A sufficient condition for a pair of bi-Lipschitz homeomorphic manifolds to be diffeomorphic*

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Abstract

We prove that if a bi-Lipschitz map between compact manifolds and its inverse map have no singular point in the sense of F.H. Clarke, respectively, then they are diffeomorphic.

1 Introduction

We can not always approximate a homeomorphism between smooth manifolds by a diffeomorphism. The clear difference between the two is due to the existence of exotic structures. E.g., Milnor’s exotic spheres [M], Donaldson’s exotic 4-dimensional spaces [D], etc. Nevertheless, it is natural to have our expectation for a pair of bi-Lipschitz homeomorphic manifolds to be diffeomorphic, because the Lipschitz category is “differentially close” to the diffeomorphic one, which means that the differential or the Jacobi matrix of a Lipschitz map exists almost everywhere as a consequence of Rademacher’s theorem [R].

Shikata would be the first researcher who carried out the expectation in the differentiable pinching problem. Note that the homeomorphisms constructed in the proofs of a few sphere theorems are bi-Lipschitz ones. In [Sh1], he introduced a pseudo distance for a pair of compact differentiable manifolds which are bi-Lipschitz homeomorphic and proved that if the distance between such a pair is smaller than a certain positive constant, then the bi-Lipschitz map can be approximated via diffeomorphisms, i.e., the manifolds are diffeomorphic. Moreover, as application to the differentiable pinching problem, he proved in [Sh2] that there exists a certain constant $\delta(n) \in (1/4, 1)$ depending on a number $n$ such that if sectional curvature of a simply connected, compact Riemannian manifold $M$ of dimension $n$ is $\delta(n)$-pinched, then $M$ is diffeomorphic to the standard sphere. What astonishes us is that he defined such a distance between two manifo lds for getting the differentiable sphere theorem more than ten years before Gromov’s Hausdorff distance in [G]. Recently, Brendle and Schoen [BS] completely solved the differentiable pinching

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problem. Their main tool was the Ricci flow method and they showed that the method is very powerful in the proof.

The purpose of this article is to carry out the expectation above from the non-smooth analysis’s standpoint. The non-smooth analysis, i.e., a non-singular point for a Lipschitz function/map, was established by F.H. Clarke in [C1] and [C2]. It has been a strong tool in the optimal control theory (cf. [C3]): Let $M, N$ be complete Riemannian manifolds, respectively, and let $F : M \to N$ be a Lipschitz map. In case $N = \mathbb{R}$, let $f := F : M \to \mathbb{R}$. By Rademacher’s theorem [R], there exists a set $E_F \subset M$ of measure zero such that the differential $dF$ of $F$ exists on $M \setminus E_F$. Then, for each point $x$, there exists a sequence $\{x_i\}$ of $x_i \in M \setminus E_F$ convergent to $x$, and hence we can define the generalized Jacobian $\partial F(x)$ of $F$ at $x \in M$ as follows:

$$\partial F(x) := \text{Conv}(\{\lim_{i \to \infty} dF_{x_i} \mid dF_{x_i} \text{ exists as } x_i \in M \setminus E_F \to x\}),$$

where “Conv(·)” means “convex hull”. In case $N = \mathbb{R}$, we call $\partial f(x)$, as alternate terminology, the generalized gradient of $f$ at $x$, i.e.,

$$\partial f(x) := \text{Conv}(\{\lim_{i \to \infty} \nabla f(x_i) \mid \nabla f(x_i) \text{ exists as } x_i \in M \setminus E_f \to x\}).$$

Here $\nabla f$ denotes the gradient vector field of $f$. Note that both of $\partial F(x)$ and $\partial f(x)$ are compact convex sets.

**Definition 1.1** ([C1], [C2]) Let $M, N$ be the same above, and $U \subset M$ an open set. Then

1. A point $x \in U$ is called non-singular of a Lipschitz map $F : U \to N$, if every matrix in $\partial F(x)$ is of maximal rank;

2. A point $x \in U$ is called non-critical of a Lipschitz function $f : U \to \mathbb{R}$, if

$$o \notin \partial f(x),$$

where $o$ denotes the zero tangent vector at $x$.

**Remark 1.2** It should be emphasized that the reason why Clarke introduced this notion was not for approximations of a bi-Lipschitz map via diffeomorphism, but for the inverse function theorem for a Lipschitz map, which contains the classical one as a special case. In fact, he proved the existence of a local Lipschitzian inverse at a non-singular point of a Lipschitz map (See [C2] Theorem 1]).

**Example 1.3** We will give three examples of Definition 1 with a few related remarks:

1. Consider two functions $f_1(x) := x^2$, $f_2(x) := x + 2$ on $(-2, 3)$, respectively. Define a Lipschitz function $f(x) := \max\{f_1(x), f_2(x)\}$ on $(-2, 3)$, i.e.,

$$f(x) = \begin{cases} x^2 & \text{on } (-2, -1) \cup (2, 3) \\ x + 2 & \text{on } [-1, 2] \end{cases}$$
Note that $f$ is not differentiable at $x = -1, 2$. Since
\[
\lambda \frac{df_1}{dx}(-1) + (1 - \lambda) \frac{df_2}{dx}(-1) = -3\lambda + 1
\]
for all $\lambda \in [0, 1]$, we have $\partial f(-1) = [-2, 1]$. As well as above, we have $\partial f(2) = [1, 4]$. Since $0 \in \partial f(-1)$ and $0 \notin \partial f(2)$, $x = -1$ is a critical point of $f$ and $x = 2$ is not that of $f$. Note that $f(-1) = 1$ is the minimum value of $f$.

(2) \text{(C2, Remark 1)} Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a Lipschitz map defined by $F(x, y) := (|x| + y, 2x + |y|)$. Note that $F$ is not differentiable at $(x, y) = (0, 0)$. Then,
\[
\partial F(0, 0) = \text{Conv}\left( \left\{ \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \right\} \right)
\]
\[
= \left\{ \begin{pmatrix} s & 1 \\ 2 & t \end{pmatrix} \middle| |s| \leq 1, |t| \leq 1 \right\}.
\]
Thus, $(0, 0)$ is non-singular of $F$. Hence, $F$ admits a local Lipschitzian inverse around the origin.

(3) \text{(Critical points of Grove-Shiohama [GS])} Let $M$ be the same above, and let $d$ be the distance function of $M$. Recall that, for a fixed point $p \in M$, $x \in M \setminus \{p\}$ is called a critical point of $d_p(\cdot) := d(p, \cdot)$ (or critical point for $p$) if, for every nonzero tangent vector $v \in T_p M$ at $x$, we find a minimal geodesic segment $\gamma$ emanating from $x$ to $p$ satisfying $\angle(v, \dot{\gamma}(0)) \leq \pi/2$. Here $\angle(v, \dot{\gamma}(0))$ is the angle between $v$ and $\dot{\gamma}(0)$. Assume that $q \in M$ is not a critical point of $d_p$. By definition, there exists $w \in T_q M \setminus \{0\}$ such that $\angle(w, \dot{\gamma}(0)) > \pi/2$ holds for any minimal geodesic segment $\gamma$ emanating from $q$ to $p$. Hence, $o \notin \partial d_p(q)$. Note that a point $q \neq p$ is a critical point of $d_p$ if and only if $q$ is a critical point of $d_p$ in the sense of Clarke. Since $\partial d_p(p)$ equals the unit sphere in the tangent space $T_p M$ at $p$, $p$ is a critical point of $d_p$ in the sense of Clarke. Remark that, even if $o \notin \partial d_p(q)$, it is possible to occur that $q$ is a cut point of $p$ in general.

Now, our main theorems are stated as follows:

**Main Theorem A.** Let $F$ be a bi-Lipschitz homeomorphism from a compact Riemannian manifold $M$ onto a Riemannian manifold $N$. If $F$ and $F^{-1}$ have no singular point on $M$ and $N$ (in the sense of Clarke), respectively, then $M$ and $N$ are diffeomorphic.

Main Theorem A is an easy consequence of the following Main Theorem B.

**Main Theorem B.** Let $F : M \rightarrow N$ be a Lipschitz map from a compact Riemannian manifold $M$ into a complete Riemannian manifold $N$. Assume that $F$ has no singular point on $M$. If $\dim M \leq \dim N$, then for any $\eta > 0$, there exists an (smooth) immersion $f_\eta$ from $M$ into $N$ such that $\max_{p \in M} d_N(f_\eta(p), F(p)) < \eta$ and the Lipschitz constant $\text{Lip}(f_\eta)$ of $f_\eta$ is not greater than $\text{Lip}(F)(1 + \eta)$. Here, $\text{Lip}(F)$ denotes the Lipschitz constant of $F$ defined by
\[
\text{Lip}(F) := \sup \left\{ \frac{d_N(F(x), F(y))}{d_M(x, y)} \middle| x, y \in M, x \neq y \right\},
\]
where $d_M$ and $d_N$ are the distance functions of $M$ and $N$, respectively.
2 Approximations of a Lipschitz map via immersions

We treat a Lipschitz map $F$ from a compact Riemannian manifold $M$ of dimension $n$ into a complete Riemannian manifold $N$ of dimension $k$. And we prove Main Theorem B as Theorem 2.12. Our technique here looks that in [GS] at a first glance. However, there is a big difference between us and them: Our smoothing technique bases on the partition of unity, while that in [GS] depends on the center of mass technique constructed by Grove and Karcher [GK]. Our spirits would rather be in Shikata’s approach ([Sh1]) than [GS].

Take any point $p \in M$, and fix it. Since the exponential map $\exp_p$ on the tangent space $T_pM$ of $M$ at $p$ is a diffeomorphism from $\mathbb{B}_r(o_p)$ onto $B_r(p)$ for a sufficiently small $r > 0$, we denote by $\exp_p^{-1}$ the inverse map of $\exp_p|_{\mathbb{B}_r(o_p)}$. Here we set $\mathbb{B}_r(o_p) := \{ v \in T_pM \mid \|v\| < r \}$ and $B_r(p) := \{ q \in M \mid d_M(p,q) < r \}$, where $o_p$ denotes the origin of $T_pM$ and $d_M$ denotes the distance function of $M$. In what follows, we identify $T_pM$ with Euclidean $n$-dimensional space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, and let $N$ be embedded into Euclidean $m$-dimensional space $(\mathbb{R}^m, \langle \cdot, \cdot \rangle)$, where $m \geq k + 1$. We may assume that $N$ is isometrically embedded into $\mathbb{R}^m$ by introducing the induced metric from the space. The Lipschitz map $F$ above is therefore a map from $M$ into $\mathbb{R}^m$. Then, we may define a smooth approximation of $F$ on a convex ball $B_r(p)$ of radius $r$, centered at each point $p \in M$.

**Definition 2.1** For each $\varepsilon > 0$, let $F_{\varepsilon}^{(p)} : B_r(p) \rightarrow \mathbb{R}^m$ denote the map defined by

$$
F_{\varepsilon}^{(p)}(q) := \int_{\mathbb{R}^n} \rho_\varepsilon(y)F(q(y))dy = \int_{\mathbb{R}^n} \rho_\varepsilon(\exp_p^{-1}q - y)F(\exp_p(y))dy,
$$

where $q(y) := \exp_p(\exp_p^{-1}q - y)$ and $\rho_\varepsilon$ denotes the mollifier.

**Lemma 2.2** For any $\varepsilon > 0$ and any $q \in B_r(p)$,

$$
\|F_{\varepsilon}^{(p)}(q) - F(q)\| \leq \varepsilon \cdot \text{Lip}(F) \cdot \text{Lip}(\exp_p|_{\mathbb{B}_{r+\varepsilon}(o_p)})
$$

holds, where $\| \cdot \|$ denotes the Euclidean norm of $\mathbb{R}^m$ and $\text{Lip}(\exp_p|_{\mathbb{B}_{r+\varepsilon}(o_p)})$ denotes the Lipschitz constant of $\exp_p|_{\mathbb{B}_{r+\varepsilon}(o_p)}$, i.e.,

$$
\text{Lip}(\exp_p|_{\mathbb{B}_{r+\varepsilon}(o_p)}) := \sup \left\{ \frac{d_M(\exp_p v, \exp_p w)}{|v - w|} \mid v, w \in \mathbb{B}_{r+\varepsilon}(o_p), \ v \neq w \right\}.
$$

**Proof.** Take any $y \in \mathbb{B}_\varepsilon(o_p)$, and fix it. By the triangle inequality,

$$
\| \exp_p^{-1}q - y\| \leq \| \exp_p^{-1}q \| + \| y \| < r + \varepsilon.
$$

Since two vectors $\exp_p^{-1}q - y, \exp_p^{-1}q \in \mathbb{B}_{r+\varepsilon}(o_p)$ for all $y \in \mathbb{B}_\varepsilon(o_p)$, we have

$$
d_M(q(y), q) \leq \text{Lip}(\exp_p|_{\mathbb{B}_{r+\varepsilon}(o_p)})\| (\exp_p^{-1}q - y) - \exp_p^{-1}q \|
$$

$$
= \text{Lip}(\exp_p|_{\mathbb{B}_{r+\varepsilon}(o_p)})\| y \|.
$$

Since $\int_{\mathbb{R}^n} \rho_\varepsilon(y)dy = 1$, we have $F(q) = \int_{\|y\| \leq \varepsilon} F(q)\rho_\varepsilon(y)dy$. Hence,

$$
\|F_{\varepsilon}^{(p)}(q) - F(q)\| \leq \int_{\|y\| \leq \varepsilon} \| F(q(y)) - F(q) \| \rho_\varepsilon(y)dy
$$

$$
\leq \text{Lip}(F) \int_{\|y\| \leq \varepsilon} d_M(q(y), q) \rho_\varepsilon(y)dy.
$$
where note that \( \|y\| < \varepsilon \) with \( \rho_\varepsilon(y) \neq 0 \). Combining (2.3) with (2.4), we get (2.2). \( \square \)

Since \( M \) is compact, we can choose finitely many convex balls \( B_{r_i}(p_i), i = 1, 2, \ldots, \ell, \) which cover \( M \). Take a partition of unity \( \varphi_i \), subordinate to \( \{ B_{r_i}(p_i) \} \) so that \( \text{supp} \varphi_i \subset B_{r_i}(p_i) \) for each \( i \). Then, for each \( \varepsilon > 0 \), we define a global approximation \( F_\varepsilon \) of \( F \) by

\[
F_\varepsilon(q) = \sum_{i=1}^\ell \varphi_i(q) F^{(p_i)}_\varepsilon(q).
\]

**Lemma 2.3** For any \( \varepsilon > 0 \) and any \( q \in M \), we have

\[
\|F_\varepsilon(q) - F(q)\| \leq \varepsilon \cdot \text{Lip}(F) \sum_{i=1}^\ell \varphi_i(q) \text{Lip}(\exp_{p_i}|_{B_{r_i+\varepsilon}(p_i)}).
\]

**Proof.** Since \( \sum_{i=1}^\ell \varphi_i = 1 \), we get \( F(q) = \sum_{i=1}^\ell \varphi_i(q) F(q) \). Hence, by Lemma 2.2 and the triangle inequality, we have the desired inequality. \( \square \)

In what follows, for a pair of points \( p \) and \( q \) of \( M \) or \( N \) admitting a unique minimal geodesic segment \( \gamma \), we denote by \( \tau_q^p \) the parallel transformation from the tangent space at \( p \) onto the tangent space at \( q \) along \( \gamma \).

**Lemma 2.4** Let \( q \in \text{supp} \varphi_i \). Then, for any \( \tilde{u} \in S_q^{n-1} := \{ v \in T_qM \mid \|v\| = 1 \} \),

\[
\|dF^{(p_i)}_\varepsilon(\tilde{u})\| \leq \text{Lip}(F) (1 + \sup_{y \in B_\varepsilon(o_{p_i})} \|Y_{y}(\tilde{u}) - \tau_{q_i(y)}^q(\tilde{u})\|),
\]

holds, where \( q_i(y) := \exp_{p_i}(\exp_{p_i}^{-1} q - y) \), and \( Y_{y}(\tilde{u}) := \frac{\partial}{\partial s_{y}}|_{s=0} \exp_{p_i} t \left( \exp_{p_i}^{-1}(\exp_{q_i} s\tilde{u}) - y \right) \)

is a Jacobi field along the geodesic \( \exp_{p_i} t \left( \exp_{p_i}^{-1} q - y \right) \).

**Proof.** From (2.1), it is easy to obtain

\[
dF^{(p_i)}_\varepsilon(\tilde{u}) = \int_{\mathbb{R}^n} \rho_\varepsilon(y) dF_{q_i(y)}(Y_{y}(\tilde{u}))(1) dy
\]

and

\[
dF_{q_i(y)}(Y_{y}^{{\tilde{u}}}(1)) = dF_{q_i(y)}(Y_{y}^{{\tilde{u}}}(1) - \tau_{q_i(y)}^q(\tilde{u}))(1) + dF_{q_i(y)}(\tau_{q_i(y)}^q(\tilde{u})).
\]

Hence, by the triangle inequality, we get (2.6). \( \square \)

**Lemma 2.5** For any \( \eta > 0 \), there exists a number \( \varepsilon_i(\eta) > 0 \) such that

\[
\sup\{ \| Y_{y}^{{\tilde{u}}}(1) - \tau_{q_i(y)}^q(\tilde{u})\| \mid q \in \text{supp} \varphi_i, \tilde{u} \in S_q^{n-1}, y \in B_{\varepsilon_i(\eta)}(o_{p_i}) \} < \eta.
\]

**Proof.** Let \( q \in \text{supp} \varphi_i \). Then, take any \( \tilde{u} \in S_q^{n-1} \), and fix it. Since \( Y_{y}^{{\tilde{u}}}(1) = \tau_{q_i(y)}^q(\tilde{u}) = \tilde{u} \) for \( y = o_{p_i} \), there exists a number \( \varepsilon(p_i, q, \tilde{u}) > 0 \) such that \( \| Y_{y}^{{\tilde{u}}}(1) - \tau_{q_i(y)}^q(\tilde{u})\| < \eta \) holds for all \( y \in B_{\varepsilon(p_i, q, \tilde{u})}(o_{p_i}) \). Since \( \text{supp} \varphi_i \) and \( S_q^{n-1} \) are compact, respectively, we get the desired number \( \varepsilon_i(\eta) > 0 \). \( \square \)
Lemma 2.6 For any \( \eta > 0 \), there exists a constant number \( \varepsilon(\eta) > 0 \) such that

\[
\|dF_\varepsilon(\tilde{u})\| \leq (1 + \eta) \text{Lip}(F)
\]

holds for all \( \varepsilon \in (0, \varepsilon(\eta)) \) and all unit tangent vectors \( \tilde{u} \) on \( M \).

Proof. Take any \( \tilde{u} \in S^{n-1}_q \). Since \( \sum_{i=1}^{\ell} \varphi_i = 1 \) on \( M \), we get \( \sum_{i=1}^{\ell} d\varphi_i(\tilde{u}) = 0 \). Hence,

\[
dF_\varepsilon(\tilde{u}) = \sum_{i=1}^{\ell} \varphi_i(q) dF^{(\varphi_i)}_\varepsilon(\tilde{u}) + \sum_{i=1}^{\ell} d\varphi_i(\tilde{u})(F^{(\varphi_i)}_\varepsilon(q) - F(q)).
\]

By applying the triangle inequality to the equation above, we have

\[
\|dF_\varepsilon(\tilde{u})\| \leq \sum_{i=1}^{\ell} \varphi_i(q) \|dF^{(\varphi_i)}_\varepsilon(\tilde{u})\| + \sum_{i=1}^{\ell} |d\varphi_i(\tilde{u})| \cdot \|F^{(\varphi_i)}_\varepsilon(q) - F(q)\|.
\]

By Lemmas 2.2, 2.4 and 2.5 we get (2.9) for all sufficiently small \( \varepsilon > 0 \). \( \square \)

From now on, we assume that \( n = \dim M \leq \dim N = k \).

Lemma 2.7 For each non-singular point \( p \in M \) of \( F \), there exist positive constants \( r(p) \) and \( \delta(p) \) such that, for any \( u \in S^{n-1}_p \), there exists a local unit vector field \( V \) on a neighborhood of \( F(p) \) satisfying

\[
\langle dF_q(\tau^p_q(u)), V_{F(q)} \rangle \geq \delta(p)
\]

for almost all \( q \in B_{2r(p)}(p) \).

Proof. Choose convex balls \( B_{r_1}(p) \) and \( B_{r_2}(F(p)) \) respectively so as to satisfy \( F(B_{r_1}(p)) \subset B_{r_2}(F(p)) \). Since the point \( p \) is a non-singular point of \( F \), there exist positive numbers \( r(p) \in (0, r_1/2) \) and \( \delta(p) \) satisfying the following property. For any \( u \in S^{n-1}_p \) at the point \( p \), there exists a unit vector \( v \) at \( F(p) \) such that \( \langle \tau^{F(q)}_p \circ dF_q \circ \tau^p_q(v), u \rangle \geq \delta(p) \) for almost all \( q \in B_{2r(p)}(p) \). Hence, we get \( \langle dF(\tau^p_q(u)), V_{F(q)} \rangle \geq \delta(p) \) for almost all \( q \in B_{2r(p)}(p) \), where \( V_{F(q)} := \tau^{F(p)}_q(v) \). \( \square \)

Henceforth, we fix any non-singular point \( p \in M \) of \( F \) and any \( \tilde{u} \in S^{n-1}_q \) at any point \( q \in B_{r(p)}(p) \). Here, we also fix an integer \( i \in \{1, 2, \ldots, \ell\} \) satisfying \( q \in \text{supp } \varphi_i \subset B_{r_i}(p_i) \).

Lemma 2.8 There exists a unit vector field \( V \) on a neighborhood of \( F(p) \) such that

\[
\langle dF^{(\varphi_i)}_\varepsilon(\tilde{u}), V_{F(q)} \rangle \geq -\text{Lip}(F)(\sup_{y \in B_{\varepsilon_i(\varphi_i)}} \|Y^{(\varphi_i)}_y(1) - U_{\varphi_i(y)}\| + \sup_{y \in B_{\varepsilon_i(\varphi_i)}} \|V_{F(q)} - V_{F(q_{\varphi_i(y)})}\|) + \delta(p)
\]

for all \( \varepsilon \in (0, \varepsilon_i(p)) \), where \( \varepsilon_i(p) := \min\{r_i, r(p)/\text{Lip}(\exp_{p_i}|_{B_{2r_i}(p_i)})\} \) and \( U_{\varphi_i(y)} := \tau^p_{\varphi_i(y)} \circ \tau^\varphi_i(\tilde{u}) \).
Lemma 2.9 For any $\eta > 0$, there exists a positive constant $\varepsilon_i(p, \eta)$ such that

$$\sup\{\|\gamma_y^{(u)}(1) - U_{q_i(y)}\| \mid q \in \text{supp } \varphi_i \cap \overline{B_{r_i(p)}(p)}, \, \bar{u} \in \mathbb{S}^{n-1}, \, y \in \mathbb{B}_{\varepsilon_i(p, \eta)}(o_{p_i})\} < \eta$$

and

$$\sup\{\|V_F(q) - V_{F(q_i(y))}\| \mid q \in \text{supp } \varphi_i \cap \overline{B_{r_i(p)}(p)}, \, v \in \mathbb{S}^{k-1}_{F(p)}, \, y \in \mathbb{B}_{\varepsilon_i(p, \eta)}(o_{p_i})\} < \eta,$$

where $\overline{B_{r_i(p)}(p)} := \{q \in M \mid d(p, q) \leq r(p)\}$ and $V_F(q) := \tau^{F(q)}_F(v)$.

Proof. Imitate the proof of Lemma 2.5.

Lemma 2.10 There exists $V_F(q) \in \mathbb{S}^{k-1}_{F(q)}$ such that $\langle (dF_\varepsilon^{(p_i)})_q(\bar{u}), V_F(q) \rangle \geq 2\delta(p)/3$ for all $\varepsilon \in (0, \varepsilon(p))$ and all $p_i$ with $q \in \text{supp } \varphi_i$. Here we set $\varepsilon(p) := \min\{\varepsilon_i(p), \varepsilon_i(p, \eta_0) \mid \text{supp } \varphi_i \cap B_{r_i(p)}(p) \neq \emptyset\}$, where $\eta_0 := \delta(p)/6 \text{ Lip}(F)$.

Proof. The inequality is immediate from Lemmas 2.8 and 2.9.

Lemma 2.11 There exist $V_F(q) \in \mathbb{S}^{k-1}_{F(q)}$ and a number $\varepsilon_0(p) > 0$ such that

$$\langle (dF_\varepsilon)_q(\bar{u}), V_F(q) \rangle \geq \frac{\delta(p)}{3}$$

for all $\varepsilon \in (0, \varepsilon_0(p))$. 

Proof. It follows from Lemma 2.7 that for the unit tangent vector $u := \tau^p_0(\bar{u})$, there exists a unit vector field $V$ on a neighborhood of $F(p)$ satisfying (2.10). By the triangle inequality, $\|\exp_{p_i}^{-1} q - y\| < r_i + \|y\| < 2r_i$ for all $y \in \mathbb{B}_{r_i(o_{p_i})}$. Thus, we get

$$d_M(q, q_i(y)) \leq \|y\| \cdot \text{Lip}(\exp_{p_i} |_{\mathbb{B}_{2r_i(o_{p_i})}})$$

for all $y \in \mathbb{B}_{r_i(o_{p_i})}$. Then, from the triangle inequality, we obtain

$$d_M(p, q_i(y)) \leq d_M(p, q) + d_M(q, q_i(y)) < 2r(p)$$

for all $y \in \mathbb{B}_{\varepsilon_i(p, o_{p_i})}$. Thus, by Lemma 2.7

(2.12) $$\langle dF_{q_i(y)}(U_{q_i(y)}), V_F(q_i(y)) \rangle \geq \delta(p)$$

for almost all $y \in \mathbb{B}_{\varepsilon_i(p, o_{p_i})}$, where $U_{q_i(y)} := \tau^p_{q_i(y)}(u)$. It is clear to see that for almost all $y \in \mathbb{B}_{\varepsilon_i(p, o_{p_i})},$

(2.13) $$\langle dF_{q_i(y)}(U_{q_i(y)}), V_F(q) \rangle
\leq \langle dF_{q_i(y)}(U_{q_i(y)}), V_F(q_i(y)) \rangle + \langle dF_{q_i(y)}(U_{q_i(y)}), V_F(q_i(y)) \rangle
\geq -\text{Lip}(F) \sup_{y \in \mathbb{B}_{\varepsilon_i(o_{p_i})}}\|V_F(q) - V_F(q_i(y))\| + \langle dF_{q_i(y)}(U_{q_i(y)}), V_F(q_i(y)) \rangle.$$

Therefore, we get (2.11) from (2.7), (2.8), (2.12) and (2.13).
Proof. Since \( \sum_{i=1}^{\ell} \varphi_i = 1 \) on \( M \), \( \sum_{i=1}^{\ell} d\varphi_i(\tilde{u}) = 0 \) holds. Then, we have

\[
(2.15) \quad dF_\varepsilon(\tilde{u}) = \sum_{i=1}^{\ell} \varphi_i(q) dF_\varepsilon^{(p_i)}(\tilde{u}) + \sum_{i=1}^{\ell} d\varphi_i(\tilde{u})(F_\varepsilon^{(p_i)}(q) - F(q)).
\]

By Lemma 2.10 for all \( \varepsilon \in (0, \varepsilon(p)) \),

\[
(2.16) \quad \sum_{i=1}^{\ell} \varphi_i(q) \langle dF_\varepsilon^{(p_i)}(\tilde{u}), V_{F(q)} \rangle \geq \frac{2}{3} \delta(p).
\]

From Lemma 2.2, we may choose a number \( \varepsilon_0(p) \in (0, \varepsilon(p)) \) satisfying

\[
(2.17) \quad \left| \sum_{i=1}^{\ell} d\varphi_i(\tilde{u})(F_\varepsilon^{(p_i)}(q) - F(q), V_{F(q)}) \right| < \frac{1}{3} \delta(p)
\]

for all \( \varepsilon \in (0, \varepsilon_0(p)) \). Thus, by combining (2.15), (2.16) and (2.17), we get (2.14) for all \( \varepsilon \in (0, \varepsilon_0(p)) \).

Theorem 2.12 Let \( F : M \rightarrow N \) be a Lipschitz map from a compact Riemannian manifold \( M \) into a complete Riemannian manifold \( N \). Assume that \( F \) has no singular point. If \( \dim M \leq \dim N \), then for any \( \eta > 0 \), there exists an (smooth) immersion \( f_\eta \) from \( M \) into \( N \) such that \( \max_{p \in M} d_N(f_\eta(p), F(p)) < \eta \) and \( \text{Lip}(f_\eta) \leq \text{Lip}(F)(1 + \eta) \).

Proof. Choose a small neighborhood \( U_N \) of the manifold \( N \) in \( \mathbb{R}^m \) so that each point \( x \) of \( U_N \) admits a unique nearest point \( \pi_N(x) \) of \( N \). Hence, the well-defined map \( \pi_N : U_N \rightarrow N \) is smooth. Since \( F(M) \) is compact, it follows from Lemma 2.3 that for all sufficiently small \( \varepsilon > 0 \), the image of the map \( F_\varepsilon \) defined by (2.5) is a subset of \( U_N \). Then, we may define a smooth map \( \pi_N \circ F_\varepsilon \) from \( M \) into \( N \) for all sufficiently small \( \varepsilon > 0 \). It is easy to check that \( (d\pi_N)_x \) is an orthogonal projection to the tangent space \( T_xN \) for each \( x \in U_N \cap N \). Therefore, from Lemma 2.11, the map \( \pi \circ F_\varepsilon : M \rightarrow N \) is an immersion for all sufficiently \( \varepsilon > 0 \). Combining Lemmas 2.3, 2.6 and the argument above, we get the assertion.

3 Proof of Main Theorem A

Using Theorem 2.12, we prove Main Theorem A as Theorem 3.2 in this section. For this, we need one more lemma. In what follows, \( d_M \) denotes the distance function on a given Riemannian manifold \( M \).

Lemma 3.1 (Key Lemma). Let \( M \) be a compact Riemannian manifold, and let \( F : M \rightarrow M \) be \( C^\infty \) and a locally diffeomorphism. If \( \max_{x \in M} d_M(F(x), x) \) is sufficiently small, then \( F \) is injective. In particular, \( F \) is a diffeomorphism.

Proof. By assumptions, for any \( x \in M \), there exists a constant number \( \delta \) such that

\[
(3.1) \quad 0 < \|dF_x\| \leq \delta
\]
holds on \( M \). Since \( M \) is compact and \( a := \max_{x \in M} d_M(F(x), x) \ll 1 \), we may assume that

\[
a < \frac{1}{4\delta} \cdot i(M),
\]

where \( i(M) \) denotes the injectivity radius of \( M \). Now, we consider \( F \) to be a covering map. Suppose that the covering \( F \) is not injective. Then, we are going to obtain a contradiction: For any point \( x \in M \), there exist distinct points \( \tilde{x}_i \in M \) \((2 \leq i < \infty)\) such that \( \tilde{x}_i \in F^{-1}(x) \). To simplify this situation, we fix two points \( \tilde{x}_1, \tilde{x}_2 \in F^{-1}(x) \). Let \( \tilde{c} \) be a unit speed minimal geodesic segment emanating from \( \tilde{x}_1 = \tilde{c}(0) \) to \( \tilde{x}_2 = \tilde{c}(d_M(\tilde{x}_1, \tilde{x}_2)) \). Since \( M \) is compact, it follows from the above assumption that there exists a geodesic loop \( \gamma \) at \( x = \gamma(0) \) on \( M \) satisfying

\[
\frac{1}{2} i(M) \leq L(\gamma) \leq L(F \circ \tilde{c});
\]

otherwise \( F \) is injective, where \( L \) denotes the length of loops \( \gamma \) and \( F \circ \tilde{c} \), respectively. Moreover, by \((3.1)\),

\[
L(F \circ \tilde{c}) \leq \delta \cdot L(\tilde{c}).
\]

On the other hand, by the triangle inequality,

\[
L(\tilde{c}) = d_M(\tilde{x}_1, \tilde{x}_2) \leq d_M(\tilde{x}_1, F(\tilde{x}_1)) + d_M(\tilde{x}_2, F(\tilde{x}_2)) \leq 2a,
\]

where note that \( x = F(\tilde{x}_1) = F(\tilde{x}_2) \). Hence, by \((3.3)\), \((3.4)\), and \((3.5)\), we get

\[
2a \geq L(\tilde{c}) \geq \frac{1}{\delta} \cdot L(F \circ \tilde{c}) \geq \frac{1}{2\delta} \cdot i(M).
\]

This inequality contradicts \((3.2)\). \( \square \)

**Theorem 3.2** Let \( M \) and \( N \) be compact Riemannian manifolds, respectively, and let \( F \) be a bi-Lipschitz homeomorphism from \( M \) onto \( N \). If \( F \) and \( F^{-1} \) have no singular point on \( M \) and \( N \), respectively, then \( M \) and \( N \) are diffeomorphic.

**Proof.** By Theorem \( 2.12 \) for any \( \eta > 0 \), there exist two \( C^\infty \)-immersions \( f_\eta \) from \( M \) into \( N \) and \( g_\eta \) from \( N \) into \( M \) such that \( \max_{p \in M} d_N(f_\eta(p), F(p)) < \eta \), \( \text{Lip}(f_\eta) \leq \text{Lip}(F)(1+\eta) \), and that \( \max_{q \in N} d_M(g_\eta(q), F^{-1}(q)) < \eta \), \( \text{Lip}(g_\eta) \leq \text{Lip}(F^{-1})(1+\eta) \), respectively. Hence,

\[
\text{Lip}(g_\eta \circ f_\eta) \leq \text{Lip}(g_\eta) \text{Lip}(f_\eta) \leq \text{Lip}(F) \text{Lip}(F^{-1})(1+\eta)^2.
\]

Moreover, by the triangle inequality, we get

\[
d_M(g_\eta \circ f_\eta(p), p) \leq d_M(g_\eta(f_\eta(p)), F^{-1}(f_\eta(p))) + d_M(F^{-1}(f_\eta(p)), F^{-1}(F(p))).
\]

Since \( d_M(g_\eta(f_\eta(p)), F^{-1}(f_\eta(p))) < \eta \) and \( d_M(F^{-1}(f_\eta(p)), F^{-1}(F(p))) < \eta \text{Lip}(F^{-1}) \) for all \( p \in M \), we obtain

\[
\max_{p \in M} d_M(g_\eta \circ f_\eta(p), p) < \eta(1 + \text{Lip}(F^{-1})).
\]

Therefore, by Lemma \( 3.1 \) for all sufficiently small \( \eta > 0 \), \( g_\eta \circ f_\eta \) is a diffeomorphism on \( M \). This implies that \( f_\eta \) and \( g_\eta \) are injective for all sufficiently small \( \eta \), and hence \( M \) and \( N \) are diffeomorphic. \( \square \)
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