Minimal Controllability Problem on Linear Structural Descriptor Systems With Forbidden Nodes

Shun Terasaki and Kazuhiro Sato, Member, IEEE

Abstract—We consider a minimal controllability problem (MCP), which determines the minimum number of input nodes for a descriptor system to be structurally controllable. We investigate the "forbidden nodes" in descriptor systems, denoting nodes that are unable to establish connections with input components. The three main results of this work are as follows. First, we show a solvability condition for the MCP with forbidden nodes using graph theory such as a bipartite graph and its Dulmage–Mendelsohn decomposition. Next, we derive the optimal value of the MCP with forbidden nodes. The optimal value is determined by an optimal solution for constrained maximum matching, and this result includes that of the standard MCP in the previous work. Finally, we provide an efficient algorithm for solving the MCP with forbidden nodes based on an alternating path algorithm.

Index Terms—Bipartite graph, Dulmage–Mendelsohn (DM) decomposition, descriptor system, large-scale system, structural controllability.

I. INTRODUCTION

Controllability analysis for large-scale network systems, such as multiagent systems [1], [2], brain networks [3], [4], and power networks [5], [6], has received a great deal of interest in recent years, because it can be used to find important nodes [7]. Controllability analysis problems include the following:

1) quantitative problems: the maximization problems of controllability metrics [8], [9], [10], [11], [12];
2) qualitative problems: selecting input problems that render the system controllable [7], [13], [14], [15], [16].

The quantitative problems require the system parameters, which are not precisely determined in practical systems. In addition, quantitative problems often become computationally intractable when the state dimension becomes large. Conversely, the structural information of a system, i.e., the nonzero patterns of system parameters, is usually known. This is an advantage for qualitative problems that deal only with nonzero patterns. Also, it is known that the structural controllability [17] of a structural system can be checked efficiently using graph algorithms. Thus, for large-scale network systems, it is more appropriate to consider qualitative problems.

Therefore, we consider a minimal controllability problem (MCP) of the following structural descriptor network system with state $x(t) \in \mathbb{R}^n$ and input $u(t) \in \mathbb{R}^m$, where $\mathbb{R}$ is the set of real numbers:

$$F \dot{x}(t) = Ax(t) + Bu(t),$$

(1)

where $F, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $F \in \mathbb{R}^{n \times n}$ can be a singular matrix. That is, we assume that although the specific elements of $F$, $A$, and $B$ remain unknown, their nonzero patterns are known. The descriptor formulation aptly models practical systems featuring algebraic constraints, such as those found in electric circuit systems [18], [19], [20]. MCPs can be divided into the following two main problems [21]:

1) MCP0: A problem that finds an $n \times m$ matrix $B$ for system (1) to be structurally controllable where $m$ is minimum. For $F = I_n$ or $F \neq I_n$, efficient algorithms exist for solving MCP0 [7], [13], [14], [16].

2) MCP1: A problem that finds an $n \times n$ diagonal matrix $B$ for system (1) to be structurally controllable and the number of nonzero elements in $B$ is minimum [14]. Although a polynomial time algorithm exists [15] for a special case of (1) with $F \neq I_n$, MCP1 is known to be NP-hard [16] in general.

It should be noted that, as mentioned in [22, Sec. 3.2], there are only a few papers on MCP0 or MCP1 for system (1) with $F \neq I_n$ although just structural controllability analysis under the assumption of a given $(F, A, B)$ has been studied in [19], [23], and [24].

The previous work in [16] on structural descriptor system (1) has not considered constraints on the input destination. There is a gap between practical situations since most physical systems have state variables to which inputs cannot be directly connected. For instance, consider a system in which the position $x(t)$ and velocity $v(t)$ of an object are the state variables. In this case, the state equation involves $\dot{x}(t) = v(t)$, but it does not make practical sense to add an input to this equation. Thus, specifying forbidden targets that cannot be connected to inputs is an important practical constraint. Therefore, Olshesky [13] introduced forbidden nodes to MCP1 for system (1) with $F = I_n$. However, no work applies MCPs with $F \neq I_n$.

In this article, we address MCP0 with forbidden nodes for structural descriptor system (1), because, in general, MCP1 for system (1) with $F \neq I_n$ is NP-hard, as shown in [16]. Here, the forbidden nodes correspond to the indices of "equations," while those in [13], which studied (1) with $F = I_n$, correspond to the indices of "variables." This naturally generalizes to $F \neq I_n$. In fact,

1) for $F = I_n$, the time evolution of $x_i$ is characterized by the $i$th equation of (1). Thus, in this case, we can regard the index of equations as that of variables.

2) for $F \neq I_n$, the time evolution of $x_i$ is not characterized by the $i$th equation of (1). Thus, in contrast to the case of $F = I_n$, here we cannot regard the index of equations as that of variables.

The contributions of this study can be summarized as follows.

1) We show a necessary and sufficient condition for the existence of the optimal solution of MCP0 with forbidden equations for structural descriptor system (1), which is described in the language of graph theory. This result is also useful in constructing the optimal solution.
Algorithm 1: Algorithm for DM Decomposition.
1. Find a maximum matching $M$ on $G$.
2. Construct an auxiliary directed graph $\tilde{G}_M$.
3. $V_0 := \{ v \in V^+ \cup V^- \mid \exists u \in V^+ \setminus \partial^+ M \ u \rightarrow_{\tilde{G}_M} v \}$,
   where $u \rightarrow_{\tilde{G}_M} v$ indicates the existence of a directed path on
   $\tilde{G}_M$ from $u$ to $v$.
4. $V_\infty := \{ v \in V^+ \cup V^- \mid \exists u \in V^- \setminus \partial^- M \ v \rightarrow_{\tilde{G}_M} u \}$.
5. Let $G'$ be the subgraph of $\tilde{G}_M$ defined by deleting all nodes
   of $V_0 \cup V_\infty$ and all edges adjacent to the nodes.
6. Let $V_k (k = 1, \ldots, b)$ be the SCCs of $G'$. Let undirected
   graph $G_k (k = 0, 1, \ldots, b, \infty)$ be the subgraph of $G$ induced
   on $V_k (k = 0, 1, \ldots, b, \infty)$.

2) We provide the optimal value of MCP0 with forbidden equations for
   structural descriptor system (1), by employing the graph-theoretic
   properties of the system, such as a bipartite graph and its Dulmage–
   Mendelsohn (DM) decomposition. The optimal value shows that
   the minimum number of input nodes is determined by a variant of
   the maximum matching problem with constraints on the matched
   nodes. This result includes that of the MCP0 without forbidden
   equations for descriptor system (1) [16].

3) We also provide an efficient algorithm for solving the aforementioned
   special matching problem by using the alternating path
   algorithm. The time complexity of this algorithm is $O(|V| + |
   E|\sqrt{|V|})$, which is on par with the algorithm for the MCP0
   [7], [16] without forbidden equations, where $|V|$ and $|E|$ are the
   numbers of nodes and edges of the bipartite graph corresponding to
   descriptor system (1), respectively.

The rest of this article is organized as follows. The basic concepts of
graph theory are summarized in Section II. The formulation of MCP0
with forbidden equations for structural descriptors is described in
Section III. In Section IV, we provide the analysis and the algorithm
of MCP0 with forbidden equations for the descriptor system (1).
Finally, Section V concludes this article.

II. BASIC CONCEPTS OF GRAPH THEORY

In this section, we present a comprehensive overview of the funda-
mental concepts of graph theory that are used in this article.

A strongly connected component (SCC) of a directed graph
with node set $V$ is a maximal subset $C \subseteq V$ whose nodes
$u, v \in C$ can be connected by a directed path on the graph.

Let $G = (V^+, V^-; E)$ be a bipartite graph. For an edge
$e = (v^+, v^-) \in E$, $\partial^+ e$ and $\partial^- e$ denote the nodes
of $v^+ \in V^+$ and $v^- \in V^-$, respectively. That is, $\partial^+ e : E \rightarrow V^+$
and $\partial^- e : E \rightarrow V^-$. The edge set $M \subseteq E$ is a matching if it does not share nodes of each edge.

A matching $M$ is termed maximum matching if $M$ contains the largest
possible number of edges. We define symbol $\nu(G)$ as the size of a
maximum matching of $G$.

We introduce the DM decomposition for a bipartite graph $G =
(V^+, V^-; E)$, which is the unique decomposition algorithm for bi-
partite graphs (see [19] for details). Algorithm 1 describes DM
decomposition. We define $M$ as a maximum matching of $G$ and an auxiliary
directed graph $\tilde{G}_M$. The edges of $\tilde{G}_M$ are oriented from $V^+$ to $V^-$
except for $M$. The decomposition is illustrated in Fig. 1.

Subgraphs $G_k (k = 0, \ldots, b, \infty)$ in step 6 of Algorithm 1 are called
DM components of $G$. $G_k (k = 1, \ldots, b)$ are called consistent DM
components; $G_0$ and $G_\infty$ are called inconsistent DM components. The

order $G_i \leq G_j$ for the consistent DM components $G_i$ and $G_j$ is defined as

"There is a directed path on $\tilde{G}_M$ from $G_i$ to $G_j"$

and the order between the consistent DM component $G_i$ and inconsis-
tent DM components $G_0$ and $G_\infty$ are defined as $G_0 \leq G_i$ and $G_i \leq G_\infty$;
respectively. Then, $\leq$ is a partial order. Moreover, the
decomposition constructed by Algorithm 1 does not depend on an
initially chosen maximum matching $M$ of $G$ in step 1), as shown in
[19, Lemma 2.3.35]. For example, in Fig. 1, there is a directed edge from
$G_2$ to $G_1$, and thus

$G_0 \leq G_1 \leq G_2 \leq G_\infty$.

The greatest computational bottleneck in the construction of the
DM decomposition is to find the maximum matching of $G$. This can
be achieved in $O(|E|\sqrt{|V|})$ by using the augmentation path algo-
rithm [25]. Thus, the computational complexity of DM decomposition is
$O(|E|\sqrt{|V|})$.

III. PROBLEM SETTINGS

In this section, we formulate MCP0 with forbidden equations for the
descriptor system (1).

First, we assume that system (1) is solvable, i.e., for any initial state
$x_0(t)$ with an admissible input $u_0(t)$, there exists a unique solution $x(t)$
to (1). This condition is equivalent to

$$\text{rank}(A - sF) = n$$

where $s$ is an indeterminant [19], [26].

In this article, we call system (1) controllable if for any admissible
initial state $x_0(t)$ that satisfies (1), there exists an input $u_0(t)$ and a final
time $T \geq 0$ such that $x(T) = 0$. This controllability is usually referred to as R-controllability. Additional controllability concepts can be found within the framework of descriptor system theory [27].

An algebraic characterization of controllability can be described as follows [26], [27].

**Proposition 1:** Descriptor system (1) is controllable if and only if

$$\text{rank} [A - zF \mid B] = n,$$

(3)

where $z$ is any complex number.

System (1) is termed structurally controllable if condition (3) in Proposition 1 holds for (1) with generic matrices $F$, $A$, and $B$. It should be noted that a matrix is considered generic if each nonzero element is an independent parameter. For a more precise definition of the generic matrix, see [19].

We now introduce MCP0 with forbidden equations for descriptor system (1). Let $R = \{e_1, \ldots, e_n\}$ be a set of equation indices and $F$ be a subset of $R$ that denotes forbidden equations. Then, MCP0 with forbidden equations for descriptor system (1) can be formalized as

$$\begin{cases}
\text{minimize} & m \\
\text{subject to} & \text{I) system (1) is structurally controllable} \\
& \text{II) indices of nonzero rows of } B \text{ are in } R \setminus F,
\end{cases}$$

(4)

where $\mathcal{G}^{n \times m}$ denote the set of all $n \times m$ generic matrices. The significant difference between the standard MCP0 and Problem (4) is II in (4). This constraint limits the input destination.

For instance, consider descriptor system (1) with

$$F = \begin{bmatrix}
0 & f_1 & 0 & 0 & f_2 \\
f_3 & 0 & 0 & 0 & 0 \\
0 & f_4 & 0 & 0 & 0 \\
0 & 0 & f_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
A = \begin{bmatrix}
a_1 & 0 & 0 & 0 & 0 \\
a_2 & 0 & 0 & 0 & 0 \\
a_3 & 0 & 0 & 0 & 0 \\
a_4 & 0 & 0 & 0 & 0 \\
a_5 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

(5)

Then, $R = \{e_1, e_2, \ldots, e_5\}$. Let $\mathcal{F} = \{e_3, e_4\}$ be the set of indices of forbidden equations. The matrices

$$B_1 = \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5
\end{bmatrix}^T,
B_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & b_2 \\
0 & b_1 & 0 & 0 & 0
\end{bmatrix}^T$$

(6)

satisfy condition II.

**IV. ANALYSIS AND ALGORITHM**

In this section, we provide the analysis and the algorithm of MCP0 with forbidden equations for descriptor system (1) with generic matrices using a graph-theoretic approach.

To this end, we first describe the graph representation of the descriptor system (1) with generic matrices and graph-theoretical controllability. Although there are several graph representations for descriptor system (1) [19], [24], [28], we used the bipartite graph representation [19] in this study. While the directed graph representation is common for system (1) with $F = I_n$, the bipartite graph representation is often used for $F \neq I_n$ [23].

The bipartite graph $G = (V^+, V^-; E)$ associated with descriptor system (1) is defined as follows: the node sets $V^+$ and $V^-$ are defined as

$$V^+ := X \cup U,
V^- := \{e_1, \ldots, e_n\},$$

where the state node set $X$ and the input node set $U$ are defined as

$$X := \{x_1, \ldots, x_n\},
U := \{u_1, \ldots, u_m\},$$

respectively, and $e_i$ in $V^-$ corresponds to the $i$th equation of system (1). That is, $V^+$ consists of state variables and inputs, and $V^-$ is the set of indices of equations. Then, the edge set $E$ is defined as

$$E := E_A \cup E_F \cup E_B$$

(7)

with $E_A := \{(e_i, x_j) \mid A_{ij} \neq 0\}$, $E_F := \{(e_i, x_j) \mid F_{ij} \neq 0\}$, and $E_B = \{(e_i, u_j) \mid B_{ij} \neq 0\}$. An edge belonging to $E_F$ is termed an s-arc. We also define important subgraphs of $G$ as $G_{A-F} = (X, V^+; E_A \cup E_F)$, $G_{A-B} = (V^-, V^-; E_A \cup E_B)$, and $G_A = (X, V^+; E_A)$. For example, the bipartite representation of descriptor system (1) with $B_2$ in (6), and its subgraphs are illustrated in Fig. 2.

Furthermore, for a DM component of $\mathcal{G}$ that has s-arcs [16], we call it a DM s-component, where $\tilde{G}$ is $G$ or $G_{A-F}$. Let $G_k$ $(k = 0, \ldots, b, \infty)$ be DM components of $\tilde{G}$, and $G_k$ $(k = 1, \ldots, b)$ are called consistent DM components; $G_0$ and $G_\infty$ are termed inconsistent DM components. The maximal consistent DM s-component is a consistent DM s-component $G_k$ of $\tilde{G}$ such that no other DM s-components are greater than $G_k$ related to the partial order $\preceq$. Note that there can be multiple maximal consistent DM s-components.

For instance, consider descriptor system (1) with parameters given in (5). The corresponding DM decomposition of the bipartite representation $G_{A-F}$ is depicted in Fig. 3. $G_1$, $G_2$, and $G_3$ are the consistent DM components of $G_{A-F}$, and $G_2 \preceq G_3 \preceq G_1$. That is, a maximal consistent DM s-component of this graph is $G_1$, because there is no edge to enter $G_1$ from other DM components.

The following graph-theoretic characterization of structural controllability for descriptor system (1) is found in [19].

**Proposition 2:** Descriptor system (1) is structurally controllable if and only if

1) $\nu(G_{A-F}) = n$;
2) $\nu(G_{A-B}) = n$;
3) no consistent DM components of $G$ contain s-arcs;

where $\nu(G)$ is the size of a maximum matching of a bipartite graph $G$.

Condition 1) represents the solvability of the system described in (2) in Section III, and the latter two conditions correspond to (3) in Proposition 1. From assumption (2), condition 1) in Proposition 2 is always satisfied. Assumption (2) also implies that there are no inconsistent DM components $G_0$ and $G_\infty$ of $G_{A-F}$. This is because if we choose a maximum matching $M$ with a size of $n$ in $G_{A-F}$, then $V_0$ and $V_n$ in steps 3 and 4 in Algorithm 1 are both empty.

The following lemma describes the characterization of the consistent DM components of $G_{A-F}$ and $G$.

**Lemma 1:** Consider descriptor system (1) that satisfies solvability condition (2). Then, the following conditions are equivalent.

1) A node $v \in X \cup V^-$ of a consistent DM component $G_i$ in $G_{A-F}$ belongs to an inconsistent DM component $G_0$ in $G$.
2) There is a directed path on $G_M$ from an input node $u \in U$ to $v \in X \cup V^-$ in step 2 of Algorithm 1 for $G$.

**Proof:** From assumption (2), condition 1) in Proposition 2 holds. This means that we can find the same maximum matching $M$ with size of $n$ in step 1 of Algorithm 1 for $G_{A-F}$ and $G$. Then, in step 2
Lemma 1 gives another characterization of the structural controllability condition for system (1) that satisfies solvability condition (2). This property will be used in the proof of Theorem 1 in the next section.

**Corollary 1:** Consider descriptor system (1) that satisfies solvability condition (2). Then, descriptor system (1) is structurally controllable if and only if

1') for each maximal consistent DM s-component $G_i$ in $G_{A-F}$, which is a subgraph of $G$, there is a directed path on $G_M$ from $U$ to $G_i$ in step 2 of Algorithm 1 for $G$.

**Proof:** From assumption (2), condition 1) in Proposition 2 holds. Condition 2') is equivalent to condition 2) in Proposition 2. Also, condition 3) in Proposition 2 holds if and only if all nodes $v \in X \cup V^-$ of consistent DM s-components $G_i$ in $G_{A-F}$ belong to an inconsistent DM s-component $G_0$ in $G$, since consistent DM s-components contain s-arcs. Using Lemma 1, this is equivalent to the following:

3') for each maximal consistent DM s-component $G_i$ in $G_{A-F}$, which is a subgraph of $G$, there is a directed path on $G_M$ from $U$ to $G_i$. This condition is also equivalent to 3') in Corollary 1, since the nonmaximal consistent DM components in $G_{A-F}$ have a directed path from the maximal consistent DM component in $G_{A-F}$ from the definition of the partial order $\preceq$.

Consider descriptor system (1) with $A$ and $F$ given in (5) and $B_1$ in (6). In $G_{A-F}$, there is no directed path from $u_1$ to $G_1$. This means that condition 3') in Corollary 1 does not hold. Thus, descriptor system (1) with (5) and $B_1$ in (6) is not structurally controllable.

**A. Existence of Solution to Problem (4)**

Unlike the traditional MCP0, it is not obvious whether or not an optimal solution exists for MCP0 with forbidden equations. By using Proposition 2 and Corollary 1, we obtain the following theorem that characterizes the existence of solutions to Problem (4). This theorem is employed to construct the optimal solution to Problem (4) in Section IV-D.

**Theorem 1:** Problem (4) has a solution if and only if both of the following conditions hold simultaneously.

a) For each node set $V_i$ of maximal consistent DM s-components $G_i$ in $G_{A-F}$, there exists $e \in V_i \cap V^-$ such that $e \not\in F$.

b) The graph-theoretic problem

$$
\begin{align*}
\text{maximize} \ & |M| \\
\text{subjectto} \ & M \text{ is matching of } G_A \\
& F \subseteq \partial^+ M
\end{align*}
$$

Note that 1) and 2) are equivalent.

For instance, consider descriptor system (1) with (5) and $B_1$ in (6). The DM decomposition of the bipartite representation $G$ is depicted in Fig. 4. $G_0$ is the inconsistent DM component, $G_1$ is the consistent DM component, and $G_0 \preceq G_1$. That is, $G_1$ is the maximal consistent DM s-component. Then, the consistent DM component $G_1$ in $G_{A-F}$ (see Fig. 3) is also consistent in $G$ (see Fig. 4), since there is no directed path from $u_1$ to $e_3$ or $x_3$ in $G$. Note that, in general, there is a directed path from $u \in U$ to $x \in X$, because edges in a matching are undirected.
is feasible.

Proof: We assume that \( B \in G_{n \times m} \) is a solution to Problem (4), and prove that conditions a) and b) hold. Note that this \( B \) defines \( E_B \) in (7).

Suppose that a maximal consistent DM s-component \( G_i \) of \( G_{A \times F} \), which is a subgraph of \( G \), is not connected to \( U \), where \( U \) denotes the set of input nodes. This means that there is no directed path on \( G_M \) for \( G \) from \( U \) to \( G_i \) in step 2 of Algorithm 1. From Corollary 1, this implies that system (1) with the given \( B \) is not structurally controllable, which contradicts the assumption. Thus, \( G_{A \times F} \), for the all node set \( V_i \) of maximal consistent DM s-components \( G_i \), there exists an equation node \( e \in V^- \cap V_i \), which is connected to an input node \( u \in U \). This means that \( e \notin F \) and condition a) holds. Moreover, condition 2) in Proposition 2 implies that the maximum matching \( M^* \) of \( G_{[A \mid B]} = (V^+, V^- \cup U; E \cup E_B) \) and \( |M^*| = n \) exists. Then, \( M^* \setminus E_B \) is a matching in \( G_A \) and \( \partial(M^* \setminus E_B) \subseteq V^- \setminus F \) (see Fig. 5). Thus, \( M^* \setminus E_B \) is a feasible solution to Problem (8), and condition b) holds.

Next, we assume that conditions a) and b) hold, and prove that Problem (4) has a solution. Consider the input nodes \( U \) with a size of \( n - |M^*| \), where \( M^* \) is the optimal solution to Problem (8). Then, we can connect each node in \( V_i \) to an input node in \( U \). This means that the set of equation nodes \( E_B \) in (7) with size \( |E_B| = n - |M^*| \) was constructed and forms a part of a matching of \( G_{[A \mid B]} \). That is, \( M^* \cup E_B \) is a maximum matching of \( G_{[A \mid B]} \); and thus, condition 2') in Corollary 1 holds. Also, for each maximal consistent DM s-component \( V_i \) of \( G_{A \times F} \), we can connect an equation node \( e \in V_i \setminus F \) to an input node from condition a). That is, all maximal consistent DM s-components of \( G_{A \times F} \) have directed paths from the input node set \( U \). This means that condition 3') in Corollary 1 holds. Thus, Problem (4) has a solution.

Theorem 1 implies that the existence of an optimal solution to MCP0 with forbidden equations for system (1) can be verified using pure graph theory.

For instance, consider descriptor system (1) with (5) depicted in Figs. 2 and 3.

1) Consider the case \( F = \{e_1, e_2\} \). Then, MCP0 with \( F \) is not feasible, since condition b) in Theorem 1 does not hold. In fact, the maximal consistent DM s-component \( G_1 \) in \( G_{A \times F} \) has a node \( e_2 \) that does not belong to \( F \). This means that condition a) in Theorem 1 holds. However, since a set of nodes that connect to \( e_1 \) or \( e_3 \) is \( \{x_1\} \) in \( G_A \) in Fig. 2(a), there is no matching \( M \) of \( G_A \) that satisfies \( F \subseteq \partial M \). Thus, Problem (8) is not feasible.

2) Consider the case \( F = \{e_2\} \). Then, MCP0 with \( F \) is not feasible, since condition a) does not hold. In fact, we can choose a feasible solution for Problem (8) as \( M_F := \{(e_2, x_1)\} \), which satisfies \( \partial M_F = \{e_2\} \subseteq F \). Thus, condition b) holds. However, a node of a maximal consistent DM s-component \( G_1 \) in \( G_{A \times F} \) is only \( e_2 \) and \( e_2 \notin F \).

B. Optimal Value for Problem (4)

Using the solution to Problem (8), we can compute the optimal value for Problem (4).

Theorem 2: Suppose that an optimal solution to Problem (4) exists. If the optimal value of Problem (8) is \( m^* \), then the minimum number of inputs that satisfy the constraints of Problem (4) is

\[ n_D = \max\{m - m^*, 1\}. \]  \hspace{1cm} (9)

Proof: We assume that \( B \) is an optimal solution with \( n_D \) column size to Problem (4), and prove that \( n_D \geq n - m^* \), where \( m^* \leq n \) by the definition. To this end, we assume that \( n_D < n - m^* \). From condition 2) in Proposition 2, \( \nu(G_{[A \mid B]} - n \) holds. Let \( M \) be a maximum matching of \( G_{[A \mid B]} \). Then, \( M^* := M \setminus E_B \) is a feasible solution to Problem (8) and \( |M^*| \geq n - n_D \). Thus, we obtain \( |M^*| > m^* \). This contradicts the maximality of \( m^* \).

If \( n_D = 0 \), the corresponding system of form (1) does not satisfy condition I) of Problem (4). Thus, \( n_D \geq \max\{n - m^*, 1\} \).

To prove that the equality holds, that is, (9) holds, we assume that \( M_F \) is an optimal solution to Problem (8) such that \( |M_F| = m^* \), and construct a system that satisfies the constraints of Problem (4) as follows.

1) Connect each input node to a node of \( V^- \setminus \partial M_F \). Then, \( \nu(G_{[A \mid B]} - n \) is satisfied and, i.e., condition 2') in Corollary 1 holds.

2) Connect the input nodes to all maximal consistent DM s-components of \( G_{A \times F} \). Then condition 3') in Corollary 1 is satisfied without increasing the number of input nodes. A system constructed by 1) and 2) satisfies condition I) of Problem (4). Moreover, a system constructed by 1) satisfies condition II), which is equivalent to that \( \partial E_{B} \subseteq V^- \setminus F \). This is because \( F \subseteq \partial M_F \). Note that in 2), we can choose nodes so that condition II) is satisfied, because condition a) in Theorem 1 implies that all maximal consistent DM s-components have at least one node that is not in \( F \). Therefore, this system of form (1) satisfies the constraints of Problem (4) and \( n_D \) is given by (9).

If the set \( F \) of forbidden equations is empty, (9) can be written as \( n_D = \max\{n - \nu(G_A), 1\} \). This result is consistent with the result of the standard MCP0 for system (1) [16].

The argument in the proof of Theorem 2 will be used to develop an algorithm for Problem (4) in Section IV-D.

C. Algorithm for Problem (8)

Algorithm 2 describes an algorithm for solving Problem (8) based on an alternating path algorithm [25]. Steps 1–4 check the feasibility of solving Problem (8). This is because if there is a feasible solution \( M \) to Problem (8), by restricting the matching nodes in \( \partial M \) to \( F \cap \partial M \), a matching \( M_F \) of \( G_F = (V^+, F; E_A) \) is obtained that satisfies \( |F| = |M_F| \) (see Fig. 6). The converse is shown to hold by the following alternating path algorithm in steps 5–9, as shown in the proof of Theorem 3.

The alternating path algorithm that is used in steps 5–9 is explained in Sections IV-D. The argument in the proof of Theorem 2 will be used to develop an algorithm for Problem (4) in Section IV-D.
Algorithm 2: Algorithm for Solving Problem (8).

1: Find the maximum matching $M^*_2$ of the graph $G_F := (V^+, F; E_A)$.
2: if $|F| > |M^*_2|$ then
3: Problem (8) is infeasible.
4: end if
5: Find the maximum matching $M^*$ of $G_A$.
6: for $v^- \in F \setminus \partial^+ M^*$ do
7: Find an alternating path $P = \{p_1, \ldots, p_{2l}\}$ which starts with $v^-$ and ends with $v^\prime_1 \in V^- \setminus F$.
8: $M^* := M^* \cup \{p_1, p_3, \ldots, p_{2l-1}\} \setminus \{p_2, p_4, \ldots, p_{2l}\}$
9: end for
10: Output $M^*$.

Algorithm 3: Algorithm for Solving Problem (4).

1: if condition a) in Theorem 1 is not satisfied then
2: Problem (4) is infeasible.
3: end if
4: Run Algorithm 2 and let $M^*$ be its output that is an optimal solution to Problem (8).
5: Connect inputs $U := \{u_1, \ldots, u_{n_D}\}$ to each equation node of $V^\prime \setminus \partial^+ M^*$, where $n_D$ is (9) in Theorem 2.
6: Connect inputs $U$ to an arbitrary node of each maximal consistent DM s-components of $G_{A,F}$, which does not belong to $F$.

Fig. 6. $G_A$ and $G_F = (V^+, F; E_A)$. The bold edges on the left are a feasible solution $M$ to Problem (8) while the right is a matching $M^*_2$ of $G_F$ that satisfies $|F| = |M^*_2|$.

Fig. 7. Alternating path algorithm.

Algorithm 3: Algorithm for Solving Problem (4).

1: if condition a) in Theorem 1 is not satisfied then
2: Problem (4) is infeasible.
3: end if
4: Run Algorithm 2 and let $M^*$ be its output that is an optimal solution to Problem (8).
5: Connect inputs $U := \{u_1, \ldots, u_{n_D}\}$ to each equation node of $V^\prime \setminus \partial^+ M^*$, where $n_D$ is (9) in Theorem 2.
6: Connect inputs $U$ to an arbitrary node of each maximal consistent DM s-components of $G_{A,F}$, which does not belong to $F$.

Fig. 8. Alternating path $P = \{p_1, p_2, \ldots, p_{2l-1}\}$. $p_1, p_3, \ldots, p_{2l-1}$ are in $M^*_2$, and $p_2, p_4, \ldots, p_{2l-2}$ are in $M^*$.
D. Algorithm for Problem (4)

Algorithm 3 shows an algorithm for solving Problem (4) and has the following property.

Theorem 4: If Problem (4) has a feasible solution, then Algorithm 3 outputs the optimal solution to Problem (4). If Problem (4) is infeasible, Algorithm 3 determines the infeasibility. Furthermore, the time complexity is $O(|V| + |E|\sqrt{|V|})$.

Proof: Algorithm 3 is based on the argument in the proof of Theorem 2. Steps 1–4 determine whether or not Problem (4) is feasible, and steps 1–3 are computed by Algorithm 1. Step 4 checks condition b) in Theorem 1 and outputs an optimal solution $M^*$ of Problem (8). Steps 5 and 6 in Algorithm 3 output the optimal solution to Problem (4), as shown in the proof of Theorem 2.

Next, we show that the time complexity is $O(|V| + |E|\sqrt{|V|})$. Steps 1–3 in Algorithm 3 can be checked by Algorithm 1, its time complexity is $O(|E|\sqrt{|V|})$. The time complexity of step 4 is $O(|V| + |E|\sqrt{|V|})$. In fact, in steps 1–5 in Algorithm 2 are computed by using the Hopcroft–Karp algorithm [25]; the time complexity is $O(|E|\sqrt{|V|})$. Moreover, steps 6–9 in Algorithm 2 can be computed by the breadth-first search algorithm in $O(|V| + |E|)$, as mentioned already. Steps 5 and 6 in Algorithm 3 can be computed in $O(|V|)$ at most. Therefore, the time complexity of Algorithm 2 is $O(|V| + |E|\sqrt{|V|})$.

Theorem 4 means that more general problems than those of [7] and [16] can be solved with the same computational complexity. In fact, the problem addressed by [7] is a special case of Problem (4) with $F = \emptyset$ and $F = I_n$, and it can be solved in $O(|V| + |E|\sqrt{|V|})$. Moreover, the algorithm proposed in [16] deals with Problem (4) with $F = \emptyset$, and the time complexity is $O(|V| + |E|\sqrt{|V|})$.

To illustrate how Algorithm 3 works, consider descriptor system (1) with (5) and forbidden equations $F = \{e_3, e_4\}$. In this case, steps 1–4 determine that Problem (4) is feasible. In fact, the maximal consistent DM s-component $G_1$ in Fig. 3 has a node $e_3$ that does not belong to $F$. Also, in step 4, we obtain $M^* = \{(e_2, x_1), (e_4, x_3), (e_5, x_4)\}$ from the discussion of how Algorithm 2 works in Section IV-C. Thus, Problem (4) is feasible, since conditions a) and b) in Theorem 1 hold. Furthermore, steps 5 and 6 produce

$$B = \begin{bmatrix} b_2 & 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 & 0 \end{bmatrix}^\top$$

that is an optimal solution to Problem (4). In fact, from Theorem 2, we have the minimum number of inputs $n_D = n - |M^*| = 2$. Also, $V^\setminus\partial M^* = \{e_1, e_2\}$. Thus, connecting $u_1$ to $e_1$, and $u_2$ to $e_2$, we have $B$ as (10).

V. CONCLUSION

In this study, we introduced the forbidden equations to MCP0 for structural descriptor systems. We gave a necessary and sufficient condition for the existence of solutions to the problem and provided the solution to MCP0. The algorithm for solving the problem can be computed in polynomial time as for the standard MCP0. That is, our proposed algorithm is well positioned for applications to large-scale descriptor systems.

In this article, we focused on MCP0 for a structural descriptor system since, for a structural descriptor system, MCP1 is NP-hard in general, as shown in [16]. Thus, finding a solvable condition for MCP1 with forbidden equations in polynomial time would be a future project.

Furthermore, structural controllability considered in this article requires that the system parameters be algebraically independent. This means that all nonzero system parameters are free, which may be a strong assumption for practical situations. To avoid this assumption, strong structural controllability has been proposed in [29], and studied from a graph-theoretic perspective in [30]. A strong structural controllability problem version in this article is one of the future works.

REFERENCES

[1] M. Mesbahi and M. Egerstedt, Graph Theoretic Methods in Multiagent Networks. Princeton, NJ, USA: Princeton Univ. Press, 2010.
[2] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, “Controllability of multi-agent systems from a graph-theoretic perspective,” SIAM J. Control Optim., vol. 48, no. 1, pp. 162–186, 2009.
[3] S. Gu et al., “Controllability of structural brain networks,” Nature Commun., vol. 6, no. 1, pp. 1–10, 2015.
[4] S. F. Muldoon et al., “Stimulation-based control of dynamic brain networks,” PLoS Comput. Biol., vol. 12, no. 9, 2016, Art. no. 1050576.
[5] G. A. Pagani and M. Aiello, “The power grid as a complex network: A survey,” Physica A, Stat. Mech. Appl., vol. 392, no. 11, pp. 2688–2700, 2013.
[6] D.-S. Yang, Y.-H. Sun, B.-W. Zhou, X.-T. Gao, and H.-G. Zhang, “Critical nodes identification of complex power systems based on electric cactus structure,” IEEE Syst. J., vol. 14, no. 3, pp. 4477–4488, Sep. 2020.
[7] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, “Controllability of complex networks,” Nature, vol. 473, no. 7346, pp. 167–173, 2011.
[8] F. Pasqualetti, S. Zampieri, and F. Bullo, “Controllability metrics, limitations and algorithms for complex networks,” IEEE Trans. Control Netw. Syst., vol. 1, no. 1, pp. 40–52, Mar. 2014.
[9] T. H. Summers, F. L. Cortesi, and J. Lygeros, “On submodularity and controllability in complex dynamical networks,” IEEE Trans. Control Netw. Syst., vol. 3, no. 1, pp. 91–101, Mar. 2016.
[10] A. Clark, B. Alomair, L. Bushnell, and R. Poovendran, “Submodularity in input node selection for networked linear systems: Efficient algorithms for performance and controllability,” IEEE Control Syst. Mag., vol. 37, no. 6, pp. 52–74, Dec. 2017.
[11] L. Romao, K. Margellos, and A. Papachristodoulou, “Distributed actuator selection: Achieving optimality via a primal-dual algorithm,” IEEE Control Syst. Lett., vol. 2, no. 4, pp. 779–784, Oct. 2018.
[12] K. Sato and A. Takeda, “Controllability maximization of large-scale systems using projected gradient method,” IEEE Control Syst. Lett., vol. 4, no. 4, pp. 821–826, Oct. 2020.
[13] A. Olshovsky, “Minimal controllability problems,” IEEE Trans. Control Netw. Syst., vol. 1, no. 3, pp. 249–258, Sep. 2014.
[14] S. Pequito, S. Kar, and A. P. Aguiar, “A framework for structural input/output and control configuration selection in large-scale systems,” IEEE Trans. Autom. Control, vol. 61, no. 2, pp. 303–318, Feb. 2016.
[15] A. Clark, B. Alomair, L. Bushnell, and R. Poovendran, “Input selection for performance and controllability of structured linear descriptor systems,” SIAM J. Control Optim., vol. 55, no. 1, pp. 457–485, 2017.
[16] S. Terasaki and K. Sato, “Minimal controllability problems on linear structural descriptor systems,” IEEE Trans. Autom. Control, vol. 67, no. 5, pp. 2522–2528, May 2022.
[17] C.-T. Lin, “Structural controllability,” IEEE Trans. Autom. Control, vol. 19, no. 3, pp. 201–208, Jun. 1974.
[18] L. Dai, Singular Control Systems. Berlin, Germany: Springer, 1989.
[19] K. Murota, *Matrices and Matroids for Systems Analysis*. Berlin, Germany: Springer, 2009.
[20] G.-R. Duan, *Analysis and Design of Descriptor Linear Systems*. Berlin, Germany: Springer, 2010.
[21] Y.-Y. Liu and A.-L. Barabási, “Control principles of complex systems,” *Rev. Modern Phys.*, vol. 88, 2016, Art. no. 035006.
[22] G. Ramos, A. P. Aguiar, and S. Pequito, “An overview of structural systems theory,” *Automatica*, vol. 140, 2022, Art. no. 110229.
[23] K. Murota, “Structural controllability of a system in descriptor form expressed in terms of bipartite graphs (in Japanese),” *Trans. Soc. Instrum. Control Engineers*, vol. 20, no. 3, pp. 272–274, 1984.
[24] K. J. Reinschke and G. Wiedemann, “Digraph characterization of structural controllability for linear descriptor systems,” *Linear Algebra Appl.*, vol. 266, pp. 199–217, 1997.
[25] B. Korte, J. Vygen, B. Korte, and J. Vygen, *Combinatorial Optimization*, vol. 2. Berlin, Germany: Springer, 2012.
[26] E. Yip and R. Sincovec, “Solvability, controllability, and observability of continuous descriptor systems,” *IEEE Trans. Autom. Control*, vol. AC-26, no. 3, pp. 702–707, Jun. 1981.
[27] T. Berger and T. Reis, “Controllability of linear differential-algebraic systems—A survey,” in *Surveys in Differential-algebraic Equations I*. Berlin, Germany: Springer, 2013, pp. 1–61.
[28] T. Yamada and D. Luenberger, “Generic controllability theorems for descriptor systems,” *IEEE Trans. Autom. Control*, vol. AC-30, no. 2, pp. 144–152, Feb. 1985.
[29] H. Mayeda and T. Yamada, “Strong structural controllability,” *SIAM J. Control Optim.*, vol. 17, no. 1, pp. 123–138, 1979.
[30] J. Jia, H. J. Van Waarde, H. L. Trentelman, and M. K. Camlibel, “A unifying framework for strong structural controllability,” *IEEE Trans. Autom. Control*, vol. 66, no. 1, pp. 391–398, Jan. 2021.