Higher order terms of spectral heat content for killed subordinate and subordinate killed Brownian motions related to symmetric $\alpha$-stable processes in $\mathbb{R}$

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May 27, 2020

Abstract

We investigate the 3rd term of spectral heat content for killed subordinate and subordinate killed Brownian motions on a bounded open interval $D = (a, b)$ in a real line when the underlying subordinators are stable subordinators with index $\alpha \in (1, 2)$ or $\alpha = 1$. We prove that in the 3rd term of spectral heat content, one can observe the length $b - a$ of the interval $D$.

1 Introduction

The classical spectral heat content $Q^{(2)}_D(t)$ measures a total heat that remains on a domain $D$ with Dirichlet boundary condition and unit initial heat. The spectral heat content can be written in probabilistic terms and it can be defined as

$$Q^{(2)}_D(t) = \int_D \mathbb{P}_x(\tau^{(2)}_D > t) dx,$$

where $\tau^{(2)}_D = \inf\{t > 0 : W_t \notin D\}$ is the first exit time of $D$ of Brownian motions $W$. When the Brownian motions are replaced by other Lévy processes, the corresponding quantity is called a spectral heat content for the Lévy processes. It was recently studied intensively in [1, 2, 9].

One of the most commonly used jump type Lévy processes is symmetric stable processes of index $\alpha \in (0, 2]$. When $\alpha = 2$, they are Brownian motions whose sample paths are continuous with the characteristic exponent $\mathbb{E}[e^{i\xi W_t}] = e^{-t\xi^2}$. When $\alpha \in (0, 2)$ they are pure-jump processes. Stable processes are in fact a special case of subordinate Brownian motions which are time-changed Brownian motions whose time change is given by stable subordinators $S^{(\alpha/2)}_t$ with Laplace exponent

$$\mathbb{E}[e^{-\lambda S^{(\alpha/2)}_t}] = e^{-t\lambda^{\alpha/2}}, \quad \lambda > 0.$$

When one studies the spectral heat content of subordinate Brownian motions, one needs to consider a time-change by a subordinator and killing the process when it first exits the domain under consideration. When we first do time-change and kill the processes, it is called killed subordinate Brownian motions and when we first kill the Brownian motions when they first exit the domain and
do time-change into the killed Brownian motions, it is called *subordinate killed Brownian motions*. These two processes are closely related, and sometimes understanding spectral heat content of one process help understand the other. The spectral heat content for killed subordinate Brownian motions when the subordinators are stable subordinators (killed stable processes) were studied in [1, 2] and the spectral heat content for subordinate killed Brownian motions were studied in [10]. In those papers, the authors found the asymptotic expansion of the spectral heat content up to the 2nd terms.

The purpose of this paper is to refine these results and find the 3rd terms of spectral heat content for subordinate killed Brownian motions and killed subordinate Brownian motions in a bounded open interval $D = (a, b) \subset \mathbb{R}$ when the subordinators are stable subordinators for $\alpha \in [1, 2)$. The main results of this paper are followings (see Section 2 for notations):

**Theorem 1.1** Let $D = (a, b) \subset \mathbb{R}^1$ with $b - a < \infty$.

(1) Let $\alpha \in (1, 2)$. Then,

$$|D| - Q_D^{(\alpha)}(t) = \mathbb{E}[X_1^{(\alpha)}]|\partial D|t^{1/\alpha} - \frac{2^\alpha \Gamma(\frac{1+\alpha}{2})}{(\alpha - 1)\pi^{1/2}\Gamma(1 - \frac{\alpha}{2})} (b - a)^{-1}\frac{1}{2}t + o(t),$$

as $t \to 0$, where $|\partial D| = 2$.

(2) Let $\alpha = 1$. Then,

$$|D| - Q_D^{(1)}(t) = \frac{1}{\pi} |\partial D|t \ln\left(\frac{1}{t}\right) - \mathbb{E}[X_1] |\partial D| t^{1/\alpha} + \frac{\ln(b - a)}{\pi} + \int_1^{\infty} \left(\mathbb{P}(X_1 > u) - \frac{1}{\pi u}\right) du t + o(t),$$

as $t \to 0$, where $|\partial D| = 2$.

**Theorem 1.2** Let $D = (a, b) \subset \mathbb{R}^1$ with $b - a < \infty$.

(1) Let $\alpha \in (1, 2)$. Then,

$$|D| - \tilde{Q}_D^{(\alpha)}(t) = \mathbb{E}[\overline{W}_1^{(1/2)}]|\partial D|t^{1/\alpha} - \frac{2^\alpha \int_0^{\infty} \mathbb{P}(\overline{W}_1 \geq u) u^{\alpha - 1} du}{(\alpha - 1)\Gamma(1 - \frac{\alpha}{2}) (b - a)^{-1}}t + o(t),$$

as $t \to 0$, where $|\partial D| = 2$.

(2) Let $\alpha = 1$. Then,

$$|D| - \tilde{Q}_D^{(1)}(t) = \frac{2}{\pi} |\partial D|t \ln\left(\frac{1}{t}\right) - \mathbb{E}[\overline{W}_1^{(1/2)}] |\partial D| t^{1/\alpha} + \frac{2 \ln(b - a)}{\pi} + \int_1^{\infty} \mathbb{P}(\overline{W}_1^{(1/2)} > u) - \frac{2}{\pi u} du t + o(t),$$

as $t \to 0$, where $|\partial D| = 2$.  

2
Studying higher order terms is not only an interesting question in itself, but we could also observe that there are some different patterns in the asymptotic expansion of spectral heat content for Brownian motions and Lévy processes by studying higher order terms. For Brownian motions, it is well-known that for smooth domains $D$ the spectral heat content has an asymptotic expansion of the form $|D| - Q_1^2(t) \sim \sum_{n=1}^{\infty} a_n t^{\frac{2}{n}}$ where $a_n$ has some geometric information about the domain $D$ such as perimeter or mean curvature. Hence, it is natural to conjecture that at least when $\alpha \in (1, 2)$ the spectral heat content for stable processes is of the form $|D| - Q_1^{(\alpha)}(t) \sim \sum_{n=1}^{\infty} b_n t^{\frac{2}{\alpha}}$. Theorems 1.1 and 1.2 say this is not the case and the asymptotic expansion involves terms that cannot be written as $t^{\frac{2}{\alpha}}$. Also, we observe that the 3rd term involves the length $b-a$ of the underlying interval $D = (a, b)$, hence one can determine the domain $D$ uniquely up to locations when $D$ is a bounded open interval in $\mathbb{R}^1$.

In this paper we focus on spectral heat content in dimension 1. The geometry of open intervals in $\mathbb{R}^1$ is simple enough to allow detailed computations possible and this could be helpful to extend results of this paper into more general settings such as spectral heat content in higher dimensions or with respect to more general processes. These problems will be studied in forthcoming projects.

In order to prove the first part of Theorem 1.1 ($\alpha \in (1, 2)$) we analyze the difference $|D| - Q_1^{(\alpha)}(t) - \mathbb{E}[(X_1^{(\alpha)}) \cdot \partial D]|^{1/\alpha}$ directly and prove that it is of order $t$. Hence, the proof is quite straightforward in this case. For the second part of Theorem 1.1 ($\alpha = 1$) the computation becomes delicate because of the log term $t \ln(1/t)$. We utilize the exact form of the density of supremum process $X_t^{(1)} = \sup_{s \leq t} X_s^{(1)}$ in [8], and compute the difference $\mathbb{P}(X_1^{(1)} > u) - \frac{1}{\pi u}$ for large $u$ and prove that main terms of order $t \ln(1/t)$ cancel out and the remaining terms are of order $t$. In order to prove Theorem 1.2 we follow a similar path as Theorem 1.1. For the first part of Theorem 1.2 ($\alpha \in (1, 2)$) we reprove [10, Theorem 1.1] when $D = (a, b)$ and $\alpha \in (1, 2)$ using a probabilistic argument in Theorem 1.3 which is similar to [2]. We would like to mention that in Theorem 1.3 we express the 2nd coefficient of $|D| - Q_1^{(\alpha)}(t)$ in a probabilistic term $\mathbb{E}[W_{S^{(\alpha/2)}_t}]$, which is more natural than previously known (compare it with [10, Theorem 1.1]). In order to prove the second part of Theorem 1.2 ($\alpha = 1$), we establish the tail probability $\mathbb{P}(W_{S^{(\alpha/2)}_t} > u)$ for $u > 1$ in Proposition 1.7 which is an amusingly simple expression. Once having established Proposition 1.7 it is straightforward to compute the difference $\mathbb{P}(W_{S^{(\alpha/2)}_t} > u) - \frac{2}{\pi u}$ for large $u$ and we prove that main terms of order $t \ln(1/t)$ cancel out again and the remaining terms are of order $t$.

The organization of this paper is as follows. In Section 2 we introduce notations and recall some preliminary facts. In Section 3 we study the spectral heat content for killed subordinate Brownian motions and prove Theorem 1.1. The first part of Theorem 1.1 is proved in the subsection 3.1 and the second part of Theorem 1.1 is proved in the subsection 3.2. In Section 4 we study the spectral heat content for subordinate killed Brownian motions and prove first and second parts of Theorem 1.2 in subsections 4.1 and 4.2, respectively.
2 Preliminaries

In this section, we introduce notations that will be used in later sections. All stochastic processes and domains will be in one dimension $\mathbb{R}^1$ in this paper.

Let $W_t$ be Brownian motions in $\mathbb{R}^1$. The density of $W_t$ is $p(t, x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ and the characteristic function is given by

$$\mathbb{E}[e^{i\xi W_t}] = e^{-t\xi^2}, \quad \xi \in \mathbb{R}. $$

The supremum process $W_t$ is defined by $W_t = \sup_{s \leq t} W_s$. It is well-known that $|W_t|$ and $W_t$ are equal in distribution.

Let $S_{t}^{(\alpha/2)}$ be stable subordinators whose Laplace exponent is $\mathbb{E}[e^{-\lambda S_{t}^{(\alpha/2)}}] = e^{-t\lambda^{\alpha/2}}$, $\lambda \in \mathbb{R}$.

By doing an elementary integral, it is easy to check that

$$\lambda^{\alpha/2} = \frac{\alpha/2}{\Gamma(1 - \frac{\alpha}{2})} \int_0^\infty (1 - e^{-\lambda t})t^{-1-\frac{\alpha}{2}}dt, \quad \lambda > 0, \alpha \in (0, 2).$$

This shows that the Lévy density $j_{SS}(u)$ for $S_{t}^{(\alpha/2)}$ is

$$j_{SS}(u) = \frac{\alpha/2}{\Gamma(1 - \frac{\alpha}{2})} u^{-\frac{\alpha}{2}}, \quad u > 0. \quad (2.2)$$

It follows from [10, Equation (2.3)] or [7, Equation (18)] that the density $g^{(\alpha/2)}(1, x)$ of $S_{t}^{(\alpha/2)}$ exists and is given by

$$g^{(\alpha/2)}(1, x) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma(1 + \frac{\alpha n}{2})}{n!} \sin\left(\frac{\pi \alpha n}{2}\right) x^{-\frac{\alpha n}{2} - 1}, \quad x > 0. \quad (2.3)$$

It follows from the scaling property (2.1) that the transition density $g^{(\alpha/2)}(t, x)$ is equal to $t^{-2/\alpha} g^{(\alpha/2)}(1, \frac{x}{t^{2/\alpha}})$.

Now we define subordinate Brownian motions. Let $W_t$ and $S_{t}^{(\alpha/2)}$ be Brownian motions and stable subordinators defined on some probability space. Assume that they are independent. Then, the subordinate Brownian motions by subordinator $S_{t}^{(\alpha/2)}$ are the following time-changed Brownian motions:

$$X_{t}^{(\alpha)} := W_{S_{t}^{(\alpha/2)}}.$$ 

By conditioning on $S_{t}^{(\alpha/2)}$ one can observe that the characteristic function of time changed process $X_{t}^{(\alpha)} := W_{S_{t}^{(\alpha/2)}}$ is given by

$$\mathbb{E}[e^{i\xi X_{t}^{(\alpha)}}] = \mathbb{E}[e^{i\xi W_{S_{t}^{(\alpha/2)}}}] = \mathbb{E}[e^{-S_{t}^{(\alpha/2)}\xi^2}] = e^{-t\xi^2}, \quad \xi \in \mathbb{R},$$

4
and this shows that \( X_t^{(\alpha)} \) are symmetric stable processes of index \( \alpha \). The Lévy density \( j^{SSP}(x) \) of \( X_t^{(\alpha)} \) is given by (see [4, Equation (1.3) and (1.22)])

\[
j^{SSP}(x) = \frac{A_{1,\alpha}}{|x|^{1+\alpha}}, \quad A_{1,\alpha} = \frac{\alpha^{2-\alpha} \Gamma(\frac{1+\alpha}{2})}{\pi^{1/2} \Gamma(1-\frac{\alpha}{2})}.
\]

(2.4)

Let \( D \) be an open set in \( \mathbb{R}^1 \) and define \( \tau_D^{(\alpha)} = \inf\{t > 0 : X_t^{(\alpha)} \notin D\} \) be the first exit time. The killed processes \( X^{(\alpha),D} \) are defined by

\[
X_t^{(\alpha),D} = \begin{cases} X_t^{(\alpha)} & \text{if } t < \tau_D^{(\alpha)}, \\ \partial & \text{if } t \geq \tau_D^{(\alpha)}, \end{cases}
\]

where \( \partial \) is a cemetery state. The process \( X_t^{(\alpha),D} \) will be called \textit{killed subordinate Brownian motions} (by stable subordinators \( S_t^{(\alpha/2)} \)) since we first subordinate (time-change) Brownian motions, then kill the process when they exit the domain. We can exchange the order of time-change and killing and the corresponding process will be called \textit{subordinate killed Brownian motions} (by stable subordinators \( S_t^{(\alpha/2)} \)). More precisely, let \( \tau_D^{(2)} = \inf\{t > 0 : W_t \notin D\} \) be the first exit time of Brownian motions \( W_t \). Define killed Brownian motions \( W_t^D \) as

\[
W_t^D = \begin{cases} W_t & \text{if } t < \tau_D^{(2)}, \\ \partial & \text{if } t \geq \tau_D^{(2)}. \end{cases}
\]

Now the killed subordinate Brownian motions \( (W^D)_{S_t^{(\alpha/2)}} \) are defined by

\[
(W^D)_{S_t^{(\alpha/2)}} = \begin{cases} W_{S_t^{(\alpha/2)}} & \text{if } S_t^{(\alpha/2)} < \tau_D^{(2)}, \\ \partial & \text{if } S_t^{(\alpha/2)} \geq \tau_D^{(2)}. \end{cases}
\]

Let \( \zeta \) be the life time of \( (W^D)_{S_t^{(\alpha/2)}} \). Then, we have

\[
\{\zeta > t\} = \{\tau_D^{(2)} > S_t^{(\alpha/2)}\}.
\]

Clearly, we have \( \{\zeta > t\} \subset \{\tau_D^{(\alpha)} > t\} \) and the inclusion can be strict.

We define the supremum processes \( \overline{X}_t \) as

\[
\overline{X}_t^{(\alpha)} := \sup_{u \leq t} X_u^{(\alpha)} = \sup_{u \leq t} W_{S_t^{(\alpha/2)}}.
\]

(2.5)

It is noteworthy to mention that even though two expressions \( X_t^{(\alpha)} \) and \( W_{S_t^{(\alpha/2)}} \) mean the same object, stable processes of index \( \alpha \), the supremum notations \( \overline{X}_t^{(\alpha)} \) and \( \overline{W}_{S_t^{(\alpha/2)}} \) are different. The supremum processes \( \overline{W}_{S_t^{(\alpha/2)}} \) are defined by

\[
\overline{W}_{S_t^{(\alpha/2)}} = \sup_{u \leq S_t^{(\alpha/2)}} W_u.
\]

(2.6)
Note that two expressions (2.5) and (2.6) are different and we always have $\tilde{X}_t^{(\alpha)} \leq \tilde{W}_{t^{(\alpha)/2}}$. The infimum processes $\tilde{W}_t$, $X_t^{(\alpha)}$, and $\tilde{W}_{t^{(\alpha)/2}}$ are defined in a similar way with the supremum being replaced by the infimum.

Finally, we define spectral heat content $Q_D^{(\alpha)}(t)$ and $\tilde{Q}_D^{(\alpha)}(t)$ for killed subordinate Brownian motions and subordinate killed Brownian motions, respectively. We define

$$Q_D^{(\alpha)}(t) := \int_D \mathbb{P}_x (\tilde{\tau}_D^{(\alpha)} > t) dx,$$

and

$$\tilde{Q}_D^{(\alpha)}(t) := \int_D \mathbb{P}_x (\zeta > t) dx = \int_D \mathbb{P}_x (\tau_D^{(2)} > S_t^{(\alpha)/2}) dx.$$

3 Spectral heat content for killed subordinate Brownian motions

3.1 $\alpha \in (1, 2)$

We start with a simple lemma.

**Lemma 3.1** Suppose that $1 < \alpha < 2$. Then

$$\lim_{t \to 0} \frac{t^{1/\alpha} \int_{(b-a)t^{-1/\alpha}}^{\infty} \mathbb{P}(X_t^{(\alpha)} > u) du}{t^{1-1/\alpha}} = \frac{2^{\alpha-1}\Gamma(\frac{1+\alpha}{2})}{(\alpha-1)\pi^{1/2}(1 - \frac{\alpha}{2})(b-a)^{\alpha-1}}.$$

**Proof.** It follows from L’Hôpital’s rule, the scaling property of $X_t^{(\alpha)}$, \[3\] Proposition VIII.1 4], \[11\] Corollary 8.9], and (2.1)

$$\lim_{t \to 0} \frac{\int_{(b-a)t^{-1/\alpha}}^{\infty} \mathbb{P}(X_t^{(\alpha)} > u) du}{t^{1-1/\alpha}} = \lim_{t \to 0} \frac{(b-a) \mathbb{P}(X_t^{(\alpha)} > (b-a)t^{-1/\alpha})}{t^{1/\alpha}} = \lim_{t \to 0} \frac{(b-a) \mathbb{P}(X_t^{(\alpha)} > (b-a)t^{-1/\alpha})}{t^{1/\alpha}} = \lim_{t \to 0} \frac{(b-a) \mathbb{P}(X_t^{(\alpha)} > (b-a)t^{-1/\alpha})}{t^{1/\alpha}} = \text{2}^{\alpha-1}\Gamma(\frac{1+\alpha}{2})/(\alpha-1)\pi^{1/2}(1 - \frac{\alpha}{2})(b-a)^{\alpha-1}.$$  

\[\square\]

Let $p^{(\alpha)}(t, x)$ be the transition density (heat kernel) for $X_t^{(\alpha)}$. Note that the following heat kernel estimate is well-known (see \[5\]);

$$C^{-1}(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}) \leq p^{(\alpha)}(t, x) \leq C(t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}) \quad (3.1)$$

for some constant $C > 1$.  

6
Lemma 3.2 Let $\alpha \in (1, 2)$. Then, there exists $t_0 > 0$ such that for any $t \in (0, t_0]$}

$$
\int_a^b \mathbb{P}_x(X_t^{(a)} > b \text{ and } X_t^{(a)} < a)dx \leq \frac{ct^{1+\frac{1}{\alpha}}}{(b-a)^\alpha} \mathbb{E}[X_1^{(a)}],
$$

for some constant $c > 0$.

**Proof.** Define

$$
\tau := \inf\{t : X_t^{(a)} > b \text{ or } X_t^{(a)} < a\}.
$$

Clearly $\tau$ is a stopping time with respect to the natural filtration $\mathcal{F}_t$ and we have

$$
\mathbb{P}_x(X_t^{(a)} > b \text{ and } X_t^{(a)} < a)
\leq \mathbb{P}_x(\tau < t, \sup_{\tau \leq s \leq t} X_s^{(a)} - X_{\tau}^{(a)} > b - a \text{ or } \inf_{\tau \leq s \leq t} X_s^{(a)} - X_{\tau}^{(a)} < -(b - a))
\leq \mathbb{P}_x(\tau < t, \sup_{\tau \leq s \leq t} X_s^{(a)} - X_t^{(a)} > b - a) + \mathbb{P}_x(\tau < t, \inf_{\tau \leq s \leq t} X_s^{(a)} - X_t^{(a)} \leq -(b - a)). \quad (3.2)
$$

It follows from the strong Markov property at $\tau$ and independent increment of the Lévy processes $X_t^{(a)}$

$$
\mathbb{P}_x(\tau < t, \sup_{\tau \leq s \leq t} X_s^{(a)} - X_t^{(a)} > b - a)
= \mathbb{P}_x\left(\tau < t, \mathbb{P}_{X_t^{(a)}}\left(\sup_{s \leq t-\tau} X_s^{(a)} > b - a\right)\right)
\leq \mathbb{P}_x(\tau < t)\mathbb{P}(\sup_{s \leq t} X_s^{(a)} > b - a)
\leq \mathbb{P}_x(X_t^{(a)} > b \text{ or } X_t^{(a)} < a)\mathbb{P}(\sup_{s \leq t} X_s^{(a)} > b - a)
= \left(\mathbb{P}_x(X_t^{(a)} > b) + \mathbb{P}_x(X_t^{(a)} < a) - \mathbb{P}_x\left(X_t^{(a)} > b \text{ and } X_t^{(a)} < a\right)\right) \mathbb{P}(\sup_{s \leq t} X_s^{(a)} > b - a)
\leq 2\mathbb{P}_x(X_t^{(a)} > b)\mathbb{P}(\sup_{s \leq t} X_s^{(a)} > b - a).
$$

From [3] Proposition VIII.1 4] and [3] there exists $t_0 > 0$ such that for any $t \leq t_0$

$$
\mathbb{P}(\sup_{s \leq t} X_s^{(a)} > b - a) = \mathbb{P}(X_1^{(a)} > \frac{b-a}{t^{1/\alpha}}) \leq 2\mathbb{P}(X_1^{(a)} > \frac{b-a}{t^{1/\alpha}}) \leq \frac{ct}{(b-a)^\alpha}.
$$

Hence by the scaling property we have

$$
\int_a^b \mathbb{P}_x(\tau < t, \sup_{\tau \leq s \leq t} X_s^{(a)} - X_t^{(a)} > b - a)dx \leq \frac{2ct}{(b-a)^\alpha} \int_a^b \mathbb{P}_x(X_t^{(a)} > b)dx
\leq \frac{2ct^{1+\frac{1}{\alpha}}}{(b-a)^\alpha} \int_a^b \mathbb{P}(X_1^{(a)} > u)du \leq \frac{2ct^{1+\frac{1}{\alpha}}}{(b-a)^\alpha} \mathbb{E}[X_1^{(a)}].
$$

The second expression in [3.2] can be handled in a similar way and this establishes the claim of this lemma. \hfill \Box
Now we are ready to prove the first part of Theorem 1.1.

**Proof of (1.1)**

By the scaling property of $X_t^{(\alpha)}$ we have

$$|D| - Q_D^{(\alpha)}(t) - E[X_1^{(\alpha)}]|\partial D|^{1/\alpha}$$

$$= \int_a^b P(\tau_D^{(\alpha)} \leq t)dx - E[X_1^{(\alpha)}]|\partial D|^{1/\alpha}$$

$$= 2^{t^{1/\alpha}} \int_0^{b-a} P(x) > u)du - \int_a^b P(x) > b \text{ and } X_t^{(\alpha)} < a)dx - 2t^{1/\alpha} \int_0^\infty P(x) > u)du$$

$$= 2^{t^{1/\alpha}} \int_0^{b-a} P(x) > u)du - \int_a^b P(x) > b \text{ and } X_t^{(\alpha)} < a)dx. \quad (3.3)$$

Now the conclusion follows immediately from Lemmas 3.1 and 3.2 \[\square\]

### 3.2 $\alpha = 1$

In this subsection, we study the asymptotic behavior of the spectral heat content for killed subordinate Brownian motions (killed stable processes) when $\alpha = 1$. We start with a lemma that is similar to Lemma 3.2.

**Lemma 3.3**

$$\int_a^b P_x(X_t^{(1)} > b \text{ and } X_t^{(1)} < a)dx = O(t^2 \ln(1/t)) \quad \text{as } t \to 0.$$ 

**Proof.** The proof is similar to the proof of Lemma 3.2 and we only explain the difference. As in the proof of Lemma 3.2 we have

$$\int_a^b P_x(\tau < t, \sup_{\tau \leq s \leq t} X_s^{(1)} > X_t^{(1)} > b - a)dx$$

$$\leq \frac{ct}{(b-a)^\alpha} \int_a^b P_x(X_t^{(1)} > b)dx \leq \frac{ct^2}{(b-a)^\alpha} \int_0^{b-a} P(X_1^{(1)} > u)du = O(t^2 \ln(1/t)),$$

where the last part comes from [2, Proposition 4.3.(i)]. \[\square\]

There was an error in the paragraph right above [2, Remark 5.1]. The density for $X_1^{(1)}$ exists and it is given by (see [8])

$$f(x) = \frac{1}{\pi x^{1/2}(1 + x^2)^{3/4}} \exp \left( -\frac{1}{\pi} \int_0^{1/x} \frac{\ln v}{1 + v^2} dv \right), \quad x > 0. \quad (3.4)$$

We note that there is also a minor error in the exact expression of $f(x)$ in [8] and the upper bound of the integral should be written as $\frac{1}{x}$ instead of $x$.

Now we are ready to prove the second part of Theorem 1.1.
Proof of (1.2)

Note that as in (3.3) we have

\[ |D| - Q^1_D(t) = \int_a^b \mathbb{P}(\tau_D^1 \leq t) dx = 2t \int_0^{b-a} \mathbb{P}(X^1_1 > u) du - \int_a^b \mathbb{P}(X^1_t > b) \text{ and } X^1_t < a) dx. \]

It follows from Lemma 3.3

\[ \lim_{t \to 0} \frac{\int_0^b \mathbb{P}(X^1_t > b) \text{ and } X^1_t < a) dx}{t} = 0. \]

Note that from [2, Proposition 4.3.(i)] we have

\[ \lim_{t \to 0} \frac{2t \int_0^{b-a} \mathbb{P}(X^1_1 > u) du}{t \ln(1/t)} = \frac{2}{\pi}. \]

We will show that

\[ \lim_{t \to 0} \frac{t \int_0^{b-a} \mathbb{P}(X^1_1 > u) du}{t \ln(1/t)} = \lim_{t \to 0} \int_0^{b-a} \mathbb{P}(X^1_1 > u) du - \frac{\ln(1/t)}{\pi}. \]

Note that

\[ \int_0^{b-a} \mathbb{P}(X^1_1 > u) du - \frac{\ln(1/t)}{\pi} \]

\[ = \int_0^1 \mathbb{P}(X^1_1 > u) du + \frac{\ln(b - a)}{\pi} + \int_1^\infty \left( \mathbb{P}(X^1_1 > u) - \frac{1}{\pi u} \right) du. \]

It follows from (3.4) and the change of variable we have \( y = \frac{1}{u} \) we have

\[ \mathbb{P}(X^1_1 > u) = \int_u^\infty \frac{1}{\pi x^{1/2}(1 + x^2)^{3/4}} \exp \left( \frac{1}{\pi} \int_x^\infty \ln y \frac{1}{1 + y^2} dy \right) dx. \]

We will show that for all sufficiently large \( u \) we have

\[ \left| \mathbb{P}(X^1_1 > u) - \frac{1}{\pi u} \right| \leq 4 \frac{\ln u}{u^2}, \]

so that by the Lebesgue dominated convergence theorem

\[ \lim_{t \to 0} \int_1^{b-a} \left( \mathbb{P}(X^1_1 > u) - \frac{1}{\pi u} \right) du = \int_1^\infty \left( \mathbb{P}(X^1_1 > u) - \frac{1}{\pi u} \right) du. \]
For \( u \geq 1 \) and \( x \geq u \) we have \( \exp \left( \frac{1}{\pi} \int_{x}^{\infty} \frac{\ln y}{1+y^2} dy \right) \geq e^0 = 1 \) and

\[
\int_{u}^{\infty} \frac{1}{\pi x^{1/2}(1+x^2)^{3/4}} \exp \left( \frac{1}{\pi} \int_{x}^{\infty} \frac{\ln y}{1+y^2} dy \right) dx \\
\geq \int_{u}^{\infty} \frac{1}{\pi x^{1/2}(1+x^2)^{3/4}} dx \geq \int_{u}^{\infty} \frac{1}{\pi(1+x^2)} dx \\
= \frac{1}{\pi} \left( \frac{\pi}{2} - \arctan u \right) = \frac{1}{\pi} \arctan(1/u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi(2n+1)} \frac{1}{u^{2n+1}},
\]

where we used an elementary identity \( \arctan u + \arctan(1/u) = \frac{\pi}{2} \). Hence, there exists \( U_1 > 0 \) such that

\[
\mathbb{P}(X^{(1)}_1 > u) - \frac{1}{\pi u} \geq -\frac{1}{2\pi} \frac{1}{u^3}, \text{ for all } u \geq U_1. \tag{3.6}
\]

Now we focus on establishing the upper bound. From Karamata’s Theorem ([1, Theorem 1.5.11 (ii)]) we have

\[
\int_{x}^{\infty} \frac{\ln y}{1+y^2} dy = \int_{x}^{\infty} y^{-2} y^2 \frac{\ln y}{1+y^2} dy \sim -\frac{x \ln x}{1+x^2} \text{ as } x \to \infty.
\]

Hence, there exists \( U_2 > 0 \) such that for all \( x \geq u \geq U_2 \) we have

\[
\int_{x}^{\infty} \frac{\ln y}{1+y^2} dy \leq \frac{2x \ln x}{1+x^2}. \tag{3.7}
\]

By an elementary calculus we see that \( e^u \leq 1 + 2u \) for all \( 0 \leq u \leq \ln 2 \), and take \( U_3 \) so that

\[
\frac{2}{\pi} \frac{x \ln x}{1+x^2} \leq \ln 2 \text{ for all } x \geq u \geq U_3. \tag{3.8}
\]

It follows from (3.7) and (3.8) for \( u \geq \max(U_2, U_3) \) we have

\[
\mathbb{P}(X^{(1)}_1 > u) - \frac{1}{\pi u} \leq \int_{u}^{\infty} \frac{1}{\pi x^{1/2}(1+x^2)^{3/4}} \exp \left( \frac{2}{\pi} \frac{x \ln x}{1+x^2} \right) dx - \frac{1}{\pi u} \\
\leq \int_{u}^{\infty} \frac{1}{\pi x^{1/2}(1+x^2)^{3/4}} \frac{4}{\pi} \frac{x \ln x}{1+x^2} dx - \frac{1}{\pi u} \\
\leq \int_{u}^{\infty} \frac{1}{\pi x^2} dx + \int_{u}^{\infty} \frac{1}{\pi x^{1/2}(1+x^2)^{3/4}} \frac{4}{\pi} \frac{x \ln x}{1+x^2} dx - \frac{1}{\pi u} \\
= \frac{4}{\pi^2} \int_{u}^{\infty} \frac{x^{1/2} \ln x}{(1+x^2)^{7/4}} dx \leq \frac{4}{\pi^2} \int_{u}^{\infty} \frac{\ln x}{x^3} dx.
\]

Again, it follows from [1, Theorem 1.5.11 (ii)] we have

\[
\int_{u}^{\infty} \frac{\ln x}{x^3} dx \sim \frac{2}{u^2} \ln u \text{ as } u \to \infty,
\]
and we can take a constant $U_4 \geq \max(U_2, U_3)$ such that $\int_u^\infty \frac{\ln x}{x^3} dx \leq \frac{\ln u}{u^2}$ for all $u \geq U_4$. Hence, for $u \geq U_4$
\[ \mathbb{P}(X_1(1)^4 > u) - \frac{1}{\pi u} \leq \frac{4 \ln u}{\pi^2 u^2}. \] (3.9)
Hence, it follows from (3.6) and (3.9) there exists $U_5 \geq \max(U_1, U_4)$ such that (3.5) holds for all $u \geq U_5$.
\[ \square \]

4 Spectral heat content for subordinate killed Brownian motions

In this section we study the 3rd term of the spectral heat cont ent for subordinate killed Brownian motions and prove Theorem 1.2.

4.1 $\alpha \in (1, 2)$

Lemma 4.1 For any $\alpha \in (0, 2)$, there exists a constant $c = c(\alpha) > 0$ such that
\[ \mathbb{P}(\overline{W}_{\xi_t^{(\alpha)}} > b - a) \leq ct \quad \text{for all } t > 0. \]

Proof. By the scaling property and the fact that $|W_t|$ is equal to $\overline{W}_t$ in distribution, we have
\[ \mathbb{P}(\sup_{u \leq \xi_t^{(\alpha)}} W_u > b - a) = \mathbb{P}(|W_{\xi_t^{(\alpha)}}| > b - a) = \mathbb{P}(S_t^{(\alpha/2)})^{1/2}|W_1 > b - a) \]
\[ = \mathbb{P}(t^{1/\alpha}(S_1^{(\alpha/2)})^{1/2} > \frac{b - a}{|W_1|}) = \mathbb{P}(S_1^{(\alpha/2)} > \frac{(b - a)^2}{t^{2/\alpha} |W_1|^2}). \]
Hence, we have
\[ \mathbb{P}(\sup_{u \leq \xi_t^{(\alpha)}} W_u > b - a) = 2 \int_0^\infty \mathbb{P}(S_1^{(\alpha/2)} > \frac{(b - a)^2}{t^{2/\alpha} x^2}) \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} dx \]
\[ = 2 \int_0^\infty \left( \frac{(b - a)^2}{t^{2/\alpha} x^2} \right)^{\alpha/2} \mathbb{P}(S_1^{(\alpha/2)} > \frac{(b - a)^2}{t^{2/\alpha} x^2}) \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} \left( \frac{b - a}{t^{2/\alpha} x^2} \right)^{-\alpha/2} dx \]
\[ = \frac{t}{\sqrt{\pi(b - a)^\alpha}} \int_0^\infty \left( \frac{(b - a)^2}{t^{2/\alpha} x^2} \right)^{\alpha/2} \mathbb{P}(S_1^{(\alpha/2)} > \frac{(b - a)^2}{t^{2/\alpha} x^2}) \times x^{\alpha} e^{-\frac{x^2}{4}} dx. \]
It follows from [10] Equation (2.8)] there exists a constant $c_1$ such that for all $u \in (0, \infty)$
\[ u^{\alpha/2} \mathbb{P}(S_1^{(\alpha/2)} > u) \leq c_1. \]
Hence, we have
\[ \mathbb{P}(\sup_{u \leq \xi_t} W_u > b - a) \leq \frac{c_1 t}{\sqrt{\pi(b - a)^\alpha}} \int_0^\infty x^{\alpha} e^{-\frac{x^2}{4}} dx. \]
\[ \square \]
Lemma 4.2 Let $\alpha \in (1, 2)$. Then, there exists a constant $c = c(\alpha) > 0$ such that

$$
\int_a^b \mathbb{P}_x \left( \sup_{u \leq S_t^{(\alpha/2)}} W_u > b \text{ and } \inf_{u \leq S_t^{(\alpha/2)}} W_u < a \right) dx \leq c \mathbb{E}[W_{S_t^{(\alpha/2)}}] t^{1 + \frac{1}{\alpha}}.
$$

Proof. The proof is similar to the proof of Lemma 3.2 and we provide the details for the reader’s convenience. Define

$$
\tau := \inf \{ t : \sup_{u \leq S_t^{(\alpha/2)}} W_u > b \text{ or } \inf_{u \leq S_t^{(\alpha/2)}} W_u < a \}.
$$

Clearly $\tau$ is a stopping time with respect to the natural filtration $\mathcal{F}_t$ and we have

$$
\mathbb{P}_x(\sup_{u \leq S_t^{(\alpha/2)}} W_u > b \text{ and } \inf_{u \leq S_t^{(\alpha/2)}} W_u < a) \\
\leq \mathbb{P}_x\left( \tau < S_t^{(\alpha/2)}, \left( \sup_{\tau \leq s \leq S_t^{(\alpha/2)}} W_s - W_\tau > b - a \text{ or } \inf_{\tau \leq s \leq S_t^{(\alpha/2)}} W_s - W_\tau < -(b - a) \right) \right) \\
\leq \mathbb{P}_x(\tau < S_t^{(\alpha/2)}, \sup_{\tau \leq s \leq S_t^{(\alpha/2)}} W_s - W_\tau > b - a) + \mathbb{P}_x(\tau < S_t^{(\alpha/2)}, \inf_{\tau \leq s \leq S_t^{(\alpha/2)}} W_s - W_\tau < -(b - a)).
$$

It follows from the strong Markov property at $\tau$, the independence increments, and symmetry we have

$$
\mathbb{P}_x(\tau < S_t^{(\alpha/2)}, \sup_{\tau \leq s \leq S_t^{(\alpha/2)}} W_s - W_\tau > b - a) \\
= \mathbb{P}_x\left( \tau < S_t^{(\alpha/2)}, \mathbb{P}_{W_\tau}\left( \sup_{s \leq S_t^{(\alpha/2)} - \tau} W_s > b - a \right) \right) \\
\leq \mathbb{P}_x(\tau < S_t^{(\alpha/2)}) \mathbb{P}(\sup_{s \leq S_t^{(\alpha/2)}} W_s > b - a) \\
= \mathbb{P}_x(\sup_{u \leq S_t^{(\alpha/2)}} W_u > b \text{ or } \inf_{u \leq S_t^{(\alpha/2)}} W_u < a) \mathbb{P}(\sup_{s \leq S_t^{(\alpha/2)}} W_s > b - a) \\
\leq 2 \mathbb{P}_x(\sup_{u \leq S_t^{(\alpha/2)}} W_u > b) \mathbb{P}(\sup_{s \leq S_t^{(\alpha/2)}} W_s > b - a) \\
\leq 2 \mathbb{E}_x(\sup_{u \leq S_t^{(\alpha/2)}} W_u > b),
$$

where we used Lemma 4.1 at the end.

Note that it follows from the scaling property of $S^{(\alpha/2)}$, $W$, and the spacial homogeneity of Lévy processes

$$
\int_a^b \mathbb{P}_x \left( \sup_{s \leq S_t^{(\alpha/2)}} W_s \geq b \right) dx \\
= t^{1/\alpha} \int_0^{(b-a)t^{-1/\alpha}} \mathbb{P}(\sup_{s \leq S_t^{(\alpha/2)}} W_u \geq y) dy
$$

12
Let \( \alpha > 0 \) and the other term can be handled in a similar way.

\[
\int_0^{(b-a)t^{1/\alpha}} \mathbb{P}(\sup_{u \leq v} W_u \geq y) dy \leq \mathbb{E}[\sup_{u \leq v} W_u] = v^{1/2} \mathbb{E}[W_1].
\]

Since \( \alpha > 1 \) it follows from [11 Proposition 2.1] \( \mathbb{E}[(S_1^{(\alpha/2)})^{1/2}] < \infty. \) Hence, by applying the Lebesgue dominated convergence theorem we arrive at

\[
\lim_{t \to 0} \frac{t^{1/\alpha} \int_0^{(b-a)t^{1/\alpha}} \mathbb{P}(\sup_{u \leq v} W_u \geq y) dy \mathbb{P}(S_1^{(\alpha/2)} \in dv)}{t^{1/\alpha}} = \int_0^{\infty} \mathbb{E}[W_1] v^{1/2} \mathbb{P}(S_1^{(\alpha/2)} \in dv) = \mathbb{E}[W_1] \mathbb{E}[(S_1^{(\alpha/2)})^{1/2}].
\]

Note that by independence the expression above can be written as

\[
\mathbb{E}[W_1] \mathbb{E}[(S_1^{(\alpha/2)})^{1/2}] = \mathbb{E}[W_1] \cdot (S_1^{(\alpha/2)})^{1/2} = \mathbb{E}[W_{S_1^{(\alpha/2)}}].
\]

This shows that

\[
\int_a^b \mathbb{P}_x(\sup_{s \leq S_t^{(\alpha/2)}} W_s \geq b) dx \leq t^{1/\alpha} \mathbb{E}[W_{S_t^{(\alpha/2)}}],
\]

and the other term can be handled in a similar way. \( \square \)

Now, we reprove the following theorem using the probabilistic argument similar to [2].

**Theorem 4.3** Let \( \alpha \in (1, 2) \) and \( D = (a, b) \) an open interval with finite length. Then, we have

\[
\lim_{t \to 0} \frac{|D| - Q_D^{(\alpha)}(t)}{t^{1/\alpha}} = \frac{2\Gamma(1 - \frac{1}{\alpha})}{\pi} |\partial D| = \mathbb{E}[W_{S_t^{(\alpha/2)}}]|\partial D|.
\]

**Proof.** The proof is similar to [2 Theorem 1.1 a)]. Note that

\[
\{\tau_D^{(2)} \leq S_t^{(\alpha/2)}\} = \{W_s \geq b \text{ or } W_s \leq a \text{ for some } s \leq S_t^{(\alpha/2)}\}.
\]

Hence

\[
|D| - Q_D^{(\alpha)}(t) = \int_D \mathbb{P}_x(\tau_D^{(2)} \leq S_t^{(\alpha/2)}) dx = \int_D \mathbb{P}_x \left(\mathbb{W}_{S_t^{(\alpha/2)}} \geq b \text{ or } \mathbb{W}_{S_t^{(\alpha/2)}} \leq a\right) dx
\]

\[
= \int_D \mathbb{P}_x \left(\mathbb{W}_{S_t^{(\alpha/2)}} \geq b\right) + \int_D \mathbb{P}_x \left(\mathbb{W}_{S_t^{(\alpha/2)}} \leq a\right) - \int_D \mathbb{P}_x \left(\mathbb{W}_{S_t^{(\alpha/2)}} \geq b \text{ and } \mathbb{W}_{S_t^{(\alpha/2)}} \leq a\right) dx \tag{4.2}
\]

Now the rest of proof is similar to proof of Lemma 4.2. \( \square \)

Next, we need the following technical computations.
Lemma 4.4

\[ \mathbb{P}(W_1 > u) \sim \frac{2}{\sqrt{\pi u}} e^{-\frac{u^2}{4}} \text{ as } u \to \infty. \]

Proof. Note that

\[ \mathbb{P}(W_1 > u) = \mathbb{P}(|W_1| > u) = 2 \int_{\infty}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} dx. \]

Now it follows from the L'Hôpital's rule we have

\[ \lim_{u \to \infty} 2 \int_{\infty}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} dx = \lim_{u \to \infty} -\frac{e^{-\frac{u^2}{4}}}{2} = 1. \]

\[ \blacksquare \]

Lemma 4.5 Let \( \alpha \in (1, 2) \). Then, we have

\[ \lim_{t \to 0} \frac{\int_{(b-a)t^{1/\alpha}}^{\infty} \int_{0}^{\frac{v}{\sqrt{t}}} \mathbb{P}(W_1 \geq \frac{v}{\sqrt{t}}) \mathbb{P}(S_{1}^{(\alpha/2)} \in dv)dy}{t^{1-\frac{1}{\alpha}}} = \frac{\alpha}{(\alpha - 1) \Gamma(1 - \frac{\alpha}{2})} \int_{1}^{\infty} \mathbb{P}(W_1 \geq u) u^{\alpha - 1} du. \]

Proof. By L'Hôpital's rule, the change of variable \( u = \frac{(b-a)t^{1/\alpha}}{\sqrt{t}} \), the scaling property \( g^{(\alpha/2)}(t, x) = t^{-2/\alpha} g^{(\alpha/2)}(1, \frac{x}{t^{1/\alpha}}) \), the Lebesgue dominated convergence theorem using Lemma 4.4 [11], Corollary 8.9, and (2.2) we have

\[ \lim_{t \to 0} \frac{\int_{(b-a)t^{1/\alpha}}^{\infty} \int_{0}^{\frac{v}{\sqrt{t}}} \mathbb{P}(W_1 \geq \frac{v}{\sqrt{t}}) \mathbb{P}(S_{1}^{(\alpha/2)} \in dv)dy}{t^{1-\frac{1}{\alpha}}} = \lim_{t \to 0} \frac{2(b-a)^3}{(\alpha - 1)t} \int_{1}^{\infty} \mathbb{P}(W_1 \geq u) g^{(\alpha/2)}(1, \frac{(b-a)^2 t^{-2/\alpha}}{u^2}) t^{-2/\alpha} u^{-3} du \]

Recall that it follows from [10] Equation (2.5)

\[ \lim_{x \to \infty} g^{(\alpha/2)}(1, x)x^{1+\frac{\alpha}{2}} = \frac{\alpha}{2\Gamma(1 - \frac{\alpha}{2}).} \]
Lemma 4.6 Let $\alpha \in (1, 2)$. Then, we have

$$
\lim_{t \to 0} \frac{\int_{(b-a)t^{1-\alpha}}^{\infty} \int_{y^2}^{\infty} P(\overline{W}_1 \geq y^{\alpha/2} \mathbb{P}(S_1^{(\alpha/2)} \in dv) dy}{t^{1-\alpha}} = \frac{\alpha}{(\alpha - 1) \Gamma(1 - \frac{\alpha}{2})(b-a)^{\alpha-1}} \int_0^1 P(\overline{W}_1 \geq w)w^{\alpha-1} dw.
$$

**Proof.** By the change of variable $w = \frac{y}{v^{1/2}}$ the inner integral in the numerator can be written as

$$
\int_{y^2}^{\infty} P(\overline{W}_1 \geq y^{\alpha/2}) \mathbb{P}(S_1^{(\alpha/2)} \in dv) dy = \int_{y^2}^{\infty} P(\overline{W}_1 \geq y^{\alpha/2}) g^{(\alpha/2)}(1, v) dv = \int_0^1 P(\overline{W}_1 \geq w) g^{(\alpha/2)}(1, \frac{y^2}{w^2}) \frac{2y^2}{w^3} dw.
$$

Since $g^{(\alpha/2)}(1, x) \leq c x^{-1 - \frac{\alpha}{2}}$ for $x \geq 1$, the integral is finite.

By the L'Hôpital's rule, Lebesgue dominated convergence theorem, and (4.3) we have

$$
\lim_{t \to 0} \frac{\int_{(b-a)t^{1-\alpha}}^{\infty} \int_{y^2}^{\infty} P(\overline{W}_1 \geq y^{\alpha/2}) (1, \frac{y^2}{w}) \frac{2y^2}{w^3} dw dy}{t^{1-\alpha}} = \lim_{t \to 0} \frac{2(b-a)}{(b-a)(b-a)^{\alpha-1}} \int_0^1 P(\overline{W}_1 \geq w) g^{(\alpha/2)} \left(1, \left(\frac{(b-a)t^{1-\alpha}}{w}\right)^2\right) \left(\frac{(b-a)t^{1-\alpha}}{w}\right)^{\alpha-1} w^{\alpha-1} dw
$$

$$
= \frac{2}{(b-a)(b-a)^{\alpha-1}} \int_0^1 P(\overline{W}_1 \geq w) \frac{\alpha}{2\Gamma(1 - \frac{\alpha}{2})} w^{\alpha-1} dw.
$$

Now we are ready to prove the first part of Theorem 1.2.

**Proof of (1.3)**

Note that from (4.1) and (4.2) we have

$$
|D| - \tilde{Q}_D^{(\alpha)}(t) = 2t^{1/\alpha} \int_0^\infty \int_{0}^{\overline{W}_1^{(\alpha/2)}} \mathbb{P}(\sup_{u \leq v} W_u \geq y) dy \mathbb{P}(S_1^{(\alpha/2)} \in dv) - \int_D \mathbb{P}_x(\overline{W}_{S_1^{(\alpha/2)}} > b \text{ and } \overline{W}_1^{(\alpha/2)} < a) dx.
$$

(4.4)

It follows from Lemma 4.2

$$
\int_D \mathbb{P}_x(\overline{W}_{S_1^{(\alpha/2)}} > b \text{ and } \overline{W}_1^{(\alpha/2)} < a) dx = O(t^{1+\frac{1}{\alpha}}).
$$

Now we focus on the first integral in (4.4). Note that we have

$$
2t^{1/\alpha} \int_0^\infty \int_{0}^{\overline{W}_1^{(\alpha/2)}} \mathbb{P}(\sup_{u \leq v} W_u \geq y) dy \mathbb{P}(S_1^{(\alpha/2)} \in dv) - 2t^{1/\alpha} \int_0^\infty \mathbb{P}(\sup_{u \leq S_1^{(\alpha/2)}} W_u \geq y) dy
$$

$$
= 2t^{1/\alpha} \int_0^\infty \int_{0}^{\overline{W}_1^{(\alpha/2)}} \mathbb{P}(\sup_{u \leq v} W_u \geq y) dy \mathbb{P}(S_1^{(\alpha/2)} \in dv) - 2t^{1/\alpha} \int_0^\infty \mathbb{P}(\sup_{u \leq v} W_u \geq y) \mathbb{P}(S_1^{(\alpha/2)} \in dv) dy
$$

15
\[= -2t^{1/\alpha} \int_0^\infty \int_{(b-a)t^{-1/\alpha}}^\infty \mathbb{P} \left( \sup_{u \leq v} W_u \geq y \right) \mathbb{P} (S_1^{(\alpha/2)} \in dv)\]
\[= -2t^{1/\alpha} \int_0^\infty \int_{(b-a)t^{-1/\alpha}}^\infty \mathbb{P} (W_1 \geq \frac{y}{\sqrt{v}}) \mathbb{P} (S_1^{(\alpha/2)} \in dv)\]
\[= -2t^{1/\alpha} \int_{(b-a)t^{-1/\alpha}}^\infty \int_0^y \mathbb{P} (W_1 \geq \frac{y}{\sqrt{v}}) \mathbb{P} (S_1^{(\alpha/2)} \in dv) dy - 2t^{1/\alpha} \int_{(b-a)t^{-1/\alpha}}^\infty \int_y^\infty \mathbb{P} (W_1 \geq \frac{y}{\sqrt{v}}) \mathbb{P} (S_1^{(\alpha/2)} \in dv) dy\]

Now the conclusion follows immediately from Lemmas 4.5 and 4.6.

\[\square\]

4.2 \(\alpha = 1\)

In this subsection, we study the spectral heat content for subordinate killed Brownian motions when the underlying subordinator is \(S_t^{(1/2)}\).

**Proposition 4.7** For any \(u > 1\), we have
\[\mathbb{P} (W_{S_t^{(1/2)}} > u) = \frac{2}{\pi} \arctan(1/u).\] (4.5)

**Proof.** It follows from (2.3) that the density of \(S_t^{(1/2)}\) is given by
\[g^{(1/2)}(1, x) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n+1} \frac{\Gamma(n + \frac{1}{2})}{(2n-1)!} x^{-n-\frac{1}{2}}, \quad x > 0.\]

It is easy to check \((n!)^2 \leq (2n-1)!\) for all \(n \geq 1\) and \(\Gamma(n + \frac{1}{2}) \leq \Gamma(n + 1)\) for all \(n \geq 2\). Hence, we have
\[\left| \sum_{n=1}^\infty (-1)^{n+1} \frac{\Gamma(n + \frac{1}{2})}{(2n-1)!} x^{-n-\frac{1}{2}} \right| \leq \sum_{n=1}^\infty \frac{\Gamma(n + \frac{1}{2})}{(2n-1)!} x^{-n-\frac{1}{2}} \leq \Gamma \left( \frac{3}{2} \right) x^{-\frac{3}{2}} + \sum_{n=2}^\infty \frac{\Gamma(n + 1)}{(2n-1)!} x^{-n-\frac{1}{2}} \leq \Gamma \left( \frac{3}{2} \right) x^{-\frac{3}{2}} + \sum_{n=2}^\infty \frac{x^{-\frac{1}{2}}}{n!} = \Gamma \left( \frac{3}{2} \right) x^{-\frac{3}{2}} + x^{-\frac{1}{2}} (e^{1/x} - 1 - \frac{1}{x}) = O \left( \frac{1}{x^{3/2}} \right)\] as \(x \to \infty\).

Hence, by the Lebesgue convergence theorem we have
\[\mathbb{P} (S_t^{(1/2)} > \frac{u^2}{x^2}) = \int_{\frac{u^2}{x^2}}^\infty g^{(1/2)}(1, v) dv = \int_{\frac{u^2}{x^2}}^\infty \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n+1} \frac{\Gamma(n + \frac{1}{2})}{(2n-1)!} v^{-n-\frac{1}{2}} dv = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n+1} \frac{\Gamma(n + \frac{1}{2})}{(2n-1)!} \frac{1}{n - \frac{1}{2}} x^{2n-1} u^{2n-1}.\]
Note that
\[ P(W_1 > a) = P(|W_1| > a) = 2 \int_{a}^{\infty} \frac{1}{\sqrt{4\pi}} \int_{a}^{\infty} e^{-\frac{x^2}{4}} dx = \frac{1}{\sqrt{\pi}} \int_{a}^{\infty} e^{-\frac{x^2}{4}} dx. \]

Hence, we have
\[
P(W_{S_1^{(1/2)}} > u) = \int_{0}^{\infty} P(W_y > u)P(S_1^{(1/2)} \in dy)
= \int_{0}^{\infty} \frac{1}{\sqrt{\pi}} \int_{\frac{u}{\sqrt{y}}}^{\infty} e^{-\frac{x^2}{4}} dx P(S_1^{(1/2)} \in dy)
= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \int_{\frac{u}{\sqrt{y}}}^{\infty} P(S_1^{(1/2)} \in dy)e^{-\frac{x^2}{4}} dx
= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \left( \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma(n + \frac{1}{2})}{(2n - 1)!} \frac{1}{u^{2n-1}} \right) \int_{0}^{\infty} x^{2n-1} e^{-\frac{x^2}{4}} dx
= \frac{2}{\pi^{3/2}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma(n + \frac{1}{2})}{(2n - 1)!} \frac{1}{2n - 1} \int_{0}^{\infty} x^{2n-1} e^{-\frac{x^2}{4}} dx
= \frac{2}{\pi^{3/2}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Gamma(n + \frac{1}{2})}{(2n - 1)!} \frac{1}{2n - 1} \frac{\Gamma(n)2^{2n-1}}{u^{2n-1}}, \quad (4.6)
\]
where we used \( \int_{0}^{\infty} x^{2n-1} e^{-\frac{x^2}{4}} dx = \Gamma(n)2^{2n-1} \), and the interchange of the infinite sum and integral is valid because of the exponential decay term and the fact \( u > 1 \). By the Legendre duplication formula we have \( \Gamma(n)\Gamma(n + \frac{1}{2}) = 2^{1-2n}\sqrt{\pi}\Gamma(2n) \). By the Taylor expansion of \( \arctan(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1} \) for \( |x| < 1 \), (4.6) can be simplified to
\[ \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n - 1} \frac{1}{u^{2n-1}} = \frac{2}{\pi} \arctan\left( \frac{1}{u} \right), \quad \text{for } u > 1. \]

\[
\text{Remark 4.8} \quad \text{Even though it is not necessary for our purpose, it would be interesting to see if (4.5) holds for all } u > 0.
\]

Lemma 4.9
\[ \int_{a}^{b} P_x(W_{S_1^{(1/2)}} > b \text{ and } W_{S_1^{(1/2)}} < a) dx = O(t^2 \ln(1/t)). \]

Proof. The proof is almost identical to the proof of Lemma 3.3 using Lemma 4.1. It follows from Proposition 4.7 \( P(W_{S_1^{(1/2)}} > u) \sim \frac{2}{\pi u} \) as \( u \to \infty \) and this shows that \( \int_{0}^{\infty} P(W_{S_1^{(1/2)}} > u) du = O(\ln(1/t)) \). \qed
Now we are ready to prove the second part of Theorem 1.2

Proof of (1.4)

Note that

\[ |D| - \tilde{Q}_D^{(1)}(t) = \int_a^b P_x(\tau_D^{(2)} \leq S_t^{(1/2)}) dx \]

\[ = 2 \int_a^b P_x(W_{S_t^{(1/2)}} > b) dx - \int_a^b P_x(W_{S_t^{(1/2)}} > b) dx \]

It follows from Lemma 4.9 the second term is \(O(t^2 \ln(1/t))\).

Now the first expression above can be written as

\[ 2 \int_a^b P_x(W_{S_t^{(1/2)}} > b) dx = 2 \int_a^b P_x(\sup_{u \leq t^2 S_t^{(1/2)}} W_{ut^{-2}} > b) dx \]

\[ = 2 \int_a^b P_x(\sup_{v \leq S_t^{(1/2)}} W_v > b/t) dx = 2t \int_0^{b-a} P(\sup_{v \leq S_t^{(1/2)}} W_v > u) du = 2t \int_0^{b-a} P(W_{S_t^{(1/2)}} > u) du. \]

Hence, we have

\[ 2t \int_0^{b-a} P(W_{S_t^{(1/2)}} > u) du - \frac{4}{\pi} t \ln(1/t) \]

\[ = 2t \left( \int_0^{b-a} P(W_{S_t^{(1/2)}} > u) du - \frac{2}{\pi} \ln(1/t) \right) \]

\[ = 2t \left( \int_0^{b-a} P(W_{S_t^{(1/2)}} > u) du + \frac{2 \ln(b-a)}{\pi} + \int_1^{b-a} P(W_{S_t^{(1/2)}} > u) - \frac{2}{\pi u} du \right). \]

From Proposition 4.7, \(P(W_{S_t^{(1/2)}} > u) - \frac{2}{\pi u} = O(\frac{1}{u^3})\), hence it is integrable on \((1, \infty)\). Hence, it follows from the monotone convergence theorem

\[ \lim_{t \to 0} \left( \int_0^1 P(W_{S_t^{(1/2)}} > u) du + \frac{2 \ln(b-a)}{\pi} + \int_1^\infty P(W_{S_t^{(1/2)}} > u) - \frac{2}{\pi u} du \right) \]

\[ = 2 \left( \int_0^1 P(W_{S_t^{(1/2)}} > u) du + \frac{2 \ln(b-a)}{\pi} + \int_1^\infty P(W_{S_t^{(1/2)}} > u) - \frac{2}{\pi u} du \right). \]

\[ \square \]

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