A NATURAL CONNECTION ON SOME CLASSES OF ALMOST CONTACT MANIFOLDS WITH B-METRIC

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Abstract
Almost contact manifolds with B-metric are considered. A special linear connection is introduced, which preserves the almost contact B-metric structure on these manifolds. This connection is investigated on some classes of the considered manifolds.

Key words: almost contact manifold, B-metric, indefinite metric, natural connection, parallel structure.

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1. Introduction
In this work we consider an almost contact B-metric manifold denoted by \((M, \varphi, \xi, \eta, g)\), i.e. a \((2n+1)\)-dimensional differentiable manifold \(M\) equipped with an almost contact structure \((\varphi, \xi, \eta)\) and a B-metric \(g\). The B-metric \(g\) is a pseudo-Riemannian metric of signature \((n, n+1)\) with the opposite compatibility of \(g\) and \((\varphi, \xi, \eta)\) by comparison with the compatibility for the known almost contact metric structure. Moreover, the B-metric is an odd-dimensional analogue of the Norden (or anti-Hermitian) metric on almost complex manifolds.

Recently, manifolds with neutral metrics and various tensor structures have been an object of interest in mathematical physics.

The geometry of almost contact B-metric manifolds is the geometry of the structures \(g\) and \((\varphi, \xi, \eta)\). The linear connections, with respect to which \(g\) and \((\varphi, \xi, \eta)\) are parallel, play an important role in this geometry. The structures \(g\) and \((\varphi, \xi, \eta)\) are parallel with respect to the Levi-Civita connection \(\nabla\) of \(g\) if and only if \((M, \varphi, \xi, \eta, g)\) belongs to the class \(\mathcal{F}_0 : \nabla \varphi = 0\). Therefore, outside of the class \(\mathcal{F}_0\), the Levi-Civita connection \(\nabla\) is no longer a connection with respect to which \(g\) and \((\varphi, \xi, \eta)\) are parallel. In the general case, on \((M, \varphi, \xi, \eta, g)\) there exist a countless number of linear connections with respect to which these structures are parallel. They are the so-called natural connections.

In this paper we introduce a natural connection on \((M, \varphi, \xi, \eta, g)\) which we call a \(\varphi\)B-connection. It is an odd-dimensional analogue of the known B-connection on almost complex manifolds with Norden metric introduced in \([1]\).

The \(\varphi\)B-connection is studied in \([2]\), \([3]\) and \([4]\) on the classes \(\mathcal{F}_1\), \(\mathcal{F}_4\), \(\mathcal{F}_5\), \(\mathcal{F}_{11}\). These are the classes, where \(\nabla \varphi\) is expressed explicitly by the structures.
In this paper we consider the $\varphi$B-connection on other classes of almost contact B-metric manifolds.

The paper is organized as follows. In Sec. 2 we furnish some necessary facts about the considered manifolds. In Sec. 3 we define the $\varphi$B-connection. On some classes of almost contact B-metric manifolds we characterize the torsion tensor and the curvature tensor of this connection. In Sec. 4, on a 5-dimensional Lie group considered as an almost contact B-metric manifold in a basic class, we establish that the $\varphi$B-connection is flat and it has a parallel torsion.

2. Almost contact manifolds with B-metric

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact manifold with B-metric or an almost contact B-metric manifold, i.e. $M$ is a $(2n+1)$-dimensional differentiable manifold with an almost contact structure $(\varphi, \xi, \eta)$ consisting of an endomorphism $\varphi$ of the tangent bundle, a vector field $\xi$, its dual 1-form $\eta$ as well as $M$ is equipped with a pseudo-Riemannian metric $g$ of signature $(n, n+1)$, such that the following algebraic relations are satisfied

$$\varphi \xi = 0, \quad \varphi^2 = -Id + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1,$$

$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y)$$

for arbitrary $x, y$ of the algebra $\mathfrak{X}(M)$ on the smooth vector fields on $M$. The structural group of $(M, \varphi, \xi, \eta, g)$ is $G \times I$, where $I$ is the identity on $\text{span}(\xi)$ and $G = GL(n; \mathbb{C}) \cap O(n, n)$.

Further, $x, y, z, w$ will stand for arbitrary elements of $\mathfrak{X}(M)$ or vectors in the tangent space $T_pM$, $p \in M$.

In [5], a classification of the almost contact manifolds with B-metric is made with respect to the $(0,3)$-tensor $F(x, y, z) = g((\nabla_x \varphi) y, z)$, where $\nabla$ is the Levi-Civita connection of $g$. The tensor $F$ has the properties

$$F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi).$$

The following relations are valid (6):

$$F(x, \varphi y, \xi) = (\nabla_x \eta) y = g(\nabla_x \xi, y), \quad \eta(\nabla_x \xi) = 0.$$

The basic 1-forms associated with $F$ are:

$$\theta(z) = g^{ij} F(e_i, e_j, z), \quad \theta^*(z) = g^{ij} F(e_i, \varphi e_j, z), \quad \omega(z) = F(\xi, \xi, z).$$

Furthermore, $g^{ij}$ are the components of the inverse matrix of $g$ with respect to a basis $\{e_i, \xi\}$ of $T_pM$.

The basic classes in the classification from [5] are $F_1, F_2, \ldots, F_{11}$. Their intersection is the class $F_0$ determined by the condition $F = 0$. It is clear that $F_0$ is the class of almost contact B-metric manifolds with $\nabla$-parallel basic structures, i.e. $\nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = 0$.

In [6], it is proved that the class $\mathcal{U} = F_4 \oplus F_5 \oplus F_6 \oplus F_7 \oplus F_8 \oplus F_9$ is defined by the conditions

$$F(x, y, z) = \eta(y)F(x, z, \xi) + \eta(z)F(x, y, \xi), \quad F(\xi, y, z) = 0.$$
Further, we consider the subclasses \( U_1, U_2 \) and \( U_3 \) of \( U \), defined as follows:

a) The subclass \( U_1 = F_4 \oplus F_5 \oplus F_6 \oplus F_9 \subset U \) is determined by (2.2) and the condition \( d\eta = 0 \);

b) The subclass \( U_2 = F_4 \oplus F_5 \oplus F_6 \oplus F_7 \subset U \) is determined by (2.2) and the condition

\[
F(x, y, \xi) = -F(\varphi x, \varphi y, \xi);
\]

c) The subclass \( U_3 = F_4 \oplus F_5 \oplus F_6 \subset U \) is determined by (2.2) and the conditions

\[
F(x, y, \xi) = F(y, x, \xi) = -F(\varphi x, \varphi y, \xi).
\]

The classes \( F_4, F_5 \) and \( F_6 \) are determined in \( U_1 \) by the conditions \( \theta^* = 0, \theta = 0 \) and \( \theta = \theta^* = 0 \), respectively.

3. The \( \varphi B \)-connection

**Definition 3.1** (2). A linear connection \( \nabla' \) is called a natural connection on \((M, \varphi, \xi, \eta, g)\) if the almost contact structure \((\varphi, \xi, \eta)\) and the B-metric \( g \) are parallel with respect to \( \nabla' \), i.e. \( \nabla' \varphi = \nabla' \xi = \nabla' \eta = \nabla' g = 0 \).

**Proposition 3.1** (7). A linear connection \( \nabla' \) is natural on \((M, \varphi, \xi, \eta, g)\) if and only if \( \nabla' \varphi = \nabla' g = 0 \).

A natural connection exists on any almost contact manifold with B-metric and coincides with the Levi-Civita connection only on a \( F_0 \)-manifold.

If \( \nabla \) is the Levi-Civita connection, generated by \( g \), then we denote

\[
\nabla'_x y = \nabla_x y + Q(x, y).
\]

Furthermore, we use the denotation \( Q(x, y, z) = g(Q(x, y), z) \).

**Proposition 3.2** (8). The linear connection \( \nabla' \), determined by (3.1), is a natural connection on a manifold \((M, \varphi, \xi, \eta, g)\) if and only if

\[
Q(x, y, \varphi z) - Q(x, \varphi y, z) = F(x, y, z),
\]

\[
Q(x, y, z) = -Q(x, z, y).
\]

As an odd-dimensional analogue of the B-connection on almost complex manifolds with Norden metric, introduced in [1], we give the following

**Definition 3.2.** A linear connection \( \nabla' \), determined by

\[
\nabla'_x y = \nabla_x y + \frac{1}{2}\{(\nabla_x \varphi) \varphi y + (\nabla_x \eta) y \cdot \xi\} - \eta(y)\nabla_x \xi
\]
on an almost contact manifold with B-metric \((M, \varphi, \xi, \eta, g)\), is called a \( \varphi B \)-connection.

According to **Proposition 3.1**, we establish that the \( \varphi B \)-connection is a natural connection.

Let us remark that the connection \( \nabla' \) determined by (3.4) is studied in [2], [3] and [4] on manifolds from the classes \( F_1, F_4, F_5, F_11 \). In this paper
we consider the $\varphi$-B-connection on manifolds from $U$ and its subclasses $U_1$, $U_2$, $U_3$.

Further $\nabla'$ will stand for the $\varphi$-B-connection.

3.1. **Torsion properties of the $\varphi$-B-connection.** Let $T$ be the torsion of $\nabla'$, i.e. $T(x, y) = \nabla'_x y - \nabla'_y x - [x, y]$ and the corresponding $(0, 3)$-tensor $t$ determined by $T(x, y, z) = g(T(x, y), z)$.

The connection $\nabla'$ on $(M, \varphi, \xi, \eta, g)$ has a torsion tensor and the torsion forms as follows:

$$T(x, y, z) = -\frac{1}{2} \{ F(x, \varphi y, \varphi^2 z) - F(y, \varphi x, \varphi^2 z) \} + \eta(x) F(y, \varphi z, \xi) - \eta(y) F(x, \varphi z, \xi) + \eta(z) \{ F(x, \varphi y, \xi) - F(y, \varphi x, \xi) \},$$

$$t(x) = \frac{1}{2} \{ \theta^*(x) + \theta^*(\xi) \eta(x) \}, \quad t^*(x) = -\frac{1}{2} \{ \theta(x) + \theta(\xi) \eta(x) \},$$

$$\dot{t}(x) = -\omega(\varphi x),$$

where the torsion forms are defined by

$$t(x) = g^{ij} T(x, e_i, e_j), \quad t^*(x) = g^{ij} T(x, e_i, \varphi e_j), \quad \dot{t}(x) = T(x, \xi, \xi).$$

In [7], we have obtained the following decomposition in 11 factors $T_{ij}$ of the vector space $T = \{ T(x, y, z) | T(x, y, z) = -T(y, x, z) \}$ of the torsion $(0,3)$-tensors on $(M, \varphi, \xi, \eta, g)$. This decomposition is orthogonal and invariant with respect to the structural group $G \times I$.

**Theorem 3.3** ([7]). The torsion $T$ of $\nabla'$ on $(M, \varphi, \xi, \eta, g)$ belongs to the subclass $T_{12} \oplus T_{13} \oplus T_{14} \oplus T_{21} \oplus T_{22} \oplus T_{31} \oplus T_{32} \oplus T_{33} \oplus T_{41}$ of $T$, where

- $T_{12}$ : $T(\xi, y, z) = T(x, y, \xi) = 0$,
  $$T(x, y, z) = -T(\varphi x, \varphi y, z) = T(\varphi x, y, \varphi z);$$

- $T_{13}$ : $T(\xi, y, z) = T(x, y, \xi) = 0$,
  $$T(x, y, z) - T(\varphi x, \varphi y, z) = \mathcal{O} \quad T(x, y, z) = 0;$$

- $T_{14}$ : $T(\xi, y, z) = T(x, y, \xi) = 0$,
  $$T(x, y, z) - T(\varphi x, \varphi y, z) = \mathcal{O} \quad T(\varphi x, y, z) = 0;$$

- $T_{21}$ : $T(x, y, z) = \eta(z) T(\varphi^2 x, \varphi^2 y, \xi)$, $T(x, y, \xi) = -T(\varphi x, \varphi y, \xi);$

- $T_{22}$ : $T(x, y, z) = \eta(z) T(\varphi^2 x, \varphi^2 y, \xi)$, $T(x, y, \xi) = T(\varphi x, \varphi y, \xi);$

- $T_{31}$ : $T(x, y, z) = \eta(x) T(\xi, \varphi^2 y, \varphi^2 z) - \eta(y) T(\xi, \varphi^2 x, \varphi^2 z)$,
  $$T(\xi, y, z) = T(\xi, z, y) = -T(\xi, \varphi y, \varphi z);$$

- $T_{32}$ : $T(x, y, z) = \eta(x) T(\xi, \varphi^2 y, \varphi^2 z) - \eta(y) T(\xi, \varphi^2 x, \varphi^2 z)$,
  $$T(\xi, y, z) = -T(\xi, z, y) = -T(\xi, \varphi y, \varphi z);$$

- $T_{33}$ : $T(x, y, z) = \eta(x) T(\xi, \varphi^2 y, \varphi^2 z) - \eta(y) T(\xi, \varphi^2 x, \varphi^2 z)$,
  $$T(\xi, y, z) = T(\xi, z, y) = T(\xi, \varphi y, \varphi z);$$
The torsion \( T(x, y, z) = \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) - \eta(y)T(\xi, \varphi^2 x, \varphi^2 z), \) 

\[ T(\xi, y, z) = -T(\xi, z, y) = T(\xi, \varphi y, \varphi z); \] 

\[ \mathcal{T}_{34} : \quad T(x, y, z) = \eta(z) \{ \eta(y)\hat{t}(x) - \eta(x)\hat{t}(y) \}. \]

Bearing in mind (2.2), the \( \varphi \)-B-connection on a manifold in \( \mathcal{U} \) has the form

\[ \nabla'_x y = \nabla x y + (\nabla x \eta) y, \xi - \eta(y)\nabla x \xi \]

and for its torsion and torsion forms are valid:

\[ T(x, y, z) = \eta(x)F(y, \varphi z, \xi) - \eta(y)F(x, \varphi z, \xi) + \eta(z)d\eta(x, y), \]

\[ t(x) = \theta^*(\xi)\eta(x), \quad t^*(x) = -\theta(\xi)\eta(x), \quad \hat{t}(x) = 0. \]

Thus we establish that \( t \) and \( t^* \) coincide with \( \theta^* \) and \( -\theta \), respectively, on the considered manifolds.

Bearing in mind (3.6), we obtain \( T(\xi, y, z) = F(y, \varphi z, \xi) \) in \( \mathcal{U} \) and thus

\[ T(x, y, z) = \eta(x)T(\xi, y, z) - \eta(y)T(\xi, x, z) \]

\[ + \eta(z)T(\xi, x, y) - \eta(z)T(\xi, y, x). \]

Using Theorem 3.3, we establish

**Proposition 3.4.** The torsion \( T(x, \varphi y, \varphi z, \xi, \eta, g) \in \mathcal{U} \) belongs to the class \( \mathcal{T}_{31} \oplus \mathcal{T}_{32} \oplus \mathcal{T}_{33} \oplus \mathcal{T}_{34} \). Moreover, if \( (M, \varphi, \xi, \eta, g) \in \mathcal{U}_1, \mathcal{U}_2 \) and \( \mathcal{U}_3 \), then \( T \in \mathcal{T}_{31} \oplus \mathcal{T}_{33} \), \( \mathcal{T}_{31} \oplus \mathcal{T}_{32} \) and \( \mathcal{T}_{31}, \) respectively.

The 1-form \( \eta \) is closed on the manifolds in \( \mathcal{U}_1 \) and then (3.6) implies

\[ T(x, y, z) = \eta(x)F(y, \varphi z, \xi) - \eta(y)F(x, \varphi z, \xi). \]

According to (3.1) and (3.5), we have in \( \mathcal{U} \) the expression

\[ Q(x, y, z) = F(x, \varphi y, \xi)\eta(z) - F(x, \varphi z, \xi)\eta(y). \]

The latter two equalities imply the following

**Proposition 3.5.** The torsion \( T(x, \varphi y, \varphi z, \xi, \eta, g) \in \mathcal{U}_1 \) has the properties \( Q(x, y, z) = T(z, y, x) \) and \( \nabla x y z T(x, y, z) = 0. \)

Since \( g(x, \varphi y) = g(\varphi x, y) \) holds and (2.2) is valid on a manifold from \( \mathcal{U}_2 \), then we obtain the following property using (2.1)

\[ \nabla x y z \xi = \varphi \nabla x \xi. \]

**Proposition 3.6.** The torsion \( T(x, \varphi y, \varphi z, \xi, \eta, g) \in \mathcal{U}_3 \) has the following properties:

\[ T(\varphi x, y, z) + T(x, \varphi y, z) - T(x, y, \varphi z) = 0; \]

\[ T(x, y, z) + T(\varphi x, y, \varphi z) + T(x, \varphi y, \varphi z) = 0; \]

\[ T(x, \varphi y, \varphi z) = T(x, \varphi z, \varphi y); \]

\[ T(x, y, z) - T(x, z, y) = \eta(y)T(x, \xi, z) - \eta(z)T(x, \xi, y). \]
Proof. Property (3.10) for $U_3 = U_1 \cap U_2$ follows from (3.8) for $U_1$ and (3.9) for $U_2$. By virtue of (3.8) we obtain

$$T(x, \varphi y, \varphi z) = \eta(x)F(\varphi y, \varphi^2 z, \xi) = -\eta(x)F(y, \varphi z, \xi),$$

$$T(\varphi x, y, \varphi z) = -\eta(y)F(\varphi x, \varphi^2 z, \xi) = \eta(y)F(x, \varphi z, \xi),$$

bearing in mind (2.3) for the class $U_2$. Summing up the equalities in the last two lines and giving an account of (3.8) again, we obtain (3.11). Because of (3.14), (2.1) and $d\eta = 0$, we prove (3.12). Equality (3.12) and $T(x, y, \xi) = 0$ imply property (3.13). \hfill \Box

3.2. Curvature properties of the $\varphi$B-connection. Let $R = [\nabla, \nabla] - \nabla_{[\cdot, \cdot]}$ be the curvature (1,3)-tensor of $\nabla$. We denote the curvature (0,4)-tensor by the same letter:

$$R(x, y, z, w) = g(R(x, y, z, w)).$$

The scalar curvature $\tau$ for $R$ as well as its associated quantity $\tau^*$ are defined respectively by

$$\tau = g^{ij}g^{kl}R(e_k, e_i, e_j, e_l)$$

and

$$\tau^* = g^{ij}g^{kl}R(e_k, e_i, e_j, \varphi e_l).$$

Similarly, the curvature tensor $R'$, the scalar curvature $\tau'$ and its associated quantity $\tau'^*$ for $\nabla'$ are defined.

According to (3.1) and (3.3), the curvature tensor $R'$ of a natural connection $\nabla'$ has the following form [9]

$$R'(x, y, z, w) = R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) + g(Q(x, z), Q(y, w)) - g(Q(y, z), Q(x, w)).$$

Then, using (3.5), we obtain the following

Proposition 3.7. For the curvature tensor $R'$ and the scalar curvature $\tau'$ of $\nabla'$ on $(M, \varphi, \xi, \eta, g) \in U$ we have

$$R'(x, y, z, w) = R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) + g(Q(x, z), Q(y, w)) - g(Q(y, z), Q(x, w)).$$

where $\|\nabla \xi\|^2 = g^{ij}g(\nabla e_i \xi, \nabla e_j \xi)$.

A curvature-like tensor $L$, i.e. a tensor with properties

$$L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z),$$

$$\bigotimes_{x, y, z} L(x, y, z, w) = 0,$$

we call a $\varphi$-Kähler-type tensor on $(M, \varphi, \xi, \eta, g)$ if the following identity is valid [2]

$$L(x, y, \varphi z, \varphi w) = -L(x, y, z, w).$$

The latter property is characteristic for $R$ on an $F_0$-manifold.

It is easy to verify that $R'$ of the $\varphi$B-connection satisfies (3.17) and (3.19) but it is not a $\varphi$-Kähler-type tensor because of the lack of (3.18). Because of (3.15), the condition $\bigotimes_{x, y, z} R'(x, y, z) = 0$ holds if and only if
\[ \mathcal{G} \{ \eta(x)R(y, z)\xi \} = \mathcal{G} \{ d\eta(x, y)\nabla_z \xi \} \] is valid. Hence it follows the equality \( \mathcal{G} \{ \eta(x)R(y, z)\xi \} = 0 \) for the class \( \mathcal{U}_1 \), which implies \( R(\varphi x, \varphi y)\xi = 0 \). Then, using (3.9) for \( \mathcal{U}_2 \), we obtain in \( \mathcal{U}_3 = \mathcal{U}_1 \cap \mathcal{U}_2 \) the following form of \( R \)

\[ R(x, y, z, \xi) = \eta(x)g(\nabla_y \xi, \nabla_z \xi) - \eta(y)g(\nabla_x \xi, \nabla_z \xi). \]

Vice versa, (3.20) implies \( \mathcal{G} \{ \eta(x)R(y, z)\xi \} = 0 \) and then we have

**Theorem 3.8.** The curvature tensor of \( \nabla' \) is a \( \varphi \)-Kähler-type tensor on \( (M, \varphi, \xi, \eta, g) \in \mathcal{U}_1 \) if and only if \( (M, \varphi, \xi, \eta, g) \in \mathcal{U}_3 \).

4. A Lie group as a 5-dimensional \( \mathcal{F}_6 \)-manifold

Let us consider the example given in [10]. Let \( G \) be a 5-dimensional real connected Lie group and let \( \mathfrak{g} \) be its Lie algebra. Let \( \{ e_i \} \) be a global basis of left-invariant vector fields of \( G \).

**Theorem 4.1** ([10]). Let \( (G, \varphi, \xi, \eta, g) \) be the almost contact B-metric manifold, determined by

\[
\begin{align*}
\varphi e_1 &= e_3, \quad \varphi e_2 = e_4, \quad \varphi e_3 = -e_1, \quad \varphi e_4 = -e_2, \quad \varphi e_5 = 0; \\
\xi &= e_5; \\
\eta(e_i) &= 0 \ (i = 1, 2, 3, 4), \quad \eta(e_5) = 1; \\
g(e_1, e_1) &= g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) = g(e_5, e_5) = 1, \\
g(e_i, e_j) &= 0, \quad i \neq j, \quad i, j \in \{1, 2, 3, 4, 5\}; \\
[e_1, \xi] &= \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4, \\
[e_2, \xi] &= \mu_1 e_1 - \lambda_1 e_2 + \mu_3 e_3 - \lambda_3 e_4, \\
[e_3, \xi] &= -\lambda_3 e_1 - \lambda_4 e_2 + \lambda_1 e_3 + \lambda_2 e_4, \\
[e_4, \xi] &= -\mu_3 e_1 + \lambda_3 e_2 + \mu_1 e_3 - \lambda_1 e_4.
\end{align*}
\]

Then \( (G, \varphi, \xi, \eta, g) \) belongs to the class \( \mathcal{F}_6 \).

Further, \( (G, \varphi, \xi, \eta, g) \) will stand for the \( \mathcal{F}_6 \)-manifold determined by the conditions of Theorem 4.1

Using (3.23), we compute the components of the \( \varphi \)-B-connection \( \nabla' \) on \( (G, \varphi, \xi, \eta, g) \). The non-zero components of \( \nabla' \) are the following:

\[
\begin{align*}
\nabla'_1 e_1 &= -\frac{1}{2} (\lambda_2 - \mu_1) e_2 - \frac{1}{2} (\lambda_4 - \mu_3) e_4, \\
\nabla'_1 e_2 &= \frac{1}{2} (\lambda_2 - \mu_1) e_1 + \frac{1}{2} (\lambda_4 - \mu_3) e_3, \\
\nabla'_1 e_3 &= \frac{1}{2} (\lambda_4 - \mu_3) e_2 - \frac{1}{2} (\lambda_2 - \mu_1) e_4, \\
\nabla'_1 e_4 &= -\frac{1}{2} (\lambda_4 - \mu_3) e_1 + \frac{1}{2} (\lambda_2 - \mu_1) e_3.
\end{align*}
\]

By virtue of the latter equalities we establish that the basic components of the curvature tensor \( R' \) of \( \nabla' \) are zero and then we have

**Proposition 4.2.** The manifold \( (G, \varphi, \xi, \eta, g) \) has a flat \( \varphi \)-B-connection.
Bearing in mind (3.7), the torsion of $\nabla'$ is determined only by the components $T_{ij} = T(\xi, e_i, e_j)$. We compute them using $T(\xi, e_i, e_j) = (\nabla e_i) e_j$, $T(\xi, e_i) = \nabla e_i \xi$ and the equalities (10)

\[
\begin{align*}
\nabla e_1 \xi &= \lambda_1 e_1 + \frac{1}{2} (\lambda_2 + \mu_1) e_2 + \lambda_3 e_3 + \frac{1}{2} (\lambda_4 + \mu_3) e_4, \\
\nabla e_2 \xi &= \frac{1}{2} (\lambda_2 + \mu_1) e_1 - \lambda_1 e_2 + \frac{1}{2} (\lambda_4 + \mu_3) e_3 - \lambda_3 e_4, \\
\nabla e_3 \xi &= -\lambda_3 e_1 - \frac{1}{2} (\lambda_4 + \mu_3) e_2 + \lambda_1 e_3 + \frac{1}{2} (\lambda_2 + \mu_1) e_4, \\
\nabla e_4 \xi &= -\frac{1}{2} (\lambda_4 + \mu_3) e_1 + \lambda_3 e_2 + \frac{1}{2} (\lambda_2 + \mu_1) e_3 - \lambda_1 e_4.
\end{align*}
\]

Thus we get that the non-zero components are:

\[
(4.2)
\begin{align*}
T_{511} &= -T_{522} = -T_{533} = T_{544} = \lambda_1, \\
T_{512} &= T_{521} = -T_{534} = -T_{543} = \frac{1}{2} (\lambda_2 + \mu_1), \\
T_{513} &= -T_{524} = T_{531} = -T_{542} = -\lambda_3, \\
T_{514} &= T_{523} = T_{532} = T_{541} = -\frac{1}{2} (\lambda_4 + \mu_3).
\end{align*}
\]

Using (4.1) and (4.2), we obtain the following

**Proposition 4.3.** The $\varphi$-$B$-connection $\nabla'$ on $(G, \varphi, \xi, \eta, g)$ has a parallel torsion $T$ with respect to $\nabla'$.