sum_{i=1}^{n} a_{ki}x_{i}\right) \text{ for } 1 \leq k \leq n, \hspace{1cm} V(x)_{k} = x_{k} \text{ for } k \geq n + 1$$  \hspace{1cm} (4.1)

Thus, the investigation of dynamics on $V(x)$ reduces to $V(x)_{k}$ for $1 \leq k \leq n$ which defines a finite Volterra QSO $\tilde{V}$ on $S_{r}^{n-1}$ for any $r \in (0, 1]$. Here $\tilde{V}$ is associated to skew-symmetric matrix $A = (a_{ij})_{i,j=1}^{n}$. Since all faces of the simplex $S_{r}^{n-1}$ are invariant with respect to $V$, therefore it is enough to study $\tilde{V}$ on $riS_{r}^{n-1}$. Thanks to Proposition 3.3 and Theorem 3.2 we conclude that for any $x_{0} = (x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}) \in riS_{r}^{n-1}$ and $x_{0} \notin Fix(\tilde{V})$,

$$\omega_{\tilde{V}}(x_{0}) \in \partial S_{r}^{n-1}$$  \hspace{1cm} (4.2)

Define $y_{0} = (x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}, y_{n+1}^{0}, \ldots) \in riS$ such that $\sum_{k=n+1}^{\infty} y_{k}^{0} = 1 - r$. Due to (4.1) and (4.2) one has

$$\omega_{V}(y_{0}) \in \partial S$$

Note that, the limiting points with respect to $\ell^{1}$ and point-wise are the same because of (4.1).

Now, we want to establish Lyapunov function for infinite dimensional Volterra QSO which has the same form as given by (3.1).
Theorem 4.2. Let $V$ be an infinite dimensional Volterra QSO such that $\mathbb{P}$ is non-empty. If there exist finitely many $p_k > 0$ i.e., there is $m$ such that

$$p = (p_1, p_2, \ldots, p_m, 0, \ldots) \in \mathbb{P}; \quad p_i \geq 0 \quad for \quad 1 \leq i \leq m \quad (4.3)$$

then the functional

$$\varphi_p(x) = \prod_{k=1}^{\infty} x_k^{p_k}$$

is a Lyapunov function for $V$.

Proof. Using the same argument as before, we can choose $x \in riS$ without the loss of generality. Next, one sees that

$$\varphi_p(V(x)) = \prod_{k=1}^{\infty} \left( 1 + \sum_{i=1}^{\infty} a_{ki} x_i \right)^{p_k} = \varphi_p(x) \prod_{k=1}^{\infty} \left( 1 + \sum_{i=1}^{\infty} a_{ki} x_i \right)^{p_k} \quad (4.4)$$

Since $p$ satisfies (4.3), then

$$\prod_{k=1}^{m} \left( 1 + \sum_{i=1}^{\infty} a_{ki} x_i \right)^{p_k} = \prod_{k=1}^{m} \left( 1 + \sum_{i=1}^{\infty} a_{ki} x_i \right)^{p_k}$$

Due to the facts $x \in riS$ and $a_{kk} = 0$, we have

$$1 + \sum_{i=1}^{\infty} a_{ki} x_i > 0 \quad for \quad any \quad k \in \mathbb{N}. \quad (4.5)$$

Therefore Young’s Inequality and (4.3) produce

$$\prod_{k=1}^{m} \left( 1 + \sum_{i=1}^{\infty} a_{ki} x_i \right)^{p_k} \leq \sum_{k=1}^{m} p_k \left( 1 + \sum_{i=1}^{\infty} a_{ki} x_i \right) = 1 + \sum_{k=1}^{m} p_k \sum_{i=1}^{\infty} a_{ki} x_i \quad (4.6)$$

Further, let show that $\sum_{k=1}^{\infty} p_k \sum_{i=1}^{\infty} a_{ki} x_i \leq 0$. Indeed, the series $\sum_{k=1}^{\infty} p_k \sum_{i=1}^{\infty} a_{ki} x_i$ converges absolutely.

From simple algebra and the fact $a_{ki} = -a_{ik}$ , we have

$$\sum_{k=1}^{\infty} p_k \sum_{i=1}^{\infty} a_{ki} x_i = \sum_{i=1}^{\infty} x_i \sum_{k=1}^{\infty} a_{ki} p_k = -\sum_{i=1}^{\infty} x_i \sum_{k=1}^{\infty} a_{ik} p_k$$

Since $p \in \mathbb{P}$, then we infer that $\sum_{k=1}^{\infty} a_{ik} p_k \geq 0$ for any $i \in \mathbb{N}$. Hence, we show that

$$\sum_{k=1}^{\infty} p_k \sum_{i=1}^{\infty} a_{ki} x_i = -\sum_{i=1}^{\infty} x_i \sum_{k=1}^{\infty} a_{ik} p_k \leq 0 \quad (4.7)$$
Equation (4.6) and (4.7) yield
\[ \prod_{k=1}^{m} \left( 1 + \sum_{i=1}^{\infty} a_{ki}x_i \right)^{p_k} \leq 1 \]

Hence, from (4.4) one concludes that
\[ \varphi_p(V(x)) \leq \varphi_p(x) \]
for any \( x \in riS \). Therefore, \( \lim_{n \to \infty} \varphi_p(V^{(n)}(x)) \) exist. This completes the proof. \( \square \)

From the last theorem, it seems hard to choose \( p \in \mathbb{P} \) that satisfied condition (4.3) due to infiniteness. This problem is solved by the next theorem.

**Theorem 4.3.** Let \( V \) be an infinite dimensional Volterra QSO associated with a skew-symmetric matrix \( A = (a_{ki}) \). Assume there exists \( m \) such that
\[ a_{ij} \geq 0 \quad \text{for} \quad i \geq n + 1, \quad j = 1, m \] \( (4.8) \)

Let \( x \in riS \), then for any \( p \in \mathbb{P}^{m-1} \), the following functional
\[ \varphi_p(x) = \prod_{k=1}^{m} x_k^{p_k} \]
is a Lyapunov function for \( V \).

**Proof.** For any \( x \in riS \), \( \varphi_p(V(x)) \) can be written as follows (see (4.4))
\[ \varphi_p(V(x)) = \varphi_p(x) \prod_{k=1}^{m} \left( 1 + \sum_{i=1}^{\infty} a_{ki}x_i \right)^{p_k} \]

Keeping in mind (4.5), hence using Young’s Inequality and fact \( a_{ki} = -a_{ik} \), we have
\[ \prod_{k=1}^{m} \left( 1 + \sum_{i=1}^{\infty} a_{ki}x_i \right)^{p_k} \leq \sum_{k=1}^{m} p_k \left( 1 + \sum_{i=1}^{\infty} a_{ki}x_i \right) = 1 - \sum_{i=1}^{m} x_i \sum_{k=1}^{m} a_{ik}p_k - \sum_{i=m+1}^{\infty} x_i \sum_{k=1}^{m} a_{ik}p_k \quad (4.9) \]

Due to \( p \in \mathbb{P}^{m-1} \) one infers
\[ \sum_{i=1}^{m} x_i \sum_{k=1}^{m} a_{ik}p_k \geq 0 \quad (4.10) \]

and assumption (4.8) yields
\[ \sum_{i=m+1}^{\infty} x_i \sum_{k=1}^{m} a_{ik}p_k \geq 0 \quad (4.11) \]

From (4.9), (4.10) and (4.11) we conclude that
\[ \varphi_p(V(x)) \leq \varphi_p(x) \]

Therefore, the limit
\[ \lim_{n \to \infty} \varphi_p(V^{(n)}(x)) \]
exists. This completes the proof. \( \square \)
Indeed, the set of Volterra QSOs that satisfies Theorem 4.3 is non-empty. Consider the following example.

**Example 4.4.** Let \( V \) be a Volterra QSO associated with the following skew-symmetric matrix

\[
A = \begin{bmatrix}
0 & -a & b & a_{14} & \cdots \\
-\frac{1}{a} & 0 & -c & a_{24} & \cdots \\
-b & c & 0 & a_{34} & \cdots \\
a_{41} & a_{42} & a_{43} & 0 & \cdots \\
& & & & \ddots \\
& & & & & \ddots
\end{bmatrix}
\]

where \( 0 < a, b, c \leq 1 \) and \( a_{ij} > 0 \) for all \( i \geq 4, j = 1,3 \). One sees that the set

\[
P^2 = \left\{ \frac{c}{a + b + c}, \frac{b}{a + b + c}, \frac{a}{a + b + c} \right\}
\]

Hence,

\[
\varphi_p(x) = \left( x_1^c, x_2^b, x_3^a \right) \frac{1}{a + b + c}
\]

is a Lyapunov function for \( V \).

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