Minimal length corrections to magnetic birefringence in vacuum

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Abstract We consider the modification to Kruglov’s nonlinear electrodynamics in the presence of minimal length ($\ell_p$), by considering a specific formulation of the generalized uncertainty principle. The presence of minimal length has been motivated by several candidate theories of quantum gravity (both perturbative and non-perturbative). We show that such minimal length modifications could lead to momentum-dependent bound on zero-point electric field strength, for the aforementioned nonlinear model of electrodynamics. We further show that the existence of minimal length modifies the measure of vacuum birefringence $\Delta n$ in a constant and uniformly strong, external magnetic field. We, therefore, suggest the presence of a frictional mechanism of fundamental origin, acting in the parallel direction of polarization of photons (in the presence of a strong magnetic field), as an additional effect of the quantized background.

1 Introduction

A consistent theoretical framework for quantum gravity, features a domain of extreme scales (where quantum-gravitational effects are predominant) that are inevitable, leading to profound epistemological and experimental challenges for physicists [1]. The current empirical limitations and theoretical freedom, have led to some candidate theories (LQG, string theory, quantum geometrodynamics, and others) that are fundamentally different [2]. As suggested and motivated in string theory, non-commutative geometries, and LQG [3–11], the existence of a minimal length $\ell_p$, may be a physical requirement in the regime of quantum gravity. Such rationale and proposals also stem from the analysis of spectral dimension within the framework of causal dynamical triangulation[12], as an implication from asymptotically safe quantum gravity[13–15], and the inherent spatial discreteness that forms the foundational tenet of the causal-set approach to quantum gravity[16]. One possible analysis of phenomenological aspects (in the presence of minimal length—for a detailed discussion, see[17]), can be done by deforming Heisenberg’s uncertainty principle in the perturbative regime [18], which eventually leads to quasi-locality [19] and modified Heisenberg (Kempf) algebra, as a function of deformation parameters $\beta$ and $\beta'$[20]. In particular, by considering such algebraic modifications, a minimal length formulation of electrodynamics in vacuum (under a specific deformation constraint, $\beta' = 2\beta$—for reference, see section (2)), has been studied by Moayedi et al.[21] which includes the modified Maxwell’s equations (in free space) and dispersion relation. However, in the realm of modified uncertainty principles, it is also important to note several caveats, namely, the problem of Hamiltonian choice, the intricate adjustment of corrections to the uncertainty principle for composite systems, the absence of a consistent generalization to Heisenberg’s algebra through deformations of QFT and several other challenges[22].

After the 1930s, a series of theoretical investigations were carried out by W. Heisenberg and H. Euler [23] which suggested that, under the influence of a sufficiently strong magnetic perturbation, change in the linear behavior of quantum vacuum occurs, which results in vacuum birefringence (analogous to Cotton–Mouton effect in liquids & gases where the birefringence is linear). Several nonlinear models have been proposed for studying the strong magnetic birefringence effect in vacuum [24]. From analysis, it is found that the difference between the refractive indices of parallel and transverse polarization directions, i.e., $\Delta n = n_\parallel - n_\perp$, is related to the strength of the external magnetic field producing birefringence. Such vacuum birefringence predictions in formidable magnetic fields [25], could be a possible observable in the domain of astrophysics of isolated neutron stars [26], where Schwinger limit of the magnetic field is often exceeded. However, it is questionable whether quantum-gravitational effects still preserve similar behavior for $\Delta n$ in the same range. In this regard, we thus explore the possible corrections to vacuum birefringence.

In the following sections, we firstly review the covariant deformed algebra (Sect. 2). Then, we analyze the Lagrangian and the modified field equations for minimally corrected electromagnetism (Sect. 3). Lastly, we consider a generic nonlinear model of electrodynamics (Sect. 4) as suggested by Kruglov [24], where vacuum birefringence is theoretically investigated in the presence of
a constant, strong magnetic perturbation, after incorporating the ingredients of the minimal length from the generalized uncertainty principle or modified Heisenberg’s uncertainty principle. The zero-point field strength is also studied in this context.

In this article, we therefore conclude that, on imposing minimal length corrections to Kruglov’s formulation of nonlinear electrodynamics[24], a correction is obtained for $\Delta n$. Moreover, we obtain that this correction is dependent on the strength of the photon momentum. Hence, we argue that the functional behavior of $\Delta n$ with a greater value might be a signature of quantum-gravitational, resistive force, possibly arising from the discrete background.

We shall use the metric convention $\text{diag}(1, -1, -1, -1)$ and the Levi–Civita convention $\epsilon_{0123} = +1$ all throughout. Greek indices run over space-time components whereas, Latin indices run over space components alone. In the final part of the computation, we have considered $c = 1$, $\mu_0 = 1$ & $\epsilon_0 = 1$ for simplicity, which has been mentioned where needed.

2 Covariant deformed algebra in minimal length background (Kempf-Queyne-Tkachuk algebra)

A formal discussion on modification to Heisenberg’s uncertainty principle in the presence of a minimal length, and the resulting deformed algebra was done by Kempf [19]. Later, Queyne and Tkachuk obtained a covariant generalization of the same. The generalized uncertainty principle in 1+D-dimensional quantized space-time[20], which is often motivated from perturbative string theory[6] and LQG is the following:  

$$\Delta X^i \Delta P^i \geq \frac{\hbar}{2} \left(1 + f_d^\beta(\beta', \beta)\right),$$  

where, $f_d^\beta(\beta', \beta) = \beta' \left((\Delta P)^2 + (P')^2\right) - \beta \left((P^0)^2 - \sum_{j=1}^{D} \left[ (\Delta P)^2 + (P')^2 \right] \right),$  

$$\text{and } i = 1, 2, \ldots, D,$$  

for isotropic uncertainties—$\Delta P^i = \Delta P; \forall i$. The above inequality is retained up to the first order in deformation parameters—$\beta$ and $\beta'$ which are related to minimal length (from eq.:3) and have the dimension—(momentum)$^{-2}$. A deformed Lorentz covariant algebra $K_{\text{cov}}(\beta, \beta')$ is thus obtained as follows [20]:

$$[X^\mu, P^\nu] = -i\hbar\left[(1 - \beta P^\rho P_\rho)p^{\mu\nu} - \beta' P^\mu P^\nu\right],$$

$$[X^\mu, X^\nu] = i\hbar\left[\frac{2\beta - \beta'}{1 - \beta P^\rho P_\rho} (P^\mu X^\nu - P^\nu X^\mu) + \mathcal{O}(\beta^2, \beta\beta', \beta')\right],$$

$$[P^\mu, P^\nu] = 0.$$  

It is obvious that the commutativity of $X^\mu$ and $X^\nu$, i.e., $[X^\mu, X^\nu] = 0$, is maintained in the first order for deformation constraint $\beta' = 2\beta$. We shall follow this prescription all throughout.

Now, from the generalized uncertainty principle expressed as a formal expansion in $l_p^2$ [21]:

$$\Delta X^i \Delta P^i \geq \frac{\hbar}{2} \left(1 + a^i(D)\right) \left(\frac{l_p}{\hbar}\right)^2 \left(\Delta P^2 + \mathcal{O}(l_p^4, (\Delta P)^4)\right)$$

and Eq. (1), the deformation parameter $\beta$ can be defined as a quadratic function of $l_p$ in the leading order, i.e., $\beta = \mathcal{O}(l_p^2)$.

From Eq. (2) and the deformation prescription, a possible representation of $X^\mu$ and $P^\nu$ is obtained as follows[21]:

$$X^\mu = x^\mu \text{ and } P^\nu = (1 - \beta p^\rho p_\rho)p^\nu,$$  

where $x^\mu$ and $p^\mu = i\hbar \frac{\partial}{\partial x^\mu} \equiv i\hbar \partial^\mu$, are the usual Heisenberg representation of canonical pair of operators. It is to be mentioned that, in the position representation, $X^\mu$ and $P^\nu$ are given as:

$$X^\mu = x^\mu \text{ and } P^\nu = i\hbar \Box^\nu = i\hbar(1 + \beta \hbar^2 \Box_D)\partial^\nu,$$  

where $\Box_D \equiv \partial_\mu \partial^\mu$ is the 1+D-dimensional d’Alembertian.

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\[1\] The following specific form of generalized uncertainty principle can also be motivated from duality principle of the zero-point length of space-time [27].

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3 Minimal length formulation of electrodynamics in vacuum by effective Lagrangian

In order to study the effects on electromagnetic waves, moving in a quantized 1+3-dimensional background in the absence of external charges (that may influence the field), we need to formulate a suitable description of electromagnetism in the presence of minimal observable length. The Lagrangian for free electromagnetic field in the absence of minimal length is given by:

$$\mathcal{L}_{EM}^{(0)} = -\frac{1}{4\mu_0} F^{\alpha\beta} F_{\alpha\beta},$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the Faraday tensor and $A^\alpha \equiv (\phi/c, \mathbf{A})$ is the 4-potential. It is to be noted that the entire treatment retains the classicality of the fields.

Now, using (5) and by straight-forward substitution as done by Moayedi et al.\[21\]:

$$\chi^\mu \to X^\mu, \quad \partial^\nu \to \mathcal{D}^\nu,$$

it can be shown that:

$$\mathcal{L}_{EM}^{(0)} \to \mathcal{L}_{EM}^{(0)} - \frac{1}{4\mu_0} a^2 F_{\alpha\beta} \Box F^{\alpha\beta} \equiv \mathcal{L}_{EM}^{(a^2)}.$$

where $\Box \equiv \Box_\lambda$, which is Podolsky’s characteristic length. The alteration introduced in (7) incorporates a higher-derivative component into the action and the resulting above form of the Lagrangian (7) is related to the formulation of generalized electrodynamics by Podolsky, capable of describing Landé-Thomas theory as a special case \[28\]. Given that this modification is purely perturbative in nature, it is foreseeable that the complete field theory, when subjected to Generalized Uncertainty Principle (GUP) deformation, will exhibit an infinite series of terms involving d’Alembertian operators, rendering it nonlocal. The utilization of GUP in our case represents a perturbative simplification of nonlocal quantum field theory, a field that has been extensively explored in the existing literature\[29–32\].

It is noteworthy to mention that a theoretical cut-off results in the existence of a maximum value for $a$. Precisely, the bound $|a| < 4.72 \times 10^{-16}$ cm $\equiv a_m$ \[33, 34\] sets an upper bound to the deformation parameter $\beta$. To be precise, the upper bound on $|\beta_0|$ is $\mathcal{O}(10^{16})$. To underscore its robustness, it is important to compare it with the most rigorous limit established in \[28\].

Now, using (7), one arrives at the modified Maxwell’s equations \[21\]:

$$\Box F^{\mu\nu} + a^2 \Box \partial_\mu F^{\mu\nu} = 0.$$  \hspace{2cm} (8)

We thus obtain the modified Maxwell’s equations by considering the modified action up to $\mathcal{O}(a^2)$. Further, the following modified wave equations are directly obtained from the minimally corrected Maxwell’s equations (eq.- 8) \[21\]:

$$\Box \mathbf{E} + a^2 \Box \Box \mathbf{E} = 0 \quad \hspace{2cm} (9a)$$

$$\Box \mathbf{B} + a^2 \Box \Box \mathbf{B} = 0 \quad \hspace{2cm} (9b)$$

The higher-order derivatives within the field equations, as described above Eq. (9) are a consequence of introducing higher derivatives into the action. It is a well-established fact that such evolution can lead to Ostrogradsky instabilities or their quantum counterparts, commonly referred to as “ghosts”. Ghosts represent propagating degrees of freedom with imaginary mass (for details, see \[36\]). In fact, (9) implies the presence of a massless photon and an additional boson with a mass of $M_{gh} = i/a$. In the context of truncations applied in nonlocal field theories, such as in our case, these entities can be considered as artifacts of the approximation. Consequently, they can be safely disregarded as external states when dealing with effective field theory below the Planck mass scale.

Considering (9), it is now straightforward to obtain the solution for $\mathbf{B}$ from above, after plugging in the following ansatz:

$$\mathbf{B} = \mathcal{B} e^{-i p_x x / \hbar}.$$  \hspace{2cm} (10)

We, therefore, obtain the plane wave solution in a minimal setting (similar arguments are valid for $\mathbf{E}$):

$$\mathbf{B}^{(\alpha)}(x, t) = \mathcal{B} \ e^{i p_x x / \hbar} e^{-i \frac{\hat{E}^{(\alpha)}}{\hbar} t} = \mathbf{B}^{(0)}(x, t) e^{i (E_p - \hat{E}^{(\alpha)}) t / \hbar} \equiv \mathbf{B}^{(0)}(x, t) e^{i \hat{E}^{(\alpha)} t / \hbar},$$

where,
is the effective (modified) dispersion relation for massive (effectively) case [21],

\[ \mathbf{B}^{(0)}(x, t) = \mathcal{B} e^{i p x / \hbar} e^{-i E^{(0)}_p t / \hbar} = \mathcal{B} e^{i p x \pm |p| c t / \hbar} \]

are the unmodified free field solutions that satisfy:

\[ \square \mathbf{B}^{(0)} = 0 \quad , \]

and \( |\hbar \alpha_p^{(a)}| = |p| |c| \left( 1 - \sqrt{1 + \frac{1}{2 a^2 |p|^2}} \right) \).

If we consider the expansion of the above expression, in the region of complex plane \( \Gamma \), of the dimensionless parameter \( \Lambda_a \equiv \frac{a}{a_m - a} \) where \( \frac{\hbar}{|p|} \ll a \), it can be shown that the residue of \( \alpha_p^{(a)} \) is related to \( a_m \) (upper bound of Podolsky’s characteristic length) in terms of the following contour integral defined on the boundary of the region \( \partial \Gamma \):

\[
\frac{\hbar}{2 \pi |p| c} \oint_{\partial \Gamma} d \Lambda_a i \alpha_p^{(a)}(\Lambda_a) = \sum_{n=1}^{\infty} (-1)^n \frac{2n}{1 - 2n} \frac{2n}{\sqrt{\frac{\hbar}{|p|}}} \frac{2n}{a_m} a_m^{-2n}
\]

From the previous results Eq. (10), we also observe that:

\[
\square \mathbf{B}^{(a)}(x, t) + \left( \frac{\alpha_p^{(a)}}{c} \right)^2 \mathbf{B}^{(a)}(x, t) = 0
\]

\[
\square \mathbf{E}^{(a)}(x, t) + \left( \frac{\alpha_p^{(a)}}{c} \right)^2 \mathbf{E}^{(a)}(x, t) = 0
\]

The above set of equations dictates the propagation of the modified magnetic and electric field solutions. Next, we look into the modification of a proposed nonlinear model [24] in a minimal length setting.

4 Modification to Kruglov’s nonlinear electrodynamics

The post-Maxwellian/perturbative approach to QED establishes some successful theoretical predictions that were experimentally verified later. Although, vacuum birefringence in the presence of a strong external field, is just another theoretical prediction that lacks sufficient empirical justification. The external field that provokes such a quantum process cannot be generated in physical laboratories. Extremely strong external magnetic field is often generated in some astrophysical objects. However, such fields often exceed the perturbative (Schwinger) limit, demanding a non-perturbative approach [37].

Several semi-classical treatments of vacuum birefringence are available in the literature to date, which feature nonlinearities in the classical electromagnetic theory arising from quantum vacuum effects. We shall now consider a hybrid nonlinear model that reduces to different known models under suitable approximations.

In order to study vacuum birefringence under a constant external magnetic field, a nonlinear model of electrodynamics with three parameters was suggested by Kruglov [24]. The parameters introduced in the model are directly associated with the physical bounds of the electric field. We first express the Lagrangian in its usual form:

\[
\mathcal{L}^{(0)}_k = -\mathcal{F} - \frac{\alpha \mathcal{F}}{2 b \mathcal{F} + 1} + \frac{\gamma}{2} \mathcal{G}^2
\]

where the bilinear Lorentz invariants are \( \mathcal{F} = \frac{1}{4 \mu_0} F_{\alpha \beta} F^{\alpha \beta} = \frac{1}{2 \mu_0} |B|^2 - \frac{\epsilon_0}{2} |E|^2 \), \( \mathcal{G} = \frac{1}{4 \mu_0} F_{\alpha \beta} F^{\alpha \beta} = \frac{1}{c \mu_0} E \cdot B \) and \( F_{\alpha \beta} = \frac{1}{2} \epsilon_{\alpha \beta \gamma} \epsilon^{\delta \gamma} F_{\delta \gamma} \) is the Hodge dual of \( F_{\alpha \beta} \). It is to be noted that, \( \alpha \) is the only dimensionless parameter in this description whereas, \( b \) has the dimension of (energy/length)\(^{-2}\) and \( \gamma \mathcal{G} \) is dimensionless. In order to incorporate the modifications (see Sect. 2), we note that under \( \mathcal{K}_{\text{cov}}(\beta, 2\beta) \)

\[
\mathcal{F} \rightarrow \mathcal{F} + \frac{\alpha^2}{4 \mu_0} F_{\alpha \beta} \Box F^{\alpha \beta} \quad \text{and}
\]

\[
\mathcal{G} \rightarrow \mathcal{G} + \frac{\alpha^2}{4 \mu_0} F_{\alpha \beta} \Box F^{\alpha \beta}
\]
Thus, we obtain the following modified Lagrangian:

\[
L^{(a^2)}_K = -F - \frac{a^2}{4\mu_0} F_{ab} \Box F^{ab} = \frac{\alpha F + \frac{\gamma a^2}{4\mu_0} F_{ab} \Box F^{ab}}{1 + 2bF + \frac{ba^2}{2\mu_0} F_{ab} \Box F^{ab}} \]

If \(2bF + \frac{ba^2}{2\mu_0} F_{ab} F^{ab} \ll 1\), then the Lagrangian simply becomes:

\[
L^{(a^2)}_K \approx -(1 + \alpha)F + 2abF^2 + \frac{\gamma}{2}a^2
\]

\[
+ \frac{a^2}{4\mu_0} \left[-(1 + \alpha)F_{ab} \Box F^{ab} + 4a_{\beta}F_{\alpha e} \Box F^{\beta e} + \gamma \tilde{G} F_{ab} \Box F^{ab} \right] + O(a^4) .
\]

Here, we observe the presence of nonlinear \(F^2\) term in the Lagrangian that is motivated by considering nonlinear effects on quantum vacuum in higher (multi) dimensional Born–Infeld electrodynamics[38]. Thus, in the weak \(b\) approximation and \(\gamma = 0\), all the nonlinear terms are expected to subside leaving the minimal length correction for \(F\). In order to have a standard quadratic term, we can redefine the above Lagrangian in terms of the renormalized tensor \(F'_{ab} \equiv \sqrt{(1 + \alpha)}F_{ab}\) as:

\[
L^{(a^2)}_K (R\mathbf{E}\mathbf{N}) \approx -(1 + \alpha)F + 2abF^2 + \frac{\gamma}{2}G^2
\]

\[
+ \frac{a^2}{4\mu_0} \left[-F'_{ab} \Box F'^{ab} + 4a_{\beta}F'_{\alpha e} \Box F'^{\beta e} + \gamma (1 - 2\alpha)G^2 F'_{ab} \Box F'^{ab} \right] + O(a^4) .
\]

for the natural assumption of \(\alpha << 1\).

It is straightforward to obtain the equations of motion for the modified Lagrangian (in Eq. 16):

\[
\partial_{\mu} \left[ (F^{\mu\nu} + a^2 \Box F^{\mu\nu}) + \left( \frac{\alpha(F^{\mu\nu} + a^2 \Box F^{\mu\nu})}{1 + 2bF + \frac{ba^2}{2\mu_0} F_{ab} \Box F^{ab}} \right) - \gamma \left( \tilde{G} F^{\mu\nu} + a^2 \tilde{G} \Box F^{\mu\nu} + \frac{a^2}{4\mu_0} \tilde{G} F_{ab} \Box F^{ab} \right) \right] + O(a^4) = 0 .
\]

In the absence of the \(\gamma\) term, the modified two-parameter model [39] has the following equations of motion:

\[
\partial_{\mu} \left[ (F^{\mu\nu} + a^2 \Box F^{\mu\nu}) + \alpha (F^{\mu\nu} + a^2 \Box F^{\mu\nu}) \right] = 0 .
\]

The above equations are self-consistent due to the nonlinearity of the theory which is severe when a minimal length prescription is incorporated. In other words, after taking the derivative w.r.t. \(\partial^\mu\), the equations can be re-expressed in the form as follows:

\[
\Box F^{\mu\nu} = f^{\mu\nu}(a, b, \gamma; F^{\delta\rho}, a^2 \Box F^{\delta\rho}, F, \tilde{G}) ,
\]

where \(f^{\mu\nu}\) is a tensorial form of function that can be obtained from the equations of motion discussed above. We, therefore, seek an approximation that is reasonable enough for obtaining \(F^{\mu\nu}\). Such approximation is motivated by the study of linear electrodynamics with minimal length as discussed before, which must also relate to the limiting behavior of \(a\). We therefore make the following leading order approximation as well as the assertion:

\[
a^2 \Box F^{\delta\rho} \approx -a^2 \left( \frac{\alpha^{(a)}}{c} \right)^2 F^{\delta\rho}
\]

\[
\text{where}, \left( \alpha^{(a)} \right)^2 \equiv \left( \alpha^{(a)} \right)^2 - \frac{c^2}{a^2}
\]

As a consequence, we have:

\[
\lim_{a \to 0} \alpha^{(a)^2} F^{\delta\rho} \approx 0 ,
\]

which is consistent theoretically as the minimal length corrections drop out in this limit and the equations of motion are similar to the ones obtained by Kruglov [24, 39].
Under the above consideration, the equations of motion are devoid of any self-consistency and take the following form:

\[ \Box F^{\mu \nu} = f_{\mu \nu}^{\alpha, b, \gamma} F^{\beta \rho} - a^2 \left( \frac{\partial (\bar{\rho}_\mu)}{c} \right)^2 \left( \bar{\rho}_\mu, \bar{\rho}_\rho, \bar{H} \right). \] (21)

More precisely, we obtain:

\[ \partial_\mu \left[ \left( F^{\mu \nu} - a^2 \left( \frac{\partial (\bar{\rho}_\mu)}{c} \right)^2 F^{\mu \nu} \right) + \frac{\alpha (F^{\mu \nu} - a^2 \left( \frac{\partial (\bar{\rho}_\mu)}{c} \right)^2 F^{\mu \nu})}{(1 + 2b \bar{F} - 2ba^2 \left( \frac{\partial (\bar{\rho}_\mu)}{c} \right)^2 \bar{F})^2} - \gamma (\bar{G} F^{\mu \nu} - 2a^2 \left( \frac{\partial (\bar{\rho}_\mu)}{c} \right)^2 \bar{G} F^{\mu \nu}) \right] \]

(22)

\[ + \mathcal{O}(a^4) = 0. \]

and for \( \gamma = 0 \) (two-parameter model), we have the following equations of motion:

\[ \partial_\mu \left[ \left( F^{\mu \nu} - a^2 \left( \frac{\partial (\bar{\rho}_\mu)}{c} \right)^2 F^{\mu \nu} \right) + \frac{\alpha (F^{\mu \nu} - a^2 \left( \frac{\partial (\bar{\rho}_\mu)}{c} \right)^2 F^{\mu \nu})}{(1 + 2b \bar{F} - 2ba^2 \left( \frac{\partial (\bar{\rho}_\mu)}{c} \right)^2 \bar{F})^2} \right] \]

(23)

\[ + \mathcal{O}(a^4) = 0. \]

Equations (22) along with the Bianchi identity \( \partial_\nu \bar{F}^{\mu \nu} = 0 \), correspond to the modified Kruglov-Maxwell’s equations in the presence of minimal length. From Eq. (22), we can therefore read out the electric induction field as follows:

\[ \mathbf{D}^{(a^2)} = \frac{\partial \mathbf{E}^{(a^2)}}{\partial \mathbf{E}} \]

Here, the electric induction field takes the following form:

\[ \mathbf{D}^{(a^2)} = \epsilon^{(a^2)} \mathbf{E} \]

where, \( \epsilon^{(a^2)}_{ij} = \epsilon^{(a^2)} \delta_{ij} + \Gamma^{(a^2)}_{ij} \mathbf{B}, \mathbf{B} \),

\[ \epsilon^{(a^2)} = \left[ 1 - a^2 \left( \frac{\partial (\bar{\rho}_\mu)}{c} \right)^2 \right] + \frac{\alpha \left[ 1 - a^2 \left( \frac{\partial (\bar{\rho}_\mu)}{c} \right)^2 \right]}{\left( 1 - b \bar{\epsilon}_0 |\mathbf{E}|^2 + \frac{\mu}{\mu_0} |\mathbf{B}|^2 \right)} \left( \frac{\partial (\bar{\rho}_\mu)}{c} \right)^2 \left( \bar{\rho}_\mu \right)^2 |\mathbf{B}|^2 \]

(24)

\[ \text{and}, \quad \Gamma^{(a^2)}_{ij} = \frac{\gamma}{\epsilon \mu_0} \left[ 1 - 2a^2 \left( \frac{\partial (\bar{\rho}_\mu)}{c} \right)^2 \right]. \]

For the ease of computation, we choose \( c = 1, \mu_0 = 1 \& \epsilon_0 = 1 \), which can later be recovered from dimensional analysis.

In addition, it is also possible to define the magnetic field \( \mathbf{H} \) from the following canonical/standard relation:

\[ \mathbf{H}^{(a^2)} = -\frac{\partial \mathbf{E}^{(a^2)}}{\partial \mathbf{B}}, \]

as well as the magnetic permeability tensor from classical electromagnetism [39] in minimal length background:

\[ \mu^{(a^2)}_{ij} = \frac{1}{\epsilon^{(a^2)}} \delta_{ij} - \frac{\Gamma^{(a^2)}_{ij}}{\epsilon^{(a^2)} \left( \Gamma^{(a^2)} \right. |\mathbf{E}|^2 - \epsilon^{(a^2)} \right)} \mathbf{E}_j \]

(25)

Consequently, we have:

\[ \mathbf{H}^{(a^2)}_{ij} = \left( \mu^{(a^2)} \right)_{ij}^{-1} \mathbf{B}_j. \]

At this stage of analysis, it is possible to investigate the physical bound on the electric field [39] by considering the two-parameter model. So, we take the specific model (see Eq. (23) for \( \gamma = 0 \), where the electric permittivity tensor \( \epsilon^{(a^2)}_{ij} \) becomes isotropic as \( \Gamma^{(a^2)}_{ij} \) being a linear function of \( \gamma \) identically goes to zero, i.e., \( \Gamma^{(a^2)}_{ij} = 0 \).

Further, if we consider the electrostatic approximation in the presence of a point source (\( \mathbf{H} = \mathbf{B} = 0 \)), we get the modified Gauss’ law as follows:

\[ \nabla \cdot \mathbf{D}_0^{(a^2)} = q \delta (\mathbf{r}) \].
Above schematic depicts the momentum-dependent variation of field strength occurring from possible perturbative quantum gravity effects on nonlinear electrodynamics. Plots for different values of $a$ in the units of $10^{-17}$ cm are constructed for comparison. The horizontal line depicts the maximal field strength in the absence of any characteristic minimum length\[39\]

All the normalized distributions $y(x)$ with minimal length modification asymptotically goes to $1/\sqrt{2}$ for $|p| \to \infty$. For large $a$, we can have $|E_0| \approx 1/\sqrt{2b}$

$$\left[\Theta(x-x_c) + \sqrt{2} \Theta(x_c-x)\right],$$

where $x_c \approx 0^+$ corresponds to the value of $|p|$ for which, the distribution inflects sharply

which yields:

$$D_{(a^2)}^{(r)} = \frac{q}{4\pi r^2} r .$$

The critical field strength can be investigated while considering the limit—$\lim_{r \to 0} |D_{(a^2)}^{(r)}(r)|$ where a consistent solution is obtained for:

$$1 - b|E_0|^2 + ba^2 \left(\frac{\omega_p}{a^2} \right)^2 |E_0|^2 = 0$$

or, $|E_0|^{(a^2)} = \frac{1}{\sqrt{b}} \sqrt{\frac{1}{2 - \frac{a^2|p|^2}{b^2} \left(1 - \frac{a^2|p|^2}{b^2} \right)}} .

(26)$

which takes a constant value $\frac{1}{\sqrt{b}}$ in the limit $a \to 0$ \[39\] and $\frac{1}{\sqrt{2b}}$, when $|p| \to \infty$. It is to be noted that the above expression is obtained in the light of the leading order approximation to self-consistent equations (Eq. 18).

We further observe the field strength variation at the origin with the strength of photon momentum and hence the energy spectrum from dispersion relation. Such variation can hence be a possible consequence of minimal length due to the coupling between $a$ and $|p|$. It is further expected that such variations would persist for higher-order approximations as well. We can thus study the above variation as a distribution of field strength values in $|p|$-space by taking $x \equiv \frac{|p|}{b}$ and $y \equiv |E_0|^{(a^2)} \sqrt{b} $ (see Figs. 1 and 2).

Quite often, phenomenological analyses rely on fixing spectral bounds to tame divergences, which can be considered as a method of mere computational expediency. It can, however, be asserted that such bounds are physical and are of cosmological origin \[40\] or might be possibly related to the micro-structure of the background. A similar type of bound is associated with the limitation of perturbative approaches in QED (e.g.- Schwinger limit in perturbative QED) which demands nonlinear modification to the Maxwellian field theory \[41\].
For parallel polarization in the direction of the external field, we get:

\[ \Delta n \approx 1 \]

and the phase velocity remains equal to the speed of light, i.e., \( v_\perp = c = 1 \) (by our choice). Such unequal velocity in mutually perpendicular, polarization directions is captured in the following expression for \( \Delta n = n_\parallel - n_\perp \):

\[ \Delta n^{(\alpha^2)} = \sqrt{1 + \gamma \frac{\varepsilon^2_{\gamma} K_2 |\mathbf{B}_{\text{ext}}|^2 \left(1 + b K_1 |\mathbf{B}_{\text{ext}}|^2\right)^2}{\alpha K_1 + \left(1 + b K_1 |\mathbf{B}_{\text{ext}}|^2\right)^2}} - 1 \]

with,

\[ K_1 = 1 - a^2 \left(\frac{\phi_{\perp}}{\phi_{\parallel}}\right)^2 \]

and

\[ K_2 = 1 - 2a^2 \left(\frac{\phi_{\perp}}{\phi_{\parallel}}\right)^2 \].
It is to be noted that $\Delta n$ depends on photon momentum due to minimal length modifications. Moreover, we observe that the expected value of $\Delta n$ is larger than the value predicted in the absence of minimal length (see Figs. 3 and 4).

As a result, the parallel component of phase velocity of the disturbance of the EM field (along the applied magnetic field), suffers a resistive force which can be modeled as: $f_{\parallel} \sim |p|^\lambda |B_{\text{ext}}|^\delta$ (for some $\lambda > 0$ and $0 < \delta < 1$, which must come from a fundamental theoretical description), possibly from the quantized property of the background.

5 Discussion

A possible modification to Kruglov’s nonlinear electrodynamics is suggested in the light of GUP. By, considering perturbative quantum gravity effects in leading order, we make modifications to the model, whereupon we obtain the zero-point field strength. The behavior of zero-point field strength depicts a variation in photon momentum. Lastly, by studying vacuum birefringence in the presence of an external, constant, and uniform magnetic field, we conclude a momentum dependency in the behavior of $\Delta n$. The minimal length modifications also suggest that the phase velocity along the parallel direction of polarization is lower than the usual value, i.e., $v_{\parallel} < v_1$. This is due to the presence of an unavoidable coupling between the momentum of the field quanta (photon) and the minimum permissible length associated with the quanta of the background.

Such theoretical implications suggest the presence of a frictional mechanism (slowing down of velocity in the direction of the applied field), possibly originating from the discreteness of the background. The existence of similar quantum-gravitational friction, affecting photons moving in space-time foam has also been discussed by J. Ellis et al. [47].

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