RELATIVISTIC RADIATIVE TRANSFER
FOR SPHERICAL FLOWS

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Abstract

We present a new complete set of Lagrangian relativistic hydrodynamical equations describing the transfer of energy and momentum between a standard fluid and a radiation fluid in a general non-stationary spherical flow. The new set of equations has been derived for a particular application to the study of the cosmological Quark–Hadron transition but can also be used in other contexts.
1. Introduction

An important problem in the study of relativistic plasmas is the correct treatment of the coupling between radiation and matter. The assumptions of the classical theory of radiative transfer are no longer adequate near to astrophysical compact objects or for some cosmological situations and a relativistic theory of radiation hydrodynamics is required for these.

In this paper we derive a new system of Lagrangian relativistic hydrodynamic equations for describing general non-stationary spherical flows in which a transfer of energy and momentum occurs between a standard perfect fluid and a generalized “radiation fluid” (i.e. a fluid composed of any effectively zero rest-mass particles having longer mean-free-path than those of the standard fluid). Our primary interest is in application of these equations to cosmological phase transitions and to spherical accretion onto black holes. This paper presents the basic system of equations; further details of the applications will be given in subsequent papers.

Consistent treatments of relativistic radiation hydrodynamics have only been developed fairly recently and among these, particular interest has been focussed on the PSTF tensor formalism devised by K.S. Thorne in 1981 [1, 2] and the covariant flux-limited diffusion theory (FLDT) approach proposed by A.M. Anile et al. [3]. Both methods represent approximations to a problem which is too complicated to handle in full detail at the present time and there has been some debate about their respective merits. Since the PSTF method has already been used for a number of successful calculations [4, 5, 6], we have followed this approach here but we await with interest results from applications of the alternative method. We will comment in subsequent sections on our reasons for thinking that the approach which we are following is a satisfactory one in the present context.

In a recent paper, M.-G. Park [7] has presented a system of relativistic radiation hydrodynamic equations for treating a time-dependent spherical accretion flow onto a black hole or neutron star. The approach there is rather different from that of the present work and, in particular, Park writes the fluid equations in an Eulerian (fixed) frame. While this form is convenient for some purposes, the Lagrangian formulation has great advantages for many practical applications in computations of one-dimensional time-dependent flows (having planar, cylindrical or spherical symmetry).

In Section 2 we briefly describe the fundamentals of the PSTF formalism and in Sections 3 and 4 we discuss the derivation of the hydrodynamical equations
for the radiation fluid and the standard fluid. We here follow the approach of the earlier papers [8, 9, 10] which are relevant background for the present work. The subsequent sections introduce further additional equations which are necessary for treating the problem of bubble growth during the cosmological Quark–Hadron transition. Section 5 contains a short description of the scenario for the transition and in Section 6 we present interface junction conditions and characteristic equations for the treatment of a discontinuity surface between two phases of the standard fluid. Discussion of the numerical implementation of the formalism presented here will be delayed until a subsequent paper [11] but, as evidence that a successful implementation can be made, some sample results are shown in Section 7.

In view of our application to a problem involving input from particle physics, we use in this paper a system of units in which $c = \hbar = k_B = 1$ rather than $c = G = 1$ as is usual for calculations in general relativity. This also has the advantage that the gravitational source terms are clearly identified because of retention of the constant $G$ in the equations. (Note that Thorne [1, 2] adopted the slightly different convention $c = h = 1$). We use a space-like signature ($-$, $+$, $+$, $+$). Greek indices are taken to run from 0 to 3 and Latin indices from 1 to 3. Covariant derivatives are denoted with a semi-colon and partial derivatives with a comma.

2. The PSTF Tensor Formalism

The PSTF tensor formalism is a technique for solving the general relativistic form of the radiative transfer equation, which describes the variation of the radiation field as it propagates through a standard fluid. The relativistic form of this is straightforward and can be written as

$$\frac{dN}{dl} = \Sigma,$$  \hspace{1cm} (1)

where $N$ is the distribution function for the photons (a relativistic invariant), $l$ is a non-affine parameter measuring the proper spatial distance travelled by the photons as seen from the standard fluid, and $\Sigma$ is a source function. Note that the total derivative is taken not just in the space-time but rather in the phase-space since $N = N(x^\alpha, p^\alpha)$, where $p$ is the photon four-momentum.

The fundamental idea of the PSTF method consists of replacing equation (1) (which is in a concise form but embodies enormous complexity) by a hierarchy of moment equations written in terms of Projected Symmetric and Trace-Free
tensors which are suitably defined at each point in the projected tangent space to the fluid four-velocity $u$. The $k$-th moments of $N$ and $\Sigma$ are

$$M_{\nu}^{\alpha_1 \ldots \alpha_k} = \left( \int \frac{N \delta(2\pi\nu + p \cdot u)}{(-p \cdot u)^{k-2}} p^{\alpha_1} \ldots p^{\alpha_k} dV_p \right)^{PSTF},$$  \hspace{1cm} (2)$$

$$S_{\nu}^{\alpha_1 \ldots \alpha_k} = \left( \int \frac{\Sigma \delta(2\pi\nu + p \cdot u)}{(-p \cdot u)^{k-2}} p^{\alpha_1} \ldots p^{\alpha_k} dV_p \right)^{PSTF},$$  \hspace{1cm} (3)$$

where $\nu$ is the specific frequency under consideration, $dV_p$ is the invariant momentum-space volume element on the light cone and $\delta(y)$ is the Dirac delta function. In the following, we will refer to $M_{\nu}^{\alpha_1 \ldots \alpha_k}$ and $S_{\nu}^{\alpha_1 \ldots \alpha_k}$ simply as the $k$-th moment and source moment respectively.

The expressions (2) and (3) can be integrated over frequency and a clear physical interpretation can be given for the first three integrated moments of the hierarchy: $M$ (the zero-th moment) is the energy density of the radiation, $M^{\alpha}$ (the first moment) is the radiative energy flux, and $M^{\alpha\beta}$ (the second moment) is the shear stress tensor of the radiation fluid (each quantity being measured in the local rest frame of the standard fluid). The stress-energy tensor for the radiation $T^{\alpha\beta}_R$ is completely defined in terms of the first three moments and higher order moments do not enter into this definition. The expression for it is

$$T^{\alpha\beta}_R = M^{\alpha} u^{\beta} + 2M^{(\alpha} u^{\beta)} + M^{\alpha\beta} + \frac{1}{3} MP^{\alpha\beta},$$  \hspace{1cm} (4)$$

where $P^{\alpha\beta}(\equiv g^{\alpha\beta} + u^{\alpha} u^{\beta})$ is the projection operator orthogonal to $u$. A consequence of this is that, if the hierarchy is truncated at the second order, it is possible to derive the equations governing the hydrodynamics of the radiation fluid in a particularly simple way by starting from the conservation laws of energy and momentum. If, on the other hand, orders higher than the second are retained, it is necessary to make direct use of the appropriate hierarchy of equations derived from (1).

In the case of planar or spherical symmetry, the $2k + 1$ independent components of each $k$ rank tensor depend on a single scalar variable so that the tensor formalism reduces to a purely scalar one. This simplification has made it possible for the method to be used for a number of astrophysical applications. The hierarchy of integrated scalar moment equations, into which equation (1) is recast, has the property that, for any $k$, the first $k$ equations involve the first $k + 1$ moments. In order to use this scheme for making calculations, it is necessary to truncate
the moment hierarchy at some finite order by introducing a closure relation which specifies the value of the highest moment used in terms of lower ones and which is derived on the basis of physical considerations.

In the next section we use the PSTF approach to derive a new set of hydrodynamical equations describing the coupling between radiation and matter in the case of a non-stationary spherically symmetric flow, with the moment equation hierarchy being truncated at the second order.

3. Hydrodynamics of the Radiation Fluid

We consider a spherically symmetric flow in which a transfer of energy and momentum occurs between a generalized “radiation fluid” and a standard fluid. It is convenient to use a Lagrangian frame comoving with the standard fluid and the spherically symmetric line element

\[ ds^2 = -a^2 dt^2 + b^2 d\mu^2 + R^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \],

(5)

where \( \mu \) is a comoving radial coordinate and \( R \) is an associated Eulerian coordinate (the Schwarzschild circumference coordinate).

We here describe the radiation hydrodynamics using the first two moment equations together with a closure relation and the calculations then involve the first three moments, together with the first two source moments. The maximum errors in the calculated values of the radiation variables which arise when truncating at the second order, are typically \( \sim 15\% \) [8]. In view of the fact that other uncertainties in the specification of the problem are of a comparable order, we regard this level of approximation as acceptable for the present purposes. Similar degrees of accuracy are reported for calculations using FLD schemes [12].

For spherical symmetry the first three moments can be written as

\[ \mathcal{M} = w_0, \]

(6)

\[ \mathcal{M}^\alpha = w_1 e_\alpha^\circ, \]

(7)

and

\[ \mathcal{M}^{\alpha\beta} = w_2 \left( e_\alpha^\circ e_\beta^\circ - \frac{1}{2} e_\alpha^\circ e_\beta^\theta - \frac{1}{2} e_\alpha^\circ e_\beta^\phi \right), \]

(8)

where \( w_0, w_1 \) and \( w_2 \) are the scalar moments and \((e_\circ, e_r, e_\theta, e_\phi)\) is the orthonormal tetrad carried by an observer comoving with the standard fluid.
The quantities $w_0$, $w_1$ and $w_2$ all have direct physical interpretations corresponding to those of the related tensor moments: $w_0$ and $w_1$ are the energy density and flux of the radiation in the fluid rest frame, while $w_2$ is the shear stress scalar of the radiation. The scalar source moments $s_0$ and $s_1$, defined in a similar way to $w_0$ and $w_1$, also have direct physical interpretations, representing the transfer of energy and momentum between the two fluids. We use the following expressions for these scalar source moments

$$s_0 = \frac{1}{\lambda} (\epsilon - w_0) + (s_0)_{SC},$$

$$s_1 = -\frac{w_1}{\lambda},$$

where $\lambda$ is the effective mean-free-path of the radiation particles as they move through the standard fluid, $(s_0)_{SC}$ is a term expressing the contribution due to scatterings, whose form depends on the specific problem, and $\epsilon$ is the energy density for radiation in thermal equilibrium with the standard fluid. Assuming that it follows a black-body law, $\epsilon$ can be written as

$$\epsilon = g_R \left( \frac{\pi^2}{30} \right) T_F^4,$$

with $g_R$ being the number of degrees of freedom of the radiation fluid and $T_F$ the temperature of the standard fluid.

Having restricted ourselves to the use of the first three scalar moments and the first two scalar source moments, it is convenient to derive the radiation hydrodynamical equations by means of the standard conservation laws for the energy and momentum of the radiation fluid applied to the stress-energy tensor (4). Following this procedure, which is equivalent to using the scalar moment equations, we then write the following three radiation hydrodynamical equations

$$-u_\alpha T^\alpha_{\beta ; \beta} = s_0,$$

$$n_j P^j \alpha T^\alpha_{\beta ; \beta} = \frac{s_1}{b},$$

$$w_2 = f_E w_0,$$

where $\mathbf{n}$ is a radial spacelike unit vector normal to $\mathbf{u}$.

The term $f_E$ in the closure relation (14) is a variable Eddington factor and indicates the degree of anisotropy of the radiation. It can take values ranging
from 0 for complete isotropy (which could, for example, be caused by the medium being extremely optically thick) to 2/3 for complete anisotropy. A key point in the present technique is that an expression for \( f \) has to be supplied, constructed on the basis of physical considerations and how this is done is, to some extent, \textit{ad hoc}. However, experience has shown that as long as the expression has the correct asymptotic behaviour in any relevant limits, results do not normally depend sensitively on the precise form used as long as it gives a suitably smooth join between the limits [4]. This is something which needs to be checked in any particular application but provided that the outcome of such a check is satisfactory, it is reasonable to proceed with confidence.

Making use of the stress-energy tensor (4) and of the line element (5), equations (12) and (13) can be written explicitly as

\begin{align}
(w_0),_t + a \left( \frac{1}{b} w_1, \mu \right) + 4 \left( \frac{b,t}{b} - \frac{R,t}{R} \right) w_0 + 2a \left( \frac{a, \mu}{a} + \frac{R, \mu}{R} \right) w_1 + \left( \frac{b,t}{b} - \frac{R,t}{R} \right) w_2 &= a s_0, \\
(w_1),_t + a \left( \frac{1}{3} w_0 + w_2 \right),_\mu + 4a \left( \frac{1}{3b} w_0 + w_2 \right), \mu + 2 \left( \frac{b,t}{b} + \frac{R,t}{R} \right) w_1 + a \left( \frac{a, \mu}{a} + \frac{3R, \mu}{R} \right) w_2 &= a s_1,
\end{align}

(15)

(16)

(see Appendix A for details). Equations (14) – (16) are our final form of the hydrodynamical equations for the radiation fluid and they need to be solved together with the corresponding hydrodynamical equations for the combined fluids which will be discussed in the next section.

### 4. Hydrodynamics of the Combined Fluids

The derivation of the hydrodynamical equations for the combined fluids (i.e. standard fluid and radiation fluid) is, in principle, more straightforward. We follow the conventions and notation of the earlier papers [9, 10] (except that here we always write partial derivatives using a comma) and start from the Einstein field equations

\[ R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}, \]

(17)

with \( T^{\alpha\beta} = T_R^{\alpha\beta} + T_F^{\alpha\beta} \) being the \textit{total} stress-energy tensor, and treat the standard fluid as perfect so that \( T_F^{\alpha\beta} = (e + p)u^\alpha u^\beta + p g^{\alpha\beta} \). The four independent equations which follow from (17) can be used to write out explicitly the conservation
equations for energy and momentum of the combined fluids and the continuity equation for the standard fluid

\[-u_\alpha T^{\alpha\beta}_{;\beta} = 0,\]  
\[n_j P^j_\alpha T^{\alpha\beta}_{;\beta} = 0,\]  
\[(pu^\alpha)_{;\alpha} = 0,\]

(18)  
(19)  
(20)

where \(\rho\) is the relative compression factor expressing the variation in the proper volume of co-moving elements of the standard fluid. (For a classical standard fluid composed of non-relativistic particles, \(\rho\) can represent the rest-mass density.)

After some algebra, whose relevant steps are shown in Appendix A, it is possible to write the following set of hydrodynamical equations

\[u_{,t} = -a \left[ \frac{\Gamma}{b} \left( \frac{p_\mu + bs_1}{e + p} \right) + 4\pi GR \left( p + \frac{1}{3}w_0 + w_2 \right) + \frac{GM}{R^2} \right],\]  
\[R_{,t} = au,\]  
\[\frac{(pR^2)_{,t}}{\rho R^2} = -a \left( \frac{u_\mu - 4\pi Gb Rw_1}{R_{,\mu}} \right),\]  
\[e_{,t} = wp_{,t} - as_0,\]  
\[\frac{(aw)_{,\mu}}{aw} = -\frac{wp_{,\mu} - e_{,\mu} + bs_1}{\rho w},\]  
\[M_{,\mu} = 4\pi R^2 R_{,\mu} \left( e + w_0 + \frac{u}{\Gamma} w_1 \right),\]  
\[\Gamma = \left( 1 + u^2 - \frac{2GM}{R} \right)^{1/2} = \frac{1}{b} R_{,\mu},\]  
\[b = \frac{1}{4\pi R^2 \rho}.\]

(21)  
(22)  
(23)  
(24)  
(25)  
(26)  
(27)  
(28)

Here \(u\) is the radial component of fluid four velocity in the associated Schwarzschild (Eulerian) frame, \(\Gamma\) is the general relativistic analogue of the Lorentz factor, and \(w\) is the specific enthalpy \((w = (e + p)/\rho)\). The generalized mass function \(M\) can also be calculated using the alternative equation

\[M_{,t} = -4\pi R^2 R_{,t} \left( p + \frac{1}{3}w_0 + \frac{\Gamma}{u} w_1 + w_2 \right).\]

(29)

The set of hydrodynamical equations (14) – (16) and (21) – (28) needs to be supplemented by an equation of state, relating the energy density \(e\), the pressure \(p\) and temperature \(T\) of the standard fluid. The basic set of equations is
then completed and can be used for describing the transfer of energy and momentum between a standard fluid and a radiation fluid in a general non-stationary relativistic flow.

It is worthy of note that the gravitational source terms appearing in the combined set of equations introduce only a very minor additional complication; for spherical symmetry, a general relativistic calculation is only marginally more complicated than a special relativistic one.

We next turn to the introduction of the further equations necessary for treating the problem of bubble growth during the cosmological Quark–Hadron transition. Here, there are two different phases of the standard fluid separated by a phase interface (the bubble surface) and so the above system of equations (which continue to hold for the bulk of each phase) needs to be supplemented by a treatment of the interface. Before discussing how this is done (in Section 6), Section 5 gives a brief introduction to the scenario within which we are working.

5. The Cosmological Quark–Hadron Transition

According to the standard Hot Big-Bang model, the early universe experienced a succession of phase transitions and breakings of symmetry which would have influenced the subsequent evolution. The last transition in this succession (which includes Inflation and the Electro-Weak transition) is usually thought to have been the Quark–Hadron transition at which strongly interacting matter passed from being a plasma of unconfined quarks and gluons to a plasma in which the quarks and gluons were confined within hadrons. Since it is thought to have been the last of the cosmological transitions, any remnants which it left behind could have had particular significance and so it is of considerable interest to carry out a detailed hydrodynamical study in order to see how it would have proceeded.

Chronologically, the transition is estimated to have taken place about 10$\mu$s after the Big-Bang, when the temperature of the universe was $\approx 200$ MeV, the mean density was $\approx 10^{15}$ g cm$^{-3}$ and the horizon scale was $\approx 10$ km. At present, QCD lattice calculations carried out to investigate the nature of the transition seem to indicate that it may well be a continuous one [13]. However, these calculations are extremely complicated and the results are not yet definitive. In view of the very interesting consequences which could arise if it turns out to be a first order (discontinuous) transition [14], extensive investigations have been carried
out into this latter picture. The work reported here is within this scenario in the
case where the transition starts with the nucleation of hadronic bubbles within
a slightly supercooled quark-gluon plasma. The bubbles of the new phase then
grow (the hadron phase is thermodynamically favoured) and subsequently coalesce
until eventually the universe is filled with the new phase except for remaining dis-
connected droplets of the quark-gluon plasma. These then shrink and can either
vanish completely, possibly leaving a significant inhomogeneity in the baryon num-
ber density [15], or perhaps attain a stable configuration in a new hypothetical
ground state for the strongly interacting matter (strange quark matter) [16].

The present paper is associated with an ongoing research programme
[9, 10, 11, 17] which has so far focussed on studying the hydrodynamics of the
growth of single spherical hadronic bubbles within a surrounding quark-gluon
plasma. While the phase transition directly involves only the strongly interacting
particles, a crucial role is also played by electromagnetically interacting particles
(photons, electrons, muons and their antiparticles) and particles which interact
only weakly (neutrinos and antineutrinos). These can couple to the quark and
hadronic plasmas and provide a mechanism for long-range transport of energy
and momentum on account of having longer mean-free-paths than the strongly
interacting particles. The effect of this starts to be significant when the radius of
the growing bubble becomes comparable with the relevant mean-free-path (∼ 10³
fermi for the electromagnetic interaction and ∼ 1 cm for the weak interaction) and
eventually produces an effective total coupling between the respective components
when the radius of the bubble is much greater than the relevant mean-free-path.

Because of the large difference between these mean-free-paths, long-range
transport can become important at two different stages during the growth of a
bubble. However, the effects on the hydrodynamics will be exactly the same
in each case and so we will omit any distinction between them here. The term
“standard fluid” will always be taken to refer to all of those particles having mean-
free-paths small compared with the current relevant scale-length for changes in the
flow of the strongly-interacting matter.

As mentioned in Section 4, the hydrodynamical equations need to be supple-
mented by equations of state for each of the two phases. For simplicity, we have
taken the equation of state for the hadronic matter to be that for an ideal gas of
massless point-like pions

\[ e_h = g_h \left( \frac{\pi^2}{30} \right) T_h^4 \]

\[ p_h = \frac{1}{3} e_h, \quad (30) \]
where $T$ is the temperature and $g$ is the degeneracy factor (the subscripts $h$ and $q$ indicate quantities in the hadron and quark phases respectively). The deconfined quarks and gluons cannot be considered as entirely free and for these we have used the phenomenological expression given by the M.I.T. Bag Model \[18]\]

$$
\begin{align*}
e_q &= g_q \left( \frac{\pi^2}{30} \right) T^4_q + B \\
p_q &= g_q \left( \frac{\pi^2}{90} \right) T^4_q - B,
\end{align*}
$$

(31)

where a positive constant (the “Bag” constant $B$) is added to the energy density and subtracted from the pressure so as to take into account the complex effects of confinement. When the photons and relativistic leptons are completely coupled to the strongly interacting matter, their contribution can be included by incrementing $g_h$ and $g_q$ by the relevant number of additional degrees of freedom.

A final comment should be made about the expression adopted in our calculations for the term $(s_0)_{SC}$, which, as mentioned in Section 3, represents the transfer of energy by means of scatterings. While complete expressions for this are available for simpler applications \[4\], the lack of detailed knowledge of the interaction processes in the present context has led us to use a phenomenological approach, expressing $(s_0)_{SC}$ as equal to the absorption and emission term, (i.e. $(\epsilon - w_0)/\lambda$), multiplied by an adjustable coefficient ranging between zero and one. Fortunately, the results of our calculations (which will be presented in paper \[11\]) turn out not to depend sensitively on the value chosen for this coefficient.

6. Treatment of the Phase Interface

As long as the radius of the bubble is large compared with the strong-interaction length scale, it is reasonable to treat the interface between the hadron and quark phases as an exact discontinuity surface with the variables on either side of it being linked by junction conditions. The phase interface could, in principle, move either supersonically (as a detonation front) or subsonically (as a deflagration front) with respect to the medium ahead. In practice, however, the bubbles are almost certain to expand subsonically in the present situation (see \[9, 19\]) and we restrict our attention to this case here.

In contrast with the situation for calculations of detonation fronts, for a deflagration it is necessary to supplement the hydrodynamical equations with an independent expression for the rate at which quark matter is converted into hadronic
matter, derived from considerations of the physical processes occurring at the interface. A simple expression for this rate was presented in [9] and setting this equal to the hydrodynamical flux across the interface, gives the additional equation

\[ \frac{a w \dot{\mu}_s}{4\pi R_s^2 (a^2 - b^2 \dot{\mu}_s^2)} = \left( \frac{\alpha}{4} \right) g_h \left( \frac{\pi^2}{30} \right) (T_q^4 - T_h^4) \]  

(32)

where \( \mu_s \) is the interface location, \( \dot{\mu}_s = d\mu_s/dt \) and \( \alpha \) is an accommodation coefficient \((0 \leq \alpha \leq 1)\).

Also, it is very important to pay attention to a correct calculation of the causal structure in the vicinity of the interface and it seems that the only satisfactory way of accomplishing this, when the interface is treated as a discontinuity, is by making use of a characteristic method [20]. This involves rewriting the system of partial differential equations as a system of ordinary differential equations along characteristic curves in the space-time (which can be physically interpreted as the worldlines of sonic perturbations in the standard fluid and radiation fluid and the flowlines of the standard fluid). The computational technique employed is then to use the continuum equations of the previous sections for the bulk of each phase and to track the interface continuously through the finite difference grid using a characteristic method together with junction conditions (see [10, 11]).

For each two moment equations which are retained from the infinite hierarchy, there are two families of corresponding characteristic curves with associated characteristic speeds. While including a larger number of moments in general increases accuracy, the role and relevance of the speeds associated with moments beyond the first two is controversial. We note that since we are using only the first two moment equations in the present work, these difficulties do not arise here.

In order to write the equations in a characteristic form, it is convenient to make use of the following equalities coming from the constraint equations obtained from (17) and from the conservation of momentum

\[ \frac{b_t}{b} = a \frac{u_{\mu}}{R_{\mu}} - \frac{4\pi G a b R}{R_{\mu}} w_1, \]  

(33)

\[ \frac{a_{\mu}}{a} = - \frac{p_{\mu} + b s_1}{\rho w}, \]  

(34)

and then to rewrite equations (15), (16), (21), (23), (24) as

\[ u_{,t} + \frac{a \Gamma}{b \rho w} p_{,\mu} + B_0 = 0, \]  

(35)
\[ p, t + \frac{c_s^2 a \rho w}{b \Gamma} u, \mu + B_1 = 0, \]  
(36)

\[ \rho, t + \frac{a \rho}{b \Gamma} u, \mu + B_2 = 0, \]  
(37)

\[ (w_0), t + \frac{a}{b} (w_1), \mu + \frac{a}{b \Gamma} \left( \frac{4}{3} + f_E \right) w_0 u, \mu - \frac{2a}{b \rho w} w_1 p, \mu + B_3 = 0, \]  
(38)

\[ (w_1), t + \frac{a}{b} \left( \frac{1}{3} + f_E \right) (w_0), \mu - \frac{a}{b \rho w} \left( \frac{4}{3} + f_E \right) w_0 p, \mu + \frac{2a}{b \Gamma} w_1 u, \mu + B_4 = 0, \]  
(39)

where

\[ B_0 = a \left\{ 4 \pi GR \left[ p + \left( \frac{1}{3} + f_E \right) w_0 \right] + \frac{GM}{R^2} + \frac{\Gamma}{\rho w} s_1 \right\}, \]  
(40)

\[ B_1 = a c_s^2 \left[ s_0 + \rho w \left( \frac{2u}{R} - \frac{4 \pi GR}{\Gamma} w_1 \right) \right], \]  
(41)

\[ B_2 = a \rho \left( \frac{2u}{R} - \frac{4 \pi GR}{\Gamma} w_1 \right), \]  
(42)

\[ B_3 = 2a \left( \frac{\Gamma}{R} - \frac{1}{\rho w} s_1 \right) w_1 + \frac{a R}{R} \left( \frac{8}{3} - f_E \right) w_0 - \frac{4 \pi Ga R}{\Gamma} \left( \frac{4}{3} + f_E \right) w_0 w_1 - as_0, \]  
(43)

\[ B_4 = -\frac{a}{\rho w} \left( \frac{4}{3} + f_E \right) w_0 s_1 + 2a \left( \frac{u}{R} - \frac{4 \pi GR}{\Gamma} w_1 \right) w_1 + \frac{3a \Gamma}{R} f_E w_0 - as_1 + \frac{a}{b} w_0 (f_E), \mu. \]  
(44)

In deriving equations (35) – (39), the relations \( R, t = au \) and \( R, \mu = b \Gamma \) have been used and we have introduced the local sound speed in the standard fluid \( c_s = (\partial p/\partial e)^{1/2} \). If we now define the state vector

\[
U = \begin{pmatrix}
    u \\
p \\
\rho \\
w_0 \\
w_1
\end{pmatrix},
(45)
\]

equations (35) – (39) can be written in the symbolic form

\[
\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial \mu} + B = 0,
(46)
\]

where \( B \) is the vector whose components are given by \( (40) – (44) \).
If the expression chosen for the Eddington factor $f_E$ is dependent on components of the state vector $U$, it is necessary to rewrite the partial derivative of $f_E$ in (44) in terms of the derivatives of the component variables. In doing this the elements of the matrix $A$ are obviously modified. For our specific application, the expression which we have chosen for the Eddington factor is the following (see Appendix B for a discussion of this)

$$f_E = \frac{8u^2/9}{(1 + 4u^2/3)} \left( \frac{\lambda_{h,q}}{\lambda_{h,q} + R} \right).$$

Equation (39) then becomes

$$\left( w_1 \right)_{,t} + \frac{a}{b} \left( \frac{1}{3} + f_E \right) \left( w_0 \right)_{,\mu} - \frac{a}{b \rho w} \left( \frac{4}{3} + f_E \right) w_0 p_{,\mu} + \frac{2a}{b \Gamma} \left( 1 + K \right) w_1 u_{,\mu} + B_4 = 0,$$

with

$$B_4 = -\frac{a}{\rho w} \left( \frac{4}{3} + f_E \right) w_0 s_1 + 2a \left( \frac{u}{R} - \frac{4\pi GR}{\Gamma} \right) w_1 +$$

$$+ \frac{3a \Gamma}{R} f_E w_0 - a s_1 - \frac{a \Gamma f_E w_0}{\lambda_{h,q} + R},$$

where, for compactness, we have defined

$$K = f_E \frac{\Gamma w_0}{u(1 + 4u^2/3)w_1}.$$  

The matrix $A$ then takes the form

$$A = \begin{pmatrix}
0 & a \Gamma / b \rho w & 0 & 0 & 0 \\
c_2^2 a \rho w / b \Gamma & 0 & 0 & 0 & 0 \\
a \rho / b \Gamma & 0 & 0 & 0 & 0 \\
a w_0 (4/3 + f_E) / b \Gamma & -2aw / b \rho w & 0 & 0 & a / b \\
2aw_1 (1 + K) / b \Gamma & -aw_0 (4/3 + f_E) / b \rho w & a (1/3 + f_E) / b & 0 & 0
\end{pmatrix}.$$  

Next, we introduce $l_i$, the set of left eigenvectors of $A$, and $\lambda_i$, the corresponding eigenvalues satisfying the relations

$$l_i A = \lambda_i l_i.$$
Equation (52) has five distinct eigenvalues (the system is hyperbolic)

\[ \lambda_0 = 0, \]
\[ \lambda_{1,2} = \pm \frac{a}{b} c_s, \]
\[ \lambda_{3,4} = \pm \frac{a}{b} \sqrt{\frac{1}{3} + f_E}, \]

to which correspond the five eigenvectors

\[ l_0 \equiv k \left( 0, \pm \frac{1}{c_s^2 w}, 1, 0, 0 \right), \]
\[ l_{1,2} \equiv k \left( \pm \frac{c_s \rho w}{\Gamma}, 1, 0, 0, 0 \right), \]
\[ l_{3,4} \equiv k \left( \pm \frac{2(c_s^2 - 1 - K)(1/3 + f_E)^{1/2} w_1 + 1/\Gamma \left( \frac{4}{3} + f_E \right) w_0}{\rho w(c_s^2 - 1/3 - f_E)} \right), \]

where \( k \) is an arbitrary constant.

Equation (46) can then be multiplied on the left by \( l_i \) so as to obtain the symbolic expression

\[ l_i \left[ \frac{\partial U}{\partial t} + \lambda_i \frac{\partial U}{\partial \mu} \right] + l_i B = 0. \]

in which each component involves derivatives only along the characteristic direction given by \( d\mu/dt = \lambda_i \). Writing out system (59) explicitly we then get the following characteristic form of the equations (35) – (39)

\[ du \pm \frac{\Gamma}{\rho w c_s} dp + a \left\{ \frac{\Gamma}{\rho w} (s_1 \pm c_s s_0) + 4\pi GR \left[ p + \left( \frac{1}{3} + f_E \right) w_0 \mp c_s w_1 \right] + GM \frac{R}{R^2} \pm \frac{2\Gamma u c_s}{R} \right\} dt = 0, \]
which are to be solved along the forward and backward characteristics of the standard fluid $d\mu = \pm (a/b)c_s \, dt$,

\[ dw_1 \pm \left( \frac{1}{3} + f_E \right)^{1/2} \, dw_0 + \]

\[ + \left[ \left( \frac{4}{3} + f_E \right) w_0 \pm \frac{2(c_s^2 - 1 - K)(1/3 + f_E)^{1/2}}{c_s^2 - 1/3 - f_E} \right] \frac{1}{\Gamma} \, du + \]

\[ + \left[ \frac{2(f_E - 2/3 - K)}{\rho w(c_s^2 - 1/3 - f_E)} \right] w_1 \, dp + a \left\{ \left( \frac{2u}{R} - \frac{4\pi GRw_1}{\Gamma} \right) \times \right. \]

\[ \left. \left[ \frac{2[(1/3 + f_E)(c_s^2 - 1) - Kc_s^2]}{c_s^2 - 1/3 - f_E} \right] w_1 \pm \left( \frac{4}{3} + f_E \right) \left( \frac{1}{3} + f_E \right)^{1/2} w_0 \right] + \]

\[ + \left( \frac{4\pi R}{p + w_0 \left( \frac{1}{3} + f_E \right)} \right) + \left( \frac{M}{R^2} \right) \times \]

\[ \times \left[ \left( \frac{4}{3} + f_E \right) w_0 \pm \frac{2(c_s^2 - 1 - K)(1/3 + f_E)^{1/2}}{c_s^2 - 1/3 - f_E} \right] \frac{G}{\Gamma} + \]

\[ - \frac{Ku(1 + 4u^2/3)}{\lambda_{h,q}(1 + R/\lambda_{h,q})} \, w_1 + \]

\[ - \frac{1}{R} \left\{ 3f_E \left[ \pm \left( \frac{1}{3} + f_E \right)^{1/2} u - \Gamma \right] w_0 - 2 \left[ \pm \left( \frac{1}{3} + f_E \right)^{1/2} \Gamma - u \right] w_1 \right\} + \]

\[ + \left[ \frac{2c_s^2}{\rho w(c_s^2 - 1/3 - f_E)} \left( f_E - \frac{2}{3} - K \right) \right] w_1 \pm \left( \frac{1}{3} + f_E \right)^{1/2} \right] s_0 + \]

\[ + \left[ \pm \frac{2}{\rho w(c_s^2 - 1/3 - f_E)} \left( f_E - \frac{2}{3} - K \right) \left( \frac{1}{3} + f_E \right)^{1/2} w_1 - 1 \right] s_1 \right\} \, dt = 0, \]

\[ \text{(61)} \]

which are to be solved along the forward and backward characteristics of the radiation fluid $d\mu = \pm (a/b)(1/3 + f_E)^{1/2} \, dt$, and

\[ d\rho - \frac{1}{c_s^2 w} \, dp - \frac{a s_0}{w} \, dt = 0, \]

\[ \text{(62)} \]

which is an advective equation and is to be solved along the flowlines of the standard fluid $d\mu = 0$. Finally, $R$ and $M$ are calculated from advective equations

\[ dR = au \, dt, \]

\[ \text{(63)} \]

\[ dM = -4\pi R^2 au \left[ p + \left( \frac{1}{3} + f_E \right) w_0 + \frac{G}{u} w_1 \right] \, dt, \]

\[ \text{(64)} \]
and the metric coefficient $a$ is calculated from (25) which is a constraint equation on the constant $t$ hypersurface (i.e. it is to be integrated along the direction $dt = 0$).

The configuration of characteristic curves adjacent to the interface is shown in Figure 1 for evolution of the system from time level $t$ to a subsequent time level $t + \Delta t$. The dashed lines represent the forward and backward characteristics for the radiation fluid $r$, the full narrow lines are the equivalent characteristics for the standard fluids $f$, the vertical dotted line is the advective characteristic for strongly interacting matter in the quark phase and the heavy line is the worldline of the interface.

Figure 1. The configuration of characteristic curves near the phase interface drawn in the Lagrangian coordinate frame.

The difference between the sound speeds in the standard fluid ($c_s$) and in the radiation fluid (\((1/3 + f_E)^{1/2}\)) is large when the former is non-relativistic but, in the present case, the standard fluid is relativistic (with $c_s \rightarrow 1/\sqrt{3}$) and the difference between the sound speeds is frequently small. This leads to some serious complications in numerical solution of the equations which we will be discussing in paper [11].

The junction conditions linking the values of quantities on either side of the interface need to take account of the surface tension and surface energy within it. A complete relativistic treatment of this problem can be given most conveniently by using the Gauss-Codazzi formalism [21, 22]. In view of some recent comments [23], it is worth stressing that this is an economical way to proceed when working within a relativistic Lagrangian framework even in situations where gravity can be neglected.
Following the approach described in [9], we can write the following equations expressing continuity across the interface of $R$, $dR/dt$ and $ds$

\[ [R]^{\pm} = 0, \]  
\[ [au + b\dot{\mu}_s \Gamma]^{\pm} = 0, \]  
\[ [a^2 - b^2 \dot{\mu}_s^2]^{\pm} = 0, \]

where $[A]^{\pm} = A^+ - A^-$, $\{A\}^{\pm} = A^+ + A^-$ and the superscripts $\pm$ indicate quantities immediately ahead of and behind the interface.

The junction conditions for the energy and momentum of the standard fluids are

\[ [(e + p)ab]^{\pm} = 0, \]  
\[ [eb^2 \dot{\mu}_s^2 + pa^2]^{\pm} = -\frac{\sigma f^2}{2} \left\{ \frac{1}{ab} \frac{d}{dt} \left( \frac{b^2 \dot{\mu}_s}{f} \right) + \frac{f \mu}{ab} + \frac{2}{f R} (b \dot{\mu}_s u + a \Gamma) \right\}^{\pm}, \]

where $\sigma$ is the surface tension (taken here to be independent of temperature) and $f = (a^2 - b^2 \dot{\mu}_s^2)^{1/2}$. Since the thickness of the interface is much smaller than the mean-free-path of the particles of the radiation fluid, it is reasonable to neglect any interaction of the latter with the interface, so that the energy and momentum junction conditions for the radiation fluid simply reduce to the continuity equations

\[ \left[ ab\dot{\mu}_s \left( \frac{4}{3} + f_E \right) w_0 - (a^2 + b^2 \dot{\mu}_s^2)w_1 \right]^{\pm} = 0, \]  
\[ \left\{ a^2 \left( \frac{1}{3} + f_E \right) + b^2 \dot{\mu}_s^2 \right\} w_0 - 2ab\dot{\mu}_s w_1 \right]^{\pm} = 0. \]

The mass function $M$ receives a contribution from the surface energy. At the time of nucleation of the bubble, conditions are essentially Newtonian so that

\[ [M]^{\pm} = 4\pi R^2 \sigma, \]

and the subsequent time evolution is given by

\[ \frac{d}{dt} [M]^{\pm} = 4\pi R^2 \left[ b\Gamma \dot{\mu}_s \left\{ e + w_0 + \frac{u w_1}{\Gamma} \right\} - au \left\{ p + \left( \frac{1}{3} + f_E \right) w_0 + \frac{\Gamma w_1}{u} \right\} \right]^{\pm}. \]

Adding conditions (65) – (73) to equations (14) – (16), (21) – (31), and using the characteristic method (equations (60) – (64)) to give a correct treatment of the causal structure, a satisfactory calculation of the growth of a cosmological
hadronic bubble in the presence of long range energy and momentum transfer can then be made.

7. Numerical Computations of Bubble Growth

A computer code has been constructed in order to implement the formalism presented in the previous sections so as to make calculations of the growth of hadronic bubbles during the cosmological Quark-Hadron transition. As mentioned previously, the code advances the solution from one time-level to the next using a standard Lagrangian finite-difference form of the continuum equations for the bulk of each phase, while the interface is tracked continuously through the grid using a characteristic method together with imposition of junction conditions. This code is a further development of the one described in reference [10]. A complete account of the numerical techniques used for these new calculations and of the results obtained will be presented in our forthcoming paper [11].

![Figure 2. Time evolution of the radiation energy density $w_0$. The sequence of graphs shows the change in the profile of $w_0$ as the bubble expands. The horizontal dashed line shows the initial profile at the time of bubble nucleation.](image)

As a sample of the results, Figure 2 shows the progressive coupling together of the radiation fluid and the standard fluid as a hadronic bubble expands with the mean-free-path of the radiation particles becoming progressively smaller compared...
with the radius of the bubble. Eventually, the coupling becomes essentially total, the radiative transfer of energy and momentum ceases to operate and the flow tends towards self-similarity.

8. Conclusion

Many astrophysical and cosmological situations involve non-stationary spherical relativistic flows in which a net transfer of energy and momentum occurs between a radiation fluid and a standard fluid. In order to describe this kind of flow, we have presented here a new set of Lagrangian hydrodynamical equations in which the radiative transfer problem is solved using the PSTF tensor formalism truncated at the second order. The equations obtained in this way describe the behaviour of the radiation variables and need to be coupled with the set of hydrodynamical equations derived from standard conservation laws.

One application of particular interest to us concerns bubble growth during the cosmological Quark–Hadron transition. We have used our set of equations in a numerical study of this problem, investigating the effects of energy and momentum transfer by weakly and electromagnetically interacting particles. In doing this we have treated the interface between the two phases of the standard fluid as an exact discontinuity surface and have tracked it continuously through the finite-difference grid using a characteristic method together with imposition of junction conditions derived with the aid of the Gauss-Codazzi equations. The interface moves subsonically and, within our scheme, use of the characteristic equations is then particularly important in order to give a correct treatment of the causal structure. A computer code has been constructed for solving this system of equations and a full account of the techniques used and results obtained will be presented in a subsequent paper [11].

The set of equations presented here is general in nature and it can be applied directly to other situations of interest. Work is now in progress on applying the equations of Sections 3 and 4 to the study of non-stationary spherical accretion onto a black hole. A computer code has been constructed for this and is at present under test.
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Appendix A

In this Appendix we give a brief sketch of the calculations leading from equations (12) – (13) and (17) – (20) to the hydrodynamical equations (15), (16) and (21) – (29). For clarity, we make a separation into two sub-sections, one for the radiation fluid alone and the other for the combined fluids.

THE RADIATION FLUID

As mentioned in Section 3, we use the spherically symmetric line element (5) and expression (4) for the stress-energy tensor of the radiation fluid. The first three PSTF moments are written as

\[ M = w_0, \]

\[ M^\alpha = w_1 e^\alpha_\hat{r}, \]

and

\[ M^{\alpha\beta} = w_2 \left( e^\alpha_\hat{r} e^\beta_\hat{r} - \frac{1}{2} e^\alpha_\hat{\theta} e^\beta_\hat{\theta} - \frac{1}{2} e^\alpha_\hat{\phi} e^\beta_\hat{\phi} \right), \]

where \( w_0, w_1 \) and \( w_2 \) can be physically interpreted as explained in Section 3 and \((e_\hat{0}, e_\hat{r}, e_\hat{\theta}, e_\hat{\phi})\) is the orthonormal tetrad carried by an observer comoving with the standard fluid. The components of this tetrad are

\[ e^\alpha_\hat{0} = \left( \frac{1}{a}, 0, 0, 0 \right), \]

\[ e^\alpha_\hat{r} = \left( 0, \frac{1}{b}, 0, 0 \right), \]

\[ e^\alpha_\hat{\theta} = \left( 0, 0, \frac{1}{R}, 0 \right), \]

\[ e^\alpha_\hat{\phi} = \left( 0, 0, 0, \frac{1}{R \sin \theta} \right). \]

The comoving observer’s four-velocity \( u \) and four-acceleration \( g \) are given by

\[ u^\alpha = \left( \frac{1}{a}, 0, 0, 0 \right) = e^\alpha_\hat{0}, \]

and

\[ g^\alpha = \left( 0, \frac{a \mu}{ab^2}, 0, 0 \right), \]
(so that \( e_\hat{\phi}^\alpha = g^\alpha / g \), where \( g = (g^\alpha g_\alpha)^{1/2} \)). The covariant derivative of the radiation fluid stress energy tensor is

\[
T^{\alpha \beta}_{R; \gamma} = \frac{4}{3} \left( M_{\gamma; \beta} u^\alpha u^\beta + M u^\alpha ; \beta u^\beta + M u^{\alpha; \beta} \right) + \frac{1}{3} M^{\alpha +} + \\
+ M^{\alpha; \beta} u^\gamma + M^{\alpha u^\alpha; \beta} + M^{\beta u^\alpha; \beta} + M^{\alpha; \beta},
\]

(A10)

where \( \Theta = u^{\alpha; \alpha} \) is the expansion and \( u^{\alpha; \beta} u^\beta = g^\alpha \). The contraction of expression (A10) with \( u^{\alpha; \alpha} \) then gives

\[
u_\alpha T^{\alpha \beta}_{R; \beta} = -M_{\gamma; \beta} u^\beta - \frac{4}{3} M \Theta - \frac{g^\beta}{g} w_{1; \beta} + \\
w_1 \left( \frac{g^\alpha}{g} \right) u^\beta u_\alpha - w_1 \left( \frac{g^\beta}{g} \right) + M^{\alpha; \beta} u_\alpha.
\]

(A11)

After some further manipulation of expression (A11) and using the equality

\[
u_\alpha M^{\alpha; \beta} = -\frac{(w_2)_\beta}{2} \left( e_\hat{\theta}^\alpha e_\hat{\theta}^\beta u_\alpha + e_\hat{\phi}^\alpha e_\hat{\phi}^\beta u_\alpha \right) + \\
w_2 \left[ \left( \frac{g^\alpha}{g} \right) ; \beta \right] g^\beta u_\alpha - \frac{1}{2} \left( e_\hat{\theta}^\alpha e_\hat{\theta}^\beta + e_\hat{\phi}^\alpha e_\hat{\phi}^\beta \right) ; \beta u_\alpha
\]

(A12)

it is then possible to rewrite equation (12) in the form (15). We proceed in the same way with the derivation of equation (13). Bearing in mind that \( n_j P^{j \alpha}_\alpha = \delta^1_{\alpha} \), it follows from (A10) that

\[
n_j P^{j \alpha}_\alpha T^{\alpha \beta}_{R; \beta} = \frac{4}{3} M g^1 + \frac{1}{3} M_{11} g^{11} + \\
+ M_{1; \beta} u^\beta + w_1 \left( \frac{g^1}{g} \right) \Theta + M^{\beta u^1; \beta} + M^{1; \beta},
\]

(A13)

Writing out each of the terms explicitly and using the expression

\[
M^{1; \beta} = \frac{1}{b^2} (w_2)_{, \mu} + \frac{w_2}{b^2} \left( a_{, \mu} + 3 \frac{R_{, \mu}}{R} \right),
\]

(A14)

equation (A13) can finally be recast in the form (16).
THE COMBINED FLUIDS

Expressions for the non-zero components of the total stress-energy tensor \( T_{\alpha\beta} = T_{\alpha\beta}^R + T_{\alpha\beta}^F \) can be calculated using equation (4) and expressions (A4) – (A7), and this gives

\[
T^{00} = \frac{1}{a^2}(e + w_0), \quad (A15)
\]
\[
T^{01} = \frac{w_1}{ab}, \quad (A16)
\]
\[
T^{11} = \frac{1}{b^2}\left(p + \frac{w_0}{3} + w_2\right), \quad (A17)
\]
\[
T^{22} = \frac{1}{R^2}\left(p + \frac{w_0}{3} - \frac{w_2}{2}\right), \quad (A18)
\]
\[
T^{33} = \frac{1}{R^2\sin^2\theta}\left(p + \frac{w_0}{3} - \frac{w_2}{2}\right), \quad (A19)
\]

where it is easy to see the contributions coming from the standard fluid and the radiation fluid respectively.

We will not write down here the expressions for the non-zero Christoffel symbols and the relevant components of the Ricci tensor, which are obtained by straightforward but particularly tedious calculations, but proceed directly to the form of the four independent Einstein equations

\[
\left(T_0^0\right)
8\pi G(e + w_0)R^2R_{\mu} + \frac{8\pi GbR^2R_t}{a}w_1 = \left\{ R\left( \frac{R_t}{a} \right)^2 - \left( \frac{R_{\mu}}{b} \right)^2 + 1 \right\}, \quad (A20)
\]

\[
\left(T_1^1\right)
8\pi G\left(p + \frac{w_0}{3} + w_2\right) + \frac{8\pi GaR^2R_{\mu}}{b}w_1 = -\left\{ R\left( \frac{R_t}{a} \right)^2 - \left( \frac{R_{\mu}}{b} \right)^2 + 1 \right\}, \quad (A21)
\]

\[
\left(T_2^2 \equiv T_3^3\right)
8\pi G\left(p + e + \frac{4}{3}w_0 - 2w_2\right) = \frac{1}{ab}\left[\left( \frac{a_{\mu}}{b} \right)_{,\mu} - \left( \frac{b_t}{a} \right)_{,t} \right] + \\
\quad + \frac{1}{R^2}\left[\left( \frac{R_t}{a} \right)^2 - \left( \frac{R_{\mu}}{b} \right)^2 + 1 \right], \quad (A22)
\]

\[
\left(T_1^0\right)
8\pi G\frac{b}{a}w_1 = -\frac{2}{a^2R}\left( \frac{a_{\mu}}{a}R_t + \frac{b_t}{b}R_{,\mu} - R_{,t\mu} \right). \quad (A23)
\]
Equation (A23) is a constraint equation which, in the form
\[ \frac{b_t}{b} = -\frac{1}{R_{,\mu}} \left( R_{,\mu t} - \frac{a_{,\mu}}{a} R_{,t} - 4\pi G \alpha b R w_1 \right) \] (A24)
has been used frequently in the course of the calculations outlined in this Appendix. In particular, writing out explicitly the time derivative on the right-hand side of equation (A21) and making use of the (A24), the expression
\[ \Gamma_{,t} = \frac{R_{,\mu}}{b} \left[ 4\pi G R w_1 - \frac{R_{,t}}{b} \left( \frac{p_{,\mu} + b s_1}{e + p} \right) \right], \] (A25)
can be obtained and this can then be further transformed, by means of equation (27), so as to arrive at the form (21).

Next we turn to writing out explicitly the hydrodynamic conservation equations (18) – (20) which then take the form
\[ -u_{,\alpha} T^{\alpha\beta}_{,\beta} = 0 = s_0 + (e + p)_{,\beta} u^{\beta} + (e + p) \Theta - p_{,\beta} g^{\alpha\beta} u_{,\alpha} , \] (A26)
\[ n_j P_{,\beta} T^{\alpha\beta}_{,\beta} = 0 = \frac{1}{b} s_1 + (e + p) n_j u^{\beta}_{,\beta} u^{\beta} + (g^{\alpha\beta} + u^{\alpha} u^{\beta}) p_{,\beta} , \] (A27)
\[ (\rho u^\alpha)_{,\alpha} = 0 = \rho_{,\alpha} u^\alpha + \rho u^\alpha_{,\alpha} + \rho u^\alpha \frac{\left( \sqrt{\det(-g^{\alpha\beta})} \right)}{\sqrt{\det(-g^{\alpha\beta})}}. \] (A28)
Using the expressions \( \Gamma = R_{,\mu}/b \) and \( u = R_{,t}/a \) together with (A24), equations (A26) – (A28) can be converted to the final form given in equations (23) – (25).

Finally, we note that if we rewrite equations (A20) and (A21) as
\[ \left\{ \frac{R}{2G} \left[ \left( \frac{R_{,t}}{a} \right)^2 - \left( \frac{R_{,\mu}}{b} \right)^2 + 1 \right] \right\}_{,\mu} = 4\pi R^2 R_{,\mu} (e + w_0 + \frac{u}{\Gamma} w_1) = M_{,\mu} , \] (A29)
and
\[ \left\{ \frac{R}{2G} \left[ \left( \frac{R_{,t}}{a} \right)^2 - \left( \frac{R_{,\mu}}{b} \right)^2 + 1 \right] \right\}_{,t} = -4\pi R^2 R_{,t} \left( p + \frac{1}{3} w_0 + \frac{\Gamma}{u} w_1 + w_2 \right) = M_{,t} , \] (A30)
this gives expressions (26) and (29) for the generalized mass function \( M \). These reduce to the familiar expressions for the standard mass function when the radiation terms are omitted.
Appendix B

As mentioned in Section 3, a particularly important point in the PSTF approach is the truncation of the infinite hierarchy of moment equations by means of a suitably defined closure relation derived on the basis of physical considerations. We have used the closure equation (14) and, for our particular problem, an appropriate expression for $f_E$ is given by equation (47). This appendix gives our justification for making this choice.

A key point in choosing an expression for $f_E$ is that it should have the correct asymptotic behaviour in the optically-thin and optically-thick limits. We are considering the case of a single spherical hadronic bubble which is initially nucleated at rest with a radius small compared with the mean-free-path of the radiation. Under these circumstances, the radiation field will be everywhere rather accurately uniform and isotropic (unless there is some other perturbing influence) and since the bubble radius is very small compared with the horizon scale, it is also a good approximation to neglect cosmological expansion. Since $w_0$, $w_1$ and $w_2$ are all measured with respect to the local rest-frames of the standard fluid, the values which they take during the early part of the bubble expansion are those produced by motion of the fluid rest frames with respect to the essentially uniform radiation field.

When the radiation is isotropic in its mean rest frame, its stress-energy tensor takes the perfect fluid form

$$T^\alpha_\beta = (e_R + p_R)v^\alpha v^\beta + p_R g^{\alpha\beta}, \quad (B1)$$

where $e_R$ and $p_R$ are the radiation energy density and pressure ($p_R = e_R/3$), measured in the mean rest frame of the radiation, and $v^\alpha = dx^\alpha/d\tau$ is the four-velocity of this frame relative to some specific observer. For purely radial motion in our metric

$$d\tau^2 = -ds^2 = \left(1 - \frac{b^2 \dot{\mu}^2}{a^2}\right)a^2 dt^2 = \frac{a^2}{\gamma^2} dt^2, \quad (B2)$$

and the non-zero components of four-velocity are

$$v^\alpha = \left(\frac{dt}{d\tau}, \frac{d\mu}{d\tau}\right) = \frac{\gamma}{a}(1, \dot{\mu}), \quad (B3)$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2}} \quad \text{and} \quad v = \frac{b\dot{\mu}}{a}. \quad (B4)$$
To find the value of $\dot{\mu}$ for the radiation frame with respect to the standard fluid, we note that since each element of the radiation fluid is remaining at a constant value of $R$

$$dR = R_t dt + R_\mu d\mu = audt + b\Gamma d\mu = 0,$$  \hspace{1cm} (B5)

and so

$$\dot{\mu} = -\frac{au}{b\Gamma},$$  \hspace{1cm} (B6)

which leads to the following expressions

$$v = -\frac{u}{\Gamma} \quad \gamma^2 = \frac{\Gamma^2}{\Gamma^2 - u^2}.$$  \hspace{1cm} (B7)

If we now write the stress-energy tensor (B1) in the frame comoving with the standard fluid, we can compare the new expressions for the components with the ones appearing in equation (4) and so obtain the following system of equations

$$T_{\mu\nu}^{00} = \frac{w_0}{a^2} = \frac{4}{3}e_R \left( \frac{\gamma^2}{a^2} \right) + \frac{1}{3}e_R \left( -\frac{1}{a^2} \right),$$  \hspace{1cm} (B8)

$$T_{\mu\nu}^{01} = \frac{w_1}{ab} = \frac{4}{3}e_R \left( \frac{\gamma^2}{a^2} \dot{\mu} \right),$$  \hspace{1cm} (B9)

$$T_{\mu\nu}^{11} = \frac{1}{\Gamma^2} \left( \frac{w_0}{3} + w_2 \right) = \frac{4}{3}e_R \left( \frac{\gamma^2}{a^2} \dot{\mu}^2 \right) + \frac{1}{3}e_R \left( \frac{1}{\Gamma^2} \right).$$  \hspace{1cm} (B10)

The solution of this system then leads to the expressions

$$w_0 = \frac{\gamma^2}{3}(3 + v^2)e_R,$$  \hspace{1cm} (B11)

$$w_1 = \frac{4}{3} \gamma^2 v e_R,$$  \hspace{1cm} (B12)

$$w_2 = \frac{\gamma^2}{3}(1 + 3v^2)e_R - \frac{w_0}{3}.$$  \hspace{1cm} (B13)

If we now define $(w_0)_N = e_R$ to be the radiation energy density at the bubble nucleation time (when there is no fluid motion), equations (B11) – (B13) can be suitably transformed so as to give expressions for the energy density, flux and shear scalar of the radiation as seen from the standard fluid

$$w_0 = \left( 1 + \frac{4}{3} \frac{u^2}{\Gamma^2 - u^2} \right) (w_0)_N \simeq \left( 1 + \frac{4}{3} u^2 \right) (w_0)_N,$$  \hspace{1cm} (B14)

$$w_1 = \frac{4}{3} \frac{\Gamma u}{\Gamma^2 - u^2} (w_0)_N \simeq -\frac{4}{3} \Gamma u (w_0)_N,$$  \hspace{1cm} (B15)

$$w_2 = \frac{8}{9} \frac{u^2}{\Gamma^2 - u^2} (w_0)_N \simeq \frac{8}{9} u^2 (w_0)_N.$$  \hspace{1cm} (B16)
The approximate forms of the expressions (B14) – (B16) result from noting that since the dimensions of the bubble are small compared with the horizon scale, \( \Gamma^2 - u^2 \simeq 1 \).

Equations (B14) and (B16) together with the definition (14), give the following analytic expression for the variable Eddington factor during the first stages of the bubble expansion

\[
f_E = \frac{8u^2 / 9}{(1 + 4u^2 / 3)}.
\]

(B17)

This is the “optically thin” limit. At the other extreme, the “optically thick” limit arises when the radius of the bubble is large compared with the radiation mean-free-path and complete coupling has been attained between the radiation and the standard fluid over length-scales comparable with the radius of the bubble. When this happens, interactions make the radiation isotropic in the local fluid rest frame so that \( w_2 \to 0 \) and \( f_E \to 0 \). A suitable smooth join is required in between the two asymptotic limits and to do this we have multiplied the expression in (B17) by \( \lambda_{h,q} / (\lambda_{h,q} + R) \) which then gives equation (47). Experiment has shown that reasonable variation in the form of the join makes an insignificant change in the results obtained.