SKYRMIONS, SPECTRAL FLOW AND PARITY DOUBLES

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Abstract

It is well-known that the winding number of the Skyrmion can be identified as the baryon number. We show in this paper that this result can also be established using the Atiyah-Singer index theorem and spectral flow arguments. We argue that this proof suggests that there are light quarks moving in the field of the Skyrmion. We then show that if these light degrees of freedom are averaged out, the low energy excitations of the Skyrmion are in fact spinorial. A natural consequence of our approach is the prediction of a $\frac{1}{2}^{-}$ state and its excitations in addition to the nucleon and delta. Using the recent numerical evidence for the existence of Skyrmions with discrete spatial symmetries, we further suggest that the the low energy spectrum of many light nuclei may possess a parity doublet structure arising from a subtle topological interaction between the slow Skyrmion and the fast quarks. We also present tentative experimental evidence supporting our arguments.

1 Introduction

The modern understanding of high energy physics is based on the Standard Model, the strong interaction sector of which is described by QCD. It is however difficult to make reliable statements about the low energy behavior of QCD where its coupling constant becomes large and perturbation theory breaks down. Several phenomenological models have been developed in an attempt to understand its low energy dynamics, like for example the chiral model (see for example [1] and references therein). In this model, the chiral field $U$, which is valued in $SU(N_f)$ for $N_f$ flavors, describes the low energy

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excitations about the QCD ground state. The baryon octet can be identified with the solitons of the Skyrme Lagrangian \[2\], the baryon number \( B \) being the winding number of the soliton \[3, 4, 5, 6\].

In the original research, static solutions of the Skyrme Lagrangian \[2\] for the winding number \( \pm 1 \) sectors were found and these solitons used for the description of \( |B| = 1 \) baryons. Since that work, there has been a considerable amount of interest in searching for Skyrme solitons with higher winding numbers as well. In particular, a \( |B| = 2 \) solution appropriate to describe the H-dibaryon \[7, 8, 9, 10, 11, 12\] and also numerical solutions with the symmetry of discrete sub-groups of \( SO(3) \) \[13, 14, 15\] have been found.

In this paper, we largely restrict ourselves to two flavors. A standard parity-invariant manner in which the quarks couple to the background \( U \)-field is via the Yukawa coupling,

\[
\mathcal{L}_{\text{int}} = -m[\bar{\psi}_L U^\dagger \psi_R + \bar{\psi}_R U \psi_L] \quad (1.1)
\]

\[
\equiv -m\overline{\psi}e^{-i\gamma_5 \vec{\tau} \cdot \vec{\phi}} \psi, \quad (1.2)
\]

\[
U = e^{i\vec{\tau} \cdot \vec{\phi}}, \quad \tau_i = \text{Pauli matrices} \quad (1.3)
\]

where \( \psi_R = \frac{1+\gamma_5}{2} \psi, \psi_L = \frac{1-\gamma_5}{2} \psi \). The different winding number sectors of this \( U \)-field are precisely the Skyrmions. There are of course no histories of \( U \) that can change its winding number. We will consider a generalization of this Lagrangian that will effectively allow us to study winding number changing histories. Using the index theorem and the associated spectral flow picture, we will show that the one-fermion state in the background of the soliton corresponds to the fermionic vacuum with no soliton. This leads to a natural interpretation for the baryon number of the Skyrme soliton. Specifically, we will show that the nucleon (with quantum numbers \( I(J^P) = \frac{1}{2}(1^+) \)) is described by taking the even combination of this fermionic state and its parity transform. The parity-odd combination describes some excited states with \( I(J^P) = \frac{1}{2}(1^-) \), perhaps the \( N(1535) \) or \( N(1650) \).

This argument also leads to two interesting consequences. First of all, since there are quarks created in the process of creating a Skyrmion, they may stick to the Skyrmion. Thus one can think of the baryon as made up of not just the “soliton lump”, but also of these fast quarks whizzing around it. As these quark degrees of freedom are of considerably higher energies than those of the collective motion of the soliton, the low energy analysis of the baryon can be based on the Born-Oppenheimer (B-O) approximation. If these quark degrees of freedom are now averaged out, the low energy
excitations of the Skyrmion become spinorial. This is a novel argument to show that the Skyrmion is a fermion. (A related argument, demonstrating the effect on the Dirac sea of a $2\pi$-rotation of the Skyrmion, has been discussed by [16].)

The second interesting consequence arises if one considers Skyrmions with discrete symmetries. The general theory for doing the B-O approximation for systems with discrete symmetries was discussed by two of the authors in [17, 18]. It was shown that for such systems, the B-O approximation leads to quite subtle quantum effects. In particular, in previous work [19] and here, it has been argued that many shapes lead to anomalous violation of parity in quantum theory if care is not taken to judiciously include aspects of microscopic degrees of freedom as well. On including these high energy degrees of freedom, it was argued that the low energy excitation spectrum of the system should display a “parity doublet structure”. An example of such a doublet is the pair of closely spaced levels of the ammonia molecule used in the construction of the ammonia maser. It was suggested in [17] that such doubles should occur among bound states involving quarks as well. We now argue that parity doubles of this kind occur in Skyrmion physics too and make specific phenomenological predictions. Parity doubles among light baryons have also been discussed by [20, 21].

The organization of this article is as follows: In Section 2, using the appropriate index theorem, we show that as the Skyrmion is created, quarks are created as well. We then argue in Section 3 that if these fast quark degrees of freedom are averaged out, the low energy excitations of the Skyrmion become in fact spinorial. In Section 4, we consider Skyrmions with discrete symmetry groups, and suggest that some of these systems possess a parity doublet structure for their low energy spectrum. We also present tentative experimental evidence for the existence of parity doublets in the low energy excitation spectrum of several light nuclei. Section 5 is devoted to concluding remarks.

2 Spectral Flow and Particle Creation

Let us consider the two-flavored chiral model, so that $U(\vec{x}, t) \in SU(2)$. The Skyrme Lagrangian is

$$\mathcal{L}_{\text{Skyrme}} = \frac{f_\pi^2}{16} \text{Tr} U^\dagger \partial_\mu U U^\dagger \partial^\mu U + \frac{1}{32e^2} \text{Tr}[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2. \quad (2.1)$$
A standard parity invariant interaction term for an $SU(2)$ doublet of Dirac fermions in the background of the chiral field $U$ is of the form (1.1). We can identify $\psi$ with the quark fields $q^\alpha$ after attaching a color index $\alpha$.

Let us now redefine the fields $\psi_L$ and $\psi_R$ as follows:

$$
\Psi_L = \psi_L, \\
\Psi_R = U^\dagger \psi_R. 
$$

(2.2)

The fermion Lagrangian density (ignoring the Skyrme Lagrangian for the moment) then becomes

$$
L_F = \bar{\Psi}_L i \gamma^\mu \partial_\mu \Psi_L + \bar{\Psi}_R i \gamma^\mu (\partial_\mu + U^\dagger \partial_\mu U) \Psi_R - m[\bar{\Psi}_R \Psi_L + h.c.].
$$

(2.3)

We can treat the mass term here as a perturbation and thus initially ignore it. Therefore, to zeroth order in $m$, the Lagrangian of interest is

$$
L_F = \bar{\Psi}_L i \gamma^\mu \partial_\mu \Psi_L + \bar{\Psi}_R i \gamma^\mu (\partial_\mu + U^\dagger \partial_\mu U) \Psi_R.
$$

(2.4)

For fixed time, $U$ is a map from $\mathbb{R}^3$ into $SU(2)$, with the condition that $U(x) \to 1$ as $|\vec{x}| \to \infty$ ($x \equiv \vec{x}, t$), that is, $U \in \text{Maps}(S^3 \to SU(2))$.

The space of $U$'s is disconnected, since

$$
\pi_0[\text{Maps}(S^3 \to SU(2))] = \pi_3[SU(2)] = \mathbb{Z}. 
$$

(2.5)

The different disconnected pieces or sectors are labeled by the “winding number” $n$ of $U$:

$$
n = \frac{1}{24\pi^2} \int Tr(U^\dagger dU)^3. 
$$

(2.6)

Here we are using the notation of differential forms, and the wedge product between differential forms is being omitted.

Clearly, $U$-fields with different winding numbers give rise to different quantum fermionic systems via (2.3). We are interested in understanding the effect on the fermionic system as the winding number is changed. Specifically, we would like to follow the energy levels of the quantum fermionic Hamiltonian as the winding number is changed. In order to do this, let us consider a slightly more general Lagrangian

$$
L' = \bar{\Psi}_L i \gamma^\mu \partial_\mu \Psi_L + \bar{\Psi}_R i \gamma^\mu (\partial_\mu + A^R_\mu) \Psi_R.
$$

(2.7)
This Lagrangian is invariant under gauge transformations of the form

\[ \begin{align*}
\Psi_R & \rightarrow g^{-1}\Psi_R, \\
A^R_\mu & \rightarrow g^{-1}A^R_\mu g + g^{-1}\partial_\mu g, \\
\Psi_L & \rightarrow \Psi_L, \\
g(x) & \in SU(2).
\end{align*} \]

Had we added the kinetic energy term

\[
\text{constant} \times \int \text{Tr}[F^{\mu\nu}R F^R_{\mu\nu}]
\]

where \( F^{\mu\nu}_R \) is the curvature of \( A^R_\mu \), we would have a chiral \( SU(2) \) gauge theory.

For flat connections, (2.7) reduces to (2.4).

The advantage of using the more general Lagrangian (2.7) is that there is no need to restrict to flat connections. For this reason, we have the possibility of studying winding number changing histories as we now show. Note that such histories do not exist in the space of \( U \)’s.

With this more general Lagrangian, we can now imagine continuous paths (in the configuration space of the gauge field \( A^R_\mu \)) connecting the two flat connections \( A^R_\mu = 0 \) and \( A^R_\mu = U^\dagger \partial_\mu U \). If \( U \) has a non-trivial winding number \( n \), then these paths necessarily go through configurations having non-trivial gauge field strengths. If we choose instanton-like interpolating fields, the spectral flow of the relevant Dirac Hamiltonian can be computed, as we shall see.

Let us consider a static Skyrme field \( U, U(x) = U_c(\vec{x}) = e^{i\theta_c(r)\vec{\tau}\cdot\hat{x}} = \cos \theta_c(r) + i\vec{\tau}\cdot\hat{x}\sin \theta_c(r) \), of winding number \( n = 1 \), where \( r = \|\vec{x}\| \) and \( \hat{x} = \vec{x}/\|\vec{x}\| \). While there are no continuous paths that interpolate between \( U \)’s of different winding numbers, we can find continuous paths interpolating between any two connections. Let us thus look at instanton-like configurations of the gauge field \( A^R \) (in the temporal gauge \( A^R_0 = 0 \)) of instanton charge \( q_R \equiv 1/16\pi^2 \int \text{Tr}(F^R \wedge F^R) = -1 \), connecting \( A^R_i = 0 \) to \( A^R_i = U^\dagger_c \partial_i U_c \). Since the winding number of a field \( g \) is defined as \( 1/24\pi^2 \int \text{Tr}(g^{-1}dg)^3 \) [cf. (2.10)], we can see that the instanton charge \( q_R \) is simply the difference between winding numbers of the fields \( 1 \) and \( U_c \).

Let us look at the 4-dimensional (Euclidean) Dirac equation in the presence of such an instanton field,

\[
\begin{bmatrix}
0 & \hat{L} \\
L & 0
\end{bmatrix}
\begin{bmatrix}
\chi_R \\
\chi_L
\end{bmatrix} = 0,
\]

(2.12)
where

\[ \Psi_R = \begin{bmatrix} \chi_R \\ 0 \end{bmatrix}, \Psi_L = \begin{bmatrix} 0 \\ \chi_L \end{bmatrix} \]  

(2.13)

in the basis in which \( \gamma_5 \) is diagonal. Also \( L \) and \( \tilde{L} \) are defined as

\[ L = i\bar{\alpha}^\mu(\partial_\mu + A_\mu^R), \quad \tilde{L} = i\alpha_\mu \partial_\mu \]  

(2.14)

where \( \alpha^\mu = (-i\vec{\sigma}, 1) \) and \( \bar{\alpha}^\mu = (i\vec{\sigma}, 1) \). We are using Dirac matrices of the form

\[ \gamma^\mu = \begin{bmatrix} 0 & \alpha^\mu \\ \bar{\alpha}^\mu & 0 \end{bmatrix} \]

With this choice of representation, \( \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \text{diag}(1, -1) \) is diagonal. The equation (2.12) is a consequence of (2.7).

Let us look for non-trivial solutions, or the zero modes of the equations

\[ L\chi^R = 0, \]  

(2.15)

\[ \tilde{L}\chi^L = 0. \]  

(2.16)

Multiplying (2.13) by \( L^\dagger \) and (2.16) by \( \tilde{L}^\dagger \), we obtain

\[ ((\partial_\mu + A_\mu^R)^2 + 2i\vec{\sigma}^{\mu\nu}F_{R\mu\nu})\chi^R = 0, \]  

(2.17)

\[ 1(\partial_\mu \partial_\mu)\chi^L = 0, \]  

(2.18)

where \( \vec{\sigma}^{\mu\nu} = (1/4i)(\alpha^\mu \bar{\alpha}^\nu - \alpha^\nu \bar{\alpha}^\mu) \). In general, it is difficult to know the number of solutions of (2.17). However, following [22] we can settle this number if we use an anti self-dual instanton field for \( A^R \) (that is, \( F_{\mu\nu}^R = -1/2\epsilon_{\mu\nu\rho\lambda}F^{R\rho\lambda} \) (and it is always possible to find one [23]). Since (2.18) has only the trivial solution \( \chi^L = 0 \), the Atiyah-Singer index theorem tells us that the number of linearly independent solutions of (2.17) is exactly \(|q_R|\), which is 1 in our case.

Following Witten [24], let us show that the number of zero modes of (2.13) is in fact equal to the spectral flow of the Dirac Hamiltonian

\[ H_R = -i\sigma^k(\partial_k + A^R_k) \]  

(2.19)

for \( \chi_R \). Rewriting (2.13) as

\[ \partial_4\chi^R = -i\sigma^k(\partial_k + A^R_k)\chi^R, \]  

(2.20)
let us adiabatically change the gauge field $A^R(\vec{x}, x_4)$ as the Euclidean time $x_4$ changes from $-\infty$ (where $A^R = U_c^\dagger dU_c$) to $+\infty$ (where $A^R = 0$). For $\chi^R$, let us make the ansatz

$$\chi^R(\vec{x}, x_4) = F(x_4)\phi^R_{x_4}(\vec{x}), \quad (2.21)$$

where $\phi^R_{x_4}(\vec{x})$ is an eigenfunction of (2.19):

$$H_R\phi^R_{x_4}(\vec{x}) = \lambda(x_4)\phi^R_{x_4}(\vec{x}). \quad (2.22)$$

Substituting the above ansatz for the zero mode of (2.15) in (2.20), we get

$$F(x_4) = F(0) \exp \left( \int_0^{x_4} d\tau' \lambda(\tau') \right). \quad (2.23)$$

It is obvious that $\chi^R(\vec{x}, x_4)$ is normalizable only if $\lambda$ is negative for $\tau \to \infty$ and positive for $\tau \to -\infty$. Thus the existence of the zero mode of (2.15) necessarily implies that one right-handed fermionic level of the $H_R$ flows from positive to negative value. In other words, we find that the 1-particle state in the background field $U_c$ gets mapped to the vacuum state of zero winding number. We can therefore unambiguously say that fermionic vacuum in the background of the trivial winding number is the single particle state in the background field of winding number one.

We need to remark on an important issue of the quantum numbers of this fermionic state. First of all, the QCD Lagrangian (and its effective low energy description) is invariant under parity. So, under a parity transformation of the above right-handed fermionic state, we get a left-handed state as well. The correct description of the nucleon in the ground state is via the parity-even combination of these two states. The parity-odd combination corresponds to excited states with quantum numbers $I(J^P) = \frac{1}{2}(-\frac{1}{2})$. Secondly, QCD is a theory with $N_c = 3$ colors. So on second quantization, the above state must be populated by three quarks forming a color singlet.

Let us take stock of the implication of such a spectral flow for our situation. Recall that we started from the Lagrangian (2.4), which we generalized to (2.7). Using the instantons available in this more general Lagrangian, we found that we could study paths connecting winding numbers 0 and 1. In terms of the original Lagrangian (2.4), what this means is that one can relate the fermionic spectrum for a winding number zero background $U$-field to that
for a winding number 1 $U$-field. In particular, there is a history which maps the vacuum of the first situation to the one-particle state of the second.

Higher winding number Skyrmions can be similarly discussed by using instantons of higher instanton number $|q|$. This results in $|q|$ energy levels flowing from negative to positive values.

Thus we can unambiguously say that processes that change the winding number of the $U$-field by $n$ are accompanied by the “binding” of $n$ Dirac fermions to the soliton. The winding number can thus be interpreted as the fermion (or baryon) number.

3 Skyrmion Spin from B-O approximation

The spectral flow argument presented in the previous section strongly suggests that there are quarks moving in the background of the Skyrme field $U$. The system thus naturally lends itself to a separation into the “fast” quark degrees of freedom, and the “slow” Skyrme degrees of freedom. It is therefore plausible that the simplest attempt to quantize this system would be via the Born-Oppenheimer approximation or some version thereof.

The bundle-theoretical formulation of B-O approximation has been reviewed and elaborated in [25, 18]. Let us recall it briefly. The B-O approximation is used to describe systems whose Hamiltonian $\mathcal{H}$ conveniently splits into $\mathcal{H}_f + \mathcal{H}_s$, where we can roughly think of $\mathcal{H}_f$ as dictating the “fast” dynamics, and $\mathcal{H}_s$ the “slow” dynamics. Let us assume that this Hamiltonian $\mathcal{H}$ has eigenfunctions that are sections of a trivial bundle over the slow configuration space $Q_{\text{slow}}$. [We can modify this hypothesis and hence the subsequent discussion by substituting twisted for trivial bundles, but this change is not appropriate for Skyrmions.] The fast wave function $\psi_f$ is an eigenfunction of $\mathcal{H}_f$ (or more generally some linear combination of the eigenfunctions of $\mathcal{H}_f$) and is a function on both the fast configuration space $Q_{\text{fast}}$ and the slow configuration space $Q_{\text{slow}}$. In particular, the fast wave function is the section of some bundle (generally twisted) over $Q_{\text{slow}}$. The slow wave function is calculated not from $\mathcal{H}_s$ but by averaging $\mathcal{H}_s$ over $\psi_f$ to give the emergent “slow” B-O Hamiltonian $\mathcal{H}_s$. This Hamiltonian generally contains a connection, and eigenfunctions of $\mathcal{H}_s$ are also sections of a (generally) twisted bundle over $Q_{\text{slow}}$ determined by this connection. But it has a twist exactly “opposite” to that of $\psi_f$ in the sense that the product $\psi_f \psi_s$ can be projected onto the trivial bundle over $Q_{\text{slow}}$. 

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Let us elaborate on the system under consideration in order to apply the above ideas. We have a $U$-field that describes a soliton and has some non-trivial winding number, and there are some quarks (the number of quarks being given by the topological arguments of the previous section, and by color confinement) moving in the vicinity of this soliton. The quarks are to be quantized with this $U$-field in the background. (Quark wave functions in the background of a Skyrmion have been studied by [26, 27] in a slightly different context.) For a complete quantum-mechanical description of low energy physics, we must now quantize the collective motion of the Skyrmion as well. We will demonstrate now that in the B-O approximation, when the fast quarks are averaged out, the low energy excitations of the Skyrmion are spinorial.

For simplicity, let us begin with the hedgehog configuration $U_c(\vec{x}) = \cos \theta (r)1 + i \vec{r} . \hat{x} \sin \theta (r)$. Under an isospin transformation by $A \in SU(2)$, we get in general a different configuration $AU_c(\vec{x})A^\dagger$, but with the same energy. (Note that since a transformation by $A$ or $-A$ give the same configuration, the space of these configurations is actually diffeomorphic to $SU(2)/\mathbb{Z}_2 = SO(3)$.) Similarly, under a non-trivial spatial rotation $\vec{x} \to R \vec{x}$, we get another configuration $U_c(R\vec{x})$ that is distinct from $U_c(\vec{x})$, but still with the same energy. We call all these configurations that are distinct from $U_c$ but with the same energy as the “collective or zero modes”. It would seem that the space of zero modes is $SO(3) \times SO(3)$, but this is not so. This is because of the identity $A\tau_i A^\dagger = \tau_j R_{ji}(A)$, which means that every spatial rotation is equivalent to some isospin transformation. The space of zero modes is thus topologically $SO(3)$.

We will occasionally refer to the space of $A$’s, which is topologically $S^3 \approx SU(2)$ as the space of zero modes although the latter is in reality obtained from $SU(2)$ only after identifying $A$ with $-A$.

According to the standard prescription of collective coordinate quantization, we can assume that $U$ is given by

$$U(\vec{x}, t) = A(t)U_c(\vec{x})A^\dagger, \quad A(t) \in SU(2). \quad (3.1)$$

The configuration space $Q_{slow}$ of the $U$’s can be identified with that of the collective modes.

How does $A$ transform under spatial and isospin rotations? Recall that under a spatial rotation, the hedgehog ansatz $U_c(\vec{x})$ changes to $U_c(R\vec{x}) = g_R U_c(\vec{x}) g_R^\dagger$, where $R$ is determined by $g_R$ via the relation $g_R \tau_i g_R^\dagger = \tau_j R_{ji}$.
This is equivalent to the transformation \( A \to Ag_R \) by (3.1). Similarly, under an isospin transformation by \( g_I, U(\vec{x}) \to g_I^\dagger U(\vec{x})g_I \). This is equivalent to the transformation \( A \to g_I^\dagger A \). Thus spatial rotations act on the spaces of \( A \)'s by right multiplication, while isospin rotations act by left multiplication \(^{28}\).

The transformations on the space of \( A \)'s are therefore done by operators \( R_s \) and \( I_s \) as follows:

\[
R_s(g_R)AR_s^\dagger(g_R) = Ag_R, \\
I_s(g_I)AI_s^\dagger(g_I) = g_I^\dagger A.
\]

(3.2)

The subscript \( s \) for \( R_s \) and \( I_s \) is to indicate that these operators act on the slow degrees of freedom.

Let us describe the “fast” wave function \( \psi_f \) in a little more detail. This \( \psi_f \) is an eigenfunction of the Dirac Hamiltonian, with the connections \( A_i^L = 0 \) and \( A_i^R = U^\dagger \partial_i U \). The ansatz \( U \) is invariant under combined spatial and isospin rotations, but not under separate spatial rotations or separate isospin transformations. This in turn implies that the symmetry group for the Hamiltonian \( \mathcal{H}_f \) of \( \psi_f \) is generated only combined spatial and isospin rotations. Thus we can arrange for \( \psi_f \) to be an eigenfunction of \( \vec{K} \equiv \vec{I}_f + \vec{J}_f \), where \( \vec{I}_f \) is the isospin and \( \vec{J}_f \) the total angular momentum of \( \psi_f \). [These do not include isospin and angular momentum of the \( U \)-field.] The total angular momentum \( \vec{J}_f \) of \( \psi_f \) is \( \vec{S}_f + \vec{L}_f \), \( \vec{S}_f \) being the spin and \( \vec{L}_f \) the orbital angular momentum. Now, \( \vec{K}^2 \) has eigenvalues \( k(k + 1) \) with \( k = 0, 1/2, 1... \).

The eigenvalue \( k(k + 1) \) is \((2k + 1)\)-fold degenerate. However, in the previous section, we argued that when a Skyrmion of winding number 1 is “created”, a single fermion state is also created. Thus, consistency with the spectral flow picture requires us to choose \( \vec{K} = 0 \), as that is the only possibility without degeneracy. But this in turn implies that \( \vec{J}_f = 1/2 \). Now, a \( \vec{J}_f = 1/2 \) state can be created by combining \( \vec{S}_f \) with \( \vec{L}_f = 0 \) or 1. The correct even parity of the nucleon is reproduced only by choosing \( \vec{L}_f = 0 \), since parity is given by \((-1)^{L_f} \). States with non-zero \( L_f \) would have higher energy because of the centrifugal barrier, so choosing \( \vec{L}_f = 1 \) would provide the description of the excited states \( I(J^P) = \frac{1}{2} (\frac{1}{2}^-) \).

Let us examine what \( \psi_f \) looks like for \( U = U_c \), that is, for \( A = 1 \). For our arguments, it is only the spin and isospin dependences of \( \psi_f \) that matters. Purely from symmetry considerations, we can find out these dependences.
When $\vec{K} = 0$, $\psi_f$ is of the form

$$\psi_f := \psi_f(1) = \left( \begin{array}{c} \psi_f^+(1) \\ \psi_f^-(1) \end{array} \right),$$

where

$$\psi_f^+(1) = (\psi_f^+(1)_{\alpha\beta}),$$

and $\alpha, \beta$ are spin and isospin indices respectively:

$$(\psi_f^+(1)_{\alpha\beta}) = \epsilon_{\alpha\beta} \eta_f^+(1),$$

where

$$\eta_f^+(1) = \begin{cases} |S = 1/2, S_z = 1/2 > | I = 1/2, I_z = -1/2 > \\ -|S = 1/2, S_z = -1/2 > | I = 1/2, I_z = 1/2 > \\ = \epsilon^{ab}|1/2, S_a > |1/2, I_b >, \end{cases}$$

where $a, b = 1, 2$ and $S_1 = I_1 = 1/2, S_2 = I_2 = -1/2$. The wave function $\eta_f^+(1)$ has the desired behavior under combined spin-isospin transformations. (We have suppressed the $\vec{x}$-dependence of this wave function since it is unnecessary for our arguments.) The state $\psi_f^-(1)$ is similarly defined. Note that the wave function (3.3) must be even under parity, and so $\eta_f^-(1) = -\eta_f^+(1)$.

The wave function $\psi_f(A)$ for arbitrary $A$ follows from (3.6) by isospin rotation and is given by

$$\psi_f(A) = \left( \begin{array}{c} \psi_f^+(A) \\ \psi_f^-(A) \end{array} \right),$$

where

$$(\psi_f^+(A))_{\alpha\beta} = \epsilon_{\alpha\beta} \eta_f^+(A),$$

$$\eta_f^+(A) = \epsilon^{ab}|1/2, S_a > |1/2, I_c > A_{cb}. \tag{3.9}$$

There are also similar expressions for $\psi_f^-(A)$ and $\eta_f^-(A)$.

The transformation properties of $\psi_f(A)$ are determined by those of $\eta_f^+(A)$.

Under spatial and isospin transformations, $\eta_f^+(1)$ transforms as

$$\mathcal{R}_f(g_R) \eta_f^+(1) = \epsilon^{ab}|1/2, S_c > |1/2, I_b > (g_R)_{ca}, \tag{3.10}$$

$$\mathcal{I}_f(g_I) \eta_f^+(1) = \epsilon^{ab}|1/2, S_a > |1/2, I_c > (g_I)_{cb}, \tag{3.11}$$
where the subscript \( f \) for \( \mathcal{R}_f \) and \( \mathcal{I}_f \) is to indicate that these operators act on the fast degrees of freedom.

We will show that \( \psi_f(A) \) is invariant under the action of total spatial rotations as well as total isospin rotations, where these act on both \( \psi_f \) and on the Skyrmion zero mode \( A \). In other words, if we rotate the Skyrmion (by the action of rotations on \( A \)) and at the same time rotate the functions \( \eta_f^\pm (A) \) using (3.10), we find that \( \psi_f(A) \) is invariant [18]. The same goes for isospin rotations as well.

The proof is as follows. Under total rotation by \( g_R \),

\[
\eta_f^+(A) \rightarrow \mathcal{R}_s(g_R)A\mathcal{R}_s(g_R^\dagger)\mathcal{R}_f(g_R)\eta_f^+(1) \\
= \epsilon^{ab}|1/2, S_c > |1/2, I_d > (g_R)_{ca}(Ag_R)_{cb}, \\
= \epsilon^{ab}|1/2, S_a > |1/2, I_d > A_{db}, \\
= \eta_f^+(A).
\]

Similarly, under total isorotation by \( g_I \),

\[
\eta_f^+(A) \rightarrow \mathcal{I}_s(g_I)A\mathcal{I}_s(g_I^\dagger)\mathcal{I}_f(g_I)\eta_f^+(1) \\
= \epsilon^{ab}|1/2, S_a > |1/2, I_c > (g_I)_{cd}(g_I^\dagger)_{db}, \\
= \epsilon^{ab}|1/2, S_a > |1/2, I_c > A_{cb}, \\
= \eta_f^+(A).
\]

Similarly, \( \eta_f^-(A) \) is invariant under total rotations and total isorotations as well.

The wave function \( \psi_f(A) \) defines the section of a bundle twisted over \( Q_{slow} \). This follows from the definition (3.7) of \( \psi_f(A) \) which shows that

\[
\psi_f(-A) = -\psi_f(A).
\]

Let us now discuss the wave function of the slow part, that is, of the collective modes of the Skyrmion. On substituting the ansatz (3.1) in the Skyrme Lagrangian (2.1), we can easily find the Lagrangian for the “zero-modes” \( A(t) \),

\[
L_{\text{Skyrme}} = \int d^3x \mathcal{L}_{\text{Skyrme}} = -\frac{1}{2}a(U_c)\text{Tr}(A^\dagger \dot{A})^2 - E(U_c),
\]

(3.15)
where $a(U_c)$ and $E(U_c)$ are positive constants (see \cite{28} for details). The Hamiltonian calculated from above is
\begin{equation}
H_{\text{slow}} = \frac{2}{a(U_c)}L_\alpha L_\alpha + E(U_c),
\end{equation}
where $L_\alpha$ are the “left” rotation generators acting on sections of the trivial bundle over $Q_{\text{slow}}$. They consist of even functions of $A$ and carry integer angular momenta.

However, as emphasized earlier, $H_{\text{slow}}$ is not the B-O Hamiltonian. The correct B-O Hamiltonian is obtained by averaging the $H_{\text{slow}}$ over $\psi_f(A)$. By the general arguments presented, we know that this averaging produces a connection, which can affect the bundle-theoretic character of its eigenstates.

This bundle is actually twisted, as can be seen by parallel transporting the full wave function $\psi_s,\psi_f$ in a closed non-contractible loop, defined by \{$Ae^{i\pi\alpha/2}, \alpha \in [0, 2\pi] \}$. By hypothesis, the full wave function is unchanged under this transport. Therefore, since $\psi_f$ changes its sign, so must $\psi_s$. Transporting $\psi_f(A)$ in this closed loop cannot be interpreted as a sequence of isospin transformations from 0 to $2\pi$. But as far as $\psi_s$ is concerned, this transport in the closed loop is in fact such a sequence of isospin transformations on the slow variables $A$, generated by its isospin operators. The slow wave function is thus an isospinor. Similar arguments show that it is a spinor under spatial rotations as well. The total wave function in view of (3.12) and (3.13) thus transforms as a spinor under spatial and isospin rotations separately.

A very interesting picture emerges from the above discussion. On the one hand, the Skyrmion can be thought of the classical lump $U$ with quarks sticking to it, as suggested by the spectral flow. This composite has the correct quantum numbers of a nucleon. (with $N_c = 3$). A dual description can also be constructed, in which the quarks are averaged out. In this picture, the slow wave function is seen to be spinorial. The two descriptions are in fact the two “limits” of the unified description involving both the fast and the slow degrees of freedom. In the unified picture, the Skyrmion is only a part of the full description. The full wave function $\psi_{\text{total}} = \psi_f(A, \vec{x}).\psi_s(A)$ is also spinorial, and thus describes the nucleon.

4 Parity Doubles in Light Nuclei
4.1 Non-spherical Symmetries and Static Fields

Stable ansatze for $U$ minimizing energy and with winding numbers $|n| \geq 2$ have been numerically found by various groups [13, 14, 15, 29]. Energy densities for these static configurations have been plotted and the remarkable fact has been discovered that they are invariant under discrete subgroups $G_R$ of the spatial rotation group $SO(3)_R$. In this section, we briefly explore the theoretical and possible phenomenological implications of this discovery.

The group $G_R$ is the symmetry group of energy density. It is not necessarily the invariance group of the static $U$-field. Published work does not report on the symmetry of the latter.

The two-flavor chiral Lagrangian is invariant under $SO(3)_R \times SO(3)_I$ where $SO(3)_I$ is the isospin group. The former acts on the operator $U(x)$ according to $U_R(s_R)U(x)U_R^{-1}(s_R) = U(s_R\vec{x},t)$ while the latter does so according to $U_I(\gamma_I)U(x)U_I^{-1}(\gamma_I) = \gamma_I^\dagger U(x)\gamma_I$. Here $s_R \in SO(3)_R, \gamma_I \in SU(2)_I$ (the covering group of $SO(3)_I$) and $U_R(s_R), U_I(\gamma_I)$ are their quantum operators. We also identify $SO(3)$ and $SU(2)$ with their standard defining matrix groups.

Suppose that $U_c$ is a static configuration with its energy density having the symmetry of $G_R$. Then $U_c$ can be invariant under any subgroup of $SO(3)_R \times SO(3)_I$ which restricted to the first factor is $G_R$. That is because the energy density being invariant under $SO(3)_I$ will have the symmetry $G_R$.

There can be subgroups $G_I$ of $SO(3)_I$ alone, in addition to subgroups involving also $G_R$, which leave $U_c$ invariant. There is no need for $G_I$ too to be discrete.

If $G_I \subset SU(2)_I$ is the double cover of $G_I$, its elements commute with $U_c$. If $G_I$ is irreducible, then $U_c$, being in $SU(2)$, is just $\pm 1$ by Schur’s lemma. So $G_I$ must be reducible, and hence $G_I$ and $G_I$ must be $\mathbb{Z}_N$ or $U(1)$ groups. We can think of them if necessary after an $SU(2)$-conjugation to consist of elements of the form $e^{i\alpha \xi}$. Excluding the trivial case $G_I = \{\pm 1\}$, we can thus see that $U_c(\vec{x}) = e^{i\alpha \theta(\vec{x})}$ and that $G_I$ and $G_I$ are in fact $U(1)$.

Let $G_R \subset SU(2) \equiv SU(2)_R$ be the inverse image of $G_R$ for the homomorphism $SU(2)_R \rightarrow SO(3)_R$ with $\alpha_R \rightarrow a_R$. Then there must also exist a subgroup

$$G_R = \{(\alpha_R, L(\alpha_R)) : L \text{ a homomorphism of } G_R \text{ into } SU(2)_I\}$$

(4.1)
of $SU(2)_R \times SU(2)_I$ leaving $U_c$ invariant:
\[
\mathcal{L}(\alpha_R)U_c(a_R^{-1}\vec{x})\mathcal{L}(\alpha_R)^{-1} = U_c(\vec{x}).
\] (4.2)

Note that if $L(a_R)$ is the image of $\mathcal{L}(\alpha_R)$ in $SO(3)_I$, we can write
\[
\mathcal{L}(\alpha_R)\tau_j\mathcal{L}(\alpha_R)^{-1} = \tau_k L(a_R)_{kj}.
\] (4.3)

The full symmetry group of $U_c$ is
\[
\mathcal{D} = \mathcal{G}_R \times \mathcal{G}_I,
\] (4.4)
where
\[
\mathcal{G}_I = \{1\} \times \mathcal{G}_I.
\] (4.5)

We have already found those $U_c$ invariant under $\mathcal{G}_I$. Let us now do the same job for $\mathcal{G}_R$ and $\mathcal{D}$.

We begin with simple preliminary observations: Any function $\chi(\vec{x})$ has the expansion
\[
\sum c_{lm}(r)Y_{lm}(\hat{x}), \hat{x} = \vec{x}/r.
\]
If $s_R \in SO(3)$ and we identify $\hat{x}$ with its third column, $\hat{x} = (s_R)_{i3}$, then also, $Y_{lm}(\hat{x}) = \text{constant} \times D^l_{mn}(s_R)$, where $D^l(s_R)$ are the rotation matrices in the conventional basis. Hence $\chi(\vec{x}) = \sum c_{lm}(r)D^l_{m0}(s_R)$.

Now suppose that $\mathcal{L}$ is the trivial homomorphism, that is, $\mathcal{L} : \mathcal{G}_R \to \{1\}$. In that case, the ansatz for $U_c$ can be found as follows. Write $U_c(\vec{x}) = e^{i \sum \phi_j(\vec{x})}$, $\phi_j(\vec{x}) \in \mathbb{R}$. Then $\phi_j(a_R^{-1}\vec{x}) = \phi_j(\vec{x})$. Expanding $\phi_j$,
\[
\phi_j(\vec{x}) = \sum c^j_{lm}(r)D^l_{m0}(s_R),
\] (4.6)
we find that
\[
c^j_{lm}(r) = 0 \quad \text{unless} \quad D^l_{m0}(a_R^{-1}s_R) = D^l_{mm'}(a_R^{-1})D^l_{m'0}(s_R) \quad \text{is} \quad D^l_{m0}(s_R).
\] (4.7)

In other words, only those indices $m$ transforming trivially under the action of $a_R$ $[D^l_{mm'}(a_R^{-1}) = \delta_{mm'}]$ survive in (4.6).

If $\mathcal{L}$ is non-trivial, we still have the expansion (4.6), but the conditions on $c^j_{lm}$ are different, being
\[
[\tau_k L(a_R)_{kj}]c^j_{lm}(r)D^l_{m0}(a_R^{-1}s_R)] = \tau_k c^k_{lm}(r)D^l_{m0}(s_R).
\] (4.8)
Let \( \bar{m}, \bar{m}' \) denote indices transforming by the representation \( L \) of \( G_R \):

\[
\mathcal{L}(\alpha_R) \tau_{\bar{m}} \mathcal{L}(\alpha_R)^{-1} = \tau_{\bar{m}'} L(a_R) \bar{m}' \\
D^l_{\bar{m}0}(a_R^{-1} s_R) = L(a_R^{-1}) \bar{m} \bar{m}' D^l_{\bar{m}'}(s_R).
\]

(4.9) \hspace{1cm} (4.10)

There could be such indices for several values of \( l \). Even for one \( l \), the representation \( L \) could occur with multiplicity, so we can write \( (\bar{m}, \nu) \), where \( \nu \) accounts for this multiplicity. The requirement of \( G_R \)-invariance is now seen to be accounted for by writing

\[
U_c(\vec{x}) = e^{i \sum_{\bar{m}} \tau_{\bar{m}} \phi_{\bar{m}}(\vec{x})}, \\
\phi_{\bar{m}}(\vec{x}) = \sum_{l, \nu} a_{\nu}(r) D^l_{(\bar{m}, \nu)}(s_R).
\]

(4.11) \hspace{1cm} (4.12)

This formula is perfectly general if \( L \) is irreducible, that is if \( G_R \neq \mathbb{Z}_N \) or \( D_2 \). If \( G_R \) is \( \mathbb{Z}_N \), then \( L \) splits into three irreducible representations (IRR’s), the image of the generator \( z \) of \( G_R \) in these IRR’s being \( e^{2\pi i/N}, e^{-2\pi i/N} \) and 1. The index \( \bar{m} \) can be taken to be +, − and 3 for these IRR’s. With \( \tau_{\pm} = \tau_1 \pm i\tau_2 \), we now have the freedom to generalize to

\[
U_c(\vec{x}) = e^{i\left[c_1 \tau_1 + \phi_+(\vec{x}) + \text{complex conjugate}\right] + c_3 \tau_3 \phi_3(\vec{x})}, \hspace{0.5cm} c_3^* = c_3.
\]

(4.13)

A similar generalization exists for \( D_2 \).

There is a difference between \( \hat{G}_R \) and \( \hat{D} \) if \( G_I = U(1) \). But then \( U_c(\vec{x}) = e^{i\tau_3 \theta(\vec{x})} \), and \( L(a_R) \) can only be a \( \mathbb{Z}_N \subset U(1) \). As it can be absorbed in \( U(1) \), we are just in a special case of having a trivial \( L \).

### 4.2 Collective Coordinates

As the symmetry of the Hamiltonian is \( SO(3)_R \times SO(3)_I \), collective coordinates are introduced using its action on \( U_c \), that is by writing

\[
U(x) = \gamma_I(t) U_c[T_R(t)^{-1} x] \gamma_I(t)^{-1}, \hspace{0.5cm} T_R(t) \in SO(3)_R, \gamma_I(t) \in SU(2)_I
\]

(4.14)

and quantizing \( T_R \) and \( \gamma_I \). The actions

\[
\hat{G}_R \ni (\alpha_R, \mathcal{L}(\alpha_R)) : (T_R, \gamma_I) \rightarrow (T_R a_R, \gamma_I \mathcal{L}(\alpha_R)), \\
\hat{G}_I \ni (\{1, \beta_I\}) : (T_R, \gamma_I) \rightarrow (T_R, \gamma_I \beta_I)
\]

(4.15) \hspace{1cm} (4.16)

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of the symmetry group $D$ leaves $U$ invariant.

The eigenfunctions $\psi$ of the Skyrme Hamiltonian are functions on $SU(2) \times SU(2)_I$, the universal cover of $SO(3)_R \times SO(3)_I$. We can denote its coordinates by $(\tau_R, \gamma_I)$, and identify $T_R$ as the image of $\tau_R$ under the homomorphism $SU(2)_R \to SO(3)_R$ \cite{28, 30}.

The group $D$ acts on $(\tau_R, \gamma_I)$,

$$ (\tau_R, \gamma_I) \to (\tau_R \alpha_R, \gamma_I \mathcal{L}(\alpha_R)), \quad (4.17) $$

$$ (\tau_R, \gamma_I) \to (\tau_R, \gamma_I \beta_I) \quad (4.18) $$

and hence also on $\psi$. A basic result \cite{28, 30} is that $\psi$’s are vector-valued, $\psi = (\psi_m)$ and transform on the index $m$ by an irreducible representation $\rho$ of $D$:

$$ \psi_m(\tau_R \alpha_R, \gamma_I \mathcal{L}(\alpha_R)) = \psi_n(\tau_R, \gamma_I)\rho_{nm}(\alpha_R, \mathcal{L}(\alpha_R)), \quad (4.19) $$

$$ \psi_m(\tau_R, \gamma_I \beta_I) = \psi_n(\tau_R, \gamma_I)\rho_{nm}((1, \beta_I)). \quad (4.20) $$

There are also restrictions on $\rho$ from the underlying quark model: $\psi$ must be spinorial under both spin and isospin rotations, or it must be tensorial under both.

### 4.3 What Parity Does to Symmetries

An adaptation of our previous work \cite{19, 17, 18} to the present situation shows that the parity transform $\mathcal{P} U_c$ of $U_c$ can be obtained by applying a particular element $(s_P, \iota_P)$ of $SO(3)_R \times SU(2)_I$:

$$ (\mathcal{P} U_c)(\vec{x}) = U_c(-\vec{x})^\dagger = \iota_P U_c(s_P^{-1}\vec{x})\iota_P^{-1}. \quad (4.21) $$

That is because the parity transform of the energy density can be obtained by applying a spatial rotation.

Let $\zeta_P$ be an inverse image of $s_P$ in $SU(2)$. It is uncertain up to a sign, so we pick one and call it $\zeta_P$. The symmetry group of $\mathcal{P} U_c$ is then $(\zeta_P^{-1}, \iota_P^{-1})D(\zeta_P, \iota_P)$. Also as $\mathcal{P}$ commutes with $SO(3)_R \times SO(3)_I$, we have the fundamental result

$$ (\zeta_P^{-1}, \iota_P^{-1})D(\zeta_P, \iota_P) = D. \quad (4.22) $$

The parity transform $\mathcal{P} \psi$ of $\psi$ is specified by

$$ (\mathcal{P} \psi)_m(\tau_R, \gamma_I) = \psi_m(\tau_R \zeta_P, \gamma_I \iota_P). \quad (4.23) $$
Its transformation under \( D \) is given, according to (4.19,4.20) by
\[
(P\psi)_m(\tau_R \alpha_R, \gamma_1 L(\alpha_R)) = (P\psi)_n(\tau_R, \gamma_1) \rho_{nm}(\zeta P^{-1} \alpha_R \zeta P, \iota P L(\alpha_R)) \tag{4.24}
\]
\[
(P\psi)_m(\tau_R, \gamma_1 \beta_I) = (P\psi)_n(\tau_R, \gamma_1) \rho_{nm}(\{1, \iota P^{-1} \beta_I \iota P\}). \tag{4.25}
\]

The \( \rho \) with its arguments here defines the parity transform \( P\rho \) of the representation \( \rho \):
\[
P\rho : (\alpha_R, L(\alpha_R)) \rightarrow \rho(\zeta P^{-1} \alpha_R \zeta P, \iota P L(\alpha_R)), \tag{4.26}
\]
\[
(\{1, \beta_I\} \rightarrow \rho(\{1, \iota P^{-1} \beta_I \iota P\}). \tag{4.27}
\]

It can happen that \( P\rho \) is inequivalent to \( \rho \), \( P\rho \neq \rho \). That can be the case for example when \( G_R \) is a dihedral or a \( \mathbb{Z}_N \) group [19, 17, 18]. When that is so, our previous work predicts a pair of approximately degenerate states.

Actually when \( P\rho \neq \rho \), what is predicted in the absence of quarks is a breakdown of parity by a mechanism like that of the QCD \( \theta \)-angle. But with quarks, parity is restored, but we are left with parity doubles.

In a similar manner, we can establish the existence of time-reversal doubles if \( \rho \) is a complex representation. Time reversal \( T \) is actually broken in the absence of quarks, but is restored when quarks are included.

In molecular physics, it is the electrons which restore the \( P \)- and \( T \)-symmetries ruined by the underlying quantum shapes. The role of electrons is thus assumed here by quarks.

The mechanism restoring \( \mathcal{P} \) and \( T \) via the B-O approximation is the same as that discussed in the last section. Let us recall it, adapting it appropriately. By assumption, the domain \( V^{(\rho_0)} \) of the full Hamiltonian \( \mathcal{H} \) must be associated with the trivial representation \( \rho_0 \). An eigenstate \( \psi_f^{(\mathcal{P})} \) of \( \mathcal{H}_f \) is a section of a vector bundle over \( Q \) in the B-O approximation (the superscripts on the wave functions indicate the UIR). When the B-O Hamiltonian \( \mathcal{H}_s \) is calculated as discussed in the previous section, it can be shown to contain a connection and has a domain associated with the UIR \( \rho \), the complex conjugate of \( \mathcal{P} \). So an eigenstate \( \psi_s^{(\rho)} \) of \( \mathcal{H}_s \) corresponds to \( \rho \) and the product wave function \( \psi = \psi_s^{(\rho)} \psi_f^{(\mathcal{P})} \) corresponds to \( \rho \otimes \mathcal{P} \). But \( \mathcal{H} \) and \( \mathcal{H}_s \) act on the total wave function and their domain can only correspond to \( \rho_0 \). That is now easily arranged as \( \rho_0 \) occurs in the reduction of \( \rho \otimes \mathcal{P} \). The correct total wave function in the B-O approximation is thus the orthogonal projection \( \chi^{(\rho_0)} = P[\psi_s^{(\rho)} \psi_f^{(\mathcal{P})}] \) of \( \psi \) to \( V^{(\rho_0)} \). Now it can happen that the parity transform \( \mathcal{P}\rho \) of \( \rho \) is \( \mathcal{P} \) and hence that of \( \mathcal{P} \) is \( \rho \). The parity transform \( \mathcal{P}\chi^{(\rho_0)} \) of
\( \chi^{(\rho_0)} \) is then of the form \( P[\psi^{(\rho)} \psi^{(\rho)}] \in V^{(\rho_0)} \). It is still in the domain of \( \mathcal{H} \) and \( \mathcal{H}_s \), so there is no question of \( \mathcal{P} \)-violation. The same goes for \( \mathcal{T} \). But there is a doubling of states. The doubles with definite \( \mathcal{P} \), for example, in the leading approximation are linear combinations of \( \chi^{(\rho_0)} \) and \( \mathcal{P}\chi^{(\rho_0)} \).

In this manner, we can see that when quantum Skyrmions violate \( \mathcal{P} \) or \( \mathcal{T} \), then we may have \( \mathcal{P} \) or \( \mathcal{T} \) doubles and no \( \mathcal{P} \) or \( \mathcal{T} \) symmetry breakdown after fast variables (in this case the quarks) are included.

Generally, some higher order interactions will lift the degeneracy between these parity doublets, and so what we expect to see experimentally are pairs of approximately degenerate states with opposite parity, all other quantum numbers (except energy) being identical. A candidate mechanism for lifting this degeneracy was discussed in [18].

We can now ask if any of the light nuclei exhibit any parity doubling in the low energy spectra. The spectra of \(^9\text{Be}\) and \(^9\text{B}\) show clear evidence of such doubling [31]. There are at least three low-lying pairs of states with all quantum numbers identical except parity (and energy) for \(^9\text{Be}\). Similarly, there are at least two such pairs for \(^9\text{B}\). The typical separation between parity partners is of the order of 0.5 to 5 MeV. The \( B = 9 \) Skyrmion has a tetrahedral symmetry, and may thus be a good candidate for illustrating the mechanism discussed in this paper.

### 5 Conclusions and Outlook

Using spectral flow arguments, we have shown that we can strengthen the identification [3, 4] of the topological winding number of the Skyrmion with the fermion number. In addition, our arguments naturally lead to the existence of quarks moving about in the field of this Skyrmion. The parity-even combination of the quark states correctly describes the nucleon and delta, whereas the parity-odd combination describes states \( I(J^P) = \frac{1}{2}(\frac{1}{2}^-) \) and their excitations. There are candidates for such states in the Particle Data booklet, such as \( N(1535) \) and \( N(1650) \). Using the B-O approximation, we also showed that when the fast quarks are averaged out, the Skyrmion itself becomes spinorial. As discussed in [15] (where it was argued that systems with certain discrete symmetries must exhibit a parity doublet structure), we also argue that some nuclei may show such a structure in their low energy spectra. We suggest the case of the \( B = 9 \) nuclei to support our claim.

Our focus has been on the case of two flavors. To discuss \( N_f \geq 3 \), we start
by looking at a generalized spherically symmetric ansatz that is analogous to
the hedgehog ansatz (See for example, Ch.17 of [28]). The B-O treatment of
this situation involves non-flat bundles. As discussed in [18], the application
of our methods to this case is straightforward.

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