Optimal Estimation of Sensor Biases for Asynchronous Multi-Sensor Registration

Wenqiang Pu, Ya-Feng Liu, Member, IEEE, Junkun Yan, Member, IEEE, Hongwei Liu, Member, IEEE, Zhi-Quan Luo, Fellow, IEEE

Abstract

An important step in the asynchronous multi-sensor registration problem is to estimate sensor range and azimuth biases from their noisy asynchronous measurements. The estimation problem is generally very challenging due to highly nonlinear transformation between the global and local coordinate systems as well as measurement asynchrony from different sensors. In this paper, we propose a novel nonlinear least square (LS) formulation for the problem by only assuming that a reference target moves with an unknown constant velocity. We also propose a block coordinate decent (BCD) optimization algorithm, with a judicious initialization, for solving the problem. The proposed BCD algorithm alternately updates the range and azimuth bias estimates by solving linear least square problems and semidefinite programs (SDPs). The proposed algorithm is guaranteed to find the global solution of the problem and the true biases in the noiseless case. Simulation results show that the proposed algorithm significantly outperforms the existing approaches in terms of the root mean square error (RMSE).

Index Terms
I. INTRODUCTION

A. Background and Motivation

In recent years, there is an increasing interest in integrating stand-alone sensors into the multi-sensor systems for command, control, and communications [2]. Instead of developing expensive high-performance sensors, directly fusing data from existing multiple inexpensive sensors is a more cost-effective approach to improving the performance of tracking and surveillance systems. An important process in multi-sensor integration is registration (or alignment) [3], whereby the multi-sensor data is expressed in a common reference frame, by removing the sensor biases caused by antenna orientation and improper alignment [4]. Since sensor biases change slowly with time [4], they can be treated as constants during a relatively long period of time. Consequently, the sensor registration problem is usually an estimation problem for sensor biases.

In the last three decades, both synchronous and asynchronous sensor registration problems have been studied [3]–[19]. In the synchronous case, all sensors simultaneously observe the target at the same time instances, whereas in the asynchronous case, the sensors observe the target at different time instances.

For the synchronous registration problem, various approaches have been studied in references [3]–[15]. Among them, [4] identified various factors which dominate the alignment errors in the multi-sensor system and established a sensor bias model. Under the assumption that there exists a bias-free sensor, a maximum likelihood (ML) estimation formulation [4], [5] and a nonlinear least square (LS) estimation formulation [6] were presented for this problem, and various algorithms were proposed to approximately solve the problem including the expectation maximization (EM) algorithm [5] and the successive linearization (least square) algorithm [6]. In the absence of a bias-free sensor, several approaches [3], [7]–[15] have also been proposed. These approaches, from the perspective of parameter estimation point, can be divided into the following three types: LS, ML and Bayesian estimation. Both of LS and ML approaches treat biases as deterministic variables while the Bayesian approach treats biases as random variables. Note that there are two different kinds of variables, target positions and sensor biases, in the sensor registration problem. The difference between LS and ML approaches lies in the way of dealing with target positions. In the LS approaches [3], [7], [8], target positions are approximately represented and eliminated in the LS formulation and finally only sensor biases are estimated. In contrast, the ML approaches jointly estimate target positions and sensor biases [9], [10]. An iterative two-step optimization algorithm with some approximations was proposed in [9] for solving the ML formulation, with one step for estimating...
target positions and the other for estimating sensor biases. The Bayesian approach treats sensor biases as random variables and integrates target tracking and biases estimation in a Bayesian estimation framework [11]–[15]. Different strategies were developed to decouple the two estimation tasks and to deal with the nonlinearity, including two-stage Kalman filter (KF) [11], EM-KF [14], and EM particle filter [13].

In practice, sensors are often not synchronized in time due to different data rates. The asynchronous sensor measurements make the estimation problem underdetermined. To overcome this difficulty, researchers have exploited a priori knowledge of the target motion model such as the nearly constant velocity motion [20]. Based on this side information, recursive Bayesian approaches for jointly estimating target states and sensor biases have been proposed in [16]–[19]. More specifically, a special case with two asynchronous sensors was studied in [16], [17] under the assumption of linear measurement model. However, these two approaches require the sensor biases to be very small. In references [18], [19], a unified Bayesian framework for target tracking and biases estimation was proposed based on probability hypothesis density (PHD) [21]. This approach relies on the implementation of particle PHD filter which can result in high computational cost when the number of sensors is large.

B. Our Contributions

In this paper, we consider the sensor registration problem and focus on a general scenario where all sensors are biased and work asynchronously. The main contributions of the paper are as follows:

(1) A New nonlinear LS Formulation: We propose a new nonlinear LS formulation for the asynchronous multi-sensor registration problem by only assuming the a priori information that the target moves along a straight line with an unknown constant velocity. Our proposed formulation is in sharp contrast to existing problem formulations [16]–[19] from Bayesian point of view.

(2) Separation Property of Range Bias Estimation: We reveal an interesting separation property of the range bias estimation problem. More specifically, we show that the solution of the problem of minimizing the LS error over the range bias for each sensor can be decoupled from its azimuth bias. This property sheds an important insight that each sensor can estimate its range bias from its local measurements.

(3) An Efficient BCD Algorithm and Exact Recovery: We fully exploit the special structure of the proposed nonlinear LS formulation and develop an efficient block coordinate decent (BCD) algorithm, with a judicious initialization by using the separation property of range bias estimation. The proposed algorithm alternately updates the range and azimuth biases by solving linear least square problems and semidefinite programs (SDPs). We show, in Theorem 2 that solving the original non-convex problem with respect to the azimuth biases is equivalent to solving an SDP, which itself is interesting. We also show exact
recovery of the proposed BCD algorithm in the noiseless case, i.e., our algorithm is able to find the true (range and azimuth) biases if there is no noise in the measurements.

The notations we adopted in this paper are listed in Table I.

| Notation | Description |
|----------|-------------|
| $x$      | Scalar number |
| $\mathbf{x}$ | Vector |
| $\mathbf{X}$ | Matrix |
| $\mathbf{X}^T$ | Transpose of $\mathbf{X}$ |
| $\mathbf{X}^\dagger$ | Conjugate transpose of $\mathbf{X}$ |
| $\mathbf{X}^{-1}$ | Inverse of invertible $\mathbf{X}$ |
| $\text{Tr}(\mathbf{X})$ | Trace of $\mathbf{X}$ |
| $\mathbf{X} \succ 0$ | $\mathbf{X}$ is a positive definite matrix |
| $\mathbf{X} \succeq 0$ | $\mathbf{X}$ is a positive semidefinite matrix |
| $\text{diag}(\mathbf{X})$ | Vector formed by diagonal entries of $\mathbf{X}$ |
| $\text{vec}(\mathbf{X})$ | Vector sequentially stacked by columns of $\mathbf{X}$ |
| $\lambda_{\text{min}}(\mathbf{X})$ | Smallest eigenvalue of $\mathbf{X}$ |
| $\text{Diag}(\mathbf{X})$ | Diagonal matrix formed by all components of $\mathbf{x}$ |
| $\mathbf{x}_n$ | $n$-th component of $\mathbf{x}$ |
| $\mathbf{x}_{n:m}$ | Vector formed by components of $\mathbf{x}$ from index $n$ to $m$ |
| $\|\mathbf{x}\|$ | Euclidean norm of $\mathbf{x}$ |
| $E_w\{\cdot\}$ | Expectation operation w.r.t. random variable $w$ |
| $\angle c$ | Phase of a complex scalar $c$ |
| $\text{Re}\{c\}$ | Real part of a complex scalar $c$ |
| $\text{Im}\{c\}$ | Imaginary part of a scalar number $c$ |
| $\mathbf{1}(0)$ | All-one (all-zero) vector of proper size |
| $\mathbf{I}_M$ | Identity matrix of size $M$ |
| $\mathbb{R}^M$ | Set of $M$-dimensional real vectors |
| $\mathbb{S}^M$ | Set of $M \times M$ symmetric matrices |
| $\mathbb{C}^M$ | Set of $M$-dimensional complex vectors |
| $\mathbb{H}^M$ | Set of $M \times M$ Hermitian matrices |

II. THE ASYNCHRONOUS MULTI-SENSOR REGISTRATION PROBLEM

Consider a multi-sensor system consisting of $M > 1$ sensors located distributively on a 2-dimensional plane with known positions. Suppose that there is a reference target (e.g., a civilian airplane) moving in the surveillance space at an unknown constant velocity, and sensors measure the relative range and
azimuth between the target and sensors themselves in an asynchronous work mode. For ease of notation and presentation, measurements from different sensors are mapped onto a common time axis at the fusion center, indexed by \( k \). Furthermore, we assume that, at time instance \( k \), only one sensor observes the target and the corresponding sensor is denoted as \( s_k \in \{1, 2, \ldots, M\} \). See Fig. 1 for an illustration of the asynchronous work mode with \( M = 2 \) sensors.

![Fig. 1: An illustration of the asynchronous work mode with 2 sensors.](image)

Let \( \xi_k = [x_k, y_k]^T \) denote the target position in the common coordinate system at time instance \( k \). Then, the measured range \( \rho_k \) and azimuth \( \phi_k \) at sensor \( s_k \) are

\[
\mathbf{z}_k = \begin{bmatrix} \rho_k \\ \phi_k \end{bmatrix} = h^{-1}(\xi_k - \mathbf{p}_{s_k}) - \bar{\theta}_{s_k} + \mathbf{w}_k.
\]

In the above, \( h^{-1}(\cdot) \) is the inverse function of the 2-dimensional spherical-to-Cartesian transformation function \( h(\cdot) \); \( \mathbf{p}_m = [p_{mx}, p_{my}]^T \) is the position of sensor \( m \); \( \bar{\theta}_m = [\Delta \rho_m, \Delta \phi_m]^T \) is the true range and azimuth biases of sensor \( m \); \( \mathbf{w}_k = [w_{k\rho}, w_{k\phi}]^T \) is an uncorrelated Gaussian random noise vector with zero mean \([20]\), i.e.,

\[
\mathbf{w}_k \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \sigma_{\rho}^2 & 0 \\ 0 & \sigma_{\phi}^2 \end{bmatrix}\right).
\]

The biased measurements for a target with a constant velocity \( \mathbf{v} = [\bar{v}_x, \bar{v}_y]^T \neq \mathbf{0} \) are illustrated in Fig. 2.

The asynchronous multi-sensor registration problem considered in this paper is to estimate sensor biases \( \{\bar{\theta}_m\}_{m=1}^M \) from the noisy measurements \( \{\mathbf{z}_k\}_{k=1}^K \), where \( K \) is the total number of measurements. There are \( 2K + 2M \) unknown parameters \( \{\xi_k\}_{k=1}^K \) and \( \{\bar{\theta}_m\}_{m=1}^M \) but in total only \( 2K \) measurements \( \{\mathbf{z}_k\}_{k=1}^K \) and thus the estimation problem is underdetermined. Assuming that the target moves along a straight line with the constant velocity and by exploiting this side information, we can estimate \( \{\bar{\theta}_m\}_{m=1}^M \) from measurements \( \{\mathbf{z}_k\}_{k=1}^K \).
Specifically, we assume that the reference target moves with an \textit{unknown} constant velocity $\bar{v}$. Then, the target positions at time instances $k+1$ and $k$ satisfy $\xi_{k+1} = \xi_k + T_k \bar{v}$. 

where $T_k$ is the time difference between the time instances $k+1$ and $k$. Notice that (1) can be equivalently rewritten as

$\xi_k = h(z_k - w_k + \bar{\theta}_{s_k}) + p_{s_k}$. \hspace{1cm} (4)

Combining (3) and (4), we can see that the asynchronous measurements $z_{k+1}$ and $z_k$ must satisfy

$h(z_{k+1} - w_{k+1} + \bar{\theta}_{s_{k+1}}) + p_{s_{k+1}}$

$= h(z_k - w_k + \bar{\theta}_{s_k}) + p_{s_k} + T_k \bar{v}$.

To handle Gaussian noise $w_k$, we make use of the following \textit{unbiased} spherical-to-Cartesian coordinate transformation $[23]$:

$\tilde{h}(z_k) = \begin{bmatrix} \lambda^{-1} \rho_k \cos \phi_k \\ \lambda^{-1} \rho_k \sin \phi_k \end{bmatrix}$, \hspace{1cm} (5)

where $\lambda = e^{-\sigma^2/2}$ is the noise compensation factor. The unbiasedness refers to $E_{w_k} \{ \tilde{h}(z_k) \} = h \{ E_{w_k} \{ z_k \} \}$, which further implies

$\xi_k = E_{w_k} \{ \tilde{h}(z_k + \bar{\theta}_{s_k}) \} + p_{s_k}$. \hspace{1cm} (6)

\footnote{For ease of presentation, we assume a deterministic target motion model in this paper. In cases where there is uncertainty in the target motion model such as the nearly-constant-velocity model [22], our main results (e.g., Theorems [1] and [2]) still hold.}
By combining (3) and (6), we can formulate the problem of estimating sensor range biases $\Delta \rho = [\Delta \rho_1, \Delta \rho_2, \ldots, \Delta \rho_M]^T$, azimuth biases $\Delta \phi = [\Delta \phi_1, \Delta \phi_2, \ldots, \Delta \phi_M]^T$, and target velocity $v$ as the following nonlinear LS problem

$$\min_{\Delta \rho, \Delta \phi, v} f(\Delta \rho, \Delta \phi, v)$$

(7)

where

$$f(\Delta \rho, \Delta \phi, v) = \sum_{k=1}^{K-1} \| g_{k+1}(\theta_{s_{k+1}}) - g_k(\theta_{s_k}) - T_k v \|^2,$$

and

$$g_k(\Delta \phi_{s_k}) = \bar{h}(z_k + \theta_{s_k}) + p_{s_k}, \; k = 1, 2, \ldots, K,$$

$$\theta_{s_k} = [\Delta \rho_{s_k}, \Delta \phi_{s_k}]^T, \; k = 1, 2, \ldots, K.$$

Problem (7) is a non-convex problem due to the nonlinearity of $\bar{h}(\cdot)$. In the sequel, we shall develop efficient algorithms for solving problem (7). In particular, we first study the single-sensor case of problem (7) with $M = 1$ in Section III and then study the multi-sensor case of problem (7) with $M \geq 2$ in Section IV.

III. SINGLE-SENSOR CASE: SEPARATION PROPERTY

Consider problem (7) with $M = 1$. With a slight abuse of notations, we still use $\{z_k\}_{k=1}^K$ to denote the measurements from the sensor and use $\theta = [\Delta \rho, \Delta \phi]^T$ to denote the range and azimuth biases to be estimated.

Problem (7) with only one sensor can be simplified to:

$$\min_{\theta, v} f(\theta, v) \triangleq \sum_{k=1}^{K-1} \| \bar{h}(z_{k+1} + \theta) - \bar{h}(z_k + \theta) - T_k v \|^2.$$

(8)

In light of (5), $f(\theta, v)$ is a convex quadratic function with respect to $\Delta \rho$ and $v$, but nonlinear and non-convex in terms of $\Delta \phi$. For any fixed $\Delta \phi$, problem (8) can be equivalently rewritten as the following convex quadratic program with respect to $\Delta \rho$ and $v$:

$$\min_{\Delta \rho, v} \| H_{\Delta \phi} \begin{bmatrix} \Delta \rho \\ v \end{bmatrix} - y_{\Delta \phi} \|^2,$$

(9)
where $H_{\Delta \phi} = [H_0, H_1] \in \mathbb{R}^{2(K-1) \times 3}$ and $y_{\Delta \phi} \in \mathbb{R}^{2(K-1)}$ depend on the value of $\Delta \phi$, the sensor measurements $\{z_k\}_{k=1}^K$, and time differences $\{T_k\}_{k=1}^{K-1}$, i.e.,

$$H_0 = \begin{bmatrix} c_1 \\ s_1 \\ c_2 \\ s_2 \\ \vdots \\ c_{K-1} \\ s_{K-1} \end{bmatrix}, \quad H_1 = \begin{bmatrix} -T_1 \\ 0 \\ -T_2 \\ 0 \\ \vdots \\ -T_{K-1} \\ 0 \end{bmatrix},$$

$$y_{\Delta \phi} = \begin{bmatrix} y_{c_1} \\ y_{s_1} \\ y_{c_2} \\ y_{s_2} \\ \vdots \\ y_{c_{K-1}} \\ y_{s_{K-1}} \end{bmatrix}^T.$$

In the above,

$$c_k = \lambda^{-1} [\cos(\phi_{k+1} + \Delta \phi) - \cos(\phi_k + \Delta \phi)],$$

$$s_k = \lambda^{-1} [\sin(\phi_{k+1} + \Delta \phi) - \sin(\phi_k + \Delta \phi)],$$

and

$$y_{c_k} = \lambda^{-1} [\rho_{k+1} \cos(\phi_{k+1} + \Delta \phi) - \rho_k \cos(\phi_k + \Delta \phi)],$$

$$y_{s_k} = \lambda^{-1} [\rho_{k+1} \sin(\phi_{k+1} + \Delta \phi) - \rho_k \sin(\phi_k + \Delta \phi)].$$

Suppose $K \geq 3$ (such that $H_{\Delta \phi}$ is generically full column rank), then the unique optimal solution of problem (9) is

$$\begin{bmatrix} \Delta \rho^* \\ v^* \end{bmatrix} = (H_{\Delta \phi}^T H_{\Delta \phi})^{-1} H_{\Delta \phi}^T y_{\Delta \phi}.$$  \hspace{1cm} (15)

The following Theorem 1 shows, somewhat surprisingly, that $\Delta \rho^*$ in (15) does not depend on $\Delta \phi$ and hence is optimal to problem (8).

**Theorem 1.** For any given $\Delta \phi$, problems (8) and (9) have the same optimal $\Delta \rho^*$ given by (15) and the same optimal objective value. However, the optimal $v^*$ in (15) depends on $\Delta \phi$.

The proof of Theorem 1 is given in Appendix A. Here we give an intuitive explanation of Theorem 1 by Fig. 3 below. Suppose that there is no noise. Given the original measurements (green points), problem (8) aims at finding an azimuth bias $\Delta \phi$, a range bias $\Delta \rho$, and a velocity vector $v$ to minimize the
matching errors (corresponding to the square sum of the length of those black segments in Fig. [3]). As shown in Fig. [3], when we rotate green points to blue points by \(\Delta \phi\) or to blue circles by \(\Delta \phi'\), the relative positions of the obtained points (circles) do not change and neither the optimal \(\Delta \rho\) and the optimal value of problem (8). However, the optimal velocity of problem (8) indeed changes, i.e., it changes from \(v\) to \(v'\) when the rotation changes from \(\Delta \phi\) to \(\Delta \phi'\).

![Fig. 3: A geometrical explanation of Theorem 1](image)

Based on Theorem 1, we have the following corollary.

**Corollary 1.** If there is no measurement noise, i.e., \(\sigma_{\rho}^2 = \sigma_\phi^2 = 0\), solving problem (9) can recover the true range bias, \(\Delta \rho^* = \Delta \bar{\rho}\), where \(\Delta \bar{\rho}\) is the true range bias.

**Proof.** In the absence of measurement noise, we have, from measurement model (1), target motion model (3), and definition of \(\bar{h}(\cdot)\), that

\[
\bar{h}(z_{k+1} + \bar{\theta}) - \bar{h}(z_k + \bar{\theta}) - T_k v = 0.
\]

In other words, the optimal value of problem (8) is zero and thus \(\bar{\theta}\) is its global minimizer. Suppose that \(\Delta \phi\) in problem (9) is \(\Delta \bar{\phi}\). Then, \(\Delta \rho^*\) in (15) should be equal to \(\Delta \bar{\rho}\). By Theorem 1, \(\Delta \rho^*\) is independent of \(\Delta \phi\). Therefore, for any fixed \(\Delta \phi\), \(\Delta \rho^* = \Delta \bar{\rho}\) always holds. \(\square\)

Theorem 1 tells us that each sensor can estimate its range bias \(\Delta \rho\) independently by solving problem (8) (or problem (9)). Moreover, in the absence of measurement noise, the range bias can be exactly recovered by (15), as shown in Corollary 1. However, each sensor cannot estimate its azimuth bias \(\Delta \phi\) and target velocity \(v\) independently by solving problem (8) due to the ambiguity of \(\Delta \phi\) and \(v\) in problem...
problem (8) has multiple optimal pairs \((\Delta \phi, v)\). The ambiguity of \(\Delta \phi\) and \(v\) arising in the single-sensor case can be solved by combining measurements from different sensors. In particular, both range and azimuth biases as well as target velocity can be estimated by solving our proposed nonlinear LS formulation (7) as shown in Section IV.

We conclude this section by using Fig. 4 which illustrates how the ambiguity of \(\Delta \phi\) and \(v\) in the single-sensor case can be resolved by combining measurements from two different sensors. Assume that the noise is absent, by solving problem (8) (or problem (9)), each sensor can recover its true range bias. Their local measurements can be aligned onto a straight line (corresponding to the black dash lines in Fig. 4) by compensating range biases. Since each sensor’s measurements are from one target with a constant velocity, there is only one possibility to rotate those dash lines onto a straight line (corresponding to the black solid line in Fig. 4). Therefore, there is no ambiguity of azimuth biases and target velocity in the two-sensor case.

Fig. 4: A geometrical explanation of resolving ambiguity of azimuth biases and target velocity by combining measurements from 2 different sensors.

IV. MULTI-SENSOR CASE: A BLOCK COORDINATE DESCENT ALGORITHM

In this section, we study problem (7) in the multi-sensor case, i.e., \(M \geq 2\). In this case, the optimal \(\Delta \rho\) of problem (7) depends on \(\Delta \phi\); see Eq. (17) further ahead. This is different from the case where the optimal \(\Delta \rho\) in (15) does not depend on \(\Delta \phi\) when there is only one sensor (see Theorem 1).
The non-convexity of problem (7) comes from the nonlinear terms \( \Delta \rho_m \sin \Delta \phi_m \) and \( \Delta \rho_m \cos \Delta \phi_m \). To handle such difficulty, we propose to use the BCD algorithm to alternately minimize \( f(\Delta \rho, \Delta \phi, v) \) with respect to two blocks \( \Delta \rho \) and \( (\Delta \phi, v) \) as follows:

\[
\Delta \rho^{t+1} = \arg \min_{\Delta \rho} f(\Delta \rho, \Delta \phi^t, v^t),
\]

\[
(\Delta \phi^{t+1}, v^{t+1}) = \arg \min_{\Delta \phi, v} f(\Delta \rho^{t+1}, \Delta \phi, v),
\]

where \( t \geq 1 \) is the iteration index. Next, we will show that both subproblems (16a) and (16b) can be solved globally and efficiently (under mild conditions) in Sections IV-A and IV-B and then give our BCD algorithm for solving problem (7) in Section IV-C.

A. Solution for Subproblem (16a)

As an unconstrained convex quadratic problem, subproblem (16a) has a closed-form solution \([24]\), and is given by (we omit the iteration index here for notational simplicity)

\[
\Delta \rho^* = (G^T G)^{-1} G^T \mathbf{z}.
\]

In the above,

\[
\mathbf{z} = \begin{bmatrix}
z_c^1 + (p_{s_2}^x - p_{s_1}^x) - T_1 v_x \\
z_s^1 + (p_{s_2}^y - p_{s_1}^y) - T_1 v_y \\
\vdots \\
z_c^{K-1} + (p_{s_K}^x - p_{s_{K-1}}^x) - T_{K-1} v_x \\
z_s^{K-1} + (p_{s_K}^y - p_{s_{K-1}}^y) - T_{K-1} v_y
\end{bmatrix} \in \mathbb{R}^{2(K-1)},
\]

and

\[
z_c^k = \lambda^{-1} \left[ \rho_{k+1} \cos(\phi_{k+1} + \Delta \phi_{s_{k+1}}) - \rho_k \cos(\phi_k + \Delta \phi_{s_k}) \right],
\]

\[
z_s^k = \lambda^{-1} \left[ \rho_{k+1} \sin(\phi_{k+1} + \Delta \phi_{s_{k+1}}) - \rho_k \sin(\phi_k + \Delta \phi_{s_k}) \right];
\]

and the \((n,m)\)-th entry of \( G \in \mathbb{R}^{2(K-1) \times M} \) is

\[
[G]_{nm} = \begin{cases} 
\lambda^{-1} \cos(\phi_{k+1}), & \text{if } n \text{ is odd and } m = s_{k+1}, \\
-\lambda^{-1} \cos(\phi_k), & \text{if } n \text{ is odd and } m = s_k, \\
\lambda^{-1} \sin(\phi_{k+1}), & \text{if } n \text{ is even and } m = s_{k+1}, \\
-\lambda^{-1} \sin(\phi_k), & \text{if } n \text{ is even and } m = s_k, \\
0, & \text{otherwise},
\end{cases}
\]

where \( k = \lfloor \frac{n}{2} \rfloor \) and \( \lfloor \cdot \rfloor \) denotes the floor operation.
B. Solution for Subproblem (16b)

To efficiently solve subproblem (16b), we first reformulate it as an equivalent complex quadratically constrained quadratic program (QCQP) as follows:

\[
\min_{x \in \mathbb{C}^M, v \in \mathbb{C}} \| \mathbf{H}x - tv + c \|^2
\]
\[
\text{s.t.} \quad |x_m|^2 = 1, \quad \forall \ m,
\]

where \( x_m \) denotes the azimuth bias of sensor \( m \) in the sense that \( \Delta \phi_m = \angle x_m \), and the complex scalar \( v = v_x + jv_y \) represents the constant velocity. In (19), matrix \( \mathbf{H} \in \mathbb{C}^{(K-1) \times M} \) are determined by sensor measurements \( \{z_k\}_{k=1}^{K-1} \) as follows:

\[
[H]_{km} = \begin{cases} 
\lambda^{-1}(\rho_{k+1} + \Delta \rho_{s_{k+1}}) e^{j\phi_{k+1}}, & \text{if } m = s_{k+1}, \\
-\lambda^{-1}(\rho_k + \Delta \rho_{s_k}) e^{j\phi_k}, & \text{if } m = s_k, \\
0, & \text{otherwise};
\end{cases}
\]

(20)

vector \( t \in \mathbb{R}^{K-1} \) is related to time differences \( \{T_k\}_{k=1}^{K-1} \) as follows:

\[ t = [T_1, T_2, \ldots, T_{K-1}]^T, \]

and \( c \in \mathbb{C}^{K-1} \) is related to sensor positions \( \{p_m\}_{m=1}^{M} \) as follows:

\[ c_k = (p_{s_{k+1}}^x - p_{s_k}^x) + j(p_{s_{k+1}}^y - p_{s_k}^y), \quad k = 1, 2, \ldots, K - 1. \]

As an unconstrained quadratic program in \( v \), problem (19) has a closed-form solution given by

\[ v = (t^\dagger t)^{-1} t^\dagger (\mathbf{H}x + c). \]

(21)

Plugging (21) into (19), we get

\[
\min_{x \in \mathbb{C}^M} \| \mathbf{PH}x + \mathbf{P}c \|^2
\]
\[
\text{s.t.} \quad |x_m|^2 = 1, \quad m = 1, 2, \ldots, M,
\]

(22)

where \( \mathbf{P} = \mathbf{I} - tt^\dagger/\|t\|^2 \).

Problem (22) is a non-convex QCQP, and such class of problems is known to be NP-hard in general [25]. One efficient convex relaxation technique for solving such class of problems, semidefinite relaxation (SDR), has shown its effectiveness in signal processing and communication communities [26]. We also apply the SDR technique to solve problem (22). To do so, we further reformulate problem (22) in a homogeneous form as follows:

\[
\min_{x \in \mathbb{C}^{M+1}} \quad x^\dagger \mathbf{C}x
\]
\[
\text{s.t.} \quad |x_m|^2 = 1, \quad m = 1, 2, \ldots, M + 1,
\]

(23)
where
\[
C = \begin{bmatrix}
H^\dagger PH & H^\dagger Pc \\
\eta^\dagger PH & 0
\end{bmatrix}.
\]

It is simple to show that problems (22) and (23) are equivalent in the sense that \(x^* \in \mathbb{C}^{M+1}\) is the optimal solution for problem (23) if and only if \(x^*_1:x^*_M+1 \in \mathbb{C}^M\) is the optimal solution for problem (22).

The SDP relaxation of problem (23) is
\[
\begin{aligned}
\min_{X \in \mathbb{H}^{M+1}} & \quad \text{Tr}(CX) \\
\text{s.t.} & \quad \text{diag}(X) = 1, \\
& \quad X \succeq 0.
\end{aligned}
\] (24)

Problem (24) can be efficiently solved by the interior-point algorithm [27]. If the optimal solution \(X^*\) for problem (24) is of rank one, i.e., \(X^* = x^*(x^*)^\dagger\), then the optimal solution for problem (7) is obtained as follows:
\[
\Delta \phi^*_m = \angle x^*_m, \quad m = 1, \ldots, M,
\] (25)
and
\[
v^* = [\text{Re}\{v^*\}, \text{Im}\{v^*\}]^T, \quad v^* = \left(t^\dagger t\right)^{-1}t^\dagger \left(H \frac{x^*_1:x^*_M+1 + c}{x^*_M+1}\right).
\] (26)

The dual problem of SDP (24) is [28]
\[
\min_{y \in \mathbb{R}^{M+1}} \quad 1^T y \\
\text{s.t.} \quad C + \text{Diag}(y) \succeq 0.
\]

The following lemma states sufficient conditions that SDP (24) admits a unique rank-one solution.

**Lemma 1.** SDP (24) has a unique minimizer \(X\) of rank one if there exists \(X \in \mathbb{H}^{M+1}\) and \(y \in \mathbb{R}^{M+1}\) such that
1. \(\text{diag}(X) = 1\) and \(X \succeq 0\);
2. \(C + \text{Diag}(y) \succeq 0\);
3. \([C + \text{Diag}(y)]X = 0\); and
4. \(H^\dagger PH + \text{Diag}(y_{1:M}) \succ 0\).

**Proof.** Notice that conditions 1, 2, and 3 are the Karush-Kuhn-Tucker (KKT) conditions of SDP (24) [24]. Since both primal and dual problems are strictly feasible, i.e., \(I_{M+1}\) is strictly feasible for the primal problem and \((|\lambda_{\min}(C)| + \epsilon)1\) with any \(\epsilon > 0\) is strictly feasible for the dual problem, it follows that Slater’s condition holds true. Then, the KKT conditions 1, 2, and 3 are sufficient and necessary
for optimality of primal and dual problems. Condition 4 implies $H\dagger PH + \text{Diag}(y_{1:M})$ is nonsingular and thus $\text{rank}(C + \text{Diag}(y)) \geq M$. This, together with Sylvester’s rank inequality \cite{29} and condition 3, immediately shows $\text{rank}(X) \leq 1$. Moreover, since $X$ is non-zero (by condition 1), we have $\text{rank}(X) = 1$ and $\text{rank}(C + \text{Diag}(y)) = M$, which further implies $C + \text{Diag}(y)$ is dual nondegenerate \cite{30}. Therefore, it follows from Theorem 10 in \cite{30} that $X$ is unique.

Lemma 1 gives sufficient conditions on the existence and uniqueness of rank-one solution of SDP (24). However, these conditions are not always satisfied, because they indeed depend on the structure of $H$, $P$, $c$, and the true azimuth biases $\Delta \phi = [\Delta \phi_1, \Delta \phi_2, \ldots, \Delta \phi_M]^T$ (as shown in Theorem \cite{2} later). In the following, we will present a mild condition such that SDP (24) admits a unique minimizer of rank one. To begin with, we divide $H$ in (19) into two parts as follows:

$$H = \tilde{H} + \Delta H.$$  \hspace{1cm} (27)

In (27), $\tilde{H}$ denotes the true part of $H$ and is defined as

$$[\tilde{H}]_{km} = \begin{cases} (\tilde{\rho}_{k+1} + \Delta \tilde{\rho}_{s_{k+1}})e^{j \tilde{\phi}_{k+1}}, & \text{if } m = s_{k+1}, \\ - (\tilde{\rho}_k + \Delta \tilde{\rho}_{s_k})e^{j \tilde{\phi}_k}, & \text{if } m = s_k, \\ 0, & \text{otherwise}, \end{cases} \hspace{1cm} (28)$$

where $\tilde{\rho}_k = \rho_k - w^\rho_k$ and $\tilde{\phi}_k = \phi_k - w^\phi_k$ ($w^\rho_k$ and $w^\phi_k$ are measurement noise defined in \cite{2}); $\Delta H = H - \tilde{H}$ represents the error part of $H$ caused by the measurement noise and possibly the inaccuracy of the fixed $\Delta \rho$. Notice that if there is no measurement noise ($w_k = 0$ and $\lambda = 1$), and $\Delta \rho$ in subproblem (16b) (equivalent to (19)) is the true range biases $\Delta \tilde{\rho} = [\Delta \tilde{\rho}_1, \Delta \tilde{\rho}_2, \ldots, \Delta \tilde{\rho}_M]^T$, then $\Delta H = 0$ and $H = \tilde{H}$. In this case, SDP (24) has the following exact recovery property.

**Corollary 2.** Suppose $\Delta H = 0$. Then, SDP (24) always has a unique minimizer of rank one, i.e., $X^* = x^*(x^*\dagger)$. Furthermore, $x^*$ exactly recovers the azimuth biases

$$\Delta \tilde{\phi}_m = \angle \frac{x_{m}^{*}}{x_{M+1}^{*}}, \hspace{1cm} m = 1, 2, \ldots, M,$$

where $\Delta \tilde{\phi}_m$ denotes the true azimuth bias of sensor $m$.

**Proof.** The proof consists of two parts. We first construct a pair of primal and dual solutions $X^*$ and $y^*$ and then prove their optimality and exact recovery property.

Without loss of generality, we assume that each sensor has sufficient number of measurements such that $PH$ is of full column rank and $H\dagger P\dagger PH = H\dagger PH$ is invertible. Let

$$X^* = x^*(x^*\dagger), \hspace{1cm} x^* = \begin{bmatrix} -(H\dagger PH)^{-1}H\dagger Pc \\ 1 \end{bmatrix} \in \mathbb{C}^{M+1},$$  \hspace{1cm} (29)
\[ y^* = \begin{bmatrix} 0 \\ \text{c}^\dagger \text{PH}(\text{H}^\dagger \text{PH})^{-1} \text{H}^\dagger \text{Pc} \end{bmatrix} \in \mathbb{R}^{M+1}. \] (30)

Notice that problem (22) is a reformulation of problem (7) with fixed \( \Delta \rho \). In the absence of measurement noise, the optimal objective value of problem (7) is zero and the optimal solutions are true sensor biases \( \Delta \vec{\phi} = [\Delta \vec{\phi}_1, \Delta \vec{\phi}_2, \ldots, \Delta \vec{\phi}_M]^T \) and true target velocity \( \vec{v} \). Therefore, if \( H = \tilde{H} \), then the following equation holds true:

\[
P \tilde{H} \begin{bmatrix} e^{j\vec{\phi}_1} \\ e^{j\vec{\phi}_2} \\ \vdots \\ e^{j\vec{\phi}_M} \end{bmatrix} + \text{Pc} = 0. \] (31)

Since \( H = \tilde{H} \) and \( \text{PH} \) is of full column rank, it follows

\[
\begin{bmatrix} e^{j\vec{\phi}_1} \\ e^{j\vec{\phi}_2} \\ \vdots \\ e^{j\vec{\phi}_M} \end{bmatrix} = - (\text{H}^\dagger \text{PH})^{-1} \text{H}^\dagger \text{Pc}.
\]

Consequently, \( X^* \) in (29) satisfies condition 1 in Lemma 1.

Recall the definitions of \( C \) and \( y^* \). Then,

\[ C + \text{Diag}(y^*) = \begin{bmatrix} \text{H}^\dagger \text{PH} & \text{H}^\dagger \text{Pc} \\ \text{c}^\dagger \text{PH} & \text{c}^\dagger \text{PH}(\text{H}^\dagger \text{PH})^{-1} \text{H}^\dagger \text{Pc} \end{bmatrix}. \]

Since \( \text{H}^\dagger \text{PH} \succ 0 \) and its Schur complement is zero, we know that \( C + \text{Diag}(y^*) \succeq 0 \), which shows that condition 2 in Lemma 1 is true. Moreover, it is simple to check \( [C + \text{Diag}(y^*)] x^* = 0 \), which implies condition 3 in Lemma 1. Since \( \text{H}^\dagger \text{PH} + \text{Diag}(y^*_{1:M}) = \text{H}^\dagger \text{PH} \succ 0 \), condition 4 in Lemma 1 also holds. Therefore, the constructed solutions \( X^* \) in (29) and \( y^* \) in (30) satisfy all conditions in Lemma 1. Hence \( X^* \) is the unique solution of SDP (24) and \( x^* \) exactly recovers the true azimuth biases. \( \square \)

Corollary 2 shows that, if \( \Delta H = 0 \) in (27), the solution of SDP (24) is rank one and exactly recovers the true azimuth biases. The following Theorem 2 generalizes this result to sufficient small \( \Delta H \neq 0 \).

In other words, if \( \Delta H \) is sufficiently small, SDP (24) always admits a unique solution of rank one. The proof of Theorem 2 is relegated to Appendix B. A sufficient condition on how \( \Delta H \) depends on the problem data and how small it should be to guarantee the unique rank-one solution of SDP (24) is given in Claim 2 of Appendix B.

**Theorem 2.** If \( \Delta H \) is sufficiently small, then SDP (24) admits a unique solution of rank one.
C. Proposed BCD Algorithm

Now, we present our BCD algorithm for solving the asynchronous multi-sensor registration problem (7); see Algorithm 1 below.

Algorithm 1 The Proposed BCD Algorithm for Problem (7)

Input: Measurements \( \{z_k\}_{k=1}^K \) collected by all sensors.

1: for \( t = 0, 1, 2, \ldots, \) do
2:     if \( t = 0 \) then
3:         Obtain \( \Delta \rho^{t+1}_m \) by (15), \( m = 1, 2, \ldots, M \);
4:     else
5:         Obtain \( \Delta \rho^{t+1} \) by (17);
6:     end if
7:     Solve SDP (24) to obtain \( X^* \);
8:     Extract \( \{\Delta \phi^*_m\}_{m=1}^M \) and \( v^* \) by (25) and (26);
9: end for

Output: Estimated biases \( \{\theta^*_m\}_{m=1}^M \) and velocity \( v^* \).

The performance of the BCD algorithm generally depends on the choice of the initial point (especially when it is applied to solve the non-convex optimization problems). In our proposed BCD algorithm, we initialize it with (15) by using the separation property of the range bias estimation in the single-sensor case. Furthermore, global convergence of our proposed algorithm can be established by using similar arguments in [31]. Based on Corollary 1 and Corollary 2, we further have the following result.

Corollary 3. If there is no measurement noise, Algorithm 1 exactly recovers the true biases of all sensors and the true velocity of the target.

In the noiseless case, solving the sensor registration problem is equivalent to solving a set of nonlinear equations (1) and (3). Corollary 3 shows that our proposed Algorithm 1 is able to exactly solve these equations. This exact recovery property distinguishes our proposed algorithm from the existing approaches [2]–[7], [9], [10], [16], [17], which use the first-order approximation to handle the nonlinearity in the registration problem and cannot recover the sensor biases even if the noise is absent.
V. NUMERICAL SIMULATION

A. Single-Sensor Case

In this subsection, we evaluate the estimation performance of single-sensor registration. The influence of the noise level and the relative locations between the target and the sensor are investigated via simulations. Since the estimation of the range bias is independent of that of the azimuth bias (Theorem 1), we only consider the scenario where the sensor is located at the origin and the target moves horizontally to the right. Moreover, there is no need to set the azimuth bias for the sensor. Hence, we only specify the range bias in this simulation. We use the root mean square error (RMSE) as the performance metric and use the Cramer Rao lower bound (CRLB) [32] as the benchmark. The RMSE is averaged over 500 independent Monte Carlo runs.

| Scenario | Starting Position | Ending Position | Velocity | $\Delta \tilde{\rho}$ |
|----------|------------------|----------------|----------|------------------|
| Scenario 1 | $[-10, 5]^T$ km | $[10, 5]^T$ km |          |                  |
| Scenario 2 | $[-20, 5]^T$ km | $[-10, 5]^T$ km | $[200, 0]^T$ m/s | 1 km |
| Scenario 3 | $[-10, 10]^T$ km | $[10, 10]^T$ km |          |                  |

Three scenarios with different starting positions of the target are considered in this simulation; see Table II for details and an illustration in Fig. 5. In the above three scenarios, the target velocity $\bar{v}$ is $[200, 0]^T$ m/s, the sensor observes the target every 10 seconds; and the total observation time is 100 seconds. In total, there are 11 measurements.

Fig. 6 plots the comparison of the RMSE of the estimated range bias and the CRLB under different noise levels $(\sigma_\rho, \sigma_\phi)$. In the absence of measurement noise, the range bias is exactly recovered in all three scenarios, which further verifies the exact recovery property in Corollary 1. As the noise level increases, the gap between RMSE and the corresponding CRLB generally increases but the performance of the estimator (15) in the three scenarios is quite different from each other. More specifically, the performance of the estimator is robust to the noise level in Scenario I while the performance of the estimator is sensitive to the noise level in Scenario II and Scenario III. In summary, our simulation results show that the estimator (15) for the range bias is very effective in all three scenarios when the other scenarios can be equivalently transformed into this scenario by rotating all measurements with an appropriate angle.
noise level is small and its performance in terms of robustness to the noise depends on different scenarios.

B. Multi-Sensor Case

In this subsection, we present some simulation results to evaluate the effectiveness of Algorithm [1] for estimating sensor biases. We consider a scenario with 3 sensors and a target moving with velocity $\bar{v} = [200, 0]^T\ m/s$ starting from position $[-10, 0]^T\ km$. Each sensor observes the target every 10 seconds with different starting time. The observation lasts 107 seconds in total and each sensor has 11 measurements. Detailed simulation setups are listed in Table [III]. We compare our proposed algorithm with the two-stage approach [1] and a linearized LS approach for problem [7]. The linearized LS approach is a direct extension of approaches [3], [6], [7] to solve our proposed nonlinear LS formulation [7], where the first-order approximation is applied to represent sensor measurements in the reference coordinate system. We compare the performance of these approaches under different levels of the measurement noise. In our numerical simulations, SDP (24) is solved by CVX [33] and it is observed that the obtained solution is always of rank one. Again, we use the RMSE as our performance metric and use the CRLB as our benchmark. The results are obtained by averaging over 500 Monte Carlo runs.

The RMSE of three approaches for the three sensors’ range and azimuth biases and the corresponding
Fig. 6: RMSE and CRLB of the range bias in three scenarios under different noise levels.

TABLE III: Simulation setup for the 3-sensor case.

| Position       | Starting Time | $\Delta \rho$ | $\Delta \phi$ |
|----------------|---------------|---------------|---------------|
| Sensor 1       | $[-5, -5]^T$ km | 0 s           | 1 km          | $-3^\circ$    |
| Sensor 2       | $[5, -5]^T$ km | 3.5 s         | $-1.2$ km     | $4^\circ$     |
| Sensor 3       | $[0, 5]^T$ km  | 7 s           | $-0.8$ km     | $3^\circ$     |

CRLB are plotted as Figs. 7, 8, 9, 10, 11, 12. We can observe from these figures that the linearized LS approach cannot find the true biases even when there is no measurement noise ($\sigma_{\rho} = \sigma_{\phi} = 0$). This is due to the model mismatch error introduced by the first-order approximation in the linearized LS approach. As the noise level increases, our proposed approach always achieves the smallest RMSE among the three approaches and its RMSE is quite close to the CRLB. The two-stage approach performs worst in terms of the range biases (as the noise level increases). The reason for this is because the two-stage approach estimates range biases only based on each sensor’s local measurements while the other two approaches use all sensors’ measurements to estimate range biases.
Fig. 7: RMSE and CRLB of sensor 1’s range bias under different $\sigma_\rho$ and $\sigma_\phi$.

VI. CONCLUSION

In this paper, we proposed an effective algorithm to the asynchronous multi-sensor registration problem by using a reference target moving in an unknown constant velocity. Unlike the existing algorithms, our proposed algorithm is capable of exactly recovering sensor biases when there is no measurement noise. Our simulation results show the effectiveness of our proposed algorithm in both noisy and noiseless scenarios.

APPENDIX A

PROOF OF THEOREM 1

We first reformulate problem (9) as

$$\min_{\Delta \rho} \min_v \| H_0 \Delta \rho + H_1 v - y_\Delta \phi \|^2. \tag{32}$$

The closed-form solution of the above problem with respect to $v$ is

$$v^*(\Delta \rho, \Delta \phi) = - (H_1^T H_1)^{-1} H_1^T (H_0 \Delta \rho - y_\Delta \phi). \tag{33}$$

Substituting (33) into (32), we obtain the following linear least square problem with respect to $\Delta \rho$:

$$\min_{\Delta \rho} \| (I_{2(K-1)} - \bar{H}_1) (H_0 \Delta \rho - y_\Delta \phi) \|^2. \tag{34}$$
where \( \hat{H}_1 = H_1 (H_1^T H_1)^{-1} H_1^T \). By the definitions of \( H_0 \), \( H_1 \), and \( y_{\Delta \phi} \) in (10), problem (34) can be further rewritten as

\[
\min_{\Delta \rho} \sum_{k=1}^{K-1} \| a_k \Delta \rho - b_k \|^2,
\]

where

\[
a_k = \left[ \sum_{\ell=1}^{K-1} \bar{T}_{k\ell} c_{\ell}, \sum_{\ell=1}^{K-1} \bar{T}_{k\ell} s_{\ell} \right]^T,
\]

\[
b_k = \left[ \sum_{\ell=1}^{K-1} \bar{T}_{k\ell} y_{\ell}, \sum_{\ell=1}^{K-1} \bar{T}_{k\ell} y_{s_{\ell}} \right]^T,
\]

and

\[
\bar{T}_{k\ell} = \begin{cases} 
1 - \frac{T_{k}^2}{\sum_{i=1}^{K-1} T_{i}^2}, & \ell = k, \\
- \frac{T_{k} T_{\ell}}{\sum_{i=1}^{K-1} T_{i}^2}, & \ell \neq k.
\end{cases}
\]

To prove Theorem 1 it suffices to show that \( a_k^T a_k \) and \( a_k^T b_k \) for all \( k = 1, 2, \ldots, K-1 \) are independent of \( \Delta \phi \). Next, we show that this is indeed true. It is simple to compute, for \( k = 1, 2, \ldots, K \),

\[
a_k^T a_k = \sum_{\ell=1}^{K-1} \sum_{j=1}^{K-1} \bar{T}_{k\ell} \bar{T}_{kj} (c_{k\ell} c_{kj} + s_{k\ell} s_{kj}),
\]
Fig. 9: RMSE and CRLB of sensor 2’s range bias under different $\sigma_\rho$ and $\sigma_\phi$.

Moreover, by the definitions of $c_k, s_k, y_k^c, \text{ and } y_k^s$ in (11), (12), (13), and (14) and the fact

$$
\cos(\alpha_1 + \Delta \phi) \cos(\alpha_2 + \Delta \phi) + \sin(\alpha_1 + \Delta \phi) \sin(\alpha_2 + \Delta \phi)
$$

$$
= \cos(\alpha_1 - \alpha_2),
$$

we can obtain, for $\ell, j = 1, 2, \ldots, K - 1$,

$$
c_\ell c_j + s_\ell s_j = \lambda^{-2} [\cos (\phi_{\ell+1} - \phi_{j+1}) + \cos (\phi_{\ell} - \phi_j)
- \cos (\phi_{\ell+1} - \phi_j) - \cos (\phi_{j+1} - \phi_{\ell})],
$$

$$
c_\ell y_j^c + s_\ell y_j^s = \lambda^{-2} [\rho_{j+1} \cos (\phi_{\ell+1} - \phi_{j+1}) + \rho_j \cos (\phi_{\ell} - \phi_j)
- \rho_j \cos (\phi_{\ell+1} - \phi_j) - \rho_{j+1} \cos (\phi_{j+1} - \phi_{\ell})].
$$

This immediately shows that $a_k^T a_k$ and $a_k^T b_k$ for all $k = 1, 2, \ldots, K - 1$ are independent of $\Delta \phi$. This completes the proof of Theorem 1.
Fig. 10: RMSE and CRLB of sensor 2’s azimuth bias under different $\sigma_\rho$ and $\sigma_\phi$.

**APPENDIX B**

**PROOF OF THEOREM 2**

To show that SDP (24) admits a unique solution of rank one, it is sufficient to show that, if $\Delta \mathbf{H}$ is sufficiently small, there always exists $\mathbf{W} = \text{Diag}(\mathbf{w})$ with $|w_m| = 1$ for all $m = 1, 2, \ldots, M$ such that $\mathbf{1}^T \in \mathbb{H}^{M+1}$ is the unique solution of the following SDP

$$
\begin{align}
\min_{\mathbf{X} \in \mathbb{H}^{M+1}} & \quad \text{Tr}(\hat{\mathbf{C}} \mathbf{X}) \\
\text{s.t.} & \quad \text{diag}(\mathbf{X}) = 1, \\
& \quad \mathbf{X} \succeq 0,
\end{align}
$$

where

$$
\hat{\mathbf{C}} = \begin{bmatrix}
\mathbf{W}^\dagger \mathbf{H}^\dagger \mathbf{P} \mathbf{H} & \mathbf{W}^\dagger \mathbf{H}^\dagger \mathbf{P} \mathbf{c} \\
\mathbf{c}^\dagger \mathbf{P} \mathbf{H} & 0
\end{bmatrix} \in \mathbb{H}^{M+1}.
$$

By Lemma 1, we only need to show that, if $\Delta \mathbf{H}$ is sufficiently small, there always exists $\mathbf{W} = \text{Diag}(\mathbf{w})$ with $|w_m| = 1$ for all $m = 1, 2, \ldots, M$ such that $\mathbf{1}^T \in \mathbb{H}^{M+1}$ and some $\mathbf{y} \in \mathbb{R}^{M+1}$ jointly satisfy

$$
\hat{\mathbf{C}} + \text{diag}(\mathbf{y}) = \begin{bmatrix}
\mathbf{W}^\dagger \mathbf{H}^\dagger \mathbf{P} \mathbf{H} + \text{Diag}(\mathbf{y}_1: M) & \mathbf{W}^\dagger \mathbf{H}^\dagger \mathbf{P} \mathbf{c} \\
\mathbf{c}^\dagger \mathbf{P} \mathbf{H} & \mathbf{y}_{M+1}
\end{bmatrix} \succeq 0,
$$

October 10, 2017
Fig. 11: RMSE and CRLB of sensor 3’s range bias under different $\sigma_\rho$ and $\sigma_\phi$.

\[
\begin{bmatrix}
W^\dagger H^\dagger PW + \text{Diag}(y_{1:M}) & W^\dagger H^\dagger Pc \\
C_{PHW} & y_{M+1}
\end{bmatrix}
1 = 0 \in \mathbb{R}^{M+1},
\]

and

\[
W^\dagger H^\dagger PW + \text{Diag}(y_{1:M}) > 0.
\]

Notice that the above (36), (37), and (38) correspond to conditions 2, 3, and 4 in Lemma 1 respectively, and $11^T \in S^{M+1}$ obviously satisfies condition 1 in Lemma 1. Moreover, we rewrite (37) as follows:

\[
\begin{bmatrix}
W^\dagger H^\dagger PW + \text{Diag}(y_{1:M}) \\
C_{PHW}
\end{bmatrix}
1 + W^\dagger H^\dagger Pc = 0 \in \mathbb{R}^M,
\]

and

\[
\begin{bmatrix}
C_{PHW}
\end{bmatrix}
1 + y_{M+1} = 0.
\]

Next, we prove that, if $\Delta H$ is sufficiently small, there always exist $W = \text{Diag}(w)$ with $|w_m| = 1$ for all $m = 1, 2, \ldots, M$ and $y \in \mathbb{R}^{M+1}$ that satisfy (36), (39a), (39b), and (38). We divide the proof into two parts. More specifically, we first show that, if there exist $W = \text{Diag}(w)$ with $|w_m| = 1$ for all $m = 1, 2, \ldots, M$ and $y \in \mathbb{R}^{M+1}$ satisfying (39a) and (38), then we can construct $y_{M+1}$ that simultaneously satisfies (36) and (39b). Then, we show that, if $\Delta H$ is sufficiently small, there indeed exist $W = \text{Diag}(w)$ with $|w_m| = 1$ for all $m = 1, 2, \ldots, M$ and $y_{1:M}$ that satisfy (39a) and (38).
Fig. 12: RMSE and CRLB of sensor 3's azimuth bias under different $\sigma_\rho$ and $\sigma_\phi$.

Let us first assume that $W = \text{Diag}(w)$ (with $|w_m| = 1$ for all $m = 1, 2, \ldots, M$) and $y_{1:M}$ satisfy (38) and (39a). We show that there exists $y_{M+1}$ that simultaneously satisfies (36) and (39b). Let

$$y_{M+1} = c^\dagger PHW \left[ W^\dagger H^\dagger PHW + \text{Diag}(y_{1:M}) \right]^{-1} W^\dagger H^\dagger P c.$$  \hspace{1cm}(40)

This, together with (38), immediately implies (36). Moreover, from (39a), we have

$$1 = - \left[ W^\dagger H^\dagger PHW + \text{Diag}(y_{1:M}) \right]^{-1} W^\dagger H^\dagger P c \in \mathbb{R}^M.$$  \hspace{1cm}(40)

Combining the above and (40), immediately yields (39b).

Now, let us argue that, if $\Delta H$ is sufficiently small, then there always exist $W = \text{Diag}(w)$ (with $|w_m| = 1$ for all $m = 1, 2, \ldots, M$) and $y_{1:M}$ that satisfy (39a) and (38). In particular, the following Claim 1 and Claim 2 guarantee the existence of $W = \text{Diag}(w)$ (with $|w_m| = 1$ for all $m = 1, 2, \ldots, M$) and $y_{1:M}$ that satisfy (39a) and (38), respectively.

**Claim 1.** There always exist a neighborhood $\mathcal{H} \subseteq \mathbb{C}^{(K-1)\times M}$ containing $\tilde{H}$ and two unique continuously differentiable functions $d_\psi : \mathbb{C}^{(K-1)\times M} \mapsto \mathbb{R}^M$ and $d_y : \mathbb{C}^{(K-1)\times M} \mapsto \mathbb{R}^M$ such that (39a) holds true for all $H \in \mathcal{H}$ with $W = \text{Diag} \left( [e^{ij\psi_1}, e^{ij\psi_2}, \ldots, e^{ij\psi_M}]^T \right)$, $\psi = d_\psi(H)$, and $y = d_y(H)$.

**Proof.** To prove Claim 1, we first reformulate complex equation (39a) as an equivalent real form; then we apply the implicit function theorem \cite{34} based on the equivalent form to show the existence of the
neighborhood \( \mathcal{H} \) and two unique continuously differentiable functions \( d_\psi \) and \( d_\gamma \).

The equivalent form of (39a) is

\[
\begin{align*}
 f ( \psi, y, H) &= 0,
\end{align*}
\]

(41)

where \( f ( \psi, y, H) : \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{C}^{(K-1) \times M} \mapsto \mathbb{R}^{2M} \) is

\[
\begin{align*}
f ( \psi, y, H) &= \begin{bmatrix}
\text{Re} \left\{ \left[ W^\dagger H^\dagger PHW + \text{diag}(y_{1:M}) \right] 1 + W^\dagger H^\dagger Pc \right\} \\
\text{Im} \left\{ \left[ W^\dagger H^\dagger PHW + \text{diag}(y_{1:M}) \right] 1 + W^\dagger H^\dagger Pc \right\}
\end{bmatrix} \\
&= \begin{bmatrix}
C_\psi \text{Re}\{A_H\} C_\psi + S_\psi \text{Im}\{A_H\} C_\psi \\
C_\psi \text{Re}\{A_H\} S_\psi + S_\psi \text{Im}\{A_H\} S_\psi
\end{bmatrix} 1 \\
&\quad + \begin{bmatrix}
-C_\psi \text{Im}\{A_H\} S_\psi + S_\psi \text{Re}\{A_H\} S_\psi \\
C_\psi \text{Im}\{A_H\} C_\psi - S_\psi \text{Re}\{A_H\} C_\psi
\end{bmatrix} 1 \\
&\quad + \begin{bmatrix}
C_\psi \text{Re}\{b_H\} + S_\psi \text{Im}\{b_H\} \\
C_\psi \text{Im}\{b_H\} - S_\psi \text{Re}\{b_H\}
\end{bmatrix} + y.
\end{align*}
\]

(42)

In the above, \( C_\psi \) and \( S_\psi \) are diagonal matrices related to \( \psi \) as follows:

\[
\begin{align*}
C_\psi &= \text{Diag} \left( [\cos \psi_1, \ldots, \cos \psi_M]^T \right), \\
S_\psi &= \text{Diag} \left( [\sin \psi_1, \ldots, \sin \psi_M]^T \right);
\end{align*}
\]

(43)

and \( A_H \in \mathbb{H}^M \) and \( b_H \in \mathbb{C}^M \) are determined by \( H \) as follows:

\[
A_H = H^\dagger PH, \quad b_H = H^\dagger Pc.
\]

Now, we apply the implicit function theorem \cite{34} Theorem 5\) to Eq. (41). Notice that it is the same to define \( f ( \psi, y, H) \) in (42) over \( (\psi, y, H) \) or define \( f (\psi, y, \text{Re}\{H\}, \text{Im}\{H\}) : \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^{(K-1) \times M} \times \mathbb{R}^{(K-1) \times M} \mapsto \mathbb{R}^{2M} \) over \( (\psi, y, \text{Re}\{H\}, \text{Im}\{H\}) \). For notational simplicity, we will keep using (42) (but we can think it as \( f (\psi, y, \text{Re}\{H\}, \text{Im}\{H\}) \)) in our proof. First of all, \( f (\psi, y, H) \) is continuously differentiable (i.e., \( f (\psi, y, \text{Re}\{H\}, \text{Im}\{H\}) \) is continuously differentiable). Moreover, it follows from (31) that

\[
\begin{align*}
f \left( \tilde{\psi}, \tilde{y}, \tilde{H} \right) &= 0,
\end{align*}
\]

(44)

where \( \tilde{\psi} = \Delta \bar{\phi} \), \( \tilde{y} = 0 \), and \( \tilde{H} \) is defined in (28). If the Jacobian matrix \( D_{(\psi, y)} f \) is invertible at point \( (\tilde{\psi}, \tilde{y}, \tilde{H}) \), then it follows from the implicit function theorem \cite{34} Theorem 5\) that the desired \( \mathcal{H}, d_\psi, \) and \( d_\gamma \) in Claim \( I \) must exist. It remains to prove that the Jacobian matrix \( D_{(\psi, y)} f \) at point \( (\tilde{\psi}, \tilde{y}, \tilde{H}) \) is invertible.
By some calculations, we obtain

\[
D_{(\psi,y)} f = \begin{bmatrix} \frac{\partial f}{\partial \psi} & \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} * & I_M \\ G & 0 \end{bmatrix} \in \mathbb{C}^{2M \times 2M},
\]

(45)

where

\[
G = C_\psi \text{Re}\{A_H\} C_\psi - S_\psi \text{Diag}(\text{Re}\{A_H\} s_\psi) + S_\psi \text{Im}\{A_H\} C_\psi + C_\psi \text{Diag}(\text{Im}\{A_H\} s_\psi) - C_\psi \text{Im}\{A_H\} S_\psi - S_\psi \text{Diag}(\text{Im}\{A_H\} c_\psi) + S_\psi \text{Re}\{A_H\} S_\psi - C_\psi \text{Diag}(\text{Re}\{b_H\}) + C_\psi \text{Diag}(\text{Re}\{b_H\}).
\]

(46)

By (44), we have

\[
\text{Re}\{A_\tilde{H}\} c_\tilde{\psi} - \text{Im}\{A_\tilde{H}\} s_\tilde{\psi} = \text{Re}\{A_\tilde{H}\},
\]

\[
\text{Im}\{A_\tilde{H}\} c_\tilde{\psi} + \text{Re}\{A_\tilde{H}\} s_\tilde{\psi} = \text{Im}\{b_\tilde{H}\}.
\]

(47)

Substituting (47) into (46), we obtain

\[
G|_{(\tilde{\psi}, \tilde{y}, \tilde{H})} = \begin{bmatrix} C_\tilde{\psi} & S_\tilde{\psi} \end{bmatrix} \begin{bmatrix} \text{Re}\{A_\tilde{H}\} & -\text{Im}\{A_\tilde{H}\} \\ \text{Im}\{A_\tilde{H}\} & \text{Re}\{A_\tilde{H}\} \end{bmatrix} \begin{bmatrix} C_\tilde{\psi} \\ S_\tilde{\psi} \end{bmatrix}.
\]

Since \( A_\tilde{H} = \tilde{H}^\dagger P \tilde{H} \) is positive definite and the matrix \([C_\tilde{\psi}, S_\tilde{\psi}]\) is of full row rank, we immediately have

\[
x^T G|_{(\tilde{\psi}, \tilde{y}, \tilde{H})} x > 0, \quad \forall \ x \in \mathbb{R}^M,
\]

which further implies that \( G|_{(\tilde{\psi}, \tilde{y}, \tilde{H})} \) is invertible. Consequently, the Jacobian matrix \( D_{(\psi,y)} f \) in (45) at point \((\tilde{\psi}, \tilde{y}, \tilde{H})\) is invertible. The proof of Claim 1 is completed.

Claim 2. Consider \( H \in \mathcal{H} \subseteq \mathbb{C}^{(K-1) \times M} \), where \( \mathcal{H} \) is the one in Claim 1. Suppose

\[
\lambda_{\min}(H^\dagger P H) > \min_{1 \leq m \leq M} \left\{ -\text{Re}\left\{ D^{-1}_m \text{vec} \left( H - \tilde{H} \right) \right\}_{M+m} \right\}.
\]

(48)

Then, \( W \) and \( y \) defined in Claim 1 satisfy (38).

Proof. It follows from Claim 1 that

\[
y_m(H) := [d_y(H)]_m, \quad m = 1, 2, \ldots, M
\]
are continuously differentiable functions with respect to \( H \in \mathcal{H} \subseteq \mathbb{C}^{(K-1)\times M} \). Therefore, for \( m = 1, 2, \ldots, M \), we get

\[
y_m(H) = y_m(\tilde{H}) + \text{Re} \left\{ \text{Tr} \left( \nabla y_m(\tilde{H}) \left( H - \tilde{H} \right) \right) \right\} + o(\| H - \tilde{H} \|) \\
= -\text{Re} \left\{ \left[ D^{-1} F \text{vec}(H - \tilde{H}) \right]_{M+m} \right\} + o(\| H - \tilde{H} \|),
\]

(49)

where (49) is due to the differentiation rule of the implicit function and \( y_m(\tilde{H}) = 0 \). Combining (49) and (48), we immediately have

\[
W^\dagger H^\dagger PHW + \text{Diag}(y_{1:M}) \\
= W^\dagger \left[ H^\dagger PH + \text{Diag}(y_{1:M}) \right] W > 0,
\]

(50)

which shows that \( W \) and \( y \) satisfy (38). The proof of Claim 2 is completed.

\[ \square \]

ACKNOWLEDGMENT

The authors would like to thank Professor Stephen Boyd of Stanford University for several useful discussions.

REFERENCES

[1] W. Pu, Y.-F. Liu, J. Yan, S. Zhou, H. Liu, and Z.-Q. Luo, “A two-stage optimization approach to the asynchronous multi-sensor registration problem,” in 2017 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), Mar. 2017, pp. 3271–3275.

[2] R. E. Helmick and T. R. Rice, “Removal of alignment errors in an integrated system of two 3-D sensors,” IEEE Transactions on Aerospace and Electronic Systems, vol. 29, no. 4, pp. 1333–1343, Oct. 1993.

[3] M. P. Dana, “Registration: A prerequisite for multiple sensor tracking,” Multitarget-Multisensor Tracking: Advanced Applications, 1990.

[4] W. L. Fischer, C. E. Muehe, and A. G. Cameron, “Registration errors in a netted air surveillance system,” DTIC Document, Tech. Rep., 1980.

[5] S. Fortunati, F. Gini, A. Farina, and A. Graziano, “On the application of the expectation-maximisation algorithm to the relative sensor registration problem,” IET Radar Sonar Navigation, vol. 7, no. 2, pp. 191–203, Feb. 2013.

[6] S. Fortunati, A. Farina, F. Gini, A. Graziano, M. S. Greco, and S. Giompapa, “Least squares estimation and Cramér-Rao type lower bounds for relative sensor registration process,” IEEE Transactions on Signal Processing, vol. 59, no. 3, pp. 1075–1087, Mar. 2011.

[7] H. Leung, M. Blanchette, and C. Harrison, “A least squares fusion of multiple radar data,” in Proceedings of RADAR, vol. 94, 1994, pp. 364–369.

[8] D. C. Cowley and B. Shafai, “Registration in multi-sensor data fusion and tracking,” in Proc. 1993 American Control Conference, June 1993, pp. 875–879.
[9] Y. Zhou, H. Leung, and P. C. Yip, “An exact maximum likelihood registration algorithm for data fusion,” *IEEE Transactions on Signal Processing*, vol. 45, no. 6, pp. 1560–1573, Jun. 1997.

[10] B. Ristic and N. Okello, “Sensor registration in ECEF coordinates using the MLR algorithm,” in *Proceedings of the Sixth International Conference of Information Fusion*, 2003, pp. 135–142.

[11] C.-S. Hsieh and F.-C. Chen, “Optimal solution of the two-stage Kalman estimator,” *IEEE Transactions on Automatic Control*, vol. 44, no. 1, pp. 194–199, Jan. 1999.

[12] N. Nabaa and R. H. Bishop, “Solution to a multisensor tracking problem with sensor registration errors,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 35, no. 1, pp. 354–363, Jan. 1999.

[13] A. Zia, T. Kirubarajan, J. P. Reilly, D. Yee, K. Punithakumar, and S. Shirani, “An EM algorithm for nonlinear state estimation with model uncertainties,” *IEEE Transactions on Signal Processing*, vol. 56, no. 3, pp. 921–936, Mar. 2008.

[14] Z. Li, S. Chen, H. Leung, and E. Bosse, “Joint data association, registration, and fusion using EM-KF,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 46, no. 2, pp. 496–507, April 2010.

[15] N. N. Okello and S. Challa, “Joint sensor registration and track-to-track fusion for distributed trackers,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 40, no. 3, pp. 808–823, July 2004.

[16] Y. Zhou, “A Kalman filter based registration approach for asynchronous sensors in multiple sensor fusion applications,” in *Proc. 2004 IEEE International Conference on Acoustics, Speech, and Signal Processing*, vol. 2, May 2004.

[17] X. Lin, Y. Bar-Shalom, and T. Kirubarajan, “Multisensor multitarget bias estimation for general asynchronous sensors,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 41, no. 3, pp. 899–921, July 2005.

[18] R. Mahler and A. El-Fallah, “Bayesian unified registration and tracking,” in *SPIE Defense, Security, and Sensing*. International Society for Optics and Photonics, 2011.

[19] B. Ristic, D. E. Clark, and N. Gordon, “Calibration of multi-target tracking algorithms using non-cooperative targets,” *IEEE Journal of Selected Topics in Signal Processing*, vol. 7, no. 3, pp. 390–398, June 2013.

[20] X. R. Li and V. P. Jilkov, “Survey of maneuvering target tracking: III. Measurement models,” pp. 423–446, 2001.

[21] R. P. S. Mahler, “Multitarget bayes filtering via first-order multitarget moments,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 39, no. 4, pp. 1152–1178, Oct. 2003.

[22] X. R. Li and V. P. Jilkov, “Survey of maneuvering target tracking. I. dynamic models,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 39, no. 4, pp. 1333–1364, Oct. 2003.

[23] L. Mo, X. Song, Y. Zhou, Z.-K. Sun, and Y. Bar-Shalom, “Unbiased converted measurements for tracking,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 34, no. 3, pp. 1023–1027, July 1998.

[24] S. P. Boyd and L. Vandenberghe, *Convex Optimization*. New York, NY, U.S.A.: Cambridge University Press, 2004.

[25] Z.-Q. Luo and T.-H. Chang, “SDP relaxation of homogeneous quadratic optimization: approximation bounds and applications,” in *Convex Optimization in Signal Processing and Communications*, 2010, pp. 117–165.

[26] Z.-Q. Luo, W. K. Ma, A. M. C. So, Y. Ye, and S. Zhang, “Semidefinite relaxation of quadratic optimization problems,” *IEEE Signal Processing Magazine*, vol. 27, no. 3, pp. 20–34, May 2010.

[27] C. Helmberg, F. Rendl, R. J. Vanderbei, and H. Wolkowicz, “An interior-point method for semidefinite programming,” *SIAM Journal on Optimization*, vol. 6, no. 2, pp. 342–361, 1996.

[28] L. Vandenberghe and S. Boyd, “Semidefinite programming,” *SIAM Review*, vol. 38, no. 1, pp. 49–95, 1996.

[29] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985, ch. 0, pp. 12–14.

[30] F. Alizadeh, J.-P. A. Haeberly, and M. L. Overton, “Complementarity and nondegeneracy in semidefinite programming,” *Mathematical Programming*, vol. 77, no. 1, pp. 111–128, April 1997.
[31] L. Grippo and M. Sciandrone, “On the convergence of the block nonlinear gauss–seidel method under convex constraints,” *Operations Research Letters*, vol. 26, no. 3, pp. 127–136, April 2000.

[32] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, 1993, ch. 3, pp. 27–77.

[33] M. Grant, S. P. Boyd, and Y. Ye, “CVX: Matlab software for disciplined convex programming,” 2008.

[34] K. Border, “Notes on the implicit function theorem,” *California Institut of Technology, Division of the Humanities and Social Sciences*, 2013. [Online]. Available: [http://people.hss.caltech.edu/~kcb/Notes/IFT.pdf](http://people.hss.caltech.edu/~kcb/Notes/IFT.pdf)