Limits on classical communication from quantum entropy power inequalities

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Almost all modern communication systems rely on electromagnetic fields. The additive white Gaussian noise (AWGN) channel is often a good approximate description of such a system, and its information-carrying capacity is given by a simple formula. The quantum analogue of AWGN channels, the bosonic Gaussian noise channel, accurately describes many quantum optical communication systems of interest. Estimating its capacity is significantly more difficult; although some simple coding strategies are known, whether or not more sophisticated techniques could dramatically improve communication rates has been unknown. Here, we present strong new upper bounds for the classical capacity of bosonic Gaussian noise channels. These results imply that known coding techniques are typically close to optimal. Our main technical tool is an entropy power inequality bounding the entropy produced as two quantum signals combine at a beamsplitter. Its proof relies on a quantum diffusion process which smooths arbitrary states towards Gaussians. 

\[ C(\mathcal{E}_{A_{N_2}}, N_1) \geq \left[ g(N_1 (1 - \lambda) N_2) - g((1 - \lambda) N_2) \right] / \ln 2 \]  

(1)

where \( g(x) = (x + 1) \ln(x + 1) - x \ln x \) (refs 6,7) and the capacity is measured in bits per channel use. This communication rate is achievable with a simple classical modulation scheme of displaced coherent states8, and exceeding it would require entangled modulation schemes. Our goal is to explore the usefulness of such novel, fundamentally quantum, strategies. We find tight bounds on any possible strategy for exceeding equation (1) for a wide range of parameters (Fig. 3). We show that such strategies are essentially useless for \( \lambda = 1/2 \). Overall, we find, for a wide range of practical channels, that good old classical modulation of coherent states cannot be improved upon substantially by using quantum tricks.

\[ e^{2H(X+Y)} \geq e^{2H(X)} + e^{2H(Y)} \]  

(2)

Figure 1 | Two independent quantum signals combined at a beamsplitter. Both X and Y are n-mode systems with quadratures \( R_i = (Q_i^x, P_i^x, \ldots, Q_i^n, P_i^n) \) and \( R_i = (Q_i^y, P_i^y, \ldots, Q_i^n, P_i^n) \). The output Z, which we denote \( X||Y \), has quadratures \( R_i = \sqrt{\lambda} R_i^x + \sqrt{1 - \lambda} R_i^y \), while the quadratures of W are \( R_i^w = \sqrt{\lambda} R_i^x - \sqrt{1 - \lambda} R_i^y \). Our main technical result is a proof that no matter what product state is prepared on X and Y, the beamsplitter always increases entropy: \( S(Z) \geq S(X) + (1 - \lambda) S(Y) \). For \( \lambda = 1/2 \), we prove the stronger constraint, equation (5). These fundamental inequalities are the natural quantum generalization of the two-classically equivalent entropy power inequalities, equations (2) and (3), and lead to strong new upper bounds on the classical communication capacity of additive bosonic channels.
controlled the entropy production as two statistically independent signals are combined. Shannon’s arguments were incomplete, but a full proof of the EPI was given by Stam⁹ and Blachman¹⁰. Generalizations of the EPI have been found, and recently there has been renewed interest in streamlining their proofs⁶,¹¹,¹². EPIs are a fundamental tool in information theory, crucial for bounding capacities of noisy channels in various scenarios¹³,¹⁴. Although equation (2) is the most commonly cited form, there are several equivalent statements¹⁵. The following formulation will be most convenient:

\[
H(\sqrt{\lambda}X + \sqrt{1 - \lambda}Y) \geq \lambda H(X) + (1 - \lambda)H(Y) \quad \text{for} \quad \lambda \in [0, 1]
\]  
(3)

A single mode of an electromagnetic field can be described in terms of its field quadratures, P and Q. When independent modes X and Y with quadratures (Qₓ, Pₓ) and (Qᵧ, Pᵧ) are combined at a beamsplitter of transmissivity λ (Fig. 1), the signal in one output mode is given by \(\sqrt{\lambda}Qₓ + \sqrt{1 - \lambda}Qᵧ, \sqrt{\lambda}Pₓ + \sqrt{1 - \lambda}Pᵧ\), a process that we denote \(X \boxplus Y\). Our main result is a quantum analogue of equation (3) adapted to this setting, namely,

\[
S(X \boxplus Y) \geq \lambda S(X) + (1 - \lambda)S(Y)
\]  
(4)

for independent states on X and Y. This inequality applies unchanged when X and Y are n-mode systems. Here, \(S(X) = -\text{tr} \rho X \ln \rho_X\) is the von Neumann entropy of the state of system X, \(\rho_X\), with \(S(Y)\) and \(S(X \boxplus Y)\) defined similarly. While equations (2) and (3) are classically equivalent, the analogous quantum inequalities do not seem to be. So, in addition to equation (4), we also prove a quantum analogue of equation (2), valid for beamsplitters of transmissivity 1/2:

\[
e^{\frac{1}{2}S(X \boxplus Y)} \geq \frac{1}{2}e^{\frac{1}{2}S(X)} + \frac{1}{2}e^{\frac{1}{2}S(Y)}
\]  
(5)

Quantum mechanics on infinite dimensional spaces has many mathematical subtleties, particularly when it comes to differentiability and the interchange of limits. These difficulties usually arise on pathological examples that are not physically relevant. As a result, it is standard to simply assume, when necessary, the required regularity to take derivatives, expand a density matrix in a basis of functions, and so on (see, for example, refs 6,16,17), and we follow this convention here. In a similar fashion, smoothness requirements are rarely considered in proofs of the classical EPI¹², and although the proof of the classical EPI is generally attributed to Stam⁷ and Blachman¹⁶, a full consideration justifying the interchange of derivatives and integrals seems to have first been given in ref. 18. Note,
the late time limit, all three states approach the same thermal state, and so satisfy equation (4) with equality. The convexity of Fisher information can be

tions\(^2\) gives the upper bound

\[
E_{\infty}(\lambda E) \leq \frac{1}{\ln 2} \ln \left( \frac{\lambda N + 1}{(1 - \lambda)N_r + 1} \right)
\]

However, their implications for the classical capacity of additive quantum channels.

Applications to classical capacity

Before proving our entropy power inequalities, we consider their implications for the classical capacity of a thermal noise channel with average photon number \(N_E\) and transmissivity \(\lambda\), \(E_{\lambda N_r}\) (Fig. 2). Equation (1) is the best-known achievable rate for classical communication over this channel with average signal photon number \(N\) (ref. 6). In general, the capacity exceeds the Holevo information, which corresponds to an enhanced communication capability from entangled signal states\(^6\). However, for the pure loss channel, \(E_{\lambda 0}\), the bound is tight, giving capacity

\[
C(E_{\lambda 0}, N) \leq g(\lambda N)/(1 - \lambda)N_r + 1)
\]

Although equation (6) follows from an elementary argument (Supplementary Information), to the best of our knowledge it is new.

Closely related to capacity, the minimum output entropy is a measure of a channel's noisiness\(^{21,22}\). Indeed, the classical capacity of any channel \(E\) satisfies

\[
C(E, N) \leq \left[ S_{\max}(E, N) - \lim_{\lambda \to \infty} \frac{1}{\ln 2} S_{\max}(E^{\otimes n}) \right] \frac{1}{\ln 2} \quad (7)
\]

where \(S_{\max}(E, N) = \max_{H(H) = 2N + 1} S(E(H))\) is the maximum output entropy with photon number constraint \(H = (1/2)[P^2 + Q^2]\) is the harmonic oscillator Hamiltonian and \((H - 1)/2\) is the number operator), \(S_{\max}(E) = \min_r S(E(r))\) is the minimum output entropy, and \(E^{\otimes n}\) is the \(n\)-fold tensor product representing \(n\) parallel uses of the channel. The difficulty in applying this upper bound is the infinite limit in the second term, which prevents us from evaluating the right-hand side. However, for additive noise channels our EPIs give lower bounds on \(S_{\min}(E^{\otimes n})\), allowing simple upper bounds on the capacity. Our work differs crucially from earlier studies in that we prove an additive lower bound on \(S_{\min}(E^{\otimes n})\), giving a single-letter expression that can be calculated explicitly. (The earlier work of refs. 17,23 and references therein use techniques that are fundamentally single-use and thus give no information on the channel capacity.) From equation (4) we thus find that the capacity of a thermal noise channel with environment photon number \(N_E\), and signal photon number \(N\) satisfies

\[
C(E_{\lambda N_r}, N) \leq \left[ g(\lambda N + (1 - \lambda)N_r) \right] \frac{1}{\ln 2} \quad (8)
\]
while for \( \lambda = 1/2 \) equation (5) implies the stronger

\[
C(E_{1/2,N_2}; N) \leq \left[ \frac{1}{2} (N + N_2) \right] - \ln(1 + e^{\delta(N_2)}) \right] \frac{1}{\ln 2} + 1 \quad (9)
\]

This bound differs from the Holevo lower bound by no more than 0.06 bits (Fig. 3).

**Divergence-based quantum Fisher information**

Fisher information is a key tool in the proof of the classical EPI\(^9,10\); however, there is no unique quantum Fisher information\(^\text{24}\). We introduce a particular quantum Fisher information defined in terms of the quantum divergence, \( S(p|x;\sigma) = H(p) - I(p;\sigma) \). Given a smooth family of states \( \rho_p \), we define the divergence-based quantum Fisher information as the second derivative of divergence along the path:

\[
j_f(p_\theta; \theta)|_{\theta = 0} = d^2 S(\rho_\theta \| \rho_p)|_{\theta = 0} \quad (10)
\]

This is non-negative \( (j_f(p_\theta; \theta) \geq 0) \), additive \( (j_f(p_\theta \otimes p'_\theta; \theta) = j_f(p_\theta; \theta) + j_f(p'_\theta; \theta)) \) and satisfies data processing \( (j_f(E(p_\theta); \theta) \leq j_f(p_\theta; \theta)) \) for any physical map \( E \). It also satisfies the reparametrization formulas \( j_f(p_\theta; \theta)|_{\theta = 0} = c^2 j_f(p_\theta; \theta)|_{\theta = 0} \) and \( j_f(p_\theta \otimes p_\theta; \theta)|_{\theta = 0} = j_f(p_\theta; \theta) \) if \( \rho \) is separable (see ref. 25).

**Quantum diffusion**

Fisher information appears in the classical EPI proof because of its relation to the entropy production rate under the addition of Gaussian noise via the de Bruijin identity,

\[
\frac{dH(X + \sqrt{t}Z)}{dt}|_{t=0} = \frac{1}{2} f(X) \quad (11)
\]

Here, \( X \) is an arbitrary variable, \( Z \) is an independent normal variable with unit variance, and \( f(X) \) is the classical Fisher information of the ensemble \( \{X + \theta : \theta \in \mathbb{R}\} \). The variable \( X + \sqrt{t}Z \) arises from a diffusion with initial state \( X \) running for time \( t \).

To explain our quantum de Bruijin identity, we must first discuss quantum diffusion processes. A quantum Markov process is associated with a Liouvillean \( L(\rho) \) and governed by a Markovian master equation

\[
\frac{d\rho}{dt} = L(\rho) \quad (12)
\]

Our process of interest has \( L(\rho) = -(1/4) \sum [R_i, [R_i, \rho]] \) and corresponds to adding Gaussian noise in phase space\(^25\) (see Fig. 1 for definitions of the quadratures \( R_i \)). We denote the action of running this process for time \( t \) on initial state \( \rho_0 \) by \( e^{tL}(\rho_0) \). We want to relate the entropy production rate of our quantum diffusion to a Fisher information, but what ensemble should we use? We choose \( 2n \) separate ensembles of states,

\[
\rho_{0i}^j = D_{R_i}(\theta) \rho_0 D_{R_i}^\dagger(\theta) \quad (13)
\]

where \( D_{R_i} \) is a displacement operator along the \( R_i \) axis in phase space. We then find for sufficiently smooth trajectory \( \rho_t \) that

\[
\frac{dS(e^{tL}(\rho_0))}{dt}|_{t=0} = \sum_{i=1}^{2n} j_f(\rho_0^i; \theta) := j(\rho_0) \quad (14)
\]

**Proof of quantum entropy power inequality**

Our path to the quantum entropy power inequality combines the quantum de Bruijin identity, equation (14), with a convexity property of the quantum Fisher information. In particular, we require that the Fisher information of the output of a beamsplitter satisfy

\[
j_f(\rho^Y \otimes \rho^Y) \leq \lambda j_f(\rho^X) + (1 - \lambda) j_f(\rho^Y) \quad (15)
\]

The proof of this relation relies on the elementary properties of \( j \), and follows the analogous classical proof\(^27\).

Roughly speaking, equation (4) is proven by subjecting inputs \( p_X \) and \( p_Y \) to a quantum diffusion for time \( t \). As \( t \to \infty \), both initial states approach a thermal state with average photon number \( (t - 1)/2 \), as does the combination of the two states at a beamsplitter. Because both inputs, as well as the beamsplitter’s output, approach the same state in the limit, equation (4) is satisfied with equality. We then use equation (6) together with the quantum de Bruijin identity to show that any violation of equation (4) would be amplified as \( t \) grows. Because in the limit \( t \to \infty \) the violation is zero, we conclude that no such violation exists. This argument also applies to multi-mode systems, so equation (4) is true for these too.

The proof of equation (5) is similar in spirit to our proof of equation (4) and Blachman’s proof\(^20\) of equation (2). Rather than convexity, we use a quantum version of Stam’s inequality:

\[
\frac{1}{2} \left( \frac{1}{f(p^X)} + \frac{1}{f(p^Y)} \right) \geq \frac{1}{2} \left( \frac{1}{f(p^X \otimes p^Y)} \right) \quad (16)
\]

and consider a ratio rather than a difference (see Methods or ref. 25 for more details).

**Discussion and outlook**

Some authors have hoped that the lower bound of equation (1) be equal to the capacity\(^23,28\). There is evidence both for\(^7,23,28,29\) and against\(^30\) this conjecture. It has been related to an ‘entropy photon-number inequality’ which, if true, would imply this equivalence, but despite concerted effort no proof has been found. Our quantum EPIs more closely resemble the classical inequalities than does the proposed inequality of ref. 23, often allowing us to rely on classical proof strategies.

We expect our results to find a variety of applications to bosonic systems. The analysis of classical network models like broadcast\(^11\) and interference channels\(^31\) relies on EPIs, so network quantum information theory is a good place to start\(^32\). Quantum EPIs may also find applications in the development of non-commutative central limit theorems\(^33,34\).

There are many potential generalizations for our inequalities. For example, one could follow Costa\(^35\) and show that \( \exp(1/n) S(e^{tL}(\rho)) \) is concave as a function of \( t \). Foremost, however, is proving the analogue of equation (5) for \( \lambda \neq 1/2 \). One would hope that

\[
e^{\lambda SX + 1 - \lambda SY} \geq \lambda e^{\lambda SX} + (1 - \lambda) e^{\lambda SY} \quad (17)
\]

but we have not yet found a proof. Such a result would give bounds on the capacity of the thermal and classical noise channels to within 0.16 bits, answering the capacity question for all practical purposes.

**Methods**

**Details of proof of equation (4).** We would like to show that, given input states \( \rho_X \) and \( \rho_Y \),

\[
S(\rho_X \otimes \rho_Y) \geq S(\rho_X) + (1 - \lambda) S(\rho_Y) \quad (18)
\]

where \( B(\rho_X \otimes \rho_Y) \) denotes the map from inputs to outputs of a beamsplitter with transmissivity \( \lambda \). To do so, we let

\[
\mu(t) = S(e^{tL}(\rho_X \otimes \rho_Y)) - S(e^{tL}(\rho_X)) - (1 - \lambda) S(e^{tL}(\rho_Y)) \quad (19)
\]

be the difference between the two sides of the desired inequality. Because as \( t \to \infty \) all states involved approach a Gaussian state with photon number \( (t - 1)/2 \), one expects that \( \lim_{t \to \infty} \mu(t) = 0 \), and indeed this is the case\(^36\). Furthermore, using the

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quantum de Bruijn identity to differentiate, we find
\begin{equation}
\dot{s}(t) = \frac{1}{2} \left( \frac{d}{dt} \left( \mathcal{L}(B_s(p_0 \otimes p_t)) \right) \right) - \lambda \left( \mathcal{L}(Z) \right)
\end{equation}
(20)

Finally, using the fact that
\begin{equation}
\mathcal{L}(B_s(p_0 \otimes p_t)) = B_s(\mathcal{L}(p_0) \otimes \mathcal{L}(p_t))
\end{equation}
(21)

we find that
\begin{equation}
\dot{s}(t) = \frac{1}{2} \left( \frac{d}{dt} \left( \mathcal{L}(B_s(p_0 \otimes p_t)) \right) \right) - \lambda \left( \mathcal{L}(Z) \right)
\end{equation}
(22)

so that by equation (6), we have \( s(t) \leq 0 \). Since \( \lim_{t \to \infty} s(t) = 0 \) (ref. 25) and \( s(t) \) is monotonically decreasing, we thus find that \( s(0) \geq 0 \). In other words, we obtain equation (18).

Proof sketch of equation (5). As mentioned above, to establish equation (5), rather than using convexity, we appeal to a quantum version of Stam’s inequality:
\begin{equation}
\frac{2}{f(p^0_t)} + \frac{1}{f(p^0_0)} \geq \frac{1}{f(p_t)}
\end{equation}
(23)

whose proof along the lines of ref. 27 can be found in ref. 25. In fact, we let \( X \) evolve according to a quantum diffusion for time \( f(t) \) and \( Y \) evolve for \( G(t) \) with \( \lim_{t \to \infty} f(t) = \lim_{t \to \infty} G(t) = \infty \). Then, letting \( p_t^0 = e^{f(t)Z} (p_0) \), \( \rho^0_0 = e^{G(t)Z} (p_0) \) and \( \rho^0_2 = e^{G(t)Z} (p_0) \), we can show that as \( t \to \infty \) the ratio
\begin{equation}
h(t) = \frac{1}{2} \exp \left( -\frac{1}{n} S(p_t^0) \right) + \frac{1}{2} \exp \left( -\frac{1}{n} S(p_0^0) \right)
\end{equation}
(24)

approaches 1. Using the quantum de Bruijn identity to evaluate \( h(t) \) allows us to find a differential equation for \( F \) and \( G \) that ensures, together with equation (23), \( h(t) \geq 0 \). This allows us to conclude that \( s(0) \geq 1 \), which implies equation (5).

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R.K. and G.S. designed and carried out the research and wrote the paper.

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This has now been corrected in the HTML and PDF versions of the Article.