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Abstract. Motivated by a theorem of Groves and Wilton, we propose the study of the lattice of numberings of isomorphism classes of marked groups as a rigorous and comprehensive framework to study global decision problems for finitely generated groups. We establish the Rice and Rice-Shapiro Theorems for recursive presentations, and establish similar results for co-recursive presentations. We give an algorithmic characterization of finitely presentable groups in terms of semi-decidability of two decision problems: the word problem and the marked quotient problem, which we introduce. We explain how this result can be used to define algorithmic generalizations of finite presentations. Finally, we discuss how the Adian-Rabin Theorem provides incomplete answers in several respects.

1. Introduction

The study of decision problems for groups began with Max Dehn who, in 1911 ([Deh11], see [Deh87] for a translation), formulated the three famous problems which are now associated with his name: the word problem, the conjugacy problem and the isomorphism problem. His motivation for introducing these came from topology, in particular from the study of the fundamental group, which had been introduced not long before by Poincaré. Because of this, he defined his problems only for finitely presented groups: his article begins by stating that his goal is to understand the structure of “the general discontinuous groups [...] given by \(n\) generators and \(m\) relations between them”.

Even though Max Dehn defined the word problem and the conjugacy problem only for finitely presented groups, this restriction is unnecessary, because these problems are local decision problems: they are set inside a single group, and concern the elements of this group. It is not necessary to suppose that a finitely generated group \(G\) has a finite presentation, nor any kind of finite description, for the word problem to make sense in it.

The situation is different for the isomorphism problem: this is a global decision problem, and for it to make sense, one must be able to provide finite descriptions of possibly infinite groups.

The goal of this article is to discuss several research directions that concern global decision problems about finitely generated groups given by descriptions other than finite presentations. Because finite presentations of groups provide a very good basis for the study of global decision problems for groups, this may seem like a gratuitous endeavor. Our main motivation lies in the following result of Daniel Groves and Henry Wilton:

**Theorem 1** ([GW09]). There exists an algorithm that, given as input a presentation for a group \(G\) and a solution to the word problem in \(G\), determines whether or not \(G\) is free.

This result is non-trivial and its proof is not elementary, even though it answers a basic question. It relies on a good understanding of the universal theory of free groups (Makanin’s algorithm to decide whether a universal sentence is true in all free groups [Mak85]) and of the groups that are models of this theory (the limit groups [Sel01]), including results of [BKM07, Wil08, DG08].

One could go as far as stating that this result is proof of the fact that the theory of global decision problems for groups should not be built using finite presentations as the basic type of finite description of a group, but using instead finite presentations supplemented by word problem algorithms.

In any case, this result at least shows that there are interesting problems about global decision problems for groups outside of finite presentations.

1.1. Undecidability results as non-classification results. A first remark is that many descriptions other than finite presentations are already commonly used to study computability with groups: matrices for linear groups [DF19], automata [EPC92], L-presentations [Bar03], different notions of “computable presentations” for countable groups that need not be finitely generated [Mel14] (these are not presentations in the sense of generators and relations).

However, our goal here is to discuss the ways in which it is possible to obtain a theory of decision problems that serves the same purpose as the one based on finite presentations, in a very precise sense: we want to be able to consider that the algorithmic tractability of a class of groups is a (weak) form of explicit classification of these
groups. This is very much in the spirit of Max Dehn, who introduced his problems as a step towards developing a comprehensive theory of finitely presented groups.

For instance, the isomorphism problem for cyclic groups given as computable groups is unsolvable [Loc81]. One will of course not consider that this result proves that there can exist no classification of the set of cyclic groups. On the other hand, unsolvability of the isomorphism problem for finitely presented groups shows that there cannot exist a classification of finitely presented groups similar to that of finite simple groups. As another example, the fact that the set of finitely presented residually finite groups is not computably enumerable (Proposition 26) can be considered as proving that there is no good answer to the question “what are the finitely presentable residually finite groups”.

We summarize this as a problem. While its statement is informal, it will be useful throughout this article, and we use it only in unambiguous cases, as above with cyclic groups.

**Problem 2.** Define descriptions of groups that contain enough information so that undecidability results can be seen as non-classification results.

For such descriptions, we expect many global decision problems to be undecidable in general, but to become decidable when restricted to tame classes of groups (finitely generated abelian groups, finite groups).

### 1.2. General setting: numberings of marked groups.

Our first step is to provide a framework in which decision problems for different types of descriptions of groups can be studied rigorously.

In particular, the theorem of Groves and Wilton quoted above, Theorem 1, was written here exactly as it appeared in [GW09], but its statement is not completely unambiguous. In fact, Groves and Wilton themselves felt this had to be adressed, and together with Manning, they revisited it in [GMW12], proposing a precise notion, that of being *computable modulo the word problem*, in order to formalize it. We discuss this formalism and its similarity with Banach-Mazur computability in Section 3.4.

The general setting we present is the study of the lattice of equivalence classes of subnumberings of isomorphism classes of marked groups.

The notion of numbering was introduced by Malcev in [Mal61] (translated in [Mal71]). A *numbering* of a set \( X \) is a partial surjection \( \nu : \subseteq \mathbb{N} \to X \). A *subnumbering* of \( X \) is a numbering of a subset of \( X \). A function \( f : X \to Y \) between subnumbered sets \((X, \nu)\) and \((Y, \mu)\) is called \((\nu, \mu)\)-computable when there is a partial computable map \( F : \subseteq \mathbb{N} \to \mathbb{N} \) (so computable in the original sense of Church and Turing) such that for all \( n \) where \( \nu \) is defined, \( F \) is also defined and \( f(\nu(n)) = \mu(F(n)) \).

Whenever mathematical objects admit some type of finite descriptions, it is possible to define an associated numbering, the natural number mapped to a point \( x \) encoding a description of \( x \). A natural number that encodes a description of \( x \) is called a *name* of \( x \). The definition of a computable function is then understood as follows: a function is computable when, given the name of a point, it is possible to compute a name of its image.

Subnumberings are considered up to equivalence: two subnumberings \( \nu \) and \( \mu \) of \( X \) are equivalent when the identity of \( X \) is both \((\nu, \mu)\)- and \((\mu, \nu)\)-computable. The set of equivalence classes of subnumberings of \( X \) is a lattice with meet and join operations corresponding respectively to the conjunction \( \land \) and the disjunction \( \lor \): the name of a point for \( \nu \land \mu \) encodes both a name of this point for \( \nu \) and a name of this point for \( \mu \), and the name of a point for \( \nu \lor \mu \) is either a name of this point for \( \nu \), or a name of this point for \( \mu \).

In Section 3.3, we define numberings \( \nu_{FP}, \nu_{RP}, \nu_{co-RP} \) and \( \nu_{WP} \), associated respectively to groups given by finite presentations, recursive presentations, co-recursive presentations and word problem algorithms.

Finally, with these, we can give a precise statement for Theorem 1: freeness is a \( \nu_{FP} \land \nu_{WP} \)-decidable property.

In fact, all four numberings \( \nu_{FP}, \nu_{RP}, \nu_{co-RP} \) and \( \nu_{WP} \) are naturally attached to marked groups. A *k-marked group* is pair \( (G, S) \), where \( G \) is a group and \( S \in G^k \) is a tuple of elements of \( G \) that generate it.

The fact that the use of marked groups is both natural and useful is rendered apparent by several results of this article. In particular, the Rice-Shapiro Theorem for recursive presentations (Theorem 4) is interpreted as characterizing the topology of \( \nu_{RP} \)-semi-decidable sets as the Scott topology on the lattice of \( k \)-marked groups (see Section 2), and the algorithmic characterization of finitely presented groups (Theorem 7) is an effective version of the fact that finitely presented groups are the compact elements of this lattice (Lemma 13). In Section 9, we look at the Adian-Rabin Theorem from the point of view of marked groups, and describe several ways in which it provides incomplete answers.

### 1.3. Decision problems with respect to local descriptions.

We remark that the distinction between local and global problems is mirrored by a distinction between descriptions that provide local information and descriptions that provide global information.

A description provides local information if it allows to compute with the elements of the given group. For instance, it could be the solution to the word problem.
A description provides global information if it relates the given group to other groups. For instance, finite presentations of groups provide global information: a group is given by a presentation \((S \mid R)\) when it is the greatest group in which the relations of \(R\) hold. And thus any group which satisfies the relations of \(R\) will be a quotient of \((S \mid R)\).

Our first results are aimed at showing that decision problems for groups given by local information are not going to be able to satisfy the criterion given in Problem 2. We establish the Rice and Rice-Shapiro theorems for groups given by recursive presentations.

Lemma 3. Suppose that \((G, S)\) and \((H, S')\) are two recursively presented marked groups, and that \((H, S')\) is a strict marked quotient of \((G, S)\). Then no algorithm that takes as input recursive presentations of either \((G, S)\) or \((H, S')\) can stop exactly on the presentations that define \((G, S)\).

Theorem 4 (Rice-Shapiro theorem for recursive presentations). If \(P\) is property of marked groups that is semi-decidable from recursive presentations, then there exists a computably enumerable sequence of finite presentations, such that a group satisfies \(P\) if and only if it is a marked quotient of a group defined by one of these presentations.

We explain that this result shows that the topology generated by semi-decidable properties of recursively presented groups is the Scott topology on the set of isomorphism classes of marked groups ordered by morphisms.

Corollary 5 (Rice theorem for recursive presentations). There is no non-trivial decidable marked group property for groups given by recursive presentations.

Out of these three results, Lemma 3 is the most problematic one regarding Problem 2, because it can be applied even in restricted settings.

Mann coined in [Man82] the term co-recursive presentation: a group \(G\) with a generating family \(S\) is co-recursively presented if there exists an algorithm that enumerates all words of \((S \cup S^{-1})^*\) that define non-identity elements of \(G\). We establish results about co-recursive presentations that are similar to these quoted above about recursive presentations, in particular on the impossibility to distinguish between a group and a strict quotient of it, and a Rice theorem which states that there is no decidable property of groups given by co-recursive presentations. These appear in Section 5. We note however that there exist dissymmetries between recursive and co-recursive presentations (the lattice operations of the set of \(k\)-marked groups are computable from recursive presentations, but only the join is computable from co-recursive presentations (Proposition 35), and there is no notion of finite co-presentation).

1.4. Finite presentations and marked quotient algorithms. Moving on, it seems that purely local information will not permit to obtain a theory of decision problems for groups that respects the criterion described in Problem 2. To obtain descriptions of groups that may have a chance to satisfy this criterion, we start by analyzing the case of finite presentations.

Definition 6 (Marked quotient problem). Let \((G, S)\) be a marked group. The marked quotient problem for \((G, S)\) is the problem of deciding if a marked group \((H, S')\) given by a recursive presentation is a marked quotient of \((G, S)\).

When a residually recursively presented marked group \((G, S)\) has a semi-decidable marked quotient problem, an algorithm that semi-decides this problem will constitute a finite description of \((G, S)\). This allows to define a numbering \(\nu_{MQA}\). We prove:

Theorem 7 (Algorithmic characterization of finitely presented groups). A marked group \((G, S)\) is finitely presented if and only if it has semi-decidable word problem and semi-decidable marked quotient problem. And this result is uniform: it states an equivalence of numberings

\[
\nu_{FP} \equiv \nu_{RP} \land \nu_{MQA}.
\]

Finitely presented groups can thus be characterized in terms of solvability of decision problems. Note that, as finite presentations of groups play a central role in the theory of decision problems for groups, it seems only fair that this theory can in turn account for why this should be the case.

Theorem 7 is not difficult to prove, it is obtained as a corollary to the Rice-Shapiro Theorem for recursive presentations. Its main interest lies in the fact that it renders explicit the local and global information given by finite presentations: semi-decidability of the word problem is a purely local form of information, which is embodied in the numbering \(\nu_{RP}\) associated to recursive presentations, whereas semi-decidability of the marked quotient problem is a purely global form of information.

And what may be the most important remark of this article is that there is an imbalance between these two types of informations: semi-decidability of the word problem a very weak form of local information, whereas semi-decidability of the marked quotient problem is a very strong form of global information.
The study of decision problems for groups given by word problem algorithms together with finite presentations, as begun by Groves and Wilton in [GW09], can be seen as a rebalancing of the numbering of finite presentations:

$$\nu_{FP} \equiv \nu_{RP} \land \nu_{MQA}$$

is replaced by

$$\nu_{WP} \land \nu_{FP} \equiv \nu_{WP} \land \nu_{MQA},$$

and thus the left hand side numbering is strengthened.

But at the same time, it is also possible to weaken the right hand side numbering that is associated to marked quotient algorithms. Indeed, global decision problems for groups are always considered in relation to their possible restrictions to different classes of groups. When working in a certain set of groups, it is very natural to restrict our attention to the marked quotient problem relative to $\mathcal{C}$. That is to say that we consider the following problem associated to a marked group $(G, S)$:

**Definition 8** (Relative marked quotient problem). Let $(G, S)$ be a marked group and $\mathcal{C}$ a set of marked groups. The **marked quotient problem for $(G, S)$ relative to $\mathcal{C}$** is the problem of deciding if a marked group $(H, S')$ that belongs to $\mathcal{C}$ given by a recursive presentation is a marked quotient of $(G, S)$.

There are already several examples in the literature of infinitely presented groups with marked quotient algorithms relative to certain classes of groups. For instance, in [Har11] and [BEH08] it was shown that groups that admit L-presentations have marked quotient algorithms with respect to finite and nilpotent groups.

In Section 8, we investigate the relation between notions of finite presentation modulo a certain class of groups and existence of marked quotient algorithms. We show that the existence of a marked quotient algorithm with respect to a group variety corresponds exactly to the notion of finite presentation modulo the laws of the variety (Proposition 53). We give the example of the lamplighter group as a group that has a marked quotient algorithm relative to finite groups while not being finitely presented as a residually finite group. Finally we ask:

**Problem 9.** Find a class $\mathcal{C}$ of recursively presented groups such that all groups in $\mathcal{C}$ admit marked quotient algorithms relative to $\mathcal{C}$ but such that not all groups in $\mathcal{C}$ are finitely presented as residually $\mathcal{C}$ groups.

2. **Lattice of $k$-marked groups and Scott topology**

Here, we introduce the lattice of $k$-marked groups and the Scott topology on it.

2.1. **The lattice of $k$-marked groups.** Fix $k \in \mathbb{N}$. A **$k$-marked group** is a finitely generated group together with a $k$-tuple of elements that generate it.

A **morphism of $k$-marked groups** from $(G, S)$ to $(H, S')$ is a group morphism between $G$ and $H$ that maps $S$ to $S'$ respecting the order. Such a morphism is an isomorphism if the underlying group morphism is a group isomorphism, and marked groups are considered up to isomorphism.

Note that there is at most one morphism from a $k$-marked group to another $k$-marked group, and that if there are morphisms $(G, S) \to (H, S')$ and $(H, S') \to (G, S)$, then $(G, S)$ and $(H, S')$ are isomorphic as marked groups: isomorphism classes of marked groups form a partially ordered set for the quotient relation.

Let $(\mathcal{G}_k, \to)$ be the poset of isomorphism classes of $k$-marked groups.

Let $(\mathcal{G}, \to)$ be the poset of isomorphism classes of marked groups, obtained by taking the disjoint union of the sets $(\mathcal{G}_k, \to)$.

Note that we consider that two groups marked by families of different cardinalities are incomparable for $\to$, contrary to what is customary (see for instance [CG05]): we do not identify a $k$-marked group $(G, (s_1, ..., s_k))$ with the $k + 1$-marked group $(G, (s_1, ..., s_k, 1_G))$. The reason for this is that the different descriptions of marked groups that we consider all provide explicitly the number $k$ of generators in a marking. The problem “is a given marked group a $k$-marked group?” is thus always decidable, and the set of $k$-marked groups should be clopen in the Scott topology of $(\mathcal{G}, \to)$. See also Section 9.1 where the fact that we do not identify $(G, (s_1, ..., s_k))$ with $(G, (s_1, ..., s_k, 1_G))$ plays a role.

The poset $(\mathcal{G}_k, \to)$ is in fact a lattice: any two marked groups admit both a sup and an inf with respect to the order. We define its lattice operations.

**Proposition 10.** Any pair of $k$-marked groups has an infimum in $(\mathcal{G}_k, \to)$.

**Proof.** Let $(G, S)$ and $(H, S)$ be two $k$-marked groups, generated by families which we identify via the canonical bijection. Consider two presentations $\pi_1 = (S | R_1)$ and $\pi_2 = (S | R_2)$ that define respectively $(G, S)$ and $(H, S)$. Define $(G, S) \wedge (H, S)$ to be the group given by the presentation $(S | R_1, R_2)$. It is immediate to check that this group constitutes the infimum of $\{(G, S), (H, S)\}$ for $\to$. □
Proposition 11. Any pair of $k$-marked groups has a supremum in $(\mathcal{G}_k, \to)$.

Proof. Let $(G, S)$ and $(H, S')$ be two $k$-marked groups, and denote $S = \{s_1, \ldots, s_k\}$ and $S' = \{s_1', \ldots, s_k'\}$. Denote by $(G, S) \vee (H, S')$ the subgroup of the group $G \times H$ generated by the elements $\{(s_1, s_1'), \ldots, (s_k, s_k')\}$. It is easy to check that this group constitutes the supremum of $\{(G, S), (H, S')\}$ for $\to$. □

Another way to define the meet is by expressing $(G, S)$ and $(H, S')$ as quotients of a free group: if $(\mathcal{F}_S, S)$ maps onto $(G, S)$ with kernel $N_1$ and onto $(H, S')$ with kernel $N_2$, then $(G, S) \vee (H, S')$ is given by $(\mathcal{F}_S/(N_1 \cap N_2), S)$.

In fact, it is easy to extend both constructions above to arbitrary suprema and infima, simply by making the same constructions as above, but with infinitely many marked groups. A lattice is bounded if it has a least and a greatest element.

The following thus follows.

Theorem 12. The ordered set $(\mathcal{G}_k, \to)$ is a complete bounded lattice with minimum the trivial group (equipped with its unique $k$-marking) and with maximum a rank $k$-free group (which also has a unique $k$-marking).

2.2. Scott topology of the lattice of $k$-marked groups. A subset $A$ of a partially ordered set $(L, \leq)$ is called Scott open if it is an upper set, i.e. $\forall x \in L, \forall y \in A, y \leq x \implies x \in A$, and if it is inaccessible by directed joins, i.e. if $D \subseteq L$ is a directed set with a supremum $\sup D$, and if there exists $y \in A$ with $y \leq \sup D$, then there exists $x \in D$ with $y \leq x$.

Define the way below relation $\ll$ on $L$ by $x \ll y$ if and only if for every directed subset $D$ of $L$ that has a supremum $\sup D$, if $y \leq \sup D$, then there exists $z$ in $D$ with $x \leq z$.

An element $x$ of $L$ is compact if $x \ll x$. The following lemma is immediate.

Lemma 13. A marked group is compact in $(\mathcal{G}_k, \rightarrow)$ if and only if it is finitely presented.

Proof. Suppose that $(G, S)$ is finitely presented. Then if $(G, S) \to \bigwedge_{n \in \mathbb{N}} (H_n, S_n)$, there must $N$ such that the finitely many relations that define $(G, S)$ already hold in some $\bigwedge_{n \leq N} (H_n, S_n)$, so that $(G, S) \to \bigwedge_{n \leq N} (H_n, S_n)$ holds.

Suppose now that $(G, S)$ is not finitely presentable, and let $\langle S \mid r_1, r_2, \ldots \rangle$ be an infinite presentation of it. Then

$$(G, S) = \bigwedge_{n \in \mathbb{N}} \langle S \mid r_1, r_2, \ldots, r_n \rangle,$$

and yet $(G, S) \to \langle S \mid r_1, r_2, \ldots, r_n \rangle$ never holds. □

A discriminating family [dCGP07] for a group $G$ is a subset of $G$ that does not contain the identity element and which intersects every non-trivial normal subgroup of $G$. A group is finitely discriminating if it has a finite discriminating family.

It is clear that a finitely discriminating is compact in $(\mathcal{G}_k, \rightarrow)$. We ask:

Problem 14. Are the compact elements of $(\mathcal{G}_k, \rightarrow)$ exactly the finitely discriminating groups?

3. Lattice of numbering types

In this section, we define numberings, the conjunction and disjunction operations on numberings. While using this formalism may seem needlessly technical, we give in Section 3.4 two examples that justify the fact that at least having access to this formalism is beneficial.

3.1. Definitions of the lattice of equivalence classes of numberings.

Definition 15. Let $X$ be a set. A numbering of $X$ is a surjection $\nu$ that maps a subset $A$ of $\mathbb{N}$ onto $X$. A subnumbering of $X$ is a numbering of a subset of $X$. We denote this by: $\nu : \subseteq \mathbb{N} \to X$.

The use of subnumberings reflects the fact that, while two types of finite description of marked groups may be applicable to different sets of marked groups, we can still consider that they live in a common set, the set of subnumberings of the set of marked groups.

The pair $(X, \nu)$ is a subnumbered set. The domain of $\nu$ is a subset of $\mathbb{N}$ denoted by $\text{dom}(\nu)$.

The image $\nu(\text{dom}(\nu))$ of $\nu$ is called the set of $\nu$-computable points of $X$, and denoted $X_\nu$. Given a point $x$ in $X$, an integer $n$ such that $\nu(n) = x$ is called a $\nu$-name of $x$.

Let $(X, \nu)$ and $(Y, \mu)$ be subnumbered sets. A function $f : X \to Y$ is called $(\nu, \mu)$-computable if there exists a partial computable function $F : \subseteq \mathbb{N} \to \mathbb{N}$ such that for all $n$ in the domain of $\nu$, $f \circ \nu(n) = \mu \circ F(n)$. That is to say, there exists $F$ computable which renders the following diagram commutative:
Proposition 17. \[ \text{Let } (n, m) \mapsto (n, m) \text{ designate Cantor’s pairing function.} \]

Definition 16. If \( \nu \) and \( \mu \) are subnumberings of a set \( X \), denote by \( \nu \leq \mu \) the fact that the identity of \( X \) is \((\nu, \mu)\)-computable.

In other words, given the \( \nu \)-name of a point of \( X \), it is possible to compute a \( \mu \)-name of this point. This formalizes the fact that \( \nu \)-names contain more information on points than \( \mu \)-names.

It is easy to see that \( \leq \) defines a pre-order. We call subnumbering types the equivalence classes induced by this pre-order.

Proposition 17. The set of subnumbering types on \( X \) equipped with the order \( \leq \) is a lattice.

Let \( (n, m) \mapsto (n, m) \) designate Cantor’s pairing function.

Proof. We define the meet and join operations, which correspond to the disjunction \( \lor \) and to the conjunction \( \land \).

Let \( \nu \) and \( \mu \) be subnumberings of \( X \).

Define \( \nu \lor \mu \) by setting, for any natural number \( k \), \( \nu \lor \mu(2k) = \nu(k) \) and \( \nu \lor \mu(2k + 1) = \mu(k) \). The domain of \( \nu \lor \mu \) is the set \( \{2k, k \in \text{dom}(\nu)\} \cup \{2k + 1, k \in \text{dom}(\mu)\} \).

Define a subnumbering \( \nu \land \mu \) by the following:

\[
\text{dom}(\nu \land \mu) = \{(n, m) \in \mathbb{N}, n \in \text{dom}(\nu), m \in \text{dom}(\mu), \nu(n) = \mu(m)\},
\]

\[
\forall (n, m) \in \text{dom}(\nu \land \mu), \nu \land \mu((n, m)) = \nu(n).
\]

In each case, it is straightforward to check that the defined operations do constitute meet and join operations. \( \square \)

Remark 18. The conjunction operation is the more important one of the two operations. Indeed, a function is computable with respect to a disjunction \( \nu \lor \mu \) if and only if it is computable for both \( \nu \) and \( \mu \). Thus the study of \( \nu \lor \mu \) amounts to that of both \( \nu \) and \( \mu \). On the contrary, while it is true that if a function is either computable for \( \nu \), or computable for \( \mu \), it will be \( \nu \land \mu \) computable, this is not an equivalence. A striking example of this fact is given by the Groves-Wilton theorem quoted in the introduction of this article, Theorem 1.

3.2. Semi-decidable sets, Ershov topology. Let \( (X, \nu) \) be a numbered set.

A subset \( A \subseteq X \) is \( \nu \)-semi-decidable if there is an algorithm that, given the \( \nu \)-name of a point \( x \) in \( X \), stops if and only if \( x \) belongs to \( A \).

A set is \( \nu \)-co-semi-decidable if its complement is \( \nu \)-semi-decidable.

A set is \( \nu \)-decidable if it is both semi-decidable and co-semi-decidable.

The intersection of finitely many \( \nu \)-semi-decidable sets is again \( \nu \)-semi-decidable, and the union of a sequence of uniformly \( \nu \)-semi-decidable sets is again \( \nu \)-semi-decidable. Thus the set of \( \nu \)-semi-decidable sets resembles a topology. The actual topology generated by the semi-decidable sets on a numbered set \( (X, \nu) \) is called the Ershov topology [Spr98]. We will characterize the Ershov topology of the numbering of recursive presentations as the Scott topology on the lattice of marked groups. The Ershov topology associated to finite presentations is the discrete topology.

The set \( X \) is computably enumerable for \( \nu \) if it is empty or if there is a \((\text{id}_\mathbb{N}, \nu)\)-computable surjection \( f: \mathbb{N} \to X \). This is one of several possible definitions for “being computably countable”. This means that it is possible to algorithmically produce a list that contains at least one \( \nu \)-name for each element of \( X \).

A \( \nu \)-computable sequence in \( X \) is a \((\text{id}_\mathbb{N}, \nu)\)-computable function \( f: \mathbb{N} \to X \).

3.3. Numberings associated to finite presentations, recursive presentations, etc. Whenever we consider objects given by finite data, it is possible to encode this data with natural numbers, and to define the associated numbering. In most cases, there is little to no benefit in defining precisely these numberings. Because we are considering numberings up to equivalence, the choice of a way of encoding the considered objects is inconsequential.

In practical implementation of algorithms, the choice of an encoding for the considered objects becomes important with respect to time complexity issues. While this is an interesting problem, this is not a problem that will be solved by precisely defining how to encode the considered objects with natural numbers from an abstract standpoint, without using an explicit programming language. This is another reason to claim that defining precisely numberings is not useful.

In any case, we still define here precisely the numberings that are used in this paper, simply to show that it is possible.
We use a pairing function \((n, m) \mapsto \langle n, m \rangle\), which is a computable bijection \(\mathbb{N}^2 \to \mathbb{N}\) with a computable inverse. Define a numbering \(\Delta\) of finite subsets of \(\mathbb{N}\) by the following:
\[
\Delta(\langle n, m \rangle) = \{u_1, ..., u_n\},
\]
where \(m = \langle u_1, \langle u_2, ..., \langle u_{n-1}, u_n \rangle \ldots \rangle\).

Fix also a countably infinite alphabet \(\mathcal{A} = \{x_1, x_2, ...\}\). For each \(n\), define \(\mathcal{A}_n = \{x_1, ..., x_n\}\), let \(\mathcal{A}_n^{-1} = \{x_1^{-1}, ..., x_n^{-1}\}\) be formal inverses to the elements of \(\mathcal{A}_n\). Let \((\mathcal{A}_n \cup \mathcal{A}_n^{-1})^*\) be the set of words on the alphabet \(\mathcal{A}_n \cup \mathcal{A}_n^{-1}\). Finally, let \(\theta_n\) be a natural bijection \(\theta_n : \mathbb{N} \to (\mathcal{A}_n \cup \mathcal{A}_n^{-1})^*\) (for instance ordering \((\mathcal{A}_n \cup \mathcal{A}_n^{-1})^*\) by the shortlex order).

Composition \(\theta_n\) and \(\Delta\) yields a numbering of finite subsets of \((\mathcal{A}_n \cup \mathcal{A}_n^{-1})^*\).

**Definition 19.** The numbering \(\nu_{FP}\) of finitely presented marked groups is defined by:
\[
\nu_{FP}(\langle n, m \rangle) = \langle \mathcal{A}_n \mid \theta_n \circ \Delta(m) \rangle
\]

To define the numbering associated to recursive presentations, we use the canonical numbering of all partial computable functions, as first defined by Church Turing and Kleene: \(\varphi_0, \varphi_1, \varphi_2, ...\) is an enumeration of the set \(\mathcal{P}\mathcal{R}\) of partial computable functions. The standard numbering of computably enumerable subset of \(\mathbb{N}\) is given by \(W_i = \text{dom}(\varphi_i)\).

**Definition 20.** We define the numbering \(\nu_{RP}\) associated to recursive presentations by:
\[
\nu_{RP}(\langle n, i \rangle) = \langle \mathcal{A}_n \mid W_i \rangle.
\]

We will similarly define the numbering associated to co-recursive presentations. Note that, contrary to what the name indicates, a co-recursive presentation is not a group presentation, in the sense of a set of generators and of relators. A **co-recursive presentation** [Man82] is an algorithmic enumeration of all the words that define non-identity elements in a given marked group, i.e. an enumeration of the co-word problem in this marked group.

A set of words \(R \subseteq (\mathcal{A}_n \cup \mathcal{A}_n^{-1})^*\) is the co-word problem of a group if and only if the complement \(R^c\) of \(R\) in \((\mathcal{A}_n \cup \mathcal{A}_n^{-1})^*\) is a normal subgroup in the free group over \(\mathcal{A}_n\).

**Definition 21.** We define the numbering \(\nu_{co-RP}\) associated to co-recursive presentations by:
\[
\text{dom}(\nu_{co-RP}) = \{(n, i) \in \mathbb{N}, \theta_n(W_i) \text{ is a co-word-problem}\};
\]
\[
\nu_{co-RP}(\langle n, i \rangle) = \langle \mathcal{A}_n \mid \theta_n(W_i)^c \rangle.
\]

Note that the domain of this numbering is not a computably enumerable subset of \(\mathbb{N}\), this cannot be avoided.

Recall from the introduction that a marked quotient algorithm for a marked group \((G, S)\) is an algorithm that stops exactly on the codes for recursive presentations of marked quotients of \((G, S)\). We define the associated numbering by relying on our previous definition of the numbering associated to recursive presentations.

**Definition 22.** We define a numbering \(\nu_{MQA}\) of recursively presented groups associated to marked quotient algorithms by:
\[
\nu_{MQA}(\langle n, i \rangle) = (G, S) \iff (\nu_{RP}(\langle n, j \rangle) \text{ is a marked quotient of } (G, S) \iff j \in W_i).
\]

The domain of \(\nu_{MQA}\) is defined to be exactly those \((n, i)\) that do define a marked quotient algorithm.

We prove in Section 7 that the above definition is correct, that is to say that the marked group \((G, S)\) is indeed uniquely determined by the above formula.

Finally, a word problem algorithm is simply a recursive presentation together with a co-recursive presentation.

**Definition 23.** The numbering \(\nu_{WP}\) associated to word problem algorithms is given by \(\nu_{WP} \equiv \nu_{RP} \land \nu_{co-RP}\).

### 3.4. Examples where the explicit use of numberings is beneficial.

#### 3.4.1. First example: recursive presentations.
Our first example comes from a result of Lockhart from [Loc81]. Here is stated the following: "There is a r.e. class of recursive presentations with uniformly solvable word problem for which the properties of freeness and finiteness are unrecognizable". The groups in question are finite and infinite cyclic groups.

But note that by Lemma 33, the word problem can never be uniformly solvable from recursive presentations in a class of marked groups that contain a marked group and a strict quotient of it. This seems to be contradictory with what is written above.

The reason why there is no contradiction is that in the result of Lockhart, the “r.e. class of recursive presentations” is not a set of presentations, but a certain set of algorithms that produce these presentations. The word problem would not be uniformly solvable on these groups if we allowed arbitrary algorithms that still produced the correct presentations.
And in fact, the result of Lockhart concerns groups given by word problem algorithms, and not groups given by recursive presentations. Formulating this result in terms of numberings makes its ambiguities disappear.

3.4.2. Second example: decidability modulo the word problem and Banach-Mazur computability. In [GMW12], the authors define a precise notion of group property recursive modulo the word problem, in order have a rigorous concept that formalizes the idea that a marked group is given by a finite presentations and a solution to its word problem.

Note that using the formalism of numberings, this amounts simply to defining a numbering via the conjunction operation: if $\nu_{FP}$ is the numbering associated to finite presentations, and $\nu_{WP}$ the numbering associated to word problem algorithms, this is the study of the decidable properties of $\nu_{WP} \land \nu_{FP}$.

We want to note here that the formalism introduced in [GMW12] is very similar to that of Banach-Mazur computability.

A real $x$ is computable if there is a computable map $f : \mathbb{N} \to \mathbb{Q}$ which on input $k$ produces a rational $q$ with $|x - q| < 2^{-k}$. A computable sequence of computable reals is a sequence $(u_n) \in \mathbb{R}^\mathbb{N}$ such that there is a computable map $f$ of two variables which, given $n$ and $k$, produces a rational approximation of the real $u_n$ precise within $2^{-k}$.

Let $\mathbb{R}_c$ be the set of computable reals.

Definition 24. A function $f : \mathbb{R}_c \to \mathbb{R}_c$ is Banach-Mazur computable if it maps computable sequences of computable reals to computable sequences of computable reals.

This notion of computability was introduced by Banach and Mazur right after Church and Turing formalized computability for functions defined on the natural numbers [BM37]. In modern computable analysis, Banach-Mazur computability is not considered to be “the good” notion of computability on the real numbers, although it still constitutes an important tool to study other notions of computability. For instance, Hertling showed in [Her02] that the classical effective continuity theorem of Kreisel-Lacombe-Schoenfield-Ceitin [Ceit67] does not hold for Banach-Mazur computable functions on the real numbers: this increases our understanding of the original theorem.

There are two main problems related to the notion of Banach-Mazur computability. The first is that it does not permit to define a Cartesian closed category: a Banach-Mazur computable function is not associated to a finite description, and so we cannot ask what is “a Banach-Mazur computable function defined on the set of Banach-Mazur computable functions”. The second one is that in practice, whenever proving that some function $f$ is computable, we always prove more than Banach-Mazur computability: we provide a single method to compute the function $f$, which is uniform on all computable sequences. And thus working with Banach-Mazur computability, one ends up systematically proving strictly more than what one states.

The definition of recursive modulo the word problem considered in [GMW12] is as follows.

Definition 25 ([GMW12], Definition 2.4). A class of finitely presented groups $C$ is said to be recursive modulo the word problem if, whenever $D$ is a set of finite presentations on which the word problem is uniformly solvable, there exist two computably enumerable sets of finitely presented groups $X$ and $Y$ such that $C \cap D \subseteq X$, $D \setminus C \subseteq Y$, $C \cap D \cap Y = \emptyset$ and $(C \setminus D) \cap X = \emptyset$.

The common feature of this and of Banach-Mazur computability is the common non-effective quantification: a problem is considered solvable if it is solvable on all sets with a certain property (in the first case, on computable sequences, here on sets where the word problem is uniformly solvable from finite presentations), but the solutions on these different sets may be different, and do not have to depend uniformly on the considered set.

The difference between the above and Banach-Mazur computability lies in the fact that the set $D$ is not supposed to be computably enumerable. But in fact, we want to argue that the above definition would be improved by restricting the quantification on $D$ to c.e. sets of finitely presented groups. Our argument lies in the following proposition, which follows from the work of Kharlampovich, Miasnikov and Sapir [KMS17]:

Proposition 26. The set of finitely presented residually finite groups is not computably enumerable (for $\nu_{FP}$), and furthermore it cannot be contained in a $\nu_{FP}$-c.e. set of finitely presented groups with uniformly solvable word problem.

We will use the following lemma. Recall that time complexity of the solution to the word problem inside a group with solvable word problem is a group invariant up to standard asymptotic equivalence of functions [Sap11].

Lemma 27. Let $C$ be a set of finitely presented groups in which the word problem is uniformly solvable. If $C$ is $\nu_{FP}$-c.e., then there exists a computable function which is a common asymptotic upper bound to the time complexity for the word problem for groups in $C$. 
Proof. By standard embedding methods (see [Dar15]), it is possible to embed the restricted direct product of all the groups in \( C \) in a finitely generated group with solvable word problem. The time complexity of the word problem in this group constitutes the desired asymptotic upper bound. \( \square \)

Proof of Proposition 26. By [McK43], the word problem is uniformly solvable in finitely presented residually finite groups. By [KMS17], there exist finitely presented residually finite groups of arbitrarily high time complexity (i.e. above any given computable function). We can conclude by Lemma 27. \( \square \)

Remark that the above lemma would also apply to the set of finitely presented simple groups if the famous Boone-Higman [BBMZ23] conjecture were true:

**Proposition 28.** If the Boone-Higman conjecture holds, then the set of finitely presented simple groups cannot be contained in a c.e. set of finitely presented groups with uniformly solvable word problem.

Now consider the consequences of Proposition 26 on Definition 25: to prove that a set \( C \) is recursive modulo the word problem, applying the definition with \( D \) being the set of residually finite groups, one needs to find c.e. sets \( X \) and \( Y \) that, in particular, must cover \( D \). But then the word problem cannot be uniformly solvable in \( X \cup Y \), and thus one ends up working in sets where the word problem algorithm is not uniformly solvable to show that something is recursive modulo the word problem.

If on the contrary we modify Definition 25, and only allow \( D \) to be a c.e. set, then the resulting new definition of computable modulo the word problem is exactly Banach-Mazur computability. And, as we explained above, while this notion is useful for theoretical purposes, it suffers from the fact that in practice, when proving something Banach-Mazur computable, one always proves that the considered function is computable for a more restrictive notion of computability. For instance, one could rewrite the results of [GMW12], replacing «computable modulo the word problem» by «computable for the numbering \( \nu_{FP} \land \nu_{FP} \)», and leaving all proof ideas unchanged. The results thus stated would be strictly stronger.

4. Rice-Shapiro for recursive presentations

Here we establish:

**Theorem 29** (Rice-Shapiro theorem for recursive presentations). If \( P \) is property of marked groups that is semi-decidable from recursive presentations, then there exists a computably enumerable sequence of finite presentations, such that a group satisfies \( P \) if and only if it is a marked quotient of a group defined by one of these presentations.

**Corollary 30.** The topology generated by sets that are semi-decidable from recursive presentations is the Scott topology on the lattice of marked groups.

And thus:

**Corollary 31.** Any property of marked groups that is semi-decidable from recursive presentations is quotient-stable.

Finally, the Rice theorem follows immediately from the Rice-Shapiro theorem.

**Corollary 32** (Rice theorem for recursive presentations). There is no non-trivial decidable group property for groups given by recursive presentations.

**Proof.** Let \( P \) be a decidable property. Either \( P \) or its complement contains the free group. Then, by Corollary 31, the one that contains it is in fact the set of all marked groups. \( \square \)

The proof of Theorem 29 follows closely the usual one [Rog87], we simply exchange the lattice of subsets of \( N \) by the lattice of marked groups. It is based on two intermediate lemmas.

**Lemma 33.** Suppose that \( (G,S) \) and \( (H,S') \) are two recursively presented marked groups, and that \( (H,S') \) is a strict marked quotient of \( (G,S) \). Then no algorithm that takes as input recursive presentations of either \( (G,S) \) or \( (H,S') \) can stop exactly on the presentations that define \( (G,S) \).

In other words, \( \{ (G,S) \} \) is not a \( \nu_{RF} \)-semi-decidable subset of \( \{ (G,S),(H,S') \} \).

**Proof.** Consider a presentation \( (S \mid r_1,r_2,...) \) of \( (G,S) \) such that \( r_1,r_2,... \) is a computable sequence, and a presentation \( (S' \mid q_1,q_2,...) \) of \( (H,S') \), \( q_1,q_2,... \) also being a computable sequence of relations. Note that in these presentations we can in fact identify \( S \) and \( S' \) without changing the considered marked groups.

Consider an effective enumeration of all partial computable functions \( \phi_0, \phi_1, \phi_2, \phi_3,... \). We write \( \phi_n(k) \uparrow \) if a run of \( \phi_n \) on input \( k \) does not end in less than \( t \) steps, and \( \phi_n(k) \downarrow \) if it halts in exactly \( t \) steps.
For each \( n \), we define a computable enumeration \( E_n = (s_1, s_2, \ldots) \) of relations by the following: \( s_t = r_t \) if \( \varphi_n(n) \uparrow \), and \( s_t = q_{t-t_0} \) if \( \varphi_n(n) \downarrow t_0 \).

By the smn-theorem, the sequence \((E_n)_{n \in \mathbb{N}}\) defines a computable sequence of recursive presentations. And \( E_n \) defines a presentation of \((H, S')\) if and only if \( \varphi_n(n) \downarrow \), and a presentation of \((G, S)\) if and only if \( \varphi_n(n) \uparrow \). We conclude by unsolvability of the halting problem.

This result implies in particular that if the word problem is uniformly solvable on a set of recursively presented groups, this set does not contain a group and a strict quotient of it. This condition is not superfluous, since recursively presented simple groups have uniformly solvable word problem (and the set of simple group does not contain a group and a strict quotient of it).

**Lemma 34.** Let \((G, S)\) be a recursively presented marked group that is not finitely presentable, and let \((r_0, r_1, \ldots)\), \(r_i \in (S \cup S^{-1})^*\), be a computable sequence of relators for it. Let \((G_n, S)\) be the marked group given by the truncated presentation \(\langle S | r_0, r_1, \ldots, r_n \rangle\).

Then no algorithm that takes as input recursive presentations of marked groups in \(\{(G, S)\} \cup \{(G_n, S), n \in \mathbb{N}\}\) can stop exactly on the presentations that define \((G, S)\).

In other words, \(\{(G, S)\}\) is not a \(\nu_{RP}\)-semi-decidable subset of \(\{(G, S)\} \cup \{(G_n, S), n \in \mathbb{N}\}\).

**Proof.** We use as in the proof of Lemma 33 an effective enumeration of all partial computable functions \(\varphi_0, \varphi_1, \varphi_2, \ldots\).

For each \( n \), we define a computable enumeration \( E_n = (s_1, s_2, \ldots) \) of relations by the following: \( s_t = r_t \) if \( \varphi_n(n) \uparrow \), and \( s_t = q_{t-t_0} \) if \( \varphi_n(n) \downarrow t_0 \).

Thus \( E_n \) gives a presentation of \((G, S)\) if and only if \( \varphi_n(n) \uparrow \), and a presentation of one of the marked groups \((G_n, S)\) if and only if \( \varphi_n(n) \downarrow \). We conclude by unsolvability of the halting problem.

Putting the above lemmas together yields a proof of Theorem 29.

**Proof of Theorem 29.** Let \( P \) be a property of marked groups semi-decidable from recursive presentations. Consider the set of finitely presented marked groups that belong to \( P \). This set is computably enumerable, since it is possible to enumerate all finite presentations, and to select from these those that satisfy \( P \).

All that we have to justify is that a group has \( P \) if and only if it is a quotient of one of these finitely presented marked groups.

Suppose that a marked group \((H, S')\) is a strict marked quotient of a group \((G, S)\) that has \( P \). Consider the restriction of \( P \) to \(\{(H, S'), (G, S)\}\). It remains a \(\nu_{RP}\)-semi-decidable property. It contains \((G, S)\). By Lemma 33, it cannot be equal to \(\{(G, S)\}\). Thus it must be all of \(\{(H, S'), (G, S)\}\), and \((H, S')\) also has \( P \).

Suppose now that \((G, S)\) is belongs to \( P \). If \((G, S)\) is finitely presentable, there is nothing to do. Otherwise, let \( B \) be the set of finitely presented marked groups that have \((G, S)\) as a quotient. By Lemma 34, \(\{(G, S)\}\) is not a \(\nu_{RP}\)-semi-decidable subset of \(\{(G, S)\} \cup B\), and thus some group in \( B \) must belong to \( P \). And thus \((G, S)\) is indeed a quotient of a finitely presentable group with \( P \).

5. Rice theorem for co-recursive presentations

One might expect to have a Rice-Shapiro theorem for groups given by co-recursive presentations that mirrors exactly the Rice-Shapiro theorem for recursive presentations, based on the exact same arguments, simply using the lattice of marked groups with reverse order, and the corresponding Scott topology. However, the proof of Theorem 29 in fact relied crucially on the set of finitely presented groups, which is a \(\nu_{RP}\)-computable sequence (all finite presentations can be enumerated, and a finite presentation can be seen as a recursive presentation, i.e. \(\nu_{FP} \leq \nu_{RP}\)) of the compact elements of \((\mathcal{G}_k, \rightarrow)\).

Note however that there is no notion of “finite co-presentation”. For instance, there is no least marked group defined on a single generator \( a \) with \( a^2 \neq 1 \). Indeed, the marked groups \(\langle a | a^3 \rangle\) and \(\langle a | a^5 \rangle\) both satisfy that \( a^2 \neq 1 \), and yet their inf in the lattice \((\mathcal{G}_1, \rightarrow)\) is the trivial group. The closest notion is that of finite discriminating family, which leads to the notion of absolute presentation of Neumann [Neu73], see [dCGP07] for a modern account. It was asked by Mann in [Man82] whether it is possible to algorithmically list all absolute presentations of groups, this problem seems to still be open today.

The following proposition also illustrates the fact that the meet and join operations of \((\mathcal{G}_k, \rightarrow)\) are not symmetrical from the point of view of computability.

**Proposition 35.** The meet and join operations of \((\mathcal{G}_k, \rightarrow)\) are computable for recursively presented groups.

The meet is computable for finitely presented groups, but the join is not, since the join of two finitely presented groups does not have to be finitely presented. Even: the join of two free groups does not have to be finitely presented.
The join operation is computable for co-recursively presented groups, but the meet is not, since the meet of two co-recursively presented groups does not have to be co-recursively presented.

Proof. Meet operation.
Consider two recursively presented groups given by $\pi_1 = \langle S | R_1 \rangle$ and $\pi_2 = \langle S | R_2 \rangle$. Their join is given by $\pi_1 = \langle S | R_1, R_2 \rangle$, which is indeed recursively presented. The same holds modulo changing recursive for finite in the above.

Consider now a finitely presented group $\langle S | R \rangle$ with unsolvable word problem. Let $R = (r_1, \ldots, r_n)$ and consider the one-relator groups $\langle S | r_i \rangle$, for $1 \leq i \leq n$. These have solvable word problem by a theorem of Magnus. Consider the meets $\langle S | r_1 \rangle \land \langle S | r_2 \rangle \land \ldots \land \langle S | r_i \rangle$, for each $i \leq n$. For $i = n$, it has unsolvable word problem. The least $i$ for which this is the case gives the meet of two groups with solvable word problem that has unsolvable word problem.

Join operation.
Consider two $k$-marked groups $(G, S)$ and $(H, S')$, and identify $S$ and $S'$ via the canonical bijection. Let $F_S$ be the free group over $S$, and express $(G, S)$ and $(H, S')$ as marked quotients of the free group: $(G, S) = (F_S/N_1, S)$ and $(H, S') = (F_S/N_2, S')$. The meet $(G, S) \lor (H, S')$ is given by $(F_S/(N_1 \cap N_2), S)$.

If $(G, S)$ and $(H, S')$ are recursively presented, then $N_1$ and $N_2$ are c.e. sets, and so is their intersection, thus $(G, S) \lor (H, S')$ is also recursively presented. And this is uniform.

If $(G, S)$ and $(H, S')$ are co-recursively presented, then the complements of $N_1$ and $N_2$ are c.e. sets, and so is their union, thus $(G, S) \lor (H, S')$ is also co-recursively presented. And this is uniform.

Finally, the fact that $(G, S)$ and $(H, S')$ can be finitely presented without $(G, S) \lor (H, S')$ being so follows from well known results on the Mihailova subgroup. Let $\langle s_1, \ldots s_k | r_1, \ldots, r_n \rangle$ be a finitely presentable group. Consider two $k+n$-markings of the free group $F_S$, namely
\[(F_S, (s_1, \ldots, s_k, r_1, \ldots, r_n)),\]
\[(F_S, (s_1, \ldots, s_k, 1, \ldots, 1)).\]
The meet of these two markings is precisely the Mihailova subgroup [Mih68], which by a theorem of Grunewald [Gru78] is not finitely presentable as soon as $\langle s_1, \ldots s_k | r_1, \ldots, r_n \rangle$ defines an infinite group. □

All this makes it so that we leave as an open problem the Rice-Shapiro Theorem for co-recursive presentation. We still establish the Rice Theorem.

Proposition 36. If $P$ is a marked group property that is semi-decidable from co-recursive presentations, than any group that has a marked quotient with $P$ also has $P$.

This proposition follows directly from Lemma 38. As an immediate corollary we get:

Corollary 37 (Rice theorem for co-recursive presentations). There is no non-trivial decidable group property for groups given by co-recursive presentations.

Proof. Let $P$ be a $\nu_{co-RP}$-decidable property. Either $P$ or its complement contains the trivial group. Then, by Proposition 36, the one that contains it is in fact the set of all groups. □

The following lemma is essentially identical to Lemma 33.

Lemma 38. Suppose that $(G, S)$ and $(H, S')$ are two co-recursively presented marked groups, and that $(H, S')$ is a strict marked quotient of $(G, S)$. Then no algorithm that takes as input co-recursive presentations of either $(G, S)$ or $(H, S')$ can stop exactly on the presentations that define $(H, S')$.

Proof. Define a co-recursive presentation associated to a run of $\varphi_n(n)$: enumerate the co-word problem of $(H, S')$ while this run last, if it stops, start enumerating the co-word problem of $(G, S)$ instead. □

The following result is the equivalent of Lemma 34, replacing recursive presentations by co-recursive presentations. Its proof is identical to that of Lemma 34, we thus omit it. The difference between this lemma and Lemma 34 is that, here, we suppose that is given a computable sequence of co-recursively presented groups whose supremum is $(G, S)$. In Lemma 34, such a sequence (with reversed arrows) was constructed from $(G, S)$ by taking truncated presentations. A similar construction “truncated” co-recursive presentations is not available here.

Lemma 39. Suppose that $(G, S)$ is a co-recursively presented marked group. Let $(G_n, S_n)$ be a $\nu_{co-RP}$-computable sequence of marked quotients of $(G, S)$, with for each $n$ $(G_{n+1}, S_{n+1}) \rightarrow (G_n, S_n)$, and whose supremum is $(G, S)$.

Then no algorithm that takes as input co-recursive presentations of marked groups in $\{(G, S)\} \cup \{(G_n, S_n), n \in \mathbb{N}\}$ can stop exactly on the presentations that define $(G, S)$. □
**Problem 40** (Rice-Shapiro for co-recursive presentations). Is it true that if $P$ is a property of marked groups that is semi-decidable from co-recursive presentations, then there exists a computable sequence $(A_n)_{n \in \mathbb{N}}$ of finite sets of relations, such that a marked group $(G, S)$ satisfies $P$ if and only if there exists $n$ so that no relation of $A_n$ holds in $(G, S)$?

### 6. Decision Problem from Word Problem Algorithms

By taking the conjunction $\nu_{RP} \land \nu_{co-RP}$, we obtain the numbering associated to word problem algorithms, which we call $\nu_{WP}$.

In [Rau21], we investigate in details decision problems for groups given by word problem algorithms. The topology that is relevant to study such decision problems is the topology of the space of marked groups, which can be defined thanks to a metric defined as follows: two $k$-marked groups are at distance less or equal to $2^{-n}$ if and only if they satisfy exactly the same relations of length at most $n$.

We want to note here that the relationship between the topology of the space of marked groups and decision problems for groups given by word problem algorithms is looser than the one that relates problems about recursive presentations and the Scott topology. Indeed, it is false that a property semi-decidable for recursive presentations for groups given by word problem algorithms is looser than the one that relates problems about recursive presentations.

### 7. Marked Quotient Algorithms and Finite Presentations

7.1. Algorithmic characterization of finitely presented groups. Recall from the introduction that a marked group $(G, S)$ has a marked quotient algorithm if there is a procedure that stops exactly on recursive presentations of its marked quotients, i.e. if the set \{$(H, S')$, $(G, S) \rightarrow (H, S')$\} is $\nu_{RP}$-semi-decidable.

**Lemma 41.** Any finitely presented group admits a marked quotient algorithm. And this is uniform.

**Proof.** To check whether the relation $(G, S) \rightarrow (H, S')$ holds, it suffices to check whether the finitely many relations of $(G, S)$ hold in $(H, S')$, this can be semi-decided thanks to a recursive presentation.

Note that any non recursively presentable simple group has a marked quotient algorithm. When restricting our attention to recursively presented groups, having a marked quotient algorithm will characterize finitely presentable groups. Recall that $\nu_{MQA}$ is the numbering of finitely presented groups associated to marked quotient algorithms.

**Theorem 42.** A group is finitely presented if and only if it admits both a recursive presentation and a marked quotient algorithm. And this statement is uniform.

**Proof.** Lemma 41 provides the obvious direction.

Suppose now that we have a recursive presentation $(S \mid r_1, r_2, ...)$ and a marked quotient algorithm for a marked group $(G, S)$. Let $(G_n, S)$ be the marked group given by the truncated presentation $(S\mid r_0, r_1, r_2, ..., r_n)$.

By Lemma 34, no algorithm that accepts recursive presentations of groups in \{$(G, S)\} \cup \{(G_n, S), n \in \mathbb{N}\}$ can stop exactly on those that define $(G, S)$.

But the marked quotient algorithm of $(G, S)$ does accept every presentation that defines $(G, S)$, since $(G, S)$ is a marked quotient of itself. Thus there must be another group in \{$(G, S)\} \cup \{(G_n, S), n \in \mathbb{N}\}$ which is a marked quotient of $(G, S)$. But for each $n$ we have a morphism of marked groups $(G_n, S) \rightarrow (G, S)$. Thus if a morphism exists in the other direction, it must be that $(G, S)$ is isomorphic to some $(G_n, S)$, and thus it is finitely presented.

We now have to justify that the above proof is effective, in that some $n$ with $(G, S) = (G_n, S)$ can be computed from the recursive presentation of $(G, S)$ and its marked quotient algorithm.

Going back to the proof of Lemma 34, we see that was constructed a computable sequence of recursive presentations $(E_n)_{n \in \mathbb{N}}$, such that if $\varphi_n(n) \uparrow$, $E_n$ is the infinite presentation of $(G, S)$, and if $\varphi_n(n) \downarrow$, $E_n$ defines $(G_n, S)$.

The sets \{\{n \in \mathbb{N}, \varphi_n(n) \uparrow\}\} and \{n \in \mathbb{N}, (G, S) \rightarrow E_n\} are two recursively enumerable sets, by the above non-effective argument they intersect, thus it is possible to algorithmically produce an element in their intersection. This will precisely provide a finite presentation of $(G, S)$. \[\Box\]
7.2. **Decision problems from marked quotient algorithms.** In Section 4, we established the Rice-Shapiro theorem for recursive presentations, which shows that very little can be said of groups given by recursive presentations. We have now decomposed the numbering associated to finite presentations into a disjunction, \( \nu_{FP} \equiv \nu_{RP} \land \nu_{MQA} \).

We will now show that this disjunction is non-trivial, we must show that \( \nu_{FP} \neq \nu_{MQA} \) (since it follows from Section 4 that \( \nu_{FP} \neq \nu_{RP} \)). We will thus show that very little can be said about a group given by a marked quotient algorithm. Note that in the following, we only consider finitely presented groups, but these are given by descriptions that are much weaker than finite presentations.

**Lemma 43.** Suppose that \((G, S)\) and \((H, S')\) are two finitely presented marked groups, and that \((H, S')\) is a strict marked quotient of \((G, S)\). Then no algorithm that takes as input marked quotient algorithms of either \((G, S)\) or \((H, S')\) can stop exactly on those that define \((H, S')\).

In other words, \(\{(G, S)\}\) is not a \(\nu_{MQA}\)-semi-decidable subset of \(\{(G, S), (H, S')\}\).

**Proof.** The proof is similar to that of Lemma 33, we only sketch it. Associated to a run of \(\phi_n(n)\), construct an algorithm that recognizes the quotients of \((H, S')\) while this run last, and that also accepts the quotients of \((G, S)\) if this run stops.

Call a property \(P\) of marked groups upward closed when a marked group that has a quotient in \(P\) also has \(P\).

**Corollary 44** (Rice-Shapiro for marked quotient algorithms). The \(\nu_{MQA}\)-semi-decidable properties are exactly the \(\nu_{FP}\)-semi-decidable properties that are upward closed in \((G, \rightarrow)\).

**Proof.** Let \(P\) be a property of finitely presented marked groups that is \(\nu_{MQA}\)-semi-decidable. That it is semi-decidable from finite presentations is immediate: finite presentations provide strictly more information than marked quotient algorithms. That it is upward closed follows from Lemma 43.

Suppose now that \(P\) is semi-decidable from finite presentations. Then, the property “having a marked quotient in \(P\)” is semi-decidable from marked quotient algorithms, since to check if a marked group \((G, S)\) has a marked quotient in \(P\), it suffices to enumerate all finite presentations of groups in \(P\), and to apply to all of these in parallel the marked quotient algorithm of \((G, S)\). When \(P\) is upward closed, “having a marked quotient in \(P\)” is simply \(P\).

8. **Marked quotient algorithms relative to a class of groups**

In this section, we introduce relative marked quotient algorithms, and explain that the cases where these are most interesting is when they do not rely on notions of finite presentations modulo a certain class of groups.

8.1. **Marked quotients algorithms and semi-decidable equality.**

**Definition 45.** Let \(\mathcal{C}\) be a class of marked groups. We say that a marked group \((G, S)\) has a marked quotient algorithm relative to \(\mathcal{C}\) if there is an algorithm that takes as input recursive presentations for groups in \(\mathcal{C}\) and stops exactly on those that define marked quotients of \((G, S)\).

See Proposition 54 for an example of a group that has a relative marked quotient algorithm for finite groups without being finitely presented.

One easily defines a numbering \(\nu_{MQA}^{\mathcal{C}}\) associated to \(\mathcal{C}\)-marked quotient algorithms, modifying the definition of \(\nu_{MQA}\) accordingly (see Definition 22). (In cases where a group is not uniquely determined by its marked quotients in \(\mathcal{C}\), when it is not residually \(\mathcal{C}\), the marked quotient algorithm in \(\mathcal{C}\) may not determine a marked group uniquely. In this case, \(\nu_{MQA}^{\mathcal{C}}\) is a multi-numbering of marked groups, i.e. a partial multi-function \(\nu_{MQA}^{\mathcal{C}} : \subseteq \mathbb{N} \rightarrow \mathcal{C}\). We will however not need that notion here.)

Note that if we suppose that \(\mathcal{C}\) is a class of groups, i.e. is closed under isomorphism, then “having a marked quotient relative to \(\mathcal{C}\)” becomes a group property.

**Lemma 46.** Let \(\mathcal{C}\) be a class of abstract groups. If a group \(G\) admits a marked quotient algorithm relative to \(\mathcal{C}\) with respect to some marking, then it also admits one with respect to any marking.

**Proof.** Let \(S\) and \(T\) be two finite generating sets of a group \(G\). We suppose that we have access only to the marked quotient algorithm for \(G\) with respect to \(S\). Fix for each \(s\) in \(S\) an expression \(s = t_1^{\alpha_1} \ldots t_k^{\alpha_k}\), with \(\alpha_i \in \{-1, 1\}\) and \(t_i \in T\), that gives \(s\) as a product of elements of \(T\) and of their inverses, and for each \(t\) in \(T\) an expression \(t = s_1^{\beta_1} s_2^{\beta_2} \ldots s_k^{\beta_k}\) that describes \(t\) in terms of the generators of \(S\) and their inverses.

Consider a marked group \((H, T')\) given by a recursive presentation, with \(T'\) of same cardinality as \(T\). We want to determine whether \((H, T')\) is a marked quotient of \((G, T)\), thus whether the bijection \(f : T \rightarrow T'\) can be extended to a group morphism \(\tilde{f}\).
Let $S'$ be the family defined, in $H$, by the same formulas as $S$ is in $G$, i.e. for each $s = t_1^{\alpha_1} \ldots t_k^{\alpha_k}$ in $S$, we define $s' = f(t_1)^{\alpha_1} \ldots f(t_k)^{\alpha_k}$. Notice that if $f$ does define a morphism $\tilde{f}$ of $G$ onto $H$, $S'$ should be the image of the family $S$ by $\tilde{f}$, and thus it should be a generating family of $H$. We can therefore, using the recursive presentation of $(H, T')$, look for an expression of the elements of $T'$ in terms of the elements of $S'$ in $H$.

If such an expression does not exist, our procedure will not stop, but then $(H, T')$ is not a quotient of $(G, T)$, thus this result is coherent.

Otherwise we can use the formulas just found to obtain a recursive presentation of $H$ with respect to $S'$. From it, we can ask whether the natural bijection $S \to S'$ defines a group morphism, thanks to the marked quotient algorithm of $G$ on $S$. If this procedure does not end, $(H, T')$ is not a marked quotient of $(G, T)$.

If it terminates, the bijection $S \to S'$ does define a group morphism $\psi : G \to H$. But we still have to check that $\psi$ maps $T$ to $T'$. This is done using the expressions of the form $t = s_1^{\beta_1} s_2^{\beta_2} \ldots s_k^{\beta_k}$, that define, in $G$, the elements of $T$ in terms of those of $S$.

Compute the formal images of these elements inside of $H$. This yields for each $t$ in $T$ an expression of $\psi(t)$ in terms of the elements of $S'$.

It then follows that $f$ extend to a group homomorphism if and only if for each $t$ of $T$, $\psi(t) = f(t)$ in $H$. This can be semi-decided thanks to the recursive presentation of $H$.

The following is an easy consequence of the fact that the quotient relation is an order for isomorphism classes of marked groups:

**Fact 47.** Isomorphism of marked groups is semi-decidable for groups in $C$ that are given by $\nu_{RP} \land \nu_{MQA}^C$-names.

*Proof.* Given $\nu_{RP} \land \nu_{MQA}^C$-names for two marked groups, it suffices to ask whether the first is a marked quotient of the second, and whether the second is a marked quotient of the first one.

*Remark 48.* The above fact can be extended to semi-decidability of the actual isomorphism relation when $C$ is closed under group isomorphism, by noticing that Lemma 46 is effective.

Having a semi-decidable isomorphism problem seems to be a very good criterion to ensure that a certain numbering $\nu$ is a good candidate for Problem 2. In particular, it guarantees that the isomorphism problem will always be solvable on finite sets of groups. It also guarantees that precise classification theorems (such as that of finitely generated abelian groups) immediately translate into an effective result. (And thus while the proof of the classification of finitely generated abelian groups is effective, the statement of this theorem itself is sufficient to argue that any numbering $\nu$ of finitely generated abelian groups, which provides less information than the finite presentation numbering, i.e. for which $\nu \geq \nu_{FP}$, and which has a semi-decidable isomorphism problem, will permit to actually solve the isomorphism problem for finitely generated abelian groups.)

### 8.2. Finite presentation modulo a class of groups

Theorem 42 states that being finitely presentable is equivalent to being recursively presentable and having a marked quotient algorithm.

A similar phenomenon sometimes occur for relative marked quotient algorithms, when we replace finite presentations by other appropriate notions of “finite presentation relative to a class of groups”.

In fact, the introduction of relative quotient algorithms will be most interesting when there is no underlying notion of relative finite presentation.

A marked group $(G, S)$ is called residually $\mathcal{C}$ when any non-trivial element of it has a non-trivial image in a marked quotient that belongs to $\mathcal{C}$. Every marked group has a greatest (for the order of the lattice of marked groups) marked quotient which is residually $\mathcal{C}$, called its residually $\mathcal{C}$ image.

For $(G, S)$ a marked group and $\mathcal{C}$ a set of marked groups, let $\mathcal{E}((G, S), \mathcal{C})$ be the set of morphisms that exist between $(G, S)$ and groups in $\mathcal{C}$.

**Definition 49.** Let $C$ be a class of marked groups, and $(G, S)$ a marked group. The **residually $\mathcal{C}$ image** of $(G, S)$ is the group

$$G/\bigcap_{\phi \in \mathcal{E}((G, S), \mathcal{C})} \ker(\phi).$$

Thanks to the above, we can define finite presentations modulo $\mathcal{C}$.

**Definition 50.** A residually $\mathcal{C}$ marked group $(G, S)$ has a **finite presentation as a residually $\mathcal{C}$ group** if there exists a finitely presented group whose residually $\mathcal{C}$ image is $(G, S)$.

The following provides the obvious case for when a marked group has a marked quotient algorithm relative to a class $\mathcal{C}$. 


Proposition 51. Suppose that the residually C image of \((G, S)\) is finitely presentable as a residually C group. Then \((G, S)\) has a marked quotient relative to C.

Proof. Just as Lemma 41.

Example 52. By equational noetherianity of free groups, for every marked group \((G, S)\), there is a finitely presented group \((H, S')\) such that \((G, S)\) and \((H, S')\) have exactly the same marked free quotients. See for instance [Hou07, Theorem 2.7]. Thus every group has a marked quotient algorithm for free groups. Similarly, every group has a marked quotient algorithm relative to abelian groups.

We can also show that with respect to group varieties, all marked quotient algorithms in fact rely on finite presentations. If \(V\) is a group variety, a residually \(V\) group is simply a group in \(V\), because group varieties are closed under unrestricted direct products and subgroups. And a finite presentation as a residually \(V\) group is simply a finite presentation which omits the infinitely many relations that are given by the laws of the variety. We call these finite presentations modulo \(V\) rather than finite presentation of residually \(V\) groups.

Proposition 53. Let \(V\) be a group variety defined by a c.e. set of laws. A recursively presented marked group \((H, S)\) of \(V\) admits a \(V\)-marked quotient algorithm if and only if it is finitely presented modulo \(V\).

What's more, this statement is uniform: it states that the numbering associated to finite presentations modulo \(V\) is equivalent to the conjunction \(\forall_{\mathbb{P}} \land \forall_{\text{MQA}}\).

Proof. The proof is identical to that of Theorem 42, except that one needs to add the laws that define \(V\) when building the presentations \(E_n\).

8.3. A relative marked quotient algorithm that does not rely on a finite presentation. We end this section by giving an example of a marked quotient algorithm that does not rely on a finite presentation. The fact that the lamplighter group has a marked finite quotient algorithm follows from results of Hartung [Har11], but we give a short proof as an illustration.

Proposition 54. The Lamplighter group \(\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}\) has a marked quotient algorithm with respect to the set of finite groups.

Proof. Indeed, it admits the following presentation:

\[ \langle a, \varepsilon \mid \varepsilon^2, [\varepsilon, a^{-n}\varepsilon a^n], n \in \mathbb{Z} \rangle \]

To see whether a finite group \(F\) generated by two elements \(a_1\) and \(\varepsilon_1\) is a quotient of it, find a multiple \(N\) of the order of \(a_1\) using the recursive presentation of \(F\). Then, notice that \((F, (a_1, \varepsilon_1))\) is a quotient of \((L, (a, \varepsilon))\) if and only if it is a quotient of the group obtained from \((L, (a, \varepsilon))\) by adding the relation \(a^N\). But the quotient \(L/\langle a^N\rangle\) is in fact the finite wreath product \(\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}/N\mathbb{Z}\), which admits the finite presentation:

\[ \langle a, \varepsilon \mid \varepsilon^2, a^N, [\varepsilon, a^{-n}\varepsilon a^n], 0 \leq n \leq N \rangle \]

Thus the problem is reduced to checking finitely many relations in \((F, (a_1, \varepsilon_1))\), this is semi-decidable.

Proposition 55. The lamplighter group is not finitely presented as a residually finite group.

Proof. Because the lamplighter group is residually finite, we just have to prove that no finitely presented group has \(\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}\) as its residually finite image.

Suppose that we have a marked group \((G, (a_1, \varepsilon_1))\), given by a presentation \(\langle a_1, \varepsilon_1 | r_1, ..., r_p \rangle\), and a morphism \(\phi : (G, (a_1, \varepsilon_1)) \rightarrow (\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}, (a, \varepsilon))\) which satisfies that any morphism \(h\) from \((G, (a_1, \varepsilon_1))\) to a finite group \((F, (a_2, \varepsilon_2))\) factors through \(\phi\).

Since \((\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}, (a, \varepsilon))\) is a quotient of \((G, (a_1, \varepsilon_1))\), it must satisfy the finitely many relations of \((G, (a_1, \varepsilon_1))\). These relations must in turn be consequences of a finite number of the relations of \((\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}, (a, \varepsilon))\). In particular, there must be a natural number \(N\) such that the first \(N\) relations of \((\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}, (a, \varepsilon))\) imply those of \(G\). Consider \((H, (a, \varepsilon))\) given by the presentation

\[ \langle a, \varepsilon \mid \varepsilon^2, [\varepsilon, a^{-n}\varepsilon a^n], 0 \leq n \leq N \rangle. \]

Then \((G, (a_1, \varepsilon_1)), (H, (a, \varepsilon))\) and \((\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}, (a, \varepsilon))\) must have exactly the same marked finite quotients.

\[
\begin{array}{c}
G \\
\phi_1 \\
\downarrow h_0 \\
H \\
\phi_2 \\
\downarrow h_1 \\
L \\
\downarrow h_2 \\
F
\end{array}
\]
To end the proof, we find a contradiction, by finding a finite group which satisfies the relations of \((H, (a, \epsilon))\), but not those of \((\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}, (a, \epsilon))\). We define a subgroup of the group \(S_{5N}\) of permutations on \(\{1, \ldots, 5N\}\). Consider the element \(\sigma_0\) of \(S_{5N}\), defined by the following formula:

\[
\sigma_0(i) = \begin{cases} 
  i + 2 & \text{if } i \leq 5N - 2, \\
  i + 2 - 5N & \text{if } i \geq 5N - 1.
\end{cases}
\]

Let \(\sigma_1\) be the product of the transpositions \((1, 2)\) and \((2N + 4, 2N + 5)\). It is then easy to see that the following relations hold between \(\sigma_0\) and \(\sigma_1\):

\[
\sigma_1^2 = id,
\]

\[
[\sigma_1, \sigma_0^n \sigma_1 \sigma_0^n] = id, 1 \leq n \leq N,
\]

\[
[\sigma_1, \sigma_0^{-N-1} \sigma_1 \sigma_0^{N+1}] \neq id.
\]

The subgroup of \(S_{5N}\) generated by \(\sigma_0\) and \(\sigma_1\) is thus a marked quotient of \((H, (a, \epsilon))\), but not of \((\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}, (a, \epsilon))\). This contradicts the supposition that \((\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}, (a, \epsilon))\) is finitely presented as a residually finite group. \(\square\)

The following problem asks for an instance of a set of marked groups with a semi-decidable equality that comes from a genuinely computational generalization of the notion of finite presentation.

**Problem 56.** Find a class \(\mathcal{C}\) of recursively presented groups such that all groups in \(\mathcal{C}\) admit marked quotient algorithms relative to \(\mathcal{C}\) (so that equality is \(\nu_{RP} \land \nu_{MQA}'\)-semi-decidable) but such that not all groups in \(\mathcal{C}\) are finitely presented as residually \(\mathcal{C}\) groups.

9. **On the Adian-Rabin theorem**

In this section, we remark how the Adian-Rabin provides incomplete answers to certain natural problems. We first recall its proof.

**Definition 57.** A Markov property is a group property \(P\) which admits a positive witness, any finitely presented group with \(P\), and a negative witness, which is any finitely presented group that does not embed in any group with \(P\).

**Theorem 58** (Adian-Rabin). A Markov property is \(\Sigma_1\)-hard, and thus not co-semi-decidable, for groups given by finite presentations.

We sketch the proof given in [Mil92], omitting the main technical lemma, which is the actual content of the proof, but our purpose is to render explicit what is proved exactly.

**Proof.** Fix \(G_+\) and \(G_-\) the positive and negative witnesses of a Markov property.

Firstly, given a finitely presented marked group \((H, S)\) with unsolvable word problem, a family of finite presentations \(\pi_w, w \in (S \cup S^{-1})^*\), is constructed, such that: \(\pi_w\) can be effectively constructed from \(w\), and \(\pi_w\) defines the trivial group if and only if \(w = 1\) in \(H\). Whatever \(w\), \(\pi_w\) is defined on two generators \(u\) and \(v\). Finally, when \(w \neq 1\) in \((H, S)\), \(\pi_w\) contains a copy of \(G_-\).

Consider a presentation for the free product of \(G_+\) and of the group defined by \(\pi_w\).

This defines a group that contains \(G_-\) when \(w \neq 1\), and on the contrary this defines exactly \(G_+\) when \(w = 1\) in \((H, S)\). \(\square\)

9.1. **From the point of view of marked groups.** Consider the problem of recognizing a single marked group from a finite presentation.

This problem is always semi-decidable.

This problem is decidable for free groups, when the marking is via a basis: a finite presentation defines a free group marked by a basis if and only if it has no relations modulo free reductions. It is also decidable for cyclic groups marked by families with one generators.

What we can easily extract from the proof of the Adian-Rabin is the following:

**Proposition 59.** Consider a finitely presented marked group \((G, (s_1, \ldots, s_k))\). The problem of deciding whether a finite presentation defines the marked group \((G, (s_1, \ldots, s_k, 1_G, 1_G))\) is not co-semi-decidable.

**Proof.** This follows immediately from the proof of the Adian-Rabin Theorem described above, by taking \(G_+ = G\). The two additional generators corresponding to the identity appear in the free product with the presentation \(\pi_w\). \(\square\)

In [BMI07], Baumslag and Miller establish the following result:
Theorem 60 ([BMI07], Theorem B). If \( n \) is chosen sufficiently large, then there is no algorithm to determine of a finite presentation with \( n \) generating symbols whether or not the group presented is free abelian of rank \( n \).

This is, to the best of our knowledge, the only result that provides more information than the Adian-Rabin construction about recognition of marked groups thanks to finite presentations.

Problem 61. Bridge the “two additional generators gap” that appears in Proposition 59, and describe exactly the marked groups that define decidable properties of groups given by finite presentations.

9.2. Classification of properties in the arithmetical hierarchy. The Adian-Rabin theorem does not permit to precisely classify properties of finitely presented groups in the arithmetical hierarchy, except at the very first levels.

Most of the group properties whose precise location in the arithmetical hierarchy is known are properties that are semi-decidable but not co-semi-decidable. The Adian-Rabin Theorem is used to prove the “not co-semi-decidable” part, and semi-decidability is proved directly. This includes: being abelian, trivial, finite, nilpotent, virtually nilpotent, hyperbolic, having Kazhdan’s Property (T) [Oza14]. Note that “having a non-trivial finite quotient” is semi-decidable and not decidable, but that this does not follow from the Adian-Rabin Theorem [BW15].

However, when it comes to classifying properties that are higher in the arithmetical hierarchy, almost nothing is known. In particular, the group properties one naturally encounters are usually Markov or co-Markov properties, but not both. And thus the Adian-Rabin falls short.

“Having solvable word problem” is known to be \( \Sigma^0_3 \)-complete [BR66].

“Being torsion free” is known to be \( \Pi^0_4 \)-complete, as was shown by Lempp [Lem97]. The proof of this result amounts to noticing that the Higman Embedding Theorem preserves torsion-freeness, it is thus easy to adapt to other properties that known Higman embeddings preserve, like “having a non-trivial center” which is \( \Sigma^0_3 \)-complete [OH07], but hard to adapt to other properties. (For instance, establishing that being solvable is not semi-decidable is probably easier than establishing a Higman Embedding Theorem for soluble groups.)

Note also that, as a corollary of [BDM83], it is proved in [DM08] that deciding whether two given finitely presented groups have the same integral homology sequence is \( \Sigma^0_3 \)-complete -thus not even in the arithmetical hierarchy.

We gave a proof in Proposition 26 of the fact that “being residually finite” is not semi-decidable. Note that this property is Markov, so it was already known to be not co-semi-decidable, but it is not co-Markov.

The following problem goes back to Miller [Mil92] and Lempp [Lem97].

Problem 62. Classify group properties higher up in the arithmetical hierarchy, for groups given by finite presentations. Prove that the following properties are not semi-decidable from finite presentations: being solvable, or being \( k \)-solvable for a fixed \( k \geq 2 \), being simple, being torsion, being amenable.

Note that the Adian-Rabin Theorem provides a result similar to half of the Rice-Shapiro Theorem: it corresponds to Lemma 33 adapted to groups given by finite presentations. No equivalent of the other half of the Rice-Shapiro Theorem, Lemma 34, was ever proved for groups given by finite presentations. In other words, we can make statements such as: “this property is not semi-decidable, because checking it would involve proving that a relation is not satisfied in the given group, which is impossible”, but we cannot make statements such as “this property is not semi-decidable, because checking it would involve proving simultaneously that infinitely many independent relations are satisfied in the given group, which is impossible”. Although presumably, at least for some of the properties quoted above, this will be true.

9.3. Classifying properties for groups given by finite presentations and word problem algorithms. The proof given above of the Adian-Rabin Theorem in fact relies on a single marking of the positive witness, the group \( G_+ \). We already noted that this marking must be extended by twice the identity as new generators for the construction to work. What we want to note here is that the fact that this construction relies on a single marking of \( G_+ \) makes it so that it provides no information on decision problems for groups given by word problem algorithms and recursive presentations.

Indeed, we have the obvious fact:

Proposition 63. The marked isomorphism problem is decidable for \( \nu_{WP} \cup \nu_{FP} \).

Proof. To check whether two finite presentations \( \langle S \mid R \rangle \) and \( \langle S \mid R' \rangle \) define the same marked group, it suffices to decide whether the relations of \( R \) hold in \( \langle S \mid R' \rangle \), and if those of \( R' \) hold in \( \langle S \mid R \rangle \). This can be decided if we can solve the word problem in these groups.

And thus any result that establishes that a certain property \( P \) is not \( \nu_{WP} \cup \nu_{FP} \)-semi-decidable will have to rely on using infinitely many different markings of groups without \( P \), and infinitely many markings of groups with \( P \).
The Adian-Rabin construction uses infinitely many markings of groups without \( P \), but only a single marking of a group with \( P \).

For instance, in \([Mil72, \text{Theorem } 26, \text{Chapter IV}]\), Miller proved that the isomorphism problem is unsolvable for finitely presented residually finite groups by building a finitely presented residually finite group \( G \) such that the set \( \{G\} \) of all markings of \( G \) is \( \text{FP-semi-decidable} \) but not \( \text{FP-decidable} \) amongst residually finite groups. Because the word problem is uniformly solvable for finitely presented residually finite groups, the marked isomorphism problem is solvable for these. Thus a necessary feature of Miller’s construction is that infinitely many markings of \( G \) are used. Even more: fixing any generating family \( S \) of \( G \), the average word length with respect to \( S \) of the different generating families of \( G \) that are used grows faster than every recursive functions.

We thus end this paragraph by the following problem:

**Problem 64.** Establish a theorem that gives sufficient conditions on a group property for it to be undecidable for groups given by finite presentations and word problem algorithms.

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