Approximate Controllability of Second-Order Evolution Differential Inclusions in Hilbert Spaces

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Abstract. In this paper, we consider a class of second-order evolution differential inclusions in Hilbert spaces. This paper deals with the approximate controllability for a class of second-order control systems. First, we establish a set of sufficient conditions for the approximate controllability for a class of second-order evolution differential inclusions in Hilbert spaces. We use Bohnenblust–Karlin’s fixed point theorem to prove our main results. Further, we extend the result to study the approximate controllability concept with nonlocal conditions and also extend the result to study the approximate controllability for impulsive control systems with nonlocal conditions. An example is also given to illustrate our main results.

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1. Introduction

Controllability is one of the elementary concepts in mathematical control theory. This is a qualitative property of dynamical control systems and is of particular importance in control theory. Roughly speaking, controllability generally means that it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Most of the criteria, which can be met in the literature, are formulated for finite dimensional systems. It should be pointed out that many unsolved problems still exist as far as controllability of infinite dimensional systems are concerned. In the case of infinite dimensional systems, two basic concepts of controllability can be distinguished which are exact and approximate controllability. This is strongly related to the fact that in infinite dimensional spaces, there exist linear subspaces, which are not closed. Exact
controllability enables to steer the system to an arbitrary final state, while approximate controllability means that the system can be steered to an arbitrary small neighborhood of the final state. In other words, approximate controllability gives the possibility of steering the system to states which form a dense subspace in the state space.

Recently, in [38], Mahmudov et al. studied the approximate controllability of second-order neutral stochastic evolution equations using semigroup methods, together with the Banach fixed point theorem. In [46], Sakthivel et al. studied the approximate controllability of second-order systems with state-dependent delay using Schauder’s fixed point theorem. In [21], Henríquez studied the existence of solutions of non-autonomous second-order functional differential equations with infinite delay using Leray–Schauder’s alternative fixed point theorem. In [60], Yan studied the approximate controllability of fractional neutral integro-differential inclusions with state-dependent delay in Hilbert spaces using Dhage’s fixed point theorem. In [57], Vijayakumar et al. discussed the approximate controllability for a class of fractional neutral integro-differential inclusions with state-dependent delay using Dhage’s fixed point theorem. In [44], Sakthivel et al. studied the approximate controllability of fractional nonlinear differential inclusions with initial and nonlocal conditions using Bohnenblust–Karlin’s fixed point theorem. Very recently, in [4], Arthi et al. established the sufficient conditions for controllability of second-order impulsive evolution systems with infinite delay using Leray–Schauder’s fixed point theorem and in [5] proved the existence and controllability results for second-order impulsive stochastic evolution systems with state-dependent delay using Leray–Schauder’s fixed point theorem. In [20], Guendouzi investigated the approximate controllability for a class of fractional neutral stochastic functional integro-differential inclusions using Bohnenblust–Karlin’s fixed point theorem.

Converting a second-order system into a first-order system and studying its controllability may not yield the desired results due to the behavior of the semigroup generated by the linear part of the converted first-order system. So, in many cases, it is advantageous to treat the second-order abstract differential system directly rather than to convert that into a first-order system. In the past decades, many papers have been published about the controllability of nonlinear systems, in which the authors effectively used the fixed point technique. In recent years, controllability problems for various types of nonlinear dynamical systems in infinite dimensional spaces by using different kinds of approaches have been considered in many publications; see [1, 4, 5, 13, 20, 24, 31, 33–38, 42, 44–48, 56–61] and the references therein. The literature related to approximate controllability of second-order nonlocal abstract Cauchy problems with impulses remains limited. Moreover, to the authors’ knowledge, no results are available for the approximate controllability for a class of second-order evolution differential systems in Hilbert spaces. This fact is the main motivation of our work.

Motivated by the above consideration, in this paper, we establish sufficient conditions for the approximate controllability for a class of second-order evolution differential inclusions in Hilbert spaces of the form
\[ x''(t) \in A(t)x(t) + F(t, x(t)) + Bu(t), \quad t \in I = [0, b], \]  
\[ x(0) = x_0 \in X, \quad x'(0) = y_0 \in X \]  
In this equation, \( A(t) : D(A(t)) \subseteq X \rightarrow X \) is a closed linear operator on a Hilbert space \( X \) and the control function \( u(\cdot) \in L^2(I, U) \), a Hilbert space of admissible control functions. Further, \( B \) is a bounded linear operator from \( U \) to \( X \), and \( F : I \times X \rightarrow 2^X \setminus \{\emptyset\} \) is a nonempty, bounded, closed and convex multivalued map.

The fixed point method can be adjudged as a very powerful and important tool to study the controllability of nonlinear systems. The fixed point approach seems appropriate for the solution of many problems in control theory, because it is constructive and provides an associated convergence theory. The fixed point approach which has found wide applications in both the theory and numerical aspect of differential equations has not yet been extensively applied to the field of stochastic impulsive control system. In this method, the controllability problem is transferred into a fixed point problem for an appropriate nonlinear operator in a function space. An essential part of this method is to guarantee the existence of fixed point for the appropriate operator. The fixed point method is the most effective one to study the existence and controllability of differential systems and fractional differential systems. Due to its importance, several researchers have studied the problems represented by evolution equations using different kinds of fixed point theorems [19,39]. We mainly employ the Bohnenblust–Karlin’s fixed point theorem to investigate the approximate controllability for a class of second-order evolution differential inclusions.

This paper is organized as follows. In Sect. 3, we establish a set of sufficient conditions for the approximate controllability for a class of second-order evolution differential inclusions in Hilbert spaces. In Sect. 4, we establish a set of sufficient conditions for the approximate controllability for a class of second-order evolution differential inclusions with nonlocal conditions in Hilbert spaces. In Sect. 5, we establish a set of sufficient conditions for the approximate controllability for a class of second-order impulsive evolution differential inclusions with nonlocal conditions in Hilbert spaces. An example is presented in Sect. 6 to illustrate the theory of the obtained results.

### 2. Preliminaries

In this section, we mention a few results, notations and lemmas needed to establish our main results. We introduce certain notations which will be used throughout the article without any further mention. Let \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) be Banach spaces, and \( \mathcal{L}(Y, X) \) be the Banach space of bounded linear operators from \( Y \) into \( X \) equipped with its natural topology; in particular, we use the notation \( \mathcal{L}(X) \) when \( Y = X \). By \( \rho(A) \), we denote the resolvent set of a linear operator \( A \). Throughout this paper, \( B_r(x, X) \) will denote the closed ball with center at \( x \) and radius \( r > 0 \) in a Hilbert space \( X \). We denote by \( C \), the Hilbert space \( C(J, X) \) endowed with supnorm given by \( \|x\|_C \equiv \sup_{t \in J} \|x(t)\| \), for \( x \in C \).
In recent times, there has been an increasing interest in studying the abstract non-autonomous second-order initial value problem

\[ \begin{align*}
    x''(t) &= A(t)x(t) + f(t), \quad 0 \leq s, t \leq b, \\
    x(s) &= x_0, \quad x'(s) = y_0,
\end{align*} \tag{3} \]

where \( A(t) : D(A(t)) \subseteq X \to X, t \in I = [0, b] \) is a closed densely defined operator and \( f : I \to X \) is an appropriate function. Equations of this type have been considered in many papers. The reader is referred to \([7,16,30,40,41]\) and the references mentioned in these works. In most of the works, the existence of solutions to the problem \((3)\) and \((4)\) is related to the existence of an evolution operator \( S(t, s) \) for the homogeneous equation

\[ x''(t) = A(t)x(t), \quad 0 \leq s, t \leq b. \tag{5} \]

Let us assume that the domain of \( A(t) \) is a subspace \( D \) dense in \( X \) and independent of \( t \), and for each \( x \in D \) the function \( t \mapsto A(t)x \) is continuous.

Following Kozak \([26]\), in this work we will use the following concept of evolution operator.

**Definition 2.1.** A family \( S \) of bounded linear operators \( S(t, s) : I \times I \to \mathcal{L}(X) \) is called an evolution operator for \((5)\) if the following conditions are satisfied:

- (Z1) For each \( x \in X \), the mapping \([0, b] \times [0, b] \ni (t, s) \mapsto S(t, s)x \in X \) is of class \( C^1 \) and
  
  \( \begin{align*}
    (i) & \quad \text{for each } t \in [0, b], \; S(t, t) = 0, \\
    (ii) & \quad \text{for all } t, s \in [0, b], \text{ and for each } x \in X, \\
    & \quad \frac{\partial}{\partial t} S(t, s)x \bigg|_{t=s} = x, \quad \frac{\partial}{\partial t} S(t, s)x \bigg|_{t=s} = -x.
  \end{align*} \)

- (Z2) For all \( t, s \in [0, b] \), if \( x \in D(A) \), then \( S(t, s)x \in D(A) \), the mapping \([0, b] \times [0, b] \ni (t, s) \mapsto S(t, s)x \in X \) is of class \( C^2 \) and
  
  \( \begin{align*}
    (i) & \quad \frac{\partial}{\partial t^2} S(t, s)x = A(t)S(t, s)x, \\
    (ii) & \quad \frac{\partial}{\partial s} S(t, s)x = S(t, s)A(s)x, \\
    (iii) & \quad \frac{\partial}{\partial s} \frac{\partial}{\partial t} S(t, s)x \bigg|_{t=s} = 0.
  \end{align*} \)

- (Z3) For all \( t, s \in [0, b] \), if \( x \in D(A) \), then \( \frac{\partial}{\partial s} S(t, s)x \in D(A) \), as well as \( \frac{\partial}{\partial t} S(t, s)x, \quad \frac{\partial^2}{\partial s^2} S(t, s)x \) and\( \frac{\partial}{\partial t} \frac{\partial}{\partial s} S(t, s)x \) and
  
  \( \begin{align*}
    (i) & \quad \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial s} S(t, s)x = A(t)\frac{\partial}{\partial s} S(t, s)x, \\
    (ii) & \quad \frac{\partial^2}{\partial s^2} \frac{\partial}{\partial t} S(t, s)x = \frac{\partial}{\partial t} S(t, s)A(s)x, \\
    \end{align*} \)

and the mapping \([0, b] \times [0, b] \ni (t, s) \mapsto A(t)\frac{\partial}{\partial s} S(t, s)x \) is continuous.

Throughout this work, we assume that there exists an evolution operator \( S(t, s) \) associated with the operator \( A(t) \). To abbreviate the text, we introduce the operator \( C(t, s) = -\frac{\partial S(t, s)}{\partial s} \). In addition, we set \( N \) and \( \tilde{N} \) for positive constants such that \( \sup_{0 \leq t, s \leq b} ||S(t, s)|| \leq N \) and \( \sup_{0 \leq t, s \leq b} ||C(t, s)|| \leq \tilde{N} \). Furthermore, we denote by \( N_1 \) a positive constant such that

\[ ||S(t + h, s) - S(t, s)|| \leq N_1|h|, \]
for all \( s, t, t + h \in [0, b] \). Assuming that \( f : I \to X \) is an integrable function, the mild solution \( x : [0, b] \to X \) of the problem (3) and (4) is given by

\[
x(t) = C(t, s)x_0 + S(t, s)y_0 + \int_0^t S(t, \tau)h(\tau)d\tau.
\]

In the literature, several techniques have been discussed to establish the existence of the evolution operator \( S(\cdot, \cdot) \). In particular, a very studied situation is that \( A(t) \) is the perturbation of an operator \( A \) that generates a cosine operator function. For this reason, below we briefly review some essential properties of the theory of cosine functions. Let \( A : D(A) \subseteq X \to X \) be the infinitesimal generator of a strongly continuous cosine family of bounded linear operators \( (C(t))_{t \in \mathbb{R}} \) on Hilbert space \( X \). We denote by \( (S(t))_{t \in \mathbb{R}} \) the sine function associated with \( (C(t))_{t \in \mathbb{R}} \) which is defined by

\[
S(t)x = \int_0^t C(s)xds, \quad x \in X, \quad t \in \mathbb{R}.
\]

We refer the reader to [17, 52, 53] for the necessary concepts about cosine functions. Next, we only mention a few results and notations about this matter needed to establish our results. It immediately follows that

\[
C(t)x - x = A \int_0^t S(s)xds,
\]

for all \( X \). The notation \([D(A)]\) stands for the domain of the operator \( A \) endowed with the graph norm \( \|x\|_A = \|x\| + \|Ax\|, x \in D(A) \). Moreover, in this paper the notation \( E \) stands for the space formed by the vectors \( x \in X \) for which the function \( C(\cdot)x \) is a class \( C^1 \) on \( \mathbb{R} \). It was proved by Kisyński [25] that the space \( E \) endowed with the norm

\[
\|x\|_E = \|x\| + \sup_{0 \leq t \leq 1} \|AS(t, 0)x\|, \quad x \in E,
\]

is a Hilbert space. The operator-valued function

\[
G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}
\]

is a strongly continuous group of linear operators on the space \( E \times X \) generated by the operator \( \mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} \), defined on \( D(A) \times E \). It follows from this that \( AS(t) : E \to X \) is a bounded linear operator such that \( AS(t)x \to 0 \) as \( t \to 0 \), for each \( x \in E \). Furthermore, if \( x : [0, \infty) \to X \) is a locally integrable function, then \( z(t) = \int_0^t S(t, s)x(s)ds \) defines an \( E \)-valued continuous function.

The existence of solutions for the second-order abstract Cauchy problem

\[
x''(t) = Ax(t) + h(t), \quad 0 \leq t \leq b, \\
x(0) = x_0, \quad x'(0) = y_0,
\]

where \( h : [0, b] \to X \) is an integrable function, has been discussed in [52]. Similarly, the existence of solutions of the semilinear second-order Cauchy
problem has been treated in [53]. We only mention here that the function
\( x(\cdot) \) given by
\[
x(t) = C(t-s)x_0 + S(t-s)y_0 + \int_s^t S(t-\tau)h(\tau)d\tau, \quad 0 \leq t \leq b,
\]  
(8)
is called the mild solution of \( (6) \) and \( (7) \) and that when \( x_0 \in E \), \( x(\cdot) \) is
continuously differentiable and
\[
x'(t) = AS(t-s)x_0 + C(t-s)y_0 + \int_s^t C(t-\tau)h(\tau)d\tau, \quad 0 \leq t \leq b.
\]
In addition, if \( x_0 \in D(A) \), \( y_0 \in E \) and \( f \) is a continuously differentiable
function, then the function \( x(\cdot) \) is a solution of the initial value problem \( (6) \) and \( (7) \).

Assume now that \( A(t) = A + \tilde{B}(t) \) where \( \tilde{B}(\cdot) : \mathbb{R} \to L(E, X) \) is a
map, such that the function \( t \to \tilde{B}(t)x \) is continuously differentiable in \( X \) for
each \( x \in E \). It has been established by Serizawa [49] that for each \( (x_0, y_0) \in D(A) \times E \)
the nonautonomous abstract Cauchy problem
\[
x''(t) = (A + \tilde{B}(t))x(t), \quad t \in \mathbb{R},
\]
\[
x(0) = x_0, \quad x'(0) = y_0,
\]
(9) (10)
has a unique solution \( x(\cdot) \), such that the function \( t \mapsto x(t) \) is continuously
differentiable in \( E \). It is clear that the same argument allows us to conclude that
equation \( (9) \) with the initial condition \( (7) \) has a unique solution \( x(\cdot, s) \),
such that the function \( t \mapsto x(t, s) \) is continuously differentiable in \( E \). It follows
from \( (8) \) that
\[
x(t, s) = C(t-s)x_0 + S(t-s)y_0 + \int_s^t S(t-\tau)\tilde{B}(\tau)x(\tau, s)d\tau.
\]
In particular, for \( x_0 = 0 \) we have
\[
x(t, s) = S(t-s)y_0 + \int_s^t S(t-\tau)\tilde{B}(\tau)x(\tau, s)d\tau.
\]
Consequently,
\[
\|x(t, s)\|_1 \leq \|S(t-s)\|_{L(X,E)}\|y_0\|
+ \int_s^t \|S(t-s)\|_{L(X,E)}\|\tilde{B}(\tau)\|_{L(X,E)}\|x(s, \tau)\|_1d\tau
\]
and, applying the Gronwall–Bellman lemma, we infer that
\[
\|x(t, s)\|_1 \leq \tilde{M}\|y_0\|, \quad s, t \in I.
\]
We define the operator \( S(t,s)y_0 = x(t,s) \). It follows from the previous estimate
that \( S(t,s) \) is a bounded linear map on \( E \). Since \( E \) is dense in \( X \), we
can extend \( S(t,s) \) to \( X \). We keep the notation \( S(t,s) \) for this extension. It is
well known that, except in the case \( \dim(X) < \infty \), the cosine function \( C(t) \)
cannot be compact for all \( t \in \mathbb{R} \). In contrast, for the cosine functions that
arise in specific applications, the sine function \( S(t) \) is very often a compact
operator for all \( t \in \mathbb{R} \).
Theorem 2.2 [21, Theorem 1.2]. Under the preceding conditions, \( S(\cdot, \cdot) \) is an evolution operator for (9) and (10). Moreover, if \( S_0(t) \) is compact for all \( t \in \mathbb{R} \), then \( S(t, s) \) is also compact for all \( s \leq t \).

We also introduce some basic definitions and results of multivalued maps. For more details on multivalued maps, see the books of Deimling [14] and Hu and Papageorgious [50].

A multivalued map \( G : X \to 2^X \setminus \{\emptyset\} \) is convex (closed) valued if \( G(x) \) is convex (closed) for all \( x \in X \). \( G \) is bounded on bounded sets if \( G(C) = \bigcup_{x \in C} G(x) \) is bounded in \( X \) for any bounded set \( C \) of \( X \), i.e.,

\[
\sup_{x \in C} \left\{ \sup \{\|y\| : y \in G(x)\} \right\} < \infty.
\]

Definition 2.3. \( G \) is called upper semicontinuous (u.s.c. for short) on \( X \) if for each \( x_0 \in X \), the set \( G(x_0) \) is a nonempty closed subset of \( X \), and if for each open set \( C \) of \( X \) containing \( G(x_0) \), there exists an open neighborhood \( V \) of \( x_0 \) such that \( G(V) \subseteq C \).

Definition 2.4. \( G \) is called completely continuous if \( G(C) \) is relatively compact for every bounded subset \( C \) of \( X \).

If the multivalued map \( G \) is completely continuous with nonempty values, then \( G \) is u.s.c., if and only if \( G \) has a closed graph, i.e., \( x_n \to x^* \), \( y_n \to y^* \), and \( y_n \in Gx_n \) imply \( y^* \in Gx^* \). \( G \) has a fixed point if there is a \( x \in X \) such that \( x \in G(x) \).

Definition 2.5. A function \( x \in C \) is said to be a mild solution of system (1)–(2) if \( x(0) = x_0 \), \( x'(0) = y_0 \) and there exists \( f \in L^1(I, X) \), such that \( f(t) \in F(t, x(t)) \) on \( t \in I \) and the integral equation

\[
x(t) = C(t, 0)x_0 + S(t, 0)y_0 + \int_0^t S(t - s)f(s)ds + \int_0^t S(t - s)Bu(s)ds, \quad t \in I
\]

is satisfied.

To address the problem, it is convenient at this point to introduce two relevant operators and basic assumptions on these operators:

\[
\Upsilon^b_0 = \int_0^b S(b - s)BB^*S^*(b - s)ds : X \to X,
\]

\[
R(a, \Upsilon^b_0) = (aI + \Upsilon^b_0)^{-1} : X \to X,
\]

where \( B^* \) denotes the adjoint of \( B \), \( S^*(t) \) is the adjoint of \( S(t) \). It is straightforward that the operator \( \Upsilon^b_0 \) is a linear bounded operator.

To investigate the approximate controllability of the system (1) and (2), we impose the following condition:

\( H_0 \) \( aR(a, \Upsilon^b_0) \to 0 \) as \( a \to 0^+ \) in the strong operator topology.

In view of [33], Hypothesis \( H_0 \) holds if and only if the linear system

\[
x''(t) = Ax(t) + (Bu)(t), \quad t \in [0, b], \tag{11}
\]

\[
x(0) = x_0 \quad x'(0) = y_0, \tag{12}
\]

is approximately controllable on \([0, b]\).
Some of our results are proved using the next well-known results.

**Lemma 2.6** [28, Lasota and Opial]. Let \( I \) be a compact real interval, \( \text{BCC}(X) \) be the set of all nonempty, bounded, closed and convex subset of \( X \), and \( F \) be a multivalued map satisfying \( F : I \times X \to \text{BCC}(X) \) measurable to \( t \) for each fixed \( x \in X \), u.s.c. to \( x \) for each \( t \in I \), and for each \( x \in C \) the set

\[
S_{F,x} = \{ f \in L^1(I, X) : f(t) \in F(t, x(t)), \ t \in I \}
\]

is nonempty. Let \( I \) be a compact real interval, \( \text{BCC}(X) \) be the set of all nonempty, bounded, closed and convex subset of \( X \), and \( F \) be a multivalued map satisfying \( F : I \times X \to \text{BCC}(X) \) measurable to \( t \) for each fixed \( x \in X \), u.s.c. to \( x \) for each \( t \in I \), and for each \( x \in C \) the set

\[
S_{F,x} = \{ f \in L^1(I, X) : f(t) \in F(t, x(t)), \ t \in I \}
\]

is nonempty. Let \( \mathcal{F} \) be linear continuous from \( L^1(I, X) \) to \( \text{C}(\text{BCC}(C)) \), then the operator

\[
\mathcal{F} \circ S_F : C \to \text{BCC}(C), \ x \to (\mathcal{F} \circ S_F)(x) = \mathcal{F}(S_{F,x}),
\]

is a closed graph operator in \( C \times C \).

**Lemma 2.7** [9, Bohnenblust and Karlin]. Let \( D \) be a nonempty subset of \( X \), which is bounded, closed, and convex. Suppose \( G : D \to 2^X \setminus \{\emptyset\} \) is u.s.c. with closed, convex values, such that \( G(D) \subseteq D \) and \( G(D) \) are compact. Then \( G \) has a fixed point.

### 3. Approximate Controllability Results

In this section, first we establish a set of sufficient conditions for the approximate controllability for a class of second-order evolution differential inclusions of the form (1)–(2) in Hilbert spaces using Bohnenblust–Karlin’s fixed point theorem. To establish the result, we need the following hypotheses:

**H1** \( S_0(t), t > 0 \) is compact.

**H2** For each positive number \( r \) and \( x \in C \) with \( \|x\|_C \leq r \), there exists \( L_{f,r}(\cdot) \in L^1(I, \mathbb{R}^+) \) such that

\[
\sup \{ \|f\| : f(t) \in F(t, x(t)) \} \leq L_{f,r}(t),
\]

for a.e. \( t \in I \).

**H3** The function \( s \to L_{f,r}(s) \in L^1([0, t], \mathbb{R}^+) \) and there exists a \( \gamma > 0 \) such that

\[
\lim_{r \to \infty} \frac{\int_0^t L_{f,r}(s) ds}{r} = \gamma < +\infty.
\]

It will be shown that the system (1)–(2) is approximately controllable, if for all \( a > 0 \), there exists a continuous function \( x(\cdot) \) such that

\[
x(t) = C(t, 0)x_0 + S(t, 0)y_0 + \int_0^t S(t, s)f(s) ds
\]

\[
+ \int_0^t S(t, s)Bu(s, x) ds, \quad f \in S_{F,x},
\]

\[
u(t, x) = B^*S(b, t)R(a, \Upsilon^b_0)p(x(\cdot)),
\]

where

\[
p(x(\cdot)) = x_b - C(b, 0)x_0 - S(b, 0)y_0 - \int_0^t S(b, s)f(s) ds.
\]
Theorem 3.1. Suppose that the hypotheses $H_0 - H_3$ are satisfied. Assume also

$$\tilde{N} \gamma \left[ 1 + \frac{1}{\alpha} \tilde{N}^2 M_B^2 b \right] < 1,$$

where $M_B = \|B\|$. Then the system (1) and (2) has a solution in $I$.

Proof. The main aim in this section is to find conditions for solvability of system (1) and (2) for $a > 0$. We show that, using the control $u(x, t)$, the operator $\Gamma : C \to 2^C$, defined by

$$\Gamma(x) = \left\{ \varphi \in C : \varphi(t) = C(t, 0)x_0 + S(t, 0)y_0 
+ \int_0^t S(t, s)[f(s) + Bu(s, x)]ds, \ f \in S_{F,x} \right\},$$

has a fixed point $x$, which is a mild solution of system (1) and (2). We observe that $x_b \in (\Gamma x)(b)$, which means that $u(t, x)$ steers system (1) and (2) from $x_0$ to $x_b$ in finite time $b$.

We now show that $\Gamma$ satisfies all the conditions of Lemma 2.7. For the sake of convenience, we subdivide the proof into five steps.

Step 1 $\Gamma$ is convex for each $x \in C$.

In fact, if $\varphi_1, \varphi_2$ belong to $\Gamma(x)$, then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in I$, we have

$$\varphi_i(t) = C(t, 0)x_0 + S(t, 0)y_0 + \int_0^t S(t, s)b_i(s)ds
+ \int_0^t S(t, s)BB^*S^*(b, t)R(a, \Upsilon_b) \left[ x_b - C(b, 0)x_0 
- S(b, 0)y_0 - \int_0^b S(b, \eta)f_i(\eta)d\eta \right] (s)ds, \ i = 1, 2.$$

Let $\lambda \in [0, 1]$. Then for each $t \in I$, we get

$$\lambda \varphi_1(t) + (1 - \lambda) \varphi_2(t) = C(t, 0)x_0 + S(t, 0)y_0 + \int_0^t S(t, s)[\lambda f_1(s) + (1 - \lambda) f_2(s)]ds
+ \int_0^t S(t, s)BB^*S^*(b, t)R(a, \Upsilon_b) \left[ x_b - C(b, 0)x_0 - S(b, 0)y_0 
- \int_0^b S(b, \eta)[\lambda f_1(\eta) + (1 - \lambda) f_2(\eta)]d\eta \right] (s)ds.$$

It is easy to see that $S_{F,x}$ is convex, since $F$ has convex values. So, $\lambda f_1 + (1 - \lambda) f_2 \in S_{F,x}$. Thus,

$$\lambda \varphi_1 + (1 - \lambda) \varphi_2 \in \Gamma(x).$$

Step 2 For each positive number $r > 0$, let $\mathcal{B}_r = \{ x \in C : \|x\| \leq r \}$. Obviously, $\mathcal{B}_r$ is a bounded, closed and convex set of $C$. We claim that there exists a positive number $r$ such that $\Gamma(\mathcal{B}_r) \subseteq \mathcal{B}_r$. 
If this is not true, then for each positive number \( r \), there exists a function \( x^r \in \mathfrak{B}_r \), but \( \Gamma(x^r) \) does not belong to \( \mathfrak{B}_r \), i.e.,
\[
\|\Gamma(x^r)\|_C \equiv \sup \left\{ \|\varphi^r\|_C : \varphi^r \in (\Gamma x^r) \right\} > r
\]
and
\[
\varphi^r(t) = C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)f^r(s)ds + \int_0^t S(t,s)Bu^r(s,x)ds,
\]
for some \( f^r \in S_{F,x^r} \). Using \( \text{H}_1 - \text{H}_3 \), we have
\[
r < \|\Gamma(x^r)(t)\|
\]
\[
\leq \|C(t,0)x_0\| + \|S(t,0)y_0\| + \int_0^t \|S(t,s)f^r(s)\|ds + \int_0^t \|S(t,s)Bu^r(s,x)\|ds
\]
\[
\leq \left[ N\|x_0\| + \tilde{N}\|y_0\| + \tilde{N} \int_0^t L_f,r(s)ds \right]
\]
\[
+ \frac{1}{\alpha}\tilde{N}^2 M_B^2 b \left[ \|x_b\| + N\|x_0\| + \tilde{N}\|y_0\| + \tilde{N} \int_0^b L_f,r(s)ds \right].
\]
Dividing both sides of the above inequality by \( r \) and taking the limit as \( r \to \infty \), using \( \text{H}_3 \), we get
\[
\tilde{N}\gamma \left[ 1 + \frac{1}{\alpha}\tilde{N}^2 M_B^2 b \right] \geq 1.
\]
This contradicts with the condition (15). Hence, for some \( r > 0 \), \( \Gamma(\mathfrak{B}_r) \subseteq \mathfrak{B}_r \).

Step 3 \( \Gamma \) sends bounded sets into equicontinuous sets of \( C \). For each \( x \in \mathfrak{B}_r \), \( \varphi \in \Gamma(x) \), there exists a \( f \in S_{F,x} \) such that
\[
\varphi(t) = C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)f(s)ds + \int_0^t S(t,s)Bu(s,x)ds.
\]
Let \( 0 < \varepsilon < 0 \) and \( 0 < t_1 < t_2 \leq b \), then
\[
|\varphi(t_1) - \varphi(t_2)| = |C(t_1,0) - C(t_2,0)||x_0| + |S(t_1,0) - S(t_2,0)||\eta|
\]
\[
+ \left| \int_0^{t_1-\varepsilon} [S(t_1,s) - S(t_2,s)]f(s)ds \right| + \left| \int_{t_1-\varepsilon}^{t_1} [S(t_1,s) - S(t_2,s)]f(s)ds \right|
\]
\[
+ \left| \int_0^{t_1} S(t_1,s)f(s)ds \right| + \left| \int_{t_1-\varepsilon}^{t_1} [S(t_1,\eta) - S(t_2,\eta)]Bu(\eta,x)d\eta \right|
\]
\[
+ \left| \int_0^{t_1} [S(t_1,\eta) - S(t_2,\eta)]Bu(\eta,\eta) d\eta \right| + \left| \int_{t_1}^{t_2} S(t_2,\eta)Bu(\eta,x)d\eta \right|
\]
\[
+ \left| \int_0^{t_1} [S(t_1,\eta) - S(t_2,\eta)]Bu(\eta,\eta)d\eta \right|
\]
\[
\leq |C(t_1,0) - C(t_2,0)||x_0| + |S(t_1,0) - S(t_2,0)||\eta|
\]
\[
+ \int_0^{t_1-\varepsilon} |S(t_1,s) - S(t_2,s)|L_{f,r}(s)ds
\]
\[
+ \int_{t_1}^{t_2} |S(t_1,s) - S(t_2,s)|L_{f,r}(s)ds
\]
\[
+ \tilde{N} \int_0^{t_1} L_{f,r}(s)ds + \tilde{N} \int_{t_1}^{t_2} L_{f,r}(s)ds
\]
\[ + M_B \int_0^{t_1 - \varepsilon} |S(t_1, \eta) - S(t_2, \eta)||u(\eta, x)|d\eta \]
\[ + M_B \int_{t_1 - \varepsilon}^{t_1} |S(t_1, \eta) - S(t_2, \eta)||u(\eta, x)|d\eta + \tilde{N}M_B \int_{t_1}^{t_2} \|u(\eta, x)\|d\eta. \]

The right-hand side of the above inequality tends to zero independently of \(x \in B_r\) as \((t_1 - t_2) \to 0\) and \(\varepsilon\) are sufficiently small, since the compactness of the evolution operator \(S(t, s)\) implies the continuity in the uniform operator topology. Thus, \(\Gamma(x^r)\) sends \(B_r\) into an equicontinuous family of functions.

**Step 4** The set \(\Pi(t) = \{\varphi(t) : \varphi \in \Gamma(B_r^r)\}\) is relatively compact in \(X\).

Let \(t \in (0, b]\) be fixed and \(\varepsilon\) a real number satisfying \(0 < \varepsilon < t\). For \(x \in B_r\), we define

\[ \varphi_\varepsilon(t) = C(t, 0)x_0 + S(t, 0)y_0 + \int_0^{t - \varepsilon} S(t, s)f(s)ds + \int_0^{t - \varepsilon} S(t, \eta)Bu(\eta, x)d\eta. \]

Since \(S(t)\) is a compact operator, the set \(\Pi_\varepsilon(t) = \{\varphi_\varepsilon(t) : \varphi_\varepsilon \in \Gamma(B_r)\}\) is relatively compact in \(X\) for each \(\varepsilon, 0 < \varepsilon < t\). Moreover, for each \(0 < \varepsilon < t\), we have

\[ |\varphi(t) - \varphi_\varepsilon(t)| \leq \tilde{N}_M \int_{t - \varepsilon}^{t} L_{f, r}(s)ds + \tilde{N}M_B \int_{t - \varepsilon}^{t} \|u(\eta, x)\|d\eta. \]

Hence, there exist relatively compact sets arbitrarily close to the set \(\Pi(t) = \{\varphi(t) : \varphi \in \Gamma(B_r)\}\), and the set \(\Pi(t)\) is relatively compact in \(X\) for all \(t \in [0, b]\). Since it is compact at \(t = 0\), \(\Pi(t)\) is relatively compact in \(X\) for all \(t \in [0, b]\).

**Step 5** \(\Gamma\) has a closed graph.

Let \(x_n \to x_*\) as \(n \to \infty\), \(\varphi_n \in \Gamma(x_n)\), and \(\varphi_n \to \varphi_*\) as \(n \to \infty\). We will show that \(\varphi_* \in \Gamma(x_*)\). Since \(\varphi_n \in \Gamma(x_n)\), there exists a \(f_n \in S_{F, x_n}\) such that

\[ \varphi_n(t) = C(t, 0)x_0 + S(t, 0)y_0 + \int_0^t S(t, s)f_n(s)ds + \int_0^t S(t, \eta)BB^*S^*(b, t)R(a, \Upsilon_0^b) \]
\[ \times \left[ x_b - S(b, 0)x_0 - \int_0^b S(b, \eta)f_n(\eta)d\eta \right] (s)ds. \]

We must prove that there exists a \(f_* \in S_{F, x_*}\) such that

\[ \varphi_*(t) = C(t, 0)x_0 + S(t, 0)y_0 + \int_0^t S(t, s)f_*(s)ds + \int_0^t S(t, s)BB^*S^*(b, t) \]
\[ \times \left[ y_0 - S(b, 0)x_0 - \int_0^b S(b, \eta)f_*(\eta)ds \right] (s)ds. \]

Set

\[ \overline{u}_{x_n}(t) = B^*S^*(b, t)[x_b - C(b, 0)x_0 - S(b, 0)y_0](t). \]

Then,

\[ \overline{u}_{x_n}(t) \to \overline{u}_{x_*}(t), \quad \text{for } t \in I, \text{ as } n \to \infty. \]

Clearly, we have

\[ \left\| \varphi_n - C(t, 0)x_0 + S(t, 0)y_0 - \int_0^t S(t, s)BB^*S^*(b, t)R(a, \Upsilon_0^b) [x_b - S(b, 0)x_0 \]
\[ - \int_0^t S(t, s)f_n(s)ds - \int_0^t S(t, \eta)BB^*S^*(b, t)R(a, \Upsilon_0^b) \right\|. \]
Consider the operator 
\[
\begin{aligned}
\tilde{F} : L^1(I, X) \rightarrow C,
\end{aligned}
\]
\[
(\tilde{F}f)(t) = \int_0^t S(t, s) \left[ f(s) - BB^* S^*(b, t) \left( \int_0^b S(b, \eta) f(\eta) d\eta \right)(s) \right] ds.
\]

We can see that the operator \( \tilde{F} \) is linear and continuous. From Lemma 2.7 again, it follows that \( \tilde{F} \circ S_F \) is a closed graph operator. Moreover,
\[
\begin{aligned}
\varphi_n - C(t, 0)x_0 - S(t, 0)y_0 - \int_0^t S(t, s)BB^* S^*(b, t) [ x_b - S(b, 0)x_0 ] - \int_0^b S(b, \eta) f_n(\eta) d\eta(s) \in F(S_{F,x_n}).
\end{aligned}
\]

In view of \( x_n \rightarrow x_* \) as \( n \rightarrow \infty \), it follows again from Lemma 2.7 that
\[
\begin{aligned}
\varphi_n - C(t, 0)x_0 + S(t, 0)y_0 - \int_0^t S(t, s)BB^* S^*(b, t) R(a, \Upsilon_0^b)[ x_b - S(b, 0)x_0 ] - \int_0^b S(b, \eta) f_n(\eta) d\eta(s) \in F(S_{F,x_*}).
\end{aligned}
\]

Therefore, \( \Gamma \) has a closed graph.

As a consequence of Steps 1–5 together with the Arzela–Ascoli theorem, we conclude that \( \Gamma \) is a compact multivalued map, u.s.c. with convex closed values. As a consequence of Lemma 2.7, we can deduce that \( \Gamma \) has a fixed point \( x \) which is a mild solution of system (1) and (2).

**Definition 3.2.** The control system (1) and (2) is said to be approximately controllable on \( I \) if \( R(b) = X \), where \( R(b) = \{ x(b; u) : u \in L^2(I, U) \} \) and \( x(\cdot, u) \) is a mild solution of the system (1) and (2).

**Theorem 3.3.** Suppose that the assumptions \( H_0 - H_3 \) hold. Assume additionally that there exists \( N \in L^1(I, [0, \infty)) \) such that \( \sup_{x \in X} \| F(t, x) \| \leq N(t) \) for a.e. \( t \in I \), then the system (1) and (2) is approximately controllable in \( I \).

**Proof.** Let \( \tilde{x}^a(\cdot) \) be a fixed point of \( \Gamma \) in \( \mathcal{B}_r \). By Theorem 3.1, any fixed point of \( \Gamma \) is a mild solution of (1) and (2) under the control
\[
\tilde{u}^a(t) = B^* S^*(b, t) R(a, \Upsilon_0^b)p(\tilde{x}^a)
\]
and satisfies the following inequality
\[
\tilde{x}^a(b) = x_b + aR(a, \Upsilon_0^b)p(\tilde{x}^a).
\]

Moreover by assumption on \( F \) and Dunford–Pettis theorem, we have that the \( \{ f^a(s) \} \) is weakly compact in \( L^1(I, X) \), so there is a sub-sequence, denoted by \( \{ f^a(s) \} \), that converges weakly to say \( f(s) \) in \( L^1(I, X) \). Define
\[
w = x_b - C(b, 0)x_0 - S(b, 0)y_0 - \int_0^b S(b, s)f(s) ds.
\]
Now, we have
\[
\|p(\tilde{x}^a) - w\| = \left\| \int_0^b S(b, s)[f(s, \tilde{x}^a(s)) - f(s)] ds \right\|
\leq \sup_{t \in I} \left\| \int_0^t S(t, s)[f(s, \tilde{x}^a(s)) - f(s)] ds \right\|. \tag{16}
\]

Using infinite-dimensional version of the Ascoli–Arzela theorem, one can show that an operator \(l(\cdot) \to \int_0^\cdot S(\cdot, s) l(s) ds : L^1(I, X) \to C\) is compact. Therefore, we obtain that \(\|p(\tilde{x}^a) - w\| \to 0\) as \(a \to 0^+\). Moreover, from (16), we get
\[
\|\tilde{x}^a(b) - x_b\| \leq \|aR(a, \Upsilon^b_0)(w)\| + \|aR(a, \Upsilon^b_0)\| \|p(\tilde{x}^a) - w\|
\leq \|aR(a, \Upsilon^b_0)(w)\| + \|p(\tilde{x}^a) - w\|.
\]

It follows from assumption \(H_0\) and the estimation (16) that \(\|\tilde{x}^a(b) - x_b\| \to 0\) as \(a \to 0^+\). This proves the approximate controllability of system (1) and (2). \(\square\)

4. Second-Order Control Systems with Nonlocal Conditions

There exist an extensive literature of differential equations with nonlocal conditions. Since it is demonstrated that the nonlocal problems have better effects in applications than the classical ones, differential equations with nonlocal problems have been studied extensively in the literature. The result concerning the existence and uniqueness of mild solutions to abstract Cauchy problems with nonlocal initial conditions was first formulated and proved by Byszewski; see [10–12]. Since the appearance of these papers, several papers have addressed the issue of existence and uniqueness of nonlinear differential equations. Existence and controllability results of nonlinear differential equations and fractional differential equations with nonlocal conditions have been studied by several authors for different kind of problems [2,3,15,18,23,29,34,56].

Inspired by the above works, in this section, we discuss the approximate controllability for a class of second-order evolution differential inclusions with nonlocal conditions in Hilbert spaces of the form
\[
x''(t) \in A(t)x(t) + F(t, x(t)) + Bu(t), \quad t \in I = [0, b], \tag{17}
\]
\[
x(0) + g(x) = x_0 \in X, \quad x'(0) + h(x) = y_0 \in X, \tag{18}
\]
where \(g, h : C \to X\) is a given function which satisfies the following condition:

**H_4** There exists a constant \(L_g > 0, L_h > 0\) such that
\[
|g(x) - g(y)| \leq L_g \|x - y\|, \quad \text{for } x, y \in C
\]
\[
|h(x) - h(y)| \leq L_h \|x - y\|, \quad \text{for } x, y \in C.
\]

for all \(x, y \in C\).

**Definition 4.1.** A function \(x \in C\) is said to be a mild solution of system (17) and (18) if \(x(0) + g(x) = x_0, x'(0) + h(x) = y_0\) and there exists \(f \in L^1(I, X)\) such that \(f(t) \in F(t, x(t))\) on \(t \in I\) and the integral equation
\[ x(t) = C(t, 0)(x_0 - g(x)) + S(t, 0)(y_0 - h(x)) + \int_0^t S(t, s)f(s)ds \]
\[ + \int_0^t S(t, s)Bu(s)ds, \quad t \in I \]
is satisfied.

**Theorem 4.2.** Assume that the assumptions of Theorem 3.1 are satisfied. Further, if the hypothesis \( H_4 \) is satisfied, then the system (17) and (18) is approximately controllable on \( I \) provided that
\[
\tilde{N}\gamma \left[ 1 + \frac{1}{\alpha} \tilde{N}^2 M_B^2 b \right] < 1
\]
where \( M_B = \|B\| \).

**Proof.** For each \( a > 0 \), we define the operator \( \hat{\Gamma}_a \) on \( X \) by
\[ (\hat{\Gamma}_a x) = z, \]
where
\[ z(t) = C(t, 0)(x_0 - g(x)) + S(t, 0)(y_0 - h(x)) + \int_0^t S(t, s)f(s)ds \]
\[ + \int_0^t S(t, s)Bu(s, x)ds, \]
\[ v(t) = B^*S^*(b, t)R(a, \Upsilon_0^b)p(x(\cdot)), \]
\[ p(x(\cdot)) = x_b - C(t, 0)(x_0 - g(x)) - S(t, 0)(y_0 - g(x)) - \int_0^t S(t, s)f(s)ds, \]
where \( f \in S_{F,x} \).

It can be easily proved that if for all \( a > 0 \), the operator \( \hat{\Gamma}_a \) has a fixed point by implementing the technique used in Theorem 3.1. Then, we can show that the second-order control system (17) and (18) is approximately controllable. The proof of this theorem is similar to that of Theorems 3.1 and 3.3, and hence it is omitted.

\[ \square \]

5. **Second-Order Impulsive Nonlocal Control Systems**

Impulsive dynamical systems are characterized by the occurrence of an abrupt change in the state of the system, which occur at certain time instants over a period of negligible duration. The dynamical behavior of such systems is much more complex than the behavior of dynamical systems without impulse effects. The presence of impulse means that the state trajectory does not preserve the basic properties which are associated with non-impulsive dynamical systems. In fact, the theory of impulsive differential equations has found extensive applications in realistic mathematical modeling of a wide variety of practical situations [32] and has emerged as an important area of investigation in recent years. It is known that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology,
optimal control models in economics, pharmacokinetics and frequency modulation systems exhibit impulse effects; see the monographs of Samoilenko and Perestyuk [43], Lakshmikantham et al. [27], Bainov et al. [6] and Benchohra et al. [8] and the references cited therein. The literature on this type of problem is vast, and different topics on the existence and qualitative properties of solutions are considered. Concerning the general motivations, relevant developments and the current status of the theory, we refer the reader to [1, 4, 5, 22, 23, 26, 41, 45, 46, 51, 54, 55].

Inspired by the above works, in this section, we discuss the approximate controllability for a class of second-order impulsive evolution differential inclusions with nonlocal conditions in Hilbert spaces of the form

\[
\begin{align*}
x''(t) &\in A(t)x(t) + F(t, x(t)) + Bu(t), \quad t \in I = [0, b]; \\
x(0) + g(x) &\in x_0, \quad x'(0) + h(x) = y_0 \in X, \\
\Delta x(t_i) &= I_i(x(t_i)), \\
\Delta x'(t_i) &= J_i(x(t_i)),
\end{align*}
\]

where \(0 < t_1 < t_2 < \cdots < t_n < b\) are fixed numbers and \(I_i : X \to X, J_i : X \to X, i = 1, \ldots, n\) are suitable functions and the symbol \(\Delta \xi(t)\) represents the jump of the function \(\xi\) at \(t\), which is defined by \(\Delta \xi(t) = \xi(t^+) - \xi(t^-)\).

Consider the following assumption

\(H_5\) The maps \(I_i : X \to X, J_i : X \to X\) are completely continuous and uniformly bounded. In the sequel, we set \(M_i = \sup\{\|I_i(x)\| : x \in X\}\) and \(\tilde{M}_i = \sup\{\|J_i(x)\| : x \in X\}\).

**Definition 5.1.** A function \(x \in C\) is said to be a mild solution of system (19)–(22) if \(x(0) + g(x) = x_0, x'(0) + h(x) = y_0\) and there exists \(f \in L^1(I, X)\) such that \(f(t) \in F(t, x(t))\) on \(t \in I\) and the integral equation

\[
x(t) = C(t, 0)(x_0 - g(x)) + S(t, 0)(y_0 - h(x)) + \int_0^t S(t, s)f(s)ds \\
+ \int_0^t S(t, s)Bu(s)ds + \sum_{0 < t_i < t} C(t, t_i)I_i(u(t_i)) \\
+ \sum_{0 < t_i < t} S(t, t_i)J_i(u(t_i)), \quad t \in I
\]

is satisfied.

**Theorem 5.2.** Assume that the assumptions of Theorems 3.1 and 4.2 are satisfied. Further, if the hypothesis \(H_5\) is satisfied, then the system (19)–(22) is approximately controllable on \(I\) provided that

\[
\tilde{N} \gamma \left[1 + \frac{1}{\alpha} \tilde{N}^2 M_B^2 b\right] < 1
\]

where \(M_B = \|B\|\).

**Proof.** For each \(\alpha > 0\), we define the operator \(\hat{\Gamma}_\alpha\) on \(X\) by

\[
(\hat{\Gamma}_\alpha x) = z,
\]
where
\[
\begin{align*}
z(t) &= C(t,0)(x_0 - g(x)) + S(t,0)(y_0 - h(x)) \\
&\quad + \int_0^t S(t,s)f(s)ds + \int_0^t S(t,s)Bu(s,x)ds \\
&\quad + \sum_{0 < t_i < t} C(t,t_i)I_i(u(t_i)) + \sum_{0 < t_i < t} S(t,t_i)J_i(u(t_i)),
\end{align*}
\]
\[
v(t) = B^*S^*(b,t)R(a,\Upsilon b_0)p(x(\cdot)) ,
\]
\[
p(x(\cdot)) = x_b - C(t,0)(x_0 - g(x)) - S(t,0)(y_0 - g(x)) - \int_0^t S(t,s)f(s)ds \\
&\quad + \sum_{i=1}^n C(t,t_i)I_i(u(t_i)) + \sum_{i=1}^n S(t,t_i)J_i(u(t_i)),
\]
in which \( f \in S_{F,x} \).

It can be easily proved that if for all \( a > 0 \), the operator \( \hat{\Gamma}_a \) has a fixed point by implementing the technique used in Theorem 3.1. Then, we can show that the second-order control system (19)–(22) is approximately controllable. The proof of this theorem is similar to that of Theorems 3.1, 3.3 and 4.2, and hence it is omitted. □

6. An Application

In this section, we apply our abstract results to a concrete partial differential equation. To establish our results, we need to introduce the required technical tools. Following the equations (9) and (10), here we consider \( A(t) = A + \tilde{B}(t) \) where \( A \) is the infinitesimal generator of a cosine function \( C(t) \) with associated sine function \( S(t) \), and \( \tilde{B}(t) : D(\tilde{B}(t)) \to X \) is a closed linear operator with \( D \subseteq D(\tilde{B}(t)) \) for all \( t \in I \).

We model this problem in the space \( X = L^2(T,\mathbb{C}) \), where the group \( T \) is defined as the quotient \( \mathbb{R}/2\pi\mathbb{Z} \). We will use the identification between functions on \( T \) and \( 2\pi \)-periodic functions on \( \mathbb{R} \). Specifically, in what follows we denote by \( L^2(T,\mathbb{C}) \) the space of \( 2\pi \)-periodic 2-integrable functions from \( \mathbb{R} \) into \( \mathbb{C} \). Similarly, \( H^2(T,\mathbb{C}) \) denotes the Sobolev space of \( 2\pi \)-periodic functions \( x : \mathbb{R} \to \mathbb{C} \) such that \( x'' \in L^2(T,\mathbb{C}) \).

We consider the operator \( Ax(\xi) = x''(\xi) \) with domain \( D(A) = H^2(T,\mathbb{C}) \). It is well known that \( A \) is the infinitesimal generator of a strongly continuous cosine function \( C(t) \) on \( X \). Moreover, \( A \) has discrete spectrum; the spectrum of \( A \) consists of eigenvalues \( -n^2 \) for \( n \in \mathbb{Z} \), with associated eigenvectors
\[
w_n(\xi) = \frac{1}{\sqrt{2\pi}}e^{in\xi}, \quad n \in \mathbb{Z},
\]
the set \( \{w_n : n \in \mathbb{Z}\} \) is an orthonormal basis of \( X \). In particular,
\[
Ax = -\sum_{n=1}^{\infty} n^2 \langle x, w_n \rangle w_n
\]
for \( x \in D(A) \). The cosine function \( C(t) \) is given by

\[
C(t)x = \sum_{n=1}^{\infty} \cos(nt) \langle x, w_n \rangle w_n, \quad t \in \mathbb{R},
\]

with associated sine function

\[
S(t)x = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle x, w_n \rangle w_n, \quad t \in \mathbb{R}.
\]

It is clear that \( \|C(t)\| \leq 1 \) for all \( t \in \mathbb{R} \). Thus, \( C(\cdot) \) is uniformly bounded on \( \mathbb{R} \).

Consider the second-order Cauchy problem with control

\[
\frac{\partial^2}{\partial t^2} z(t, \tau) \in \frac{\partial^2}{\partial \tau^2} z(t, \tau) + b(t) \frac{\partial}{\partial t} z(t, \tau) + \mu(t, \tau) + Q(t, z(t, \tau)) \quad (23)
\]

for \( t \in I, \ 0 \leq \tau \leq \pi \), subject to the initial conditions

\[
z(t, 0) = z(t, \pi) = 0, \quad t \in I, \quad (24)
\]

\[
z(0, \tau) = z_0(\tau), \quad 0 \leq \tau \leq \pi, \quad \frac{\partial}{\partial t} z(0, \tau) = z_1(\tau), \quad (25)
\]

where \( b : \mathbb{R} \to \mathbb{R} \) is a continuous function. We fix \( a > 0 \) and set \( \beta = \sup_{0 \leq t \leq a} |b(t)| \).

We take \( \tilde{B}(t)x(\tau) = b(t)x'(\tau) \) defined on \( H^1(\mathbb{T}, \mathbb{C}) \). It is easy to see that \( A(t) = A + \tilde{B}(t) \) is a closed linear operator. Initially, we will show that \( A + \tilde{B}(t) \) generates an evolution operator. It is well known that the solution of the scalar initial value problem

\[
q''(t) = -n^2 q(t) + p(t),
\]

\[
q(s) = 0, \quad q'(s) = q_1
\]

is given by

\[
q(t) = \frac{q_1}{n} \sin n(t - s) + \frac{1}{n} \int_{s}^{t} \sin n(t - \tau)p(\tau)d\tau.
\]

Therefore, the solution of the scalar initial value problem

\[
q''(t) = -n^2 q(t) + i n b(t) q(t), \quad (26)
\]

\[
q(s) = 0, \quad q'(s) = q_1, \quad (27)
\]

satisfies the integral equation

\[
q(t) = \frac{q_1}{n} \sin n(t - s) + i \int_{s}^{t} \sin n(t - \tau)b(\tau)q(\tau)d\tau.
\]

Applying the Gronwall–Bellman lemma, we can affirm that

\[
|q(t)| \leq \frac{|q_1|}{n} e^{\beta(t-s)} \quad (28)
\]

for \( s \leq t \). We denote by \( q_n(t, s) \) the solution of (26) and (27). We define

\[
S(t, s)x = \sum_{n=1}^{\infty} q_n(t, s) \langle x, w_n \rangle w_n.
\]
It follows from the estimate (28) that $S(t,s): X \to X$ is well defined and satisfies the conditions of Definition 2.1.

Put $z(t) = z(t,\cdot)$, that is $z(t)(\tau) = z(t,\tau)$, $t \in I$, $\tau \in [0,\pi]$ and $u(t) = \mu(t,\cdot)$; here, $\mu: I \times [0,\pi] \to [0,\pi]$ is continuous. Let us define $f: I \times X \to X$ as

$$F(t,z)(\tau) = Q(t,z(\tau)), \quad z \in X, \quad \tau \in [0,\pi]$$

and the bounded linear operator $B: U \to X$ by

$$Bu(t)(\tau) = \mu(t,\tau), \quad t \in I, \quad \tau \in [0,\pi].$$

Assume that these functions satisfy the requirement of hypotheses. From the above choices of the functions and evolution operator $A(t)$ with $B = I$, the system (23)–(25) can be formulated as the system (1)–(2) in $X$. Since all hypotheses of Theorem 3.3 are satisfied, approximate controllability of system (23)–(25) on $I$ follows from Theorem 3.3.

7. Conclusion

Approximate controllability for a class of evolution differential inclusions in Hilbert spaces is discussed in this paper. By using Bohnenblust–Karlin’s fixed point theorem, we proved our main results. Further, we extended our result to study the approximate controllability concept with nonlocal conditions and also to study the approximate controllability for impulsive control systems with nonlocal conditions. Finally, we have given an example for illustration of the presented theory.

In future work, we will try to focus our study on the approximate controllability for a class of second-order neutral impulsive evolution differential inclusions with nonlocal conditions in Hilbert spaces by using Bohnenblust–Karlin’s fixed point theorem.

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References

[1] Abada, N., Benchohra, M., Hammouche, H.: Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions. J. Differ. Equ. 246(10), 3834–3863 (2009)
[2] Aizicovici, S., McKibben, M.: Existence results for a class of abstract nonlocal Cauchy problems. Nonlinear Anal. 39, 649–668 (2000)
[3] Aizicovici, S., Lee, H.: Nonlinear nonlocal Cauchy problems in Banach spaces. Appl. Math. Lett. 18, 401–407 (2005)
[4] Arthi, G., Balachandran, K.: Controllability of second-order impulsive evolution systems with infinite delay. Nonlinear Anal. Hybrid Syst. 11, 139–153 (2014)
| Reference | Details |
|-----------|---------|
| [5]       | Arthi, G., Park, J.H., Jung, H.Y.: Existence and controllability results for second-order impulsive stochastic evolution systems with state-dependent delay. Appl. Math. Comput. **248**, 328–341 (2014) |
| [6]       | Bainov, D.D., Simeonov, P.S.: Impulsive Differential Equations: Periodic Solutions and Applications. Longman Scientific and Technical Group, England (1993) |
| [7]       | Batty, C.J.K., Chill, R., Srivastava, S.: Maximal regularity for second order non-autonomous Cauchy problems. Studia Math. **189**, 205–223 (2008) |
| [8]       | Benchohra, M., Henderson, J., Ntouyas, S.K.: Impulsive Differential Equations and Inclusions, Contemporary Mathematics and Its Applications. Hindawi Publishing Corporation, New York (2006) |
| [9]       | Bohnenblust, H.F., Karlin, S.: On a Theorem of Ville. In: Contributions to the Theory of Games, vol. I, pp. 155–160. Princeton University Press, Princeton (1950) |
| [10]      | Byszewski, L.: Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. J. Math. Anal. Appl. **162**, 494–505 (1991) |
| [11]      | Byszewski, L., Akca, H.: On a mild solution of a semilinear functional-differential evolution nonlocal problem. J. Appl. Math. Stoc. Anal. **10**(3), 265–271 (1997) |
| [12]      | Byszewski, L., Lakshmikantham, V.: Theorem about the existence and uniqueness of solutions of a nonlocal Cauchy problem in a Banach space. Appl. Anal. **40**, 11–19 (1990) |
| [13]      | Dauer, J.P., Mahmudov, N.I., Matar, M.M.: Approximate controllability of backward stochastic evolution equations in Hilbert spaces. J. Math. Anal. Appl. **323**(1), 42–56 (2006) |
| [14]      | Deimling, K.: Multivalued Differential Equations. De Gruyter, Berlin (1992) |
| [15]      | Ezzinbi, K., Fu, X., Hilal, K.: Existence and regularity in the a-norm for some neutral partial differential equations with nonlocal conditions. Nonlinear Anal. **67**, 1613–1622 (2007) |
| [16]      | Faraci, F., Iannizzotto, A.: A multiplicity theorem for a perturbed second-order non-autonomous system. Proc. Edinb. Math. Soc. **49**, 267–275 (2006) |
| [17]      | Fattorini, H.O.: Second Order Linear Differential Equations in Banach Spaces, vol. 108. North-Holland Math. Stud., Amsterdam (1985) |
| [18]      | Fu, X., Ezzinbi, K.: Existence of solutions for neutral functional differential evolution equations with nonlocal conditions. Nonlinear Anal. **54**, 215–227 (2003) |
| [19]      | Grans, A., Dugundji, J.: Fixed Point Theory. Springer, New York (2003) |
| [20]      | Guendouzi, T., Bousmaha, L.: Approximate controllability of fractional neutral stochastic functional integro-differential inclusions with infinite delay. Qual. Theory Dyn. Syst. **13**, 89–119 (2014) |
| [21]      | Henríquez, H.R.: Existence of solutions of non-autonomous second-order functional differential equations with infinite delay. Nonlinear Anal. TMA. **74**, 3333–3352 (2011) |
| [22]      | Henríquez, H.R., Genaro, Castillo: The Kneser property for the second order functional abstract Cauchy problem. Integ. Equ. Oper. Theory **52**, 505–525 (2005) |
[23] Henríquez, H.R., Hernández, E.: Existence of solutions of a second order abstract functional Cauchy problem with nonlocal conditions. Ann. Polon. Math. 88(2), 141–159 (2006)
[24] Henríquez, H.R.: Approximate controllability of linear distributed control systems. Appl. Math. Lett. 21(10), 1041–1045 (2008)
[25] Kisyński, J.: On cosine operator functions and one parameter group of operators. Studia. Math. 49, 93–105 (1972)
[26] Kozak, M.: A fundamental solution of a second order differential equation in a Banach space. Univ. Iagellon. Acta Math. 32, 275–289 (1995)
[27] Laksmikantham, V., Bainov, D., Simenov, P.S.: Theory of Impulsive Differential Equations, Series in Modern Applied Mathematics, vol. 6. World Scientific Publishing Co., Inc., Teaneck (1989)
[28] Lasota, A., Opial, Z.: An application of the Kakutani-Ky-Fan theorem in the theory of ordinary differential equations or noncompact acyclic-valued map. Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. 13, 781–786 (1965)
[29] Liang, J., Liu, J.H., Xiao, T.J.: Nonlocal Cauchy problems governed by compact operator families. Nonlinear Anal. 57, 183–189 (2004)
[30] Lin, Y.: Time-dependent perturbation theory for abstract evolution equations of second order. Stud. Math. 130, 263–274 (1998)
[31] Liu, Z., Lv, J.: Approximate controllability of fractional functional evolution inclusions with delay in Hilbert spaces. IMA J. Math. Cont. Inform. 31, 363–383 (2013)
[32] Miller, B.M., Rubinovich, E.Y.: Impulsive Control in Continuous and Discrete-Continuous Systems. Kluwer Academic/Plenum Publishers, New York (2003)
[33] Mahmudov, N.I., Denker, A.: On controllability of linear stochastic systems. Int. J. Cont. 73, 144–151 (2000)
[34] Mahmudov, N.I.: Approximate controllability of evolution systems with nonlocal conditions. Nonlinear Anal. TMA. 68(3), 536–546 (2008)
[35] Mahmudov, N.I., Zorlu, S.: On the approximate controllability of fractional evolution equations with compact analytic semigroup. J. Comput. Appl. Math. 259, 194–204 (2014)
[36] Mahmudov, N.I.: Approximate controllability of some nonlinear systems in Banach spaces. Bound. Val. Prob 2013(1), 1–13 (2013)
[37] Mahmudov, N.I., Denker, A.: On controllability of linear stochastic systems. Int. J. Control. 73, 144–151 (2000)
[38] Mahmudov, N.I., McKibben, M.A.: Approximate controllability of second-order neutral stochastic evolution equations. Dyn. Contin. Discrete Impuls. Syst. 13(5), 619–634 (2006)
[39] Martin, R.H.: Nonlinear Operators and Differential Equations in Banach Spaces. Robert E Krieger Publishing Co, Florida (1987)
[40] Obrecht, E.: Evolution operators for higher order abstract parabolic equations. Czech. Math. J. 36, 210–222 (1986)
[41] Peng, Y., Xiang, X.: Second-order nonlinear impulsive time-variant systems with unbounded perturbation and optimal controls. J. Ind. Manag. Optim. 4, 17–32 (2008)
[42] Ren, Y., Dai, H., Sakthivel, R.: Approximate controllability of stochastic differential systems driven by a Lévy process. Int. J. Cont. 86, 1158–1164 (2013)
Vol. 13 (2016) Approximate Controllability 3453

[43] Samoilenko, A.M., Perestyuk, N.A.: Impulsive Differential Equations. World Scientific, Singapore (1995)

[44] Sakthivel, R., Ganesh, R., Anthoni, S.M.: Approximate controllability of fractional nonlinear differential inclusions. Appl. Math. Comput. 225, 708–717 (2013)

[45] Sakthivel, R., Mahmudov, N.I., Kim, J.H.: Approximate controllability of nonlinear impulsive differential systems. Report. Math. Phy. 60(1), 85–96 (2007)

[46] Sakthivel, R., Anandhi, E.R., Mahmudov, N.I.: Approximate controllability of second-order systems with state-dependent delay. Numer. Funct. Anal. Optim. 29(11–12), 1347–1362 (2008)

[47] Sakthivel, R., Ren, Y.: Complete controllability of stochastic evolution equations with jumps. Rep. Math. Phy. 68, 163–174 (2011)

[48] Sakthivel, R., Ren, Y.: Approximate controllability of fractional differential equations with state-dependent delay. Res. Math. 63, 949–963 (2013)

[49] Serizawa, H., Watanabe, M.: Time-dependent perturbation for cosine families in Banach spaces. Hous. J. Math. 12, 579–586 (1986)

[50] Hu, S., Papageorgiou, N.S.: Handbook of Multivalued Analysis (Theory). Kluwer Academic Publishers, Dordrecht (1997)

[51] Sivasankaran, S., Mallika Arjunan, M., Vijayakumar, V.: Existence of global solutions for second order impulsive abstract partial differential equations. Nonlinear Anal. TMA. 74(17), 6747–6757 (2011)

[52] Travis, C.C., Webb, G.F.: Compactness, regularity, and uniform continuity properties of strongly continuous cosine families. Houston J. Math. 3(4), 555–567 (1977)

[53] Travis, C.C., Webb, G.F.: Cosine families and abstract nonlinear second order differential equations. Acta. Math. Acad. Sci. Hungar. 32, 76–96 (1978)

[54] Vijayakumar, V., Sivasankaran, S., Mallika Arjunan, M.: Existence of solutions for second order impulsive partial neutral functional integrodifferential equations with infinite delay. Nonlinear Stud. 19(2), 327–343 (2012)

[55] Vijayakumar, V., Sivasankaran, S., Mallika Arjunan, M.: Existence of global solutions for second order impulsive abstract functional integrodifferential equations. Dyn. Contin. Discrete Impuls. Syst. 18, 747–766 (2011)

[56] Vijayakumar, V., Ravichandran, C., Murugesu, R.: Nonlocal controllability of mixed Volterra-Fredholm type fractional semilinear integro-differential inclusions in Banach spaces. Dyn. Contin. Discrete Impuls. Syst. 20(4), 485–502 (2013)

[57] Vijayakumar, V., Ravichandran, C., Murugesu, R.: Approximate controllability for a class of fractional neutral integro-differential inclusions with state-dependent delay. Nonlinear Stud. 20(4), 511–530 (2013)

[58] Vijayakumar, V., Selvakumar, A., Murugesu, R.: Controllability for a class of fractional neutral integro-differential equations with unbounded delay. Appl. Math. Comput. 232, 303–312 (2014)

[59] Vijayakumar, V., Ravichandran, C., Murugesu, R., Trujillo, J.J.: Controllability results for a class of fractional semilinear integro-differential inclusions via resolvent operators. Appl. Math. Comput. 247, 152–161 (2014)

[60] Yan Z.: Approximate controllability of fractional neutral integro-differential inclusions with state-dependent delay in Hilbert spaces. IMA J. Math. Cont. Inform. (2012). doi:10.1093/imamci/dns033
[61] Yan, Z.: Approximate controllability of partial neutral functional differential systems of fractional order with state-dependent delay. Int. J. Cont. 85(8), 1051–1062 (2012)

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