A prime geodesic theorem for higher rank spaces

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Abstract. A prime geodesic theorem for regular geodesics in a higher rank locally symmetric space is proved. An application to class numbers is given. The proof relies on a Lefschetz formula for higher rank torus actions.

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Introduction

The prime geodesic theorem gives a growth asymptotic for the number of closed geodesics counted by their lengths. It has hitherto only been proven for manifolds of strictly negative curvature. For manifolds containing higher dimensional flats it is not a priori clear what a prime geodesic theorem might look like. In this paper we give such a theorem for locally symmetric spaces of arbitrary rank, i.e., they may contain higher dimensional flats. The regular geodesics in such a space give points in a higher dimensional Weyl cone. Therefore it is natural to expect a prime geodesic theorem which gives asymptotics in several variables. We introduce a new analytical function which could be viewed as higher dimensional analogue of the logarithmic derivative of the Selberg zeta function. Of this we get analytic continuation only in a certain translate of the positive Weyl chamber, but this actually suffices to prove the prime geodesic theorem.

We describe the main result of the paper. One of the various equivalent formulations of the prime geodesic theorem for locally symmetric spaces of rank one is the following. Let $\tilde{X}$ be a compact locally symmetric space with universal covering of rank one. For $T > 0$ let

$$\psi(T) = \sum_{c : l(c) \leq T} l(c_0).$$

Here the sum runs over all closed geodesics $c$ such that $e^{l(c)} \leq T$, where $l(c)$ is the length of the geodesic $c$, and $c_0$ is the prime geodesic underlying $c$. Then, under a suitable scaling of the metric, as $T \to \infty$,

$$\psi(T) \sim T.$$ 

We now replace the space $\tilde{X}$ by an arbitrary compact locally symmetric space which is a quotient of a globally symmetric space $X = G/K$ where $G$ is a semisimple Lie group of split-rank $r$ and $K$ a maximal compact subgroup. A closed geodesic $c$ gives rise to a point $a_c$ in the closure of the negative Weyl chamber $A^{-}$ of a maximal split torus $A$. We assume that the geodesic is regular, i.e., the element $a_c$ lies in the interior of the Weyl chamber. (In rank one every geodesic is regular.) For $T_1, \ldots, T_r > 0$ let

$$\psi(T_1, \ldots, T_r) = \sum_{c : a_{c,j} \leq T_j} \lambda_c,$$
where $\lambda_c$ is the volume of the unique maximal flat $c$ lies in and $a_{c,j}$ are the coordinates of $a_c$ with respect to a canonical coordinate system given by the roots. The sum runs over all regular closed geodesics modulo homotopy. The main result of this paper is that, as $T_j$ tends to infinity for every $j$, 

$$\psi(T_1, \ldots T_r) \sim T_1 \cdots T_r.$$ 

The proof is based on a Lefschetz formula that generalizes the pioneering work of Andreas Juhl [15]. The formula can be interpreted as a dynamical Lefschetz formula as follows. The choice of a negative Weyl chamber $A^-$ induces a stable foliation $\mathcal{F}$. The torus $A$ acts on the reduced tangential cohomology $\bar{H}^\bullet(\mathcal{F})$ of this foliation and the Lefschetz formula states that there is an identity of distributions on the negative Weyl chamber $A^-$,

$$\sum_{q=0}^{\text{rank} \mathcal{F}} (-1)^q \text{tr} (a | \bar{H}^q(\mathcal{F})) = \sum_c \text{ind}(c) \delta(a_c),$$

where the sum on the right hand side runs over all regular closed geodesics, $\delta(a_c)$ is the $\delta$-distribution at $a_c$ and $\text{ind}(c)$ is a certain index which incorporates monodromy data of $c$.

A specific test function is constructed so that the local side of the Lefschetz formula gives a high derivative of the generalized Dirichlet series $L(s) = \sum_c \text{ind}(c) a_c^s$. The Lefschetz formula allows for a detailed analysis of this derivative of $L(s)$. By methods of analytical number theory one finally gets the prime geodesic theorem.

The methods of this paper will not give a similar result for non-regular geodesics. Though there is a Lefschetz formula for non-regular geodesics, in general it contains Euler characteristics which may be positive or negative. The Wiener-Ikehara Theorem used, however, relies on positivity.

In an appendix we give an application to class numbers in totally real number fields.

1 The Lefschetz formula

In this section we prove a Lefschetz formula for higher rank groups. We give a more general version then actually needed in section 2 as we allow twists
by nontrivial $M$-representations $\sigma$ (see below).

Let $G$ be a connected semisimple Lie group with finite center and choose a maximal compact subgroup $K$ with Cartan involution $\theta$, i.e., $K$ is the group of fixed points of $\theta$. Let $P$ be a minimal parabolic subgroup with Langlands decomposition $P = MAN$. Modulo conjugation we can assume that $A$ and $M$ are stable under $\theta$. Then $A$ is a maximal split torus of $G$ and $M$ is a subgroup of $K$. The centralizer of $A$ is $AM$. Let $W(A, G)$ be the Weyl group of $A$, i.e. $W$ is the quotient of the normalizer of $A$ by the centralizer. This is a finite group acting on $A$.

We have to fix Haar measures. We use the normalization of Harish-Chandra \cite{10}. Note that this normalization depends on the choice of an invariant bilinear form $B$ on $g_{\mathbb{R}}$ which we keep at our disposal until later. Changing $B$ amounts to scaling the metric of the symmetric space. Note further that in this normalization of Haar measures the compact groups $K$ and $M$ have volume 1.

We write $g_{\mathbb{R}}, t_{\mathbb{R}}, a_{\mathbb{R}}, m_{\mathbb{R}}, n_{\mathbb{R}}$ for the real Lie algebras of $G, K, A, M, N$ and $g, k, a, m, n$ for their complexifications. $U(g)$ is the universal enveloping algebra of $g$. This algebra is isomorphic to the algebra of all left invariant differential operators on $G$ with complex coefficients. Pick a Cartan subalgebra $t$ of $m$. Then $h = a \oplus t$ is a Cartan subalgebra of $g$. Let $W(h, g)$ be the corresponding absolute Weyl group.

Let $a^*$ denote the dual space of the complex vector space $a$. Let $a_{\mathbb{R}}^*$ be the real dual of $a_{\mathbb{R}}$. We identify $a_{\mathbb{R}}^*$ with the real vector space of all $\lambda \in a^*$ that map $a_{\mathbb{R}}$ to $\mathbb{R}$. Let $\Phi \subset a^*$ be the set of all roots of the pair $(a, g)$ and let $\Phi^+$ be the subset of positive roots with respect to $P$. Let $\Delta \subset \Phi^+$ be the set of simple roots. Then $\Delta$ is a basis of $a^*$. The open negative Weyl chamber $a^-_{\mathbb{R}} \subset a_{\mathbb{R}}$ is the cone of all $X \in a_{\mathbb{R}}$ with $\alpha(X) < 0$ for every $\alpha \in \Delta$. Let $a^-_{\mathbb{R}}$ be the closure of $a^-_{\mathbb{R}}$. The $W(A, G)$-translates $wa^-_{\mathbb{R}}$ of $a^-_{\mathbb{R}}$ are pairwise disjoint and their union equals $a_{\mathbb{R}}$ minus a finite number of hyperplanes. This is called the regular set,

$$a^-_{\mathbb{R}} \overset{\text{def}}{=} \bigcup_{w \in W(A, G)} wa^-_{\mathbb{R}}.$$

Let $A^\text{reg} = \exp(a^\text{reg}_{\mathbb{R}})$ be the regular set in $A$. The elements are called the regular elements of $A$. A given $a \in A$ lies in $A^\text{reg}$ if and only if the centralizer of $a$ in $G$ equals the centralizer of $A$ in $G$ which is $AM$. 
The bilinear form $B$ is indefinite on $\mathfrak{g}_\mathbb{R}$, but the form
\[ \langle X, Y \rangle \overset{\text{def}}{=} -B(X, \theta(Y)) \]
is positive definite, i.e., an inner product on $\mathfrak{g}_\mathbb{R}$. We extend it to an inner product on the complexification $\mathfrak{g}$. Let $\|X\| = \sqrt{\langle X, X \rangle}$ be the corresponding norm. The form $B$, being nondegenerate, identifies $\mathfrak{g}$ to its dual space $\mathfrak{g}^*$. In this way we also define an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm on $\mathfrak{g}^*$. Furthermore, if $V \subset \mathfrak{g}$ is any subspace on which $B$ is nondegenerate, then $B$ gives an identification of $V^*$ with $V$ and so one gets an inner product and a norm on $V^*$. This in particular applies to $V = \mathfrak{h}$, a Cartan subalgebra of $\mathfrak{g}$, which is defined over $\mathbb{R}$.

Let $\Gamma \subset G$ be a discrete, cocompact, torsion-free subgroup. We are interested in the closed geodesics on the locally symmetric space $X_\Gamma = \Gamma \backslash X = \Gamma \backslash G/K$. Every such geodesic $c$ lifts to a $\Gamma$-orbit of geodesics on $X$ and gives a $\Gamma$-conjugacy class $[\gamma_c]$ of elements closing the particular geodesics. This induces a bijection between the set of all homotopy classes of closed geodesics in $X_\Gamma$ and the set of all non-trivial conjugacy classes in $\Gamma$.

An element $am$ of $AM$ is called split regular if $a$ is regular in $A$. An element $\gamma$ of $\Gamma$ is called split regular if $\gamma$ is in $G$ conjugate to a split regular element $a_\gamma m_\gamma$ of $AM$. In that case we may (and do) assume that $a_\gamma$ lies in the negative Weyl chamber $A^- = \exp(\mathfrak{a}_{\mathbb{R}}^-)$ in $A$. Let $\mathcal{E}(\Gamma)$ denote the set of all conjugacy classes in $\Gamma$ that consist of split regular elements. Via the above correspondence the set $\mathcal{E}(\Gamma)$ can be identified with the set of all homotopy classes of regular closed geodesics in $X_\Gamma$, $[\mathcal{L}]$.

Let $[\gamma] \in \mathcal{E}(\Gamma)$. There is a closed geodesic $c$ in the Riemannian manifold $\Gamma \backslash G/K$ which gets closed by $\gamma$. This means that there is a lift $\tilde{c}$ to the universal covering $\tilde{G}/K$ which is preserved by $\gamma$ and $\gamma$ acts on $\tilde{c}$ by a translation. The closed geodesic $c$ is not unique in general but lies in a unique maximal flat submanifold $F_\gamma$ of $\Gamma \backslash G/K$. Let $\lambda_\gamma$ be the volume of that flat, 
\[ \lambda_\gamma \overset{\text{def}}{=} \text{vol}(F_\gamma). \]

Let $\Gamma_\gamma$ and $G_\gamma$ denote the centralizers of $\gamma$ in $\Gamma$ and $G$ respectively. The flat $F_\gamma$ is the image of $\Gamma_\gamma \backslash G_\gamma/K_\gamma$ in $\Gamma \backslash G/K$, where we assume that $K$ is chosen so that $K_\gamma = G_\gamma \cap K$ is a maximal compact subgroup of $G_\gamma$. In
Harish-Chandra’s normalization $G_\gamma$ is equipped with the Haar measure that satisfies $\int_{G_\gamma} = \int_{G_\gamma/K_\gamma}$, where $K_\gamma$ has the Haar measure with $\text{vol}(K_\gamma) = 1$ and $G_\gamma/K_\gamma$ gets the measure induced by the metric of $G/K$. Therefore

$$\lambda_\gamma = \text{vol}(\Gamma_\gamma \backslash G_\gamma / K_\gamma) = \text{vol}(\Gamma_\gamma \backslash G_\gamma).$$

Note that for not maximally split elements $\gamma$ the relation between $\lambda_\gamma$ and $\text{vol}(\Gamma_\gamma \backslash G_\gamma)$ is more complicated [4, 6].

Let $n$ denote the complexified Lie algebra of $N$. For any $n$-module $V$ let $H_q(n, V)$ and $H^q(n, V)$ for $q = 0, \ldots, \dim n$ be the Lie algebra homology and cohomology [2]. Let $\hat{G}$ denote the unitary dual of $G$, i.e., the set of isomorphism classes of irreducible unitary representations of $G$. For $\pi \in \hat{G}$ let $\pi_K$ be the $(\mathfrak{g}, K)$-module of $K$-finite vectors. If $\pi \in \hat{G}$, then $H_q(n, \pi_K)$ and $H^q(n, \pi_K)$ are finite dimensional $AM$-modules [12]. Note that they are a priori only are $(\mathfrak{a} \oplus \mathfrak{m}, M)$-modules, but since $A$ is isomorphic to its Lie algebra they are $AM$-modules.

Note that $AM$ acts on the Lie algebra $n$ of $N$ by the adjoint representation.

Let $[\gamma] \in \mathcal{E}(\Gamma)$. Since $a_\gamma \in A^-$ it follows that every eigenvalue of $a_\gamma m_\gamma$ on $n$ is of absolute value $< 1$. Therefore $\det(1 - a_\gamma m_\gamma | n) \neq 0$.

For $[\gamma] \in \mathcal{E}(\Gamma)$ let

$$\text{ind}(\gamma) = \frac{\lambda_\gamma}{\det(1 - a_\gamma m_\gamma | n)} > 0.$$  

Since $\Gamma$ is cocompact, the unitary $G$-representation on $L^2(\Gamma \backslash G)$ splits discretely with finite multiplicities

$$L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \hat{G}} N_{\Gamma}(\pi)\pi,$$

where $N_{\Gamma}(\pi)$ is a non-negative integer and $\hat{G}$ is the unitary dual of $G$. Fix a finite dimensional irreducible representation $\sigma$ of $M$ and denote by $\hat{\sigma}$ the dual representation. A quasi-character of $A$ is a continuous group homomorphism to $\mathbb{C}^\times$. Via differentiation the set of quasi-characters can be identified with the dual space $\mathfrak{a}^*$. For $\lambda \in \mathfrak{a}^*$ we write $a \mapsto a^\lambda$ for the corresponding quasicharacter on $A$. We denote by $\rho \in \mathfrak{a}^*$ the modular shift with respect to $P$, i.e., for $a \in A$ we have $\det(a | n) = a^{2\rho}$. 
For a complex vector space $V$ on which $A$ acts linearly and $\lambda \in \mathfrak{a}^*$ let $(V)_{\lambda}$ denote the generalized $(\lambda + \rho)$-eigenspace, i.e.,

$$(V)_{\lambda} = \{v \in V \mid (a - a^{\lambda+\rho}Id)^n v = 0 \text{ for some } n \in \mathbb{N}\}.$$

Since $H^p(n, \pi_K)$ is finite dimensional, the Jordan Normal Form Theorem implies that

$$H^p(n, \pi_K) = \bigoplus_{\nu \in \mathfrak{a}^*} H^p(n, \pi_k)_{\nu}.$$

Let $T$ be a Cartan subgroup of $M$ and let $\mathfrak{t}$ be its complex Lie algebra. Then $AT$ is a Cartan subgroup of $G$. Let $\Lambda_\pi \in (\mathfrak{a} \oplus \mathfrak{t})^*$ be a representative of the infinitesimal character of $\pi$. By Corollary 3.32 of [12] it follows,

$$H^p(n, \pi_K) = \bigoplus_{\nu = w_{\Lambda_\pi}|_{\mathfrak{a}}} H^p(n, \pi_K)_\nu,$$

where $w$ ranges over $W(g, h)$.

**Lemma 1.1** For $0 \leq p \leq d = \dim(n)$ we have

$$H^p(n, \pi_K) \cong H^{d-p}(n, \pi_K) \otimes \det(n),$$

where the determinant of a finite dimensional space is the top exterior power.

So $\det(n)$ is a one dimensional $AM$-module on which $AM$ acts via the quasi-character $am \mapsto \det(am|n) = a^{2\rho}$. This in particular implies

$$H^p(n, \pi_K) = \bigoplus_{\nu = w_{\Lambda_\pi}|_{\mathfrak{a}}} H^p(n, \pi_K)_{\nu-2\rho}.$$

**Proof:** The first part follows straight from the definition of Lie algebra cohomology. The second part by Corollary 3.32 of [12].

For $\lambda \in \mathfrak{a}^*$ and $\pi \in \hat{G}$ let

$$m^\sigma_\lambda(\pi) = \sum_{q=0}^{\dim n} (-1)^{q+\dim n} \dim (H^q(n, \pi_K)_{\lambda} \otimes \bar{\sigma})^M,$$

where the superscript $M$ indicates the subspace of $M$-invariants. Then $m^\sigma_\lambda(\pi)$ is an integer and by the above, the set of $\lambda$ for which $m^\sigma_\lambda(\pi) \neq 0$ for a given $\pi$ has at most $|W(g, h)|$ many elements.

For $\mu \in \mathfrak{a}^*$ and $j \in \mathbb{N}$ let $C^{j, \mu, -}(A)$ denote the space of functions $\varphi$ on $A$ which
are \( j \)-times continuously differentiable on \( A \),

- are zero outside \( A^- \),

- are such that \( a^{-\mu}D\varphi(a) \) is bounded on \( A \) for every invariant differential operator \( D \) on \( A \) of degree \( \leq j \).

For every invariant differential operator \( D \) of degree \( \leq j \) let \( N_D(\varphi) = \sup_{a \in A} |a^{-\mu}D\varphi(a)| \). Then \( N_D \) is a seminorm. Let \( D_1, \ldots, D_n \) be a basis of the space of invariant differential operators of degree \( \leq j \), then \( N(\varphi) = \sum_{j=1}^n N_{D_j}(\varphi) \) is a norm that makes \( C^{j,\mu,-}(A) \) into a Banach space. A different choice of basis will give an equivalent norm.

**Theorem 1.2 (Lefschetz Formula)**

There exists \( j \in \mathbb{N} \) and \( \mu \in \mathfrak{a}^* \) such that for any \( \varphi \in C^{j,\mu,-}(A) \) we have

\[
\sum_{\pi \in \hat{G}} N_\pi(\varphi) \sum_{\lambda \in \mathfrak{a}^*} m^\pi_\lambda(\varphi) \int_{A^-} \varphi(a) a^{\lambda+\rho} da = \sum_{[\gamma] \in \mathcal{E}(\Gamma)} \text{ind}(\gamma) \text{tr} \sigma(\rho_\gamma) \varphi(a_\gamma),
\]

where all sums and integrals converge absolutely. The inner sum on the left is always finite, more precisely it has length \( \leq |W(h,g)| \). The left hand side is called the global side and the other the local side of the Lefschetz Formula. Both sides of the formula give a continuous linear functional on the Banach space \( C^{j,\mu,-}(A) \).

**Proof:** The Selberg trace formula \cite{Selberg} says that for a compactly supported function \( f \) on \( G \) that is \((\dim G + 1)\)-times continuously differentiable one has

\[
\sum_{\pi \in \hat{G}} N_\pi(\pi) \text{tr} \pi(f) = \sum_{[\gamma]} \text{vol}(\Gamma \backslash G_\gamma) \mathcal{O}_\gamma(f),
\]

where the sum on the right hand side runs over the conjugacy classes of \( \Gamma \), and \( \mathcal{O}_\gamma(f) \) is the orbital integral \( \mathcal{O}_\gamma(f) = \int_{G/G_\gamma} f(x\gamma x^{-1})dx \). We need to extend the trace formula beyond compactly supported functions.

For \( d \in \mathbb{N} \) let \( C^{2d}(G) \) be the space of all functions \( f \) on \( G \) which are \( 2d \)-times continuously differentiable and satisfy \( Df \in L^1(G) \) for every \( D \in U(g) \) with \( \deg D \leq 2d \).
Lemma 1.3 Let \( d > \frac{\dim G}{2} \) and let \( f \in C^{2d}(G) \). Then the trace formula is valid for \( f \).

Proof: In [4] Lemma 2.6 the lemma is proven in the following two cases:

(a) \( f \geq 0 \), or

(b) the geometric side of the trace formula converges with \( f \) replaced by \( |f| \).

Now let \( f \) be arbitrary. First there is \( \tilde{f} \in C^{2d}(G) \) such that \( \tilde{f} \geq |f| \). To construct such a function one proceeds as follows. First choose a smooth function \( b : \mathbb{R} \to [0, \infty) \) with \( b(x) = |x| \) for \( |x| \geq 1 \) and \( b(x) \geq |x| \) for every \( x \). Next let \( b(\varepsilon, x) = \varepsilon b(x/\varepsilon) \) for \( \varepsilon > 0 \). Choose a function \( h \in C^{2d}(G) \) with \( 0 < h(x) \leq 1 \) and let \( \tilde{f}(x) = b(h(x), f(x)) \). Then \( \tilde{f} \in C^{2d}(G) \) and \( \tilde{f} \geq |f| \).

By (a) the trace formula is valid for \( \tilde{f} \), hence the geometric side converges for \( \tilde{f} \), so it converges for \( |f| \), so, by (b) the trace formula is valid for \( f \). □

We now take \( j \in \mathbb{N} \) and \( \mu \in \mathfrak{a}^* \) and keep these at our disposal. We let \( \varphi \in C^{j,\mu,-}(A) \) and we construct a test function \( f \) which has the following orbital integrals for semisimple \( y \in G \). If \( y \) is not conjugate to an element of \( A^{-M} \), then \( O_y(f) = 0 \). For \( am \in A^{-M} \) one has

\[
O_{am}(f) = \frac{\text{tr} \sigma(m)}{\det(1 - am | n)} \varphi(a).
\]

The construction of such a function \( f \) is rather straightforward. One chooses a function \( \eta \in C^\infty_c(N) \) with \( \eta \geq 0 \) and \( \int_N \eta(n) dn = 1 \). Then one sets

\[
f(kn am (kn)^{-1}) = \eta(n) \text{tr} \sigma(m) \frac{\varphi(a)}{\det(1 - am | n)},
\]

for \( am \in A^{-M}, k \in K \) and \( n \in N \); further, one sets \( f(x) = 0 \) if \( f \) is not conjugate to an element of \( A^{-M} \). We have to show that \( f \) is well defined. This follows from the next lemma.

Lemma 1.4 Let \( am, a'm' \in A^{-M} \) and \( x \in G \) suppose that \( a'm' = xamx^{-1} \). Then \( a = a' \) and \( x \in AM \).
Proof: Every eigenvalue of $\text{Ad}(am)$ on $n$ is less than 1 in absolute value. The space $n$ is the maximal subspace of $g$ with this property. The same holds for $am$ replaced by $a'm'$. Therefore $\text{Ad}(x)$ must preserve $n$ and so $x$ lies in the normalizer of $n$ which is $P = NMA$. Suppose $x = nm_1a_1$ and write $\hat{m} = m_1mm^{-1}$. Then $xamx^{-1} = namn^{-1} = a\hat{m}((a\hat{m})^{-1}nam)n^{-1}$. Since this lies in $A^{-}M$ and $AM \cap N = \{1\}$ we infer that $(a\hat{m})^{-1}nam = n$, which implies that $n = 1$. The lemma follows. □

Let $d \in \mathbb{N}$. We will show that for $\text{Re}(\mu)$ and $j$ sufficiently large the function $f$ lies in $C^{2d}(G)$.

Note that the factor $1/\det(1-am|n)$ has a pole at $am = 1$ of order equal to the dimension of $n$. To make this more precise let $r = \text{dim } A$ and let $\beta_1, \ldots, \beta_r$ be the simple roots of $(A,P)$. There are $q_1, \ldots, q_r \in \mathbb{N}$ such that $2\rho = q_1\beta_1 + \cdots + q_r\beta_r$. Define coordinates $a_1, \ldots, a_r$ on $A$ by $a_j = -\beta_j(\log a)$. Then $a$ lies in $A^{-}$ if and only if the coordinates $a_j$ are all $> 0$. The function on $A$ given by $a \mapsto a^{q_1 \cdots a^q r}$ extends continuously to the boundary of $A^{-}$, and the $q_j$ are minimal with this property. So, if we assume $j > \text{dim } n$, the function $f$ will on $AM$ be $(j - 1 - \text{dim } n)$-times continuously differentiable. To investigate the differentiability on $G$ we need to look at the conjugation map.

Consider the map

$$F: K \times N \times M \times A \to G \quad (k, n, m, a) \mapsto knam(kn)^{-1}.$$

Then $f$ is a $j - 1 - \text{dim}(n)$-times continuously differentiable function on $K \times N \times M \times A$ which factors over $F$. To compute the order of differentiability as a function on $G$ we have to take into account the zeroes of the differential of $F$. So we compute the differential $F_{*}$ of $F$ which we view as a map on tangent spaces. Let at first $X \in \mathfrak{t}$, then

$$F_{*}(X)f(knam(kn)^{-1}) = \frac{d}{dt}|_{t=0}f(k \exp(tX)namn^{-1}\exp(-tX)kn^{-1}),$$

which implies the equality

$$F_{*}(X)x = (\text{Ad}(k)(\text{Ad}(am)^{-1}n^{-1}) - 1)X_x,$$

when $x$ equals $knam(kn)^{-1}$. Similarly for $X \in \mathfrak{n}$ we get that

$$F_{*}(X)x = (\text{Ad}(kn)(\text{Ad}(am)^{-1}) - 1)X_x.$$
and for $X \in a \oplus m$ we finally have $F_*(X)_x = (\text{Ad}(kn)X)_x$. From this it becomes clear that $F_*$, regular on $K \times N \times M \times A^-$, may on the boundary have vanishing differential of order $\dim(n) + \dim(t)$. Together we get that $f$ is $(j - 1 - 2 \dim n - \dim t)$-times continuously differentiable on $G$. So we assume $j \geq 2 \dim(n) + \dim(t) + 1$ from now on. In order to show that $f$ goes into the trace formula for $\text{Re}(\mu)$ and $j$ large we fix $d > \frac{\dim G}{2}$. We have to show that $Df \in L^1(G)$ for any $D \in U(g)$ of degree $\leq 2d$. For this we recall the map $F_*$ and our computation of its differential. Let $q \subset k$ be a complementary space to $kM$. On the regular set $F_*$ is surjective and it becomes bijective on $q \oplus n \oplus a \oplus m$. Fix $x = knam(kn)^{-1}$ in the regular set and let $F^{-1}_*,x$ denote the inverse map of $F_*,x$ which maps to $q \oplus n \oplus a \oplus m$. Introducing norms on the Lie algebras we get an operator norm for $F^{-1}_*,x$ and the above calculations show that $\| F^{-1}_*,x \| \leq P(am)$, where $P$ is a class function on $AM$, which, restricted to any Cartan $H = AB$ of $AM$ is a linear combination of quasi-characters. Supposing $j$ and $\text{Re}(\mu)$ large enough we get for $D \in U(g)$ with deg$(D) \leq 2d$:

$$|Df(knam(kn)^{-1})| \leq \sum_{D_1} P_{D_1}(am)|D_1f(k,n,a,m)|,$$

where the sum runs over a finite set of $D_1 \in U(t \oplus n \oplus a \oplus m)$ of degree $\leq 2d$ and $P_{D_1}$ is a function of the type of $P$. On the right hand side we have considered $f$ as a function on $K \times N \times A \times M$. This discussion uses the facts that $K$ is compact, $N$ is unipotent, and $\det(\text{Ad}(n^{-1}m^{-1}) - 1) = \det(\text{Ad}(am^{-1}) - 1)$. Finiteness of the sum in the inequality above follows from the Poincaré-Birkhoff-Witt Theorem.

It follows that if $j, \text{Re(}\mu) \gg 0$ then $(D_1f)PP_{D_1}$ lies in $L^1(K \times N \times A \times M)$ for any $D_1$. We have proven that the trace formula is valid for $f$.

We plug this test function into the trace formula and we claim that the result is precisely the Lefschetz formula. To start with the geometric side recall that $\Gamma$, being cocompact, only contains semisimple elements and that the orbital integrals over $f$ vanish except for $[\gamma] \in E(\Gamma)$, where we get

$$O_\gamma(f) = \frac{\text{tr} \sigma(m_\gamma)}{\det(1 - a_\gamma m_\gamma | n)} \varphi(a_\gamma).$$

For $[\gamma] \in E(\Gamma)$ we have $\lambda_\gamma = \text{vol}(G_\gamma/\Gamma_\gamma)$. So we see that

$$\text{vol}(\Gamma_\gamma \setminus G_\gamma)O_\gamma(f) = \text{ind}(\gamma) \text{tr} \sigma(m_\gamma) \varphi(a_\gamma),$$
which means that the geometric side of the trace formula is the local side of
the local side of the Lefschetz formula.

To compute the spectral side of the trace formula let $\pi \in \hat{G}$. Harish-Chandra
showed that there is a locally integrable function $\Theta^G_\pi$ on $G$, called the global
character of $\pi$, such that $\text{tr} \pi(h) = \int_G h(x) \Theta^G_\pi(x) dx$ for every $h \in C_c^\infty$.
It follows that $\Theta^G_\pi$ is invariant under conjugation. Hecht and Schmid have
shown in [12] that for $am \in A^-M$,

$$\Theta^G_\pi = \sum_{q=0}^{\dim \mathfrak{n}} (-1)^q \Theta^{AM}_{H_q(n,\pi_K)}(am) \overline{\det(1-am|\mathfrak{n})},$$

where $\Theta^{AM}$ is the corresponding global character on the group $AM$.

Let $g \in L^1(G)$ be supported in the set $\{xamx^{-1}|am \in A^-M, x \in G\}$. Then,
as a consequence of the Weyl integration formula or by direct proof one gets
that $\int_G g(x)dx$ equals

$$\int_{KN} \int_{A^-M} g(knam(kn)^{-1}) |\det(1-am|n \oplus \bar{n})| dadmdkdn.$$ 

We apply this to $g(x) = \Theta^G_\pi(x)f(x)$ to get

$$\text{tr} \pi(f) = \int_B \Theta^G_\pi(x)f(x)dx = \int_{A^-M} \Theta^G_\pi(am) \text{tr} \sigma(m) \varphi(a) |\det(1-am|\bar{n})| dadm.$$ 

Using the result of Hecht and Schmid we see that this equals

$$\int_{A^-M} \text{tr} \sigma(m) \varphi(a) \sum_{p=0}^{\dim \mathfrak{n}} (-1)^p \Theta^{AM}_{H_p(n,\pi_K)}(am) \overline{\det(1-am|\bar{n})} \det(1-am|\mathfrak{n}) dadm.$$ 

For $am \in A^-M$ we have

$$|\det(1-am|\bar{n})| = (-1)^{\dim \mathfrak{n}} \det(1-am|\bar{n})$$

$$= (-1)^{\dim \mathfrak{n}} a^{-2\rho \text{det}(a^{-1} - m|\bar{n})}$$

$$= (-1)^{\dim \mathfrak{n}} a^{-2\rho \text{det}((am)^{-1} - 1|\bar{n})}$$

$$= a^{-2\rho \text{det}(1 - (am)^{-1}|\bar{n})}$$

$$= a^{-2\rho \text{det}(1 - am|\mathfrak{n})}$$
So that
\[ \text{tr } \pi(f) = \int_{A^{-M}} \text{tr } \sigma(m_{\gamma}) \sum_{p=0}^{\dim n} (-1)^p \Theta_{H^p(n, \pi_K)}^{AM}(am) a^{-2\rho} \varphi(a) da dm. \]

Lemma 1.1 implies
\[ \sum_{p=0}^{\dim n} (-1)^p \Theta_{H^p(n, \pi_K)}^{AM}(am) a^{-2\rho} = \sum_{p=0}^{\dim n} (-1)^{p+\dim n} \Theta_{H^p(n, \pi_K)}^{AM}(am). \]

And so
\[
\text{tr } \pi(f) = \int_{A^{-M}} \sum_{p=0}^{\dim n} (-1)^{p+\dim n} \Theta_{H^p(n, \pi_K)}^{AM}(am) \varphi(a) \text{tr } \sigma(m) da dm \\
= \sum_{\lambda \in \mathfrak{a}^*} \sum_{p=0}^{\dim n} (-1)^{p+\dim n} \dim(H^p(n, \pi_K)_{\lambda} \otimes \tilde{\sigma})^M \int_{A^{-}} a^{\lambda+\rho} \varphi(a) da.
\]

the convergence of the trace formula implies that for given \( \lambda \in \mathfrak{a}^* \) the number
\[ N_\Gamma(\pi) \sum_{p=0}^{\dim n} (-1)^{p+\dim n} \dim(H^p(n, \pi_K)_{\lambda} \otimes \tilde{\sigma})^M \]

is nonzero only for finitely many \( \pi \in \hat{G} \). Thus the spectral side of the trace formula gives the global side of the Lefschetz formula which therefore is proven. The continuity also follows from the proof. \[ \square \]

2 The Dirichlet series

Let \( r = \dim A \) and for \( k = 1, \ldots, r \) let \( \alpha_k \) be a positive real multiple of a simple root of \((A, P)\) such that the modular shift \( \rho \) satisfies
\[ 2\rho = \alpha_1 + \cdots + \alpha_r. \]

This defines \( \alpha_1, \ldots, \alpha_r \) uniquely up to order. We fix a Haar measure (i.e., a form \( B \)) such that the subset of \( A \),
\[ \{ a \in A \mid 0 \leq \alpha_k(\log a) \leq 1, \ k = 1, \ldots, r \} \]
has volume 1.

For $a \in A$ and $k = 1, \ldots, r$ let $l_k(a) = |\alpha_k(\log a)|$ and $l(a) = l_1(a) \cdots l_r(a)$. For $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$ and $j \in \mathbb{N}$ define

$$L_j^j(s) = \sum_{[\gamma] \in E(\Gamma)} \text{ind}(\gamma) l(a_\gamma)^j a_\gamma^{s - \alpha}$$

where $s \cdot \alpha = s_1 \alpha_1 + \cdots + s_r \alpha_r$. We will show that this series converges if $\text{Re}(s_k) > 1$ for $k = 1, \ldots, r$. Let $D$ denote the differential operator

$$D = (-1)^r \left( \frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_r} \right).$$

Let $\hat{G}(\Gamma)$ denote the set of all $\pi \in \hat{G}$, $\pi \neq \text{triv}$ with $N_\Gamma(\pi) \neq 0$. For given $\pi \in \hat{G}$ let $\Lambda(\pi)$ denote the set of all $\lambda \in \mathfrak{a}^*$ with $m_{\lambda - \rho}(\pi) \neq 0$. Then $\Lambda(\pi)$ has at most $|W(\mathfrak{h}, \mathfrak{g})|$ elements.

Let $\lambda \in \mathfrak{a}^*$. Since $\alpha_1, \ldots, \alpha_r$ is a basis of $\mathfrak{a}^*$ we can write $\lambda = \lambda_1 \alpha_1 + \cdots + \lambda_r \alpha_r$ for uniquely determined $\lambda_k \in \mathbb{C}$.

Let $R_k(s), k \in \mathbb{N}$ be a sequence of rational functions on $\mathbb{C}^r$. For an open set $U \subset \mathbb{C}^r$ let $N(U)$ be the set of natural numbers $k$ such that the pole-divisor of $R_k$ does not intersect $U$. We say that the series

$$\sum_k R_k(s)$$

converges weakly locally uniformly on $\mathbb{C}^r$ if for every open $U \subset \mathbb{C}^r$ the series

$$\sum_{k \in N(U)} R_k(s)$$

converges locally uniformly on $U$.

**Theorem 2.1** For $j \in \mathbb{N}$ large enough the series $L_j^j(s)$ converges locally uniformly in the set

$$\{ s \in \mathbb{C} : \text{Re}(s_k) > 1, \ k = 1, \ldots, r \}.$$
The function $L^j(s)$ can be written as Mittag-Leffler series,

$$L^j(s) = D^{j+1} \frac{1}{(s_1 - 1) \cdots (s_r - 1)} + \sum_{\pi \in \hat{G}(\Gamma)} N_{\Gamma}(\pi) \sum_{\lambda \in \Lambda(\pi)} m_{\lambda-\rho}(\pi) D^{j+1} \frac{1}{(s_1 + \lambda_1) \cdots (s_r + \lambda_r)}.$$ 

The double series converges weakly locally uniformly on $\mathbb{C}^r$. For $\pi \neq \text{triv}$ and $\lambda \in \Lambda(\pi)$ we have $\text{Re}(\lambda_k) \geq -1$ for $k = 1, \ldots, r$ and there is $k \in \{1, \ldots, r\}$ with $\text{Re}(\lambda_k) > -1$. So in particular, the double series converges locally uniformly on $\{\text{Re}(s_k) > 1\}$.

The proof will occupy the rest of this section. We will show that the series $L^j(s)$ converges if the real parts $\text{Re}(s_k)$ are sufficiently large for $k = 1, \ldots, r$. Since $L^j(s)$ is a Dirichlet series with positive coefficients, the convergence in the set $\{\text{Re}(s_k) > 1\}$ will follow, once we have established holomorphy there. This holomorphy will in turn follow from the convergence of the Mittag-Leffler series.

Let $a^{*+}_R = \{\lambda_1\alpha_1 + \cdots + \lambda_r\alpha_r | \lambda_1, \ldots, \lambda_r > 0\}$ be the dual positive cone. Let $\overline{a^{*+}_R}$ be the closure of $a^{*+}_R$ in $a^*_R$.

**Proposition 2.2** Let $\pi \in \hat{G}$, $\lambda \in a^*$ with $m_{\lambda}(\pi) \neq 0$. Then $\text{Re}(\lambda)$ lies in the set

$$C = -3\rho + \overline{a^{*+}_R}.$$ 

For $\pi \in \hat{G}$ and $\text{Re}(\lambda) = -3\rho$ we have $m_{\lambda}(\pi) = 0$ unless $\pi$ is the trivial representation and $\lambda = -3\rho$ in which case $m_{\lambda}(\pi) = 1$.

**Proof:** We introduce a partial order on $a^*$ by

$\mu > \nu \iff \mu - \nu$ is a linear combination,

with positive integral coefficients, of roots in $\Phi^+$. 

**Lemma 2.3** Let $p \in \mathbb{N}$, let $\pi \in \hat{G}$ and $\mu \in a^*$ such that $H_p(n, \pi_K)_\mu \neq 0$. Then there exists $\nu \in a^*$ with $\nu < \mu$ and $H_0(n, \pi_K)_\nu \neq 0$.

Equivalently, if $0 \leq p < d = \dim(n)$ and $H^p(n, \pi_K)_\mu \neq 0$, then there exists $\eta \in a^*$ with $\eta < \mu$ and $H^d(n, \pi_K)_\eta \neq 0$. 


Proof: The first assertion is a weak version of Proposition 2.32 in [12] and the second follows from the first and Lemma 1.1. □

Theorem 4.25 of [12] states that the set of $\nu \in \mathfrak{a}^*$ with $H_0(n, \pi_K)_\nu \neq 0$ which are minimal with respect to $<$ are precisely the leading exponents of the asymptotic of matrix coefficients of $\pi$. Since $\pi$ is unitary, its matrix coefficients are bounded and thus its leading coefficients in the normalization of Theorem 4.16 of [12] all lie in the set $-\rho + \mathfrak{a}^*_R$. Now Lemma 2.3 and Lemma 1.1 imply the first statement of Proposition 2.2.

It remains to consider the case $\text{Re}(\lambda) = -3\rho$. So let $\lambda = -3\rho + i\mu$ for $\mu \in \mathfrak{a}_R^*$. Using the definition of Lie algebra homology it is easy to show that $m_{-3\rho}(\text{triv}) = 1$. Let $\pi \in \hat{G}$ and assume that $H^\bullet(n, \pi_K)_M^{\rho + i\mu} \neq 0$. The claim will follow, if we show that this implies $\pi = \text{triv}$.

Since $H^\bullet(n, \pi_K) \cong H_\bullet(n, \pi_K) \otimes \Lambda_{\text{top}} n$ we find that our condition is equivalent to $H_\bullet(n, \pi_K)_{-\rho + i\mu} \neq 0$.

Let $\xi$ be an irreducible representation of $M$ and let $\nu \in \mathfrak{a}^*$. let $\pi_{\xi,\nu}$ be the induced principal series representation. Recall that if $\text{Re}(\nu) \in \mathfrak{a}_R^{*+}$, then $\pi_{\xi,\nu}$ has a unique irreducible quotient, its Langlands quotient. The representation dual to $\pi_{\xi,\nu}$ is $\pi_{\xi,-\nu}$, where $\check{\xi}$ is the dual to $\xi$. Therefore $\pi_{\xi,-\nu}$ has a unique irreducible subrepresentation. Replacing $\xi$ by $\check{\xi}$ and $\nu$ by $-\nu$ it follows that for $\text{Re}(\nu) \in -\mathfrak{a}_R^{*+}$ the principal series representation $\pi_{\xi,\nu}$ has a unique irreducible subrepresentation, namely the dual of the Langlands quotient of $\pi_{\xi,-\nu}$.

Frobenius reciprocity says that

$$\text{Hom}_G(\pi, \pi_{\xi,\nu}) \cong \text{Hom}_{AM}(H_0(n, \pi_K), \xi \otimes (\nu + \rho)).$$

So in particular,

$$\text{Hom}_G(\pi, \pi_{1,\nu}) \cong \text{Hom}_A(H_0(n, \pi_K)^M, \nu + \rho).$$

If $H_p(n, \pi_K)^M_{-\rho + i\mu} \neq 0$ for some $p > 0$ then by Schmid’s vanishing result it follows that $\pi$ must have a leading coefficient with real part $<-3\rho$ which we already have ruled out. So it follows that $H_0(n, \pi_K)^M_{-\rho + i\mu} \neq 0$ and hence $\text{Hom}_G(\pi, \pi_{1,-\rho + i\mu}) \neq 0$. None of the representations $\pi_{1,-\rho + i\mu}$, however, has a unitary subrepresentation, except for $\mu = 0$. 

The representation $\pi_{1,-\rho}$ has a unique irreducible subrepresentation which is the trivial representation. This implies $\pi = \text{triv}$. □

For $a \in A$ set

$$\varphi(a) = l(a)^{j+1} a^{s_\alpha}.$$  

For $\text{Re}(s_k) >> 0$, $k = 1, \ldots, r$ the Lefschetz formula is valid for this test function. The local side of the Lefschetz formula for $\sigma = \text{triv}$ equals

$$\sum_{[\gamma] \in E(\Gamma)} \text{ind}(\gamma) l(a_{\gamma})^{j+1} a_{\gamma}^{s_\alpha} = L_j^j(s).$$

The convergence assertion in the Lefschetz formula implies that the series converges absolutely if $\text{Re}(s_k)$ is sufficiently large for every $k = 1, \ldots, r$. Since $L_j^j(s)$ is a Dirichlet series with positive coefficients it will converge locally uniformly for $s$ in some open set. We will show that it extends to a holomorphic function in the set $\text{Re}(s_k) > 1$, $k = 1, \ldots, r$. Again, since it is a Dirichlet series with positive coefficients it must therefore converge in that region.

With our given test function and the Haar measure chosen we compute

$$\int_{A^-} \varphi(a) a^\lambda da = (-1)^r (j+1) \int_{A^-} (\alpha_1 (\log a) \cdots \alpha_r (\log a))^{j+1} a^{s_\alpha + \lambda} da$$

$$= (-1)^r (j+1) \int_0^\infty \cdots \int_0^\infty (t_1 \cdots t_r)^{j+1} e^{-(s_1 + \lambda_1)t_1 + \cdots + (s_r + \lambda_r)t_r} dt_1 \cdots dt_r$$

$$= D^{j+1} \int_0^\infty \cdots \int_0^\infty e^{-(s_1 + \lambda_1)t_1 + \cdots + (s_r + \lambda_r)t_r} dt_1 \cdots dt_r$$

$$= D^{j+1} \frac{1}{(s_1 + \lambda_1) \cdots (s_r + \lambda_r)}.$$  

Writing $m_\lambda = m^{\text{triv}}_\lambda$ and performing a $\rho$-shift we see that the Lefschetz formula gives

$$L_j^j(s) = \sum_{\pi \in \hat{G}} N_\Gamma(\pi) \sum_{\lambda \in a^\sigma} m_{\lambda-\rho}(\pi) D^{j+1} \frac{1}{(s_1 + \lambda_1) \cdots (s_r + \lambda_r)}$$

$$= \sum_{\pi \in \hat{G}} N_\Gamma(\pi) \sum_{\lambda \in a^\sigma} m_{\lambda-\rho}(\pi) \frac{(j+1)!^r}{(s_1 + \lambda_1)^{j+2} \cdots (s_r + \lambda_r)^{j+2}}$$
for Re($s_k$) $>> 0$. For every $\pi \in \hat{G}$ we fix a representative $\Lambda_\pi \in (a + t)^*$ of the infinitesimal character of $\pi$. According to Lemma 1.1 if $m_{\lambda - \rho}(\pi) \neq 0$, then $\lambda - w_\Lambda \pi_{\mid \alpha} - \rho$ for some $w \in W(h, g)$. By abuse of notation we will write $w_\Lambda \pi$ instead of $w_\Lambda \pi_{\mid \alpha}$. Hence we get

$$L^j(s) = \sum_{\pi \in \hat{G}} N_\Gamma(\pi) \sum_{w \in W(h, g)} m_{w_\Lambda \pi_{\mid \alpha} - \rho}(\pi) D^{j+1} \frac{1}{(s_1 + \lambda_1) \cdots (s_r + \lambda_r)}.$$  

For $\lambda \in a^*$ let $\| \lambda \|$ be the norm given by the form $B$ as explained in the beginning of section II.

**Proposition 2.4** There are $m \in \mathbb{N}$, $C > 0$ such that for every $\pi \in \hat{G}$ and every $\lambda \in a^*$ one has

$$|m_{\lambda - \rho}(\pi)| \leq C(1 + \| \lambda \|)^m.$$  

**Proof:** Recall from the first section that for $a \in A^-$,

$$\sum_{\lambda \in a} m_{\lambda - \rho}(\pi) a^\lambda = \int_M \sum_{p=0}^{\dim n} (-1)^p \Theta_{H^p(n, \pi_K)}^{AM}(am) \, dm$$

$$= \left( -1 \right)^{\dim n} a^{-2\rho} \int_M \sum_{p} (-1)^p \Theta_{H^p(n, \pi_K)}^{AM}(am) \, dm$$

$$= \left( -1 \right)^{\dim n} a^{-2\rho} \int_M \det(1 - am | n) \Theta_{\pi}^G(\pi) \, dm$$

Choose a set of positive roots $\phi^+(t, m) \subset \phi(t, m)$ compatible to the choice of $P$. Let $\rho_M = \frac{1}{2} \sum_{\alpha \in \phi^+(t, m)} \alpha$. For $t \in T$ set

$$D_T(t) \overset{\text{def}}{=} t^{\rho_M} \prod_{\alpha \in \phi^+(t, m)} (1 - t^{-\alpha}).$$

This is the Weyl denominator for $T$. The Weyl integral formula gives

$$\sum_{\lambda \in a^*} m_{\lambda - \rho}(\pi) a^\lambda = \int_{T^{\rho_M}} (-1)^{\dim n} a^{-2\rho} \det(1 - am | n) |D_T(t)|^2 \Theta_{\pi}^G(at) \, dt.$$
The function \((-1)^{\dim \mathfrak{a}} a^{-2\rho} \det(1 - a \mathfrak{n}) D_T(t)\) equals the Weyl denominator for \(H = AT\). By Theorems 10.35 and 10.48 of \[14\] there are constants \(c_w, w \in W(h, g)\) such that

\[
(-1)^{\dim \mathfrak{a}} a^{-2\rho} \det(1 - a \mathfrak{n}) D_T(t) \Theta_G^G(at) = \sum_{w \in W(h_0)} c_w (at)^{w \Lambda_\pi}.
\]

We thus have proved the following Lemma.

**Lemma 2.5** For \(a \in A^+\),

\[
\sum_{\lambda \in \mathfrak{a}^*} m_{\lambda - \rho}(\pi) a^\lambda = \sum_{w \in W(h, g)} c_w a^{w \Lambda_\pi} \int_{T^{\text{reg}}} t^{w \Lambda_\pi - \rho_M} \prod_{\alpha \in \phi^+ (t, m)} (1 - t^\alpha) \, dt.
\]

Proposition 2.4 will follow from explicit formulae for the global character \(\Theta_\pi^G\) (see below) which give bounds on the \(c_w\). Another remarkable consequence of Lemma 2.5 is the fact that there is a finite set \(E \subset \mathfrak{t}^*\) such that whenever \(m_{\lambda - \rho}(\lambda) \neq 0\) for some \(\lambda \in \mathfrak{a}^*\) it follows \(\Lambda_\pi|_t \in E\). Hence Proposition 2.4 will follow from the estimate

\[
|m_{\lambda - \rho}(\pi)| \leq C(1 + \|\Lambda_\pi\|^m).
\]

In \[9\] Harish-Chandra gives an explicit formula for characters of discrete series representations which imply the sharper estimate \(|m_{\lambda - \rho}(\pi)| \leq C\) for the discrete series representations. From Harish-Chandra’s paper a similar formula can be deduced for limit of discrete series representations. Alternatively, one can use Zuckerman tensoring (Prop. 10.44 of \[16\]) to deduce the estimate for limits of discrete series representations. Next, if \(\pi = \pi_{\sigma, \nu}\) is induced from some parabolic \(P_1 = M_1 A_1 N_1\), then the character of \(\pi\) can be computed from the character of \(\sigma\) and \(\nu\), see formula (10.27) in \[16\]. From this it follows that the claim holds for standard representations, i.e. admissible representations which are induced from discrete series or limit of discrete series representations.

**Lemma 2.6** There are natural numbers \(n, m\) and a constant \(d > 0\) such that for every \(\pi \in \hat{G}\) there are standard representations \(\pi_1, \ldots, \pi_n\) and integers \(c_1, \ldots, c_n\) with

\[
\Theta_\pi = \sum_{k=1}^n c_k \Theta_{\pi_k}.
\]
and $|c_k| \leq d(1 + \| \Lambda_\pi \|^m)$ for $j = 1, \ldots, n$.

**Proof:** By the Langlands classification, $\pi$ is a quotient $p/q$ of a standard representation $p$. Then $\pi$ and $p$ and thus every constituent of $q$ share the same infinitesimal character $\Lambda_\pi$. The $A$-parameter of every constituent of $q$ is smaller than the one of $\pi$, so one can perform an induction on this $A$-parameter as suggested by ([16], Problem 10.2) to get the linear combination as above. The length of this linear combination is globally bounded because the combination only contains characters of representations that share a common infinitesimal character and the number of such is globally bounded. The coefficients are bounded by the multiplicity that a constituent $\pi$ of an induced representation $\pi_{\sigma,\nu}$ might have, where $\sigma$ is a discrete series or limit of discrete series representation. Now assume that $\pi$ contains a $K$-type $\tau$. Then the multiplicity of $\pi$ in $\pi_{\sigma,\nu}$ is bounded by the multiplicity of $\tau$ in $\pi_{\sigma,\nu}$ which in turn is bounded by $\dim \tau$. It remains to show that every irreducible admissible $\pi$ contains a minimal $K$-type whose dimension is bounded by a power of $\| \Lambda_\pi \|$. By Weyl’s character formula this is equivalent to say that the norm of the infinitesimal character of $\tau$ is bounded by a power of $\| \Lambda_\pi \|$. This indeed follows from the minimal $K$-type formula ([16], Theorem 15.1) together with ([16], Theorem 15.10). So we have shown that the $m_{\lambda-\rho}(\pi)$ grow at most like a power of $\| \Lambda_\pi \|$ and Lemma 2.6 follows. By the above, this also implies Proposition 2.4.

It remains to deduce Theorem 2.1. Since the coefficients $m_{\lambda-\rho}(\pi)$ grow at most like a power of $\| \Lambda_\pi \|$, the convergence assertion in Theorem 2.1 will be implied by the following lemma.

**Lemma 2.7** Let $S$ denote the set of all pairs $(\pi, \lambda) \in \hat{G} \times a^*$ such that $m_{\lambda-\rho}(\pi) \neq 0$. There is $m_1 \in \mathbb{N}$ such that

$$
\sum_{(\pi, \lambda) \in S} \frac{N_\Gamma(\pi)}{(1 + \| \lambda \|)^{m_1}} < \infty.
$$

**Proof:** By the remark following Lemma 2.5 it suffices to show that there is $m \in \mathbb{N}$ such that

$$
\sum_{(\pi, \lambda) \in S} \frac{N_\Gamma(\pi)}{(1 + \| \Lambda_\pi \|)^{m_1}} < \infty.
$$
Let \( \pi \in \hat{G} \). The restriction of \( \pi \) to the maximal compact subgroup \( K \) decomposes into finite dimensional isotypes

\[
\pi|_K = \bigoplus_{\tau \in \hat{K}} \pi(\tau).
\]

Let \( C_K \) be the Casimir operator of \( K \) and let

\[
\Delta_G \overset{\text{def}}{=} -C + 2C_K.
\]

Then \( \Delta_G \) is the Laplacian on \( G \) given by the left invariant metric which at the point \( e \in G \) is given by \( \langle ., . \rangle = -B(., \theta(\cdot)) \). Since \( \Delta_G \) is left invariant it induces an operator on \( \Gamma \backslash G \) denoted by the same letter. This operator is \( \geq 0 \) and elliptic, so there is a natural number \( k \) such that \( (1 + \Delta_G)^{-k} \) is of trace class on \( L^2(\Gamma \backslash G) \). Hence

\[
\infty > \text{tr} (1 + \Delta_G)^{-k} = \sum_{\pi \in \hat{G}} N_\Gamma(\pi) \sum_{\tau \in \hat{K}} \dim \pi(\tau) \sum_{\tau \in \hat{K}} (1 - \pi(C) + 2\tau(C_K))^{-k} \]

\[
\geq \sum_{\pi \in \hat{G}} N_\Gamma(\pi) \frac{(1 - \pi(C) + 2\tau_\pi(C_K))^{-k}}{(1 + \Delta_G)^{-k}},
\]

where for each \( \pi \in \hat{G} \) we fix a minimal \( K \)-type \( \tau_\pi \). Since the infinitesimal character of the minimal \( K \)-type grows like the infinitesimal character of \( \pi \) the Lemma follows. \( \square \)

Finally, to prove Theorem 2.11 let \( U \in \mathcal{C}^* \) be open. Let \( S(U) \) be the set of all pairs \( (\pi, \lambda) \in \hat{G} \times \mathfrak{a}^* \) such that \( m_{\lambda - \rho}(\pi) \neq 0 \) and the pole divisor of

\[
\frac{1}{(s_1 + \lambda_1) \cdots (s_r + \lambda_r)}
\]

does not intersect \( U \). Let \( V \subset U \) be a compact subset. We have to show that for some \( j \in \mathbb{N} \) which does not depend on \( U \) or \( V \),

\[
\sup_{s \in V} \sum_{(\pi, \lambda) \in S(U)} \left| \frac{N_\Gamma(\pi) \ m_{\lambda - \rho}(\pi)}{(s_1 + \lambda_1)^{j+2} \cdots (s_r + \lambda_r)^{j+2}} \right| < \infty.
\]
Let $m$ be as in Lemma 2.4 and $m_1$ as in Lemma 2.7. Then let $j \geq m + m_1 - 2$. Since $V \subset U$ and $V$ is compact there is $\varepsilon > 0$ such that $s \in V$ and $(\pi, \lambda) \in S(U)$ implies $|s_k + \lambda_k| \geq \varepsilon$ for every $k = 1, \ldots, r$. Hence there is $c > 0$ such that for every $s \in V$ and every $(\pi, \lambda) \in S(U)$,

$$|(s_1 + \lambda_1) \cdots (s_r + \lambda_r)| \geq c(1 + \| \lambda \|).$$

This implies,

$$\left| \frac{m_{\lambda - \rho}(\pi)}{(s_1 + \lambda_1)^{j+2} \cdots (s_r + \lambda_r)^{j+2}} \right| \leq \frac{1}{c^{j+2}} \frac{|m_{\lambda - \rho}(\pi)|}{(1 + \| \lambda \|)^{j+2}} \leq \frac{C}{c^{j+2}} \frac{1}{(1 + \| \lambda \|)^{j+2 - m}} \leq \frac{C}{c^{j+2}} \frac{1}{(1 + \| \lambda \|)^{m_1}}.$$

The claim now follows from Lemma 2.7. The proof of Theorem 2.1 is finished.

\[\square\]

3 The prime geodesic theorem

We now give the main result of the paper.

**Theorem 3.1 (Prime Geodesic Theorem)**

For $T_1, \ldots, T_r > 0$ let

$$\Psi(T_1, \ldots, T_r) = \sum_{[\gamma] \in \varepsilon(r)} \lambda_\gamma.$$ 

Then, as $T_k \to \infty$ for $k = 1, \ldots, r$ we have

$$\Psi(T_1, \ldots, T_r) \sim T_1 \cdots T_r.$$

The proof of the theorem will occupy the rest of the section.
For $x_1, \ldots, x_r > 0$ let
\[
A(x) = A(x_1, \ldots, x_r) = \sum_{[\gamma] \in \mathcal{E}(\Gamma)} l(a_\gamma)^{j+1} \text{ind}(\gamma).
\]
Let $\mathbb{R}_+$ be the set of positive real numbers. Then $L_j(s) = \int_{\mathbb{R}_+^r} A(x) e^{-sx} dx$.

We need a higher dimensional analogue of the Wiener-Ikehara Theorem. For this we introduce a partial order on $\mathbb{R}^r$. We define
\[
(x_1, \ldots, x_r) \geq (y_1, \ldots, y_r) \iff x_1 \geq y_1 \text{ and } \ldots \text{ and } x_r \geq y_r.
\]
A real valued function $F$ on a subset of $\mathbb{R}^r$ is called monotonic if $x \geq y$ implies $F(x) \geq F(y)$. The partial order also gives sense to the assertion that a sequence $x^n$ tends to $+\infty$. It does so if all its components $x^n_k$ tend to $+\infty$ separately.

**Theorem 3.2** (Higher dimensional version of the Wiener-Ikehara Theorem) Let $A \geq 0$ be a monotonic measurable function on $\mathbb{R}^r_+$. Suppose that the integral $L(s) = \int_{\mathbb{R}_+^r} A(x) e^{-sx} dx$ converges for $\text{Re}(s_k) > 1$, $k = 1, \ldots, r$.

Suppose further that there is a countable set $I$ and for each $i \in I$, there is $\theta_i \in \mathbb{C}^r$ such that $\text{Re}(\theta_{i,k}) \leq 1$ for $k = 1, \ldots, r$. Assume that for each $i \in I$ there is $k \in \{1, \ldots, r\}$ such that $\text{Re}(\theta_{i,k}) < 1$ and that there are integers $c_i$ such that the function
\[
L(s) - D^{j+1} \frac{1}{(s_1 - 1) \cdots (s_r - 1)} - \sum_{i \in I} c_i D^{j+1} \frac{1}{(s_1 - \theta_{i,1}) \cdots (s_r - \theta_{i,r})}
\]
extends to a holomorphic function on the set of all $s \in \mathbb{C}^r$ with $\text{Re}(s_k) \geq 1$ for $k = 1, \ldots, r$. Here we assume that the sum converges weakly locally uniformly absolutely on $\mathbb{C}^r$. Under these circumstances we can conclude
\[
\lim_{x \to +\infty} A(x) (x_1 \cdots x_r)^{-(j+1)} e^{-(x_1 + \cdots + x_r)} = 1.
\]

The proof of this theorem is a fairly straightforward application of methods from analytic number theory. We include it for the convenience of the reader.

**Proof:** Let $A$ as in the theorem and let
\[
B(x) = A(x) (x_1 \cdots x_r)^{-(j+1)} e^{-(x_1 + \cdots + x_r)}.
\]
To simplify notations we will frequently identify a complex number \( z \) with the vector \( (z, \ldots, z) \in \mathbb{C}^r \). Since

\[
\frac{1}{(s_1 - 1) \cdots (s_r - 1)} = \int_{\mathbb{R}_+^r} e^{-(s-1) \cdot x} \, dx
\]

we get

\[
D^{j+1} \frac{1}{(s_1 - 1) \cdots (s_r - 1)} = \int_{\mathbb{R}_+^r} (x_1 \cdots x_r)^{j+1} e^{-(s-1) \cdot x} \, dx.
\]

Likewise,

\[
D^{j+1} \frac{1}{(s_1 - \theta_{i,1}) \cdots (s_r - \theta_{i,r})} = \int_{\mathbb{R}_+^r} (x_1 \cdots x_r)^{j+1} e^{-(s-\theta_i) \cdot x} \, dx.
\]

Let \( f \) be a smooth function of compact support of \( \mathbb{R} \) which is real valued and even. Then its Fourier transform \( \hat{f} \) will also be real valued. We further assume \( f \) to be of the form \( f = f_1 * f_1 \) for some \( f_1 \). Then \( \hat{f} = (\hat{f}_1)^2 \) is positive on the reals. Let \( I(f) \) be the set of \( i \in I \) such that \( \text{Im}(\theta_i) \in (\text{supp} f)^r \). Then the function

\[
g(s) = L(s) - D^{j+1} \frac{1}{(s_1 - 1) \cdots (s_r - 1)} - \sum_{i \in I(f)} c_i D^{j+1} \frac{1}{(s_1 - \theta_{i,1}) \cdots (s_r - \theta_{i,r})}
\]

extends to an analytic function on \( \{ \text{Re}(s_k) \geq 1, \text{Im}(s_k) \in \text{supp} f \} \). It follows

\[
g(s) = \int_{\mathbb{R}_+^r} (B(x) - 1) (x_1 \cdots x_r)^{j+1} e^{-(s-1) \cdot x} \, dx
\]

\[
- \sum_{i \in I(f)} c_i \int_{\mathbb{R}_+^r} (x_1 \cdots x_r)^{j+1} e^{-(s-\theta_i) \cdot x} \, dx
\]

Let \( \varepsilon > 0 \). For \( y \in \mathbb{R} \) the integral

\[
\int_{\mathbb{R}^r} g(1 + \varepsilon + it) f(t_1) \cdots f(t_r) e^{iyt} \, dt_1 \cdots dt_r
\]
equals
\[
\int_{\mathbb{R}^r} f(t_1) \cdots f(t_r) e^{iyt} \int_{\mathbb{R}_+^r} (B(x) - 1) (x_1 \cdots x_r)^{j+1} e^{-(\varepsilon + it) \cdot x} \, dx \, dt
\]
\[
- \sum_{i \in I(f)} c_i \int_{\mathbb{R}^r} f(t_1) \cdots f(t_r) e^{iyt} \int_{\mathbb{R}_+^r} (x_1 \cdots x_r)^{j+1} e^{-(1-\theta_i) \cdot x} e^{-\varepsilon x} \, dx \, dt.
\]
We want to interchange the order of integration. This only causes a problem for the summand involving \(B(x)\). To justify the interchange, note that by the monotonicity of \(A\) we have for real \(s\), and \(x \in \mathbb{R}_+^r\),
\[
L(s) = \int_{\mathbb{R}_+^r} A(u) e^{-s \cdot u} \, du \geq A(x) \int_{x+\mathbb{R}_+^r} \int_{x+\mathbb{R}_+^r} e^{-s \cdot u} \, du = \frac{A(x) e^{-s \cdot x}}{s_1 \cdots s_r}.
\]
In other words, \(A(x) \leq s_1 \cdots s_r L(s) e^{x \cdot s}\). Therefore \(A(x) = O(e^{-s \cdot x})\) for every \(s_1, \ldots, s_r > 1\) which implies \(A(x) = o(e^{-s \cdot x})\) for every \(s_1, \ldots, s_r > 1\). So for \(\delta > 0\), \(B(x) (x_1 \cdots x_r)^{j+1} e^{-\delta x} = A(x) e^{-(1+\delta) \cdot x} = o(1)\) for every \(\delta > 0\). This implies that the integral
\[
\int_{\mathbb{R}_+^r} (B(x) - 1) (x_1 \cdots x_r)^{j+1} e^{-(\varepsilon + it) \cdot x} \, dx
\]
converges locally uniformly in \(t\). So we can interchange the order of integration to obtain that
\[
\int_{\mathbb{R}^r} g(1 + \varepsilon + it) f(t_1) \cdots f(t_r) e^{iyt} \, dt
\]
equals
\[
\int_{\mathbb{R}_+^r} (B(x) - 1) (x_1 \cdots x_r)^{j+1} e^{-\varepsilon x} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx
\]
\[
- \sum_{i \in I(f)} c_i \int_{\mathbb{R}_+^r} (x_1 \cdots x_r)^{j+1} e^{-(1-\theta_i) \cdot x} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) e^{-\varepsilon x} \, dx.
\]
Since \(g(s)\) is analytic in the set \(\text{Re}(s_1), \ldots, \text{Re}(s_r) \geq 1\) we can let \(\varepsilon \to 0\) to obtain that
\[
\int_{\mathbb{R}^r} g(t) f(t_1) \cdots f(t_r) e^{iyt} \, dt
\]
equals
\[ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^r_+} (B(x) - 1) (x_1 \cdots x_r)^{j+1} e^{-\varepsilon x} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx. \]

\[ - \sum_{i \in I(f)} c_i \int_{\mathbb{R}^r_+} (x_1 \cdots x_r)^{j+1} e^{-(1 - \theta_i)x} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx. \]

Since \( \hat{f} \) is rapidly decreasing the limit for \( \varepsilon \to 0 \) of
\[ \int_{\mathbb{R}^r_+} (x_1 \cdots x_r)^{j+1} e^{-\varepsilon x} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx \]
exists and equals
\[ \int_{\mathbb{R}^r_+} (x_1 \cdots x_r)^{j+1} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx. \]

For the sum over \( I(f) \) recall that the imaginary parts of the \( \theta \)'s are in a compact set, therefore the real parts must tend to \(-\infty\) and so the convergence is uniform in \( \varepsilon \), i.e., the limit can be interchanged with the summation. Hence also the limit
\[ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^r_+} B(x) (x_1 \cdots x_r)^{j+1} e^{-\varepsilon x} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx \]
exists. Since we assume \( \hat{f} \geq 0 \) the integrand is nonnegative and monotonically increasing as \( \varepsilon \to 0 \). Therefore the limit may be drawn under the integral sign. We conclude that
\[ \int_{\mathbb{R}^r} g(t) f(t_1) \cdots f(t_r) e^{igt} \, dt \]
equals
\[ \int_{\mathbb{R}^r_+} (B(x) - 1) (x_1 \cdots x_r)^{j+1} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx. \]

\[ - \sum_{i \in I(f)} c_i \int_{\mathbb{R}^r_+} (x_1 \cdots x_r)^{j+1} e^{-(1 - \theta_i)x} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx. \]
By the Riemann-Lebesgue Lemma this tends to zero as $y \to \infty$. For $y >> 0$ we estimate

$$
\int_{\mathbb{R}^r} (x_1 \cdots x_r)^{j+1} e^{(\theta-1)x} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx
\leq \int_{\mathbb{R}^r} (x_1 \cdots x_r)^{j+1} e^{(\theta-1)x} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx
= \int_{\mathbb{R}^r} ((x_1 + y_1) \cdots (x_r + y_r))^{j+1} e^{(\theta-1)(x+y)} \hat{f}(-x_1) \cdots \hat{f}(-x_r) \, dx
\leq (\text{const}) (y_1 \cdots y_r)^{j+1} e^{(\theta-1)y}.
$$

This implies that the sum over $I(f)$ tends to zero as $y \to \infty$.

Therefore

$$
\lim_{y \to \infty} \int_{\mathbb{R}^r} (B(x) - 1) (x_1 \cdots x_r)^{j+1} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx = 0.
$$

Lemma 3.3 For every $k = 0, 1, 2, \ldots$,

$$
\lim_{y \to \infty} \frac{1}{y^k} \int_0^\infty x^k \hat{f}(y - x) \, dx = 2\pi f(0).
$$

Proof: Start with $k = 0$. Then

$$
\lim_{y \to \infty} \int_0^\infty \hat{f}(y - x) \, dx = \lim_{y \to \infty} \int_{-\infty}^\infty \hat{f}(y - x) \, dx
= \int_{-\infty}^\infty \hat{f}(-x) \, dx = 2\pi f(0)
$$

For $k \mapsto k + 1$ consider

$$
\lim_{y \to \infty} \frac{1}{y^{k+1}} \int_0^\infty x^{k+1} \hat{f}(y - x) \, dx = \lim_{y \to \infty} \frac{1}{y^{k+1}} \int_{-\infty}^\infty x^{k+1} \hat{f}(y - x) \, dx
= \lim_{y \to \infty} \frac{1}{y^{k+1}} \int_{-\infty}^\infty (x + y)^{k+1} \hat{f}(-x) \, dx
= 2\pi f(0).
$$
This lemma implies that
\[
\lim_{y \to \infty} \int_{\mathbb{R}_+^r} B(x) \left( \frac{x_1 \cdots x_r}{y_1 \cdots y_r} \right)^{j+1} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx = (2\pi f(0))^r.
\]

Let \( S > 0 \). Since \( A(x) \) is monotonic we have \( A(y - S) \leq A(x) \leq A(y + S) \) whenever \( y - S \leq x \leq y + S \). In that range we then have
\[
B(y - S)((y_1 - S) \cdots (y_r - S))^{j+1} e^{(y-S)1} \leq B(x)(x_1 \cdots x_r)^{j+1} e^{x1}
\leq B(y + S)((y_1 + S) \cdots (y_r + S))^{j+1} e^{(y+S)1}.
\]
The first inequality implies
\[
B(x)(x_1 \cdots x_r)^{j+1} \geq B(y - S)((y_1 - S) \cdots (y_r - S))^{j+1} e^{(y-x-S)1}
\geq B(y - S)((y_1 - S) \cdots (y_r - S))^{j+1} e^{-2S1}.
\]
So for \( y \geq S \),
\[
e^{-2rS} B(y - S) \left( \frac{(y_1 - S) \cdots (y_r - S)}{y_1 \cdots y_r} \right)^{j+1} \int_{y-S}^{y+S} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx
\leq \int_{y-S}^{y+S} B(x) \left( \frac{x_1 \cdots x_r}{y_1 \cdots y_r} \right)^{j+1} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx
\leq \int_{\mathbb{R}_+^r} B(x) \left( \frac{x_1 \cdots x_r}{y_1 \cdots y_r} \right)^{j+1} \hat{f}(y_1 - x_1) \cdots \hat{f}(y_r - x_r) \, dx.
\]
This implies
\[
\limsup_{y \to \infty} B(y) \leq e^{2rS} \frac{(2\pi f(0))^2}{\int_{-S}^{S} \hat{f}(x_1) \cdots \hat{f}(x_r) \, dx}.
\]
We vary \( f \) so that \( \hat{f} \) is small outside \([-S, S]\). In this way we get
\[
\limsup_{y \to \infty} B(y) \leq e^{2rS}.
\]
Since this is true for any \( S > 0 \) it follows
\[
\limsup_{y \to \infty} B(y) \leq 1.
\]
The inequality $\liminf_{y \to \infty} B(y) \geq 1$ is obtained in a similar fashion. The Wiener-Ikehara Theorem is proven.

Let
\[
\phi(T) = \phi(T_1, \ldots, T_r) \overset{\text{def}}{=} \sum_{\substack{[\gamma] \in \mathcal{E}(\Gamma) \\ a_\gamma^{\alpha_k} \leq T_k}} \frac{\lambda_\gamma}{\det(1 - a_\gamma m_\gamma | n)}.
\]

**Lemma 3.4** As $T_1, \ldots, T_r \to \infty$ we have
\[
\phi(T_1, \ldots, T_r) \sim T_1 \cdots T_r.
\]

**Proof:** Let
\[
\phi_j(T) \overset{\text{def}}{=} \sum_{\substack{[\gamma] \in \mathcal{E}(\Gamma) \\ a_\gamma^{\alpha_k} \leq T_k}} \frac{\lambda_\gamma (l_1(a_\gamma) \cdots l_r(a_\gamma))^{j+1}}{\det(1 - a_\gamma m_\gamma | n)}.
\]

By the Wiener-Ikehara theorem we know that
\[
\frac{\phi_j(T)}{T_1 \cdots T_r (\log T_1)^{j+1} \cdots (\log T_r)^{j+1}}
\]
tends to 1 as $T \to \infty$. Obviously
\[
\frac{\phi_j(T)}{(\log T_1)^{j+1} \cdots (\log T_r)^{j+1}} \leq \phi(T),
\]
which implies
\[
\liminf_{T \to \infty} \frac{\phi(T)}{T_1 \cdots T_r} \geq 1.
\]

Let $0 < \mu < 1$. Then
\[
\phi_j(T) \geq \sum_{\substack{[\gamma] \in \mathcal{E}(\Gamma) \\ T_k^\mu < a_\gamma^{\alpha_k} \leq T_k}} \text{ind}(\gamma) (l_1(a_\gamma) \cdots l_r(a_\gamma))^{j+1}
\]
\[
\geq \mu^{r(j+1)} ((\log T_1) \cdots (\log T_r))^{j+1} \sum_{\substack{[\gamma] \in \mathcal{E}(\Gamma) \\ T_k^\mu < a_\gamma^{\alpha_k} \leq T_k}} \text{ind}(\gamma)
\]
\[
= \mu^{r(j+1)} ((\log T_1) \cdots (\log T_r))^{j+1} (\phi(T) - \phi(T^\mu))
\]
From this we get
\[
\frac{\phi(T)}{T_1 \cdots T_r} \leq \mu^{-r(j+1)} \frac{\phi_j(T)}{T_1 \cdots T_r (\log T_1)^{j+1} \cdots (\log T_r)^{j+1}} + \frac{\phi(T^\mu)}{T_1^\mu \cdots T_r^\mu (T_1 \cdots T_r)^{1-\mu}}.
\]

Assume first that $\phi(T)/T_1 \cdots T_r$ tends to infinity as $T \to \infty$. Then there is a sequence $nT$ in $\mathbb{R}_+^r$, tending to infinity such that $\phi(nT)/nT_1 \cdots nT_r$ tends to infinity and
\[
\frac{\phi(nT)}{nT_1 \cdots nT_r} \geq \frac{\phi(S)}{S_1 \cdots S_r}
\]
for every $S \leq nT$. In particular, one can choose $S = (nT)^a$. Then
\[
\frac{\phi(nT)}{nT_1 \cdots nT_r} \leq \mu^{-r(j+1)} \frac{\phi_j(nT)}{nT_1 \cdots nT_r (\log nT_1)^{j+1} \cdots (\log nT_r)^{j+1}} + \frac{\phi(nT)}{nT_1 \cdots nT_r (nT_1 \cdots nT_r)^{1-\mu}} + \frac{\phi(nT)}{nT_1 \cdots nT_r (nT_1 \cdots nT_r)^{1-\mu}},
\]
so that
\[
\frac{\phi(nT)}{nT_1 \cdots nT_r} \leq \mu^{-r(j+1)} \frac{\phi_j(nT)}{nT_1 \cdots nT_r (\log nT_1)^{j+1} \cdots (\log nT_r)^{j+1}} + \frac{1}{(nT_1 \cdots nT_r)^{1-\mu}}.
\]

Since the right hand side converges we reach a contradiction. This implies that
\[
L \overset{\text{def}}{=} \limsup_{T \to \infty} \frac{\phi(T)}{T_1 \cdots T_r} = \limsup_{T \to \infty} \frac{\phi(T^\mu)}{T_1^\mu \cdots T_r^\mu}
\]
is finite. We get
\[
L \leq \mu^{-r(j+1)} + L \limsup_{T \to \infty} \frac{1}{(T_1 \cdots T_r)^{1-\mu}} = \mu^{-r(j+1)}.
\]

Since $\mu$ is arbitrary we get $L \leq 1$. The lemma follows. \qed

We now finish the proof of the theorem. We have that $\frac{\phi(T_1 \cdots T_r)}{T_1^\mu \cdots T_r^\mu}$ tends to 1 as $T_k \to \infty$. Since $a \in A^-$ has only eigenvalues of absolute value < 1 on $n$ it follows that $0 < \det(1 - am \mid n) < 1$ for every $am \in A^-M$. Let $0 < \varepsilon < 1$. Let $\psi_\varepsilon(T)$ and $\phi_\varepsilon(T)$ denote the same sums as above extended only over those classes $[\gamma]$ with $1 - \varepsilon < \det(1 - a_\gamma m_\gamma \mid n) < 1$
Lemma 3.5 As $T_1, \ldots, T_r \to \infty$ we have

$$\frac{\phi(T_1, \ldots, T_r) - \phi_\varepsilon(T_1, \ldots, T_r)}{T_1 \cdots T_r} \to 0,$$

and

$$\frac{\psi(T_1, \ldots, T_r) - \psi_\varepsilon(T_1, \ldots, T_r)}{T_1 \cdots T_r} \to 0.$$

Proof: Note that $\det(1 - a_\gamma m_\gamma \mid n)$ can only be $\leq 1 - \varepsilon$ if $a_\gamma$ is close to a wall in $A^-$. In other words, $a_\gamma$ has to lie in one of the segments

$$S_\alpha^c = \{a \in A^- \mid a^{-\alpha} < c(\varepsilon, \alpha)\}$$

for some $\alpha \in \Delta$ and some $c(\varepsilon, \alpha) > 1$. With $c = \max(c(\varepsilon, \alpha), \alpha \in \Delta)$ it follows that

$$\phi(T) - \phi_\varepsilon(T) \leq \sum_{j=1}^{r} \phi(T_1, \ldots, c, \ldots, T_r).$$

The $k$th summand on the right hand side does not depend upon $T_k$. This implies the first assertion. For the second note that

$$\phi(T) - \phi_\varepsilon(T) \geq \frac{1}{1 - \varepsilon}(\psi(T) - \psi_\varepsilon(T)).$$

The lemma follows. \qed

It follows that

$$\frac{\phi_\varepsilon(T_1, \ldots, T_r)}{T_1 \cdots T_r} \to 1 \quad \text{as } T_j \to \infty.$$

For each $[\gamma]$ in this sum we have

$$1 - \varepsilon < \det(1 - a_\gamma m_\gamma \mid n) < 1,$$

and so

$$1 - \varepsilon < \frac{1}{\det(1 - a_\gamma m_\gamma \mid n)} < \frac{1}{\det(1 - a_\gamma m_\gamma \mid n)}.$$

Summing up we get

$$\frac{\phi_\varepsilon(T_1, \ldots, T_r)}{T_1 \cdots T_r}(1 - \varepsilon) < \frac{\psi_\varepsilon(T_1, \ldots, T_r)}{T_1 \cdots T_r} < \frac{\phi_\varepsilon(T_1, \ldots, T_r)}{T_1 \cdots T_r}.$$
Since \( \phi_{\epsilon}(T_1, \ldots, T_r) \) tends to 1 it follows that

\[
1 - \epsilon \leq \liminf_{T_j \to \infty} \frac{\psi_{\epsilon}(T_1, \ldots, T_r)}{T_1 \cdots T_r} \leq \limsup_{T_j \to \infty} \frac{\psi_{\epsilon}(T_1, \ldots, T_r)}{T_1 \cdots T_r} \leq 1.
\]

The second part of Lemma 3.5 gives

\[
1 - \epsilon \leq \liminf_{T_j \to \infty} \frac{\psi(T_1, \ldots, T_r)}{T_1 \cdots T_r} \leq \limsup_{T_j \to \infty} \frac{\psi(T_1, \ldots, T_r)}{T_1 \cdots T_r} \leq 1.
\]

Since \( \epsilon \) is arbitrary the prime geodesic theorem follows.

We will finish this section with a conjecture. For each \( [\gamma] \in \mathcal{E}(\Gamma) \) pick a closed geodesic \( g_\gamma \) which is closed by \( \gamma \). Note that \( \lambda_\gamma \) equals the volume of the unique maximal flat that contains the geodesic \( g_\gamma \). The contribution of all geodesics that lie in a given flat \( F \) grows like \( \left(\log T_1\right) \cdots \left(\log T_r\right) \frac{\text{vol}(F)}{T_1 \cdots T_r} \), as \( T_1, \ldots, T_r \to \infty \). This motivates the following conjecture.

Conjecture 3.6 Let

\[
\pi(T_1, \ldots, T_r) \overset{\text{def}}{=} \# \{ [\gamma] \in \mathcal{E}(\Gamma) \mid a_1^{a_1} \leq T_1, \ldots, a_r^{a_r} \leq T_r \}.
\]

Then, as \( T_1, \ldots, T_r \to \infty \),

\[
\pi(T_1, \ldots, T_r) \sim \frac{T_1}{\log T_1} \cdots \frac{T_r}{\log T_r}.
\]

It is not hard to show that the conjecture holds for products of rank one spaces.

A Appendix: An application to class numbers

In this section we give a new asymptotic formula for class numbers of orders in number fields. It is quite different from known results like Siegel’s Theorem ([Π], Thm 6.2). The asymptotic is in several variables and thus contains more
Let $d$ be a prime number $\geq 3$. Let $S$ be a finite set of primes with $|S| \geq 2$. Let $C(S)$ be the set of all totally real number fields $F$ with the property $p \in S \Rightarrow p$ is non-decomposed in $F$.

Let $O(S)$ denote the set of all orders $O$ in number fields $F \in C(S)$ which are maximal at each $p \in S$. For such an order $O$ let $h(O)$ be its class number, $R(O)$ its regulator and $\lambda_S(O) = \prod_{p \in S} f_p$, where $f_p$ is the inertia degree of $p$ in $F = O \otimes \mathbb{Q}$. Then $f_p \in \{1, d\}$ for every $p \in S$.

For $\lambda \in O^*$ let $\rho_1, \ldots, \rho_d$ denote the real embeddings of $F$ ordered in a way that $|\rho_k(\lambda)| \geq |\rho_{k+1}(\lambda)|$ holds for $k = 1, \ldots, d - 1$.

For $k = 1, \ldots, d - 1$ let

$$\alpha_k(\lambda) \overset{\text{def}}{=} k(d-k) \log \left( \frac{|\rho_k(\lambda)|}{|\rho_{k+1}(\lambda)|} \right).$$

Let

$$c = (\sqrt{2})^{1-d} \left( \prod_{k=1}^{d-1} 2k(d-k) \right).$$

So $c > 0$ and it comes about as correctional factor between the Haar measure normalization used in the Prime Geodesic Theorem and the normalization used in the definition of the regulator.

**Theorem A.1** For $T_1, \ldots, T_r > 0$ set

$$\vartheta_S(T) \overset{\text{def}}{=} \sum_{\lambda \in O^*/\pm 1, O \in O(S) \atop 0 < \alpha_k(\lambda) \leq T_k \atop k=1,\ldots,d-1} R(O) \ h(O) \ \lambda_S(O).$$

Then we have, as $T_1, \ldots, T_{d-1} \to \infty$,

$$\vartheta(T_1, \ldots, T_{d-1}) \sim \frac{c}{\sqrt{d}} T_1 \cdots T_{d-1}.$$
**Proof:** For given $S$ there is a division algebra $M$ over $\mathbb{Q}$ of degree $d$ which splits exactly outside $S$. Fix a maximal order $M(\mathbb{Z})$ in $M$ and for any ring $R$ define $M(R) \overset{\text{def}}{=} M(\mathbb{Z}) \otimes R$. Let $\det : M(R) \to R$ denote the reduced norm then

$$\mathcal{G}(R) \overset{\text{def}}{=} \{ x \in M(R) \mid \det(x) = 1 \}$$

defines a group scheme over $\mathbb{Z}$ with $\mathcal{G}(\mathbb{R}) \cong \text{SL}_d(\mathbb{R}) = G$. Then $\Gamma = \mathcal{G}(\mathbb{Z})$ is a cocompact discrete torsion-free subgroup of $G$ (see [5]). As can be seen in [5], Theorem A.1 can be deduced from the prime geodesic theorem.

□

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