Research Article

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Duplication in a model of rock fracture with fractional derivative without singular kernel

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Abstract: We provide a mathematical analysis of a break-up model with the newly developed Caputo-Fabrizio fractional order derivative with no singular kernel, modeling rock fracture in the ecosystem. Recall that rock fractures play an important role in ecological and geological events, such as groundwater contamination, earthquakes and volcanic eruptions. Hence, in the theory of rock division, especially in eco-geology, open problems like phenomenon of shattering, which remains partially unexplained by classical models of clusters’ fragmentation, is believed to be associated with an infinite cascade of breakup events creating a ‘dust’ of stone particles of zero size which, however, carry non-zero mass. In the analysis, we consider the case where the break-up rate depends of the size of the rock breaking up. Both exact solutions and numerical simulations are provided. They clearly prove that, even with this latest derivative with fractional order and no singular kernel, the system describing crushing and grinding of rocks contains (partially) duplicated fractional poles. According to previous investigations, this is an expected result that provides the new Caputo-Fabrizio derivative with a precious and promising recognition.

Keywords: Caputo-Fabrizio fractional derivative, Differentiation without singular kernel, Duplicated poles, Break-up dynamics, Generalized functions, Numerical approximations

MSC: 26A33, 37N25, 33E12

1 Motivation and Introduction

In the recent years, a high number of works have been undertaken to determine both the mechanism and rates of rock fracture and degradation under various conditions. In a geological point of view, rock fracture is any scission in a geologic formation, like a fault or a joint that splits the piece of rock into two, three or more smaller pieces. In general, a fracture causes a discontinuity in rock or soil mass, and happens during the construction of, for instance, a tunnel, a foundation or during the exploitation of a deposit of mine or oil. A fracture will sometimes create a deep fissure in the rock (or soil) that might have a huge influence on its mechanical balance and behavior, weakening its strength and causing deformations in eco and geological systems.

Indeed, rock fractures have been proved to play a significant role in many ecological and geological phenomena, such as earthquakes, volcanic eruptions, plate tectonics and fluid transport in the Earth’s crust. Today, rock fracture is seen as one of the precursors to volcanic eruptions around the world. The mechanism has been used in various studies analysing seismicity and ground deformation to forecast volcanic eruptions after a long repose interval [5, 15]. It is proven that even in the cases of relatively small numbers of fractures, seismic energy can propagate and affect the whole balance of a region’s underground.

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Various techniques and (seismic) methods are used to locate fracture zones in rocks that sometimes serve as reservoirs, among which are petroleum reservoirs. Most of these reservoirs are developed by forming fracture networks or extending existing networks to provide pathways for fluid circulation between wells. Many petroleum reservoirs form highly fractured rocks, where fracture properties such as density and orientation are crucial to reservoir economics. In these cases the fractures are usually important because of permeability rather than porosity. Rock fracture also plays a major role in groundwater contamination [2, 5]. Indeed, fissures in a rock can create pathways for the introduction of all kinds of poisons, chemicals and toxic wastes that are emplaced in or released to the underground environment.

Thus, it is clear that rock fracture is essential for the foundation of structural geology, including earth structure, and has a high impact on the balance of the geological system of the planet. It is therefore important to understand the basic mechanisms of the fractures, predict the following ones together with the consequences for the environment and people living in. The phenomenon of shattering observed in fragmentation (and degradation) processes in general [9, 18, 20] and in rock fracture in particular, is seen as an explosive or dishonest Markov process, see e.g. [3, 20, 21]. It is believed to be associated with an infinite cascade of breakup events creating a ‘dust’ of stone particles of zero size which, however, carry non-zero mass. Understanding and modeling such a complex process is not easy for scientists and geologists. Accordingly, classical fragmentation and degradation models (with first order derivative \( \frac{d}{dt} \)) cannot fully describe this bizarre phenomenon of shattering.

Recall that [10] the normal derivative \( \frac{d}{dt} \) gives the rate of accumulation or loss in the system; that is, gain rate minus loss rate, at infinitesimal bounded space. However, if that infinitesimal space is having traps (of various sizes), where the variable under study is temporarily parked, then will the \( \frac{d}{dt} \) duplicate the real picture of accumulation or loss? Similarly these trap pictures could be islands or forbidden zones in the infinitesimal space where the variable (particle, mass, density, flux etc.) cannot reside; accordingly, the rate of accumulation or loss will be different than \( \frac{d}{dt} \). Hence, the fractional differentiation \( \frac{d^\alpha}{dt^\alpha} \), with \( \alpha \in \mathbb{R} \) or \( \mathbb{C} \) may give the sub- or super-rate of accumulation or loss with index \( \alpha \), symbolizing the heterogeneity distribution of the infinitesimal space (traps or islands). We can understand now why there is an increasing volition to extend classical models to models with fractional derivative and investigate them with various and different techniques in order to establish broader outlooks on the real phenomena they describe.

## 2 Model and analysis

The classical kinetics of rock fracture integrodifferential process was extended to dynamics with fractional order derivative [8, 9], using the classical Caputo derivative in the following model

\[
^C D_t^\alpha g(x,t) = -g(x,t) \int_0^x H_\alpha(y, x - y)dy + 2 \int_x^\infty g(y, t)H_\alpha(x, y - x)dy, \quad x, t > 0.
\]  

subject to the initial condition

\[
g(x, 0) = g_0(x), \quad x > 0.
\]

where

\[
^C D_t^\alpha (g(x, t)) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} \frac{\partial}{\partial \tau} g(x, \tau)d\tau,
\]

with \( 0 < \alpha \leq 1 \). Here \( g(x, t) \) represents the density of \( x \)-mass rock (i.e. rock with mass of size \( x \)) at time \( t \) and \( H(x, y) \) gives the average fragmentation rate, that is the average number at which clusters of size \( x + y \) undergo splitting to form an \( x \)-mass rock and a \( y \)-mass rock. This model is applicable in many branches of natural sciences including physics, chemistry, engineering, biology, ecology and numerous domains of applied sciences like depolymerization, chain degradation, cellulose degradation, rock fracture and break of droplets. Various types of fragmentation equations have been comprehensively analyzed in numerous works (see, e.g., [4, 8, 9, 11, 13, 15–17, 19–21]). The author in [9] used the model (1)-(2) to perform a biomathematical analysis of cellulose degradation...
process and pointed out that the system describing fractional cellulose degradation contains replicated fractional poles caused by a combination of higher transcendental functions, namely the Mittag-Leffler function, the further generalized $G$-function and the Pochhammer polynomial. A similar model was investigated in [8] to finally show that there exists a solution operator to the full fractional model that is positive and contractive. In their paper in [4], the authors used the inverse process of the break-up dynamics, the coagulation process to find and investigate the size distribution of a similar model. The domain of rock fracture dynamics has also been of considerable interest, since it results in fragmentation process, see [15, 21] and can also been analysed via alternative methods [12, 14].

To enhance the mathematical modeling in fractional calculus and address some open problems related to systems of classical viscoelastic materials, thermal media, electromagnetic, and so on, Caputo and Fabrizio [7] recently proposed the following definition of fractional time derivative without any singularity:

$$\frac{c_f}{0}D_t^\alpha g(x,t) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t \frac{\partial}{\partial \tau} g(x, \tau) \exp \left( \frac{-\alpha(t-\tau)}{1-\alpha} \right) d\tau,$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. Note that using fractional order differential equations usually reduces the number of errors arising from the neglected parameters when modeling real life phenomena, like for instance, the fracture of rock in nature. We are interested in numerical approach to investigate the evolution of the number density of rock particles described by the break-up integrodifferential model given by:

$$\frac{c_f}{0}D_t^\alpha g(x,t) = -g(x,t) \int_0^x \int_0^\infty yg(y,t)H_{\alpha}(x, y-x)dy + 2 \int_x^\infty g(y,t)H_{\alpha}(x, y-x)dy, \quad x, \ t > 0. \quad (5)$$

subject to the initial condition

$$g(x,0) = g_0(x), \quad x > 0. \quad (6)$$

For the reason of simplicity and without loss of generality, we take $M(\alpha) = 1$ and $\frac{c_f}{0}D_t^\alpha$ is noted $D_t^\alpha$ as defined in (4). The Laplace transform of the new Caputo-Fabrizio fractional derivative (5) is given by [7]

$$\mathcal{L}\left(D_t^\alpha g(x,t), s \right) = \frac{s\tilde{g}(x, s) - g_0(x)}{s + \alpha(1-s)} \quad (7)$$

where $\tilde{g}(x, s)$ is the Laplace transform $\mathcal{L}(g(x,t), s)$ of $g(x,t)$.

### 3 Mathematical analysis of the rock fracture process

#### 3.1 Exact solutions

To determine the exact solution of the model (5)-(6), we assume that the breakup rate $H_{\alpha}$ is proportional to the size of the rock fracturing, that is

$$H_{\alpha}(x, y) = x + y.$$  

Hence, (5) is reduced to

$$D_t^\alpha g(x,t) = -x^2 g(x,t) + 2 \int_x^\infty yg(y,t)dy, \quad x, \ t > 0. \quad (8)$$

Application of Laplace transform on both side of (8) yields

$$\mathcal{L}\left(D_t^\alpha (g(x,t)), s \right) = \mathcal{L}\left(-x^2 g(x,t) + 2 \int_x^\infty yg(y,t)dy, s \right)$$
where

\[
\mathcal{L} \left( -x^2 g(x, t) + 2 \int_x^\infty y g(y, t) dy, s \right) = -x^2 \tilde{g}(x, s) + 2 \int_x^\infty y \tilde{g}(y, s) dy.
\]

Substituting the later equation into the former and rearranging finally leads to

\[
g_0(x) = \tilde{g}(x, s) \left( s + x^2 (s + \alpha (1 - s)) \right) + 2 (s + \alpha (1 - s)) \int_x^\infty y \tilde{g}(y, s) dy.
\]

We can convert (9) into a partial differential equation by putting

\[
Z(x, s) = 2 \theta \int_x^\infty y \tilde{g}(y, s) dy,
\]

with

\[
\theta = s + \alpha (1 - s).
\]

**Remark 3.1.** To provide (8) with a meaningful mathematical signification, we require the following mappings \( y \rightarrow g(y, t), y \rightarrow f(y) \) and therefore \( y \rightarrow \tilde{g}(y, s) \) to be Lebesgue integrable on any \( [\xi, \infty) \) for \( \xi > 0 \) and almost every \( x > 0 \).

Hence, the integral in (10) exists in any interval \( [\epsilon, \infty) \) and is absolutely continuous at each \( x > 0 \). Thus, differentiating (10) gives the partial differential equation

\[
g_0(x) = -Z(x, s) - \frac{s + x^2 \theta}{2x} \frac{Z(x, s)}{\theta_x}.
\]

Choosing the constant of integration so as to have vanishing solutions at \( \infty \), we obtain the solution of (12) given by

\[
Z(x, s) = 2 \theta \exp \left( -\sigma_{s, \alpha}(x) \right) \int_x^\infty \frac{rg_0(r)}{s + r^2 \theta} \exp \left( \sigma_{s, \alpha}(r) \right) dr.
\]

where

\[
\sigma_{s, \alpha}(x) = \int_0^x \frac{2r \theta}{s + r^2 \theta} dr = \ln \left( \frac{s + x^2 \theta}{s} \right).
\]

The solution of (9) is therefore

\[
\tilde{g}(x, s) = \frac{s (2 - 2 \alpha) + 2 \alpha \sqrt{s (1 + x^2 - \alpha x^2)} + \alpha x^2 \frac{g_0(x)}{s (1 + x^2 - \alpha x^2) + \alpha x^2}}{\sqrt{s (1 + x^2 - \alpha x^2) + \alpha x^2}},
\]

where we have used (10) and (11).

Put

\[
\theta = \frac{\alpha x^2}{1 + x^2 - \alpha x^2}.
\]

Applying the inverse laplace transform \( \mathcal{L}^{-1}(\tilde{g}(x, s, t)) = g(x, t) \) to each term of the later equation:

\[
\mathcal{L}^{-1} \left( \frac{1}{(s + \theta)^2}, t \right) = te^{-\theta t} = te^{-\left( \frac{\alpha x^2}{1 + x^2 - \alpha x^2} \right) t}
\]

\[
\mathcal{L}^{-1} \left( \frac{s}{(s + \theta)^2}, t \right) = G_{1, 1, 2}(\theta, t)
\]

and

\[
\mathcal{L}^{-1} \left( \frac{1}{s + \theta}, t \right) = e^{-\theta t}.
\]
We finally obtain the exact solution of the fractional model (8):

\[
g(x, t) = \left[\frac{2 - 2\alpha}{(1 + x^2 - \alpha x^2)^2} G_{1,1,2}(\theta, t) + \frac{2t\alpha}{(1 + x^2 - \alpha x^2)^2} e^{-\theta t}\right] \int_x^\infty r g_0(r) dr + \frac{g_0(x)e^{-\theta t}}{1 + x^2 - \alpha x^2}. \tag{15}
\]

Here, \(G_{q,\beta,r}(\theta, t)\) is the higher transcendental generalized \(G_{q,\beta,r}\) function defined by

\[
G_{q,\beta,r}(\theta, t) = \sum_{j=0}^\infty \frac{(r)_j (-\theta)^j (r+j)^{q-\beta-1}}{(1+j)^\Gamma(r+j)\Gamma(q-\beta)}
\tag{16}
\]

with

\((r)_j = (r)(r - 1) \cdots (r - (j - 1))\)

the Pochhammer polynomial.

### 3.2 Numerical approximations

The numerical approximation in space and time of the new Caputo-Fabrizio fractional derivative was recently proposed in [1]. Let us recall here the main points. Indeed, for the finite difference scheme, a positive integer \(K \in \mathbb{N}\) is considered to define the sizes of grids as

\[l = \frac{1}{K}\]

and the time grid points \(t_k = kl\) taken in the time interval \([0, T]\) with \(k = 0, 1, 2, \cdots N\). The value of the function \(g\) at the grid point \(t_k\) is noted \(g^k = g(t_k)\). Making use of the following formula at \(t_k\),

\[D_\alpha^\theta g(t_k) = \frac{M(\alpha)}{(1-\alpha)} \int_0^{t_k} \hat{g}(\tau) \exp\left(-\frac{\alpha}{1-\alpha}(t_k - \tau)\right) d\tau
\]

and using the first order approximation \(\frac{dg}{dt} = \frac{g(t_{k+1}) - g(t_k)}{l} + O(l)\) to

\[D_\alpha^\theta g(t_k) = \frac{M(\alpha)}{(1-\alpha)} \left[ \sum_{i=1}^k \int_{i-1}^i \left(\frac{g^{i+1} - g^i}{k} + O(l)\right) \exp\left(-\frac{\alpha}{1-\alpha}(t_k - \tau)\right) d\tau \right]
\]

yield the following proposition that was proved in [1]:

**Proposition 3.2.** Let \(g : (a, b) \rightarrow \mathbb{R}\) an arbitrary real and locally integrable function, \(t_k \in (a, b)\) and \(\alpha\) a number in \([0, 1]\) then, the first order approximation of the new Caputo-Fabrizio fractional derivative with no singular kernel (4) at a point \(t_k\) is given by

\[D_\alpha^\theta g(t_k) = \frac{M(\alpha)}{\alpha} \left[ \sum_{i=1}^k \left(\frac{g^{i+1} - g^i}{l}\right) \theta_{i,l} \right] + O(l^2), \tag{17}
\]

where

\[\theta_{i,l} = \exp\left(-\frac{\alpha l}{1-\alpha}(k-i)\right) - \exp\left(-\frac{\alpha l}{1-\alpha}(k-i+1)\right). \tag{18}\]

Now, we can consider again the case where the breakup rate \(H_\alpha\) is proportional to the size of the rock fragmenting. Applying (17), approximate solutions together with exact solutions are illustrated in Fig. 1 and Fig. 2. They are plotted according to different values of initial condition \(g_0(x)\) with \(\alpha = 0.8\), \(N = 100\) and \(l = 0.02\). Fig. 3 and Fig. 4 are respectively the cross section and longitudinal section of the surface solution plotted for some specific values of the rock size \(x\) and time \(t\).
Fig. 1. Exact (top) and approximate (bottom) solution \( g(x, t) \) when \( \alpha = 0.8 \) and \( g_0(x) = \frac{\ln x}{(x^2 + 1)^{\frac{3}{2}}} \).

Fig. 2. Exact (top) and approximate (bottom) surface solution \( g(x, t) \) when \( \alpha = 0.8 \) and \( g_0(x) = \frac{\sin x}{x^4} \).

Fig. 3. Exact (top) and approximate (bottom) cross section solution \( g(t) \) when \( \alpha = 0.8 \) and \( g_0(x) = e^{-x^2} \).
4 Concluding remarks

We have used analytical and computational approaches to investigate a break-up model of rock fracture in the ecosystem, endowed with the new Caputo-Fabrizio fractional order derivative with no singular kernel. Both exact and approximate solutions show that even with the new derivative with no singular kernel, the system describing rocks breaking up contains (partially) duplicated fractional poles (Figs 1–4). This research improves the preceding ones since it is the first instance where numerical and analytical methods are applied to that new derivative in the same work. The obtained results provide us with enough arguments to grant a certain reconnoitring to that new Caputo-Fabrizio fractional derivative which seems to be really promising in addressing open problems like those related to non usual behavior observed in systems of fragmentation, classical viscoelastic materials, thermal media, electromagnetic and so on.

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