Beyond $\Sigma^2_1$ Absoluteness

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Abstract

There have been many generalizations of Shoenfield’s Theorem on the absoluteness of $\Sigma^1_2$ sentences between uncountable transitive models of ZFC. One of the strongest versions currently known deals with $\Sigma^2_1$ absoluteness conditioned on CH. For a variety of reasons, from the study of inner models and from simply combinatorial set theory, the question of whether conditional $\Sigma^2_1$ absoluteness is possible at all, and if so, what large cardinal assumptions are involved and what sentence(s) might play the role of CH, are fundamental questions. This article investigates the possibilities for $\Sigma^2_1$ absoluteness by extending the connections between determinacy hypotheses and absoluteness hypotheses.

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1. Absoluteness and strong logics

There have been many generalizations of Shoenfield’s Theorem on the absoluteness of $\Sigma^1_2$ sentences between uncountable transitive models of ZFC. Absoluteness theorems are meta-mathematically interesting since they identify levels of complexity where the technique of forcing cannot be used to establish independence.

A sentence, $\phi$, is a $\Sigma^2_1$-sentence if for some $\Sigma^1_2$-formula, $\psi(x)$, $\phi$ is provably equivalent in ZFC, Zermelo Frankel set theory with the Axiom of Choice, to the assertion that $\psi[\mathbb{R}]$ holds. While this is not the standard definition, for the purposes of this article it is equivalent.

**Theorem 1.1** Suppose that $\phi$ is a $\Sigma^2_1$ sentence, there exists a proper class of measurable Woodin cardinals and that $\text{CH}$ holds. Suppose $\mathbb{P}$ is a partial order and that $V^\mathbb{P} \models \text{CH}$. Then $V \models \phi$ if and only if $V^\mathbb{P} \models \phi$.

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This is $\Sigma^2_1$ generic absoluteness conditioned on CH. Because CH is itself a $\Sigma^2_1$ sentence, this conditional form of $\Sigma^2_1$ generic absoluteness is the best one can hope for. The meta-mathematical significance of this kind of absoluteness result is simply this. If a problem is expressible as a $\Sigma^2_1$ sentence, and there are many such examples from analysis, then it is likely settled by CH (augmented by modest large cardinal hypotheses). The technique of forcing cannot be used to demonstrate otherwise.

Absoluteness theorems can be naturally reformulated using strong logics. For generic absoluteness the relevant logic is $\Omega^*-logic$.

**Definition 1.2 ($\Omega^*-logic$) Suppose that there exists a proper class of Woodin cardinals and that $\phi$ is a sentence. Then**

$$ZFC \vdash_{\Omega^*} \phi$$

if for all ordinals $\alpha$ and for all partial orders $P$ if $V^{\alpha}_P \models ZFC$, then $V^{\alpha}_P \models \phi$.

The theorem on $\Sigma^2_1$-absoluteness and CH can be reformulated as follows.

**Theorem 1.3** Suppose there exists a proper class of measurable Woodin cardinals. Then for each $\Sigma^2_1$ sentence $\phi$, either $ZFC + CH \vdash_{\Omega^*} \phi$; or $ZFC + CH \vdash_{\Omega^*} (\neg \phi)$. □

But there is another natural strong logic; $\Omega$-logic, the definition of $\Omega$-logic involves universally Baire sets of reals.

**Definition 1.4** A set $A \subseteq \mathbb{R}^n$ is universally Baire if for any continuous function, $F : \Omega \to \mathbb{R}^n$, where $\Omega$ is a compact Hausdorff space, the preimage of $A$,

$$\{ p \in X \mid F(p) \in A \},$$

has the property of Baire in $\Omega$; i.e. is open in $\Omega$ modulo a meager set. □

Every borel set $A \subseteq \mathbb{R}^n$ is universally Baire. More generally the universally Baire sets form a $\sigma$-algebra closed under preimages by borel functions

$$f : \mathbb{R}^n \to \mathbb{R}^m.$$  

The universally Baire sets have the classical regularity properties of the borel sets, for example they are Lebesgue measurable and have the property of Baire. If there exists a proper class of Woodin cardinals then the universally Baire sets are closed under projection and every universally Baire set is determined.

Suppose that $A \subseteq \mathbb{R}$ in universally Baire and that $V[G]$ is a set generic extension of $V$. Then the set $A$ has canonical interpretation as a set

$$A_G \subseteq \mathbb{R}^{V[G]}.$$  

The set $A_G$ is defined as

$$A_G = \cup \{ \text{range}(\pi_G) \mid \pi \in V, \text{range}(\pi) = A \};$$

here $\pi$ is a function, $\pi : \lambda^\omega \to \mathbb{R}$, which satisfies the uniform continuity requirement that for $f \neq g$:

$$| \pi(f) - \pi(g) | < 1/(n + 1)$$
where \( n < \omega \) is least such that \( f(n) \neq g(n) \). If there exists a proper class of Woodin cardinals then
\[
(H(\omega_1), \in) \prec (H(\omega_1)^{V[G]}, A_G, \in).
\]

**Definition 1.5** Suppose that \( A \subseteq \mathbb{R} \) is universally Baire and that \( M \) is a transitive set such that \( M \models \text{ZFC} \). Then \( M \) is \( A \)-closed if for each partial order \( P \in M \), if \( G \subset P \) is \( V \)-generic then in \( V[G] \): \( A_G \cap M[G] \in M[G] \).

**Definition 1.6 (\( \Omega \) logic)** Suppose that there exists a proper class of Woodin cardinals and that \( \phi \) is a sentence. Then \( ZFC \vdash \Omega \phi \) if there exists a universally Baire set \( A \subseteq \mathbb{R} \) such that if \( M \) is any countable transitive set such that \( M \models \text{ZFC} \) and such that \( M \) is \( A \)-closed, then \( M \models \phi \).

Both \( \Omega \)-logic and \( \Omega^* \)-logic are definable and generically invariant.

A natural question, given the theorem on generic absoluteness for \( \Sigma^2_1 \) is the following question:

**Suppose there exists a proper class of measurable Woodin cardinals. Does it follow that for each \( \Sigma^2_1 \) sentence \( \phi \), either \( ZFC + \text{CH} \vdash \Omega \phi \); or \( ZFC + \text{CH} \vdash \Omega (\neg \phi) \)?**

The answer is yes if “iterable” models with measurable Woodin cardinals exist.

**\( \Omega \) Conjecture:**

**Suppose that there exists a proper class of Woodin cardinals and that \( \phi \) is a \( \Pi^2_2 \) sentence. Then \( ZFC \vdash \Omega \phi \) if and only if \( ZFC \vdash \Omega \phi \).**

It is immediate from the definitions and Theorem 1.1, that the \( \Omega \) Conjecture settles the question above affirmatively. But the consequences of the \( \Omega \)-Conjecture are far more reaching. If the \( \Omega \) Conjecture is true, then generic absoluteness is equivalent to absoluteness in \( \Omega \)-logic and this in turn has significant metamathematical implications.

We fix some conventions. A formula, \( \phi(x) \), is a \( \Sigma^2_2 \)-formula if for some \( \Sigma^2_2 \)-formula, \( \psi(x) \), the formula \( \phi(x) \) is provably equivalent in \( \text{ZFC} \) to the formula
\[
"x \in H(c^+) \text{ and } (H(c^+), \in) \models \psi[x]."
\]

Finally \( \phi(x) \) is a \( \Sigma^2_2(\mathcal{I}_{NS}) \)-formula if for some \( \Sigma^2_2 \)-formula, \( \psi(x) \), the formula \( \phi(x) \) is provably equivalent in \( \text{ZFC} \) to the formula
\[
"x \in H(c^+) \text{ and } (H(c^+), \mathcal{I}_{NS}, \in) \models \psi[x]."
\]

where \( \mathcal{I}_{NS} \) denotes the nonstationary ideal on \( \omega_1 \).

There is a limit to the possible extent of absoluteness in \( \Omega \)-logic. One version is given by the following theorem.

**Theorem 1.7** Suppose that there exist a proper class of Woodin cardinals, \( \Psi \) is a sentence and that for each \( \Sigma^2_2(\mathcal{I}_{NS}) \) sentence \( \phi \), either \( \text{ZFC} + \Psi \vdash \Omega \phi \); or \( \text{ZFC} + \Psi \vdash \Omega (\neg \phi) \). Then \( \text{ZFC} + \Psi \) is \( \Omega \)-inconsistent. \( \square \)

In short:
\[ \Sigma_2^2(I_{NS}) \] absoluteness is not possible in \( \Omega \)-logic. If the \( \Omega \) Conjecture holds then generic absoluteness for \( \Sigma_2^2(I_{NS}) \) sentences is not possible.

So for absoluteness in \( \Omega \)-logic the most one can hope for is that there exist a sentence \( \Psi \) such that for each \( \Sigma_2^2 \) sentence \( \phi \), either \( \text{ZFC} + \Psi \vdash \Omega \phi \), or \( \text{ZFC} + \Psi \vdash \Omega (\neg \phi) \). In particular if the \( \Omega \) Conjecture holds then \( \Sigma_2^2 \) generic absoluteness is the most one can hope for.

Suppose that \( \Psi \) is a sentence such that for each \( \Sigma_2^2 \) sentence \( \phi \), either \( \text{ZFC} + \Psi \vdash \Omega \phi \), or \( \text{ZFC} + \Psi \vdash \Omega (\neg \phi) \). Then \( \text{ZFC} + \Psi \vdash 2^{\aleph_0} < 2^{\aleph_1} \). A natural conjecture is that in fact \( \text{ZFC} + \Psi \vdash \Omega \text{CH} \).

In any case from this point on we shall consider absoluteness just in the context of \( \text{CH} \).

Generic absoluteness is closely related to determinacy. The statement of a theorem which illustrates one aspect of this requires the following definition.

**Definition 1.8** Suppose that there exists a proper class of Woodin cardinals. A set \( A \subseteq \mathbb{R} \) is \( \Omega^* \)-recursive if there exists a formula \( \phi(x) \) such that:

1. \( A = \{ r \mid \text{ZFC} \vdash \Omega \phi[r] \} \);
2. For all partial orders, \( P \), if \( G \subseteq P \) is \( V \)-generic then for each \( r \in \mathbb{R}^{[G]} \), either
   \[ V[G] \models \text{“ZFC} \vdash \Omega \phi[r]” \]
   or
   \[ V[G] \models \text{“ZFC} \vdash \Omega (\neg \phi)[r]” \]. \qed

**Theorem 1.9** Suppose that there exists a proper class of Woodin cardinals. Suppose that \( A \subseteq \mathbb{R} \) is \( \Omega^* \)-recursive. Then \( A \) is determined. \qed

On the other hand there are many examples where suitable determinacy assumptions imply generic absoluteness. Our main results deal with generalizations of these connections to \( \Sigma_2^1 \) and \( \Sigma_2^2 \) in the context of \( \text{CH} \).

2. **Absoluteness and determinacy**

We fix a reasonable coding of elements of \( \text{H}(\omega_1) \) by reals. This is simply a surjection

\[ \pi : \text{dom}(\pi) \to \text{H}(\omega_1) \]

where \( \text{dom}(\pi) \subseteq \mathbb{R} \). All we require of \( \pi \) is that \( \pi \in \text{L}(\mathbb{R}) \); the natural choice for \( \pi \) is definable in \( \text{H}(\omega_1) \). For each set \( X \subseteq \text{H}(\omega_1) \) let

\[ X^* = \{ x \in \mathbb{R} \mid \pi(x) \in X \} \]

Suppose \( X \subseteq \{0,1\}^{\omega_1} \). Associated to \( X \) is a game of length \( \omega_1 \). The convention is that Player I plays first at limit stages. Strategies are functions:

\[ \tau : \{0,1\}^{<\omega_1} \to \{0,1\} \]

Suppose that \( \Gamma \subseteq \mathcal{P} (\mathbb{R}) \). Then \( X \) is \( \Gamma \)-clopen if there exist sets \( Y \subset \text{H}(\omega_1) \) and \( Z \subset \text{H}(\omega_1) \) such that
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1. $Y \cap Z = \emptyset$,
2. for all $a \in \{0, 1\}^{\omega_1}$ there exists $\alpha < \omega_1$ such that either $a|\alpha \in Y$ or $a|\alpha \in Z$,
3. $X$ is the set of $a \in \{0, 1\}^{\omega_1}$ such that there exists $\alpha < \omega_1$ such that $a|\alpha \in Y$
   and such that $a|\beta \notin Z$ for all $\beta < \alpha$,
4. $Y^* \in \Gamma$ and $Z^* \in \Gamma$.

The first result on the determinacy of $\Gamma$-clopen sets is due to Itay Neeman. One version is the following.

**Theorem 2.1** Suppose that there exists a Woodin cardinal which is a limit of Woodin cardinals. Suppose that $X \subseteq \{0, 1\}^{\omega_1}$ and $X$ is $\Pi^1_1$-clopen.

Then $X$ is determined. □

The proof of Neeman’s theorem combined with techniques from the fine structure theory associated to $\text{AD}^+$ yields the following generalization which we shall need.

**Theorem 2.2** Suppose that there is a proper class of Woodin cardinals which are limits of Woodin cardinals. Let $\Gamma^\infty$ be the set of all $A \subseteq \mathbb{R}$ such that $A$ is universally Baire and suppose that $X \subseteq \{0, 1\}^{\omega_1}$ is such that $X$ is $\Gamma^\infty$-clopen.

Then $X$ is determined. □

Suppose $X \subseteq \{0, 1\}^{\omega_1}$ and that $\Gamma \subseteq \mathcal{P}(\mathbb{R})$. Then $X$ is $\Gamma$-open if there exist sets $Y \subset H(\omega_1)$ such that

1. $X$ is the set of $a \in \{0, 1\}^{\omega_1}$ such that there exists $\alpha < \omega_1$ such that $a|\alpha \in Y$ .
2. $Y^* \in \Gamma$.

John Steel has proved that under fairly general conditions, if $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is such that all $\Gamma$-open sets are determined then for each $X \subseteq \{0, 1\}^{\omega_1}$, if $X$ is $\Gamma$-open and if Player I wins the game given by $X$, then there is a winning strategy for Player I which is (suitably) definable from parameters in $\Gamma$: [4]. The following is a straightforward corollary:

**Corollary 2.3** Suppose that there exists a proper class of Woodin cardinals. Let $\Gamma^\infty$ be the set of all $A \subseteq \mathbb{R}$ such that $A$ is universally Baire and suppose that for each $A \in \Gamma^\infty$, $\text{ZFC} \vdash \Omega \text{ " All } \Sigma^1_1(A)\text{-open games are determined".}$

Then for each $A \in \Gamma^\infty$, for each $\Sigma^1_2$-formula $\phi(x)$; either $\text{ZFC + CH} \vdash \Omega \phi[A]$ or $\text{ZFC + CH} \vdash \Omega (\neg \phi)[A]$. □

Using the theorem on the determinacy of $\Gamma^\infty$-clopen games one obtains the converse.

**Theorem 2.4** Suppose that there exists a proper class of Woodin cardinals which are limits of Woodin cardinals. Let $\Gamma^\infty$ be the set of all $A \subseteq \mathbb{R}$ such that $A$ is universally Baire. Then the following are equivalent.

1. For each $A \in \Gamma^\infty$, $\text{ZFC} \vdash \Omega \text{ " All } \Sigma^1_1(A)\text{-open games are determined".}$
2. For each $A \in \Gamma^\infty$, for each $\Sigma^1_2$-formula $\phi(x)$, either $\text{ZFC + CH} \vdash \Omega \phi[A]$ or $\text{ZFC + CH} \vdash \Omega (\neg \phi)[A]$. □
A set $A \subseteq \mathbb{N}$ is $\Omega$-recursive if there exists a formula $\phi(x)$ such that for all $k \in \mathbb{N}$, either $\text{ZFC} \vdash_\Omega \phi[k]$ or $\text{ZFC} \vdash_\Omega (\neg \phi)[k]$; and such that $A = \{k \in \mathbb{N} \mid \text{ZFC} \vdash_\Omega \phi[k]\}$.

The question of whether there exists a sentence $\Psi$ such that for each $\Sigma^2_2$-sentence $\phi$, either $\text{ZFC} + \text{CH} + \Psi \vdash_\Omega \phi$, or $\text{ZFC} + \text{CH} + \Psi \vdash_\Omega (\neg \phi)$, and such that $\text{ZFC} + \text{CH} + \Psi$ is $\Omega$-consistent; can be reformulated as:

*Suppose there exists a proper class of Woodin cardinals and that $\text{CH}$ holds. Let $T$ be the set of all $\Sigma^2_2$-sentences, $\phi$, such that $V \models \phi$. Can $T$ be $\Omega$-recursive?*

**Theorem 2.5** Suppose that there exists a proper class of inaccessible limits of Woodin cardinals. Let $\Gamma^\infty$ be the set of all $A \subseteq \mathbb{R}$ such that $A$ is universally Baire and suppose that all $\Gamma^\infty$-open games are determined. Let $T_{\text{max}}$ be the set of all $\Sigma^2_2$ sentences $\phi$ such that $\text{ZFC} + \text{CH} + \phi$ is $\Omega$-consistent.

Then $\text{ZFC} + \text{CH} + T_{\text{max}}$ is $\Omega$-consistent. $\square$

The following conjecture can be proved from rather technical assumptions on the existence of an inner model theory for the large cardinal hypothesis: $\kappa$ is $\delta$ supercompact where $\delta > \kappa$ and $\delta$ is a Woodin cardinal. The conjecture is:

*Suppose that there exists a proper class of supercompact cardinals. Let $T_{\text{max}}$ be the set of all $\Sigma^2_2$ sentences $\phi$ such that $\text{ZFC} + \text{CH} + \phi$ is $\Omega$-consistent. Then $T_{\text{max}}$ is $\Omega$-recursive.*

While the plausibility of this conjecture is some evidence that $\Sigma^2_2$ absoluteness is possible, it does not connect $\Sigma^2_2$ absoluteness with any determinacy hypothesis.

From inner model theory considerations any such determinacy hypothesis must be beyond the reach of superstrong cardinals. In fact, Itay Neeman has defined a family of games whose (provable) determinacy is arguably beyond the reach of superstrong cardinals; 

### 3. Neeman games

For each formula $\phi(x_1, \ldots, x_n)$, let $X_\phi$ be the set of all $a \in \{0, 1\}^{\omega_1}$ such that there exists a closed, unbounded set $C \subseteq \omega_1$ such that for all $\alpha_1 < \cdots < \alpha_n$ in $C$,

$$\langle H(\omega_1), a, \varepsilon \rangle \models \phi[\alpha_1, \ldots, \alpha_n].$$

The game given by $X_\phi$ is a **Neeman game**. Are Neeman games determined? Surprisingly the consistency strength of the determinacy of all Neeman games is relatively weak.

**Lemma 3.1** Suppose that all $\Delta^1_3$-clopen games are determined. Then there exists $A \subseteq \omega_1$ such that in $L[A]$ if $X \subseteq \{0, 1\}^{\omega_1}$ is definable an $\omega$-sequence of ordinals, then $X$ is determined. $\square$
One can easily introduce additional predicates for sets of reals.
For each formula, $\phi(x_1, \ldots, x_n)$, and for each set $A \subseteq \mathbb{R}$ let $X_{(\phi,A)}$ be the set of all $a \in \{0, 1\}^{\omega_1}$ such that there exists a closed, unbounded set $C \subseteq \omega_1$ such that for all $\alpha_1 < \cdots < \alpha_n$ in $C$, $(H(\omega_1), a, A, \in) \models \phi[\alpha_1, \ldots, \alpha_n]$. The game given by $X_{(\phi,A)}$ is an $A$-Neeman game.

**Definition 3.2** $\circ_G$: For each $\Sigma^2_2$ sentence, $\phi$, $V \models \phi$ if and only if $V^{\text{Coll}(\omega_1, \mathbb{R})} \models \phi$. \hfill $\Box$

The principle, $\circ_G$, is a generic form of $\circ$. The next theorem gives a connection between versions of $\Sigma^2_2$ absoluteness and determinacy specifically the determinacy of Neeman games. In this theorem it is the principle, $\circ_G$, which plays the role of CH in the theorem on $\Sigma^1_2$ absoluteness.

**Theorem 3.3** Suppose that there exists a proper class of supercompact cardinals. Let $\Gamma^\infty$ be the set of all $A \subseteq \mathbb{R}$ such that $A$ is universally Baire. Then the following are equivalent.

1. For each $A \in \Gamma^\infty$, ZFC + $\circ_G$ $\vdash$ $\Omega$ “All $A$-Neeman games are determined”.
2. For each $A \in \Gamma^\infty$, for each $\Sigma^2_2$-formula $\phi(x)$, either ZFC + $\circ_G$ $\vdash$ $\Omega$ $\phi[A]$ or ZFC + $\circ_G$ $\vdash$ $\Omega$ ($\neg \phi)[A]$. \hfill $\Box$

We note the following trivial lemma which simply connects the results here with the earlier “evidence” that $\Sigma^4_2$ absoluteness is possible; cf. the discussion after Theorem 2.5.

**Lemma 3.4** Suppose that there exists a proper class of inaccessible limits of Woodin cardinals and suppose that for each $\Sigma^2_2$-sentence $\phi$, either ZFC + $\circ_G$ $\vdash$ $\Omega$ $\phi$ or ZFC + $\circ_G$ $\vdash$ $\Omega$ ($\neg \phi$). Then for each $\Sigma^2_2$ sentence $\phi$ the following are equivalent: 

1. ZFC + $\circ_G$ $\vdash$ $\Omega$ $\phi$;
2. ZFC + CH + $\phi$ is $\Omega$-consistent. \hfill $\Box$

The next theorem suggests that $\Sigma^2_2$ absoluteness conditioned simply on $\circ$ might actually follow from some large cardinal hypothesis. Such a theorem would certainly be a striking generalization of Theorem 1.1 and its proof might well yield fundamental new insights into the combinatorics of subsets of $\omega_1$.

**Theorem 3.5** Suppose that there exists a proper class of supercompact cardinals. Let $\Gamma^\infty$ be the set of all $A \subseteq \mathbb{R}$ such that $A$ is universally Baire and suppose that for each $A \in \Gamma^\infty$, ZFC $\vdash$ $\Omega$ “All $A$-Neeman games are determined”. Then for each $A \in \Gamma^\infty$, for each $\Sigma^2_2$-formula $\phi(x)$, either ZFC + $\circ$ $\vdash$ $\Omega$ $\phi[A]$ or ZFC + $\circ$ $\vdash$ $\Omega$ ($\neg \phi)[A]$. \hfill $\Box$

Given Theorem 3.5, the natural conjecture is that Theorem 3.3 holds with $\circ_G$ replaced by $\circ$. The missing ingredient in proving such a conjecture seems to be a lack of information on the nature of definable winning strategies for Neeman games and more fundamentally on the lack of any genuine determinacy proofs whatsoever for Neeman games.
In an exploration of the combinatorial aspects of Neeman games it is useful to consider a wider class of games. This class we now define.

For each formula, $\phi(x_1, \ldots, x_n)$, and for each stationary set $S \subseteq \omega_1$ let $Y_\phi$ be the set of all $a \in \{0, 1\}^{\omega_1}$ such that there exists a stationary set $S \subseteq \omega_1$ such that for all $\alpha_1 < \cdots < \alpha_n$ in $S$, $\langle H(\omega_1), a, \in \rangle \models \phi[\alpha_1, \ldots, \alpha_n]$. The game given by $Y_\phi$ is a stationary Neeman game.

Can some large cardinal hypothesis imply that all stationary Neeman games are determined? Given the impossibility of $\Sigma^2_2(I_{NS})$-absoluteness, modulo failure of the $\Omega$ Conjecture one would naturally conjecture that the answer is “no”. This is simply because there is no apparent candidate for an absoluteness theorem which would correspond to the (provable) determinacy of all stationary Neeman games.

We define two games of length $\omega_1$. The first is a Neeman game and the second is a stationary Neeman game. Rather than have the moves be from $\{0, 1\}$ it is more convenient to have the moves be from $H(\omega_1)$.

**The canonical function game:** Player I plays $<a_\alpha : \alpha < \omega_1>$ and Player II plays $<b_\alpha : \alpha < \omega_1>$ subject to the rules: $a_{\alpha+1} \subseteq \alpha \times \alpha$ and $b_\alpha$ is a countable ordinal.

Player I wins if there exists a set $A \subseteq \omega_1 \times \omega_1$ such that $A$ is a wellordering of $\omega_1$ and such that there exists a closed unbounded set $C \subseteq \omega_1$ such that for all $\alpha \in C$: $a_{\alpha+1} = A \cap (\alpha \times \alpha)$ and $b_\alpha < \text{rank}(a_{\alpha+1})$.

**The stationary canonical function game:** Player I plays $<a_\alpha : \alpha < \omega_1>$ and Player II plays $<b_\alpha : \alpha < \omega_1>$ subject to the rules: $a_{\alpha+1} \subseteq \alpha \times \alpha$ and $b_\alpha$ is a countable ordinal.

Player I wins if there exists a set $A \subseteq \omega_1 \times \omega_1$ such that $A$ is a wellordering of $\omega_1$ and such that there exists a stationary set $S \subseteq \omega_1$ such that for all $\alpha \in S$: $a_{\alpha+1} = A \cap (\alpha \times \alpha)$ and $b_\alpha < \text{rank}(a_{\alpha+1})$.

In models where $L$-like condensation principles hold these games are easily seen to be determined.

**Lemma 3.6** Suppose $\diamondsuit$ holds. Then Player II has a winning strategy in the canonical function game.

**Lemma 3.7** Suppose $\diamondsuit^+$ holds. Then Player II has a winning strategy in the stationary canonical function game.

In contrast to the previous lemma, the following theorem shows that it is consistent that Player I has a winning strategy in the stationary canonical function game, at least if fairly strong large cardinal hypotheses are assumed to be consistent.

**Theorem 3.8** Suppose there is a huge cardinal. Then there is a partial order, $\mathbb{P}$, such that in $V^\mathbb{P}$, Player I has a winning strategy in the stationary canonical function game.

These two results strongly suggest that no large cardinal hypothesis can imply that the stationary canonical function game is determined. In fact from consistency of a relatively weak large cardinal hypothesis, one does obtain the consistency that the stationary canonical function game is not determined. Note that if the stationary canonical function game is not determined then every function, $f : \omega_1 \rightarrow \omega_1$, is
bounded by a canonical function on a stationary set and so the consistency of some large cardinal hypothesis is necessary.

**Theorem 3.9** Suppose there is a measurable cardinal. Then there is a partial order, $P$, such that in $V^P$ the stationary canonical function game is not determined. $\square$

There are many open problems about the canonical function games. Here are several.

1. Is it consistent that Player I has a winning strategy in the canonical function game?
2. Is it consistent that Player II does not have a winning strategy in the canonical function game?
3. Is it consistent that Player I has a winning strategy in the stationary canonical function game on each stationary set?
4. How strong is the assertion that Player I has a winning strategy in the stationary canonical function game?

For each formula, $\phi(x_1, \ldots, x_n)$, for each sequence $S = \langle S_\alpha : \alpha < \omega_1 \rangle$ of pairwise disjoint stationary subsets of $\omega_1$ and that $A \subseteq \mathbb{R}$, let $Y^S_{(\phi, A)}$ be the set of all $a \in \{0, 1\}^{\omega_1}$ such that there exists a stationary set $S \subseteq \omega_1$ such that for all $\alpha_1 < \cdots < \alpha_n$ in $S$,

$$\langle H(\omega_1), a, A, \in \rangle \models \phi[\alpha_1, \ldots, \alpha_n],$$

and such that $S \cap S_\alpha$ is stationary for all $\alpha < \omega_1$.

**Theorem 3.10** Suppose that there exists a proper class of supercompact cardinals. Let $\Gamma^\infty$ be the set of all $A \subseteq \mathbb{R}$ such that $A$ is universally Baire.

Suppose that $A \in \Gamma^\infty$, $\phi(x_1, \ldots, x_n)$ is a formula and that

$$\text{ZFC} \vdash \Omega \ " \text{The Neeman game } X_{(\phi, A)} \text{ is determined}".$$

Then either:

1. $\text{ZFC} \vdash \Omega \ " \text{I wins the game } X_{(\phi, A)}"$, or;
2. $\text{ZFC} \vdash \Omega \ " \text{For all } S, \text{ II wins the game } Y^S_{(\phi, A)}"$. $\square$

The determinacy hypothesis: All Neeman games are determined; is relatively weak in consistency strength (the consistency strength is at most that of the existence of a Woodin cardinal which is a limit of Woodin cardinals). However the determinacy hypothesis:

For each formula $\phi$, either Player I wins the game $X_\phi$, or for each sequence,

$$S = \langle S_\alpha : \alpha < \omega_1 \rangle,$$

of pairwise disjoint stationary subsets of $\omega_1$, Player II wins $Y^S_{(\phi, \emptyset)}$;

seems plausibly very strong.
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