On Kadell’s two Conjectures for the $q$-Dyson Product

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Abstract

By extending Lv-Xin-Zhou’s first layer formulas of the $q$-Dyson product, we prove Kadell’s conjecture for the Dyson product and show the error of his $q$-analogous conjecture. With the extended formulas we establish a $q$-analog of Kadell’s conjecture for the Dyson product.

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1 Introduction

In 1962, Freeman Dyson [3] conjectured the following constant term identity.

Theorem 1.1 (Dyson’s Conjecture). For nonnegative integers $a_0, a_1, \ldots, a_n$,

$$\text{CT}_x \prod_{0 \leq i < j \leq n} \left(1 - \frac{x_i}{x_j}\right)^{a_i} = \frac{a!}{a_0! a_1! \cdots a_n!},$$

where $a := a_0 + a_1 + \cdots + a_n$ and $\text{CT}_x f(x)$ means to take constant term in the $x$’s of the series $f(x)$.

The conjecture was quickly proved independently by Gunson [6] and by Wilson [15]. An elegant recursive proof was published by Good [5], and a combinatorial proof was given by Zeilberger [16]. In 1975, George Andrews [1] came up with a $q$-analog of the Dyson conjecture.

Theorem 1.2. (Zeilberger-Bressoud). For nonnegative integers $a_0, a_1, \ldots, a_n$,

$$\text{CT}_x \prod_{0 \leq i < j \leq n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{x_j}{x_i}\right)_{a_j} = \frac{(q)_a}{(q)_{a_0} (q)_{a_1} \cdots (q)_{a_n}},$$

where $(z)_m := (1 - z)(1 - zq) \cdots (1 - zq^{m-1})$. 

The Laurent polynomials in the above two theorems are respectively called the *Dyson product* and the *q-Dyson product* and respectively denoted by $D_n(x, a)$ and $D_n(x, a, q)$, where $x := (x_0, \ldots, x_n)$ and $a := (a_0, \ldots, a_n)$.

The Zeilberger-Bressoud q-Dyson Theorem was first proved, combinatorially, by Zeilberger and Bressoud [17] in 1985. Recently, Gessel and Xin [4] gave a very different proof by using the properties of the formal Laurent series and of the polynomials. The coefficients of the Dyson and the q-Dyson product were researched in [2, 7, 8, 9, 11, 12, 13]. In the equal parameter case, the identity reduces to Macdonald’s constant term conjecture [10] for root systems of type $A$. In 1988 Stembridge [14] gave the first layer formulas of the q-Dyson product in the equal parameter case.

Let $I = \{i_1, \ldots, i_m\}$ be a proper subset of $\{0, 1, \ldots, n\}$ and $J = \{j_1, \ldots, j_m\}$ be a multi-subset of $\{0, 1, \ldots, n\} \setminus I$, where $0 \leq i_1 < \cdots < i_m \leq n$ and $0 \leq j_1 \leq \cdots \leq j_m \leq n$.

Our first objective in this paper is to prove the following conjecture of Kadell [7].

**Conjecture 1.3.** For nonnegative integers $a_0, a_1, \ldots, a_n$ we have

$$\left(1 + a - \sum_{k \in I} a_k\right) \left(\frac{1}{x_i} \prod_{k=1}^{m} \left(1 - \frac{x_j}{x_{i_k}}\right)^{a_i} \prod_{0 \leq i \neq j \leq n} \left(1 - \frac{x_i}{x_j}\right)^{a_i}\right) = \left(1 + a\right) \frac{a!}{a_0! a_1! \cdots a_n!} \quad (1.1)$$

In the same paper, Kadell also gave a q-analogous conjecture, we restate it as follows.

**Conjecture 1.4.** Let $P = \{(i_k, j_k) | i_k \in I, j_k \in J, k = 1, 2, \ldots, m\}$. Then for nonnegative integers $a_0, a_1, \ldots, a_n$ we have

$$\left(1 - q^{1+a - \sum_{k \in I} a_k}\right) \left(\frac{1}{x_i} \prod_{0 \leq s < t \leq n} \left(\frac{x_s}{x_t}\right)^{a_i} \prod_{a_j + \chi((t,s) \in P) \leq a_i + \chi((s,t) \in P)} \left(\frac{x_s}{x_t}\right)^{a_i}\right) = \left(1 - q^{1+a}\right) \frac{(q)_a}{(q)_{a_0}(q)_{a_1} \cdots (q)_{a_n}}, \quad (1.2)$$

where the expression $\chi(S)$ is 1 if the statement $S$ is true, and 0 otherwise.

In trying to prove Conjecture 1.4, we find that the conjectured formula is incorrect. One way to modify the conjecture is to evaluate the left-hand side of (1.2). This can be done by writing it as a linear combination of some first layer coefficients of the q-Dyson product, and then applying the formulas of [8]. Unfortunately, we are not able to derive a nice formula.

Our second objective is to contribute a q-analogous formula of (1.1), which is motivated by the proof of (1.1), and is stated in Theorem 4.1.

This paper is organized as follows. In Section 2 we reformulate the main result in [8] and give an extended form of it. In Section 3 we prove Conjecture 1.3 and give an example to show the error of Conjecture 1.4. In Section 4 based on Conjecture 1.4 we give our main theorem.
2 Basic results

Let \( T = \{t_1, \ldots, t_d\} \) be a \( d \)-element subset of \( I \) with \( t_1 < \cdots < t_d \). Define

\[
w_i = \begin{cases} 
  a_i, & \text{for } i \notin T; \\
  0, & \text{for } i \in T.
\end{cases}
\]  

(2.1)

Let \( S \) be a set and \( k \) be an element in \( \{0, 1, \ldots, n\} \). Define \( N(k, S) \) as the number of the elements in \( S \) which are not larger than \( k \), i.e.,

\[
N(k, S) = \left| \{i \leq k \mid i \in S\} \right|.
\]  

(2.2)

In particular, \( N(k, \emptyset) = 0 \).

The first layer formulas of the \( q \)-Dyson product can be restated as follows.

**Theorem 2.1.** [8] Let \( I, J \) be defined as in Conjecture [63]. Then for nonnegative integers \( a_0, a_1, \ldots, a_n \) and fixed \( i_1 = 0 \) we have

\[
C_T x_{j_1} x_{j_2} \cdots x_{j_m} D_n(x, a, q) = \frac{(q)_{a_n}}{(q)_{a_0} \cdots (q)_{a_n}} \sum_{\emptyset \neq T \subseteq I} (-1)^d q^{L(T \mid I)} \frac{1 - q^{\sum_{k \in T} a_k}}{1 - q^{1 + a - \sum_{k \in T} a_k}},
\]  

(2.3)

where

\[
L(T \mid I) = \sum_{k=0}^{n} \left[ N(k, I) - N(k, J) \right] w_k.
\]  

(2.4)

We need the explicit formula for the case \( i_1 \neq 0 \) for our calculation. As stated in [8], the formula for this case can be derived using an action \( \pi \) on Laurent polynomials:

\[
\pi(F(x_0, x_1, \ldots, x_n)) = F(x_1, x_2, \ldots, x_n, x_0/q).
\]

By iterating, if \( F(x_0, x_1, x_2, \ldots, x_n) \) is homogeneous of degree 0, then

\[
\pi^{n+1}(F(x_0, x_1, \ldots, x_n)) = F(x_0/q, x_1/q, x_2/q, \ldots, x_n/q) = F(x_0, x_1, x_2, \ldots, x_n),
\]

so that in particular \( \pi \) is a cyclic action on \( D_n(x, a, q) \). We use the following lemma to derive an extended form of Theorem 2.1.

**Lemma 2.2.** [8] Let \( L(x) \) be a Laurent polynomial in the \( x \)'s. Then

\[
C_T x L(x) D_n(x, a, q) = C_T x \pi(L(x)) D_n(x, (a_n, a_0, \ldots, a_{n-1}), q).
\]  

(2.5)

By iterating (2.5) and renaming the parameters, evaluating \( C_T x L(x) D_n(x, a, q) \) is equivalent to evaluating \( C_T x \pi^k(L(x)) D_n(x, a, q) \) for any integer \( k \).

Assume for some \( t \) we have \( j_t < i_1 \) and \( j_{t+1} > i_1 \). Let \( J^- = \{j_1, \ldots, j_t\} \) and \( J^+ = \{j_{t+1}, \ldots, j_m\} \).
Theorem 2.3. For nonnegative integers \(a_0, a_1, \ldots, a_n\) we have

\[
\text{CT} \times \frac{x_{j_1} \cdots x_{j_m}}{x_{i_1} \cdots x_{i_m}} D_n(x, a, q) = \frac{(q)_a}{(q)_{a_0} \cdots (q)_{a_n}} \sum_{\emptyset \neq T \subseteq I} (-1)^d q^{L^*(T | I)} \frac{1 - q^{\sum_{k \in T} a_k}}{1 - q^{1 + a - \sum_{k \in T} a_k}}, \tag{2.6}
\]

where

\[
L^*(T | I) = t + \sum_{k = 1}^n \left[ N(k, I) - N(k, J^+) \right] w_k + \sum_{k = 1}^{i_1 - 1} \left[ t - N(k, J^-) \right] a_k. \tag{2.7}
\]

We remark that there is not the restriction \(i_1 = 0\) in the above theorem. The idea to prove this theorem is by iterating Lemma 2.2 to transform the random \(i_1\) in (2.6) to zero and then applying Theorem 2.3. But in the proof there are many tedious transformations of the parameters, so we put the proof to the appendix for those who are interested in.

Note that \(w_k\) only occurs in the first summation of (2.7), so only the first summation of (2.7) depends on \(T\).

Letting \(q \to 1^-\) in Theorem 2.3 we get

Corollary 2.4. [8] For nonnegative integers \(a_0, \ldots, a_n\) we have

\[
\text{CT} \times \frac{x_{j_1} \cdots x_{j_m}}{x_{i_1} \cdots x_{i_m}} \prod_{0 \leq i \neq j \leq n} \left(1 - \frac{x_i}{x_j}\right)^{a_i} = \frac{a!}{a_0! \cdots a_n!} \prod_{\emptyset \neq T \subseteq I} (-1)^d \frac{\sum_{k \in T} a_k}{1 + a - \sum_{k \in T} a_k}. \tag{2.8}
\]

This result also follows from [8, Theorem 1.7] by permuting the variables. Note that the right-hand side of (2.8) is independent of \(j\)’s.

3 Proof of Conjecture 1.3

Now we are ready to prove Conjecture 1.3.

Proof of Conjecture 1.3. If \(I = \emptyset\) then Conjecture 1.3 reduces to the Dyson Theorem, which is also the case when \(m = 0\) in Corollary 2.4. So we assume that \(I \neq \emptyset\). Expanding the first product of (1.1) gives

\[
\text{CT} \times \prod_{i = 1}^m \left(1 - \frac{x_{i_k}}{x_{i_k}}\right) \prod_{0 \leq i \neq j \leq n} \left(1 - \frac{x_i}{x_j}\right)^{a_i} = \text{CT} \times \left[1 + \sum_{l = 1}^m (-1)^l \sum_{\emptyset \neq I_l \subseteq I} \frac{x_{u_1} \cdots x_{u_l}}{x_{u_1} \cdots x_{u_l}} \prod_{0 \leq i \neq j \leq n} \left(1 - \frac{x_i}{x_j}\right)^{a_i}\right],
\]

where \(I_l = \{u_1, \ldots, u_l\}\) ranges over all subsets of \(I\) except the empty set and \(\{v_1, \ldots, v_l\}\) is the corresponding subset of \(J\). Denote the left constant term in the above equation by \(LC\). Applying Corollary 2.4 we get

\[
LC = \left[1 + \sum_{l = 1}^m (-1)^l \sum_{\emptyset \neq I_l \subseteq I} \left(\sum_{\emptyset \neq T \subseteq I_l} (-1)^d \frac{\sum_{k \in T} a_k}{1 + a - \sum_{k \in T} a_k}\right)\frac{a!}{a_0! \cdots a_n!}\right]. \tag{3.1}
\]
where \( d = |T| \). Changing the order of the summations, and observing that for any fixed set \( T \) there are \( \binom{m-d}{l-d} \) such \( I_l \) satisfying \( T \subseteq I_l \subseteq I \), we obtain

\[
LC = \left[ 1 + \sum_{\varnothing \neq T \subseteq I} \sum_{l=d}^{m} (-1)^{l+d} \binom{m-d}{l-d} \frac{\sum_{k \in T} a_k}{1+a-\sum_{k \in T} a_k} \right] a! = \left( 1 + \frac{\sum_{k \in I} a_k}{1+a-\sum_{k \in I} a_k} \right) a! \cdot \ldots \cdot a_1!,
\]

(3.2)

where we used the easy fact that for \( d \neq m \)

\[
\sum_{l=d}^{m} (-1)^{l+d} \binom{m-d}{l-d} = \sum_{l=0}^{m-d} (-1)^l \binom{m-d}{l} = (1-x)^{m-d} \big|_{x=1} = 0.
\]

The conjecture then follows by multiplying both sides of (3.2) by \( 1 + a - \sum_{k \in I} a_k \).

For the \( q \)-case, Conjecture 1.4 does not hold even for \( m = 1 \). To see this take \( n = 2, I = \{0\}, J = \{1\} \) and \( a_0 = a_1 = a_2 = 1 \). For these values the left-hand side of (1.2) is

\[
(1-q^3) CT \left( 1 - \frac{x_0}{x_1} \right) \left( 1 - \frac{x_1}{x_0} \right) \left( 1 - q^2 \frac{x_0}{x_1} \right) \left( 1 - \frac{x_0}{x_2} \right) \left( 1 - \frac{x_1}{x_2} \right) \left( 1 - \frac{x_2}{x_1} \right) \left( 1 - \frac{x_1 x_2}{x_0} \right) \left( 1 - \frac{x_0 x_2}{x_1} \right) \\
= (1-q^3)(1+2q+3q^2+2q^3).
\]

While the right-hand side of (1.2) equals \( (1-q^4)(1+q)(1+q+q^2) \), which is not equal to the left-hand side.

### 4 A \( q \)-analog of Kadell’s conjecture

#### 4.1 Motivation and presentation of the main theorem

In this section we will construct a \( q \)-analog of Conjecture 1.4. The new identity is motivated by the proof of Conjecture 1.4 in the last section, where massive cancelations happen. We hope for similar cancelations in the \( q \)-case.

Our first hope is to modify Conjecture 1.4 to obtain a formula of the form:

\[
\left( 1-q^{1+a-\sum_{k \in I} a_k} \right) \frac{CT}{x} \prod_{k=1}^{m} \left( 1-q^{L_k i_k} x_{i_k} \right) D_n(x, a, q) = \left( 1-q^{1+a} \right) \frac{(q)_a}{(q)_{a_0} (q)_{a_1} \cdots (q)_{a_n}},
\]

(4.1)

where \( L_k \) is an integer depending on \( i_k, j_k \) and \( a \).

It is intuitive to consider the \( m = 2 \) case, so take \( I = \{i_1, i_2\} \). We need to choose appropriate \( L_1 \) and \( L_2 \) such that

\[
\left( 1-q^{1+a-i_{1i}}-a_{i_2} \right) \frac{CT}{x} \left( 1-q^{L_{i_1} i_{1i}} x_{i_1} \right) \left( 1-q^{L_{i_2} i_{2i}} x_{i_2} \right) D_n(x, a, q) = \left( 1-q^{1+a} \right) \frac{(q)_a}{(q)_{a_0} (q)_{a_1} \cdots (q)_{a_n}}.
\]

(4.2)
By applying Theorem 2.3 the left-hand side of (4.2) becomes

\[
\left(1 - q^{1+a-a_1-a_{12}}\right) \left(1 + q^{L_1+L^*(\{i_1\})\{i_1\}} \right) \frac{1 - q^{a_1}}{1 - q^{1+a-a_1}} + q^{L_2+L^*(\{i_2\})\{i_2\}} \frac{1 - q^{a_{12}}}{1 - q^{1+a-a_{12}}}
- q^{L_1+L_2+L^*(\{i_1\})\{i_1,i_2\}} \frac{1 - q^{a_1}}{1 - q^{1+a-a_1}} - q^{L_1+L_2+L^*(\{i_2\})\{i_1,i_2\}} \frac{1 - q^{a_{12}}}{1 - q^{1+a-a_{12}}}
+ q^{L_1+L_2+L^*(\{i_1,i_2\})\{i_1,i_2\}} \frac{1 - q^{a_{11}+a_{12}}}{1 - q^{1+a-a_1-a_{12}}}
\]

\[
\frac{(q)_a}{(q)_{a_0} (q)_{a_1} \cdots (q)_{a_n}}.
\]

It is natural to have the following requirements to get (4.2).

\[
\begin{align*}
q^{L_1+L^*(\{i_1\})\{i_1\}} - q^{L_1+L_2+L^*(\{i_1\})\{i_1,i_2\}} &= 0, \\
q^{L_2+L^*(\{i_2\})\{i_2\}} - q^{L_1+L_2+L^*(\{i_2\})\{i_1,i_2\}} &= 0, \\
q^{L_1+L_2+L^*(\{i_1,i_2\})\{i_1,i_2\}} &= q^{1+a-a_1-a_{12}}.
\end{align*}
\]

This is actually a linear system and has no solution, so our first hope broke.

Looking closer at (4.3), we see that the first two equalities must be satisfied to have a nice formula. Agreeing with this, for general \( I \) with \( |I| = m \) we will need \( 2^m - 1 \) restrictions for massive cancelations as in the proof of Conjecture 1.3. More precisely, by applying Theorem 2.3, the left-hand side of (4.1) will be written as

\[
\left(1 - q^{1+a-\sum_{k \in T} a_k}\right) \left(1 + \sum_T B_T \frac{1 - q^{\sum_{k \in T} a_k}}{1 - q^{1+a-\sum_{k \in T} a_k}} \frac{(q)_a}{(q)_{a_0} \cdots (q)_{a_n}}\right),
\]

where \( T \) ranges over all subsets of \( I \) except the empty set. We need to have \( B_T = 0 \) for all \( T \) except for \( T = I \). This is why using only \( m \) unknowns dooms to fail.

We hope for some nice \( A_T \) such that the constant term of

\[
\sum_T A_T \frac{x_{v_1} \cdots x_{v_l}}{x_{a_1} \cdots x_{a_l}} D_n(x,a,q)
\]

has the desired cancelations. We are optimistic because from the view of linear algebra, such \( A_T \) exists but is difficult to solve and might only be rational in \( q \). Amazingly, it turns out that in many situations, the \( A_T \) may be chosen to be \( \pm q^{\text{integer}} \). Our formula for \( A_T \) is inspired by the proof of Conjecture 1.3. To present our result, we need some notations.

Fix a subset \( I = \{i_1, \ldots, i_m\} \) and a multi-subset \( J = \{j_1, \ldots, j_m\} \) of \( \{0,1,\ldots,n\} \), where \( i_1 < \cdots < i_m, j_1 \leq \cdots \leq j_m \) and \( I \cap J = \emptyset, 0 \leq m \leq n \). Given an \( l \)-element subset \( I_l = \{u_1, \ldots, u_l\} \) of \( I \), we say \( J_l = \{v_1, \ldots, v_l\} \) is the pairing set of \( I_l \) if \( u_k = i_t (1 \leq k \leq l) \) for some \( t \) implies that \( v_k = j_t \). Write \( I \setminus I_l = \{i_{r_1}, \ldots, i_{r_{m-l}}\}, r_1 < \cdots < r_{m-l} \). We use \( A \overset{i}{\rightarrow} B \) to denote \( B = A \cup \{i\} \), and define a sequence of sets:

\[
I_l = I_{m-l+1} \overset{i_{r_{m-l}}}{\rightarrow} I_{m-l} \overset{i_{r_{m-l-1}}}{\rightarrow} I_{m-l-1} \overset{i_{r_{m-l-2}}}{\rightarrow} \cdots \overset{i_{r_1}}{\rightarrow} I_1 = I. \tag{4.5}
\]

For a set \( S \) of integers, we denote by \( \min S \) the smallest element of \( S \). Define \( J^*_k(J_l) \) to be the set \( \{j_s : j_s \in J_l \cup \{j_k\}\} \), we use \( J^*_k \) as an abbreviation for \( J^*_k(J_l) \).

Our \( q \)-analog of Conjecture 1.3 can be stated as follows.
Theorem 4.1. (Main Theorem) For nonnegative integers $a_0, a_1, \ldots, a_n$ and $I, J$ as above, if there is no $s, t, u$ such that $1 \leq s < t < u \leq m$ and $j_t < i_s < j_u < i_t$, then

\[
\left(1-q^{1+a-\sum_{k \in I} a_k}\right) \frac{C(I)}{x} \left[1 + \sum_{\emptyset \neq I \subseteq I} (-1)^{|I|} q^{C(I)} \frac{x_{v_1} \cdots x_{v_j}}{x_{u_1} \cdots x_{u_i}} D_a(x, a, q)\right]
\]

\[
= \left(1-q^{1+a}\right)^{(q)_{a}} \frac{(q)_{a}(q)_{a_1} \cdots (q)_{a_n}}{(q)_{a_n}} ,
\]

where, with $L^*(I | I)$ defined as in (2.7),

\[
C(I) = 1 + a - \sum_{k \in I} a_k + \sum_{k=1}^{m-l} \left[N(i_{r_k}, I_l) - N(i_{r_k}, J^*_k)\right] a_{r_k} - L^*(I | I_l). \tag{4.7}
\]

We remark that there is no analogous simple formula if the $u$'s and the $v$'s are not paired up, and that the sum $1 + \sum_{\emptyset \neq I \subseteq I} (-1)^{|I|} q^{C(I)} \frac{x_{v_1} \cdots x_{v_j}}{x_{u_1} \cdots x_{u_i}}$ in (4.6) does not factor.

4.2 Factorization and cancelation lemma

To prove the main theorem, we need some lemmas.

Let $U$ be a subset of $I_l$, $|U| = d$ and $I \setminus U = \{i_1, \ldots, i_{m-d}\}$, $t_1 < \cdots < t_{m-d}$. For fixed $I_l$, suppose that min $I_l = i_v$. By tedious calculation we can get the following lemma.

Lemma 4.2. Let $U, C(I_l), L^*(U | I_l)$ be as described. Then for $i_{t_s} \in I_l$ but $i_{t_s} \notin U \cup \{i_v\}$ we have

\[
C(I_l) + L^*(U | I_l) - C(I_l \setminus \{i_{t_s}\}) - L^*(U | I_l \setminus \{i_{t_s}\})
\]

\[
= -\sum_{k=v}^{s-1} \chi(i_{t_k} > j_{t_s} > i_v) a_{i_{t_k}} + \sum_{k=s+1}^{m-d} \chi(i_{t_k} > j_{t_s} > i_v) a_{i_{t_k}}, \tag{4.8}
\]

where $\chi(i_{t_k} > j_{t_s} > i_v) := 1 - \chi(i_{t_k} > j_{t_s} > i_v)$.

We denote $-\sum_{k=v}^{s-1} \chi(i_{t_k} > j_{t_s} > i_v) a_{i_{t_k}} + \sum_{k=s+1}^{m-d} \chi(i_{t_k} > j_{t_s} > i_v) a_{i_{t_k}}$ by $g(i_{t_s})$.

Lemma 4.3. For $n \geq 2$, every term in the expansion of $\prod_{s=1}^{n} \sum_{k \neq s} a(s, k)$ has $a(k, r)a(s, l)$ as a factor for some $k, r, s, l$ satisfying $1 \leq r \leq s < k < l \leq n$.

Proof. Construct a matrix $A$ with $0$'s in the main diagonal as follows.

\[
A = \begin{pmatrix}
0 & a(1,2) & \cdots & a(1,n) \\
a(2,1) & 0 & \cdots & a(2,n) \\
\vdots & \vdots & \ddots & \vdots \\
a(n,1) & a(n,2) & \cdots & 0
\end{pmatrix}.
\]

Then each term in the expansion of $\prod_{s=1}^{n} \sum_{k \neq s} a(s, k)$ corresponds to picking out one entry except for the $0$'s from each row of $A$. We prove by contradiction.
Suppose we choose \( a(1, k_1) \) \((k_1 \geq 2)\) from the first row. Then we can not choose \( a(2, 1) \), for otherwise \( a(2, 1) a(1, k_1) \) forms the desired factor. Now from the second row, we have to choose \( a(2, k_2) \) \((k_2 \geq 3)\). It then follows that \( a(3, 1) \) and \( a(3, 2) \) can not be chosen, for otherwise \( a(3, e) a(2, k_2), e = 1, 2 \) forms the desired factor. Repeat this discussion until the \( n - 1 \)th row, where we have to choose \( a(n - 1, n) \). But then our \( n \)th row element \( a(n, e) \) (with \( 1 \leq e \leq n - 1 \)) together with \( a(n - 1, n) \) forms the desired factor, a contradiction. \( \square \)

The following factorization and cancelation lemma plays an important role and it is our main discovery in this paper.

**Lemma 4.4.** For fixed set \( U \neq I \) and integer \( i_v \leq \min U \) we have the following factorization

\[
\sum_{I_l} (-1)^{l+d} q^{C(I_l)+L^*(U \cup I_l)} = (-1)^{\min U \neq i_v} q^{C(U \cup \{i_v\})+L^*(U \cup I_v)} \prod_{i_v \in I \setminus \{i_1, \ldots, i_v\}} (1 - q^{g(i_v)}),
\]

(4.9)

where \( I_l \) ranges over all supersets of \( U \) with the restriction \( \min I_l = i_v \). Furthermore, if there is no \( s, t, u \) such that \( 1 \leq s < t < u \leq m \) and \( j_t < i_s < j_u < i_t \), then

\[
\prod_{i_v \in I \setminus \{i_1, \ldots, i_v\}} (1 - q^{g(i_v)}) = 0,
\]

(4.10)

with the only exceptional case when \( I \setminus U \setminus \{i_1, \ldots, i_v\} = \emptyset \).

**Proof.** We prove this lemma in two parts.

1. Proof of (4.9).

Notice that \( I_l = U \cup \{i_v\} \) is the smallest set which satisfies \( \min I_l = i_v \) and \( U \subseteq I_l \). So first we extract the common factor \( q^{C(U \cup \{i_v\})+L^*(U \cup \{i_v\})} \) from the summation of (4.9). Thus we need to calculate

\[
C(I_l) + L^*(U \mid I_l) - C(U \cup \{i_v\}) - L^*(U \mid U \cup \{i_v\}).
\]

By Lemma 4.2 we have

\[
C(I_l) + L^*(U \mid I_l) - C(I_l \setminus \{i_{t_s}\}) - L^*(U \mid I_l \setminus \{i_{t_s}\}) = g(i_{t_s}),
\]

(4.11)

where \( i_{t_s} \in I_l \) but \( i_{t_s} \notin U \cup \{i_v\} \). Thus iterating (4.11) we get

\[
C(I_l) + L^*(U \mid I_l) - C(U \cup \{i_v\}) - L^*(U \mid U \cup \{i_v\}) = \sum_{i_{t_s} \in I_l \setminus U \cup \{i_v\}} g(i_{t_s}).
\]

(4.12)

So extracting the common factor \( q^{C(U \cup \{i_v\})+L^*(U \cup \{i_v\})} \) from the left-hand side of (4.9) and by (4.12) we have

\[
\sum_{I_l} (-1)^{l+d} q^{C(I_l)+L^*(U \cup \{i_v\})} = q^{C(U \cup \{i_v\})+L^*(U \cup \{i_v\})} \sum_{i_v \in I_l \setminus U \cup \{i_v\}} g(i_{t_s}),
\]

(4.13)

where \( I_l \) ranges over all supersets of \( U \) with the restriction \( \min I_l = i_v \).
Second we prove the following factorization.

\[
\sum_{I_t} (-1)^{l+d} q^{\sum_{i_s \in I_t \setminus \{i_v\}} g(i_s)} = (-1)^{\chi(\min U \neq i_v)} \prod_{i_s \in I_t \setminus \{i_1, \ldots, i_v\}} (1 - q^{g(i_s)}), \tag{4.14}
\]

where \(I_t\) ranges over all supersets of \(U\) and we restrict \(\min I_t = i_v\).

If \(\min U = i_v\), then the sign in the right-hand side of (4.14) is positive. Every term in the expansion of the right-hand side of (4.14) is of the form \((-1)^{|G|} \prod_{i_s \in G} q^{g(i_s)}\), where \(G\) is a subset of \(I \setminus U \setminus \{i_1, \ldots, i_v\}\). Thus expanding the product of (4.14) we get

\[
\prod_{i_s \in I_t \setminus \{i_1, \ldots, i_v\}} (1 - q^{g(i_s)}) = \sum_{G \subseteq I_t \setminus \{i_1, \ldots, i_v\}} (-1)^{|G|} \prod_{i_s \in G} q^{g(i_s)}. \tag{4.15}
\]

Notice that \(I_t \setminus U \setminus \{i_v\}\) reduces to \(I_t \setminus U\) when \(\min U = i_v\). Substitute \(I_t \setminus U\) by \(G'\) in the left-hand side of (4.14). Then \(G'\) ranges over all subsets of \(I \setminus U \setminus \{i_1, \ldots, i_v\}\) if \(I_t\) ranges over all supersets of \(U\) with the restriction \(\min I_t = i_v\). Notice that \((-1)^{|G'|} = (-1)^{l-d} = (-1)^{l+d}\), thus the left-hand side of (4.14) can also be written as the right hand side of (4.15). Hence (4.14) holds when \(\min U = i_v\). The case \(\min U \neq i_v\) is similar.

Therefore (4.9) follows from (4.13) and (4.14).

2. Under the assumption that there is no \(s, t, u\) such that \(1 \leq s < t < u \leq m\) and \(j_t < i_s < j_u < i_t\) we need to prove (4.10).

If \(\min I_t = \min U = i_v\), recall that \(I \setminus U = \{i_1, \ldots, i_{m-d}\}\) and \(t_1 < \cdots < t_{m-d}\), then \(t_k = k\) for \(k = 1, \ldots, v - 1\) and \(t_v > v\). Thus \(t_v \in I \setminus U \setminus \{i_1, \ldots, i_v\}\). It follows that

\[
\prod_{i_s \in I_t \setminus \{i_1, \ldots, i_v\}} (1 - q^{g(i_s)}) = \prod_{s=v}^{m-d} (1 - q^{g(i_s)}).
\]

If \(\min I_t \neq \min U\), then \(t_v = v\). It follows that \(t_v \notin I \setminus U \setminus \{i_1, \ldots, i_v\}\). Thus we have

\[
\prod_{i_s \in I_t \setminus \{i_1, \ldots, i_v\}} (1 - q^{g(i_s)}) = \prod_{s=v}^{m-d} (1 - q^{g(i_s)})\]

\(\chi(i_v > j_s > i_v)\) and \(\chi(i_v > j_s > i_v) = \chi(i_v > j_s > i_v) = 0\). In this case \(g(i_v)\) reduces to

\[g(i_v) = -\sum_{k=v+1}^{s-1} \chi(i_k > j_s > i_v) a_{i_k} + \sum_{k=s+1}^{m-d} \chi(i_k > j_s > i_v) a_{i_k}.
\]

We only prove (4.10) when \(\min I_t = \min U\), the case \(\min I_t \neq \min U\) is similar.

We can write the left-hand side of (4.10) as \(\prod_{s=v}^{m-d} (1 - q^{g(i_s)})\) when \(\min I_t = \min U\). To prove \(\prod_{s=v}^{m-d} (1 - q^{g(i_s)}) = 0\), it is sufficient to prove \(\prod_{s=v}^{m-d} g(i_s) = 0\).

Taking \(a(s, k) = -\chi(i_k > j_s > i_v) a_{i_k}\) for \(s > k\) and \(a(s, k) = \chi(i_k > j_s > i_v) a_{i_k}\) for \(s < k\), by the definition of \(g(i_s)\) we can write \(\prod_{s=v}^{m-d} g(i_s)\) as \(\prod_{s=v}^{m-d} \sum_{k \neq s} a(s, k)\). By Lemma 4.3 each term in the expansion of \(\prod_{s=v}^{m-d} g(i_s)\) has a factor of the form \(-\chi(i_r > j_k > i_v) a_{i_r} a_{i_k}\), where \(v \leq r \leq s < k \leq l \leq m - d\). Thus

\[
\prod_{s=v}^{m-d} g(i_s) = -\sum_{v \leq r \leq s < k \leq l \leq m-d} \chi(i_r > j_k > i_v) \chi(i_l > j_s > i_v) a_{i_r} a_{i_k} \cdot \Delta, \tag{4.16}
\]
where $\Delta$ is the product of some $a(s, k)$’s.

Next we prove each $\chi(it_r > j_t > i_v)\chi(it_l > j_s > i_v) = 0$ by contradiction under the assumption that there is no $s, t, u$ such that $1 \leq s < t < u \leq m$ and $j_t < i_s < j_u < i_t$.

Suppose $\chi(it_r > j_t > i_v)\chi(it_l > j_s > i_v) = 1$ for some $v < r < v < k < \leq m - d$. Then $\chi(it_r > j_t > i_v) = \chi(it_l > j_s > i_v) = 1$. By $\chi(it_r > j_t > i_v) = 1$ we have

$$it_r > j_t > i_v.$$  \hspace{1cm} (4.17)

By $\chi(it_l > j_s > i_v) = 1$ we obtain

$$it_l < j_s \quad \text{or} \quad j_s < i_v \quad \text{or} \quad it_l < i_v.$$  \hspace{1cm} (4.18)

Since $l > v$, we have $t_l \geq l > v$ and $it_l > i_v$. Thus the last inequality of (4.18) can not hold. Because $l > r, k > s$ and $it_r > j_t$ in (4.17), we have $it_l > i_v > j_t > j_s$. So the first inequality of (4.18) can not hold too. Thus by (4.17) and the middle inequality of (4.18) we obtain that if $\chi(it_r > j_t > i_v)\chi(it_l > j_s > i_v) = 1$ then $j_s < i_v < j_t < it_l$. It follows that $j_s < i_v < j_t < i_s$ since $r < s$. Because $v < s < k$, we have $v < t_v \leq t_s < t_k$. Thus for $v < t_s < t_k$ the fact $j_s < i_v < j_t < i_s$ conflicts with our assumption. \hfill \square

**Lemma 4.5.** If $U$ is of the form $\{i_h, i_{h+1}, \ldots, i_m\}$, then

$$q^{C(U) + L^*(U|U)} - q^{C(U \cup \{i_{h-1}\}) + L^*(U \cup \{i_{h-1}\})} = 0.$$  \hspace{1cm} (4.19)

**Proof.** By the formula of $C(I_t)$ in (4.7) we have

$$C(U) + L^*(U | U) = 1 + a - \sum_{k \in U} a_k + \sum_{k=1}^{h-1} [N(i_{r_k}, U) - N(i_{r_k}, V_k^*)] a_{i_{r_k}},$$

where $V_k^* = \{j_s > i_k | j_s \in V_1 \cup \{j_{r_k}\}\}$ and $V_1 = \{j_h, \ldots, j_m\}$ is the pairing set of $U$. Since $U$ is of the form $\{i_h, i_{h+1}, \ldots, i_m\}$, we have $i_{r_k} = i_k$ for $k = 1, \ldots, h - 1$. Hence $N(i_{r_k}, U) = N(i_{r_k}, V_k^*) = 0$ for $k = 1, \ldots, h - 1$. It follows that $C(U) + L^*(U | U) = 1 + a - \sum_{k \in U} a_k$.

Meanwhile

$$C(U \cup \{i_{h-1}\}) + L^*(U | U \cup \{i_{h-1}\})$$

$$= 1 + a - \sum_{k \in U} a_k - a_{i_{h-1}} + \sum_{k=1}^{h-2} [N(i_{r_k}, U \cup \{i_{h-1}\}) - N(i_{r_k}, V_k^*)] a_{i_{r_k}}$$

$$- L^*(U \cup \{i_{h-1}\} | U \cup \{i_{h-1}\}) + L^*(U | U \cup \{i_{h-1}\}),$$

where $V_k^* = \{j_s > i_k | j_s \in V_2 \cup \{j_{r_k}\}\}$ and $V_2 = \{j_{h-1}, \ldots, j_m\}$. Since $U \cup \{i_{h-1}\}$ is of the form $\{i_{h-1}, i_h, \ldots, i_m\}$, we have $i_{r_k} = i_k$ for $k = 1, \ldots, h - 2$. Hence $N(i_{r_k}, U \cup \{i_{h-1}\}) = N(i_{r_k}, V_k^*) = 0$ for $k = 1, \ldots, h - 2$. And by the definition of $L^*(T | I)$ in (2.7) we have

$$- L^*(U \cup \{i_{h-1}\} | U \cup \{i_{h-1}\}) + L^*(U | U \cup \{i_{h-1}\}) = a_{i_{h-1}}.$$  \hspace{1cm} (4.19)

Therefore $C(U \cup \{i_{h-1}\}) + L^*(U | U \cup \{i_{h-1}\})$ has the same value as $C(U) + L^*(U | U)$.

\hfill \square
4.3 Proof of the main theorem

Having Lemma 4.4 and Lemma 4.5 we are ready to prove the main theorem.

Proof of Theorem 4.1. If \( m = 0 \), then the theorem reduces to the \( q \)-Dyson Theorem. So we assume that \( m \geq 1 \).

Applying Theorem 2.3 to the constant term in the left-hand side of (4.6) yields

\[
CT \left[ \left( 1 + \sum_{\varnothing \neq I \subseteq I} (-1)^l q^{C(I)} \frac{x_{v_1} \cdots x_{v_l}}{x_{u_1} \cdots x_{u_l}} \right) D_n(x, a, q) \right]
\]

\[
= \frac{(q)_a}{(q)_{a_0} \cdots (q)_{a_n}} \left( 1 + \sum_{\varnothing \neq U \subseteq I} \sum_{v_i = i_1}^{\min U} \sum_{I_i} (-1)^{d + l} q^{C(I_i) + L^*(U|I_i)} \frac{1 - q^{k \sum_{k \in U \setminus a_k}}}{1 - q^{1 + a - \sum_{k \in U \setminus a_k}}} \right),
\]

(4.20)

where \( l = |I| \) and \( d = |U| \).

Because \( U \) is a subset of \( I \), we have \( \min I_i = i_v \leq \min U \). By changing the summation order, the right-hand side of (4.20) can be rewritten as

\[
\frac{(q)_a}{(q)_{a_0} \cdots (q)_{a_n}} \left( 1 + \sum_{\varnothing \neq U \subseteq I} \sum_{i_v = i_1}^{\min U} \sum_{I_i} (-1)^{d + l} q^{C(I_i) + L^*(U|I_i)} \frac{1 - q^{\sum_{k \in U \setminus a_k}}}{1 - q^{1 + a - \sum_{k \in U \setminus a_k}}} \right),
\]

(4.21)

where \( I_i \) ranges over all supersets of \( U \) with the restriction \( \min I_i = i_v \).

If \( U \neq I \), then by Lemma 4.4 under the assumption that there is no \( s, t, u \) such that \( 1 \leq s < t < u \leq m \) and \( j_t < i_s < j_u < i_t \) we have

\[
\sum_{I_i} (-1)^{d + l} q^{C(I_i) + L^*(U|I_i)} = 0,
\]

(4.22)

with the only exceptional case when \( I \setminus U \setminus \{i_1, \ldots, i_v\} = \varnothing \), where \( I_i \) ranges over all supersets of \( U \) and we restrict \( \min I_i = i_v \).

If \( I \setminus U \setminus \{i_1, \ldots, i_v\} = \varnothing \), then \( U \) is of the form \( \{i_h, i_{h+1}, \ldots, i_m\} \) and \( i_v \) is either \( i_h \) or \( i_{h-1} \), and in this case \( I_i = U \) or \( I_i = U \cup \{i_{h-1}\} \) respectively. Thus by Lemma 4.5 we have

\[
q^{C(U) + L^*(U|U)} - q^{C(U \cup \{i_{h-1}\}) + L^*(U \cup \{i_{h-1}\})} = 0.
\]

(4.23)

By (4.22) and (4.23) the summands in (4.21) cancel with each other except for the summand when \( U = I_i = I \). It follows that (4.21) reduces to

\[
\frac{(q)_a}{(q)_{a_0} \cdots (q)_{a_n}} \left( 1 + q^{C(I) + L^*(I|I)} \frac{1 - q^{\sum_{k \in I \setminus a_k}}}{1 - q^{1 + a - \sum_{k \in I \setminus a_k}}} \right).
\]

(4.24)

By the formula of \( C(I) \) in (4.7) we get \( C(I) = 1 + a - \sum_{k \in I \setminus a_k} - L^*(I \mid I) \). Substituting \( C(I) \) into (4.24) and multiplying the equation by \( 1 - q^{1 + a - \sum_{k \in I \setminus a_k}} \) we can obtain the right-hand side of (4.6). \( \square \)
5  Remark

If there exist some \( s, t, u \) such that \( s < t < u \) and \( j_t < i_s < j_u < i_t \), then our main theorem does not lead to the desired cancelations. As stated in Section 4.1, we can solve for \( A_T \) such that the constant term of \( \sum_T A_T \frac{x_{u_1} \cdots x_{u_l}}{x_{i_1} \cdots x_{i_m}} D_n(x, a, q) \) has the desired cancelations. However, experiments show that there is no nice form for \( A_T \) in this situation.

Another possibility to let the \( u \)'s and the \( v \)'s be not paired up. Some of the cases can be established by applying the operator \( \pi \) defined in Section 2 to our main theorem. But not all the un-paired up cases can be obtained in this way.

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6  Appendix: Proof of Theorem 2.3

Proof. By the definition of \( \pi \), it is easy to deduce that

\[
\pi^k x_i = \begin{cases} x_{i+k}, & \text{for } i+k \leq n; \\ x_{i+k-n-1/q}, & \text{for } i+k > n. \end{cases}
\]  

(6.1)

Iterating Lemma 2.2 \( n - i_1 + 1 \) times, i.e., acting with \( \pi^{n-i_1+1} \), we obtain

\[
\text{CT} \frac{x_{j_1} \cdots x_{j_m}}{x_{i_1} \cdots x_{i_m}} D_n(x, a, q) = \text{CT} \prod_{l=1}^{t} x_{j_l}^{n-i_l+1} \prod_{l=t+1}^{m} x_{j_l-i_l} q^{-(m-t)} x_0 x_{i_2-i_1} \cdots x_{i_m-i_1} q^{-m} D_n(x, (b_0, \ldots, b_n), q),
\]  

(6.2)

where

\[
b_k = \begin{cases} a_{k+1}, & \text{for } k = 0, \ldots, n - i_1; \\ a_{k-(n-i_1+1)}, & \text{for } k = n - i_1 + 1, \ldots, n. \end{cases}
\]  

(6.3)

To apply Theorem 2.1 we define \( \bar{I} = \{0, i_2 - i_1, \ldots, i_m - i_1\} \), and \( \bar{J}^- = \{j_1 + n - i_1 + 1, \ldots, j_t + n - i_1 + 1\} \), \( \bar{J}^+ = \{j_{t+1} - i_1, \ldots, j_m - i_1\} \). Then by Theorem 2.1 we have

\[
\text{CT} \frac{x_{j_1} \cdots x_{j_m}}{x_{i_1} \cdots x_{i_m}} D_n(x, a, q) = q^t \frac{(q)_a}{(q)_{a_0} \cdots (q)_{a_n}} \sum_{\emptyset \neq \bar{T} \subseteq \bar{I}} (-1)^d q^{L(\bar{T})} \frac{1 - q^{\sum_{k \in \bar{T}} b_k}}{1 - q^{1+a-\sum_{k \in \bar{T}} b_k}},
\]

where \( |\bar{T}| = d \) and

\[
L(\bar{T} | \bar{I}) = \sum_{k=0}^n [N(k, \bar{I}) - N(k, \bar{J})] w_k,
\]  

(6.4)
in which \( \tilde{w}_k \) is \( b_k \) if \( k \notin \tilde{T} \) and 0 otherwise.

There is a natural one-to-one correspondence between \( I \) and \( \tilde{I} \): \( I \xrightarrow{f} \tilde{I}, f(a) = a - i_1, \ a \in I \). This correspondence clearly applies between their subsets \( T \) and \( \tilde{T} \).

Since the largest element in \( \tilde{T} \) is not larger than \( i_m - i_1 \) and \( i_m - i_1 \leq n - i_1 \), by the definition of \( b_k \) we have
\[
\sum_{k \in \tilde{T}} b_k = \sum_{k \in \tilde{T}} a_{k+i_1} = \sum_{k \in T} a_k.
\]

Next we have to rewrite (6.4) in terms of \( w_k \), \( N(k, I) \) and \( N(k, J) \) to get \( L^*(T \mid I) \).

Because the largest element in \( \tilde{I} \) is \( i_m - i_1 \leq n - i_1 \), so if \( k > n - i_1 \) then \( k \notin \tilde{T} \). It follows that
\[
\tilde{w}_k = b_k = a_{k-(n-i+1)}, \quad \text{(6.5)}
\]
If \( k \leq n - i_1 \), then
\[
\tilde{w}_k = \begin{cases} b_k = a_{k+i_1}, & \text{if } k \notin \tilde{T}; \\ 0, & \text{if } k \in \tilde{T}, \end{cases} \quad \text{(6.6)}
\]
which is in fact \( w_{k+i_1} \).

It is straightforward to check that
\[
N(k, \tilde{I}) = N(k+i_1, I), \quad \text{(6.7)}
\]
\[
N(k, \tilde{J}^-) = N(k-(n-i_1+1), J^-), \quad N(k, \tilde{J}^+) = N(k+i_1, J^+), \quad \text{(6.8)}
\]
\[
N(k, \tilde{J}) = N(k, \tilde{J}^-) + N(k, \tilde{J}^+) \quad \text{(6.9)}
\]

Substituting (6.5) and (6.6) into (6.4) we have
\[
L(\tilde{T} \mid \tilde{I}) = \sum_{k=0}^{n-i_1} \left[ N(k, \tilde{I}) - N(k, \tilde{J}) \right] w_{k+i_1} + \sum_{k=n-i_1+1}^{n} \left[ N(k, \tilde{I}) - N(k, \tilde{J}) \right] a_{k-(n-i+1)}.
\]
By (6.7)–(6.9) the above equation becomes
\[
L(\tilde{T} \mid \tilde{I}) = \sum_{k=0}^{n-i_1} \left[ N(k+i_1, I) - N(k-(n-i_1+1), J^-) - N(k+i_1, J^+) \right] w_{k+i_1}
\]
\[
+ \sum_{k=n-i_1+1}^{n} \left[ N(k+i_1, I) - N(k-(n-i_1+1), J^-) - N(k+i_1, J^+) \right] a_{k-(n-i+1)}. \quad \text{(6.10)}
\]

If \( k \in [0, n - i_1] \) then \( k - (n - i_1 + 1) < 0 \). Thus \( N(k - (n - i_1 + 1), J^-) = 0 \).

If \( k \in [n - i_1 + 1, n] \) then \( k + i_1 > n \). Thus \( N(k + i_1, I) = m \) and \( N(k + i_1, J^+) = m - t \).
Therefore (6.10) reduces to

\[
L(T | I) = L\left(\tilde{T} | \tilde{I}\right) \\
= \sum_{k=0}^{n-i_1} \left[N(k + i_1, I) - N(k + i_1, J^+)\right] w_{k+i_1} + \sum_{k=n-i_1+1}^{n} \left[t - N(k - (n - i_1 + 1), J^-)\right] a_{k-(n-i_1+1)} \\
= \sum_{k=i_1}^{n} \left[N(k, I) - N(k, J^+)\right] w_k + \sum_{k=0}^{i_1-1} \left[t - N(k, J^-)\right] a_k.
\]

Then we obtain

\[
L^*(T | I) = t + L(T | I) = t + \sum_{k=i_1}^{n} \left[N(k, I) - N(k, J^+)\right] w_k + \sum_{k=0}^{i_1-1} \left[t - N(k, J^-)\right] a_k.
\]

\[\square\]

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