Stability of black holes from hydrodynamics

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Abstract. It has been well known from old days as a “membrane paradigm” that black holes exhibit laws analogous to laws of fluids. Recent development of AdS/CFT correspondence enhances this similarity into an exact duality, according to which the fluid lumps solving Navier-Stokes equations corresponds holographically to black holes in Scherk-Schwarz (SS) AdS spacetime solving Einstein’s equations in the bulk. Black rings in SS-AdS are considered to be gravitational duals of stationary and axisymmetric fluid annuli found by Lahiri and Minwalla. We will present a study on the stability of higher dimensional black holes via this dual configurations and further discuss new multiple plasma configurations.

1. Preliminaries
Higher dimensional black holes are of much interest in various points of view. Among other things, stability against gravitational perturbations is a very fundamental but difficult issue to be answered [1]. The AdS/CFT correspondence may provide a useful insight to this problem in a new approach. In the AdS/CFT viewpoint, certain conformal field theory of finite temperature is dual to a black hole immersed in AdS spacetime. At high energies, field theories have an effective hydrodynamic description: an approximation at long-wavelength expansion. The similarity of physics between black holes and hydrodynamics has been known as a “membrane paradigm,” which is now enhanced to exact duality within the context of AdS/CFT duality. For example, it was demonstrated in [2] that large black holes in AdS space is mapped to the conformal fluid (filling the boundary) which solves the Navier-Stokes equation on the boundary. Furthermore, Aharony et al. have shown that the lumps of plasma are holographic duals of black holes in the SS-compactified AdS\(d+2\) space [3]. Topology of these black holes is given by the \(S^1\)-fibre over the corresponding plasma lump. To be specific, the plasmaballs and plasmarings in \(d\)-dimensions are dual to the black holes and black rings in SS-AdS\(d+2\) [4]. These bounded objects appear in field theories that admit deconfinement phase transition of first-order. Lumps of fluids are used as a testbed to gain insight into black holes in AdS bulk (see [5, 6]). The analysis of stability is a primary concern in this short article.

2. Stability of plasmaballs
Let us consider equilibrium fluid lumps in \(d = n + 2\) dimensional flat space, 
\[
ds^2 = -dt^2 + dr^2 + r^2d\Omega_n^2,
\]
where \(d\Omega_n^2\) denotes the line element of a unit \(n\)-sphere. Dynamical variables in hydrodynamics are the number flux \(N^a\) and the stress-energy tensor \(T^{ab}\) of fluid, which obey divergence-type equation of motion 
\[
\nabla_a N^a = \nabla_a T^{ab} = 0
\]
and the constitutive equations 
\[
\nabla_a X^{abc} = I^{bc},
\]
where \(X^{abc}\) and \(I^{ab}\) are functional of \(N^a\) and \(T^{ab}\) with \(X^{ab} = X^{a[bc]} = I^{[ab]} = 0\).
The dissipative effects (entropy generation) are encoded in the constitutive equations. The positive divergence of an entropy current, \( S^\alpha \), specifies the direction of time.

We mean by ‘equilibrium’ as a state fulfilling the following 4-conditions: (i) vanishing entropy generation \( \nabla_\alpha S^\alpha = 0 \), (ii) there exists a unique hydrodynamic velocity \( u^\alpha \) for which no spatially preferable directions, (iii) \( P := T^i_i/(d-1) \) satisfies Euler’s relation (no bulk viscous pressure), (iv) density entropy takes a maximum value. Condition (i) implies that all vector quantities are proportional to \( u^\alpha \): \( N^\alpha = n u^\alpha \), \( S^\alpha = S u^\alpha \), etc. Condition (ii) means that the stress-tensor is diagonalizable and condition (iv) leads to \( \nabla_\alpha (\delta S^\alpha) = 0 \), implying \( \nabla_\alpha \alpha = 0 = \nabla_\alpha (\beta_\alpha) \), where \( \alpha := (P+\rho)/nT -(S/n) \) is the chemical potential per temperature and \( \beta_\alpha := T^{-1}u_\alpha \) is the inverse temperature flow. Constant \( \alpha \) is indicative of Tolmann’s effect. It follows that the equilibrium condition is necessarily stationary (\( \mathcal{L}_\beta g_{ab} = 0 \) and \( \beta^a \beta_\beta < 0 \), which forbids dissipative terms (expansion and shear). Hence, we shall confine attention to the stationary perfect fluid.

Away from the boundary of the fluid, the fluid lumps satisfy the continuity equation and the relativistic Euler’s equations

\[
\begin{align*}
\quad u^\alpha \nabla_\alpha \rho + (P + \rho) \nabla_\alpha u^\alpha &= 0, \\
(P + \rho) u^b \nabla_b u^a &= -(g^{ab} + u^a u^b) \nabla_b P.
\end{align*}
\]

The plasma lumps are sustained by surface tension \( \sigma \) approximated by a delta-function, which contributes to the Young-Laplace equation,

\[
P_\lt - P_\gt = \sigma K, \quad K = h^{ab} \nabla_a n_b.
\]

Here \( n^a \) is a unit outward normal to the boundary. The SS-equation of state reads \( P = \frac{\frac{1}{2} \rho - \frac{d+1}{d+2} \rho_0}{\rho} \), where \( \rho_0 \) is a constant [3].

Let us discuss the static plasmaballs with an \( SO(n) \)-symmetry specified by \( r = R_0 \) surface, for which the field equations are solved by

\[
u^\alpha = (\partial/\partial t)^\alpha, \quad K = \frac{n}{R_0}, \quad P = n \frac{\sigma}{R_0}.
\]

Specifically, the boundary surface is described by a minimal surface. The static plasmaball is characterized by conserved charges \( \tilde{E}, \tilde{S} \) (“tilde” is used for the dimensionless quantities) given by

\[
\begin{align*}
\tilde{E} &= \frac{\rho_0^n}{\nu_\alpha \sigma^{n+1}} E = (n + 3) \tilde{R}_0^{n+1} + (n^2 + 3n + 1) \tilde{R}_0^{n} \\
\tilde{S} &= \frac{\rho_0^{(n^2 + 3n + 1)/(n + 3)}}{\nu_\alpha \sigma^{1/(n + 3)}} S = (n + 3) \tilde{R}_0^{n+1} \left( 1 + \frac{n}{R_0} \right)^{(n+2)/(n+3)},
\end{align*}
\]

satisfying the first law,

\[
\begin{align*}
d\tilde{E} &= \tilde{T} d\tilde{S}, \\
\tilde{T} &= \left( \frac{\alpha}{\rho_0} \right)^{n+3} T = \left( 1 + \frac{n}{R_0} \right)^{1/(n+3)},
\end{align*}
\]

where \( \alpha \) is a constant [for \( d = 3, \alpha = \pi^2 N^2/(8T_c) \)]. It then immediately follows that Smarr’s formula is reproduced in the limit of large spacetime dimensions, \( n \to \infty \).

We now turn to the analysis of linearized stability of these plasmaballs. We shall consider the perturbations \( \delta \rho, \delta P, \delta u^\alpha \) of an equilibrium plasmaball (hereafter 0th-order quantities are marked by naught), where \( \delta \rho \) and \( \delta P \) are related to the linearized SS-equation of state. We shall decompose the fluctuation modes in terms of a spherical harmonics, \( Y_k \), of a unit \( n \)-sphere satisfying \( (D^i D_i + k^2) Y_k (\Omega_n) = 0 \), where \( D_i \) is a covariant derivative of \( d\Omega^2_\alpha \) and \( k^2 = \ell(\ell + n - 1) \).
is an eigenvalue. The “vector” and “tensor” modes turn out to be trivial. Then, the perturbed quantities \( \delta Q = \{ \delta u^a, \delta \rho, \delta P \} \) of “scalar-type” are expressed as
\[
\delta Q = \delta Q_k(r)e^{\omega t}Y_k(\Omega_n); \quad \delta \bar{Q} = \{ \delta u^i, \delta u^r, \delta \rho, \delta P \}, \quad \delta u^i = \delta u_k(r)e^{\omega t}r^{-2}D^iY(\Omega_n). \tag{7}
\]
In what follows we omit the index \( k \) specifying the each mode. Under the above first-order fluctuations, the boundary of the plasma lump changes also at first order. We specify it by
\[
r = R(t, r, \Omega_n) = R_0 + \epsilon e^{\omega t}Y_k(\Omega_n). \tag{8}
\]
These perturbed quantities \( \delta Q \) evolve according to the linearized energy conservation equation and the linearized Euler’s equations, subjected to the linearized boundary conditions,
\[
\delta P|_{\text{boundary}} = \sigma \left( K(R(t, r, \Omega_n)) - K(R_0) \right) - \left[ \left( P^{(0)} - \bar{P}^{(0)} \right)_{R(t, r, \Omega)} - \left( P^{(0)}_0 \right) \right]_R_0, \tag{9}
\]
and \( \delta n_a u^a_0 + n_{a(0)}\delta u^a = 0 \). Express \( \delta u^i \) and \( \delta u^r \) in terms of \( \delta P \) by solving linearized Euler’s equations, and insert the result into the linearized continuity equation, we obtain
\[
\frac{d^2}{dr^2}\delta P + \frac{n}{r}\frac{d}{dr}\delta P - \left( \frac{k^2}{r^2} + (n + 2)\omega^2 \right) \delta P = 0. \tag{10}
\]
In terms of a modified Bessel function of the first kind, \( I_\ell(x) \), this equation solves as
\[
\delta P(r) = A \frac{1}{r^{(n-1)/2}I_{\ell+(n-1)/2}(pr)}, \quad p = \sqrt{n+2}\omega. \tag{11}
\]
Here, \( A \) is a constant to be determined by the boundary conditions. Substitute (11) into the linearized boundary conditions (9), we obtain the dispersion relation,
\[
-\omega^2 = C \left( \frac{pI_{\ell+(n+1)/2}(pR_0)}{I_{\ell+(n-1)/2}(pR_0)} + \ell \right) \left( \omega^2 + (\ell - 1)(\ell + n) \frac{R_0^2}{p^2} \right), \tag{12}
\]
where \( C = \sigma/[\left( P^{(0)} + \rho^{(0)} \right)R_0] \). This equation does not allow solutions of real \( \omega \). Hence, one concludes that the static fluid lump is stable under linear perturbations in arbitrary dimensions. For the static mode \( (\omega = 0) \), equation (10) is solved as,
\[
\delta P(r) = C_1 r^{\alpha_+} + C_2 r^{\alpha-}, \quad \alpha_\pm = \frac{1}{2} \left( -n + 1 \pm \sqrt{(n-1)^2 + 4k^2} \right) \geq 0. \tag{13}
\]
Hence, either of two linearly independent solutions is divergent at origin or at infinity unless \( \ell = 0 \). That is, the static mode with \( \ell \geq 1 \) is not normalizable. Whereas the pressure perturbation for the static s-wave is constant, i.e., it is not a dynamical perturbation.

In the nonrelativistic limit \( \omega R_0 \ll 1 \) and \( P^{(0)}/\rho^{(0)} \ll 1 \), the above equation (12) reduces to
\[
-\omega^2 = \frac{\sigma \ell(\ell - 1)(\ell + n)}{\rho^{(0)}R_0^2}. \tag{14}
\]
It then follows that the \( \ell = 0, 1 \) modes are not excited in this limit since they amount to the trivial addition of mass and change of the center of mass coordinate, respectively.

Since the static plasmballs are linearly stable against perturbations, the corresponding (yet unknown) static black holes in SS-AdS\(_{d+2}\) are expected to be stable. Moreover, the lack of normalizable static modes implies that the static SS-AdS\(_{d+2}\) black holes are perturbatively unique [7].
Figure 1. Phase diagram of plasma lumps in 2+1 dimensions. The red and blue curves denote a plasmaball and plasmaring, respectively. We write two saturns with distinct mass distribution between ball and ring. The magenta curve is $R_b = 2$ corresponding to $(E_b = 26, E_r = 74)$, while the green curve for $R_b = 3$ corresponding to $(E_b = 51, E_r = 49)$ has larger entropy.

3. Lumps of multiple plasma

In higher dimensions, black holes are able to remain equilibrium without any external forces supplied by electromagnetic force or conical singularity, just by the finely tuned balance between gravity and angular momentum. These multiple black objects reflect the solitonic character of circular system for vacuum Einstein’s equations.

We can construct corresponding duals of multiple plasma lumps. To illustrate, let us consider a simplest example of a plasmasaturn (coexistence of a plasmaball and a plasmaring enclosing it) with inner plasmaball being non-rotating. As for the black saturn, plasmasaturns exhibit a two-fold non-uniqueness corresponding to the distribution of total mass and total angular momentum. Figure 1 depicts an entropy vs angular momentum diagram with fixed energy $\tilde{E} = 100$. As the ball becomes large, the entropy tends to increase. As argued in [8], the saturn configuration with thin ring may be entropically favoured among balls and rings. Also, these saturn configurations necessarily have distinct temperature on the ball and on the ring.

4. Concluding remarks

We have explored the stability of plasmaballs in arbitrary dimensions and showed that they are indeed stable. The AdS/CFT dictionary tells us that the corresponding higher-dimensional black holes are stable. We have also constructed a stationary multi-plasma objects. We believe that further analysis in this direction provides intriguing aspects of higher-dimensional gravity and updates a new entry in the holographic dictionary.

Acknowledgments

The author is grateful to Umpei Miyamoto for discussions.

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