On the Curved Patterns Seen in the Graph of PPTs

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Abstract

Dr. Ron Knott constructed a graph of all Primitive Pythagorean Triples (PPTs) with legs up to length 10,000, using Mathematica. The patterns are very interesting, suggesting conic sections. We show that they indeed are parabolic curves which follow in a natural way from the mathematics of the subject matter.

A Pythagorean Triple (PT) is a set of 3 positive integers \((a,b,c)\) which satisfy the Pythagoras Equation \(a^2+b^2 = c^2\). The numbers \(a, b, c\) are associated with a right triangle. For example, \((3,4,5)\) is a Pythagorean Triple, since the right triangle \(\Delta ABC\) with sides of lengths 3, 4, 5, satisfies \(3^2+4^2 = 5^2\).

Given a PT \((a,b,c)\), if the integers \(a, b, c\) are relatively prime, then call the triple \((a,b,c)\) a Primitive Pythagorean Triple (PPT). For example \((3,4,5)\) is a PPT, but the PT \((6,8,10)\) is not.

We are interested in analyzing a graph of PPTs, which we construct in a certain way.

Given a PT \((a,b,c)\) there is a way to graph it which allows us to compare PTs. Choose the ordered pair \((a,b)\) and graph this in the 1st quadrant of the xy-plane, see the graph here.

Then the right triangle \(\Delta ABC\) with sides \(a, b, c\) is congruent to the triangle formed by the origin \(O\), the point \((a,0)\), and the point \((a,b)\). Thus the point \((a,b)\) determines a congruent copy of \(\Delta ABC\).

Another congruent copy of \(\Delta ABC\) is provided by the graph of the point \((b,a)\). The point \((b,a)\) is the reflection of \((a,b)\) about the line \(y = x\).

We will assume that \(a<b<c\), when referring to the PT \((a,b,c)\), but occasionally it will happen, by way of an argument, that we have the PT \((b,a,c)\) as in the triangles figures just above. The line \(y = x\) then divides the first quadrant into two regions, one which contains the points \((a, b), a<b\), and one which contains the points \((b, a), a<b\). Since the
points \((a, b)\) and \((b, a)\) are symmetric about the line \(y = x\), we call \((b, a)\) the \textit{reflected point} of \((a, b)\).

Observe that if \((a,b,c)\) is a PPT, then one of \(a, b\) must be odd, and the other must be even, while \(c\) is always odd.

If we graph all PPTs with \(0 < a, b < 10,000\) using the method above, we get the amazing graph shown here. This graph was obtained by Dr. Ron Knott using Mathematica [3]. The red region is the set of points \((a, b)\), where \(a < b\), and the black region is the set of reflected points \((b, a)\). The two regions are symmetric about the line \(y = x\). We call this the ‘Main Graph’.
In order to analyze this graph, consider the following set of the first 18 PPTs ordered by the size of the short leg $a$.

\[
\begin{align*}
3, 4, 5 & \quad 5, 12, 13 & \quad 7, 24, 25 & \quad 8, 15, 17 & \quad 9, 40, 41 & \quad 11, 60, 61 \\
12, 35, 37 & \quad 13, 84, 85 & \quad 15, 112, 113 & \quad 16, 63, 65 & \quad 17, 144, 145 & \quad 19, 180, 181 \\
20, 21, 29 & \quad 20, 99, 101 & \quad 21, 220, 221 & \quad 23, 264, 265 & \quad 24, 143, 145 & \quad 25, 312, 313
\end{align*}
\]

Those PPTs which are in bold print are to call your attention to them. These particular 12 PPTs all have terms $a$ which are odd numbers, and they also have the form $(a, b, b+1)$, that is $b$ and $c$ are consecutive numbers, $c = b+1$.

It is given that these triples all satisfy the Pythagorean equation $a^2 + b^2 = c^2$, and the terms are obviously relatively prime, since consecutive numbers $b, b+1$ cannot have any common divisors, but they also satisfy the equivalent equation $b = (a^2 - 1)/2$, and therefore $b+1 = (a^2 + 1)/2$. This means that $a^2$ (and thus $a$) must be an odd integer, so that $b$ is then an even integer.

What we have discovered is that if we are given an odd positive integer $a$, then the triple $(a, b, b+1)$ is a PPT, whenever $b = (a^2 - 1)/2$. So these PPTs all have the form $(a, (a^2-1)/2, (a^2+1)/2)$, for $a$ an odd positive integer. This determines a one-to-one correspondence between the odd positive integers $a$ and those PPTs which have the form $(a, b, b+1)$.

This is not a new result, it is well known. According to Proclus (410-485 AD), this was known to the Pythagoreans (570-495 BCE), and perhaps before that [1]. This does not seem to prevent the result from being ‘rediscovered’ occasionally.

We can graph these PPTs in the $xy$-plane using the graphing method above, by graphing the points $(a, (a^2 - 1)/2)$, and the reflected points $((a^2 - 1)/2, a)$, for $a$ an odd positive integer.

For example, the odd integer 3 determines the points $(3, 4)$ and $(4, 3)$, and the odd integers 5, 7, 9, ..., 25, determine the points corresponding to the PPTs given in bold in the list above.

A graph of these points and their reflected points in the $xy$-plane in the range $0 < a,b < 120$ is given here. It suggests parabolic shaped curves about the positive $x$ and $y$ axes. The red points have the form $(a, (a^2 - 1)/2)$, so they satisfy the parabola equation $y = (x^2 - 1)/2$, and the reflected points have the form $((a^2 - 1)/2, a)$, so they satisfy the reflected parabola equation $x = (y^2 - 1)/2$.

The first parabola opens about the positive $y$-axis, with focus at $O$, and vertex at $(0, -1/2)$. The second parabola opens about the positive $x$-axis, has the focus at $O$, and vertex at $(-1/2, 0)$, see the figure.
Now, looking back at the list of the first 18 PPTs above, notice that several of the PPTs have the form \((a, b, b+2)\). We denote these PPTs by \(d = 2\), since the ‘difference’ \(c-b = 2\). Denote the PPTs we studied above by \(d = 1\). The set of PPTs which satisfy \(d = 2\) also satisfy the equivalent equation \(b = (a^2 - 4)/4\). This means that \(a^2\), and thus \(a\), must be even, and \(b\) must be odd. The numbers \(a, b, c\) are relatively prime, since the odd terms \(b, c\), are 2 units apart, so they cannot have any common divisors. Proclus attributes this result to Plato (429-347 BCE) [1].

From the list of \(d = 2\) PPTs above, we see that \(a\) is even and \(a = 0 \bmod 4\). But if \(a\) is even, and not a multiple of 4, \(a = 2 \bmod 4\), then the triple is a PT, and not a PPT, so it is not on the list.

Thus, if \(a = 0 \bmod 4\), then \(a = 4k, k > 0\), and \((a, b, b+2) = (4k, 4k^2 -1, 4k^2 +1)\), a PPT. But if \(a = 2 \bmod 4\), then \(a = 4k +2\), and \(b = 4k^2 +4k\), \(b+2 = 4k^2 +4k +2\). This PT is not a PPT, as all terms are even. So every other even term \(a\) determines a PPT. The curves which are for these \(d = 2\) PPTs are the parabolas \(y = (x^2 -4)/4\), and \(x = (y^2 -4)/4\). See the figures here.

In general, the parabolas for arbitrary values of \(d > 0\) are given by the equations \(y = (x^2 -d^2)/2d\), and \(x = (y^2 -d^2)/2d\), where \(a\) and \(d\) are either both odd or both even positive integers, and \(d\) divides \(a\). However we find that a lot of the values of \(d\) determine PTs, but not PPTs.

There is another \(d\)-value associated with the PPT \((a,b,c)\), which is determined by the form \((b,a,c)\), namely \(d' = c-a\). The points \((a,b)\) (and \((b,a)\)) will occur at the intersection of the parabolas for \(d\) and \(d'\). For example, the graph below shows the point \((3,4)\) at the intersection of \(d = 1\) and \(d' = 2\), and \((4,3)\) at the intersection of \(d' = 1\), and \(d = 2\).

Any integer point \((s,t)\) on a parabola \(y = (x^2 -d^2)/2d\) for some \(d\) will satisfy \(t = (s^2 -d^2)/2d\), and \((s,t,u)\) is a PT for that \(d\), where \(u = (x^2 +d^2)/2d\). Each integer point in the the graph of the PPTs above is on a parabola for some value of \(d\).

The value \(d = 3\) does not determine any PPTs, for all such PTs have a common value of 3 in their coordinates. The same thing happens for the values \(d = 4, 5, 6, 7,\)
10, 11, and 12. The next PPTs, after \( d = 1, 2 \), occur when \( d = 8 \) and 9, and these have mixed results similar to those we found for \( d = 2 \).

The list of \( d \)-values which give PPTs begins as follows: 1, 2, 8, 9, 18, 25, 32, 49, 50, 72, 81, 98, …, [2]. This is the OEIS sequence A096033. The graph of some of the representative points and parabolic curves for some of these \( d \) and \( d' \) values is shown below.

If you compare this graph with the Main Graph on p. 2 you can see where some of the curved patterns are coming from. We are only looking at a very small part of the lower left corner of the Main Graph, but the general picture is starting to take shape.
If you study the main graph on p. 2 further, you may notice that there also appear to be parabolic curve patterns which open about the negative $x$ and $y$ axes. These curves are mentioned in general in [3]. The equations for these curves are formed in a different manner. We will first consider the parabolic curves which open about the negative $y$-axis.

An example is the parabola $y = -(x^2/3^2 - 3^2)/2$, on the graph shown here in orange. This parabola has vertex at $y = 4.5$, and $x$-intercepts $x = \pm 9$. It also contains the point $(3,4)$.

If you study the graphs above, and below, you may notice that certain sets of points seem to be in a parabolic shaped pattern which opens down. For example, the points $(20,99)$, $(60,91)$, $(140,51)$, and $(180,19)$ appear to form a parabolic curve which opens about the negative $y$-axis. Note that the 1st coordinates of these points are all multiples of 10. The LH point $(20,99)$ is on the $d = 2$ parabola, and is from the triple $(20,99,101)$. The equation for this parabola is $y = -(x^2/(2 \cdot 10^2) - 2 \cdot 10^2)/2$. The vertex point is at $y = 100 = 10^2$, and the $x$-intercepts are $x = \pm 2 \cdot 10^2$.

For another example, consider the points $(28,45)$, $(56,33)$, and $(84,13)$. They also appear to be on a parabola which opens about the negative $y$-axis. The 1st coordinates here are all multiples of 28. The LH point $(28,45)$ is on the parabola $d = 8$ parabola, and is from the triple $(28,45,53)$. The equation for this parabola is $y = -(x^2/(2 \cdot 7^2) - 2 \cdot 7^2)/2$. The vertex point is at $y = 49 = 7^2$, and the $x$-intercepts are $x = \pm 2 \cdot 7^2$.

These examples help give us a pattern for the equations of the parabolas which open about the negative $y$-axis. Given a series of points with even 1st coordinates all multiples of, say $2 \cdot n^2$, the general equation of the parabolas is $y = -(x^2/(2 \cdot n^2) - 2 \cdot n^2)/2$, where $n = a/k$, for $a$ the 1st coordinate in the LH point in the series, and $k = 2$, if this point is on the $d = 2$ parabola, and $k = 4$, if it’s on the $d = 8$ parabola. Thus every point from a PPT on the $d = 2$ and $d = 8$ parabolas (the middle and RH red dashed lines in the graph below) determines a parabola in this set. These curves are shown in orange in the graph below.

If $a$ is odd, as in the example with the point $(3, 4)$ above, then the equation is derived as in the following example. The points $(13,84)$, $(39,80)$, $(65,72)$, $(91,60)$, $(117,44)$, and $(143,24)$, appear to be on a parabolic curve which opens about the negative $y$-axis. The 1st coordinates of the points are all multiples of 13, the LH point $(13,84)$ is on the parabola $d = 1$, and it’s from the triple $(13,84,85)$. The equation for this parabola is $y = -(x^2/(13^2) - 13^2)/2$. It has vertex at $y = 84.5$, and $x$-intercepts $x = \pm 13^2$.

For another example, consider the points $(11,60)$, $(33,56)$, $(55,48)$, $(77,36)$, and $(99,20)$. They also appear to be on a parabola which opens about the negative $y$-axis. All of the 1st coordinates of the points are multiples of 11, the LH point $(11,60)$ is on the parabola $d = 1$, and it is from the triple $(11,60,61)$. The equation for this parabola is $y = -(x^2/(11^2) - 11^2)/2$, the vertex point is at $y = 60.5$. The $x$-intercepts are $x = \pm 11^2$. 
The general form for the equations of these parabolas which open about the negative $y$-axis, when $a$ is odd, is $y = -(x^2/a^2 - a^2)/2$, where $a$ is the value of the $x$-coordinate of the first LH point $(a,b)$ on the parabola $d = 1$, in the series of points being considered (the LH red dashed line).

Thus every point $(a,b)$, from a PPT point $(a,b,c)$, on the $d = 1$ parabola determines a parabola in this set. These curves are shown in orange in the graph above.

When these 2 sets of parabolas, for $a$ even or $a$ odd, are reflected about the line $y = x$, we have the corresponding parabolas for the $d'$-values. These parabolas open about the negative $x$-axis, and are shown in dark blue in the graph above. Not all of the curves which exist in this range are shown here.
That every parabola shown which opens about the negative y-axis has a point \((a,b)\) from a PPT \((a,b,c)\) on the parabolas \(d = 1\), or \(d = 2\) or \(8\), follows from the observation that these parabolas are the closest ones to the y-axis for even or odd values of \(d\). The \(d = 9\) points are not usable, as they are covered by the \(d = 1\) points. Similarly for the \(d = 18\) and the \(d = 8\) points.

References

[1] T. Heath, Euclid, The Thirteen Books of the Elements, Vol. 1, Dover, 2nd ed., 1956.
[2] D. Joyce, Primitive Pythagorean Triples, Clark U., March 2010, https://mathcs.clarku.edu/~djoyce/numbers/pyth.pdf
[3] R. Knott, Pythagorean Right Triangles, http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Pythag/pythag.html#section3