Aspects of Instanton Dynamics in AdS/CFT Duality

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Abstract

We consider aspects of instanton dynamics in the large-$N$ limit using the AdS/CFT duality for D0/D4 bound states. In the supergravity picture of wrapped D0-brane worldlines on D4-branes, we find the single-instanton measure and discuss its dependence on compactification finite-size effects, as well as its matching to perturbative results. In the non-supersymmetric case, the same dynamical effects that produce the theta-angle dependence perturbatively in $1/N$, render the instantons unstable, although approximate instantons of very small size still exist.

The smeared D0/D4 black-brane supergravity solution can be interpreted as dual to a field theory configuration of an instanton condensate in the vacuum. In this case, we derive a holographic relation between the bare theta angle and the topological charge density of the instanton condensate.

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1 Introduction

As approximation schemes for gauge dynamics, instanton calculus \([1]\) and ‘t Hooft’s 1/\(N\) expansion \([2]\) do not seem to combine in a useful fashion. Since effects of a charge \(k\) instanton sector are of \(\mathcal{O}(e^{-8\pi^2 k/g^2}) = \mathcal{O}(e^{-N})\), it would seem that they are always irrelevant in the large-\(N\) limit unless they control the leading contribution to some observable (for instance, because of supersymmetry non-renormalization arguments), or somehow the integral over instanton moduli space is ill-defined. Such non-commutativity of the large-\(N\) limit and the instanton sum is assumed to be behind well-known instances of theta-angle dependence at perturbative order in the 1/\(N\) expansion, notably in the context of large-\(N\) chiral dynamics \([3]\).

On the other hand, it is known that some toy models \([4]\) completely suppress instanton-like excitations once the large-\(N\) limit has been taken. In other words, the ‘effective action’ resulting after large-\(N\) diagram summation does not support instantons any more. So, one may wonder whether the large-\(N\) ‘master field’ always loses its discrete topological structure. Recently, the AdS/CFT correspondence of \([5, 6]\) has provided a new set of non-trivial master fields for some gauge theories. In particular, the theta-angle dependence of \(\mathcal{N} = 4\) Super Yang–Mills (SYM\(_4\)) in four dimensions can be studied in the large-\(N\) expansion via perturbative Type IIB string theory in AdS\(_5\) × S\(_5\). It is saturated by instantons, which appear in the supergravity description as D-instantons \([7]\). So, the radical view that an instanton gas is incompatible with the large-\(N\) limit is not vindicated in this case.

In this paper, we investigate these questions in a QCD cousin introduced by Witten \([8]\), which admits a supergravity description of its master field, while removing the constraints of extended supersymmetry and conformal invariance. More precisely, we would like to learn under what conditions some kind of topological configurations (instantons) still give the leading semiclassical effects of \(\mathcal{O}(e^{-N})\), even after the planar diagrams have been summed over. We shall focus on the most elementary case of the dilute limit, i.e. the single-instanton sector.

One description of this theory is in terms of \(N\) D4-branes wrapped on a circle S\(_1\)\(_{\beta}\) of length \(\beta\), with thermal boundary conditions. At weak coupling, the low-energy theory on the D4-branes is a perturbative five-dimensional Super Yang–Mills theory (SYM\(_{4+1}\)), which reduces to non-supersymmetric, \(SU(N)\) Yang–Mills theory in four euclidean dimensions (YM\(_4\)), at distance scales much larger than the inverse temperature \(T^{-1} = \beta\). Since five-dimensional gauge fields originate from massless open strings, their coupling scales as \(g^2_5 \sim g_s \sqrt{\alpha'}\) with \(g_s = \exp(\phi_\infty)\) denoting the string coupling constant. Therefore, the four-dimensional coupling at the cut-off scale \(T\) is given by \(g^2 \sim g_s T \sqrt{\alpha'}\).

The weak-coupling description of instantons in this set up is in terms of D0/D4-branes bound states. Due to the Wess–Zumino coupling between the type IIA Ramond–Ramond (RR) one-form and the gauge fields on the D4-branes world-volume, \(\mathcal{L}_{\text{WZ}} = C_{D0} \wedge F \wedge F\), a D0-brane ‘inside’ a D4-brane carries the instanton charge. The action of an euclidean
world-line wrapped on a circle $S^1$ is
\[ S_{D0} = M_{D0} \cdot \beta = \frac{\beta}{\sqrt{\alpha'} g_s} = \frac{8\pi^2}{g^2} = \frac{8\pi^2 N}{\lambda}. \] (1.1)

Incidentally, this relation also fixes the numerical conventions in the definition of the four-dimensional coupling $g$. We have also introduced the standard notation for the large-$N$ 't Hooft coupling $\lambda \equiv g^2 N$.

The moduli of these instantons are encoded in the quantum mechanical zero-modes of the D0–D0 and D0–D4 strings. For a standard compactification, the D0-branes (i.e. the 'instanton particles' of the gauge theory) describe standard instantons of $\mathcal{N} = 4$ SYM$_4$ (see [1] for some generalizations). If the circle breaks supersymmetry, the instanton fermionic zero modes should be lifted accordingly to mass of $O(T)$, and one should get essentially a Yang–Mills instanton with no fermionic zero modes. Other one-loop effects would incorporate the perturbative running of the coupling constant in the standard way.

The supergravity framework for SYM$_{4+1}$ at finite temperature is given by the black D4-brane solution [8, 10]. The full metric in the string frame is
\[ ds^2 = H_4^{-1/2} (h \, d\tau^2 + d\vec{y}^2) + H_4^{1/2} \left( d\tau^2 / h + r^2 d\Omega_4^2 \right) \] (1.2)

where
\[ H_4 = 1 + (r_{Q4}/r)^3, \quad h = 1 - (r_0/r)^3. \] (1.3)

There are two length scales associated with this metric: the Schwarzschild radius, $r_0$, related to the Hawking temperature $T$ by $T^{-1} = \beta = (4\pi/3) r_0 [H_4(r_0)]^{1/2}$, and the charge radius $r_{Q4}$, given by
\[ r_{Q4}^3 = -\frac{1}{2} r_0^3 + \sqrt{\frac{1}{4} r_0^6 + \alpha'^3 (\pi g_s N)^2}, \] (1.4)

In the Maldacena or gauge-theory limit, one scales $\alpha' \to 0$ with $r/\alpha' = u$ and $r_0/\alpha' = u_0$ fixed. The new coordinate $u$ has dimensions of energy and the scaling properties of the Higgs expectation value. In this limit, only the combination
\[ \alpha'^2 H_4 \to \frac{\pi g_s N \sqrt{\alpha'}}{u^3} = \frac{\lambda \beta}{8\pi u^3} \] (1.5)
is relevant. In the supergravity picture, the D4-branes have disappeared in favour of the 'throat geometry' $X_{\text{bb}}$ [12], i.e. we have no open strings and the description is fully gauge invariant. The black-brane manifold $X_{\text{bb}}$, with topology $\mathbb{R}^2 \times \mathbb{R}^4 \times S^4$, has a boundary at $u = \infty$ of topology $S^1 \times \mathbb{R}^4$, which is interpreted as the SYM$_5$ space-time (the $(\tau, \vec{y})$ space). The physical interpretation is that asymptotic boundary conditions for the supergravity fields at $u = \infty$ represent coupling constants of microscopic operators in the gauge theory [13].

The same boundary conditions are satisfied by the extremal D4-brane metric with thermal boundary conditions. This is the 'vacuum' manifold, denoted $X_{\text{vac}}$, with topology

\[^1\text{See, for example, [11] and references therein for a review of metrics relevant to this paper.}\]
$S^1 \times \mathbb{R}^5 \times S^4$, obtained from (1.2) by setting $u_0 = 0$, with fixed $\beta$. However, one can show [8, 12] that $X_{\text{vac}}$ is suppressed by a relative factor of $\mathcal{O}(e^{-N^2})$ with respect to $X_{\text{bb}}$, in the large-$N$ limit. In other words, the $\mathcal{O}(N^2)$ actions satisfy
\[
I(X_{\text{bb}}) - I(X_{\text{vac}}) = -KN^2 \lambda VT^4 < 0
\]
for any $T > 0$, with $K$ a positive constant, i.e. there is no Hawking-Page transition [13, 8].

Unlike the case of $N = 4$ SYM$_4$, the dilaton is not constant in the supergravity description. It becomes strongly coupled at radial coordinates of order $u \sim \mathcal{O}(N^4 / \beta \lambda)$, where one has $e^\phi = g_s (H_4)^{-1/4} = \mathcal{O}(1)$. Beyond this point, one should use a dual picture in terms of a wrapped M5-brane in M-theory, i.e. a quotient of $\text{AdS}_7 \times S^4$. For the purposes of the discussions in this paper, we are studying the theory at fixed energy scales of $\mathcal{O}(1)$ in the ’t Hooft’s large-$N$ limit, with fixed $\lambda = g^2 N$. Therefore, such non-perturbative thresholds effectively decouple in the regime of interest, and we shall formally extend the D4-brane manifold all the way up to $u = \infty$.

From a physical point of view, $\alpha'$-corrections to the classical geometry pose a more serious limitation to the supergravity description. The curvature at the horizon scales as $(u_0 g_s N \sqrt{\alpha'})^{-1/2} \sim \lambda^{-1}$, in string units, so that the supergravity description is accurate only for large bare ’t Hooft coupling $\lambda \gg 1$. On the other hand, the glueball mass gap [14] in this theory is of order $M_{\text{glue}} \sim \beta^{-1} = T$, while inspection of the Wilson loop expectation value gives a four-dimensional string tension [13] of order $\sigma \sim \lambda T^2$, i.e. hierarchically larger in the supergravity regime. This lack of scaling indicates that the supergravity picture is far from the ‘continuum limit’ of the YM$_4$ theory, a suspicion already clear from the existence of non-QCD states of Kaluza–Klein origin at the same mass scale as the glueballs: $M_{\text{KK}} \sim T \sim M_{\text{glue}}$.

2 The Localized Instanton

The natural candidate for an instanton excitation in the large-$N$ supergravity picture is a D0-brane probe wrapped around the thermal circle. For the supersymmetric case, this is indeed the T-dual configuration to the D-instantons in $\text{AdS}_5 \times S^5$ discussed in [7]. Wrapped D0-branes have the correct quantum numbers to be interpreted as Yang–Mills instantons in the effective four-dimensional theory. The topological charge is interpreted as the wrapping number on the thermal circle $S^1_\beta$. In the large-$N$ limit, it is justified to take the D0-brane as a probe, neglecting its back-reaction on the supergravity fields, since the gravitational radius is sub-stringy: $(r_{\text{probe}})^7 \sim \alpha'^7/2 e^\phi \sim \mathcal{O}(1/N)$, although for instanton numbers of $\mathcal{O}(N)$ with identical moduli we may need a supergravity description for the instanton dynamics in terms of the D0-branes near-horizon geometry (i.e. a T-dual of the limit in [10], or the solution of section 3 below). We shall postpone these interesting complications by working in the single-instanton sector, and with instanton moduli of $\mathcal{O}(1)$ in the ’t Hooft large-$N$ limit.

One important ingredient of the the instanton/D0-brane mapping is a physical interpretation in gauge-theory language of the wrapped D0 world-line’s radial position. For
this purpose, we use the generalized UV/IR connection as discussed in [17]. According to this, a radial coordinate $u$ is associated to a length scale in the SYM $p+1$ gauge theory of order $\ell \sim \sqrt{g_{p+1}^2 N/u^{5-p}}$. Thus, in our case, the size parameter $\rho$ of the instanton satisfies:

$$\rho^2 = \frac{\beta \lambda}{u}. \quad (2.1)$$

We will assume this relation as the definition of the instanton’s size modulus.

We will now discuss both manifolds with $S^1 \times \mathbb{R}^4$ boundary conditions at $u = \infty$, in spite of the fact that eq. (1.6) ensures the dynamical dominance of $X_{bb}$. The reason for considering also the ‘vacuum’ manifold is first that we find interesting differences between the manifolds, and that $X_{vac}$ is the only relevant manifold for supersymmetric compactification, with which we can make contact with the $\text{AdS}_5 \times S^5$ case.

### 2.1 Vacuum Manifold

The $S^1$ factor on the boundary extends to the bulk of $X_{vac}$ becoming singular as $u \to 0$, since $\text{Vol}(S^1_u) = \beta \sqrt{g_{\tau\tau}} = \beta (H_4)^{-1/4} \to 0$. However, the action of the instanton is constant, due to the dilaton dependence in the Dirac–Born–Infeld action:

$$S_{D0} = M_{D0} \int_0^\beta d\tau (g_s e^{-\phi}) \sqrt{g_{\tau\tau}} = \frac{8\pi^2}{g^2}. \quad (2.2)$$

Thus, the size $\rho$ is an exact modulus in the supergravity description on $X_{vac}$. On general grounds, the path integral of a D-particle in a curved background $X$ contains an ultralocal term in the measure of the form $DX^\mu [\text{det}(g_{\mu\nu})]^{1/2}$, to ensure invariance under target-space diffeomorphisms. In the description of instantons on the manifold $X$, we concentrate on the zero-mode part which then leads to a single-instanton measure

$$d\mu(X) = C_{N,\lambda} d\eta \int_{S^1_\beta \times S^4_\lambda} (\alpha')^{-5} d\text{Vol}(X), \quad (2.3)$$

where $d\eta$ is the measure over fermionic zero-modes (sixteen in the supersymmetric case), and $C_{N,\lambda}$ is a constant to be determined by the matching to the perturbative measure. We have produced a measure in the physical space-time and scale-parameter space by averaging over $S^1_\beta \times S^4_\lambda$. The result for $X_{D4} = X_{vac}$, using the UV/IR connection (2.1) is:

$$d\mu(X_{D4}) = C_{N,\lambda} \lambda^5 (\rho T)^{-6} \rho^{-5} d\rho d\tilde{y} d\eta. \quad (2.4)$$

We see that the presence of the dimensionful scale $T$ explicitly violates the conformal invariance of the measure, which we must take as a concrete prediction of the supergravity approach. As such, it is valid at large $N$ and $\lambda$.

The singularity of $X_{vac}$ as $u \to 0$ is not relevant. At $u \sim u_s = T\lambda^{1/3}$ the size of the world-line is of $O(1)$ in string units. So, for $u \ll u_s$ we must use the T-dual metric of $N$ D3-branes smeared over the dual circle of coordinate length $\beta = 4\pi^2 \alpha' / \bar{\beta}$:

$$ds^2(X_{D3}) = H_4^{-\frac{1}{2}} d\bar{y}^2 + H_4^{\frac{1}{2}} \left( d\tau^2 + dr^2 + r^2 d\Omega_3^2 \right), \quad (2.5)$$
with $\tilde{r} \equiv \tilde{r} + \tilde{\beta}$. In the T-dual metric the size of $S^1_u$ grows with decreasing $u$.

In fact, the metric (2.3) is unstable if any small amount of energy is added. It collapses to the array solution of localized D3-branes [18]:

$$ds^2(X_{D3}) = H_3^{1/2}d\bar{y}^2 + H_3^{1/2}(d\tilde{r}^2 + dr^2 + r^2d\Omega_4^2), \quad \text{with} \quad H_3 = 1 + \sum_n \frac{4\pi \bar{g}_s \alpha'^2}{|r^2 + (n\beta)^2|^2}.$$  

By the T-duality rules and our coupling conventions: $4\pi \bar{g}_s = 8\pi^2 g_s \sqrt{\alpha'}/\beta = g^2$. In the regime $r \gg \beta$ we can approximate the discrete sum over images by a continuous integral, and we recover the smeared metric (2.4) as an approximation. On the other hand, for $r \ll \beta$ we can instead neglect the images and approximate the sum by the $n = 0$ term. The result is of course the standard $AdS_5 \times S^5$ metric corresponding to D3-branes at strong coupling. Indeed, the UV/IR relation for D-instantons in D3-branes [7], $\rho = \sqrt{\lambda}/u$, matches the five-dimensional one (2.4) precisely at $u = u_{loc} = 1/\beta$, which is equivalent to $r = r_{loc} = \beta/4\pi^2$.

The instanton measure (2.3) matches across these finite-size transitions to the corresponding measures for the new manifolds $X_{D3}$ and $X_{D3}$, because the definition (2.3) applies in general and the volume form matches across the transitions at $u = u_s$ and $u = u_{loc}$. The resulting measures are (both up to $O(1)$ numerical factors):

$$d\mu(X_{D3}) = C_{N,4} \lambda^4 (\rho T)^{-3} \rho^{-5} d\rho d\bar{y} d\eta,$$

$$d\mu(X_{D3}) = C_{N,5} \lambda^{5/2} \rho^{-5} d\rho d\bar{y} d\eta.$$  \hspace{1cm} (2.6)

This last measure is conformally invariant, and coincides with that of refs. [7] for D-instantons in $AdS_5 \times S^5$.

Finally, as pointed out in the introduction, the validity of the supergravity picture is limited by the requirement that we can control the $\alpha'$-corrections. The curvature of the D4-brane metrics is of $O(1)$ in string units at the ‘correspondence line’ $u_c \sim (\beta \lambda)^{-1}$ [13]. For the D3-brane metrics, the condition is simply $\lambda \sim 1$. This implies that, for $\rho < \beta$, we have a correspondence line for instanton sizes $\rho = \rho_c = \beta \lambda$. For $\rho > \beta$ the correspondence line is independent of $\rho$ and lies at $\lambda \sim 1$. Below the correspondence line the system is better described in Yang–Mills perturbation theory, although we lose the analytic control over the $1/N$ expansion.

The geometrical D-instanton measure in $AdS_5 \times S^5$ has been matched to the perturbative instanton measure in the $N = 4$ SYM$_4$ theory in great detail, including multi-instanton terms [14]. In particular, this allows us to fix the coupling-dependent constant as $C_{N,4} = N^{-7/2} \lambda^{3/2}$. This is rather remarkable, since the geometrical measure holds at large $\lambda$, whereas the perturbative measure is derived in Yang–Mills perturbation theory, valid for $\lambda \ll 1$. This robustness of the instanton measure in this case might be due to the high degree of supersymmetry and/or conformal symmetry. For instance, the analogous matching between the D4-brane supergravity measure (2.4) and the perturbative description of the ‘instanton particles’ of SYM$_5$, through the correspondence line $\rho = \rho_c = \beta \lambda$,

\[\text{Notice that the UV/IR connection (2.1) remains unchanged by T-duality, as the new metric only differs by } g_{\pi\pi} \to 1/g_{\pi\pi}, \text{ with the } u, \bar{y} \text{ components of the metric unaffected.}\]
fails by one power of $\lambda$. This means that the very precise matching of [20] for $\text{AdS}_5$ D-instantons is probably a consequence of conformal invariance.

This discussion may be summarized in Fig. 1, where the finite-size transitions, as well as the correspondence lines are depicted as a function of the ‘t Hooft coupling and the instanton size.

### 2.2 Black-brane Manifold

Although the wrapping charge of D0-branes is well defined in $X_{\text{vac}}$, the thermal circle being non-contractible, this is not the case for $X_{\text{bb}}$, whose $(\tau, u)$ subspace has $\mathbb{R}^2$ topology. Therefore, the thermal circle at fixed radial coordinate $S^1_u$, is contractible, being the boundary of a disc: $S^1_u = \partial D_u$, i.e. we can ‘unwrap’ the D0-brane instanton through the horizon. Thus, while exact instanton charges can be identified in the supersymmetric case, no quantized topological charge seems to survive in the non-supersymmetric case, due to the dynamical dominance of $X_{\text{bb}}$ in the large-$N$ limit (1.6).

Still, we can talk of approximate or ‘constrained’ instantons, provided the probe D0-brane world-line wraps far away from the horizon. In this case the un-wrapping costs a large action. In order to estimate the action as a function of $u$ (or the instanton size $\rho$), we calculate the Dirac–Born–Infeld action of the probe D0-brane:

$$S_{D0} = M_{D0} \int_0^\beta d\tau \left( g_s e^{-\phi} \right) \sqrt{g_{\tau\tau}} = \frac{8\pi^2}{g_s^2} \sqrt{h} = \frac{8\pi^2}{g_s^2} \sqrt{1 - (\rho/\beta)^6},$$

(2.7)

where we have used the UV/IR relation (2.1) in the last step. Thus, $\rho$ is not an exact
Figure 2: Instanton phase diagram for the compactified D4 theory on a thermal circle of size $\beta$. We have continued the glueball mass scale curve $\rho \Lambda_{\text{QCD}} \sim 1$ to weak coupling in a way tentatively consistent with asymptotic freedom.

modulus, as instantons tend to grow. For an instanton of the order of the glueball’s Compton wave-length $\rho \sim \beta$, the action is comparable to the vacuum action, and the instanton has disappeared (un-wrapped).

In the far ultraviolet, we can use the approximate instantons of very small size $\rho \ll \beta$, to measure a ‘running effective theta angle’, by requiring that the approximate instanton is weighed by a phase $\exp(i \theta_{\text{eff}})$, with $\theta_{\text{eff}}(u = \infty) = \theta$, the bare theta angle of the four-dimensional YM$_4$ theory. Following Witten [21], a bare theta angle is associated to a RR two-form

$$f_{D0} = dC_{D0} = \overline{\theta} \frac{3}{\pi \zeta^7} d\zeta \wedge d\psi,$$

where, in the notation of [21], $\zeta^2 = u/u_0$, and $\psi = 2\pi \tau/\beta$. The bare theta angle, measured at $u = \infty$, is $\theta = \overline{\theta}$ (mod $2\pi$), due to the multiplicity of meta-stable vacua as described in [21], i.e. $f_{D0} \propto \overline{\theta} = (\theta + 2\pi n)$ in the $n$-th vacuum (see also [22] for another geometric approach to this question). In what follows, we shall obviate this technicality by working in the $n = 0$ vacuum, so that $\theta = \overline{\theta}$. The effective theta angle at throat radius $u$ is

$$\theta_{\text{eff}}(u) = \int_{S^1_u} C_{D0} = \int_{D_u} f_{D0} = \theta \left( 1 - 6 \int_{\zeta(u)}^\infty d\zeta \zeta^{-7} \right) = \theta h(u) = \theta \left( 1 - (\rho/\beta)^6 \right).$$

The ‘correspondence line’ $u_c \sim (\beta \lambda)^{-1}$ [19], controlling $\alpha'$-corrections is also defined in $X_{bb}$. In terms of instanton sizes, for $\rho < \beta$, we have a correspondence line at $\rho_c = \beta \lambda$. Since no instantons survive for $\rho > \beta$ in the supergravity picture, the finite-size effects related to T-duality in $S^1_\beta$ and localization effects are absent for $X_{bb}$, i.e. there is no phase of D-instantons in $\text{AdS}_5 \times S^5$. The situation can be summarized by Fig. 2.
3 The Smeared Instanton Solution

In the previous section we have seen that probe D0-branes wrapping the thermal circle of a black D4-brane in the far ultraviolet are dual to (unstable) small-size instantons. Vice-versa, there exists a different supergravity solution dual to a field-theory configuration which can be interpreted as containing a condensate of large instantons.

Indeed, the smeared, black D0/D4-brane solution is interpreted (as in ref. [23] for the supersymmetric T-dual case) to be dual to a Yang-Mills theory with a non-vanishing self-dual background. The self-duality of the background implies that it can be related to instantons, and the smeariness of the D0-branes can be interpreted very heuristically as the fact that the instantons are ‘smooth’ and then ‘large’.

In fact, in real time, the D0-branes are smeared on the D4-branes as soon as they ‘fall behind’ the horizon, due to the no-hair property (this corresponds to $u = u_0$ or, using (2.1), to $\rho = \beta$.) This statement has only a heuristic value because, in the euclidean time configurations we are considering, space-time effectively ends at $u = u_0$. Still, the effects of the source D0-branes can be detected on the long-range fields such as the metric, dilaton, and RR fields. In this section, we pursue this view of the smeared D0-branes not as probes, as in the previous section, but as background data.

The string-frame metric outside a system of $k$ D0-branes smeared over the volume of $N$ D4-branes differs from that in (1.2) by one more harmonic function $H_0$:

$$ds^2 = H_0^{-\frac{3}{4}} H_4^{-\frac{3}{2}} h d\tau^2 + H_0^\frac{1}{4} H_4^{-\frac{1}{2}} d\tilde{y}^2 + H_0^\frac{1}{4} H_4^\frac{1}{2} \left( dr^2 / h + r^2 d\Omega_4^2 \right). \quad (3.1)$$

In the gauge-theory limit, this function is given by

$$H_0(u) = 1 + \left( \frac{u_0}{u} \right)^{\frac{3}{2}} - \frac{1}{2} \left( \frac{u_0}{u} \right)^3 + \frac{1}{4} \left( \frac{u_0}{u} \right)^6 + \frac{1}{6} \left( \frac{u_k}{u} \right)^6. \quad (3.2)$$

It depends on a new energy scale $u_k$, related to the number density of D0-branes $k/V \equiv \kappa$ by

$$u_k^3 = \kappa \frac{2\pi^3 \beta \lambda}{N}. \quad (3.3)$$

The new scale is small ($u_k^3 = \mathcal{O}(1/N)$) in the large-$N$ limit with fixed instanton charge density. In this paper, we are interested in the physics at energies of $\mathcal{O}(1)$ in the large-$N$ limit, so that $u_k \ll u_0$ and $H_0$ may be approximated by $H_0 = 1 + \left( \frac{u_k^2}{u_0} \right) + \mathcal{O}(1/N^4)$. The dilaton profile also receives $\kappa$-dependent corrections, $g_s e^{-\phi} = (H_4/H_0^3)^{1/4}$, as well as the Hawking temperature:

$$T^{-1} = \beta = \frac{4\pi}{3} r_0 \sqrt{H_0(r_0)} H_4(r_0) = \left( \frac{2\pi \lambda \beta}{9 u_0} H_0(u_0) \right)^{1/2}. \quad (3.4)$$

This yields an equation for $u_0$ that can be solved iteratively in powers of $(u_k/\lambda T)^6$.

\(^3\)At very low energies, $u_0 \ll u_k$, the smeared solution is $X_6 \times T^4$, with $X_6$ conformal to $\text{AdS}_2 \times S^4$ in the sense of [24]. It is presumably related to quantum mechanics in the large-$k$ instanton moduli space \([23]\).
The relation between the smeared D0-brane number density $\kappa$ and the running theta angle is obtained from the supergravity solution for the RR two form:

$$f_{D0} = \frac{c\kappa}{u^4 (H_0)^2} \, du \wedge d\tau,$$

(3.5)

with $c$ a known numerical constant. As before, a wrapped D0-brane probe can be used to measure an effective theta angle whose value at $u = \infty$ defines the bare theta angle. Plugging (3.5) into (2.9) we obtain:

$$\theta_{\text{eff}}(u) = \int_{D_u} f_{D0} = \theta \frac{u^3 - u_0^3}{u^3 + u_0^3} = \theta \frac{h(u)}{H_0(u)}, \quad \text{where} \quad \theta = \frac{\beta c\kappa}{3 u_0^3 + u_0^3}. \quad (3.6)$$

The two-form solution found by Witten (2.8) corresponds to the $u_0 \gg u_k$ regime of (3.5). This provides a relation between the number density $\kappa$ of smeared D0-branes and the bare theta angle, valid in the large-$N$ limit with fixed $\kappa$:

$$\theta = \frac{\beta c\kappa}{3 u_0^3 H_0(u_0)} = \kappa \cdot \frac{9e}{\lambda^3 T^4} \cdot \left(\frac{3}{2\pi}\right)^3 + O(1/N^2), \quad (3.7)$$

where we have used $u_0 = 2\pi\lambda T/9 + O(1/N^2)$, from (3.4). In interpreting this relation, it is important to remember that we are working in the $n = 0$ vacuum, out of the $O(N)$ metastable vacua mentioned in section 2.2, i.e. the actual values of the parameters are such that the r.h.s of (3.7) is smaller than $2\pi$.

Equation (3.7) is a very suggestive relation, holographic in nature, in which the bare theta angle is obtained in a ‘mean-field’ picture from the parameters of a kind of ‘instanton condensate’. We should stress that (3.7) is only valid in the non-supersymmetric case. The extremal (supersymmetric) solution has a non-contractible $S^1$ so that we can add an arbitrary harmonic piece to $C_{D0}$, thereby changing the asymptotic value of $\theta$ independently of $k$ and $\beta$ (i.e. we cannot use Stokes’s theorem as we do in (2.9) and (3.6)).

An interesting application of this connection is the computation of topological charge correlations to the leading order in the large-$N$ and large $\lambda$ limit. In view of (3.7), this can be done by studying the $\kappa$-dependence of the vacuum energy of the YM$_4$ theory (or equivalently the thermal free energy of the SYM$_5$ theory.) For example, the action can be calculated as

$$I = \beta E_{\text{YM}} - S_{\text{BH}}, \quad \text{with} \quad S_{\text{BH}} = (A_{\text{horizon}})/4G_{10}$$

the black-brane entropy and $E_{\text{YM}} = M_{\text{ADM}} - NVT_{D4}$, the ADM mass above extremality. One obtains

$$I = 3 \frac{\text{Vol}(S^4) \beta V}{16\pi G_{10}} r_0^3 \left( H_0(r_0) - \frac{7}{6} \right) = N^2 \frac{4VT}{\pi \lambda^2} \frac{u_0^3}{u_0^3} \left( H_0(u_0) - \frac{7}{6} \right). \quad (3.8)$$

Solving $\theta$ from (3.7) and using the relation

$$\left(\frac{u_k}{u_0}\right)^3 = 6\pi \frac{3}{e} \frac{H_0(u_0) \cdot \lambda \theta}{N}, \quad (3.9)$$

combined with (3.2) and (3.4), we learn that the functional form of the $n = 0$ vacuum energy is given by

$$I(\theta)_{n=0} = N^2 VT^4 \lambda f(\lambda \theta/N), \quad (3.10)$$
with \( f(x) \) an even function (as expected from considerations of CP symmetry), whose Taylor expansion around \( \theta = 0 \) may be determined by solving (3.4) iteratively. These selection rules determine the large-\( N \) and large \( \lambda \) scaling of the topological charge correlators at \( \theta = 0 \):

\[
\langle (Q_{\text{top}})^{2m} \rangle_{\text{connected}}^{\theta=0} = \left( \frac{d}{d\theta} \right)^{2m} I(\theta) \sim VT^4 \frac{\lambda^{2m+1}}{N^{2m-2}}.
\]  

(3.11)

For the standard topological susceptibility, \( m = 1 \), the scaling agrees with ref. [26].

### 4 Concluding Remarks

Within the AdS/CFT correspondence, the large-\( N \) master field of the gauge theory is encoded in the gravitational saddle-points of the supergravity description, subject to boundary conditions.

In the model of ref. [8], which has a good supergravity description for large \( N \) and large \( 't \) Hooft coupling \( \lambda = g^2 N \), there are two ‘master fields’, or generalized large-\( N \) saddle-points, given by the two manifolds \( X_{\text{vac}} \) and \( X_{\text{bb}} \), with \( S^1 \times \mathbb{R}^4 \) boundary. We find that \( X_{\text{vac}} \) supports instantons in the form of wrapped D0-branes, and leads to exponentially suppressed theta-angle dependence, very much like in the \( \text{AdS}_5 \times S^5 \) case, to which it is dual through a set of T-duality and localization transitions that we discuss in some detail, including the matching of the single-instanton measure.

However, \( X_{\text{vac}} \) is only the dominant master field in the supersymmetric case. The large-\( N \) dynamics in the non-supersymmetric case is dominated by \( X_{\text{bb}} \), which does not support finite-action topological excitations with the instanton charge. Therefore, the dominant master field shows perturbative (in \( 1/N \)) theta-angle dependence, but has no ‘instanton topology’, very much like in the two-dimensional toy models of refs. [4]. Instead, we can identify approximate (constrained) instantons of size \( \rho \ll \beta \), merging with the vacuum at sizes of the order of the glueball’s Compton wave-length \( \rho \sim \beta \), which for this model coincides with the Kaluza–Klein threshold.

The approximate equivalence of \( X_{\text{vac}} \) and \( X_{\text{bb}} \) in the ultraviolet regime \( u \to \infty \), poses the question of whether the approximate small instantons of \( X_{\text{bb}} \) are really artifacts of the regularization of the Yang–Mills theory by a hot five-dimensional supersymmetric theory. Unfortunately, this question cannot be settled with present techniques, since \( M_{\text{glue}} \sim M_{\text{KK}} \) in the supergravity approximation, \( \lambda \gg 1 \), and we lack a regime in which we could follow the instantons as genuine four-dimensional configurations. It would be very interesting to see if the non-supersymmetric gravity duals based on Type 0 D-branes [27] provide a more vantageous point to study this question.

Heuristically, according to the UV/IR relation, an instanton of size \( \rho \gg \beta \) would be associated to a D0-brane ‘inside’ the horizon of the black D4-brane. Because of the no-hair properties, such a configuration would have the D0-charge completely de-localized over the horizon (see [28] for a recent discussion in the extremal case). Therefore, such configurations should be interpreted as homogeneous self-dual backgrounds in the gauge theory, and the supergravity description involves the ‘smeared’ D0/D4 solution. Although this

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picture cannot be held literally in the euclidean solutions, which lack an ‘interior region’ behind the horizon, we can still identify the RR two-form generated by the D0-branes ‘dissolved’ in the D4-brane horizon. This RR flux is in turn responsible for the generation of a theta angle, via the AdS/CFT rules of [3]. Therefore, we obtain a holographic relation between the theta angle and the smeared instanton charge. Although the general relation between background fields and theta angle is not new (see [29, 4] for explicit two-dimensional examples), we find it interesting that in our case the background field is explicitly associated to an instanton condensate, with quantized topological charge (equal to the number \( k \) of smeared D0-branes). This is reminiscent of the instanton liquid models, where the instanton density is fixed self-consistently (see for instance [30]).

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