Uncertainty In Quantum Computation

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Abstract
We examine the effect of previous history on starting a computation on a quantum computer. Specifically, we assume that the quantum register has some unknown state on it, and it is required that this state be cleared and replaced by a specific superposition state without any phase uncertainty, as needed by quantum algorithms. We show that, in general, this task is computationally impossible.

1 Introduction
Quantum computing algorithms (for example, [3, 4]) work in two ways: (i) they apply an appropriate unitary transformation on a superposition state; (ii) they increase the amplitude of the state that represents the solution to the problem so that, upon interaction of a measuring device with the quantum register, the solution is available with a high probability. This conception proceeds without any uncertainty.

But there are uncertainties in all quantum description, which is why quantum information is not subject to the same rules as classical information [5]. When a measurement is made on one qubit (which may be imagined to be a photon with an unknown angle of polarization) that has not been examined before, it is as likely to be a 0 as a 1; when measured again from a different orientation, its uncertainty remains. If $\Delta I$ represents the change in information and $\Delta t$ represents the measurement step (counted in integers), one may speak of the following uncertainty constraint for information:
\[ \Delta I + \Delta t \geq 1 \]

This says that any unknown qubit will yield one bit of information, and when checked again, the result may be the same, but at the expense of the additional interaction. If observed frequently enough, the state will become frozen (the quantum Zeno effect).

There is also uncertainty related to the preparation of the initial state. In earlier papers [3, 4, 5], it was argued that the quantum computing model leaves out several elements of physical constraints and the problem of initial state preparation, making the paradigm computationally unrealistic [6]. In this analysis, it was assumed that somehow the quantum register will be constituted as qubits become available, one by one.

In the present paper, we consider the problem from a slightly different perspective, where the quantum register is supposed to have already been used for the solution of some problem. The task now is to prepare this register to start a new computation. The standard quantum algorithms require that the register be in a definite superposition state with, at worst, an unknown global phase.

Since the quantum register – howsoever it is implemented – is a single system, the question arises: Can one determine the state of this system, so that the appropriate unitary transformation can be carried out on it to take it to the desired initial state of the next computation? According to quantum mechanics, it is impossible to determine the unknown quantum wavefunction of a single system [1], so we must first interact with the register to reduce its wavefunction to some convenient eigenstate. But steering this eigenstate to the desired starting point of the computation is much more difficult than may be imagined. The precision of the macroscopic measurement device is limited because its own state can never be completely known and its interaction with the quantum register represents a many-to-one mapping (many register states mapped to the same measuring device state [10]). We show that, in general, one cannot steer the eigenstate to a specific superposition state without any unknown phases between the components.
2 The initial superposition state

Let there be a total number of \( n \) qubits. Without going into the question of quantum statistical constraints, the total number of component (eigenstates) states of interest from the point of view of computation is \( N = 2^n \).

When \( n = 2 \), for example, the number of component states is four, being \(|00\rangle, |01\rangle, |10\rangle, |11\rangle\). These states may be represented by \((1,0,0,0), (0,1,0,0), (0,0,1,0), \) and \((0,0,0,1)\), and they may be combined in an infinite variety of combinations by the use of complex weights to create superposition state functions.

The initial state, \(|R_i\rangle\), is normally taken to be \((1,0,0...0)\), whereas the final superposition state is taken to be \((1,1,1...1)\). However, no distinction is made between

\[
[R_i] = (1,1,1,...,1) \tag{1}
\]

and

\[
[R_i] = (1,e^{i\theta_1}, e^{i\theta_2},...,e^{i\theta_{N-1}}) \tag{2}
\]
even though they are distinct states.

To eliminate states with nonzero \( \theta \)'s, one confronts the problem of estimating an unknown wavefunction, a problem that is insoluble. What one can do is to test the wavefunction with an appropriate measuring device for its a priori values.

For example, consider the states \((1,1)\) and \((1,i)\). If viewed as photons, they each represent polarization at \(45^\circ\). If we perform the following unitary transformation:

\[
\begin{bmatrix}
\sqrt{3}/2 & -i/2 \\
 i/2 & -\sqrt{3}/2
\end{bmatrix}
\]

we are led to new superposition states with equal probabilities for \(|0\rangle\) and \(|1\rangle\) if the starting state is \((1,1)\); but probabilities of 93\% and 7\% for \(|0\rangle\) and \(|1\rangle\) if the starting state is \((1,i)\).

Clearly, we must insist on a distinction between various states, such as \((1)\) and \((2)\), if they have different phases amongst their components. But, there is no way of ensuring that a given superposition does not have relative phases, as in \((2)\), unless one does additional tests.
If one knew \textit{a priori} what the relative phase was, then one could remove it, as in the example where the qubit is $(1, e^{i\theta})$. Such a qubit can be aligned back if the unitary transformation
\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{e^{-i\theta}}{\sqrt{2}} \\
\frac{e^{i\theta}}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{bmatrix}
\] (4)
is employed. But since there is no way of knowing this unknown $\theta$, such an operation cannot be performed. It is a chicken-and-egg problem: If one knew what the phases were, one could get rid of them, but there is no way of \textit{a priori} knowing these phases.

3 Number of unitary operations

The state of the macroscopic measuring device, $M$, cannot be completely known. Upon measurement of (2) by it, the state on the register will be some random

$(0, 0, ..., 1, ..., 0)$

where the randomness is with regard to the location of the 1 in the vector. In other words, the initial measurement to “erase” the old information on the quantum register reduces the wavefunction to one of the $2^n$ component states. One would need an appropriate transformation out of a total of $2^n$ unitary transformations to steer this component state into the specific starting state of the quantum computer.

The cost of achieving the appropriate transformation increases exponentially with respect to the number of qubits.

If the erasing apparatus is not in \textit{perfect} alignment with the apparatus used to implement the unitary transformation and the final measurement, then we have a further complication. The reduced state will now be a superposition of the components of the quantum register, with $N - 1$ unknown phases. To steer this state to the starting superposition state would be \textit{impossible}, because the number of cases would be infinite.

The measuring apparatus, $M$, and the quantum register, $R$, have a joint state function that is a product of the individual states. When the measurement produces some eigenvalue of $M$, the register wavefunction is correspondingly reduced. But the state of the macroscopic apparatus can only be
determined incompletely, therefore there will be a residual uncertainty with regard to the knowledge obtained about the quantum register.

4 Conclusions

We have shown that if the measurement apparatus is perfectly aligned with the quantum register, the task of erasing the old information requires the use of one of $2^n$ unitary transformations. When the alignment is not perfect, the task is impossible. Since the measuring apparatus and the register cannot, in general, be aligned perfectly because of limitations of precision and noise, the problem of initializing the register is impossible to solve.

The information obtained about the register wavefunction is indirect, through observations on the macroscopic measuring apparatus $M$. But since information about it must remain incomplete, there exists a corresponding incompleteness in our knowledge of the register wavefunction.

Alter and Yamamoto assert[1]: “The information that can be obtained about the quantum wavefunction of a single system in a series of measurements cannot account for the physical reality (i.e., ontological meaning) of the wavefunction, and that the quantum wavefunction is limited to having a statistical (i.e., epistemological) meaning only.” This is another way of looking at the difficulty of erasing past information from a quantum register, emphasizing the fact that the wavefunction provides meaning when used for an ensemble of systems.

The precision with which one can know the wavefunction depends on the maximum information that can flow from the system into the measuring device. As the device can get into one of $2^n$ states, the total information that can be known about the wavefunction is $\log_2 2^n = n$ bits. The total uncertainty associated with the wavefunction, given that there exist $(N - 1)$ relative phases of arbitrary value, is without bound. Therefore, it is impossible to completely know the register wavefunction. This is in accord with the observation of Einstein, Tolman and Podolsky[2] that “the principle of the quantum mechanics must involve an uncertainty in the description of the past events which is analogous to the uncertainty in the prediction of future events.”
References

1. O. Alter and Y. Yamamoto, *Quantum Measurement of a Single System*. John Wiley, New York, 2001.

2. A. Einstein, R.C. Tolman, and B. Podolsky, “Knowledge of past and future in quantum mechanics,” *Physical Review* 37, 780 (1931).

3. A. Ekert and R. Jozsa, “Quantum computation and Shor’s factoring algorithm,” *Reviews of Modern Physics* 68, 733 (1996).

4. L.K. Grover, “Quantum mechanics helps in searching for a needle in a haystack,” *Physical Review Letters* 79, 325 (1997).

5. S. Kak, “Quantum information in a distributed apparatus,” *Foundations of Physics* 28, 1005 (1998). LANL Archive quant-ph/9804047.

6. S. Kak, “The initialization problem in quantum computing,” *Foundations of Physics* 29, 267 (1999). LANL Archive quant-ph/9805002.

7. S. Kak, “Rotating a qubit,” *Information Sciences* 128, 149 (2000). LANL Archive quant-ph/9910107.

8. S. Kak, “Statistical constraints on state preparation for a quantum computer,” *Pramana* 57, 683 (2001). LANL Archive quant-ph/0010109.

9. S. Kak, “Are quantum computing models realistic?” LANL Archive quant-ph/0110040.

10. H.D. Zeh, “On the interpretation of measurement in quantum theory,” *Foundations of Physics* 1, 69 (1970).