Unfurling Khovanov-Lauda-Rouquier algebras

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Abstract. One famous difficulty of studying algebras and categories presented with explicit generators and relations is the difficulty of checking that they have the “expected size,” for example, that an obvious spanning set is in fact a basis. One particular example which has resisted a simple proof of a basis theorem is the 2-category $U(g)$ categorifying a symmetrizable Kac-Moody algebra $g$, defined by Khovanov, Lauda, and Rouquier. Khovanov and Lauda gave a conjectured basis for the 2-morphism spaces in this category, as well as a conjectured isomorphism of its Grothendieck group to $\tilde{U}(g)$, the idempotented universal enveloping algebra in 2008. This has been proven in finite types, but has remained open in the general case.

In this paper, we give a proof of the conjectures above, exploiting the technique of deforming representations of this 2-category; by showing that these representations have the expected size at the generic point, we can confirm that they are not smaller at the special point of this deformation. We achieve this by a more general study of the behavior of categorical actions of a Lie algebra $g$ under the deformation of their spectra. We give conditions under which the general point of a family of categorical actions of $g$ carry an action of a larger Lie algebra $\tilde{g}$, which we call an unfurling of $g$. This is closely related to the folding of Dynkin diagrams, but to avoid confusion, we think it is better to use a different term.

1. Introduction

The categorification of Lie algebras and their representations has proven to be a rich and durable subject since its introduction roughly a decade ago. This theory produces a 2-category $\mathcal{U}$ depending on the choice of a Cartan datum and a choice of parameters; a representation of this 2-category is called a categorical Lie algebra action of the Kac-Moody algebra $g$ corresponding to the Cartan datum. Many interesting categories carry a categorical Lie algebra action, though it must be admitted that most of the interesting examples are for a Cartan datum of (affine) type A.

However, since the 2-quantum group $\mathcal{U}$ was first defined by Khovanov-Lauda [KL10] and Rouquier [Rou08], it has been haunted by a serious problem: Since it is

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1Supported by the NSF under Grant DMS-1151473. This research was supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Research, Innovation and Science.
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presented by generators and relations, it is hard to check that it is not smaller than expected. Specifically, in [KL10], it has been proven that there is a surjective map from the modified quantum group $\hat{U}$ to the Grothendieck group of $U$ and that the dimension of 2-morphism spaces between 1-morphisms in $U$ is bounded above by a variation on Lusztig’s bilinear form on the modified quantum group $\hat{U}$. If equality holds in this bound, then $\hat{U}$ is isomorphic to the Grothendieck group and we call the corresponding categorification non-degenerate.

As a general rule, proving non-degeneracy depends on constructing appropriate representations where one can show that no unexpected relations exist between 2-morphisms in $U$. This is done for $\mathfrak{sl}_n$ in [KL10], using an action on the cohomology of flag varieties. The next major step was independent proofs by Kang-Kashiwara [KK12] and the author [Web17a] that the simple highest weight representations of $\mathfrak{g}$ possess categorifications which are non-degenerate in an appropriate sense; this is sufficient to prove the non-degeneracy for finite type Cartan data.

This allowed significant progress, but along with many other techniques (such as connections to quiver varieties studied in [CKL13; Rou12; Web17b]), it has an unfortunate defect. Recall that the open Tits cone of a Cartan datum is the elements in the orbit of a dominant weight under the Weyl group; for example, if $\mathfrak{g}$ is affine, then the open Tits cone is the set of weights of positive level. This set is convex and every weight of a highest integrable representation lies inside the open Tits cone. Similarly, weights of lowest weight representations lie in the negative of the Tits cone. If the Cartan datum is of infinite type, then no information about a weight $\lambda$ outside the open Tits cone and its negative (in the affine case, these are level 0 representations) is contained in any of these representations, since the corresponding idempotent $1_{\lambda} \in \hat{U}$ kills any integrable highest weight representation.

Thus, if we are to understand non-degeneracy for weights outside the open Tits cone and its negative, we must have access to representations which are not highest or lowest weight. For our purposes, the most promising are those given by a tensor product of highest and lowest weight representations (perhaps many of each type). A construction of such categorifications $X^{\lambda-\mu}$ of the tensor product of the simple modules with highest weight $\lambda$ and lowest weight $-\mu$ was given in [Web15], but the non-degeneracy proof given there is only valid for weights inside the Tits cone. Thus, to access these other weights, we must give a new argument for the non-degeneracy of these tensor product categorifications, which will then imply the non-degeneracy of $U$. In particular, we prove that:

**Theorem A** (Theorem 3.6) Fix any commutative ring $k$ and consider any Cartan datum $(\mathcal{I}, \langle-,-\rangle)$ and choice of the polynomials $Q_{ij}(u,v) \in k[u,v]$ which is homogeneous (in the sense discussed in Section 2.7). The associated 2-quantum group $U$ is non-degenerate and the Grothendieck group of $U$ is $\hat{U}$. 

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While an interesting theorem in its own right, the techniques introduced here to prove this result have a significance of their own. A simple, but underappreciated, technique for proving this type of non-degeneracy argument is the upper semicontinuity of dimension under deformation. Perhaps calling this “underappreciated” is unfair, since it is certainly a well-known and much-used trick, but at least this author wishes he had learned earlier to exploit it systematically.

Thus, much of this paper will be dedicated to an exploration of the behavior of categorical actions under deformation. Let $R$ be the KLR algebra of the Cartan datum $(I, \langle -, - \rangle)$ (defined in Section 2.1). We wish to consider quotients of this algebra where the dots (the elements usually denoted $y_k \in R$) have a fixed spectrum; of course, all of these quotients can be packaged together into a completion $\hat{R}$. Most often, people have studied representations where the elements $y_k$ act nilpotently (all gradable finite-dimensional representations have this property), but we can also have them act with certain fixed non-zero eigenvalues. Given a choice of spectrum for the dots, we have an associated graph with the vertex set $\tilde{I}$ and associated Cartan datum. There is a natural map $\tilde{I} \to I$, which one can informally think of as a “branched cover” of the Cartan datum $(I, \langle -, - \rangle)$.

This is closely related to the phenomenon of folding of Dynkin diagrams, but due to some technical differences, we think it would be misleading to use the term “folding” here. Thus, we call $\tilde{I}$ an unfurling of $I$ (and $I$ a furling of $\tilde{I}$). Note that while $I$ is not necessarily symmetric as a Cartan datum (it may have roots of different lengths), we will define $\tilde{I}$ in such a way as to be symmetric. To give the reader a sense of this operation, let us discuss some examples:

- If $I$ is simply laced, then $\tilde{I}$ will be a topological cover of $I$, such as an $A_\infty$ graph covering an $n$-cycle or the trivial cover $\tilde{I} \cong I \times U$.
- If $I$ is not simply laced, we can arrange for $\tilde{I}$ to be given by a simply-laced Cartan datum with an isomorphism $\tilde{g} \cong g^{\sigma}$ for some diagram automorphism $\sigma$. This means that $I$ is the Langlands dual of what is usually called a folding of $\tilde{I}$ for the automorphism $\sigma$.

At the moment, it is unclear to the author what, if any, is the relationship between this work and that of McNamara [McN19] and Elias [Eli17], which also combine the ideas of categorification and folding. Obviously, this would be an interesting topic for future consideration.

We always have a map of Lie algebras $g \hookrightarrow \tilde{g}$ (after appropriate completion if $\tilde{I}$ is infinite), and this map has a categorical analogue:

**Theorem B** We have a functor $U_g \to \tilde{U}_{\tilde{g}}$, the completion of the additive closure of $U_{\tilde{g}}$ where dots are nilpotent. We can always choose $\tilde{I}$ so that the categorification $X^{\lambda - \mu}$ flatly deforms to the pullback of the categorification of a simple under this functor.
Since we know a basis theorem for the morphism spaces in categorified simples, this allows us to prove a similar theorem for $\chi^\lambda_{-\mu}$.

We’ll first give the most straightforward proof of Theorem A based on this result, and then proceed to some complementary results. In particular, Theorem B has a converse, where we can define a $\mathcal{U}_g$-action on a category with a $\mathcal{U}_g$-action where the dots are not nilpotent (Proposition 4.10).

Finally, we apply these results to studying the categorifications of tensor products introduced in [Web15, §4]. The theory developed there is ultimately dependent on certain non-degeneracy results that we were only able to prove using the deformation techniques of this paper.

We also include an appendix discussing the theory of valued graphs and how it relates to the structure of the graphs $\hat{I}$. This material is not strictly required anywhere in the main body of the paper, but connects some of the combinatorics we use with existing constructions.

Since this paper has been substantially rewritten since it was first posted on the arXiv, we feel some discussion is needed to clarify how this paper is different from earlier versions. The ideas at the center of the first version of this paper ultimately led to a very fruitful collaboration with Jon Brundan and Alistair Savage [BSW23; BSW20b; BSW20a; BSW22; BSW21]. In particular, very analogous ideas were central to the new proof of the basis theorem for Heisenberg categories in [BSW23, Th. 6.4] and the relation between Heisenberg and Kac-Moody categorifications in [BSW20a, Th. A]. The aim of this revision is to apply similar ideas, replacing Heisenberg categorifications by Kac-Moody categorifications. Thus, Section 3 is largely new, and heavily based on [BSW23; BSW20a]. This has created a slightly tangled relationship; those papers were inspired by the original version of this paper, but the underlying ideas and calculations were refined a great deal in the process of writing them (mostly due to the positive influence of my coauthors) and I apply those improvements here to reorganize this paper. In particular, there is a strong analogy between:

1. Theorem 4.10 and [BSW20a, Th. A]. Both take an action of a 2-category (Kac-Moody or Heisenberg), apply a spectral decomposition to the functors, and find a new, more “homogeneous” 2-category action.
2. Theorem B and [BSW23, Th. 5.4] (and also [BSW20a, Th. 5.22]). Both define a functor from one 2-category into a localization of another, allowing for the construction of “large” modules over the source 2-category via pullback.
3. Theorem A and [BSW23, Th. 6.4]. Both use the “large” modules constructed via the pullback above to check a basis theorem for the morphisms in the 2-category.

Acknowledgements. As discussed in the introduction above, the current version of this paper would have been impossible without what I’ve learned through the wisdom and hard work of my collaborators Jon Brundan and Alistair Savage. A
mathematician could not wish for finer colleagues. I would also like to thank Eric Vasserot for pointing out to me the difficulties which arise from the Tits cone; Ben Elias for some very helpful comments; Chris Leonard for pointing out a very silly mistake; and all the people (Wolfgang Soergel, Raphael Rouquier, and Catharina Stroppel among them) who taught me the importance of deforming things.

2. Background

Throughout, we’ll fix a (possibly infinite) set $I$, and a Cartan datum on this set. That is, the free abelian group generated by the simple roots $\alpha_i$ for $i \in I$ carries a symmetric bilinear form $\langle -,- \rangle$ such that $\langle \alpha_i, \alpha_i \rangle \in 2\mathbb{Z}_{>0}$ and $C = \begin{pmatrix} c_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} \end{pmatrix}$ is a symmetrizable locally finite Cartan matrix. By locally finite, we mean that for $i \in I$, we have that $c_{ij} \neq 0$ for only finitely many $j$; that is, the associated Dynkin diagram is a locally finite graph. Note that $d_i = \langle \alpha_i, \alpha_i \rangle / 2$ are symmetrizing coefficients for this Cartan matrix—we have an equality $d_i c_{ij} = d_j c_{ji} = \langle \alpha_i, \alpha_j \rangle$ for all $i, j$. Before we turn to a more detailed discussion of constructing Kac-Moody algebras from this data, let us attempt to reassure the reader: these details are not important and will not play an important role in our proofs. Rather, we want to work in the greatest possible generality to show that our results don’t depend on any fussy details of the construction of Kac-Moody algebras, as we show below in Lemma 2.7.

Let $\mathbb{F}$ be a field of characteristic 0. We will choose a realization of the Cartan datum discussed above over $\mathbb{F}$. That is, we choose:

1. An $\mathbb{F}$-vector space $\mathfrak{h}$ (possibly of infinite dimension).
2. Elements $\alpha_i^\vee \in \mathfrak{h}$. Let $\mathfrak{w} = \mathfrak{h}^\ast$ be the restricted dual of $\mathfrak{h}$, the subspace of the dual of $\mathfrak{h}$ consisting of functions which vanish on all but finitely many $\alpha_i^\vee \in I$ such that $\alpha_i^\vee (\alpha_j) = c_{ij}$.
3. Elements $\alpha_i \in \mathfrak{w}$ for $i \in I$.

As usual, we can define a Kac-Moody Lie algebra $\mathfrak{g}$ with Cartan $\mathfrak{h}$ generated by formal symbols $E_i$ and $F_i$ satisfying $[E_i, F_i] = \alpha_i^\vee$ and the Serre relations for the Cartan matrix.

For a given Cartan matrix, there are 4 canonical choices of realization (up to isomorphism), which we can derive from the free vector spaces $\mathfrak{h}_0 = \mathbb{F}^I$ and $\mathfrak{w}_0 = \mathbb{F}^I$. The Cartan matrix defines a natural pairing between these spaces $\langle h, w \rangle = \sum_{i,j \in I} h_i c_{ij} w_j$. Let $e_i^\vee, e_j$ be the coordinate unit vectors in $\mathfrak{h}_0, \mathfrak{w}_0$.

1. We can take

$$
\begin{align*}
\mathfrak{h} &= \mathfrak{h}_0 / \{ h \in \mathfrak{h}_0 | \langle h, w \rangle = 0 \, \forall w \in \mathfrak{w}_0 \} & e_i^\vee &\mapsto \alpha_i^\vee \\
\mathfrak{w} &= \mathfrak{w}_0 / \{ w \in \mathfrak{w}_0 | \langle h, w \rangle = 0 \, \forall h \in \mathfrak{h}_0 \} & e_i &\mapsto \alpha_i.
\end{align*}
$$

This corresponds to a Kac-Moody algebra with no grading elements (so possibly infinite weight multiplicities) and which has trivial center. The sets $\alpha_i^\vee$ and $\alpha_i$ obviously span $\mathfrak{h}$ and $\mathfrak{w}$, but if $C$ is degenerate, they will not be linearly independent. An affine example would be the loop algebra $\mathfrak{g}(\mathfrak{g})$ for $\mathfrak{g}$ finite-dimensional.
We can take $h = h_0$ with $\alpha_i^\vee = e_i^\vee$; we must take $\alpha_i$ to be the function defined by $h \mapsto \epsilon(h, e_i)$. If $C$ is degenerate, the roots $\alpha_i$ will both fail to span and to be linearly independent. This is the algebra we obtain if we naively extend the Serre presentation of finite-dimensional simple Lie algebras to more general Cartan matrices. An affine example would be the universal central extension of $g(t)$.

We’ll see below that this choice of realization is often the most convenient; since it will be useful to refer to it later, we’ll call it the universal derived Lie algebra for this Cartan matrix. One useful property it has is that it is initial amongst all realizations over $F$—if $g$ is the Kac-Moody algebra with this realization, and $g'$ any other Kac-Moody algebra for the same Cartan matrix, then there is a canonical homomorphism $g \rightarrow g'$ whose image is $[g', g']$. (3) We can reverse the roles of roots and coroots, and take $h = w_0^*$ with $\alpha_i = e_i$, which similarly requires that $\alpha_i^\vee$ be the function $w \mapsto \epsilon(e_i^\vee, w)$. In this case, the coroots $\alpha_i^\vee$ might fail to span and to be linearly independent. An affine example would be $F \times \ltimes g((t))$, the loop with a grading element added to the Cartan.

(4) We can require that both $\alpha_i$ and $\alpha_i^\vee$ are linearly independent at the cost that neither will span their respective spaces if $C$ is degenerate. This is most often the version of the Kac-Moody algebra considered. An affine example would be the universal central extension of $F \times \ltimes g((t))$.

Note that in the case where $C$ is finite rank and non-degenerate, these realizations are all the same. In particular, this difference is only relevant for infinite-dimensional Kac-Moody algebras.

**Definition 2.1** The weight lattice $Y = \{ \lambda \in w \mid \alpha_i^\vee(\lambda) \in \mathbb{Z} \}$ of $g$ is the subgroup of $w$ on which $\alpha_i^\vee$ has integer value. Elements of $Y$ are called weights.

If the coroots $\alpha_i^\vee$ span $h$, then the map $\alpha_i^\vee : Y \rightarrow \mathbb{Z}^I$ sending $\lambda \mapsto (\alpha_i^\vee(\lambda))$ is injective, and so $Y$ is a free abelian group. On the other hand, if this map is not injective, its kernel is a positive-dimensional $F$-vector space. As a prelude to letting the reader completely forget the discussion above, the abelian group $Y$ together with the map $\alpha_i^\vee$ will be the only aspect of the Kac-Moody algebra above and the corresponding realization of the Cartan matrix that are relevant for the construction of the categorification below.

Our construction of KLR algebras and categorified quantum groups will depend on the choice of Cartan matrix $C$ and weight lattice $Y$ above, a choice of field $k$ which we will assume is algebraically closed of characteristic coprime to all $d_i$, and for each $i \neq j \in I$, a choice of polynomial $Q_{ij}(x, y) = Q_{ji}(y, x) \in k[x, y]$ which is homogeneous of degree $-2\langle \alpha_i, \alpha_j \rangle = -2d_i c_{ij} = -2d_j c_{ji}$ when $x$ is given degree $2d_i$ and $y$ degree $2d_j$. We’ll assume throughout that $Q_{ij}(1, 0)$ is non-zero for all $i, j$. 


This homogeneity implies that we define a sensible notion of the order of vanishing
of $Q_{ij}(x, y)$ at $x = u, y = u'$ for $u, u' \in \mathbb{k} \setminus \{0\}$. This is the same as the order of vanishing
of $Q_{ij}(u, y)$ at $y = u'$ or $Q_{ij}(x, u')$ at $x = u$.

Fix a subset (finite or infinite) $U_i \subset \mathbb{k} \setminus \{0\}$ for each $i \in I$.

**Definition 2.2** Let $\bar{I}$ be the oriented graph whose vertex set is the pairs $\{(i, u) \in I \times \mathbb{k} | u \in U_i\}$ with the number of edges oriented from $(i, u)$ to $(i', u')$ being the order of
vanishing of $Q_{ij}(x, y)$ at $x = u, y = u'$ as defined above. Let $\tilde{g}$ be the Kac-Moody
algebra associated to this graph.

We call a choice of the sets $U_i$ complete if, whenever $Q_{ij}(u, u') = 0$ for $u \in U_i$, then
$u' \in U_j$.

When the choice of $U_i$’s is complete, we call the graph $\bar{I}$ an unfurling of $I$. We’ll
show that there is a natural homomorphism from $U(g)$ to a suitable completion $\hat{U}(\tilde{g})$
(Prop. A.5) which sends:

\[ (2.1) \quad F_i \mapsto \sum_{u \in U_i} F_{i,u} \quad E_i \mapsto \sum_{u \in U_i} E_{i,u} \quad H_i \mapsto \sum_{u \in U_i} H_{i,u}. \]

We can think of Theorem B as a categorification of this result, but first we must
introduce the relevant categories.

### 2.1. The KLR algebra

Let $\mathbb{k}$ and $Q_{ij}$ be as above.

**Definition 2.3** Let $R_n$ denote the KLR algebra with generators given by:

- The idempotent $e_i$ which is straight lines labeled with $(i_1, \ldots, i_n) \in I^n$.
- The element $y^i_k$ which is just straight lines with a dot on the $k$th strand.
- The element $\psi^i_j$ which is a crossing of the $i$th and $i + 1$st strands.
and relations:

\begin{align}
(2.2a) \quad \begin{array}{ccc}
\begin{array}{c}
\includegraphics[width=1cm]{sum1.png}
\end{array} & - & \begin{array}{c}
\includegraphics[width=1cm]{sum2.png}
\end{array}
\end{array} = \begin{array}{c}
\includegraphics[width=1cm]{sum3.png}
\end{array} - \begin{array}{c}
\includegraphics[width=1cm]{sum4.png}
\end{array} = \delta_{ij} \downarrow_j \uparrow_i,
\end{align}

\begin{align}
(2.2b) \quad \begin{array}{c}
\includegraphics[width=1cm]{sum5.png}
\end{array} = \begin{cases}
0 & \text{if } j = i, \\
Q_{ij} \begin{array}{c}
\includegraphics[width=1cm]{sum6.png}
\end{array} & \text{if } i \neq j,
\end{cases}
\end{align}

\begin{align}
(2.2c) \quad \begin{array}{ccc}
\begin{array}{c}
\includegraphics[width=1cm]{sum7.png}
\end{array} & - & \begin{array}{c}
\includegraphics[width=1cm]{sum8.png}
\end{array}
\end{array} = \begin{cases}
\tilde{Q}_{ij} \begin{array}{c}
\includegraphics[width=1cm]{sum9.png}
\end{array} & \text{if } i = k, \\
0 & \text{otherwise},
\end{cases}
\end{align}

where

\[ \tilde{Q}_{ij}(u, v, w) = \frac{Q_{ij}(u, v) - Q_{ij}(w, v)}{u - w}. \]

Choose polynomials \( P_{ij}(x, y) \in \mathbb{k}[x, y] \) such that \( Q_{ij}(x, y) = P_{ij}(x, y)P_{ji}(y, x) \). Let \( p_{ij} = P_{ij}(1, 0) \) and \( t_{ij} = Q_{ji}(0, 1)^{-1} \).

Following Rouquier [Rou08, Prop. 3.12], we have a polynomial action of \( R_n \) on a sum of polynomial rings \( \mathbb{k}[Y_1, \ldots, Y_n] \) in \( n \) variables, one for each \( i \in \mathbb{I}^n \). Since the idempotents \( e_i \) act by projection to the different summands, we can write this sum as

\[ \mathcal{P} = \bigoplus_{i \in \mathbb{I}^n} \mathbb{k}[Y_1, \ldots, Y_n]e_i, \]

with the action defined by

\begin{align}
(2.3a) \quad y_i f(Y_1, \ldots, Y_n) e_i &= Y_k f(Y_1, \ldots, Y_n) e_i \\
(2.3b) \quad \psi_i f(Y_1, \ldots, Y_n) &= \begin{cases}
\begin{array}{c}
\includegraphics[width=1cm]{sum10.png}
\end{array} & \text{if } i \neq i_{k+1} \\
\begin{array}{c}
\includegraphics[width=1cm]{sum11.png}
\end{array} & \text{if } i = i_{k+1}
\end{cases}
\end{align}

2.2. Categorical actions. We’ve considered the KLR algebra \( R \) with an eye toward studying categorical actions of Lie algebras. By “a categorical action of a Lie algebra” we mean a representation of a specific 2-category \( \mathcal{U} \) defined by Khovanov-Lauda [KL10] and Rouquier [Rou08] (the equivalence of these 2-categories is proven in [Bru16]). For technical reasons, we’ll use a slightly different version of this 2-category. The Kac-Moody 2-category \( \mathcal{U}(\mathfrak{g}) \) is the strict \( \mathbb{k} \)-linear 2-category with

(i) objects formal direct sums (with arbitrary indexing set) of the elements of \( X \),

(ii) generating 1-morphisms \( E_i1_{\lambda} = \uparrow_{\lambda} : \lambda \rightarrow \lambda + \alpha_i \) and \( F_i1_{\lambda} = \downarrow_{\lambda} : \lambda \rightarrow \lambda - \alpha_i \) for \( I \in \mathbb{I} \) and \( \lambda \in X \), and
(iii) generating 2-morphisms

\[(2.4) \quad \lambda : F_i 1_{1,\lambda} \Rightarrow F_i 1_{1,\lambda}, \quad \bigwedge_i \lambda : 1_{1,\lambda} \Rightarrow F_i 1_{1,\lambda}, \quad \bigcap_i \lambda : 1_{E_i 1_{1,\lambda}} \Rightarrow 1_{1,\lambda}, \quad (2.5) \quad i : F_j F_i 1_{1,\lambda} \Rightarrow F_i F_j 1_{1,\lambda}, \quad \bigwedge_j \lambda : 1_{1,\lambda} \Rightarrow F_i 1_{1,\lambda}, \quad \bigcap_i \lambda : F_i 1_{E_i 1_{1,\lambda}} \Rightarrow 1_{1,\lambda}. \]

Note that one would usually just take the set of 0-morphisms to be \(X\) itself, but it will be more convenient later to consider direct sums of these objects. As usual, a 1-morphism between direct sums is a column-finite matrix of 1-morphisms, and 2-morphisms between these are defined entrywise. The sideways crossings and downward dots are defined by

\[(2.6) \quad \bigwedge_j \lambda := \bigcap_j \bigwedge j \lambda, \quad \bigcap_j \lambda := \bigcup_j \bigwedge j \lambda, \]

and there are negatively dotted bubbles defined by

\[(2.7) \quad \begin{align*}
\bigwedge_i n - \alpha_i^{\vee}(\lambda) - 1 &:= \begin{cases} 
(-1)^n \det \left( \sum_{r,s=1,\ldots,n} r - s + \alpha_i^{\vee}(\lambda) \bigwedge_i \lambda \right) & \text{if } \alpha_i^{\vee}(\lambda) \geq n > 0, \\
1_{1,\lambda} & \text{if } \alpha_i^{\vee}(\lambda) \geq n = 0, \\
0 & \text{if } \alpha_i^{\vee}(\lambda) \geq n < 0,
\end{cases} \\
\bigcap_i n + \alpha_i^{\vee}(\lambda) - 1 &:= \begin{cases} 
(-1)^n \det \left( \sum_{r,s=1,\ldots,n} r - s - \alpha_i^{\vee}(\lambda) \bigwedge_i \lambda \right) & \text{if } -\alpha_i^{\vee}(\lambda) \geq n > 0, \\
1_{1,\lambda} & \text{if } -\alpha_i^{\vee}(\lambda) \geq n = 0, \\
0 & \text{if } -\alpha_i^{\vee}(\lambda) \geq n < 0.
\end{cases}
\]

The generating 2-morphisms are subject to the following relations \((2.2a-2.2c)\) and the additional relations:

\[(2.8) \quad \bigwedge_i \lambda = \bigwedge_i \lambda, \quad \bigcap_i \lambda = \bigcap_i \lambda, \quad (2.9) \quad \bigwedge_i \lambda = \bigwedge_i \lambda, \quad \bigcap_i \lambda = \bigcap_i \lambda, \quad (2.10) \quad \bigwedge_i \lambda = \bigwedge_i \lambda.\]
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(2.11)  
\[
\lambda^i = -\delta_{\alpha^i_0,0} \lambda^i \text{ if } \alpha^i_0(\lambda) \leq 0,
\]

(2.12)  
\[
\lambda^i = \delta_{\alpha^i_0,0} \lambda^i \text{ if } \alpha^i_0(\lambda) \geq 0,
\]

(2.13)  
\[
n + \alpha^i_0(\lambda) - 1 \lambda^i = \delta_{n,0} \lambda^i \text{ if } -\alpha^i_0(\lambda) < n \leq 0,
\]

(2.14)  
\[
n - \alpha^i_0(\lambda) - 1 \lambda^i = \delta_{n,0} \lambda^i \text{ if } \alpha^i_0(\lambda) < n \leq 0,
\]

(2.15)  
\[
\lambda^i = (-1)^{b_{ij}} \lambda^j + \delta_{i,j} \sum_{r,s \geq 0} \lambda^i \lambda^j_{-r-s-2},
\]

(2.16)  
\[
\lambda^i = (-1)^{b_{ij}} \lambda^j + \delta_{i,j} \sum_{r,s \geq 0} \lambda^i \lambda^j_{-r-s-2}.
\]

Remark 2.4. These relations are different from those used in [BSW20a], since (2.6) incorporates the scalar $t_{ij}$. Our relations do match those of [Bru16] where

\[
Q_{ij}(u, v) = t_{ij}^{-1} t_{ji}^{-1} (t_{ij} u^{c_{ij}} + t_{ji} v^{c_{ji}} + \sum_{p,q} s_{pq}^{ij} u^p v^q).
\]

In particular, the scalar $t_{ij}$ matches the way it is used in loc. cit., though we must negate all crossings where the labels are the same in order to match [Bru16, (2.6)] with (2.2a). While this complicates some calculations, it has the advantage that when all strands are the same color, we get the usual categorified $\mathfrak{sl}_2$ relations, while in [BSW20a], the relations on bubbles, such as (2.13), are modified by a scalar depending on the label on strands.

The Main Theorem of [Bru16] states that:

**Theorem 2.5** Given a decomposition $C \cong \bigoplus \mu C_{\mu}$ and functors $E_i : C_{\mu} \to C_{\mu + \alpha_i}, F_i : C_{\mu} \to C_{\mu - \alpha_i}$, define a representation of $\mathcal{U}_g$ if and only if:

(KM1) there are prescribed adjunctions $(E_i, F_i)$ for all $i \in I$;

(KM2) For $m \geq 0$ there is an action of the KLR algebra $R_m$ with $m$ strands for $l$ on the $m$th power of the functor $E := \bigoplus_{i \in I} E_i$.
(KM3) If \( k_i = \alpha_i(\mu) \geq 0 \) then there is an isomorphism \( \mathcal{E}_i \mathcal{F}_i|_{C_\mu} \Rightarrow F_i \mathcal{E}_i|_{C_\mu} \oplus \text{Id}_{C_k}^{\oplus k_i} \) induced by the column vector

\[
\begin{pmatrix}
1 \\
1 \\
\_ \\
\_ \\
\vdots \\
\_ \\
\_ \\
1 \\
\end{pmatrix}
\]

If \( k_i \leq 0 \), there is an isomorphism \( \mathcal{E}_i \mathcal{F}_i|_{C_\mu} \oplus \text{Id}_{C_k}^{\oplus -k_i} \Rightarrow F_i \mathcal{E}_i|_{C_\mu} \) induced by the row vector

\[
\begin{pmatrix}
1 \\
1 \\
\_ \\
\_ \\
\vdots \\
\_ \\
\_ \\
1 \\
\end{pmatrix}
\]

As in [BSW23, BSW20a], we will often use generating functions when working with elements of an algebra \( A \). This means that we will work with formal Laurent series \( f(z) \in A((z^{-1})) \) in an indeterminate \( z \) (or \( v, w, \ldots \)). We write \( [f(z)]_{z'} \) for the \( z' \)-coefficient of such a series, \( [f(z)]_{z=0} \) for \( \sum_{r<0} [f(z)]_{z'} z^r \), \( [f(z)]_{z=0} \) for \( \sum_{r>0} [f(z)]_{z'} z^r \) (which is a polynomial), and so on. To give an example, suppose that

\[
f(z) = \sum_{r \geq 0} f_r z^{1-r} \in z^k A + z^{k-1} A[z^{-1}]
\]

for some \( f_r \in A \). Then we can define new elements \( g_r \in A \) by declaring that

\[
g(z) = \sum_{r \geq 0} g_r z^{-k-r} \in z^{-k} A + z^{-k-1} A[z^{-1}]
\]

is the inverse of the formal Laurent series \( f(z) \). In fact, setting \( f_r := 0 \) for \( r < 0 \), we have

\[
g_r = \det(-f_{s+t+1})_{s,t = 1, \ldots, r}.
\]

This identity is valid even if \( A \) is non-commutative providing the determinant is interpreted as a suitably ordered Laplace expansion. We let

\[
\begin{align*}
\lambda \circ \lambda (z) := & \sum_{i \in \mathbb{Z}} \lambda \bigcirc_{i} z^{-r-1} \in z^{n_1(\lambda)} 1_{1_1} + z^{n_1(\lambda)-1} \text{End}(1_{1})[z^{-1}], \\
\lambda \odot \lambda (z) := & \sum_{i \in \mathbb{Z}} \lambda \smallcirc_{i} z^{-r-1} \in z^{-n_1(\lambda)} 1_{1_1} + z^{-n_1(\lambda)-1} \text{End}(1_{1})[z^{-1}].
\end{align*}
\]

Copying the notation of [BSW20a], we interpret a dot with an adjacent power series \( p(z, y) \in k[[z^{-1}], y] \) as the expression obtained by substituting in a dot on the adjacent strand for \( y \), leaving \( z \) as a formal variable. We can then rewrite the relations \( (2.11)-(2.14) \) as

\[
\lambda \smallcirc (z) \circ \lambda (z) \lambda = 1_{1_1},
\]
\[(2.23) \quad (z-y)^{-1} \bigcirc \ = \ \left[ \bigcirc_i (z-y)^{-1} \right]_{x=0}, \quad (z-y)^{-1} \bigcirc_i = \left[ (z-y)^{-1} \bigcirc_i \right]_{x=0} \]

\[(2.24) \quad \bigcirc_i (z) \downarrow_{\lambda} = (z-y)^{-2} \bigcirc_i (z), \quad \bigcirc_i (z) = \bigcirc_i (z) \downarrow (z-y)^{-2} \]

Since we are working with arbitrary \(Q_i\)'s we have to be careful about rewriting the bubble slides in the case where \(i \neq j\), but we can rewrite [Web17a, Prop 2.8] as this form as:

\[(2.25) \quad \bigcirc_i (z) \downarrow_{\lambda} = \iota_{\beta Q_i(z,y)} \bigcirc_i (z), \quad \bigcirc_i (z) = \bigcirc_i (z) \downarrow \iota_{\beta Q_i(z,y)} \]

**Lemma 2.6** ([BSW20a, Lem. 3.5]) For a polynomial \(p(z) \in \mathbb{k}[z]\), we have that:

\[(2.26) \quad \lambda \bigcirc_i p(z) = \left[ \lambda \bigcirc_i p(z) \right]_{z^{-1}}, \quad \lambda \bigcirc_i p(z) = \left[ \lambda \bigcirc_i p(z) \right]_{z^{-1}}, \quad \lambda \bigcirc_i p(z) = \left[ \lambda \bigcirc_i p(z) \right]_{z^{-1}}, \quad \lambda \bigcirc_i p(z) = \left[ \lambda \bigcirc_i p(z) \right]_{z^{-1}} \]

A representation of \(\mathcal{U}\) is a 2-functor from \(\mathcal{U}\) to the 2-category of exact categories with exact functors. This sends \(\mu\) to a category \(C_{\mu}\) and a direct sum of 0-morphisms to the direct sum of the corresponding categories \(C_{\mu}\). We always assume that \(C_{\mu}\) is locally finite abelian or Schurian over the field \(\mathbb{k}\), as defined in [BSW20a, §2.2]:

- We call \(C_{\mu}\) **locally finite abelian** if every object has finite length and every morphism space is finitely dimensional over \(\mathbb{k}\).
- We call \(C_{\mu}\) **Schurian** if it is isomorphic to the locally finite-dimensional, locally unital modules over a locally finite-dimensional, locally unital algebra \(A\).

Since \(I\) might potentially be infinite, we also add the restriction that:

\[(\dagger) \quad \text{on any given object } M \text{ we only have } \mathcal{E}_i M \neq 0 \text{ or } \mathcal{F}_i M \neq 0 \text{ for finitely many } i \in I. \]

Since the definition of \(\mathcal{U}\) uses the set of weights \(Y\), it depends on the choice of realization, not just the Cartan matrix \(C\). However, we can simplify the dependence on realization, by noting that the relations between 2-morphisms depend only on the scalars \(\alpha_i^\gamma\). Thus, if \(\lambda \in \mathfrak{h}^\ast\) is perpendicular to all \(\alpha_i^\gamma\), then there is an equivalence of 2-categories that acts on 0-morphisms by \(\mu \mapsto \mu + \lambda\) and by the obvious identifications of 1-morphisms and 2-morphisms.
Let $g' = [g, g] \subset g$ be the derived subgroup. The torus of $g'$ is the subspace of the Cartan $h' \subset h$ spanned by the elements $H_i = \alpha_i^\vee$. We have a functor $\mathcal{U}_g \to \mathcal{U}_{g'}$ by pulling back weights and the obvious identification on 1-morphisms and 2-morphisms.

This is not full on 1-morphisms—if the Cartan matrix is degenerate, then we could have a sum of roots which is perpendicular to all $\alpha_i^\vee$, but which was non-zero in $h$. However, this functor is fully-faithful on categories of 1-morphisms; in particular, two monomials $E_i, E_j$ which have different weights for $h$ might have the same weight under $h'$ but there will still be no 2-morphisms between them. In particular, the question of non-degeneracy of a categorification will be unchanged by going from $g$ to $g'$, and so we can assume that $g = g'$.

Similarly, note that if there are linear relations between $\alpha_i^\vee$ in $g$, then we can write $g$ as a quotient of its universal central extension $g''$. In this case, we have a 2-functor $\mathcal{U}_g \to \mathcal{U}_{g''}$ (note the slightly surprising direction of this functor!) which acts on weights by pullback and by the obvious identification of 1-morphisms and 2-morphisms. This is fully faithful on 1-morphisms and 2-morphisms, so if our categorification for $g''$ is non-degenerate, then the categorification for $g$ is non-degenerate.

Some of our constructions will be a little easier if we use a realization of this type. Thus it will be useful later to summarize this discussion:

**Lemma 2.7** If we check that non-degeneracy holds when $g$ is the universal derived Lie algebra for a Cartan matrix (where simple coroots are linearly independent and span $h$), then it holds for all other realizations of this Cartan matrix.

Thus, for Sections 3 and 4, we will assume that $g$ is the universal derived Kac-Moody algebra attached to our chosen Cartan matrix.

### 3. The proof of non-degeneracy

#### 3.1. The $\mathfrak{sl}_2$ coproduct functor.

As in [BSW23], we can define a functor from $\mathcal{U}_{\mathfrak{sl}_2}$ to the outer tensor product of two copies of itself, with a morphism inverted. As in [BSW23], we will use the convention that the 1-morphisms $\mathcal{E}, F$ in our first factor are colored blue and those in the second factor are colored red. This might seem a peculiar idea if one is just now encountering it for the first time, but it is quite powerful when employed in non-degeneracy proofs. The reasoning behind the definition of this map is explained in more detail below (6.10).

Let $\mathcal{U} \overline{\otimes} \mathcal{U}$ be the strict 2-category obtained from the tensor product $\mathcal{U} \otimes \mathcal{U}$ of additive 2-categories by localizing the morphism $\bullet - \bullet$. This means that we adjoin a two-sided inverse to this morphism, which we denote as a dumbbell

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\end{array} := \left( \begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\end{array} \right)^{-1}.
\]
Note that since we take tensor product of additive 2-categories, an object in $\mathcal{U} \otimes \mathcal{U}$ is a direct sum of pairs of integers $(k, \ell)$.

**Theorem 3.1** We can define a functor $\Delta$ from $\mathcal{U}$ to $\mathcal{U} \otimes \mathcal{U}$ sending:

$$
\begin{align*}
n &\mapsto \bigoplus_{k+\ell=n} (k, \ell) \\
\mathcal{E} &\mapsto \mathcal{E} \oplus \mathcal{E} \\
\mathcal{F} &\mapsto \mathcal{F} \oplus \mathcal{F}
\end{align*}
$$

which we should interpret as sending $\mathcal{E} : n \to n + 2$ to the matrix of 1-morphisms where the entry with column $(k, \ell)$ and row $(k + 2, \ell)$ is $\mathcal{E}$ and the entry with column $(k, \ell)$ and row $(k, \ell + 2)$ is $\mathcal{E}$. On 2-morphisms, this acts by

$$
\begin{align*}
\begin{array}{c}
\bullet \\
\circ \\
\end{array} &\mapsto \begin{array}{c}
\bullet \\
\circ \\
\end{array} + \\
\begin{array}{c}
\circ \\
\bullet \\
\end{array}, \\
\begin{array}{c}
\circ \\
\circ \\
\end{array} &\mapsto \begin{array}{c}
\circ \\
\circ \\
\end{array} + \\
\begin{array}{c}
\circ \\
\circ \\
\end{array}, \\
\begin{array}{c}
\circ \\
\bullet \\
\end{array} &\mapsto \begin{array}{c}
\circ \\
\bullet \\
\end{array} +
\end{align*}
$$

$$
\begin{align*}
\begin{array}{c}
\circ \\
\circ \\
\end{array} &\mapsto \begin{array}{c}
\circ \\
\circ \\
\end{array} + \begin{array}{c}
\circ \\
\circ \\
\end{array} - \\
\begin{array}{c}
\circ \\
\circ \\
\end{array} + \\
\begin{array}{c}
\circ \\
\circ \\
\end{array} - \\
\begin{array}{c}
\circ \\
\bullet \\
\end{array} + \\
\begin{array}{c}
\circ \\
\bullet \\
\end{array}.
\end{align*}
$$

Due to its length, we give the proof of this result on page 33.

### 3.2. Iterated coproducts.

We can also iterate this map to obtain a functor $\mathcal{U} \to \mathcal{U} \otimes \cdots \otimes \mathcal{U}$ for an arbitrary number of factors. This is given by the formulas (3.3–3.4), where we sum each term which is all red over the $n$ factors of the tensor product, and each term that contains red and blue over all pairs of factors. Since it is hard to choose a consistent set of colors to represent the elements of a set of arbitrary size, we will instead label copies by elements of a subset $A \subset \mathbb{k}^\times$. We will distinguish the morphisms $\mathcal{E}_a$ and $\mathcal{F}_a$ by writing $a$ at the beginning or end of the strand. We recognize that this is potentially confusing when such labels usually stand for elements of the Dynkin diagram, but we can think of the set $A$ with no arrows as the Dynkin diagram of $\mathfrak{sl}_2[A]$.

In Section 3.1, we worked with localizations of our categories, but for certain constructions we wish to do below, we require something a bit stronger. First, we’ll want to consider representations of $\mathcal{U}$ where on any object, only finitely many $\mathcal{E}_i$’s are non-zero. Furthermore, we’ll wish to assume that our dots are topologically nilpotent, that is, the limit $\lim_{n \to \infty} (\mathcal{E}_i)^n = 0$.

These properties can be captured on the level of 2-categories by using completion. The graded vector space $\text{Hom}_{\mathcal{U}}(u, v)$ has a natural **grading topology** where the elements $\text{Hom}_{\mathcal{U}}^{\leq k}(u, v)$ of degree $\geq k$ is an open neighborhood of the identity, and these form a basis of such neighborhoods. Of course, any 2-morphism of positive degree in $\text{Hom}_{\mathcal{U}}(u, u)$ is topologically nilpotent in this topology. We need to modify this topology a bit to account for the fact that $I$ might be infinite. For a finite set $I_0 \subset I$, we have a subspace $\text{Hom}_{\mathcal{U}}^{I_0}(u, v)$ defined by the 2-morphisms that factor through a monomial of length $\leq r$ which contains one of the symbols $\mathcal{E}_{i, a}$ with $i \notin I_0$. Note that for any object $X \in \text{Ob}(C)$ in a representation $C$, we assume that (†) holds, that is, the set of vertices such that $\mathcal{E}_{i, a} X$ are not both zero is finite. Applying the same principle to the images of $X$, there are only finitely many monomials of length $\leq r$ such that
\( e_i X \neq 0 \), and thus, the set \( I_0 \) of indices appearing in these monomials is 0. Thus, the 2-morphisms in \( \text{Hom}^{br}_U(u, v) \) acts trivially on \( X \). This shows that the action of \( U \) is continuous in the **cofinite topology** where the sets \( \text{Hom}^{br}_U(u, v) \) form a basis of the identity, giving morphism spaces in the representation the discrete topology.

Now, consider the join of the grading and cofinite topologies, which we’ll call the **GC topology**; this topology is generated by the spans \( \text{Hom}^{\geq k}_U(u, v) + \text{Hom}^{br}_U(u, v) \) for all \( k, r \in \mathbb{Z}_{\geq 0} \) and finite sets \( I_0 \subset I \).

**Definition 3.2** Let \( \hat{U} \) be the 2-category enriched on the level of 2-morphisms by topological vector spaces, where

- 0-morphism spaces are unchanged from \( U \).
- 1-morphism spaces are summands of (potentially infinite) formal Cartesian products of monomials \( e_i \) in 1-morphisms where:
  - there is an upper bound \( r \) on the length of all monomials appearing, and
  - for any finite subset \( I_0 \subset I \), there only finitely monomials \( e_i \) in which only functors \( e_{xi} \) for \( i \in I_0 \) appear.
- 2-morphism spaces \( \text{Hom}_{\hat{U}}(\prod u_\alpha, \prod v_\beta) \) are the completion of the Cartesian product \( \prod_{a,b} \text{Hom}_U(u, v) \) with respect to the GC topology (i.e., they are matrices of 2-morphisms with no finiteness conditions on rows or columns).

It is not manifest that the composition in this category is well-defined, but we confirm it below:

1. For 1-morphisms, note that for each finite set \( I_0 \), there are only finitely monomials in which \( e_{xi} \) for \( i \in I_0 \) appears in both \( u \) and \( v \), so the same is true of their composition.
2. For 2-morphisms, we have to be concerned about the fact that we are composing matrices where there are potentially infinitely many non-zero entries in both rows and columns. That is, when we consider the composition

\[
\text{Hom}_{\hat{U}}(\prod u_\alpha, \prod v_\beta) \times \text{Hom}_{\hat{U}}(\prod v_\beta, \prod w_\gamma) \to \text{Hom}_{\hat{U}}(\prod u_\alpha, \prod w_\gamma),
\]

the matrix entry for \( u_\alpha \to w_\gamma \) is potentially an infinite sum of morphisms factoring through \( v_\beta \). However, for any \( I_0 \), all but finitely many of these terms lie in \( \text{Hom}^{br}_U(u, v) \) where \( r \) is the upper bound on the length of monomials in \( \prod v_\beta \), so this composition is well-defined in the completion.

This might seem a little strange, but the simple way to think about this completion is that it acts on any graded \( U \)-module category satisfying \((\dagger)\) where the grading on \( \text{Hom}(\mu M, \nu M) \) for an object \( M \) is bounded above, and in fact, this action is continuous when the morphisms in this module category are endowed with the discrete topology. In this category, we can thus make sense of \( \gamma_a(\hat{z}) \) for any power series \( \gamma_a(z) = \)
Unfurling Khovanov-Lauda-Rouquier algebras

\[ a + a^{(1)}z + a^{(2)}z^2 + \cdots \] with \( a \neq 0 \) and \( a^{(1)} \neq 0 \). Note that the power series

\[ q_a(\begin{array}{c} \downarrow \\ \uparrow \end{array}) = \frac{\gamma_a(\downarrow) - \gamma_a(\uparrow)}{\gamma_a(\downarrow) - \gamma_a(\uparrow)} \]

is well-defined and a unit in the power series ring, since its reciprocal has constant term \( a^{(1)} \).

We’ll want to consider the category \( \hat{U}_A \) where \( A \subset \mathbb{k} \) is a set with the disconnected Cartan datum, that is, \( \langle \alpha_a, \alpha_b \rangle = 2\delta_{a,b} \). The corresponding Kac-Moody algebra is the direct sum \( \mathfrak{sl}_2 \oplus A \).

If \( A = \{a_1, \ldots, a_n\} \subset \mathbb{k} \) is finite, the 2-category \( \hat{U}_A \) receives a functor from the iterated localized tensor product \( U \bigodot \cdots \bigodot U \) where on the \( k \)th factor, we apply the morphism

(3.5) \[ \begin{array}{c} \downarrow \\ \uparrow \end{array} \mapsto \gamma_{a_k}(\begin{array}{c} \downarrow \\ \uparrow \end{array}), \quad \begin{array}{c} \bigwedge \\ \bigwedge \end{array} \mapsto \begin{array}{c} \bigwedge \\ \bigwedge \end{array}, \quad \begin{array}{c} \times \\ \times \end{array} \mapsto q_{a_k}(\begin{array}{c} \downarrow, \downarrow \\ \uparrow, \uparrow \end{array}). \]

Taking the composition \( U \rightarrow U \bigodot \cdots \bigodot U \rightarrow \hat{U}_A \), this maps:

(3.6) \[ n \mapsto \bigoplus_{\sum_{a \in A} a \equiv n} \lambda \quad \mathcal{E} \mapsto \bigoplus_{i=1}^{k} \mathcal{E}_a \]

Note that this maps each integer \( n \), thought of as a weight, to an infinite direct sum of weights, which we can think of as functions \( A \rightarrow \mathbb{Z} \) whose values sum of \( n \). On 2-morphisms, this functor maps

(3.7) \[ \begin{array}{c} \downarrow \\ \uparrow \end{array} \mapsto \sum_{a \in A} \gamma_{a}(\begin{array}{c} \downarrow \\ \uparrow \end{array}), \quad \begin{array}{c} \bigwedge \\ \bigwedge \end{array} \mapsto \sum_{a \in A} \begin{array}{c} \bigwedge \\ \bigwedge \end{array}, \quad \begin{array}{c} \times \\ \times \end{array} \mapsto \sum_{a \in A} \begin{array}{c} \times \\ \times \end{array}, \]

(3.8) \[ \begin{array}{c} \times \\ \times \end{array} \mapsto \sum_{a \in A} q_a(\begin{array}{c} \downarrow, \downarrow \\ \uparrow, \uparrow \end{array}) + \sum_{a \neq b \in A} f_{a,b}(\begin{array}{c} \downarrow, \downarrow, \downarrow, \downarrow \\ \uparrow, \uparrow, \uparrow, \uparrow \end{array}) + \sum_{a \neq b \in A} f_{a,b}(\begin{array}{c} \downarrow, \downarrow, \downarrow, \downarrow \\ \uparrow, \uparrow, \uparrow, \uparrow \end{array}) \]

where \( f_{a,b} = (\gamma_a(x_1) - \gamma_b(x_2))^{-1} \). These formulas make sense in the completion \( \hat{U}_A \), since modulo any power of the dot, it becomes a finite sum.

Above, we’ve assumed that \( A \) is finite, but in fact we can relax this assumption:

**Lemma 3.3** The formulas (3.6)-(3.8) define a functor \( U \rightarrow \hat{U}_A \) for an arbitrary subset \( A \subset \mathbb{k} \).

**Proof.** Checking each relation involves only finitely many indices, so assuming these maps make sense, the relations follow automatically from the finite case. On the 1-morphisms \( \mathcal{E}, \mathcal{F} \), these are sent to a sum of length 1 monomials, where each \( i \) appears
once, so in fact only finitely many terms use \( i \in I_0 \) for any finite \( I_0 \). The result for all other 1-morphisms follows by composition.

Now consider 2-morphisms. For each \( I_0 \), in (3.7–3.8) only finitely many terms involve only indices from \( I_0 \) so all but finitely many lie in \( \Hom_{\tilde{U}}(u,v) \). Thus, the functor is well-defined. □

Consider a collection of non-zero representations \( X_a \) for \( a \in A \) of \( U(\mathfrak{sl}_2) \) in which \( \mathfrak{e} \) acts nilpotently on \( \mathcal{E}M \) for all \( M \). Also, assume that all but finitely many of these are the trivial representation where we take one copy of \( \text{Vect} \) in degree 0 with \( \mathcal{E}, \mathcal{F} \) both acting trivially. The tensor product \( \tilde{U}_A \) acts on the Deligne tensor products of these representations \( \bigotimes_{a \in A} X_a \); we can think of these as a finite Deligne product by only considering \( a \) where \( X_a \) is non-trivial. Note that the collections of locally finite abelian and Schurian abelian categories are both closed under Deligne tensor product. This is shown in the locally finite abelian case in [Lop13, Prop. 22], and in the Schurian case, we simply take the tensor product of the corresponding locally finite-dimensional locally unital algebras. By Lemma 3.3, we obtain an induced \( U \)-representation on \( \bigotimes_{a \in A} X_a \).

The most natural example is the cyclotomic quotient for some highest weight \( \lambda_a \) for \( a \in A \) with all but finitely many of these being \( \lambda_a = 0 \); this is the unique categorification generated by a single object \( V_a \) of weight \( \alpha_i \) where \( y: \mathcal{F}V_a \to \mathcal{F}V_a \) satisfies \( x^\lambda_a = 0 \). In this case, the tensor product is generated by a single object \( V = \bigotimes_{a \in A} V_a \) such that \( y: \mathcal{F}V \to \mathcal{F}V \) satisfies

\[
\prod_{a \in A} (y - a)^{\lambda_a} = 0.
\]

We’ll show in Lemma 5.9 that, in fact, the tensor product has an induced equivalence to the deformed cyclotomic quotient with the equation (3.9). Of course, for \( U(\mathfrak{sl}_2) \), it’s easy to construct this cyclotomic quotient by hand and so the discussion above seems like an unnecessarily roundabout method, but for higher rank, it will pay significant dividends.

3.3. Extension to higher rank. Now consider \( U_\mathfrak{g} \) for a general Cartan datum. As mentioned previously, we assume in this section that \( \mathfrak{g} \) is the universal derived algebra for this Cartan matrix. By Lemma 2.7, we will lose no generality by doing this.

As before, we choose a spectrum \( U_i \) for each \( i \in I \) which we assume to be complete. As defined in Definition 2.2, we have a graph \( \tilde{I} \) naturally mapping to \( I \) whose vertices are pairs \((i,u)\) for \( u \in U_i \). Our previous section discussed an example of this construction when \( I \) is a single point with no edges. In this case, \( \tilde{I} \) was an arbitrary number of points, identified with the set \( U_\bullet \). In order to understand \( \tilde{I} \), let us give a more detailed analysis of the polynomials \( P_{ij} \). Let \( g_{ij} = \gcd(-c_{ij}, -c_{ji}) \) and \( h_{ij} = -c_{ij}/g_{ij} \). We
must have that
\[ P_{ij}(x, y) = p_{ij} \prod_{a_{ij}^{(k)} \in A_{ij}} (x^{1/d_i} - a_{ij}^{(k)} y^{1/d_i}) \]
where \( A_{ij} = \{a_{ij}^{(k)}\} \) is the multiset of roots of \( P_{ij}(x^{d_i}, 1) \), considered with multiplicity. Let \( B_{ij} = \{b_{ij}^{(k)} = (a_{ij}^{(k)})^{-1}\} \) be the reciprocals of these numbers. Note that
\[ Q_{ij}(x, y) = t_{ij}^{-1} \prod_{a_{ij}^{(k)} \in A_{ij}} (x^{1/d_i} - a_{ij}^{(k)} y^{1/d_i}) \prod_{b_{ij}^{(k)} \in B_{ij}} (x^{1/d_i} - b_{ij}^{(k)} y^{1/d_i}) \]

Homogeneity requires that \( P_{ij}(x^{1/h_{ij}}, 1) \) be a polynomial. We’ll also let \( A_{ij} = \{a_{ij}^{(k)}\} \) be the roots of \( P_{ij}(x^{1/h_{ij}}, 1) \), again considered with multiplicity; note that the elements of \( A_{ij} \) are the \( d_i h_{ij} \)th roots of the elements of \( A_{ij} \). Furthermore, we have that \( d_i h_{ij} = d_j h_{ji} = \text{lcm}(d_i, d_j) \). We also let \( B_{ij} = \{b_{ij}^{(k)} = (a_{ij}^{(k)})^{-1}\} \). As before, the elements of \( B_{ij} \) are the \( d_i h_{ij} \)th roots of the elements of \( B_{ij} \).

The order of vanishing of \( P_{ij} \) at \( x = u, y = u' \) can be interpreted in these terms as the number of \( k \) such that \( u^{h_{ij}} = a_{ij}^{(k)} (u')^{h_{ij}} \), which shows that it is well-defined.

Associated to the graph \( \bar{I} \), we have polynomials
\[ P_{(i,u),(j,u')}((x, y) = (x - y)^{(h(i,u) - h(j,u'))} \in \mathbb{k}[x, y] \]
These are the “geometric coefficients” for a KLR algebra of this graph.

**Definition 3.4** Consider the KLR algebra \( R \) of \( \bar{I} \) with the symmetric Cartan datum associated to its graph structure and the polynomials \( P_{(i,u),(j,u')} \). To avoid confusion, we’ll denote the elements \( \psi_k, y_k, \mathfrak{e}_1 \) of this algebra with sans serif letters.

Let \( \tilde{\mathfrak{u}} \) be the derived universal symmetric Kac-Moody algebra attached to the graph \( \bar{I} \). Let \( U_{\tilde{\mathfrak{u}}} \) be the 2-category categorifying \( \tilde{\mathfrak{u}} \) with the polynomials \( P_{(i,u),(j,u')} \).

We want to extend Lemma 3.3 to give a 2-functor \( U_{\mathfrak{u}} \to \tilde{U}_{\tilde{\mathfrak{u}}} \). First, we must be careful about how we set up these functors on the individual \( sl' \)'s. In order to define these functors on strands with label \( i \) via the formulas (3.6)-(3.8) using the power series
\[ g'_{u,j}(z) = u(z + 1)^{d_i} \]

In order to accomplish this extension, we only need to describe the image of the crossing of differently labeled strands. To understand the formulas for this image, note that by (3.10), we have that
\[ P_{ij}(g'_{u,j}(x), g'_{u',j}(y)) = p_{ij} \prod_{a_{ij}^{(k)} \in A_{ij}} (u^{1/d_i}(x + 1) - a_{ij}^{(k)} (u')^{1/d_i}(y + 1)) \]
for \( u^{1/d_i} \) and \( (u')^{1/d_i} \) any roots of \( u, u' \in \mathbb{k} \). The product above is independent of this choice. Note that each of the factors in the product above is either:
invertible if \( u_{ij}^{1/d_i} \neq a_{ij}^{(k)}(u')^{1/d_i} \) or

(2) a multiple of \( x - y \) if \( u_{ij}^{1/d_i} = a_{ij}^{(k)}(u')^{1/d_i} \).

and that the number of the latter factors is exactly the number of arrows \((i, u) \rightarrow (j, u')\). Thus, we can factor

\[
P_{ij}(y_{u,i}(x), y_{u',j}(y)) = P_{(i,u),(j,u')}(x, y)P_{ij}^\circ uu'(x, y)
\]

for an invertible power series \( P_{ij}^\circ(x, y) \in \mathbb{k}[x, y] \). We find that:

**Lemma 3.5** We have a 2-functor \( \mathcal{U}(\mathfrak{g}) \rightarrow \hat{\mathcal{U}}(\hat{\mathfrak{g}}) \) acting on 0-morphisms and 1-morphisms by:

\[
(3.12) \quad \lambda \mapsto \bigoplus_{\mu | \mu_h = \lambda} \mu \quad \mathcal{E}_i \mapsto \bigoplus_{u \in \mathcal{U}_i} \mathcal{E}_{i,u}
\]

The action on 2-morphisms is determined by the formulas (3.7–3.8) and

\[
(3.13) \quad \sum_{\lambda, \mu \in \mathcal{P} \mathcal{L}_\lambda} \mu \rightarrow \sum_{\lambda, \mu \in \mathcal{P} \mathcal{L}_\lambda} \mu
\]

define a functor \( \mathcal{U}(\mathfrak{g}) \rightarrow \hat{\mathcal{U}}(\hat{\mathfrak{g}}) \).

We give the proof of this result on page 37.

3.4. **Non-degeneracy.** This provides us with the tools we need to prove non-degeneracy for categorified Kac-Moody algebras. In [KL10, §3.2.3], Khovanov and Lauda define a spanning set \( B_{i,i',\lambda} \) of the Hom space between two 1-morphisms in \( \mathcal{U}_\mathfrak{g} \) that they hypothesize are a basis. This set is not truly canonical, but it is well-defined up to strictly lower triangular change of basis.

This set is constructed as follows:

1. for each way of dividing the set \( i \cup -i' \) into pairs of matching elements (an \((i, i')\)-pairing, in the terminology of [KL10]), choose a diagram which connects the terminals for the two elements of each pair, with a minimal number of crossings and no bubbles or dots.

2. for each diagram of the above, and each arc joining two terminals, we fix a position on that arc (avoiding the crossings with any others). The full list of diagrams in \( B \) are indexed by diagrams as above, a choice of non-negative integer for each arc joining two terminals, and a monomial in the unnested bubbles (a basis vector of the ring \( \Pi_{\lambda} \) in the notation of [KL10, §3.2.1]). We construct the corresponding diagram by adding the corresponding number of dots on each arc and putting the bubbles to the left of the diagram.
In this paper, we have assumed that $k$ is an algebraically closed field. In fact, we can work over an arbitrary commutative ring $k'$ together with a choice of $Q_{ij}(x, y)$ such that $t_{ij}$ and $t_{ji}$ are multiplicative inverses of the scalars $Q_{ij}(0, 1), Q_{ij}(1, 0)$.

**Theorem 3.6**  In the category $\mathcal{U}$ defined over $k'$, Khovanov and Lauda’s spanning set $B_{i, i', \lambda}$ is a basis of the Hom space $\text{HOM}_{\mathcal{U}}(i, i')$ as a free $k'$-module.

We’ll prove this by exploiting Lemma 3.5 to construct examples of representations where any hypothetical relation must act trivially. We give the proof of this result on page 37. Note that while our proof will use the universal derived Kac-Moody algebra of our Cartan matrix, Lemma 2.7 shows that this will imply the result for all different choices of realization.

### 4. Spectral analysis of actions

#### 4.1. Eigenfunctors.  
In this section, we’ll reverse the construction of $\mathcal{U}_g$-representations from $\mathcal{U}_\beta$-representations. As mentioned in the introduction, our approach is very similar to that of [BSW20a, §4], only requiring occasional changes due to the different nature of Heisenberg and Kac-Moody categorifications.

In much of the literature, the representations of $\mathcal{U}$ considered have had the action of the dots and bubbles be nilpotent; this is necessary in a graded 2-representation which is locally finite abelian or Schurian. However, it can be a very powerful technique to deform these representations in such a way as to break this assumption. Consider a representation of the 2-category $\mathcal{U}$, sending $\lambda \mapsto C_\lambda$ such that $C_\lambda$ is $k$-linear locally finite abelian or Schurian. Since we assume that $k$ is algebraically closed, the endomorphisms of a simple object in such a category will always be the base field $k$.

For simplicity, we’ll start by working in the case of $\mathfrak{g} = \mathfrak{sl}_2$. By the usual identification of the weight lattice of $\mathfrak{sl}_2$ with $\mathbb{Z}$, if $C$ carries a categorical $\mathfrak{g}$-action, we can write $C \cong \bigoplus_{k \in \mathbb{Z}} C_k$.

**Definition 4.1**  For $u \in k$, let $E_u$ and $F_u$ be the subfunctors of $E$ and $F$ defined on $V \in C_k$ by declaring that $E_u V$ and $F_u V$ are the generalized $u$-eigenspaces of the endomorphisms $\downarrow |V$ and $\downarrow |V$, respectively.

As discussed in [BSW20a, §4.1], it suffices to define these functors on finitely generated objects. For a finite generated object, we have minimal polynomials $m_V(z), n_V(z) \in k[z]$ of the endomorphisms $\downarrow |V$ and $\downarrow |V$, respectively. Then there
are injective homomorphisms
\[(4.1) \quad \mathbb{k}[z]/(m_V(z)) \hookrightarrow \text{End}_C(EV), \quad \mathbb{k}[z]/(n_V(z)) \hookrightarrow \text{End}_C(FV), \]
\[p(z) \mapsto p(x) \updownarrow, \quad p(z) \mapsto p(x) \updownarrow. \]

Also let $\varepsilon_u(V)$ and $\phi_u(V)$ denote the multiplicities of $u \in \mathbb{k}$ as a root of the polynomials $m_V(z)$ and $n_V(z)$, respectively. By the Chinese remainder theorem, we have that
\[
(4.2) \quad \mathbb{k}[z]/(m_V(z)) \cong \bigoplus_{u \in \mathbb{k}} \mathbb{k}[z]/(z - u)^{\varepsilon_u(V)}, \quad \mathbb{k}[z]/(n_V(z)) \cong \bigoplus_{u \in \mathbb{k}} \mathbb{k}[z]/(z - u)^{\phi_u(V)}. \]

There are corresponding decompositions $1 = \sum_{u \in \mathbb{k}} e_u$ of and $1 = \sum_{u \in \mathbb{k}} f_u$ of the identity elements of these algebras as a sum of mutually orthogonal idempotents. The functors $E_uV$ and $F_uV$ can be defined as the images of the idempotents $e_u$ and $f_u$ in the summands of $EV$ and $FV$.

We will represent the identity endomorphisms of the functors $E_u$ and $F_u$ by vertical strings colored by $u$ as in the diagrams below. The inclusions $E_u \hookrightarrow E$ and $F_u \hookrightarrow F$ are depicted by the second pair of diagrams below. The projections $E \twoheadrightarrow E_u$ and $F \twoheadrightarrow F_u$ are the final pair.

To illustrate the notation, the natural transformation $\delta_{u,v} : E \Rightarrow E$ is the projection of $E$ onto its summand $E_{u,v}$, while
\[
(4.3) \quad \delta_{u,v} : E \Rightarrow E \updownarrow. \]

It is also clear from the definition that the endomorphisms of $E$ and $F$ defined by the dots restrict to endomorphisms of the summands $E_u$ and $F_u$. Representing these restrictions simply by drawing the dots on a string colored by $i$, we have that
\[
(4.4) \quad \bullet = \bullet, \quad \bullet = \bullet, \quad \bullet = \bullet, \quad \bullet = \bullet. \]

Since the downwards dot is both the left and right mate of the upwards dot, the adjunctions $(E, F)$ and $(F, E)$ induce adjunctions $(E_u, F_u)$ and $(F_u, E_u)$ for all $u \in U$. We draw the units and counits of these adjunctions using cups and caps colored by $i$. Again, the various inclusions and projections commute with these morphisms:
\[
(4.5) \quad \hat{\cup} = \hat{\cup}, \quad \hat{\cup} = \hat{\cup}, \quad \hat{\cup} = \hat{\cup}, \quad \hat{\cup} = \hat{\cup}. \]
The situation with crossings is more interesting. For \(u, v, u', v' \in k\), define

\[
(4.6) \quad v u' \diamond v' u := v' u' \diamond v u
\]

Thus, these natural transformations are defined by first including the summand \(E_v E_u\) into \(E E\), then applying natural transformation \(E E \Rightarrow E E\) defined by the usual crossing (positive or negative in the quantum case), then projecting \(E E\) onto the summand \(E_{v'} E_{u'}\). The defining relations plus (4.4) imply that

\[
(4.7) \quad v u' \diamond v' u := v' u' \diamond v u + \delta_{u,u'} \delta_{v,v'}
\]

There are also sideways and downwards versions of the new crossings which may be defined in a similar way, or equivalently by “rotating” the upwards ones using (4.5). The same proofs as [BSW20a, Lem. 4.1–2] show that:

**Lemma 4.2** If \([u, v] \neq [u', v']\) then the natural transformation (4.6) is zero. The same holds for the rotated versions of these crossings.

If \(v = v' \neq u = u'\), we have that

\[
\begin{align*}
\begin{diagram}
\node{(v \ u \ v' \ u')}{\bullet}{\bullet}{\bullet}\end{diagram} & = \begin{diagram}
\node{(v' \ u')}{\bullet}{\bullet}{\bullet}\end{diagram} (z_2 - z_1)^{-1}
\end{align*}
\]

4.2. **Bubbles and central characters.** Any dotted bubble defines an endomorphism of the identity functor \(\text{Id}_k\), i.e., an element of the center of the category \(C_k\). In particular, for \(V \in C_k\), dotted bubbles evaluate to elements of the center \(Z_V\) of the endomorphism algebra \(\text{End}_{C_k}(V)\). It is convenient to work with all of these endomorphisms at once in terms of the generating function

\[
(4.8) \quad \Theta_V(z) := \frac{\bigcirc(z)}{z} = \left(\frac{\bigcirc(z)}{z}\right)^{-1}.
\]

Recalling (2.20) and (2.21), we have \(\Theta_V(z) \in z^k + z^{k-1}Z_V[z^{-1}]\) when \(V\) has weight \(k\). In the following lemma, given a polynomial \(p(z) = \sum_{s=0}^r z^s z^{r-s} \in Z_V[z]\), we let

\[
\begin{align*}
\begin{diagram}
\node{p(z)}{\bullet}{\bullet}{\bullet}\end{diagram} & := \sum_{s=0}^r z^s \bigcirc, \\
\begin{diagram}
\node{p(x)}{\bullet}{\bullet}{\bullet}\end{diagram} & := \sum_{s=0}^r x^s \bigcirc.
\end{align*}
\]

Lemma 2.6 obviously extends to the setting of coefficients in \(Z_V\).

Most categorical \(\mathfrak{sl}_2\)-actions with which readers are familiar will assume that \(\bigcirc\) is nilpotent, but this is by no means necessary. On any object \(V\), the endomorphism \(\bigcirc\) must satisfy a polynomial relation with coefficients in \(k\), and thus has a finite
Let $U = \cup_V U_V$ be the union of these spectra for all objects.

Repeating the proof of [BSW20a, Lem. 4.3–4] with minor modifications shows that:

**Lemma 4.3** Let $V \in C_k$ be any object of weight $k$:

1. If $f(z) \in Z_V[z]$ is a monic polynomial such that $f(z) \not|_V = 0$, then $g(z) := O_V(z)f(z)$ is a monic polynomial in $Z_V[z]$ of degree $\deg f(z) + k$ such that $g(z) \not|_V = 0$.

2. If $g(z) \in Z_V[z]$ is a monic polynomial such that $g(z) \not|_V = 0$, then $f(z) := O_V(z)^{-1}g(z)$ is a monic polynomial in $Z_V[z]$ of degree $\deg g(z) - k$ such that $f(z) \not|_V = 0$.

For any simple module $L \in C_k$, we have that

$$O_L(z) = n_L(z)/m_L(z).$$

Now we will finally encounter a point where there is a substantive difference with [BSW20a] (though a strong analogy remains):

**Lemma 4.4** Suppose that $L \in C_\mu$ is an irreducible object and let $K$ be an irreducible subquotient of $E_u L$ for some $u \in \mathbb{k}$. Then

$$O_K(z) = O_L(z)(z - u)^2.$$  

**Proof.** This follows from the bubble slides (2.24):

$$\Phi(z) \bigg|_u \bigg|_L = (z-x)^2 \bigg|_u \bigg|_L = O_L(z)(z-x)^2 \bigg|_u \bigg|_L.$$  

When we pass to the irreducible subquotient $K$ of $E_u L$, we can replace the occurrences of $x$ in the expression on the right-hand side here with $i$, and the lemma follows. \hfill \Box

Just as in [BSW20a, §4.2], we can refine the direct sum decomposition of $C$ in a way that is compatible with the functors $E_u$ and $F_u$: for any simple $L$, let $k_u(L) = \phi_u(L) - \varepsilon_u(L)$ for each $i \in I$. In other words, $k_u(L) \in \mathbb{Z}$ is the multiplicity of $u = i$ as a zero or pole of the rational function $O_L(z) \in \mathbb{k}(z)$ for each $u \in U$.

**Definition 4.5** For a fixed $\mathbf{m} = (m_u) \in \mathbb{Z}^{|U|}$, we let $C_\mathbf{m}$ be the Serre subcategory of $C$ consisting of the objects $V$ such that every irreducible subquotient $L$ of $V$ satisfies $k_u(L) = m_u$.

Since the endomorphisms of any indecomposable finitely generated object in $C$ is a local algebra with residue field $\mathbb{k}$, any indecomposable object $M$ lies in one of these categories, determined by the image of $O_V(z)$ under the unique $\mathbb{k}$-algebra map
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End(\(V\)) \(\rightarrow \mathbb{k}\). Using the general theory of blocks in our two sorts of Abelian category, it follows that

\[
C = \begin{cases} 
\bigoplus_{\lambda \in \mathcal{X}} C_{\lambda} & \text{if } C \text{ is locally finite Abelian}, \\
\prod_{\lambda \in \mathcal{X}} C_{\lambda} & \text{if } C \text{ is Schurian}.
\end{cases}
\]

(4.10)

In terms of categorifications, we think of \(U\) as a Dynkin diagram with no edges, thus corresponding to the Lie algebra \(\tilde{g} = sl_U^2\). It’s worth noting the contrast here the similar analysis in \([BSW20a, \S 4.2]\); in that case we naturally ended up with a non-trivial Dynkin diagram. This is most obviously spotted by comparing (4.9) with \([BSW20a, (4.11)]\); in both cases, the entries of the Cartan matrix can be read off from the poles and zeros of the ratio of the bubble power series acting on the two simples. Thus, we can think of \(m\) as a weight over \(\tilde{g} = sl_U^2\), and in particular, we have the simple roots \(\alpha_u = (0, \ldots, 2, \ldots, 0)\). By Lemma 4.4, for \(m \in \mathbb{Z}^U\) and \(u \in U\), the restrictions of \(E_u\) and \(F_u\) to \(C_{\lambda}\) give functors

\[
E_u|_{C_{\lambda}} : C_{\lambda} \to C_{\lambda}^{k_{i,u} \alpha_u}, \\
F_u|_{C_{\lambda}} : C_{\lambda} \to C_{\lambda}^{-k_{i,u} \alpha_u},
\]

**Proposition 4.6** The functors \(E_u\) give a categorical action of \(sl_U^2\).

We give the proof of this result on page 39.

4.3. **Higher rank.** Given that most of the difficult relations in \(U_0\) only concern strands with a single label, it is simple to extend this result to higher rank. So, as in Section 4.1 we consider a Schurian action of \(U_0\) on a category \(C = \bigoplus_{\lambda \in \mathcal{X}} C_{\lambda}\).

We apply the constructions of the previous two sections for each \(i \in I\). In particular, we consider the generalized eigenspaces as in Definition 4.1

**Definition 4.7** Let \(E_{i,u}\) be the \(u\)-generalized eigenspace of \(y\) acting on \(E_i\), and similarly, \(F_{i,u}\) the analogous eigenspace for \(F_i\).

Let \(U_i = \{u \in \mathbb{k} \mid E_{i,u} \neq 0\} \subset \mathbb{k}\); note that since the corresponding functors are adjoint, \(E_{i,u}\) or \(F_{i,u}\) can be used symmetrically in this definition.

Consider the locally finite graph \(\bar{I}\) with vertices given by pairs \((i, u)\) with \(u \in U_i\) constructed from the polynomials \(P_{ij}\) as in Definition 2.2. For each \(i\) and simple \(L \in C\), we have the statistic \(k_{i,u}(L)\) defined as the vanishing order of \(\Theta_{i,u}(z)\) at \(z = u\).

We assign to each simple object \(L \in C\) the unique weight \(\mu_L\) in the weight lattice \(\mathcal{X}\) for \(\bar{I}\) such that \(\alpha_{i,u}(\mu_L) = k_{i,u}(L)\).

**Definition 4.8** Let \(C(\mu)\) be the Serre subcategory of objects whose simple subquotients \(L\) all satisfy \(\mu_L = \mu\).

Since every indecomposable module lies in one of these subcategories, we can define a block decomposition of our category \(C = \bigoplus_{\mu} C(\mu)\).
Lemma 4.9  The functors $E_{i,u}, F_{i,u}$ send objects in $C_{(\mu)}$ to $C_{(\mu \pm \alpha_i, u)}$.

Proof. This follows immediately from the bubble slides. It follows immediately from (2.24) that $k_{i,u}(E_{i,L}) = 2\delta_{i,u'}$, and $k_{i,u'}(F_{i,L}) = -2\delta_{u,u'}$ as expected. On the other hand, (2.25) shows that $k_{j,u'}(E_{i,L})$ decreases by exactly the vanishing order of $Q_{ji}(x, y)$ at $x = u'$ and $y = u$, which is the same as the number of edges joining $(i, u)$ and $(j, u')$ in $\tilde{I}$.

Thus, we have a direct sum decomposition

$$C = \bigoplus_{\mu} C_{(\mu)}$$

indexed by weights of $\tilde{g}$. The eigenspace functors $E_{i,u}, F_{i,u}$, defined as in Definition 4.7, act as expected on weights.

Theorem 4.10  The functors $E_{i,u}$ and $F_{i,u}$ and the weight space categories $C_{(\mu)}$ define a categorical action of $\tilde{g}$.

Proof. We will use Theorem 2.5 to show that we have a categorical action. Checking the conditions given in that Theorem:

(KM1) The adjunction of $E_{i,u}$ and $F_{i,u}$ follows immediately from the adjunction of $E_i$ and $F_i$ and the equation (2.7).

(KM2) As part of the structure of a categorical action, $R_n$ acts on the $n$th power $E^n_i$. Since the action of any dot on $E_i$ satisfies a polynomial relation with roots in $U_i$, this extends to an action of $\tilde{R}_n$. By transport of structure using the isomorphism $\nu$ of Proposition 6.4, we have an induced action of $R_n$ such that $y$ is nilpotent.

(KM3) This is simply Proposition 4.6.

This completes the proof. □

5. Deformed tensor product algebras

5.1. The definition. In [Web15, §5], we introduced natural categorifications $X^A$ for tensor products of highest and lowest weight representations. These categorifications have natural deformations, which we wish to study in the context of the previous section. As mentioned in the introduction, we require certain non-degeneracy results for these algebras which are analogous to Theorem 3.6, in order to show that $X^A$ have the expected Grothendieck group and Hom spaces. In an earlier version of this article, we required the results of this section to prove Theorem 3.6. We still believe this is a better way to think about the proof, although the version in Section 3 requires less technical overhead.

In [Web15, §4], we introduced the notion of tricolore diagrams, which we will quickly recall here (see [Web15, Def. 4.2] for the definition).
Definition 5.1 A tricolore diagram is a collection of finitely many oriented curves in $\mathbb{R} \times [0, 1]$ that satisfy the usual genericity relations. Each curve is either

- colored red and labeled with a dominant weight of $g$, or
- colored blue and labeled with an anti-dominant weight of $g$, or
- colored black and labeled with $i \in \Gamma$ and decorated with finitely many dots.

together with a labeling of these regions by weights consistent with the rules

\[
\begin{align*}
\mu & \downarrow \mu + \lambda & \mu & \uparrow \mu - \lambda & \mu & \downarrow \mu - \alpha_i \\
\lambda & \downarrow -\lambda & \lambda
\end{align*}
\]

Since this labeling is fixed as soon as one region is labeled, we will typically not draw in the weights in all regions in the interest of simplifying pictures.

The red and blue strands must map diffeomorphically to $[0, 1]$ under forgetting $x$-value and are forbidden to intersect with any other red or blue strand. The black strands are allowed to close into circles, self-intersect, intersect red and blue strands, etc.

As usual, we will want to record the horizontal slices at $y = 0$ and $y = 1$, the bottom and top of the diagram. This will be encoded as a tricolore quadruple, consisting of

- A sequence $i \in (\pm \Gamma)^n$ of simple roots and their negatives on black strands, read from the left;
- A sequence $\lambda \in (\mathcal{Y}^\pm)^\ell$ of dominant or anti-dominant weights on red and blue strands, read from the left;
- A weakly increasing function $\kappa : [1, \ell] \to [0, n]$ such that $\kappa(m)$ is the number of black strands left of $m$th red or blue strand (both counted from the left). By convention, we write $\kappa(i) = 0$, if the $i$th red or blue strand is left of all black strands.
- Weights $\mathcal{L}$ and $\mathcal{R}$ at the far left and right of the diagram. These are related by

\[
\mathcal{L} + \sum_{k=1}^{\ell} \lambda_k + \sum_{m=1}^{n} \alpha_{i_m} = \mathcal{R}.
\]

Tricolore diagrams are endowed with horizontal and vertical composition operations, just like KL and DS diagrams; similarly tricolore quadruples are endowed with a horizontal composition. These naturally make tricolore quadruples the 1-morphisms and tricolore diagrams the 2-morphisms of a 2-category $\tilde{T}$ whose objects are the integral weights $\mathcal{Y}$.

\[\text{The reader might wonder if these red and blue lines have anything to do with the red and blue copies of } \mathcal{U} \text{ used in Section 3.1. It would make a lot of sense if that were the case, but there is no connection. We are just very unoriginal about choosing colors.}\]
The categorifications $\mathcal{X}_\lambda$ are natural subquotients of this category, and our deformed categorifications arise from a straightforward deformation of the relations from [Web15, §4], which we present below:

**Definition 5.2** Let $\mathcal{T}$ be the quotient of $\tilde{T} \otimes \mathbb{k}[\beta_1, \ldots, \beta_\ell]$ by the relations (2.2a, 2.2c, 2.8–2.16) on black strands and (5.1a–5.1h) below relating red and blue strands to black. Note that the relations (5.1a–5.1h) are deformations of the relations of $\mathcal{T}$ in [Web15, p. 4.3]; thus we will recover the category $\mathcal{T}$ if we specialize $\beta_i = 0$.

\[
\begin{align*}
(5.1a) & & \lambda_i - \mu & = & \lambda_i - \mu \\
(5.1b) & & \lambda_k & = & (x - \beta_k) \lambda_k \\
(5.1c) & & (x - \beta_k) \lambda_k & = & (x - \beta_k) \lambda_k \\
(5.1d) & & & & \\
\end{align*}
\]
The reader should read the label $\lambda_k$ in this diagram to indicate that the strand shown is the $k$th of the red and blue strands from the left. In particular, $\beta_k$ is connected to this $k$th strand, and could be thought of as a new endomorphism of the tricolore triple with a single red or blue strand and $i = \emptyset$.

We let $\mathcal{X}_\mathcal{A}$ be the idempotent completion of the quotient of the category of tricolore quadruples $(\Lambda, i, \kappa, \ell)$ in $\mathcal{T}$ by the tricolore quadruples where $\kappa(\ell) < n$. That is, we consider 1-morphisms with label 0 at the right, where we fix the labels of the red and blue strands as well as their order to match $\Lambda$, but allow arbitrary black strands. We then take the quotient of this category of 1-morphisms by killing the diagrams with a black line at the far right.

Remark 5.3. Note that we have switched the role of left and right from [Web15, Def. 4.3], where we killed diagrams at the far left. These two approaches give equivalent categories. We can see this by extending the functor $\tilde{\sigma}$ on $\mathcal{U}$ defined in [KL10, §3.3.3] which reflects diagrams in the $y$-axis while negating the label on each region and
multiplying by $-1$ raised with the number of crossings with the same labels. This extension must make replace a red or blue strand with label $\lambda$ with one of the opposite color with label $-\lambda$. The only interesting relation check is that the image of (5.1g) is the negation of (5.1h) and vice versa.

The definition of $X^A_{\lambda}$ has precisely the same form as that of $X^A_{-\lambda}$, with the only difference being the relations (5.1a-5.1h) in place of the relations in [Web15, Def. 4.3] and the left/right reversal noted in Remark 5.3.

From the definition, it’s clear that there is a 2-functor $U \to \mathcal{T}$, since (2.2a-2.16) are simply the relations of $U$. Thus, horizontal composition on the left induces a $U$ action on $X^A_{\lambda}$.

Note that we’ll need to know the relations for bubble slides in $\mathcal{T}$.

**Lemma 5.4** In $\mathcal{T}$, we have the bubble slides:

\[
\begin{align*}
\mathcal{O}(z)_{\lambda_k} & = (z-\beta_k)^{\lambda_k} \mathcal{O}(z), & \mathcal{O}(z)_{-\lambda_k} & = (z-\beta_k)^{-\lambda_k} \mathcal{O}(z).
\end{align*}
\]

**Definition 5.5** Given a $k[\beta_1, \cdots, \beta_l]$-algebra $K$, we let $X^A_{-\lambda} \otimes_{k[\beta_1, \cdots, \beta_l]} K$ be the idempotent completion of the extension of scalars $X^A_{-\lambda} \otimes_{k[\beta_1, \cdots, \beta_l]} K$.

The main examples we’ll want to consider are $K = k(\beta_1, \cdots, \beta_l)$ and the algebraic closure $\bar{K}$.

5.2. **Spectral analysis.**

**Definition 5.6** Define sets $U_i \subset \bar{K}$ as follows: if for some $k$, we have $\alpha_i^\gamma(\lambda_k) \neq 0$, then $\beta_k \in U^{(0)}_i$, and all elements of $U^{(0)}_i$ are of this type. Now we inductively define $U^{(N)}_i$ to be the union of $U^{(N-1)}_i$ with the elements $u$ of $\bar{K}$ that satisfy $Q_{ij}(u, u') = 0$ for $u' \in U^{(N-1)}_j$, and $U_i = \bigcup_{N \in \mathbb{Z}} U^{(N)}_i$.

Let $U'_i$ be the union of $U_i$ with the set of elements in $\bar{K}$ that appear in the spectrum of the elements $y_k e_i$ with $i_k = i$ acting on objects in the category $X^A_{\lambda} \otimes K$; that, is the eigenvalues that appear when dots on strands with the label $i$ act.

It might seem strange that we add the elements of $U_i$ to $U'_i$ by definition, but this simplifies matters for us, since we have not yet established that $X^A_{\lambda} \otimes K$ is non-zero. Thus, we have not yet established that there are any elements of the spectrum of $y$ acting on objects in this category. We will ultimately see that $U_i = U'_i$, and these both coincide with the union of spectra discussed above.
We let $\tilde{I}, \tilde{I}$ be the graphs constructed from these sets as before and $\tilde{g}'$ and $\tilde{g}$ the corresponding Kac-Moody algebras. We let $\tilde{\lambda}$ be the weight for $\tilde{g}'$ such that $\alpha_{(i,u)}^{\tilde{g}'}(\tilde{\lambda}) = \sum_m \delta_{u,\beta_m} \alpha_i^{\tilde{g}'}(\lambda_m)$.

Since the elements $\beta_m$ are algebraically independent from each other, every element of $u \in U_i$ is algebraically dependent on exactly one $\beta_m$. We denote this index $m(u)$. In many cases that interest us, there is exactly one component of $\tilde{I}$ for each of these indices, but if $\alpha_i^{\tilde{g}'}(\lambda_m) \neq 0$ for several elements $i$, the pairs $(i, \beta_m)$ can lie in different components for different $i$.

We can define formal power series valued in the center of the category $\mathcal{X}^\Lambda$ which act on the object $(\Lambda, \iota, \kappa)$ as

$$y_i(z) := \prod_{i_k = \pm i} (z - y_k)^{\pm 1}, \quad Q_{ji}(z) = \prod_{i_k = \pm j} \left(t_{ij}Q_{ij}(z, y_k)\right)^{\pm 1},$$

where $y_k$ is the dot acting on the $k$th strand from the left.

These are supersymmetric polynomials (in the sense of [Ste85]) in pairs of alphabets for each $k \in I$ given by dots on upward-oriented $k$ strands and dots on downward-oriented $k$ strands. Any such polynomial commutes with all upward or downward-oriented diagrams by [KL09, p. 2.9], since each coefficient is symmetric in the corresponding variables. It commutes with a cup or cap joining the $k$th strand to the $k + 1$st since multiplying by $(z - y_k)^{\pm 1}$ at one end of the cup or cap cancels with $(z - y_{k+1})^{\mp 1}$ at the other (this is a restatement of the supersymmetric property). Note that the bubble slides (2.24, 2.25, 5.2, 5.3) and the triviality of bubbles at the far right show that at the far left of the diagram, we have:

$$\bigodot(z) = y_i(z)^2 \prod_{j \neq i} Q_{ji}(z) \prod_{m=1}^\ell (z - \beta_m)^{-\lambda_m^i}$$

(5.4)

$$\bigodot(z) = y_i(z)^2 \prod_{j \neq i} Q_{ji}(z)^{-1} \prod_{m=1}^\ell (z - \beta_m)^{\lambda_m^i}.$$

(5.5)

Let $\tilde{\mu} = \tilde{\lambda} - \sum a_{i,u} \alpha_{i,u}$ be a weight of $\tilde{I}'$. We can define subcategories $\mathcal{V}_{(\tilde{\mu})} \subset \mathcal{X}^\Lambda_{\tilde{K}}$ by the vanishing orders of $\bigodot(z)$ as in Definition 4.8. By Theorem 4.10 the functors $\mathcal{F}_{i,u}$ and their adjoints $\mathcal{E}_{i,u}$ induce a categorical action of $\tilde{g}'$ on $\mathcal{X}^\Lambda_{\tilde{K}}$, with weight decomposition given by $\mathcal{X}^\Lambda_{\tilde{K}} \cong \oplus_{\tilde{\mu}} \mathcal{V}_{(\tilde{\mu})}$.

Given a triple $\tilde{\lambda} = (\lambda_i)$ with $i = (i_1, \ldots, i_n)$ considered as an object in in $\mathcal{X}^\Lambda_{\tilde{K}}$ (recall that we will often exclude $\lambda$ from the notation when it is unlikely to be confused), we can thus decompose it according to the spectrum of the dots $y_i$. For a sequence $j_k = (i_k, u_k) \in \tilde{I}'$ for $k = 1, \ldots, n$, we let $(i, \kappa)_{u_k}$ be the simultaneous stable kernel of $y_k - u_k$ for all $1 \leq k \leq n$. 

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Lemma 5.8 We have that

\[(5.6) \quad (i, \kappa)_u \cong \mathcal{E}_{i_1, u_1} \cdots \mathcal{E}_{i_n, u_n}(\emptyset, 0) \]

if \(u_k \in U_k\) and \(k > \kappa(m(u_k))\) for each \(k\), and \((i, \kappa)_u = 0\) otherwise.

In particular, we have \(U_i = U'_i\) for all \(i\) and \(\tilde{g} = \tilde{g}'\), and the category \(X^A_k\) is generated by the tricolore triple \((\emptyset, 0)\) as a categorical module over \(\tilde{g}\).

Proof. First, we note that if \(u \in U_i\) and \(u' \notin U_j\), then \(Q_{ij}(y_1, y_2)\) acts on \(\mathcal{E}_{i,u_1} \mathcal{E}_{j,u_2}\) with its only eigenvalue \(Q_{ij}(u, u') \neq 0\) (by the definition of \(U_i\)). Thus the crossing \(\psi\) induces an isomorphism \(\mathcal{E}_{i,u} \mathcal{E}_{j,u'} \cong \mathcal{E}_{j,u'} \mathcal{E}_{i,u}\). Similarly, \(\mathcal{T}_{j,u'}\) commutes past all red and blue strands since \(y - \beta_k\) is invertible, with its only eigenvalue \(u' - \beta_k\); in fact, this still follows for the \(k\)th red/blue strand if \(u' \neq \beta_k\) (in particular, if \(k \neq m(u')\)).

We establish the result by induction on \(n\) and \(\ell\) (that is, on the total number of strands). If \(n = 0\), the result is tautological. Otherwise, the leftmost strand in the idempotent for the object \((i, \kappa)\) is one of black, blue, or red. If it is black, then \((i, \kappa) = \mathcal{E}_{i_1}(i^-, \kappa)\) for some \(i \in \pm I\), and \(i^- = (i_2, \ldots, i_n)\). Decomposing with respect to the eigenvalues of \(y\), we have \(\mathcal{E}_{i}(i^-, \kappa) \cong \oplus \mathcal{E}_{i,u}(i^-, \kappa)\) where \(u\) ranges over the roots of the minimal polynomial of \(y\) acting on \(\mathcal{E}_{i}(i^-, \kappa)\). By induction \((i^-, \kappa)\) is a sum of modules obtained from \((\emptyset, 0)\) by the functors \(\mathcal{E}_{j,u'}\) and \(\mathcal{T}_{j,u'}\) for \(u' \in U_j\). If \(u\) is not in \(U_i\), then all these functors commute with \(\mathcal{E}_{i,u}\) (as argued above), and \(\mathcal{E}_{i,u}(\emptyset, 0) = 0\), so \(\mathcal{E}_{i,u}(i^-, \kappa) = 0\), and \(\mathcal{T}_{i}(i^-, \kappa) \cong \oplus_{u \in U_i} \mathcal{E}_{i,u}(i^-, \kappa)\). By induction, this establishes the result.

On the other hand, if the leftmost strand is blue or red, we simply apply induction with the tricolore triple \((A^-, i, \kappa^-)\) with this strand removed. By induction, \((A^-, i, \kappa^-) \cong \oplus (A^-, i, \kappa^-)_u\) with \(k > \kappa(m(u_k))\) and \(m(u_k) < \ell\). Since adding in the \(\ell\)th blue or red strand does not change the eigenvalues of the dots, we also have \((A, i, \kappa) \cong \oplus (A, i, \kappa)_u\) with \(u\) ranging over the same set. This shows equation \((5.6)\), and that \(U'_i = U_i\). \(\square\)

Thus \(X^A_k\) is generated by a single object, which is highest weight for the components of \(\tilde{I}\) with \(\lambda_{m(u)}\) dominant and lowest weight for those with \(\lambda_{m(u)}\) anti-dominant. Alternatively, we can easily choose a Borel for which this representation is straightforwardly highest weight. To distinguish objects which are highest weight for this Borel, we call them **signed highest weight**. We can write each weight \(\lambda\) uniquely as a sum \(\bar{\lambda} = \bar{\lambda}_1 + \cdots + \bar{\lambda}_\ell\) where \(\bar{\lambda}_m\) is supported on components with \(m(u) = m\).

Let \(X^A\) be the category over the base field \(K\) defined in [Web15, §5] for the singleton \((\bar{\lambda})\) and the Dynkin diagram \(\bar{I}\), and \(X^A\) this category for the sequence \(\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_\ell)\). Both of these are defined using the signed highest weight Borel, so for example, \(X^A\) is defined using a single colored strand at right (whether it is red, blue or purple is a matter of taste) that satisfies the red version of the relations \((5.1a, 5.1h)\) for a black strand with label \((i, u)\) such that \(\lambda_{m(u)}\) is dominant, and the blue version if \(\lambda_{m(u)}\) is anti-dominant. These categories are equivalent via the obvious
functor \( \lambda^\lambda \rightarrow \lambda^\lambda \). Both categories can be defined over \( \mathbb{k} \), but we will be more often interested in their base extensions to \( \overline{\mathbb{k}} \).

The important fact we will need about the category \( \lambda^\lambda \) is that unlike the general case, we already know the non-degeneracy results we need for this category, based on [Web17a], since we’ve deformed to a signed highest-weight representation. Thus our strategy for proving non-degeneracy for \( \lambda^\lambda \) is to compare it with this category whose Grothendieck group and Hom spaces are known by [Web17a, Cor. 3.20 & 3.22]. By applying the automorphism \( \tilde{\omega} \) from [KL10, §3.3.2] which swaps the functors \( E_i \) and \( F_i \) to all components where \( \tilde{\lambda}_i \) is anti-dominant, we can reduce to the case where all \( \tilde{\lambda}_i \) are dominant. In this case, the endomorphisms of the sum of all objects in \( \lambda^\lambda \) is precisely the algebra \( DR^\lambda \) defined in [Web17a, Def. 3.1]. By the Morita equivalence of this ring to the cyclotomic quotient \( R^\lambda \) ( [Web17a, Def. 3.3]) shown in [Web17a, Cor. 3.20], we can compute the dimension of morphism spaces in \( \lambda^\lambda \). In fact, we will only need the basic observation that in this category

(5.7) \( \text{Hom}_{\lambda^\lambda}(((i, u), 0), ((i, u), 0)) \cong \mathbb{k}[y]/(y^{\lambda_{(0)}}) \),

that is, if we have a single strand, then the dots satisfy the cyclotomic relation and nothing else; this is obvious in the cyclotomic quotient \( R^\lambda \). Of course, this is readily extended to the case where there is only one strand with label in any single component; up to Morita equivalence, one just obtains a tensor product of these endomorphism algebras.

**Lemma 5.9** There is a strongly equivariant functor

(5.8) \( \Phi: \lambda^\lambda \otimes_\mathbb{k} \overline{\mathbb{k}} \cong \lambda^\lambda \otimes_\mathbb{k} \overline{\mathbb{k}} \rightarrow \lambda^\lambda \)

sending \((\emptyset, 0) \mapsto (\emptyset, 0)\). The functor \( \Phi \) is an equivalence.

We give the proof of this result on page 40.

5.3. **Non-degeneracy.** Given two tricolore triples \((\underline{A}, i, \kappa)\) and \((\underline{A}, i', \kappa')\), we define a set \( B \) of diagrams in \( \text{HOM}_{\overline{\mathbb{T}}}((\underline{A}, i, \kappa), (\underline{A}, i', \kappa')) \) which generalizes the set \( B_{i', i, \lambda} \) defined in [KL10, §3.2.3]. In fact, we can define \( B \) to be the set of diagrams where the black strands trace out an element of Khovanov and Lauda’s spanning set \( B_{i', i, \lambda} \) and red and blue strands are added as required by the functions \( \kappa \) and \( \kappa' \). That is, it is a set satisfying following conditions:

1. for each way of dividing the set \( i \cup -i' \) into pairs of matching elements (an \( (i, i') \)-pairing, in the terminology of [KL10]), there is a unique diagram in \( B \) which connects the terminals for the two elements of each pair, with a minimal number of crossings and no bubbles or dots.

2. for each diagram of the above, and each arc joining two terminals, we fix a position on that arc (avoiding the crossings with any others). The full list of diagrams in \( B \) are indexed by diagrams as above, a choice of non-negative
integer for each arc joining two terminals, and a monomial in the unnested bubbles (a basis vector of the ring $\Pi_{\lambda}$ in the notation of [KL10, §3.2.1]). We construct the corresponding diagram by adding the corresponding number of dots on each arc and putting the bubbles to the left of the diagram.

**Theorem 5.10** For two tricolore triples $(\lambda, i, \kappa)$ and $(\lambda, i', \kappa')$, the set $B$ is a basis over $\mathbb{C}[z]$ for the morphism space $\text{HOM}_T((\lambda, i, \kappa), (\lambda, i', \kappa'))$.

We give the proof of this result on page 41.

6. Proofs

**Proof of Theorem 3.1.** While the morphisms given in the statement of Theorem 3.1 fix the 2-functor, to prove that we have a 2-functor, it is useful to also describe the image of the functor on some additional morphisms:

(6.1) $\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \mapsto \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$ , $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \mapsto \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$ , $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \mapsto \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$ , $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \mapsto \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$

where we use the notation of internal bubbles:

(6.2) $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \mapsto \sum_{b \in \mathbb{Z}} b \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$ , $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \mapsto \sum_{b \in \mathbb{Z}} b \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$.

Note that since red and blue 1-morphisms commute, it does not matter which side the bubbles above are drawn on.

While the Heisenberg and $\mathfrak{sl}_2$ Kac-Moody categories are quite different, because the relations Eqs. (2.22), (2.23) and (2.2a) are almost identical to corresponding relations in the Heisenberg category [BSW20a, (3.14–3.17)], many of the same relations hold. One unfortunate proviso here: the signs are off in several of these relations, so our proofs will be nearly identical but with sign differences. That is one can check that:

(6.3) $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} = \sum_{a \geq 0} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$ , $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} = \sum_{a \geq 0} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$

(6.4) $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} = \sum_{a \geq 0} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$

(6.5) $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} = \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}\right)^{-1}$ $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$ , $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} = \sum_{b \in \mathbb{Z}} b \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$ for all $a \geq 0$.

(6.6) $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} = \sum_{a, b \geq 0} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$

(6.7) $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} = \sum_{a \geq 0} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}$
Unfurling Khovanov-Lauda-Rouquier algebras

\[ (6.8) \quad \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig1} \end{array} = - \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig2} \end{array} + \sum_{a, b \geq 0 \atop c \in \mathbb{Z}} c \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig3} \end{array} \]

\[ (6.9) \quad \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig4} \end{array} = \sum_{a, b \geq 0 \atop c \in \mathbb{Z}} \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig5} \end{array} - a - b - c - 3 \]

\[ (6.10) \quad \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig6} \end{array} = \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig7} \end{array} + \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig8} \end{array} \]

Since the proofs are identical to those of [BSW20a, Lem. 5.6–12] except for minor sign differences, we omit the proofs.

First, we check the relations of the nilHecke algebra \((2.2a-2.2c)\) in the case where all labels are the same. This is not difficult to do "by hand," but it might be clearer to explain more conceptually why this works. Fix an integer \(n\) and consider the direct sum of one copy of \(k[y_1, \ldots, y_n]\) for each sequence \(c \in \{r, b\}^n\), and localize this ring at the multiplicative set \(\{y_i - y_j\}\) for \(i, j\) ranging over all pairs where \(c_i \neq c_j\). Let \(S_n\) act on this direct sum by permuting variables and entries of \(c\) simultaneously. We have an action of the nilHecke algebra on this direct sum which sends \(\psi_k\) to \(\frac{1}{y_i - x_{i+1}}(1 - s_k)\); if \(c_k = c_{k+1}\), this is just the usual divided difference operator on the corresponding polynomial ring, whereas if \(c_k \neq c_{k+1}\), this operator corresponds to the last 4 terms in \((3.4)\), and in particular, relies on using the fact that we have inverted \(x_k - x_{k+1}\). Thus, checking the relations of the nilHecke algebra \((2.2a, 2.2c)\) simply consists of translating the proof that these divided difference operators satisfy the nilHecke relations into diagrams.

For instance, equation \((2.2a)\) reduces to the equations below and their mirror images:

\[
\begin{array}{c} \includegraphics[width=0.5\textwidth]{fig9} \end{array} = \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig10} \end{array} + \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig11} \end{array}, \quad \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig12} \end{array} = \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig13} \end{array}, \quad \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig14} \end{array} = \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig15} \end{array} + \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig16} \end{array}.
\]

Only the first relation of \((2.2b)\) will appear, and this quadratic relation follows by:

\[
\begin{array}{c} \includegraphics[width=0.5\textwidth]{fig17} \end{array} - \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig18} \end{array} + \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig19} \end{array} - \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig20} \end{array} + \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig21} \end{array} - \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig22} \end{array} + \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig23} \end{array} - \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig24} \end{array} + \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig25} \end{array} - \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig26} \end{array} + \begin{array}{c} \includegraphics[width=0.5\textwidth]{fig27} \end{array} = 0 .
\]

The braid type relation \((2.2c)\) is a similar calculation; we will spare the reader the diagrammatic proof of it, but note that it corresponds to the proof that \(\psi_k \psi_{k+1} \psi_k = \psi_{k+1} \psi_k \psi_{k+1}\) in the polynomial representation discussed at the start of the proof.

Now, we turn to the relations \((2.8-2.16)\). The relations \((2.8)\) are immediate and \((2.9)\) follow from \((6.5)\). The proof of \((2.11, 2.12)\) is exactly as in [BSW23, Th. 5.4]: the LHS
is sent to
\[\begin{align*}
\sum_{a,b, c\in \mathbb{Z}} & \delta_{a,b, c} = \delta_{k,0} + \delta_{k,0}.
\end{align*}\]
where the equality follows from (6.7).

Finally, we turn to the proof of (2.15–2.16). Under our functor, the sideways crossings will be sent to:
\[\begin{align*}
\sum_{a,b, c\in \mathbb{Z}} & \delta_{a,b, c} = \delta_{k,0} + \delta_{k,0}.
\end{align*}\]

Thus, the image of the LHS of (2.15) is:
\[\begin{align*}
(6.11) \quad & A = B + C \\
A = & \quad \frac{\sum_{a,b, c\in \mathbb{Z}} \delta_{a,b, c}}{}
\end{align*}\]
\[\begin{align*}
B = & \quad \frac{\sum_{a,b, c\in \mathbb{Z}} \delta_{a,b, c}}{}
\end{align*}\]
\[\begin{align*}
C = & \quad \frac{\sum_{a,b, c\in \mathbb{Z}} \delta_{a,b, c}}{}
\end{align*}\]

We can simplify $A$ by applying (6.8) to the portion of the diagram above the lowest internal bubble, simplify $B$ using (6.10 to the portion above the bottom crossing, and $C$ using (6.9) on the central part of the diagram. The results are:
\[\begin{align*}
(6.13) \quad & A = \sum_{a,b, c\in \mathbb{Z}} \delta_{a,b, c} \\
B = & \quad \sum_{a,b, c\in \mathbb{Z}} \delta_{a,b, c} \\
C = & \quad \sum_{a,b, c\in \mathbb{Z}} \delta_{a,b, c}
\end{align*}\]

This immediately completes the proof of (2.15), and the proof of (2.16) is identical. \qed

Before the proof of Lemma 3.5, let us state some lemmata:

**Lemma 6.1** The formulas (3.7–3.13) define an injective homomorphism $\gamma: R \to \hat{R}$.

**Proof.** The algebras $R$ and $\hat{R}$ are equipped with the polynomial representations $P$ and $P$ as in (2.3a–2.3b). Equation (3.7) defines a homomorphism $\gamma_p: P \to \hat{P}$ on the underlying polynomial representations. For each generator of $R$, its action on the polynomial representation is intertwined with its image in $\hat{R}$ by this map between polynomials, that is, we have $\gamma(r)\gamma_p(p) = \gamma_p(rp)$ for all generators $r$ and all $p \in P$. Since the action of $R$ on $P$, the action of $R$ and $P$, and the action of the completion of $\hat{R}$ on the completion $\hat{P}$ are faithful (the last of these by [Web20, Lem. 2.5]), this means that $\gamma$ must extend to an injective homomorphism. \qed

In fact, one can relatively easily show that this map becomes an isomorphism if we replace $R$ by an appropriate completion.

For any commutative subalgebra $S \subset R$ and any finite-dimensional quotient $R/J$, we can decompose $R/J$ over the maximal spectrum of $S$. That is, we can diagonalize
the semi-simple part of each element of $S$. For each maximal ideal $m$, there is a unique idempotent $e$ in $R/J$ whose image is the vectors killed by $m^N$ for some $N$.

For any finite subset $T \subset \text{MaxSpec}(S)$, we can define the completion of $R$ at $T$ as the inverse limit of the directed system of all quotients which are supported in the spectral decomposition above on $T$. This is the same as saying that as coherent sheaves on $\text{MaxSpec}(S)$, they are supported on $T$.

In $R_\gamma$, we have a commutative subring $S$ generated by the idempotents $e(i)$ for $i \in I^n$ and the dots $y_1, \ldots, y_n$; this is a direct sum of polynomial rings, so its MaxSpec is a disjoint union of copies of $\mathbb{A}^n$. Let $A^n_i$ be the component of $\text{MaxSpec}(S)$ on which $e(i)$ is not zero.

**Definition 6.2** Let $\widehat{R}_n$ be the completion of $R_n$ with respect to the subset $T$ defined by the union of the subsets

$$T = \bigcup_{i \in I^n} \{ a \in A^n_i \mid a_k \in U_i \}.$$  

That is, this is the completion at the directed system of all quotients where the spectrum of $y_i e(i)$ lies in $U_i$.

**Definition 6.3** For a given $j = (j_1 = (i_1, u_1), \ldots, j_n = (i_n, u_n))$, let $e_j$ be the idempotent corresponding to the point $u$ in $A^n_i$, that is, the maximal ideal generated by $e_j$ for $i' \neq i$ and by $y_k - u_k$. In particular, $e_i e_j = e_j e_i = e_j$ and $(y_j - u_j) e_j$ is topologically nilpotent.

**Proposition 6.4** The map $\gamma$ induces an isomorphism $\widehat{R} \cong \widehat{R}$ sending $e_i$ to $e_j$.

**Proof.** First, note that $\gamma_p$ induces an isomorphism of the corresponding completion of polynomial rings. Since the action of $\widehat{R}$ on $\widehat{R}$ is faithful, we have an injective map $\widehat{R} \rightarrow \widehat{R}$. Note that by uniqueness, $\gamma(e_i) = e_j$. Furthermore, we can easily use these to solve for the inverse map to $\gamma$. On dots, we use the inverse power series

$$\gamma^{-1}_{u_k, d}(x) = \sqrt[1/d]{x/a} - 1 = \frac{1}{d} \left( \frac{x}{a} - 1 \right) + \left( \frac{1}{d^2} \right) \left( \frac{x}{a} - 1 \right)^2 + \cdots$$

using Taylor expansion at $x/a = 1$. Comparing with (3.8) and (3.13), we find that the inverse is given by

$$\gamma^{-1}(y_k e_i) = \gamma^{-1}_{u_k, d_{i_k}}(y_k) e_j$$

and

$$\gamma^{-1}(e_i \psi_k) = \left\{ \begin{array}{ll}
\frac{p_{i, j_k}^{y_k}}{p_{i, j_k+1}^{y_k}} \left( \gamma^{-1}_{u_k, d_{i_k}}(y_k), \gamma^{-1}_{u_k, d_{i_k+1}}(y_k) \right)^{-1} e_j \psi_k & i_k \neq i_{k+1} \\
((y_{k+1} - y_k) \psi_k + 1) e_j & i_k = i_{k+1}, u_k \neq u_{k+1} \\
\gamma^{-1}_{u_k, d_{i_k}}(y_k) \psi_k e_j & i_k = i_{k+1}, u_k = u_{k+1}
\end{array} \right.$$  

This completes the proof. \qed
Proof of Lemma 3.5 The proof that the finiteness conditions of $\mathring{U}_\emptyset$ hold is the same as in Lemma 3.3.

As usual, we use Theorem 2.5. The conditions (KM1) and (KM3) involve only a single color on strands, and so follow by Lemma 3.5. Thus, it suffices to check (KM2) which follows from Lemma 6.1.

Proof of Theorem 3.6 The strategy is simple at heart—if we produce a large number of representations $\mathcal{U}_g$ representations which are “big enough,” we can use the categorical action of Lemma 3.5 to show that we have the desired linear independence.

Reduction to algebraically closed fields: First, let us show that we can reduce to the case where the coefficient field is an algebraically closed field. Let $k'$ be the ring given by polynomials over $\mathbb{Z}$ with $t_{ij}^{\pm}$ and the other coefficients of $Q_{ij}$ adjoined as formal variables. If we prove that $B_{i',j',\lambda}$ is a basis of the Hom space $\text{HOM}_{u}(i',i')$ as a free $k'$-module, it will follow for every other choice of $k$ by base change.

Furthermore, note that $\text{HOM}_{U}(i,i')$ is filtered by the finitely generated submodules of elements of degree $\leq k$ for each $k \in \mathbb{Z}$, and it is equivalent to showing that the elements of $B_{i',j',\lambda}$ that lie in this space are a basis for it as a free module. As usual, it is enough to prove this after localization, so we can assume that $k'$ is a regular local domain. By Nakayama’s lemma, a finite set of elements is a free basis for a module over a local domain if and only if they are a basis after base change to the residue field and the fraction field. Thus, we can assume that $k'$ is a field, but the result will be unchanged by passing to a field extension, and we can assume without loss of generality that $k' = \bar{k}'$ is algebraically closed.

Construction of representations: By Lemma 2.7, we can assume that $g$ is universal derived. For a given $m$, consider the field of rational functions on an alphabet with $2m$ elements $z_{i,\pm 1}, \ldots, z_{i,\pm m}$ for each $i \in I$. Let $U_i$ be the complete choice of spectra in the algebraic closure of the field $k(z_{i,\pm k})_{\text{el}, k \in [1,m]}$ produced by starting with $z_{i,\pm k} \in U_i$, and then completing as necessary. Note that all elements connected to $z_{i,k}$ in $I$ are in the algebraic closure of $k(z_{i,k})$, and thus are algebraically independent from all other $z_{i,t}$. In particular, $\tilde{I}$ divides into two disjoint subgraphs $\tilde{I}_\pm$ given by the components that contain $z_{i,\pm k}$ for $i \in I, k \in [1,m]$.

Consider a weight $\lambda$, which we can think of as just the function $I \to \mathbb{Z}$ sending $i$ to $\alpha_i^\vee(\lambda)$. Consider weights $\mu_\pm$ that satisfy, for all $i \in I$:

$$\alpha_{i,z_i,\lambda}(\mu_+) = m + \max(0, \alpha_i^\vee(\lambda))$$

$$\alpha_{i,z_i,-1}(\mu_-) = -m + \min(0, \alpha_i^\vee(\lambda))$$

$$\alpha_{i,z_i,\pm k}(\mu_\pm) = \pm m$$

for $k \in [2, m]$.

The 2-category $\mathcal{U}_\emptyset$ has a generalized cyclotomic quotient representation $X^{\mu_+ + \mu_-}$, generated by an object $V$ of weight $\mu_+ + \mu_-$ which satisfies that $E_{(i,u)}V = 0$ if $(i,u) \in I_+$ and $F_{(i,u)}V$ if $(i,u) \in I_-$. Let $\tilde{\omega}_- : \mathcal{U}_0 \to \mathcal{U}_\emptyset$ be the automorphism which applies Khovanov and Lauda’s automorphism $\hat{\omega}$ (which reverses the orientation of strands; see [KL10, §3.3.3]) to...
strands with labels in $\tilde{I}_-$, leaving those in $\tilde{I}_+$ unchanged. The category $X^{\mu_+,-\mu_-}$ is the same as the projective modules over the deformed cyclotomic quotient $\tilde{R}^{\mu_+,-\mu_-}$ (defined in [Web17a, Def. 3.24]), with the categorical action twisted by $\tilde{\omega}_-$, so that the highest and lowest weight objects for the subalgebra corresponding to $\tilde{I}_-$ are swapped.

This means that by [KK12, Th. 6.2] or [Web17a, Cor. 3.22], we know the dimensions of Hom spaces in this category in terms of the Shapovalov form on the corresponding representation. One particularly important consequence is that the endomorphisms $Z = \text{End}(V)$ of $V$ are freely generated by the fake bubbles in the categorification; for each $i \in I, k \in [1, m]$, we obtain two polynomial rings in $m$ variables, which are the coefficients of the minimal polynomials of the induced morphisms

$$
\begin{align*}
\downarrow_{i, z, k} & : \mathcal{T}_{(i, z, k)}V \to \mathcal{T}_{(i, z, k)}V \\
\uparrow_{i, z, -k} & : \mathcal{E}_{(i, z, k)}V \to \mathcal{E}_{(i, z, k)}V,
\end{align*}
$$

and $Z$ is the tensor product of all these rings. The ring $\tilde{R}^{\mu_+,-\mu_-}$ is free as a module over $Z$.

By Lemma 3.5 we have a $U_{\tilde{g}}$-action on $X^{\mu_+,-\mu_-}$. This category is generated (as a Karoubian additive category) by the objects $\mathcal{E}_i V$. By construction, these have a decomposition $\mathcal{E}_i V \equiv \bigoplus_{u \in \prod U_{i|u}} \mathcal{E}_{(i, u)} V$, where to avoid confusion, we let $\mathcal{E}_*$ denote the categorical action of $U_{\tilde{g}}$. The projections in this direct sum decomposition can be constructed purely in terms of the action of $U_{\tilde{g}}$—they are the projections $\epsilon_{i, u}$ where we fix the spectrum of the dots on each strand.

**Triviality of a relation:** Consider a relation in this spanning set $B_{i, i', \lambda}$ for 1-morphisms $i, i': \mu \to \mu'$. Amongst the $(i, i')$ pairings appearing, fix a pairing $\pi$ which contributes at least one diagram with non-trivial coefficient and has a maximal number of crossings with respect to this property.

Let $m$ be 1 plus the maximum of:

1. The degrees of all bubbles appearing in diagrams for $\pi$.
2. The number of arcs appearing in one of these diagrams, that is, $(|i| + |i'|)/2$.
3. The number of dots that appear on any of these arcs.

By assumption, we can choose an injection from the set of arcs to $[2, m]$, and choose $j$ and $j'$ by letting $j_k = (i_k, z_{i_k, q})$ where $q$ is the index corresponding to the arc that touches this terminal and similarly for $j'$. This means that each second index appears at exactly two terminals in $j$ and $j'$, which are at opposite ends of an arc—assuming that it appears at all.

Now, consider the action of our relation on the generating object $V$ in the category $X^{\mu_+,-\mu_-}$. This defines a map $\mathcal{E}_i V \to \mathcal{E}_{i'} V$. We can compose this with the inclusion of and projection to a summand to obtain a morphism $\mathcal{E}_j V \to \mathcal{E}_{j'} V$. Note that $\pi$ is the only matching where the two ends of each arc have the correct labels; there are no other elements of $\text{HOM}_{X^{\mu_+,-\mu_-}}(\mathcal{E}_j V, \mathcal{E}_{j'} V)$.
In fact, this Hom space is easy to describe—it has a basis over \( \mathbb{Z} \) given by the diagram \( D \) with a number of dots between 0 and \( m - 1 \). This is essentially Khovanov and Lauda’s basis with a constraint on the number of dots, which comes from the cyclotomic relation. Since we only use one label from each component of \( \bar{I} \) once, this calculation reduces to the cyclotomic quotient for \( sl_2 \) with 1 strand. In this case, the monomials \( 1, y, \ldots, y^{m-1} \) are indeed a basis.

By the assumption that we started with a relation, the resulting morphism must be 0. Note also that the functor defined by Eqs. (3.7), (3.8) and (3.13) sends the diagram for any pairing to a sum of diagrams for that pairing and for pairings with strictly fewer crossings. Thus, any diagram which does not come from \( \pi \) which appears in the relation has image under this functor consisting only of diagrams that do not come from \( \pi \); as noted above, this means that the resulting map \( \tilde{E}_j V \to \tilde{E}_{j'} V \) is necessarily 0.

Thus, when we consider the image of our relation, it is the same as the image of only the diagrams that match according to \( \pi \). Let \( D \) be such a diagram that has a minimal number of dots that appears in our relation. By construction, under Eqs. (3.7), (3.8) and (3.13), every diagram is sent a multiple of that diagram by a power series in dots with non-zero constant term. Thus \( D \) is sent to \( cD + \cdots \) where all other terms have more dots. The diagram \( D \) will be one of the elements of our basis of diagrams with \( < m \) on each arc by assumption. Thus, when expanded in this basis, the diagram \( D \) cannot appear in the image of any other diagram, by the minimality of the number of dots. Thus, \( cD \) cannot be cancelled, and this contradicts the assumption that our relation was non-zero. This shows the desired non-degeneracy. \( \square \)

**Proof of Proposition 4.6.** We need only verify the conditions of Theorem 2.5.

The axiom (KM1) is already established by the left and right adjunctions that we have constructed.

Since there are no edges in the graph \( U \), the polynomials \( Q_{uv} \) are simply 1 for all \( u \neq v \). One can easily check that \( R_m \) is Morita equivalent to the sum of tensor products of nilHecke algebras

\[
\bigoplus_{\sum_m m_u = m} \bigotimes_{i \in I} NH_{m_u}
\]

with each summand corresponding to the subalgebra where \( m_u \) strands have the label \( u \). Taking \( \gamma'_{u}(x) = x - u \) and applying Proposition 6.4, we find that the axiom (KM2) holds as well. That is, there is a homomorphism \( R_m \to \text{End}(F^m) \) which sends

\[
\begin{cases}
  v & \leftrightarrow \begin{cases}
    v & \text{if } u = v \\
    x_2 - x_1 & \text{if } u \neq v
  \end{cases} \\
  u & \leftrightarrow \begin{cases}
    u & \text{if } u = v \\
    -u & \text{if } u \neq v
  \end{cases}
\end{cases}
\]

(6.16)
Thus, we need only establish (KM3). This is identical to the proof of [BSW20a, Lem. 4.9]. □

**Proof of Lemma 5.9.**  
**Existence of the functor:** Both the source and the target are generated by a signed highest weight object, so [Web17a, Prop. 3.25] shows that a strongly equivariant functor is induced whenever there is a map

\[
\text{End}_{\mathcal{X}' \otimes_k \bar{\mathbb{K}}}(\emptyset, 0) \cong \bar{\mathbb{K}} \to \text{End}_{\mathcal{X}_{\bar{\mathbb{K}}}}(\emptyset, 0)
\]

compatible with the action of fake bubbles. Since \(y: \mathcal{F}_{i,u} \to \mathcal{F}_{i,u}\) is nilpotent, the fake bubbles act trivially in both cases, and the \(\bar{\mathbb{K}}\)-algebra structure on \(\text{End}_{\mathcal{X}_{\bar{\mathbb{K}}}}(\emptyset, 0)\) induces the functor. The functor is an equivalence if and only if the map of (6.17) is an isomorphism. This can only fail if \(\chi_{\bar{\mathbb{K}}} = 0\).

**Inverse functor:** Thus, we need to rule out the possibility that \(\chi_{\bar{\mathbb{K}}} = 0\). We will do this by defining a functor

\[
\Xi: \chi_{\bar{\mathbb{K}}} \to \chi_{\bar{\mathbb{K}}} \otimes_k \bar{\mathbb{K}}
\]

which sends \((i, \kappa)\) to the sum \(\bigoplus(j, \kappa)\) where \(j\) ranges over sequences with \(j_k = (i_k, u_k)\). If \(j\) satisfies \(k > \kappa(m(u_k))\), then we have \((j, \kappa) \cong (j, 0)\), and otherwise the corresponding object is 0, so Lemma 5.8 shows that this is quasi-inverse to \(\Phi\) on the level of 1-morphisms.

First, let us explain how this will complete the proof. By [Web17a, Prop. 3.25], this functor is an equivalence if \(\text{End}(\emptyset, 0) \cong \bar{\mathbb{K}}\). Thus, the only issue is that the object \((\emptyset, 0)\) might simply be 0 (in which case the entire category \(\chi_{\bar{\mathbb{K}}} = 0\)). The functor \(\Xi\) sends \((\emptyset, 0)\) to \((\emptyset, 0)\). The existence of this functor establishes that \(\chi_{\bar{\mathbb{K}}} = 0\), so \(\Phi\) must be an equivalence. In fact, we can easily see that \(\Xi\) is strongly equivariant for \(\tilde{g}\), so it must be quasi-inverse to \(\Phi\) when composed with the equivalence \(\chi_{\bar{\mathbb{K}}} \otimes_k \bar{\mathbb{K}} \cong \chi_{\bar{\mathbb{K}}} \otimes_k \bar{\mathbb{K}}\).

\(\Xi\) on morphisms: Thus, we turn to considering the construction of \(\Xi\). We have described its behavior on objects, so we must only define how \(\Xi\) acts on morphisms. First, note that by Theorem 4.10, we have a categorical action of \(g\) on \(\chi_{\bar{\mathbb{K}}} \otimes_k \bar{\mathbb{K}}\). On purely black diagrams, \(\Xi\) simply employs this action; that is, on upward oriented diagrams, it follows the formulas (6.14–6.15). Since left (or right) adjunctions are unique up to isomorphism, we can send the leftward cup and cap in \(\mathcal{U}_g\) to any adjunction we choose. For simplicity, we simply match leftward oriented cups and caps as below:

\[
i \longmapsto \sum_{i \in \mathcal{U}_l} (i, u) \quad \text{(i, u)}
\]

For rightward oriented cups, the formula is quite complicated, but is fixed by the choices we have made thus far, and the existence of a consistent choice follows from the existence of the \(g\)-action. Thus, we need only define this action on diagrams with red/blue strands, with formulas given below (6.19–6.20).
Checking relations: Now, we need to prove that this assignment defines a functor, that is, it is compatible with all the relations on morphisms. The equations that can be stated purely using upward or downward diagrams, that is, (2.2a–2.2c), (5.1b–5.1c), (5.1g–5.1e) all follow by straightforward calculations as in the proof of Proposition 6.4.

Thus, we only need to argue for the relations involving right cups and caps. The way we have defined the right cup/cap means that the relations ((2.2a–2.16) are automatic. The remaining relations (5.1b, 5.1a) are actually redundant when the right cap and cup are defined in terms of the left cup and cap. From this perspective, the relation (5.1b) is the definition of the upward red/black or downward blue/black crossings. For the relation (5.1a), assume that we are considering the red version; the blue version follows similarly. We must consider two different cases. Let \( \mu \) be the label of the region at the left of the picture.

- If \( \mu^i \geq 0 \), then we have a loop at the left with \( \mu^i \) dots. Pulling this through and applying (5.1b,5.1c), then undoing this bubble, we obtain the desired relation.
- If \( \mu^i \leq 0 \), then we start with the diagram with a leftward cup at the bottom and rightward cup at top, and compare the result of applying (2.15) to these two strands to the left and right of the red strand. Using the relations (5.1c,5.1b), we can move the bigon to the right side of the red line, and using (5.1b,5.1c) to remove bigons between red and downward strands. The left- and right-hand sides now have the same pattern of black strands, but in one the upward strands make a bigon with the red strand, and in the other, they do not. This can only hold if (5.1a) is true.

\[ \lambda_m \rightarrow \sum_{i \in U_i} (i, u) \lambda_m \]

\[ \lambda_m \rightarrow \sum_{i \in U_i \setminus \{i, \beta \}} (y - \beta_m + u) \lambda_m \]

\[ \lambda_m \rightarrow \sum_{i \in U_i \setminus \{i, \beta \}} (y - \beta_m + u) \lambda_m \]

Proof of Theorem 5.10: \( B \) spans: The proof that these are a spanning set is essentially equivalent to that of \([KL10, \text{Prop. 3.11}]\). First, note that any two minimal diagrams for the same matching are equivalent modulo those with fewer crossings (using the relations (2.2c, 5.1g, 5.1h)). Similarly, moving the dots to the chosen positions only introduces diagrams with fewer crossings.

Thus, we only need to show that minimal diagrams span. Of course, if a diagram is non-minimal, then it can be rewritten in terms of the relations in terms of ones with fewer crossings, using the relations to clear all strands out from a bigon, and then the relations (2.2b, 2.15, 2.16, 5.1b, 5.1c) to remove it. Thus, by induction, this process
must terminate at an expression in terms of minimal diagrams. Thus, these elements span, and it suffices to show that these elements are linearly independent when $z$ are generic, that is, after base change to $\overline{K}$.

**Linear independence:** Assume $\mathcal{L} = \mu$. We consider how the elements in $B$ act on a quadruple with $i = \emptyset$ in the deformed category $\mathcal{X}^\lambda$ with $\lambda$ chosen so that $\sum \lambda_i = \mu$.

It suffices to check that these elements act linearly independently on $\mathcal{X}^\lambda_{\overline{K}}$ for some $\lambda$; in the course of the proof we’ll modify $\lambda$ as necessary to achieve this. Note the enormous advantage obtained by having both dominant and anti-dominant weights, as we can add canceling pairs of these without changing the total sum.

As in [Web17a, p. 4.17], we can compose with the diagram $\eta_\kappa$ pulling all black strands to the left and $\overline{\eta_\kappa}$, its vertical reflection. This will send a non-trivial relation between the diagrams in $B$ to a non-trivial relation between diagrams where $\kappa(i) = n$ for all $i$.

**Choice of eigenvalues:** We can now project this relation to the subspace where we fix the eigenvalue of each dot acting at the top and bottom. The formulas (6.14–6.15) and (6.18–6.20) defining the functor $\Xi$ show that this projection is the image under $\Phi$ of a diagram with an equal or smaller number of crossings, and we can only have equality if we choose eigenvalues so that they coincide at opposite ends of a strand. Now, fix a matching $D$ such that an associated basis vector appears in our relation, and the corresponding diagram has a maximal number of crossings among those that appear.

Let us number the arcs $\{a_1, \ldots, a_n\}$ in the diagram $D$, and let $i_{am}$ be the index that labels this arc. As noted before, we are free to add cancelling pairs $(\lambda, -\lambda)$ of highest and lowest weights to $\lambda$, and to reorder our weights, so we can assume that $\lambda_m$ is

1. dominant if the corresponding arc $a$ is downward oriented at its lefthand edge and $\lambda_a$ anti-dominant if the arc $a$ is upward oriented at its lefthand edge,
2. $\lambda_m \neq 0$ for all $m$, that is, $\lambda_m$ is not perpendicular to the coroot corresponding to the arc $a_m$.

For simplicity, let $\lambda_{am} = \lambda_m$ and $\beta_{am} = \beta_m$.

For example, if

$$D = \begin{array}{c}
  i & i & j \\
  3 & & 2 \\
  1 & & 2 \\
  i & j & i
\end{array}$$

---

3We use $\eta$ instead of $\theta$ here since we are pulling left rather than right.
with the numbering as shown, then we must have $\lambda_1$ antidominant, and $\lambda_2, \lambda_3$ dominant, with
$$\lambda_1^i < 0 \quad \lambda_2^j > 0 \quad \lambda_3^i > 0,$$
and can take $\lambda_4 = -\lambda_1, \lambda_5 = -\lambda_2, \lambda_6 = -\lambda_3$.

**Projection to eigenspaces:** Now, let us take the projection to the subspace where the eigenvalue of the dot at each end of the arc a is the variable $\beta_a$. Let $j$ and $j'$ be the associated sequences in $\tilde{I}$ at the bottom and top of the diagram. In the above example, we would have $j = (i, \beta_1, (-j, \beta_2), (-i, \beta_1))$ and $j' = ((-i, \beta_3), (i, \beta_3), (j, \beta_2))$.

Note that $D$ gives the only way to match the terminals in $j$ and $j'$ to produce a legal tricolore diagram. Thus, all diagrams in our relation that give a different matching from $D$ project to 0, since there is no matching which has the same eigenvalue at both ends of each strand and fewer crossings than $D$.

Therefore, this must be the projection of a relation in $\mathcal{X}_K$ where all terms have the underlying matching $D$ with some number $d_a$ of dots on the arc $a$, times some monomial $M$ in the bubbles at the left of the diagram. If we show that no such relation exists, then for each choice of $d_a$ and $M$, the corresponding term must have had coefficient 0 in the original relation. It follows that the original relation must have been trivial.

**Ruling out projected relations:** First, we consider bubbles. Any bubble at the left of the diagram evaluates to a scalar, using the relations (5.1a) and (5.1b–5.1c). The clockwise bubbles with the label $i$ evaluate to the coefficients of the power series $\prod_i (z - \beta_k)^{\lambda_i^i}$ and counterclockwise bubbles to the coefficients of its formal inverse $\prod_i (z - \beta_k)^{-\lambda_i^i}$. By adding new pairs of red and blue strands with labels $\nu$ and $-\nu$ for a strictly dominant weight $\nu$, we can assure that any finite set of monomials in clockwise bubbles are sent to elements of $\tilde{K}$ which are algebraically independent over $k$.

Furthermore, we can explicitly describe the space $\text{HOM}_T((\lambda, j, \kappa), (\lambda, j', \kappa'))$; it has a basis over $\tilde{K}$ given diagrams with matching $D$ and with $d_a < |\lambda_a^i|$. This follows from (5.7): using the fact that functors labeled by different components commute, and adjunction as appropriate, we can reduce to the case where $D$ is a set of vertical strands with no crossings, where this is just the obvious basis of a tensor product of truncated polynomial rings. Thus, for any finite number of ways of choosing $d_a$ and $M$, we can choose $\lambda'$'s so that the corresponding diagrams lie in this basis. Since these diagrams remain linearly independent after acting in $\mathcal{X}_K$, they must be linearly independent, so we must have that the coefficient of this diagram in $\tilde{K}$ is 0. This, in turn, supplies a polynomial relation between the values of the clockwise bubbles. We can rule out this possibility by choosing $\lambda$ so that the bubbles which appear in the relation are algebraically independent. Thus, we see that the relation we chose is trivial. This establishes the linear independence of our prospective basis and establishes the result. □
Appendix A. Valued graphs

We’ll follow the conventions of Lemay [Lem12] in this section. For simplicity, “graph” will always mean a graph without loops.

Definition A.1 A relatively valued graph is an oriented graph with vertex set $I$ with a pair of positive integers $(\eta_e, \nu_e)$ assigned to each edge such that there exist $d_i \in \mathbb{Z}_{>0}$ for each $i \in I$ such that $d_i \eta_e = d_j \nu_e$ for $e: i \to j$.

An absolutely valued graph is an oriented graph as above with a fixed choice of integers $d_i$ for each vertex $i$ and $m_e = d_i \eta_e = d_j \nu_e$ for each edge $e$.

Each absolutely valued graph has an associated relatively valued graph with $\eta_e = \frac{m_e}{d_i}$, $\nu_e = \frac{m_e}{d_j}$ for an edge $e: i \to j$, and every relatively valued graph has this form. The values $\eta_e$ and $\nu_e$ will be integers if $d_i$ and $m_e$ are and lcm$(d_i, d_j)$ divides $m_e$ for an edge $e: i \to j$.

Note that relatively valued graphs have a natural notion of Langlands duality, given by switching $\eta_e$ and $\nu_e$. We attach a Cartan matrix to each such graph without loops, with $c_{ii} = 2$ and

$$c_{ij} = -\sum_{e: i \to j} \eta_e - \sum_{e: j \to i} \nu_e = -\frac{1}{d_i} \sum_{e: i \to j} m_e - \frac{1}{d_j} \sum_{e: j \to i} m_e.$$ 

Note that Langlands duality transposes this Cartan matrix.

Having chosen $P_{ij}(x, y)$ for each pair $i, j \in I^2$, we can canonically associate an absolutely valued graph with vertex set $I$ where we add an edge $i \to j$ whenever $P_{ij}(x, y)$ is non-constant. The values $d_i$ are as before, and $m_e = \deg P_{ij}(x^{d_i}, 0) = \deg P_{ij}(0, x^{d_j})$. In the associated relatively valued graph, the values we add to this edge are $(\deg P_{ij}(x, 0), \deg P_{ij}(0, x))$. The Cartan matrix of the result is our original Cartan matrix $C$.

Given a graph homomorphism between two valued graphs, we can consider various forms of compatibility between the valuings on the two graphs. One notion considered by Lemay [Lem12] is a morphism of valued graphs: this is a homomorphism of graphs where the appropriate statistics $(\eta_e, \nu_e, m_e, d_e)$ are preserved; this is too inflexible for our purposes. Instead, we’ll consider a set of maps which are more analogous to topological covers.

Definition A.2 We call a map $f: X \to Y$ of relatively valued graphs a furling if given any $y, y' \in Y$, and $x \in f^{-1}(y)$, we have that for each edge $d: y \to y'$ and each edge $e: y' \to y$,

$$v_d = \sum_{x' \in f^{-1}(y')} \sum_{\substack{d': y' \to x' \atop f(d') = d}} v_{d'}$$

$$\eta_e = \sum_{x' \in f^{-1}(y')} \sum_{\substack{e': x' \to x \atop f(e') = e}} \eta_{e'}.$$
The notion of a furling is very closely related to a “folding,” but we won’t use this term, since it usually applies to the Langlands dual of the operation above, and implies the existence of a group action. We’ll call $Y$ a **furling** of $X$ and $X$ an **unfurling** of $Y$. A morphism of relatively valued graphs which is also a topological cover is a furling.

Note that since this is a condition on preimages, it is satisfied when $X = \emptyset$; however, it does require that if $y \in Y$ has a preimage, then any $y'$ adjacent to it along an edge has a preimage. Thus, the image of any furling is a union of connected components.

Note that if $X$ has an compatible absolute valued structure such that $d_x$ and $m_e$ is constant on the fibers of $f$ and these fibers are finite, then for an edge $e: y' \to y$, we have that

$$
\eta_e = \sum_{x' \in f^{-1}(y')} \frac{1}{d_{x'}} \sum_{x': x' \to x} m_{e'},
$$

$$
\nu_e = \sum_{x \in f^{-1}(y)} \frac{1}{d_x} \sum_{x': x \to x} m_{e'}.
$$

Thus, we can choose an absolute valued structure on $Y$ such that

$$
d_y = \frac{d_x}{|f^{-1}(y)|},
$$

$$
m_e = \frac{\sum_{f(e') = e} m_{e'}}{|f^{-1}(y)| \cdot |f^{-1}(y')|}.
$$

As defined here, $d_y$ and $m_e$ may not be integers, but if $Y$ is finite, then we can always just multiply every $d_y$ and $m_e$ by lcm($|f^{-1}(y)|$) to clear denominators.

One special case of particular interest is when $X$ is given the trivial valuation $d_x = m_e = \nu_e = \eta_e = 1$ and is equipped with an admissible automorphism $\sigma$; recall that we call an automorphism of a graph admissible if no edges connect two vertices in the same orbit under the action. We let $Y$ be the quotient graph $X/\sigma$ and $f: X \to Y$ the obvious projection map. In this case, we have

$$
d_y = \frac{1}{|f^{-1}(y)|},
$$

$$
m_e = \frac{|f^{-1}(e)|}{|f^{-1}(y)| \cdot |f^{-1}(y')|}.
$$

This is the Langlands dual of the “folding” discussed in [Lem12, §1] (which is the more common way of associating a Cartan matrix to a graph with automorphism).

**Lemma A.3** Given a furling $f: X \to Y$, for any fixed $y, y' \in Y$ and $x' \in f^{-1}(y')$ we have that

$$
c_{yy'} = \sum_{x \in f^{-1}(y)} c_{xx'}.
$$
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Proof.

\[ c_{yy'} = - \sum_{e: y \to y'} \eta_e - \sum_{e: y' \to y} \nu_e \]

\[ = - \sum_{x \in f^{-1}(y)} \left( \sum_{e: x \to x'} \eta_e - \sum_{e: x' \to x} \nu_e \right) \]

\[ = \sum_{x \in f^{-1}(y)} c_{xx'} \]

We assume from now on for all valued graphs appearing that the matrix \( C \) is a generalized Cartan matrix (in particular, all off-diagonal entries are negative integers).

**Definition A.4** Given a valued graph \( X \), let \( \hat{g}_X \) be the Lie algebra generated by \( E_i, F_i, H_i \) with the relations

\[ [H_i, E_j] = c_{ij} E_j \quad [H_i, F_j] = -c_{ij} F_j \quad [E_i, F_j] = \delta_{ij} H_i \quad [H_i, H_j] = 0 \]

We also have the associated universal derived Kac-Moody algebra \( g_X \) where we further quotient by the relations

\[ \text{ad}_{E_i}^{1-c_{ij}} = \text{ad}_{F_i}^{1-c_{ij}} = 0. \]

Since \( I \) might be infinite, we need to define a cofinite topology on the Lie algebras \( \hat{g}_X, g_X \). Both of these algebras are spanned by iterated Lie brackets of the elements \( E_i, F_i, H_i \). Let \( \hat{g}_X^{I_0,r}, g_X^{I_0,r} \) for a finite subset \( I_0 \subset I \) be the span of all monomials of length \( \leq r \) in these elements that involve at least one \( E_i, F_i, \) or \( H_i \) for \( i \notin I_0 \). Let the cofinite topology be the topology on \( \hat{g}_X, g_X \) with these sets as a basis of neighborhoods of the identity.

Note the analogy to the finiteness conditions of 1-morphisms in \( \hat{U} \). For any representation of \( g_X \) where on any given vector \( v \), we have \( E_i v = F_i v = H_i v = 0 \) for all but finitely many \( i \in I \), we will have \( g_X^{I_0,r} v = 0 \) for some \( I_0 \), so the action on any such representation factors through the completion \( g_X \) with respect to this topology.

A straightforward extension of [Kac90, p. 7.9] shows that:

**Proposition A.5** If \( f: X \to Y \) is a furling of valued graphs, there is an induced homomorphism of Kac-Moody algebras \( g_Y \to \hat{g}_X \) given by the formulas:

\[ F_y \mapsto \sum_{x \in f^{-1}(y)} F_x \quad E_y \mapsto \sum_{x \in f^{-1}(y)} E_x \quad H_y \mapsto \sum_{x \in f^{-1}(y)} H_x. \]

**Proof.** The relations (1.2) are straightforward computations using Lemma [A.3] We have that:

\[ [ \sum_{x \in f^{-1}(y)} H_x, \sum_{x' \in f^{-1}(y')} E_{x'}] = \sum_{x' \in f^{-1}(y')} \left( \sum_{x \in f^{-1}(y)} c_{xx'} \right) E_{x'} = \sum_{x' \in f^{-1}(y')} c_{yy'} E_{x'} \]
\[ \left[ \sum_{x \in f^{-1}(y)} H_{x}, \sum_{x' \in f^{-1}(y')} F_{x'} \right] = \sum_{x' \in f^{-1}(y')} \left( \sum_{x \in f^{-1}(y)} -c_{xx'} \right) F_{x'} = \sum_{x' \in f^{-1}(y')} -c_{yy'} F_{x'} \]

This shows that we have a homomorphism \( \tilde{\gamma} \to \tilde{\delta} \), which we wish to show descends. If there are any components of \( Y \) with no preimages, this map kills all the attached \( E_i, H_i, F_i \), so this is just augmentation map and indeed descends.

Thus, without loss of generality, we may assume that \( X \to Y \) is surjective. Consequently, the induced map on Cartan subalgebras \( h_Y \to h_X \) is injective, and so this map sends elements of non-zero weight to elements of non-zero weight. In particular, the kernel of the map \( \tilde{\gamma} \to \tilde{\delta} \) is an ideal of this Lie algebra with no non-zero vectors of weight zero. Thus, the same is true of its image in \( \tilde{\delta}_X \). By the Gabber-Kac theorem [GK81, Cor. 2], any such ideal lies in the ideal generated by the relations (1.3), so this completes the proof. \( \square \)

We wish to consider the “order of vanishing” of \( Q_{ij}(x, y) \) at \( x = u, y = u' \); of course, this is not well-defined for a 2-variable polynomial, but because of the homogeneity, we can make sense of it in this case as the vanishing order of \( Q_{ij}(u, y) \) at \( y = u' \) or of \( Q_{ij}(x, u') \) at \( x = u \). For a general 2-variable polynomial, these will not be the same, but in our case, it will be the number of elements of \( A_{ij} \) such that \( u^{1/d_i} = a_{ij}^{(k)} (u')^{1/d_i} \). These solutions are also in bijection with solutions to \( u^{h_{ij}} = a_{ij}^{(k)} (u')^{h_{ij}} \) for \( a_{ij}^{(k)} \in A_{ij} \) (which tend to be slightly easier to count).

We will consider this as an absolutely/relatively valued graph with trivial valuation \( \eta_e = \nu_e = d_i = m_e = 1 \).

Note that if all \( a_{ij}^{(k)} \)'s are \( m \)th roots of unity for some \( m \), then any finite choice of \( U_i \)'s can be made complete by adding finitely many elements (for example, all products of elements of \( U_i \) and \( m \)th roots of unity will suffice). Any countable choice of spectra can be completed to a countable complete choice of spectra; we let \( V_{i}^{(0)} = U_i \) and define \( V_{i}^{(k)} = \{ v \in \mathbb{K} \mid Q_{ij}(v, v') = 0 \text{ for } v' \in V_{j}^{(k-1)} \} \). The union \( V_{i}^{(\infty)} = \bigcup_{i=0}^{\infty} V_{i}^{(k)} \) is a complete choice of spectra.

**Proposition A.6** If \( U_i \) is a complete choice of spectra, then the map \( \tilde{I} \to I \) is a furling of valued graphs.

**Proof.** The unique edge \( e: i \to i' \) in \( I \) has preimages corresponding to each element \( u \in U_i \) and each root of \( P_{i'}(u, x) \) as a polynomial in \( x \). Thus the number of preimages \( e' \) is the degree of this polynomial in \( x \), and each has \( \nu_{e'} = 1 \), so this agrees with \( \nu_{e} = \deg P_{i'}(0, x) \). Similarly, if we consider the edge \( d: i' \to i \), the edges are in bijection with the roots of \( P_{i'}(x, u) \), and \( \eta_d = \deg P_{i'}(x, 0) \). This completes the proof. \( \square \)
The most important example is the so-called “geometric” parameters for the symmetric Cartan matrix for an oriented graph, where $P_{ii'}(x, y) = (x - y)^{\#i\to i'}$. In this case, $a_{iir}^{(k)} = 1$ for all $k$ and $(i, u)$ is connected to $(i', u')$ by the same number of edges as $i$ and $i'$ if $u = u'$ and none otherwise. Thus, if we choose $U_i = U$ for some fixed set $U \subset \mathbb{k}$, this is a complete choice of spectra and $\bar{I} = I \times U$ with the obvious graph structure.

If $I$ is simply laced, but not simply-connected, then we can obtain non-trivial covers as $\bar{I}$. For example, if $I$ in an $n$-cycle with its vertex set identified with $\mathbb{Z}/n\mathbb{Z}$ with edges $i \to i + 1$. Fix some $q \in \mathbb{k}$ and choose $Q_{i,i+1}(x, y) = qx - y$. If we fix $U_0 \subset \mathbb{k}$ to be any subset closed under multiplication by $q^n$, then we have a complete choice of spectra with $U_i = q^i U_0$. The components of the graph $\bar{I}$ correspond to the orbits of multiplication by $q$; these will be cycles if $q$ is a root of unity, or $A_\infty$ graphs if $q$ is not.

On the other hand, if we have a nonsymmetric Cartan matrix, then we may find a more interesting result. If $Q_{12}(x, y) = x^2 - y$ and $d_1 = 1, d_2 = 2$ (so $g = sp_3$), then the factorization shown earlier is that $Q_{12}(x, y) = (x + y^{1/2})(x - y^{1/2})$, and so $a_{12}^{(1)} = 1$ and $a_{12}^{(2)} = -1$. Thus, the number of edges joining $(1, x)$ to $(2, y)$ is given by the number of solutions to $x = \pm \sqrt{y}$. Thus, every component of $\bar{I}$ is a subgraph of an $A_3$ formed by $(1, x) \to (2, x^2) \leftrightarrow (1, -x)$ (assuming $1 \neq -1$).

More generally, let $d = \text{lcm}(d_i)_{i \in I}$. Let $U_i$ be the $d/d_i$th roots of unity; there are $d/d_i$ distinct roots of unity since $d$ is coprime to the characteristic of $\mathbb{k}$. Assume that each $\alpha_{ij}^{(k)}$ is a $d$th root of unity for all $i, j, k$. For example, we can assume that

\begin{equation}
Q_{ij}(x, y) = \pm (x^{h_{ij}} - y^{k_{ij}})^{\xi_{ij}}
\end{equation}

in which case, $Q_{ij}(x^{d_i}, 1) = \pm (x^{d_i h_{ij}} - 1)^{\xi_{ij}}$, so the multiset of $\alpha_{ij}^{(k)}$ and $b_{ij}^{(k)}$ is given by the $d_i h_{ij} = d_j h_{ji} = \text{lcm}(d_i, d_j)$ roots of unity each with multiplicity $g_{ij}$. In this case, we have $\alpha_{ij}^{(k)} = b_{ij}^{(k)} = 1$ for any $k$.

We have that each $u \in U_i$ is connected by $c_{ji}$ edges to elements of $U_j$, given by the $h_{ji}$th roots of $\alpha_{ij}^{(k)} u^{h_{ij}}$. If $Q_{ij}$ is as in (1.4), then for each $d/d_i h_{ij}$th root of unity $\xi$, we connect each $h_{ij}$th root of $\xi$ in $U_i$ to each $h_{ji}$th root of $\xi$ in $U_j$ with $g_{ij}$ edges (with orientation depending on $P_{ij}$). Let $\zeta$ be a primitive $d$th root of unity.

**Proposition A.7** Assuming $a_{ij}^{(k)}$ is a $d$th root of unity for all $i, j, k$ and $U_i$ is the $d/d_i$th roots of unity, the map $\sigma: (i, u) \mapsto (i, \zeta^{d_i} u)$ is an admissible automorphism of the graph $\bar{I}$; the map $\bar{I} \to I = \bar{I}/\sigma$ induces the relative valued structure on $I$ associated to the polynomials $P_{ij}$.

Note that the absolute weighting of (1.1) is the symmetrization we have chosen for our Cartan matrix divided by $d$, since $|f^{-1}(i)| = d/d_i$
This index of notation gives a brief description of the main notation used in the paper, together with the section and page where the notation is defined.

| §   | Symbol | Description                                                                 | Page |
|-----|--------|------------------------------------------------------------------------------|------|
| 2.0 | \(I, g\) | The indexing set of simple roots in a Cartan datum and associated Kac-Moody algebra. | 5    |
|     | \(c_{ij}, C\) | The entries \(c_{ij} = \frac{\langle \alpha_{i}, \alpha_{j} \rangle}{\langle \alpha_{i}, \alpha_{i} \rangle}\) of the Cartan matrix \(C\). | 5    |
|     | \(d_{i}\) | The symmetrizing coefficients \(d_{i} = \frac{\langle \alpha_{i}, \alpha_{i} \rangle}{\langle \alpha_{i}, \alpha_{i} \rangle}\) satisfying \(d_{i}c_{ij} = d_{j}c_{ji} = \frac{\langle \alpha_{i}, \alpha_{j} \rangle}{\langle \alpha_{i}, \alpha_{i} \rangle}\). | 5    |
|     | \(F\) | The base field of characteristic 0 for the Lie algebra \(g\). | 5    |
|     | \(k\) | The base field for KLR algebras, which is algebraically closed of characteristic coprime to all \(d_{i}\). | 6    |
|     | \(Q_{ij}\) | The polynomials that appear in the definition of the KLR algebra. | 6    |
|     | \(U_{i}\) | A subset of \(k\) attached to a vertex \(i \in I\). Typically the spectrum of a natural transformation of an associated functor (Definition 2.2). | 7    |
|     | \(\tilde{I}, \tilde{g}\) | The graph with vertex set pairs of vertices \(i \in I\) and scalars \(u \in U_{i} \subset k\) and the Kac-Moody algebra attached to this graph. | 7    |
| 2.1 | \(R_{n}\) | The KLR algebra attached to the Cartan datum \(I\) and polynomials \(Q_{ij}\) over \(k\). | 7    |
|     | \(P_{ij}\) | Polynomials \(P_{ij}(x, y) \in k[x, y]\) such that \(Q_{ij}(x, y) = P_{ij}(x, y)P_{ji}(y, x)\). | 8    |
| 2.2 | \(U(q)\) | The Kac-Moody 2-category categorifying \(\hat{U}(q)\). | 9    |
|     | \(\bigodot(z), \bigodot(z)\) | The bubble power series defined in (2.20–2.21). | 11   |
| 3.1 | \(U \otimes U\) | The strict 2-category obtained from the tensor product \(U \otimes U\) of additive 2-categories by adjoining the barbell (3.1). | 13   |
| 3.2 | GC topology | The topology generated by spans \(\text{Hom}_{\text{U}(U)}^{\geq k}(u, v) + \text{Hom}_{\text{U}(U)}^{l}(u, v)\), the 2-morphisms of degree \(\geq I\) and factoring through monomials of length \(\leq r\) containing \(E_{\pm i}\) with \(i \notin I_{0}\). | 15   |
|     | \(\widetilde{U}\) | The completion of the 2-morphisms \(U\) by the GC topology. | 15   |
|     | \(\gamma_{a}\) | A power series \(\gamma_{a}(z) = a + a^{(1)}z + a^{(2)}z^{2} + \cdots\) with \(a \neq 0\) and \(a^{(1)} \neq 0\) used in the definition of the functor of Lemma 3.3. | 16   |
| 3.3 | \(g_{ij}, h_{ij}\) | The greatest common denominator of the Cartan matrix entries \(g_{ij} = \gcd(-c_{ij}, -c_{ji})\) and the ratio \(h_{ij} = \frac{c_{ij}}{g_{ij}}\). | 18   |
### Unfurling Khovanov-Lauda-Rouquier algebras

| § | Symbol | Description | Page |
|---|---|---|---|
| | $A_{ij}, B_{ij}$ | The multiset $A_{ij} = \{a_{ij}^{(k)}\}$ of roots of $P_{ij}(x^d, 1)$ and its reciprocals $B_{ij} = \{b_{ij}^{(k)} = (a_{ij}^{(k)})^{-1}\}$. | 18 |
| | $P_{(i,a),(j,a')}$ | The polynomials corresponding to “geometric coefficients” for the graph $\tilde{I}$. | 18 |
| | $p_{ij}^{o,uu'}$ | The invertible power series such that $P_{ij}(\gamma_u(x), \gamma_{u'}(y)) = P_{(i,a),(j,a')}(x, y)p_{ij}^{o,uu'}(x, y)$. | 19 |
| 3.4 | $B_{i,i',\lambda}$ | The basis of $\text{HOM}_{\mathcal{U}}(i, i')$ constructed in Section 3.4. | 19 |
| 4.1 | $\mathcal{E}_{i,u}, \mathcal{F}_{i,u}$ | The summands of $\mathcal{E}_i$ and $\mathcal{F}_i$ given by generalized eigenspaces of dots. | 20 |
| 4.2 | $\mathcal{O}_V(z)$ | The action of the power series $\bigodot(z)$ on the object $V$. | 22 |
| 5.1 | $\mathcal{X}_{\lambda}$ | The deformed categorification of a tensor product of highest and lowest weight representations. | 28 |
| | $\mathbb{K}, \bar{\mathbb{K}}$ | The function field $\mathbb{K} = \mathbb{k}(\beta_1, \ldots, \beta_\ell)$ and its algebraic closure. | 29 |
| | $\mathcal{X}_{\lambda}^{\bar{\mathbb{K}}}$ | The idempotent completion of the extension of scalars $\mathcal{X}_{\lambda} \otimes_{\mathbb{K}[\beta_1, \ldots, \beta_\ell]} \bar{\mathbb{K}}$. | 29 |
| 5.2 | $\lambda, m(u)$ | The weight for $\tilde{\mathcal{G}}'$ defined in Definition 5.7. | 30 |
| | | The unique index $m$ such that $u$ and $\beta_m$ are algebraically dependent. | 30 |

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