MATHEMATICS OF LEARNING

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ABSTRACT. We study the convergence properties of a pair of learning algorithms (learning with and without memory). This leads us to study the dominant eigenvalue of a class of random matrices. This turns out to be related to the roots of the derivative of random polynomials (generated by picking their roots uniformly at random in the interval [0, 1], although our results extend to other distributions). This, in turn, requires the study of the statistical behavior of the harmonic mean of random variables as above, which leads us to delicate question of the rate of convergence to stable laws and tail estimates for stable laws. The reader can find the proofs of most of the results announced here in [KR2001a].

The original motivation for the work in this paper was provided by the first-named author’s research in learning theory, specifically in various models of language acquisition (see [KNN2001, NKN2001, KN2001]) and more specifically yet by the analysis of the speed of convergence of the memoryless learner algorithm. Curiously, our methods also result in a complete analysis of learning with full memory, as shown in some detail in section 3.2. The setup is described in section 3.1, so here we will just recall the essentials. There is a collection of concepts $R_1, \ldots, R_n$ and words which refer to these concepts, sometimes ambiguously. The teacher generates a stream of words, referring to the concept $R_1$. This is not known to the student, but he must learn by, at each steps, guessing some concept $R_i$ and checking for consistency with the teacher’s input. The memoryless learner algorithm consists of picking a concept $R_i$ at random, and sticking by this choice, until it is proven wrong. At this point another concept is picked randomly, and the procedure repeats. Learning with full memory follows the same general process with the important difference that once a concept is
rejected, the student never goes back to it. It is clear that once the student hits on the right answer $R_1$, this will be his final answer, so the question is then:

*How quickly do the two methods converge to the truth?*

Since the first method is *memoryless*, as the name implies, it is clear that the learning process is a *Markov process*, and as is well-known the convergence rate is determined by the gap between the top (Perron-Frobenius) eigenvalue and the second largest eigenvalue. However, we are also interested in a kind of a *generic* behavior, so we assume that the sizes of overlaps between concepts are *random*, with some (sufficiently regular) probability density function supported in $[0, 1]$, and that the number of concepts is large. This makes the transition matrix random, though of a certain restricted kind, as described in detail in section 3.1. The analysis of convergence speed then comes down to a detailed analysis of the size of the second-largest eigenvalue and also of the properties of the eigenspace decomposition. The analysis for learning with full memory is quite different, but the results have a very similar form. We summarize below:

**Theorem 0.1.** Let $N_\Delta$ be the number of steps it takes for the student to have probability $1 - \Delta$ of learning the concept. Then we have the following estimates for $N_\Delta$:

- if the distribution of overlaps is uniform, or more generally, the density function $f(1 - x)$ at 0 has the form $f(x) = c + O(x^\delta)$, $\delta, c > 0$, then there exist positive constants $C_1, C_2, C'_1, C'_2$ such that
  \[
  \lim_{n \to \infty} P \left( C_1 < \frac{N_\Delta}{|\log \Delta| n \log n} < C_2 \right) = 1
  \]
  for the memoryless algorithm and
  \[
  \lim_{n \to \infty} P \left( C'_1 < \frac{N_\Delta}{(1 - \Delta)^2 n \log n} < C'_2 \right) = 1
  \]
  when learning with full memory;
- if the probability density function $f(1 - x)$ is asymptotic to $cx^\beta + O(x^{\beta + \delta})$, $\delta, \beta > 0$, as $x$ approaches 0, then for the two algorithms we have respectively
  \[
  \lim_{n \to \infty} P \left( c_1 < \frac{N_\Delta}{|\log \Delta| n} < c_2 \right) = 1,
  \]

1Another important learning algorithm is the so-called *batch learner*. This is analysed completely in [R2001].
and
\[
\lim_{n \to \infty} P \left( c_1' < \frac{N\Delta}{(1 - \Delta)^2n} < c_2' \right) = 1
\]
for some positive constants \(c_1, c_2, c_1', c_2';\)

- if the asymptotic behavior is as above, but \(-1 < \beta < 0,\) then

\[
\lim_{x \to \infty} P \left( \frac{1}{x} < \frac{N\Delta}{\log \Delta |n^{1/(1+\beta)}|} < x \right) = 1
\]
for the memoryless learning algorithm, and similarly

\[
\lim_{x \to \infty} P \left( \frac{1}{x} < \frac{N\Delta}{(1 - \Delta)^2n^{1/(1+\beta)}} < x \right) = 1
\]
for learning with full memory.

It should be said that our methods give quite precise estimates on the constants in the asymptotic estimate, but the rate of convergence is rather poor – logarithmic – so these precise bounds are of limited practical importance.

1. **Eigenvalues and Polynomials**

In order to calculate the convergence rate of the learning algorithm described above, we need to study the spectrum of a class of random matrices. The matrices have the following form:

\[
T_{ij} = \begin{cases} a_i & i = j, \\ \frac{(1-a_i)}{n-1} & \text{otherwise}, \end{cases}
\]

where

\[
a_1 = 1, \quad 0 \leq a_i < 1, \quad 2 \leq i \leq n.
\]

Let \(B = \frac{2-n}{n}(I - T),\) so that the eigenvalues of \(T, \lambda_i,\) are related to the eigenvalues of \(B, \mu_i\) by \(\lambda_i = 1 - \left[n/(n-1)\right] \mu_i.\) We show the following amusing

**Lemma 1.1.** Let \(p(x) = (x - x_1) \ldots (x - x_n),\) where \(x_i = 1 - a_i.\) Then the characteristic polynomial \(p_B\) of \(B\) satisfies:

\[
p_B(x) = \frac{x}{n} \frac{dp(x)}{dx}.
\]

From lemma [1.1], the second largest eigenvalue of the matrix \(T, \lambda_*,\) and the smallest root of \(p'(x),\) which we denote as \(\mu_*,\) are related as

\[
\lambda_* = 1 - \frac{n}{n-1} \mu_*.
\]
Therefore, we need to study the distribution of the \textit{smallest} root of \(p'(x)\), given that the smallest root of \(p(x)\) is fixed at 0. Letting the roots of \(p(x)\) be \(0 = x_1 < x_2 < \cdots < x_n\), and letting

\[
H(x_2, \ldots, x_n) = \frac{(n - 1)}{\sum_{i=2}^{n} 1/x_i}
\]

be the \textit{harmonic mean} of the nontrivial roots of \(p(x)\), we have

**Theorem 1.2.** The smallest root \(\mu_*\) of \(p'(x)\) satisfies:

\[
\frac{1}{2} H(x_2, \ldots, x_n) \leq (n - 1) \mu_* \leq H(x_2, \ldots, x_n).
\]

We can see that the study of the distribution of \(\mu_*\) entails the study of the distribution of the asymptotic behavior of the harmonic mean of a sample from a distribution on \([0, 1]\).

2. \textsc{Statistics of the harmonic mean.}

The arithmetic, harmonic, and geometric means are examples of the “conjugate means”, given by

\[
m_{\mathcal{F}}(x_1, \ldots, x_n) = \mathcal{F}^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \mathcal{F}(x_i) \right),
\]

where \(\mathcal{F}(x) = x\) for the arithmetic mean, \(\mathcal{F}(x) = \log(x)\) for the geometric mean, and \(\mathcal{F}(x) = 1/x\) for the harmonic mean. The interesting situation is when \(\mathcal{F}\) has a singularity in the support of the distribution of \(x\), and this case seems to have been studied very little, if at all. Here we will devote ourselves to the study of harmonic mean.

Given \(x_1, \ldots, x_n\) – a sequence of independent, identically distributed in \([0, 1]\) variables (with common probability density function \(f\)), the nonlinear nature of the harmonic mean leads us to consider first the random variable

\[
X_n = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i}
\]

Since the variables \(1/x_i\) are easily seen to have infinite expectation and variance, our prospects seem grim at first blush, but then we notice that the variable \(1/x_i\) falls straight into the framework of the “stable laws” of Lévy – Khintchine, which is briefly presented below.
2.1. **Stable limit laws.** Consider an infinite sequence of independent identically distributed random variables \( y_1, \ldots, y_n, \ldots \), with probability distribution function \( \mathcal{F} \). Typical questions studied in probability theory are the following.

Let \( S_n = \sum_{j=1}^{n} y_j \). How is \( S_n \) distributed? What can we say about the distribution of \( S_n \) as \( n \to \infty \)?

The best known example is one covered by the Central Limit Theorem: if \( \mathcal{F} \) has finite mean \( \mu \) and variance \( \sigma^2 \), then \( (S_n - n\mu)/(\sqrt{n}\sigma) \) converges in distribution to the normal distribution [Norris1940]. Similarly, we say that the variable \( X \) belongs to the domain of attraction of a non-singular distribution \( G \), if there are constants \( a_1, \ldots, a_n, \ldots \) and \( b_1, \ldots, b_n, \ldots \) such that the sequence of variables \( Y_k = a_k S_k - b_k \) converges in distribution to \( G \). It was shown by Lévy and by Khintchine that having a domain of attraction constitutes severe restrictions on the distribution as well as the norming sequences \( \{a_k\} \) and \( \{b_k\} \). To wit, one can always pick \( a_k = k^{-1/\alpha}l(k), \quad 0 < \alpha \leq 2 \), where \( l(k) \) is a slowly varying function (in the sense of Karamata). In that case, \( G \) is called a **stable distribution of exponent** \( \alpha \). If the variable \( y \) belongs to the domain of a stable distribution of exponent \( \alpha > 1 \), then \( y \) has an expectation \( \mu \); just as in the case \( \alpha = 2 \), we can choose \( b_k = k^{1-1/\alpha}\mu \).

When \( \alpha < 1 \), the variable \( y \) has no mean, and it turns out that we can take \( b_k \equiv 0 \); for \( \alpha = 1 \), we can take \( b_n = c\log n \), where \( c \) is some constant depending on \( \mathcal{F} \). In particular, the normal distribution is a stable distribution of exponent 2 (and is unique, up to scale and shift). This is one of the few cases where we have an explicit expression for the density of a stable distribution. The Fourier transforms of the densities are explicitly known; the reader can find them in [FellerV2, Chapter XVII]. The stable distribution of a given exponent are parameterized by parameters \( p, q, C \) defined below:

\[
\lim_{x \to -\infty} \frac{1 - \mathcal{F}(x)}{1 - \mathcal{F}(x) + \mathcal{F}(-x)} = Cp,
\]

\[
\lim_{x \to -\infty} \frac{\mathcal{F}(-x)}{1 - \mathcal{F}(x)} = Cq,
\]

and \( p + q = 1 \). We will say that the stable law is **unbalanced** if \( p = 1 \) or \( q = 1 \) above. This will happen if the support of the variable \( y \) is positive – this will be the only case we will consider in the sequel. Note that this *does not* mean that the stable distribution is supported away from \(-\infty\), though that is true for exponents smaller than 1.

2.2. **Limiting distribution of the harmonic mean.** Which particular stable law comes up in the study of the variable \( X_n \) in \( \mathcal{F} \), depends
on the distribution function $f(x)$. Let us assume that $$f(x) \approx cx^\beta,$$
as $x \to 0$ (for the uniform distribution $\beta = 0, c = 1$). (The notation $b \approx a$ means that $a$ is asymptotically the same as $b$, i.e. there exist constants $c_1, c_2, d_1, d_2$, so that $c_1 b + d_1 \leq a \leq c_2 b + d_2$.) Then we have

**Theorem 2.1.** If $\beta = 0$, then let $Y_n = X_n - \log n$. The variables $Y_n$ converge in distribution to the variable $Y$ distributed in accordance to the unbalanced stable law $G(\alpha)$ with $\alpha = 1$. If $\beta > 0$, then $X_n$ converges in distribution to $\delta(x - \mu)$, where $\mu = E(1/x_i)$ (since the $x_i$ are identically distributed the value of the index $i$ is not relevant). If $-1 < \beta < 0$, then $n^{1/(1+\beta)}X_n$ converges in distribution to the variable $X$ distributed in accordance to the stable law with exponent $\alpha = 1 + \beta$.

**Remark.** In the case when the variables $x_1, \ldots, x_i$ have positive and continuous density at 0, the variables $X_n$ above converge to the Cauchy distribution (the symmetric stable distribution of exponent 1). This is the content of exercise 7.6 in [Durrett91], though the (necessary) condition of positivity of the density at 0 is inadvertently omitted there.

The Theorem 2.1 points us in the right direction, since it allows us to guess the form of the following results ($H_n$ is the harmonic mean of the variables):

**Theorem 2.2.** Let $H_n = 1/X_n$ and $\beta = 0$. Then there exists a constant $C_1$ such that $$\lim_{n \to \infty} E(H_n \log n) = C_1.$$

**Theorem 2.3.** Suppose $\beta > 0$, let $y = 1/x$, and let $\mu$ be the mean of the variable $y$. Then $$\lim_{n \to \infty} E(\mu H_n) = 1.$$ Finally,

**Theorem 2.4.** Suppose $\beta < 0$. Then there exists a constant $C_2$ such that $$E(H_n/n^{1-1/(1+\beta)}) = C_2.$$

We also have the following laws of large numbers:

**Theorem 2.5.** Laws of large numbers for harmonic mean. Let $\beta = 0$ and let $a > 0$. Then $$\lim_{n \to \infty} P(|H_n \log n - C_1| > a) = 0.$$ If $\beta > 0$, and $\mu$ is as in the statement of Theorem 2.3, then
\[ \lim_{n \to \infty} \mathbb{P}(\left| H_n - \frac{1}{\mu} \right| > a) = 0. \]

The proofs of the above results use a variety of estimates; the reader is referred to [KR2001a]. In addition to the laws of large numbers, we also have the following limiting distribution results:

**Theorem 2.6.** For \( \alpha = 1 \), the random variable \( \log n (H_n \log n - \mathfrak{c}_1) \) converges to \( 1 - G(-x/\mathfrak{c}_1^2) \), where \( G \) is the limiting distribution (of exponent \( \alpha = 1 \)) of variables \( Y_n = X_n - c \log n \) and \( \mathfrak{c}_1 = 1/c \).

**Theorem 2.7.** For \( \alpha > 1 \), the random variable \( n^{1-\alpha}(H_n - 1/2) \) converges in distribution to the variable \( \mathcal{H} \) with the distribution function \( 1 - G(-x\mathfrak{c}_2) \), where \( G \) is the unbalanced stable law of exponent \( \alpha \).

**Theorem 2.8.** For \( 0 < \alpha < 1 \), the random variable \( H_n/n^{1-\alpha} \) converges in distribution to the variable \( \mathcal{H} \), with the distribution function \( 1 - G(1/x) \), where \( G \) is the distribution function of the unbalanced stable law of exponent \( \alpha \).

3. A pair of learning algorithms

3.1. **The memoryless learner algorithm.** Suppose there are \( n \) intersecting sets, \( R_1, \ldots, R_n \), and \( n \) probability measures, \( \nu_1, \ldots, \nu_n \), each defined on its set (so that \( \nu_i(R_i) = 1 \)). The similarity matrix \( A \) is given by \( a_{ij} = \nu_i(R_j) \). It follows that \( 0 \leq a_{ij} \leq 1 \) and \( a_{ii} = 1 \) for all \( i \) and \( j \).

Let us consider a typical problem of learning theory. A teacher generates a sequence of points which belong to one of these sets, say to set \( R_1 \). The total length of the sequence is \( N \). The learner’s task is to guess what set is the teacher’s set after receiving \( N \) points. For simplicity we assume here that \( a_{ij} < 1 \) for \( i \neq j \), which means that no set is a subset of another set. Many different algorithms are available to the learner, one given by the so-called memoryless learner algorithm [Niyogi1998], a favorite with learning theorists. It works in the following way. The learner starts by (randomly) choosing one of the \( n \) sets as an initial state. Then \( N \) sample points are received from the teacher. For each sampling, the learner checks if the point belongs to its current set. If it does, no action is taken; otherwise, the learner randomly picks a different set. The initial probability distribution of the learner is uniform: \( p^{(0)} = (1/n, \ldots, 1/n)^T \), i.e. each of the sets has the same chance to be picked at the initial moment. The discrete time evolution of the vector \( p^{(t)} \) is a Markov process with transition matrix \( T \), which depends on
the similarity matrix, \( A \). The transition matrix is given by Eqs. (1), (2) with \( a_i = \nu_i(R_i) \).

After \( N \) samplings, the probability of learning the correct set is given by \( Q_{11} = [((p(0))^T T^N)]_{11} \). It is clear that the convergence rate of the memoryless algorithm can be determined if we study properties of the matrix \( T \). We are interested in the rate of convergence as a function of \( n \), the number of possible sets.

We define the convergence rate of the method as the difference \( 1 - Q_{11} \). In order to evaluate the convergence rate of the memoryless learner algorithm, let us represent the matrix \( T \) as

\[
T^N = V \Lambda^N W.
\]

Let us arrange the eigenvalues so that \( \lambda_1 = 1 \) and \( \lambda_2 \equiv \lambda_* \) is the second largest eigenvalue. If \( N \) is large, we have \( \lambda_i^N \ll \lambda_*^N \) for all \( i \geq 3 \), so only the first two largest eigenvalues need to be taken into account. This means that in order to evaluate \( T^N \) we only need the following eigenvectors: \( v_1 = (1/n, 1/n, \ldots, 1/n)^T \), \( v_2 \), \( w_1 = (n, 0, 0, \ldots, 0) \), and \( w_2 \). The result is:

\[
Q_{11} = 1 - C \lambda_*^N,
\]

where \( C = -\sum_{j=1}^n [v_2]_j [w_2]_j / n \). It follows, therefore, that the convergence rate of the memoryless learner algorithm can be estimated if we estimate \( \lambda_* \) and \( C \). It turns out that once we understand \( \lambda_* \), we can also estimate \( C \).

Our results can be summarized as follows. For large \( n \), the quantity \( C \) is bounded from above and below by some constants. From formulas (3) and (4) we can see that in order for the learner to pick up the correct set with probability \( 1 - \Delta \), we need to have at least

\[
N_\Delta \sim |\log \Delta| / \mu_*
\]

sampling events (Theorem 2.5 tells us that \( \mu_* = o(1/n) \), and so we have the right to replace log(1 - \( \mu_* \)) by \( -\mu_* \)). Using the relationship between \( \mu_* \) and the harmonic mean (5), and our results for \( H_n \) from Theorem 2.5, we obtain the following estimate:

\[
N_\Delta \sim |\log \Delta| h(n),
\]

where \( h(n) \) is \( n \log n \) if the overlaps are uniformly distributed (in other words, the entries \( a_{ij} \) of the similarity matrix, as random variables, are
uniformly distributed in $[0,1)$, and $h(n)$ is $n$ if the density of overlaps at 1 goes to 0. Estimate (11) should be understood in the sense that the right hand side of (10) converges in probability to the right hand side of (11). If the density grows at 1 as $(1-x)^\beta$, $-1 < \beta < 0$, then
\[
\lim_{x \to \infty} P\left(\frac{1}{x} < \frac{N_\Delta}{\log \Delta |n^{1/(1+\beta)}|} < x\right) = 1.
\]

3.2. A better algorithm. Consider the following improvement on the previous learning algorithm: the student keeps a list of the sets he has not rejected, and when the time comes to switch, he picks uniformly among those sets only. It is clear that this algorithm ("learning with full memory") should perform better than the memoryless learner algorithm described in the last section, but how much better?

Since the analysis is quite simple, we present it here. There are two questions which need to be answered (we always assume that the correct answer is the first set, $G_1$):

**Question 1.** Suppose the student has picked the set $G_i$, $i \neq 1$. What is the expected number of turns before he is forced to reject $G_i$ and jump to a different set?

**Question 2.** What is the probability that the student will change his mind exactly $k$ times before guessing the right answer?

We answer the second question first, by

**Lemma 3.1.** The probability that the set $G_1$ is encountered on the $k$-th turn is independent of $k$ (and so equals $1/n$.)

**Proof.** Suppose the student starts by picking a set $G_{i_1}$ at random, and then keeps picking sets $G_{i_2}, G_{i_3}, \ldots, G_{i_n}$, until there are none left, and making sure never to repeat a set. The sequence $i_1, \ldots, i_n$ is a permutation of the sequence $1, \ldots, n$, and it is clear (for reasons of symmetry) that every permutation is equally likely. Since for any $k$, precisely $(n-1)!$ permutations have 1 in the $k$-th position, the lemma is proved. \hfill \Box

Question 1 is also easily answered, by

**Lemma 3.2.** If $\nu_1(G_i) = a_i$, then the expected number of turns before switching is $1/(1 - a_i)$.

**Proof.** Let $P_k$ be the probability of switching on the $k$-th step or earlier. Then we have the equation:
\[
P_{k+1} = P_k + (1 - P_k)(1 - a_i) = a_i P_k + (1 - a_i).
\]
Since $P_0 = 0$, it is easy to check that $P_j = 1 - a_j^i$. If $p_k$ is the probability of switching on the $k$-th turn, then $p_k = a_k - 1 - a_k$, and the expected time of switching is

$$\sum_{j=1}^{\infty} j(a_j^i - a_i^j) = \sum_{j=0}^{\infty} a_i^j = \frac{1}{1 - a_i},$$

the first equality being obtained by telescoping the sum. \(\square\)

From the two lemmas, it follows that given the probabilities $a_2, \ldots, a_n$, the expected time taken by the improved learner is

$$T = \frac{1}{n} \sum_{k=1}^{n-1} \left( \frac{n-2}{k} \right) \sum_{i \in S_k} \frac{1}{1 - a_i},$$

where the middle summation is over all subsets $S_k$ of $2, \ldots, n$ which have size $k$. Since for any $i$, the number of subsets of $2, \ldots, n$ of size $k$ containing $i$ equals $(n-2)_{k-1}$, the above expression can be rewritten as

$$T = \frac{1}{n} \sum_{k=1}^{n-1} \left( \frac{n-2}{k} \right) \sum_{i \in S_k} \frac{1}{1 - a_i}$$

$$= \sum_{i=2}^{n} \frac{1}{1 - a_i} \sum_{k=1}^{n-1} \frac{k}{n(n-1)} = \frac{1}{2} \sum_{i=2}^{n} \frac{1}{1 - a_i} = \frac{n-1}{2H_{n-1}},$$

where $H_{n-1}$ is defined in (4) with $x_i = 1 - a_i$. These computations can be easily adapted to solve the following problem: suppose that we want to be $1 - \Delta$ sure of getting to the right answer. How many steps do we need? Notice that we will need to take $(1 - \Delta)n$ jumps, so the computation as above gives us:

$$N_\Delta = \frac{1}{n} \sum_{k=1}^{(1-\Delta)n} \left( \frac{n-2}{k} \right) \sum_{i=2}^{n} \frac{1}{1 - a_i}$$

$$= \sum_{i=2}^{n} \frac{1}{1 - a_i} \sum_{k=1}^{(1-\Delta)n} \frac{k}{n(n-1)} \to \frac{n(1 - \Delta)^2}{2H_n}.$$
REFERENCES

[Durett91] Durrett, R. (1991) Probability: Theory and examples., Wadsworth and Brooks/Cole.

[FellerV2] Feller, W. (1971) An Introduction to Probability Theory and its Applications, vol. 2, John Wiley and Sons.

[IbLin1971] Ibragimov and Linnik. Independent and Stationary sequences of Random Variables, Wolters-Noordhoff Publishing, Groningen, 1971.

[KNN2001] Komarova, N. L., Niyogi, P. and Nowak, M. A. (2001) The evolutionary dynamics of grammar acquisition, J. Theor. Biology, 209(1), pp. 43-59.

[KN2001] Komarova, N. L. and Nowak, M. A. (2001) Natural selection of the critical period for grammar acquisition, Proc. Royal Soc. B, to appear.

[KR2001a] Komarova, N. L. and Rivin, I. (2001) Harmonic mean, random polynomials and stochastic matrices, preprint.

[Niyogi1998] Niyogi, P. (1998). The Informational Complexity of Learning. Boston: Kluwer.

[Norris1940] Norris, N. (1940) The standard errors of the geometric and harmonic means and their applications to index numbers, Ann. Math. Statistics 11, pp 445-448.

[NKN2001] Nowak, M. A., Komarova, N. L., Niyogi, P. (2001) Evolution of universal grammar, Science 291, 114-118.

[R2001] Rivin, I. (2001) Yet another zeta function and learning, ArXiv.org preprint cs.LG/0107033.

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