Proper local complete intersection morphisms preserve perfect complexes

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Abstract

Let \( f : X \to Y \) be a proper and local complete intersection morphism of schemes. We prove that \( Rf_* \) preserves perfect complexes, without any projectivity or noetherian assumptions. This provides a different proof of a theorem by Neeman and Lipman (see [Li-Ne]) based on techniques from derived algebraic geometry to proceed a reduction to the noetherian case.

Introduction

In [SGA6, Exp. XIV] Grothendieck and its collaborators present a huge list of open problems around the most general version of intersection theory on schemes and of the Grothendieck-Riemann-Roch formula. An important part of these problems concern the question of proving the GRR formula for proper morphisms of schemes without assuming the existence of a global factorization and/or without any noetherian assumption. One of the most basic questions is the existence of push-forward operations in algebraic K-theory of schemes in the most general setting. Directly related to this specific question is the following conjecture.

Conjecture 0.1 (See [SGA6, ExpIII-2.1]) Let \( f : X \to Y \) be a proper and pseudo-coherent morphism of schemes. Then, the derived direct image

\[
Rf_* : D_{qcoh}(X) \to D_{qcoh}(Y)
\]

preserves pseudo-coherent complexes.

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This conjecture has been proven in [Ki], and more recently A. Neeman and J. Lipman deduced from it the following theorem.

**Theorem 0.2** ([Li-Ne, Ex. 2.2 (a)]) Let \( f : X \to Y \) be a proper and perfect morphism of schemes. Then, the derived direct image
\[
\mathbb{R}f_* : D^\text{qcoh}(X) \to D^\text{qcoh}(Y)
\]
preserves perfect complexes.

The purpose of this work is to propose a new proof of the above theorem by assuming further that \( f \) is a local complete intersection morphism.

**Theorem 0.3** Let \( f : X \to Y \) be a proper and local complete intersection morphism of schemes. Then, the derived direct image
\[
\mathbb{R}f_* : D^\text{qcoh}(X) \to D^\text{qcoh}(Y)
\]
preserves perfect complexes.

The proof we propose in this work is based on a reduction to the noetherian case (no surprise) but in the somehow unexpected context of derived algebraic geometry. The general strategy proceeds as follows. The statement being local on \( Y \) we can reduce it to the case where \( Y \) is an affine scheme \( \text{Spec} A \). We then write \( A \) as a filtered colimit of noetherian rings \( A_i \) and we try to descend the whole situation over one of the ring \( A_i \). The unexpected fact is that even thought the scheme \( X \) descend as a scheme \( X_i \) proper and of finite type over some \( A_i \), it does not seem to be true that \( X_i \) can be chosen to be itself a local complete intersection. Our main observation is that \( X_i \) can be chosen to be a proper and local complete intersection derived scheme. This fact can then be used to prove the theorem 0.3 by using some standard facts about derived categories of derived schemes, particularly the base change property (see 1.4), which is a statement specifically true in the derived setting and wrong in the underived setting without extra flatness assumptions. This method of proof also shows that the theorem 0.3 remains true for \( X \) and \( Y \) being themselves derived schemes, even thought we do not make this statement explicit in this work.

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## 1 Derived schemes

We have collected in this preliminary section some facts concerning derived schemes that we will use in our proof of the theorem 0.3. They belong to the general properties of derived schemes and derived stacks and are certainly well known to experts. Some of these statements are probably established in the topological setting in [Liu, §2]. We note however some differences between the theory of derived schemes we will be using and the theory of spectral schemes used in [Liu] (notably cotangent complexes are not quite the same). We have therefore included proofs for the three main statements we will be using: base
change, continuity and noetherian approximation.

We recall from [To-Ve2, To1, To-Va] the existence of an $\infty$-category $dSt$, of derived stacks, and its full subcategory $dSch \subset dSt$ of derived schemes. The $\infty$-category $dSch$ contains as a full sub-category $Sch$, the category of schemes. The inclusion functor $Sch \hookrightarrow dSch$ possesses a right adjoint $h^0 : dSch \to Sch$, the truncation functor. We note that even thought both $\infty$-categories $Sch$ and $dSch$ admit finite limits, $Sch$ is not stable by finite limits in $dSch$.

Affine derived schemes are of the form $Spec A$ for some commutative simplicial ring $A \in sComm$. The $Spec$ $\infty$-functor defines an equivalence of $\infty$-categories

$$Spec : sComm^{op} \simeq dAff,$$

where $dAff \subset dSch$ is the full sub $\infty$-category of derived affine schemes, and $sComm^{op}$ is the $\infty$-categories of simplicial commutative rings. Restricted to affine objects the inclusion $\infty$-functor from schemes to derived schemes, and its right adjoint $h^0$, is equivalent to the adjunction of $\infty$-categories

$$\pi_0 : sComm \leftrightarrow Comm : i,$$

where $i$ sends a ring to the constant simplicial ring and $\pi_0$ is the connected component $\infty$-functor.

For all derived scheme $X$, the scheme $h^0(X)$, considered as an object in $dSch$, has an adjunction morphism $j : h^0(X) \to X$ which is a closed immersion. When $X = Spec A$ is affine, this morphism is equivalent to $Spec \pi_0(A) \to Spec A$ corresponding to the natural projection $A \to \pi_0(A)$. The higher homotopy groups $\pi_i(A)$ are $\pi_0(A)$-modules, and thus define quasi-coherent sheaves on $Spec \pi_0(A)$. These sheaves will be denoted by $h^{-i}(X) := \pi_i(A)$. This construction glue in the non-affine case: for any derived scheme $X$ the scheme $h^0(X)$ carries natural quasi-coherent sheaves $h^i(X) := \pi_i(O_X)$.

Any derived scheme $X$ possesses a cotangent complex $L_X \in L_{qcoh}(X)$ (see [To-Ve2] \S 1.4, [To1] \S 4.2), which is a quasi-coherent complex on $X$ (see our next subsection for quasi-coherent complexes and the definition of $L_{qcoh}(X)$). When $X = Spec A$ is affine, $L_X$ is the cotangent complex of $A$ introduced by Quillen (it is a simplicial $A$-module than can be turned into a dg-module over the normalization of $A$ to consider it as an object in $L_{qcoh}(X)$). For a morphism of derived schemes $f : X \to Y$ we define the relative cotangent complex $L_f$, or $L_{X/Y}$, as the cone, in the dg-category $L_{qcoh}(X)$, of the natural morphism

$$f^*(L_Y) \to L_X.$$

**Definition 1.1**

1. Let $X$ be a derived scheme. We say that $X$ is quasi-compact (resp. quasi-separated, resp. separated) if the scheme $h^0(X)$ is so.

2. Let $f : X \to Y$ be a morphism of derived schemes. We say that $f$ is proper if $h^0(f) : h^0(X) \to h^0(Y)$ is a proper morphism of schemes.

3. Let $f : X \to Y$ be a morphism of derived schemes. We say that $f$ is locally of finite presentation if $h^0(f) : h^0(X) \to h^0(Y)$ is a morphism locally of finite presentation of schemes and if the relative cotangent complex $L_f$ is perfect on $X$.

We note that the definition above of morphism locally of finite presentation possesses several equivalent versions (see [To-Ve2] Prop. 2.2.2.4)). In particular, a morphism of derived affine schemes $f : Spec A \to
for $i$-modules. The heart is the full sub-category of $D$ this defines an aisle of a non-degenerate t-structure on $X$ Locally on dg-category $L$ whose heart is canonically equivalent to the category of quasi-coherent sheaves on the scheme Zariski topology on $X$ structure (by which we mean that the associated triangulated category $\text{Spec } A$ take a rather simple colimit description by taking all the $\text{Spec } B$ is locally of finite presentation if and only if $\text{Spec } B$ is equivalent to $\text{Spec } A$ if it is such that $\text{Spec } B$ is equivalent to a retract of a finite cell commutative $\text{Spec } A$ is equivalent to $\text{Spec } A$ consists of dg-module $\pi$-modules. An object $E$ in $L(A)$ is declared to belong to $L(A) \leq 0$ if it is such that $H^i(E) = 0$ for all $i \geq 0$. It is easy to see that this defines an aisle of a non-degenerate t-structure on $D(A) = [L(A)]$, the derived category of $\text{Spec } A$ to belong to a basis of opens for the Zariski topology on $X$. Finally, the dg-category $L_{\text{qcoh}}(X)$ of any derived scheme $X$ possesses a natural non-degenerate t-structure (by which we mean that the associated triangulated category $[L_{\text{qcoh}}(X)]$ has such a t-structure), whose heart is canonically equivalent to the category of quasi-coherent sheaves on the scheme $h^0(X)$. Locally on $X$, over an affine open $U = \text{Spec } A \subset X$ this t-structure can be described as follows. The dg-category $L_{\text{qcoh}}(U)$ is identified with $L(A)$ the dg-category of cofibrant $N(A)$-dg-modules. An object $E$ in $L(A)$ is declared to belong to $L(A) \leq 0$ if it is such that $H^i(E) = 0$ for all $i \geq 0$. It is easy to see that this defines an aisle of a non-degenerate t-structure on $D(A) = [L(A)]$, the derived category of $N(A)$-dg-modules. The heart is the full sub-category of $D(A)$ consisting of dg-module $E$ with $H^i(E) = 0$ except for $i = 0$. This sub-category is equivalent, via the functor $E \mapsto H^0(E)$, to the category of $\pi_0(A)$-modules.

**Definition 1.2** Let $X = \text{Colim } X_\alpha$ be a colimit in the $\infty$-category $\text{dSch}$ of derived schemes. We will say that the colimit is strong if it is also a colimit in the bigger $\infty$-category $\text{dSt}$ of derived stacks.

### 1.1 Review of derived categories of derived schemes

According to [To2] §2, for any derived stack $X \in \text{dSt}$, we have a natural (Z-linear) dg-category $L_{\text{qcoh}}(X)$ of quasi-coherent complexes over $X$. This can be made into an $\infty$-functor

$$L_{\text{qcoh}} : \text{dSt}^{op} \to \text{dg-Cat},$$

where $\text{dg-Cat}$ is the $\infty$-category of (locally presentable, see [To2]) dg-categories. For a morphism of derived stacks $f : X \to Y$, we have an adjunction of dg-categories

$$f^* : L_{\text{qcoh}}(Y) \to L_{\text{qcoh}}(X) : f_*.$$

When $X = \text{Spec } A$ is an affine derived scheme, the dg-category $L_{\text{qcoh}}(X)$ can be explicitly described, up to a natural equivalence, as follows. The commutative simplicial ring $A$ possesses a normalisation $N(A)$, which is a (commutative) dg-ring. The dg-category $L(A)$ of cofibrant $N(A)$-dg-modules is naturally equivalent to $L_{\text{qcoh}}(X)$. This description is moreover functorial in the following way. A morphism of derived affine schemes $f : X = \text{Spec } A \to Y = \text{Spec } B$ corresponds to a morphism of commutative simplicial rings $B \to A$, and thus to a morphism of dg-rings $N(B) \to N(A)$. The adjunction

$$f^* : L_{\text{qcoh}}(Y) \to L_{\text{qcoh}}(X) : f_*$$

is then equivalent to the following adjunction

$$- \otimes_A B : L(A) \leftrightarrow L(B) : f_*,$$

where we have written $\otimes_A$ symbolically for $\otimes_{N(A)} N(B)$, and where $f$ is the forgetful functor.

In the general case, we write a general derived stack $X$ as a colimit in $\text{dSt}$, $X = \text{Colim } \text{Spec } A_\alpha$, and we have

$$L_{\text{qcoh}}(X) \simeq \text{Lim } L_{\text{qcoh}}(\text{Spec } A_\alpha) \simeq \text{Lim } L(A_\alpha),$$

where the limit here is taken in the $\infty$-category of dg-categories. When $X$ is a derived scheme we can take a rather simple colimit description by taking all the $\text{Spec } A_\alpha$ to belong to a basis of opens for the Zariski topology on $X$. Finally, the dg-category $L_{\text{qcoh}}(X)$ of any derived scheme $X$ possesses a natural non-degenerate t-structure (by which we mean that the associated triangulated category $[L_{\text{qcoh}}(X)]$ has such a t-structure), whose heart is canonically equivalent to the category of quasi-coherent sheaves on the scheme $h^0(X)$. Locally on $X$, over an affine open $U = \text{Spec } A \subset X$ this t-structure can be described as follows. The dg-category $L_{\text{qcoh}}(U)$ is identified with $L(A)$ the dg-category of cofibrant $N(A)$-dg-modules. An object $E$ in $L(A)$ is declared to belong to $L(A) \leq 0$ if it is such that $H^i(E) = 0$ for all $i \geq 0$. It is easy to see that this defines an aisle of a non-degenerate t-structure on $D(A) = [L(A)]$, the derived category of $N(A)$-dg-modules. The heart is the full sub-category of $D(A)$ consisting of dg-module $E$ with $H^i(E) = 0$ except for $i = 0$. This sub-category is equivalent, via the functor $E \mapsto H^0(E)$, to the category of $\pi_0(A)$-modules.
Definition 1.3 Let $X$ be a derived stack. An object $E \in L_{qcoh}(X)$ is perfect if for all affine derived scheme $Z = \text{Spec } A$ and all morphism $f : Z \rightarrow X$, the pull-back $f^*(E)$ is a compact object in $L_{qcoh}(Z)$.

The full sub-dg-category of $L_{qcoh}(X)$ consisting of perfect complexes will be denoted by $L_{par f}(X)$.

Note that for $Z = \text{Spec } A$ affine, the compact objects in $L_{qcoh}(Z)$ can also be described in several different ways: as strongly dualizable objects or as retracts of finite cell $A$-dg-modules (see discussion after [To-Va, Def. 2.3]).

Suppose now that we have a cartesian square of derived stacks

$$
\begin{array}{ccc}
X' & \xrightarrow{g} & Y' \\
\downarrow{q} & & \downarrow{p} \\
X & \xrightarrow{f} & Y,
\end{array}
$$

by adjunction there is, for any object $E \in L_{qcoh}(Y')$, a natural morphism in the dg-category $L_{qcoh}(X)$

$$
\rho_E : f^*p_*(E) \longrightarrow q_*g^*(E).
$$

Proposition 1.4 If

$$
\begin{array}{ccc}
X' & \xrightarrow{g} & Y' \\
\downarrow{q} & & \downarrow{p} \\
X & \xrightarrow{f} & Y,
\end{array}
$$

is a cartesian square of quasi-compact and quasi-separated derived schemes, then for all $E \in L_{qcoh}(Y')$ the morphism $\rho_E : f^*p_*(E) \longrightarrow q_*g^*(E)$ above is an equivalence.

Proof: Localising on the Zariski topology on $X$ we can assume that $X$ and $Y$ are affine derived schemes. Let us write the cartesian square as follows

$$
\begin{array}{ccc}
Z_B := Z \times_{\text{Spec } A} \text{Spec } B & \xrightarrow{g} & Y' = Z \\
\downarrow{q} & & \downarrow{p} \\
X = \text{Spec } B & \xrightarrow{f} & Y = \text{Spec } A.
\end{array}
$$

For $E \in L_{qcoh}(Z)$, the object $f^*p_*(E)$ is $H(Z, E) \otimes_A^L B$, where $H(Z, E)$ is the cohomology $A$-dg-module of cohomology of $Z$ with coefficients in $E$. The object $q_*g^*(E)$ is $H(Z_B, g^*(E))$, the cohomology $B$-dg-module of $Z_B$ with coefficients in $g^*(E)$. The morphism $\rho_E$ is then the natural morphism $g^* : H(Z, E) \longrightarrow H(Z_B, g^*(E))$, extended to $H(Z, E) \otimes_A^L B$ by linearity.

As $Z$ is quasi-compact and quasi-separated it belongs to the smallest full sub $\infty$-category of $\text{dSch}$ containing affines and which is stable by finite strong colimits in $\text{dSt}$ (see definition [1.2]). Therefore, to prove our proposition it is enough to prove the following two individual statements:

1. If $Z$ is affine, then for all $E \in L_{qcoh}(Z)$, the morphism $\rho_E$ is an equivalence.
2. The full sub $\infty$-category of objects $Z \in \mathbf{dSch}/\text{Spec } A$ for which $\rho_E$ is an equivalence for all $E \in L_{qcoh}(Z)$ is stable by finite strong colimits.

The property (1) follows directly from the shape of fiber products of derived affine schemes (see [To-Ve2, Prop. 1.1.0.8]). For the property (2), let $Z \simeq \text{Colim } Z_i$ be a finite strong colimit in $\mathbf{dSch}$ for which we know that the proposition holds for all the derived schemes $Z_i$. We let $Z_{i,B} = Z_i \times_{\text{Spec } A} \text{Spec } B$, and we notice that as colimits are universal in $\mathbf{dSt}$ (because it is an $\infty$-topos, see [To-Ve1, To-Ve2, To1]), we have an induced strong colimit $Z_B \simeq \text{Colim } Z_{i,B}$. As the colimit is strong we have moreover

$$H(Z, E) \simeq \text{Lim } H(Z_i, E|_{Z_i}),$$

where $\text{Lim}$ stands for the limit in the $\mathbf{dg}$-category $L_{qcoh}(\text{Spec } A)$. As this limit is finite, we have

$$H(Z, E) \otimes^L_A B \simeq \text{Lim } (H(Z_i, E|_{Z_i}) \otimes^L_A B) \simeq \text{Lim } H(Z_{i,B}, g^*(E)|_{Z_i}) \simeq H(Z_B, g^*(B)).$$

The second formal property we will need is continuity, relating the $\mathbf{dg}$-category of quasi-coherent complexes of certain limits of derived schemes to the colimit (inside the $\infty$-category of $\mathbf{dg}$-categories) of the $\mathbf{dg}$-categories of quasi-coherent complexes on each individual derived schemes.

For this we let $A$ be a commutative ring which is written as a filtered colimit

$$A = \text{Colim } A_i.$$

We will suppose that the indexing category $I$ has an initial object $0 \in I$. Let $X_0 \to \text{Spec } A_0$ be a derived scheme, and let $X_i := X_0 \times_{\text{Spec } A_0} \text{Spec } A_i$ its base change to $A_i$. We will also denote by $X := X \times_{\text{Spec } A_0} \text{Spec } A$ the base change to $A$ itself. In this situation we have a natural morphism of $\mathbf{dg}$-categories

$$\text{Colim } L_{qcoh}(X_i) \to L_{qcoh}(X).$$

**Proposition 1.5** With the same notations as above, if the derived scheme $X_0$ is quasi-compact and quasi-separated then the morphism

$$\text{Colim } L_{parf}(X_i) \to L_{parf}(X)$$

is an equivalence of $\mathbf{dg}$-categories.

**Proof:** It follows the same lines as the proof of the proposition [1.4]. We reduce the proposition to the following two individual statements.

1. Proposition [1.5] holds for $X_0$ affine.

2. The full sub $\infty$-category of objects $X_0 \in \mathbf{dSch}/\text{Spec } A_0$ for which the proposition holds is stable by strong finite colimits.

The property (1) simply states that for a filtered colimit of commutative simplicial rings $B = \text{Colim } B_i$, we have

$$\text{Colim } L_{parf}(\text{Spec } B_i) \simeq L_{parf}(\text{Spec } B),$$

6
which is a particular case of \cite[Lem. 2.10]{To-Va}. The second property is proven as follows. Let $X_0 = \text{Colim} X_{0,\alpha}$ be a finite strong colimit in $\mathbf{dSch}$ such that the proposition \cite[Lem. 2.9]{Bi} holds for all $X_{0,\alpha}$. We let $X_{i,\alpha} := X_i \times_{X_0} X_{0,\alpha}$ and $X_\alpha = X \times_{X_0} X_{0,\alpha}$. As the colimit is filtered, it commutes with finite limits, and thus we have

\[
\text{Colim} L_{parf}(X_i) \simeq \text{Colim} \text{Lim}_\alpha L_{parf}(X_{i,\alpha}) \simeq \text{Lim}_\alpha \text{Colim} L_{parf}(X_{i,\alpha}) \simeq \text{Lim}_\alpha L_{parf}(X_\alpha).
\]

\[\square\]

\section{1.2 Noetherian approximation for derived schemes}

We let $A$ be a commutative ring which is written as a filtered colimit $A = \text{Colim} A_i$. The $\infty$-category of derived schemes over $\text{Spec} A_i$ (resp. over $\text{Spec} A$) will be denoted by $\mathbf{dSch}_{A_i}$ (resp. $\mathbf{dSch}_A$). We will study the $\infty$-functor

\[
\text{Colim} \mathbf{dSch}_{A_i} \rightarrow \mathbf{dSch}_A.
\]

For this we denote by $\mathbf{dSch}_{A_i}^{\leq n}$ the full sub $\infty$-category of $\mathbf{dSch}_A$ consisting of derived schemes $f : X \rightarrow \text{Spec} A$, for which $f$ is of locally of finite presentation, quasi-compact, quasi-separated, and such that the cotangent complex $L_f$ is of amplitude $[-n, 0]$ (see \cite[§2.4]{To-Va}). We use the same notation for $\mathbf{dSch}_{A_i}^{\leq n} \subset \mathbf{dSch}_{A_i}$. The base change $\infty$-functors

\[
\mathbf{dSch}_{A_i} \rightarrow \mathbf{dSch}_A
\]

preserves cotangent complexes (see \cite[Lem. 1.4.16 (2)]{To-Ve}), and thus restrict to $\infty$-functors

\[
\mathbf{dSch}_{A_i}^{\leq n} \rightarrow \mathbf{dSch}_A^{\leq n}.
\]

\begin{proposition}

The $\infty$-functor

\[
\text{Colim} \mathbf{dSch}_{A_i}^{\leq n} \rightarrow \mathbf{dSch}_A^{\leq n}
\]

is an equivalence of $\infty$-categories.

\end{proposition}

\begin{proof}
The proof again goes along the same general lines as for the proofs of propositions \cite[Lem. 2.9]{Bi} and \cite[Lem. 1.5]{Bi}. We start by considering $\mathbf{dAff}_{A_i}^{\leq n} \subset \mathbf{dSch}_{A_i}^{\leq n}$ the full sub $\infty$-category consisting of affine objects. We define in the same way $\mathbf{dAff}_{A_i}^{\leq n} \subset \mathbf{dSch}_{A_i}^{\leq n}$.

\begin{lemma}

The $\infty$-functor

\[
\text{Colim} \mathbf{dAff}_{A_i}^{\leq n} \rightarrow \mathbf{dAff}_A^{\leq n}
\]

is an equivalence.

\end{lemma}

\begin{proof}

Proof of the lemma: We start by proving that the $\infty$-functor is fully faithful. Let $B$ and $C$ be two simplicial $A_i$-algebras of finite presentation (and with cotangent complexes relative to $A_i$ of amplitude $[-n, 0]$). As $B$ is a finitely presented $A_i$-algebra, we have

\[
\text{Map}_A(B \otimes_{A_i}^\mathbb{L} A, C \otimes_{A_i}^\mathbb{L} A) \simeq \text{Map}_{A_i}(B, \text{Colim}_{j \geq i} (C \otimes_{A_i}^\mathbb{L} A_j)) \simeq \text{Colim}_{j \geq i} \text{Map}_{A_i}(B, C \otimes_{A_i}^\mathbb{L} A_j),
\]

\end{proof}
where we have denoted by $\text{Map}_A$ the mapping spaces of the $\infty$-category of commutative simplicial $A$-algebras (and similarly for $A_j$). This proves fully faithfulness.

Now, let $B$ be a commutative simplicial $A$-algebra of finite presentation and with cotangent complex $L_{B/A}$ of amplitude in $[-n,0]$. We know that $B$ is equivalent to a retract of a finite cell commutative $A$-algebra $B'$. In particular there is an index $i$ and a finite cell commutative $A_i$-algebra $B'_i$ such that $B' \simeq B'_i \otimes_{A_i}^L A$.

The object $B$ defines a projector up to homotopy on $B'$, that is a projector $p$ on $B'$ considered as an object in the homotopy category $\text{Ho}(A-CAlg)$ of commutative $A$-algebras. By choosing $i$ big enough we can moreover assume that this projector is induced by a projector $p_i$ on $B'_i$ in $\text{Ho}(A_i-sComm)$. By [163, Sublemma 3] the category $\text{Ho}(A_i-sComm)$ is Karoubian closed, so $p_i$ splits as a composition in $\text{Ho}(A_i-sComm)$

$$p = vu : B'_i \xrightarrow{u} B_i \xrightarrow{v} B'_i$$

for some commutative $A_i$-algebra $B_i$ and with $uv = id$.

The $A_i$-algebra $B_i$ is of finite presentation (because it is a retract of a finite cell commutative $A_i$-algebra) and we have $B_i \otimes_{A_i}^L A \simeq B$. It remains to show that $i$ can be chosen so that the cotangent complex $L_{B_i/A_i}$ has amplitude contained in $[-n,0]$. We let $Z_i := \text{Spec} B_i$, $X_i := \text{Spec} A_i$, $Z := \text{Spec} B$ and $X := \text{Spec} A$. The locus in $Z_i$ in which $L_{B_i/A_i}$ has amplitude contained in $[-n,0]$ is an open derived sub-scheme $U_i \subset Z_i$. Moreover, we have $U_i \times X_i X_j \simeq U_j$ for all $j \geq i$, because cotangent complexes are stable by base changes. As $U_i \times X_i X_j \simeq X_j$ (we can cover $U_i$ by elementary opens $\text{Spec} B_i[f_a^{-1}]$, and as 1 is a linear combination of the $f_a^n$s in $\pi_0(B)$ it must be so in some $\pi_0(B_j)$). This finishes the proof of the lemma. \hfill $\square$

**Lemma 1.8** Let $X_i, Y_i \in \text{dSch}_{\leq n}^{A_i}$, and denote by $X_j := X_i \times_{\text{Spec} A_i} \text{Spec} A_j$ and $Y_j := Y_i \times_{\text{Spec} A_i} \text{Spec} A_j$ for $j \geq i$, and $X := X_i \times_{\text{Spec} A_i} \text{Spec} A$, $Y := Y_i \times_{\text{Spec} A_i} \text{Spec} A$. Then, the natural morphism

$$\text{Colim}_{j \geq i} \text{Map}_{\text{dSch}_{A_j}}(X_j, Y_j) \longrightarrow \text{Map}_{\text{dSch}_A}(X, Y)$$

is an equivalence.

**Proof of the lemma:** Let us first assume that $X_i = \text{Spec} B_i$ is affine. We have $X = \text{Spec} B$ with $B = \text{Colim} B_i$. We thus have

$$\text{Map}_{\text{dSch}_A}(X, Y) \simeq \text{Map}_{\text{dSch}_{A_i}}(X, Y_i) \simeq \text{Colim}_{j \geq i} \text{Map}_{\text{dSch}_{A_j}}(X_j, Y_i) \simeq \text{Colim}_{j \geq i} \text{Map}_{\text{dSch}_{A_j}}(X_j, Y_j)$$

because $Y$ is locally of finite presentation over $\text{Spec} A$. To pass from the case where $X_i$ is affine to the general case we use the same argument as for the proof of propositions 1.4 and 1.5. The full sub $\infty$-category of $\text{dSch}_{\leq n}^{A_i}$ for which the lemma is true contains affine and is stable by finite strong colimits. Therefore it contains all quasi-compact and quasi-separated derived schemes. \hfill $\square$

To finish proposition 1.6 it remains to prove essential surjectivity.

**Lemma 1.9** Let $f_i : X_i \longrightarrow Y_i$ be a morphism in $\text{dSch}_{\leq n}^{A_i}$. Let $f : X \longrightarrow Y$ be the induced morphism by base change along $\text{Spec} A \longrightarrow \text{Spec} A_i$. If $f$ is a Zariski open immersion then there is $j \geq i$ for which

$$f_j : X_j = X_i \times_{\text{Spec} A_i} \text{Spec} A_j \longrightarrow Y_j = Y_i \times_{\text{Spec} A_i} \text{Spec} A_j$$

is so.
Proof of the lemma: As $f$ is a Zariski open immersion it is an étale monomorphism. Therefore, the cotangent complex $L_f$ is zero. By compatibility of cotangent complexes by base changes, and by the proposition [1.6] we must have $L_{f_j}$ for some $j \geq i$. Therefore, there is $j \geq i$ such that $f_j$ is étale. Moreover, as $f$ is a monomorphism the diagonal morphism $X \rightarrow X \times_Y X$ is an equivalence. By the lemma [1.8] we must have a $j \geq i$ such that $X_j \rightarrow X_j \times_Y X_j$ is an equivalence, or in other words that $f_j$ is a monomorphism of derived schemes. Therefore, there is a $j \geq i$ such that $f_j$ is an étale monomorphism and thus an open immersion.

We finally finish the proof of the proposition [1.6]. By lemma [1.7] we know that affine derived schemes in $\mathsf{dSch}_{\mathbf{A}}^{\leq n}$ belongs to the essential image. We first extend this to any quasi-compact quasi-affine derived scheme $X$ (i.e. any quasi-compact open of a derived affine scheme). Indeed, any such object is the image of a finite family of Zariski open affines

$$\{\text{Spec } B_{\alpha} \subset \text{Spec } B\}_{\alpha}.$$ 

By the lemmas [1.7] and [1.9] this family is induced by a finite family of opens

$$\{\text{Spec } B_{i,\alpha} \subset \text{Spec } B_{i}\}_{\alpha},$$

in $\mathsf{dSch}_{\mathbf{A}_i}^{\leq n}$ for some $i$. The image of this family defines a derived scheme $X_i \in \mathsf{dSch}_{\mathbf{A}_i}^{\leq n}$ such that $X_i \times_{\text{Spec } A_{i}} \text{Spec } A \simeq X$.

Finally, we proceed by induction on the number of affines in an open covering. We assume that we have proven that all $X \in \mathsf{dSch}_{\mathbf{A}}^{\leq n}$ covered by $k$ affine opens are in the essential image. If $X \in \mathsf{dSch}_{\mathbf{A}}$ is covered by $(k + 1)$ affine opens, we can form a push-out square in $\mathsf{dSch}_{\mathbf{A}}^{\leq n}$

$$\begin{array}{ccc} W & \rightarrow & U \\ \downarrow & & \downarrow \\ V & \rightarrow & X, \end{array}$$

where all morphisms are Zariski open immersions, $V$ is affine and $U$ can be covered by $k$ affine opens. By what we have seen for quasi-affines, by induction and by lemmas [1.7] and [1.9] this push-out square descent to a diagram of open immersions in $\mathsf{dSch}_{\mathbf{A}_i}^{\leq n}$ for some $i$

$$\begin{array}{ccc} W_i & \rightarrow & U_i \\ \downarrow & & \downarrow \\ V_i. \end{array}$$

Taking the push-out in $\mathsf{dSch}_{\mathbf{A}_i}$ defines a derived scheme $X_i \in \mathsf{dSch}_{\mathbf{A}_i}^{\leq n}$ such that $X_i \times_{\text{Spec } A_{i}} \text{Spec } A \simeq X$.

\[\square\]

2 Proof of the main theorem

We are now ready to prove the main theorem of this work.
Theorem 2.1 Let $f : X \to Y$ be a local complete intersection and proper morphism schemes, then $f_* : L_{qcoh}(X) \to L_{qcoh}(Y)$ preserves perfect complexes.

Proof: The statement is local on the Zariski topology of $Y$, so we can assume that $Y = \text{Spec } A$ is affine (note that when doing so $X$ and $Y$ become automatically quasi-compact and separated). We write $A = \text{Colim } A_i$, a filtered colimit such that $A_i$ is noetherian for all $i$. As $f$ is lci, $L_f$ is perfect of amplitude $[-1, 0]$, so $X$ lies in $\mathbf{dSch}_{A}^{\leq 1}$. By proposition 1.6 there is an index $i$ and an object $X_i \in \mathbf{dSch}_{A_i}^{\leq 1}$ with $X_i \times_{\text{Spec } A} \text{Spec } A \simeq X$. On the level of truncations we have a fiber product in the category of schemes $h^0(X_i) \times_{\text{Spec } A_i} \text{Spec } A_i \simeq h^0(X)$. We can thus use [EGAIV-3, Thm. 8.10.5] to show that if $i$ is taken big enough the scheme $h^0(X_i)$ is proper over $\text{Spec } A_i$. Therefore, $X_i \to \text{Spec } A_i$ is a proper lci morphism of derived schemes.

Lemma 2.2 Let $f : X \to S = \text{Spec } A$ be a proper and lci morphism of derived schemes with $A$ a noetherian ring. Then $f_* : L_{qcoh}(X) \to L_{qcoh}(S)$ preserves perfect complexes.

Proof of the lemma: We will first need to recall the following local structure theorem for derived schemes whose cotangent complexes have amplitude in $[-1, 0]$ (also called quasi-smooth in the literature).

Sublemma 2.3 With the same notation as above the quasi-coherent sheaves $h^i(X)$ are coherent on $h^0(X)$ and only a finite number of them are non-zero.

Proof of the sublemma: This is a local statement so we can assume that $X = \text{Spec } B$ with $B$ a retract of a finite cell commutative simplicial $A$-algebra with $L_{B/A}$ of amplitude in $[-1, 0]$. As the statement we would like to prove is stable by retracts, we can even assume that $B$ has a finite cell decomposition:

$$B_0 = A \to B_1 \to B_2 \to \ldots \to B_k = B.$$ 

For all $i$, there is a push-out of commutative simplicial $A$-algebras

$$B_i \to B_{i+1} \leftarrow \otimes A[\partial \Delta^i] = \otimes A[\Delta^{i+1}],$$

where $A[K]$ denotes the free commutative simplicial $A$-algebra generated by a simplicial set $K$ and $\alpha_i$ is the number of $i$-dimensional cells. Let us consider the morphism $p : B_1 \to B$. This morphism induces an isomorphism on $\pi_0$ and an epimorphism on $\pi_1$. Its (homotopy) fiber is therefore connected. Using a Postnikov decomposition of the morphism $p$, obstruction theory (see [To-Ve2, Lem. 2.2.1.1]), and the fact
that $L_{B/A}$ as amplitude $[-1,0]$, we see that the morphism $p$ as a section up to homotopy. In other words, the derived scheme $\text{Spec } B$ is a retract of $\text{Spec } B_{(1)}$. Moreover, by definition of cell algebras, $\text{Spec } B_{(1)}$ sits into a cartesian square

$$
\begin{array}{ccc}
\text{Spec } B_{(1)} & \longrightarrow & \mathbb{A}_A^{\alpha_0} \\
\downarrow & & \downarrow u \\
\{0\} & \longrightarrow & \mathbb{A}_A^{\alpha_1},
\end{array}
$$

where the morphism $u$ is determined by the attaching map

$$\otimes^{\alpha_1} A[\partial \Delta^1] \longrightarrow B_{(0)} = A[X_1, \ldots, X_{\alpha_0}].$$

This shows that $\text{Spec } B_{(1)}$ satisfies the conclusion of the sublemma, and thus so does $\text{Spec } B$. This finishes the proof of 2.3.

The sublemma and the fact that $E$ is perfect implies that the sheaves $H^i(E)$ are easily seen to be coherent and only a finite number of them are non-zero. Therefore, by Grothendieck’s finiteness theorem and dévissage we have that $f_*(E)$ is a bounded coherent complex on $S$. Moreover, for a closed point $s : \text{Spec } k(s) \longrightarrow S$, the proposition 1.4 implies that $s^*(f_*(E)) \simeq H(X_s, E_s)$, where $X_s$ is the fiber of $f$ at $s$ and $E_s$ the pull-back of $E$ on $X_s$. Another application of the sublemma implies that $h^i(X_s)$ is coherent and non-zero only for a finite number of indices $i$. Therefore, perfect complexes on $X_s$ are also with coherent and bounded cohomology sheaves. As $X_s$ is proper the complex of $k(s)$-vector spaces $\mathbb{H}(X_s, E_s)$ is cohomologically bounded with finite dimensional cohomology. The fibers of the bounded coherent complex $f_*(E)$ at every closed point is thus cohomologically bounded. It is therefore of finite Tor dimension and is thus a perfect complex on $S$.

This finishes the proof of theorem 2.1.

References

[Du] B. Duma, Vers la forme générale du théorème de Grothendieck-Riemann-Roch., thesis, university Paris7, September 2012.

[EGAIV-3] A. Grothendieck, Eléments de géométrie algébrique. IV. Etude locale des schémas et des morphismes de schémas. III. Inst. Hautes Études Sci. Publ. Math. No. 28 1966 255 pp.

[Ki] R. Kiehl, Ein ”Descente”-Lemma und Grothendiecks Projektionssatz für nichtnoethersche Schemata. Math. Ann. 198 (1972), 287316.

[Li-Ne] J. Lipman, A. Neeman, Quasi-perfect scheme-maps and boundedness of the twisted inverse image functor. Illinois J. Math. 51 (2007), no. 1, 209236.

[Lu] J. Lurie, Proper Morphisms, Completions, and the Grothendieck Existence Theorem., preprint available at http://www.math.harvard.edu/~lurie/
[Ne] A. Neeman, *The Grothendieck duality theorem via Bousfield’s techniques and Brown representability*. J. Amer. Math. Soc. **9** (1996), no. 1, 205-236.

[SGA6] *Théorie des intersections et théorème de Riemann-Roch*. Séminaire de Géométrie Algébrique du Bois-Marie 19661967 (SGA 6). Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre. Lecture Notes in Mathematics, Vol. **225**. Springer-Verlag, Berlin-New York, 1971.

[Tho] R. Thomason, T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*. The Grothendieck Festschrift, Vol. III, 247-435, Progr. Math., **88**, Birkhuser Boston, Boston, MA, 1990.

[To1] B. Toën, *Higher and derived stacks: a global overview*. Algebraic geometrySeattle 2005. Part 1, 435-487, Proc. Sympos. Pure Math., **80**, Part 1, Amer. Math. Soc., Providence, RI, 2009.

[To2] B. Toën, *Derived Azumaya algebras and generators for twisted derived categories*. Invent. Math. **189** (2012), no. 3, 581-652.

[To3] B. Toën, *Anneaux de définition des dg-algèbres propres et lisses*. Bull. Lond. Math. Soc. **40** (2008), no. 4, 642-650.

[To-Va] B. Toën, M. Vaquié, *Moduli of objects in dg-categories*. Ann. Sci. Ecole Norm. Sup. (4) **40** (2007), no. 3, 387-444.

[To-Ve1] B. Toën, G. Vezzosi, *Homotopical algebraic geometry. I. Topos theory*. Adv. Math. **193** (2005), no. 2, 257-372.

[To-Ve2] B. Toën, G. Vezzosi, *Homotopical algebraic geometry. II. Geometric stacks and applications*. Mem. Amer. Math. Soc. **193** (2008), no. 902, x+224 pp.