Noether conservation laws in quantum mechanics

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Abstract. Being quantized, conserved Noether symmetry functions are represented by Hermi-
tian operators in the space of solutions of the Schrödinger equation, and their mean values are
conserved.

Classical non-relativistic time-dependent mechanics can be described as a particular
field theory on a fibre bundle $Q \to \mathbb{R}$ over the time axis $\mathbb{R}$ [3, 4, 5]. Its configuration
space $Q$ is equipped with bundle coordinates $(t, q^i)$, where $t$ is the Cartesian coordinate
on $\mathbb{R}$ possessing transition functions $t' = t + \text{const}$. A fibre bundle $Q \to \mathbb{R}$ is trivial, but
its different trivializations correspond to different non-relativistic reference frames.

Noether conservation laws in Hamiltonian mechanics issue from the invariance of an
integral invariant of Poincaré–Cartan under one-parameter groups of bundle isomorphisms
of a configuration space $Q \to \mathbb{R}$ [6]. Therefore, being quantized, conserved Noether
symmetry functions commute with the Schrödinger operator. They act in the space of
solutions of the Schrödinger equation, and their mean values are conserved.

1 Noether conservation laws in classical Hamiltonian mechanics

The momentum phase space of non-relativistic mechanics is the vertical cotangent bundle
$V^*Q$ of $Q \to \mathbb{R}$ equipped with holonomic coordinates $(t, q^i, p_i)$ [3, 4, 5]. The cotangent
bundle $T^*Q$ of $Q \to \mathbb{R}$ coordinated by $(t, q^i, p, p_i)$ plays a role of the homogeneous
momentum phase space. It is provided with the canonical Liouville form $\Xi = pdt + p_i dq^i$ and
the canonical symplectic form $\Omega = d\Xi$. The corresponding Poisson bracket reads

$$\{f, f'\}_T = \partial^p f \partial_t f' + \partial^i f \partial_i f' - \partial_t f \partial^p f' - \partial_i f \partial^i f', \quad \partial^p = \partial/\partial p, \quad f, f' \in C^\infty(T^*Q).$$ (1)
There is the trivial affine bundle

$$\zeta : T^*Q \to V^*Q.$$  \hfill (2)

Due to this fibration, the vertical cotangent bundle $V^*Q$ is provided with the canonical Poisson bracket

$$\{f, f'\}_V = \partial^i f \partial^i f' - \partial_i f \partial^i f', \quad f, f' \in C^\infty(V^*Q),$$  \hfill (3)

such that

$$\zeta^*\{f, f'\}_V = \{\zeta^* f, \zeta^* f'\}_T,$$  \hfill (4)

where $\zeta^* f$ denotes the pull-back onto $T^*Q$ of a function $f$ on $V^*Q$.

A Hamiltonian of non-relativistic time-dependent mechanics is defined as a section

$$h : V^*Q \to T^*Q, \quad p \circ h = -\mathcal{H}(t, q^i, p_i)$$  \hfill (5)

of the fibre bundle (2). The pull-back $h^*\Xi$ onto $V^*Q$ of the Liouville form $\Xi$ by means of a section $h$ (5) is the well-known integral invariant of Poincaré–Cartan

$$H = p_i dq^i - \mathcal{H}dt.$$  \hfill (6)

We agree to call it a Hamiltonian form. There exists a unique vector field $\gamma_H$ on $V^*Q$ such that

$$d_t|\gamma_H = 1, \quad \gamma_H|dH = 0,$$

$$\gamma_H = \partial_t + \partial^i \partial_t q^i - \partial_i \mathcal{H} \partial^i.$$  \hfill (7)

It defines the first order Hamilton equation

$$d_t q^i = \partial^i \mathcal{H}, \quad d_t p_i = -\partial_i \mathcal{H}$$  \hfill (8)

on $V^*Q$, where $d_t = \partial_t + q^i \partial_i + p_i \partial^i$ is the total derivative written with respect to the adapted coordinates $(t, q^i, p_i, q^i_t, p_i_t)$ on the jet manifold $J^1V^*Q$ of the fibre bundle $V^*Q \to \mathbb{R}$. Accordingly, a smooth real function $f$ on $V^*Q$ is an integral of motion if its Lie derivative

$$L_{\gamma_H} f = \gamma_H|f = (\partial_t + \partial^i \partial_t q^i - \partial_i \mathcal{H} \partial^i)f$$  \hfill (9)

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along $\gamma_H$ vanishes.

In an equivalent way, let us consider the pull-back $\zeta^* H$ of the Hamiltonian form $H$ (6) onto $T^* Q$, and let us define the function

$$H = \partial_t \Xi - \zeta^* H = p + H$$

(10)
on $T^* X$. Then, the relation

$$\zeta^* (L_\gamma H) = \{H, \zeta^* f\}_T$$

(11)
holds for any smooth real function $f$ on $V^* Q$. In particular, $f$ is an integral of motion iff the bracket $\{H, \zeta^* f\}_T$ vanishes. We agree to call $H$ (10) the homogeneous Hamiltonian.

For the sake of simplicity, we will further denote the pull-back $\zeta^* f$ onto $T^* Q$ of a function $f$ on $V^* Q$ by the same symbol $f$, and will identify the Poisson algebra $C^\infty(V^* Q)$ with the Poisson subalgebra $\zeta^* C^\infty(V^* Q)$ of $C^\infty(T^* Q)$.

A Noether conservation law in Hamiltonian mechanics issue from the invariance of a Hamiltonian form $H$ (6) under a one-parameter groups of bundle automorphisms of a configuration space $Q \to \mathbb{R}$ [3, 4, 6]. Its infinitesimal generator is a projectable vector field

$$u = u^i \partial_i + u^i(t, q^j) \partial_i$$

(12)
on $Q \to \mathbb{R}$, where $u^i = 0, 1$ because time reparametrizations are not considered. If $u^i = 0$, we have a vertical vector field $u = u^i \partial_i$ which takes its values into the vertical tangent bundle $VQ$ of $Q \to \mathbb{R}$. If $u^i = 1$, a vector field $u$ (12) is a connection on the configuration bundle $Q \to \mathbb{R}$. Note that connections

$$\Gamma = \partial_t + \Gamma^i(t, q^j) \partial_i$$

(13)
on $Q \to \mathbb{R}$ make up an affine space modelled over the vector space of vertical vector fields on $Q \to \mathbb{R}$, i.e., the sum of a connection and a vertical vector field is a connection, while the difference of two connections is a vertical vector field on $Q \to \mathbb{R}$.

Any projectable vector field $u$ (12) on $Q \to \mathbb{R}$ admits the canonical lift

$$\tilde{u} = u^i \partial_i + u^i \partial_i - p_j \partial_i u^j \partial^i$$

(14)
on $V^* Q$. It generates a one-parameter group of holonomic bundle automorphisms of the momentum phase space $V^* Q \to \mathbb{R}$. The Hamiltonian form $H$ (6) is invariant under
this group of automorphisms iff its Lie derivative
\[ \mathbf{L}_u H = \tilde{u} \] \[ H + d(\tilde{u} \] \[ H) = (\partial_i(p_i u^i - u^i H) - u^i \partial_i H + p_j \partial_i u^j \partial^i H)dt \] (15)
along \( \tilde{u} \) (14) vanishes. There is equality
\[ \mathbf{L}_u H = -\gamma_H]d\mathbf{T}_u, \] (16)
where
\[ \mathbf{T}_u = u^i \mathcal{H} - u^i p_i \] (17)
is called the Noether symmetry function associated to a vector field \( u \). If the Lie derivative \( \mathbf{L}_u H \) vanishes, then
\[ \gamma_H]d\mathbf{T}_u = \{ H, \mathbf{T}_u \}_T = 0, \] (18)
and the symmetry function \( \mathbf{T}_u \) (17) is an integral of motion. One can treat the equality as the Noether conservation law \( d_t \mathbf{T}_u \approx 0 \) on the shell (8) [3, 6].

Since \( u^i = 0, 1 \), there are the following two types of Noether symmetry functions (17). If \( u = v = v^i \partial_i \) is a vertical vector field on \( Q \to \mathbb{R} \), then the symmetry function
\[ -\mathbf{T}_v = v^i p_i \] (19)
is the momentum along \( v \).

If \( u = \Gamma = \partial_t + \Gamma^i \partial_i \) is a connection, the corresponding symmetry function
\[ \mathbf{T}_\Gamma = \mathcal{H} - p_i \Gamma^i. \] (20)
is an energy function. However, it need not be a true physical energy. There are different energy functions \( \mathbf{T}_\Gamma \) (20) corresponding to different connections \( \Gamma \) on \( Q \to \mathbb{R} \). Moreover, if an energy function \( \mathbf{T}_\Gamma \) is an integral of motion and the momentum \( \mathbf{T}_v \) (19) is so, the energy function \( \mathbf{T}_{\Gamma + v} = \mathbf{T}_\Gamma + \mathbf{T}_v \) is also an integral of motion.

**Example 1.** Let us consider a one-dimensional motion of a point particle subject to friction. It is described by the dynamic equation
\[ q_{tt} = -k q_t, \quad k > 0, \] (21)
on the configuration space \( Q = \mathbb{R}^2 \to \mathbb{R} \) coordinated by \( (t, q) \). This equation is equivalent to the Lagrange equation of the Lagrangian
\[ L = \frac{1}{2} \exp[kt]q_t^2 dt. \]
It is a hyperregular Lagrangian. The unique associated Hamiltonian form reads
\[ H = pdq - \frac{1}{2} \exp[-kt]p^2 dt. \] (22)
The corresponding Hamilton equation
\[ q_t = \exp[-kt]p, \quad p_t = 0 \]

is equivalent to the dynamic equation (21). Let us consider the vector field
\[ \Gamma = \partial_t - \frac{k}{2} q \partial_q \]
on $Q \to \mathbb{R}$. Its prolongation (14) onto $V^*Q$ reads
\[ \bar{\Gamma} = \partial_t - \frac{k}{2} q \partial_q + \frac{k}{2} p \partial_p. \]

It is readily observed that the Lie derivative of the Hamiltonian form (22) with respect to this vector field vanishes. Then, the energy function
\[ T_{\bar{\Gamma}} = \frac{1}{2}(\exp[-kt]p^2 + kqp) \] (23)
is an integral of motion. Another integral of motion is the momentum $T_u = -p$ along the vertical vector field $u = \partial_q$.

Note that, given a connection $\Gamma$ (13), there exist bundle coordinates on $Q \to \mathbb{R}$ such that $\Gamma_\gamma = 0$. A glance at the expression (15) shows that the Lie derivative $L_{\bar{\Gamma}}H$ vanishes iff the Hamiltonian $\mathcal{H}$ written with respect to these coordinates is independent of time.

In Example 1, such a coordinate is
\[ q' = \exp\left[\frac{k}{2} t\right] q. \] (24)

Accordingly, we have
\[ p' = \exp\left[-\frac{k}{2} t\right] p. \] (25)
The Hamiltonian form (22) with respect to these coordinates is
\[ H = p' dq' - \frac{1}{2} (p'^2 + q' p') dt. \] (26)
2 Quantum Noether conservation laws

In order to quantize non-relativistic time-dependent mechanics, we provide geometric quantization of the cotangent bundle $T^*Q$ with respect to vertical polarization which is the vertical tangent bundle $VT^*Q$ of $T^*Q \to Q$ (see [1, 2] for a detailed exposition). This quantization is compatible with the Poisson algebra monomorphism $C^\infty(V^*Q) \to C^\infty(T^*Q)$. The corresponding quantum algebra consists of smooth functions which are affine in momenta $p, p^i$. Moreover, we restrict our consideration to its subalgebra $A$ of functions

$$f = a^t p + a^i(t, q^j)p_i + b(t, q^j), \quad a^t = 0, 1. \tag{27}$$

These functions are represented by the operators

$$\hat{f} = -ia^t \partial_t - ia^k \partial_k - i\frac{2}{2} \partial_k a^k + b \tag{28}$$

which act in the space $E$ of sections $\psi$ of the fibre bundle $D \to Q$ whose restriction $\psi_t$ to each fibre $Q_t$ of $Q \to \mathbb{R}$ are complex half-forms of compact support on $Q_t$. This space is provided with the structure of a pre-Hilbert $C^\infty(\mathbb{R})$-module with respect to the non-degenerate Hermitian forms

$$\langle \psi_t | \psi'_t \rangle_t = \int_{Q_t} \psi_t \psi'_t.$$ 

One can use the formal coordinate expression

$$\psi = \psi(t, q^i)(dq^1 \wedge \cdots \wedge dq^m)^{1/2}.$$ 

The operators (28) obey the Dirac condition

$$[\hat{f}, \hat{f}'] = -i\{\hat{f}, \hat{f}'\}_T.$$ 

They are Hermitian because of the equality

$$\langle \hat{f} \psi_t | \psi_t \rangle_t - \langle \psi_t | \hat{f} \psi_t \rangle_t =$$

$$\int_{Q_t} [\psi(-ia^t \partial_t - ia^k \partial_k - i\frac{2}{2} \partial_k a^k + b)\psi - \psi(ia^t \partial_t + ia^k \partial_k + i\frac{2}{2} \partial_k a^k + b)\overline{\psi}] =$$

$$-ia^t \partial_t \langle \psi_t | \psi_t \rangle_t - i \int_{Q_t} \partial_k (a^k \psi \overline{\psi}) = -ia^t \partial_t \langle \psi_t | \psi_t \rangle_t.$$


It should be emphasized that the homogeneous Hamiltonian \( H \) (10) need not belong to the quantum algebra \( \mathcal{A} \), unless \( H \) is affine in momenta \( p^i \). Let us further assume that \( H \) is a polynomial of momenta. One can show that, in this case, \( H \) can be represented by a product of affine functions (27) and, consequently, can be quantized as a Hermitian element

\[
\hat{H} = -i\partial_t + \hat{H} \quad (29)
\]
of the enveloping algebra of the Lie algebra \( \mathcal{A}[1,2] \). However, this representation and the corresponding quantization (29) fail to be unique. The operator (29) yields the Schrödinger equation

\[
\hat{H}\psi = (-i\partial_t + \hat{H})\psi = 0, \quad \psi \in E. \quad (30)
\]

Given quantizations (28) and (29), any Noether symmetry function \( T_u \) (17) is quantized by the Hermitian operator

\[
\hat{T}_u = u^t \hat{H} - u^t \hat{p}_i \quad (31)
\]
in the pre-Hilbert module \( E \). If \( T_u \) is an integral of motion, the operator \( \hat{T}_u \) (31) commutes with the Schrödinger operator \( \hat{H} \), and acts in the subspace \( \ker \hat{H} \subset E \) of solutions of the Schrödinger equation (30).

In particular, let \( u = v = v^k \partial_k \) and \( T_v = -v^k(t,q^j)p_k \). Since the operator \( \hat{T}_v = iv^k(t,q^j)\partial_k \) commutes with the Schrödinger operator \( \hat{H} \) (30), we obtain the equalities

\[
\partial_t \langle \hat{T}_v \psi | \psi \rangle = \int_{Q_t} \left[ \langle i\partial_t v^k \partial_k \psi | \psi \rangle + \langle iv^k \partial_k \partial_t \psi | \psi \rangle + (iv^k \partial_k \psi) \partial_t \psi \rangle \right] = \\
\int_{Q_t} \left[ \langle i\partial_t v^k \partial_k \psi | \psi \rangle + v^k \partial_k (\hat{H}\psi | \psi) \rangle - (v^k \partial_k \psi) (\hat{H}\psi) \rangle \right] = - \int_{Q_t} (\hat{H}, \hat{T}_v | \psi \rangle \psi = 0
\]
on the \( \ker \hat{H} \). It follows that the mean values of the operator \( \hat{T}_v = iv^k \partial_k \) on solutions of the Schrödinger equation (30) are conserved.

Let \( u = \Gamma \) and \( T_\Gamma = \hat{H} - p_k \Gamma^k \). Let us choose bundle coordinates \( q^j \) such that \( \Gamma^k = 0 \). Then, \( \hat{T}_\Gamma = \hat{H} \) is independent of time. Since \( \hat{H} = -i\partial_t + \hat{T}_\Gamma \), the Schrödinger equation (30) comes to the conservative one

\[
\psi = \exp(-i\mathcal{E}t)\phi_E, \quad \hat{T}_\Gamma \phi_E = \mathcal{E} \phi_E. \quad (32)
\]
For instance, the Schrödinger equation for a point particle subject to friction in Example 1 reads

$$-\frac{1}{2}(\partial_{q'}^2 + iq'\partial_{q'})\phi_E = \mathcal{E}\phi_E.$$ 

References

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