DP-3-coloring of planar graphs without certain cycles

Mengjiao Rao  Tao Wang*
Institute of Applied Mathematics
Henan University, Kaifeng, 475004, P. R. China

Abstract

DP-coloring is a generalization of list-coloring, which was introduced by Dvořák and Postle. Zhang showed that every planar graph with neither adjacent triangles nor 5-, 6-, 9-cycles is 3-choosable. Liu et al. showed that every planar graph without 4-, 5-, 6- and 9-cycles is DP-3-colorable. In this paper, we show that every planar graph with neither adjacent triangles nor 5-, 6-, 9-cycles is DP-3-colorable, which generalizes these results. Yu et al. gave three Bordeaux-type results by showing that: (i) every planar graph with the distance of triangles at least three and no 4-, 5-cycles is DP-3-colorable; (ii) every planar graph with the distance of triangles at least two and no 4-, 5-, 6-cycles is DP-3-colorable; (iii) every planar graph with the distance of triangles at least two and no 5-, 6-, 7-cycles is DP-3-colorable. We also give two Bordeaux-type results in the last section: (i) every planar graph with neither 5-, 6-, 8-cycles nor triangles at distance less than two is DP-3-colorable; (ii) every planar graph with neither 4-, 5-, 7-cycles nor triangles at distance less than two is DP-3-colorable.

1 Introduction

All graphs considered in this paper are finite, simple and undirected. A planar graph is a graph that can be embedded into the plane so that its edges meet only at their ends. A plane graph is a particular embedding of a planar graph into the plane. We set a plane graph $G = (V,E,F)$ where $V,E,F$ are the sets of vertices, edges, and faces of $G$, respectively. A vertex $v$ and a face $f$ are incident if $v \in V(G)$. Two faces are adjacent if they have at least one common edge. Call $v \in V(G)$ a $k$-vertex, or a $k^+$-vertex, or a $k^-$-vertex if its degree is equal to $k$, or at least $k$, or at most $k$, respectively. The notions of a $k$-face, a $k^+$-face and a $k^-$-face are similarly defined.

A proper $k$-coloring of a graph $G$ is a mapping $f: V(G) \to [k]$ such that $f(u) \neq f(v)$ whenever $uv \in E(G)$, where $[k] = \{1,2,\ldots,k\}$. The smallest integer $k$ such that $G$ has a proper $k$-coloring is called the chromatic number of $G$, denoted by $\chi(G)$. Vizing [21], and independently Erdős, Rubin, and Taylor [5] introduced list-coloring as a generalization of proper coloring. A list-assignment $L$ gives each vertex $v$ a list of available colors $L(v)$. A graph $G$ is $L$-colorable if there is a proper coloring $\phi$ of $G$ such that $\phi(v) \in L(v)$ for each $v \in V(G)$. A graph $G$ is $k$-choosable if $G$ is $L$-colorable for each $L$ with $|L(v)| \geq k$. The minimum integer $k$ such that $G$ is $k$-choosable is called the list-chromatic number $\chi_l(G)$.

For ordinary coloring, since every vertex has the same color set $[k]$, the operation of vertex identification is allowed. For list-coloring, the vertices may have different lists, so it is infeasible to identify vertices in general. To overcome this difficulty, Dvořák and Postle [4] introduced DP-coloring under the name “correspondence coloring”, showing that every planar graph without cycles of lengths 4 to 8 is 3-choosable.

*Corresponding author: wangtao@henu.edu.cn; iwangtao8@gmail.com
Definition 1. Let $G$ be a simple graph and $L$ be a list-assignment for $G$. For each vertex $v \in V(G)$, let $L_v = \{v\} \times L(v)$; for each edge $uv \in E(G)$, let $\mathcal{M}_{uv}$ be a matching between the sets $L_u$ and $L_v$, and let $\mathcal{M} = \bigcup_{uv \in E(G)} \mathcal{M}_{uv}$, called the matching assignment. The matching assignment is called $k$-matching assignment if $L(v) = [k]$ for each $v \in V(G)$. A cover of $G$ is a graph $H_{v,\mathcal{M}}$ (simply write $H$) satisfying the following two conditions:

(C1) the vertex set of $H$ is the disjoint union of $L_v$ for all $v \in V(G)$;
(C2) the edge set of $H$ is the matching assignment $\mathcal{M}$.

Note that the matching $\mathcal{M}_{uv}$ is not required to be a perfect matching between the sets $L_u$ and $L_v$, and possibly it is empty. The induced subgraph $H[L_v]$ is an independent set for each vertex $v \in V(G)$.

Definition 2. Let $G$ be a simple graph and $H$ be a cover of $G$. An $\mathcal{M}$-coloring of $H$ is an independent set $I$ in $H$ such that $|I \cap L_v| = 1$ for each vertex $v \in V(G)$. The graph $G$ is DP-$k$-colorable if for any list-assignment $|L(v)| \geq k$ and any matching assignment $\mathcal{M}$, it has an $\mathcal{M}$-coloring. The DP-chromatic number $\chi_{DP}(G)$ of $G$ is the least integer $k$ such that $G$ is DP-$k$-colorable.

We mainly concentrate on DP-coloring of planar graphs in this paper. Dvořák and Postle [4] noticed that $\chi_{DP}(G) \leq 5$ if $G$ is a planar graph, and $\chi_{DP}(G) \leq 3$ if $G$ is a planar graph with girth at least five. Several groups have given sufficient conditions for a planar graph to be DP-3-colorable, which extends the 3-choosability of such graphs.

Theorem 1.1 (Liu et al. [16]). A planar graph is DP-3-colorable if it satisfies one of the following conditions:

1. it contains no 3, 6, 7, 8-cycles.
2. it contains no 3, 5, 6-cycles.
3. it contains no 4, 5, 6, 9-cycles.
4. it contains no 4, 5, 7, 9-cycles.
5. the distance of triangles is at least two and it contains no 5, 6, 7-cycles.

Theorem 1.2 (Liu et al. [15]). If $a$ and $b$ are distinct values from $\{6, 7, 8\}$, then every planar graph without 4-, $a$-, $b$-, 9-cycles is DP-3-colorable.

Zhang and Wu [32] showed that every planar graph without 4-, 5-, 6- and 9-cycles is 3-choosable. Zhang [28] generalized this result by showing that every planar graph with neither adjacent triangles nor 5-, 6- and 9-cycles is 3-choosable. Liu et al. [16] showed that every planar graph without 4-, 5-, 6- and 9-cycles is DP-3-colorable. In this paper, we first extend these results by showing the following theorem.

Theorem 1.3. Every planar graph with neither adjacent triangles nor 5-, 6- and 9-cycles is DP-3-colorable.

The distance of two triangles $T$ and $T'$ is defined as the value $\min\{\text{dist}(x, y) : x \in T$ and $y \in T'\}$, where dist$(x, y)$ is the distance of the two vertices $x$ and $y$. In general, we use $\text{dist}'$ to denote the minimum distance of two triangles in a graph. Yin and Yu [26] gave the following Bordeaux condition for planar graphs to be DP-3-colorable.

Theorem 1.4 (Yin and Yu [26]). A planar graph is DP-3-colorable if it satisfies one of the following two conditions:

1. the distance of triangles is at least three and it contains no 4,5-cycles.
Table 1: List-3-coloring and DP-3-coloring.

| dist$^\beta$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | list-3-coloring | DP-3-colorable       |
|-------------|---|---|---|---|---|---|---|----|----------------|----------------------|
| ≤2          |   |   |   |   |   |   |   |    |                |                      |
| ≥2          |   |   |   |   |   |   |   |    |                |                      |
| ≥3          |   |   |   |   |   |   |   |    |                |                      |
| ≥4          |   |   |   |   |   |   |   |    |                |                      |
| ≥5          |   |   |   |   |   |   |   |    |                |                      |
| ≥6          |   |   |   |   |   |   |   |    |                |                      |
| ≥7          |   |   |   |   |   |   |   |    |                |                      |
| ≥8          |   |   |   |   |   |   |   |    |                |                      |
| ≥9          |   |   |   |   |   |   |   |    |                |                      |
| ≥10         |   |   |   |   |   |   |   |    |                |                      |

(2) the distance of triangles is at least two and it contains no 4, 5, 6-cycles.

Theorem 1.4 implies the following new results on 3-choosability.

**Corollary 1.5.** A planar graph is 3-choosable if it satisfies one of the following conditions:

1. the distance of triangles is at least three and it contains no 4, 5-cycles.
2. the distance of triangles is at least two and it contains no 4, 5, 6-cycles.

The following are two Bordeaux-type results on 3-choosability.

**Theorem 1.6** (Zhang and Sun [30]). Every planar graph with neither 5-, 6-, 8-cycles nor triangles at distance less than two is 3-choosable.

**Theorem 1.7** (Han [6]). Every planar graph with neither 4-, 5-, 7-cycles nor triangles at distance less than two is 3-choosable.

In the last section, we give two Bordeaux-type results on DP-3-coloring. The first one improves Theorem 1.6 and the second one improves Theorem 1.7.

**Theorem 1.8.** Every planar graph with neither 5-, 6-, 8-cycles nor triangles at distance less than two is DP-3-colorable.
Theorem 1.9. Every planar graph with neither 4-, 5-, 7-cycles nor triangles at distance less than two is DP-3-colorable.

It is observed that every $k$-degenerate graph is DP-$(k+1)$-colorable. Theorem 1.9 can be derived from the following Theorem 1.10.

Theorem 1.10. Every planar graph with neither 4-, 5-, 7-cycles nor triangles at distance less than two is 2-degenerate.

For more results on DP-coloring of planar graphs, we refer the reader to [1, 7, 10, 13, 14, 17]. For convenience, we collect some results on list-2-degenerate.

In this short section, some preliminary results are given, and these results can be used separately elsewhere. Liu et al. [16] showed the “nearly $(k-1)$-degenerate” subgraph is reducible for DP-$k$-coloring.

Lemma 2.1 (Liu et al. [16]). Let $k \geq 3$, $K$ be a subgraph of $G$ and $G' = G - V(K)$. If the vertices of $K$ can be ordered as $v_1, v_2, \ldots, v_t$ such that the following hold:

1. $|V(G') \cap N_G(v_1)| < |V(G') \cap N_G(v_t)|$;
2. $d_G(v_i) \leq k$ and $v_1v_t \in E(G)$;
3. for each $2 \leq i \leq t-1$, $v_i$ has at most $k-1$ neighbors in $G - \{v_{i+1}, v_{i+2}, \ldots, v_t\}$,

then any DP-$k$-coloring of $G'$ can be extended to a DP-$k$-coloring of $G$.

A graph is minimal non-DP-$k$-colorable if it is not DP-$k$-colorable but every subgraph with fewer vertices is DP-$k$-colorable. We give more specific reducible “nearly 2-degenerate” configuration for DP-3-coloring.

Lemma 2.2. Suppose that $G$ is a minimal non-DP-3-colorable graph and it has no adjacent 4-cycles. Let $C$ be an $m$-cycle $v_1v_2\ldots v_m$, let $X = \{ i : d(v_i) = 4, \ 1 \leq i \leq m \}$ and $E^+ = \{ v_i v_{i+1} : i \in X \} \cup \{ v_m v_1 \}$. If $v_m$ is a 3-vertex and $v_m v_1$ controls a 3-cycle $v_m v_1 u$ or a 4-cycle $v_m v_1 u v$, then $G$ contains no configuration satisfying all the following conditions:

(i) every edge $e$ in $E^+$ controls a 4-cycle $C_e$;

(ii) all the vertices on $C$ and the other vertices on cycles controlled by $E^+$ are distinct;

(iii) every vertex on $C$ is a 4-vertex;
(iv) every vertex on cycles controlled by $E^+$ but not on $C$ is a 3-vertex;

(v) the vertex $u$ has a neighbor neither on $C$ nor on the cycles controlled by $E^+$.

**Proof.** Suppose to the contrary that there exists such a configuration. For the path $P = v_1v_2 \ldots v_m$, replace each edge $v_iv_{i+1}$ in $E(P) \cap E^+$ by the other part of the controlled cycle, and append $v_mu$ (when $v_mv_1u$ is the controlled 3-cycle) or $v_muw$ (when $v_mv_1uw$ is the controlled 4-cycle) at the end. This yields a path starting at $v_1$ and ending at $u$. This path trivially corresponds to a sequence of vertices, and the sequence satisfies the condition of Lemma 2.1 with $k = 3$, a contradiction. 

By the definition of minimal non-DP-$k$-colorable, it is easy to obtain the following lemma.

**Theorem 2.1.** If $G$ is a minimal non-DP-$k$-colorable graph, then $\delta(G) \geq k$.

The following structural result for minimal non-DP-$k$-colorable graphs is a consequences of Theorem in [18].

**Theorem 2.2.** Let $G$ be a graph and $B$ be a 2-connected induced subgraph of $G$ with $d_G(v) = k$ for all $v \in V(B)$. If $G$ is a minimal non-DP-$k$-colorable graph, then $B$ is a cycle or a complete graph.

### 3 Proof of Theorem 1.3

Recall that our first main result is the following.

**Theorem 1.3.** Every planar graph with neither adjacent triangles nor 5-, 6- and 9-cycles is DP-3-colorable.

**Proof.** Let $G$ be a counterexample to the theorem with fewest number of vertices. We may assume that $G$ has been embedded in the plane. Thus, it is a minimal non-DP-3-colorable graph with $\delta(G) \geq 3$, and

1. $G$ is connected;
2. $G$ is a plane graph without adjacent triangles and 5-, 6-, 9-cycles;
3. $G$ is not DP-3-colorable;
4. any subgraph with fewer vertices is DP-3-colorable.

A **poor face** is a 10-face incident with ten 3-vertices, adjacent to one 4-face and four 3-faces. A **bad face** is a 10-face incident with ten 3-vertices and adjacent to five 3-faces. A **bad vertex** is a 3-vertex on a bad face. A **bad edge** is an edge on the boundary of a bad face. A **special face** is a $(3, 3, 3, 3, 4, 3, 3, 4, 3, 4)$-face adjacent to six 3-faces. A **semi-special face** is a $(3, 3, 3, 3, 4, 3, 3, 4, 3, 4, 3)$-face adjacent to five 3-faces and one 4-face as depicted in Fig. 1d. An illustration of these faces is in Fig. 1.

By Theorem 2.2, we can easy obtain the following structural result.

**Lemma 3.1.** Let $f$ be a 10-face bounded by a cycle in $G$. If $f$ is incident with ten 3-vertices and it controls a 4$^-$-face, then the controlled 4$^-$-face is incident with at least one 4$^+$-vertex.

By Lemma 3.1 and the definitions of poor faces and bad faces, we have the following consequences.

**Lemma 3.2.**

1. There are no adjacent poor faces.
2. There are no adjacent bad faces.
(a) poor face  (b) bad face  (c) special face  (d) semi-special face

Fig. 1: Some 10-faces in Theorem 1.3.

(a) A 7-face adjacent to a 3-face  (b) A 7-face adjacent to a 4-face

Fig. 2: A 7-face adjacent to a 4−-face.

(iii) There are no poor faces adjacent to bad faces.

The following structural results will be frequently used.

**Lemma 3.3.**

(a) Every 7−-cycle is chordless.

(b) Every 3-cycle is not adjacent to 6−-cycle.

(c) Every 7-face is adjacent to at most one 4−-face; the possible situations see Fig. 2. Consequently, there are no bad faces adjacent to 7-faces.

(d) No 8-face is adjacent to a 3-face; no 9-face is adjacent to a 3-face.

(e) There are no adjacent 6−-faces; thus every 3-vertex is adjacent to at most one 4−-face.

**Proof.** (a) If a 4-cycle has a chord, then there are two adjacent triangles. Note that 5-cycles and 6-cycles are excluded in $G$. If a 7-cycle has a chord, then there is a 5- or 6-cycle, a contradiction.

(b) Note that 5-cycles and 6-cycles are excluded in $G$. If a 3-cycle is adjacent to a 3- or 4-cycle, then it contradicts Lemma 3.3(a).

(c) Let $f$ be a 7-face and $C$ be its boundary. (i) Suppose that $C$ is a cycle. If $w_1w_2w_3w_4$ is on the boundary and $w_2w_3$ is incident with a 4-face $u_1w_2w_3u_4$, then none of $u_1$ and $u_4$ is on $C$ because $C$ is chordless and $\delta(G) \geq 3$, but $C$ and $u_1w_2w_3u_4$ form a 9-cycle, a contradiction. Suppose that $f$ is adjacent to two 3-faces $uw$ and $u'w'$ with $w, u'v'$ on $C$. If $w = w'$, then there are two adjacent triangles or a 5-cycle, a contradiction; and if $w \neq w'$, then there is a 9-cycle, a contradiction. (ii) Suppose that $C$ is not a cycle, and thus it consists of a 3-cycle and a 4-cycle. Hence, $f$ cannot be adjacent to any 3-face by Lemma 3.3(b). If $f$ is adjacent to a 4-face, then it can only be shown in Fig. 2b. Therefore, $f$ is adjacent to at most one 4−-face.

(d) If an 8-face is bounded by a cycle, then it cannot be adjacent to a 3-face, otherwise they form a 9-cycle or a 8-cycle with two chords, a contradiction. Suppose that the boundary of an 8-face is not a cycle but it is adjacent to a 3-face. By Lemma 3.3(b), the boundary of the 8-face must contain a 7+-cycle, but this is impossible.
Since there is no 9-cycle, the boundary of a 9-face is not a cycle. Suppose the boundary of a 9-face is adjacent to a 3-face. By Lemma 3.3(b), the boundary of the 9-face must contain a 7+-cycle, but this is impossible.

(e) Since there is no 6-cycle, the boundary of a 6-face consists of two triangles. It is easy to check that there are no adjacent 6+-faces.

Lemma 3.4. Each (3, 3, 3+, 4+)-face \( f \) is adjacent to at most one poor face.

Proof. Since every poor face is incident with ten 3-vertices, \( f \) can only be adjacent to poor faces via (3, 3)-edges. Let \( f = v_1v_2v_3v_4 \) with \( d(v_1) = d(v_2) = 3, d(v_3) \geq 3 \) and \( d(v_4) \geq 4 \). If \( d(v_3) \geq 4 \), then \( f \) is incident with exactly one (3, 3)-edge, and then it is adjacent to at most one poor face. Suppose that \( d(v_3) = 3 \) and \( f \) is adjacent to two poor faces \( f_1 \) and \( f_2 \) via \( v_1v_2 \) and \( v_2v_3 \). Since \( v_2 \) is a 3-vertex, the poor face \( f_1 \) is adjacent to the poor face \( f_2 \), but this contradicts Lemma 3.2.

Lemma 3.5. Each bad face is adjacent to at most two special faces.

Proof. Let \( f = v_1v_2 \ldots v_{10} \) be a bad face and incident with five 3-faces \( v_1v_2v_4, v_3v_4u_3, v_5v_6u_5, v_7v_8u_7, v_9v_{10}u_9 \). Suppose that \( f \) is adjacent to \( f_i \) via edge \( v_iv_{i+1} \) for \( 1 \leq i \leq 10 \), where the subscripts are taken modulo 10. Suppose to the contrary that \( f \) is adjacent to at least three special faces. Then there exist two special faces \( f_m \) and \( f_n \) such that \( |m - n| = 2 \) or \( 8 \), where \( \{m, n\} \subset \{2, 4, 6, 8, 10\} \). Without loss of generality, assume that \( f_2 \) and \( f_4 \) are the two special faces. By Lemma 2.2 and the definition of special face, \( d(u_1) = d(u_4) = 4 \).

Let \( x_3 \) and \( x_4 \) be the neighbors of \( u_3 \) other than \( v_3 \) and \( v_4 \). Since \( f_2 \) and \( f_4 \) are special faces, we have that \( x_3x_4 \in E(G) \) and \( d(x_3) = d(x_4) = 3 \), but this contradicts Lemma 2.2.

Lemma 3.6. Suppose that \( f \) is a 10+-face and it is not a bad face. Let \( t \) be the number of incident bad edges, and \( t \geq 1 \). Then \( 3t \leq d(f) \). Moreover, if \( d(f) > 3t \), then \( f \) is incident with at least \( (t + 1) \) 4+-vertices (repeated vertices are counted as the number of appearance on the boundary).

Proof. Suppose that \( f \) is adjacent to a bad face through \( uv \). Let \( x \) be the neighbor of \( u \) on \( f \) and \( y \) be the neighbor of \( v \) on \( f \). Then \( u \) and \( v \) are bad vertices and the faces controlled by \( f \) through \( xu \) and \( vy \) are all 3-faces. By Lemma 3.1 and the definition of bad face, \( d(x) \geq 4 \) and \( d(y) \geq 4 \). It is observed that two bad edges are separated by at least two other edges along the boundary of \( f \), this implies that \( 3t \leq d(f) \).

By the above discussion, every bad vertex has a 4+-neighbor along the boundary. Since \( 3t < d(f) \), there are two bad edges separated by at least two 4+-vertices, thus \( f \) is incident with at least \( (t + 1) \) 4+-vertices.

To prove the theorem, we are going to use discharging method. Define the initial charge function \( \mu(x) \) on \( V \cup F \) to be \( \mu(v) = d(v) - 6 \) for \( v \in V \) and \( \mu(f) = 2d(f) - 6 \) for \( f \in F \). By Euler’s formula, we have the following equality,

\[
\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2d(f) - 6) = -12.
\]

We design suitable discharging rules to change the initial charge function \( \mu(x) \) to the final charge function \( \mu'(x) \) on \( V \cup F \) such that \( \mu'(x) \geq 0 \) for all \( x \in V \cup F \), this leads to a contradiction and completes the proof.

The following are the needed discharging rules.

R1 Each 4-face sends \( \frac{1}{2} \) to each incident 3-vertex.

R2 Each 6-face sends 1 to each incident vertex.

R3 Each 7-face sends \( \frac{3}{2} \) to each incident semi-rich 3-vertex, 1 to each other incident vertex.
R4 Each 8-face sends $\frac{5}{4}$ to each incident vertex.

R5 Each 9-face sends $\frac{4}{3}$ to each incident vertex.

R6 Suppose that $v$ is a 3-vertex incident with a $10^+$-face $f$ and two other faces $g$ and $h$.

(a) If $v$ is incident with three $5^+$-faces, then $f$ sends 1 to $v$.

(b) If $v$ is incident with a 4-face, then $f$ sends $\frac{5}{4}$ to $v$;

(c) If $f$ is a bad face, $g$ is a 3-face and $h$ is not a special face, then $f$ sends $\frac{4}{3}$ to $v$ and $h$ sends $\frac{2}{3}$ to $v$.

(d) Otherwise, $f$ sends $\frac{2}{5}$ to $v$.

R7 Let $v$ be a 4-vertex on a $10^+$-face $f$.

(a) If $v$ is a rich vertex or a poor vertex of $f$, then $f$ sends 1 to $v$.

(b) Otherwise, $f$ sends $\frac{1}{2}$ to $v$.

R8 Let $v$ be a 5-vertex on a $10^+$-face $f$.

(a) If $v$ is incident with two $4^-$-face, then $f$ sends $\frac{1}{3}$ to $v$.

(b) If $v$ is incident with exactly one $4^-$-face, then $f$ sends $\frac{1}{4}$ to $v$.

(c) Otherwise, $f$ sends $\frac{1}{5}$ to $v$.

R9 Each $(3,3,3^+,4^+)$-face sends $\frac{1}{2}$ to adjacent poor face.

R10 Each $(3,4,3^+,4^+)$-face and $(3,4,4^+,3^+)$-face send $\frac{1}{4}$ to each adjacent semi-special face.

It remains to check that the final charge of every element in $V \cup F$ is nonnegative.

(1) Let $v$ be an arbitrary vertex of $G$.

By Theorem 2.1, $G$ has no $2^+$-vertices. If $v$ is a $6^+$-vertex, then $\mu'(v) \geq \mu(v) = d(v) - 6 \geq 0$. We may assume that $3 \leq d(v) \leq 5$.

Suppose that $v$ is a 3-vertex. By Lemma 3.3(e), $v$ is incident with at most one $4^-$-face. If $v$ is incident with no $4^-$-face, then it receives at least 1 from each incident face, and then $\mu'(v) \geq 3 - 6 + 3 \times 1 = 0$. If $v$ is incident with a 4-face, then it receives at least $\frac{5}{4}$ from each incident $7^+$-face, and then $\mu'(v) \geq 3 - 6 + 2 \times \frac{5}{4} + \frac{1}{2} = 0$. If $v$ is incident with a 5-face and a $7$-face, then the other incident face is not a bad face by Lemma 3.3(c), and then $\mu'(v) = 3 - 6 + 2 \times \frac{3}{2} = 0$. By Lemma 3.3(d), if $v$ is incident with a 3-face, then it is not incident with any 8- or 9-face. If $v$ is incident with a 3-face and two $10^+$-faces, then $\mu'(v) \geq 3 - 6 + \min \{\frac{4}{3} + \frac{5}{2}, 2 \times \frac{3}{2}\} = 0$.

Suppose that $v$ is a 4-vertex. By Lemma 3.3(e), $v$ is incident with at most two $4^-$-faces. If $v$ is incident with exactly one $4^-$-face, then $\mu'(v) \geq 4 - 6 + 2 \times \frac{1}{2} + 1 = 0$. If $v$ is incident with two $4^-$-faces, then $\mu'(v) \geq 4 - 6 + 2 \times 1 = 0$. If $v$ is incident with no $4^-$-face, then $\mu'(v) \geq 4 - 6 + 4 \times 1 > 0$.

Suppose that $v$ is a 5-vertex. By Lemma 3.3(e), $v$ is incident with at most two $4^-$-faces. If $v$ is incident with no $4^-$-face, then it receives at least $\frac{1}{5}$ from each incident $5^+$-face, and $\mu'(v) \geq 5 - 6 + 5 \times \frac{1}{5} = 0$. If $v$ is incident with exactly one $4^-$-face, then it receives at least $\frac{1}{4}$ from each incident $5^+$-face, and $\mu'(v) \geq 5 - 6 + 4 \times \frac{1}{4} = 0$. If $v$ is incident with two $4^-$-faces, then it receives at least $\frac{1}{3}$ from each incident $5^+$-face, and $\mu'(v) \geq 5 - 6 + 3 \times \frac{1}{3} = 0$.

(2) Let $f$ be an arbitrary face in $F(G)$.

If $f$ is a 3-face, then $\mu'(f) = \mu(f) = 0$. Suppose that $f$ is a 4-face. If $f$ is incident with four 3-vertices, then $\mu'(f) = 2 - 4 \times \frac{1}{2} = 0$. If $f$ is incident with exactly one $4^+$-vertex, then it is adjacent to at most one poor face by Lemma 3.4, and then $\mu'(f) \geq 2 - 3 \times \frac{1}{2} - \frac{1}{2} = 0$. If $f$ is a $(3,3,4^+,4^+)$-face, then it is adjacent
to at most one poor face and at most two semi-special faces, and then $\mu'(f) \geq 2 - 2 \times \frac{1}{2} - \frac{1}{2} - 2 \times \frac{1}{4} = 0$. If $f$ is a $(3,4^+,3,4^+)$-face, then it sends at most $\frac{1}{4}$ to each adjacent face, and $\mu'(f) \geq 2 - 2 \times \frac{3}{4} - 4 \times \frac{1}{4} = 0$.

If $f$ is adjacent with exactly three $4^+$-vertices, then it is adjacent to at most two semi-special faces, and $\mu'(f) \geq 2 - \frac{1}{2} - 2 \times \frac{1}{4} > 0$. If $f$ is incident with four $4^+$-vertices, then $\mu'(f) = \mu(f) = 2$.

If $f$ is a 6-face, then $\mu'(f) = 6 - 6 \times 1 = 0$. Suppose that $f$ is a 7-face. By Lemma 3.3(c), $f$ is adjacent to at most one $4^-$-face. If $f$ is adjacent to a $4^-$-face (see Fig. 2), then $f$ is incident with at most two semi-rich 3-vertices, which implies that $\mu'(f) \geq 8 - 2 \times \frac{3}{2} - 5 \times 1 = 0$. If $f$ is not adjacent to any $4^-$-face, then $f$ sends 1 to each incident vertex, and $\mu'(f) = 8 - 7 \times 1 > 0$. If $f$ is an 8-face, then $\mu'(f) = 10 - 8 \times \frac{3}{4} = 0$. If $f$ is a 9-face, then $\mu'(f) = 12 - 9 \times \frac{4}{3} = 0$.

Suppose that $f$ is a $10^+$-face. Let $t$ be the number of incident bad edges. Hence, $f$ is incident with exactly 2$t$ bad vertices. By Lemma 3.6, $f$ is incident with at least $t$ $4^+$-vertices. Thus, $\mu'(f) \geq 2d(f) - 6 - 2t \times \frac{5}{3} - t \times 1 - (d(f) - 3t) \times \frac{3}{2} = \frac{1}{2}d(f) - 6 + \frac{t}{6}$. If $d(f) \geq 12$, then $\mu'(f) \geq 12 \times \frac{1}{2} - 6 + \frac{t}{6} \geq 0$. So it suffices to consider 10-faces and 11-faces.

**Suppose that $f$ is an 11-face.** (i) $t = 0$. It follows that $f$ is not incident with any bad vertex, and it sends at most $\frac{3}{2}$ to each incident vertex. If $f$ is incident with a $4^+$-vertex, then $\mu'(f) \geq 16 - 10 \times \frac{3}{2} = 1 = 0$. Suppose that $f$ is a 3-regular face. By Lemma 3.3(e), every vertex on $f$ is incident with at most one $4^-$-face. Since $d(f)$ is odd, $f$ must be incident with a rich 3-vertex. This implies that $\mu'(f) \geq 16 - 10 \times \frac{3}{2} = 1 = 0$. (ii) $t \geq 1$. It follows that $f$ is incident with exactly 2$t$ bad vertices and at least $(t + 1)$ $4^+$-vertices, and then $\mu'(f) \geq 16 - 2t \times \frac{5}{3} - (t + 1) \times 1 - (11 - (3t + 1)) \times \frac{3}{2} = \frac{t}{6} > 0$.

Finally we may assume that $f$ is a 10-face. If $f$ is a special face, then $\mu'(f) = 14 - 8 \times \frac{3}{2} - 2 \times 1 = 0$.

If $f$ is a bad face, then it is adjacent to at most two special faces by Lemma 3.5, which implies that $\mu'(f) \geq 14 - 4 \times \frac{3}{2} - 6 \times \frac{1}{2} = 0$.

So we may assume that $f$ is neither a bad face nor a special face. By Lemma 3.6, $t \leq \lfloor \frac{d(f)}{3} \rfloor = 3$.

- **$t = 0$.** It follows that $f$ is not incident with any bad vertex. Hence, $f$ sends at most $\frac{3}{2}$ to each incident 3-vertex, at most 1 to each incident 4-vertex, and at most $\frac{1}{2}$ to each incident 5-vertex. If $f$ is incident with a $5^+$-vertex, then $\mu'(f) \geq 14 - 9 \times \frac{3}{2} - \frac{1}{2} > 0$. If $f$ is incident with at least two 4-vertices, then $\mu'(f) \geq 14 - 8 \times \frac{3}{2} - 2 \times 1 = 0$.

So we may assume that $f$ is incident with at most one 4-vertex and no $5^+$-vertices. If $f$ is incident with a semi-rich 4-vertex, then $\mu'(f) \geq 14 - 9 \times \frac{3}{2} - 1 = 0$. If $f$ is incident with a rich 4-vertex and nine 3-vertices, then at least one of the incident 3-vertices is rich, and then $\mu'(f) \geq 14 - 1 - 8 \times \frac{3}{2} - 1 = 1 = 0$. If $f$ is incident with a poor 4-vertex, then there exists a rich 3-vertex incident with $f$, and $\mu'(f) \geq 14 - 1 - 8 \times \frac{3}{2} - 1 = 1 = 0$.

Suppose that $f$ is incident with ten 3-vertices. If $f$ is adjacent to at most four $4^-$-faces, then $\mu'(f) \geq 14 - 8 \times \frac{3}{2} - 2 \times 1 = 0$. If $f$ is adjacent to at least two $4^-$-faces, then $\mu'(f) \geq 14 - 6 \times \frac{3}{2} - 4 = \frac{5}{4} = 0$. If $f$ is adjacent to four $3$-faces and one 4-face, then $f$ must be a poor face and the 4-face must be $(3,3,3^+,4^+)$-face, and then $\mu'(f) = 14 - 8 \times \frac{3}{2} - 2 \times \frac{5}{4} + \frac{1}{2} = 0$. If $f$ is adjacent to five 3-faces, then it is a bad face, so we are done.

- **$t = 1$.** It follows that $f$ is incident with exactly two bad vertices and at least two $4^+$-vertices. If $f$ is incident with a rich 3-vertex or at least three 4-vertices, then $\mu'(f) \geq 14 - 2 \times \frac{3}{2} - 3 \times 1 - 5 \times \frac{3}{2} > 0$. If $f$ is incident with a $5^+$-vertex, then $\mu'(f) \geq 14 - 2 \times \frac{5}{3} - 1 - \frac{1}{2} - 6 \times \frac{1}{4} > 0$. If $f$ is incident with a semi-rich 4-vertex, then $\mu'(f) \geq 14 - 2 \times \frac{3}{2} - 1 - \frac{1}{2} - 6 \times \frac{3}{2} > 0$. So we may assume that $f$ is incident with two poor 4-vertices and eight semi-rich 3-vertices. Thus $f$ is a $(3,4,3,4,3,4,3,3,3,3)$-face $w_1w_2\ldots w_{10}$, where $w_3w_4$ is a bad edge, each of $w_2w_3$ and $w_4w_5$ controls a 3-face, each of $w_1w_2$ and $w_5w_6$ controls a $4^-$-face. If $f$ controls at least one 4-face by a $(3,3)$-edge, then $\mu'(f) \geq 14 - 2 \times \frac{3}{2} - 2 \times 1 - 2 \times \frac{3}{2} - 4 \times \frac{3}{2} > 0$. So we may further assume that each of $w_7w_8$ and $w_9w_{10}$ controls a 3-face. If each of $w_1w_2$ and $w_5w_6$ controls a 4-face, then $\mu'(f) \geq 14 - 2 \times \frac{3}{2} - 2 \times 1 - 2 \times \frac{3}{2} - 4 \times \frac{3}{2} > 0$. If each of $w_1w_2$ and $w_5w_6$ controls a 3-face, then $f$
must be a special face, a contradiction. If exactly one of $w_1w_2$ and $w_5w_6$ controls a 4-face, then $f$ must be a semi-special face. In this case the controlled 4-face must be a $(3, 4, 3^+, 4^+)$-face or $(3, 4, 4^+, 3^+)$-face due to Lemma 2.2. Thus $\mu'(f) \geq 14 - 2 \times \frac{5}{3} - 2 \times 1 - \frac{5}{2} - 5 \times \frac{3}{2} + \frac{1}{4} \geq 0$.

• $t = 2$. It follows that $f$ is incident with exactly four bad vertices and at least three $4^+$-vertices. If $f$ is incident with at least four $4^+$-vertices, then $\mu'(f) \geq 14 - 4 \times \frac{5}{3} - 4 \times 1 - (10 - 4 - 4) \times \frac{3}{2} > 0$. Thus, $f$ is incident with exactly four bad vertices and exactly three $4^+$-vertices. If there is a semi-rich $4^+$-vertex, then $\mu'(f) \geq 14 - 4 \times \frac{5}{3} - 3 \times 1 - 2 \times \frac{3}{2} - 1 > 0$.

• $t = 3$. It follows that $f$ is incident with six bad vertices and four $4^+$-vertices. Thus, $\mu'(f) \geq 14 - 6 \times \frac{5}{3} - 4 \times 1 = 0$.

4 Distance of triangles at least two

In this section, we give two Bordeaux type results on planar graphs with distance of triangles at least two.

4.1 Planar graphs without 5-, 6- and 8-cycles

Recall that our second main result is the following.

**Theorem 1.8.** Every planar graph with neither 5-, 6-, 8-cycles nor triangles at distance less than two is DP-3-colorable.

**Proof.** Suppose to the contrary that $G$ is a counterexample with the number of vertices as small as possible. We may assume that $G$ has been embedded in the plane. Thus, $G$ is a minimal non-DP-3-colorable graph.

**Lemma 4.1.**

(a) There are no 5-faces and no 6-faces.

(b) A 3-face cannot be adjacent to an 8+ -face.

(c) There are no adjacent 6+ -faces.

**Proof.** Since every 5-face is bounded by a 5-cycle, but there is no 5-cycles in $G$, this implies that there is no 5-faces in $G$. Since there is no 6-cycles in $G$, the boundary of every 6-face consists of two triangles, thus the distance of these triangles is zero, a contradiction. Therefore, there is no 5-faces and no 6-faces in $G$.

It is easy to check that every 7+ -cycle is chordless. Let $f$ be a 3-face. If $f$ is adjacent to a 4-face $g$, then they form a 5-cycle with a chord, a contradiction. Suppose that $g$ is a 7-face. Then $g$ may be bounded by a cycle or a closed walk with a cut-vertex. If $g$ is bounded by a cycle and it is adjacent to $f$, then these two cycles form an 8-cycle with a chord, a contradiction. If the boundary of $g$ contains a cut-vertex, then the
boundary consists of a 3-cycle and a 4-cycle, and neither the 3-cycle nor the 4-cycle can be adjacent to the 3-face \( f \). If \( g \) is an 8-face, then the boundary of \( g \) consists of two 4-cycles, or two triangles and a cut-edge, but no edge on such boundary can be adjacent to the 3-face \( f \).

By the hypothesis and fact that a 3-face cannot be adjacent to an \( 8^- \)-face, it suffices to prove that there is no adjacent 4-faces. Since every 4-cycle has no chords, two adjacent 4-faces must form a 6-cycle with a chord, a contradiction.

A 7-face \( f \) is **special** if \( f \) is incident with six semi-weak 3-vertices and a poor 4-vertex, see Fig. 3a. A 7-face \( f \) is **poor** if \( f \) is incident with six semi-weak 3-vertices and a strong 3-vertex, see Fig. 3b. Note that a 7-face is not adjacent to any 3-face by Lemma 4.1(b).

**Lemma 4.2.** Each poor 7-face is adjacent to three \((3, 3, 3^+, 4^+)\)-faces.

**Proof.** Suppose that \( f = w_1w_2 \ldots w_7 \) is a poor 7-face and it is adjacent to a 4-face \( g = u_1w_2w_3u_4 \). Since \( f \) is incident with seven 3-vertices, it must be bounded by a 7-cycle. Note that every 7-cycle has no chords, we have that \( \{u_1, u_2\} \cap \{w_1, w_2, \ldots, w_7\} = \emptyset \). The subgraph induced by \( \{w_1, w_2, \ldots, w_7\} \cup \{u_1, u_2\} \) is 2-connected, and it is neither a complete graph nor a cycle. By Theorem 2.2, \( g \) must be incident with a \( 4^- \)-vertex.

Applying Lemma 2.2 to a special 7-face, we get the following result.

**Lemma 4.3.** If \( f \) is a special 7-face and it controls a \((3, 3, 3)\)-face, then each of the face controlled by \((3, 4)\)-edge has at least two \( 4^+ \)-vertices.

Define the initial charge function \( \mu(x) \) on \( V \cup F \) to be \( \mu(v) = d(v) - 6 \) for \( v \in V \) and \( \mu(f) = 2d(f) - 6 \) for \( f \in F \). By Euler’s formula, we have the following equality,

\[
\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2d(f) - 6) = -12.
\]

We give some discharging rules to change the initial charge function \( \mu(x) \) to the final charge function \( \mu'(x) \) on \( V \cup F \) such that \( \mu'(x) \geq 0 \) for all \( x \in V \cup F \), which leads to a contradiction.

The following are the discharging rules.

**R1** Each 4-face sends \( \frac{1}{2} \) to each incident 3-vertex.

**R2** Each \( 7^- \)-face sends \( \frac{3}{4} \) to each incident weak 3-vertex, \( \frac{5}{4} \) to each incident semi-weak 3-vertex, 1 to each incident strong 3-vertex.

**R3** Each \( 7^+ \)-face sends 1 to each incident poor 4-vertex, \( \frac{3}{4} \) to each incident semi-rich 4-vertex, \( \frac{1}{2} \) to each incident rich 4-vertex.

**R4** Each \( 7^+ \)-face sends \( \frac{1}{5} \) to each incident 5-vertex.

**R5** Each \((3, 3, 3^+, 4^+)\)-face sends \( \frac{1}{4} \) to each adjacent poor 7-face and each adjacent special 7-face through \((3, 3)\)-edge, respectively.

**R6** Each \((3, 4, 3^+, 4^+)\)-face and \((3, 4, 4^+, 3^+)\)-face sends \( \frac{1}{4} \) to each adjacent special 7-face through \((3, 4)\)-edge.

It remains to check that the final charge of every element in \( V \cup F \) is nonnegative.

(1) Let \( v \) be an arbitrary vertex of \( G \).

By Theorem 2.1, \( G \) has no \( 2^- \)-vertices. If \( v \) is a \( 6^- \)-vertex, then it is not involved in the discharging procedure, hence \( \mu'(v) = \mu(v) = d(v) - 6 \geq 0 \). Next, we may assume that \( 3 \leq d(v) \leq 5 \).
Suppose that \( v \) is a 3-vertex. If \( v \) is incident with no \( 4^- \)-face, then it is incident with three \( 7^+ \)-faces, and then \( \mu'(v) = \mu(v) + 3 \times 1 = 0 \). If \( v \) is incident with a 3-face, then the other two incident faces are \( 9^+ \)-faces by Lemma 4.1(b), and then \( \mu'(v) = \mu(v) + 2 \times \frac{3}{2} = 0 \). If \( v \) is incident with a 4-face, then the other two incident faces are \( 7^+ \)-faces by Lemma 4.1(c), and then \( \mu'(v) = \mu(v) + 2 \times \frac{3}{2} + \frac{1}{2} = 0 \).

Suppose that \( v \) is a 4-vertex. By Lemma 4.1(c), \( v \) is incident with at most two \( 4^- \)-faces. If \( v \) is incident with no \( 4^- \)-face, then it is incident with four \( 7^+ \)-faces, and then \( \mu'(v) = \mu(v) + 4 \times \frac{1}{2} = 0 \). If \( v \) is incident with exactly one \( 4^- \)-face, then \( \mu'(v) = \mu(v) + 2 \times \frac{3}{4} + \frac{1}{2} = 0 \). If \( v \) is incident with exactly two \( 4^- \)-faces, then \( \mu'(v) = \mu(v) + 2 \times 1 = 0 \).

Suppose that \( v \) is a 5-vertex. By Lemma 4.1(c), \( v \) is incident with at most two \( 4^- \)-faces. Therefore, it is incident with at least three \( 7^+ \)-faces, and \( \mu'(v) \geq \mu(v) + 3 \times \frac{1}{3} = 0 \).

(2) Let \( f \) be an arbitrary face in \( F(G) \).

Since the distance of triangles is at least two, each \( k \)-face is adjacent to at most \( \lfloor \frac{k}{3} \rfloor \) triangular-faces, thus \( f \) contains at most \( 2 \times \lfloor \frac{k}{3} \rfloor \) weak 3-vertices. As observed above, \( G \) has no 5-face and 6-face. If \( f \) is a 3-face, then it is not involved in the discharging procedure, and then \( \mu'(f) = \mu(f) = 0 \).

Suppose that \( f \) is a 4-face. If \( f \) is incident with four 3-vertices, then \( \mu'(f) = \mu(f) - 4 \times \frac{1}{2} = 0 \). If \( f \) is incident with exactly one \( 4^- \)-vertex, then \( f \) sends at most \( \frac{1}{4} \) through each incident \( (3,3) \)-edge, and then \( \mu'(f) \geq \mu(f) - 3 \times \frac{1}{2} - 2 \times \frac{1}{4} = 0 \). If \( f \) is incident with at least two \( 4^- \)-vertices, then \( \mu'(f) \geq \mu(f) - 2 \times \frac{1}{2} - 4 \times \frac{1}{4} = 0 \).

If \( f \) is an 8-face, then it is not adjacent to any 3-face and it sends at most \( \frac{5}{4} \) to each incident vertex, and then \( \mu'(f) \geq \mu(f) - 8 \times \frac{5}{4} = 0 \).

Suppose that \( f \) is a 9-face. Recall that \( f \) is incident with at most six weak 3-vertices. If \( f \) is incident with exactly six weak 3-vertices, then \( f \) sends at most 1 to each other incident vertex, and then \( \mu'(f) \geq \mu(f) - 6 \times \frac{3}{2} - (9 - 6) \times 1 = 0 \). If \( f \) is incident with exactly five weak 3-vertices, then \( f \) must be adjacent to three 3-faces and one of the six incident vertices on triangles must be a \( 4^- \)-vertex (see Fig. 4), and then \( \mu'(f) \geq \mu(f) - 5 \times \frac{3}{2} - 1 - \frac{5}{4} - 2 \times \frac{1}{2} > 0 \). If \( f \) is incident with exactly four weak 3-vertices and at least one \( 4^- \)-vertex, then \( \mu'(f) \geq \mu(f) - 4 \times \frac{3}{2} - 1 - (9 - 4 - 1) \times \frac{5}{4} = 0 \). If \( f \) is incident with exactly four weak 3-vertices and no \( 4^- \)-vertex, then \( f \) is incident with at least one \( 3 \)-vertex and at most four semi-weak 3-vertices, and then \( \mu'(f) \geq \mu(f) - 4 \times \frac{3}{2} - 4 \times \frac{5}{4} - 1 = 0 \). If \( f \) is incident with at most three weak 3-vertices, then \( \mu'(f) \geq \mu(f) - 3 \times \frac{3}{2} - (9 - 3) \times \frac{5}{4} = 0 \).

If \( f \) is a 10-vertex, then \( \mu'(f) \geq \mu(f) - 2 \times \left( d(f) \left\lfloor \frac{d(f)}{3} \right\rfloor \right) - \left( d(f) - 2 \times \left\lfloor \frac{d(f)}{3} \right\rfloor \right) \times \frac{5}{4} \geq 0 \).

Suppose that \( f \) is a 7-face. By Lemma 4.1(b), \( f \) is not incident with any weak 3-vertex. It is observed that \( f \) is incident with at most six semi-weak 3-vertices. If there is an incident vertex receives at most \( \frac{1}{2} \) from \( f \), then \( \mu'(f) \geq \mu(f) - \frac{1}{2} - (7 - 1) \times \frac{5}{4} = 0 \). So we may assume that \( f \) is incident with seven \( 4^- \)-vertices and no rich \( 4^- \)-vertex. If \( f \) is incident with at most four semi-weak 3-vertices, then \( \mu'(f) \geq \mu(f) - 4 \times \frac{5}{4} - 3 \times 1 = 0 \). So we may further assume that \( f \) is incident with at least five semi-weak 3-vertices and at most two \( 4^- \)-vertices. If \( f \) is incident with two semi-rich \( 4^- \)-vertices, then \( \mu'(f) = \mu(f) - 2 \times \frac{3}{4} - (7 - 2) \times \frac{5}{4} > 0 \). If \( f \) is incident
with a semi-rich 4-vertex and a poor 4-vertex, then \( \mu'(f) = \mu(f) - \frac{2}{3} - 1 - (7 - 2) \times \frac{5}{4} = 0 \). It is impossible that \( f \) is incident with five semi-weak 3-vertices and two poor 4-vertices.

In the following, assume that \( f \) is incident with at most one 4-vertex and at least five semi-weak 3-vertices. If \( f \) is incident with a semi-rich 4-vertex, then it is incident with at most five semi-weak 3-vertices, and then \( \mu'(f) \geq \mu(f) - \frac{2}{3} - 5 \times \frac{5}{4} - 1 = 0 \). Suppose that \( f \) is incident with a poor 4-vertex, then it must be adjacent to six semi-weak 3-vertices, i.e., \( f \) is a special 7-face, see Fig. 3a. If \( f \) controls two \((3,3,3^+,4^+)\)-faces through \((3,3)\)-edges, then \( \mu'(f) = \mu(f) - 1 - 6 \times \frac{5}{4} + 2 \times \frac{1}{4} = 0 \). Then we may assume that \( f \) controls at least one \((3,3,3,3)\)-face. By Lemma 4.3, \( f \) controls two 4-faces incident with at least two \( 4^+ \)-vertices through \((3,4)\)-edges, thus \( \mu'(f) = \mu(f) - 1 - 6 \times \frac{5}{4} + 2 \times \frac{1}{4} = 0 \). Finally, we may assume that \( f \) is incident with seven 3-vertices. In this case, \( f \) can only be a poor face, see Fig. 3b. By Lemma 2.2, \( f \) is incident with three \((3,3,3^+,4^+)\)-faces, thus \( \mu'(f) = \mu(f) - 1 - 6 \times \frac{5}{4} + 3 \times \frac{1}{4} > 0 \).

\[ \square \]

### 4.2 Planar graphs without 4-, 5- and 7-cycles

The third main result can be derived from the following theorem on degeneracy.

**Theorem 1.10.** Every planar graph with neither 4-, 5-, 7-cycles nor triangles at distance less than two is 2-degenerate.

**Proof.** Suppose that \( G \) is a planar graph satisfying all the hypothesis but the minimum degree is at least three. Without loss of generality, we may assume that \( G \) is connected and it has been embedded in the plane.

**Lemma 4.4.**

(a) There is no 4-, 5-, 7-faces. Every 6-face is bounded by a 6-cycle.

(b) A 3-face cannot be adjacent to a 7^-face.

**Proof.** (a) Since every 4-face must be bounded by a 4-cycle, but there is no 4-cycles in \( G \), this implies that there is no 4-faces in \( G \). Similarly, there is no 5-faces in \( G \). Since there is no 7-cycles in \( G \), there is no 7-face bounded by a cycle, and then the boundary of every 7-face must consist of a triangle and a 4-cycle, but this contradicts the absence of 4-cycles. If the boundary of a 6-face is not a cycle, then it must consist of two triangles, and the distance of these two triangles is zero, a contradiction. Therefore, every 6-face is bounded by a 6-cycle.

(b) It is easy to check that every 8^-cycle is chordless. Since there is no two triangles at distance less than two, there is no two adjacent 3-faces. Every 6-face is bounded by a 6-cycle and it is chordless, thus a 3-face cannot be adjacent to a 6-face, for otherwise they form a 7-cycle with a chord, a contradiction. \[ \square \]

Define the initial charge function \( \mu(x) \) on \( V \cup F \) to be \( \mu(v) = d(v) - 6 \) for \( v \in V \) and \( \mu(f) = 2d(f) - 6 \) for \( f \in F \). By Euler’s formula, we have the following equality,

\[
\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2d(f) - 6) = -12.
\]

Next, we define some discharging rules to change the initial charge function \( \mu(x) \) to the final charge function \( \mu'(x) \) on \( V \cup F \) such that \( \mu'(x) \geq 0 \) for all \( x \in V \cup F \). This leads to a contradiction, and then we complete the proof.

**R1.** Each 6^+-face sends 1 to each incident strong 3-vertex, \( \frac{1}{2} \) to each incident rich 4-vertex, \( \frac{1}{4} \) to each incident 5-vertex.
R2 Each $8^+$-face sends $\frac{3}{2}$ to each incident weak 3-vertex, $\frac{3}{4}$ to each incident semi-rich 4-vertex.

It remains to check that the final charge of every element in $V \cup F$ is nonnegative.

- Let $v$ be an arbitrary vertex of $G$.

  If $v$ is a $6^+$-vertex, then it is not involved in the discharging procedure, hence $\mu'(v) = \mu(v) - 6 \geq 0$. We may assume that $3 \leq d(v) \leq 5$. Since there is no two triangles at distance less than two, every vertex is incident with at most one 3-face.

  Suppose that $v$ is a 3-vertex. If $v$ is not incident with any 3-face, then it is incident with three $6^+$-faces, and then $\mu'(v) = \mu(v) + 3 \times 1 = 0$. If $v$ is incident with a 3-face, then the other two incident faces are $8^+$-faces by Lemma 4.4(b), and then $\mu'(v) = \mu(v) + 2 \times \frac{3}{2} = 0$.

  Suppose that $v$ is a 4-vertex. If $v$ is not incident with any 3-face, then it is incident with four $6^+$-faces, and then $\mu'(v) = \mu(v) + 4 \times \frac{3}{2} = 0$. If $v$ is incident with a 3-face, then $\mu'(v) = \mu(v) + 2 \times \frac{3}{2} + \frac{3}{2} = 0$.

  Suppose that $v$ is a 5-vertex. Since $v$ is incident with at most one 3-face, it is incident with at least four $6^+$-faces, so $\mu'(v) \geq \mu(v) + 4 \times \frac{3}{4} = 0$.

- Let $f$ be an arbitrary face in $F(G)$.

  Note that there is no 4-, 5-, 7-faces. Since every 3-face $f$ is not involved in the discharging procedure, we have that $\mu'(f) = \mu(f) = 0$. By Lemma 4.4(b), every 6-face $f$ is adjacent to six $6^+$-faces, thus $\mu'(f) \geq \mu(f) - 6 \times 1 = 0$. Suppose that $f$ is a $d$-face with $d \geq 8$. Since the distance of triangles is at least two, we have that $f$ is adjacent to at most $\lfloor \frac{d}{3} \rfloor$ triangular-faces, thus it is incident with at most $2 \times \lfloor \frac{d}{3} \rfloor$ weak 3-vertices. Hence, $\mu'(f) \geq 2d - 6 - 2 \times \lfloor \frac{d}{3} \rfloor \times \frac{3}{2} - (d - 2 \times \lfloor \frac{d}{3} \rfloor) \times 1 = d - 6 - \lfloor \frac{d}{3} \rfloor \geq 0$. \hfill $\Box$

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