Role of local response in manipulating the elastic properties of disordered solids by bond removal

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We explore the range over which the elasticity of disordered spring networks can be manipulated by the removal of selected bonds. By taking into account the local response of a bond, we demonstrate that the effectiveness of pruning can be improved so that auxetic (i.e., negative Poisson’s ratio) materials can be designed without the formation of cracks even while maintaining the global isotropy of the network. The bulk, $B$, and shear, $G$, moduli scale with the number of bonds removed and we estimate the exponents characterizing these power laws. We also find that there are spatial correlation lengths in the change of $B$ and $G$ upon removing different bonds that diverge as the network approaches the isostatic limit where the excess coordination number $\Delta Z \to 0$.

I. INTRODUCTION

Manipulating the elastic properties of solids is an important problem with broad applications $[1,2]$. The most common approach in designing mechanical metamaterials is based on a periodically repeating unit cell that is carefully constructed to yield a given elastic property or function. Recently, a novel design principle has been introduced based on “pruning” disordered spring networks $[3]$. This exploits the broad distribution of how different bonds contribute to the elastic moduli in such systems – by selectively removing a bond that contributes more to one modulus than to another, one can prune a system to achieve desired elastic properties. Disorder provides two clear advantages over periodic lattices: 1) disordered systems are isotropic on large scales; 2) disorder allows materials to be designed with inhomogeneous and even local responses $[4,7]$

To demonstrate the potential flexibility that pruning provides, consider the effect on an elastic modulus of the removal of a single bond in a spring network with $N_b$ bonds. The modulus could characterize the cost of a global deformation, such as compression, or a local deformation, such as the pinching together of two nodes. If the system is periodic, removing a bond in one unit cell will result in the same change in the elastic moduli as the removing the equivalent bond in any other unit cell. However, for a disordered network, removing different bonds leads to different responses. Naively, the removal of the first bond results in $N_b$ different possible responses while removing $N_r$ bonds leads to $N_b!/N_r!(N_b-N_r)!$ which even for small $N_r$ can be an enormous number.

To realize this large range of possible designs it is important for the system to obey two properties. First, removing a bond, $i$, affects different moduli differently in an uncorrelated way. Indeed this was shown to be the case for bulk deformations: removal of bond $i$ changes the bulk modulus by $\Delta B_i$ and the shear modulus by $\Delta G_i$, where $\Delta B_i$ and $\Delta G_i$ have nearly vanishing correlations $[3]$. Second, the change in moduli upon the removal of a bond must have a broad range. This is also the case for bulk deformations, where $\Delta B_i$ and $\Delta G_i$ at small values scale as a power-law $[8]$.

The systems we design are based on disordered networks derived from jammed packings $[9,10]$. Soft repulsive spherical particles are placed randomly in space and the energy is minimized to attain force balance. The centers of the spheres are then connected by springs to form a network and the equilibrium spring length is set to the equilibrium distance between nodes, thus removing all stresses. For simplicity, all the spring constants, $k$, are chosen to be equal. We characterize ensembles of such networks by the coordination number per node, $Z$, and the excess coordination number, $\Delta Z = Z - Z_c$, where $Z_c$ is the critical value of $Z$ at which rigidity is lost: in an infinite system, $Z_c = 2d$ $[11,13]$.

If a bond length, $r_i$, between two nodes is different from the equilibrium length, $r_i^0$, there is an energy cost of $\frac{1}{2} k \delta r_i$ and a tension $\tau_i = k \delta r_i$ where $\delta r_i \equiv r_i - r_i^0$. Since the networks are initially unstressed, compressing the system results in an energy $\frac{1}{2} B \epsilon_B^2$, where $B$ is the bulk modulus and $\epsilon_B$ is the compression strain; similarly the energy cost of a shear is $\frac{1}{2} G \epsilon_G^2$ where $G$ is the shear modulus and the $\epsilon_G$ is the shear strain. Since the energy

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is additive, \( U = \sum_i \frac{1}{2} k \delta^2 r_i \), this allows us to decompose the bulk and shear modulus into a sum over their single bond contributions: \( B = \sum_i B_i \) and \( G = \sum_i G_i \) where \( B_i \) and \( G_i \) are the contributions to \( B \) and \( G \) respectively of bond \( i \).

In an isotropic system, the Poisson’s ratio, \( \nu \), is a monotonically decreasing function of \( G/B \). In the method introduced by Ref. [3], bonds are pruned that target either \( G \) or \( B \) to yield the desired value of \( \nu \). To attain, for example, a large \( G/B \), two strategies may be envisioned. Bonds that contribute a large amount to the bulk modulus (large values of \( B_i \)) can be successively removed; due to the relatively weak correlations between \( B_i \) and \( G_i \), this results in only a moderate decrease in \( B \) but a large drop in \( B \). Alternatively, bonds that contribute little to \( G \) (small values of \( G_i \)) can be removed so that \( G \) does not change appreciably but again due to the weak correlations, \( B \) decreases more steeply. The first approach has the undesired effect of creating cracks [14] since the removed bonds carry a lot of stress under the deformation, which is distributed to its neighbors when removed.

We recently pointed out [8] that there is a difference between the contribution of a given bond \( i \) to the modulus, \( M_i \), and the change of the modulus if bond \( i \) is removed, \( \Delta M_i \). Clearly, the evolution of the \( M \) under pruning depends on the latter quantity. Ref. [3] uses \( B_i \) and \( G_i \) as proxies for predicting \( \Delta B_i \) and \( \Delta G_i \) in order to tune the values of \( B \) and \( G \). This approach was quite successful—it was found that a large \( G/B \) can be attained by removing the bonds with the largest \( B_i \). Similarly, a very small \( G/B \) can be attained by either removing the bonds with maximal \( G_i \) or minimal \( B_i \). However, removing the minimal \( G_i \) strategy fails and both \( B \) and \( G \) change in a correlated manner as the bonds are removed. Here, we show that this failure results from the use of \( G_i \) as a proxy for \( \Delta G_i \), and that when minimal \( \Delta G_i \) bonds are removed, one can obtain very large values of \( G/B \).

Moreover, consideration of \( \Delta M_i \) instead of \( M_i \) allows us to estimate theoretically the scaling exponents of \( G/B \) as a function of the number of bonds removed. We also report correlations in \( \Delta B_i \) and \( \Delta G_i \) as a function of the distance between bonds with correlation lengths \( \xi_{\Delta B} \) and \( \xi_{\Delta G} \) respectively. These correlation lengths diverge as power laws as the network connectivity decreases towards the rigidity threshold \( \Delta Z = 0 \).

II. TUNING \( G/B \)

Here we tune \( G/B \) by removing bonds based on \( \Delta B_i \) and \( \Delta G_i \). We explore different pruning strategies that target bonds that either have a maximal or minimal value of \( \Delta B_i \) or \( \Delta G_i \).

The quantities \( M_i \) and \( \Delta M_i \) are related via a linear-response relation [8]:

\[
\Delta M_i = M_i/S_i^2, \tag{1}
\]

where \( kS_i^2 \) is local modulus characterizing the cost of a change in the equilibrium length of bond \( i \).

To employ Eq. 1 to tune \( G/B \), one needs to compute \( B_i \), \( G_i \) and \( S_i^2 \) for each bond. Evaluation of \( S_i^2 \) for each bond requires \( N_i \) calculations, however this is done only once for the initial unpruned network. Thereafter, the evolution of the spring network is efficiently computed using methods described in Appendix A. We note that there are \( \frac{d(d+1)}{2} \) independent shear moduli in \( d \) dimensions, which are denoted as \( G^{(j)} \). Pruning based on their average, denoted by \( G = \frac{1}{\max_{i=1}^N} \sum_j G^{(j)} \), allows tuning of \( G/B \) in a manner that leaves the system isotropic. We note that \( \Delta G_i \) is a linear average over all \( \Delta G^{(j)}_i \), which will be important for determining its distribution.

In Fig. 1 the evolution of \( G/B \) is shown for different pruning strategies in which we remove bonds with: \( \max \Delta B_i \), \( \min \Delta B_i \), \( \max \Delta G_i \) or \( \min \Delta G_i \). Results in dimensions \( d = 2 \) and \( d = 3 \) are shown. The number of removed bonds is characterized by \( \Delta Z = \Delta Z_0 - 2N_r/N \) where \( \Delta Z_0 \) is the initial excess coordination number and \( N_r \) are the number of bonds removed. The pruning procedures based on \( \min \Delta B_i \) and \( \max \Delta G_i \) result in a very small \( G/B \) ratio while those based on \( \max \Delta B_i \) and \( \min \Delta G_i \) result in a very large \( G/B \). We emphasize that only a few percent of bonds are removed, yet the change in \( G/B \) can be almost ten orders of magnitude.

![Figure 1. The evolution of \( G/B \) as a function of \( \Delta Z \) which signifies the number of bonds removed (a) in two dimensions. (b) in three dimensions.](image)
Table I. A comparison of the exponents, defined by \( \frac{G}{B} \propto \Delta Z^\alpha \), when bonds are selected based on \( \Delta B_i \) and \( \Delta G_i \) values versus when they are selected by their \( B_i \) and \( G_i \) values, taken from Ref. [3]. The subscript of \( \mu \) designates the targeted modulus while \(+\) \((-\) marks the maximal (minimal) values targeted.

| pruning method | \( 2D \) | \( 3D \) | \( 2D \) | \( 2D \) | \( 3D \) | \( 3D \) | \( 3D \) |
|----------------|---------|---------|---------|---------|---------|---------|---------|
| \( B_i, G_i \) | -5.36   | 1.27    | 3.05    | -7.90   | 1.01    | 1.82    | -        |
| \( \Delta B_i, \Delta G_i \) | -10.5   | 1.0     | 5.5     | -1.3    | -11.3   | 1.0     | 3.1     |
| \( -0.95 \) |

Equation [1] explains why the procedure based on removing the bond with \( \text{min } G_i \) was unsuccessful. Because of the appearance of \( S_i^2 \) in the denominator, a bond with a small \( G_i \) does not in general have a small \( \Delta G_i \). Reference [8] shows that \( G_i \propto S_i^2 \) at small \( S_i^2 \). Thus bonds that seem unimportant and carry little stress may in fact be important; their removal can vary \( G \) significantly.

To highlight the difference between \( \Delta G_i \) and \( G_i \), we now consider their distributions. Ref. [3] shows that the distribution of \( G_i \) at small values scales as a power-law, which to a good approximation in three dimensions is given by, \( G_i \propto -0.38 \). This suggests that there are many bonds which can be removed with little change to the shear modulus. We now argue that this is not the case, as will be inferred from the \( \Delta G_i \) distribution.

To compute the distribution of \( \Delta G_i \) we employ the analysis of Ref. [8] which studied the distribution of \( \Delta G_i^{(j)} \) for any shear direction. It was shown numerically, and supported by theoretical arguments that to a good approximation:

\[
P \left( \frac{\Delta G_i^{(j)}}{\langle \Delta G_i \rangle} = y \right) = \frac{1}{\sqrt{2\pi} y^{\frac{3}{2}}} e^{-\frac{y}{2}}. \tag{2}
\]

The distribution of \( \Delta G_i \) is then given by the sum over \( n = \frac{d(d+1)}{2} - 1 \) different shear directions, \( \Delta G_i^{(j)} \). Assuming that these are independent, the distribution of \( \Delta G_i \) is computed in [6] and found to be a Gamma distribution:

\[
P \left( \frac{\Delta G_i}{\langle \Delta G_i \rangle} = y \right) = \left( \frac{n}{2} \right)^{n/2} \frac{1}{\Gamma \left( \frac{n}{2} \right)} y^{\frac{n}{2} -1} e^{-\frac{ny}{2}}, \tag{3}
\]

where \( \Gamma \left( \frac{n}{2} \right) \) is the Gamma function. The important observation is that at small values \( P \left( \Delta G_i \right) \propto \Delta G_i^{\frac{d(d+1)}{4} - \frac{2}{3}} \), so that in three dimensions \( P \left( \Delta G_i \right) \propto \Delta G_i^{2.6} \) in contrast to the \( P \left( G_i \right) \propto G_i^{-0.38} \) which has the opposite sign in the exponent. Thus, most bonds thought to be unimportant based on their \( G_i \) actually lead to a substantial decrease in \( G \).

Figure 2. The distribution of \( \Delta B_i \), \( \Delta G_i \) and \( \Delta G_i^{(j)} \) in both two and three dimensions. The dashed curves overlaying \( P \left( \Delta G_i^{(j)} \right) \) is the prediction from Ref. [8]: \( \frac{1}{\sqrt{2\pi}} y^{\frac{3}{2}} e^{-\frac{2}{3}}. \) The dashed curves overlaying \( P \left( \Delta G_i \right) \) in two and three dimension are the prediction based on Eq. [3].

III. ESTIMATING THE EXPONENTS

To understand why the curves of \( G/B \) versus \( \Delta Z \) look like approximate power laws for the four different pruning strategies, we approximate Eq. [1] as a differential equation:

\[
dM = -\alpha \frac{M}{\Delta Z}, \tag{4}
\]

where \( d \left( \Delta Z \right) = \frac{\tau_i^2}{\bar{\tau}} \) is the change in the coordination number when a bond is pruned and

\[
\alpha = \Delta Z \frac{N}{2} \frac{\tau_i^2}{\bar{\tau}^2}. \tag{5}
\]

Here \( \tau_i^2 = M_i/M \) is proportional to the energy on bond \( i \), but normalized so that \( \sum_i \tau_i^2 = 1 \). Typically, the stresses due to a global deformation are not localized and therefore \( \tau_i^2 \sim \frac{1}{N} \). We also note that \( [8, 15] S_i^2 \propto \Delta Z \). For these two reasons, \( \alpha \) should not depend on the system size or \( \Delta Z \). If \( \alpha \) is constant then the solution to this equation is \( M \propto \Delta Z^{-\alpha} \). If bonds are chosen with some specific rule and the distribution of \( \alpha \) is stationary, then its average remains constant as the system is pruned. In this case, \( M \propto \Delta Z^{-\alpha} \), where \( \bar{\tau} \) denotes the average of \( \alpha \) and depends on the pruning procedure.

In Ref. [8] the distributions of \( \Delta B_i \) and \( \Delta G_i^{(j)} \) were measured for different pruning procedures and it was shown that for different pruning strategies the distribution of \( \Delta G_i^{(j)} \) is universal and is given by Eq. [2]. While the starting distribution of \( \Delta B_i \) is initially different, it evolves to this universal distribution for all pruning strategies discussed here except for the case when the
bonds with \( \min \Delta B_i \) are targeted. Based on the definition of \( \alpha \) in Eq. \[ \alpha \propto \Delta M_i, \] and therefore \( \alpha \) will have the same distribution

\[
P(\alpha) = \frac{1}{\sqrt{2\pi \alpha_0}} \alpha^{-1/2} \exp \left(-\frac{\alpha}{2\alpha_0}\right).
\] (6)

The only free parameter is \( \alpha_0 \) and it can be evaluated by noting that if bonds are removed randomly, \( \bar{\alpha} = 1 \). By requiring that \( \int d\alpha P(\alpha) = 1 \) we find that \( \alpha_0 = 1 \).

We begin by considering the exponents associated with pruning the \( \min \Delta M_i \). Numerically, it is found that in all cases the change \( \Delta M \) is very small. Due to the weak correlation between \( \Delta B_i \) and \( \Delta G_i \), pruning the \( \min \Delta B_i \) (\( \min \Delta G_i \)) results in the decrease in \( G(\Delta) \) as if a random bond is removed. Therefore, \( G \propto \Delta Z \approx 1.0 \) when the \( \min \Delta B_i \) bonds are removed and \( B \propto \Delta Z \approx 1.0 \) when the \( \min \Delta G_i \) bonds are removed.

The case of pruning \( \max \Delta M_i \) is very different. In general, these bonds carry a lot of stress which is then redistributed upon their removal. Ref. [14] shows that these stresses are redistributed on the length scale of \( \zeta \) which diverges in the limit of \( \Delta Z \rightarrow 0 \). Therefore, if the system size is smaller than \( \zeta \), then bonds are removed approximately homogeneously throughout the system, while for systems larger than \( \zeta \) a system spanning crack forms. All our data and analysis concerns the first, homogeneous case.

We begin by estimating the exponent associated with the change of the bulk modulus \( B \propto \Delta Z \bar{\alpha}_{\max \Delta B_i} \) for the \( \max \Delta B_i \) procedure. This requires a calculation of the average maximal value of \( \alpha \), which depends on the number of independent bonds. Assuming \( N \) independent random variables, the distribution of \( \alpha_{\max} \) is given by:

\[
Q(\alpha_{\max}) = N \left( \int_0^{\alpha_{\max}} dy P(y) \right)^{N-1} P(\alpha_{\max}).
\] (7)

Its average can be estimated numerically, yielding \( \bar{\alpha}_{\max \Delta B_i} \approx 14.6 \) for \( N = 4000 \) for comparison to simulations. This is somewhat greater than the measured value of 11.5 in two dimensions and 12.3 in three dimensions.

A similar estimate can be found for the case of pruning \( \max \Delta G_i \). In two dimensions the distribution of \( \alpha \) is then given by \( P(\alpha) = \exp (\alpha) \), where the average is chosen to be unity. The distribution \( Q(\alpha_{\max}) \) can be computed analytically

\[
Q(\alpha_{\max}) = N \left( 1 - e^{-\alpha_{\max}} \right)^{N-1} e^{-\alpha_{\max}} \approx Ne^{-\alpha_{\max}} e^{-\alpha_{\max}}.
\] (8)

where the asymptotic form is the Gumbel distribution, with \( \bar{\alpha}_{\max} \approx \log N + \gamma \), and \( \gamma \approx 0.5772 \) is the Euler–Mascheroni constant. This an instance of extreme value theory which predicts that when \( N \) is large, \( Q(\alpha_{\max}) \) will always be given by a Gumbel distribution as long as \( P(\alpha) \) decays fast enough [16]. In 2d the shear modulus should therefore scale as \( G \propto \Delta Z \bar{\alpha}_{\max \Delta G_i} \), with the exponent \( \bar{\alpha}_{\max \Delta G_i} \approx 8.87 \) (for \( N = 4000 \) as used in simulations). This is the same order but 36% larger than the value \( \approx 6.5 \) measured numerically. A similar analysis in three dimensions yields \( \bar{\alpha}_{\max \Delta G_i} \approx 5.0 \) which is also 22% greater than the value of 4.1 found numerically.

Since in all cases the estimated exponents are larger than those found numerically we test the validity of our assumptions. First, we consider the distribution of \( \alpha \) and check if it is indeed stationary, focusing on the large \( \alpha \) values crucial for the \( \max \Delta M_i \) pruning. To probe the tail of \( P(\alpha) \) we measure \( \bar{\alpha}_{\max} \) as a function of bonds removed. This is found to be a constant, after several bonds are removed.

Interestingly, the \( \bar{\alpha}_{\max} \) is found to be smaller than that predicted from Eq. \[ \gamma \] We note that \( \bar{\alpha}_{\max} \) has a logarithmic dependence on the system size, \( N \), since it samples the exponential tail of \( P(\alpha) \). Indeed, we find numerically that the \( \bar{\alpha}_{\max \Delta B_i} \) grows slowly when the system size is increased. However, contrary to our assumption not all bonds are independent, and as demonstrated in Section [IV] there are significant spatial correlations in the system. We believe these correlation reduce \( \bar{\alpha}_{\max} \), since they re-normalize \( N \) to a smaller value. Naively, if the spatial correlations are the dominant contribution then the independent number of bonds should scale as \( (N/\xi^d f_\xi) \), where \( \xi \) is the correlation length associated with \( \Delta B_i \) or \( \Delta G_i \) and \( f_\xi \) are the number of independent bonds within \( \xi^d \).

We conclude by noting that despite the system size dependence there are features that are universal, stemming from the \( \alpha \) distribution being independent of the pruning protocol. This is demonstrated by comparing pruning \( \max \Delta B_i \) to \( \max \Delta G_i \), where \( G^{(1)} \) is the simple shear modulus. Since these share the same \( P(\alpha) \), we expect that \( B(\Delta Z) \) should have the same behavior as \( G^{(1)}(\Delta Z) \) after several bonds are removed. Indeed, in Fig. [9] these two curves shown to be almost parallel in the small \( \Delta Z \) regime.
the cutting length \( \ell^* \propto \Delta Z^{-1} \) and the shear modulus with the transverse length scale \( \ell^t \propto \Delta Z^{-\frac{2}{5}} \).

**IV. SPATIAL CORRELATIONS**

In this section we discuss spatial correlations of \( \Delta B_i \) and \( \Delta G_i \). This is most easily done in Fourier space, for which we define:

\[
\langle |\Delta M(q)|^2 \rangle = \frac{1}{N\sigma^2_M} \left| \sum_i \Delta M_i e^{-iqr} \right|^2
\]

where \( \sigma^2_M \) is the variance of \( \Delta M_i \) which is a convenient normalization. The average is performed over all directions in \( q \) space and over about 10 realizations of disordered unpruned networks. Aside from normalization, this is the Fourier transform of the correlation function \( \langle \Delta M(r) \Delta M(0) \rangle - \langle \Delta M \rangle^2 \). Since identifying growing length scales requires systems with a large linear dimension, we focus on two-dimensional systems.

Figure 4 shows \( \langle |\Delta B(q)|^2 \rangle \) and \( \langle |\Delta G(q)|^2 \rangle \) in two dimensions for different values of \( \Delta Z \). There are two regimes. At large values of \( q \) both quantities vary as \( q^{-\gamma} \) with \( \gamma_{\Delta B} \approx 1.5 \) for the bulk modulus and \( \gamma_{\Delta G} \approx 1.25 \) for the shear modulus. This indicates spatial power-law correlations which scale as, \( r^{-d+\gamma} \) for \( r \) smaller than a correlation length, \( \xi \). At small \( q \) there is a crossover to a constant which signals a transition to an uncorrelated state. The crossover value of \( q \) is identified as the inverse correlation length and its \( \Delta Z \) dependence is found by a data collapse where the two axes are scaled with a power of \( \Delta Z \). We find that the \( \Delta B_i \) correlation length is given by \( \xi_{\Delta B} \propto \Delta Z^{-0.5 \pm 0.15} \) while the \( \Delta G_i \) correlation length scales as \( \xi_{\Delta G} \propto \Delta Z^{-1.0 \pm 0.15} \). This differs from the usual picture \[17\], in which the bulk modulus is associated with

![Figure 3](image)

Figure 3. A comparison between \( B(\Delta Z) \) for the max\( \Delta B_i \) pruning to the \( G^{(1)}(\Delta Z) \) for the max\( \Delta G^{(1)} \) pruning. After several bonds are removed these are almost parallel, suggesting universality. Here \( G^{(1)} \) denotes the simple shear modulus and \( B_0 \) and \( G_0^{(1)} \) denote the bulk and shear modulus before any pruning.

![Figure 4](image)

Figure 4. The spatial correlations in two dimensions measured by the angle average of \( \Delta M(q) \) defined in Eq. 10. In panel (a) and (b) we show the unscaled distributions while in panel (c) and (d) we collapse by rescaling the axis with powers of \( \Delta Z \). The \( \Delta B_i \) correlation length is consistent with \( \xi_{\Delta B} \propto \Delta Z^{-0.5} \) while the \( \Delta G_i \) correlation length is consistent with \( \xi_{\Delta G} \propto \Delta Z^{-1.0} \).

Since \( \Delta M_i \) depends on \( S_i^2 \) we also measure the correlations of \( S_i^2 \). Using the same definition in Eq. 10 we compute \( \langle |\Delta S_i^2(q)|^2 \rangle \) by replacing \( \Delta M_i \) with \( S_i^2 \). These correlations are shown in Fig. 9 and the diverging length scale is consistent with \( \Delta Z^{-0.5 \pm 0.15} \). Interestingly, at large \( q \), \( \langle |\Delta S_i^2(q)|^2 \rangle \propto q^{-1.5} \) as in the case of \( \langle |\Delta B(q)|^2 \rangle \). This suggests that these correlations have the same source. We also note the length scale \( \Delta Z^{-0.5} \) has been observed in the tension profile resulting from squeezing a bond \[18\][19]. This should be the same length scale.
Figure 5. The spatial correlations in two dimensions of $S^2_2$ measured using $\langle |\Delta S^2_2(q)|^2 \rangle$. The correlation length agrees with $\Delta Z^{-\frac{1}{2}}$.

V. CONCLUSIONS

In this paper we have shown that pruning bonds with maximum or minimum values of $\Delta B_i$ and $\Delta G_i$ provides an effective method for designing disordered metamaterials with targeted properties. Previous protocols relied on targeting the contributions to the moduli, $B_i$ and $G_i$. The current approach targets the change in the moduli when a bond is removed (i.e., targeting $\Delta B_i$ and $\Delta G_i$). This procedure allows us to tune the network to a nearly maximally negative Poisson’s ratio by targeting bonds with very small $\Delta G_i$. This was not possible with the previous approach and is a significant improvement because it allows the network to be pruned to the auxetic limit without developing cracks.

We have also shown that if the system is kept isotropic, the possibility of removing a bond which contributes very little to the shear modulus is greatly reduced. This problem is more severe in higher dimension, yet even in three dimensions there are enough small $\Delta G_i$ bonds to yield an auxetic material for min$\Delta G_i$ pruning.

We have provided a rationale for why moduli tend to scale as power laws with $\Delta Z$ when bonds are removed. We estimated the exponents for different pruning strategies, and found agreement to within 30% of the measured values. Our analysis suggests that the exponents have some universal features, however, they depend weakly on the system size. We also argue that spatial correlations may reduce the exponents.

Finally we have examined the spatial correlations of $\Delta B_i$ and $\Delta G_i$ in two dimensions and identified diverging length scales. The correlation length for $\Delta B_i$ is given by $\xi_{\Delta B} \propto \Delta Z^{-0.5}$ while the correlation length for $\Delta G_i$ is given by $\xi_{\Delta G} \propto \Delta Z^{-1.0}$. This contrasts with the intuition $[17]$ that associates the $\ell^* = \Delta Z^{-1}$ with the bulk modulus and $\ell^\dagger = \Delta Z^{-\frac{1}{2}}$ with the shear modulus. We also find that $S^2_2$ has a correlation length of $\Delta Z^{-0.5}$. These spatial correlation reduce the range of possible designs in the system.

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VI. APPENDIX

A. An efficient algorithm for recomputing the elastic properties resulting from bond removal

In this section we provide an efficient algorithm for recomputing the elastic response due to the removal of a bond. We employ the notation of Ref. $[8]$ $[20]$. The energy cost of a deformation is given by:

$$U = \frac{1}{2} k \sum_{i,j} e_i [S_i]_j e_j$$

and the tension in a bond is given by:

$$t_i = k \sum_{j=1}^N e_j [S_i]_j .$$ (11)

Here $e_i$ is the affine extension of a bond and $[S_i]_j$ is the tension in bond $j$ resulting from a unit change in the equilibrium length of bond $i$. All information regarding the elastic behavior depends only on $[S_i]_j$ and therefore we would like to compute how it changes with the removal of a bond. In particular $S^2_i \equiv [S_i]_j$ and $M_i = \frac{2t_i^2}{V}$ where $V$ is the volume and $\epsilon$ is the strain. Eq. 5 of Ref. $[8]$ allows to compute the change in the tension when a bond is removed for any deformation. The corresponding affine extension for this deformation is given by $e_j = \delta_{ij}$. Assuming bond $k$ is removed the element $[S_i]_j'$ is then given by:

$$[S_i]_j' = [S_i]_j - [S_i]_k [S_k]_j .$$

Thus the evolution of the whole elastic response is easily computed when a bond is removed, including $B_i$, $G_i$ and $S^2_i$. 

\[\text{Figure 5. The spatial correlations in two dimensions of } S^2_2 \text{ measured using } \langle |\Delta S^2_2(q)|^2 \rangle. \text{ The correlation length agrees with } \Delta Z^{-\frac{1}{2}}.\]
B. Distribution of the sum of random variables with the universal form

We compute the sum of \( n \) independent random variables, \( Z = \sum_{i=1}^{n} y_i \), where \( P(y_i) = \frac{1}{\sqrt{2\pi y_0}} y_i^{-1/2} \exp \left( -\frac{y_i}{2y_0} \right) \). We exploit the fact that \( x_i = \sqrt{y_i} \) is a Gaussian random variable.

\[
P(x_i) = \frac{1}{\sqrt{2\pi y_0}} \exp \left( -\frac{x_i^2}{2y_0} \right). \tag{12}
\]

To compute the distribution of \( Z \), we first compute the distribution of \( R = \sqrt{Z} = \sqrt{\sum x_i^2} \) and then use a transformation of variables to compute \( P(Z) \). The distribution of \( R \) is the sum of Gaussian variables and therefore straightforward:

\[
P(R) = \frac{S(n)}{(2\pi y_0)^{n/2}} R^{n-1} \exp \left( -\frac{R^2}{2y_0} \right) \tag{13}
\]

where \( S(n) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \) is the surface of a \( n \)-dimensional hypersphere and \( \Gamma \) is the Gamma function. A transformation of variables results in:

\[
P(Z) = P(R) \frac{dR}{dZ} = \frac{1}{2} \frac{S(n)}{(2\pi y_0)^{n/2}} Z^{n/2-1} \exp \left( -\frac{Z}{2y_0} \right). \tag{14}
\]

\[
= \frac{1}{(2y_0)^{n/2} \Gamma \left( \frac{n}{2} \right)} Z^{n/2-1} \exp \left( -\frac{Z}{2y_0} \right). \tag{15}
\]

The average of \( Z \) can be computed by noting that it is a sum of \( n \) identical random variables, and thus is given by \( \langle y_i \rangle = ny_0 \).

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