Greenberger-Horne-Zeilinger paradox for continuous variables

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We show how to construct states for which a Greenberger-Horne-Zeilinger type paradox occurs if each party measures either the position or momentum of his particle. The paradox can be ascribed to the anticommutation of certain translation operators in phase space. We then rephrase the paradox in terms of modular and binary variables. The origin of the paradox is then due to the fact that the associativity of addition of modular variables is true only for c-numbers but does not hold for operators.

The Greenberger-Horne-Zeilinger (GHZ) paradox [1,2] exhibits in a very simple way the non locality inherent to quantum mechanics. The paradox is based on the following state

$$\Psi_{GHZ} = \frac{|↑A\downarrow B\downarrow C⟩ + |↓A\downarrow B\uparrow C⟩}{\sqrt{2}}$$

(1)

of three spin 1/2 particles shared among three spatially separated parties A, B, C. The GHZ state is an eigenstate of four commuting operators, the GHZ operators:

$$\sigma_x^A \sigma_x^B \sigma_x^C \Psi_{GHZ} = - \Psi_{GHZ},$$

$$\sigma_x^A \sigma_y^B \sigma_y^C \Psi_{GHZ} = + \Psi_{GHZ},$$

$$\sigma_y^A \sigma_x^B \sigma_x^C \Psi_{GHZ} = + \Psi_{GHZ},$$

$$\sigma_y^A \sigma_y^B \sigma_x^C \Psi_{GHZ} = + \Psi_{GHZ}.$$  

(2)

The paradox arises when one compares the predictions of quantum mechanics and of local hidden variable theories for local measurements of \(\sigma_x\) and \(\sigma_y\) by the three parties. By carrying out such local measurements, one can measure any of the four GHZ operators that appear on the left hand side of (2). Quantum mechanics predicts that for the state \(\Psi_{GHZ}\) measuring any of the four GHZ operators must yield a result equal to the eigenvalue appearing on the right hand side of (2). On the other hand in a local hidden variable theory one must, prior to the measurement, assign values to all six of the operators \(\sigma_{x,y}^j\), \(j = A, B, C\). These values must be equal to one of the eigenvalues of the operators, i.e. to \(\pm 1\). But it is impossible to assign values to all six operators \(\sigma_{x,y}^j\) and be compatible with the predictions of quantum mechanics. Indeed taking the product of the 4 GHZ equations one obtains +1 when the \(\sigma_{x,y}^j\) are replaced by c-numbers. Quantum mechanically one obtains −1 because \(\sigma_x^j\) and \(\sigma_y^j\) anti-commute.

The aim of the present paper is to show how one can generalize in a natural way the GHZ paradox to continuous variables such as position and momentum. Our analysis is inspired by the analogy between the EPR state for continuous variables \(\Psi_{EPR} = \int dx |x_A, -x_B⟩\) and the singlet state for discrete systems \(\Psi^- = (|↑A\downarrow B⟩ - |↓A\uparrow B⟩)/\sqrt{2}\). These states can be defined in terms of the operators of which they are eigenstates: \(x_A + x_B\) \(\Psi_{EPR}⟩ = 0\), \((p_A - p_B)\) \(\Psi_{EPR}⟩ = 0\) and \(\sigma_x^A \sigma_x^B |\Psi^-⟩ = - |\Psi^-⟩\), \(\sigma_y^A \sigma_y^B |\Psi^-⟩ = - |\Psi^-⟩\). This suggests that the way to pass from discrete variables to continuous variables is to replace products of Pauli matrices by sums of position and momentum operators. But then one does not see how to obtain a GHZ paradox of the form eq. (3) since addition of operators is always commutative. We shall show that there are two equivalent ways to circumvent this. The first is to work with products of translation operators of the form \(\exp(\imath \alpha x)\) and \(\exp(\imath \beta p)\); the second is to work with sums of modular variables (first introduced in [3]). The origin of the paradox is in one case the non-commutativity of translation operators and in the second the non associativity of the modulo operation for operators. Finally we introduce a new kind of variable, which we call binary variables, in terms of which the continuous variable GHZ paradox can be mapped onto the the GHZ paradox for spins eq. (4).

Multi-particle non local states for continuous variables have been considered previously by van Loock and Braunstein [4]. One of the interests of the states considered by van Loock and Braunstein is that they can be easily constructed using squeezed states and beam splitters. But the measurements that exhibit the non locality are complicated and cannot be realized in the laboratory at present. On the other hand the states we discuss here seem significantly more complicated to construct than those considered by van Loock and Braunstein (in particular they cannot be constructed using squeezers and beam splitters). But the measurements that exhibit the non locality are simple position and momentum measurements.

Let us consider a set of unitary operators \(X_j, Y_j, j = A, B, C\) acting on A, B and C’s particles respectively. Using these operators we construct the following 4 GHZ...
satisfied. This follows from the identity

\[ \exp(i\pi \tilde{x}_j \alpha) \exp(i\pi \tilde{p}_j \beta) = \exp(i\pi \tilde{p}_j \beta) \exp(i\pi \tilde{x}_j \alpha) \exp(-i\pi \alpha \beta) . \] (7)

In particular we see that if the product \( \alpha \beta \) is an even integer, \( \exp(i\pi \tilde{x}_j \alpha) \) and \( \exp(i\pi \tilde{p}_j \beta) \) commute, whereas if \( \alpha \beta \) is an odd integer they anti-commute.

Thus any common eigenstate of \( V_1, V_2, V_3, V_4 \) with \( X_j, Y_j \) given in eq. (6) yields a GHZ paradox if the three parties carry out local measurements of either \( x \) or \( p \). Before describing the states that yield the GHZ paradox, we pursue the algebraic analysis of these equations.

By taking the logarithm of the GHZ equations and dividing by \( i\pi \), we obtain the following form in terms of hermitian operators

\[ (\tilde{x}_A + \tilde{x}_B + \tilde{x}_C) \text{mod} 2 |\Psi_{GHZ}'\rangle = \eta_1 |\Psi_{GHZ}'\rangle , \]
\[ (-\tilde{x}_A + \tilde{p}_B - \tilde{p}_C) \text{mod} 2 |\Psi_{GHZ}'\rangle = \eta_2 |\Psi_{GHZ}'\rangle , \]
\[ (-\tilde{p}_A - \tilde{x}_B + \tilde{p}_C) \text{mod} 2 |\Psi_{GHZ}'\rangle = \eta_3 |\Psi_{GHZ}'\rangle , \]
\[ (\tilde{p}_A - \tilde{p}_B - \tilde{x}_C) \text{mod} 2 |\Psi_{GHZ}'\rangle = \eta_4 |\Psi_{GHZ}'\rangle , \] (8)

where the eigenvalues \( \eta_k \in [0,2] \) obey the relation

\[ (\eta_1 + \eta_2 + \eta_3 + \eta_4) \text{mod} 2 = 1 \] (9)

We recall that if \( |a\rangle \) is an eigenvector of the operator \( A \) with eigenvalue \( a \), then the modular operator \( (A)\text{mod}k \) is defined by \((A)\text{mod}k |a\rangle = (a)\text{mod}k |a\rangle \). Modular variables where introduced in [8] as a general tool to study non-locality in quantum mechanics. It is interesting that they reappear in the context of the GHZ paradox.

In the case of eq. (8), the paradox arises because the associativity of the modulo operation \((A + B)\text{mod}k = ((A)\text{mod}k + (B)\text{mod}k)\text{mod}k\) which holds for c-numbers is in general not valid when \( A \) and \( B \) are non-commuting operators. Thus in a local hidden variable theory one must assign real values to \( (\tilde{x}_j)\text{mod}2 \) and to \( (\tilde{p}_j)\text{mod}2 \). Then taking the sum of the 4 equations (8) and using the associativity property of modulo for c-numbers one finds \((\eta_1 + \eta_2 + \eta_3 + \eta_4)\text{mod}2 = 0\) in contradiction with the quantum condition eq. (8).

We shall now rephrase the GHZ paradox eq. (8) in a different way by using binary variables. Consider a position eigenstate \( |x\rangle \) and write its eigenvalue in base 2 as

\[ x = (-1)^{\text{sgn} x} L \sqrt{\pi} \sum_{n=-\infty}^{\infty} [\tilde{x}]_n 2^n . \] (10)

This allows us to introduce the sign operator \( \text{sgn} x \) and the binary operators \([\tilde{x}]_n \) defined by
Using the base 2 decomposition of momentum

\[ \text{sgn}_n \left| x \right\rangle = \text{sgn}_n \left| x \right\rangle, \]
\[ \left| \tilde{x} \right|_n \left| x \right\rangle = \left| \tilde{x} \right|_n \left| x \right\rangle. \]  
\[ (11) \]

The modular position is then written as

\[ (\tilde{x}) \text{mod} 2^k = \sum_{n=-\infty}^{k-1} \left| \tilde{x} \right|_n 2^n \]  
\[ (12) \]

Similarly we can introduce the operators \( \text{sgn}_n \) and \( \left| \tilde{p} \right|_n \) using the base 2 decomposition of momentum

\[ p = (-1)^{\text{sgn}_p} \frac{\sqrt{n}}{L} \sum_{n=-\infty}^{+\infty} \left| \tilde{p} \right|_n 2^n. \]  
\[ (13) \]

Using these definitions we have the relation

\[ (z) \text{mod} 2^{k+1} = (z) \text{mod} 2^k + [z]_k 2^k \]  
\[ (14) \]

for \( z = \tilde{x}, \tilde{p} \) (from now on we drop the over operators since it will be clear from the context whether \( x, p \) denote operators or c-numbers). We can then rewrite eq. \( (8) \) as

\[ (\tilde{x}_A) \text{mod} 1 + (\tilde{x}_B) \text{mod} 1 + (\tilde{x}_C) \text{mod} 1 + [\tilde{x}_A]_0 + [\tilde{x}_B]_0 + [\tilde{x}_C]_0 \text{ mod } 2 \left| \Psi_{\text{GHZ}} \right\rangle = \eta_1 \left| \Psi_{\text{GHZ}} \right\rangle, \]
\[ (\tilde{x}_A) \text{mod} 1 + (\tilde{p}_B) \text{mod} 1 + (\tilde{p}_C) \text{mod} 1 + [\tilde{x}_A]_0 + [\tilde{p}_B]_0 + [\tilde{p}_C]_0 \text{ mod } 2 \left| \Psi_{\text{GHZ}} \right\rangle = \eta_2 \left| \Psi_{\text{GHZ}} \right\rangle, \]
\[ (\tilde{p}_A) \text{mod} 1 + (\tilde{x}_B) \text{mod} 1 + (\tilde{x}_C) \text{mod} 1 + [\tilde{p}_A]_0 + [\tilde{x}_B]_0 + [\tilde{x}_C]_0 \text{ mod } 2 \left| \Psi_{\text{GHZ}} \right\rangle = \eta_3 \left| \Psi_{\text{GHZ}} \right\rangle, \]
\[ (\tilde{p}_A) \text{mod} 1 + (\tilde{p}_B) \text{mod} 1 + (\tilde{x}_C) \text{mod} 1 + [\tilde{p}_A]_0 + [\tilde{p}_B]_0 + [\tilde{x}_C]_0 \text{ mod } 2 \left| \Psi_{\text{GHZ}} \right\rangle = \eta_4 \left| \Psi_{\text{GHZ}} \right\rangle, \]  
\[ (15) \]

where we have used the fact that \( [-z]_0 = [z]_0 \).

In order to understand the structure of eq. \( (12) \), we note that \( (\tilde{x}) \text{mod} 2^k \) and \( (\tilde{p}) \text{mod} 2^k \) commute if \( k + l \leq 1 \). Indeed, \( (\tilde{x}) \text{mod} 2^k \) and \( e^{i \frac{\pi}{2} \tilde{x}} \) have the same eigenstates and similarly \( (\tilde{p}) \text{mod} 2^k \) and \( e^{i \tilde{p}} \) commute. Thus if \( e^{i \frac{\pi}{2} \tilde{x}} \) and \( e^{i \tilde{p}} \) commute, they share a common basis of eigenstates, and therefore \( (\tilde{x}) \text{mod} 2^k \) and \( (\tilde{p}) \text{mod} 2^k \) do too. But from \( (\tilde{x}) \text{mod} 2^k \) and \( (\tilde{p}) \text{mod} 2^k \) commute only if \( k + l \leq 1 \). Using \( (12) \), we also have that \( (\tilde{x}) \text{mod} 2^k \) and \( (\tilde{p}) \text{mod} 2^k \) commute if \( k + m \leq 0 \); \( (\tilde{x}) \text{mod} 2^k \) and \( (\tilde{x}) \text{mod} 2^k \) commute if \( l + m \leq 0 \); \( (\tilde{x}) \text{mod} 2^k \) and \( (\tilde{p}) \text{mod} 2^k \) commute if \( n + m \leq -1 \).

From these properties we deduce that the mod 1 terms on the left hand side of the four equations \( (12) \) commute with all the other terms that appear in these equations. These terms are therefore not essential to the paradox and can be dropped. Omitting them yields the following simple form for eq. \( (12) \)

\[ \left( (\tilde{x}_A)_0 + [\tilde{x}_B]_0 + [\tilde{x}_C]_0 \right) \text{mod } 2 \left| \Psi_{\text{GHZ}} \right\rangle = b_1 \left| \Psi_{\text{GHZ}} \right\rangle, \]
\[ \left( (\tilde{x}_A)_0 + [\tilde{p}_B]_0 + [\tilde{p}_C]_0 \right) \text{mod } 2 \left| \Psi_{\text{GHZ}} \right\rangle = b_2 \left| \Psi_{\text{GHZ}} \right\rangle, \]
\[ \left( [\tilde{p}_A]_0 + [\tilde{p}_B]_0 + [\tilde{x}_C]_0 \right) \text{mod } 2 \left| \Psi_{\text{GHZ}} \right\rangle = b_3 \left| \Psi_{\text{GHZ}} \right\rangle, \]
\[ \left( [\tilde{p}_A]_0 + [\tilde{p}_B]_0 + [\tilde{x}_C]_0 \right) \text{mod } 2 \left| \Psi_{\text{GHZ}} \right\rangle = b_4 \left| \Psi_{\text{GHZ}} \right\rangle, \]  
\[ (16) \]

where \( b_{1,2,3,4} \in \{0, 1\} \) satisfy

\[ (b_1 + b_2 + b_3 + b_4) \text{mod } 2 = 1 \]  
\[ (17) \]

Once again the paradox is due to the associativity of the modulo operation which holds for c-numbers but does not hold for operators. (As an example the sum of two even integers is an even integer, but if one takes the sum of two operators both of whose eigenvalues are even integers, one does not in general obtain an operator whose eigenvalues are only even integers).

Exponentiating eq. \( (10) \) we obtain

\[ e^{i \pi \tilde{x}_A} e^{i \pi \tilde{x}_B} e^{i \pi \tilde{x}_C} \left| \Psi_{\text{GHZ}} \right\rangle = e^{i \pi b_1} \left| \Psi_{\text{GHZ}} \right\rangle, \]
\[ e^{i \pi \tilde{x}_A} e^{i \pi \tilde{p}_B} e^{i \pi \tilde{p}_C} \left| \Psi_{\text{GHZ}} \right\rangle = e^{i \pi b_2} \left| \Psi_{\text{GHZ}} \right\rangle, \]
\[ e^{i \pi \tilde{p}_A} e^{i \pi \tilde{x}_B} e^{i \pi \tilde{p}_C} \left| \Psi_{\text{GHZ}} \right\rangle = e^{i \pi b_3} \left| \Psi_{\text{GHZ}} \right\rangle, \]
\[ e^{i \pi \tilde{p}_A} e^{i \pi \tilde{p}_B} e^{i \pi \tilde{x}_C} \left| \Psi_{\text{GHZ}} \right\rangle = e^{i \pi b_4} \left| \Psi_{\text{GHZ}} \right\rangle. \]  
\[ (18) \]

We now show that this is identical to the original GHZ paradox for spins. Indeed the operators \( X = e^{i \pi \tilde{x}_A}, \)
\( Y = e^{i \pi \tilde{p}_B}, \) and \( Z = -iXY \) are a representation of \( su(2) \) that obey the usual commutation relations \( [X, Y] = 2iZ \) and cyclic permutations. This can be verified using the eigenstates of \( Z \):

\[ \left| \uparrow \right\rangle_{x_0, p_0} = \frac{1}{\sqrt{2}} \left( \sum_{k=-\infty}^{\infty} e^{i \pi \tilde{x}_0} \left| \tilde{x} = x_0 + 2k \right\rangle \right. \]
\[ + i \sum_{k=-\infty}^{\infty} e^{i \pi \tilde{x}_0} \left| \tilde{p} = p_0 + k \right\rangle \right) \]
\[ = \frac{1}{\sqrt{2}} \left( \sum_{k=-\infty}^{\infty} e^{-i \pi \tilde{x}_0} \left| \tilde{x} = x_0 + 2k + 1 \right\rangle \right. \]
\[ + i \sum_{k=-\infty}^{\infty} e^{-i \pi \tilde{x}_0} \left| \tilde{p} = p_0 + k \right\rangle \right) \]  
\[ (19) \]
These states form a basis of the Hilbert space. The action of an
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Define
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space.

where \( \hat{x}_0 \) and \( \hat{p}_0 \) ∈ \([0, 1]\) and where we have defined
\[
|\hat{x}| = |x = \sqrt{\pi} L \hat{x}\rangle, \quad |\hat{p}| = |p = \sqrt{\pi} p/L\rangle
\] (21)

These states form a basis of the Hilbert space. The action of
X, Y and Z on them is

\[
X |\uparrow\rangle_{\hat{x}_0, \hat{p}_0} = |\downarrow\rangle_{\hat{x}_0, \hat{p}_0}, \quad X |\downarrow\rangle_{\hat{x}_0, \hat{p}_0} = |\uparrow\rangle_{\hat{x}_0, \hat{p}_0},
Y |\uparrow\rangle_{\hat{x}_0, \hat{p}_0} = i |\downarrow\rangle_{\hat{x}_0, \hat{p}_0}, \quad Y |\downarrow\rangle_{\hat{x}_0, \hat{p}_0} = -i |\uparrow\rangle_{\hat{x}_0, \hat{p}_0},
Z |\uparrow\rangle_{\hat{x}_0, \hat{p}_0} = |\uparrow\rangle_{\hat{x}_0, \hat{p}_0}, \quad Z |\downarrow\rangle_{\hat{x}_0, \hat{p}_0} = - |\downarrow\rangle_{\hat{x}_0, \hat{p}_0}.
\] (22)

This shows that the GHZ paradox we have constructed
for continuous variables is not fundamentally different
from the original paradox. It is only expressed in terms
of an \( su(2) \) sub-algebra of an infinite dimensional Hilbert
space.

To conclude we now describe the common eigenstate of
the four GHZ operators in the case of eqs. (8) and (18).
Define

\[
|b_i\rangle_j = |\uparrow\rangle_{\hat{x}_j, \hat{p}_0} \text{ if } b_i \text{ mod } 2 = 0
= |\downarrow\rangle_{\hat{x}_j, \hat{p}_0} \text{ if } b_i \text{ mod } 2 = 1
\] (23)

where \( j = A, B, C \). Using this notation, the state

\[
|\psi(b, z)\rangle = \frac{1}{\sqrt{2}} \left( |b_2\rangle_A |b_3\rangle_B |b_4\rangle_C 
+ (-1)^{b_1} |b_2 + 1\rangle_A |b_3 + 1\rangle_B |b_4 + 1\rangle_C \right)
\] (24)

depending on the variables \( b = (b_1, b_2, b_3, b_4) \) and \( z = (\hat{x}_0^A, \hat{x}_0^B, \hat{x}_0^C, \hat{p}_0^A, \hat{p}_0^B, \hat{p}_0^C) \) is a solution of eq. (18). The
general eigenstate of eq. (18) is then of the form

\[
|\Psi_{GHZ}(b)\rangle = \int dz \ f(z) |\psi(b, z)\rangle
\] (25)

where \( f(z) \) is some normalized function.

Finally, we can use this expression to obtain the eigenstates
of eq (8) (or equivalently (15)). It is easy to check that

\[
\left[ (z_A) \text{ mod } 1 + (z_B) \text{ mod } 1 + (z_C) \text{ mod } 1 \right] |\psi(b, z)\rangle
= \left[ z_A^0 + z_B^0 + z_C^0 \right] |\psi(b, z)\rangle
\] (26)

where \( z \) stands for \( \hat{x} \) or \( \hat{p} \). Therefore inserting the state
(24) into eq. (13) we obtain the equations

\[
\begin{align*}
(\hat{x}_0^A + \hat{x}_0^B + \hat{x}_0^C + b_1) \text{ mod } 2 &= \eta_1 \\
(-\hat{x}_0^A + \hat{p}_0^B - \hat{p}_0^C + b_2) \text{ mod } 2 &= \eta_2 \\
(-\hat{p}_0^A - \hat{x}_0^B + \hat{p}_0^C + b_3) \text{ mod } 2 &= \eta_3 \\
(\hat{p}_0^A - \hat{p}_0^B - \hat{x}_0^C + b_4) \text{ mod } 2 &= \eta_4 .
\end{align*}
\] (27)

The general solution of eq. (13) is then an arbitrary
superposition of states (24) for which \( b, z \) obey the con-
straints (27). If we denote by \( \Xi \) the set of solutions of
(27), then the general solution can be written

\[
|\Psi_{GHZ}'\rangle = \int_{\Xi} d\xi \ g(\xi) \ |\psi(\xi)\rangle
\] (28)

where \( \xi = (b, z) \).

In summary we have exhibited a large class of states that
satisfy a GHZ paradox. Exhibiting the paradox is very
simple: it requires only that the parties carry out local
measurements of position and momentum. However con-
structing these states is a much more difficult task. For
instance it seems impossible to even approximately con-
struct them using squeezers and beam splitters, contrary
to the states described in [3]. Constructing them explicit-
itly will therefore require more sophisticated quantum
state engineering than we presently possess.

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