SOBOLEV MAPPINGS BETWEEN NONRIGID CARNOT GROUPS

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Abstract. We consider mappings \( f : G \supset U \to G' \) where \( G \) and \( G' \) are Carnot groups. In this paper, which is a continuation of [KMX20], we focus on Carnot groups which are nonrigid in the sense of Ottazzi-Warhurst. We show that quasisymmetric homeomorphisms are reducible in the sense that they preserve a special type of coset foliation, unless the group is isomorphic to \( \mathbb{R}^n \) or a real or complex Heisenberg group (where the assertion fails). We use this to prove the quasisymmetric rigidity conjecture for such groups. The starting point of the proof is the pullback theorem established in [KMX20].

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This is the second in a series of papers \[ \text{KMX20, KMXb, KMX21b, KMX21a} \] on geometric mapping theory in Carnot groups, in which we establish new regularity, rigidity, and partial rigidity results for bilipschitz, quasiconformal, or more generally, Sobolev mappings, between Carnot groups. Our motivation comes from the analytical theory of quasiconformal homeomorphisms, geometric group theory (especially rigidity phenomena and Gromov hyperbolic spaces), analysis on metric spaces, the differential geometry of subriemannian manifolds, and the literature on rigidity and oscillatory solutions to partial differential equations (and relations).

While our main interest in this paper is in quasiconformal homeomorphisms – and more generally Sobolev mappings – between open subsets of Carnot groups, for context we first consider the special case of quasiconformal diffeomorphisms.

For the remainder of the introduction $G$ will be a step $s$ Carnot group with Lie algebra $\mathfrak{g}$, grading $\mathfrak{g} = \bigoplus_{j=1}^{s} V_j$, dilation group $\{ \delta_r : G \to G \}_{r \in (0, \infty)}$, and homogeneous dimension $\nu = \sum j \dim V_j$. Without explicit mention, $G$ will be equipped with Haar measure and a Carnot-Caratheodory metric denoted generically by $d_{CC}$. We will use the same notation with primes for another Carnot group $G'$.

It is easy to see that a diffeomorphism $f : U \to U'$ between open subsets $U, U' \subset G$ is locally quasiconformal if and only if it is contact, i.e. the differential $Df$ preserves the horizontal subbundle $V_1 \subset TG$. The study of contact diffeomorphisms has a long and fascinating history intertwined with the theory of Lie pseudogroups, overdetermined systems, and $G$-structures (in the sense of E. Cartan); the literature extends back to the 19th century, with major contributions in 1900-10 by Cartan, in 1955-70 by Kuranishi, Singer, Sternberg, Guillemin, Quillen, and Tanaka (among many others); there has been a resurgence of interest in recent decades, coming from new connections with geometric group theory and quasiconformal mappings. The literature on this topic is substantial, so we will mention just a few points which are directly relevant to our setting, and refer the interested reader to \[ \text{SS65, Tan70, CO15} \] for references and more discussion. The contact condition is a nonlinear system of PDEs which is formally overdetermined except when $G$ is the Engel group or a product $\mathbb{H}_k \times \mathbb{R}^\ell$ for some $k, \ell \geq 0$, and hence one expects some form of rigidity in the generic case; here $\mathbb{H}_k$ denotes the $k^{th}$ Heisenberg group. However, the analytical character of the condition is quite different for different groups:
• When $G = \mathbb{H} \times \mathbb{H}$ contact diffeomorphisms must be products (locally) but otherwise are quite flexible [CR03].

• When $G = \mathbb{H}_n^C$ is the complexification of $\mathbb{H}_n$ the contact condition is locally flexible, but still “hypoelliptic”, i.e. contact diffeomorphisms are holomorphic or antiholomorphic [RR00].

• When $G$ is an $H$-type group with center of dimension at least 3, (e.g. one of the Carnot groups studied by Pansu) or a free Carnot group of step $s \geq 3$, then the smooth contact embeddings $G \supset U \to G$ form a finite dimensional family when $U$ is a connected open subset [Pan89, Rei01, War07].

Although it has been formalized in different ways, there is a dichotomy of contact systems into finite (or rigid) type and infinite (or nonrigid) type according to whether the contact diffeomorphisms are locally determined by finitely many, or infinitely many parameters [Car04, Kur59, GS64, SS65, Tan70]. This was greatly clarified by Ottazzi-Warhurst [OW11b] who proved a fundamental result characterizing rigidity/nonrigidity of the Carnot group $G$ in terms of the complexification $g^C$ of the Lie algebra $g$, and showed that $C^2$ contact diffeomorphisms of rigid Carnot groups are $C^\infty$ (which readily implies that they are real analytic).\(^1\) The prolongation theory in [Tan70] gives an algebraic approach to determine the local contact diffeomorphisms, at least in principle; however, as it stands currently, the general theory is based on the Cartan-Kähler theorem and therefore only provides a satisfactory picture for real analytic diffeomorphisms. In particular, to our knowledge apart from some specific groups which have been analyzed [RR00, War03, War05], in the nonrigid case there has been limited progress toward understanding $C^k$ contact diffeomorphisms, $1 \leq k \leq \infty$; even a conjectural description is lacking.

We now drop the smoothness assumption. Motivated by the above regularity result in the smooth case, for Ottazzi-Warhurst rigid Carnot groups we have (cf. [OW11a, OW11b, p.2]):

**Regularity Conjecture 1.1.** If $G$ is a rigid Carnot group, then any quasiconformal homeomorphism $G \supset U \to U' \subset G$ is $C^\infty$.

Together with earlier work of Tanaka [Tan70], the conjecture implies a description of quasiconformal mappings in algebraic terms, at least in principle. Conjecture [11] has been proven for groups whose graded automorphisms act conformally on the first layer, by subelliptic regularity

\(^{1}\)Recently Jonas Lelmi improved this by replacing the $C^2$ regularity assumption with $C^1$ (or even Euclidean bilipschitz); the same result was shown by Alex Austin for the $(2, 3, 5)$ distribution [Le] [Aus].
It follows from \cite{KMX20} that the conjecture holds for a product of Carnot groups $\prod G_i$ if it holds for all of the factors $G_i$. We expect that Conjecture 1.1 holds under weaker regularity assumptions, for instance for suitably nondegenerate $W_{\text{loc}}^{1,p}$-mappings when $p$ is strictly larger than the homogeneous dimension $\nu$ of $G$; on the other hand it is unclear what to expect when $p$ is small (or $p = 1$), i.e. whether regularity/rigidity holds, or if such mappings are more flexible (cf. \cite[p.2]{OW11b}). We will present further results on rigid groups elsewhere \cite{KMXa}, and now turn to the case of nonrigid groups.

We recall that a quasiconformal homeomorphism between open subsets of $G$ is a Sobolev mapping with respect to the Carnot distance – it belongs to $W_{\text{loc}}^{1,p}$ for some $p > \nu$. More generally, let $f : G \supset U \to G'$ be a $W_{\text{loc}}^{1,p}$-mapping for some $p > \nu$, where $U \subset G$ is open. Then $f$ has a well-defined Pansu differential at a.e. $x \in U$; this is a graded group homomorphism $D_P f(x) : G \to G'$ which we often conflate with the associated homomorphism of graded Lie algebras $D_P f(x) : g \to g'$. If $f$ is a quasiconformal homeomorphism, then $\det(D_P f(x)) \neq 0$ for a.e. $x \in U$, and has the same sign as the local degree of $f$. See \cite{Pan89} or Theorem 2.12 in \cite{KMX20}.

Our main result is that apart from some exceptional cases, for nonrigid Carnot groups one always has partial rigidity – mappings (virtually) preserve a foliation.

**Theorem 1.2 (See Section 3 for definitions).** If $G$ is a nonrigid Carnot group (in the sense of Ottazzi-Warhurst) with homogeneous dimension $\nu$, then one of the following holds:

1. $G$ is isomorphic to $\mathbb{R}^n$ or to a real or complex Heisenberg group $\mathbb{H}_n$, $\mathbb{H}_n^\mathbb{C}$ for some $n \geq 1$.
2. There is a closed horizontally generated subgroup $\{e\} \subseteq H \subseteq G$, a constant $K$, and a finite set $A$ of graded automorphisms of $G$ with the following properties:
   - For every $p > \nu$, $x \in G$, $r \in (0, \infty)$, and every $W_{\text{loc}}^{1,p}$-mapping $f : G \supset B(x,r) \to G$ such that the sign of $\det(D_P f)$ is constant almost everywhere, then for some $\Phi \in A$ the restriction of the composition $\Phi \circ f$ to the subball $B(x,\frac{r}{K})$ preserves the coset foliation of $H$. In particular, the conclusion holds for quasiconformal homeomorphisms.
   - The Lie algebra of $H$ is generated by a linear subspace $\{0\} \subseteq W \subseteq V_1$ with $[W,V_j] = \{0\}$ for all $j \geq 2$. 

\[\text{Cap99}, \text{CC06}\]
Thus, apart from the exceptional cases in (1), quasiconformal homeomorphisms preserve a foliation, up to post-composition with a graded automorphism. This suggests that for the (nonexceptional) nonrigid cases there may be a more detailed description of quasiconformal homeomorphisms along the lines of [Xie15] for model filiform groups. To our knowledge such a description is not known even for smooth contact diffeomorphisms, either locally or globally.

Remark 1.3. Using more refined version of the pullback theorem from [KMX6] rather than the version in [KMX20] applied here, in some cases one can reduce the requirements on the Sobolev exponent $p$ in Theorem 1.2.

Another aspect of quasiconformal rigidity/flexibility has to do with the Sobolev exponent, i.e. higher integrability of the derivative. Quasiconformal homeomorphisms $f : G \supset U \to U' \subset G$ are always in $W^{1,p}_{\text{loc}}$ for some $p$ strictly larger than the homogeneous dimension of $G$, where $p$ depends on $G$ and the quasiconformal distortion of $f$ [HK98]. However, except for $\mathbb{R}^n$ and the Heisenberg groups $\mathbb{H}_n$ (see for example [Bal01]), all known examples of quasiconformal homeomorphisms are in $W^{1,\infty}_{\text{loc}}$, i.e. are locally bilipschitz. This inspired the following:

Conjecture 1.4 (Xie). If $G$ is a Carnot group other than $\mathbb{R}^n$ or $\mathbb{H}_n$ for some $n$, then every quasiconformal homeomorphism $f : G \supset U \to U' \subset G$ is locally bilipschitz; moreover, if $U = G$ then $f$ is bilipschitz.

Using Theorem 1.2 and a variation on [SX12, Xie13b, LDX16] we show:

Theorem 1.5. Conjecture 1.4 holds for nonrigid Carnot groups.

See Theorem 3.1 for a more precise statement. Note that for rigid groups Conjecture 1.4 would follow from the Regularity Conjecture 1.1 and [CO15].

We now give some indication of the proof of Theorem 1.2.

Let $G$ be a nonrigid Carnot group with graded Lie algebra $\mathfrak{g}$. By [OW11b, Theorem 1] (see also [DR10]), the first layer $V^C_1$ of the complexification $\mathfrak{g}^C$ contains a nonzero element $Z$ such that $[Z, \mathfrak{g}^C]$ is a subspace of dimension at most 1. The first phase of the proof of Theorem 1.2 which is implemented in Section 4 is to work out the algebraic implications of this, which leads to the following trichotomy:

(a) $G$ is isomorphic to $\mathbb{R}^n$, $\mathbb{H}_n$, or $\mathbb{H}_n^C$ for some $n \geq 1$.

\footnote{The trichotomy is a variation on unpublished work of the third author [Xie13a].}
(b) There is a linear subspace \(\{0\} \subsetneq W \subset V_1\) which is invariant under graded automorphisms of \(\mathfrak{g}\), such that \([W, V_j] = \{0\}\) for all \(j \geq 2\).

(c) \(G\) is a product quotient: it is of the form \(G = \tilde{G}/\exp(K)\) where \(\tilde{G} = \prod_{j=1}^n \tilde{G}_j\) and the \(\tilde{G}_j\)'s are copies of \(\mathbb{H}_m\) or \(\mathbb{H}_m^C\) for some \(m \geq 1\), and \(K \subset \tilde{\mathfrak{g}}\) is a linear subspace of the second layer \(\tilde{V}_2 \subset \tilde{\mathfrak{g}}\) which satisfies several conditions, see Definition 4.18.

In case (a) assertion (1) of Theorem 1.2 holds, so we are done. In case (b), letting \(\mathfrak{h} \subset \mathfrak{g}\) be the Lie subalgebra generated by \(W\), and \(H := \exp(\mathfrak{h})\), the \(\text{Aut}(\mathfrak{g})\)-invariance of \(W\) implies that the Pansu differential of \(f\) respects the coset foliation of \(H\), and then (2) follows by an integration argument. In case (c), the main obstacle is that there are automorphisms \(\Phi : \tilde{G}/\exp(K) = G \to G = \tilde{G}/\exp(K)\) induced by an automorphism \(\tilde{\Phi} : \tilde{G} \to \tilde{G}\) which permutes the factors of \(\tilde{G} = \prod_j \tilde{G}_j\), and one has to show that the Pansu differentials of \(f:\ B(x, r) \to G\) at different points in \(B(x, r)\) induce the same permutation of the factors. The easiest case is when \(K = \{0\}\), i.e. \(G = \tilde{G}\) is a product, and then the constancy of the permutation was addressed in the product rigidity theorem in [KMX20]. This follows easily by applying the Pullback Theorem from [KMX20] (See Subsection 2.1) to the volume forms coming from the factors of \(\tilde{G}\). The treatment of case (c) in general breaks down into three main subcases depending on the dimension of \(K\) and the type of the factors \(\tilde{G}_j\). The arguments are lengthy and hard to summarize briefly; we refer the reader to Section 5 and to the beginning of Sections 6-8 for more explanation; see also below for suggestions on how to read this paper.

Suggestions for reading.
We suggest reading the paper in the following order:

- The Pullback Theorem, Subsection 2.1
- Subsection 3.1: precise statement of main results, and reduction to results proven later in the paper.
- The statements of the Theorem 4.9 and Lemma 4.17 and the definition of product quotient, Definition 4.18
- Examples 4.20-4.24
- The proof of product rigidity from [KMX20, Subsection 7.1].
- Section 6 on product quotients with \(\text{dim } K = 1\).
- Section 8 on product quotients of complex Heisenberg groups and higher Heisenberg groups. This could also be read after Section 7, but the proof is relatively simple and independent of Sections 6 and 7.
• Lemma 4.27 giving a canonical decomposition of a product quotient in conformal product quotients.
• Section 7 on conformal product quotients with \( \dim K \geq 2 \).

2. Preliminaries

We refer to [KMX20] for most of the preliminaries, which are common to both papers; this includes background as well as notation and conventions. Apart from the pullback theorem, the results collected here are specific to the needs of this paper.

2.1. The Pullback Theorem. We give a concise presentation here, and refer the reader to [KMX20] for more details.

Let \( G \) be a step \( s \) Carnot group with Lie algebra \( \mathfrak{g} \), grading \( \mathfrak{g} = \bigoplus_{j=1}^{s} V_j \), dilation group \( \{ \delta_r : G \to G \}_{r \in (0, \infty)} \), dimension \( N \), and homogeneous dimension \( \nu = \sum_j j \dim V_j \). We let \( \{ X_j \}_{1 \leq j \leq N} \) be a graded basis for \( \mathfrak{g} \), and \( \{ \theta_j \}_{1 \leq j \leq N} \) be the dual basis.

The weight of a subset \( I \subset \{ 1, \ldots, N \} \) is given by

\[
\text{wt} I := -\sum_{i \in I} \deg i
\]

where \( \deg : \{ 1, \ldots, N \} \to \{ 1, \ldots, s \} \) is defined by \( \deg i = j \) iff \( X_i \in V_j \). For a non-zero left-invariant form \( \alpha = \sum_I a_I \theta_I \) we define \( \text{wt}(\alpha) = \max\{ \text{wt} I : a_I \neq 0 \} \) and set \( \text{wt}(0) := -\infty \); here \( \theta_I \) denotes the wedge product \( \Lambda_{i \in I} \theta_i \).

We now let \( G' \) be a second Carnot group, use primes for the associated structure.

Fix \( p > \nu \), and let \( f : U \to G' \) be a \( W^{1,p}_{\text{loc}} \)-mapping for some open subset \( U \subset G \). If \( \omega \in \Omega^k(G') \) is a differential \( k \)-form with continuous coefficients, the Pansu pullback of \( \omega \) is the \( k \)-form with measurable coefficients \( f_P^* \omega \) given by

\[
f_P^* \omega(x) := (D_P f(x))^* \omega(f(x)),
\]

where \( D_P f(x) : \mathfrak{g} \to \mathfrak{g}' \) is the Pansu differential of \( f \) at \( x \in U \).

We will use the following special case of [KMX20] Theorem 4.2:

**Theorem 2.1** (Pullback Theorem (special case)). Suppose \( \varphi \in C_0^\infty(U) \) and that \( \alpha \) and \( \beta \) are closed left invariant forms which satisfy

\[
(2.2) \quad \deg \alpha + \deg \beta = N - 1 \quad \text{and} \quad \text{wt}(\alpha) + \text{wt}(\beta) \leq -\nu + 1.
\]
Then
\begin{equation}
\int_U f_P^*(\alpha) \wedge d(\varphi \beta) = 0.
\end{equation}

2.2. **Exterior algebra and interior products.** Here we recall some basic facts about interior products. This will be useful for constructing closed left invariant forms of low codegree that will be useful in the application of the Pullback Theorem. In this subsection we work purely in the setting of multilinear algebra. Exterior differentiation of forms will be considered in the next subsection.

Given a finite dimensional vector space $V$ over $\mathbb{R}$ we denote the space of $m$-vectors by $\Lambda_m V$ and the space of alternating $m$-linear forms by $\Lambda^m V$. The elements of $\Lambda^m V$ can be identified with linear functionals on $\Lambda_m V$ (or $m$-covectors).

The interior product of a vector $X \in V$ and a $p$-form $\alpha \in \Lambda^p V$ is the $(p-1)$-form $i_X \alpha$ given by
\begin{equation}
i_X \alpha(X_2, \ldots, X_p) = \alpha(X, X_2, \ldots, X_p).
\end{equation}

Using the dual pairing of vectors and covectors this can be rewritten as
\[i_X \alpha(Z) = \alpha(X \wedge Z)\]
for $q \leq p$, $X \in \Lambda_q V$ and $\alpha \in \Lambda^p V$ one defines $i_X \alpha \in \Lambda^{p-q} V$ by
\begin{equation}
(i_X \alpha)(Z) = \alpha(X \wedge Z) \quad \forall Z \in \Lambda_{p-q} V.
\end{equation}

Then for $X \in \Lambda_q$ and $Y \in \Lambda_{q'}$
\[(i_X i_Y \alpha)(Z) = i_Y \alpha(X \wedge Z) = \alpha(Y \wedge X \wedge Z)\]
and thus
\begin{equation}
i_X \circ i_Y = i_{Y \wedge X} \quad \forall X \in \Lambda_q, \ Y \in \Lambda_{q'}.
\end{equation}

The interior product satisfies the graded Leibniz rule
\begin{equation}
i_X (\beta \wedge \gamma) = i_X \beta \wedge \gamma + (-1)^{\deg \beta} \beta \wedge i_X \gamma \quad \forall X \in V.
\end{equation}

If $\omega$ is a top-degree form on $V$, then
\begin{equation}
\alpha \wedge i_X \omega = i_X \alpha \wedge \omega = \alpha(X) \omega, \quad \forall \alpha \in \Lambda^p V, \ X \in \Lambda_p V.
\end{equation}

This can easily be checked using the standard basis and dual basis and verifying the assertion for $X$ and $\beta$ being simple vectors in terms of these basis vectors.

We finally briefly discuss the interior product on quotient spaces. Let $\tilde{V}$ be a finite-dimensional vector space over $\mathbb{R}$, let $K \subset \tilde{V}$ be a subspace and let $V = \tilde{V}/K$. We say that $\beta \in \Lambda^p \tilde{V}$ annihilates $K$ if
\[ \beta(X_1, \ldots, X_p) = 0 \text{ whenever at least one of the } X_i \text{ satisfies } X_i \in K. \]

If \( \tilde{\alpha} \in \Lambda^p \bar{V} \) annihilates \( K \) then it gives rise to a unique form \( \alpha \in \Lambda^p V \) defined by
\[
(\alpha(X_1 + K, \ldots, X_p + K) = \tilde{\alpha}(X_1, \ldots, X_p) \quad \forall X_i \in \bar{V}.)
\]

Conversely for every \( \alpha \in \Lambda^p V \) there exists a unique lift \( \tilde{\alpha} \in \Lambda^p \bar{V} \) which annihilates \( K \) and satisfies (2.9). Consider such \( \alpha \) and \( \tilde{\alpha} \) as well as \( X \in \tilde{V} \). Then
\[
\tilde{i}_{X+K}\alpha = i_X\tilde{\alpha}.
\]

To simplify the notation we sometimes use the notation \( i_X\alpha \) for the form \( i_{X+K}\alpha \) and identify \( i_X\alpha \) and \( i_X\tilde{\alpha} \).

2.3. Exterior derivatives of left invariant forms with low codegree. In this subsection we recall some useful formulae for exterior derivatives of forms on unimodular Lie groups, particularly for low codegree forms defined using interior products of vectors with the volume form. Recall that Carnot groups are unimodular [CG90, Theorem 1.2.10]; we will only use the results in this section in the case of Carnot groups, but for the sake of logical clarity we have formulated them for unimodular groups, because they hold in such generality.

Let \( G \) be a Lie group. We identify the corresponding Lie algebra \( g \) with the space of left invariant vector fields on \( G \), i.e., with the left invariant sections in the tangent bundle \( T G \). Similarly we identify the space of 1-forms \( \Lambda^1 g \) with the space of left invariant sections of the cotangent bundle \( T^*G \) and the spaces of \( p \)-vectors in \( g \) and \( p \)-forms on \( g \) with the spaces of left invariant sections of the corresponding tensor bundles.

For a vector field \( X \) let \( \mathcal{L}_X \) denote the Lie derivative with respect to \( X \). We will use Cartan’s formula
\[
\mathcal{L}_X\alpha = d(i_X\alpha) + i_Xd\alpha
\]
as well as the relation
\[
i_{[X,Y]} = [\mathcal{L}_X, i_Y],
\]
see, e.g., [Mic08, Theorem 9.9].

Lemma 2.12. Let \( G \) be a unimodular group and let \( \omega \) be a bi-invariant volume form on \( G \). Then
\[
d(i_X\omega) = \mathcal{L}_X\omega = 0 \quad \forall X \in g,
\]
\[
d(i_Xi_Y\omega) = i_{[X,Y]}\omega \quad \forall X, Y \in g,
\]
\[
d(i_Xi_Yi_Z\omega) = i_A\omega \quad \forall X, Y, Z \in g, \quad \text{where}
\]
\[
A = X \wedge [Y, Z] + Y \wedge [Z, X] + Z \wedge [X, Y].
\]
Finally if
\[ (2.17) \quad X, X', X'', Z \in g \quad \text{and} \quad [X, Z] = [X', Z] = [X'', Z] = 0 \]
then
\[ (2.18) \quad d(i_Z i_{X''} i_{X'} i_X \omega) = -i_Z d(i_{X''} i_{X'} i_X \omega) \]
\[ \text{Proof.} \quad \text{Since } \omega \text{ has top degree it is closed and thus Cartan’s formula gives} \]
\[ (2.19) \quad d(i_X \omega) = \mathcal{L}_X \omega \quad \forall X \in g. \]
Since the flow of a left invariant vector field is given by right translation and since \( \omega \) is also right invariant we have \( \mathcal{L}_X \omega = 0 \).
To show (2.14) we use Cartan’s formula and (2.13) to get
\[ d(i_X i_Y \omega) = \mathcal{L}_X (i_Y \omega) - i_X d(i_Y \omega) = \mathcal{L}_X (i_Y \omega) = \mathcal{L}_X (i_Y \omega) - i_Y (\mathcal{L}_X \omega) = i_{[X, Y]} \omega. \]
To prove (2.15) we start again from Cartan’s formula
\[ d(i_X i_Y i_Z \omega) = \mathcal{L}_X (i_Y i_Z \omega) - i_X d(i_Y i_Z \omega) = [\mathcal{L}_X, i_Y] i_Z \omega + i_Y \mathcal{L}_X i_Z \omega - i_X i_Y i_Z \omega = i_{[X, Y]} i_Z \omega + i_Y [\mathcal{L}_X, i_Z] \omega + i_Y i_Z \mathcal{L}_X \omega - i_X i_Y i_Z \omega. \]
Now the formula for \( A \) follows from (2.11) and (2.13).
To prove (2.18) we use that \( Z \) commutes with each of \( X'', X', X \). Thus successive application of (2.11) and (2.13) give
\[ \mathcal{L}_Z (i_{X''} i_{X'} i_X \omega) = i_{X''} \mathcal{L}_Z (i_{X'} i_X \omega) = \ldots = i_{X''} i_{X'} i_X \mathcal{L}_Z \omega = 0. \]
Now (2.18) follows from Cartan’s formula (2.10).
\[ \text{□} \]

The exterior derivative of a left invariant form is left invariant since left translation is smooth and hence commutes with exterior differentiation. Thus the exterior derivative induces a linear map on \( \Lambda^* g \) via
\[ (2.20) \quad d\alpha := (d\tilde{\alpha})(e). \]
where \( \tilde{\alpha} \) is the left invariant extension of \( \alpha \).

\[ \textbf{Lemma 2.21.} \quad \text{Let } G \text{ be a Lie group and let } \alpha \in \Lambda^k g. \text{ Then} \]
\[ (2.22) \quad d\alpha(X_0, \ldots, X_k) = \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k). \]
\[ \text{Proof.} \quad \text{See [Mic08, Lemma 14.14].} \quad \text{□} \]
For one forms this reduces to
\begin{equation}
(2.23) \quad d\alpha(X_0, X_1) = -\alpha([X_0, X_1]).
\end{equation}

Let $\Phi \in \text{Aut}(g)$. We define the pullback of $\alpha \in \Lambda^k g$ by $(\Phi^*\alpha)(X_1, \ldots, X_k) := \omega(\Phi(X_1), \ldots, \Phi(X_k))$ for $X_i \in g$. Then (2.22) implies that
\begin{equation}
(2.24) \quad d\Phi^*\omega = \Phi^*d\omega.
\end{equation}

Note that if $\Phi$ is the tangent map of group homomorphism, then the above definition of pullback agrees with the usual one, i.e. one first extends $\alpha \in \Lambda^k g$ to a left invariant form and applies the usual pullback operation.

2.4. Miscellaneous. The following Lemma is the distributional version of the statement: “$g$ constant along a line implies $h \circ g$ constant on this line”.

We recall that the flow $\tau_s$ of a left invariant vector field $X$ is given by right translation by the 1-parameter group generated by $X$, i.e. $\tau_s(p) = p \exp(sX)$ see [Lee09], Thm. 5.74 or [Mic08, 4.17]. We also recall that Carnot groups are unimodular, since they are nilpotent. See, for example, [CG90, Theorem 1.2.10].

**Lemma 2.25.** Let $G$ be a unimodular Lie group with bi-invariant Haar measure $\mu$, $U \subset G$ be an open subset, $g \in L^1_{\text{loc}}(U; \mathbb{R}^l)$. For every $X \in g$, if $\tau := r_{\exp sX}$ denotes 1-parameter group of right translations generated by $X$, then:

1. We have
\begin{equation}
(2.26) \quad \int_U g X\varphi d\mu = 0 \quad \forall \varphi \in C^\infty_c(U)
\end{equation}
if and only if for every compact set $V \subset U$ there exists an $\varepsilon > 0$ such that
\begin{equation}
(2.27) \quad g \circ \tau_s = g \quad \text{a.e. on } V \quad \forall s \in (-\varepsilon, \varepsilon).
\end{equation}

2. Let $h : \mathbb{R}^l \to \mathbb{R}$ be a Borel function with $|h(t)| \leq C_1|t| + C_2$. Then (2.26) implies that
\begin{equation}
(2.28) \quad \int_U (h \circ g) X\varphi d\mu = 0 \quad \forall \varphi \in C^\infty_c(U).
\end{equation}

**Proof.** The main point is that (2.26) is equivalent to
\begin{equation}
(2.29) \quad \int_U g (\varphi \circ \tau_s - \varphi) d\mu = 0 \quad \forall \varphi \in C^\infty_c(U), s \in (-\varepsilon, \varepsilon).
\end{equation}
where \( \varepsilon_\varphi \) depends on the support of \( \varphi \), \( U \) and \( X \). Since \( \mu \) is bi-invariant (2.29) is equivalent to
\[
\int_U (g \circ \tau_s - g) \varphi \, d\mu = 0 \quad \forall \varphi \in C^\infty_c(U), s \in (-\varepsilon_\varphi, \varepsilon_\varphi)
\]
or to \( g \circ \tau_s = g \) a.e. This concludes the proof of the first assertion.

To prove the second assertion note that the condition \( h \) is Borel ensures that \( h \circ g \) is measurable. The growth condition implies that \( h \circ g \in L^1_{\text{loc}}(U) \). Now apply \( h \) to both sides of (2.27) and use the equivalence of (2.27) and (2.26). \( \square \)

Now let \( G, G' \) be Carnot groups, where \( G \) has homogeneous dimension \( \nu \). Let \( \mathfrak{h} \subset \mathfrak{g}, \mathfrak{h}' \subset \mathfrak{g}' \) be Lie subalgebras, where \( \mathfrak{h} \) is horizontally generated (i.e. \( \mathfrak{h} \) is generated by \( \mathfrak{h} \cap V_1 \)), and let \( H := \exp(\mathfrak{h}) \), \( H' := \exp(\mathfrak{h}') \). Note that \( H, H' \) are closed subgroups since the exponential maps are diffeomorphisms.

**Lemma 2.30.** There exists \( \Lambda \in [1, \infty) \) such that if \( x \in G, r \in (0, \infty) \), \( f : B(x, \Lambda r) \to G' \) is a \( W^{1,p}_{\text{loc}} \)-mapping for some \( p > \nu \), and the Pansu differential satisfies \( D_P f(x)(\mathfrak{h}) \subset \mathfrak{h}' \), then for every \( y \in G \) the image of \( yH \cap B(x, r) \) under \( f \) is contained in a single left coset of \( H' \).

**Proof.** This is a slight variation on [Xie13b, Proposition 3.4].

Since \( \mathfrak{h} \) is horizontally generated, it is invariant under the dilation \( \delta_r : G \to G \) for every \( r \in (0, \infty) \). Therefore there is a direct sum decomposition \( \mathfrak{h} = \bigoplus_j V^\mathfrak{h}_j \) where \( V^\mathfrak{h}_j := \mathfrak{h} \cap V_j \), and \( (H, \mathfrak{h}) = \bigoplus_j V^\mathfrak{h}_j \) is a Carnot group. We may equip the first layer \( V^\mathfrak{h}_1 \) with the restriction of the inner product on \( V_1 \), and let \( d^H_{CC} \) be the corresponding Carnot-Caratheodory metric on \( H \).

Note that there is a constant \( C_1 \) such that
\[
d_{CC}(x, x') \leq d^H_{CC}(x, x') \leq C_1 d_{CC}(x, x');
\]
the first inequality follows directly from the definitions, while the second follows for pairs \((x, x') = (e, y)\) with \( d_{CC}(e, y) = 1 \) by compactness, and for general pairs by applying translation and scaling.

Let \( \Lambda := 2C_1\Lambda_0 + 1 \), where \( \Lambda_0 \) is the constant given by Lemma 2.31. Suppose \( x, r, \) and \( f \) are as in the statement of Lemma 2.30 by rescaling and left translation we may assume that \( x = e \) and \( r = 1 \). Arguing as in [Xie13b, Proposition 3.4], for every path \( \gamma : [0, 1] \to B(e, \Lambda) \) with constant horizontal velocity and with image lying in a coset of \( H \), the image \( f(\gamma([0, 1])) \) is contained in a single coset of \( H' \subset G' \).
Choose \( y \in G \), and \( x, x' \in yH \cap B(e, 1) \). Then \( d_{CC}(x, x') < 2 \), so \( d_{CC}^H(x, x') < 2C_1 \). Therefore by Lemma 2.31 for every \( \varepsilon > 0 \) there is a piecewise horizontal path \( \gamma : [0, 1] \to yH \cap B(e, \Lambda) \) with \( \gamma(0) = x \) and \( \gamma(1) \in B(x', \varepsilon) \); since \( f(\gamma([0, 1])) \) lies in a single coset of \( H' \) and \( \varepsilon \) is arbitrary, by continuity it follows that \( x \) and \( x' \) belong to the same coset of \( H' \). □

Lemma 2.31. Let \( H \) be a Carnot group. Then there exists \( \Lambda_0 > 1 \) with the following property.

Let \( \Gamma \) be the collection of paths \( \gamma : [0, 1] \to H \) such that \( \gamma(0) = e \) and \( \gamma' \) is piecewise constant and horizontal, i.e. there exists a partition \( 0 = t_0 < \ldots < t_n = 1 \), and for every \( 1 \leq j \leq n \) elements \( X_j \in V_1 \), \( h_j \in H \), such that

\[
\gamma(t) = h_j \exp((t - t_{j-1})X_j) \quad \text{for all} \quad t \in [t_{j-1}, t_j].
\]

Then the set

\[
\{ \gamma(1) \mid \gamma \in \Gamma \, , \, \gamma([0, 1]) \subset B(e, \Lambda_0) \} \cap B(e, 1)
\]

is dense in \( B(e, 1) \).

Proof. Let \( \hat{H} \) be the closure of the subgroup generated by the collection of horizontal 1-parameter subgroups. Then \( \hat{H} \) is a Lie subgroup, and its Lie algebra \( \mathfrak{h} \) contains \( V_1 \), so \( H = \hat{H} \). It follows that the collection \( \{ \gamma(1) \mid \gamma \in \Gamma \} \) is dense in \( H \).

Now choose a finite collection \( \Gamma_1 = \{ \gamma_1, \ldots, \gamma_n \} \subset \Gamma \) such that the endpoints are \( \frac{1}{2} \)-dense in \( B(e, 1) \), i.e. for every \( x \in B(e, 1) \) we have

\[
d(x, \{ \gamma_j(1) \}_{1 \leq j \leq n}) < \frac{1}{2}.
\]

Then for some \( \Lambda_1 \) and every \( 1 \leq j \leq n \) the image \( \gamma_j([0, 1]) \) is contained in the ball \( B(e, \Lambda_1) \).

Pick \( y \in B(e, 1) \). By induction, there is a sequence \( y_0 = e, y_1, \ldots \) and for every \( j \geq 1 \) an element \( i_j \in \{ 1, \ldots, n \} \) such that \( d(y_j, y) < 2^{-j} \) for all \( j \geq 0 \), and the path \( \ell_{y_j} \circ \delta_{2^{-j}} \circ \gamma_{i_j} \) joins \( y_j \) to \( y_{j+1} \). By concatenating, for every \( j \) we obtain a path \( \gamma \in \Gamma \) where \( \gamma(1) = y_j \) and \( \gamma([0, 1]) \subset B(e, \Lambda_0) \), where \( \Lambda_0 := 2\Lambda_1 \). □

3. Quasisymmetric rigidity of nonrigid Carnot groups

In this section we state our main rigidity results, and prove them using results that will be established later in the paper. We also prove
a localized version of a result of Le Donne-Xie which deduces quasisymmetric rigidity (i.e. automatic improvement to bilipschitz regularity) for mappings that preserve certain types of foliations.

### 3.1. Statement of results and some initial reductions.

**Theorem 3.1** (Quasisymmetric rigidity). Let $G$ be a Carnot group that is nonrigid in the sense of Ottazzi-Warhurst, and assume that $G$ is not isomorphic to $\mathbb{R}^n$ or a real Heisenberg group. Let $f : G \supset U \to U' \subset G$ be an $\eta$-quasisymmetric homeomorphism between open subsets, where $U = B(x, r)$ for some $x \in G$, $r \in (0, \infty)$. Then, modulo post-composition with a Carnot rescaling, the restriction of $f$ to the subball $B(x, \frac{r}{K})$ is a $K$-bilipschitz homeomorphism onto its image, where $K = K(\eta)$.

Consequently, modulo post-composition with a Carnot rescaling every locally $\eta$-quasisymmetric homeomorphism between open subsets of $G$ is locally $K = K(\eta)$-bilipschitz, and any global $\eta$-quasisymmetric homeomorphism onto its image $G \to U' \subset G$ is a (surjective) $K = K(\eta)$-bilipschitz homeomorphism $G \to G$.

**Remark 3.2.** If desired, one may replace $\frac{r}{K}$ with $\frac{r}{2}$ in the statement of Theorem 3.1 at the cost of increasing $K$; this follows readily by combining $\eta$-quasisymmetry with the bilipschitz control on small balls.

We will show that apart from the complex Heisenberg case, quasisymmetric homeomorphisms are “reducible” in the following sense.

**Definition 3.3.** Let $H$ be a closed subgroup of a Lie group $G$, and $f : G \supset U \to U' \subset G$ be a homeomorphism between open subsets. Then $f$ preserves the coset foliation of $H$ if for every $g \in G$ there is a $g' \in G$ such that $f(U \cap gH) = U' \cap g'H$.

**Proof of Theorem 3.1 using Theorem 1.2.** By [KMX20, Theorem 2.12] we know that $f$ is a $W^{1,p}_\text{loc}$-mapping for some $p > \nu$. Also, since $B(x, r)$ is connected, the sign of the determinant of Pansu differential $D_P f(x) : \mathfrak{g} \to \mathfrak{g}$ is constant almost everywhere. Theorem 3.1 now follows from [KMX20, Corollary 8.3] when $G$ is a complex Heisenberg group; otherwise it follows from Theorem 1.2 and Proposition 3.4 below.

---

**Proof of Theorem 1.2.** Let $f : G \supset B(x, r) \to G$ be a $W^{1,p}$-mapping as in the statement of the theorem.
Case 1. There is an $\text{Aut}(g)$-invariant linear subspace $\{0\} \neq W \subset V_1$ such that $[W, V_i] = \{0\}$ for $i \geq 2$. Let $\mathfrak{h}$ be the subalgebra generated by $W$, and $H$ the (closed) subgroup with Lie algebra $\mathfrak{h}$. Then the Pansu differentials of $f$ preserve the tangent space of the coset foliation of $H$, so by Lemma 2.30 the restriction of $f$ to the subball $B(x, \frac{r}{K})$ preserves the cosets of $H$, for $K = K(G)$. So we are done in this case.

Case 2. There is no subspace $W \subset V_1$ as in Case 1. In this situation, by using the fact that $G$ is a nonrigid group and analyzing the algebraic implications, we are able to deduce that $G$ has a very special structure. Specifically, we may apply Theorem 4.9 so conclusions (1)-(5) hold; moreover, since $G$ is not a complex Heisenberg group, we have $n \geq 2$. This implies that $G$ is a product quotient $\tilde{G}/\exp(K)$ where $\tilde{G}$ has at least two factors, see Definition 4.18, Lemma 4.17, and Remark 4.19.

Let $W = V_{1,i}$ for some $1 \leq i \leq n$, and let $H$ be the subgroup generated by $W$; note that since $G$ is a step 2 Carnot group, the Lie algebra $\mathfrak{h}$ of $H$ satisfies $[\mathfrak{h}, V_i] = \{0\}$ for all $i \geq 2$. We may apply Theorem 5.1 and Lemma 2.30 to obtain that the restriction of $f$ to the subball $B(p, \frac{r}{K})$ preserves the cosets of $H$, after possibly post-composing with an automorphism which permutes the subalgebras $g_1, \ldots, g_n$; this theorem is proved by means of a subtle analysis based on the pullback theorem (Sections 5-7). Here we are using the fact that the permutation of the subalgebras $g_i$ induced by the Pansu differential is locally constant (by Theorem 5.1), and therefore and therefore constant since the ball $B(p, r)$ is connected.

3.2. Promoting locally quasisymmetric homeomorphisms to locally bilipschitz homeomorphisms. We recall that it was shown in [LDX16] (see also [SX12, Xie13b]) that quasisymmetric homeomorphisms are necessarily bilipschitz provided they preserve certain types of foliations. The purpose of this subsection is to prove a localized version of this assertion in certain cases, which include the ones arising in this paper. These results may be viewed as refinements of the elementary observation that if $f_1 : \mathbb{R}^k \to \mathbb{R}^k$, $f_2 : \mathbb{R}^l \to \mathbb{R}^l$ are quasisymmetric homeomorphisms, then the product

$$f = (f_1, f_2) : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k \times \mathbb{R}^l$$

is quasisymmetric only if $f_1$ and $f_2$ (and hence also $f$) are bilipschitz. The rough idea of [SX12, Xie13b, LDX16], in the simplest case (as above) where the leaves of the foliation are parallel, is that for any
Proposition 3.4. Let $G$ be a Carnot group with graded Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^{r} V_i$, and let $W \subset V_1$ be a subspace of the first stratum. Assume that $\{0\} \neq W \subsetneq V_1$ and

$$[W, V_i] = \{0\} \quad \forall i \geq 2. \quad (3.5)$$

Let $\mathfrak{h}$ be the subalgebra generated by $W$ and let $H$ be the subgroup with Lie algebra $\mathfrak{h}$. Then there is an $L = L(\eta)$ such that any $\eta$-quasisymmetric homeomorphism

$$f : G \ni B(x, r) \rightarrow U \subset G$$

which respects the coset foliation of $H$ (see Definition 3.3) is, modulo postcomposition with a suitable Carnot rescaling, $L$-bilipschitz on the subball $B(x, L^{-1} r)$.

The following result and its proof were inspired by Lemma 4.4 in [LDX16] where it is shown that cosets are either parallel or diverge at $\infty$.

Proposition 3.6. Let $\mathfrak{h}$, $\mathfrak{g}$, $H$ and $G$ be as in Proposition 3.4. Then there exists a constant $c > 0$ with the following property. If

$$x_0, y \in G, \quad x_0^{-1} y \notin H \quad \text{and} \quad X \in W \quad (3.7)$$

then at least one of the following holds:

$$d(x_0 \exp(tX), yH) \geq cd(x_0, yH) \quad \forall t \geq 0 \quad (3.8)$$

or

$$d(x_0 \exp(tX), yH) \geq cd(x_0, yH) \quad \forall t \leq 0. \quad (3.9)$$

Replacing $y$ by $x_0^{-1} y$ we see that it suffices to prove the proposition for $x_0 = e$. The heart of the matter is to estimate the distance from $\exp(tX)^{-1} y^{-1} \exp(tX)$ to $H$ from below. The key calculation is contained in the following proposition. Recall that the exponential map $\exp : \mathfrak{g} \rightarrow G$ is a diffeomorphism, so we may pull back the group operation to the algebra. We denote by $*$ the resulting operation, i.e. $\exp(A \ast B) = \exp A \exp B$. For $A \in \mathfrak{g}$ we denote by $A_i$ the projection to the $i$-th stratum.

Lemma 3.10. Let $\mathfrak{h}$ be as in Proposition 3.6. Define

$$N(A, B) := A \ast B - (A + B).$$
Then
\[(3.11) \quad [[W, W], g] = \{0\} \quad \text{and hence} \quad h = W \oplus [W, W].\]

Moreover,
\[(3.12) \quad [X, \oplus_{i \geq 2} V_i] = \{0\} \quad \implies \quad [[X, g], \oplus_{i \geq 2} V_i] = \{0\},\]
and for all \(A, A' \in h\) and \(B \in g\)
\[(3.13) \quad (-A) * B * A = B - [A_1, B_1],\]
\[(3.14) \quad N(A', B) = N(A'_1, B_1),\]
\[(3.15) \quad A' * (-A) * B * A = A' + B - [A_1, B_1] + N(A'_1, B_1).\]

**Proof.** The identities in (3.11) and (3.12) follow from the Jacobi identity. To prove (3.13) recall that \((-A) * B * A = e^{-ad A} B\) and use that \((ad A)^2 = 0\) since \([A, \oplus_{i \geq 2} V_i] = 0\). The Baker-Campbell-Hausdorff formula implies that \(N(A', B)\) is a linear combination of iterated commutators of \(A'\) and \(B\). Since \([A', \oplus_{i \geq 2} V_i] = 0\) only commutators of the form \((ad B)^j A'_1\) appear. Now (3.11) and inductive application of (3.12) yield \((ad B)^j A' = (ad B)^j A'_1 = (ad B_1)^j A'_1\). This implies (3.14). Finally (3.15) follows by applying (3.14) with \(B\) replaced with \((-A) * B * A\) and using (3.13). \(\square\)

Recall that we may define a quasinorm on \(g\) by
\[
\|X\| := \sum_j \|X_j\|_0^{\frac{1}{2}},
\]
where \(\| \cdot \|_0\) denotes some fixed Euclidean norm on \(g\), and \(X = \sum_j X_j\) is the layer decomposition of \(X\). The distance \(d\) on \(G\) is equivalent to the quasinorm, i.e. \(C^{-1}\|A\| \leq d(exp A, e) \leq C\|A\|\), see [Hei95 2.15 and 2.19] or [HK00 Proposition 11.15].

**Proof of Proposition 3.6.** We may assume \(x_0 = e\). Using a good representative of \(yH\) and a suitable dilation we may further assume that \(d(e, y) = d(e, yH) = 1\). Let \(x(t) = \exp(tX)\). Then
\[
(3.16) \quad d(x(t), yH) = d(y^{-1}x(t), H) = d(x(t)^{-1}y^{-1}x(t), H) = \min_{x' \in H} d(x'^{-1}x(t)^{-1}y^{-1}x(t), e).
\]
Let \(y = \exp Y\) and \(x' = \exp X'\). Then (3.15) implies that
\[
(3.17) \quad \exp^{-1}(x'^{-1}x(t)^{-1}y^{-1}x(t)) = (-X') * (-tX) * (-Y) * (tX) = -Y - X' + t[X, Y_1] + N(-X'_1, -Y_1).
\]
Thus, if the assertion is false then there exist
\[ Y_n \in \mathfrak{g}, \quad X_n \in W, \quad s_n > 0, \quad t_n > 0, \quad X_n^\pm \in \mathfrak{h} \]
such that
\begin{equation}
(3.18) \quad d(e, y_n) = d(e, y_nH) = 1, \quad y_n = \exp Y_n
\end{equation}
and
\begin{align*}
(3.19) \quad &\| -Y_n - X_n^+ + s_n[X_n, Y_{n,1}] + N(-X_{n,1}^+, -Y_{n,1})\| \to 0, \\
(3.20) \quad &\| -Y_n - X_n^- - t_n[X_n, Y_{n,1}] + N(-X_{n,1}^-, -Y_{n,1})\| \to 0.
\end{align*}
Changing the sign of \( X_n \) if needed we may assume that \( \lambda_n := t_n/s_n \leq 1. \)
By assumption \( d(\exp Y_n, e) = 1. \) Hence \( \|Y_n\| \) is bounded. Passing to subsequences we may assume that
\begin{equation}
Y_n \to \bar{Y}, \quad \lambda_n = t_n/s_n \to \lambda \in [0, 1].
\end{equation}
Using (3.19) and (3.20) we get for the first layer quantities
\begin{equation}
X_{n,1}^\pm \to -\bar{Y}_1 \quad \text{and} \quad \bar{Y}_1 \in \mathfrak{h}.
\end{equation}
Hence
\begin{equation}
N(-X_{n,1}^\pm, -Y_{n,1}) \to N(\bar{Y}_1, -\bar{Y}_1) = 0.
\end{equation}
Note that \( X_n \in V_1 \) and thus \( [X_n, Y_{n,1}] \in V_2. \) Moreover \( X_n^\pm \in \mathfrak{h} \subset V_1 \oplus V_2. \) Thus (3.19) and (3.22) imply
\begin{equation}
\bar{Y}_i = 0 \quad \forall i \geq 3.
\end{equation}
It remains to analyze the behaviour in the second stratum. Set \( a_n = s_n[X_n, Y_{n,1}] = s_n[X_{n,1}, Y_{n,1}] \). Equations (3.19), (3.20) and (3.22) and the definition \( \lambda_n = t_n/s_n \) imply that
\begin{equation}
(3.24) \quad -Y_{n,2} - X_{n,2}^+ + a_n \to 0, \quad -Y_{n,2} - X_{n,2}^- - \lambda_n a_n \to 0.
\end{equation}
Let \( P^+: V_2 \to V_2 \) denote the orthogonal projection onto the orthogonal complement of \( \mathfrak{h} \cap V_2 = [W, W]. \) Then we get
\[ P^+ a_n \to P^+ \bar{Y}_2, \quad \lambda_n P^+ a_n \to -P^+ \bar{Y}_2. \]
Since \( \lambda_n \to \lambda \in [0, 1] \) we deduce that \( \lambda P^+ \bar{Y}_2 = -P^+ \bar{Y}_2 \) and hence \( P^+ \bar{Y}_2 = 0. \) Together with (3.21) and (3.23) it follows that \( Y_n \to \bar{Y} \in \mathfrak{h} \) and hence \( (y_n)^{-1} = \exp(-Y_n) \to \exp(-\bar{Y}) \in H. \) This contradicts the assumption \( d((y_n)^{-1}, H) = d(e, y_nH) = 1. \) \( \square \)
**Proof of Proposition 3.4.** It suffices to consider the case when $x = e$, $r = 1$, $f(e) = e$, and the image $f(B(e, 1))$ has diameter 1. We will show that for $L = L(\eta)$, and every $x, y \in B(e, L^{-1})$, we have $d(f(x), f(y)) \geq L^{-1}d(x, y)$. The lemma then follows by applying this estimate to the inverse homeomorphism, and adjusting $L$.

To prove the assertion by contradiction, assume that there is a sequence $\{f_j : B(e, 1) \to U \subset G\}$ of $\eta$-quasisymmetric homeomorphisms with $f(e) = e$, diam$(f_j(B(e, 1))) = 1$, and there are sequences $\{x_j\}, \{y_j\} \subset B(e, j^{-1})$ with $d(f_j(x_j), f_j(y_j)) \leq j^{-1}d(x_j, y_j)$. By $\eta$-quasisymmetry and our normalization, it follows that $r_j := d(x_j, y_j) \to 0$ and $d(f_j(x_j), e) \to 0$. Let $K = \{g \in G : g^{-1}hg \in H \forall h \in H\}$ be the normalizer of $H$ in $G$ and let $\rho_j = d(f_j(x_j), f_j(y_j))$.

**Claim.** There exist $z_j$ such that

\[
d(x_j, z_jH) = r_j, \quad d(f_j(x_j), f_j(z_j)H) \leq C_1\rho_j
\]

where $C_1 = \eta(1)$ and

\[
f_j(x_j)^{-1}f_j(z_j) \in K.
\]

Note that (3.26) implies that the cosets $f_j(x_j)H$ and $f_j(z_j)H$ are parallel and in particular

\[
d(f_j(x_j)h, f_j(z_j)H) = d(f_j(x_j), f_j(z_j)H)
\]

\[
\leq C_1\rho_j \leq C_1j^{-1}r_j \quad \forall h \in H.
\]

**Proof of claim.** We first note that $K \neq H$; this holds more generally see, e.g., [LDX16, Lemma 4.2], however in our case one can see this as follows. By equation (3.5) from the hypotheses, the normalizer contains $\bigoplus_{i \geq 2}V_i$, so we are done unless $\mathfrak{h}$ contains $\bigoplus_{i \geq 2}V_i$, in which case $\mathfrak{h}$ is an ideal, so its normalizer is all of $\mathfrak{g}$.

To see that such a $z_j$ exists, note that there exists $\zeta \in K$ such that $d(\zeta, H) = d(\zeta, e) \neq 0$. Applying a Carnot dilation we may assume in addition that $d(\zeta, e) = 1$. For $s > 0$ let $\delta_s$ denote the Carnot dilation by $s$ and define

\[
g(s) := d(x_j, f_j^{-1}(f_j(x_j)\delta_s\zeta)H).
\]

If there exists an $s_j \in [0, C_1\rho_j]$ such that $g(s_j) = r_j$ then $z_j = f_j^{-1}(f_j(x_j)\delta_{s_j}\zeta)$ has the desired properties as $\delta_s(K) = K$. Now $g$ is continuous, $g(0) = 0$ and so it suffices to show that $g(C_1\rho_j) \geq r_j$. Assume $g(C_1\rho_j) < r_j$ and let $s = C_1\rho_j$. Then there exist $\tilde{z}_j \in f_j^{-1}(f_j(x_j)\delta_s\zeta)H$ such that $d(x_j, \tilde{z}_j) < r_j$. By quasisymmetry we get a contradiction: $d(x_j, \tilde{z}_j) < r_j = d(x_j, y_j)$, so $s = d(f_j(x_j), f_j(\tilde{z}_j)H) \leq d(f_j(x_j), f_j(\tilde{z}_j)) < C_1d(f_j(x_j), f_j(y_j)) = s$. 

\[
\]
Here we are using the fact that the distortion function \( \eta : [0, \infty) \to [0, \infty) \) is chosen as strictly increasing (i.e. a homeomorphism). □

Let \( X \in h \cap V_1 \) with \( \|X\| = 1 \). After passing to a subsequence and replacing \( X \) by \(-X\) if necessary, by Proposition 3.6 we may assume that

\[
d(x_j \exp(tX), z_j H) \geq cd(x_j, z_j H) = cr_j \quad \forall t \in [0, \infty)
\]

where \( c = c(H, G) \). Let \( t \in [0, \frac{1}{2}] \) and let \( x'_j = x_j \exp(tX) \). Then \( x'_j \in B(e, \frac{3}{4}) \) (if \( j > 4 \)). Note that since the open ball \( B(x'_j, cr_j) \) is disjoint from the coset \( z_j H \), the image \( f_j(B(x'_j, cr_j)) \) is disjoint from \( f_j(z_j H \cap B(e, 1)) \), which has distance at most \( C_1j^{-1}r_j \) from \( f_j(x'_j) \).

It follows that \( \text{diam}(f_j(B(x'_j, cr_j))) \leq C_2j^{-1}r_j \) for \( C_2 = C_2(\eta) \). For large \( j \), we may find a finite sequence of points \( x_j = x_{j,1}, \ldots, x_{j,N_j} \) of the form \( x_{j,i} = x_j \exp(t_i X) \) with \( t_i \in [0, \frac{1}{2}] \), where \( N_j \leq c^{-1}r_j^{-1} \), \( d(x_{j,k}, x_{j,k+1}) \leq cr_j \), and \( t_{N_j} = \frac{1}{2} \) so that

\[
d(e, x_{j,N_j}) \geq d(x_j, x_j \exp \frac{1}{2}X) - j^{-1} \geq \frac{1}{4}.
\]

Here we have used the fact that \( d(e, \exp \frac{1}{2}X) = \frac{1}{2} \) for \( X \in V_1 \) and \( \|X\| = 1 \), which follows from the definition of the Carnot distance. It follows that

\[
d(f_j(x_j), f_j(x_{j,N_j})) \leq \sum_k d(f_j(x_{j,k}), f_j(x_{j,k+1}))
\leq c^{-1}r_j^{-1} \cdot C_2j^{-1}r_j \leq c^{-1}C_2j^{-1} \to 0
\]
as \( j \to \infty \). Thus \( d(e, f_j(x_{j,N_j})) \to 0 \). Since \( d(e, x_{j,N_j}) \geq \frac{1}{4} \) (for large \( j \)) we have \( f_j(B(e, 1)) \subset B(e, \eta(4)d(e, f_j(x_{j,N_j}))) \). This contradicts our normalization \( \text{diam}(f_j(B(e, 1))) = 1 \).

□

4. Structure of nonrigid Carnot groups

In this section we analyze the algebraic structure of graded Lie algebras corresponding to Carnot groups which are nonrigid in the sense of Ottazzi-Warhurst. Starting from the fact that the first layer must contain nontrivial elements with rank at most 2 [OW11b] (see also [DR10]), we establish a trichotomy: either the first layer contains a special type of automorphism invariant subspace, or the Carnot group is \( \mathbb{R}^n \) or a real Heisenberg group, or the Carnot group must have a very special structure – it must be a quotient of a product of real or
complex Heisenberg groups by a specific type of subgroup (see Theorem 4.9). This result is a variation of an unpublished classification theorem of the third author [Xie13a]. In the concluding section we show that such product quotients admit a canonical graded product decomposition into factors whose second layers admit automorphism invariant conformal structures.

After some preliminaries, the structure theorem (Theorem 4.9) will be given in Subsection 4.2. We then discuss the characterization in terms of quotients in Subsection 4.3, and in Subsection 4.4 we show that product quotients have a graded direct sum decomposition into conformal product quotients.

4.1. Preliminaries.

In what follows, $\mathbb{F}$ will always be either $\mathbb{R}$ or $\mathbb{C}$, and if $X$ is an element of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, then $\text{rank}_{\mathbb{F}} X := \text{rank}_{\mathbb{F}} \text{ad}_X = \dim_{\mathbb{F}} [X, \mathfrak{g}]$. Given an $\mathbb{F}$-linear subspace $W \subset \mathfrak{g}$, and $r \geq 0$, we let $\text{rank}_{\mathbb{F}}(r, W)$ and $\text{rank}_{\mathbb{F}}(\leq r, W)$ be the collections of elements $X \in W$ with $\text{rank}_{\mathbb{F}} X = r$, and $\text{rank}_{\mathbb{F}} X \leq r$, respectively. Although the field $\mathbb{F}$ is implicit, and therefore strictly speaking the subscript is redundant, in what follows we will sometimes have algebras over $\mathbb{R}$ and $\mathbb{C}$ in the same context, and so prefer to have the subscript to eliminate any potential ambiguity.

We denote the complexification $V \otimes \mathbb{C}$ of an $\mathbb{R}$-vector space by $V^\mathbb{C}$.

Lemma 4.1 (Complexification of complex Lie algebras). Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, with the complex multiplication denoted by $J$; we let $\bar{\mathfrak{g}}$ be the Lie algebra over $\mathbb{C}$ with the same underlying Lie algebra over $\mathbb{R}$, but with complex multiplication given by $-J$. Viewing $\mathfrak{g}$ as a Lie algebra over $\mathbb{R}$, we let $\mathfrak{g}^\mathbb{C} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of $\mathfrak{g}$, considered as a vector space over $\mathbb{C}$ with complex multiplication denoted by $i$. We denote by $J^\mathbb{C} := J \otimes \text{id}_{\mathbb{C}} : \mathfrak{g}^\mathbb{C} \to \mathfrak{g}^\mathbb{C}$ the map induced by $J$.

Then:

1. $\mathfrak{g}^\mathbb{C}$ decomposes as a direct sum $\mathfrak{g}^\mathbb{C} = \mathfrak{g}_{+}^\mathbb{C} \oplus \mathfrak{g}_{-}^\mathbb{C}$, of eigenspaces of $J^\mathbb{C}$, which are interchanged by complex conjugation.
2. $\mathfrak{g}_{+}^\mathbb{C}$ and $\mathfrak{g}_{-}^\mathbb{C}$ are Lie subalgebras isomorphic over $\mathbb{C}$ to $\mathfrak{g}$ and $\bar{\mathfrak{g}}$, respectively; the isomorphisms are given by taking twice the real part.
3. $[\mathfrak{g}_{+}^\mathbb{C}, \mathfrak{g}_{-}^\mathbb{C}] = \{0\}$.

Proof. Note that $V_{\pm i} := \{X \neq iJX \mid X \in \mathfrak{g}\} \subset \mathfrak{g}^\mathbb{C}$.
is a \( C \)-linear subspace contained in the \( \pm i \) eigenspace of \( J^C \), and complex conjugation interchanges \( V_i \) and \( V_{-i} \). Since we have \( \mathfrak{g}^C = V_i \oplus V_{-i} \), (1) follows.

If \( Z \in \mathfrak{g}^C_{\pm i} \), \( Z' \in \mathfrak{g}^C \), then
\[
J^C[Z, Z'] = [J^C Z, Z'] = [\pm i Z, Z'] = \pm i [Z, Z'],
\]
so \( [Z, Z'] \in \mathfrak{g}^C_{\pm i} \). Hence \( \mathfrak{g}^C_{\pm i} \) is an ideal, assertion (3) holds, and we have a direct sum decomposition \( \mathfrak{g}^C = \mathfrak{g}^C_i \oplus \mathfrak{g}^C_{-i} \). Taking real parts gives (up to a factor) the isomorphisms \( \mathfrak{g}^C_i \simeq \mathfrak{g}, \mathfrak{g}^C_{-i} \simeq \bar{\mathfrak{g}} \). \( \square \)

Remark 4.2. If in Lemma 4.1 the graded algebra \( \mathfrak{g} \) itself happens to be the complexification of some graded algebra over \( \mathbb{R} \) – for instance if \( \mathfrak{g} \) is a complex Heisenberg algebra – then there is a \( C \)-linear graded isomorphism \( \mathfrak{g} \to \bar{\mathfrak{g}} \) given by complex conjugation. Therefore in this case all four graded algebras \( \mathfrak{g}, \bar{\mathfrak{g}}, \mathfrak{g}^C_i, \mathfrak{g}^C_{-i} \) are graded isomorphic over \( \mathbb{C} \).

Lemma 4.3. Suppose \((\mathfrak{g}, \{V_j\}_{j=1}^s)\) is a graded Lie algebra over a field \( \mathbb{F} \). If \( X \in V_1 \) and \( \dim_{\mathbb{F}}[X, \mathfrak{g}] \leq 1 \), then \([X, V_j] = \{0\}\) for all \( j \geq 2 \).

Proof. Suppose \([X, V_1] = \{0\}\). Since the centralizer of an element is a subalgebra and \( V_1 \) generates \( \mathfrak{g} \), we get \([X, \mathfrak{g}] = \{0\}\) and we're done in this case. Now suppose \([X, V_1] \neq \{0\}\). Since \( \dim_{\mathbb{F}}[X, \mathfrak{g}] \leq 1 \) we deduce that \([X, \mathfrak{g}] = [X, V_1] \subset V_2 \), and for all \( j \geq 2 \) we have \([X, V_j] \subset V_{j+1} \cap V_2 = \{0\}\). \( \square \)

Lemma 4.4. Suppose \((\mathfrak{g}, \{V_j\}_{j=1}^2)\) is a step 2 graded Lie algebra over a field \( \mathbb{F} \), such that \( \dim_{\mathbb{F}} V_2 = 1 \) and the center of \( \mathfrak{g} \) intersects \( V_1 \) trivially:

\[
\{ X \in V_1 | [X, \mathfrak{g}] = \{0\}\} = \{0\}.
\]

Then \( \mathfrak{g} \) is graded isomorphic to a Heisenberg algebra over \( \mathbb{F} \).

Proof. Identifying \( V_2 \) with a copy of \( \mathbb{F} \) by an \( \mathbb{F} \)-linear isomorphism, the Lie bracket defines a skew-symmetric bilinear form \( [\cdot, \cdot] : V_1 \times V_1 \to \mathbb{F} \), which is nondegenerate. Such forms are \( \mathbb{F} \)-linearly equivalent to a direct sum of standard 2-dimensional symplectic forms by a straightforward induction argument (see for example [Jac85, p.349]). This yields the isomorphism to the Heisenberg algebra. \( \square \)
Lemma 4.5. For $F \in \{\mathbb{R}, \mathbb{C}\}$, let $\text{Sym}(m, F)$ be the symplectic group, i.e. the stabilizer of the standard non-degenerate skew form on $\mathbb{F}^{2m}$. For $n \geq 1$, consider an $\mathbb{R}$-linear subspace $K \subset (\mathbb{F}^{2m})^n$ which is invariant under the product action $(\text{Sym}(m, F))^n \cap (\mathbb{F}^{2m})^n$. Then for some subset $J \subset \{1, \ldots, n\}$, the subspace $K$ is the span of the factors indexed by $J$, i.e.

$$K = \{(x_1, \ldots, x_n) \in (\mathbb{F}^{2m})^n \mid x_k = 0 \text{ if } k \notin J\}.$$ 

Proof. Suppose for some $j \in \{1, \ldots, n\}$ we have $x_j \neq 0$ for some $x \in K$. Choosing $T \in \text{Sym}(m, F)$ such that $Tx_j \neq x_j$, we get that $y \in K$, where $y_k = x_k$ if $k \neq j$, and $y_j = Tx_j$. Then $y - x \in K$ has precisely one nonzero component, namely $Tx_j - x_j$. Applying $\text{Sym}(m, F)$ and taking the linear span over $\mathbb{R}$, we get that the $j^{th}$ factor of $(\mathbb{F}^{2m})^n$ is contained in $K$.

Applying the above to every $j \in \{1, \ldots, n\}$, the lemma follows. \hfill $\Box$

Lemma 4.6. Let $h_m$ be the $m^{th}$ Heisenberg algebra, and $h_m^\mathbb{C}$ be the complexification, i.e. the $m^{th}$ complex Heisenberg algebra. Let $\text{Aut}_\mathbb{R}(h_m^\mathbb{C})$ and $\text{Aut}_\mathbb{C}(h_m^\mathbb{C})$ denote the groups of graded $\mathbb{R}$-linear and $\mathbb{C}$-linear automorphisms, respectively. Then $\text{Aut}_\mathbb{R}(h_m^\mathbb{C})$ is generated by $\text{Aut}_\mathbb{C}(h_m^\mathbb{C})$ and complex conjugation. In particular, if $\phi : h_m^\mathbb{C} \to h_m^\mathbb{C}$ is a graded $\mathbb{R}$-linear automorphism, then either:

- $\phi$ is $\mathbb{C}$-linear and $\det_{\mathbb{R}} \phi \big|_{V_2} > 0$.
- $\phi$ is $\mathbb{C}$-antilinear and $\det_{\mathbb{R}} \phi \big|_{V_2} < 0$.

Proof. The $m = 1$ case appears in [RR00, Section 6], although their argument clearly works for general $m$.

We give a short argument here.

Let $h := h_m^\mathbb{C}$, and let $g = V_1 \oplus V_2$ be the grading. Viewing this as a graded algebra over $\mathbb{R}$, we use Lemma 4.1 to see that $g^\mathbb{C} = g^\mathbb{C}_i \oplus g^\mathbb{C}_{-i}$, and the first layer elements of $\mathbb{C}$-rank $\leq 1$ in $g^\mathbb{C}$ are precisely $V_1^\mathbb{C} \cap (g^\mathbb{C}_i \cup g^\mathbb{C}_{-i})$.

If $\phi \in \text{Aut}_\mathbb{R}(g)$, then $\phi^\mathbb{C} : g^\mathbb{C} \to g^\mathbb{C}$ must preserve the subset $V_1^\mathbb{C} \cap (g^\mathbb{C}_i \cup g^\mathbb{C}_{-i})$. Since $\phi^\mathbb{C}$ is a $\mathbb{C}$-linear automorphism it must preserve the collection $\{g^\mathbb{C}_\pm i\}$, so we have two cases: either $\phi^\mathbb{C}$ maps $g^\mathbb{C}_{\pm i}$ to itself, or it interchanges the two. In the first case $\phi^\mathbb{C}$ induces a $\mathbb{C}$-linear map from $g^\mathbb{C}_i$ to itself. Since the real part map $\Re : g^\mathbb{C}_i \to g$ is $\mathbb{C}$-linear it follows that $\phi$ is $\mathbb{C}$-linear on $g$. In the other case $\Re : g^\mathbb{C}_{-i} \to g$ is $\mathbb{C}$-linear, so $\phi : g \to g$ is $\mathbb{C}$-linear, which means that $\phi : g \to g$ is $\mathbb{C}$-antilinear.

Finally note that an invertible $\mathbb{C}$-linear map on the real two-dimensional space $V_2$ has positive determinant and an invertible $\mathbb{C}$-antilinear map has negative determinant. \hfill $\Box$
We now give a generalization of Lemma 4.6.

Let \( g = \bigoplus_{j=-1}^s V_j \) be a Carnot Lie algebra over \( \mathbb{C} \) and \( I \subset \{1, \cdots, s\} \).

Set \( V_I = V_{I,g} := \bigoplus_{j \in I} V_j \). For \( X \in g \), let \( \text{rank}_{I,g}(X) = \dim_{\mathbb{C}}(\text{ad}_X(V_I)) \).

Define \( r_I = r_{I,g} := \min\{\text{rank}_{I,g}(X) | X \in V_I \setminus \{0\} \} \) and \( R_I = R_{I,g} := \max\{\text{rank}_{I,g}(X) | X \in V_I\} \).

**Lemma 4.7.** Let \( g \) be the complexification of a Carnot Lie algebra. Assume there is some \( I \subset \{1, \cdots, s\} \) such that \( R_I < 2r_I \). Then every graded \( \mathbb{R} \)-linear automorphism of \( g \) is either \( \mathbb{C} \)-linear or \( \mathbb{C} \)-antilinear.

**Proof.** By Lemma 4.1, the complexification \( g^\mathbb{C} \) of \( g \) admits a decomposition \( g^\mathbb{C} = g^\mathbb{C}_I + g^\mathbb{C}_{I'} \) into a direct sum of two graded subalgebras over \( \mathbb{C} \) and \([g^\mathbb{C}_I, g^\mathbb{C}_{I'}] = 0\). Let \( \phi \in \text{Aut}_\mathbb{R}(g) \). Then \( \phi^\mathbb{C} : g^\mathbb{C} \to g^\mathbb{C} \) is a graded \( \mathbb{C} \)-linear automorphism. We shall show that \( \phi^\mathbb{C} \) either maps \( g^\mathbb{C}_I \) to itself or switches the two. The lemma then follows by the argument in the proof of Lemma 4.6.

Let \( 0 \neq X \in V_{1,g^\mathbb{C}_I} \cup V_{1,g^\mathbb{C}_{I'}} \). Since both \( g^\mathbb{C}_I \) and \( g^\mathbb{C}_{I'} \) are isomorphic to \( g \) and \([g^\mathbb{C}_I, g^\mathbb{C}_{I'}] = 0\), we have \( \text{rank}_{I,g^\mathbb{C}}(X) \leq R_I < 2r_I \). On the other hand, we have \( \text{rank}_{I,g^\mathbb{C}}(X) \geq 2r_I \) for any \( X \in V_{1,g^\mathbb{C}} \setminus (V_{1,g^\mathbb{C}_I} \cup V_{1,g^\mathbb{C}_{I'}}) \). To see this, we write \( X = X_1 + X_2 \) with \( 0 \neq X_1 \in V_{1,g^\mathbb{C}_I}, 0 \neq X_2 \in V_{1,g^\mathbb{C}_{I'}} \).

Then
\[
\text{ad}_X(V_{I,g^\mathbb{C}}) = \text{ad}_{X_1}(V_{I,g^\mathbb{C}_I}) \oplus \text{ad}_{X_2}(V_{I,g^\mathbb{C}_{I'}})
\]
and so
\[
\text{rank}_{I,g^\mathbb{C}}(X) = \text{rank}_{I,g^\mathbb{C}_I}(X_1) + \text{rank}_{I,g^\mathbb{C}_{I'}}(X_2) \geq 2r_I.
\]
Here we used the facts that \( g^\mathbb{C}_I \) and \( g^\mathbb{C}_{I'} \) are isomorphic to \( g \) and \([g^\mathbb{C}_I, g^\mathbb{C}_{I'}] = 0\). Since for any graded \( \mathbb{C} \)-linear isomorphism \( f : g_1 \to g_2 \) of complex Carnot algebras we have \( \text{rank}_{I,g_1}(X) = \text{rank}_{I,g_2}(f(X)) \), we see that \( \phi^\mathbb{C} \) maps \( V_{1,g^\mathbb{C}_I} \cup V_{1,g^\mathbb{C}_{I'}} \) to itself. Since \( \phi^\mathbb{C} \) is \( \mathbb{C} \)-linear, \( \phi^\mathbb{C} \) either maps \( g^\mathbb{C}_I \) to itself or switches the two. \( \square \)

The assumption in Lemma 4.7 is satisfied for the complexification \( g \) of the following classes of Carnot algebras:

(1) model filiform algebras; in this case, \( R_{\{1\}} = r_{\{1\}} = 1 \).

(2) free nilpotent Lie algebras; more generally, quotients of free nilpotent Lie algebras by graded ideals contained in the direct sum of higher layers \( V_j, j \geq 3 \); in this case, \( R_{\{1\}} = r_{\{1\}} > 0 \).

(3) Carnot algebras \( h = \bigoplus_j V_j \) satisfying \([X, V_i] = V_{i+1}\) for all \( 0 \neq X \in V_1 \) and some fixed \( i \geq 1 \) with \( \dim(V_{i+1}) \) odd. These include nilpotent Lie algebras satisfying Métivier’s hypothesis (H) (in particular, \( H \) type algebras) whose centers have odd dimension. To see that \( g := h^\mathbb{C} = \bigoplus_j V_j^\mathbb{C} \) satisfies the assumption of Lemma 4.7, we denote
dim$_\mathbb{R}(V_{i+1}) = 2k + 1$. Then dim$_\mathbb{C}(V_{i+1}^C) = 2k + 1$ and so $R_{(i)} \leq 2k + 1$. We show that rank$_{(i)}g(X) \geq k + 1$ for any $0 \neq X \in V_{i+1}^C$, which implies $r_{(i)} \geq k + 1$ and so $R_{(i)} < 2r_{(i)}$. Write $X = X_1 + iX_2$ with $X_1, X_2 \in V_1$. We may assume $X_1 \neq 0$. We have $\Re[X, V_i^C] \supset \Re[X, V_i] = V_{i+1}$. Hence dim$_\mathbb{R}([X, V_i^C]) \geq$ dim$_\mathbb{R}(V_{i+1}) = 2k + 1$, which implies rank$_{(i),g}(X) = \text{dim}_\mathbb{C}([X, V_i^C]) \geq k + 1$.

Lemma 4.8. Let $(g, \{V_j\})_{j=1}^2$ be a step 2 graded Lie algebra over $\mathbb{F}$. Suppose rank$_\mathbb{F}(0, V_1) = \{0\}$, and span$_\mathbb{F}$(rank$_\mathbb{F}(1, V_1)) = V_1$. Then there is a collection $g_1, \ldots, g_n$ of graded subalgebras (over $\mathbb{F}$) of $g$ such that:

1. Each $g_j$ is graded isomorphic over $\mathbb{F}$ to some Heisenberg algebra over $\mathbb{F}$.
2. The first layers of the $g_j$s define a direct sum decomposition of $V_1$:

   $$V_1 = \oplus_j (V_1 \cap g_j).$$

3. The $g_j$s commute with one another: [g$_j$, g$_k$] = 0 for $1 \leq j \neq k \leq n$.
4. $g_j \cap g_k = \{0\}$ for $1 \leq j \neq k \leq n$.
5. The collection is permuted by the graded automorphism group Aut(g).

Moreover conditions (1)-(4) determine $n$ and the collection $g_1, \ldots, g_n$ uniquely.

Proof. Let $\{L_j\}_{j \in J}$ be the collection of 1-dimensional subspaces of $V_2$ of the form $[X, g]$, where $X \in \text{rank}_\mathbb{F}(1, V_1)$; here the index set $J$ might be infinite a priori. For every $j \in J$, let $K_j := \{X \in V_1 \mid [X, g] \subset L_j\}$. Then $K_j$ is a subspace of $V_1$, and by our assumption that span$_\mathbb{F}$($1, V_1$) = $V_1$, the $K_j$s span $V_1$.

Note that if $j \neq j'$ then $[K_j, K_{j'}] \subset L_j \cap L_{j'} = \{0\}$. Therefore for every $j_0 \in J$ we have $[K_{j_0}, \sum_{j \neq j_0} K_j] = \{0\}$. It follows that if $X \in K_{j_0} \cap \sum_{j \neq j_0} K_j$ then $[X, K_j] = \{0\}$ for all $j$, and so $[X, g] = \{0\}$, forcing $X = 0$ by assumption. Hence we have a direct sum decomposition $V_1 = \oplus_j K_j$, and in particular $J$ is finite.

For every $j \in J$, let $g_j := K_j \oplus L_j$. Since $g$ has step 2, $g_j$ is a graded subalgebra of $g$. Since $V_1 \setminus \{0\}$ has no rank zero elements, and $[g_j, g_k] = \{0\}$ for $k \neq j$, it follows that $g_j$ has no rank zero first layer elements. By Lemma 4.4, $g_j$ is isomorphic to a Heisenberg algebra over $\mathbb{F}$, for every $j$.

If $j \neq k$, we get $g_j \cap g_k = \{0\}$ from the fact that its projections to both layers are $\{0\}$.
To prove uniqueness, we observe that if $X \in V_1 \cup \bigcup_j g_j$, then $X$ has rank at least 2. Thus if $g_1', \ldots, g_k'$ is another collection of subalgebras satisfying (1)-(4), then each $g_j' \cap V_1$ must be contained in $g_j \cap V_1$ for some $1 \leq j \leq n$, and vice-versa. Since the $g_j$s are determined by their first layers, this gives uniqueness and consequently assertion (5) as well.

\[ \square \]

4.2. The classification. We recall that from [OW11b, DR10], a Carnot group $G$ is nonrigid if and only if the first layer of its complexification contains an element $X \neq 0$ with rank$_C X \leq 1$. Hence the following theorem yields a dichotomy for nonrigid graded Lie algebras: either the first layer contains a special type of automorphism invariant subspace, or the graded algebra has a very special structure.

**Theorem 4.9.** Let $(\mathfrak{g}, \{V_j\}_{j=1}^n)$ be a (real) graded Lie algebra, and $(\mathfrak{g}^C, \{V_j^C\}_{j=1}^n)$ be the complexification with its induced grading. Suppose:

(a) There is no Aut($\mathfrak{g}$)-invariant subspace $\{0\} \neq W \subset V_1$ such that $[W, V_i] = \{0\}$ for all $i \geq 2$.

(b) There is a nonzero element $Z \in (V_1^C) \setminus \{0\}$ such that rank$_C Z \leq 1$ (or equivalently, $(\mathfrak{g}, \{V_j\}_{j=1}^n)$ is the graded Lie algebra of a nonrigid Carnot group).

Then either $\mathfrak{g}$ is abelian, or it has step 2, and for some $F \in \{\mathbb{R}, \mathbb{C}\}$, $n \geq 1$, there is a collection $g_1, \ldots, g_n$ of graded subalgebras of $\mathfrak{g}$ with the following properties:

1. For some $m$, each $g_j$ is graded isomorphic over $\mathbb{R}$ to the $m$-th Heisenberg algebra over $F$ (viewed as a graded Lie algebra over $\mathbb{R}$).

2. The first layers of the $g_j$s define a direct sum decomposition of $V_1$:

$$ V_1 = \bigoplus_j (V_1 \cap g_j). $$

3. The $g_j$s commute with one another: $[g_j, g_k] = 0$ for $1 \leq j \neq k \leq n$.

4. If $F = \mathbb{R}$, then the second layers are distinct: $g_j \cap V_2 \neq g_k \cap V_2$ for $1 \leq j \neq k \leq n$. If $F = \mathbb{C}$, then the second layers need not be distinct. However, for each $j$ we have a (graded) decomposition of the complexification $g_j^C = (g_j^C)_1 \oplus (g_j^C)_{-1}$ from Lemma 4.7; the second layers of $(g_j^C)_{\pm 1}$ are distinct and interchanged by complex conjugation, and we obtain distinct pairs of second layers as $j$ varies.

5. Aut($\mathfrak{g}$) preserves the collection $\{g_j\}_{j=1}^n$, and permutes the $g_j$s transitively.
Moreover the field $\mathbb{F}$, and the collection $g_1, \ldots, g_n$ are uniquely determined by $g$.

Remark 4.10. Note that in (1) for $\mathbb{F} = \mathbb{C}$ the definition of $(g_j^{\mathbb{C}})_{\pm i}$ requires a complex multiplication on $g_j$. Such a multiplication is induced by a choice of an $\mathbb{R}$-linear graded isomorphism from the $m$-th complex Heisenberg group to $g_j$ which exist by (1). In view of Lemma 4.6 different choices of an $\mathbb{R}$-linear graded isomorphism yield the same complex structure on $g_j$, up to a possible change of sign. Changing the sign of the complex structure just exchanges $(g_j^{\mathbb{C}})_i$ and $(g_j^{\mathbb{C}})_{-i}$. Thus the collection $\{(g_j^{\mathbb{C}})_i, (g_j^{\mathbb{C}})_{-i}\}$ is uniquely determined by $g_j$.

Remark 4.11. The reader may wonder why Hypothesis (a) in Theorem 4.9 includes the restriction that $[W, V_i] = \{0\}$ for all $i \geq 2$. The theorem would remain true if this restriction is dropped, since, due to the negation, it would make this a logically stronger hypothesis; however, for the results in Section 3 we need the stronger result of Theorem 4.9.

Proof. We first sketch the overall logic before beginning the formal proof.

By our assumptions, it follows readily that $V_1 \setminus \{0\}$ contains an element $X$ with $\text{rank}_{\mathbb{R}}(X) \in \{0, 1, 2\}$. We then deal with the three possibilities by elimination. If $\text{rank}_{\mathbb{R}}(0, V_1) \neq \{0\}$, then one concludes that $g$ is abelian. Assuming $\text{rank}_{\mathbb{R}}(0, V_1) = \{0\}$ and $\text{rank}_{\mathbb{R}}(1, V_1) \neq \emptyset$, we reduce to Lemma 4.8. Finally, assuming $\text{rank}_{\mathbb{R}}(\leq 1, V_1) = \{0\}$, we analyze the situation by complexifying, applying Lemma 4.8 to the complexification, and interpreting the results back in the original graded algebra $g$.

We now return to the proof.

Suppose $\text{rank}_{\mathbb{R}}(0, V_1) \neq \{0\}$. Then $\text{rank}_{\mathbb{R}}(0, V_1)$ is a nontrivial, $\text{Aut}(g)$-invariant subspace of $V_1$ which commutes with $g$. Thus by hypothesis (a) we have $\text{rank}_{\mathbb{R}}(0, V_1) = V_1$.

Since $V_1$ generates $g$, it follows that $g$ is abelian, and we are done. Therefore we may assume that

\begin{equation}
\text{rank}_{\mathbb{R}}(0, V_1) = \{0\}.
\end{equation}

Now suppose $\text{rank}_{\mathbb{R}}(1, V_1) \neq \emptyset$. Then $W_1 := \text{span}_{\mathbb{R}} \text{rank}_{\mathbb{R}}(1, V_1)$ is a nontrivial, $\text{Aut}(g)$-invariant subspace of $V_1$. By Lemma 4.3, we have $[W_1, V_j] = \{0\}$ for all $j \geq 2$. Thus hypothesis (a) implies that $W_1 = V_1$. 


Using Lemma 4.3 again we see that \([V_1, \oplus_{j \geq 2} V_j] = \{0\}\) and we conclude that \(g\) has step 2.

By Lemma 4.8 we get graded subalgebras \(g_1, \ldots, g_n\) satisfying (1)-(4) of Lemma 4.8 in this case, and by (5) of Lemma 4.8 they are permuted by \(\text{Aut}(g)\). By hypothesis (a) they must be permuted transitively, since otherwise an orbit would give rise to a nontrivial \(\text{Aut}(g)\)-invariant subspace of \(V_1\) which commutes with \(\oplus_{j \geq 2} V_j\). Thus (5) holds. In particular, all the \(g_j\) are isomorphic to the same Heisenberg algebra and we are done in this case.

We now assume in addition that \(\text{rank}_R(1, V_1) = \emptyset\), i.e.

\[
(4.13) \quad \dim_R [X, g] \geq 2 \quad \text{for every} \quad X \in V_1 \setminus \{0\}.
\]

Recall that \(\text{rank}_C(1, V_1^C)\) is the collection of \(Z \in V_1^C\) such that \(\text{rank}_C Z = 1\).

**Claim.** Suppose \(0 \neq Z \in \text{rank}_C(\leq 1, V_1^C) \subset g^C\), and let

\[
\begin{align*}
P^R_Z &:= \text{span}_R(\Re Z, \Im Z) \\
P^C_Z &:= \text{span}_C(Z, \bar{Z}) = CZ + \mathbb{C}\bar{Z}.
\end{align*}
\]

Then

\[
\begin{align*}
(1) \quad &\dim_C [Z, g^C] = 1. \\
(2) \quad &P^R_Z \otimes \mathbb{C} = P^C_Z. \\
(3) \quad &\dim_R P^R_Z = \dim_C P^C_Z = 2. \\
(4) \quad &[\bar{Z}, g^C] \neq [Z, g^C]. \\
(5) \quad &\dim_C [P^R_Z, g] = \dim_C [P^C_Z, g^C] = 2.
\end{align*}
\]

**Proof of claim.** We cannot have \(\dim_C [Z, g^C] = 0\), since then we would have \(\Re Z, \Im Z \in V_1\) and \([\Re Z, g] = [\Im Z, g] = 0\), contradicting (4.12). Hence \(\dim_C [Z, g^C] = 1\) and (1) holds.

(2) is immediate from \(\Re Z = \frac{1}{2}(Z + \bar{Z}), \Im Z = \frac{1}{2i}(Z - \bar{Z})\).

Note that \(P^C_Z = P^R_Z \otimes \mathbb{C}\) implies \(\dim_R P^R_Z = \dim_C P^C_Z\). We cannot have \(\dim_C P^C_Z = 1\), since then \(P^C_Z\) would contain a nonzero real element with complex rank 1, and hence real rank 1, contradicting (4.13). Thus \(\dim_C P^C_Z = 2\), and (3) holds.

We now consider the action of complex conjugation \(Z \mapsto \bar{Z}\). Suppose \([\bar{Z}, g^C] = [Z, g^C]\). Then \(P^C_Z\) is a \(\mathbb{C}\)-linear subspace of \(V_1^C\) that is complex conjugation invariant, and \([P^C_Z, g^C] = [Z, g^C] = \bar{Z}, g^C\) is a \(\mathbb{C}\)-linear, complex conjugation invariant subspace with \(\dim_C [P^C_Z, g^C] = 1\). Therefore the real points \(P^R_Z = \Re P^C_Z\) have the property that \([P^R_Z, g]\) lies in the real points of \([P^C_Z, g^C]\), which form an \(\mathbb{R}\)-linear subspace of dimension 1 over \(\mathbb{R}\). This contradicts (4.13). It follows that \([\bar{Z}, g^C] \neq [Z, g^C]\), so (4) holds.
Finally, (4) implies that \( \dim_{C}[P_{Z}^{C}, g] = 2 \). Since \([P_{Z}^{R}, g] \otimes \mathbb{C} \subset [P_{Z}^{C}, g] \), we get \( \dim_{R}[P_{Z}^{R}, g] \leq \dim_{C}[P_{Z}^{C}, g] = 2 \). By (4.13) we cannot have \( \dim_{R}[P_{Z}^{R}, g] \leq 1 \), so (5) follows.

Let \( \hat{W}_{1} := \text{span}_{\mathbb{R}}\{P_{Z}^{R} | Z \in \text{rank}_{\mathbb{C}}(1, V_{1}^{C})\} \). This is a non-trivial \( \text{Aut}(g) \) invariant subspace of \( V_{1} \). If \( Z \in \text{rank}_{\mathbb{C}}(1, V_{1}^{C}) \), then for \( j \geq 2 \) we have

\[
[P_{Z}^{R}, V_{j}] \subset [P_{Z}^{C}, V_{j}^{C}] = [Z, V_{j}^{C}] + [Z, V_{j}^{C}] = \{0\}
\]

by Lemma 4.3 and the identity \( \dim_{C}[\hat{Z}, g^{C}] = \dim_{C}[Z, g^{C}] \). Thus by hypothesis (a) we get \( \hat{W}_{1} = V_{1} \). This implies that \( g \) is a step 2 graded algebra, and hence \( g^{C} \) also has step 2. Furthermore,

\[
V_{1}^{C} = V_{1} \otimes \mathbb{C} = \hat{W}_{1} \otimes \mathbb{C} = \text{span}_{\mathbb{C}}\{P_{Z}^{R} \otimes \mathbb{C} | Z \in \text{rank}_{\mathbb{C}}(1, V_{1}^{C})\}
\]

\[
= \text{span}_{\mathbb{C}}\{P_{Z}^{C} | Z \in \text{rank}_{\mathbb{C}}(1, V_{1}^{C})\}.
\]

Thus \( \text{rank}_{\mathbb{C}}(1, V_{1}^{C}) \) spans \( V_{1}^{C} \), and we can apply Lemma 4.8 to obtain a collection \( \{\hat{g}_{j}\}_{j \in J_{0}} \) of graded subalgebras (over \( \mathbb{C} \)) of \( g^{C} \) as in that lemma. By the uniqueness assertion in Lemma 4.8, the collection \( \{\hat{g}_{j}\}_{j \in J_{0}} \) is preserved by both complex conjugation and the action \( \text{Aut}(g) \triangleleft g^{C} \). Hence by (4) of the claim, complex conjugation will act freely on the collection \( \{V_{2}^{C} \cap \hat{g}_{j}\} \) of second layers (here we are using the fact that \( [\hat{g}_{j}, g^{C}] = [\hat{g}_{j}, V_{1}^{C}] = [\hat{g}_{j}, \hat{g}_{j}] = V_{2}^{C} \cap \hat{g}_{j} \)). Picking one representative from each orbit under complex conjugation, we get a subcollection \( \{\hat{g}_{j}\}_{j \in J} \) such that we get a direct sum decomposition

\[
V_{1}^{C} = \bigoplus_{j \in J} (\hat{g}_{j} \oplus \overline{\hat{g}_{j}}) \cap V_{1}^{C}
\]

and also for every \( j \in J \) we have

\[
(4.14) \quad \hat{g}_{j} \cap \overline{\hat{g}_{j}} = \{0\}
\]

by (4) of the claim. For \( j \in J \), define

\[
(4.15) \quad g_{j} := \Re(\hat{g}_{j} \oplus \overline{\hat{g}_{j}}) = \{X + \hat{X} | X \in \hat{g}_{j}\};
\]

here we have used the fact that the real subspace is the \( \mathbb{R} \)-linear subspace of vectors which are fixed under complex conjugation. In view of (4.14) the map \( X \rightarrow X + \hat{X} \) is an \( \mathbb{R} \)-linear graded automorphism from \( \hat{g}_{j} \) to \( g_{j} \). Using this graded isomorphism, we may pass the complex multiplication on \( \hat{g}_{j} \) to \( g_{j} \); then by applying Lemma 4.1 we have identifications \( \hat{g}_{j} = (g^{C})_{j}, \overline{\hat{g}_{j}} = (g^{C})_{-j} \). Note that we have direct sum decomposition \( V_{1} = \bigoplus_{j \in J}(g_{j} \cap V_{1}) \). Since the action \( \text{Aut}(g) \triangleleft g^{C} \) preserves the collection \( \{g_{j}\}_{j \in J_{0}} \) and commutes with complex conjugation, it follows from (4.15) that \( \text{Aut}(g) \triangleleft g \) preserves \( \{g_{j}\}_{j \in J} \). By hypothesis (a), \( \text{Aut}(g) \) permutes the \( g_{j} \) transitively. Indeed a non-trivial orbit
would generate a non-trivial $\text{Aut}(g)$ invariant subspace of $V_1$. This subspace trivially commutes with $V_2$ since we have already shown that $g$ has step 2. Transitivity of the action implies that the $g_j$s are isomorphic to the $m$-th complex Heisenberg algebra for a fixed $m$ independent of $j$. Assertions (1)-(4) now follow from the above discussion.

Thus we have established the existence of $F$, $n$, and the collection $\{g_j\}_{j=1}^n$ satisfying (1)-(5). We now prove uniqueness. Suppose $F'$, $n'$, and $\{g'_j\}_{j=1}^{n'}$ also satisfy (1)-(5). Note that if either $F = \mathbb{R}$ or $F' = \mathbb{R}$, then $\text{rank}_\mathbb{R}(1, V_1) \neq \emptyset$, while if $F = \mathbb{C}$ or $F' = \mathbb{C}$ then $\text{rank}_\mathbb{R}(1, V_1) = \emptyset$ because $\dim_\mathbb{R}[X, g] \geq 2$ for every $X \in V_1 \setminus \{0\}$. Hence we must have $F = F'$. If $F' = F = \mathbb{R}$, then Lemma 4.8 implies that the collections of subalgebras $\{g_j\}$, $\{g'_j\}$ coincide. If $F' = F = \mathbb{C}$, then by looking at the complexification $\mathfrak{g}^c$, and using condition (4), we may argue as in the proof of Lemma 4.8 to see that the collections of pairs $\{(g_j^c)_{\pm i}\}$, $\{(g'_j)^c)_{\pm i}\}$ coincide. Taking real parts we see that the collections $\{g_j\}$, $\{g'_j\}$ are the same. □

4.3. A characterization using quotients. We now show that there is an alternative description of the graded algebras which satisfy hypotheses of Theorem 4.9 as a certain type of quotient of a product.

Let $(\mathfrak{g}, \{V_j\}_{j=1}^2)$ be a nonabelian graded Lie algebra as in Theorem 4.9, and let $F$ and $g_1, \ldots, g_n$ be the uniquely determined graded subalgebras as supplied by that theorem.

Let $(\tilde{g}, \{\tilde{V}_j\}_{j=1}^2) := (\bigoplus_{k=1}^n g_k, \{\bigoplus_{k=1}^n V_{j,k}\}_{j=1}^2)$, where $g_k = V_{1,k} \oplus V_{2,k}$ is the layer decomposition of $g_k$; here $\bigoplus_{k=1}^n g_k$ denotes the abstract direct sum (the subalgebras are typically not independent in $\mathfrak{g}$). Then we get an epimorphism of graded Lie algebras $\pi : \tilde{g} \rightarrow g$ by sending $g_k$ to $g$ by the inclusion. Since the $g_k$s are unique, they are permuted by $\text{Aut}(g)$; hence we get an induced action $\text{Aut}(g) \curvearrowright \tilde{g}$, and $\pi$ is $\text{Aut}(g)$-equivariant.

Let $K := \ker(\pi)$. Since $\pi$ is $\text{Aut}(g)$-equivariant, it follows that $K$ is $\text{Aut}(g)$-invariant. To summarize:

**Lemma 4.16.** We have a canonical embedding $\text{Aut}(g) \hookrightarrow \text{Aut}(\tilde{g})$, and the image is precisely $\text{Stab}(K, \text{Aut}(\tilde{g}))$, the stabilizer of $K$ in $\text{Aut}(\tilde{g})$.

Henceforth we will sometimes identify $\text{Aut}(g)$ with $\text{Stab}(K, \text{Aut}(\tilde{g}))$.

The first layers $g_j \cap V_1$ yield a direct sum decomposition of $V_1$, and therefore $K \subset \tilde{V}_2$. Taken together, the conditions (2)-(5) from Theorem 4.9 imply that the following properites hold for the projection $\pi : \tilde{g} \rightarrow g = \tilde{g}/K$:
(1) The restriction of \( \pi \) to \( \mathfrak{g}_k \) is injective.

(2) If \( \mathbb{F} = \mathbb{R} \), then for \( j \neq k \), the second layers \( V_{2,j}, V_{2,k} \) project under \( \pi \) to distinct subspaces of \( V_2 \cong \tilde{V}_2 / K \). If \( \mathbb{F} = \mathbb{C} \), then for each \( j \) we have a (graded) decomposition of the complexification

\[
\mathfrak{g}_j^C = (\mathfrak{g}_j^C)_i \oplus (\mathfrak{g}_j^C)_{-i} = (V_{1,j}^C)_i \oplus (V_{2,j}^C)_i \oplus (V_{1,j}^C)_{-i} \oplus (V_{2,j}^C)_{-i}
\]

given by Lemma 4.17 the projections \( \pi((V_{2,j}^C)_i), \pi((V_{2,j}^C)_{-i}) \) are distinct subspaces which are interchanged by complex conjugation, and yield distinct pairs as \( j \) varies.

(3) The action \( \text{Aut}(\mathfrak{g}) \circlearrowright \mathfrak{g} \) permutes the summands \( \{\mathfrak{g}_k\} \) transitively.

The converse holds:

**Lemma 4.17.** Suppose \( \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}, m \geq 1 \), and we are given the graded direct sum \( \mathfrak{g} = \bigoplus_{k=1}^{n} \mathfrak{g}_k \), where the \( \mathfrak{g}_k \)'s are graded isomorphic over \( \mathbb{R} \) to the \( m \)-th Heisenberg algebra over \( \mathbb{F} \), and an \( \mathbb{R} \)-linear subspace \( K \subset \tilde{V}_2 \), such that the following hold:

(4) \( K \cap \tilde{\mathfrak{g}}_k = \{0\} \) for all \( 1 \leq k \leq n \).

(5) If \( \mathbb{F} = \mathbb{R} \) then we have \( K + (\mathfrak{g}_j \cap \tilde{V}_2) \neq K + (\mathfrak{g}_k \cap \tilde{V}_2) \) for \( j \neq k \). If \( \mathbb{F} = \mathbb{C} \), then for each \( j \) we have a (graded) decomposition of the complexification \( \mathfrak{g}_j^C = (\mathfrak{g}_j^C)_i \oplus (\mathfrak{g}_j^C)_{-i} = (V_{1,j}^C)_i \oplus (V_{2,j}^C)_i \oplus (V_{1,j}^C)_{-i} \oplus (V_{2,j}^C)_{-i} \) given by Lemma 4.17 the projections \( \pi((V_{2,j}^C)_i), \pi((V_{2,j}^C)_{-i}) \) are distinct subspaces which are interchanged by complex conjugation, and yield distinct pairs as \( j \) varies.

(6) The stabilizer of \( K \) in \( \text{Aut}(\mathfrak{g}) \) permutes the factors \( \{\mathfrak{g}_k\} \) transitively.

Then the quotient \( \mathfrak{g} := \tilde{\mathfrak{g}} / K \) with the grading \( \{	ilde{V}_1, \tilde{V}_2 / K\} \) induced from \( \tilde{\mathfrak{g}} \) satisfies the hypotheses of Theorem 4.9.

**Proof.** Let \( \pi : \tilde{\mathfrak{g}} \to \mathfrak{g} \) be the quotient map.

Suppose \( X \in V_1 \setminus \{0\} \). Pick \( \bar{X} \in \tilde{V}_1 \) with \( \pi(\bar{X}) = X \), and let \( \bar{X}_1 + \ldots + \bar{X}_n \) be the decomposition induced by \( \bar{X}_k = \oplus_k \bar{\mathfrak{g}}_k \). Then \( \bar{X}_k \neq 0 \) for some \( k \), and there is a \( \bar{Y} \in \tilde{\mathfrak{g}}_k \) such that \( [\bar{X}, \bar{Y}] \in V_{2,k} \setminus \{0\} \). Now \( [X, \pi(\bar{Y})] = \pi([\bar{X}, \bar{Y}]) = \pi([X_k, \bar{Y}]) \neq 0 \) by (4). Therefore \( V_1 \) contains no nonzero elements of rank zero.

If \( \mathbb{F} = \mathbb{R} \), then \( \text{rank}_{\mathbb{R}}(1, \tilde{V}_1) \) spans \( \tilde{V}_1 \), so \( \text{rank}_{\mathbb{C}}(1, V_1^C) \) spans \( V_1^C \). If \( \mathbb{F} = \mathbb{C} \), then by Lemma 4.17 we have the decomposition

\[
\mathfrak{g}^C = \tilde{\mathfrak{g}}^C / K^C = (\bigoplus_{k=1}^{n} \tilde{\mathfrak{g}}_k^C) / K^C = (\bigoplus_{k=1}^{n} ([\tilde{\mathfrak{g}}_k^C]_i \oplus (\tilde{\mathfrak{g}}_k^C)_{-i}]) / K^C ,
\]

so \( \text{rank}_{\mathbb{C}}(1, V_1^C) \) also spans \( V_1^C \) in this case.
To see that Aut(\(g\)) acts irreducibly on the first layer suppose that \(W \subset V_1\) is a nontrivial subspace invariant under Aut(\(g\)) and consider first the case \(\mathbb{F} = \mathbb{R}\). Then in particular \(W\) is invariant under the action of the subgroup of Aut(\(\tilde{g}\)) consisting of elements that act trivially on \(V_2\). So we may apply Lemma 4.5 to conclude that \(W = \oplus_{j \in J} V_{1,j}\) for some nonempty subset \(J \subset \{1, \ldots, n\}\). Now (6) implies that \(J = \{1, \ldots, n\}\). Now consider the case \(\mathbb{F} = \mathbb{C}\). Then the map \(\Phi\) which acts as complex multiplication by \(i\) on the first layer and multiplication by \(-1\) on the second layer belong to Aut(\(\tilde{g}\)). Since it preserves \(K\) it also defines an element of Aut(\(g\)). Thus \(W\) is a \(\mathbb{C}\)-linear subspace of \(V_1\). Now we can conclude as before by considering the subgroup of Aut(\(\tilde{g}\)) consisting of \(\mathbb{C}\)-linear elements that act trivially on \(V_2\). Thus \((g, \{V_j\}_{j=1}^2)\) is a nonrigid graded algebra such that Aut(\(g\)) acts irreducibly on the first layer; in particular hypothesis (a) of Theorem 4.9 holds. This concludes the proof.

\[
\text{Definition 4.18. A product quotient is a Carnot group } G \text{ of the form } G/\exp(K), \text{ where the graded Lie algebra } \tilde{g} \text{ of } G \text{ and } K \subset \tilde{g} \text{ satisfy the assumptions of Lemma 4.17, and in the case } \mathbb{F} = \mathbb{R} \text{ we have } n \geq 2. \text{ We will use the term product quotient to refer to both Carnot groups and their graded Lie algebras.}
\]

\[
\text{Remark 4.19. By virtue of Theorem 4.9 and Lemma 4.17, product quotients are precisely the Carnot groups whose associated graded algebra satisfies the hypotheses of Theorem 4.9 with the exception of abelian groups and real Heisenberg groups. We have chosen to exclude the latter two cases from the definition because our main objective is to study rigidity phenomena which fail to hold in those cases. We emphasize that a complex Heisenberg group is a product quotient, while a real Heisenberg group is not.}
\]

We close this subsection by discussing a few examples of product quotients.

The simplest examples of product quotients are simply products, i.e. they are of the form \(\tilde{g} = \tilde{g} = \oplus_{j=1}^n \tilde{g}_j, K = \{0\}\), where each summand \(\tilde{g}_j\) is a copy of a fixed Heisenberg algebra over \(\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}\), and \(n \geq 2\) if \(\mathbb{F} = \mathbb{R}\).

\[
\text{Example 4.20. (Diagonal product quotient) Let } \tilde{g} = \oplus_{j=1}^n \tilde{g}_j, \text{ where } n \geq 3, \text{ } \tilde{g}_j \text{ is a copy of the first real Heisenberg group, with basis }
\]
\(\tilde{X}_{3j-2}, \tilde{X}_{3j-1}, \tilde{X}_{3j}, [\tilde{X}_{3j-2}, \tilde{X}_{3j-1}] = -\tilde{X}_{3j}\). Let \(K \subset \tilde{V}_2\) be the diagonal subspace \(K = \text{span}(\tilde{X}_3 + \ldots + \tilde{X}_{3n})\). Then the permutation action \(S_n \curvearrowright \tilde{g}\) leaves \(K\) invariant, and hence Lemma 4.17(6) holds. We may obtain similar examples by using the \(m\)-th real or complex Heisenberg group, instead of the first real Heisenberg group.

We emphasize that it is necessary to have \(n \geq 3\) in Example 4.20 for the conditions in Lemma 4.17 to be satisfied, because otherwise Lemma 4.17(5) would be violated. We also point out that if \(n = 2\) and \(K \subset \tilde{V}_2\) is the diagonal subspace, then:

- If the \(\tilde{g}_j\)s are copies of the first real Heisenberg group, then \(K\) does not satisfy the assumptions of Lemma 4.17, the quotient of \(\tilde{g}/K\) is the second real Heisenberg algebra which satisfies Theorem 4.9 but which is not a product quotient, since the real Heisenberg algebras are excluded.
- If the \(\tilde{g}_j\)s are copies of the first complex Heisenberg group, then although \(K\) does not satisfy the assumptions of Lemma 4.17, the quotient \(\tilde{g}/K\) is the second complex Heisenberg algebra, which satisfies Theorem 4.9 and is a product quotient. Note also that the same graded algebra has an alternate description as \(h_\mathbb{C}^2/K'\), where \(K' := \{0\}\), and in particular \(\text{Aut}(\tilde{g}/K)\) acts transitively on the first layer.

**Example 4.21.** (\(\mathbb{Z}_5\)-product quotients) Let \(\tilde{g} = \bigoplus_{j=1}^{5}\tilde{g}_j\), where \(\tilde{g}_j\) is a copy of the first real Heisenberg group with basis \(X_{3j-2}, X_{3j-1}, X_{3j}, [X_{3j-2}, X_{3j-1}] = -X_{3j}\). Then we have the permutation action \(S_5 \curvearrowright \tilde{g}\), and we restrict this to the subgroup \(\mathbb{Z}_5\) which permutes the summands cyclically. Then the representation \(\mathbb{Z}_5 \curvearrowright \tilde{V}_2\) is a copy of the permutation representation \(\mathbb{Z}_5 \curvearrowright \mathbb{R}^5\), which is isomorphic to the left regular representation of \(\mathbb{Z}_5\). This decomposes as a direct sum \(\tilde{V}_2 = K_1 \oplus K_2 \oplus K_3\), where \(K_1 = \text{span}(\tilde{X}_3 + \ldots + \tilde{X}_{15})\) is the diagonal, and

\[
K_2 = \text{span}\{-aX_3 + aX_6 + X_9 - X_{15}, -aX_3 - X_6 + X_{12} + aX_{13}\},
\]

\[
K_3 = \text{span}\{X_3 - X_6 + aX_9 - aX_{15}, -aX_3 + X_6 - X_9 + aX_{12}\},
\]

with \(a = (\sqrt{5} - 1)/2\), are 2-dimensional irreducible representations, where the generator of \(\mathbb{Z}_5\) acts as rotation by \(\frac{2\pi}{5}\) and \(\frac{4\pi}{5}\), respectively. Then \(K_i\) satisfies the assumptions of Lemma 4.17.

**Example 4.22.** (Permutation product quotients) We may generalize the two preceding examples as follows. Let \(\tilde{g} = \bigoplus_{j=1}^{n}\tilde{g}_j\), where \(n \geq 2\) and \(\tilde{g}_j\) is a copy of the \(m\)-th real or complex Heisenberg algebra. Then
permutation of the copies of $\tilde{g}_j$ gives an action $S_n \acts \tilde{g}$, and we may seek an $S_n$-invariant $\mathbb{R}$-linear subspace $K \subset \tilde{V}_2$ which satisfies (4) and (5).

**Example 4.23.** Let $\tilde{g} = \tilde{g}_1 \oplus \tilde{g}_2$ where $\tilde{g}_i$ is a copy of the complex Heisenberg graded algebra $\mathfrak{h}_m^\mathbb{C}$. Let $\tilde{V}_2$ be the second layer of $\tilde{g}_i$, so $\tilde{V}_2 = \tilde{V}_{2,1} \oplus \tilde{V}_{2,2}$. Letting $\phi : \tilde{V}_{2,1} \to \tilde{V}_{2,2}$ be an $\mathbb{R}$-linear isomorphism, and $K \subset \tilde{V}_2 = \tilde{V}_{2,1} \oplus \tilde{V}_{2,2}$ be its graph, we obtain an example satisfying Lemma 4.17, see [HSX15] for details.

**Example 4.24.** (Decomposable product quotients) Suppose $g = \tilde{g}/K$ is a product quotient, and let $g'$ be the direct sum of $N \geq 2$ copies of $g$. Then $g'$ is also a product quotient, as the conditions of Lemma 4.17 clearly carry over to $g'$. We shall show in the next subsection that every product quotient can be written canonically as a direct sum of $N \geq 1$ so called conformal product quotients (see Definition 4.25).

### 4.4. Decomposition of product quotients

The goal of this subsection is Lemma 4.27, which asserts that every product quotient admits a canonical decomposition as a direct sum of indecomposable summands which are conformal, in the sense of Definition 4.25 below.

Let $(g, \{V_j\}_{j=1}^2)$ be a product quotient as discussed in the preceding subsection. Hence there are canonically defined graded subalgebras $g_1, \ldots, g_n \subset g$, and a graded epimorphism $\pi : \tilde{g} := \oplus_j g_j \to g$, where $\tilde{g}$ has the direct sum grading, and the kernel $K \subset \tilde{V}_2$ satisfies (4), (6) above. We have canonical actions

$$\text{Aut}(g) = \text{Stab}(K, \text{Aut}(\tilde{g})) \acts \tilde{g}$$

and $\text{Aut}(g) \acts \{1, \ldots, n\}$.

**Definition 4.25.** A product quotient $(g, \{V_j\}_{j=1}^2)$ is **conformal** if the action $\text{Aut}(g) \acts \tilde{g}$ preserves a conformal structure on $\tilde{V}_2$, i.e. if there is an inner product on $\tilde{V}_2$ that is $\text{Aut}(g)$-invariant up to scale.

**Remark 4.26.** We emphasize to the reader that graded automorphisms of conformal product quotients act conformally on the second layer, but not necessarily on the first layer.

**Lemma 4.27.** Every product quotient $g$ has graded direct sum decomposition $g = \oplus_k \hat{g}_k$, where the summands $\hat{g}_k$ are conformal product quotients, and the decomposition is respected by $\text{Aut}(g)$, i.e. each $\Phi \in \text{Aut}(g)$ induces a permutation of the summands. Moreover, this decomposition is unique.
The proof of the lemma will occupy the remainder of this subsection. We now fix a product quotient \((g, \{V_j\}_j=1)\).

Our approach to proving Lemma 4.27 is to find a partition of \(\{1, \ldots, n\}\) that is compatible with \(K\) and the action \(\text{Aut}(g) \rhd \{1, \ldots, n\}\).

**Definition 4.28.** A partition \(\{1, \ldots, n\} = J_1 \sqcup \cdots \sqcup J_\ell\) is \(K\)-compatible if the direct sum decomposition

\[
\tilde{V}_2 = (\oplus_{j \in J_1} \tilde{V}_{2,j}) \oplus \cdots \oplus (\oplus_{j \in J_\ell} \tilde{V}_{2,j})
\]

is compatible with \(K\):

\[
K = (K \cap \oplus_{j \in J_1} \tilde{V}_{2,j}) \oplus \cdots \oplus (K \cap \oplus_{j \in J_\ell} \tilde{V}_{2,j}).
\]

**Lemma 4.30.**

1. A partition \(\{1, \ldots, n\} = J_1 \sqcup \cdots \sqcup J_\ell\) is \(K\)-compatible iff \(K\) is invariant under each projection \(\pi_{J_k} : \tilde{V}_2 \to \oplus_{j \in J_k} V_{2,j}\).
2. A \(K\)-compatible partition gives rise to graded decomposition of \(g\). If the partition is invariant under the action \(\text{Aut}(g) \rhd \{1, \ldots, n\}\), then the resulting graded decomposition of \(g\) is \(\text{Aut}(g)\)-invariant.
3. There exists a unique finest \(K\)-compatible partition.
4. The partition in (3) induces an \(\text{Aut}(g)\)-invariant graded decomposition of \(g\) into product quotients.

**Proof.** (1) and (2) are straightforward.

(3) If \(J_1 \sqcup \cdots \sqcup J_\ell\) and \(J'_1 \sqcup \cdots \sqcup J'_{\ell'}\) are both \(K\)-compatible then (1) implies that \(K\) is invariant under \(\pi_{J_k \cap J'_{k'}} = \pi_{J_k} \circ \pi_{J'_{k'}}\) for all \(k \in \{1, \ldots, \ell\}\), \(k' \in \{1, \ldots, \ell'\}\). Hence the common refinement \(J''_1 \sqcup \cdots \sqcup J''_{\ell''}\) is also \(K\)-compatible by (1). This implies (3).

(4) This follows readily from the definitions. \(\square\)

We now let \(\{1, \ldots, n\} = J_1 \sqcup \cdots \sqcup J_\ell\) be the finest \(K\)-compatible partition given by the lemma. Letting

\[
\hat{V}_k := \oplus_{j \in J_k} \tilde{V}_{2,j}, \quad \hat{K}_k := K \cap \hat{V}_k, \quad \hat{g}_k := (\oplus_{j \in J_k} V_{1,j}) \oplus (\hat{V}_k / \hat{K}_k) \subset g,
\]

we have \(\text{Aut}(g)\)-invariant decompositions

\[
\hat{V}_2 = \oplus_k \hat{V}_k, \quad K = \oplus_k \hat{K}_k, \quad g = \oplus_k \hat{g}_k.
\]

We consider the action \(\text{Aut}(g) \rhd \hat{g}\). Let \(D \subset \text{Aut}(g)\) be the subgroup that leaves each summand \(g_j \subset \hat{g}\) invariant, and acts on each summand by \(F\)-linear automorphisms. Then \(D\) has finite index in \(\text{Aut}(g)\) because by Lemma 4.6 the \(F\)-linear automorphisms of \(g_j\) have index at most 2 in its full automorphism group (as a graded algebra over \(R\)). Thus
$D$ acts on $\tilde{V}_2 = \bigoplus_j \tilde{V}_{2,j} \simeq F^n$ by linear transformations that are diagonalizable over $F$, and even when $F = \mathbb{C}$, these admit generalized eigenspace decompositions over $\mathbb{R}$ (i.e. direct sum decompositions into 2-dimensional irreducible subspaces).

**Lemma 4.31.** If $A \in D$ and $k \in \{1, \ldots, \ell\}$, then the restriction of $A$ to $\hat{V}_k$ has a single eigenvalue if $F = \mathbb{R}$ or eigenvalues with the same modulus if $F = \mathbb{C}$.

**Proof.** Suppose $A|_{\hat{V}_k}$ has distinct eigenvalues if $F = \mathbb{R}$, or two eigenvalues with distinct modulus, if $F = \mathbb{C}$. Then the (generalized) eigenspaces of $A|_{\hat{V}_k}$ on $\hat{V}_k$ give a nontrivial direct sum decomposition of $\hat{V}_k$ corresponding to a nontrivial partition $J_k = \bar{J}_1 \sqcup \ldots \sqcup \bar{J}_\ell$:

$$\hat{V}_k = (\bigoplus_{j \in \bar{J}_1} \tilde{V}_{2,j}) \oplus \ldots \oplus (\bigoplus_{j \in \bar{J}_\ell} \tilde{V}_{2,j}).$$

Notice that $\text{id}_{\hat{V}_k} = \pi_1 + \ldots + \pi_\ell$ where the $\pi_i$s are idempotents realizing the decomposition (4.32), and each $\pi_j$ is a polynomial in $A$ with coefficients in $\mathbb{R}$. Since $K$ is $D$-invariant, we therefore obtain a nontrivial decomposition

$$\hat{K}_k = (\hat{K}_k \cap \bigoplus_{j \in \bar{J}_1} \tilde{V}_{2,j}) \oplus \ldots \oplus (\hat{K}_k \cap \bigoplus_{j \in \bar{J}_\ell} \tilde{V}_{2,j}).$$

This contradicts the assumption that $J_1 \sqcup \ldots \sqcup J_\ell$ is the finest $K$-compatible decomposition of $\tilde{V}_2$. \hfill \Box

Pick $k \in \{1, \ldots, \ell\}$. Choose an inner product $\langle \cdot, \cdot \rangle_k$ on $\hat{V}_k$ such that:

- The subspaces $\{V_{2,j}\}_{j \in J_k}$ are $\langle \cdot, \cdot \rangle_k$-orthogonal.
- If $F = \mathbb{C}$, then complex multiplication is $\langle \cdot, \cdot \rangle_k$-orthogonal.

In view of the definition of $D$ and Lemma 4.31, $D$ acts conformally on $\langle \cdot, \cdot \rangle_k$. Since $D$ has finite index in the stabilizer $\text{Stab}(J_k, \text{Aut}(\frak{g}))$, there is an inner product $\langle \cdot, \cdot \rangle_k'$ on $\hat{V}_k$ on which $\text{Stab}(J_k, \text{Aut}(\frak{g}))$ acts conformally.

We now turn to the uniqueness assertion.

We will show that the summands of any decomposition as in the statement of Lemma 4.27 are nonabelian and indecomposable; it then follows from [KMX20, Lemma 7.3] that the decomposition is unique. The invariance under $\text{Aut}(\frak{g})$ follows from uniqueness.

We claim that for every $k$, the summand $\hat{\frak{g}}_k$ cannot be decomposed nontrivially as a direct sum of graded algebras. Arguing indirectly, let $\hat{\frak{g}}_k = \frak{h}_1 \oplus \frak{h}_2$ be a nontrivial decomposition of $\hat{\frak{g}}_k$ as a direct sum. The 2-parameter family of graded automorphisms arising from independent scalings of the summands will contradict the fact that $\hat{\frak{g}}_k$ is conformal,
unless one of the two summands $\mathfrak{h}_i$ has trivial second layer. However, this would mean that $\mathfrak{h}_i$ is abelian, and moreover is an ideal of $\mathfrak{g}$ contained in the first layer. This is impossible, since every nonzero element of the first layer of $\mathfrak{g}$ has rank $\geq 1$. This proves the claim.

This completes the proof of Lemma 4.27.

5. RIGIDITY OF PRODUCT-QUOTIENTS

The remainder of the paper is concerned with the rigidity of mappings $G \supset U \to U' \subset G$ where $G$ is a product quotient – one of the nonrigid Carnot groups that emerged from the structure theorem in Section 4. In this section we will state the main rigidity result for such groups (Theorem 5.1), and give the proof using results from Sections 6–8.

For the remainder of the paper we let $G = \tilde{G}/\exp(K)$ be a product quotient (Definition 4.18), and will use the notation from Lemma 4.17 for $\tilde{\mathfrak{g}} = \bigoplus_{i=1}^n \tilde{\mathfrak{g}}_i$, $K \subset \tilde{V}_2$, etc.

**Theorem 5.1.** Let $G = \tilde{G}/\exp(K)$ be a product quotient of homogeneous dimension $\nu$, and $\mathfrak{g} = V_1 \oplus V_2$ be the layer decomposition of its graded Lie algebra. Suppose $p > \nu$, $U \subset G$ is open, and $f : G \supset U \to G$ is a $W^{1,p}_{\text{loc}}$-mapping such that the sign of the determinant of $D_Pf(x) : \mathfrak{g} \to \mathfrak{g}$ is constant almost everywhere. Then there is a locally constant assignment of a permutation $U \ni x \mapsto \sigma_x \in S_n$ such that:

1. For a.e. $x \in U$, the Pansu derivative $D_Pf(x)$ permutes the subalgebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_n \subset \mathfrak{g}$ in accordance with the permutation $\sigma$:

   $$D_Pf(x)(\mathfrak{g}_i) = \mathfrak{g}_{\sigma(x)(i)}.$$

2. For every $x \in U$, cosets of $G_i$ are mapped to cosets of $G_{\sigma_x(i)}$ locally near $x$: there is neighborhood $V_x$ of $x$ such that for every $1 \leq j \leq n$, and every $y \in V$,

   $$f(V_x \cap yG_i) \subset f(y)G_{\sigma_x(i)}.$$

**Proof.** Note that (2) follows from (1) by Lemma 2.30.

We may assume that $n \geq 2$ since otherwise the theorem is trivial.

**Case 1.** Each $\tilde{\mathfrak{g}}_i$ is either a copy of a complex Heisenberg algebra or a real Heisenberg algebra $\mathfrak{h}_m$ with $m \geq 2$. This case is handled in Section 8.

**Case 2.** Each $\tilde{\mathfrak{g}}_i$ is a copy of the first real Heisenberg algebra for all $i$.

**Case 2a.** $\mathfrak{g}$ does not have a nontrivial decomposition as a direct sum of graded Lie algebras. Then $\dim K \geq 1$, and the cases when $\dim K = 1$ and $\dim K \geq 2$ are treated in Sections 8 and 9 respectively.
Case 2b. \( \mathfrak{g} \) has a nontrivial direct sum decomposition \( \mathfrak{g} = \oplus_{j} \hat{\mathfrak{g}}_j \) where the \( \hat{\mathfrak{g}}_j \)'s are indecomposable. By [KMX20, Theorem 1.1], after shrinking the domain and postcomposing with an isometry \( G \to \tilde{G} \) which permutes the factors, we may assume that \( f \) is a product mapping. Applying Case 2a to each factor mapping, the theorem follows. \( \square \)

We remark that the proofs in Cases 1, 2a, and 2b are all ultimately based on an application of the Pullback Theorem (Theorem 2.1),

\[
\int_U f^*_P(\alpha) \wedge d(\varphi \beta) = 0,
\]

with \( \varphi \in C^\infty_c(U) \) and suitable closed left invariant forms \( \alpha \) and \( \beta \) which satisfy the conditions \( \deg \alpha + \deg \beta = N - 1 \) and \( \text{wt}(\alpha) + \text{wt}(\beta) \leq -\nu + 1 \). A more detailed motivation for the choice of the forms \( \alpha \) and \( \beta \) is given at the beginning of Sections 6-8 where the three different cases are treated. Here we just outline some guiding principles which helped us to identify good classes of left invariant forms.

(i) The pullback of \( \alpha \) by the Pansu differential \( D_P f(x) \) (or its lift to \( D_P \tilde{f}(x) \in \text{Stab}(K, \text{Aut} \hat{\mathfrak{g}}) \), see Lemma 4.16) should detect the permutation \( \sigma_x \) without depending on too detailed information on \( D_P f \);

(ii) We should work with closed forms \( \alpha \) modulo exact forms (see [KMX20, Lemma 4.8]);

(iii) There should be a sufficient supply of “complementary” forms \( \beta \) which are closed but not exact.

Regarding (i) we found it useful to look for forms which are related to the action of \( D_P f \) on the second layer \( V_2 \) (or its lift \( \tilde{V}_2 \)) as this action captures the permutation but is in general much simpler. This would suggest to take \( \alpha \) as linear combination of wedge products of forms in \( \Lambda^1 V_2 \). Such forms are, however, in general not closed: the exterior derivative maps \( \Lambda^1 V_2 \to \Lambda^2 V_1 \) and in fact to \( \bigoplus_{i=1}^n \Lambda^2 V_{1,i} \). Thus we consider linear combinations of wedge products of \( \Lambda^2 V_{1,i} \) and \( \Lambda^1 V_2 \), and look for closed forms modulo exact forms.

6. Product-quotients with \( \dim K = 1 \)

In this section and the next we prove Theorem 5.1 for product quotients \( G = \tilde{G} / \exp(K) \) where \( \tilde{G} \) is a product of copies of the first Heisenberg group; we remark that in this case it suffices to assume that the Pansu differential is an isomorphism almost everywhere – the sign condition on the determinant will not be used. In Subsection 6.1 we fix notation that will remain in force for this section and the next. In
Subsection [6.2] we show that the case when \( \dim K = 1 \) reduces to the case that \( K \) is the diagonal subspace \( \sum_m \tilde{Y}_m \). In Subsection [6.3] we carry out the proof of the rigidity theorem, after first motivating the argument by comparing with the product case.

6.1. Notation for product quotients of the first Heisenberg group. Let \( \mathfrak{h} \) denote the first real Heisenberg algebra. We denote by \( \tilde{X}_1, \tilde{X}_2, \tilde{X}_3 \) its standard basis with \([\tilde{X}_1, \tilde{X}_2] = -\tilde{X}_3\). As usual we identify elements of the algebra with left invariant vector fields of the first real Heisenberg group \( H \). The elements of dual basis are denoted by \( \tilde{\theta}_i \) and satisfy \( d\tilde{\theta}_3 = \tilde{\theta}_1 \wedge \tilde{\theta}_2 \).

Let \( \tilde{\mathfrak{g}} = \bigoplus_{i=1}^p \tilde{\mathfrak{g}}_i \) where \( \tilde{\mathfrak{g}}_i \) is a copy of \( \mathfrak{h} \), and let \( \tilde{X}_1, \ldots, \tilde{X}_{3n} \) be the basis of \( \tilde{\mathfrak{g}} \), where \( \tilde{X}_{3i-2}, \tilde{X}_{3i-1}, \tilde{X}_{3i} \) form a standard basis for \( \tilde{\mathfrak{g}}_i \), i.e. \([\tilde{X}_{3i-2}, \tilde{X}_{3i-1}] = -\tilde{X}_{3i}\). The dual basis is \( \tilde{\theta}_1, \ldots, \tilde{\theta}_3 \) with \( d\tilde{\theta}_3 = \tilde{\theta}_{3i-2} \wedge \tilde{\theta}_{3i-1} \). The first layer subspace \( \tilde{V}_{1,i} \) is spanned by \( \tilde{X}_{3i-2} \) and \( \tilde{X}_{3i-1} \) and the second layer subspace \( \tilde{V}_{2,i} \) is spanned by \( \tilde{X}_{3i} \). To shorten notation, we will identify \( \tilde{V}_{1,i} \) with \( V_{1,i,j} \), and drop the tilde when denoting first layer vectors and forms. We further introduce the shorthand notation

\begin{equation}
\tilde{\tau}_i = \tilde{\theta}_{3i}, \quad \tilde{Y}_i = \tilde{X}_{3i}
\end{equation}

and

\begin{equation}
\gamma_i = \theta_{3i-2} \wedge \theta_{3i-1}, \quad Z_i = X_{3i-2} \wedge X_{3i-1}.
\end{equation}

We also consider

\begin{equation}
\tau, \quad \text{a volume form on } V_2 = \tilde{V}_2/K,
\end{equation}

and

\begin{equation}
\omega = \gamma_1 \wedge \ldots \wedge \gamma_n \wedge \tau, \quad \text{a volume form on } \tilde{\mathfrak{g}}.
\end{equation}

6.2. Reduction to convenient \( K \) by anisotropic dilation. For the remainder of this section, we will assume that \( \dim K = 1 \).

We now show that without loss of generality we may take \( K = \text{span}(\sum_{i=1}^n \tilde{Y}_i) \).

Lemma 6.5. There exists a graded automorphism \( \Psi: \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}} \) such that \( \Psi(\text{span}(\sum_{i=1}^n \tilde{Y}_i)) = K \). In particular, \( \Psi \) induces an isomorphism of graded algebras \( \tilde{\mathfrak{g}}/(\sum_i \tilde{Y}_i) \to \tilde{\mathfrak{g}}/K \).

Proof. Suppose that \( K \subset \bigoplus_{i \neq i'} V_{2,i} \) for some \( i' \). Since \( \text{Stab}(K, \text{Aut}(\tilde{\mathfrak{g}})) \) acts transitively on the collection of subspaces \( \{V_{1,i}\}_{1 \leq i \leq n} \) and hence also on the collection \( \{V_{2,i}\}_{1 \leq i \leq n} \) we deduce that \( K \subset \bigoplus_{i \neq i'} V_{2,i} \) for all \( i' \), so \( K \subset \bigcap_{i'} (\bigoplus_{i \neq i'} V_{2,i}) = \{0\} \). This contradicts the fact that \( \dim K = 1 \).

Therefore we have \( K = \text{span}(\sum_i \mu_i \tilde{Y}_i) \) where \( \mu_i \neq 0 \) for all \( i \).
For every \( i \), let \( \Psi_i : h \simeq \tilde{g}_i \to \tilde{g}_i \) be a graded automorphism with \( \Psi_i|_{\tilde{V}_2,i} = \mu_i \text{id}_{\tilde{V}_2,i} \), and define \( \Psi := \oplus \Psi_i : \tilde{g} \to \tilde{g} \). Then \( \Psi(\sum_i \tilde{Y}_i) = \sum_i \mu_i \tilde{Y}_i \) as desired.

In view of Lemma 6.5 we will from now on assume without loss of generality that

\[
K = \text{span}(\sum_{i=1}^n \tilde{Y}_i).
\]

**Proposition 6.7.** If \( \Phi \in \text{Aut}(g) \) and \( \tilde{\Phi} \in \text{Stab}(K, \text{Aut}(\tilde{g})) \) is the unique lift provided by Lemma 4.16, then there exist a \( \lambda \neq 0 \) and a permutation \( \sigma \in S_n \) such that

\[
\tilde{\Phi}^* \tilde{\tau}_i = \lambda \tilde{\tau}_{\sigma^{-1}(i)}, \quad \tilde{\Phi}^* \tilde{\gamma}_i = \lambda \tilde{\gamma}_{\sigma^{-1}(i)} \quad \forall i = 1, \ldots, n,
\]

or, equivalently,

\[
\tilde{\Phi}^* \tilde{Y}_i = \lambda \tilde{Y}_{\sigma(i)}, \quad \tilde{\Phi}^* \tilde{Z}_i = \lambda \tilde{Z}_{\sigma(i)}, \quad \Phi^* (Z_i) = \lambda Z_{\sigma(i)} \quad \forall i = 1, \ldots, n
\]

where \( Z_i = X_{3i-2} \wedge X_{3i-1} \).

**Proof.** Since \( \Phi \) permutes the collections \( \{\tilde{V}_{1,i}\}, \{\tilde{V}_{2,i}\} \) there exist \( \mu_i \) such that

\[
\Phi \tilde{Y}_i = \mu_i \tilde{Y}_{\sigma(i)}.
\]

Since \( \Phi \) preserves \( K \) the vector \( \sum_{j=1}^n \tilde{Y}_j \) must be an eigenvector of \( \Phi \)

\[
\Phi \sum_{j=1}^n \tilde{Y}_j = \lambda \sum_{j=1}^n \tilde{Y}_j.
\]

Comparison gives \( \mu_i = \lambda \) for all \( i = 1, \ldots, n \). From this identity all the assertions follow easily.

**6.3. Local constancy of the permutation.** Let \( f : G \supset U \to G \) be a \( W_{\text{loc}}^{1,p} \)-mapping for some \( p > \nu \), and assume that the Pansu differential \( D_P f(x) : g \to g \) is an isomorphism for a.e. \( x \in U \).

Our approach to controlling \( f \) is motivated by the treatment of products \( \prod_j H_j \), where we applied the pullback theorem to the wedge product \( f_P^*(d\tilde{\tau}_j \wedge \tilde{\tau}_j) \wedge d(\varphi \hat{\beta}) \) where \( 1 \leq j \leq n \) and \( \hat{\beta} \) is a suitably chosen closed codegree 4 form. The form \( \hat{\beta} \) may be expressed as \( i_X \beta \) where

\[
\beta = i_{Y_m} i_{X_{3m-2}} i_{X_{3m-1}} \omega = \pm d\tilde{\tau}_1 \wedge \tilde{\tau}_1 \wedge \ldots \wedge d\tilde{\tau}_m \wedge \tilde{\tau}_m \wedge \ldots \wedge d\tilde{\tau}_n \wedge \tilde{\tau}_n
\]

for some \( m \in \{1, \ldots, n\} \) and \( X \in \oplus_{j \neq m} V_{1,j} \). 
Note that in the present context the forms $\tilde{\tau}_i$ are not well-defined because they do not descend to the quotient $\mathfrak{g} = \tilde{\mathfrak{g}}/K$. We observe that the differences $\tilde{\tau}_{i,j} := \tilde{\tau}_i - \tilde{\tau}_j$ descend to well-defined forms $\tau_{i,j}$ for all $i \neq j$. This motivates our choice of the 3-forms

$$
\omega_{ij} := (\gamma_i + \gamma_j) \wedge (\tau_{i,j})
$$

whose pullback to $\tilde{\mathcal{G}}$ is $(d\tilde{\tau}_i + d\tilde{\tau}_j) \wedge (\tilde{\tau}_i - \tilde{\tau}_j)$ for $1 \leq i \neq j \leq n$. We will apply the pullback theorem to $f^*_P \omega_{ij} \wedge d(\varphi i_X \beta)$ where $\beta$ is a closed codegree 3 form given below. It follows from Proposition 6.7 that

$$
f^*_P(\omega_{ij}) = \lambda^2(x)(\omega_{\sigma^{-1}_j(i)\sigma^{-1}_j(j)}).
$$

At the heart of the matter are the following two identities.

**Lemma 6.12.** Let $m \in \{1, \ldots, n\}$ and set

$$
\beta = i_{Y_m} i_{X_{3m-2}} i_{X_{3m-1}} \omega
$$

where $Y_m := \tilde{Y}_m + K$ is the image of $\tilde{Y}_m$ under the projection $\tilde{\mathfrak{g}} \to \mathfrak{g}$. Then

$$
d(i_X \beta) = 0 \quad \forall X \in \oplus_{j \neq m} V_{1,j}
$$

and $i_X \beta$ is a form of codegree 4 and weight $\leq -\nu + 5$. Moreover,

$$
\omega_{kl} \wedge d(\varphi i_X \beta) = \begin{cases} 
0 & \text{if } k \neq m \text{ and } l \neq m \\
-(X\varphi)\omega & \text{if } k = m \text{ and } l \neq m \\
(X\varphi)\omega & \text{if } k \neq m \text{ and } l = m.
\end{cases}
$$

**Proof.** Using \(2.6\), \(2.18\), \(2.15\), and the identity $[X_{3m-2}, X_{3m-1}] = -Y_m$ we get

$$
d(i_X \beta) = -d(i_{Y_m} i_{X_{3m-2}} i_{X_{3m-1}} i_X \omega) = i_{Y_m} d(i_{X_{3m-2}} i_{X_{3m-1}} i_X \omega)
$$

$$
= -i_{Y_m} i_X i_{X_{3m-2}} i_{X_{3m-1}} \omega = i_{Y_m} i_X i_Y \omega = 0
$$

since $i_{Y_m} \circ i_{Y_m} = 0$.

Applying \(2.6\) and \(2.8\), we get:

$$
\omega_{kl} \wedge d(\varphi i_X \beta)
= \omega_{kl} \wedge d\varphi \wedge i_X i_{Y_m} i_{X_{3m-2}} i_{X_{3m-1}} \omega
= (\omega_{kl} \wedge d\varphi)(X_{3m-1}, X_{3m-2}, Y_m, X) \omega
= \omega_{kl}(X_{3m-1}, X_{3m-2}, Y_m) d\varphi(X) \omega.
$$

In the last step we used that

$$
\omega_{kl}(X, X_{3m-2}, X_{3m-1}) = (\gamma_k + \gamma_l) \wedge \tau_{k,l}(X, X_{3m-2}, X_{3m-1}) = 0
$$
because \( \tau_{k,l} \) vanishes on \( V_1 \), and \( \omega_{k,l}(X, X_{3m-i'}, \cdot) = 0 \) for \( i' \in \{1, 2\} \)

because \( \gamma_j(X, X') = 0 \) for \( X' \in V_{m,1} \) and \( X \notin V_{m,1} \). Since \( \gamma_j \) vanishes on \( V_2 \) and \( \tau_{k,l} \) vanishes on \( V_1 \) we get

\[
\omega_{k,l}(X_{3m-1}, X_{3m-2}, Y_m) = (\gamma_k + \gamma_l)(X_{3m-1}, X_{3m-2}) \tau_{k,l}(Y_m)
\]

\[
= -(\delta_{km} + \delta_{lm})(\delta_{km} - \delta_{lm})
\]

This concludes the proof of (6.15). \( \square \)

We now choose \( X \) and \( \beta \) as in Lemma 6.12 and let

(6.16)

\[
P_{mi}(x) = \delta_{\sigma^{-1}(i)} = \delta_{\sigma(m)i}.
\]

Since \( i_X \beta \) is closed, has codegree 4 and weight \( \leq -\nu + 5 \), while \( \omega_{ij} \) is closed, has degree 3 and weight \(-4\) we can apply Theorem 2.1 to \( f^\ast(\omega_{ij}) \wedge d(\varphi i X \beta) \) and using (6.11) and (6.15) we get

(6.17)

\[
X[\lambda^2(P_{mi} - P_{mj})] = 0
\]

in the sense of distributions, for every \( i, j, m \in \{1, \ldots, n\} \) and every \( X \in V_{1,\ell} \) with \( l \neq m \).

**Lemma 6.18.** The permutation \( \sigma \) and function \( \lambda^2 \) are constant almost everywhere.

**Proof.** Let \( h : \mathbb{R} \to [0, \infty) \) be given by \( h(t) = \max(t, 0) \). Then by (6.17) and Lemma 2.25

\[
Xh(\lambda^2(P_{mi} - P_{mj})) = 0
\]

for every \( i, j, m \in \{1, \ldots, n\} \) and every \( X \in V_{1,\ell} \) with \( l \neq m \). Now assume \( j \neq i \) and note that \( P_{mi} - P_{mj} \in \{-1, 0, 1\} \) and that \( P_{mi} - P_{mj} = 1 \) if and only if \( P_{mi} = 1 \). Thus

\[
h(\lambda^2(P_{mi} - P_{mj})) = \lambda^2 P_{mi}.
\]

Hence

\[
\forall i, m \forall l \neq m \forall X \in V_{1,\ell} \quad X(\lambda^2 P_{mi}) = 0.
\]

Since \( \sum_{i=1}^n P_{mi} = 1 \) we get

\[
X\lambda^2 = 0
\]

for all \( X \in V_{1,\ell} \) and \( l \neq m \). Choosing \( m = 1 \) and \( m = 2 \) we conclude that \( X\lambda^2 = 0 \) for all \( X \in V_1 \) and hence that \( \lambda^2 \) is constant a.e. and

\[
\forall i, m \forall l \neq m \forall X \in V_{1,\ell} \quad XP_{mi} = 0.
\]

In particular

\[
XP_{m'i} = 0 \quad \text{for } m' \neq m \text{ and } X \in V_{1,m}.
\]
Since $P_{mi} = 1 - \sum_{m' \neq m} P_{m'i}$ we deduce that $XP_{mi} = 0$ for all $X \in V_1$. Thus the matrix $P$ and the permutation $\sigma$ are constant a.e. $\square$

7. Indecomposable product-quotients with $\dim K \geq 2$

In this section we will retain the notation and conventions from Subsection 6.1. The goal of this section is to prove Theorem 5.1 for indecomposable product quotients $G = (\prod_{i=1}^n G_i)/\exp(K)$ where $G_i$ is a copy of the first Heisenberg group and $\dim K \geq 2$. The argument in this case is significantly more complicated than the case when $\dim K = 1$, so we will first give some motivation and an overview in Subsection 7.1.

7.1. Overview of the proof. We recall that in the $\dim K = 1$ case, our approach was to apply the pullback theorem to the closed left invariant forms $\omega_{ij}$ and $i_X \beta$, which are of degree 3 and codegree 4, respectively, and have appropriate weight. It turns out that when $\dim K \geq 2$ such forms typically do not exist, so we use a different strategy.

We first reduce to the case when the lift $\tilde{\Phi} \in \text{Stab}(K, \text{Aut}(\hat{g}))$ of every graded automorphism $\Phi \in \text{Aut}(\mathfrak{g})$ acts orthogonally on $\tilde{V}_2$ with respect to the inner product induced by the basis $\{\tilde{Y}_1, \ldots, \tilde{Y}_n\} \subset \tilde{V}_2$. In particular letting $\tilde{D}_P f(x)$ denote the lift of the Pansu differential $D_P f(x)$ of our Sobolev mapping $f : U \rightarrow G$, we note that $f$ is somewhat reminiscent of a conformal mapping, in the sense that its differential is conformal, although only when restricted to the second layer.

The above observation suggests that we look to the proof of Liouville’s theorem for inspiration. In its simplest form it states that a Lipschitz map $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ which satisfies $\nabla f \in SO(n)$ a.e. has locally constant gradient. The key idea is to use the relations $\nabla \nabla f = 0$ and $\text{div cof} \nabla f = 0$ for every Lipschitz map. In connection with the pointwise identity $\text{cof} \nabla f = \nabla f$ a.e. they imply that $f$ is weakly harmonic and hence smooth. The differential constraints on $f$ can be derived by pulling back constant differential forms of degree 1 and codegree 1. In our setting we are interested in the action of the Pansu differential on the second layer. This action can be captured by using special degree 2 forms in the first layer (namely elements of $\bigoplus_{i=1}^n \Lambda^2 V_{1,i}$). Thus the analogy with the proof of Liouville’s theorem suggests the use of the closed degree 2 forms $\gamma_i = \theta_{3i-2} \wedge \theta_{3i-1}$ and closed codegree 2 forms of the type $i_Z \omega$ where $\omega$ is a volume form on $G$ (see (6.4)) and $Z$ is a two-vector in $\bigoplus_{i=1}^n \Lambda^2 V_{1,i}$. Indeed Lemma 7.14 can be seen as a counterpart of the property $\nabla f = 0$ while (7.25) can be viewed as a counterpart of the identity $\text{div cof} \nabla f = 0$. 
The use of these forms will show that the lift $\tilde{D}_P f(x) \in \operatorname{Stab}(K, \operatorname{Aut}(\tilde{g}))$ of the Pansu differential $D_P f(x)$ acts as a constant on $K$ (see Theorem 7.24 below). We then look at the group $H'$ of graded automorphisms of $\tilde{g} = \oplus_{i=1}^n g_i$ which act as the identity on $K$. If the induced action $H' \acts I = \{1, \ldots, n\}$ (which reflects the permutation of the collection $\{V_{2,i}\}$ by such a graded automorphism) is trivial we are done. If not, then we have the induced decomposition $I = I_1 \sqcup \ldots I_k$ into $H'$-orbits, and it is easy to see that the projection of $K$ onto each of the corresponding subspaces $V_{i_j} = \oplus_{k \in I_j} V_{2,k}$ is one-dimensional (see Lemma 7.26 below). Then we can use the argument for the case $\dim K = 1$ to show that the permutation induced by $D_P f(x)$ is locally constant on each orbit and hence locally constant (see Section 7.5 below).

7.2. Set-up. Retaining the notation from Section 6.1 we consider $\tilde{g} = \sum_{i=1}^n g_i$, where $g_i$ is a copy of the first Heisenberg algebra, and let $g = \tilde{g}/K$ be an indecomposable product-quotient. By Lemma 4.27 there exists a scalar product $b$ on $\tilde{V}_2$ such that the action of $\operatorname{Stab}(K, \operatorname{Aut}(\tilde{g}))$ is conformal, i.e., invariant up to scaling; in other words, $g$ is a conformal product quotient, see Definition 4.25.

**Proposition 7.1.** There exists an anisotropic dilation $\delta_\mu$ such that the action of $\operatorname{Stab}(\delta_\mu K, \operatorname{Aut}(\tilde{g}))$ is conformal with respect to the standard scalar product on $V_2$ which is characterized by the identities $(\tilde{Y}_i, \tilde{Y}_j) = \delta_{ij}.$

**Proof.** We first note that for every anisotropic dilation $\delta_\mu$ the quotient $\tilde{g}/\delta_\mu K$ is an indecomposable product quotient and that the push-forward scalar product $(\delta_\mu)_* b$ is $\operatorname{Aut}(\tilde{g}/\delta_\mu K)$ invariant up to scaling. Indeed, $\delta_\mu$ is a graded automorphism of $\tilde{g}$ that sends the ideal $K$ to $\delta_\mu K$. Thus $\delta_\mu$ induces a graded isomorphism from $\tilde{g}/K$ to $\tilde{g}/\delta_\mu K$. Moreover $\delta_\mu : (\tilde{V}_2, b) \to (\tilde{V}_2, (\delta_\mu)_* b)$ is a linear isometry and $\operatorname{Stab}(\delta_\mu K, \tilde{g}) = \delta_\mu \circ \operatorname{Stab}(K, \tilde{g}) \circ \delta_\mu^{-1}$. Hence $\operatorname{Stab}(\delta_\mu K, \tilde{g})$ preserves $(\delta_\mu)_* b$ up to scaling.

We now choose a special dilation by taking $\mu_i = (b(Y_i, Y_i))^{1/2}.$ Then

$$\tag{7.2} (\delta_\mu)_* b(\tilde{Y}_i, \tilde{Y}_i) = b(\delta_\mu^{-1} \tilde{Y}_i, \delta_\mu^{-1} \tilde{Y}_i) = \mu_i^{-2} b(\tilde{Y}_i, \tilde{Y}_i) = 1.$$ 

The action of $\Phi \in \operatorname{Stab}(\delta_\mu K, \operatorname{Aut}(\tilde{g}))$ on the second layer can be written as a product of a permutation and an invertible diagonal operator. In view of (7.2) permutations preserve $(\delta_\mu)_* b.$ Since $\Phi$ preserves $(\delta_\mu)_* b$ up to a scalar factor the diagonal operator must also preserve that scalar product up to a scalar factor. It follows that the norm of
the eigenvalues of the diagonal operator is constant. Then the diagonal operator also preserves the standard scalar product up to a scalar factor. Since permutations preserve the standard scalar product the assertion follows. □

To simplify the notation we will assume from now on without loss of generality

\[ \text{Stab}(K, \text{Aut}(\tilde{g})) \text{ acts conformally on the second layer (7.3) with respect to the standard scalar product.} \]

**Proposition 7.4.** Let \( \Phi \in \text{Stab}(K, \text{Aut}(\tilde{g})) \) and let \( \sigma = \sigma_\Phi \) be the induced permutation of the subspaces \( V_{1,i} \) (and \( V_{2,i} \)). Then there exist \( \lambda_\Phi \in (0, \infty) \), a diagonal matrix \( S_\Phi \) with entries \( s_i \in \{-1, 1\} \), and a permutation matrix \( P_\Phi \) with the following properties. If \( m \in \mathbb{R}^n \) and

\[
Y = \sum_{j=1}^{n} m_j Y_j, \quad Z = \sum_{j=1}^{n} m_j Z_j
\]

then

\[
\Phi(Y) = \sum_{i=1}^{n} (\lambda_\Phi S_\Phi P_\Phi m)_i Y_i, \quad \Phi^*(Z) = \sum_{i=1}^{n} (\lambda_\Phi S_\Phi P_\Phi m)_i Z_i.
\]

Moreover \((P_\Phi)_{ij} = \delta_{i\sigma_\Phi(j)}\) and \( S_\Phi \) and \( \lambda_\Phi \) are uniquely determined and

\[ S_\Phi \in O(n), \quad P_\Phi \in O(n). \]

(Note that this definition of \( P \) differs from the one in Sections 8 and 9 by a transpose.)

**Proof.** There exist \( \mu_i \neq 0 \) such that \( \Phi(Y_j) = \mu_{\sigma(j)} Y_{\sigma(j)} \). It follows from (7.3) that there exist a \( \lambda \in (0, \infty) \) such that \( |\mu_{\sigma(j)}| = \lambda \) for all \( j = 1, \ldots, n \). Set \( s_i = \mu_i/\lambda \). Then \( s_i \in \{-1, 1\} \) and \( \Phi(Y'_j) = \sum_{i=1}^{n} (\lambda S P)_{ij} Y_i \). Thus the formula for \( \Phi(Y) \) follows by linearity.

The formula for \( \Phi(Z) \) is proved similarly. Alternatively one can use the following facts. There is a unique linear map \( L_{[\_]} : \Lambda_2 V_1 \to V_2 \) such that \( L_{[\_]}(X \wedge Y) = [X, Y] \). This map commutes with every graded homomorphism \( \Phi \), i.e., \( \Phi \circ L_{[\_]} = L_{[\_]} \circ \Phi^* \). For the first Heisenberg group the restriction of \( L_{[\_]} \) to \( \Phi_{i=1}^{n} \Lambda_2 V_{1,i} \) is a linear isomorphism onto \( V_2 \) and \( L_{[\_]} Z_i = -Y_i \).

Uniqueness of \( \lambda = |\Phi(Y_j)| \) and the \( s_i \) is verified easily. The property (7.5) follows directly from the definition of \( P \) and \( S \). □
7.3. Restrictions on the Pansu differential arising from the pullback of 2-forms.

For the remainder of Section 7 we fix for some $p > \nu$ a $W^{1,p}_{\text{loc}}$-mapping $f : G \supset U \to G$, where $U$ is open and the sign of the determinant of the Pansu differential $D_Pf(x) : g \to g$ is constant almost everywhere.

At each point $x \in U$ of Pansu differentiability the Pansu differential, viewed as an automorphism of the algebra $g = \tilde{g}/K$, can be identified with an element of $\text{Stab}(K, \text{Aut}(\tilde{g}))$, which we denote by

$$D_Pf(x) \in \text{Stab}(K, \text{Aut}(\tilde{g})).$$

Recalling the notation in Proposition 7.4 we set

$$\lambda(x) = \lambda_{D_Pf(x)}, \quad S(x) = S_{D_Pf(x)}, \quad P(x) = P_{D_Pf(x)}.$$

We also define

$$\hat{K} := \{m \in \mathbb{R}^n : \sum_{i=1}^n m_i Y_i \in K\}.$$

Note that $\tilde{D}_Pf(x)$ preserves $K$ and thus by Proposition 7.4

$$S(x)P(x)\hat{K} = \hat{K}.$$

The key calculation is contained in the following result. Recall that

$$\gamma_i = \theta_{2i-1} \wedge \theta_{2i}$$

and $Z_i = X_{3i-2} \wedge X_{3i-1}$.

**Lemma 7.10.** Let $k \in \{1, \ldots, n\}$, $X \in V_{1,k}$ and $Z = \sum_{i=1}^n m_i Z_i$. Then

$$d(i_Z\omega) = 0 \quad \forall m \in \hat{K},$$

$$d(i_Xi_Z\omega) = 0 \quad \forall m \in \hat{K} \cap e_k^\perp.$$

For $\varphi \in C_0^\infty(U)$ we have

$$f^*_p(\gamma_i) \wedge d(\varphi i_Xi_Z\omega) = (\lambda SPm)_i X \varphi \omega \quad \forall m \in \hat{K} \cap e_k^\perp.$$

**Proof.** Since $Z_i = X_{3i-2} \wedge X_{3i-1}$ and $[X_{3i-2}, X_{3i-1}] = -\hat{Y}_i$ we get from (2.6) and (2.14)

$$d(i_Z\omega) = d(i_{X_{3i-1}}i_{X_{3i-2}}\omega) = i_{Y_i}\omega$$

where $i_{Y_i}\omega$ is a shorthand for $i_{Y_i+K}\omega$ (see the end of Section 2.2). If $m \in \hat{K}$ then $Y := \sum_{i=1}^n m_i Y_i \in K$ and thus

$$d(i_Z\omega) = i_Y\omega = 0$$

since $\omega$ annihilates $K$. Similarly to prove (7.12) we use (2.15) and get

$$d(i_Xi_Z\omega) = -i_Xi_Y\omega \quad \forall i \neq k.$$

Multiplying by $m_i$, summing over $i$ and using that $m_k = 0$ we get $d(i_Xi_Z\omega) = -i_Xi_Y\omega = 0$ since $Y \in K$. 

To show (7.13) we compute
\[ f^*_P(\gamma_i) \wedge d(\varphi i_X i_Z \omega) = (f^*_P(\gamma_i) \wedge d\varphi)(Z \wedge X) \omega = f^*_P(\gamma_i)(Z) d\varphi(X) \omega. \]

To get the second equality we used the identity \( i_X \circ i_Z = i_Z \wedge X \) (see (2.6)) and (2.8). For the last equality we used \( m_k = 0 \) and \( f^*_P(\gamma_i)(X, Y) = 0 \) if \( X \in V_{1,k} \) and \( Y \in V_{1,j} \) with \( j \neq k \) since \( f^*_P(\gamma_i) \) is a linear combination of the forms \( \gamma_j \). Now it follows from Proposition 7.4 that
\[ f^*_P(\gamma_i)(Z) = \gamma_i((D_P f)_* Z) = \sum_{j=1}^m (\lambda SP)_{ij} m_j = (\lambda SP m)_i. \]

This concludes the proof of (7.13). \( \square \)

**Lemma 7.14.** Let \( k \in \{1, \ldots, n\} \). Then
\begin{equation}
X(\lambda SP m) = 0 \quad \forall X \in V_{1,k}, \ m \in \hat{\mathcal{K}} \cap e^\perp_k,
\end{equation}
in the sense of distributions on \( U \) and
\begin{equation}
\lambda \text{ is locally constant almost everywhere in } U.
\end{equation}

**Proof.** By Lemma 7.10 the form \( i_X i_Z \omega \) is a closed codegree 3 form with weight \( \leq -\nu + 3 \). Thus \( d(\varphi i_X i_Z \omega) = d\varphi \wedge i_X i_Z \omega \) is a codegree 2 form with weight \( \leq -\nu + 2 \). Moreover \( \gamma_i \) is a closed 2-form of weight \( -2 \). Hence the identity (7.15) follows from the Theorem 2.1 and equation (7.13).

To prove (7.16) it suffices to show \( X \lambda = 0 \) for all \( X \in V_{1,k} \) and all \( k = 1, \ldots, n \). Fix \( k \). Since \( \dim K \geq 2 \) the set \( \hat{\mathcal{K}} \cap e^\perp_k \) contains an element \( m \) with \( |m| = 1 \). Since \( P \) and \( S \) are isometries and \( \lambda > 0 \) we have \( \lambda = |\lambda SP m| \). Thus the desired assertion \( X \lambda = 0 \) follows from Lemma 2.25 by taking \( h(v) = |v| \). \( \square \)

### 7.4. Restrictions from codegree 2 forms and constant action on \( K \)

We continue the analysis of the Sobolev map \( f : U \to G \). The map \( SP \) is an isometry of \( \mathbb{R}^n \) and maps \( \hat{\mathcal{K}} \) to itself. Hence \( (SP)^T = (SP)^{-1} \) also maps \( \hat{\mathcal{K}} \) to itself. Thus (7.13) is equivalent to
\begin{equation}
X((\lambda SP)^T a, m) = 0 \quad \forall a \in \hat{\mathcal{K}}, \ m \in \hat{\mathcal{K}} \cap e^\perp_k, \ X \in V_{1,k}.
\end{equation}

We already know that \( \lambda \) is locally constant. To show that \( SP|_{\hat{\mathcal{K}}} \) is locally constant we need to show that in addition
\begin{equation}
X((SP)^T a, e_k) = 0 \quad \forall a \in \hat{\mathcal{K}}, \ X \in V_{1,k}.
\end{equation}
This will be achieved by pulling back suitable closed codegree 2 forms. The main identities are contained in the following lemma.

Lemma 7.19. Let \( \Phi \in \text{Stab}(K, \text{Aut}(\tilde{g})) \), viewed as an element of \( \text{Aut}(g) \), let \( \lambda_\Phi, S_\Phi \) and \( P_\Phi \) be the quantities introduced in Proposition 7.4. Let \( m \in \hat{K} \) and set \( Z = \sum_{i=1}^n m_i Z_i \). Then

\[
\det \Phi = \lambda_\Phi^{2n-\dim K},
\]

and for every two-form \( \alpha \)

\[
\Phi^* (i_Z \omega) \wedge \alpha = \det \Phi (\Phi^{-1} Z) \omega.
\]

Proof. The determinant of \( \Phi \) is characterised by the identity \( \Phi^* (\omega) = \det \Phi \omega \). It follows from Proposition 7.4 that

\[
\Phi^* (\bigwedge_{i=1}^n \gamma_i) = \lambda^n \det (SP) \bigwedge_{i=1}^n \gamma_i = \pm \lambda^n \bigwedge_{i=1}^n \gamma_i.
\]

Moreover \( V_2 = \tilde{V}_2 / K \) can be identified with \( K^\perp \) and \( \lambda^{-1} \Phi \), viewed as a map on \( \tilde{V}_2 \), is an isometry and preserves \( K \) and hence \( K^\perp \). Thus \( \lambda^{-1} \Phi \) restricted to \( K^\perp \) has determinant \( \pm 1 \) and hence

\[
\Phi^* (\tau) = \pm \lambda^{n-\dim K} \tau
\]

for every volume form \( \tau \) on \( V_2 \). Together with (7.23) this implies (7.20).

The second assertion is just (7.11).

To show (7.22) we compute

\[
\Phi^* (i_Z \omega) \wedge \alpha = \Phi^* (i_Z \omega \wedge (\Phi^{-1})^* \alpha)
\]

\[
= \det \Phi i_Z \omega \wedge (\Phi^{-1})^* \alpha
\]

\[
= \det \Phi ((\Phi^{-1})^* \alpha) (Z) \omega = \det \Phi \alpha (\Phi^{-1} Z) \omega.
\]

\[\square\]

Theorem 7.24. The restriction of \( \lambda SP \) to \( \hat{K} \) is locally constant almost everywhere in \( U \).

Proof. Let \( Z \) be as in Lemma 7.19 and pick \( k \in \{1, \ldots, n\} \). Let \( \eta = \varphi \theta_{3k-1} \) and \( \varphi \in C_\infty^\infty (U) \). Thus \( d\eta = d\varphi \wedge \theta_{3k-1} \). Now \( i_Z \omega \) has codegree 2 and weight \( -\nu + 2 \) while the form \( d\eta \) has weight \( \leq -2 \). Thus the pullback theorem applies to \( f_P^* (i_Z \omega) \wedge d\eta \). Since the sign of \( \det D_P f \) is constant almost everywhere by hypothesis, after post-composing with a graded automorphism if necessary, we may assume without loss of generality that \( \det D_P f \) is positive a.e. and then Lemma 7.19 implies that

\[
f_P^* (i_Z \omega) \wedge d\eta = \lambda^{2n-\dim K} (d\varphi \wedge \theta_{3k-1}) ((D_P f)^{-1} (Z)) \omega.
\]
Since \((D_P f)^{-1}(Z) \in \oplus_{i=1}^n \Lambda_2 V_{1,i}\) only the term with \(i = k\) contributes and we get
\[
(d\varphi \wedge \theta_{3k-1})((D_P f)^{-1}(Z)) = (X_{3k-2} \varphi) \gamma_k ((D_P f)^{-1} Z)
= (X_{3k-2} \varphi) \lambda^{-1} ((SP)^{-1} m, e_k).
\]
Since \(\lambda\) is locally constant it follows that
\[
X_{3k-2}((SP)^{-1} m, e_k) = 0 \quad \forall m \in \hat{K}.
\]
Using the form \(\eta = \varphi \theta_{3k-2}\) we get the same assertion with \(X_{3k-2}\) replaced by \(X_{3k-1}\). Using that \((SP)^{-1} = (SP)^T\) we thus conclude that
\[
X((SP)^T a, e_k) = 0 \quad \forall a \in \hat{K}, X \in V_{1,k}.
\]
Combining this with (7.17) we deduce that \(X(SP)^T a = 0\) for all \(a \in \hat{K}\), all \(X \in V_{1,k}\) and all \(k\). Therefore \(\lambda SP\) is locally constant, as desired. \(\square\)

### 7.5. Constancy of the permutation.

It follows from Theorem 7.24 that after composing \(f\) with a graded automorphism of \(G\) and shrinking \(U\) we may assume without loss of generality that \(\tilde{D}_P f\) acts as the identity on \(K\). In this subsection we analyse the graded automorphisms which act as the identity on \(K\) and then show that the permutation induced by \(\tilde{D}_P f\) is locally constant.

To simplify notation, we let \(H := \text{Aut}(\mathfrak{g})\) be the graded automorphism group of \(\mathfrak{g}\). By Lemma 4.16 we know that \(H\) canonically embeds in \(\text{Aut}(\tilde{\mathfrak{g}})\) as the stabilizer of \(K \subset \tilde{V}_2 \subset \tilde{\mathfrak{g}}\). Therefore we have actions \(H \curvearrowright \mathfrak{g}\), \(H \curvearrowright \tilde{\mathfrak{g}}\) by graded automorphisms, as well as the induced action \(H \curvearrowright I = \{1, \ldots, n\}\) on the (set of indices of the) summands \(\tilde{\mathfrak{g}}_i\), and the restriction action \(H \curvearrowright \hat{K}\). We let
\[
H' := \{h \in H \mid h \cdot W = W, \forall W \in K\} \subset H
\]
be the ineffective kernel of the action \(H \curvearrowright K\); since \(H'\) is the kernel of a homomorphism, it is a normal subgroup of \(H\).

We let \(I = I_1 \sqcup \ldots \sqcup I_k\) be the decomposition of \(I\) into the distinct orbits of the action \(H' \curvearrowright I\); as \(H'\) is normal in \(H\), the \(H'\)-orbit decomposition is respected by \(H\).

We let \(\tilde{V}_2 = \oplus_{i \in I_j} \tilde{V}_{2,i}\), so we have direct sum decomposition \(\tilde{V}_2 = \oplus_j \tilde{V}_{2,I_j}\). For every \(j \in \{1, \ldots, k\}\), let \(K_j := \pi_{\tilde{V}_{2,I_j}}(K)\).

**Lemma 7.26.** \(\dim K_j = 1\) for all \(j \in \{1, \ldots, k\}\).
Proof. We first claim that $H$ preserves the collection $\{K_j\}_{1 \leq j < k}$, and acts transitively on it. To see this, note that $H$ preserves $K$ and the direct sum decomposition $\tilde{V}_2 = \bigoplus_j \tilde{V}_{2, I_j}$. Therefore if $h \in H$ and $h(I_j) = I_j'$, then $h \circ \pi_{\tilde{V}_{2, I_j}} = \pi_{\tilde{V}_{2, I_j'}} \circ h$, so $h(K_j) = K_j'$. Since $H \subset \{I_1, \ldots, I_k\}$ is transitive, this proves the claim.

It follows that all the $K_j$s have the same dimension, and since $K \subset \bigoplus_j K_j$ we must have $\dim K_j \geq 1$ for all $j$.

Now suppose $\dim K_j \geq 2$ for some $j$. Choose $i_0 \in I_j$. Since $|I_j| \geq \dim K_j \geq 2$, the set $I_j \setminus \{i_0\}$ is nonempty, and it follows that $K_j$ intersects the hyperplane $\bigoplus_{i \in I_j \setminus \{i_0\}} \tilde{V}_{2, i}$ in some vector $W \neq 0$. Let $J := \{i \in I_j \mid \pi_{\tilde{V}_{2, i}}(W) \neq 0\}$. Note that $J$ is invariant under $H'$, since $H'$ fixes $W$ and preserves the direct sum decomposition $\tilde{V}_2 = \bigoplus_i \tilde{V}_{2, i}$.

Since $I_j$ is an $H'$-orbit, we must have $J = I_j$. This contradicts the fact that $J \subset I_j \setminus \{i_0\}$. \hfill $\square$

Using an argument similar to Lemma 6.5 up to a graded isomorphism, we may assume without loss of generality that each $K_j \subset V_{2, I_j}$ is “diagonal”, i.e. $K_j = \text{span}(\sum_{i \in I_j} Y_i)$ for all $1 \leq j \leq k$.

Theorem 7.27. The permutation $\sigma_x$ induced by the Pansu differential $\tilde{D}_p f(x)$ is locally constant almost everywhere in $U$.

Proof. Recall that we have reduced to the case when $\tilde{D}_p f(x)$ acts as the identity on $K$ and $K_j = \text{span}(\sum_{i \in I_j} Y_i)$ for all $j \in \{1, \ldots, k\}$. Thus $\tilde{D}_p f(x) \in H'$ for a.e. $x \in U$. If $H' = \{\text{id}\}$ we are done. So we assume that $H' \neq \{\text{id}\}$ and will show that $\tilde{D}_p f(x)$ acts as a constant permutation on each $H'$-orbit $I_j$, for $j \in \{1, \ldots, k\}$. Then we can argue as in the case $\dim K = 1$ to deduce constancy of the permutation.

We provide some details for the convenience of the reader. Let $I$ be an orbit and $\tilde{V}_{2, I} = \bigoplus_{i \in I} \tilde{V}_{2, i}$. The map $\tilde{D}_p f(x)$ preserves $\tilde{V}_{2, I}$ and acts as the identity on the subspace $K_I = \text{span}(\sum_{i \in I} \tilde{Y}_{2, i})$. Thus the argument in the proof of Proposition 6.7 shows that the identities in that proposition hold with $\lambda = 1$. Recall that $P_{mi}(x) = \delta_{m\sigma_x^{-1}(i)}$. Applying the pullback theorem to the forms $f_p^* \omega_{ij} \wedge d(\phi_i X \beta)$ where $\omega_{ij}$ and $\beta$ are as in (6.10) and (6.13) with $i, j, m \in I$ and where $X \in V_{1, l}$ with $l \neq m$ we get as in the proof of Lemma 6.18 the distributional identity $XP_{mi} = 0$ for all $m, i \in I$ and all $X \in V_{1, l}$ with $l \neq m$. Since $\sigma_x(I) = I$ we have $\sum_{m \in I} P_{mi} = 1$ for all $i \in I$. Now for $X \in V_{1, m}$ we get $XP_{mi} = X(1 - \sum_{m' \in I \setminus \{m\}} P_{m' i}) = 0$. Thus $P_{mi}$ is constant for $m, i \in I$ and hence $\sigma_x|_I$ is constant. \hfill $\square$
8. Product quotients of complex Heisenberg groups and higher real Heisenberg groups

In this section we prove Theorem 5.1 for product quotients $G = \tilde{G}/\exp(K)$ where $\tilde{G}$ is a product of copies of a higher real Heisenberg group or complex Heisenberg group. We retain the notation for product quotients from Section 5. The approach used in this section is different, and conceptually simpler, than the ones used in Sections 6 and 7. The main difference between these two situations is that for the higher real and the complex Heisenberg group there exist linearly independent vectors $X, Y$ in $V_{1,i}$ with $[X, Y] = 0$ while this is not the case for the first Heisenberg group. More systematically, consider the map

$$L[] : \Lambda_2 V_1 \rightarrow V_2, \quad L[](X \wedge Y) = [X, Y].$$

For the complex Heisenberg groups and higher real Heisenberg groups

$$k_i := \ker L[] \cap V_{1,i}$$

is non-trivial (as follows from a dimension count) while for the first Heisenberg group $L[] : \Lambda_2 V_{1,i} \rightarrow V_2$ is injective. Any graded automorphism – in particular the Pansu differential – permutes the subspaces $\ker_i$. Now for complex or higher real Heisenberg product quotients we can pull back a collection of forms $\alpha \in \Lambda^2 V_{1,i}$ which restrict to a basis of the dual space $\ker_i'$ and take $\beta = i_X i_Z \omega$ where $X \in \ker_k$ and $Z \in \oplus_{k' \neq k} V_{1,k'}$. The choice of this collection of forms $\alpha$ may look somewhat adhoc at this point, but it is in fact related to our general strategy to look for closed forms modulo exact forms. The point is that the map $L[]$ is the dual of the exterior differential $d : \Lambda^1 V_2 \rightarrow \Lambda^2 V_1 \simeq (\Lambda_2 V_1)'$, see Remark 8.14 after the end of the proof. With the above choice of $\alpha$ and $\beta$ a short argument similar to the one used for products shows that the permutation of the subspaces induced by $D_P f$ must be locally constant. The subspace $K$ plays no role in this argument. In fact the argument can also be used to obtain an alternative proof for products of the complex or the higher real Heisenberg groups.

We now turn to the formal proof. Note first that (2.15), (2.16), the relation $[V_{1,i}, V_{1,j}] = 0$ for $i \neq j$ and (2.6) imply that

$$d(i_Y i_Y, i_Z \omega) = -i_Z i_{[Y,Y']} \omega \quad \forall \ Y, Y' \in V_{1,i}, \ Z \in V_{1,j} \text{ with } i \neq j.$$  

Lemma 8.4. Let $k \neq k', \ Y, Y' \in V_{1,k}, \ Z \in V_{1,k'}$ and $\gamma \in \oplus_{j=1}^n \Lambda^2(V_{1,j})$. For $U \subset G$ open let $\varphi \in C^1(U)$. Then the following identities hold in
(8.5) \[ d(\varphi Y_i Y_i' Y_i Z \omega) = (Y \varphi) Y_i Y_i' Y_i Z \omega + (Z \varphi) Y_i Y_i' Y_i Z \omega + \varphi d(Y_i Y_i' Y_i Z \omega), \]

(8.6) \[ \gamma \wedge d(\varphi Y_i Y_i' Y_i Z \omega) = -\gamma(Y, Y') (Z \varphi) \omega + \varphi \gamma \wedge (i Y_i Y_i' Y_i Z \omega). \]

If \( X \in \Lambda_2 V_{1,k} \) then

(8.7) \[ \gamma \wedge d(\varphi X_i Y_i Y_i' Y_i Z \omega) = \gamma(X) (Z \varphi) \omega + \varphi \gamma \wedge (i X_i Y_i Y_i' Y_i Z \omega). \]

**Proof.** We have

\[
\begin{align*}
d(\varphi Y_i Y_i' Y_i Z \omega) - \varphi d(Y_i Y_i' Y_i Z \omega) &= d\varphi \wedge i Y_i Y_i' Y_i Z \omega, \\
&= (Y \varphi) Y_i Y_i' Y_i Z \omega - i_Y(d\varphi \wedge Y_i Y_i' Y_i Z \omega) \\
&= (Y \varphi) Y_i Y_i' Y_i Z \omega - i_Y((Y' \varphi) Y_i Y_i' Y_i Z \omega) + i_Y Y_i Y_i' Y_i Z \omega(d\varphi \wedge Y_i Y_i' Y_i Z \omega) \\
&= (Y \varphi) Y_i Y_i' Y_i Z \omega - (Y \varphi) Y_i Y_i' Y_i Z \omega + (Z \varphi) Y_i Y_i' Y_i Z \omega.
\end{align*}
\]

In the last step we used (2.8). This proves (8.5).

The assertion (8.6) now follows from (2.8) and the fact that \( \gamma(Y', Z) = \gamma(Y, Z) = 0 \) since \( Z \) lies in a different first layer subspace than \( Y \) and \( Y' \). Note that (2.6) gives the identity \( i_Y \circ i_Y' = -i_{Y \wedge Y'} \) and thus \( i_Y i_Y' \gamma = -\gamma(Y, Y') \).

Finally for \( X = -Y \wedge Y' \) (8.7) is the same as (8.6) and for general \( X \) the identity (8.7) follows by linearity. \( \square \)

We now assume that \( f : G \supset U \to G \) is a \( W^{1,p}_{\text{loc}} \)-mapping for some \( p > \nu \) such that \( D_P f(x) \) is an isomorphism for a.e. \( x \in U \). Recall that \( \ker_i \) is defined in (8.2). The main point in the proof of Theorem 5.1 for product quotients of the higher groups is to show the following identity

(8.8) \[
\int_U \alpha((D_P f)_* X) (Z \varphi) = 0 \quad \forall \varphi \in C_c^\infty(U),
\]

\[ \forall \alpha \in \bigoplus_{j=1}^n \Lambda^2 V_{1,j}, \quad X \in \ker_k, \quad Z \in V_{1,k'} \quad \text{with } k' \neq k. \]

Then we can deduce easily that the permutation induced by \( D_P f \) is locally constant.

To prove (8.8) fix \( \alpha, X, Z \) and \( \varphi \) and consider the codegree 3 forms

\[ \beta = i_{X \wedge Z} \omega \quad \text{and} \quad \eta = \varphi \beta. \]
It follows from (8.3) and linearity that $\beta$ is closed. Moreover $\text{wt}(\beta) = -\nu + 3$. Thus Theorem 2.1 implies that

$$\int f^*_{P_\alpha} \wedge d\eta = 0.$$  

(8.9)

Since $D_P f$ permutes the first layer subspaces $V_{1,j}$ it follows that $f^*_{P_\alpha} \in \oplus_{j=1}^n \Lambda^2 V_{1,j}$. Thus we get from (8.7) and (8.3)

$$f^*_{P_\alpha} \wedge d\eta = f^*_{P_\alpha}(X)(Z \varphi) \omega.$$  

This yields (8.8).

To deduce properties of the permutation $\sigma_x$ induced by $D_P f(x)$ recall that $D_P f(x)$ maps $\ker_k \subset \Lambda_2 V_{1,k}$ to $\ker_{\sigma_x(k)}$. Let

$$P_{kl}(x) = \delta_{\sigma_x(k)l}.$$  

If we choose basis vectors $X_{k,m}$ of $\ker_k$ then there exist measurable functions $G_{mm'} : U \to \mathbb{R}$ such that

$$(D_P f)_* X_{k,m} = \sum_{l=1}^n P_{kl} \sum_{m'} G_{m'm} X_{l,m'}$$

and the matrix $G(x)$ is invertible for a.e. $x$. Let $\alpha_{k,m'} \in \Lambda^2 V_{1,k}$ be chosen such that the restrictions to $\ker_k$ yield a basis of the dual space $\ker'_k$ which is dual to the basis $X_{k,m'}$, i.e. $\alpha_{k,m'}(X_{k,m}) = \delta_{mm'}$ for each $k = 1, \ldots, n$. Applying (8.8) with $X = X_{k,m}$ and $\alpha = \alpha_{l,m'}$ we get for each $k$ and $l$

$$\int_U P_{kl} G(Z \varphi) \omega = 0 \quad \forall \varphi \in C^\infty_c(U), \ Z \in V_{1,k'} \quad \text{with } k' \neq k.$$  

(8.10)

Define a function $h$ on the space of matrices by $h(G) = 1$ if $\det G \neq 0$ and $h(G) = 0$ else. Then $h$ is a Borel function. Since the matrix $G$ in (8.10) is invertible a.e. in $U$, we have $h(P_{kl} G) = P_{kl}$ a.e. and it follows from Lemma 2.25 that

$$\int_U P_{kl} (Z \varphi) \omega = 0 \quad \forall \varphi \in C^\infty_c(U), \ Z \in V_{1,k'} \quad \text{with } k' \neq k.$$  

(8.11)

In other words,

$$Z P_{kl} = 0 \quad \forall l, \ \forall Z \in V_{1,k'}, \ k' \neq k$$

(8.12)

in the sense of distributions. By exchanging $k$ and $k'$ we get

$$Z P_{kl} = 0 \quad \forall l, \ \forall Z \in V_{1,k}, \ k' \neq k.$$  

(8.13)

Since $P_{kl} = 1 - \sum_{k' \neq k} P_{k'l}$ we deduce that $Z P_{kl} = 0$ for all $Z \in V_1$ and all $k, l$. Thus $P_{kl}$ is locally constant and hence the permutation of the first layer subspaces induced by $D_P f(x)$ is locally constant. This completes the proof.
Remark 8.14. The choice of the forms \( \alpha_{k,l} \) can be motivated by our guiding principle to look for closed left invariant forms modulo exact left invariant forms (see [KMX20, Lemma 4.8]). First note by (2.22) we have \( d\alpha(X,Y) = -\alpha([X,Y]) \) for \( \alpha \in \Lambda^1 V_2 \). If we identify two-forms on \( V_1 \) with the dual space of two-vectors by setting \( \alpha(X \wedge Y) = \alpha(X,Y) \), then we see that exterior differentiation \( d : \Lambda^1 V_2 \to (\Lambda^2 V_1)' \) is dual map of \( -L_\emptyset \). Now consider first the case that \( g = \bigoplus_i h_i \) is a direct sum of \( n \) copies of the same (higher) Heisenberg group. A natural space of closed left invariant forms to detect the permutation \( \sigma_x \) induced by the Pansu differential is the space \( \bigoplus_i \Lambda^2 V_{1,i} \). In this case the relevant space of exact forms is given by \( d\Lambda^1 V_2 = \bigoplus_i d\Lambda^1 V_{2,i} \). By the duality between \( d \) and \( -L_\emptyset \) all elements of \( d\Lambda^1 V_2 \) vanish on all the space \( \ker_i \) and the sets \( \ker_i \) are characterized by this condition. Thus to find a basis for the quotient space \( \bigoplus_i \Lambda^2 V_{1,i}/d\Lambda^1 V_2 \) it is natural to look for elements of \( \Lambda^2 V_{1,i} \) whose restriction to \( \ker_i \) does not vanish. This leads to the basis \( \alpha_{k,l} \) used in the proof. The fact that there exist non-zero \( X \in \ker_i \) is crucial for finding closed codegree forms \( \beta = i_Z i_X \omega \) which interact well with the forms \( \alpha_{k,l} \).

If \( g \) is a product quotient \( \bigoplus \tilde{g}_i/K \) then it is still true that each element of \( d\Lambda^1 V_2 \) vanishes on each space \( \ker_i \). Hence the collection of cosets \( \alpha_{k,l} + d\Lambda^1 V_2 \) is still linearly independent in the quotient space and the proof shows that it is sufficiently rich to detect the permutation \( \sigma_x \).

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