On the Rates of Convergence in Learning of Optimal Temporally Fair Schedulers

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Abstract

Opportunistic multi-user scheduling is necessary to fully exploit the multiplexing gains in non-orthogonal multiple access (NOMA) and full-duplex (FD) systems. Multi-user schedulers are designed to achieve optimal average system utility (e.g. throughput) subject to a set of fairness criteria. In this work, scheduling under temporal fairness constraints is considered. Prior works have shown that a class of scheduling strategies called threshold based strategies (TBSs) achieve optimal system utility under temporal fairness constraints. A TBS assigns a threshold value to each user in the network. The subset of users to be activated at each resource block is a function of the resulting throughput — which varies stochastically at different resource blocks — and the assigned thresholds. The value of the optimal thresholds depend on the channel statistics. However, the scheduler does not typically have prior knowledge of the channel statistics, and instead, it learns the optimal thresholds based on the empirical observations of the channel realizations. In this work, the rate of convergence of the TBS thresholds to the optimal value and the effect on the resulting system utility is investigated. It is shown that the best estimate of the threshold vector is at least $\omega\left(\frac{1}{\sqrt{t}}\right)$ away from the optimal value, where $t$ is the number of observations of the channel realizations. Furthermore, the scheduler may achieve an average utility that is higher than the optimal long-term utility by violating the fairness criteria for a long initial period. Consequently, the resulting system utility may converge to its optimal long-term value from above. The results are verified by providing simulations of practical scheduling scenarios.

I. Introduction

Opportunistic multi-user scheduling in cellular communications is a topic of significant interest [1]–[5]. Opportunistic scheduling strategies are designed to achieve maximum system utility by exploiting...
the channel state information subject to a set of fairness criteria. Various criteria have been proposed to model and evaluate fairness of scheduling strategies. These criteria are categorized as temporal [1]–[3] utilitarian [6], [7], and proportional [8], [9] fairness criteria.

In this work, we consider scheduling under temporal fairness, where the fraction of the resource blocks in which each user is activated is required to be within some predetermined upper and lower bounds. Temporally fair schedulers provide each user with a minimum temporal share in order to control the average delay [4], and restrict the maximum power drain of users by placing upper-bounds on their temporal shares [1].

In [2], [10], we consider the user scheduling problem for non-orthogonal multiple access (NOMA) and full-duplex (FD) systems under temporal fairness constraints, and show that optimal average system utility is achieved using a class of scheduling strategies called Threshold Based Strategies (TBSs). Furthermore, we show that any optimal scheduling strategy is equivalent to a TBS. In other words, an optimal scheduling strategy can always be represented in the form of a TBS. A TBS assigns real-valued thresholds to each of the users in the network. At each resource block a subset of users is activated based on the resulting system utility — which is calculated based on the channel realizations in that time-slot — and the thresholds assigned to each of the users. The optimal thresholds assigned to each user in the TBS are functions of the users’ channel statistics. One of the main challenges in the design of TBSs in practical systems is the fact that the scheduler does not typically have prior knowledge of the users’ channel statistics. Rather, it gains an empirical estimate of the statistics using its observations of the prior channel realizations obtained through channel estimation and feedback techniques. Consequently, the scheduler cannot calculate the optimal thresholds prior to the start of communication, and it has to update the thresholds in an online fashion as it gains a more accurate estimate of the channel statistics by accumulating empirical observations [2], [3]. A problem of interest which arises from this observation is the characterization of the rate of convergence (RoC) of the thresholds as a function of the length of the channel estimation phase, and the effect on the resulting average system utility.

In this paper, we investigate the RoC of the thresholds used in the TBS strategy to the optimal threshold values as well as the RoC of the resulting average system utility to the optimal utility. We show that the threshold values converge with rate at most \( O(\frac{1}{\sqrt{t}}) \), where \( t \) is the number of the prior empirical observations of the users’ channel statistics. We argue that this upper bound on the RoC is in agreement with the well-known lower bounds on the RoC of stochastic approximation algorithms [11],

\[ f(x) = O(g(x)) \text{ if } \lim_{x \to \infty} \frac{f(x)}{g(x)} < \infty. \]
which suggests that the bound is tight. We also investigate the RoC for the average system utility, and show that one can design a scheduling strategy whose utility converges to the optimal long-term utility from above. Loosely speaking, this is achieved by violating the temporal fairness constraints for a long initial period while accumulating additional system utility. After this initial period, having gained an accurate estimate of the channel statistics, the scheduler operates at near-optimal utility while satisfying the fairness constraints. We verify the results by providing simulations of several practical scenarios.

Notation: The set of numbers \( \{1, 2, \cdots, n\} \) is represented by \( [n] \). The vector \( (x_1, x_2, \cdots, x_n) \) is written as \( x^n \). The \( m \times t \) matrix \( [g_{i,j}]_{i \in [m], j \in [t]} \) is denoted by \( g_{m \times t} \). For an event \( \mathcal{A} \), the random variable \( 1_{\mathcal{A}} \) is the indicator function. For the continuous random variable \( X \) whose probability density function (PDF) depends on the model parameter \( \theta \), we write the PDF as \( f_X(x; \theta), x \in \mathbb{R} \).

II. Preliminaries

In this paper, we focus on opportunistic NOMA scheduling under temporal fairness constraints which was formulated in [2]. However, the formulation and main results may be extended to orthogonal multiple access (OMA) and FD systems with minor modifications [10].

A. System Model

We consider user scheduling in a single-cell with \( n \) users and one base-station (BS). Only specific subsets of users may be activated simultaneously. For instance, in NOMA communication systems — due to restricted computation complexity at network terminals — a limited number of users can be activated either in uplink (UL) or downlink (DL) at each time-slot. Another example is FD systems where at each resource block, a pair of users are activated, one in UL and one in DL transmission. Subsets of users \( \mathcal{V}_j, j \in [m] \) which can be activated simultaneously are called virtual users, where \( m \leq 2^n \). The set of all virtual users is denoted by \( \mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_m\} \). The choice of the active virtual user at a given time-slot determines the resulting system utility at that time-slot. The vector of system utilities due to activating each of the virtual users is called the performance vector. For instance, the performance vector can be taken as the vector of sum-rates resulting from activating each of the virtual users. In this case, the performance vector is random and its value depends on the realization of the underlying time-varying channel. In a given time-slot, the system utilities due to activating different virtual users may depend on each other. However, the performance vectors in different time-slots are assumed to be independent of each other, for example due to independence of the channels.
**Definition 1 (Performance Vector).** The vector of jointly continuous variables \((R_{1,t}, R_{2,t}, \ldots, R_{m,t}), t \in \mathbb{N}\) is the performance vector of the virtual users at time \(t\). The sequence \((R_{1,t}, R_{2,t}, \ldots, R_{m,t}), t \in \mathbb{N}\) is a sequence of independent vectors distributed identically according to the joint density \(f_{R^m}\).

**Example 1 (OMA Performance Vector).** Consider an OMA downlink scenario where only a single user may be activated at each time-slot. There are \(m = n\) virtual users in this case, where \(\mathcal{V}_i = \{u_i\}, i \in [n]\). The system utility is defined as the network throughput. Let \(H_{i,t} = \beta_i G_{i,t}\) be the propagation channel coefficient between user \(u_i\) and the BS at time-slot \(t\). In this model, \(\beta_i\) captures large-scale channel variations such as distance-dependent path loss and shadowing which mainly depend on the location of the user \(u_i\). Furthermore, \(G_{i,t}\) captures the small-scale variations of the channel caused by multi-path fading which depends on the scattering profile of the propagation environment. We assume that \(\beta_i\) is constant over the time interval of interest and \(G_{i,t}, i \in [n], t \in \mathbb{N}\) follows a complex normal distribution as in Rayleigh fading model. Additionally, it is assumed that \(G_{i,t}, i \in [n]\) are independent over time. Consequently, channel coefficients \(H_{i,t}, i \in [n]\) are also independent over time as \(\beta_i, i \in [n]\) are constant. The resulting signal to noise ratio (SNR) of user \(u_i\) at time-slot \(t\) is defined as \(\text{SNR}_{i,t} = p|H_{i,t}|^2/\sigma^2\) where, \(p\) and \(\sigma^2\) denote the downlink transmit power and noise power, respectively. Consequently, the performance value for virtual user \(\mathcal{V}_i\) at time-slot \(t\) is defined by \(R_{i,t} = \max\{\log_2(1 + \text{SNR}_{i,t}), \gamma_{\text{max}}\}\), where \(\gamma_{\text{max}}\) models the maximum spectral efficiency in the system. As a result, the PDF of the performance vector \(R^m\), i.e. \(f_{R^m}\), depends on the distribution of the underlying propagation channels which themselves depend on a set of model parameters such as the users’ locations. Generally, we assume that the statistics of the performance vector is parametrized by some fixed model parameters \((\theta_1, \theta_2, \ldots, \theta_n)\), so that the PDF of \(R^m\) is written as \(f_{R^m}(r^m; \theta_1, \theta_2, \ldots, \theta_n), r^m \in \mathbb{R}^m\).

Temporal fairness requires that the fraction of time-slots in which each user is activated be bounded from below (above). The vector of lower (upper) bounds \(\underline{w}^n\) (\(\overline{w}^n\)) is called the lower (upper) temporal demand vector. The objective is to design a scheduling strategy satisfying the temporal fairness constraints while maximization the resulting system utility. Accordingly, a scheduling strategy is defined as follows.

**Definition 2 (Scheduler).** Consider the scheduling setup parametrized by \((n, \mathcal{V}, \underline{w}^n, \overline{w}^n, f_{R^m})\). A scheduling strategy \(Q = (Q_t)_{t \in \mathbb{N}}\) is a family of (possibly stochastic) functions \(Q_t : \mathbb{R}^{mx\mathcal{V}} \rightarrow \mathcal{V}, t \in \mathbb{N}\), where:

- The input to \(Q_t, t \in \mathbb{N}\) is the matrix of performance vectors \(R^m_{\mathcal{V}t}\) which consists of \(t\) independently and identically distributed column vectors with distribution \(f_{R^m}\).

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• The temporal demand constraints are satisfied:

\[ P\left( w_i \leq A_{i,t}^Q \quad \& \quad A_{i,t}^Q \leq \bar{w}_i, i \in [n] \right) = 1, \]

where, the temporal share of user \( u_i, i \in [n] \) up to time \( t \in [s] \) is defined as

\[ A_{i,t}^Q = \frac{1}{t} \sum_{k=1}^{t} 1_{\{u_i \in Q_k(R_m \times k)\}}, \forall i \in [n], t \in \mathbb{N}, \]

\[ A_i^Q = \liminf_{t \to \infty} A_{i,t}^Q, \quad \bar{A}_i^Q = \limsup_{t \to \infty} A_{i,t}^Q. \]

We consider homogeneous systems where the scheduler is allowed to activate subsets of at most \( N_{\text{max}} \) users at each time-slot. More precisely, for a homogeneous multi-user system with \( n \) users and maximum number of active users \( N_{\text{max}} \leq n \), the set of virtual users is defined as

\[ \mathcal{V} = \left\{ \mathcal{V}_j \subset \mathcal{U} | |\mathcal{V}_j| \leq N_{\text{max}} \right\}. \]

We write \( (n, N_{\text{max}}, \bar{w}^a, \bar{w}^b, f_{\text{re}}) \) instead of \( (n, \mathcal{V}, \bar{w}^a, \bar{w}^b, f_{\text{re}}) \) to characterize a homogeneous system. A scheduling setup where the user temporal shares are required to take a specific value, i.e. \( A_{i,t}^Q = w_i, i \in [n] \), is called a setup with equality temporal constraints and is parametrized by \( (n, N_{\text{max}}, w^a, w^b, f_{\text{re}}) \). The average system utility of a scheduler is defined as:

**Definition 3 (System Utility).** For an \( s \)-scheduler \( Q \):

• The average system utility up to time \( t \), is defined as

\[ U_i^Q = \frac{1}{t} \sum_{k=1}^{t} \sum_{j=1}^{m} R_{j,k} 1_{\{Q_k(R_m \times k) = \mathcal{V}_j\}}. \]

• The variable \( U^Q \) is called the average system utility for the scheduler, where

\[ U^Q = \liminf_{t \to \infty} U_i^Q. \]

A scheduler \( Q^* \) is optimal if and only if \( Q^* \in \text{argmax}_{Q \in \mathcal{Q}} U^Q \), where \( \mathcal{Q} \) is the set of all temporally fair schedulers for the scheduling setup. The optimal utility is denoted by \( U^* \).

**B. Prior Literature**

In [2], we showed that a class of scheduling strategies called TBSs achieve the optimal system utility subject to temporal fairness constraints. A TBS is formally defined below.
Definition 4 (TBS). For the scheduling setup \((n, N_{\text{max}}, \bar{w}^n, \overline{w}^n, f_{R^n})\) a threshold based strategy (TBS) is characterized by the vector \(\lambda^n \in \mathbb{R}^n\). The strategy \(Q_{\text{TBS}}(\lambda^n) = (Q_{\text{TBS},t})_{t \in \mathbb{N}}\) is defined as:

\[
Q_{\text{TBS}}(R_{m\times t}) = \arg\max_{\forall V_j \in \mathcal{V}} S(V_j, R_{j,t}), \ t \in \mathbb{N},
\]

where \(S(V_j, R_{j,t}) = R_{j,t} + \sum_{i=1}^{n} \lambda_i \mathbb{1} \{ u_i \in V_j \} \) is the ‘scheduling measure’ corresponding to the virtual user \(V_j\). The resulting temporal shares are represented as \(A_{i}^{Q_{\text{TBS}}} \), \(i \in [n]\). The utility of the TBS is written as \(U_{w^n}(\lambda^n)\). The space of all threshold based strategies is denoted by \(Q_{\text{TBS}}\).

It can be shown that any optimal scheduling strategy can be represented as a TBS. In other words, any optimal strategy is equivalent to a TBS, where equivalence is defined in [2].

Theorem 1 ([2]). For the scheduling setup \((n, N_{\text{max}}, \bar{w}^n, \overline{w}^n, f_{R^n})\), assume that \(Q \neq \emptyset\). Then, there exists an optimal threshold based strategy \(Q_{\text{TBS}}\). Furthermore, for any optimal strategy \(Q\), there exists a threshold based strategy \(Q'\) such that \(Q \sim Q'\).

Theorem 1 proves the existence of optimal TBSs. However, the question of how to construct such TBSs is not addressed by this theorem. An iterative algorithm based on the Robins-Monro method was proposed in [2] to construct the optimal thresholds using the BS’s observations of the channel realizations at each time-slot. It was shown that the output of the algorithm converges to the optimal thresholds under mild assumptions on the channel statistics. In this paper, we are interested in providing upper and lower bounds on the RoC of the thresholds and the effect on average system utility.

C. Rates of Convergence

The optimal threshold vector \(\lambda^*\) is a function of the channel statistics. More precisely, it can be shown that for the scheduling problem with equality constraints \((n, N_{\text{max}}, \bar{w}^n, \overline{w}^n, f_{R^n})\), the optimal threshold vector \(\lambda^*\) is the unique vector for which the following fairness constraints hold:

\[
P \left( \max_{V_j: u_i \in V_j} S(V_j, R_{j,t}) \geq \max_{V_j: u_i \in V_j} S(V_j, R_{j,t}) \right) = w_i, \ i \in [n],
\]

where the probability is taken with respect to \(f_{R^n}\).

In theory, if the BS has access to the distribution \(f_{R^n}\), it may solve Equation (6) to derive the optimal threshold vector. However, in practice, the BS does not have access to the statistics of the performance vector \(R^n\). Rather at time \(t\), the BS estimates the optimal threshold vector using the realization \(R_{m\times t} = r_{m\times t}\) of the performance vector up to time \(t\). Let \(\hat{\lambda}^n_t\) be the estimate of the optimal threshold vector at time \(t\).
In this work, we are interested in deriving upper and lower bound on the RoC of the sequence of vectors \( \tilde{\lambda}_t^n \) to \( \lambda^n \).

**Definition 5 (Threshold RoC).** Consider the set of scheduling setups with equality constraints \((n, N_{\text{max}}, w^n, w^n, f_{R^n})\), \( f_{R^n} \in \mathcal{P} \), where \( \mathcal{P} \) is a set of PDFs. Define the space of all mapping from \( \mathbb{R}^{m\times t} \) to \( \mathbb{R}^n \) as \( \mathcal{G}_t = \{ g : \mathbb{R}^{m\times t} \to \mathbb{R}^n \} \). The optimal threshold RoC is defined as

\[
\alpha^* = \sup_{(g_t)_{t\in \mathbb{N}}} \inf_{\mathcal{G}_t} \sup_{\alpha \geq 0} \lim_{t \to \infty} \frac{\mathbb{E}_{R^{m\times t}}(\|\lambda^n - \tilde{\lambda}_t^n\|_2^2)}{t^{-2\alpha}} < \infty,
\]

where \( \tilde{\lambda}_t^n \triangleq g_t(R^{m\times t}) \) and \( \| \cdot \|_2 \) is the \( \ell_2 \) norm.

The long-term-fair utility RoC (LTU-RoC) is defined as:

**Definition 6 (LTU-RoC).** Consider the set of scheduling setups with equality constraints \((n, N_{\text{max}}, w^n, w^n, f_{R^n})\), \( f_{R^n} \in \mathcal{P} \), where \( \mathcal{P} \) is a set of PDFs. Define the space of all mappings from \( \mathbb{R}^{m\times t} \) to \( [m] \) as \( \mathcal{Q}_t = \{ Q_t : \mathbb{R}^{m\times t} \to [m] \} \).

The optimal LTU-RoC is defined as

\[
\zeta^* = \sup_{(Q_t)_{t \in \mathbb{N}}} \inf_{\mathcal{Q}_t} \sup_{f_{R^n} \in \mathcal{P}} \lim_{t \to \infty} |\mathbb{E}_{R^{m\times t}}(U^* - \frac{1}{t} \sum_{i \in [t]} \tilde{U}_i^n)|_+ < \infty,
\]

where \( \tilde{U}_i^n \) is the utility due to the scheduler \( Q_t \) and \(|x|_+ = x \cdot 1_{x>0}\).

Note that in the above definition \( Q_t \) need not be a TBS.

### III. Bounds on the Rates of Convergence

We provide bounds on the RoC of the TBS thresholds and the system utility. We show that the threshold RoC is at most \( \frac{1}{2} \), so that the threshold vector is at least\(^2 \omega(\frac{1}{\sqrt{t}}) \) away from the optimal value, where \( t \) is the number of observed channel realizations. In the case of LTU-RoC, we show that \( \zeta^* \) can be arbitrarily large.

#### A. Threshold Rate of Convergence

As a first step, we investigate the RoC of the best estimate of the optimal threshold vector, when the BS has access to \( t \in \mathbb{N} \) prior observations of the users’ channel realizations. We derive an upper-bound

\(^2\text{We write } f(x) = \omega(g(x)) \text{ if } \lim_{x \to \infty} \frac{f(x)}{g(x)} < \infty.\)
on the RoC assuming the observations of channel realizations are noiseless (i.e. perfect channel state information). It is straightforward to argue that the upper-bound holds for the case of noisy observations as well. The arguments presented in this section build upon the following extension of the Cramér-Rao bound for parameter estimation.

**Lemma 1** ([12]). Let $X^{mx}$ be a matrix of random variables with distribution $f_{X^{mx}}(X^{mx}; \theta, \alpha)$, where $\alpha$ and $\theta$ are deterministic model parameters and $f_{X^{mx}}(X^{mx}; \theta, \alpha)$ is twice differentiable with respect to $\alpha$ for any fixed $\theta$. Then, the variance of any unbiased estimator $\hat{\alpha}(X^{mx})$ of $\alpha$ is bounded as follows:

$$\sigma_{\alpha}^2 \geq \mathbb{E}_{\theta, X^{mx}} \left( \frac{\partial^2}{\partial \alpha^2} \ln \left( f_{X^{mx}}(X^{mx}; \theta, \alpha) \right) \right).$$

(7)

**Remark 1.** The right hand side of Equation (7) is called the Fisher information of the variable $\alpha$ and is denoted by $I_{X^{mx}}(\alpha)$.

The optimal threshold vector $\lambda^n$ is a function of the PDF of the performance vector $R^n$. As in Example 1, the PDF of the performance vector is parameterized by a set of model parameters such as the users’ locations. Generally, we assume that the PDF is parametrized by the real-valued vector, $(\theta_1, \theta_2, \cdots, \theta_n)$, so that the statistics of $R^n$ can be expressed as $f_{R^n}(R^n; \theta_1, \theta_2, \cdots, \theta_n)$. In order to derive an upper-bound on the threshold RoC, we consider a genie-assisted BS which is given the values of the model parameters $\theta_i, i \neq i'$ and optimal thresholds $\lambda^*_i, i \neq i'$, for some $i' \in [n]$. Consequently, the genie-assisted BS does not know the model parameter $\theta_i$ and the optimal threshold $\lambda^*_i$. Furthermore, we assume that the BS can accurately calculate $f_{R^n}(R^n; \theta_1, \theta_2, \cdots, \theta_n)$ and hence $\lambda^n$ provided that it has access to $\theta_1, \theta_2, \cdots, \theta_n$. Under these assumptions, the problem of finding the optimal threshold vector is related to the well-studied quantile estimation problem [13]. To elaborate, note that from Equation (6) finding the optimal $\lambda^*_i$ requires solving the following equation:

$$w_f = \mathbb{P} \left( \max_{V_j, u_i \in V_j} (R_{j, t} + \sum_{u_i \in V_j} \lambda^*_i) - \max_{V_j, u_i \in V_j} (R_{j, t} + \sum_{u_i \in V_j} \lambda^*_i) \geq -\lambda^*_f \right).$$

(8)

Define $\overline{R} \triangleq \max_{V_j, u_i \in V_j} (R_{j, t} + \sum_{u_i \in V_j} \lambda^*_i) - \max_{V_j, u_i \in V_j} (R_{j, t} + \sum_{u_i \in V_j} \lambda^*_i)$ and let $f_{\overline{R}}(\overline{R}; \theta_1, \theta_2, \cdots, \theta_n)$ be the underlying probability measure. Equation (8) can be written in the form of the following integral equation which describes a quantile estimation problem:

$$w_f = \int_{-\lambda^*_f}^{\infty} f_{\overline{R}}(r; \theta_1, \theta_2, \cdots, \theta_n) dr.$$

(9)
Consequently, $\lambda^*_i$ is a function of $\theta_i$ given $\theta_i \neq i'$. We assume that $\lambda^*_i(\theta_i)$ is a smooth, differentiable function. For a given $\theta_i = \theta$, assume that $\frac{\partial}{\partial \theta_i} \lambda^*_i(\theta_i) > 0$, without loss of generality. Then, it is well-known that there exists a local neighborhood $B = [\theta - \epsilon, \theta + \epsilon], \epsilon > 0$, for which $\lambda^*_i(\theta_i), \theta_i \in B$ is increasing in $\theta_i$. Particularly, $\lambda^*_i$ is one-to-one as a function of $\theta_i$ over the interval $B$. As a result, the PDF of $\tilde{R}$ can be parametrized by $\theta_1, \theta_2, \cdots, \theta_{i-1}, \lambda^*_i, \theta_{i+1}, \cdots, \theta_n$ in this neighborhood. We assume that $f_\hat{R}(\tilde{R}; \theta_1, \theta_2, \cdots, \theta_{i-1}, \lambda^*_i, \theta_{i+1}, \cdots, \theta_n)$ is twice differentiable in $\lambda^*_i$. Note that this property can be verified in conventional models of cellular communication systems such as the one described in Example 1.

The BS needs to estimate the parameter $\lambda^*_i$ using the observations $R^{m_{\text{out}}}$. This resembles the problem described in Lemma 1. Using the Cramér-Rao bound for parameter estimation, we prove the following theorem which provides upper-bounds on the threshold RoC when unbiased estimators are used.

**Theorem 2.** Let $\sigma^*_i$ denote the minimum mean square error (MMSE) in estimating $\lambda^*_i$ for the genie-assisted setup described above. The following holds $\sigma^*_i \geq I_\hat{R}(\lambda^*_i)$, where

$$I_\hat{R}(\lambda^*_i) = \frac{1}{t} E_{\hat{R}}^{-1} \left( \frac{\partial^2}{\partial \lambda^*_i} \ln(f_\hat{R}(\tilde{R}; \theta)) \right),$$

where $\theta = (\theta_1, \theta_2, \cdots, \theta_{i-1}, \lambda^*_i, \theta_{i+1}, \cdots, \theta_n)$. Particularly, we have $\alpha^* \leq \frac{1}{2}$, where $\alpha^*$ is the threshold RoC.

**Proof.** From Lemma 1, we have:

$$E_{R^{m\text{out}+}}(||\lambda^n - \lambda^*_i||^2) \geq E_{\hat{R}}^{-1} \left( \frac{\partial^2}{\partial \lambda^*_i} \ln(f_\hat{R}(\tilde{R}; \theta)) \right) = \frac{1}{t} E_{\hat{R}}^{-1} \left( \frac{\partial^2}{\partial \lambda^*_i} \ln(f_\hat{R}(\tilde{R}; \theta)) \right),$$

where in the last equality we have used the fact that the channel realizations at different time-slots are independent to conclude that $f_\hat{R}(\tilde{R}; \theta) = \prod_{i=1}^n f_\hat{R}(\tilde{R}; \theta)$, and the fact that the channel realizations are identically distributed in different time-slots to write $E_{\hat{R}} \left( \frac{\partial^2}{\partial \lambda^*_i} \ln(f_\hat{R}(\tilde{R}; \theta)) \right) = t E_{\hat{R}} \left( \frac{\partial^2}{\partial \lambda^*_i} \ln(f_\hat{R}(\tilde{R}; \theta)) \right)$. Consequently, we have shown that $E_{R^{m\text{out}+}}(||\lambda^n - \lambda^*_i||^2) \geq \frac{c}{2}$ for some constant $c > 0$. As a result, from Definition 5, we conclude that $\alpha^* \leq \frac{1}{2}$ since

$$\lim_{t \to \infty} \frac{E_{R^{m\text{out}+}}(||\lambda^n - \lambda^*_i||^2)}{t^{-2\alpha}} \geq \lim_{t \to \infty} \frac{c t^{-1}}{t^{-2\alpha}} = \infty, \forall \alpha > \frac{1}{2}.$$ 

\[\square\]

**Remark 2.** Theorem 2 provides bounds on the threshold RoC when unbiased estimators are used to estimate $\lambda^*_i$. Similar bounds can potentially be derived when biased estimators are employed using an alternative median estimation argument based on the order statistics of the vectors $R^n(1), R^n(2), \cdots, R^n(t)$ as in [14].

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B. Utility Rate of Convergence

So far, we have derived an upper-bound on the threshold RoC. In the following, we show that the LTU-RoC is arbitrarily large. In fact, it is shown that there exists a scheduler construction strategy which produces the scheduler \((Q_i)_{i \in \mathbb{N}}\) using the empirical observations of the users’ channels for which \([U^* - \frac{1}{t} \sum_{i=1}^{t} \hat{U}^n_i]_+ = 0, \forall t \in \mathbb{N}\). So that the utility of the scheduler approaches the long-term optimal utility from above as the scheduling window increases asymptotically.

**Theorem 3.** Consider the set of scheduling setups \((n, N_{\text{max}}, \underline{w}_n, \overline{w}_n, f_{R^n})\), \(f_{R^n} \in \mathcal{P}\), where \(\mathcal{P}\) is a compact set of bounded and differentiable PDFs on \(\mathbb{R}^n\). Assume that \(\alpha^* > 0\) for this set of scheduling setups. There exists a scheduling strategy \(Q^* = (Q_i^*)_{i \in \mathbb{N}}\) for which \([U^* - \frac{1}{t} \sum_{i=1}^{t} \hat{U}^n_i]_+ = 0\) for all large enough \(t \in \mathbb{N}\). Particularly, we have \(\zeta^* = \infty\).

**Proof.** From the theorem statement, the threshold RoC is equal to \(\alpha^* > 0\). So, there exists a family of threshold construction functions \(f_i : \mathbb{R}^{m \times t} \to \mathbb{R}^n\) and constant \(c \in \mathbb{R}\) for which \(\mathbb{E}^\text{RoC}(\|\lambda^n - \bar{\lambda}_i^n\|_2) \leq \frac{1}{\alpha^*}\) for any \(f_{R^n} \in \mathcal{P}\) and large enough \(t \in \mathbb{N}\), where \(\bar{\lambda}_i^n = f(R^{m \times t})\). Consequently, by the Chebyshev inequality, we have:

\[
P(\|\lambda^n - \bar{\lambda}_i^n\|_2 > \frac{\sqrt{c}}{\rho^{0.5\alpha^*}}) \leq \frac{\mathbb{E}^\text{RoC}(\|\lambda^n - \bar{\lambda}_i^n\|_2)}{\left(\frac{\sqrt{c}}{\rho^{0.5\alpha^*}}\right)^2} \leq \frac{1}{\rho^{\alpha^*}}. \tag{10}
\]

Note that \(c\) and \(\alpha^*\) are universal constants which do not depend on the underlying distribution \(f_{R^n}\).

We assume that the scheduler can calculate these constants. Let \(V_i(\lambda^n)\) be the utility due to the TBS with threshold \(\lambda^n \in \mathbb{R}^n\) at time \(t \in \mathbb{N}\). By definition, we have \(V_i(\lambda^n) = R_j\), where \(J = \arg\max_{V_j \in \mathcal{V}} S_{\lambda^n}(V_j, R_{j,t})\), where \(S_{\lambda^n}(V_j, R_{j,t})\) is the scheduling measure of the \(j\)th user when the threshold vector is \(\lambda^n\). Note that \(S_{\lambda^n}(V_j, R_{j,t}) - S_{\lambda^n}(V_j, R_{j,t}) \leq n \times \max_{i \in \mathbb{N}} |\lambda_i - \lambda^n_i| \leq n \times \|\lambda^n - \lambda^n\|_2\), where \(\lambda^n, \lambda^n \in \mathbb{R}^n\). Consequently, from Equation (10), we have:

\[
P(|V_i(\lambda^n) - V_i(\bar{\lambda}_i^n)|) \leq \frac{n \sqrt{c}}{\rho^{0.5\alpha^*}} \geq 1 - \frac{1}{\rho^{\alpha^*}}.
\]

It follows that \(\mathbb{E}^\text{RoC}(|U(\lambda^n) - U(\bar{\lambda}_i^n)|) \leq \frac{c'}{\rho^{0.5\alpha^*}}\), where \(U(\lambda^n)\) is the average system utility due to the TBS with threshold vector \(\lambda^n\) and \(c' > 0\) is constant in \(t\). Furthermore, \(U(0^n) > U(\lambda^n), \forall \lambda^n \neq 0^n\) with probability one since the TBS with the all-zero threshold vector always activates the virtual user which gives the maximum system utility. Let \(\mathbb{E}(U(0^n)) - \mathbb{E}(U(\lambda^n)) = \epsilon > 0\). Let us fix a natural number \(M > 2\). We describe the operation of the scheduler in the resource blocks \(t \in [\sum_{j=1}^{k-1} M^j(1 + M^{\omega_j^*}) + 1, \sum_{j=1}^{k} M^j(1 + M^{\omega_j^*})]\) for any \(k \in \mathbb{N}\). In the first \(M^k\) resource blocks, the scheduler uses the TBS with threshold vector \(0^n\) for scheduling and in the next \(M^{k+1}\) it uses the TBS with threshold vector \(\hat{\lambda}_M^{n_{M+\omega}}\). It is straightforward to
show that the expected cumulative utility in the first $M^k$ resource blocks is $\mathbb{E}((U^* + \epsilon)M^k)$ and in the next $M^{k+\frac{k\epsilon}{M^k}}$ it is at least $\mathbb{E}((U^* - \frac{\epsilon}{M^k})M^{k+\frac{k\epsilon}{M^k}})$. So, we have

$$|\mathbb{E}(tU^* - \sum_{i \in [t]} \tilde{U}_i^n)|_+ \leq | - \sum_{i \in [k]} (\epsilon - M^{-\frac{\epsilon}{M^k}})M^{i}|_+,$$

with probability one for asymptotically large $k$, where $t = \sum_{i \in [k]} M^{i}(1 + M^{-\frac{\epsilon}{M^k}})$. The right hand side is equal to 0 for large enough $k$. This completes the proof.

IV. Simulation Results

In this section, we provide simulation results to verify the derivations presented in Section III and investigate the RoC of the Robbins Monro based iterative algorithm for finding the optimal thresholds of TBS proposed in [2, Algorithm 2]. We refer to this algorithm as the threshold learning algorithm (TLA).

We consider a time-slotted single small-cell scenario with a BS located in the center and four users distributed uniformly at random in a ring around the BS with inner and outer radii of 20 m and 100 m, respectively. We investigate two communication settings. In the first setting (OMA setting), an individual user is scheduled at each time-slot, i.e. $N_{\text{max}} = 1$. In the second setting (NOMA setting) an individual user or a pair of users are scheduled at each time-slot, i.e. $N_{\text{max}} = 2$. Additionally, we only consider lower temporal demand for each user and assume that there are no upper temporal demand constraints. The network utility is modeled by the truncated Shannon sum-rate as in [2] with maximum allowed spectral efficiency of $\gamma_{\text{max}} = 6 \text{ bps/Hz}$. At each time-slot, prior to scheduling, a max-min power optimization is performed for each virtual user [15]. For a given virtual user, we find the transmit power which maximizes the minimum individual user rates in that virtual user. This max-min optimization allows for a balanced rate allocation within the virtual user. It can be shown that the max-min optimization is quasi-concave. Consequently, quasi-concave programming methods such as bisection search can be used to find the optimal transmit powers [15]. Maximum BS transmit power constraint is chosen such that the average SNR of 10 dB is achievable when a single user is active on the boundary of the cell. Furthermore, we use TLA to obtain an estimation of the optimal user thresholds. The total number of time-slots is set to $5 \times 10^6$. According to [2], the step-size $s_t$ used to update the thresholds at time-slot $t$ should satisfy the following constraints: i) $s_t > 0$, ii) $\lim_{t \to \infty} s_t = 0$, and iii) $\sum_{t=1}^{\infty} s_t = \infty$, $\sum_{t=1}^{\infty} s_t^2 < \infty$. We take the step-size to be $s_t = t^{-0.7}$ which satisfies the required conditions.

We investigate the convergence of the thresholds in OMA and NOMA settings. We consider a random user distribution and assume that the lower temporal demand vector is $w^d = [0.1, 0.2, 0.3, 0.4]$ which can
Fig. 1: The evolution of the user thresholds in time for (a) OMA setting and (b) NOMA setting. The horizontal axis is the sampled time-slot index, where the sampling parameter $H$ is set to $10^{-2}$. 

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be shown to be feasible for both communication settings as $\sum_i w_i \leq 1$ [2]. Figure 1 depicts the time-evolution of the user thresholds. We observe that the thresholds converge in both settings.

Next, we investigate the RoC for the thresholds and system utility in OMA and NOMA settings. Figure 2 illustrates the RoC Measure as a function of time for OMA and NOMA communication settings. We define RoC measure at time-slot $t$, as $x_t = t^{1/2} \mathbb{E}_{\{\lambda^n\}}(||\lambda^n - \hat{\lambda}_n||_\infty)$ and $y_t = t^{1/2} \mathbb{E}_{\{U^n\}}(||U^n - \hat{U}_n||_\infty)$ for the thresholds and the system utility, respectively. We approximate the expectation by empirical average over 100 channel realizations. Furthermore, an estimation of the optimal thresholds and system utility is obtained by running TLA for $5 \times 10^6$ time-slots. Figure 2 shows the $x_t$ and $y_t$ for the first $2 \times 10^5$ time-slots where the sampling parameter $H$ is set to $10^{-1}$, i.e. one sample per $10^2$ time-slots. We observe that the sequence $x_t, t \in \mathbb{N}$ converges to a constant value, verifying the fact that the threshold RoC is $\alpha^* = \frac{1}{2}$. On the other hand, the sequence $y_t$ is decreasing pointing to the fact that $\zeta^*$ may be more than $\frac{1}{2}$ in this case. This confirms the results presented in Theorems 2 and 3. Additionally, the heuristic TLA method achieves the optimal threshold RoC in this scenario.

V. Conclusion

We have considered multi-user scheduling under temporal fairness constraints. We have investigated the rates of convergence of the best mean square estimates of the scheduling threshold vector, when threshold based strategies are used. We have shown that the rate of convergence is at most $\frac{1}{2}$, so that the best threshold vector estimate is at least $\omega(\frac{1}{\sqrt{t}})$ away from the optimal threshold vector, where $t$ is the number of instances of users’ channel realizations used in estimation. We have further studied the RoC for the average system utility, and shown that the utiliy RoC is arbitrarily large under long-term fairness constraints. We have provided several simulations of practical scenarios verifying our results.

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Fig. 2: RoC measures $x_t$ (thresholds) and $y_t$ (utility) as a function of time for the thresholds and system utility in the (a) OMA setting and (b) NOMA setting. The horizontal axis is the sampled time-slot index, where the sampling parameter $H$ is set to $10^{-2}$. 
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