Empirical spectral distributions of high-dimensional matrix-valued processes driven by fractional Brownian motion

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Abstract: In this article, we study a class of symmetric or Hermitian random matrices, whose entries are generated from the solution $X_t$ of stochastic differential equation driven by fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. Three types of matrices are considered: (1) Wigner-type matrices; (2) matrices with entries being local sums of i.i.d. copies of $X_t$; (3) Wishart-type matrices. When the dimension of the matrix-valued processes grows to infinity, we characterize the limit of the empirical measure-valued processes of their eigenvalues.

As a byproduct, we obtain pathwise Hölder continuity for the solution of the SDE driven by fractional Brownian motion under proper conditions, and provide a general tightness criterion for probability measures on the space $C([0, T], \mathbb{P}(\mathbb{R}))$ of continuous measure-valued processes.

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1. Introduction

There has been increasing research activity on matrix-valued stochastic differential equations (SDEs) in recent years. A prominent example is the class of generalized Wishart processes introduced in [8]. This class generalizes classical examples of symmetric Brownian motion [7],
the Wishart process [4] and the symmetric matrix-valued process whose entries are independent Ornstein-Uhlenbeck processes [5]. Recent works on generalized Wishart processes include [20,21] and [9,10]. A common feature of these processes is that they are all driven by independent Brownian motions.

In contrast, the study of SDE matrices driven by fractional Brownian motions (fBm) has a shorter history and less literature. To our best knowledge, the first paper is [15], where the symmetric fractional Brownian matrix was studied. The SDE for the associated eigenvalues and conditions for non-collision of the eigenvalues were obtained when the Hurst parameter $H > 1/2$ by using fractional calculus and Malliavin calculus. The convergence in distribution of the empirical spectral measure of a scaled symmetric fractional Brownian matrix was established in [17] by using Malliavin calculus and a tightness argument. These results were generalized to centered Gaussian process in [13]. Besides, [18] obtained SDEs for the eigenvalues, conditions for non-collision of eigenvalues and the convergence in law of the empirical eigenvalue measure processes for a scaled fractional Wishart matrix (the product of an independent fractional Brownian matrix and its transpose). In the present paper, by developing a different approach, we obtain stronger results on the convergence of empirical measures of eigenvalues for a significantly larger class of matrix-valued processes driven by fBms (see Remark 3.1).

More precisely, consider the following 1-dimensional SDE

$$dX_t = \sigma(X_t) \circ dB^H_t + b(X_t)dt, \quad t \geq 0,$$  \hspace{1cm} (1.1)

with initial value $X_0$ independent of $B^H_t = (B^H_s)_{s \geq 0}$. Here, $B^H_t$ is a fractional Brownian motion with Hurst parameter $H \in (1/2,1)$, and the differential $\circ dB^H_t$ is in the Stratonovich sense. It has been shown in [14] that there is a unique solution to SDE (1.1) if the coefficient functions $\sigma$ and $b$ have bounded derivatives which are Hölder continuous of order greater than $1/H - 1$.

Let $\{X_{ij}(t)\}_{i,j \geq 1}$ be i.i.d. copies of $X_t$ and $Y^N(t) = (Y^N_{ij}(t))_{1 \leq i,j \leq N}$ be a symmetric $N \times N$ matrix with entries

$$Y^N_{ij}(t) = \begin{cases} \frac{1}{\sqrt{N}} X_{ij}(t), & 1 \leq i < j \leq N, \\ \frac{\sqrt{2}}{\sqrt{N}} X_{ii}(t), & 1 \leq i \leq N. \end{cases}$$  \hspace{1cm} (1.2)

Let $\lambda^N_1(t) \leq \cdots \leq \lambda^N_N(t)$ be the eigenvalues of $Y^N(t)$ and

$$L_N(t)(dx) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda^N_{i}(t)}(dx)$$  \hspace{1cm} (1.3)

be the empirical distribution of the eigenvalues.

In this paper, we aim to study, as the dimension $N$ tends to infinity, the limiting behavior of the empirical measure-valued processes $\{L_N(t), t \in [0,T]\}$ defined by (1.3) and the empirical measure-valued processes arising from other related matrix-valued processes in the space $C([0,T], \mathbf{P}(\mathbb{R}))$ of continuous measure-valued processes, with $\mathbf{P}(\mathbb{R})$ being the space of probability measures equipped with its weak topology. A summary of the contents of the paper is as follows.

In Section 2, we study the solution of SDE (1.1). In [11], a pathwise upper bound for $|X_t|$ on the time interval $[0,T]$ was obtained under some boundedness and smoothness conditions on the coefficient functions. We adapt the techniques used in [11] to study the increments $|X_t - X_s|$ for $t,s \in [0,T]$ and obtain the pathwise Hölder continuity for $X_t$. Moreover, under some integrability conditions on the initial value $X_0$, the Hölder norm of $X_t$ is shown to be $L^p$-integrable. This result, which seems new, has its own interest and may have applications in other studies of the SDE (1.1).

In Section 3, we prove the almost sure high-dimensional convergence in $C([0,T], \mathbf{P}(\mathbb{R}))$ for the empirical spectral measure-valued process (1.3) of the Wigner-type matrix (1.2). We also obtain the PDE for the Stieltjes transform $G_t(z)$ of the limiting measure process $\mu_t$, which turns to
be the complex Burgers’ equation up to a change of variable (see Remark 3.2 and Remark 3.3). This generalizes the results of [17], where the special case of \( X_t = B_t^H \) was studied (note that the convergence obtained therein is in law). A complex analogue is also studied at the end of this section.

In Section 4, we extend the model of Wigner-type matrix to the locally dependent symmetric matrix-valued stochastic process \( R^N(t) = (R_{ij}^N(t))_{1 \leq i,j \leq N} \), where each \( R_{ij}^N(t) \) is a weighted sum of i.i.d. \( \{X_{(k,l)}(t)\} \) for \( (k, l) \) in a fixed and bounded neighborhood of \( (i, j) \) (see (4.1) for the definition). We also establish the almost sure convergence of the empirical spectral measure-valued process in \( C([0,T], \mathcal{P}(\mathbb{R})) \). Moreover, the limiting measure-valued process \( \mu_t \) is characterized by an equation satisfied by its Stieltjes transform (see Theorem 4.1). It is worth noticing that for the proofs of the almost sure convergence for the empirical spectral measure-valued processes in both Section 3 and Section 4, we follow the strategy used in [21] (which was inspired by [1]).

In Section 5, we study the high-dimensional limit of the empirical spectral measure-valued processes of the Wishart-type matrix-valued processes given by (5.1) and its complex analogue (5.9). In the special case where \( X_t = B_t^H \), we recover the convergence result in [18]. We also obtain the PDE for the Stieltjes transform \( G_t(z) \) of the limit measure process \( \mu_t \) (see Remarks 5.4 and 5.6).

For the Wishart-type matrix-valued processes, we are not able to obtain the almost sure convergence in \( C([0,T], \mathcal{P}(\mathbb{R})) \) for the empirical spectral measure-valued processes as done in Section 3 and Section 4 since the method used there heavily relies on the independence of the upper triangular entries. Instead, we obtain the convergence in law inspired by a tightness argument on \( C([0,T], \mathcal{P}(\mathbb{R})) \) used in [13, 17, 18]. Note that, however, this argument of tightness was only sketched very briefly in these references (see Remark 5.1) and seems vague. In Appendix B, we provide a tightness criterion (Theorem B.1) for subsets in \( C([0,T], \mathcal{P}(\mathbb{R})) \), which is of interest in itself and also provides a rigorous justification for the argument used in [13, 17, 18].

Finally, some preliminaries on random matrices that will be used in the proofs are provided in Appendix A for the reader’s convenience.

2. Hölder continuity of the solution to the SDE (1.1)

Some notations are in order. The Hölder norm of a Hölder continuous function \( f \) of order \( \beta \) is

\[
\|f\|_{a,b;\beta} = \sup_{a \leq x < y \leq b} \frac{|f(x) - f(y)|}{|x - y|^{\beta}}.
\]

We also use

\[
\|M\|_F = \left( \sum_{i,j=1}^N |M_{ij}|^2 \right)^{1/2},
\]

to denote the Frobenius norm (also known as the Hilbert-Schmidt norm or 2-Schatten norm) of a \( N \times N \) matrix \( M = (M_{ij})_{1 \leq i,j \leq N} \).

2.1. Preliminaries on fractional calculus and fractional Brownian motion

In this subsection, we recall some basic results in fractional calculus. See [19] for more details. Let \( a, b \in \mathbb{R} \) with \( a < b \) and let \( \alpha > 0 \). The left-sided and right-sided fractional Riemann-Liouville integrals of \( f \in L^1([a,b]) \) of order \( \alpha \) are defined for almost all \( t \in (a,b) \) by

\[
I_{a+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds,
\]

and

\[
I_{b-}^\alpha f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) \, ds,
\]

where \( \Gamma(\cdot) \) is the Gamma function.
respectively, where \((-1)^{-\alpha} = e^{-i\pi\alpha}\) and \(\Gamma(\alpha) = \int_0^\infty r^{\alpha-1}e^{-r}dr\) is the Euler gamma function.

Let \(I_{a+}^\alpha(L^p)\) (resp. \(I_{b-}^\alpha(L^p)\)) be the image of \(L^p([a, b])\) by the operator \(I_{a+}^\alpha\) (resp. \(I_{b-}^\alpha\)). If \(f \in I_{a+}^\alpha(L^p)\) (resp. \(f \in I_{b-}^\alpha(L^p)\)) and \(\alpha \in (0, 1)\), then the left-sided and right-sided fractional derivatives are defined as

\[
D_{a+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(t)}{(t-a)^\alpha} - \frac{\int_a^t f(t) - f(s) }{(t-s)^{\alpha+1}} ds \right),
\]

\[
D_{b-}^\alpha f(t) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left( \frac{f(t)}{(b-t)^\alpha} - \frac{\int_t^b f(t) - f(s) }{(s-t)^{\alpha+1}} ds \right)
\]

(2.1)

respectively, for almost all \(t \in (a, b)\).

Let \(C^\alpha([a, b])\) denote the space of \(\alpha\)-Hölder continuous functions of order \(\alpha\) on the interval \([a, b]\). When \(\alpha p > 1\), then we have \(I_{a+}^\alpha(L^p) \subset C^{\alpha - \frac{1}{p}}([a, b])\). On the other hand, if \(\beta > \alpha\), then \(C^{\beta}([a, b]) \subset I_{a+}^\alpha(L^p)\) for all \(p > 1\).

The following inversion formulas hold:

\[
I_{a+}^\alpha(I_{a+}^\beta f) = I_{a+}^{\alpha + \beta} f, \quad f \in L^1;
\]

\[
D_{a+}^\alpha(I_{a+}^\beta f) = f, \quad f \in L^1;
\]

\[
I_{a+}^\alpha(I_{a+}^\beta f) = f, \quad f \in I_{a+}^\alpha(L^1);
\]

\[
D_{a+}^\alpha(D_{a+}^\beta f) = D_{a+}^{\alpha + \beta} f, \quad f \in I_{a+}^{\alpha + \beta}(L^1), \quad \alpha + \beta \leq 1.
\]

Similar inversion formulas hold for the operators \(I_{b-}^\alpha\) and \(D_{b-}^\alpha\) as well.

We also have the following integration by parts formula.

**Proposition 2.1.** If \(f \in I_{a+}^\alpha(L^p), g \in I_{b-}^\beta(L^q)\) and \(\frac{1}{p} + \frac{1}{q} = 1\), we have

\[
\int_a^b (D_{a+}^\alpha f)(s)g(s)ds = \int_a^b f(s)(D_{b-}^\beta g)(s)ds.
\]

(2.2)

The following proposition indicates the relationship between Young’s integral and Lebesgue integral.

**Proposition 2.2.** Suppose that \(f \in C^\lambda(a, b)\) and \(g \in C^\mu(a, b)\) with \(\lambda + \mu > 1\). Let \(\lambda > \alpha\) and \(\mu > 1 - \alpha\). Then the Riemann-Stieltjes integral \(\int_a^b f dg\) exists and it can be expressed as

\[
\int_a^b f dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f(t) D_{b-}^{1-\alpha} g_b(t) dt,
\]

(2.3)

where \(g_b(t) = g(t) - g(b)\).

It is known that almost all the paths of \(B^H\) are \((H - \varepsilon)\)-Hölder continuous for \(\varepsilon \in (0, H)\). By the Fernique Theorem, we have the following estimation for its Hölder norm.

**Lemma 2.1.** There exists a positive constant \(\alpha = \alpha(H, \varepsilon, T)\) depending on \((H, \varepsilon, T)\), such that

\[
E\left[\alpha^{\|B^H\|_{\beta, T; H-\varepsilon}}\right] < \infty.
\]

### 2.2. Hölder continuity

By [16, Theorem 2.1], under proper conditions, the paths of \(X_t\) are a.s. \((H - \varepsilon)\)-Hölder continuous for any \(\varepsilon \in (0, H)\), a.s. In this subsection, we provide some estimations for the Hölder norm of \(X\), following the approach developed in [11, Theorem 2].
\section*{Theorem 2.1.} Suppose that the coefficient functions $\alpha$ and $\beta$ are bounded and have bounded derivatives which are Hölder continuous of order greater than $1/(H-\varepsilon)-1$ for some $\varepsilon \in (0, H - \frac{1}{2})$. Then there exists a constant $C(H, \varepsilon)$ that depends on $(H, \varepsilon)$ only, such that for all $T > 0$ and $0 \leq s < t \leq T$,

$$|X_t - X_s| \leq C(H, \varepsilon) \|\sigma\| \left[ \|B^H\|_{0, T; H-\varepsilon}(t-s)^{H-\varepsilon} \vee \|B^H\|_{0, T; H-\varepsilon} \|\sigma\|^{\frac{1}{1/(H-\varepsilon)-1}}(t-s) \right] + 2\|b\|_{\infty}(t-s).$$

(2.4)

Consequently, there exists a random variable $\xi$ with $E|\xi|^p < \infty$ for all $p > 1$ such that for $0 \leq s < t \leq T$,

$$|X_t(\omega) - X_s(\omega)| \leq \xi(\omega)|s-t|^{H-\varepsilon}, \text{ a.s.}$$

Proof. Fix $\alpha \in (1 - H + \varepsilon, 1/2)$. By Proposition 2.2, for $0 \leq s \leq t \leq T$,

$$\left| \int_s^t \sigma(X_r) \circ dB^H_r \right| \leq \int_s^t D_\alpha^s \sigma(X_r) D_\alpha^{t-s}(B_t^H - B_s^H) \, dr.$$

(2.5)

By (2.1) and Lemma 2.1, for $r \in [s, t]$, we have

$$|D_\alpha^s \sigma(X_r)| \leq \frac{1}{\Gamma(\alpha)} \frac{|\sigma(X_r)|}{|r-s|^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_s^r \frac{|\sigma(X_r) - \sigma(X_u)|}{|r-u|^{\alpha+1}} \, du$$

(2.6)

and

$$|D_\alpha^s \sigma(X_r)| \leq \frac{1}{\Gamma(1-\alpha)} |r-s|^{-\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_s^r \frac{|X_r - X_u|}{|r-u|^{\alpha+1}} \, du$$

(2.7)

By (2.5), (2.6) and (2.7), we have

$$\left| \int_s^t \sigma(X_r) \circ dB^H_r \right|$$

$$\leq \int_s^t \left( \frac{\|\sigma\|}{\Gamma(1-\alpha)} |r-s|^{-\alpha} + \frac{\alpha\|\sigma'\|}{\Gamma(1-\alpha)} \frac{|X_s|}{(H-\varepsilon-\alpha)} |r-s|^{H-\varepsilon-\alpha} \right)$$

$$\left( \frac{1}{\Gamma(\alpha)} + \frac{1}{(H-\varepsilon+\alpha-1)\Gamma(\alpha)} \right) \|B^H\|_{s, t; H-\varepsilon} \|t-r|^{H-\varepsilon-1+\alpha} dr$$

$$+ \left( \frac{1}{\Gamma(\alpha)} + \frac{1}{(H-\varepsilon+\alpha-1)\Gamma(\alpha)} \right) \frac{\alpha\|\sigma'\|}{\Gamma(1-\alpha)} \|X_s|_{s, t; H-\varepsilon} \|B^H\|_{s, t; H-\varepsilon} \int_s^t |t-r|^{H-\varepsilon-1+\alpha} |r-s|^{-\alpha} dr$$
\[ \begin{align*}
&= \left( \frac{1}{\Gamma(\alpha)} + \frac{1 - \alpha}{(H - \varepsilon + \alpha - 1)\Gamma(\alpha)} \right) \frac{\|\sigma\|_\infty}{\Gamma(1 - \alpha)} \|B^H\|_{s,t:H - \varepsilon(t - s)^{H - \varepsilon}} \beta(H - \varepsilon + \alpha, 1 - \alpha) \\
&\quad + \left( \frac{1}{\Gamma(\alpha)} + \frac{1 - \alpha}{(H - \varepsilon + \alpha - 1)\Gamma(\alpha)} \right) \frac{\alpha\|\sigma'\|_\infty}{\Gamma(1 - \alpha)} \|X\|_{s,t:H - \varepsilon} \|B^H\|_{s,t:H - \varepsilon(t - s)^{2H - 2\varepsilon}} \\
&\quad \times \beta(H - \varepsilon + \alpha, H - \varepsilon - \alpha + 1) \\
\leq& C_1(H, \varepsilon)\|B^H\|_{0,T:H - \xi} \left[ \|\sigma\|_\infty(t - s)^{H - \varepsilon} + \|\sigma'\|_\infty \|X\|_{s,t:H - \varepsilon(t - s)^{2H - 2\varepsilon}} \right], \tag{2.8}
\end{align*} \]

where \( \beta(p, q) \) is the Beta function, and \( C_1(H, \varepsilon) \) is a constant depending on \((H, \varepsilon)\) only.

Hence, by (2.8), for \( 0 \leq s \leq t \leq T \),

\[ |X_t - X_s| \leq \left| \int_s^t \sigma(X_r) \circ dB^H_r \right| + \left| \int_s^t b(X_r) \, dr \right| \]
\[ \leq C_1(H, \varepsilon)\|B^H\|_{0,T:H - \varepsilon} \left[ \|\sigma\|_\infty(t - s)^{H - \varepsilon} + \|\sigma'\|_\infty \|X\|_{s,t:H - \varepsilon(t - s)^{2H - 2\varepsilon}} \right] \]
\[ + \|b\|_\infty(t - s), \tag{2.9} \]

and therefore,

\[ \|X\|_{s,t:H - \varepsilon} \leq C_1(H, \varepsilon)\|B^H\|_{0,T:H - \varepsilon} \left[ \|\sigma\|_\infty + \|\sigma'\|_\infty \|X\|_{s,t:H - \varepsilon(t - s)^{H - \varepsilon}} \right] \]
\[ + \|b\|_\infty(t - s)^{1 - H + \varepsilon}. \tag{2.10} \]

Choose \( \Delta \) such that

\[ \Delta^{H - \varepsilon} = \frac{1}{2C_1(H, \varepsilon)\|B^H\|_{0,T:H - \varepsilon} \|\sigma'\|_\infty}, \]

and if the denominator vanishes, we set \( \Delta = \infty \).

If \( \Delta \geq t - s \), (2.10) yields

\[ \|X\|_{s,t:H - \varepsilon} \leq C_1(H, \varepsilon)\|B^H\|_{0,T:H - \varepsilon} \|\sigma\|_\infty + \frac{1}{2} \|X\|_{s,t:H - \varepsilon} + \|b\|_\infty(t - s)^{1 - H + \varepsilon}, \]

and thus,

\[ |X_t - X_s| \leq (t - s)^{H - \varepsilon} \|X\|_{s,t:H - \varepsilon} \]
\[ \leq 2C_1(H, \varepsilon)\|B^H\|_{0,T:H - \varepsilon} \|\sigma\|_\infty(t - s)^{H - \varepsilon} + 2\|b\|_\infty(t - s). \tag{2.11} \]

If \( \Delta < t - s \), we divide the interval \([s, t]\) into \( n = \lfloor (t - s)/\Delta \rfloor + 1 \) subintervals, whose lengths are smaller than \( \Delta \). Let \( s = u_0 < u_1 < \cdots < u_n = t \) be the endpoints of the subintervals. Then \( u_i - u_{i-1} \leq \Delta \) for \( 1 \leq i \leq n \). Hence, by (2.11),

\[ |X_t - X_s| \leq \sum_{i=1}^{n} |X_{u_i} - X_{u_{i-1}}| \]
\[ \leq 2C_1(H, \varepsilon)\|B^H\|_{0,T:H - \varepsilon} \|\sigma\|_\infty \sum_{i=1}^{n} (u_i - u_{i-1})^{H - \varepsilon} + 2\|b\|_\infty \sum_{i=1}^{n} (u_i - u_{i-1}) \]
\[ \leq 2C_1(H, \varepsilon)\|B^H\|_{0,T:H - \varepsilon} \|\sigma\|_\infty n\Delta^{H - \varepsilon} + 2\|b\|_\infty(t - s) \]
\[ \leq 4C_1(H, \varepsilon)\|B^H\|_{0,T:H - \varepsilon} \|\sigma\|_\infty(t - s)^{H - \varepsilon - 1} + 2\|b\|_\infty(t - s) \]
\[ \leq 2^{1 - H}(H - \varepsilon)C_1(H, \varepsilon)^{1/(H - \varepsilon)}\|B^H\|_{0,T:H - \varepsilon} \|\sigma\|_\infty\|\sigma'\|_\infty^{1/(H - \varepsilon) - 1}(t - s) + 2\|b\|_\infty(t - s). \tag{2.12} \]

The desired result follows from (2.11) and (2.12), and the proof is concluded. \( \square \)
For the case where the coefficient functions $\sigma$ or $b$ are unbounded, we have the following estimation.

**Theorem 2.2.** Suppose that the coefficient functions $\sigma$ and $b$ have bounded derivatives which are Hölder continuous of order greater than $1/(H - \varepsilon) - 1$ for some $\varepsilon \in (0, H - \frac{1}{2})$. Moreover, if $\|\sigma\|_{\infty} + \|b\|_{\infty} > 0$, then for all $T > 0$, for all $0 \leq s < t \leq T$,

$$\|X_t - X_s\| \leq C(H, \varepsilon, \sigma, b, T)^{1 + (\|B^H\|_{0,T,H-\varepsilon})^{1/(H-\varepsilon)}}(|X_0| + 1)(t - s)^{H-\varepsilon}, \quad (2.13)$$

where $C(H, \varepsilon, \sigma, b, T)$ is a constant depending only on $(H, \varepsilon, \sigma, b, T)$.

**Proof.** Obviously the function $g(t) = t$ is Hölder continuous of any order $\beta \in (0, 1]$ with the Hölder norm $\|g\|_{0,T;\beta} = T^{1-\beta}$. Hence, by [11, Theorem 2 (i)], we have

$$\sup_{t \in [0,T]} |X_t| \leq 2^{1 + C(H, \varepsilon)T^{1 + (\|B^H\|_{0,T,H-\varepsilon})^{1/(H-\varepsilon)}}}(|X_0| + 1). \quad (2.14)$$

In this proof, $C(H, \varepsilon)$ and $C(H, \varepsilon, \sigma, b, T)$ are generic positive constants depending only on $(H, \varepsilon)$ and $(H, \varepsilon, \sigma, b, T)$, respectively, and they may vary in different places.

The estimations (2.5) and (2.6) are still valid. Instead of (2.7), we have

$$|D^\alpha_{s+}\sigma(X_r)| = \frac{1}{\Gamma(1 - \alpha)} \left| \frac{\sigma(X_r) - \sigma(X_u)}{(r - s)^\alpha} + \alpha \int_s^r \frac{\sigma(X_r) - \sigma(X_u)}{(r - u)^{\alpha+1}} du \right| \leq \frac{1}{\Gamma(1 - \alpha)} \left| \frac{\sigma(X_r) - \sigma(0)}{|r - s|^\alpha} \right| + \frac{\alpha}{\Gamma(1 - \alpha)} \int_s^r \frac{|\sigma(X_r) - \sigma(X_u)|}{|r - u|^\alpha} du \leq \frac{1}{\Gamma(1 - \alpha)} \left| \frac{\|\sigma\|_{\|\sigma\|_{\infty}|X_r| + |\sigma(0)|}}{|r - s|^\alpha} \right| + \frac{\alpha\|\sigma\|_{\|\sigma\|_{\infty}}\|X\|_{s,t;H-\varepsilon}}{\Gamma(1 - \alpha)} \int_s^r \frac{|X_r - X_u|}{|r - u|^\alpha} du \leq \frac{1}{\Gamma(1 - \alpha)} \left| \frac{\|\sigma\|_{\|\sigma\|_{\infty}|X_r| + |\sigma(0)|}}{|r - s|^\alpha} \right| + \frac{\alpha\|\sigma\|_{\|\sigma\|_{\infty}}\|X\|_{s,t;H-\varepsilon}}{\Gamma(1 - \alpha)} \int_s^r \frac{1}{|r - u|^\alpha} \frac{1}{(H - \varepsilon - \alpha)} dr \quad (2.15)$$

By (2.5), (2.6), (2.15) and (2.14), we have

$$\left| \int_s^t \sigma(X_r) \circ dB^H_r \right| \leq \int_s^t \left( \frac{1}{\Gamma(1 - \alpha)} \left( \frac{(1 - \alpha)}{(H - \varepsilon + \alpha + 1)\Gamma(1 - \alpha)} \right) \left| \frac{\sigma(X_r) - \sigma(X_u)}{|r - s|^\alpha} \right| + \frac{\alpha\|\sigma\|_{\|\sigma\|_{\infty}}\|X\|_{s,t;H-\varepsilon}}{\Gamma(1 - \alpha)} \int_s^r \frac{1}{|r - u|^\alpha} \frac{1}{(H - \varepsilon - \alpha)} dr \right) dr \leq \frac{(H - \varepsilon)\|B^H\|_{s,t;H-\varepsilon}}{(1 - \alpha)\Gamma(1 - \alpha)(H - \varepsilon + \alpha + 1)} \left( \|\sigma\|_{\|\sigma\|_{\infty}|X_r| + |\sigma(0)|} \int_s^r \frac{1}{|r - s|^\alpha} \frac{1}{(H - \varepsilon - \alpha)} dr \right) + \frac{H - \varepsilon}{(1 - \alpha)\Gamma(1 - \alpha)(H - \varepsilon + \alpha + 1)} \left( \|\sigma\|_{\|\sigma\|_{\infty}}\|X\|_{s,t;H-\varepsilon} \int_s^r \frac{1}{|r - s|^\alpha} \frac{1}{(H - \varepsilon - \alpha)} dr \right) \leq C(H, \varepsilon)\|B^H\|_{s,t;H-\varepsilon} \left( \|\sigma\|_{\|\sigma\|_{\infty}|X_r| + |\sigma(0)|} \int_s^r \frac{1}{|r - s|^\alpha} \frac{1}{(H - \varepsilon - \alpha)} dr \right), \quad (2.16)$$
Similarly, it is easy to see that
\[
\left| \int_s^t b(X_r)dr \right| \leq |t-s| \left[ \|b'\|_\infty \sup_{r \in [0,T]} |X_r| + |b(0)| \right]
\leq |t-s|^{H-\varepsilon} T^{1-(H-\varepsilon)} \left[ \|b'\|_\infty \sup_{r \in [0,T]} |X_r| + |b(0)| \right].
\] (2.17)

Hence, by (2.16) and (2.17), there exists a constant \(C_0(H, \varepsilon)\) such that
\[
|X_t - X_s| \leq \left| \int_s^t \sigma(X_r) \circ dB_r^H \right| + \left| \int_s^t b(X_r)dr \right|
\leq C_0(H, \varepsilon) \left( \|B^H\|_{s,t:H-\varepsilon} + T^{1-(H-\varepsilon)} \right)|t-s|^{H-\varepsilon} \left[ \|\sigma'\|_\infty + \|b'\|_\infty \sup_{r \in [0,T]} |X_r| \right]
+ |\sigma(0)| + |b(0)| + \|\sigma'\|_\infty \|X\|_{s,t:H-\varepsilon} |t-s|^{H-\varepsilon},
\] and consequently,
\[
\|X\|_{s,t:H-\varepsilon} \leq C_0(H, \varepsilon) \left( \|B^H\|_{0,T:H-\varepsilon} + T^{1-(H-\varepsilon)} \right) \left[ \|\sigma'\|_\infty + \|b'\|_\infty \sup_{r \in [0,T]} |X_r| \right]
+ |\sigma(0)| + |b(0)| + \|\sigma'\|_\infty \|X\|_{s,t:H-\varepsilon} |t-s|^{H-\varepsilon}.
\] (2.18)

Fix a positive constant \(\Delta\) such that
\[
\Delta \leq \left( \frac{1}{3C_0(H, \varepsilon)\|\sigma'\|_\infty (\|B^H\|_{0,T:H-\varepsilon} + T^{1-(H-\varepsilon)})} \right)^{1/(H-\varepsilon)}.
\]

If \(t-s \leq \Delta\), we have
\[
\|X\|_{s,t:H-\varepsilon} \leq \frac{3}{2} C_0(H, \varepsilon) \left( \|B^H\|_{0,T:H-\varepsilon} + T^{1-(H-\varepsilon)} \right) \left[ (|\sigma(0)| + |b(0)| + (\|\sigma'\|_\infty + \|b'\|_\infty) \sup_{r \in [0,T]} |X_r|) \right].
\] (2.19)

Then by (2.14) and (2.19), we have
\[
\|X\|_{s,t:H-\varepsilon} \leq C(H, \varepsilon, \sigma, b, T)^{1+\|B^H\|_{s,t:H-\varepsilon}} (||X_0|| + 1).
\] (2.20)

If \(t-s > \Delta\), Similar to the proof of Theorem 2.1, we divide the interval \([s,t]\) into \(n = \lceil (t-s)/\Delta \rceil + 1\) subintervals, whose lengths are smaller than \(\Delta\). Let \(s = u_0 < u_1 < \cdots < u_n = t\) be endpoints of the subintervals. Then by (2.14) and (2.19), we have
\[
|X_t - X_s| \leq \sum_{i=1}^{n} |X_{u_i} - X_{u_{i-1}}|
\leq C(H, \varepsilon, \sigma, b, T)^{1+\|B^H\|_{s,t:H-\varepsilon}} (||X_0|| + 1) \sum_{i=1}^{n} (u_i - u_{i-1})^{H-\varepsilon}
\leq C(H, \varepsilon, \sigma, b, T)^{1+\|B^H\|_{s,t:H-\varepsilon}} (||X_0|| + 1) n^{H-\varepsilon}
\leq C(H, \varepsilon, \sigma, b, T)^{1+\|B^H\|_{s,t:H-\varepsilon}} (||X_0|| + 1) (t-s) \Delta^{H-\varepsilon-1}
\leq C(H, \varepsilon, \sigma, b, T)^{1+\|B^H\|_{s,t:H-\varepsilon}} (||X_0|| + 1) (t-s).
\] (2.21)

The desired results (2.13) comes from (2.20) and (2.21).

Combining Theorem 2.1, Theorem 2.2, and Lemma 2.1 with dominated convergence theorem, one can prove the following corollary.

**Corollary 2.1.** Assume that \(\mathbb{E}[|X_0|^p] < \infty\) for \(p \geq 2\). If the conditions in Theorem 2.1 or in Theorem 2.2 hold, then \(\mathbb{E}[|X_t|^p]\) is a continuous function of \(t\).
3. High-dimensional limit for Wigner-type matrices

3.1. Relative compactness of empirical spectral measure-valued processes

Denote by $P(\mathbb{R})$ the space of probability measures equipped with weak topology, then $P(\mathbb{R})$ is a Polish space. For $T > 0$, let $C([0, T], P(\mathbb{R}))$ be the space of continuous $P(\mathbb{R})$-valued processes. In this subsection, under proper conditions, we obtain the almost sure relative compactness in $C([0, T], P(\mathbb{R}))$ for the set $\{L_N(t), t \in [0, T]\}_{N \in \mathbb{N}}$ of empirical spectral measures of $Y^N(t)$ with entries given in (1.2).

**Theorem 3.1.** Assume one of the following hypotheses holds:

- **H1.** Conditions in Theorem 2.1 hold;
- **H2.** Conditions in Theorem 2.2 hold and $E|X_0|^4 < \infty$.

Assume that there exists a positive function $\varphi(x) \in C^1(\mathbb{R})$ with bounded derivative, such that $\lim_{|x| \to \infty} \varphi(x) = \infty$ and $\sup_{N \in \mathbb{N}} \langle \varphi, L_N(0) \rangle \leq C_0$,

for some positive constant $C_0$ almost surely. Then for any $T > 0$, the sequence $\{L_N(t), t \in [0, T]\}_{N \in \mathbb{N}}$ is relatively compact in $C([0, T], P(\mathbb{R}))$ almost surely.

**Proof.** Under hypothesis H1 (hypothesis H2, resp.), by Theorem 2.1 (Theorem 2.2, resp.), we have

$$|X_t - X_s| \leq \xi|t - s|^{H-\varepsilon}$$

where $\xi$ is a random variable with finite moment of any order (with finite 4-th order moment, resp.).

By (1.3) and Lemma A.1, for any $f \in C^1(\mathbb{R})$ with bounded derivative,

$$\frac{1}{N} \sum_{i=1}^{N} \left| f(\lambda^N_i(t)) - f(\lambda^N_i(s)) \right|^2 \leq \frac{1}{N} \sum_{i=1}^{N} \left| f(\lambda^N_i(t)) - f(\lambda^N_i(s)) \right|^2 \leq \frac{1}{N} \left\| f' \right\|_{\infty}^2 \sum_{i=1}^{N} \left| \lambda^N_i(t) - \lambda^N_i(s) \right|^2$$

$$\leq \frac{1}{N} \left\| f' \right\|_{\infty}^2 \text{Tr}(Y^N(t) - Y^N(s))^2 = \frac{1}{N} \left\| f' \right\|_{\infty}^2 \sum_{i,j=1}^{N} (Y^N_{ij}(t) - Y^N_{ij}(s))^2$$

$$= \left\| f' \right\|_{\infty}^2 \left[ \frac{2}{N^2} \sum_{i < j} (X^N_{ij}(t) - X^N_{ij}(s))^2 + \frac{2}{N^2} \sum_{i=1}^{N} (X^N_{ii}(t) - X^N_{ii}(s))^2 \right]$$

$$\leq \left\| f' \right\|_{\infty}^2 (t-s)^{2H-2\varepsilon} \left[ \frac{2}{N^2} \sum_{i < j} \xi^2_{ij} + \frac{2}{N^2} \sum_{i=1}^{N} \xi^2_{ii} \right]$$

$$\leq \frac{4}{N(N+1)} \left\| f' \right\|_{\infty}^2 (t-s)^{2H-2\varepsilon} \sum_{i \leq j} \xi^2_{ij}, \quad (3.1)$$

where $\{\xi_{ij}\}$ are i.i.d copies of $\xi$.

Recall that by the Arzela-Ascoli Theorem, for any number $M > 0$, the set

$$\bigcap_{n=1}^{\infty} \left\{ g \in C([0, T], \mathbb{R}) : \sup_{s,t \in [0,T],|t-s| \leq \eta_n} |g(t) - g(s)| \leq \varepsilon_n, \sup_{t \in [0,T]} |g(t)| \leq M \right\},$$
where \( \{\varepsilon_n, n \in \mathbb{N}\} \) and \( \{\eta_n, n \in \mathbb{N}\} \) are two sequences of positive real numbers going to zero as \( n \) goes to infinity, is compact in \( C([0, T], \mathbb{R}) \).

Define
\[
K = \left\{ \nu \in \mathbf{P}(\mathbb{R}) : \int \varphi(x)\nu(dx) \leq C_0 + M_0 \right\},
\]
where \( M_0 \) is a positive number that will be determined later. Then \( K \) is compact in \( \mathbf{P}(\mathbb{R}) \). Note that there exists a sequence of \( C^1_0(\mathbb{R}) \) functions \( \{f_i\}_{i \in \mathbb{N}} \) that is dense in \( C_0(\mathbb{R}) \). Choose a positive integer \( p_0 \), such that \( p_0(H - \varepsilon) > 1 \), and define
\[
C_T(f_i) = \bigcap_{n=1}^{\infty} \left\{ \nu \in C([0, T], \mathbf{P}(\mathbb{R})) : \sup_{s,t \in [0, T], |t-s| \leq n^{-p_0}} |\langle f_i, \nu_t \rangle - \langle f_i, \nu_s \rangle| \leq M_n n^{-p_0(H-\varepsilon)} \right\},
\]
where \( \{M_i\}_{i \in \mathbb{N}} \) is a sequence of positive number that are independent of \( n \) and will be determined later. Denote
\[
\mathcal{C} = \{ \nu \in C([0, T], \mathbf{P}(\mathbb{R})) : \nu_t \in K, \forall t \in [0, T] \} \cap \bigcap_{i=1}^{\infty} C_T(f_i),
\]
then \( \mathcal{C} \) is compact in \( C([0, T], \mathbf{P}(\mathbb{R})) \), according to [1, Lemma 4.3.13]. Hence, it is enough to show
\[
\mathbb{P} \left( \lim_{N \to \infty} \left\{ L_N(t) \in \mathcal{C} \right\} \right) = 1.
\]

Note that \( \xi_{ij}^2 - \mathbb{E}[\xi_{ij}^2] := \xi_{ij}^2 - m \) are i.i.d. with mean zero and finite variance denoted by \( \sigma^2 \) for \( 1 \leq i \leq j \leq N \). By (3.1) and the Markov inequality, we have
\[
\mathbb{P} \left( L_N(t) \not\in \left\{ \nu \in C([0, T], \mathbf{P}(\mathbb{R})) : \nu_t \in K, \forall t \in [0, T] \right\} \right)
= \mathbb{P} \left( \exists t \in [0, T], L_N(t) \not\in K \right)
= \mathbb{P} \left( \sup_{t \in [0, T]} \langle \varphi, L_N(t) \rangle > C_0 + M_0 \right)
\leq \mathbb{P} \left( \sup_{t \in [0, T]} |\langle \varphi, L_N(t) \rangle - \langle \varphi, L_N(0) \rangle| > M_0 \right)
\leq \mathbb{P} \left( \frac{4}{N(N+1)} \|\varphi\|_\infty^2 T^{2H-2\varepsilon} \sum_{i \leq j} \xi_{ij}^2 > M_0^2 \right)
= \mathbb{P} \left( \frac{2}{N(N+1)} \sum_{i \leq j} \xi_{ij}^2 - m > \frac{M_0^2}{2\|\varphi\|_\infty^2 T^{2H-2\varepsilon} - m} \right)
\leq \left( \frac{M_0^2}{2\|\varphi\|_\infty^2 T^{2H-2\varepsilon} - m} \right)^{-2} \mathbb{E} \left[ \left( \frac{2}{N(N+1)} \sum_{i \leq j} \xi_{ij}^2 - m \right)^2 \right]
= \left( \frac{M_0^2}{2\|\varphi\|_\infty^2 T^{2H-2\varepsilon} - m} \right)^{-2} \frac{4}{N^2(N+1)^2} \mathbb{E} \left[ \left( \sum_{i \leq j} \xi_{ij}^2 \right)^2 \right]
= \left( \frac{M_0^2}{2\|\varphi\|_\infty^2 T^{2H-2\varepsilon} - m} \right)^{-2} \frac{4}{N^2(N+1)^2} \mathbb{E} \left[ \left( \sum_{i \leq j} (\xi_{ij}^2 - m) \right)^2 \right]
= \left( \frac{M_0^2}{2\|\varphi\|_\infty^2 T^{2H-2\varepsilon} - m} \right)^{-2} \frac{2}{N(N+1)} \sigma^2.
\]
(3.2)
If we choose $M_0 = 2\|\varphi\|_\infty T^{H-\epsilon}(1 + m)$, then (3.2) becomes
\[
\mathbb{P} \left( L_N(t) \notin \left\{ \nu \in C([0, T], \mathbb{P}(\mathbb{R})) : \nu_t \in K, \forall t \in [0, T] \right\} \right) \
\leq \frac{2\sigma^2}{N(N + 1)(2m^2 + 3m + 2)^2} \leq \frac{8\sigma^2}{N(N + 1)}. \tag{3.3}
\]
Similarly, by (3.1) and the Markov inequality, we have
\[
\mathbb{P} \left( L_N(t) \notin C_T(f_i) \right) \
\leq \sum_{n=1}^\infty \mathbb{P} \left( L_N(t) \notin \left\{ \nu \in C([0, T], \mathbb{P}(\mathbb{R})) : \sup_{s, t \in [0, T], |t-s| \leq n^{-p_0}} |\langle f_i, \nu_t \rangle - \langle f_i, \nu_s \rangle| \leq M_n n^{-p_0(H-\epsilon)} \right\} \right) \
= \sum_{n=1}^\infty \mathbb{P} \left( \sup_{s, t \in [0, T], |t-s| \leq n^{-p_0}} |\langle f_i, L_N(t) \rangle - \langle f_i, L_N(s) \rangle| > M_n n^{-p_0(H-\epsilon)} \right) \
\leq \sum_{n=1}^\infty \mathbb{P} \left( 2\|f_i\|_\infty^2 n^{-p_0(2H-2\epsilon)} \left( \frac{2}{N(N + 1)} \sum_{i \leq j} \xi_{ij}^2 \right) > M_n^2 n^{2-2p_0(H-\epsilon)} \right) \
= \sum_{n=1}^\infty \mathbb{P} \left( \frac{2}{N(N + 1)} \sum_{i \leq j} \xi_{ij}^2 - m > \frac{M_n^2 n^2}{2\|f_i\|_\infty^2} - m \right) \
\leq \sum_{n=1}^\infty \left( \frac{M_n^2 n^2}{2\|f_i\|_\infty^2} - m \right)^{-2} \mathbb{E} \left[ \left( \frac{2}{N(N + 1)} \sum_{i \leq j} \xi_{ij}^2 - m \right)^2 \right] \
= \sum_{n=1}^\infty \left( \frac{M_n^2 n^2}{2\|f_i\|_\infty^2} - m \right)^{-2} \frac{4}{N^2(N + 1)^2} \mathbb{E} \left[ \sum_{i \leq j} (\xi_{ij}^2 - m)^2 \right] \
\leq \sum_{n=1}^\infty \left( \frac{M_n^2 n^2}{2\|f_i\|_\infty^2} - m \right)^{-2} \frac{2}{N(N + 1)} \sigma^2. \tag{3.4}
\]
Now, we choose $M_i = 2\|f_i\|_\infty \gamma$, where $\gamma$ is a positive real number such that $2\gamma^2 > m$. Then (3.4) becomes
\[
\mathbb{P} \left( L_N(t) \notin C_T(f_i) \right) \leq \sum_{n=1}^\infty \frac{2\sigma^2}{N(N + 1)(2n^2\gamma^2 - m)^2}. \tag{3.5}
\]
Hence, by the definition of $\mathcal{C}$, (3.3) and (3.5), we have
\[
\sum_{N=1}^\infty \mathbb{P} \left( L_N(t) \notin \mathcal{C} \right) \leq \sum_{N=1}^\infty \mathbb{P} \left( L_N(t) \notin \left\{ \nu \in C([0, T], \mathbb{P}(\mathbb{R})) : \nu_t \in K, \forall t \in [0, T] \right\} \right) \
+ \sum_{N=1}^\infty \sum_{i=1}^\infty \mathbb{P} \left( L_N(t) \notin C_T(f_i) \right) \
\leq \sum_{N=1}^\infty \frac{8\sigma^2}{N(N + 1)} + \sum_{N=1}^\infty \sum_{i=1}^\infty \sum_{n=1}^\infty \frac{2\sigma^2}{N(N + 1)(2n^2\gamma^2 - m)^2} \
< \infty. \tag{3.6}
\]
Therefore, by the Borel-Cantelli Lemma and (3.6), we have
\[
\mathbb{P} \left( \limsup_{N \to \infty} \{ L_N(t) \notin \mathcal{C} \} \right) = 0.
\]
The proof is concluded. \qed
3.2. Limit of empirical spectral distributions

Recall that the celebrated semicircle distribution $\mu_{sc}(dx)$ on $[-2,2]$ has density function

$$p_{sc}(x) = \frac{\sqrt{4 - x^2}}{2\pi}1_{[-2,2]}(x).$$

(3.7)

**Theorem 3.2.** Suppose that the conditions in Theorem 3.1 hold. We also assume that $\mathbb{E}[|X_0|^2] < \infty$ and denote $m_t = \mathbb{E}[X_t]$ and $d_t = (\mathbb{E}[|X_t|^2] - m_t^2)^{1/2}$. Then for any $T > 0$, the sequence $\{L_N(t), t \in [0,T]\}_{N \in \mathbb{N}}$ converges to $\mu = \{\mu_t, t \in [0,T]\}$ in $C([0,T], \mathbb{P}(\mathbb{R}))$ almost surely, as $N \rightarrow \infty$. The limit measure $\mu_t$ has density $p_t(x) = p_{sc}(x/d_t)/d_t$, where $p_{sc}(x)$ is given by (3.7).

**Proof.** First, for fixed $t \in [0,T]$, we prove the almost sure weak convergence of the empirical measure $L_N(t)$.

Note that by Corollary 2.1, $m_t$ and $d_t$ are continuous functions of $t$ on $[0,T]$. Let $\tilde{Y}_N(t) = \left(\tilde{Y}_{ij}^N(t)\right)_{1 \leq i,j \leq N}$ be a symmetric matrix with entries

$$\tilde{Y}_{ij}^N(t) = \frac{Y_{ij}^N(t) - m_t/\sqrt{N}}{d_t}, \quad 1 \leq i \leq j \leq N.$$  

(3.8)

Then $\tilde{Y}_N(t)$ is a Wigner matrix. Let $\tilde{\lambda}_{1}^N(t) \leq \cdots \leq \tilde{\lambda}_{N}^N(t)$ be the eigenvalues of $\tilde{Y}_N(t)$ and $\tilde{L}_N(t)(dx) = \sum_{i=1}^{N} \delta_{\tilde{\lambda}_i^N(t)}(dx)/N$ be the empirical spectral measure. By Lemma A.2, $\tilde{L}_N(t)(dx)$ converges weakly to the semicircle distribution $p_{sc}(x)dx$ almost surely for all $0 \leq t \leq T$. Hence the empirical measure of the eigenvalues of $d_t\tilde{Y}_N(t)$ converges weakly to $\frac{1}{dt}p_{sc}\left(\frac{x}{dt}\right)dx$ almost surely.

Note that by (3.8), $Y_N(t) = d_t\tilde{Y}_N(t) + \frac{m_t}{\sqrt{N}}E_N$, where $E_N$ is an $N \times N$ matrix with unit entries. Then by [22, Exercise 2.4.4], we can conclude that the empirical distribution $L_N(t)(dx)$ converges weakly to $\frac{1}{dt}p_{sc}\left(\frac{x}{dt}\right)dx$, almost surely, for all $t \in [0,T]$.

Now, we show that $\{L_N(t), t \in [0,T]\}_{N \in \mathbb{N}}$ converges weakly to $\{\mu_t, t \in [0,T]\}_{N \in \mathbb{N}}$ almost surely.

Let $\{L_N(t), t \in [0,T]\}_{N \in \mathbb{N}}$ be an arbitrary convergent subsequence with the limit $\{\mu_t, t \in [0,T]\}$, then for $t \in [0,T]$, $\mu_t(dx) = p_{sc}(x/d_t)/d_tdx$. Therefore, noting that Theorem 3.1 yields that the sequence $\{L_N(t), t \in [0,T]\}_{N \in \mathbb{N}}$ is relatively compact almost surely in $C([0,T], \mathbb{P}(\mathbb{R}))$, any subsequence of $\{L_N(t), t \in [0,T]\}_{N \in \mathbb{N}}$ has a convergent subsequence with the unique limit $\{\mu_t(dx) = p_{sc}(x/d_t)/d_tdx, t \in [0,T]\}$. This implies that the total sequence $\{L_N(t), t \in [0,T]\}_{N \in \mathbb{N}}$ converges in $C([0,T], \mathbb{P}(\mathbb{R}))$ with the limit $\{\mu_t(dx) = p_{sc}(x/d_t)/d_tdx, t \in [0,T]\}$, almost surely. \[\square\]

**Remark 3.1.** If $\sigma(x) = 1$, $b(x) = 0$ and $X_0 = 0$, then the solution to the SDE (1.1) is the fractional Brownian motion $X_t = B^H_t$, and Theorem 3.2 implies that the empirical spectral measure converges weakly to the scaled semicircle distribution with density $p_t(x) = p_{sc}(x/t^H)/t^H$ almost surely. This improves the results in [17], where the convergence of the empirical spectral measure in law is obtained.

**Remark 3.2.** The Stieltjes transform $G_t(z)$ of the limiting measure $\mu_t$ is

$$G_t(z) = \int \frac{\mu_t(dx)}{z-x} = \int \frac{p_{sc}(x/d_t)}{z-x}dx = \int \frac{p_{sc}(x)}{z-d_tx}dx = \frac{1}{d_t}G_{sc}(z/d_t),$$

where

$$G_{sc}(z) = \int \frac{p_{sc}(x)}{z-x}dx,$$

is the stieltjes transform of the semi-circle distribution. If we assume that the variance $d_t^2$ of the solution $X_t$ is continuously differentiable on $(0,T)$, then we have

$$\partial_z G_t(z) = \partial_t \int \frac{p_{sc}(x)}{z-d_tx}dx = d_t \int \frac{xp_{sc}(x)}{(z-d_tx)^2}dx.$$
Here, \( \{ \text{differentiable functions and } B \} \)

Denoting \( F \) the following conditions,

\[
\text{By [2, Lemma 2.11], it is easy to get}
\]

\[
d_t^2 G_t(z)^2 = z G_t(z) - 1,
\]

and by (3.9),

\[
\partial_t G_t(z) = -d_t' (G_t(z) + z \partial_z G_t(z)) = -d_t' (z G_t(z) - 1)
\]

\[
= -d_t' \partial_z (G_t(z)^2) = -(d_t^2)' G_t(z) \partial_z G_t(z).
\]

For the case \( X_t = B_t^H \) with \( d_t^2 = t^{2H} \), equation (3.10) becomes

\[
\partial_t G_t(z) = -2 H t^{2H-1} G_t(z) \partial_z G_t(z).
\]

Denoting \( F_t(z) = G_{1/2H}(z) \), by change of variable and (3.11), one can deduce that \( F_t(z) \) satisfies the complex Burgers’ equation

\[
\partial_t F_t(z) = -F_t(z) \partial_z F_t(z).
\]

This relationship was obtained in [13].

### 3.3. Complex case

In this subsection, we consider the following 2-dimensional SDE for \( Z_t = (Z_t^{(1)}, Z_t^{(2)}) \),

\[
dZ_t = \tilde{\sigma}(Z_t) \circ dB_t^H + \tilde{b}(Z_t)dt,
\]

with initial value \( Z_0 \) that is independent of \( B_t^H \). Here \( \tilde{b} : \mathbb{R}^2 \to \mathbb{R}^2, \tilde{\sigma} : \mathbb{R}^2 \to \mathbb{R}^{2 \times 2} \) are continuously differentiable functions and \( B_t^H \) is a 2-dimensional fractional Brownian motion. By [14], there exists a unique solution to SDE (3.12), if \( \tilde{\sigma} \) and \( \tilde{b} \) have bounded derivatives which are Hölder continuous of order greater than \( 1/H - 1 \).

Denote by \( i \) the imaginary unit. Let \( \{ Z_{kl} \}_{k,l \geq 1} \) be i.i.d. copies of \( Z_t^{(1)} + i Z_t^{(2)} \) and \( Z^N(t) = (Z^N_{kl}(t))_{1 \leq k,l \leq N} \) be a Hermitian \( N \times N \) matrix with entries

\[
Z^N_{kl}(t) = \begin{cases} 
1/N & Z_{kl}(t), \quad 1 \leq k < l \leq N, \\
1/\sqrt{N} & Z_{kk}(t), \quad 1 \leq k \leq N.
\end{cases}
\]

Here, \( \{ X_{kk}(t) \} \) are i.i.d. copies of the real-valued process \( X_t \) satisfying (1.1) and independent of the family \( \{ Z_{kl}(t) \}_{k,l \geq 1} \). Let \( \lambda^N_1(t) \leq \cdots \leq \lambda^N_N(t) \) be the eigenvalues of \( Z^N(t) \) and denote the empirical spectral measure by

\[
L_N(t)(dx) = \frac{1}{N} \sum_{k=1}^{N} \delta_{\lambda^N_k(t)}(dx).
\]

**Theorem 3.3.** Suppose that the coefficient functions \( \tilde{\sigma}, \tilde{b}, \sigma \) and \( b \) have bounded (partial) derivatives which are Hölder continuous of order greater than \( 1/(H - \varepsilon) - 1 \). Besides, assume that among the following conditions,

(\( a_1 \)) \( \tilde{\sigma} \) and \( \tilde{b} \) are bounded and \( \mathbb{E}[\|Z_0\|^2] < \infty \);
\[(a_2) \quad \| \tilde{\sigma}_t \|_{L^\infty(\mathbb{R}^2)} + \| b_t \|_{L^\infty(\mathbb{R}^2)} > 0, \quad \mathbb{E}[\| Z_0 \|^4] < \infty; \tag{b_1} \]
\[\| \sigma \|_\infty + \| b' \|_\infty > 0, \quad \mathbb{E}[\| X_0 \|^2] < \infty; \tag{b_2} \]
\[(a_1) \text{ or } (a_2) \text{ holds and } (b_1) \text{ or } (b_2) \text{ holds. Furthermore, suppose that there exists a positive function } \varphi(x) \in C^1(\mathbb{R}) \text{ with bounded derivative, such that } \lim_{|x| \to \infty} \varphi(x) = +\infty \text{ and} \]
\[\sup_{N \in \mathbb{N}} \langle \varphi, L_N(0) \rangle \leq C_0, \]
for some positive constant \(C_0\) almost surely.

Then for any \(T > 0\), \(\mathbb{E}[\| X_t \|^2 + \| Z_t \|^2] < \infty\) for \(t \in [0, T]\), and the sequence \(\{L_N(t), t \in [0, T]\}_{N \in \mathbb{N}}\) converges to \(\mu = \{\mu_t, t \in [0, T]\} \in C([0, T], \mathcal{P}(\mathbb{R}))\) almost surely. The limiting measure \(\mu_t\) has density \(p_t(x) = p(x/d_Z(t))/d_Z(t)\), where \(d_Z^2(t)\) is the variance of the solution \(Z_t\) to SDE \((3.12)\).

**Proof.** The proof is similar to the real case, which is sketched below.

First of all, following the proof of Theorem 2.1 or Theorem 2.2, we can still establish the pathwise Hölder continuity for the solution \(Z_t = (Z_t^{(1)}, Z_t^{(2)})\) to the SDE \((3.12)\). More specifically, for the case that condition \((a1)\) holds, we can obtain

\[
\begin{align*}
Z_t^{(1)} - Z_s^{(1)} + Z_t^{(2)} - Z_s^{(2)} \leq C(H, \varepsilon)\| \tilde{\sigma} \|_{\infty} \left[ \| B^H \|_{0, T; H-\varepsilon} (t-s)^{H-\varepsilon} \vee \| B^H \|_{0, T; H-\varepsilon} (t-s)^{1/(H-\varepsilon)} \right] \\
+ 2\| b \|_{\infty} (t-s),
\end{align*}
\]

and for the case of \((a2)\),

\[
\begin{align*}
\left| Z_t^{(1)} - Z_s^{(1)} \right| + \left| Z_t^{(2)} - Z_s^{(2)} \right| \leq C(H, \varepsilon, \tilde{\sigma}, b, T)^{1+\| B^H \|_{0, T; H-\varepsilon} \left( \| Z_0 \| + 1 \right)} (t-s)^{H-\varepsilon}.
\end{align*}
\]

for all \(T > 0\), for all \(0 \leq s < t \leq T\). Thus, by Lemma 2.1 and the moment assumption on \(Z_0\), we have that for both cases, there exists a positive random variable \(\zeta\) with finite second moment, such that

\[
\begin{align*}
|Z_t^{(1)} - Z_s^{(1)}|^2 + |Z_t^{(2)} - Z_s^{(2)}|^2 \leq \zeta (t-s)^{2H-2\varepsilon}, \quad \text{a.s.} \tag{3.14}
\end{align*}
\]

Similar to (3.1), we have

\[
|\langle f, L_N(t) \rangle - \langle f, L_N(s) \rangle|^2 \leq \frac{1}{N^2} \| f' \|^2_{\infty} (t-s)^{2H-2\varepsilon} \left( \sum_{k=1}^{N} \xi^2_{kk} + 2 \sum_{k<l} \xi^2_{kl} \right),
\]

for any \(f \in C^1(\mathbb{R})\) with bounded derivation. Then following the same approach used in the proof of Theorem 3.1, we can show that the sequence of empirical spectral measures \(\{L_N(t), t \in [0, T]\}_{N \in \mathbb{N}}\) is relatively compact in \(C([0, T], \mathcal{P}(\mathbb{R}))\) almost surely.

Following the proof of Corollary 2.1, it is easy to see that the mean \(m_Z = \mathbb{E}[Z_t]\) and the variance \(d_Z^2(t) = \mathbb{E}[|Z_t - m_Z(t)|^2]\) are continuous functions of \(t\).

Next, we introduce a Hermitian matrix \(\tilde{Z}_N(t) = \left( \tilde{Z}_{kl}^N(t) \right)_{1 \leq k, l \leq N}\) satifying

\[
Z_N(t) = \frac{m_Z(t)}{\sqrt{N}} (E_N - I_N) + \frac{m_t}{\sqrt{N}} I_N + d_Z(t) \tilde{Z}_N(t).
\]

Hence, \(\tilde{Z}_N(t)\) is a Hermitian Wigner matrix for all \(t \in [0, T]\). Finally, by [22, Exercise 2.4.3, Exercise 2.4.4], we can conclude that the almost-sure limit of the empirical distribution of the eigenvalues of \(Z_N(t)\) coincides with that of \(d_Z \tilde{Z}_N(t)\) for all \(t \in [0, T]\). Therefore, by Lemma A.3, \(\{L_N(t), t \in [0, T]\}_{N \in \mathbb{N}}\) converges towards \(\{\mu_t, t \in [0, T]\} \in C([0, T], \mathcal{P}(\mathbb{R}))\) with density \(\mu_t(dx) = p_t(x)dx = p(x/d_Z(t))/d_Z(t)dx\).

**Remark 3.3.** Similar to Remark 3.2, the Stieltjes transform of the limiting measure \(\mu_t\) satisfies the differential equation \((3.10)\) with \(d_i\) replaced by \(d_Z(t)\).
4. High-dimensional limit for symmetric matrices with dependent entries

Let \{X_{(i,j)}(t)\}_{i,j \in \mathbb{Z}} be i.i.d. copies of \(X_t\), the solution of (1.1). Fix a finite set \(I \subset \mathbb{Z}^2\) that is symmetric in the sense that \((i, j) \in I\) if and only if \((j, i) \in I\). Also fix a family of constants \(\{a_r : r \in I\}\). Let \(|I| = \max\{|i| \vee |j| : (i, j) \in I\}\) and \#I\ be the cardinality of \(I\). Note that \#I \leq (2|I| + 1)^2. Let \(R^N(t) = (R^N_{ij}(t))_{1 \leq i, j \leq N}\) be an \(N \times N\) real symmetric matrix with entries

\[
R^N_{ij}(t) = \frac{1}{\sqrt{N}} \sum_{r \in I} a_r X_{(i,j)+r}(t), \quad 1 \leq i \leq j \leq N. \tag{4.1}
\]

Let \(\lambda_1^N(t) \leq \cdots \leq \lambda_N^N(t)\) be the eigenvalues of \(R^N(t)\), and

\[
L_N(t)(dx) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i^N(t)}(dx)
\]

be the empirical spectral measure of \(R^N(t)\).

**Theorem 4.1.** Suppose that the conditions in Theorem 3.1 hold. Then for any \(T > 0\), the sequence \(\{L_N(t), t \in [0, T]\}_{N \in \mathbb{N}}\) converges to \(\{\mu_t, t \in [0, T]\}\) in \(C([0, T], \mathcal{P}(\mathbb{R}))\) almost surely. The Stieltjes transform \(S_t(z) = \int (z-x)^{-1} \mu_t(dx)\) of the limit measure is given by, for \(z \in \mathbb{C} \setminus \mathbb{R}\),

\[
S_t(z) = \int_0^1 h_t(x, z)dx,
\]

where \(h_t(x, z)\) is the solution to the equation

\[
h_t(x, z) = \left(-z + \int_0^1 f_t(x, y)h_t(y, z)dy\right)^{-1},
\]

with

\[
f_t(x, y) = \sum_{k, l \in \mathbb{Z}} \gamma_t(k, l)e^{-2\pi i (kx + ly)},
\]

where \(\gamma_t(k, l) = \gamma_t(l, k) = d_t^2 \sum_{r \in I \cap (I+1+(k,l))} a_r a_{r-(k,l)}\) for \(k \leq l\).

**Proof.** By Theorem 2.1 and Theorem 2.2, we have the estimate

\[
|X_{(i,j)}(t) - X_{(i,j)}(s)| \leq \xi_{(i,j)}|t - s|^{H-\varepsilon},
\]

where \(\xi_{(i,j)}\) are i.i.d. copies of \(\xi\) with \(\mathbb{E}|\xi|^4 < \infty\). Thus,

\[
|R^N_{ij}(t) - R^N_{ij}(s)| \leq \frac{1}{\sqrt{N}} \sum_{r \in I} |a_r| |X_{(i,j)+r}(t) - X_{(i,j)+r}(s)|
\leq \frac{1}{\sqrt{N}} \sum_{r \in I} |a_r| \xi_{(i,j)+r}|t - s|^{H-\varepsilon}.
\]

Define

\[
F_{ij} = \left( \sum_{r \in I} |a_r| \xi_{(i,j)+r} \right)^2.
\]

Then all \(F_{ij}\)’s for \(i, j \in \mathbb{Z}\) are distributed identically with finite second moment. Analogous to (3.1), we have

\[
|\langle f, L_N(t) \rangle - \langle f, L_N(s) \rangle|^2 = \frac{1}{N} \|f^\prime\|^2 \sum_{i,j=1}^{N} (R^N_{ij}(t) - R^N_{ij}(s))^2
\]

for all \(f\) in the domain of \(L_N(t)\), and

\[
\mathbb{E}|\langle f, L_N(t) \rangle - \langle f, L_N(s) \rangle|^2 = \frac{1}{N} \|f^\prime\|^2 \sum_{i,j=1}^{N} \mathbb{E}(R^N_{ij}(t) - R^N_{ij}(s))^2
\]

for all \(f\) in the domain of \(L_N(t)\).
Noting that the mutual independence among \( \{\xi_{ij}\} \) implies the independence between \( F_{ij} \) and \( F_{kl} \) if \( k \notin [i-|I|, i+|I|] \) or \( l \notin [j-|I|, j+|I|] \), we have

\[
\begin{align*}
\mathbb{E} \left[ \left( \sum_{i \leq j} (F_{ij} - \mathbb{E}[F_{ij}]) \right)^2 \right] &= \sum_{i \leq j} \sum_{k \leq l} \mathbb{E} \left[ (F_{ij} - \mathbb{E}[F_{ij}]) (F_{kl} - \mathbb{E}[F_{kl}]) \right] \\
&= \sum_{i \leq j} \sum_{k=-|I|}^{i+|I|} \sum_{l=-|I|}^{j+|I|} \mathbb{E} \left[ (F_{ij} - \mathbb{E}[F_{ij}]) (F_{kl} - \mathbb{E}[F_{kl}]) \right] \\
&\leq \sum_{i \leq j} \sum_{k=-|I|}^{i+|I|} \sum_{l=-|I|}^{j+|I|} \left( \mathbb{E} \left[ (F_{ij} - \mathbb{E}[F_{ij}])^2 \right] \mathbb{E} \left[ (F_{kl} - \mathbb{E}[F_{kl}])^2 \right] \right)^{1/2} \\
&= \frac{(2|I|+1)^2 N(N+1)}{2} \left( \mathbb{E}[F_{ij}^2] - (\mathbb{E}[F_{ij}])^2 \right) .
\end{align*}
\]

Thus, following the proof of Theorem 3.1, we may get estimations analogous to (3.2) and (3.4) therein, and then obtain the almost sure relatively compactness of the empirical spectral measure \( \{L_N(t), t \in [0, T]\}_{N \in \mathbb{N}} \).

Now, let \( \tilde{R}^N(t) = \left( \tilde{R}^N_{ij}(t) \right)_{1 \leq i, j \leq N} \) be a symmetric matrix with entries

\[
\tilde{R}^N_{ij}(t) := R^N_{ij}(t) - \mathbb{E}[R^N_{ij}(t)] = \frac{1}{\sqrt{N}} \sum_{r \in I} a_r \left( X_{(i,j)+r}(t) - \mathbb{E} \left[ X_{(i,j)+r}(t) \right] \right) , \quad 1 \leq i, j \leq N.
\]

Let \( \tilde{L}_N(t) \) be the empirical spectral measure of \( \tilde{R}^N(t) \). Then by Lemma A.6 ([3, Theorem 3]), for each \( t \in [0, T] \), \( \tilde{L}_N(t) \) converges to a deterministic probability measure \( \mu_t \) almost surely. Moreover, the Stieltjes transform of the limit measure \( \mu_t \) is given by

\[
S_t(z) = \int_0^1 h_t(x, z) dx ,
\]

where \( h_t(x, z) \) is the solution to the equation

\[
h(x, z) = \left( -z + \int_0^1 f(x, y) h(y, z) dy \right)^{-1} ,
\]

with

\[
f(x, y) = \sum_{k,l \in \mathbb{Z}} \gamma_{k,l} e^{-2\pi i (kx + ly)} ,
\]

where, for \( k \leq l ,

\[
\gamma_{k,l} = \gamma_{l,k} = \mathbb{E} \left[ \sum_{r \in I} a_r \left( X_r(t) - \mathbb{E}[X_r(t)] \right) \sum_{r' \in I} a_{r'} \left( X_{(k,l)+r'}(t) - \mathbb{E} \left[ X_{(k,l)+r'}(t) \right] \right) \right] \\
= \mathbb{E} \left[ \sum_{r \in I} a_r \left( X_r(t) - \mathbb{E}[X_r(t)] \right) \sum_{r' \in I \cup \{k,l\}} a_{r'-(k,l)} \left( X_{r'}(t) - \mathbb{E}[X_{r'}(t)] \right) \right] .
\]
\[ \begin{align*}
&= \sum_{r \in I^N(1 + (k,l))} a_r a_{r-(k,l)} \mathbb{E} \left[ (X_r(t) - \mathbb{E}[X_r(t)])^2 \right] \\
&= d_t^2 \sum_{r \in I^N(1 + (k,l))} a_r a_{r-(k,l)}.
\end{align*} \]

Finally, by [22, Exercise 2.4.4], the empirical spectral measure \( L_N(t)(dx) \) of \( R^N(t) \) converges to the same limit \( \mu_1 \) almost surely. The proof is concluded. \( \square \)

5. High-dimensional limit for Wishart-type matrices

5.1. Real case

Recall that \( \{X_{ij}(t)\}_{i,j \geq 1} \) are i.i.d. copies of \( X_t \) which is the solution to (1.1). Let

\[ \hat{U}^N(t) = \left( \hat{U}_{ij}^N(t) \right)_{1 \leq i \leq p, 1 \leq j \leq N} \]

be a \( p \times N \) matrix with entries \( \hat{U}_{ij}^N(t) = X_{ij}(t) - \mathbb{E}[X_{ij}(t)] \). Here, \( p = p(N) \) is a positive integer that depends on \( N \). Let

\[ U^N(t) = \frac{1}{N} \hat{U}^N(t)\hat{U}^N(t)^\top \]

be a \( p \times p \) symmetric matrix with \( p \) eigenvalues \( \lambda_i^N(t) \leq \cdots \leq \lambda_p^N(t) \), and

\[ L_N(t)(dx) = \frac{1}{p} \sum_{i=1}^p \delta_{\lambda_i^N(t)}(dx) \]

be the empirical spectral measure of \( U^N(t) \).

**Theorem 5.1.** Suppose that one of the following conditions holds,

(i) Conditions in Theorem 2.1 hold and \( \mathbb{E}[|X_0|^2] < \infty \);

(ii) Conditions in Theorem 2.2 hold and \( \mathbb{E}[|X_0|^4] < \infty \).

Assume that there exists a positive function \( \varphi(x) \in C^1(\mathbb{R}) \) with bounded derivative, such that

\[ \lim_{|x| \to \infty} \varphi(x) = +\infty \text{ and} \]

\[ \sup_{N \in \mathbb{N}} \langle \varphi, L_N(0) \rangle \leq C_0, \]

for some positive constant \( C_0 \) almost surely. Furthermore, assume that there exists a positive constant \( c \), such that \( p/N \to c \) as \( N \to \infty \).

Then for any \( T > 0 \), \( \mathbb{E}[|X_i|^2] < \infty \) for \( t \in [0,T] \), and the sequence \( \{L_N(t), t \in [0,T]\}_{N \in \mathbb{N}} \) converges in probability to \( \{\mu_t, t \in [0,T]\} \) in \( C([0,T], \mathcal{P}(\mathbb{R})) \), where \( \mu_t(dx) = \mu_{MP}(c,d_i)(dx) \) with \( \mu_{MP} \) given in (A.1).

**Proof.** Noting that \( \mathbb{E}[|X_i(t)|^2] \) exists finitely for all \( 1 \leq i \leq p, 1 \leq j \leq N \), we have that \( \hat{U}_{ij}^N(t) \) has mean 0 and finite second moment \( d_t^2 := \mathbb{E}[|\hat{U}_{ij}^N(t)|^2] \). Then by Lemma A.4, for any \( t \in [0,T] \), almost surely, the empirical distribution

\[ L_N(t)(dx) \to \mu_{MP}(c,d_i)(dx) \]

weakly as \( N \to \infty \). Thus, it remains to obtain the tightness of \( \{L_N(t), t \in [0,T]\}_{N \in \mathbb{N}} \) in the space \( C([0,T], \mathcal{P}(\mathbb{R})) \).

By Theorem 2.1 and Theorem 2.2, we have that \( |X_{ij}(t) - X_{ij}(s)| \leq \xi_{ij}|t - s|^{H-\epsilon} \), where \( \{\xi_{ij}\}_{1 \leq i \leq p, 1 \leq j \leq N} \) are i.i.d. copies of \( \xi \) with \( \mathbb{E}[|\xi|^4] < \infty \). Thus,

\[ \left| \hat{U}_{ij}^N(t) - \hat{U}_{ij}^N(s) \right| \leq |X_{ij}(t) - X_{ij}(s)| + |\mathbb{E}[X_{ij}(t)] - \mathbb{E}[X_{ij}(s)]| \]
and for

Without loss of generality, we assume that $p$ is a positive random variable that has finite second moment which is given by

\[
\|\mathbf{X}_{ij}(t) - \mathbf{X}_{ij}(s)\| + \mathbb{E} \|\mathbf{X}_{ij}(t) - \mathbf{X}_{ij}(s)\|
\]

\[
\leq (\xi_{ij} + \mathbb{E} [\xi_{ij}]) (t - s)^{H - \varepsilon}.
\] (5.3)

Hence, by (5.3), for $0 \leq s < t \leq T$,

\[
\left| \hat{U}_{ik}^{N}(t) \hat{U}_{jk}^{N}(t) - \hat{U}_{ik}^{N}(s) \hat{U}_{jk}^{N}(s) \right|^2 \\
\leq 2 \left| \hat{U}_{ik}^{N}(t) - \hat{U}_{ik}^{N}(s) \right|^2 \left| \hat{U}_{jk}^{N}(t) \right|^2 + 2 \left| \hat{U}_{ik}^{N}(s) \right|^2 \left| \hat{Y}_{jk}^{N}(t) - \hat{Y}_{jk}^{N}(s) \right|^2 \\
\leq 2 (\xi_{ik} + \mathbb{E} [\xi_{ik}])^2 \left( \left| \hat{U}_{jk}^{N}(0) \right| + (\xi_{jk} + \mathbb{E} [\xi_{jk}]) T^{H-\varepsilon} \right)^2 (t - s)^{2H-2\varepsilon} \\
+ 2 (\xi_{jk} + \mathbb{E} [\xi_{jk}])^2 \left( \left| \hat{U}_{ik}^{N}(0) \right| + (\xi_{ik} + \mathbb{E} [\xi_{ik}]) T^{H-\varepsilon} \right)^2 (t - s)^{2H-2\varepsilon} \\
= E_{H,T,\varepsilon}^{(i,j;k)} (t - s)^{2H-2\varepsilon}.
\] (5.4)

Here, $E_{H,T,\varepsilon}^{(i,j;k)}$ is a positive random variable that has finite second moment which is given by

\[
E_{H,T,\varepsilon}^{(i,j;k)} = 2 (\xi_{ik} + \mathbb{E} [\xi_{ik}])^2 \left( \left| \hat{U}_{jk}^{N}(0) \right| + (\xi_{jk} + \mathbb{E} [\xi_{jk}]) T^{H-\varepsilon} \right)^2 \\
+ 2 (\xi_{jk} + \mathbb{E} [\xi_{jk}])^2 \left( \left| \hat{U}_{ik}^{N}(0) \right| + (\xi_{ik} + \mathbb{E} [\xi_{ik}]) T^{H-\varepsilon} \right)^2.
\]

Let, for $i \neq j$,

\[
E_1 = \mathbb{E} \left[ E_{H,T,\varepsilon}^{(i,j;k)} \right] = 4 \mathbb{E} \left[ (\xi + \mathbb{E} [\xi])^2 \right] \mathbb{E} \left[ (|X_0 - \mathbb{E} [X_0]| + (\xi + \mathbb{E} [\xi]) T^{H-\varepsilon})^2 \right],
\]

and for $i = j$,

\[
E_2 = \mathbb{E} \left[ E_{H,T,\varepsilon}^{(i,i;k)} \right] = 4 \mathbb{E} \left[ (\xi + \mathbb{E} [\xi])^2 \right] \left( |X_0 - \mathbb{E} [X_0]| + (\xi + \mathbb{E} [\xi]) T^{H-\varepsilon} \right)^2.
\]

Then $E_1, E_2$ are two positive numbers depending only on $(H, T, \varepsilon)$.

Without loss of generality, we assume that $\frac{1}{N^2} \leq c + 1$. Using the Cauchy-Schwarz inequality twice, the mean value theorem, Lemma A.1, and (5.4), we can obtain

\[
\mathbb{E} \left[ \langle f, L_N(t) \rangle - \langle f, L_N(s) \rangle \right]^2 = \mathbb{E} \left[ \frac{1}{p} \sum_{i=1}^{p} f(\lambda_X^N(t)) - f(\lambda_X^N(s)) \right]^2 \\
\leq \frac{1}{p} \mathbb{E} \left[ \sum_{i=1}^{p} \left( f(\lambda_X^N(t)) - f(\lambda_X^N(s)) \right)^2 \right] \\
\leq \frac{\|f\|_{\infty}^2}{p} \sum_{i,j=1}^{p} \left( U_{ij}^N(t) - U_{ij}^N(s) \right)^2 \\
= \frac{\|f\|_{\infty}^2}{p} \sum_{i \neq j} \mathbb{E} \left[ \left( \frac{1}{N} \sum_{k=1}^{N} \left[ \hat{U}_{ik}^{N}(t) \hat{U}_{jk}^{N}(t) - \hat{U}_{ik}^{N}(s) \hat{U}_{jk}^{N}(s) \right] \right)^2 \right] \\
+ \frac{\|f\|_{\infty}^2}{p} \sum_{i=1}^{p} \mathbb{E} \left[ \left( \frac{1}{N} \sum_{k=1}^{N} \left[ \hat{U}_{ik}^{N}(t)^2 - \hat{U}_{ik}^{N}(s)^2 \right] \right)^2 \right] \\
\leq \frac{\|f\|_{\infty}^2}{pN^2} \sum_{i \neq j} \sum_{k=1}^{N} \mathbb{E} \left[ \left( \hat{U}_{ik}^{N}(t) \hat{U}_{jk}^{N}(t) - \hat{U}_{ik}^{N}(s) \hat{U}_{jk}^{N}(s) \right)^2 \right] \\
+ \frac{\|f\|_{\infty}^2}{pN} \sum_{i=1}^{p} \sum_{k=1}^{N} \mathbb{E} \left[ \left( \hat{U}_{ik}^{N}(t)^2 - \hat{U}_{ik}^{N}(s)^2 \right)^2 \right].
\]
\[
\frac{\|f\|_\infty^2}{pN^2} \sum_{i \neq j} \sum_{k=1}^N E \left[ E^{(i,j;k)}_{H,T} \right] (t-s)^{2H-2\varepsilon} + \frac{\|f\|_2^2}{pN} \sum_{i=1}^N \sum_{k=1}^N E \left[ E^{(i,i;k)}_{H,T} \right] (t-s)^{2H-2\varepsilon}
= \frac{\|f\|_\infty^2 (p-1)}{N} E_1 (t-s)^{2H-2\varepsilon} + \|f\|_2^2 E_2 (t-s)^{2H-2\varepsilon}
\leq ((c+1)E_1 + E_2) \|f\|_\infty^2 (t-s)^{2H-2\varepsilon}
\] (5.5)

for any \( f \in C^1(\mathbb{R}) \) with bounded derivative. Hence, by Theorem B.1 (with Remark B.1) and (5.2), we can conclude that the sequence \( \{L_N(t), t \in [0,T]\} \) converges in law to \( \{\mu_t = \mu_{MP}(c, d_t), t \in [0,T]\} \). Finally, noting that the limit measure \( \{\mu_t, t \in [0,T]\} \) is deterministic, the convergence in law actually coincides with the convergence in probability.

The proof is concluded.

**Remark 5.1.** In contrast, the convergences of the empirical measure-valued processes obtained in Theorem 3.1 and other subsequent results in Section 3 are almost-sure convergence, which is stronger than the in-probability convergence obtained in Theorem 5.1.

In section 3, we construct a compact set in \( C([0,T], \mathbb{P}(\mathbb{R})) \) and show that the sequence \( \{L_N(t), t \in [0,T]\} \) is that compact set almost surely. However, in the Wishart case, we are not able to get an estimation analogous to (3.6) which is the key ingredient to get the almost-sure convergence, due to the lack of the independence for the upper triangular entries. Instead, we obtain the tightness in \( C([0,T], \mathbb{P}(\mathbb{R})) \) for \( \{L_N(t), t \in [0,T]\} \) thanks to Theorem B.1, and then the convergence in law follows consequently. Note that a similar argument has appeared in related literature such as [13, 17, 18] but seems rather vague (see, e.g., the last paragraph in the proof of Theorem 5 on page 359 in [18], or the last paragraph in the proof of Proposition 1 on page 8 in [17]). Also note that in [13, Proposition 4.1], ‘almost surely’ in the statement of this proposition should be removed.

**Remark 5.2.** Let \( \sigma(x) = 1, b(x) = 0 \) and \( X_0 = 0 \), then the solution to (1.1) is the fractional Brownian motion \( X_t = B^H_t \). Then we have the convergence in law of the empirical spectral measures towards the scaled Marchenko-Pastur law \( \mu_{MP}(c, t^H) (dx) \), which recovers the results obtained in [18].

**Remark 5.3.** Let \( \tilde{Y}^N(t) = (X_{ij}(t))_{1 \leq i \leq p, 1 \leq j \leq N} \). Then under the conditions in Theorem 5.1, the sequence of empirical measures of the eigenvalues of \( \frac{1}{N} \tilde{Y}^N(t)\tilde{Y}^N(t)^\top \) converges in probability to \( \mu_{MP}(c, d_t)(dx) \) in \( C([0,T], \mathbb{P}(\mathbb{R})) \). Indeed, by the Lidskii inequality in [22, Exercise 1.3.22 (ii)], we have

\[
\left| F_{A}(\frac{1}{N} \tilde{Y}^N(t)\tilde{Y}^N(t)^\top) - F_{A}(\frac{1}{N} \tilde{U}^N(t)\tilde{U}^N(t)^\top) \right| \leq \frac{1}{N} \text{rank} \left( \frac{1}{N} \tilde{Y}^N(t)\tilde{Y}^N(t)^\top - \frac{1}{N} \tilde{U}^N(t)\tilde{U}^N(t)^\top \right),
\]

where \( F_A(x) \) is the number of the eigenvalues of \( A \) that are smaller than \( x \). Noting that the rank of

\[
\frac{1}{N} \tilde{Y}^N(t)\tilde{Y}^N(t)^\top - \frac{1}{N} \tilde{U}^N(t)\tilde{U}^N(t)^\top
= \frac{1}{N} \left( \tilde{U}^N(t) + m_t E_N \right) \left( \tilde{U}^N(t)^\top + m_t E_N \right) - \frac{1}{N} \tilde{U}^N(t)\tilde{U}^N(t)^\top
= \frac{m_t}{N} E_N \tilde{U}^N(t)^\top + \frac{m_t^2}{N} E_N + m_t^2 E_N
\]

is at most 3 for all \( t \in [0,T] \), the convergence in probability of \( \{L_N(t)\} \) towards \( \mu_{MP}(c, d_t)(dx) \) implies that the empirical spectral measures of \( \frac{1}{N} \tilde{Y}^N(t)\tilde{Y}^N(t)^\top \) converges to the same limit in probability.

**Remark 5.4.** The Stieltjes transform \( G_t(z) \) of the limiting measure \( \mu_t \) is

\[
G_t(z) = \int \frac{\mu_t(dx)}{z-x} = \int \frac{p_{MP}(c,d_t)(x)}{z-x} dx = \int \frac{p_{MP}(c,1)(x)}{z-d_t^2 x} dx,
\]
where \( p_{MP}(c, d_i)(x) \) is the probability density of the Marchenko-Pastur distribution \( \mu_{MP}(c, d_i) \) given in (A.1). Assuming that the variant \( d_i^2 \) of the solution \( X_t \) is continuously differentiable on \((0, T)\), we have

\[
\partial_t G_t(z) = \partial_t \int \frac{p_{MP}(c, 1)(x)}{z - d_i^2 x} dx = (d_i^2) \int \frac{x p_{MP}(c, 1)(x)}{(z - d_i^2 x)^2} dx \\
= \frac{(d_i^2)^{\prime}}{d_i^2} \int \left( \frac{z}{(z - d_i^2 x)^2} - \frac{1}{z - d_i^2 x} \right) p_{MP}(c, 1)(x) dx \\
= \frac{(d_i^2)^{\prime}}{d_i^2} (\partial_z G_t(z) + G_t(z)). \tag{5.6}
\]

On the other hand, [Bai and Silverstein, Lemma 3.11] and some computation yield

\[
ckd_i^2 G_t(z)^2 = G_t(z) (z - d_i^2 (1 - c)) - 1.
\]

Taking partial derivative with respect to \( z \), we have

\[
ckd_i^2 G_t(z)^2 + 2ckd_i^2 G_t(z) \partial_z G_t(z) = G_t(z) + \partial_z G_t(z) (z - d_i^2 (1 - c)). \tag{5.7}
\]

Therefore, by (5.6) and (5.7), we have

\[
\partial_t G_t(z) = -(d_i^2)^{\prime} (ckd_i^2 G_t(z)^2 + 2ckd_i^2 G_t(z) \partial_z G_t(z) + (1 - c) \partial_z G_t(z)). \tag{5.8}
\]

5.2. Complex case

Recall that \( Z = (Z^{(1)}, Z^{(2)}) \) is the solution to (3.12). Let \( \tilde{W}^N(t) = \left( \tilde{W}^N_{ij}(t) \right) \) be a \( p \times N \) matrix with entries \( \tilde{W}^N_{ij}(t) = Z_{ij}(t) - E[Z_{ij}(t)] \), where \( Z_{ij} \) are i.i.d. copies of \( Z^{(1)} + iZ^{(2)} \) and \( p = p(N) \) is a positive integer depending on \( N \). Let

\[
W^N(t) = \frac{1}{N} \tilde{W}^N(t) \tilde{W}^N(t)^* \tag{5.9}
\]

be a \( p \times p \) symmetric matrix with eigenvalue empirical measure \( L_N(t)(dx) \).

**Theorem 5.2.** Suppose that the coefficient functions \( \tilde{\sigma}, \tilde{b} \) have bounded derivatives which are Hölder continuous of order greater than \( 1/(H - z) - 1 \). Besides, assume that one of the following conditions holds,

(a) \( \| (\tilde{\sigma}_x, \tilde{\sigma}_y) \|_{L^\infty}(\mathbb{R}^2) + \| (\tilde{b}_x, \tilde{b}_y) \|_{L^\infty}(\mathbb{R}^2) > 0, \) \( E[|Z_0|^4] < \infty. \)

(b) \( \tilde{\sigma} \) and \( \tilde{b} \) are bounded and \( E[|Z_0|^2] < \infty, \)

Moreover, suppose that there exists a positive function \( \varphi(x) \in C^1(\mathbb{R}) \) with bounded derivative, such that \( \lim_{|x| \to \infty} \varphi(x) = +\infty \) and

\[
\sup_{N \in \mathbb{N}} \langle \varphi, L_N(0) \rangle \leq C_0,
\]

for some positive constant \( C_0 \) almost surely. Furthermore, suppose that there exists a positive constant \( c \), such that \( p/N \to c \) as \( N \to \infty. \)

Then for any \( T > 0 \), \( E[|Z_t|^2] < \infty, \) and the sequence \( \{L_N(t), [0, T]\} \) \( N \in \mathbb{N} \) converges in probability to \( m_{MP}(c, dZ(t))(dx) \) in \( C([0, T], \mathbb{P}(\mathbb{R})) \).

**Proof.** The proof is similar to the proofs of Theorem 5.1 and Theorem 3.3, which is sketched below.

From the proof of Theorem 3.3, we can obtain the finiteness of the mean \( m_{Z}(t) \) and \( d_{Z(t)}^2 \). Analogous to (5.2), by using Lemma A.5, we have the almost-sure convergence

\[
L_N(t)(dx) \to m_{MP}(c, dZ(t))(dx). \tag{5.10}
\]
Note that the estimation (3.14) in the proof Theorem 3.3 is still valid. Similar to the estimation (5.4) and (5.5) in the proof of Theorem 5.1, we can obtain
\[
\mathbb{E} \left[ |⟨f, L_N(t)⟩ - ⟨f, L_N(s)⟩|^2 \right] \leq C∥f′∥_∞^2(t - s)^{2H-2\varepsilon}.
\]
Then following the argument at the end of the proof of Theorem 5.1, we can obtain the tightness of the sequence \(\{L_N(t)\}_{N∈\mathbb{N}}\), which implies the convergence in distribution and hence the convergence in probability, with the deterministic limit given in (5.10).

**Remark 5.5.** Let \(\tilde{W}^N(t) = (Z_{ij}(t))_{1≤i≤p, 1≤j≤N}\). Then under the conditions in Theorem 5.2, the sequence of empirical spectral measures of \(\frac{1}{N}\tilde{W}^N(t)\tilde{W}^N(t)^T\) converges in probability to \(μ_{MP}(c, d_{\mathbb{Z}})(dx)\) in \(C([0, T], \mathbb{P}(\mathbb{R}))\).

**Remark 5.6.** Similar to Remark 5.4, the Stieltjes transform of the limit measure \(μ_t\) satisfies the differential equation (5.8) with \(d_t\) replaced by \(d_{\mathbb{Z}}(t)\).

### Appendix A: Preliminaries on (random) matrices

The following is the Hoffman-Wielandt lemma, which can be found in [1, Lemma 2.1.19], see also [22].

**Lemma A.1 (Hoffman-Wielandt).** Let \(A = (A_{ij})_{1≤i,j≤N}\) and \(B = (B_{ij})_{1≤i,j≤N}\) be \(N × N\) Hermitian matrices, with ordered eigenvalues \(λ^A_1 ≤ λ^A_2 ≤ \cdots ≤ λ^A_N\) and \(λ^B_1 ≤ λ^B_2 ≤ \cdots ≤ λ^B_N\). Then
\[
\sum_{i=1}^{N} |λ^A_i - λ^B_i|^2 \leq \text{Tr} [(A - B)(A - B)^*] = \sum_{i,j=1}^{N} |A_{ij} - B_{ij}|^2.
\]

The next two lemmas are the famous Wigner semi-circle law for the real case and complex case respectively (see, e.g., [22]).

**Lemma A.2.** Let \(M_N\) be the top left \(N × N\) minors of an infinite Wigner matrix \((ξ_{ij})_{i,j≥1}\), which symmetric, the upper-triangular entries \(ξ_{ij}, i > j\) are i.i.d. real random variables with mean zero and unit variance, and the diagonal entries \(ξ_{ii}\) are i.i.d. real variables, independent of the upper-triangular entries, with bounded mean and variance. Then the empirical spectrum distributions \(μ_{M_N/\sqrt{N}}\) converge almost surely to the Wigner semicircular distribution
\[
μ_{sc}(dx) = \frac{√4 - x^2}{2\pi} 1_{[-2,2]}(x)dx.
\]

**Lemma A.3.** Let \(M_N\) be the top left \(N × N\) minors of an infinite complex Wigner matrix \((ξ_{ij})_{i,j≥1}\), which Hermitian, the upper-triangular entries \(ξ_{ij}, i > j\) are i.i.d. complex random variables with mean zero and unit variance, and the diagonal entries \(ξ_{ii}\) are i.i.d. real variables, independent of the upper-triangular entries, with bounded mean and variance. Then the conclusion of Lemma A.2 holds.

The next two lemmas concern the celebrated Marchenko-Pastur law, which was introduced in [2].

**Lemma A.4.** Let \(X_N\) be the top left \(p(N) × N\) minors of an infinite random matrix, whose entries are i.i.d. real random variable with mean zero and variate \(σ^2\). Here, \(p(N)\) is a positive integer such that \(p(N)/N → c \in (0, ∞)\) as \(N → ∞\). Then the empirical distribution of the eigenvalues of the \(p × p\) matrix
\[
Y^N = \frac{1}{N}X_NX_N^T
\]
converges weakly to the Marchenko-Pastur distribution
\[
\mu_{MP}(c, \sigma)(dx) = \frac{1}{2\pi d^2 c} \sqrt{\sigma^2(1 + \sqrt{c})^2 - x} \left( \frac{x - \sigma^2(1 - \sqrt{c})^2}{\sigma^2(1 - \sqrt{c})^2} \right) 1_{[\sigma^2(1 - \sqrt{c})^2, \sigma^2(1 + \sqrt{c})^2]}(x) dx
\]
\[+ \left(1 - \frac{1}{c}\right) \delta_0(x) dx 1_{[c, 1]}, \tag{A.1}
\]
almost surely, where \(\delta_0\) is the point mass at the origin.

**Lemma A.5.** Let \(X_N\) be the top left \(p(N) \times N\) minors of an infinite random matrix, whose entries are i.i.d. complex random variables with mean zero and variance \(\sigma^2\). Here, \(p(N)\) is a positive integer such that \(p(N)/N \to c \in (0, \infty)\) as \(N \to \infty\). Then the empirical distribution of the \(p \times p\) matrix

\[
Y^N = \frac{1}{N} X_N X_N^\star
\]
converges almost surely to the Marchenko-Pastur distribution \(\mu_{MP}(c, \sigma)(dx)\) described in Lemma A.4.

The following result characterizes the limiting empirical spectral distribution of the symmetric random matrix with correlated entries, which is a direct corollary of [3, Theorem 3].

**Lemma A.6.** Let \((\xi_{i,j})_{i,j}\in\mathbb{Z}^2\) be an array of i.i.d. real-valued random variables with finite second moment. Let \(I\) be a finite subset of \(\mathbb{Z}^2\), \(\{a_r : r \in I\}\) be a family of constant and

\[
X_{i,j} = \sum_{r \in I} a_r \xi_{(i,j)+r}, \quad 1 \leq i \leq j.
\]

Suppose that \(\mathbb{E}[X_{0,0}] = \sum_{r \in I} a_r \mathbb{E}[\xi_{0,0}] = 0\). Denote \(\gamma_{k,l} = \gamma_{l,k} = \mathbb{E}[X_{0,0} X_{k,l}]\) for all \(k \leq l\). Let \(X^N = (X^N_{i,j})_{1 \leq i,j \leq N}\) be a symmetric matrix with entries \(X^N_{i,j} = X_{i,j}/\sqrt{N}\) for \(1 \leq i \leq j \leq N\). Then the empirical spectral measure of \(X^N\) converges to a nonrandom probability measure \(\mu_c\) with Stieltjes transform \(S_c(z) = \int_0^1 h(x, z) dx\), where \(h(x, z)\) is the solution to the equation

\[
h(x, z) = \left( -z + \int_0^1 f(x, y) h(y, z) dy \right)^{-1} \quad \text{with} \quad f(x, y) = \sum_{k,l \in \mathbb{Z}} \gamma_{k,l} e^{-2\pi i(kx+ly)}.
\]

**Appendix B: A tightness criterion for probability measures on \(C([0, T], \mathbb{P}(\mathbb{R}))\)**

Recall that \(\mathbb{P}(\mathbb{R})\) is the set of probability measures on \(\mathbb{R}\) endowed with its weak topology, and that \(C([0, T], \mathbb{P}(\mathbb{R}))\) is the space of continuous probability-measure-valued processes, both of which are Polish spaces. Denote by \(C_0(\mathbb{R})\) the set of continuous functions vanishing at infinity, which is also a Polish space.

The following lemma ([1, Lemma 4.3.13]) provide an approach to construct compact subsets in \(C([0, T], \mathbb{P}(\mathbb{R}))\).

**Lemma B.1.** Let \(K\) be a compact subset of \(\mathbb{P}(\mathbb{R})\), let \(\{f_i\}_{i \geq 0}\) be a sequence of bounded continuous functions that is dense in \(C_0(\mathbb{R})\), and let \(C_i\) be compact subsets of \(C([0, T], \mathbb{R})\) for all \(i \geq 0\). Then the sets

\[
\mathcal{H} = \left\{ \mu_t : \forall t \in [0, T], \mu_t \in K \right\} \bigcap \{t \to \mu_t(f_i) \in C_i \}
\]
are compact subsets of \(C([0, T], \mathbb{P}(\mathbb{R}))\).

Actually, we can generalize this result by letting the compact set \(K\) be depending on \(t\):
Lemma B.2. Let $D$ be a countable dense subset of $[0, T]$, let $\{K_t\}_{t \in D}$ be a family of compact subsets of $\mathbb{P}(\mathbb{R})$, let $\{f_i\}_{i \geq 0}$ be a countable dense subset of $C_0(\mathbb{R})$, and let $C_i$ be compact subsets of $C([0, T], \mathbb{R})$ for all $i \geq 0$. Then the sets
\[
K = \{ \mu_t : \mu_t \in K_t, \forall t \in D \} \cap \bigcap_{i \geq 0} \{ t \mapsto \mu_t(f_i) \in C_i \}
\]
are compact subsets of $C([0, T], \mathbb{P}(\mathbb{R}))$.

Proof. The proof is similar to that of Lemma 4.3.12 in [1]. Noting that $K$ is a closed subset of $C([0, T], \mathbb{P}(\mathbb{R}))$ which is a Polish space, it suffices to prove that $K$ is sequentially compact.

Take a sequence $\mu^n \in \mathcal{H}$ then $\mu^n_t \in K_t$ and $t \mapsto \mu^n_t(f_i) \in C_i$. Then by the diagonal procedure and the compactness of $C_i$ and $K_t$, we can find a subsequence $\mu^n_{i(t)}(n)$ such that $t \mapsto \mu^n_{i(t)}(f_i)$ converges in $C_i$ for all $i$ and $\mu^n_t(f_i)$ converge in $K_t$ for all $t \in D$ as $n$ tends to infinity. Denote
\[
\lim_{n \to \infty} \mu^n_{i(t)}(f_i) = \varphi_i(t) \in C_i \text{ for all } i \in \mathbb{N}, t \in [0, T] \text{ and } \lim_{n \to \infty} \mu^n_t(f_i) = \mu_t \in K_t \text{ for all } t \in D.
\]
The weak convergence of the measures implies that $\varphi_i(t) = \mu_t(f_i)$ for all $i \in \mathbb{N}$ and $t \in D$. Note that $\{f_i\}_{i \in \mathbb{N}}$ is dense in $C_0(\mathbb{R})$ and $\varphi_i(t)$ is continuous, $\{\mu_t, t \in D\}$ can be uniquely extended to a family of probability measures $\{\mu_t, t \in [0, T]\}$ that is continuous with respect to $t$. Therefore,
\[
\lim_{n \to \infty} \mu^n_t(f_i) = \nu \in C([0, T], \mathbb{P}(\mathbb{R})).
\]

To construct a compact set $K$ in $\mathbb{P}(\mathbb{R})$, we usually employ the following lemma (see, e.g., [6, Theorem 3.2.14]).

Lemma B.3. If there is a non-negative function $\varphi$ so that $\varphi(x) \to \infty$ as $|x| \to \infty$ and
\[
C = \sup_{n \in \mathbb{N}} \int \varphi(x) \mu_n(dx) < \infty,
\]
then the sequence of probability measures $\{\mu_n\}_{n \in \mathbb{N}}$ is tight.

By the Arzela-Ascoli Theorem, we have the following lemma:

Lemma B.4. A set of the form
\[
C = \bigcap_{n \in \mathbb{N}} \left\{ g \in C([0, T], \mathbb{R}) : \sup_{t, s \in [0, T], |t-s| \leq \varepsilon_n} |g(t) - g(s)| \leq \varepsilon_n, \sup_{t \in [0, T]} |g(t)| \leq M \right\},
\]
is compact in $C([0, T], \mathbb{P}(\mathbb{R}))$, where $M$ is a positive constant and $\{\varepsilon_n, n \in \mathbb{N}\}$ and $\{\eta_n, n \in \mathbb{N}\}$ are two sequences of positive numbers going to zero as $n$ goes to infinity.

The following is the main result in this section, which provides a tightness criterion for probability measures on $C([0, T], \mathbb{P}(\mathbb{R}))$.

Theorem B.1. For a fixed number $T > 0$, let $\{\mu_t^{(n)}, t \in [0, T]\}_{n \in \mathbb{N}}$ be a sequence of probability-measure-valued stochastic processes. Assume that there exists a nonnegative function $\varphi(x)$ such that $\varphi(x) \to \infty$ as $|x| \to \infty$ and that the family $\{\varphi, \mu_t^{(n)}\}_{n \in \mathbb{N}, t \in [0, T]}$ is uniformly integrable.

Moreover, suppose that there exists a countable set $\{f_i\}_{i \geq 0}$ that is dense in $C_0(\mathbb{R})$, such that there exist positive constants $\alpha, \beta$,
\[
\mathbb{E} \left[ \left| f_t \mu_t^{(n)} - \langle f_t, \mu_t^{(n)} \rangle \right|^{1+\alpha} \right] \leq C_{f, T}|t-s|^{1+\beta}, \quad \forall t, s \in [0, T], \forall n \in \mathbb{N}, \quad (B.1)
\]
for all $f \in \{f_i\}_{i \geq 0}$, where $C_{f, T}$ is a constant depending only on $f$ and $T$. Then the set $\{\mu_t^{(n)}, t \in [0, T]\}_{n \in \mathbb{N}}$ of probability-measure-valued processes induces a tight family of measures on $C([0, T], \mathbb{P}(\mathbb{R}))$.

Proof. By the Kolomogorov tightness criterion, see [12, Theorem 4.2 and Theorem 4.3], the stochastic process sequence of $\{f_t^{(n)}\}$ is tight for any $f \in \{f_i\}_{i \geq 0}$. Then for any $\varepsilon > 0$, there exist compact sets $C_i^\varepsilon$ of $C([0, T], \mathbb{R})$, such that for each $i \in \mathbb{N}$,
\[
P \left( \{ f_i^{(n)} \} \in C_i^\varepsilon, \forall n \in \mathbb{N} \right) \geq 1 - \varepsilon/2^i,
\]
and hence,

\[
P \left( \bigcap_{i \geq 0} \left\{ (f, \mu_i^{(n)}) \in C_i^\varepsilon, \forall n \in \mathbb{N} \right\} \right) \geq 1 - \sum_{i \geq 0} P \left( \{ (f, \mu_i^{(n)}) \in C_i^\varepsilon, \forall n \in \mathbb{N} \}^c \right) \geq 1 - \varepsilon. \tag{B.2}
\]

Let \( D = \{ t_i \}_{i \geq 0} \) be a countable dense subset of \([0, T]\). For any \( \varepsilon > 0 \) and \( i \in \mathbb{N} \), there exists a positive number \( M_i^\varepsilon \) that depends on \( \varepsilon \) and \( i \) only, such that

\[
P \left( \langle \varphi, \mu_i^{(n)} \rangle > M_i^\varepsilon, \forall n \in \mathbb{N} \right) \leq \frac{\varepsilon}{2^i}.
\]

due to the uniform integrability of \( \{ \langle \varphi, \mu_i^{(n)} \rangle \}_{n \in \mathbb{N}, t \in [0, T]} \). Define the compact sets,

\[K_i^\varepsilon = \{ \nu \in \mathbf{P}(\mathbb{R}) : \langle \varphi, \nu \rangle \leq M_i^\varepsilon \}, \quad i \in \mathbb{N}.
\]

Thus,

\[
P \left( \mu_i^{(n)} \in K_i^\varepsilon, \forall i \geq 0, \forall n \in \mathbb{N} \right) \geq 1 - \sum_{i \geq 0} P \left( \mu_i^{(n)} \notin K_i^\varepsilon, \exists n \in \mathbb{N} \right) = 1 - \sum_{i \geq 0} P \left( \langle \varphi, \mu_i^{(n)} \rangle > M_i^\varepsilon, \exists n \in \mathbb{N} \right) \geq 1 - \varepsilon. \tag{B.3}
\]

Finally, note that by Lemma B.2, the set

\[K(\varepsilon) = \{ \mu_i : \mu_i \in K_i^\varepsilon, \forall i \geq 0 \} \cap \bigcap_{i \geq 0} \{ t \rightarrow \langle f_i, \mu_i \rangle \in C_i^\varepsilon \},\]

is compact in \( C([0, T], \mathbf{P}(\mathbb{R})) \) for any \( \varepsilon > 0 \). By (B.2) and (B.3), we have

\[
P \left( \mu_i^{(n)} \in K(\varepsilon), \forall n \in \mathbb{N} \right) \geq 1 - 2\varepsilon.
\]

This implies the tightness of the probability measures on \( C([0, T], \mathbf{P}(\mathbb{R})) \) induced by the stochastic probability-measure-valued processes \( \{ \mu_i^{(n)}, t \in [0, T] \}_{n \in \mathbb{N}} \), and the proof is concluded. \( \square \)

**Remark B.1.** One sufficient condition that yields the uniform integrability of \( \{ \langle \varphi, \mu_i^{(n)} \rangle \}_{n \in \mathbb{N}, t \in [0, T]} \) is

\[
\sup_{n \in \mathbb{N}, t \in [0, T]} \mathbb{E} \left[ \left| \langle \varphi, \mu_i^{(n)} \rangle \right|^{1+\alpha} \right] < \infty,
\]

for some \( \alpha > 0 \), which can be obtained, for instance, if we assume that

\[
\sup_{n \in \mathbb{N}} \int \varphi(x)\mu_0^{(n)}(dx) < M_0
\]

holds almost surely for some \( M_0 > 0 \) and that (B.1) holds for \( \varphi \).

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