Counting attractors in synchronously updated random Boolean networks

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Despite their apparent simplicity, random Boolean networks display a rich variety of dynamical behaviors. Much work has been focused on the properties and abundance of attractors. We here derive an expression for the number of attractors in the special case of one input per node. Approximating some other non-chaotic networks to be of this class, we apply the analytic results to them. For this approximation, we observe a strikingly good agreement on the numbers of attractors of various lengths. Furthermore, we find that for long cycle lengths, there are some properties that seem strange in comparison to real dynamical systems. However, those properties can be interesting from the viewpoint of constraint satisfaction problems.

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INTRODUCTION

Random Boolean networks have long enjoyed the attention of researchers, both in their own right and as simplistic models, in particular for gene regulatory networks. The properties of these networks have been studied for a variety of network architectures, distributions of Boolean rules, and even for different updating strategies. The simplest and most commonly used strategy is to synchronously update all nodes. Networks of this kind have been investigated extensively, see e.g. [1, 2, 3, 4, 5].

The networks we consider are, generally speaking, such where the inputs to each node are chosen randomly with equal probability among all nodes, and where the Boolean rules of the nodes are picked randomly and independently from some distribution. The network state is updated synchronously, with the state of a node at any time being a function of the state of its inputs at the previous time step.

In this work we determine analytically the numbers of attractors of different lengths in networks with connectivity (in-degree) one. We compare these results to networks of higher connectivity and find a remarkable degree of agreement, meaning that networks of single-input nodes can be employed to approximate more complicated networks, even for small systems. For large networks, a reasonable level of correspondence is expected. See [6] on effective connectivity for critical networks, and [5] on the limiting numbers of cycles in subcritical networks.

Random Boolean networks with connectivity one have been investigated analytically in earlier work [1, 2]. In those papers, a graph-theoretical approach was employed. In this work, we use another approach [10] which is based on the ideas of fixed point analysis. Our approach is a powerful tool for counting the average numbers of attractors, but it does not give any direct information on the properties of the attractor basins.

For long cycles, especially in large networks, there are some artefacts that make comparisons to real networks difficult. For example, the integer divisibility of the cycle length is important, see e.g. [4, 5, 8, 10, 11]. Also, the total number of attractors grows superpolynomially with system size in critical networks [10], and most of the attractors have tiny attractor basins as compared to the full state space. In this work, such artefacts become particularly apparent, and we think that long cycles are hard to connect to real dynamical systems.

Comparisons to real dynamical systems, on the other hand, still seem to be relevant with regard to fixed points and some stability properties [6, 12]. An interesting way to make more realistic comparisons regarding cycles is to consider those attractors that are stable with respect to repeated infinitesimal changes in the timing of updating events [13].

Furthermore, the problem of finding attractors with small attractor basins, for a given network, can be seen as a hard constraint satisfaction problem. Those properties that are artefacts when comparing to dynamical systems, could be keys to the understanding of real life combinatorial optimization problems.

Throughout this paper, \( N \) denotes the number of nodes in the network, and \( L \) the length of an attractor, be it a cycle \( (L > 1) \) or a fixed point \( (L = 1) \). For brevity we use the term \( L \text{-cycle} \), and understand this to mean an attractor such that taking \( L \) time steps forward produces the initial state. When \( L \) is the smallest positive integer fulfilling this, we speak of a \textit{proper} \( \text{\text{-}} \text{-cycle} \). We denote the number of proper \( \text{\text{-}} \text{-cycles} \) with \( C_L \). The average over networks of a certain size is \( \langle C_L \rangle_N \), so the average number of network states that are part of a proper \( \text{\text{-}} \text{-cycle} \) is \( \langle C_L \rangle_N \). Related to this is \( \langle N_L \rangle_N \), the number of states that reappear after \( L \) time steps and hence are part of any \( \text{\text{-}} \text{-cycle} \), proper or not.

To calculate the average number of attractors, we use the same approach as in [10], whereby we first transform the problem of finding an \( L \text{-cycle} \) into a fixed point problem, and then find a mathematical expression for the average number of solutions to that problem.
In the case that every node has one input, \( \langle \Omega_L \rangle_N \) can be calculated analytically for any \( N \). The limit \( \langle \Omega_L \rangle_{\infty} \) has a relatively simple form, and this limit can be mapped to the 1-input case for any subcritical network of multi-input nodes, as shown in [2]. Furthermore, critical networks of multi-input nodes seem to show strong similarities to the corresponding networks of 1-input nodes.

**THEORY**

Assume that a Boolean network performs a proper \( L \)-cycle. Then, each node performs one of \( m = 2^L \) series of output values. We call these \( L \)-cycle series. We associate each \( L \)-cycle series with an index \( i \), such that \( i \in \{0, 1, \ldots, m - 1\} \). For convenience, we let the constant \( L \)-cycle series have the indicies 0 and 1, in such a way that a constant \textit{false} output has index 0 whereas a constant \textit{true} output has index 1.

If we view each \( L \)-cycle series as a state, an \( L \)-cycle of the entire network turns into a fixed point (in this enlarged state space). \( L \langle C_L \rangle_N \) is then the average number of input states (choices of \( L \)-cycle series), for the whole network, such that the output is the same as the input.

It is inconvenient to work directly with \( \langle C_L \rangle_N \). Let \( \langle \Omega_L \rangle_N \) denote the average number of states that reappear after \( L \) timesteps. Such a state is part of a proper \( L \)-cycle where \( L \) is a divisor to \( L \). \( \langle C_L \rangle_N \) can be calculated from \( \langle \Omega_L \rangle_N \), using the set theoretic inclusion–exclusion principle.

**Expressing \( \langle C_L \rangle_N \) in terms of \( \langle \Omega_L \rangle_N \)**

Let \( C_L \) denote the set of network state sequences that represent proper \( L \)-cycles. Similarly, let \( \omega_L \) denote the non-proper counterpart to \( C_L \), meaning that \( \omega_L = \bigcup_{\ell \mid L} C_{\ell} \), where \( \ell \mid L \) means that \( \ell \) divides \( L \).

Consider a given network, and let \( Q \) denote the set of sequences of states, \( Q \), that are consistent with the network dynamics. Let \( \hat{\omega}_L = \omega_L \cap Q \) and \( \hat{C}_L = C_L \cap Q \). Then, \( \Omega_L = \vert \hat{\omega}_L \vert \), and the number of proper \( L \)-cycles of that network is given by \( C_L = \frac{1}{L} \vert \hat{C}_L \vert \). To calculate \( C_L \), we start by expressing \( \hat{C}_L \) in terms of \( \hat{\omega}_L \). We see that

\[
\hat{C}_L = \hat{\omega}_L \setminus \bigcup_{1 \leq \ell < L, \ell \mid L} \hat{\omega}_\ell \setminus \bigcup_{d \text{ prime}} \hat{\omega}_L/d, \tag{1}
\]

because any positive \( \ell \) dividing \( L \) is also a divisor to a number of the form \( L/d \), where \( d \) is a prime.

Then, the inclusion–exclusion principle, applied to eq. (1), yields

\[
C_L = \frac{1}{L} \sum_{s \in \{0, 1\}^n} (-1)^s \Omega_{L/d_L(s)}, \tag{2}
\]

where \( s = \sum_{i=1}^n s_i \), \( d_L(s) = \prod_{i=1}^n (d_L^i)^{s_i} \) and \( d_L^1, \ldots, d_L^L \) are the prime divisors to \( L \). For averages over randomly chosen \( N \)-node networks, we get

\[
\langle C_L \rangle_N = \frac{1}{L} \sum_{s \in \{0, 1\}^n} (-1)^s \Omega_{L/d_L(s)}. \tag{3}
\]

**Basic expressions for \( \langle \Omega_L \rangle_N \)**

Consider what a rule does when it is subjected to \( L \)-cycle series on its inputs. It performs some Boolean operation, but it also delays the output, giving a one step difference in phase for the output \( L \)-cycle series. Let \( A_L^i(x) \) denote the probability that a randomly selected rule will output \( L \)-cycle series \( i \), given that the input series are selected from the distribution \( x = (x_0, \ldots, x_{m-1}) \). Then, we get

\[
\langle \Omega_L \rangle_N = \sum_{n \in N} \left( \frac{N}{n} \right)^{m-1} \prod_{i=0}^{n-1} \left[ A_L^i(n/N) \right]^{n_i}, \tag{4}
\]

where \( n = n_0 + \cdots + n_{m-1} \). This is the same expression as presented in [2].

Let \( e_0 \) be the vector with elements \( e_0^i = \delta_{i0} \), where \( \delta \) is the Kronecker delta. For networks of 1-input nodes, \( A_L^i(x) \) is affine and can be expressed as

\[
A_L^i(x) = (x - e_0) \cdot \nabla A_L^i + A_L^i(e_0) \tag{5}
\]

\[
= x \cdot \nabla A_L^i + \delta_{i0} c_0 + \delta_{i1} c_1, \tag{6}
\]

where \( r_C \) and \( r_I \) are the fractions of nodes that copy and invert their input, respectively, while \( c_0 \) and \( c_1 \) are the fractions of constant nodes that output \textit{false} and \textit{true}, respectively. The elements \( \partial_j A_L^i \) of the gradient \( \nabla A_L^i \) are nonzero only if the \( L \)-cycle series \( i \) is the output of a node receiving the \( L \)-cycle series \( j \) on its input. This is true if the series \( j \) is the series \( i \), or the inverse of \( i \), rotated one step backwards in time. Let \( \partial_C^j(i) \) and \( \partial_I^j(i) \), respectively, denote those values of \( j \). Then,

\[
x \cdot \nabla A_L^i = r_C^i x \partial_C^i(i) + r_I^i x \partial_I^i(i). \tag{7}
\]

Let \( B_L \) be an \( m \times m \) matrix with elements

\[
B_{lj}^j = \partial_j A_L^i + \delta_{i0} (c_0 - 1) + c_1 \delta_{i1}. \tag{8}
\]

Then,

\[
A_L(x) = B_L x + e_0 \tag{9}
\]

for all \( x \) such that \( x_0 + \cdots + x_{m-1} = 1 \). Furthermore,

\[
\sum_{i=0}^{m-1} (B_L x + e_0)_i = 1 \tag{10}
\]

for all \( x \). This property will be important later on.
From the theory of linear algebra, we know that any matrix can be transformed to a triangular matrix. Thus, \( B_L \) can be rewritten as
\[
B_L = C_L^{-1} D_L C_L ,
\]
where \( C_L \) is an invertible matrix and \( D_L \) is a triangular matrix. As we shall see, it turns out to be sufficient to know \( \det(1 - \zeta D_L) \) as a function of \( \zeta \) to calculate \( \langle \Omega_L \rangle^N \).

Furthermore,
\[
\det(1 - \zeta D_L) = \det(1 - \zeta B_L)
\]
because
\[
(1 - \zeta B_L) = C_L^{-1} (1 - \zeta D_L) C_L .
\]
Thus, we want to calculate \( \det(1 - \zeta B_L) \).

**The structure of \( B_L \)**

To understand the structure of \( B_L \), consider subspaces spanned by \( L \)-cycle series indices of the type \( \{ i, \phi_L^C(i), \phi_L^1(i), \phi_L^C \circ \phi_L^C(i), \phi_L^C \circ \phi_L^1(i), \ldots \} \), containing all possible results of repeatedly applying \( \phi_L^C \) and \( \phi_L^1 \) to \( i \).

We call those sets \textit{invariant sets of} \( L \)-cycle series, which is the same as the \textit{invariant sets of} \( L \)-cycle patterns in \[10\], but formulated with respect to \( L \)-cycle series instead of \( L \)-cycle patterns. Let \( \rho_L^0, \ldots, \rho_L^{H_L-1} \) denote the invariant sets of \( L \)-cycle series, where \( H_L \) is the number of such sets.

For convenience, let \( \rho_L^0 \) be the invariant set \( \{0, 1\} \). These definitions are consistent with the notation in \[10\], meaning that \( H_L \) is the same number and \( \rho_0 \) corresponds to the same invariant set.

The elements of \( B_L \) are given by
\[
B_L^{ij} = r^C \delta_{j, \phi_L^C(i)} + r^1 \delta_{j, \phi_L^1(i)} + \delta_{i0} (c_0 - 1) + \delta_{i1} c_1 ,
\]
which means that \( B_L \) and \( 1 - \zeta B_L \) are block triangular matrices, with blocks corresponding to the invariant sets. Consequently, \( \det(1 - \zeta B_L) \) is the product of the determinants of all blocks on the diagonal.

Let \( r = r^C + r^1 \) and \( \Delta r = r^C - r^1 \). Then, the determinant of the block corresponding to \( \rho_L^0 \) is
\[
\det \left[ \begin{array}{cc} 1 - \zeta (r^C + c_0 - 1) & -\zeta (r^1 + c_0 - 1) \\ -\zeta (r^1 + c_1) & 1 - \zeta (r^C + c_1) \end{array} \right] = 1 - \Delta r \zeta ,
\]
because \( r^C + r^1 + c_0 + c_1 = 1 \).

To calculate the determinants of the blocks corresponding to the other invariant sets, we need to explore the structure of the invariant sets. Consider an invariant set of \( L \)-cycle series, \( \rho_L^i \). Let \( \ell \) be the \textit{length} of \( \rho_L^i \), meaning that \( \ell \) is the lowest number such that, for \( i \in \rho_L^i \), \( (\phi_L^C)^I(i) \) is either \( i \) or the index of series \( i \) inverted. If \( (\phi_L^C)^I(i) = i \), we say that the \textit{parity} of \( \rho_L^i \) is positive. Otherwise the parity is negative. The structure of an invariant set of \( L \)-cycle series is fully determined by its length and its parity. Such a set can be enumerated on the form \( \{ \phi_L^C(i), \ldots, (\phi_L^C)^{I-1}(i), \phi_L^1(i), \ldots, (\phi_L^C)^{\ell-1}(i) \} \) and for positive parity \( (\phi_L^C)^I(i) = i \), while \( \phi_L^1 \circ (\phi_L^C)^{\ell-1}(i) \) for negative parity.

Let strings of \( T \) and \( F \) denote specific \( L \)-cycle series. Then \( \phi_L^C(FFT) = fftf \) and \( \phi_L^1(FFT) = tftf \). Examples of invariant sets of \( 4 \)-cycle series are \{FFFT, FFTF, FTFF, TFFT, TFFF, TTFF, TTFT, TFFT, TFFF, TTTT\} and \{FFTF, FTFT\}. The first example has length 4 and positive parity, while the second has length 1 and negative parity.

Let \( R_L^+ \) and \( R_L^- \) denote the blocks in the diagonal of \( B_L \) that correspond to invariant sets of length \( \ell \) with positive and negative parity, respectively. Before we present the form of \( R_L^\pm \) for general \( \ell \), we consider a few examples.

The invariant set \{FTTT, TFFT\} corresponds to
\[
R_1^+ = \begin{pmatrix} r^1 & r^C \\ r^1 & r^C \end{pmatrix} ,
\]
where the first row corresponds to the 4-cycle series FFFF while the second row corresponds to TFFT. Correspondingly, we define
\[
R_1^- = \begin{pmatrix} 0 & R_1^- & 0 & 0 \\ 0 & 0 & R_1^- & 0 \\ 0 & 0 & 0 & R_1^- \\ 0 & 0 & 0 & 0 \end{pmatrix} ,
\]
with rows and columns connected to the 4-cycle series FFFF, FFTF, FTFF, TFFT, TFFF, TFFFF, TTTT, in that order.

For general \( \ell \) and parity, we get
\[
R_\ell^+ = \begin{pmatrix} 0 & R_\ell^+ & 0 & \cdots & 0 \\ 0 & 0 & R_\ell^+ & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & R_\ell^+ \\ R_\ell^+ & 0 & 0 & \cdots & 0 \end{pmatrix} ,
\]
where order of the rows (and columns) is given by index sequences of the type \( \phi_L^C(i), \phi_L^1(i), \ldots, (\phi_L^C)^{I-1}(i), \phi_L^1 \circ (\phi_L^C)^{\ell-1}(i) \).

The unitary tranformation matrix
\[
S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]
transforms \( R_1^\pm \) according to
\[
R_1^\pm = S_1^\dagger R_1^\pm S_1 = \begin{pmatrix} r & 0 \\ 0 & \pm \Delta r \end{pmatrix} .
\]
Let
\[ S_ℓ = \begin{pmatrix} S_1 & 0 & \cdots & 0 \\ 0 & S_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_1 \end{pmatrix}. \] (22)

Then,
\[
\det(1 - ζ R_ℓ^±) = \det[S_ℓ (1 - ζ R_ℓ^±) S_ℓ] = \det(1 - ζ S_ℓ R_ℓ^± S_ℓ)
\]
\[
= \det\left( \begin{array}{cccc} 1 & -ζ R_ℓ^± & 0 & \cdots & 0 \\ 0 & 1 & -ζ R_ℓ^± & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -ζ R_ℓ^± \end{array} \right)
\]
\[
= \det\left[ 1 - (rζ)^ℓ [1 + (Δrζ)^ℓ] \right]. \] (26)

To calculate \( \det(1 - ζ B_L) \), we need to find the distribution of lengths and parities of the invariant sets of \( L \)-cycle series. If an \( L \)-cycle series belongs to an invariant set with length \( ℓ \) it will be itself or itself inverted after \( ℓ \) timesteps, depending on whether the invariant set is of positive or negative parity. This gives a periodicity of \( ℓ \) or \( 2ℓ \) timesteps. Hence, for \( L \)-cycles, there will be invariant sets of length \( ℓ \) and positive parity if \( ℓ | L \), whereas sets of negative parity are present if \( 2ℓ | L \).

If invariant sets of a specific length and parity are present, their number is independent of \( L \), because the basic form of the series in such sets does not change with \( L \). Only the number of basic repetitions differs. Let \( J_ℓ^± \) and \( J_ℓ^- \) denote the numbers of invariant sets of length \( ℓ \) with positive or negative parity, respectively. Then,
\[
\det(1 - B_L ζ) = \prod_{ℓ | L} [1 - (rζ)^ℓ - δ_{ℓ1} [1 - (Δrζ)^ℓ] J_ℓ^±]
\]
\[
\times \prod_{2ℓ | L} [1 - (rζ)^ℓ] J_ℓ^- \left[ 1 + (Δrζ)^ℓ \right] J_ℓ^-, \] (27)

where the Kronecker delta, \( δ_{ℓ1} \), takes care of the special case of the block of \( B_L \), corresponding to \( L^0 \). This block has the determinant \( 1 - Δrζ \), instead of the \( (1 - rζ)(1 - Δrζ) \) it would have if it obeyed eq. (26).

\textbf{Calculation of} \( ⟨ Ω_L ⟩_N \) via tensor calculus

Here, we derive how \( ⟨ Ω_L ⟩_N \) can be determined from \( \det(1 - B_L ζ) \). From eq. (4) and eq. (9), we see that
\[
⟨ Ω_L ⟩_N = \sum_{n ∈ N^m} N^{m-1} \prod_{i=0}^{m-1} [(B_L n/N + e_0)_i]^{n_i}. \] (28)

Eq. (28) can be rewritten, using \((m + m)\)-dimensional tensors such that each element is indexed by two of the vectors \( \tilde{κ}, \tilde{λ}, \tilde{µ}, \tilde{ν} ∈ N^m \). Each such vector corresponds to a distribution of \( L \)-cycle patterns, and \( κ, λ, μ, ν \) denote the sums of the elements in each vector. We get
\[
⟨ Ω_L ⟩_N = \text{Tr}(G_L), \] (29)

where
\[
(G_L)_{\mu}^{\nu} ≡ δ_{μN} \left( \sum_{j=0}^{m-1} [(B_L ν/N + e_0)_j]^{μ_j} \right), \] (30)

and the trace operator is defined as
\[
\text{Tr}(X) ≡ \sum_{0 ≤ μ_0, ..., μ_{m-1}} X_{\mu}^{\nu}, \] (31)

for an \((m + m)\)-dimensional tensor \( X \).

For convenience, we define the operation \( n^k \) as
\[
n^k ≡ \prod_{i=0}^{k-1} (n - i). \] (32)

We also choose the convention to interpret \( n^0 \) as an empty product, independently of the value of \( n \). This means that \( 0^0 = 1 \), and division must be treated with special care. Let
\[
M_{κ}^{ρ} ≡ \prod_{j=0}^{m-1} μ_j^{ρ_j}, \] (33)

and
\[
\tilde{M}_{κ}^{ρ} ≡ \prod_{j=0}^{m-1} ρ_j^{μ_j}. \] (34)

\( \tilde{M} \) is triangular in the sense that \( κ_0 ≤ μ_0, ..., κ_{m-1} ≤ μ_{m-1} \) for all non-zero elements \( \tilde{M}_{κ}^{ρ} \). Hence, \( \tilde{M} \) has an inverse that obeys \( μ_0 ≤ κ_0, ..., μ_{m-1} ≤ κ_{m-1} \) for all non-zero elements \( \tilde{M}^{-1}_{κ}^{ρ} \).

Letting \( \tilde{M} \) act on \( G_L \) yields
\[
\tilde{M}_{κ}^{ρ} (G_L)_{μ}^{ν} = \sum_{μ_0 = κ_0}^{∞} \sum_{μ_{m-1} = κ_{m-1}}^{∞} δ_{μN} N^{κ} \left( \frac{N - κ}{μ - κ} \right)
\]
\[
× \prod_{j=0}^{m-1} [(B_L ν/N + e_0)_j]^{μ_j}, \] (35)

\[
= N^κ \prod_{j=0}^{m-1} [(B_L ν/N + e_0)_j]^{κ_j}, \] (36)

and
\[
= N^κ \prod_{j=0}^{m-1} [(B_L ν/N + e_0)_j]^{κ_j}. \] (37)
because \( \sum_{j=0}^{m-1} (B_L x + e_0)_j = 1 \). Eq. (38) can be rewritten to

\[
\tilde{M}^\mu_\nu(G_L)_{\mu\nu} = \frac{N^\nu}{N^\mu} \prod_{j=0}^{m-1} [(B_L \bar{\nu})_j + N \delta_{j0}]^{\kappa_j}
\]

\[
= \frac{N^\nu}{N^\mu} \sum_{\lambda=0}^{\kappa_0} \left( \frac{\kappa_0}{\lambda} \right) [B_L \bar{\nu}]^{\lambda} N^{\kappa_0 - \lambda} \prod_{j=1}^{m-1} [(B_L \bar{\nu})_j]^{\kappa_j}. \quad (39)
\]

Let \( Z^\bar{\mu}(X) \), for an arbitrary \( m \times m \) matrix \( X \), denote the coefficients of the formal expansion of \( \prod_{j=0}^{m-1} [(Xz)_j]^{\lambda_j} \) in such a way that

\[
\prod_{j=0}^{m-1} [(Xz)_j]^{\lambda_j} = \sum_{0 \leq \kappa_0, \ldots, \kappa_{m-1}} \sum_{j=0}^{m-1} Z^\bar{\mu}(X) \prod_{j=0}^{m-1} z_j^{\kappa_j}. \quad (40)
\]

Then, we see that the operator \( Z \) satisfies the condition

\[
Z(XY) = Z(X)Z(Y) \quad (41)
\]

for all \( m \times m \) matrices \( X \) and \( Z \). Furthermore, \( Z(X) \) is block diagonal for any \( X \), in the sense that \( \kappa = \lambda \) for all nonzero elements \( Z^\bar{\mu}(X) \).

Define \( \delta_{\bar{\mu}\bar{\nu}} \) according to

\[
\delta_{\bar{\mu}\bar{\nu}} = \prod_{j=0}^{m-1} \delta_{\nu_j \bar{\mu}_j} \quad (42)
\]

for arbitrary \( m \)-vectors \( \bar{\nu} \) and \( \bar{\mu} \). Now, eq. (34) can be rewritten as

\[
\tilde{M}G_L = NEZ(B_L)M, \quad (43)
\]

where

\[
N^\bar{\mu} = \delta_{\bar{\mu} \bar{\kappa}} \frac{N^\kappa}{N^\lambda} \quad (44)
\]

and

\[
E^\bar{\lambda} = N^{\kappa_0 - \lambda_0} \prod_{j=1}^{m-1} \delta_{\lambda_j, \gamma_j}. \quad (45)
\]

The block diagonal structure of \( Z(X) \) yields

\[
NZ(X) = Z(X)N \quad (46)
\]

for all \( X \), because \( N \) is diagonal with diagonal elements that are constant for constant \( \kappa \). According to eq. (11), \( B_L \) can be written as

\[
B_L = C^{-1}_L D_L C_L \quad (47)
\]

where \( D_L \) is triangular. Thus,

\[
\tilde{M}G_L = NEZ(C^{-1}_L D_L C_L)M \quad (48)
\]

\[
Z(C_L) \tilde{M}G_L = NE\tilde{L}Z(D_L)Z(C_L)M \quad (49)
\]

\[
Z(C_L) \tilde{M}G_L \tilde{M}^{-1}Z(C_L^{-1}) = NE\tilde{L}Z(D_L)Z(C_L)\tilde{M}^{-1}Z(C_L^{-1}) \quad (50)
\]

where \( \tilde{E}_L = Z(C_L)EZ(C_L^{-1}) \). Because \( \tilde{M}^{-1} \) is triangular with the lower indices (acting to the left) less than or equal to the upper indices, right multiplication with \( \tilde{M}^{-1} \) always yields a convergent result. Similarly, \( Z(C_L) \) and \( Z(C_L^{-1}) \) are well-behaved with respect to both left and right multiplication, as a consequence of their block diagonal structure. Thus, both sides of the equality in eq. (34) are well-defined.

Let \( T_L = Z(C_L)\tilde{M}^{-1}Z(C_L^{-1}) \), which yields

\[
Z(C_L) \tilde{M}G_L \tilde{M}^{-1}Z(C_L^{-1}) = NE\tilde{L}Z(D_L)T_L. \quad (51)
\]

The tensor \( \tilde{M}^{-1} \) tells how to express moments in terms of combinatorial moments. Each moment can be expressed as a sum of the combinatorial moment of the same order and a linear combination of combinatorial moments of lower order. Hence,

\[
M^\mu_\nu(\tilde{M}^{-1})^\bar{\lambda}\bar{\kappa} = \delta_{\kappa \bar{\lambda}} \quad (52)
\]

This property is conserved as \( \tilde{M}^{-1} \) is transformed by \( Z(C_L) \), because

\[
Z^\bar{\mu}(C_L) = 0 \quad \text{for} \ \kappa \neq \nu, \quad (53)
\]

meaning that

\[
T^\bar{\mu} = \delta_{\bar{\mu}\bar{\nu}} \quad \text{for} \ \mu \geq \nu. \quad (54)
\]

This is also true for \( \tilde{E}_L \), i.e.,

\[
\tilde{E}_L^\bar{\lambda} = E^\bar{\lambda}_L = \delta_{\bar{\lambda} \bar{\kappa}} \quad \text{for} \ \lambda \geq \kappa. \quad (55)
\]

Because \( T_L \) and \( \tilde{E}_L \) are triangular (with the same orientation) with unitary block diagonal, and \( N \) and \( Z(D_L) \) are block diagonal, we get

\[
\langle \Omega_L \rangle_N = \text{Tr}(G_L) = \text{Tr}[NE\tilde{L}Z(D_L)T_L] = \text{Tr}[NZ(D_L)]. \quad (56)
\]

The triangular structure of \( D_L \) yields that the diagonal elements of \( Z(D_L) \) are given by

\[
Z^\bar{\mu}_\nu(D_L) = \prod_{j=0}^{m-1} (D^\nu_{ij})^{\kappa_j}. \quad (57)
\]

Thus,

\[
\text{Tr}[NZ(D_L)] = \sum_{0 \leq \kappa_0, \ldots, \kappa_{m-1}} \frac{N^\nu}{N^\lambda} \prod_{j=0}^{m-1} D^\nu_{jj} \quad (58)
\]
and

$$\langle \Omega_L \rangle_N = \sum_{\nu=0}^{N} \frac{N^\nu}{N^\nu} \sum_{0 \leq \kappa_0, \ldots, \kappa_{m-1}} \delta_{\nu \nu} \prod_{j=0}^{m-1} (D_{L_j}^{j})^{\kappa_j}$$  \hspace{1cm} (59)

$$= \sum_{\nu=0}^{N} \frac{N^\nu}{N^\nu} \frac{1}{d^\nu} \left| \sum_{0 \leq \kappa_0, \ldots, \kappa_{m-1}} \prod_{j=0}^{m-1} (D_{L_j}^{j})^{\kappa_j} \right|$$  \hspace{1cm} (60)

$$= \sum_{\nu=0}^{N} \frac{1}{N^\nu} \left( \frac{N}{\nu} \right) \frac{d^\nu}{d^\nu \nu} \left| \sum_{0 \leq \kappa_0, \ldots, \kappa_{m-1}} \prod_{j=0}^{m-1} \frac{1}{1 - D_{L_j}^{j} \zeta} \right|$$  \hspace{1cm} (61)

$$= \left( 1 + \frac{1}{N} \frac{d}{d \zeta} \right)^N \left| \sum_{0 \leq \kappa_0, \ldots, \kappa_{m-1}} \prod_{j=0}^{m-1} \frac{1}{1 - D_{L_j}^{j} \zeta} \right|$$  \hspace{1cm} (62)

Because

$$\prod_{j=0}^{m-1} (1 - D_{L_j}^{j} \zeta) = \det(1 - D_L \zeta)$$  \hspace{1cm} (63)

$$= \det(1 - B_L \zeta)$$  \hspace{1cm} (64)

we get

$$\langle \Omega_L \rangle_N = \left( 1 + \frac{1}{N} \frac{d}{d \zeta} \right)^N \left| \frac{1}{\det(1 - B_L \zeta)} \right|$$  \hspace{1cm} (65)

where

$$\det(1 - B_L \zeta) = \prod_{\ell \mid L} \left[ 1 - (r\zeta)^\ell \right]^{J_\ell^+} - \hat{\delta}_L \left[ 1 - (\Delta r\zeta)^\ell \right]^{J_\ell^-}$$

$$\times \prod_{2 \mid L} \left[ 1 - (r\zeta)^\ell \right]^{J_\ell^+} \left[ 1 + (\Delta r\zeta)^\ell \right]^{J_\ell^-}$$  \hspace{1cm} (66)

**Calculation of $J_\ell^\pm$**

Let $\psi_\ell^+$ and $\psi_\ell^-$, respectively, denote the sets of (infinite) time series of TRUE and FALSE, such that each series is identical to, or the inverse of, itself after $\ell$ time steps. Then, the set of time series that are part of an invariant set of $L$-cycles with length $\ell$ and negative parity, $J_\ell^-$, is given by

$$J_\ell^- = \psi_\ell^- \setminus \bigcup_{d \text{ odd prime } \ell/d} \psi_{\ell/d}^-$$  \hspace{1cm} (67)

For positive parity, we get

$$J_\ell^+ = \psi_\ell^+ \setminus \left( \bigcup_{d \text{ prime } \ell/d} \psi_{\ell/d}^+ \right) \setminus J_{\ell/2}^-$$  \hspace{1cm} (68)

where $J_{\ell/2}^-$ is the empty set if $\ell/2$ is not an integer.

The numbers of elements in $J_\ell^\pm$ are given by $2\ell J_\ell^\pm$, where $J_\ell^\pm$ are the numbers of invariant sets with length $\ell$. Then, the inclusion–exclusion principle yields

$$J_\ell^- = \frac{1}{2\ell} \sum_{s \in \{0, 1\}^{\ell/2}} (-1)^s \psi_\ell^/d(s)$$  \hspace{1cm} (69)

where $s = \sum_{i=1}^{\ell/2} s_i$, $\hat{d}(s) = \prod_{i=1}^{\ell/2} (\hat{d}_i^{s_i})$, and $\hat{d}_1, \ldots , \hat{d}_\ell$ are the odd prime divisors to $\ell$. Similarly

$$J_\ell^+ = \frac{1}{2\ell} \sum_{s \in \{0, 1\}^{\ell/2}} \sum_{\sigma = 0}^{\ell/2} (-1)^{s_0 + s_\sigma} 2^{\ell/2} d(s, s_0) - \frac{1}{2} J_{\ell/2}^-$$  \hspace{1cm} (70)

where $d(s, s) = 2^s \hat{d}(s)$ and $J_{\ell/2}^- = 0$ if $\ell/2$ is not an integer. Insertion of Eq. 69 into Eq. 70 yields

$$J_\ell^+ = \frac{1}{2\ell} \sum_{s \in \{0, 1\}^{\ell/2}} \sum_{\sigma = 0}^{\ell/2} (1 + s_0)(-1)^{s_0 + s_\sigma} 2^{\ell/2} d(s, s_0, s_\sigma)$$  \hspace{1cm} (71)

which also can be written as

$$J_\ell^+ = J_\ell^- - J_{\ell/2}^-$$  \hspace{1cm} (72)

**Calculation of $\langle \Omega_L \rangle_N$ for small and large $N$**

Iterative application of eq. 72 and the identity

$$[1 - (\Delta r\zeta)^\ell][1 + (\Delta r\zeta)^\ell] = 1 - (\Delta r\zeta)^{2\ell}$$  \hspace{1cm} (73)

to eq. 60 yields

$$\det(1 - B_L \zeta) =$$

$$\prod_{\ell \mid L} \left[ 1 - (r\zeta)^\ell \right]^{J_\ell^+} - \hat{\delta}_L \left[ 1 - (\Delta r\zeta)^\ell \right]^{J_\ell^-}$$

$$\times \prod_{2 \mid L} \left[ 1 - (r\zeta)^\ell \right]^{J_\ell^+} \left[ 1 + (\Delta r\zeta)^\ell \right]^{J_\ell^-}$$

$$\times \prod_{L/\ell \text{ even}} \left[ 1 - (r\zeta)^\ell \right]^{2J_\ell^-} - \hat{\delta}_L \left[ 1 - (\Delta r\zeta)^\ell \right]^{J_\ell^-}$$  \hspace{1cm} (74)

where $J_\ell^-$ is given by eq. 69.

The simplest way to calculate $\langle \Omega_L \rangle_N$ is to work with power series expansions. If

$$\sum_{k=0}^{\infty} \eta_k \zeta^k = \frac{1}{\det(1 - B_L \zeta)}$$  \hspace{1cm} (75)

$\langle \Omega_L \rangle_N$ is given by

$$\langle \Omega_L \rangle_N = \sum_{k=0}^{N} N_k \sum_{k=0}^{N_k} c_k \zeta^k$$  \hspace{1cm} (76)

where the operation $N_k$ is defined in eq. 62.
For \( r < 1 \), \( \zeta = 1 \) is within the convergence radius of the power series in eq. (76). Hence, we get

\[
\lim_{N \to \infty} \left( 1 + \frac{1}{N} \frac{d}{d\zeta} \right)^N \bigg|_{\zeta=0} \frac{1}{\det(1 - B_L \zeta)} = \frac{1}{\det(1 - B_L)},
\]

which means that the large \( N \) limit of \( \langle \Omega_L \rangle_N \) is given by

\[
\langle \Omega_L \rangle_\infty = \frac{1}{\det(1 - B_L)}. \tag{77}
\]

This result is consistent with the calculations in eq. (70), which also are valid for subcritical networks with many inputs per node.

For \( r = 1 \), the dominant contribution to \( \langle \Omega_L \rangle_N \) comes from the pole at \( \zeta = 1 \). For \( |\Delta r| < 1 \) and large \( N \), we get

\[
\langle \Omega_L \rangle_N \approx \left( 1 + \frac{1}{N} \frac{d}{d\zeta} \right)^N \bigg|_{\zeta=0} \frac{\gamma_L}{(1 - \zeta)^{H_L-1}} \tag{79}
\]

where \( H_L \) is the total number of invariant \( L \)-cycle sets and \( \gamma_L \) is a constant. For large \( N \), a power series expansion of \( 1/(1 - \zeta)^{H_L-1} \) yields

\[
\langle \Omega_L \rangle_N \approx \gamma_L \sum_{k=0}^{N} \frac{N!}{N^k} \left( H_L - 2 + k \right) \tag{80}
\]

\[
\approx \tilde{\gamma}_L \int_0^\infty dk \frac{H_L - 2}{e^{-k^2/N}}, \tag{81}
\]

where \( \tilde{\gamma}_L \) is a constant. The dominant terms in eq. (80) satisfy \( H_L \ll k \ll N \) in the large \( N \) limit, and Stirling’s formula has been applied to those. From eq. (81), we see that the asymptotic behavior of \( \langle \Omega_L \rangle_N \) is the scaling

\[
\langle \Omega_L \rangle_N \propto N^{(H_L-1)/2}, \tag{82}
\]

for critical \( r = 1 \) networks with \( |\Delta r| < 1 \). This also means that

\[
\langle C_L \rangle_N \propto N^{(H_L-1)/2} \tag{83}
\]

for large \( N \).

If \( |\Delta r| = 1 \), the network only consists of copy operators (if \( \Delta r = 1 \)) or inverters (if \( \Delta r = -1 \)). Then, \( H_L \) should be replaced by the number of sets of \( L \)-cycle series that are invariant under the present choice of operators. This number will be as least as large as \( H_L \), because the invariant set can split, but not merge, when one operator is removed.

One can also use complex analysis to retrieve the operator identity

\[
\left( 1 + \frac{1}{N} \frac{d}{d\zeta} \right)^N \bigg|_{\zeta=0} \frac{1}{2\pi iN} \oint_{|\zeta|=\epsilon} \frac{e^{N\zeta}}{\zeta^{N+1}} d\zeta = \left( 1 + \frac{1}{N} \frac{d}{d\zeta} \right)^N \bigg|_{\zeta=0}, \tag{84}
\]

where \( \epsilon \) is a positive constant small enough to keep all poles of the given function outside \( |\zeta| \leq \epsilon \). This is especially useful to calculate the total number of states in attractors; \( \langle \Omega_0 \rangle_N \).

Eq. (70) can be rewritten as the exponential of a power series expansion. The expansion

\[
-\ln(1 - z) = \sum_{j=1}^{\infty} \frac{x^j}{j} \tag{85}
\]

yields

\[
-\ln \det(1 - B_L \zeta) = \sum_{\ell \mid L} \left[ J_{\ell}^{\text{odd}} \sum_{j=1}^{\infty} \frac{(r\zeta)^j}{j\ell} + J_{\ell} \sum_{j=1}^{\infty} \frac{(\Delta r\zeta)^j}{j\ell} \right] \tag{86}
\]

\[
+ \sum_{\ell \mid L} \left[ J_{\ell}^{\text{even}} \sum_{j=1}^{\infty} \frac{(r\zeta)^j}{j\ell} \right],
\]

where \( J_{\ell}^{\text{odd}} = J_{\ell} - J_{\ell/2 - \delta\ell} \) and \( J_{\ell}^{\text{even}} = 2J_{\ell} - J_{\ell/2 - \delta\ell} \).

Let \( (a,b) \) denote the greatest common divisor of \( a \) and \( b \). Then, the substitution \( k = j\ell \) and reordered summation gives

\[
-\ln \det(1 - B_L \zeta) = \sum_{k \mid (L,k)} \frac{\zeta^k}{k} \sum_{\ell \mid (k,L)} \left[ -r^k + (\Delta r)^k \right] J_{\ell} \tag{87}
\]

\[
= \sum_{k \geq 0} \frac{\zeta^k}{k} \sum_{\ell \mid (k,L)} \left[ -r^k + (\Delta r)^k \right] J_{\ell} \tag{88}
\]

The inner sums can be calculated using the connection between \( J_{\ell} \) and \( \psi_{\ell} \). Thus,

\[
-\ln \det(1 - B_L \zeta) = \sum_{k \mid (L,k)} \frac{\zeta^k}{k} \sum_{\ell \mid (k,L)} \left[ -r^k + (\Delta r)^k \right] 2^{(k,L) - 1} \tag{89}
\]

For \( L = 0 \), we get

\[
-\ln \det(1 - B_0 \zeta) = \sum_{k \mid (L,k)} \frac{(2r\zeta)^k}{k} - \frac{(r\zeta)^k}{k} \tag{90}
\]
which means that
\[
\frac{1}{\det(1 - B_0 \zeta)} = \frac{1 - r \zeta}{1 - 2r \zeta} \tag{91}
\]
and
\[
\langle \Omega_0 \rangle_N \approx \begin{cases} 
\frac{1 - r}{1 - 2r} & \text{for } r < 1/2 \\
\frac{1}{2} \sqrt{\frac{r}{N}} & \text{for } r = 1/2 \\
\frac{1}{2} \sqrt{\frac{r}{N}} e^{(\ln 2r - 1 + 1/2r)N} & \text{for } r > 1/2
\end{cases} \tag{93}
\]

RESULTS

Our main results from the analytical calculations are the expressions that yield \( \langle \Omega_L \rangle_N \) and \( \langle C_L \rangle_N \) for finite \( N \), and their asymptotic growth. See eqs. 88, 89, 90 and 91 on expressions for general \( N \), and eqs. 92, 93 and 94 on expressions valid for the high \( N \) limit. Fig. 4 shows the numbers of attractors with short length as a function of system size, for different \( r \), but with \( \Delta r = 0 \). For critical networks, with \( r = 1 \), the asymptotic growth of \( \langle C_L \rangle_N \) is a power law, while \( \langle C_L \rangle_N \) approaches a constant for subcritical networks as \( N \) goes to infinity.

For networks with \( \Delta r \neq 0 \), the prevalences of copy operators and inverters are not the same. Cycles of even length are in general more prevalent than cycles of odd length. An increased number of inverters strengthens this difference, while a low fraction of inverters makes the difference less pronounced. See fig. 4 which shows the symmetric case \( \Delta r = 0 \) and the extreme cases \( \Delta r = \pm r \).

The total number of attractors, \( \langle C \rangle_N \), and the total number of states in attractors, \( \langle \Omega_0 \rangle_N \), can diverge for large \( N \), even though the number of attractors of any fixed length converges. This is true for subcritical networks with \( r > 1/2 \). See figs. 3 and 4. The growth of \( \langle \Omega_0 \rangle_N \) is exponential if \( r > 1/2 \). Interestingly, there is no qualitative difference in the growth of \( \langle \Omega_0 \rangle_N \) when comparing the critical case, \( r = 1 \), to the subcritical ones with \( 1 > r > 1/2 \).

For \( r < 1/2 \), both \( \langle C \rangle_N \) and \( \langle \Omega_0 \rangle_N \) converge to constants for large \( N \). In the border line case \( r = 1/2 \), \( \langle \Omega_0 \rangle_N \) diverges like a square root of \( N \), but \( \langle C \rangle_N \) seems to approach a constant. See fig. 3.

The quantity \( \langle \Omega_0 \rangle_N \) has an interesting graph theoretical interpretation. Let \( N_{\text{active}} \) be the fraction of nodes that are part of a loop of non-constant nodes. Then,
\[
\langle \Omega_0 \rangle_N = \langle \exp(N_{\text{active}} \ln 2) \rangle \tag{94}
\]
which means that \( \ln \langle \exp(N_{\text{active}} \ln 2) \rangle \) grows linearly with \( N \) if \( 1/2 < r < 1 \). This stands in sharp contrast to
FIG. 2: The average number of proper $L$-cycles for networks with $N = 100$ and $r = 3/4$, as function of $L$. $\Delta r = 0$ (thick solid line) and $\Delta r = -3/4$ (dotted line). Note the importance of what numbers divide $L$.

FIG. 3: $\langle \Omega_0 \rangle_N$ (solid lines) and $\langle C \rangle_N$ (dotted lines) for $r = 1/2$, $r = 3/4$ and $r = 1$, in that order, from the bottom to the top of the plot. $\Delta r = 0$ in all cases. Note that $\langle \Omega_0 \rangle_N$ is independent of $\Delta r$. Due to the double logarithmic, $\langle C_L \rangle_N$ appears to approach $\langle \Omega_L \rangle_N$ for $r = 3/4$ and $r = 1$. This is not the case — have in mind that a constant translation in a double logarithmic scale corresponds to a rise to a constant power.

FIG. 4: The number of attractors with lengths $L \leq L_{\text{max}}$ in networks with $N$ single-input nodes, for different values of $N$. In (a) $N = 10, 10^2, \ldots, 10^5$ for $r = 1$ (thin solid lines) and $N = 10, \ldots, 10^4$ for $r = 3/4$ (thin dotted lines). In (b) $N = 10, 10^2, 10^3$ (thin solid lines) for $r = 1/2$ and $N = 10$ for $r = 1/4$ (thin dotted line). For all cases, $\Delta r = 0$. The thick lines in (a) and (b) show the limiting number of attractors when $N \to \infty$. The arrowhead in (b) marks this limit for $L_{\text{max}} = 10^7$ for $r = 1/2$. The small increase in the number of attractors when $L_{\text{max}}$ is changed from $10^3$ to $10^7$ indicates that $\langle C \rangle_N$ converges when $N \to \infty$. Note the drastic change in the $y$-scale between the case $r > 1/2$ and $r \leq 1/2$.

striking difference between the average number of fixed points and the same quantity for a typical network.

All the properties above are derived and calculated for networks with one input per node, but they seem to a large extent to be valid for networks with multi-input nodes. For such networks, we define $r$ and $\Delta r$ as measures of disturbance propagation according to the following procedure:

Find the mean field equilibrium fraction of nodes that
we have the value TRUE. Pick a random state from this equilibrium as an initial configuration. Let the system evolve one time step, with and without first toggling the value of a randomly selected node. The average fraction of nodes that in both cases copy or invert the state of the selected node are \( r^C \) and \( r^I \), respectively. Finally, let \( r = r^C + r^I \) and \( \Delta r = r^C - r^I \). For a more detailed description, see Fig. 7.

From Fig. 7, we know that for subcritical networks the limit of \( \langle C_L \rangle_N \) as \( N \to \infty \) is only dependent on \( r \) and \( \Delta r \). Hence, we can expect that \( \langle C_L \rangle_N \) for a subcritical network with multi-input nodes can be approximated with \( \langle C_L \rangle'_N \), calculated for a network with single-input nodes, but with the same \( r \) and \( \Delta r \). For the networks in Fig. 7, with a power law in-degree distribution, this approximation fits surprisingly well. See Fig. 6.

For the critical Kauffman model with in-degree 2, one can do a similar comparison. The number of nodes that are non-constant grows like \( N^{2/3} \) for large \( N \). See Figs. 7b. Furthermore, the effective connectivity between the non-constant nodes approaches 1 for large \( N \). Hence, one can expect that this type of \( N \)-node Kauffman networks can be mimicked by networks with \( N' = N^{2/3} \) 1-input nodes. For those networks, we choose \( r = 1 \) and \( \Delta r = 0 \), which are the same values as for the Kauffman networks.

For large \( N \), \( \langle C_L \rangle_N \) in the Kauffman networks grows like \( N^{(H_L - 1)/3} \), where \( H_L \) is the number of invariant \( L \)-cycle sets. For the selected networks with 1-input nodes, we have \( \langle C_L \rangle'_N \propto N^{r(H_L - 1)/2} \propto N^{(H_L - 1)/3} \) for large \( N' \), see eq. 8. This confirms that the choice \( N' = N^{2/3} \) is reasonable, but it does not indicate whether the proportionality factor in \( N' \propto N^{2/3} \) is close to 1. This factor could also be dependent on \( L \), as can be seen from the calculations in Fig. 7b. However, this initial guess turns out to be surprisingly good, see Fig. 6b.
From the good agreement for short cycles, one can expect a similar agreement in the total number of attractors. This is investigated in fig. 6. For networks with up to about 100 nodes, the agreement is good, and the extremely fast growth of \( \langle C_L \rangle_N \) for larger \( N \) is consistent with the slow convergence in the simulations.

**SUMMARY AND DISCUSSION**

Using analytical tools, we have investigated random Boolean networks with single-input nodes. We extract the exact distributions of cycles with lengths up to 1000 in networks with up to \( 10^5 \) nodes. Furthermore, we find some interesting scaling properties that hold for large \( N \).

As has been pointed out in earlier work [14], we see that a small fraction of the networks have many more cycles than a typical network. This property becomes more pronounced as the system size grows, and changes the scaling of the average number of states belonging to cycles drastically. For networks that have the stability parameter \( r > 1/2 \), the average number of states in attractors grows exponentially with the system size \( N \), whereas this number grows exponentially with \( \sqrt{N} \) for a typical network if \( r = 1 \), and approaches a constant if \( r < 1 \).

The dynamics in random Boolean networks with multi-input nodes can to a large extent be understood in terms of the simpler single-input case. In a direct comparison to critical Kauffman networks of connectivity two and to subcritical networks with power law in-degree, the agreement is surprisingly good.

Our results highlight some previously observed artefacts in random Boolean networks. The synchronous updates lead to dynamics that largely is governed by integer divisibility effects. Furthermore, when counting attractors in large networks, most of them are found in highly atypical networks and have attractor basins that are extremely small compared to the full state space.

In [12], a new concept of stability in attractors of Boolean networks is presented. To only consider that type of stable attractors is one way to make more relevant comparisons to real systems. Another way is to focus on fixed points and stability properties as in [11]. Furthermore, the limit of large systems may not always make sense in comparison with real systems. Small Boolean networks may tell more about real systems than large networks would.

Although there are problems in making direct comparisons between random Boolean networks and real systems, we think that insight in the dynamics of Boolean networks will improve the general understanding of complex systems. For example, can real systems have lots of attractors that are never found due to small attractor basins, and what implications would such attractors have on the system?

Another interesting viewpoint is to compare with combinatorial optimization problems, e.g. scheduling and digital circuit design. The question of finding an attractor in a random Boolean network is similar to such problems. For those problems, an attractor with a small attractor basin corresponds to a solution that is hard to find. Carrying the analogy further, we should not be surprised if a combinatorial problem that seem to be impossible to solve has plenty of solutions. Random Boolean networks could provide a playground for algorithms that handle such problems.

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