Poisson gauge models and Seiberg-Witten map

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ABSTRACT: The semiclassical limit of full non-commutative gauge theory is known as Poisson gauge theory. In this work we revise the construction of Poisson gauge theory paying attention to the geometric meaning of the structures involved and advance in the direction of a further development of the proposed formalism, including the derivation of Noether identities and conservation of currents. For any linear non-commutativity, $\Theta_{ab}(x) = f_{abc}^\ast x^c$, with $f_{abc}^\ast$ being structure constants of a Lie algebra, an explicit form of the gauge Lagrangian is proposed. In particular a universal solution for the matrix $\rho$ defining the field strength and the covariant derivative is found. The previously known examples of $\kappa$-Minkowski, $\lambda$-Minkowski and rotationally invariant non-commutativity are recovered from the general formula. The arbitrariness in the construction of Poisson gauge models is addressed in terms of Seiberg-Witten maps, i.e., invertible field redefinitions mapping gauge orbits onto gauge orbits.

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1 Introduction

Recent investigations on noncommutative field theory [1, 2] have proposed a novel approach where gauge connections and field strengths are defined on the basis of two main requests. First, the commutative limit has to be well defined and to give back the standard Maxwell theory. Second, the set of infinitesimal gauge transformations has to close a Lie algebra in such a way to be compatible with the underlying noncommutative spacetime, namely, a homomorphism has to exist between the latter and the noncommutative algebra of gauge parameters. These requests lead to a non-linear modification of Maxwell theory, solely dictated by compatibility of the noncommutative picture of space-time and theoretical consistency of electrodynamics.

Interestingly, non-linearly modified Maxwell models, mainly based on phenomenological reasons, go back to Born and Infeld in 1933 [3] and a few years later by Euler and
Heisenberg [4]. Since then, many other models of non-linear electrodynamics have been proposed and analysed in many areas of theoretical physics including gravity, cosmology, string theory and condensed matter. A recent review can be found in [5]. It would be certainly interesting and worth to compare our findings with the latter, which in principle are based on different premises and seem to be completely unrelated. We plan to come back to this issue in a future analysis.

For the present paper, we shall deal with an $n$-dimensional manifold, $\mathcal{M}$ representing space-time, locally described by $x^a$, $a = 0, \cdots, n - 1$, a set of local coordinates. Non-commutative deformations of space-time may be characterised by the Kontsevich star product of functions on $\mathcal{M}$, [6], which for each given Poisson bivector $\Theta^{ij}(x)$, reads

$$ f \star g = f \cdot g + \frac{i}{2} \{ f, g \} + \ldots $$

where

$$ \{ f, g \} = \Theta^{ij}(x) \partial_i f \partial_j g $$

stands for the Poisson bracket associated with the Poisson tensor $\Theta^{ij}(x)$, while the remaining terms, denoted through “…” , contain higher derivatives of the functions $f$ and $g$.

For constant noncommutativity the deformation is provided by the usual Moyal-Weyl star-product and its siblings [7, 8] and the construction of noncommutative gauge theories is standard and well-known (see [9, 10] for a review), although not completely satisfactory because of the known renormalizability issues. However, coordinate-dependent noncommutativity is non-trivial already at the classical level, because a differential calculus is needed, which ought to be compatible with the associated $\star$ product. In the context of derivation based differential calculus the issue is discussed in [11–15]. For an approach with twisted differential calculus see for example [16–18].

The framework adopted in the present research is the one proposed in [2], where the noncommutative gauge theory is constructed by requiring compatibility of the gauge algebra with space-time noncommutativity, namely infinitesimal gauge transformations should close the non-commutative algebra

$$ [\delta_f, \delta_g] = \delta_{-i\{f,g\} \star} , \quad f, g \in C^\infty(\mathcal{M}) , $$

with

$$ [f, g]_\star = f \star g - g \star f = i\{f, g\} + \ldots $$

the star-commutator of the gauge parameters $f$ and $g$. Moreover, the commutative limit is requested to be well defined and standard.

For a better understanding of the physical implications and mathematical structures, we work in a semiclassical approximation of spacetime non-commutativity. Therefore, star commutators will be replaced by imaginary unit times the Poisson brackets. In this approximation, the full non-commutative algebra of gauge parameters defines the Poisson gauge algebra, $[\delta_f, \delta_g] = \delta_{\{f,g\} \star}$, introduced in [19]. In the same work it was proposed an approach to the construction of (almost)-Poisson gauge transformations and the corresponding gauge algebra, based on the symplectic embedding of (almost)-Poisson structures.
In a subsequent work [20] Poisson gauge theory was further developed, as a dynamical field theoretical model with the Poisson gauge algebra representing its gauge symmetries.

As we shall explain in detail in section 2, the Poisson gauge theory is based on two essential ingredients. For a given Poisson bivector \( \Theta \) on \( \mathcal{M} \) one has to construct:

- the \( n \) by \( n \) matrix \( \gamma(A, x) \), whose components \( \gamma^a_{\phantom{a}b} \) satisfy the first master equation
  \[
  \gamma^j_{\phantom{j}i} \partial^i_A \gamma^k_{\phantom{k}l} - \gamma^k_{\phantom{k}i} \partial^i_A \gamma^j_{\phantom{j}l} + \Theta^{ij} \partial^i \gamma^k_{\phantom{k}l} - \gamma^a_{\phantom{a}b} \partial^i \Theta^{ij} = 0,
  \]
  \[(1.5)\]
  - and the \( n \) by \( n \) matrix \( \rho(A, x) \), which obeys the second master equation,
  \[
  \gamma^j_{\phantom{j}l} \partial^l_A \rho^i_{\phantom{i}a} + \rho^a_{\phantom{a}l} \partial^i_A \gamma^j_{\phantom{j}l} + \Theta^{il} \partial_i \rho^j_{\phantom{j}a} = 0.
  \]
  \[(1.6)\]

The former defines the deformed gauge transformations [2], whilst the latter allows to introduce a covariant derivative and a covariant field strength [20]. Hereafter we are using the notation \( \partial^i_A \equiv \frac{\partial}{\partial A^i(x)} \). In order to avoid confusions, we emphasise that in these equations the partial derivatives over coordinates act on the explicit dependence on \( x \) only, whilst \( A(x) \) is considered as an independent variable.

The solution of the first master equation has been constructed for an arbitrary Poisson bivector \( \Theta \), which is linear in \( x \) [19], in terms of a single matrix-valued function. From now on we address this solution as the “universal” one. In the same work the deformed gauge transformation was expressed via symplectic geometric quantities, and the role of the matrix \( \gamma \) in this geometric construction was clarified. However, the role of the matrix \( \rho \) in the geometric formalism was unclear. Moreover, the universal solution of the second master equation for the matrix \( \rho \), in the same spirit as the one for \( \gamma \) [19], was also missing.

The present project fills both the gaps. First, we interpret the deformed field strength in the symplectic geometric entries and clarify the role of the matrix \( \rho \) within this geometric construction. Second, we construct the matrix \( \rho \) for an arbitrary Poisson bivector \( \Theta \), which is linear in \( x \), establishing that

\[
\rho^{-1} = \gamma - i \hat{A}.
\]

(1.7)

In this formula \( \gamma \) is given by the universal solution mentioned above, and \( \hat{A} \) is the \( n \) by \( n \) matrix, constructed out of the gauge field \( A \) and the structure constants \( f^a_{\phantom{a}b} = \partial_c \Theta^{ab} \), whose components, by definition, read

\[
[\hat{A}]^b_c = -if^a_{\phantom{a}b} A_a,
\]

(1.8)

see section 4 for details. This universal solution for \( \rho \), accompanied by the universal solution for \( \gamma \), allows to construct the dynamical equations of motion for any Poisson bivector, linear in \( x \). As we shall see in detail, there are other solutions of the master equations, i.e. for a given Poisson bivector \( \Theta \) one can build different Poisson gauge theories. In the present article we will discuss in more details the arbitrariness in the definition of the Poisson gauge theory and provide the corresponding Seiberg-Witten maps in some particular examples, confronting the solutions, obtained in this paper with the ones, known before.

The paper is organized as follows. In sections 2–3 we review in detail the formulation of Poisson gauge theory by clarifying its geometric content and discussing Noether identities.
and conserved currents. The main novelties of section 2 are the expression for the deformed field strength and gauge covariant derivative in terms of symplectic geometric structures, together with the role of the matrix \( \rho \) within the geometric construction. Starting from section 4 we concentrate on linear noncommutative structures. In particular we obtain a universal solution for the matrix \( \rho \), eq. (1.8), which enters the definition of the covariant derivatives and the field strength. In sections 5 and 6 we illustrate our general findings in two noteworthy cases, namely the \( \mathfrak{su}(2) \)-like non-commutativity and the \( \kappa \)-Minkowski space. For the latter we obtain new solutions for the matrices \( \gamma \) and \( \rho \), different from those previously obtained in [21]. The reason for that is non-uniqueness of the solution of corresponding master equations. Section 7 discusses this arbitrariness in terms of Seiberg-Witten maps and exhibits explicit maps for the linear cases considered. We conclude with a discussion section and an appendix.

2 Poisson gauge theory

In this section we will describe the main points of the construction of the Poisson gauge theory based on the symplectic embedding of Poisson manifolds. The main emphasis will be on the geometric meaning of all appearing objects. First we review the known results, regarding the gauge transformations and the matrix \( \gamma \). After that we discuss the deformed field strength and the deformed covariant derivatives. In particular, we present new expressions of these objects in terms of geometric structures and we outline the role of the matrix \( \rho \) in the symplectic geometric construction.

2.1 Gauge transformations

The Poisson gauge transformations \( \delta_f A \) should satisfy the following two conditions: they should close the gauge algebra,

\[
[\delta_f, \delta_g] A = \delta_{\{f,g\}} A,
\]

and reproduce the standard \( \text{U}(1) \) gauge transformations in the classical limit,

\[
\lim_{\Theta \to 0} \delta_f A_a = \partial_a f.
\]

If \( \Theta^{ij} \) is constant, one may easily see that the expression,

\[
\delta^{\text{con}}_f A = df + \{A, f\}_{\text{can}},
\]

satisfies (2.1). However, for non-constant \( \Theta^{ij} \) the standard Leibniz rule with respect to the partial derivative is violated, \( \partial_a \{f, g\} \neq \{\partial_a f, g\} + \{f, \partial_a g\} \), therefore the same expression will not close the algebra (2.1) anymore, it being

\[
[\delta_f, \delta_g] A = d\{f, g\} + \{A, \{f, g\}\} - d\Theta^{ij}(x) \partial_i f \partial_j g.
\]

To overcome this difficulty one has to modify the expression for the gauge transformations (2.3) introducing corrections which compensate the last unwanted term in (2.4). A suitable modification has the following form [2]:

\[
\delta_f A_a = \gamma^\alpha_a(A) \partial_x f(x) + \{A_a(x), f(x)\},
\]
where the matrix $\gamma$ satisfies the first master equation (1.5). Such a construction has an elegant geometric interpretation in terms of symplectic embeddings and constraints, described below. The idea of symplectic embeddings of Poisson manifolds is quite general and widely used in many different contexts. It may be traced back to the so called symplectic realizations introduced by Weinstein [22] and further developed in [23]. The approach followed here is due to [24].

Both the gauge transformation (2.5) and the first master equation (1.5) can be obtained in a simple way, considering an extended space, by means of symplectic embedding techniques [19]. The formalism is essentially based on two ingredients. The first key ingredient of the construction is the symplectic embedding itself: to each coordinate $x^i$ of the initial space-time $\mathcal{M}$ one associates a conjugate variable $p_i$, in such a way that the corresponding extended Poisson brackets on $T^*\mathcal{M}$,

$$\{x^i, x^j\} = \Theta^{ij}(x), \quad \{x^i, p_j\} = \gamma^i_j(x, p), \quad \{p_i, p_j\} = 0,$$

(2.6)
satisfy the Jacobi identity, under the condition that the matrix $\gamma^i_j(x, p)$ be non-degenerate. For constant $\Theta^{ij}$ one finds $\gamma^i_j(x, p) = \delta^i_j$, so that $\{f(x), p_i\} = \partial_i f(x)$, namely the Poisson bracket with the auxiliary variable $p_i$ is just a partial derivative of $f$. For $\Theta^{ij}(x)$ not constant the expression for $\gamma^i_j(x, p)$ is more complicated. The Jacobi identity for the algebra (2.6) implies the partial differential equation [24],

$$\gamma^i_j \partial_p \gamma^k_a - \delta_a^i \partial_p \gamma^k_a + \Theta^{lm} \partial_m \gamma^k_a - \Theta^{km} \partial_m \gamma^l_a - \gamma^m_a \partial_m \Theta^{lk} = 0$$

(2.7)

with $\partial_m = \partial/\partial x^m$ and $\partial_p = \partial/\partial p_b$. The second key ingredient of the construction is the set of constraints

$$\Phi_a := p_a - A_a(x), \quad a = 1, \ldots, n \quad (2.8)$$

which allows to get rid of the auxiliary variables $p$. Imposing this constraint on (2.7) we get exactly eq. (1.5). In other words, the first master equation is simply the Jacobi identity in the extended space, obtained via the symplectic embedding.

In order to understand how this construction allows for the generalization (2.5) of gauge transformations in presence of a non-trivial Poisson bracket in space-time, let us review the procedure, following [25]. We first consider the standard setting of $U(1)$ gauge theory, with $\Theta = 0$. Then, the cotangent bundle $T^*\mathcal{M}$ is endowed with the canonical symplectic form $\omega_0$ which locally reads $dp_i \wedge dx^i$. The gauge field $A$ is a local one-form on $\mathcal{M}$. It is therefore associated with a local section of the cotangent space $T^*\mathcal{M}$, $s_A : \mathcal{U} \to T^*\mathcal{U}$, through a local trivialisation, $\psi^{-1}_U(s_A(x)) = (x, A(x))$, where $\mathcal{U}$ is a local chart on $\mathcal{M}$. Let

$$\xi_A = \lambda_0 - \pi^* A$$

(2.9)

be a one-form on $T^*\mathcal{U}$ with $\lambda_0$ the Liouville form (locally equal to $p_i dx^i$) and $\pi : T^*\mathcal{M} \to \mathcal{M}$ the usual projection map. The one-form $\xi_A$ vanishes locally, through the pullback $s_A^*$

$$s_A^*(\xi_A) = 0$$

(2.10)

because $s_A^*(\lambda_0) = A = (\pi \circ s_A)^*(A)$. This means that $\xi_A$ vanishes exactly on $\text{im}(s_A) \subset T^*\mathcal{U}$. Therefore the latter is identified by the constraint (2.10). This amounts to fix $p,$
which is the fibre coordinate at \(x\), to its value \(A(x)\) identified by the section \(s_A\), i.e. the relation (2.10) corresponds to the constraint (2.8), rewritten in the language of differential geometry. Then, the infinitesimal gauge transformation of the gauge potential \(A\), with gauge parameter \(f\), may be defined in terms of the canonical Poisson bracket \(\omega^{-1}\) as follows

\[
\delta_f A_n(x) = s_A^*\{\pi^* f, \xi_{An}\}_{\omega^{-1}} = \frac{\partial f}{\partial x^m} \frac{\partial \Phi_n}{\partial p_m} = \partial_n f ,
\]

that is to say, loosely speaking, one first performs the Poisson bracket in \(T^*U\), then goes to its local form through \(s_A\), recovering the standard infinitesimal gauge variation of the potential.

The complicated procedure described above, certainly redundant in the canonical case \(\Theta = 0\), is extremely useful and constructive for the case \(\Theta \neq 0\). One first performs a symplectic embedding of \((M, \Theta)\) as described above, with symplectic one-form \(\omega\) given by the inverse of the non-degenerate Poisson bracket (2.6), with Poisson tensor

\[
\Pi(x,p) = \Theta_{rs} \partial_i \partial_j + \gamma_{rs}(x,A) \partial_i \partial^j p ,
\]

while the image of \(U \subset M\) through the local section \(s_A\) is still defined by the constraint (2.9). Then, the infinitesimal gauge transformation of the gauge potential is formally defined in the same way as in the previous case, (2.11), except for the fact that the canonical Poisson bracket is to be replaced by the Poisson bracket (2.6). Therefore we find

\[
\delta_f A_a := s_A^*\{\pi^* f, \xi_{Aa}\}_{\Pi} = \Theta_{rs} \frac{\partial f}{\partial x^r} \frac{\partial \xi_{Aa}}{\partial x^s} + \gamma_{rs}(x,A) \frac{\partial f}{\partial x^r} \frac{\partial \xi_{Aa}}{\partial p_s} = \{A_a, f\}_\Theta + \gamma_{rs}(x,A) \frac{\partial f}{\partial x^r} ,
\]

what is nothing but eq. (2.5). The result can be restated in a simpler language, by replacing the constraint (2.10) with its local form (2.8)

\[
\delta_f A_a := \{f, \Phi_a\}_{\Phi_a = 0} = \{A_a(x), f(x)\} + \gamma_{rs}(A) \partial_r f(x) .
\]

Summarising, we see that the nontrivial gauge transformation (2.5) simply corresponds to the Poisson bracket (2.14) of the gauge parameter with the constraint (2.8) in the extended space, obtained via a symplectic embedding. The role of the matrix \(\gamma\) is also clear: it defines the symplectic embedding (2.6).

### 2.2 Field strength and covariant derivative

Now let us proceed to the definition of the field strength. For constant Poisson bracket \(\Theta\) one may simply set,

\[
F_{ab}^{\text{can}} := \{p_a - A_a(x), p_b - A_b(x)\} = \partial_a A_b - \partial_b A_a + \{A_a, A_b\}_{\text{can}} .
\]

This quantity transforms covariantly under the gauge transformation (2.3), namely, \(\delta_f^{\text{can}} F_{ab}^{\text{can}} = \{F_{ab}^{\text{can}}, f\}_{\text{can}}\), and reproduces the standard U(1) field strength in the commutative limit, with \(\lim_{\Theta \to 0} F_{ab}^{\text{can}} = \partial_a A_b - \partial_b A_a\). So, following the logic of the symplectic
embedding it would be reasonable to test the same structure for coordinate-dependent Poisson brackets, by posing

$$F_{ab} := \{ \Phi_a, \Phi_b \}_\Phi = 0 = \gamma_a^i(A) \partial_i A_b - \gamma_b^i(A) \partial_i A_a + \{ A_a(x), A_b(x) \}.$$  \hspace{1cm} (2.16)

However, by checking its behaviour under Poisson gauge transformation (2.14) we find (see details in appendix A)

$$\delta_f F_{ab} = \{ F_{ab}, f \} + \left( \partial_c^p \{ f, p_a \} \right)_{f=0} F_{cb} - \left( \partial_c^p \{ f, p_b \} \right)_{f=0} F_{ca}.$$ \hspace{1cm} (2.17)

The first term is exactly what we need for the gauge covariance condition, but the other two terms are undesirable.

Interestingly, a solution may be found by performing a transformation in the basis of constraints

$$\Phi_a \to \Phi'_a := \rho_m^a(A) \Phi_m ,$$ \hspace{1cm} (2.18)

with $\rho_m^a(A)$ a non-degenerate matrix to be determined by the covariance request. The non-degeneracy ensures that

$$\Phi'_a = 0, \quad \Leftrightarrow \quad \Phi_a = 0 .$$ \hspace{1cm} (2.19)

The field strength is thus defined by means of the new basis of constraints according to

$$\mathcal{F}_{ab} := \{ \Phi'_a, \Phi'_b \}_\Phi = 0 = \rho_{ma}^m(A) \rho_{ba}^p(A) F_{mn} .$$ \hspace{1cm} (2.20)

By computing its gauge variation we find

$$\delta_f \mathcal{F}_{ab} = \delta_f (\rho_{ma}^m(A)) \rho_{ba}^p(A) F_{mn} + \rho_{ma}^m(A) \delta_f (\rho_{ba}^p(A)) F_{mn} + \rho_{ma}^m(A) \rho_{ba}^p(A) \delta_f F_{mn} .$$ \hspace{1cm} (2.21)

Observing that,

$$\delta_f \rho_{ma}^m(A) = \partial_c^p \rho_{ma}^m(A) \{ f, p_b \} \{ f, \Phi_b \}_\Phi = 0 = \{ f, \rho_{ma}^m(p) - \rho_{ma}^m(A) \} \Phi = 0 ,$$ \hspace{1cm} (2.22)

and using (2.17) we obtain

$$\delta_f \mathcal{F}_{ab} = \{ \mathcal{F}_{ab}, f \} + \left[ \{ f, \rho_{ma}^m(p) \} + \rho_{ma}^m(p) \partial_a^p \{ f, p_c \} \right]_{f=0} \rho_{ba}^p(A) F_{mn} + \rho_{ma}^m(A) \left[ \{ f, \rho_{ba}^p(p) \} + \rho_{ba}^p(p) \partial^m_c \{ f, p_c \} \right]_{f=0} F_{mn} .$$

We thus conclude that the field strength (2.20) transforms covariantly,

$$\delta_f \mathcal{F}_{ab} = \{ \mathcal{F}_{ab}, f \} ,$$ \hspace{1cm} (2.23)

if $\rho_{ma}^m(x, p)$ satisfies the equation

$$\{ f(x), \rho_{ma}^m(x, p) \} + \rho_{ma}^m(x, p) \partial_a^m \{ f(x), p_b \} = 0 , \quad \forall f(x) .$$ \hspace{1cm} (2.24)

\footnote{Notice that the field strength was already defined in previous works as a deformation of the classical one; the two definitions yield exactly the same expression, the advantage being here in the geometric interpretation that appears to be more natural.}
In local coordinates this yields

\[ \gamma^j_b \partial^b p^j + \rho^b_a \partial^b \gamma^j_a + \Theta^b \partial_b \rho^j_a = 0. \]  

(2.25)

Imposing the constraint (2.8), i.e. setting \( \rho^a_A(x,A(x)) = \rho^a_A(x,p) \Phi=0 \) we thus recover the second master equation (1.6).

Let us now discuss the covariant derivative. To this, let \( \psi \) be a field which transforms upon the deformed gauge transformations according to the rule

\[ \delta_f \psi := \{ f, \psi \}. \]  

(2.26)

Using the new basis of constraints we can define the covariant derivative \( D \psi \) in the following way

\[ D_a \psi := \{ \psi, \Phi^a \}_\Phi=0 \]  

(2.27)

which explicitly yields

\[ D_a \psi = \rho^m_a (A) (\gamma^\ell_m(A) \partial_\ell \psi + \{ A_m, \psi \}). \]  

(2.28)

Notice that this result was already found in [20] by imposing covariance under gauge transformation, but the interpretation in terms of the new constraint was missing. By direct computation it can be checked [20] that it transforms correctly, namely

\[ \delta_f (D_a \psi) = \{ D_a \psi, f \}, \]  

(2.29)

and reproduces the commutative limit

\[ \lim_{\Theta \to 0} D_a \psi = \partial_a \psi. \]  

(2.30)

Summarising, we see that the deformed field strength and the deformed covariant derivative are respectively defined through the Poisson brackets (2.20) and (2.27) of the “rotated” constraints (2.18) with themselves and with the gauge parameter. The role of the matrix \( \rho \) is clear as well: it performs a redefinition of the constraints, in the extended space, obtained via the symplectic embedding.

As a final remark to this section, one may ask what if we start all over and replace the transformation of gauge potentials (2.14) with an analogous definition in terms of the newly defined constraints (2.18),

\[ \delta_f' A^a := \{ f, \Phi^a \}_\Phi'=0 = \rho^m_a (A) \delta_f A_m. \]  

(2.31)

It can be checked that the latter does not close the desired gauge algebra, yielding

\[ \left[ \delta_f', \delta_g' \right] \neq \delta_f' \delta_g'. \]  

(2.32)

So, as matter of fact, there are two sets of constraints, one, given by eq. (2.8), which is needed for the definition of Poisson gauge transformations (2.14), the other, represented by
eq. (2.18), which allows for the definition of the field strength and the covariant derivative, with the desired covariance property (2.23).

As we have seen in this section, the key ingredients of the Poisson gauge theory, viz. the deformed gauge transformation, the deformed field strength, and the deformed covariant derivative, are completely determined in terms of the matrices $\gamma^b_i(x,A)$ and $\rho^i_b(x,A)$. In the next section we will show how to put these objects together in order to construct the dynamical equations of motion, which exhibit the deformed version of the first pair of the Maxwell’s equations. The corresponding Lagrangian formulation will allow us to obtain new results for the Noether identities.

We will also consider the deformed version of the second pair of the Maxwell’s equations. The original derivation, presented in [20], was actually based on a brute-force deformation of the corresponding commutative counterpart without clear connection to the symplectic embeddings. The new derivation, instead, being more elegant, exploits the symplectic geometric construction, discussed above. We start from the latter.

3 Maxwell-Poisson equations

The two Maxwell equations which correspond to gauge constraints, namely those summarised by the Bianchi identity, descend from the Jacobi identity for the modified constraints (2.18),

$$\{\Phi'_a, \{\Phi'_b, \Phi'_c}\} + \text{cycl}(abc) = 0 .$$

Indeed, using the identity (see (A.2))

$$\{F(x,p), G(x,p)\}_{\Phi=0} - \{F(x,p), G(x,A(x))\}_{\Phi=0} = \left(\partial^m_p G(x,p)\right)_{\Phi=0} \{F(x,p), \Phi_m\}_{\Phi=0} ,$$

and cyclic permutations one finds,

$$\{\Phi'_a, \{\Phi'_b, \Phi'_c\}\}_{\Phi=0} + \text{cycl}(abc) = D_a (F_{bc}) + \rho^i_b (A) \left[\{\Phi, \rho^i_c (A)\}_{\Phi=0} \rho^j_c (A) \{\Phi, \rho^k_c (A)\}_{\Phi=0}\right] \{\Phi_j, \Phi_k\}_{\Phi=0}$$

$$+ \rho^i_b (A) \rho^j_c (A) \rho^k_e (A) \left(\partial^m_p \{\Phi_j, \Phi_k\}\right)_{\Phi=0} \{\Phi_m, \Phi_i\}_{\Phi=0} + \text{cycl}(abc) .$$

By explicitly computing the Poisson brackets of constraints (see [20], section 5.1 for details) one arrives at

$$D_a (F_{bc}) - F_{ad} B_{b}^{de} F_{ec} - (K_{ab} e - K_{ba} e) F_{ec} + \text{cycl}(abc) = 0 ,$$

where,

$$B_{b}^{de} (A) = \left(\rho^{-1}\right)^d_j \left(\partial^m_A \rho^i_b (A) - \partial^m_A \rho^i_j (A)\right) \left(\rho^{-1}\right)^e_m ,$$

$$K_{ab} e (A) = \rho^i_a (A) \gamma^m_i (A) \left(\partial_m \rho^b_0 (x,p)\right)_{\Phi=0} \left(\rho^{-1}\right)^e_j .$$

Both $B$ and $K$ tend to zero in the classical limit $\Theta \to 0$, so that eq. (3.4) reproduces the correct classical result $\partial_a F_{bc} + \text{cycl}(abc) = 0$. Moreover, the first term of the identity is
gauge covariant by construction, implying that the whole expression is gauge covariant (which could be however checked by direct inspection).

Now we turn to the gauge covariant deformation of the remaining Maxwell equations, namely those with true dynamical content, $\partial_a F_0^{ab} = 0$. Taking into account (2.23) and (2.29) the natural candidate reads,

$$\mathcal{E}_N^b := \mathcal{D}_a F^{ab} = 0,$$

where the subscript $N$ stands for "natural". This quantity transforms covariantly, $\delta f \mathcal{E}_N^b = \{ \mathcal{E}_N^b, f \}$, and reproduces the dynamical Maxwell equations in the classical limit, $\mathcal{E}_N^b \rightarrow \partial_a F_0^{ab}$ for $\Theta \rightarrow 0$.

An alternative way of obtaining the deformed equations of motion was proposed in [2], starting from an action principle. Basically the idea is the following: having in hands the gauge covariant Poisson field strength (2.20) there is a natural gauge covariant deformation of the standard Lagrangian, which reads

$$L_g = -\frac{1}{4} F^{ab} F_{ab}, \quad \text{with} \quad \delta f L_g = \{ L_g, f \}. \quad (3.8)$$

Then, introducing an appropriate measure $\mu(x)$, such that for any two Schwartz functions $f$ and $g$ the following holds

$$\int d^n x \mu(x) \{ f, g \} = 0 \quad \Leftrightarrow \quad \partial_i \left( \mu(x) \Theta^{lk}(x) \right) = 0, \quad (3.9)$$

one constructs the gauge invariant action,

$$S_g = \int d^n x \mu(x) L_g, \quad \text{such that}, \quad \delta f S_g = 0. \quad (3.10)$$

The corresponding Euler-Lagrange equations,

$$\mathcal{E}_c^{b EL} := \frac{\delta S_g}{\delta A_c} = 0,$$

are gauge covariant by construction. Without going into the tedious calculations which can be found in [20] we write here,

$$\mathcal{E}_c^{b EL} = \rho_b^c \left[ \mu \mathcal{D}_a \left( F^{ac} \right) + \frac{1}{2} F^{bc} B_b^{de} F_{de} - \mu F^{bc} B_b^{ce} F_{de} + \left( \rho^{-1} \right)_k^c F^{ab} \partial_i \left( \mu \rho^l_c \rho^k_b \gamma^i_l \right) \right]. \quad (3.11)$$

It is worth mentioning here that for some specific choices of the space-time Poisson structure the equations of motion constructed according to (3.7) and (3.12) are equivalent. In particular, for the $su(2)$-like Poisson structure, $\Theta^{ab}(x) = 2 \alpha \varepsilon^{abc} x^c$, the integration measure is constant $\mu(x) = 1$ and the additional terms in (3.12) vanish in such a way that,\(^2\)

$$\mathcal{E}_c^{b EL} = \rho_b^a \mathcal{E}_c^{a N}. \quad (3.13)$$

\(^2\)In this particular situation the matrix $\rho_b^a(A)$ plays the role of a Lagrangian multiplier in the sense of non-Lagrangian systems [27]. The original equations of motion, $\mathcal{E}_N^b = 0$, are non-Lagrangian, however the multiplication by the non-degenerate matrix $\rho_b^a(A)$ transforms them into an equivalent set of Euler-Lagrange equations, that is (3.11), for the action (3.10).
On the other hand, for the \( \kappa \)-Minkowski non-commutativity the integration measure \( \mu(x) \) is non-trivial [21, 26], so the relation (3.13) does not hold. In sections 5 and 6 we will discuss these two cases in more detail.

The advantage of working with the action principle formalism is that in a reasonably simple way we may introduce sources and derive the corresponding conservation equations, as well as the Noether identities for the original equations of motion. Following the standard approach we introduce the current \( j_{a}(x) \) adding the term,

\[
S_{\text{int}} = - \int d^{n}x \mu(x) j^{a} A_{a}, \tag{3.14}
\]

to the action (3.10). The resulting Euler-Lagrange equations become,

\[
E_{a}^{\text{EL}} = j^{a}. \tag{3.15}
\]

The interaction term should be gauge invariant, i.e.,

\[
\delta_{f} S_{\text{int}} = - \int d^{n}x \mu(x) j^{a} \left( \gamma_{a}^{l}(A) \partial l f(x) + \{ A_{a}(x), f(x) \} \right) \tag{3.16}
\]

\[
= \int d^{n}x \left[ \partial l \left( \mu j^{a} \gamma_{a}^{l}(A) \right) + \mu \{ A_{a}, j^{a} \} \right] f \equiv 0. \tag{3.17}
\]

Thus gauge invariance of the action implies the current conservation equation,

\[
\partial l \left( \mu j^{a} \gamma_{a}^{l}(A) \right) + \mu \{ A_{a}, j^{a} \} = 0. \tag{3.18}
\]

The same logic applied to the action (3.10) results in the Noether identities for the equations of motion,

\[
\partial l \left( \mu E_{a}^{\text{EL}} \gamma_{a}^{l}(A) \right) + \mu \{ A_{a}, E_{a}^{\text{EL}} \} = 0. \tag{3.19}
\]

In the classical limit \( \Theta \to 0 \) both (3.18) and (3.19) reduce to the standard relations, \( \partial_{a} j^{a} = 0 \) and \( \partial_{a} \partial_{b} F^{ab} = 0 \).

The current conservation condition (3.18), together with explicit solutions of the equations of motion (3.15) deserve further investigation. In order to understand their physical meaning, a starting approach could be to address the problem within specific space-time models. We plan to come back to this issue in future research.

Once the prescription for the construction of Poisson gauge theory is established, we are interested in the explicit solutions of the master equations (1.5) and (1.6). In previous works we have investigated several particular examples of linear non-commutative structures, such as, the kappa-Minkowski, the \( su(2) \) and the \( \lambda \)-Minkowski cases. In all these cases, the master equations were solved explicitly. As we announced in the Introduction, the universal solution of the first master equation is known for any linear non-commutativity [19]. In the subsequent section we shall obtain a universal solution of the second master equation.

## 4 Linear Poisson structures

Consider a linear Poisson structure

\[
\Theta^{ab} = f_{c}^{ab} x^{c}. \tag{4.1}
\]
The quantities $f_{c}^{ab}$, are structure constants, satisfying the Jacobi identity
\[ f_{i}^{kl} f_{j}^{ila} + f_{i}^{jl} f_{a}^{ikl} + f_{i}^{al} f_{j}^{k} = 0. \] (4.2)

In this case, one can consider solutions which do not depend on $x$ explicitly, so the master equations (2.7) and (2.25) reduce to
\[ \gamma_{i}^{j} \partial_{A} \gamma_{i}^{k} - \gamma_{i}^{k} \partial_{A} \gamma_{i}^{j} - \gamma_{i}^{j} f_{i}^{jk} = 0 \] (4.3)
and
\[ \gamma_{i}^{j} \partial_{A} \rho_{a}^{j} + \rho_{a}^{j} \partial_{A} \gamma_{i}^{j} = 0 \] (4.4)
respectively. Interestingly, for any $\Theta$ of the form (4.1), eq. (4.3) has been solved in terms of a universal matrix function [19]. The latter may be conveniently described by introducing the following notation
\[ \gamma(A) = G(\hat{A}), \quad \hat{A} \equiv A_{a} \epsilon^{a}, \] (4.5)
with the function $G$ given by,
\[ G(p) := \frac{ip}{2} + \frac{p}{2} \cot \frac{p}{2} = \sum_{n=0}^{\infty} (ip)^{n} B_{n}^{-}. \] (4.6)

Here $B_{n}^{-}$, $n \in \mathbb{Z}_{+}$ stand for the Bernoulli numbers (with the index “minus”), and $\epsilon^{a}$, $a = 0, \ldots, n - 1$ are $n \times n$ matrices so defined:
\[ [\epsilon^{a}]_{bc} = -if_{c}^{ab}. \] (4.7)

When all $\epsilon^{a}$ are linearly independent, they are the generators of the adjoint representation of an $n$-dimensional Lie algebra $g$, defined by the structure constants $f_{c}^{ab}$. In particular, the commutation relation
\[ [\epsilon^{a}, \epsilon^{b}] = if_{c}^{ab} \epsilon^{c}, \] (4.8)
is equivalent to the Jacobi identity (4.2). In order to solve the second master equation, eq. (4.4), we make the hypothesis that, similarly to the first master equation, there exists a universal function $F(p)$, such that the matrix function,
\[ \rho(A) = F(\hat{A}), \] (4.9)
is a solution for any Lie-Poisson bivector $\Theta$. In order to prove the existence of $F$, we start from the $su(2)$ case, elaborating on a known solution.

\[ ^{\mathbf{3}} \text{Ref. [19] operates with the function } \chi(u) = \sqrt{\frac{2}{u}} \cot \sqrt{\frac{2}{u}} - 1 \text{ and the notation } M = -\hat{A}^{2}. \]

\[ ^{\mathbf{4}} \text{More precisely, the linear Poisson bracket (4.1) becomes the Kirillov-Souriau-Konstant bracket which is defined on the dual of } g, g^{*} \equiv \mathbb{R}^{n}. \text{ Lie algebra type Poisson brackets are also referred to as Lie-Poisson brackets, not to be confused with Poisson-Lie brackets, which appear in the semi-classical limit of quantum groups.} \]
4.1 Candidate solution

For the $\mathfrak{su}(2)$-case,\footnote{Let us recall that a star product associated with this kind of noncommutativity was originally introduced in [28].} 
\[ f^{jk}_l = 2\alpha \varepsilon^{jk}_l, \quad \varepsilon^{jk}_l := \varepsilon^{ks} \delta_{sl}, \]  
(4.10)
a solution of the second master equation was found in [20]. It reads 
\[ \rho^i_a(A) = \delta^i_a + \alpha \varepsilon^{ik}_a A_k \zeta(z \alpha^2) - \alpha^2 (\delta^i_a z - A^i A_a) \tau(z \alpha^2), \quad z := A_i A^i, \quad A^j := \delta^{jk} A_k. \]  
(4.11)
Let us show that this solution can be put in the form (4.9) for a specific function $F$. Both form factors in (4.11) 
\[ \zeta(v) := -\frac{\sin \sqrt{v}}{\sqrt{v}} = \sum_{k=0}^{\infty} \zeta_k v^k, \quad \tau(v) = -\frac{1}{v} \left( \frac{\sin 2\sqrt{v}}{2\sqrt{v}} - 1 \right) = \sum_{k=0}^{\infty} \tau_k v^k, \]  
(4.12)
are analytic functions of the variable $v$. As for the third term in (4.11), introducing the projector 
\[ \hat{M}^j_i := \delta^i_j - \frac{A^j A_i}{z}, \quad \hat{M}^2 = \hat{M}, \]  
(4.13)
we obtain:\footnote{A similar computational trick has been used in [25].} 
\[ -\alpha^2 (\delta^i_a z - A^i A_a) \tau(z \alpha^2) = -\hat{M}^i_a \cdot (s \tau(s))|_{s=z\alpha^2} = -\sum_{k=0}^{\infty} \tau_k (z \alpha^2)^{k+1} \hat{M}^i_a |_{[\hat{M}^{k+1}]_a} \]  
(4.14)
\[ = -\sum_{k=0}^{\infty} \tau_k [(z \alpha^2 \hat{M})^{k+1}]_a - \left[ (s \tau(s))|_{s=z\alpha^2 \hat{M}} \right]_a^{\frac{i}{4}} =: T(\hat{A})_a^i, \]  
(4.15)
where we took into account the fact that\footnote{We remind that $\hat{A}^i_k = -if^{ik} A_j$, see eq. (4.5) and eq. (4.7).} 
\[ \hat{A}^2 = z \alpha^2 \hat{M}, \]  
and the form factor $T$ is given by 
\[ T(p) := \frac{\sin p}{p} - 1. \]  
(4.15)
Now we elaborate the second term of eq. (4.11). For the $\mathfrak{su}(2)$ case one can easily prove, e.g. by induction, the equalities 
\[ \hat{A}^i_j (4z \alpha^2)^n = \left[ \hat{A}^{2n+1} \right]_j^i, \quad \forall n \in \mathbb{Z}_+. \]  
(4.16)
Therefore for a generic even analytic function $Q(w) = \sum_{k=0}^{\infty} c_k w^{2k}$ the following relation holds: 
\[ \hat{A}^i_j Q(2\sqrt{|z|}) = \left[ \hat{A} Q(\hat{A}) \right]_j^i. \]  
(4.17)
On choosing $Q(w) = \zeta(w^2/4)$, we rewrite the second term of (4.11) as follows: 
\[ \alpha \varepsilon^{ik}_a A_k \zeta(z \alpha^2) = -\frac{1}{2} \hat{A}^i_a \cdot \zeta(z \alpha^2) = Z(\hat{A})^i_a, \]  
(4.18)
where
\[ U(p) = 2i \cdot \frac{(\sin \frac{p}{2})^2}{p}. \] (4.19)

Finally, noticing that
\[ 1 + T(p) = \int_0^1 d\beta \cos (\beta p), \quad U(p) = i \int_0^1 d\beta \sin (\beta p), \] (4.20)
we get a simple answer for the undetermined function in eq. (4.9).
\[ F(p) = 1 + T(p) + U(p) = \int_0^1 d\beta e^{ip\beta} = e^{ip} - 1. \] (4.21)

Now we demonstrate that, in the \( su(2) \)-case, there is a simple connection between \( \gamma \) and \( \rho \). Calculating the inverse of the expression (4.21), and comparing with eq. (4.6), one can easily establish the following relation between the functions \( F \) and \( G \),
\[ \frac{1}{F(p)} = \frac{ip}{e^{ip} - 1} = -\frac{ip}{2} + \frac{p}{2} \cot \frac{p}{2} = -ip + G(p), \] (4.22)
which implies an intriguing connection (1.8) between the matrices \( \gamma \) and \( \rho \), announced in the Introduction. Below we shall prove that this relation holds for an arbitrary linear Poisson structure, which, for \( \gamma \) given by (4.5), is equivalent to prove the conjecture (4.9).

4.2 Universality of the solution for \( \rho \)
Since \( \rho \) is non degenerate by hypothesis, instead of eq. (4.4) we can equivalently write, for \( \rho^{-1} \),
\[ \gamma_i^k \partial_A^j [\rho^{-1}]^l_j - [\rho^{-1}]^l_i \partial_A^k = 0. \] (4.23)
On using (1.8) and setting \( \gamma \) equal to the universal solution (4.5) we obtain
\[ \gamma_i^k \partial_A^j [\rho^{-1}]^l_j - [\rho^{-1}]^l_i \partial_A^k = \gamma_i^k \partial_A^j \gamma_i^k - \gamma_i^j \partial_A^k \gamma_i^k - i\gamma_i^k \partial_A^k \hat{A}^l + i\hat{A}^j \partial_A^k \gamma_i^k \]
\[ = -\{G(\hat{A})\}^k_i f^i j - [G(\hat{A})]^i j f^i k + i\hat{A}^j \partial_A^k [G(\hat{A})]^k_i \] (4.24)
where eq. (4.3) has been used. But this can be seen to be equal to zero, due to the fact that eq. (4.6) defines an analytic function at a point \( p = 0 \), together with the following result:

**Proposition 4.25.** For any arbitrary function \( S(p) \), which is analytic at \( p = 0 \), it holds:
\[ i\hat{A}^j \partial_A^k [S(\hat{A})]^k_i = [S(\hat{A})]^k_i f^i j + [S(\hat{A})]^i j f^i k. \] (4.26)

**Proof.** Expanding \( S \) in Taylor series, we see that eq. (4.26) is equivalent to the following relations:
\[ i\hat{A}^j \partial_A^k [\hat{A}^n]^k_i = [\hat{A}^n]^k_i f^i j + [\hat{A}^n]^i j f^i k, \quad \forall n \in \mathbb{Z}^+ \] (4.27)
which can be verified by induction. At \( n = 0 \) this equation becomes:
\[ 0 = \delta^k_i f^i j + \delta^i_j f^i k, \] (4.28)
what is obviously true, since \( f_{i}^{kj} = -f_{i}^{jk} \). Now, assuming (4.27) to be true for \( n - 1 \), namely that
\[
\hat{A}_{i}^{j} \partial_{A} [\hat{A}^{n-1}]_{s}^{k} = [\hat{A}^{n-1}]_{s}^{k} f_{i}^{ij} + [\hat{A}^{n-1}]_{s}^{k} f_{i}^{jk}, \quad n \in \mathbb{N}. \tag{4.29}
\]
we write the l.h.s. of (4.27) at \( n \) as follows
\[
i \hat{A}_{i}^{j} \partial_{A} [\hat{A}^{n}]_{i}^{k} = i \hat{A}_{i}^{j} \partial_{A} ([\hat{A}^{n-1}]_{s}^{k} \cdot \hat{A}_{i}^{s}) = i \hat{A}_{i}^{j} \partial_{A} [\hat{A}^{n-1}]_{s}^{k} \cdot \hat{A}_{i}^{s} + i \hat{A}_{i}^{j} [\hat{A}^{n-1}]_{s}^{k} \cdot \partial_{A} \hat{A}_{i}^{s} = [\hat{A}^{n-1}]_{s}^{k} f_{i}^{ij} \cdot \hat{A}_{i}^{s} + [\hat{A}^{n-1}]_{s}^{k} f_{i}^{jk} \cdot \hat{A}_{i}^{s} + \hat{A}_{i}^{l} f_{i}^{ls} [\hat{A}^{n-1}]_{s}^{k}, \tag{4.30}
\]
where the assumption (4.29) has been used. The second term in the last line of eq. (4.30) reproduces the second term of the r.h.s. of (4.27): whilst the remaining terms can be rewritten as follows
\[
[\hat{A}^{n-1}]_{s}^{k} f_{i}^{ij} \cdot \hat{A}_{i}^{s} + \hat{A}_{i}^{l} f_{i}^{ls} [\hat{A}^{n-1}]_{s}^{k} = [\hat{A}^{n-1}]_{s}^{k} (f_{i}^{sj} \hat{A}_{i}^{j} + f_{i}^{js} \hat{A}_{i}^{j}) = [\hat{A}^{n-1}]_{s}^{k} \hat{A}_{i}^{s} f_{i}^{ij} + [\hat{A}^{n-1}]_{s}^{k} (f_{i}^{sj} \hat{A}_{i}^{j} + f_{i}^{js} \hat{A}_{i}^{j} - \hat{A}_{i}^{s} f_{i}^{ij}) = [\hat{A}^{n-1}]_{s}^{k} f_{i}^{ij} - i A_{r} [\hat{A}^{n-1}]_{s}^{k} (f_{i}^{rs} f_{i}^{rj} + f_{i}^{is} f_{i}^{jr} + f_{i}^{js} f_{i}^{ir}) \]
\[
= [\hat{A}^{n}]_{i}^{k} f_{i}^{ij}, \tag{4.31}
\]
where use has been made of the relation \( \hat{A}_{i}^{s} f_{i}^{ij} = -i f_{i}^{jk} A_{j} \). This is the first term in the r.h.s. of (4.27), thus proving eq. (4.26) by induction.

Therefore, we can conclude that eq. (4.9), or equivalently (1.8), hold true for any linear Poisson bracket.

We have therefore found the universal solutions for both the matrices \( \gamma \) and \( \rho \) in terms of matrix-valued functions. In the following two sections we illustrate on a few nontrivial examples how these matrix-valued functions can be calculated explicitly.

5 A family of four-dimensional spaces with commutative time

As an application of the general formulae obtained in the previous section we consider here a family of four dimensional models with three-dimensional non commutativity, introduced in [25]. We use the Greek letters \( \mu, \nu, \ldots \), and the Latin letters \( a, b, c, \ldots \), to denote respectively the four-dimensional and the three-dimensional (i.e. the spatial) coordinates. The two-parameter family of Poisson structures to be considered reads:
\[
\Theta^{0 \mu} = 0 = \Theta^{\mu 0}, \quad \Theta^{jk} = -\lambda \varepsilon^{jks} \tilde{\beta}_{sl} x^{l}, \tag{5.1}
\]
where the \( 3 \times 3 \) matrix \( \tilde{\beta} \) is given by:
\[
\tilde{\beta} := \text{diag} \{1, 1, \beta\}, \quad \beta \in \mathbb{R}. \tag{5.2}
\]
At \( \beta = 0 \) we get the Poisson structure which corresponds to angular noncommutativity [29–35], at \( \beta = 1 \) the three-dimensional bivector \( \Theta^{jk} \) is nothing but the Poisson structure of
the \(su(2)\) case [28], and \(\beta = -1\) corresponds to the Lie algebra \(su(1,1)\). The structure constants read
\[
f_{\mu}^{\nu\rho} = -\lambda \delta_{\rho}^{\mu} \delta_{k}^{\nu} \delta_{l}^{\rho} \varepsilon^{jks} \tilde{\beta}_{jkl},
\]
(5.3)
where
\[
\delta_{\mu}^{\nu} := \delta_{\mu}^{\nu} - \delta_{\mu}^{0} \delta_{0}^{\nu},
\]
(5.4)
is a projector on the three-dimensional space. The four by four matrix \(\hat{A}\) is given by:
\[
\hat{A}_{0}^{0} = \hat{A}_{0}^{j} = \hat{A}_{j}^{0} = 0, \quad \hat{A}_{k}^{l} = i\lambda \varepsilon^{ks} \tilde{\beta}_{jkl}.
\]
(5.5)
On introducing the operator
\[
[\hat{M}]_{\mu}^{\nu} = \delta_{\mu}^{\nu} - \frac{A_{\mu} A_{\nu}}{Z_{\beta}}, \quad Z_{\beta} = A_{\mu} A_{\mu} = \beta \cdot (A_{1})^{2} + \beta \cdot (A_{2})^{2} + (A_{3})^{2},
\]
(5.6)
with
\[
A_{\mu} := \delta_{\mu}^{\nu} A_{\nu}, \quad A_{\mu}^{\nu} := \tilde{\beta}_{\mu} A_{\nu}, \quad \tilde{\beta} := \text{diag}\{0, \beta, \beta, 1\}.
\]
(5.7)
one can easily check that
\[
[\hat{A}]^{2n}_{\mu} = (\lambda^{2} Z_{\beta})^{n} \hat{M}_{\beta}, \quad n \in \mathbb{N},
\]
(5.8)
The matrix \(\hat{M}_{\beta}\) is a projector, i.e. \(\hat{M}_{\beta}^{2} = \hat{M}_{\beta}\), and hence,
\[
\hat{M}_{\beta}^{n} = \hat{M}_{\beta}, \quad \forall n \in \mathbb{N}.
\]
(5.9)
It is worth noting that at \(\beta = 0\), the quantities \(Z_{0}\) and \(\hat{M}_{0}\) coincide with \(z\) and \(\hat{M}\) introduced in section 4.1. Using the property (5.9), together with \(\hat{M} \cdot \hat{A} = \hat{A}\), one can easily check that
\[
\hat{A}^{2n} = (\lambda^{2} Z_{\beta})^{n} \hat{M}_{\beta}, \quad n \in \mathbb{N},
\]
\[
\hat{A}^{2n+1} = (\lambda^{2} Z_{\beta})^{n} \hat{A}, \quad n \in \mathbb{N}.
\]
(5.10)
Therefore, for any even function \(S_{\text{even}}(p)\), which is analytic at \(p = 0\), and which vanishes at \(p = 0\)
\[
S_{\text{even}}(\hat{A}) = S_{\text{even}}(\lambda \sqrt{Z_{\beta}}) \cdot \hat{M}_{\beta},
\]
(5.11)
and for any odd function \(S_{\text{odd}}(p)\), which is analytic at \(p = 0\),
\[
S_{\text{odd}}(\hat{A}) = \frac{S_{\text{odd}}(\lambda \sqrt{Z_{\beta}})}{\lambda \sqrt{Z_{\beta}}} \cdot \hat{A}.
\]
(5.12)
Thus, for \(S_{\text{odd}}(p) = ip/2\) and \(S_{\text{even}}(p) = (p/2) \cot(p/2) - 1\), as in eq. (4.6), we get
\[
G(\hat{A}) = \frac{i}{2} \cdot \hat{A} + 1 + \left( (\lambda \sqrt{Z_{\beta}}/2) \cdot \cot \left( \lambda \sqrt{Z_{\beta}}/2 \right) - 1 \right) \cdot \hat{M}_{\beta},
\]
(5.13)
therefore
\[
\gamma_{\mu}^{\nu}(A) = \delta_{\mu}^{\nu} - \frac{1}{2} f_{\mu}^{\nu \xi} : A_{\xi} + \frac{1}{Z_{\beta}} \cdot \left( (\lambda \sqrt{Z_{\beta}}/2) \cdot \cot \left( \lambda \sqrt{Z_{\beta}}/2 \right) - 1 \right) \cdot (\delta_{\mu}^{\nu} Z_{\beta} - A_{\mu}^{\nu} A_{\mu}),
\]
(5.14)
where, we remind, the structure constants are defined by eq. (5.3). At $\beta = 1$ and $\lambda = -2\alpha$ the three-dimensional part $\gamma^i_j$ reproduces the result for the $\mathfrak{su}(2)$-case, derived in [2]. The calculation, in a slightly different form has been presented in [25]. Applying eqs. (5.11) and (5.12) to $S_{\text{odd}}(p) = 2i (\sin^2 p/2)/p$ and $S_{\text{even}}(p) = \sin (p)/p-1$, and using the definition (4.21), we arrive at

$$ F(\hat{A}) = 1 + 2i \cdot \frac{\sin^2(\lambda \sqrt{Z\beta}/2)}{\lambda \sqrt{Z\beta}} \cdot \hat{A} + \left( \frac{\sin (\lambda \sqrt{Z\beta})}{\lambda \sqrt{Z\beta}} - 1 \right) \cdot \hat{M}_\beta, \quad (5.15) $$

therefore

$$ \rho^\mu_\nu(A) = \delta^\mu_\nu - \frac{2 \sin^2(\lambda \sqrt{Z\beta}/2)}{\lambda^2 Z\beta} f^\rho_\mu A_\rho + \frac{1}{Z\beta} \cdot \left( \frac{\sin (\lambda \sqrt{Z\beta})}{\lambda \sqrt{Z\beta}} - 1 \right) \cdot (\delta^\mu_\nu Z_\beta - A^\nu_\alpha A_\alpha). \quad (5.16) $$

At $\beta = 1$ and $\lambda = -2\alpha$ the three-dimensional part $\rho^i_j$ reproduces the known formula (4.11) for the $\mathfrak{su}(2)$-case, derived in [20].

### 5.1 Comments on the presence of commutative coordinates

The previous example has a peculiar feature - the presence of the fourth commutative coordinate, which we associate with time, whilst the noncommutativity is essentially three-dimensional. The $4 \times 4$ matrices $\rho^\mu_\nu$ and $\gamma^\mu_\nu$ are given by a simple generalisation of the corresponding three-dimensional results, since

$$ \gamma^{\mu0} = \gamma^{0\mu} = \delta^{\mu0}, \quad \rho^{\mu0} = \rho^{0\mu} = \delta^{\mu0}, \quad (5.17) $$

whilst the $3 \times 3$ $\gamma^i_j$ and $\rho^i_j$ solve the three-dimensional master equations. On the other hand, it has been shown in [25], that the deformed field strength $F_{\mu\nu}$ exhibits a highly nontrivial behaviour in the four-dimensional case, namely it is not a simple generalisation of the three dimensional one. In particular the components $F_{0j}$ are nonlinear in the gauge potential $A$, even in the simplest case of a spatially homogeneous situation, i.e. when $A$ does not depend on the spatial coordinates $x^j$.

Below we demonstrate that the simple behaviour (5.17) of the universal solution, indeed implies the nontrivial behaviour of $F_{0j}$, mentioned above. We have

$$ F_{ab} = (\rho^m_a \rho^m_b - \rho^m_b \rho^m_a)\gamma^l_m \partial_l A_n + \frac{1}{2} (\rho^m_a \rho^m_b - \rho^m_b \rho^m_a) \{A_m, A_n\} \quad (5.18) $$

therefore, for $A$ independent on spatial coordinates, we find

$$ F_{0j} = (\delta^m_0 \rho^m_j - \delta^m_j \rho^m_0)\gamma^l_m \partial_l A_n + \frac{1}{2} (\delta^m_0 \rho^m_j - \delta^m_j \rho^m_0) \{A_m, A_n\} $$

$$ = \delta^0_j \partial_l A_n - \gamma^l_j \partial l_0 A_n + \{A_0, A_j\} = \rho^0_j \partial_0 A_n, \quad (5.19) $$

namely, the electrical component of the field strength is non-trivially modified. Thus, by adding commutative coordinates, it is certainly true that $\gamma$ and $\rho$ are given by a trivial generalization of the corresponding lower dimensional results. But this is definitely an intermediate stage. The field strength, which is the true dynamical object, gets instead non-trivial contributions, already in the simple hypothesis of spatial homogeneity. These conclusions go beyond the example and in general will apply to any non-commutative spacetime with commutative directions.
Another important application of the results of section 4 is the $\kappa$-Minkowski case in $N$ dimensions [36–41], where the Poisson bivector is given by
\[ \Theta^{ij} = 2(\omega^i x^j - \omega^j x^i), \] (6.1)
where $\omega^i, i = 1, \ldots, N$ are deformation parameters. Substituting the outcome structure constants
\[ f_k^{aj} = 2(\omega^a \delta_j^k - \omega^j \delta_k^a), \] (6.2)
and taking into account eq. (4.5) and eq. (4.7), we obtain:
\[ \hat{A}_k^j = -i f_k^{aj} A_a = -2i (\omega \cdot A) \hat{P}_k^j, \] (6.3)
where, by definition,
\[ \omega \cdot A = \omega^j A_j, \] (6.4)
and
\[ \hat{P}_k^j = \delta_k^j - (\omega \cdot A)^{-1} \cdot \omega^j A_k. \] (6.5)
The matrix $\hat{P}$ is a projector, i.e. $\hat{P}^2 = \hat{P}$. Therefore, using eq. (4.5) and eq. (4.9), we can immediately calculate the matrices $\gamma$ and $\rho$. For this purpose, we represent the functions $G$ and $F$, defined by eq. (4.6) and eq. (4.21) as follows
\[ G(p) = 1 + \tilde{G}(p), \]
\[ F(p) = 1 + \tilde{F}(p), \] (6.6)
where the functions $\tilde{G}(p)$ and $\tilde{F}(p)$ are analytic and vanishing at $p = 0$. For any such function, say $S$, the following matrix identity can be easily checked:
\[ S(\lambda \hat{P}) = S(\lambda) \hat{P}, \] (6.7)
with $\lambda$ any complex number. Therefore,
\[ \gamma_j^i(A) = \delta_j^i + \tilde{G}(-2i (\omega \cdot A)) \hat{P}_j^i = \delta_j^i + ((\omega \cdot A) + (\omega \cdot A) \coth (\omega \cdot A) - 1) \hat{P}_j^i, \] (6.8)
and
\[ \rho_j^i(A) = \delta_j^i + \tilde{F}(-2i (\omega \cdot A)) \hat{P}_j^i = \delta_j^i + \frac{1}{2} (\omega \cdot A)^{-1} (e^{2(\omega \cdot A)} - 1 - 2(\omega \cdot A)) \hat{P}_j^i. \] (6.9)
Restoring the original notations, we get our final expressions for the matrices $\gamma$ and $\rho$ in the form:
\[ \gamma_j^i(A) = (\omega \cdot A) [1 + \coth (\omega \cdot A)] \delta_j^i + \frac{1 - (\omega \cdot A) - (\omega \cdot A) \coth (\omega \cdot A)}{\omega \cdot A} \cdot \omega^i A_j, \]
\[ \rho_j^i(A) = \frac{e^{2(\omega \cdot A)} - 1}{2(\omega \cdot A)} \delta_j^i + \frac{1 + 2(\omega \cdot A) - e^{2(\omega \cdot A)}}{2(\omega \cdot A)^2} \omega^i A_j. \] (6.10)
These results are different from the ones obtained in [20, 21]. In the next section we address the non-uniqueness of the construction. In particular, we will show that the new and the old expressions for $\gamma$ and $\rho$ are related via the Seiberg-Witten map.
7 Arbitrariness of the solutions and the Seiberg-Witten map

It has already been noticed that solutions of the master equations (1.5) and (1.6), defining the Poisson gauge field theoretical model, are not unique. It is easy to see that for any invertible field redefinition

\[ A \to \tilde{A}(A), \]  

the quantities

\[ \tilde{\gamma}^i_j(\tilde{A}) = \left( \gamma^i_k(A) \cdot \frac{\partial \tilde{A}_i}{\partial A_k} \right)_{A = A(\tilde{A})}, \]  

and

\[ \tilde{\rho}^i_a(\tilde{A}) = \left( \frac{\partial A_s}{\partial A_i} \cdot \rho^s_a(A) \right)_{A = A(\tilde{A})}, \]

are again solutions of the master equations. Therefore, one may construct another gauge transformation corresponding to \( \tilde{\gamma}^i_a, \)

\[ \delta_f \tilde{A}_a = \tilde{\gamma}^i_a(\tilde{A}) \partial_i f + \{ \tilde{A}_a, f \}, \]  

which will close the same gauge algebra (2.1). Upon the field redefinition (7.1) the gauge orbits of the original fields and the gauge orbits of the new fields are mapped onto each other. Indeed, the Seiberg-Witten condition [42],

\[ \tilde{A}(A + \delta_f A) = \tilde{A}(A) + \delta_f \tilde{A}(A), \]

is trivially satisfied up to the linear order in \( f. \) Therefore the invertible field redefinitions are nothing but the Seiberg-Witten maps.

As we have seen above, Poisson gauge models are actually based on symplectic embeddings. The ambiguity, discussed above, corresponds to the freedom in choosing different symplectic embeddings for the Poisson manifolds [19, 20]. Finally, we recall that, in the equivalent approach in terms of \( L_\infty \) bootstrap, it was shown in [43] that Seiberg-Witten maps for the Poisson gauge algebra under analysis correspond in that picture to \( L_\infty \)-quasi-isomorphisms which describe the arbitrariness in the definition of the related \( L_\infty \) algebra. In what follows, we derive the Seiberg-Witten map for some noteworthy cases with linear non-commutativity.

7.1 The \( \kappa \)-Minkowski case

One can check by a straightforward calculation that the solutions of the master equations for the \( \kappa \)-Minkowski non-commutativity, presented in [20, 21], viz

\[ \tilde{\gamma}^i_a(A) = \left[ \sqrt{1 + (\omega \cdot A)^2 + (\omega \cdot A)} \right] \delta^i_a - \omega^k A_a, \]

and

\[ \tilde{\rho}^i_a(A) = \left[ \sqrt{1 + (\omega \cdot A)^2 + (\omega \cdot A)} \right] \delta^i_a - \frac{1}{\sqrt{1 + (\omega \cdot A)^2 + (\omega \cdot A)}} \omega^k A_a, \]
can be obtained from the new ones (6.10), using the formulae (7.2) and (7.3). The new fields are defined in the following way
\[
\tilde{A}_a = \sinh(\omega \cdot A) \frac{\omega \cdot A}{\omega \cdot A}_a.
\]
(7.8)

Such a Seiberg-Witten map is invertible, and the inverse transformation reads:
\[
A_a = \arcsinh (\omega \cdot \tilde{A}) \frac{\omega \cdot \tilde{A}}{\omega \cdot \tilde{A}}_a.
\]
(7.9)

### 7.2 The su(2) case

As we mentioned above, setting \(\beta = 1\) and \(\lambda = -2\alpha\) in the three-dimensional component of eq. (5.14) and eq. (5.16), we get the “standard” solution of the master equations (2.7) and (2.25) for the su(2)-case:
\[
\gamma_k^i (A) = \delta_k^i - \alpha \varepsilon^{ul} A_l + \frac{1}{Z_1} (\alpha \sqrt{Z_1} \cot (\alpha \sqrt{Z_1}) - 1) (\delta_k^i Z_1 - A^i A_k)
\]
(7.10)
\[
\rho_a^i (A) = \delta_a^i - \alpha \varepsilon^{ik} A_k \frac{1}{Z_1} \sin^2 \sqrt{Z_1} + \frac{1}{Z_1} \left( \sin \left( \frac{2 \alpha \sqrt{Z_1}}{\alpha \sqrt{Z_1}} \right) - 1 \right) (\delta_a^i Z_1 - A^i A_a)
\]
which have been previously presented in [20] and [2]. We remind that the quantity \(Z_1\) is given by
\[
Z_1 := (A_1)^2 + (A_2)^2 + (A_3)^2,
\]
(7.11)
see eq. (5.6) at \(\beta = 1\).

An alternative solution of the first master equation (2.7) can be obtained using the results of [44] in the following form:
\[
\tilde{\gamma}_k^i (A) = \delta_k^i - \alpha \varepsilon^{ul} A_l + \alpha^2 A^i A_k.
\]
(7.12)
It is easily checked that the “standard” and the new solution are related via the field redefinition
\[
\tilde{A}_i (A) := A_i \cdot \frac{\tan (\alpha \sqrt{Z_1}) \alpha \sqrt{Z_1}}{\alpha \sqrt{Z_1}}
\]
(7.13)
according to the relation (7.2). The inverse field redefinition reads:
\[
A_j (\tilde{A}) = \frac{\arctan (\alpha \sqrt{Z_1})}{\alpha \sqrt{Z_1}},
\]
(7.14)
where we set by definition
\[
\tilde{Z}_1 := \left( \tilde{A}_1 \right)^2 + \left( \tilde{A}_2 \right)^2 + \left( \tilde{A}_3 \right)^2.
\]
(7.15)

Using these transformations together with the relation (7.3), one can easily construct the new solution for the second master equation (2.25)
\[
\tilde{\rho}_a^i (A) = \frac{1}{1 + \alpha^2 Z_1} \cdot (\delta_a^i - \alpha \varepsilon^{ik} A_k).
\]
(7.16)

---

\(^8\) Applying the universal formulas (4.5) and (4.9), (4.21), one gets exactly these matrices \(\gamma\) and \(\rho\).

\(^9\) Note that in [45] this map was interpreted as a coordinate transformation relating two different representation of a sphere \(S^3\).
7.3 The $\lambda$-Minkowski case

For the sake of simplicity we restrict ourselves to the three-dimensional $\lambda$-Minkowski case. Indeed, since the fourth coordinate is commutative, the four-dimensional generalisation of $\gamma$ and $\rho$ is trivial, see the discussion in section 5.1. Setting $\beta = 0$ in the three-dimensional parts of eq. (5.14) and eq. (5.16), we arrive at the following "standard" solutions of the master equations:

$$
\gamma(A) = \begin{pmatrix}
\frac{A_3\lambda}{2} \cot \left( \frac{A_3\lambda}{2} \right) & -\frac{A_3\lambda}{2} & 0 \\
\frac{A_3\lambda}{2} & \frac{A_3\lambda}{2} \cot \left( \frac{A_3\lambda}{2} \right) & 0 \\
-\cot \left( \frac{A_3\lambda}{2} \right) \lambda A_1 - \frac{2A_1}{\lambda} & -\cot \left( \frac{A_3\lambda}{2} \right) \lambda A_2 + \frac{2A_2}{\lambda} & 1
\end{pmatrix},
$$

(7.17)

and

$$
\rho(A) = \begin{pmatrix}
\frac{\sin(A_3\lambda)}{A_3\lambda} & -2\frac{\sin \left( \frac{A_3\lambda}{2} \right)^2}{A_3\lambda} & 0 \\
2\frac{\sin \left( \frac{A_3\lambda}{2} \right)^2}{A_3\lambda} & \frac{\sin(A_3\lambda)}{A_3\lambda} & 0 \\
-\frac{2A_2 \left( \sin \left( \frac{A_3\lambda}{2} \right) \right)^2}{A_3\lambda} + \left( 1 - \frac{\sin(A_3\lambda)}{A_3\lambda} \right) A_1 & \frac{2A_1 \left( \sin \left( \frac{A_3\lambda}{2} \right) \right)^2}{A_3\lambda} + \left( 1 - \frac{\sin(A_3\lambda)}{A_3\lambda} \right) A_2 & 1
\end{pmatrix}.
$$

(7.18)

The results of [44] suggest a new (much simpler!) solution for the first master equation, that is:

$$
\tilde{\gamma}(A) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-\theta A_2 & \theta A_1 & 1
\end{pmatrix}.
$$

(7.19)

One can show that the two gamma matrices are related through the relation (7.2), where the new fields are defined as follows:

$$
\begin{align*}
\tilde{A}_1 &= \frac{\sin(A_3\lambda)}{A_3\lambda} A_1 - \frac{2\sin^2 \left( \frac{A_3\lambda}{2} \right)}{A_3\lambda} A_2, \\
\tilde{A}_2 &= \frac{\sin(A_3\lambda)}{A_3\lambda} A_2 + \frac{2\sin^2 \left( \frac{A_3\lambda}{2} \right)}{A_3\lambda} A_1, \\
\tilde{A}_3 &= A_3.
\end{align*}
$$

(7.20)

The inverse transformation reads:

$$
\begin{align*}
A_1 &= \frac{\tilde{A}_3\lambda}{2} \cot \left( \frac{\tilde{A}_3\lambda}{2} \right) \tilde{A}_1 + \frac{\tilde{A}_3\lambda}{2} \tilde{A}_2, \\
A_2 &= \frac{\tilde{A}_3\lambda}{2} \cot \left( \frac{\tilde{A}_3\lambda}{2} \right) \tilde{A}_2 - \frac{\tilde{A}_3\lambda}{2} \tilde{A}_1, \\
A_3 &= \tilde{A}_3.
\end{align*}
$$

(7.21)
This Seiberg-Witten map allows to construct a new solution of the second master equation, which corresponds to $\tilde{\gamma}$, via the relation (7.3):

$$
\tilde{\rho}(A) = \begin{pmatrix}
\cos (A_3 \lambda) - \sin (A_3 \lambda) & 0 \\
\sin (A_3 \lambda) & \cos (A_3 \lambda) & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

(7.22)

Though these new solutions of the master equations do not obey the intriguing relation (1.8), they satisfy another interesting identity:

$$
\tilde{\rho} \tilde{\gamma} + \mathbb{1} = \tilde{\rho} + \tilde{\gamma}.
$$

(7.23)

8 Conclusions

As we explained in the review sections of this paper, Poisson gauge modelss are completely determined by the matrices $\gamma$ and $\rho$, which solve the two master equations. The new results of the present article are the following.

- We obtained the universal solution (1.8) of the second master equation, which is valid for any Poisson bivector, linear in coordinates. This solution for $\rho$, together with the previously known universal solution (4.5) for $\gamma$, allows to build the Poisson gauge theory completely. This is the most important result of our paper.

- The expressions for $\rho$ and $\gamma$, which come out from the universal solutions, do not in general coincide with the expressions that have been obtained previously in the literature for the same noncommutativity. However, the corresponding Poisson gauge theories are connected with each other through Seiberg-Witten maps. We have constructed these maps explicitly, see eq. (7.8), eq. (7.13) and (7.21).

- Using the Seiberg-Witten maps, we obtained the new simple expressions (7.16) and (7.22) for the matrix $\rho$ in the $su(2)$ and in the $\lambda$-Minkowski cases respectively. The matrices (7.19) and (7.22), related to the $\lambda$-Minkowski noncommutativity, provide the simplest nontrivial solutions of the master equations, which have been found so far.

- Another set of interesting results concerns the geometric interpretation of Poisson gauge models in terms of symplectic embeddings and constrains in the extended space. In eq. (2.20) and (2.27) we expressed the deformed field strength and the deformed gauge covariant derivatives through Poisson brackets of the “rotated” constrains in the extended space, obtained via the symplectic embedding. Moreover, we clarified the role of the matrix $\rho$ in this geometric approach: it performs non-linear transformations of the constrains in the extended space, see eq. (2.18). These results, accompanied by the previously known symplectic geometric interpretation of the matrix $\gamma$ and of the deformed gauge transformation, complete the geometric interpretation of Poisson gauge theory.

- Finally, we derived the deformed Noether identities (3.19).
A Useful formulae

Let us first calculate,

\[
\{ f(x), g(x), p_a \}_\Phi = 0 - \{ f(x), g(x), p_a \}_\Phi = 0 = \{ f(x), g(x), p_a \}_\Phi = 0
\]

(A.1)

\[
= \{ f(x), \gamma^a_b(x, p) \partial g(x) \}_\Phi = 0 - \{ f(x), \gamma^a_b(x, A(x)) \partial g(x) \}_\Phi = 0
\]

= \left( \partial^b_p \{ g(x), p_a \} \right)_\Phi = 0 \{ f(x), A_b(x) \}_\Phi = 0
\]

(A.2)

\[
= \left( \partial^b_p \{ g(x), p_a \} \right)_\Phi = 0 \{ f(x), \Phi_b \}_\Phi = 0.
\]

Observe that the Poisson bracket of two functions of coordinates only does not depend on \( p \)-variables, so that

\[
\{ f(x), g(x) \}_\Phi = 0 = \{ f(x), g(x) \}.
\]

(A.3)

The combination of (A.1) and (A.2) implies,

\[
\{ f(x), \{ g(x), \Phi_a \} \}_\Phi = 0 - \{ f(x), g(x), \Phi_a \}_\Phi = 0 = \left( \partial^d_p \{ g(x), \Phi_a \} \right)_\Phi = 0 \{ f(x), \Phi_d \}_\Phi = 0.
\]

(A.4)

One may also check that,

\[
\{ \{ f(x), \Phi_b \}, \Phi_a \}_\Phi = 0 - \{ \{ f(x), \Phi_b \}, \Phi_a \}_\Phi = 0 = \left( \partial^d_p \{ f(x), \Phi_b \} \right)_\Phi = 0 \{ \Phi_d, \Phi_a \}_\Phi = 0
\]

and,

\[
\{ \{ \Phi_c, \Phi_b \}, \Phi_a \}_\Phi = 0 - \{ \{ \Phi_c, \Phi_b \}, \Phi_a \}_\Phi = 0 = \left( \partial^d_p \{ \Phi_c, \Phi_b \} \right)_\Phi = 0 \{ \Phi_d, \Phi_a \}_\Phi = 0.
\]

(A.5)

Proof of eq. (2.17). We compute

\[
\delta_f F_{ab} = \left( \partial^c_p \{ \Phi_a, \Phi_b \} \right)_\Phi = 0 \{ f, \Phi_c \}_\Phi = 0 - \{ f, \{ \Phi_a, \Phi_b \}_\Phi = 0, \Phi_c \}_\Phi = 0 - \{ \Phi_a, \{ f, \Phi_b \}_\Phi = 0 \}_\Phi = 0
\]

(A.6)

Using eqs. (A.2)–(A.5), we rewrite the r.h.s. as,

\[
\text{r.h.s.} = \left( \partial^c_p \{ \Phi_a, \Phi_b \} \right)_\Phi = 0 \{ f, \Phi_c \}_\Phi = 0 - \{ f, \{ \Phi_a, \Phi_b \}_\Phi = 0, \Phi_c \}_\Phi = 0 - \{ \Phi_a, \{ f, \Phi_b \}_\Phi = 0 \}_\Phi = 0
\]

\[
= \left( \partial^c_p \{ f, \Phi_a \} \right)_\Phi = 0 \{ \Phi_c, \Phi_b \}_\Phi = 0 - \left( \partial^c_p \{ f, \Phi_b \} \right)_\Phi = 0 \{ \Phi_c, \Phi_a \}_\Phi = 0.
\]

(A.7)

Using in the first line the Jacobi identity and eqs. (A.2)–(A.5) one more time, we end up with

\[
\text{r.h.s.} = \{ \{ \Phi_a, \Phi_b \}_\Phi = 0, f \}_\Phi = 0 + \left( \partial^c_p \{ f, \Phi_a \} \right)_\Phi = 0 \{ \Phi_c, \Phi_b \}_\Phi = 0 - \left( \partial^c_p \{ f, \Phi_b \} \right)_\Phi = 0 \{ \Phi_c, \Phi_a \}_\Phi = 0
\]

We therefore have proved that,

\[
\delta_f F_{ab} = \{ F_{ab}, f \}_\Phi = 0 + \left( \partial^c_p \{ f, \Phi_a \} \right)_\Phi = 0 F_{cb} - \left( \partial^c_p \{ f, \Phi_b \} \right)_\Phi = 0 F_{ca}.
\]

(A.8)
Now remember that
\[ \partial_c \{ f, \Phi_a \} = \partial_c \{ f, p_a \}, \quad (A.9) \]
since \( \{ f(x), A_a(x) \} \) does not depend on \( p \). Also the fact that \( F_{ab} \) is a function of \( x \) only implies that the Poisson bracket of two functions of \( x \), \( \{ F_{ab}, f \} \) does not depend on \( p \)-variables, so \( \{ F_{ab}, f \}_{\Phi=0} = \{ F_{ab}, f \} \). By substituting this result in eq. (A.8) we have proved (2.17).

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