Spherically symmetric perfect fluid in area-radial coordinates

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Abstract

We study the spherically symmetric collapse of a perfect fluid using area-radial coordinates. We show that analytic mass functions describe a static regular centre in these coordinates. In this case, a central singularity cannot be realized without an infinite discontinuity in the central density. We construct mass functions involving fluid dynamics at the centre and investigate the relationship between those and the nature of the singularities.

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1. Introduction

Gravitational collapse is one of the main issues of gravitational physics. The nature of spacetime singularities formed in gravitational collapse has been studied for a long time. In these studies, some sort of spacetime symmetry, such as spherical symmetry, is commonly assumed.

The final fate of spherically symmetric dust collapse has been extensively studied. In this case, a general exact solution to Einstein’s field equations is available. It was proved that a collapsing dust ball results in naked or covered singularities [1]. The condition for both outcomes has been derived in terms of the initial data [2–4] and the structure of naked singularities formed in this system has also been studied in detail [5, 6].

As for more realistic fluids with equations of state, such as $p = \alpha \epsilon$, where the pressure $p$ is proportional to the energy density $\epsilon$, several important results have been obtained so far. There are self-similar collapse solutions which result in naked singularity formation [7, 8]. The generic appearance of naked singularities has been strongly suggested numerically for
a very small value of $\alpha$ [9, 10]. The critical collapse appears at the black-hole threshold for $0 < \alpha \leqslant 1$, which can be identified with the naked singularity [11, 12]. In spite of these findings, a comprehensive understanding of the endstates of gravitational collapse in this system has not been achieved yet. This is partly due to the fact that we do not know a general exact solution for this system, unlike for the dust cases. It should be noted that there are several papers related to this work by Christodoulou [13].

Recently, Giamb`o et al gave an interesting approach to the spherically symmetric system of a perfect fluid with the barotropic equation of state $p = p(\epsilon)$ [14, 15]. They adopted the so-called area-radial coordinates and derived a second-order quasi-linear partial differential equation (PDE) for a mass function, which is identical to the Misner–Sharp quasi-local mass. Due to the presence of pressure the mass function is time dependent. A solution to this PDE gives a solution to the whole set of Einstein’s field equations. Using mass functions which are analytical around the centre, they have examined the spacetime singularities resulting from gravitational collapse. In particular, they claimed that there are always naked singularities for that class of mass function.

In this paper, we analyse the behaviour of the metric around the regular centre and show that analytic mass functions imply staticity around the centre. This means that, in this case, a central singularity cannot be formed without an infinite discontinuity in the fluid density. We then present a more general analysis of the final fate of collapse in this system by (i) studying other classes of mass functions and (ii) investigating naked singularity formation within such classes.

The plan of this paper is as follows. In section 2, we briefly recall the formulation of spherically symmetric spacetimes in area-radial coordinates. In section 3, we clarify the behaviour of metric functions during the regular evolution, which is key to the present problem. We investigate what type of collapse can be represented by the analytic mass functions given by Giamb`o et al [14, 15] in section 4. In section 5, we investigate the Friedmann solution and construct other dynamical mass functions. We investigate the relationship between the mass functions and naked singularity formation in section 6. In section 7, we give concluding remarks. We adopt units in which $G = c = 1$.

2. Formulation

2.1. Comoving coordinates

The line element of a spherically symmetric spacetime in comoving coordinates can be written as

$$ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(1)

where $\nu = \nu(r,t)$, $\lambda = \lambda(r,t)$ and $R = R(r,t)$ are of class $C^2$. We consider a perfect fluid as a matter field and denote the energy density and the isotropic pressure of the fluid by $\epsilon$ and $p$, respectively. Then, the Einstein field equations can be reduced to

$$\Psi' = 4\pi \epsilon R^2 R',$$

(2)

$$\Psi = -4\pi p R^2 \dot{R},$$

(3)

$$\dot{R}' = \dot{R} v' + R' \dot{\lambda},$$

(4)

$$p' = -(\epsilon + p)v',$$

(5)

where a dot and a prime denote derivatives with respect to $t$ and $r$, respectively, and $\Psi = \Psi(r,t)$ is the Misner–Sharp quasi-local mass,

$$\Psi(r,t) = \frac{R}{2} [1 - R^2 e^{-2\lambda} + R^2 e^{-2\nu}].$$

(6)
The matter (number) density is given by

$$\rho = \frac{e^{-\lambda}}{4\pi E R^2}, \quad \text{(7)}$$

where $E$ is an arbitrary positive function of $r$. Here we assume a barotropic perfect fluid with a linear equation of state

$$p = \alpha \epsilon , \quad \text{(8)}$$

where $\alpha > 0$ is constant. This together with $p = \rho \frac{d\epsilon}{d\rho} - \epsilon$ implies $\epsilon = \rho^{\alpha + 1}$, where the constant of proportion was absorbed into the function $E$. Therefore, from equation (5) we get

$$e^\nu = \rho^{-\alpha}, \quad \text{(9)}$$

where the scaling of $t$ is chosen so that the above equation holds. Using the freedom in the scaling of $r$, we can set at the regular initial moment

$$R(r, 0) = r, \quad \text{(10)}$$
$$R'(r, 0) = 1, \quad \text{(11)}$$

if there is no apparent horizon initially. In the following we choose this scaling.

### 2.2. Area-radial coordinates

Using area-radial coordinates $(r, R)$, we find from equations (2), (3) and (8) that both $R'$ and the energy density are related to the mass function as follows:

$$R' = -\frac{\alpha}{\alpha + 1} \frac{\Psi_r}{\Psi_R}, \quad \text{(12)}$$
$$\epsilon = \rho^{\alpha + 1} = -\frac{\Psi_R}{4\pi \alpha R^2}, \quad \text{(13)}$$

where a subscript denotes the partial derivative with respect to the indicated variable in the area-radial coordinates $(r, R)$. In these coordinates, the line element (1) can be written as

$$ds^2 = -\frac{1}{u^2} \left[ dR^2 - 2R' dR dr + \left( R' \frac{Y}{\sqrt{Y}} \right)^2 \left( 1 - \frac{2\Psi}{R} \right) dr^2 \right] + R^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad \text{(14)}$$

where

$$Y = R' e^{-\lambda}, \quad \text{(15)}$$
$$u^2 = \frac{2\Psi}{R} + Y^2 - 1. \quad \text{(16)}$$

Using equations (7), (12) and (13), the function $Y$ can also be written in terms of the mass function as

$$Y = \frac{E \Psi_r}{(\alpha + 1) \rho^2}. \quad \text{(17)}$$

We can now use equations (9), (12), (13) and (17) in equation (4) to reduce Einstein’s field equations (2)–(5) to the following PDE:

$$-\frac{1}{\alpha + 1} \left( \frac{2\Psi}{R} - 1 + (1 - \alpha) Y^2 \right) \frac{\Psi_{rr}}{\Psi_r} - 2Y^2 \frac{\Psi_{rR}}{\Psi_r} + \frac{\alpha + 1}{\alpha} \frac{\Psi_{rR}}{\Psi_r} Y^2 \frac{\Psi_{rr}}{\Psi_r} - \frac{1}{\alpha + 1} \left( \frac{2\Psi}{R} - 1 + (1 - \alpha) Y^2 \right) \frac{2\alpha}{R} - 2Y^2 \frac{\Psi}{R^2} + \frac{\Psi}{\alpha R} + \frac{\Psi_{r}}{\Psi_r} \frac{E_f}{E} Y^2 = 0. \quad \text{(18)}$$
Now, by eliminating $E(r)$ from equation (18), using two arbitrary functions $\Psi_1(r) = \Psi_1(r, r)$ and $Y_0(r) = Y(r, r)$, one can obtain the following PDE for $\Psi_1$:

\[
-\frac{1}{\alpha + 1} \left( \frac{2\Psi}{R} - 1 + (1 - \alpha)Y^2 \right) \frac{\Psi_{RR}}{\Psi_R} - 2Y^2 \frac{\Psi_{xR}}{\Psi_x} + \frac{\alpha + 1}{\alpha} \frac{\Psi_{xR}}{\Psi_x} \frac{\Psi_{xx}}{\Psi_x} \\
- \frac{2\alpha}{\alpha + 1} \left( \frac{2\Psi}{R} + 2Y^2 - 1 \right) \frac{1}{R} + \frac{\Psi}{R^2} + \frac{\Psi_{xR}}{\alpha R} \\
+ \frac{\alpha + 1}{\alpha} \frac{\Psi_{xR}}{\Psi_x} \left( \frac{Y_0'}{Y_0} - \frac{1}{\alpha + 1} \frac{\Psi_0'}{\Psi_0} - \frac{2\alpha}{(\alpha + 1)r} \right) Y^2 = 0,
\]

(19)

and

\[
Y(r, R) = \frac{\Psi_x(r, R)}{\Psi_x(r, r)} \left[ \frac{\Psi_{xR}(r, r) R^2}{\Psi_{xR}(r, R) R^2} \right]^{\frac{\alpha}{\alpha + 1}} Y_0(r).
\]

(20)

In the following, we call the above quasi-linear second-order PDE Giambò–Giannoni–Magli–Piccione (GGMP) equation. Because we adopt the scaling of the comoving coordinate $r$ which is given by equations (10) and (11), the mass function must also satisfy equation (12) at the initial surface, i.e.,

\[
\Psi_{xR}(r, r) = -\frac{\alpha}{\alpha + 1} \Psi_x(r, r).
\]

(21)

3. Behaviour of the metric tensor around the regular centre

First we consider the comoving coordinate system. The regularity condition at $r = 0$ requires

\[
R(0, t) = 0,
\]

(22)

\[
R'(0, t) = e^{\lambda(0, t)},
\]

(23)

during the evolution. It should be noted that these conditions are used in Giambò et al [14].

Also, in the following analysis, it is sufficient to assume lower differentiability, $C^1$ class, for the functions $R(r, t), R'(r, t), e^{\lambda(r, t)}$ and $\rho(r, t)$ on $t = \text{const}$ slice.

An estimate of equation (7) at $t = 0$ with a regular centre yields

\[
E(r) \approx \frac{1}{4\pi \rho_0(0, 0) r^2},
\]

(24)

where we have used (10), (11) and (23) and the weak equality is used with the following meaning:

\[
A(r) \approx B(r) \iff \lim_{r \to 0} \frac{A(r)}{B(r)} = 1.
\]

(25)

Then, substituting equation (24) into equation (7), and taking condition (23) into account, we get

\[
\rho(r, t) \approx \rho(0, 0) \frac{r^2}{R^2 R'},
\]

(26)

where both sides depend on $r$ and $t$ and the weak equality ‘$\approx$’ is used with the following meaning:

\[
A(r, t) \approx B(r, t) \iff \lim_{r \to 0} \frac{A(r, t)}{B(r, t)} = 1.
\]

(27)
i.e. the limit is taken as \( t \) is fixed. Integrating equation (26) for small \( r \) as \( t \) is fixed, we have

\[
R^3 \approx \frac{\rho(0, 0)}{\rho(0, t)} r^3 + C(t),
\]

(28)

where \( C(t) \) is an arbitrary function. From the regularity condition (22), we conclude that \( C(t) = 0 \). Then, we finally get

\[
R \approx \left[ \frac{\rho(0, 0)}{\rho(0, t)} \right]^{1/3} r \quad \text{(29)}
\]

\[
R' \approx \left[ \frac{\rho(0, 0)}{\rho(0, t)} \right]^{1/3} . \quad \text{(30)}
\]

This implies that the area radius \( R \) is proportional to the comoving coordinate \( r \) in lowest order as \( t \) is fixed. It should be noted that the above discussion does not prove uniform convergence.

Next we move on to the area-radial coordinate system. The above condition can be translated into the area-radial coordinates. A family of \( t = \text{const} \) regular spacelike hypersurfaces in the comoving coordinates is transformed to a one-parameter family of curves \( R = R_t(r) \) in the \( rR \) plane, which cross the origin \( R = r = 0 \) along straight lines \( R = \tau r \) with \( 0 < \tau < \infty \). If the fluid is static, the family of \( t = \text{const} \) spacelike hypersurfaces is transformed to one degenerate straight line \( R = r \) in the \( rR \) plane. If we consider models in which the central density changes monotonically with time, this slope parameter \( \tau \) parametrizes the family of evolution curves. Moreover, if we focus on monotonically collapsing situations, we can assume \( 0 < \tau \leq 1 \). The initial hypersurface \( t = 0 \) corresponds to the straight line \( R = r \) with \( 45^\circ \) in the \( rR \) plane. As time proceeds, the slope of the evolution curves at the origin \( R = r = 0 \) gradually decreases towards zero. The \( r \) axis corresponds to the singularity curve. This means that we can approach a central singularity along a curve \( R \approx r^n \) with \( n > 1 \). The regularity condition (23) is now translated into the area-radial coordinates in terms of the function \( Y \) as

\[
\lim_{r \to 0} Y(r, \tau r) = 1, \quad \text{for } 0 < \tau < \infty \text{, where the limit is taken as } \tau \text{ is fixed.}
\]

4. Analytic mass functions

In this section, we would like to investigate what type of collapse can be represented by the mass functions given by Giambò et al [14, 15]. We recall that in [14] it is assumed that \( \Psi \) is an analytic function with respect to \( r \) and \( R \) which can be expanded as

\[
\Psi(r, R) = \sum_{k=0}^{\infty} \sum_{i+j=3+2k} \Psi_{ij} r^i R^j ,
\]

(32)

where \( i \) and \( j \) are non-negative integers, and it is proved that the regularity condition (31) requires the following structure for the lowest order terms:

\[
\Psi(r, R) = \frac{h}{2} \left( r^3 - \frac{\alpha}{\alpha + 1} R^3 \right) + \sum_{k=1}^{\infty} \sum_{i+j=3+2k} \Psi_{ij} r^i R^j .
\]

(33)

In addition, it is assumed that the coefficient \( \Psi_{41} \) vanishes in the proof of lemma 3.5 of [14]. Using this form of the mass function, it is claimed that the central singularity is naked in
theorem 3.4 of [14]. In the following, however, we show that their solution has a static centre in the regular evolution.

Let us consider that we have regular evolution after the initial moment. During the regular evolution, we can use the relation \( R \approx \tau r (0 < \tau \leq \infty) \) for small \( r \), as discussed in section 3. Then, it can be shown that the central density is

\[
\lim_{r \to 0} \rho^{(1+\alpha)}(r, \tau r) = \frac{3h}{8\pi(\alpha + 1)},
\]

by substituting equation (33) into equation (13), that is, the central density does not depend on \( \tau \). This guarantees, through equation (9), that the function \( e^\nu \) is constant with time at the centre. This also guarantees, through equations (29) and (30), that \( R \approx r \) and \( R' \approx 1 \). We can also obtain the same result from equation (12). Equation (12) with the relation \( R = \tau r \) (0 < \( \tau < \infty \)) implies

\[
R' \approx \left( \frac{r}{R} \right)^2.
\]

Since this relation is satisfied at any time \( t \), this can be integrated as \( t \) is fixed. The result is

\[
R^3(r, t) \approx r^3 + D(t),
\]

where \( D \) is an arbitrary function of \( t \). The regularity condition (22) requires \( D(t) = 0 \) and therefore \( R \approx r \) and \( R' \approx 1 \). Hence the regularity condition (23) implies \( e^\lambda \approx 1 \). Therefore, the symmetric centre cannot become singular without an infinite discontinuity in \( \rho \).

To construct mass functions which can describe genuine dynamical situations, we shall include terms of negative power of \( R \) in an expansion of the mass function. In the next section we shall construct such mass functions.

5. Mass functions involving fluid dynamics

We have pointed out some problems associated with the choice of mass functions (33). In this section, we construct other forms for \( \Psi \) which can be used to study models of gravitational collapse as well as cosmological models.

To begin with, we consider the Friedmann solution as the simplest solution of a dynamical fluid. In this case, the matter density is homogeneous and satisfies

\[
\rho \propto \frac{1}{a(t)} \propto \frac{1}{R^3},
\]

where \( a \) is a scale factor. Substituting this relation into equation (13), we obtain,

\[
\rho_0 \left( \frac{R}{r} \right)^{-3} = \left( -\frac{\Psi_R}{4\pi \alpha R^2} \right)^{1/\alpha},
\]

where \( \rho_0 \) is a constant. After integrating equation (38) with respect to \( R \), the mass function can be written as

\[
\Psi(r, R) = \frac{4\pi \rho_0^{\alpha+1}}{3} \frac{r^{3(\alpha+1)}}{R^{\alpha+1}} + f(r),
\]

where \( f(r) \) is an arbitrary function. From the assumption that \( R = r \) is the initial regular slice, we find that \( R = \tau r \) is also a regular slice, where 0 < \( \tau < 1 \) is a constant. When we adopt the regularity condition for \( \Psi \), we find \( f(0) = 0 \). Furthermore, from equation (21), we obtain \( f' = 0 \) and then

\[
f(r) = 0.
\]
Therefore, the mass function (39) of the Friedmann solution results in
\[ \Psi(r, R) = \Psi_3 \frac{r^{3(\alpha+1)}}{R^{3\alpha}}, \] (41)
where \( \Psi_3 = 4\pi \rho_0^{\alpha+1}/3 \).

In order to recover the familiar form of the Friedmann solution, we note that the metric functions in comoving coordinates \( e^\nu \) and \( e^\xi \) can be written in terms of the scale factor \( a \) using equation (15), \( Y = Y_0(r) \) and \( e^\nu \propto \rho^{-\alpha} \). Then, after an appropriate coordinate transformation we get
\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{Y_0^2(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right). \] (42)

Next, we would like to investigate more general situations. A physically reasonable mass function should satisfy not only the PDE (19) but also relation (21) on the initial slice as well as the regularity conditions at the regular centre on the regular slice \( R = r \tau \). In addition to these conditions, we can impose condition (A) that the mass function is the same as in the Friedmann solution (41) at the lowest order around the symmetric centre. To satisfy condition (A), the generalized mass function is written as
\[ \Psi(r, R) = \Psi_3 \frac{r^{3(\alpha+1)}}{R^{3\alpha}} + \Pi(r, R), \] (43)
where \( \Pi(r, R) \) is a higher-order term on the regular slice, related to the inhomogeneity of mass distribution. The functional form of \( \Pi(r, R) \) is important for the final fate of the collapse, that is, whether it is naked or covered.

In addition to the above restrictions on the mass function, we may impose condition (B) that the mass function recovers an appropriate dust limit as \( a \to 0 \), in which the mass function depends only on the comoving radial coordinate \( r \). We can consider that the mass function \( \Psi \) is given by the following series:
\[ \Psi = \sum_{k=1}^{\infty} \Psi_{2k+1} r^{2k+1} \left( \frac{R}{r} \right)^{\beta_{2k+1}}, \] (44)
where \( \{\Psi_{2k+1}; k = 1, 2, \ldots \} \) are constant coefficients and \( \{\beta_{2k+1}; k = 1, 2, \ldots \} \) are constants which depend only on \( k \) and \( \alpha \). Equation (21) yields \( \beta_{2k+1} = -(2k + 1)\alpha \). This satisfies both conditions (A) and (B). From this form of expansion, we can conclude that \( \Psi = \Psi(r^{\alpha+1}/R^\alpha) \). This together with equation (12) yields \( R = \tau(t)r \), where \( \tau \) is an arbitrary function of \( t \). This implies that the singularity is simultaneous and therefore covered.

A potentially more interesting form for the mass function results from a slight generalization of (44):
\[ \Psi = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \Psi_{2k+1,i} r^{2k+1} \left( \frac{R}{r} \right)^{\beta_{2k+1,i}}, \] (45)
where \( \Psi = \{\Psi_{2k+1,i}; k = 1, 2, \ldots, i = 1, 2, \ldots \} \) is a constant coefficient matrix and \( \{\beta_{2k+1,i}; k = 1, 2, \ldots, i = 1, 2, \ldots \} \) are constants which depend only on \( \alpha, k \), and \( i \). Condition (A) yields \( \beta_{3i} = -3\alpha \) and \( \beta_{3i} = 0 \) for \( i \geq 2 \). Condition (B) is satisfied if and only if \( \beta_{2k+1,i} \to 0 \) as \( \alpha \to 0 \). Equation (21) yields
\[ \sum_{i=1}^{\infty} \Psi_{2k+1,i} [\beta_{2k+1,i} + (2k + 1)\alpha] = 0. \] (46)
This implies that if the coefficient matrix \( \Psi \) is regular, \( \beta_{2k+1,i} = -(2k + 1)\alpha \) is obtained, which eventually results in the same situation as the simplest model. Therefore, if the mass function
which admits the expansion (45) describes naked singularity formation, then the coefficient
matrix \( \Psi \) must be singular, or \( \det \Psi = 0 \). In this case, there exists at least a set of values for \( k \)
and \( i \) such that \( \beta_{2k+1,i} \neq -(2k + 1)\alpha \).

We note that while the emphasis here has been given to models of collapse, the mass
functions \( \Psi \) can also be used in cosmological models. In particular, mass functions (41) and
(44) can be associated with a simultaneous big bang and can be used in the study of the early
universe.

6. Naked singular mass functions

In this section, we investigate mass functions which involve spacetimes with naked
singularities. We follow a technique used by Mena and Nolan [5] and developed by Giambò
et al [16]. To show the existence of null geodesics emanating from the central singularity,
we show the existence of the region from which future-directed outgoing null geodesics
cannot get out when they are traced back to the centre. This region is between two curves
\( R = R_1(r) \) and \( R = R_2(r) \) (\( 0 < R_1(r) < R_2(r) \)) both emanating from the central singularity.

Defining \( \varphi(r, R) \) as the first-order derivative \( dR/dr \) along a future-directed outgoing radial
null geodesic, i.e.

\[
\varphi(r, R) \equiv -\frac{\alpha}{\alpha + 1} \frac{\Psi_r}{\Psi_R} \left( 1 - \frac{u}{\gamma} \right),
\]

(47)

the curve \( R_1 \) satisfies,

\[
\frac{dR_1(r)}{dr} \geq \varphi(r, R_1),
\]

(48)

while \( R_2 \) satisfies

\[
\frac{dR_2(r)}{dr} \leq \varphi(r, R_2).
\]

(49)

In the following analysis, we assume that only one term in the expansion of \( \Pi(r, R) \) in (43),
say

\[
\Pi_0 r^{5-m} R^m,
\]

(50)

where \( m \) is a constant, is important to examine the existence of naked singularities. Under
this assumption we will derive the conditions on \( \Pi_0 \) and \( m \) for the existence of radial null
geodesics emanating from the singularity which forms from gravitational collapse. As we
have already mentioned, to construct a solution we may need another term

\[
\Pi_1 r^{5-m} R^m,
\]

(51)

which is of the same order as the term of equation (50) on the regular slice.

In this analysis, an important role is played by the apparent horizon \( R = R_h(r) \), where

\[
R_h(r) = 2\Psi(r, R_h(r)).
\]

(52)

Under the above assumption, \( R_h \) behaves around the centre as

\[
R_h = hr^m.
\]

(53)
where the index \( n_h \geq 1 \) depends on the value of \( m \) and \( h \) is a positive constant. After a careful analysis, we obtain \( n_h \) and \( h \) in terms of the value of \( m \),

\[
n_h = \frac{3(\alpha + 1)}{3\alpha + 1}, \quad h = (2\Psi_3)^{\frac{1}{3\alpha + 1}} \quad \text{for} \quad m > -6\alpha - 1, (54)
\]

\[
n_h = \frac{3(\alpha + 1)}{3\alpha + 1}, \quad h = \left( \Psi_3 + \sqrt{\Psi_3^2 + 2\Pi_0} \right)^{\frac{1}{3\alpha + 1}} \quad \text{for} \quad m = -6\alpha - 1, (55)
\]

\[
n_h = \frac{5 - m}{1 - m}, \quad h = (2\Pi_0)^{\frac{1}{3\alpha + 1}} \quad \text{for} \quad m < -6\alpha - 1. (56)
\]

It can easily be shown that

\[
\psi(r, R_h) = 0 < \frac{dR_h}{dr} = n_h h r^{n_h - 1}. (57)
\]

Therefore we can identify \( R_h \) with \( R_1 \). If there is a curve \( R = R_x(r) \) such that

\[
\text{(a) } R_x(r) > R_h(r) \quad \text{and} \quad \text{(b) } \frac{dR_x}{dr} \leq \psi(r, R_x(r)), (58)
\]

we can identify \( R_x \) as \( R_2 \) and show the existence of radial null geodesics emanating from the central singularity. Here we assume that the curve \( R = R_x(r) \) behaves as

\[
R_x = x r^{n_x} (59)
\]

with \( n_x > 1 \), which assures that \( R_x \) emerges out of the central singularity. Condition (a) in equation (58) can be rewritten as

\[
\text{(a1) } n_x < n_h \quad \text{or} \quad \text{(a2) } n_x = n_h \quad \text{and} \quad x > h. (60)
\]

In the rest of this section, we investigate the behaviour of \( R'(r, R_x(r)) \) along the line \( R_x \),

\[
R'(r, R_x(r)) = \frac{3\alpha(\alpha + 1)\Psi_3 + \alpha(5 - m)\Pi_0 x^{3\alpha + m, (3\alpha + m) n_x + 3\alpha - m r^2}}{3\alpha(\alpha + 1)\Psi_3 - (\alpha + 1)m\Pi_0 x^{3\alpha + m, (3\alpha + m) n_x + 3\alpha - m r^2}} x r^{n_x - 1} (61)
\]

and the limit,

\[
\lim_{r \to 0} \left( \frac{1 - u(r, R_x(r))}{Y(r, R_x(r))} \right) = C_x, (62)
\]

where \( C_x \) is a constant for each \( x \). Then, \( \psi(r, R_x(r)) \) can be represented as

\[
\psi(r, R_x(r)) \approx C_x R'(r, R_x(r)) (63)
\]

around \( r = 0 \).

When the inequality

\[
3\alpha + m - 2 < (3\alpha + m) n_x (64)
\]

holds, the first terms of the denominator and the numerator are leading terms and then the right-hand side of equation (61) is approximated as \( R'(r, R_x) \approx x r^{n_x - 1} \). Also, it can be shown that \( Y(r, R_x(r)) \approx 1 \) around the centre. Therefore, outside the apparent horizon, it can be easily shown that the limit \( C_x \) is less than or equals 1. Therefore the following relation holds:

\[
\psi(r, R_x(r)) \approx C_x x r^{n_x - 1} < n_x x r^{n_x - 1} = \frac{dR_x}{dr}. (65)
\]

As a result, condition (b) in equation (58) cannot be satisfied and the central singularity would not be naked in this case.
Next we consider the case
\[ 3\alpha + m - 2 = (3\alpha + m)n_z, \]  
where \( R'(r, R_z) \) behaves as
\[ R'(r, R_z) \approx \frac{3\alpha(\alpha + 1)\Psi_3 + \alpha(5 - m)\Pi_0 x^{3\alpha - m}}{3\alpha(\alpha + 1)\Psi_3 - (\alpha + 1)m\Pi_0 x^{3\alpha - m}} x^{n_z - 1}, \]  
around the centre. Also, the following relations hold around the centre:
\[ Y(r, R_z(r)) \approx 1 + \frac{3\alpha(\alpha + 1)/\Psi_3}{1 - \frac{\Pi_0 x^{3\alpha - m}}{3\alpha\Psi_3}} x^{3\alpha - m} \]  
\[ \frac{2\Psi(r, R_z(r))}{R_z(r)} \approx \left( \frac{2\Psi_3}{x^{3\alpha + 1} + 2\Pi_0 x^{m - 1}} \right) x^{3(\alpha + 1) - (3\alpha + 1)n_z}. \]  
When \( m > -6\alpha - 1 \), condition (a) means
\[ n_z \leq 1 + \frac{2}{3\alpha + 1}. \]  
Using this inequality and equation (66), it can be shown that
\[ m = -3\alpha - \frac{2}{n_z - 1} \leq -6\alpha - 1, \]  
which is inconsistent with \( m > -6\alpha - 1 \).

When \( m = -6\alpha - 1 \), we obtain from equation (66)
\[ n_z = \frac{3(\alpha + 1)}{3\alpha + 1} = n_h. \]  
To satisfy condition (a2) the following relation should hold:
\[ x > h = \left( \Psi_3 + \sqrt{\Psi_3^2 + 2\Pi_0} \right)^{\frac{1}{3\alpha + 1}}. \]  
The condition (b) can be rewritten in
\[ \frac{3(\alpha + 1)}{3\alpha + 1} \leq \left\{ 1 - \frac{1 - \left( 1 - 2\Psi_3 x^{-3\alpha - 1} - 2\Pi_0 x^{-3\alpha - 1} \right) \left( 1 + \frac{6\alpha + 1}{3\alpha\Psi_3} x^{-3\alpha - 1} \right)^{\frac{2\alpha + 1}{\alpha - 1}}}{1 - \frac{2\Pi_0 x^{-3\alpha - 1}}{3\alpha\Psi_3}} \right\} \times \frac{1 + \frac{2\Pi_0 x^{-3\alpha - 1}}{3\alpha\Psi_3}}{1 + \frac{(6\alpha + 1)\Pi_0}{3\alpha\Psi_3} x^{-3\alpha - 1}}. \]  
To satisfy this inequality, the right-hand side of it should be larger than unity. Therefore we need
\[ \frac{1 + \frac{2\Pi_0 x^{-3\alpha - 1}}{3\alpha\Psi_3}}{1 + \frac{(6\alpha + 1)\Pi_0}{3\alpha\Psi_3} x^{-3\alpha - 1}} > 1 \]  
at least. This inequality holds when \( \Pi_0 < 0 \) and
\[ x^{3\alpha + 1} > \frac{(6\alpha + 1)\Pi_0}{3\alpha\Psi_3}. \]
We note that the positive matter density condition $\Psi_{r,R} < 0$ and the non-shell crossing condition $\Psi_{r,r}(r, R) > 0$, which are derived from equations (13) and (12), are satisfied in these cases. Now we can rewrite equation (74) as

$$-(1 - 2\Psi_3 x^{-3a-1} - 2\Pi_0 x^{-6a-2}) \left( 1 + \frac{(6a + 1)\Pi_0}{3a\Psi_3} x^{-3a-1} \right)$$

This inequality can hold, for example, if $\Pi_0$ is given by (76). To be compatible with equation (73), an additional condition for $\Pi_0$ may be needed as

$$-\frac{6a(3a + 1)}{(6a + 1)^2} < \Pi_0 < 0.$$  \hspace{1cm} (78)

Similarly to the previous case, we can show the existence of the parameter set which satisfies (79) when $\alpha < 1$ and $\Pi_0 < 0$. In this case, $\Pi_0$ does not have a lower limit.

The last case to analyse is

$$3\alpha + m - 2 > (3\alpha + m)n_x,$$  \hspace{1cm} (80)

where $\frac{m(\alpha - 1)}{3\alpha + m}$ which satisfy (79) for $\alpha < 1$ and $\Pi_0 < 0$. In this case, $\Pi_0$ does not have a lower limit.

In summary, there are parameter sets of $m$, $\Psi_3 \Pi_0$, and $x$ which satisfy $m = -6\alpha - 1$ and equations (73) and (74) or $m = -6\alpha - 1$ and equation (79). In these cases, there are curves $R = R_h(r)$ and $R = R_s(r)$, where $R_h$ satisfies equation (57) and $R_s$ satisfies equation (58). Therefore, when the mass function with such parameters is a solution of the GGMP equation, for which there is a central singularity after a finite time, we can show the existence of radial null geodesics which emanate from the central singularity.
for each case, the consistency between the expanded form of mass function and the GGMP equation. After that we may use the above results to obtain a restriction on $\alpha$ which produces a naked singularity. Another way to do this would be to investigate whether the solution of equation (19) admits an expansion where the second-order term is given by equation (50) with $m \leq -6\alpha - 1$ and the existence $x$ which satisfies equations (73) and (74) or equation (79).

However, in general these tasks are much more difficult. As a final remark, we also note that while using the above procedure it should be checked, for each $\Psi$, whether the behaviour of $R_h$ and $\varphi(r, R(x(r)))$ is in fact determined by the second-order term of the mass function.

7. Concluding remarks

In this paper, we have investigated the collapse of spherically symmetric perfect fluids with the equation of state $p = \alpha \epsilon$ using area-radial coordinates. In this coordinate system, the physical variables are written in terms of the mass function which obeys a second-order PDE, which we call the GGMP equation. We have found that the assumption of analyticity of the mass function has the problem of implying staticity around the regular centre. In this case, the only way to form a central singularity is to permit an infinite jump of a physical variable. Furthermore, in order to have genuine dynamics, we argued that the expansion of the mass function around the centre should have negative powers of $R$.

We have then constructed other forms for the mass function, starting with the Friedmann solution as the simplest dynamical solution of the GGMP equation. We have then considered classes of mass functions which represent inhomogeneous dynamical fluids. Such mass functions approach the Friedmann solution at the lowest order around the symmetric centre. We have also considered a condition for which the solution has a meaningful dust limit. We note that the power indices and the coefficients of the mass function expansions were not given a priory and should be determined, in each case, by the GGMP equation.

Using generalized forms for the mass function, we have investigated the conditions for the appearance of naked singularities resulting from collapse. We have proved that if (i) $\Psi$ is a solution of the GGMP equation and is given by (45) with the second-order term proportional to $r^{5-m} R^m$ with $m \leq -6\alpha - 1$, for which there is a central singularity after a finite time and (ii) there are $x$ which satisfy equations (73) and (74) or equation (79), then there are naked singularities. One should be cautioned, however, that this result does not fully prove the existence of naked singularities. The work towards a complete proof of this is in progress.

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