ALMOST PERIODIC SOLUTIONS OF PERIODIC LINEAR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

VU TRONG LUONG AND NGUYEN VAN MINH

Abstract. We study conditions for the abstract periodic linear functional differential equation \( \dot{x} = Ax + F(t)x_t + f(t) \) to have almost periodic with the same structure of frequencies as \( f \). The main conditions are stated in terms of the spectrum of the monodromy operator associated with the equation and the frequencies of the forcing term \( f \). The obtained results extend recent results on the subject. A discussion on how the results could be extended to the case when \( A \) depends on \( t \) is given.

1. Introduction

In this paper we consider the existence and uniqueness of almost periodic solutions with the same structure of spectrum as \( f \) in equations of the following form

\[
\frac{dx(t)}{dt} = Ax(t) + F(t)x_t + f(t), \quad x \in \mathbb{X}, t \in \mathbb{R},
\]

where the (unbounded) linear operator \( A \) generates a strongly continuous semigroup and the bounded linear operator \( F(t) \) is periodic and is defined as follows, \( x_t \in C_r := C([-r, 0], \mathbb{X}) \), \( x_t(\theta) := x(t + \theta) \), \( r > 0 \) is a given positive real number, \( F(t)\varphi := \int_{-r}^{0} d\eta(t, s)\varphi(s), \forall \varphi \in C_r, \eta(t, \cdot) : C_r \to L(\mathbb{X}) \) is periodic in \( t \), of bounded variation, and \( \sup_t \| F(t) \| < \infty \), and \( f \) is a \( \mathbb{X} \)-valued almost periodic function. A discussion on how the results could be extended to the case when \( A \) depends on \( t \) periodically will be given at the end of the paper.

In the theory of ordinary differential equations one of the questions that are of interest to many researchers is when exist periodic solutions to equations of the form

\[
\frac{dx}{dt} = B(t)x + f(t), t \in \mathbb{R}, x \in \mathbb{C}^n,
\]

(F)

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where $f$ is periodic, and $B(t)$ is a $n \times n$-matrix that is periodic with the same period as $f(t)$. A famous Massera’s Theorem ([11]) says that Eq. (F) has a periodic solution with the same period as $B$ and $f$ if and only if it has a solution that is bounded on the positive half line. In addition, the periodic solution is unique if 1 is not an eigenvalue of the monodromy operator. Since then there have been many efforts to extend this classic result to various classes of equations and functions (see e.g. [1, 2, 5, 6, 16, 17, 18, 19, 20, 23]). We refer the reader to some recent developments [5, 8, 19, 20, 23] and their references for more recent information in this direction. We note that the results on the existence of periodic solutions are usually proved via the existence of fixed points of the monodromy operator (or, period map) (see e.g. [2, 10, 23]). Among the research methods used in this direction we note that when $f$ is almost periodic the monodromy operator method is no longer applicable because the system is no longer periodic. Instead, one uses a new method that is based on the concept of *evolution semigroups* associated with the evolutionary processes generated by the equations. Also, the requirement that the period of the solutions be the same as that of the forcing term $f$ will be understood as a requirement on the frequencies of the solutions that are not more than those of $f$. This justifies the introduction of the concept of *spectrum of a function* that allows us to measure the set of frequencies of a function on the real line. As is known, a fundamental technique of research in the ODE and FDE is variation-of-constant formulas (VCF) in the phase space. In the case of abstract functional differential equations, the VCF in the phase space is no longer valid. Instead, a weak version may make sense. In this short paper we will recall briefly these concepts and related results in the next section. We will present an extension of the Massera’s Theorem for almost periodic solutions of Eq. (1.1) (Theorems 3.3 and 3.4). We prove that the condition of existence of bounded solutions could be removed and the equations always have a unique almost periodic solutions with frequencies as $f$ if the part of spectrum of the monodromy operator on the unit circle does not intersect the spectrum of $f$. To our best knowledge the results obtained in this paper extends some previous ones in [1, 5, 18], and complements many other results in [1, 15, 16, 17, 19, 22]. In [17] the authors showed that if $A$ generates a compact $C_0$-semigroup the existence of almost periodic solutions to Eq. (1.1) could be reduced to the finite dimensional case of ODE, so the problem could be thoroughly studied. The novelty of our results obtained in this paper is that we study the problem when $A$ generates any $C_0$-semigroup, (and even more generally, when $A$ is a family of operators that generates a periodic evolutionary process). This makes the part of spectrum on the unit circle more complicated and the nature of the problem is not of finite dimension. Finally, we give a discussion on how the obtained results could be extended to the case when $A$ may depend on time $t$ periodically. In this case without the
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variation-of-constants in the phase space the main results are still true though their proofs will be adjusted.

2. Preliminaries

2.1. Notation. Throughout the paper we will use the following notations: \( N, Z, \mathbb{R}, \mathbb{C} \) stand for the sets of natural, integer, real, complex numbers, respectively. \( \Gamma \) denotes the unit circle in the complex plane \( \mathbb{C} \). For any complex number \( z \) the notation \( \Re z \) stands for its real part. \( X \) will denote a given complex Banach space. Given two Banach spaces \( X, Y \) by \( L(X, Y) \) we will denote the space of all bounded linear operators from \( X \) to \( Y \). As usual, \( \sigma(T), \rho(T), R(\lambda, T) \) are the notations of the spectrum, resolvent set and resolvent of the operator \( T \). The notations \( BC(\mathbb{R}, X), BUC(\mathbb{R}, X), AP(X) \) will stand for the spaces of all \( X \)-valued bounded continuous, bounded uniformly continuous functions on \( \mathbb{R} \) and its subspace of almost periodic (in Bohr’s sense) functions, respectively.

2.2. Circular Spectrum of Functions. Below we will introduce a transform of a function \( g \in L^\infty(\mathbb{R}, X) \) on the real line that leads to a concept of spectrum of a function. This spectrum coincides with the set \( e^{\text{sp}(g)} \) if in addition \( g \) is uniformly continuous, where \( \text{sp}(g) \) denotes the Beurling spectrum of \( g \). All results mentioned below on the circular spectrum of a function could be found in [14].

Let \( g \in L^\infty(\mathbb{R}, X) \). Consider the complex function \( Sg(\lambda) \) in \( \lambda \in \mathbb{C}\setminus\Gamma \) defined as

\[
Sg(\lambda) := R(\lambda, S)g, \quad \lambda \in \mathbb{C}\setminus\Gamma.
\]

Since \( S \) is a translation, this transform is an analytic function in \( \lambda \in \mathbb{C}\setminus\Gamma \).

Definition 2.1. The circular spectrum of \( g \in L^\infty(\mathbb{R}, X) \) is defined to be the set of all \( \xi_0 \in \Gamma \) such that \( Sg(\lambda) \) has no analytic extension into any neighborhood of \( \xi_0 \) in the complex plane. This spectrum of \( g \) is denoted by \( \sigma(g) \) and will be called for short the spectrum of \( g \) if this does not cause any confusion. We will denote by \( \rho(g) \) the set \( \Gamma\setminus\sigma(g) \).

Proposition 2.2. Let \( \{g_n\}_{n=1}^\infty \subset L^\infty(\mathbb{R}, X) \) such that \( g_n \to g \in L^\infty(\mathbb{R}, X) \), and let \( \Lambda \) be a closed subset of the unit circle. Then the following assertions hold:

i) \( \sigma(g) \) is closed.
ii) If \( \sigma(g_n) \subset \Lambda \) for all \( n \in \mathbb{N} \), then \( \sigma(g) \subset \Lambda \).
iii) \( \sigma(\mathcal{A}g) \subset \sigma(g) \) for every bounded linear operator \( \mathcal{A} \) acting in \( BUC(\mathbb{R}, X) \) that commutes with \( S \).
iv) If \( \sigma(g) = \emptyset \), then \( g = 0 \).
Proof. For i), ii) and iv) the proofs are given in [14]. For iii) the proof is obvious from the definition of the circular spectrum.

\[ \square \]

Corollary 2.3. Let \( \Lambda \) be a closed subset of the unit circle and \( \mathcal{F} \) be one of the function spaces \( BUC(\mathbb{R}, \mathbb{X}), AP(\mathbb{X}) \). Then, the set

\[ \Lambda_{\mathcal{F}}(\mathbb{X}) := \{ g \in \mathcal{F} \mid \sigma(g) \subset \Lambda \} \]

is a closed subspace of \( \mathcal{F} \).

Lemma 2.4. Let \( \Lambda \) be a closed subset of the unit circle and \( \mathcal{F} \) be one of the function spaces \( BUC(\mathbb{R}, \mathbb{X}), AP(\mathbb{X}) \). Then, the translation operator \( S \) leaves the space \( \Lambda_{\mathcal{F}}(\mathbb{X}) \) invariant. Moreover,

\[ \sigma(S|_{\Lambda_{\mathcal{F}}(\mathbb{X})}) = \Lambda. \]

Below we will recall the concept of Beurling spectrum of a function. We denote by \( F \) the Fourier transform, i.e.

\[ (Ff)(s) := \int_{-\infty}^{+\infty} e^{-ist}f(t)dt \]

\((s \in \mathbb{R}, f \in L^1(\mathbb{R}))\). Then the Beurling spectrum of \( u \in BUC(\mathbb{R}, \mathbb{X}) \) is defined to be the following set

\[ sp(u) := \{ \xi \in \mathbb{R} : \forall \epsilon > 0 \ \exists f \in L^1(\mathbb{R}), \supp Ff \subset (\xi - \epsilon, \xi + \epsilon), f \ast u \neq 0 \} \]

where

\[ f \ast u(s) := \int_{-\infty}^{+\infty} f(s-t)u(t)dt. \]

The following result is a consequence of the Weak Spectral Mapping Theorem that relates the circular spectrum and Beurling spectrum of a uniformly continuous function.

Corollary 2.5. Let \( g \in BUC(\mathbb{R}, \mathbb{X}) \). Then

\[ \sigma(g) = e^{isp(g)}. \]
2.3. **Almost periodic functions.** A subset \( E \subset \mathbb{R} \) is said to be *relatively dense* if there exists a number \( l > 0 \) (inclusion length) such that every interval \([a, a+l]\) contains at least one point of \( E \). Let \( f \) be a continuous function on \( \mathbb{R} \) taking values in a complex Banach space \( X \). \( f \) is said to be *almost periodic in the sense of Bohr* if to every \( \epsilon > 0 \) there corresponds a relatively dense set \( T(\epsilon, f) \) (of \( \epsilon \)-periods) such that

\[
\sup_{t \in \mathbb{R}} \| f(t + \tau) - f(t) \| \leq \epsilon, \ \forall \tau \in T(\epsilon, f).
\]

If \( f \) is almost periodic function, then (approximation theorem [9, Chap. 2]) it can be approximated uniformly on \( \mathbb{R} \) by a sequence of trigonometric polynomials, i.e., a sequence of functions in \( t \in \mathbb{R} \) of the form

\[
P_n(t) := \sum_{k=1}^{N(n)} a_{n,k} e^{i\lambda_{n,k} t}, \quad n = 1, 2, \ldots; \lambda_{n,k} \in \mathbb{R}, a_{n,k} \in X, t \in \mathbb{R}.
\]

Of course, every function which can be approximated by a sequence of trigonometric polynomials is almost periodic. Specifically, the exponents of the trigonometric polynomials (i.e., the reals \( \lambda_{n,k} \) in (2.6)) can be chosen from the set of all reals \( \lambda \) (*Fourier exponents*) such that the following integrals (*Fourier coefficients*)

\[
a(\lambda, f) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\lambda t} dt
\]

are different from 0. As is known, there are at most countably such reals \( \lambda \), the set of which will be denoted by \( \sigma_b(f) \) and called *Bohr spectrum* of \( f \). Throughout the paper we will use the relation \( sp(f) = \sigma_b(f) \).

If \( g \in BUC(\mathbb{R}, X) \) with countable \( \sigma(g) \), then its Beurling spectrum \( sp(g) \) is also countable by Corollary 2.5. Therefore, if \( X \) does not contain any space isomorphic to \( c_0 \) (the space of all numerical sequences converging to zero), the function \( g \) is almost periodic (see e.g. [9]). If \( X \) is convex it does not contain \( c_0 \).

2.4. **Evolutionary processes and the associated evolution semigroups.**

**Definition 2.6.** Let \((U(t, s))_{t \geq s}\) be a two-parameter family of bounded operators in a Banach space \( X \). Then, it is called an evolutionary process if

i) \( U(t, t) = I \) for all \( t \in \mathbb{R} \),

ii) \( U(t, s)U(s, r) = U(t, r) \) for all \( t \geq s \geq r \),

iii) The map \((t, s) \mapsto U(t, s)x \) is continuous for every fixed \( x \in X \),

iv) \( \|U(t, s)\| < Ne^{\omega(t-s)} \) for some positive \( N, \omega \) independent of \( t \geq s \).
An evolutionary process is called \textit{1-periodic} if
\[ U(t + 1, s + 1) = U(t, s), \quad \text{for all } t \geq s. \]

Recall that for a given 1-periodic evolutionary process \((U(t, s))_{t \geq s}\), the following operator
\[ M(t) := U(t, t - 1), \quad t \in \mathbb{R} \]
is called \textit{monodromy operator} (or sometimes \textit{period map}, \textit{Poincaré map}). Thus we have a family of monodromy operators. We will denote \( M := M(0) \). The nonzero eigenvalues of \( M(t) \) are called \textit{characteristic multipliers}. An important property of monodromy operators is stated in the following lemma whose proof can be found in \cite{7, 8}.

**Lemma 2.7.** Under the notation as above the following assertions hold:

i) \( M(t + 1) = M(t) \) for all \( t \); characteristic multipliers are independent of time, i.e. the nonzero eigenvalues of \( M(t) \) coincide with those of \( M \),

ii) \( \sigma(M(t)) \setminus \{0\} = \sigma(M) \setminus \{0\} \), i.e., it is independent of \( t \),

iii) If \( \lambda \in \rho(M) \), then the resolvent \( R(\lambda, M(t)) \) is strongly continuous,

iv) If \( M \) denotes the operator of multiplication by \( M(t) \) in any one of the function spaces \( BUC(\mathbb{R}, X) \) or \( AP(X) \), then
\[
\sigma(M) \setminus \{0\} \subset \sigma(M) \setminus \{0\}. \tag{2.7}
\]

Given an evolutionary process \((U(t, s))_{t \geq s}\), the following semigroup \((T^h)_{h \geq 0}\) is called its associated evolution semigroup
\[
T^h g := U(t, t - h)g(t - h), \quad t \in \mathbb{R}, \ g \in BUC(\mathbb{R}, X). \tag{2.8}
\]

In general, the evolution semigroup associated with a 1-periodic evolutionary process may not be strongly continuous in the whole space \( BUC(\mathbb{R}, X) \), but in a closed subspace \( F \) that includes all elements of \( AP(X) \) and mild solutions in the above sense (see e.g. \cite{1, 18}).

To describe the evolution semigroup associated with a given \((U(t, s))_{t \geq s}\) we consider the following integral equation
\[
u(t) = U(t, s)u(s) + \int_s^t U(t, \xi)f(\xi)d\xi, \quad \text{for all } t \geq s, \tag{2.9}
\]

where \( f \) is an element of \( BUC(\mathbb{R}, X) \). We recall the following linear operator \( L : D(L) \subset BUC(\mathbb{R}, X) \to BUC(\mathbb{R}, X) \), where \( D(L) \) consists of all solutions of Eq.\( (2.9) \) \( u(\cdot) \in BUC(\mathbb{R}, X) \) with some \( f \in BUC(\mathbb{R}, X) \). If \( u \in D(L) \), then we define \( Lu(\cdot) := f \). This operator \( L \) is well defined as a singled-valued operator and is obviously an extension of the differential operator \( d/dt - A \) (see e.g. \cite{16}). Below, by abuse of notation, we will use the same notation
\[ L \] to designate its restriction to closed subspaces of \( BUC(\mathbb{R}, X) \) if this does not make any confusion.

If \((T(t))_{t \geq 0}\) is a \(C_0\)-semigroup in a Banach space \(X\), then \(U(t, s) := T(t - s)\) determines a 1-periodic evolutionary process.

2.5. Mild solutions of Eq. (1.1) and a variation of constants formula.

**Definition 2.8.** A continuous function \(u(\cdot)\) on \(\mathbb{R}\) is said to be a mild solution on \(\mathbb{R}\) of Eq. (1.1) with initial \(\phi \in C_r\), and is denoted by \(u(\cdot, s, \phi, f)\) if \(u_s = \phi\) and for all \(t > s\)

\[
(2.10) \quad u(t) = T(t - s)\phi(0) + \int_s^t T(t - \xi)[F(\xi)u_\xi + f(\xi)]d\xi.
\]

A function \(u \in BC(\mathbb{R}, X)\) is said to be a mild solution of (1.1) on \(\mathbb{R}\) if

\[
(2.11) \quad u(t) = T(t - s)u(s) + \int_s^t T(t - \xi)[F(\xi)u_\xi + f(\xi)]d\xi, \quad \text{for all } t \geq s.
\]

Below we will denote by \(F\) the operator acting on \(BUC(\mathbb{R}, X)\) defined by the formula

\[
F u(\xi) := F(\xi)u_\xi, \quad \forall u \in BUC(\mathbb{R}, X).
\]

The following results can be verified directly following the lines in [1, 12, 18].

**Lemma 2.9.** Let \((T^h)_{h \geq 0}\) be the evolution semigroup associated with a given strongly continuous semigroup \((T(t))_{t \geq s}\) and \(S\) denote the space of all elements of \(BUC(\mathbb{R}, X)\) at which \((T^h)_{h \geq 0}\) is strongly continuous. Then the following assertions hold true:

i) Every mild solution \(u \in BUC(\mathbb{R}, X)\) of Eq. (1.1) is an element of \(S\);

ii) \(AP(X) \subset S\),

iii) For the infinitesimal generator \(G\) of \((T^h)_{h \geq 0}\) in the space \(S\) one has the relation:

\[
Gg = -Lg \quad \text{if } g \in D(G).
\]

For bounded uniformly continuous mild solutions \(x(\cdot)\) the following characterization is very useful:

**Theorem 2.10.** \(x(\cdot)\) is a bounded uniformly continuous mild solution of Eq. (1.1) if and only if \(Lx(\cdot) = Fx(\cdot) + f\).

As is well known, the homogeneous equation associated with (1.1) generates an evolutionary process \((U(t, s))_{t \geq s}\) in the space \(C_r = C([-r, 0], X)\). In fact,

\[
(2.12) \quad U(t, s) : C_r \ni \phi \mapsto u_t \in C_r,
\]
where \( u \) is the solution of the equation

\[
\begin{align*}
    u(\tau) &= T(\tau - s)\phi(0) + \int_{s}^{\tau} T(\tau - \xi)F(\xi)u_{\xi}d\xi, \quad \tau \geq s, \\
    u_{s} &= \phi.
\end{align*}
\]

We introduce a function \( \Gamma^n \) defined by

\[
\Gamma^n(\theta) = \begin{cases} 
(n\theta + 1)I, & -1/n \leq \theta \leq 0 \\
0, & \theta < -1/n,
\end{cases}
\]

where \( n \) is any positive integer and \( I \) is the identity operator on \( X \). Since the evolutionary process \((U(t, s))_{t \geq s}\) is strongly continuous, the \( C_r \)-valued function \( U(t, s)\Gamma^n f(s) \) is continuous in \( s \in (-\infty, t] \) whenever \( f \in BC(\mathbb{R}, X) \).

The following theorem, whose proof could be found in [17], is a variation of constant formula for solutions of (1.1) in the phase space \( C_r \):

**Theorem 2.11.** The segment \( u_t(s, \phi; f) \) of solution \( u(\cdot, s, \phi, f) \) of (1.1) satisfies the following relation in \( C_r \):

\[
(2.13) \quad u_t(s, \phi; f) = U(t, s)\phi + \lim_{n \to \infty} \int_{s}^{t} U(t, \xi)\Gamma^n f(\xi)d\xi, \quad t \geq s.
\]

Moreover, the above limit exists uniformly for bounded \( |t - s| \).

### 3. Existence of almost periodic solutions of Eq. (1.1)

The result below is an upper estimate of the spectrum of a mild solution to (1.1) that is a key to understand the behavior of a bounded and uniformly continuous mild solution of (1.1).

**Lemma 3.1.** Let \( u \) be a bounded and uniformly continuous mild solution of the equation (1.1). Then, the following estimate holds

\[
(3.1) \quad \sigma(u) \subset \sigma_{T}(M) \cup \sigma(f).
\]

where \( \sigma_{T}(M) := \{ z \in \mathbb{C} : |z| = 1, z \in \sigma(M) \} \).

**Proof.** By the formula (2.13)

\[
(3.2) \quad u_t = U(t, t - 1)u_{t - 1} + \lim_{n \to \infty} \int_{t - 1}^{t} U(t, s)\Gamma^n f(s)ds,
\]
and the limit exists uniformly for all bounded \( t \). First, as \( f \) is uniformly continuous and bounded we can see that the function

\[
A : \mathbb{R} \ni t \mapsto \lim_{n \to \infty} \int_{t-1}^{t} U(t, s) \Gamma^n f(s) ds \in C_r
\]

is also bounded and uniformly continuous. We can check easily the validity of the identity

\[
\lambda R(\lambda, S)S(-1) = R(\lambda, S) + S(-1),
\]

for any \( |\lambda| \neq 1 \), where \( S(t) \) stands for the translation group, and \( S := S(1) \). Note that the operator \( \mathcal{M} \) of multiplication by \( M(t) \) commutes with \( S \) since the evolutionary process \((U(t, s))_{t \geq s}\) is 1-periodic. Below we will denote by \( \omega \) the function \( \mathbb{R} \ni t \mapsto u_t \in C_r \). Then, from the identity (3.2) one has (for all \( \lambda \neq 0 \) and \( |\lambda| \neq 1 \))

\[
\lambda R(\lambda, S)\omega = \lambda R(\lambda, S)\mathcal{M}S(-1)\omega + \lambda R(\lambda, S)A.
\]

Therefore,

\[
\lambda R(\lambda, S)\omega - \mathcal{M}R(\lambda, S)\omega = \mathcal{M}S(-1)\omega + \lambda R(\lambda, S)A,
\]

\[
(\lambda - \mathcal{M})R(\lambda, S)\omega = \mathcal{M}S(-1)\omega + \lambda R(\lambda, S)A.
\]

As shown in [14, Lemma 5.3] for each fixed \( n \in \mathbb{N} \)

\[
\sigma(G_{n}f) \subset \sigma(f),
\]

where

\[
G_{n}f(t) := \int_{t-1}^{t} U(t, s) \Gamma^n f(s) ds.
\]

As the limit in the formula (2.13) is uniform in \( t \) we can see that \( \sigma(A) \subset \sigma(f) \). Finally, if \( \lambda_0 \notin (\sigma_T(M) \cup \sigma(f)) \), then near \( \lambda_0 \) the following holds

\[
(3.4) \quad R(\lambda, S)\omega = R(\lambda, \mathcal{M})(\mathcal{M}S(-1)\omega + \lambda R(\lambda, S)A).
\]

This shows that the complex function \( R(\lambda, S)\omega \) is defined as an analytic function in a neighborhood of \( \lambda_0 \).

We will show further that this yields that the function \( R(\lambda, S)\omega(0) \) is also defined and analytic in a neighborhood of \( \lambda_0 \). In fact, before we proceed that we introduce \( p : C_r \to \mathbb{X} \) defined as \( p(w) := w(0) \). If so, with our above notations \( p \circ \omega = u \), and \( p \circ S^k \omega = S^k u \) for
all \( k \in \mathbb{N} \). If \( |\lambda| > 1 \) we have

\[
p \circ R(\lambda, S)\omega = \lambda^{-1}p \circ (I - S/\lambda)^{-1}\omega
\]
\[
= \lambda^{-1} p \circ \left( \sum_{k=0}^{\infty} S^k/\lambda^k \right) \omega
\]
\[
= \lambda^{-1} \left( \sum_{k=0}^{\infty} S^k/\lambda^k \right) u
\]
\[
= R(\lambda, S)u.
\]

Note that for simplicity we make an abuse of notation by denoting also by \( S \) the translation in the function space \( BUC(\mathbb{R}, \mathbb{X}) \) as well as in \( BUC(\mathbb{R}, C_r) \). Similarly, for \( \lambda \neq 0 \) and \( |\lambda| < 1 \) we can show that \( p \circ R(\lambda, S)\omega = R(\lambda, S)u \). Hence, the transform \( R(\lambda, S)u \) of the function \( u \) has \( p \circ R(\lambda, S)\omega \) as an analytic extension in a neighborhood of \( \lambda_0 \). This shows that (3.1) holds true, finishing the proof of the lemma.

Next, we recall some concepts and results in [20]. Note that although the proofs could be found in [20] we would like to give some new ones that seem to be simpler and would be more convenient to the reader.

Let us consider the subspace \( \mathcal{N} \subset BUC(\mathbb{R}, \mathbb{X}) \) (or \( AP(\mathbb{X}) \), respectively) consisting of all functions \( v \in BUC(\mathbb{R}, \mathbb{X}) \) (or \( AP(\mathbb{X}) \), respectively) such that

(3.5) \[ \sigma(v) \subset S_1 \cup S_2, \]

where \( S_1, S_2 \) are disjoint closed subsets of the unit circle \( \Gamma \).

**Lemma 3.2.** Under the above notations and assumptions the function space \( \mathcal{N} \) can be split into a direct sum \( \mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2 \) such that \( v \in \mathcal{N}_i \) if and only if \( \sigma(v) \subset S_i \) for \( i = 1, 2 \). Moreover, any bounded linear operator in \( BUC(\mathbb{R}, \mathbb{X}) \) (or \( AP(\mathbb{X}) \), respectively), that commutes with the translation \( S \), leaves invariant \( \mathcal{N} \) as well as \( \mathcal{N}_j, j = 1, 2 \).

**Proof.** By Lemma 2.4 and the Riezs spectral projection the space \( \mathcal{N} \) could be split into the direct sum \( \mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2 \) with \( \mathcal{N}_1 \) is the image of the projection

\[
P := \frac{1}{2i\pi} \int_\gamma R(\lambda, S|_{\mathcal{N}})d\lambda,
\]

where \( \gamma \) is a positively oriented contour enclosing \( S_1 \) and disjoint from \( S_2 \). We have

\[ \sigma(S|_{\mathcal{N}_1}) \subset S_1; \sigma(S|_{\mathcal{N}_2}) \subset S_2. \]
Therefore, if \( v \in \mathcal{N}_i, (i = 1, 2) \) by the definition of the circular spectrum it is easy to see that

\[
\sigma(v) \subset \sigma(S|\mathcal{N}_i) \subset S_i.
\]

The second claim is obvious as any bounded linear operator in \( BUC(\mathbb{R}, X) \) (or \( AP(X) \), respectively) that commutes with \( S \) must commute with \( P \), so it leaves the spaces \( \mathcal{N}, \mathcal{N}_1, \mathcal{N}_2 \) invariant.

\[\square\]

**Theorem 3.3.** (Decomposition Theorem) Let the following condition be satisfied

i) Eq. \((1.1)\) has a mild solution \( u \in BUC(\mathbb{R}, X) \) (or in \( AP(X) \), respectively)

ii)

\[
(3.6) \quad \sigma_{\Gamma}(M) \setminus \sigma(f) \text{ be closed.}
\]

Then there exists a mild solution \( w \) of Eq. \((1.1)\) in \( BUC(\mathbb{R}, X) \) (or \( AP(X) \), respectively) such that

\[
(3.7) \quad \sigma(w) \subset \sigma(f),
\]

that is unique if

\[
(3.8) \quad \sigma_{\Gamma}(M) \cap \sigma(f) = \emptyset.
\]

**Proof.** By Lemma \ref{Lemma 3.1}

\[
(3.9) \quad \sigma(u) \subset \sigma_{\Gamma}(M) \cup \sigma(f).
\]

Let us denote by \( \Lambda \) the set \( \sigma_{\Gamma}(M) \cup \sigma(f) \), \( S_1 \) the set \( \sigma(f) \) and \( S_2 \) the set \( \sigma_{\Gamma}(M) \setminus \sigma(f) \), respectively. Thus, these two sets are closed and disjoint subsets of the unit circle \( \Gamma \), so by Lemma \ref{Lemma 3.2} there exists the projection \( P \) from \( \mathcal{N} \) onto \( \mathcal{N}_1 \) which is commutative with \( \mathcal{F} \) and \( T^h \). Since \( u \) is a mild solution of \((1.1)\) if and only if \( u \in D(\mathcal{L}) \) and

\[
(3.10) \quad \mathcal{L}u = \mathcal{F}u + f,
\]

by Lemma \ref{Lemma 2.9} we have

\[
\mathcal{L}u = -\mathcal{G}u,
\]
so this yields

\[ PLu = -PGu \]
\[ = - \lim_{h \to 0^+} \frac{T^h u - u}{h} \]
\[ = - \lim_{h \to 0^+} \frac{P T^h u - u}{h} \]
\[ = - \lim_{h \to 0^+} \frac{T^h Pu - Pu}{h} \]
\[ = - GPu \]
\[ = \mathcal{L} Pu. \]

Since \( Pf = f \) and \( P \) commutes with \( \mathcal{F} \),

\[ PLu = PFu + Pf \]
\[ \mathcal{L} Pu = \mathcal{F} Pu + f. \]

By Theorem 2.10 this shows \( w := Pu \in \mathcal{N}_1 \) is a mild solution of Eq. (1.1) that has circular spectrum \( \sigma(Pu) \subset S_1 = \sigma(f) \). Next, if condition (3.8) holds, then the uniqueness of such a solution in \( \mathcal{N}_1 \) is clear. In fact, suppose that there is another mild solution \( v \in BUC(\mathbb{R}, X) \) (or in \( AP(X) \), respectively) to Eq. (1.1) such that \( \sigma(v) \subset \sigma(f) \), then \( w - v \) is a mild solution of the homogeneous equation corresponding to Eq. (1.1), so \( \sigma(w - v) \subset \sigma(M) \). As \( \sigma(v) \subset \sigma(f) \), by (3.8) this yields that \( \sigma(w - v) = \emptyset \), and because of this \( w - v = 0 \). This completes the proof of the theorem.

Recall that the set of all real numerical sequences that are convergent to zero is a Banach space with sup-norm that is denoted by \( c^0 \). As a consequence of the above theorem we obtain the following main result of the paper.

**Theorem 3.4.** Assume that Eq. (1.1) has a bounded uniformly continuous mild solution \( u \), and Condition (3.6) of Theorem 3.3 is satisfied. Moreover, let the space \( X \) not contain \( c^0 \) and \( \sigma(f) \) be countable. Then there exists an almost periodic mild solution \( w \) to Eq. (1.1) such that \( \sigma(w) \subset \sigma(f) \). Furthermore, if (3.8) holds, then such a solution \( w \) is unique.

**Proof.** The proof is obvious in view of [9, Theorem 4, p.92] and Theorem 3.3. \( \square \)

Below we will relax the condition on the existence of a bounded uniformly continuous mild solutions when a condition (3.8) is satisfied.

**Theorem 3.5.** Under the above notation assume that

\[ \sigma(M) \cap \sigma(f) = \emptyset \]  

(3.11)
holds. Then there exists a unique almost periodic mild solution \( w \) to Eq. (1.1) such that \( \sigma(w) \subset \sigma(f) \).

**Proof.** Consider the difference equation

\[(3.12) \quad w(t) = M(t)w(t - 1) + g(t), \quad t \in \mathbb{R}, \]

where for all \( t \in \mathbb{R} \)

\[ M(t) := U(t, t - 1), \]

\[ g(t) := \lim_{n \to \infty} \int_{t-1}^{t} U(t, s)\Gamma^n f(s)ds. \]

First, we note that \( g \) is almost periodic function taking values in \( C_r \). In fact, for each \( n \in \mathbb{N} \) the function

\[ F_n : \mathbb{R} \ni t \mapsto \Gamma^n f(t) \in C_r \]

is an almost periodic function with \( \sigma(F_n) \subset \sigma(f) \). Next, by [14, Lemma 5.3] the function

\[ F : \mathbb{R} \ni t \mapsto \int_{t-1}^{t} U(t, \xi)F_n(\xi)d\xi \]

is also almost periodic, and \( \sigma(F) \subset \sigma(F_n) \subset \sigma(f) \). Therefore, \( g \) is almost periodic and \( \sigma(g) \subset \sigma(f) \).

By [14, Theorem 4.7] if (3.11) holds there exists a unique almost periodic solution \( w \) to Eq. (3.12) such that \( \sigma(w) \subset \sigma(f) \). Our next goal is to prove that there exists a mild solution \( u \) of Eq. (1.1) such that \( u_n = w(n) \) for all \( n \in \mathbb{Z} \). For each fixed \( n \in \mathbb{Z} \) consider the unique mild solution to Eq. (1.1) on the interval \([n, n+1]\) that is generated by the equation

\[ u(t) = T(t-n)[w(n)](0) + \int_{n}^{t} T(t-\eta)[F(\eta)u(\eta) + f(\eta)]d\eta, \quad t \in [n, n+1], \]

\[ u_n = w(n). \]

This solution exists uniquely on the interval \([n, n+1]\) for each \( n \in \mathbb{Z} \). By the Variation-of-Constants formula (2.13)

\[(3.13) \quad u_t = U(t, n)w(n) + \lim_{m \to \infty} \int_{n}^{t} U(t, s)\Gamma^m f(s)ds, \quad t \geq n.\]

Therefore, if \( t = n + 1 \) we have that \( u_{n+1} = w(n + 1) \). This means that we obtain a mild solution \( u \) of Eq. (1.1) that is defined on each interval \([n, n+1]\) by (3.13) so that it coincides with \( w \) at each integer \( n \). Therefore, the sequence \( w(n) = u_n \) is almost periodic. This yields that \( u(n) = u_n(0) \) is an almost periodic sequence. We are going to prove that \( u \) is almost
As $w(\cdot)$ and $f$ are almost periodic, so is the function $g : \mathbb{R} \ni t \mapsto (w(t), f(t)) \in C \times X$ (see [9, p.6]). As is known, the sequence $\{g(n)\} = \{(w(n), f(n))\}$ is almost periodic. Hence, for every positive $\epsilon$ the following set is relatively dense (see [4, p. 163-164])

$$T := \mathbb{Z} \cap T(g, \epsilon),$$

where $T(g, \epsilon) := \{\tau \in \mathbb{R} : \sup_{t \in \mathbb{R}} \|g(t + \tau) - g(t)\| < \epsilon\}$, i.e., the set of $\epsilon$ periods of $g$. Hence, for every $m \in T$ we have

$$\|f(t + m) - f(t)\| < \epsilon, \forall t \in \mathbb{R},$$

$$\|w(n + m) - w(n)\| < \epsilon, \forall n \in \mathbb{Z}.$$
4. Discussion: Variation-of-constant formula in the phase spec and further extension

Our results in the previous section could be extended to a bit more general case of periodic equations. Namely, let us consider equations of the form

\[
\frac{du}{dt} = A(t)u + F(t)u_t + f(t), \quad t \in \mathbb{R},
\]

where the family of (possibly unbounded) operators \(A(t)\) generates a 1-periodic evolutionary process and \(F(t)\) is a 1-periodic family of bounded operators as in (1.1), and \(f\) is an almost periodic function taking values in \(X\).

The presentation of our proofs of the results in the previous section relies on the variation-of-constants formula (2.13) in the phase space \(C_r\) that allows us to easily outline the ideas. In turn, we have made use of the formula available in the case when \(A(t)\) is independent of \(t\) although our results could be true even if \(A(t)\) may depend on \(t\) periodically with the same period as that of \(F(t)\).

As shown in [5, Lemma 4.1], there is a way to get around with the variation-of-constant formula (2.13). Below is a version of Lemma 4.1 from [5] that could be used to extend our results in the previous section to the general case of equations (4.1). We consider the following Cauchy Problem for each given \(t \in \mathbb{R}\)

\[
\begin{align*}
y(\xi) &= \int_{t-1}^{\xi} V(\xi, \eta)[F(\eta)y_{\eta} + f(\eta)]d\eta, \quad \xi \geq t - 1, \\
y_{t-1} &= 0 \in C_r,
\end{align*}
\]

where \((V(t, s))_{t \geq s}\) is a 1-periodic evolutionary process generated by the homogeneous equation

\[
\frac{du}{dt} = A(t)u,
\]

Let us define \(v : \mathbb{R} \ni t \mapsto y_t \in C_r\). We define the operator \(L : BUC(\mathbb{R}, X) \ni f \mapsto v\).

**Lemma 4.1.** The operator \(L\) is well defined operator in \(BUC(\mathbb{R}, X)\) that is linear and continuous and commutes with the translation \(S\).

**Proof.** Since the proof could be easily adapted from that of [5, Lemma 4.1] details will be omitted. \(\Box\)

From the definition of the function \(v\) we can verify that if \(u\) is a mild solution of (1.1) on the real line, then

\[
u_t = U(t, t - 1)u_{t-1} + v(t), \quad t \in \mathbb{R}.
\]
Therefore, the circular spectrum of $u$ could be estimated as below

**Lemma 4.2.**

\[ \sigma(u) \subset \sigma_T(M) \cup \sigma(f). \]

**Proof.** Since $L$ is linear, bounded and commutes with $S$ we have $\sigma(v) = \sigma(Lf) \subset \sigma(f)$. The rest of the proof is similar to that of Lemma 3.1. \[ \square \]

All main results of the previous section, Theorems 3.3, 3.4 and 3.5 will follow if we adjust the technique of decomposition as discussed in [20] to periodic evolutionary processes.

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**E-mail address:** vutrongluong@gmail.com

**E-mail address:** mvnguyen1@ualr.edu