Spin-entangled Squeezed State on a Bloch Four-hyperboloid

Kazuki Hasebe
National Institute of Technology, Sendai College, Ayashi, Sendai, 989-3128, Japan
khasebe@sendai-nct.ac.jp

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Abstract

The Bloch hyperboloid $H^2$ underlies the quantum geometry of the original $SO(2, 1)$ squeezed states. In [23], the author utilized a non-compact 2nd Hopf map and a Bloch four-hyperboloid $H^{2,2}$ to explore an $SO(2, 3)$ extension of the squeezed states. In the present paper, we further pursue the idea to derive an $SO(4, 1)$ version of squeezed vacuum based on the other Bloch four-hyperboloid $H^4$. We show that the obtained $SO(4, 1)$ squeezed vacuum is a particular four-mode squeezed state not quite similar to the previous $SO(2, 3)$ squeezed vacuum. In view of the Schwinger’s formulation of angular momentum, the $SO(4, 1)$ squeezed vacuum is interpreted as a superposition of an infinite number of maximally entangled spin-pairs of all integer spins. We clarify basic properties of the $SO(4, 1)$ squeezed vacuum, such as von Neumann entropy of spin entanglement, spin correlations and uncertainty relations with emphasis on their distinctions to the original $SO(2, 1)$ case.
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1 Introduction

Squeezed state [1, 2, 3, 4, 5, 6] is a particularly important quantum state in quantum optical information and practical applications (see [8, 9] as nice reviews.) In quantum optical information, two-mode squeezed light realizes an entangled state of photons and it has been employed in state-of-the-art experiments of quantum mechanics (see [7] for instance and references therein). Also in practical application to quantum telecommunication, the squeezed state has attracted a lot of attention due to its intrinsic anisotropic property of quantum noise. The uncertainty region of squeezed state is not isotropic unlike the coherent state (laser) and an unfavorable quantum noise is reduced in a specific direction.

Up to now, many theoretical extensions of the squeezed state have been proposed [11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. In an attempt of extending the formalism of the squeezed state, mathematical aesthetics and logically natural extension should be appreciated. One fascinating way to formulate a theory of physics may resort to geometry. Needless to say, most successful theories of physics in this approach may be Einstein’s general relativity of Riemann geometry and Yang-Mills theory of fibre-bundle geometry. Also in quantum information theory, there is a reasonable fact that we can believe that geometric approach plays a key role, since qubit, the fundamental object in quantum information, is described by the geometry of Bloch sphere associated with the Hopf map [21, 22], and geometric structure seems to be inherent in the formulation.

As Bloch sphere provides a geometric description of qubit, Bloch hyperboloid plays a similar role in description of squeezed state [see [23] and references therein]. Mathematically, Bloch sphere realizes the basemanifold of the 1st Hopf map, and similarly Bloch hyperboloid denotes the basemanifold of the non-compact 1st Hopf map. For 2D hyperboloids, we have two kinds of hyperboloids, two-leaf hyperboloid and one-leaf hyperboloid, which are related to the two kinds of the non-compact 1st Hopf maps:

$$H^2 \simeq H^{2,1}/S^1, \quad H^{1,1} \simeq H^{2,1}/H^1. \quad (1)$$

In both cases, the total bundle space is $H^{2,1}$. The base-manifolds are $H^2$ and $H^{1,1}$ with compact fibre $S^1$ and non-compact fibre $H^1$ respectively. The geometric origin of the original squeezed state is accounted for by the left of (1) with the Bloch hyperboloid $H^2$. Interestingly, the 1st non-compact Hopf maps have their higher dimensional cousins, i.e. the 2nd and 3rd non-compact Hopf maps [24, 25]. This observation led the author to propose an extension of squeezed state based on the geometry of higher dimensional hyperboloids [23]. As a higher dimensional analogue of (1), we have the 2nd non-compact Hopf maps:

$$H^4 \simeq H^{4,3}/S^3, \quad H^{2,2} \simeq H^{4,3}/H^{2,1}. \quad (6)$$

As in (1), the total space in both cases of (6) is given by $H^{4,3}$, and $H^4$ and $H^{2,2}$ respectively realize the base-manifolds with compact fibre $S^3$ and non-compact fibre $H^{2,1}$. We utilized $H^{2,2}$ (the right of (6))

1 The non-compact Hopf maps have already been applied to several subjects, such as twistorial quantum Hall effect [26], non-Hermitian topological insulators [27, 28], and cosmological models of matrix model [29, 30, 31]. Higher dimensional hyperboloid $H^{p,q}$ is defined as

$$\sum_{i=1}^{p} x^i x^i - \sum_{i=p+1}^{p+q+1} x^i x^i = -1. \quad (2)$$

As a coset space, $H^{p,q}$ is represented as

$$H^{p,q} \simeq SO(p,q+1)/SO(p,q), \quad (3)$$

and the topology is

$$H^{p,q} \simeq \mathbb{R}^p \otimes S^q. \quad (4)$$

In low dimensions, $H^{p,q}$ realizes Anti-de Sitter, de Sitter and Euclidean anti-de Sitter spaces:

$$H^{2,0} = EAdS^2, \quad H^{1,1} = dS^2 = AdS^2,$$
$$H^{4,0} = EAdS^4, \quad H^{3,1} = AdS^4, \quad H^{1,3} = dS^4. \quad (5)$$
to construct an $Sp(4; \mathbb{R}) \simeq SO(2, 3)$ extension of squeezed state \[23\]. In the present paper, we propose an $SO(4, 1)$ version of squeezed state based on the $H^4$ geometry (the right of (10)). It turns out that the $SO(4, 1)$ squeezed vacuum attains a more natural generalization of the original squeezed vacuum than the previous $SO(2, 3)$ case in several respects. The $SO(4, 1)$ squeezed vacuum accommodates an interesting spin structure related to the hierarchy of the non-compact 2nd Hopf map:

$$H^4 \rightarrow S^3 \rightarrow \frac{S^1}{H^4} \rightarrow S^2$$

$H^{4:3}$ can be regarded as a fibre-bundle of base-manifold $H^4$ with $S^3$-fibre which itself denotes the fibre-bundle space of 1st Hopf map of the Bloch sphere. This implies the existence of inherent spin geometry in the present $SO(4, 1)$ formulation.

Before proceeding to detail discussions, we mention differences between the previous $SO(2, 3)$ formulation \[23\] and the present $SO(4, 1)$ formulation in a group theory point of view. For the original $SO(2, 1)$ squeezed state, the associated two-hyperboloid is given by

$$H^2 \simeq SO(2, 1)/SO(2) \simeq SU(1, 1)/U(1) \simeq Sp(2; \mathbb{R})/U(1) \simeq U(1; \mathbb{H})/U(1).$$

(7)

The symmetry group $SO(2, 1)$ allows Majorana representation as well as the Dirac representation, which leads to one-mode and two-mode squeeze operators. The holonomy group is a compact group $SO(2) \simeq U(1)$. Meanwhile in the previous work \[23\], we utilized a Bloch four-hyperboloid (right of (6)):

$$H^{2:2} \simeq SO(2, 3)/SO(2, 2) \simeq Sp(4; \mathbb{R})/(SU(1, 1) \otimes SU(1, 1)) \simeq U(2; \mathbb{H})/(U(1; \mathbb{H}) \otimes U(1; \mathbb{H})).$$

(8)

The symmetry group $SO(2, 3)$ also accommodates Majorana representation as well as the Dirac representation, and the corresponding two-mode and four-mode $SO(2, 3)$ squeezed states are constructed. A crucial difference to the original case is the holonomy group $SO(2, 2) \simeq SU(1, 1) \otimes SU(1, 1)$, which is non-compact unlike $SO(2)$. In the present work, we will adopt the other Bloch four-hyperboloid (left of (6)):

$$H^4 \simeq SO(4, 1)/SO(4) \simeq USp(2, 2)/(SU(2) \otimes SU(2)) \simeq U(1, 1; \mathbb{H})/(U(1; \mathbb{H}) \otimes U(1; \mathbb{H})).$$

(9)

The symmetry group $SO(4, 1)$ does not accommodate Majorana representation, and so the two-mode realization of $SO(4, 1)$ squeezed state is not possible, and hence the $SO(4, 1)$ squeezed operator can be realized only by a four-mode Dirac representation. An analogous point to the original case is the holonomy group, which is given by a compact group $SO(4) \simeq SU(2) \otimes SU(2)$ like $SO(2)$.

This paper is organized as follows. In Sec.2 we review spin-coherent state and squeezed state with emphasis on their relations to the geometry of Bloch sphere and hyperboloid. We also discusses interesting relationship between spin-coherent states and squeezed states. In Sec.3 the $SO(4, 1)$ squeeze operator is introduced based on the geometry of the Bloch four-hyperboloid $H^4$. In Sec.4 we propose a physical interpretation of the $SO(4, 1)$ squeezed vacuum in spin geometry and explore its basic properties, such as von Neumann entropy, spin correlations and uncertainty relations. Sec.5 is devoted to summary and discussions.

2 Bloch hyperboloid and squeezed states

We begin with a review of the Bloch sphere and Bloch hyperboloid and their associated coherent states. The Schwinger operator formulation is not only useful to represent spin state with arbitrary spin but also
closely related to the squeeze operator.

### 2.1 Bloch sphere and spin-coherent state

The geometry of the Bloch sphere \[^{21}\] stems from the 1st Hopf map:\[^{2}\]

\[
S^2 \xrightarrow{S^1} S^2, \tag{10}
\]

and the Hopf map is explicitly realized as the map from the 2-component normalized spinor \((\psi \dagger \psi = 1 : S^3)\) to a normalized three component vector \((\sum_{i=1}^{3} x_i = 1 : S^2)\) \[^{[22]}\]:

\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to x_i = \psi_i \sigma_i \psi. \tag{11}
\]

Here, \(\sigma_i (i = 1, 2, 3)\) are the Pauli matrices:

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}
\]

When we parameterize \(x_i \in S^2\) by using the usual polar coordinates

\[
x_1 = \sin \theta \cos \phi, \quad x_2 = \sin \theta \sin \phi, \quad x_3 = \cos \theta,
\]

\(\psi\) can be expressed as

\[
\psi(\phi, \theta, \chi) = \begin{pmatrix} \cos(\frac{\theta}{2}) e^{-i\frac{\chi}{2}} \\ \sin(\frac{\theta}{2}) e^{i\frac{\chi}{2}} \end{pmatrix} e^{-i\frac{i}{2} \chi}. \tag{14}
\]

Here, \(e^{-i\frac{i}{2} \chi}\) denotes the overall \(U(1)\) phase that corresponds to the \(S^1\) fibre in \[^{10}\]. Importantly, \(\psi\) has a physical meaning as spin-coherent state \[^{[33, 34, 35]}\] that is aligned to the direction of \((\theta, \phi)\) on the \(S^2\) (see the left of Fig.1):

\[
(x(\theta, \phi) \cdot \sigma)\psi = \psi. \tag{15}
\]

Applying the Euler angle rotation \[^{[36]}\] to the spin state aligned to \(z\) direction, we can readily reproduce \[^{[14]}\] as

\[
\psi(\phi, \theta, \chi) \equiv D^{(1/2)}(\phi, \theta, \chi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos(\frac{\theta}{2}) e^{-i\frac{\chi}{2}(\chi+\phi)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\frac{\theta}{2}) e^{-i\frac{\chi}{2}(\chi-\phi)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{16}
\]

The rotation matrix in \[^{[10]}\] is known as the Wigner’s (spin 1/2) D-matrix of the \(SU(2)\) group:

\[
D^{(1/2)}(\phi, \theta, \chi) = e^{-i\frac{\chi}{2} \sigma_z} = e^{-i\frac{\chi}{2} \sigma_z} e^{-i\frac{\chi}{2} \sigma_z} = \begin{pmatrix} \cos(\frac{\theta}{2}) e^{-i\frac{\chi}{2}(\chi+\phi)} & -\sin(\frac{\theta}{2}) e^{i\frac{\chi}{2}(\chi-\phi)} \\ \sin(\frac{\theta}{2}) e^{i\frac{\chi}{2}(\chi-\phi)} & \cos(\frac{\theta}{2}) e^{-i\frac{\chi}{2}(\chi+\phi)} \end{pmatrix}, \tag{17}
\]

where the range of the parameters are

\[
0 \leq \phi < 2\pi, \quad 0 \leq \theta < \pi, \quad 0 \leq \chi < 4\pi. \tag{18}
\]

For later convenience, we also introduce coset representative of the \(SU(2)\) group:\[^{[3]}\] As \(S^2\) is expressed as a coset

\[
S^2 \simeq S^3/S^1 \simeq SU(2)/U(1), \tag{19}
\]

\[^{2}\] See \[^{[22]}\] for the roles of the 2nd Hopf map in the context of entanglement of qubits.

\[^{3}\] Coset representative (see \[^{[38]}\] for instance) is also referred to as the non-linear realization \[^{[39, 40, 41]}\].
the corresponding coset representative is obtained as

\[ e^{i\theta \sum_{m=1,2} y_m \sigma_{m3}} = e^{-\frac{i\phi}{2}(e^{-i\phi} \sigma^+ - e^{i\phi} \sigma^-)} = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} e^{-i\phi} \\ \sin \frac{\theta}{2} e^{i\phi} & \cos \frac{\theta}{2} \end{pmatrix}, \]

(20)

where

\[ y_1 = \cos \phi, \quad y_2 = \sin \phi, \]

and

\[ \sigma_{ij} = \frac{1}{2} \delta_{ij} \sigma_k, \quad \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]

(21)

(20) can be factorized as

Euler decomposition : \[ e^{i\theta \sum_{m=1,2} y_m \sigma_{m3}} = e^{-\frac{i\phi}{2}\sigma^+} e^{-\frac{i\phi}{2}\sigma_z} e^{i\phi/2} \sigma_z = D^{(1/2)}(\phi, \theta, -\phi) \]

(23)

and

Gauss decomposition : \[ e^{i\theta \sum_{m=1,2} y_m \sigma_{m3}} = e^{-\frac{i\phi}{2}\sigma^+} e^{-\tan(\frac{\theta}{2})\sigma_z} e^{\frac{i\phi}{2}\sigma_z} \]

(24)

Notice that the Euler decomposition (23) is realized as a special case of the D-matrix and the physical meaning of the coset representative is transparent in this decomposition.

2.1.1 Schwinger operator realization

We construct the spin-coherent state with arbitrary spin magnitude using the Schwinger operator formalism \[37\,36\]. With the Schwinger boson operator

\[ \hat{\psi} \equiv \begin{pmatrix} a \\ b \end{pmatrix} \]

(25)

subject to

\[ [a, a^\dagger] = [b, b^\dagger] = 1, \quad [a, b] = [a, b^\dagger] = 0, \]

(26)

the spin operators are expressed as

\[ \hat{S}_i = \hat{\psi}^\dagger \frac{1}{2} \sigma_i \hat{\psi}. \]

(27)

In detail,

\[ \hat{S}_x = \frac{1}{2} (a^\dagger b + b^\dagger a), \quad \hat{S}_y = -i \frac{1}{2} (a^\dagger b - b^\dagger a), \quad \hat{S}_z = \frac{1}{2} (a^\dagger a - b^\dagger b), \]

(28)
and so
\[ \hat{S}^2 = \hat{S}(\hat{S} + 1) \]  
with
\[ \hat{S} = \frac{1}{2}(\hat{n}_a + \hat{n}_b) = \frac{1}{2}(a^\dagger a + b^\dagger b). \]
The spin states \(|S, S_z\rangle\) that satisfy
\[ \hat{S}^2 |S, S_z\rangle = S(S + 1) |S, S_z\rangle, \quad \hat{S}_z |S, S_z\rangle = S_z |S, S_z\rangle \]  
are constructed as
\[ |S, S_z\rangle \equiv |n_a, n_b\rangle = \frac{1}{\sqrt{n_a!n_b!}} (a^\dagger)^{n_a} (b^\dagger)^{n_b} |0, 0\rangle, \]
where the relations between the boson numbers \((n_a, n_b)\) and spin values \((S, S_z)\) are depicted in Fig. 2, and hereafter we utilize \(|n_a, n_b\rangle\) and \(|S, S_z\rangle\) interchangeably.

Figure 2: The SU(2) irreducible representation with fixed \(S\) is specified by each of the oblique blue lines of \(n_a + n_b = 2S\).

We introduce D-operator as
\[ \hat{D}(\phi, \theta, \chi) \equiv e^{-i\phi \hat{S}_z} e^{-i\theta \hat{S}_y} e^{-i\chi \hat{S}_z} \]  
whose matrix representation is the D-matrix:
\[ \langle S, S_z | \hat{D}(\phi, \theta, \chi) | S', S'_z \rangle = D^{(S)}(\phi, \theta, \chi)_{S, S_z} \cdot \delta_{S, S'}, \]
i.e.,
\[ D^{(S)}(\phi, \theta, \chi) = e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\chi S_z}. \]
In (35), \(S_{x,y,z}\) denote the SU(2) matrices with spin magnitude \(S\). Under the SU(2) transformation of \(\hat{D}(\phi, \theta, \chi)\), (29) transforms as an SU(2) spinor
\[ \hat{D}(\phi, \theta, \chi) |\psi\rangle = D^{(1/2)}(\phi, \theta, \chi) |\psi\rangle. \]
In a similar manner to (16), we can readily construct the spin-coherent state with arbitrary spin magnitude:
\[ |S, S_z(\phi, \theta, \chi)\rangle = \hat{D}(\phi, \theta, \chi) |S, S_z\rangle = e^{-iS_z \chi} \hat{D}(\phi, \theta, 0) |S, S_z\rangle, \]
which satisfies
\[ \hat{S}^2 |S, S_z(\phi, \theta, \chi)\rangle = \frac{1}{4}(n_a + n_b)(n_a + n_b + 2) |S, S_z(\phi, \theta, \chi)\rangle, \]
\[ (\mathbf{x}(\theta, \phi) \cdot \hat{S}) |S, S_z(\phi, \theta, \chi)\rangle = \frac{1}{2}(n_a - n_b) |S, S_z(\phi, \theta, \chi)\rangle. \]
With (34), (37) can be concisely expressed as a linear combination of the spin states with expansion coefficients of the components of D-matrix:

$$|S, S_z(\phi, \theta, \chi)\rangle = \sum_{S_z'=S}^{S} D^{(S)}(\phi, \theta, \chi)_{S_z'S_z} |S, S_z'\rangle,$$  \hspace{1cm} (39)

which is a ket state expression of (16) generalizing the spin $1/2$ to arbitrary spin magnitude. In the Schwinger boson notation, (39) is represented as

$$|S, S_z(\phi, \theta, \chi)\rangle = \frac{1}{2} \sum_{n_{a},n_{b}=0}^{2S} D^{(S=\frac{1}{2}(n_{a}+n_{b}))}(\phi, \theta, \chi)_{\frac{1}{2}(n_{a}'-n_{b}'), \frac{1}{2}(n_{a}-n_{b})} |n_{a}', n_{b}'\rangle, \hspace{1cm} (40)$$

2.2 Bloch hyperboloid and pseudo-spin-coherent state

Here, we extend the above discussions to the case of hyperboloid. The $SU(1,1)$ version of D-operator realizes the squeeze operator in its special form.

2.2.1 $SU(1,1)$ algebra and the Schwinger operator realization

The non-compact Hopf map is given by

$$H^{2,1} \xrightarrow{S^1} H^2,$$  \hspace{1cm} (41)

and, in the same sense of the Bloch sphere, we refer to the base-manifold $H^2$ as the Bloch hyperboloid. $H^{2,1}$ is a group manifold of the $SU(1,1)$ whose invariant matrix is $\sigma_z$. The $SU(1,1)$ matrix generators can be realized as the following non-Hermitian matrices

$$\frac{1}{2} \tau^i = \{i \frac{1}{2} \sigma_x, \frac{1}{2} \sigma_y, \frac{1}{2} \sigma_z\}, \hspace{1cm} (42)$$

which satisfy

$$\{\tau^i, \tau^j\} = 2\eta^{ij}, \quad [\tau^i, \tau^j] = -2i\epsilon^{ijk}\tau_k, \hspace{1cm} (43)$$

with

$$\eta_{ij} = \text{diag}(-, -, +), \quad \epsilon^{123} = 1. \hspace{1cm} (44)$$

With two-component spinor subject to $\psi^\dagger \sigma_z \psi = |\psi_1|^2 - |\psi_2|^2 = 1 : H^{2,1}$, the non-compact Hopf map is explicitly realized as

$$\psi \rightarrow x^i = \psi^\dagger \kappa^i \psi, \hspace{1cm} (45)$$

where $\kappa^i$ are Hermitian matrices made of $\tau^i$:

$$\kappa^i = \sigma_z \tau^i = \{-\sigma_y, \sigma_x, 1\}, \hspace{1cm} (46)$$

$x^i$ (45) satisfy

$$\eta_{ij} x^i x^j = x^i x_i = -x^1 x^1 - x^2 x^2 + x^3 x^3 = (\psi^\dagger \kappa \psi)^2 = 1, \hspace{1cm} (47)$$

and denote the coordinates on $H^2$. When $x^i$ are parameterized as

$$x^1 = \sinh \rho \sin \phi, \quad x^2 = \sinh \rho \cos \phi, \quad x^3 = \cosh \rho, \hspace{1cm} (48)$$

Interestingly, D-matrices themselves have a physical meaning; $D^{(S)}(\phi, \theta, \chi)_{S_z'S_z}$ denote $2S + 1$ degenerate eigenstates of $(S - |S_z|)$th Landau level in the monopole background with magnetic charge $S_z$ (see [44] for instance).
ψ can be realized as
\[
\psi(\phi, \rho, \chi) = \begin{pmatrix}
\cosh(\frac{\phi}{2}) e^{i\frac{\chi}{2}} \\
\sinh(\frac{\phi}{2}) e^{-i\frac{\chi}{2}}
\end{pmatrix} e^{i\frac{\rho}{2}},
\]
(49)
which also acts as the pseudo-spin-coherent state \[34, 42, 43\] (see the right of Fig.1):
\[(x^i(\rho, \phi) \tau_i) \psi = \psi.\]
(50)
ψ can be derived by acting the SU(1, 1) rotation to the pseudo-spin state aligned to z-direction:
\[
\psi(\phi, \rho, \chi) = D^{(1/2)}(\phi, \rho, \chi) \begin{pmatrix}
1 \\
0
\end{pmatrix},
\]
(51)
where \(D^{(1/2)}(\phi, \rho, \chi)\) is the SU(1, 1) version of the D-matrix\[5\]
\[
D^{(1/2)}(\phi, \rho, \chi) = e^{i\frac{\rho}{2} z^z} e^{-i\frac{\phi}{2} z^z} e^{i\frac{\chi}{2} z^z} = \begin{pmatrix}
\cosh(\frac{\phi}{2}) e^{i\frac{\chi}{2}(\chi+\phi)} & \sinh(\frac{\phi}{2}) e^{-i\frac{\chi}{2}(\chi-\phi)} \\
\sinh(\frac{\phi}{2}) e^{i\frac{\chi}{2}(\chi-\phi)} & \cosh(\frac{\phi}{2}) e^{-i\frac{\chi}{2}(\chi+\phi)}
\end{pmatrix}.
\]
(53)
The ranges of the parameters are
\[
\rho = [0, \infty), \quad \phi = [0, 2\pi), \quad \chi = [0, 4\pi).
\]
(54)
With the Schwinger boson operators, we can construct the SU(1, 1) operators as
\[
\hat{T}^i = \hat{\psi}^\dagger \kappa^i \hat{\psi}.
\]
(55)
Here, \(\hat{\psi}\) is defined as
\[
\hat{\psi} \equiv \begin{pmatrix}
a \\
b \end{pmatrix},
\]
(56)
whose components satisfy
\[
[\hat{\psi}_\alpha, \hat{\psi}_\beta^\dagger] = (\sigma_z)_{\alpha\beta}, \quad [\hat{\psi}_\alpha, \hat{\psi}_\beta] = 0.
\]
(57)
The SU(1, 1) Hermitian operators are represented as \[45, 46, 47\]
\[
\hat{T}^x = i\frac{1}{2}(a^\dagger b - ab), \quad \hat{T}^y = i\frac{1}{2}(a^\dagger b^\dagger + ab), \quad \hat{T}^z = \frac{1}{2}(a^\dagger a + b^\dagger b) + \frac{1}{2},
\]
(58)
and the ladder operators are
\[
\hat{T}^+ = a^\dagger b^\dagger, \quad \hat{T}^- = ab.
\]
(59)
The SO(2, 1) Casimir is derived as
\[
C = \hat{T}_i \hat{T}^i = -(\hat{T}^x)^2 - (\hat{T}^y)^2 + (\hat{T}^z)^2 = \frac{1}{4}(\hat{\psi}^\dagger \sigma_z \hat{\psi}) (\hat{\psi}^\dagger \sigma_z \hat{\psi} + 2),
\]
(60)
where
\[
\hat{\psi}^\dagger \sigma_z \hat{\psi} = a^\dagger a - bb^\dagger = a^\dagger a - b^\dagger b - 1.
\]
(61)
The SU(1, 1) D-operator is given by
\[
\hat{D}(\phi, \rho, \chi) = e^{i\phi \hat{T}_z} e^{-i\rho \hat{T}_x} e^{i\chi \hat{T}_z}.
\]
(62)
Under the SU(1, 1) transformation of \(\hat{D}(\phi, \rho, \chi)\), \(\hat{\psi}\) transforms as an SU(1, 1) spinor
\[
\hat{D}(\phi, \rho, \chi)^\dagger \hat{\psi} \hat{D}(\phi, \rho, \chi) = D^{(1/2)}(\phi, \rho, \chi) \hat{\psi}.
\]
(63)
2.2.2 $SU(1,1)$ coset representative and the squeeze operator

In the parameterization
\[ y_1(\phi) = \sin \phi, \quad y_2(\phi) = \cos \phi \in S^1, \] (64)
an $SU(1,1)$ coset representative for $H^2 \simeq SU(1,1)/U(1)$ is obtained as
\[ M(\rho, \phi) = e^{i \frac{\rho}{2} t_m n m(\phi)} \tau^n = e^{-\frac{\rho}{2} e^{i\phi} t^+ + \frac{\rho}{2} e^{-i\phi} t^-} = \begin{pmatrix} \cosh(\frac{\rho}{4}) & -\sinh(\frac{\rho}{4}) e^{i\phi} \\ -\sinh(\frac{\rho}{4}) e^{-i\phi} & \cosh(\frac{\rho}{4}) \end{pmatrix}, \] (65)
with $\epsilon_{12} = -\epsilon_{21} = 1$ and
\[ t^+ = -\frac{1}{2} \tau^x + \frac{1}{2} \tau^y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t^- = -\frac{1}{2} \tau^x + \frac{1}{2} \tau^y = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \] (66)

(65) can be factorized as
\[ \text{Euler decomposition : } M(\rho, \phi) = e^{i \frac{\rho}{2} \tau^x} e^{i \frac{\phi}{2} \tau^y} e^{-i \frac{\phi}{2} \tau^x} = D^{(1/2)}(\phi, -\rho, -\phi), \] (67)
and
\[ \text{Gauss decomposition : } M(\rho, \phi) = e^{-\tanh(\frac{\rho}{4})} e^{i\phi} \tau^+ e^{-i\ln(\cosh(\frac{\rho}{2})) \tau^+} e^{\tanh(\frac{\rho}{4})} e^{-i\phi} \tau^- \]
\[ = e^{-\tanh(\frac{\rho}{4}) y_m(\frac{1}{2} \tau_m - i \tau m_3)} e^{-\log(\cosh(\frac{\rho}{2})) \tau_z} e^{\tanh(\frac{\rho}{4}) y_m(\frac{1}{2} \tau_m + i \tau m_3)}, \] (68)
where
\[ \tau^{ij} = -\frac{1}{4} [\tau^i, \tau^j] = -\frac{1}{2} \epsilon^{ijk} \tau_k. \] (69)

In the Gauss decomposition (68), the Poincaré Disc coordinates, $\tanh(\frac{\rho}{4}) y_m(\phi)$, appear on the shoulder of the exponent. The operator that corresponds to the coset representative (68) is constructed as
\[ \hat{S}(\rho, \phi) = e^{i \frac{\rho}{2} \tau^x} e^{i \frac{\phi}{2} \tau^y} e^{-i \frac{\phi}{2} \tau^x} = \exp \left( -\frac{\rho}{2} e^{i\phi} a^\dagger b + \frac{\rho}{2} e^{-i\phi} ab \right). \] (70)

Notice that (70) is nothing but the two-mode squeeze operator in quantum optics [2, 3, 4]. Thus, the $SU(1,1)$ D-operator realizes the squeeze operator in its special form. Since the $SU(1,1)$ D-operator signifies hyperbolic rotation, the squeeze operator can be interpreted as a hyperbolic rotation operator. As exemplified in the above, (70) is factorized as
\[ \hat{S}(\rho, \phi) = \hat{D}(\phi, -\rho, -\phi) = e^{i\phi \hat{T}^z} e^{i\rho \hat{T}^x} e^{-i\phi \hat{T}^z} \]
\[ = e^{i \frac{\phi}{2} (\hat{n}_a + \hat{n}_b)} e^{-\frac{\rho}{2} (a^\dagger b + ab)} e^{-i \frac{\phi}{2} (\hat{n}_a + \hat{n}_b)}, \] (71)
and
\[ \text{Gauss decomposition : } \hat{S}(\rho, \phi) = e^{-\tanh(\frac{\rho}{4})} e^{i\phi} \tau^+ e^{-2 \ln(\cosh(\frac{\rho}{2})) \tau^+} e^{\tanh(\frac{\rho}{4})} e^{-i\phi} \tau^- \]
\[ = \frac{1}{\cosh^2(\frac{\rho}{4})} e^{-\tanh(\frac{\rho}{4})} e^{i\phi} a^\dagger b^\dagger \left( \frac{1}{\cosh(\frac{\rho}{4})} \right) e^{\tau^+} \tau_z \left( e^{\tanh(\frac{\rho}{4})} \right) e^{-i\phi} ab. \] (72)

The Euler decomposition (71) demonstrates the physical meaning of the squeeze operation, i.e., the $SU(1,1)$ hyperbolic rotation. Meanwhile, the Gauss decomposition is very useful in deriving the number state expansion of squeezed state [17], as (72) is realized as an almost normal ordered form.

The phase $\phi$ can be absorbed in the phase redefinition of the Schwinger boson operators:
\[ a \rightarrow e^{-i\alpha} a, \quad b \rightarrow e^{-i\beta} b, \] (73)
\[ \text{The commutation relations (66) are immune to the phase redefinition of each Schwinger operator (73).} \]
the squeeze operator is transformed as

\[ \hat{S}(\rho, \phi) = \exp \left( -\frac{\rho}{2} e^{i\phi} a^\dagger b^\dagger + \frac{\rho}{2} e^{-i\phi} ab \right) \rightarrow S(\rho) = \exp \left( -\frac{\rho}{2} a^\dagger b^\dagger + \frac{\rho}{2} ab \right), \]

(74)

where

\[ \alpha = \frac{1}{2} \phi + \gamma, \quad \beta = \frac{1}{2} \phi - \gamma. \]

(75)

(The degree of freedom of \( \gamma \) still remains.)

### 2.3 Two-mode \( SO(2,1) \) squeeze vacuum and entangled spin-coherent states

#### 2.3.1 \( SO(2,1) \) two-mode squeeze operator and the interaction Hamiltonian

With the Gauss decomposition (72), the two-mode squeezed vacuum \( |\text{sq}(\rho, \phi)\rangle \) is readily obtained as

\[ |\text{sq}(\rho, \phi)\rangle = \hat{S}(\rho, \phi)|0\rangle \otimes |0\rangle = \frac{1}{\cosh \frac{\rho}{2}} \sum_{n=0}^{\infty} \left( -\tanh \left( \frac{\rho}{2} \right) \right)^n e^{in\phi} |n\rangle \otimes |n\rangle, \]

(76)

where

\[ |n\rangle \otimes |n\rangle = \frac{1}{\sqrt{n!}} a^n |0\rangle \otimes \frac{1}{\sqrt{n!}} b^n |0\rangle. \]

(77)

|\text{sq}(\rho, \phi)\rangle \) signifies an entangled state of the number states. Removing the right \( U(1) \) factor of (71), we introduce the Schwinger-type squeeze operator as \[ \hat{S}(\rho, \phi) = e^{i\phi T^z} e^{i\rho T^x}. \]

(78)

The difference between the squeezed vacuum of the Dirac-type and that of the Schwinger-type is just the phase factor:

\[ |\text{sq}(\rho, \phi)\rangle = \hat{S}(\rho, \phi)|0\rangle \otimes |0\rangle = e^{-i\frac{\rho}{2}} \hat{S}(\rho, \phi)|0\rangle \otimes |0\rangle, \]

(79)

and so the Dirac-type and Schwinger-type squeezed vacua are physically identical. In the interaction picture, the two-mode Hamiltonian realizes as the time-evolution operator of the squeeze operator (70), which is

\[ \hat{H}^{\text{int}}(\rho, \phi) = -i\frac{\rho}{2} e^{i\phi} a^\dagger b^\dagger + i\frac{\rho}{2} e^{-i\phi} ab. \]

(80)

#### 2.3.2 Two two-mode squeezed vacua as entangled spin-coherent states

For later convenience, we point out an interesting correspondence between the \( SU(2) \) spin-coherent states and the \( SU(1,1) \) squeezed states. Let us consider a direct product of two independent squeezed vacua with same squeeze parameter \( \rho \). We also assume that a squeezed vacuum is realized in the Schwinger boson space of \( a \) and \( c \), and another in the Schwinger boson space of \( b \) and \( d \):

\[ |\text{sq}(\rho, \phi)\rangle_{a,c} \otimes |\text{sq}(\rho, \phi')\rangle_{b,d} = \frac{1}{\cosh^2 \frac{\rho}{2}} e^{-\frac{\rho}{2} e^{i\phi} a^\dagger c^\dagger} e^{-\frac{\rho}{2} e^{-i\phi'} b^\dagger d^\dagger} |0,0,0,0\rangle = \frac{1}{\cosh^2 \frac{\rho}{2}} \sum_{n,m=0}^{\infty} (-\tanh \left( \frac{\rho}{2} \right))^{n+m} e^{i(n\phi+m\phi')} |n, m, n, m\rangle, \]

(81)

7See Appendix [A] about basic properties of the two-mode squeezed vacuum.

8(81) is a four-mode analogue of the two-mode \( SU(1,1) \otimes SU(1,1) \) state of [18].
where \( |n, m, n, m\rangle \equiv |n\rangle_a |m\rangle_b |n\rangle_c |m\rangle_d \). Recall that the Schwinger interpretation of spin states, in which \( |n, m\rangle \) is interpreted as the spin state \( |S, S_z\rangle \), and the sum of all particles numbers is replaced with the sum of all possible spin degrees of freedom (see Fig. 2):

\[
\sum_{n,m=0,1,2,3,\cdots} \rightarrow \sum_{S=0,1/2,1,3/2,\cdots}^{S} S_z = -S .
\]

With this replacement, (81) can be rewritten as

\[
|\text{sq}(\rho, \phi)\rangle_{a,c} \otimes |\text{sq}(\rho, \phi')\rangle_{b,d} = \frac{1}{\cosh^2(\rho/2)} \sum_{S=0,1/2,1,\cdots} (-\tanh(\rho/2))^{2S} e^{iS(\phi + \phi')} \sum_{S_z=-S}^{S} e^{iS_z(\phi - \phi')} |S, S_z\rangle_{a,b} \otimes |S, S_z\rangle_{c,d} ,
\]

which denotes a linear combination of direct products of two identical spin states with all possible spins. In each spin sector, we have a maximally entangled spin state:

\[
\sum_{S_z=-S}^{S} e^{iS_z(\phi - \phi')} |S, S_z\rangle_{a,b} \otimes |S, S_z\rangle_{c,d} .
\]

Though in (83) the left and right-hand sides are mathematically identical thorough the Schwinger operator formalism, they describe totally different physics: The left-hand side denotes a separable state of two squeezed states, while the right-hand side stands for a maximally entangled spin state.

### 3 Bloch four-hyperboloid and \( SO(4,1) \) squeeze operator

In this section, we exploit the four-hyperboloid geometry (Fig. 3) and the \( SO(4,1) \) coset representative. The geometric structures of \( H^2 \) and \( H^4 \) are respectively given by

\[
H^2 \simeq \mathbb{R}^2 \simeq S^1 \otimes \mathbb{R}_+ , \quad H^4 \simeq \mathbb{R}^4 \simeq S^3 \otimes \mathbb{R}_+ .
\]

\( H^4 \) can be regarded as a natural generalization of \( H^2 \) whose latitude is expanded from \( S^1 \) to \( S^3 \). Such an \( S^3 \) structure brings an internal spin structure to the \( SO(4,1) \) formulation. Adopting a set of four Schwinger operators as an \( SO(4,1) \) representation, we derive an \( SO(4,1) \) squeeze operator and discuss its internal spin structure.

#### 3.1 \( SO(4,1) \) algebra and Bloch four-hyperboloid

**3.1.1 \( H^4 \) and the \( SO(4,1) \) algebra**

The four-hyperboloid \( H^4 \) is given by

\[
\sum_{a,b=1}^{5} \eta_{ab} x^a x^b = - \sum_{m=1}^{4} x^m x^m + x^5 x^5 = 1
\]

with

\[
\eta^{ab} = \eta_{ab} = \text{diag}(-,-,-,-,+) .
\]

As a coset, \( H^4 \) is represented as

\[
H^4 \simeq SO(4,1)/SO(4) .
\]
The $SO(4,1)$ gamma matrices $\gamma^a$ ($a = 1, 2, 3, 4, 5$) are introduced so as to satisfy
\[ \{ \gamma^a, \gamma^b \} = 2\eta^{ab}, \] (89)
and the $SO(4,1)$ generators $\Sigma^{ab}$ are constructed as
\[ \Sigma^{ab} = -i\frac{1}{4}[\gamma^a, \gamma^b], \] (90)
which satisfy
\[ [\Sigma^{ab}, \Sigma^{cd}] = i\eta^{ac}\Sigma^{bd} - i\eta^{ad}\Sigma^{bc} + i\eta^{bd}\Sigma^{ac} - i\eta^{bc}\Sigma^{ad}. \] (91)

Hereafter, we take the $SO(4,1)$ gamma matrices and the generators as
\[ \gamma^m = \begin{pmatrix} 0 & -q^m \\ q^m & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \] (92)
with $q^m$ being quaternions,
\[ q^m = \{-i\sigma_i, 1_2\}, \quad \bar{q}^m = \{i\sigma_i, 1_2\}, \] (93)
and then\footnote{The independent generators of $\Sigma^{ab}$ are}
\[ \Sigma^{mn} = -\frac{1}{2} \begin{pmatrix} \eta_{mn} & 0 \\ 0 & \bar{\eta}_{mn} \end{pmatrix}, \quad \Sigma^{m5} = -\frac{1}{2} \begin{pmatrix} 0 & \bar{q}^m \\ q^m & 0 \end{pmatrix}. \] (97)

Among $\Sigma^{ab}$, $\Sigma^{mn}$ are Hermitian matrices, while $\Sigma^{m5}$ are anti-Hermitian matrices. $i\Sigma^{ab}$ \footnote{are recognized as the $U(1,1;\mathbb{H})$ generators, since their components are quaternions}
\[ i\Sigma^{ab} = \begin{pmatrix} \frac{1}{2} q_i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ q_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \bar{q}^m \\ q^m & 0 \end{pmatrix}, \] (95)
and they satisfy
\[ (i\Sigma^{ab})^\dagger \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} i\Sigma^{ab}. \] (96)
Here, \( \eta_{mn} \) and \( \bar{\eta}_{mn} \) are the 't Hooft symbols:

\[
\eta_{mn} = \epsilon^{mn4} + \delta^{m4} \delta^{n4} - \delta^{m4} \delta^{n4}, \quad \bar{\eta}_{mn} = \epsilon^{mn4} - \delta^{m4} \delta^{n4} + \delta^{m4} \delta^{n4},
\]

with \( \epsilon^{1234} = 1 \). Notice that \( \gamma^a \) and \( \Sigma^{ab} \) amount to fifteen matrices that satisfy the \( \text{SO}(4,2) \simeq \text{SU}(2,2) \) algebra. With the \( \text{SU}(2,2) \) invariant matrix

\[
k = \gamma^5,
\]

we can “hermitianize” the \( \text{SU}(2,2) \) generators as

\[
k^a \equiv k \gamma^a, \quad k^{ab} = k \Sigma^{ab},
\]

or

\[
k^m = \begin{pmatrix}
0 & -\bar{q}^m \\
q^m & 0
\end{pmatrix} = \left\{ \begin{pmatrix}
0 & -i \sigma_i \\
i \sigma_i & 0
\end{pmatrix}, \begin{pmatrix}
0 & -12 \\
-12 & 0
\end{pmatrix} \right\}, \quad k^5 = \begin{pmatrix}
12 & 0 \\
0 & 12
\end{pmatrix}.
\]

Also note that neither the Hermitian matrices \( k^a \) satisfy (89) nor \( k^{ab} \) satisfy the \( \text{SO}(4,1) \) algebra (91).

### 3.1.2 Bloch four-hyperboloid and \( \text{SO}(4,1) \) coset representative

The 2nd non-compact (hybrid) Hopf map is [25]

\[
H^{4,3} \xrightarrow{S^3} H^4,
\]

which is explicitly given by

\[
\Psi \rightarrow x^a = \Psi^\dagger k^a \Psi.
\]

Here \( \Psi = (\Psi_1 \; \Psi_2 \; \Psi_3 \; \Psi_4)^t \) denotes a four-component spinor subject to

\[
\Psi^\dagger k \Psi = |\Psi_1|^2 + |\Psi_2|^2 - |\Psi_3|^2 - |\Psi_4|^2 = 1 : H^{4,3},
\]

and \( x^a \) automatically satisfy

\[
\eta_{ab} x^a x^b = -\sum_{m=1}^4 x^m x^m + x^5 x^5 = (\Psi^\dagger k \Psi)^2 = 1 : H^4.
\]

We can realize \( \Psi \) as

\[
\Psi = \frac{1}{\sqrt{2(1 + x^5)}} \begin{pmatrix}
1 + x^5 \\
x^m q_m
\end{pmatrix} \psi
\]

with two-component spinor \( \psi = (\psi_1 \; \psi_2)^t \) subject to

\[
\psi^\dagger \psi = 1 : S^3.
\]

With polar angle coordinate representation [10]

\[
x^1 = \sinh \rho \sin \chi \sin \theta \cos \phi, \quad x^2 = \sinh \rho \sin \chi \sin \theta \sin \phi, \quad x^3 = \sinh \rho \sin \chi \cos \theta, \\
x^4 = \sinh \rho \cos \chi, \quad x^5 = \cosh \rho,
\]

The ranges of the parameters are

\[
0 \leq \rho < \infty, \quad 0 \leq \chi, \theta \leq \pi, \quad 0 \leq \phi < 2\pi.
\]
\[ \Psi = \left( \begin{array}{c} \cosh(\frac{x}{2}) \ 1 \\ \sinh(\frac{x}{2}) \ y^m q_m \end{array} \right) \psi. \] (110)

Here, \( y^m (\sum_{m=1}^{4} y^m y^m = 1) \) denote coordinates on the normalized \( S^3 \) "circle" at latitude \( \rho \) on \( H^4 \) (see Fig. 3):

\[ y^m = \frac{1}{\sinh \rho} x^m = \{ \sin \chi \sin \theta \cos \phi, \ \sin \chi \sin \theta \sin \phi, \ \sin \chi \cos \theta, \ \cos \chi \}, \] (111)

and so

\[ \sum_{m=1}^{4} y^m q_m = -\sum_{m=1}^{4} y^m q_m = \left( -\cos(\chi - i \sin \chi \cos \theta) \ i \sin \chi \sin \theta e^{i\phi} \ -\cos(\chi + i \sin \chi \cos \theta) \right) = \left( \sum_{m=1}^{4} y^m q_m \right)^\dagger. \] (112)

Notice that \( \Psi \) acts as the \( SO(4,1) \) pseudo-spin-coherent state on \( H^4 \):

\[ (x^a \gamma_a) \ \Psi = \Psi, \] (113)

which suggests

\[ \Psi = M(\rho, \chi, \theta, \phi) \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \] (114)

where \( M(\rho, \chi, \theta, \phi) \) denotes an \( SO(4,1) \) rotation.\[11\]

\[ M(\rho, \chi, \theta, \phi) = \begin{pmatrix} \cosh(\frac{x}{2}) & \frac{1}{2} \sinh(\frac{x}{2}) \sum_{m=1}^{4} y^m q_m \\ \sinh(\frac{x}{2}) \sum_{m=1}^{4} y^m q_m & \cosh(\frac{x}{2}) \end{pmatrix} = e^{i\rho \Sigma_{m=1}^{4} y^m \Sigma_{m=5}}, \] (116)

with

\[ \Sigma_{m=5} = -\Sigma_{m=5} = \frac{1}{2} \begin{pmatrix} 0 & q_m \\ q_m & 0 \end{pmatrix}. \] (117)

The last expression of (116) implies \( M(\rho, \chi, \theta, \phi) \) is a coset representative of \( SO(4,1) \) group, since \( \Sigma_{m=5} \) act as the broken generators associated with the symmetry breaking, \( SO(4,1) \rightarrow SO(4) \). Generalizing the \( SO(5) \) coset matrix for \( S^4 \) \[13\], the Euler decomposition of (116) is given by

\[ M(\rho, \chi, \theta, \phi) = H(\chi, \theta, \phi)^\dagger \cdot e^{-i\rho \Sigma_{m=5}} \cdot H(\chi, \theta, \phi). \] (118)

Here, \( H(\chi, \theta, \phi) \) denotes an \( SU(2) \) matrix of the form

\[ H(\chi, \theta, \phi) = e^{i\chi \Sigma_{12}} \cdot e^{i\theta \Sigma_{13}} \cdot e^{-i\phi \Sigma_{12}} = \begin{pmatrix} H_L(\chi, \theta, \phi) & 0 \\ 0 & H_R(\chi, \theta, \phi) \end{pmatrix}, \] (119)

with

\[ H_L(\chi, \theta, \phi) = H_R(-\chi, \theta, \phi) = e^{-i\frac{1}{2} \sigma_3} \cdot e^{i\frac{1}{2} \sigma_2} \cdot e^{i\frac{1}{2} \sigma_3} = \begin{pmatrix} \cos(\frac{\chi}{2}) e^{-i\frac{1}{2} (\chi-\phi)} & \sin(\frac{\chi}{2}) e^{-i\frac{1}{2} (\chi+\phi)} \\ -\sin(\frac{\chi}{2}) e^{i\frac{1}{2} (\chi+\phi)} & \cos(\frac{\chi}{2}) e^{i\frac{1}{2} (\chi-\phi)} \end{pmatrix}. \] (120)

\[11\]In the polar coordinate representation, (116) is expressed as

\[ M(\rho, \chi, \theta, \phi) = \begin{pmatrix} \cosh(\frac{x}{2}) & 0 & -\sinh(\frac{x}{2}) (\cos \chi + i \sin \chi \cos \theta) & -i \sinh(\frac{x}{2}) \sin \chi \sin \theta e^{-i\phi} \\ 0 & \cosh(\frac{x}{2}) & -i \sinh(\frac{x}{2}) \sin \chi \sin \theta e^{i\phi} & \sinh(\frac{x}{2}) (\cos \chi - i \sin \chi \cos \theta) \\ -\sinh(\frac{x}{2}) (\cos \chi - i \sin \chi \cos \theta) & i \sinh(\frac{x}{2}) \sin \chi \sin \theta e^{-i\phi} & \cosh(\frac{x}{2}) & 0 \\ i \sinh(\frac{x}{2}) \sin \chi \sin \theta e^{i\phi} & -\sinh(\frac{x}{2}) (\cos \chi + i \sin \chi \cos \theta) & 0 & \cosh(\frac{x}{2}) \end{pmatrix}. \] (115)
Obviously (118) is a natural generalization of the SU(1,1) case (67) with replacement of the U(1) factor $e^{-i\frac{\phi}{2}z}$ with the SU(2) factor $H(\chi, \theta, \phi)$ (119). The Gauss decomposition is also given by

\[ M(\rho, \chi, \theta, \phi) = \exp\left(\tanh\left(\frac{\rho}{2}\right) \begin{pmatrix} 0 & y^m q_m \\ 0 & 0 \end{pmatrix}\right) \cdot \exp\left(-\ln(\cosh\left(\frac{\rho}{2}\right)) \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}\right) \cdot \exp\left(\tanh\left(\frac{\rho}{2}\right) \begin{pmatrix} 0 & 0 \\ y^m q_m & 0 \end{pmatrix}\right) \]

\[ = \exp\left(-\tanh\left(\frac{\rho}{2}\right) y^m \cdot (\frac{1}{2} \gamma_m - i \Sigma_{m5})\right) \cdot \exp\left(-\ln(\cosh\left(\frac{\rho}{2}\right)) \gamma_5\right) \cdot \exp\left(\tanh\left(\frac{\rho}{2}\right) y^m \cdot (\frac{1}{2} \gamma_m + i \Sigma_{m5})\right), \tag{121} \]

which realizes a natural generalization of (68), again.

### 3.2 $SO(4, 1)$ squeeze operator

#### 3.2.1 $SO(4, 1)$ Schwinger operator realization and $SO(4, 2)$ operators

As the $SO(4, 1)$ group does not accommodate the Majorana representation\textsuperscript{12} two-mode $SO(4, 1)$ squeeze operator does not exist unlike the $SO(2, 3)$ case \textsuperscript{23}. We then consider a set of four Schwinger operators in correspondence with the Dirac representation:

\[ \hat{\Psi} = \begin{pmatrix} a \\ b \\ c^\dagger \\ d^\dagger \end{pmatrix} \tag{126} \]

whose components satisfy

\[ [\hat{\Psi}_\alpha, \hat{\Psi}_\beta^\dagger] = k_{\alpha\beta}, \quad [\hat{\Psi}_\alpha, \hat{\Psi}_\beta] = 0. \tag{127} \]

Sandwiching (100) with (126), we introduce the following Hermitian operators:

\[ \hat{X}^a = \hat{\Psi}^\dagger k_a \hat{\Psi}, \quad \hat{X}^{ab} = \hat{\Psi}^\dagger k^{ab} \hat{\Psi}. \tag{128} \]

\[ \hat{X}^a \]

are explicitly

\[ \hat{X}^1 = -i(a^\dagger d^\dagger + b^\dagger c^\dagger - ad - bc), \quad \hat{X}^2 = -(a^\dagger d^\dagger - b^\dagger c^\dagger + ad - bc), \]

\[ \hat{X}^3 = -i(a^\dagger c^\dagger - b^\dagger d^\dagger - ac + bd), \quad \hat{X}^4 = -(a^\dagger c^\dagger + b^\dagger d^\dagger + ac + bd), \]

\[ \hat{X}^5 = a^\dagger a + b^\dagger b + c^\dagger c + d^\dagger d + 2, \tag{129} \]

\textsuperscript{12}The $SO(4, 1)$ gamma matrices (92) and generators (94) satisfy

\[ (\gamma^a)^* = C\gamma^a C^{-1}, \quad (\Sigma^{ab})^* = -C\Sigma^{ab} C^{-1}, \tag{122} \]

where $C$ denotes the charge conjugation matrix

\[ C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{123} \]

With $C$ \textsuperscript{123} the Majorana condition is imposed as

\[ \Psi_M^* = C\Psi_M. \tag{124} \]

and so

\[ (\Psi_M^*)^* = C^* C\Psi_M = -\Psi_M. \tag{125} \]

Under the usual definition of the complex conjugation $(\Psi^*)^* = \Psi$, \textsuperscript{123} implies $\Psi_M = 0$, which means that the Majorana spinor does not exist in the $SO(4, 1)$ group.
\( \hat{X}_{ab} \) consist of the \( SO(4) \) particle conserving operators \( \hat{X}_{mn} \)

\[
\begin{align*}
\hat{X}^{12} &= -\frac{1}{2}(a^\dagger a - b^\dagger b - c^\dagger c + d^\dagger d), \\
\hat{X}^{13} &= i\frac{1}{2}(-a^\dagger b + b^\dagger a - c^\dagger d + d^\dagger c), \\
\hat{X}^{14} &= -\frac{1}{2}(a^\dagger b + b^\dagger a + c^\dagger d + d^\dagger c), \\
\hat{X}^{23} &= \frac{1}{2}(a^\dagger b + b^\dagger a - c^\dagger d - d^\dagger c), \\
\hat{X}^{24} &= -\frac{1}{2}(a^\dagger b - b^\dagger a - c^\dagger d + d^\dagger c), \\
\hat{X}^{34} &= \frac{1}{2}(a^\dagger a - b^\dagger b + c^\dagger c - d^\dagger d), \\
\end{align*}
\]

and the remaining particle number non-conserving operators \( \hat{X}_{m5} \)

\[
\begin{align*}
\hat{X}^{15} &= \frac{1}{2}(a^\dagger d^\dagger + b^\dagger c^\dagger + ad + bc), \\
\hat{X}^{25} &= -\frac{1}{2}(a^\dagger d^\dagger - b^\dagger c^\dagger - ad + bc), \\
\hat{X}^{35} &= \frac{1}{2}(a^\dagger c^\dagger - b^\dagger d^\dagger + ac - bd), \\
\hat{X}^{45} &= -\frac{1}{2}(a^\dagger c^\dagger + b^\dagger d^\dagger - ac - bd). \\
\end{align*}
\]

Notice that the Hermitian matrices \( k^{ab} \) do not satisfy the \( SO(4,1) \) algebra, but the corresponding Hermitian operators \( \hat{X}^{ab} \) do satisfy the \( SO(4,1) \) algebra. \( SO(4,1) \) invariant operators are constructed as

\[
\hat{X}^a \hat{X}_a = \frac{1}{2} \sum_{a < b} \hat{X}^{ab} \hat{X}_{ab} = (\hat{\Psi}^\dagger k \hat{\Psi})(\hat{\Psi}^\dagger k \hat{\Psi} + 4),
\]

where

\[
\hat{\Psi}^\dagger k \hat{\Psi} = a^\dagger a + b^\dagger b - cc^\dagger - dd^\dagger = n_a + n_b - n_c - n_d - 2.
\]

Furthermore, \( \hat{X}^{ab} \) and \( \hat{X}^{6a} = -\hat{X}^{6a} \equiv -\frac{1}{2} \hat{X}^a \) constitute the \( SU(2,2) \simeq SO(4,2) \) algebra in total:

\[
[\hat{X}^{AB}, \hat{X}^{CD}] = i\eta^{AC} \hat{X}^{BD} - i\eta^{AD} \hat{X}^{BC} + i\eta^{BD} \hat{X}^{AC} - i\eta^{BC} \hat{X}^{AD}, \quad (A,B,C,D = 1,2,3,4,5,6)
\]

with

\[
\eta^{AB} = \text{diag}(-, -, -, +, +, +).
\]

The corresponding \( SU(2,2) \simeq SO(4,2) \) Casimir is constructed as

\[
\hat{X}^a \hat{X}_a + 4 \sum_{a < b} \hat{X}^{ab} \hat{X}_{ab} = 3(\hat{n}_a + \hat{n}_b - \hat{n}_c - \hat{n}_d - 2)(\hat{n}_a + \hat{n}_b - \hat{n}_c - \hat{n}_d + 2) = 3(\hat{\Psi}^\dagger k \hat{\Psi})(\hat{\Psi}^\dagger k \hat{\Psi} + 4).
\]

The eigenstates are given by

\[
|n_a, n_b, n_c, n_d\rangle = \frac{1}{\sqrt{n_a! n_b! n_c! n_d!}} (a^\dagger)^{n_a} (b^\dagger)^{n_b} (c^\dagger)^{n_c} (d^\dagger)^{n_d} |0, 0, 0, 0\rangle.
\]

Obviously, the particle numbers, \( n_a, n_b, n_c \) and \( n_d \), uniquely specify the eigenstates.

### 3.2.2 Subgroup decomposition of the \( SO(4,1) \) squeeze operator

From a view of the \( SO(4,2) \) group theory, we delve into internal spin and pseudo-spin structures. The \( SO(4,2) \) group accommodates two important subgroups, the \( SO(4) \) group for spins and the \( SO(2,2) \) group for pseudo-spins:

\[
\begin{align*}
SO(4) &\simeq SU(2)_L \otimes SU(2)_R \subset SO(4,1) \subset SO(4,2), \\
SO(2,2) &\simeq SU(1,1)_T \otimes SU(1,1)_B \subset SO(2,3) \subset SO(4,2).
\end{align*}
\]

For oscillator realization of the \( SU(2,2) \) group and its representation, one may consult Refs. [54, 55, 56, 57, 58, 59, 60].
For $SO(4) \cong SU(2)_L \otimes SU(2)_R$, the corresponding left and right $SU(2)$ spin operators are constructed as

\[
\tilde{L}_i = \frac{1}{4} \eta_{mn} \tilde{x}^{mn} = \left( \begin{array}{c} a \\ b \end{array} \right) \frac{1}{2} \sigma_i \left( \begin{array}{c} a \\ b \end{array} \right), \quad \tilde{R}_i = \frac{1}{4} \eta_{mn} \tilde{x}^{mn} = \left( \begin{array}{c} c \\ d \end{array} \right) \frac{1}{2} \sigma_i \left( \begin{array}{c} c \\ d \end{array} \right),
\]

which satisfy

\[
\sum_{i=1}^{3} \tilde{L}_i \tilde{L}_i = \hat{L}(\hat{L} + 1), \quad \sum_{i=1}^{3} \tilde{R}_i \tilde{R}_i = \hat{R}(\hat{R} + 1)
\]

with

\[
\hat{L} = \frac{1}{2}(\hat{n}_a + \hat{n}_b), \quad \hat{R} = \frac{1}{2}(\hat{n}_c + \hat{n}_d).
\]

The $SU(2,2)$ eigenstate (137) can be specified by the spin group quantum numbers,

\[
\hat{L} = \frac{1}{2}(\hat{n}_a + \hat{n}_b), \quad \hat{R} = \frac{1}{2}(\hat{n}_c + \hat{n}_d), \quad \hat{L}_z = \frac{1}{2}(\hat{n}_a - \hat{n}_b), \quad \hat{R}_z = \frac{1}{2}(\hat{n}_c - \hat{n}_d).
\]

Meanwhile for $SO(2,2) \cong SU(1,1)_T \otimes SU(1,1)_B$, the corresponding top and bottom $SU(1,1)$ pseudo-spin operators are introduced as

\[
\hat{T}^i = \left( \begin{array}{c} a \\ c^\dagger \end{array} \right) \frac{1}{2} \kappa_i \left( \begin{array}{c} a \\ c^\dagger \end{array} \right), \quad \hat{B}^i = \left( \begin{array}{c} b \\ d^\dagger \end{array} \right) \frac{1}{2} \kappa_i \left( \begin{array}{c} b \\ d^\dagger \end{array} \right),
\]

which satisfy

\[
\sum_{i,j=1}^{3} \eta_{ij} \hat{T}^i \hat{T}^j = \hat{T}(\hat{T} + 1), \quad \sum_{i=1}^{3} \eta_{ij} \hat{B}^i \hat{B}^j = \hat{B}(\hat{B} + 1),
\]

with $\eta_{ij} = \text{diag}(-, -, +)$ and

\[
\hat{T} = \frac{1}{2}(\hat{n}_a - \hat{n}_c - 1), \quad \hat{B} = \frac{1}{2}(\hat{n}_b - \hat{n}_d - 1).
\]

In detail,

\[
\hat{T}^x = \frac{i}{2}(a^\dagger c^\dagger - ac) = \frac{1}{2} \hat{x}^{35} - \frac{1}{4} \hat{x}^3, \quad \hat{T}^y = \frac{i}{2}(a^\dagger c^\dagger + ac) = -\frac{1}{2} \hat{x}^{35} - \frac{1}{4} \hat{x}^4, \\
\hat{T}^z = \frac{1}{2}(a^\dagger a + c^\dagger c) + \frac{1}{2} = \frac{1}{2} \hat{x}^{34} + \frac{1}{4} \hat{x}^5, \quad \hat{B}^x = \frac{i}{2}(b^\dagger d^\dagger - bd) = \frac{1}{2} \hat{x}^{45} + \frac{1}{4} \hat{x}^3, \\
\hat{B}^y = \frac{1}{2}(b^\dagger d^\dagger + bd) = \frac{1}{2} \hat{x}^{35} - \frac{1}{4} \hat{x}^4, \quad \hat{B}^z = \frac{1}{2}(b^\dagger b + d^\dagger d) + \frac{1}{2} = -\frac{1}{2} \hat{x}^{34} + \frac{1}{4} \hat{x}^5.
\]

The $SU(2,2)$ eigenstate (137) can be specified by the pseudo-spin quantum numbers of $\hat{T}$, $\hat{B}$, $\hat{T}^z$ and $\hat{B}^z$.

### 3.2.3 $SO(4,1)$ squeeze operator and the interaction Hamiltonian

The existence of spin and pseudo-spin groups in the $SO(4,1)$ group implies that the $SO(4,1)$ formalism is large enough to incorporate both spin and squeezed state structures. Replacing the non-Hermitian matrices in (116) with the corresponding Hermitian operators, we introduce the $SO(4,1)$ squeeze operator:

\[
\hat{S}(\rho, \chi, \theta, \phi) = e^{i\rho \sum_{m=1}^{4} y^m (\chi, \theta, \phi) \hat{x}^m} = e^{-i\rho \sum_{m=1}^{4} y^m (\chi, \theta, \phi) \hat{x}^m}.
\]

The interaction Hamiltonian is then given by

\[
\hat{H}^{\text{int}}(\rho, \chi, \theta, \phi) = -\rho \sum_{m=1}^{4} y^m \hat{x}_m
\]

\[
= \frac{\rho}{2} \left( \sin \chi (\cos \theta (a^\dagger c^\dagger - b^\dagger d^\dagger) + \sin \theta (e^{-i\phi} a^\dagger c^\dagger + e^{i\phi} b^\dagger d^\dagger)) - i \cos \chi (a^\dagger c^\dagger + b^\dagger d^\dagger) \right) + \text{h.c.}
\]

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The SO(4,1) squeeze operator is a unitary operator that satisfies
\[ \hat{S}(\rho, \chi, \theta, \phi) \dagger = \hat{S}(\rho, \chi, \theta, \phi)^{-1} = \hat{S}(\rho, \chi, \theta, \phi). \] (149)
In \[148\], we can easily see that the phase \( \phi \) can be absorbed in the phase redefinition of the Schwinger operators:
\[ a \rightarrow e^{-i\alpha} a, \quad b \rightarrow e^{i\beta} b, \quad c \rightarrow e^{i\alpha} c, \quad d \rightarrow e^{-i\beta} d, \] (150)
where
\[ \alpha + \beta = \phi. \] (151)
To grasp physical meaning of the SO(4,1) squeeze operator \[167\], we rewrite the SO(4,1) squeeze operator in the language of the SU(2) spin and the SU(1,1) pseudo-spin groups. For this purpose, we apply the Euler decomposition \[118\] to the squeeze operator \[14\]:
\[ \hat{S}(\rho, \chi, \theta, \phi) = \hat{H}(\chi, \theta, \phi) \dagger \cdot \hat{S}(\rho) \cdot \hat{H}(\chi, \theta, \phi), \] (154)
where
\[ \hat{H}(\chi, \theta, \phi) \equiv e^{i\chi \hat{X}^3} \cdot e^{i\theta \hat{X}^1} \cdot e^{-i\phi \hat{X}^2} = \hat{D}_L(\chi, -\theta, -\phi) \cdot \hat{D}_R(\chi, -\theta, \phi) \] (155)
and
\[ \hat{S}(\rho) \equiv e^{-i\rho \hat{X}^5} = \hat{S}_T(\rho) \cdot \hat{S}_B(\rho). \] (156)
Here,
\[ \hat{D}_L(\chi, \theta, \phi) \equiv e^{-i\chi \hat{L}^x} e^{-i\theta \hat{L}^y} e^{-i\phi \hat{L}^z}, \quad \hat{D}_R(\chi, \theta, \phi) \equiv e^{-i\chi \hat{R}^z} e^{-i\theta \hat{R}^y} e^{-i\phi \hat{R}^x}, \quad \hat{S}_T(\rho) \equiv e^{i\rho \hat{R}^z}, \quad \hat{S}_B(\rho) \equiv e^{i\rho \hat{R}^x}. \] (157)
Consequently, the SO(4,1) squeeze operator can be expressed as \[15\]
\[ \hat{S}(\rho, \chi, \theta, \phi) = \hat{D}_L(\phi, \theta, -\chi) \cdot \hat{D}_R(-\phi, \theta, -\chi) \cdot \hat{S}_T(\rho) \cdot \hat{S}_B(\rho) \cdot \hat{D}_L(\chi, -\theta, -\phi) \cdot \hat{D}_R(\chi, -\theta, \phi), \] (158)
Removing the right SU(2) factor of \[154\], we introduce the Schwinger-type SO(4,1) squeeze operator \[15\]
\[ \hat{S}(\rho, \chi, \theta, \phi) = \hat{H}(\chi, \theta, \phi) \dagger \cdot \hat{S}(\rho) \equiv \hat{S}(\rho, \chi, \theta, \phi). \] (160)

---

\[^{14}\text{One may of course utilize the Gauss decomposition of } \hat{S}(\rho, \chi, \theta, \phi) \text{ in deriving the number state expansion of the SO(4,1) squeezed vacuum:}\]
\[ \hat{S}(\rho, \chi, \theta, \phi) = \exp \left( -\tan(\chi) y^m \left( \frac{1}{2} \hat{X}_m - i \hat{X}_m^5 \right) \right) \cdot \exp \left( -\log(\cosh(\frac{\rho}{2}) \hat{X}_5) \right) \cdot \exp \left( \tan(\chi) y^m \left( \frac{1}{2} \hat{X}_m + i \hat{X}_m^5 \right) \right), \] (152)
where
\[ y^m \left( \frac{1}{2} \hat{X}_m - i \hat{X}_m^5 \right) = i \sin \chi \sin \theta e^{-i\phi} a^d d^l + i \sin \chi \sin \theta e^{i\phi} b^c c^l + (\cos \chi + i \sin \chi \cos \theta) a^c c^c + (\cos \chi - i \sin \chi \cos \theta) b^d d^d, \]
\[ \hat{X}_5 = \hat{n}_a + \hat{n}_b + \hat{n}_c + \hat{n}_d + 2. \] (153)

---

\[^{15}\text{We used } \hat{D}(\chi, \theta, \phi)^\dagger = \hat{D}(-\phi, -\theta, -\chi) \text{ in the derivation of (155).}\]

\[^{16}\text{The corresponding squeeze matrix of the Schwinger-type is given by}\]
\[ \mathcal{M}(\rho, \chi, \theta, \phi) = H(\chi, \theta, \phi) \dagger \cdot e^{-i\rho \Sigma_{45}} \]
\[ = \begin{pmatrix}
\cosh \frac{\rho}{2} \cos \frac{\chi}{2} e^{\frac{1}{2}(x+\phi)} & -\cosh \frac{\rho}{2} \sin \frac{\chi}{2} e^{\frac{1}{2}(x+\phi)} e^{-i\frac{1}{2}(x+\phi)} & -\sinh \frac{\rho}{2} \cos \frac{\chi}{2} e^{\frac{1}{2}(x+\phi)} e^{i\frac{1}{2}(x+\phi)} & \sinh \frac{\rho}{2} \sin \frac{\chi}{2} e^{-i\frac{1}{2}(x+\phi)} e^{i\frac{1}{2}(x+\phi)} \\
\cosh \frac{\rho}{2} \sin \frac{\chi}{2} e^{\frac{1}{2}(x+\phi)} & \cosh \frac{\rho}{2} \cos \frac{\chi}{2} e^{-\frac{1}{2}(x+\phi)} e^{i\frac{1}{2}(x+\phi)} & -\sinh \frac{\rho}{2} \sin \frac{\chi}{2} e^{\frac{1}{2}(x+\phi)} e^{-i\frac{1}{2}(x+\phi)} & -\sinh \frac{\rho}{2} \cos \frac{\chi}{2} e^{i\frac{1}{2}(x+\phi)} e^{-i\frac{1}{2}(x+\phi)} \\
-\sinh \frac{\rho}{2} \cos \frac{\chi}{2} e^{-\frac{1}{2}(x+\phi)} & \sinh \frac{\rho}{2} \sin \frac{\chi}{2} e^{\frac{1}{2}(x+\phi)} e^{-i\frac{1}{2}(x+\phi)} & \cosh \frac{\rho}{2} \cos \frac{\chi}{2} e^{-\frac{1}{2}(x+\phi)} e^{i\frac{1}{2}(x+\phi)} & \cosh \frac{\rho}{2} \sin \frac{\chi}{2} e^{-i\frac{1}{2}(x+\phi)} e^{i\frac{1}{2}(x+\phi)} \\
-\sinh \frac{\rho}{2} \sin \frac{\chi}{2} e^{-\frac{1}{2}(x-\phi)} & -\sinh \frac{\rho}{2} \cos \frac{\chi}{2} e^{\frac{1}{2}(x-\phi)} e^{i\frac{1}{2}(x-\phi)} & \cosh \frac{\rho}{2} \sin \frac{\chi}{2} e^{-\frac{1}{2}(x-\phi)} e^{-i\frac{1}{2}(x-\phi)} & \cosh \frac{\rho}{2} \cos \frac{\chi}{2} e^{i\frac{1}{2}(x-\phi)} e^{-i\frac{1}{2}(x-\phi)}
\end{pmatrix}. \] (159)
4 \ SO(4, 1) squeezed vacuum and its basic properties

Armed with the spin and pseudo-spin based expression of the \SO(4, 1) squeezed vacuum operator, we now derive a mathematical expression of the \SO(4, 1) squeezed vacuum and discuss its physical properties.

4.1 \SO(4, 1) squeezed vacuum

4.1.1 Spin interpretation

We simply construct the \SO(4, 1) squeezed vacuum by applying the \SO(4, 1) squeeze operator to the vacuum:

\[
|\text{Sq}(\rho, \chi, \theta, \phi)\rangle = \hat{S}(\rho, \chi, \theta, \phi)|0, 0, 0, 0\rangle.
\] (161)

The squeezed vacuum is identical for the Dirac-type \(\hat{S}\) and the Schwinger-type \(\hat{S}\):

\[
|\text{Sq}(\rho, \chi, \theta, \phi)\rangle = \hat{D}_L(\phi, \theta, -\chi) \cdot \hat{D}_R(-\phi, \theta, -\chi) \cdot \hat{S}_T(\rho) \cdot \hat{S}_B(\rho)|0, 0, 0, 0\rangle
= \hat{S}(\rho, \chi, \theta, \phi)|0, 0, 0, 0\rangle.
\] (162)

This is a property similar to the original \SO(2, 1) case, in which the squeezed vacua in the Dirac-type and the Schwinger-type are physically equivalent \([72]\). (In the case of the \(\text{Sp}(4; \mathbb{R}) \simeq \SO(2, 3)\) squeezed states \([23]\), the Dirac-type and Schwinger-type squeezed vacua are physically different.) Thus, the \SO(4, 1) squeezed vacuum inherits the uniqueness of the squeezed vacuum. Expanding the two squeeze operators, \(\hat{S}_T\) and \(\hat{S}_B\), in terms of the number operators, we can express the squeezed vacuum \(\text{Eq}(162)\) as

\[
|\text{Sq}(\rho, \chi, \theta, \phi)\rangle = \frac{1}{\cosh^2(\frac{\rho}{2})} \sum_{n, m = 0}^{\infty} (-\tanh(\frac{\rho}{2}))^{n+m} \hat{D}_L(\phi, \theta, -\chi) \cdot \hat{D}_R(-\phi, \theta, -\chi) |n, m\rangle \otimes |n, m\rangle.
\] (163)

With the spin interpretation \([32]\), we can rewrite \(\text{Eq}(163)\) as

\[
|\text{Sq}(\rho, \chi, \theta, \phi)\rangle = \frac{1}{\cosh^2(\frac{\rho}{2})} \sum_{S = 0, 1/2, 1/2, \ldots} S (-\tanh(\frac{\rho}{2}))^{2S} \hat{D}_L(\phi, \theta, -\chi) \cdot \hat{D}_R(-\phi, \theta, -\chi) |S, S_z\rangle \otimes |S, S_z\rangle
= \frac{1}{\cosh^2(\frac{\rho}{2})} \sum_{S = 0, 1/2, 1/2, \ldots} S (-\tanh(\frac{\rho}{2}))^{2S} \sum_{S_z = -S}^{S} |S, S_z(\phi, \theta, -\chi)\rangle \otimes |S, S_z(-\phi, \theta, -\chi)\rangle.
\] (164)

Here, we changed the summation from \(n\) and \(m\) to \(S\) and \(S_z\) according to \([32]\). Thus, the \SO(4, 1) four-mode squeezed vacuum can be understood as a linear combination of two-body spin-coherent states. In \(\text{Eq}(163)\), the first spin direction is specified by \((\theta, -\phi)\) while the second spin direction is specified by \((\theta, \phi)\), and then, by tuning \(\theta\) and \(\phi\), we can manipulate relative configurations of two spins\(^{17}\). Extracting the \(U(1)\) phase factor of the spin state

\[
|S, S_z(\phi, \theta, \chi)\rangle = e^{-i\chi S_z} |S, S_z(\theta, \phi)\rangle,
\] (165)

we can express \(\text{Eq}(164)\) as

\[
|\text{Sq}(\rho, \chi, \theta, \phi)\rangle = \frac{1}{\cosh^2(\frac{\rho}{2})} \sum_{S = 0, 1/2, 1/2, \ldots} (-\tanh(\frac{\rho}{2}))^{2S} \sum_{S_z = -S}^{S} e^{2i\chi S_z} |S, S_z(\theta, \phi)\rangle \otimes |S, S_z(-\phi, \theta, -\chi)\rangle.
\] (166)

\(^{17}\)In the four-mode interpretation, due to the independent phase choice of the four-mode operators, the angle \(\phi\) can be “gauged” away \([161]\), but in the spin interpretation, only the overall phase of \(a\) and \(b\) and that of \(c\) and \(d\) are allowed to change, since the angular momentum operators \([130]\) are just immune to overall phase redefinition of these operators. Thus in the spin interpretation, \(\phi\) cannot be gauged away and is indeed related the relative direction of two spins.
where

$$|S, S_z(\theta, \phi)\rangle \equiv \hat{D}(\phi, \theta, 0)|S, S\rangle.$$ (167)

One may find that (166) naturally generalizes the number-state expansion of the original squeezed vacuum (76) [Fig.4].

**The SO(2,1) squeezed vacuum**

$$|\text{Sq}(\rho, \phi)\rangle = \frac{1}{\cosh \frac{\rho}{2}} \sum_{n=0}^{\infty} \left( - \tanh \frac{\rho}{2} \right)^n e^{in\phi} |n\rangle \otimes |n\rangle$$

$$= \frac{1}{\cosh \frac{\rho}{2}} \left( \begin{array}{ccc}
1 & 1 & 2 \\
1 & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 3/2 \\
1 & 1 & 3/2 \\
\end{array} \right)$$

**The SO(4,1) squeezed vacuum**

$$|\text{Sq}(\rho, \chi, \theta, \phi)\rangle = \frac{1}{\cosh^2 \frac{\rho}{2}} \sum_{S=0,1/2,1/2,\ldots} \left( - \tanh \frac{\rho}{2} \right)^{2S} \sum_{S_z=-S}^{S} e^{2i\chi S} |S, S_z(\theta, \phi)\rangle \otimes |S, S_z(\theta, -\phi)\rangle$$

$$= \frac{1}{\cosh^2 \frac{\rho}{2}} \left( \begin{array}{ccc}
0 & 0 & 1/2 \\
1/2 & 1/2 & 1/2 \\
3/2 & 3/2 & 1 \\
\end{array} \right)$$

Figure 4: The original SO(2,1) squeezed vacuum is an entangled state of infinite (separable) pairs of number-states. The SO(4,1) squeezed vacuum is an entangled state of infinite (maximally entangled) spin-pairs.

### 4.1.2 Spin entanglement

We can represent (164) as

$$|\text{Sq}(\rho, \chi, \theta, \phi)\rangle = \frac{1}{\cosh^2 \left( \frac{\rho}{2} \right)} \sum_{S=0,1/2,1,3/2,\ldots} \sqrt{2S+1} \left( - \tanh \left( \frac{\rho}{2} \right)^{2S} |S \otimes (\chi, \theta, \phi)\rangle, \right.$$ (168)

where $|S \otimes (\chi, \theta, \phi)\rangle$ denotes a (normalized) spin-pair state:

$$|S \otimes (\chi, \theta, \phi)\rangle \equiv \frac{1}{\sqrt{2S+1}} \sum_{S_z=-S}^{S} |S, S_z(\phi, \theta, -\chi)\rangle \otimes |S, S_z(-\phi, \theta, -\chi)\rangle$$

$$= \frac{1}{\sqrt{2S+1}} \sum_{S_z=-S}^{S} \sum_{S_z=-S}^{S} \sum_{S_z=-S}^{S} D(S)(\phi, \theta, -\chi)_{L_z} D(S)(-\phi, \theta, \chi)_{R_z} |S, L_z\rangle \otimes |S, R_z\rangle$$

$$= \frac{1}{\sqrt{2S+1}} \sum_{S_z=-S}^{S} \sum_{S_z=-S}^{S} (D(S)(\phi, \theta, -\chi) D(S)(-\chi, -\theta, -\phi))_{L_z} |S, L_z\rangle \otimes |S, R_z\rangle.$$ (169)

In the last equation, we used the property of the D-matrix:

$$D^{(S)}(\chi, \theta, \phi)^t = D^{(S)}(\phi, -\theta, \chi).$$ (170)
exhibits a duplicate entanglement which we refer to as the hybrid entanglement:

1. An infinite number of spin-pairs are entangled with the squeeze parameter $\rho$. This manifests a natural generalization of the entanglement of the original $SO(2,1)$ squeezed vacuum.

2. In each spin-sector, two spins are maximally entangled. This is a unique feature in the $SO(4,1)$ case. In the original $SO(2,1)$ squeezed vacuum, the corresponding state is simply a product state, $|n\rangle \otimes |n\rangle$.

The spin part \[169\] can be expressed as a superposition of the integer total spin states only:

$$I \equiv S \otimes S = 0 \oplus 1 \oplus 2 \oplus \cdots \oplus 2S.$$  (171)

This is a generalization of the fact that even total number states $|n\rangle \otimes |n\rangle$ only appear in the expansion of the two-mode $SO(2,1)$ squeezed vacuum. Using the Clebsch-Gordan coefficient $C_{j_1,m_1,j_2,m_2}$, we can construct total integer spin states

$$|I, I_z\rangle = \sum_{L_z=-S}^{S} \sum_{R_z=-S}^{S} C_{I,L_z+S,L_z}^{I,I_z} |S, L_z\rangle \otimes |S, R_z\rangle,$$  (172)

and \[169\] is expressed as

$$|S \otimes S (\chi, \theta, \phi)\rangle = \frac{1}{\sqrt{2S+1}} \sum_{I=0}^{2S} \sum_{I_z=-I}^{I} C_{S,I_z}^{I,L_z}(\chi, \theta, \phi) |I, I_z\rangle$$  (173)

where

$$C_{S,I_z}^{I,L_z}(\chi, \theta, \phi) = \sum_{L_z=-S}^{S} \sum_{R_z=-S}^{S} C_{I,L_z+S,L_z}^{I,I_z} (D^{(S)}(\phi, \theta, -\chi) D^{(S)}(-\chi, -\theta, -\phi))_{L_z,R_z}.$$  (174)

Consequently, the $SO(4,1)$ squeezed vacuum \[168\] is actually expressed as a linear combination of an infinite number of spin-pairs with total integer spins:

$$|\text{Sq} (\rho, \chi, \theta, \phi)\rangle = \sum_{I=0,1,2,\ldots} \sum_{I_z=-I}^{I} C_{I,I_z}^{I,I_z}(\rho, \chi, \theta, \phi) |I, I_z\rangle,$$  (175)

with

$$C_{I,I_z}^{I,I_z}(\rho, \chi, \theta, \phi) = \frac{1}{\cosh^{2}(\frac{\rho}{2})} \sum_{S=\frac{I}{2}, \frac{I}{2}+1, \frac{I}{2}+2, \ldots} \sum_{R_z=-S}^{S} (-\tanh(\frac{\rho}{2}))^{2S} C_{S,I_z}^{I,I_z}(\chi, \theta, \phi).$$  (176)

Up to $O(\tanh(\frac{\rho}{2}))$, \[175\] is represented as

$$|\text{Sq} (\rho, \chi, \theta, \phi)\rangle = \frac{1}{\cosh^{2}(\frac{\rho}{2})} \left( |0,0\rangle - \tanh(\frac{\rho}{2}) \left( (\cos \chi + i \sin \chi \cos \theta) |1,1\rangle + \sqrt{2}i \sin \chi \sin \theta \cos \phi |1,0\rangle 
+ (\cos \chi - i \sin \chi \cos \theta) |1,-1\rangle + \sqrt{2} \sin \chi \sin \theta \sin \phi |0,0\rangle \right) + O(\tanh^{2}(\frac{\rho}{2})).$$  (177)

### 4.2 Dimensional Reduction

Here, we discuss reductions of the squeezed vacuum associated with lower dimensional geometries of the Bloch four-hyperboloid. First, we consider a 0D reduction: We focus on the the lowest point of the upper leaf of $H^4$, i.e., $(x^1, x^2, x^3, x^4, x^5) = (0, 0, 0, 0, 1)$, which corresponds to no squeezing $\rho = 0$, and so the squeeze operator \[147\] becomes trivial:

$$\hat{S} = 1.$$  (178)
The squeezed vacuum \([166]\) is reduced to the trivial vacuum:

\[ |Sq\rangle = |0, 0, 0, 0\rangle. \] (179)

Next, let us consider a 2D reduction, \(H^4 \rightarrow H^2\), which is realized at \(\theta = 0\):

\[(x^1, x^2, x^3, x^4, x^5) = (0, 0, \sin \chi \sinh \rho, \cos \chi \sinh \rho, \cosh \rho). \] (180)

The interaction Hamiltonian \([148]\) is reduced to

\[ \hat{H}_{int} = -i \frac{\rho}{2} (a^\dagger c^\dagger e^{i\chi} - b^\dagger d^\dagger e^{-i\chi} + \text{h.c.}) = \hat{H}_{int}^T(\rho, \chi) + \hat{H}_{int}^B(\rho, -\chi) \]

and then the squeezed vacuum \([166]\) becomes

\[ |Sq\rangle = \frac{1}{\cosh^\frac{\rho}{2}} \sum_{S=0,1/2,1,\cdots} \sqrt{2S+1} (-\tanh(\frac{\rho}{2}))^{2S} \sum_{S_z=-S}^S |S, S_z\rangle \otimes |S, -S_z\rangle. \] (181)

which is obvious from \([83]\). Lastly, let us consider 1D reductions. For \(\chi = \theta = \phi = \pi\), i.e.

\[(x^1, x^2, x^3, x^4, x^5) = (0, \sinh \rho, 0, 0, \cosh \rho), \]

we have

\[ \hat{H}_{int} = -i \frac{\rho}{2} (a^\dagger d^\dagger - ad) + i \frac{\rho}{2} (b^\dagger c^\dagger - bc) = \hat{H}_{int}^T(\rho, 0) + \hat{H}_{int}^B(\rho, \pi), \] (182)

and

\[ |Sq\rangle = \frac{1}{\cosh^\frac{\rho}{2}} \sum_{S=0,1/2,1,\cdots} \sqrt{2S+1} (-\tanh(\frac{\rho}{2}))^{2S} \sum_{S_z=-S}^S (\tanh^2 \frac{\rho}{2})^{S-S_z} |S, S_z\rangle \otimes |S, -S_z\rangle. \] (183)

Similarly for \(\chi = \theta = \phi = \pi\), i.e.

\[(x^1, x^2, x^3, x^4, x^5) = (0, 0, \sin \rho, 0, \cosh \rho), \]

we have

\[ \hat{H}_{int} = -i \frac{\rho}{2} (a^\dagger d^\dagger - ad) + i \frac{\rho}{2} (b^\dagger c^\dagger - bc) = \hat{H}_{int}^T(\rho, 0) + \hat{H}_{int}^B(\rho, \pi), \] (184)

and

\[ |Sq\rangle = \frac{1}{\cosh^\frac{\rho}{2}} \sum_{S=0}^\infty \sqrt{2S+1} (-\tanh(\frac{\rho}{2}))^{2S} \sum_{S_z=-S}^S (\tanh^2 \frac{\rho}{2})^{S-S_z} |S, S_z\rangle \otimes |S, -S_z\rangle. \] (185)

Each spin sector of \([185]\) realizes a maximally entangled spin-singlet state

\[ \frac{1}{\sqrt{2S+1}} \sum_{S_z=-S}^S (\tanh^2 \frac{\rho}{2})^{S-S_z} |S, S_z\rangle \otimes |S, -S_z\rangle, \]

which is utilized in the context of violations of Bell inequality for arbitrary spins \([49, 50, 51, 52]\).
4.3 Statistical properties

4.3.1 Statistics about spins

The probability function to find a state \(|S, L_z\rangle \otimes |S, R_z\rangle\) in the \(SO(4, 1)\) squeezed vacuum is \(^{18}\)

\[
P_{S, L_z, R_z} (\rho, \chi, \theta, \phi) \equiv |\langle S, L_z | \otimes \langle S, R_z |\rangle \rangle_{\text{Sq.}(\rho, \chi, \theta, \phi)}|^2 = \frac{1}{\cosh^4 \left( \frac{\rho}{2} \right)} \tanh^{4S} \left( \frac{\rho}{2} \right) |(D^{(S)}(\phi, \theta, -\chi) D^{(S)}(-\chi, -\theta, -\phi))_{L_z, R_z}|^2.
\]

We also introduce the following probability function to find the two spin-coherent state with \(L = R = S\):

\[
P_S (\rho) = \sum_{L_z = -S}^{S} \sum_{R_z = -S}^{S} P_{S, L_z, R_z} (\rho, \chi, \theta, \phi) = \frac{2S + 1}{\cosh^4 \left( \frac{\rho}{2} \right)} \tanh^{4S} \left( \frac{\rho}{2} \right).
\]

Meanwhile, the probability function for the original \(SO(2, 1)\) squeezed vacuum is (see Appendix A)

\[
P_n (\rho) = \frac{1}{\cosh^2 \left( \frac{\rho}{2} \right)} \tanh^{2n} \left( \frac{\rho}{2} \right),
\]

which monotonically decreases as the squeeze parameter increases. \(^{190}\) exhibits non-trivial behavior with respect to \(S\), and its maximum value is attained generally at \(S \neq 0\) [Fig. 5]. This is because of the internal spin degrees of freedom that appears as the numerator in \(^{190}\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Behaviors of the probability functions for the \(SO(2, 1)\) (blue) and \(SO(4, 1)\) (red) for \(\rho = 2\) and \(\rho = 3\). The maximum value of \(P_S (\rho = 2, 3)\) is attained at \(2S = 1, 4\), while the maximum value of \(P_n (\rho = 2, 3)\) is attained at \(n = 0\). For \(\rho \gg 1\), the maximum value is attained at \(2S \simeq \frac{1}{4} e^\rho\).}
\end{figure}

It is straightforward to derive

\[
\langle \hat{S} \rangle = \sum_{S=0,1/2,1,\ldots} S P_S (\rho, \chi, \theta, \phi) = \sinh^2 \left( \frac{\rho}{2} \right),
\]

\[
\langle \hat{S}^2 \rangle = \sum_{S=0,1/2,1,\ldots} S^2 P_S (\rho, \chi, \theta, \phi) = \frac{3}{2} \sinh^4 \left( \frac{\rho}{2} \right) + \frac{1}{2} \sinh^2 \left( \frac{\rho}{2} \right).
\]

\(^{18}\)Since product of two D-matrices should be an \(SU(2)\) group element, it satisfies

\[
\sum_{L_z = -S}^{S} |(D^{(S)}(\phi, \theta, -\chi) D^{(S)}(-\chi, -\theta, -\phi))_{L_z, R_z}|^2 = 2S + 1.
\]

From the formula, \(\sum_{S=0,1/2,1,\ldots} (2S + 1) \tanh^{4S} \left( \frac{\rho}{2} \right) = \cosh^4 \left( \frac{\rho}{2} \right)\), one may see that the summation of \(^{18}\) indeed becomes unity:

\[
\sum_{S=0,1/2,1,3/2,\ldots} \sum_{L_z = -S}^{S} P_{S, L_z, R_z} (\rho, \chi, \theta, \phi) = 1.
\]
and the variation of spin-magnitude is

\[ \langle \Delta \hat{S}^2 \rangle = \langle \hat{S}^2 \rangle - \langle \hat{S} \rangle^2 = \frac{1}{8} \sinh^2 \rho. \]  \hspace{1cm} (193)

The ratio of the spin fluctuation

\[ \sqrt{\frac{\langle \Delta \hat{S}^2 \rangle}{\langle \hat{S} \rangle}} = \frac{1}{\sqrt{2}} \coth \left( \frac{\rho}{2} \right) \]  \hspace{1cm} (194)

monotonically decreases from \( \infty \) to \( \frac{1}{\sqrt{2}} \) as \( \rho \) increases from 0 to \( \infty \) (left of Fig.6). The expectation values of the left and right spins are (see Appendix B.1)

\[ \langle \hat{L} \rangle = \langle \hat{R} \rangle = \sinh^2 \left( \frac{\rho}{2} \right) = \langle \hat{S} \rangle, \] \[ \langle \hat{L}^2 \rangle = \langle \hat{R}^2 \rangle = \frac{3}{2} \sinh^4 \left( \frac{\rho}{2} \right) + \frac{1}{2} \sinh^2 \left( \frac{\rho}{2} \right) = \langle \hat{S}^2 \rangle. \]  \hspace{1cm} (195)

In the expansion of the squeezed vacuum (196), the expansion coefficient in front of the spin-pair, \( \tanh \left( \frac{\rho}{2} \right)^{2S} \), will become dominant for higher \( S \) as \( \rho \) increases, and so higher spin states will become significant. The correlation between the left and right spins is obtained as

\[ \langle \hat{L} \cdot \hat{R} \rangle = \frac{3}{2} \sinh^4 \left( \frac{\rho}{2} \right) + \frac{1}{2} \sinh^2 \left( \frac{\rho}{2} \right) = \langle \hat{S}^2 \rangle. \]  \hspace{1cm} (196)

The linear correlation coefficient, \( J^{SO(4,1)}(L, R) = \frac{\text{cov}(L, R)}{\sqrt{\langle \Delta L^2 \rangle \langle \Delta R^2 \rangle}} \), takes the maximum value

\[ J^{SO(4,1)}(L, R) = 1, \]  \hspace{1cm} (198)

which should be compared with that for the \( SO(2,1) \) two-mode squeezed vacuum (233b):

\[ J^{SO(2,1)}(n_a, n_b) = 1. \]  \hspace{1cm} (199)

As the original \( SO(2,1) \) squeezed vacuum exhibits maximal correlation, the \( SO(4,1) \) squeezed vacuum has the maximal correlation between \( L \) and \( R \) spins. This result is reasonable since, without the spin degrees of freedom, the \( SO(4,1) \) squeeze vacuum is almost equivalent to the original \( SO(2,1) \) squeezed vacuum (Fig.3) and two-mode correlations are analogous in the \( SO(4,1) \) and \( SO(2,1) \) states. For \( z \)-components of left and right spins, a bit of calculation shows (see Appendix B.1)

\[ \langle \hat{L}_z \rangle = \langle \hat{R}_z \rangle = 0, \] \[ \langle \Delta \hat{L}_z^2 \rangle = \langle \Delta \hat{R}_z^2 \rangle = \frac{1}{8} \sinh^2 \rho, \] \[ \text{cov}(\hat{L}_z, \hat{R}_z) = \langle \hat{L}_z \cdot \hat{R}_z \rangle - \langle \hat{L}_z \rangle \langle \hat{R}_z \rangle = \frac{1}{8} \sinh^2 \rho \left( 1 - 2 \sin^2 \chi \sin^2 \theta \right), \]  \hspace{1cm} (200)

and then

\[ J^{SO(4,1)}(\hat{L}_z, \hat{R}_z) = \frac{\text{cov}(\hat{L}_z, \hat{R}_z)}{\sqrt{\langle \Delta \hat{L}_z^2 \rangle \langle \Delta \hat{R}_z^2 \rangle}} = 1 - 2 \sin^2 \chi \sin^2 \theta. \]  \hspace{1cm} (201)

At \( \chi = \theta = \frac{\pi}{4} \), the linear correlation (201) takes maximum negative correlation \( J^{SO(4,1)}(\hat{L}_z, \hat{R}_z) = -1 \), while either at \( \chi = 0, \pi \) or \( \theta = 0, \pi \), (201) takes the maximum positive correlation \( J^{SO(4,1)}(\hat{L}_z, \hat{R}_z) = +1 \) (Right of Fig.6). There is no correlation for \( \chi \) and \( \theta \) that satisfy \( \sin^2 \chi \sin^2 \theta = 1/2 \).
Figure 6: Left: Behaviors of the spin magnitude (blue), spin fluctuation (red) and normalized spin fluctuation (magenta) and with respect to the squeeze parameter. Right: The angular coordinate dependence of $J^{SO(4,1)}(L_z, R_z)$. (The blue plane denotes $J = 0$ for comparison.)

### 4.3.2 von Neumann entropy

With the usual definition of the density operator, $\hat{\rho} \equiv \langle Sq(\rho, \chi, \theta, \phi) | Sq(\rho, \chi, \theta, \phi) \rangle$, we construct the reduced density operator

$$\hat{\rho}^{(\text{Red})} \equiv \sum_{S=0,1/2,1,\cdots}^{\infty} \sum_{R_z=-S}^{S} \langle S, R_z(-\phi, \theta, -\chi) | \hat{\rho} | S, R_z(-\phi, \theta, -\chi) \rangle$$

$$= \frac{1}{\cosh^4(\frac{\rho}{2})} \sum_{S=0,1/2,1,\cdots}^{\infty} \sum_{R_z=-S}^{S} \tanh^{4S}(\frac{\rho}{2}) \tanh^{4S}(\frac{\rho}{2}) \log \left( \frac{\cosh^4(\frac{\rho}{2})}{\tanh^{4S}(\frac{\rho}{2})} \right).$$

and so the von Neumann entropy is given by

$$S_{vN}^{SO(4,1)} = -\text{tr}(\hat{\rho}^{(\text{Red})} \log \hat{\rho}^{(\text{Red})}) = \frac{1}{\cosh^4(\frac{\rho}{2})} \sum_{S=0,1/2,1,3/2,\cdots}^{\infty} (2S + 1) \tanh^{4S}(\frac{\rho}{2}) \log \left( \frac{\cosh^4(\frac{\rho}{2})}{\tanh^{4S}(\frac{\rho}{2})} \right).$$

The coefficient $(2S + 1)$ on the last right-hand side of (203) comes from the $SU(2)$ spin degrees of freedom. Meanwhile, the von Neumann entropy of the original squeezed vacuum (235) is given by

$$S_{vN}^{SO(2,1)} = \frac{1}{\cosh^2(\frac{\rho}{2})} \sum_{n=0,1,2,\cdots}^{\infty} \tanh^{2n}(\frac{\rho}{2}) \log \left( \frac{\cosh^2(\frac{\rho}{2})}{\tanh^{2n}(\frac{\rho}{2})} \right).$$

The von Neumann entropy of the $SO(4,1)$ (203) grows rapidly than that of the $SO(2,1)$ (204), because of the internal spin degrees of freedom (Fig.7).
Figure 7: The von Neumann entropies of the SO(2,1) (blue) and SO(4,1) (red) squeezed vacua.

4.4 SO(4,1) uncertainty relations

From the SO(4,1) Schwinger operators, we introduce non-commutative 4D coordinates

\[ X_1 = \frac{1}{2\sqrt{2}}(a + a^\dagger + c + c^\dagger), \quad X_2 = -i\frac{1}{2\sqrt{2}}(a - a^\dagger + c - c^\dagger), \]
\[ X_3 = \frac{1}{2\sqrt{2}}(b + b^\dagger + d + d^\dagger), \quad X_4 = -i\frac{1}{2\sqrt{2}}(b - b^\dagger + d - d^\dagger), \] (208)

which satisfy the Heisenberg-Weyl algebra:

\[ [X_1, X_2] = [X_3, X_4] = i\frac{1}{2}, \quad [X_1, X_3] = [X_1, X_4] = [X_2, X_3] = [X_2, X_4] = 0. \] (209)

From the expectation values for the squeezed vacuum (see Appendix B), we have

\[ \langle X_1 \rangle = \langle X_2 \rangle = \langle X_3 \rangle = \langle X_4 \rangle = 0, \] (210)

and the variants

\[ \langle (\Delta X_m)^2 \rangle \equiv \langle (X_m)^2 \rangle - \langle X_m \rangle^2 = \langle (X_m)^2 \rangle \quad (m = 1, 2, 3, 4) \] (211)

as

\[ \langle (\Delta X_1)^2 \rangle = \langle (\Delta X_3)^2 \rangle = \frac{1}{4}(\cosh \rho - \sinh \rho \cos \chi), \]
\[ \langle (\Delta X_2)^2 \rangle = \langle (\Delta X_4)^2 \rangle = \frac{1}{4}(\cosh \rho + \sinh \rho \cos \chi). \] (212)

Consequently,

\[ \frac{\langle (\Delta X_1)^2 \rangle \cdot \langle (\Delta X_2)^2 \rangle}{\langle (\Delta X_3)^2 \rangle \cdot \langle (\Delta X_4)^2 \rangle} = \frac{\langle (\Delta X_3)^2 \rangle \cdot (\Delta X_3)^2}{\langle (\Delta X_4)^2 \rangle \cdot (\Delta X_4)^2} = \frac{1}{16} + \frac{1}{16} \sin^2 \rho \sin^2 \chi \geq \frac{1}{16}. \] (213)

\footnote{One can adopt other 4D coordinates:

\[ X_1 = \frac{1}{2\sqrt{2}}(a + a^\dagger + b + b^\dagger), \quad X_2 = -i\frac{1}{2\sqrt{2}}(a - a^\dagger + b - b^\dagger), \]
\[ X_3 = \frac{1}{2\sqrt{2}}(c + c^\dagger + d + d^\dagger), \quad X_4 = -i\frac{1}{2\sqrt{2}}(c - c^\dagger + d - d^\dagger), \] (205)

which satisfy same algebra as (209). However, their expectation values for the SO(4,1) squeezed vacuum are

\[ \langle X_1 \rangle = \langle X_2 \rangle = \langle X_3 \rangle = \langle X_4 \rangle = 0, \quad \langle (\Delta X_1)^2 \rangle = \langle (\Delta X_2)^2 \rangle = \langle (\Delta X_3)^2 \rangle = \langle (\Delta X_4)^2 \rangle = \frac{1}{4} \cosh \rho, \] (206)

and so

\[ \frac{\langle (\Delta X_1)^2 \rangle \cdot \langle (\Delta X_2)^2 \rangle}{\langle (\Delta X_3)^2 \rangle \cdot \langle (\Delta X_4)^2 \rangle} = \frac{\langle (\Delta X_3)^2 \rangle \cdot (\Delta X_3)^2 \rangle}{\langle (\Delta X_4)^2 \rangle \cdot (\Delta X_4)^2} = \frac{1}{16} \cosh^2 \rho \geq \frac{1}{16}. \] (207)

Therefore, for (205), only the trivial vacuum (\( \rho = 0 \)) saturates the uncertainty bound.}
As the squeeze parameter $\rho$ increases, the uncertainty monotonically increases. The inequality is saturated at $\chi = 0, \pi$:

$$\langle (\Delta X_1)^2 \rangle = \frac{1}{4} e^{+\rho}, \quad \langle (\Delta X_2)^2 \rangle = \frac{1}{4} e^{-\rho}, \quad \langle (\Delta X_3)^2 \rangle = \frac{1}{4} e^{+\rho}, \quad \langle (\Delta X_4)^2 \rangle = \frac{1}{4} e^{-\rho}. \quad (214)$$

To remove the $\chi$-dependence in (212), let us apply the following coordinate rotation (same as in the original case (225)):

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \frac{\chi}{2} & -\sin \frac{\chi}{2} \\ \sin \frac{\chi}{2} & \cos \frac{\chi}{2} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \begin{pmatrix} X_3 \\ X_4 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \frac{\chi}{2} & \sin \frac{\chi}{2} \\ -\sin \frac{\chi}{2} & \cos \frac{\chi}{2} \end{pmatrix} \begin{pmatrix} X_3 \\ X_4 \end{pmatrix}, \quad (215)$$

but we end up with

$$\langle (\Delta X_1)^2 \rangle = \langle (\Delta X_3)^2 \rangle = \frac{1}{4} (\cosh \rho - \sinh \rho (\cos^2 \chi + \sin^2 \chi \cos \theta)), \quad \langle (\Delta X_2)^2 \rangle = \langle (\Delta X_4)^2 \rangle = \frac{1}{4} (\cosh \rho + \sinh \rho (\cos^2 \chi + \sin^2 \chi \cos \theta)). \quad (216)$$

As mentioned around (151), we can remove $\phi$-dependence by the redefinition of the Schwinger operators but not $\chi$, so the angle dependence in (212) is not generally removal unlike the original $SO(2,1)$ case (see Appendix A). However, for the special case $\theta = 0$, $\chi$-dependence in (210) can be removed:

$$\langle (\Delta X_1)^2 \rangle = \langle (\Delta X_3)^2 \rangle \rightarrow \frac{1}{4} e^{-\rho}, \quad \langle (\Delta X_2)^2 \rangle = \langle (\Delta X_4)^2 \rangle \rightarrow \frac{1}{4} e^{\rho}. \quad (217)$$

Recall that for $\theta = 0$ the $SO(4,1)$ is simply reduced to the direct product of two $SO(2,1)$ squeezed states with opposite angles, $\chi$ and $-\chi$ (182), and in each $SO(2,1)$ squeezed vacuum, the minimum uncertainty is realized (Fig.8).

![Figure 8: The uncertainty regions for $\theta = 0$.](image)

5 Summary

In this work, we proposed a generalized framework of the squeezed state that includes the spin degrees of freedom based on the Bloch four-hyperboloid (Table 1). While the obtained $SO(4,1)$ squeezed vacuum is a four-mode squeezed state in the photon picture, it can be interpreted as an entangled spin state with the Schwinger’s angular momentum operator formalism. The four parameters of the $SO(4,1)$ squeezed vacuum have a clear geometric origin in the Bloch four-hyperboloid: The squeeze parameter corresponds to the radial coordinate and the three internal spin parameters correspond to the three angular coordinates on
SO(2, 1) squeezed vacuum

| Symmetry group  | SO(2, 1) = Sp(2; R) = U(1; ℍ⁺) |
|-----------------|----------------------------------|
| Representation  | Majorana/Dirac                   |
| Topological map | 1st non-compact Hopf map          |
| Quantum manifold| Bloch two-hyperboloid $H^2$       |
| Degrees of freedom | One/two-mode photons           |

SO(4, 1) squeezed vacuum

| Symmetry group  | SO(4, 1) = U(1, 1; ℍ) |
|-----------------|------------------------|
| Representation  | Dirac                  |
| Topological map | 2nd non-compact (hybrid) Hopf map |
| Quantum manifold| Bloch four-hyperboloid $H^4$ |
| Degrees of freedom | Four-mode photons or two-mode spins |

Table 1: Comparison between the SO(2, 1) squeezed vacuum and the SO(4, 1) squeezed vacuum.

S^3-latitude. As the original SO(2, 1) squeezed vacuum is expressed by a superposition of identical number two-mode photons, the SO(4, 1) squeezed vacuum is realized as a superposition of identical magnitude spin-pairs. The SO(4, 1) squeezed vacuum exhibits a hybrid entanglement: One is about the infinite set of the spin-pairs and the other is about two spins in each spin-pair. The linear correlation coefficient between the left and right spins realizes the maximum correlation. The statistical properties of the SO(4, 1) squeezed vacuum are qualitatively different to the original SO(2, 1) squeezed vacuum in several respects, such as the spin magnitude dependence of the probability function and the angular coordinate dependence of the linear correlation between the left and right third-spin-components. The uncertainty relation was also generalized in 4D with coordinate dependence of the Bloch four-hyperboloid.

The Bloch four-hyperboloid has provided a formalism that accommodates higher spins and squeezed states on the same footing. Since non-compact geometry can incorporate compact geometry as its internal structure, theory based on non-compact geometry generally provides a more comprehensive framework than that on compact geometry. While the importance of compact geometry in quantum information has already been appreciated [61], non-compact or indefinite signature geometry has less been exploited so far. We hope that significance of non-compact geometry in quantum information will be further unveiled in future developments.

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A Basic properties of the SO(2, 1) two-mode squeezed vacuum

This section is mainly based on Sec.7.7 of Gerry and Knight [7].

A.1 One-mode SO(2, 1) squeezed vacuum (Majorana representation)

For the one-mode SO(2, 1) squeezed vacuum

$$|\text{sq}(\rho, \phi)\rangle = e^{-\frac{\rho}{4} e^{i\phi} a^2 + \frac{\rho}{4} e^{-i\phi} a^2} |0\rangle,$$

we have

$$\langle \hat{n} \rangle = \sinh^2 \left( \frac{\rho}{2} \right), \quad \langle \hat{n}^2 \rangle = \sinh^4 \left( \frac{\rho}{2} \right) + \frac{1}{2} \sinh^2 \rho.$$

The variant of the number is

$$\langle \Delta \hat{n}^2 \rangle = \frac{1}{2} \sinh^2 \rho,$$

and then

$$\frac{\sqrt{\langle \Delta \hat{n}^2 \rangle}}{\langle n_a \rangle} = \sqrt{2} \coth \left( \frac{\rho}{2} \right) \geq \sqrt{2}.$$
The non-commutative coordinates which satisfy

\[ [X_1, X_2] = i \frac{1}{2} \]  

are defined as

\[ X_1 = \frac{1}{2}(a + a^\dagger), \quad X_2 = -i \frac{1}{2}(a - a^\dagger). \]  

The relevant expectation values are

\[ \langle X_1 \rangle = \langle X_2 \rangle = 0, \quad \langle (\Delta X_1)^2 \rangle = \frac{1}{4}(\cosh \rho - \sinh \rho \cos \phi), \quad \langle (\Delta X_2)^2 \rangle = \frac{1}{4}(\cosh \rho + \sinh \rho \cos \phi). \]  

An appropriate choice of the rotation

\[ \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \frac{\phi}{2} & -\sin \frac{\phi}{2} \\ \sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \]  

can remove the angle dependence of the variants:

\[ \langle (\Delta X_1)^2 \rangle = \frac{1}{4}e^{-\rho}, \quad \langle (\Delta X_2)^2 \rangle = \frac{1}{4}e^{\rho}. \]  

A.2 Two-mode \( SO(2,1) \) squeezed vacuum (Dirac representation)

For the \( SO(2,1) \) two-mode squeezed vacuum

\[ |sq(\rho, \phi)\rangle = e^{-\frac{\phi}{2} a^\dagger b^\dagger + \frac{\phi}{2} ab}|0\rangle, \]  

the probability function is given by

\[ P_{n_a, n_b}(\rho, \phi) = |\langle n_a, n_b|sq(\rho, \phi)\rangle|^2 = \delta_{n_a, n_b} \cdot \frac{1}{\cosh^2(\frac{\rho}{2})} \tanh^2 n_a(\frac{\rho}{2}), \]  

which satisfies

\[ \sum_{n_a, n_b=0}^{\infty} P_{n_a, n_b}(\rho) = \sum_{n=0}^{\infty} P_n(\rho) = 1. \]  

Here, \( P_n \) denotes the probability to find \( |n\rangle \otimes |n\rangle \) in the squeezed vacuum:

\[ P_n(\rho) = \equiv P_{n,n}(\rho) = \frac{1}{\cosh^2(\frac{\rho}{2})} \tanh^2 n(\frac{\rho}{2}). \]  

The number averages are evaluated as

\[ \langle \hat{n}_a \rangle = \langle \hat{n}_b \rangle = \sum_{n=0}^{\infty} n \cdot P_n(\rho, \phi) = \sinh^2(\frac{\rho}{2}), \]  

\[ \langle \hat{n}_a^2 \rangle = \langle \hat{n}_b^2 \rangle = \sum_{n=0}^{\infty} n^2 \cdot P_n(\rho, \phi) = \sinh^4(\frac{\rho}{2}) + \frac{1}{4} \sinh^2 \rho, \]  

and the variants and covariants are

\[ \langle (\Delta n_a)^2 \rangle = \langle (\Delta n_b)^2 \rangle = \text{cov}(n_a, n_b) = \frac{1}{4} \sinh^2 \rho. \]
The ratio of the number fluctuation and the linear correlation are respectively given by

\[
\frac{\sqrt{\langle \Delta n_a^2 \rangle}}{\langle n_a \rangle} = \coth\left(\frac{\rho}{2}\right) \geq 1,
\]

(233a)

\[
J(n_a, n_b) = \frac{\text{cov}(n_a, n_b)}{\sqrt{\langle \Delta n_a^2 \rangle \langle \Delta n_b^2 \rangle}} = 1.
\]

(233b)

The ratio (233a) monotonically decreases to unity, as the squeeze parameter increases. The statistics of the number is a super-Poissonian statistics.

With the density operator \(\hat{\rho} \equiv |\text{sq}(\rho, \phi)\rangle \langle \text{sq}(\rho, \phi)|\), the reduced density operator is defined as

\[
\hat{\rho}^{(\text{Red})} = \text{tr}_R \hat{\rho} = \frac{1}{\cosh^2\left(\frac{\rho}{2}\right)} \sum_{n=0,1,2,\ldots} \tanh^{2n}\left(\frac{\rho}{2}\right) |n_L\rangle \langle n_L|,
\]

(234)

and the associated von Neumann entropy is

\[
S_{\text{vN}} = -\text{tr}(\hat{\rho}^{(\text{Red})} \log \hat{\rho}^{(\text{Red})}) = \frac{1}{\cosh^2\left(\frac{\rho}{2}\right)} \sum_{n=0}^{\infty} \tanh^{2n}\left(\frac{\rho}{2}\right) \log \left(\frac{\cosh^2\left(\frac{\rho}{2}\right)}{\tanh^{2n}\left(\frac{\rho}{2}\right)}\right).
\]

(235)

Expectation values for the non-commutative coordinates

\[
X_1 = \frac{1}{2\sqrt{2}}(a + a^\dagger + b + b^\dagger), \quad X_2 = -\frac{i}{2\sqrt{2}}(a - a^\dagger + b - b^\dagger),
\]

(236)

are given by

\[
\langle X_1 \rangle = \langle X_2 \rangle = 0, \quad \langle (\Delta X_1)^2 \rangle = \frac{1}{4}(\cosh \rho - \sinh \rho \cos \phi), \quad \langle (\Delta X_2)^2 \rangle = \frac{1}{4}(\cosh \rho + \sinh \rho \cos \phi),
\]

(237)

which are identical to the one-mode case (224).

B  Expectation values of the SO(4, 1) and SO(2, 3) squeezed vacua

Even without knowing a simple number state expansion of squeezed vacuum, we can readily evaluate squeezed vacuum expectation values by the covariance of the Schwinger operator:

\[
\hat{S}^\dagger \hat{\Psi} \hat{S} = M\hat{\Psi},
\]

(238)

which implies the following useful formula

\[
\langle O(\hat{\Psi}) \rangle = \langle \text{sq}(O(\hat{\Psi}))\text{sq} \rangle = \langle 0|\hat{S}^\dagger O(\hat{\Psi})\hat{S}|0\rangle = \langle 0|O(\hat{S}^\dagger \hat{\Psi} \hat{S})|0\rangle = \langle 0|O(M\hat{\Psi})|0\rangle.
\]

(239)

B.1  SO(4, 1) squeezed vacuum

For the SO(4, 1) squeezed vacuum (161), we can derive

\[
\langle 0|\hat{S}^\dagger a^2 \hat{S}|0\rangle = \langle 0|\hat{S}^\dagger c^2 \hat{S}|0\rangle = \langle 0|\hat{S}^\dagger ac \hat{S}|0\rangle = 0,
\]

\[
\langle 0|\hat{S}^\dagger a^4 \hat{S}|0\rangle = |M_{11}|^2 + |M_{12}|^2 = \cosh^2\left(\frac{\rho}{2}\right), \quad \langle 0|\hat{S}^\dagger c^4 \hat{S}|0\rangle = |M_{31}|^2 + |M_{32}|^2 = \cosh^2\left(\frac{\rho}{2}\right),
\]

\[
\langle 0|\hat{S}^\dagger a^2 c \hat{S}|0\rangle = |M_{13}|^2 + |M_{14}|^2 = \sinh^2\left(\frac{\rho}{2}\right), \quad \langle 0|\hat{S}^\dagger c^2 a \hat{S}|0\rangle = |M_{31}|^2 + |M_{32}|^2 = \sinh^2\left(\frac{\rho}{2}\right),
\]

\[
\langle 0|\hat{S}^\dagger ac \hat{S}|0\rangle = M_{11}M_{13}^* + M_{12}M_{14}^* = -\frac{1}{2} \sinh \rho (\cos \chi + i \sin \chi \cos \theta),
\]

(240)
where $M(\rho, \chi, \theta, \phi)$ is (114). With these results, the expectation value of
\[
(X_1)^2 = \frac{1}{8} (a + a^\dagger + c + c^\dagger)^2 = \frac{1}{8} (a^2 + a a^\dagger + a^\dagger a + a^\dagger a^2 + c^2 + c c^\dagger + c^\dagger c + c^2 + 2ac + 2ac^\dagger + 2a^\dagger c + 2a^\dagger c^\dagger),
\]
is evaluated as
\[
\langle (X_1)^2 \rangle = \frac{1}{4} (\cosh \rho - \sinh \rho \cos \chi).
\]
In a similar manner, we have
\[
\langle \hat{n}_a \rangle = \langle \hat{n}_b \rangle = \langle \hat{n}_c \rangle = \langle \hat{n}_d \rangle = \sinh^2(\frac{\rho}{2}),
\]
\[
\langle \hat{n}_a^2 \rangle = \langle \hat{n}_b^2 \rangle = \langle \hat{n}_c^2 \rangle = \langle \hat{n}_d^2 \rangle = 2 \sinh^2(\frac{\rho}{2}) + \sinh^2(\frac{\rho}{2}),
\]
\[
\langle \Delta n_a \rangle = \langle \Delta n_b \rangle = \langle \Delta n_c \rangle = \langle \Delta n_d \rangle = \frac{1}{4} \sinh^2 \rho,
\]
which are exactly equal to (231) and (232). Furthermore with
\[
\langle \hat{n}_a \hat{n}_b \rangle = \langle \hat{n}_a \hat{n}_d \rangle = \sinh^4(\frac{\rho}{2}),
\]
\[
\langle \hat{n}_a \hat{n}_c \rangle = \langle \hat{n}_b \hat{n}_d \rangle = \frac{1}{4} \sinh^2 \rho \left( \cos^2 \chi + \sin^2 \chi \cos^2 \theta \right) + \sinh^4(\frac{\rho}{2}),
\]
\[
\langle \hat{n}_a \hat{n}_d \rangle = \langle \hat{n}_b \hat{n}_c \rangle = \frac{1}{4} \sinh^2 \rho \left( \sin^2 \chi \sin^2 \theta \right) + \sinh^4(\frac{\rho}{2}),
\]
the covariances, $\text{cov}(n_a, n_b) = \langle n_a n_b \rangle - \langle n_a \rangle \langle n_b \rangle$, are derived as
\[
\text{cov}(n_a, n_b) = \text{cov}(n_c, n_d) = 0,
\]
\[
\text{cov}(n_a, n_c) = \text{cov}(n_b, n_d) = \frac{1}{4} \sinh^2 \rho \left( 1 - \sin^2 \chi \sin^2 \theta \right),
\]
\[
\text{cov}(n_a, n_d) = \text{cov}(n_b, n_c) = \frac{1}{4} \sinh^2 \rho \left( \sin^2 \chi \sin^2 \theta \right),
\]
The linear correlation coefficients, $J(n_a, n_b) = \frac{1}{4} \text{cov}(n_a, n_b) / \sqrt{\langle \Delta n_a \rangle \langle \Delta n_b \rangle}$, are
\[
J(n_a, n_b) = J(n_c, n_d) = 0,
\]
\[
J(n_a, n_c) = J(n_b, n_d) = 1 - \sin^2 \chi \sin^2 \theta,
\]
\[
J(n_a, n_d) = J(n_b, n_c) = \sin^2 \chi \sin^2 \theta.
\]
The linear correlation coefficients (240) depend on the angular coordinates, while for the original squeezed vacuum they are constant (233). The linear correlation coefficients (240) become the maximum value 1 at $\chi = 0, \pi$ or $\theta = 0, \pi$, while they vanish either at $\chi = \theta = \pi$. The expectation values for the left and right spins are evaluated as
\[
\langle \hat{L} \rangle = \langle \hat{R} \rangle = \frac{1}{2} \langle \hat{n}_a \rangle + \frac{1}{2} \langle \hat{n}_b \rangle = \sinh^2(\frac{\rho}{2}),
\]
\[
\langle \hat{L}_z \rangle = \langle \hat{R}_z \rangle = 0,
\]
and
\[
\langle \hat{L}^2 \rangle = \langle \hat{R}^2 \rangle = \frac{1}{4} \langle (\hat{n}_a + \hat{n}_b)^2 \rangle = \frac{3}{2} \sinh^4(\frac{\rho}{2}) + \frac{1}{2} \sinh^2(\frac{\rho}{2}),
\]
\[
\langle \hat{L} \cdot \hat{R} \rangle = \frac{1}{4} \langle (\hat{n}_a + \hat{n}_b) (\hat{n}_c + \hat{n}_d) \rangle = \frac{1}{8} \sinh^2 \rho + \sinh^4(\frac{\rho}{2}),
\]
\[
\langle \hat{L}_z^2 \rangle = \langle \hat{R}_z^2 \rangle = \frac{1}{4} \langle (\hat{n}_a - \hat{n}_b)^2 \rangle = \frac{1}{8} \sinh^2 \rho,
\]
\[
\langle \hat{L}_z \cdot \hat{R}_z \rangle = \frac{1}{4} \langle (\hat{n}_a - \hat{n}_b) (\hat{n}_c - \hat{n}_d) \rangle = \frac{1}{8} \sinh^2 \rho \left( 1 - 2 \sin^2 \chi \sin^2 \theta \right).
\]
Consequently,

\[ \langle \Delta L^2 \rangle = \langle \hat{L}^2 \rangle - \langle \hat{L} \rangle^2 = \frac{1}{4}\langle(\Delta n_a)^2 + 2\text{cov}(n_a, n_b) + (\Delta n_b)^2 \rangle = \frac{1}{8}\sinh^2 \rho = \langle \Delta R^2 \rangle, \]

\[ \langle \Delta L_z^2 \rangle = \langle \hat{L}_z^2 \rangle - \langle \hat{L}_z \rangle^2 = \frac{1}{8}\sinh^2 \rho = \langle \Delta R_z^2 \rangle, \tag{249} \]

and

\[ \text{cov}(L, R) = \langle \hat{L} \cdot \hat{R} \rangle - \langle \hat{L} \rangle \langle \hat{R} \rangle = \frac{1}{4}(\text{cov}(n_a, n_c) + \text{cov}(n_a, n_d) + \text{cov}(n_b, n_c) + \text{cov}(n_b, n_d)) \]

\[ = \frac{1}{8}\sinh^2 \rho, \]

\[ \text{cov}(L_z, R_z) = \langle \hat{L}_z \cdot \hat{R}_z \rangle - \langle \hat{L}_z \rangle \langle \hat{R}_z \rangle = \frac{1}{4}(\text{cov}(n_a, n_c) - \text{cov}(n_a, n_d) - \text{cov}(n_b, n_c) + \text{cov}(n_b, n_d)) \]

\[ = \frac{1}{8}\sinh^2 \rho (1 - 2 \sin^2 \chi \sin^2 \theta). \tag{250} \]

B.2 \( SO(2, 3) \sim \text{Sp}(4; \mathbb{R}) \) squeezed vacua

For comparison, we also derive expectation values for the \( SO(2, 3) \) squeezed vacua [23]. In the \( SO(2, 3) \) case, we have two-mode and four-mode squeezed states, since \( SO(2, 3) \) allows Majorana representation and Dirac representation. The \( SO(2, 3) \) squeezed vacua are also physically distinct for the Dirac-type and the Schwinger-type. In this section, we utilize the following parameterization of the coordinates on \( H^{2,2} \):

\[ x^1 = \sin \theta \cos \chi \sinh \rho, \quad x^2 = \sin \theta \sin \chi \sinh \rho, \quad x^3 = \cos \phi \sin \theta \cosh \rho, \]
\[ x^4 = \sin \phi \sin \theta \cosh \rho, \quad x^5 = \cos \theta, \tag{251} \]

which satisfy

\[ (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2 - (x^5)^2 = -1. \tag{252} \]

B.2.1 Two-mode \( SO(2, 3) \) squeezed vacuum (Majororana representation)

- Dirac-type

\[ \langle \hat{n}_a \rangle = \langle \hat{n}_b \rangle = \sinh^2 \rho \sin^2 \left( \frac{\theta}{2} \right), \]

\[ \langle \Delta n_a^2 \rangle = \langle \Delta n_b^2 \rangle = \sinh^2 \rho \sin^2 \left( \frac{\theta}{2} \right) \left( 1 + \cosh(2\rho) \sin^2 \left( \frac{\theta}{2} \right) \right), \]

\[ \text{cov}(n_a, n_b) = \frac{1}{4}\sinh^2 \rho \sin^2 \theta, \]

\[ J(n_a, n_b) = \frac{1}{4(1 + \cosh(2\rho) \sin^2 \left( \frac{\theta}{2} \right))}. \tag{253} \]

Similar to the \( SO(4, 1) \) case (see Appendix 111), the expectation values for the Dirac-type \( SO(2, 3) \) squeezed vacuum have angle dependence but in a different manner.

- Schwinger-type
\[ \langle \hat{n}_a \rangle = \langle \hat{n}_b \rangle = \sinh^2 \left( \frac{\rho}{2} \right), \]
\[ \langle \Delta n_a^2 \rangle = \langle \Delta n_b^2 \rangle = \langle \Delta n_c^2 \rangle = \langle \Delta n_d^2 \rangle = \frac{1}{2} \sinh^2 \rho, \]
\[ \text{cov}(n_a, n_b) = 0, \]
\[ J(n_a, n_b) = 0. \]  
\[(254)\]

The Schwinger-type \( SO(2, 3) \) squeezed vacuum is just given by a direct product of two \( SO(2, 1) \) squeezed vacuum states \[23\]. Therefore, (254) is just given by two copies of (219) and (220), and there is no correlation between \( a \) and \( b \) modes.

B.2.2 Four-mode \( SO(2, 3) \) squeezed vacuum (Dirac representation)

- Dirac-type

\[ \langle \hat{n}_a \rangle = \langle \hat{n}_b \rangle = \langle \hat{n}_c \rangle = \langle \hat{n}_d \rangle = \sinh^2 \rho \sin^2 \left( \frac{\theta}{2} \right), \]
\[ \langle \Delta n_a^2 \rangle = \langle \Delta n_b^2 \rangle = \langle \Delta n_c^2 \rangle = \langle \Delta n_d^2 \rangle = \frac{1}{4} \sinh^2 \rho \sin^2 \left( \frac{\theta}{2} \right)(1 + \sin^2 \rho \sin^2 \left( \frac{\theta}{2} \right)), \]
\[ \text{cov}(n_a, n_b) = \text{cov}(n_c, n_d) = \frac{1}{4} \sinh^2 \rho \sin^2 \theta, \]
\[ \text{cov}(n_a, n_c) = \text{cov}(n_a, n_d) = \text{cov}(n_b, n_c) = \text{cov}(n_b, n_d) = 0, \]
\[ J(n_a, n_b) = J(n_c, n_d) = \frac{1}{1 + \sin^2 \rho \sin^2 \left( \frac{\theta}{2} \right)} \cosh^2 \rho \sin^2 \left( \frac{\theta}{2} \right), \quad J(n_a, n_c) = J(n_b, n_d) = 0, \]
\[ J(n_a, n_d) = J(n_b, n_c) = \frac{1}{1 + \sin^2 \rho \sin^2 \left( \frac{\theta}{2} \right)} \cos^2 \left( \frac{\theta}{2} \right). \]  
\[(255)\]

As in the two-mode case \[254\], \[255\] exhibits non-trivial angle dependence but in a different manner. As \( \rho \) increases, \( J(n_a, n_b) \) and \( J(n_c, n_d) \) monotonically increase while \( J(n_a, n_d) \) and \( J(n_b, n_c) \) monotonically decrease.

- Schwinger-type

\[ \langle \hat{n}_a \rangle = \langle \hat{n}_b \rangle = \langle \hat{n}_c \rangle = \langle \hat{n}_d \rangle = \sinh^2 \left( \frac{\rho}{2} \right), \]
\[ \langle \Delta n_a^2 \rangle = \langle \Delta n_b^2 \rangle = \langle \Delta n_c^2 \rangle = \langle \Delta n_d^2 \rangle = \frac{1}{4} \sinh^2 \rho, \]
\[ \text{cov}(n_a, n_b) = \text{cov}(n_c, n_d) = 1, \quad \text{cov}(n_a, n_c) = \text{cov}(n_a, n_d) = \text{cov}(n_b, n_c) = \text{cov}(n_b, n_d) = 0, \]
\[ J(n_a, n_b) = J(n_c, n_d) = 1, \quad J(n_a, n_c) = J(n_a, n_b) = J(n_b, n_c) = J(n_b, n_d) = 0. \]  
\[(256)\]

This is a straightforward generalization of the two-mode case \[251\].

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