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Subarsha Banerjee*

Laplacian spectrum of comaximal graph of the ring $\mathbb{Z}_n$

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Abstract: In this paper, we study the interplay between the structural and spectral properties of the comaximal graph $\Gamma(\mathbb{Z}_n)$ of the ring $\mathbb{Z}_n$ for $n > 2$. We first determine the structure of $\Gamma(\mathbb{Z}_n)$ and deduce some of its properties. We then use the structure of $\Gamma(\mathbb{Z}_n)$ to deduce the Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ for various $n$. We show that $\Gamma(\mathbb{Z}_n)$ is Laplacian integral for $n = p^a q^b$, where $p, q$ are primes and $a, \beta$ are non-negative integers and hence calculate the number of spanning trees of $\Gamma(\mathbb{Z}_n)$ for $n = p^a q^b$. The algebraic and vertex connectivity of $\Gamma(\mathbb{Z}_n)$ have been shown to be equal for all $n$. An upper bound on the second largest Laplacian eigenvalue of $\Gamma(\mathbb{Z}_n)$ has been obtained, and a necessary and sufficient condition for its equality has also been determined. Finally, we discuss the multiplicity of the Laplacian spectral radius and the multiplicity of the algebraic connectivity of $\Gamma(\mathbb{Z}_n)$. We then investigate some properties and vertex connectivity of an induced subgraph of $\Gamma(\mathbb{Z}_n)$. Some problems have been discussed at the end of this paper for further research.

Keywords: comaximal graph, Laplacian eigenvalues, vertex connectivity, algebraic connectivity, Laplacian spectral radius, finite ring

MSC 2020: 05C25, 05C50

1 Introduction

Let $G$ be a finite simple undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. If two vertices $v_1, v_2$ are adjacent, we denote it by $v_1 \sim v_2$. The join of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ denoted by $G_1 \vee G_2$ is a graph obtained from $G_1$ and $G_2$ by joining each vertex of $G_1$ to all vertices of $G_2$. The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ denoted by $G_1 \cup G_2$ is a graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The adjacency matrix of $G$ denoted by $A(G) = (a_{ij})$ is an $n \times n$ matrix defined as $a_{ij} = 1$ when $v_i \sim v_j$ and 0 otherwise. The Laplacian matrix $L(G)$ of $G$ is defined as $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of $G$ and $D(G)$ is the diagonal matrix of vertex degrees. Since the matrix $L(G)$ is a real, symmetric, and a positive semi-definite matrix, all its eigenvalues are real and non-negative. Also 0 is an eigenvalue of $L(G)$ with eigenvector $[1, 1, 1, \ldots, 1]^T$ whose multiplicity equals the number of connected components in the graph $G$. Let the eigenvalues of $L(G)$ be denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$. The largest eigenvalue $\lambda_1$ is known as the spectral radius of $G$, and the second smallest eigenvalue $\lambda_{n-1}$ is known as the algebraic connectivity of $G$. Also $\lambda_{n-1} > 0$ if and only if $G$ is connected. The term algebraic connectivity was given by Fiedler in [1]. A separating set in a connected graph $G$ is a set $S \subset V(G)$ such that $V(G) \setminus S$ has more than one connected component. The vertex connectivity of $G$ denoted by $\kappa(G)$ is defined as $\kappa(G) = \min\{|S| : S$ is a separating set of $G\}$. The papers [1] and [2] list several interesting properties of $\lambda_{n-1}$ and $\kappa$. Readers may refer to [3] for a survey on the Laplacian matrix of a graph $G$. A graph $G$ is called Laplacian integral if all the eigenvalues of $L(G)$ are integers. We follow [4] for definitions of standard terms in graph theory.

* Corresponding author: Subarsha Banerjee, Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kol-700019, West Bengal, India, e-mail: subarshabnrj@gmail.com

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Let $R$ be a commutative ring with unity $1 \neq 0$. The comaximal graph of a ring $R$ denoted by $\Gamma(R)$ was introduced by Sharma and Bhatwadekar in [5]. The vertices of $\Gamma(R)$ are the elements of the ring $R$, and two distinct vertices $x, y$ of $\Gamma(R)$ are adjacent if and only if $Rx + Ry = R$. They proved that $R$ is a finite ring if and only if the chromatic number of $\Gamma(R)$ denoted by $\chi(\Gamma(R))$ is finite. It was further shown that $\chi(\Gamma(R))$ satisfies $\chi(\Gamma(R)) = t + 1$, where $t$ denotes the number of maximal ideals of $R$ and $l$ denotes the number of units of $R$. A lot of research has been done on the comaximal graph of a ring $R$ over the last few decades. For some literature on $\Gamma(R)$, readers may refer to the works [6,7] and [8]. In this paper, the Laplacian spectrum of the comaximal graph of the finite ring $\mathbb{Z}_n$, denoted by $\Gamma(\mathbb{Z}_n)$, has been studied for various $n$. In Section 2, we provide the preliminary theorems that have been used throughout the paper. In Section 3, we discuss the structure of $\Gamma(\mathbb{Z}_n)$ and investigate some structural properties of $\Gamma(\mathbb{Z}_n)$ and find the characteristic polynomial of $\Gamma(\mathbb{Z}_n)$ for $n > 2$. We then explicitly determine the spectrum of $\Gamma(\mathbb{Z}_n)$ for $n = p^a q^b$, where $p, q$ are distinct primes and $a, b$ are non-negative integers and conclude that $\Gamma(\mathbb{Z}_n)$ is Laplacian integral for $n = p^a q^b$. In Section 4, we discuss the vertex and the algebraic connectivity of $\Gamma(\mathbb{Z}_n)$. In Section 5, we find an upper bound on the second largest eigenvalue of $\Gamma(\mathbb{Z}_n)$ and determine a necessary and sufficient condition when it attains its bounds. We use it to determine the multiplicity of the spectral radius of $\Gamma(\mathbb{Z}_n)$. We also determine the multiplicity of the algebraic connectivity of $\Gamma(\mathbb{Z}_n)$. In Section 6, we study an induced subgraph of $\Gamma(\mathbb{Z}_n)$ formed by the non-zero non-unit elements of $\mathbb{Z}_n$. Finally, in Section 7, we provide some problems for further research.

2 Preliminaries

In this section, we will provide some preliminary theorems that will be required in our subsequent sections. Throughout this paper by eigenvalues and characteristic polynomial of a given graph $G$, we shall mean the eigenvalues and characteristic polynomial of the Laplacian matrix $L(G)$ of $G$. Also, the characteristic polynomial and the multiset of eigenvalues of $G$ have been denoted by $\mu(G, x)$ and $\sigma(G)$, respectively. Thus, $\lambda_i(G)$ shall denote the $i$th eigenvalue of $L(G)$.

**Theorem 2.1.** [4, Theorem 1.2.18] A graph $G$ is bipartite if and only if it has no odd cycle.

**Theorem 2.2.** [9, Corollary 3.7] Let $G_1 \vee G_2$ denote the join of two graphs $G_1$ and $G_2$. Then,

$$\mu(G_1 \vee G_2, x) = \frac{x(x - n_1 - n_2)}{(x - n_1)(x - n_2)}\mu(G_1, x - n_2)\mu(G_2, x - n_1),$$

where $n_1$ and $n_2$ are orders of $G_1$ and $G_2$ respectively.

**Theorem 2.3.** [9, Theorem 3.1] Let $G$ be the disjoint union of the graphs $G_1, G_2, \ldots, G_k$. Then,

$$\mu(G, x) = \prod_{i=1}^{k}\mu(G_i, x).$$

**Theorem 2.4.** [9, Theorem 2.2] If $G$ is a simple graph on $n$ vertices, then the largest eigenvalue $\lambda_1$ of $G$ satisfies $\lambda_1 \leq n$, where the equality holds if and only if its complement $\overline{G}$ is disconnected.

**Definition 2.5.** [10, Definition 3.9.1] Given a graph $G$ with vertex set $V(G)$, a partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ is said to be an equitable partition of $G$ if every vertex in $V_i$ has the same number of neighbors $b_{ij}$ in $V_j$ where $1 \leq i, j \leq k$. 
Theorem 2.6. [11, Theorem 2.1] Let \( G \) be a non-complete, connected graph on \( n \) vertices. Then \( \kappa(G) = \lambda_{n-1}(G) \) if and only if \( G \) can be written as \( G = G_1 \vee G_2 \), where \( G_1 \) is a disconnected graph on \( n - \kappa(G) \) vertices and \( G_2 \) is a graph on \( \kappa(G) \) vertices with \( \lambda_{n-1}(G) \geq 2\kappa(G) - n \).

Definition 2.7. [12, see p. 15] Let \( H \) be a graph with vertex set \( V(H) = \{1, 2, \ldots, k\} \). Let \( G_i \) be disjoint graphs of order \( n_i \) with vertex sets \( V(G_i) \), where \( 1 \leq i \leq k \). The \( H \)-join of graphs \( G_1, G_2, \ldots, G_k \), denoted by \( H[G_1, G_2, \ldots, G_k] \), is formed by taking the graphs \( G_i \) and any two vertices \( v_i \in G_i \) and \( v_j \in G_j \) are adjacent if \( i \) is adjacent to \( j \) in \( H \).

Theorem 2.8. [13, Theorem 8] Let us consider a family of \( k \) graphs \( G_j \) of order \( n_j \), with \( j \in \{1, 2, \ldots, k\} \) having Laplacian spectrum \( \sigma(G_j) \). If \( H \) is a graph such that \( V(H) = \{1, 2, \ldots, k\} \), then the Laplacian spectrum of \( H[G_1, G_2, \ldots, G_k] \) is given by

\[
\sigma(H[G_1, G_2, \ldots, G_k]) = \left( \bigcup_{j=1}^{k} (N_j \oplus \sigma(G_j) \setminus \{0\}) \right) \cup \sigma(M),
\]

where

\[
M = \begin{bmatrix}
N_1 & -\rho_{1,2} & \cdots & -\rho_{1,k} \\
-\rho_{1,k} & N_2 & \cdots & -\rho_{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
-\rho_{l,k} & -\rho_{2,k} & \cdots & N_k
\end{bmatrix},
\]

\( \rho_{a,b} = \rho_{b,a} = \begin{cases} \sqrt{n_a n_b} & \text{if } ab \in E(H) \\ 0 & \text{otherwise} \end{cases} \)

and \( N_j = \begin{cases} \sum_{i \in \mathbb{N}_{0}(j)} n_i & \text{if } N_{\Omega}(j) \neq \emptyset \\ 0 & \text{otherwise}. \end{cases} \)

Here, \( N_{\Omega}(j) = \{i : ij \in E(H)\} \).

3 Structure of \( \Gamma(Z_n) \)

We denote the elements of the ring \( \mathbb{Z}_n \) by \( \{0, 1, 2, \ldots, n-2, n-1\} \). If \( x \in \mathbb{Z}_n \), then \( \langle x \rangle \) will denote the ideal generated by \( x \). We follow [14] for standard definitions in ring theory.

In this section, we describe the structure of \( \Gamma(Z_n) \). We show that \( \Gamma(Z_n) \) can be expressed as the join and union of certain induced subgraphs of \( \Gamma(Z_n) \). We then investigate the Laplacian spectra of \( \Gamma(Z_n) \) for various \( n \). We first find an equivalent condition for adjacency of two vertices in \( \Gamma(Z_n) \).

By using the adjacency criterion for any two vertices in \( \Gamma(R) \), we find that two vertices \( v_i, v_j \in \Gamma(Z_n) \) are adjacent if and only if \( \mathbb{Z}_n v_i + \mathbb{Z}_n v_j = \mathbb{Z}_n \). Now since \( \mathbb{Z}_n \) is a principal ideal ring (PIR), so \( \mathbb{Z}_n v_i = \langle v_i \rangle = \langle \gcd(v_i, n) \rangle \). Since sum of two ideals is again an ideal, so the adjacency criterion in \( \Gamma(Z_n) \) becomes the following:

\[ v_i \text{ is adjacent to } v_j \text{ in } \Gamma(Z_n) \iff \gcd(v_i, v_j, n) = 1. \tag{2} \]

We have \( V(\Gamma(Z_n)) = S \cup T \), where \( S = \{a : \gcd(a, n) = 1\} \) and \( T = \mathbb{Z}_n \setminus S \). Clearly, for all \( v \in S \), \( \deg(v) = n - 1 \), and \( \deg(0) = \varphi(n) \). Let \( G_1 \) denote the induced subgraph of \( \Gamma(Z_n) \) on the set \( S \) and \( G_2' \) denote the induced subgraph of \( \Gamma(Z_n) \) on the set \( T \). We have,

\[ \Gamma(Z_n) = G_1 \vee G_2' \equiv K_{\varphi(n)} \vee G_2'. \]

Again, if we let \( G_2 \) to be the induced subgraph of \( \Gamma(Z_n) \) on the set \( T \setminus \{0\} \), then

\[ \Gamma(Z_n) \equiv K_{\varphi(n)} \vee G_2' \equiv K_{\varphi(n)} \vee (G_2 \cup K_1). \tag{3} \]

We can make the following observations as applications of equation (3).
Proposition 3.1. \( \Gamma(\mathbb{Z}_n) \) is complete if and only if \( n \) is prime.

Proof. Using equation (3), \( \Gamma(\mathbb{Z}_n) \) is complete if and only if \( K_1 \cup G_2 \) is complete. Now \( K_1 \cup G_2 \) is complete if and only if \( G_2 \) is a null graph. Now, \( G_2 \) is a null graph if and only if every non-zero element in \( \mathbb{Z}_n \) is a unit, which in turn implies that \( \mathbb{Z}_n \) is a field. Now since \( \mathbb{Z}_n \) is a field if and only if \( n \) is a prime, we conclude that \( \Gamma(\mathbb{Z}_n) \) is complete if and only if \( n \) is a prime number. \( \square \)

Proposition 3.2. If \( n > 2 \), then \( \Gamma(\mathbb{Z}_n) \) is not bipartite.

Proof. Since \( n \geq 3 \), so \( \varphi(n) \geq 2 \). We take \( v_1, v_2 \in K_{\varphi(n)} \), and \( v_3, v_4 \in G_2 \). Now, \( v_1 \sim v_2 \sim v_3 \sim v_4 \) forms a cycle of length 3. By using Theorem 2.1, we conclude that \( \Gamma(\mathbb{Z}_n) \) is not bipartite. \( \square \)

Theorem 3.3. The characteristic polynomial of \( \Gamma(\mathbb{Z}_n) \) is \( \mu(G_1 \cup G_2) = x(x - \varphi(n))\mu(G_2, x - \varphi(n)) \), where \( G_2 \) is given by equation (3).

Proof. By using equation (3) and Theorems 2.2 and 2.3, we obtain,

\[
\mu(G_1 \cup G_2) = x(x - \varphi(n))\mu(G_2, x - \varphi(n)) = x(x - \varphi(n))(x - (n - \varphi(n)))\mu(G_2, x - \varphi(n)) = x(x - \varphi(n))(x - (n - \varphi(n)))(x - n)\mu(G_2, x - \varphi(n)) = x(x - \varphi(n))\mu(G_2, x - \varphi(n)).
\]

The following observations about \( \mu(\Gamma(\mathbb{Z}_n)) \) are evident:

Corollary 3.4. If \( n > 2 \), then \( n \) is an eigenvalue of \( \Gamma(\mathbb{Z}_n) \) with multiplicity at least \( \varphi(n) \).

Corollary 3.5. If \( n = p \), where \( p \) is a prime number, then \( p \) and \( 0 \) are eigenvalues of \( \Gamma(\mathbb{Z}_n) \) with multiplicity \( p - 1 \) and 1, respectively.

From equation (4) of Theorem 3.3, we find that the eigenvalues of \( \Gamma(\mathbb{Z}_n) \) are known if the spectrum of \( G_2 \) given in equation (3) is completely determined. We thus proceed to study the graph \( G_2 \) in more detail.

### 3.1 Structure of \( G_2 \)

Let \( n = p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k} \) be a prime factorization of \( n \), where \( p_1 < p_2 < \cdots < p_k \) are primes and \( a_i \) are positive integers. We say that \( d \) is a proper divisor of \( n \) if \( d \) divides \( n \) and \( d \notin \{1, n\} \). The total number of positive divisors of \( n \) equals \( (a_1 + 1)(a_2 + 1)\cdots(a_k + 1) \). The total number of proper positive divisors of \( n \) will be given by \( w = (a_1 + 1)(a_2 + 1)\cdots(a_k + 1) - 2 \).

Let \( d_1 < d_2 < \cdots < d_w \) be the set of all proper divisors of \( n \) arranged in increasing order. For each \( d_i \), where \( 1 \leq i \leq w \), we define

\[
A_{d_i} = \{x : \gcd(x, n) = d_i\}.
\]

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Any element of $A_d$ is of the form $zd$, where $\gcd\left(z, \frac{n}{z}\right) = 1$ and hence the number of elements of $A_d$ is $\varphi\left(\frac{n}{z}\right)$. Thus, $|A_d| = \varphi\left(\frac{n}{z}\right)$. Clearly, $V(G_2) = \bigcup_{z\in \mathbb{Z}_n} A_d$.

**Lemma 3.6.** $x_i \in A_d$ is adjacent to $x_j \in A_d$ if and only if $\gcd(d_i, d_j) = 1$.

**Proof.** Assume that $x_i \in A_d$ is adjacent to $x_j \in A_d$. Using equation (2), $x_i$ adjacent to $x_j$ implies that either $\gcd(x_i, x_j) = 1$ or $\gcd(x_i, x_j)$ is a unit in $\mathbb{Z}_n$. We consider the following two cases:

Case 1: $\gcd(x_i, x_j) = 1$. Let $\gcd(d_i, d_j) = d$, then $d|d_i$ and $d|d_j$. Since $\gcd(x_i, n) = d_i$ and $\gcd(x_j, n) = d_j$, we have $d|x_i$ and $d|x_j$, which in turn implies that $d|x_i$ and $d|x_j$. Again since $\gcd(x_i, x_j) = 1$, $d = 1$ and hence $\gcd(d_i, d_j) = 1$.

Case 2: $\gcd(x_i, x_j)$ is a unit in $\mathbb{Z}_n$. Let $\gcd(x_i, x_j) = a$ which is a unit in $\mathbb{Z}_n$ and hence, $\gcd(a, n) = 1$. Let $\gcd(d_i, d_j) = d_i$, then $d|d_i$ and $d|d_j$. Since $\gcd(x_i, n) = d_i$ and $\gcd(x_j, n) = d_j$, we have $d|x_i$, $d|n$, which in turn implies that $d|x_i$, $d|x_j$ and $d|n$. Since $\gcd(x_i, x_j) = a$, $d|a$. Since $\gcd(a, n) = 1$, from the facts that $d|a$ and $d|n$, it follows that $d = 1$. Hence, $\gcd(d_i, d_j) = 1$.

Thus if $x_i \in A_d$ is adjacent to $x_j \in A_d$, then $\gcd(d_i, d_j) = 1$.

Conversely, we now assume that $\gcd(d_i, d_j) = 1$. Let $d = \gcd(x_i, x_j)$. We claim that either $d = 1$ or $d$ is a unit in $\mathbb{Z}_n$. Assume the contrary, then $d > 1$ and $d$ is not a unit in $\mathbb{Z}_n$, which implies $d > 1$ divides $n$.

If $d = \gcd(x_i, x_j)$, then $d|x_i$ and $d|x_j$

$⇒ (d|x_i, d|n)$ and $(d|x_j, d|n)$

$⇒ d|d_i = \gcd(x_i, n)$ and $d|d_j = \gcd(x_j, n)$

$⇒ d|\gcd(d_i, d_j) = 1$, which is a contradiction.

Thus, either $d = 1$ or $d$ is a unit in $\mathbb{Z}_n$, and hence, $\langle x_i \rangle + \langle x_j \rangle = \langle d \rangle = \mathbb{Z}_n$ which implies by equation (2) that $x_i \in A_d$ is adjacent to $x_j$ in $A_d$. \qed

**Lemma 3.7.** If $v_i \in A_d$ is adjacent to $v_j \in A_d$ for some $i \neq j$, then $v_i$ is adjacent to $v_j$ for all $v_j \in A_d$.

**Proof.** Let $v_i \in A_d$ is adjacent to $v_j \in A_d$ for some $i \neq j$, then using Lemma 3.6 $\gcd(d_i, d_j) = 1$. Let $v_j \neq v_i$ be another member of $V_j$, then $\gcd(v_j, n) = d_j$. Using the fact that $\gcd(d_i, d_j) = 1$ and Lemma 3.6, we conclude that $v_j$ is adjacent to $v_i$. \qed

**Lemma 3.8.** No two members of the set $A_d$ are adjacent.

**Proof.** If $v_i, v_j \in A_d$, then $\gcd(v_i, n) = \gcd(v_j, n) = d_i$. Using Lemma 3.6, the proof follows. \qed

If $v_i \in A_d$, using Lemma 3.7, we observe that the number of neighbors of $v_i$ in $A_d$, where $j \neq i$ is fixed, i.e., either the number of neighbors of $v_i$ in $A_d$ equals 0 or $|A_d|$. Also using Lemma 3.8, the number of neighbors of $v_i$ in $A_d$ equals 0 for all $1 \leq i \leq w$. If we denote $V_i = A_d$, where $1 \leq i \leq w$, then using Definition 2.5, we find that $V_1 \cup V_2 \cup \cdots \cup V_w$ is an equitable partition of graph $G_2$.

Thus, we have the following theorem:

**Theorem 3.9.** For any $n \geq 2$, the induced subgraph $G_2$ of $\Gamma(Z_n)$ with vertex set $V(G_2)$ has an equitable partition as $V(G_2) = \bigcup_{i=1}^w A_d$, where $w$ denotes the total number of positive proper divisors of $n$ and the sets $A_d$ have been defined as in equation (5).

Using Theorems 3.9 and 2.7, it is evident that $G_2$ is the $H$-join of the graphs $G_d$, where $G_d$ is the induced subgraph of $\Gamma(Z_n)$ on $A_d$, and $H$ can be obtained as follows:
Construction of $H$: $V(H) = \{d_i : 1 \leq i \leq w\}$, where $d_i$ is a positive proper divisor of $n$. The vertices $d_i$, $d_j$ are adjacent in $H$ if and only if $\gcd(d_i, d_j) = 1$. Thus, $E(H) = \{d_i d_j : \gcd(d_i, d_j) = 1\}$.

We use Theorem 2.8 to determine the spectrum of $G_2$. We find that $G_2$ is the $H$-join of $G_{d_i}$, where $G_{d_i}$ is a null graph on $\varphi (\frac{n}{d_i})$ vertices. Hence, $\sigma(G_{d_i}) = \{0\}$. Also, $N_{H}(d_i) = \{d_i : \gcd(d_i, d_j) = 1\}$, and hence,

$$N_{d_i} = \sum_{d_{d_i} \in N_{H}(d_i)} n_i = \sum_{d_{i} : \gcd(d_{i}, d_{j}) = 1} \varphi \left( \frac{n}{d_{i}} \right).$$

Moreover, $N_{d_i} = \varphi \left( \frac{n}{d_{i}} \right)$, where $1 \leq i \leq w$.

**Example 3.10.** If $n = pqr$, where $p$, $q$, and $r$, are primes with $p < q < r$, then the proper positive divisors of $n$ are $p, q, r, pq, pr, qr$. Using the construction of $H$ given earlier, we find that $G_2$ is the $H$-join of $G_p$, $G_q$, $G_r$, $G_{pq}$, $G_{pr}$, $G_{qr}$, where $H$ is given by Figure 1.

Now we have,

$$N_p = \varphi \left( \frac{pqr}{q} \right) + \varphi \left( \frac{pqr}{r} \right) + \varphi \left( \frac{pqr}{qr} \right)$$

$$= \varphi(p) + \varphi(pr) + \varphi(p)$$

$$= (p - 1)(r - 1) + (p - 1)(q - 1) + (p - 1)$$

$$= (p - 1)(r - 1 + q - 1 + 1)$$

$$= (p - 1)(q + r - 1).$$

Similarly, $N_q = (q - 1)(p + r - 1)$, $N_r = (r - 1)(p + q - 1)$

$N_{pq} = (p - 1)(q - 1)$, $N_{pr} = (p - 1)(r - 1)$ and $N_{qr} = (q - 1)(r - 1)$.

Also

$$n_p = (q - 1)(r - 1), n_q = (p - 1)(r - 1), n_r = (p - 1)(q - 1)$$

$$n_{pq} = r - 1, n_{pr} = q - 1, n_{qr} = p - 1.$$

Using Theorem 2.8, we find that the eigenvalues of $G_2$ are $(p - 1)(q + r - 1)$ with multiplicity $qr - r - q$, $(q - 1)(p + r - 1)$ with multiplicity $pr - r - p$, $(r - 1)(p + q - 1)$ with multiplicity $pq - p - q$, $(p - 1)(q - 1)$ with multiplicity $r - 2$, $(p - 1)(r - 1)$ with multiplicity $q - 2$, and $(q - 1)(r - 1)$ with multiplicity $p - 2$ and

\[\begin{align*}
\text{Figure 1: } H \text{ for } n = pqr.
\end{align*}\]
remaining eigenvalues are the eigenvalues of $6 \times 6$ matrix $M$ (equation (1)) whose entries can be determined from equations (6) and (7).

We now find the spectrum of $\Gamma(\mathbb{Z}_n)$ for $n = p^\alpha q^\beta$, where $p, q$ are primes, and $\alpha, \beta$ are nonnegative integers.

**Theorem 3.11.** When $n = p^m$, where $p$ is a prime and $m > 1$ is a positive integer, then the eigenvalues of $\Gamma(\mathbb{Z}_n)$ are $n$ with multiplicity $\varphi(n)$, $\varphi(n)$ with multiplicity $n - \varphi(n) - 1$ and 0 with multiplicity 1.

**Proof.** When $p$ is a prime and $m > 1$ is a positive integer, the proper divisors of $p^m$ are $p, p^2, p^3, \ldots, p^{m-2}, p^{m-1}$. We partition the vertex set $V(G_2)$ of $G_2$ as $V_1, V_2, \ldots, V_{m-2}, V_{m-1}$, where $V_i = A_{p^i} = \{x : \gcd(x, n) = p^i\}$.

Since $\gcd(p^i, p^j) = p^{\min(i, j)} \neq 1$, using Lemmas 3.6 and 3.8, we find that $x_i \in V_i$ is not adjacent to $x_j \in V_j$ for all $1 \leq i, j \leq m - 1$.

Thus, no two vertices in the graph $G_2$ are adjacent and hence $G_2 = \mathbb{K}_{n-\varphi(n)-1}$. By using equation (4), we obtain $\mu(\Gamma(\mathbb{Z}_n)) = x(x - n)^{\varphi(n)}(x - \varphi(n))^{n-\varphi(n)-1}$. □

**Theorem 3.12.** If $n = p^\alpha q^\beta$, where $p, q$ are primes with $p < q$ and $\alpha, \beta$ are positive integers, then the eigenvalues of $\Gamma(\mathbb{Z}_n)$ are $n$ with multiplicity $\varphi(n)$, $(t + 1)(p - 1) + \varphi(n)$ with multiplicity $(t + 1)(q - 1) - 1$.

![Figure 2: $G_2$ for $n = p^\alpha q^\beta$.](image-url)
(t + 1)(q - 1) + \varphi(n) \text{ with multiplicity } (t + 1)(p - 1) - 1, \varphi(n) \text{ with multiplicity } t + 1, \text{ and } (t + 1)(p + q - 2) + \varphi(n), 0 \text{ each with multiplicity } 1, \text{ where } t = p^{a-1}q^{b-1} - 1.

**Proof.** If \( n = p^a q^b \) where \( p, q \) are primes with \( p < q \) and \( \alpha, \beta \) are positive integers, then the proper divisors of \( n \) are \( p^i q^j \), where \( 0 \leq i \leq \alpha, \ 0 \leq j \leq \beta \) with \( i + j \neq 0, \alpha + \beta \). We partition the vertex set \( V(G_2) \) as follows:

\[
V(G_2) = (A_p \cup A_{p^2} \cup \cdots \cup A_{p^\alpha}) \cup (A_q \cup A_{q^2} \cup \cdots \cup A_{q^\beta}) \cup (\cup_{j=1}^\beta A_{p^j q^j}) \cup (\cup_{j=1}^\beta A_{p^j q^j}) \cup \cdots \cup (\cup_{j=1}^\beta A_{p^j q^j}).
\]  

(8)

If \( 1 \leq i \leq \alpha, 1 \leq j \leq \beta \), then \( \text{gcd}(p^i, q^j) = 1 \). Using Lemmas 3.6 and 3.7, we find that every vertex of \( A_{p^i} \) is adjacent to every vertex of \( A_{q^j} \).

Also Lemma 3.6 indicates that if \( 1 \leq i \leq \alpha, 1 \leq j \leq \beta \) with \( i + j \neq \alpha + \beta \), then no vertex of \( A_{p^i q^j} \) is adjacent to any other vertex of \( G_2 \). If we draw the graph \( G_2 \) with the vertex partitions as given in equation (8), it looks like Figure 2. (A solid line in the figure indicates that each vertex of \( A_{di} \) is adjacent to each vertex of \( A_{dj} \). No line between two nodes \( A_{di} \) and \( A_{dj} \) indicates that no vertex of \( A_{di} \) is adjacent to any vertex of \( A_{dj} \).

Let \( G_{21} \) be the induced subgraph of \( G_2 \) on the set \( A_p \cup A_{p^2} \cup \cdots \cup A_{p^\alpha} \) and \( G_{22} \) be the induced subgraph of \( G_2 \) on the set \( A_q \cup A_{q^2} \cup \cdots \cup A_{q^\beta} \).

Now the number of elements in \( A_p \cup A_{p^2} \cup \cdots \cup A_{p^\alpha} \) is \( \sum_{i=1}^\alpha |A_{p^i}| \). Hence,

\[
\sum_{i=1}^\alpha |A_{p^i}| = |A_p| + |A_{p^2}| + \cdots + |A_{p^\alpha}|
\]

\[
= \varphi(p^{a-1}q^b) + \varphi(p^{a-2}q^{b^2}) + \cdots + \varphi(q^b)
\]

\[
= p^{a-1}\left(1 - \frac{1}{p}\right)q^b\left(1 - \frac{1}{q}\right) + p^{a-2}\left(1 - \frac{1}{p}\right)q^b\left(1 - \frac{1}{q}\right) + \cdots + \left(1 - \frac{1}{p}\right)q^b\left(1 - \frac{1}{q}\right) + q^b\left(1 - \frac{1}{q}\right)
\]

\[
= q^b\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{p}\right) + q^b\left(1 - \frac{1}{q}\right) + q^b\left(1 - \frac{1}{q}\right)
\]

\[
= q^b\left(1 - \frac{1}{q}\right)\left(p^{a-1} - 1 + 1\right)
\]

\[
= q^b\left(p^{a-1} - 1 + 1\right)
\]

Again the number of elements in the set \( A_p \cup A_{p^2} \cup \cdots \cup A_{p^\alpha} \) is \( \sum_{i=1}^\beta |A_{q^j}| \). By using similar calculations as in equation (9), we find that

\[
\sum_{i=1}^\beta |A_{q^j}| = p^{a-1}q^{b-1}(p - 1).
\]

(10)

The vertices of \( G_2 \) that are not adjacent to any other vertex in \( G_2 \) are the members of the set \( (\cup_{j=1}^\beta A_{p^j q^j}) \cup (\cup_{j=1}^\beta A_{p^j q^j}) \cup \cdots \cup (\cup_{j=1}^\beta A_{p^j q^j}) \). By using equations (9) and (10), the number of such vertices denoted by \( t \) equals

\[
t = p^aq^b - \varphi(p^aq^b) - 1 - p^{a-1}q^{b-1}(p - 1) - p^{a-1}q^{b-1}(q - 1) - p^{a-1}q^{b-1}(p - 1) - p^{a-1}q^{b-1}(q - 1) - p^{a-1}q^{b-1}(p - 1) - p^{a-1}q^{b-1}(q - 1) - p^{a-1}q^{b-1} - 1.
\]

Clearly, the induced subgraph of \( G_2 \) on \( p^{a-1}q^{b-1} - 1 \) vertices is a null graph. Since every vertex of the graph \( G_{21} \) is adjacent to every vertex of the graph \( G_{22} \) and the remaining vertices of \( G_2 \) are not adjacent to any other vertex, the following is evident

\[
G_2 = (G_{21} \lor G_{22}) \cup \overline{K_t}.
\]

(11)
By using equations (9) and (10) and Theorem 2.2, we obtain

\[
\mu((G_1 \cup G_2), x) = x(x - p^{a-1}q^{b-1}(q - 1))^{\lambda_n-1} x^{(t+1)(p-1)}(x - p^{a-1}q^{b-1}(p - 1))^{\lambda_n-1} x^{((t+1)(q-1)-1)}(x - (p^{a-1}q^{b-1}(p + q - 2))).
\]  

(12)

By using Theorem 2.3, equations (11) and (12) we obtain,

\[
\mu(G_2, x) = x^t \times \mu((G_1 \cup G_2), x)
\]

\[
\mu(G_1 \cup G_2, x)
\]

\[
x^{(t+1)(x - p^{a-1}q^{b-1}(q - 1))^{\lambda_n-1} x^{(t+1)(p-1)}(x - p^{a-1}q^{b-1}(p - 1))^{\lambda_n-1} x^{((t+1)(q-1)-1)}(x - (p^{a-1}q^{b-1}(p + q - 2))).
\]

(13)

By using equation (13) in equation (4), we have

\[
\mu(\Gamma(Z_n), x) = x(x - n)^{\phi(n)} \mu(G_2, x - \varphi(n))
\]

\[
= x(x - n)^{\phi(n)}(x - \varphi(n))^{(t+1)(x - p^{a-1}q^{b-1}(q - 1))^{\lambda_n-1} x^{(t+1)(p-1)}(x - p^{a-1}q^{b-1}(p - 1))^{\lambda_n-1} x^{((t+1)(q-1)-1)}(x - (p^{a-1}q^{b-1}(p + q - 2)) - \varphi(n))).
\]

Thus, the eigenvalues of \(\Gamma(Z_n)\) are \(n\) with multiplicity \(\varphi(n)\), \((t+1)(p-1) + \varphi(n)\) with multiplicity \((t+1)(q-1)-1\), \((t+1)(q-1) + \varphi(n)\) with multiplicity \((t+1)(p-1) - 1, \varphi(n)\) with multiplicity \(t+1\), and \((t+1)(p+q-2) + \varphi(n)\), 0 each with multiplicity 1.

By using Corollary 3.5 and Theorems 3.11 and 3.12, the following is evident.

**Theorem 3.13.** If \(n = p^a q^b\), where \(p, q\) are primes and \(a, b\) are non-negative integers, then \(\Gamma(Z_n)\) is Laplacian integral.

By using Theorem 3.12 and Kirchhoff's matrix tree theorem [10], we have

**Theorem 3.14.** If \(n = p^a q^b\), where \(p, q\) are primes and \(a, b\) are non-negative integers, then the number of spanning trees of \(\Gamma(Z_n)\) is expressed as follows:

\[
st(\Gamma(Z_n)) = ((t+1)pq)^{(t+1)(p-1)}(t+1)(p-1)\varphi(q-1)\varphi(p-1)(t+1)(p-1)(p-1)^{(t+1)(p-1)}
\]

\[
\times ((t+1)(p-1)(q-1)^{(t+1)(p-1)}(t+1)(p-1))\text{ where } t = p^{a-1}q^{b-1} - 1.
\]

4 **Algebraic connectivity and vertex connectivity of \(\Gamma(Z_n)\)**

In this section, we investigate the algebraic connectivity \((\lambda_{n-1})\) and vertex connectivity \((\kappa)\) of \(\Gamma(Z_n)\) for any \(n > 2\). We also show that \(\lambda_{n-1}\) and \(\kappa\) are equal for any \(n > 2\).

**Lemma 4.1.** If \(n > 2\), then \(\varphi(n)\) is an eigenvalue of \(\Gamma(Z_n)\) with multiplicity at least 1.

**Proof.** Since 0 is always an eigenvalue of the Laplacian matrix of a given graph \(G\), so the Laplacian matrix of the graph \(G_2\) also has 0 as an eigenvalue. By using equation (4), \(x - \varphi(n)\) is a factor of \(\mu(G_2, x - \varphi(n))\), which in turn implies

\[
\mu(\Gamma(Z_n), x) = x(x - n)^{\phi(n)} \mu(G_2, x - \varphi(n)) = x(x - n)^{\phi(n)}(x - \varphi(n))g(x - \varphi(n)),
\]

(14)

where \(g(x)\) is a polynomial of degree \(n - \varphi(n) - 2\). Hence, \(\varphi(n)\) is an eigenvalue of \(\Gamma(Z_n)\) with multiplicity at least 1.
Theorem 4.2. $\lambda_{n-1}(\Gamma(Z_n)) = \varphi(n)$.

Proof. Using Lemma 4.1, $\varphi(n)$ is an eigenvalue of $\Gamma(Z_n)$. Since the smallest root of the polynomial $g(x - \varphi(n))$ in equation (14) is $\varphi(n)$ and $0 < \varphi(n) < n$, we conclude that the second smallest root of $\mu(\Gamma(Z_n), x)$ is $\varphi(n)$, which implies that $\lambda_{n-1}(\Gamma(Z_n)) = \varphi(n)$.

Theorem 4.3. For all $n > 2$, $\kappa(\Gamma(Z_n)) = \lambda_{n-2}(\Gamma(Z_n)) = \varphi(n)$.

Proof. By using equation (3), we find that $\Gamma(Z_n) = (G_2 \cup K_i) \cup K_{\varphi(n)}$. If we take $G_1 = G_2 \cup K_i$ and $G_2 = K_{\varphi(n)}$, we find that $G_1$ is a disconnected graph on $n - \varphi(n)$ vertices and $G_2$ is a graph on $\varphi(n)$ vertices. Clearly, $\lambda_{n-1}(G_2) = \lambda_{n-1}(K_{\varphi(n)}) = \varphi(n)$. We find that if we assume $\kappa(\Gamma(Z_n)) = \varphi(n)$, then all the conditions of Theorem 2.6 along with the inequality $\lambda_{n-1}(G_2) \geq 2\kappa(G) - n$ are satisfied. Hence, we conclude that $\kappa(\Gamma(Z_n)) = \lambda_{n-1}(\Gamma(Z_n)) = \varphi(n)$.

5  Largest and second largest eigenvalue of $\Gamma(Z_n)$

In this section, we discuss the second largest eigenvalue $\lambda_2$ of $\Gamma(Z_n)$, which in turn helps us to find certain information about the largest eigenvalue $\lambda_1$ of $\Gamma(Z_n)$.

We first study the connectivity of $G_2$.

Theorem 5.1. The graph $G_2$ is connected if and only if $n$ is a product of distinct primes.

Proof. Let $n = p_1^{a_1}p_2^{a_2} \cdots p_m^{a_m}$, where $p_i$ are distinct primes and $a_i$ are positive integers, $1 \leq i \leq m$.

We first assume that $G_2$ is connected. To show that $n$ is a product of distinct primes, we prove that $a_i = 1$ for all $1 \leq i \leq m$. Assume the contrary that $a_i > 1$ for at least one $i$. Without loss of generality, we take $a_1 > 1$. We consider the vertex $a = p_1^{a_1}p_2^{a_2} \cdots p_m^{a_m}$ of $G_2$. Clearly, $a \neq 0$ as $a_1 > 1$. Consider any other vertex of $G_2$ say $w$. Since $\Gamma(G_2) = \bigcup_{a \in A_d} A_d$, where $A_d$ has been defined in equation (5), $w \in A_d$ for some positive proper divisor $d_i$ of $n$. Thus, $\gcd(w, n) = d_i$. Also $a \in A_{dp_1p_2 \cdots p_m}$. Since $\gcd(d_i, p_1^{a_1}p_2^{a_2} \cdots p_m^{a_m}) \neq 1$, by using Lemma 3.6, we conclude that $w$ is not adjacent to $a$. Since $w$ is arbitrary, we find that the vertex $a$ is not adjacent to any other vertex of $G_2$, which contradicts the fact that $G_2$ is connected. Hence, our assumption that $a_1 > 1$ is false. Thus, $a_i = 1$ for all $1 \leq i \leq m$, which proves that $n$ is a product of distinct primes.

Conversely, we assume that $n$ is a product of distinct primes. To show that $G_2$ is connected, we choose two arbitrary distinct vertices $x_i, x_j \in G_2$. Then, $x_i \in A_{d_i}$ and $x_j \in A_{d_j}$ for some proper positive divisor $d_i, d_j$ of $n$. We consider the following two cases, which may arise.

Case 1. $\gcd(d_i, d_j) = 1$

Using Lemma 3.6, $x_i$ and $x_j$ are adjacent in $G_2$.

Case 2. $\gcd(d_i, d_j) \neq 1$

Since $\gcd(d_i, d_j) \neq 1$, $d_i$ and $d_j$ have a prime factor in common. Since $n$ is a product of distinct primes, so there exists a prime factor $p_1$ of $n$ such that $\gcd(d_i, p_1) = 1$. Also it is possible to choose another prime factor $p_j \neq p_1$ of $n$ such that $\gcd(d_j, p_j) = 1$. Since $\gcd(p_1, p_2) = 1$, if we choose $x_{p_1} \in A_{p_1}$ and $x_{p_2} \in A_{p_2}$, then $x_{p_1}$ is adjacent to $x_{p_2}$. Thus, using Lemma 3.6, we obtain a path of length 3 from $x_i$ to $x_j$ given by $x_i \sim x_{p_1} \sim x_{p_2} \sim x_j$.

By combining cases 1 and 2, we find that any two vertices of $G_2$ are either adjacent or there exists a path between them, which implies that $G_2$ is connected when $n$ is a product of distinct primes.

Thus, $G_2$ is connected if and only if $n$ is a product of distinct primes.

We now investigate the connectivity of $\overline{G_2}$ when $n$ is a product of distinct primes.
When \( n \) is a product of two distinct primes, i.e., \( n = pq \), then \( n \) has only two distinct proper positive divisors namely \( p \) and \( q \). Thus \( V(G_2) = A_p \cup A_q \). Since \( \gcd(p, q) = 1 \), using Lemma 3.6, \( G_2 \) becomes as shown in Figure 3. (Here, the solid line indicates that each vertex of \( A_p \) is adjacent to each vertex of \( A_q \)).

Clearly, \( G_2 \) is disconnected when \( n = pq \).

In the next theorem, we investigate the connectivity of \( G_2 \) when \( n \) is a product of more than two distinct primes.

**Theorem 5.2.** If \( n \) is a product of more than two distinct primes, then \( G_2 \) is connected.

**Proof.** Let \( n = p_1 p_2 p_3 \cdots p_m \), where \( p_i \) are distinct primes and \( m > 2 \). Let \( x_i, x_j \) be two distinct vertices of \( G_2 \). Then, \( x_i \in A_{d_i} \) and \( x_j \in A_{d_j} \), where \( d_i, d_j \) are positive proper divisors of \( n \). We consider the following two cases:

1. \( \gcd(d_i, d_j) \neq 1 \).
   Using Lemma 3.6, \( x_i \in A_{d_i} \) is not adjacent to \( x_j \in A_{d_j} \) in \( G_2 \), which implies that \( x_i \in A_{d_i} \) is adjacent to \( x_j \in A_{d_j} \) in \( G_2 \).
2. \( \gcd(d_i, d_j) = 1 \).
   Using Lemma 3.6, \( x_i \in A_{d_i} \) is not adjacent to \( x_j \in A_{d_j} \) in \( G_2 \). Let \( p_1 \) be a prime factor of \( d_i \) and \( p_2 \) be a prime factor of \( d_j \). Since \( n \) is a product of more than two distinct primes, so \( p_1 p_2 \) is a positive proper divisor of \( n \). Hence, using Lemma 3.6, there exists \( y \in A_{p_1 p_2} \) such that \( x_i, x_j \) are not adjacent to \( y \) in \( G_2 \). Thus, \( y \) is adjacent to both \( x_i \) and \( x_j \) in \( G_2 \), and hence, there exists a path of length 2 given by \( x_i \rightarrow y \rightarrow x_j \) from \( x_i \) to \( x_j \) in \( G_2 \).

Combining cases 1 and 2, we find that any two vertices of \( G_2 \) are either adjacent or there exists a path between them which implies that \( G_2 \) is connected when \( n \) is a product of more than two distinct primes. □

**Theorem 5.3.** \( \lambda_2(\Gamma(Z_n)) \leq n - 1 \), where equality holds if and only if \( n \) is a product of two distinct primes.

**Proof.** Let \( \lambda(G_2) \) denote the largest eigenvalue of the Laplacian matrix of \( G_2 \). By using equation (4) of Theorem 3.3, it is evident that the second largest eigenvalue of \( \Gamma(Z_n) \) is the largest eigenvalue of the Laplacian matrix of \( G_2 \), which implies

\[
\lambda_2(\Gamma(Z_n)) = \lambda_2(G_2) + \varphi(n).
\]

Since \( G_2 \) is a graph on \( n - \varphi(n) - 1 \) vertices, using Theorem 2.4, we have \( \lambda_2(G_2) \leq n - \varphi(n) - 1 \), where equality holds if and only if \( G \) is connected and \( G_2 \) is disconnected.

By using Theorems 5.1 and 5.2, we find that \( G_2 \) is connected if and only if \( n \) is a product of distinct primes and \( G_2 \) is disconnected if \( n \) is a product of two primes. Thus,

\[
\lambda_2(\Gamma(Z_n)) = \lambda_2(G_2) + \varphi(n) \leq (n - \varphi(n) - 1) + \varphi(n) = n - 1,
\]

where equality holds if and only if \( n \) is a product of two primes. □

**Theorem 5.4.** For any \( n > 2 \), \( \lambda_1(\Gamma(Z_n)) = n \) has multiplicity exactly \( \varphi(n) \).
Proof. By using Theorem 5.3, \( \lambda_d(\Gamma(Z_n)) \leq n - 1 \). Thus, from equation (4) of Theorem 3.3, we conclude that \( \lambda_1 = n \) has multiplicity exactly \( q(n) \).

\[ \text{Theorem 5.5.} \quad \text{If } n = \prod_{i=1}^{m} p_i^{a_i}, \text{ where } p_i \text{ are distinct primes and } a_i \text{ are positive integers, then } q(n) \text{ is an eigenvalue of } \Gamma(Z_n) \text{ with multiplicity } \frac{n}{\prod_{i=1}^{m} p_i}. \]

Proof. Let us first assume that \( n \) is a product of distinct primes, i.e., \( n = p_1p_2 \cdots p_m \). By using Lemma 5.1, \( G_2 \) is connected, and hence, 0 is an eigenvalue of \( L(G_2) \) with multiplicity 1, which in turn using equation (4) implies that \( q(n) \) is an eigenvalue of \( \Gamma(Z_n) \) with multiplicity 1. Since \( \frac{n}{\prod_{i=1}^{m} p_i} = 1 \), the theorem holds true.

We now assume that \( n \) is not a product of distinct primes, i.e., \( a_i > 1 \) for at least one \( 1 \leq i \leq m \). The set of vertices of \( G_2 \) in \( \{p_1p_2 \cdots p_m\} \setminus \{0\} \) are not adjacent to any other vertex in \( G_2 \). Since the set \( \{p_1p_2 \cdots p_m\} \setminus \{0\} \) has \( n - \prod_{i=1}^{m} p_i - 1 \) elements, the graph \( G_2 \) has \( n - \prod_{i=1}^{m} p_i \) connected components. Hence, 0 is an eigenvalue of \( L(G_2) \) with multiplicity \( \frac{n}{\prod_{i=1}^{m} p_i} \), which in turn using equation (4) implies that \( q(n) \) is an eigenvalue of \( \Gamma(Z_n) \) with multiplicity \( \frac{n}{\prod_{i=1}^{m} p_i} \). \[ \Box \]

6 Properties of \( G_2 \)

In this section, we discuss some properties of \( G_2 \) defined in equation (3), which is an induced subgraph of \( \Gamma(Z_n) \). In the next theorem, we find the values of \( n \) for which \( G_2 \) is bipartite.

\[ \text{Theorem 6.1.} \quad G_2 \text{ is bipartite if and only if } n = p^a q^b, \text{ where } p, q \text{ are primes and } a, b \text{ are positive integers.} \]

Proof. If \( n = p^a q^b \), the positive proper divisors of \( n \) are \( p, p^2, \ldots, p^{a-1}, p^a, q, q^2, \ldots, q^{b-1}, q^b, \) and \( p^i q^j \), where, \( 1 \leq i \leq a, 1 \leq j \leq b, i + j \neq a + b. \)

Since \( G_2 \) is the induced subgraph of \( \Gamma(Z_n) \) on the set \( T \setminus \{0\} \) and \( T \setminus \{0\} = \cup_d A_d \), where \( d_i \) is a positive proper divisor of \( n \), by using Lemma 3.6, we find that any vertex \( x \in A_{d_i} \), where \( 1 \leq i \leq a \) is adjacent to every vertex of \( A_{d_j} \), where \( 1 \leq j \leq b \) (Figure 4).

Also the vertices in the sets \( A_{d_i} \), where \( 1 \leq i \leq a, 1 \leq j \leq b, i + j \neq a + b \) are isolated.\(^1\)

By using the aforementioned information, we partition the vertex set of \( G_2 \) in the following way:

Let \( C \) be the set containing those vertices of \( \Gamma(Z_n) \), which belong to \( A_{p_i} \), where \( 1 \leq i \leq a \) and let \( D \) be the set consisting of those vertices of \( \Gamma(Z_n) \), which belong to the sets \( A_{d_i} \), where \( 1 \leq j \leq b \). We also put the remaining vertices of \( \Gamma(Z_n) \), which belong to \( G_2 \) in either \( C \) or \( D \) according to our wish.

Thus, \( G_2 = C \cup D \) and for any edge \( xy \) in \( G \), either \( x \in C, y \in D \) or \( x \in D, y \in C \), which implies that \( G_2 \) is bipartite.

Conversely, suppose that prime decomposition of \( n \) has more than two prime factors. Let \( n = \prod_{i=1}^{m} p_i^{a_i}, \) where \( p_i \) are distinct primes and \( a_i \) are positive integers. Let us further assume that \( m \geq 3 \).

We consider the vertices \( p_1, p_2, \) and \( p_3 \) of \( \Gamma(Z_n) \). Since \( \text{gcd}(p_i, n) = p_i \neq 1 \) for all \( 1 \leq i \leq 3 \), we find that \( p_i \) are vertices of \( G_2 \).

Clearly, \( p_1 \in A_{p_i} \) for all \( 1 \leq i \leq 3 \). By using Lemma 3.6, we find that \( p_1 \)'s are adjacent to each other and hence form a triangle. Thus, when \( m \geq 3, G_2 \) contains a 3-cycle. By using Theorem 2.1, we conclude that \( G_2 \) is not bipartite. Thus, \( G_2 \) is not bipartite when the prime decomposition of \( n \) has more than two prime factors.

Hence, \( G_2 \) is bipartite if and only if \( n = p^a q^b \), where \( p, q \) are primes and \( a, b \) are positive integers and the result follows. \[ \Box \]

---

\(^1\) A vertex is isolated if it has degree 0.
We now discuss the vertex connectivity of $G_2$. Since $G_2$ is connected if and only if $n$ is a product of distinct primes, we discuss $\kappa(G_2)$ when $n = p_1 p_2 \cdots p_m$, where $p_i, 1 \leq i \leq m$ are distinct primes. We first give an example to illustrate $\kappa(G_2)$.

**Example 6.2.** Suppose $n = 3 \times 5 \times 7$. Consider the vertex 15 in $G_2$. Consider the set $\{7, 14, 28, 49, 56, 77, 91, 98\}$. Thus, the set $\{7, 14, 28, 49, 56, 77, 91, 98\}$ is a separating set of $G_2$.

The elements of the set $\{7, 14, 28, 49, 56, 77, 91, 98\}$ are of the form $\{7k : 1 \leq k \leq 14, \gcd(k, 14) = 1\}$. We find that $\kappa(G_2) \leq 8 = \varphi(15)$.

We prove the aforementioned formally in the following theorem:

**Theorem 6.3.** If $n = p_1 p_2 \cdots p_m$, then $\kappa(G_2) \leq \varphi(p_1 p_2 p_3 \cdots p_{m-1})$.

**Proof.** We first verify the result when $n$ is a product of two distinct primes. The graph of $G_2$ when $n$ is a product of two distinct primes has been shown in Figure 3. If $n = p_1 p_2$, then $G_2$ is the join of two disconnected graphs having vertex sets as $A_{p_1}$ and $A_{p_2}$, and hence, $\kappa(G_2) = \min|V_1|, |V_2| = \min(p_1 - 1, p_2 - 1) = p_1 - 1 = \varphi(p_1)$, and hence, our result holds.

When $n = \prod_{i=1}^m p_i$ where $m > 2$, then the vertex $p_1 p_2 \cdots p_{m-1}$ of the graph $G_2$ is adjacent only to those members $a$ of the set $\langle p_m \rangle \setminus \{0\}$ such that $\gcd(a, p_1 p_2 \cdots p_{m-1}) = 1$. The number of those elements $a$ such that $\gcd(a, p_1 p_2 \cdots p_{k-1}) = 1$ equals $\varphi(p_1 p_2 \cdots p_{m-1})$. Since the vertices $a$ for which $\gcd(a, p_1 p_2 \cdots p_{m-1}) = 1$ becomes a separating set of the graph $G_2$, the result follows. \hfill $\square$

### 7 Problems

In this section, we pose some problems for further research.

By using Theorem 3.13, we observe that $\Gamma(Z_n)$ is Laplacian integral for $p^\alpha q^\beta$, where $p$, $q$ are primes and $\alpha$, $\beta$ are non-negative integers. Since it is quite motivating to find those graphs that are Laplacian integral, we ask the following:

**Problem 7.1.** Is it true that $\Gamma(Z_n)$ is Laplacian integral if and only if $n = p^\alpha q^\beta$, where $p$, $q$ are primes and $\alpha$, $\beta$ are non-negative integers? If not, then find all $n$ such that $\Gamma(Z_n)$ is Laplacian integral.
Again, in Section 6, we have provided an upper bound on the vertex connectivity of the graph $G_2$, which is an induced subgraph of $\Gamma(\mathbb{Z}_n)$. Although we have provided an upper bound on $\kappa(G_2)$, the readers are encouraged to calculate the exact value of $\kappa(G_2)$ if possible. Thus, we ask the following:

**Problem 7.2.** If $n = p_1 p_2 \cdots p_m$ where $p_1 < p_2 < \cdots < p_m$ are primes, find $\kappa(G_2)$.

### 8 Conclusion

In this paper, we have determined the Laplacian spectrum of the comaximal graph of the ring of integers modulo $n$, where $n \geq 2$. We first derive some structural properties and then determine an expression for the characteristic polynomial of $\Gamma(\mathbb{Z}_n)$. We have also calculated the vertex connectivity and algebraic connectivity of $\Gamma(\mathbb{Z}_n)$ and shown them to be equal for all $n$. An upper bound on the second largest Laplacian eigenvalue of $\Gamma(\mathbb{Z}_n)$ has been obtained and a necessary and sufficient condition for its equality has also been determined. We further discuss the multiplicity of the Laplacian spectral radius and the multiplicity of the algebraic connectivity of $\Gamma(\mathbb{Z}_n)$. In the end, we discuss the vertex connectivity of an induced subgraph of $\Gamma(\mathbb{Z}_n)$ and also provide some problems for further research.

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