Implications of $\mathcal{N} = 1$ supersymmetry for QCD conformal operators.

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Abstract

We prove a set of identities for the anomalous dimensions of the quark and gluon conformal operators in the flavour singlet channel in QCD. These relations arise from the graded commutator algebra of the $\mathcal{N} = 1$ superconformal group. We evaluate the rotation matrices for the quantities under study from the conventional dimensional regularization to the supersymmetry preserving regularization scheme. Using them we verify the equalities in two-loop approximation employing the results for the NLO anomalous dimensions of the conformal operators in the minimal subtraction scheme derived earlier.

Keywords: superconformal algebra, Ward identities, dimensional reduction, anomalous dimensions, conformal operators

PACS numbers: 11.10.Gh, 11.30.Pb, 12.38.Bx

1Alexander von Humboldt Fellow.
1. Introduction. In spite of a number of attractive theoretical features of supersymmetric gauge theories [1] in view of unification of the fundamental forces of Nature, the manifestation of their predictions was not observed experimentally so far. Nevertheless, they provide an excellent technical playground for the exploration of new physical concepts. It is remarkable that due to the high symmetry of the underlying Lagrangian the theory enjoys the property of reducibility of independent parameters such as gauge couplings and field renormalization constants. Since the structure of the corresponding Lagrangians resembles that of ordinary gauge theories with an adjusted fermion sector it can serve as a technical tool for deriving relationships between observables. Using heuristic arguments of this kind several relations have been derived in Ref. [2] between the anomalous dimensions of the local quark and gluon operators without total derivatives which appear in the description of the deep inelastic scattering via the operator product expansion. One of them is the empirically established Dokshitzer relation [3].

In this paper we address the issue of constraints on the anomalous dimensions of the conformal operators from the point of view of graded commutator algebra of the superconformal group and Ward identities for the Green functions with composite operator insertion. Presently we consider restricted \( Q \)-supersymmetry transformations which provide relations for the scale anomalies of composite operators with total derivatives, i.e. their anomalous dimensions. We make use of the subalgebra of the full superconformal algebra [4, 5] which includes scale transformation and consists of the relations, apart from the usual ones,

\[
[Q, P_\mu] = 0, \quad [Q, M_{\mu\nu}] = \frac{1}{2} \sigma_{\mu\nu} Q, \quad [Q, \bar{Q}] = 2 \gamma_\mu P_\mu, \quad [Q, D] = \frac{i}{2} Q. \tag{1}
\]

In the infinitesimal form their action on the field operator \( \phi \) is defined as \( \delta^Q \phi \equiv i[\phi, G] \) with \( G = P_\mu, M_{\mu\nu}, D \) for even generators and \( G = \bar{\zeta} Q \) for odd \( Q \) with \( \zeta \) being a Majorana Grassman valued spinor. One can equally include special supertransformations [4, 5] which would allow to derive relations for the special conformal anomalies [6] by means of the anomalous superconformal Ward identities.

2. \( \mathcal{N} = 1 \) QCD. The Lagrangian for the \( \mathcal{N} = 1 \) super-Yang-Mills theory [7] in the Wess-Zumino gauge takes the form which resembles ordinary one-flavour QCD with Majorana fermions, \( \psi = C \bar{\psi}^T \), in the adjoint representation of the \( SU(N_c) \) group

\[
\mathcal{L}_{cl} = -\frac{1}{4} (G^a_{\mu\nu})^2 + \frac{i}{2} \bar{\psi}^a \mathcal{D}^{ab} \psi^b + \frac{1}{2} (D^a)^2. \tag{2}
\]

The Wess-Zumino gauge breaks linear \( \mathcal{N} = 1 \) supersymmetry but the remaining short supermultiplet \( (B^a_\mu, \psi^a, D^a) \) respects the restricted non-linear supersymmetric transformation laws

\[
\delta^Q \psi^a = \frac{i}{2} C^a_{\mu\nu} \sigma_{\mu\nu} \zeta - i D^a \gamma_5 \zeta, \quad \delta^Q B^a_\mu = -i \bar{\zeta} \gamma_\mu \psi^a, \quad \delta^Q D^a = \bar{\zeta} \mathcal{D}^{ab} \gamma_5 \psi^b. \tag{3}
\]
The action transforms w.r.t. these transformations as \( \delta^Q S = - \int d^4 x \left\{ \partial_\mu \tilde{\zeta} Q_\rho - \sum_\phi (\delta S/\delta \phi) \delta^Q \phi \right\} \) with the non-anomalous (on quantum level) \([6]\) supersymmetry current \( Q_\rho = \frac{1}{2} C^a_{\mu \nu} \sigma_{\mu \nu} \gamma_\rho \psi^a \). It is conserved on the mass shell: \( \partial_\mu \tilde{\zeta} Q_\rho = \sum_\phi (\delta S/\delta \phi) \delta^Q \phi \) and, thus, \( \delta^Q S = 0 \).

Eqs. \([3]\) form, however, a modified algebra which contains apart from the usual terms on the r.h.s. of \([3]\) also a gauge transformation \( \delta_{\text{gauge}} \), i.e. \( [\delta^Q_1, \delta^Q_2] = -2i a_\mu \delta^Q_\mu + \delta_{\text{gauge}} \), with field dependent gauge parameter, \( 2i \tilde{C}_1 B^a \zeta_2 \), and translation vector \( a_\mu = \tilde{\zeta}_1 \gamma_\mu \zeta_2 \). Note, however, that they define the ordinary SUSY operator algebra \([8]\) in the space spanned on gauge invariant objects.

In our consequent discussion we will be most interested in the commutator algebra restricted to the action on the “good” light-cone components \([12]\) of the field operators introduced as follows \(^3 \)

with projectors \( \Pi_\pm = \frac{1}{2} \gamma_\mp \gamma_\pm \) for fermion fields: the \( \pm \)-components of which are defined via 
\[ \psi^a_\pm = \Pi_\pm \psi^a; \]
and with the two-dimensional metric tensor \( g^\pm_{\mu \nu} = g_{\mu \nu} - n_\mu n^*_\nu - n^* n_\nu \), for bosons: 
\[ B^a_\pm = g^\pm_{\mu \nu} B^a_{\mu \nu}. \]
We also impose a restriction on the constant Majorana fermion \( \zeta_+ \equiv \Pi_+ \zeta = 0 \), so that \( \delta^Q B^a_+ = 0 \). Then a peculiar feature of the supersymmetry transformations \([3]\) becomes manifest, namely, that they do not involve the auxiliary field, \( D^a \), provided we restrict ourselves to the “good” components — the ones which enter the quasi-partonic conformal operators defined below in Eq. \([3]\). Explicitly \([13, 14]\)

\[ \delta^Q \psi^a_+ = -G^a_+ \gamma_- \gamma^\perp \zeta, \quad \delta^Q B^a_+ = -i \bar{\zeta} \gamma^\perp \psi^a_+. \]  

Moreover, they form a representation of the unaltered supersymmetry algebra \([10]\).

3. Supermultiplet of conformal operators. To derive the relations for the anomalous dimensions we have first to find an irreducible representations of the superalgebra in the basis of the conformal operators \([5]\):

\[
\begin{pmatrix}
Q \sigma^V \\
Q \sigma^A
\end{pmatrix}
_{jl} = \frac{1}{2} \bar{\psi}_+(i \partial_+)^l \gamma_7 C^{j/2} \begin{pmatrix}
\tilde{\mathcal{D}}_+ \\
\partial_+
\end{pmatrix} \psi^a_+, \\
\begin{pmatrix}
\tilde{\mathcal{C}} \sigma^V \\
\tilde{\mathcal{C}} \sigma^A
\end{pmatrix}
_{jl} = G^a_+ \gamma_- \gamma^\perp \zeta, \\
\begin{pmatrix}
\tilde{\mathcal{C}} \sigma^V \\
\tilde{\mathcal{C}} \sigma^A
\end{pmatrix}
_{jl} = \frac{1}{2} \bar{\psi}_+(i \partial_+)^l \gamma_7 C^{j/2} \begin{pmatrix}
\tilde{\mathcal{D}}_+ \\
\partial_+
\end{pmatrix} G^a_+,
\]

where \( \partial = \partial + \bar{\partial} \) and \( \mathcal{D} = \mathcal{D} - \bar{\mathcal{D}} \), and the factor \( \frac{1}{2} \) in front of the fermion operator serves to avoid double counting of the same components of the spinors due to their Majorana nature.

According to the appendix one can introduce the following combinations of the conformal operators

\[
\begin{pmatrix}
S^1 \\
\mathcal{P}^1
\end{pmatrix}
_{jl} = \frac{6}{j} \tilde{\mathcal{C}} \sigma^V_{jl} + \tilde{\mathcal{C}} \sigma^A_{jl}, \\
\begin{pmatrix}
S^2 \\
\mathcal{P}^2
\end{pmatrix}
_{jl} = \frac{6}{j + 1} \tilde{\mathcal{C}} \sigma^V_{jl} - \frac{j + 3}{j + 1} \tilde{\mathcal{C}} \sigma^A_{jl},
\]

\(^3\)The + and – components of any vector are obtained by contraction with the two light-like vectors \( n \) and \( n^* \), such that \( n^2 = n^{*2} = 0 \) and \( nn^* = 1 \).
with \( \Gamma = V(A) \) for the upper (lower) entry on the l.h.s. of these equations. By construction they transform covariantly w.r.t. supertransformations and together with

\[
\left\{ \mathcal{V} \right\}_{jl} = \frac{(j+2)(j+3)}{(j+1)} G_{\gamma\mu}^{a\perp} (i \partial_\gamma) \partial_j (\gamma_\mu^+(\partial_+) \gamma_\mu^-) \langle 1 \rangle \psi^a, \tag{10}
\]

form an irreducible representation with the transformation laws:

\[
\begin{align*}
\delta^Q S_{jl}^1 &= \frac{1}{2} [1 - (-1)^j] \bar{\zeta} \mathcal{V}_{j-1l}, \\
\delta^Q S_{jl}^2 &= \frac{1}{2} [1 - (-1)^j] \bar{\zeta} \mathcal{V}_{jl}, \\
\delta^Q \mathcal{P}_{jl}^1 &= \frac{1}{2} [1 + (-1)^j] \bar{\zeta} \mathcal{U}_{j-1l}, \\
\delta^Q \mathcal{P}_{jl}^2 &= \frac{1}{2} [1 + (-1)^j] \bar{\zeta} \mathcal{U}_{jl}, \\
\delta^Q \mathcal{V}_{j-1l} &= -\gamma - \zeta \left\{ S_{jl}^1 + S_{jl-1}^2 \right\} - \gamma - \zeta \left\{ \mathcal{P}_{jl}^1 + \mathcal{P}_{jl-1}^2 \right\}.
\end{align*}
\]

The transformation law for the operator \( \mathcal{U}_{jl} \) follows from the observation \( \mathcal{U}_{jl} = -\gamma_5 \mathcal{V}_{jl} \) and the \( \text{Eq. (11)} \).

4. Ward identities and commutator constraints. To proceed with our derivation of the Ward identities we have to fix the remaining ordinary gauge degrees of freedom of the classical Lagrangian (3), i.e. to add a gauge fixing term together with an associated ghost piece. Provided we would work in the superfield formalism this can be done in a way which preserves linear supersymmetry with SUSY Fermi-Feynman gauge. However, then we should proceed with the complete gauge supermultiplet at an expense of a number of auxiliary fields on top of the dynamical \( \psi \) and \( B \).

To achieve this goal in the Wess-Zumino supergauge without explicit breaking of the supersymmetry on the Lagrangian level one is forced to use the light-cone gauge\(^4\) so that \( \mathcal{L} = \mathcal{L}_{cl} + \mathcal{L}_{gf} + \mathcal{L}_{gh} \), where \( \mathcal{L}_{gf} = -\frac{1}{2 \xi} \left( B_a^\mu \right)^2 \), \( \mathcal{L}_{gh} = \bar{\omega}^a \mathcal{D}_{\gamma}^\mu \omega^b \). Ghosts decouple from all Green functions since \( n_\mu \langle B^a_\mu(x_1) B^b_\mu(x_2) \rangle |_{\xi \to 0} = 0 \). However, because the Lorentz invariance is violated by fixing a particular direction in the Minkowski space with the vector \( n_\mu \), different (i.e. “good” and “bad”) components of the field operators renormalize with different renormalization constants and, moreover, non-local counterterms are required to cure all divergencies of the classical action, i.e.\( \text{[20]} \)

\[
B^a_\mu \to Z^{1/2}_3 \left( B^a_\mu - (1 - \bar{Z}_3^{-1}) n_\mu \Omega^a \right), \quad \psi \to Z^{1/2}_2 \left( \Pi_+ + \bar{Z}_2 \Pi_- \right) \psi, \quad g \to Z_g g, \quad \xi \to Z_\xi \xi.
\]

\(^4\)Covariant gauges inevitably break SUSY since the necessary condition for keeping symmetry of the Lagrangian intact is that the algebra of the gauge and rigid symmetries has the form of a semi-direct product \([13]\). However, this breaking is only due to the BRST exact operator \( \delta^\text{BRST} \delta^Q (\bar{\omega}^a \partial_\mu B^a_\mu) \) (since \( [\delta^\text{BRST}, \delta^Q] = 0 \) \([11, 16]\)) and will not affect physical quantities. We shall address this question within the present context elsewhere. Next, the axial gauge breaks the supersymmetry of the Lagrangian but leads to supersymmetric counterterms for the gauge multiplet \([17, 18]\). However, the latter property is violated as soon as matter superfields are taken into account \([18]\). In spite of the fact that we are considering only gauge multiplet we prefer to deal with supersymmetry preserving gauge fixing on the Lagrangian level.
where $\Omega^a = \left( D_+^{-1} G_{++} \right)^a$ is the only non-local structure involved. This instance complicates the derivation and the use of the conformal Ward identities. However, the fact that only “good” components are relevant suggests to integrate out the “bad” ones in favour of the “good” ones in the path integral for the Green functions with operator insertions, $O$, 

$$\langle O' \rangle = \lim_{\xi \to 0} \left( \int D\phi \, e^{iS} \right)^{-1} \int D\phi \, O e^{iS}, \quad (12)$$

where $S = \int d^d x \mathcal{L}$ and $X$ is a monomial of the “good” components of the field operators, i.e. $X = \prod_i \psi_+ (x_i) \prod_j \psi_+ (x_j) \prod_k B^k (x_k)$. In spite of the fact that this procedure results in the non-local form of the action, the Lagrangian manifests explicit supersymmetry w.r.t. renormalized transformations (4) since from the Slavnov-Taylor identities it follows that $Z_3 = Z_2 \equiv Z_\phi \quad (14)$. To maintain this property in perturbation theory it is necessary to deal with UV divergencies in a supersymmetric way, i.e. by regularization via, e.g. Siegel’s dimensional reduction (DRED) [22, 23, 24]. Provided we have done this, the light-cone gauge preserves SUSY (4) on the quantum level as well and the transformations (4) remain unaffected even for the renormalized fields. This was checked in one-loop approximation for the principal value (PV) [19] and the Mandelstam-Leibbrandt (ML) [14] prescription on auxiliary $1 \epsilon_x$-pole in the gluon propagator. In the latter case $Z_g = Z_3^{-1/2}$, while in the former $Z_{2,3}$-factors are $\epsilon$-dependent due to the fact that the PV prescription violates power counting. This can be traced back to the breaking of rescaling invariance of the gluon density matrix.

It is obvious, that in the course of renormalization the operators $S^i$ ($P^i$) mix with each other. We introduce the renormalized operators according to

$$[O_{jl}] = \sum_{k=0}^{j} \{ Z_O \}_{jk} Z_\phi O^{(0)}_{kl}, \quad \text{with} \quad Z_O = \begin{pmatrix} 11 Z_O & 12 Z_O \\ 21 Z_O & 22 Z_O \end{pmatrix}, \quad Z_\phi = \begin{pmatrix} Z^{-1}_\phi & 0 \\ 0 & Z^{-1}_\phi \end{pmatrix}, \quad (13)$$

where $O_{jl}$ stands for a two-dimensional vector $S = (S^1, S^2)$, and similar for $P^i$. The superscript $(0)$ means that the operator is written in terms of bare fields. And the anomalous dimensions are defined via the renormalization group equations

$$\mu \frac{d}{d\mu} [O_{jl}] = -\sum_{k=0}^{j} \gamma_{jk} O_{kl}, \quad \text{with} \quad \gamma^O = \begin{pmatrix} 11 \gamma^O & 12 \gamma^O \\ 21 \gamma^O & 22 \gamma^O \end{pmatrix} \quad \text{for} \quad O = S, P, \quad (14)$$

$$\mu \frac{d}{d\mu} [V_{jl}] = -\sum_{k=0}^{j} \lambda_{jk} V_{kl}.$$  

The fermionic operators $V$ and $U$ evolve with the same anomalous dimensions, $\lambda_{jk}$, since $U_{jl} = -\gamma_5 V_{jl}$. Note, that from symmetry properties of the operators it follows that the anomalous

\footnote{This is not a fully self-consistent regularization method [24] but at least its reliability as a scheme which respects supersymmetry has been checked on two-loop level [23].}
dimensions for the parity even/odd operators vanish for even/odd \( j \), while they admit both values for fermionic case.

For the derivation of the Ward identities we proceed with the regularization of the action via dimensional reduction since otherwise \( \delta^D \mathcal{L}_{cl} = -\frac{\kappa}{2} f^a_{abc} \psi^a \gamma_\mu \psi^b \zeta \gamma_\mu \psi^c \) due to inapplicability of the 4-dimensional Fiertz identity. Although it is a legitimate procedure to use a symmetry breaking regulator, but this would result to an addendum in the Ward identity and as a result in the relations we are going to derive. With Siegel’s method at hand the supersymmetry Ward identity simply reads

\[
\langle [\mathcal{O}_{jl}]^{\delta\mathcal{Q}} \mathcal{X} \rangle = -\langle \delta^{\mathcal{Q}} [\mathcal{O}_{jl}] \mathcal{X} \rangle,
\]

with \( \delta^Q \mathcal{O}_{jl} \) given by Eqs. (8)-(10). Finally, the dilatation Ward identity \( \mathbb{I} \) adjusted to the present scheme is

\[
\langle [\mathcal{O}_{jl}]^{\delta\mathcal{D}} \mathcal{X} \rangle = \sum_{k=0}^{j} \{ (l + 3) \mathbb{I} + \gamma^O \}_j \langle [\mathcal{O}_{kl}] \mathcal{X} \rangle + \sum_i \mathcal{F}^i(g) \langle i[\mathcal{O}_{jl}\Delta^\mathcal{X}] \mathcal{X} \rangle,
\]

where the last term stands for Green functions with the renormalized operator insertions, i.e. differential vertex operator insertion \([\Delta^\mathcal{O}] = g \frac{\partial}{\partial g} \mathcal{L}\) and equation of motion operators \( \Omega_{\phi} = \phi \frac{\delta S}{\delta \phi} \). Here \( l + 3 = l - 1 + 2(d^\text{can}_G + 1) = l + 2d^\text{can}_Q \), for glionic and fermionic operators, respectively, with canonical dimension \( d^\text{can}_\phi \) of the field \( \phi \). Thus from the last commutator \( \mathbb{I} \) in Eq. (11) acting on the Green function with conformal operator insertion \( \mathbb{I} \) \( \langle [\mathcal{O}_{jl}]^{\delta\mathcal{Q}} \mathcal{D} \mathcal{X} \rangle \) \( \mathbb{I} \) and from the Ward equalities displayed above we can easily obtain a set of identities for the anomalous dimensions. For this one has to note that \( \langle [\mathcal{O}_{jl}\Delta^\mathcal{O}] \mathcal{X} \rangle = g \frac{\partial}{\partial g} \langle [\mathcal{O}_{jl}] \mathcal{X} \rangle \) and \( \sum_{\phi=\psi,G} \gamma_{\phi} \langle [\mathcal{O}_{jl}\Omega_{\phi}] \mathcal{X} \rangle = i\gamma_\psi \mathcal{V} \langle [\mathcal{O}_{jl}] \mathcal{X} \rangle \mathbb{I} \) with \( \gamma_\psi = \gamma_G = \gamma_\phi \) due to the gauge we have chosen. \( N \) is the total number of fields in \( \mathcal{X} \). Therefore, these terms drop out from the commutator of the scale and supersymmetry transformations and thus do not show up in the constraints. Namely, for \( \mathcal{O} = \mathcal{S} \) we have in components

\[
\sum_{m=0}^{n} \{ 11 \gamma_{2n+1,2m+1}^{S} [\mathcal{V}_{2m,l}] + 12 \gamma_{2n+1,2m+1}^{S} [\mathcal{V}_{2m+1,l}] \} = \sum_{m=0}^{n} \lambda_{2n,2m} [\mathcal{V}_{2m,l}] + \sum_{m=0}^{n-1} \lambda_{2n,2m+1} [\mathcal{V}_{2m+1,l}],
\]

\[
\sum_{m=0}^{n} \{ 21 \gamma_{2n+1,2m+1}^{S} [\mathcal{V}_{2m,l}] + 22 \gamma_{2n+1,2m+1}^{S} [\mathcal{V}_{2m+1,l}] \} = \sum_{m=0}^{n} \lambda_{2n+1,2m} [\mathcal{V}_{2m,l}] + \sum_{m=0}^{n-1} \lambda_{2n+1,2m+1} [\mathcal{V}_{2m+1,l}],
\]

while for \( \mathcal{O} = \mathcal{P} \)

\[
\sum_{m=0}^{n-1} \{ 11 \gamma_{2n,2m+2}^{P} [\mathcal{U}_{2m+1,l}] + 12 \gamma_{2n,2m+2}^{P} [\mathcal{U}_{2m+1,l}] \} = \sum_{m=0}^{n} \lambda_{2n-1,2m} [\mathcal{U}_{2m,l}] + \sum_{m=0}^{n-1} \lambda_{2n-1,2m+1} [\mathcal{U}_{2m+1,l}],
\]

\[
\sum_{m=0}^{n-1} \{ 21 \gamma_{2n,2m+2}^{P} [\mathcal{U}_{2m+1,l}] + 22 \gamma_{2n,2m+2}^{P} [\mathcal{U}_{2m+1,l}] \} = \sum_{m=0}^{n} \lambda_{2n,2m} [\mathcal{U}_{2m,l}] + \sum_{m=0}^{n-1} \lambda_{2n,2m+1} [\mathcal{U}_{2m+1,l}].
\]

\( ^6 \)Obviously, only “good” components are endowed with well defined canonical dimensions.

\( ^7 \)Note that its nonzero r.h.s. just serves to shift the canonical scale dimension of the elementary field \( \phi \) by \( \frac{1}{2} \) in mass units, thus changing, for instance \( \delta^\text{can}_G \rightarrow \delta^\text{can}_G + \frac{1}{2} = \delta^\text{can}_Q \).
Then, using linear independence of conformal operators we get finally the following relations
\begin{align}
11\gamma_{2n+1,2m+1}^S &= 22\gamma_{2n,2m}^P = \lambda_{2n,2m}, & m \leq n, \\
12\gamma_{2n+1,2m+1}^S &= 21\gamma_{2n,2m+1}^P = \lambda_{2n,2m+1}, & m \leq n - 1, \\
21\gamma_{2n+1,2m+1}^S &= 12\gamma_{2n+2,2m}^P = \lambda_{2n+1,2m}, & m \leq n, \\
22\gamma_{2n+1,2m+1}^S &= 11\gamma_{2n+2,2m+2}^P = \lambda_{2n+1,2m+1}, & m \leq n, \\
12\gamma_{2n+1,2m+1}^S &= 0 , & 12\gamma_{2n,2m}^P = 0 .
\end{align}

This is the most general set of equations for the anomalous dimensions of operators with total derivatives which arise in a theory with $\mathcal{N} = 1$ supersymmetry.

5. **Reduced supersymmetry relations.** From the above relations for the anomalous dimensions of the operators which form a supermultiplet we can deduce identities for the familiar anomalous dimensions of quark and gluon operators. For this purpose note that they are related to each other via the following matrix equation
\begin{align}
\begin{pmatrix}
\frac{1}{k} & 11\gamma_{jk} \\
\frac{1}{k+1} & 12\gamma_{jk} \\
\frac{1}{k} & 21\gamma_{jk} \\
\frac{1}{k+1} & 22\gamma_{jk}
\end{pmatrix} &= \frac{1}{2k+3}
\begin{pmatrix}
1 & \frac{k+3}{6} & \frac{6}{j} & \frac{k+3}{j} \\
-1 & \frac{k}{6} & \frac{6}{j+1} & \frac{k}{j+1} \\
-j+3 & \frac{(k+3)(j+3)}{6(j+1)} & \frac{6}{j+1} & \frac{k+3}{j+1} \\
j+3 & \frac{k(j+3)}{6(j+1)} & \frac{6}{j+1} & \frac{k}{j+1}
\end{pmatrix}
\begin{pmatrix}
QQ_{\gamma_{jk}} \\
GQ_{\gamma_{jk}} \\
QG_{\gamma_{jk}} \\
GG_{\gamma_{jk}}
\end{pmatrix}.
\end{align}

Then the transformation to the usual operator basis becomes trivial. Let us consider some particular limits of Eqs. (18)-(21). Namely, from Eqs. (22) we obtain the well-known Dokshitzer relations empirically established in the original paper [3] for the vector channel (here and below $\gamma_{jj} \equiv \gamma_{j}$)
\begin{align}
QQ_{\gamma_{j+1}} + \frac{6}{j+1}GQ_{\gamma_{j+1}} = \frac{j}{6}GQ_{\gamma_{j}} + GQ_{\gamma_{j}}, & i = V, A.
\end{align}

From (18) and (21), reduced to the forward case, we have
\begin{align}
QQ_{\gamma_{j+1}} + \frac{6}{j+1}GQ_{\gamma_{j+1}} = QQ_{\gamma_{j}} - \frac{j}{6}GQ_{\gamma_{j}} & \quad \text{and} \quad QQ_{\gamma_{j+1}} + \frac{6}{j+1}GQ_{\gamma_{j+1}} = QQ_{\gamma_{j}} - \frac{j}{6}GQ_{\gamma_{j}}.
\end{align}

Finally, from Eqs. (19) and (20) a relation follows between the diagonal (read forward) and non-diagonal elements of the anomalous dimensions of the conformal operators
\begin{align}
\frac{6}{j}GQ_{\gamma_{j}} - \frac{j}{6}GQ_{\gamma_{j}} &= \frac{j+1}{2j+1}A_{j+1,j-1}, & \frac{6}{j}GQ_{\gamma_{j}} - \frac{j}{6}GQ_{\gamma_{j}} &= \frac{j+1}{2j+1}A_{j+1,j-1}, \\
\Delta_{j+1,j-1}^{V} &\equiv \frac{j-1}{j+1}GQ_{\gamma_{j+1,j-1}} + \frac{j-1}{6}GQ_{\gamma_{j+1,j-1}} - \frac{6}{j+1}GQ_{\gamma_{j+1,j-1}} - QQ_{\gamma_{j+1,j-1}}.
\end{align}

Obviously, in the leading order conformal operators do not mix and the r.h.s. of the equations (27) are zero. However, beyond one loop [4] these equations provide a non-trivial check of existing results.
Figure 1: Generic form of Feynman diagrams for $z$ projected onto the tensor structures given in the text.

6. Transformation from SUSY preserving to conventional DREG schemes. Let us address now the question of explicit checks of the above predictions beyond leading order of QCD perturbation theory. Since all above equations hold only for the entities evaluated by means of supersymmetry preserving dimensional reduction we have to compute rotation matrices to the conventional dimensional regularization (DREG) — a scheme used in practical QCD calculations where all higher order results are available. Note that the supersymmetric limit of ordinary QCD can be achieved by equating the Casimir operators $C_A = C_F = 2T_F = N_c$.

The change of the scheme is achieved via the following finite transformation of the quark-gluon operator $\mathcal{O} = \left(\begin{array}{c} Q \end{array}\right)$ renormalized according to the conventional DREG to the DRED scheme $\left[\mathcal{O}\right]^{\text{DRED}} = z\left[\mathcal{O}\right]^{\text{DREG}}$. Thus, the quark-gluon anomalous dimension matrix, $\gamma$, for the regularization with DRED are related to the DREG one via

$$\gamma^{\text{DRED}} = z\gamma^{\text{DREG}} z^{-1} - \beta(g) \frac{\partial}{\partial g} z \cdot z^{-1}. \quad (28)$$

In order to check the relations we have derived above in the two-loop approximation the problem is thus reduced to the computation of $z$ at $\mathcal{O}(\alpha_s)$ (see Fig. 1). Let us add few remarks on this calculation which has been performed with ordinary QCD Feynman rules identifying afterwards the Casimir operators. Since the Clifford algebra is considered as 4-dimensional in the DRED as well as in conventional DREG the projectors used are the same in both schemes $Q_{\mu}^{\text{(V,A)}} = \frac{1}{2}(1, \gamma_5)\gamma_-$ for the parity even and odd sectors, respectively. On the other hand the gluon polarization vectors are treated as 4-dimensional in DRED and $d$-dimensional in DREG. We have then for parity even case $G_{\mu}^{\text{V}} = (d - 2)^{-1}g_{\mu}^{\perp(d)}$ with $d = 4$ for DRED and $d = 4 - 2\epsilon$ for conventional DREG, while for odd parity $G_{\mu}^{\text{A}} = i(d - 2)^{-1}(d - 3)^{-1}\epsilon_{\mu\rho\sigma}$. We have used the HVBM scheme [32] for dealing with $\gamma_5$ and $\epsilon_{\mu\rho\sigma}$ which are pure 4-dimensional objects. These procedure has proved to be the most reliable for these purpose. Moreover, as a cross check we have adopted also Larin’s prescription [33] according to which we have used the substitution $\gamma_5 = -\frac{i}{4}\epsilon_{\mu\rho\sigma}\gamma_{\mu}\gamma_{\rho}\gamma_{\sigma}$ in quark-gluon diagrams and the resulting product of two $\epsilon$-tensors has

Remarkable that the number of different transformation matrices in the forward case equals to the number of independent investigations on the subject [25]-[31]. Diversity of opinions is welcome but outside physics.
been understood as $\epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} = -4! g_\mu^\alpha g_\nu^\beta g_\rho^\gamma g_\sigma^\delta$ with the metric tensors being $d$-dimensional for DREG and 4-dimensional for DRED. Note, however, that we have been able to obtain the same results as for the HVBM recipe only for $QG$ and $GG$ graphs and have failed for others. Due to gauge invariance of the rotation matrices we have done the calculations using covariant Feynman and non-covariant light-cone gauges with indeed identical final results.

Finally, we have found the finite part of the anomalous dimensions of the quark and gluon conformal operators, obtained from the difference of the supersymmetry preserving and the conventional dimensional regularization schemes, which reads

$$z_{jk} = \mathbb{I}_j \delta_{jk} + \frac{\alpha_s}{2\pi} N_c \left\{ z_j^D \delta_{jk} + z_j^{ND} \theta_{j-2,k} [1 + (-1)^{j-k}] \right\}, \quad (29)$$

with the following diagonal matrices

$$z_j^{D,V} = \begin{pmatrix} -\frac{j(j+3)}{2(j+1)(j+2)} & \frac{12}{j(j+2)(j+3)} \\ -\frac{j}{6(j+2)} & -\frac{1}{6} \end{pmatrix}, \quad z_j^{D,A} = \begin{pmatrix} -\frac{j(j+3)}{2(j+1)(j+2)} & \frac{12}{j(j+1)(j+2)} \\ -\frac{j}{3(j+1)(j+2)} & -\frac{1}{6} - \frac{4}{(j+1)(j+2)} \end{pmatrix}, \quad (30)$$

for the vector and axial channels, respectively, and the universal non-diagonal part

$$z_j^{ND} = \begin{pmatrix} 0 & \frac{6(2k+3)}{k(k+1)(k+2)} \\ -\frac{6(2k+3)}{(2k+3)(k-k)(k+k+3)} & -\frac{k(k+1)(k+2)(k+3)}{k(k+1)(k+2)(k+3)} \end{pmatrix}. \quad (31)$$

In the course of the calculation we have clarified the reason for a difference in the rotation matrices given in the literature for forward scattering. For instance, the recent results of Ref. [31] can be reproduced in practically all cases (except axial $GQ$ channel) provided we discard the contributions of $\epsilon$-scalars in the external lines. We can hardly advocate this recipe since the $z$-matrices have to be considered as insertions into the internal virtual lines, so that the $\epsilon$-scalars contribute on equal footing with the $d$-dimensional gauge particles. The diagonal eigenvalues, $z_j^D$, coincide with [25, 28] for parity even and with [30] for parity odd operators.

Note that only diagonal elements of these matrices are required to fulfill the supersymmetry relations in Eqs. (24) and (25) which can be explicitly checked transforming the results of Refs. [29, 30, 34] with the help of the rotation matrices (30) to the DRED scheme. In Eq. (26) the off-diagonal elements [6] enter which have to be rotated with (31) to the supersymmetry preserving scheme as well. The appearance of these turns out to be the reason why the authors of Ref. [31] have found the violation of the third SUSY relation in NLO — they have used an equation of [2] which corresponds to our Eq. (26) but with zero r.h.s.

Moreover, we have found that the general relations (18-21) are satisfied when transformed with the rotation matrices we have derived presently. This provides a confirmation for the correctness on the two-loop anomalous dimensions for the QCD composite operators available in the literature [6, 29, 30, 34].
7. Summary. To summarize we have derived in the present study a general set of relations for the anomalous dimensions of the bosonic and fermionic conformal composite operators in the Yang-Mills theory with $\mathcal{N} = 1$ supersymmetry. In their reduced form two of them were known earlier while the last one (Eq. (26)) contains a novel non-vanishing r.h.s. with the non-diagonal elements. This explains the difficulties in the check of their validity beyond leading order of perturbation theory observed before. We have thus supported our results for the non-diagonal NLO anomalous dimensions of the conformal operators derived in [6] and we now have a stronger evidence for the supersymmetric nature of the universality of the special conformal anomalies of conformal operators conjectured there. This issue is currently under study [36].

This work was supported by BMBF and the Alexander von Humboldt Foundation (A.B.).

Appendix. Here we briefly describe the main steps for the construction of an irreducible representation of supersymmetry in the basis of the conformal composite operators. The variation of the bosonic conformal operators in Eq. (5) leads to

\[
\delta^Q \left\{ \frac{Q^V}{Q^A} \right\}_{jl} = \left[ 1 - (-1)^j \right] \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\} \frac{(-1)(j + 2)}{2} \zeta G_{+\mu}^a(i\partial_\pm)^l P_j^{(1,1)} \left( \frac{\overleftrightarrow{\partial}_+}{\partial_+} \right) \gamma_\mu \left\{ 1 \right\} \psi^a_+,
\]

\[
\delta^Q \left\{ \frac{G^V}{G^A} \right\}_{jl} = \left[ 1 - (-1)^j \right] \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\} \frac{(j + 2)(j + 3)}{12} \zeta G_{+\mu}^a(i\partial_\pm)^{l-1} i\overleftrightarrow{\partial}_+ P_{j-1}^{(2,2)} \left( \frac{\overleftrightarrow{\partial}_+}{\partial_+} \right) \gamma_\mu \left\{ 1 \right\} \psi^a_+.
\]

On the other hand since the fermionic operator, $(G\psi)_{jj}$, like the bosonic ones (5) has to be the highest weight vector of the corresponding conformal tower, $[(G\psi)_{jj}, \mathcal{K}_-] = 0$, with the generator of the special conformal transformation $\mathcal{K}_\mu$, this requirement fixes the indices of the Jacobi polynomial to be $d_\phi + s_\phi - 1$. Here $d_\phi$ and $s_\phi$ are the canonical scale dimension and spin of the field $\phi$, respectively. For the case at hand we have thus $P_j^{(2,1)}$. This condition implies severe constraints on the coefficients with which the operators (5) can enter $Q_j^C Q_{jl} + G_j^C G_{jl}$, which form an irreducible representation of the supersymmetry algebra. The corresponding equation for $C_j$ has two solutions. They are given in Eq. (5).

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