THE BAR-NATAN HOMOLOGY AND UNKNOTTING NUMBER

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Abstract. We show that the order of torsion homology classes in Bar-Natan deformation of Khovanov homology is a lower bound for the unknotting number. We give examples of knots that this is a better lower bound than $|s(K)/2|$, where $s(K)$ is the Rasmussen $s$ invariant defined by the Bar-Natan spectral sequence.

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1. INTRODUCTION

In [Kho00], Khovanov introduced a knot (and link) invariant which categorifies the Jones polynomial, now known as Khovanov homology. This invariant is constructed by applying a specific TQFT to the cube of resolutions corresponding to a projection of the knot. Using a different TQFT, Bar-Natan defined a deformation of Khovanov homology in [BN05], which we will work with in this paper. The goal is to describe a lower bound for the unknotting number in terms of the $h$-torsion in the Bar-Natan chain complex.

A homology class $\alpha \in \mathcal{H}_{BN}(K)$ is called torsion if $h^n\alpha = 0$ for a positive integral $n$. The smallest $n$ with this property is called order of $\alpha$, denoted by $\text{ord}(\alpha)$. Let $T_{BN}(K)$ denotes the set of torsion classes in $\mathcal{H}_{BN}(K)$.
Definition 1.1. For an oriented knot $K$ in $\mathbb{R}^3$, we define
\[ u(K) := \max_{\alpha \in T_{BN}(K)} \text{ord}(\alpha). \]

Theorem 1.2. For any oriented knot $K$, $u(K)$ is a lower bound for the unknotting number of $K$.

Let $K_+$ and $K_-$ be knot diagrams that differ in a single crossing $c$, which is a positive crossing in $K_+$ and a negative crossing in $K_-$. We prove Theorem 1.2 in by introducing chain maps
\[ f_c^+: \mathcal{C}_{BN}(K_+) \to \mathcal{C}_{BN}(K_-) \text{ and } f_c^-: \mathcal{C}_{BN}(K_-) \to \mathcal{C}_{BN}(K_+) \]
such that the induced maps by $f_c^- \circ f_c^+$ and $f_c^+ \circ f_c^-$ on $\mathcal{H}_{BN}(K_+)$ and $\mathcal{H}_{BN}(K_-)$, respectively, are equal to multiplication by $h$. In [AD], Dowlin and the author introduce similar chain maps for Lee homology and prove Knight Move Conjecture [Kho00, BN02] for knots with unknotting number smaller than 3.

Despite the algebraic definition of the chain maps (1), we show that they can be described in terms of cobordism maps associated to specific cobordisms from $K_+$ to $K_- \# H$, and $K_- \to K_+ \# mH$, where $H$ is the right-handed Hopf link and $mH$ is its mirror. In [AE], Eftekhary and the author use corresponding cobordism maps for knot Floer homology to deduce a lower bound for the unknotting number, in terms of the order of torsion classes in variants of knot Floer homology.

This paper is organized as follows. Section 2 reviews Bar-Natan chain complex and collects some results we will need later. Section 3 proves Theorem 1.2. Section 4 gives a geometric description, using cobordism maps, for the chain maps, defined algebraically, in the process of proving Theorem 1.2 in Section 3. Finally, Section 5 gives examples of knots for which our invariant (Definition 1.1) is a better lower bound comparing to the $s$-invariant i.e. $u(K) > |s(K)|/2$.

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2. Background

In this section we review the Bar-Natan chain complex, describe its module structure and discuss some of its basic properties.

2.1. Bar-Natan’s deformation of Khovanov homology. Let $K$ be an oriented knot or link diagram in $\mathbb{R}^2$ with $n$ crossings. Denote the set of crossings in $K$ by $\mathcal{C} = \{c_1, ..., c_n\}$. Each crossing can be resolved in two different ways, the 0-resolution and the 1-resolutions, see Figure 1.

For any vertex $v$ of $\{0, 1\}^n$, let $K_v$ denote the complete resolution obtained by replacing the crossing $c_i$ by its $v_i$-resolution. Let $k_v$ denote the number of connected components of $K_v$. 
There is a partial order on \( \{0,1\}^n \) by setting \( u \leq v \) if \( u_i \leq v_i \) for all \( 1 \leq i \leq n \). Denote \( u < v \) if \( u < v \) and \( |v| - |u| = 1 \), where \( |v| \) denotes \( \sum_i v_i \). Corresponding to each edge of the cube, i.e. a pair \( u \prec v \), there is an embedded cobordism in \( \mathbb{R}^2 \times [0,1] \) from \( K_u \) to \( K_v \), constructed by attaching an embedded one handle near the crossing \( c_i \) where \( u_i < v_i \). If \( k_u > k_v \), the cobordism merges two circles, otherwise splits two circles.

Set \( F = \mathbb{Z}/2\mathbb{Z} \). Let \( A \) denotes the 2-dimensional Frobenius algebra over \( F[h] \) with basis \( \{x_+,x_-\} \) and multiplication and comultiplication defined as:

\[
\begin{align*}
    x_+ \otimes x_+ &\mapsto x_+ \\
    x_- \otimes x_+ &\mapsto x_- \\
    x_+ \otimes x_- &\mapsto x_- \Delta \xrightarrow{x_+} x_+ \otimes x_+ + x_- \otimes x_+ + hx_+ \otimes x_+ \\
    x_- \otimes x_- &\mapsto hx_-
\end{align*}
\]

The Bar-Natan chain complex is obtained by applying the \((1+1)\)-dimensional TQFT corresponding to \( A \) to the above cube of cobordisms for \( K \). More precisely, corresponding to a vertex \( v \in \{0,1\}^n \), a Khovanov generator is a labeling of the circles in \( K_v \) by \( x_+ \) or \( x_- \). The module \( C_{BN}(K_v) \) is defined as the free \( F[h] \)-module generated by the Khovanov generators corresponding to \( v \) and

\[
C_{BN}(K) := \bigoplus_{v \in \{0,1\}^n} C_{BN}(K_v).
\]

The differential \( \delta_{BN} \) decomposes along the edges; for any \( u < v \) the component

\[
\delta_{BN}^{uv} : C_{BN}(K_u) \rightarrow C_{BN}(K_v)
\]

is defined by the above multiplication if \( K_v \) is obtained from \( K_u \) by merge, otherwise, it is defined by comultiplication. The Bar-Natan chain complex, \((C_{BN}(K), \delta_{BN})\), was studied by Bar-Natan in [BN05]; the homology is denoted by \( H_{BN}(K) \). For simplicity, we denote the differential by \( \delta \).

The chain complex is bigraded; homological grading \( \text{gr}_h \) and an internal grading \( \text{gr}_q \) called quantum grading. The homological grading for each summand \( C_{BN}(K_v) \) of \( C_{BN}(K) \) is given by \( |v| - n_- \), where \( n_\bullet \) denotes the number of \( \bullet \)-crossings in \( K \) for \( \bullet \in \{+,-\} \). The quantum grading for each Khovanov generator \( x \) at a vertex \( v \) is given by

\[
\text{gr}_q(x) = n_+ - 2n_- + |v| + k_v^+ - k_v^- 
\]
where $k_{x^*}$ denote the number of circles labelled by $x_*$ in $K_v$, for $* = +, -$. Furthermore, the formal variable $h$ has homological grading $0$ and quantum grading $-2$.

2.2. Module structure on Bar-Natan homology and basepoint action. Let $K$ be a knot diagram and $p$ be a point on $K$ away from the crossings. The choice of $p$, induces a module structure on the Khovanov homology of $K$, described in [Kho03]. Let us recall this structure for Bar-Natan homology. Choose a small unknot $U$ near $p$ and disjoint from $K$ such that merging the unknot with $K$ gives a knot or link diagram isotopic to $K$. Then, attaching the corresponding embedded one handle to $K \sqcup U$ gives an embedded cobordism in $\mathbb{R}^3 \times I$ from $K \sqcup U$ to $K$ and its associated cobordism map, denoted by $m_p$,

$$m_p : C_{BN}(K \sqcup U) = C_{BN}(K) \otimes F[h] A \to C_{BN}(K)$$

is given by the multiplication map $m$ of $A$. More precisely, for a Khovanov generator $x \in C_{BN}(K_v)$, $m_p(x \otimes x_*)$ is the Khovanov generator obtained from $x$ by multiplying the label of the circle containing $p$ with $x_*$.

Similarly, let

$$\Delta_p : C_{BN}(K) \to C_{BN}(K) \otimes F[h] A$$

denote the cobordism map associated to the inverse cobordism from $K$ to $K \sqcup U$.

If $K$ is related to another knot diagram $K'$ by a Reidemeister move away from the basepoint $p$, it is straightforward that the chain homotopy equivalence between $C_{BN}(K)$ and $C_{BN}(K')$, defined in [BN05], commutes with $m_p$. On the other hand, any Reidemeister move which crosses $p$ is equivalent to a sequence of Reidemeister moves away from $p$. So it induces an $A$-module structure on the Bar-Natan homology of the underlying knot $K$.

For a point $p$ on $K$, let

$$x_p : C_{BN}(K) \to C_{BN}(K)$$

be the chain map $x_p(a) = m_p(a \otimes x_-)$, defined as in [HN13]. Therefore, for a Khovanov generator $x \in C_{BN}(K_v)$ if the circle containing $p$ is labeled by $x_+$ then $x_p(x)$ is the Khovanov generator obtained from $x$ by changing the label of this circle to $x_-$, otherwise $x_p(x) = hx$. Thus, $x_p \circ x_p = h x_p$ and $x_p$ reduces the quantum grading by 2.

In contrast to Khovanov homology, the chain homotopy type of the module multiplication map $x_p$ is not independent of the marked point $p$. In fact, [HN13, Lemma 2.3] may be generalized to describe the difference of $x_p$ and $x_q$ when $p$ and $q$ lie on opposite sides of a crossing as follows.

**Lemma 2.1.** Let $p, q \in K$ be points away from the crossings, that lie on the opposite sides of a single crossing as in Figure 2. Then, $x_p + x_q$ is homotopy equivalent to multiplication by $h$. 
First we need to prove another lemma. Assume $K$ and $L$ be oriented link diagrams such that $L$ is obtained from $K$ by an oriented saddle move as in Figure 3. This saddle move represents an oriented, embedded saddle cobordism in $\mathbb{R}^3 \times [0,1]$ from $K$ to $L$. Let

$$f : C_{BN}(K) \to C_{BN}(L) \quad \text{and} \quad \tilde{f} : C_{BN}(L) \to C_{BN}(K)$$

denote the chain maps on the Bar-Natan chain complex associated with this cobordism and its inverse, respectively.

**Figure 3.** Saddle move: $q$ and $q'$ are the attaching points of the corresponding one-handle.

Lemma 2.2. With the above notation fixed, for any $a \in C_{BN}(K)$ we have

$$\tilde{f} \circ f(a) = h a + x_q(a) + x_{q'}(a).$$

**Proof.** Given a vertex $v \in \{0,1\}^n$, if $q$ and $q'$ lie on the same connected component of the complete resolution $K_v$, then $x_q|_{C_{BN}(K_v)} = x_{q'}|_{C_{BN}(K_v)}$. Thus, for any $a \in C_{BN}(K_v)$ the Equality (2) follows from

$$m \Delta(x_+) = hx_+, \quad \text{and} \quad m \Delta(x_-) = hx_-.$$

Otherwise, if $q$ and $q'$ belong to distinct connected components of $K_v$, for any Khovanov generator $x \in C_{BN}(K_v)$ the statement follows from one of the following relations, depending on the labels of the circles containing $q$ and $q'$.

$$\begin{align*}
\Delta m(x_+ \otimes x_+) &= x_+ \otimes x_- + x_- \otimes x_+ + hx_+ \otimes x_+ \\
\Delta m(x_- \otimes x_+) &= x_- \otimes x_- = x_- \otimes x_+ + m(x_- \otimes x_-) \otimes x_+ + hx_- \otimes x_+ \\
\Delta m(x_+ \otimes x_-) &= x_+ \otimes x_- = x_- \otimes x_- + x_+ \otimes m(x_- \otimes x_-) + hx_+ \otimes x_- \\
\Delta m(x_- \otimes x_-) &= hx_- \otimes x_- = hx_- \otimes x_- + m(x_- \otimes x_-) \otimes x_- + x_- \otimes m(x_- \otimes x_-)
\end{align*}$$

$\square$
Proof. (Lemma 2.1) The proof is similar to the proof of [HN13, Lemma 2.3]. Denote the crossing between $p$ and $q$ by $c$. Let $K_*$ be the diagram obtained from $K$ after applying the $*$-resolution at $c$. One may orient $K_0$ and $K_1$ such that $K_1$ is obtained from $K_0$ by an oriented saddle move, and up to appropriate grading shifts, $C_{BN}(K)$ is given by the mapping cone $f: C_{BN}(K_0) \to C_{BN}(K_1)$ where $f$ is the corresponding cobordism map. Let $\bar{f}$ denote the cobordism map associated with the inverse cobordism from $K_1$ to $K_0$. Under this decomposition we define $H(a_0, a_1) := (\bar{f}(a_1), 0)$.

Then, $\delta H(a_0, a_1) + H\delta(a_0, a_1) = \delta(\bar{f}(a_1), 0) + H(\delta a_0, f(a_0) + \delta a_1) = \delta(\bar{f}(a_1), \bar{f}(a_1)) + (\bar{f}(f(a_0) + \delta a_1), 0) = (\bar{f}(a_0), \bar{f}(a_1))$, and thus it follows from lemma 2.2 that $H$ is a chain homotopy between $x_p + x_q$ and multiplication by $h$.

Corollary 2.3. Assume $K$ and $L$ are oriented link diagrams so that $L$ is obtained from $K$ by an oriented saddle move. If the attaching points of the saddle lie on the same connected component of $K$, then $\bar{f} \circ f$ is chain homotopic to multiplication by $h$. As before, $f$ and $\bar{f}$ denote the chain maps associated with the corresponding saddle cobordism and its inverse, respectively.

Proof. Let $p$ and $q$ be the attaching points of the saddle. Consider an arc $\alpha \subset K$ connecting $p$ to $q$. Moving the point $p$ along $\alpha$, it would cross an even number of crossings till it gets to $q$; thus Lemma 2.1 implies that $x_p$ is homotopy equivalent to $x_q$. Then, by Lemma 2.2 we have $\bar{f}f$ is homotopy equivalent to multiplication by $h$.

2.3. Connected sum with a Hopf link. A connected sum formula for Khovanov homology has been studied in [Kho00]. In this section, we recall a special case of this formula, taking a connected sum with Hopf link, for Bar-Natan homology.

Let $H$ be the right-handed Hopf link. Choose an arbitrary point $p \in K$. We obtain an oriented diagram $L$ for the link $K \# H$ by changing $K$ locally in a neighborhood of $p$, as in Figure 4.

Denote the new crossings by $c$ and $c'$. For $i, j \in \{0, 1\}$, let $L_{ij}$ denote the diagram obtained from $L$ by applying $i$- and $j$-resolutions at the crossings $c$ and $c'$, respectively. We orient these diagrams such that their orientations
coincide with the orientation of $K$ outside the above neighborhood of $p$. The oriented diagrams $L_0$ and $L_1$ are isotopic to $K$, while $L_0'$ and $L_1'$ are isotopic to a disjoint union of $K$ with an unknot near the point $p$. The chain complex $C_{BN}(L)$ is given by the mapping cone:

$$
\begin{array}{ccc}
C_{BN}(L_0) = C_{BN}(K) & \xrightarrow{\Delta_p} & C_{BN}(L_1) = C_{BN}(K) \otimes F[h] A \\
\downarrow{m_p} & & \downarrow{m_p} \\
C_{BN}(L_0') = C_{BN}(K) \otimes F[h] A & \xrightarrow{m_p} & C_{BN}(L_0) = C_{BN}(K)
\end{array}
$$

From now on, we use this decomposition to show any $a \in C_{BN}(L)$ as $a = (a_0, a_1, a_{10}, a_{11})$ where $a_{ij} \in C_{BN}(L_{ij})$ and so $a_{00}, a_{11} \in C_{BN}(K) \otimes A$ while $a_{10}, a_{01} \in C_{BN}(K)$.

We define chain maps $i : C_{BN}(K) \to C_{BN}(L)$ and $p : C_{BN}(L) \to C_{BN}(K)$ as

\begin{equation}
(3) \quad i(a) = (0, 0, 0, a \otimes x_+) \quad \text{and} \quad p(a_0, a_1, a_{10}, a_{11}) = a_{00}
\end{equation}

where $a_{00} = a_{00} \otimes x_+ + a_{00} \otimes x_-$. It is straightforward that both $i$ and $p$ are chain maps.

**Lemma 2.4.** The sequence

$$
0 \to \mathcal{H}_{BN}(K) \xrightarrow{i_*} \mathcal{H}_{BN}(L) \xrightarrow{p_*} \mathcal{H}_{BN}(K) \to 0
$$

is a split exact sequence.

**Proof.** First, we prove that $\text{im}(i_*) = \ker(p_*)$. Consider a homology class $\alpha$ in $\ker(p_*)$. Any such class can be represented by a cycle $a$ such that $a_{00} = 0$. Since, $\delta(a) = 0$, $m_p(a_0) + \delta(a_{01}) = 0$ and thus

$$a + \delta(a_{01} \otimes x_+, 0, 0, 0) = (0, 0, a_{01} + a_{10}, a_{11}).$$

Again, follows from $\delta a = 0$ that

$$\delta a_{11} = \Delta_p(a_{10} + a_{01}) = (a_{10} + a_{01}) \otimes x_+ + (x_p(a_{10} + a_{01}) + h(a_{10} + a_{01})) \otimes x_+.$$

Therefore, $\delta a_{11} = a_{10} + a_{01}$ where $a_{11} = a_{11}^+ \otimes x_+ + a_{11}^- \otimes x_-$ and so

$$(0, 0, a_{10} + a_{01}, a_{11}) + \delta(0, 0, a_{11}, 0) = (0, 0, 0, a_{11} + \Delta_p a_{11}^-) = i(a_{11}^+ + x_p(a_{11}) + h a_{11}^-).$$
Then, let \( r : \mathcal{C}_{BN}(L) \to \mathcal{C}_{BN}(K) \) and \( s : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(L) \) be the chain maps defined as
\[
 r(a_{00}, a_{01}, a_{10}, a_{11}) = a_{11}^+ x_p(a_{11}^-) + h a_{11}^-
\]
and
\[
 s(a) = (x_p(a) \otimes x_+ + a \otimes x_- , 0 , 0 , 0)
\]
where \( a_{11} = a_{11}^+ \otimes x_+ + a_{11}^- \otimes x_- \). It is straightforward that \( r \) and \( s \) are chain maps such that \( r \circ i = \text{id} \) and \( p \circ s = \text{id} \). So \( i_* \) and \( p_* \) are injective and surjective, respectively, and the sequence splits.

The homomorphism \( p_* \) preserves homological grading and decreases the quantum grading by 1, while \( i_* \) increases homological grading by 2 and quantum grading by 5.

Similarly, we may define chain maps \( i \) and \( p \) for the connected sum of \( K \) with the left-handed trefoil, \( K \# mH \), so that the induced homomorphism on homology gives a split exact sequence. The only difference is that \( p_* \) increases the homological grading by 2 and quantum grading by 5, while \( i_* \) preserves the homological grading and decreases the quantum grading by 1.

Finally, we define the chain maps \( i : \mathcal{C}_{BN}(K) \to \mathcal{C}_{BN}(L) \) and \( p : \mathcal{C}_{BN}(L) \to \mathcal{C}_{BN}(K) \) as
\[
 i = i + s \quad \text{and} \quad p = p + r
\]
where the chain maps \( i \) and \( p \) are defined in Equation (3) and \( r \) and \( s \) are defined in the proof of Lemma 2.4. Note that for \( L = K \# mH \), one may define similar chain maps, abusing the notation, we denote these maps by \( i \) and \( p \), too.

3. Lower bound for unknotting number

The goal of this section is to prove Theorem 1.2.

Let \( C \) be a chain complex of \( \mathbb{F}[h] \)-modules. Recall that a homology class \( \alpha \in H_*(C) \) is called torsion if \( h^n \alpha = 0 \) for some positive \( n \), and the smallest such \( n \) is called the order of \( \alpha \), denoted by \( \text{ord}(\alpha) \). Let \( T(C) \) be the set of torsion homology classes in \( H_*(C) \) and define
\[
 u(C) := \max_{\alpha \in T(C)} \text{ord}(\alpha).
\]

**Lemma 3.1.** Given chain complexes \( C \) and \( C' \) of \( \mathbb{F}[h] \)-modules, together with chain maps
\[
 f : C \to C' \quad \text{and} \quad g : C' \to C
\]
so that both \( f_* \circ g_* \) and \( g_* \circ f_* \) are equal to multiplication by \( h^n \) for some \( n > 0 \), then
\[
 |u(C) - u(C')| \leq n.
\]

**Proof.** For any homology class \( \alpha \in T(C) \) we have \( f_*(\alpha) \in T(C') \), and it follows from \( g_* \circ f_*(\alpha) = h^n \alpha \) that
\[
 \text{ord}(h^n \alpha) \leq \text{ord}(f_*(\alpha)) \leq \text{ord}(\alpha).
\]
Thus, \( \text{ord}(\alpha) \leq \text{ord}(f_\lambda(\alpha)) + n \), and so \( u(C) \leq u(C') + n \). Similarly, \( u(C') \leq u(C) + n \) which proves the result.

\[ \square \]

Suppose \( K_+ \) and \( K_- \) be oriented knot diagrams so that \( K_- \) is obtained from \( K_+ \) by changing one positive crossing, denoted by \( c \), to a negative crossing. For \( i = 0, 1 \), denote the \( i \)-resolution of \( K_+ \) at the crossing \( c \) by \( K_i \). We orient \( K_0 \) and \( K_1 \) such that they are related by an oriented saddle move. Let \( f \) and \( \bar{f} \) denote the cobordism maps corresponding to the saddle cobordism from \( K_0 \) to \( K_1 \) and its inverse from \( K_1 \) to \( K_0 \), respectively. The Bar-Natan chain complexes \( C_{BN}(K) \) and \( C_{BN}(\bar{K}) \), upto grading shifts, are given by the mapping cones of \( f \) and \( \bar{f} \) respectively.

Choose points \( p \) and \( q \) on the opposite sides of the crossing \( c \), as in Figure 2 and define

\begin{align*}
(5) \quad f_+^c : C_{BN}(K_+) & \rightarrow C_{BN}(K_-) \\
\quad f_-^c : C_{BN}(K_-) & \rightarrow C_{BN}(K_+) \\
\quad f_+^c(a_0, a_1) = ((x_p + x_q)(a_1), a_0) \quad \text{and} \quad \quad f_-^c(a_1, a_0) = ((x_p + x_q)(a_0), a_1)
\end{align*}

where \( a_i \in C_{BN}(K_i) \).

**Lemma 3.2.** Both \( f_+^c \) and \( f_-^c \) are chain maps.

**Proof.** Let \( \delta \) denote the differential of \( C_{BN}(K_i) \) for \( i \in \{0, 1, +, -\} \). It follows from \((x_p + x_q) \circ f = 0 \) and \( f \circ (x_p + x_q) = 0 \) that

\[ f_+^c \delta(a_0, a_1) = ((x_p + x_q)(\delta(a_1) + f(a_0)), \delta(a_0)) = ((x_p + x_q)\delta(a_1), \delta(a_0)) = \delta_- f_+^c(a_0, a_1). \]

The proof for \( f_-^c \) is similar.

\[ \square \]

**Corollary 3.3.** With the above notation fixed, \( |u(K_+) - u(K_-)| \leq 1 \).

**Proof.** By Lemma 2.1 the induced maps on homology by both \( f_+^c \circ f_-^c \) and \( f_-^c \circ f_+^c \) are equal to multiplication by \( h \). Thus the claim follows from Lemma 3.1.

\[ \square \]

**Proof.** (Theorem 1.2) Consider an oriented diagram for \( K \) such that we get an oriented diagram for the unknot after switching \( N \) crossings \( \{c_1, ..., c_N\} \), where \( N \) is the unknotting number of \( K \). Abusing the notation we denote the diagram by \( K \). For any \( i = 1, ..., N \), let \( K_i \) be the diagram obtained from \( K \) after switching the crossings \( c_1, ..., c_i \). The diagrams \( K_{i-1} \) and \( K_i \) differ in a single crossing for each \( i \), so it follows from Corollary 3.3 that \( |u(K_{i-1}) - u(K_i)| \leq 1 \). Therefore, \( |u(K) - u(\text{Unknot})| = u(K) \leq N \).

\[ \square \]
Remark 3.4. Setting $h = 1$, one may think of $(\mathcal{C}_{BN}(K), \delta)$ as a filtered chain complex of $\mathbb{F}$-modules, where the differential increases homological grading by 1 and does not decrease quantum grading. This gives a spectral sequence from Khovanov homology with coefficients in $\mathbb{F}$ of $K$ to $\mathbb{F} \oplus \mathbb{F}$, called Bar-Natan spectral sequence \cite{Tur06}. If this spectral sequence collapses in the $n$-th page, then $u(K) = n - 1$.

4. A geometric interpretation of chain maps

Suppose $K$ and $K'$ are oriented pointed knots i.e. oriented knots with marked points on them, such that $K'$ is obtained from $K$ by a sequence of crossing changes. To any such sequence, Eftekhary and the author associate a decorated cobordism from $K$ to a connected sum of $K'$ with some right- or left-handed Hopf links. Then, by the corresponding cobordism maps for knot Floer homology \cite{AE Section 8.2], we define chain maps between the knot Floer chain complexes of $K$ and $K'$ satisfying the assumptions of Lemma 3.1 \cite{AE Section 8.3}. As a result, one gets a lower bound for the unknotting number in term of the $u$-torsion in knot Floer homology.

Following the approach in \cite{AE} one may use cobordism maps for Bar-Natan homology to define chain maps between $\mathcal{C}_{BN}(K)$ and $\mathcal{C}_{BN}(K')$ which satisfy the assumptions of Lemma 3.1. The goal of this section is to show that for any crossing change process, these chain maps are equal to the ones defined in Section 3.

As before, let $K_+$ be an oriented knot diagram with a specific positive crossing $c$, and $K_-$ be the oriented knot diagram obtained from $K_+$ by changing $c$ into a negative crossing. As in Figure 5 after a Reidemeister II move near the crossing $c$ on $K_+$, followed by an oriented saddle move, one gets a diagram for $K_- \# H$. Here $H$ is the right-handed Hopf link.

\begin{figure}[h!]
\centering
\includegraphics[scale=0.5]{figure5.png}
\caption{}
\end{figure}

As in Figure 5 we denote the knot diagram obtained from $K_+$ by the specified Reidemester II move by $\widetilde{K}_+$. Further, let $h : \mathcal{C}_{BN}(K_+) \to \mathcal{C}_{BN}(\widetilde{K}_+)$ and $h : \mathcal{C}_{BN}(\widetilde{K}_+) \to \mathcal{C}_{BN}(K_+)$ be the chain homotopy equivalences corresponding to this move as defined in \cite{BN05 Section 4.3}. For the reader’s convenience, we recall the definition of $h$ and $\tilde{h}$. For $i, j = 0, 1$, let $\mathcal{C}_{BN}^{ij}(\widetilde{K}_+)$ denotes the direct sum of the summands of $\mathcal{C}_{BN}(\widetilde{K}_+)$ corresponding to vertices $v$ of the cube so that $v(c_2) = i, v(c_3) = j$. Also, let $h^{ij} = \pi_{ij} \circ h$ and
\( \tilde{h}^{ij} = \tilde{h} \circ \pi_{ij} \) where \( \pi_{ij} \) is the projection of \( C_{BN}(\tilde{K}_+) \) on \( C^i_{BN}(\tilde{K}_+) \) and \( \pi_{ij} \) is the inclusion of \( C^i_{BN}(\tilde{K}_+) \) in \( C_{BN}(\tilde{K}_+) \). Then, \( h^{00} = h^{11} = \tilde{h}^{00} = \tilde{h}^{11} = 0 \) and \( h^{10} = \tilde{h}^{10} = \text{id} \). Further, \( \tilde{h}^{01} = g \otimes x_+ \), where \( g \) is the cobordism map corresponding to the saddle move as in Figure 6 i.e. \( \tilde{h}^{01} \) is a chain map for the cobordism which is union of a saddle and a cup. Finally, \( \tilde{h}^{01} \) is the cobordism map for the inverse of the saddle move in Figure 6 union a cap.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6}
\caption{Figure 6.}
\end{figure}

Let \( f : C_{BN}(\tilde{K}_+) \rightarrow C_{BN}(K_- \# H) \) and \( \tilde{f} : C_{BN}(K_- \# H) \rightarrow C_{BN}(\tilde{K}_+) \) denote the cobordism maps for the saddle move in Figure 5.

**Theorem 4.1.** With the above notation fixed, \( f^+_c = p \circ \tilde{f} \circ h \) and \( f^-_c = \tilde{h} \circ \tilde{f} \circ i \), where \( i \) and \( p \) are the chain maps defined in (4).

**Proof.** We prove both equalities by looking at the cube of resolutions for the three crossings \( c_1, c_2 \) and \( c_3 \). For \( i, j, k = 0, 1 \), let \( C^i_{BN}(\bullet) \) denotes the direct sum of the summands of \( C_{BN}(\bullet) \) corresponding to vertices \( v \) of the cube so that \( v(c_1) = i, v(c_2) = j \) and \( v(c_3) = k \) for \( \bullet = \tilde{K}_+, K_- \# H \). Similarly, \( C^i_{BN}(K_+) \) and \( C^i_{BN}(K_-) \) denote the summands of \( C_{BN}(K_+) \) and \( C_{BN}(K_-) \), respectively, corresponding to the \( i \)-resolution at \( c \).

Assume \( a = (a_0, a_1) \) be an element in \( C_{BN}(K) \) so that \( a_i \in C^i_{BN}(K_+) \). It follows from the definition of \( h \) that \( h^{ij}(a_j) = 0 \) for any \( i, j = 0, 1 \). Further, considering the definition of \( p \), it is enough to compute \( f(h^{01}(a_0)) \) and \( f(h^{10}(a_1)) \). As in Figure 7, \( h^{01}(a_0) = \Delta_p(a_0) \otimes x_+ \) and so

\begin{equation}
(6) \quad f(h^{01}(a_0)) = \Delta_p(a_0) = a_0 \otimes x_- + (x_p(a_0) + h a_0) \otimes x_+
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7}
\caption{Figure 7.}
\end{figure}

Similarly, Figure 8 shows that \( h^{10}(a_1) = a_1 \) and

\begin{equation}
(7) \quad f(h^{10}(a_1)) = \Delta_q(a_1) = a_1 \otimes x_- + (x_q(a_1) + h a_1) \otimes x_+
\end{equation}
Thus, by equalities (6) and (7) we have \( p(f \circ h(a)) = (x_p(a_1), x_q(a_1), a_0) \).

![Figure 8](image)

Let \( a = (a_0, a_1) \) be an element in \( C_{BN}(K_-) \) where \( a_i \in C_{BN}^i(K_-) \) for \( i = 0, 1 \). Denote the components of \( i(a_*) \) in \( C_{BN}^{jk}(K_- \# H) \) by \( i_{jk}(a) \). It follows from the definitions of \( \tilde{h} \) and \( i \) that to compute \( \tilde{h}(\bar{f} \circ i(a)) \), it is enough to compute:

\[
\tilde{f}(i^{11}(a_0)) = \tilde{f}(a_0 \otimes x_+) = a_0
\]

and

\[
\tilde{f}(i^{00}(a_1)) = \tilde{f}(x_q(a_1) \otimes x_+ + a_1 \otimes x_-) = \Delta_q(x_q(a_1) \otimes x_+ + a_1 \otimes x_-).
\]

It is not hard to see that \( \tilde{h}(\bar{f} \circ i^{11}(a_0)) = a_0 \) and \( \tilde{h}(\bar{f} \circ i^{00}(a_1))) = x_p(a_1) + x_q(a_1) \). This completes the proof.

**Remark 4.2.** Similarly, one may change \( K_- \) by a Reidemeister II move near \( c \) to get a diagram \( \tilde{K}_- \), so that \( K_+ \# mH \) is obtained from \( \tilde{K}_- \) by a saddle move. Let \( h : C_{BN}(K_-) \to C_{BN}(\tilde{K}_-) \) and \( \tilde{h} : C_{BN}(\tilde{K}_-) \to C_{BN}(K_-) \) be the corresponding chain homotopy equivalences and \( f : C_{BN}(K_-) \to C_{BN}(K_+ \# mH) \) and \( \tilde{f} : C_{BN}(K_+ \# mH) \to C_{BN}(\tilde{K}_-) \) be the cobordism maps for the saddle move. Then, by the same argument, one can show that \( f_c^- = p \circ \bar{f} \circ h \) and \( f_c^+ = \bar{h} \circ \tilde{f} \circ i \).

5. Examples

The Rasmussen’s \( s \) invariant gives a lower bound for the slice genus, \(|s(K)|/2\), and thus the unknotting number [Ras10]. We used Cotton Seed’s package, Knotkit [Sec], to compute \( u \) and \( s \) (defined using the Bar-Natan spectral sequence) for some knots with more than 12 crossings. We found some example that \( u \) is a better lower bound comparing to \(|s|/2\), for instance, \(|s|/2\) for the knots 13n689, 13n1166, 13n2504 and 13n2807 is equal to 1, while \( u \) is equal to 2.
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