A FOURIER RESTRICTION THEOREM FOR HYPERSURFACES WHICH ARE GRAPHS OF CERTAIN REAL POLYNOMIALS

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Abstract. We will extend the Fourier restriction inequality for quadratic hypersurfaces obtained by Strichartz. We will consider the case where the hypersurface is a graph of a certain real polynomial which is a sum of one-dimensional monomials. It is essential to examine the decay of a one-dimensional oscillatory integral.

1. Introduction

Let $S$ be a hypersurface in $\mathbb{R}^n$, $n \geq 2$. We consider the Fourier restriction inequality

$$\left( \int_S |\hat{\phi}(\xi)|^2 d\mu_n(\xi) \right)^{1/2} \leq C_p \|\phi\|_{L^p(\mathbb{R}^n)} \quad \text{for } \phi \in L^p(\mathbb{R}^n),$$

where the measure $d\mu_n$ on $S$ is defined as follows:

$$d\mu_n(\xi) = \left| \frac{\partial \tilde{R}}{\partial \xi_n}(\xi) \right|^{-1} d\xi_1 \ldots d\xi_{n-1}$$

when $S$ is written as $S = \{ \xi \in \mathbb{R}^n; \tilde{R}(\xi) = r \}$ with a constant $r \in \mathbb{R}$ and a real-valued function $\tilde{R} \in C^0(\mathbb{R}^n)$ which is partially differentiable with respect to $\xi_n$ and $(\partial \tilde{R}/\partial \xi_n)(\xi) \neq 0$ for almost every $\xi \in \mathbb{R}^n$. For $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, we write $\xi' = (\xi_1, \ldots, \xi_{n-1})$. In particular, if $S$ is the form of $S = \{ \xi \in \mathbb{R}^n; \xi_n = R(\xi') \}$, then (1.1) becomes

$$\left( \int_{\mathbb{R}^{n-1}} |\hat{\phi}(\xi', R(\xi'))|^2 d\xi' \right)^{1/2} \leq C_p \|\phi\|_{L^p(\mathbb{R}^n)} \quad \text{for } \phi \in L^p(\mathbb{R}^n).$$

For $p \in [1, \infty)$ and a subset of a Euclidean space $\Omega$, set

$$\|f\|_{L^p(\Omega)} = \left( \int_{\Omega} |f(x)|^p dx \right)^{1/p}$$

and let $L^p(\Omega)$ denote the set of all Lebesgue measurable functions $f$ on $\Omega$ such that $\|f\|_{L^p(\Omega)} < \infty$. Let $i$ always denote the imaginary unit. We define the Fourier transform

\[\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} dx\]

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in $x \in \mathbb{R}^n$ and the inverse Fourier transform in $\xi \in \mathbb{R}^n$ by setting

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx,$$

$$\hat{f}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}d\xi,$$

respectively. Those of a generalized function are also denoted by the same notation.

In [3], Strichartz determined the optimal range of the exponent $p$ for which (1.1) holds for all quadratic hypersurfaces $S$. A nondegenerate quadratic hypersurface $S$ which is not contained in an affine hyperplane is transformed into one of the following three types under an affine transformation.

(1) $S = \{\xi \in \mathbb{R}^n; \xi_n = \xi_1^2 + \cdots + \xi_s^2 - \xi_{s+1}^2 - \cdots - \xi_{n-1}^2\}$, where $1 \leq s \leq n - 1$.

(2) $S = \{\xi \in \mathbb{R}^n; \xi_1^2 + \cdots + \xi_s^2 - \xi_{s+1}^2 - \cdots - \xi_{n}^2 = 0\}$, where $1 \leq s \leq n - 1$.

(3) $S = \{\xi \in \mathbb{R}^n; \xi_1^2 + \cdots + \xi_s^2 - \xi_{s+1}^2 - \cdots - \xi_{n}^2 = 1\}$, where $1 \leq s \leq n$.

The results obtained by Strichartz for the first type are the following.

**Theorem 1.1** (Strichartz [3, Theorem 1]). Let $n \geq 2$, and let $S$ be a hypersurface $S = \{\xi \in \mathbb{R}^n; \xi_n = \xi_1^2 + \cdots + \xi_s^2 - \xi_{s+1}^2 - \cdots - \xi_{n-1}^2\}$, where $1 \leq s \leq n - 1$. Then, (1.2) holds with $C_p$ independent of $f$ if and only if

$$p = \frac{2(n+1)}{n-3}.$$

Strichartz also gives an estimate of solutions to the inhomogeneous Schrödinger evolution equations as an application of the Fourier restriction theorem.

**Theorem 1.2** (Strichartz [3 Corollary 1]). Let $n \geq 1$. Let

$$p = \frac{2(n+2)}{n+4},$$

and assume $\phi \in L^2(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^{1+n})$. Let $u(t, x)$ be a solution to the initial value problem for inhomogeneous partial differential equations

$$D_t u + \Delta u = f(t, x) \text{ in } \mathbb{R}^{1+n},$$

$$u(0, x) = \phi(x) \text{ in } \mathbb{R}^n,$$

where

$$D_t = -i\frac{\partial}{\partial t}, \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

Then,

$$\|u\|_{L^p/(p-1)(\mathbb{R}^{1+n})} \leq C(\|\phi\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^{1+n})})$$

holds with $C$ independent of $\phi$, $f$ and $u$.

The purpose of this paper is to study generalization of Theorems 1.1 and 1.2. We will consider the case where $S$ is the graph of a certain real polynomial. We will introduce a method due to Strichartz, and extend Theorem 1.1 to prove the following.
Theorem 1.3. Let $n \geq 2$, and let $S$ be a hypersurface $S = \{\xi \in \mathbb{R}^n; \xi_n = R(\xi')\}$, where

\begin{equation}
R(\xi') = \sum_{j=1}^{n-1} a_j \xi_j^{k_j},
\end{equation}

$a_j \in \mathbb{R} \setminus \{0\}$ and $k_j \in \{2, 3, 4, \ldots\}$ for all $j = 1, \ldots, n - 1$. Then, (1.2) holds with $C_p$ independent of $f$ if and only if

\[ p = 2 - \frac{2}{2 + \sum_{j=1}^{n-1} \frac{1}{k_j}}. \]

The following theorem is our corresponding application of Theorem 1.3:

Theorem 1.4. Let $n \geq 1$. Let

\[ a(\xi) = \sum_{j=1}^{n} a_j \xi_j^{k_j}, \]

where $a_j \in \mathbb{R} \setminus \{0\}$ and $k_j \in \{2, 3, 4, \ldots\}$ for all $j = 1, \ldots, n$. Let

\[ p = 2 - \frac{2}{2 + \sum_{j=1}^{n} \frac{1}{k_j}}, \]

and assume $\phi \in L^2(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^{1+n})$. Let $u(t, x)$ be a solution to the initial value problem for inhomogeneous partial differential equations

\begin{align}
D_t u - a(D)u &= f(t, x) \text{ in } \mathbb{R}^{1+n}, \\
u(0, x) &= \phi(x) \text{ in } \mathbb{R}^n,
\end{align}

where

\[ D_t = -i \frac{\partial}{\partial t}, \quad D = (D_1, \ldots, D_n), \quad D_j = -i \frac{\partial}{\partial x_j}. \]

Then,

\[ \|u\|_{L^p((p-1)(\mathbb{R}^{1+n}))} \leq C (\|\phi\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^{1+n})}) \]

holds with $C$ independent of $\phi$, $f$ and $u$.

An essential matter in proving Theorem 1.3 is to examine the decay of the Fourier transform of $\exp(itR(x'))$, that is,

\begin{equation}
\int_{\mathbb{R}^{n-1}} \exp(ix' \cdot \xi' + itR(\xi'))d\xi'
\end{equation}

for large $t$. Since $R$ is a sum of one-dimensional monomials, an estimate of (1.6) is effectively reduced to that of a one-dimensional oscillatory integral

\begin{equation}
\int_{-\infty}^{\infty} \exp(ix \xi + it\xi^k)d\xi,
\end{equation}
where \( k_j \in \{2, 3, 4, \ldots \} \). We will bound (1.7) by \( 12|t|^{-1/k} \). If the region of integration is a bounded interval, then

\[
\left| \int_{\alpha}^{\beta} \exp(i x \xi + i t \xi^k) d\xi \right| \leq C_{\alpha, \beta}|t|^{-1/k}
\]

immediately follows from the Van der Corput lemma.

The organization of this paper is as follows. In Section 2, we estimate the one-dimensional oscillatory integral (1.7), and give a proof of Theorem 1.3. Section 3 describes a proof of Theorem 1.4.

2. A Fourier restriction theorem

In this section, we estimate the one-dimensional oscillatory integral (1.7), and give a proof of Theorem 1.3. We use a method due to Strichartz. Let \( \Gamma \) denote the Gamma function. For \( s \in \mathbb{R} \), let \( s^+ \) denote its positive part: \( s^+ = \max\{s, 0\} \).

**Proposition 2.1** (Strichartz [3, Lemma 2]). Let \( n \geq 2 \). We assume that a hypersurface \( S \) is written as \( S = \{ \xi \in \mathbb{R}^n; \tilde{R}(\xi) = r \} \) with a constant \( r \in \mathbb{R} \) and a real-valued function \( \tilde{R} \in C^0(\mathbb{R}^n) \). Let

\[
G_z(\xi) = \frac{(\tilde{R}(\xi) - r)^z}{\Gamma(z + 1)}
\]

for \( z \in \mathbb{C} \). Moreover, we assume that for some \( \lambda > 1 \), \( \tilde{G}_{-\lambda + i\eta} \) is bounded:

\[
\|\tilde{G}_{-\lambda + i\eta}\|_{L^\infty(\mathbb{R}^n)} \leq C_\eta,
\]

and that there exists \( b < \pi \) such that

\[
\sup_{\eta \in \mathbb{R}} e^{-b|\eta|} \log C_\eta < \infty.
\]

Then, (1.1) holds for

\[
p = \frac{2\lambda}{\lambda + 1}.
\]

We use the following formula of integration later.

**Lemma 2.2** ([1 p. 360]).

\[
\frac{(2\pi)^{1/2}}{\Gamma(z + 1)}(\xi^z)^+(x) = \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} e^{-\varepsilon \xi z} \frac{e^{i\varepsilon \xi}}{\Gamma(z + 1)} d\xi = ie^{i\varepsilon/2}(x + i0)^{-z-1}
\]

for all \( z \in \mathbb{C} \).

Next, for \( k_j \in \{2, 3, 4, \ldots \} \), we define a one-dimensional oscillatory integral

\[
A_k(x) = \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \exp(-\varepsilon|x|^k + ixs + is^k) ds
\]

for \( x \in \mathbb{R} \). Changing the variables yields

\[
\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \exp(-\varepsilon|x|^k + ixs + it \xi^k) d\xi = \begin{cases} t^{-1/k}A_k(t^{-1/k}x) & \text{if } t > 0, \\ (-t)^{-1/k}A_k((-t)^{-1/k}x) & \text{if } t < 0, \end{cases}
\]
and then,

\[
(2.1) \quad \left| \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \exp(-\varepsilon|\xi|^k + ix\xi + it\xi^k)d\xi \right| = |t|^{-1/k} |A_k(|t|^{-1/k}x \text{ sgn } t)|
\]

for \( t \in \mathbb{R} \setminus \{0\} \). Here, \( \text{sgn} \) denotes the signature function: \( \text{sgn } s = s/|s| \) if \( s \in \mathbb{R} \setminus \{0\} \), \( \text{sgn } s = 0 \) if \( s = 0 \).

The bounds of \( A_k \) are the following.

**Proposition 2.3.**

\[
\left| \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \exp(-\varepsilon|\xi|^k + ix\xi + it\xi^k)d\xi \right| \leq 12 |t|^{-1/k}
\]

for any \( k_j \in \{2, 3, 4, \ldots \} \) and \( t \in \mathbb{R} \setminus \{0\} \).

In view of (2.1), Proposition 2.3 immediately follows from the following.

**Lemma 2.4.** \(|A_k(x)| \leq 12 \) for any \( k_j \in \{2, 3, 4, \ldots \} \).

**Proof.** \(|A_2(x)| = 2\sqrt{\pi} \) follows from the well-known formula

\[
\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \exp(-\varepsilon \xi^2 + ix\xi + it\xi^2)d\xi = \frac{2\sqrt{\pi}}{\sqrt{it}} e^{-ix^2/4t}.
\]

Moreover,

\[
A_3(x) = \frac{2\pi}{\sqrt{3}} \text{Ai}(x/\sqrt{3}),
\]

where \( \text{Ai} \) denotes the Airy function. The boundedness of \( \text{Ai} \) yields that of \( A_3 \).

Now, we will observe the boundedness of the functions \( A_k \). The proof depends on whether \( k \) is even or odd. Let \( \varepsilon > 0 \). First, suppose that \( k \) is even. Set

\[
x^* = \left( \frac{|1-x|}{k} \right)^{1/(k-1)} \text{ sgn}(1-x), \quad x_* = \left( \frac{|-1-x|}{k} \right)^{1/(k-1)} \text{ sgn}(-1-x).
\]

Note that these numbers satisfy

\[
x + k(x^*)^{k-1} = 1, \quad x + k(x_*)^{k-1} = -1,
\]

and

\[
0 \leq x^* - x_* \leq \frac{2}{k^{1/(k-1)}} \leq 2.
\]
Integrating by parts on the intervals \((-\infty, x_*]\) and \([x*, \infty)\), we have
\[
\int_{-\infty}^\infty \exp(-s^k + iks + is^k) ds = \int_{-\infty}^{x_*} \exp(-s^k + iks + is^k) ds
- i\frac{\exp(-s(x_*)^k + ixx_* + i(x_*)^k)}{i\varepsilon k(x_*)^{k-1} - 1} + i\frac{\exp(-s(x_*^*)^k + ixx_*^* + i(x_*^*)^k)}{i\varepsilon k(x_*^*)^{k-1} + 1}
- ik(k - 1)(1 + i\varepsilon) \int_{-\infty}^{x_*} \frac{s^{k-2} \exp(-s^k + iks + is^k)}{(i\varepsilon k s^{k-1} + x + ks^{k-1})^2} ds
- ik(k - 1)(1 + i\varepsilon) \int_{x_*}^\infty \frac{s^{k-2} \exp(-s^k + iks + is^k)}{(i\varepsilon k s^{k-1} + x + ks^{k-1})^2} ds.
\]
Changing the variables \(x + ks^{k-1} = \tilde{s}\) in the second and the third integrals, and using (2.2), we have
\[
\left| \int_{-\infty}^\infty \exp(-s^k + iks + is^k) ds \right| \leq 2 + \int_{x_*}^{x_*^*} ds + k(k - 1)(1 + \varepsilon) \int_{-\infty}^{x_*^*} \frac{s^{k-2}}{(x + ks^{k-1})^2} ds
+ k(k - 1)(1 + \varepsilon) \int_{x_*}^\infty \frac{s^{k-2}}{(x + ks^{k-1})^2} ds
= 2 + x_* - x_* + (1 + \varepsilon) \int_{-\infty}^{-1} \frac{d\tilde{s}}{s^2} + (1 + \varepsilon) \int_{1}^{\infty} \frac{d\tilde{s}}{s^2}
\leq 2 + x_* - x_* + 2(1 + \varepsilon)
\leq 6 + 2\varepsilon.
\]
Therefore, we obtain \(|A_k(x)| \leq 6\). This completes the proof in the case where \(k\) is even.

Second, suppose that \(k\) is odd. Set
\[
x_* = \left(\frac{1 - x}{k}\right)^{1/(k-1)} \quad \text{for } x \leq 1, \quad x_* = \left(\frac{-1 - x}{k}\right)^{1/(k-1)} \quad \text{for } x \leq -1
\]
this time. Note that these numbers satisfy
\[
x + k(\pm x^*)^{k-1} = 1, \quad x + k(\pm x_*)^{k-1} = -1,
\]
and
\[
x_* \leq \left(\frac{2}{k}\right)^{1/(k-1)} \leq 1, \quad 0 \leq x_* - x_* \leq \left(\frac{2}{k}\right)^{1/(k-1)} \leq 1.
\]
When \(x \leq -1\), integrating by parts on the intervals \((-\infty, -x_*] \), \([-x_*, x_*]\) and \([x^*, \infty)\) yields \(|A_k(x)| \leq 12\). When \(-1 \leq x \leq 0\), integrating by parts on the intervals \((-\infty, -x_*]\) and \([x^*, \infty)\) yields \(|A_k(x)| \leq 6\). When \(0 \leq x \leq 1\), integrating by parts on the intervals \((-\infty, -1]\) and \([1, \infty)\) yields \(|A_k(x)| \leq 10/3\). When \(x \geq 1\), integrating by parts on the whole interval yields \(|A_k(x)| \leq 2\). This completes the proof in the case where \(k\) is odd.

Incidentally, we state an estimate of another oscillatory integral.
Corollary 2.5.
\[ \left| \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \exp(-\varepsilon|\xi|^K + ix\xi + it|\xi|^K) d\xi \right| \leq 10|t|^{-1/K} \]
for any \( K > 1 \) and \( t \in \mathbb{R} \setminus \{0\} \).

We can verify Corollary 2.5 by similar calculus to the proof of Lemma 2.4 in the case where \( k \) is even.

Now, we prove the Fourier restriction theorem.

**Proof of Theorem 1.3.** We argue as in [3, Proof of Theorem 1, Case I]. Set
\[ R_0(\xi) = \sum_{j=1}^{n-1} |\xi_j|^{k_j}, \quad \frac{1}{q} = \sum_{j=1}^{n-1} \frac{1}{k_j} \]
for short. Using Lemma 2.2 we have
\[ \hat{G}_z(x) = (2\pi)^{-n/2} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{-\varepsilon(\xi_0'-|\xi_0-R(\xi)'|)} G_z(\xi) e^{ix\cdot\xi} d\xi \]
\[ = (2\pi)^{-n/2} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \frac{e^{-\varepsilon(\xi_0'-|\xi_0-R(\xi)'|)}(\xi_n - R(\xi)')_{+} e^{ix\cdot\xi}}{\Gamma(z+1)} d\xi \]
\[ = (2\pi)^{-n/2} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n-1}} \exp(-\varepsilon R_0(\xi') + ix' \cdot \xi') \int_{-\infty}^{\infty} \frac{e^{-\varepsilon(\xi_n-R(\xi)')} e^{ixn\xi_n}}{\Gamma(z+1)} d\xi_n d\xi' \]
\[ = i(2\pi)^{-n/2} e^{iz\pi/2} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n-1}} \exp(-\varepsilon R_0(\xi') + ix_n R(\xi') + ix' \cdot \xi') d\xi' \]
\[ = i(2\pi)^{-n/2} e^{iz\pi/2} \prod_{j=1}^{n-1} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \exp(-\varepsilon|\xi_j|^{k_j} + ix_j \xi_j + ia_j x_n^{k_j}) d\xi_j. \]

For \( z \in \mathbb{C}, \Re z \) and \( \Im z \) denote its real part and imaginary part, respectively. Therefore, using Proposition 2.3, we have
\[ |\hat{G}_z(x)| = (2\pi)^{-n/2} e^{-\Im z \pi /2} |x_n|^{-\Re z - 1} \prod_{j=1}^{n-1} \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \exp(-\varepsilon|\xi_j|^{k_j} + ix_j \xi_j + ia_j x_n^{k_j}) d\xi_j \]
\[ \leq 12^{n-1} (2\pi)^{-n/2} e^{-\Im z \pi /2} |x_n|^{-\Re z - 1 - 1/q} \prod_{j=1}^{n-1} |a_j|^{1/k_j} \]
for \( x_1, \ldots, x_n \neq 0 \). Namely, we obtain
\[ |\hat{G}_{-(1+1/q)+i\eta}(x)| \leq Ce^{-\eta \pi /2} \]
for \( x_1, \ldots, x_n \neq 0 \) and all \( \eta \in \mathbb{R} \) with \( C \) depending only on \( n \) and \( a_1, \ldots, a_{n-1} \). Now, we can apply Proposition 2.1 with \( \lambda = 1 + 1/q \). Then we obtain the desired sufficient condition on the exponent \( p \) for \([1,2]\).
In the rest of the proof, we also argue in essentially the same way as \cite[Proof of Theorem 1]{3}. We use a homogeneity argument with respect to the nonisotropic dilations 

\[ d_s \phi(x) = \phi(s^{1/k_1}x_1, \ldots, s^{1/k_{n-1}}x_{n-1}, sx_n) \]

for \( s > 0 \). On one hand,

\[
(d_s \phi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \phi(s^{1/k_1}x_1, \ldots, s^{1/k_{n-1}}x_{n-1}, sx_n) e^{-ix \cdot \xi} dx
\]

\[
= (2\pi)^{-n/2} s^{-(1+1/q)} \int_{\mathbb{R}^n} \phi(x) \times \exp(-is^{-1/k_1}x_1\xi_1 - \cdots - is^{-1/k_{n-1}}x_{n-1}\xi_{n-1} - is^{-1}x_n\xi_n) dx
\]

\[
= s^{-(1+1/q)} d_{s^{-1}} \hat{\phi}(\xi).
\]

On the other hand,

\[
\int_{\mathbb{R}^{n-1}} |d_{s^{-1}} \hat{\phi}(\xi', R(\xi'))|^2 d\xi' = s^{1/q} \int_{\mathbb{R}^{n-1}} |\hat{\phi}(\xi', R(\xi'))|^2 d\xi'.
\]

Now using

\[
\|d_s \phi\|_{L^p(\mathbb{R}^n)} = s^{-(1+1/q)/p} \|\phi\|_{L^p(\mathbb{R}^n)},
\]

and applying (1.2) for the function \( d_s \phi \), we have

\[
\left( \int_{\mathbb{R}^{n-1}} |\hat{\phi}(\xi', R(\xi'))|^2 d\xi' \right)^{1/2} = s^{-1/2q} \left( \int_{\mathbb{R}^{n-1}} |d_{s^{-1}} \hat{\phi}(\xi', R(\xi'))|^2 d\xi' \right)^{1/2}
\]

\[
= s^{1+1/2q} \left( \int_{\mathbb{R}^{n-1}} |(d_s \phi)^\wedge(\xi', R(\xi'))|^2 d\xi' \right)^{1/2}
\]

\[
\leq C_p s^{1+1/2q} \|d_s \phi\|_{L^p(\mathbb{R}^n)}
\]

\[
= C_p s^{1+1/2q - (1+1/q)/p} \|\phi\|_{L^p(\mathbb{R}^n)}.
\]

Therefore, to obtain (1.2) for any \( s > 0 \), we must have \( 1 + 1/2q - (1 + 1/q)/p = 0 \). Then the desired necessary condition on the exponent \( p \) for (1.2) follows. This completes the proof. \( \square \)

In view of Corollary 2.5 we can prove Theorem 1.3 with replacing some \( \xi_j^{k_j} \) in (1.3) by \( |\xi_j|^{K_j} \) where \( K_j > 1 \).

For the case where \( R \) is homogeneous (\( k_1 = \cdots = k_{n-1} \)), see \cite[Chapter 8, \S 5.17]{2}.

3. An application to partial differential equations

Finally, we prove Theorem 1.4

Proof of Theorem 1.4. We argue as in \cite[Proof of Corollary 1]{3}. We may assume \( \phi \in \mathcal{S}(\mathbb{R}^n) \) and \( f \in \mathcal{S}(\mathbb{R}^{1+n}) \), where \( \mathcal{S} \) denotes the Schwartz class. Set

\[
R_0(\xi) = \sum_{j=1}^n |\xi_j|^{k_j}, \quad \frac{1}{q} = \sum_{j=1}^n \frac{1}{k_j}
\]
By duality, (1.2) is equivalent to
\[ u(t, x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot \xi} e^{ita(\xi)} \hat{\phi}(\xi) d\xi \]
\[ + (2\pi)^{-n/2} \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}^n} e^{-\varepsilon R_0(\xi)} e^{ix\cdot \xi} e^{i(t-s)a(\xi)} (f(s, \cdot))^{(\xi)} d\xi ds. \]

By duality, (1.2) is equivalent to
\[ \| (Fd\mu_n) \|_{L^p/(p-1)(\mathbb{R}^n)} \leq C_{p/(p-1)} \left( \int_{\mathbb{R}^n} |F(x', \tilde{R}(\xi'))|^2 d\xi' \right)^{1/2} \text{ for } F \in L^2(d\mu_n). \]

Now, let \( S = \{(t, \xi) \in \mathbb{R}^{1+n}; t + a(\xi) = 0\} \). Replacing \( n \) by \( n + 1 \) and \( x_{n+1} \) by \( t \), we have
\[ \| \mathcal{F}_{t,x}[Fd\mu_t] \|_{L^p/(p-1)(\mathbb{R}^{1+n})} \leq C_{p/(p-1)} \left( \int_{\mathbb{R}^n} |F(-a(x), x)|^2 dx \right)^{1/2} \text{ for } F \in L^2(d\mu_t), \]
where \( \mathcal{F}_{t,x}[f] \) denotes the Fourier transform of \( f \) in \( (t, x) \in \mathbb{R}^{1+n} \), that is,
\[ \mathcal{F}_{t,x}[f](\tau, \xi) = (2\pi)^{-(1+n)/2} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} f(t, x) e^{-it\tau - ix\cdot \xi} dt dx. \]

Applying this, we have
\[ \left\| \int_{\mathbb{R}^n} e^{ix\cdot \xi} e^{ita(\xi)} \hat{\phi}(\xi) d\xi \right\|_{L^p/(p-1)(\mathbb{R}^{1+n})} = \left\| \int_{\mathbb{R}^n} e^{ix\cdot \xi} e^{-it\tau} \hat{\phi}(\xi) d\mu_t(\tau, \xi) \right\|_{L^p/(p-1)(\mathbb{R}^{1+n})} \]
\[ = (2\pi)^{(1+n)/2} \| \mathcal{F}_{t,x}[\hat{\phi}(\xi) d\mu_t](t, -x) \|_{L^p/(p-1)(\mathbb{R}^{1+n})} \]
\[ \leq C \left( \int_{\mathbb{R}^n} |\hat{\phi}(x)|^2 dx \right)^{1/2} \]
\[ = C \| \phi \|_{L^2(\mathbb{R}^n)}. \]

Next, set
\[ T_s f(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{-\varepsilon R_0(\xi)} e^{ix\cdot \xi} e^{i(t-s)a(\xi)} \hat{f}(s, \xi) d\xi \]
for \( s \in \mathbb{R} \). Here, we write \( \hat{f}(s, \xi) = (f(s, \cdot))^{(\xi)} \). It follows from the Plancherel theorem that
\[ \| T_s f \|_{L^2(\mathbb{R}^n)} = \| f(s, \cdot) \|_{L^2(\mathbb{R}^n)}. \]

Now, we have
\[ T_s f(x) \]
\[ = (2\pi)^{-n/2} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{-\varepsilon R_0(\xi)} e^{ix\cdot \xi} e^{i(t-s)a(\xi)} \int_{\mathbb{R}^n} f(s, y) e^{-iy\xi} dy d\xi \]
\[ = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(s, y) \left( \prod_{j=1}^n \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \exp(-\varepsilon |\xi|^{k_j} + i(x_j - y_j)\xi_j + i\alpha_j(t-s)\xi_j^{k_j}) d\xi_j \right) dy. \]
Using Proposition 2.3, we have
\[
|T_s f(x)| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(s, y)| \prod_{j=1}^{n} \int_{-\infty}^{\infty} \exp(i(x_j - y_j)\xi_j + ia_j(t - s)\xi_j^{k_j})d\xi_j \ dy
\]
\[
\leq 12^n (2\pi)^{-n/2} |t - s|^{-1/q} \|f(s, \cdot)\|_{L^1(\mathbb{R}^n)} \prod_{j=1}^{n} |a_j|^{-1/k_j}
\]
for \(s \neq t\). Namely,
\[
(3.2) \quad \|T_s f\|_{L^\infty(\mathbb{R}^n)} \leq C |t - s|^{-1/q} \|f(s, \cdot)\|_{L^1(\mathbb{R}^n)}.
\]
Interpolating (3.1) into (3.2), we have
\[
\|T_s f\|_{L^p/(p-1)(\mathbb{R}^n)} \leq C |t - s|^{-2(p-1)/p} \|f(s, \cdot)\|_{L^p(\mathbb{R}^n)}
\]
\[
= C |t - s|^{-2(p-1)/p} \|f(s, \cdot)\|_{L^p(\mathbb{R}^n)}.
\]
Now, using the Hardy-Littlewood-Sobolev inequality (see [2] page 354 for instance), we have
\[
\left\| \int_0^t \int_{\mathbb{R}^n} e^{i\xi \cdot \xi} e^{i(t-s)\alpha(\xi)} \hat{f}(s, \xi) d\xi ds \right\|_{L^p/(p-1)(\mathbb{R}^{1+n})}
\]
\[
= \left\| \int_0^t T_s f(x) ds \right\|_{L^p/(p-1)(\mathbb{R}^{1+n})}
\]
\[
\leq \left\| \int_0^t \|T_s f\|_{L^p/(p-1)(\mathbb{R}^n)} ds \right\|_{L^p/(p-1)(\mathbb{R}^n)}
\]
\[
\leq C \left\| \int_{-\infty}^{\infty} \|T_s f\|_{L^p/(p-1)(\mathbb{R}^n)} ds \right\|_{L^p/(p-1)(\mathbb{R}^n)}
\]
\[
\leq C \left\| f(t, \cdot) \right\|_{L^p(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^{1+n})}
\]
Thus, we obtain
\[
\|u\|_{L^p/(p-1)(\mathbb{R}^{1+n})} \leq C \left( \|\phi\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^{1+n})} \right).
\]
This completes the proof.

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