

DERIVED CATEGORIES OF QUOT SCHEMES OF ZERO-DIMENSIONAL QUOTIENTS ON CURVES

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Abstract. We prove the existence of semiorthogonal decompositions of derived categories of Quot schemes of zero-dimensional quotients on curves in terms of derived categories of symmetric products of curves. The above result is a categorical analogue of a similar formula for the class of Quot schemes in the Grothendieck ring of varieties by Bagnarol-Fantechi-Perroni. It is a special case of a more general Quot formula of relative dimension one, which is regarded as a Bosonic counterpart of the Quot formula conjectured by Jiang and proved by the author. The proof involves categorical wall-crossing formula for framed one loop quiver, which itself is motivated and has applications to categorical wall-crossing formula of Donaldson-Thomas invariants.

1. Introduction

1.1. Derived categories of Quot schemes over curves. For \((r,d) \in \mathbb{Z}^2_{\geq 0}\), let \(\text{Gr}(r,d)\) be the Grassmannian variety parameterizing quotients \(C' \rightarrow Q\) with \(\dim Q = d\). In [Kap84], Kapranov proved the existence of a full strong exceptional collection

\[ D^b(\text{Gr}(r,d)) = \langle E_\alpha : \alpha \in \mathbb{B}(r,d) \rangle \]

where \(\mathbb{B}(r,d)\) is the set of Young diagrams with width \(\leq r - d\) and height \(\leq d\).

The Grassmannian \(\text{Gr}(r,d)\) is regarded as a Grothendieck Quot scheme over the point. By replacing the point by a curve, we obtain the Quot scheme of points over a curve. Let \(C\) be a smooth projective curve over \(\mathbb{C}\), and \(E \rightarrow C\) a vector bundle on it of rank \(r\). We consider the Quot scheme parameterizing zero-dimensional quotients of \(E\) with length \(d\)

\[ \text{Quot}_C(E,d) = \{E \rightarrow Q : \dim \text{Supp}(Q) = 0, \text{length}(Q) = d\} \]

The Quot scheme \(\text{Quot}_C(E,d)\) is a smooth projective variety of dimension \(rd\). When \(r = 1\), it is isomorphic to the symmetric product \(\text{Sym}^d(C)\). In this paper, we prove the following structure of the derived category of \(\text{Quot}_C(E,d)\), which gives a one dimensional analogue of (1.1):

**Theorem 1.1.** There is a semiorthogonal decomposition of the form

\[ D^b(\text{Quot}_C(E,d)) = \bigoplus \text{Sym}^{d_1}(C) \times \cdots \times \text{Sym}^{d_r}(C) : d_1 + \cdots + d_r = d \]

Here the order of the above semiorthogonal summands is given by a lexicographic order of \((d_1, \ldots, d_r) \in \mathbb{Z}_{\geq 0}^r\).

In [BFP20 Proposition 4.5], Bagnarol-Fantechi-Perroni proved the following identity of the class of \(\text{Quot}_C(E,d)\) in the Grothendieck ring of varieties (also see [BPr89]):

\[ [\text{Quot}_C(E,d)] = \sum_{d_1 + \cdots + d_r = d} [\text{Sym}^{d_1}(C)] \times \cdots \times [\text{Sym}^{d_r}(C)] \times [\mathbb{A}^1]^{l_r} \]

where \(l_r := \sum_{i=1}^{r}(i-1)d_i\). The semiorthogonal decomposition in Theorem 1.1 gives a categorical analogue of the above identity. We also refer to [OP] on enumerative geometry related to \(\text{Quot}_C(E,d)\). We also note that each semiorthogonal summand (1.2) may be further decomposed (see Remark [4.10]).
1.2. Quot formula of relative dimension one. The result of Theorem 1.1 is a special case of the Quot formula of relative dimension one, which is described below. Let $S$ be a smooth quasi-projective scheme and

$$\pi : C \to S$$

be a smooth projective morphism of relative dimension one. For $E \in \text{Coh}(C)$ with rank $r$, we consider the $S$-relative Quot scheme

$$\text{Quot}_{C/S}(E, d) \to S$$

whose fiber over $s \in S$ is the Quot scheme $\text{Quot}_{C_s}(E_s, d)$ over the curve $C_s$. As we do not require $E$ to be locally free nor $S$-flat, several geometric properties of $\text{Quot}_{C_s}(E_s, d)$ (e.g. dimension, smoothness) may depend on $s \in S$.

Let $H := \text{Ext}_C^1(E, \mathcal{O}_C)$. As a dual side of (1.3), we also consider the $S$-relative Quot scheme

$$\text{Quot}_{C/S}(H, d) \to S$$

whose classical truncations are $\text{Quot}_{C/S}(E, d)$, $\text{Quot}_{C/S}(H, d)$, with virtual dimensions $\dim S + rd$, $\dim S - rd$, respectively (if non-empty). We also denote by

$$\text{Sym}^k(C/S) := (C \times_S \cdots \times_S C) / \mathfrak{S}_k \to S$$

the relative symmetric product, whose fiber at $s \in S$ is $\text{Sym}^k(C_s)$. We prove the following:

**Theorem 1.2.** There is a semiorthogonal decomposition of the form

$$D^b(\text{Quot}_{C/S}(E, d)) = \left\langle D^b(\text{Sym}^{d_1}(C/S) \times_S \cdots \times_S \text{Sym}^{d_r}(C/S) \times_S \text{Quot}_{C/S}(H, d - \sum_{i=1}^r d_i)) : (d_1, \ldots, d_r) \in \mathbb{Z}_{\geq 0}^r \right\rangle.$$

When $C = S$, the following Quot formula is conjectured in [Jia] and proved in [Todd]:

$$D^b(\text{Quot}_S(E, d)) = \left\langle \binom{r}{k} \text{-copies of } D^b(\text{Quot}_S(H, d - k)) : 0 \leq k \leq \min\{d, r\} \right\rangle.$$  

(1.4)

The result of Theorem 1.2 gives a relative dimension one analogue of the above semiorthogonal decomposition. Note that Theorem 1.1 is obtained from Theorem 1.2 by taking $S$ to be the point and $E$ to be a vector bundle. In some sense, the semiorthogonal decomposition in Theorem 1.2 may be regarded as a bosonic counterpart of the semiorthogonal decomposition (1.4) (see Remark 1.8).

1.3. Categorical wall-crossing formula for framed one loop quiver. The proof of Theorem 1.2 is similar to the proof of the formula (1.4) in [Todd]. Namely we construct relevant functors using global categorified Hall products, and reduce the problem to a local situation. In [Todd], the required local statement is semiorthogonal decomposition under Grassmannian flip proved in [Todd]. In the case of Theorem 1.2, the required local statement is semiorthogonal decomposition under wall-crossing of framed one loop quivers.

We denote by $Q$ the quiver with one vertex $\{1\}$ and one loop. For $a, b \in \mathbb{Z}_{\geq 0}$ with $r := a - b \geq 0$, we denote by $Q_{a,b}$ the quiver with vertices $\{\infty, 1\}$, where there are $a$-arrows from $\infty$ to $1$, $b$-arrows from $1$ to $\infty$:

$$Q = \bullet_1 \quad Q_{2,1} = \bullet_\infty \quad Q_{2,1} = \bullet_1$$
For $d \in \mathbb{Z}$, we denote by $\mathcal{M}_Q(d)$ the moduli stack of $Q$-representations with dimension vector $d$, $\mathcal{G}_{a,b}(d)$ the $C^*$-rigidified moduli stack of $Q_{a,b}$-representations with dimension vector $(1, d)$. There are two GIT stable loci $G_{a,b}^+(d) \subset \mathcal{G}_{a,b}(d)$ with respect to the characters $\chi_{0}^\pm$, where $\chi_{0} : \text{GL}(d) \to \mathbb{C}^*$ is the determinant character. They are related by a flip $(a > b)$, flop $(a = b)$

Here $\mathcal{G}_{a,b}(d) \to G_{a,b}(d)$ is the good moduli space. The D/K principle by Bondal-Orlov [BO] and Kawamata [Kaw02] predicts a fully-faithful functor $D^b(G_{a,b}(d')) \to D^b(G_{a,b}(d))$. This is not difficult to prove, but we need more: we need to describe its semiorthogonal complements in terms of $D^b(G_{a,b}(d'))$ for $d' < d$. Similarly to the argument in [To1], we prove it using categorified Hall products introduced in [Pâda]. It is a functor

\[(1.5)\quad * : D^b(\mathcal{M}_Q(d_1)) \boxtimes D^b(\mathcal{G}_{a,b}(d_2)) \to D^b(\mathcal{G}_{a,b}(d_1 + d_2)).\]

The above functor is defined by the stack of short exact sequences of $Q_{a,b}$-representations, and categorifies cohomological Hall algebras in [KS11].

We will use the following two subcategories for $c \in \mathbb{Z}$: \[(1.6)\quad \mathbb{W}(d) \subset D^b(\mathcal{M}_Q(d)), \quad \mathbb{W}_c(d) \subset D^b(\mathcal{G}_{a,b}(d)).\]

The first one is the subcategory introduced in [Pâda], which gives an approximation of BPS sheaf in the context of cohomological DT theory in [DM20] (in [PT], a similar category for the triple loop quiver with a super-potential is called a quasi-BPS category). For a general symmetric quiver it seems difficult to investigate it, but in our situation of the quiver $Q$ the structure of $\mathbb{W}(d)$ is very simple: it is just generated by $\mathcal{O}(\mathcal{M}_Q(d))$ and equivalent to $D^b(\mathcal{M}_Q(d))$ for the good moduli space

\[\mathcal{M}_Q(d) \to M_Q(d) = \text{Sym}^d(A^1) \cong A^d.\]

The second one is a window subcategory introduced and studied in [HL15] [BFK19] [HLS20]. It has the property that the composition functors

\[\mathbb{W}_c(d) \hookrightarrow D^b(\mathcal{G}_{a,b}(d)) \to D^b(G_{a,b}^\pm(d))\]

are equivalences for $(c = a, +)$, $(c = b, -)$. The definitions of the subcategories (1.6) are similar, but because of the generic stabilizers of $\mathcal{M}_Q(d)$ their play different roles in this paper.

We show that, for each $(d_1, \ldots, d_r) \in \mathbb{Z}_{\geq 0}^r$, the categorified Hall product induces the fully-faithful functor

\[\mathbb{W}_c(d) \boxtimes (\mathbb{W}(d_2) \otimes \chi_0) \boxtimes \cdots \boxtimes (\mathbb{W}(d_1) \otimes \chi_0^{r-1}) \boxtimes (\mathbb{W}_b(d - \sum_{i=1}^r d_i) \otimes \chi_0^r) \to \mathbb{W}_d(d)\]

whose essential images form a semiorthogonal decomposition (see Theorem 3.12). In particular, we have the following:

**Theorem 1.3.** (Corollary 3.13) There is a semiorthogonal decomposition of the form

\[D^b(G_{a,b}^\pm(d)) = \left\langle D^b(\mathcal{M}_Q(d_1)) \boxtimes \cdots \boxtimes D^b(\mathcal{M}_Q(d_r)) \boxtimes D^b(G_{a,b}^-(d - \sum_{i=1}^r d_i)) : (d_1, \ldots, d_r) \in \mathbb{Z}_{\geq 0}^r \right\rangle.\]

In [To1], a categorical wall-crossing formula for framed zero-loop quivers is applied to give a categorical wall-crossing formula for the resolved conifold. In [PT], a categorical wall-crossing formula for framed triple-loop quiver is obtained to give a categorical wall-crossing formula for $\mathbb{C}^3$. The result of Theorem 1.3 can be also used to give a categorical wall-crossing formula in other situations, e.g. a CY 3-fold which contracts a divisor to a curve.
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1.5. Notation and convention. In this paper, all the schemes and (derived) stacks are defined over \( \mathbb{C} \). For a derived stack \( \mathcal{X} \), we denote by \( D^b(\mathcal{X}) \) the homotopy category of \( \infty \)-category of quasi-coherent sheaves on \( \mathcal{X} \) with coherent cohomologies. For a scheme \( S \) and derived stacks \( \mathcal{X}_i \to S \) over \( S \) for \( i = 1, 2 \), we denote by \( D^b(\mathcal{X}_1) \boxtimes_S D^b(\mathcal{X}_2) = D^b(\mathcal{X}_1 \times_S \mathcal{X}_2) \). For triangulated subcategories \( \mathcal{C}_i \subset D^b(\mathcal{X}_i) \), we denote by \( \mathcal{C}_1 \boxtimes_S \mathcal{C}_2 \) the triangulated subcategory of \( D^b(\mathcal{X}_1) \boxtimes_S D^b(\mathcal{X}_2) \) split generated by \( \mathcal{C}_1 \boxtimes \mathcal{C}_2 \) for \( \mathcal{C}_i \in \mathcal{C}_i \). For a morphism of schemes \( T \to S \) and an Artin stack \( \mathcal{X} \to S \), we write \( \mathcal{X}_T := \mathcal{X} \times_T \mathcal{S} \). For a perfect complex \( E \) on \( \mathcal{X} \), we denote by \( \mathcal{E}_T \) its pull-back to \( \mathcal{X}_T \).

Let \( S = \text{Spec} \mathcal{R} \) for a complete local \( \mathbb{C} \)-algebra \( \mathcal{R} \), and \( T_i = \text{Spec} \mathcal{A}_i \) for complete local \( \mathcal{R} \)-algebras \( \mathcal{A}_i \) for \( i = 1, 2 \). We denote by \( \mathcal{A}_1 \boxtimes \mathcal{R} \mathcal{A}_2 \) the complete tensor product, and write \( T_1 \times_R T_2 := \text{Spec}(\mathcal{A}_1 \boxtimes_R \mathcal{A}_2) \). For derived stacks \( \mathcal{X}_i \to T_i \) over \( T_i \), we denote by \( \mathcal{X}_1 \times_R \mathcal{X}_2 \to T_1 \times_R T_2 \) the pull-back of \( \mathcal{X}_1 \times \mathcal{X}_2 \to T_1 \times \mathcal{T}_2 \) via \( T_1 \times_R T_2 \to T_1 \times \mathcal{T}_2 \). We denote by \( D^b(\mathcal{X}_1) \boxtimes_R D^b(\mathcal{X}_2) = D^b(\mathcal{X}_1 \times_R \mathcal{X}_2) \). The triangulated subcategory \( \mathcal{C}_1 \boxtimes_R \mathcal{C}_2 \) in \( D^b(\mathcal{X}_1) \boxtimes_R D^b(\mathcal{X}_2) \) is defined to be split generated by \( \mathcal{C}_1 \boxtimes \mathcal{C}_2 \) for \( \mathcal{C}_i \in \mathcal{C}_i \). These notation also apply to categories of \( (\mathbb{C}^* \text{-equivariant}) \) factorizations in Subsection 2.3.2 in an obvious way.

Let \( G \) be a reductive algebraic group with maximal torus \( T \). We denote by \( M \) the character lattice of \( T \) and \( N \) the cocharacter lattice of \( T \). For a \( G \)-representation \( Y \), we denote by \( \text{wt}_T(Y) \subset M \) the set of \( T \)-weights of \( Y \). There is a perfect pairing
\[
\langle -, - \rangle : N \times M \to \mathbb{Z}.
\]
The Weyl group of \( G \) is denoted by \( W \), and \( M^W \subset M \) is defined to be the fixed part of \( W \)-action on \( M \). We fix a Borel subgroup \( B \subset G \) and set roots of \( B \) to be negative roots. We denote by \( M^+ \subset M \) the dominant chamber, and for \( \chi \in M^+ \) we denote by \( V(\chi) \) the irreducible \( G \)-representation with highest weight \( \chi \). We also define \( \rho \in M_Q \) to be the half sum of positive roots. For \( \chi \in M \) and \( w \in W \), define
\[
w \cdot \chi := w(\chi + \rho) - \rho.
\]
If \( \chi + \rho \) has a trivial stabilizer in \( W \), there is a unique \( w \in W \) such that \( w \cdot \chi \in M^+ \), and in that case we set \( \chi^+ := w \cdot \chi \). Otherwise we set \( V(\chi^+) \) to be zero.

2. Preliminary

2.1. Attracting loci. Let \( Y \) be a smooth affine scheme with an action of a reductive algebraic group \( G \). For a one parameter subgroup \( \lambda : \mathbb{C}^* \to G \), let \( Y^{\lambda \geq 0}, Y^{\lambda = 0} \) be defined by
\[
Y^{\lambda \geq 0} := \{ y \in Y : \lim_{t \to 0} \lambda(t)(y) \text{ exists} \},
\]
\[
Y^{\lambda = 0} := \{ y \in Y : \lambda(t)(y) = y \text{ for all } t \in \mathbb{C}^* \}.
\]
The Levi subgroup and the parabolic subgroup
\[
G^{\lambda = 0} \subset G^{\lambda \geq 0} \subset G
\]
are also similarly defined by the conjugate \( G \)-action on \( G \), i.e. \( g \cdot (-) = g(-)g^{-1} \). The \( G \)-action on \( Y \) restricts to the \( G^{\lambda \geq 0} \)-action on \( Y^{\lambda \geq 0} \), and the \( G^{\lambda = 0} \)-action on \( Y^{\lambda = 0} \). We note that \( \lambda \) factors through \( \lambda : \mathbb{C}^* \to G^{\lambda = 0} \), and it acts on \( Y^{\lambda = 0} \) trivially. So we have the decomposition into fixed \( \lambda \)-weight subcategories
\[
D^b(Y^{\lambda = 0}/G^{\lambda = 0}) = \bigoplus_{j \in \mathbb{Z}} D^b((Y^{\lambda = 0}/G^{\lambda = 0}))_{\lambda \cdot \text{wt} = j}.
\]
Moreover by setting the slope to be 1 for each 1

\[ Z = \lim_{\lambda \to 0} [Y^\lambda = 0/G^\lambda = 0]. \]

Here \( p_\lambda \) is induced by the inclusion \( Y^{\lambda \geq 0} \subset Y \), and \( q_\lambda \) is given by taking the \( t \to 0 \) limit of the action of \( \lambda(t) \) for \( t \in \mathbb{C}^* \). The morphism \( \sigma_\lambda \) is a section of \( q_\lambda \) induced by inclusions \( Y^{\lambda = 0} \subset Y^{\lambda \geq 0} \) and \( G^{\lambda = 0} \subset G^{\lambda \geq 0} \). We will use the following lemma:

**Lemma 2.1.** ([HL15, Corollary 3.17, Amplification 3.18])

(i) For \( \mathcal{E}_i \in D^b([Y^\lambda \geq 0/G^\lambda \geq 0]) \) with \( i = 1, 2 \), suppose that

\[ \sigma_\lambda^i \mathcal{E}_1 \in D^b([Y^{\lambda = 0}/G^{\lambda = 0}])_{\lambda \text{-wt} \geq j}, \quad \sigma_\lambda^2 \mathcal{E}_2 \in D^b([Y^{\lambda = 0}/G^{\lambda = 0}])_{\lambda \text{-wt} < j} \]

for some \( j \). Then \( \text{Hom}(\mathcal{E}_1, \mathcal{E}_2) = 0 \).

(ii) For \( j \in \mathbb{Z} \), the functor

\[ q_\lambda^*: D^b([Y^{\lambda = 0}/G^{\lambda = 0}])_{\lambda \text{-wt} = j} \to D^b([Y^{\lambda \geq 0}/G^{\lambda \geq 0}]) \]

is fully-faithful.

2.2. Kempf-Ness stratification. Here review Kempf-Ness stratifications associated with GIT quotients of reductive algebraic groups following the convention of [HL15, Section 2.1]. Let \( Y \) and \( G \) be as in the previous subsection. For an element \( l \in \text{Pic}([Y/G])_\mathbb{R} \), we have the open subset of \( l \)-semistable points

\[ Y^{l \text{-ss}} \subset Y \]

characterized by the set of points \( y \in Y \) such that for any one parameter subgroup \( \lambda: \mathbb{C}^* \to G \) such that the limit \( z = \lim_{t \to 0} \lambda(t)(y) \) exists in \( Y \), we have \( \text{wt}(l|_z) \geq 0 \). Let \( |*| \) be the Weyl-invariant norm on \( N_\mathbb{R} \). The above subset of \( l \)-semistable points fits into the Kempf-Ness (KN) stratification

\[ Y = S_1 \cup S_2 \cup \cdots \cup S_N \sqcup Y^{l \text{-ss}}. \]

Here for each \( 1 \leq i \leq N \) there exists a one parameter subgroup \( \lambda_i: \mathbb{C}^* \to T \subset G \), an open and closed subset \( Z_i \) of \( (Y \setminus \cup_{j<i} S_j)/G^{\lambda_i = 0} \) (called center of \( S_i \)) such that

\[ S_i = G \cdot Y_i, \quad Y_i := \{ y \in Y^{\lambda_i \geq 0} : \lim_{t \to 0} \lambda_i(t)(y) \in Z_i \}. \]

Moreover by setting the slope to be

\[ \mu_i := -\frac{\text{wt}(l|_{Z_i})}{|\lambda_i|} \in \mathbb{R} \]

we have the inequalities \( \mu_1 > \mu_2 > \cdots > 0 \). We have the following diagram (see [HL15, Definition 2.2])

\[ [Y_i/G^{\lambda_i \geq 0}] \xrightarrow{q_i} [S_i/G][Y_i/G^{\lambda_i = 0}]. \]

Here the left vertical arrow is given by taking the \( t \to 0 \) limit of the action of \( \lambda_i(t) \) for \( t \in \mathbb{C}^* \), and \( \tau_i, q_i \) are induced by the embedding \( Z_i \to Y, S_i \to Y \) respectively.

Halpern-Leistner [HL15] extended the above notion of Kempf-Ness stratifications to \( \Theta \)-stratifications for more general Artin stacks (see [HL15, Definition 2.2]). Let \( \mathcal{N} \) be a classical Artin stack locally of finite type and with affine stabilizers. Suppose that it admits a good moduli space \( \mathcal{N} \to N \) (see [Alp13]). Then for any \( l \in \text{Pic}(\mathcal{N})_\mathbb{Q} \) and a positive definite \( b \in H^4(\mathcal{N}, \mathbb{Q}) \) (which corresponds to
Weyl-invariant norm on $N_R$ in the above setting), there is an associated $\Theta$-stratification (see [HLa, Theorem 4.1.3])

$$N = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_N \sqcup N_{t\text{-ss}}.$$ 

Here $b$ is called positive definite if for any non-degenerate $f : BC^* \to N$, we have $q^{-2}f^*b > 0$, where $q$ is the generator of $H^*(BC^*) = \mathbb{Q}[g]$. Similarly to KN stratifications, there are associated closed substacks $Z_i \subset S_i$ (called center of $S_i$) with canonical $C^*$-stabilizers at each point of $Z_i$, and a diagram of attracting loci similar to (2.4)

\[
\begin{array}{c}
S_i \ar[r] & N \ar[r] & N \ar[r] & Z_i. \\
\end{array}
\] (2.5)

2.3. Window theorem. In the diagram (2.4), let $\eta_i \in \mathbb{Z}$ be defined by

$$\eta_i := \text{wt}_{\lambda_i}(\det(N_{S_i/Y}|z_i)).$$

In the case that $Y$ is a $G$-representation, it is also written as

$$\eta_i = (\lambda_i, (Y^\vee)^{\lambda_i} > 0 - (g^\vee)^{\lambda_i} > 0).$$

Here for a $G$-representation $W$ and a one parameter subgroup $\lambda : \mathbb{C}^* \to T$, we denote by $W^{\lambda > 0} \in K(BT)$ the subspace of $W$ spanned by weights which pair positively with $\lambda$. We will use the following version of window theorem:

**Theorem 2.2.** ([HL15, BFK19]) For each $i$, we take $m_i \in \mathbb{R}$. Let

$$\mathbb{W}_{m_i}^i([Y/G]) \subset D^b((Y/G))$$

be the subcategory of objects $P$ satisfying the condition

$$\tau^*(P) \in \bigoplus_{j \in [m_i, m_i + \eta_i]} D^b([Z_i/G^{\lambda_i=0}]_{\lambda_i, \text{wt}=j})$$

for all $1 \leq i \leq N$. Then the composition functor

$$\mathbb{W}_{m_i}^i([Y/G]) \hookrightarrow D^b((Y/G)) \twoheadrightarrow D^b([Y^\text{t-ss}/G])$$

is an equivalence.

Suppose that $Y$ is a symmetric $G$-representation, i.e. $Y \cong Y^\vee$ as $G$-representations. In this case, there is another version of window theorem [HLS20], called magic window theorem. We denote by $\Sigma \subset M_R$ the subset

$$\Sigma = \sum_{\gamma \in \text{wt}_{T}(Y)} [0,1] \cdot \gamma \subset M_R.$$ 

Namely $\Sigma$ is the convex hull of $T$-weights of $\wedge^+(Y)$. For $\delta \in M^W_R$, the magic window subcategory

$$\mathbb{W}_\delta \subset D^b((Y/G))$$

is defined to be split generated by $V(\chi) \otimes \mathcal{O}_Y$ for $\chi \in M^+$ satisfying

$$\chi + \rho \in \frac{1}{2} \Sigma + \delta.$$ 

An element in $M^W_R$ is called $\Sigma$-generic if it lies in the linear span of $\Sigma$ but is not parallel to any face of $\Sigma$. 

Theorem 2.3. ([HLS20 Theorem 3.2]) We take \( \delta, l \in M^W_{\mathbb{R}} \) such that \( \partial(\Sigma/2 + \delta) \cap M^+ = \emptyset \). Then the composition functor

\[ \mathcal{W}_\delta \to D^b([Y/G]) \to D^b([Y^{\text{ss}}/G]) \]

is fully-faithful. If there is a \( \Sigma \)-generic element in \( M^W_{\mathbb{R}} \), then the above functor is an equivalence whenever \( Y^{\text{ss}} = Y^{\text{st}} \), where \( Y^{\text{st}} \) is the l-stable part.

2.4. Categorified Hall product. In the setting of the diagram (2.1), since \( p_\chi \) is proper we have the functor

\[ p_\lambda \ast q_\lambda^!: D^b([Y^{\lambda=0}/G^{\lambda=0}]) \to D^b([Y^{\lambda \geq 0}/G^{\lambda \geq 0}]) \to D^b([Y/G]). \]

In the case that \( Y \) is a moduli stack of representations of quivers, the above functor gives a categorified Hall product \( \mathcal{P} \). The image of \( V(\chi) \otimes \mathcal{O}_{Y^{\lambda=0}} \) under the above functor is calculated using Borel-Weil-Bott theorem. We will use the following fact:

Proposition 2.4. ([HLS20 Proposition 3.8]) For \( \chi \in M^+ \), the object

\[ p_\lambda \ast q_\lambda^!(V(\chi) \otimes \mathcal{O}_{Y^{\lambda=0}}) \in D^b([Y/G]) \]

is a successive extension of objects of the form \( V((\chi - \sigma_1)^+) \otimes \mathcal{O}_Y[2 I - l(I)] \). Here \( I \) is a finite subset

\[ I \subset \{ \beta \in \text{wt}_T(Y) : \langle \beta, \lambda \rangle < 0 \}, \]

\( \sigma_1 \) is the sum of \( \beta \in I \) and \( l(I) \) is the length of \( w \in W \) with \( w \ast (\chi - \sigma_1) \in M^+ \). Moreover if \( (\chi - \sigma_1)^+ = \chi \) implies \( 2 I = 0 \), then \( V(\chi) \otimes \mathcal{O}_Y \) appears exactly once.

2.5. The category of factorizations. Let \( \mathcal{Y} \) be a smooth noetherian algebraic stack over \( \mathbb{C} \) and take \( w \in \Gamma(\mathcal{O}_{\mathcal{Y}}) \). A (coherent) factorization of \( w \) consists of

\[ (2.12) \quad \mathcal{P}_0 \circlearrowleft_{\alpha_1} \mathcal{P}_1, \quad \alpha_0 \circ \alpha_1 = \cdot w, \quad \alpha_1 \circ \alpha_0 = \cdot w, \]

where each \( \mathcal{P}_i \) is a coherent sheaf on \( \mathcal{Y} \) and \( \alpha_i \) is a morphism of coherent sheaves. The category of coherent factorizations naturally forms a dg-category, whose homotopy category is denoted by \( \text{HMF}(\mathcal{Y}, w) \). The subcategory of absolutely acyclic objects

\[ \text{Acy}^{\text{abs}} \subset \text{HMF}(\mathcal{Y}, w) \]

is defined to be the minimum thick triangulated subcategory which contains totalizations of short exact sequences of coherent factorizations of \( w \). The triangulated category of factorizations of \( w \) is defined by (cf. [Orl12 EP15 PV11])

\[ \text{MF}(\mathcal{Y}, w) := \text{HMF}(\mathcal{Y}, w)/\text{Acy}^{\text{abs}}. \]

If \( \mathcal{Y} \) is an affine scheme, then \( \text{MF}(\mathcal{Y}, w) \) is equivalent to Orlov’s triangulated category of matrix factorizations of \( w \) [Orl02].

Let \( \mathcal{Y} = [Y/G] \) for a smooth quasi-projective scheme \( Y \) and \( G \) is an affine algebraic group acting on \( Y \). Suppose that there is an auxiliary \( \mathbb{C}^* \)-action on \( Y \) which commutes with the \( G \)-action, and the regular function \( w \in \Gamma(\mathcal{O}_{\mathcal{Y}}) \) is of \( \mathbb{C}^* \)-weight one. A \( \mathbb{C}^* \)-equivariant (coherent) factorization of \( w \) consists of (2.12) such that \( \alpha_0 \) is of \( \mathbb{C}^* \)-weight zero and \( \alpha_1 \) is of \( \mathbb{C}^* \)-weight one. The triangulated category of \( \mathbb{C}^* \)-equivariant factorizations of \( w \) is also similarly defined, and denoted by

\[ \text{MF}^{\mathbb{C}^*}(\mathcal{Y}, w). \]

If \( \mathbb{C}^* \) acts on \( Y \) trivially and \( w = 0 \), then \( \text{MF}^{\mathbb{C}^*}(\mathcal{Y}, 0) \) is equivalent to \( D^b(\mathcal{Y}) \).
2.6. **Koszul duality.** For a derived Artin stack \( \mathcal{M} \), its \((-1)\)-shifted cotangent is defined by

\[
\Omega_{\mathcal{M}}[-1] := \text{Spec Sym}(\mathcal{T}_{\mathcal{M}}[1]).
\]

Here \( \mathcal{T}_{\mathcal{M}} \) is the tangent complex of \( \mathcal{M} \). In the case that \( \mathcal{M} \) is a derived complete intersection, the classical truncation of \( \Omega_{\mathcal{M}}[-1] \) has the following critical locus description. Let \( \mathcal{Y} = [Y/G] \) for a smooth quasi-projective scheme \( Y \) and \( G \) is an affine algebraic group acting on \( Y \). Let \( \mathcal{F} \to \mathcal{Y} \) be a vector bundle on it with a section \( s \). Suppose that \( \mathcal{M} \) is a derived zero locus of \( s \). Let \( w \) be the function

\[
w: \mathcal{F}^\vee \to \mathbb{A}^1, \ w(y,v) = \langle s(y), v \rangle
\]

for \( y \in \mathcal{Y} \) and \( v \in \mathcal{F}^\vee|_y \). Then \( t_0(\mathcal{T}_{\mathcal{M}}[-1]) \) is isomorphic to the critical locus \( \text{Crit}(w) \) (see [Hir17, Proposition 2.8], [Todd, Lemma 2.5]). The above construction is summarized in the following diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{w} & \mathbb{A}^1 \\
\downarrow & & \downarrow \sim \\
\mathcal{Y}^i & \xrightarrow{F^i} & \mathcal{Y}, \\
\mathcal{M} & \xrightarrow{t_0(\mathcal{T}_{\mathcal{M}}[-1])} & \text{Crit}(w) \\
\end{array}
\]

Let \( \mathbb{C}^\ast \) acts on fibers of \( \mathcal{F}^\vee \to Y \) with weight one. In the above setting, the Koszul duality equivalence in [Hir17, Proposition 4.8] (also see [Isi13, Shi12, Todd]) is the following:

**Theorem 2.5.** ([Hir17, Isi13, Shi12, Todd]) There is an equivalence

\[
D^b(\mathcal{M}) \xrightarrow{\sim} \text{MF}^{\mathbb{C}^\ast}(\mathcal{F}^\vee, w).
\]

Let \( \psi: F_0 \to F_1 \) be a morphism of \( G \)-equivariant vector bundles on \( Y \) and set \( \mathcal{F}_i = [F_i/G] \). We denote by

\[
(F_0 \xrightarrow{\psi} F_1) \in D^b(\mathcal{Y})
\]

the associated two term complex. We set \( F^{-i} = F_i^\vee \), \( F^{-i} = F_i^\vee \), and consider the following total space

\[
\mathcal{F}_0 \times_Y F^{-1} = \left( [F_0 \times_Y F^{-1}/G] \right).
\]

There is a regular function \( w \) on it

\[
w: \mathcal{F}_0 \times_Y F^{-1} \to \mathbb{A}^1, \ (y,u,v) \mapsto \langle \psi|_y(u), v \rangle = \langle u, \psi|_y^\vee(v) \rangle.
\]

Here \( y \in Y \), \( u \in F_0|_y \), \( v \in F^{-1}|_y \), and \( \psi^\vee: F^{-1} \to F^0 \) is the dual of \( \psi \). We have two auxiliary \( \mathbb{C}^\ast \)-actions on \( \mathcal{F}_0 \times_Y F^{-1} \) which commute with the \( G \)-action: the one is acting on the fibers of \( F^{-1} \to Y \) by weight one, and the other is acting on the fibers of \( F_0 \to Y \) by weight one. The function \( w \) is of weight one with respect to both of the above actions, so we obtain two \( \mathbb{C}^\ast \)-equivariant categories of factorizations

\[
\text{MF}^{\mathbb{C}^\ast}(\mathcal{F}_0 \times_Y F^{-1}, w), \ \text{MF}^{\mathbb{C}^\ast}(\mathcal{F}_0 \times_Y F^{-1}, w)' \]

where the left one is defined by the \( \mathbb{C}^\ast \)-action on \( F^{-1} \), and the right one is defined by the \( \mathbb{C}^\ast \)-action on \( F_0 \).

Suppose that there is a one dimensional subtorus \( \mathbb{C}^\ast \subset G \) which lies in the center of \( G \) and acts on \( Y \) trivially. Then \( \mathbb{C}^\ast \subset G \) acts on fibers of \( F_i \to Y \), and we assume that they are of weight one. In this situation, the following lemma is implicit in [KT21, Subsection 2.2].

**Lemma 2.6.** In the above situation, there is an equivalence

\[
\text{MF}^{\mathbb{C}^\ast}(\mathcal{F}_0 \times_Y F^{-1}, w) \simeq \text{MF}^{\mathbb{C}^\ast}(\mathcal{F}_0 \times_Y F^{-1}, w)'.
\]
Proof. The isomorphism of algebraic groups
\[ (2.16) \quad G \times \mathbb{C}^* \cong G \times \mathbb{C}^* \quad (g, t) \mapsto (t^{-1} g, t) \]
gives an isomorphism of stacks
\[ (2.17) \quad [(F_0 \times_Y F^{-1})/(G \times \mathbb{C}^*)] \cong [(F_0 \times_Y F^{-1})/(G \times \mathbb{C}^*)] \]
Here in the left hand side the second \( \mathbb{C}^* \) acts on the fibers of \( F^{-1} \to Y \) by weight one, and in the right hand side it acts on the fibers of \( F_0 \to Y \) by weight one. Since \( (2.16) \) commutes with the second projection, the isomorphism \( (2.17) \) induces the equivalence \( (2.15) \). \( \square \)

Remark 2.7. Let \( \mathcal{U}, \mathcal{U}' \) be the derived zero loci
\[ \mathcal{U} = (\psi = 0) \subset F_0, \quad \mathcal{U}' = (\psi' = 0) \subset F^{-1}. \]
Then Theorem 2.5 together with Lemma 2.6 implies an equivalence \( D^b(\mathcal{U}) \cong D^b(\mathcal{U}') \), which recovers linear Koszul duality in \( [MR10] \).

3. Categorical wall-crossing for framed one loop quiver

3.1. One loop quiver. We denote by \( Q \) the one loop quiver, i.e. it consists of one vertex \( \{1\} \) and one loop:

\[ (3.1) \quad Q = \bullet \xrightarrow{1} \]

For \( d \in \mathbb{Z}_{\geq 0} \), let \( V \) be a \( d \)-dimensional vector space. The quotient stack
\[ \mathcal{M}_Q(d) := [\text{End}(V)/\text{GL}(V)] \]
is the moduli stack of \( Q \)-representations of dimension \( d \). We have the good moduli space
\[ (3.2) \quad \pi_Q : \mathcal{M}_Q(d) \to \mathcal{M}_Q(d) := \text{End}(V)/\text{GL}(V) \cong \mathbb{A}^d. \]
Here the last isomorphism is given by assigning \( A \in \text{End}(V) \) to the coefficients of its characteristic polynomial.

Remark 3.1. The stack \( \mathcal{M}_Q(d) \) is isomorphic to the stack of zero-dimensional sheaves \( Q \) on \( \mathbb{A}^1 \) with \( \chi(Q) = d \). The morphism \( (3.2) \) is identified with the Hilbert-Chow morphism sending \( Q \) to the support of \( Q \) in \( \text{Sym}^d(\mathbb{A}^1) \cong \mathbb{A}^d \).

We fix a basis of \( V \), and a maximal torus \( T \subset \text{GL}(V) \) to be consisting of diagonal matrices. We also set a Borel subgroup \( B \subset \text{GL}(V) \) to be consisting of upper triangular matrices, where roots of \( B \) are set to be negative roots. The character lattice \( M \) for \( T \) is given by \( M = \mathbb{Z}^d \), and the dominant chamber \( M^+ \subset M \) is given by
\[ M^+_B = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_1 \leq x_2 \leq \cdots \leq x_d\}. \]
We also often denote the standard basis of \( M \) as \( \{e_1, \ldots, e_d\} \) and write an element of \( M_\mathbb{R} \) as \( x_1 e_1 + \cdots + x_d e_d \). The half sum of positive roots \( \rho \) is given by
\[ \rho = \frac{1}{2} \sum_{i > j} (e_i - e_j) = \left( -\frac{d-1}{2}, -\frac{d-3}{2}, \ldots, -\frac{1}{2}, \frac{d-1}{2} \right). \]
The Weyl group of \( \text{GL}(V) \) is the symmetric group \( S_d \), and the Weyl-invariant part \( M^W \) is generated by \( \chi_0 = (1, \ldots, 1) \), where \( \chi_0 \) corresponds to the character
\[ (3.3) \quad \chi_0 : \text{GL}(V) \to \mathbb{C}^*, \quad g \mapsto \det(g). \]
Let $\Sigma(d) \subset M_R$ be the subset in (2.9) for the symmetric $\text{GL}(V)$-representation $\text{End}(V)$. Explicitly it is

$$\Sigma(d) = \sum_{-1 \leq c_{ij} \leq 1, i > j} c_{ij} \cdot (e_i - e_j).$$

For $\delta = t\chi_0 \in M^+_R$ with $t \in \mathbb{R}$, the subcategory

$$(3.4) \quad \mathbb{W}_\delta(d) \subset D^b(\mathcal{M}_Q(d))$$

is defined as in (2.11), i.e. it is split generated by $V(\chi) \otimes \mathcal{O}_{M^+_Q(d)}$, where $\chi \in M^+$ satisfies $\chi + \rho \in \Sigma(d)/2 + \delta$.

**Lemma 3.2.** The triangulated subcategory $(3.4)$ is non-zero if and only if $\delta = t\chi_0$ for $t \in \mathbb{Z}$. In this case, it is split generated by $\chi_0^{\otimes t}$. Here we have regarded $\chi_0$ as a line bundle on $\mathcal{M}_Q(d)$.

**Proof.** Note that we have

$$\frac{1}{2} \Sigma(d) - \rho + t\chi_0 = \sum_{-1 \leq c_{ij} \leq 0, i > j} c_{ij} \cdot (e_i - e_j) + \sum_{i=1}^d t \cdot e_i.$$  

Therefore any element $\chi$ in the LHS is written as

$$\chi = (-c_{21} - \cdots - c_{dd} + t)e_1 + (c_{21} - c_{32} - \cdots - c_{dd} + t)e_2 + \cdots + (c_{d1} + \cdots + c_{dd-1} + t)e_d.$$  

If $\chi$ lies in the dominant chamber, we have $-c_{21} - \cdots - c_{dd} \leq c_{d1} + \cdots + c_{dd-1}$, hence $c_{d1} = \cdots = c_{d1} = c_{d2} = \cdots = c_{dd-1} = 0$ as $c_{ij} \leq 0$ for all $i, j$. By applying the same argument for other coefficients of $\chi$, we conclude that $c_{ij} = 0$ for all $i, j$. Therefore we have

$$\left( \frac{1}{2} \Sigma(d) - \rho + t\chi_0 \right) \cap M^+_R = t\chi_0,$$  

and the lemma holds.  \qed

Below we set

$$\mathbb{W}(d) := \mathbb{W}_{\delta=0}(d) \subset D^b(\mathcal{M}_Q(d)),$$

which is generated by $\mathcal{O}_{M^+_Q(d)}$ by Lemma 3.2. Note that for $t \in \mathbb{Z}$, we have $\mathbb{W}_{\delta=0}(d) = \mathbb{W}(d) \otimes \chi_0^{\otimes t}$.

**Lemma 3.3.** The pull-back by the morphism $(3.2)$ induces the equivalence

$$(3.5) \quad \pi^*_Q : D^b(\mathcal{M}_Q(d)) \cong \mathbb{W}(d).$$

**Proof.** As $\pi_Q$ is a good moduli space morphism, it induces the isomorphism

$$\pi^*_Q : \text{Hom}_{M^+_Q(k)}(\mathcal{O}_{M^+_Q(k)}, \mathcal{O}_{M^+_Q(k)}) \cong \text{Hom}_{M^+_Q(k)}(\mathcal{O}_{M^+_Q(k)}, \mathcal{O}_{M^+_Q(k)}).$$

Since the triangulated category $D^b(\mathcal{M}_Q(d))$ is generated by $\mathcal{O}_{M^+_Q(d)}$, it follows that the functor $(3.5)$ is fully-faithful. By Lemma 3.2, the functor $(3.5)$ is also essentially surjective.  \qed

For a one parameter subgroup $\lambda : \mathbb{C}^* \to \text{GL}(V)$, we have the diagram of attracting loci

$$(3.6) \quad \mathcal{M}_Q(d)^{\lambda \geq 0} \xrightarrow{p^\lambda} \mathcal{M}_Q(d)$$

$$\xrightarrow{q_{\lambda}} \mathcal{M}_Q(d)^{\lambda = 0}.$$  

For $d = d_1 + d_2$, let $\lambda : \mathbb{C}^* \to T \subset \text{GL}(V)$ be given by

$$\lambda(t) = (t, \ldots, t, 1, \ldots, 1).$$  

$$\xrightarrow{d_1} \xrightarrow{d_2}$$
Then we have
\[ \mathcal{M}_Q(d)^{\lambda = 0} = \mathcal{M}_Q(d_1) \times \mathcal{M}_Q(d_2) \]
and the functor (2.11) gives the associative categorified Hall product
\[ (3.8) \quad * = p_{\lambda} q_{\lambda}^*: D^b(\mathcal{M}_Q(d_1)) \boxtimes D^b(\mathcal{M}_Q(d_2)) \rightarrow D^b(\mathcal{M}_Q(d)). \]

**Remark 3.4.** Using Proposition 2.4, one can check that the functor (3.8) induces the functor
\[ p_\lambda \times q_\lambda: \mathbb{W}(1)^{\oplus d} \rightarrow \mathbb{W}(d) \]
such that \( \mathbb{W}(d) \) is split generated by the image of the above functor. However the above functor is not fully-faithful.

### 3.2. Moduli stacks of representations of framed one loop quiver.

Let us take \((a,b) \in \mathbb{Z}^2_{\geq 0}, \ r := a - b \geq 0.\)

We denote by \(Q_{a,b}\) the extended quiver of \(Q\), that is the vertex \(\{\infty\}\), the \(a\)-arrows from \(\infty\) to \(1\) and the \(b\)-arrows from \(1\) to \(\infty\) are added:

\[ Q_{2,1} = \bullet \xrightarrow{\alpha} \infty \xleftarrow{\beta} \bullet \]

Let \(A, B, V\) be vector spaces such that \(\dim A = a, \ \dim B = b, \ \dim V = d.\)

We set the following \(\text{GL}(V)\)-representation
\[ (3.9) \quad Y_{a,b}(d) := \text{Hom}(A, V) \oplus \text{Hom}(V, B) \oplus \text{End}(V), \]
and form the following quotient stack
\[ \mathcal{G}_{a,b}(d) := \left[ (\text{Hom}(A, V) \oplus \text{Hom}(V, B) \oplus \text{End}(V)) / \text{GL}(V) \right]. \]

The above stack is the \(\mathbb{C}^*\)-rigidified moduli stack of \(Q_{a,b}\)-representations with dimension vector \((1, d)\). For a one parameter subgroup \(\lambda: \mathbb{C}^* \rightarrow T\), we use the following notation for the diagram of attracting loci
\[ (3.10) \quad \mathcal{G}_{a,b}(d)^{\lambda \geq 0} \xrightarrow{p_\lambda} \mathcal{G}_{a,b}(d) \]
\[ \quad \mathcal{G}_{a,b}(d)^{\lambda = 0}. \]

There exist two GIT quotients with respect to \(\chi_{\lambda}^{\pm 1}\) given by open substacks
\[ G_{a,b}(d)^{\pm} \subset \mathcal{G}_{a,b}(d), \]
which are smooth quasi-projective varieties. Here \(\chi_0\)-semistable locus \(G_{a,b}(d)^{+}\) consists of
\[ (\alpha, \beta, \gamma) \in \text{Hom}(A, V) \oplus \text{Hom}(V, B) \oplus \text{End}(V) \]
such that, by setting \(V_\gamma\) to be the \(\mathbb{C}[Q]\)-module structure on \(V\) determined by \(\gamma\), the image of \(\alpha: A \rightarrow V\) generates \(V_\alpha\) as a \(\mathbb{C}[Q]\)-module. Similarly \(\chi_0^{-1}\)-semistable locus \(G_{a,b}(d)^{-}\) consists of \((\alpha, \beta, \gamma)\) such that the image of \(\beta_\gamma: B_\gamma \rightarrow V_\gamma\) generates \(V_\gamma\) as a \(\mathbb{C}[Q]\)-module (see [Toda] Lemma 5.1.9)).
Let $G_{a,b}(d)$ be the good moduli space for $G_{a,b}(d)$:

$$G_{a,b}(d) := (\text{Hom}(A, V) \oplus \text{Hom}(V, B) \oplus \text{End}(V)) / \text{GL}(V).$$

We have the diagram

$$(3.11)\quad \begin{array}{ccc}
G^+_{a,b}(d) & \xrightarrow{\sim} & G^-_{a,b}(d) \\
\downarrow & & \downarrow \\
G_{a,b}(d) & & G_{a,b}(d)
\end{array}$$

which is a flip if $a > b > 0$, flop if $a = b > 0$ (see [Toda] Lemma 7.11).

**Remark 3.5.** When $b = 0$, then $G^+_{a,0}(d) = \emptyset$ and $G^+_{a,0}(d)$ is the Quot scheme on $\mathbb{A}^1$ which parametrizes quotients $\mathcal{O}_{\mathbb{A}^1} \to Q$ such that $Q$ is zero-dimensional of length $d$.

We take the Weyl-invariant norm on $N_\mathbb{R} = \mathbb{R}^d$ to be $|\lambda|^2 = \lambda_1^2 + \cdots + \lambda_d^2$. By [Toda] Lemma 5.1.9, we have the KN stratifications with respect to $(\chi^+_0, |\cdot|)$

$$(3.12)\quad G_{a,b}(d) = S^+_{0} \sqcup S^+_{1} \sqcup \cdots \sqcup S^+_{d-1} \sqcup G^+_{a,b}(d)$$

where $S^+_{i}$ consists of $(\alpha, \beta, \gamma)$ such that the image of $\alpha: A \to V$ generates $i$-dimensional $\mathbb{C}[Q]$-submodule of $V$, $S^+_{0}$ consists of $(\alpha, \beta, \gamma)$ such that the image of $\beta^\vee: B^\vee \to V^\vee$ generates $i$-dimensional $\mathbb{C}[Q]$-submodule of $V^\vee$. The associated one parameter subgroups $\lambda^+_{i}: \mathbb{C}^* \to T$ are taken as

$$(3.13)\quad \lambda^+_{i}(t) = (1, \ldots, 1, t^{-i}, \ldots, t^{-1}), \quad \lambda^-_{i}(t) = (t, \ldots, t, 1, \ldots, 1)$$

with associated slopes (2.3) to be $\mu^+_i = \sqrt{d - i}$.

### 3. Window subcategories.

For $c \in \mathbb{Z}$, we set

$$(3.14)\quad \mathbb{B}_c(d) := \{(x_1, x_2, \ldots, x_d) \in M^+: 0 \leq x_i \leq c - 1\}.$$

We define the triangulated subcategory

$$(3.15)\quad \mathbb{W}_c(d) \subset D^b(G_{a,b}(d))$$

to be the smallest thick triangulated subcategory which contains $V(\chi) \otimes \mathcal{O}_{G_{a,b}(d)}$ for $\chi \in \mathbb{B}_c(d)$. Note that $V(\chi)$ is the Schur power of $V$ associated with the Young diagram corresponding to $\chi$.

**Proposition 3.6.** The following composition functors are equivalences

$$(3.16)\quad \mathbb{W}_c(d) \subset D^b(G_{a,b}(d)) \to D^b(G^+_{a,b}(d)),$$

$$(3.17)\quad \mathbb{W}_c(d) \subset D^b(G_{a,b}(d)) \to D^b(G^-_{a,b}(d)).$$

**Proof.** Let $\lambda^+_{i}$ be the one parameter subgroup in (3.13). Then $\eta^+_i$ given in (2.6) is

$$(\lambda^+_{i}, (\text{Hom}(A, V)^\vee \oplus \text{Hom}(V, B)^\vee \oplus \text{End}(V)^\vee)_{\lambda^+_i > 0} - \text{End}(V)^\vee_{\lambda^+_i > 0}\} = a(d - i).$$

Let $\chi' = (x'_1, \ldots, x'_d)$ be a $T$-weight of $V(\chi)$ for $\chi \in \mathbb{B}_a(d)$. Then we have $0 \leq x'_j \leq a - 1$ for $1 \leq j \leq d$, so

$$-\eta^+_i = -a(d - i) < \langle \chi', \lambda^+_i \rangle = - \sum_{j=i+1}^{d} x'_j \leq 0.$$ 

Therefore by setting $m_i = -\eta^+_i + \varepsilon$ for $0 < \varepsilon \ll 1$ and $l = \chi_0$ in (2.7), we have

$$(3.17)\quad \mathbb{W}_a(d) \subset \mathbb{W}_m^{\chi_0}(G_{a,b}(d)) \subset D^b(G_{a,b}(d)).$$
Therefore we have
\[ Y_{\chi}(2.9) \text{ for } V \]

Therefore any element \( \chi \) in the above set is written as
\[ \chi = (-c_{21} - \cdots - c_{dd} + c_1)e_1 + (c_{21} - c_{32} - \cdots - c_{d2} + c_2)e_2 + \cdots + (c_{d1} + \cdots + c_{dd-1} + c_d)e_d. \]

We write it as \( \chi = \alpha_1 e_1 + \cdots + \alpha_d e_d \). If it lies in \( M^+_R \), then as \( c_{ij} \leq 0 \) we have
\[ t - \frac{a}{2} \leq c_1 \leq \cdots \leq c_d \leq t + \frac{a}{2}. \]

It follows that we have
\[ \left( \frac{1}{2} \Sigma_a(d) - \rho + t \chi_0 \right) \cap M^+_R = \left( \sum_{t-a/2 \leq c_i \leq t+a/2} c_i \cdot e_i \right) \cap M^+_R. \]

By setting \( t = a/2 - \varepsilon \) for \( 0 < \varepsilon \ll 1 \), we conclude that
\[ (3.18) \quad \left( \frac{1}{2} \Sigma_a(d) - \rho + t \chi_0 \right) \cap M^+ = \{ (x_1, x_2, \ldots, x_d) \in M : 0 \leq x_1 \leq \cdots \leq x_d \leq a - 1 \}. \]

Since \( \chi_0 \) is \( \Sigma_a(d) \)-generic and \( \Sigma_a(d)/2 - \rho + t \chi_0 \) does not contain integer points on the boundary, the composition (3.16) is an equivalence by Theorem 3.16 when \( a = b \).

When \( a > b \), we fix a decomposition into the direct sum \( A = B \oplus B' \). Then the projection \( A \twoheadrightarrow B \) and the inclusion \( B \hookrightarrow A \) define the projections \( p \) and the zero sections \( i \).

Since the \( \chi_0 \)-stability (resp. \( \chi_0^{-1} \)-stability) on \( G_{a,b}(d) \) does not impose constraint on \( \text{Hom}(V,B) \)-factor (resp. \( \text{Hom}(A,V) \)-factor), we have Cartesian squares

Here the vertical arrows are open immersions. Note that each morphism \( p \) is an affine bundle. We have the functors
\[ i^* : D^b(G^+_a, b)(d) \to D^b(G^+_a, b)(d), \quad p^* : D^b(G^-_a, b)(d) \to D^b(G^-_a, b)(d). \]

Since the images of the above functors generate \( D^b(G^+_a, b)(d) \), \( D^b(G^-_a, b)(d) \) respectively, from the essentially surjectivity of the functors (3.16) for \( a = b \), we also have the essentially surjectivity of (3.16) for \( a > b \).
Remark 3.7. The proof of Proposition 3.6 implies that the first inclusion in (3.17) is an equal, i.e.
\[ \mathbb{W}_a(d) = \mathbb{W}_{m_i=-n_i^+ + \varepsilon}(\mathcal{G}_{a,b}(d)), \quad \mathbb{W}_b(d) = \mathbb{W}_{m_i=0}(\mathcal{G}_{a,b}(d)). \]
This fact will be used in Lemma 4.5.

3.4. Computations of categorified Hall products. Let \( \lambda : \mathbb{C}^* \to T \subset \text{GL}(V) \) be the one-parameter subgroup given by
\[ \lambda(t) = (t, \ldots, t, 1, \ldots, 1). \]
Then we have
\[ \mathcal{G}^{\lambda=0}_{a,b}(d) = \mathcal{M}_Q(d_1) \times \mathcal{G}_{a,b}(d - d_1). \]
We have the diagram of attracting loci (3.10) and the associated categorified Hall product
\[ : D^b(\mathcal{M}_Q(d_1)) \boxtimes D^b(\mathcal{G}_{a,b}(d - d_1)) \to D^b(\mathcal{G}_{a,b}(d)). \]
By the iteration, we have the categorified Hall product
\[ : D^b(\mathcal{M}_Q(d_1)) \boxtimes \cdots \boxtimes D^b(\mathcal{M}_Q(d_i)) \boxtimes D^b(\mathcal{G}_{a,b}(d - d_1 - \cdots - d_i)) \to D^b(\mathcal{G}_{a,b}(d)). \]
We remark that, for \( A_i \in D^b(\mathcal{M}_Q(d_i)) \) and \( B \in D^b(\mathcal{G}_{a,b}(d - d_1 - \cdots - d_i)) \), the above functor satisfies that
\[ (A_1 * \cdots * A_i * B) \otimes \chi_0 \cong (A_1 \otimes \chi_0) * \cdots (A_i \otimes \chi_0) * (B \otimes \chi_0). \]
Here \( \chi_0 \) in the LHS is the determinant character for \( \text{GL}(d) \) and by abuse of notation \( \chi_0 \) in the RHS are determinant characters for \( \text{GL}(d_i) \) and \( \text{GL}(d - d_1 - \cdots - d_i) \). The above isomorphism follows immediately from the definition of categorical Hall products.

We fix \( c > b \). For \( d' < d \), we fix the following embedding
\[ \mathbb{B}_c(d') \hookrightarrow \mathbb{B}_c(d), \quad (x_{d-d'+1}, \ldots, x_d) \mapsto (x_1 = \cdots = x_{d-d'} = 0, x_{d-d'+1}, \ldots, x_d). \]
We regard an element of \( \mathbb{B}_c(d') \) as an element of \( \mathbb{B}_c(d) \) by the above embedding. For \( 0 \leq k \leq d \), let \( \mathbb{B}_{c,k}(d) \subset \mathbb{B}_c(d) \) be the subset defined by
\[ \mathbb{B}_{c,k}(d) = \{ (x_1, \ldots, x_d) \in \mathbb{B}_c(d) : x_1 = \cdots = x_k = 0, x_{k+1} > 0 \}. \]
Note that we have \( \mathbb{B}_{c,d}(d) = \{ 0 \} \) and
\[ \mathbb{B}_{c,0}(d) = \mathbb{B}_{c-1}(d) + \chi_0 \]
as \( \chi_0 = (1,1,\ldots,1) \). We have the decomposition into the disjoint union
\[ \mathbb{B}_c(d) = \mathbb{B}_{c,0}(d) \sqcup \mathbb{B}_{c,1}(d) \sqcup \cdots \sqcup \mathbb{B}_{c,d}(d). \]
For \( d' < d \) and \( d - d' \leq k \leq d \), the embedding (3.23) induces the bijection
\[ \mathbb{B}_{c,k-d+d'}(d') \cong \mathbb{B}_{c,k}(d). \]
We define the subcategory
\[ \mathbb{W}_{c,k}(d) \subset \mathbb{W}_c(d) \]
to be split generated by \( V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,k}(d)} \) for \( \chi \in \mathbb{B}_{c,k}(d) \).

Proposition 3.8. For \( 1 \leq k \leq d \) and \( \chi \in \mathbb{B}_{c,0}(d - k) \), the object
\[ \mathcal{O}_{\mathcal{M}_Q(k)} * (V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d-k)}) \]
is generated by \( V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)} \), where \( \chi \) is regarded as an element of \( \mathbb{B}_{c,k}(d) \) by (3.24), and \( V(\chi') \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)} \) for \( \chi' \in \mathbb{B}_{c,k}(d) \). Moreover \( V(\chi) \otimes \mathcal{O}_{\mathcal{G}_{a,b}(d)} \) appears exactly once.
Proof. Let $Y_{a,b}(d)$ be the GL($V$)-representation (3.9), and $\lambda$ the one parameter subgroup (3.19) for $d_1 = k$. Then we have
\[
\{ \beta \in \text{wt}_T(Y_{a,b}(d)) : (\lambda, \beta < 0) \} = \bigcup_{1 \leq i \leq k} \{ -e_i, \ldots, -e_i \} \bigcup_{1 \leq j \leq k, k < i \leq d} \{ (e_i - e_j) \}.
\]

Let $I$ be a subset of weights in the above set. Then in the notation of Proposition 2.4 for $\chi = (0, \ldots, 0, x_{k+1}, \ldots, x_d) \in \mathbb{B}_{c,k}(d)$, the element $\chi - \sigma_I + \rho$ is of the form
\[
\chi - \sigma_I + \rho = \sum_{i=k+1}^d x_ie_i + \sum_{i=1}^k s_ie_i - \sum_{1 \leq i < j \leq k} s_{ij}(e_i - e_j) + \frac{1}{2} \sum_{i > j} (e_i - e_j)
\]
for some $s_i \in \mathbb{Z}$ with $0 \leq s_i \leq b$ and $s_{ij} \in \{0, 1\}$. Therefore we have
\[
\chi - \sigma_I + \rho \in \left( \sum_{-c/2 \leq \epsilon_i \leq c/2} c_ie_i + \sum_{i > j, -1 \leq \epsilon_{ij} \leq 0} c_{ij}(e_i - e_j) \right) + t\chi_0
\]
for $t = c/2 - \epsilon$ with $0 < \epsilon \ll 1$. Suppose that $(\chi - \sigma_I)^+ \in M^+$ is defined. By its definition, there is unique $w \in S_d$ such that
\[
w(\chi - \sigma_I + \rho) = (\chi - \sigma_I)^+ + \rho.
\]
Since the right hand side of (3.26) is invariant under the Weyl group action, we have
\[
(\chi - \sigma_I)^+ \in \left( \sum_{-c/2 \leq \epsilon_i \leq c/2} c_ie_i + \sum_{i > j, -1 \leq \epsilon_{ij} \leq 0} c_{ij}(e_i - e_j) \right) \cap M^+
\]
Here the last identity follows as in (3.18). Therefore we have $(\chi - \sigma_I)^+ \in \mathbb{B}_c(d)$.

We show that $(\chi - \sigma_I)^+ \in \mathbb{B}_{c,k}(d)$, and $(\chi - \sigma_I)^+ \in \mathbb{B}_{c,k}(d)$ if and only if $I = \emptyset$. Then the proposition follows from Proposition 2.4. Let us write
\[
(\chi - \sigma_I)^+ = \sum_{i=1}^d x'_i e_i, \quad 0 \leq x'_1 \leq \cdots \leq x'_d \leq c - 1.
\]
Then by setting $y_i = x'_i - d/2 - 1/2 + i$, we have
\[
(\chi - \sigma_I)^+ + \rho = \sum_{i=1}^d y_ie_i, \quad \frac{d}{2} - \frac{1}{2} \leq y_1 < y_2 < \cdots < y_d < c + \frac{d}{2} - \frac{1}{2}.
\]
Note that $(\chi - \sigma_I)^+ \in \mathbb{B}_{c,0}(d)$ if and only if $y_i > -(d - 1)/2$ for all $i$.

Let $(\chi - \sigma_I + \rho)_{e_i}$ be the coefficient of $(\chi - \sigma_I + \rho)$ at $e_i$. From (3.25), for $1 \leq j \leq k$ we have
\[
(\chi - \sigma_I + \rho)_{e_j} = -\frac{d}{2} - \frac{1}{2} + j + s_j + \sum_{i=k+1}^d s_{ij} \geq \frac{-d - 1}{2},
\]
and the equality holds if and only if $j = 1$, $s_1 = 0$ and $s_{ij} = 0$ for all $k < i \leq d$. Also for $k < i \leq d$, we have
\[
(\chi - \sigma_I + \rho)_{e_i} = -\frac{d}{2} - \frac{1}{2} + i + x_i - \sum_{j=1}^k s_{ij} > -\frac{d - 1}{2} + i - k - 1 \geq \frac{-d - 1}{2}.
\]
Here the first inequality is strict since $x_i > 0$. Therefore by the identity (3.19), we have either $(\chi - \sigma_I)^+ \in \mathbb{B}_{c,0}(d)$, or $s_1 = s_{ij} = 0$ for all $k < i \leq d$. 

Suppose that $s_1 = s_{i1} = 0$ for all $k < i \leq d$, so that $y_1 = -(d-1)/2$. For $2 \leq j \leq k$, from (3.28) we have

$$(\chi - \sigma_1 + \rho)_{e_j} \geq -\frac{d-3}{2}$$

and the equality holds only if $j = 2$, $s_2 = 0$ and $s_{i2} = 0$ for $k < i \leq d$. Moreover for $k < i \leq d$, the inequality (3.29) is improved as

$$(\chi - \sigma_1 + \rho)_{e_i} = -\frac{d}{2} - \frac{1}{2} + i + x_i - \sum_{j=2}^{k} s_{ij} > -\frac{d-3}{2} + i - k - 1 \geq -\frac{d-3}{2}.$$ 

It follows that we have either $y_2 > -(d-3)/2$, i.e. $(\chi - \sigma_1)^+ \in \mathbb{B}_{c,1}(d)$, or $s_2 = s_{i2} = 0$ for all $k < i \leq d$.

Repeating the above argument, we conclude that $(\chi - \sigma_1)^+ \in \mathbb{B}_{c,k}(d)$, or $s_1 = \cdots = s_k = 0$ and $s_{ij} = 0$ for all $1 \leq j \leq k$ and $k < i \leq d$. In the latter case, we have $I = \emptyset$ and $(\chi - \sigma_1)^+ = \chi \in \mathbb{B}_{c,k}(d)$. □

**Lemma 3.9.** The subcategory $\mathcal{W}_c(d) \subset D^b(G_{a,b}(d))$ is generated by $\mathcal{W}(k) \ast (\mathcal{W}_{c-1}(d-k) \otimes \chi_0)$ for $0 \leq k \leq d$.

**Proof.** Since $\mathcal{W}_{c-1}(d-k) \otimes \chi_0 = \mathcal{W}_{c,0}(d-k)$ and $\mathcal{W}(k)$ is generated by $O_{M_2}(k)$ by Lemma 3.2, we have

(3.30) $\mathcal{W}(k) \ast (\mathcal{W}_{c-1}(d-k) \otimes \chi_0) \subset \mathcal{W}_c(d)$, $0 \leq k \leq d$

by Proposition 3.8. It is enough to show that for any $\chi \in \mathbb{B}_c(d)$ the object $V(\chi) \otimes O_{G_{a,b}}(d)$ is generated by the LHS in (3.30) for $0 \leq k \leq d$. If $\chi \in \mathbb{B}_{c,0}(d)$, then $V(\chi) \otimes O_{G_{a,b}}(d)$ is an object in the LHS in (3.30) for $k = 0$. For $\chi \in \mathbb{B}_{c,k}(d)$ with $k > 0$, by Proposition 3.8 $V(\chi) \otimes O_{G_{a,b}}(d)$ is generated by $\mathcal{W}(k) \ast \mathcal{W}_{c,0}(d-k) = \mathcal{W}(k) \ast (\mathcal{W}_{c-1}(d-k) \otimes \chi_0)$ and $V(\chi') \otimes O_{G_{a,b}}(d)$ for $\chi' \in \mathbb{B}_{c,k}(d)$. Therefore by the induction of $k$, $V(\chi) \otimes O_{G_{a,b}}(d)$ is generated by the LHS in (3.30). □

**Proposition 3.10.** The subcategory $\mathcal{W}_c(d) \subset D^b(G_{a,b}(d))$ is generated by the subcategories

(3.31) $\mathcal{W}(d_*) := \mathcal{W}(d_1) \ast (\mathcal{W}(d_2) \otimes \chi_0) \ast \cdots \ast (\mathcal{W}(d_l) \otimes \chi_0^{l-1}) \ast (\mathcal{W}(d-l_d - \cdots - d_l) \otimes \chi_0^l)$

for $l = c - b > 0$ and $(d_1, \ldots, d_l) \in \mathbb{Z}_{\geq 0}^l$.

**Proof.** Suppose that the proposition holds for $c - 1$. Then for any $d_1 \geq 0$, the category $\mathcal{W}_{c-1}(d_d - d_1)$ is generated by

$\mathcal{W}(d_2) \ast (\mathcal{W}(d_3) \otimes \chi_0) \ast \cdots \ast (\mathcal{W}(d_l) \otimes \chi_0^{l-1}) \ast (\mathcal{W}(d_l - d_1 - \cdots - d_l) \otimes \chi_0^l)$

for $(d_2, \ldots, d_l) \in \mathbb{Z}_{\geq 0}^{l-1}$. Then by Lemma 3.9, $\mathcal{W}_c(d)$ is generated by

$\mathcal{W}(d_1) \ast \{ (\mathcal{W}(d_2) \ast (\mathcal{W}(d_3) \otimes \chi_0) \ast \cdots \ast (\mathcal{W}(d_l) \otimes \chi_0^{l-2}) \ast (\mathcal{W}(d_l - d_1 - \cdots - d_l) \otimes \chi_0^{l-1})) \otimes \chi_0 \}$

$= \mathcal{W}(d_1) \ast (\mathcal{W}(d_2) \otimes \chi_0) \ast \cdots \ast (\mathcal{W}(d_l) \otimes \chi_0^{l-1}) \ast (\mathcal{W}(d_l - d_1 - \cdots - d_l) \otimes \chi_0^l)$

for $(d_1, d_2, \ldots, d_l) \in \mathbb{Z}_{\geq 0}^l$. Then the proposition holds by the induction of $c$. □

### 3.5. Semiorthogonal decomposition.

**Proposition 3.11.** For each $0 \leq k \leq d$, the functor

(3.32) $\ast : \mathcal{W}(k) \boxtimes (\mathcal{W}_{c-1}(d-k) \otimes \chi_0)) \rightarrow \mathcal{W}_c(d)$

is fully-faithful, such that we have the semiorthogonal decomposition

$\mathcal{W}_c(d) = (\mathcal{W}(d) \ast (\mathcal{W}_{c-1}(0) \otimes \chi_0), \mathcal{W}(d-1) \ast (\mathcal{W}_{c-1}(1) \otimes \chi_0), \cdots, \mathcal{W}_{c-1}(d) \otimes \chi_0)$. 
Proof. The generation is proved in Lemma 3.9. It is enough to show that the functor is fully-faithful, and the images of the above functors are semimorphothogonal.

Let us take \( k \leq k' \) and \( \chi \in B_{c-1,0}(d-k) \), \( \chi' \in B_{c-1,0}(d-k') \). Let \( \lambda, \lambda' \) be the one parameter subgroups \( C^* \to T \subset GL(V) \) given by

\[
\lambda(t) = (t, \ldots, t, 1, \ldots, 1), \quad \lambda'(t) = (t, \ldots, t, 1, \ldots, 1).
\]

We have the diagrams of attracting loci

\[
\begin{array}{ccc}
\mathcal{G}_{a,b}(d)^{\lambda \geq 0} & \xrightarrow{p_\lambda} & \mathcal{G}_{a,b}(d) \\
q_\lambda & & q_\lambda \\
\mathcal{M}_Q(k) \times \mathcal{G}_{a,b}(d-k), & & \mathcal{M}_Q(k') \times \mathcal{G}_{a,b}(d-k').
\end{array}
\]

Then we have

\[
\begin{aligned}
\text{Hom}(\mathcal{O} \ast (V(\chi) \otimes \mathcal{O}), \mathcal{O} \ast (V(\chi') \otimes \mathcal{O})) \\
\cong \text{Hom}(p_{\lambda*}, q_\lambda^*(\mathcal{O} \boxtimes (V(\chi) \otimes \mathcal{O})), p_{\lambda*}, q_\lambda^*(\mathcal{O} \boxtimes (V(\chi') \otimes \mathcal{O}))).
\end{aligned}
\]

(3.34)

The object \( p_{\lambda*}q_\lambda^*(\mathcal{O} \boxtimes (V(\chi) \otimes \mathcal{O})) \) is generated by \( V(\chi) \otimes \mathcal{O} \) and \( V(\chi') \otimes \mathcal{O} \) for \( \chi'' \in B_{c;k}(d) \) by Proposition 3.3. If \( k < k' \), then any \( T \)-weight of \( V(\chi) \) and \( V(\chi') \) pairs positively with \( \lambda' \). Since any \( T \)-weight of \( V(\chi') \) pairs zero with \( \lambda' \), it follows that (3.34) is zero by Lemma 2.1 (i). Therefore the images of the functors 3.11 are semimorphothogonal.

Suppose that \( k = k' \). Since any \( T \)-weight of \( V(\chi') \) pairs positively with \( \lambda \), we have

\[
\begin{aligned}
\text{Hom}(\mathcal{O} \ast (V(\chi) \otimes \mathcal{O}), \mathcal{O} \ast (V(\chi') \otimes \mathcal{O})) \\
\cong \text{Hom}(p_{\lambda*}, q_\lambda^*(\mathcal{O} \boxtimes (V(\chi) \otimes \mathcal{O})), q_\lambda^*(\mathcal{O} \boxtimes (V(\chi') \otimes \mathcal{O}))).
\end{aligned}
\]

(3.34)

Here the second isomorphism follows since \( \mathcal{G}_{a,b}(d)^{\lambda \geq 0} \) is the stack of exact sequences

\[
0 \to \mathcal{V}^{\lambda > 0} \to \mathcal{V} \to \mathcal{V}^{\lambda = 0} \to 0
\]

of \( Q_{a,b} \)-representations, so the only Schur power power from \( \mathcal{V}^{\lambda = 0} \) survives by Lemma 2.1 (i). The last isomorphism follows from Lemma 2.1 (ii). Therefore the functor is fully-faithful. \( \Box \)

For \( d_\bullet = (d_1, \ldots, d_l) \in \mathbb{Z}_{\geq 0}^l \) and \( d'_\bullet = (d'_1, \ldots, d'_l) \in \mathbb{Z}_{\geq 0}^l \), we take the following lexicographic order: \( d'_i \succ d_i \) if and only if there is some \( m \) so that \( d_i = d_i^m \) for \( i < m \) and \( d_m > d'_m \).

Theorem 3.12. We take \( c > b \) with \( l := c - b > 0 \). Then for each \( d_\bullet = (d_1, \ldots, d_l) \in \mathbb{Z}_{\geq 0}^l \), the categorified Hall product \( \mathcal{C}(d_\bullet) \) induces the fully-faithful functor

\[
*: \mathcal{W}(d_1) \boxtimes (\mathcal{W}(d_2) \otimes \chi_0) \boxtimes \cdots \boxtimes (\mathcal{W}(d_l) \otimes \chi_{l-1}^{l-1}) \boxtimes (\mathcal{W}_b(d - \sum_{i=1}^l d_i) \otimes \chi_0^l) \to \mathcal{W}_c(d)
\]

such that, by setting \( \mathcal{C}(d_\bullet) \subset D^b(\mathcal{G}_{a,b}(d)) \) to be the essential image of the above functor, we have the semimorphothogonal decomposition

\[
\mathcal{W}_c(d) = \langle \mathcal{C}(d_\bullet) : d_\bullet \in \mathbb{Z}_{\geq 0}^l \rangle.
\]

Here \( \text{Hom}(\mathcal{C}(d'_\bullet), \mathcal{C}(d_\bullet)) = 0 \) for \( d'_\bullet \succ d_\bullet \).

Proof. The result is proved for \( c = b + 1 \) in Proposition 3.11. Then similarly to the proof of Proposition 3.10, the theorem easily follows by the induction of \( c \). \( \Box \)

By applying Theorem 3.12 to \( c = a \) and \( r = a - b \), we obtain the following:
Corollary 3.13. There is a semiorthogonal decomposition of the form
\[ D^b(G^-_{a,b}(d)) = \left\langle D^b(M_Q(d_1)) \boxtimes \cdots \boxtimes D^b(M_Q(d_r)) \boxtimes D^b(G^+_{a,b}(d - \sum_{i=1}^r d_i)) : (d_1, \ldots, d_r) \in \mathbb{Z}_{\geq 0}^r \right\rangle. \]

Proof. The corollary follows from Theorem 3.12, Proposition 3.6 and Lemma 3.3. □

3.6. The case of multiple quivers. For \( m \geq 1 \), let \( Q^{(m)} \) be the quiver with \( m \)-vertices \( \{1, \ldots, m\} \) with one loop at each vertex. We also define \( Q^{(m)}_{a,b} \) to be the quiver with vertices \( \{\infty, 1, \ldots, m\} \) with one loop at each vertex \( \{1, \ldots, m\} \), \( a \)-arrows from \( \infty \) to each \( i \in \{1, \ldots, m\} \), \( b \)-arrows from each \( i \in \{1, \ldots, m\} \) to \( \infty \). See the following picture for \( Q^{(3)}_{2,1} \):
We have the KN stratification of $G_{a,b}^{(m)}(d(\ast))$ with respect to $(\chi_0^{\pm 1}, |\epsilon|)$

\[(3.35) \quad \mathcal{G}_{a,b}^{(m)}(d(\ast)) = \mathcal{S}_0^+ \sqcup \cdots \sqcup \mathcal{S}_{|d(\ast)|-1}^+ \sqcup G_{a,b}^{(m)+}(d(\ast)).\]

A strata $\mathcal{S}_i^\pm$ corresponds to $Q_{a,b}^{(m)}$-representations such that the images of arrows from the vertex $\infty$ (resp. duals of arrows going to $\infty$) generates i-dimensional $Q^{(m)}$-representations (see [Todo], Lemma 5.1.9). Explicitly, by setting

\[G_{a,b}(d(j)) = \mathcal{S}_0^{(j)\pm} \sqcup \cdots \sqcup \mathcal{S}_{|d(j)|-1}^{(j)\pm} \sqcup G_{a,b}^{(m)\pm}(d(j))\]

to be KN stratifications \[(3.12)\] for $d(j)$, we have

\[\mathcal{S}_i = \prod_{i_1 + \cdots + i_m = i} \mathcal{S}_{i_j}^{(j)\pm}.\]

The corresponding one parameter subgroup is given by \{\lambda_{i_j}^{\pm}\}_{1 \leq j \leq m}, where $\lambda_{i_j}^{\pm}$ is given in \[(3.13)\], with slope \[(2.3)\] given by $\mu_i^\pm = \sqrt{|d(\ast)| - i}$.

We set

\[(3.36) \quad \mathbb{W}(d(\ast)) := \bigotimes_{j=1}^m \mathbb{W}(d(j)) \subset D^b(M Q^{(m)}(d(\ast))), \]
\[\mathbb{W}_{c}(d(\ast)) := \bigotimes_{j=1}^m \mathbb{W}_{c}(d(j)) \subset D^b(G_{a,b}^{(m)}(d(\ast))).\]

By Proposition \[(3.6)\] the following composition functors are equivalences:

\[\mathbb{W}_a(d(\ast)) \subset D^b(G_{a,b}^{(m)}(d(\ast))) \to D^b(G_{a,b}^{(m)+}(d(\ast))),\]
\[\mathbb{W}_b(d(\ast)) \subset D^b(G_{a,b}^{(m)}(d(\ast))) \to D^b(G_{a,b}^{(m)-}(d(\ast))).\]

For a decomposition $d(\ast) = d_1^{(\ast)} + d_2^{(\ast)}$, let $\lambda : \mathbb{C}^* \to \prod_{j=1}^m \text{GL}(V(j))$ be the one parameter subgroup given by

\[\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)}), \quad \lambda^{(j)}(t) = (t, \ldots, t, 1, \ldots, 1).\]

We have the diagram of attracting loci

\[(3.37) \quad \prod_{j=1}^m G_{a,b}(d(j))^{\lambda^{(j)} = 0} \xrightarrow{q^\lambda} G_{a,b}^{(m)}(d(\ast))^{\lambda = 0} \xrightarrow{p^\lambda} G_{a,b}^{(m)}(d(\ast))\]

which gives the categorized Hall product

\[(3.38) \quad \ast = q^\lambda \cdot p^\lambda : D^b(M Q^{(m)}(d_1^{(\ast)})) \boxtimes D^b(G_{a,b}^{(m)}(d_2^{(\ast)})) \to D^b(G_{a,b}^{(m)}(d(\ast))).\]

From Theorem \[(3.12)\] we have the following:

**Theorem 3.14.** We take $c > b$ with $l := c - b > 0$. Then for each $d_\ast = (d_1, \ldots, d_l) \in \mathbb{Z}_{\geq 0}^l$, the categorized Hall product induces the fully-faithful functor

\[\ast : \bigoplus_{(d_1^{(\ast)}, \ldots, d_l^{(\ast)})} (\mathbb{W}(d_1^{(\ast)}) \boxtimes (\mathbb{W}(d_2^{(\ast)}) \otimes \chi_0) \boxtimes \cdots \boxtimes (\mathbb{W}(d_l^{(\ast)}) \otimes \chi_0^{l-1}) \boxtimes (\mathbb{W}_b(d(\ast) - \sum_{i=1}^l d_i^{(\ast)}) \otimes \chi_0^l) \to \mathbb{W}_c(d(\ast))\]
such that, by setting \( C(d_{\bullet}) \subset D^b(\mathcal{G}^{(m)}_{a,b}(d_{\bullet})) \) to be the essential image of the above functor, we have the semiorthogonal decomposition

\[
\mathcal{W}_c(d_{\bullet}) = \langle C(d_{\bullet}) : d_{\bullet} \in \mathbb{Z}_{\geq 0}^l \rangle.
\]

Here \( \text{Hom}(C(d_{\bullet}'), C(d_{\bullet})) = 0 \) for \( d_{\bullet}' > d_{\bullet} \).

**Proof.** The categorified Hall product \( (\mathbb{K}, \mathbb{S}) \) fits into the commutative diagram

\[
\begin{array}{ccc}
D^b(\mathcal{M}_{Q^{(m)}}(d_{1}^{(\bullet)})) \boxtimes D^b(\mathcal{G}^{(m)}_{a,b}(d_{2}^{(\bullet)})) & \xrightarrow{\ast} & D^b(\mathcal{G}^{(m)}_{a,b}(d_{\bullet}^{(\bullet)})) \\
\downarrow & & \downarrow \\
\boxtimes_{j=1}^{m} D^b(\mathcal{M}_{Q}(d_{j}^{(\bullet)})) \boxtimes D^b(\mathcal{G}_{a,b}(d_{2}^{(\bullet)})) & \xrightarrow{\boxtimes \ast} & D^b(\mathcal{G}_{a,b}(d_{\bullet}^{(\bullet)})).
\end{array}
\]

Here the left vertical arrow is the exchange of factors. Therefore from Theorem 3.12 for each \((d_{1}^{(\bullet)}, \ldots, d_{l}^{(\bullet)})\) the functor

\[ 
\ast : \mathcal{W}(d_{1}^{(\bullet)}) \boxtimes (\mathcal{W}(d_{2}^{(\bullet)}) \otimes \chi_0) \boxtimes \cdots \boxtimes (\mathcal{W}(d_{l}^{(\bullet)}) \otimes \chi_0^{-1}) \boxtimes (\mathcal{W}_b(d_{\bullet}^{(\bullet)}) - \sum_{i=1}^{l} d_{i}^{(\bullet)} \otimes \chi_0) \rightarrow D^b(\mathcal{G}_{a,b}^{(m)}(d_{\bullet}^{(\bullet)}))
\]

is fully-faithful such that, by setting \( C(d_{\bullet}^{(\bullet)}) \) to be essential image of the above functor, we have the semiorthogonal decomposition of the form

\[ 
\mathcal{W}_c(d_{\bullet}^{(\bullet)}) = \langle C(d_{\bullet}^{(\bullet)}) : d_{\bullet}^{(\bullet)} \in \mathbb{Z}_{\geq 0}^m \rangle.
\]

For \( d_{\bullet}^{(\bullet)} \in \mathbb{Z}_{\geq 0}^m \), we set \( |d_{\bullet}^{(\bullet)}| := (|d_{1}^{(\bullet)}|, \ldots, |d_{l}^{(\bullet)}|) \in \mathbb{Z}_{\geq 0}^l \). Then for \( d_{\bullet}^{(\bullet)} \neq d_{\bullet}^{(\bullet)'} \) with \( |d_{\bullet}^{(\bullet)}| = |d_{\bullet}^{(\bullet)'}| \), the subcategories \( C(d_{\bullet}^{(\bullet)}) \) and \( C(d_{\bullet}^{(\bullet)'}) \) are orthogonal. Indeed in this case there exist \( j, j' \) such that \( d_{j}^{(\bullet)} < d_{j}^{(\bullet)'} \) and \( d_{j}^{(\bullet)} > d_{j}^{(\bullet)'} \), so the orthogonality follows from the last statement of Theorem 3.12. Similarly if \( |d_{\bullet}^{(\bullet)'}| > |d_{\bullet}^{(\bullet)}| \), then there is \( j \) such that \( |d_{j}^{(\bullet)'}| > |d_{j}^{(\bullet)}| \) so we have \( \text{Hom}(C(d_{\bullet}^{(\bullet)'}) , C(d_{\bullet}^{(\bullet)})) = 0 \). Therefore the theorem holds. \( \square \)

### 3.7. Some versions for categories of factorizations.

We will use some variants of Theorem 3.14 for categories of factorizations on some formal completions. Let

\[ 
\mathcal{M}_{Q^{(m)}}(d_{\bullet}^{(\bullet)}) \rightarrow M_{Q^{(m)}}(d_{\bullet}^{(\bullet)})
\]

be the good moduli space. Let \( R \) be a complete local \( \mathbb{C} \)-algebra with closed point \( 0 \in \text{Spec} \, R \). We denote by \( \tilde{M}_{Q^{(m)}}(d_{\bullet}^{(\bullet)})_{R} \) the following formal completion of \( M_{Q^{(m)}}(d_{\bullet}^{(\bullet)}) \times \text{Spec} \, R \)

\[ 
\tilde{M}_{Q^{(m)}}(d_{\bullet}^{(\bullet)})_{R} := \text{Spec} \, \hat{O}_{M_{Q^{(m)}}(d_{\bullet}^{(\bullet)}) \times \text{Spec} \, R, (0,0)},
\]

and take the following formal fibers

\[
\begin{array}{ccc}
\tilde{G}^{(m)}_{a,b}(d_{\bullet}^{(\bullet)})_{R} & \xrightarrow{\sim} & \mathcal{G}^{(m)}_{a,b}(d_{\bullet}^{(\bullet)}) \times \text{Spec} \, R \\
\downarrow & & \downarrow \\
\mathcal{M}_{Q^{(m)}}(d_{\bullet}^{(\bullet)})_{R} & \xrightarrow{\sim} & \mathcal{M}_{Q^{(m)}}(d_{\bullet}^{(\bullet)}) \times \text{Spec} \, R \\
\downarrow & & \downarrow \\
\tilde{M}_{Q^{(m)}}(d_{\bullet}^{(\bullet)})_{R} & \xrightarrow{\sim} & \mathcal{M}_{Q^{(m)}}(d_{\bullet}^{(\bullet)}) \times \text{Spec} \, R.
\end{array}
\]

Here the upper right vertical arrow is the projection. We consider an auxiliary \( \mathbb{C}^* \)-action on \( \mathcal{G}^{(m)}_{a,b}(d_{\bullet}^{(\bullet)}) \) acting on maps from \( \infty \) to each vertex \( \{1, \ldots, m\} \) with weight one, and acts on \( \text{Spec} \, R \).
trivially. Then it induces the \( C^\ast \)-action on \( \mathcal{G}_{a,b}^{(m)}(d^{(s)})_R \). We take a regular function
\[
(3.40) \quad w: \mathcal{G}_{a,b}^{(m)}(d^{(s)})_R \to \mathbb{A}^1
\]
which is of weight one with respect to the above \( C^\ast \)-action. We consider the triangulated category of \( C^\ast \)-equivariant factorizations \( MF_{C^\ast}^{\ast}(\mathcal{G}_{a,b}^{(m)}(d^{(s)})_R, w) \).

The diagram (3.37) extends to the diagram
\[
(3.41) \quad \mathcal{G}_{a,b}^{(m)}(d^{(s)})_{\lambda \geq 0} \times \text{Spec } R \rightarrow \mathcal{G}_{a,b}^{(m)}(d^{(s)}) \times \text{Spec } R
\]

\[\vdots\]
\[\mathcal{M}_{Q^{(m)}}^{\ast}(d_1^{(s)}) \times \mathcal{G}_{a,b}^{(m)}(d_2^{(s)}) \times \text{Spec } R \rightarrow M_{Q^{(m)}}(d^{(s)}) \times \text{Spec } R.\]

Here the right vertical arrow and the left bottom vertical arrow are compositions in the right vertical arrow in (3.39). By taking pull-back via the bottom horizontal arrow in (3.39), and taking account of the function (3.40), we obtain the diagram
\[
(3.42) \quad \mathcal{G}_{a,b}^{(m)}(d^{(s)})_{\lambda \geq 0} \rightarrow \mathcal{G}_{a,b}^{(m)}(d^{(s)})_R
\]

\[\vdots\]
\[\mathcal{M}_{Q^{(m)}}^{\ast}(d_1^{(s)}) \times \mathcal{G}_{a,b}^{(m)}(d_2^{(s)}) \rightarrow \mathcal{M}_{Q^{(m)}}^{\ast}(d^{(s)})_R 
\]

Here in the above diagram, the function \( w \) descends to a function of the form \((0, w')\) in the middle horizontal arrow by the \( C^\ast \)-weight one condition of \( w \). By the abuse of notation, we also denote \( w' \) by \( w \). From the above diagram, the Hall product (3.38) induces the one for formal completions
\[
(3.43) \quad \ast: D^h(\mathcal{M}_{Q^{(m)}}^{\ast}(d_1^{(s)})_R) \otimes_{\mathcal{O}} MF_{C^\ast}^{\ast}(\mathcal{G}_{a,b}^{(m)}(d_2^{(s)})_R, w) \rightarrow MF_{C^\ast}^{\ast}(\mathcal{G}_{a,b}^{(m)}(d^{(s)})_R, w).
\]

The window subcategories
\[
(3.44) \quad \mathcal{W}(d^{(s)}) \subset D^h(\mathcal{M}_{Q^{(m)}}^{\ast}(d^{(s)})_R), \mathcal{W}_c(d^{(s)}) \subset MF_{C^\ast}^{\ast}(\mathcal{G}_{a,b}^{(m)}(d^{(s)})_R, w)
\]
are also defined similarly to (3.36): when \( m = 1 \) and \( d^{(s)} = d \), they are the smallest thick triangulated subcategories which contain \( \mathcal{O}_{\mathcal{M}_{Q^{(m)}}^{\ast}(d^{(s)})_R} \), factorizations with entries direct sums of \( V(\chi) \otimes \mathcal{O}_{BC^\ast}(j) \otimes \mathcal{O} \) for \( \chi \in \mathcal{B}_c(d) \) and \( j \in \mathbb{Z} \), respectively. Here \( \mathcal{O}_{BC^\ast}(j) \) is the one dimensional \( C^\ast \)-representation with weight \( j \). For \( m > 1 \), they are defined to be the box-products of window subcategories of each factors as in (3.36). The result of Theorem 3.14 immediately implies the following variant of it (for example see the argument of [Toda, Corollary 4.22]):

**Theorem 3.15.** We take \( c > b \) with \( l := c - b > 0 \). For each \( d_\bullet = (d_1, \ldots, d_l) \in \mathbb{Z}_{\geq 0}^l \), the functors (3.44) induce the fully-faithful functor
\[
\ast: \bigoplus_{(d_1^{(s)}, \ldots, d_l^{(s)}) \mid |d_1^{(s)}| = d_1} \mathcal{W}(d_1^{(s)}) \otimes_{\mathcal{O}} \mathcal{W}(d_2^{(s)}) \otimes_{\mathcal{O}} \mathcal{W}(d_3^{(s)}) \cdots \otimes_{\mathcal{O}} \mathcal{W}(d_l^{(s)}) \otimes_{\mathcal{O}} \chi_0^{l-1} \otimes_{\mathcal{O}} \mathcal{W}(d - \sum_{i=1}^l d_i) \otimes \chi_0^l
\]

\[
\rightarrow \mathcal{W}_c(d^{(s)})
\]
such that, by setting \( \hat{C}(d_\bullet) \subseteq \mathcal{M} \sim \mathcal{Q}^{\mathrm{m}}(\nu, w) \) to be the essential image of the above functor, we have the semiorthogonal decomposition

\[
\hat{W}_C(d^{(s)}) = \langle \hat{C}(d_\bullet) : d_\bullet \in \mathbb{Z}_{\geq 0} \rangle.
\]

Here \( \text{Hom}(\hat{C}(d'_\bullet), \hat{C}(d_\bullet)) = 0 \) for \( d'_\bullet \succ d_\bullet \).

4. Quot formula of relative dimension one

4.1. Relative Quot schemes. Let \( S \) be a smooth quasi-projective scheme and

\[
\pi : \mathcal{C} \to S
\]

be a smooth projective morphism of relative dimension one. The stack of \( S \)-relative zero-dimensional sheaves of length \( d \) is given by the 2-functor

\[
\mathcal{M}_{\mathcal{C}/S}(d) : \text{(Sch}/S) \to \text{(Groupoid)}
\]

which sends \( T : S \) to the groupoid of \( T \)-flat sheaves \( \mathcal{P} \in \text{Coh}(\mathcal{C}_T) \) such that for any \( t \in T \), the object \( \mathcal{P}_t \in \text{Coh}(\mathcal{C}_t) \) is zero-dimensional of length \( d \). The stack \( \mathcal{M}_{\mathcal{C}/S}(d) \) is a smooth Artin stack such that the structure morphism

\[
\mathcal{M}_{\mathcal{C}/S}(d) \to S
\]

is smooth whose fiber at \( s \in S \) is the stack of zero-dimensional sheaves of length \( d \) on the smooth curve \( \mathcal{C}_s \).

For \( \mathcal{E} \in \text{Coh}(\mathcal{C}) \) with \( r := \text{rank}(\mathcal{E}) \geq 0 \), let \( \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}, d) \) be the 2-functor

\[
\mathcal{M}_{\mathcal{C}/S}(\mathcal{E}, d) : \text{(Sch}/S) \to \text{(Groupoid)}
\]

by sending \( T \) to the groupoid of pairs \((\mathcal{P}, u)\) where \( \mathcal{P} \) is a \( T \)-valued point of \( \mathcal{M}_{\mathcal{C}/S}(d) \) and \( u : \mathcal{E}_T \to \mathcal{P} \) is a morphism. The set of isomorphisms is given by commutative diagrams

\[
\begin{array}{ccc}
\mathcal{E}_T & \xrightarrow{u} & \mathcal{P} \\
\downarrow & \cong & \downarrow \\
\mathcal{E}_T & \xrightarrow{u'} & \mathcal{P}'.
\end{array}
\]

The 2-functor (4.1) is an Artin stack with morphisms

\[
\mathcal{M}_{\mathcal{C}/S}(\mathcal{E}, d) \to \mathcal{M}_{\mathcal{C}/S}(d) \to S
\]

where the first arrow is forgetting \( u \).

The stack \( \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}, d) \) contains an open substack

\[
\text{Quot}_{\mathcal{C}/S}(\mathcal{E}, d) \subseteq \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}, d)
\]

corresponding to \((\mathcal{P}, u)\) such that \( u : \mathcal{E}_T \to \mathcal{P} \) is surjective. The substack \( \text{Quot}_{\mathcal{C}/S}(\mathcal{E}, d) \) is the \( S \)-relative Grothendieck Quot scheme of \( \mathcal{E} \), and it is a projective scheme over \( S \). The fiber at \( s \in S \) is the Quot scheme which parameterizes surjections \( \mathcal{E}_s \to \mathcal{Q} \) where \( \mathcal{Q} \in \text{Coh}(\mathcal{C}_s) \) is zero-dimensional of length \( d \).

If \( \mathcal{E} \) is locally free, then the first arrow in (4.2) is the total space of a vector bundle over \( \mathcal{M}_{\mathcal{C}/S}(d) \). Indeed let

\[
\mathcal{Q} \in \text{Coh}(\mathcal{C} \times_S \mathcal{M}_{\mathcal{C}/S}(d))
\]

be the universal zero-dimensional sheaf, and \( p_1, p_2 \) be the projections from \( \mathcal{C} \times_S \mathcal{M}_{\mathcal{C}/S}(d) \) onto the corresponding factors. Then \( p_{2*}(\text{Hom}(p_1^*\mathcal{E}, \mathcal{Q})) \) is a locally free sheaf on \( \mathcal{M}_{\mathcal{C}/S}(d) \), and we have

\[
\mathcal{M}_{\mathcal{C}/S}(\mathcal{E}, d) = \text{Tot}(p_{2*}(\text{Hom}(p_1^*\mathcal{E}, \mathcal{Q}))) = \text{Spec}_{\mathcal{M}_{\mathcal{C}/S}(d)} \text{Sym}(p_{2*}(\mathcal{E}^\vee \boxtimes \mathcal{Q})^\vee).
\]

In particular in this case \( \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}, d) \) is smooth of relative dimension \( rd \).
Let \( W \) be a \( d \)-dimensional vector space, and \( \text{GL}_S(W) := \text{GL}(W) \times S \to S \) be the group scheme over \( S \). Let
\[
\text{Quot}^0_{\mathcal{C}/S}(W \otimes \mathcal{O}_C, d) \subset \text{Quot}_{\mathcal{C}/S}(W \otimes \mathcal{O}_C, d)
\]
be the open subscheme whose \( T \)-valued points correspond to \( u: W \otimes \mathcal{O}_C \to \mathcal{P} \) such that the induced morphism \( W \otimes \mathcal{O}_T \to \pi_T^* \mathcal{P} \) is an isomorphism. Since any zero-dimensional sheaf is globally generated, the first morphism in (4.2) induces the isomorphism over \( S \)
\[
[\text{Quot}^0_{\mathcal{C}/S}(W \otimes \mathcal{O}_C, d) / \text{GL}_S(W)] \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}/S}(d).
\]

4.2. Derived structures of relative Quot schemes. Suppose that \( \mathcal{E} \) has homological dimension less than or equal to one. Then there is a locally free resolution
\[
0 \to \mathcal{E}^{-1} \xrightarrow{\phi} \mathcal{E}^0 \to \mathcal{E} \to 0
\]
such that
\[
\mathcal{H} := \mathcal{E} \mathcal{H}^0_{\mathcal{D}_C}(\mathcal{E}, \mathcal{O}_{\mathcal{C}}) = \text{Cok}(\phi^\vee: \mathcal{E}_0 \to \mathcal{E}_1).
\]
Here we have set \( \mathcal{E}_0 := (\mathcal{E}^0)^\vee \) and \( \mathcal{E}_1 := (\mathcal{E}^{-1})^\vee \). The morphism \( \phi \) induces the morphism of vector bundles on \( \mathcal{M}_{\mathcal{C}/S}(d) \)
\[
\phi: \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}^0, d) \to \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}^{-1}, d)
\]
which, on \( T \)-valued points, is defined by
\[
(u: \mathcal{E}_T \to \mathcal{P}) \mapsto (u \circ \phi: \mathcal{E}_T^{-1} \xrightarrow{\phi^\vee} \mathcal{E}_T^0 \xrightarrow{\phi} \mathcal{P}).
\]
We define the derived stack \( \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}, d) \) by the derived Cartesian square
\[
\begin{array}{ccc}
\mathcal{M}_{\mathcal{C}/S}(\mathcal{E}, d) & \xrightarrow{\phi} & \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}^{-1}, d) \\
\mathcal{M}_{\mathcal{C}/S}(\mathcal{E}_0, d) & \xrightarrow{0} & \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}^0, d)
\end{array}
\]
Here the right vertical arrow is the zero section of the vector bundle \( \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}^{-1}, d) \to \mathcal{M}_{\mathcal{C}/S}(d) \).
The classical truncation of \( \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}, d) \) is isomorphic to \( \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}_0, d) \).
The surjection \( \mathcal{E}^0 \to \mathcal{E} \) induces the closed immersion
\[
\text{Quot}_{\mathcal{C}/S, d}(\mathcal{E}) \hookrightarrow \text{Quot}_{\mathcal{C}/S, d}(\mathcal{E}^0),
\]
where the target is an open substack of \( \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}_0, d) \). We define the quasi-smooth derived scheme \( \text{Quot}_{\mathcal{C}/S}(\mathcal{E}, d) \) over \( S \) by the Cartesian square
\[
\begin{array}{ccc}
\text{Quot}_{\mathcal{C}/S}(\mathcal{E}, d) & \xrightarrow{\phi^\vee} & \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}, d) \\
\text{Quot}_{\mathcal{C}/S}(\mathcal{E}_0, d) & \xrightarrow{0} & \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}_0, d)
\end{array}
\]
Here horizontal arrows are open immersions. Note that \( \text{Quot}_{\mathcal{C}/S}(\mathcal{E}, d) \) is a derived open substack of \( \mathcal{M}_{\mathcal{C}/S}(\mathcal{E}, d) \), with virtual dimension \( \text{dim } S + rd \), and whose classical truncation is \( \text{Quot}_{\mathcal{C}/S}(\mathcal{E}, d) \).

By taking the dual of the sequence (4.5), we obtain the exact sequence
\[
\mathcal{E}_0 \xrightarrow{\phi^\vee} \mathcal{E}_1 \to \mathcal{H} \to 0.
\]
Similarly to (1.8) and (1.10), we define $\mathbf{M}_{C/S}(\mathcal{H}, d)$ and $\text{Quot}_{C/S}(\mathcal{H}, d)$ by the derived Cartesian squares

\[
\begin{array}{ccc}
\text{Quot}_{C/S}(\mathcal{H}, d) & \overset{\square}{\longrightarrow} & \mathbf{M}_{C/S}(\mathcal{H}, d) \\
\downarrow & & \downarrow \\
\text{Quot}_{C/S}(\mathcal{E}_1, d) & \overset{\phi^\vee}{\longrightarrow} & \mathcal{M}_{C/S}(\mathcal{E}_0, d).
\end{array}
\]

Here the bottom right arrow is defined similarly to (4.7) from $\phi^\vee : \mathcal{E}_0 \rightarrow \mathcal{E}_1$. The derived stack $\text{Quot}_{C/S}(\mathcal{H}, d)$ is a derived open substack of $\mathbf{M}_{C/S}(\mathcal{H}, d)$ with virtual dimension $\dim S - rd$, and its classical truncation is $\text{Quot}_{C/S}(\mathcal{H}, d)$.

### 4.3. $(-1)$-shifted cotangent construction

By applying the construction in (2.14) for the morphism of vector bundles (4.10), we obtain the stack $\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d)$ with a regular function $w$

\[
\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d) := \mathcal{M}_{C/S}(\mathcal{E}^0, d) \times_{\mathcal{M}_{C/S}(d)} \mathcal{M}_{C/S}(\mathcal{E}^{-1}, d)^\vee \xrightarrow{w} \mathbb{A}^1.
\]

By the Grothendieck duality, the $T$-valued points of the stack $\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d)$ consist of

\[
(\mathcal{E}^0 \xrightarrow{\eta} \mathcal{P} \xrightarrow{\eta^-} \mathcal{E}^{-1}_T \otimes \omega_{C_T/T}[1])
\]

where $\mathcal{P}$ is a $T$-valued point of $\mathcal{M}_{C/S}(d)$. For an affine $T$, the function $w$ on the $T$-valued point (4.12) is given by

\[
\text{Tr}(\mathcal{E}^0 \xrightarrow{\eta} \mathcal{P} \xrightarrow{\eta^-} \mathcal{E}^{-1}_T \otimes \omega_{C_T/T}[1] \xrightarrow{\phi^\vee} (\mathcal{E}^0 \otimes \omega_{C_T/T}[1]) \in H^1(C_T, \omega_{C_T/T}) = \mathcal{O}_T.
\]

By the construction, we have the isomorphism (see Subsection 2.6)

\[
t_0(\Omega_{\mathcal{M}_{C/S}(\mathcal{E}, d)}[-1]) \cong \text{Crit}(w) \subset \mathcal{N}_{C/S}(\mathcal{E}^\bullet, d).
\]

By applying the similar construction for the bottom right arrow in (4.11), we obtain the stack $\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d)$ with a regular function $w^\vee$ on it

\[
\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d) := \mathcal{M}_{C/S}(\mathcal{E}^1, d) \times_{\mathcal{M}_{C/S}(d)} \mathcal{M}_{C/S}(\mathcal{E}_0, d)^\vee \xrightarrow{w^\vee} \mathbb{A}^1.
\]

The $T$-valued points of the stack $\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d)$ consist of

\[
((\mathcal{E}^1) \rightarrow \mathcal{P} \xrightarrow{\psi^\vee} (\mathcal{E}_0) \otimes \omega_{C_T/T}[1])
\]

where $\mathcal{P}$ is a $T$-valued point of $\mathcal{M}_{C/S}(d)$. For an affine $T$, the function $w^\vee$ on the $T$-valued point (4.13) is given by

\[
\text{Tr}((\mathcal{E}^1) \rightarrow \mathcal{P} \xrightarrow{\psi^\vee} (\mathcal{E}_0) \otimes \omega_{C_T/T}[1] \xrightarrow{\phi^\vee} (\mathcal{E}^1) \otimes \omega_{C_T/T}[1]) \in H^1(C_T, \omega_{C_T/T}) = \mathcal{O}_T.
\]

We also have the isomorphism

\[
t_0(\Omega_{\mathcal{M}_{C/S}(\mathcal{H}, d)}[-1]) \cong \text{Crit}(w^\vee) \subset \mathcal{N}_{C/S}(\mathcal{E}^\bullet, d).
\]

For a $T$-valued point $\mathcal{P} \in \text{Coh}(C_T)$ of $\mathcal{M}_{C/S}(d)$, we define

\[
\mathcal{P}^\vee := R\text{Hom}_{C_T}(\mathcal{P}, \omega_{C_T/T}[1]) = \text{Ext}_C^1(\mathcal{P}, \omega_{C_T/T}) \in \text{Coh}(C_T).
\]

The above object also gives a $T$-valued point of $\mathcal{M}_{C/S}(d)$, so we obtain the involution isomorphism

\[
\mathbb{D} : \mathcal{M}_{C/S}(d) \xrightarrow{\cong} \mathcal{M}_{C/S}(d), \quad \mathcal{P} \mapsto \mathcal{P}^\vee.
\]

**Lemma 4.1.** There is an isomorphism

\[
\mathbb{D} : \mathcal{N}_{C/S}(\mathcal{E}^\bullet, d) \xrightarrow{\cong} \mathcal{N}_{C/S}(\mathcal{E}^\bullet, d)
\]
such that the following diagram commutes:

\[
\begin{array}{ccc}
A^1 & \xrightarrow{w} & \mathcal{N}_{C/S}(\mathcal{E}^\bullet, d) \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{M}_{C/S}(d) & \xrightarrow{\pi} & \mathcal{M}_{C/S}(d).
\end{array}
\]

Here the vertical arrows are projections.

**Proof.** By applying \( R\text{Hom}_{C/T}(-, \omega_{C/T}[1]) \) to (4.12), we obtain

\[
(4.19) \quad ((\mathcal{E}_T)_1 \xrightarrow{\nu} \mathcal{P} \xrightarrow{\eta} (\mathcal{E}_T)_0 \otimes \omega_{C/T}[1]).
\]

The isomorphism (4.17) is obtained by assigning (4.12) with (4.19). Then the diagram (4.18) obviously commutes. \( \Box \)

4.4. \( \Theta \)-stratification. The stack \( \mathcal{M}_{C/S}(d) \) is the \( S \)-relative moduli stack of zero-dimensional semistable sheaves on \( C \), so it admits a good moduli space (which is nothing but the symmetric product)

\[
(4.20) \quad \pi_\mathcal{M}: \mathcal{M}_{C/S}(d) \to \mathcal{M}_{C/S}(d) = \text{Sym}^d(C/S).
\]

A point \( p \in \text{Sym}^d(C/S) \) corresponds to a point \( s \in S \) and an effective divisor on \( C_s \) of degree \( d \)

\[
(4.21) \quad p = \sum_{j=1}^{m} d(j)p^{(j)},
\]

where \( p^{(1)}, \ldots, p^{(m)} \) are distinct points in \( C_s \) and \( d(j) \geq 0 \) with \( d^{(1)} + \cdots + d^{(m)} = d \). Let \( R \) be the complete local \( C \)-algebra \( R = \mathcal{O}_{S,s} \). Formally locally at \( p \in \mathcal{M}_{C/S}(d) \), the stack \( \mathcal{M}_{C/S}(d) \) is described by the following commutative diagram

\[
(4.22) \quad \begin{array}{ccc}
\tilde{\mathcal{M}}_{Q^{(m)}}(d^{(\bullet)})_R & \xrightarrow{\pi_\mathcal{M}} & \mathcal{M}_{C/S}(d) \\
\downarrow \cong & & \downarrow \cong \\
\tilde{\mathcal{M}}_{Q^{(m)}}(d^{(\bullet)})_R & \xrightarrow{\pi_\mathcal{M}} & \mathcal{M}_{C/S}(d).
\end{array}
\]

Here the middle vertical arrow is the formal fiber at \( p \), and we have used the notation in Subsection 3.7 for the left vertical arrow.

The morphism \( \mathcal{N}_{C/S}(\mathcal{E}^\bullet, d) \to \mathcal{M}_{C/S}(d) \) is an affine morphism, so it also admits a good moduli space \( \mathcal{N}_{C/S}(\mathcal{E}^\bullet, d) \) together with a commutative diagram

\[
(4.23) \quad \begin{array}{ccc}
\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d) & \xrightarrow{\mathcal{M}_{C/S}(d)} & \mathcal{M}_{C/S}(d) \\
\downarrow g & & \downarrow \\
\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d) & \xrightarrow{\mathcal{M}_{C/S}(d)} & \mathcal{M}_{C/S}(d).
\end{array}
\]

Indeed \( \mathcal{N}_{C/S}(\mathcal{E}^\bullet, d) = \text{Spec} g_* \mathcal{O}_{\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d)} \), where \( g \) is the clockwise composition in the diagram (4.22).

For the universal sheaf \( \mathcal{Q} \) in (4.3), we set

\[
\mathcal{L} := \text{det}(p_2, \mathcal{Q}) \in \text{Pic}(\mathcal{M}_{C/S}(d)), \quad b := \text{ch}_2(p_2, \mathcal{Q}) \in H^4(\mathcal{M}_{C/S}(d), \mathbb{Q}).
\]
We also regard them as elements of $\text{Pic}(\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d))$, $H^2(\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d), \mathbb{Q})$ by pulling back them via $\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d) \to \mathcal{M}_{C/S}(d)$.

**Lemma 4.2.** The element $b \in H^4(\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d), \mathbb{Q})$ is positive definite.

**Proof.** Let $f: BC^* \to \mathcal{N}_{C/S}(\mathcal{E}^\bullet, d)$ be a non-degenerate morphism. It corresponds to a point $s \in S$ and a diagram

$$(\mathcal{E}_s^0 \to \mathcal{P}_0 \to \mathcal{E}_s^{-1} \otimes \omega_{C_s}[1]) \oplus \bigoplus_{j \neq 0} (0 \to \mathcal{P}_j \to 0)$$

where $\mathcal{P}_j$ are zero-dimensional sheaves on $C_s$ and have $C^*$-weight $j$. As $f$ is non-degenerate, we have $\mathcal{P}_j \neq 0$ for some $j \neq 0$. Then we have

$$q^{-2}f^*b = \sum_j \frac{1}{2} j^2 \cdot \chi(\mathcal{P}_j) > 0.$$

By the above lemma, there exist associated $\Theta$-stratifications with respect to $(\mathcal{L}^{\pm 1}, b)$

$$(4.24) \quad \mathcal{N}_{C/S}(\mathcal{E}^\bullet, d) = \mathcal{S}_0^{\pm} \sqcup \mathcal{S}_1^{\pm} \sqcup \cdots \sqcup \mathcal{S}_N^{\pm} \sqcup \mathcal{N}_{C/S}(\mathcal{E}^\bullet, d)^{\pm \infty}.$$

The stack $\mathcal{N}_{C/S}(\mathcal{E}^\bullet, d)$ and its $\Theta$-stratifications are described formally locally on $M_{C/S}(d)$ in the following way. For $p \in M_{C/S}(d)$, we write it as $(1.21)$, and set $R = \hat{O}_{S,*}$. We have the commutative diagram

$$(4.25) \quad \hat{\mathcal{G}}_{a,b}^{(\lambda)}(d^{ss})_R \xrightarrow{\sim} \hat{\mathcal{N}}_{C/S}(\mathcal{E}^\bullet, d)_p \xrightarrow{\sim} \mathcal{N}_{C/S}(\mathcal{E}^\bullet, d)_p \xrightarrow{\sim} \hat{M}_{C/S}(d)_p \xrightarrow{\sim} M_{C/S}(d).$$

Here $a := \text{rank} \mathcal{E}^0$, $b := \text{rank} \mathcal{E}^{-1}$, and we have used the notation in Subsection 3.7 for the left vertical arrow. Since $(\mathcal{L}, b)$ pulls back to $(\lambda_0, |\lambda|)$ under the top arrows in (4.25), the $\Theta$-stratification pulls back to the KN stratifications in (3.35). In particular, we have $N^+ = d - 1$.

**4.5. Window subcategories.** For the good moduli space morphism (4.20), since $M_{C/S}(d)$ is smooth the functor

$$(4.26) \quad \pi^*_M: D^b(M_{C/S}(d)) \to D^b(\mathcal{M}_{C/S}(d))$$

is well-defined and it is fully-faithful by the definition of good moduli space. We define the triangulated subcategory

$$(4.27) \quad \mathcal{W}_{\text{glob}}(d) \subset D^b(\mathcal{M}_{C/S}(d))$$

to be the essential image of the functor (4.20).

**Lemma 4.3.** An object $\mathcal{E} \in D^b(\mathcal{M}_{C/S}(d))$ lies in $\mathcal{W}_{\text{glob}}(d)$ if and only if for any $p \in M_{C/S}(d)$ as in (4.21), we have $\iota^*_p \mathcal{E} \in \hat{\mathcal{W}}(d^{ss})$. Here $\iota_p$ is given in (4.22) and $\hat{\mathcal{W}}(d^{ss})$ is given in (3.44).

**Proof.** An object $\mathcal{E} \in D^b(\mathcal{M}_{C/S}(d))$ lies in $\mathcal{W}_{\text{glob}}(d)$ if and only if the adjunction morphism

$$\pi^*_M \pi^*_M \mathcal{E} \to \mathcal{E}$$

is an isomorphism. Let $\mathcal{F}$ be the cone of the above morphism. Then the last condition is equivalent to that $\mathcal{F} = 0$. This property is formally local on $M_{C/S}(d)$, so it is equivalent to that $\iota^*_p \mathcal{F} = 0$ for any $p \in M_{C/S}(d)$. By the base change, it is also equivalent to that

$$\pi^*_p \pi^*_p (\iota^*_p \mathcal{E}) \to \iota^*_p \mathcal{E}.$$
is an isomorphism, which is equivalent to that \( i_p^* \mathcal{E} \in \mathcal{W}(d^{(*)}) \).

We take the \( \mathbb{C}^* \)-action on \( N_{C/S}(\mathcal{E}^\bullet, d) \) by the weight one action on the factor \( \mathcal{M}_{C/S}(\mathcal{E}^0, d) \), i.e. \( t \in \mathbb{C}^* \) acts on (4.12) by

\[
(\mathcal{E}_T^0 \xrightarrow{t} \mathcal{P} \xrightarrow{\nu} \mathcal{E}_T^{-1} \otimes \omega_{C/T}[1]).
\]

The function (4.13) is of weight one with respect to the above \( \mathbb{C}^* \)-action. So we have the triangulated category of \( \mathbb{C}^* \)-equivariant factorizations of \( w \)

\[
\text{MF}^{\mathbb{C}^*}(N_{C/S}(\mathcal{E}^\bullet, d), w).
\]

Let us consider \( \Theta \)-stratifications (4.24). For each \( 1 \leq i \leq N^\pm \), let us take \( m_i^{\pm} \in \mathbb{R} \). Similarly to (2.17), the window subcategory

\[
\mathcal{W}_{m_i^\pm}(d) \subset \text{MF}^{\mathbb{C}^*}(N_{C/S}(\mathcal{E}^\bullet, d), w)
\]

is defined to be consisting of objects \( \mathcal{P} \) so that for each \( 1 \leq i \leq N^\pm \) and center \( Z_i^\pm \subset S_i^\pm \), we have the weight condition with respect to the canonical \( \mathbb{C}^* \)-stabilizer groups in \( Z_i^\pm \):

\[
\text{wt}(\mathcal{P}|_{Z_i^\pm}) \subset [m_i^+, m_i^- + \eta_i^\pm)
\]

Here \( \eta_i^\pm \in \mathbb{Z} \) are defined as in (2.6), i.e. the weight of the conormal bundle of \( S_i^\pm \) inside \( N_{C/S}(\mathcal{E}^\bullet, d) \) restricted to \( Z_i^\pm \).

**Proposition 4.4.** There exist equivalences

\[
\mathcal{W}_{m_i^\pm}(d) \simeq D^b(\text{Quot}(\mathcal{E}, d)), \quad \mathcal{W}_{m_i^\pm}(d) \simeq D^b(\text{Quot}(\mathcal{H}, d)).
\]

**Proof.** A version of window theorem implies that the composition functors

\[
\mathcal{W}_{m_i^\pm}(d) \hookrightarrow \text{MF}^{\mathbb{C}^*}(N_{C/S}(\mathcal{E}^\bullet, d), w) \rightarrow \text{MF}^{\mathbb{C}^*}(N_{C/S}(\mathcal{E}^\bullet, d)^{\mathcal{L}^{-1}\text{-ss}}, w)
\]

are equivalences (see [HLa, Prop. 2.4.2] for perfect complexes, and then use the argument of [HL15, Proposition 5.5] to deduce the result for factorization categories). From the formal local description of \( \Theta \)-stratifications by the diagram (4.25), it follows that (4.12) is a \( T \)-valued point of \( N_{C/S}(\mathcal{E}^\bullet, d)^{\mathcal{L}-ss} \) if and only if \( u: \mathcal{E}_T \rightarrow \mathcal{P} \) is surjective. Therefore from Theorem 2.5 the last category in (4.30) for \( \mathcal{L} \)-semistable part is equivalent to \( D^b(\text{Quot}(\mathcal{E}, d)) \), so the first equivalence in (4.29) follows.

Under the isomorphism (4.16), the element line bundle \( \mathcal{L} \) pulls back to \( \mathcal{L}^{-1} \). Therefore from the diagram (4.18), the isomorphism (4.17) restricts to the isomorphism

\[
\mathcal{D}: N_{C/S}(\mathcal{E}^\bullet, d)^{\mathcal{L}^{-1}\text{-ss}} \cong N_{C/S}(\mathcal{E}^\bullet, d)^{\mathcal{L}-ss}.
\]

Similarly to above, the \( T \)-valued points of the right hand side consist of (4.14) such that \( u ': (\mathcal{E}_1)_T \rightarrow \mathcal{P} \) is surjective. Also note that, under the isomorphism (4.14), the diagonal torus \( \mathbb{C}^* \subset \text{GL}(W) \) acts on \( \text{Quot}_{C/S}(W \otimes \mathcal{O}_C, d) \) trivially and acts on fibers of \( \mathcal{M}_{C/S}(\mathcal{E}_i, d) \rightarrow \mathcal{M}_{C/S}(d) \) by weight one. Therefore by Theorem 2.5 and Lemma 2.6, the last category in (4.30) for \( \mathcal{L}^{-1}\text{-semistable part is equivalent to } D^b(\text{Quot}(\mathcal{H}, d)) \), so the second equivalence in (4.29) follows.

We take the special choices of \( m_i^\pm \) by \( m_i^+ = -\eta_i + \varepsilon \) for \( 0 < \varepsilon \ll 1 \) and \( m_i^- = 0 \), and define

\[
\mathcal{W}^+_{\text{glob}}(d) := \mathcal{W}_{m_i^+ = -\eta_i + \varepsilon}(d), \quad \mathcal{W}^-_{\text{glob}}(d) := \mathcal{W}_{m_i^- = 0}(d).
\]

**Lemma 4.5.** An object \( \mathcal{E} \in \text{MF}^{\mathbb{C}^*}(N_{C/S}(\mathcal{E}^\bullet, d), w) \) lies in \( \mathcal{W}^+_{\text{glob}}(d) \) (resp. \( \mathcal{W}^-_{\text{glob}}(d) \)) if and only if for any \( p \in M_{C/S}(d) \) as in (4.27), the object \( i_p^* \mathcal{E} \in \text{MF}^{\mathbb{C}^*}(\mathcal{G}_{a,b}^{(m)}(d^{(*)})_R, t_p^*w) \) lies in \( \mathcal{W}_{c}(d^{(*)}) \) for \( c = a \) (resp. \( c = b \)). Here \( t_p \) is given in (4.24) and \( \mathcal{W}_{c}(d^{(*)}) \) is given in (4.44).

**Proof.** The defining condition of \( \mathcal{W}^+_{\text{glob}}(d) \) is local on \( M_{C/S}(d) \), so \( \mathcal{E} \) is an object in \( \mathcal{W}^+_{\text{glob}}(d) \) if and only if \( i_p^* \mathcal{E} \) satisfies the same weight condition (4.28) with respect to the \( \Theta \)-stratifications (4.24) pulled back via \( t_p \). As we already mentioned, they coincide with KN stratifications in (3.35). Therefore the argument of Proposition 3.6 implies the lemma (see Remark 3.7). □
4.6. **Categorified Hall product.** For a decomposition \( d = d_1 + d_2 \), we define the stack over \( S \)

\[ \mathcal{E}_{x_{C/S}}(d_1, d_2): (\text{Sch}/S) \to (\text{Groupoid}) \]

whose \( T \)-valued points consist of exact sequences

\[
0 \to \mathcal{P}_1 \to \mathcal{P}_3 \to \mathcal{P}_2 \to 0
\]

(4.32)

where \( \mathcal{P}_i \) are \( T \)-valued points of \( \mathcal{M}_{C/S}(d_i) \) with \( d_3 = d \). The stack \( \mathcal{E}_{x_{C/S}}(d_1, d_2) \) is a smooth Artin stack of finite type over \( S \). Indeed we have the obvious evaluation morphisms

\[
\mathcal{E}_{x_{C/S}}(d_1, d_2) \xrightarrow{\text{ev}_2} \mathcal{M}_{C/S}(d_3)
\]

(4.33)

\[ (\text{ev}_1, \text{ev}_2) \]

\[ \mathcal{M}_{C/S}(d_1) \times_S \mathcal{M}_{C/S}(d_2) \]

The left vertical arrow is smooth because of the vanishing of \( \text{Ext}^2 \) on curves, so \( \mathcal{E}_{x_{C/S}}(d_1, d_2) \) is also smooth. Since every stacks in (4.33) are smooth over \( S \) and \( \text{ev}_3 \) is proper, we have the functor

\[
* := \text{ev}_3(\text{ev}_1, \text{ev}_2)^* : D^b(\mathcal{M}_{C/S}(d_1)) \boxtimes S D^b(\mathcal{M}_{C/S}(d_2)) \to D^b(\mathcal{M}_{C/S}(d))
\]

(4.34)

giving categorified Hall algebra structure on

\[
\bigoplus_{d \geq 0} D^b(\mathcal{M}_{C/S}(d)).
\]

(4.35)

We also define the stack over \( S \)

\[ \mathcal{E}_{x_{C/S}}(E^\bullet, d_1, d_2): (\text{Sch}/S) \to (\text{Groupoid}) \]

whose \( T \)-valued points consist of diagrams

\[
\begin{array}{ccc}
0 & \xrightarrow{u_2} & 0 \\
\downarrow & & \downarrow \\
\mathcal{E}_T^0 & \xrightarrow{u_3} & \mathcal{P}_3 \\
\downarrow & & \downarrow \\
\mathcal{E}_T^1 & \xrightarrow{v_3} & \mathcal{E}_T^{-1} \otimes \omega_{C/T}[1] \\
\downarrow & & \downarrow \\
\mathcal{E}_T^2 & \xrightarrow{v_2} & \mathcal{P}_2 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

(4.36)

where \( \mathcal{P}_i \) are \( T \)-valued points of \( \mathcal{M}_{C/S}(d_i) \) with \( d_3 = d \), the vertical arrows are exact sequences, the middle and the bottom arrows are \( T \)-valued points of \( N_{C/S}(E^\bullet, d_i) \) for \( i = 3, 2 \).

The stack \( \mathcal{E}_{x_{C/S}}(E^\bullet, d_1, d_2) \) is a smooth Artin stack of finite type. Indeed given an exact sequence (4.32), giving a diagram (4.36) is equivalent to giving morphisms \( \mathcal{E}_T^0 \to \mathcal{P}_3 \) and \( \mathcal{P}_2 \to \mathcal{E}_T^{-1} \otimes \omega_{C/T}[1] \), so \( \mathcal{E}_{x_{C/S}}(E^\bullet, d_1, d_2) \) is constructed as a Cartesian square

\[
\begin{array}{ccc}
\mathcal{E}_{x_{C/S}}(E^\bullet, d_1, d_2) & \xrightarrow{\text{ev}_2} & \mathcal{E}_{x_{C/S}}(d_1, d_2) \\
\downarrow & & \downarrow \\
\mathcal{M}_{C/S}(E^{-1}, d_2) \times_S \mathcal{M}_{C/S}(E^0, d_3) & \xrightarrow{\text{ev}_2 \times \text{ev}_3} & \mathcal{M}_{C/S}(d_2) \times_S \mathcal{M}_{C/S}(d_3).
\end{array}
\]

Since \( \mathcal{E}_{x_{C/S}}(d_1, d_2) \) is smooth and the bottom horizontal arrow is a vector bundle, the above construction in particular implies that \( \mathcal{E}_{x_{C/S}}(E^\bullet, d_1, d_2) \) is smooth over \( S \).
We have the obvious evaluation morphisms, commuting with super-potentials in (4.13)

\[(4.37)\]

\[\mathcal{E}_{C/S}(\mathcal{E}^\bullet, d_1, d_2) \xrightarrow{\text{ev}_3} N_{C/S}(\mathcal{E}^\bullet, d_3)\]

\[\xrightarrow{(ev_1, ev_2)}\]

\[M_{C/S}(d_1) \times_S N_{C/S}(\mathcal{E}^\bullet, d_2) \xrightarrow{(0, w)} \mathbb{A}^1\]

\[M_{C/S}(d_1) \times_S M_{C/S}(d_2) \oplus M_{C/S}(d_1).

Here the right vertical arrow is the morphism \(g\) in (4.23), and the left bottom vertical arrow is the product of (4.20) with \(g\). We have an auxiliary \(C^\ast\)-action on \(\mathcal{E}_{C/S}(\mathcal{E}^\bullet, d_1, d_2)\) acting on (4.36) by \((u_3, v_3, u_2, v_2) \mapsto (tv_3, v_3, tu_2, v_2)\). Then the diagram (4.37) is equivariant under the auxiliary \(C^\ast\)-actions. Also note that every stacks in the upper left diagram in (4.37) are smooth and \(\text{ev}_3\) is proper as any fiber is a closed subscheme of a Quot scheme. Therefore we have the categorified Hall product

\[(4.38)\]

\[\ast := \text{ev}_3 \circ (ev_1, ev_2)^\ast : D^b(M_{C/S}(d_1)) \boxtimes_S \text{MF}^C(N_{C/S}(\mathcal{E}^\bullet, d_2), w) \to \text{MF}^C(N_{C/S}(\mathcal{E}^\bullet, d), w)\]

which gives a left module structure of (4.36) on

\[\bigoplus_{d \geq 0} \text{MF}^C(N_{C/S}(\mathcal{E}^\bullet, d), w)\]

4.7. **Semiorthogonal decomposition.** Recall that \(r = \text{rank}(\mathcal{E})\), \(a = \text{rank} \mathcal{E}^0\), \(b = \text{rank} \mathcal{E}^{-1}\) so that \(r = a - b\). We prove the following proposition:

**Proposition 4.6.** For \((d_1, \ldots, d_r) \in \mathbb{Z}_{\geq 0}^r\), the categorified Hall product (4.38) induces the fully-faithful functor

\[(4.39)\]

\[\ast : W_{\text{glob}}(d_1) \boxtimes_S (W_{\text{glob}}(d_2) \otimes \mathcal{L}) \boxtimes_S \cdots \boxtimes_S (W_{\text{glob}}(d_r) \otimes \mathcal{L}^{r-1}) \boxtimes_S (W_{\text{glob}}(d - \sum_{i=1}^{r} d_i) \otimes \mathcal{L}^r) \rightarrow W_{\text{glob}}^+(d)\]

such that, by setting \(C_{\text{glob}}^-(d_\ast)\) to be the essential image of the above functor, we have the semiorthogonal decomposition

\[W_{\text{glob}}^+(d) = (C_{\text{glob}}^-(d_\ast)) : d_\ast \in \mathbb{Z}_{\geq 0}^r\]

Here \(\text{Hom}(C_{\text{glob}}(d_\ast'), C_{\text{glob}}(d_\ast)) = 0\) for \(d_\ast' > d_\ast\).

**Proof.** Following the argument of [Toen, Theorem 5.16], it is enough to show that the functor (4.39) is fully-faithful and forms a semiorthogonal decomposition formally locally on \(M_{C/S}(d)\). Let us take a closed point \(p \in M_{C/S}(d)\) written as in (4.21), and set \(R = \widehat{O}_{S, s}\). We write the formal completion \(\widehat{M}_{C/S}(d)_p\) as \(\widehat{M}_{C/S}(d)_{d_\ast p(\ast)}\). Let \(d = d_1 + d_2\) be a decomposition. Then we have the Cartesian square

\[
\begin{array}{ccc}
\prod_{d_1^{(\ast)} + d_2^{(\ast)} = d^{(\ast)}, |d_1^{(\ast)}| = d_1} \widehat{M}_{C/S}(d_1)_{d_1^{(\ast)} p(\ast)} \times_R \widehat{M}_{C/S}(d_2)_{d_2^{(\ast)} p(\ast)} & \longrightarrow & \widehat{M}_{C/S}(d)_{d^{(\ast)} p(\ast)} \\
\downarrow & & \\
M_{C/S}(d_1) \times_S M_{C/S}(d_2) & \oplus & M_{C/S}(d).
\end{array}
\]
Together with the diagrams (4.22), (4.25), the base change of the diagram (4.37) via \( \hat{M}_{C/S}(d)_p \to M_{C/S}(d) \) gives the diagram

\[
\begin{align*}
\hat{\mathcal{E}}x(\mathcal{E}^*, d_1, d_2)_p & \to \hat{\mathcal{G}}^{(m)}_{a,b} (d^{(*)})_R \\
\prod_{d_1^{(*)} + d_2^{(*)} = d^{(*)}, |d^{(*)}| = d_1} \hat{M}_{Q^{(m)}} (d_1^{(*)})_R \times_R \hat{M}_{Q^{(m)}} (d_2^{(*)})_R & \to A^1 \\
\prod_{d_1^{(*)} + d_2^{(*)} = d^{(*)}, |d^{(*)}| = d_1} \hat{M}_{Q^{(m)}} (d_1^{(*)})_R \times_R \hat{M}_{Q^{(m)}} (d_2^{(*)})_R & \oplus \hat{M}_{Q^{(m)}} (d^{(*)})_R.
\end{align*}
\]

By comparing with the diagram (3.42), we see that the base change of the product functor (4.38) via \( \hat{M}_{C/S}(d)_p \to M_{C/S}(d) \) is identified with the direct sum of (3.43) for all the decompositions \( d^{(*)} = d_1^{(*)} + d_2^{(*)} \) with \( |d_i^{(*)}| = d_i \).

For \( (d_1, \ldots, d_r) \in \mathbb{Z}_{\geq 0}^r \), the above argument shows that the base change of the product functor

\[ * : D^b(\mathcal{M}_{C/S}(d_1)) \boxtimes S \cdots \boxtimes S D^b(\mathcal{M}_{C/S}(d_r)) \boxtimes S MF^{C^*}(N_{C/S}(\mathcal{E}^*, d - \sum_{i=1}^r d_i), w) \to MF^{C^*}(N_{C/S}(\mathcal{E}^*, d), w) \]

via \( \hat{M}_{C/S}(d)_p \to M_{C/S}(d) \) gives the Hall product in Subsection 3.7

\[ * : \bigoplus_{(d_1^{(*)}, \ldots, d_r^{(*)})} D^b(\hat{M}_{Q^{(m)}}(d_1^{(*)})_R) \boxtimes_R \cdots \boxtimes_R D^b(\hat{M}_{Q^{(m)}}(d_r^{(*)})_R) \boxtimes_R MF^{C^*}(\hat{G}^{(m)}_{a,b} (d^{(*)}) - \sum_{i=1}^r d_i^{(*)})_R, \iota_p^* w) \to MF^{C^*}(\hat{G}^{(m)}_{a,b} (d^{(*)})_R, \iota_p^* w). \]

Then from Lemma 4.3 and Lemma 4.5, the required formal local statement is given in Theorem 4.15.

The following is the main result in this paper:

**Theorem 4.7.** There is a semiorthogonal decomposition of the form

\[ D^b(\text{Quot}_{C/S}(\mathcal{E}, d)) = \left< D^b(\text{Sym}^{d_1}(C/S)) \times_s \cdots \times_s \text{Sym}^{d_r}(C/S) \times_s \text{Quot}_{C/S}(\mathcal{H}, d - \sum_{i=1}^r d_i) : (d_1, \ldots, d_r) \in \mathbb{Z}_{\geq 0}^r \right>. \]

**Proof.** The theorem follows from Proposition 4.4, Proposition 4.5 and the fact that \( \mathcal{W}_{\text{glob}}(d) \) is equivalent to \( D^b(\text{Sym}^{d}(C/S)) \) by its definition. \( \square \)

**Remark 4.8.** The number of semiorthogonal summands in Theorem 4.7 involving \( \text{Quot}_{C/S}(\mathcal{H}, d - k) \) is \( \dim \text{Sym}^k(C^*) \), while that involving \( \text{Quot}_{C/S}(\mathcal{H}, d - k) \) in (1.3) is \( \dim \text{Sym}^k(C^*[1]) \). So in some sense the semiorthogonal decompositions in Theorem 4.7 and (1.4) are related by boson-fermion correspondence (see the related phenomena for cohomological DT theory in [Dav]).

For a smooth projective curve \( C \), the Quot scheme of zero-dimensional quotients \( \text{Quot}_{C}(\mathcal{E}, d) \) is smooth of expected dimension \( rd \). Therefore Theorem 4.7 implies the following:

**Corollary 4.9.** There is a semiorthogonal decomposition of the form

\[ D^b(\text{Quot}_{C}(\mathcal{E}, d)) = \left< D^b(\text{Sym}^{d_1}(C) \times \cdots \times \text{Sym}^{d_r}(C)) : d_1 + \cdots + d_r = d \right>. \]
Remark 4.10. The semiorthogonal decomposition \((4.40)\) may be further decomposed. Indeed for each \(\delta > 0\), there is a semiorthogonal decomposition proved in [Todd21 Corollary 5.11]

\[
D^b(\text{Sym}^{g-1+\delta}(C)) = (D^b(\text{Sym}^{g-1-\delta}(C)), J(C), \ldots, J(C))
\]

where \(g\) is the genus of \(C\) and \(J(C)\) is the Jacobian of \(C\). So if \(d_i > g - 1\) for some \(i\) in \((4.40)\), it can be further decomposed. In particular if \(C = \mathbb{P}^1\), then \((4.40)\) implies the existence of a full exceptional collection whose cardinality is

\[
\sum_{d_1 + \cdots + d_r = d} \prod_{i=1}^r (d_i + 1) = \binom{2r + d - 1}{d}.
\]

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