Singularity structure of the two point function of the free Dirac field on a globally hyperbolic spacetime

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Abstract

We give an introduction to the techniques from microlocal analysis that have successfully been applied in the investigation of Hadamard states of free quantum field theories on curved spacetimes. The calculation of the wave front set of the two point function of the free Klein-Gordon field in a Hadamard state is reviewed, and the polarization set of a Hadamard two point function of the free Dirac field on a curved spacetime is calculated.
1 Introduction

The general framework of this work is that of quantum field theory on curved spacetime. Since we still do not have a full theory of quantum gravity, as a first step we try to include gravitational effects into the quantum field theoretical description of our world by a semi-classical approximation. The gravitational field is considered as a fixed classical background spacetime on which matter is described by a quantum field theory. Since the Planck length, at which effects from quantum gravity are expected to become important, is very small, this approximation should have a wide range of validity and should be appropriate to predict quantum effects near black holes or in the early universe. One famous result in this framework is the Hawking radiation of black holes.

The main problem in the quantization of a field theory on a curved background is that one cannot make use of Poincaré invariance like on Minkowski spacetime, where this symmetry fixes a unique vacuum state. On a curved spacetime, however, there is no such natural choice of a state, and the central task is to single out the physically acceptable ones.

For free, linear models it has turned out that these acceptable states are the quasi-free Hadamard states. They are determined by their two point function which has got a prescribed singularity structure fixed by geometry. For the free Klein-Gordon field it was recently discovered by Radzikowski that this Hadamard condition, which at first sight is a global condition, can be reformulated as a local condition in terms of wave front sets. This allows one to apply the elegant and powerful mathematical techniques from microlocal analysis. In this way it could be proved that certain states known for a long time are indeed Hadamard states. A microlocal spectrum condition for interacting quantum field theories was formulated and used to define Wick polynomials and to prove the renormalizability of interacting scalar field theories on globally hyperbolic spacetimes.

For more realistic quantum field theories, however, first of all the Dirac field, more or less nothing was done, not because of conceptual problems, but because of the greater complexity due to the multi-component nature of the Dirac field. In this work the singularity structure of the two point function of the free Dirac field will be investigated. We will make use of the polarization set which was introduced by Dencker in order to analyze the singularities of vector-valued distributions. Applying Dencker’s results, we will show that the singularities of solutions of the Dirac equation propagate in a simple way. This will be used to calculate the polarization set of the two point function for Hadamard states of the Dirac field. Our analysis makes use of unpublished results by Radzikowski [18], and our results are in agreement with those independently obtained by Hollands [23] using a slightly different approach.

This paper is organized as follows: The first chapter after the introduction gives an overview of the physical problems we are interested in and fixes the notation used throughout this work. We introduce the Klein-Gordon and Dirac fields on a curved
spacetime and give a definition of Hadamard states. After that, we come to the mathematics and review the methods from microlocal analysis on which our analysis is based. The central definitions are the wave front and polarization sets, and the main results are the theorems on the propagation of singularities. As a side result, we will rederive the well-known properties of light propagation in a curved spacetime in a mathematically very elegant way. In the fourth chapter these methods are applied to our physical objects, and we state our main results. Finally we give an outlook and an overview about possible applications of our results.

2 Quantum field theory on curved spacetime

2.1 Notions from general relativity

In the general theory of relativity, for an introduction see [1], spacetime is represented by a differentiable manifold \((\mathcal{M}, g)\) equipped with a Lorentzian metric \(g\), i.e. a differentiable, symmetric, and non-degenerate type \((0, 2)\) tensor field of signature \((+ − − −)\). For simplicity, we restrict ourselves to the physical four dimensions.

As usual, the tangent bundle will be denoted \(T\mathcal{M}\), and the cotangent bundle is \(T^*\mathcal{M}\). For the set \(\{(x, \xi) \in T^*\mathcal{M} | \xi \neq 0\}\) we will shortly write \(T^*\mathcal{M} \setminus 0\). The canonical projection will always be denoted \(\pi : T^*\mathcal{M} \to \mathcal{M}\). Any vector bundle \(E\) over \(\mathcal{M}\) can be pulled back to a vector bundle \(\pi^*E\) over \(T^*\mathcal{M}\). The set of smooth sections of a vector bundle \(E\) over \(\mathcal{M}\) will be denoted \(\Gamma(\mathcal{M}, E)\), sections having compact support form the set \(\Gamma_0(\mathcal{M}, E)\).

The Levi-Civita connection is the unique connection on \(T\mathcal{M}\) that is symmetric and compatible with the metric. The associated covariant derivative is denoted \(\nabla\). The Riemann tensor is the curvature tensor of this connection which is contracted to the Ricci tensor and finally to the scalar curvature \(R\).

We will restrict ourselves to globally hyperbolic spacetimes. These are spacetimes that admit a Cauchy surface, that is a spacelike hypersurface \(\Sigma \subset \mathcal{M}\) which is intersected by any nonextendable causal curve in \(\mathcal{M}\) exactly once. In other words, \(\mathcal{M} = D(\Sigma)\), where we have introduced the domain of causal dependence \(D(\Sigma)\) which is the set of all points \(x \in \mathcal{M}\) such that any causal curve through \(x\) intersects \(\Sigma\). In the globally hyperbolic case the manifold is topologically of the simple form \(\mathcal{M} = \mathbb{R} \times \Sigma\). Many physically interesting spacetimes like flat Minkowski spacetime, Schwarzschild spacetime as a black hole model, or Robertson-Walker spacetimes as cosmological models fall into this class.

In order to include more general models like anti-de Sitter spacetime, this restriction can be somewhat weakened. Crucial for the following is mainly the existence of unique propagators as well as the existence of a spinor bundle. Since our constructions are purely local, they immediately carry over to all globally hyperbolic submanifolds of any spacetime.
Globally hyperbolic spacetimes are time orientable, such that the light cone, i.e. the set of all nonvanishing timelike covectors in \( T^*_x \mathcal{M} \), can be separated into a forward and backward light cone, \( V^+_x \) and \( V^-_x \), continuously in \( x \). The closed light cones \( V^\pm_x \) include the lightlike covectors. The time orientation induces the separation of the causal future \( J^+(x) \) and past \( J^-(x) \) of a point \( x \in \mathcal{M} \) which are the sets of all points that can be reached from \( x \) by a future (past) directed causal curve.

While the tangent bundle \( T \mathcal{M} \) and the cotangent bundle \( T^* \mathcal{M} \) naturally exist for any spacetime \( \mathcal{M} \), the existence of spinors is less trivial. In order to define spinors, the bundle of orthonormal frames, which is a principal fibre bundle whose structure group is the proper orthochronous Lorentz group, has to be lifted to a principal fibre bundle with the universal covering group SL(2, \( \mathbb{C} \)) as structure group, called the spin structure. The Dirac spinor bundle is then defined as the associated \( \mathbb{C}^4 \)-vector bundle. It can be shown that for any globally hyperbolic spacetime there exists such a spinor bundle \( D \mathcal{M} \) which in this case is just a trivial bundle over \( \mathcal{M} \). The construction of the spinor bundle leads naturally to a covariant derivative on \( D \mathcal{M} \) and the dual bundle \( D^* \mathcal{M} \) which will also be denoted \( \nabla \). In local coordinates one has \( \nabla_\mu = \partial_\mu + \sigma_\mu \), where the \( 4 \times 4 \) matrices \( \sigma_\mu \) can be expressed in terms of the Christoffel symbols and the Dirac matrices. For details see [9].

2.2 The free Klein-Gordon field

The simplest example of a quantum field theory is the free scalar field, satisfying the Klein-Gordon equation

\[
(\Box_g + m^2)\phi(x) = 0. \tag{1}
\]

This is already the covariant field equation on a generic spacetime \((\mathcal{M}, g)\) if \( \Box_g \) is the covariant wave operator \( \Box_g = g^{\mu\nu} \nabla_\mu \nabla_\nu \) of the metric \( g \). The positive real constant \( m \) plays the role of a mass.

Because of the global hyperbolicity of our spacetime, the Cauchy problem for the Klein-Gordon operator has a unique solution, and there exist unique retarded and advanced propagators, \( \Delta_{\text{ret}} \) and \( \Delta_{\text{adv}} \), such that

\[
(\Box_g + m^2)\Delta_{\text{ret,adv}} = \Delta_{\text{ret,adv}}(\Box_g + m^2) = 1 \tag{2}
\]

and \( \text{supp}(\Delta_{\text{ret,adv}}f) \subset J^\pm(\text{supp}f) \). We define the commutator function \( \Delta = \Delta_{\text{ret}} - \Delta_{\text{adv}} \) and identify the operator \( \Delta \) with the distribution

\[
\Delta : (f, g) \mapsto \Delta(f, g) = \int f(x)(\Delta g)(x) d\mu(x), \tag{3}
\]

where \( d\mu \) is the volume element on \( \mathcal{M} \).

In the quantum theory the field \( \phi \) is represented as an operator valued distribution such that we have smeared field operators \( \phi(f), f \in \mathcal{C}_0^\infty(\mathcal{M}) \), acting on some
Hilbert space with a vacuum vector $\Omega$. They are required to satisfy the Klein-Gordon equation and the canonical commutation relations

$$[\phi(f), \phi(g)] = i\Delta(f, g) \cdot 1.$$  (4)

Since we investigate a free theory, we expect that all vacuum expectation values of products of field operators are determined by the two point function

$$\Lambda(x, y) = \langle \Omega, \phi(x)\phi(y)\Omega \rangle$$  (5)

which is assumed to be a distribution $\Lambda \in D'(M \times M)$.

Canonical quantization of the free Klein-Gordon field can be carried through in close analogy to quantization on Minkowski spacetime (see for example [2]), and one ends up with a Fock space representation of the field $\phi$ in terms of creation and annihilation operators. However, the problem is that there is no preferred mode expansion in terms of plain waves, because the Fourier transform is not invariant under general coordinate transformations. The crucial point is the ambiguous division into positive- and negative-frequency modes. Therefore this construction is highly non-unique, and there is a large amount of unitarily inequivalent representations, as was first observed by Fulling [3]. A priori we do not know the ‘real’ vacuum state $\Omega$, and as a consequence there is no preferred interpretation of the theory in terms of particles. If the spacetime is asymptotically flat, one can carry over the particle interpretation from the flat part to the whole spacetime, but on a generic spacetime one has to face the fact that quantum field theory is primarily a theory of fields, and no unique particle interpretation can be expected.

From an axiomatic viewpoint, the vacuum state on Minkowski spacetime is singled out by the requirement of its Poincaré invariance and the spectrum condition which is the requirement that the energy-momentum operator (the generator of translations) takes its spectrum in the closed forward light cone. In the absence of Poincaré invariance on a generic spacetime these conditions no longer make any sense. While one will still demand invariance under the symmetries of the spacetime (if there are any), there is no simple analogue to the spectrum condition since there are no generators of translations. Thus one central task of quantum field theory on curved spacetime is to characterize physical states by finding a generalized spectrum condition for generic spacetimes.

Usually the construction sketched above is performed in the algebraic approach to quantum field theory. There the local net of observable algebras can be defined in a unique way, but one still has to fix a state on the algebra (which corresponds to the preparation of the system). Again one restricts oneself to quasifree states that are fixed once a two point distribution is specified, and via the GNS construction the state induces a Hilbert space representation of the observable algebra. For details on this approach we refer to [4].
2.3 Hadamard states

In order to single out a class of acceptable quantum states, one physical requirement is that we should be able to formulate the semi-classical Einstein equations in order to describe the back reaction of the quantum field on the spacetime geometry:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \langle T_{\mu\nu} \rangle. \] (6)

Here, \( \langle T_{\mu\nu} \rangle \) is the expectation value of the energy momentum tensor of the quantum field. Thus physical states have to allow a unique definition of this expectation value.

It could be shown \([5]\) that this requirement is fulfilled if we demand that the two point function takes the following form which was first given in \([6]\) and since then intensively studied (see \([7]\) and the references therein):

\[ \Lambda(x, y) = \frac{1}{4\pi^2} \left( \frac{u(x, y)}{\sigma(x, y)} + v(x, y) \ln |\sigma(x, y)| + w(x, y) \right), \] (7)

where \( \sigma(x, y) \) denotes the quadratic geodesic distance, and \( u, v, \) and \( w \) are some smooth functions.

A state whose two point function is of this form is called a Hadamard state. The field equation and the commutation relations fix \( u \) and \( v \), and the remaining freedom in the definition of a Hadamard state lies in the choice of the smooth function \( w \).

In order to properly define a distribution, one has to modify (7) by an \( i\epsilon \)-prescription in the denominator. Also, since \( \sigma \) is not defined globally, one must specify some neighbourhood in which (7) is supposed to hold, and it is also understood that no further singularities exist for spacelike separated points. To this end, one demands the convergence of a series of distributions to \( \Lambda \) in a causal normal neighbourhood (which is basically a sufficiently small neighbourhood of a Cauchy surface). Thereby the Hadamard condition becomes a global condition. Although the choice of a Cauchy surface enters into the definition, it can be shown that the Hadamard condition is independent of this choice. All these details were worked out in \([8]\), but most of them are not essential for this work. What should be kept in mind is that the vacuum state on Minkowski spacetime is Hadamard, that Cauchy evolution respects the Hadamard singularity structure, and that two Hadamard distributions differ only by a smooth part.

The main lesson of this rather sketchy definition is that the Hadamard condition prescribes the singular short distance behaviour of the two point distribution, and it is obvious that it would be very useful to have at hand some powerful mathematical methods to characterize this singularity structure.

2.4 The free Dirac field

We will now briefly outline the construction of the free Dirac field on a curved spacetime. For details see \([9]\).
Just like the Klein-Gordon equation, we can generalize the Dirac equation to a spinor bundle over an arbitrary curved spacetime by replacing the derivatives and the Dirac matrices by their covariant counterparts. For a spinor field $\psi \in \Gamma(\mathcal{M}, D\mathcal{M})$ and a cospinor field $\bar{\psi} \in \Gamma(\mathcal{M}, D^*\mathcal{M})$ it reads

\[
\begin{align*}
(-i\nabla + m)\psi &= 0, & \nabla \psi &= \gamma^\mu \nabla_\mu \psi; \\
(i\nabla + m)\bar{\psi} &= 0, & \nabla \bar{\psi} &= (\nabla_\mu \psi)\gamma^\mu.
\end{align*}
\] (8)

Here, $\gamma^\mu$ are the generalized Dirac matrices which satisfy the anticommutation relations

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot 1.
\] (10)

It can be shown that they form a covariant section $\gamma \in \Gamma(\mathcal{M}, T\mathcal{M} \otimes D\mathcal{M} \otimes D^*\mathcal{M})$. Like in (10), we will always choose a local frame and suppress the spinor indices referring to $D_x\mathcal{M} \otimes D^*_y\mathcal{M}$ such that $\gamma(x)$ is treated like a vector of $4 \times 4$ matrices. We can then define matrices $\sigma^{\mu\nu}, \gamma^5, \gamma^\mu \gamma^5$ and $1$ in the usual way, which together with $\gamma^\mu$ give a basis for $D\mathcal{M} \otimes D^*\mathcal{M}$. A further important property of $\gamma$ is that it is covariantly constant: $\nabla \gamma = 0$. For any covector field $\xi \in \Gamma(\mathcal{M}, T^*\mathcal{M})$ we define $\xi$ as contraction with $\gamma$, $\xi \equiv \xi_\mu \gamma^\mu$.

For the quantization of the theory we again need propagators $S_{\text{ret}}$ and $S_{\text{adv}}$. For globally hyperbolic spacetimes it can be shown that these propagators exist and are unique [9]. Their difference defines the anticommutator function $S = S_{\text{ret}} - S_{\text{adv}}$.

In the quantized theory the spinor field operator now has to be smeared with a smooth cospinor field of compact support in order to give an operator on the Hilbert space,

\[
\psi(v), \quad v \in \Gamma_0(\mathcal{M}, D^*\mathcal{M}); \quad \bar{\psi}(u), \quad u \in \Gamma_0(\mathcal{M}, D\mathcal{M}).
\] (11)

The field operators are required to solve the Dirac equation and obey the canonical anticommutation relations,

\[
\{\psi(v), \bar{\psi}(u)\} = iS(v, u).
\] (12)

Again, quasi-free states are completely characterized by their two point distribution

\[
\omega^+(x, y) = \langle \Omega, \psi(x)\bar{\psi}(y)\Omega \rangle
\] (13)

which is now a vector-valued distribution $\omega^+ \in D'(\mathcal{M} \times \mathcal{M}, D\mathcal{M} \boxtimes D^*\mathcal{M})$ taking values in the bispinor bundle $D\mathcal{M} \boxtimes D^*\mathcal{M}$. This denotes the outer tensor product: the fibre over $(x, y) \in \mathcal{M} \times \mathcal{M}$ is $D_x\mathcal{M} \otimes D^*_y\mathcal{M}$, and the first factor in the tensor
product transforms like a spinor in the point \( x \), while the second factor transforms like a cospinor in \( y \).

In the quantization procedure on Minkowski spacetime one often makes use of the fact that the squared Dirac operator gives back the Klein-Gordon operator. On a curved spacetime this is expressed by Lichnerowicz’s identity:

\[
(-i\nabla + m)(i\nabla + m)\psi = (\Box - \frac{1}{4} R + m^2)\psi.
\]  

(14)

Here, \( R \) is the curvature scalar, and \( \Box = g^{\mu\nu}\nabla_\mu \nabla_\nu \) is the spinorial wave operator acting on sections of \( D_M \). Note that only the principal part of \( \Box \) gives the scalar wave operator, and in general it contains non-diagonal terms. Nevertheless, one obtains the propagators \( S_i \) for the Dirac equation by applying the Dirac operator to the propagators \( \Delta_i \) for this spinorial wave operator,

\[
S_i = (i\nabla + m)\Delta_i. \quad (15)
\]

Analogously, one demands that the two point function \( \omega^+ \) can be extracted from an auxiliary two point function \( \tilde{\omega} \),

\[
\omega^+(x, y) = (i\nabla_x + m)\tilde{\omega}(x, y). \quad (16)
\]

Then \( \tilde{\omega} \) is a solution of the spinorial Klein-Gordon equation,

\[
\left( \Box_{x,y} - \frac{1}{4} R + m^2 \right)\tilde{\omega}(x, y) = 0, \quad (17)
\]

and in order to define Hadamard states for the Dirac field one can make the same ansatz as in the scalar case,

\[
\tilde{\omega} = \frac{1}{4\pi^2} \left( \frac{\tilde{u}}{\sigma} + \tilde{v} \ln |\sigma| + \tilde{w} \right), \quad (18)
\]

but now \( \tilde{u}, \tilde{v}, \) and \( \tilde{w} \) are smooth bispinor-valued functions. The rigorous definition was given by Köhler [10]. Again, the singular part of the two point function is fixed by geometry, and the freedom in the choice of a Hadamard state lies in the choice of a smooth function \( \tilde{w} \).

3 Microlocal analysis

Let us now introduce the mathematical tools we need to investigate the singularities of distributions. This theory of ‘microlocal analysis’ was developed by Hörmander and Duistermaat in the seventies for their analysis of partial differential equations, see their original papers [11, 12] or the monographs [13, 14]. For physical applications
see for example [13, 21]. It should be noted that similar tools were independently developed by Bros and Iagolnitzer [16] in the context of analyticity properties of the S-matrix.

The generalization to vector-valued distributions and the definition of the polarization set was accomplished by Dencker in the eighties [17], but it seems that is has not found its way into the physical literature yet.

3.1 Scalar distributions – the wave front set

As test function space we will take \( \mathcal{D}(X) = \mathcal{C}_0^\infty(X) \), the space of infinitely differentiable functions on an open subset \( X \subset \mathbb{R}^n \), equipped with the usual topology. Its dual is the space of distributions, \( \mathcal{D}'(X) \).

The roughest characterization of the singularities of a distribution \( u \in \mathcal{D}'(X) \) is the singular support, \( \text{sing supp} u \). It is the set of all points \( x \in X \) such that \( u \) does not correspond to a smooth function in any neighbourhood of \( x \).

But this definition can be very much refined. The following statements show that much information about the singularities is contained in the Fourier transform:

1. If \( u \) is of compact support, then there exists a Fourier transform \( \hat{u} \) which furthermore is a smooth function.

2. A distribution \( u \) of compact support corresponds to a smooth function if and only if \( \hat{u} \) decays faster than any power, that is if for every \( m \in \mathbb{N} \) there is a constant \( C_m \in \mathbb{R} \) such that
   \[
   |\hat{u}(\xi)| \leq C_m (1 + |\xi|)^{-m} \quad \forall \xi \in \mathbb{R}^n.
   \]  

Thus in order to investigate the singularities of a distribution \( u \in \mathcal{D}'(X) \) locally at the point \( x \in X \), we are led to analyze the Fourier transform \( \hat{\phi u} \) for test functions \( \phi \in \mathcal{C}_0^\infty(X) \) with \( \phi(x) \neq 0 \) which cut off the singularities far away from \( x \). Then \( x \in \text{sing supp } u \) iff there is no \( \phi \) such that \( \hat{\phi u} \) falls off rapidly in all directions.

This immediately suggests two refinements of the singular support: we may ask

- If \( \hat{\phi u} \) does not decay rapidly in all directions, which are those directions in Fourier space which are responsible for the singularity?

- If \( \hat{\phi u} \) does not fall off rapidly, how fast does it decay?

The first question leads directly to the notion of the wave front set, while the second question motivates the definition of local Sobolev spaces \( H^s(x) \). In combination one can define \( H^s \)-wave front sets, but in the following we will only be interested in the \( \mathcal{C}^\infty \)-case:
**Definition 1.** Let \( u \in \mathcal{D}'(X), \; X \subset \mathbb{R}^n \). A point \((x, \xi) \in X \times (\mathbb{R}^n \setminus \{0\})\) is called a regular directed point of \( u \) if there is a function \( \phi \in C_0^\infty(X) \), not vanishing at \( x \), such that for any \( m \in \mathbb{N} \) there is a constant \( C_m \in \mathbb{R} \) with
\[
|\hat{\phi}u(\xi')| \leq C_m (1 + |\xi'|)^{-m}
\]
for all \( \xi' \) in a conic neighbourhood \( \Gamma \subset \mathbb{R}^n \setminus \{0\} \) of \( \xi \). (A neighbourhood \( \Gamma \) is called conic if with \( \xi \in \Gamma \) all points \( t \cdot \xi, \; t \in \mathbb{R}^+ \), are contained in \( \Gamma \).)

The wave front set of \( u \), denoted \( WF(u) \), is the complement in \( X \times (\mathbb{R}^n \setminus \{0\}) \) of the set of regular directed points of \( u \).

**Example.** The wave front set of the \( \delta \)-distribution \( \delta \in \mathcal{D}'(\mathbb{R}^n) \) is
\[
WF(\delta) = \{(0, \xi) \mid \xi \in \mathbb{R}^n \setminus \{0\}\}.
\]
This can directly be seen from its Fourier transform
\[
\hat{\delta}(\xi) = \delta(e^{-i<\xi, \cdot>}) = \delta(0)
\]
which does not decay rapidly in any direction.

Our second example shows that the wave front set does not always contain the whole Fourier space for all points of the singular support:

**Example.** The distribution \( u \in \mathcal{D}'(\mathbb{R}), \; u(x) = \frac{1}{x+i\epsilon} \), has got the wave front set
\[
WF(u) = \{(0, \xi) \mid \xi \in \mathbb{R}^+ \setminus \{0\}\}
\]
which can be seen from its Fourier transform
\[
\hat{u}(\xi) = -i\sqrt{2\pi} \theta(\xi).
\]

We will now collect some basic properties of the wave front set \([\text{III}]\):

**Theorem 1.**

1. \( WF(u) \) is a closed conic subset of \( X \times (\mathbb{R}^n \setminus \{0\}) \).

2. The wave front set is a refinement of the singular support:
\[
\pi(WF(u)) = \text{sing supp } u.
\]

In particular, the wave front set of a smooth function is empty.

3. For any partial differential (or more generally any pseudo-differential) operator \( P \) we have the pseudolocal property
\[
WF(Pu) \subseteq WF(u).
\]
4. Let \( U, V \subset \mathbb{R}^n \), \( u \in \mathcal{D}'(V) \), and \( \chi : U \to V \) a diffeomorphism such that \( \chi^*u \in \mathcal{D}'(U) \) is the distribution pulled back by \( \chi \): \( \chi^*u(f) = u(f \circ \chi^{-1}) \). Then

\[
WF(\chi^*u) = \chi^*WF(u) \equiv \{(\chi^{-1}(x), \lambda(\chi^{-1}(x))\xi) \mid (x, \xi) \in WF(u)\}.
\] (27)

In particular, under coordinate transformations the elements of the wave front set transform like covectors. Thus the wave front set can be defined for distributions on differentiable manifolds by gluing together wave front sets over the coordinate patches, and for \( u \in \mathcal{D}'(M) \) we have \( WF(u) \subset T^*M \setminus 0 \).

The wave front set can be defined in a second way which will be important for the investigation of solutions of partial differential equations as well as for the generalization to vector-valued distributions. But first we need some notions from the theory of partial differential operators (PDOs):

A PDO \( A \) is a polynomial of partial derivatives, \( A = P(x, \partial) \). In momentum space it acts like multiplication by the same polynomial of momenta \( P(x, i\xi) = \sigma_A(x, \xi) \) which is called the symbol of \( A \). The leading order \( a(x, \xi) \) of \( \sigma_A \) is called the principal symbol \( \sigma_P(A) \). The generalization to a manifold \( M \) via local charts is obvious, and it should be noted that the principal symbol is then a well-defined function on \( T^*M \).

Usually this is generalized to pseudo-differential operators (ΨDOs), where \( \sigma_A(x, \xi) \) may be a more general but still well-behaved function, not necessarily a polynomial (for details see [14]). In this work we will only deal with classical ΨDOs whose symbol is an asymptotic sum of homogeneous terms. The set of classical ΨDOs on \( M \) of order \( m \) will be denoted \( L^m(M) \).

The characteristic set of a ΨDO \( A \in L^m(M) \) with principal symbol \( a = \sigma_P(A) \) is defined as

\[
\text{char}(A) = \{(x, \xi) \in T^*M \setminus 0 \mid a(x, \xi) = 0\}
\] (28)

and can be interpreted as the set of all directions \( \xi \) suppressed by \( A \) to leading order, at a point \( x \).

Now the following theorem makes precise the intuitive statement that the singular directions of a distribution (which make up the wave front set) are just those directions that have to be suppressed to leading order by any operator that maps the distribution to a smooth function.

**Theorem 2.** For the wave front set of a distribution \( u \in \mathcal{D}'(M) \) we have

\[
WF(u) = \bigcap_{A \in \mathcal{C}^\infty(M)} \text{char}(A),
\] (29)

where the intersection is taken over all pseudo-differential operators \( A \in L^0(M) \) with \( Au \in \mathcal{C}^\infty(M) \).
This property of the wave front set is particularly useful if we know such an operator $A$ with $Au \in C^\infty$, especially if $u$ is a solution of a partial differential equation. The first part of the following theorem is then obvious, but for certain operators one can make an even stronger statement which goes under the name ‘propagation of singularities theorem’ and was first proved by Duistermaat and Hörmander [12]:

**Theorem 3.** Let $P \in L^m(M)$ be a $\Psi$DO on $M$ with principal symbol $p$. If $u \in \mathcal{D}'(M)$ such that $Pu \in C^\infty(M)$, then

$$\text{WF}(u) \subset \text{char}(P).$$

(30)

If furthermore $P$ is of real principal type, then $\text{WF}(u)$ is invariant under the flow generated by the Hamiltonian vector field of $p$.

Here we have imposed a natural restriction on $P$ in order to obtain a real Hamiltonian and a non-degenerate Hamiltonian flow:

**Definition 2.** A pseudo-differential operator $P \in L^m(M)$ is said to be of real principal type if its principal symbol $p(x, \xi)$ is real and for $p = 0$ the Hamiltonian vector field $H_p$,

$$H_p(x, \xi) = \sum_{\mu} \partial_{\xi_\mu} p(x, \xi) \partial_{x_\mu} - \sum_{\mu} \partial_{x_\mu} p(x, \xi) \partial_{\xi_\mu},$$

(31)

does not vanish nor does it have the radial direction, that is $H_p \neq -\frac{\partial p}{\partial x^\mu} \frac{\partial}{\partial \xi_\mu}$.

Thus the wave front set of a distribution $u$ with $Pu = 0$ is made up of integral curves of $H_p$ in $\text{char}(P)$ which are also called the null bicharacteristics of $P$. Their projections onto $M$ are called the bicharacteristic curves of $P$ and constitute the singular support of $u$.

To illustrate this theorem, let us take a look at a physical example, the wave operator on a curved spacetime:

**Example.** Let $(M, g)$ be a spacetime, $f, b \in C^\infty(M)$, such that $0 \neq f(x) \in \mathbb{R}$ $\forall x \in M$, and $a \in \Gamma(M, T^*M)$. The operator

$$P = f\Box + a^\mu \nabla_\mu + b, \quad \Box = g^{\mu\nu} \nabla_\mu \nabla_\nu,$$

(32)

has got the principal symbol $p(x, \xi) = -f(x)g^{\mu\nu}(x)\xi_\mu\xi_\nu = -f \cdot \xi^2$. Hence, $P$ is of real principal type.

For $(x, \xi) \in \text{char}(P)$ we see that the covector $\xi$ has to be lightlike. Furthermore, for $\xi^2 = 0$ the function $p(x, \xi) = -\xi^2$ is a well-known Hamiltonian that generates the null geodesics which is not affected by the nonvanishing factor $f$. Therefore the bicharacteristic curves of $P$ are null geodesics $x(\tau)$. 

12
From Hamilton’s equations we obtain
\[ \frac{dx^\mu}{d\tau} = \frac{dp(x, \xi)}{d\xi^\mu} = -2f(x(\tau))g^{\mu\nu}\xi_\nu. \tag{33} \]
which means that \( \xi \) is tangent to the geodesic. By reparameterization of the geodesic one can achieve \( \xi(\tau) = \frac{dx}{d\tau}(\tau) \) for all values of the curve parameter \( \tau \). The factor, however, is irrelevant in what follows since the wave front set is conic in \( \xi \).

Thus the null bicharacteristics of \( P \) are curves \((x, \xi)(\tau) \in T^*M \setminus 0\) such that \( x(\tau) \) describes a lightlike geodesic and \( \xi(\tau) \) is tangent to \( x(\tau) \) for all values of \( \tau \).

Applying theorem 3, we conclude that the singular support of a distribution \( u \) which solves the wave equation, i.e. \( Pu = 0 \), is a union of null geodesics, while the wave front set additionally indicates the covectors tangent to the geodesics in the singular support. Note that the geodesics are directed and that the wave front set may contain only the future or past directed covectors.

If we take light rays as distributional solutions of the wave equation, this result can be interpreted as the well-known fact that light propagates along lightlike geodesics.

### 3.2 Vector-valued distributions – the polarization set

We will now go over to vector-valued distributions \( u \in D'(\mathcal{M}, \mathbb{C}^N) \), i.e. vectors \( u = (u_i)_{i=1\ldots N} \) of distributions \( u_i \in D'(\mathcal{M}) \). This section is mainly a review of [17].

The wave front set of a vector-valued distribution \( u = (u_i) \in D'(\mathcal{M}, \mathbb{C}^N) \) is just defined as the union of the wave front sets of all its components:
\[ \text{WF}(u) = \bigcup_{i=1}^{N} \text{WF}(u_i). \tag{34} \]

Because of theorem 2 this is nothing but
\[ \text{WF}(u) = \bigcup_{i=1}^{N} \bigcap_{Au \in C^\infty(\mathcal{M})} \text{char}(A). \tag{35} \]

The wave front set does not contain any information about the components of the distribution that are singular. In order to specify the singular directions in the vector space \( \mathbb{C}^N \), one could consider vector-valued operators that map the vector-valued distribution to a smooth scalar function, instead of just looking at scalar operators mapping the individual components to smooth functions. This approach leads to the definition of the polarization set:

**Definition 3.** The polarization set of a distribution \( u \in D'(\mathcal{M}, \mathbb{C}^N) \) is defined as
\[ \text{WF}_{pol}(u) = \bigcap_{Au \in C^\infty(\mathcal{M})} \mathcal{N}_A. \tag{36} \]
\[ \mathcal{N}_A = \{(x, \xi; w) \in (T^*\mathcal{M} \setminus 0) \times \mathbb{C}^N \mid w \in \ker a(x, \xi)\}, \]  

where the intersection is taken over all $1 \times N$ systems $A \in L^0(\mathcal{M})^N$ of classical \(\Psi\)DOs with principal symbol $a$, and $\ker a(x, \xi)$ is the kernel of the matrix $a(x, \xi)$.

Obviously, $(T^*\mathcal{M} \setminus 0) \times \{0\} \subset WF_{pol}(u)$ for any $u \in \mathcal{D}'(\mathcal{M}, \mathbb{C}^N)$. Furthermore, the polarization set is closed, linear in the fibre and conic in the $\xi$-variable.

For scalar distributions ($N = 1$) the polarization set contains the same information as the wave front set. For arbitrary $N$, we get back the wave front set by projecting the nontrivial points onto the cotangent bundle:

**Theorem 4.** Let $u \in \mathcal{D}'(\mathcal{M}, \mathbb{C}^N)$ and $\pi_{1,2} : T^*\mathcal{M} \times \mathbb{C}^N \to T^*\mathcal{M}$ be the projection onto the cotangent bundle: $\pi_{1,2}(x, \xi; w) = (x, \xi)$. Then

\[ \pi_{1,2}(WF_{pol}(u) \setminus (T^*\mathcal{M} \times \{0\})) = WF(u). \]  

In this way the polarization set is a refinement of the wave front set. In addition, it contains information about the directions in the additional vector space in which the distribution is singular.

**Example.** Let $u = (u_1, u_2) \in \mathcal{D}'(\mathcal{M}, \mathbb{C}^2)$ and $(y, \eta) \notin WF(u_1)$. Then

\[ WF_{pol}(u) \subseteq \{(x, \xi; (0, z)) \in (T^*\mathcal{M} \setminus 0) \times \mathbb{C}^2, z \in \mathbb{C}\} \]  

over a conic neighbourhood of $(y, \eta)$.

The polarization set indicates only the most singular directions, even if the projection of the distribution on other directions is also singular, as the following example shows:

**Example.** Let $u = (v, \Delta v) \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^2)$, where $v \in \mathcal{D}'(\mathbb{R}^n)$ and $\Delta$ is the Laplacian on $\mathbb{R}^n$. Then

\[ WF_{pol}(u) \subseteq \{(x, \xi; (0, z)) \in (T^*\mathbb{R}^n \setminus 0) \times \mathbb{C}^2, z \in \mathbb{C}\}, \]  

because $\Delta u_1 - u_2 = 0$ and $\ker \sigma_P(\Delta, -1)(x, \xi) = \ker(-\xi^2, 0) = \{(0, z), z \in \mathbb{C}\}$.

Our third example shows that the direction of the strongest singularities in $\mathbb{C}^N$ can depend on the singular direction in the cotangent bundle:

**Example.** Let $u = \nabla \delta^{(2)} = (\partial_{x_1} \delta^{(2)}, \partial_{x_2} \delta^{(2)}) \in \mathcal{D}'(\mathbb{R}^2, \mathbb{C}^2)$. Then

\[ WF_{pol}(u) \subseteq \{(0, \xi; \lambda \cdot \xi) \in (T^*\mathbb{R}^2 \setminus 0) \times \mathbb{C}^2, \lambda \in \mathbb{C}\}, \]  

since $u$ solves the equation $(-\partial_{x_2}, \partial_{x_1}) \cdot u = 0$, and $\ker(-i\xi_2, i\xi_1) = \mathbb{C} \cdot (\xi_1, \xi_2)$ for $\xi \neq 0$.

The polarization set has got an interesting transformation property under the action of a system of partial differential operators:
Theorem 5. Let $A$ be an $M \times N$ system of pseudo-differential operators on $\mathcal{M}$ with principal symbol $a(x, \xi)$, and $u \in \mathcal{D}'(\mathcal{M}, \mathbb{C}^N)$. Then
\[ a(WF_{pol}(u)) \subseteq WF_{pol}(Au), \tag{42} \]
where $a$ acts on the fibre: $a(x, \xi; w) = (x, \xi; a(x, \xi)w)$.

If $E$ is an $N \times N$ system of pseudo-differential operators on $\mathcal{M}$ and its principal symbol $e(x, \xi)$ is not characteristic at $(y, \eta) \in T^*\mathcal{M} \setminus 0$, i.e. $\det e(y, \eta) \neq 0$, then
\[ e(WF_{pol}(u)) = WF_{pol}(Eu) \tag{43} \]
over a conic neighbourhood of $(y, \eta)$.

Note the different order of the inclusion in (42) compared to (26). Since the polarization set only indicates the most singular directions, the projection on orthogonal directions can change the polarization set substantially.

From (43) we learn that the polarization set behaves covariantly under a change of basis in the vector space $\mathbb{C}^N$. Thus the polarization set of a distribution $u \in \mathcal{D}'(\mathcal{M}, E)$ taking values in a vector bundle $E \to \mathcal{M}$ can be defined by gluing together the polarization sets of local trivializations. This gives a well-defined subset $WF_{pol}(u) \subset \pi^*E$ of the vector bundle $E$ lifted over the cotangent bundle.

Like in the scalar case, we are interested in the propagation of singularities for a solution of a system of partial differential equations. Again it turns out that there is a powerful theorem for a restricted class of operators.

Definition 4. An $N \times N$ system $P$ of pseudo-differential operators on $\mathcal{M}$ with principal symbol $p(x, \xi)$ is said to be of real principal type at $(y, \eta) \in T^*\mathcal{M} \setminus 0$ if there is an $N \times N$ symbol $\tilde{p}(x, \xi)$ such that
\[ \tilde{p}(x, \xi)p(x, \xi) = q(x, \xi) \cdot 1 \tag{44} \]
in a neighbourhood of $(y, \eta)$, with a scalar symbol $q(x, \xi)$ of real principal type. We say that $P$ is of real principal type in $\mathcal{O} \subseteq T^*\mathcal{M} \setminus 0$ if it is at all points $(y, \eta) \in \mathcal{O}$.

The choice of $\tilde{p}$ and $q$ is not unique, but it can be shown that the only freedom is the multiplication by a smooth non-vanishing function on $T^*\mathcal{M} \setminus 0$.

From now on let $P$ be an $N \times N$ system of classical pseudo-differential operators $P$ of real principal type of order $m$, $u$ a solution of $Pu = 0$, and locally $\tilde{p}(x, \xi)$ and $q(x, \xi)$ are chosen as above such that
\[ \tilde{p}p = q \cdot 1. \tag{45} \]

We define the set
\[ \Omega_P = \{(x, \xi) \in T^*\mathcal{M} \setminus 0 \mid \det p(x, \xi) = 0\}. \tag{46} \]
Because of (45), we have \( \Omega_P = q^{-1}(0) \) locally. If \( Pu \in \mathcal{C}_\infty \), for the existence of nontrivial elements \((x, \xi; w) \in \mathcal{N}_P\) in the polarization set over a point \((x, \xi)\), by definition it is necessary that \( w \in \ker p(x, \xi) \), that is, \( \ker p(x, \xi) \neq \{0\} \). This means that \((x, \xi)\) and thus the whole set \(WF(u)\) has to be a subset of \(\Omega_P\).

It can now easily be shown that the wave front set is a union of null bicharacteristics of \(q\) in \(\Omega_P\). Though the bicharacteristics are not independent of the choice of \(q\), we will just call them the bicharacteristics in \(\Omega_P\) without referring to a special choice of \(q\). A different choice of \(q\) only leads to a multiplication of the \(\xi\)-variable by a scalar factor which is irrelevant since the wave front set is conic in \(\xi\). In fact, in general it is not even possible to choose \(q\) globally, and several coordinate patches must be glued together anyhow.

Once we know the wave front set of the distribution \(u\), the remaining task is to calculate the polarization vectors over the points in the wave front set. It turns out that these vectors follow a simple parallel transport law along the bicharacteristics that form the wave front set.

This parallel transport, which we will introduce in a second, does not only depend on the principal symbol as in the scalar case, but also the subprincipal symbol contributes. The symbol of \(P\) is a sum of homogeneous terms,

\[
\sigma(P)(x, \xi) = p(x, \xi) + p_{m-1}(x, \xi) + p_{m-2}(x, \xi) + \ldots, \tag{47}
\]

where \(p = \sigma_P(P)\) is the principal symbol, and \(p_j\) is homogeneous of order \(j\). The subprincipal symbol is now defined as

\[
p^s(x, \xi) = p_{m-1}(x, \xi) - \frac{1}{2i} \sum_\mu \frac{\partial^2 p(x, \xi)}{\partial x^\mu \partial \xi_\mu}. \tag{48}
\]

Furthermore let

\[
\{\tilde{p}, p\} = \sum_\mu \frac{\partial \tilde{p}(x, \xi)}{\partial \xi_\mu} \frac{\partial p(x, \xi)}{\partial x^\mu} - \sum_\mu \frac{\partial \tilde{p}(x, \xi)}{\partial x^\mu} \frac{\partial p(x, \xi)}{\partial \xi_\mu} \tag{49}
\]

be the Poisson bracket and

\[
H_q(x, \xi) = \sum_\mu \left( \frac{\partial q(x, \xi)}{\partial \xi_\mu} \frac{\partial}{\partial x^\mu} - \frac{\partial q(x, \xi)}{\partial x^\mu} \frac{\partial}{\partial \xi_\mu} \right) \tag{50}
\]

the Hamiltonian vector field of \(q\).

**Definition 5.** For a smooth section \(w\) of the vector bundle \((T^*\mathcal{M} \setminus 0) \times \mathbb{C}^N\), we define Dencker’s connection as

\[
D_P w = H_q w + \frac{1}{2} \{\tilde{p}, p\} w + i\tilde{p}p^s w. \tag{51}
\]
This is a partial connection along the bicharacteristics in $\Omega_P$, that is a connection on all $\mathbb{C}^N$-vector bundles over these bicharacteristics. Furthermore, it can be shown that (51) defines a partial connection in $N_P$, this means that for every vector field $w$ along a bicharacteristic $\gamma$ in $\Omega_P$ we have $D_Pw \in \ker p$ along $\gamma$ if and only if $w \in \ker p$ along $\gamma$. The equation $D_Pw = 0$ can then be solved with $(x, \xi; w) \in N_P$, which is a necessary condition for elements of the polarization set.

Again, the definition of the connection depends on the choice of $\tilde{p}$ and $q$, but it can be shown that a different choice changes a solution of the equation $D_Pw = 0$ in $N_P$ only by a scalar factor. Therefore, we define a Hamilton orbit of a system $P$ of real principal type as a line bundle $L \subseteq N_P|_\gamma$ over a bicharacteristic $\gamma$ in $\Omega_P$ which is spanned by a section $w$ satisfying the equation $D_Pw = 0$, i.e., that is parallel with respect to Dencker’s connection. These Hamilton orbits are then independent of the choice of $\tilde{p}$.

Now we are prepared to state the main result of [17], the theorem on the propagation of singularities for vector-valued distributions:

**Theorem 6.** Let $P$ be an $N \times N$ system of classical pseudo-differential operators over a manifold $M$, and $u \in \mathcal{D}'(M, \mathbb{C}^N)$. Furthermore, let $P$ be of real principal type at $(y, \eta) \in \Omega_P$, and $(y, \eta) \not\in \text{WF}(Pu)$. Then, over a neighbourhood of $(y, \eta)$ in $\Omega_P$, $WF_{pol}(u)$ is a union of Hamilton orbits of $P$.

Though it is not obvious from the definition (51), in interesting cases Dencker’s connection takes on a very simple form. We will illustrate this by a physical example that was already discussed by Radzikowski [18] in a slightly modified way.

**Example.** We investigate Maxwell’s equations on a spacetime $(M, g)$. For a vector field $A \in \Gamma(M, T^1M)$ in Lorentz gauge $(\nabla_\mu A^\mu = 0)$, in a local coordinate frame they read

$$
\square_g A^\nu - R^\nu_{\mu} A^\mu = 0, \quad \square_g = g^{\rho\sigma} \nabla_\rho \nabla_\sigma.
$$

We want to calculate Dencker’s connection for the operator

$$
P^\nu_{\mu} = \square_g A^\nu - R^\nu_{\mu} = g^{\rho\sigma}(\delta^\nu_{\mu} \partial_\rho \partial_\sigma + \Gamma^\nu_{\rho\mu} \partial_\sigma - \delta^\nu_{\mu} \Gamma^\lambda_{\rho\sigma} \partial_\lambda + \Gamma^\nu_{\sigma\mu} \partial_\rho) + \ldots,
$$

where the dots stand for lower order terms that do not contribute to Dencker’s connection.

The principal symbol of $P$ is

$$
p(x, \xi)_{\mu} = -g^{\rho\sigma}(x)\xi_\rho \xi_\sigma \delta^\nu_{\mu},
$$

so that we can choose

$$
\tilde{p}^\lambda_{\nu} = \sqrt{-g} \delta^\lambda_{\nu}, \quad q(x, \xi) = -\sqrt{-g} g^{\rho\sigma} \xi_\rho \xi_\sigma
$$

17
to get $\tilde{p}p = q \cdot 1$. The factor $\sqrt{-g}$ is introduced for no obvious reason, but it turns out that it simplifies the calculation. We see that $q$ is a scalar symbol of real principal type, therefore $P$ is of real principal type.

The bicharacteristic curves of $q$ are again the null geodesics, and the space $\Omega_P = q^{-1}(0)$ consists of all lightlike vectors in $T^*\mathcal{M} \setminus 0$. Since $p \propto 1$, the space $\ker p(x, \xi)$ on which Dencker’s connection acts is the whole tangent space.

Putting together the different terms in (51) and using well-known identities for the Christoffel symbols and Hamilton’s equations, we arrive at

$$
(D_P w)\nu = \left( H_p w + \frac{1}{2} \{\tilde{p}, p\} w + i\tilde{p}p^s w \right)\nu
$$

$$
= \frac{d w\nu}{d\tau} + \Gamma^\nu_{\rho\mu} \xi^\rho w^\mu
$$

$$
= \left( (\pi^*\nabla_\tau) w \right)\nu,
$$

(56)

for a vector field $w(x(\tau), \xi(\tau))$ along the null bicharacteristics. That is, Dencker’s connection for $P$ (with respect to our choice of $\tilde{p}$) is just the usual Levi-Civitá connection, lifted over the cotangent bundle and restricted to the bicharacteristics. Thus a vector field over one of the bicharacteristics is parallel with respect to Dencker’s connection (and thereby generates a Hamilton orbit) iff the projected vector field over the characteristic curve is parallel with respect to the Levi-Civitá connection.

Thus, following the theorem on the propagation of singularities, the polarization set of a solution of Maxwell’s equations is a union of such Hamilton orbits, that is it consists of curves $(x, \xi; w)(\tau) \in \pi^*T^*\mathcal{M}$ such that

- $x(\tau)$ describes a null geodesic,
- $\xi(\tau)$ is tangent to the geodesic and
- $w(\tau)$ is parallel transported along the geodesic.

In physical language, we have reproduced the well-known result on the propagation of light in a curved spacetime: light travels along lightlike geodesics, while the polarization vector is parallel transported along the path.

In particular this means that spacetime curvature cannot lead to double refraction which is only possible if matter effects are introduced into Maxwell’s equations. Double refraction may then occur at points where the operator $P$ is no longer of real principal type. This was also investigated by Dencker [14].

### 4 Singularity structure of the two point function

We are now going to investigate the singularity structure of the two point function of Hadamard states using the microlocal techniques that were introduced in the previous section.
The wave front set of a scalar Hadamard distribution was first calculated by Radzikowski [20]. After we have reviewed his results, we will derive an analogous result for the wave front set of the two point function of the free Dirac field and furthermore calculate its polarization set.

### 4.1 Klein-Gordon field

Let us first state the main result by Radzikowski [20]:

**Theorem 7.** A quasi-free state of the Klein-Gordon field on a globally hyperbolic spacetime \( M \) is a Hadamard state if and only if its two point distribution \( \Lambda \) has got the following wave front set:

\[
WF(\Lambda) = \{(x, y; \xi, -\eta) \in T\prime(M \times M) \setminus 0 \mid (x, \xi) \sim (y, \eta), \xi \in V^+_{x}\}. 
\]

(57)

The equivalence relation \((x, \xi) \sim (y, \eta)\) means that there is a lightlike geodesic \( \gamma \) connecting \( x \) and \( y \), such that at the point \( x \) the covector \( \xi \) is tangent to \( \gamma \) and \( \eta \) is the vector parallel transported along the curve \( \gamma \) at \( y \) which is again tangent to \( \gamma \). On the diagonal \((x, \xi) \sim (x, \eta)\) if \( \xi \) is lightlike and \( \xi = \eta \).

Thus the two point distribution of a Hadamard state is singular only for pairs of points that can be connected by a lightlike geodesic. In addition, the condition \( \xi \in V^+_{x} \) expresses the fact that only positive frequencies contribute and can be viewed as the microlocal remnant of the spectrum condition. The obvious advantage of our formalism from microlocal analysis is that the spectrum condition (for the free scalar field) is now formulated also for curved spacetimes.

This microlocal characterization of the Hadamard condition opens the door for further investigations of Hadamard states using the powerful tools from microlocal analysis. As a first example, in [21] it was shown that certain states that were known long before, the ground- and KMS-states on static spacetimes and adiabatic states of infinite order on Robertson-Walker spacetimes, are indeed Hadamard states.

We will now briefly review Köhler’s proof [10] that the two point function of a Hadamard state has got the given wave front set, since it will lead us in the generalization to the Dirac field.

First we take a look at the two point function on flat spacetime:

**Theorem 8.** On Minkowski spacetime the two point function of the free Klein-Gordon field in the vacuum state has got the wave front set

\[
WF(\Lambda) = \{(x, y; \xi, -\xi) \in T\prime(\mathbb{R}^{1,3} \times \mathbb{R}^{1,3}) \mid x \neq y, (x - y)^2 = 0, \\
\xi \parallel (x - y), \xi_0 > 0\} \\
\cup \{(x, x; \xi, -\xi) \in T\prime(\mathbb{R}^{1,3} \times \mathbb{R}^{1,3}) \mid \xi^2 = 0, \xi_0 > 0\},
\]

(58)

i.e. it is of the form (57).
A complete proof can be found in [15], or the wave front set can be read off from the well-known Fourier transform

$$\hat{\Lambda}(\xi, \eta) = (2\pi)^{-1} \delta(\xi + \eta) \theta(\xi_0) \delta(\xi^2 - m^2).$$  \hspace{1cm} (59)

The extension of this theorem to arbitrary spacetimes is done by deforming the spacetime such that the metric is flat in one part of the spacetime. The known singularity structure in the flat part can then be shifted to the curved part by applying Hörmander’s theorem on the propagation of singularities.

The proof makes use of the following theorem which was proved in [10] based on ideas from [22]:

**Theorem 9.** Let \((\mathcal{M}, g)\) be a globally hyperbolic spacetime and \(x\) a point on a Cauchy surface \(\Sigma \subset \mathcal{M}\). Then there is a neighbourhood \(U\) of \(x\) and a globally hyperbolic spacetime \((\tilde{\mathcal{M}}, \tilde{g})\) with the following properties:

1. A causal normal neighbourhood \(N\) of \(\Sigma\) with \(U \subset N\) is isometric to a neighbourhood \(\tilde{N}\) in \(\tilde{\mathcal{M}}\), and the isometry \(\rho : N \to \tilde{N}\) maps \(\Sigma\) to a Cauchy surface \(\tilde{\Sigma} \subset \tilde{N}\).

2. There is a spacelike hypersurface \(\hat{\Sigma}\) together with a neighbourhood \(\hat{U}\) in \(\tilde{\mathcal{M}}\) such that \(\tilde{g}\), restricted to \(\hat{U}\), is the Minkowski metric and \(\rho(U) \equiv \tilde{U} \subset D(\hat{\Sigma})\).

The two point function \(\Lambda\) which is given on the neighbourhood \(N\) of the Cauchy surface \(\Sigma\) and thereby also on \(\tilde{N} = \rho(N)\) now induces a Hadamard two point distribution \(\hat{\Lambda}\) on the whole deformed spacetime \(\tilde{\mathcal{M}}\) such that \(\Lambda|_{N \times N} = \rho^*(\hat{\Lambda}|_{\tilde{N} \times \tilde{N}})\).

In the flat part of \(\tilde{\mathcal{M}}\) we already know the wave front set of the distribution \(\hat{\Lambda}\). Since only local properties of the spacetime in a neighbourhood of a Cauchy surface enter into the Hadamard condition and the metric on \(\hat{U}\) is flat, on \(\hat{U} \times \hat{U}\) we have because of theorem 8:

\[
(x, y; \hat{\xi}, -\hat{\eta}) \in WF(\hat{\Lambda}|_{\hat{U} \times \hat{U}}) \iff (\hat{x}, \hat{\xi}) \sim (\hat{y}, \hat{\eta}), \hat{\xi} \in \nabla^+_{\hat{x}}.
\]  \hspace{1cm} (60)

\(\hat{\Lambda}\) solves the Klein-Gordon equation in the second variable, therefore we can apply theorem 3 on the propagation of singularities for the operator \(1 \otimes (\Box + m^2)\) to get the singular directions of \(\hat{\Lambda}\) for points in \(\hat{U} \times \hat{U}\). From our investigation of the wave operator in the previous section, we learned that two points \((x, \xi)\) and \((y, \eta)\) lie on the same null bicharacteristic for the operator \(\Box + m^2\) iff \((x, \xi) \sim (y, \eta)\). Because of \(\tilde{U} \subset D(\hat{U})\), every null geodesic through \(\hat{U}\) intersects \(\hat{U}\), and theorem 3 tells us that

\[
(x, y; \hat{\xi}, -\hat{\eta}) \in WF(\hat{\Lambda}|_{\hat{U} \times \hat{U}}) \iff (x, y; \hat{\xi}, -\hat{\eta}) \in WF(\hat{\Lambda}|_{\hat{U} \times \tilde{U}}), (y, \eta) \sim (\hat{y}, \hat{\eta})
\]

\[
\iff (y, \eta) \sim (\hat{x}, \hat{\xi}), \hat{\xi} \in \nabla^+_{\hat{x}}.
\]  \hspace{1cm} (61)
The same argument for the operator \((\Box + m^2) \otimes 1\) gives us the wave front set for points in \(\tilde{U} \times \tilde{U}\):

\[(x, y; \xi, -\eta) \in WF(\tilde{\Lambda}|_{\tilde{U} \times \tilde{U}}) \Leftrightarrow (x, \xi) \sim (\hat{x}, \hat{\xi}), (y, \eta) \sim (\hat{x}, \hat{\xi}), \hat{\xi} \in \nabla^+_x, \] (62)

and therefore

\[WF(\tilde{\Lambda}|_{\tilde{U} \times \tilde{U}}) = \{(x, y; \xi, -\eta) \in T^* (\tilde{U} \times \tilde{U}) \setminus 0 | (x, \xi) \sim (y, \eta), \xi \in \nabla^+_x \}. \] (63)

Now \(\Lambda\) is the distribution pulled back by the isometry \(\rho\), and because of theorem 1.4 also the wave front set of \(\Lambda\) in \(U \times U\) is of this form.

Using the same line of arguments again in the undeformed spacetime \(M\), we finally obtain the wave front set of \(\Lambda\) for arbitrary pairs of points \((x, y) \in M \times M\):

\[WF(\Lambda) = \{(x, y; \xi, -\eta) \in T^* (M \times M) \setminus 0 | (x, \xi) \sim (y, \eta), \xi \in \nabla^+_x \}, \] (64)

and we have finished the proof.

### 4.2 Dirac field

Let us now turn to Hadamard states of the free Dirac field. We first state our main result:

**Theorem 10.** Let \(\omega\) be a Hadamard state of the free Dirac field on a globally hyperbolic spacetime \(M\). Then its two point function \(\omega^+\) has got the following wave front and polarization sets:

\[WF(\omega^+) = \{(x, y; \xi, -\eta) \in T^* (M \times M) \setminus 0 | (x, \xi) \sim (y, \eta), \xi \in \nabla^+_x \}, \] (65)

\[WF_{pol}(\omega^+) = \{(x, y; \xi, \eta; w) \in \pi^* (D_M \boxtimes D^*_M) | (x, y; \xi, \eta) \in WF(\omega^+); (1 \otimes J_\gamma(x, y))w = \lambda \cdot \xi, \lambda \in \mathbb{C} \}. \] (66)

Here, \(J_\gamma(x, y) : D^*_y M \to D^*_x M\) denotes the parallel transport in \(D^*_M\) along the geodesic \(\gamma\) connecting \(x\) and \(y\), such that \(\xi\) is tangent to \(\gamma\) in the point \(x\).

The proof is similar to that for the scalar case: We will first calculate the polarization set on Minkowski spacetime and afterwards shift the result to our spacetime \(M\) by making use of Dencker’s theorem on the propagation of singularities.

**Theorem 11.** The two point function \(\omega^+\) of a Hadamard state of the free Dirac field on Minkowski spacetime \(M = \mathbb{R}^{1,3}\) is (up to a smooth part) of the form

\[\omega^+ = [(i\partial + m) \otimes 1] (\Lambda \cdot 1) \], \] (67)

where \(\Lambda\) is the two point function of a Hadamard state of the free Klein-Gordon field. Its polarization set is

\[WF_{pol}(\omega^+) = \{(x, y; \xi, \eta; \lambda \cdot \xi) \in \pi^* (D_M \boxtimes D^*_M) | (x, y; \xi, \eta) \in WF(\Lambda), \lambda \in \mathbb{C} \}. \] (68)
Proof. The bundle $D\mathcal{M} \boxtimes D^*\mathcal{M}$ over Minkowski spacetime $\mathcal{M} = \mathbb{R}^{1,3}$ is the trivial bundle $(\mathcal{M} \times \mathcal{M}) \times \mathbb{C}^{4 \times 4}$. Thus we can simplify our notation by identifying the spinor spaces over all points. Also, the covariant derivative coincides with the partial derivative of the individual components, and the curvature scalar vanishes. Therefore the functions $\tilde{u}$ and $\tilde{v}$ in the definition (18) of a Hadamard distribution have the form $\tilde{u} = u \cdot 1$, $\tilde{v} = v \cdot 1$ with the corresponding scalar functions $u$ and $v$. That is, the auxiliary two point function of any Hadamard state of the free Dirac field on Minkowski spacetime is (up to a smooth part) a multiple of the unit matrix, and the nonvanishing components are two point functions of scalar Hadamard states. Since these are fixed up to a smooth part, the auxiliary two point function is of the form $\tilde{\omega} = \Lambda \cdot 1$ with a scalar Hadamard distribution $\Lambda$ (up to a smooth part), and we have

$$\omega^+ = [(i\partial + m) \otimes 1](\Lambda \cdot 1).$$  (69)

From theorem 7 we know that the wave front set of $\Lambda \cdot 1$ has the form (57), and its polarization set is obviously

$$WF_{pol}(\Lambda \cdot 1) = \{(x, y; \xi, \eta; \lambda \cdot 1) | (x, y; \xi, \eta) \in WF(\Lambda), \lambda \in \mathbb{C}\}. \quad (70)$$

We obtain $\omega^+$ from $\Lambda \cdot 1$ by application of the operator

$$A = [(i\partial + m) \otimes 1]$$  (71)

with principal symbol $a(x, y; \xi, \eta) = -\xi \otimes 1$. Following theorem 8, this does not enlarge the wave front set. In addition, from theorem 5 we have the following restriction on the polarization set of $\omega^+ = A(\Lambda \cdot 1)$:

$$WF_{pol}(\omega^+) \supseteq \{(x, y; \xi, \eta; a(x, y; \xi, \eta)w) | (x, y; \xi, \eta; w) \in WF_{pol}(\Lambda \cdot 1)\}$$

$$= \{(x, y; \xi, \eta; \lambda \cdot \xi) | (x, y; \xi, \eta) \in WF(\Lambda), \lambda \in \mathbb{C}\}. \quad (72)$$

Now $\xi \neq 0$ if $(x, y; \xi, \eta) \in WF(\Lambda)$, and since the projection of the nontrivial part of the polarization set onto the cotangent bundle gives back the wave front set, we see that by the action of $A$ the wave front set does not become smaller. Thus $\omega^+$ has got the same wave front set as the scalar Hadamard distribution $\Lambda$.

The equality for the polarization set in (72), however, does not follow from theorem 8, since the operator $A$ is characteristic just in the interesting points: we have $\det a(x, y; \xi, \eta) = -(\xi^2)^2 = 0$ for $(x, y; \xi, \eta) \in WF(\Lambda)$.

Instead, we can give a direct calculation of the polarization vectors $w$ over a point $(x, y; \xi, \eta)$ in the wave front set. Because of (69), such vectors have to be a linear combination of the unit and gamma matrices, and we can set $w = \alpha 1 + \beta \gamma^\nu$.

By definition, if $a$ is the principal symbol of an operator $A$ with $A\omega^+ = 0$, a point $(x, y; \xi, \eta; w)$ can only be contained in the polarization set of $\omega^+$ if $a(x, y; \xi, \eta)w = 0$. 22
We know such an operator, since the two point distribution is a solution of the Dirac equation
\[
(-i \partial_x + m)\omega^+(x, y) = 0,
\]
and we obtain
\[
0 = \xi \cdot w = \xi_\mu \gamma^\mu (\alpha 1 + \beta \nu \gamma^\nu) = \alpha \xi_\mu \gamma^\mu + \xi_\mu \beta^\mu 1 - i \sum_{\mu > \nu} (\xi_\mu \beta_\nu - \xi_\nu \beta_\mu) \sigma^{\mu\nu}, \tag{74}
\]
where \( \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \). Hence, since \( \xi \neq 0 \), we must have \( \alpha = 0 \) and \( \beta = \lambda \cdot \xi \) with \( \lambda \in \mathbb{C} \). This means that the polarization vectors over points \( (x, y; \xi, \eta) \in WF(\omega^+) \) are indeed only the vectors of the form \( w = \lambda \cdot \xi \).

This completes the proof. \( \square \)

We are now going to give a generalization to arbitrary globally hyperbolic spacetimes by using Dencker’s theorem on the propagation of singularities of vector-valued distributions. First we calculate Dencker’s connection for the Dirac equation:

**Theorem 12.** Dencker’s connection \( D^P \) for the Dirac operator \( P = (-i \nabla + m) \)
acting on the Dirac bundle \( D^M \) over an arbitrary spacetime \( M \) can be chosen to coincide with the spin connection lifted to \( \pi^* D^M \) and restricted to the null geodesics. The analogous statement holds for the Dirac operator on the dual bundle. In particular, the connection is independent of the mass \( m \).

**Proof.** A first (rather involved) proof for the Dirac equation for spinor half densities was given by Radzikowski [18]. Here we give a simple proof that holds for the Dirac equation on \( D^M \).

Once we have chosen coordinate frames for the tangent and cotangent bundle, the operator
\[
-i \nabla + m = -i\gamma^\mu (\partial_\mu + \sigma_\mu) + m 1 \tag{75}
\]
has principal and subprincipal symbols
\[
p(x, \xi) = \xi_\mu \gamma^\mu(x), \quad p^s(x, \xi) = -i\gamma^\mu(x)\sigma_\mu(x) + m 1 - \frac{1}{2i} \partial_\mu \gamma^\mu(x). \tag{76}
\]
We choose
\[
\tilde{p}(x, \xi) = \sqrt{-g(x)} \xi_\mu \gamma^\mu(x), \quad q(x, \xi) = \sqrt{-g(x)} g^{\mu\nu}(x) \xi_\mu \xi_\nu, \tag{77}
\]
and see that \( \tilde{p}p = q \cdot 1 \), thereby the operator is of real principal type. The characteristic curves of \( q \) are again the null geodesics. Dencker’s connection acts on spinor fields \( w \) along the bicharacteristics that lie in the kernel of \( p(x, \xi) \) for every point \( (x, \xi) \), that is they satisfy \( \xi w(x, \xi) = 0 \).
Putting together the different terms in Dencker’s connection and using $\xi w = 0$, Hamilton’s equations and the properties of the Christoffel symbols and the Dirac matrices, yields

$$D_P w = H q + \frac{1}{2} \{ p, p \} + i \tilde{p} p^* = \frac{dw}{d\tau} + \sqrt{-g} \xi_{\nu} \left( \frac{1}{2} \gamma^\mu (\partial_\mu \gamma^\nu) - \frac{1}{2} (\partial_\mu \gamma^\nu) \gamma^\mu - \frac{1}{4} g (\partial_\mu g) \gamma^\nu \gamma^\mu \right) w$$

$$+ \sqrt{-g} \xi_{\nu} \left( \gamma^\nu \gamma^\mu \sigma_\mu + im \gamma^\nu - \frac{1}{2} \gamma^\nu (\partial_\mu \gamma^\mu) \right) w$$

$$= \left( \frac{d}{d\tau} + \hat{x}^\mu \sigma_\mu \right) w = (\pi^* \nabla_\tau) w,$$  \hspace{1cm} (78)

which is just the covariant derivative of $w$ (as a vector field along the geodesic) along the geodesic line with respect to the spin connection.

The proof for the adjoint operator is done in the same way. \hfill \Box

The proof of theorem \[11\] now proceeds like in the scalar case. The spacetime is deformed such that one part $\hat{U}$ carries the Minkowski metric. By the isometry $\rho$, the undeformed part of the deformed spacetime inherits a spin structure which can be extended to the whole spacetime, since in the globally hyperbolic case all spin structures are trivial bundles. Because the metric stays unchanged in a neighbourhood of a Cauchy surface, the two point function of the Hadamard state on the undeformed spacetime induces a Hadamard two point function on the whole deformed spacetime.

The polarization set in the flat part $\hat{U}$ was calculated above. On the diagonal we have

$$(\hat{x}, \hat{x}; \hat{\xi}, \hat{\eta}; w) \in \WF_{pol}(\omega^+|_{\hat{U} \times \hat{U}}) \iff -\hat{\eta} = \hat{\xi} \in \overline{V}_{\hat{x}}^+, \hat{\xi}^2 = 0; w = \lambda \cdot \hat{\xi}, \lambda \in \mathbb{C}. \hspace{1cm} (79)$$

In order to calculate the polarization vectors $w$ over a point $(x, x) \in \hat{U} \times \hat{U}$ in the curved part of the spacetime, we make use of the fact that the two point distribution solves the equation

$$P \omega^+ = (\langle -i \nabla + m \rangle \otimes 1 + 1 \otimes (i \nabla + m)) \omega^+ = 0.$$  \hspace{1cm} (80)

The curves $(x, x; \xi, -\xi)(\tau)$ such that $x(\tau)$ is a null geodesic and $\xi(\tau)$ is tangent to the curve $x(\tau)$ for any $\tau$ are null bicharacteristics of this operator. As a corollary to theorem \[12\], one can easily see that Dencker’s connection for $P$ is $D_P = \pi^* (\nabla_\tau \otimes \nabla_\tau)$ along these curves.

According to theorem \[3\], the polarization set of $\omega^+$ consists of Hamilton orbits for the operator $P$. These are sections $w$ of the bundle $\pi^* (D\mathcal{M} \boxtimes D\mathcal{M})$ along the null bicharacteristics that are parallel with respect to the connection $D_P$. In our case, for the bicharacteristics $(x, x, \xi, -\xi)$ this just means that the polarization vectors
$w = \lambda \cdot \xi$ are parallel transported along the geodesic curves $(x, x)(\tau)$. Now we have $\nabla \cdot \xi = 0$ along the geodesics, since $\xi$ is covariantly constant (the tangent vector of a geodesic is parallel transported along the curve) as well as the Dirac matrices. Thus the polarization vectors retain their form $w = \lambda \cdot \xi$, and the polarization set over the diagonal in the curved part is of the same form as in the flat part of the spacetime and consists of points $(x, x; \xi, -\xi; \lambda \cdot \xi)$ such that $\xi$ is lightlike and in the forward light cone, with arbitrary $\lambda \in \mathbb{C}$.

Because of theorem 5, this form of the polarization set is conserved when the two point distribution is pulled back to the undeformed spacetime via the isometry $\rho$.

Finally, we obtain the polarization vectors away from the diagonal by shifting the second spinor to the second point: for the operator $P = 1 \otimes (i \nabla + m)$ Dencker’s connection is simply $D_P = 1 \otimes \nabla$. Thus a point $(x, y; \xi, -\eta; w)$ is contained in the polarization set of $\omega^+$ if and only if the parallel transport along the null geodesic connecting $x$ and $y$, such that $\xi$ and $\eta$ are tangent to the geodesic, shifts a polarization vector $\lambda \cdot \xi$ over the point $(x, x, \xi, -\xi)$ to $w$.

This is just the statement of theorem 11.

To summarize, we have proved that, like the wave front set, the polarization set of the two point function of a Hadamard state is uniquely fixed by the underlying geometry, and that the fibre over each point of the wave front set is one-dimensional. Thus the polarization set takes on the minimal possible form.

It seems now natural to define a Hadamard state by the polarization set of its two point distribution. A first step in this direction was undertaken by Hollands [23] who takes equation (65) for the wave front set, which is a result in our approach, as the definition of a Hadamard state for the free Dirac field. It is then shown that this condition on the wave front set already fixes the polarization set.

5 Outlook and conclusion

In this paper we have demonstrated the usefulness of a microlocal characterization of the distributions that are of interest in quantum field theory on curved spacetime. However, this is not confined to the investigation of quasifree Hadamard states.

In the naive perturbative construction of interacting quantum field theories one encounters formal products of distributions that are a priori not well-defined. Over the years, one has learned how to deal with the divergencies in these formal expressions by different renormalization techniques, but the understanding of the singularity structure of the distributions which are involved can lead to some further insight where these divergencies originate.

For example, in scalar $\phi^4$-theory the perturbative construction of the $S$-matrix involves the time-ordered two point distribution, whose formal expansion in terms
of Wick products of free fields contains terms like
\[
T_2(x,y) = c_1(i\Delta_F(x,y))^3 :\phi(x)\phi(y) : + c_2(i\Delta_F(x,y))^2 :\phi^2(x)\phi^2(y) : + \ldots,
\]  
(81)

where
\[
\Delta_F = i\Lambda + \Delta^+
\]  
(82)
is the Feynman propagator.

Because of the singularities of distributions, it is impossible to define a reasonable product on the whole space of distributions, especially the products of Feynman propagators in (81) do not exist as well-defined distributions. This is the reason for the well-known ultraviolet divergencies in these expressions.

For certain well-behaved distributions, however, it is possible to define a product, and it can be shown that a simple condition on the wave front sets of the factors is sufficient for its existence [11]: If the singular directions in the wave front sets of the two factors over the same point do not add up to zero, the product can be defined.

The wave front set of the Feynman propagator \(\Delta_F\) of a Hadamard state has also been calculated by Radzikowski [20],
\[
WF(\Delta_F) = \{(x,y;\xi,-\eta) \in T^*(\mathcal{M} \times \mathcal{M}) \setminus 0 \mid (x,\xi) \sim (y,\eta), x \neq y, \xi \in V^\pm_x \text{ if } x \in J^\pm(y)\}
\]
\[
\cup \{(x,x;\xi,-\xi) \in T^*(\mathcal{M} \times \mathcal{M}) \setminus 0\}.
\]  
(83)
The condition \(\xi \in V^\pm_x \text{ if } x \in J^\pm(y)\) now ensures the existence of the products in (81) at all points away from the diagonal, while over the points of the diagonal the wave front sets of the factors are too large. Therefore the only ambiguity in the definition of the product lies in the extension of the resulting distribution to the diagonal, and renormalization can be viewed as the process of extending the time-ordered distributions to the whole spacetime.

This approach to renormalization theory has not only the advantage of being mathematically elegant and rigorous, but it also works entirely in configuration space. Thus it can be extended to a generic spacetime, where all the powerful renormalization techniques in momentum space cannot be applied because of the lack of a global momentum space. Indeed, in [24] it could be shown that scalar field theories on curved spacetimes can be renormalized under the same conditions as on Minkowski spacetime.

As the main new result of this work, we have calculated the polarization set for the two point function of a Hadamard state of the free Dirac field. The Feynman propagator can be investigated in the same way, and on Minkowski spacetime it can easily be shown that the polarization vectors are of the same form as for the two point function. The generalization to a curved spacetime, however, is complicated by the fact that the Feynman propagator does not solve the homogeneous Dirac equation.
Therefore Dencker's theorem on the propagation of singularities can only be used to determine the polarization vectors away from the diagonal. Nevertheless, we expect that the Feynman propagator for physical states does not have a larger polarization set than on Minkowski spacetime. The knowledge of the singularity structure of the Feynman propagator should then contribute to a better understanding of interacting quantum field theories containing fermions.

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