Rainbow tensor model with enhanced symmetry and extreme melonic dominance

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Abstract

We introduce and briefly analyze the rainbow tensor model where all planar diagrams are melonic. This leads to considerable simplification of the large $N$ limit as compared to that of the matrix model: in particular, what are dressed in this limit are propagators only, which leads to an oversimplified closed set of Schwinger-Dyson equations for multi-point correlators. We briefly touch upon the Ward identities, the substitute of the spectral curve and the AMM/EO topological recursion and their possible connections to Connes-Kreimer theory and forest formulas.

Introduction. Tensor models are thought of as straightforward generalizations of matrix models from matrices to tensors \cite{1}, and are the most natural objects to consider in the study of non-linear algebra \cite{2}. For a variety of reasons, they did not long attract the attention they deserve, though a lot of important work was performed by numerous research groups (see \cite{3, 4} and especially \cite{5}-\cite{10} for incomplete set of references).

The current surge of interest to tensor models \cite{11, 12, 14, 15} is partly explained by their simplified large $N$ behavior as compared to the matrix models: instead of all planar diagrams, what contribute in some cases are only melonic ones. This makes the Schwinger-Dyson equation (SDE) in the large $N$ limit almost as simple as in the leading logarithmic approximation in QED: it is enough to dress the propagator, while the vertices remain undressed. The only complication is that, in this limit, the SDE is no longer linear and solved by a geometric progression: instead it turns into a higher order equation, but the semi-infinite Bogoliubov chain does not arise.

In this letter, we consider the rainbow tensor model which possesses these properties in extreme: it selects the melonic diagrams out of all planar by symmetry, all others simply do not exist as its Feynman diagrams. In exchange, it clarifies to some extent what is the price to pay for this "simplification": already the SDE becomes much more involved when one considers a generic point in the moduli space of external parameters (colorings). The model can be easily formulated for tensors of any rank $D$, but all its features are already seen at the level of $D = 3$, which we use in most considerations to simplify the presentation. We mostly repeat in slightly different words and with slightly different accents the elementary basic observations, which were already made in the above cited references, and try to minimize deviations from the previously introduced notation. We concentrate on the matrix model aspects of the story and do not consider time-dependence \cite{11, 12, 14, 15}, relation to SYK model \cite{16, 17}, holography \cite{17}.

We also do not dwell upon intriguing similarity to arborescent calculus in knot theory \cite{18}. It is, however, reflected in some of the terminology below.
The tetrahedron/starfish model is

\[
Z = \prod_{l=0}^{D} d^2 A_l e^{-M_l A_l A_l} \exp \left( g \prod_{l=0}^{D} A_l + g \prod_{l=0}^{D} \bar{A}_l \right)
\]

(1)

Each $A_l$ is a rank $r$ tensor, and $\bar{A}_l$ is obtained by exchanging covariant and contravariant indices. In $A_l \bar{A}_l$, the indices are convoluted in an obvious way, while the interaction term is a "tetrahedron" or, generically, a "starfish" vertex, with indices contracted in a very special way:

\[
A_i^j B_j^k C_k^l \quad A_{\alpha}^j B_{\beta}^j C_{k}^\alpha D_l^\beta
\]

\[
A_0^i B_1^j C_2^l \quad A_{\alpha}^j B_{\beta}^j C_{k}^\alpha D_l^\beta
\]

\[
A_0^i B_1^j C_2^l \quad A_{\alpha}^j B_{\beta}^j C_{k}^\alpha D_l^\beta
\]

In general, the indices here belong to different groups (tensors are "rectangular"):

- $i = 1, \ldots, N_{\text{green}}$
- $j = 1, \ldots, N_{\text{red}}$
- $k = 1, \ldots, N_{\text{orange}}$
- $l = 1, \ldots, N_{\text{yellow}}$
- $\alpha = 1, \ldots, N_{\text{blue}}$
- $\beta = 1, \ldots, N_{\text{violet}}$
- $a = 1, \ldots, N_{\text{brown}}$
- $b = 1, \ldots, N_{\text{pink}}$

When one draws a vertex for $D = 3$ as contraction of four rank $D$ tensors, it looks like a tetrahedron:

\[
\begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\]

but if one inserts it as a vertex into Feynman diagrams, the lines at the corners do not merge and the picture looks more like a starfish as above, which explains the two names.

In the simplest model of this type [13], one makes no difference between the $D$ fields ($A = B = C = \ldots$) and does not distinguish the upper and lower indices. This model possesses the $O(N)^D$ symmetry with limited coloring, as we have $D$ instead of $\frac{D(D+1)}{2}$ colors. It was studied in [14] under the name of "uncolored", though in our context "$D$-colored" or "single-field" seems somewhat more appropriate. Still, in the large $N$ limit, it also represents the universality class of models which are *dominated* by the melonic diagrams.

The rainbow model. An opposite option is to provide each line in these pictures with its own coloring, i.e. to endow the model with the symmetry

\[
U_D = \prod_{a=1}^{D(D+1)/2} U(N_a)^{N_a} U(N)^{\frac{D(D+1)}{2}}
\]

(2)

For $D = 3$, this gives six different colorings, thus the name *rainbow*, and we sometimes denote these colorings by $r$ (red), $o$ (orange), $y$ (yellow), $g$ (green), $b$ (blue) and (missing indigo) $v$ (violet). Accordingly, the vertices have three ($D$) colorings, but not every triple is allowed: there are just four ($D+1$) permitted combinations

\[
I \in \{ A = \bar{g}br, \ B = \bar{r}vo, \ C = \bar{o}by, \ D = \bar{y}vg \} = \{ A = \bar{g}br, \ B = \bar{r}vo, \ C = \bar{o}by, \ D = \bar{y}vg \}
\]

(3)

Each field transforms under the action of the corresponding three ($D$) groups:

\[
A_{i_j k_j} \rightarrow \sum_{j_s=1}^{N_r} \sum_{j_b=1}^{N_s} U_{i_j}^{j_s} U_{j_s}^{j_b} A_{j_s k_s} ;
\]

\[
B_{i_v} \rightarrow \sum_{j_r=1}^{N_v} \sum_{j_b=1}^{N_s} U_{i_v}^{j_r} U_{j_r}^{j_s} B_{j_s} ;
\]
\[ C_{\lambda_\gamma}^{i_\alpha i_\delta} \rightarrow \sum_{j_\gamma=1}^{N_r} \sum_{j_\delta=1}^{N_g} \sum_{j_\epsilon=1}^{N_b} U_{i_\lambda j_\gamma}^{i_\alpha} \tilde{U}_{j_\epsilon j_\delta}^{i_\delta} \overline{C}_{j_\epsilon j_\delta}^{j_\gamma}, \]
\[ \overline{D}_{\lambda_\gamma}^{i_\alpha i_\delta} \rightarrow \sum_{j_\gamma=1}^{N_r} \sum_{j_\delta=1}^{N_g} \sum_{j_\epsilon=1}^{N_b} U_{i_\lambda j_\gamma}^{i_\alpha} \overline{U}_{j_\epsilon j_\delta}^{i_\delta} \overline{D}_{j_\epsilon j_\delta}^{j_\gamma}. \]

It is often convenient to distinguish between the "external" and "internal" lines in the pictures, and denote the interaction vertex by
\[
\prod_{l=0}^{3} A_l = A_{i_\lambda k_\gamma}^{i_\alpha k_\delta} B_{i_\nu k_\gamma}^{i_\beta k_\mu} \overline{C}_{i_\lambda j_\gamma}^{i_\alpha j_\delta} \overline{D}_{i_\nu j_\gamma}^{i_\beta j_\mu} = A_{i_\lambda k_\gamma}^{i_\alpha k_\delta} B_{i_\nu k_\gamma}^{i_\beta k_\mu} \overline{C}_{i_\lambda j_\gamma}^{i_\alpha j_\delta} \overline{D}_{i_\nu j_\gamma}^{i_\beta j_\mu} = \text{Tr} A B \overline{C} \overline{D}, \tag{5}
\]
i.e. we consider the fields as matrices with respect to the indices \( i_{r, o, y, g} \), while the additional indices \( k_b \) and \( k_v \) correspond to the additional (internal) lines in the Feynman graphs.

To avoid possible confusion, we emphasize that our colorings distinguish different components of the symmetry group: they are not the colors labeling different elements of the fundamental representation of a single \( SU(N) \) (like quark colors in QCD). One could rather associate them with a kind of flavors labeling elements of representation of a global symmetry \( O \left( \frac{D(D+1)}{2} \right) \), but the closest is the analogy with the constituents of the gauge group in the quiver models of \( \cite{19} \). The difference from the \( D \)-colored model of \( \cite{14} \) is that, in the rainbow model, one has \( D + 1 \) different types of fields and propagators labeled by a set \( I \) of indices.

**Planar and melonic diagrams.** This latter difference, however, has a profound effect on the calculus based on diagrams. With the interaction vertices above and the above notation, one can formally separate the "external" and "internal" colorings and draw Feynman diagrams as the ordinary matrix model double-line (fat graph) diagrams for the "external" ones. In other words, the propagators are still rather bands than tubes (not all the colorings are on equal footing). Extra internal lines are inserted at the next stage and provide an additional structure. The point is that even if the fat graphs are planar, the loop calculus for internal lines can still be different, and this leads to distinguishing the class of melonic diagrams from others. Specifics of the rainbow model is that the possible contractions of internal lines are strongly limited, so that only the melonic planar fat graphs are actually contributing, while the non-melonic planar graphs are not just damped by powers of \( N_{\text{int}} \) (as in the case of the \( D \)-colored model), but they are simply absent. The phenomenon is clear from the following picture:

planar melonic            planar non-melonic (trefoil)

\[ N_{\text{ext}}^4 N_{\text{int}}^2 = N_r N_o N_y N_g \cdot N_b N_v = N_r N_o N_y N_g \cdot N_b N_v \]
\[ N_{\text{ext}}^5 N_{\text{int}}, \text{ coloring impossible} \]

It is clear that, in the second non-melonic picture, the internal line makes just one loop instead of three, which causes a damping factor \( N_{\text{int}}^{-2} \). However, such an internal line can not be ascribed any definite coloring (it self-intersects, thus can be neither blue nor violet), i.e. such a diagram simply does not exist in the rainbow model (and exists in the \( D \)-colored model, where it is dumped in large \( N \) limit). This is an illustration of the general statement: while in all tetrahedron/starfish models beginning from the \( D \)-colored one, the melonic diagrams are the only surviving in the large \( N \) limit, while in the rainbow model, all the planar diagrams are automatically melonic.

Moreover, even the right internal circle in the trefoil picture can not be ascribed any definite coloring: should it be orange or green? However, this is because the trefoil is not just non-melonic, but it also has an odd number of vertices, thus it is forbidden in the rainbow model by a much simpler reason: each diagram should have equal number of chiral and anti-chiral vertices \( \text{Tr} A \overline{B} \overline{C} \overline{D} \) and \( \text{Tr} D \overline{C} B A \). We have, however, emphasized another argument related to the absence of \( \text{Tr} A \overline{C} B \overline{D} \) vertices with another contraction of indices,
which remains applicable to the case of non-melonic diagrams with an even number of vertices as well: the following diagram

\[ N_{ext}^{4(6)} N_{int}^{0(1)}, \text{ coloring impossible} \]

is also an impossible picture in the rainbow model: there is no way to ascribe colors (blue and violet) to the black internal line. The powers in brackets correspond to the case of the vacuum diagram, when the external lines are connected to form two additional loops, and non-melonic damping is then due to the first (instead of the third) power of \( N_{int} \).

**SDE in the melonic limit.** The dominance of melonic diagrams leads to a very simple Schwinger-Dyson equation for the dressed propagator:

\[ = \]

The lines in this picture are drawn thick to emphasize that they are now \( D \)-colored, i.e. labeled by indices \( I \). However, the tensor structure of lines is simply \( \delta_{IJ} \), while the change of multi-colorings in the vertices is described as above. Simplicity of the SDE equation is due to the lack of any dressing at vertices. If we denote the dressed propagators by \( G_I \), then the SDE in the large \( N \) approximation turns into

\[
G_I = 1 + g^2 \prod_{a \in I} N_a \cdot G_I \prod_{J \neq I} G_J = 1 + \frac{g^2}{N_I} \cdot \prod_{a=1}^{D(D+1)/2} N_a \cdot \prod_{J=1}^D G_J \tag{6}
\]

where \( N_I = \prod_{a \in I} N_a \). If one does not differentiate the colorings, then this leads us to the large \( N \) behavior of the type \( G \sim N^{-(D-1)/2D} D^3 \sim N^{-3/4} \), and the term at the l.h.s does not contribute. However, when colorings are different, as in the rainbow model, things are not so simple: the factor in the bracket at the r.h.s. is independent of \( I \), while the coefficient in front, \( N_I^{-1} \) depends on \( I \), thus, the solution with the l.h.s. neglected can not exist.

To see what happens in this case, one can consider the simplest formal example of \( \#I = 2 \)

\[
G_1 = 1 - \alpha_1 G_1 G_2 \\
G_2 = 1 - \alpha_2 G_1 G_2 \tag{7}
\]

where the solution is just

\[
G_1 = 1 - \frac{\alpha_1 + \alpha_2 + 1 \pm \sqrt{(\alpha_1 - \alpha_2)^2 + 2(\alpha_1 + \alpha_2) + 1}}{2\alpha_2} \tag{8}
\]

\[
G_2 = 1 - \frac{\alpha_1 + \alpha_2 + 1 \pm \sqrt{(\alpha_1 - \alpha_2)^2 + 2(\alpha_1 + \alpha_2) + 1}}{2\alpha_1} \tag{9}
\]

Then, for the sake of definiteness, choosing the plus sign in front of the square root and \( \alpha_1 > \alpha_2 \), we obtain

\[
G_1 \sim 1 - \frac{\alpha_1}{\alpha_2} + O\left(\alpha^{-1}\right) \\
G_2 \sim \frac{1}{\alpha_1 - \alpha_2} + O\left(\alpha^{-2}\right) \tag{10}
\]
or vice versa, under $\alpha_1 \leftrightarrow \alpha_2$ or under the other sign. We are interested in the limit of large $\alpha$, and we see that the solution depends strongly on the ratio $\alpha_1/\alpha_2$. For bigger values of $D(\geq 3)$, more different branches appear. At the transition lines from one to the other branch, there is a multicritical behaviour. In the example of $D = 2$, it is just the critical point\(^{[7]} \alpha_1 = \alpha_2 = \alpha$ where both

$$G_i \sim -\frac{1}{2\alpha} + O(\alpha^{-2})$$

(11)

At $D = 3$, there are already a few critical lines (such as $\alpha_2 = \alpha_3$, $\alpha_2^2 - \alpha_1 \alpha_2 - \alpha_1 \alpha_3 + \alpha_2 \alpha_3$ etc). This means that changing the size of the gauge group is related to a relevant operator, and there are some marginal deformations that act along the critical surfaces. One should proceed further with studying the wall-crossing phenomenon in this case. Thus, the large $N$ limit in the rainbow model is very singular: it non-analytically depends on the ratios of the colorings. On one hand, this looks very similar to singularities of conformal blocks in the vicinity of singular points (zeros of Kac determinant) considered in \[22\]. On another hand, this looks like a peculiar feature of the dominance of melonic diagrams, which allows dressing of propagators only (not vertices) and leads to the oversimplified SDE. In large $N$ matrix (not tensor) models, which are dominated by all planar diagrams, like ABC-model at $D = 2$, such singularities seem to be absent. Simple examples are the vector models, which are of course very distinguished among the models of rectangular matrices \[23\], but this is a limit of $N_1 \ll N_2$, not $N_1 \sim N_2 \gg 1$. Likewise, such singularities are most probably absent in the quiver Yang-Mills models. In this sense, the tensor models can get much closer to conformal theories (possessing a simpler large $N$ limit) as compared to the matrix and Yang-Mills models, which can also be a manifestation of their close relation to holography \[17\].

The melonic dominance (which, in the case of rainbow model, becomes absolute) implies that dressed in the large $N$ limit are only propagators, not vertices, which makes this part of the theory essentially Abelian (and similar to QED in the leading logarithmic approximation). In particular, all multi-point correlators can be easily expressed through the dressed propagators, as it is already discussed in \[14\] (where the propagators are not-quite-trivial given the time-dependence such as in \[11\], and this would presumably put the model in the universality class of the quenched SYK models \[16\] \[17\].)

**Single-trace operators.** A very first step in the matrix-model-based study of effective actions \[24\] is division of all observables (gauge invariant operators) into three classes. First of all, the set of operators forms a ring, where addition and multiplication are defined (the latter one, operator product expansion can be quite involved beyond matrix models), and one distinguishes linearly generated operators from multiplicatively generated ones: roughly speaking, linearly independent are all gauge invariant multi-trace combinations $\prod_{a=1}^m \text{Tr} M^{k_a}$ with any $m$ (we denote the corresponding couplings in the generating functional of all correlators as $t_{k_1, k_2, \ldots, k_m}$), while multiplicatively independent ones are just single traces: $\text{Tr} M^k$. From the point of view of effective actions, the difference between them is that dependence on the first ones is easily reduced to that on the second:

$$\frac{\partial Z}{\partial t_{k_1, k_2, \ldots, k_m}} = \left\langle \prod_{a=1}^m \text{Tr} M^{k_a} \right\rangle = \frac{\partial^m Z}{\partial t_{k_1} \partial t_{k_2} \ldots \partial t_{k_m}}$$

(12)

i.e. introduction of all the $t_{k_1, k_2, \ldots, k_m}$ into the set of couplings is superficial: the set $\{ t_k \}$ is quite enough. Note that this does not necessarily mean that the averages factorize as this happens only in the large $N$ limit:

$$\left\langle \prod_{a=1}^m \text{Tr} M^{k_a} \right\rangle \text{ at large } N \sim \prod_{a=1}^m \left\langle \text{Tr} M^{k_a} \right\rangle$$

(13)

Hence, there is no need to introduce multi-trace couplings in order to generate all correlators at any $N$.

Dependence on single-trace operators is more involved. It is controlled by the Ward identities, the quantum substitute of classical equations of motion in matrix models, reflecting the invariance of (functional) integral under the change of integration variables. These are nicknamed as Virasoro/W-constraints, because they are just these in the simplest possible examples like

$$\left\langle \text{Tr} W'(M) + k t_k \text{Tr} M^{k+n} + \sum_{a+b=n} \text{Tr} M^a \text{Tr} M^b \right\rangle = 0, \quad n \geq -1$$

(14)

\(^{[1]}\)In terms of the group ranks, this critical point corresponds to $N_1 = N_2$. This reminds the phase transition when the number of SYK Majorana fermions matches the number of free Majorana fermions in \[20\], and also the phase transition in \[21\] when the number of Majorana fermions matches the number of replicas. We are grateful to the reviewer of our paper who paid our attention at this fact.
for the Hermitian matrix model. Here $W(M)$ is the potential (action) of the model. What makes it non-trivial is the last term, which disappears only when there are multiplicative dependence among the "single-trace" operators. This happens for vector models, when the rectangular matrices $M$ degenerate to vectors, and $\text{Tr}(MM)^k = (V^2)^k$. The situation is a little more involved and interesting for rectangular matrices of generic size.

Unfortunately, in multi-matrix models, of which the rainbow tensor model is a generalization, the set of single-trace operators becomes unacceptably big. Already for the $ABC$-model, the naive set consists of "chiral" ones $\text{Tr}(ABC)^k$, "antichiral" ones $\text{Tr}(\bar{A}B\bar{C})^k$, non-chiral ones $\text{Tr}(A\bar{A})^k$, $\text{Tr}(B\bar{B})^k$, $\text{Tr}(C\bar{C})^k$ and their arbitrarily sophisticated mixtures such as

$$\text{Tr} \left\{ (ABC)^{k_1}(A\bar{A})^{m_1} \left( AB(C\bar{C})^lC \right)^{m_1} (A\bar{A})^{m_1} (ABC)^{k_2}(A\bar{A})^{m_2} \ldots \right\}$$

(15)

Thus, one is led to further division of single-trace operators into two classes: chiral and non-chiral.

**Chirality.** The interaction vertex in the $ABC$-model is chiral: it contains the fields $A, B, C$, but not their conjugates $\bar{A}, \bar{B}, \bar{C}$. Non-chiral instead are the kinetic terms $\text{Tr}A\bar{A}$, $\text{Tr}B\bar{B}$ and $\text{Tr}C\bar{C}$. The point is that non-chiral single-trace operators (15) can be obtained by the contraction of chiral and anti-chiral vertices with the help of inverted kinetic terms (propagators), i.e. they can be produced by tree Feynman diagrams. In terms of the Ward identities (or SDE), this is a corollary of the equations of motion

$$\begin{align*}
\bar{A} + gBC &= 0, \\
\bar{B} + gCA &= 0, \\
\bar{C} + gAB &= 0
\end{align*}$$

(16)

Consider just one example, of how $\text{Tr}(ABC)(\bar{C}\bar{B}\bar{A})$ emerges from joining two chiral and two antichiral vertices via three propagators:

![Diagram](image)

Chiral operators $\text{Tr}(ABC)^k$ are not produced by a change of fields (integration variables) from any other chiral operators, in this sense they are "cohomological". Instead, they are variations of non-chiral kinetic terms,

$$\text{Tr}(ABC)^k = \delta(\text{Tr}C\bar{C}) \quad \text{for} \quad \delta\bar{C} = (ABC)^{k-1}(AB)$$

(17)

which have unit Jacobians. Therefore, in the chiral sector, the Ward identities degenerate into relations of the trivial type and, in this sense, the dependence of effective action on the chiral couplings is almost absent. In fact, the chiral couplings rather play the role of the holomorphic moduli or "boundary conditions" describing different solutions to the Virasoro constraints describing the corresponding integrability properties are not yet understood in universal terms. While solutions of the Virasoro constraints are usually integrable $\tau$-functions of non-chiral (non-cohomological) couplings, i.e. are fully controlled by Lie algebra representation theory, dependence on the chiral (cohomological) couplings is rather similar to that on the "point of the Grassmannian". It is presumably encoded in associated quasiclassical (Whitham) $\tau$-functions and controlled by the WDVV equations. Also, it is often described in a much poorer language of chiral rings, where chiral operators like $\text{Tr}(ABC)^k$ form a multiplicative basis. The theory of chiral effective actions is one of the directions in which study of the rainbow tensor models can produce considerable progress, simply because the rainbow models are the first example among the topical matrix/tensor models which possesses a non-trivial set of chiral operators.

To avoid a terminological confusion: in a sense, chirality is no longer respected in the higher rank tetrahedron/starfish models. For example, for $D = 3$ the fields can not be tensors of the type $(3,0)$ and $(0,3)$ (with all incoming or all outgoing arrows) only: e.g., the tetrahedron vertex requires $(2,1)$ and $(1,2)$ fields and has a shape $ABCD$ with two conjugate fields. However, if one calls chirality a counterpart of the above explained "tree irreducibility" (which is the one relevant to the SUSY non-renormalization), then, actually chiral are $A, B, C, D$, while $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are anti-chiral fields, and in this sense the tetrahedron/starfish interaction vertex is always chiral.

**Ward identities.** A counterpart of the Virasoro constraints in the tensor models was already studied in by a direct generalization of the standard line of. Needed for the derivation is a choice of shifts of...
integration variable (or considering a full derivative in the integrand) as this defines a basis in the set of Ward identities. A clever choice could be gradients of the chiral single trace operators. Therefore, the Ward identities are closely related to the Connes-Kreimer Hopf algebra, which appears in the description of diffeomorphisms of the coupling constant space in terms of Feynman graphs, in particular, in renormalization theory, and is best understood and described in the basic matrix model terms. Similarly, an example of tensor model was shown to be perturbatively renormalizable, and the corresponding Connes-Kreimer Hopf algebra was constructed in

**Spectral curve and AMM/EO topological recursion.** The most popular choice of generating functions in matrix models is in terms of resolvents:

\[
\text{Tr} \frac{1}{z - M} = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \text{Tr} M^k
\]

In the large \(N\) limit (or some subtler limits), the Ward identities can be often rewritten as a closed algebraic (or differential/difference) equation for the average of the resolvent, then, it is called spectral curve (or quantum spectral curve, see the case of Hermitean one matrix model)

\[
W'(z) \left( \text{Tr} \frac{1}{z - M} \right) + \left( \text{Tr} \frac{1}{z - M} \right)^2 = \sum_{a,b} T_{a+b+2} z^a \langle \text{Tr} M^b \rangle = f_W(z)
\]

with \(W(z) = \sum_k T_k z^k\) being the potential of matrix model. The entire 1/\(N\) expansion then can be reformulated in terms of effective field theory over the spectral curve and expressed through various quantities on the spectral bundle over the moduli space. This procedure is known as AMM/EO topological recursion

However, the resolvent is not the only and often not the most clever choice: for example, the use of the Wilson loops

\[
\text{Tr} e^{sM} = \sum_k \frac{s^k}{k!} \text{Tr} M^k
\]

and especially of the Harer-Zagier generating functions

\[
\sum_{N,k} \frac{\lambda^N z^k}{(2k-1)!!} \text{Tr}_{N \times N} M^{2k}
\]

leads to far more explicit expressions for matrix model averages. Moreover, there is no direct generalization of resolvent from matrix to tensor models, thus, one should consider generating functions in a somewhat more abstract way. In general, the Ward identities are provided by a non-infinitesimal change of the integration variable, which can be substituted by a certain shift of the coupling constants in the partition function

\[
Z \left( T + v(T) \right) = : e^\hat{v}(T) : = Z(T),
\]

with \(\hat{v}(T) = v(T) \frac{\partial}{\partial T}\). In every particular model, one can work out particular relevant diffeomorphisms \(v(T)\) which leave \(Z\) intact, but nice formulas usually require elimination of the normal ordering, which is provided by a somewhat complicated Bogolubov-Zimmermann forest formula, see details (see also for an interesting related issue). Namely, without normal ordering the action of the vector field \(V\) involves a sum repeated actions of \(\hat{V}\) on itself:

\[
Z \left( T + v(T) \right) = e^\hat{V} Z(T) = \left( 1 + \sum \text{forests} \left( \sum \text{trees} \frac{1}{\text{Tree}(\mathcal{F})!} \prod_{T \in \mathcal{F}} \frac{\hat{V}_T}{\sigma_T T!} \right) \right) Z(T) = : e^\hat{V}(T) : \cdot Z(T)
\]

with certain \(V\)-independent combinatorial coefficients \(\sigma_T\) and \(\mathcal{T}\). This formula expresses \(\hat{v}\) through \(\hat{V}\). The inverse transformation is best described, when the partition function is expressed in terms of Feynman graphs, \(Z(T) = \sum_T F_T Z_T(T)\), then

\[
e^\hat{V} F_T = \sum_{n=0}^\infty \left( \sum_{\text{non-intersecting box-subgraphs}} \hat{v}_{\gamma_1} \ldots \hat{v}_{\gamma_n} F_T/\langle \gamma_1, \ldots, \gamma_n \rangle \right)
\]
Conclusion. We made a brief review of the currently popular tensor models from the perspective of matrix model theory a la [24]. Nowadays interest is concentrated on the tensor models of a very special class with peculiar tetrahedron/starfish interactions, where distinguished in the large $N$ limit are peculiar melonic diagrams, while non-trivial planar diagrams are suppressed leading to the drastic simplifications. We have considered a universal model of this type, by making all possible indices independent and extending the symmetry to its extreme, which is a product of $\frac{(D+1)!}{D!}^2$ unitary groups $U(N_n)$. In this rainbow model, the non-melonic diagrams are not just suppressed among the planar ones, but they simply do not contribute irrespective of the value of $N$. The large $N$ limit instead acquires another type of non-triviality: a non-analytical dependence on ratios of different color numbers, which is obscured and actually absent in rectangular matrix models and quiver Yang-Mills theories, but which is present in non-perturbative description of conformal theories. The derivation of Ward identities is technically straightforward, but, at a conceptual level, requires a suitable definition of generating functions properly associated with the diffeomorphisms group of the moduli space. This puts the story within the context of the Bogoliubov-Zimmerman/Connes-Kreimer theory. It is still a question whether a pronounced simplicity of the rainbow tensor models helps us to consider a new avenue of quantum gravity, but the study in this framework seems certainly important for development of group theory, integrabilities and non-linear algebra as well.

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