ABSTRACT

For the generalized chiral Schwinger model defined on the circle, a direct calculation of the zero curvature part of the vacuum Berry phase connection is given. Although this part does not contribute to the curvature, it is shown to attach several features to the total connection and to produce a physical background of linearly rising electric fields.
1. Berry phase [1] plays an important role in gauge models with anomaly [2]. A common topological nature of this phase and gauge anomalies was shown in [3]. It was proved that a $U(1)$ connection related to the vacuum Berry phase contributes to the Hamiltonian and that just this contribution makes the theory gauge invariant [4, 5]. In [6] an interrelation between the nonvanishing vacuum Berry phase and anomaly was demonstrated explicitly for the generalized chiral Schwinger model (or chiral $QED_2$).

Usually the Berry phase connection is not calculated directly. It is not well defined globally on a manifold of all static gauge field configurations and is not invariant under gauge field dependent redefinitions of the phases of the states acquiring the Berry phase. That is why we first compute the corresponding $U(1)$ curvature tensor and then deduce from it an expression for the connection.

However, in this way the Berry phase connection is not determined uniquely. We can not fix that part of the connection which has vanishing curvature. Namely, if $A$ is a connection deduced from the curvature $F$, then $A + A^0$ where $A^0$ is an arbitrary connection with zero curvature also corresponds to $F$. Since the vacuum Berry phase connection represents a background in which the physical degrees of freedom of anomalous models are moving, a direct calculation of its zero curvature part is very important for understanding the dynamics of such models.

In the present note, we aim to calculate the zero curvature part of the Berry phase connection for the vacuum Fock state in the generalized chiral $QED_2$ defined on the circle $S^1$. We work in the temporal gauge $A_0 = 0$ and use the system of units where $c = 1$. In the generalized chiral $QED_2$ a $U(1)$ gauge field is coupled with different charges to both chiral components of a fermionic field. Only matter fields are quantized, while the gauge field is handled as a classical field. We show that the zero curvature Berry phase connection is nothing else than the linearly rising electric field found previously in [7, 8].

2. In the temporal gauge $A_0 = 0$, the fermionic Hamiltonian density of the generalized chiral $QED_2$ is [6, 8]

$$H_F = H_+ + H_-,$$

$$H_\pm \equiv \psi_\pm^d d_\pm \psi_\pm = \mp \psi_\pm^d (i\hbar \partial_1 + e_\pm A_1) \psi_\pm,$$

where $\psi_+$ and $\psi_-$ are correspondingly positive and negative chirality matter fields. We suppose that space is a circle of length $L$, $-\frac{L}{2} \leq x < \frac{L}{2}$, and that the fields are periodic on the circle.

The eigenfunctions and the eigenvalues of the first quantized fermionic Hamiltonians are

$$d_\pm \langle x|n; \pm \rangle = \pm \varepsilon_{n,\pm} \langle x|n; \pm \rangle,$$

where

$$\langle x|n; \pm \rangle = \frac{1}{\sqrt{L}} \exp \left\{ \frac{i}{\hbar} e_\pm \int_{-L/2}^x dz A_1(z) + \frac{i}{\hbar} \varepsilon_{n,\pm} \cdot x \right\},$$

$$\varepsilon_{n,\pm} = \frac{2\pi}{L} \left( n\hbar - \frac{e_\pm b L}{2\pi} \right),$$

and

$$b \equiv \frac{1}{L} \int_{-L/2}^{L/2} dx A_1(x)$$

is the gauge field zero mode.
Now we introduce the second quantized positive and negative chirality Dirac fields. At time $t = 0$, in terms of the eigenfunctions of the first quantized fermionic Hamiltonians the second quantized ($\zeta$-function regulated) fields have the expansion:

$$\psi_+^s(x) = \sum_{n \in \mathbb{Z}} a_n \langle x | n; + \rangle |\lambda \varepsilon_{n,+}|^{-s/2}$$
$$\psi_-^s(x) = \sum_{n \in \mathbb{Z}} b_n \langle x | n; - \rangle |\lambda \varepsilon_{n,-}|^{-s/2}.$$ \hspace{1cm} (2)

Here $\lambda$ is an arbitrary constant with dimension of length which is necessary to make $\lambda \varepsilon_{n,\pm}$ dimensionless, while $a_n, a_n^\dagger$ and $b_n, b_n^\dagger$ are correspondingly positive and negative chirality fermionic annihilation and creation operators which fulfill the commutation relations

$$[a_n, a_m^\dagger]_+ = [b_n, b_m^\dagger]_+ = \delta_{m,n}.$$ 

The vacuum state

$$\left| \text{vac}; A \right> = \left| \text{vac}; A; + \right> \otimes \left| \text{vac}; A; - \right>$$

is defined such that all negative energy levels are filled and the others are empty:

$$a_n \left| \text{vac}; A; + \right> = 0 \quad \text{for} \quad n > \left[ \frac{e_+ b L}{2 \pi \hbar} \right],$$
$$a_n^\dagger \left| \text{vac}; A; + \right> = 0 \quad \text{for} \quad n \leq \left[ \frac{e_+ b L}{2 \pi \hbar} \right],$$ \hspace{1cm} (3)

and

$$b_n \left| \text{vac}; A; - \right> = 0 \quad \text{for} \quad n \leq \left[ \frac{e_- b L}{2 \pi \hbar} \right],$$
$$b_n^\dagger \left| \text{vac}; A; - \right> = 0 \quad \text{for} \quad n > \left[ \frac{e_- b L}{2 \pi \hbar} \right],$$ \hspace{1cm} (4)

where $\left[ \frac{e_+ b L}{2 \pi \hbar} \right]$ and $\left[ \frac{e_- b L}{2 \pi \hbar} \right]$ are integer parts of $\frac{e_\pm b L}{2 \pi \hbar}$.

Next we define the fermionic parts of the second-quantized Hamiltonian as

$$\hat{H}_s^\pm = \int_{-L/2}^{L/2} dx \hat{H}_s^\pm(x) = \frac{1}{2} \int_{-L/2}^{L/2} dx (\psi_s^\dagger \psi_s^s - \psi_s^s \dagger \psi_s^s).$$

Substituting (2) into these expressions, we obtain

$$\hat{H}_+ = \lim_{s \to 0} \sum_{k \in \mathbb{Z}} \varepsilon_{k,+} a_k^\dagger a_k |\lambda \varepsilon_{k,+}|^{-s},$$
$$\hat{H}_- = \lim_{s \to 0} \sum_{k \in \mathbb{Z}} \varepsilon_{k,-} b_k b_k^\dagger |\lambda \varepsilon_{k,-}|^{-s}.$$ 

The operators $\hat{H}_\pm$ are well defined when acting on finitely excited states which have only a finite number of excitations relative to the Fock vacuum.

3. In the adiabatic approach [9, 10], the dynamical variables are divided into two sets, one which we call fast variables and the other which we call slow variables. In our case, we treat the fermions as fast variables and the gauge fields as slow variables.
Let $\mathcal{A}^1$ be a manifold of all static gauge field configurations $A_1(x)$. On $\mathcal{A}^1$ a time-dependent gauge field corresponds to a path and a periodic gauge field to a closed loop.

The second-quantized fermionic Hamiltonian $\hat{H}_F := \hat{H}_+ + \hat{H}_-$ is normal ordered with respect to the vacuum state depends on $t$ through the background gauge field $A_1$ and so changes very slowly with time. Let us assume that the background gauge field $A_1(x,t)$ is periodic ($0 \leq t < T$). After a time $T$ the periodic field returns to its original value: $A_1(x,0) = A_1(x,T)$, so that $\hat{H}_\pm(0) = \hat{H}_\pm(T)$.

At each instant $t$ we define eigenstates for $\hat{H}_\pm(t)$ by

$$\hat{H}_\pm(t)|F, A(t); \pm\rangle = \varepsilon_{F,\pm}(t)|F, A(t); \pm\rangle.$$

The Fock states $|F, A(t)\rangle = |F, A(t); +\rangle \otimes |F, A(t); -\rangle$ depend on $t$ only through their implicit dependence on $A_1$. They are assumed to be orthonormalized and nondegenerate.

According to the adiabatic approximation, upon parallel transport around a closed loop on $\mathcal{A}^1$ the Fock states $|F, A(t); \pm\rangle$ can acquire only a phase. This phase consists of two parts, the usual dynamical phase and an extra phase discovered by Berry. Whereas the dynamical phase provides information about the duration of the evolution, the Berry’s phase

$$\gamma^{\text{Berry}}_{F,\pm} = \int_0^T dt \int_{-L/2}^{L/2} dx \dot{A}_1(x,t) \mathcal{A}_{F,\pm}(x,t),$$

which is integrated exponential of the $U(1)$ connections

$$\mathcal{A}_{F,\pm}(x,t) = \langle F, A(t); \pm | i \frac{\delta}{\delta A_1(x,t)} | F, A(t), \pm \rangle,$$

reflects the nontrivial holonomy of the Fock states on $\mathcal{A}^1$.

The connections $\mathcal{A}_{F,\pm}$ can be defined only locally on $\mathcal{A}^1$, in regions where $[\frac{e b_L}{2\pi \hbar}]$ are fixed. If $[\frac{e a_L}{2\pi \hbar}]$ change, then there is a nontrivial spectral flow, i.e. some energy levels of the first quantized fermionic Hamiltonians cross zero and change sign. This means that the definition of the Fock vacuum of the second quantized fermionic Hamiltonian changes. Since the creation and annihilation operators $a^\dagger, a$ (and $b^\dagger, b$) are continuous functionals of $A_1(x)$, the definition of all excited Fock states is also discontinuous. The connections $\mathcal{A}_{F,\pm}$ are not therefore well defined globally.

Moreover, $\mathcal{A}_{F,\pm}$ are not invariant under $A$-dependent redefinitions of the phases of the Fock states. For these reasons, we usually compute first the $U(1)$ curvature tensors

$$\mathcal{F}^{\pm}_{F}(x,y,t) \equiv \frac{\delta}{\delta A_1(x,t)} \mathcal{A}_{F,\pm}(y,t) - \frac{\delta}{\delta A_1(y,t)} \mathcal{A}_{F,\pm}(x,t)$$

and then deduce $\mathcal{A}_{F,\pm}$.

In particular, for the vacuum states the curvature tensors are

$$\mathcal{F}^{\pm}_{\text{vac}}(x,y,t) = \pm \frac{e^2}{2\pi \hbar^2} \left( \frac{1}{2} \epsilon(x-y) - \frac{1}{L} (x-y) \right),$$

so the corresponding connections are deduced as

$$\mathcal{A}_{\text{vac},\pm}(x,t) = \mathcal{A}_{0,\pm}(x,t) - \frac{1}{2} \int_{-L/2}^{L/2} dy \mathcal{F}^{\pm}_{\text{vac}}(x,y,t) A_1(y,t),$$

where $\mathcal{A}_{0,\pm}(x,t)$ are arbitrary connections which have zero curvature and can not be therefore fixed by this procedure. In $[\mathbb{F}]$ we put $\mathcal{A}_{0,\pm}(x,t) = 0$. 
Let us show that the zero curvature part of the connections (6) can be computed directly by using a Fock vacuum whose definition is globally single-valued. We introduce the new Fock vacuum $|\text{vac}; A; \pm \rangle = |\text{vac}; A; + \rangle \otimes |\text{vac}; A; - \rangle$ defined as

$$a_n |\text{vac}; A; + \rangle = 0 \quad \text{for} \quad n > 0,$$
$$a_n^\dagger |\text{vac}; A; + \rangle = 0 \quad \text{for} \quad n \leq 0,$$

and

$$b_n |\text{vac}; A; - \rangle = 0 \quad \text{for} \quad n \leq 0,$$
$$b_n^\dagger |\text{vac}; A; - \rangle = 0 \quad \text{for} \quad n > 0.$$ 

The new Fock vacuum is defined such that for all values of $[e^{+b L}]_{2\pi \hbar}$ only the levels with energy lower than (or equal to) the energy of the level $n = 0$ are filled and the others are empty, i.e. the new definition does not depend on $[e^{+b L}]_{2\pi \hbar}$ and remains unchanged as the gauge configuration changes. The definition of all excited states constructed over the new Fock vacuum is also globally single-valued.

In the region where $[e^{+b L}]_{2\pi \hbar} = 0$ the old and the new Fock vacuums coincide,

$$|\text{vac}; A; \pm \rangle_{(0)} = |\text{vac}; A; \pm \rangle,$$

where the subscript $(0)$ indicates that $[e^{+b L}]_{2\pi \hbar}$ vanish.

In regions with nonzero $[e^{+b L}]_{2\pi \hbar}$ the positive chirality vacuums are connected as

$$|\text{vac}; A; + \rangle_{(k_+)} = a_{k_+}^\dagger \cdots a_2^\dagger a_1^\dagger |\text{vac}; A; + \rangle$$

for $[e^{+b L}]_{2\pi \hbar} \equiv k_+ > 0$, and

$$|\text{vac}; A; + \rangle_{(k_-)} = a_{k_-+1} \cdots a_{-1} a_0 |\text{vac}; A; + \rangle$$

for $k_- < 0$.

For $k_+ = \pm 1$, the old and the new Fock vacuums of positive chirality are compared in Figures 1 and 2.

In the negative chirality sector we have analogously

$$|\text{vac}; A; - \rangle_{(k_-)} = b_{k_-} \cdots b_2 b_1 |\text{vac}; A; - \rangle$$

for $[e^{+b L}]_{2\pi \hbar} \equiv k_- > 0$, and

$$|\text{vac}; A; - \rangle_{(k_-)} = b_{k_-+1}^\dagger \cdots b_{-1}^\dagger b_0^\dagger |\text{vac}; A; - \rangle$$

for $k_- < 0$.

Next we define the vacuum Berry phase connections in the regions of different values of $k_+$:

$$\mathcal{A}_{\text{vac}, +}^{(k_+, q_+)}(x, t) \equiv (k_+) \langle \text{vac}; A; + | i \frac{\delta}{\delta A_1(x, t)} |\text{vac}; A; + \rangle_{(q_+)}. $$

For simplicity, we start with the positive chirality sector and positive values of $k_+, q_+$.

Making transition to the new Fock vacuum, we get

$$\mathcal{A}_{\text{vac}, +}^{(k_+, q_+)}(x, t) = \langle \text{vac}; A; + | a_1 a_2 \cdots a_{k_+} \cdot i \frac{\delta}{\delta A_1(x, t)} \cdot a_{q_+}^\dagger \cdots a_2^\dagger a_1^\dagger |\text{vac}; A; + \rangle. $$

(7)
For \( k_+ \neq q_+ \), the number of creation operators in (7) is not equal to the number of annihilation ones. Those creation (or annihilation) operators which have not their annihilation (or creation) counterparts annihilate the vacuum state \( \langle \text{vac; } A; + | \) (or \( | \text{vac; } A; + \rangle \)), so that the connections \( \mathcal{A}_{\text{vac,+}}^{(k_+, q_+)} \) vanish.

For \( k_+ = q_+ \), all the creation and annihilation operators in (7) can be paired. In the \( \zeta \)-function regularization scheme, the action of the functional derivative \( \delta / \delta A_1 \) on the operators \( a_n, a_n^\dagger \) is given by [11]

\[
\frac{\delta}{\delta A_1} a_n = -\lim_{s \to 0} \sum_{m \in \mathbb{Z}} \langle n; + | \frac{\delta}{\delta A_1} | m; + \rangle a_m | \lambda \varepsilon_{m,+} |^{-s/2},
\]

\[
\frac{\delta}{\delta A_1} a_n^\dagger = \lim_{s \to 0} \sum_{m \in \mathbb{Z}} \langle m; + | \frac{\delta}{\delta A_1} | n; + \rangle a_m^\dagger | \lambda \varepsilon_{m,+} |^{-s/2}.
\]

Eq.(7) is then rewritten as

\[
\mathcal{A}_{\text{vac,+}}^{(k_+, q_+)} = \mathcal{A}_{\text{vac,+}} + \lim_{s \to 0} \sum_{n=1}^{k_+} \langle n; + | \frac{\delta}{\delta A_1} a_n | n; + \rangle | \lambda \varepsilon_{n,+} |^{-s/2},
\]

where

\[
\mathcal{A}_{\text{vac,+}}(x, t) \equiv \langle \text{vac; } A; + | \frac{\delta}{\delta A_1} (x, t) | \text{vac; } A; + \rangle
\]

is the Berry phase connection for the new vacuum of positive chirality.

By a direct calculation of the expectation values of \( i \frac{\delta}{\delta A_1} \) for the first quantized kets and bras we obtain

\[
\mathcal{A}_{\text{vac,+}}^{(k_+, q_+)} = \mathcal{A}_{\text{vac,+}} + \frac{1}{\hbar} e_+ \left( x - \frac{L}{2} \right) k_+.
\] (8)

For negative values of \( k_+, q_+ \), we get the same result, namely, if \( k_+ \neq q_+ \) including the cases \( k_+ > 0, q_+ < 0 \) and \( k_+ < 0, q_+ > 0 \), then \( \mathcal{A}_{\text{vac,+}}^{(k_+, q_+)} \) vanish, while for \( k_+ = q_+ \) the connections are given by Eq.(8).

In the negative chirality sector, the connections

\[
\mathcal{A}_{\text{vac,-}}^{(k_-, q_-)}(x, t) \equiv (q_-) \langle \text{vac; } A; - | i \frac{\delta}{\delta A_1} (x, t) | \text{vac; } A; - \rangle
\]

are computed analogously, being nonzero as before only for \( k_- = q_- \):

\[
\mathcal{A}_{\text{vac,-}}^{(k_-, q_-)} = \mathcal{A}_{\text{vac,-}} - \frac{1}{\hbar} e_- \left( x - \frac{L}{2} \right) k_-,
\] (9)

where

\[
\mathcal{A}_{\text{vac,-}} \equiv \langle \text{vac; } A; - | i \frac{\delta}{\delta A_1} | \text{vac; } A; - \rangle.
\]

Using the definition of \( k_\pm \), we immediately reproduce from Eqs.(8) and (9) the vacuum Berry connections defined on whole manifold \( \mathcal{A}^1 \):

\[
\mathcal{A}_{\text{vac,±}} = \mathcal{A}_{\text{vac,±}} \pm \frac{1}{\hbar} e_\pm \left( x - \frac{L}{2} \right) \frac{e_\pm bL}{2\pi\hbar}.
\] (10)
Only the first term in (10) contributes to the vacuum curvature tensor, \( F_{\text{vac}}^\pm = \overline{F}_{\text{vac}}^\pm \), and can be therefore deduced from the curvature, so we write

\[
A_{\text{vac}}^\pm(x, t) = -\frac{1}{2} \int_{-L/2}^{L/2} dy F_{\text{vac}}^\pm(x, y, t) A_1(y, t).
\] (11)

The second term has vanishing curvature, and we identify it with \( A_0^\pm(x, t) :\)

\[
A_0^\pm(x, t) = \pm \frac{1}{\hbar} e^\pm \left[ \frac{e^\pm b L}{2\pi\hbar} \right].
\] (12)

We see that just the zero curvature part of the vacuum Berry phase connections is discontinuous on \( A_1 \). In the transition between regions of different \( \frac{e^\pm b L}{2\pi\hbar} \) the zero curvature part changes by a multiple of \( \pm \frac{1}{\hbar} e^\pm (x - \frac{L}{2}) \).

In the models with anomaly the vacuum Berry connection is added to the electric field operator in order to keep gauge invariant the full quantum theory with both matter and gauge fields quantized [4]. This connection represents a background for the quantum physical degrees of freedom. In our case, the background corresponding to the zero curvature connections (12) consists of linearly rising electric fields

\[
E_\pm(x) = \pm \frac{e_\pm}{L} x \left[ \frac{e^\pm b L}{2\pi\hbar} \right],
\]

so that

\[
A_{0,\pm}(x, t) = \frac{1}{\hbar} \left( E_\pm(\frac{L}{2}) - E_\pm(x) \right).
\]

4. In conclusion, we have calculated the zero curvature part of the vacuum Berry phase connections directly by using the globally single-valued definition for the Fock vacuum.

The zero curvature vacuum Berry connections have several features. They depend only on the zero mode of the gauge field, change discontinuously on \( A_1 \) and define a background of linearly rising electric fields.

That part of the vacuum Berry phase connections which contributes to the curvature is continuous on \( A_1 \) and depends on the gauge field non-zero modes. For any gauge field defined on the circle, its zero mode represents the only global physical degree of freedom, while the non-zero ones are gauge variant and can be removed by the gauge transformations. The background defined by the zero curvature part of the Berry phase connections is therefore a physical one and survives in the physical sector of the full quantum theory. This indicates that the zero curvature vacuum Berry connections play an essential role in the construction of the physical quantum picture of the anomalous models.

For the chiral Schwinger model, the existence of the background of the linearly rising electric field \( E(x) = E_+(x) + E_-(x) \) was proved previously in [1, 8]. In the present paper, we have obtained the same background in the framework of the adiabatic approximation.

It would be of interest to calculate directly the connections \( A_{\text{vac}}^\pm \) as well. Then we could check whether the separation of the zero and non-zero curvature parts in (11) is complete or not.

Both the zero and non-zero curvature parts of the vacuum Berry phase connections are associated with anomaly. The motion of the physical degrees of freedom in the linearly rising background electric field represents a new type of interactions which are absent in the nonanomalous models [8]. For the standard Schwinger model with \( e_+ = e_- \), the zero curvature parts of the Berry phase connections for the vacuums of positive and negative chiralities are opposite in sign and so cancel each other. The total non-zero curvature part of the vacuum Berry connections is related to 1-cocycle of the gauge group projective representation responsible for anomaly and also vanishes for \( e_+ = e_- \) [3].
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Figure Captions

**FIG. 1.** Schematic representation of vacuum states for $\frac{e_{e^{bL}}}{2\pi\hbar} = 1$ : (a): $|\text{vac}; A; +\rangle$, (b): $|\text{vac}; A; +\rangle$. Only the positive chirality sector is shown.

**FIG. 2.** Schematic representation of vacuum states for $\frac{e_{e^{bL}}}{2\pi\hbar} = -1$ : (a): $|\text{vac}; A; +\rangle$, (b): $|\text{vac}; A; +\rangle$. Only the positive chirality sector is shown.
FIG. 1.
FIG. 2.