Subordination of functions in subclass of Bazilevič Functions $B_1(\alpha, \beta)$

Marjono
Faculty of Mathematics and Natural Sciences, University of Brawijaya, Malang, Jawa Timur 65145, Indonesia.
E-mail: marjono@ub.ac.id

Abstract. Let $f$ be analytic in the unit disc $D = \{ z : |z| < 1 \}$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and for $\alpha \geq 0$ and $0 < \beta \leq 1$, let $B_1(\alpha, \beta)$, denote for the class of Bazilevič functions satisfying the expression $|\arg z^{1-\alpha} f'(z) f(z)^{1-\alpha}| < \frac{\beta \pi}{2}$. We give sharp estimates for various coefficient problems for functions in $B_1(\alpha, \beta)$, which unify and extend well-known results for starlike functions, strongly starlike functions and functions whose derivative has positive real part in domain $D$.

1. Definitions and Some Preliminaries
We will first recall the Bazilevič functions $B(\alpha)$ as follows [4].

Let $S$ be the class of analytic and normalized univalent functions $f$ defined in $z \in D = \{ z : |z| < 1 \}$ and given by the following

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

Then for $\alpha \geq 0$, $f \in B(\alpha) \subset S$ if, and only if, there exists a starlike function $g$ in $z \in D$, such that

$$\text{Re} \frac{zf'(z)}{f(z)^{1-\alpha} g(z)^{\alpha}} > 0.$$  

Taking $g(z) \equiv z$ gives the class $B_1(\alpha)$ of Bazilevič Functions with logarithmic growth. We note that $B_1(0)$ is the class of starlike functions $S^*$, and $B_1(1)$ the well-known class $R$ of functions whose derivative has positive real part in $D$.

Thus $f \in B_1(\alpha)$, if and only if,

$$\text{Re} \frac{zf'(z)}{f(z)^{1-\alpha} z^{\alpha}} > 0.$$
We know that various best possible properties of these problems have been obtained for the class $B_1(\alpha)$. Amongst other results in this topic, fortunately, Singh [16], found sharp estimates for the moduli of the first four coefficients, and he also obtained the solution to the expression known as the Fekete-Szegö problem. The author of [22] has recently obtained sharp bounds for the second Hankel determinant, the initial coefficients of the function with the form $\log f(z)$, and also obtained the initial coefficients of the inverse function $f^{-1}$. Distortion theorems and some length-area results were also obtained in London, Singh, and Thomas [13, 19, 20].

For $0 < \beta \leq 1$, let $B_1(\alpha, \beta)$ be the set of functions $f$, given by (1) satisfying

$$\left| \arg \frac{zf'(z)}{f(z)^{1-\alpha}z^\alpha} \right| < \frac{\beta \pi}{2}. \quad (2)$$

Then $B_1(0, \beta)$ is the class of strongly starlike functions $SS^*$ and $B_1(1, \beta)$ the class of functions such that $Re \ f'(z)$ lies in a sector, which are extensions of the classical sets of starlike functions $S^*$ and $R$ respectively.

It follows from (2) that we can write

$$z^{1-\alpha}f'(z) = f(z)^{1-\alpha}h(z)^\beta, \quad (3)$$

where $h \in P$, the class of function satisfying $Re \ h(z) > 0$ for $z \in D = \{z : |z| < 1\}$.

Let

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

We shall use the following results, see e.g. [1, 2, 6, 9]

**Lemma 1**

If $h \in P$ with coefficients $c_n$ as above, then for some complex valued $x$ with $|x| \leq 1$ and some complex valued $\zeta$ with $|\zeta| \leq 1$,

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\zeta.$$  

**Lemma 2**

If $p \in P$, then

$$\left| p_2 - \frac{\mu}{2}p_1^2 \right| \leq \max\{2, 2|\mu - 1|\} = \begin{cases} 2, & 0 \leq \mu \leq 2, \\ 2|\mu - 1|, & \text{elsewhere}. \end{cases} \quad (4)$$

**Lemma 3**

Let $h \in P$ with coefficients $c_n$ as above, then if $0 \leq \lambda \leq 1$ and $\lambda(2\lambda - 1) \leq \delta < \lambda$, 

$$\frac{z}{h(z)} \leq \frac{z}{f(z)}.$$
\[ |c_3 - 2\lambda c_1 c_2 + \delta c_1^3| \leq 2. \]

**Lemma 4**

Let \( h \in P \) with coefficients \( c_n \) as above, then,

\[ |c_3 - c_1 c_2 + \frac{1}{2} c_1^3| \leq 2. \]

## 2. Initial Coefficients

We first extend the coefficient results of Singh [16] for the coefficients of \( B_1(\alpha) \), and the results of Brannan, Clunie, and Kirwan [3], Ali [1], Ali and Singh [2] and Krisna Adilia [10] for strongly starlike functions as follows:

**Theorem 1** For \( f \in B_1(\alpha, \beta) \) and given by (1), then

\[
|a_2| \leq \frac{2\beta}{1+\alpha} \text{ for } \alpha \geq 0 \text{ and } 0 < \beta \leq 1, \\
|a_3| \leq \frac{2\beta}{2+\alpha} \text{ for } 0 \leq \alpha \leq 1 \text{ and } 0 < \beta \leq \frac{(1+\alpha)^2}{3+\alpha}, \\
\leq \frac{2(3+\alpha)\beta^2}{(1+\alpha)^2(2+\alpha)} \text{ for } 0 \leq \alpha \leq 1 \text{ and } \frac{(1+\alpha)^2}{3+\alpha} \leq \beta \leq 1, \\
\leq \frac{2\beta}{2+\alpha} \text{ for } \alpha \geq 1 \text{ and } 0 < \beta \leq 1, \\
|a_4| \leq \frac{2\beta}{3+\alpha} \text{ for } 0 \leq \alpha \leq 1 \text{ and } 0 < \beta \leq \sqrt{\frac{2 + 7\alpha + 9\alpha^2 + 5\alpha^3 + \alpha^4}{17 + 6\alpha + \alpha^2}} \\
\leq \frac{2\beta}{3+\alpha} - \frac{4\beta(2 + 5\alpha^3 + \alpha^4 - 17\beta^2 + \alpha(7 - 6\beta^2) - \alpha^2(-9 + \beta^2))}{3(1+\alpha)^3(2+\alpha)(3+\alpha)} \\
\text{ for } 0 \leq \alpha \leq 1 \text{ and } \sqrt{\frac{2 + 7\alpha + 9\alpha^2 + 5\alpha^3 + \alpha^4}{17 + 6\alpha + \alpha^2}} \leq \beta \leq 1, \\
\leq \frac{2\beta}{3+\alpha} \text{ for } \alpha \geq 1 \text{ and } 0 < \beta \leq 1. 
\]

All inequalities are sharp.

*Proof.* Equating coefficients in (3) gives
\[(1 + \alpha)a_2 = \beta c_1,\]
\[(2 + \alpha)a_3 = \beta c_2 - \frac{\beta(1 + \alpha^2 - \alpha(-2 + \beta) - 3\beta)c_1^2}{2(1 + \alpha)^2},\]
\[(3 + \alpha)a_4 = \beta c_3 + \frac{\beta(-2 - \alpha^2 + \alpha(-3 + \beta) + 5\beta)c_1 c_2}{(1 + \alpha)(2 + \alpha)} + \frac{\beta(4 + 2\alpha^4 + \alpha^2(10 - 3\beta) - 15\beta + 17\beta^2 + \alpha^2(18 - 21\beta + \beta^2) + \alpha(14 - 33\beta + 6\beta^2))c_1^3}{6(1 + \alpha)^3(2 + \alpha)}.\] (4)

The first inequality in Theorem 1 follows at once since \(|c_1| \leq 2\).

For \(|a_3|\), we use Lemma 2. Write

\[|a_3| = \frac{\beta}{(2 + \alpha)} \left| c_2 - \frac{(1 + \alpha^2 - \alpha(-2 + \beta) - 3\beta)c_1^2}{2(1 + \alpha)^2} \right|.\]

Then in Lemma 2, let

\[\mu = \frac{(1 + \alpha^2 - \alpha(-2 + \beta) - 3\beta)}{2(1 + \alpha)^2},\]

so that \(0 \leq \mu \leq 2\) provided \(0 \leq \alpha \leq 1\) and \(0 < \beta \leq \frac{(1 + \alpha)^2}{(3 + \alpha)}\), and when \(\alpha \geq 1\) and \(0 < \beta \leq 1\).

Applying Lemma 2 now gives the inequalities for \(|a_3|\).

For the coefficient \(a_4\), we use Lemmas 2, 3, and 4.

From (4) we obtain

\[|a_4| = \frac{\beta}{3 + \alpha} \left| c_3 - \frac{2 + \alpha^2 - \alpha(-3 + \beta) - 5\beta}{(1 + \alpha)(2 + \alpha)} c_1 c_2 + \frac{4 + 2\alpha^4 + \alpha^3(10 - 3\beta) - 15\beta + 17\beta^2 + \alpha^2(18 - 21\beta + \beta^2) + \alpha(14 - 33\beta + 6\beta^2)}{6(1 + \alpha)^3(2 + \alpha)} c_1^3 \right|.\] (5)

We now can apply Lemma 3 with

\[\lambda = \frac{2 + \alpha^2 - \alpha(-3 + \beta) - 5\beta}{2(1 + \alpha)(2 + \alpha)},\]

and
\[
\delta = 4 + 2\alpha^4 + \alpha^3(10 - 3\beta) - 15\beta + 17\beta^2 + \alpha^2(18 - 21\beta^2 + \alpha(14 - 33\beta + 6\beta^2))
\]

so that the conditions \(0 \leq \lambda \leq 1\) and \(\lambda(2\lambda - 1) \leq \delta \leq \lambda\) are satisfied whenever \(0 \leq \alpha \leq 1\) and \(0 < \beta \leq \sqrt{\frac{2 + 7\alpha + 9\alpha^2 + 5\alpha^3 + \alpha^4}{17 + 6\alpha + \alpha^2}}\), and when \(\alpha \geq 1\) and \(0 < \beta \leq 1\).

By applying Lemma 3 gives the first and last inequalities for \(|a_4|\).

To prove the second inequality we will use Lemmas 2 and 4.

Write \(c_3 - 2\lambda c_1 c_2 + \delta c_1^2 = c_3 - c_1 c_2 + \frac{1}{2} c_1^3 + (1 - 2\delta)c_1\left(c_2 + \frac{c_1^2(-1 + 2\delta)}{2(1 - 2\lambda)}\right)\).

We now substitute for \(\lambda\) and \(\delta\), and add and subtract \(\frac{1}{2} c_1^2\) into the last expression to obtain the following

\[
(1 - 2\delta)c_1\left(c_2 + \frac{c_1^2(-1 + 2\delta)}{2(1 - 2\lambda)}\right) = \left(\frac{\beta(5 + \alpha)}{(1 + \alpha)(2 + \alpha)} c_1\right)
\]

\[
\left(c_2 - \frac{1}{2} c_1^2 - \frac{(2 + 5\alpha^3 + \alpha^4 - 17\beta^2 + \alpha(7 - 6\beta^2) - \alpha^2(-9 + \beta^2))}{6(1 + \alpha)^2(5 + \alpha)\beta} c_1^2\right)
\]

Next note that \(- \frac{2 + 5\alpha^3 + \alpha^4 - 17\beta^2 + \alpha(7 - 6\beta^2) - \alpha^2(-9 + \beta^2)}{6(1 + \alpha)^2(5 + \alpha)\beta} \geq 0\), when \(0 \leq \alpha \leq 1\) and \(\sqrt{\frac{2 + 7\alpha + 9\alpha^2 + 5\alpha^3 + \alpha^4}{17 + 6\alpha + \alpha^2}} < \beta \leq 1\). We now apply Lemma 2 to the last bracket in (7), noting that the resulting expression inside the last bracket increase for \(0 \leq |c_1| \leq 2\), so that the maximum value occurs when \(|c_1| = 2\). Thus applying Lemma 4 in (6), and noting that \(|c_1| \leq 2\), gives the bound for \(|a_4|\).

The inequality for \(|a_2|\) is sharp when \(c_1 = 2\). The first and third inequalities for \(|a_3|\) are sharp when \(c_1 = 0\) and \(c_2 = 2\), and the second inequality for \(|a_3|\) is sharp when \(c_1 = c_2 = 2\). For \(|a_4|\), the first and third inequalities are sharp when \(c_1 = c_2 = 0\) and \(c_3 = 2\) and the second inequality is sharp when \(c_1 = c_2 = c_3 = 2\).

The proof is complete.
3. Fekete-Szegő Theorems

We next establish sharp Fekete-Szegő results for $B_1(\alpha, \beta)$, which extend that given in [16].

**Theorem 2**

For $B_1(\alpha, \beta)$ and real $\nu$,

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{2\beta}{2 + \alpha} + \frac{2\beta^2(1 - \alpha - 2\nu)}{(1 + \alpha)^2} & \text{if } \nu \leq \frac{\beta(3 + \alpha) - (1 + \alpha)^2}{2\beta(2 + \alpha)}, \\ \frac{2\beta(2\beta(2 + \alpha) - (1 + \alpha)^2 - \beta(3 + \alpha))}{(1 + \alpha)^2(2 + \alpha)} & \text{if } \nu \leq \frac{\beta(3 + \alpha) + (1 + \alpha)^2}{2\beta(2 + \alpha)}. \end{cases}$$

**Proof.** We use Lemma 2. From (4)

$$|a_3 - \nu a_2^2| = \frac{\beta}{2 + \alpha} \left| c_2 - \frac{1 + \alpha^2 + \beta(-3 + 4\nu) + \alpha(2 + \beta(-1 + 2\nu))}{2(1 + \alpha)^2} c_1 \right|.$$

Write

$$c_2 - \frac{1 + \alpha^2 + \beta(-3 + 4\nu) + \alpha(2 + \beta(-1 + 2\nu))}{2(1 + \alpha)^2} c_1 = c_2 - \frac{1}{2} c_1^2 + \frac{\beta(3 + \alpha - 2\nu(2 + \alpha))}{2(1 + \alpha)^2} c_1.$$

Assume first that $3 + \alpha - 2\nu(2 + \alpha) \geq 0$, then using Lemma 2 with $\mu = 1$, we obtain

$$\left| c_2 - \frac{1 + \alpha^2 + \beta(-3 + 4\nu) + \alpha(2 + \beta(-1 + 2\nu))}{2(1 + \alpha)^2} c_1 \right| \leq 2 - \frac{1}{2} |c_1^2| + \frac{\beta(3 + \alpha - 2\nu(2 + \alpha))}{2(1 + \alpha)^2} |c_1|^2.$$

Simple calculus now shows that this expression has maximum $\frac{2\beta(3 + \alpha - 2\nu(2 + \alpha))}{(1 + \alpha)^2}$ at $|c_1| = 2$ for $\alpha \geq 0$ and $0 < \beta \leq 1$, provided $\nu \leq \frac{\beta(3 + \alpha) - (1 + \alpha)^2}{2\beta(\beta + \alpha)}$, and a maximum of 2 at $|c_1| = 0$ for $\alpha \geq 0$ and $0 < \beta \leq 1$, provided $\nu \geq \frac{\beta(3 + \alpha) - (1 + \alpha)^2}{2\beta(\beta + \alpha)}$.

We next assume that $3 + \alpha - 2\nu(2 + \alpha) \leq 0$ and again use Lemma 2 as follows.

First note that in Lemma 2, $\mu \leq 2$, if $\alpha \geq 0$ and $0 < \beta \leq 1$ provided $\nu \leq \frac{\beta(3 + \alpha) + (1 + \alpha)^2}{2\beta(2 + \alpha)}$, and that $\mu \geq 2$, if $\alpha \geq 0$ and $0 < \beta \leq 1$ provided $\nu \geq \frac{\beta(3 + \alpha) + (1 + \alpha)^2}{2\beta(2 + \alpha)}$. 

6
Applying Lemma 2 and using simple calculus, establishes the remaining two inequalities, which completes the proof of Theorem 2. The proof is complete.

**Theorem 3** For \( B_1(\alpha, \beta) \) and any complex number \( \nu \),

\[
|a_3 - \nu a_2^2| \leq \max \left\{ \frac{2\beta}{2 + \alpha}, \frac{2\beta^2|3 + \alpha - 2\nu(2 + \alpha)|}{(1 + \alpha)^2(2 + \alpha)} \right\}.
\]

**Proof.** A simple application of Lemma 2 gives the result. The proof is complete.

**Remark 1**

Obtaining the sharp upper bounds for \( |a_n| \) for all \( n \geq 5 \) for \( f \in B_1(\alpha, \beta) \) remains an open question, even in the case \( \beta = 1 \). It was shown in [17], Marjono [15] and Sa’adatul Fitri [18] that when \( \beta = 1 \), \( |a_n| \leq |B_n| \), for \( \alpha = \frac{1}{N} \), where \( N \geq 2 \) is a positive integer and where \( B_n \) is the general coefficient of the extreme function \( \phi \) for the class \( B_1(\alpha, 1) \) given by

\[
\phi(z) = (\alpha \int_0^z t^{\alpha-1} \frac{1 + t}{1 - t} dt)^{1/\alpha}.
\]

4. The Second Hankel Determinant

The \( q^{th} \) Hankel determinant of \( f \) is defined for \( q \geq 1 \) and \( n \geq 1 \) as follows, and has been extensively studied e.g. [8, 9, 15, 17].

\[
H_q(n) = \begin{vmatrix}
a_n & a_{n+1} & \cdots & a_{n+q+1} \\
a_{n+1} & \cdots & \vdots \\
\vdots \\
a_{n+q-1} & \cdots & a_{n+2q-2}
\end{vmatrix}.
\]

We prove the following, which extends the result in [22], noting that the result is valid for \( \alpha \geq 0 \).

**Theorem 4** Let \( f \in B_1(\alpha, \beta) \), then

\[
H_2(2) = |a_2 a_4 - a_3^2| \leq \frac{4\beta^2}{(2 + \alpha)^2}.
\]

The inequality is sharp.

**Proof.** Using (4) and simplifying, we obtain
\[ H_2(2) = |a_2a_4 - a_3^2| = \frac{\beta^2(4 + 6\alpha^3 + \alpha^4 - 13\beta^2 - \alpha^2(-13 + \beta^2) - 2\alpha(-6 + 5\beta^2))c^4}{12(1 + \alpha)^3(2 + \alpha)^2(3 + \alpha)} + \frac{(-1 + \beta)\beta^2c_1c_2}{(1 + \alpha)(2 + \alpha)^2(3 + \alpha)} - \frac{\beta^2c_1^2}{(2 + \alpha)^2} + \frac{\beta^2c_1c_3}{(1 + \alpha)(3 + \alpha)}. \]

We now use Lemma 1 to write \( c_2 \) and \( c_3 \) in terms of \( c_1 \), and, without loss in generality, take \( c_2 = c \), where \( c \in [0, 2] \). Also for simplicity, we write \( X = 4 - c^2 \) and \( Z = (1 - |x|^2)\zeta \), to obtain

\[ H_2(2) = |\beta^2\Delta_1(\alpha, \beta)c^4| \leq \frac{\beta^3c_2X}{2(1 + \alpha)(2 + \alpha)^2(3 + \alpha)} + \frac{\beta^2|\Delta_1(\alpha, \beta)|c_2^2}{4(1 + \alpha)(3 + \alpha)} \leq \frac{\beta^3c_2X}{2(1 + \alpha)(3 + \alpha)}. \]

where \( \Delta_1(\alpha, \beta) := 4 + 6\alpha^3 + \alpha^4 - 13\beta^2 - \alpha^2(-13 + \beta^2) - 2\alpha(-6 + 5\beta^2) \).

We now use the triangle inequality to obtain

\[ H_2(2) \leq \frac{\beta^2|\Delta_1(\alpha, \beta)|c_2^2}{4(1 + \alpha)(3 + \alpha)} + \frac{\beta^3c_2X}{2(1 + \alpha)(3 + \alpha)} + \frac{\beta^2|\Delta_1(\alpha, \beta)|c_3^2}{4(2 + \alpha)^2} + \frac{\beta^3cX(1 - |x|^2)}{2(1 + \alpha)(3 + \alpha)} := \phi(c, |x|). \]

Thus we need to maximise \( \phi(c, |x|) \) over the rectangle \( I = [0, 2] \times [0, 1] \).

First assume that there is a critical point at \((c_0, |x_0|)\) inside \( I \). Since each term of the derivative of \( \phi(c, |x|) \) with respect to \( |x| \) contains the expression \( 4 - c^2 \), equating to zero gives a contradiction. Thus any maximum point must occur on the boundary of \( I \).

On \( c = 0 \),

\[ \phi(0, |x|) = \frac{4\beta^2|\Delta_1(\alpha, \beta)|c_2^2}{(2 + \alpha)^2} \leq \frac{4\beta^2}{(2 + \alpha)^2}. \]

On \( c = 2 \),

\[ \phi(2, |x|) = \frac{4\beta^2|\Delta_1(\alpha, \beta)|c_3^2}{3(1 + \alpha)^3(2 + \alpha)^2(3 + \alpha)} \leq \frac{4\beta^2}{(2 + \alpha)^2}. \]

On \( |x| = 0 \),

...
\[ \phi(c, 0) = \frac{\beta^2 |\Delta_1(\alpha, \beta)|c^4}{12(1 + \alpha)^3(2 + \alpha)^2(3 + \alpha)} + \frac{\beta^2 cX}{2(1 + \alpha)(3 + \alpha)} \leq \frac{4\beta^2}{(2 + \alpha)^2} \]
onumber

on \([0, 2]\).

Finally when \(|x| = 1\),

\[ \phi(c, 1) = \frac{\beta^2 |\Delta_1(\alpha, \beta)|c^4}{12(1 + \alpha)^3(2 + \alpha)^2(3 + \alpha)} + \frac{\beta^3 c^2 X}{2(1 + \alpha)(2 + \alpha)^2(3 + \alpha)} + \frac{\beta^2 c^2 X}{4(1 + \alpha)(3 + \alpha)} + \frac{\beta^2 X^2}{4(2 + \alpha)^2}. \]

This quadratic expression in \(c^2\) is minimum when \(c = 0\), with maximum values at positive and negative values on the \(c\) axis. Thus taking \(c = 2\) in the above it is easily seen once more that \(\phi(c, 1) \leq \frac{4\beta^2}{(2 + \alpha)^2}\) on \(0 \leq c \leq 2\).

We note finally that equality is attained when \(c_1 = c_3 = 0\) and \(c_2 = 2\).

The proof is complete.

**Remark 2**

We note that the case \(\alpha = 1\) corresponds to the class \(R\) of functions whose derivative has positive real part in a sector, and when \(\alpha = 0\) and \(\beta = 1\) to the starlike functions [7]. The case \(\alpha = 0\) and \(0 < \beta \leq 1\) corresponds to the strongly starlike functions [5].

We finally note that it is easily seen from (3) that when \(f \in B_1(\alpha, \beta)\) and \(0 < \beta < 1\), \(M(r, f)\) is bounded, which implies that \(n|a_n|\) is also bounded.

**References**

[1] Ali R M 2003 Coefficients of the Inverse of Strongly Starlike Functions *Bull. Malaysian Math. Sc. Soc.* 26 pp 63–71

[2] Ali R M and Singh V A 1996 On the fourth and fifth coefficients of strongly starlike functions *Results in Mathematics* 29 pp 197–202

[3] Brannan D A, Clunie J, and Kirwan W E 1970 Coefficient estimates for a class of starlike functions *Can. J. Math.* XXII no 3 pp 476–485

[4] Bazilevič I E 1955 On a case of integrability in quadratures of the Lowner-Kufarev equation *Mat. Sb.* 37 (79) pp 471–476 (Russian) MR 17 356

[5] Deekonda V K and Thoutreddy R 2014 An upper bound to the second Hankel determinant for a subclass of analytic functions *Bull. of the International Mathematical Virtual Institute* 4 pp 17–26

[6] Duren P L 1983 In *Univalent Functions* (Springer-Verlag) pp 114–115 Mat. Sb. 37 (79)(195) pp 471–476 (Russian) MR 17 356

[7] Girela D 2000 Logarithmic coefficients of univalent functions *Annals Acad. Sci. Fenn. Math. Sb* 25 (61) pp 337–350

[8] Hayman W K 1968 On the second Hankel determinant of mean univalent functions *Proc. Lond. Math. Soc.* 3 no 18 (1968) pp 77–94

[9] Janteng A, Halim S, and Darus M 2007 Hankel Determinants for Starlike and Convex Functions *Int. Journal. Math. Analysis* 1 no 13 pp 619–625

[10] Daniswara K A, Marjono, and Wibowo R B E 2020 The Fifth Coefficients of Strongly Convex Functions *Journal of Physics Conference Series* 1562:012006; DOI: 10.1088/1742-6596/1562/1/012006
[11] Libera R J and Zlotkiewicz E J 1983 Coefficient bounds for the inverse of a function with derivative in \( P \)
Proc. Amer. Math. Soc. 87 no 2 pp 251–257
[12] Libera R J and Zlotkiewicz E J 1984 Coefficient Bound for the Inverse of a Function with Derivative in \( P \)
Proc. Amer. Math. Soc. 92 no 1 pp 58–60
[13] London R R and Thomas D K 1988 On the Derivative of Bazilevič Functions Proc. Amer. Math. Soc. 104
no 1 pp 235–238
[14] Löwner C 1923 Untersuchungen über schlichte konforme Abbildungen des Einheitskreises J. Math. Ann 89
pp 103–121
[15] Marjono, Fitri S, and Daniswara K A 2020 The Higher Coefficients for Bazlevic Functions B1(\( \alpha \))
Australian Journal of Mathematical Analysis and Applications 17 2 pp 1–11
[16] Noonan J W and Thomas D K 1976 On the second Hankel determinant of areally, mean p-valent functions
Trans. Amer. Math. Soc. 223 (2) pp 337–346
[17] Pommerenke Ch 1967 On the Hankel determinants of univalent functions Mathematika (London) 16 no 13
pp 108–112
[18] Fitri S, Marjono, Thomas D K, and Wibowo R B E 2020 Coefficients inequalities for subclass of Bazilevic
Functions Demonstratio Mathematica 53 1 pp 27–37
[19] Singh R 1973 On Bazlevi ć Functions Proc. Amer. Math. Soc. 38 no 2 pp 261–271
[20] Thomas D K 1985 In On a Subclass of Bazilevič function Int. J. Math. & Math. Sci. 8 no 4 pp 779–783
[21] Thomas D K 1991 In New Trends in Geometric Function Theory and Applications (World Scientific) pp
146–158
[22] Thomas D K On the Coefficients of Bazlevič Functions with Logarithmic Growth Indian Journal of
Mathematics to appear
[23] Ye Z 2008 The logarithmic coefficients of close to convex functions Bull. Inst. Math., Acad. Sin. (N.S.) 3
no 23 pp 445–452