Spectral energy dynamics in magnetohydrodynamic turbulence

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Spectral direct numerical simulations of incompressible MHD turbulence at a resolution of up to 1024^4 collocation points are presented for a statistically isotropic system as well as for a setup with an imposed strong mean magnetic field. The spectra of residual energy, \( E_k^R = |E_k^M - E_k^S| \), and total energy, \( E_k = E_k^R + E_k^M \), are observed to scale self-similarly in the inertial range as \( E_k^R \sim k^{-7/3} \), \( E_k \sim k^{-5/3} \) (isotropic case) and \( E_k^R \sim k_{i,-}^{-5/3} \), \( E_k \sim k_{i,-}^{-3/2} \) (anisotropic case, perpendicular to the mean field direction). A model of dynamic equilibrium between kinetic and magnetic energy, based on the corresponding evolution equations of the eddy-damped quasi-normal Markovian (EDQNM) closure approximation, explains the findings. The assumed interplay of turbulent dynamo and Alfvén effect yields \( E_k^R \sim k E_k^M \) which is confirmed by the simulations.

The nonlinear behavior of turbulent plasmas gives rise to a variety of dynamical effects such as self-organization of magnetic confinement configurations in laboratory experiments \([1]\), generation of stellar magnetic fields \([2]\) or structure formation in the interstellar medium \([3]\). The understanding of these phenomena is incomplete as the same is true for many inherent properties of the underlying turbulence.

Large-scale low-frequency plasma turbulence is treated in the magnetohydrodynamic (MHD) approximation describing the medium as a viscous and electrically resistive magnetofluid neglecting additional kinetic effects. Incompressibility of the flow is assumed for the sake of simplicity. In this setting the nature of the turbulent energy cascade is a central and still debated issue with different phenomenologies being proposed \([4, 5, 6, 7, 8]\) (cf. \([9]\) for a review). The associated spectral dynamics of kinetic and magnetic energy, in spite of its comparable importance, has received less attention (as an exception (cf. \([9]\) for a review). The ratio of kinetic and magnetic energy are initially equal with \( E^K = E^M = 0.5 \). The ratio \( E^K/E^M \) decreases in time taking on values of 0.28 to 0.23 in the period considered (cf. \([12]\)). The ratio of kinetic and magnetic energy dissipation rate, \( \varepsilon^K/\varepsilon^M \), with \( \mu = \eta = 1 \times 10^{-4} \) also decreases during turbulence decay from 0.7 to about 0.6, the difference in dissipation rates reflecting the imbalance of the related energies. The Reynolds number \( Re = (E^K)/\varepsilon^M \) at \( t = 6 \) is about 2700 and slightly diminishes during the run. Magnetic, \( H^C = \frac{1}{2} \int_V \mathbf{d} \mathbf{a} \cdot \mathbf{b} \), \( \mathbf{b} = \nabla \times \mathbf{a} \), and cross helicity, \( H^C = \frac{1}{2} \int_V \mathbf{d} \mathbf{v} \cdot \mathbf{b} \), are negligible with \( H^C \) showing a dynamically unimportant increase from 0.03 to 0.07 during the simulation. The run covers 9 eddy turnover times defined as the time required to reach the maximum of dissipation from \( t = 0 \). The large-scale rms magnetic field decays from initially 0.7 to 0.3.

Case II is a 1024^2 \times 256 forced turbulence simulation with an imposed constant mean magnetic field of strength \( b_0 = 5 \) in units of the large-scale rms magnetic field \( b_{\text{rms}} \approx \nu_{\text{rms}} \approx 1 \). The forcing, which keeps the ratio of fluctuations to mean field approximately constant, is implemented by freezing modes with \( k \leq k_f = 2 \). The simulation with \( \mu = \eta = 9 \times 10^{-5} \) has been brought into quasi-equilibrium over 20 eddy-turnover times at a resolution of 512^2 \times 256 and spans about 5 eddy turnover times of quasi-stationary turbulence with 1024^2 \times 256

\[ \begin{align*}
\partial_t \omega &= \nabla \times [\mathbf{v} \times \omega - \mathbf{b} \times (\nabla \times \mathbf{b})] + \mu \Delta \omega \\
\partial_t \mathbf{b} &= \nabla \times (\mathbf{v} \times \mathbf{b}) + \eta \Delta \mathbf{b} \\
\nabla \cdot \mathbf{v} &= \nabla \cdot \mathbf{b} = 0.
\end{align*} \]
Fourier modes and $\text{Re} \approx 2300$ (based on field perpendicular fluctuations). Kinetic and magnetic energy as well as the ratio $E^K/E^M$ are approximately unity with a slight excess of $E^M$. Perpendicular to the imposed field, large-scale magnetic fluctuations with $b_{\text{rms}} \approx 0.4$ are observed. Correspondingly, $\varepsilon^K/\varepsilon^M \approx 0.95$ during the simulation. The system has relaxed to $H^C \approx 0.15$ with a fluctuation level of about 30% and $H^M \approx 0.2H^M_{\text{max}}$ with $H^M_{\text{max}} \sim E^M/k_f$.

Fourier-space-angle integrated spectra of total, magnetic, and kinetic energy for case I are shown in Fig. 1. To neutralize secular changes as a consequence of turbulence decay, amplitude normalization is used assuming a Kolmogorov total energy spectrum, $E_k \rightarrow E_k/(\varepsilon k^5)$, $\varepsilon = -\partial_k E$, with wavenumbers given in inverse multiples of the associated dissipation length, $t_D \approx (\mu^3/\varepsilon)^{1/4}$. The quasi-stationary normalized spectra are time averaged over the period of self-similar decay, $t = 6 - 8.9$. As in previous numerical work \cite{12,14} and also observed in solar wind measurements \cite{12}, Kolmogorov scaling applies for the total energy in the well-developed inertial range, $0.01 \lesssim k \lesssim 0.1$. However, here the remarkable growth of excess magnetic energy with decreasing wavenumber is of interest. Qualitatively similar behavior is observed with large scale forcing exerted on the system. We note that no pile-up of energy is seen at the dissipative fall-off contrary to other high-resolution simulations \cite{12,14}. Apart from different numerical techniques and physical models this difference might be due to the limited simulation period at highest resolution namely 5 \cite{14} and 4.3 \cite{17} large-eddy-turnover times. Depending on initial conditions the energy spectrum at $1024^3$-resolution is still transient at that time.

In case II, pictured in Fig. 2 strong anisotropy is generated due to turbulence depletion along the mean magnetic field, $b_0$, (cf. also \cite{18,12,20,21,22}). This is visible when comparing the normalized and time-averaged field-perpendicular one-dimensional spectrum, $E_{k_\perp} = \int \int d\varepsilon dk_\perp E(k_\perp, k_1, k_2)$ (solid line) with the field-parallel spectrum, defined correspondingly and adumbrated by the dash-dotted line in Fig. 2. The fixed $k_\perp$-axis is chosen arbitrarily in the $k_1$-$k_2$-plane perpendicular to $b_0$ where fluctuations are nearly isotropic. For the strong $b_0$ chosen here, field-parallel and -perpendicular energy spectra do not differ notably from the ones found \cite{18} and with numerical studies claiming to support the GS picture \cite{23,24}. How- ever, the stiffness of magnetic field lines. The numerical reso- lution in the parallel direction can, therefore, be reduced.

While there is no discernible inertial range in the parallel spectrum, its perpendicular counterpart exhibits an interval with Iroshnikov-Kraichnan scaling, $E_{k_\perp} \sim k_\perp^{-3/2}$ (Note that due to identical energy scales in Figs. 1 and 2 the absolute difference between Kolmogorov and Iroshnikov-Kraichnan scaling is the same as in Fig. 1). This is in contradiction to the anisotropic cascade phenomenology of Goldreich and Sridhar for strong turbulence predicting $E_{k_\perp} \sim k_\perp^{-5/3}$ \cite{7} and with numerical studies claiming to support the GS picture \cite{23,24}. However, the strength of $b_0$ in these simulations is of the order of the turbulent fluctuations and consequently much weaker than for the anisotropic system considered here.
The geometric coefficients mean component, \( b_0 / b \sim 3 \times 10^2 \). The authors of the aforementioned paper, however, are unsure whether they observe a numerical artefact or physical behavior.

The strongly disparate spectral extent of field-parallel and -perpendicular fluctuations suggests that Alfvén waves propagating along the mean field do not have a significant influence on the perpendicular energy spectrum (in the sense of Goldreich-Sridhar, cf. also \[21\]). Instead, the strong \( b_0 \) constrains turbulence to quasi-two-dimensional field-perpendicular planes as is well known and has been shown for this particular system \[23\].

Another intriguing feature of system II is that \( E_k^R \approx E_k^M \) with only slight dominance of \( E^M \) (cf. Fig. 2) in contrast to the growing excess of spectral magnetic energy with increasing spatial scale for case I. Since both states are dynamically stable against externally imposed perturbations (as has been verified numerically), they presumably represent equilibria between two competing nonlinear processes: field-line deformation by turbulent motions on the spectrally local time scale \( \tau_{NL} \sim \ell / \nu \sim (k^3 E_k^R)^{-1/2} \) leading to magnetic field amplification (turbulent small-scale dynamo) and energy equipartition by shear Alfvén waves with the characteristic time \( \tau_A \sim \ell / b_0 \sim (kb_0)^{-1} \) (Alfvén effect). The conjecture can be verified via the EDQNM closure approximation \[26\] which yields evolution equations for kinetic and magnetic energy spectra \[27\] by including a phenomenological eddy-damping term for third-order moments. The spectral relaxation equation for the signed residual energy, \( E^R = E^M - E^R \), in the case of negligible cross helicity reads \[28\]:

\[
(\partial_t + (\mu + \eta) k^2) E_k^R = \int_{\Delta} dp dq \Theta_{kpq} (T_{res}^R + T_{crs}^R + T_{Dyn}^R) \]

with the spectral energy flux contributions

\[
T_{res}^R = m_{kpq} k^2 E_p E_q + r_{kpq} E_p^2 E_q^2 + s_{kpq} E_p E_q^2 \]

\[
T_{crs}^R = -m_{kpq} p E_p E_k + t_{kpq} E_p E_k \]

\[
T_{Dyn}^R = \frac{\Theta_{kpq}}{k^2} (k^2 E_p E_q - p^2 E_q E_k) \]

The geometric coefficients \( m_{kpq}, r_{kpq}, s_{kpq}, t_{kpq} \), a consequence of the solenoidality constraints \[3\], are given in \[28\]. The ‘\( \Delta \)’ restricts integration to wave vectors \( k, p, q \) which form a triangle, i.e. to a domain in the \( p-q \) plane which is defined by \( q = |p + k| \). The time \( \Theta_{kpq} \) is characteristic of the eddy damping of the nonlinear energy flux involving wave numbers \( k, p, q \). It is defined phenomenologically but its particular form does not play a role in the following arguments.

Local triad interactions with \( k \sim p \sim q \) are dominating the hydrodynamic turbulent energy cascade and lead to Kolmogorov scaling of the associated spectrum (cf., for example, \[24\]). In contrast, the nonlinear interaction of Alfvén waves includes non-local triads with, e.g., \( k \ll p \sim q \). In this case a simplified version of equation \[14\] can be derived:

\[
\partial_t E_k^R = -\Gamma_k k E_k^R \equiv T_{Alf}^R. \tag{5}
\]

with \( \Gamma_k = \frac{4}{3} k \int_0^1 dq \Theta_{kpq} E_k^M \). It is now assumed that the right hand side of \[14\] can be written as \( T_{Alf}^R + T_{Dyn}^R \[14\]. This states that the residual energy is a result of a dynamic equilibrium between turbulent dynamo and Alfvén effect. For stationary conditions and in the inertial range, dimensional analysis of \[14\] and \[5\] yields \( k^3 E_k^2 \sim k^2 E^M E_k^R \) which can be re-written as

\[
E_k^R \sim \sim k E_k^2. \tag{6}
\]

The relaxation time, \( \Theta \), appears as a factor on both sides of the relation and, consequently, drops out. We note that with \( \tau_A \sim (kb_0)^{-1} \), where \( b_0 \) is the mean magnetic field carried by the largest eddies, \( b_0 \approx (E^M)^{1/2} \), and by re-defining \( \tau_{NL} \sim \ell / (v^2 + \ell^2)^{1/2} \sim (k^3 E_k^{-1})^{1/2} \) (for system II all involved quantities are based on field-perpendicular fluctuations) relation \[14\] can be obtained in the physically more instructive form

\[
E_k^R \sim \frac{\tau_A}{\tau_{NL}} E_k. \tag{7}
\]

The modification of \( \tau_{NL} \) is motivated by considering that gradients of the Alfvén speed contribute to nonlinear transfer as much as velocity shear (see, e.g., \[30\]).

For the examined setups relation \[14\] is consistent with the underlying physical idea of dynamical equilibrium between Alfvén and dynamo effect. At small scales with \( k \gg k_0 \) (for system II: \( k_0 \approx k_f \)). Alfvén interaction always dominates the energy exchange since \( \tau_A \ll \tau_{NL} \) (e.g. at \( k = 0.3 f D \) for system I: \( \tau_A \approx 5 \times 10^{-2}, \tau_{NL} \approx 0.2 \), for system II: \( \tau_A \approx 1 \times 10^{-2}, \tau_{NL} \approx 0.1 \)) which results in approximate spectral equipartition of kinetic and magnetic energy. At larger spatial scales the Alfvén effect becomes less efficient in balancing the transformation of kinetic to magnetic energy by the small-scale dynamo with \( \tau_A \approx \tau_{NL} \) (e.g. at \( k = 0.01 f D \) for system I: \( \tau_A \approx 0.9, \tau_{NL} \approx 0.8 \), at \( k = 3 \times 10^{-3} f D \) for system II: \( \tau_A \approx 1.2, \tau_{NL} \approx 0.9 \)) allowing larger deviations from equipartition.

An interesting consequence of \[14\] is that the difference between possible spectral scaling exponents, which is typically small and hard to measure reliably, is enlarged by a factor of two in \( E_k^R \). Even with the limited Reynolds numbers in today’s simulations such a magnified difference is clearly observable (e.g. dash-dotted lines in Figs. \[1\]and\[3\]). For system I with Kolmogorov scaling, \( E_k \sim k^{-5/3} \) (Fig. \[1\]), relation \[14\] predicts \( E_k^R \sim k^{-7/3} \) in agreement with the simulation (Fig. \[3\]). In the case of Iroshnikov-Kraichnan behavior, \( E_k^R \sim k^{-3/2} \) as realized in system II (Fig. \[2\]), \( E_k^R \sim k^{-5/2} \) is obtained. This result is confirmed by the residual energy spectrum shown in
In summary, based on the structure of the EDQNM closure equations for incompressible MHD a model of the nonlinear spectral interplay between kinetic and magnetic energy is formulated. Throughout the inertial range a quasi-equilibrium of turbulent small-scale dynamo and Alfvén effect leads to the relation, \( E_k \sim kE_k^2 \), linking total and residual energy spectra, in particular \( E_k^R \sim k^{-7/3} \) for \( E_k \sim k^{-5/3} \) and \( E_k^R \sim k^{-2} \) for \( E_k \sim k^{-3/2} \). Both predictions are confirmed by high-resolution direct numerical simulations, limiting the possible validity of the Goldreich-Sridhar phenomenology to MHD turbulence with moderate mean magnetic fields.

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[1] S. Ortolani and D. D. Schnack, *Magnetohydrodynamics of Plasma Relaxation* (World Scientific, Singapore, 1993).
[2] Y. B. Zeldovich, A. A. Ruzmaikin, and D. D. Sokoloff, *Magnetic Fields In Astrophysics* (Gordon and Breach Science Publishers, New York, 1983).
[3] D. Biskamp, *Magnetohydrodynamic Turbulence* (Cambridge University Press, Cambridge, 2003).
[4] A. N. Kolmogorov, Proceedings of the Royal Society A 434, 9 (1991), [Dokl. Akad. Nauk SSSR, 30(4), 1941].
[5] P. S. Iroshnikov, Soviet Astronomy 7, 566 (1964). [Astron. Zh., 40:742, 1963].
[6] R. H. Kraichnan, Physics of Fluids 8, 1385 (1965).
[7] P. Goldreich and S. Sridhar, Astrophysical Journal 485, 680 (1997).
[8] S. Sridhar and P. Goldreich, Astrophysical Journal 432, 612 (1994).
[9] W.-C. Müller and D. Biskamp, in *Turbulence and Magnetic Fields in Astrophysics*, edited by E. Falgarone and T. Passot (Springer Berlin, 2002), vol. 614 of Lecture Notes in Physics, pp. 3–27.
[10] R. Grappin, A. Pouquet, and J. Léorat, Astronomy and Astrophysics 126, 51 (1983).
[11] A. Vincent and M. Meneguzzi, Journal of Fluid Mechanics 225, 1 (1991).
[12] D. Biskamp and W.-C. Müller, Physical Review Letters 83, 2195 (1999).
[13] W.-C. Müller and D. Biskamp, Physical Review Letters 84, 475 (2000).
[14] N. E. L. Haugen, A. Brandenburg, and W. Dobler, Physical Review E 70, 016308 (2004).
[15] R. J. Leamon, C. W. Smith, N. F. Ness, W. H. Matthaeus, and H. K. Wong, Journal of Geophysical Research 103, 4775 (1998).
[16] M. L. Goldstein and D. A. Roberts, Physics of Plasmas 6, 4154 (1999).
[17] Y. Kanda, T. Ishihara, M. Yokokawa, K. Itakura, and A. Uno, Physics of Fluids 15, L21 (2003).
[18] W.-C. Müller, D. Biskamp, and R. Grappin, Physical Review E 67, 066302 (2003).
[19] R. Grappin, Physics of Fluids 29, 2433 (1986).
[20] J. V. Shebalin, W. H. Matthaeus, and D. Montgomery, Journal of Plasma Physics 29, 525 (1983).
[21] R. M. Kinney and J. C. McWilliams, Physical Review E 57, 7111 (1998).
[22] S. Oughton, W. H. Matthaeus, and S. Ghosh, Physics of Plasmas 5, 4235 (1998).
[23] J. Cho, A. Lazarian, and E. T. Vishniac, Astrophysical Journal 564, 291 (2002).
[24] J. Cho and E. T. Vishniac, Astrophysical Journal 539, 273 (2000).
[25] J. Maron and P. Goldreich, Astrophysical Journal 554, 1175 (2001).
[26] S. A. Orszag, Journal of Fluid Mechanics 41, 363 (1970).
[27] A. Pouquet, U. Frisch, and J. Léorat, Journal of Fluid Mechanics 77, 321 (1976).
[28] R. Grappin, U. Frisch, J. Léorat und A. Pouquet, Astronomy and Astrophysics 105, 6 (1982).
The other definition of $E^R_k$ involving the modulus operator avoids case differentiations since the applied dimensional analysis is unable to predict the sign of $E^M_k$. However, the physical picture underlying Eqs. (6) and (7) implies $E^M_k \geq E^R_k$ as it expresses an equilibrium between magnetic energy amplification and equipartition of $E^R_k$ and $E^M_k$. 