DEFORMATIONS OF CALABI–YAU THREEFOLDS AND THEIR MODULI OF VECTOR BUNDLES

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ABSTRACT. We describe deformations of the noncompact Calabi–Yau threefolds $W_k = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k-2))$ for $k = 1, 2, 3$, as well as their moduli of holomorphic vector bundles of rank 2. Deformations are computed concretely by calculations of $H^1(W_k, TW_k)$. Information about the moduli of vector bundles is obtained by analysing bundles that are extensions of line bundles. We show that for each $k = 1, 2, 3$ the associated structures are qualitatively different, and we also comment on their difference from the analogous structures for the simpler noncompact twofolds $\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$ which had been studied previously by the authors.

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1. Motivation

Our motivation to study deformations of Calabi–Yau threefolds comes from mathematical physics. In fact, deformations of complex structures of Calabi–Yau threefolds enter as terms of the integrals defining the action of the theories of Kodaira–Spencer gravity [B]. As we shall see, in general our threefolds will have infinite-dimensional deformation spaces, thus allowing for rich applications. Here we describe their deformation theory and features of their moduli spaces of holomorphic vector bundles.
We consider smooth Calabi–Yau threefolds $W_k$ containing a line $\ell \cong \mathbb{P}^1$. For the applications we have in mind for future work it will be useful to observe the effect of contracting the line to a singularity. The existence of a contraction of $\ell$ imposes heavy restrictions on the normal bundle $\text{[Jim]}$, namely $N_{\ell/W}$ must be isomorphic to one of

(a) $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$, (b) $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$, or (c) $\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

Conversely, Jiménez states that if $\mathbb{P}^1 \cong \ell \subset W$ is any subspace of a smooth threefold $W$ such that $N_{\ell/W}$ is isomorphic to one of the above, then:

- in (a) $\ell$ always contracts,
- in (b) either $\ell$ contracts or it moves, and
- in case (c) there exists an example in which $\ell$ does not contract nor does any multiple of $\ell$ (i.e. any scheme supported on $\ell$) move.

$W_1$ is the space appearing in the basic flop. Let $X$ be the cone over the ordinary double point defined by the equation $xy - zw = 0$ on $\mathbb{C}^4$. The basic flop is described by the diagram:

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
  \node (W) at (0,0) {$W$};
  \node (Wm) at (-2,-2) {$W^-_1$};
  \node (Wp) at (2,-2) {$W^+_1$};
  \node (X) at (0,-4) {$X$};
  \draw[->] (W) -- (Wm) node [midway, left] {$p_1$};
  \draw[->] (W) -- (Wp) node [midway, right] {$p_2$};
  \draw[->] (Wm) -- (X) node [midway, below] {$\pi_1$};
  \draw[->] (Wp) -- (X) node [midway, below] {$\pi_2$};
\end{tikzpicture}
\end{array}
\end{align*}

Here $W := W_{x,y,z,w}$ is the blow-up of $X$ at the vertex $x = y = z = w = 0$, $W^-_1 := Z_{x,z}$ is the small blow-up of $X$ along $x = z = 0$ and $W^+_1 := Z_{y,w}$ is the small blow-up of $X$ along $y = w = 0$. The basic flop is the rational map from $W^-$ to $W^+$. It is famous in algebraic geometry for being the first case of a rational map that is not a blow-up.

Thus, we will focus on the Calabi–Yau cases

$$W_k := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k-2))$$

for $k = 1, 2, 3$.

We observe that from the point of view of moduli of vector bundles the cases $k \geq 4$ behave quite similarly to the case $k = 3$. We will also consider surfaces of the form

$$Z_k := \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k))$$

for comparison in Sections 3 and 4.

## 2. Statements of results

We describe deformations and moduli of vector bundles for complex surfaces and threefolds which are the total spaces of (sums of) line bundles on the complex projective line $\mathbb{P}^1$.

Regarding surfaces, in contrast to what happens in the case of $Z_k$ with $k > 0$, where all holomorphic vector bundles are algebraic [G1, Lem. 3.1, Thm. 3.2], we present in Prop. 4.2 a holomorphic vector bundle on $Z_{(-1)}$ that is not algebraic. Moreover, we prove that the deformations of the
surfaces $Z_k$, described in [BG], can be obtained from the deformations of the Hirzebruch surfaces $F_k$, Lem. 4.4.

For the case of the Calabi–Yau threefolds $W_k$, Thm. 5.3 shows that the generic part of the moduli of algebraic bundles of splitting type $(j, -j)$ (see Def. 5.2) on $W_k$ is smooth and of dimension $4j - 5$. Thm. 5.4 shows that all holomorphic bundles on $W_1$ are algebraic; a detailed treatment appears in [K]. In contrast, we present a holomorphic bundle on $W_3$ that is not algebraic, Cor. 4.3. For $W_1$ the moduli of holomorphic bundles is finite-dimensional, Cor. 5.5. For $W_2$, however, the moduli spaces are infinite-dimensional, Thm. 5.6, with greater detail appearing in [R].

Our results on deformations of the threefolds $W_k$ are as follows. We show that $W_1$ has no deformations, Thm. 6.1, whereas $W_2$ has an infinite-dimensional deformation space, Thm. 7.1. Furthermore, we exhibit a deformation $W_2$ of $W_2$ which turns out to be a non-affine manifold, a very different case from that of surfaces $Z_k$, $k > 0$, where all the deformations are affine varieties. Finally, we give an infinite-dimensional family of deformations of $W_3$ which is not universal, but is semiuniversal, Cor. 8.4. The case $W_2$ is quite different from $W_1$, $W_2$, or the surfaces. The tools used so far to describe deformation spaces and moduli have not been sufficient for $W_3$, therefore must we look for more effective techniques. We know from Cor. 4.3 that $W_3$ contains properly holomorphic bundles, and that we will have infinite-dimensional moduli spaces. The cases $k \geq 3$ present similar features; we will continue their study in future work.

3. Comparison with the deformation theory of surfaces

Several results are known for the case of deformations of the surfaces $Z_k$. It turned out rather interestingly that the results we obtained for threefolds are not at all analogous to the ones for surfaces.

[BGK2, Thm. 4.11] showed that the holomorphic vector bundles on $Z_k$ with splitting type $(-j, j)$ (see Def. 5.2) are quasiprojective varieties of dimension $2j - k - 2$. In contrast, we will see that moduli spaces of holomorphic bundles on the threefolds $W_2$ and $W_3$ are infinite-dimensional.

[BG, Thm. 6.11] showed that the moduli spaces of vector bundles on a nontrivial deformation of $Z_k$ are zero-dimensional. Thus classical deformations of $Z_k$ do not give rise to deformations of their moduli of vector bundles. This will not be the case for $W_k$.

Regarding applications to mathematical physics, the deformations of surfaces turned out rather disappointing, because instantons on $Z_k$ disappear under a small deformation of the base [BG, Thm. 7.3]. This resulted from the fact that deformations of $Z_k$ are affine varieties. The case of threefolds is a lot more promising, since for $k > 1$, $W_k$ has deformations which are not affine.
Nevertheless, deformations of the surfaces $Z_k$ turned out to have an interesting application to a question motivated by the Homological Mirror Symmetry conjecture. [BBGGS, Sec. 2] showed that the adjoint orbit of $s_l(2, \mathbb{C})$ has the complex structure of the nontrivial deformation of $Z_2$, and it used this structure to construct a Landau–Ginzburg model that does not have projective mirrors. Further applications to mirror symmetry give us another motivation to study deformation theory for Calabi–Yau threefolds.

4. SOME RESULTS ABOUT SURFACES

In this section we prove some results about the surfaces $Z_k$ that will be used in the development of the theory for threefolds.

4.1. A holomorphic bundle on $Z_{(-1)}$ that is not algebraic. By definition $Z_{(-1)} = \text{Tot}(\mathcal{O}_P^1(+1))$, and in canonical coordinates $Z_{(-1)} = U \cup V$, where $U = \{(z, u)\}$ and $V = \{ (\xi, v) \}$, $U \cap V \cong \mathbb{C}^* \times \mathbb{C}$, with change of coordinates given by:

$$(\xi, v) \mapsto (z^{-1}, z^{-1} u).$$

Lemma 4.1. $H^1(Z_{(-1)}, \mathcal{O}(-2))$ is infinite-dimensional, generated as a vector space over $\mathbb{C}$ by the monomials $z^l u^i$ with $l = -2, -1$ and $i = 1, 2, \ldots$.

Proof. A 1-cocycle $\sigma$ can be written in the form

$$\sigma = \sum_{i=0}^{+\infty} \sum_{l=-\infty}^{+\infty} \sigma_{i,l} z^l u^i.$$ 

Since monomials containing nonnegative powers of $z$ are holomorphic in $U$, these are coboundaries, thus

$$\sigma \sim \sum_{i=0}^{+\infty} \sum_{l=-\infty}^{-1} \sigma_{i,l} z^l u^i,$$

where $\sim$ denotes cohomological equivalence. Changing coordinates, we obtain

$$T \sigma = z^2 \sum_{i=0}^{+\infty} \sum_{l=-\infty}^{-1} \sigma_{i,l} z^{l+2} u^i = \sum_{i=0}^{+\infty} \sum_{l=-\infty}^{-1} \sigma_{i,l} z^{l+2} u^i,$$

where terms satisfying $l + 2 \leq -1$ are holomorphic on $V$. Thus, the non-trivial terms on $H^1(Z_{(-1)}, \mathcal{O}(-2))$ are all those that have either $l = -2$ or $l = -1$. Hence

$$H^1(Z_{(-1)}, \mathcal{O}(-2)) = \langle z^l u^i : l = -2, -1, \ i \geq 1 \rangle.$$

□

Proposition 4.2. The bundle $E$ over $Z_{(-1)}$ defined in canonical coordinates by the matrix

$$(4.1) \quad \begin{bmatrix} z^1 & z^{-1} e^u \\ 0 & z^{-1} \end{bmatrix}$$

is holomorphic but not algebraic.
Proof. This bundle $E$ can be represented by the element
\[ z^{-1} e^u \in \text{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) \cong H^1(Z_{(-1)}, \mathcal{O}(-2)). \]
We have
\begin{equation}
(4.2) \quad \begin{bmatrix} z^1 & z^{-1} e^u \\ 0 & z^{-1} \end{bmatrix} = \begin{bmatrix} z^1 \sigma \\ 0 \ z^{-1} \end{bmatrix}
\end{equation}
with $z^{-2} e^u = \sigma \in H^1(Z_{(-1)}, \mathcal{O}(-2))$, see [Har, p. 234]. Observe that
\[ z^{-2} e^u = z^{-2} \left( 1 + u + \frac{u^2}{2} + \cdots + \frac{u^n}{n!} + \cdots \right) = z^{-2} + z^{-2} \left( \underbrace{u + \frac{u^2}{2} + \frac{u^3}{6} + \cdots + \frac{u^n}{n!} + \cdots}_{\gamma} \right), \]
where the monomials in $\gamma \in \langle z^l u^i : l = -2, -1, i \geq 1 \rangle$ represent pairwise distinct nontrivial classes in $H^1(Z_{(-1)}, \mathcal{O}(-2))$ as shown in Lemma 4.1. Consequently, the class $z \sigma \in \text{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1))$ corresponding to the bundle $E$ cannot be represented by a polynomial, hence $E$ is holomorphic but not algebraic. \hfill $\square$

Corollary 4.3. The threefold $W_3$ has holomorphic bundles that are not algebraic.

Proof. Consider the map $p: W_3 \to Z_{(-1)}$ given by projection on the first and third coordinates, that is, in canonical coordinates as in (8.1) we see $Z_{(-1)}$ as cut out inside $W_3$ by the equation $u_1 = 0$. Then the pullback bundle $p^* E$ is holomorphic but not algebraic on $W_3$. \hfill $\square$

4.1.1. A similar bundle on $Z_1$. It is instructive to verify the result of defining a bundle by the same matrix, but over the surface $Z_1$ instead. Recall that $Z_1 = U \cup V$, with change of coordinates given by:

\[ (\xi, v) \mapsto (z^{-1}, z u) \]

Consider the bundle $E$ on $Z_1$, given by transition matrix
\begin{equation}
(4.3) \quad \begin{bmatrix} z^1 & z^{-1} e^u \\ 0 & z^{-1} \end{bmatrix}.
\end{equation}

Note that this is the same matrix used in (4.1). Thus $E$ corresponds to the element $z^{-1} e^u \in \text{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) = H^1(Z_1, \mathcal{O}(-2))$. Consequently, we may rewrite the transition function
\begin{equation}
(4.4) \quad \begin{bmatrix} z^1 & z^{-1} e^u \\ 0 & z^{-1} \end{bmatrix} = \begin{bmatrix} z^1 \sigma \\ 0 \ z^{-1} \end{bmatrix}
\end{equation}
where $z^{-2} u = \sigma \in H^1(Z_1, \mathcal{O}(-2))$. But $\sigma = \xi^3 \nu$ is holomorphic on the $V$ chart, and hence a coboundary. Thus $\sigma = 0 \in H^1(Z_1, \mathcal{O}(-2))$, and accordingly $z^{-1} e^u = 0 \in \text{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1))$. Therefore the extension splits and
\[ E = \mathcal{O}(-1) \oplus \mathcal{O}(1). \]
4.2. **Deformations of** $Z_k$. [BG, Thm. 5.3] construct a $(k - 1)$-dimensional semiuniversal deformation space $\mathcal{Z}$ for $Z_k$ given by

\begin{equation}
(\xi, v, t_1, \ldots, t_{k-1}) = (z^{-1}, z^k u + t_{k-1} z^{k-1} + \cdots + t_1 z, t_1, \ldots, t_k).
\end{equation}

**Lemma 4.4.** Deformations of $Z_k$ can be obtained from deformations of $\mathbb{F}_k$. Thus, the family $\mathcal{Z}$ is not universal.

**Proof.** We compare deformations of the surfaces $Z_k$ with those of the Hirzebruch surfaces. Let us first rewrite them as homogeneous manifolds. The surface $Z_k = \text{Tot}(\mathcal{O}(-k))$ can also be written as the quotient

$$Z_k = \frac{(\mathbb{C}^2 - \{0\}) \times \mathbb{C}}{\mathbb{C} - \{0\}}.$$ 

where the action is given by

$$(l_0, l_1, t) \sim (\lambda l_0, \lambda l_1, \lambda^{-k} t),$$

with $\lambda \in \mathbb{C} - \{0\}$. For $k \in \mathbb{Z}_+$, the Hirzebruch surface $\mathbb{F}_k$ can also be written as the quotient

$$\mathbb{F}_k = \frac{(\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^2 - \{0\})}{(\mathbb{C} - \{0\}) \times (\mathbb{C} - \{0\})}.$$ 

where the action is given by

$$(l_0, l_1, t_0, t_1) \sim (\lambda l_0, \lambda l_1, \lambda^k \mu t_0, \mu t_1),$$

with $\lambda, \mu \in \mathbb{C} - \{0\}$. Choose coordinates $(t_1, \ldots, t_{k-1}, [l_0, l_1], [x_0, \ldots, x_{k+1}])$ for the product $\mathbb{C}_t^{k-1} \times \mathbb{P}_1 \times \mathbb{P}_x^{k+1}$. [M, Chap. II] shows that the Hirzebruch surface $\mathbb{F}_k$ has a $(k - 1)$-dimensional semiuniversal deformation space given by the smooth subvariety $M \subset \mathbb{C}_t^{k-1} \times \mathbb{P}_1 \times \mathbb{P}_x^{k+1}$ cut out by the equations

\begin{equation}
l_0(x_1, x_2, \ldots, x_k) = l_1(x_2 - t_1 x_0, \ldots, x_k - t_{k-1} x_0, x_{k+1}).
\end{equation}

Let $\mathcal{Z}$ and $M$ denote the deformations given by 4.5 and 4.6, respectively. Now consider the following map:

$$f : \mathcal{Z} \rightarrow M$$

$$(z, u, t_1, \ldots, t_{k-1}) \mapsto (t_1, \ldots, t_{k-1}, [1, z], [-1, z_1, \ldots, z_k, u])$$

$$(\xi, v, t_1, \ldots, t_{k-1}) \mapsto (t_1, \ldots, t_{k-1}, [\xi, 1], [-1, v_1, v_2, \ldots, v_{k+1}]).$$

where we used the following notation:

\begin{align*}
z_1 &= z^k u + t_{k-1} z^{k-1} + \cdots + t_1 z & \xi_2 &= \xi v - t_1 \\
z_2 &= z^{k-1} u + t_{k-1} z^{k-2} + \cdots + t_2 z & \xi_3 &= \xi^2 v - t_1 \xi - t_2 \\
&\vdots & \vdots \\
z_{k-1} &= z^2 u + t_{k-1} z & \xi_k &= \xi^{k-1} v - t_1 \xi^{k-2} - \cdots - t_{k-1} \\
z_k &= z u & \xi_{k+1} &= \xi^k v - t_1 \xi^{k-1} - \cdots - t_{k-1} \xi
It turns out that this map is injective and satisfies $f(Z_t) \subset M_t$ for all $t \in \mathbb{C}^{k-1}$. Notice that, for each $t \in \mathbb{C}^{k-1}$, we can decompose $M_t$ as

$$M_t = A_t \cup B_t,$$

where $A_t = \{ p \in M_t, x_0 = 0 \}$ and $B_t = \{ p \in M_t, x_0 \neq 0 \}$. It then follows that

- $B_t = f(Z_t)$, and
- $A_t$ is the boundary of $B_t$,

implying as a corollary that: $M_t = M_{t'}$ if and only if $Z_t = Z_{t'}$.

So we conclude that each $Z_k$ has as many deformations as $F_k$, specifically, $\lfloor k/2 \rfloor$. In particular, the deformation family of $Z_k$ is not universal.

\[\square\]

5. \textbf{THE THREEFOLDS $W_k$ AND THEIR MODULI OF VECTOR BUNDLES}

The threefolds $W_k = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k)) \oplus \mathcal{O}_{\mathbb{P}^1}(k-2)$ can be given canonical coordinate charts as follows.

\textbf{Notation 5.1.} We fix once and for all coordinate charts on $W_k$, to which we will refer as \textit{canonical coordinates},

\begin{equation}
U = \mathbb{C}^3 = \{(z, u_1, u_2)\} \quad \text{and} \quad V = \mathbb{C}^3 = \{ (\xi, v_1, v_2) \},
\end{equation}

such that on the intersection $U \cap V = \mathbb{C} - \{0\} \times \mathbb{C} \times \mathbb{C}$ they satisfy

\begin{equation}
(\xi, v_1, v_2) = (z^{-1}, z^k u_1, z^{2-k} u_2).
\end{equation}

\textbf{Definition 5.2.} Let $E$ be a holomorphic rank-$r$ vector bundle on $W_k$ (or $Z_k$), and consider the restriction of $E$ to the distinguished line $\mathbb{P}^1 \subset W_k$ (or $\mathbb{P}^1 \subset Z_k$). By Grothendieck’s splitting principle there are integers $a_i$ such that $E|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$. We call $(a_1, \cdots, a_r)$ the \textit{splitting type} of $E$.

Köppe studied moduli of \textit{algebraic} rank-2 vector bundles on $W_k$ for $k = 1, 2, 3$. The variety formed by vector bundles whose extension class is nontrivial on the first infinitesimal neighbourhood of the $\mathbb{P}^1$ forms what can be regarded as the generic part of the moduli space $\mathcal{M}_j(W_k)$ of bundles on $W_k$ with splitting type $(-j, j)$.

\textbf{Theorem 5.3.} [K, Prop. 3.20] For $k = 1, 2, 3$, the generic part of the moduli of algebraic bundles $\mathcal{M}_j(W_k)$ is smooth of dimension $4j - 5$.

We observe that the cases of moduli of algebraic bundles on $W_k$ for $k > 3$ have not been described in the literature, but it seems most likely that they present a similar behaviour as the case $k = 3$ with the same dimension for the generic part of the moduli of rank-2 algebraic bundles. Thus, the generic part of these moduli of vector bundles does not provide any tool for distinguishing these threefolds from one another. We will see that the situation is quite the opposite with respect to their deformation theory. The situation changes a bit when we consider holomorphic bundles. We have:
Theorem 5.4. [K, Thm. 3.10] Holomorphic bundles on $W_1$ are filtrable and algebraic.

Corollary 5.5. Moduli spaces of holomorphic bundles on $W_1$ are finite-dimensional.

Theorem 5.6. $W_2$ has infinite-dimensional moduli of holomorphic bundles.

Proof. For brevity we give just an example. Consider the moduli space that contains the tangent bundle of $W_2$. The Zariski tangent space of this moduli space at $TW_2$ is given by the cohomology $H^1(W_2, \text{End}(TW_2))$, which is infinite-dimensional. Indeed, Čech cohomology calculations show that $H^1(W_2, \text{End}(TW_2))$ is generated as a $\mathbb{C}$-vector space by the following cocycles:

$$(0, \ldots, 0, z^{-1}u_1 u_2^k, 0, \ldots, 0), (0, \ldots, 0, z^{-i}u_2^k, 0, \ldots, 0) \text{ for } i = 1, 2, 3,$$

$$(0, \ldots, 0, z^{-1}u_2^k, 0, \ldots, 0), (0, \ldots, 0, z^{-1}u_2^k, 0, \ldots, 0) \text{ for } k \geq 0.$$  

□

6. RIGIDITY OF $W_1$

Theorem 6.1. [R] $W_1$ is rigid, that is, its complex structure has no deformations.

Proof. Deformations of complex structures are parametrised by first cohomology with coefficients in the tangent bundle. Direct calculation of Čech cohomology shows that $H^1(W_1, TW_1) = 0$. □

7. DEFORMATIONS OF $W_2$

Theorem 7.1. [R] $W_2$ has an infinite-dimensional family of deformations.

Proof. The proof will follow from Lemmas 7.2 and 7.3 below. First we show that the first cohomology with tangent coefficients is infinite-dimensional. Then we show that its cocycles are integrable, and thus they parametise deformations of $W_2$. □

Lemma 7.2. $H^1(W_2, TW_2)$ is generated as a vector space over $\mathbb{C}$ by cocycles of the form $(0, z^{-1}u_2^j, 0), j \geq 0$ (written in canonical coordinates).

Proof. Recall that $W_2$ can be covered by

$$U = \{(z, u_1, u_2)\} \quad \text{and} \quad V = \{ (\xi, v_1, v_2) \},$$

with $U \cap V = \mathbb{C} - \{0\} \times \mathbb{C} \times \mathbb{C}$ and transition function given by:

$$((\xi, v_1, v_2)) = (z^{-1}, z^2u_1, u_2).$$
We have then that the transition function for $TW_2$ is

$$A = \begin{bmatrix} -z^{-2} & 0 & 0 \\ 2zu_1 & z^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $\sigma$ be a 1-cocycle, i.e. a holomorphic function on $U \cap V$:

$$\sigma = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} \begin{bmatrix} a_{ij} \\ b_{ij} \\ c_{ij} \end{bmatrix} z^l u_i u_j.$$

But

$$\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \begin{bmatrix} a_{ij} \\ b_{ij} \\ c_{ij} \end{bmatrix} z^l u_i u_j$$

is a coboundary, so

$$\sigma \sim \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} \begin{bmatrix} a_{ij} \\ b_{ij} \\ c_{ij} \end{bmatrix} z^l u_i u_j = \sigma',$$

where $\sim$ denotes cohomological equivalence. So

$$A\sigma' = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} \begin{bmatrix} -a_{ij}z^{-2} \\ 2a_{ij}z u_1 + b_{ij}z^2 \\ c_{ij} \end{bmatrix} z^l u_i u_j$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} \begin{bmatrix} -a_{ij}z^{-4} \\ 2a_{ij}z^3(u_1^2 + b_{ij}z^2) \\ c_{ij}z^{-2} \end{bmatrix} z^{2i-l-2} u_1^i u_2^j$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} \begin{bmatrix} -a_{ij}\xi^4 \\ 2a_{ij}\xi^3 v_1 + b_{ij} \xi^2 \\ c_{ij}\xi^2 \end{bmatrix} \xi^{2i-l-2} v_1^i v_2^j.$$

Except for the case where $l = -1$ and $i = 0$, we have that $2i - l - 2 \geq 0$, thus the corresponding monomials are holomorphic in $V$ and hence coboundaries. It follows that

$$A\sigma' \sim \sum_{j=0}^{\infty} \begin{bmatrix} -a_j\xi^4 \\ 2a_j\xi^3 v_1 + b_j \\ c_j\xi^2 \end{bmatrix} \xi^{-1} v_2^j$$

$$\sim \sum_{j=0}^{\infty} \begin{bmatrix} 0 \\ b_j \\ 0 \end{bmatrix} \xi^{-1} v_2^j,$$
where we omit the indices \(-1\) for \(l\) and \(0\) for \(i\) for simplicity. We conclude then that \(H^1(W_2, TW_2)\) is infinite-dimensional, generated by the sections

\[
\sigma_j = \begin{bmatrix}
0 \\
z^{-1}u^j \\
0
\end{bmatrix}
\]

for \(j \geq 0\).

\[\square\]

**Lemma 7.3.** All cocycles in \(H^1(W_2, TW_2)\) are integrable.

**Proof.** We can write the transition of \(W_2\) as:

\[
\begin{bmatrix}
\xi \\
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix}
z^{-1} \\
z^2u_1 \\
0
\end{bmatrix} = \begin{bmatrix}
z^{-2} & 0 & 0 \\
0 & z^2 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
z \\
u_1 \\
u_2
\end{bmatrix}.
\]

As we computed in Lemma 7.2, \(H^1(W_2, TW_2)\) is generated by the sections

\[
\begin{bmatrix}
0 \\
z^{-1}u^j \\
0
\end{bmatrix}
\]

for \(j \geq 0\). Then we can express the deformation family for \(W_2\) as

\[
\begin{bmatrix}
\xi \\
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix}
z^{-2} & 0 & 0 \\
0 & z^2 & 0 \\
0 & 0 & 1
\end{bmatrix} \left( \begin{bmatrix}
z \\
u_1 \\
u_2
\end{bmatrix} + \sum_{j \geq 0} t_j \begin{bmatrix}
z^{-1}u^j \\
0
\end{bmatrix} \right)
\]

i.e. we have an infinite-dimensional deformation family given by

\[
U = \mathbb{C}^3_{z, u_1, u_2} \times \mathbb{C}[t_j] \quad \text{and} \quad V = \mathbb{C}^3_{\xi, v_1, v_2} \times \mathbb{C}[t_j]
\]

with

\[
(\xi, v_1, v_2, t_0, t_1, \ldots) = (z^{-1}, z^2u_1 + \sum_{j \geq 0} t_jzu^j_2, u_2, t_0, t_1, \ldots)
\]

on the intersection \(U \cap V = (\mathbb{C} - \{0\}) \times \mathbb{C} \times \mathbb{C}[t_j].\)

\[\square\]

**7.1. A non-affine deformation.** The proof of 7.3 gives us that deformations of \(W_2\) are threefolds given by change of coordinates of the form

\[
(\xi, v_1, v_2) = (z^{-1}, z^2u_1 + \sum_{j \geq 0} t_jzu^j_2, u_2).
\]

We consider now the example \(W_2\) that occurs when \(t_1 = 1\) and all \(t_j\) vanish for \(j \neq 1\), that is, the one with change of coordinates

\[
(\xi, v_1, v_2) = (z^{-1}, z^2u_1 + zu_2, u_2).
\]

**Lemma 7.4.** \(H^1(W_2, O(-4)) \neq 0\).
Proof. Consider the 1-cocycle $\sigma$ written in the $U$ coordinate chart as $\sigma = z^{-1}$. Suppose $\sigma$ is a coboundary, then we must have

$$\sigma = \alpha + T^{-1}\beta$$

where $\alpha \in \Gamma(U)$ and $\beta = \Gamma(V)$. Consequently

$$z^{-1} = \alpha(z, u_1, u_2) + z^{-4}\beta(z^{-1}, z^2u_1 + zu_2, u_2).$$

But $\alpha$ has only positive powers of $z$, and the highest power of $z$ appearing on $z^{-4}\beta$ is $-4$, hence the right-hand side has no terms in $z^{-1}$ and the equation is impossible, a contradiction. \qed

Corollary 7.5. $W_2$ is not affine.

Remark 7.6. Note that this result contrasts with the situation for surfaces, since [BG, Thm. 6.15] prove that all nontrivial deformations of $Z_k$ are affine.

8. Deformations of $W_3$

We start by computing the group $H^1(W_3, TW_3)$ which parametrises deformations of $W_3$. Recall that $W_3$ can be covered by $U = \{(z, u_1, u_2)\}$ and $V = \{(\xi, v_1, v_2)\}$, with $U \cap V = C - \{0\} \times C^2$ and transition function given by:

$$\begin{aligned}
(\xi, v_1, v_2) = (z^{-1}, z^3u_1, z^{-1}u_2)
\end{aligned}$$

Theorem 8.1. There is a versal deformation space $\mathcal{W}$ for $W_3$ parametrised by cocycles of the form

$$\begin{bmatrix}
a_{ij} \\
b_{ij} \\
c_{ij}
\end{bmatrix} z^l u_1^i u_2^j \quad 3i - 3 - l - j < 0.$$

Proof. In canonical coordinates, the transition matrix for the tangent bundle $TW_3$ is given by

$$T = \begin{bmatrix}
-z^{-2} & 0 & 0 \\
3z^2u_1 & z^3 & 0 \\
-z^{-2}u_2 & 0 & z^{-1}
\end{bmatrix} \cong \begin{bmatrix}
z^{-1} & 0 & -z^{-2}u_2 \\
0 & z^3 & 3z^2u_1 \\
0 & 0 & -z^{-2}
\end{bmatrix},$$

where $\cong$ denotes isomorphism, and the latter expression is handier for calculations. A 1-cocycle can be expressed in $U$ coordinates in the form

$$\begin{aligned}
\sigma &= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} \begin{bmatrix}
a_{ij} \\
b_{ij} \\
c_{ij}
\end{bmatrix} z^l u_1^i u_2^j \\
&\sim \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=-\infty}^{\infty} \begin{bmatrix}
a_{ij} \\
b_{ij} \\
c_{ij}
\end{bmatrix} z^l u_1^i u_2^j,
\end{aligned}$$
where \( \sim \) denotes cohomological equivalence. Changing coordinates we obtain
\[
T\sigma = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \sum_{l=-\infty}^{-1} \left[ a_{ij}z^{-1} - c_{lij}z^{-2}u_2 \right] \begin{bmatrix}
3a_{ij}z^2u_1 + b_{lij}z^3 \\
-c_{lij}z^{-2}
\end{bmatrix} z^lu_1^i u_2^j
\]
where all terms inside the matrix are holomorphic on \( V \) except for
\[
\begin{bmatrix}
ob_{lij}z^3 \\
0
\end{bmatrix}.
\]
These impose the condition for a cocycle to be nontrivial. Since we have
\[
z^3z^l u_1^i u_2^j = z^{l+3-3i+j}(z^3u_1)^i (z^{-1}u_2)^j = \xi^{3i-3-l-j} u_1^i u_2^j,
\]
a nontrivial cocycle satisfies \( 3i - 3 - l - j < 0 \).

We now give a partial description of deformations of \( W_3 \).

**Lemma 8.2.** The sections
\[
\sigma_1 = \begin{bmatrix}
0 \\
z^{-1} \\
0
\end{bmatrix} \quad \text{and} \quad \sigma_2 = \begin{bmatrix}
0 \\
z^{-2} \\
0
\end{bmatrix}
\]
are nonzero cocycles on \( H^1(W_3, TW_3) \).

**Proof.** Let
\[
\sigma_l = \begin{bmatrix}
o \\
z^{-l} \\
0
\end{bmatrix},
\]
for \( l = 1, 2 \). Then \( \sigma_l \) is not a coboundary on the chart \( U \). We change coordinates by multiplying by the transition \( T \) given in 8.2,
\[
T\sigma_l = \begin{bmatrix}
0 \\
z^{l+3} \\
0
\end{bmatrix} = \begin{bmatrix}
o \\
\xi^{-l-3} \\
0
\end{bmatrix},
\]
which is not holomorphic on the chart \( V \) and therefore not a coboundary.

**Lemma 8.3.** The following 2-parameter family of deformations of \( W_3 \) is contained in \( \mathcal{W} \):
\[
(\xi, v_1, v_2) = (z^{-1}, z^3 u_1 + t_2 z^2 + t_1 z, z^{-1} u_2)
\]

**Proof.** The transition for \( W_3 \) is given by,
\[
(\xi, v_1, v_2) = (z^{-1}, z^3 u_1, z^{-1} u_2).
\]
In matrix form:

$$
\begin{bmatrix}
\xi \\
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
z^{-2} & 0 & 0 \\
0 & z^3 & 0 \\
0 & 0 & z^{-1}
\end{bmatrix}
\begin{bmatrix}
z \\
u_1 \\
u_2
\end{bmatrix}
$$

So we can construct a deformation family for \( W_3 \) using the cocycles from Lemma 8.2:

$$
\begin{bmatrix}
\xi \\
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
z^{-2} & 0 & 0 \\
0 & z^3 & 0 \\
0 & 0 & z^{-1}
\end{bmatrix}
\begin{bmatrix}
z \\
u_1 \\
u_2
\end{bmatrix}
+ t_2
\begin{bmatrix}
0 \\
z^{-1} \\
0
\end{bmatrix}
+ t_1
\begin{bmatrix}
0 \\
z^{-2} \\
0
\end{bmatrix}
$$

Now it suffices to observe that, by Lemma 8.2, \( \sigma_1 \) and \( \sigma_2 \) are nontrivial directions in \( \mathcal{W} \).

\[ \square \]

**Corollary 8.4.** *The family presented in Theorem 8.3 is semiuniversal but not universal.*

**Proof.** As a consequence of Lemma 8.3 and Corollary 4.4, we have that the deformations in the directions of the cocycles of Lemma 8.2 are isomorphic. Indeed, these deformations are induced by \( Z_3 \) which, as \( \mathbb{F}_3 \), only has one nontrivial direction of deformation.

\[ \square \]

**Acknowledgements**

Results of this paper were presented by Gasparim and Suzuki in their talks at the Geometry and Physics session of the *V Congreso Latinoamericano de Matemáticas*. These authors thank UMALCA, Universidad del Norte and the Colombian Mathematical Society for the financial support and hospitality. Gasparim thanks also the Vice Rectoría de Investigación y Desarrollo tecnológico at Universidad Católica del Norte (Chile). Suzuki acknowledges support from the Beca Doctorado Nacional – Folio 21160257.

The authors thank Bernardo Uribe for the invitation to organise a session at the congress as well as for giving us the opportunity to submit our contribution to these proceedings.

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