The Layered Structure of Tensor Estimation and its Mutual Information

Jean Barbier†♦, Nicolas Macris† and Léo Miolane*

† Laboratoire de Théorie des Communications, Faculté Informatique et Communications, Ecole Polytechnique Fédérale de Lausanne, 1015, Suisse.
♦ Probability and Applications Group, School of Mathematical Sciences, Queen Mary University of London, United-Kingdom.
* INRIA & Ecole Normale Supérieure, 75005, Paris, France.

Abstract—We consider rank-one non-symmetric tensor estimation and derive simple formulas for the mutual information. We start by the order 2 problem, namely matrix factorization. We treat it completely in a simpler fashion than previous proofs using a new type of interpolation method developed in [1]. We then show how to harness the structure in “layers” of tensor estimation in order to obtain a formula for the mutual information for the order 3 problem from the knowledge of the formula for the order 2 problem, still using the same kind of interpolation. Our proof technique straightforwardly generalizes and allows to rigorously obtain the mutual information at any order in a recursive way.

I. INTRODUCTION

In the last two decades tensor estimation (also called tensor factorization or decomposition) has found many applications in signal processing, high dimensional statistics, data mining and machine learning [2–4]. In this contribution we consider simple versions of the problem within a Bayesian framework. One observes a noisy version of an n-dimensional, rank-one, order p tensor $U_1 \otimes U_2 \otimes \cdots \otimes U_p$, and the goal is to provide an estimate of the n-dimensional random vectors $U_i$, $i = 1, \ldots, p$ constituting the tensor. We consider additive Gaussian noise and in the Bayesian formulation the variance of the noise as well as the priors on the vectors to be estimated are supposed to be known. A central quantity is the average mutual information, or log-partition function, associated to the Bayes posterior. Indeed from this quantity one can typically determine phase transitions as well as performance measures related to minimum mean-square-errors (MMSE). There are very precise conjectures within this framework that come from analytical computations based on the replica method [5] of statistical physics and message-passing methods providing the so-called approximate message-passing (AMP) algorithm [6, 7].

These calculations have allowed to derive phase diagrams predicting structural phase transitions inherent to the problem, and to compare them to the algorithmic phase transitions [8]. The main finding is that there is a region of parameters where AMP performs (in an MMSE sense) as well as the optimal Bayesian estimator, but there also typically exist large regions of parameters where AMP is sub-optimal or cannot even estimate better than a pure random guess.

We point out that this phenomenology seems to be quite universal and is found in many other problems related to Bayesian inference [9].

In this contribution we provide a rigorous analysis of the mutual information for rank-one, order p tensor estimation in the asymptotic regime $n \to \infty$. Computing the mutual information (or log-partition sum) a priori involves intractable n-dimensional integrals or sums. We reduce the problem to low dimensional (typically of order p) variational expressions which can in principle be solved on a computer. These variational problems also lead to interesting questions that are not fully solved, and we provide related conjectures.

Our method of analysis is based on a recently developed adaptive interpolation method [1] together with an inherent layered structure that underpins the tensor estimation problem: We will relate the solution of the order $p+1$ problem to that of the order $p$ one and provide recursive variational formulas. The case $p = 2$, the so-called “matrix factorization” problem, forms the base case and will be presented first as a pedagogical example of our interpolation technique. We then explicitly show how to go from $p = 2$ to $p = 3$. The generalization as well as other details of our analysis are left to a longer forthcoming contribution.

Let us briefly say a word on the history of interpolation techniques which are central to this work. They first originated in the seminal works of Guerra and Toninelli [10, 11] which paved the way towards Talagrand’s proof [12] of the Parisi formula [13] for the free energy of the Sherrington-Kirkpatrick spin glass. Not only these methods have allowed to obtain many more rigorous results on mean-field spin-glasses [14], but remarkably, they have found numerous applications in coding theory, signal processing and theoretical computer science problems. So far, Bayesian inference has provided the most fertile ground and replica (symmetric) formulas for mutual informations are completely proved in many such cases. A non-exhaustive list of examples is: Cod-
ing theory [15], random linear estimation [16, 17] and matrix factorization [18–20]. All these works combine the original Guerra-Toninelli interpolation with some other methodology such as spatial coupling [15, 18] or the Aizenman-Sims-Starr principle [20, 21]. The present contribution uses a refined form of interpolation which is self-contained and provides what we believe is a much simpler and unified approach. This approach has also been successfully used very recently for non-linear estimation [22]. Finally, as pointed out above, a special feature of the present problem is the layered structure of tensor estimation and we believe that this aspect can be leveraged to analyze other relevant multilayered problems.

II. SETTING AND RESULTS

A. Non-symmetric tensor estimation

1) Order 2 tensor estimation: We use the notation $X \sim \mathcal{P}$ to express that the vector (or tensor) $X$ has i.i.d. components distributed according to $\mathcal{P}$. The order $p = 2$ rank-one tensor estimation problem, or matrix factorization, is the task of inferring the vectors $U \in \mathbb{R}^{a \times n}$ and $V \in \mathbb{R}^{c \times n}$ (all $\alpha$’s are fixed) from the matrix $Y \in \mathbb{R}^{a \times c \times n}$ obtained from the following observation model

$$Y_{ij} = \sqrt{\lambda} U_{i} V_{j} + Z_{ij},$$

for $1 \leq i \leq \alpha_a n$ and $1 \leq j \leq \alpha_c n$. Here $Z \sim \mathcal{N}(0,1)$ is a Gaussian noise matrix. $\lambda$ is the signal-to-noise ratio and the normalization $1/\sqrt{n}$ makes the estimation problem non-trivial. We suppose that $U \sim P_u$ and $V \sim P_v$ where the probability distributions $P_u$ and $P_v$ are known by the statistician. We assume that $P_u$ and $P_v$ have a bounded support (this boundedness hypothesis can be relaxed at the likelihood of the (component-wise conditionally independent) observation matrix $Y$ is

$$P(Y|u, v) = (2\pi)^{-\frac{nc}{2}} e^{-\frac{1}{2n} \sum_{i,j=1}^{nc} (Y_{ij} - \sqrt{\lambda} u_i v_j)^2},$$

and from the Bayes formula we get the posterior distribution ($\Theta$ is the set of quenched variables, in this case $U, V$ and $Z$)

$$P(u, v|Y) = \frac{P_u(u) P_v(v) e^{-\mathcal{H}(u,v,\Theta)}}{Z(\Theta)},$$

where we slightly abuse notation by writing $P_u(u) = \prod_{i=1}^{a} P_u(u_i)$ and so forth. We employ the vocabulary of statistical physics and call

$$\mathcal{H}(u, v; \Theta) = \lambda \sum_{i,j=1}^{nc} \left( \frac{(u_i v_j)^2}{2n} - \frac{u_i v_j U_i V_j}{n} - \frac{u_i v_j Z_{ij}}{\sqrt{\lambda} n} \right)$$

the Hamiltonian of the model (for obtaining the posterior we replaced $Y$ by $\Theta$ and simplified the terms independent of $u, v$ when normalizing it). The partition function

$$Z(\Theta) = \int dP_u(u) dP_v(v) e^{-\mathcal{H}(u,v,\Theta)}$$

is the posterior normalization factor (we keep its $\lambda$ dependency implicit).

Our principal quantity of interest is the average free energy (the superscript stands for order $p = 2$)

$$f^{(2)}_n(\lambda) = \frac{1}{n} \mathbb{E} \ln Z(\Theta)$$

where $\mathbb{E}$ always means expectation w.r.t. the quenched random variables appearing inside an expression. It is equal up to an additive constant to the Shannon entropy $H(Y)$ of $P(Y)$. This object is related to the mutual information through

$$I(U, V; Y) = f^{(2)}_n(\lambda) + \frac{\lambda}{n^2} \alpha_u \alpha_v \rho_u \rho_v.$$

Its limit $\lim_{n \to \infty} f^{(2)}_n(\lambda)$ contains interesting information such as the location of phase transitions corresponding to its non analyticity points. Of particular interest is its first (as $\lambda$ is decreased from infinity) non-analyticity point sometimes called the information theoretic threshold $\lambda_{\text{opt}}$. It is the lowest signal-to-noise ratio such that inference of $(U, V)$ from $Y$ is information theoretically “possible”. Indeed the optimal value of performance measures, such as the minimum mean-square-errors of the vectors, is typically low only above $\lambda_{\text{opt}}$. We refer to [8, 16, 17, 19, 20] for more motivations for computing free energies, including algorithmic aspects.

Remark 2.1 (Channel universality): The Gaussian noise setting (1) is actually sufficient to completely characterize the generic model where $Y$ is observed through a noisy element-wise (possibly non-linear) output probabilistic channel $P_{\text{out}}(Y_{ij} | u_i v_j / \sqrt{n})$. This is made possible by a theorem of channel universality [23] (conjectured in [24] and already proven for community detection in [25]). The same remark applies to higher order tensor estimation models [8, 24, 26].

2) Order 3 tensor estimation: We now observe the order 3 tensor $F$ with entries

$$F_{ijk} = \frac{\sqrt{\lambda}}{n} U_i V_j W_k + Z_{ijk}$$

for $1 \leq i \leq \alpha_a n, 1 \leq j \leq \alpha_c n$ and $1 \leq k \leq \alpha_w n$. The normalization dividing the product of vector is $n^{(p-1)/2}$ for an order $p$ tensor problem; again this normalization makes the problem non-trivial. There is now an additional $W \in \mathbb{R}^{\alpha_w n}$ to infer. It has i.i.d. components drawn from the known prior $P_w$ with bounded support and with second moment $\rho_w$.

Now $Z \sim \mathcal{N}(0, 1)$ is a Gaussian noise tensor.

As for matrix estimation one can define the posterior $P(u, v, w|F)$ similarly as (3) but with the additional $w$ dependence. The associated Hamiltonian $\mathcal{H}(u, v, w; \Theta)$ is equal to

$$\lambda \sum_{i,j,k=1}^{\alpha_a \alpha_c \alpha_w} \left( \frac{(u_i v_j w_k)^2}{2n^2} - \frac{u_i v_j w_k U_i V_j W_k}{n^2} - \frac{u_i v_j w_k Z_{ijk}}{\sqrt{\lambda} n} \right).$$

Then the average free energy $f^{(3)}_n(\lambda)$ for this model is defined similarly as (4), but with the partition function (the normalization of $P(u, v, w|F)$) being now $Z(\Theta) =$
The average free energy is related to the mutual information through
\[ I(U, V; F) \]
\[ \frac{1}{n} = f_n^{(3)}(\lambda) + \frac{1}{2} \alpha u_0 \alpha w \rho v \rho w. \]

Note the following recursive, or “layered”, structure linking the order 2 and 3 versions of tensor estimation problems under Gaussian noise. Consider the estimation problem as
\[ B. \text{ Variational formulas for the mutual information} \]

An important role in our proof is played by simple scalar estimation problems under Gaussian noise. Consider the estimation of the scalar r.v \( X \sim P_u \) from the observation \( Y = \sqrt{m} X + Z \) where \( Z \sim N(0, 1) \) and \( m \) plays the role of a signal-to-noise ratio. Then the average free energy for this problem is
\[ f_u(m) = -E \ln \int dP_u(x) e^{-m(\frac{1}{2} - xZ/\sqrt{m})}. \]

It is related to minus the expected logarithm of the normalization of the posterior \( P(x|Y) \). Define similarly \( f_v \) and \( f_w \) as the r.h.s of (6) but with \( x, X \sim P_v \) or \( P_w \) respectively.

Define the potential for matrix estimation as
\[ f_{pol}^{(2)}(m_u, m_v; \lambda) = \frac{\lambda u_0 \alpha v}{2} m_u m_v + \alpha u f_u(\lambda u_0 m_v) + \alpha v f_v(\lambda u_0 m_u). \]

In the next section we prove using the adaptive interpolation method the following result, first proven in [20] using a more technical strategy based on a rigorous version of the cavity method [5,9], the so-called Aizenman-Sims-Starr scheme [21]. In order to state the result we need to introduce the set of critical points of the potential (7):
\[ \Gamma_2(\lambda) = \left\{ (m_u, m_v) \in \mathbb{R}_+^2 \mid m_u = -2 f_u(\lambda u_0 m_v), m_v = -2 f_v(\lambda u_0 m_u) \right\}. \]

These equations are known as “replica-symmetric equations” in spin glass theory (see [9,27] for instance) or “state evolution equations” in the context of approximate message-passing algorithms [28,29].

**Theorem 2.2 (Free energy of matrix estimation):** Fix \( \lambda > 0 \). The average free energy of model (1) verifies
\[ \lim_{n \to \infty} f_n^{(3)}(\lambda) = \inf_{\Gamma_2(\lambda)} f_{pol}^{(2)}(m_u, m_v; \lambda) \]
\[ \quad = \sup_{m_u m_v} f_{pol}^{(2)}(m_u, m_v; \lambda) \]
where this optimization is over \( m_u, m_v \in [0, \rho_u], m_v \in [0, \rho_v] \).

Define the potential of the order 3 problem as
\[ f_{pol}^{(3)}(m_u, m_v, m_w; \lambda) = \lambda u_0 \alpha v \alpha w m_u m_v m_w + \alpha u f_u(\lambda u_0 \alpha w m_v m_w) + \alpha v f_v(\lambda u_0 \alpha w m_u m_w) + \alpha w f_w(\lambda u_0 \alpha w m_u m_w) \]

and the corresponding set of critical points:
\[ \Gamma_3(\lambda) = \left\{ (m_u, m_v, m_w) \in \mathbb{R}_+^3 \mid m_u = -2 f_u(\lambda u_0 \alpha w m_v m_w), m_v = -2 f_v(\lambda u_0 \alpha w m_u m_w), m_w = -2 f_w(\lambda u_0 \alpha w m_u m_v) \right\}. \]

Once Theorem 2.2 proven, we will use it for obtaining

**Theorem 2.3 (Free energy of tensor estimation):** Fix \( \lambda > 0 \). The average free energy of model (5) verifies
\[ \lim_{n \to \infty} f_n^{(3)}(\lambda) = \inf_{\Gamma_3(\lambda)} f_{pol}^{(3)}(m_u, m_v, m_w; \lambda). \]

**Remark 2.4:** The fact that the sets \( \Gamma_2(\lambda) \) and \( \Gamma_3(\lambda) \) are not empty follows from the fact that the functions \( -2 f_u, -2 f_v, -2 f_w \) are continuous, bounded and non-negative (see Lemma 39 in [20]) and from an application of Brouwer’s theorem.

**III. PROOFS**

The main ingredient of our proof is the adaptive interpolation method recently introduced by two of us in [1]. Note that in contrast with the discrete version of the method presented in [1] we will here use it in a continuous form which is even more straightforward for the present problem (yet equivalent). The main difference with the canonical interpolation method developed by Guerra and Toninelli in the context of spin glasses [11,30] is the following: The interpolating estimation model that we introduce next is parametrized by “trial interpolating functions” instead of a single trial parameter. These will allow for much more flexibility when choosing the interpolation path, and will actually permit us to select an “optimal” interpolation path.

**A. Initializing the recursion: Proof of Theorem 2.2**

1) The interpolating model: Consider the joint estimation of \( (U, V) \) from the three following types of “time-dependent” observations
\[ Y_{(t)} \]
\[ Y_{u}^{(t)} = \sqrt{\frac{1-t}{n}} U_i V_j + Z_{ij}, \]
\[ Y_{v}^{(t)} = \lambda u_0 \int_0^1 m_v(s) ds U_i + Z_{i}^{(v)}, \]
\[ Y_{w}^{(t)} = \lambda w_0 \int_0^1 m_w(s) ds V_j + Z_{j}^{(w)}, \]
for \( 1 \leq i \leq n_u \) and \( 1 \leq j \leq n_v \). Again \( U \nsubseteq P_u \), \( V \nsubseteq P_v \) and \( Z, Z^{(v)}, Z^{(w)} \nsubseteq N(0, 1) \). The interpolation parameter (or “time”) \( t \in [0, 1] \) and the interpolating functions \( m_u(s) \in [0, \rho_u], m_v(s) \in [0, \rho_v] \) are to be fixed later. This model interpolates between the matrix factorization model at \( t = 0 \) to a model composed of two independent scalar Gaussian channels (one for \( U \), one for \( V \)) at \( t = 1 \). The \( \lambda(1-t) \) appearing in the first set of observations in (6), \( \lambda u_0 \int_0^1 m_v(s) ds \) and \( \lambda w_0 \int_0^1 m_w(s) ds \) appearing in the second and third sets all play the role of signal-to-noise ratios, with \( t \) giving more and more “power” (or weight) to the scalar inference channels.
when increasing. Here is a crucial and novel ingredient of our interpolation scheme. In classical interpolations, these signal-to-noise ratios (snr) would all take a trivial form (i.e. would be linear in $t$) but here, the additional degree of freedom gained from the non-trivial (integral) dependency in $t$ of the two latter snr through the introduction of the interpolating functions will be essential.

Let us define the following interpolating Hamiltonian $\mathcal{H}_t = \mathcal{H}_t(u, v)$ (from now on we do not indicate anymore the dependence w.r.t quenched variables to ease the notations) associated with $\mathcal{H}$

$$\mathcal{H}_t = \lambda \alpha c \int_0^1 m_u(s) ds \sum_{i=1}^{n_u} \left( \frac{u_i^2}{2} - u_i U_i - \frac{u_i Z_i^{(u)}}{\sqrt{\lambda \alpha c}} \int_0^1 m_u(s) ds \right) + \lambda \alpha v \int_0^1 m_v(s) ds \sum_{j=1}^{n_v} \left( \frac{v_j^2}{2} - v_j V_j - \frac{v_j Z_j^{(v)}}{\sqrt{\lambda \alpha v}} \int_0^1 m_v(s) ds \right) + \lambda (1-t) \sum_{i,j=1}^{n_{\text{snr}}} \left( \frac{(u_i v_j)^2}{2n} - \frac{u_i v_j U_i V_j}{n} - \frac{u_i v_j Z_{ij}}{\sqrt{(1-t) n}} \right).$$

We note that for $t=0$ this Hamiltonian is $\mathcal{H}$. This Hamiltonian relates to the $t$-dependent posterior of the interpolating model through

$$P_t(u, v) = \frac{P_u(u) P_v(v) e^{-\mathcal{H}_t}}{Z_t},$$

where $Z_t$ is the normalization. To (10) is associated a Gibbs bracket $\langle \cdot \rangle_t$ defined as $\langle A \rangle_t = \int dP_t(u, v) A(u, v)$. Moreover the interpolating free energy is

$$f_n(t) = -\frac{1}{n} \mathbb{E} \ln Z_t = -\frac{1}{n} \mathbb{E} \ln \int dP_u(u) dP_v(v) e^{-\mathcal{H}_t},$$

where $\mathbb{E}$ is the expectation w.r.t $U, V, Z, Z^{(u)}$ and $Z^{(v)}$.

2) Overlap concentration: Let us define the overlaps $Q_u = n_u^{-1} \sum_{i=1}^{n_u} u_i U_i$ and $Q_v = n_v^{-1} \sum_{j=1}^{n_v} v_j V_j$ where $\mathbf{u}, \mathbf{v}$ are jointly drawn from the posterior (10). The next lemma plays a key role in our proof. Essentially it states that the overlaps concentrate around their mean, a behavior called “replica symmetric” in statistical physics. Similar results have been obtained in the context of the analysis of spin glasses [27]. Here we use a formulation tailored to Bayesian inference problems as developed in the context of LDPC codes, linear estimation and Nishimori symmetric spin glasses [31–33].

In order to prove this concentration we need to introduce a “small” perturbation of the interpolating model by adding

$$\sum_{i=1}^{n_u} (\epsilon_u u_i^2 / 2 - \epsilon_u u_i U_i - \epsilon_u u_i Z_i^{(u)}) + \sum_{j=1}^{n_v} (\epsilon_v v_j^2 / 2 - \epsilon_v v_j V_j - \epsilon_v v_j Z_j^{(v)})$$

to the Hamiltonian $\mathcal{H}_t$, where $Z_i^{(u)}, Z_j^{(v)} \sim N(0, 1)$. These terms can be interpreted as having extra observations coming from Gaussian side-channels $\tilde{Y}_i^{(u)} = \sqrt{\epsilon_u} U_i + \tilde{Z}_i^{(u)}$ and similarly for $\mathbf{v}$. The new Hamiltonian $\mathcal{H}_{t, \epsilon_u, \epsilon_v}$ defines a new Gibbs bracket $\langle \cdot \rangle_{t, \epsilon_u, \epsilon_v}$ and free entropy $f_{n, \epsilon_u, \epsilon_v}(t)$, and all the set up of the previous section trivially extends. This perturbation induces only a small change in the free energy, namely of the order of $\epsilon_u, \epsilon_v$. Indeed, a simple computation gives

$$f_n(t) = -\frac{1}{n} \mathbb{E} \ln \left\{ \prod_{t=1}^{t_{\text{snr}}} \left( f_{n, \epsilon_u, \epsilon_v}(t) \right) \right\}.$$

Replacing (13) and (14) in the fundamental theorem of calculus and using the concentration (12) for $Q_u, Q_v$ together with the Cauchy-Schwarz inequality leads to

$$f_n^{(2)}(\lambda) = \alpha_u \tilde{f}_u(\lambda \alpha_u \tilde{m}_u) + \alpha_v \tilde{f}_v(\lambda \alpha_v \tilde{m}_u),$$

where $\tilde{m}_u = \int_0^1 m_u(s) ds \in [0, \rho_u]$ and $\tilde{m}_v = \int_0^1 m_v(s) ds \in [0, \rho_v]$. So at $t=0$ one recovers the average free energy $\mathcal{F}$ of the original model, while at $t=1$ appear two terms of the potential $\mathcal{F}$. This is precisely the reason for the introduction of the scalar channels in (10). In order to compare $f_n^{(2)}(\lambda)$ with the potential we use the fundamental theorem of calculus $f_n(0) = f_n(1) - \int_0^1 f'_n(t) dt$. It is thus natural to compute (see sec. [11] for the proof)

$$f'_n(t) = -\frac{\lambda \alpha_u \alpha_v}{2} \left\{ m_u(t) m_v(t) - \mathbb{E} \left( (Q_u - m_u(t))(Q_v - m_v(t)) \right) \right\}.$$
Cauchy-Schwarz and (12). Now we choose the constant function \( m_u(t) = m_u \in [0, \rho_u] \) and thus \( m_u = m_u \). Using (7) allows then to recognize
\[
\begin{align*}
    f_n^{(2)}(\lambda) &= f_{\text{pot}}^{(2)}(m_u, m_v; \lambda) + \sigma_n(1) \\
    &- \frac{\lambda \alpha_u \alpha_v}{2} \int_0^t dt (E(Q_u)_t - m_u)(E(Q_v)_t - m_v(t)). \quad (15)
\end{align*}
\]

Now let \( m_v(t) = \mathbb{E}(Q_v)_t = g_n(t, \int_0^t m_v(s)ds, m_u) \),
where \( g_n \) is a \( C^1 \) bounded non-negative function. As \( \mathbb{E}(Q_v)_t \in [0, \rho_v] \), a direct application of the parametric Cauchy-Lipschitz theorem gives that there exists a unique solution \( m_v(m_u)(t) \) and that the mapping
\[ M_n : m_u \in [0, \rho_u] \rightarrow \int_0^1 m_v(m_u)(s)ds \in [0, \rho_v] \]
is continuous. With this choice \( m_v(t) = m_v(m_u)(t) \) we cancel the remainder (the integral appearing in (15)) and obtain
\[
    f_n^{(2)}(\lambda) = f_{\text{pot}}^{(2)}(m_u, M_n(m_u); \lambda) + \sigma_n(1). \quad (16)
\]
This is really the crucial and novel point of the adaptive interpolation method that makes it so powerful: In previous interpolations, the remainder always remains and if by luck it has an obvious sign, then we can compare the left and right hand sides of identities like (15) in order to obtain bounds. But with our new method, the remainder can be directly canceled which allows to obtain much stronger results irrespective of the sign of the remainder.

4) Obtaining matching bounds: Equation (16) is true for any \( m_u \in [0, \rho_u] \). Consequently
\[
    \lim_{n \to \infty} \sup_{m_u, m_v} f_n^{(2)}(\lambda) \leq \inf_{m_u, m_v} f_{\text{pot}}^{(2)}(m_u, m_v; \lambda), \quad (17)
\]
where the optimization is over \( m_u \in [0, \rho_u], m_v \in [0, \rho_v] \).

For \( m \geq 0 \), write \( F_u(m) = -2f_u(\lambda \alpha_u, m) \).

Using the basic properties of the free entropy of the scalar channel \( Y = \sqrt{m} X + Z \), (see for instance sec. 7.1 in [20]) \( F_u \) is continuous and takes values in \( [0, \rho_u] \). Consequently, \( F_u \circ M_n \) is a continuous map from \( [0, \rho_u] \) into itself: It thus admits at least one fixed point \( m_v^* \). Plugging it into (16) we get
\[
    f_n^{(2)}(\lambda) = f_{\text{pot}}^{(2)}(m_v^*, M_n(m_v^*); \lambda) + \sigma_n(1)
    = f_{\text{pot}}^{(2)}(F_u(m_v^*), M_n(m_v^*); \lambda) + \sigma_n(1)
    = f_{\text{pot}}^{(2)}(F_u(m_v^*), m_v^*; \lambda) + \sigma_n(1), \quad (18)
\]
where \( m_v^* = M_n(m_v^*) \in [0, \rho_v] \). By Lemma 39 in [20], \( f_u \) is concave, thus \( f_{\text{pot}}^{(2)}(F_u(m_v^*), m_v; \lambda) \) is concave in \( m_v \). Its derivative is
\[
    \frac{\partial f_{\text{pot}}^{(2)}(F_u(m_v^*), m_v; \lambda)}{\partial m_v} = \frac{\lambda \alpha_u \alpha_v}{2} (F_u(m_v^*) - F_u(m_v))
\]
which thus cancels at its maximum attained at \( m_v = m_v^* \).

Therefore
\[
    f_n^{(2)}(F_u(m_v^*), m_v^*; \lambda) = \sup_{m_u, m_v} f_n^{(2)}(F_u(m_v^*), m_v; \lambda)
    \geq \inf_{m_u, m_v} f_n^{(2)}(m_u, m_v; \lambda).
\]

Thus (18) gives
\[
    \lim_{n \to \infty} \inf_{m_u, m_v} f_n^{(2)}(m_u, m_v; \lambda) \geq \inf_{m_u, m_v} f_{\text{pot}}^{(2)}(m_u, m_v; \lambda),
\]
where the optimization is over \( m_u \in [0, \rho_u], m_v \in [0, \rho_v] \). This proves Theorem 2.3 when combined with (17). \( \blacksquare \)

5) Proof of (14). Let us show how the derivative of the interpolating free energy is obtained. Is is given by
\[
    f_n(t) = \frac{1}{n} \mathbb{E} \left( \frac{d\mathcal{H}_t}{dt} \right)_t
    = \frac{\lambda \alpha \alpha_v}{n} \mathbb{E} \left( \sum_{i=1}^{n_v} \left( u_i^2 - u_i U_i - \frac{u_i Z_i^{(u)}}{2 \sqrt{\lambda \alpha \alpha_v} \int_0^t m_v(s)ds} \right) \right)_t
    + \frac{\lambda \alpha \alpha_u}{n} \mathbb{E} \left( \sum_{j=1}^{n_u} \left( v_j^2 - v_j V_j - \frac{v_j Z_j^{(v)}}{2 \sqrt{\lambda \alpha \alpha_u} \int_0^t m_u(s)ds} \right) \right)_t
    - \frac{\lambda}{n} \mathbb{E} \left( \sum_{i,j=1}^{n_u, n_v} \left( \frac{(u_i v_j)^2}{2n} - \frac{u_i v_i U_i V_j}{n} - \frac{u_i v_i Z_{ij}}{2 \sqrt{\lambda \alpha \alpha_u} \int_0^t m_u(s)ds} \right) \right)_t.
\]

Let \((u', v')\) be jointly drawn from the posterior (10) and this independently from \((u, v)\), itself also drawn from the same posterior. We now integrate by part the Gaussian noise variables using the elementary formula \( \mathbb{E}[Za(Z)] = \mathbb{E}[a'(Z)] \) for \( Z \sim \mathcal{N}(0, 1) \) and for continuously differentiable \( a \) such that these expectations are well-defined. This leads to
\[
    f_n(t) = \frac{\lambda \alpha \alpha_v}{n} \mathbb{E} \left( \sum_{i=1}^{n_v} \left( - u_i U_i + \frac{u_i u_i'}{2} \right) \right)_t
    + \frac{\lambda \alpha \alpha_u}{n} \mathbb{E} \left( \sum_{j=1}^{n_u} \left( - v_j V_j + \frac{v_j v_j'}{2} \right) \right)_t
    - \frac{\lambda}{n} \mathbb{E} \left( \sum_{i,j=1}^{n_u, n_v} \left( - \frac{u_i v_i U_i V_j}{n} + \frac{u_i v_i v_i'}{2n} \right) \right)_t.
\]

We now use the following identities \( \mathbb{E}(u_i U_i)_t = \mathbb{E}(u_i u_i')_t \) and \( \mathbb{E}(v_j V_j)_t = \mathbb{E}(v_j v_j')_t \). These follow directly from the following identity which is nothing more than a direct consequence of Bayes formula (see [17, 20] for a proof): \( \mathbb{E}(g(u, v, U, V)) = \mathbb{E}(g(u, v, u', v')) \) for any continuous bounded function \( g \). Thus
\[
    f_n'(t) = -\frac{\lambda \alpha \alpha_v}{n} \mathbb{E} \left( \sum_{i=1}^{n_v} u_i U_i \right)_t
    - \frac{\lambda \alpha \alpha_u}{n} \mathbb{E} \left( \sum_{j=1}^{n_u} v_j V_j \right)_t
    + \frac{\lambda}{n} \mathbb{E} \left( \sum_{i,j=1}^{n_u, n_v} u_i v_i U_i V_j \right)_t
    - \frac{\lambda \alpha \alpha_u}{2} \mathbb{E} \langle m_v(t)Q_u + m_u(t)Q_v - Q_u Q_v \rangle_t
\]
which is (14). \( \blacksquare \)

B. From \( p = 2 \) to \( p = 3 \): Proof of Theorem 2.3

Let us now prove the second theorem using our previous findings, using again the adaptive interpolation method. We will start by proving an alternative version of the limit of the
free energy, using an auxiliary potential:

\[ f_{\text{aux}}^{(3)}(m_u, m_v, m_w, m_{uw}; \lambda) = \frac{\lambda \alpha_u \alpha_v \alpha_w m_{uw} m_w}{2} + \alpha_w \bar{f}_w(\lambda \alpha_u \alpha_v m_{uw}) + f_{\text{pot}}^{(2)}(m_u, m_v; \lambda \alpha_u m_w). \]

**Proposition 3.2 (Auxiliary free energy formula):** Fix \( \lambda > 0 \). The average free energy of model \( 5 \) verifies

\[ \lim_{n \to \infty} f_n^{(3)}(\lambda) = \inf_{m_w} \inf_{m_u, m_v} f_{\text{aux}}^{(3)}(m_u, m_v, m_w, m_{uw}; \lambda) \]

where the optimization is over \( m_w \in [0, \rho_w] \), \( m_u \in [0, \rho_u] \), and \( m_v \in [0, \rho_v] \) and \( m_{uw} \in [0, \rho_{uw}] \). Once Proposition 3.2 will be proved, Theorem 2.3 will simply follow from Lemma 1.3 presented in appendix \( f_{uv}, f_v \) and \( f_w \) are indeed strictly concave, differentiable, Lipschitz, non-increasing functions over \( \mathbb{R}_+ \) by Lemma 39 from [20].

1) The “layered” interpolating model: Consider this time the following observation model

\[
\begin{align*}
F(t)_{ij} &= \sqrt{\frac{\lambda(1-t)}{n}} U_i V_j W_k + \bar{Z}_{ijk}, \\
Y_{uv}(t, w) &= \sqrt{\frac{\lambda}{n}} w_{uv} m_{uw}(s) ds U_i V_j + \bar{Z}_{(uv)}, \\
Y_{(w)}(t) &= \sqrt{\frac{\lambda \alpha_u \alpha_v}{n}} m_{uw}(s) ds W_k + \bar{Z}_{(w)},
\end{align*}
\]

for \( 1 \leq i \leq n_u, 1 \leq j \leq n_v, \) and \( 1 \leq k \leq n_w \). Here \( U \overset{i.i.d.}{\sim} P_u, \quad V \overset{i.i.d.}{\sim} P_v, \quad W \overset{i.i.d.}{\sim} P_w \) and \( Z, Z^{(uv)}, Z^{(w)} \overset{i.i.d.}{\sim} N(0,1) \). Again \( t \in [0,1] \) and the interpolating functions \( m_{uw}(s) \in [0, \rho_{uw}] \), \( m_u(s) \in [0, \rho_u] \), and \( m_v(s) \in [0, \rho_v] \) are to be fixed later. This model interpolates between an order \( p+1 = 3 \) tensor estimation at \( t = 0 \) to a model combined of a scalar estimation problem over \( W \) under Gaussian noise and an order \( p = 2 \) tensor joint estimation problem over \( (U, V) \) at \( t = 1 \). This model is “layered” in the sense that one decoupled scalar estimation problem is considered in addition of the order \( p = 2 \) joint estimation problem that has already been treated analytically, see Theorem 2.2.

As previously, we associate to this model its posterior distribution given by \( P_t(u, v, w) = Z_{n}^{-1} P_u(u) P_v(v) P_w(w) \exp(-\mathcal{H}_t) \) where the interpolating Hamiltonian \( \mathcal{H}_t = \mathcal{H}_t(u, v, w) \) (again quenched variables are not indicated explicitly) is

\[ \mathcal{H}_t = \lambda(1-t) \sum_{i,j,k=1}^{n_u, n_v, n_w} \frac{(u_{ij} v_{jk} w_k)^2}{2n} + \frac{u_{ij} v_{jk} Z_{ijk}}{\sqrt{\lambda n^2 (1-t)}} + \lambda \alpha_w \int_0^t m_w(s) ds \sum_{i,j=1}^{n_u, n_v} \frac{(u_{ij})^2}{2n} \\
- \frac{u_{ij} v_{jk} Z_{ij}}{\sqrt{\lambda n^2 (1-t)}} - \frac{u_{ij} v_{jk} Z_{ij}}{n} - \frac{u_{ij} v_{jk} Z_{ij}}{n} + \lambda \alpha_w \int_0^t m_w(s) ds \sum_{i,j=1}^{n_u, n_v} \frac{(u_{ij})^2}{2n} - w_k \bar{Z}_{(w)} - \lambda \alpha_v \int_0^t m_{uw}(s) ds \sum_{k=1}^{n_w} \frac{w_k^2}{2} - w_k W_k \]

The Gibbs bracket \( \langle \cdot \rangle_t \) is, as before, the expectation w.r.t. this posterior. Finally the interpolating free energy is

\[ f_n(t) = -\frac{1}{n} \mathbb{E} \ln \int dP_u(u) dP_v(v) dP_w(w) e^{-\mathcal{H}_t}. \]

2) Adaptive interpolation: The steps that we follow now are similar to sec. III-A.3. Define \( \tilde{m}_w = \int_0^t m_w(s) ds \in [0, \rho_w] \), \( m_{uw} = \int_0^t m_{uw}(s) ds \in [0, \rho_{uw}] \). \( f_n(t) \) verifies

\[ \begin{cases}
  f_n(0) = f_n^{(3)}(\lambda), \\
  f_n(1) = f_n^{(2)}(\lambda \alpha_w \tilde{m}_w + \alpha_w \bar{f}_w(\lambda \alpha_u \alpha_v \tilde{m}_w)).
\end{cases} \] (19)

Here clearly appears the recursive construction of our proof that exploits the layered structure of the problem: Theorem 2.2 allows to compute \( f_n^{(3)}(\lambda \alpha_w \tilde{m}_w) \) (note the “effective” signal-to-noise \( \lambda \alpha_w \tilde{m}_w \), appearing in the expression) that we will then use to obtain \( f_n^{(3)}(\lambda) \) through the adaptive interpolation method. By the very same steps as in sec. III-A.5 we get

\[ f_n^{(t)} = -\frac{\lambda \alpha_u \alpha_v \alpha_w}{2} \left( m_{uw}(t) m_w(t) \right) \\
- \mathbb{E}\left\{ (\mathcal{Q}_w \mathcal{Q}_v - m_w(t)) (\mathcal{Q}_w - m_w(t)) \right\} \right\}. \] (20)

We choose the constant function \( m_w(t) = m_{uw}(t) \in [0, \rho_w] \) and thus \( \tilde{m}_w = m_{uw} \). As mentionned in sec. III-A.2, the concentration of overlaps generalizes to the present setting. Thus by plugging (19) and (20) in the fundamental theorem of calculus and then using the concentration of \( \mathcal{Q}_w, \mathcal{Q}_v, \mathcal{Q}_w \) (more precisely, we use (12) combined with Cauchy-Schwarz, we reach

\[ f_n^{(3)}(\lambda) = f_n^{(2)}(\lambda \alpha_u \tilde{m}_w) + \alpha_w \bar{f}_w(\lambda \alpha_u \alpha_v \tilde{m}_w) \]

\[ + \frac{\lambda \alpha_u \alpha_v}{2} M_n(m_{uw}) m_w + c_n(1), \]

where by Theorem 2.2

\[ f_{n}^{(2)}(\lambda \alpha_u \tilde{m}_w) = \lim_{n \to \infty} f_{n}^{(2)}(\lambda \alpha_u \tilde{m}_w) \]

\[ = \inf \sup_{m_{uw}} f_{n}^{(2)}(m_u, m_v; \lambda \alpha_u \tilde{m}_w). \]

Equation (22) has exactly the same structure than (16). Indeed, \( f_w \) now plays the role of \( f_n \) and \( f_{n}^{(2)} \) plays the role of \( f_v \). Therefore, using the same arguments as in sec. III-A.4

\[ \lim_{n \to \infty} f_n^{(3)}(\lambda) = \inf \sup_{m_{uw}} \left\{ f_{n}^{(2)}(\lambda \alpha_u \tilde{m}_w) + \alpha_w \bar{f}_w(\lambda \alpha_u \alpha_v \tilde{m}_w) \right\} \]

\[ + \frac{\lambda \alpha_u \alpha_v}{2} m_{uw} \]

\[ = \inf \sup \sup_{m_{uw}} f_{n}^{(3)}(m_u, m_v, m_{uw}; \lambda). \]
where the optimization is restricted over \( m_u \in [0, \rho_u], m_v \in [0, \rho_v], m_w \in [0, \rho_w] \) and \( m_{uv} \in [0, \rho_u \rho_v] \). This proves Proposition 3.2 and thus Theorem 2.3. 

C. A conjectured \( p \)-letter variational expression

An inspection of the structure of \( f^{(3)}_{\text{aux}} \) in Proposition 3.2 shows that we can replace \( \inf_{m_u} \sup_{m_v} \inf_{m_w} f^{(p)}_{\text{pot}}(m_u, m_v, m_w) \) by \( \inf_{m_u} \inf_{m_v} \sup_{m_w} f^{(p)}_{\text{pot}}(m_u, m_v, m_w) \). Then, replacing \( m_{uv} \) by \( m_u m_v \), we immediately obtain the lower bound

\[
\lim_{n \to \infty} f^{(3)}_{\beta}(\lambda) \geq \inf_{m_u} \inf_{m_v} \sup_{m_w} f^{(p)}_{\text{pot}}(m_u, m_v, m_w; \lambda). 
\]

We conjecture that in fact the equality holds. More generally we conjecture that the mutual information for order \( p \) tensor estimation is given by the natural generalization of the r.h.s in (23) involving an optimization \( \inf_{m_u} \cdots \inf_{m_{p-1}} \sup_{m_p} f^{(p)}_{\text{pot}}(m_u, \ldots, m_{p}; \lambda) \) over \( p \) variables. In view of the (provable) generalization of Theorem 2.3 to the order \( p \) problem it would suffice to prove:

Conjecture 3.3: Let \( k \geq 2 \) and \( \psi_1, \ldots, \psi_k \) be \( k \) strictly convex, differentiable, non-decreasing and Lipschitz functions on \( \mathbb{R}_+ \). Define

\[
\phi : \mathbb{R}_+^k \to \left\{ \sum_{i=1}^{k} \psi_i \left( \prod_{j \neq i} q_j \right) - (k - 1) \right\},
\]

and \( \Gamma = \{ q \in \mathbb{R}_+^k \mid \forall i \in \{1, \ldots, k\}, \psi_i \left( \prod_{j \neq i} q_j \right) \} \). Then \( \sup_{q_2 \cdots q_{p-1} \geq 0} \inf_{q_1 \geq 0} \phi(q) = \sup_{q_1 \geq q_2 \cdots q_{p-1} \geq 0} \phi(q) \) and both extremas are achieved at the same points.

From a \( 2p-2 \) variational problem generalizing Proposition 3.2 the same argument leading to (23) implies a one-sided inequality. The converse bound is an open problem.

APPENDIX

This appendix gathers some technical results regarding the manipulation of “sup-inf” expressions. The first lemma comes from [22] (Appendix D).

Lemma 1.1: Let \( f \) and \( g \) be two convex, non-decreasing Lipschitz functions on \( \mathbb{R}_+ \). Suppose that \( g \) is strictly convex and differentiable. For \( q_1, q_2 \in \mathbb{R}_+ \), we define \( \psi(q_1, q_2) = f(q_1) + g(q_2) - q_1 q_2 \). Then

\[
\sup_{q_1 \geq 0} \inf_{q_2 \geq 0} \psi(q_1, q_2) = \sup_{q_2 \geq 0} \inf_{q_1 \geq 0} \psi(q_1, q_2) = \sup_{q_1 = g'(q_2)} \psi(q_1, q_2) = \sup_{q_2 = f'(q_1)} \psi(q_1, q_2).
\]

(24)

Moreover, the above extremas are achieved precisely on the same couples \((q_1, q_2)\) and \( f \) is differentiable at \( q_1 \).

Lemma 1.2: Let \( f \) and \( g \) be two convex, non-decreasing Lipschitz functions on \( \mathbb{R}_+ \). Suppose that \( f \) and \( g \) are differentiable and strictly convex. Then the function

\[
\phi : t \geq 0 \mapsto \sup_{q_1 \geq 0} \inf_{q_2 \geq 0} f(t q_1) + g(t q_2) - t q_1 q_2
\]

is convex, Lipschitz and non-decreasing. Moreover \( \phi'(0^+) = f'(0^+) g'(0^+) \) and for all \( t > 0 \):

\[
\phi'(t^-) = \min\{q_1'(t) q_2(t), q_1(t) q_2'(t)\} \text{ optimal couple in } (25),
\]

\[
\phi'(t^+) = \max\{q_1'(t) q_2(t), q_1(t) q_2'(t)\} \text{ optimal couple in } (25).
\]

Proof: Let \( g^* : x \in \mathbb{R} \mapsto \sup_{y \in \mathbb{R}_+} \{xy - g(y)\} \in \mathbb{R} \cup \{+\infty\} \) be the Fenchel-Legendre transform of \( g \). For \( t \geq 0 \)

\[
\phi(t) = \sup_{q_1 \geq 0} f(t q_1) - g^*(q_1)
\]

(26)

(this is true for \( t > 0 \) and one can verify easily that it is also true for \( t = 0 \) because \( g^* \) is non-decreasing). \( \phi \) is thus a suprema of convex functions and is therefore convex.

Let \( 0 < a < b \). For all \( t \in [a, b] \), Lemma 1.1 gives that the supremum (26) is achieved on a compact set (that does not depend on \( t \), but only on \( a, b \)). Thus Corollary 4 from [34]

gives that

\[
\phi'(t^-) = \min\{q_1'(t) f'(t q_1'(t)), q_1'(t) q_2'(t)\} \text{ optimal in } (26),
\]

\[
\phi'(t^+) = \max\{q_1'(t) f'(t q_1'(t)), q_1'(t) q_2'(t)\} \text{ optimal in } (26).
\]

Using Lemma 1.1 one see that \( f'(t q_1'(t)) \) is equal to the \( q_2'(t) \) from the proposition. \( \phi'(0^+) \) is computed analogously.

The fact that \( \phi \) is Lipschitz and non-decreasing follows from the expression of its left- and right-derivatives. Indeed, we know by Lemma 1.1 that the optimal couples on (23) are in \([0, \sup_{x \geq 0} g'(x)] \times [0, \sup_{x \geq 0} f'(x^+)]\).

Lemma 1.3: Let \( f_1, f_2, f_3 \) be 3 strictly convex, non-decreasing, differentiable, Lipschitz functions from \( \mathbb{R}_+ \) to \( \mathbb{R} \). Then

\[
\sup_{q_3 \geq 0} \sup_{q_2 \geq 0} \min\{f_1(q_2 q_3) + f_2(q_1 q_3) - q_1 q_2 q_3\} = \sup_{q_1 \geq 0} f_1(q_2 q_3) + f_2(q_1 q_3) - 2 q_1 q_2 q_3,
\]

\[
q_1 = f_1(q_2 q_3),
\]

\[
q_2 = f_2(q_1 q_3),
\]

\[
q_3 = f_3(q_1 q_3).
\]

Proof: Let us define

\[
\psi : q_3 \geq 0 \mapsto \sup_{q_1 \geq 0, q_2 \geq 0} \{f_1(q_2 q_3) + f_2(q_1 q_3) - q_1 q_2 q_3\}.
\]

We know by Lemma 1.2 that \( \psi \) is convex, Lipschitz and non-decreasing over \( \mathbb{R}_+ \).

We will first prove that in the setting of Lemma 1.1 all the quantities of (24) are equal to \( \sup_{q_1 = g'(q_2)} \psi(q_1, q_2) \). Obviously,

\[
\sup_{q_1 = g'(q_2)} \psi(q_1, q_2) \geq \sup_{q_1 = g'(q_2)} \psi(q_1, q_2).
\]

(27)

Now, let \( q_1, q_2 \geq 0 \) such that \( q_1 = g'(q_2) \). The function \( r \mapsto \psi(q_1, r) \) is convex and its derivative at \( r = q_2 \) vanishes. Thus

\[
\psi(q_1, q_2) = \inf_{r \geq 0} \psi(q_1, r) \leq \sup_{r_1 \geq 0, r_2 \geq 0} \inf_{r \geq 0} \psi(r_1, r_2),
\]

which combined with (27) and (23) gives that \( \sup_{q_1 = g'(q_2)} \psi(q_1, q_2) \) is equal to (24). We now apply this result twice to obtain

\[
\sup_{q_1 \geq 0} \sup_{q_2 \geq 0} \min\{f_3(r q_1) + f_2(q_1 q_3) - q_1 q_2 q_3\} - r q_3 \geq 0 = \sup_{q_3 \geq 0} \sup_{q_2 \geq 0} \{f_3(r) + f_1(q_2 q_3) - q_1 q_2 q_3 - r q_3 \},
\]

\[
q_3 = f_3(r q_1).
\]

Let us add two more constraints to the last supremums, namely \( r = q_1 q_2 \) and \( q_2 = f_2'(q_1 q_3) \). Adding constraints to a supremum cannot increase it, therefore
Let us now prove the converse bound. We apply Lemma 2 twice to obtain

\[
\sup_{q_3 \geq 2} \inf_{q_2 \geq 2} \left\{ f_3(r) + \sup_{q_1 \geq 2} \left\{ f_1(q_2 q_3) + f_2(q_1 q_3) - q_1 q_3 \right\} - r q_3 \right\} 
\geq \sup_{q_3 \geq 2} \left\{ f_3(r) + f_1(q_2 q_3) + f_2(q_1 q_3) - q_1 q_3 \right\} 
- \sup_{q_1 \geq 2} \left\{ f_1(q_2 q_3) + f_2(q_1 q_3) - q_1 q_3 \right\} 
- \sup_{q_2 \geq 2} \left\{ f_2(q_1 q_3) - q_1 q_3 \right\} 
= \sup_{q_3 \geq 2} \left\{ f_3(r) + f_1(q_2 q_3) + f_2(q_1 q_3) - q_1 q_3 \right\} 
- \sup_{q_1 \geq 2} \left\{ f_1(q_2 q_3) + f_2(q_1 q_3) - q_1 q_3 \right\} 
- \sup_{q_2 \geq 2} \left\{ f_2(q_1 q_3) - q_1 q_3 \right\} 
= \sup_{q_3 \geq 2} \left\{ f_3(r) + f_1(q_2 q_3) + f_2(q_1 q_3) - q_1 q_3 \right\} 
- \sup_{q_1 \geq 2} \left\{ f_1(q_2 q_3) + f_2(q_1 q_3) - q_1 q_3 \right\} 
- \sup_{q_2 \geq 2} \left\{ f_2(q_1 q_3) - q_1 q_3 \right\} 
\]

which concludes the proof.

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