Extension of quantum information theory to curved spacetimes

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The representation of measurements by positive operator valued measures and the description of the most general state transformations by means of completely positive maps are two basic concepts of quantum information theory. These concepts can be trivially extended to field theories in curved spacetime if all the representations of canonical commutation or anticommutation relations are unitarily equivalent. We show that both concepts can be applied even when there is no such unitary equivalence.

I. INTRODUCTION

Quantum information theory has many faces and uses variety of techniques. Among them two concepts are of outstanding importance, as can be seen from the standard monographs [1-6]. The first concept is a generalization of the elementary representation of measurements by projection valued measures, associated with self-adjoint operators (this description is variously labeled as von Neumann measurement, standard measurement, or PVM). The most general description of a measurement is given by positive operator valued measures (POVM, or generalized measurement). Next, state transformations are given in terms of completely positive maps (superoperators), which include the measurement induced transformation. An extension of this formalism to curved spaces is the subject of this paper.

While many quantum information experiments are performed with photons, most discussions are restricted to non-relativistic quantum mechanics in general, and finite-dimensional Hilbert spaces in particular [1, 2]. Undoubtedly this is sufficient for conventional quantum information processing. Nevertheless, there is nothing in the mathematical structure of the theory that prevents its application to objects that are described by quantum field theories in flat or curved spaces. Both concepts (generalized measurements and state transformations) could and should be applied in this context. One of the initial stimuli to the investigations that led to quantum information theory was quantum optics [1, 2]. Moreover, C* and W* algebras are employed both in studies of information dynamics [3, 4] and in field theories [5, 6, 7].

Several aspects of the POVM formalism in relativistic quantum theory were recently discussed. It was applied to construct unsharp position observables for field theories in Minkowski space time [11, 12], that were investigated also from the algebraic point of view [13]. Causality properties of general state transformations and measurements were also investigated, [4, 15, 16].

In this paper we show that the POVM formalism and general state transformations can be successfully applied to quantum field theories in general curved spacetimes, even in the case when different Hilbert space representations of canonical commutation or anticommutation relations are unitarily inequivalent. First we present some essential facts about generalized measurements and state transformations. Next, we review how algebraic field theory deals with the extraction of unambiguous physical predictions from unitarily inequivalent representations. Finally we use this result to extend the basic notions of quantum information theory in general globally hyperbolic spacetime.

II. BASICS OF QUANTUM INFORMATION

We begin from the standard mathematical formalism of a measurement [1, 2]. A PVM description of quantum measurements, which is only a particular case of generalized measurements in quantum information theory, is still used in the monographs on quantum field theory [8, 9]. An observable is represented by a self-adjoint operator $A$ on a Hilbert space $\mathcal{H}$. Its expectation value on a state $\rho$ is given by $\text{tr} \rho A$. The spectral theorem [17] gives a correspondence between self-adjoint operators and projection valued measures (PVM):

$$ A = \int_{-\infty}^{\infty} x P(dx), $$

where $P(\cdot)$ is a PVM. The probability that a measurement of $A$ on the state $\rho$ will give a result in (the Borel set $X$ is

$$ p^A_\rho(X) = \text{tr}[\rho P(X)]. $$

This description is adequate for the prediction of energy levels, scattering cross sections and many other physical properties. On the other hand, joint measurements of conjugate observables, time-of-arrival measurements, position measurements in relativistic theory and other detection problems in information theory require more general formalism.

The existence of self-adjoint operators is not an essential part of the formalism. What is really needed is a method to construct a probability measure. Thus a generalized description of measurement by a normalized positive operator valued measure (POVM) is defined as follows. If $\Omega$ is a (locally compact) set with a $\sigma$-algebra

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F then a POVM is a map $E : \mathcal{F} \to \mathcal{L}^+(\mathcal{H})$ (a map from the set of possible results to the set of positive bounded operators on the Hilbert space) such that

$$E(X) \geq E(\emptyset) = 0,$$

for all $X \in \mathcal{F}$. For a countable collection of disjoint sets in $\mathcal{F}$, we have

$$E \left( \bigcup_{n=1}^{\infty} X_n \right) = \sum_{n=1}^{\infty} E(X_n),$$

and

$$E(\Omega) = I,$$

while the probability is calculated according to

$$p^E_\rho(X) = \text{tr}[\rho E(X)].$$

Therefore POVMs are extensively involved in information problems in finite-dimensional Hilbert spaces. In relativistic quantum theory they are indispensable for the construction of reasonable position observables. Leaving aside the issue of actual hardware realization of a given POVM we consider the following important result (Neumark’s theorem).

Let $E$ be a POVM on a Borel $\sigma$-algebra of a compact metrisable space $\Omega$, $E(X) \in \mathcal{L}(\mathcal{H})$. Then, there is a PVM $P(\cdot)$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that if $P$ is the projection of $\mathcal{K}$ onto $\mathcal{H}$, then $E(X)$ is the restriction of $PP(E)P$ to $\mathcal{K}$ for all Borel sets $X \subseteq \Omega$. For a separable $\mathcal{H}$ the space $\mathcal{K}$ can be taken separable.

Kraus’s ‘second representation theorem’ is based on this result. It guarantees that any POVM can be realized by adjoining auxiliary system and performing unitary transformations and standard von Neumann measurements.

The most general state transformation $T[\rho]$ is a normal completely positive linear map. It can be represented by a (non-unique) set of bounded operators via the ‘first representation theorem’:

$$T[\rho] = \sum_{n \in N} A_n^\dagger \rho A_n,$$

where the indexing set $N$ may be taken countable if $\mathcal{H}$ is separable. The operators $A_n$ satisfy $\sum A_n^\dagger A_n \leq 1$, with the equality holding when the operation that is represented by $T$ is non-selective (i.e., accomplished with certainty).

The only strong mathematical requirement in both representation theorems is a demand for Hilbert space to be separable. This requirement may appear problematic, since there is a persistent belief that quantum field theory necessarily implies non-separable Hilbert spaces. It is not so: this misconception was dismissed on general grounds a long time ago. Fock spaces built from one-particle spaces of free field theories are separable.

Actual Hilbert spaces in various important curved spacetime models, such as Hawking radiation or Unruh effect, are indeed separable.

Transformations that are described by Eq. (7) form a semigroup. A Lindblad equation governs the time evolution of the states (or in dual picture, of the observables). Typically in the relativistic theory $\rho$ is the input and $T[\rho]$— the output scattering state. Usually $T$ is given by the $S$-matrix, which is unitary (either rigorously or formally). Hawking has introduced a non-unitary super-scattering operator that maps density matrices describing the initial situation to density matrices describing the final situation for the process of black hole formation and evaporation. Such a step has been considered controversial in the conventional framework, but it is quite natural in quantum information theory.

### III. A RESULT IN ALGEBRAIC FIELD THEORY

It is well-known that different representations of canonical commutation relations (CCR) or canonical anticommutation relations (CAR) in field theories lead to unitarily inequivalent representations. In the case of Minkowski spacetime existence of preferred vacuum state enables to define a unique Hilbert space representation. A similar construction is possible in stationary curved spacetimes. However, in a general globally hyperbolic spacetime this is impossible and one is faced with multiple inequivalent representations. The simplest example is the Unruh effect, where the operators of Bogolyubov transformation are unitary only formally. The algebraic approach to the field theory makes it possible to overcome this difficulty.

Algebraic quantum field theory is presented in the books of Haag and Araki and mathematical exposition of its results is given in the monograph of Baumgärtel and Wellenberg. It can be naturally applied to quantum field theory in curved spacetime. We mention here only some simple facts. A basic structure of the theory is an algebra of local operators (i.e. ones confined to some open regions of spacetime), that actually are suitably smeared field operators. States are normalized positive linear functionals on the algebra. A representation of an algebra is obtained using suitably defined inner product via the so-called GNS (Gelfand, Neumark, Segal) construction. After having an one-particle space (in the case of free field theories), a Fock space is constructed in the usual way. Canonical commutation relations lead to unbounded operators, so to be able to utilize the machinery of $C^*$-algebras it is customary to consider the unitary Weyl algebra that is constructed by formal exponentiation of the CCR algebra.

Since not all the observables are included in this algebra, the set of states defined by being positive linear functionals of unit norm is too large. The most obvious (and
painful) example is the expectation value of the quantum energy-momentum tensor: a renormalization procedures actually lead to a restriction to ‘physical’ states [10, 24]. This restriction can be given a rigorous mathematical meaning [10, 24].

The algebraic approach to field theory leads to the conclusion that physical predictions of the theory are independent of the choice of a representation (in the case of a PVM description of measurements) [8, 10]. This proof is built in several steps that will be now outlined. Our claim that both POVM and general state transformation formalism are independent of representation will be based on it.

First, it should be noted that any measurement can actually be performed only with a finite accuracy, a finite number of outcomes, and a finite number of times. Suppose that we measure the value \( q \) of the observable \( Q \) and among \( N \) runs a value \( q_j \) is obtained \( n_j \) times. A relative frequency \( w_j = n_j/N \) is used to extract a probability estimate or it is taken at face value and interpreted as the estimate. Thus the information about a state \( \rho \) can be formulated as [10, 27]

\[
|p^Q(q_j) - w_j| < \epsilon_j,
\]

for some positive \( \epsilon_j \). These inequalities induce a natural topology on the state space, which is called a ‘physical topology’ [4]. Mathematically, they define a weak*-topology on the state space and weak topology on the set of observables [3, 4, 7].

The second important fact is that CAR C*-algebra and the Weyl C*-algebra that represents CCR are simple [28], i.e. do not contain maximal non-trivial ideals. Therefore, their representations are faithful (i.e. have zero kernel). If the algebra is considered to be only a C*-algebra, it includes all the projections associated with its algebra. The reason is that in our POVM analysis we had

IV. EXTENSION OF THE FORMALISM

However, what about using a POVM? The problem of (an approximate) localization of particles requires it. When there is a preferred representation of the algebra, or several unitarily equivalent representations, the generalization is straightforward. What happens when there is no unitary equivalence? Fortunately, since a position POVM is constructed from suitably smeared field operators and related projections, the discussion in the previous section is sufficient. They can be approximated arbitrarily close by polynomial functions of algebra elements, and thus belong either to an algebra (von Neumann case) or to its weak closure. In any case, the choice of the representation remains irrelevant.

In particular, in the case of Unruh effect both inertial and accelerated observer would agree on the localization of the detected particle in the phase space, i.e., its unsharp position and momenta. A simpler example of such an agreement is a position detection [24]. An application of the formal unitary operator by \( U_E U^\dagger \), suitably truncated [10, 24], gives its Minkowskian counterpart.

Considering POVMs in general, a question arises whether we can give to all the operators on a particular Hilbert space an invariant meaning. It turns out that this is always possible.

We start from some representation \((\mathcal{H}, \pi)\) of the algebra \( \mathcal{A} \). We adjoin to it the set of all POVM \( E(X) \) operators, and consider the smallest algebra that contains the union of \( \pi(\mathcal{A}) \) and the set of all POVMs. It is easy to see that this algebra is again simple. It is a representation of some simple abstract algebra \( \mathcal{A} \), with \( \mathcal{A} \) being a proper subalgebra. Then \( \mathcal{A} \setminus \mathcal{A} \) itself is described by the action of algebraic states on its elements. We just invert final steps in the GNS construction of the representation space [3, 4, 7, 10]. For any POVM operator \( E \) on \( \mathcal{H} \) that cannot be expressed as a function of elements of \( \pi(\mathcal{A}) \) we define the action of the state \( \omega_\alpha \) by

\[
\omega(\pi^{-1}(E)) := \text{tr}(\rho_\omega E),
\]

where \( \rho_\omega \) is a density matrix on \( \mathcal{H} \) that corresponds to the algebraic state \( \omega \) via GNS construction. Thus we arrive to the minimal algebra that contains the canonical relations of the theory and all possible observables in it. From this point the derivation leading to Eq. (1) is straightforward.

Incidentally, we get an argument in favor of considering an initial CCR/CAR algebra \( \mathcal{A} \) as a von Neumann algebra. The reason is that in our POVM analysis we had
to expand this initial algebra. By doing so we have adjoined also all the projections operators that correspond to self-adjoint operators of $\mathcal{A}$ (a PVM is a particular case of POVM).

Now we are able to turn to the question of state transformation. We show that to any transformation of density matrices on a Hilbert space we can define an algebraic state transformation. Since the converse of the statement is obvious, we establish a correspondence between such transformations on different representation spaces. We define an algebraic transformation $\Phi$ via its dual action on the elements of $\mathcal{A}$, following a usual practice in Banach spaces [3, 17]:

$$\Phi[\omega](A) := \omega(\Phi^\dagger[A]), \quad \forall A \in \mathcal{A}. \quad (12)$$

When there is a single Hilbert space a definition of the adjoint map $T^\dagger$ is given by

$$\text{tr}(\rho T^\dagger[\mathcal{B}]) := \text{tr}(T[\rho] \mathcal{B}), \quad (13)$$

where $\mathcal{B} \in \mathcal{L}(\mathcal{H})$. To reconstruct $\Phi$ from its representation $\pi(\Phi) = T$ on the representation space $\mathcal{H}$ we limit ourselves to $\mathcal{B} \in \pi(\mathcal{A})$. We set

$$\omega(\Phi^\dagger[A]) := \text{tr}(\rho, T^\dagger[\pi(A)]). \quad (14)$$

To ensure that a continuity of $\Phi$ follows from the continuity of $T$ we have to take $\mathcal{A}$ to be a von Neumann algebra [25]. It is straightforward to check that $\Phi$ has all the properties required from a valid state transformation (of course, whether this transformation is compatible with the requirements of causality is a different question [14, 15]).

V. SUMMARY

We showed that the basic structures of quantum information theory can be applied to field theory in curved spacetimes. It gives us a potentially powerful tool for the analysis of the relations of information, entropy and black holes. In particular, discussion of the information loss paradox, superscattering operator, etc., gets a different perspective.

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