LINEARIZATION OF QUASIPERIODICALLY FORCED CIRCLE FLOW BEYOND BRJUNO CONDITION

RAPHAËL KRIKORIAN, JING WANG, JIANGONG YOU, AND QI ZHOU

Abstract. We prove that an analytic quasiperiodically forced circle flow with a not super-Liouvillean base frequency and which is close enough to some constant rotation is $C^\infty$ rotations reducible, provided its fibered rotation number is Diophantine with respect to the base frequency. As a corollary, we obtain that among such systems, the linearizable ones and those displaying mode-locking are locally dense for the $C^\infty$-topology.

1. Introduction

An analytic quasiperiodically forced (qpf) circle flow is the flow defined by a differential equation of the form
\begin{align}
\dot{\theta} &= f(\theta, \varphi) \\
\dot{\varphi} &= \omega
\end{align}
where $f : \mathbb{T}^1 \times \mathbb{T}^d \to \mathbb{R}$ is analytic (here $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$), and $\omega \in \mathbb{T}^d$ is rationally independent. We denote by $(\omega, f)$ the flow of the vector field (1.1). Its discrete version is the qpf circle diffeomorphism
\begin{align}
(\tilde{\omega}, f) &= \tilde{f} : (\theta, \varphi) \mapsto (f_\varphi(\theta), \varphi + \tilde{\omega})
\end{align}
where $\theta \mapsto f_\varphi(\theta)$ are orientation preserving circle diffeomorphisms and $(1, \tilde{\omega}) \in \mathbb{T}^d$ is rationally independent.

The qpf circle flow (1.1) and its discrete version (1.2) have been extensively studied as a source of examples of interesting dynamics and as models of fundamental phenomena in mathematical physics. One well known example is the quasiperiodically forced Arnold circle map, which serves as a simple model for oscillators forced with two or more incommensurate frequencies [11]. Another example comes from the projective action of quasi-periodic cocycles: a typical example is the Harper map, appearing in the study of quasi-periodic crystals and which can be seen as the projective action of the cocycle associated to an almost Mathieu operator [6, 7].

One of the classical questions about system $(\omega, f)$ is whether there exist a $C^r$ map $H : \mathbb{T}^1 \times \mathbb{T}^d \to \mathbb{T}^1 \times \mathbb{T}^d$ and $\tilde{\rho} \in \mathbb{R}$, such that $H$ conjugates the system $(\omega, \rho + f)$ to $(\omega, \tilde{\rho})$. If it is the case, we say the system $(\omega, \rho + f)$ is $C^r$ ($r = \infty$, $\omega$) linearizable, or $C^r$ reducible.

Date: September 26, 2016.
We first review the corresponding linearization results in the case \( d = 1 \). In that case, this means \((1, 1)\) is periodically forced or \((1, 2)\) is unforced, i.e. is a circle map. One of the most important results for these maps is Poincaré-Denjoy’s classification theorem: if the rotation number of an orientation preserving homeomorphism of the circle \( f \) is rational, then \( f \) has periodic orbits; if the rotation number is irrational, then \( f \) is semi-conjugate to the corresponding irrational (rigid) rotation; if the rotation number is irrational and furthermore \( f \) admits a derivative with bounded variation, then the conjugacy is a homeomorphism. Subsequently, Arnold [1] proved that if the rotation number is Diophantine, and if the diffeomorphism is \( C^\omega \) close to a rotation, then it is \( C^\omega \) linearizable. Later in [14, 22], Arnold’s result was improved obtaining a global linearization result in the \( C^\infty \) setting: if the rotation number of the diffeomorphism is Diophantine, then it is linearizable. The \( C^\omega \) version was treated by Yoccoz [23], in fact, Yoccoz was able to prove more: the Brjuno condition is optimal for the local linearization result, and \( \mathcal{H} \) condition \(^1\) is optimal for the global linearization.

In this paper, we focus our attention on the case \( d = 2 \), and without loss of generality, we just assume \( \omega = (1, \alpha) \) with \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). From the topological point of view, Jäger and Stark [17] first gave a Poincaré-like classification theorem: if the quasiperiodically forced circle diffeomorphism \((\alpha, f)\) is \( \rho \)-bounded, then either there exists a \( p,q \)-invariant strip and \( \rho_f, \alpha \) and \( 1 \) are rationally dependent or \((\alpha, f)\) is semi-conjugate to the irrational torus translation \((\theta, \varphi) \mapsto (\theta + \rho_f, \varphi + \alpha)\); if \((\alpha, f)\) is \( \rho \)-unbounded, then neither of these alternatives can occur and the map is always topologically transitive. Here, \( \rho_f = \rho(\alpha, f) \) is the fibred rotation number of \((\alpha, f)\) (see section 2.3 for details). However, the analytic part is not completely satisfactory, the only result in that direction being Herman’s local linearization result [14], which states that the qpf flow \((\omega, \rho + f)\)

\[
\begin{align*}
\dot{\theta} &= \rho + f(\theta, \varphi) \\
\dot{\varphi} &= \omega
\end{align*}
\]

with \( \rho \in \mathbb{R}, \ f : T^1 \times T^2 \to \mathbb{R} \) analytic and \((\omega, \rho_f)\) satisfying the Diophantine condition

\[
|\langle k, \omega \rangle + l \rho_f| \geq \frac{\gamma}{(|k| + |l|)\tau} \quad \text{for all} \quad (k, l) \in \mathbb{Z}^2 \times \mathbb{Z}, \quad |k| + |l| \neq 0,
\]

with \( \gamma > 0, \tau > 2, \) is \( C^\omega \) linearizable provided \( f \) is sufficiently small (how small depending on \( \tau \) and \( \gamma \)).

We note that if \( l = 0 \), then (1.4) implies that \( \omega \) is Diophantine with exponent \( \tau \) and constant \( \gamma \) which we denote by \( \omega \in DC(\gamma, \tau) \). By Yoccoz’s result [23], Brjuno condition is optimal for local linearizaton of analytic circle diffeomorphism. It is a long-standing question whether Brjuno condition is also optimal for local linearization problems of other nonlinear systems. The essential contribution of this paper is that we can really say something if \( \omega \)

\(^1\)See [23] for the definition of \( \mathcal{H} \).
is not Brjuno or even if it is in some Liouvillean class. It is clear the notion of linearization for (1.3) with Liouvillean frequencies is too restrictive, since even \((\omega, g(\varphi))\) is in general not linearizable if \(\omega\) is Liouvillean. The suitable notions in this regard are rotations reducibility and almost reducibility. We say a qpf circle flow \((\omega, f(\theta, \varphi))\) is \(C^\omega\) rotations reducible if it can be \(C^\omega\) conjugated to \((\omega, g(\varphi))\). We say that \((\omega, f(\theta, \varphi))\) is \(C^\omega\) almost reducible, if there exist sequences \((\rho_n)_{n\in\mathbb{N}}\in\mathbb{R}\) such that \((f_n)_{n\in\mathbb{N}}\in (C^\omega(T \times \mathbb{T}^2, \mathbb{R} \times \mathbb{R}^2))^N\) and \((f_n)_{n\in\mathbb{N}}\in (C^\omega(T \times \mathbb{T}^2, \mathbb{R}))^N\) such that \((f_n)\) goes to 0 in the \(C^\omega\) topology and \(H_n\) conjugates the system \((\omega, f(\theta, \varphi))\) to \((\omega, \rho_n + f_n(\theta, \varphi))\). In the analytic category one should be more careful about the choice of topology. Let \(W_{s,r}(\mathbb{T} \times \mathbb{T}^2) = \{(\theta, \varphi) \in (\mathbb{T} \oplus i\mathbb{R}) \times (\mathbb{T}^2 \oplus i\mathbb{R}^2) : |3m\theta| < s, |3m\varphi| < r\} \). For a bounded analytic (possibly matrix valued) function \(f: \mathbb{T} \times \mathbb{T}^2 \to \mathbb{R}\) defined on \(W_{s,r}(\mathbb{T} \times \mathbb{T}^2)\), let 

\[
\|f\|_{s,r} = \sup_{(\theta, \varphi) \in W_{s,r}} |f(\theta, \varphi)|.
\]

We denote by \(C^\omega_{s,r}(\mathbb{T} \times \mathbb{T}^2, \ast)\) the set all these bounded analytic functions, where \(\ast\) will usually denote \(\mathbb{R}, \mathbb{R} \times \mathbb{R}^2\). We say that \((\omega, f(\theta, \varphi))\) is \(C^\omega\) almost reducible, if there exist \(s > 0, r > 0\) and sequences \(\rho_n \in \mathbb{R}, H_n \in C^\omega_{s,r}(\mathbb{T} \times \mathbb{T}^2, \mathbb{R} \times \mathbb{R}^2)\) and \(f_n \in C^\omega_{s,r}((\mathbb{T} \times \mathbb{T}^2, \mathbb{R}))\) such that \(\lim_{n \to \infty} \|f_n\|_{s,r} = 0\) and \(H_n\) conjugates the system \((\omega, f(\theta, \varphi))\) to \((\omega, \rho_n + f_n(\theta, \varphi))\). Since this will be important later on, we notice that if one cannot guaranty the convergence of \((f_n)\) to 0 on fixed complex strips of width \(s > 0, r > 0\) but instead knows that there exist sequences \((s_n)_{n\in\mathbb{N}}, (r_n)_{n\in\mathbb{N}}, s_n > 0, r_n > 0\) such that for any \(p \in \mathbb{N}, \|f_n\|_{s_n,r_n} = O((s_n r_n)^p)\), then \((\omega, \rho + f)\) is \(C^\infty\) almost reducible (this is a consequence of the classical Cauchy estimates)\(^2\).

Apart from the above motivations, our main inspiration comes from reducibility theory of quasi-periodic \(SL(2, \mathbb{R})\) cocycle,

\[
(\alpha, A) : \mathbb{T}^1 \times \mathbb{R}^2 \to \mathbb{T}^1 \times \mathbb{R}^2
\]

\[
(\theta, v) \mapsto (\theta + \alpha, A(\theta) \cdot v).
\]

The cocycle \((\alpha, A)\) is called reducible (resp. rotations reducible), if there exist \(B \in C^\omega(2\mathbb{T}, SL(2, \mathbb{R}))\) and \(A_n \in SL(2, \mathbb{R})\) (resp. \(A_n \in C^\omega(T, SO(2, \mathbb{R}))\)) such that \(B\) conjugates \((\alpha, A)\) to \((\alpha, A_n)\) which means that \((0, B) \circ (\alpha, A) \circ (0, B)^{-1} = (\alpha, A_n)\) or equivalently \(B(\cdot + \alpha) A(\cdot) B(\cdot)^{-1} = A_n(\cdot)\). Similarly, we can define \(C^\infty\) (or \(C^\omega\)) almost reducibility of the cocycles. If \(\alpha\) is Diophantine, the reducibility results are well developed [10, 12]. In fact, Eliasson [12] proved that if \(\alpha\) is Diophantine, then any cocycle which is close to constant is \(C^\infty\) almost reducible. A breakthrough came recently, based on the “cheap trick” developed in [13], where Avila, Fayad and Krikorian [5] obtained a local positive measures set of \(C^\omega\) rotations reducible result for any base forcing frequency \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\). Meanwhile, for the continuous time

\(^2\)Let us mention, though we shall not go in that direction in this paper, that another possible notion of almost reducibility would be to consider the convergence of \((f_n)\) in the analytic inductive limit topology.
version, by using KAM and Floquet theory, Hou and You [16] proved that if the system is close to constant ones, then it is $C^\infty$ almost reducible, and for full measure set of rotation numbers, the system is $C^\omega$ rotations reducible. Readers can consult [24, 25] for related results. The highlight is Avila’s global theory of one-frequency $SL(2, \mathbb{R})$ cocycles [2], especially his almost reducibility conjecture (recent solved by him in [3, 4]), which says that if the cocycle has subexponentially growth in a fixed analytic strip, then the cocycle is strongly almost reducible.

If a quasi-periodic $SL(2, \mathbb{R})$ cocycle is reducible (resp. almost reducible), then its corresponding qpf circle diffeomorphism is linearizable (resp. almost reducible). However, the linearization of general quasiperiodically forced circle flows is more difficult due to their nonlinear nature. In this paper, we will overcome these difficulties and generalize the linear result [5, 16] to general nonlinear quasiperiodically forced circle flows.

To state our main theorem precisely, we first introduce some notations that will be used in the sequel. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\tilde{U}(\alpha) := \sup_{n>0} \frac{\ln \ln q_n^{n+1}}{\ln q_n}$. If $\tilde{U}(\alpha) < \infty$, we say $\alpha$ is not super-Liouvillean. Then our main results can be stated as the following:

**Theorem 1.1.** Let $\tilde{\rho} \in \mathbb{R}$, $\gamma > 0$, $s > 0$, $r > 0$, $\tau > 2$, $\omega = (1, \alpha)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\tilde{U}(\alpha) < \infty$. Assume that $\rho(\omega, \tilde{\rho} + f(\theta, \varphi)) = \rho_f \in DC^\omega(\gamma, \tau)$ in the sense

$$|\langle k, \omega \rangle + l\rho_f| \geq \frac{\gamma}{(|k| + |l|)^r} \text{ for all } (k, l) \in \mathbb{Z}^2 \times \mathbb{Z}, \ l \neq 0.$$  

Then there exists $\varepsilon = \varepsilon(\tau, \gamma, s, r, \tilde{U}) > 0$ such that if $\|f(\theta, \varphi)\|_{s, r} \leq \varepsilon$, the following holds:

(a) The system $(\omega, \tilde{\rho} + f(\theta, \varphi))$ is $C^\infty$ rotations reducible.

(b) Moreover, $(\omega, \tilde{\rho} + f(\theta, \varphi))$ is $C^\infty$ accumulated by $C^\omega$ linearizable quasiperiodically forced circle flow.

**Remark 1.1.** As readers will see from the proof, the system is in fact $C^\infty$ almost reducible w.r.t $\theta$, and $C^\omega$-almost reducible w.r.t $\varphi$. Also note that $C^\infty$ rotations reducibility immediately implies $C^\infty$ almost reducibility.

**Remark 1.2.** Comparing Theorem 1.1 with the result of Herman [14], we handle more frequencies $\omega$ including many Liouvillean ones. Actually, our result applies even to those $\omega$ whose best approximations contain a subsequence of $(q_n)$ such that $q_n = O(e^{\gamma n-1})$. However, it is still an open question whether the results hold for all rationally independent $\omega$.

These are the results when the fibred rotation number is non-resonant with the forcing frequency. The reader can consult [21] for the totally resonant case. The reader can also compare this result with the following recent
result of Krikorian [19], who proved that any circle diffeomorphism which is Hölder conjugated to a rigid rotation is $C^\infty$ almost reducible.

Besides the linearization of a system, another celebrated question is whether the rotation number as a function of the twist parameter is a “devil’s staircase”, which means mode-locking is dense in the parameter interval, and for quasiperiodic cocycle this is equivalent to the fact that uniform hyperbolicity is dense in the given parameter space. When formulated in the framework of Schrödinger operators, it means the spectrum of the corresponding operator is a Cantor set. Yet, existing results are still restricted to almost Mathieu operators, for which the question is known as the “Ten Martini Problem” [6, 8, 20]. Although it is still an open question whether uniformly hyperbolic is dense in the category of $C^\omega$ quasi-periodic SL(2, R)-cocycles which are homotopic to the identity, it is true for cocycles which are close to constant ones [18]. As a corollary of our main theorem, the nonlinear version of [18] is also true, which is the following:

**Corollary 1.1.** Under the assumptions of Theorem 1.1, the qpf flow $(\omega, \tilde{\rho} + f(\theta, \varphi))$ is $C^\infty$ accumulated by mode-locked quasiperiodically forced circle flows.

**Remark 1.3.** To the best knowledge of the authors, this result is the first result of denseness about mode-locking in the nonlinear setting.

**Remark 1.4.** All these results are also valid for the discrete case, i.e., quasiperiodically forced circle maps.

Finally, we want to point out that in this paper the method of diagonally dominant operators we developed is new in itself (consult more discussions in section 3). The method has recently been proved to be very powerful: it can be used to solve several interesting problems, and we will come back to theses issues in forthcoming papers.

2. Notations and preliminaries

2.1. Continued fraction expansion. Let $\alpha \in (0, 1)$ be irrational. Define $a_0 = 0, a_0 = \alpha$, and inductively for $k \geq 1$,

$$a_k = [\alpha_{k-1}^{-1}], \quad \alpha_k = G(\alpha_{k-1}) = \alpha_{k-1}^{-1} - a_k = \{\alpha_{k-1}^{-1}\}.$$  

We define

$$p_0 = 0, \quad p_1 = 1,$$

$$q_0 = 1, \quad q_1 = a_1,$$

and inductively,

$$p_k = a_k p_{k-1} + p_{k-2},$$

$$q_k = a_k q_{k-1} + q_{k-2}.$$  

---

3 Consult section 2.3 for more information.
Then the sequence \((q_n)\) is the sequence of denominators of the best rational approximations for \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\), since it satisfies
\[
\forall 1 \leq k < q_n, \quad \|k\alpha\|_T \geq \|q_{n-1}\alpha\|_T,
\]
and
\[
\frac{1}{q_n + q_{n+1}} \leq \|q_n\alpha\|_T \leq \frac{1}{q_{n+1}},
\]
where we use the notation
\[
\|x\|_T = \inf_{p \in \mathbb{Z}} |x - p|.
\]

2.2. CD bridge. For any \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\), we will fix in the sequel a particular subsequence \((q_{n_k})\) of the denominators of the continued fraction expansion for \(\alpha\), which is denoted by \((Q_k)\), and the subsequence \((q_{n_k+1})\) is denoted by \((\overline{Q}_k)\).

**Definition 2.1.** [5] Let \(0 < A \leq B \leq C\). We say that the pair of denominators \((q_l, q_n)\) \((l < n)\) forms a CD\((A, B, C)\) bridge if
\begin{itemize}
  \item \(q_{i+1} \leq q_l^A, \quad \forall i = l, \ldots, n-1\)
  \item \(q_l^C \geq q_n \geq q_l^B\).
\end{itemize}

**Lemma 2.1.** [5] For any \(A > 0\), there exists a subsequence \((Q_k)\) such that \(Q_0 = 1\) and for each \(k \geq 0\), \(Q_{k+1} \leq \overline{Q}_k^A\). Furthermore, either \(\overline{Q}_k \geq Q_k^A\), or the pairs \((\overline{Q}_{k-1}, Q_k)\) and \((Q_k, Q_{k+1})\) are both CD\((A, A, A^3)\) bridges.

In the sequel, we assume \(A = 8\), and \((Q_n)\) is the selected subsequence in Lemma 2.1 accordingly. As an immediate corollary of Lemma 2.1, we have

**Corollary 2.1.** If \(\tilde{U}(\alpha) < \infty\), then we have \(Q_n \geq Q_{n-1}^A\) for every \(n \geq 1\). Furthermore, we have
\[
\sup_{n > 0} \frac{\ln \ln Q_{n+1}}{\ln Q_n} \leq U(\alpha),
\]
where \(U(\alpha) := \tilde{U}(\alpha) + 4 \frac{\ln A}{\ln 2} < \infty\).

**Proof.** For \(n = 1\), it is obvious that \(Q_1 \geq Q_0^A\) since \(Q_0 = 1\). For \(n \geq 2\), we distinguish two cases below. If \(\overline{Q}_{n-1} \geq Q_{n-1}^A\), then \(Q_n \geq \overline{Q}_{n-1} \geq Q_{n-1}^A\). Otherwise, the pairs \((\overline{Q}_{n-2}, Q_{n-1})\) and \((Q_{n-1}, Q_n)\) are both CD\((A, A, A^3)\) bridges. Thus, we get \(Q_n \geq Q_{n-1}^A\) by definition.

For the second statement, we have
\[
\frac{\ln \ln Q_{n+1}}{\ln Q_n} \leq \frac{\ln A^4 + \ln \ln \overline{Q}_n}{\ln Q_n} \leq \tilde{U} + 4 \frac{\ln A}{\ln 2} = U,
\]
since for each \(n \geq 0\), \(Q_{n+1} \leq \overline{Q}_n^A\). \(\square\)
2.3. The fibred rotation number and mode-locking. Suppose $(\omega, f)$ is a qpf circle flow. Let

$$\rho(\omega, f) = \lim_{t \to \infty} \frac{\Phi^t_{\varphi}(\hat{\theta})}{t}$$

be the fibred rotation number associated with $(\omega, f)$, where $\Phi^t_{\varphi}(\hat{\theta}) : \mathbb{R}_+^1 \times \mathbb{R} \times \mathbb{T}_2 \to \mathbb{R}_+^1$, via $(t, \hat{\theta}, \varphi) \mapsto \Phi^t_{\varphi}(\hat{\theta})$ denotes the lift of the flow of $(\omega, f)$ of the first variable. The limit exists and is independent of $(\hat{\theta}, \varphi)$ [15]. As a direct consequence of the definition, we have the following well-known results:

**Lemma 2.2.** [15] Let $\omega = (1, \alpha), \check{\rho} \in \mathbb{R}$. If $\|f(\theta, \varphi)\|_{C^0} \leq \varepsilon$, then

$$|\rho(\omega, \check{\rho} + f(\theta, \varphi)) - \rho(\omega, \check{\rho})| \leq \varepsilon$$

**Lemma 2.3.** [15] Suppose $(\omega, f)$ is a qpf circle flow and $H \in C^0(\mathbb{T} \times \mathbb{T}^2, \mathbb{T} \times \mathbb{T}^2)$ is homotopic to the identity. Then the fibred rotation number of $(\omega, f)$ remains the same under the transformation $H$.

If the rotation number of a qpf circle flow $(\omega, f)$ remains constant under all sufficiently small $C^0$-perturbations of $f$, we say $(\omega, f)$ is mode-locked. For our purpose, we need the following proposition, which was proved in [9].

**Proposition 2.1.** [9] Suppose the quasiperiodically forced circle flow $(\omega, f)$ is given by the projective action of a quasi-periodic $sl(2, \mathbb{R})$-flow. Then $(\omega, f)$ is mode-locked if and only if the quasi-periodic $sl(2, \mathbb{R})$-flow is uniformly hyperbolic.

Readers can consult [9] for more interesting properties of mode-locking.

3. Outline of the proof

In this section, we will outline the main ideas of the proof, and show the principal difference between our method and previous works. We first introduce some notations that will be used in the sequel. If $f \in C^\omega(\mathbb{T}^2, \mathbb{R})$, we denote $\hat{f}(k) = \int_{\mathbb{T}^2} f(\varphi)e^{-i(k, \varphi)}d\varphi$. For $f \in C^\omega(\mathbb{T} \times \mathbb{T}^2, \mathbb{R})$, we define its fourier coefficients by $f_k^l = \int_{\mathbb{T} \times \mathbb{T}^2} f(\theta, \varphi)e^{-il\theta}e^{-i(k, \varphi)}d\theta d\varphi$. For any $N > 0$, we denote the truncation and projection operators $T_N$ and $R_N$ by

$$T_N(f) = \sum_{|k| + |l| < N} f_k^l e^{i(k, \varphi)}, \quad R_N(f) = \sum_{|k| + |l| \geq N} f_k^l e^{i(k, \varphi)}.$$

The proof is based on a modified KAM scheme. Considering the quasiperiodically forced circle flow $(\omega, \rho + f(\theta, \varphi))$, if we want to eliminate the nonresonant terms of $f(\theta, \varphi)$ as in usual KAM steps, then the classical homological equation on the fibre reads as

$$f(\theta, \varphi) - \partial_\omega h - \rho \frac{\partial h}{\partial \theta} = 0.$$
Checking the Fourier coefficients, we have

\[ h^k_l = \frac{f^k_l}{i(k, \omega + l\rho)}. \]

Since we have no Diophantine condition on \( \omega \), the homological equation may have no analytic solution. This is the essential difference compared to the classical KAM theorem, which also means the resonant terms \( f^k_0 e^{i(k, \varphi)} \) cannot be solved at all. So we rewrite the system as \( (\omega, \rho + g(\varphi) + f(\theta, \varphi)) \), with \( \int_{T^2} f(\theta, \varphi) d\varphi = 0 \). In this case, the homological equation on the fibre is

\[ \partial_\omega h + (\rho + g(\varphi)) \frac{\partial h}{\partial \theta} = f(\theta, \varphi). \]

In order to get desired result, we distinguish three steps. The first step is to eliminate the non-resonant terms of \( g(\varphi) \) by solving \( \partial_\omega h(\varphi) = T_Q g(\varphi) \). Notice that \( \|h(\varphi)\| \) may be very large. However, in this step, we use a small trick to control \( \|\Im h(\varphi)\| \) at the cost of reducing the analytic radius greatly.

The second step is to solve the homological equation

\[ \partial_\omega h + (\rho + \tilde{g}(\varphi)) \frac{\partial h}{\partial \theta} = f(\theta, \varphi), \]

where \( \|\tilde{g}(\varphi)\| = O(\|f\|) \). By introducing diagonally dominant operators, we can solve the approximate equation

\[ \partial_\omega h + \rho \frac{\partial h}{\partial \theta} + T_K (\tilde{g}(\varphi) \frac{\partial h}{\partial \theta}) = T_K f(\theta, \varphi) \]

and then make the perturbation \( f(\theta, \varphi) \) as small as we can by iteration.

Using these two steps, we can already prove our almost reducibility result. However, to obtain the rotations reducibility result, in the end of one KAM step we need to inverse the first step; the conjugation we get is then close to the identity.

4. The inductive step

4.1. Basic proposition. In this subsection, we will show how to use the method of diagonally dominant operators to solve the homological equation

\[ \partial_\omega h + (\rho + g(\varphi)) \frac{\partial h}{\partial \theta} = f(\theta, \varphi). \]

We should point out that the following proposition holds, irrespective of any arithmetical property on the base frequency \( \omega \), and this is fundamental in our reduction.

**Proposition 4.1.** Let \( \gamma > 0, \tau > 2, r > \sigma > 0, s > \delta > 0, \sigma \leq \delta/2, \rho \in DC_\omega(\gamma, \tau), g(\varphi) \in C^r(\mathbb{T}^2, \mathbb{R}), f(\theta, \varphi) \in C^s_r(\mathbb{T} \times \mathbb{T}^2, \mathbb{R}) \) with \( \int_{\mathbb{T}^2} f(\theta, \varphi) d\theta = 0 \). There exist \( 0 < \eta \leq \eta \leq 1 \), such that if \( \|g(\varphi)\|_r \leq \eta, \|f(\theta, \varphi)\|_{s,r} \leq \tilde{\eta} \) and

\[ K = \left[ \frac{1}{\sigma} \ln \frac{1}{\eta} \right] < \left( \frac{\gamma^2}{\eta} \right)^{\frac{1}{r+s}}, \]
then the homological equation (4.1) has an approximate solution \( h(\theta, \varphi) \) with estimate
\[
\|h\|_{s-\delta,r-\sigma} \leq \frac{2\gamma}{\gamma^2 + r},
\]
and the error term \( \tilde{P} = R_K(-g(\varphi)\frac{\partial h}{\partial \varphi} + f(\theta, \varphi)) \) satisfies
\[
\|\tilde{P}\|_{s-\delta,r-\sigma} \leq \frac{4\gamma^2}{\gamma^2 + r}.
\]

**Proof.** Since the homological equation (4.1) may have no analytic solution, we solve its approximate equation
\[
(4.3)\quad \partial_\omega h + \rho \frac{\partial h}{\partial \theta} + T_K(g(\varphi)\frac{\partial h}{\partial \varphi}) = T_K f(\theta, \varphi).
\]
Let
\[
f(\theta, \varphi) = \sum_l f_l(\varphi)e^{i\theta}, \quad f_l(\varphi) = \sum_k f_l^k e^{i(k, \varphi)},
\]
\[
h(\theta, \varphi) = \sum_{0 < |l| < K} h_l(\varphi)e^{i\theta}, \quad h_l(\varphi) = \sum_{|k| < K-|l|} h_l^k e^{i(k, \varphi)}.
\]
In order to solve (4.3), it is equivalent to solve
\[
(4.4)\quad \partial_\omega h_l(\varphi) + il\rho h_l(\varphi) + il\Gamma_{K-|l|} (g(\varphi) h_l(\varphi)) = \Gamma_{K-|l|} f_l(\varphi)
\]
for \( 0 < |l| < K \), where for \( f(\varphi) \in C^\omega(\mathbb{T}^2, \mathbb{R}) \), \( N > 0 \),
\[
\Gamma_N f(\varphi) = \sum_{0 \leq |k| < N} \hat{f}(k)e^{i(k, \varphi)}.
\]
For any fixed \( l \), (4.4) can be viewed as a matrix equation
\[
(A_l + G_l) h_l = \tilde{f}_l,
\]
where
\[
h_l = (h_l^k)_{|k| < K-|l|}, \quad A_l = diag(i\langle k, \omega \rangle + il\rho : |k| < K - |l|),
\]
\[
\tilde{f}_l = (f_l^k)_{|k| < K-|l|}, \quad G_l = (i\tilde{\gamma}(p-q))_{|p|,|q| < K-|l|}.
\]
If we denote \( \Omega_{l,r'} = diag(\cdots, e^{i|k|r'}, \cdots)_{|k| < K-|l|} \) for any \( r' \leq r \), then
\[
\Omega_{l,r'}(A_l + G_l)\Omega_{l,r'}^{-1} h_l = \Omega_{l,r'} \tilde{f}_l.
\]
Rewrite it as
\[
(\tilde{A}_{l,r'} + \tilde{G}_{l,r'}) h_{l,r'} = \tilde{f}_{l,r'},
\]
where
\[
\tilde{A}_{l,r'} = \Omega_{l,r'} A_l \Omega_{l,r'}^{-1}, \quad \tilde{G}_{l,r'} = \Omega_{l,r'} G_l \Omega_{l,r'}^{-1},
\]
\[
\tilde{h}_{l,r'} = \Omega_{l,r'} h_l, \quad \tilde{f}_{l,r'} = \Omega_{l,r'} f_l.
\]
with estimate
\[
\|\tilde{G}_{l,r'}\| \leq |l|(K - |l|)^2 \parallel g \parallel_{r'}.
\]
Since \( \rho \in DC_{\omega}(\gamma, \tau) \), then together with (4.2), we have
\[
|\langle k, \omega \rangle + lp| > (|l| (K - |l|)^2 \|g\|_r)^{1/2}
\]
for all \( 0 < |k| + |l| < K, l \neq 0 \). As a result, the diagonally dominant operators \( \tilde{A}_{l,r} + \tilde{G}_{l,r} \) has a bounded inverse and \( \| (I + \tilde{A}_{l,r}^{-1} \tilde{G}_{l,r})^{-1} \|_{op(l^1)} < 2 \), where \( \| \cdot \|_{op(l^1)} \) denotes the operator norm associated to the \( l^1 \) norm \( |u|_{l^1} = \sum_{|k|<K-|l|} |u^k| \) (indeed, if \( M \) is a matrix, \( \| M \|_{op(l^1)} \leq C(M) \) with \( C(M) = \sum_i \max_j |M_{ij}| \) and from the above estimates it is clear that \( C(\tilde{A}_{l,r}^{-1} \tilde{G}_{l,r}) < 1/2 \).

If \( \sigma \leq \delta/2 \), then we can estimate
\[
\| h(\theta, \varphi) \|_{s-\delta/2,r-\sigma} \leq \sum_{0 < |l| < K} \| h^k_l |e^{[k(\sigma-r)]}\|_{(s-\delta/2)} = \sum_{0 < |l| < K} \| \tilde{h}_{l,r-\sigma} |l\|_{e^{[l(\delta-\sigma)]}}(s-\delta/2)
\]
\[
= \sum_{0 < |l| < K} \| (I + \tilde{A}_{l,r-\sigma}^{-1} \tilde{G}_{l,r-\sigma})^{-1} \tilde{A}_{l,r-\sigma}^{-1} \tilde{f}_{l,r-\sigma} |l\|_{e^{[l(\delta-\sigma)]}}(s-\delta/2)
\]
\[
\leq \sum_{0 < |l| < K} \| (I + \tilde{A}_{l,r-\sigma}^{-1} \tilde{G}_{l,r-\sigma})^{-1} \|_{op(l^1)} \cdot \| \tilde{A}_{l,r-\sigma}^{-1} \tilde{f}_{l,r-\sigma} |l\|_{e^{[l(\delta-\sigma)]}}(s-\delta/2)
\]
\[
\leq \sum_{0 < |l| < K} \sum_{|k| < K-|l|} \frac{2(|k| + |l|)}{\gamma} f^k_l |e^{[k(\sigma-r)]}\|_{(s-\delta/2)}
\]
\[
\leq \frac{2\tilde{\eta}}{\gamma^{\sigma^2 + \tau}}.
\]
Consequently by Cauchy estimates, we get the control of the error term:
\[
\| \tilde{P} \|_{s-\delta,r-\sigma} \leq \| R_K f(\theta, \varphi) \|_{s-\delta,r-\sigma} + \| R_K (g(\varphi) \frac{\partial h}{\partial \theta}) \|_{s-\delta,r-\sigma}
\]
\[
\leq \frac{\tilde{\eta} e^{-K\sigma}}{\sigma^2} + \frac{\eta \| h(\theta, \varphi) \|_{s-\delta/2,r-\sigma} e^{-K\sigma}}{\sigma}
\]
\[
\leq \frac{\tilde{\eta}^2}{\sigma^2} + \frac{2\tilde{\eta}^2}{\gamma^{\sigma^2 + \tau}} < \frac{4\tilde{\eta}^2}{\gamma^{\sigma^2 + \tau}}.
\]
\( \square \)

4.2. KAM step. We give details about one step of the KAM iteration. For simplicity, we introduce some notations which will be used in this section. For any \( r, s, \eta, \tilde{\eta} > 0, \rho_f \in \mathbb{R} \), we define
\[
\mathcal{F}_{s,r}(\rho_f, \eta, \tilde{\eta}) = \left\{ \tilde{\rho} + g(\varphi) + f(\theta, \varphi) \in C^\omega_{s,r}(T \times T^2, \mathbb{R}) \left| \begin{array}{c}
\rho(\omega, \tilde{\rho} + g(\varphi) + f(\theta, \varphi)) = \rho_f \\
\| g(\varphi) \|_r \leq \eta, \| f(\theta, \varphi) \|_{s,r} \leq \tilde{\eta}
\end{array} \right. \right\}.
\]
Let \( \alpha \in \mathbb{R} \setminus Q \), with \( U = \tilde{U}(\alpha) + 12 < \infty \), \( (Q_n) \) is the selected sequence of \( \alpha \) by Lemma 2.1 with \( \mathcal{A} = 8 \). Let \( r_0, s_0, \gamma > 0, \tau > 2 \) and \( Q_s \) be the smallest
Q \in \mathbb{N}_0 \text{ such that } 
ln Q < \frac{Q^{1/4}r_0}{40crU}.

Suppose that \( \varepsilon_0 \) is small enough such that
\[
(4.6) \quad \varepsilon_0 < \min \left\{ \frac{(r_0s_0\gamma)^{12(\tau+3)}}{\tau!Q_1^{2\tau U}}, e^{-2\tau U}e^{-40(\ln Q_*)^{2\tau U}} \right\} \quad \text{and} \quad \ln \frac{1}{\varepsilon_0} < \left( \frac{1}{\varepsilon_0} \right)^{12(\tau+3)}.
\]

where \( c \) is a global constant with \( c > 10(\tau + 3)/\tau \). For any given \( r_0, s_0, \varepsilon_0 \), we inductively define some sequences depending on \( r_0, s_0, \varepsilon_0 \) for \( j \geq 1 
\]
\[
\Delta_1 = \frac{s_0}{10}, \quad \Delta_j = \frac{\Delta_1}{2^{j-1}},
\]
\[
(4.7) \quad r_j = \frac{r_0}{4Q_j^3}, \quad s_j = s_{j-1} - \Delta_j,
\]
\[
\varepsilon_j = \frac{\varepsilon_{j-1}}{Q_j^{2j+1/2}}, \quad \tilde{\varepsilon}_j = \sum_{m=0}^{j-1} \varepsilon_m,
\]
\[
K(j) = \left( \frac{s_j^2}{4\tilde{\varepsilon}_j} \right)^{1/2}
\]

Since \( \frac{1}{Q_j} \) goes to 0 much faster than \( \Delta_j \), we can just assume that \( 4r_j < \Delta_j \) without loss of generality.

4.2.1. Eliminate the non-resonant terms.

Lemma 4.1. Given qpf circle flows
\[
(4.8) \quad \begin{cases} 
\dot{\theta} = \rho_f + g(\varphi) + f(\theta, \varphi) \\
\dot{\varphi} = \omega = (1, \alpha)
\end{cases}
\]
if \( \rho_f + g(\varphi) + f(\theta, \varphi) \in \mathcal{F}_{s_{n-1}, r_{n-1}}(\rho_f, 4\tilde{\varepsilon}_{n-1}, \varepsilon_{n-1}) \), then, if we denote \( \tilde{\tau}_n = s_{n-1} - \Delta_n, \quad \tau_n = \frac{\tilde{\tau}_n}{Q_n} \), there exists \( h(\varphi) \) with \( \|h(\varphi)\|_{\tau_n} \leq Q_n^{1/2} \varepsilon_0^1 \), such that the transformation \( \theta = \overline{\theta} + h(\varphi) \) (mod 1) conjugates the system (4.8) into
\[
(4.9) \quad \begin{cases} 
\tilde{\theta} = \rho_f + \overline{f}(\varphi) + \overline{f}(\overline{\theta}, \varphi) \\
\tilde{\varphi} = \omega = (1, \alpha)
\end{cases}
\]
with \( \rho_f + \overline{f}(\varphi) + \overline{f}(\overline{\theta}, \varphi) \in \mathcal{F}_{s_n, \tau_n}(\rho_f, \varepsilon_{n-1}^{1/2}, \varepsilon_{n-1}). \)

Proof. Under the transformation \( \theta = \overline{\theta} + h(\varphi) \) (mod 1), the fibred equation becomes
\[
\tilde{\theta} = \rho_f - \partial_\omega h + g(\varphi) + f(\overline{\theta} + h(\varphi), \varphi).
\]
Let \( \partial_\omega h(\varphi) = \mathcal{T}_{Q_n} g(\varphi) \), then the fibred equation is
\[
\tilde{\theta} = \tilde{f}(\overline{\theta}, \varphi) = \rho_f + \tilde{g}(0) + \mathcal{R}_{Q_n} g(\varphi) + f(\overline{\theta} + h(\varphi), \varphi).
\]
Since for $0 < |k| < Q_n$, $|\langle k, \omega \rangle| > \frac{1}{2Q_n}$, it is clear that

$$\|h(\varphi)\|_{\ell^{\frac{\alpha}{2}}} < 2Q_n \sum_{0 < |k| < Q_n} \|g\|_{\ell^{r_{n-1}}} e^{-|k|^{\frac{\alpha - 1}{2}}} \leq \frac{64Q_n\varepsilon_0}{r_{n-1}^2} \leq Q_n^{\frac{7}{4}} \varepsilon_0^{\frac{1}{4}}.$$  

Observe that if $\varepsilon_0^{-1/2} < Q_n^{7/4}$, we may lose the control of the norm of $f(\vec{\theta} + h(\varphi), \varphi)$.

However, in order to estimate $f(\vec{\theta} + h(\varphi), \varphi)$, it is sufficient to control the imaginary part of $h(\varphi)$. To fulfill this, we need a small trick, which says that $|\Re h(\varphi)|$ can be well controlled at the cost of reducing the analytic radius greatly.

Let $\varphi = \varphi_1 + i\varphi_2$ for $\varphi_1 \in \mathbb{T}^2$, $\varphi_2 \in \mathbb{R}^2$ and define

$$h_1(\varphi_1) = \sum_{0 < |k| < Q_n} \frac{\hat{g}(k)}{i(k, \omega)} e^{i(k, \varphi_1)}, \quad h_2(\varphi) = h(\varphi) - h_1(\varphi_1).$$

Since $g(\varphi)$ is real analytic, we have $\Re h_1(\varphi_1) = 0$. Moreover, by Corollary 2.1, we have $Q_n \geq Q_n^{8}$, which implies the following estimate

$$\Re h(\varphi) = \Re h_2(\varphi) = \Re h_2(\varphi_1)^n \leq \Re h_2(\varphi) \leq 2Q_n \sum_{0 < |k| < Q_n} |\hat{g}(k)| e^{-|k|^{\alpha + 2} - 1}$$

$$< 2Q_n \sum_{0 < |k| < Q_n} \|g\|_{\ell^{r_{n-1}}} e^{-|k|^{\alpha} + 2} \leq 2Q_n \sum_{0 < |k| < Q_n} \|g\|_{\ell^{r_{n-1}}} e^{-|k|^{\alpha} + 2} \|e^{-|k|^{\alpha}}\|_{\ell^{r_{n-1}}} < \frac{C\varepsilon_0}{Q_n^{1/2}} \frac{\varepsilon_0^{1/2}}{Q_n^{1/2}} < \frac{\varepsilon_0}{3}.$$  

As a consequence, we have

$$\|f(\vec{\theta} + h(\varphi), \varphi)\|_{\ell^{\alpha} + r_{n-1}} \leq \|f(\theta, \varphi)\|_{\ell^{\alpha} + r_{n-1}} \leq \varepsilon_{n-1},$$

$$\|\Re_Q g(\varphi)\|_{\ell^{\alpha} + r_{n-1}} \leq CQ_n^{2\varepsilon_0} e^{-Q_n^{1/2}} \leq \varepsilon_{n-1}^{1/3},$$

by (4.6). Lemma 2.3 implies $\rho(\omega, f(\vec{\varphi})) = \rho_f$. Then by Lemma 2.2, it follows that

$$|\hat{g}(0)| \leq \|f(\vec{\theta} + h(\varphi), \varphi)\|_{\ell^{\alpha} + r_{n-1}} + \|\Re_Q g(\varphi)\|_{\ell^{\alpha} + r_{n-1}} \leq 2\varepsilon_{n-1}^{1/3}.$$  

Let $\vec{\varphi} = \hat{g}(0) + \Re_Q g(\varphi)$ and $\vec{f}(\varphi) = f(\vec{\theta} + h(\varphi), \varphi)$. Then the result follows. \(\square\)

4.2.2. Reduction by diagonally dominant operators. In this section, we will apply Proposition 4.1 to make the perturbation as small as we can.

**Lemma 4.2.** Under the assumptions of Lemma 4.1, if furthermore $\rho_f \in DC_{\omega}(\gamma, \tau)$, then there exists $\vec{\Pi} \in C^n_{s_{n+1}, r_{n+1}}(\mathbb{T} \times \mathbb{T}^2, \mathbb{T} \times \mathbb{T}^2)$ with estimates

$$\|\vec{\Pi} - id\|_{s_{n+1}, r_{n+1}} \leq 4\varepsilon_{n-1}^{\frac{3}{2}}.$$  

$$\|\vec{\Pi} - id\|_{s_{n+1}, r_{n+1}} \leq 4\varepsilon_{n-1}^{\frac{3}{2}}.$$
(4.12) \[ \|D(\bar{\Pi} - id)\|_{s_{n+},r_{n+}} \leq 4\varepsilon_{n-1}^2, \]

such that \(\bar{\Pi}\) conjugates the system (4.9) to

(4.13) \[
\begin{align*}
\dot{\theta}_+ &= \rho_f + \bar{g}_+(\varphi) + \bar{f}_+(\bar{\theta}_+, \varphi) \\
\dot{\varphi} &= \omega = (1, \alpha)
\end{align*}
\]

with \(\rho_f + \bar{g}_+(\varphi) + \bar{f}_+(\bar{\theta}_+, \varphi) \in F_{s_{n+},r_{n+}}(\rho_f, 2\varepsilon_{n-1}^{1/2}, \varepsilon_{n})\) and \(\|\bar{g}_+ - \bar{g}\|_{r_{n+}} \leq 4\varepsilon_{n-1},\) where we denote \(r_{n+} = \frac{r_0}{2Q_n^2}, s_{n+} = s_n - \frac{\Delta_n}{3}.\)

**Proof.** To avoid ambiguous notations, in the proof of this lemma, we fix \(n\), and denote temporarily \(\bar{r} = \bar{r}_n = \frac{r_0}{2Q_n^2}, \bar{s} = \bar{s}_n = s_{n-1} - \frac{\Delta_n}{3}, \eta = 2\varepsilon_{n-1}, \bar{\eta} = 2\varepsilon_{n-1}^{1/2}.\) We will prove this lemma by iteration. First we define sequences:

\[
\begin{align*}
\bar{r}_0 &= \bar{r}, & \bar{s}_0 &= \bar{s}, & \eta_\nu &= \eta(\bar{\eta})^\nu, \\
\sigma_1 &= \frac{\bar{r}}{4} = \frac{r_0}{4Q_n^2}, & \sigma_{\nu+1} &= \frac{1}{2^\nu} \sigma_1, \\
\delta_1 &= \frac{\Delta_n}{6}, & \delta_{\nu+1} &= \frac{1}{2^\nu} \delta_1, \\
\bar{r}_\nu &= \bar{r}_{\nu-1} - \sigma_\nu, & \bar{s}_\nu &= \bar{s}_{\nu-1} - \delta_\nu, \\
K_\nu &= \left[ \frac{1}{\sigma_\nu} \ln \frac{1}{\eta_{\nu-1}} \right].
\end{align*}
\]

Since \(\frac{1}{Q_n}\) goes to 0 much faster than \(\Delta_n\), we can assume that \(\bar{r} < \Delta_n/3\) without loss of generality. Then as a consequence \(\sigma_\nu \leq \delta_\nu/2.\)

Let \(N = \left[2^n c_1 \tau U \ln Q_n \right] + 1\), where \(c_1 = \frac{c}{32(\tau + 3) \ln 3}\). Assume for \(j = 1, 2, \cdots, \nu - 1 < N\), there are \(h_j \in C^\omega_{\tilde{s}_{j-1}, \tilde{s}_j} \times T^2\), \(\tilde{f}_j \in C^\omega_{\tilde{s}_{j-1}, \tilde{s}_j} \times T^2\), \(\tilde{g}_j \in C^\omega_{\tilde{s}_{j-1}, \tilde{s}_j} \times T^2\), with \(\|f\|_{\tilde{s}_{j-1}, \tilde{s}_j} \leq \eta_j, \|h_j\|_{\tilde{s}_{j-1}, \tilde{s}_j} \leq \eta_j, \|\tilde{f}_j\|_{\tilde{s}_{j-1}, \tilde{s}_j} \leq \eta_j\), such that the transformation \(\tilde{\theta}_{j-1} = \tilde{\theta}_j + h_j(\tilde{\theta}_j, \varphi)\) conjugates the system

(4.14) \[
\begin{align*}
\dot{\tilde{\theta}}_{j-1} &= \rho_f + \tilde{g}_{j-1}(\varphi) + \tilde{f}_{j-1}(\tilde{\theta}_{j-1}, \varphi) \\
\dot{\varphi} &= \omega = (1, \alpha)
\end{align*}
\]

to

\[
\begin{align*}
\dot{\tilde{\theta}}_j &= \rho_f + \tilde{g}_j(\varphi) + \tilde{f}_j(\tilde{\theta}_j, \varphi) \\
\dot{\varphi} &= \omega = (1, \alpha)
\end{align*}
\]

with \(\tilde{g}_j = \tilde{g}_{j-1} + [\tilde{f}_{j-1}(\tilde{\theta}_j, \varphi)]_{\tilde{\theta}_j}\), where \([\tilde{f}_{j-1}]_{\tilde{\theta}_j}\) means the mean value of \(\tilde{f}_{j-1}\) in \(\tilde{\theta}_j\) over \(T\).

When \(j = \nu\), under the transformation \(\bar{\theta}_{\nu-1} = \bar{\theta}_\nu + \nu(\bar{\theta}_\nu, \varphi) \mod 1,\) the fibre equation of (4.14) becomes

(4.15) \[
\begin{align*}
\dot{\bar{\theta}}_\nu &= \rho_f + \bar{g}_{\nu-1}(\varphi) + \bar{f}_{\nu-1}(\bar{\theta}_\nu, \varphi) - \partial_\omega \nu - (\rho_f + \bar{g}_{\nu-1}(\varphi)) \frac{\partial h_\nu}{\partial \bar{\theta}_\nu} + h.o.t
\end{align*}
\]

\]
where \( h.o.t \) means higher order terms of \( \mathcal{T}_{\nu^{-1}} \) and \( h_{\nu} \). So the homological equation is

\[
\partial_\nu h_{\nu} + (\rho_f + \mathcal{g}_{\nu^{-1}}(\varphi)) \frac{\partial h_{\nu}}{\partial \theta_{\nu}} = \mathcal{T}_{\nu^{-1}}(\bar{\theta}_{\nu}, \varphi) - \mathcal{T}_{\nu^{-1}}(\mathcal{G}_{\nu}, \varphi)]\mathcal{g}_{\nu},
\]

By the definition of \( \bar{g}_j, j = 1, 2, \cdots, \nu \), we have \( \|\bar{g}_\nu\|_{\hat{r}_\nu^{-1}} \leq \|\bar{g}\|_{\hat{r}_0} + \sum_{j=0}^{\nu-1} \eta_j < 2\varepsilon_{n-1}^{1/2} \). Moreover, by our selection of \( \varepsilon_0 \) and the definition of \( \varepsilon_{n-1}, \bar{r}_{\nu^{-1}}, \bar{\nu}_{\nu^{-1}}, \eta_{\nu^{-1}} \), we have

\[
K_{\nu} = \left[ \frac{1}{\sigma_{\nu}} \ln \frac{1}{\eta_{\nu^{-1}}} \right] \leq \frac{4 \cdot 3^{\nu^{-1}} Q_n^3}{2 \varepsilon_{n^{-1}}} \frac{1}{r_0} \ln \frac{1}{\varepsilon_{n^{-1}}}
\]

\[
< \left( \frac{\gamma}{2 \varepsilon_{n^{-1}}} \right) = K^{(n-1)}
\]

for \( \nu \leq N \). By the assumption that \( \rho_f \in DC_\omega(\gamma, \tau) \), the conditions of Proposition 4.1 are satisfied. We can apply this proposition, obtaining an approximate solution of the homological equation (4.16) with estimates

\[
\|h_{\nu}\|_{\hat{s}_{\nu^{-1}} - \frac{\hat{r}_{\nu^{-1}}}{2} - \frac{\hat{a}_{\nu}}{2}} < \frac{3}{2} \eta_{\nu^{-1}} < \frac{\delta_1}{2\nu} = \frac{\delta_\nu}{2},
\]

and the error satisfying

\[
\|\bar{P}_{\nu}\|_{\hat{s}_{\nu^{-1}} - \frac{\hat{r}_{\nu^{-1}}}{2} - \frac{\hat{a}_{\nu}}{2}} \leq \frac{4 \eta_{\nu^{-1}}^2}{\gamma(\sigma_{\nu}/2)^{3+\tau}} \leq \frac{2 \eta_{\nu^{-1}}^2}{5}.
\]

Now (4.15) becomes

\[
\dot{\bar{\theta}}_{\nu} = \rho_f + \mathcal{g}_{\nu}(\varphi) + \bar{f}_{\nu}(\bar{\theta}_{\nu}, \varphi),
\]

where \( \mathcal{g}_{\nu} = \mathcal{g}_{\nu^{-1}} + [\mathcal{T}_{\nu^{-1}}(\bar{\theta}_{\nu}, \varphi)]\mathcal{g}_{\nu}, \) and

\[
\bar{f}_{\nu}(\bar{\theta}_{\nu}, \varphi) + \bar{f}_{\nu}(\bar{\theta}_{\nu}, \varphi) \frac{\partial h_{\nu}}{\partial \theta_{\nu}}
\]

\[
= (\mathcal{T}_{\nu^{-1}}(\bar{\theta}_{\nu} + h_{\nu}(\bar{\theta}_{\nu}, \varphi), \varphi) - \mathcal{T}_{\nu^{-1}}(\mathcal{G}_{\nu}, \varphi)]\mathcal{g}_{\nu}, \frac{\partial h_{\nu}}{\partial \theta_{\nu}}
\]

By the mean value theorem and the Cauchy estimate, we have

\[
\|\mathcal{T}_{\nu^{-1}}(\bar{\theta}_{\nu} + h_{\nu}(\bar{\theta}_{\nu}, \varphi), \varphi) - \mathcal{T}_{\nu^{-1}}(\bar{\theta}_{\nu}, \varphi)\|_{\hat{s}_{\nu^{-1}} - \frac{\hat{r}_{\nu^{-1}}}{2} - \frac{\hat{a}_{\nu}}{2}} \leq \frac{\eta_{\nu^{-1}}}{\delta_{\nu} + \gamma/2} \|h_{\nu}\|_{\hat{s}_{\nu^{-1}} - \frac{\hat{r}_{\nu^{-1}}}{2} - \frac{\hat{a}_{\nu}}{2}} \leq \frac{\eta_{\nu^{-1}}}{5}.
\]
It follows that
\[
\|f_\nu(\tilde{\theta}_\nu, \varphi)\|_{\tilde{s}_\nu, \tilde{r}_\nu} \leq (1 + \eta_{\nu-1}^2) \left( \frac{\eta_{\nu-1}^2}{5} + \frac{2\eta_{\nu-1}^2}{5} + \frac{\eta_{\nu-1}^2}{5} \right) \leq \eta_{\nu-1}^2 = \eta_\nu.
\]

To finish the proof, we estimate the size of \(f_N(\tilde{\theta}_N, \varphi)\). By the choice of \(U = U(\alpha)\) (cf. Corollary 2.1), we have \(Q_n^U \geq \ln Q_{n+1}\). As a consequence,
\[
\left(\frac{3}{2}\right)^N - 1 \geq \left(\frac{3}{2}\right)^{2n+1} \ln Q_n - 1 \geq Q_n^{2n+1} \ln(3/2).
\]
Therefore,
\[
\|f_N\|_{\tilde{s}_N, \tilde{r}_N} \leq \eta (\frac{3}{2})^N = \eta e^{-(\frac{3}{2})^N - 1} \leq \eta e^{-Q_n^{2n+1} \ln(3/2)} < \frac{1}{Q_n^{2n+1} \ln(3/2)} = \varepsilon_n,
\]
which is possible since by our selection \(\varepsilon_0 < e^{-2c_\tau U}\).

For simplicity, we just denote \(\tilde{\theta}_N\) by \(\theta_+\). We denote \(H_\nu(\tilde{\theta}_\nu, \varphi) = (\theta_\nu + h_\nu(\tilde{\theta}_\nu, \varphi) \mod 1, \varphi)\) for \(1 \leq \nu \leq N\). Since all these transformations are close to the identity, then by Lemma 2.3, we have \(\rho(\omega, \rho_f + f(\theta, \varphi)) = \rho_f\).

Let
\[
\overline{H}_\nu(\tilde{\theta}_\nu, \varphi) = H_1 \circ \cdots \circ H_{\nu-1} \circ H_\nu(\tilde{\theta}_\nu, \varphi).
\]
Then \(\overline{H}_\nu(\tilde{\theta}_\nu, \varphi)\) is analytic in \(D(\tilde{s}_\nu, \tilde{r}_\nu)\) and
\[
\|\partial_{\theta_\nu} (\Pi_1 \circ \overline{H}_\nu(\tilde{\theta}_\nu, \varphi))\|_{\tilde{s}_\nu, \tilde{r}_\nu} \leq \prod_{j=1}^{\nu} (1 + \eta_j^2).
\]

If we rewrite \(\overline{H}_N(\theta_+, \varphi) = (\theta_+ + h(\theta_+, \varphi) \mod 1, \varphi)\), then we have
\[
\|h(\theta_+, \varphi)\|_{\tilde{s}_{N-1}, \tilde{r}_{N-1} - \frac{\sigma}{2}} = \|\Pi_1 \circ \overline{H}_N(\theta_+, \varphi) - \theta_+\|_{\tilde{s}_{N-1}, \tilde{r}_{N-1} - \frac{\sigma}{2}} \leq \|\Pi_1 \circ (\overline{H}_N(\theta_+, \varphi) - \overline{H}_{N-1}(\theta_+, \varphi))\|_{\tilde{s}_{N-1}, \tilde{r}_{N-1} - \frac{\sigma}{2}}
\]
\[
+ \cdots + \|\Pi_1 \circ (\overline{H}_{N-2}(\theta_+, \varphi) - \overline{H}_{N-3}(\theta_+, \varphi))\|_{\tilde{s}_{N-2}, \tilde{r}_{N-2} - \frac{\sigma}{2}}
\]
\[
+ \cdots + \|\Pi_1 \circ (\overline{H}_1(\theta_+, \varphi) - \overline{H}_0(\theta_+, \varphi))\|_{\tilde{s}_0, \tilde{r}_0 - \frac{\sigma}{2}}
\]
\[
\leq \sum_{\nu=1}^{N} \prod_{j=1}^{\nu} (1 + \eta_j^2) \|h_\nu\|_{\tilde{s}_\nu, \tilde{r}_\nu} \leq 2\eta^2 < 4\varepsilon_{n-1}^{3/4}.
\]

Similarly, we have
\[
(4.18) \quad \|\frac{\partial h}{\partial \theta_+}(\tilde{\theta}_+, \varphi)\|_{\tilde{s}_N, \tilde{r}_N} \leq 4\varepsilon_{n-1}^{3/4},
\]
where \(\Pi_1 : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}\) denotes the natural projection to the first variable.
In conclusion, let \( \bar{\mathbf{g}}_+(\varphi) = \mathbf{g}_N(\varphi), \bar{T}_+(\bar{\theta}_+, \varphi) = T_N(\bar{\theta}_+, \varphi) \). Then \( \bar{\Pi}(\bar{\theta}_+, \varphi) = (\bar{\theta}_+ + \bar{h}(\bar{\theta}_+, \varphi) \mod 1, \varphi) \) transforms the system (4.9) to

\[
\begin{align*}
\dot{\bar{\theta}}_+ &= \rho_f + \bar{g}_+(\varphi) + \bar{T}_+(\bar{\theta}_+, \varphi) \\
\dot{\varphi} &= \omega = (1, \alpha)
\end{align*}
\]

with estimate

\[
\|\bar{g}_+(\varphi) - g(\varphi)\|_{r_n} \leq \sum_\nu \|\bar{f}_\nu\|_{\bar{s}_\nu, \bar{r}_\nu} < \sum_\nu \eta_\nu < 4\varepsilon_{n-1}.
\]

It follows that \( \rho_f + \bar{g}_+(\varphi) + \bar{T}_+(\bar{\theta}_+, \varphi) \in \mathcal{F}_{s_{n+}, r_{n+}}(\rho_f, 2\varepsilon_{n-1}^{1/2}, \varepsilon_n) \). \( \square \)

4.2.3. End of one step. In the first step, we eliminate the non-resonant terms of \( g(\varphi) \) and as a result the transformation we obtain is not close to the identity. In order to get rotations reducibility results, we need to inverse the first step, which means conjugating back by the transformation of the first step.

**Lemma 4.3.** Under the assumptions of Lemma 4.2, there exists \( \bar{H} \in C^{\omega}_{s_n, r_n}(\mathbb{T} \times \mathbb{T}^2, \mathbb{T} \times \mathbb{T}^2) \) with estimates

\[
\|\bar{H} - \text{id}\|_{s_{n}, r_{n}} \leq 4\varepsilon_{n-1}^{3/4},
\]

\[
\|D(\bar{H} - \text{id})\|_{s_{n}, r_{n}} \leq 4\varepsilon_{n-1}^{3/4},
\]

such that \( \bar{H} \) conjugates the system (4.8) to

\[
\begin{align*}
\dot{\bar{\theta}}_+ &= \rho_f + g_+(\varphi) + f_+(\theta_+, \varphi) \\
\dot{\varphi} &= \omega = (1, \alpha)
\end{align*}
\]

with \( \rho_f + g_+(\varphi) + f_+(\theta_+, \varphi) \in \mathcal{F}_{s_{n+}, r_{n+}}(\rho_f, 4\varepsilon_{n}, \varepsilon_n) \).

**Proof.** By Lemma 4.1, there exists \( h(\varphi) \in C^{\omega}_{s_n}(\mathbb{T}^2, \mathbb{R}) \) such that the system (4.8) can be conjugated to (4.9). Set \( H(\bar{\theta}, \varphi) = (\bar{\theta} + h(\varphi) \mod 1, \varphi) \). Then we apply Lemma 4.2 to get \( \bar{h} \in C^{\omega}_{s_{n+}, r_{n+}}(\mathbb{T} \times \mathbb{T}^2, \mathbb{R}) \) such that \( \bar{H}(\bar{\theta}_+, \varphi) = (\bar{\theta}_+ + \bar{h}(\bar{\theta}_+, \varphi) \mod 1, \varphi) \) conjugates the system (4.9) to (4.13) without changing the fibred rotation number.

Let \( g_+(\varphi) = T_Q g(\varphi) + \bar{g}_+(\varphi) \), \( f_+(\theta_+, \varphi) = \bar{T}_+(\theta_+ - h(\varphi), \varphi) \), then under the transformation \( \theta_+ = \bar{\theta} - h(\varphi) \) (mod 1), the fibred equation of system (4.13) becomes

\[
\begin{align*}
\dot{\bar{\theta}}_+ &= \rho_f + \bar{g}_+(\varphi) + \bar{T}_+(\bar{\theta}_+ - h(\varphi), \varphi) + \partial_\omega h(\varphi) \\
\dot{\varphi} &= \omega = (1, \alpha)
\end{align*}
\]

Then we have

\[
\|g_+(\varphi) - g(\varphi)\|_{r_n} \leq \|\bar{g}_+(\varphi) - \bar{g}(\varphi)\|_{r_{n+}} < 4\varepsilon_{n-1}.
\]
Therefore, we have $\|g_+(\varphi)\|_{r_n} \leq 4\bar{\varepsilon}_{n-1} + 4\bar{\varepsilon}_{n-1} = 4\bar{\varepsilon}_n$. Using the same argument as in Lemma 4.1, we have

$$\rho f + g_+(\varphi) + f_+(\theta_+, \varphi) \in \mathcal{F}_{s_n, r_n}(\rho f, 4\bar{\varepsilon}_n, \varepsilon_n).$$

Let $\tilde{H} = H \circ \tilde{H} \circ H^{-1}$. Then

$$\Pi_1 \circ \tilde{H}(\theta_+, \varphi) = \theta_+ + \tilde{h}(\theta_+ - h(\varphi), \varphi) \mod 1,$$

and $\tilde{H}(\theta_+, \varphi)$ conjugates the system (4.8) to (4.21). Furthermore, the estimates

$$\|\tilde{H} - id\|_{s_n, r_n} \leq \|\tilde{h}(\theta_+, \varphi)\|_{s_n, r_n} < 4\varepsilon_{n-1}^3,$$

$$\|D(\tilde{H} - id)\|_{s_n, r_n} \leq \|D\tilde{h}\|_{s_n, r_n} < 4\varepsilon_{n-1}^4$$

follows from (4.11) and (4.12). \qed

4.3. Iteration and convergence. Summarizing conclusions of section 4.2, we have the following iteration lemma.

**Lemma 4.4.** For any $\varepsilon_0 > 0, r_0 > 0, s_0 > 0, \gamma > 0, \tau > 2$, $\omega = (1, \alpha)$ with $U = U(\alpha) < \infty$, $\varepsilon_n, \bar{\varepsilon}_n, r_n, s_n$ are defined as in (4.7). Suppose $\varepsilon_0$ is small enough such that it satisfies (4.6). Then the following holds for all $n \geq 1$: If the system

$$\begin{cases}
\dot{\theta} = \rho f + g_n(\varphi) + f_n(\theta, \varphi) \\
\dot{\varphi} = \omega = (1, \alpha)
\end{cases}$$

satisfies $\rho f + g_n(\varphi) + f_n(\theta, \varphi) \in \mathcal{F}_{s_n, r_n}(\rho f, 4\bar{\varepsilon}_n, \varepsilon_n)$, then there exists $H_n : T \times T^2 \to T \times T^2$ with estimates

- $\|H_n - id\|_{s_n+1, r_n+1} \leq 4\varepsilon_n^3$
- $\|D(H_n - id)\|_{s_n+1, r_n+1} \leq 4\varepsilon_n^4$

such that it transforms the system (4.22) to

$$\begin{cases}
\dot{\theta} = \rho f + g_{n+1}(\varphi) + f_{n+1}(\theta, \varphi) \\
\dot{\varphi} = \omega = (1, \alpha)
\end{cases}$$

with $\rho f + g_{n+1}(\varphi) + f_{n+1}(\theta, \varphi) \in \mathcal{F}_{s_{n+1}, r_{n+1}}(\rho f, 4\bar{\varepsilon}_{n+1}, \varepsilon_{n+1})$.

With carefully checking of the proof, more precisely if we only do the first two steps as in section 4.2.1 and 4.2.2, then the following iteration lemma holds; it will be the basis for the proof of almost reducibility.

**Lemma 4.5.** Under the assumptions of Lemma 4.4, the following holds for all $n \geq 1$: If the system

$$\begin{cases}
\dot{\theta} = \rho f + g_n(\varphi) + f_n(\theta, \varphi) \\
\dot{\varphi} = \omega = (1, \alpha)
\end{cases}$$

with $\rho f + g_n(\varphi) + f_n(\theta, \varphi) \in \mathcal{F}_{s_n, r_n}(\rho f, 4\bar{\varepsilon}_n, \varepsilon_n)$. 

satisfy \( \rho_f + g_n(\varphi) + f_n(\theta, \varphi) \in \mathcal{F}_{s_n, r_n}(\rho_f, 2\varepsilon_n^{1/2}, \varepsilon_n) \), then there exists \( \tilde{H}_n : \mathbb{T} \times \mathbb{T}^2 \to \mathbb{T} \times \mathbb{T}^2 \) with estimate \( \| \tilde{H}_n - \text{id} \|_{s_{n+1}; r_{n+1}} \leq Q_{n+1}^2 \varepsilon_0 \) such that it transforms the system (4.24) to

\[
\begin{align*}
\dot{\theta} &= \rho_f + g_{n+1}(\varphi) + f_{n+1}(\theta, \varphi) \\
\dot{\varphi} &= \omega = (1, \alpha)
\end{align*}
\]

with \( \rho_f + g_{n+1}(\varphi) + f_{n+1}(\theta, \varphi) \in \mathcal{F}_{s_{n+1}, r_{n+1}}(\rho_f, 2\varepsilon_n^{1/2}, \varepsilon_{n+1}) \).

5. Proof of Theorem 1.1

In this section, we first use Lemma 4.4 and Lemma 4.5 to prove Theorem 1.1 (a). Then as corollaries, we prove the local denseness of linearization (Theorem 1.1 (b)) and local denseness of mode-locking (Corollary 1.1).

5.1. Proof of Theorem 1.1 (a): Select \( \varepsilon_0 \) such that it satisfies (4.6) with \( r_0 = r, s_0 = s \). We first prove that if \( \rho(\omega, \tilde{\rho} + f(\theta, \varphi)) = \rho_f \in DC_\omega(\gamma, \tau) \) and \( \| f \|_{s, r} \leq \varepsilon_0/2 \), then the system \((\omega, \tilde{\rho} + f(\theta, \varphi))\) is \( C^\infty \) rotations linearizable.

Without loss of generality, we can rewrite the system \((\omega, \tilde{\rho} + f(\theta, \varphi))\) as

\[
\begin{align*}
\dot{\theta} &= \rho_f + \tilde{g}(\varphi) + \tilde{f}(\theta, \varphi) \\
\dot{\varphi} &= \omega = (1, \alpha)
\end{align*}
\]

with \( \rho_f + \tilde{g}(\varphi) + \tilde{f}(\theta, \varphi) \in \mathcal{F}_{s, r}(\rho_f, \varepsilon_0, \varepsilon_0) \). Since \( \rho_f \in DC_\omega(\gamma, \tau) \) and \( \varepsilon_0 \) satisfies the inequality (4.6), we can apply Lemma 4.2 (in fact without using Lemma 4.1 and Lemma 4.3), and thus get \( H_0 \in C^{\omega}_{s_1, r_1}(\mathbb{T} \times \mathbb{T}^2, \mathbb{T} \times \mathbb{T}^2) \) which conjugates (5.1) to

\[
\begin{align*}
\dot{\theta} &= \rho_f + g_1(\varphi) + f_1(\theta, \varphi) \\
\dot{\varphi} &= \omega = (1, \alpha)
\end{align*}
\]

with \( \rho_f + g_1(\varphi) + f_1(\theta, \varphi) \in \mathcal{F}_{s_1, r_1}(\rho_f, 4\varepsilon_1, \varepsilon_1) \). Then we apply Lemma 4.4, and inductively we obtain sequence \( H_i \in C^{\omega}_{s_{i+1}, r_{i+1}}(\mathbb{T} \times \mathbb{T}^2, \mathbb{T} \times \mathbb{T}^2), i = 1, \ldots, n-1 \) such that \( H_i = H_{i-1} \circ H_1 \cdots \circ H_{n-1} \) conjugates the system (1.3) to \((\omega, \rho_f + g_n(\varphi) + f_n(\theta, \varphi))\) with \( \rho_f + g_n(\varphi) + f_n(\theta, \varphi) \in \mathcal{F}_{s_n, r_n}(\rho_f, 4\varepsilon_n, \varepsilon_n) \).

Let \( H = \lim_{n \to \infty} H^{(n)} \), \( g_\infty = \lim_{n \to \infty} g_n \). Then the system \((\omega, \tilde{\rho} + f(\theta, \varphi))\) is conjugated by \( H \) to \((\omega, \rho_f + g_\infty(\varphi))\). The remaining task is to prove that the transformation \( H \) is actually in \( C^{\infty} \). Since

\[
\| DH^{(n)} \|_{s_n, r_n} \leq \| DH_0 \|_{s_1, r_1} \cdot \| DH_1 \|_{s_2, r_2} \cdots \cdot \| DH_{n-1} \|_{s_n, r_n} \leq \prod_{i=0}^{n-1} (1 + 4 \varepsilon_i^{3/4}) < 2,
\]

then we get

\[
\| H^{(n+1)} - H^{(n)} \|_{s_{n+1}, r_{n+1}} \leq \| DH^{(n)} \|_{s_n, r_n} \cdot \| H_n - \text{id} \|_{s_{n+1}, r_{n+1}} \leq 8 \varepsilon_n^{3/4}.
\]

By the definition of \( (\varepsilon_n) \in \mathbb{N} \), we know that for any \( j \in \mathbb{Z}_+^2 \), there exists some \( N \in \mathbb{N} \), so that for any \( n \geq N \), we have \( Q_n^4 |j| < \varepsilon_n^{-1/4} \), that is

\[
Q_n^4 |j| \varepsilon_n^{3/4} < \varepsilon_n^{1/2}, \quad \forall n \geq N.
\]
Then by Cauchy estimates, if we denote \( x := (\theta, \varphi) \in \mathbb{T}^3 \), we have
\[
\left| \frac{\partial^{[j]}(H^{(n+1)} - H^{(n)})}{\partial x} \right| \leq r_n^{-[j]} \|H^{(n+1)} - H^{(n)}\|_{s_n+1, r_n+1} \leq Q_{n+1}^{4[j]} \varepsilon_n^{3/4} < \varepsilon_n^{1/2}
\]
for any \( n \geq N - 1 \). This guarantees the limit \( H = \lim_{n \to \infty} H^{(n)} \) belongs to \( C^\infty \). As a consequence, we have \( g_\infty \in C^\infty(\mathbb{T}^2, \mathbb{R}) \).

The proof of the \( C^\infty \) almost reducibility is exactly the same as the proof of the \( C^\infty \) rotations reducibility. The only difference is that we use Lemma 4.5 instead of Lemma 4.4. Note in this case, if \( \tilde{H} = \lim_{n \to \infty} \tilde{H}_0 \circ \cdots \circ \tilde{H}_{n-1} \), then \( \tilde{H} \) actually diverges. \( \square \)

5.2. Proof of Theorem 1.1 (b): Suppose that \( \varepsilon_0 \) is selected according to (4.6) with \( r_0 = r, s_0 = s \). Consider the system \((\omega, \rho + f(\theta, \varphi))\) with \( \rho(\omega, \tilde{\rho} + f(\theta, \varphi)) = \rho f \in DC_{\omega}(\gamma, \tau) \) and \( ||f||_{s, r} < \varepsilon_0/2 \). For any \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( \frac{1}{Q_{N-1}^{3\varepsilon r U}} < \varepsilon \). Then by Lemma 4.5, there exists a map \( H \in C^\infty(\mathbb{T} \times \mathbb{T}^2, \mathbb{T} \times \mathbb{T}^2) \) such that the system \((\omega, \rho + f(\theta, \varphi))\) can be conjugated to \((\omega, \rho + \tilde{g}(\varphi) + \tilde{f}(\theta, \varphi))\) with
\[
||\tilde{g}||_{\tau} < \varepsilon^{1/2}, \quad ||\tilde{f}||_{\pi, \tau} < \frac{1}{Q_N^{2\varepsilon r U}}.
\]
\[
||\Pi_1 \circ H - id||_{\pi, \tau} < Q_N^{2\varepsilon r U}, \quad ||\frac{\partial}{\partial \theta}(\Pi_1 \circ H - id)||_{\pi, \tau} < Q_N^{2\varepsilon r U}.
\]
where \( \tau = s_{N-1}, \Pi_1 = r_{N-1} \). Rewrite \((\omega, \rho f + \tilde{g}(\varphi) + \tilde{f}(\theta, \varphi))\) as \((\omega, \tilde{\rho} + \tilde{g}(\varphi) + \tilde{f}(\theta, \varphi))\), where \( \tilde{\rho} = \rho f + \tilde{g}(0), \tilde{g}(\varphi) = T_{Q_N} g(\varphi), \) and \( \tilde{f}(\theta, \varphi) = \tilde{f}(\theta, \varphi) + R_{Q_N} g(\varphi) \). Then let \( \tilde{s} = \tilde{s}, \tilde{r} = \frac{\tau}{2} \), we have
\[
(5.3) \quad ||\tilde{\Pi}_1 \circ \tilde{H}||_{\tilde{s}, \tilde{r}} \leq ||\tilde{g}||_{\pi, \tau} + ||R_{Q_N} g(\varphi)||_{\pi, \tau} < 2\frac{1}{Q_N^{2\varepsilon r U}}.
\]
We consider the reference system \((\omega, \tilde{\rho} + \tilde{g}(\varphi))\): it must be \( C^\infty \) linearizable. The reason is that since \( \tilde{g}(\varphi) \) is a trigonometric polynomial, if we let \( \partial_\omega \tilde{h}(\varphi) = \tilde{g}(\varphi) \), then \( \tilde{h}(\varphi) \) is analytic.

If we conjugate back both \((\omega, \tilde{\rho} + \tilde{g}(\varphi))\) and \((\omega, \tilde{\rho} + \tilde{g}(\varphi) + \tilde{f}(\theta, \varphi))\) by the above transformation \( H \), we obtain \((\omega, \tilde{f}(\theta, \varphi))\) and \((\omega, \tilde{\rho} + f(\theta, \varphi))\). Here, \((\omega, \tilde{f}(\theta, \varphi))\) is clearly \( C^\infty \) linearizable, since linearization is conjugacy invariant. As a consequence of (5.3), we have
\[
||\tilde{\rho} + f(\theta, \varphi) - \tilde{f}(\theta, \varphi)||_{\tilde{s}, \tilde{r}} \leq ||\frac{\partial}{\partial \theta}(\Pi_1 \circ H)||_{\tilde{s}, \tilde{r}} \cdot ||\tilde{f}||_{\tilde{s}, \tilde{r}} < \varepsilon.
\]
That is to say, \((\omega, \tilde{\rho} + f(\theta, \varphi))\) is \( C^\infty \)-accumulated by the qpf circle flow \((\omega, \tilde{f}(\theta, \varphi))\), while \((\omega, \tilde{f}(\theta, \varphi))\) is \( C^\infty \) linearizable. \( \square \)
5.3. Proof of Corollary 1.1. To complete the picture in the local situation, we give the following proposition:

**Proposition 5.1.** Any rigid rotation flow \((\omega, \rho)\) can be \(C^\infty\) accumulated by flows \((\omega, \hat{f}(\theta, \varphi))\), where \((\omega, \hat{f}(\theta, \varphi))\) is mode-locked.

**Proof.** For any \(\rho > 0, \varepsilon > 0\), there exists \(k \in \mathbb{Z}^2\) such that \(|\langle k, \omega \rangle - \rho| < \frac{\varepsilon}{2}\), since \(\omega\) is rationally independent. Under the transformation \(\theta = \overline{\theta} + \langle k, \varphi \rangle \pmod{1}\), \((\omega, \langle k, \omega \rangle)\) is conjugated to \((\omega, 0)\). Now we choose a qpf circle flow \((\omega, \frac{\varepsilon}{4\pi} \sin 4\pi \varphi)\), which can be viewed as projective action of a uniformly hyperbolic \(sl(2, \mathbb{R})\) quasi-periodic flow:

\[
\begin{cases}
\dot{x} = \left( \begin{array}{cc}
\frac{\varepsilon}{4\pi} & 0 \\
0 & -\frac{\varepsilon}{4\pi}
\end{array} \right) x \\
\dot{\varphi} = \omega
\end{cases}
\]

By Proposition 2.1, \((\omega, \frac{\varepsilon}{4\pi} \sin 4\pi \varphi)\) is mode-locked. Invert the transformation \(\theta = \overline{\theta} + \langle k, \varphi \rangle \pmod{1}\) both for \((\omega, 0)\) and \((\omega, \frac{\varepsilon}{4\pi} \sin 4\pi \varphi)\), we get \((\omega, \langle k, \omega \rangle)\) and \((\omega, \hat{f})\). A direct calculation shows that \(\|\hat{f}(\theta, \varphi) - \rho\|_{s,r} < \frac{\varepsilon}{2}\), while \((\omega, \hat{f})\) is mode-locked.

Now we can finish the proof of Corollary 1.1. For any qpf circle flow \((\omega, \hat{\rho} + f(\theta, \varphi))\) with \(\rho(\omega, \hat{\rho} + f(\theta, \varphi)) = \rho_f \in DC_\omega(\gamma, \tau)\) and \(\|f\|_{r,s} < \varepsilon_0/2\), first we perturb \((\omega, \hat{\rho} + f(\theta, \varphi))\) to \(C^\omega\) linearizable qpf circle flow \((\omega, \tilde{f}(\theta, \varphi))\) by Theorem 1.1 (b). Then \((\omega, \tilde{f}(\theta, \varphi))\) is \(C^\infty\)-accumulated by mode-locked qpf circle flow by Proposition 5.1. \(\square\)

**Acknowledgements**

R.Krikorian was supported by the “Chaire d’Excellence” Labex MME-DII at the University of Cergy-Pontoise and by the ANR project BEKAM (ANR-15-CE40-0001). J. Wang has been supported by a research fellowship of the Alexander-Humboldt-Foundation, NNSF of China (11601230) and Natural Science Foundation of Jiangsu Province, China(BK20160816). J. You was supported by NNSF of China (11471155) and 973 projects of China (2014CB340701). Q. Zhou was supported by Fondation Sciences Mathématiques de Paris (FSMP) and NNSF of China (11671192).

**References**

[1] V.I. Arnold, Small denominators I. On the mapping of a circle into itself. Akad.Nauk.Math, 25 (1961), 21-86.
[2] A. Avila, Global theory of one-frequency Schrödinger operators, Acta Math. 215, 1-54 (2015).
[3] A. Avila, Almost reducibility and absolute continuity, http://w3.impa.br/avila/
[4] A. Avila, KAM, Lyapunov exponents and the spectral dichotomy for one-frequency schrödinger operators. In preparation.
[5] A. Avila, B. Fayad and R. Krikorian, A KAM scheme for SL(2, \mathbb{R}) cocycles with Liouvillean frequencies. Geom. Funct. Anal. 21 (2011), 1001-1019.

[6] A. Avila and S. Jitomirskaya, The ten Martini problem, Annals of Mathematics. 170 (2009) 303-342.

[7] A. Avila and R. Krikorian Reducibility or non-uniform hyperbolicity for quasiperiodic Schrödinger cocycles. Annals of Mathematics. 164 (2006), 911-940.

[8] J. Béllissard and B. Simon. Cantor spectrum for the almost Mathieu equation. J. Funct. Anal., 48 (3) (1982), 408-419.

[9] K. Bjerklöv and T. Jäger, Rotation numbers for quasiperiodically forced circle maps - mode-locking vs strict monotonicity. J. Am. Math. Soc. 22 (2009), 353-362.

[10] E. Dinaburg and Ya. Sinai, The one-dimensional Schrödinger equation with a quasi-periodic potential. Funct. Anal. Appl. 9 (1975), 279-289.

[11] M. Ding, C. Grebogi, E. Ott, Evolution of attractors in quasiperiodically forced systems: From quasiperiodic to strange nonchaotic to chaotic, Physical Review A, 39(5) (1989) 2593-2598.

[12] H. Eliasson, Floquet solutions for the one-dimensional quasiperiodic Schrödinger equation. Comm.Math.Phys. 146 (1992), 447-482.

[13] B. Fayad and R. Krikorian, Rigidity results for quasiperiodic SL(2, \mathbb{R})-cocycles. Journal of Modern Dynamics. 3, (2009), no. 4. 479-510.

[14] M. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, Inst. Hautes Etudes Sci. Publ. Math. (1979) 5-233.

[15] M. Herman, Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d’un théorème d’Arnold et de Moser sur le tore de dimension 2, Comment. Math. Helv. 58 (1983) 453-502.

[16] X. Hou and J. You, Almost reducibility and non-perturbative reducibility of quasiperiodic linear systems. Invent. Math. 190 (2012) 209-260.

[17] T. Jäger, J. Stark, Towards a classification for quasiperiodically forced circle homeomorphisms, Journal of the LMS, 73(3) (2006) 727-744.

[18] R. Krikorian, Reducibility, differentiable rigidity and Lyapunov exponents for quasiperiodic cocycles on \mathbb{T} \times SL(2, \mathbb{R}), http://arxiv.org/abs/math/0402333.

[19] R. Krikorian, On almost reducibility of circle diffeomorphisms, in preparation.

[20] J. Puig, Cantor spectrum for the Almost Mathieu operator. Comm.Math.Phys. 244 (2004), 297-309.

[21] J. Wang, Lower dimensional invariant tori for quasiperiodically forced circle diffeomorphisms, J. Diff. Equations 253 (2012) 1489-1543.

[22] J. C. Yoccoz, Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne, Ann. Sci. Ecole Norm. Sup. 17 (1984) 333-359.
[23] J. C. Yoccoz, Analytic linearization of circle diffeomorphisms, in Dynamical Systems and Small divisors(Cetraro, 1998), Lecture Notes in Math. 1784, Springer-Verlag, New York, (2002) 125-173.

[24] J. You and Q. Zhou, Embedding of analytic quasi-periodic cocycles into analytic quasi-periodic linear systems and its applications. Comm. Math. Phys. 323(2013), 975-1005.

[25] Q. Zhou and J. Wang, Reducibility results for quasiperiodic cocycles with Liouvillean frequency. J. Dyn. Diff. Equat., 24 (2012) 61-83.

Department of Mathematics, CNRS UMR 8088, Université de Cergy-Pontoise, 2, av. Adolphe Chauvin F-95302 Cergy-Pontoise, France
E-mail address: raphael.krikorian@u-cergy.fr

Department of Mathematics, Nanjing University of Science and Technology, 210094, China
E-mail address: jing.wang@njust.edu.cn

Chern Institute of Mathematics, Nankai University, Tianjin 300071, China
E-mail address: jyou@nju.edu.cn

Department of Mathematics, Nanjing University, Nanjing 210093, China
E-mail address: qizhou@nju.edu.cn