Spherical varieties and integral representations of $L$-functions.

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Abstract
We present a conceptual and uniform interpretation of the methods of integral representations of $L$-functions (period integrals, Rankin-Selberg integrals). This leads to: (i) a way to classify of such integrals, based on the classification of certain embeddings of spherical varieties (whenever the latter is available), (ii) a conjecture which would imply a vast generalization of the method, and (iii) an explanation of the phenomenon of “weight factors” in the relative trace formula.

Contents

1 Introduction .............................................. 2
  1.1 Goals .............................................. 2
  1.2 Background on the methods .......................... 3
  1.3 Schwartz spaces and $X$-Eisenstein series ......... 5
  1.4 Comments and acknowledgements ................... 6

2 Elements of the theory of spherical varieties ........ 7
  2.1 Invariants associated to spherical varieties ....... 7
  2.2 Spherical embeddings and affine spherical varieties 10
  2.3 Stratification ....................................... 12

3 Geometric models and local Schwartz spaces .......... 16
  3.1 Goals .............................................. 16
  3.2 Geometric models ................................... 17
  3.3 Local Schwartz space ................................ 19

4 The automorphic pairing and $X$-Eisenstein series .... 22
  4.1 Global Schwartz space ................................ 22
  4.2 Pseudo-$X$-Eisenstein series and the automorphic pairing 22
  4.3 Spectral transform and $X$-Eisenstein series ...... 25

5 Connection to usual Eisenstein series ................. 27
  5.1 Certain stacks and sheaves related to flag varieties 27
  5.2 The corresponding functions ....................... 30
  5.3 Connection to Eisenstein series .................... 32
1 Introduction

1.1 Goals

The study of automorphic $L$-functions (and their special values at distinguished points, or $L$-values) is very central in many areas of present-day number theory, and an incredible variety of methods has been developed in order to understand the properties of these mysterious objects and their deep links with seemingly unrelated arithmetic invariants. Oddly enough, notwithstanding their elegant and very general definition by Langlands in terms of Euler products, virtually all methods for studying them depart from an integral construction of the form:

$$\text{A suitable automorphic form (considered as a function on the automorphic quotient } G(k)\backslash G(A_k), \text{ integrated against a suitable distribution on } G(k)\backslash G(A_k), \text{ is equal to a certain } L\text{-value.}$$

For “geometric” automorphic forms, such an integral can often be expressed as a pairing between elements in certain homology and cohomology groups, but the essence remains the same. Given the importance of such methods, it appears as a paradox that there is no general theory of integral representations of $L$-functions and, in fact, they are often considered as “accidents”.

The purpose of the present article is to give a conceptual interpretation to the most mysterious, maybe, of these methods, which have been called “Rankin-Selberg” integrals. This interpretation leads to the first systematic classification of such integrals, based on the classification of certain spherical varieties. More importantly, it naturally gives rise to a very general conjecture, whose proof would lead to a vast extension of the method and would allow us to study many more $L$-functions than are within our reach at this moment. Finally, it explains phenomena which have been observed in the theory of the relative trace formula, in a way that is well-suited to the geometric methods employed in the proof of the fundamental lemma by Ngo.

Let me sketch the main ideas, before giving more details on each of them: Let $G$ be a connected reductive algebraic group over a global field $k$. A spherical variety for $G$ is, by definition, a normal variety with a $G$-action such that, over
the algebraic closure, the Borel subgroup of $G$ has a dense orbit. Let $X$ be an affine spherical variety, and denote by $X^+$ the open $G$-orbit on $X$. We want to define a “Schwartz space” $S(X(\mathbb{A}_k))$ of functions on $X^+$ (where $\mathbb{A}_k$ denotes the ring of adeles of $k$) which reflect the geometry and singularities of $X$, and a pairing $\mathcal{P}_X : S(X(\mathbb{A}_k)) \otimes \pi \to \mathbb{C}$ for every cuspidal automorphic representation $\pi$ of $G$ with “sufficiently positive” central character. We conjecture that this pairing admits meromorphic continuation to all $\pi$. Then, assuming that the spectrum of this Schwartz space is multiplicity-free, in a suitable sense\footnote{Motivated by the work of Jacquet [Ja01], the “multiplicity-freeness” property can be relaxed to a “stable multiplicity-freeness” property.}, one expects the pairing to be associated to some $L$-value of $\pi$.

If our variety is of the form $H\backslash G$ with $H$ a reductive subgroup of $G$ then from this construction we recover the period integral of automorphic forms over $H(k)\backslash H(\mathbb{A}_k)$. More generally, if $X$ is fibered over such a variety and the fibers are (related to) flag varieties, then we can prove meromorphic continuation using the meromorphic continuation of Eisenstein series, and we recover integrals of “Rankin-Selberg” type. Thus, we reduce the problem of finding Rankin-Selberg integrals to the problem of classifying affine spherical varieties with a certain geometry. For smooth affine spherical varieties, this geometric problem has been solved by Knop and Van Steirteghem [KS06]. By inspection of their tables, we recover some of the best-known constructions, such as those of Rankin and Selberg, Godement and Jacquet, Bump and Friedberg, all spherical period integrals, as well as some new ones.

We give an example, involving the tensor product $L$-function of $n$ cuspidal representations on $GL_2$, to support the point of view that the basic object giving rise to an Eulerian integral related to an $L$-function is the spherical variety $X$ and not a geometry related to flag varieties. Finally, we apply these ideas to the relative trace formula to show that certain “weight factors” which have appeared in examples of this theory and are often considered an “anomaly” can, in fact, be well-understood using the notion of Schwartz spaces.

1.2 Background on the methods

To an automorphic representation $\pi \simeq \bigotimes_v \pi_v$ of a reductive group $G$ over a global field $k$, and to an algebraic representation $\rho$ of its Langlands dual group $L^\vee G$, Langlands attached a complex $L$-function $L(\pi, \rho, s)$, defined for $s$ in some right-half plane of the complex plane as the product, over all places $v$, of local factors $L_v(\pi_v, \rho, s)$\footnote{At ramified places and for most $\rho$, the definition still depends on the local functoriality conjectures.}.

Despite the beauty of its generality, the definition is of little use when attempting to prove analytic properties of $L$-functions, such as their meromorphic continuation and functional equation. Such properties are usually obtained by integration techniques, namely presenting the $L$-function as some integral transform of a function in the space of the automorphic representation. Such methods in fact predate Langlands by more than a century, but the most definitive con-
struction (as every automorphic $L$-function should be a $GL_n$ $L$-function) was studied by Godement and Jacquet [GJ72] (generalizing Tate’s construction for $GL_1$, [Ta67]), who proved the analytic continuation and functional equation of $L(\pi, s) := L(\pi, \text{std}, s)$, where $\pi$ is an automorphic representation of $G = GL_n$ and $\text{std}$ is the standard representation of $L G = GL_n(\mathbb{C}) \times \text{Gal}(\bar{k}/k)$. Their method relies on proving the equality:

$$L(\pi, s - \frac{1}{2}(n - 1)) = \int_{GL_n(\mathbb{A}_k)} \left\langle \pi(g), \phi \right\rangle \Phi(g) |\det(g)|^s dg$$  \hspace{1cm} (1.1)

where $\phi$ is a suitable vector in $\pi$, $\tilde{\phi}$ a suitable vector in its contragredient and $\Phi$ a suitable function in $S(\text{Mat}_n(\mathbb{A}_k))$, the Schwartz space of functions on $\text{Mat}_n(\mathbb{A}_k)$.

The main analytic properties of $L(\pi, \rho, s)$, then, follow from Fourier transform on the Schwartz space and the Poisson summation formula.

Going several decades back in history, Hecke showed that the standard $L$-function of a cuspidal automorphic representation on $GL_2$ (with, say, trivial central character) has a presentation as a period integral, which in adelic language reads:

$$L(\pi, s + \frac{1}{2}) = \int_{k^* \backslash \mathbb{A}_k^*} \phi \left( \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) \right) |a|^s da$$  \hspace{1cm} (1.2)

where, again, $\phi$ is a suitable vector in the automorphic representation under consideration.

Period integrals (by which we mean integrals over the orbit of some subgroup on the automorphic space $G(k) \backslash G(\mathbb{A}_k)$, possibly against a character of that subgroup) have since been studied extensively, although there are still many open conjectures about their relation to $L$-functions (cf., for instance, [II]). Still, they form perhaps the single class of examples where we have a general principle answering the question: How to write down an integral with good analytic properties, which is related to some $L$-function (or $L$-value)? Piatetski-Shapiro discussed this in [PS75], and suggested that the period integral of a cusp form on a group $G$ over a subgroup $H$ (against, perhaps, an analytic family $\delta_s$ of characters of $H$ as in (1.2)) should always be related to some $L$-value if the subgroup $H$ enjoys a “multiplicity-one” property: $\dim \text{Hom}_{H(\mathbb{A}_k)}(\pi, \delta_s) \leq 1$ for every irreducible representation $\pi$ of $G(\mathbb{A}_k)$ and (almost) every $s$.

The method of periods usually fails when the subgroup $H$ is non-reductive, the reason being that, typically, the orbit of $H(\mathbb{A}_k)$ on $G(k) \backslash G(\mathbb{A}_k)$ is not closed. Therefore there is no a priori reason that the period integral should have nice analytic properties (as the character $\delta_s$ varies), and one can in fact check in examples that for values of $s$ such that the period integral converges, it does not represent an $L$-function.

In a different vein, Rankin and Selberg independently discovered an integral representing the tensor product $L$-function of two cuspidal automorphic representations of $GL_2$. The integral uses as auxilliary data an Eisenstein series on $GL_2$ and has the following form:

$$L(\pi_1 \times \pi_2, \otimes, s) = \int_{\text{PGL}_2(k) \backslash \text{PGL}_2(\mathbb{A}_k)} \phi_1(g) \phi_2(g) E(g, s) dg$$
with suitable $\phi_1 \in \pi_1, \phi_2 \in \pi_2$.

Later, this method was taken up by Jacquet, Piatetski-Shapiro, Shalika, Rallis, Gelbart, Ginzburg, Bump, Friedberg and many others, in order to construct numerous examples of automorphic $L$-functions expressed as integrals of cusp forms against Eisenstein series, with important corollaries for every such expression discovered. Despite the abundance of examples, however, there has not been a systematic understanding of how to produce an integral representing an $L$-function.

### 1.3 Schwartz spaces and $X$-Eisenstein series

While it was shown by Piatetski-Shapiro and Rallis [GPSR87] that the method of Godement and Jacquet can also be phrased in the language of Rankin-Selberg integrals, the fact that no systematic theory of these constructions exists has led many authors to consider them as coincidental and/or to seek direct generalizations of [GJ72], as a “more canonical” construction (cf. [BK00]). We adopt a different point of view which treats Godement-Jacquet, Rankin-Selberg, and period integrals as parts of the same concept, in fact a concept which should be much more general!

The basic object here is an affine spherical variety $X$ of the group $G$. The reason that such varieties are suitable is that they are related to the “multiplicity-free” property discussed above. For instance, in the category of algebraic representations, the ring of regular functions $k[X]$ of an affine $G$-variety is multiplicity-free if and only if the variety is spherical. In the $p$-adic setting and for unramified representations, questions of multiplicity were systematically examined in [Sa08, Sa2], and of course in special cases such questions have been examined in much greater detail (see, for example, [Pr90]).

The main idea is to associate to every affine spherical variety a space of distributions on $G(k) \backslash G(\mathbb{A}_k)$ which should have “good analytic properties”. For reasons of convenience we set up our formulations in such a way that the analytic problem does not have to do with varying a character of some subgroup $H$ (the isotropy subgroup of a “generic” point on $X$), but with varying a cuspidal automorphic representation of $G$. For instance, to the Hecke integral (for $\text{PGL}_2$) we do not associate the variety $\mathbb{G}_m \backslash \text{PGL}_2$, but the variety $X = \text{PGL}_2$ under the $G = \mathbb{G}_m \times \text{PGL}_2$-action. Our distributions (in fact, smooth functions) on $G(k) \backslash G(\mathbb{A}_k)$ come from a “Schwartz space” of distributions on $X^+(\mathbb{A}_k)$ via an “pseudo-Eisenstein series” construction (i.e. summation over $k$-points of $X^+$). Here $X^+$ denotes the open $G$-orbit on $X$. The main conjecture, then, (Conjectures 4.3.1 and 4.3.2) states that the integral of these “pseudo-$X$-Eisenstein series” against central idele class characters (I call this integral an $X$-Eisenstein series), originally defined in some domain of convergence, has meromorphic continuation everywhere. Under additional assumptions on $X$ (related to the “multiplicity-freeness” property mentioned above), the pairings of pseudo-$X$-Eisenstein series with automorphic forms should be related, in a suitable sense, to automorphic $L$-functions or special values of those.

Having identified the problem as that of finding the “correct” functions on
$X^+(A_k)$, the answer (at least part of it) comes “for free” from the Geometric Langlands program [BG02, BFGM02] and the work of Braverman and Kazhdan [BK08, BK02] on the special case that $X$ is the affine closure of $[P, P] \backslash G$, where $P$ is a parabolic subgroup. Let us discuss this work: The prototype here is the case $X = U \backslash SL_2 = \mathbb{A}^2 \backslash \{0\}$ (where $U$ denotes a maximal unipotent subgroup), $X = \mathbb{A}^2$ (two-dimensional affine space). The Schwartz space is the usual Schwartz space on $X(A_k)$ which, by definition, is the restricted tensor product $\mathcal{S}(X(A_k)) := \bigotimes_v (\mathcal{S}(k_v^2) : \Phi^0_v)$, where for finite places $k_v$ with rings of integers $\mathfrak{o}_v$, the “basic vectors” $\Phi^0_v$ are the characteristic functions of $\mathfrak{o}_v^2$. There is a natural meromorphic family of morphisms: $\mathcal{S}(X(A_k)) \to I_B^G(A_k)(\chi)$ (where $I_B^G$ denotes normalized parabolic induction from the parabolic $P$, $B$ denotes the Borel subgroup), and for idèle class characters $\chi$ the composition with the Eisenstein series morphism: $\text{Eis}_\chi : I_B^G(A_k)(\chi) \to C^K(G(k) \backslash G(A_k))$ provides meromorphic sections of Eisenstein series, whose functional equation can be deduced from the Poisson summation formula on $\mathbb{A}^2_k$ – in particular, the $L$-factors which appear in the functional equation of “usual” sections are absent here.

This was found to be the case more generally in [BK98, BG02, BFGM02, BK02]: One can construct “normalized” sections of Eisenstein series from certain “Schwartz spaces” of functions on $[P, P] \backslash G(A_k)$ (or $U_P \backslash G(A_k)$), where $U_P$ is the unipotent radical of $P$. These Schwartz spaces should be defined as tensor products over all places, restricted with respect to some “basic vector”; and the “basic vector” should be the function-theoretic analog of the intersection cohomology sheaf of some geometric model for the space $X(\mathfrak{o}_v)$. For instance, if $X$ is smooth then the intersection cohomology sheaf is constant, which means that $\Phi^0_v$ is the characteristic function of $X(\mathfrak{o}_v)$; this explains the distributions in Tate’s thesis, the work of Godement and Jacquet, and the case of period integrals. (In the latter, the characteristic function of $X(\mathfrak{o}_v) = H_1 \backslash G(\mathfrak{o}_v)$ is obtained as the “smoothening” of the delta function at the point $H1 \in X$.)

Such geometric models where recently defined by Gaitsgory and Nadler [GN] for every affine spherical variety. They provide us with the data necessary to formulate our conjecture and general picture. It should be noted, however, that even to define the “correct” functions on $X^+(A_k)$ out of these geometric models one has to rely on certain natural conjectures on them – therefore the problem of finding an independent or unconditional definition should be considered part of the steps which need to be taken towards establishing our conjecture.

### 1.4 Comments and acknowledgements

Most of the ingredients in the present work are not new. Experts in the Rankin-Selberg method will recognize in our method, to a lesser of greater extent, the heuristics they have been using to find new integrals. The idea that geometric models and intersection cohomology should give rise to the “correct” space of functions on the $p$-adic points of a variety comes straight out of the Geometric Langlands program and the work of Braverman and Kazhdan. I have nothing to
offer in this direction and, in fact, I can only define part of the Schwartz space, in a very ad hoc way, and conditional on certain folklore conjectures on these geometric models (Assumption 3.3.2). It should be noted that Braverman and Kazhdan have managed to define the whole Schwartz space in the case where $X$ is the affine closure of $[P,P]\backslash G$, in an independent way.

However, the mixture of these ingredients is new and I think that there is enough evidence that it is the correct one. For the first time, a precise criterion is formulated on how to construct a “Rankin-Selberg” integral, reducing the problem to a purely geometric one – classifying certain embeddings of spherical varieties. And evidence shows that there should be a vast generalization which does not depend on such embeddings. I prove no “hard” theorems and, in particular, I do not know how to establish the meromorphic continuation of the $X$-Eisenstein series. Hence, I do not know whether I am putting the cart before the horse – however, the distributions defined here are completely geometric and have nothing to do a priori with $L$-functions, which leaves a lot of space for hope. Finally, this point of view proves useful in explaining the phenomenon of “weight factors” in the relative trace formula.

This work started more than four years ago during a semester at New York University and was put aside for most of the time since. I am very grateful to Joseph Bernstein, Daniel Bump, Dennis Gaitsgory, David Ginzburg, Hervé Jacquet, David Nadler and Akshay Venkatesh for many useful discussions and encouragement.

2 Elements of the theory of spherical varieties

2.1 Invariants associated to spherical varieties

A spherical variety for a connected reductive group $G$ over a field $k$ is a normal variety $X$ together with a $G$-action, such that over the algebraic closure the Borel subgroup of $G$ has a dense orbit.

We denote throughout by $k$ a number field and, unless otherwise stated, we make the following assumptions on $G$ and $X$:

- $G$ is a split, connected, reductive group,
- $X$ is affine.

The open $G$-orbit in $X$ will be denoted by $X^+$, and the open $B$-orbit by $X^+$ (where $B$ is a fixed Borel subgroup of $G$, whose unipotent radical we denote by $U$).\(^3\)

The assumption that $G$ is split is certainly unnecessary, but it is enough to demonstrate our point of view, and convenient because of many geometric and representation-theoretic results which have been established in this case. We will discuss affine spherical varieties in more detail later, but I just mention

\(^3\)Notice that this is different from that of [GN], but compatible with the notation used in [Sa08, Sa2, SV].
here that a common source of examples is when $X^+ = H \backslash G$, a quasi-affine homogeneous variety, and $X = \overline{H \backslash G}^{\text{aff}} = \text{spec } k[H \backslash G]$, the affine closure of $H \backslash G$, cf. §2.2.

Let us discuss certain invariants associated to a spherical variety. First of all, for any algebraic group $\Gamma$ we denote by $\mathcal{X}(\Gamma)$ its character group, and for any variety $Y$ with an action of $\Gamma$ we denote by $\mathcal{X}_\Gamma(Y)$ the group of $\Gamma$-eigencharacters appearing in the action of $\Gamma$ on $k(Y)$. If $\Gamma$ is our fixed Borel subgroup $B$, then we will denote $\mathcal{X}_B(Y)$ simply by $\mathcal{X}(Y)$. The multiplicative group of non-zero eigenfunctions (semiinvariants) for $B$ on $k(Y)$ will be denoted by $k(Y)^{(B)}$. If $Y$ has a dense $B$-orbit, then we have a short exact sequence: $0 \to k^* \to k(Y)^{(B)} \to \mathcal{X}(Y) \to 0$.

For a finitely generated $\mathbb{Z}$-module $M$ we denote by $M^*$ the dual module $\text{Hom}_\mathbb{Z}(M, \mathbb{Z})$. For our spherical variety $X$, we let $\Lambda_X = \mathcal{X}(X)^*$ and $\mathcal{Q} = \Lambda_X \otimes_{\mathbb{Z}} \mathbb{Q}$. A $B$-invariant valuation on $k(X)$ which is trivial on $k^*$ induces by restriction to $k(X)^{(B)}$ an element of $\Lambda_X$. We let $\mathcal{V} \subset \mathcal{Q}$ be the cone\(^4\) generated by $\mathcal{G}$-invariant valuations which are trivial on $k^*$, cf. [Kn91, Corollary 1.8]. We denote by $\Lambda_X^\mathcal{V}$ the intersection $\Lambda_X \cap \mathcal{V}$. Notice that under the quotient map $\mathcal{X}(A)^* \otimes \mathcal{Q} \to \mathcal{Q}$, $\mathcal{V}$ contains the image of the negative Weyl chamber of $G$ [Kn91, Corollary 5.3]. We say that $X$ is a wavefront spherical variety if $\mathcal{V}$ is precisely equal to the image of the negative Weyl chamber. Symmetric varieties, in particular, are wavefront [Kn91], but not, for instance, the variety $U \backslash G$ under the action of $G$.

The $G$-automorphism group of a homogeneous $G$-variety $X^+ = H \backslash G$ is equal to the quotient $\mathcal{N}(H)/H$. It is known [Lo08, Lemma 7.17] that for $X^+$ spherical the $G$-automorphisms of $X^+$ extend to any affine completion $X$ of $X^+$. Moreover, it is known that $\text{Aut}^G(X)$ is diagonalizable; the cocharacter group of its connected component can be canonically identified (by considering the scalars by which an automorphism acts on rational $B$-eigenfunctions) with $\Lambda_X \cap \mathcal{V} \cap (-\mathcal{V})$. It will be convenient many times to replace the group $G$ by a central extension thereof, so that the connected component of $\text{Aut}^G(X)$ is induced by the action of the center of $G$.

The associated parabolic to $X$ is the standard parabolic $P(X) := \{ p \in G \mid X^+, p = X^+ \}$. Make once and for all a choice of a point $x_0 \in X^+(k)$ and let $H$ denote its stabilizer; hence $X^+ = H \backslash G$, and $HB$ is open in $G$. There is the following “good” way of choosing a Levi subgroup $L(X)$ of $P(X)$: Pick $f \in k[X]$, considered by restriction as an element of $k[G]^H$, such that the set-theoretic zero locus of $f$ is $X \setminus X^+$. Its differential $df$ at $1 \in G$ defines an element in the coadjoint representation of $G$, and the centralizer $L(X)$ of $df$ is a Levi subgroup of $P(X)$. We fix throughout a maximal torus $A$ in $B \cap L(X)$. We define $A_X$ to be the torus: $L(X)/(L(X) \cap H) = A/(A \cap H)$; its cocharacter group is $\Lambda_X$. We consider $A_X$ as a subvariety of $X^+$ via the orbit map on $x_0$.

\(^4\)A cone in a $\mathbb{Q}$-vector space is a subset which is closed under addition and under multiplication by $\mathbb{Q}_{\geq 0}$, its relative interior is its interior in the vector subspace that it spans, and a face of it is the zero set, in the cone, of a linear functional which is non-negative on the cone – hence, the whole cone is a face as well.
The cone $\mathcal{V}$ is the fundamental domain for a finite reflection group $W_X \subset \text{End}(\mathcal{Q})$, called the \textit{little Weyl group} of $X$. The set of simple roots of $G$ corresponding to $B$ and the maximal torus $A \subset B$ will be denoted by $\Delta$. Consider the (strictly convex) cone dual to $\mathcal{V}$: $\mathcal{V}^\perp = \{ \chi \in \mathcal{X}(X) \otimes \mathbb{Q} | \langle \chi, v \rangle \leq 0 \text{ for every } v \in \mathcal{V} \}$. The generators of the intersections of its extremal rays with $\mathcal{X}(X)$ are called the \textit{(simple) spherical roots}\footnote{The work of Gaitsgory-Nadler [GN] and Sakellaridis-Venkatesh [SV] suggests that for representation-theoretic reasons one should slightly modify this definition of spherical roots. However, the lines on which the modified roots lie are still the same, and for the purposes of the present article this is enough.} of $X$ and their set is denoted by $\Delta_X$. They are known to form the set of simple roots of a based root system with Weyl group $W_X$. We will denote by $\Delta(X)$ the subset of $\Delta$ consisting of simple roots in $L(X)$, and by $W_{L(X)} \subset W$ the Weyl groups of $L(X)$, resp. $G$. There is a canonical way [Kn94b, Theorem 6.5] to identify $W_X$ with a subgroup of $W$, which normalizes $W_{L(X)}$ (and intersects it trivially).

For the sake of simplicity, and because it is almost a prerequisite for the applications to $L$-functions which we want to describe, we make throughout (unless otherwise noted) the following assumptions on $X$:

- The $B$-stabilizer $B_{x_0}$ of a point $x_0 \in X^+$ is connected. Equivalently, the map $\mathcal{X}(A)^\epsilon \to \Lambda_X$ is surjective.

- $X$ is \textit{wavefront}.

We notice that the first of these conditions also implies that the $G$-stabilizer $H$ of a point $x_0 \in X^+$ is connected. Indeed, if $H^0$ is the connected component of $H$ then $H^0 \cap G$ is also spherical and the component group $\pi_0(H)$ acts by $G$-automorphisms on it, hence preserves the open $B$-orbit. The covering $B_{x_0} \backslash B \to B_{x_0} \backslash B$ being finite (where $x_0$ is a preimage of $x_0$ in the open $B$-orbit of $H^0 \cap G$), this implies that $B_{x_0} \backslash B$ is not connected unless $(H : H^0) = 1$.

Rephrasing the above conditions, the first one says that $X^+(k)$ forms a single $B(k)$-orbit (the first Galois cohomology of the stabilizer is trivial), and the second condition implies that no other Borel orbit has the same character group as $X^+$ (cf. [Kn95]). These two conditions conjecturally ensure that $(G(F), H(F))$ is a Gelfand pair, for every completion $F$ of $G$, that is: $\dim \text{Hom}_{H(F)}(\pi, G) \leq 1$ for every irreducible admissible representation $\pi$ of $G(F)$. (For $F$ a non-archimedean completion and unramified representations in general position this was proven in [Sa08], and for every unramified representation under some additional conditions on $X$ in [Sa2].) Therefore, they are suitable for our discussion of integral representations of $L$-functions, which are usually expected to be related to some “multiplicity-one” property. However, most of the discussion and conjectures in this paper can easily be formulated without the last two assumptions.

Finally, we note that we will be using standard and self-explanatory notation for varieties and algebraic groups; e.g. $\mathcal{N}(H)$, $\mathcal{Z}(H)$, $H^0$ will be, respectively, the normalizer, center and connected component of a (sub)group $H$, $\hat{Y}$ will be the closure of a subvariety $Y$, etc. The isotropy group of a point $x$ under a $G$-action
will be denoted by $G_x$ and the fiber over $y \in Y$ of a morphism $X \to Y$ by $X_y$. The base change of an $S$-scheme $Y$ with respect to a morphism $T \to S$ will be denoted by $Y_T$, but if $v$ denotes a completion of a number field and $Y$ is defined over $k$ then we will be denoting by $Y_v$ the set $Y(k_v)$.

2.2 Spherical embeddings and affine spherical varieties

We will use the word “embedding” or “completion” of a spherical $G$-variety $X$ to refer to a spherical $G$-variety $\bar{X}$ (not necessarily complete) with an open equivariant embedding: $X \to \bar{X}$. A spherical embedding is called simple if it contains a unique closed $G$-orbit. Spherical embeddings have been classified by Luna and Vust [LV83]; our basic reference for this theory will be [Kn91]. We will now recall the main theorem classifying simple spherical embeddings.

For now, we are working over an algebraically closed field in characteristic zero, although one can easily describe the modifications needed to work over any field of characteristic zero when $G$ is split. Let $X$ be a spherical variety and let $X^+$ be its open $G$-orbit. The colors of $X$ are the closures of the $B$-stable prime divisors of $X^+$; their set will be denoted by $D$. For every $B$-stable divisor $D$ in any completion $X'$ of $X^+$ we denote by $\rho(D)$ the element of $Q$ induced by the valuation defined by $D$. A strictly convex colored cone is a pair $(C, F)$ with $C \subset Q$, $F \subset D$ such that:

1. $C$ is a strictly (i.e. not containing lines) convex cone generated by $\rho(F)$ and finitely many elements of $\mathcal{V}$,

2. the intersection of $\mathcal{V}$ with the relative interior of $C$ is non-empty,

3. $0 \notin \rho(F)$.

If $X$ is a simple embedding of $X^+$ with closed orbit $Y$, we let $\mathcal{F}(X)$ denote the set of $D \in D$ such that $D \supseteq Y$, and we let $\mathcal{C}(X)$ denote the cone in $Q$ generated by all $\rho(D)$, where $D$ is a $B$-invariant divisor (possibly also $G$-invariant) in $X$ containing $Y$.

2.2.1 Theorem ([Kn91, Theorem 3.1]). The association $X \to (\mathcal{C}(X), \mathcal{F}(X))$ is a bijection between isomorphism classes of simple embeddings of $X^+$ and strictly convex colored cones.

Now let us focus on affine and quasi-affine spherical varieties. We recall from [Kn91, Theorem 6.7]:

2.2.2 Theorem. A spherical variety $X$ is affine if and only if $X$ is simple and there exists a $\chi \in \mathcal{A}(X)$ with $\chi|_{\mathcal{V}} \geq 0$, $\chi|_{\mathcal{C}(X)} = 0$ and $\chi|_{\rho(D)} < 0$. In particular, $H \setminus G$ is affine if and only if $\mathcal{V}$ and $\rho(D)$ are separated by a hyperplane, while it is quasi-affine if and only if $\rho(D)$ does not contain zero and spans a strictly convex cone.

Recall [BG02, §1.1] that a variety $Y$ over a field $k$ is called strongly quasi-affine if the algebra $k[Y]$ of global functions on $Y$ is finitely generated and the
natural map \( Y \to \text{spec}\, k[Y] \) is an open embedding. Then the variety \( Y^{\text{aff}} := \text{spec}\, k[Y] \) is called the affine closure of \( Y \).

2.2.3 Proposition. A homogeneous quasi-affine spherical variety \( Y = H \backslash G \) is strongly quasi-affine. If \( X := H \backslash G^{\text{aff}} \) then the data \( (C(X), F(X)) \) can be described as follows: Consider the cone \( R \subset C(X) \otimes \mathbb{Q} \) generated by the set of \( \chi \in C(X) \) such that \( \chi|_{\mathcal{D}} \geq 0 \), \( \chi|_{\mathcal{D}'} \leq 0 \). Choose a point \( \chi \) in the relative interior of \( R \). Then \( F(X) = \{ D \in \mathcal{D} | \rho(D)(\chi) = 0 \} \) and \( C(X) \) is the cone generated by \( F(X) \).

2.2.4 Remark. The first statement of the proposition generalizes a result of Hochschild and Mostow [HM73] for the variety \( U_P \backslash G \), where \( U_P \) is the unipotent radical of a parabolic subgroup \( P \) of \( G \). Indeed, this variety is spherical under the action of \( M \times G \), where \( M \) is the reductive quotient of \( P \).

Proof. As a representation of \( G \), \( k[Y] \) is locally finite and decomposes:

\[
k[Y] = \bigoplus \lambda V_\lambda
\]

where \( V_\lambda \) is the isotypic component corresponding to the representation with highest weight \( \lambda \), and the sum is taken over all \( \lambda \) with \( V_\lambda \neq 0 \). Since the variety is spherical, each \( V_\lambda \) is isomorphic to one copy of the representation with highest weight \( \lambda \). Moreover, the multiplicative monoid of non-zero highest-weight vectors \( k[Y]|^H \) is the submonoid of \( k(Y)|^H \) (the group of non-zero rational \( B \)-eigenfunctions) consisting of regular functions. Regular \( B \)-eigenfunctions are precisely those whose eigencharacter satisfies \( \chi|_{\mathcal{D}} \geq 0 \); since the set \( \mathcal{D} \) is finite, the monoid of \( \lambda \) appearing in the decomposition (2.1) is finitely-generated. Since the multiplication map: \( V_\mu \otimes V_\nu \) has image in the sum of \( V_\lambda \) with \( \lambda \leq \mu + \nu \), and composed with the projection: \( k[Y] \to V_{\mu + \nu} \) it is surjective, it follows that the sum of the \( V_\lambda \), for \( \lambda \) in a set of generators for the monoid of \( \lambda \)'s appearing in (2.1), generates \( k[Y] \).

The second condition, namely that \( Y \to X \) is an open embedding, follows from the assumption that \( Y \) is quasi-affine and the homogeneity of \( Y \). Hence, \( Y \) is strongly quasi-affine.

The affine closure \( X \) has the property that for every affine completion \( X' \) of \( Y \) there is a morphism: \( X \to X' \). The description of \( (C(X), F(X)) \) now follows from Theorem 2.2.2 above and Theorem 4.1 in [Kn91], which describes morphisms between spherical embeddings. Notice that the cone \( C(X) \), as described, will necessarily contain the intersection of \( V \) with the cone generated by \( \rho(D) \) in its relative interior, therefore its relative interior will have non-empty intersection with \( V \).

Let us now discuss the geometry of affine spherical varieties. The following is a corollary of Luna’s slice theorem:

2.2.5 Theorem ([Lu73, III.1. Corollaire 2]). For any affine \( G \)-variety \( X \) there is a homogeneous affine \( G \)-variety \( Y \) and a morphism: \( X \to Y \) such that for every point \( y \in Y \) the stabilizer \( G_y \) has a fixed point on the fiber over \( y \).
If $X$ is $G$-spherical then, in particular, the fiber over any point $y \in Y$ is $G_y$-spherical and hence simple. Therefore, it contains a unique $G_y$-fixed point, and by identifying it with $y$ we can regard $Y$ as a closed subvariety of $X$. As we will see, $Y$ is a “retract” of $X$ under the action of the $G$-automorphism group:

2.2.6 Proposition. Let $X$ be an affine spherical $G$-variety and let $Y$ be as in the theorem above, considered both as a quotient and as a subvariety of $X$. Let $T$ be the maximal torus in $\text{Aut}^G(X)$ which acts trivially on $Y$. Then the closure of the $T$-orbit of every point on $X$ meets $Y$. Equivalently, $k[X]^T = k[Y]$.

Proof. This is essentially Corollary 7.9 of [Kn94a]. More precisely, let us assume that $G$ has a fixed point on $X$, i.e. $Y$ is a point. (The question is easily reduced to this case, since every $G_y$-automorphism of the fiber of $X \rightarrow Y$ over $y$ extends uniquely to a $G$-automorphism of $X$.) The proof of loc. cit. shows that for a generic point $x \in X$ there is a one-parameter subgroup $H$ of $\text{Aut}^G(X)$ such that $x : H$ contains the fixed point in its closure. Hence $k[X]^T = k$ and therefore $X$ contains a unique closed $T$-orbit. \qed

Notice that if $G$ has a fixed point on $X$ then we can embed $X$ into a finite sum $V = \bigoplus_i V_i$ of finite-dimensional representations of $G$, such that the fixed point is the origin in $V$ and there is a subtorus $T$ of $\prod_i \text{Aut}^G(V_i)$ acting on $X$ with the origin as its only closed orbit. (Simply take $V$ to be the dual of a $G$-stable, generating subspace of $k[X]$.)

2.3 Stratification

Let $\mathcal{K} = \mathbb{C}((t))$, the field of formal Laurent series over $\mathbb{C}$, and $\mathcal{D} = \mathbb{C}[[t]]$ the ring of formal power series. If $X^+$ is a homogeneous spherical variety over $\mathbb{C}$, it was proven by Luna and Vust [LV83] that:

2.3.1 Theorem. $G(\mathcal{D})$-orbits on $X^+(\mathcal{K})$ are parametrized by $\Lambda^+_X$, where to $\bar{\lambda} \in \Lambda^+_X$ corresponds the orbit through $\lambda(t) \in A_X(\mathcal{K})$.

A new proof was given by Gaitsgory and Nadler in [GN], which can be used to prove the analogous statement over $p$-adic fields. We revisit their argument, adapt it to the $p$-adic case, and extend it to determine the set of $G(\mathcal{O}_F)$-orbits on $X(\mathcal{O}_F)$, when $G$ and $X$ are affine and defined over a number field and $F$ is a non-archimedean completion (outside of a finite set of places). The argument uses compactification results of Brion, Luna and Vust. We first need to recall a few more elements of the theory of spherical varieties. The results below have appeared in the literature for $k$ an algebraically closed field in characteristic zero, but the proofs hold verbatim when $k$ is any field in characteristic zero and the groups in question are split over $k$. (The basic observation being, here, that in all proofs one gets to choose $B$-eigenfunctions in $k(X)$, and since the variety is spherical and the group is split the eigenspaces of $B$ are one-dimensional and defined over $k$, therefore the chosen eigenfunctions are $k$-rational up to $\bar{k}$-multiple.)
A toroidal embedding of $X^+$ is an embedding $X^c$ of $X^+$ in which no color (B-stable divisor which is not $G$-stable) contains a $G$-orbit. Theorem 2.2.1 implies that simple toroidal embeddings are classified by strictly convex, finitely generated subcones of $\mathcal{V}$. Moreover, the simple toroidal embedding $X^c$ obtained from a simple embedding $X$ by taking the cone $C(X^c) = \mathcal{C}(X) \cap \mathcal{V}$ comes with a proper equivariant morphism: $X^c \to X$ [Kn91, Theorem 4.1] which is surjective [Kn91, Lemma 3.2].

The local structure of a simple toroidal embedding is given by the following theorem of Brion, Luna and Vust:

2.3.2 Theorem ([BLV86, Théorème 3.5]). Let $X^c$ be a simple toroidal embedding of $X^+$ and let $X^c_B$ denote the complement of all colors. Then $X^c_B$ is an open, $P(X)$-stable, affine variety with the following properties:

1. $X^c_B$ meets every $G$-orbit.

2. If we let $Y^c$ be the closure of $A_X$ in $X^c_B$ (an affine toric subvariety), then the action map $Y^c \times U_{P(X)} \to X^c_B$ is an isomorphism.

We emphasize the structure of the affine toric variety $Y^c$: Its cone of regular characters is precisely $C(X^c)^\circ := \{\chi \in \mathcal{X}(X) \otimes \mathbb{Q} | (\chi, v) \geq 0 \text{ for all } v \in C(X^c)\}$, in other words:

$$Y^c = \text{spec } k[C(X^c)^\circ \cap \mathcal{X}(X)].$$

By the theory of toric varieties, the theorem also implies that $X^c$ is smooth if and only if the monoid $C(X^c) \cap \Lambda_X$ is generated by primitive elements in its “extremal rays” (i.e. is a free abelian monoid).

Notice that when $\mathcal{V}$ is strictly convex (equivalently: $\text{Aut}^G(X^+)$ is finite) then $X^+$ admits a canonical toroidal embedding $\tilde{X}$, with $\mathcal{C}(\tilde{X}) = \mathcal{V}$, which is complete. This is sometimes called the wonderful completion of $X^+$, although often the term “wonderful” is reserved for the case that this completion is smooth. If $\mathcal{V}$ is not strictly convex then $X^+$ still admits a (non-unique) toroidal embedding $\tilde{X}$, which is not simple, but as remarked in [GN, 8.2.7] Theorem 2.3.2 still holds, with $Y^c$ a suitable (non-affine) toric variety containing $A_X$. The fan of $Y^c$ depends on the chosen embedding $\tilde{X}$, but its support is precisely the dual cone of $\mathcal{V}$ (i.e. the set of cocharacters $\lambda$ of $A_X$ such that $\lim_{t \to 0} \lambda(t) \in Y^c$ is equal to $\Lambda_X^\circ$).

We will use Theorem 2.3.2 for two toroidal varieties: First, for a complete toroidal embedding $\tilde{X}$ of $X^+$. Secondly, for the variety $\tilde{X}$ obtained from our affine spherical variety $X$ by taking $\mathcal{C}(\tilde{X}) = \mathcal{C}(X) \cap \mathcal{V}$. Before we proceed, we discuss models of these varieties over rings of integers.

2.3.3 Models over rings of integers

We start with toric varieties. Let $\mathfrak{o}$ be an integral domain with fraction field $k$, and let $Y$ be a simple toric variety for a split torus $T$ over $k$. We endow $T$ with its smooth model $T = \mathfrak{o}[\mathcal{X}(T)]$ over $\mathfrak{o}$. Since $Y = \text{spec } k[M]$ for some saturated monoid $M \subset \mathcal{X}(T)$, the $\mathfrak{o}$-scheme $\mathcal{Y} = \text{spec } \mathfrak{o}[M]$ is a model for $Y$ over $\mathfrak{o}$ with
an action of $T$, and we will call it the *standard model*. The notion easily extends to the case where $Y$ is not necessarily affine, but defined by a fan. If $T$ and $Y$ are defined over a number field $k$ and endowed with compatible models over the $S$-integers $\mathfrak{o}_S$ for a finite set $S$ of places of $k$, then these models will coincide with the standard models over $\mathfrak{o}_{S'}$, for some finite $S' \supset S$.

Now we return to the setting where $k$ is a number field, $G$, $X$, $X^+$, $\tilde{X}$, $\tilde{X}$ are as before (over $k$). Then we can choose compatible integral models outside of a finite set of places, such that the structure theory of Brion, Luna and Vust continues to hold for these models:

**2.3.4 Proposition.** There are a finite set of places $S_0$ of $k$ and compatible flat models $\mathcal{G}$, $\mathfrak{X}$, $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{X}}$ for $G$, $X$, $\tilde{X}$ and $\tilde{X}$ over the $S_0$-integers $\mathfrak{o}_{S_0}$ of $k$ such that:

- $S_0$ contains all archimedean places;
- the chosen point $x_0 \in \tilde{\mathfrak{X}}^+(\mathfrak{o}_{S_0})$;
- $\mathcal{G}$ is reductive over $\mathfrak{o}_{S_0}$, $X^+ \to \text{spec} \mathfrak{o}_{S_0}$ is smooth and surjective;
- the statement of Theorem 2.3.2 holds for $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{X}}$ over $\mathfrak{o}_{S_0}$: namely, if we denote any one of them by $\mathfrak{X}^c$ then there is an open, $\mathcal{P}(\mathcal{X})$-stable subscheme $\mathfrak{X}^c_B$ and a toric $\mathcal{A}$-scheme $\mathcal{Y}^c$ of standard type such that the subscheme $\mathfrak{X}^c_B$ meets every $\mathcal{G}$-orbit on $\mathfrak{X}^c$ and the action map: $\mathcal{Y}^c \times U_{\mathcal{P}(\mathfrak{X})} \to \mathfrak{X}^c_B$ is an isomorphism of $\mathfrak{o}_{S_0}$-schemes.
- $\tilde{\mathfrak{X}}$ is proper over $\mathfrak{o}_{S_0}$, and the map $\tilde{\mathfrak{X}} \to \mathfrak{X}$ is proper.

**2.3.5 Remark.** By $\mathfrak{X}^c$ (resp. $\tilde{\mathfrak{X}}^c$) we denote the complement of the closure, in any of the above schemes, of the complement of $X^+$ (resp. $\tilde{X}^+$) in the generic fiber. It is implicitly part of the “compatibility” of the models that this does not depend on which of the varieties we choose to define it. We understand the statement “meets every orbit” as follows: Let $|\mathcal{Z}|$ denote the set of scheme-theoretic points of a scheme $\mathcal{Z}$. Consider the two maps: $p: \mathcal{G} \times \mathfrak{X} \to \mathfrak{X}$ (projection to the second factor) and $a: \mathcal{G} \times \mathfrak{X} \to \mathfrak{X}$ (action map). Then for every $x \in |\mathfrak{X}^c|$ the set $a(p^{-1}\{x\})$ intersects $|\mathfrak{X}^c_B|$ non-trivially.

**Proof.** For a finite set $S$ of places and a flat model $\mathfrak{X}^c$ of $X^c$ over $\mathfrak{o}_S$ (assumed proper if $X^c = \tilde{X}$), let $D$ denote the union of all colors over the generic point of $\mathfrak{o}_S$, let $\mathfrak{D}$ denote the closure of $D$ in $\mathfrak{X}^c$ and let $\mathfrak{X}^c_B$ be the complement of $\mathfrak{D}$ in $\mathfrak{X}^c$. Let $\mathcal{G}$ denote a compatible reductive model for $G$ over $\mathfrak{o}_S$. The image of $\mathcal{G} \times \mathfrak{X}^c_B \to \mathfrak{X}^c$ is open and contains the generic fiber, hence by enlarging the set $S$, if necessary, we can make it surjective.

Now define $\mathcal{Y}^c$ as the closure of $\mathcal{Y}^c$ in $\mathfrak{X}^c_B$. By enlarging the set $S$, if necessary, we may assume that $\mathcal{Y}^c$ is of standard type. The action map $\mathcal{Y}^c \times U_{\mathcal{P}(\mathfrak{X})} \to \mathfrak{X}^c_B$ being an isomorphism over the generic fiber, it is an isomorphism over $\mathfrak{o}_S$ by enlarging $S$, if necessary. \qed
From now on we fix such a finite set of places \( S_0 \) and such models. The combinatorial invariants of the above schemes are the same at all places of \( S_0 \):

**2.3.6 Proposition.** The data\(^6\) \( \mathcal{X}(X), \mathcal{V}, \mathcal{C}(X), \mathcal{C}(\overline{X}), \mathcal{C}^{\pm}(\overline{X}) \) are the same for the reductions of \( X \) etc. at all closed points of \( \mathfrak{o}_{S_0} \). The set of \( G \)-orbits on each of these varieties is the same as the set of \( \mathcal{G} \)-orbits on each of their reductions.

**Proof.** The toric scheme \( \mathcal{Y}^c \) being of the standard type, it means that \( \mathcal{X}(X) = \mathcal{X}_A(Y^c) \) is the same at all reductions. For every place \( v \) of \( \mathfrak{o}_S \) the reductions \( \mathcal{X}_{F_v}, \mathcal{X}_{G_v} \) are toroidal: Indeed, denoting by \( \mathcal{X}^c \) either of them, the complement of \((\mathcal{X}_B^c)_{F_v}\) is a \( \mathcal{B}_v \)-stable union of divisors which does not contain any \( \mathcal{G}_v \)-orbit, since \((\mathcal{X}_B^c)_{G_v}\) meets every \( \mathcal{G}_v \)-orbit. Moreover, \( \mathcal{X}_B \) meets no colors: for if it did, then a non-open \( \mathcal{A}_{F_v} \)-orbit on \( \mathcal{V}^c_{F_v} \) would belong to the open \( \mathcal{G}_v \)-orbit, and hence the open \( \mathcal{G}_v \)-orbit would belong to the closure of a non-open \( G \)-orbit over the generic point, a contradiction since by assumption \( X^+ \) is smooth and surjective. Therefore, the complement of \((\mathcal{X}_B^c)_{F_v}\) is the union of all colors of \((\mathcal{X}^c)_{F_v}\), and \( \mathcal{X}_{G_v} \) is toroidal. Moreover, the \( \mathcal{G}_v \)-invariant valuations on \( F_{F_v}(\mathcal{X}^+_{F_v}) \) whose center is in \( \mathcal{X}_{G_v}^c \) are precisely those of \( \Lambda_X \cap \mathcal{C}(\mathcal{X}^c) \) (which proves the equality of \( \mathcal{C}(\mathcal{X}^c) \) at all \( v \notin S_0 \)), and from the fact that \( \mathcal{X}_{G_v} \) is complete and \( \mathcal{C}(\mathcal{X}_{G_v}) = \Lambda_X^+ \) it follows that \( \mathcal{V} \) is precisely the cone of invariant valuations on \( F_v(\mathcal{X}^c) \). \[ \square \]

Now we are ready to apply the argument of [GN, Theorem 8.2.9] to determine the set of \( \mathcal{G}(\mathfrak{o}_F) \)-orbits on \( X^+_{\mathfrak{o}_F} \), for every completion \( F \) of \( k \) outside of \( S_0 \), and also extend it to a description of the set of orbits which are contained in \( \mathcal{X}(\mathfrak{o}_F) \). Notice that since \( \mathcal{G} \) is reductive, \( \mathcal{G}(\mathfrak{o}_F) \) is a hyperspecial maximal compact subgroup of \( G(F) \). From now on we denote our fixed models over \( \mathfrak{o}_{S_0} \) by regular script, since there will be no possibility of confusion.

**2.3.7 Theorem.** For \( F \) a completion of \( k \) outside of \( S_0 \) the set of \( G(\mathfrak{o}_F) \)-orbits on \( X^+_{\mathfrak{o}_F}(F) \) is parametrized by \( \Lambda_X^+ \), with \( \lambda \in \Lambda_X^+ \) corresponding to the orbit through \( \lambda(\mathbb{F}_F) \in A_X(F) \). The orbits contained in \( \mathcal{X}(\mathfrak{o}_F) \) are precisely those of \( \Lambda_X^+ \cap \mathcal{C}(X) \).

**2.3.8 Remark.** The theorem holds as stated under our ongoing assumption that the quotient \( \mathcal{X}(A)/\mathcal{X}(A_X) \) is torsion-free. As the proof will show, the subset \( Y(\mathfrak{o}_F) \cap A_X(F) \) will always represent all \( G(\mathfrak{o}_F) \)-orbits, and distinct \( A_X(\mathfrak{o}_F) \)-orbits on it belong to distinct \( G(\mathfrak{o}_F) \)-orbits, but it is not true, in general, that all elements in an \( A_X(\mathfrak{o}_F) \)-orbit belong to the same \( G(\mathfrak{o}_F) \)-orbit – for instance, if \( X^+ = H \backslash G \) with \( H \) not connected then the map: \( G(\mathfrak{o}_F) \ni g \mapsto x_0 \cdot g \in X^+_{\mathfrak{o}_F}(F) \) will not be surjective.

**Proof.** Denote \( \mathfrak{o}_F \) by \( \mathfrak{o} \) and the residue field by \( \mathbb{F} \). We first use the notation \( X^e, X^*_B, Y^e, \) etc. as above for the scheme \( \overline{X} \). The \( \mathfrak{o} \)-scheme \( X^c \) is proper and hence \( X^e(\mathfrak{o}) = X^e(F) \). We will first show that \( Y^e(\mathfrak{o}) \) contains representatives
for all $G(\mathfrak{o})$-orbits on $X^c(\mathfrak{o})$. Let $x \in X^c(\mathfrak{o})$ and denote by $\bar{x} \in X^c(\mathbb{F})$ its reduction. The open, $P(X)$-stable subvariety $X_B^c$ meets every $G$-orbit; for a spherical variety for a split group over an arbitrary field (denoted $\mathbb{F}$, since we will apply it to this field) the $\mathbb{F}$-points of the open $B$-orbit meet every $G(\mathbb{F})$-orbit (following the argument of [Sa08, Lemma 3.7.3]). This means that there is a $\bar{g} \in G(\mathbb{F})$ (which we can lift to a $g \in G(\mathfrak{o})$) such that $\bar{x} = \bar{g} \circ x \in X_B^c(\mathbb{F})$. Since $X_B^c$ is open, this means that $x \cdot g \in X_B^c(\mathfrak{o}) = Y^c(\mathfrak{o}) \times U_{P(X)}(\mathfrak{o})$. Acting by a suitable element of $U_{P(X)}(\mathfrak{o})$, we get a representative for the $G(\mathfrak{o})$-orbit of $x$ in $Y^c(\mathfrak{o})$.

Hence, $G(\mathfrak{o})$-orbits on $X^+(\mathbb{F})$ are represented by elements of $Y^c(\mathfrak{o}) \cap A_X(F)$.

By the description of the toric scheme $Y^c$, those are just the elements of the form $\lambda(\varpi) \cdot A_X(\mathfrak{o})$ with $\lambda \in \Lambda_X^+$, and by our ongoing assumption that the map: $\mathcal{X}(A) \to \mathcal{X}(A_X)$ is surjective, each $A_X(\mathfrak{o})$-orbit is also an $A(\mathfrak{o})$-orbit. Hence the elements $\lambda(\varpi)$ with $\lambda \in \Lambda_X^+$ represent all $G(\mathfrak{o})$-orbits.

To prove that two distinct such elements $\lambda(\varpi)$, $\lambda'(\varpi)$ belong to distinct $G(\mathfrak{o})$-orbits, the argument of Gaitsgory and Nadler carries over verbatim: If $\lambda$ and $\lambda'$ are not $\mathbb{Q}$-multiples of each other, we can construct as in [Kn91] a toroidal compactification $\bar{X}$ of $X^+$ over $\mathfrak{o}$ such that $\lambda(\varpi) \in \bar{X}(\mathfrak{o})$ but $\lambda'(\varpi) \notin \bar{X}(\mathfrak{o})$. Finally, if $\lambda$ and $\lambda'$ are proportional, then we can find a toroidal compactification $\bar{X}$ such that $\lim_{t \to 0} \lambda(t)$ belongs to some $G$-orbit $D$ of codimension one, and then the intersection numbers of $\lambda(\varpi)$ and $\lambda'(\varpi)$ (considered as 1-dimensional subschemes of $X^c$) with $D$ are different. (Notice that the constructions of [Kn91] are over a field of arbitrary characteristic, and based on Proposition 2.3.6 one can carry them over over the ring $\mathfrak{o}_F$.)

Finally, if we set $X^c = \bar{X}$ then we have a proper morphism: $X^c \to X$ which is an isomorphism on $X^+$. By the valuative criterion for properness, every point in $X(\mathfrak{o}) \cap X^+(\mathbb{F})$ lifts to a point on $X^c(\mathfrak{o})$, therefore for the last statement it suffices to determine the set of $G(\mathfrak{o})$-orbits on $X^c(\mathfrak{o}) \cap X^+(\mathbb{F})$. By the same argument as before, every $G(\mathfrak{o})$-orbit meets $Y^c(\mathfrak{o})$, and the latter intersects $A_X(F)$ precisely in the union of $A_X(\mathfrak{o})$-orbits represented by $A_X \cap C(X)$. □

2.3.9 Remark. In the case of symmetric spaces similar statements on the set of $G(\mathfrak{o}_F)$-orbits on $X(F)$ and in a more general setting – without assuming that $G$ is split – have been proven by Benoist and Oh [BO07], Delorme and Sécherre [DS].

### 3 Geometric models and local Schwartz spaces

#### 3.1 Goals

For this section we let $F$ be a local-nonarchimedean field with ring of integers $\mathfrak{o}$ and residue field $\mathbb{F}$, and let $X$ be an affine spherical scheme for a group $G$ over $\mathfrak{o}$, having the properties of Proposition 2.3.4. We denote $K = G(\mathfrak{o})$. We want to define a “Schwartz space” of functions on the $F$-points of the smooth locus of $F$. If $X$ is non-singular, then the Schwartz space should simply be the space $C^0_b(X(F))$. However, the work of Braverman and Kazhdan [BK98, BK02] and considerations arising from integral representations of $L$-functions (as we will
see) suggest that if $X$ is singular then one should consider a different space of functions reflecting the singularities of $X$. More precisely, one would like to have the following:

1. A way to think of $X(F)$ as the $\mathbb{F}$-points of an (infinite-dimensional, ind-) scheme $X'$.

2. A “Schwartz space” $\mathcal{S}(X(F))$ of smooth functions on the $F$-points of the smooth locus of $X$, which reflects the singularities of $X'$.

While the first is essentially possible by the theory of the Greenberg functor ([Gr61, Gr63]), it is not clear how to achieve the second. For the purposes of our present discussion, the only Schwartz function which matters is a distinguished element $\Phi^0 \in \mathcal{S}(X(F))^K$ which should be the function corresponding, via Grothendieck’s function-sheaf correspondence, to the “intersection cohomology sheaf” of the $\mathbb{F}$-scheme $X$ with $X(\mathbb{F}) = X(\phi)$. Since such a notion of intersection-cohomology sheaves on infinite-dimensional schemes does not exist, all that we will do here is to discuss certain finite-dimensional geometric models which have appeared in the literature in the case of local and global fields of equal characteristic. We will use these models to define the “basic function” $\Phi^0$ in a very ad-hoc way based on the generalized Cartan decomposition (Theorem 2.3.7). A proper definition of $\mathcal{S}(X(F))$ is far from my reach at this point, and should be considered as part of the conjectural program which this article attempts to outline. However, as we shall see, for certain spaces related to flag varieties there are precise formulas describing the function $\Phi^0$ obtained via our ad-hoc constructions, and these are enough to put the theory of Rankin-Selberg integrals on the correct footing.

3.2 Geometric models

The models that we are about to discuss are relevant to a spherical variety $X$ over an equal-characteristic local field $F$, and in fact the first two are not local, but global in nature.

3.2.1 The Gaitsgory-Nadler spaces [GN].

Let $X$ be an affine spherical variety over $\mathbb{C}$, and let $C$ be a smooth complete complex algebraic curve. Consider the ind-stack $Z$ of meromorphic quasimaps which, by definition, classifies data:

$$(c, \mathcal{P}_G, \sigma)$$

where $c \in C$, $\mathcal{P}_G$ is a principal $G$-bundle on $C$, and $\sigma$ is a section: $C \setminus \{c\} \to \mathcal{P}_G \times^G X$ whose image is not contained in $X \setminus X^+$. Clearly, $Z$ is fibered over $C$ (projection to the first factor). It is a stack of infinite type, however it is a union of open substacks of finite type, each being the quotient of a scheme by an affine group, and therefore one can define intersection cohomology sheaves on it without a problem.
The same definitions can be given if \( G, X \) are defined over a finite field \( F \).

To any quasimap one can associate an element of \( X^+(\mathcal{K})/G(\mathcal{O}) \) (where \( \mathcal{O} = \mathbb{C}[[t]], \mathcal{K} = \mathbb{C}((t)) \)) as follows: Choose a trivialization of \( \mathcal{P}_G \) in a formal neighborhood of \( c \) and an identification of this formal neighborhood with \( \text{spec}(\mathcal{O}) \) – then the section \( \sigma \) defines a point in \( X^+(\mathcal{K}) \), which depends on the choices made. The corresponding coset in \( X^+(\mathcal{K})/G(\mathcal{O}) \) is independent of choices.

This allows us to stratify our space according to the stratification, provided by Theorem 2.3.1, of \( X^+(\mathcal{K})/G(\mathcal{O}) \). We only describe some of the strata here: For \( \theta \in X^+_X \), let \( Z^0 \) denote the quasimaps of the form \( (c, \mathcal{P}_G, \sigma : C \smallsetminus \{c\} \to \mathcal{P}_G \times^G X^+) \) which correspond to the coset \( \theta \in X^+(\mathcal{K})/G(\mathcal{O}) \) at \( c \). Then \( Z^0 \) can be thought of as a (global) geometric model for that coset. The basic stratum \( Z^0 \) consists of quasimaps of the form \( (c \in C, \mathcal{P}_G, \sigma : C \to \mathcal{P}_G \times^G X^+) \). Notice that these sub-stacks do not depend on the compactification \( X \) of \( X^+ \). Their closure, though, does. For instance, the closure of \( Z^0 \) can be identified with an open substack in the quotient stack \( X_C/G_C \) over \( C \), namely the stack whose \( S \)-objects are \( S \)-objects of \( X_C/G_C \) but not of \( (X \smallsetminus X^+)C/G_C \). These are the quasimaps for which the corresponding point in \( X^+(\mathcal{K})/G(\mathcal{O}) \) lies in the image of \( X^+(\mathcal{K}) \cap X(\mathcal{O}) \). Hence, the closure of \( Z^0 \) should be thought of as a geometric model for \( X^+(\mathcal{K}) \cap X(\mathcal{O}) \).

Since the spaces of Gaitsgory and Nadler are global in nature, it is in fact imprecise to say that they are geometric models for local spaces. However, their singularities are expected to model the singularities of \( G(\mathcal{O}) \)-invariant subsets of \( X^+(\mathcal{K}) \).

### 3.2.2 Drinfeld’s compactifications.

The spaces of Gaitsgory and Nadler described above are (slightly modified) generalizations of spaces introduced by Drinfeld in the cases: \( X = \overline{U^P\backslash G}^{\text{aff}} \) or \( X = \overline{[P, P]\backslash G}^{\text{aff}} \), where \( P \subset G \) is a proper parabolic and \( U_P \) its unipotent radical. The corresponding spaces are denoted by \( \overline{\text{Bun}_P} \) and \( \overline{\text{Bun}_P^+} \), respectively. Our basic references here are \([BG02, BFGM02]\). The only differences between the definition of these stacks and the stacks \( Z \) of Gaitsgory and Nadler are that the section \( \sigma \) has to be defined on all \( C \), and it does not have a distinguished point \( c \). Therefore, for a quasimap in Drinfeld’s spaces and any point \( c \in C \) the corresponding element of \( X^+(\mathcal{K})/G(\mathcal{O}) \) has to belong to the cosets which belong to \( X(\mathcal{O}) \). (These will be described later when we review the computations of \([BFGM02]\)).

This particular case is very important to us because it is related to Eisenstein series, and moreover the intersection cohomology sheaf of the “basic stratum” has been computed (when \( G, X \) are defined over \( F \)).

### 3.2.3 The theorem of Grinberg and Kazhdan.

Let \( X \) be an \( F \)-variety, and let \( \mathcal{K} = \mathbb{F}((t)), \mathcal{O} = \mathbb{F}[[t]], D = \text{spf} \mathbb{F}[[t]] \). Consider the (infinite-dimensional) scheme \( \mathcal{L}(X) \) of formal arcs on \( X \) – its \( F \)-points are...
just $X(K)$. A finite-dimensional geometric model for the singularities of $X(K)$ can be obtained directly (without any global constructions and without reference to the group acting on it) from the following theorem of Drinfeld [Dr], generalizing to arbitrary characteristic a theorem of Grinberg and Kazhdan [GK00]:

**3.2.4 Theorem.** For every $x \in \mathcal{L}(X)(\mathbb{F})$ there exists a scheme $Y$ of finite type over $\mathbb{F}$ and a point $y \in Y(\mathbb{F})$ such that we have an isomorphism of formal neighborhoods: $\mathcal{L}(X)_x \cong Y_y \times D^\times$, where $D^\times$ denotes a product of countably many copies of $D$.

We will not use this theorem in the rest of the paper, but we included it in order to point out that one might be able to have a purely local definition of the Schwartz functions that we are going to define by appealing to it.

### 3.3 Local Schwartz space

#### 3.3.1 The basic function

We return to the setting where $X$ is an affine spherical scheme for a split group $G$ over the integers $\mathfrak{o}$ of a local, non-archimedean field $\mathbb{F}$ whose (finite) residue field we denote by $\mathbb{F}$. We assume that $X$, $G$ and the completions $\overset{\sim}{X}$, $\tilde{X}$ introduced before have the properties of Proposition 2.3.4 over $\mathfrak{o}$, and denote $K = G(\mathfrak{o})$.

The goal is to define the “basic function” $\Phi_0$ on $X^+(F)$, which will be $K$-invariant and supported in $X^+(\mathfrak{o})$. By the decomposition $X^+(F)/K \cong \Lambda^+_X$, this is a function on $\Lambda^+_X$. The idea is to define a function on $\Lambda^+_X$ using equal-characteristic models of $X$.

Define the Gaitsgory-Nadler stack $\mathcal{Z}$ as in §3.2.1 over $\mathbb{F}$. Since, by assumption, $X_F$ has a completion $\overset{\sim}{X}_F$ with the properties of Proposition 2.3.4 (and, hence, the same holds for the base change $X_{\mathbb{F},[[t]]}$), the stratification theorem 2.3.7 holds for $G(\mathbb{F},[[t]])$-orbits on $X^+(F(\!(t)\!))$: they are naturally parametrized by $\Lambda^+_X$. Hence the strata $\mathcal{Z}^\theta$ of $\mathcal{Z}$ are well-defined over $\mathbb{F}$. Let $IC^0$ denote the intersection cohomology sheaf of the closure of the basic stratum $\mathcal{Z}^0$ (how exactly to normalize it is not important at this point, since we will normalize the corresponding function). We will obtain the value of our function at $\lambda \in \Lambda^+_X$ as trace of Frobenius acting on the stalk of $IC^0$ at an $\mathbb{F}$-object $x_\lambda$ in the stratum $\mathcal{Z}^\lambda$. However, since these strata are only locally of finite type, and not of pure dimension, we must be careful to make compatible choices of points as $\lambda$ varies. (It is expected that $IC^0$ is locally constant on the strata – this will be discussed below.)

The compatibility condition is related to the natural requirement that the action of the unramified Hecke algebra on the functions which will be obtained from sheaves is compatible, via the function-sheaf correspondence, with the action of its geometric counterpart on sheaves. First of all, let us fix a quasimap $x_0 = (c_0, P_0, \sigma_0)$ in the $\mathbb{F}$-objects of the basic stratum $\mathcal{Z}^0$. Now consider the subcategory $\mathcal{Z}_{x_0}$ of $\mathcal{Z}$ consisting of $\mathbb{F}$-quasimaps $(c_0, P_G, \sigma)$ with the property that there exists an isomorphism $\iota : P_0|_{C^{\sim}(c_0)} \cong P_G|_{C^{\sim}(c_0)}$ (inducing isomorphisms...
between $P_0 \times^G X$ and $P_G \times^G X$, also to be denoted by $\iota$ such that $\sigma = \iota \circ \sigma_0$. Hence, the objects in $Z_{x_0}$ are those obtained from $x_0$ via meromorphic Hecke modifications at the point $c_0$ [GN, §4].

For each $\hat{\lambda} \in \Lambda_X^+$, pick an object $x_\hat{\lambda} \in Z_{x_0}$ which belongs to the stratum $Z^\hat{\lambda}$. We define the basic function $\Phi^0$ on $\Lambda_X^+$ to be:

$$
\Phi^0(\hat{\lambda}) = c \cdot \sum_i (-1)^i \text{tr}(\text{Fr}, H^i(IC^0_{x_\hat{\lambda}}))
$$

(3.1)

where $IC^0_{x_\hat{\lambda}}$ denotes the stalk of $IC^0$ at $x_\hat{\lambda}$ and $\text{Fr}$ denotes the geometric Frobenius. The constant $c$ (independent of $\hat{\lambda}$) is chosen such that $\Phi^0(0) = 1$.

Now we return to $X(F)$ and we identify $\Phi^0$ with a $K$-invariant function on $X^+(F)$ (also to be denoted by $\Phi^0$) via the stratification of Theorem 2.3.7.

One could also use the other geometric models discussed in §3.2 to define $\Phi^0$. As mentioned, the singularities of these geometric objects should depend only on the geometry of $\mathcal{X}$, the scheme of formal arcs on $X_F$, and therefore the functions obtained should be independent of the choices made. In the case of Drinfeld spaces this has been verified by computation, as we shall see. However, since we do not know a general proof of this fact, we will impose the independence from the choice of $x_\hat{\lambda}$ (and hence the well-posedness of the definition, independently of the model) as an axiom, together with a condition on the action of the Hecke algebra:

**3.3.2 Assumption.** The basic function $\Phi^0$ on $X^+(F)$ is well-defined and independent of:

- the choices of objects $x_\hat{\lambda}$;
- which model of §3.2 one uses to define them;
- the group $G$ acting on $X$; more precisely, if $G_1, G_2$ act on $X$ and we denote by $X_1^+, X_2^+$ the open orbits, then the restriction of $\Phi^0$ to $X_1^+(F) \cap X_2^+(F)$ should be the same.

We will also make an assumption on the growth of the basic function, to be used later:

**3.3.3 Assumption.** There is a $c \in \text{Hom}(\Lambda_X, \mathbb{R})$ (which, when $X$ is defined over a number field $k$, is independent of the completion $F$ of $k$ outside of $S_0$), such that $\Phi^0(x_\hat{\lambda}) = O(q^{c(\hat{\lambda})})$ (where $q = |F|$).

**3.3.4 Local Schwartz space**

**3.3.5 Definition.** We define $S(X(F))^K \subset C^\infty(X^+(F))$ to be the space generated by $\Phi^0$ under the action of the unramified Hecke algebra $\mathcal{H}(G,K)$ of compactly generated, $K$-biinvariant measures on $G(F)$. 

20
3.3.6 Remark. Similarly, we could have defined functions $\Phi^\theta$, $\theta \in \Lambda^+_X$, using the intersection cohomology sheaves of the closures of the strata $Z^a$ of Gaitsgory and Nadler. (Again, under assumptions similar to 3.3.2.) Based on the work of Gaitsgory and Nadler, and under the multiplicity-freeness assumptions of §2.1, all $\Phi^\theta$ should be contained in $S(X(F))^K$. In any case, the only space of functions which will matter for our present paper is the $\mathcal{H}(G, K)$-span of $\Phi^0$.

The space $S(X(F))^K$ depends on the group $G$ and not only on $X$. However, it is expected that this is just the $K$-invariants of a larger, $G$-invariant space of smooth functions $S(X(F))$ which does not depend on $G$, but only on $X$. Since we do not know how to define $S(X(F))$ in general, we define:

3.3.7 Definition. The mixed Schwartz space $S^m(X(F))$ is the space of functions on $X^+(F)$ generated over $\mathcal{H}(G(F))$ (the full Hecke algebra of compactly supported, smooth measures on $G(F)$) by $S(X(F))^K$ and $C^e_\varepsilon(X^+(F))$.

The reason that it is called “mixed” is that the correct Schwartz space should consist of functions of arbitrary ramification defined “geometrically” such as the unramified functions $\Phi^\theta$ above. For lack of such a definition, we have used the space $C^e_\varepsilon(X^+(F))$ which will play a role only at a finite number of places (in the integrals of automorphic forms which we are about to consider) but is not expected to give the “correct” $L$-factors at those places.

3.3.8 Remark. A priori we do not even know whether $S(X(F))^K$ contains $C^e_\varepsilon(X(F))^K$, though it is expected to be so. But, strictly speaking, $S^m(X)^K$ might be larger than $S(X(F))^K$.

3.3.9 Example. If $X$ is homogeneous and affine (hence $X = H\backslash G$ for some reductive $H$) then it is smooth and, hence, the intersection cohomology sheaf is constant. Therefore, we have $S^m(X(F)) = C^e_\varepsilon(X(F))$, which we also consider to be the “correct” Schwartz space $S(X(F))$.

3.3.10 Example. Let $X = \mathbb{A}^n$ ($n$-dimensional affine space). It can be identified with the affine closure of $P_n \backslash \text{GL}_n$ where $P_n$ is the “mirabolic” subgroup of $\text{GL}_n$. Then the (correct) Schwartz space $S(X(F))$ is the space of smooth, compactly supported functions on $F^n$. It contains $S^m(X(F))$, defined in terms of the action of the group $\text{GL}_n$.

3.3.11 Example. The previous example is (essentially) a special case of the spaces $X = [P, P] \backslash G^m$, where $P$ is a proper parabolic, considered by Braverman and Kazhdan in [BK98, BK02]. They were able to define the correct Schwartz space as the closure of $C^e_\varepsilon(X^+(F))$ under “Fourier transforms”. Unfortunately, we don’t see how to define “Fourier transforms” in the general case. Notice that for these spaces to satisfy the assumptions of §2.1 we need, in general, to consider them as spherical varieties for the group $P_{ab} \times G$, the first factor acting “on the left”.

3.3.12 Example. As we saw (Theorem 2.2.5), every affine spherical variety $X$ is fibered over a homogeneous variety $Y$ such that the fiber $X_y$ over any point in $Y$ (which here we take to be in $Y(\mathfrak{p})$) has a fixed point for the stabilizer.
$G_y$. In other words: $X = X_y \times^{G_y} G$. We can write the basic function as: $\Phi^0 = K \ast \Phi^0_{X_y}$, where $K \ast$ denotes convolution by the characteristic measure of $K$ and $\Phi^0_{X_y}$ denotes the basic function on the spherical variety $X_y$ for $G_y$, considered as a generalized function on $X_y$. Indeed, by our assumptions of §2.1 the group $K = G(\mathfrak{a})$ acts transitively on $Y(\mathfrak{a})$; the quotient stack $X/G$ is equivalent to $X_y/G_y$ and hence the closures of the basic strata $Z_0$ of the Gaistgory-Nadler spaces for $X$ and $X_y$ can be identified. This, of course, is a generalization of Example 3.3.9.

Based on the previous examples, we define the (correct) Schwartz space $S(X(\mathfrak{a}))$ of $X$ to be equal to the Braverman-Kazhdan Schwartz space, if $X$ is the affine closure of a variety of the form $[P, P] \backslash G$, and equal to $C^\infty_c(X(F))$ if $X$ is smooth affine. Moreover, in the setting of Example 3.3.12, if we know the “correct” Schwartz space for the fiber $X_y$ then we consider its elements as generalized functions on $X^+(F)$ and define $S(X(F)) = \mathcal{H}(G(F)) \ast S(X_y(F))$.

4 The automorphic pairing and $X$-Eisenstein series

4.1 Global Schwartz space

Returning to the global setting we now denote, for every $v \notin S_0$, the local mixed Schwartz space at the place $v$ by $S^m(X_v)$. For $v \in S_0$ (which includes archimedean completions) we define $S^m(X_v) = C^\infty_c(X_v)$. Whenever possible, we substitute these spaces by “correct” Schwartz spaces defined as before.

4.1.1 Definition. The global mixed Schwartz space $S^m(X(\mathfrak{a}))$ is the restricted tensor product $\bigotimes_v S^m(X_v)$ with respect to the basic vectors $\Phi^0_v$.

Since $\Phi^0_{X^+(\mathfrak{a})} \equiv 1$, it can be identified with a subspace of the space of smooth functions on $X^+(\mathfrak{a})_k$ (not, however, functions on $X(\mathfrak{a})$ in general). We define the (correct) Schwartz space $S(X(\mathfrak{a}))$ similarly in the cases where the local correct spaces are defined.

4.1.2 Remark. From the notation of the mixed Schwartz space we have suppressed the dependence on the group $G$ and the set of places $S_0$. The reason is that we plan to formulate somewhat weak statements about these spaces, which do not depend on those choices.

4.2 Pseudo-$X$-Eisenstein series and the automorphic pairing

We assume without serious loss of generality that $X^+$ carries a top-degree differential eigenform for the $G$-action. This gives rise to a positive eigenmeasure on $X(F)$, for every completion $F$ of $k$, and if $\eta$ is its eigencharacter we define a normalized action of $G$ on the space of functions on $X^+(F)$ by
\[(g \cdot \Phi)(x) = \sqrt{\eta(g)} f(x \cdot g).\] The corresponding action of an element \(h\) of the (full) Hecke algebra of \(G\) will be denoted by \(h \cdot f\).

Let \(\Phi \in \mathcal{S}(X(\mathbb{A}_k))\). We define the pseudo-\(X\)-Eisenstein series:

\[
\Psi(\Phi, g) = \sum_{\gamma \in X^+(k)} (g \cdot \Phi)(\gamma).
\] (4.1)

This sum is absolutely convergent: Indeed, the closure in \(X(\mathbb{A}_k)\) of the support of \(\Phi\) is compact, and the set \(X^+(k)\) is discrete, since \(X\) is affine. Moreover:

**4.2.1 Proposition.** The function \(\Psi(\Phi, g)\) on \(G(k) \backslash G(\mathbb{A}_k)\) is of moderate growth.

Assuming this (it will be proven in the next subsections), let \(\mathcal{S}(G(k) \backslash G(\mathbb{A}_k))\) denote the space of smooth functions on \(G(k) \backslash G(\mathbb{A}_k)\) which, together with all their derivatives, are rapidly decreasing. We define a bilinear pairing:

\[
\mathcal{P}_X : \mathcal{S}(X(\mathbb{A}_k)) \otimes \mathcal{S}(G(k) \backslash G(\mathbb{A}_k)) \to \mathbb{C}
\]

by:

\[
(\Phi, \phi) \mapsto \int_{G(k) \backslash G(\mathbb{A}_k)} \Psi(\Phi, g)\phi(g) dg.
\] (4.2)

**4.2.2 Distance functions.**

For two functions \(f_1\) and \(f_2\) we will write \(f_1 \ll^p f_2\) (where the exponent \(p\) stands for “polynomially”) if there exists a polynomial \(P\) such that \(|f_1| \leq P(|f_2|)\). We will say that \(f_1\) and \(f_2\) are polynomially equivalent if \(f_1 \ll^p f_2\) and \(f_2 \ll^p f_1\).

Recall that an automorphic function \(\Psi\) is “of moderate growth” if \(\Psi(\Phi, g) \ll^p |g|\) for some natural norm \(\| \cdot \|\) on \(G_X\). Recall that a “natural norm” is a positive function on \(G_X\), which is polynomially equivalent to \(\|\rho(g)\|\) where: \(\rho\) denotes an algebraic embedding \(G \hookrightarrow \operatorname{GL}_{n}\), and \(|g| := \max\{|g_{1}\cdots, |g_{n}|_v\} \) on \(\operatorname{GL}_{n}(\mathbb{A}_k)\) (where \(|\cdot|_v\) denotes the operator norm for the standard representation of \(\operatorname{GL}_{n}\) on \(l^2(\{1,\ldots, n\})\)).

Before we proceed with the proof of Proposition 4.2.1, we discuss the basic tool which is certain natural “distance functions”. Let \(F\) be a local field, \(Z \hookrightarrow X\) be a closed embedding of \(F\)-varieties, and denote by \(X^+\) the complement of \(Z\). By a distance function from \(Z\) we will mean a non-negative continuous function \(d_Z\) on \(X(F)\) whose zero set is \(Z(F)\).

Assume that \(X\) and \(Z\) are affine. Let \(\{f_i\}_i \subset F[X]\) be a finite set of generators for the ideal of \(Z\). Define \(d_Z(x) = \max_i |\{f_i(x)\}|\). For any two distance functions \(d_Z, d_Z'\) defined this way, and any compact neighborhood \(N\) of a point in \(Z(k)\), their inverses satisfy \(d_Z^{-1} = O(d_Z'^{-1})\) and \(d_Z^{-1} = O(d_Z'^{-1})\) on \(N\). We will call such a distance function a natural one.

Now let \(X\) and \(Z\) be affine, defined over a global field \(k\). Then we can define, locally at all places \(v\), natural distance functions \(d_{Z,v}\) using the same set of regular functions \(\{f_i\}_i \subset k[X]\). The Euler product \(d_Z(x) := \prod_v d_{Z,v}(x_v)\) is a well-defined positive function on \(X^+(\mathbb{A}_k)\), since almost all of the Euler factors are equal to one. We call it a *natural adelic distance function from \(Z\).*
4.2.3 Proof of Proposition 4.2.1.

Assume without loss of generality that $\Phi = \bigotimes_v \Phi_v$, with $\Phi_v \in S^m(X_v)$, and let $S_{\Phi} = \bigcap S_{\Phi_v}$, where $S_{\Phi_v}$ is the support of $\Phi_v$ in $X(k_v)$. Having taken $S^m(X_v) = C_c^\infty(X_\mathbb{A}_E)$ the set $S_{\Phi}$ is a compact subset of $X(\mathbb{A}_E)$.

In fact, it suffices to assume that $\Phi_v = \Phi_0$ for all places $v \notin S_0$. (Indeed, we may enlarge the set $S_0$ such that for $v \notin S_0$ the function $\Phi_v$ is in the $G$-space generated by $\Phi_0$. But then the resulting pseudo-Eisenstein series will be of moderate growth if and only if the pseudo-Eisenstein series obtained by replacing $\Phi_v$ by $\Phi_0$ is.)

Recall that we have fixed a model for $X$ over $\mathfrak{o}_{S_0}$ (Proposition 3.2). Let $Z = X \smallsetminus X^+$, and let $M \subset \mathfrak{o}_{S_0}[X]$ denote the ideal of $Z$. Assume, for simplicity, that $M$ is free over $\mathfrak{o}_{S_0}$, fix an $\mathfrak{o}_{S_0}$-basis $\{f_i\}$ of $M$ and use it to define a natural adelic distance function $d_Z$ from $Z$, as above. (In the general case, we can do this separately on each element of an open cover of $\text{spec} \mathfrak{o}_{S_0}$, and the inverses of the distance functions thus obtained are all polynomially equivalent.)

The claim follows immediately from the following properties (for $g \in G_X$ and $x \in X^+(\mathbb{A}_E)$):

1. $\#(X^+(k) \cap S_{\Phi}g) \ll_p |g|$.
2. $|\Phi(x)| \ll_p d_Z(x)^{-1}$.
3. $(\inf_{x \in X^+(k)} d_Z(xg))^{-1} \ll_p |g|$.

Indeed, assuming these properties we have:

$$
\Psi(\Phi, g) = \sum_{\gamma \in X^+(k)} (g \cdot \Phi)(\gamma) \ll \#(X^+(k) \cap S_{\Phi}g^{-1}) \cdot \sup_{x \in X^+(k)} |\Phi(xg)| \ll_p \ll_p |g| \cdot \left(\inf_{x \in X^+(k)} d_Z(xg)\right)^{-1} \ll_p |g| \cdot |g|.
$$

The first property is standard, and follows from the analogous claim for $GL_n$ (after fixing an equivariant embedding of $X$ in the vector space of a representation of $G$), since $S_{\Phi}$ is a compact subset of $X(\mathbb{A}_E)$. The others are also standard in one form or another but slightly more complicated, so we discuss them in detail for the sake of completeness.

To prove 2, we need to relate our natural distance functions $d_Z$ to the parametrization of $K_v$-orbits on $X_v^+$ by the monoid $\Lambda^+_X$. (Recall that we have assumed in 3.3.3 a growth property for the basic function $\Phi_0$.) We claim:

The local distance function $d_{Z,v}$ (for $v \notin S_0$) is $G(\mathfrak{o}_v)$-invariant.

\footnote{The modifications needed to handle the case when the “mixed” Schwartz spaces are replaced by the “correct” Schwartz spaces, both at archimedean and at non-archimedean places, are easy. For instance, in the Braverman-Kazhdan local Schwartz spaces over a non-archimedean place $v$, the growth of any function can be bounded by the growth of the basic function $\Phi_0$.}
Indeed, since the functions \( \{f_i\}_i \) used to define \( d_Z \) are an \( \mathfrak{g}_{S_0} \)-basis for \( M \), we have a homomorphism \( G \to \text{GL}(M) \) over \( \mathfrak{g}_{S_0} \) and hence the image of \( G(\mathfrak{g}_{S_0}) \) lies in \( \text{GL}(M, \mathfrak{g}_e) \) and does not change the number \( \max_i |f_i(x_v)| \).

Therefore, by Theorem 2.3.7, it suffices to prove 2 when \( x \in A_X(\mathbb{A}_k) \) and such that, for every \( v \notin S_0 \), \( x_v \) belongs to the \( A_X(\mathfrak{g}_v) \)-coset of \( \tilde{\lambda}(\pi) \) for some \( \tilde{\lambda} \in \Lambda_X \cap \mathcal{C}(X) \). The restriction of each of the functions \( f_i \in k[X] \to A_X \) can be written as a linear combination of characters of \( A_X \) which extend by zero to its closure. In particular, such characters are strictly positive on \( \mathcal{C}(X) \setminus \{0\} \).

Therefore, in the bound of Assumption 3.3.3: \( \Phi_v^0(x_\lambda) = O(q_v^{l^\infty(\{f_i\}_i)}) \) the quantity on the right is polynomially bounded (uniformly in \( v \)) by \( d_Z^{-1} \).

To prove 3 we may assume that the norm on \( G_{\mathcal{X}} \) is induced by the \( l^\infty(\{f_i\}_i) \)-operator norm on \( \text{GL}(M_x) \). (If the homomorphism \( G \to \text{GL}(M) \) is not injective, then this \( l^\infty \) norm is bounded by some natural norm on \( G_{\mathcal{X}}, \) which is enough for the proof of 3.) Then for every \( x \in X_{\mathcal{X}} \) and \( g \in G_{\mathcal{X}} \) we have:

\[
\|g\|^{-1} \cdot d_{Z,\mathcal{X}}(x) \leq d_{Z,\mathcal{X}}(x \cdot g) \leq |g| \cdot d_{Z,\mathcal{X}}(x)
\]

(while we keep assuming that \( d_Z \) is defined by a basis for \( M \)).

We apply this to points \( x \in X^+(F) \). Notice that for every \( x \in X^+(k) \) we have \( f_i(x) \in k \) and \( \neq 0 \) for at least one \( i \), hence \( d_Z(x) = \prod_v \max_i |f_i(x)|_v \geq \max_i \prod_v |f_i(x)|_v = 1 \). Therefore, we have: \( d_{Z,\mathcal{X}}(x \cdot g) \geq |g|^{-1} \cdot d_{Z,\mathcal{X}}(x) \geq \|g\|^{-1} \).

\( \square \)

4.3 Spectral transform and \( X \)-Eisenstein series

By the pairing (4.2), every \( \Phi \in \mathcal{S}(X(\mathbb{A}_k)) \) defines a functional on \( \mathcal{S}(G(k) \backslash G(\mathbb{A}_k)) \). The basic conjecture of this paper will be that the spectral transform of this functional is meromorphic everywhere. However, even to define the spectral transform is a problem which involves an enormous amount of difficulties, tantamount to the spectral expansion of Arthur’s trace formula or the relative trace formula of Jacquet. Therefore, in this paper we will confine ourselves to defining the cuspidal spectral expansion. Notice, however, that there are a lot of interesting examples which have zero cuspidal contribution, e.g. \( X = \text{Sp}_{2n} \setminus \text{GL}_{2n} \).

By a cuspidal automorphic representation of \( G(\mathbb{A}_k) \) we will mean a pair \((\tau, \nu)\), where \( \tau \) is an (abstract) irreducible admissible representation of \( G(\mathbb{A}_k) \) and \( \nu : \tau \to C_{\text{cusp}}^\infty(G(k) \backslash G(\mathbb{A}_k)) \) an embedding with cuspidal image. We think of cuspidal representations as belonging to holomorphic families, namely \( \tau \) lives in the family of representations \( \tau \otimes \omega \), where \( \omega \) ranges over the complex group \( \Theta \) of idele class characters of \( G \) which are generated by powers of absolute values of algebraic characters, and \( \nu \) gives rise to embeddings \( \nu \otimes \omega \) of \( \tau \otimes \omega \), for all \( \omega \in \Theta \). In other words, the underlying vector spaces of all \( \tau \otimes \omega \) have been identified, and their images in \( C_{\text{cusp}}^\infty(G(k) \backslash G(\mathbb{A}_k)) \) coincide up to multiplication by elements in \( \Theta \), considered as functions on \( G(k) \backslash G(\mathbb{A}_k) \).

We will be denoting by \( Z^0 := Z(G)^0 \), the connected component of the center of \( G \), and by \( \check{X}_C \) the group of complex idele class characters of \( Z^0 \) which are trivial on the stabilizer of a generic point on \( X \). We think of \( \check{X}_C \) as a
countable union of complex varieties; two characters belong to the same connected component if their quotient can be written as a product of powers of absolute values of algebraic characters. From now on and throughout the rest of the paper we will make the following assumption, which we clearly may after dividing by a subgroup of $Z^0$:

- $X_{Z^0}(X) = X(Z^0)$.

Equivalently, $Z^0$ acts faithfully on $X$. Therefore, the group $\Omega$ of characters of $Z^0(\mathbb{A}_k)$ which is generated by powers of absolute values of algebraic characters can be identified with the identity component of $\hat{X}_C$. We have a map: $\Theta \to \Omega$, by restriction of the characters to $Z^0(\mathbb{A}_k)$.

Moreover, by enlarging $G$ by a subtorus of $\text{Aut}^G(X)$, if necessary, we may and will assume that:

- $\text{Aut}^G(X)^0$ is in the image of $Z^0$ in $\text{Aut}(X)$.

Now let $(\tau, nu)$ vary over a holomorphic family of cuspidal representations, and let $\tau \mapsto \phi_\tau \in \nu(\tau)$ be a meromorphic section of vectors. Let $\Phi \in \mathcal{S}(X(\mathbb{A}_k))$. Then we consider the integral:

$$\int_{G(k)\backslash G(\mathbb{A}_k)} \phi_\tau(g) \Psi(\Phi, g) dg.$$ 

(We use throughout Tamagawa measures on $G(k)\backslash G(\mathbb{A}_k)$.) This integral, if it converges, defines a pairing: $\tau \otimes \mathcal{S}(X(\mathbb{A}_k)) \to \mathbb{C}$. We will discuss later that, under our growth assumption 3.3.3, the integral converges for $\tau$ in a certain region. The main conjecture of this paper is:

4.3.1 Conjecture. If $(\tau, nu)$ is a holomorphic family of cuspidal representations and $\tau \mapsto \phi_\tau$ a meromorphic section, the integral:

$$\int_{G(k)\backslash G(\mathbb{A}_k)} \phi_\tau(g) \Psi(\Phi, g) dg$$

admits meromorphic continuation to all $\tau$.

We can rephrase this conjecture in terms of objects which coincide with Eisenstein series in case $X = U/G^{\text{aff}}$. Let $\omega$ vary over the torus $\hat{X}_C$ and define the $X$-Eisenstein series:

$$E(\Phi, \omega, g) = \int_{Z^0(\mathbb{A}_k)} \Psi(z \cdot \Phi, g) \omega(z) dz$$

whenever this integral is convergent (which, again, we will see is true for $\omega$ in a certain region of each connected component of $\hat{X}_C$).

Denote by $\mathcal{S}(G(k)\backslash G(\mathbb{A}_k))_{\text{cusp}, f}$ the elements of $\mathcal{S}(G(k)\backslash G(\mathbb{A}_k))$ which belong to a finite sum of cuspidal components of the automorphic spectrum of $G$. (Where a “component” is, spectrally, the $\Theta$-orbit of an irreducible cuspidal representation.)

Then it is easily seen that Conjecture 4.3.1 is equivalent to:
4.3.2 Conjecture. For every $\phi \in S(G(k) \backslash G(\mathbb{A}_k))_{\text{cusp}, f}$ the integral
\[
\int_{G(k) \backslash G(\mathbb{A}_k)} \phi(g) E(\Phi, \omega, g) dg
\] (4.5)
admits meromorphic continuation to all $\omega \in \hat{X}_\mathbb{C}$.

As noted above, this is a special case of what we expect in general, namely the meromorphic continuation of $E(\Phi, \omega, g)$, in a suitable sense.

Let us fill in the missing convergence details for those integrals, and justify the term “spectral decomposition”. The subgroup of $\hat{X}_\mathbb{C}$ consisting of unitary characters will be denoted simply by $\hat{X}_\mathbb{C}$. The real part $\Re(\omega)$ of $\omega \in \hat{X}_\mathbb{C}$ is a well-defined element of $X_0(k) \backslash G(\mathbb{A}_k)$. (Simply express $\omega$ as a unitary character times characters of the form $\eta_1^{s_1} \cdots \eta_r^{s_r}$, where the $\eta_i$’s are algebraic characters; then $\Re(\omega) = \sum \Re(s_i) \eta_i$.) We will say that the character $\omega$ is sufficiently $X$-positive if $\langle \Re(\omega), \hat{\lambda} \rangle$ is large enough for every non-zero $\hat{\lambda} \in X(Z_0^0)^*$ such that $\lim_{t \to 0} \hat{\lambda}(t) \in X$. For example, if $X = G_a$ under the $G_m$ action, this means that $\omega$ is sufficiently vanishing at the origin, while if $X$ is homogeneous affine then all characters are sufficiently $X$-positive.

4.3.3 Proposition. For sufficiently $X$-positive $\omega$, the integral defining $E(\Phi, \omega, g)$ converges, the function $E(\Phi, \omega, g)$ is of moderate growth in $g$ and for every $\phi \in S(G(k) \backslash G(\mathbb{A}_k))_{\text{cusp}, f}$ we have:
\[
P_X(\Phi, \phi) = \int_{\sigma \in X} \int_{G(k) \backslash G(\mathbb{A}_k)} \phi(g) E(\Phi, \omega, g) dg d\omega
\]
where $\sigma$ is any sufficiently $X$-positive element of $\hat{X}_\mathbb{C}$.

Proof. The statement about convergence follows easily using the structure of affine spherical varieties of Proposition 2.2.6, and moderate growth is obtained as in Proposition 4.2.1.

The expression for $P_X$ follows from abelian harmonic analysis on the group $Z_0^0(k) \backslash Z_0^0(\mathbb{A}_k)$. □

In the next section we will see how our $X$-Eisenstein series are connected to usual Eisenstein series when $X = [P, P] \backslash G$ or $X = U_P \backslash G$ (where $P$ is a proper parabolic and $U_P$ its unipotent radical). Afterwards, we will use this connection to prove that the conjecture is valid for smooth affine spherical varieties, and more generally whenever the spherical variety admits the structure of a “pre-flag bundle”. In those cases the integral (4.3) is, as we shall see, nothing else but a period integral or a Rankin-Selberg integral.

5 Connection to usual Eisenstein series

5.1 Certain stacks and sheaves related to flag varieties

Let $G$ be a split connected reductive group whose derived group is simply connected, and let $P = MU_P$ be a parabolic with its Levi decomposition. We
collect results from [BFGM02], notation being as in §3.2.2. The purpose of the present subsection is not to give complete definitions of the constructions of loc. cit., but to serve as a guide for the reader who would like to extract from it the parts most relevant to our present discussion. Our goal is the following statement about the basic function \( \Phi^0 \) (locally over a non-archimedean place, which we suppress from the notation):

5.1.1 **Theorem.** For \( X = \overline{H \setminus G} \) in each of the following cases, we have:

- If \( H = U_P \): \( \Phi^0 = \sum_{i \geq 0} q^{-i} \text{Sat}_M \left( \text{Sym}(\bar{u}_P) \right) \cdot 1_{H \setminus K} = \text{Sat}_M \left( \frac{1}{\Lambda^\text{top}(1 - q^{-1} \bar{u}_P)} \right) \cdot 1_{H \setminus K}. \) \( (5.1) \)

- If \( H = [P, P] \): \( \Phi^0 = \sum_{i \geq 0} q^{-i} \text{Sat}_{M^\text{ab}} \left( \text{Sym}(\bar{u}_{P_i}) \right) \cdot 1_{H \setminus K} = \text{Sat}_{M^\text{ab}} \left( \frac{1}{\Lambda^\text{top}(1 - q^{-1} \bar{u}_{P_i})} \right) \cdot 1_{H \setminus K}. \) \( (5.2) \)

The notation will be explained below.

We denote by \( \Lambda_G, P \) the lattice of cocharacters of the torus \( M/[M, M] \) and by \( \Lambda_G^{\text{pos}} \), the sub-semigroup spanned by the images of \( \Delta \setminus \Delta_M \). For every \( \theta \in \Lambda_G^{\text{pos}} \), we have a canonical locally closed embedding: \( j_\theta : C \times \text{Bun}_P \to \overline{\text{Bun}_P} \) [BFGM02, Proposition 1.5]. The image will be denoted by \( (\theta) \overline{\text{Bun}_P} \). (Notice: This is not the same as what is denoted in loc. cit. by \( \overline{\text{Bun}_P} \), but rather what is denoted by \( \overline{\text{Bun}_P} \), when \( \overline{\text{Bun}_P} \) is the trivial partition of \( \theta \).) Its preimage in \( \text{Bun}_P \) will be denoted by \( (\theta) \text{Bun}_P \). We have a canonical isomorphism: \( \text{Bun}_P \times \text{Bun}_M \cong (\theta) \text{Bun}_P \) or \( (\theta) \text{Bun}_M \), where \( (\theta) \text{Bun}_M \) is a stack which will be described below.

5.1.2 **Remarks.**

1. If \( X = [P, P]\setminus G \) under the \( M^\text{ab} = M/[M, M] \times G \)-action, then \( \Lambda_X^+ \) can be identified with \( \Lambda_G, P \), and \( (\theta) \text{Bun}_P \) is precisely the analog of what we denoted by \( Z^{w_\theta} \) on the Gaitsgory-Nadler stacks, where \( u_0 \) is the longest element in the Weyl group of \( G \). The reason that only \( \theta \in \Lambda_G^{\text{pos}} \) appear is that, as we remarked in §3.2, the quasi-maps on Drinfeld spaces are, by definition, not allowed to have poles. For the reader who would like to trace this back to the combinatorics of quasi-affine varieties and their affine closures of §2.2 we mention that the cone spanned by \( \rho(D) \) is the cone spanned by the images of \( \Delta \setminus \Delta_M \).

2. If \( X = \overline{U_P \setminus G} \) under the \( M \times G \)-action then \( \Lambda_X^+ \cong \{ \hat{\lambda} \in \Lambda_A | \langle \hat{\lambda}, \alpha \rangle \leq 0 \text{ for all } \alpha \in \Delta_M \} \) (where we denote by \( A \) the maximal torus of \( G \) and by \( \Lambda_A \) its cocharacter lattice). There is a map: \( \Lambda_X \to \Lambda_G, P \), and \( (\theta) \overline{\text{Bun}_P} \) corresponds to the union of the strata \( Z^{w_\lambda} \) of Gaitsgory-Nadler, with \( \lambda \) ranging over all the \( M \)-dominant preimages of \( \theta \).
We have the geometric Satake isomorphism, i.e. a functor $\text{Loc} : \text{Rep}(\tilde{G}) \rightarrow \text{Perv}(\mathcal{G}_G)$ such that the irreducible representation of $\tilde{G}$ with highest weight $\lambda$ goes to the intersection cohomology sheaf of a $G(\mathfrak{g})$-equivariant closed, finite-dimensional subscheme $\mathcal{G}^\lambda_G$. We will make use of this functor for $M$, rather than $\tilde{G}$. If $V$ is a representation of $\tilde{M}$ – assumed “positive”: this has to do with the fact that we don’t allow poles, but there’s no need to explain it here – and $\theta \in \Lambda^\text{pos}_{G,P}$ then we define $\text{Loc}(\theta)(V)$ to be $\text{Loc}(V_\theta)$, where $V_\theta$ is the $\theta$-isotypic component of $V$. (We ignore a twist by $\Theta_l[1] \left( \frac{1}{2} \right)^{-1}$ introduced in [BFGM02], and modify the results accordingly.)

We now introduce relative, global versions of the above spaces. We denote by $\mathcal{H}_M$ the Hecke stack of $M$. It is related to $\mathcal{G}_M$ as follows: If we fix a curve $C$ and a point $x \in C$ then, by definition, $\mathcal{G}_M$ is the functor $\text{Schemes} \rightarrow \text{Sets}$ which associates to every scheme $S$ the set of pairs $(\mathcal{F}_M, \beta)$ where $\mathcal{F}_M$ is a principal $M$-bundle over $C \times S$ and $\beta$ is an isomorphism of it outside of $(C \smallsetminus \{x\}) \times S$ with the trivial $M$-bundle. The relative version of this, as we allow the point $x$ to move over the curve, is denoted by $\mathcal{G}_{M,C}$, and the relative version of the latter, as we replace the trivial $M$-bundle with an arbitrary $M$-bundle, is $\mathcal{H}_M$. It is fibered over $C \times \text{Bun}_M$.

In loc.cit., p. 389, certain closed, finite-dimensional subschemes $\mathcal{G}^\theta_M$ of $\mathcal{G}_M$ are defined for every $\theta \in \Lambda^\text{pos}_{G,P}$ which at the level of reduced schemes are isomorphic to $\mathcal{G}^\theta_M$, where $\iota(\theta)$ is an $M$-dominant coweight associated to $\theta$ – the “least dominant” coweight mapping to $\theta$. The relative versions of those give rise to substacks $\mathcal{H}^\theta_M$ of $\mathcal{H}_M$.

For these relative versions we have: Functors $\text{Loc}_{\text{Bun}_M,C}$ (resp. $\text{Loc}_{\text{Bun}_M,C}^\theta$) from $\text{Rep}(\tilde{M})$ to perverse sheaves on $\mathcal{H}_M$ (resp. $\mathcal{H}_M^\theta$) and $\text{Loc}_{\text{Bun}_P,C}$ (resp. $\text{Loc}_{\text{Bun}_P,C}^\theta$) to perverse sheaves on $\text{Bun}_P \times \text{Bun}_M \times \mathcal{H}_M$ (resp. $\text{Bun}_P \times \text{Bun}_M \times \mathcal{H}_M^\theta$), the latter being $IC_{\text{Bun}_P}$ along the base $\text{Bun}_P$.

Then the main theorem of [BFGM02] (Theorem 1.12) is a description of the $\ast$-restriction of $IC_{\text{Bun}_P}$ to $\text{Bun}_P \simeq \text{Bun}_P \times \text{Bun}_M \mathcal{H}_M^\theta$. Moreover, Theorem 7.3 does the same thing for $IC_{\text{Bun}_P}$ and $\text{Bun}_P \simeq C \times \text{Bun}_P$. The normalization of $IC$ sheaves is that they are pure of weight 0; i.e. for a smooth variety $Y$ of dimension $n$ we have $IC_Y \simeq \left( \Theta_l \left( \frac{1}{2} \right)[1] \right)^{\otimes n}$, where $\Theta_l \left( \frac{1}{2} \right)$ is a fixed square root of $q$.

5.1.3 Theorem ([BFGM02], Theorems 1.12 and 7.3). The $\ast$-restriction of $IC_{\text{Bun}_P}$ to $\text{Bun}_P \simeq \text{Bun}_P \times \text{Bun}_M \mathcal{H}_M^\theta$ is equal to:

$$\text{Loc}_{\text{Bun}_P,C}^\theta \left( \oplus_{i \geq 0} \text{Sym}^i(\mathcal{U}_P) \otimes \Theta_l(i)[2i] \right).$$ (5.3)

The $\ast$-restriction of $IC_{\text{Bun}_P}$ to $\text{Bun}_P \simeq C \times \text{Bun}_P$ is equal to:

$$IC_{\ast,\text{Bun}_P} \simeq \text{Loc} \left( \oplus_{i \geq 0} \text{Sym}^i(\mathcal{U}_P) \otimes \Theta_l(i)[2i] \right).$$ (5.4)
Here $\tilde{u}_P$ denotes the adjoint representation of $\tilde{M}$ on the unipotent radical of the parabolic dual to $P$. Moreover, $\tilde{u}_P'$ denotes the subspace which is fixed under the nilpotent endomorphism $f$ of a principal $\mathfrak{sl}_2$-triple $(h,e,f)$ in the Lie algebra of $\tilde{M}$. For the definition of $\text{Loc}(V)$, which takes into account the grading on $V$ arising from the $h$-action, cf. loc.cit., §7.1.

5.2 The corresponding functions

Let us fix certain normalized Satake isomorphisms. As before, our local, non-archimedean field is denoted by $F$, its ring of integers by $\mathfrak{o}_F$, and our groups are assumed to have reductive models over $\mathfrak{o}_F$. As usual, we normalize the action of $M(F)$ (resp. $M_{ab}(F)$) on functions on $(H\backslash G)(F)$ where $H = U_P$ (resp. $[P,P]$) so that it is unitary on $L^2((H\backslash G)(F))$:

$$m \cdot f(H(F)g) = \delta_P^M(m) f(H(F)m^{-1}g), \quad (5.5)$$

where $\delta_P$ is the modular character of $P$. We let $M_0 = M(\mathfrak{o}_F)$, and normalize the (classical) Satake isomorphism as follows:

- For the Hecke algebra $\mathcal{H}(M, M_0)$ in the usual way:
  $$\text{Sat}_M : \mathbb{C}[M]^H \cong \mathbb{C}[\text{Rep}\tilde{M}] \xrightarrow{\sim} \mathcal{H}(M, M_0)$$
  where $\mathbb{C}[\text{Rep}\tilde{M}]$ is the Grothendieck algebra over $\mathbb{C}$ of the category of algebraic representations of $\tilde{M}$.

- For the Hecke algebra $\mathcal{H}(M_{ab}, M_{0ab})$ we shift the usual Satake isomorphism: $\mathcal{H}(M_{ab}, M_{0ab}) \cong \mathbb{C}[\mathcal{Z}(\tilde{M})] \cong \mathbb{C}[\text{Rep}\tilde{M}]$ by $e^{-\rho_M}$, where $\rho_M$ denotes half the sum of positive roots of $M$. In other words, if $h$ is a compactly supported measure on $M(F)/M_0$, considered (canonically) as a linear combination of cocharacters of $M_{ab}$ and hence as a regular function $f$ on the center $\mathcal{Z}(\tilde{M})$ of its dual group, then we will assign to $h$ the function $z \mapsto f(e^{\rho_M}z)$ on the subvariety $e^{-\rho_M}\mathcal{Z}(\tilde{M})$ of $G$:
  $$\text{Sat}_{M_{ab}} : \mathbb{C}[e^{-\rho_M}\mathcal{Z}(\tilde{M})] \xrightarrow{\sim} \mathcal{H}(M_{ab}, M_{0ab}).$$

Let $1_{HK}$ denote the characteristic function of $H\backslash HK$ (where $K = G(\mathfrak{o}_F)$), and consider the action map: $\mathcal{H}(M,M_0) \to C^c_c((U_P \backslash G)(F))^{M_0 \times K}$, respectively $\mathcal{H}(M_{ab}, M_{0ab}) \to C^c_c(([P,P]\backslash G)(F))^{K}$ given by $h \mapsto h \cdot 1_{HK}$. The map is bijective, and identifies the module $C^c_c((H\backslash G)(F))^{M_0 \times K}$ with $\mathbb{C}[M]^H$, resp. $\mathbb{C}[e^{-\rho_M}\mathcal{Z}(\tilde{M})]$. Our normalization of the Satake isomorphism is such that this is compatible with the Satake isomorphism for $G$, Sat$_G : \mathcal{H}(G,K) = \mathbb{C}[G]^G = \mathbb{C}[\text{Rep}(\tilde{G})]$, in the sense that for $f \in \mathbb{C}[G]^G$ we have:

$$\text{Sat}_G(f) \cdot 1_{HK} = \text{Sat}_M(f) \cdot 1_{HK}.\$$

Here and later, by the symbol $\tilde{h}$ we will be denoting the adjoint of the element $h$ in a Hecke algebra. Its appearance is due to the the definition (5.5) of the
action of $M$ as a right action on the space and a left action on functions. We extend the “Sat” notation to the fraction field of $\mathbb{C}[\text{Rep} M]$ (and, respectively, of $\mathbb{C}[e^{-\rho M} Z(M)]$), where $\text{Sat}_M$ or $M^{ab}(R)$ (with $R$ in the fraction field) is thought of as a power series in the Hecke algebra.

Returning to the Drinfeld spaces discussed in the previous subsection, let $\text{Ff}(E)(x) := \sum_i (-1)^i \text{tr}(\text{Fr}, H^i(E_x))$ denote the alternating sum of the trace of Frobenius acting on the homology of the stalks of a perverse sheaf (Ff stands for “faisceaux-fonctions”). As in §3.3, we fix an object $x_0$ on the basic stratum, a point $c_0 \in C$ (recall that in the definition of Drinfeld’s spaces, quasimaps do not have distinguished points) and we evaluate $\text{Ff}(E)$, where $E = IC_{\widetilde{\text{Bun}}_P}$ or $IC_{\widetilde{\text{Bun}}_P}$, only at objects $x_\lambda$ which are obtained by $M \times G$-Hecke modifications at $c_0$. This way, and using the Iwasawa decomposition, we obtain our basic function $\Phi^0$, which is an $M_0 \times K$-invariant function on $(H \backslash G)(F)$. Recall that it is by definition normalized such that $\Phi^0(H \backslash H1) = 1$.

The study of the Hecke correspondences in [BG02] implies that

$$\text{Ff}(\text{Loc}_{\text{Bun}_P, C}(V)) = \text{Sat}_M(V) \cdot \text{Ff}(\text{Loc}_{\text{Bun}_P, C}(1))$$

if $H = U_P$, and

$$\text{Ff}(\text{Loc}(V)) = \text{Sat}_{M^{ab}}(V) \cdot \text{Ff}(\text{Loc}(1))$$

if $H = [P, P]$.

5.2.1 Remark. The “unitary” normalization of the action of $M$ is already present in the sheaf-theoretic setting as follows: Suppose that an object $x_\lambda$ belongs to $(\lambda, \text{Bun})$ and can be obtained from $x_0$ via Hecke modifications at the distinguished object of $x_0$. Then the dimension of $(\lambda, \text{Bun}) \cong C \times \text{Bun}_P$ at $x_\lambda$ is $<\lambda, 2\rho_P>$ less than that of $(0, \text{Bun}) P_\lambda$ around $x_0$, where $\rho_P$ denotes the half-sum of roots in the unipotent radical of $P$, i.e. $\delta_P = e^{2\rho_P}$. Hence, by the aforementioned normalization of $IC$ sheaves, the contribution of the factor $IC_{(\lambda, \text{Bun}) P}$ (via Theorem 5.1.3) to $\Phi^0(\lambda)$ will be $q^{<\lambda, \rho_P>}$ times the contribution of the factor $IC_{(0, \text{Bun}) P}$ to $\Phi^0(0)$. Similarly for the strata of $\text{Bun}_P$.

Thus, Theorem 5.1.3 translates to the statement of Theorem 5.1.1:

- If $H = U_P$: $\Phi^0 = \sum_{i \geq 0} q^{-i} \text{Sat}_M \left( \text{Sym}^i(\tilde{u}_P) \right) \cdot 1_{HK} = \text{Sat}_M \left( \frac{1}{\lambda^{\text{top}}(1 - q^{-1}\tilde{u}_P)} \right) \cdot 1_{HK}$.

- If $H = [P, P]$; $\Phi^0 = \sum_{i \geq 0} q^{-i} \text{Sat}_{M^{ab}} \left( \text{Sym}^i(\tilde{u}_P^f) \right) \cdot 1_{HK} = \text{Sat}_{M^{ab}} \left( \frac{1}{\lambda^{\text{top}}(1 - q^{-1}\tilde{u}_P^f)} \right) \cdot 1_{HK}$.

Notice that in the last expression $\tilde{u}_P^f$ is considered as a representation of the maximal torus $\tilde{A}$ of $\tilde{M}$ determined by the principal $\mathfrak{sl}_2$-triple $(h, e, f)$ and, by restricting its character to the subvariety $e^{-\rho M} Z(M)$, as an element of $\mathcal{H}(M^{ab}, M_0^{ab})$. This is the case studied in [BK02], and $\Phi^0$ is the function denoted by $c_{P, 0}$ there.
5.3 Connection to Eisenstein series

Now we discuss our main conjecture when the variety is $X = \overline{U P \backslash G}^\text{aff}$ or $X = [P, P] \backslash G^\text{aff}$ under the (normalized) action of $M \times G$, resp. $M^\text{ab} \times G$. In the latter case, our Eisenstein series $E(\Phi, \omega, g)$ are the usual (degenerate, if $P$ is not the Borel) principal Eisenstein series normalized as in [BK98, BK02], and hence $E(\Phi, \omega, g)$ is indeed meromorphic for all $\omega$.

It will be useful to recall how these meromorphic sections are related to the more usual sections $E(f, \omega, g)$, which are defined in the same way but with $f \in C^c_{\text{cusp}}([P, P] \backslash G)(A_k)$. We assume that $\Phi = \prod_v \Phi_v$, $f = \prod_v f_v$ and $S$ is a finite set of places (including $S_0$) such that for $v \notin S$ we have $\Phi_v = \Phi_v^0$ and $f_v = f_v^0 := 1_{U \backslash G(a_v)}$. Let us also assume for simplicity that for $v \in S$ we have $\Phi_v = f_v$ (a finite number of places certainly do not affect meromorphicity properties). Clearly, for $E(\Phi, \omega, g)$ and $E(f, \omega, g)$ to be non-zero the character $\omega$ must be unramified outside of $S$. Then by the results of the previous paragraph we have:

$$E(\Phi, \omega, g) = L^S(e^{-\rho M} \omega, \hat{u}_p^f, 1)E(f, \omega, g) \quad (5.6)$$

where $L^S(e^{-\rho M} \omega, \hat{u}_p^f, 1)$ denotes the value at 1 of the partial (abelian) $L$-function corresponding to the representation $\hat{u}_p^f$, whose local factors (at each place $v$) are considered as rational functions on the maximal torus $A \subset M$ and evaluated at the point $e^{-\rho M} \omega_v \in e^{-\rho M} G(M) \subset A$.

Now let us consider the case $X = \overline{U P \backslash G}^\text{aff}$. We let $(\tau, nu)$ vary over a holomorphic family of cuspidal representations of $M \times G$ and let $\tau \mapsto \phi_\tau$ be a meromorphic section; write $\tau = \tau_1 \otimes \tau_2$ according to the decomposition of the group $M \times G$, and assume that, accordingly, $\phi_\tau = \phi_{\tau_1} \otimes \phi_{\tau_2}$, a pure tensor. Assume for the moment that the central character of $\tau$ is sufficiently $X$-positive. If we perform the integration of Conjecture 4.3.1, but only over the factor $M(k) \backslash M(A_k)$, then this integral can be written as:

$$\int_{M(k) \backslash M(A_k)} \phi_\tau(m, g) \Psi(\Phi, (m, g)) dm =$$

$$= \phi_{\tau_2}(g) \int_{M(k) \backslash M(A_k)} \phi_{\tau_1}(m) \Psi(\Phi, (m, g)) dm. \quad (5.7)$$

It is valued in the space of functions on $G(k) \backslash G(A_k)$. If $E : \mathcal{I}_{P(A_k)}^G(\tau_1) \to C^X(G(k) \backslash G(A_k))$ denotes the usual Eisenstein operator, then by unfolding the last integral we see that it is equal to the Eisenstein series:

$$E_M(\Phi, \phi_1, g) := \text{Eis} \left( \int_{M(A_k)} \phi_{\tau_1}(m \cdot \Phi) dm \right)(g) \quad (5.8)$$

hence the connection to usual Eisenstein series.

5.3.1 Proposition. Assume that the partial $L$-function $L^S(\tau_1, \hat{u}_p, 1)$ (for some large enough $S$) has meromorphic continuation to all $\tau_1$. Then the expression (5.7) admits meromorphic continuation to all $\tau_1$. [32]
Proof. By the meromorphic continuation of Eisenstein series, it is enough to show that the integral \((\Phi, \phi_{\tau_1}) \mapsto \int_{M(\mathbb{A}_k)} \phi_{\tau_1}(m) (m \cdot \Phi) \, dm\), which represents a morphism: \(\iota_{\tau_1} : S(U_P \backslash G(\mathbb{A}_k)) \to \Gamma_{P(\mathbb{A}_k)}(\tau_1)\), admits meromorphic continuation in \(\tau_1\). This would be the case if \(\Phi\) was in \(C^\infty_c(U_P \backslash G(\mathbb{A}_k))\). The analogous morphism: \(C^\infty_c(U_P \backslash G(\mathbb{A}_k)) \to \Gamma_{P(\mathbb{A}_k)}(\tau_1)\) will also be denoted by \(\iota_{\tau_1}\).

Again, we let \(S\) be a finite set of places containing \(S_0\) and take functions \(\Phi = \prod \Phi_v \in S(U_P \backslash G(\mathbb{A}_k))\) and \(f = \prod f_v \in C^\infty_c(U_P \backslash G(\mathbb{A}_k))\) such that for \(v \notin S\) \(\Phi_v = \Phi^0\) is the basic \(M_0 \times K\)-invariant function of the previous paragraph, \(f_v = f^0_v = 1_{U_P K}\) and for \(v \in S\) we have \(\Phi_v = f_v\) (for simplicity). Moreover, we assume that \(\tau_1\) is unramified for \(v \notin S\), otherwise the integral will be zero.

We saw previously that \(\Phi^0_v = \text{Sat}_M \left( \frac{1}{1 - \tau_v q_v} \cdot f^0_v \right)\). By definition of the Satake isomorphism and the equivariance of \(\iota_{\tau_1}\), in the domain of convergence we have \(\iota_{\tau_1}(\Phi) = L^S(\tau_1, \tilde{\mu}_P, 1) \iota_{\tau_1}(f)\).

Therefore \(\text{Eis}(\iota_{\tau_1}(\Phi)) = L^S(\tau_1, \tilde{\mu}_P, 1) \text{Eis}(\iota_{\tau_1}(f))\), and the claim follows from the meromorphic continuation of \(\text{Eis}(\iota_{\tau_1}(f))\).

5.3.2 Remarks. 1. The meromorphic continuation of \(L^S(\tau_1, \tilde{\mu}_P, 1)\) is known in many cases, e.g. for \(G\) a classical group and \(\tau\) generic, by the work of Langlands, Shahidi and Kim, cf. [CKM04].

2. Notice that, as was also observed in [BK98, BK02], the Eisenstein series (5.8) has normalized functional equations without \(L\)-factors.

6 Pre-flag bundles, period integrals and the Rankin-Selberg method

6.1 Pre-flag bundles

Let \(X\) be an affine spherical variety for \(G\) and let \(T\) denote the torus \(\text{Aut}^G(X)^0\). We will say that \(X\) has the structure of a pre-flag bundle if \(G\) has a direct product decomposition: \(G = L \times G'\) and \(X\) is the affine closure of an open subvariety \(X^+\) with the following properties:

1. \(\tilde{X}^+\) is homogeneous under a reductive subgroup \(\tilde{G} \subset \text{Aut}_{L \cdot T}(X)\) which contains \(G'\).

2. the group \(L\) acts freely on \(\tilde{X}^+\) and such that \(\tilde{X}^+ / LT\) is (\(\tilde{G}\)-equivariantly) proper over an affine \(\tilde{G}\)-homogeneous variety \(Y\), called the base of the pre-flag bundle.

Hence, we have the following geometry for \(\tilde{X}^+\):
\[ \begin{array}{c}
\tilde{X}^+ \\
\tilde{X}^+/LT \\
\text{fiber over } y \in Y \text{ is a flag variety for } \tilde{G}_y
\end{array} \]

\[ Y \quad (\simeq G'/G' \simeq \tilde{G}_y \tilde{G} \text{ with } G', \tilde{G}_y \text{ reductive}). \]

Necessarily, under the above, \( G' \) acts transitively on \( Y \) since \( \text{Aut}^G(X)^0 \) acts trivially on \( Y \) (cf. Proposition 2.2.6). Of course, \( L \) can be trivial or \( G \) can be equal to \( \tilde{G} \).

Notice that \( L \) is necessarily a quotient of a Levi subgroup of \( \tilde{G}_y \); namely, if the stabilizer in \( \tilde{G}_y \) of a point on the fiber of \( \tilde{X}^+/LT \) over \( y \in Y \) is the parabolic \( \tilde{P}_y \) then \( LT \) is a reductive quotient of \( \tilde{P}_y \). We will additionally impose the condition on all pre-flag varieties, without extra mentioning it, that if \( \tilde{G}_y \) is written as a product of simple groups \( \tilde{G}_y = \tilde{G}_y \times \tilde{G}_y \), then the kernel of \( (\tilde{P}_y)_i \to LT \) either contains the commutator subgroup of \( (\tilde{P}_y)_i \) or is equal to its unipotent radical times a central subgroup of its Levi. The definition and this extra condition are just suited, of course, for making use of the analytic continuation of the Eisenstein series that we discussed in \( \S 5.3 \).

The requirement that \( \tilde{G} \) commutes with the action of \( T \) is meant to allow us to relate \( X \)-Eisenstein series, which are formed by integrating against idèle class characters of \( T \), with usual Eisenstein series on \( \tilde{G}_y \) induced from \( \tilde{P}_y \).

6.1.1 Example. The variety \( \text{Mat}_n \) for \( \text{GL}_n \times \text{GL}_m \) \((n \geq 2)\) is a pre-flag variety, and more generally so is any \( N \)-dimensional vector space \((N = n^2)\) with a linear \( G \)-action, as it is equal to the affine closure of \( P_N \setminus \text{GL}_N \) (with \( P_N \) the mirabolic subgroup). Notice, however, that an \( n + m \)-dimensional vector space \((n, m \geq 2)\) can be considered as a pre-flag variety for both \( G = \text{GL}_n \times \text{GL}_m \) and \( \tilde{G} = \text{GL}_n \times \text{GL}_m \); which one we will choose will depend on which torus action we will consider.

The notions of a pre-flag variety and a pre-flag bundle are not very good, since they are not defined in terms of the group \( G \), but in terms of another group \( \tilde{G} \). From our point of view, whether a spherical variety is a pre-flag bundle or not is a matter of “chance” and in fact should be irrelevant as far as properties of \( X \)-Eisenstein series and their applications go – the fundamental object is just \( X \) as a \( G \)-variety, and not its structure of a pre-flag bundle. We will try to provide support for this point of view in \( \S 6.4 \). However, in absence of a proof of Conjecture 4.3.2, this is the only case where its validity can be proven, and the concept of pre-flag bundles is enough to explain a good part of the Rankin-Selberg method.

We now return to Conjecture 4.3.1. We remind that it has to do with the analytic continuation of the integral:

\[ \int_{G(k) \setminus G(\mathbb{A}_k)} \phi_x(g) \Psi(\Phi, g) dg \quad (6.1) \]
as \( \tau \) ranges over a holomorphic family of cuspidal representations and \( \tau \mapsto \phi_\tau \) is a meromorphic section.

6.1.2 Theorem. Assume that \( X \) has the structure of a a pre-flag bundle, and let \((\tau, nu)\) vary over a holomorphic family of cuspidal automorphic representations of \( G \). Additionally assume that if \( \tau = \tau_1 \otimes \tau_2 \) denotes the decomposition of \( \tau \) according to \( G = L \times G' \) then for some finite set of places \( S \) the partial L-function \( L^2(\tau_1, \tilde{u}_\rho, 1) \) has meromorphic continuation everywhere.

Then the integral (6.1) of Conjecture 4.3.1 is meromorphic everywhere.

Obviously, this theorem will be proven by appealing to the analogous properties of (usual) Eisenstein series, i.e. Proposition 5.3.1. However, it is not completely trivial as in some cases we have to use the theory of spherical varieties to show that as we “unfold” this integral certain summands vanish (in the language often used in the theory of Rankin-Selberg integrals: certain \( G \)-orbits on \( X \) are “negligible”). Before we prove the theorem in this generality, let us recall a few basic examples of integral representations of \( L \)-functions expressed in this language.

6.2 Period integrals

First consider the extreme case of a pre-flag bundle with trivial fibers: Namely \( X = H \backslash G \) with \( H \) reductive. Then (locally) \( S(X_v) = C^*_c(X_v) \) and any element \( \Phi_v \in S(X_v) \) is of the form \( \phi \cdot \delta_H \) where \( \phi \in \mathcal{H}(G_v) \); in other words, the map \( C^*_c(G_v) \rightarrow C^*_c(X_v) \) given by integration over \( H_v \) is surjective. Moreover, for almost every \( v \) the basic function \( \Phi_v^0 \) is equal to \( X(\alpha_v) \), which in turn is equal to \( 1_{K_v} \cdot \delta_{H_1} \). Let \( \phi \in S(G(k) \backslash G(\mathbb{A}_k))_{\text{cusp}} \), \( \omega \in X_C \) and consider the integral:

\[
\int_{G(k) \backslash G(\mathbb{A}_k)} \phi(g) E(\Phi, \omega, g) dg.
\]

If \( S \) is a finite set of places containing \( S_0 \) and such for \( v \) that outside of \( S \) the function \( \phi \) is \( K_v \)-invariant and \( \Phi_v = \Phi_v^0 \), and if for \( v \in S \) we write \( \Phi_v = h_v \cdot \delta_{H_{v, 1}} \) then the above integral can be written:

\[
\int_{(HZ^0)(k) \backslash (HZ^0)(\mathbb{A}_k)} \phi'(g) \omega(g) dg
\]

where \( \phi' = \prod_{v \in S_0} \hat{h}_v \cdot \phi \) is another cusp form and \( \hat{h}_v \) denotes the adjoint of \( h_v \).

(Conveniently, by our assumptions, \((H \backslash HZ^0)(F)) = (Z^0 \cap H \backslash Z^0)(F) = Z^0(F)\) for any completion \( F \) of \( k \), therefore \( \omega(g) \) makes sense on \((HZ^0)(\mathbb{A}_k))\). This is called a period integral, and such integrals have been studied extensively. Hence period integrals are the special case of our pairing (4.2) which is obtained from pre-flag bundles with trivial fibers (i.e. affine homogeneous spherical varieties).

For example, when \( X = \text{GL}_2 \), \( G = \mathbb{G}_m \times \text{GL}_2 \), with \( \mathbb{G}_m \) acting as a non-central torus of \( \text{GL}_2 \) by multiplication on the left, we get the period integral of Hecke (1.2), discussed in the introduction. All spherical period integrals are included in the lists of Knop and van Steirteghem which we will discuss in the next section.
6.3 The Rankin-Selberg method

According to [Bu05, §5], the Rankin-Selberg method involves a cusp form on $G$ and an Eisenstein series on a group $\tilde{G}$, where we have either an embedding: $G \hookrightarrow \tilde{G}$ or an embedding $\tilde{G} \hookrightarrow G$, or “something more complicated”. We certainly do not claim to explain all constructions which have been called “Rankin-Selberg integrals”, but let us see how a large part of this method is covered by our constructions.

Let $(\tau, nu)$ vary in a holomorphic family of irreducible cuspidal representations of $G$, and let $\tau \mapsto \phi_\tau \in \nu(\tau)$ be a meromorphic section. Let $X$ be an affine spherical variety for $G$ which has the structure of a pre-flag bundle with base $Y$, and choose a point $y \in Y(\mathcal{O}_{S_0})$. According to the decomposition $G / \mathcal{O} \rightarrow L$, we write $\tau = \tau_1 \otimes \tau_2$ and $\phi = \phi_1 \otimes \phi_2$ (without loss of generality we only consider pure tensors).

Let $\Phi \in \mathcal{S}(X(\mathbb{A}_k))$. Recall that the pseudo-Eisenstein series $\Psi(\Phi, g)$ has been defined via a sum over $X^+(k)$, where $X^+$ denotes the open $G$-orbit on $X$. On the other hand, to relate our integrals to usual Eisenstein series, we need to sum over $\tilde{X}^+(k)$, where $\tilde{X}$ is the open $\tilde{G}$-orbit. Hence, we define:

$$\tilde{\Psi}(\Phi, g) = \sum_{\gamma \in \tilde{X}^+(k)} \Phi(\gamma \cdot g).$$

We compare the integral of Conjecture 4.3.1 with the corresponding integral when $\Psi$ is substituted by $\tilde{\Psi}$:

6.3.1 Proposition. If $\phi$ is a cusp form on $G$ (with sufficiently $X$-positive central character, so that the following integrals converge) then:

$$\int_{G(k) \backslash G(\mathbb{A}_k)} \phi(g)\tilde{\Psi}(\Phi, g)dg = \int_{G(k) \backslash G(\mathbb{A}_k)} \phi(g)\tilde{\Psi}(\Phi, g)dg$$

(6.2)

Assume this proposition for now. Write the last integral as:

$$\int_{G'(k) \backslash G'(\mathbb{A}_k)} \phi_2(g') \int_{L(k) \backslash L(\mathbb{A}_k)} \phi_1(l)\tilde{\Psi}(\Phi, (l, g'))dl dg.$$

Using the reasoning of the previous subsection, we may substitute $\Phi$ by a Schwartz function on $\tilde{X}^+(\mathbb{A}_k)$, considered as a generalized function on $\tilde{X}^+(\mathbb{A}_k)$. Then the integral becomes:

$$\int_{G'(k) \backslash G'(\mathbb{A}_k)} \phi_2(g') \int_{L(k) \backslash L(\mathbb{A}_k)} \phi_1(l)\tilde{\Psi}(\Phi, (l, g'))dl dg.$$

The inner integral is equal to the Eisenstein series $E_{L}(\Phi, \phi_1, g')$ on the group $G'_y$, in the notation of (5.8), or a degenerate Eisenstein series as in (5.6), or a product of such, and it has meromorphic continuation under the assumption

$^8$Rankin-Selberg constructions with products of Eisenstein series have often been encountered in the literature, e.g. [BFG99, GH04].
that $L^S(\tau_1, \tilde{\mu}_P, 1)$ does. Hence, we see that the integral of conjecture 4.3.1 is equal to the Rankin-Selberg integral:

\[ \int_{G'_y(k)\backslash G'_y(\mathcal{A}_k)} \phi_2(g') E_L(\Phi, \phi_1, g') dg \quad (6.3) \]

and this also completes the proof of Theorem 6.1.2.

Let us see how this gives rise to the two extreme cases described as Rankin-Selberg integrals in [Bu05]: First, if the basis $Y$ is trivial then we have an embedding: $G' \hookrightarrow \tilde{G}_y = \tilde{G}$ and the integral (6.3) is a Rankin-Selberg integral for $G'$ formed with an Eisenstein series on the larger group $\tilde{G}$. Secondly, if the basis is non-trivial but $\tilde{G} = G'$ then the integral (6.3) is a Rankin-Selberg integral formed with an Eisenstein series on the smaller group $G'_y$.

6.3.2 Proof of Proposition 6.3.1: Negligible orbits.

Proposition 6.3.1 will follow from the following statement on the structure of certain spherical varieties:

6.3.3 Proposition. If $X$ is a wavefront spherical variety for $G$ with $\text{Aut}^G(X)$ finite, then the isotropy groups of all non-open $G$-orbits contain the unipotent radical of a proper parabolic of $G$.

From this, Proposition 6.3.1 follows easily; in the domain of convergence we have:

\[ \int_{G(k)\backslash G(\mathcal{A}_k)} \phi(g) \tilde{\Psi}(\Phi, g) = \sum_{\xi \in [\tilde{X}^+(k)\backslash G(k)]} \int_{G\xi(k)\backslash G(\mathcal{A}_k)} \phi(g) \Phi(\xi g) dg \]

where $[\tilde{X}^+(k)\backslash G(k)]$ denotes any set of representatives for the set of $G(k)$-orbits on $\tilde{X}^+(k)$. Notice that, by our assumptions on $X$, the $k$-points of the open $G$-orbit form a unique $G(k)$-orbit. The summand corresponding to $\xi$ can be written:

\[ \int_{G\xi(k)\backslash G(\mathcal{A}_k)} \Phi(\xi g) \int_{G\xi(k)\backslash G\xi(\mathcal{A}_k)} \phi(hg) dh dg \]

Since $\text{Aut}^G(\tilde{X}^+/T)$ is finite, for $\xi$ in the non-open orbit the stabilizer $G\xi$ contains the unipotent radical of a proper parabolic, and since $\phi$ is cuspidal the inner integral will vanish. Therefore, only the summand corresponding to the open orbit survives, which folds back to the integral:

\[ \int_{G(k)\backslash G(\mathcal{A}_k)} \phi(g) \tilde{\Psi}(\Phi, g). \]

Proposition 6.3.3, in turn, rests on the following result of Luna. A $G$-homogeneous variety $Y$ is said to be induced from a parabolic $P$ if it is of the form $Y' \times^P G$, where $Y'$ is a homogeneous spherical variety for the Levi quotient of $P$; equivalently, $Y = H\backslash G$, where $H \subset P$ contains the unipotent radical of $P$. 

37
6.3.4 Proposition. [Lu01, Proposition 3.4] A homogeneous spherical variety $Y$ for $G$ is induced from a parabolic $P$ (assumed opposite to a standard parabolic $P$) if and only if the union of $\Delta(Y)$ with the support\footnote{The support of a subset in the span of $\Delta$ is the smallest set of elements of $\Delta$ in the span of which it lies.} of the spherical roots of $Y$ is contained in the set of simple roots of the Levi subgroup of $P$.

Proof of Proposition 6.3.3. For every $G$-orbit $Y$ in a spherical variety $X$ there is a simple toroidal variety $\tilde{X}$ with a morphism $\tilde{X} \to X$ which is birational and whose image contains $Y$. Therefore, it suffices to assume that $X$ is a simple toroidal variety.

Moreover, if $\tilde{X}$ denotes the wonderful compactification of $X^+$ (i.e. the simple toroidal compactification with $C(X) = V$) then every simple toroidal variety $X$ admits a morphism $X \to \tilde{X}$ which, again, is birational and has the property that every non-open $G$-orbit on $X$ goes to a non-open $G$-orbit in $\tilde{X}$. Indeed, any non-open $G$-orbit $Y \subset X$ corresponds to a non-trivial face of $C(X)$, and its character group $\mathcal{X}(Y)$ is the orthogonal complement of that face in $\mathcal{X}(X)$, which is of lower rank than $\mathcal{X}(X)$, therefore $Y$ has to map to an orbit of lower rank. Moreover, $Y$ is a torus bundle over its image. This reduces the problem to the case where $X$ is a wonderful variety, which we will now assume.

By Proposition 6.3.4, it suffices to show that the union of $\Delta(X)$ and the support of the spherical roots of $Y$ is not the whole set $\Delta$ of simple roots. The spherical roots of $Y$ are a proper subset of the spherical roots of $X$, and $\Delta(Y) = \Delta(X)$. It therefore suffices to prove that for any proper subset $\Theta \subset \Delta_X$ there exists a simple root $\alpha \in \Delta \setminus \Delta(X)$ such that $\alpha$ is not contained in the support of $\Theta$.

Denote $a^\ast := \mathcal{X}(A)^\ast \otimes \mathbb{Q}$, $a^\ast_{f(\gamma)} = (\Delta(X))^\perp \subset a^\ast$, and consider the canonical quotient map: $q : a \to \mathbb{Q}$. Denote by $f_\gamma \subset a^\ast$ the anti-dominant Weyl chamber in $a$. Every set of spherical roots $s \subset \Delta_X$ corresponds to a face $V_s \subset V = \mathcal{V}_\gamma \subset \mathbb{Q}$ (more precisely, $V_s$ is the face spanning the orthogonal complement of $s$), and similarly every set $r \subset \Delta$ of simple roots of $G$ corresponds to a face $f_r \subset f_\gamma$. The simple roots in the support of $\gamma \in \Delta_X$ are those corresponding to the largest face $f_\gamma$ which is contained in $q^{-1}(V_{\gamma+1})$. Notice that the maximal vector subspace $f_{\Delta}$ of $f_\gamma$ maps into the maximal vector subspace $V_{\Delta_X}$ of $V$.

By assumption, $f_\gamma$ surjects onto $V$. Moreover, since every element of $f_\gamma$ can be written as a sum of an element in $f_{\Delta(X)}$ and a non-negative linear combination of $\Delta(X) := \{\alpha \in \Delta(X)\}$, and since $\Delta(X)$ is in the kernel of $a \to \mathbb{Q}$, it follows that $f_{\Delta(X)}$ surjects onto $V$. Now let $\Theta \subset \Delta_X$ be a proper subset. Let $f_s$ be a face of $f_{\Delta(X)}$ which surjects onto $V_{\Theta}$. Since $f_s \neq f_{\Delta}$, there is an $\alpha \in \Delta \setminus \Delta(X)$ which is not in the support of $\Theta$.\qed

6.4 Tensor product $L$-functions of $GL_2$ cusp forms

In section 4 we proposed a general conjecture involving distributions which are obtained from the geometry of an affine spherical variety $X$, and in this section we saw how this conjecture is true, and gives rise to period- and Rankin-Selberg
integrals, in the case that $X$ admits the structure of a “pre-flag bundle”. It was written above that such a structure should be considered essentially irrelevant and a matter of “chance”. We now wish to provide some evidence for this point of view by recalling the known constructions of $n$-fold tensor product $L$-functions for $GL_2$, where $n \leq 3$. The point is that while these constructions seem completely different from the point of view of Rankin-Selberg integrals, from the point of view of spherical varieties they are completely analogous!

Before we consider the specific example, let us become a bit more precise about what it means that a period integral is related to some $L$-value. Assume that $G$ is semisimple. Let $\pi = \otimes \pi_v$ be an (abstract) unitary representation of $G(\mathbb{A}_k)$, the tensor product of unitary irreducible representations $\pi_v$ of $G(k_v)$ with respect to distinguished unramified vectors $u_v^0$ (for almost every place $v$) of norm 1. Let $P$ be a functional on $\pi$. In our applications the functional $P$ will arise as the composition of a cuspidal automorphic embedding $\nu : \pi \to L^2_{\text{cusp}}(G(k)\backslash G(\mathbb{A}_k))$, assumed unitary, with a period integral or, more generally, the pairing (4.2) with a fixed pseudo-$X$-Eisenstein series. Let $\rho$ be a representation of the dual group, and let $L(\pi, \rho, s)$ denote the value of the corresponding $L$-function at the point $s$. We say that $P$ is related to $L(\pi, \rho, s)$ if there exist non-zero skew-symmetric forms: $\Lambda_v : \pi_v \otimes \overline{\pi}_v \to \mathbb{C}$ for every $v$ such that for any large enough set of places $S$, and for a vector $u = \otimes_{v \not\in S} u_v^0 \otimes_{v \in S} u_v$, one has: $|P(u)|^2 = \prod_{v \not\in S} \Lambda^2(u_v, \overline{u}_v)$. (Of course, for this to happen we must have $\Lambda_v(u_v^0, \overline{u}_v^0) = L_v(\pi_v, \rho_v, s)$.) Moreover, it is required that each $\Lambda_v$ has a definition which has no reference to any other representation but $\pi_v$. The reader will notice that the last condition does not stand the test of mathematical rigor; however, not imposing it would make the rest of the statement void up to whether $P$ is zero or not. In practice, the $\Lambda_v$’s will be given by reference to some non-arithmetic model for $\pi_v$. See [I] for a precise conjecture in a specific case, and [SV] for a more general but less precise conjecture.\(^{10}\)

We can generalize this in an obvious way to reductive groups. The only additional point here is that we would like to allow for representations which are unitary only up to twisting by a character, in which case we will replace $|P(u)|^2$ by $P(u) \cdot \overline{P}(\tilde{u})$, where $\tilde{P}$ denotes another (given) functional on $\tilde{\pi}$, and the skew-symmetric forms $\Lambda_v$ by bilinear forms: $\pi_v \otimes \overline{\pi}_v \to \mathbb{C}$.

6.4.1 Example. The Whittaker period: $\phi \mapsto \int_{U(k)\backslash U(\mathbb{A}_k)} \phi(u) \psi^{-1}(u) du$ (where $\psi$ is a generic idele class character of the maximal unipotent subgroup) on cusp forms for $G = GL_n$ is related to the $L$-value:

$$\frac{1}{L(\pi, \text{Ad}, 1)}$$

cf. [Ja01]. Notice that the examples which we are about to discuss admit “Whittaker unfolding” and this factor will enter, although most references introduce a different normalization and ignore this factor.

\(^{10}\)For the sake of completeness, we should mention that when $P$ comes from a period integral one should in general modify the above conjecture by some “mild” arithmetic factors, such as the sizes of centralizers of Langlands parameters – see [I]. However, in the example we are about to discuss there is no such issue since the group is $GL_2$. 39
Now we are ready to discuss our example: Let $n$ be a positive integer, $G = (\text{GL}_2)^n \rtimes G_m$, and let $H$ be the subgroup:

$$\left\{ \left( \begin{array}{cc} a & x_1 \\ 1 & 1 \end{array} \right), \left( \begin{array}{cc} a & x_2 \\ 1 & 1 \end{array} \right), \ldots, \left( \begin{array}{cc} a & x_n \\ 1 & 1 \end{array} \right) \right\},$$

where $x_1 + x_2 + \cdots + x_n = 0$.

We let $X = \overline{H \setminus G}^{\mathrm{aff}}$. As usual, we normalize the action of $G$ on functions on $X^+$ so that it is unitary with respect to the natural measure. Let us see that for $n = 1, 2, 3$ the variety $X$ admits the structure of a pre-flag bundle, and hence the integral of Conjecture 4.3.2 can be interpreted as a Rankin-Selberg integral, as discussed above:

- **$n = 1$.** Here $\overline{H \setminus G}^{\mathrm{aff}} = H \setminus G$ and we get the integral (1.2) of Hecke. If $\tau_s = \tau \otimes |\cdot|^s$, where $\tau$ is a cuspidal representation of $\text{GL}_2$ (for simplicity: with trivial central character), the corresponding linear functional on $\tau_s \otimes \overline{\tau}_s$ is related to the $L$-value:

$$L(\tau, \frac{1}{2} + s)L(\overline{\tau}, \frac{1}{2} - s)$$

$$L(\tau, \text{Ad}, 1).$$

- **$n = 2$.** Here the projection of $H \to \text{GL}_2$ is conjugate to the mirabolic subgroup of $\text{GL}_2$ embedded diagonally. Therefore, the affine closure of $H \setminus G$ is equal to the bundle over $\text{GL}_2^{\text{diag}} \setminus (\text{GL}_2)^2$ with fiber equal to the affine closure of $U_2 \setminus \text{GL}_2$, where $U_2$ denotes a maximal unipotent subgroup of $\text{GL}_2$. Corresponding to this pre-flag bundle is a Rankin-Selberg integral “with the Eisenstein series on the smaller group” $\text{GL}_2^{\text{diag}}$, namely the classical integral of Rankin and Selberg. If $\tau = \tau_1 \otimes \tau_2 \otimes |\cdot|^s$ is a cuspidal automorphic representation of $G$ (for simplicity: with trivial central character), the corresponding integral is related to the $L$-value:

$$L(\tau_1 \otimes \tau_2, \frac{1}{2} + s)L(\overline{\tau}_1 \otimes \overline{\tau}_2, \frac{1}{2} - s)$$

$$L(\tau, \text{Ad}, 1).$$

- **$n = 3$.** In this case there is a structure of a pre-flag variety not on $X$, but on $X^0$: the corresponding spherical variety for the subgroup $G^0 = \{(g_1, g_2, g_3, a) \in G \mid \det(g_1) = \det(g_2) = \det(g_3)\}$. The structure of a pre-flag variety involves the group $\tilde{G} = \text{GSp}_6$ and the subgroup $\tilde{H} = [\tilde{P}, \tilde{P}]$, where $\tilde{P}$ is the Siegel parabolic – this is a construction of Garrett [Ga87]. The group $(\text{GL}_2^3)^0$ is embedded in $\text{GSp}_6$ as $(\text{GSp}_2^3)^0$. Then, according to [PSR87, Corollary 1 to Lemma 1.1] the group $G^0$ has an open orbit in $[\tilde{P}, \tilde{P}] \setminus \tilde{G}$ with stabilizer equal to $H$. We claim:

**6.4.2 Lemma.** The affine closure $X^0$ of $H \setminus G^0$ is equal to the affine closure of $[\tilde{P}, \tilde{P}] \setminus \tilde{G}$.

**Proof.** Denote by $Y$ the affine closure of $[\tilde{P}, \tilde{P}] \setminus \tilde{G}$. We have an open embedding: $X^0 \hookrightarrow Y$. By [PSR87, Lemma 1.1], all non-open $G$-orbits have codimension at least two. Therefore, the embedding is an isomorphism. \[\square\]
Hence, our integral for $X^0$ coincides with the Rankin-Selberg integral of Garrett. The only thing that remains to do is to compare the normalizations for the sections of Eisenstein series. From [PSR87, Theorem 3.1] one sees that our integral is related to the $L$-value:

$$
\frac{L(\tau_1 \otimes \tau_2 \otimes \tau_3, 1 + s) L(\tilde{\tau}_1 \otimes \tilde{\tau}_2 \otimes \tilde{\tau}_3, \frac{1}{2} - s)}{L(\tau, \text{Ad}, 1)}
$$

(Again, for simplicity, we assume trivial central characters. Notice that the zeta factors in [PSR87, Theorem 3.1] disappear because of the correct normalization of the Eisenstein series!)

It is completely natural to expect the corresponding integral for $n = 4$ or higher to be related to the $n$-fold tensor product $L$-function. It becomes obvious from the above example that the point of view of the spherical variety is the natural setting for such integrals, while at the same time the structure of a pre-flag bundle may not exist and, even if it exists, it has a completely different form in each case which conceals the uniformity of the construction.

7 Smooth affine spherical varieties

Given that we do not know how to prove Conjecture 4.3.2, except in the cases of pre-flag bundles, it is natural to ask the purely algebro-geometric question: Which are the spherical varieties admit the structure of a pre-flag bundle? An answer to this question would amount to a complete classification of Rankin-Selberg integrals, in the slightly restrictive sense that “Rankin-Selberg” has been used here. Such an answer has been given in the special case of smooth affine spherical varieties: These varieties automatically have the structure of a pre-flag bundle, and they have been classified by Knop and Van Steirteghem [KS06], hence can be used to produce Eulerian integrals of automorphic forms! There seems to be little point in computing every single example in the tables of [KS06], and my examination of most of the cases has not produced any striking new examples. However, we get some of the best-known integral constructions, as well as some new ones (which do not produce any interesting new $L$-functions).

7.1 Smooth affine spherical triples

By a corollary to Luna’s étale slice theorem [Lu73], every smooth affine spherical variety of $G$ (over an algebraically closed field in characteristic zero) is of the form $V \times_H G$, where $H$ is a reductive subgroup (so that $H \setminus G$ is affine) and $V$ is an $H$-module. In other words, all smooth affine spherical varieties are pre-flag bundles: The corresponding integrals include all period integrals over reductive subgroups, as well as Rankin-Selberg integrals involving mirabolic Eisenstein series (i.e. those induced from the mirabolic subgroup of $\text{GL}_n$).

In [KS06], Knop and Van Steirteghem classify all smooth affine spherical triples $(\mathfrak{g}, \mathfrak{h}, V)$, which amounts to a classification of smooth affine spherical
varieties up to coverings, central tori and $\mathbb{G}_m$-fibrations. We recall their definitions:

7.1.1 Definition. 1. Let $\mathfrak{h} \subset \mathfrak{g}$ be semisimple Lie algebras and let $V$ be a representation of $\mathfrak{h}$. For $\mathfrak{s}$, a Cartan subalgebra of the centralizer $\mathfrak{c}_g(\mathfrak{h})$ of $\mathfrak{h}$, put $\tilde{\mathfrak{h}} := \mathfrak{h} \oplus \mathfrak{s}$, a maximal central extension of $\mathfrak{h}$ in $\mathfrak{g}$. Let $\mathfrak{z}$ be a Cartan subalgebra of $\mathfrak{gl}(V)$ (the centralizer of $\mathfrak{h}$ in $\mathfrak{gl}(V)$). We call $(\mathfrak{g}, \mathfrak{h}, V)$ a spherical triple if there exists a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a vector $v \in V$ such that

(a) $\mathfrak{b} + \tilde{\mathfrak{h}} = \mathfrak{g}$ and
(b) $[(\mathfrak{b} \cap \tilde{\mathfrak{h}}) + \mathfrak{z}]v = V$ where $\mathfrak{s}$ acts via any homomorphism $\mathfrak{s} \to \mathfrak{z}$ on $V$.

2. Two triples $(\mathfrak{g}_1, \mathfrak{h}_1, V_1), i = 1, 2$, are isomorphic if there exist isomorphisms of Lie algebras resp. vector spaces $\alpha : \mathfrak{g}_1 \to \mathfrak{g}_2$ and $\beta : V_1 \to V_2$ such that

(a) $\alpha(\mathfrak{h}_1) = \mathfrak{h}_2$
(b) $\beta(\xi v) = \alpha(\xi)\beta(v)$ for all $\xi \in \mathfrak{h}_1$ and $v \in V_1$.

3. Triples of the form $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h}_1 \oplus \mathfrak{h}_2, V_1 \oplus V_2)$ with $(\mathfrak{g}_i, \mathfrak{h}_i, V_i) \neq (0, 0, 0)$ are called decomposable.

4. Triples of the form $(\mathfrak{f}, \mathfrak{f}, 0)$ and $(0, 0, V)$ are said to be trivial. A pair $(\mathfrak{g}, \mathfrak{h})$ of semisimple Lie algebras is called spherical if $(\mathfrak{g}, \mathfrak{h}, 0)$ is a spherical triple.

5. A spherical triple (or pair) is primitive if it is non-trivial and indecomposable.

Clearly, every smooth affine spherical variety gives rise to a spherical triple. Conversely, each spherical triple is obtained from a (not necessarily unique) smooth affine spherical variety, as follows by an a posteriori inspection of all spherical triples. (The non-obvious step here is that the $\mathfrak{h}$-module $V$ integrates to an $H$-module, where $H$ is the corresponding subgroup.)

The classification of all primitive spherical triples is given in [KS06], Tables 1, 2, 4 and 5, modulo the inference rules described in Table 3. The diagrams are read in the following way: The nodes in the first row correspond to the simple direct summands $\mathfrak{g}_i$ of $\mathfrak{g}$, the ones in the second row to the simple direct summands $\mathfrak{h}_i$ of $\mathfrak{h}$ and the ones in the third row to the simple direct summands $V_i$ of $V$. If $(\mathfrak{g}, \mathfrak{h})$ contains a direct summand of the form $(\mathfrak{h}_1, \mathfrak{h}_1)$ then the $\mathfrak{h}_1$ summand is omitted from the first row. There is an edge between $\mathfrak{g}_i$ and $\mathfrak{h}_j$ if $\mathfrak{h}_j \hookrightarrow \mathfrak{g}_i$ is non-zero and an edge between $\mathfrak{h}_j$ and $V_k$ if $V_k$ is a non-trivial $\mathfrak{h}_j$-module. The edges are labeled to describe the inclusion of $\mathfrak{h}$ in $\mathfrak{g}$, resp. the action of $\mathfrak{h}$ on $V$; the labels are omitted when those are the “natural” ones.

We number the cases appearing in the list of Knop and Van Steirteghem as follows: First, according to the table on which they appear (Tables 1, 2, 4, 5 in [KS06]); and for each table, numbered from left to right, top to bottom.
7.2 Eulerian integrals arising from smooth affine varieties

In what follows we will discuss a sample of the global integrals obtained from varieties in the list of Knop and Van Steirteghem. We allow ourselves to choose the spherical variety corresponding to a given spherical triple as is most convenient, and in fact we sometimes replace semisimple groups by reductive ones. Of course, the classification in [KS06] is over an algebraically closed field, which leaves a lot of freedom for choosing the precise form of the spherical variety over $k$, even when $G$ is split. In the discussion which follows we will always take both the group and generic stabilizer to be split. Many of the varieties of Knop and Van Steirteghem will not satisfy the conditions of §2.1, or will have zero cuspidal contribution. Still, this list contains some of the best-known examples of integral representations of $L$-functions. It contains also some new ones. At this point it is more convenient to not normalize the action of $G$.

In subsection §6.4 we explained what it means for a period integral $P$ to be “related to” an $L$-value, namely by considering the value of $|\mathcal{P}(\phi)|^2$, where $\phi$ is an automorphic form in the space of the given automorphic representation. To simplify our expressions here we modify our language by considering just $\mathcal{P}(\phi)$; in this sense, the Whittaker period on a cuspidal automorphic representation in $GL_n$ is “related to” $L(\pi, \text{Ad}, 1)^{-\frac{1}{2}}$.

### 7.2.1 Table 1

In this table the group $H$ is equal to $G$, i.e. the data consists of a group and a spherical representation of it. This table contains the following interesting integrals (numbered according to their occurrence in the tables of Knop and Van Steirteghem):

1. **The integrals of Godement and Jacquet.** Here the group is $GL_n \times GL_m$ with the tensor product representation (i.e. on $\text{Mat}_{n \times m}$). It is easy to see that if $m \neq n$ then the stabilizer is parabolically induced, hence the only interesting case (as far as cusp forms are concerned) is $m = n$. In this case, our integral (4.3) is that of Godement and Jacquet:

$$
\int_{Z^{\text{int}}(A_k) GL_n^{\text{diag}}(k) \backslash GL_n(A_k) \times GL_n(A_k)} \phi_1(g_1) \phi_2(g_2) \Phi(g_1^{-1} g_2) \cdot \det(g_1^{-1} g_2)^{i} d(g_1, g_2).
$$

15. **Two new integrals.** (Here there is a choice between the first and the last fundamental representation of $GL_n$. It can easily be seen that they amount to the same integral, so we will consider only $\omega_1$.)

The group is $GL_m \times GL_n$ and the representation is the direct sum $\text{Mat}_{m \times n}$ with the standard representation for $GL_n$. If $m \neq n, n - 1$ then we can easily see that the stabilizer is parabolically induced. Hence there are two interesting cases:
(i) \( m = n \). We let \( \phi_1 \in \pi_1, \phi_2 \in \pi_2 \) be two cusp forms on \( \text{GL}_n \). Then the integral is:

\[
\int_{\text{PGSp}(k) \times \text{GL}_n(n)} \phi_1(g_1)\phi_2(g_2)\Phi(g_1^{-1}g_2)\Phi'([0, \ldots, 0, 1] \cdot g_1) \quad \cdot |\det(g_1^{-1}g_2)|^{s_1} |\det(g_1)|^{s_2} dg_1 dg_2.
\]

Here \( \Phi \) is a Schwartz function on \( \text{Mat}_n(\mathbb{A}_k) \) and \( \Phi' \) is a Schwartz function on \( \mathbb{A}_k^n \).

**7.2.2 Theorem.** The above integral is Eulerian and related to the \( L \)-value:

\[
\frac{L(\pi_2 \otimes \pi_2, s_2) \cdot L(\pi_2, s_1 - \frac{1}{2}(n - 1))}{\sqrt{L(\pi_1, \text{Ad}, 1)L(\pi_2, \text{Ad}, 1)}}.
\]

(7.1)

**Proof.** It follows from the standard “unfolding” technique that the above integral, in the domain of convergence, is equal to:

\[
\int_{(U_n(\mathbb{A}_k) \setminus \text{GL}_n(\mathbb{A}_k))^2} W_1(g_1)W_2'(g_2)\Phi(g_1^{-1}g_2)\Phi'([0, \ldots, 0, 1] \cdot g_1) \quad \cdot |\det(g_1^{-1}g_2)|^{s_1} |\det(g_1)|^{s_2} dg_1 dg_2
\]

where \( W_1(g) = \int_{U_n(k)U_n(\mathbb{A}_k)} \phi_1(ug)\psi(u)du \) and \( W_2' \) the same with \( \phi_1 \) replaced by \( \phi_2 \) and \( \psi \) replaced by \( \psi^{-1} \).

The last integral is (for “factorizable data”) a product of local factors:

\[
\int_{(U_n(k_v) \setminus \text{GL}_n(k_v))^2} W_{1,v}(g_1)W_{2,v}'(g_2)\Phi_v(g_1^{-1}g_2)\Phi_v'([0, \ldots, 0, 1] : g_1) \quad \cdot |\det(g_1^{-1}g_2)|^{s_1} |\det(g_1)|^{s_2} dg_1 dg_2.
\]

Assume that \( \Phi_v = \Phi_v^0 \), the basic function of \( \mathcal{S}(\text{Mat}_n(\mathbb{k}_v)) \). Considering the action of the spherical Hecke algebra of \( G_2 \) (the second copy of \( \text{GL}_n \)) on \( \mathcal{S}(\text{Mat}_n(\mathbb{k}_v)) \), the work of Godement and Jacquet [GJ72, Lemma 6.10] proves:

\[
\Phi_v^0(x)|\det(x)|^{s_1} = \text{Sat}_{G_2} \left( \frac{1}{\lambda \cdot \left( 1 - q_v^{-s_1 + s_2} \cdot \text{std} \right)} \right) \cdot 1_{\text{GL}_n(\mathbb{k}_v)}.
\]

(7.2)

Therefore for unramified data the last integral is equal to:

\[
\frac{L(\pi_2, s_1 - \frac{1}{2}(n - 1)) \cdot \int_{(U_n(k_v) \setminus \text{GL}_n(k_v))^2} W_{1,v}(g_1)W_{2,v}'(g_2) \quad \cdot 1_{\text{GL}_n(\mathbb{k}_v)}(g_1^{-1}g_2)\Phi_v'([0, \ldots, 0, 1] \cdot g_1) |\det(g_1^{-1}g_2)|^{s_1} |\det(g_1)|^{s_2} dg_1 dg_2 =}
\]
This table contains the following interesting integrals:

1. The Bump-Friedberg integral. This group is GL_{m+n} where m = n or n+1, the subgroup H is GL_m × GL_n and the representation is the standard representation of the second factor. This is the integral examined in BF90:

\[
\int_{\text{GL}_m(k) \times \text{GL}_n(k) \backslash \text{GL}_{m+n}(k)} \Phi \left( \begin{pmatrix} g_1 & \ 0 \\
 & g_2 \end{pmatrix} \right) |\det(g_1)|^{s_1} |\det(g_2)|^{s_2} dg_1 dg_2
\]

It is related to the L-value:

\[
\frac{L(\pi, s_1 + \frac{1}{2})L(\pi, \wedge^2, s_2)}{\sqrt{L(\pi, Ad, 1)}}.
\]

(ii) m = n − 1. Notice that if V denotes the standard representation of GL_n then the space Mat_{(n-1)×n} ⊗ V can be identified under the G_1 × G_2 := GL_{n-1} × GL_n-action with the space X = Mat_n, where g ∈ G_1 acts as \( \begin{pmatrix} g^{-1} & \ 0 \\
 & 1 \end{pmatrix} \) on the left. Let \( \phi_1 \in \pi_1 \) be a cusp form on GL_{n-1} and \( \phi_2 \in \pi_2 \) a cusp form in GL_n. Then the integral is:

\[
\int_{\text{GL}^{\text{ad}}_n(k) \backslash \text{GL}_{n+1}(A_k) \times \text{GL}_n(A_k)} \phi_1(g_1) \phi_2(g_2) \cdot \Phi \left( \begin{pmatrix} g_1^{-1} & \ 0 \\
 & g_2 \end{pmatrix} \right) |\det(g_2)|^{s_1} |\det(g_1)|^{s_2} dg_1 dg_2
\]

where \( \Psi \in S(\text{Mat}_n(A_k)) \).

As before, one can prove:

**7.2.3 Theorem.** The above integral is Eulerian and related to the L-value:

\[
\frac{L(\pi_1 \otimes \pi_2, s_2 + \frac{1}{2}) \cdot L(\pi_1, s_1 - \frac{1}{2} n)}{\sqrt{L(\pi_1, Ad, 1)L(\pi_2, Ad, 1)}}.
\] (7.3)

### 7.2.4 Table 2

In this table H is smaller than G and the representation V of H is non-trivial. This table contains the following interesting integrals:

1. The Bump-Friedberg integral.
3. A new integral. The group is $GL_{m+1} \times GL_n$, and $G' = GL_m \times GL_n$ with the tensor product of the standard representations (i.e. on Mat_{m\times n}). The only interesting case is $m = n$. If $n > m$ then the stabilizer is parabolically induced, and when $m > n$ it unfolds to a parabolically induced model.

If $m = n$ we get:

\[
\int_{GL^{\text{diag}}(k) \backslash GL_n(A_k) \times GL_n(A_k)} \phi_1 \left( \begin{array}{cc} g_1 & 1 \\ 1 & 1 \end{array} \right) \phi_2(g_2) \Phi(g_1^{-1} g_2) \cdot \left| \frac{\det(g_2)}{\det(g_1)} \right|^{n_1} \left| \det(g_1) \right|^{s_2} dg_1 \cdot dg_2.
\]

As before, one can prove:

7.2.5 Theorem. The above integral is Eulerian and related to the $L$-value:

\[
\frac{L(\pi_1 \otimes \pi_2, s_1 + \frac{1}{2}) \cdot L(\pi_2, s_1 - \frac{1}{2}(n - 1))}{\sqrt{L(\pi_1, \text{Ad}, 1)L(\pi_2, \text{Ad}, 1)}}.
\]  

(7.4)

5. The classical Rankin-Selberg integral. The group is GL_n \times GL_n and the subgroup G' is GL_n^{\text{diag}} with the standard representation. This is the classical Rankin-Selberg integral:

\[
\int_{GL_n(k) \backslash GL_n(A_k)} \phi_1(g) \phi_2(g) \Phi([0, \ldots, 0, 1] \cdot g) \cdot \det g|^{s_2} dg.
\]

It is related to the $L$-value (cf. [Co03]):

\[
\frac{L(\pi_1 \otimes \pi_2, s)}{\sqrt{L(\pi_1, \text{Ad}, 1)L(\pi_2, \text{Ad}, 1)}}.
\]

7.2.6 Tables 4 and 5

Here the representation $V$ is trivial, hence we get period integrals over reductive algebraic subgroups. All known cases of multiplicity-free period integrals are contained in these tables, but there is no point in discussing them here, since I have not examined any new ones.

8 A remark on the relative trace formula

At this point we drop our assumptions of §2.1 on the groups and the spherical varieties, in order to be able to discuss non-split examples, as well as examples where $G(k)$ does not surject onto the points $X(k)$ of a homogeneous variety. The definitions of corresponding Schwartz spaces are easy to carry over in this context, and in any case these issues are not the main focus of the discussion that follows.
The relative trace formula is a method which was devised by Jacquet and his collaborators to study period integrals of automorphic forms. In its most simplistic form, it can be described as follows: Let $H_1$ and $H_2$ be two reductive spherical subgroups of $G$ (a reductive group defined over a global field $k$) and let $f \in C_c^\infty(G(\mathbb{A}_k))$. Then one builds the usual kernel function: $K_f(x, y) = \sum_{\gamma \in G(k)} f(x^{-1}\gamma y)$ for the action of $f$ on the space of automorphic functions and (ignoring analytic difficulties) defines the functional:

$$RTF_{H_1,H_2}^G(f) = \int_{H_1(k) \backslash H_1(\mathbb{A}_k)} \int_{H_2(k) \backslash H_2(\mathbb{A}_k)} K_f(h_1, h_2) dh_1 dh_2.$$ \hspace{1cm} (8.1)

The functional can be decomposed in two ways, one geometric and one spectral, and the spectral expansion involves period integrals of automorphic forms. By comparing two such RTFs (i.e. made with different choices of $H_1$, $H_2$, maybe even different groups $G$) one can deduce properties of those period integrals, such as that their non-vanishing characterizes certain functorial lifts.

The above presentation is too simplistic for several reasons: First, the correct functional has something to do with the varieties $H_i \backslash G$, rather than the spaces $H_i(k) \backslash G(k)$ – therefore, if $G(k)$ does not surject onto $H_i(k)$ one should take the sum of the above expressions over stabilizers $H_{i,e}$ of a set of representatives of $G(k)$-orbits. (This will become clearer in a reformulation which we will present below.) Even then, one may need to take a further summation of relative trace formulae over inner forms of $G$ – this phenomenon also has an explanation, but we will ignore it here. Moreover, one can consider an idele class character $\eta$ of $H_i$ and integrate against this character; we will adjust our notation accordingly, for instance: $RTF_{H_1,H_2,\eta}^G$. There are often analytic difficulties in making sense of the above integrals. And one does not have to restrict to reductive subgroups, but can consider parabolically induced subgroups together with a character on their unipotent radical (such as in the Whittaker period). However, we will ignore most of these issues and focus on another one, first noticed in [JLR93]: It seems that in certain cases, in order for the relative trace formula $RTF_{H_1,H_2}^G$ to be comparable to some other relative trace formula, the functional (8.1) is not the correct one and one has to add a “weight factor” in the definition, such as:

$$RTF_{H_1,H_2}^G(f) = \int_{H_1(k) \backslash H_1(\mathbb{A}_k)} \int_{H_2(k) \backslash H_2(\mathbb{A}_k)} K_f(h_1, h_2) \theta(h_1) dh_1 dh_2$$ \hspace{1cm} (8.2)

where $\theta$ is a suitable automorphic form on $H_1$.

Our goal here is to explain how, under the point of view developed in the present paper, the above expression is not a relative trace formula for $H_1$, $H_2$ but represents a relative trace formula for some other subgroups. We will discuss this in the context of [JLR93], though our starting point will not be (8.2) but another formula of [JLR93] from which the identities for (8.2) are derived, and which is closer to our point of view.

More precisely, let $E/F$ be a quadratic extension of number fields with corresponding idele class character $\eta$, $G = \text{Res}_{E/F} \text{PGL}_2$, $G' = \text{PGL}_2 \times \text{PGL}_2$ (over
$F$), $H \subset G$ the projectivization of the quasi-split unitary group (which is in fact split, i.e. isomorphic to $\text{PGL}_2$ over $F$), $H' = \text{the diagonal copy of PGL}_2$ in $G'$. (Compared to [JLR93], we restrict to $\text{PGL}_2$ for simplicity.) We consider $\eta$ as a character of $H$ in the natural way. Naively, one would like to compare the functional: $\text{RTF}^{G,H}_H$ to the functional $\text{RTF}^{G',H'}$ (usual trace formula for $G'$). However, it turns out that the correct comparison is between the functionals:

$$f \mapsto \int_{(H(k) \backslash H(\mathbb{A}_k))^2} K_f(h_1, h_2) E(h_1, s) \eta(h_1) dh_1 dh_2 \quad (8.3)$$

on $G$ and

$$f' \mapsto \int_{(H'(k) \backslash H'(\mathbb{A}_k))^2} K_{f'}(h'_1, h'_2) E'(h'_1, s) dh'_1 dh'_2 \quad (8.4)$$

on $G'$, where $E, E'$ are suitable Eisenstein series on $H, H'$. (More precisely, in the first case one takes the sum over the unitary groups of all $G(k)$-conjugacy classes of non-degenerate hermitian forms for $E/F$, as we mentioned above, but only in the second variable.)

Notice that we have already made a modification to the formulation of [JLR93], namely in the second case they let $G' = \text{PGL}_2$ and consider the integral:

$$\int_{\text{PGL}_2(k) \backslash \text{PGL}_2(\mathbb{A}_k)} K_{f'}(x, x) E'(x, s) dx,$$

but this is easily seen to be equivalent to our present formulation.

**Claim.** The functionals $(8.3), (8.4)$ can naturally be understood as pairings:

$$\text{RTF}^{G_m \times G, \omega}_{X_1, X_2} : \mathcal{S}(X_1(\mathbb{A}_k)) \otimes \mathcal{S}(X_2(\mathbb{A}_k)) \to \mathbb{C}$$

respectively:

$$\text{RTF}^{G_m \times G', \omega'}_{X'_1, X'_2} : \mathcal{S}(X'_1(\mathbb{A}_k)) \otimes \mathcal{S}(X'_2(\mathbb{A}_k)) \to \mathbb{C}$$

where: $X_2 = H \backslash G$, $X'_2 = H' \backslash G'$ and $X_1, X'_1$ are the affine closures of the varieties:

$$U_F \backslash G$$

respectively:

$$U'_F \backslash G'$$

where $U_F, U'_F$ are maximal unipotent subgroups of $H$ resp. $H'$.

The varieties $X_1, X'_1$ are considered here as spherical varieties under $G_m \times G$ (resp. $G_m \times G'$), where $G_m = B_2/U_2$, and we extend the $G_m$-action to the varieties $X_2, X'_2$ in the trivial way. The exponent $\omega$ in $\text{RTF}^{G_m \times G, \omega}_{X_1, X_2}$ will be explained below.

Before we explain the claim, let us go back to the simpler formula $(8.1)$ and explain how it can be considered as a pairing between $\mathcal{S}(X_1(\mathbb{A}_k))$ and $\mathcal{S}(X_2(\mathbb{A}_k))$ (where $X_i = H_i \backslash G_i$). Here we will identify Hecke algebras with spaces of functions, by choosing Haar measures. Assume that $f = f_1 * f_2$ with
Then we set: \( \Phi_1(g) = \int_{H_i(k \backslash A_k)} f_i(hg) \, dh \). By the definition of \( S(X_i(A_k)) \) when \( H_i \) is reductive, it follows that \( \Phi_1 \in S(X_i(A_k)) \). (It is at this point that one should add over representatives for \( G_i(k) \)-orbits on \( X_i(k) \), since in general the map \( C^G_\psi(G(A_k)) \rightarrow S(X_i(A_k)) \) is not surjective.) The functional \( RTF_{H_1, H_2}^G(f_1 \ast f_2) \) clearly does not depend on \( f_1, f_2 \) but only on \( \Phi_1, \Phi_2 \). Hence, it defines a \( G^{\text{diag}} \)-invariant functional:

\[
S(X_1(A_k)) \otimes S(X_2(A_k)) \rightarrow \mathbb{C}.
\]

Now let us return to the setting of the Claim, and of equations (8.3), (8.4). The product \( E(h_1, s) \eta(h_1) \) in (8.3) will be considered as an Eisenstein series on \( H(k) \backslash H(A_k) \). We have seen that suitable sections of Eisenstein series can be obtained from integrating pseudo-Eisenstein series \( \Psi^m_{U_2} \) where \( \Phi \in S(U_2 \backslash H(A_k)) \) against a character \( \omega \) of \( G_m \). Now consider \( \Phi \in S(U_2 \backslash H(A_k)) \) as a generalized function on \( U_2 \backslash H(A_k) \). Assume again that \( f = \Phi \ast \Phi \in C^G_\psi(G(A_k)) \).

Then \( \Phi_1 = f_1 \cdot \Phi \in S(U_2 \backslash H(A_k)) \) and \( \Phi_2(g) = \int_{H_i(A_k)} f(hg) \, dg \in S(H \backslash G(A_k)) \).

Again, of course, we must take many \( f \)'s and sum over representatives for \( G(k) \) on \( X_2(k) \), incidentally, our point of view explains why there is no need to sum over representatives for orbits in the first variable: because \( G(k) \) surjects on \( X_1(k) \)!

Similarly, one can explain (8.4) as a pairing between \( S(X'_1(A_k)) \otimes S(X'_2(A_k)) \), and this completes the explanation of our Claim. We have introduced the exponents \( \omega, \omega' \) in the notation, because we have already integrated against the corresponding character of \( G_m \) in order to form Eisenstein series.

Notice that this point of view is very close to the geometric interpretation of the fundamental lemma which led to its proof by Ngô [Ngo] in the case of the Arthur-Selberg trace formula. I hope that this point of view will lead to a more systematic study of the relative trace formula – at least by alleviating the impression created by weight factors that it is something “less canonical” than the Arthur-Selberg trace formula.

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52