Tsallis statistics and Langevin equation with multiplicative noise in different orders of prescription

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Abstract

Usually discussions on the question of interpretation in the Langevin equation with multiplicative white noise are limited to the Ito and Stratonovich prescriptions. In this work, a Langevin equation with multiplicative white noise and its Fokker-Planck equation are considered. From this Fokker-Planck equation a connection between the stationary solution and the Tsallis distribution is obtained for different orders of prescription in discretization rule for the stochastic integrals; the Tsallis index $q$ and the prescription parameter $\lambda$ are determined with the drift and diffusion coefficients. The result is quite general. For application, one shows that the Tsallis distribution can be described by a class of population growth models subject to the linear multiplicative white noise.

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The Langevin equation is a very important tool for describing out of equilibrium systems \[1–3\]. Moreover, this equation has been extensively investigated; many properties and analytical solutions of it have also been revealed. In particular, one considers the following Langevin equation in one-dimensional space with a multiplicative white noise term:

\[
\frac{d\xi}{dt} = h(\xi) + g(\xi)\Gamma(t), \tag{1}
\]

where \(\xi\) is a stochastic variable and \(\Gamma(t)\) is the Langevin force with the averages \(\langle \Gamma(t) \rangle = 0\) and \(\langle \Gamma(t)\Gamma(t') \rangle = 2\delta(t-t')\) [1]. \(h(\xi)\) is the deterministic drift. Physically, the additive noise \((g(\xi)\) constant) may represent the heat bath acting on the particle of the system, and the multiplicative noise term, for variable \(g(\xi)\), may represent a fluctuating barrier. For \(g = \sqrt{D}\) and \(h(\xi) = 0\), Eq. (1) describes the Wiener process and the corresponding probability distribution is described by a Gaussian function. In the case of \(g(\xi)\), some specific functions have been employed to study, for instance, turbulent flows \((g(x) \sim |x|^a)\) [4]. It should be noted that the linear Fokker-Planck equation corresponding to the Langevin equation (1) involves different orders of prescription in discretization rule for the stochastic integrals [1].

The aim of this work is to obtain a class of these Fokker-Planck equations in which their stationary solutions are described by the Tsallis distribution [5, 6]. In particular, the Tsallis distribution has been connected to a variety of natural systems (see, for instance, [7–12]), and it is given by

\[
W_q(x) = N [1 - \beta(1-q)U(x)]^{\frac{1}{1-q}}, \tag{2}
\]

where \(q\) is the Tsallis index and \(U(x)\) is the potential. Notice that this result extends the one given in Ref. [13]. Moreover, one shows that the Tsallis distribution can be described by a class of population growth models subject to the linear multiplicative white noise.

In one-dimensional space, the forward Fokker-Planck equation [1, 14–16] for the probability distribution corresponding to the Langevin equation (1) is given by

\[
\frac{\partial W(x,t)}{\partial t} = -\frac{\partial}{\partial x} [D_1(x)W(x,t)] + \frac{\partial^2}{\partial x^2} [D(x)W(x,t)], \tag{3}
\]

where \(D_1(x,t)\) and \(D(x,t)\) are the drift and diffusion coefficients given by

\[
D_1(x) = h(x) + 2\lambda \frac{dg(x)}{dx}g(x) \tag{4}
\]

and

\[
D(x) = g^2(x), \tag{5}
\]
and \( 0 \leq \lambda \leq 1 \) is the prescription parameter due to the discretization rule for the stochastic integrals. For instance, \( \lambda = 0 \) corresponds to the Ito prescription, \( \lambda = 1/2 \) corresponds to the Stratonovich prescription and \( \lambda = 1 \) corresponds to the transport or Hänggi-Klimontovich prescription \([14–16]\). Notice that Eq. (4) does not have a spurious drift only for the Ito prescription, and the prescription parameter is directly linked to the coefficient \( g(x) \) of the fluctuation. The stationary solution of Eq. (3) can be described as follows:

\[
W(x) = N \exp \left( \int \frac{dx}{D(x)} \left( D_1(x) - \frac{dD(x)}{dx} \right) \right).
\]  

(6)

Now this distribution will be equal to the Tsallis distribution (2) if the following relation is satisfied

\[
\frac{1}{D(x)} \left[ h(x) + (\lambda - 1) \frac{dD(x)}{dx} \right] = -\frac{\beta}{1 - \beta(1 - q)} U(x) \frac{dU(x)}{dx}.
\]

(7)

Note that this last relation is quite general, i.e., it is valid for a large class of drift and diffusion coefficients and any prescription. In particular, Eq. (7) has been used to show that the probability distribution in momentum space of the atom-laser interaction in the optical lattice is described by the Tsallis distribution \([17]\). For another application of Eq. (7), one considers the models of population growth which are frequently described by nonlinear differential equations without the independent variable (time) explicitly. For instance, the classical Verhulst logistic equation is a simple nonlinear model of population growth and it has been employed as a starting point to formulate various generalized models \([18–25]\).

Moreover, this logistic equation has been successfully used to model many laboratory populations such as yeast growth in laboratory cultures, growth of the Tasmanian and South Australian sheep populations \([19]\) and self-organization at macromolecular level \([26, 27]\).

Besides, the parameters involved in these models are subject to fluctuations and various types of noises which may affect the replication processes. In this work, one considers the following class of population growth models \([28]\):

\[
h(x) = r \frac{x \left[ 1 - \left( \frac{x}{K} \right)^\nu \right]}{\mu \left[ 1 - \left( 1 - \frac{\nu}{\mu} \left( \frac{x}{K} \right)^\nu \right) \right]},
\]

(8)

where \( x(t) \) is the number of population alive at time \( t \), \( \mu \) and \( \nu \) are real positive parameters, \( r \) is the intrinsic growth rate and \( K \) is the carrying capacity. The deterministic model given by Eqs. (1) and (8), with \( g(x) = 0 \), contains the classical growth models such as Verhulst logistic model (\( \mu = 1 \) and \( \nu = 1 \)), Gompertz model (\( \mu = 0 \) and \( \nu \to 0 \)), Schoener model
(μ = 0 and ν = 1), Richards model (μ = 0 and 0 ≤ ν < ∞) and Smith model (0 ≤ μ < ∞ and ν = 1). In applications, the influence of the noise (such as a change in temperature, food and water supplies) is usually considered by the linear multiplicative noise of the effective birth rate \[ \dot{x} = \epsilon x, \] where \( \epsilon \) is associated with the noise strength; in this case, the diffusion coefficient is given by \( D(x) = \epsilon^2 x^2 \). For nonlinearly coupled noise see for instance [30]. For the system described by Eqs. (1) and (8) without the influence of the noise \( g(x) = 0 \), the solution can be determined implicitly and it is given by

\[
rt = \ln \left( \frac{(x/K)^\mu \left( 1 - \left( \frac{x}{x_0} \right)^\nu \right)}{(x_0/K)^\mu \left( 1 - \left( \frac{x}{x_0} \right)^\nu \right)} \right),
\]

(9)

where \( x_0 = x(t = 0) \) is the initial value. Fig. 1 shows the evolution of population for different models; they exhibit sigmoidal shapes.

Now, with the influence of the noise one considers the linearly coupled noise; from Eq. (6) yields

\[
W(x) \sim x^{r\mu\epsilon^2 + 2(\lambda - 1)} \left[ 1 - \left( 1 - \frac{\nu}{\mu} \right) \left( \frac{x}{K} \right)^\nu \right]^{r\mu\epsilon^2 (\mu - \nu)}.
\]

(10)

In particular, for \( \mu = \nu \) one obtains

\[
W(x) \sim x^{r\mu\epsilon^2 + 2(\lambda - 1)} \exp \left( -\frac{r}{\mu^2 \epsilon^2} \left( \frac{x}{K} \right)^\mu \right).
\]

(11)

The distribution (11) is well-known and it is called the Weibull distribution. Besides, the distribution (10) presents interesting aspects. In order to retain a consistent probabilistic interpretation, the cut-off condition imposes \( W(x) = 0 \) whenever \( \mu > \nu \) and \( \left[ 1 - \left( 1 - \frac{\nu}{\mu} \right) \left( \frac{x}{K} \right)^\nu \right] < 0 \). For \( \lambda = 1 \) and positive values for \( r \) and \( \mu \) the distribution (10) is zero at the origin; in this case the populations can always survive. However, for \( \lambda \neq 1 \) the distribution (10) is divergent at the origin for

\[-1 < \frac{r}{\mu \epsilon^2} + 2 \left( \lambda - 1 \right) < 0;\]

(12)

this means that the order of prescription in the discretization rule for the stochastic integrals can prevent population extinction.

From the solution (10) the Tsallis distribution can be obtained by setting

\[
\frac{r}{\mu \epsilon^2} + 2 \left( \lambda - 1 \right) = 0, \quad \lambda \neq 1
\]

(13)

and

\[
1 - q = \frac{\mu \epsilon^2 (\mu - \nu)}{r}.
\]

(14)
Eq. (13) implies that

\[ \frac{r}{2\mu\epsilon^2} < 1 \] (15)

and

\[ \lambda = 1 - \frac{r}{2\mu\epsilon^2}. \] (16)

The expression (14) shows that the Tsallis index \( q \) can assume any real value, and it is also associated with the microscopic parameters of the system. For \( q = 1 \) yields \( \mu = \nu \), and the distribution (10) reduces to the stretched exponential for \( 0 < \mu < 1 \), exponential function for \( \mu = 1 \) and compressed exponential for \( \mu > 1 \). It is worth noting that the distribution (10) with the conditions (13) and (14) also satisfy the relation (17) for

\[ U(x) = x^\nu \] (17)

and

\[ \beta = \frac{r}{\mu^2\epsilon^2 K^\nu}. \] (18)

In summary, a Langevin equation with multiplicative white noise and its corresponding linear Fokker-Planck equation for different orders of prescription in discretization rule for the stochastic integrals have been considered. In particular, the stationary solution of the Fokker-Planck equation has been connected with the Tsallis distribution for generic drift and diffusion coefficients. As an application one showed that the stationary solution of a class of population growth models subject to the linear multiplicative white noise could be described by the Tsallis distribution for different orders of prescription. In these models, the Tsallis index \( q \) has been connected with the microscopic parameters. Interesting aspects have been achieved; the order of prescription in discretization rule for the stochastic integrals plays a key role to connect with the Tsallis distribution. Moreover, the change in the prescription parameter \( \lambda \) of a given system may lead to the extinction or survival of a population. As is expected, the prescription parameter can modify the behavior of the system considerably, i.e., different prescription may describe different behavior.

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Figure Captions

Fig. 1 - Plots of the evolution of population for different growth models described by Eq. (9), in arbitrary units. The parameter values are given by $r = 0.5$, $x_0 = 0.2$ and $K = 10$. 