Figuring the fine structure of the black hole at the Galactic Center with extremely large mass-ratio inspirals

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In the Galaxy, extremely large mass-ratio inspirals (X-MRIs) composed of brown dwarfs and the massive black hole (MBH) at the Galactic Center could be potential gravitational wave (GW) sources for space-borne detectors. The estimated event rate is considerable, making X-MRIs a promising class of targets for future GW missions. With a few X-MRI observations, astronomers could determine the fine structure of the central MBH with incredible accuracy. In this work, using a waveform model for axisymmetric black holes, we simulate the GWs from twenty X-MRI systems with varied parameters. We find that the mass, spin, and deviation parameters of the Kerr black hole could be determined accurately (∼10⁻⁵ – 10⁻⁶) with only one X-MRI event with high a signal-to-noise ratio (SNR). The measurement of the above parameters could be improved with more X-MRI observations.

I. INTRODUCTION

The first observations of GWs from the binary black hole merger and the binary neutron star inspiral ushered in a new era of GW physics and astronomy[1, 2]. Since then, the ground-based detectors have detected 90 GW events[3–5]. The detectable frequency band of current ground-based GW detectors such as Advanced LIGO[6], Advanced Virgo[7], and KAGRA[8] ranges from 10 to 10,000 Hz, which makes ground-based GW detectors unable to detect any GWs with frequencies less than 10 Hz, while abundant sources are emitting GWs in the low-frequency band[9]. The space-borne GW detectors such as LISA[10], Taiji[11], and TianQin[12], which will be launched in the 2030s, will open GW windows from 0.1 mHz to 1 Hz, and are expected to probe the nature of astrophysics, cosmology, and fundamental physics.

One of the most essential and promising GW sources for space-borne GW detectors is the extreme-mass ratio inspiral (EMRI)[9, 13]. The EMRI system is formed when a massive black hole (MBH) captures a small compact object. The small object should be compact to keep it from being tidally disrupted by the MBH so that it is unlikely to be a main-sequence star. The possible candidate could be a stellar-mass black hole (BH), neutron star, white dwarf, or other compact objects. The designed space-borne detectors will be sensitive to EMRIs that contain MBHs with the mass 10⁴ – 10⁷ M⊙ and small compact objects with stellar mass, and the fiducial mass ratio will be 10³ – 10⁶[14].

Moreover, a special kind of EMRI, extremely large mass-ratio inspirals (X-MRIs) with a mass ratio of q ∼ 10⁸ also are potential sources for space-borne GW detectors[15, 16]. The X-MRI system is formed when an MBH captures a brown dwarf (BD) with mass ∼ 10⁻² M⊙. Brown dwarfs are substellar objects with insufficient mass to sustain nuclear fusion and become main-sequence stars[17]. Brown dwarfs are denser than main-sequence stars, and their Roche limit is closer to the horizon of MBH[15, 18]. Therefore, brown dwarfs could survive very close to the MBH.

The mass of BD is relatively tiny, so space-borne GW detectors like LISA could only observe X-MRIs nearby, especially X-MRIs at the Galactic Center (GC)[15, 16]. The MBH of these X-MRIs, Sgr A*, is 8 kpc from the solar system, and its mass is about 4 × 10⁶ M⊙[19–22]. A typical X-MRI at the GC covers ∼ 10⁸ cycles, which last millions of years in the LISA band[16]. Such X-MRI could have a relatively high SNR (more than 1000), and dozens of X-MRIs might be observed during the LISA mission period[16]. Therefore, the X-MRIs at the GC offers a natural laboratory for studying the properties of BH and testing gravitational theories.

In this paper, we simulate GWs from X-MRIs at GC to show how and to what extent the fine structure of Sgr A* could be figured. According to the no-hair theorem, black holes in general relativity (GR) are characterized by their masses, spins, and electric charges. However, alternative theories of gravity predict hairy black holes. Moreover, in GR, the Kerr metric is believed to be the metric that describes the space-time around a BH. However, alternative theories of gravity predict non-Kerr metrics (KRZ metric, a model-independent parameterization metric, could be

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used to study different non-Kerr metrics that are generic stationary and axisymmetric). X-MRIs at GC would be an efficient tool for studying the above problems.

This paper is organized as follows, in section II, we review the KRZ parametrization. In section III, we introduce the "kludge" waveforms used in our work and simulate the GWs emitted by X-MRIs at GC. In section IV, based on the simulated GWs, we apply the Fisher matrix to these GWs and present the accuracy of parameter estimation of Sgr A* in future space-borne GW detection. The conclusion and outlook are given in section V. Throughout this letter, we use natural units \(G = C = 1\), greek letters \((\mu, \nu, \sigma, \ldots)\) stand for space-time indices, and Einstein summation is assumed.

II. KRZ PRAMETRIZED METRIC

In the GR framework, the Kerr metric describes the space-time of BH. However, there are other possible solutions in modified and alternative theories of gravity for black holes. In order to deal with numerous theoretical models and corresponding black holes, one may use the parameterized metric to describe the space-time of non-Kerr black holes. A model-independent framework used in parametrizing the generic black hole space-time is helpful.

There are several model-independent frameworks, one of which parametrizes the most generic black hole geometry through a finite number of adjustable quantities and is known as Johannsen-Psaltis parametrization (J-P metric) [23]. The J-P metric expresses deviations from general relativity in terms of a Taylor expansion in powers of \(M/r\), where \(M\) is the mass of BH and \(r\) is the radial coordinate. The J-P parametrization is widely adopted, but because of some unresolved difficulties, it appears not to be a robust and generic parametrization for rotating black holes [24, 25]. Notably, the parametric axisymmetric J-P metric obtained from the Janis-Newman algorithm [26] does not cover all deviations from Kerr space-time.

Another model-independent parameterization metric [24, 27], KRZ metric, is based on a double expansion in both the polar and radial directions of a generic stationary and axisymmetric metric. The KRZ metric is effective in reproducing the space-time of three commonly used rotating black hole (Kerr, rotating dilation, and Einstein-dilaton-Gauss-Bonnet black holes) with finite parameters (see Ref. [24] for more details). According to KRZ parameterization, the space-time of any axisymmetric black hole with total mass \(M\) and rotation parameter \(a\) could be expressed in the following form [24]:

\[
\begin{align*}
\text{ds}^2 &= -\frac{N^2 - W^2 \sin^2 \theta}{K^2} dt^2 - 2Wr \sin^2 \theta dt d\phi \\
&\quad + K^2 r^2 \sin^2 \theta d\phi^2 + S \left( \frac{B^2}{N^2} dr^2 + r^2 d\theta^2 \right),
\end{align*}
\]

(2.1)

where [28]

\[
\begin{align*}
S &= \frac{\Sigma}{r^2} = 1 + \frac{a^2}{r^2} \cos^2 \theta, \\
\Sigma &= r^2 + a^2 \cos^2 \theta,
\end{align*}
\]

(2.2)

(2.3)

\(N, B, W,\) and \(K\) are the functions of the radial and polar coordinates (expanded in term \(\cos \theta\)),

\[
W = \sum_{i=0}^{\infty} \frac{W_i(r)(\cos \theta)^i}{S},
\]

(2.4)

\[
B = 1 + \sum_{i=0}^{\infty} B_i(r)(\cos \theta)^i,
\]

(2.5)

\[
N^2 = \left(1 - \frac{r_0}{r} \right) A_0(r) + \sum_{i=1}^{\infty} A_i(r)(\cos \theta)^i,
\]

(2.6)

\[
K^2 = 1 + \frac{aW}{r} + \frac{a^2}{r^2} + \sum_{i=1}^{\infty} \frac{K_i(r)(\cos \theta)^i}{S},
\]

(2.7)

with

\[
B_i = b_i r_0 + \tilde{B}_i r_0^2,
\]

(2.8)
\[ \check{b}_i = \frac{b_{i1}}{1 + \frac{x_i}{r+2}}, \quad (2.9) \]
\[ W_i = b_{i0} \frac{r_0^2}{r^2} + \check{B}_i \frac{r_0^3}{r^3}, \quad (2.10) \]
\[ \check{W}_i = \frac{\omega_{i1}}{1 + \frac{x_{i2}}{r+3}}, \quad (2.11) \]
\[ K_{i>0}(r) = k_{i0} \frac{r_0^2}{r^2} + \check{K}_i \frac{r_0^3}{r^3}, \quad (2.12) \]
\[ \check{K}_i = \frac{k_{i1}}{1 + \frac{x_{i3}}{r+4}}, \quad (2.13) \]
\[ A_0(r) = 1 - \epsilon_0 \frac{r_0}{r} + (a_{00} - \epsilon_0) \frac{r_0^2}{r^2} + \check{A}_0 \frac{r_0^3}{r^3}, \quad (2.14) \]
\[ A_{i>0} = K_i(r) + \epsilon_i \frac{r_0^2}{r^2} + a_{i0} \frac{r_0^3}{r^3} + \check{A}_i \frac{r_0^4}{r^4}, \quad (2.15) \]
\[ \check{A}_i = \frac{a_{i1}}{1 + \frac{x_{i4}}{r+5}}, \quad (2.16) \]

where \( x = 1 - r_0/r \), and \( r_0 \) is the radius of the black hole horizon in the equatorial plane. The metric (2.1) is characterized by the order of expansion in radial and polar direction. The parameters \( a_{ij}, b_{ij}, \omega_{ij}, k_{ij} \) (here \( i = 0, 1, 2, 3..., j = 1, 2, 3... \)) are effectively independent. This is because that one of these functions, \( A_i(x), B_i(x), W_i(x) \) and \( K_i(x) \), is fixed by coordinate choice[28].

In the following, we show the parameterized metric with first-order radial expansion and second-order polar direction(for simplicity, here \( M = 1 \) and \( \check{a} = a/M \) stands for the spin parameter), which describe the space-time of a deformed Kerr black hole[25, 28]:

\[ B = 1 + \frac{\epsilon_2 r_0^2}{r^2} + \frac{\epsilon_3 r_0^3}{r^3} \cos^2 \theta, \quad (2.17) \]
\[ W = \frac{1}{r} \left[ \frac{\omega_0 r_0^2}{r^2} + \frac{\delta_2 r_0^3}{r^3} + \frac{\epsilon_2 r_0^3}{r^3} \cos^2 \theta \right], \quad (2.18) \]
\[ K^2 = 1 + \frac{a W}{r} + \frac{1}{r} \left[ k_{00} \frac{r_0^3}{r^3} + k_{21} \frac{r_0^3}{r^3} \cos^2 \theta \right], \quad (2.19) \]
\[ N^2 = \left( 1 - \frac{r_0}{r} \right) \left[ 1 - \frac{\epsilon_0 r_0}{r} + (k_{00} - \epsilon_0) \frac{r_0^2}{r^2} + \frac{\delta_1 r_0^3}{r^3} \right] \]
\[ + \left( k_{21} + a_{20} \right) \frac{r_0^3}{r^3} + \frac{a_{21} r_0^4}{r^4} \right] \cos^2 \theta, \quad (2.20) \]

The radius of the horizon and the Kerr parameter are

\[ r_0 = M + \sqrt{M^2 - a^2}, \quad a = J/M, \quad (2.21) \]

where \( J \) is the total angular momentum. The coefficient \( r_0, a_{20}, a_{21}, \epsilon_0, k_{00}, k_{21} \) and \( \omega_{00} \) in the KRZ metric could be expressed as follows[25, 29]

\[ r_0 = 1 + \sqrt{1 - \check{a}^2}, \quad (2.22) \]
\[ a_{20} = \frac{2 \check{a}^2}{r_0^3}, \quad (2.23) \]
\[ a_{21} = -\frac{\check{a}^4}{r_0^6} + \delta_6, \quad (2.24) \]
\[ \epsilon_0 = \frac{2}{r_0^2}, \quad (2.25) \]
\[ \omega_{00} = \frac{2 \check{a}}{r_0^3}, \quad (2.26) \]
\[ k_{00} = \frac{\tilde{a}^2}{r_0^2}, \]  
\[ k_{21} = \frac{\tilde{a}^4}{r_0^4} - 2\frac{\tilde{a}^2}{r_0^3} - \delta_6, \]  
\[ k_{22} = -\frac{\tilde{a}^2}{r_0^2} + \delta_7, \]  
\[ k_{23} = \frac{\tilde{a}^2}{r_0^2} + \delta_8. \]  

The deformation parameters \( \delta_j (j = 1, 2, ..., 8) \) represent the deviations from the Kerr metric. The physical meaning of these parameters could be summarized as follows: \( \delta_1 \) is related to deformations of \( g_{tt} \); \( \delta_2, \delta_3 \) are related to the rotational deformations of the metric; \( \delta_4, \delta_5 \) are related to deformations of \( g_{rr} \) and \( \delta_6 \) is related to the deformations of the event horizon (see Ref. [24] for more details). The KRZ metric is an appropriate tool to measure the potential deviations from the Kerr metric. As a first order approximation, in this work we mainly take account of \( \delta_1 \) and \( \delta_2 \).

### III. Waveform Model for KRZ Black Holes

Several waveform models could simulate the signal of EMRI[14, 30–34]. Among these models, the kludge model could generate waveforms quickly and have a 95% accuracy compared with the Teukolsky-based waveforms[33]. Kludge waveforms may be essential in searching for EMRIs/X-MRIs in future space-borne GW detection. We employ the kludge waveforms to simulate X-MRI waveforms[29]. Before presenting the results, we would like to review the structure and logic of the calculation. The calculation of waveforms could be summarized in the following steps:

- First, regard the brown dwarf of the X-MRI as a point particle.
- Second, use the given metric to calculate the particle’s trajectory by integrating the geodesic equations augmented with the radiation flux.
- Finally, use the quadrupole expression to get the GWs emitted from the system of the X-MRI.

To get the trajectory of the particle, we start by calculating the geodesics using the following equations:

\[ \dot{u}^\mu = -\Gamma^\mu_{\rho\sigma} u^\rho u^\sigma, \] (3.1)  
\[ \dot{x}^\mu = u^\mu, \] (3.2)

where \( x^\mu \) is the coordinate of the particle, \( u^\mu \) is the 4-velocity, and \( \Gamma^\mu_{\rho\sigma} \) is the Christoffel symbol. A set of first-order differential equations governs geodesic motion in the Kerr space-time and have the following form,

\[ \Sigma \frac{dr}{d\tau} = \pm \sqrt{V_r}, \] (3.3)  
\[ \Sigma \frac{d\theta}{d\tau} = \pm \sqrt{V_\theta}, \] (3.4)  
\[ \Sigma \frac{d\phi}{d\tau} = \pm \sqrt{V_\phi}, \] (3.5)  
\[ \Sigma \frac{dt}{d\tau} = \pm \sqrt{V_t}, \] (3.6)

where \( \tau \) is proper time along the worldline. The potential functions are defined as

\[ V_r(r) = P^2 - [r^2 + (L_z^2 - aE)^2 + Q]\Delta, \] (3.7)  
\[ V_\theta(\theta) = Q - \cos^2 \theta(a^2(1 - E^2) + (\frac{L_z}{\sin \theta})^2), \] (3.8)  
\[ V_\phi(r, \theta) = \frac{L_z}{\sin^2 \theta} - aE + \frac{aP}{\Delta}, \] (3.9)  
\[ V_t(r, \theta) = aL_z - \frac{a^2E\sin^2 \theta}{\Delta} + \frac{(r^2 + a^2)P}{\Delta}, \] (3.10)

where \( P = E(r^2 + a^2) - aL_z \). The orbital eccentricity \( e \) and semi-latus rectum \( p \) could be defined by periastron \( r_p \) and apastron \( r_a \), and the inclination angle \( \iota \) is defined in the Keplerian convention:

\[ e = \frac{r_a - r_p}{r_a + r_p}, \]  
\[ p = \frac{2r_a r_p}{r_a + r_p}, \]  
\[ \iota = \frac{\pi}{2} - \theta_{\text{min}}. \] (3.11)
where $\theta_{\text{min}}$ is the minimum of $\theta$ along the geodesic. Obviously, the solutions of $V_r$ determine $r_p$ and $r_a$, and the solutions of $V_\theta$ determines $\theta_{\text{min}}$.

A timelike Kerr geodesic is fully described by three first integrals of motion: the orbital energy $E$, the $z$ component of the orbital angular momentum $L_z$, and the quadratic Carter constant $Q$. These three integrals of motion could be defined by the following equations [29]:

\begin{align}
|u| &= g_{\mu\nu}u^\mu u^\nu = -1, \\
E &= -u_t - g_{tt}u^t - g_{t\phi}u^\phi, \\
L_z &= u_\phi = g_{t\phi}u^t + g_{\phi\phi}u^\phi.
\end{align}

(3.12) (3.13) (3.14)

In the case of Kerr space-time, the Carter constant has the following form [35],

\begin{equation}
Q = (g_{\theta\theta}u^\theta)^2 + \cos^2 \theta \left(a^2(\eta^2 - E^2) + \left(\frac{L_z}{\sin \theta}\right)^2\right).
\end{equation}

(3.15)

For a bound orbit, the geodesic may be specified by the parameters $(r_a, r_p, \theta_{\text{min}})$, which fully describe the range of motion in the radial and polar coordinates. While the orbital constants $(E, L_z, Q)$ in the above geodesic setup do not vary with time, it is convenient to work with alternative parametrizations of $(E, L_z, Q)$. The relationship between $(E, L_z, Q)$ and $(r_a, r_p, \theta_{\text{min}})$ and $(E, L_z, Q)$ is given by Eq. (3.7) and Eq. (3.8)

\begin{align}
P^2 &= \left[ r^2 + (L_z - aE)^2 + Q \right] \Delta r | r = r_a \theta = \frac{\pi}{2} = 0, \\
P^2 &= \left[ r^2 + (L_z - aE)^2 + Q \right] \Delta \theta | r = r_a \theta = \frac{\pi}{2} = 0, \\
Q &= \cos^2 \theta_{\text{min}} \left[a^2(1 - E^2) + \left(\frac{L_z}{\sin \theta}\right)^2\right].
\end{align}

(3.16) (3.17) (3.18)

In Kerr space-time, we could determine the three orbital parameters $e, p, \iota$ from $E, L_z, Q$ and vice versa [14]. For equatorial orbits in a stationary and axisymmetric general metric, while $u^\alpha u_\alpha = -1$ with $u_r = 0$ at apastron and periastron, we could analytically determine $e, p$ from $E, L_z$ and vice versa by [29]

\begin{align}
g_{tt}^{(1)} &= \left[ r^2 + (L_z - aE)^2 + Q \right] \Delta r | r = r_a \theta = \frac{\pi}{2} = 0, \\
g_{tt}^{(2)} &= \left[ r^2 + (L_z - aE)^2 + Q \right] \Delta \theta | r = r_a \theta = \frac{\pi}{2} = 0.
\end{align}

(3.19) (3.20)

The effect of radiation reaction is included by replacing the Eq. (3.1) with the following one:

\begin{equation}
\frac{\text{d}u^\mu}{\text{d}\tau} = -\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta + \mathcal{F}^\mu.
\end{equation}

(3.21)

where the radiation force $\mathcal{F}^\mu$ is connected with the adiabatic radiation fluxes $(\dot{E}, \dot{L}_z, \dot{Q})$ as

\begin{align}
\dot{E}u^t &= -g_{tt}\mathcal{F}^t - g_{t\phi}\mathcal{F}^\phi, \\
\dot{L}_zu^t &= g_{t\phi}\mathcal{F}^t + g_{\phi\phi}\mathcal{F}^\phi, \\
\dot{Q}u^t &= 2g_{\theta\theta}u^\theta \mathcal{F}^\theta + 2\cos^2 \theta_2 a^2 E \dot{E} + 2\cos^2 \theta \frac{L_z L_z}{\sin^2 \theta}, \\
g_{\mu\nu}u^\mu u^\nu &= 0.
\end{align}

(3.22)

Eq. (3.22) could be deduced by taking derivatives with respect to proper time in Eqs. (3.12)-(3.15).

Finally, after generating the trajectory, we turn to the third step – to calculate the GW from the geodesics of test particle. We start from transforming the Boyer-Lindquist coordinates $(t, r, \phi, \theta)$ into Cartesian coordinates $(t, x, y, z)$ using the relations:

\begin{align}
t &= t, \\
x &= r \sin \theta \cos \phi, \\
y &= r \sin \theta \sin \phi, \\
z &= r \cos \theta.
\end{align}

(3.23) (3.24) (3.25) (3.26)
Then we calculate the quadrupole expression (see Ref. [33])
\begin{align}
\bar{h}_{jk}(t,x) &= \frac{2}{r}[\bar{I}_{jk}'(t')]|_{t'=t-r}, \\
I_{jk}'(t') &= \int x' j' x' T^00(t',x') d^3x',
\end{align}
where $I_{jk}'(t')$ is the source's mass quadrupole moment, $T^00$ is component of the energy-momentum tensor $T^{\mu\nu}(t',x')$, and $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h^{\rho\sigma} \eta_{\rho\sigma} h_{\mu\nu}$ is the trace-reversed metric perturbation. Then we transform the waveform into the transverse-traceless gauge (see Ref. [33] for more details)
\begin{align}
\bar{h}_{TT} &= \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & h^{\Theta\Theta} - h^{\Phi\Phi} & 2h^{\Theta\Phi} \\
0 & 2h^{\Theta\Phi} & h^{\Phi\Phi} - h^{\Theta\Theta}
\end{pmatrix},
\end{align}
with
\begin{align}
h^{\Theta\Theta} &= \cos^2 \Theta \left[ h^{xx} \cos^2 \Phi + h^{xy} \sin 2\Phi + h^{yy} \sin^2 \Phi \right] \\
&\quad + h^{zz} \sin^2 \Theta - \sin 2\Theta \left[ h^{xx} \cos \Phi + h^{yz} \sin \Phi \right], \\
h^{\Phi\Theta} &= \cos \Theta \left[ -\frac{1}{2} h^{xx} \sin 2\Phi + h^{xy} \cos 2\Phi + \frac{1}{2} h^{yy} \sin 2\Phi \right] \\
&\quad + \sin \Theta \left[ h^{zz} \sin \Phi - h^{yz} \cos \Phi \right], \\
h^{\Phi\Phi} &= \left[ h^{xx} \sin^2 \Phi - h^{xy} \sin 2\Phi + h^{yy} \cos^2 \Phi \right].
\end{align}
Now we get the plus and cross components of the waveform observed at latitudinal angle $\Theta$ and azimuthal angle $\Phi$
\begin{align}
h_+ &= h^{\Theta\Theta} - h^{\Phi\Phi} \\
&= \cos^2 \Theta \left[ h^{xx} \cos^2 \Phi + h^{xy} \sin 2\Phi + h^{yy} \sin^2 \Phi \right] \\
&\quad + h^{zz} \sin^2 \Theta - \sin 2\Theta \left[ h^{xx} \cos \Phi + h^{yz} \sin \Phi \right] \\
&\quad - \left[ h^{xx} \sin^2 \Phi - h^{xy} \sin 2\Phi + h^{yy} \cos^2 \Phi \right], \\
h_\times &= 2h^{\Theta\Phi} \\
&= 2 \left\{ \cos \Theta \left[ -\frac{1}{2} h^{xx} \sin 2\Phi + h^{xy} \cos 2\Phi + \frac{1}{2} h^{yy} \sin 2\Phi \right] \\
&\quad + \sin \Theta \left[ h^{zz} \sin \Phi - h^{yz} \cos \Phi \right] \right\}.
\end{align}
Using the Eqs. (3.33)-(3.34) we could define our waveform as $h = h_+ - ih_\times$.
We could define the SNRs of the signals as
\begin{equation}
\rho := \sqrt{\langle h|h \rangle},
\end{equation}
where $\langle \cdot | \cdot \rangle$ is the standard matched-filtering inner product between two data streams. The inner product between signal $a(t)$ and template $b(t)$ is
\begin{equation}
\langle a|b \rangle = 2 \int_0^\infty \tilde{a}(f) \tilde{b}^*(f) \frac{S_n(f)}{S_n(f)} df,
\end{equation}
where $\tilde{a}(f)$ is the Fourier transform of the time series signal $a(t)$, $\tilde{a}^*(f)$ is the complex conjugate of $\tilde{a}(f)$ and $S_n(f)$ is the power spectral density of the GW detectors' noise. Throughout this paper, the power spectral density is taken to be the noise level of LISA.
In this work, we use maximized fitting factor (overlap) to quantify the differences between gravitational waveforms. The fitting factor between the signal and template is
\begin{equation}
FF(a, b) = \frac{\langle a|b \rangle}{\sqrt{\langle a|a \rangle \langle b|b \rangle}}.
\end{equation}
If we include time shift $t_s$ and phase shift $\phi_s$, the fitting factor reads
\[
ff(t_s, \phi_s, a(t), b(t)) = \frac{(a(t)|b(t + t_s)e^{i\phi_s})}{\sqrt{(a|a)(b|b)}},
\]
the maximized fitting factor is defined as
\[
FF(a, b) = \max_{t_s, \phi_s} \frac{(a(t)|b(t + t_s)e^{i\phi_s})}{\sqrt{(a|a)(b|b)}}.
\]

IV. DATA ANALYSIS

In this section, we first specify the main parameters values we used in this work. Then we use XSPEG (a software for generating GWs in the KRZ metric, provided by the authors of Ref. [29]) to calculate the gravitational waveforms and do some analysis. Finally, we employ the Fisher information matrix to estimate the parameter estimation accuracy for LISA-like GW detectors.

For X-MRI at the GC, the mass of the brown dwarfs ranges from $\sim 0.01 \, M_\odot$ to $\sim 0.08 \, M_\odot$ [36]. The parameter values for the MBH Sgr A* in this work are as follows:

- the mass of Sgr A* $M_{\text{SgrA*}} = 4 \times 10^6 \, M_\odot$ [19–21];
- the dimensionless spin parameter $a = 0.5$ [37];
- the distance between Sgr A* and the solar system $R_p = 8.3$ kpc [38];
- the latitudinal angle $\Theta = -29^\circ$ and the azimuthal angle $\Phi = 266.417^\circ$ [39].

Based on the parameters above, we first simulate the GW signals of twenty X-MRIs at the GC (see Table I). The mass ratio $q$ ranges from $5 \times 10^{-7}$ to $4.0 \times 10^{8}$, the orbit eccentricity $e$ ranges from 0.1 to 0.8, the semi-latus rectum $p$ ranges from 10.6 to 50.0, the inclination angle $i$ ranges from $-2\pi/3$ to $\pi/3$, and the duration of above signals is one year. Then, we calculate the overlaps between above GW signals and many GW series with varying parameters. Finally, we use the Fisher information matrix to provide the uncertainties of parameter estimations. To illustrate our work clearly, we divide this section into two parts: IV A and IV B.

A. The overlaps between simulated GW signals of XMRIs and GW series with varying parameters

Suppose the GW signal and corresponding GW template’s overlaps are above 0.97 [40]. In that case, we would find neither the deviations from GR nor the unusual parameters of X-MRIs, which is called the "confusion" problem [40]. The confusion problem could prevent us from getting accurate parameter estimation of the X-MRIs. To make sure there is no confusion in our study, we calculate the overlaps between different gravitational waveforms of twenty X-MRIs with varying parameters $\lambda_i$ ($\lambda_i = a, M, \delta_1, \delta_2, e, p, i$). Here ($a, M$) are the parameters of the Sgr A*, ($\delta_1, \delta_2$) are the deformations of the space-time from the Kerr solution, and ($e, p, i$) are the parameters of orbit (eccentricity, semi-latus rectum, inclination).

Because $a, M, \delta_1,$ and $\delta_2$ are the intrinsic parameters of Sgr A* and present the nature of MBH directly, we pay more attention to these four parameters. The Figs. 1-4 display the overlaps between the original waveforms and the waveforms with varying parameters $a, M, \delta_1$ and $\delta_2$. As these figures show, the overlap tends to decrease while the increment of $\lambda$ increases.

Taking the overlap value 0.97 as a criterion would give the constraints of $\lambda$. Specifically, to get the constraints $\delta \lambda_i$ by the GWs of X-MRI, we first keep the other parameters fixed and generate several waveforms with varying $\lambda_i$. Then we calculate the overlaps between the original waveform and the waveforms with varying $\lambda_i$. Finally, the corresponding value of $\lambda_i$ when overlap equals 0.97 could be regarded as the limit of $\lambda_i$. From these figures, we could see that the parameter constraint ability for different X-MRI varies.

B. Evaluate the accuracy of parameter estimation for X-MRIs

The GW signals of X-MRIs in Table I have enough SNRs. Therefore, these signals could be applied to the Fisher information matrix to estimate the accuracy of parameter estimation. We present the accuracy of parameter estimation
The bottom plane represents

FIG. 1. Overlaps between the original waveforms and the waveforms changed with spin $a$. The other parameters ($M$, $\delta_1$, $\delta_2$, $c$, $p$, $i$) of systems listed in Table I remain unchanged. The top plane represents $a$-overlap curves from the top 10 systems (X-MRI 01 to X-MRI 10). The bottom plane represents $a$-overlap curves from the last 10 systems (X-MRI 11 to X-MRI 20).

### TABLE I. Parameter setting and parameter estimation accuracy for the 20 X-MRIs at the GC

| Signal | $e$ | $p$ | $i$ | $M_{Object}$ | SNR | $\Delta a/a$ | $\Delta M/M$ | $\Delta \delta_1$ | $\Delta \delta_2$ | $\Delta R_p/R_p$ |
|--------|----|----|----|-------------|-----|------------|-------------|---------------|--------------|---------------|
| 01     | 0.617 | 10.600 | $5\pi/6$ | $2.80 \times 10^{-2}$ | 1584.363 | $2.85 \times 10^{-6}$ | $4.18 \times 10^{-7}$ | $3.63 \times 10^{-6}$ | $3.28 \times 10^{-6}$ | $7.18 \times 10^{-4}$ |
| 02     | 0.520 | 12.000 | $\pi/6$ | $2.00 \times 10^{-2}$ | 636.988 | $2.10 \times 10^{-6}$ | $9.53 \times 10^{-7}$ | $1.48 \times 10^{-5}$ | $1.48 \times 10^{-5}$ | $1.78 \times 10^{-3}$ |
| 03     | 0.300 | 14.400 | $\pi/6$ | $2.00 \times 10^{-2}$ | 224.915 | $9.39 \times 10^{-6}$ | $4.04 \times 10^{-6}$ | $3.92 \times 10^{-5}$ | $4.95 \times 10^{-5}$ | $5.03 \times 10^{-3}$ |
| 04     | 0.200 | 16.800 | $\pi/7$ | $2.72 \times 10^{-2}$ | 144.765 | $4.60 \times 10^{-5}$ | $4.54 \times 10^{-6}$ | $7.31 \times 10^{-5}$ | $1.04 \times 10^{-4}$ | $7.37 \times 10^{-3}$ |
| 05     | 0.400 | 16.800 | $\pi/7$ | $2.72 \times 10^{-2}$ | 201.217 | $1.33 \times 10^{-5}$ | $2.91 \times 10^{-6}$ | $6.17 \times 10^{-5}$ | $8.26 \times 10^{-5}$ | $5.29 \times 10^{-3}$ |
| 06     | 0.514 | 27.243 | $-\pi/12$ | $1.80 \times 10^{-2}$ | 30.518 | $2.47 \times 10^{-4}$ | $1.96 \times 10^{-5}$ | $8.48 \times 10^{-4}$ | $1.39 \times 10^{-3}$ | $3.58 \times 10^{-2}$ |
| 07     | 0.500 | 24.750 | $\pi/4$ | $3.60 \times 10^{-2}$ | 59.061 | $7.79 \times 10^{-5}$ | $3.33 \times 10^{-6}$ | $1.70 \times 10^{-4}$ | $3.44 \times 10^{-4}$ | $1.68 \times 10^{-2}$ |
| 08     | 0.600 | 19.200 | $\pi/5$ | $2.80 \times 10^{-2}$ | 148.460 | $1.92 \times 10^{-5}$ | $1.91 \times 10^{-6}$ | $1.21 \times 10^{-4}$ | $1.92 \times 10^{-4}$ | $6.92 \times 10^{-3}$ |
| 09     | 0.700 | 15.300 | $\pi/6$ | $1.00 \times 10^{-2}$ | 140.487 | $1.15 \times 10^{-5}$ | $4.43 \times 10^{-6}$ | $1.54 \times 10^{-4}$ | $1.80 \times 10^{-4}$ | $7.73 \times 10^{-3}$ |
| 10     | 0.800 | 12.600 | $\pi/8$ | $1.20 \times 10^{-2}$ | 355.391 | $4.80 \times 10^{-6}$ | $3.42 \times 10^{-6}$ | $6.45 \times 10^{-5}$ | $6.18 \times 10^{-5}$ | $3.27 \times 10^{-3}$ |
| 11     | 0.100 | 39.600 | $-\pi/6$ | $7.00 \times 10^{-2}$ | 32.112 | $5.38 \times 10^{-3}$ | $6.25 \times 10^{-5}$ | $3.66 \times 10^{-3}$ | $9.96 \times 10^{-3}$ | $5.57 \times 10^{-2}$ |
| 12     | 0.253 | 35.093 | $-\pi/3$ | $7.84 \times 10^{-2}$ | 39.303 | $7.72 \times 10^{-5}$ | $2.36 \times 10^{-7}$ | $4.13 \times 10^{-5}$ | $1.41 \times 10^{-4}$ | $2.63 \times 10^{-2}$ |
| 13     | 0.206 | 30.159 | $-\pi/4$ | $7.60 \times 10^{-2}$ | 45.575 | $1.38 \times 10^{-4}$ | $1.94 \times 10^{-6}$ | $1.56 \times 10^{-4}$ | $3.69 \times 10^{-4}$ | $2.25 \times 10^{-2}$ |
| 14     | 0.368 | 41.053 | $-\pi/7$ | $8.00 \times 10^{-2}$ | 47.910 | $1.49 \times 10^{-3}$ | $1.23 \times 10^{-5}$ | $3.00 \times 10^{-3}$ | $8.19 \times 10^{-3}$ | $3.61 \times 10^{-2}$ |
| 15     | 0.295 | 47.924 | $-\pi/9$ | $6.00 \times 10^{-2}$ | 24.535 | $1.03 \times 10^{-2}$ | $1.18 \times 10^{-4}$ | $1.37 \times 10^{-2}$ | $3.34 \times 10^{-2}$ | $7.85 \times 10^{-2}$ |
| 16     | 0.425 | 32.775 | $-\pi/11$ | $3.00 \times 10^{-2}$ | 19.701 | $6.08 \times 10^{-4}$ | $2.15 \times 10^{-5}$ | $1.04 \times 10^{-3}$ | $1.99 \times 10^{-3}$ | $5.27 \times 10^{-2}$ |
| 17     | 0.300 | 27.300 | $\pi/3$ | $3.20 \times 10^{-2}$ | 27.179 | $9.05 \times 10^{-5}$ | $1.01 \times 10^{-6}$ | $7.17 \times 10^{-5}$ | $2.40 \times 10^{-4}$ | $3.78 \times 10^{-2}$ |
| 18     | 0.133 | 44.200 | $-2\pi/3$ | $6.80 \times 10^{-2}$ | 29.731 | $9.80 \times 10^{-4}$ | $2.19 \times 10^{-6}$ | $4.83 \times 10^{-4}$ | $2.21 \times 10^{-4}$ | $3.43 \times 10^{-2}$ |
| 19     | 0.137 | 50.039 | $-\pi/3$ | $7.20 \times 10^{-2}$ | 24.736 | $1.04 \times 10^{-3}$ | $2.53 \times 10^{-6}$ | $7.36 \times 10^{-5}$ | $2.42 \times 10^{-3}$ | $4.07 \times 10^{-2}$ |
| 20     | 0.477 | 25.108 | $-3\pi/5$ | $8.00 \times 10^{-2}$ | 101.345 | $4.48 \times 10^{-5}$ | $2.05 \times 10^{-7}$ | $1.16 \times 10^{-5}$ | $3.58 \times 10^{-5}$ | $9.75 \times 10^{-3}$ |
FIG. 2. Overlaps between the original waveforms and the waveforms changed with mass $M$. The other parameters ($a$, $\delta_1$, $\delta_2$, $e$, $p$, $\iota$) of systems listed in Table I remain unchanged. The top plane represents $M$-overlap curves from the top 10 systems (X-MRI 01 to X-MRI 10). The bottom plane represents $M$-overlap curves from the last 10 systems (X-MRI 11 to X-MRI 20).

FIG. 3. Overlaps between the original waveforms and the waveforms changed with deformation parameter $\delta_1$. The other parameters ($M$, $a$, $\delta_2$, $e$, $p$, $\iota$) of systems listed in Table I remain unchanged. The top plane represents $\delta_1$-overlap curves from the top 10 systems (X-MRI 01 to X-MRI 10). The bottom plane represents $\delta_1$-overlap curves from the last 10 systems (X-MRI 11 to X-MRI 20).
FIG. 4. Overlaps between the original waveforms and the waveforms changed with deformation parameter \( \delta_2 \). The other parameters \((M, a, \delta_1, e, p, \iota)\) of systems listed in Table I remain unchanged. The top plane represents \( \delta_2 \)-overlap curves from the top 10 systems (X-MRI 01 to X-MRI 10). The bottom plane represents \( \delta_2 \)-overlap curves from the last 10 systems (X-MRI 11 to X-MRI 20).

for Sgr A* in this part using the Fisher information matrix. To better estimate the distance between Sgr A* and the solar system, we take account of the external parameter \( R_p \) and constrain it by the gravitational waveforms of the X-MRIs in Table I.

The Fisher information matrix \( \Gamma \) for a GW signal \( h \) parameterized by \( \lambda \) is given by (See Ref[41] for details)

\[
\Gamma_{i,j} = \langle \frac{\partial h}{\partial \lambda_i} \frac{\partial h}{\partial \lambda_j} \rangle, \tag{4.1}
\]

where \( \lambda_i = (a, M, \delta_1, \delta_2, e, p, \iota, R_p) \) is one of the parameters of the X-MRI system. The parameter estimation error \( \Delta \lambda \) due to Gaussian noise has the normal distribution \( N(0, \Gamma^{-1}) \) in the case of high SNR, so the root-mean-square errors in the general case could be approximated as

\[
\Delta \lambda_i = \sqrt{(\Gamma^{-1})_{ii}}. \tag{4.2}
\]

For parameter estimation errors \( \Delta \lambda_i, \Delta \lambda_j \) \((i \neq j)\), the corresponding likelihood is \([41–43]\).

\[
\mathcal{L}(\lambda) \propto e^{-\frac{1}{2} \Gamma_{i,j} \Delta \lambda_i \Delta \lambda_j}. \tag{4.3}
\]

For an X-MRI with eight parameters, we could get a Fisher matrix \( \Gamma_{i,j} \) by applying the results of these parameters’ preliminary constraints to equation (4.1). Element \( \Gamma_{i,j} \) \((i \neq j)\) in the Fisher matrix is the result of the combination of parameter \( \lambda_i \) and parameter \( \lambda_j \). With the Fisher matrix, absolute error \( \Delta \lambda_i \) of any parameter \( \lambda_i \) could be estimated by calculating the equation (4.2). Here we focus on the estimations of Sgr A*’s parameters \((a, M, \delta_1, \delta_2, R_p)\).

By using the Fisher matrix, the parameter estimation accuracy of \((a, M, \delta_1, \delta_2, R_p)\) for the twenty X-MRI signals is shown in Table I. Different X-MRI systems have different abilities to estimate the error accuracy of the same parameter. For the spin of Sgr A*, the relative errors \( \Delta a/a \) estimated by X-MRI 01, X-MRI 02, X-MRI 03, and X-MRI 10 reach a very high precision \( \sim 10^{-6} \). While \( \Delta a/a \) estimated by X-MRI 15 is only \( \sim 10^{-2} \). For the mass of Sgr A*, its relative errors \( \Delta M/M \) estimated by X-MRI 01, X-MRI 02, X-MRI 12, and X-MRI 20 reach \( \sim 10^{-7} \), and \( \Delta M/M \) estimated by X-MRI 15 is \( \sim 10^{-4} \). For the space-time deformations around Sgr A*, \( \Delta \delta_1 \) and \( \Delta \delta_2 \) estimated
by X-MRI 01 reach $\sim 10^{-6}$, while the relative error of these deformations parameter estimated by X-MRI 15 is only $\sim 10^{-2}$. For the distance $R_p$, its relative error $\Delta R_p/R_p$ estimated by X-MRI 01 reaches $\sim 10^{-4}$, while the accuracy of $\Delta R_p/R_p$ estimated by X-MRI 06, X-MRI 07, X-MRI 11, and X-MRI 19 is only $\sim 10^{-2}$. From above analysis, we find that X-MRI 01 have the stringent constraints for the five parameters $(a, M, \delta_1, \delta_2, R_p)$. Therefore, we take X-MRI 01 as an example to present their likelihoods calculated by Eqs. 4.1-4.3. As shown in the Figs. 5-7, it is obvious that the parameter estimation for X-MRI 01 may be affected by any other parameter. Thus, it is reasonable to consider the parameters of one X-MRI signal to estimate any parameter.

We further study the influence of the combination of GW signals on the parameter estimation accuracy. Here we take parameter $\alpha$ as an example to present the data processing. Firstly, we assume that there are $n$ X-MRI systems at the GC. Then, we calculate the Fisher matrices of all these signals to determine the diagonal element $\Gamma_{\alpha,\alpha}$. Then we add these matrices to get the matrix $\Gamma_{\alpha}$.

$$\Gamma_{\alpha} = \Gamma_{\alpha 1} + \Gamma_{\alpha 2} + ... + \Gamma_{\alpha n},$$

with $\Gamma_{\alpha}$, we get the estimation of absolute error from the equation

$$\Delta \alpha = \sqrt{(\Gamma_{\alpha})^{-1}_{\alpha,\alpha}}.$$  

We repeat the steps of the estimation for $\Delta \alpha$, and calculate the absolute errors of $a, M, \delta_1, \delta_2, R_p$. Then we will get the relative errors. The results are shown in Figs. 8. The accuracy gets better as the number of X-MRI increases. With all twenty X-MRI systems in Table I, the estimation accuracy for these parameters all reach higher precision. $\Delta a/a$ reaches the accuracy $\sim 10^{-7}$. $\Delta M/M$ reaches the accuracy $\sim 10^{-8}$. $\Delta \delta_1$ reaches the accuracy $\sim 10^{-6}$. $\Delta \delta_2$ reaches the accuracy $\sim 10^{-6}$. $\Delta R_p/R_p$ reaches the accuracy $\sim 10^{-4}$. The observation number of X-MRI systems does make sense for parameter estimation. With as many X-MRI systems as possible, the parameter estimation accuracy is expected to be improved.

| $\Delta a/a$   | $\Delta M/M$ | $\Delta \delta_1$ | $\Delta \delta_2$ | $\Delta R_p/R_p$ |
|---------------|--------------|-------------------|-------------------|------------------|
| $5.38 \times 10^{-7}$ | $7.02 \times 10^{-8}$ | $2.40 \times 10^{-6}$ | $2.18 \times 10^{-6}$ | $6.05 \times 10^{-4}$ |

TABLE II. Results of parameter estimation accuracy with a group of X-MRIs.

V. CONCLUSIONS AND OUTLOOK

Sgr A* is the nearest MBH for the Solar system. Therefore, it is an ideal laboratory for studying the properties of black holes and testing gravity theories. To study the structure of Sgr A*, we simulate GW signals from the twenty X-MRI systems using the KRZ metric and kludge waveform. Then we apply the Fisher information matrix method to these GW signals. With just one detected X-MRI GW event, we could get relatively accurate estimations of the spin $a$, mass $M$, and deviation parameters $\delta_1$, $\delta_2$. More X-MRI observations would improve the measurement above parameters. The estimations of $\Delta a/a$, $\Delta M/M$, $\Delta \delta_1$, $\Delta \delta_2$ and $\Delta R_p/R_p$ could reach the accuracy of $\sim 10^{-7}$, $\sim 10^{-8}$, $\sim 10^{-6}$, $\sim 10^{-6}$ and $\sim 10^{-4}$ respectively when consider all the twenty X-MRIs.

In practice, other GW signals with similar frequency (e.g. binary WDs and EMRIs) are likely to superimpose the GW signals of X-MRIs at the GC. However, GW signals from X-MRIs, which evolve very slowly, could reach large SNRs, and it is not difficult to separate GW signals from X-MRIs from weaker ones [16]. All in all, once the GW signals from X-MRIs are observed, the fine structure of MBH will be figured out accurately.

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FIG. 5. Likelihoods of $(\delta_1, \delta M/M)$, $(\delta_1, \delta a/a)$, $(\delta_1, \delta_2)$, $(\delta_2, \delta M/M)$, $(\delta_2, \delta a/a)$, $(\delta a/a, \delta M/M)$ derived from the Fisher matrix of X-MRI 01. The black dashed eclipses show the $\sigma$ confidence level. The upper and the four right-hand panels show the marginalized probability distribution for $\delta_1, \delta_2, \delta a/a$ and $\delta M/M$, respectively.

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FIG. 6. Likelihoods of \((\delta a/a, \delta e/e), (\delta a/a, \delta p/p), (\delta a/a, \delta R_p/R_p), (\delta M/M, \delta e/e), (\delta M/M, \delta p/p), (\delta M/M, \delta R_p/R_p), (\delta_1, \delta e/e), (\delta_1, \delta p/p), (\delta_1, \delta R_p/R_p), (\delta_2, \delta e/e), (\delta_2, \delta p/p), (\delta_2, \delta R_p/R_p)\) derived from the Fisher matrix of X-MRI 01. The black dashed eclipses show the 3\(\sigma\) confidence level.

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FIG. 7. Likelihoods of $(\delta e/e, \delta R_p/R_p)$, $(\delta e/e, \delta \iota/\iota)$, $(\delta e/e, \delta p/p)$, $(\delta p/p, \delta R_p/R_p)$, $(\delta p/p, \delta \iota/\iota)$, $(\delta \iota/\iota, \delta R_p/R_p)$ derived from the Fisher matrix of X-MRI 01. The black dashed eclipses show the $3\sigma$ confidence level. The upper and the four right-hand panels show the marginalized probability distribution for $\delta e/e, \delta p/p, \delta \iota/\iota$ and $\delta R_p/R_p$, respectively.

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FIG. 8. The relation between the parameter estimation accuracy and the X-MRI signals number. The parameter in the first plane is $\delta a/a$, in the second plane is $\delta M/M$, in the third plane are $\delta_1$ (red) and $\delta_2$ (blue), in the fourth plane is $\delta R_p/R_p$. 
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