Gauge Fields Out-Of-Equilibrium: A Gauge Invariant Formulation and the Coulomb Gauge

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Abstract

We study the abelian Higgs model out-of-equilibrium in two different approaches, a gauge invariant formulation, proposed by Boyanovsky et al. [1] and in the Coulomb gauge. We show that both approaches become equivalent in a consistent one loop approximation. Furthermore, we carry out a proper renormalization for the model in order to prepare the equations for a numerical implementation. The additional degrees of freedom, which arise in gauge theories, influence the behavior of the system dramatically. A comparison with results in the 't Hooft-Feynman background gauge found by us recently, shows very good agreement.

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I. INTRODUCTION

Non-equilibrium dynamics have become a very active field of research in the last few years in nearly all parts of physics. In condensed matter physics for example the description of the dynamics of non-equilibrium phase transitions plays an important role [2]. Such phase transitions occur in ferromagnets, superfluids, and liquid crystals to name only a few. They are subjects of intensive studies, both theoretical and experimental.

Also in cosmology some phenomena require the use of non-equilibrium technics. One example is the electroweak phase transition. The electroweak phase transition could lead to a possible model for baryogenesis. This problem is investigated by several groups, e.g., [3, 4], using non-equilibrium methods.

Another phenomenon in cosmology where non-equilibrium dynamics are important is the inflationary phase of the early universe which is studied intensively by different groups [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Inflation refers to an epoch during the evolution of the universe in which it underwent an accelerated expansion phase. This would resolve some of the short comings of the Standard hot Big Bang model, e.g., the flatness problem, concerning the energy density of the universe and the horizon problem, related by the large scale smoothness of the universe, indicated by the Cosmic Microwave Background Radiation (CMBR).

At lower energies heavy ion collisions are under consideration as non-equilibrium processes [15, 16, 17, 18, 19, 20]. In such heavy ion collisions a new state of matter could be reached if the short range impulsive forces between nucleons could be overcome and if squeezed nucleons would merge into each other. This new state should be a Quark Gluon Plasma (QGP), in which quarks and gluons, the fundamental constituents of matter, are no longer confined, but free to move around over a volume in which a high enough temperature and/or density reveals. Heavy ion collisions are studied experimentally at current and forthcoming accelerators, the Relativistic Heavy Ion Collider RHIC at Brookhaven and the Large Hadron Collider LHC at CERN. The occurring Quantum Chromodynamic (QCD) phase transition in these processes could be out of equilibrium and lead to formations of coherent condensates of low energy pions, so called Disoriented Chiral Condensates (DCC). Recent results reported from CERN-SPS [21] seem to indicate a strong evidence for the existence of a QGP in Pb-Pb collisions.

The underlying theories of the above described phenomena are gauge theories. Therefore, it is of great importance to implement gauge field fluctuations if one wants to deal with more realistic models. Gauge theories out of equilibrium are still poorly explored in the literature. While the aspects of various gauges and of gauge invariance have been discussed extensively in perturbation theory and for equilibrium systems, the analysis of such systems out of equilibrium is still to be developed. The analysis presented here focus on different approaches for the description of non-equilibrium gauge theories and compare these approaches.

In [22] we have presented the theoretical framework and its numerical implementation for the SU(2) Higgs model in the ’t Hooft-Feynman background gauge. Obviously, the question of gauge dependence and independence has to be explored. In [23] we investigated the gauge invariance of the one-loop effective action of a Higgs field in the SU(2)-Higgs model for a time independent problem, the bubble nucleation. An analogous analysis for a system out of equilibrium [24] has shown that the fluctuation operator decomposes in a similar way to the one derived in [23]. We have found within this analysis a close relation between the ’t Hooft-background gauge and the Coulomb gauge.
In this work we are now considering two other approaches in order to investigate gauge
theories out of equilibrium and to get some insight how the behavior of the system is in-
fluenced by the choice of gauge. Therefore, we study the Abelian Higgs model in a gauge
invariant approach which was developed by Boyanovsky et al. \cite{1, 25} in order to study non-
equilibrium aspects of the effective potential. We use their formalism in order to derive the
equations of motion in the one loop approximation. We will indicate that the gauge invariant
approach leads to IR-problems beyond the one loop approximation. As a second approach
we examine the Coulomb gauge. Here we use a gauge condition in order to eliminate the
unphysical degrees of freedom. We show how a proper renormalization can be implemented.
Therefore, we use a scheme which was developed by us for the $\phi^4$ theory \cite{26} and extended
to fermionic and gauge systems \cite{22, 27}. Furthermore, we show how the gauge invariant
approach and the Coulomb gauge are related to each other.

In order to compare the behavior of the system for different gauges quantitatively, we
reexamine the ’t Hooft-Feynman gauge and implement the ’t Hooft-Feynman gauge and
the Coulomb gauge numerically. We examine the development of the zero mode and the
different fluctuations. We also investigate the problem of energy conservation.

The paper is organized as follows. In section 2 we start our discussion by outlining the
main aspects of the gauge invariant approach of Boyanovsky et al. We derive the equation
of motion for the zero mode and the different fluctuations and we determine the energy. In
section 3 we present the equation of motion and the energy in the Coulomb gauge. The
comparison of the two approaches is investigated in section 4. In section 5 we show how
the renormalization can be done properly in order to prepare the equations for numerical
implementations. Some numerical results for the Coulomb gauge and for comparison also in
the ’t Hooft-Feynman background gauge are presented in section 5. We conclude in section
6 with a discussion of the influence of the different degrees of freedom and an outlook to
more realistic applications of the method.

II. A GAUGE INVARIANT APPROACH

A. The Formalism

We study the Abelian Higgs model in a gauge invariant formulation. The basic ideas for
this description are developed in \cite{1, 25}. We give here a short overview about the derivation
of the Hamiltonian. For details the reader is referred to the two papers mentioned above. The
Lagrangian density for the Abelian Higgs model reads

$$ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + D_\mu \phi^\dagger D^\mu \phi - \frac{\lambda}{4} (\phi^\dagger \phi - v^2)^2, \quad (2.1) $$

$$ D_\mu = \partial_\mu + ieA_\mu. \quad (2.2) $$

We want to formulate a gauge invariant Hamiltonian. Therefore, we use as in \cite{1, 25} the
canonical formulation. We have to identify the canonical field variables and constrains. The
canonical momenta conjugate to the scalar and vector fields are given by

$$ \Pi^0 = 0, \quad (2.3) $$

$$ \Pi^i = \dot{A}^i + \nabla^i A^0 = -E^i, \quad (2.4) $$

$$ \pi^\dagger = \dot{\phi} + ieA^0 \phi, \quad (2.5) $$

$$ \pi = \dot{\phi}^\dagger - ieA^0 \phi^\dagger. \quad (2.6) $$

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Therefore, the Hamiltonian reads

\[
H = \int d^3x \left\{ \frac{1}{2} \bar{\Pi} \cdot \Pi^\dagger + \pi^\dagger \pi + (\nabla \phi - ie\bar{A}\phi) \cdot (\nabla \phi^* + ie\bar{A}\phi^*) + \frac{1}{2}(\nabla \times \bar{A})^2 + \frac{\lambda}{4}(\phi^\dagger \phi - v^2)^2 + A_0 \left[ \nabla \cdot \Pi - ie(\pi \phi - \pi^\dagger \phi^*) \right] \right\} .
\] (2.7)

We will quantize this system with Dirac’s method [28]. Therefore, we have to recognize the first class constraints (mutually vanishing Poisson brackets). Then the constraints become operators in the quantum theory and are imposed onto the physical states, thus defining the physical subspace of the Hilbert space and gauge invariant operators. We have two first class constraints

\[
\Pi^0 = \frac{\delta L}{\delta A^0} = 0 ,
\] (2.8)

and Gauss’ law

\[
G(\vec{x}, t) = \nabla^i \pi^i - \rho = 0 ,
\] (2.9)

\[
\rho = ie \left( \phi \pi^* - \phi^* \pi \right) ,
\] (2.10)

with \(\rho\) being the matter field charge density. We can now quantize the system by imposing the canonical equal time commutation relations

\[
[\Pi^0(\vec{x}, t), A^0(\vec{y}, t)] = -i\delta(\vec{x} - \vec{y}) ,
\] (2.11)

\[
[\Pi^i(\vec{x}, t), A^j(\vec{y}, t)] = -i\delta^{ij}\delta(\vec{x} - \vec{y}) ,
\] (2.12)

\[
[\pi^\dagger(\vec{x}, t), \phi(\vec{y}, t)] = -i\delta(\vec{x} - \vec{y}) ,
\] (2.13)

\[
[\pi(\vec{x}, t), \phi^*\pi^* - \phi^* \pi^\dagger + \phi \pi^\dagger - \phi^* \pi] = -i\delta(\vec{x} - \vec{y}) .
\] (2.14)

In Dirac’s formulation, physical operators are those that commute with the first class constraints. Since \(\Pi^0(\vec{x}, t)\) and \(G(\vec{x}, t)\) are generators of local gauge transformations, operators that commute with the first class constraints are gauge invariant [1, 25]. As shown in [1] the fields and the conjugate momenta can be written in the following form

\[
\Phi(\vec{x}) = \phi(\vec{x}) \exp \left[ ie \int d^3y \bar{A}(\vec{y}) \cdot \nabla y G(\vec{y} - \vec{x}) \right] ,
\] (2.15)

\[
\Pi(\vec{x}) = \pi(\vec{x}) \exp \left[ -ie^2 \int d^3y \bar{A}(\vec{y}) \cdot \nabla y G(\vec{y} - \vec{x}) \right] ,
\] (2.16)

with \(G(\vec{y} - \vec{x})\) the Coulomb Green’s function that satisfies

\[
\triangle G(\vec{y} - \vec{x}) = \delta^3(\vec{y} - \vec{x}) .
\] (2.17)

They are invariant under gauge transformations [1]. The gauge field can be separated into transverse and longitudinal components

\[
\bar{A}(\vec{x}) = \bar{A}_L(\vec{x}) + \bar{A}_T(\vec{x}) ,
\] (2.18)

\[
\nabla \times \bar{A}_L(\vec{x}) = 0 ,
\] (2.19)

\[
\nabla \cdot \bar{A}_T(\vec{x}) = 0 .
\] (2.20)
Since the fields $\vec{A}_T$ and $\Phi$ and their canonical momenta commute with the constraints, they are gauge invariant. It is also possible to write the momentum canonical to the gauge field in longitudinal and transverse components

$$\vec{\Pi}(\vec{x}) = \vec{\Pi}_L(\vec{x}) + \vec{\Pi}_T(\vec{x}) , \quad (2.22)$$

where both components are gauge invariant. In [1], it is mentioned that in all matrix elements between gauge invariant states the longitudinal component can be replaced by the charge density

$$\vec{\Pi}_L(\vec{x}) \rightarrow ie \left[ \Phi(\vec{y})\Pi(\vec{y}) - \Phi^\dagger(\vec{y})\Pi^\dagger(\vec{y}) \right] = \rho . \quad (2.23)$$

Finally, the Hamiltonian becomes

$$H = \int d^3x \left\{ \frac{1}{2} \vec{\Pi}_T \cdot \vec{\Pi}_T + \Pi^\dagger \Pi + (\vec{\nabla} \Phi - ie\vec{A}_T \Phi) \cdot (\vec{\nabla} \Phi^\dagger + ie\vec{A}_T \Phi^\dagger) + \frac{1}{2} (\vec{\nabla} \times \vec{A}_T)^2 
+ \frac{\lambda}{4} (\Phi^\dagger \Phi - v^2)^2 \right\} + \frac{1}{2} \int d^3x \int d^3y \rho(\vec{x})G(\vec{x} - \vec{y})\rho(\vec{y}) \right\} . \quad (2.24)$$

The features of this Hamiltonian are discussed at length in [1, 25]. One of the striking points is the equivalence with the Hamiltonian in the Coulomb gauge. This similarity is not uncommon because, in the Coulomb gauge, only the physical degrees of freedom are taken into account. We focus our interest on the non-equilibrium aspects of the theory. In [1, 25], the effective potential was derived and some aspects of non-equilibrium dynamics were discussed. But as there explain the effective potential is not suitable for non-equilibrium dynamics. We derive here the full non-equilibrium equations. We include not only the terms which are quadratic in the zero mode, but all terms up to second order which are relevant for the one loop approximation. We consider the derivative terms of the zero mode and of its conjugate momentum. This yields the wave function renormalization which is not considered in [1, 25] or has to be introduced by hand. We will also see that the formalism does not give a clear statement about the loop order which is included. When linearized, the equations become equivalent to those obtaining in the Coulomb gauge, which we consider in detail in section [IV]. There, we show the correspondence of the Hamiltonian approach and the Coulomb gauge. As we will show, the inclusion of higher loop terms in the Hamiltonian approach leads to problems in the IR-region.

First of all, we derive the equation of motion for the fields. Therefore, we separate the expectation value of $\Phi$ and of its canonical momentum into a mean value and a fluctuation part

$$\Phi(\vec{x}, t) = \phi(t) + \varphi(\vec{x}, t) , \quad (2.25)$$
$$\Pi(\vec{x}, t) = \Pi(t) + \pi(\vec{x}, t) . \quad (2.26)$$

We also introduce real fields and canonical momenta as follows

$$\varphi = \frac{1}{\sqrt{2}}(h + i\varphi) , \quad (2.27)$$
$$\pi = \frac{1}{\sqrt{2}}(\pi_h - i\pi_\varphi) . \quad (2.28)$$

Therefore, we find as a gauge invariant Hamiltonian

$$H = \Omega \left[ \frac{1}{2} \Pi^2 + U(\phi) \right] \quad (2.29)$$

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\[
+ \int d^3x \left[ \frac{1}{2} \pi_{\perp}^2 - \frac{1}{2} (\nabla \times \vec{a}_{\perp})^2 + \frac{1}{2} \phi^2 a_{\perp}^2 + \frac{1}{2} \pi_h^2 + \frac{1}{2} \pi_{\phi}^2 \right.
\]
\[
+ \frac{1}{2} (\nabla h)^2 + \frac{1}{2} (\nabla \varphi)^2 + \frac{\lambda}{2} (\phi^2 - v^2)(h^2 + \varphi^2) + \lambda \phi^2 h^2 \bigg] \]
\[
+ \int d^3x \int d^3y \frac{e^2}{2} \left[ \phi \pi(\vec{x}) - \Pi \varphi(\vec{x}) \right] G(\vec{x} - \vec{y}) \left[ \phi \pi(\vec{y}) - \Pi \varphi(\vec{y}) \right],
\]

with
\[
U(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2.
\]

**B. Equation of Motion and Energy Density**

We are now able to derive the equations of motion for the zero mode and the fluctuation fields. In the Hamiltonian formalism we find the equation of motion by calculating the commutator between the Hamiltonian and the field. We obtain for the zero mode
\[
\dot{\phi} = \Pi \left[ 1 + \frac{e^2}{\Omega} \int d^3x \int d^3y \varphi(\vec{x}) G(\vec{x} - \vec{y}) \phi(\vec{y}) \right]
\]
\[
- \frac{e^2}{2\Omega} \phi \int d^3x \int d^3y \left[ \varphi(\vec{x}) G(\vec{x} - \vec{y}) \pi(\vec{y}) + \pi(\vec{x}) G(\vec{x} - \vec{y}) \phi(\vec{y}) \right],
\]
and for the canonical momentum
\[
\dot{\Pi} = -U'(\phi) - \frac{\phi}{\Omega} \int d^3x \left[ e^2 \vec{a}_{\perp}^2 + \lambda (h^2 + \varphi^2) + 2\lambda h^2 \right]
\]
\[
- \frac{e^2}{\Omega} \phi \int d^3x \int d^3y \pi(\vec{x}) G(\vec{x} - \vec{y}) \phi(\vec{y})
\]
\[
+ \frac{e^2}{2\Omega} \Pi \int d^3x \int d^3y \left[ \pi(\vec{x}) G(\vec{x} - \vec{y}) \varphi(\vec{y}) + \varphi(\vec{x}) G(\vec{x} - \vec{y}) \pi(\vec{y}) \right].
\]

We also need the equations of motion for the three different quantum fluctuations. The first one is the transverse gauge field. We find an equation for the field \( a_{\perp} \) itself and for its canonical momentum. We can combine these two expressions to a second order differential equation for the gauge field:
\[
\dot{\vec{a}}_{\perp} = \vec{\pi}_{\perp}, \quad \pi_{\perp} = (\Delta - e^2 \phi^2) a_{\perp}
\]
\[
\Rightarrow \ddot{a}_{\perp} = (\Delta - e^2 \phi^2) a_{\perp}.
\]

In the same way, we get the equation of motion for the real component of the Higgs fluctuation
\[
\dot{h} = \left[ \Delta - \lambda (3\phi^2 - v^2) \right] h.
\]

More difficulties arise for the Goldstone sector \( \varphi \) because the field and its canonical momentum are coupled via Green functions. We find for the field itself
\[
\dot{\varphi} = \pi_{\varphi} + \frac{e^2}{2} \int d^3x \left[ \phi G_{xy}(\phi \pi_{\varphi} - \Pi \varphi) + (\phi \pi_{\varphi} - \Pi \varphi) G_{xy} \phi \right],
\]

(2.35)
and for the momentum

\[ \hat{\pi}_\varphi = \Delta \varphi - \lambda \varphi (\dot{\varphi}^2 - v^2) - \frac{e^2}{2} \int d^3x \left[ \Pi^2 G_{x\varphi \varphi} - (\Pi \varphi + \phi \Pi) G_{x\varphi \pi} \right]. \] (2.36)

Now we transform the equations of motion for the zero mode and the fluctuations into Fourier space. Therefore, we expand the fluctuation fields in the following way

\[ \psi(x, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_0} \left\{ a_k U(k, t) e^{i\vec{k} \cdot \vec{x} - i\omega_0 t} + a_k^\dagger U^*(k, t) e^{-i\vec{k} \cdot \vec{x} + i\omega_0 t} \right\}, \] (2.37)

with the usual commutator relations for the annihilation and creation operators \( a_k, a_k^\dagger \):

\[ [a_k, a_{k'}^\dagger] = (2\pi)^3 2\omega_0 \delta^3(\vec{k} - \vec{k'}) . \] (2.38)

The \( U(k, t) \) are the mode functions for the fluctuations depending on \( k \). For convenience we omit the momentum dependence in the following. The Fourier transform for the Green function leads to a factor \( 1/k^2 \). With these expansions, the equation of motion for the zero mode reads

\[ \dot{\varphi}(t) = \Pi(t) \left[ 1 + e^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_0 k^2} |U_\varphi(t)|^2 \right] - \frac{e^2}{2} \phi(t) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_0 k^2} \left[ U_\varphi(t)U^*_{\varphi}(t) + U_{\varphi}(t)U^*_{\varphi}(t) \right], \] (2.39)

and for the canonical momentum

\[ \dot{\Pi}(t) = -U''(\phi) - \phi(t) \int \frac{d^3k}{(2\pi)^3} \left[ \frac{e^2}{\omega_0} |U_\varphi(t)|^2 + \frac{3\lambda}{2\omega_0} |U_\varphi(t)|^2 + \frac{\lambda}{2\omega_0} |U_{\varphi}(t)|^2 + \frac{e^2}{2\omega_0 k^2} |U_{\varphi}(t)|^2 \right] + \Pi(t) \int \frac{d^3k}{(2\pi)^3} \frac{e^2}{4\omega_0 k^2} \left[ U_{\varphi}(t)U^*_{\varphi}(t) + U_{\varphi}(t)U^*_{\varphi}(t) \right], \] (2.40)

where we have introduced the following frequencies

\[ \omega_{\varphi 0}^2 = \left[ \vec{k}^2 + \lambda (\dot{\phi}_0^2 - v^2) \right] / k^2 = \omega_{\varphi 0}^2 - \omega_{a 0}^2 / k^2, \] (2.41)

\[ \omega_{\varphi 0}^2 = \vec{k}^2 + \lambda (\dot{\phi}_0^2 - v^2), \] (2.42)

\[ \omega_{a 0}^2 = \vec{k}^2 + m_W^2 + e^2 (\dot{\phi}_0^2 - v^2), \] (2.43)

\[ \omega_{h 0}^2 = \vec{k}^2 + m_h^2 + 3\lambda (\dot{\phi}_0^2 - v^2), \] (2.44)

\[ m_h^2 = 2\lambda v^2, \quad m_W^2 = e^2 v^2. \] (2.45)

The index 0 indicates the choice of \( t = 0 \). Notice, that we have included a factor two for the two transverse gauge freedoms in the equation of motion for \( \dot{\Pi} \). \( \omega_{\varphi 0} \) was introduced in \( [1] \). By comparing these results with the zero mode equation in the gauge fixed theory which we have examined in \( [22] \) we find some analogies in the fluctuation integrals. The transverse gauge field and the Higgs field component \( h \) lead to the same contribution in both theories. The Goldstone channel \( \varphi \) fulfills a coupled differential equation. In the gauge invariant description we have a coupling between the field itself and its canonical momentum. In
the 't Hooft-Feynman-gauge, it couples to the time component of the gauge field. This component do not appear in the new description because they are unphysical. In the 't Hooft-Feynman gauge the unphysical degrees of freedom are compensated by the ghosts.

In the same way we have found the zero mode equation, we can derive the mode functions for the fluctuation fields. We find

\[
\begin{align*}
\left[ \frac{d^2}{dt^2} + \omega_a^2(t) \right] U_{\perp}(t) &= 0 , \quad (2.46) \\
\left[ \frac{d^2}{dt^2} + \omega_h^2(t) \right] U_h(t) &= 0 , \quad (2.47) \\
\left[ \frac{d}{dt} + e^2 k^2 \Phi(t) \right] U_\varphi(t) - \frac{\omega_\varphi^2(t)}{k^2} U_\varphi(t) &= 0 , \quad (2.48) \\
\left[ \frac{d}{dt} - e^2 k^2 \Phi(t) \right] U_{\pi \varphi}(t) + \left[ \omega_\varphi^2(t) + \frac{e^2}{k^2} \Pi^2(t) \right] U_\varphi(t) &= 0 . \quad (2.49)
\end{align*}
\]

In the mode equations for \( U_\varphi \) and \( U_{\pi \varphi} \), a problem in the IR region becomes obvious. The denominator with \( k^2 \) leads to problems for vanishing momentum. During the discussion of the comparison of the different approaches in section IV, it will become clear that this IR-instability is caused by higher loop effects. It is possible by combining the differential equation for the field (2.48) and its conjugate momentum (2.49) to find an IR-stable mode equation by neglecting all terms of higher order than one loop.

The energy density can easily be calculated from the Hamiltonian. We have to insert the expansion of the fields in dependence of the mode functions with the corresponding creators and annihilators. Then we have to take the expectation value in the ground state. The field expansion is analogous to the one for the equation of motion (2.37). We find for the energy density in the Fourier space

\[
\mathcal{E} = \frac{1}{2} \Pi^2(t) + U[\phi(t)] \\
+ \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_0} \left[ |\dot{U}_{\perp}(t)|^2 + \omega_\perp^2(t)|U_{\perp}(t)|^2 \right] \\
+ \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_0} \left[ |\dot{U}_h(t)|^2 + \omega_h^2(t)|U_h(t)|^2 \right] \\
+ \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_\varphi} \left[ |U_{\pi \varphi}(t)|^2 + \omega_\varphi^2(t)|U_\varphi(t)|^2 \right] \\
+ \int \frac{d^3k}{(2\pi)^3} \frac{e^2}{4\omega_\varphi} \left[ \phi^2(t)|U_{\pi \varphi}(t)|^2 + \Pi^2(t)|U_\varphi(t)|^2 \right. \\
\left. - \Pi(t)\Phi(t) \left[ U^*_{\pi \varphi}(t)U_{\varphi}(t) + U_{\pi \varphi}(t)U^*_{\varphi}(t) \right] \right] . \quad (2.50)
\]

It is easy to check the conservation of the energy by determining the time derivative. By using the equations of motion we can show that it vanishes.
III. COULOMB GAUGE

A. Equation of Motion and Energy Density

Our starting point is again the Lagrangian (2.1). As discussed, e.g. in [1], and in the last section it is possible to find a gauge invariant description of the Abelian Higgs model by quantizing the theory with Dirac’s method. An equivalent way is to choose the Coulomb gauge condition \( \vec{\nabla} \cdot \vec{A} = 0 \). One gets a Hamiltonian written in terms of transverse components and including the instantenous Coulomb interaction. This Coulomb interaction can be traded with a Lagrange multiplier field linearly coupled to the charge density. This leads to the Lagrangian in the Coulomb gauge

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \Phi^\dagger \partial^\mu \Phi + \frac{1}{2} \partial_{\mu} \vec{A}_T \partial^\mu \vec{A}_T - e \vec{A}_T \cdot \vec{j}_T - e^2 \vec{A}_T \cdot \vec{A}_T \Phi^\dagger \Phi \\
+ \frac{1}{2} (\nabla A_0)^2 + e^2 A_0^2 \Phi^\dagger \Phi - ie A_0 \rho - V(\Phi^\dagger \Phi),
\]

\[
\vec{j}_T = i(\Phi^\dagger \vec{\nabla}_T \Phi - \vec{\nabla}_T \Phi^\dagger \Phi), \quad \rho = -i(\Phi \dot{\Phi}^\dagger - \Phi^\dagger \dot{\Phi}),
\]

where \( A_T \) is the transverse component of the gauge field. The field \( A_0 \) is a gauge invariant Lagrange multiplier whose equation of motion is algebraic:

\[
\nabla^2 A_0(\vec{x}, t) = \rho(\vec{x}, t).
\]

Using the usual decomposition of \( \Phi \) into an expectation value and a fluctuation part, and splitting the fluctuations into a real part \( h \) and an imaginary part \( \phi \), we can derive the equations of motions for the different fluctuations and the zero mode. We give the equations in the Fourier space. The physical degrees of freedom - the transversal gauge mode and the Higgs mode \( h \) - are the same as in the gauge invariant approach and in the ‘t Hooft-Feynman gauge. We now investigate the remaining degrees of freedom, the Goldstone field \( \phi \) and the Lagrange multiplier \( a_0 \). As already mentioned, the equation that fixes \( a_0 \) is purely algebraic. The field equation reads

\[
\omega_a^2(t)a_0(t) = e \left[ \dot{\phi}(t) \phi(t) - \varphi(t) \dot{\phi}(t) \right].
\]

For the Goldstone field we find

\[
\ddot{\varphi}(t) + \left[ k^2 + \lambda(\varphi^2(t) - v^2) \right] \varphi(t) = e a_0(t) \phi(t) + 2 e a_0(t) \dot{\phi}(t).
\]

It is possible to eliminate the mode function for the Lagrange multiplier \( a_0 \) in (3.4). By using the differential equation for \( \varphi \) and the classical equation of motion

\[
\ddot{\phi} - \lambda \phi(\varphi^2 - v^2) = 0,
\]

it is easy to show that the time derivative of \( a_0 \) is given by

\[
\dot{a}_0(t) = \frac{e}{\omega_a^2(t)} \left[ \phi(t) \dot{\phi}(t) - \varphi(t) \dot{\phi}(t) - 2 e \phi(t) \dot{\phi}(t) a_0(t) \right] = -e \phi(t) \varphi(t).
\]

Inserting (3.3) and (3.4) in (3.4) leads to

\[
\mathcal{M}_{\varphi \varphi}(t) \varphi(t) = 0,
\]
where $\mathcal{M}_{\phi\phi}$ is given by
\[
\mathcal{M}_{\phi\phi}(t) = \frac{d^2}{dt^2} + k^2 + e^2\phi^2(t) + \lambda \left[ \phi^2(t) - v^2 \right] - \frac{2e^2\phi(t)}{\omega_a^2(t)} \left[ \phi(t) \frac{d}{dt} - \dot{\phi}(t) \right]. \quad (3.8)
\]
$a_0$ is a dependent mode and the physical modes are the two transversal gauge fields $a_\perp$ and the scalar Higgs mode $h$. From the Lagrangian $(3.1)$, we can also derive the equation of motion for the zero mode $\phi$. We find
\[
\ddot{\phi} + \lambda \phi (\phi^2 - v^2) + 3\lambda \phi \langle h^2 \rangle + 2e^2 \phi \langle a_\perp^2 \rangle + \lambda \phi \langle \varphi^2 \rangle - e^2 \phi \langle a_0^2 \rangle + 2e \langle a_0 \varphi \rangle + e \langle a_0 \varphi \rangle = 0, \quad (3.9)
\]
where the expectation values for the fields and their normalization are not specified yet. Now we eliminate the field $a_0$ without using the classical equation of motion. We set $\dot{\phi}$ equal to
\[
\dot{\phi} = -\lambda \phi (\phi^2 - v^2) + R(t), \quad (3.10)
\]
where $R(t)$ contains the fluctuations. The field equations are then given by
\[
a_0 = \frac{e}{\omega_a^2} (\phi \dot{\phi} - \dot{\phi} \phi), \quad (3.11)
\]
\[
\dot{a}_0 = -e \phi \varphi + \frac{e}{k^2} \varphi R, \quad (3.12)
\]
\[
\ddot{\varphi} = -\omega^2 \varphi + \frac{2e^2 \phi}{\omega_a^2} (\phi \dot{\phi} - \dot{\phi} \phi) + \frac{e^2}{k^2} \lambda \phi R \varphi, \quad (3.13)
\]
with
\[
\omega^2 \varphi = k^2 + e^2 \phi^2 + \lambda (\phi^2 - v^2). \quad (3.14)
\]
By taking the fluctuation integral $R(t)$ into account, we obtain a mode function which is of order higher than one loop. In this case we are dealing with the back reaction of the quantum fluctuations $a_0$ and $\varphi$. Obviously, these higher loop terms contain a factor $e^2/k^2$ as in the Hamiltonian approach which lead to the IR-instabilities. By choosing $R(t) = 0$ we consider only one loop effects and $(3.13)$ is identical with $(3.7)$. We will discuss this case in detail at the end of this section. Using $(3.11)$ and $(3.12)$, we find for the zero mode the equation of motion in terms of the quantum fluctuations
\[
\ddot{\phi} + \lambda \phi (\phi^2 - v^2) + 3\lambda \phi \langle h^2 \rangle + 2e^2 \phi \langle a_\perp^2 \rangle + \left( \lambda - e^2 \right) \phi \varphi^2 - e^4 \phi \dot{\phi}^2 \left( \frac{\varphi^2}{\omega_a^2} \right)
\]
\[
+ e^2 \phi \left( \frac{2\omega_a^2 - e^2 \varphi^2}{\omega_a^4} \right)
\]
\[
- 2e^2 \phi^2 \left( \frac{k^2}{\omega_a^2} \varphi \dot{\varphi} \right) + e^2 R \left( \frac{\varphi^2}{k^2} \right) = 0. \quad (3.15)
\]
Again we see here the appearance of a higher loop contribution which is IR-divergent. We can also compute the energy density in the Coulomb gauge. It is given by
\[
\mathcal{E} = \frac{1}{2} \dot{\phi}^2 + \frac{\lambda}{4} (\phi^2 - v^2)^2 + \frac{1}{2} \left[ \langle \dot{h}^2 \rangle + \langle \omega_a^2 h^2 \rangle \right] + \left[ \langle \dot{a}_\perp^2 \rangle + \langle \omega_a^2 a_\perp^2 \rangle \right]
\]
\[
+ \frac{1}{2} \left[ \langle \dot{\varphi}^2 \rangle + \langle \omega_\varphi^2 \varphi^2 \rangle \right] - \frac{1}{2} \langle \omega_a^2 a_0^2 \rangle. \quad (3.16)
\]
The component $\langle a_0^2 \rangle$ has a negative sign because of the indefinite metric of the time component of the gauge field. By inserting (3.11) into the last term of the energy, $\mathcal{E}$ can be written similar to the equation of motion in the form

$$
\mathcal{E} = \frac{1}{2} \dot{\phi}^2 + \frac{\lambda}{4} (\phi^2 - v^2)^2 + \frac{1}{2} \left[ (\dot{h}^2) + (\omega_a^2 h^2) \right] + \left[ (\dot{a}_\perp^2) + (\omega_a^2 a_\perp^2) \right] \\
+ \left( \frac{k^2}{2\omega_a^2} \phi^2 \right) + \frac{1}{2} \left( \omega_a^2 - \frac{\phi^4}{\omega_a^2} \right) \phi^2 \\
+ e^2 \phi \dot{\phi} \langle \frac{\phi}{\omega_a^2} \dot{\phi} \rangle.
$$

(3.17)

With (3.10), (3.13) and the equations of motion for $h$ and $a_\perp$, it is straightforward to show that the time derivative of $\mathcal{E}$ vanishes. Now we have to decide whether we neglect $\mathcal{R}(t)$ or not. For a well-defined one-loop approximation which we have considered in the 't Hooft-Feynman gauge in [22] we have to take it to be zero. Numerically, this leads to some problems for the energy conservation. In order to show the energy conservation, we have to compute the time derivative of $\mathcal{E}$. In the energy density terms proportional $\dot{\phi}$ appear, and, after performing the time derivative, it is necessary to insert the equation of motion for the zero mode. Since we have taken $\mathcal{R}(t) = 0$, we have to insert $\ddot{\phi} = -\lambda \phi (\phi^2 - v^2)$. Analytically, this is no problem, but numerically this equation is only solved at the initial time. By solving the equation of motion for the zero mode numerically we automatically take the fluctuation integral into account. Therefore, we cannot expect exact energy conservation. On the other hand this is a good cross check for the quality of the one loop approximation. For small coupling constant $e^2$ the numerical energy conservation has to be acceptable.

IV. EQUIVALENCE OF THE DIFFERENT APPROACHES

At this point we can compare the gauge invariant approach and the Coulomb gauge. In the gauge invariant approach the Goldstone channel was described by a combination of two first order differential equations for the field itself (2.48) and its canonical momentum (2.49). Now we show that this approach also leads in the one-loop order to the same equations as in the Coulomb gauge. If we differentiate (2.48) with respect to $t$ and use the classical equation of motion $\ddot{\phi} = \Pi$, we find a second order differential equation of the form

$$
\dddot{U}_\varphi + \frac{e^2}{k^2} \left( \phi^2 + \phi \dot{\phi} \right) U_\varphi + \frac{e^2}{k^2} \phi \ddot{\phi} U_\varphi - \frac{2e^2 \phi \dot{\phi}^{\skew1{\hbox{}} \skew1{\hbox{\hbox{-}}}} U_{\pi \varphi}}{k^2} - \frac{\omega_a^2}{k^2} \dot{U}_{\pi \varphi} = 0.
$$

(4.1)

With the relations for $U_{\pi \varphi}$ and $\dot{U}_{\pi \varphi}$

$$
U_{\pi \varphi} = \frac{k^2}{\omega_a^2} \dot{U}_\varphi + \frac{e^2 \phi \dot{\phi}}{\omega_a^2} U_\varphi,
$$

(4.2)

$$
\dot{U}_{\pi \varphi} = \frac{e^2 \phi \dot{\phi}^{\skew1{\hbox{}} \skew1{\hbox{\hbox{-}}} } \ddot{U}_\varphi + \frac{e^4 \phi^2 \dot{\phi}^{\skew1{\hbox{}} \skew1{\hbox{\hbox{-}}} } U_\varphi - \omega_a^2 U_\varphi - \frac{e^2}{k^2} \phi \dot{\phi}^{\skew1{\hbox{}} \skew1{\hbox{\hbox{-}}} } U_\varphi}{\omega_a^2},
$$

(4.3)

and again with the classical equation of motion now in the form $\ddot{\phi} = -\lambda \phi (\phi^2 - v^2)$, it is straightforward to show

$$
\mathcal{M}_{\varphi \varphi} U_\varphi = 0,
$$

(4.4)
where $\mathcal{M}_{\varphi}$ is given by (3.8). By inserting the classical equation of motion without the fluctuation part, we suppress the higher loop terms and therefore get rid of the IR problem. Therefore, we have shown that if the classical equation of motion is fulfilled, the Goldstone mode in the Coulomb gauge and in the Hamiltonian approach has the same fluctuation operator. We also want to compare the equation of motion for the zero mode $\phi$ in the Coulomb gauge and in the Hamiltonian approach. In order to shorten the notation and make a comparison easier we also write here the fluctuation integrals as expectation values and do not worry about the normalization. By taking the time derivative of the zero mode equation (2.39), we find:

$$\ddot{\phi} = \dot{\Pi} + 2e^2 \Pi \frac{\langle \varphi \dot{\varphi} \rangle}{k^2} - e^2 \phi \left[ \langle \dot{\varphi} \dot{\pi} \rangle + \langle \frac{\varphi \dot{\pi}}{k^2} \rangle \right]. \quad (4.5)$$

Now we insert the differential equation for the conjugate momentum $\dot{\Pi}$ (2.40). Since we are only interested in the one loop order we can use the classical equation of motion $\dot{\phi} = \Pi$ without fluctuations and neglect terms of higher orders arising from the product of fluctuation integrals. Then the linearized field equation reads as a second order differential equation

$$\ddot{\phi} = -U' \left( \phi \right) - \phi \left[ 2e^2 \langle a_\perp^2 \rangle + 3\lambda \langle h^2 \rangle + \lambda \langle \varphi^2 \rangle + e^2 \langle \frac{\pi^2}{k^2} \rangle \right] - e^2 U'' \left( \phi \right) \langle \varphi^2 k^2 \rangle + 2e^2 \phi \left[ \langle \frac{\varphi \dot{\varphi}}{k^2} \rangle + \langle \varphi \dot{\pi} \rangle \right]. \quad (4.6)$$

By using (4.2) and (4.3) for the conjugate momentum of the fluctuation field $\varphi$, we get the same result as in (3.15) if we choose $\mathcal{R}(t) = 0$.

V. RENORMALIZATION

We are interested in the behavior of the zero mode under the influence of the quantum fluctuations and in the energy. Therefore, we need well defined, finite relations, and we have to renormalize the equation of motion and the energy density. With a perturbative expansion of the mode functions it is possible to extract the leading divergences and to introduce counter terms independent of the initial conditions. This method was developed for non-equilibrium systems in [26] and implemented for gauge theories in [22]. We have given some details of the perturbative expansion of the mode functions in appendix A. In order to make a comparison with the ’t Hooft-Feynman gauge possible, we also introduce some renormalization conditions. We choose as renormalization point the minimum of the effective potential, which is due to Nielsen’s theorem the same point for both gauges. The renormalization has to satisfy the condition $|\Phi| = v$ or in other words the one loop effective potential has to satisfy some basic properties of the classical potential. These include the same minimum and the same curvature at the minimum for both potentials. Therefore, we have to add some finite corrections to the Lagrangian. In general the effective potential is not very useful in the context of non-equilibrium dynamics as it is discussed for example in [15]. Especially, the imaginary part of the effective potential leads to instabilities and to misleading results for the investigations of the dynamics of the system. Nevertheless, the minimum of the effective potential point seems to be a natural point for the renormalization since we are using for the non-equilibrium calculations the same approximation scheme. For
our numerical investigations we consider only initial conditions near the minimum of the potential since this is the only stable point for the one loop approximation. If we would like to enter in the instable region of the potential then we have to take the back reaction also in the mode functions into account. This is beyond the scope of this paper. We are here interested in the comparison of different gauges and the influence of additional gauge modes in general. Therefore, it is useful to restrict ourselves to the stable region of the potential.

A. Equation of Motion

First of all, we have to write the equation of motion (3.15) in terms of the mode functions. Details for the quantization of the Goldstone mode \( \varphi \) are given in appendix A. In order to deal with a well defined one loop approximation we choose \( \mathcal{R} \) equals to zero. The equation of motion for the zero mode then reads

\[
\ddot{\varphi}(t) + \lambda \left( \varphi^2(t) - v^2 \right) \varphi(t) + \mathcal{F}(t) = 0 ,
\]

with

\[
\mathcal{F}(t) = 3\lambda \phi \int \frac{d^3k}{(2\pi)^3} \frac{|U_h|^2}{2\omega_{h0}} + 2e^2 \phi \int \frac{d^3k}{(2\pi)^3} \frac{|U_\perp|^2}{2\omega_{a0}}
\]

\[+ (\lambda - e^2) \phi \int \frac{d^3k}{(2\pi)^3} \frac{\omega_a^2}{2k^2\omega_{e\lambda 0}} |\dot{U}_\varphi|^2 \]

\[+ e^2 \phi \int \frac{d^3k}{(2\pi)^3} \frac{2k^2 + e^2 \varphi^2}{2k^2\omega_{e\lambda 0}\omega_a^2} |\ddot{\varphi}|^2 - e^4 \phi \int \frac{d^3k}{(2\pi)^3} \frac{2\omega_a^2 + k^2}{2\omega_a^2\omega_{e\lambda 0}} |\tilde{\varphi}|^2 \]

\[+ e^2 \phi \int \frac{d^3k}{(2\pi)^3} \frac{e^2 \varphi^2 \omega_a^2}{2k^2\omega_{e\lambda 0}\omega_a^4} \left( \tilde{U}_\varphi \dot{\tilde{U}}_\varphi + \tilde{U}_\varphi \dot{\tilde{U}}_\varphi \right) .
\]

The first four integrals contain divergent contributions, which we can extract by introducing the perturbative expansion for the mode functions as explained in appendix A. Dimensional regularization of the divergent parts leads to the following counter terms

\[
\delta m = 3\lambda m_h^2 I_1(m_h) - 2e^2 m_h^2 I_3(m_h) + (\lambda + e^2)m_h^2 I_1(m_h) ,
\]

\[
\delta \lambda = \left[ (\lambda - e^2)^2 + 9\lambda^2 \right] I_1(m_h) + e^4 I_3(m_h) ,
\]

\[
\delta Z = 2e^2 I_3(m_h) ,
\]

with

\[
I_1(m_h) = \frac{1}{16\pi^2} \left\{ 2 \quad + \ln \frac{4\pi \mu^2}{m_h^2} - \gamma + 1 \right\} , \]

\[
I_3(m_h) = \frac{1}{16\pi^2} \left\{ 2 \quad + \ln \frac{4\pi \mu^2}{m_h^2} - \gamma \right\} .
\]

After introducing the counter terms we finally get the following finite equation of motion

\[
(1 + \delta Z)\ddot{\varphi} - \frac{1}{2}(m_h^2 + \Delta m^2)\varphi + (\lambda + \Delta \lambda)\varphi^3 + \mathcal{F}_{\text{fin}} = 0 ,
\]

(5.8)
with finite corrections for the wave function, the mass, and the coupling constant

$$\Delta Z = -2e^2 C,$$

$$\Delta m^2 = \frac{1}{64\pi^2} \left[ 6\lambda m_h^2 + 6\epsilon^2 m_W^2 - 2\epsilon^2 m_h^2 + m_t^2 (e^2 - \lambda) \ln \frac{2\lambda}{e^2} + 3\lambda m_h^2 \ln \frac{m_h^2}{m_{h0}^2} + (\lambda - e^2) m_h^2 \ln \frac{m_h^2}{m_{\phi0}^2} - 4e^2 \phi_0^2 - 18\lambda^2 \phi_0^2 - 2(\lambda + e^2)e^2 \phi_0^2 \right],$$

$$\Delta \lambda = \frac{1}{64\pi^2} \left[ 4\lambda(\lambda + e^2) + (e^2 - \lambda)^2 \ln \frac{2\lambda}{e^2} - 9\lambda^2 \ln \frac{m_h^2}{m_{h0}^2} - 2e^4 \ln \frac{m_W^2}{m_{W0}^2} - (\lambda - e^2)^2 \ln \frac{m_\phi^2}{m_{\phi0}^2} \right],$$

$$C = \frac{1}{16\pi^2} \ln \frac{m_h^2}{m_{\phi0}^2}.$$

The terms independent of the initial conditions are due to the choice of the renormalization conditions. The finite fluctuation integral reads

$$F_{\text{fin}} = 3\lambda \phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{h0}} \left( 2\text{Re} f_h + |f_h|^2 \right) + 2e^2 \phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{a0}} \left( 2\text{Re} f_\perp + |f_\perp|^2 \right) + e^2 \phi \int \frac{d^3k}{(2\pi)^3} \frac{m^2_W}{2\omega_{\alpha0}^2 \omega_{\lambda0}} - \frac{e^2}{2} \phi \int \frac{d^3k}{(2\pi)^3} \frac{V_{e\lambda} m^2_{e\lambda0}}{2k^2 \omega^3_{e\lambda0}}$$

$$- e^2 \phi \int \frac{d^3k}{(2\pi)^3} \frac{m^2_W m^2_{e\alpha0}}{2k^2 \omega^3_{e\alpha0}} \left( \frac{\omega^2}{k^2} \left( 2\text{Re} f_\varphi + |f_\varphi|^2 \right) + \frac{V_{e\lambda}}{2\omega^2_{e\lambda0}} \right)$$

$$+ (\lambda - e^2) \phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\lambda0}} \left[ \frac{\omega^2}{k^2} \left( 2\text{Re} f_\varphi + |f_\varphi|^2 \right) + \frac{V_{e\lambda}}{2\omega^2_{e\lambda0}} \right]$$

$$+ e^2 \phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 \omega^2_{a\lambda0}} \left( \left( 2k^2 + e^2 \phi^2 \right) \left[ |f_\varphi|^2 \right] - \omega^2_{e\lambda0} \left( 2\text{Re} f_\varphi + |f_\varphi|^2 \right) - \omega^2_{e\lambda0} \right)$$

$$- e^4 \phi^2 \int \frac{d^3k}{(2\pi)^3} \frac{2\omega^2 + k^2}{2\omega^2_{a\lambda0}} |\tilde{U}_\varphi|^2$$

$$+ e^2 \phi^2 \int \frac{d^3k}{(2\pi)^3} \frac{\epsilon^2 \phi^2 \omega^2}{2k^2 \omega^2_{a\lambda0} \omega^2_{e\lambda0}} \left( \tilde{U}_\varphi \tilde{U}_\varphi^* + \tilde{U}_\varphi^* \tilde{U}_\varphi \right).$$

Therefore, we have a well defined finite equation of motion which we can investigate numerically.

**B. Energy Density**

In order to find the divergent parts of the energy density we have to introduce the mode functions for the different fields in Eq. (3.17) in the same way as for the equation of motion.
This leads to

\[
\mathcal{E} = \frac{1}{2} \dot{\phi}^2 + \frac{\lambda}{4} (\phi^2 - v^2)^2 + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{h0}} \left[ |U_k^2|^2 + \omega^2_n(t)|U_n^2|^2 \right] + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{a0}} \left[ |U_\perp|^2 + \omega^2_a(t)|U_\perp|^2 \right] + \int \frac{d^3 k}{(2\pi)^3} \frac{\omega^2_a}{4k^2\omega_{e\lambda}k^2}\omega^2_n |\tilde{U}_\varphi|^2 + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{e\lambda}^2} |\tilde{U}_\varphi|^2 
\]

\[+ e^4 \phi^2 \frac{\omega^2_a}{\omega_{e\lambda}} \int \frac{d^3 k}{(2\pi)^3} 2a_k^2 + k^2 |\tilde{U}_\varphi|^2 + e^2 \frac{\omega^2_a}{\omega_{e\lambda}} (\dot{U}_\varphi \ddot{U}_\varphi + \dddot{U}_\varphi \dot{U}_\varphi). \tag{5.14}\]

As in the equation of motion the first four integrals contain divergences. With the help of the truncated mode functions \( f_h, f_\perp, f_\varphi \) we find after a lengthy but straightforward calculation in addition to the mass and coupling constant counter term a cosmological constant counter term which arise from a quartic divergence of the form

\[\delta \Lambda = \frac{m^4}{128\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m^2} - \gamma + \frac{3}{2} \right\}. \tag{5.15}\]

We also get an additional finite term

\[\Delta \Lambda = -\frac{1}{256\pi^2} \left( 5m^4_h + 6m^4_W - 4m^2_m^2_W - m^4_h \ln \frac{2\lambda}{e^2} \right) - \frac{m^4_h}{128\pi^2} \ln \frac{m^2_h}{m^2_{h0}} + \frac{1}{128\pi^2} \left\{ \left[ 2e^4 + 9\lambda^2 + (e^2 + \lambda)^2 \right] \phi_0^4 + m^2_h(e^2 + 4\lambda)\phi_0^2 \right\}, \tag{5.16}\]

so that the complete finite energy density is given by

\[\mathcal{E}_\text{fin} = \frac{1}{2} (1 + \Delta Z) \dot{\phi}^2 + \frac{\lambda}{4} (\phi^2 - v^2)^2 + \frac{1}{4} \{(\lambda + \Delta \lambda) \phi^4 + \Delta \Lambda + \Delta \Lambda' \}
\]

\[
+ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{h0}} \left[ |\dot{f}_h|^2 + V_h \left( 2\text{Re}f_h + |f_h|^2 \right) + \frac{V^2_h}{16\omega^3_{h0}} \right] + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{a0}} \left[ |\dot{f}_\perp|^2 + V_a \left( 2\text{Re}f_\perp + |f_\perp|^2 \right) + \frac{V^2_a}{16\omega^3_{a0}} \right]
\]

\[
+ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{e\lambda}k^2} \left( \frac{m^2_W(t)m^2_\varphi(t)}{k^2} + \frac{m^2_Wm^2_\varphi}{\omega^2_{a0}} \right) (2\text{Re}f_\varphi + |f_\varphi|^2)
\]

\[
+ \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{e\lambda}} \left( |\dot{f}_\varphi|^2 - \frac{V^2_\varphi}{4\omega^3_{e\lambda}} \right) + \int \frac{d^3 k}{(2\pi)^3} \frac{V_{e\lambda}}{4\omega_{e\lambda}} \left( 2\text{Re}f_\varphi + |f_\varphi|^2 + \frac{V_{e\lambda}}{2\omega^2_{e\lambda0}} \right)
\]

\[-e^2 \phi^2 \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{e\lambda}k^2} (2\text{Re}f_\varphi + |f_\varphi|^2) \]

15
\begin{equation}
+e^4 \phi^2 \dot{\phi}^2 \int \frac{d^3 k}{(2\pi)^3} \frac{2\omega_a^2 + k^2}{4\omega_{e\lambda 0} \omega_a^4} |\tilde{U}_\varphi|^2 \\
+e^2 \phi \dot{\phi} \int \frac{d^3 k}{(2\pi)^3} \frac{2k^2 + e^2 \phi^2}{4k^2 \omega_{e\lambda 0} \omega_a^2} \left( \tilde{U}_\varphi \tilde{U}_\varphi^* + \tilde{U}_\varphi^* \tilde{U}_\varphi \right), \tag{5.17}
\end{equation}

which can now be treated numerically.

VI. NUMERICAL RESULTS

In order to investigate the influence of the gauge field sector on the zero mode in a system out of equilibrium, we have carried out some numerical examples. We are interested in two different aspects: first we investigate the influence of the different gauges. Therefore, we reinvestigate the 't Hooft-Feynman background gauge which we have discussed in detail in \[22\] and we compare the results with those we find in the Coulomb gauge. Secondly, we investigate the effect of the different degrees of freedom which arise in gauge theories. For the 't Hooft-Feynman gauge we summarize the results for the equation of motion, the mode functions, and the energy density which we have published in \[22\]. For more details, especially in view of the renormalization, the reader is referred to our paper \[22\].

A. The 't Hooft-Feynman Gauge

We give a short review on the main steps to derive the equation of motion and the energy in the 't Hooft-Feynman background gauge. In opposite to our previous work \[22\] we restrict ourselves here to the abelian Higgs model. This leads to some modifications in the degeneracy factors which we have introduced in \[22\]. The Lagrangian which describes the abelian Higgs model in the 't Hooft-Feynman gauge contains three parts

\[ \mathcal{L}_{\text{tot}} = \mathcal{L} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}. \tag{6.1} \]

\( \mathcal{L} \) is given again by (2.1). We split the Higgs field into a mean value and fluctuations

\[ \Phi(\vec{x}, t) = [\phi(t) + h(\vec{x}, t) + i\varphi(\vec{x}, t)] , \tag{6.2} \]

and consider for the gauge field only fluctuations

\[ A_\mu(\vec{x}, t) = a_\mu(\vec{x}, t) . \tag{6.3} \]

The gauge fixing term has the following form

\[ \mathcal{L}_{\text{GF}} = -\frac{1}{2} F^2 , \tag{6.4} \]

with

\[ F(a_\mu, \varphi) = \partial_\mu a^\mu + e(\phi + h)\varphi . \tag{6.5} \]

The corresponding Faddeev-Popov Lagrangian which is relevant for our calculations is

\[ \mathcal{L}_{\text{FP}} = \partial_\mu \eta^\dagger \partial_\eta - e^2 \phi^2 \eta^\dagger \eta . \tag{6.6} \]
After inserting the expansions for the fields in the total Lagrangian (6.1) and neglecting all terms higher than second order in the fluctuations we derive the following equation of motion for the zero mode

\[ \ddot{\phi} + \lambda \phi (\phi^2 - v^2) + 3\lambda \phi \langle h^2 \rangle + e^2 \phi \langle a_\perp^2 \rangle \\
+ (\lambda + e^2) \phi \langle \varphi^2 \rangle - e^2 \phi \langle a_0^2 \rangle - e \partial_t \langle a_0 \varphi \rangle = 0 . \]  

(6.7)

In this notation we have not taken care of the normalization and the renormalization or the different solutions for the mode functions of the coupled channel. We only want to give an overview of the equations and the connections of the fields and do not go into technical details. As already mentioned, the mode functions for the isoscalar Higgs field and the transversal gauge field are the same as in the Coulomb gauge (2.46), (2.47). For the coupled sector \( a_0 \varphi \), they are given by

\[ \{ -\partial_t^2 - \omega^2 (t) \phi \dot{\varphi} (t) \} \{ a_0 (t) \}
\{ \partial_t^2 + \omega_{\phi \lambda} (t)^2 \} \{ \varphi (t) \} = 0 . \]  

(6.8)

The operator has two interesting features which distinguishes it from the single modes, the indefinite metric of the time component of the gauge field, and the time derivative of the zero mode in the off diagonal elements which connect the two fields. We find analogous properties in the Coulomb gauge; the fluctuation part for \( a_0 \) contributes with a negative sign to the fluctuation integral (3.9), and we find time derivatives of the zero mode in the equation of motion for the zero mode (3.15) as well as in the mode function for \( \varphi \) (3.7).

In the Feynman gauge, the mode functions for the transversal gauge field, for the longitudinal gauge field, and for the ghost fields are the same. Two of the three gauge components are canceled by the ghost fields and only one degree of freedom is left in contrast to the Coulomb gauge where we have two transverse gauge components. The energy density can be derived by integration of the equation of motion for the zero mode or from the corresponding Hamiltonian of the system. It reads (see also [22])

\[ E = \frac{1}{2} \dot{\phi}^2 + \frac{\lambda}{4} (\phi^2 - v^2)^2 \\
+ \frac{1}{2} \left[ \langle \dot{h}^2 \rangle + \langle \omega_{\phi a}^2 h^2 \rangle \right] + \frac{1}{2} \left[ \langle \dot{a}_\perp^2 \rangle + \langle \omega_{\phi a}^2 a_\perp^2 \rangle \right] \\
+ \frac{1}{2} \left[ \langle \dot{\varphi}^2 \rangle + \langle \omega_{\phi \lambda}^2 \varphi^2 \rangle \right] - \frac{1}{2} \left[ \langle \dot{a}_0^2 \rangle + \langle \omega_{\phi a}^2 a_0^2 \rangle \right] . \]  

(6.9)

This expression looks very similar to the Coulomb energy (3.16) despite the fact that \( a_0 \) is dynamical and contributes with a derivate part.

**B. Results**

For our numerical calculations, we have chosen four different sets of parameters listed in Table I. The initial value for the zero mode \( \phi \) and the Higgs mass \( m_h \) are the same for all sets. They are chosen in such a way that the zero mode evolves in the right minimum of the potential. With this choice of initial conditions, the zero mode can not evolve into the complex part of the effective potential around zero. In this region, instabilities increase
dramatically and the one loop approximation breaks down as explained for the $\phi^4$ theory for example in [15, 29].

In order to give an impression of the potential we have plotted in Fig. 1 for the Coulomb gauge and Fig. 2 for the Feynman gauge different approximations for the potential. The dashed line shows the classical potential $V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2$, the circles include finite corrections due to the renormalization, and the pluses display the effective one loop potential. Since we have chosen the renormalization conditions in such a way, that the classical and the effective potential have the same minimum and the same curvature at the minimum, they are very close to each other. The solid line shows the zero mode part of the energy versus $\phi$. The field begins to roll down the potential but the energy is not high enough for the field to reach the maximum at zero. Therefore, it starts to oscillate in the minimum. The prediction of these two plots in the context of non-equilibrium dynamics is not clear, because the effective potential is an equilibrium quantity, since the expectation value of the scalar field, that serves as order parameter, is space time independent. Nevertheless, they are instructive to get an idea of the potential by which the zero mode is influenced.

For our numerical considerations we have only varied the coupling constants $\lambda$ and $e$ and therefore the masses of the different fields. We have also given the initial masses of the three different fields in Table I.

For the first parameter set, we have chosen the same coupling constant for the Higgs field and the gauge field. The initial masses for the fields are all small but not zero. Since we have taken the initial value for the zero mode to be small, the effect of the quantum fluctuations is negligible and therefore, $\phi$ oscillates with constant amplitude. The behavior of the zero mode, which is displayed in Fig. 3 agrees in both gauges excellent.

In the second parameter set we have chosen a dominant gauge coupling, therefore the initial mass of the Higgs field becomes very small compared to $m^2_{h0}$ and $m^2_{e\lambda0}$. In this case, the zero mode is moderately damped, as shown in Fig. 4, the behavior is comparable to a purely scalar theory. The two gauges agree again very well, at late time the zero modes are slightly dephased.

The situation changes drastically, if we choose a smaller gauge coupling and therefore a nearly vanishing Goldstone mass. In Fig. 5 and Fig. 6 we show two examples for this constellation.

In the first example we have chosen a nearly vanishing gauge coupling with $e = 0.001$. In this case an instability occurs and the system breaks down. We have analyzed a similar problem occurring in the $\phi^4$ theory in [29]. There we have found that the spontaneous symmetry breaking leads to instabilities if the mass squared term of the fluctuation field becomes negative due to the decrease of the zero mode. In addition to $m^2_h(t) = m^2_h + 3\lambda(\phi^2 - v^2)$, the mass term, which is also relevant for the spontaneously broken $\phi^4$ theory, we now have to secure, that $m^2_{e\lambda}(t) = e^2\phi^2 + \lambda(\phi^2 - v^2)$ does not become negative when $\phi(t)$ decreases. This leads to the two conditions

$$m^2_h(t) > 0 \Rightarrow \phi(t) > \sqrt{\frac{m^2_h}{6\lambda}}, \quad (6.10)$$

$$m^2_{e\lambda}(t) > 0 \Rightarrow \phi(t) > \sqrt{\frac{m^2_h}{2(e^2 + \lambda)}}. \quad (6.11)$$

Therefore, the critical value for the field $\phi(t)$ for parameter set 3 is around $\phi(t) > 0.5$ (exactly 0.49999975) in order to find a stable development for the system. We have displayed in the
frame the development of the zero mode for a shorter time period. The solid line at 0.5 shows the critical value for $\phi$, for which $m_{c\lambda}(t)$ becomes negative. One sees that the average value of the zero mode is slightly smaller than this critical value. Therefore, the system breaks down.

In the second example we have chosen the gauge coupling to be $e = 0.1$, therefore slightly larger but again small. In this case the zero mode is strongly damped and settles down to the minimum. An appropriate size for the gauge coupling prevents the system to destabilize. We can influence $m_{c\lambda}(t)$ either by changing the Higgs coupling or the gauge coupling. As this example in Fig. 6 shows it is possible to find a proper relation between $\lambda$ and $e$ in order to allow the decay of $h$ into $\phi$ and gauge field and a stable development of the system in the minimum of the potential. We have also plotted the critical value for the zero mode in this case where it is obviously much smaller than the final value of the zero mode.

In order to understand this strong damping of the field more precisely, it is instructive to investigate the Feynman graphs for the problem in detail. This is especially easy in the 't Hooft-Feynman gauge. We have listed the the occurring graphs in appendix B. The relevant graph which leads to the strong damping is the following

\[\begin{array}{c}
h \quad \phi \\
\a
\end{array}\]

It allows the decay of the Higgs field into $a$ and $\phi$ for a proper relation between the masses. In the Coulomb gauge this mechanism is not as obvious as in the 't Hooft-Feynman gauge since we have eliminated the $a_0$-mode. Also the structure of the fluctuation integral for the $\phi$-mode is rather complicated and allows therefore not such a simple analysis.

In order to check our numerics we have plotted the energy density for the Coulomb gauge in Fig. 7 and the 't Hooft-Feynman gauge in Fig. 8 for the fourth parameter set. The upper line displays the fluctuation energy which increases and the lower line the zero mode part of the energy which decreases. The solid line shows the total energy. For convenience we have added in both cases a constant to the fluctuation part of the energy. Otherwise the curves are only straight lines due to their distance. The energy conservation is excellent in both cases.

Summarizing the results, we have found that the damping effect is strongest for a nearly massless Goldstone field and a massive isoscalar Higgs field. In this case the isoscalar Higgs field has the possibility to decay into the other fields. We have found an analogous behavior in the $\phi^4$ theory in the large $N$ limit. There, the damping of zero mode was due to the massless Goldstone bosons.

We have also found that the behavior of the zero mode and the energy is the same for the different gauges and that the Goldstone channel plays an important role.

**VII. CONCLUSIONS AND OUTLOOK**

We have analyzed in this paper the abelian Higgs model out of equilibrium in detail. Therefore, we have investigated a gauge invariant approach which was developed by Boyanovsky et al. [1]. We have extended their calculations to a complete set of equations, which describes the evolution of the zero mode under the influence of gauge and Higgs fluctuations. We found some problems in the IR region induced by the inclusion of terms higher then one
loop order. Since these terms were included in an uncontrollable manner, we have computed a linearized form of the equations. They are equivalent to the Coulomb gauge fixed theory in the one loop approximation. We have performed the renormalization of the system in order to get finite well defined equations which we have investigated numerically. We were also interested in the behavior of the system under the influence of different gauges. Based on [22] we have reinvestigated the ’t Hooft-Feynman gauge. The main differences between the two gauges is that we have eliminated in the Coulomb gauge all unphysical degrees of freedom while in the ’t Hooft-Feynman gauge we add extra degrees of freedoms, the ghosts fields, in order to cancel the unphysical degrees of freedom. We found a very good agreement in both gauges for the development of the zero mode and the energy. The influence of the various degrees of freedoms which arise in gauge theories have led to new and interesting results in the behavior of the zero mode. Due to the possibility for the Higgs field to decay into other fields, the zero mode was efficiently damped. This behavior was until now not observed for systems treated in the one loop approximation beside for the case of fermion decay [27]. As in the one loop approximation in the $\phi^4$ theory the evolution of the zero mode in the unstable region was impossible.

For future studies there are two different interesting aspects. The first one is more technical. In order to enter the unstable regime of the effective potential it is necessary to go beyond the one loop approximation. Therefore, the implementation of Hartree-like approximation would be very interesting. As we have shown here one has to include higher order effects in a careful way in order to avoid IR-problems. In addition one has to choose very carefully an expansion parameter in order to secure gauge invariance. Since the usual Hartree-approximation is just an ansatz and an inconsistent summation and not an expansion in the coupling this will probably lead to problems. Also the investigation of the non-abelian gauge would be very interesting. Beside these technical considerations, the implementation of our results for the gauge fields in a cosmological context would be very interesting. Together with our studies on fermionic systems out of equilibrium we now have built the foundation to examine more realistic models to describe the physics of the early universe. The recent success in detecting neutrino masses has revived the idea of grand unification. An implementation of our method in Grand Unified Theories could lead to new and interesting results. Also the implementation of Friedman-Robertson-Walker cosmology in the model we have considered is interesting in order to get a more suitable model for describing the inflationary scenario.

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APPENDIX A: PERTURBATIVE EXPANSION

In this appendix we give some details of the perturbative expansion of the mode functions which we have used for the renormalization procedure. We have developed this formalism for non-equilibrium systems in [26] and extended it for gauge theories in [22]. For more details the reader is referred to these works. First of all we have to quantize the Goldstone
field $\varphi$ and introduce the mode representation in the same way as the transversal gauge
mode and the Higgs mode in Eqs. (2.46) and (2.47). Therefore, we have to investigate the
following equation for the Goldstone field (3.7)

$$\ddot{\varphi} + \omega^2_e(t) \varphi - \frac{2e^2 \dot{\phi}}{\omega^2_a} \left[ \phi \dot{\varphi} - \dot{\phi} \varphi \right] = 0 .$$

(A1)

In order to quantize the field we have to find the conjugate momentum. We can read it of
from the Lagrangian (3.1):

$$\Pi_\varphi = \dot{\varphi} - ea_0 \varphi = \dot{\varphi} \frac{k^2}{\omega^2_a} - \frac{e^2 \phi \dot{\phi}}{\omega^2_a} \varphi .$$

(A2)

The commutation relation for the field and its momentum is given by

$$[\varphi, \Pi_\varphi] = i \delta(\vec{x} - \vec{x}') .$$

(A3)

By computation of the commutator for the field and its time derivative, we find that it is
multiplied by a factor

$$[\varphi, \dot{\varphi}] \frac{k^2}{\omega^2_a} = i \delta(\vec{x} - \vec{x}') .$$

(A4)

Now we can expand the field in terms of the mode functions and the corresponding annihi-
lation and creation operators

$$\varphi = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left( a_k U_\varphi(t) e^{i\vec{k} \cdot \vec{x}} + a_k^\dagger U_\varphi^*(t) e^{-i\vec{k} \cdot \vec{x}} \right) .$$

(A5)

In order to satisfy the commutator for $\varphi$ and $\dot{\varphi}$, the commutator for $a_k$ and $a_k^\dagger$ differs from
its common form

$$[a_k, a_k^\dagger] = 2i(2\pi)^3 \delta^3(\vec{k} - \vec{k}') \frac{1}{k^2} .$$

(A6)

The expectation value for the field is then given by

$$\langle \varphi \dot{\varphi} \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{|U_\varphi|^2}{2\omega k^2} .$$

(A7)

The mode function satisfies the differential equation

$$\ddot{U}_\varphi + \omega^2_e U_\varphi - \frac{2e^2 \dot{\phi}}{\omega^2_a} \left( \phi \dot{U}_\varphi - \dot{\phi} U_\varphi \right) = 0 .$$

(A8)

The frequency in the expectation value has to be fixed by the determination of the Wronskian
which belongs to the mode equation. Since the form of the differential equation is not
standard due to the first derivative of $U_\varphi$, the following transformation is efficient

$$U_\varphi = \omega_a \tilde{U}_\varphi .$$

(A9)

The $\dot{U}_\varphi$-terms are canceled and we find a new mode equation of the form

$$\ddot{\tilde{U}}_\varphi + \left( \omega^2_e + \frac{3e^2 \dot{\phi}^2 k^2}{\omega^4_a} + \frac{e^2 \dot{\phi} \ddot{\phi}}{\omega^2_a} \right) \tilde{U}_\varphi = 0 .$$

(A10)
Now we redefine the frequency $\omega^2_{e\lambda}$ in order to find a suitable Wronskian. We choose

$$\tilde{\omega}^2_{e\lambda} = \omega^2_{e\lambda} + \frac{3e^2\phi^2k^2}{\omega_0^4} + \frac{e^2\phi\ddot{\phi}}{\omega_0^2},$$

(A11)

and get the new differential equation for the mode function

$$\ddot{U}_\varphi + \tilde{\omega}^2_{e\lambda}\tilde{U}_\varphi = 0.$$  \hspace{1cm} (A12)

The Wronskian then has the well known form

$$\dot{\tilde{U}}_\varphi \tilde{U}_\varphi^* - \ddot{\tilde{U}}_\varphi \dot{\tilde{U}}_\varphi^* = C.$$  \hspace{1cm} (A13)

Since $\tilde{U}_\varphi$ behaves like $e^{-i\tilde{\omega}_{e\lambda 0}t}$, we can fix the constant to be $C = -2i\tilde{\omega}_{e\lambda 0}$. For the initial time, $\dot{\phi}$ and the fluctuation integral vanish, so that $\tilde{\omega}^2_{e\lambda 0}$ simplifies to

$$\tilde{\omega}^2_{e\lambda 0} = \lambda (\phi^2_0 - v^2) - \frac{e^2\lambda}{\omega^2_0} \phi^2_0 (\phi^2_0 - v^2).$$  \hspace{1cm} (A14)

For high momenta, which are important for the UV-divergences, the last term is negligible.

Now all mode equations have the similar structure

$$\left[\frac{d^2}{dt^2} + \omega_{j0}^2\right] U_j(t) = -V_j(t)U_j(t),$$  \hspace{1cm} (A15)

$$\left[\frac{d^2}{dt^2} + \tilde{\omega}_{e\lambda 0}^2\right] \tilde{U}_\varphi(t) = -\tilde{V}_{e\lambda}(t)\tilde{U}_\varphi(t),$$  \hspace{1cm} (A16)

with $j = a, h$. Here we have introduced the potentials

$$V_a(t) = e^2[\phi^2(t) - \phi^2_0],$$  \hspace{1cm} (A17)

$$V_h(t) = 3\lambda \left[\phi^2(t) - \phi^2_0\right],$$  \hspace{1cm} (A18)

$$\tilde{V}_{e\lambda}(t) = V_{e\lambda}(t) + \frac{3e^2\phi(t)^2k^2}{\omega_0^4(t)} + \frac{e^2\phi(t)\ddot{\phi}(t)}{\omega_0^2(t)} + \frac{e^2\lambda\phi_0^2(\phi^2_0 - v^2)}{\omega_0^2},$$  \hspace{1cm} (A19)

$$V_{e\lambda} = (e^2 + \lambda) \left(\phi^2 - \phi^2_0\right).$$  \hspace{1cm} (A20)

We separate the mode functions into a trivial part corresponding to the case that the potential vanishes and a function $f_j(t)$ which represents the reaction to the potential by making the ansatz

$$U_j(t) = e^{-i\omega_{j0}t}[1 + f_j(t)],$$  \hspace{1cm} (A21)

$$\tilde{U}_\varphi(t) = e^{-i\tilde{\omega}_{e\lambda 0}t}[1 + f_\varphi(t)].$$  \hspace{1cm} (A22)

Then the new mode functions satisfy the differential equation

$$\ddot{f}_j(t) - 2i\omega_{j0}\dot{f}_j(t) = -V_j(t)[1 + f_j(t)],$$  \hspace{1cm} (A23)

respectively for $f_\varphi$ with the initial conditions $f_j(0) = \dot{f}_j(0) = 0$. Expanding now $f_j(t)$ with respect to orders in the potential by writing

$$f_j(t) = f^{(1)}_j + f^{(2)}_j + f^{(3)}_j + \cdots,$$  \hspace{1cm} (A24)
we can extract the leading behavior for $f_j(t)$:

$$f_j^{(1)}(t) = -\frac{i}{2\omega_j} \int_0^t dt' V_j(t') - \frac{V_j(t)}{4\omega_j^2} + \frac{1}{4\omega_j^2} \int_0^t dt' e^{2\omega_j \Delta t} \dot{V}_j(t') + O(\omega_j^{-3}),$$

$$f_j^{(2)}(t) = -\frac{1}{4\omega_j^2} \int_0^t dt' \int_0^{t'} dt'' V_j(t') V_j(t'') + O(\omega_j^{-3}),$$

with $\Delta t = t - t'$. For $f_\phi$ the frequency and the potential have to be replaced by $\tilde{\omega}_{e\lambda}$ and $\tilde{V}_{e\lambda}$.

Inserting the leading behavior of the new mode functions into the fluctuation integral (5.13) allows us to extract the following divergences

$$F_{\text{div}} = 3\lambda \phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{h0}} \left( 1 - \frac{V_h}{2\omega_{h0}} \right) + 2e^2 \phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{a0}} \left( 1 - \frac{V_a}{2\omega_{a0}} \right) + \lambda \phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\tilde{\omega}_{e\lambda0}} \left( 1 + \frac{e^2 \phi^2}{k^2} - \frac{V_{e\lambda}}{2\tilde{\omega}_{e\lambda0}} \right) + e^2 \phi \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\tilde{\omega}_{e\lambda0}} \left( 1 + \frac{e^2 \phi^2}{k^2} - \frac{V_{e\lambda}}{2\tilde{\omega}_{e\lambda0}} \right).$$

Dimensional regularization leads to the counter terms given in Eqs. (5.3)-(5.5) and finite corrections.

In a similar way we have to treat the energy density (5.14). Here we find with the help of the truncated mode functions $f$

$$E_{\text{div}} = \int \frac{d^3k}{(2\pi)^3} \left( \frac{\omega_{h0}}{2} + \frac{V_h}{4\omega_{h0}} - \frac{V_h^2}{16\omega_{h0}^3} \right) + \int \frac{d^3k}{(2\pi)^3} \left( \frac{\omega_{a0}}{2} + \frac{V_a}{2\omega_{a0}} - \frac{V_a^2}{8\omega_{a0}^3} \right) - e^2 \phi^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{4k^2\tilde{\omega}_{e\lambda0}} + \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\tilde{\omega}_{e\lambda0}}{2} + \frac{V_{e\lambda}}{4\tilde{\omega}_{e\lambda0}} + \frac{m_e^2 m_{\psi}^2}{4k^2\tilde{\omega}_{e\lambda0}} - \frac{V_{e\lambda}^2}{16\tilde{\omega}_{e\lambda0}^3} + \frac{m_{e0}^2 m_{W0}^2}{4\tilde{\omega}_{e\lambda0}} \right]$$

which leads again to $\delta m, \delta \lambda,$ and $\delta Z$ and in addition to $\delta \Lambda$ given in Eq. (5.13) and also to an extra finite contribution.

**APPENDIX B: THE LEADING FEYNMAN DIAGRAMS**

In order to understand the behavior of the system it is very instructive to analyse the leading Feynman diagrams in detail. This is especially easy in the ’t Hooft-Feynman gauge, since we can read of the Feynman rules directly from the gauge fixed Lagrangian, which is given for example in [22]. We find four different propagators which are
1. the gauge boson propagator:

\[ i \Delta^{ab}_{\mu \nu} = - \frac{i g_{\mu \nu} \delta^{ab}}{k^2 - m_W^2 + i \epsilon}, \quad \text{(B1)} \]

with \( m_W^2 = e^2 v^2 \).

2. the propagator for the isoscalar Higgs field:

\[ i \Delta_h = i \frac{1}{k^2 - m_h^2 + i \epsilon}, \quad \text{(B2)} \]

with \( m_h^2 = 2 \lambda v^2 \).

3. the propagator for the Goldstone field:

\[ i \Delta^{ab} \phi = i \frac{\delta^{ab}}{k^2 - m_W^2 + i \epsilon}, \quad \text{(B3)} \]

4. the propagator for the ghost field:

\[ i \Delta^{ab} \eta = i \frac{\delta^{ab}}{k^2 - m_W^2 + i \epsilon}, \quad \text{(B4)} \]

and ten vertices which are relevant if we restrict ourselves to second order in the fluctuations

V1)  \hspace{2cm} V2)

\[ i \Gamma = - i \frac{3}{2} \lambda [\phi^2(t) - v^2] \hspace{2cm} i \Gamma = - i \left( \frac{\lambda}{2} + \frac{\epsilon}{2} \right) \delta^{ab} [\phi^2(t) - v^2] \]

V3)  \hspace{2cm} V4)

\[ i \Gamma = i \epsilon \frac{3}{2} \delta^{ab} g_{\mu \nu} [\phi^2(t) - v^2] \hspace{2cm} i \Gamma = - i \epsilon^2 \delta^{ab} [\phi^2(t) - v^2] \]
These lead to the following Feynman graphs up to second order in perturbation theory
The plus and minus signs at the graphs indicates the use of propagators in the CTP-formalism. Especially interesting is the last graph; it leads to the possibility of the decay of $\mathcal{h}$ into $a_0$ and $\varphi$ which causes the strong damping of the zero mode for parameter set 4 displayed in Fig. 6. The various degrees of freedom in gauge fields lead to new and interesting results in non-equilibrium quantum field theory. A detailed analysis of these graphs within the CTP-formalism is given in [24].

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$$e = \frac{g_2}{m_2^2} \phi_0, m_{\text{h0}}^2, m_{\text{f0}}^2, m_{\text{c0}}^2$$

| Parameter set 1 | $\lambda$ | $e$ | $m_2^2$ | $\phi_0$ | $m_{\text{h0}}^2$ | $m_{\text{f0}}^2$ | $m_{\text{c0}}^2$ |
|----------------|---------|-----|---------|---------|----------------|----------------|------------|
| Parameter set 1 | 1       | 1   | 0.5     | 0.51    | 0.26           | 0.53           | 0.27       |
| Parameter set 2 | 0.33    | 1.3 | 0.5     | 0.51    | 0.44           | 0.01           | 0.28       |
| Parameter set 3 | 1       | 0.001 | 0.5 | 0.51 | 2.6 $\times 10^{-7}$ | 0.53 | 0.01 |
| Parameter set 4 | 1       | 0.1  | 0.5     | 0.51    | 2.6 $\times 10^{-3}$ | 0.53 | 0.01 |

**TABLE I: Parameter sets**
FIG. 1: Potential vs. \( \phi \) in the Coulomb gauge for parameter set 4, solid line: classical energy vs. \( \phi \), dashed line: classical potential, circles: classical potential with finite corrections, pluses: effective potential.

FIG. 2: Potential vs. \( \phi \) in the Feynman gauge for parameter set 4, solid line: classical energy vs. \( \phi \), dashed line: classical potential, circles: classical potential with finite corrections, pluses: effective potential.
FIG. 3: Zero mode vs. \( t \) for parameter set 1, dashed line: Feynman gauge, dotted line: Coulomb gauge.

FIG. 4: Zero mode vs. \( t \) for parameter set 2, dashed line: Feynman gauge, dotted line: Coulomb gauge.
FIG. 5: Zero mode vs. $t$ for parameter set 3, dashed line: Feynman gauge, dotted line: Coulomb gauge; frame: zoom onto a shorter time scale, solid line: critical value for $\phi(t)$.

FIG. 6: Zero mode vs. $t$ for parameter set 4, dashed line: Feynman gauge, dotted line: Coulomb gauge, solid line: critical value for $\phi(t)$. 
FIG. 7: Energy vs. $t$ in the Coulomb gauge for parameter set 4, solid line: total energy, dashed line: classical energy, dotted line: fluctuation energy.

FIG. 8: Energy vs. $t$ in the Feynman gauge for parameter set 4, solid line: total energy, dashed line: classical energy, dotted line: fluctuation energy.