Pedagogical Content Knowledge: Teacher’s Knowledge of Students in Learning Mathematics on Limit of Function Subject

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Abstract. This research aims at describing the profile of high school teacher’s Pedagogical Content Knowledge in learning mathematics from the perspective of teaching experience. Pedagogical Content Knowledge (PCK) covers teacher’s knowledge of subject matter, knowledge of pedagogy, and knowledge of students. The subject of this research was two high school mathematics teachers who have different teaching experience. The data were obtained through interview and observation then analyzed qualitatively. The focus of this research is the novice teacher’s PCK deals with knowledge of students. Knowledge of Student is defined as teacher’s knowledge about the students’ conception and misconception on limit of function material and teacher’s ability to cope with students’ difficulty, mistake, and misconception. The result of this research shows that novice teacher’s ability in analyzing the cause of students’ difficulty, mistake, and misconception was limited. Novice teacher tended to overcome the students’ difficulty, mistake, and misconception by re-explaining the procedure of question completion which is not understood by the students.

1. Introduction

Teacher is a key component in a learning as well as in learning mathematics. A teacher should master the mathematic content taught and know how to transfer the lesson so the students can understand it easily. Knowledge of content and knowledge of pedagogic are not enough to obtain effective teaching practice without knowledge of students, curriculum, education objectives, and teaching materials. A teacher should know and comprehend a particular topic which is easy or difficult for the students and a conception or misconception which the students may have related to a particular topic. Shulman explained that Pedagogical Content Knowledge (PCK) is a kind of specific knowledge as the basic knowledge for teacher which includes the connection of various knowledge and skill of representation, analogy, examples, demonstration of a material in order to be understood by the students [1].

According to Lee, teacher’s way to transfer mathematic knowledge to students with understandable technique is the core of PCK [2]. In addition, it is explained that teacher’s PCK is an important element to be an effective mathematics teacher. Teacher’s mastery on the lesson or mathematic content is important for teacher’s success in teaching. A teacher cannot be expected to explain a mathematic concept, if he does not have a complete comprehension of that concept [3]. On the other side, mastering the lesson that will be taught is not enough to gain the objective intended on students [4]. Thus, the teacher who is expected can teach effectively is a teacher who master the lesson that will be taught and able to explain that material so it can be understood by the students.
Ball and Bass identified teacher’s knowledge deal with the students’ difficulties and leaning strategies which suitable to overcome those problems in learning mathematics as a part of PCK [5]. Meanwhile, Ma [6] gave a definition that PCK in mathematics learning is a teacher’s ability using his knowledge of mathematics to explore the mathematic topics and present the content of the topics in the methods that can build a successful mathematics learning. Furthermore, Ma explained that a teacher should be able to demonstrate the width, depth, and interconnection of content knowledge and utilize all of them to express mathematic solution. Kilic divided PCK into four components namely: (1) Knowledge of Subject Matter, (2) Knowledge of Pedagogy, (3) Knowledge of Students, and (4) Knowledge of Curriculum [7].

Based on the definitions above, it can be concluded that knowledge of content or knowledge of main material owned by a mathematics teacher should be transformed by utilizing various resources such as text book with the concept presentation that is easy to be understood by the students. Besides, a teacher in transforming knowledge of content should use different representations, help students to make connection between different representations to solve mathematic problems, identify students’ wrong thinking, be able to give respond toward students’ questions.

PCK is a specific knowledge which integrates knowledge of mathematic content, knowledge of pedagogic, and knowledge of students. This PCK is an important knowledge in teaching because it can help teacher to anticipate students’ learning misconception and mistake and be ready to give alternative model or explanation to overcome those misconception and mistake. An, Kulm, and Wu identified four aspects of PCK [8]. They are (1) building students’ idea in mathematics, (2) overcoming students’ misconception, (3) involving students in learning mathematics, and (4) supporting students’ thought of mathematics.

Novak & Gowin stated that a misconception is the interpretation of concepts in a statement which is not acceptable [9]. Then, Smith in Sessa and Roschelle (in [10]) stated that a misconception can be easily fixed through an isolation. Moore stated “one method of remediation is by “explaining” [10]. While Lucariello stated that misconception can really hamper the learning for some reasons [11]. First, the students generally do not realize that the knowledge they have is wrong. Besides, a misconception can be very strong in the students’ thought. Also, new experience that is integrated through a wrong conception can be disturbing in understanding a new information.

Next, Moore explained that the causes of misconception are: (1) the schema that is owned is wrong and fail to be changed, (2) the schema is wrong in accommodating a new information, (3) the new concept or schema corrupts the previous schema and causes confusion [10]. Another causes of misconception are the development stage of cognition is not appropriate with the concept learned, the student’s reasoning is limited and wrong, the student’s ability in grasping and understanding the concept learned, and the student’s interest to learn the concept.

Zevenbergen, Dole, and Wright explained that a good teaching involves teacher’s knowledge of students’ thought about mathematic concept and knowledge of how to direct them to construct a more complex, complete, and strong knowledge using organized activity, habit, learning environment [12]. Students’ misconception happened in mathematics learning is because of the less of students’ understanding on mathematic concept, included on the material of quadrate function. The teacher should have knowledge of the students’ mistake and misconception on the material taught so he can be more focus in teaching process by adopting appropriate model, method, strategy, and approach.

Mathematics teacher’s PCK on algebra function limit material in this research is a kind of knowledge which integrates knowledge of content, knowledge of pedagogy, and knowledge of students in learning mathematics on algebra function limit material. The components of PCK in this research are (1) Knowledge of Subject Matter, (2) Knowledge of Pedagogy, (3) Knowledge of Students. This article describes the novice teachers’ knowledge of students in overcoming the students’ misconception on algebra function limit material.
2. Method

This research is a descriptive research using qualitative approach. The subject of this research was two senior high school mathematics teachers who teach at grade XI Social in the same school. The two teachers have the qualification of Mathematics Education Bachelor. The criteria in determining the subject are: (1) teaching at grade XI Social in the same school, (2) having minimum 5 years teaching experience, (3) willing to give relevant data, included the learning observation in each class. Subject 01 with 15 years teaching experience (experienced teacher) is symbolized as S01, subject 02 with 7 years teaching experience (novice teacher) is symbolized as S02, since upraised as civil servant.

The data were collected through the observation of the learning and interview the subjects. The algebra function limit was held in five meetings according to the subjects’ Lesson Plan and the sixth meeting was the test. The researchers did the observation on five meetings of the subjects. The students’ answer sheets were examined, then it was found that the problems on the test which mostly answered wrong by the students were number 3 and number 5. The students’ answers on number 3 and 5 have different mistakes. The interview with the subjects was based on the questionnaire adapted from An, Kulm, and Wu [8]. The questionnaire contained the result of students’ work in answering the problems given. The questions given to the teachers refer to the result of students’ work in order to explore the teachers’ PCK deals with the teachers’ knowledge in overcoming the students’ misconception. The questionnaire consists of two issues based on the result of the students’ test to investigate the teachers’ PCK of the knowledge about students on the topic of algebra function limit. The supporting instrument used was in the form of questionnaire.

Problem 1: The students at grade XI Social were given problem \( \lim_{x \to 3} \frac{9-x^2}{4\sqrt{x^2+7}} \). The result of student Dp and Sr were as follow:

a. The answer of Dp: 
   \[
   \lim_{x \to 3} \frac{9-x^2}{4\sqrt{x^2+7}} = \lim_{x \to 3} \frac{9-2^2}{4\sqrt{2^2+7}} = \frac{9-9}{4\sqrt{9}} = \frac{0}{4} = 0
   \]

b. The answer of Sr: 
   \[
   \lim_{x \to 3} \frac{9-x^2}{4\sqrt{x^2+7}} = \lim_{x \to 3} \frac{9-2^2}{4\sqrt{2^2+7}} = \frac{9-4}{4\sqrt{9}} = \frac{5}{12}
   \]

Problem 2: The students worked on the problem \( \lim_{x \to \infty} \frac{(x^2+1)^2}{4x^4 - 3x + 1} \). The result of the student Al and Ni were as follow:

a. The answer of Al: 
   \[
   \lim_{x \to \infty} \frac{(x^2+1)^2}{4x^4 - 3x + 1} = \lim_{x \to \infty} \frac{(x^2 + 1)^2}{4x^4 - 3x + 1}
   \]
\[
\frac{x^4 + 2}{4x^4 - 3 + 1} = \frac{x^4 + 2}{4x^4 - 3} + 1 = 4x^8 + 0 = 4x^8
\]

b. The answer of Ni

\[
\lim_{x \to \infty} \frac{(x^2 + 1)^2}{4x^4 - 3x + 1} = \lim_{x \to \infty} \frac{(\frac{x^2}{x^4} + \frac{1}{x^4})}{\frac{4x^4}{x^4} - \frac{3x}{x^4} + \frac{1}{x^4}} = \lim_{x \to \infty} 4x^4 - 3x + 1 = \lim_{x \to \infty} 4\infty^4 - 3\infty + 1 = \infty + 1 = \infty
\]

Based on the result of the students’ work, to trace the teachers’ knowledge of the students’ mistake and misconception, the following questions were given to the subjects:

1. What is probably thought by each student?
2. What might be the cause of misconception on each student?
3. How is the alternative way to overcome the misconception of the four students?
4. What kind of question/task will be given to each student to overcome the misconception?
5. How to explain those problems to make the students easier to understand the problems?

3. Result and Discussion

The PCK of the novice teacher deals with the knowledge to overcome the students’ mistake and misconception on algebra function limit material is described as follow:

3.1. The novice teacher knowledge of the students’ mistake and misconception on algebra function limit

The novice teacher identified that the students’ mistake and misconception based on the students’ work describe what might be thought by the students. Generally, the students’ answers for problem 1 are the same because initialized with substitution. In the beginning of limit material, to determine the value of function limit, the first step done is substitution. Then the two students got the result of indefinite form \((0/0)\). They continued by choosing multiplication of complex number because the form of the function limit is fraction and its denominator contains irrational form. Thus, the steps to determine the value of function limit is already correct. But, when finishing the problem, the students did the mistake namely forget to give parenthesis in doing two syllables multiplication operation.

Student \(Dp\) was wrong when multiplied \((9 - x^2)\) with \((4 + \sqrt{x^2} + 7)\) that forget to put parenthesis so the result got was \(9 - x^2(4 + \sqrt{x^2} + 7)\) as well as the result of the multiplication of the denominator in which \(Dp\) forget to put parenthesis after the subtraction operation so the result got was \((4 - x^4 + 7)\) which should be \(4 - (x^4 + 7)\). The procedural mistake also happened that student \(Dp\) did wrong in doing calculation because forget to give parenthesis. That case also showed that student \(Dp\) did not understand mathematics basic rule about the use of parenthesis and its function. Student \(Sr\) also did the same mistake with student \(Dp\) namely calculation case. The difference was that student \(Sr\) directly did substitution after multiplying the complex number. Basically, student \(Sr\) and \(Dp\) did the same mistake because the purpose of the multiplication of complex number is make it easy to eliminate the same factor later on, but in the two students’ alternative answers there was no factor eliminated. The mistake of the two students was also a concept mistake because the function of complex number multiplication has not been understood that is when doing complex number multiplication the same factor will be later obtained, but the two students’ answers did not get to that stage.
Two students faced concept and procedure mistake on the second problem. \( Al \) and \( Ni \) did not understand that the concept of function limit for \( x \) approaches is illimitable. When substituting directly, the result got is \( \frac{2}{x} \), so the algebra manipulation should be done before direct substitution of \( x \) value approaches illimitable. For the alternative answer of the first student, \( Al \) did not show substitution way, did not divide it with the highest degree and also did not multiply it with complex factor, what is done was only combining the denominator and the numerator. The first step done by \( Al \) was trying to scatter or finish the quadrate but it was still wrong. \( Al \) was wrong in scattering \((x^2 + 1)^2\) because \( Al \) directly drew the degree so the result became \((x^2+2)^2\). \( Al \) might think that to simplify \((x^2+1)^2\) is enough by multiplying 2 with the constanta number in parenthesis. Then \( Al \) continued the problem solving by taking the denominator to the numerator. Meanwhile, \( Ni \) has actually known that the direction of the solving is dividing by the highest degree variable, but \((x^2 + 1)(x^2 + 1)\) which should be multiplied first was not done by \( Ni \). When finding \((x^2+1)^2\), it should be scattered first before dividing it by the highest degree that becomes \((x^2 + 1)\) multiplied by \((x^2 + 1)\).

The novice teacher identified the students’ mistake and misconception based on the result of their work on the test of algebra function limit by understanding what might be thought by them about algebra function limit. It is in line with Kilic’s statement that a teacher needs to identify the students’ misconception and difficulty by proposing questions or using appropriate task [7].

3.2. The novice teacher knowledge about the cause of the students’ mistake and misconception on algebra function limit material

The novice teacher described that the cause of the mistake and misconception is that the students commonly have not understood the steps to determine the value of algebra function limit, the tendency of the students to memorize the problem and its solving steps. So when the students solve the problem without seeing the example, they will forget how to solve the limit problem. As an example, the students sometimes remember that the problem like \( \lim_{x\to 3} \frac{9-x^2}{4-\sqrt{x^4+4}} \) will be multiplied with complex form but they do not understand the purpose of using complex number multiplication. As a result, they will directly substitute the limit value after multiplying it with the complex form, while indeed it should be factorized first because if it is substituted first the result will remain as indefinite value or 0/0. The misconception happened because the students have not understood the purpose of complex form multiplication so for a problem like \( \lim_{x\to 3} \frac{9-x^2}{4-\sqrt{x^4+4}} \) it is solved by directly substituting without thinking to direct it to eliminate the similar factors of the denominator and the numerator.

The next is problem 2 that was the students’ answers on the problem of calculating the value of function limit of \( \lim_{x\to 4} \frac{(x^2+1)^2}{4x^4-3x+1} \). The novice teacher described that \( Al’s \) misconception in determining the value of illimitable function was because \( Al \) did not understand the steps to solve such problem so the writing of the solving was not clear from the beginning. While \( Ni \) might forget that for a problem like \( \lim_{x\to 4} \frac{(x^2+1)^2}{4x^4-3x+1} \), \((x^2 + 1)^2\) should be scatter first becomes \((x^2 + 1)(x^2 + 1) = x^4 + 2x^2 + 1\), then it is divided by the highest degree to obtain the value of function limit \( \lim_{x\to 4} \frac{(x^2+1)^2}{4x^4-3x+1} \).

The novice teacher knew the cause of the students’ mistake and misconception based on the result of their work on the test of algebra function limit material. According to Kilic, a teacher needs to determine the source of a student’s difficulty and mistake to be corrected accurately [7]. The research result of Tanisl & Kose showed that a pre-service elementary school mathematics teacher understood the students’ thinking process related to the knowledge of variable, similarity, and equation but found it difficult to explain the cause of the students’ thought [13].
3.3. The novice teacher knowledge about the alternative way to overcome the students’ mistake and misconception on algebra function limit material

The novice teacher described that to overcome the students’ mistake and misconception in completing the problem \( \lim_{x \to 3} \frac{9-x^2}{4\sqrt{x^2+7}} \), the steps of solving the problem should be re-explained. The first thing that should be explained is the purpose of the multiplication with complex form namely

\[
\lim_{x \to 3} \frac{9-x^2}{4\sqrt{x^2+7}} = \lim_{x \to 3} \frac{(9-x^2)(4+\sqrt{x^2+7})}{16(x^2+7)} = \frac{(9-x^2)(4+\sqrt{x^2+7})}{9-9^2},
\]

after that algebra manipulation or factorization is done to obtain the same denominator and numerator that is

\[
\lim_{x \to 3} \frac{(9-x^2)(4+\sqrt{x^2+7})}{9-9^2} = \lim_{x \to 3} (4 + \sqrt{x^2 + 7}) \text{ so it can be simplify then directly substituted } = 4 + \sqrt{3^2 + 7} = 8.
\]

The novice teacher overcame the students’ mistake by re-explaining that when we substitute the result got is indefinite value or 0/0. Then, it is explained to the students that the denominator and the numerator contain a factor which results in zero when it is substituted. When we eliminate the factor which results in zero, the limit value is gained because the value resulted is no longer in indefinite value like \( \lim_{x \to 3} \frac{9-x^2}{4\sqrt{x^2+7}} \) which becomes a factor resulting zero that is (x-3) because the limit approaches 3, in order that the function form does not result in 0/0, the value (x-3) that will be eliminated by form multiplication or factorization of one function form to get the similar form (x-3) on the denominator and the numerator, so it can be eliminated or similar to one.

Next, to cope with the students’ mistake and misconception on the problem \( \lim_{x \to \infty} \frac{(x^2+1)^2}{4x^4-3x+1} \), the novice teacher re-explained the procedure of problem solving namely the first step is scattering \((x^2 + 1)^2\) to gain \(\frac{x^4 + 2x^2 + 1}{4x^4 - 3x + 1}\), then each syllable is divided by the highest degree variable,

\[
\lim_{x \to \infty} \frac{x^4 + 2x^2 + 1}{4x^4 - 3x + 1} = \lim_{x \to \infty} \frac{1}{4} = \frac{1}{4}.
\]

The students’ mistake and misconception can also be overcome by giving them another suitable problem, by reviewing the suitable solving stages so they can correct their misconception.

The novice teacher overcame the students’ mistake and misconception by re-explaining the concept or procedure of the correct problem solving and giving another example similar with the problem. This is fit with Moore’s statement that one remediation method is by explaining [10]. While Richland explained that using analogy example is one way for the teacher to fix the students’ misconception [14]. Analogical reasoning has important role and is an effective strategy in mathematics learning [15]-[19].

One aspect of PCK in this research is the novice teacher knowledge about the students which covers their mistake and misconception, the cause of the mistake and misconception, and the alternative way to cope with the students’ mistake and misconception on algebra function limit material. The research result of Lee showed that teaching experience is also important factor in mathematics teacher PCK [2]. PCK contains the element of teaching material, how to teach it, and knowledge of students. Baker & Chick emphasized that the knowledge of lesson material, knowledge of pedagogic, and knowledge of students are important parts of PCK [20]. The knowledge of students is the core of PCK and considered as one of the important component of PCK [1][8].

4. Conclusion

PCK is a kind of specific knowledge which integrates the mastery of mathematics content, how to teach it, and knowledge of students. Knowledge of students consists of teacher’s knowledge about the students’ conception and misconception on algebra function limit material. The teaching experience is one of the important factors of teacher’s PCK. The novice teacher knew the students’ mistakes and misconception based on the result of the students’ work on the test of algebra function limit material.
by understanding the students’ thought. The novice teacher knew the cause of the students’ mistake and misconception that the students did not understand the concept and solving procedure of algebra function limit material clearly. The novice teacher overcame the students’ mistake and misconception by re-explaining the concept or solving procedure correctly and giving similar example.

References
[1] Shulman, L. S. 1986. Those who understand: Knowledge growth in teaching. Educational Researcher, 15, 2, pp. 4 – 14.
[2] Lee, JooHi, 2010. Exploring Kindergarten Teachers’ Pedagogical Content Knowledge of Mathematics: Department of Curriculum and Instruction-EC4, College of Education and Health Professional, University of Texas at Arlington, Science Hall.
[3] Yeo, K.K.J. 2008. Teaching Area and Perimeter Mathematics-Pedagogical Content Knowledge in Action. In M. R. Brown & K. Maker (Eds). Proceeding of the 31th Annual Conference of the Mathematics Education Research Group Australia. Merga, pp. 621-627
[4] Isiskal, M. & Cakirogu, E. 2011. The nature of Prospective Mathematics Teachers’ Pedagogical Content Knowledge: The Case of Multiplication of Fraction. Journal Math Teacher Education. 14, 3, pp.213-230.
[5] Ball, D. L., & Bass, H. 2000. Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), Multiple perspectives on mathematics teaching and learning (pp. 83–104). Westport, UK: Ablex
[6] Ma, Liping, 1999. Knowing and Teaching Elementary Mathematics: Teachers’ Understanding of Fundamental Mathematics in China and the United States. New York: Routledge.
[7] Kilic, Hulya. 2011. Preservice Secondary Mathematics Teachers’ Knowledge of Students. Turkish Online Journal of Qualitative Inquiry, 2, 2.
[8] An, S., Kulm, G., & Wu, Z. 2004. The Pedagogical Content Knowledge of Midle School, Mathematics Teacher in China and The USA. Journal of Mathematics Teacher Education, 7, pp. 145-172
[9] Novak & Gowin. 1984. Learning How to Learn. New York: Cambridge University Press
[10] Moore, Gordon March, 2006. Mathematical Misconceptions in School Children. Identification Impact and Remediation with a Computer Based Assesment System Master’s Dissertation.
[11] Lucariello, Juan. 2004. How Do I Get. My Students Their Alternative Conception (Misconception) for Learning. City University of New York.
[12] Zevenbergen, Dole, Wright. 2004. Teaching Mathematics in primary Schools. Sydney: Allen & Unwin.
[13] Tanisli, D & Kose, N.Y. 2013. Pre-Service mathematics Teachers’ Knowledge of Students about the Algebraic Concepts. Australian Journal of Teacher Education, 38, 2.
[14] Richland, Lindsey E., Holyoak, Keith J., and Stigler, James W. 2004. Analogy Use in Eighth-Grade Mathematics Classrooms. Cognition and Instruction, 22, 1, pp. 37–60
[15] Arsyad, N., Rahman, A., & Ahmar, A. S. 2017. Developing a self-learning model based on open-ended questions to increase the students’ creativity in calculus. Global Journal of Engineering Education, 9, 2, pp. 143-147.
[16] Ahmar, A. S., & Rahman, A. 2017. Development of teaching material using an Android. Global Journal of Engineering Education, 19, 1.
[17] Mulbar, U., Rahman, A., & Ahmar, A. S. 2017. Analysis of the ability in mathematical problem-solving based on SOLO taxonomy and cognitive style. World Transactions on Engineering and Technology Education, 15, 1, pp. 74-77.
[18] Rahman, A., & Ahmar, A. S. 2016. Exploration of Mathematics Problem Solving Process Based on The Thinking Level of Students in Junior High Schoo. International Journal of Environmental & Science Education, 11, 14.
[19] Rahman, A., Ahmar, A. S., & Rusli. 2016. The Influence of Cooperative Learning Models on Learning Outcomes Based on Students’ Learning Styles. World Transactions on Engineering
and Technology Education, 14, 3, pp. 74-77.

[20] Baker, M & Chick, H. 2006. Pedagogical Content Knowledge for Teaching Primary Mathematics: A case study of two teachers.