Nonclassical Properties of Coherent States

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It is demonstrated that a weak measurement of the squared quadrature observable may yield negative values for coherent states. This result cannot be reproduced by a classical theory where quadratures are stochastic \( \psi \)-numbers. The real part of the weak value is a conditional moment of the Margenau-Hill distribution. The nonclassicality of coherent states can be associated with negative values of the Margenau-Hill distribution. A more general type of weak measurement is considered, where the pointer can be in an arbitrary state, pure or mixed.

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Harmonic oscillator coherent states were first investigated by Schrödinger, who was looking for classical-like states [1]. There are several ways in which coherent states are the “most classical” of any pure state. They keep their shape, not spreading out as they move in the harmonic oscillator potential [1]. They minimize Heisenberg’s uncertainty relation, with equal uncertainty in both quadratures. In this way, they are the closest possible quantum mechanical representation of a point in phase space. The term “coherent state” was introduced by Glauber [2]. He demonstrated that coherent states are produced when an essentially classical current interacts with the radiation field [2]. Aharonov et. al. demonstrated that coherent states are the only pure states that produce independent output when split in two [3]. Zurek et. al. have demonstrated that coherent states are natural “pointer states” for a harmonic oscillator weakly coupled to a thermal environment [4].

Glauber and Sudarshan demonstrated that any density operator can be expanded in terms of coherent states [2, 5]

\[
\hat{\rho} = \int d^2 \alpha P(\alpha) \, |\alpha\rangle \langle \alpha|.
\]  

The weight function \( P(\alpha) \) is known as the \( P \)-distribution. Glauber defined nonclassical states as those for which the \( P \)-distribution fails to be a probability density. More specifically, nonclassical states have a \( P \)-distribution which is negative or more singular than a \( \delta \)-function [2, 6–9]. This criterion is the basis of various measures of “nonclassicality” [10–13].

It is the purpose of this Letter to demonstrate that a quantum state may be nonclassical even though the \( P \)-distribution is a probability density, and that also coherent states display nonclassical characteristics. In this Letter, we associate nonclassicality with the failure of the Margenau-Hill distribution [14] to be a probability distribution. The Margenau-Hill distribution yields correct marginal distributions, just as the Wigner distribution [15]. But in contrast to the Wigner distribution, it is negative for coherent states [16]. We give an operational significance to conditional moments of the Margenau-Hill distribution by demonstrating that they can be observed in “weak measurements”. Weak measurements were proposed by Aharonov et. al. [17]. Their suggestion was initially met with criticism [18–20], but has since been confirmed in various ways (see, e.g., [21–24]). The results reported in this Letter are related to a paper by Aharonov et. al. [25], which demonstrated that a weak measurement of kinetic energy of a particle in a classically forbidden region might yield negative values.

In the original von Neumann measurement scheme [26], it was found that in order to distinguish different eigenvalues of the object, the pointer should be in a state with small uncertainty in the pointer position. Aharonov et. al. [17] proposed to define weak measurements by using a pointer with a large pointer position uncertainty. In this Letter, we abandon this condition. Instead, we assume that the interaction between the pointer and the object is sufficiently weak. Thus, the pointer can be in an arbitrary state, pure or mixed. We impose only one condition on the pointer, namely that the current density should vanish.

We consider an object and a pointer described by the density operators \( \hat{\rho}_o \) and \( \hat{\rho}_p \), respectively. Prior to the measurement interaction, the combined object plus pointer is assumed to be in a product state \( \hat{\rho}_0 = \hat{\rho}_o \otimes \hat{\rho}_p \). We wish to perform a weak measurement of an arbitrary object observable \( \hat{c} \). To this end, we shall assume that the interaction part of the Hamiltonian has the form

\[
\hat{H}_I = \epsilon \delta(t) \, \hat{c} \otimes \hat{P}.
\]  

This interaction Hamiltonian is essentially the same as proposed in Ref. [17], except that we have introduced an interaction strength \( \epsilon \). It is a generalization of the interaction Hamiltonian proposed by von Neumann [26]. It has been discussed in detail for “strong” measurements in Ref. [27]. \( \hat{P} \) is the momentum observable of the pointer. We will consistently denote observables associated with the pointer by capital letters. We assume that during the measurement interaction, the interaction part of the Hamiltonian dominates the time evolution. Nevertheless, we shall assume that the interaction between the object and pointer is weak, i.e., \( \epsilon \) is so small that we can perform

\[
\int d^2 \alpha P(\alpha) \, |\alpha\rangle \langle \alpha|
\]
a series expansion to first order in $\epsilon$. The possibility of realizing this or similar interactions experimentally will be discussed at the end of this Letter.

Because of the interaction between the object and pointer, the density operator evolves to $\hat{\rho}_e = \hat{U}_e \hat{\rho}_0 \hat{U}_e^\dagger$, where the unitary evolution operator $\hat{U}_e$ is (setting $\hbar = 1$)

$$
\hat{U}_e = e^{-i \int \hat{H}_e(t) dt} = e^{-i e\hat{Q} \hat{\rho}}.
$$

(3)

In this experiment, we are interested in the final values of the pointer position $\hat{Q}$ and the object position $\hat{q}$. The joint probability distribution for these observables is

$$
\rho_e(Q, q) = \langle q | \otimes (Q | \hat{\rho}_e | Q) \otimes | q \rangle.
$$

(4)

We require that the current density of the pointer vanishes,

$$
\langle Q | \hat{P} \hat{\rho}_e | Q \rangle + \langle Q | \hat{\rho}_e \hat{P} | Q \rangle = 0.
$$

(5)

This is our only restriction on the state of the pointer. It can then be shown that to the first order in $\epsilon$, the probability density for the pointer position $\hat{Q}$ conditioned on the object position $q$ reads [28]

$$
\rho_e(Q | q) = \frac{\rho_e(Q, q)}{\int dQ \rho_e(Q, q)} \approx \hat{T} \langle Q | \hat{\rho}_0 | Q \rangle,
$$

(6)

where

$$
\hat{T} = 1 - \epsilon \text{Re}(c_w) \frac{\partial}{\partial Q}
$$

(7)

is a first order translation operator, and where

$$
c_w(q) = \langle q | \hat{\rho}_e | q \rangle.
$$

(8)

is the weak value of $\hat{c}$ for an object preselected in a mixed state $\hat{\rho}_e$ and postselected in the eigenstate $| q \rangle$ [17, 29, 30]. This shows that the pointer position $\hat{Q}$ for a given object position $q$ has been translated by a distance $\epsilon \text{Re}(c_w)$. The basic condition for a weak measurement is that the translation of the pointer should be small compared to the standard deviation of the pointer $\sigma$, i.e. $\epsilon \text{Re}(c_w) \ll \sigma$.

We now demonstrate that a weak measurement of the positive operator $\hat{p}^2$ may yield negative values for a coherent state. Consider the quadrature representation of a coherent state $| \alpha \rangle$ (with $\omega = 1$) [31]

$$
\langle q | \alpha \rangle = \pi^{-1/4} \exp \left[ -\frac{q^2}{2} + \sqrt{2} \alpha q - \frac{1}{2} | \alpha |^2 - \frac{1}{2} 2\alpha^2 \right].
$$

(9)

The weak value of $\hat{p}^2$ for an ensemble preselected in the coherent state $| \alpha \rangle$ and postselected in the quadrature eigenstate $| q \rangle$ then is

$$
\langle p^2 \rangle_w = \frac{-\partial^2 \langle q | \alpha \rangle / \partial q^2}{\langle q | \alpha \rangle} = 1 - (q - \sqrt{2} \alpha)^2.
$$

(10)

The real part of the weak value is $\text{Re}[\langle p^2 \rangle_w] = 1 + \alpha_1^2 - (q - \alpha_r)^2$, where we have introduced the notation $\alpha = (\alpha_r + i \alpha_i)/\sqrt{2}$. We see that $\text{Re}[\langle p^2 \rangle_w]$ is negative if $(q - \alpha_r)^2 > 1 + \alpha_1^2$. The surprising conclusion is that the weak value of $\hat{p}^2$ can be negative for coherent states, although $\hat{p}^2$ has only nonnegative eigenvalues. The probability of obtaining a negative value is

$$
P = \int_{-\infty}^{\alpha_r - \sqrt{1 + \alpha_1^2}} \langle q | \alpha \rangle^2 dq
$$

$$
+ \int_{\alpha_r + \sqrt{1 + \alpha_1^2}}^{\infty} \langle q | \alpha \rangle^2 dq.
$$

(11)

This is found to be $\text{erf} \sqrt{1 + \alpha_1^2}$, where $\text{erf}(x)$ is the complementary error function. This function is plotted in Fig. 1. It has a maximum when the imaginary part of the coherent state amplitude vanishes, in which case it equals $\text{erf}(1) \approx 0.16$.

We now demonstrate that a negative $\text{Re}[\langle p^2 \rangle_w]$ is closely related to negativity of the Margenau-Hill distribution. Consider the weak value as defined in Eq. (8) for the observable $\hat{c} = \hat{p}^n$. By inserting the completeness relation $\int dp \langle p | \rho \rangle \langle p | = 1$ in the numerator, we find that the weak value of $\hat{p}^n$ can be written as

$$
\langle p^n \rangle_w = \frac{\int dp \langle p^n \rangle S(q, p)}{\langle q | \rho_s | q \rangle},
$$

(12)

where

$$
S(q, p) = \langle q | p | \rho_s | q \rangle
$$

(13)

is the standard ordered distribution [32]. This is the complex conjugate of the Kirkwood distribution [33]. The Margenau-Hill distribution is the real part of the standard ordered or Kirkwood distributions, so that

$$
\text{Re}[\langle p^n \rangle_w] = \frac{\int dp \langle p^n \rangle M(q, p)}{\langle q | \rho_s | q \rangle},
$$

(14)

where $M(q, p)$ is the Margenau-Hill distribution. $\text{Re}[\langle p^n \rangle_w]$ is a conditional moment of the Margenau-Hill distribution.

FIG. 1: The probability of observing a negative weak value for a coherent state with amplitude $(\alpha_r + i \alpha_i)/\sqrt{2}$. It is plotted as a function of the imaginary component $\alpha_i$. The probability is independent of the real component $\alpha_r$. 

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P = \int_{-\infty}^{\alpha_r - \sqrt{1 + \alpha_1^2}} \langle q | \alpha |^2 dq
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This is found to be $\text{erf} \sqrt{1 + \alpha_1^2}$, where $\text{erf}(x)$ is the complementary error function. This function is plotted in Fig. 1. It has a maximum when the imaginary part of the coherent state amplitude vanishes, in which case it equals $\text{erf}(1) \approx 0.16$.

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(14)

where $M(q, p)$ is the Margenau-Hill distribution. $\text{Re}[\langle p^n \rangle_w]$ is a conditional moment of the Margenau-Hill distribution.
nian dominates over any other terms in the Hamiltonian term (2). We again assume that the interaction Hamiltonian is the classical equivalent of the quantum interaction and assume that the interaction Hamiltonian is

\[ H_{\epsilon} = \epsilon \delta(t) c(q, p) P. \]  

This is the classical equivalent of the quantum interaction term (2). We again assume that the interaction Hamiltonian dominates over any other terms in the Hamiltonian during the short time of interaction. The equation of motion is given by the classical Liouville theorem,

\[ \frac{\partial F}{\partial t} = -\{F, H_{\epsilon}\}. \]  

Due to the interaction, the joint phase space distribution evolves to \[ F_{\epsilon}(q, p, Q, P). \] The joint probability density for the two position variables then reads

\[ \rho_{\epsilon}(Q, q) = \int dp \int dPF_{\epsilon}(q, p, Q, P). \]  

By assuming once more that the current density of the pointer vanishes,

\[ \int dP \ P F_{0}(Q, P) = 0, \]  

it can be shown that to the first order in \[ \epsilon, \] the probability density for the pointer position \[ Q \] conditioned on the object position \[ q \] is [28]

\[ \rho_{\epsilon}(Q \mid q) = \frac{\rho_{\epsilon}(Q, q)}{\int dQ \rho_{\epsilon}(Q, q)} \approx \hat{T}_{c} f_{a}(Q), \]  

where

\[ f_{a}(Q) = \int dp F_{0}(Q, P) \]  

and

\[ \hat{T}_{c} = 1 - \epsilon c_{w} \frac{\partial}{\partial Q} \]  

is a first order translation operator. Here we have introduced

\[ c_{w} = \frac{\int dp c(q, p) F_{\epsilon}(q, p)}{\int dp F_{\epsilon}(q, p)}. \]  

which is the “classical weak value” of \[ c(q, p) \]. We see that the classical weak value is the conditional expectation value of that variable. In other words, \[ c_{w} \] is simply the expectation value of \[ c(q, p) \] “given” \[ q \]. This shows that the pointer \[ Q \] has been translated by a distance \[ c_{\sigma} \]. In this case, the measurement is weak provided that \[ \epsilon c_{w} \ll \sigma \], where \[ \sigma \] is the standard deviation of the pointer position.

It follows straightforwardly from Eq. (23) that if \[ c(q, p) \geq 0 \], then due to a nonnegative integrand, \[ c_{w} \geq 0 \]. The classical weak value of a positive observable cannot be negative. However, we have just seen that this condition can be violated for positive observables on coherent states. We therefore conclude that coherent states possess nonclassical properties.

Our analysis assumed an interaction of the form (2). However, just as a standard, projective von Neumann measurement is not dependent on the specific interaction Hamiltonian proposed by von Neumann [26], it is
not to be expected that weak measurements are critically dependent on the specific form of interaction Hamiltonian proposed here. For a massive particle in a harmonic oscillator potential, it should be possible to employ almost any measurement scheme for kinetic energy provided that the interaction between the particle and the pointer is sufficiently weak. The position measurement can be provided by using a detector with spatially limited size placed in the desired position.

Since the energy of a harmonic oscillator is

$$\hat{E} = \frac{1}{2} (\hat{p}^2 + \hat{q}^2),$$

it is easily shown that

$$\text{Re}[\langle \hat{p}^2 \rangle_w(q)] = 2\text{Re}[E_w(q)] - q^2,$$

where $E_w(q)$ is the weak value of energy postselected on position. This suggests an alternative measurement strategy of $\text{Re}[\langle \hat{p}^2 \rangle_w(q)]$ by performing a weak measurement of energy postselected on the quadrature $\hat{q}$, and subsequently subtracting the squared quadrature.

In conclusion, we have investigated a general class of weak measurements where the state of the pointer could be either pure or mixed. We have also investigated classical weak measurements. We have demonstrated that weak measurements will reveal nonclassical properties of coherent states. We demonstrated that weak values are conditional moments of the Margenau-Hill distribution, and that nonclassicality of coherent states is related to negativity of the Margenau-Hill distribution.

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