Topological Recursion Relations on $\overline{M}_{3,2}$

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Abstract

In this paper, we give some new genus-3 universal equations for Gromov-Witten invariants of compact symplectic manifolds. These equations were obtained by studying new relations in the tautological ring of the moduli space of 2-pointed genus-3 stable curves. A byproduct of our search for genus-3 equations is a new genus-2 universal equation for Gromov-Witten invariants.

It is well known that relations in the tautological ring of moduli space of stable curves $\overline{M}_{g,n}$ produce universal equations for Gromov-Witten invariants of all compact symplectic manifolds. A typical genus-0 example is the associativity equation for the quantum cohomology, also known as the WDVV equation. Finding explicit higher genus universal equations is a very difficult problem. Genus-1 and genus-2 universal equations were discovered in [Ge1], [Ge2], and [BP]. Relations among these equations were studied in [L2]. For manifolds with semisimple quantum cohomology, these equations determine the genus-1 and genus-2 Gromov-Witten invariants in terms of genus-0 invariants (cf. [DZ] for the genus-1 case and [L1] for the genus-2 case). In [KL], the authors proved a genus-3 topological recursion relation by studying a tautological relation on $\overline{M}_{3,1}$. Certain topological recursion relations of all genera were proved in [LP]. Despite all these progresses, the understanding of universal equations is still very limited and unsatisfactory. For example, so far the known genus-3 equations still cannot determine the genus-3 generating function even for manifolds with semisimple quantum cohomology. Therefore it is very interesting to find more genus-3 universal equations. The main purpose of this paper is to obtain new genus-3 equations by studying tautological relations on $\overline{M}_{3,2}$.

To describe the universal equations, we need to use some operators introduced in [L1]. Let $M$ be a compact symplectic manifold. The big phase space for Gromov-Witten invariants of $M$ is a product of infinitely many copies of $H^*(M; \mathbb{C})$. We will choose a basis $\{\gamma_\alpha \mid \alpha = 1, \ldots, N\}$ of $H^*(M; \mathbb{C})$. The quantum product $\mathcal{W}_1 \circ \mathcal{W}_2$ of two vector fields $\mathcal{W}_1$ and $\mathcal{W}_2$ on the big phase space was introduced in [L1]. This is an associative product without an identity element. An operator $T$ on the space of vector fields on the big phase space was also introduced in [L1] to measure the failure of the string vector field to be an identity element with respect to this product. This operator turns out to be a very useful device to translate relations in the tautological rings of $\overline{M}_{g,n}$ into universal equations for Gromov-Witten invariants. We will write universal equations of Gromov-Witten invariants as equations among tensors $\langle \langle \mathcal{W}_1 \cdots \mathcal{W}_k \rangle \rangle_g$ which are defined to be the $k$-th covariant derivatives of the generating functions of genus-$g$ Gromov-Witten invariants.

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with respect to the trivial connection on the big phase space. We will briefly review these definitions in Section [1] for completeness.

Define

\[
Q(W_1, W_2) := \frac{2}{9} \langle\langle W_2 T(\gamma^a) \rangle\rangle_2 \langle\langle \gamma_\alpha W_1 \gamma^\beta \gamma_\beta \rangle\rangle_0 + \frac{5}{24} \langle\langle T(W_2) \gamma^a \rangle\rangle_2 \langle\langle \gamma_\alpha W_1 \gamma^\beta \gamma_\beta \rangle\rangle_0 \\
+ \frac{16}{3} \langle\langle \gamma_\alpha W_2 T(\gamma^a) \rangle\rangle_2 \langle\{ \gamma_\alpha \circ W_1 \} \rangle_1 + 5 \langle\langle T(W_2) \gamma^a \rangle\rangle_2 \langle\{ \gamma_\alpha \circ W_1 \} \rangle_1 \\
+ \frac{40}{3} \langle\langle T(\alpha) \rangle\rangle_2 \langle\{ \gamma_\alpha \circ W_1 \} \rangle_2 W_2 + \frac{1}{6} \langle\langle W_2 T(\gamma^a) \{ \gamma_\alpha \circ W_1 \} \rangle\rangle_2 \\
+ \frac{1}{2} \langle\langle T(W_2) \gamma^a \{ \gamma_\alpha \circ W_1 \} \rangle\rangle_2 + \frac{2}{9} \langle\{ W_2 \circ \gamma^a \circ \gamma_\alpha \} \rangle_2 \\
+ \frac{1}{18} \langle\langle \gamma_\alpha W_1 \gamma^a \rangle\rangle_1 \langle\langle \gamma_\beta W_2 \{ \gamma^\beta \circ \gamma_\beta \} \rangle\rangle_1 + \frac{1}{18} \langle\langle \gamma_\alpha W_1 \gamma^a \rangle\rangle_1 \langle\langle \gamma_\beta W_2 \gamma^\mu \gamma^\mu \rangle\rangle_0 \\
+ \frac{1}{30} \langle\langle \gamma^a \rangle\rangle_1 \langle\langle \gamma_\alpha W_2 \gamma^\beta \rangle\rangle_1 \langle\langle \gamma_\beta W_1 \gamma^\mu \gamma^\mu \rangle\rangle_0 + \frac{1}{30} \langle\langle \gamma^a \rangle\rangle_1 \langle\langle \gamma_\alpha W_2 \gamma^\beta \{ \gamma_\beta \circ W_1 \} \rangle\rangle_1 \\
+ \frac{9}{10} \langle\langle \gamma^a \gamma_\alpha W_2 \gamma^\beta \rangle\rangle_1 \langle\langle \gamma_\alpha \{ \gamma_\beta \circ W_1 \} \rangle\rangle_1 + \frac{1}{30} \langle\langle \gamma^a \gamma_\alpha W_2 \gamma^\beta \rangle\rangle_1 \langle\langle \{ \gamma_\beta \circ W_1 \} \rangle\rangle_1 \\
+ \frac{2}{15} \langle\langle \gamma^a \gamma_\alpha \gamma^\beta \rangle\rangle_1 \langle\{ \gamma_\beta \circ W_1 \} \rangle_2 W_2 + \frac{3}{5} \langle\langle \gamma^a \gamma_\alpha W_2 \gamma^\beta \rangle\rangle_1 \langle\langle \gamma_\alpha \gamma_\beta W_1 \gamma^\mu \rangle\rangle_0 \langle\{ \gamma^\mu \} \rangle_1 \\
+ \frac{1}{6} \langle\langle \gamma^a \gamma_\alpha \gamma^\beta \rangle\rangle_1 \langle\{ \gamma_\alpha \circ \gamma_\beta \} \rangle_2 W_2 + \frac{16}{15} \langle\langle \gamma^a \gamma^\beta \rangle\rangle_1 \langle\langle \gamma_\alpha \gamma_\beta W_1 \gamma^\mu \rangle\rangle_0 \langle\{ \gamma^\mu \} \rangle_1 \\
+ \frac{41}{90} \langle\langle \gamma^a \rangle\rangle_1 \langle\langle \gamma_\alpha W_1 \gamma^a \gamma^\beta \gamma^\mu \rangle\rangle_0 \langle\{ \gamma^\mu \} \rangle_1 + \frac{4}{5} \langle\langle \gamma^a \rangle\rangle_1 \langle\langle \gamma_\alpha W_2 \gamma^a \gamma^\beta \rangle\rangle_1 \langle\{ \gamma^\mu \} \rangle_1 \langle\{ \gamma_\alpha \gamma_\beta \gamma^\mu \gamma_1 \} \rangle_0 \\
+ \frac{1}{16} \langle\langle \gamma^a \gamma_\alpha \gamma^\beta \rangle\rangle_1 \langle\{ \gamma_\beta \circ W_2 \} \rangle_1 W_1 + \frac{92}{15} \langle\langle \gamma^a \gamma_\alpha W_2 \gamma^a \gamma^\beta \rangle\rangle_1 \langle\{ \gamma^\mu \} \rangle_1 \langle\{ \gamma_\alpha \gamma_\beta \gamma^\mu \gamma_1 \} \rangle_0 \\
+ \frac{1}{120} \langle\langle \gamma^a \gamma_\alpha \gamma^\beta \rangle\rangle_1 \langle\{ \gamma_\alpha W_1 \gamma^a \gamma^\beta \gamma^\mu \gamma^\mu \rangle\rangle_0 + \frac{1}{720} \langle\langle \gamma^a \gamma_\alpha W_2 \gamma^a \gamma^\beta \{ \gamma_\beta \circ W_1 \} \rangle\rangle_1 \\
+ \frac{1}{135} \langle\langle \gamma^a \gamma_\alpha \gamma^\beta \gamma^\mu \rangle\rangle_1 \langle\{ \gamma_\alpha \gamma_\beta \gamma^\mu \gamma_1 \} \rangle_0 + \frac{1}{40} \langle\langle \gamma^a \gamma_\alpha W_2 \gamma^a \gamma^\beta \rangle\rangle_1 \langle\{ \gamma_\alpha \gamma_\beta W_1 \gamma^a \gamma^\mu \gamma^\mu \rangle\rangle_0 \\
+ \frac{1}{120} \langle\langle \gamma^a \gamma_\alpha \gamma^\beta \gamma^\mu \gamma^\mu \rangle\rangle_0 \rangle\rangle_1.
\]

Note that \(Q(W_1, W_2)\) only involves data with genus \(\leq 2\). In this paper we will prove the following genus-3 universal equation

**Theorem 0.1** For all compact symplectic manifolds,

\[
\langle\langle T^2(W_1)T(W_2) \rangle\rangle_3 - \langle\langle T^2(W_2)T(W_1) \rangle\rangle_3 = \frac{1}{7} \{Q(W_1, W_2) - Q(W_2, W_1)\}
\]

for all vector fields \(W_1\) and \(W_2\) on the big phase space.

Note that in general

\[
\langle\langle T^2(W_1)T(W_2) \rangle\rangle_3 \neq \frac{1}{7} Q(W_1, W_2).
\]

Instead \(\langle\langle T^2(W_1)T(W_2) \rangle\rangle_3\) is given by a much more complicated genus-3 universal equation in Theorem [2.2] which is the main result of this paper. Theorem [0.1] is actually a corollary of
Theorem 2.2. We would like to point out that the genus-3 topological recursion relation in \cite{KL} also follows from the formula in Theorem 2.2 (see the remark at the end of Section 2).

During the proof of Theorem 2.2, we also discovered a new genus-2 universal equation under the assumption that the tautological ring of $\overline{M}_{3,2}$ is Gorenstein. This genus-2 equation is given in Proposition 2.3. This equation does not seem to follow from known genus-2 equations. By a result of \cite{L1}, known genus-2 equations already completely determine the genus-2 generating function for manifolds with semisimple quantum cohomology. Because of this fact, the new genus-2 equation in Proposition 2.3 is quite a surprise.

We also notice that universal equations in this paper correspond to relations in the tautological ring of $\overline{M}_{3,2}$. We will give the relation corresponding to Theorem 0.1 in Section 3.

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1 Preliminaries

Let $M$ be a compact symplectic manifold. The big phase space is by definition the infinite product

$$P := \prod_{n=0}^{\infty} H^*(M; \mathbb{C}).$$

Fix a basis $\{\gamma_0, \ldots, \gamma_N\}$ of $H^*(M; \mathbb{C})$, where $\gamma_0$ is the identity element, of the ordinary cohomology ring of $M$. Then we denote the corresponding basis for the $n$-th copy of $H^*(M; \mathbb{C})$ in $P$ by $\{\tau_n(\gamma_0), \ldots, \tau_n(\gamma_N)\}$. We call $\tau_n(\gamma_\alpha)$ a descendant of $\gamma_\alpha$ with descendant level $n$. We can think of $P$ as an infinite dimensional vector space with a basis $\{\tau_n(\gamma_\alpha) | 0 \leq \alpha \leq N, n \in \mathbb{Z}_{\geq 0}\}$.

Let $\{t_\alpha^n | 0 \leq \alpha \leq N, n \in \mathbb{Z}_{\geq 0}\}$ be the corresponding coordinate system on $P$. For convenience, we identify $\tau_n(\gamma_\alpha)$ with the coordinate vector field $\frac{\partial}{\partial t_\alpha^n}$ on $P$ for $n \geq 0$. If $n < 0$, $\tau_n(\gamma_\alpha)$ is understood to be the 0 vector field. We also abbreviate $\tau_0(\gamma_\alpha)$ by $\gamma_\alpha$.

We use $\tau_+$ and $\tau_-$ to denote the operators which shift the level of descendants by 1, i.e.

$$\tau_\pm \left( \sum_{n, \alpha} f_{n, \alpha} \tau_n(\gamma_\alpha) \right) = \sum_{n, \alpha} f_{n, \alpha} \tau_{n \pm 1}(\gamma_\alpha)$$

where $f_{n, \alpha}$ are functions on the big phase space.

We will adopt the following notational conventions: Lower case Greek letters, e.g. $\alpha$, $\beta$, $\mu$, $\nu$, $\sigma$, ..., etc., will be used to index the cohomology classes on $M$. These indices run from 0 to $N$. Lower case English letters, e.g. $i$, $j$, $k$, $m$, $n$, ..., etc., will be used to index the level of descendants. These indices run over the set of all non-negative integers, i.e. $\mathbb{Z}_{\geq 0}$. All summations are over the entire ranges of the corresponding indices unless otherwise indicated. Let

$$\eta_{\alpha\beta} = \int_M \gamma_\alpha \cup \gamma_\beta$$

be the intersection form on $H^*(M, \mathbb{C})$. We will use $\eta = (\eta_{\alpha\beta})$ and $\eta^{-1} = (\eta^{\alpha\beta})$ to lower and raise indices. For example,

$$\gamma^\alpha := \eta^{\alpha\beta} \gamma_\beta.$$

Here we are using the summation convention that repeated indices (in this formula, $\beta$) should be summed over their entire ranges.
Let

$$\langle \tau_{n_1}(\gamma_{\alpha_1}) \tau_{n_2}(\gamma_{\alpha_2}) \cdots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_{d,g} := \int_{\overline{\mathcal{M}}_{g,n}(M;d)} \sum_{d \in H_2(M;\mathbb{Z})} q^d \langle \tau_{n_1}(\gamma_{\alpha_1}) \tau_{n_2}(\gamma_{\alpha_2}) \cdots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_{d,g}$$

be the genus-\(g\), degree \(d\), descendant Gromov-Witten invariant associated to \(\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_k}\) and nonnegative integers \(n_1, \ldots, n_k\) (cf. [W], [RT], [LiT]). Here, \(\overline{\mathcal{M}}_{g,k}(M;d)\) is the moduli space of stable maps from genus-\(g\), \(k\)-pointed curves to \(M\) of degree \(d \in H_2(M;\mathbb{Z})\). \(\Psi_i\) is the first Chern class of the tautological line bundle over \(\overline{\mathcal{M}}_{g,k}(M;d)\) whose geometric fiber over a stable map is the cotangent space of the domain curve at the \(i\)-th marked point while \(\text{ev}_i : \overline{\mathcal{M}}_{g,n}(M;d) \to M\) is the \(i\)-th evaluation map for all \(i = 1, \ldots, k\). Finally, \(\overline{\mathcal{M}}_{g,n}(M;d)\) is the virtual fundamental class. The genus-\(g\) generating function is defined to be

$$F_g = \sum_{k \geq 0} \frac{1}{k!} \sum_{p_1 \ldots p_k} t_{p_1} \cdots t_{p_k} \sum_{d \in H_2(V;\mathbb{Z})} q^d \langle \tau_{n_1}(\gamma_{\alpha_1}) \tau_{n_2}(\gamma_{\alpha_2}) \cdots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_{d,g}$$

where \(q^d\) belongs to the Novikov ring. This function is understood as a formal power series in the variables \(\{t_{p_i}\}\) with coefficients in the Novikov ring.

Introduce a \(k\)-tensor \(\langle \langle \cdots \cdots \rangle \rangle_k\) defined by

$$\langle \langle \mathcal{W}_1 \mathcal{W}_2 \cdots \mathcal{W}_k \rangle \rangle_k := \sum_{m_1, \alpha_1, \ldots, m_k, \alpha_k} f^1_{m_1, \alpha_1} \cdots f^k_{m_k, \alpha_k} \frac{\partial^k \langle \langle \mathcal{W}_1 \mathcal{W}_2 \cdots \mathcal{W}_k \rangle \rangle_k}{\partial t_{m_1, \alpha_1} \cdots \partial t_{m_k, \alpha_k}} F_g,$$

for vector fields \(\mathcal{W}_i = \sum_{m, \alpha} f_{m, \alpha}^i \frac{\partial}{\partial t_m} \) where \(f_{m, \alpha}^i\) are functions on the big phase space. This tensor is called the \(k\)-point (correlation) function.

For any vector fields \(\mathcal{W}_1\) and \(\mathcal{W}_2\) on the big phase space, the quantum product of \(\mathcal{W}_1\) and \(\mathcal{W}_2\) is defined by

$$\mathcal{W}_1 \circ \mathcal{W}_2 := \langle \langle \mathcal{W}_1 \mathcal{W}_2 \gamma^\alpha \rangle \rangle_0 \gamma_\alpha.$$

Define the vector field

$$T(\mathcal{W}) := \tau_+ (\mathcal{W}) - \langle \langle \mathcal{W} \gamma^\alpha \rangle \rangle_0 \gamma_\alpha$$

for any vector field \(\mathcal{W}\). The operator \(T\) was introduced in [LI] as a convenient tool in the study of universal equations for Gromov-Witten invariants. Let \(\psi_i\) be the first Chern class of the tautological line bundle over \(\overline{\mathcal{M}}_{g,k}\) whose geometric fiber over a stable curve is the cotangent space of the curve at the \(i\)-th marked point. When we translate a relation in the tautological ring of \(\overline{\mathcal{M}}_{g,k}\) into differential equations for generating functions of Gromov-Witten invariants, each \(\psi\) class corresponds to the insertion of the operator \(T\). Let \(\nabla\) be the trivial flat connection on the big phase space with respect to which \(\tau_n(\gamma_\alpha)\) are parallel vector fields for all \(\alpha\) and \(n\). Then the covariant derivative of the quantum product satisfies

$$\nabla_{\mathcal{W}_3}(\mathcal{W}_1 \circ \mathcal{W}_2) = (\nabla_{\mathcal{W}_3} \mathcal{W}_1) \circ \mathcal{W}_2 + \mathcal{W}_1 \circ (\nabla_{\mathcal{W}_3} \mathcal{W}_2) + \langle \langle \mathcal{W}_1 \mathcal{W}_2 \mathcal{W}_3 \gamma^\alpha \rangle \rangle_0 \gamma_\alpha$$

and the covariant derivative of the operator \(T\) is given by

$$\nabla_{\mathcal{W}_2} T(\mathcal{W}_1) = T(\nabla_{\mathcal{W}_2} \mathcal{W}_1) - \mathcal{W}_2 \circ \mathcal{W}_1$$

for any vector fields \(\mathcal{W}_1, \mathcal{W}_2\) and \(\mathcal{W}_3\) (cf. [LI] Equation (8) and Lemma 1.5)). We need to use these formulas in order to compute derivatives of universal equations.
2 Topological Recursion Relations on $\overline{M}_{3,2}$

The cohomology class $\psi^2_1\psi_2$ vanishes on $M_{3,2}$ due to a result of Ionel (cf. [Io]). Furthermore, by a result of Faber and Pandharipande [FP2], $\psi^2_1\psi_2$ is equal to a class from the boundary strata which is tautological, and therefore is a linear combination of products of $\psi$ and $\kappa$ classes and fundamental classes of some boundary strata. For any curve in $\partial M_{3,2} := \overline{M}_{3,2} - M_{3,2}$ with a genus-3 component, it has a dual graph

(see Section 3 for the conventions of dual graphs). Since this graph is a tree, all such curves are of compact type. For this stratum to occur in the expression of $\psi^2_1\psi_2$, it must be multiplied by combinations of $\psi$-classes and $\kappa$-classes of degree 2 on the genus-3 component. By a result of Yang [Y], $\kappa$-classes in this expression can be replaced with combinations of $\psi$-classes and fundamental classes of boundary strata. On all other boundary strata, all components of curves must have genus at most 2, therefore $\kappa$ classes again can be represented as linear combinations of $\psi$ classes and fundamental classes of boundary strata (cf. [AC]). Therefore, it follows that $\psi^2_1\psi_2$ on $\overline{M}_{3,2}$ can be written as a linear combination of products of the $\psi$ classes and the fundamental classes of some boundary strata. By taking into consideration the genus-0 and genus-1 topological recursion relations as well as Mumford’s genus-2 relation, we can translate these results into the following universal equations for Gromov-Witten invariants with unknown constants $a_1, \ldots, a_{105}$:

$$0 = \Phi(W_1, W_2) = - \langle \langle T^2(W_1)T(W_2) \rangle \rangle_3 + a_1 \langle \langle T^2(W_1 \circ W_2) \rangle \rangle_3 + a_2 \langle \langle W_1 W_2 T(\gamma_1 \circ \gamma_1) \rangle \rangle_2$$

$$+ a_3 \langle \langle T(W_1) \gamma_1 \rangle \rangle_2 \langle \langle \gamma_1 W_2 \gamma_2 \gamma_3 \rangle \rangle_0 + a_4 \langle \langle W_1 T(\gamma_1) \rangle \rangle_2 \langle \langle \gamma_1 W_2 \gamma_2 \gamma_3 \rangle \rangle_0$$

$$+ a_5 \langle \langle T(W_1) \gamma_1 \rangle \rangle_2 \langle \langle \gamma_1 W_1 \gamma_2 \gamma_3 \rangle \rangle_0 + a_6 \langle \langle W_2 T(\gamma_1) \rangle \rangle_2 \langle \langle \gamma_1 W_2 \gamma_2 \gamma_3 \rangle \rangle_0$$

$$+ a_7 \langle \langle T(\gamma_1) \rangle \rangle_2 \langle \langle \gamma_1 W_1 W_2 \gamma_2 \gamma_3 \rangle \rangle_0 + a_8 \langle \langle W_1 T(\gamma_1) \rangle \rangle_2 \langle \langle \gamma_1 \circ W_2 \rangle \rangle_1$$

$$+ a_9 \langle \langle T(W_2) \gamma_1 \rangle \rangle_2 \langle \langle \gamma_1 \circ W_1 \rangle \rangle_1 + a_{10} \langle \langle W_2 T(\gamma_1) \rangle \rangle_2 \langle \langle \gamma_1 \circ W_1 \rangle \rangle_1$$

$$+ a_{11} \langle \langle T(\gamma_1) \rangle \rangle_2 \langle \langle \gamma_1 \circ W_1 \rangle \rangle_2 + a_{12} \langle \langle T(\gamma_1) \rangle \rangle_2 \langle \langle \gamma_1 \circ W_2 \rangle \rangle_1$$

$$+ a_{13} \langle \langle T(\gamma_1) \rangle \rangle_2 \langle \langle \gamma_1 \circ W_1 W_2 \gamma_2 \rangle \rangle_0 \langle \langle \gamma_3 \rangle \rangle_1 + a_{14} \langle \langle W_1 T(\gamma_1) \rangle \rangle_2 \langle \langle \gamma_1 \circ W_2 \rangle \rangle_2$$

$$+ a_{15} \langle \langle T(W_2) \gamma_1 \rangle \rangle_2 \langle \langle \gamma_1 \circ W_1 \rangle \rangle_2 + a_{16} \langle \langle W_2 T(\gamma_1) \rangle \rangle_2 \langle \langle \gamma_1 \circ W_1 \rangle \rangle_2$$

$$+ a_{17} \langle \langle T(\gamma_1) \gamma_3 \rangle \rangle_2 \langle \langle \gamma_1 \circ W_1 W_2 \gamma_3 \rangle \rangle_0 + a_{18} \langle \langle T(\gamma_1) \gamma_2 \rangle \rangle_0 \langle \langle \gamma_1 \circ W_1 \rangle \rangle_2$$

$$+ a_{19} \langle \langle \gamma_2 \gamma_3 \gamma_1 \circ W_1 W_2 \rangle \rangle_2 + a_{20} \langle \langle T(\gamma_1) \rangle \rangle_2 \langle \langle \gamma_1 \circ W_1 \rangle \rangle_2$$

$$+ a_{21} \langle \langle T(W_1 \circ W_2) \gamma_1 \rangle \rangle_2 \langle \langle \gamma_1 \rangle \rangle_1 + a_{22} \langle \langle W_1 \circ W_2 \rangle \rangle_1 \langle \langle W_1 \circ W_2 \rangle \rangle_1$$

$$+ a_{23} \langle \langle W_1 \circ W_2 \circ \gamma_1 \rangle \rangle_2 + a_{24} \langle \langle W_2 \circ W_1 \circ \gamma_1 \rangle \rangle_2$$

$$+ a_{25} \langle \langle W_1 \circ W_2 \rangle \rangle_2 \langle \langle \gamma_2 \circ \gamma_1 \rangle \rangle_2 + a_{26} \langle \langle \gamma_2 \gamma_3 \rangle \rangle_2 \langle \langle \gamma_1 \circ W_1 \circ W_2 \rangle \rangle_2 \gamma_2 \gamma_3 \rangle \rangle_0$$
\[+a_{27} \langle \gamma^\alpha \rangle_2 \langle \gamma_\alpha W_1 \gamma^\beta \{ \gamma_\beta \circ W_2 \} \rangle + a_{28} \langle \gamma^\alpha \rangle_2 \langle \{ \gamma_\alpha \circ \gamma^\beta \} \gamma_\beta W_1 W_2 \rangle \]
\[+a_{29} \langle \gamma^\alpha \rangle \{ \gamma_\alpha \circ W_1 \circ W_2 \} + a_{30} \langle \gamma^\alpha \rangle_2 \langle \{ \gamma_\alpha \circ \gamma^\beta \} \rangle_3 \]
\[+a_{31} \langle \gamma^\alpha \rangle_2 \{ \{ \gamma_\alpha \circ W_1 \circ W_2 \} \} + a_{32} \langle W_1 \circ W_2 \gamma^\alpha \rangle \langle \gamma_\alpha \{ \gamma_\beta \circ \gamma^\beta \} \rangle_1 \]
\[+a_{33} \langle W_1 \circ \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{34} \langle W_2 \gamma^\alpha \rangle \langle \gamma_\alpha \{ \gamma_\beta \circ \gamma^\beta \} \rangle_1 \]
\[+a_{35} \langle W_1 \circ \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{36} \langle W_2 \gamma^\alpha \rangle \langle \gamma_\alpha \{ \gamma_\beta \circ \gamma^\beta \} \rangle_1 \]
\[+a_{37} \langle \gamma^\alpha \rangle_2 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{38} \langle \gamma^\alpha \rangle_2 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{39} \langle \gamma^\alpha \rangle_2 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{40} \langle \gamma^\alpha \rangle_2 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{41} \langle W_1 \circ \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{42} \langle W_2 \gamma^\alpha \rangle \langle \gamma_\alpha \{ \gamma_\beta \circ \gamma^\beta \} \rangle_1 \]
\[+a_{43} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{44} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{45} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{46} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{47} \langle W_2 \gamma^\alpha \rangle \langle \gamma_\alpha \{ \gamma_\beta \circ \gamma^\beta \} \rangle_1 \]
\[+a_{48} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{49} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{50} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{51} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{52} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{53} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{54} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{55} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{56} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{57} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{58} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{59} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{60} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{61} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{62} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{63} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{64} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{65} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{66} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{67} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{68} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{69} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{70} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{71} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{72} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{73} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{74} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{75} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{76} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[+a_{77} \langle \gamma^\alpha \rangle_3 \{ \{ \gamma_\alpha \circ \gamma^\beta \} \} + a_{78} \langle \gamma^\alpha \rangle_3 \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle_0 \]
\[ +a_{79} \left\langle W_1 W_2 \gamma^\alpha \right\rangle_1 \left\langle \gamma^\beta \right\rangle_1 \left\langle \{ \gamma_\alpha \circ \gamma_\beta \} \right\rangle_1 + a_{80} \left\langle W_1 \gamma^\alpha \right\rangle_1 \left\langle \gamma^\beta \right\rangle_1 \left\langle \gamma^\mu \right\rangle_1 \left\langle \{ \gamma_\alpha \gamma_\beta \gamma_\mu W_2 \} \right\rangle_0 \\
+a_{81} \left\langle W_2 \gamma^\alpha \right\rangle_1 \left\langle \gamma^\beta \right\rangle_1 \left\langle \gamma^\mu \right\rangle_1 \left\langle \{ \gamma_\alpha \gamma_\beta \gamma_\mu W_1 \} \right\rangle_0 \\
+a_{82} \left\langle \gamma^\alpha \right\rangle_1 \left\langle \gamma^\beta \right\rangle_1 \left\langle \gamma^\mu \right\rangle_1 \left\langle \{ \gamma_\alpha \gamma_\beta \gamma_\mu W_1 W_2 \} \right\rangle_0 \\
+a_{83} \left\langle W_1 \gamma^\alpha \right\rangle_1 \left\langle \{ W_2 \gamma^\beta \} \right\rangle_1 \left\langle \{ \gamma_\alpha \circ \gamma_\beta \} \right\rangle_1 + a_{84} \left\langle W_1 W_2 \gamma^\alpha \gamma_\alpha \{ \gamma^\beta \circ \gamma_\beta \} \right\rangle_1 \\
+a_{85} \left\langle W_1 \gamma^\alpha \gamma_\alpha \gamma_\beta \right\rangle_1 \left\langle \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 + a_{86} \left\langle W_2 \gamma^\alpha \gamma_\alpha \gamma_\beta \right\rangle_1 \left\langle \gamma_\beta \gamma_\beta \gamma_\mu W_1 \right\rangle_0 \\
+a_{87} \left\langle \gamma^\alpha \gamma_\alpha \gamma_\beta \right\rangle_1 \left\langle \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 + a_{88} \left\langle \gamma^\alpha \gamma_\alpha \gamma_\beta \gamma_\beta \{ W_1 \circ W_2 \} \right\rangle_1 \\
+a_{89} \left\langle \gamma^\alpha \gamma_\alpha \gamma_\beta \{ \gamma_\beta \circ W_1 \} \right\rangle_1 + a_{90} \left\langle \gamma^\alpha \gamma_\alpha \gamma_\beta \{ \gamma_\beta \circ W_1 \} \right\rangle_1 \\
+a_{91} \left\langle \gamma^\alpha \gamma_\alpha \gamma_\beta \gamma_\beta \right\rangle_1 \left\langle \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 + a_{92} \left\langle \gamma_\beta \right\rangle_1 \left\langle \gamma_\beta \gamma_\beta \gamma_\mu W_1 \right\rangle_0 \\
+a_{93} \left\langle \gamma^\alpha \gamma_\alpha \gamma_\beta \gamma_\beta \gamma_\mu \right\rangle_1 \left\langle \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 + a_{94} \left\langle \gamma^\alpha \gamma_\alpha \gamma_\beta \gamma_\beta \gamma_\mu \right\rangle_1 \left\langle \gamma_\beta \gamma_\beta \gamma_\mu W_1 \right\rangle_0 \\
+a_{95} \left\langle \gamma^\alpha \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 + a_{96} \left\langle \gamma^\alpha \gamma_\beta \gamma_\beta \gamma_\mu W_1 \right\rangle_0 \\
+a_{97} \left\langle \gamma^\alpha \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 + a_{98} \left\langle \gamma^\alpha \gamma_\beta \gamma_\beta \gamma_\mu W_1 \right\rangle_0 \\
+a_{99} \left\langle \gamma^\alpha \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 + a_{100} \left\langle \gamma_\beta \right\rangle_1 \left\langle \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 \\
+a_{101} \left\langle \gamma_\beta \right\rangle_1 \left\langle \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 + a_{102} \left\langle \gamma^\alpha \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 \\
+a_{103} \left\langle \gamma^\alpha \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 + a_{104} \left\langle \gamma_\beta \right\rangle_1 \left\langle \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 \\
+a_{105} \left\langle \gamma^\alpha \gamma_\beta \gamma_\beta \gamma_\mu W_2 \right\rangle_0 \]  \\
(2)

where \( W_i \) are arbitrary vector fields on the big phase space. The following terms are omitted from this formula due to the lower genus equations:

\[
\begin{align*}
&b_1 \left\langle T(W_1)W_2 \{ \gamma_\alpha \circ \gamma_\alpha \} \right\rangle_2 + b_2 \left\langle W_1T(W_2) \{ \gamma_\alpha \circ \gamma_\alpha \} \right\rangle_2 \\
&+b_3 \left\langle T(W_1) \gamma^\alpha \{ \gamma_\alpha \circ W_2 \} \right\rangle_2 + b_4 \left\langle T(W_1) \{ W_2 \circ \gamma^\alpha \} \right\rangle_2 \left\langle \{ \gamma_\alpha \} \right\rangle_1 \\
&+b_5 \left\langle \gamma^\alpha \gamma_2 \left\langle \{ \gamma_\alpha \circ W_1 \} W_2 \gamma^\beta \gamma_\beta \right\rangle_0 + b_6 \left\langle \gamma^\alpha \gamma_2 \left\langle \{ \gamma_\alpha \circ W_2 \} W_1 \gamma^\beta \gamma_\beta \right\rangle_0 \\
&+b_7 \left\langle \gamma^\alpha \gamma_2 \left\langle \{ \gamma_\alpha \circ W_1 \} \gamma^\beta \{ \gamma_\beta \circ W_1 \} \right\rangle_0.
\end{align*}
\]

(3)

The \( b_5 \), \( b_6 \), \( b_7 \) terms in this equation are eliminated using the first derivatives of the WDVV equation. To eliminate other terms, we need the Belorousski-Pandharipande equation (abbreviated as BP equation) whose top order terms have the form

\[
\sum_{\sigma \in S_3} \left( \left\langle W_\sigma(1) T(W_\sigma(2) \circ W_\sigma(3)) \right\rangle_2 - \left\langle T(W_\sigma(1)) \{ W_\sigma(2) \circ W_\sigma(3) \} \right\rangle_2 \right) = \text{L.O.T.}
\]

for any vector fields \( W_1, W_2, W_3 \). Throughout this paper, "top order terms" refer to highest order derivatives of the highest genus generating function involved in each equation. "L.O.T." is an abbreviation for "Lower Order Terms". The \( b_4 \)-term in equation (3) is eliminated by using the BP equation applied to \( W_1, W_2, \gamma^\alpha \).
Taking covariant derivative of the BP equation with respect to $\gamma^\alpha$ and applying it to $\mathcal{W}_1, \mathcal{W}_2, \gamma_\alpha$, we obtain

$$
\langle \langle \gamma^\alpha \gamma_\alpha T(\mathcal{W}_1 \circ \mathcal{W}_2) \rangle \rangle_2 - \langle \langle \gamma^\alpha T(\gamma_\alpha) \{\mathcal{W}_1 \circ \mathcal{W}_2\} \rangle \rangle_2 \\
+ \langle \langle \gamma^\alpha \mathcal{W}_1 T(\gamma_\alpha \circ \mathcal{W}_2) \rangle \rangle_2 - \langle \langle \gamma^\alpha T(\mathcal{W}_1) \{\gamma_\alpha \circ \mathcal{W}_2\} \rangle \rangle_2 \\
+ \langle \langle \gamma^\alpha \mathcal{W}_2 T(\gamma_\alpha \circ \mathcal{W}_1) \rangle \rangle_2 - \langle \langle \gamma^\alpha T(\mathcal{W}_2) \{\gamma_\alpha \circ \mathcal{W}_1\} \rangle \rangle_2 = \text{L.O.T.}
$$

This equation can be used to eliminate the $b_3$-term in equation (3).

Taking covariant derivative of the BP equation with respect to $\mathcal{W}_2$ and applying it to $\gamma^\alpha, \gamma_\alpha, \mathcal{W}_1$, we have

$$
\langle \langle \mathcal{W}_1 \mathcal{W}_2 T(\gamma^\alpha \circ \gamma_\alpha) \rangle \rangle_2 - \langle \langle \mathcal{W}_2 T(\gamma^\alpha) \{\gamma_\alpha \circ \mathcal{W}_1\} \rangle \rangle_2 = \text{L.O.T.}
$$

We can use this equation to eliminate the $b_1$-term in equation (8). Similarly the $b_2$-term can be eliminated by using the covariant derivative of the BP equation with respect to $\mathcal{W}_1$.

Since equation (2) holds for Gromov-Witten invariants of all compact symplectic manifolds, it must be satisfied for Gromov-Witten invariants of a point and $\mathbb{P}^1$. It is well known that the Gromov-Witten invariants of these two manifolds satisfy the Virasoro constraints (see \textcite{W} for the point case, and \textcite{EHX, Gi} for the $\mathbb{P}^1$ case). A computer program for calculating such invariants based on the Virasoro constraints was written by Gathmann \textcite{Ga}. We use Gathmann’s program to compute such invariants and plug in derivatives of equation (2) to find relations among constants $a_1, \ldots, a_{105}$.

When the target manifold is a point, the degree of all stable maps must be 0. Hence, we will omit any reference to the degrees of Gromov-Witten invariants of a point. In fact, such Gromov-Witten invariants are just intersection numbers

$$
\langle \tau_{n_1} \cdots \tau_{n_k} \rangle_g = \int_{\overline{M}_{g,k}} \psi_1^{n_1} \cdots \psi_k^{n_k}
$$

over the moduli spaces $\overline{M}_{g,k}$. Since the cohomology space of a point is one dimensional, coordinates on the big phase space are simply denoted by $t_0, t_1, t_2, \ldots$. We also identify vector fields $\frac{\partial}{\partial t_m}$ with $\tau_m$ on the big phase space. Using Gromov-Witten invariants of a point, we obtained 43 linearly independent relations among constants $a_1, \ldots, a_{105}$. These relations are listed in section A.1 of the appendix as equations (7) to (19).

When the target manifold is $\mathbb{C}P^1$, the degrees of the stable maps are indexed by $H_2(\mathbb{C}P^1; \mathbb{Z}) \cong \mathbb{Z}$. The degree $d$ part of any equation for generating functions of Gromov-Witten invariants is the coefficient of $q^d$ in the Novikov ring. We choose the basis $\{\gamma_0, \gamma_1\}$ for $H^*(\mathbb{C}P^1; \mathbb{C})$ with $\gamma_0 \in H^0(\mathbb{C}P^1; \mathbb{C})$ being the identity of the ordinary cohomology ring and $\gamma_1 \in H^2(\mathbb{C}P^1; \mathbb{C})$ the Poincare dual to a point. Coordinates on the big phase space are denoted by $\{t_0^{\alpha}, t_1^{\alpha} \mid n \in \mathbb{Z}_+\}$. We identify vector fields $\frac{\partial}{\partial t_0^\alpha}$ and $\frac{\partial}{\partial t_1^\alpha}$ with $\tau_{n,0}$ and $\tau_{n,1}$ respectively. Using Gromov-Witten invariants of $\mathbb{P}^1$, we obtained 61 linearly independent relations among constants $a_1, \ldots, a_{105}$. These relations are listed in section A.2 of the appendix as equations (21) to (110).

Combining results from Gromov-Witten invariants of a point and $\mathbb{P}^1$, we obtained 104 linearly independent relations (21) – (110) among $a_1, \ldots, a_{105}$. Using these relations we can solve all $a_i$ with $i \neq 2$ in terms of $a_2$, and obtain the following formulas:
Lemma 2.1

\[ a_1 = 5, \quad a_2 \text{ is free}, \quad a_3 = 0, \quad a_4 = -\frac{1}{36} - 4a_2, \]
\[ a_5 = \frac{19}{7}, \quad a_6 = \frac{21}{2} - 4a_2, \quad a_7 = \frac{7}{4} + 6a_2, \quad a_8 = -\frac{7}{4} - 120a_2, \]
\[ a_9 = \frac{7}{4}, \quad a_{10} = \frac{21}{2} - 120a_2, \quad a_{11} = \frac{49}{10} + 240a_2, \]
\[ a_{13} = 7 + 120a_2, \quad a_{14} = -\frac{1}{10} - 6a_2, \quad a_{15} = \frac{11}{4}, \]
\[ a_{17} = \frac{11}{4} + 24a_2, \quad a_{18} = \frac{7}{4} + 6a_2, \quad a_{19} = 0, \]
\[ a_{21} = 0, \quad a_{22} = \frac{2}{5} + 120a_2, \quad a_{23} = \frac{1}{36} + 4a_2, \]
\[ a_{25} = -\frac{1}{36} - 5a_2, \quad a_{26} = \frac{47}{504} + 4a_2, \quad a_{27} = 0, \]
\[ a_{29} = -\frac{1}{36} + 6a_2, \quad a_{30} = 0, \quad a_{31} = -\frac{47}{504} + 120a_2, \]
\[ a_{33} = -\frac{1}{36} - \frac{5}{2}a_2, \quad a_{34} = \frac{11}{43} - \frac{8}{5a_2}, \quad a_{35} = \frac{49}{20} + \frac{2}{5}a_2, \]
\[ a_{36} = \frac{11}{43} + \frac{2}{5}a_2, \quad a_{37} = \frac{47}{504} - \frac{10}{3a_2}, \quad a_{38} = \frac{504}{180} + \frac{10}{3a_2}, \]
\[ a_{39} = -\frac{1}{36} + \frac{10}{3a_2}, \quad a_{40} = \frac{504}{180} - \frac{10}{3a_2}, \quad a_{41} = -\frac{1}{36} + 6a_2, \]
\[ a_{42} = \frac{21}{2} + 6a_2, \quad a_{43} = \frac{105}{14} + 6a_2, \quad a_{44} = \frac{21}{2} + 6a_2, \]
\[ a_{46} = \frac{105}{14} + 18a_2, \quad a_{47} = \frac{9}{4} + 18a_2, \quad a_{48} = -\frac{9}{4} - 36a_2, \]
\[ a_{49} = -\frac{21}{4} + 6a_2, \quad a_{50} = \frac{14}{5} + 6a_2, \quad a_{51} = \frac{14}{5} + 6a_2, \]
\[ a_{52} = \frac{8}{5} + 6a_2, \quad a_{53} = \frac{8}{5} + 6a_2, \quad a_{54} = -\frac{14}{5} - 12a_2, \]
\[ a_{55} = -\frac{7}{10} - 12a_2, \quad a_{56} = -\frac{7}{10} - 12a_2, \quad a_{57} = \frac{7}{10} + 12a_2, \]
\[ a_{58} = -\frac{7}{10} - 12a_2, \quad a_{59} = \frac{11}{10} + \frac{5}{3}a_2, \quad a_{60} = \frac{7}{10} - 12a_2, \]
\[ a_{61} = -\frac{7}{10} + 12a_2, \quad a_{62} = -\frac{7}{10} + 12a_2, \quad a_{63} = -\frac{3}{4} + \frac{5}{3}a_2, \]
\[ a_{64} = -\frac{3}{4} + \frac{5}{3}a_2, \quad a_{65} = -\frac{3}{4} + \frac{5}{3}a_2, \quad a_{66} = -\frac{3}{4} + \frac{5}{3}a_2, \]
\[ a_{67} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{68} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{69} = \frac{3}{4} - \frac{5}{3}a_2, \]
\[ a_{70} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{71} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{72} = \frac{3}{4} - \frac{5}{3}a_2, \]
\[ a_{73} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{74} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{75} = \frac{3}{4} - \frac{5}{3}a_2, \]
\[ a_{76} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{77} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{78} = \frac{3}{4} - \frac{5}{3}a_2, \]
\[ a_{79} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{80} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{81} = \frac{3}{4} - \frac{5}{3}a_2, \]
\[ a_{82} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{83} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{84} = \frac{3}{4} - \frac{5}{3}a_2, \]
\[ a_{85} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{86} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{87} = \frac{3}{4} - \frac{5}{3}a_2, \]
\[ a_{88} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{89} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{90} = \frac{3}{4} - \frac{5}{3}a_2, \]
\[ a_{91} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{92} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{93} = \frac{3}{4} - \frac{5}{3}a_2, \]
\[ a_{94} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{95} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{96} = \frac{3}{4} - \frac{5}{3}a_2, \]
\[ a_{97} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{98} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{99} = \frac{3}{4} - \frac{5}{3}a_2, \]
\[ a_{100} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{101} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{102} = \frac{3}{4} - \frac{5}{3}a_2, \]
\[ a_{103} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{104} = \frac{3}{4} - \frac{5}{3}a_2, \quad a_{105} = \frac{3}{4} - \frac{5}{3}a_2. \]

After plugging these formulas into equation (2), the coefficient of \( a_2 \) is an expression totally symmetric with respect to \( W_1 \) and \( W_2 \), which can be written as

\[
\Omega(W_1, W_2) + \Omega(W_2, W_1)
\]

where

\[
\Omega(W_1, W_2) := \frac{1}{2} \left[ \| W_1 W_2 T(\gamma_\alpha \circ \gamma_\beta) \|_2^2 - 6 \langle W_1 T(\gamma_\alpha \circ \gamma_\beta) \gamma_\alpha \rangle_2 \right. \\
+ 3 \langle T(\gamma_\alpha) \rangle_2 \left\langle \gamma_\alpha W_1 W_2 \gamma_\beta \gamma_\beta \right\rangle_0 
- 4 \langle W_1 T(\gamma_\alpha) \rangle_2 \langle \gamma_\alpha W_2 \gamma_\beta \gamma_\beta \rangle_0 \\
- 120 \langle W_1 T(\gamma_\alpha) \rangle_2 \langle \gamma_\alpha W_2 \rangle_1 + 240 \langle T(\gamma_\alpha) \rangle_2 \langle \gamma_\alpha W_1 \rangle W_2 \rangle_1 \\
+ 60 \langle T(\gamma_\alpha) \rangle_2 \langle \gamma_\alpha W_1 W_2 \gamma_\beta \rangle_0 \langle \gamma_\beta \rangle_1 + 12 \langle T(\gamma_\alpha) \gamma_\beta \rangle_2 \langle \gamma_\alpha \beta \gamma_\gamma \rangle \rangle_0
\]

\[= 9\]
\[-120 \langle \langle T(\gamma^\alpha) \rangle \rangle_2 \langle \langle \gamma_\alpha \{ W_1 \circ W_2 \} \rangle \rangle_1 + 60 \langle \langle W_1 \circ W_2 \rangle \rangle_2 \langle \langle T(\gamma^\alpha) \rangle \rangle_2 \langle \langle \gamma_\alpha \rangle \rangle_1 \\
+ 4 \langle \langle W_1 \{ W_2 \circ \gamma^\alpha \circ \gamma_\alpha \} \rangle \rangle_2 - \frac{5}{2} \langle \langle W_1 \circ W_2 \rangle \rangle_2 \langle \langle \gamma^\alpha \circ \gamma_\alpha \rangle \rangle_2 \\
+ 2 \langle \langle \gamma^\alpha \rangle \rangle_2 \langle \langle \gamma_\alpha \{ W_1 \circ W_2 \} \rangle \rangle \gamma_\beta \gamma_\beta \rangle \rangle_0 - 6 \langle \langle \gamma^\alpha \rangle \rangle_2 \langle \langle \{ \gamma_\alpha \circ \gamma^\beta \} \rangle \rangle_1 \langle \langle \gamma_\beta W_1 W_2 \rangle \rangle_0 \\
+ 60 \langle \langle \gamma^\alpha \rangle \rangle_2 \langle \langle \gamma_\alpha \circ \gamma_\alpha \circ W_1 \circ W_2 \rangle \rangle_1 - \frac{3}{10} \langle \langle W_1 W_2 \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \rangle \rangle_1 \langle \langle \gamma^\beta \circ \gamma_\beta \rangle \rangle_1 \\
- \frac{8}{5} \langle \langle W_1 \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \{ W_2 \circ \gamma^\beta \} \rangle \rangle_1 + \frac{22}{5} \langle \langle W_1 \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta \rangle \rangle_1 \langle \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle \rangle_0 \\
- \frac{11}{20} \langle \langle \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta \rangle \rangle_1 \langle \langle \gamma_\beta W_1 \gamma^\mu \gamma_\mu \rangle \rangle_0 + 6 \langle \langle W_1 \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta \{ \gamma_\beta \circ W_2 \} \rangle \rangle_1 \\
+ 6 \langle \langle \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta \{ \gamma_\beta \circ W_2 \} \rangle \rangle_1 - 12 \langle \langle \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma^\beta \gamma^\mu \rangle \rangle_1 \langle \langle \gamma_\beta \gamma_\mu W_1 W_2 \rangle \rangle_0 \\
+ 18 \langle \langle W_1 \gamma^\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\alpha \{ \gamma_\beta \circ W_2 \} \rangle \rangle - 18 \langle \langle \gamma^\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma^\mu \rangle \rangle_1 \langle \langle \gamma_\mu \gamma_\beta W_1 W_2 \rangle \rangle_0 \\
- 9 \langle \langle \gamma^\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta \{ W_1 \circ W_2 \} \rangle \rangle - 6 \langle \langle \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta \gamma_\beta \{ W_1 \circ W_2 \} \rangle \rangle_1 \\
+ 3 \langle \langle \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\beta \{ W_1 \circ W_2 \} \rangle \rangle + 6 \langle \langle \gamma^\alpha \gamma_\alpha \gamma_\beta \rangle \rangle_1 \langle \langle \{ \gamma_\beta \circ W_2 \} \rangle \rangle_1 \\
- 3 \langle \langle \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\beta \gamma_1 W_1 W_2 \gamma^\mu \rangle \rangle_0 \langle \langle \gamma_\mu \rangle \rangle_1 - 12 \langle \langle \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\beta \circ W_1 \rangle \rangle W_2_1 \\
+ \frac{18}{5} \langle \langle W_1 \{ W_2 \circ \gamma^\alpha \circ \gamma_\beta \} \rangle \rangle_1 \langle \langle \gamma_\alpha \circ \gamma_\beta \rangle \rangle_1 - \frac{54}{5} \langle \langle W_1 \gamma^\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta W_2 \gamma^\mu \rangle \rangle_0 \langle \langle \gamma_\mu \rangle \rangle_1 \\
+ \frac{18}{5} \langle \langle \gamma^\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta W_1 W_2 \gamma^\mu \rangle \rangle_0 \langle \langle \gamma_\mu \rangle \rangle_1 - \frac{54}{5} \langle \langle W_1 \gamma^\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\alpha \circ \gamma_\beta \rangle \rangle W_2_1 \\
+ \frac{36}{5} \langle \langle \gamma^\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta W_1 \gamma^\mu \rangle \rangle_0 \langle \langle \gamma_\mu \rangle \rangle_1 + \frac{18}{5} \langle \langle \gamma^\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\alpha \circ \gamma_\beta \rangle \rangle W_1 W_2_1 \\
+ \frac{77}{20} \langle \langle W_1 W_2 \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta \gamma_\beta \gamma^\mu \rangle \rangle_0 \langle \langle \gamma_\mu \rangle \rangle_1 - \frac{33}{10} \langle \langle W_1 \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha W_2 \gamma^\beta \gamma_\beta \gamma^\mu \rangle \rangle_0 \langle \langle \gamma_\mu \rangle \rangle_1 \\
- \frac{22}{5} \langle \langle W_1 \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta \gamma_\beta \gamma^\mu \rangle \rangle_0 \langle \langle \gamma_\mu \rangle \rangle_1 + \frac{11}{20} \langle \langle \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha W_1 W_2 \gamma^\beta \gamma_\beta \gamma^\mu \rangle \rangle_0 \langle \langle \gamma_\mu \rangle \rangle_1 \\
+ 72 \langle \langle \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\beta \{ W_1 \circ W_2 \} \rangle \rangle + 144 \langle \langle W_1 \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \{ \gamma_\beta \circ W_2 \} \rangle \rangle_1 \\
+ 144 \langle \langle \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha W_1 \gamma^\beta \rangle \rangle_1 \langle \langle \{ \gamma_\beta \circ W_2 \} \rangle \rangle_1 \\
- 72 \langle \langle \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\beta W_1 W_2 \gamma^\mu \rangle \rangle_0 \langle \langle \gamma_\mu \rangle \rangle_1 \\
- 288 \langle \langle \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\beta \circ W_1 \rangle \rangle W_2_1 + 48 \langle \langle W_1 W_2 \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma^\beta \rangle \rangle_1 \langle \langle \{ \gamma_\alpha \circ \gamma_\beta \} \rangle \rangle_1 \\
- 24 \langle \langle W_1 \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma^\beta \rangle \rangle_1 \langle \langle \gamma^\mu \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta \gamma_\mu W_2 \rangle \rangle_0 \\
+ 6 \langle \langle \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma^\beta \rangle \rangle_1 \langle \langle \gamma^\mu \rangle \rangle_1 \langle \langle \gamma_\alpha \gamma_\beta \gamma_\mu W_1 W_2 \rangle \rangle_0 \\
- 84 \langle \langle W_1 \gamma^\alpha \rangle \rangle_1 \langle \langle W_2 \gamma^\beta \rangle \rangle_1 \langle \langle \{ \gamma_\alpha \circ \gamma_\beta \} \rangle \rangle_1 - \frac{1}{40} \langle \langle W_1 W_2 \gamma^\alpha \gamma_{\alpha} \{ \gamma^\beta \circ \gamma_\beta \} \rangle \rangle_1 \\
+ \frac{1}{5} \langle \langle W_1 \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\beta W_2 \gamma^\mu \gamma_\mu \rangle \rangle_0 - \frac{3}{20} \langle \langle \gamma^\alpha \gamma_\alpha \gamma^\beta \rangle \rangle_1 \langle \langle \gamma_\beta W_1 W_2 \gamma^\mu \gamma_\mu \rangle \rangle_0}
Theorem 2.2 proved the following

\[-\frac{1}{8} \left\langle \left\langle \gamma^\alpha \gamma_\alpha \gamma^\beta \gamma_\beta \{W_1 \circ W_2\} \right\rangle_1 \right\rangle + \frac{1}{4} \left\langle \left\langle \gamma^\alpha \gamma_\alpha W_1 \gamma^\beta \{\gamma_\beta \circ W_2\} \right\rangle_1 \right\rangle
\]
\[-\frac{1}{2} \left\langle \left\langle \gamma^\alpha \gamma_\alpha \gamma^\beta \gamma_\beta \right\rangle_1 \right\rangle \cdot \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\mu W_2 \right\rangle_0 \right\rangle + \frac{1}{15} \left\langle \left\langle W_1 W_2 \gamma^\alpha \gamma_\beta \{\gamma_\alpha \circ \gamma_\beta\} \right\rangle_1 \right\rangle
\]
\[-\frac{8}{15} \left\langle \left\langle W_1 \gamma^\alpha \gamma_\beta \gamma^\mu \right\rangle_1 \right\rangle \cdot \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\mu W_2 \right\rangle_0 \right\rangle + \frac{2}{5} \left\langle \left\langle \gamma^\alpha \gamma^\beta \gamma^\mu \right\rangle_1 \right\rangle \cdot \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\mu W_1 W_2 \right\rangle_0 \right\rangle
\]
\[+ \frac{3}{20} \left\langle \left\langle W_1 W_2 \gamma^\alpha \gamma_\beta \right\rangle_1 \right\rangle \cdot \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\mu \mu_\gamma \right\rangle_0 \right\rangle - \frac{9}{20} \left\langle \left\langle W_1 \gamma^\alpha \gamma_\beta \right\rangle_1 \right\rangle \cdot \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_2 \gamma^\mu \mu_\gamma \right\rangle_0 \right\rangle
\]
\[+ \frac{3}{20} \left\langle \left\langle \gamma^\alpha \gamma_\beta \right\rangle_1 \right\rangle \cdot \left\langle \left\langle \gamma_\alpha \gamma_\beta W_1 W_2 \gamma^\mu \mu_\gamma \right\rangle_0 \right\rangle + \frac{3}{40} \left\langle \left\langle W_1 W_2 \gamma^\alpha \right\rangle_1 \right\rangle \cdot \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\mu \gamma_\mu \right\rangle_0 \right\rangle
\]
\[-\frac{1}{10} \left\langle \left\langle W_1 \gamma^\alpha \right\rangle_1 \right\rangle \cdot \left\langle \left\langle \gamma_\alpha W_2 \gamma^\beta \gamma_\mu \gamma_\mu \right\rangle_0 \right\rangle + \frac{1}{80} \left\langle \left\langle \gamma^\alpha \right\rangle_1 \right\rangle \cdot \left\langle \left\langle \gamma_\alpha W_1 W_2 \gamma^\beta \gamma_\mu \gamma_\mu \right\rangle_0 \right\rangle
\]
\[-72 \left\langle \left\langle \gamma^\alpha \right\rangle_1 \right\rangle \cdot \left\langle \left\langle \gamma_\alpha \{W_1 \circ W_2\} \gamma_\beta \right\rangle_1 \right\rangle \cdot \left\langle \left\langle \gamma^\beta \right\rangle_1 \right\rangle.
\]

(4)

Note that \( \Omega(W_1, W_2) \) only involves data up to genus-2. Combining the above results, we have proved the following

**Theorem 2.2** For Gromov-Witten invariants of any compact symplectic manifold, we have

\[\langle T^2(W_1)T(W_2) \rangle_3 = 5 \langle T^2(W_1 \circ W_2) \rangle_3 - \frac{1}{36} \langle W_1 T(\gamma^\alpha) \rangle_2 \langle \langle \gamma_\alpha W_2 \gamma^\beta \rangle_0 \right\rangle + \frac{5}{168} \langle T(W_2) \gamma^\alpha \rangle_2 \langle \langle \gamma_\alpha W_1 \gamma^\beta \gamma_\beta \rangle \right\rangle_0 + \frac{1}{252} \langle W_2 T(\gamma^\alpha) \rangle_2 \langle \langle \gamma_\alpha W_1 \gamma^\beta \gamma_\beta \rangle \right\rangle_0
\]
\[+ \frac{7}{24} \langle T(\gamma^\alpha) \rangle_2 \langle \langle \gamma_\alpha W_1 W_2 \gamma^\beta \gamma_\beta \rangle \right\rangle_0 - \frac{2}{3} \langle W_1 T(\gamma^\alpha) \rangle_2 \langle \langle \{\gamma_\alpha \circ W_2\} \right\rangle_1
\]
\[+ \frac{5}{7} \langle T(W_2) \gamma^\alpha \rangle_2 \langle \langle \{\gamma_\alpha \circ W_1\} \right\rangle_1 + \frac{2}{21} \langle W_2 T(\gamma^\alpha) \rangle_2 \langle \langle \{\gamma_\alpha \circ W_1\} \right\rangle_1
\]
\[+ \frac{20}{3} \langle T(\gamma^\alpha) \rangle_2 \langle \langle \{\gamma_\alpha \circ W_1\} \{W_2 \right\rangle_0 \right\rangle_0 + \frac{100}{21} \langle T(\gamma^\alpha) \rangle_2 \langle \langle \{\gamma_\alpha \circ W_2\} \rangle_2 \langle \langle \gamma_\beta \rangle \right\rangle_1
\]
\[+ \frac{1}{14} \langle T(W_2) \gamma^\alpha \{\gamma_\alpha \circ W_1\} \rangle_2 - \frac{1}{42} \langle W_2 T(\gamma^\alpha) \{\gamma_\alpha \circ W_1\} \rangle_2
\]
\[+ \frac{11}{14} \langle \langle T(\gamma^\alpha) \gamma^\beta \rangle \rangle_2 \langle \langle \gamma_\alpha \gamma_\beta W_1 W_2 \rangle \rangle_0 + \frac{1}{21} \langle T(\gamma^\alpha) \gamma_\alpha \{W_1 \circ W_2\} \rangle_2
\]
\[- \frac{100}{21} \langle T(\gamma^\alpha) \rangle_2 \langle \langle \gamma_\alpha \{W_1 \circ W_2\} \rangle_1 + \frac{2}{3} \langle \{W_1 \circ W_2\} T(\gamma^\alpha) \rangle_2 \langle \langle \gamma_\alpha \rangle \right\rangle_1
\]
\[+ \frac{1}{36} \langle W_1 \{W_2 \circ \gamma^\alpha \circ \gamma_\alpha\} \rangle_2 - \frac{1}{252} \langle W_2 \{W_1 \circ \gamma^\alpha \circ \gamma_\alpha\} \rangle_2
\]
\[- \frac{1}{36} \langle W_1 \circ W_2 \{\gamma^\alpha \circ \gamma_\alpha\} \rangle_2 - \frac{47}{504} \langle \gamma^\alpha \rangle_2 \langle \langle \gamma_\alpha \{W_1 \circ W_2\} \gamma^\beta \gamma_\beta \rangle \rangle_0
\]
\[- \frac{7}{12} \langle T(\gamma^\alpha) \rangle_2 \langle \langle \{\gamma_\alpha \circ \gamma_\beta\} \gamma_\beta W_1 W_2 \rangle \rangle_0 - \frac{11}{42} \langle \gamma^\alpha \gamma_\alpha \circ W_1 \circ W_2 \rangle \rangle_2
\]
\[- \frac{47}{21} \langle \gamma^\alpha \rangle_2 \langle \langle \{\gamma_\alpha \circ W_1 \circ W_2\} \rangle_1 + \frac{1}{168} \langle W_1 W_2 \gamma^\alpha \rangle_1 \langle \langle \gamma_\alpha \{\gamma^\beta \circ \gamma_\beta\} \rangle \rangle_1
\]
\[- \frac{1}{72} \langle W_1 \gamma^\alpha \rangle_1 \langle \langle \gamma_\alpha W_2 \{\gamma^\beta \circ \gamma_\beta\} \rangle \rangle_1 - \frac{11}{504} \langle W_2 \gamma^\alpha \rangle_1 \langle \langle \gamma_\alpha W_1 \{\gamma^\beta \circ \gamma_\beta\} \rangle \rangle_1
\]
where $W_1$ and $W_2$ are arbitrary vector fields on the big phase space, and $a_2$ is a constant (see also the remark after Proposition 2.3).

It is surprising that the coefficient of $a_2$ in the above formula is completely symmetric with respect to $W_1$ and $W_2$ but the formula itself is not symmetric (e.g. $a_3 = 0$ but $a_5 \neq 0$). It is also interesting to observe that

$$
\Omega(W_1, T(W_2)) + \Omega(T(W_2), W_1)
$$

is also symmetric with respect to $W_1$ and $W_2$. The proof of this fact, which is omitted here since it is quite long, uses all known universal equations of genus $\leq 2$. These facts give a hint that $\Omega(W_1, W_2) + \Omega(W_2, W_1)$ might be identically equal to 0. We have testified this using a computer program involving Gromov-Witten invariants of a point, $\mathbb{P}^1$, and $\mathbb{P}^2$. In fact, we can prove this under the assumption that the intersection pairing on $R^3(\overline{M}_{3,2}) \times R^5(\overline{M}_{3,2})$ is non-degenerate.

We note that this assumption follows from a well known conjecture that the tautological ring $R^*(\overline{M}_{g,n})$ should be Gorenstein (cf. [FP1] and [P]). It is also remarked in [FP1] p4 that $R^*(\overline{M}_{g})$ is indeed Gorenstein for $g \leq 3$. But currently it is not known whether $R^*(\overline{M}_{3,2})$ is Gorenstein.

**Proposition 2.3** If the intersection pairing on $R^3(\overline{M}_{3,2}) \times R^5(\overline{M}_{3,2})$ is non-degenerate, then

$$
\Omega(W_1, W_2) + \Omega(W_2, W_1) = 0
$$

for all vector fields $W_1$ and $W_2$. 
Proof: In [Y], S. Yang calculated the rank of the intersection pairing on $R^{3}(\mathcal{M}_{3,2}) \times R^{5}(\mathcal{M}_{3,2})$ to be 104. Each term in equation (2) corresponds to an element in $R^{3}(\mathcal{M}_{3,2})$. So terms in equation (2) with coefficients $a_1, \ldots, a_{105}$ give 105 elements in $R^{3}(\mathcal{M}_{3,2})$. If the intersection pairing on $R^{3}(\mathcal{M}_{3,2}) \times R^{5}(\mathcal{M}_{3,2})$ is non-degenerate, there must exist a linear relation among these 105 terms. This linear relation gives a universal equation for Gromov-Witten invariants.

We can add freely any scalar multiplication of this equation to equation (2). By Theorem 2.2, the only freedom in equation (2) is $a_2\{\Omega(W_1, W_2) + \Omega(W_2, W_1)\}$.

Therefore we must have $\Omega(W_1, W_2) + \Omega(W_2, W_1) = 0$. □

Remark: By Proposition 2.3, we can set $a_2 = 0$ in Theorem 2.2 if it is true that the intersection pairing on $R^{3}(\mathcal{M}_{3,2}) \times R^{5}(\mathcal{M}_{3,2})$ is non-degenerate.

Proof of Theorem 0.1: We can use Theorem 2.2 to compute $\langle [T^2(W_1)T(W_2)] \rangle_3$ and $\langle [T^2(W_2)T(W_1)] \rangle_3$ separately. The difference of these two terms then proves Theorem 0.1. Note that all symmetric terms in Theorem 2.2 are canceled out in $\langle [T^2(W_1)T(W_2)] \rangle_3 - \langle [T^2(W_2)T(W_1)] \rangle_3$.

This makes Theorem 0.1 a much simpler formula than Theorem 2.2. We would like to point out that the proof of Theorem 0.1 does not rely on Proposition 2.3. The reason is that $a_2\{\Omega(W_1, W_2) + \Omega(W_2, W_1)\}$ is symmetric with respect to $W_1$ and $W_2$. So this term is canceled in $\langle [T^2(W_1)T(W_2)] \rangle_3 - \langle [T^2(W_2)T(W_1)] \rangle_3$ without using Proposition 2.3. □

Remark: It turns out that the genus-3 topological recursion relation proved in [KL] can be derived using Theorem 2.2. To see this, let $S := -\sum_{m,\alpha} \tilde{t}^\alpha_m \tau_{m-1}(\gamma_\alpha)$ be the string vector field, where $\tilde{t}^\alpha_m = t^\alpha_m - \delta_{\alpha,1}\delta_{m,1}$. Then $T(S) = D := -\sum_{m,\alpha} \tilde{t}^\alpha_m \tau_m(\gamma_\alpha)$ is the dilaton vector field (cf. [L1] for a proof of this fact). Applying Theorem 2.2 for $W_1 = T(W)$ and $W_2 = S$, then getting rid of the string and dilaton vector fields by using the string and dilaton equations, we can obtain the main result of [KL]. Note that in this derivation, we also don’t need Proposition 2.3 since we can prove

$$\Omega(W, S) + \Omega(S, W) = \Omega(W, D) + \Omega(D, W) = 0$$

directly using the string and dilaton equations. The proofs of these facts are quite long and are omitted here.
3 New relations in the tautological ring of $\overline{M}_{3,2}$

Universal equations in Theorems 0.1, Theorem 2.2, and Proposition 2.3 corresponds to 3 relations in the tautological ring of $\overline{M}_{3,2}$. To describe such relations, we use dual graphs to represent strata of $\overline{M}_{g,n}$. We adopt the conventions of [Ge2] for dual graphs with a slight modification. We denote vertices of genus 0 by a hollow circle $\circ$, and vertices of genus $g \geq 1$ by $\otimes$. A vertex with an incident arrowhead denotes the $\psi$ class associated to the marked point (or a node) on the irreducible component associated to that vertex. When translating relations in the tautological ring of $\overline{M}_{g,n}$ to universal equations for Gromov-Witten invariants, we need to divide the coefficient of each stratum by the number of elements in the automorphism group of the corresponding dual graph.

Theorem 2.2 corresponds to a relation representing $\psi_1^2\psi_2$ as a linear combination of 99 boundary classes on $\overline{M}_{3,2}$. Proposition 2.3 is a linear relation among 69 boundary classes of $\overline{M}_{3,2}$. Since these two relations are very long, we will omit their dual graph representations in this paper. We only give the dual graph representation for the relation corresponding to Theorem 0.1. The other two relations can be similarly recovered from Proposition 2.3 and Theorem 2.2.

Let

$$G_{i,j} := \frac{4}{9} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{5}{12} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{16}{3} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + 5 \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{40}{3} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{1}{6} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{4}{9} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{1}{9} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{1}{9} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{1}{15} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{1}{15} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{9}{10} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{1}{15} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{4}{15} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{6}{5} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{1}{3} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array} + \frac{32}{15} \begin{array}{c} i \\ \otimes \\ i \\ \otimes \end{array}$$
Theorem 3.1 In the tautological ring of $\overline{M}_{3,2}$, the following relation holds
\[
\psi_1^2 \psi_2 - \psi_1 \psi_2^2 = \frac{1}{7} \left\{ G_{1,2} - G_{2,1} \right\}.
\]

Appendix

A Relations among constants $a_1, \ldots, a_{105}$

The formulas in Lemma 2.1 are obtained by solving a system of 104 linearly independent relations among constants $a_1, \ldots, a_{105}$ in equation (2). These relations are obtained by using Gromov-Witten theory of a point and $\mathbb{P}^1$. In this appendix, we list all these relations and indicate how they are obtained.

A.1 Relations from the Gromov-Witten invariants of a point

From $\Phi(\tau_0, \tau_5)|_{t=0} = 0$, we obtain
\[
0 = \frac{a_2}{288} + \frac{a_{15}}{1152} + \frac{a_{16}}{288} + \frac{a_{24}}{1152} + \frac{a_{84}}{24} + \frac{a_{90}}{24} + \frac{a_{92}}{24} + a_{104} - \frac{77}{414720}.
\]

From $\Phi(\tau_1, \tau_4)|_{t=0} = 0$, we obtain
\[
0 = \frac{a_2}{96} + \frac{a_5}{1152} + \frac{a_6}{384} + \frac{a_{24}}{6} + \frac{a_{84}}{24} + \frac{a_{92}}{24} + a_{104} + 5a_{104}
\]
From $\Phi(\tau_2, \tau_3)|_{t=0} = 0$, we obtain
\[ 0 = \frac{29a_2}{1440} + \frac{a_{33}}{576} + \frac{a_{62}}{576} + \frac{7a_{84}}{24} + \frac{7a_{92}}{24} + \frac{a_{98}}{24} + \frac{a_{101}}{24} + 10a_{104} - \frac{503}{1451520}. \]  

(8)

From $\Phi(\tau_3, \tau_2)|_{t=0} = 0$, we obtain
\[ 0 = \frac{29a_2}{1440} + \frac{a_{34}}{576} + \frac{a_{61}}{576} + \frac{7a_{84}}{24} + \frac{7a_{92}}{24} + \frac{a_{97}}{24} + \frac{a_{102}}{24} + 10a_{104} - \frac{503}{1451520}. \]  

(9)

From $\Phi(\tau_4, \tau_1)|_{t=0} = 0$, we obtain
\[ 0 = \frac{a_2}{96} + \frac{a_3}{1152} + \frac{a_4}{384} + \frac{a_{84}}{6} + \frac{a_{85}}{24} + \frac{a_{92}}{6} + \frac{a_{93}}{24} + 5a_{104} - \frac{77}{414720}. \]  

(10)

From $\Phi(\tau_5, \tau_0)|_{t=0} = 0$, we obtain
\[ 0 = \frac{77a_1}{414720} + \frac{5a_2}{1152} + \frac{5a_4}{1152} + \frac{5a_{15}}{1152} + \frac{5a_{16}}{1152} + \frac{5a_{18}}{1152} + \frac{5a_{19}}{1152} + \frac{a_{23}}{1152} + \frac{a_{24}}{1152} + \frac{a_{25}}{1152} + \frac{a_{29}}{1152} + \frac{a_{88}}{24} + \frac{a_{89}}{24} + \frac{a_{90}}{24} + \frac{a_{92}}{24} + a_{104} - \frac{5}{69120}. \]  

(11)

From $\tau_6\Phi(\tau_0, \tau_0)|_{t=0} = 0$, we obtain
\[ 0 = \frac{11a_2}{288} + \frac{a_7}{288} + \frac{a_4}{288} + \frac{a_{15}}{90} + \frac{5a_{16}}{288} + \frac{a_{17}}{288} + \frac{a_{24}}{288} + \frac{a_{28}}{1152} + \frac{a_{30}}{1152} + \frac{5a_{84}}{24} + \frac{a_{85}}{24} + \frac{5a_{90}}{24} + \frac{a_{91}}{24} + \frac{5a_{92}}{24} + \frac{a_{93}}{24} + \frac{6a_{104}}{24} - \frac{17}{5760}. \]  

(12)

From $\tau_5\Phi(\tau_0, \tau_1)|_{t=0} = 0$, we obtain
\[ 0 = \frac{11a_2}{288} + \frac{a_7}{384} + \frac{a_{11}}{9216} + \frac{17a_{15}}{960} + \frac{11a_{16}}{288} + \frac{11a_{24}}{1440} + \frac{a_{34}}{576} + \frac{a_{42}}{576} + \frac{a_{55}}{576} + \frac{a_{61}}{576} + \frac{11a_{84}}{24} + \frac{a_{87}}{24} + \frac{11a_{90}}{24} + \frac{11a_{92}}{24} + \frac{a_{95}}{24} + \frac{a_{97}}{24} + a_{102} + 15a_{104} - \frac{1121}{241920}. \]  

(13)
From $\tau_3\Phi(\tau_0, \tau_3)|_{t=0} = 0$, we obtain
\begin{equation}
0 = \frac{29a_2}{576} + \frac{17a_{15}}{960} + \frac{29a_{16}}{576} + \frac{29a_{24}}{2880} + \frac{a_{32}}{576} + \frac{a_{33}}{576} + \frac{a_{47}}{576} + \frac{a_{62}}{576} \\
+ \frac{a_{65}}{576} + \frac{7a_{84}}{12} + \frac{7a_{90}}{12} + \frac{7a_{92}}{24} + \frac{a_{98}}{24} + \frac{a_{99}}{24} + \frac{a_{100}}{24} + \frac{a_{101}}{24} \\
+ 20a_{104} - \frac{1121}{241920}.
\end{equation}

From $\tau_2\Phi(\tau_0, \tau_1)|_{t=0} = 0$, we obtain
\begin{equation}
0 = \frac{11a_2}{288} + \frac{a_5}{1152} + \frac{a_6}{384} + \frac{a_9}{27648} + \frac{a_{10}}{9216} + \frac{a_{15}}{90} + \frac{11a_{16}}{288} + \frac{11a_{24}}{1440} \\
+ \frac{a_{37}}{576} + \frac{a_{44}}{576} + \frac{a_{53}}{576} + \frac{a_{57}}{576} + \frac{a_{84}}{24} + \frac{a_{86}}{24} + \frac{11a_{90}}{24} + \frac{11a_{92}}{24} \\
+ \frac{a_{94}}{24} + \frac{a_{96}}{24} + \frac{a_{103}}{24} + 15a_{104} - \frac{17}{5760}.
\end{equation}

From $\tau_5\Phi(\tau_1, \tau_0)|_{t=0} = 0$, we obtain
\begin{equation}
0 = \frac{5a_2}{288} + \frac{a_5}{288} + \frac{a_6}{288} + \frac{5a_{14}}{288} + \frac{a_{17}}{288} + \frac{a_{23}}{288} + \frac{a_{27}}{1152} + \frac{a_{28}}{1152} \\
+ \frac{a_{30}}{1152} + \frac{a_{32}}{24} + \frac{a_{38}}{24} + \frac{a_{39}}{24} + \frac{a_{41}}{24} + \frac{a_{48}}{24} + \frac{a_{49}}{24} + \frac{6a_{104}}{241920}.
\end{equation}

From $\tau_4\Phi(\tau_1, \tau_1)|_{t=0} = 0$, we obtain
\begin{equation}
0 = \frac{5a_2}{96} + \frac{11a_3}{1440} + \frac{a_4}{96} + \frac{11a_5}{1440} + \frac{a_6}{96} + \frac{a_7}{96} + \frac{5a_{84}}{6} + \frac{a_{85}}{6} \\
+ \frac{a_{86}}{6} + \frac{a_{87}}{12} + \frac{5a_{92}}{6} + \frac{a_{93}}{6} + \frac{a_{94}}{6} + \frac{a_{95}}{12} + 30a_{104} - \frac{1121}{241920}.
\end{equation}

From $\tau_3\Phi(\tau_1, \tau_2)|_{t=0} = 0$, we obtain
\begin{equation}
0 = \frac{29a_2}{288} + \frac{29a_5}{2880} + \frac{29a_6}{1440} + \frac{a_{32}}{288} + \frac{a_{33}}{288} + \frac{a_{34}}{192} + \frac{a_{36}}{192} + \frac{a_{61}}{576} + \frac{a_{63}}{576} \\
+ \frac{a_{65}}{288} + \frac{35a_{84}}{24} + \frac{7a_{86}}{24} + \frac{35a_{92}}{24} + \frac{7a_{94}}{24} + \frac{a_{97}}{8} + \frac{a_{99}}{8} + \frac{a_{100}}{12} \\
+ \frac{a_{102}}{6} + 60a_{104} - \frac{583}{96768}.
\end{equation}

From $\tau_2\Phi(\tau_1, \tau_3)|_{t=0} = 0$, we obtain
\begin{equation}
0 = \frac{29a_2}{288} + \frac{11a_5}{1440} + \frac{29a_6}{1440} + \frac{a_{33}}{288} + \frac{a_{37}}{192} + \frac{a_{39}}{192} + \frac{a_{57}}{576} + \frac{a_{59}}{576} \\
+ \frac{a_{62}}{288} + \frac{35a_{84}}{24} + \frac{7a_{86}}{24} + \frac{35a_{92}}{24} + \frac{7a_{94}}{24} + \frac{a_{96}}{8} + \frac{a_{98}}{8} + \frac{a_{101}}{12} \\
+ \frac{a_{103}}{6} + 60a_{104} - \frac{1121}{241920}.
\end{equation}
From \( \tau_4 \Phi(\tau_2, \tau_0)|_{t=0} = 0 \), we obtain

\[
0 = \frac{11a_2}{288} + \frac{a_7}{384} + \frac{a_{12}}{9216} + \frac{11a_{14}}{288} + \frac{11a_{23}}{1440} + \frac{a_{33}}{576} + \frac{a_{41}}{576} + \frac{a_{56}}{576} + \frac{a_{62}}{576} + \frac{a_{87}}{24} + \frac{11a_{89}}{11a_{92}} + \frac{a_{95}}{24} + \frac{a_{99}}{24} + \frac{a_{101}}{24} + 15a_{104} - \frac{607}{241920}.
\] (22)

From \( \tau_3 \Phi(\tau_2, \tau_1)|_{t=0} = 0 \), we obtain

\[
0 = \frac{29a_2}{288} + \frac{29a_3}{2880} + \frac{29a_4}{1440} + \frac{a_{32}}{288} + \frac{a_{33}}{192} + \frac{a_{35}}{576} + \frac{a_{42}}{576} + \frac{a_{64}}{576} + \frac{a_{65}}{24} + \frac{35a_{84}}{24} + \frac{7a_{85}}{24} + \frac{7a_{92}}{24} + \frac{a_{98}}{8} + \frac{a_{99}}{8} + \frac{a_{100}}{12} + 60a_{104} - \frac{1121}{241920}.
\] (23)

From \( \tau_2 \Phi(\tau_2, \tau_2)|_{t=0} = 0 \), we obtain

\[
0 = \frac{7a_2}{48} + \frac{a_{33}}{144} + \frac{a_{34}}{144} + \frac{a_{37}}{144} + \frac{a_{57}}{144} + \frac{a_{61}}{144} + \frac{a_{62}}{144} + \frac{a_{67}}{576} + \frac{a_{69}}{576} + \frac{a_{83}}{13824} + \frac{2a_{84}}{24} + \frac{2a_{85}}{24} + \frac{a_{96}}{6} + \frac{a_{97}}{6} + \frac{a_{98}}{6} + \frac{a_{101}}{4} + \frac{a_{102}}{4} + 90a_{104} - \frac{1121}{241920}.
\] (24)

From \( \tau_3 \Phi(\tau_3, \tau_0)|_{t=0} = 0 \), we obtain

\[
0 = \frac{29a_2}{576} + \frac{29a_{14}}{576} + \frac{29a_{23}}{2880} + \frac{a_{32}}{288} + \frac{a_{34}}{576} + \frac{a_{46}}{576} + \frac{a_{61}}{576} + \frac{a_{65}}{576} + \frac{7a_{84}}{12} + \frac{7a_{85}}{12} + \frac{7a_{92}}{24} + \frac{a_{99}}{24} + \frac{a_{100}}{24} + \frac{a_{102}}{24} + 20a_{104} - \frac{241920}{503}.
\] (25)

From \( \tau_2 \Phi(\tau_3, \tau_1)|_{t=0} = 0 \), we obtain

\[
0 = \frac{29a_2}{288} + \frac{11a_3}{1440} + \frac{29a_4}{1440} + \frac{a_{34}}{288} + \frac{a_{37}}{192} + \frac{a_{38}}{576} + \frac{a_{57}}{192} + \frac{a_{58}}{576} + \frac{a_{61}}{24} + \frac{35a_{84}}{24} + \frac{7a_{85}}{24} + \frac{7a_{92}}{24} + \frac{a_{99}}{8} + \frac{a_{97}}{8} + \frac{a_{102}}{12} + 60a_{104} - \frac{17}{5760}.
\] (26)

From \( \tau_2 \Phi(\tau_4, \tau_0)|_{t=0} = 0 \), we obtain

\[
0 = \frac{11a_2}{288} + \frac{a_3}{1152} + \frac{a_4}{384} + \frac{a_5}{9216} + \frac{11a_{14}}{288} + \frac{11a_{23}}{1440} + \frac{a_{37}}{576} + \frac{a_{43}}{576} + \frac{a_{52}}{576} + \frac{a_{57}}{576} + \frac{11a_{84}}{24} + \frac{a_{85}}{24} + \frac{11a_{89}}{24} + \frac{11a_{92}}{24} + \frac{a_{93}}{24} + \frac{a_{96}}{24} + \frac{a_{103}}{24} + 15a_{104} - \frac{77}{69120}.
\] (27)
From $\tau_3 \tau_4 \Phi(\tau_0, \tau_0)|_{t=0} = 0$, we obtain

$\begin{align*}
0 &= \frac{1121a_1}{241920} + \frac{17a_2}{160} + \frac{a_7}{384} + \frac{a_{11}}{9216} + \frac{a_{12}}{9216} + \frac{17a_{14}}{160} + \frac{17a_{15}}{160} + \frac{17a_{16}}{160} \\
&\quad + \frac{17a_{18}}{160} + \frac{17a_{19}}{160} + \frac{a_{20}}{9216} + \frac{a_{23}}{17a_{24}} + \frac{a_{25}}{17a_{25}} + \frac{17a_{29}}{a_{32}} + \frac{3}{576} \\
&\quad + \frac{a_{34}}{576} + \frac{a_{41}}{576} + \frac{a_{42}}{576} + \frac{a_{43}}{576} + \frac{a_{47}}{576} + \frac{a_{50}}{576} + \frac{a_{51}}{576} \\
&\quad + \frac{a_{52}}{576} + \frac{a_{56}}{576} + \frac{a_{61}}{576} + \frac{a_{62}}{576} + \frac{a_{56}}{25a_{84}} + \frac{a_{87}}{25a_{88}} \\
&\quad + \frac{160}{24} + \frac{25a_{89}}{24} + \frac{25a_{90}}{24} + \frac{a_{95}}{24} + \frac{a_{97}}{24} + \frac{a_{98}}{24} + \frac{a_{99}}{24} + \frac{a_{100}}{24} \\
&\quad + \frac{a_{101}}{24} + \frac{a_{102}}{24} + \frac{35a_{104}}{24} - \frac{1121}{34560}. \quad (28)
\end{align*}$

From $\tau_2 \tau_5 \Phi(\tau_0, \tau_0)|_{t=0} = 0$, we obtain

$\begin{align*}
0 &= \frac{17a_{10}}{5760} + \frac{a_2}{15} + \frac{a_3}{288} + \frac{a_4}{288} + \frac{a_5}{288} + \frac{a_6}{288} + \frac{a_8}{6912} + \frac{a_9}{6912} \\
&\quad + \frac{6912}{a_{14}} + \frac{15}{a_{15}} + \frac{15}{a_{16}} + \frac{15}{a_{17}} + \frac{a_{18}}{a_{18}} + \frac{a_{19}}{a_{19}} + \frac{a_{21}}{a_{21}} \\
&\quad + \frac{a_{22}}{90} + \frac{a_{23}}{90} + \frac{a_{24}}{a_{24}} + \frac{a_{25}}{1152} + \frac{a_{26}}{a_{26}} + \frac{a_{27}}{a_{27}} + \frac{a_{28}}{a_{28}} + \frac{a_{29}}{a_{29}} \\
&\quad + \frac{a_{30}}{a_{30}} + \frac{a_{31}}{a_{31}} + \frac{a_{37}}{576} + \frac{a_{43}}{576} + \frac{a_{44}}{576} + \frac{a_{50}}{576} + \frac{a_{52}}{576} + \frac{a_{53}}{a_{53}} \\
&\quad + \frac{a_{57}}{576} + \frac{a_{84}}{24} + \frac{a_{85}}{24} + \frac{a_{86}}{24} + \frac{a_{88}}{24} + \frac{a_{89}}{24} + \frac{a_{90}}{24} + \frac{a_{91}}{24} \\
&\quad + \frac{2a_{92}}{24} + \frac{a_{93}}{24} + \frac{a_{94}}{24} + \frac{a_{96}}{24} + \frac{a_{103}}{24} + \frac{21a_{104}}{24} - \frac{119}{5760}. \quad (29)
\end{align*}$

From $\tau_3 \tau_3 \Phi(\tau_0, \tau_1)|_{t=0} = 0$, we obtain

$\begin{align*}
0 &= \frac{29a_{20}}{96} + \frac{29a_3}{576} + \frac{a_{24}}{576} + \frac{109a_{15}}{576} + \frac{29a_{16}}{96} + \frac{29a_{17}}{576} + \frac{29a_{24}}{576} + \frac{29a_{28}}{2880} \\
&\quad + \frac{2880}{a_{32}} + \frac{a_{33}}{96} + \frac{a_{35}}{576} + \frac{a_{37}}{288} + \frac{a_{42}}{96} + \frac{a_{43}}{576} + \frac{a_{47}}{576} + \frac{a_{48}}{288} + \frac{a_{64}}{288} \\
&\quad + \frac{a_{65}}{96} + \frac{7a_{84}}{2} + \frac{7a_{85}}{12} + \frac{7a_{90}}{2} + \frac{7a_{91}}{12} + \frac{7a_{92}}{12} + \frac{7a_{93}}{4} + \frac{a_{98}}{4} \\
&\quad + \frac{a_{99}}{3} + \frac{a_{100}}{a_{101}} + \frac{140a_{104}}{3} - \frac{205}{3456}. \quad (30)
\end{align*}$

From $\tau_2 \tau_4 \Phi(\tau_0, \tau_1)|_{t=0} = 0$, we obtain

$\begin{align*}
0 &= \frac{11a_2}{48} + \frac{11a_3}{288} + \frac{11a_4}{288} + \frac{11a_5}{1440} + \frac{a_6}{128} + \frac{a_7}{34560} + \frac{a_{10}}{2304} \\
&\quad + \frac{a_{11}}{4608} + \frac{a_{13}}{9216} + \frac{a_{14}}{48} + \frac{11a_{16}}{1440} + \frac{a_{17}}{96} + \frac{11a_{24}}{1440} + \frac{11a_{28}}{1440} + \frac{11a_{30}}{1440} \\
&\quad + a_{34} + \frac{a_{37}}{a_{38}} + \frac{a_{42}}{a_{44}} + \frac{a_{45}}{a_{53}} + \frac{a_{54}}{288} + \frac{a_{43}}{144} + \frac{a_{44}}{576} + \frac{a_{45}}{144} + \frac{576}{144} \\
&\quad + \frac{a_{55}}{288} + \frac{a_{57}}{a_{58}} + \frac{a_{61}}{144} + \frac{11a_{84}}{24} + \frac{11a_{85}}{24} + \frac{a_{86}}{6} + \frac{a_{87}}{8} \\
&\quad + \frac{11a_{90}}{4} + \frac{11a_{91}}{24} + \frac{11a_{92}}{4} + \frac{11a_{93}}{24} + \frac{a_{94}}{6} + \frac{a_{95}}{8} + \frac{a_{96}}{6} + \frac{a_{97}}{8} \\
&\quad + \frac{4}{24} + \frac{4}{4} + \frac{6}{24} + \frac{6}{8} + \frac{6}{6} + \frac{8}{8}.
\end{align*}$

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\[
\frac{a_{102}}{12} + \frac{5a_{103}}{24} + 105a_{104} - \frac{53}{1152}.
\]

From \(\tau_2\tau_3\Phi(\tau_0, \tau_2)\)|\(_{t=0} = 0\), we obtain
\[
0 = \frac{5a_2}{12} + \frac{29a_3}{2880} + \frac{29a_6}{1440} + \frac{29a_7}{1440} + \frac{29a_9}{69120} + \frac{29a_{10}}{34560} + \frac{29a_{11}}{34560} + \frac{109a_{15}}{576} + \frac{5a_{16}}{12} + \frac{5a_{21}}{12} + \frac{a_{32}}{72} + \frac{a_{33}}{144} + \frac{7a_{34}}{144} + \frac{a_{36}}{576} + \frac{7a_{37}}{576} + \frac{a_{40}}{576} + \frac{7a_{42}}{7a_{44}} + \frac{a_{47}}{7a_{45}} + \frac{7a_{55}}{7a_{57}} + \frac{a_{60}}{576} + \frac{7a_{61}}{a_{62}} + \frac{6a_{65}}{a_{66}} + \frac{a_{67}}{144} + \frac{a_{68}}{576} + \frac{a_{69}}{576} + \frac{a_{73}}{576} + \frac{a_{77}}{144} + \frac{a_{76}}{576} + \frac{a_{76}}{7a_{86}} + \frac{a_{86}}{7a_{86}} + \frac{a_{98}}{a_{99}} + \frac{a_{100}}{a_{101}} + \frac{a_{102}}{12} + \frac{a_{103}}{12} + \frac{205}{3456}.
\]

From \(\tau_2\tau_2\Phi(\tau_0, \tau_3)\)|\(_{t=0} = 0\), we obtain
\[
0 = \frac{5a_2}{12} + \frac{11a_5}{720} + \frac{29a_6}{17280} + \frac{11a_9}{17280} + \frac{29a_{10}}{17280} + \frac{7a_{15}}{48} + \frac{5a_{16}}{12} + \frac{5a_{24}}{72} + \frac{a_{32}}{144} + \frac{a_{33}}{144} + \frac{7a_{37}}{7a_{44}} + \frac{a_{39}}{a_{47}} + \frac{7a_{45}}{7a_{53}} + \frac{7a_{57}}{12} + \frac{a_{60}}{12} + \frac{7a_{64}}{12} + \frac{5a_{84}}{59a_{90}} + \frac{7a_{86}}{7a_{86}} + \frac{a_{98}}{a_{99}} + \frac{a_{100}}{a_{101}} + \frac{a_{103}}{4} + \frac{5a_{103}}{6} + \frac{a_{103}}{6} + \frac{210a_{104}}{1152}.
\]

From \(\tau_3\tau_3\Phi(\tau_1, \tau_0)\)|\(_{t=0} = 0\), we obtain
\[
0 = \frac{29a_2}{96} + \frac{29a_5}{576} + \frac{29a_6}{576} + \frac{29a_{14}}{96} + \frac{29a_{17}}{576} + \frac{29a_{23}}{576} + \frac{29a_{27}}{2880} + \frac{29a_{28}}{2880} + \frac{29a_{30}}{2880} + \frac{a_{32}}{288} + \frac{a_{33}}{96} + \frac{a_{36}}{96} + \frac{a_{46}}{288} + \frac{a_{48}}{96} + \frac{a_{61}}{288} + \frac{a_{63}}{96} + \frac{a_{65}}{288} + \frac{7a_{84}}{2} + \frac{7a_{86}}{2} + \frac{7a_{89}}{2} + \frac{7a_{91}}{2} + \frac{7a_{92}}{2} + \frac{7a_{94}}{2} + \frac{7a_{95}}{4} + \frac{a_{97}}{4} + \frac{a_{99}}{4} + \frac{a_{100}}{3} + \frac{a_{102}}{3} + \frac{140a_{104}}{13824}.
\]

From \(\tau_2\tau_4\Phi(\tau_1, \tau_0)\)|\(_{t=0} = 0\), we obtain
\[
0 = \frac{11a_2}{48} + \frac{11a_3}{1440} + \frac{a_4}{96} + \frac{11a_5}{288} + \frac{11a_6}{288} + \frac{a_7}{128} + \frac{a_8}{2304} + \frac{a_{12}}{4608} + \frac{a_{13}}{9216} + \frac{11a_{14}}{48} + \frac{11a_{17}}{288} + \frac{11a_{23}}{288} + \frac{11a_{27}}{1440} + \frac{11a_{28}}{1440} + \frac{11a_{30}}{1440} + \frac{a_{33}}{288} + \frac{a_{37}}{144} + \frac{a_{41}}{144} + \frac{a_{43}}{144} + \frac{a_{45}}{144} + \frac{a_{52}}{144} + \frac{a_{54}}{144} + \frac{a_{56}}{288}.
\]
\[
\begin{align*}
&+ \frac{a_{57}}{144} + \frac{a_{59}}{288} + \frac{a_{62}}{11a_{84}} + \frac{a_{85}}{6} + \frac{a_{86}}{24} + \frac{a_{87}}{8} + \frac{a_{93}}{4} + \frac{11a_{94}}{6} + \frac{a_{95}}{24} + \frac{a_{96}}{8} + \frac{a_{98}}{6} + \frac{a_{101}}{12} \\
&+ \frac{11a_{91}}{24} + \frac{a_{92}}{4} + \frac{11a_{94}}{6} + \frac{a_{95}}{24} + \frac{a_{96}}{8} + \frac{a_{98}}{6} + \frac{a_{101}}{12} \\
&+ \frac{5a_{103}}{24} + 105a_{104} - \frac{1121}{34560}. \\
\end{align*}
\]

From \(\tau_2\tau_3\Phi(\tau_1, \tau_1)|_{t=0} = 0\), we obtain

\[
0 = \frac{29a_{2}}{48} + \frac{5a_{3}}{72} + \frac{29a_{4}}{288} + \frac{5a_{5}}{72} + \frac{29a_{6}}{288} + \frac{29a_{7}}{720} + \frac{a_{32}}{96} + \frac{a_{33}}{96} \\
+ \frac{a_{34}}{96} + \frac{a_{35}}{288} + \frac{a_{36}}{288} + \frac{a_{37}}{48} + \frac{a_{38}}{192} + \frac{a_{39}}{192} + \frac{a_{40}}{288} + \frac{a_{49}}{48} \\
+ \frac{a_{58}}{192} + \frac{a_{59}}{192} + \frac{a_{60}}{288} + \frac{a_{61}}{288} + \frac{a_{62}}{96} + \frac{a_{63}}{96} + \frac{a_{64}}{288} + \frac{a_{65}}{288} \\
+ \frac{35a_{84}}{96} + \frac{35a_{85}}{24} + \frac{35a_{86}}{24} + \frac{35a_{87}}{12} + \frac{35a_{89}}{24} + \frac{35a_{93}}{24} + \frac{35a_{94}}{12} + \frac{7a_{95}}{12} \\
+ \frac{a_{96}}{2} + \frac{3a_{97}}{8} + \frac{3a_{98}}{8} + \frac{a_{99}}{2} + \frac{a_{100}}{4} + \frac{a_{101}}{3} + \frac{a_{102}}{3} + \frac{5a_{103}}{6} \\
+ 420a_{104} = \frac{205}{3456}.
\]

From \(\tau_2\tau_2\Phi(\tau_1, \tau_2)|_{t=0} = 0\), we obtain

\[
0 = \frac{7a_{2}}{8} + \frac{5a_{3}}{72} + \frac{7a_{4}}{48} + \frac{a_{32}}{72} + \frac{a_{33}}{36} + \frac{a_{34}}{36} + \frac{a_{35}}{144} + \frac{a_{36}}{18} \\
+ \frac{a_{39}}{72} + \frac{a_{37}}{18} + \frac{a_{59}}{72} + \frac{a_{61}}{36} + \frac{a_{62}}{36} + \frac{a_{63}}{144} + \frac{a_{65}}{72} + \frac{a_{66}}{144} \\
+ \frac{a_{67}}{144} + \frac{a_{68}}{96} + \frac{a_{69}}{96} + \frac{a_{70}}{3456} + \frac{a_{79}}{6912} + \frac{a_{81}}{3456} + \frac{a_{83}}{12a_{84}} \\
+ \frac{2a_{86}}{144} + \frac{2a_{92} + 2a_{94} + 4a_{96}}{3} + \frac{2a_{97}}{3} + \frac{a_{98}}{a_{101}} + \frac{a_{102}}{2} + \frac{5a_{103}}{4} + \frac{a_{104}}{2} + 630a_{104} - \frac{205}{3456}.
\]

From \(\tau_2\tau_3\Phi(\tau_2, \tau_0)|_{t=0} = 0\), we obtain

\[
0 = \frac{5a_{2}}{12} + \frac{29a_{3}}{2880} + \frac{29a_{4}}{1440} + \frac{29a_{7}}{1440} + \frac{29a_{8}}{34560} + \frac{29a_{12}}{34560} + \frac{5a_{14}}{12} + \frac{5a_{23}}{72} \\
+ \frac{a_{32}}{144} + \frac{7a_{33}}{576} + \frac{a_{34}}{144} + \frac{a_{35}}{576} + \frac{7a_{37}}{576} + \frac{a_{40}}{576} + \frac{a_{41}}{576} + \frac{7a_{43}}{576} \\
+ \frac{a_{46}}{144} + \frac{7a_{52}}{576} + \frac{7a_{56}}{576} + \frac{7a_{57}}{576} + \frac{a_{60}}{144} + \frac{a_{61}}{576} + \frac{7a_{62}}{576} + \frac{a_{64}}{576} \\
+ \frac{a_{65}}{144} + \frac{a_{67}}{576} + \frac{a_{68}}{576} + \frac{a_{69}}{576} + \frac{a_{72}}{13824} + \frac{a_{78}}{13824} + \frac{a_{83}}{13824} + \frac{59a_{84}}{12} \\
+ \frac{7a_{85}}{144} + \frac{7a_{87}}{576} + \frac{59a_{89}}{576} + \frac{59a_{92}}{13824} + \frac{7a_{93}}{13824} + \frac{7a_{95}}{13824} + \frac{7a_{96}}{13824} + \frac{a_{97}}{6} \\
+ \frac{7a_{98}}{24} + \frac{a_{100} + 5a_{101} + 5a_{104}}{12} + \frac{a_{102} + 5a_{103}}{12} + \frac{210a_{104}}{1121} - \frac{34560}{34560}.
\]

From \(\tau_2\tau_2\Phi(\tau_2, \tau_1)|_{t=0} = 0\), we obtain

\[
0 = \frac{7a_{2}}{8} + \frac{5a_{3}}{72} + \frac{7a_{4}}{48} + \frac{a_{32}}{72} + \frac{a_{33}}{36} + \frac{a_{34}}{36} + \frac{a_{35}}{144} + \frac{a_{37}}{18}
\]
From $\tau_2\tau_2\Phi(\tau, \tau_0)|_{t=0}$, we obtain
\[
0 = \frac{5a_2}{12} + \frac{11a_3}{720} + \frac{29a_4}{720} + \frac{29a_4}{17280} + \frac{5a_4}{12} + \frac{5a_5}{72} + \frac{a_5}{144} + \frac{a_6}{144} + \frac{a_6}{2} + \frac{9a_7}{72} + \frac{a_8}{144} + \frac{a_8}{144} + \frac{12a_9}{53} + \frac{a_{10}}{1152}.
\]

From $\tau_2\tau_3\tau_3\Phi(\tau, \tau_0)|_{t=0}$, we obtain
\[
0 = \frac{205a_1}{1152} + \frac{763a_2}{576} + \frac{29a_3}{576} + \frac{29a_4}{576} + \frac{29a_5}{576} + \frac{29a_6}{576} + \frac{29a_7}{720} + \frac{29a_8}{13824} + \frac{29a_9}{17280} + \frac{29a_{10}}{17280} + \frac{29a_{11}}{17280} + \frac{29a_{12}}{17280} + \frac{29a_{13}}{17280} + \frac{29a_{14}}{17280} + \frac{29a_{15}}{17280} + \frac{29a_{16}}{17280} + \frac{29a_{17}}{17280} + \frac{29a_{18}}{17280} + \frac{29a_{19}}{17280} + \frac{29a_{20}}{17280} + \frac{29a_{21}}{17280} + \frac{29a_{22}}{17280} + \frac{29a_{23}}{17280} + \frac{29a_{24}}{17280} + \frac{29a_{25}}{17280} + \frac{29a_{26}}{17280} + \frac{29a_{27}}{17280} + \frac{29a_{28}}{17280} + \frac{29a_{29}}{17280} + \frac{29a_{30}}{17280} + \frac{29a_{31}}{17280} + \frac{29a_{32}}{17280} + \frac{29a_{33}}{17280} + \frac{29a_{34}}{17280} + \frac{29a_{35}}{17280} + \frac{29a_{36}}{17280} + \frac{29a_{37}}{17280} + \frac{29a_{38}}{17280} + \frac{29a_{39}}{17280} + \frac{29a_{40}}{17280} + \frac{29a_{41}}{17280} + \frac{29a_{42}}{17280} + \frac{29a_{43}}{17280} + \frac{29a_{44}}{17280} + \frac{29a_{45}}{17280} + \frac{29a_{46}}{17280} + \frac{29a_{47}}{17280} + \frac{29a_{48}}{17280} + \frac{29a_{49}}{17280} + \frac{29a_{50}}{17280} + \frac{29a_{51}}{17280} + \frac{29a_{52}}{17280} + \frac{29a_{53}}{17280} + \frac{29a_{54}}{17280} + \frac{29a_{55}}{17280} + \frac{29a_{56}}{17280} + \frac{29a_{57}}{17280} + \frac{29a_{58}}{17280} + \frac{29a_{59}}{17280} + \frac{29a_{60}}{17280} + \frac{29a_{61}}{17280} + \frac{29a_{62}}{17280} + \frac{29a_{63}}{17280} + \frac{29a_{64}}{17280} + \frac{29a_{65}}{17280} + \frac{29a_{66}}{17280} + \frac{29a_{67}}{17280} + \frac{29a_{68}}{17280} + \frac{29a_{69}}{17280} + \frac{29a_{70}}{17280} + \frac{29a_{71}}{17280} + \frac{29a_{72}}{17280} + \frac{29a_{73}}{17280} + \frac{29a_{74}}{17280} + \frac{29a_{75}}{17280} + \frac{29a_{76}}{17280} + \frac{29a_{77}}{17280} + \frac{29a_{78}}{17280} + \frac{29a_{79}}{17280} + \frac{29a_{80}}{17280} + \frac{29a_{81}}{17280} + \frac{29a_{82}}{17280} + \frac{29a_{83}}{17280} + \frac{29a_{84}}{17280} + \frac{29a_{85}}{17280} + \frac{29a_{86}}{17280} + \frac{29a_{87}}{17280} + \frac{29a_{88}}{17280} + \frac{29a_{89}}{17280} + \frac{29a_{90}}{17280} + \frac{29a_{91}}{17280} + \frac{29a_{92}}{17280} + \frac{29a_{93}}{17280} + \frac{29a_{94}}{17280} + \frac{29a_{95}}{17280} + \frac{29a_{96}}{17280} + \frac{29a_{97}}{17280} + \frac{29a_{98}}{17280} + \frac{29a_{99}}{17280} + \frac{29a_{100}}{17280} + \frac{29a_{101}}{17280} + \frac{29a_{102}}{17280} + \frac{29a_{103}}{17280} + \frac{29a_{104}}{17280}.
\]
From $\tau_2\tau_3\Phi(\tau_0, \tau_1)|_{t=0} = 0$, we obtain

\begin{align*}
0 &= 35a_2 + \frac{5a_3}{12} + \frac{5a_4}{12} + \frac{5a_5}{36} + \frac{29a_6}{144} + \frac{29a_7}{240} + \frac{5a_9}{864} + \frac{29a_{10}}{3456} + \\
&\quad + \frac{29a_{11}}{8640} + \frac{29a_{12}}{17280} + \frac{11a_{15}}{6} + \frac{35a_{16}}{12} + \frac{5a_{17}}{12} + \frac{5a_{21}}{12} + \frac{5a_{24}}{12} + \frac{5a_{28}}{72} + \frac{5a_{30}}{72} + \\
&\quad + \frac{7a_{32}}{144} + \frac{7a_{33}}{144} + \frac{7a_{34}}{144} + \frac{a_{35}}{12} + \frac{a_{36}}{12} + \frac{35a_{37}}{12} + \frac{7a_{38}}{12} + \frac{a_{39}}{12} + \\
&\quad + \frac{a_{40}}{144} + \frac{7a_{42}}{144} + \frac{7a_{43}}{144} + \frac{7a_{44}}{144} + \frac{7a_{45}}{144} + \frac{a_{47}}{144} + \frac{a_{48}}{144} + \frac{35a_{52}}{12} + \frac{7a_{54}}{12} + \\
&\quad + \frac{a_{61}}{144} + \frac{a_{62}}{144} + \frac{a_{63}}{144} + \frac{7a_{64}}{144} + \frac{7a_{65}}{144} + \frac{a_{66}}{144} + \frac{a_{67}}{144} + \frac{a_{68}}{144} + \frac{a_{69}}{144} + \frac{a_{70}}{144} + \frac{a_{73}}{12} + \\
&\quad + \frac{2304}{6912} + \frac{35a_{86}}{12} + \frac{7a_{87}}{12} + \frac{413a_{90}}{12} + \frac{59a_{91}}{12} + \frac{413a_{92}}{12} + \frac{59a_{93}}{12} + \frac{35a_{94}}{12} + \frac{7a_{95}}{4} + \\
&\quad + \frac{35a_{96}}{12} + \frac{7a_{97}}{12} + \frac{17a_{98}}{12} + \frac{23a_{99}}{12} + \frac{17a_{100}}{12} + \frac{23a_{101}}{12} + \frac{5a_{102}}{3} + \frac{5a_{103}}{3} + \\
&\quad + 1680a_{104} - \frac{193}{288}.
\end{align*}

From $\tau_2\tau_2\Phi(\tau_0, \tau_2)|_{t=0} = 0$, we obtain

\begin{align*}
0 &= \frac{49a_2}{12} + \frac{5a_5}{24} + \frac{7a_6}{16} + \frac{7a_7}{48} + \frac{5a_9}{576} + \frac{7a_{10}}{384} + \frac{7a_{11}}{1152} + \frac{11a_{15}}{6} + \\
&\quad + \frac{49a_{16}}{12} + \frac{7a_{24}}{12} + \frac{a_{32}}{12} + \frac{a_{33}}{12} + \frac{a_{34}}{12} + \frac{a_{36}}{48} + \frac{a_{37}}{4} + \frac{a_{39}}{24}.
\end{align*}
\[
\begin{align*}
&+ \frac{a_{40}}{48} + \frac{a_{42}}{12} + \frac{a_{44}}{4} + \frac{a_{47}}{12} + \frac{a_{53}}{4} + \frac{a_{55}}{12} + \frac{a_{57}}{4} + \frac{a_{59}}{24} \\
&+ \frac{a_{60}}{48} + \frac{a_{61}}{12} + \frac{a_{62}}{12} + \frac{a_{63}}{4} + \frac{a_{65}}{12} + \frac{a_{66}}{4} + \frac{a_{67}}{12} + \frac{a_{68}}{32} \\
&+ \frac{a_{69}}{48} + \frac{a_{70}}{12} + \frac{a_{73}}{12} + \frac{a_{75}}{4} + \frac{a_{77}}{12} + \frac{a_{79}}{4} + \frac{a_{81}}{48} + \frac{a_{82}}{1152} \\
&+ \frac{a_{83}}{1152} + 48a_{84} + 6a_{86} + 2a_{87} + 48a_{90} + 48a_{92} + 6a_{94} + 2a_{95} \\
&+ 6a_{96} + 2a_{97} + 3a_{98} + 3a_{99} + 3a_{100} + 3a_{101} + \frac{15a_{102}}{4} + \frac{45a_{103}}{4} \\
&+ 2520a_{104} - \frac{193}{288}.
\end{align*}
\]

From \(\tau_2\tau_2\tau_2\Phi(\tau_1, \tau_1)|_{t=0} = 0\), we obtain

\[
0 = \frac{49a_{2}}{8} + \frac{7a_{3}}{12} + \frac{7a_{4}}{8} + \frac{7a_{5}}{12} + \frac{7a_{6}}{8} + \frac{7a_{7}}{24} + \frac{a_{32}}{8} + \frac{a_{33}}{6} \\
+ \frac{a_{34}}{12} + \frac{a_{35}}{6} + \frac{a_{36}}{24} + \frac{5a_{37}}{12} + \frac{a_{38}}{12} + \frac{a_{39}}{24} + \frac{a_{40}}{12} + \frac{5a_{57}}{12} \\
+ \frac{a_{58}}{12} + \frac{a_{59}}{6} + \frac{a_{60}}{24} + \frac{a_{61}}{12} + \frac{a_{62}}{6} + \frac{a_{63}}{24} + \frac{a_{64}}{24} + \frac{a_{65}}{8} \\
+ \frac{a_{66}}{16} + \frac{a_{67}}{12} + \frac{a_{68}}{24} + \frac{a_{69}}{8} + \frac{a_{70}}{384} + \frac{a_{71}}{1152} + \frac{a_{81}}{1152} \\
+ \frac{a_{82}}{1152} + \frac{a_{83}}{576} + 84a_{84} + 12a_{85} + 12a_{86} + 4a_{87} + 84a_{92} + 12a_{93} \\
+ 12a_{94} + 4a_{95} + 10a_{96} + 6a_{97} + 6a_{98} + 6a_{99} + \frac{9a_{100}}{2} + \frac{15a_{101}}{2} \\
+ \frac{15a_{102}}{2} + \frac{45a_{103}}{2} + 5040a_{104} - \frac{193}{288}.
\]

From \(\tau_2\tau_2\tau_2\Phi(\tau_2, \tau_2)|_{t=0} = 0\), we obtain

\[
0 = \frac{49a_{2}}{12} + \frac{5a_{3}}{24} + \frac{7a_{4}}{16} + \frac{7a_{7}}{48} + \frac{7a_{8}}{384} + \frac{7a_{12}}{1152} + \frac{49a_{14}}{48} + \frac{7a_{23}}{12} \\
+ \frac{a_{32}}{12} + \frac{a_{33}}{12} + \frac{a_{34}}{12} + \frac{a_{35}}{24} + \frac{a_{36}}{4} + \frac{a_{37}}{24} + \frac{a_{38}}{4} + \frac{a_{40}}{48} + \frac{a_{41}}{12} \\
+ \frac{a_{42}}{4} + \frac{a_{43}}{12} + \frac{a_{46}}{12} + \frac{a_{47}}{4} + \frac{a_{48}}{12} + \frac{a_{49}}{4} + \frac{a_{50}}{48} + \frac{a_{51}}{12} \\
+ \frac{a_{52}}{12} + \frac{a_{53}}{12} + \frac{a_{54}}{4} + \frac{a_{55}}{4} + \frac{a_{56}}{12} + \frac{a_{57}}{4} + \frac{a_{58}}{4} + \frac{a_{59}}{12} \\
+ \frac{a_{60}}{12} + \frac{a_{61}}{12} + \frac{a_{62}}{4} + \frac{a_{63}}{24} + \frac{a_{64}}{4} + \frac{a_{65}}{48} + \frac{a_{66}}{16} \\
+ \frac{a_{67}}{1152} + \frac{a_{68}}{576} + \frac{a_{69}}{1152} + \frac{a_{70}}{576} + \frac{a_{71}}{2304} + \frac{a_{72}}{2304} + \frac{a_{83}}{1152} + 48a_{84} \\
+ 6a_{85} + 2a_{87} + 48a_{89} + 48a_{92} + 6a_{93} + 2a_{95} + 6a_{96} + 3a_{97} \\
+ 2a_{98} + 3a_{99} + 3a_{100} + \frac{15a_{101}}{4} + 3a_{102} + \frac{45a_{103}}{4} + 2520a_{104} - \frac{53}{144}.
\]

From \(\tau_2\tau_2\tau_2\tau_3\Phi(\tau_0, \tau_0)|_{t=0} = 0\), we obtain

\[
0 = \frac{193a_{1}}{288} + \frac{44a_{2}}{3} + \frac{5a_{3}}{4} + \frac{5a_{4}}{4} + \frac{5a_{5}}{4} + \frac{5a_{6}}{4} + \frac{61a_{7}}{96} + \frac{5a_{8}}{96} \\
+ \frac{5a_{9}}{96} + \frac{5a_{10}}{96} + \frac{5a_{11}}{96} + \frac{3a_{12}}{96} + \frac{29a_{13}}{1920} + \frac{44a_{14}}{1920} + \frac{44a_{15}}{1920} + \frac{44a_{16}}{1920} \\
+ \frac{5a_{17}}{3} + \frac{44a_{18}}{3} + \frac{44a_{19}}{3} + \frac{31a_{20}}{96} + \frac{5a_{21}}{96} + \frac{5a_{22}}{96} + \frac{11a_{23}}{6} + \frac{11a_{24}}{6}
\]
\[
\begin{align*}
&+ \frac{11a_{25}}{6} + \frac{5a_{26}}{24} + \frac{5a_{27}}{24} + \frac{5a_{28}}{24} + \frac{11a_{29}}{6} + \frac{5a_{30}}{24} + \frac{5a_{31}}{576} + \frac{11a_{32}}{48} \\
&+ \frac{11a_{33}}{48} + \frac{11a_{34}}{48} + \frac{a_{35}}{24} + \frac{a_{36}}{24} + \frac{245a_{37}}{96} + \frac{7a_{38}}{96} + \frac{7a_{39}}{96} + \frac{5a_{40}}{96} \\
&+ \frac{11a_{41}}{48} + \frac{11a_{42}}{48} + \frac{59a_{43}}{96} + \frac{59a_{44}}{96} + \frac{7a_{45}}{96} + \frac{11a_{46}}{48} + \frac{a_{47}}{24} \\
&+ \frac{11a_{49}}{48} + \frac{59a_{50}}{96} + \frac{11a_{51}}{48} + \frac{59a_{52}}{96} + \frac{59a_{53}}{96} + \frac{7a_{54}}{7a_{55}} + \frac{11a_{56}}{96} + \frac{a_{57}}{48} + \frac{a_{58}}{48} \\
&+ \frac{59a_{57}}{96} + \frac{7a_{58}}{96} + \frac{7a_{59}}{96} + \frac{5a_{60}}{96} + \frac{11a_{61}}{96} + \frac{11a_{62}}{96} + \frac{a_{63}}{24} + \frac{a_{64}}{24} \\
&+ \frac{11a_{65}}{48} + \frac{7a_{66}}{96} + \frac{5a_{67}}{96} + \frac{5a_{68}}{96} + \frac{a_{69}}{24} + \frac{a_{70}}{48} + \frac{a_{71}}{48} + \frac{a_{72}}{48} \\
&+ \frac{a_{73}}{576} + \frac{7a_{74}}{2304} + \frac{7a_{75}}{2304} + \frac{7a_{76}}{2304} + \frac{a_{77}}{576} + \frac{a_{78}}{2304} + \frac{a_{79}}{2304} + \frac{a_{80}}{2304} \\
&+ \frac{a_{81}}{2304} + \frac{a_{82}}{2304} + \frac{a_{83}}{4} + \frac{605a_{84}}{4} + \frac{59a_{85}}{4} + \frac{5a_{86}}{4} + \frac{29a_{87}}{4} + \frac{605a_{88}}{4} \\
&+ \frac{605a_{89}}{4} + \frac{605a_{90}}{4} + \frac{5a_{91}}{4} + \frac{605a_{92}}{4} + \frac{59a_{93}}{4} + \frac{59a_{94}}{4} + \frac{29a_{95}}{4} + \frac{59a_{96}}{4} \\
&+ \frac{a_{97}}{29a_{98}} + \frac{29a_{99}}{35a_{100}} + \frac{35a_{101}}{4} + \frac{35a_{102}}{4} + \frac{35a_{103}}{4} + \frac{105a_{104}}{4} + \frac{105a_{105}}{4} + \frac{7560a_{106}}{4} \\
&+ \frac{7a_{105}}{2304} - \frac{193}{32}
\end{align*}
\]

From \( \tau_2\tau_2\tau_2\tau_2\Phi(\tau_0, \tau_1) \mid_{t=0} = 0 \), we obtain

\[
0 = \frac{98a_{2}}{3} + \frac{49a_{3}}{12} + \frac{49a_{4}}{3} + \frac{7a_{5}}{2} + \frac{7a_{6}}{4} + \frac{7a_{7}}{72} + \frac{7a_{8}}{48} + \frac{7a_{9}}{144} + \frac{7a_{10}}{288} + \frac{7a_{11}}{2304} + \frac{7a_{12}}{2304} + \frac{245a_{13}}{12} + \frac{98a_{16}}{12} + \frac{49a_{17}}{12} + \frac{49a_{24}}{12} + \frac{7a_{28}}{12} + \frac{7a_{30}}{12}
\]

\[
+ \frac{2a_{32}}{3} + \frac{2a_{33}}{3} + \frac{2a_{34}}{3} + \frac{a_{35}}{6} + \frac{a_{36}}{6} + \frac{2a_{37}}{3} + \frac{a_{38}}{3} + \frac{a_{39}}{3}
\]

\[
+ \frac{a_{40}}{4} + \frac{2a_{42}}{3} + \frac{2a_{44}}{3} + \frac{a_{45}}{3} + \frac{2a_{47}}{6} + \frac{a_{48}}{2a_{53}} + \frac{a_{54}}{3} + \frac{a_{55}}{3} + \frac{a_{57}}{3} + \frac{a_{58}}{3} + \frac{a_{59}}{3} + \frac{a_{60}}{3}
\]

\[
+ \frac{a_{61}}{3} + \frac{a_{62}}{3} + \frac{a_{63}}{3} + \frac{a_{64}}{3} + \frac{2a_{65}}{4} + \frac{a_{66}}{4} + \frac{a_{67}}{4} + \frac{a_{68}}{4} + \frac{a_{69}}{4} + \frac{5a_{70}}{4} + \frac{a_{73}}{4}
\]

\[
+ \frac{a_{75}}{72} + \frac{a_{76}}{288} + \frac{a_{77}}{144} + \frac{a_{79}}{72} + \frac{a_{80}}{288} + \frac{a_{81}}{192} + \frac{a_{82}}{144} + \frac{a_{83}}{4}
\]

\[
+ \frac{a_{84}}{3} + \frac{a_{85}}{4} + \frac{a_{86}}{4} + \frac{a_{87}}{4} + \frac{a_{88}}{4} + \frac{a_{89}}{4} + \frac{a_{90}}{4} + \frac{a_{91}}{4} + \frac{a_{92}}{4} + \frac{a_{93}}{4}
\]

\[
+ \frac{2a_{94}}{4} + \frac{2a_{95}}{4} + \frac{2a_{96}}{4} + \frac{2a_{97}}{4} + \frac{2a_{98}}{4} + \frac{2a_{99}}{4} + \frac{2a_{100}}{4} + \frac{2a_{101}}{4} + \frac{2a_{102}}{4} + \frac{2a_{103}}{4} + \frac{2a_{104}}{4}
\]

\[
+ \frac{30a_{102}}{4} + \frac{30a_{103}}{4} + \frac{22680a_{104}}{4} - \frac{1225}{144}.
\]

From \( \tau_2\tau_2\tau_2\tau_2\Phi(\tau_0, \tau_0) \mid_{t=0} = 0 \), we obtain

\[
0 = \frac{1225a_{1}}{144} + \frac{735a_{2}}{4} + \frac{245a_{3}}{12} + \frac{245a_{4}}{12} + \frac{245a_{5}}{12} + \frac{245a_{6}}{12} + \frac{245a_{7}}{12} + \frac{245a_{8}}{12} + \frac{245a_{9}}{288} + \frac{245a_{10}}{288} + \frac{35a_{11}}{144} + \frac{35a_{12}}{144} + \frac{35a_{13}}{144} + \frac{735a_{14}}{4} + \frac{735a_{15}}{4} + \frac{735a_{16}}{4}.
\]
A.2 Relations from the Gromov-Witten invariants of a \( \mathbb{C}P^1 \)

From the degree 0 part of \( \Phi(\tau_{0,0}, \tau_{2,1})|_{t=0} = 0 \), we obtain

\[
0 = -a_{7a} - \frac{a_{79}}{13824} - \frac{a_{75}}{13824} - \frac{a_{79}}{13824} - \frac{a_{82}}{13824} + \frac{31}{96768}. \tag{50}
\]

From the degree 0 part of \( \Phi(\tau_{0,0}, \tau_{2,3})|_{t=0} = 0 \), we obtain

\[
0 = -a_{7a} + \frac{a_9}{720} + \frac{a_{10}}{5760} + \frac{a_{37}}{2880} + \frac{a_{37}}{720} + \frac{a_{24}}{2880} + \frac{a_{28}}{288} + \frac{a_{28}}{2329} + \frac{a_{37}}{288} + \frac{a_{37}}{2329} + \frac{a_{37}}{1451520}. \tag{51}
\]

From the degree 0 part of \( \Phi(\tau_{1,0}, \tau_{1,1})|_{t=0} = 0 \), we obtain

\[
0 = -a_{79} - \frac{a_{80}}{6912} - \frac{a_{81}}{13824} - \frac{a_{82}}{6912} - \frac{a_{83}}{13824} + \frac{31}{96768}. \tag{52}
\]

From the degree 0 part of \( \Phi(\tau_{1,0}, \tau_{2,0})|_{t=0} = 0 \), we obtain

\[
0 = -a_{72} + \frac{a_{59}}{288} + \frac{a_{75}}{288} + \frac{a_{75}}{6912} + \frac{a_{79}}{6912} - \frac{a_{83}}{13824} + \frac{31}{96768}. \tag{53}
\]

From the degree 0 part of \( \Phi(\tau_{1,1}, \tau_{1,0})|_{t=0} = 0 \), we obtain

\[
0 = -a_{79} - \frac{a_{80}}{6912} - \frac{a_{81}}{13824} - \frac{a_{82}}{6912} - \frac{a_{83}}{13824} + \frac{31}{96768}. \tag{54}
\]
From the degree 0 part of $\Phi(\tau_{2,0}, \tau_{0,1})|_{t=0} = 0$, we obtain
\[
0 = -\frac{7a_8}{4060} - \frac{a_{74}}{13824} - \frac{a_{79}}{13824} - \frac{a_{82}}{13824} + \frac{31}{193536}.
\] (55)

From the degree 0 part of $\Phi(\tau_{3,0}, \tau_{0,0})|_{t=0} = 0$, we obtain
\[
0 = \frac{7a_2}{720} + \frac{a_8}{1920} + \frac{7a_{14}}{720} + \frac{7a_{23}}{2880} + \frac{a_{37}}{288} + \frac{a_{43}}{288} + \frac{a_{52}}{728} + \frac{a_{57}}{728} + \frac{a_{70}}{288} + \frac{a_{74}}{6912} + \frac{a_{79}}{6912} - \frac{290304}{290304}.
\] (56)

From the degree 0 part of $\tau_{3,1}\Phi(\tau_{0,0}, \tau_{0,0})|_{t=0} = 0$, we obtain
\[
0 = -\frac{31a_1}{9676} - \frac{7a_8}{34560} - \frac{7a_9}{34560} - \frac{7a_{10}}{34560} - \frac{7a_{21}}{34560} - \frac{7a_{22}}{34560} - \frac{7a_{31}}{138240} - \frac{a_{74}}{1875} - \frac{a_{75}}{1875} - \frac{a_{79}}{1875} - \frac{a_{82}}{1875} - \frac{a_{105}}{1875} - \frac{31}{1875}.
\] (57)

From the degree 0 part of $\tau_{4,0}\Phi(\tau_{0,0}, \tau_{0,0})|_{t=0} = 0$, we obtain
\[
0 = \frac{2320a_1}{145152} + \frac{7a_2}{576} + \frac{a_8}{1440} + \frac{a_9}{1440} + \frac{a_{10}}{1440} + \frac{7a_{14}}{576} + \frac{7a_{15}}{576} + \frac{7a_{16}}{576} + \frac{7a_{18}}{576} + \frac{7a_{19}}{576} + \frac{a_{21}}{1440} + \frac{a_{22}}{1440} + \frac{7a_{23}}{2880} + \frac{7a_{24}}{2880} + \frac{7a_{25}}{2880} + \frac{7a_{29}}{2880} + \frac{a_{31}}{a_{37}} + \frac{a_{43}}{a_{44}} + \frac{a_{50}}{a_{52}} + \frac{a_{53}}{a_{57}} + \frac{5760}{288} + \frac{288}{288} + \frac{288}{288} + \frac{288}{288} + \frac{288}{288} + \frac{288}{288} + \frac{2329}{288} + \frac{6912}{6912} + \frac{6912}{6912} - \frac{241920}{241920}.
\] (58)

From the degree 0 part of $\tau_{2,1}\Phi(\tau_{0,0}, \tau_{1,0})|_{t=0} = 0$, we obtain
\[
0 = \frac{7a_9}{23040} - \frac{7a_{10}}{11520} - \frac{7a_{11}}{46080} - \frac{7a_{13}}{46080} - \frac{a_{73}}{13824} - \frac{a_{75}}{4608} - \frac{a_{76}}{13824} - \frac{a_{77}}{4608} - \frac{a_{80}}{13824} - \frac{a_{81}}{13824} - \frac{a_{82}}{13824} - \frac{a_{83}}{13824} + \frac{31}{10752}.
\] (59)

From the degree 0 part of $\tau_{3,0}\Phi(\tau_{0,0}, \tau_{1,0})|_{t=0} = 0$, we obtain
\[
0 = \frac{7a_2}{144} + \frac{7a_3}{720} + \frac{7a_4}{720} + \frac{43a_9}{34560} + \frac{a_{10}}{480} + \frac{a_{11}}{1920} + \frac{a_{13}}{1920} + \frac{7a_{15}}{288} + \frac{7a_{16}}{144} + \frac{7a_{17}}{720} + \frac{7a_{24}}{2880} + \frac{7a_{30}}{2880} + \frac{a_{34}}{288} + \frac{a_{38}}{72} + \frac{288}{288} + \frac{288}{288} + \frac{288}{288} + \frac{288}{288} + \frac{288}{288} + \frac{288}{288} + \frac{288}{288} + \frac{288}{288} + \frac{6912}{6912} + \frac{6912}{6912} - \frac{859}{48384}.
\] (60)

From the degree 0 part of $\tau_{2,0}\Phi(\tau_{0,0}, \tau_{2,0})|_{t=0} = 0$, we obtain
\[
0 = \frac{7a_2}{96} + \frac{7a_5}{2880} + \frac{7a_6}{960} + \frac{7a_7}{960} + \frac{31a_9}{23040} + \frac{221a_{10}}{69120} + \frac{7a_{11}}{23040} + \frac{7a_{15}}{288}.
\]
\begin{align}
&+ \frac{7a_{16}}{96} + \frac{7a_{24}}{480} + \frac{a_{32}}{288} + \frac{a_{33}}{288} + \frac{a_{37}}{48} + \frac{a_{39}}{288} + \frac{a_{40}}{288} + \frac{a_{44}}{48} \\
&+ \frac{a_{37}}{288} + \frac{a_{45}}{48} + \frac{a_{59}}{48} + \frac{a_{60}}{288} + \frac{a_{62}}{288} + \frac{a_{65}}{288} + \frac{a_{66}}{288} \\
&+ \frac{a_{79}}{288} + \frac{a_{73}}{48} + \frac{a_{75}}{6912} + \frac{a_{77}}{1152} + \frac{a_{79}}{6912} + \frac{a_{81}}{2304} + \frac{a_{82}}{859} \\
&+ \frac{a_{83}}{6912} - \frac{48384}{48384}. \quad (61)
\end{align}

From the degree 0 part of $\tau_{2,1}(\tau_{1,0}, \tau_{0,0})|_{t=0} = 0$, we obtain
\begin{align}
0 &= -\frac{7a_{8}}{11520} - \frac{7a_{12}}{46080} - \frac{7a_{13}}{46080} - \frac{a_{72}}{13824} - \frac{a_{74}}{4608} - \frac{a_{76}}{13824} - \frac{a_{78}}{13824} \\
&\quad - \frac{a_{79}}{4608} - \frac{a_{80}}{13824} - \frac{a_{81}}{13824} - \frac{a_{82}}{4608} - \frac{a_{83}}{13824} + \frac{31}{16128}. \quad (62)
\end{align}

From the degree 0 part of $\tau_{2,0}(\tau_{1,0}, \tau_{1,0})|_{t=0} = 0$, we obtain
\begin{align}
0 &= \frac{7a_{2}}{48} + \frac{7a_{3}}{480} + \frac{7a_{4}}{240} + \frac{7a_{5}}{480} + \frac{7a_{6}}{240} + \frac{7a_{7}}{480} + \frac{a_{32}}{144} + \frac{a_{33}}{96} \\
&\quad + \frac{a_{34}}{96} + \frac{a_{35}}{288} + \frac{a_{36}}{288} + \frac{a_{37}}{288} + \frac{a_{38}}{96} + \frac{a_{39}}{96} + \frac{a_{40}}{144} + \frac{a_{57}}{24} \\
&\quad + \frac{a_{58}}{96} + \frac{a_{59}}{96} + \frac{a_{60}}{144} + \frac{a_{61}}{96} + \frac{a_{62}}{96} + \frac{a_{63}}{288} + \frac{a_{64}}{288} + \frac{a_{65}}{288} \\
&\quad + \frac{a_{66}}{144} + \frac{a_{67}}{96} + \frac{a_{68}}{96} + \frac{a_{69}}{24} + \frac{a_{70}}{288} + \frac{5a_{79}}{3536} + \frac{a_{80}}{576} + \frac{a_{81}}{576} \\
&\quad + \frac{a_{82}}{1152} + \frac{5a_{83}}{6912} - \frac{859}{48384}. \quad (63)
\end{align}

From the degree 1 part of $\tau_{0,1}(\tau_{0,0}, \tau_{0,0})|_{t=0} = 0$, we obtain
\begin{align}
0 &= \frac{a_{1}}{30240} - \frac{2880}{2880} - \frac{a_{9}}{11520} - \frac{a_{9}}{11520} - \frac{a_{10}}{11520} - \frac{a_{10}}{11520} + \frac{7a_{14}}{2880} + \frac{a_{15}}{192} + \frac{7a_{16}}{2880} \\
&\quad + \frac{a_{18}}{2880} + \frac{a_{19}}{192} - \frac{a_{21}}{11520} - \frac{a_{21}}{11520} + \frac{a_{22}}{960} + \frac{a_{23}}{960} + \frac{a_{24}}{960} + \frac{a_{25}}{960} + \frac{a_{29}}{960} \\
&\quad - \frac{a_{31}}{46080} - \frac{a_{37}}{288} - \frac{a_{43}}{288} - \frac{a_{44}}{288} - \frac{a_{50}}{288} - \frac{a_{52}}{288} - \frac{a_{57}}{288} - \frac{a_{57}}{288} \\
&\quad + \frac{a_{70}}{13824} + \frac{a_{74}}{13824} + \frac{a_{75}}{13824} + \frac{a_{79}}{13824} - \frac{a_{82}}{6} + \frac{a_{84}}{6} + \frac{a_{88}}{6} + \frac{a_{89}}{6} \\
&\quad + \frac{a_{90}}{6} + \frac{a_{92}}{12} - \frac{a_{103}}{13824} - \frac{a_{104}}{13824} - \frac{a_{105}}{5040} - \frac{1}{5040}. \quad (64)
\end{align}

From the degree 1 part of $\tau_{4,1}(\tau_{0,0}, \tau_{0,1})|_{t=0} = 0$, we obtain
\begin{align}
0 &= \frac{11a_{1}}{60480} + \frac{a_{2}}{960} + \frac{a_{8}}{34560} + \frac{a_{9}}{34560} - \frac{a_{10}}{11520} + \frac{19a_{14}}{2880} + \frac{a_{15}}{960} + \frac{a_{16}}{620} \\
&\quad + \frac{a_{18}}{320} - \frac{a_{19}}{960} + \frac{a_{21}}{34560} + \frac{a_{22}}{11520} + \frac{7a_{23}}{2880} + \frac{a_{24}}{960} + \frac{a_{25}}{960} + \frac{a_{29}}{960} + \frac{576}{960} \\
&\quad - \frac{46080}{a_{31}} - \frac{a_{37}}{288} - \frac{a_{44}}{288} - \frac{a_{50}}{288} - \frac{a_{52}}{288} - \frac{a_{57}}{288} - \frac{a_{57}}{288} + \frac{a_{70}}{288} \\
&\quad + \frac{a_{74}}{13824} + \frac{a_{75}}{13824} + \frac{a_{79}}{13824} + \frac{a_{82}}{13824} + \frac{a_{84}}{6} + \frac{a_{88}}{6} + \frac{a_{89}}{6} + \frac{a_{90}}{6} \\
&\quad + \frac{a_{92}}{8} - \frac{a_{103}}{24} - \frac{a_{104}}{13824} + \frac{a_{105}}{13824} - \frac{11}{20160}. \quad (65)
\end{align}
From the degree 1 part of $\tau_{5,0}(\tau_{0,0}, \tau_{0,1})|_{t=0} = 0$, we obtain

$$0 = -\frac{779 a_1}{1451520} + \frac{a_2}{120} - \frac{a_3}{34560} - \frac{13 a_8}{34560} - \frac{13 a_9}{34560}  - \frac{13 a_{10}}{60} - \frac{6 a_{14}}{120} + \frac{a_{15}}{120}$$

$$+ \frac{13 a_{16}}{720} + \frac{13 a_{18}}{720} + \frac{a_{19}}{120} - \frac{13 a_{21}}{34560} - \frac{13 a_{22}}{34560} - \frac{a_{23}}{120} + \frac{a_{24}}{180} + \frac{a_{25}}{180} - \frac{a_{29}}{144} + \frac{a_{31}}{144} - \frac{a_{37}}{144} - \frac{a_{43}}{144} - \frac{a_{44}}{144} - \frac{a_{50}}{144} - \frac{a_{52}}{144} - \frac{a_{53}}{144} - \frac{a_{57}}{144} + \frac{13824}{a_{74}} + \frac{13824}{a_{75}} + \frac{13824}{a_{79}} - \frac{a_{82}}{144} + \frac{a_{84}}{144} - \frac{a_{88}}{144} - \frac{a_{89}}{144} - \frac{13824}{a_{74}} + \frac{13824}{a_{75}} + \frac{13824}{a_{79}} + \frac{2}{3} + \frac{2}{6} + \frac{a_{90}}{2} + \frac{a_{92}}{3} + \frac{a_{93}}{12} - \frac{a_{103}}{6} + \frac{16 a_{104}}{13824} + \frac{a_{105}}{13824} + \frac{61}{34560}. \quad (66)$$

From the degree 1 part of $\tau_{4,1}(\tau_{0,0}, \tau_{1,0})|_{t=0} = 0$, we obtain

$$0 = \frac{a_2}{144} + \frac{a_3}{240} - \frac{a_4}{720} - \frac{23 a_9}{69120} - \frac{a_{10}}{3840} - \frac{a_{11}}{15360} - \frac{a_{13}}{15360}$$

$$+ \frac{7 a_{15}}{480} + \frac{a_{16}}{144} + \frac{a_{17}}{720} + \frac{a_{24}}{240} + \frac{a_{28}}{960} + \frac{a_{30}}{960} + \frac{a_{34}}{288} + \frac{a_{37}}{72} + \frac{a_{38}}{a_{42}} + \frac{a_{44}}{a_{45}} + \frac{a_{53}}{a_{54}} + \frac{a_{55}}{a_{57}}$$

$$- \frac{288}{288} + \frac{72}{72} + \frac{288}{288} - \frac{72}{72} + \frac{288}{288} - \frac{288}{288} - \frac{72}{72}$$

$$- \frac{288}{a_{70}} + \frac{a_{73}}{a_{75}} + \frac{a_{77}}{4608} + \frac{a_{79}}{4608} + \frac{a_{80}}{a_{81}} + \frac{a_{82}}{a_{83}} + \frac{5 a_{84}}{6} + \frac{a_{85}}{6} + \frac{5 a_{90}}{6}$$

$$+ \frac{a_9}{6} + \frac{5 a_{92}}{12} + \frac{a_{93}}{12} + \frac{a_{102}}{6} + \frac{5 a_{103}}{6} + \frac{1}{6} + 48 a_{104} - \frac{41}{161280}. \quad (67)$$

From the degree 1 part of $\tau_{3,1}(\tau_{0,0}, \tau_{1,1})|_{t=0} = 0$, we obtain

$$0 = -\frac{31 a_1}{96768} + \frac{7 a_3}{720} + \frac{7 a_4}{720} - \frac{7 a_8}{34560} - \frac{a_9}{13824} - \frac{a_{11}}{138240} + \frac{7 a_{13}}{138240} + \frac{7 a_{15}}{138240} + \frac{7 a_{17}}{138240} + \frac{7 a_{21}}{138240} + \frac{7 a_{22}}{138240} + \frac{7 a_{28}}{138240} + \frac{7 a_{30}}{138240} + \frac{7 a_{31}}{138240} + \frac{7 a_{37}}{138240} + \frac{7 a_{42}}{138240} + \frac{7 a_{44}}{138240} + \frac{7 a_{45}}{138240} + \frac{7 a_{53}}{138240} + \frac{7 a_{54}}{138240} + \frac{7 a_{55}}{138240} + \frac{7 a_{57}}{138240}$$

$$+ \frac{288}{288} - \frac{34560}{34560} - \frac{34560}{34560} + \frac{2880}{2880} + \frac{2880}{2880} - \frac{138240}{138240} - \frac{288}{288}$$

$$- \frac{34560}{a_{68}} + \frac{a_{72}}{a_{70}} + \frac{a_{73}}{a_{74}} + \frac{a_{75}}{a_{76}} + \frac{a_{77}}{a_{79}} + \frac{a_{79}}{6912} + \frac{a_{80}}{13824} + \frac{a_{82}}{13824} + \frac{a_{83}}{13824} + \frac{a_{84}}{13824} + \frac{3}{3} + \frac{12}{12} + \frac{4}{4}$$

$$+ \frac{a_{90}}{24} - \frac{a_{95}}{24} - \frac{a_{97}}{12} - \frac{a_{102}}{6} - \frac{2 a_{103}}{3} + \frac{32 a_{104}}{13824} - \frac{1}{4608}. \quad (68)$$

From the degree 1 part of $\tau_{4,0}(\tau_{0,0}, \tau_{1,1})|_{t=0} = 0$, we obtain

$$0 = \frac{2329 a_1}{1451520} - \frac{a_2}{576} - \frac{a_3}{30} - \frac{a_4}{30} + \frac{a_8}{1440} + \frac{19 a_9}{69120} - \frac{a_{10}}{2304} - \frac{a_{11}}{46080} - \frac{a_{13}}{46080} + \frac{7 a_{14}}{46080} + \frac{11 a_{15}}{46080} + \frac{7 a_{16}}{46080} + \frac{11 a_{17}}{46080} + \frac{7 a_{18}}{46080} + \frac{7 a_{19}}{46080} + \frac{7 a_{23}}{1440} + \frac{192}{720} + \frac{2880}{2880} + \frac{120}{2880} + \frac{2880}{120} + \frac{7560}{7560}$$

$$+ \frac{a_{24}}{288} + \frac{a_{25}}{288} + \frac{a_{28}}{720} + \frac{a_{29}}{720} + \frac{a_{30}}{720} + \frac{a_{31}}{720} + \frac{a_{32}}{288} + \frac{a_{42}}{288} + \frac{a_{43}}{288} + \frac{a_{44}}{288} + \frac{a_{45}}{288} + \frac{a_{50}}{288} \quad (68)$$
\[ a_{0} = \frac{7a_{53} - a_{54} - a_{55} - 7a_{57} - a_{58} - a_{61} + a_{70}}{288 - 288 - 144 - 144 - 288 - 144 - 144 + 288} + \frac{a_{73} + a_{74} + 5a_{75}}{13824 + 6912 + 13824 + 13824 + 13824 + 13824 + 13824 + 13824} + \frac{a_{82}}{4608} + a_{83} + 3a_{84} - \frac{a_{85} + 3a_{90}}{2 - 6 + 2 - 6 + 11a_{92} + a_{95}} + \frac{a_{97} - a_{102} - 5a_{103}}{6 - 6 + 72a_{104} + a_{105} - \frac{131}{6912}}. \] (69)

From the degree 1 part of \( \tau_{2,1}\Phi(\tau_{0,0}, \tau_{2,0})|_{\ell=0} = 0 \), we obtain
\[ 0 = \frac{31a_{1}}{96768} + \frac{1}{288 - 288 - 960 - 2880 + 2880 + 7a_{6} - 7a_{7} - 7a_{8} - a_{9} - a_{10}} + \frac{7a_{11}}{6912 - 288 + 288 + 34560 + 34560 + 96 + 138240 - 288} + \frac{7a_{15}}{a_{33} + a_{34} + a_{37} + a_{39} - a_{40} + a_{42} - a_{44} - a_{47}} + \frac{7a_{16}}{a_{53} + a_{55} + a_{57} + a_{59} - a_{60} + a_{62} - a_{65} + a_{66}} + \frac{7a_{21} + 7a_{22} + 7a_{24} + 7a_{31} - 7a_{32}}{36 + 144 + 36 + 288 - 288 - 288 + 288 + 288} + \frac{7a_{75}}{a_{67} + a_{68} + a_{70} - a_{73} + a_{74} + 7a_{75} + a_{77} + 7a_{79}} + \frac{7a_{75}}{a_{81} + a_{82} - \frac{5a_{84}}{3} + a_{87} + a_{88} + a_{89} + a_{90} + a_{92} + a_{93} - a_{100}} + \frac{5a_{103}}{6 - 3 + 112a_{104} + a_{105} - 53 - 48384}. \] (70)

From the degree 1 part of \( \tau_{2,1}\Phi(\tau_{0,0}, \tau_{2,1})|_{\ell=0} = 0 \), we obtain
\[ 0 = \frac{a_{2}}{288} + \frac{7a_{5}}{2880} + \frac{7a_{6}}{960} + \frac{7a_{7}}{960} - \frac{a_{9} - a_{10}}{13824 - 34560 + 46080 - 46080} + \frac{7a_{11}}{40 + 144 + 144 + 288 - 288 - 288 - 288 + 288 + 288} + \frac{7a_{15}}{a_{44} + a_{47} + a_{53} + a_{57} + a_{66} + a_{70} + a_{73}} + \frac{7a_{16}}{a_{75} + a_{76} + a_{77} + a_{79} + a_{80} + a_{81} + a_{82} + a_{83}} + \frac{7a_{21} + 7a_{22} + 7a_{24} + 7a_{31} - 7a_{32}}{3 + 3 + 4 + 24 + 24 + 12 + 12 - 6} + \frac{a_{101}}{6} - a_{103} + 48a_{104} - \frac{1}{4608}. \] (71)

From the degree 1 part of \( \tau_{3,0}\Phi(\tau_{0,0}, \tau_{2,1})|_{\ell=0} = 0 \), we obtain
\[ 0 = \frac{31a_{1}}{96768} + \frac{7a_{3}}{720} + \frac{7a_{4}}{720} + \frac{a_{7}}{40} - \frac{7a_{8}}{34560} - \frac{23a_{9}}{69120} - \frac{13a_{10}}{23040} + \frac{7a_{11}}{1920 + 1920 + 288 + 144 + 720 + 34560 + 34560 + 34560 + 72} + \frac{7a_{25}}{2880 + 2880 - 138240 + 144 + 144 + 288 - 288 - 288 - 288} + \frac{7a_{30}}{a_{40} + a_{42} + 5a_{44} + a_{45} + 5a_{53} + a_{54} + 5a_{55} + 5a_{57}} + \frac{a_{58}}{a_{60} + a_{61} + a_{65} + a_{67} + a_{68} + a_{70} + a_{73}} + \frac{a_{73}}{288 - 144 - 288 - 144 - 144 + 288 - 144 - 72 - 6912}. \]
From the degree 1 part of $\tau \Phi(\tau_0, \tau_3, 1)|_{t=0} = 0$, we obtain

$$0 = \frac{7a_2}{720} + \frac{a_5}{120} - \frac{a_6}{40} - \frac{a_9}{3072} - \frac{113a_{10}}{138240} + \frac{7a_{11}}{46080} + \frac{7a_{13}}{46080} + \frac{31a_{15}}{1440}.$$

From the degree 1 part of $\tau \Phi(\tau_0, \tau_3, 1)|_{t=0} = 0$, we obtain

$$0 = \frac{7a_2}{720} + \frac{a_5}{120} - \frac{a_6}{40} - \frac{a_9}{3072} - \frac{113a_{10}}{138240} + \frac{7a_{11}}{46080} + \frac{7a_{13}}{46080} + \frac{31a_{15}}{1440}.$$

From the degree 1 part of $\tau_1 \Phi(\tau_0, \tau_0)|_{t=0} = 0$, we obtain

$$0 = \frac{11a_{11}}{60480} + \frac{a_2}{960} - \frac{a_8}{11520} - \frac{a_9}{11520} + \frac{a_{10}}{11520} + \frac{a_{15}}{34560} + \frac{19a_{16}}{320} + \frac{a_{14}}{576} + \frac{5a_{15}}{2880} + \frac{a_{29}}{2880}.$$

From the degree 1 part of $\tau_5 \Phi(\tau_0, \tau_0)|_{t=0} = 0$, we obtain

$$0 = -\frac{779a_{11}}{1451520} + \frac{a_2}{120} - \frac{13a_8}{34560} - \frac{13a_9}{34560} - \frac{13a_{10}}{34560} - \frac{13a_{14}}{720} - \frac{a_{15}}{144}.$$
\[ \begin{align*}
- a_{16} & + \frac{13 a_{18}}{60} + \frac{a_{19}}{120} - \frac{13 a_{21}}{34560} - \frac{13 a_{22}}{34560} + \frac{a_{23}}{180} - \frac{a_{24}}{120} + \frac{a_{25}}{180} \\
- a_{29} & + \frac{a_{37}}{13 a_{31}} - \frac{a_{37}}{a_{43}} + \frac{a_{44}}{a_{50}} - \frac{a_{52}}{a_{53}} \\
- \frac{720}{a_{57}} & + \frac{a_{74}}{a_{75}} + \frac{a_{75}}{a_{79}} + \frac{a_{82}}{a_{84}} + \frac{a_{84}}{a_{88}} + \frac{a_{88}}{a_{89}} \\
& + \frac{a_{90}}{6} + \frac{a_{92}}{3} + \frac{a_{94}}{12} - \frac{a_{103}}{6} + \frac{16 a_{104}}{13824} + \frac{a_{105}}{13824} + \frac{779}{6} + \frac{241920}{241920}. 
\end{align*} \] (76)

From the degree 1 part of \( \tau_{3,1} \Phi(\tau_{0,1}, \tau_{0,1})|_{t=0} = 0 \), we obtain
\[ \begin{align*}
0 & = -\frac{31 a_{1}}{96768} + \frac{7 a_{17}}{1440} - \frac{7 a_{21}}{34560} - \frac{7 a_{22}}{34560} + \frac{7 a_{27}}{5760} + \frac{7 a_{28}}{5760} + \frac{7 a_{30}}{2880} \\
& + \frac{a_{37}}{a_{43}} + \frac{a_{44}}{a_{50}} + \frac{a_{52}}{a_{53}} + \frac{a_{57}}{a_{60}} - \frac{a_{70}}{a_{74}} + \frac{a_{75}}{a_{79}} + \frac{a_{82}}{a_{84}} + \frac{a_{84}}{a_{89}} + \frac{a_{89}}{a_{90}} \\
& + \frac{a_{92}}{8} - \frac{a_{95}}{24} - \frac{a_{103}}{6} + \frac{8 a_{104}}{13824}. 
\end{align*} \] (77)

From the degree 1 part of \( \tau_{4,0} \Phi(\tau_{0,1}, \tau_{0,1})|_{t=0} = 0 \), we obtain
\[ \begin{align*}
0 & = \frac{2329 a_{1}}{1451520} + \frac{a_{2}}{192} - \frac{a_{8}}{17280} - \frac{a_{9}}{17280} - \frac{a_{10}}{17280} + \frac{a_{10}}{576} + \frac{a_{14}}{576} + \frac{a_{15}}{576} + \frac{a_{16}}{576} \\
& + \frac{a_{17}}{60} + \frac{7 a_{18}}{576} + \frac{7 a_{19}}{1440} + \frac{a_{21}}{1440} + \frac{a_{22}}{2880} + \frac{7 a_{23}}{2880} + \frac{7 a_{24}}{2880} + \frac{7 a_{25}}{2880} \\
& - \frac{a_{27}}{a_{28}} + \frac{7 a_{29}}{a_{30}} + \frac{a_{31}}{a_{37}} + \frac{a_{37}}{a_{43}} + \frac{a_{44}}{a_{50}} - \frac{a_{52}}{a_{53}} - \frac{a_{57}}{a_{60}} - \frac{a_{70}}{a_{74}} + \frac{a_{75}}{a_{79}} + \frac{a_{82}}{a_{84}} + \frac{a_{84}}{a_{89}} + \frac{a_{89}}{a_{90}} \\
& + \frac{a_{92}}{3} + \frac{a_{95}}{3} + \frac{a_{103}}{3} + \frac{24 a_{104}}{6} + \frac{a_{105}}{6} - \frac{25}{6} + \frac{6912}{6} - \frac{16128}{6}. 
\end{align*} \] (78)

From the degree 1 part of \( \tau_{3,1} \Phi(\tau_{0,1}, \tau_{0,1})|_{t=0} = 0 \), we obtain
\[ \begin{align*}
0 & = -\frac{a_{9}}{4608} + \frac{7 a_{10}}{34560} + \frac{7 a_{11}}{138240} + \frac{a_{12}}{138240} + \frac{5 a_{15}}{1440} + \frac{7 a_{16}}{1440} + \frac{7 a_{17}}{14400} \\
& + \frac{7 a_{24}}{720} + \frac{7 a_{28}}{2880} + \frac{7 a_{30}}{2880} - \frac{a_{34}}{288} - \frac{a_{37}}{288} - \frac{a_{38}}{288} + \frac{a_{38}}{72} + \frac{a_{38}}{72} - \frac{288}{72} \\
& + \frac{a_{53}}{288} - \frac{a_{54}}{288} - \frac{a_{57}}{288} + \frac{a_{57}}{288} + \frac{a_{72}}{288} + \frac{a_{72}}{13824} + \frac{4068}{288} \\
& - \frac{a_{76}}{288} - \frac{a_{77}}{288} - \frac{a_{79}}{288} - \frac{a_{80}}{288} - \frac{a_{81}}{288} + \frac{a_{82}}{a_{84}} + \frac{a_{84}}{a_{89}} + \frac{a_{89}}{a_{90}} \\
& + \frac{a_{92}}{6} + \frac{5 a_{90}}{12} + \frac{a_{91}}{12} + \frac{5 a_{92}}{8} + \frac{a_{93}}{8} - \frac{a_{94}}{6} - \frac{a_{95}}{12} - \frac{a_{102}}{6} \\
& - \frac{5 a_{103}}{6} + 48 a_{104} - \frac{3}{10752}. 
\end{align*} \] (79)

From the degree 1 part of \( \tau_{4,0} \Phi(\tau_{0,1}, \tau_{1,0})|_{t=0} = 0 \), we obtain
\[ \begin{align*}
0 & = \frac{a_{2}}{72} + \frac{a_{3}}{360} + \frac{a_{4}}{360} - \frac{67 a_{9}}{69120} - \frac{13 a_{10}}{11520} - \frac{13 a_{11}}{46080} - \frac{13 a_{13}}{46080} - \frac{a_{15}}{60} \\
\end{align*} \]

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From the degree 1 part of $\tau_{2,1}\Phi(\tau_{0,1},\tau_{1,1})|_{t=0} = 0$, we obtain

$$0 = \frac{7a_7}{960} + \frac{a_9}{13824} + \frac{7a_{11}}{960} - \frac{7a_{13}}{960} - \frac{a_{15}}{576} - \frac{a_{34}}{288} + \frac{a_{37}}{288} + \frac{a_{40}}{288}$$

From the degree 1 part of $\tau_{3,0}\Phi(\tau_{0,1},\tau_{1,1})|_{t=0} = 0$, we obtain

$$0 = \frac{31a_1}{96768} + \frac{7a_3}{720} + \frac{7a_4}{720} + \frac{10a_7}{40} - \frac{7a_8}{43560} - \frac{a_9}{3456} + \frac{7a_{10}}{3456}$$

From the degree 1 part of $\tau_{2,1}\Phi(\tau_{0,1},\tau_{2,0})|_{t=0} = 0$, we obtain

$$0 = \frac{a_2}{96} + \frac{7a_5}{5760} + \frac{7a_6}{960} - \frac{7a_7}{960} - \frac{a_9}{3456} + \frac{a_10}{11520} + \frac{7a_{11}}{23040} + \frac{a_{15}}{48}$$
From the degree 1 part of $\tau_{3,0}\Phi(\tau_{0,1}, \tau_{2,0})|_{t=0} = 0$, we obtain

\begin{equation}
0 = \frac{31a_1}{96768} + \frac{7a_2}{144} + \frac{7a_3}{40} + \frac{a_7}{34560} + \frac{7a_8}{2560} + \frac{3a_9}{13824} - \frac{a_{10}}{288} - \frac{3a_{11}}{13824} - \frac{11a_{15}}{720} - \frac{a_{16}}{8} + \frac{7a_{21}}{34560} + \frac{7a_{22}}{43424} + \frac{7a_{31}}{138240} - \frac{a_{32}}{144} - \frac{a_{33}}{144} + \frac{a_{34}}{48} - \frac{7a_{37}}{144} + \frac{a_{40}}{144} + \frac{a_{42}}{144} - \frac{7a_{44}}{144} + \frac{a_{47}}{7a_{53}} + \frac{a_{55}}{7a_{57}} + \frac{a_{60}}{144} + \frac{a_{61}}{144} - \frac{a_{62}}{144} + \frac{a_{63}}{144} - \frac{a_{65}}{13824} + \frac{a_{67}}{144} - \frac{a_{68}}{144} - \frac{a_{73}}{3456} + \frac{a_{74}}{13824} + \frac{7a_{75}}{13824} + \frac{a_{77}}{3456} + \frac{7a_{79}}{13824} + \frac{7a_{82}}{13824} - \frac{a_{83}}{3} + \frac{13a_{84}}{6} - \frac{a_{87}}{13824} + \frac{13a_{84}}{15120}.
\end{equation}

From the degree 1 part of $\tau_{2,0}\Phi(\tau_{0,1}, \tau_{2,1})|_{t=0} = 0$, we obtain

\begin{equation}
0 = \frac{a_2}{96} + \frac{7a_5}{2880} + \frac{7a_6}{960} + \frac{7a_7}{960} + \frac{a_9}{8640} + \frac{a_{10}}{2560} + \frac{7a_{11}}{46080} - \frac{7a_{13}}{46080} + \frac{5a_{15}}{288} + \frac{a_{16}}{24} + \frac{7a_{24}}{480} + \frac{a_{32}}{288} + \frac{a_{33}}{288} + \frac{a_{37}}{72} + \frac{a_{39}}{288} + \frac{a_{40}}{288} + \frac{a_{44}}{288} + \frac{5a_{57}}{288} + \frac{a_{59}}{288} - \frac{a_{60}}{288} - \frac{a_{66}}{288} + \frac{a_{67}}{288} + \frac{a_{70}}{144} + \frac{a_{73}}{13824} + \frac{5a_{75}}{13824} - \frac{a_{76}}{13824} + \frac{a_{77}}{13824} + \frac{a_{79}}{13824} - \frac{a_{80}}{13824} + \frac{a_{81}}{13824} + \frac{a_{82}}{4608} + \frac{a_{83}}{13824} + \frac{7a_{84}}{6} + \frac{7a_{89}}{12} + \frac{7a_{92}}{6} - \frac{a_{96}}{24} + \frac{a_{99}}{12} - \frac{a_{98}}{6} - \frac{a_{99}}{6} - \frac{a_{100}}{6} + \frac{2a_{102}}{3} - \frac{5a_{103}}{3} + \frac{208a_{104}}{13824} + \frac{139}{15120}.
\end{equation}

From the degree 1 part of $\tau_{2,0}\Phi(\tau_{0,1}, \tau_{3,0})|_{t=0} = 0$, we obtain

\begin{equation}
0 = \frac{7a_2}{144} + \frac{a_5}{120} - \frac{35a_9}{27648} - \frac{503a_{10}}{138240} + \frac{7a_{11}}{46080} + \frac{7a_{13}}{46080} - \frac{a_{15}}{60} - \frac{a_{16}}{8} + \frac{43a_{24}}{720} + \frac{a_{32}}{144} - \frac{a_{33}}{144} + \frac{a_{37}}{36} + \frac{a_{39}}{144} + \frac{a_{44}}{24} + \frac{a_{47}}{144} - \frac{a_{53}}{18} - \frac{a_{57}}{24} + \frac{a_{62}}{144} - \frac{a_{65}}{144} - \frac{a_{66}}{72} + \frac{a_{70}}{13824} - \frac{a_{73}}{3456} \frac{a_{76}}{13824} + \frac{a_{77}}{13824} + \frac{a_{79}}{13824} + \frac{a_{80}}{13824} + \frac{a_{81}}{13824} + \frac{a_{82}}{3456} + \frac{4a_{90}}{17a_{92}} + \frac{a_{96}}{13824} + \frac{a_{98}}{6} + \frac{a_{99}}{3} - \frac{a_{101}}{6} - \frac{a_{103}}{208a_{104}} + \frac{1793}{241920}.
\end{equation}
From the degree 1 part of \( \tau_{1,0} \Phi(\tau_{1,0}, \tau_{0,0})|_{t=0} = 0 \), we obtain

\[
0 = \frac{-a_2}{144} + \frac{a_5}{240} - \frac{a_6}{720} - \frac{a_8}{3840} - \frac{a_{12}}{15360} - \frac{a_{13}}{15360} + \frac{a_{14}}{144} + \frac{a_{17}}{720} + \frac{a_{23}}{240} + \frac{a_{27}}{960} + \frac{a_{28}}{960} + \frac{a_{30}}{288} - \frac{a_{33}}{72} + \frac{a_{37}}{a_{39}} - \frac{a_{41}}{a_{43}} - \frac{a_{45}}{a_{52}} - \frac{a_{54}}{a_{56}} - \frac{a_{57}}{a_{59}} - \frac{288}{288} - \frac{288}{72} - \frac{288}{72} - \frac{288}{72} - \frac{288}{72} - \frac{a_{62}}{a_{67}} + \frac{a_{70}}{a_{72}} + \frac{a_{74}}{a_{76}} + \frac{a_{78}}{a_{80}} + \frac{a_{81}}{a_{82}} + \frac{a_{83}}{a_{84}} + \frac{a_{86}}{a_{88}} + \frac{5a_{89}}{6} + \frac{a_{91}}{6} + \frac{5}{16128}.
\]

From the degree 1 part of \( \tau_{3,1} \Phi(\tau_{1,0}, \tau_{0,1})|_{t=0} = 0 \), we obtain

\[
0 = \frac{7a_4}{34560} + \frac{7a_{12}}{138240} + \frac{7a_{13}}{138240} + \frac{7a_{14}}{288} + \frac{7a_{17}}{1440} + \frac{7a_{23}}{720} + \frac{7a_{27}}{2880} + \frac{7a_{28}}{2880} + \frac{7a_{30}}{a_{33}} + \frac{7a_{37}}{a_{39}} - \frac{7a_{52}}{a_{54}} - \frac{7a_{55}}{a_{56}} - \frac{7a_{57}}{a_{59}} + \frac{2880}{288} - \frac{288}{72} - \frac{288}{72} - \frac{288}{72} - \frac{288}{72} - \frac{a_{59}}{a_{62}} + \frac{a_{67}}{a_{70}} + \frac{a_{72}}{a_{74}} + \frac{a_{76}}{a_{78}} + \frac{a_{80}}{a_{81}} + \frac{a_{83}}{a_{84}} + \frac{5a_{86}}{6} + \frac{a_{86}}{a_{88}} + \frac{5a_{89}}{6} + \frac{5a_{92}}{12} + \frac{5a_{93}}{6} + \frac{5a_{94}}{6} + \frac{5a_{93}}{12} - \frac{5a_{101}}{6} + \frac{a_{101}}{6} + \frac{5a_{103}}{6} + \frac{48a_{104}}{16128}.
\]

From the degree 1 part of \( \tau_{4,0} \Phi(\tau_{1,0}, \tau_{0,1})|_{t=0} = 0 \), we obtain

\[
0 = \frac{a_2}{72} + \frac{a_5}{360} + \frac{a_6}{360} - \frac{13a_8}{11520} + \frac{13a_{12}}{406080} + \frac{13a_{13}}{406080} + \frac{11a_{14}}{144} - \frac{11a_{17}}{720} + \frac{30}{a_{23}} + \frac{120}{a_{27}} + \frac{120}{a_{28}} + \frac{120}{a_{30}} + \frac{120}{a_{33}} + \frac{120}{a_{37}} + \frac{120}{a_{39}} + \frac{a_{41}}{a_{43}} + \frac{a_{45}}{a_{52}} + \frac{a_{54}}{a_{56}} + \frac{a_{57}}{a_{59}} + \frac{a_{62}}{a_{72}} + \frac{a_{74}}{a_{76}} + \frac{a_{78}}{a_{80}} + \frac{a_{81}}{a_{82}} + \frac{a_{83}}{a_{84}} + \frac{5a_{86}}{6} + \frac{a_{86}}{a_{88}} + \frac{5a_{89}}{6} + \frac{a_{91}}{6} + \frac{911}{241920}.
\]

From the degree 1 part of \( \tau_{3,1} \Phi(\tau_{1,0}, \tau_{1,0})|_{t=0} = 0 \), we obtain

\[
0 = \frac{7a_2}{144} + \frac{a_5}{96} - \frac{7a_{12}}{720} + \frac{a_5}{96} - \frac{7a_{16}}{1440} + \frac{7a_{17}}{144} + \frac{a_{33}}{a_{34}} - \frac{a_{35}}{a_{36}} - \frac{a_{37}}{a_{38}} - \frac{a_{39}}{a_{40}} + \frac{96}{96} + \frac{288}{288} + \frac{24}{24} + \frac{144}{144} + \frac{5a_{58}}{57} - \frac{a_{59}}{a_{60}} + \frac{a_{61}}{a_{62}} - \frac{5a_{63}}{a_{64}} + \frac{24}{24} + \frac{96}{96} + \frac{144}{144}.
\]

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From the degree 1 part of $\tau_{2,1} \Phi(\tau_{1,0}, \tau_{1,1})|_{t=0} = 0$, we obtain

\[
0 = \frac{7a_3}{480} + \frac{7a_4}{240} + \frac{a_5}{288} + \frac{7a_7}{480} - \frac{7a_8}{11520} - \frac{7a_{12}}{46080} - \frac{7a_{13}}{46080} - \frac{a_{32}}{144} \\
- \frac{a_{33}}{a_{34}} - \frac{a_{35}}{a_{36}} - \frac{a_{37}}{a_{38}} + \frac{a_{39}}{a_{40}} + \frac{288}{72} + \frac{288}{96} - \frac{288}{144} + \frac{96}{6} + \frac{96}{6} + \frac{288}{24} \\
- \frac{a_{72}}{a_{74}} - \frac{a_{76}}{a_{78}} - \frac{5a_{79}}{a_{80}} - \frac{a_{81}}{a_{82}} + \frac{a_{83}}{4608} + \frac{a_{84}}{6} + \frac{a_{85}}{8} + \frac{a_{86}}{6} + \frac{a_{87}}{8} + \frac{a_{88}}{12} - \frac{a_{97}}{4} + \frac{a_{99}}{a_{100}} - \frac{a_{101}}{2} - \frac{2a_{102}}{3} + \frac{5a_{103}}{2} + 144a_{104} - \frac{1}{2304}.
\]

From the degree 1 part of $\tau_{3,0} \Phi(\tau_{1,0}, \tau_{1,1})|_{t=0} = 0$, we obtain

\[
0 = \frac{7a_2}{144} - \frac{43a_3}{720} - \frac{a_4}{10} + \frac{a_5}{72} - \frac{7a_6}{720} - \frac{7a_7}{20} + \frac{a_8}{480} \\
+ \frac{a_{12}}{1920} + \frac{a_{13}}{1920} - \frac{7a_{14}}{144} + \frac{7a_{17}}{720} - \frac{7a_{23}}{720} + \frac{7a_{27}}{2880} + \frac{7a_{28}}{2880} + \frac{7a_{30}}{2880} \\
- \frac{a_{32}}{72} - \frac{5a_{33}}{a_{34}} - \frac{a_{35}}{a_{36}} + \frac{a_{37}}{a_{38}} - \frac{5a_{39}}{a_{40}} + \frac{288}{48} + \frac{288}{144} + \frac{288}{72} + \frac{288}{72} + \frac{288}{72} + \frac{288}{72} \\
- \frac{a_{58}}{a_{59}} - \frac{a_{60}}{a_{61}} - \frac{5a_{62}}{a_{63}} + \frac{a_{64}}{a_{65}} + \frac{a_{67}}{a_{70}} - \frac{a_{72}}{a_{74}} - \frac{a_{76}}{a_{78}} + \frac{a_{79}}{a_{80}} + \frac{a_{81}}{1728} + \frac{a_{82}}{1728} + \frac{a_{83}}{1728} - \frac{5a_{85}}{3} + \frac{a_{86}}{3} - \frac{a_{87}}{3} + \frac{a_{88}}{3} + \frac{a_{89}}{2} + \frac{a_{90}}{2} + \frac{a_{91}}{2} + \frac{a_{92}}{2} + \frac{a_{93}}{2} + \frac{a_{94}}{2} + 336a_{104} \\
- \frac{a_{95}}{12} - \frac{a_{96}}{2} - \frac{a_{97}}{2} - \frac{a_{98}}{3} - \frac{a_{99}}{3} - \frac{a_{100}}{3} - \frac{2a_{102}}{3} + \frac{2a_{102}}{3} - \frac{2a_{102}}{3} + \frac{10a_{103}}{3} + 336a_{104} \\
- \frac{1313}{241920}.
\]

From the degree 1 part of $\tau_{2,1} \Phi(\tau_{1,0}, \tau_{2,0})|_{t=0} = 0$, we obtain

\[
0 = \frac{a_2}{120} + \frac{3a_5}{160} + \frac{a_6}{96} + \frac{7a_8}{11520} - \frac{7a_9}{138240} - \frac{7a_{10}}{46080} + \frac{7a_{12}}{46080} + \frac{7a_{13}}{46080} \\
+ \frac{a_{32}}{72} - \frac{a_{33}}{a_{34}} + \frac{a_{35}}{a_{36}} - \frac{5a_{37}}{a_{38}} - \frac{a_{39}}{a_{40}} + \frac{5a_{57}}{a_{58}} - \frac{a_{59}}{a_{60}} - \frac{a_{61}}{a_{62}} + \frac{a_{63}}{a_{64}} + \frac{a_{67}}{a_{70}} - \frac{a_{72}}{a_{74}} - \frac{a_{76}}{a_{78}} + \frac{a_{79}}{a_{80}} + \frac{a_{81}}{13824} + \frac{a_{82}}{13824} + \frac{a_{83}}{13824} - \frac{5a_{85}}{3} + \frac{a_{86}}{3} - \frac{a_{87}}{3} + \frac{a_{88}}{3} + \frac{a_{89}}{2} + \frac{a_{90}}{2} + \frac{a_{91}}{2} + \frac{a_{92}}{2} + \frac{a_{93}}{2} + \frac{a_{94}}{2} + 336a_{104} \\
- \frac{a_{95}}{12} - \frac{a_{96}}{2} - \frac{a_{97}}{2} - \frac{a_{98}}{3} - \frac{a_{99}}{3} - \frac{a_{100}}{3} - \frac{2a_{102}}{3} + \frac{2a_{102}}{3} - \frac{2a_{102}}{3} + \frac{10a_{103}}{3} + 336a_{104} \\
- \frac{1313}{241920}.
\]

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\[-\frac{a_{96}}{2} - \frac{a_{98}}{2} + \frac{2a_{100}}{3} - a_{101} + \frac{4a_{102}}{3} - 5a_{103} + 432a_{104} - \frac{79}{120960} \tag{93}\]

From the degree 1 part of $\tau_{2,0}\Phi(\tau_{1,0}, \tau_{2,1})|_{t=0} = 0$, we obtain

\[
0 = \frac{a_2}{48} + \frac{7a_{3}}{480} + \frac{7a_{4}}{240} + \frac{a_{5}}{96} - \frac{a_{6}}{240} + \frac{7a_{7}}{480} - \frac{7a_{8}}{11520} + \frac{7a_{9}}{138240} + \frac{7a_{10}}{46080} + \frac{7a_{12}}{46080} - \frac{a_{32}}{144} + \frac{a_{33}}{288} + \frac{a_{34}}{96} + \frac{a_{35}}{288}
\]
\[
+ \frac{a_{36}}{288} + \frac{7a_{37}}{144} + \frac{a_{38}}{96} + \frac{a_{39}}{96} + \frac{a_{40}}{144} - \frac{5a_{57}}{72} + \frac{a_{58}}{96} - \frac{5a_{59}}{288} + \frac{a_{60}}{144} + \frac{5a_{62}}{288} + \frac{a_{63}}{288} + \frac{a_{64}}{288} + \frac{a_{65}}{144} - \frac{a_{66}}{144} + \frac{a_{67}}{96} + \frac{a_{68}}{96} - \frac{a_{72}}{13824} - \frac{a_{74}}{4608} + \frac{a_{75}}{13824} - \frac{a_{76}}{13824} - \frac{a_{78}}{13824} + \frac{a_{79}}{3456} + \frac{a_{80}}{1728} + \frac{a_{82}}{1152} - \frac{6912}{2} + \frac{5a_{84}}{2} + \frac{a_{86}}{2} + \frac{5a_{92}}{2} + \frac{a_{94}}{2} + \frac{a_{97}}{2} + \frac{a_{98}}{2} - \frac{a_{101}}{3} - \frac{11a_{103}}{3} + 432a_{104} + \frac{1}{5376}. \tag{94}\]

From the degree 1 part of $\tau_{3,1}\Phi(\tau_{1,1}, \tau_{0,0})|_{t=0} = 0$, we obtain

\[
0 = \frac{-3a_{1}}{96708} + \frac{7a_{5}}{720} + \frac{7a_{6}}{720} - \frac{7a_{9}}{34560} + \frac{7a_{10}}{34560} - \frac{7a_{12}}{138240} + \frac{7a_{13}}{138240} + \frac{7a_{17}}{1440} + \frac{7a_{21}}{34560} - \frac{7a_{22}}{34560} + \frac{7a_{27}}{2880} + \frac{7a_{28}}{2880} - \frac{7a_{30}}{138240} - \frac{7a_{31}}{138240} - \frac{a_{33}}{288} + \frac{a_{39}}{288} + \frac{a_{40}}{144} - \frac{a_{43}}{144} + \frac{a_{52}}{144} - \frac{a_{54}}{144} + \frac{a_{55}}{144} - \frac{a_{56}}{288} + \frac{a_{72}}{13824} - \frac{a_{74}}{6912} + \frac{a_{75}}{13824} + \frac{a_{76}}{13824} + \frac{a_{78}}{13824} + \frac{a_{79}}{6912} + \frac{a_{80}}{13824} + \frac{a_{82}}{13824} - \frac{a_{83}}{3} + \frac{a_{84}}{3} + \frac{a_{91}}{3} + \frac{a_{92}}{12} + \frac{a_{94}}{24} + \frac{a_{95}}{24} + \frac{a_{98}}{12} - \frac{2a_{103}}{3} + 32a_{104} - \frac{a_{105}}{13824}. \tag{95}\]

From the degree 1 part of $\tau_{4,0}\Phi(\tau_{1,1}, \tau_{0,0})|_{t=0} = 0$, we obtain

\[
0 = \frac{2329a_{1}}{1451520} - \frac{a_{2}}{576} - \frac{a_{5}}{30} + \frac{a_{6}}{30} - \frac{a_{8}}{2304} + \frac{a_{9}}{1440} + \frac{a_{10}}{1440} + \frac{a_{11}}{1440} - \frac{a_{12}}{46080} - \frac{a_{13}}{46080} + \frac{7a_{14}}{576} + \frac{7a_{15}}{576} + \frac{7a_{16}}{576} - \frac{7a_{17}}{720} + \frac{7a_{18}}{576} + \frac{7a_{19}}{576} + \frac{7a_{21}}{1440} + \frac{7a_{24}}{192} + \frac{7a_{25}}{2880} + \frac{7a_{27}}{2880} - \frac{a_{28}}{120} + \frac{a_{29}}{120} + \frac{a_{30}}{120} - \frac{5a_{31}}{576} + \frac{a_{33}}{144} + \frac{7a_{37}}{144} - \frac{a_{41}}{144} + \frac{7a_{43}}{288} + \frac{a_{44}}{288} + \frac{a_{45}}{288} + \frac{a_{50}}{288} + \frac{7a_{52}}{288} + \frac{a_{53}}{288} - \frac{a_{54}}{144} + \frac{7a_{57}}{144} - \frac{a_{59}}{144} + \frac{a_{62}}{144} + \frac{a_{70}}{288} + \frac{5a_{74}}{288} + \frac{a_{75}}{13824} + \frac{a_{76}}{13824} + \frac{a_{78}}{13824} + \frac{5a_{79}}{13824} + \frac{a_{80}}{13824} + \frac{a_{82}}{4608} + \frac{3a_{84}}{2} + \frac{a_{86}}{2} + \frac{3a_{89}}{2} + \frac{a_{91}}{2} + \frac{11a_{92}}{12} + \frac{a_{95}}{12} - \frac{a_{98}}{12} - \frac{5a_{103}}{6} + 72a_{104} + \frac{a_{105}}{6912} - \frac{13}{8064}. \tag{96}\]
From the degree 1 part of $\tau_{2,1} \Phi(\tau_1,1,\tau_{0,1})|_{t=0} = 0$, we obtain

$$0 = \frac{7a_7 - 7a_{12}}{960} - \frac{7a_{13}}{46080} - \frac{7a_{13} - a_{33} - a_{37}}{288} - \frac{a_{40} - a_{41}}{576} - \frac{a_{43}}{576}$$

\[\vdots\]

$$+ \frac{a_{103}}{2} + 24a_{104}.\]  

(97)

From the degree 1 part of $\tau_{3,0} \Phi(\tau_1,1,\tau_{0,1})|_{t=0} = 0$, we obtain

$$0 = -\frac{31a_{12}}{96768} + \frac{7a_5}{720} - \frac{7a_6}{34560} + \frac{7a_8}{720} - \frac{7a_9}{34560} - \frac{7a_{10}}{34560}$$

\[\vdots\]

$$+ \frac{a_{105}}{48384}.\]  

(98)

From the degree 1 part of $\tau_{2,1} \Phi(\tau_1,1,\tau_{0,1})|_{t=0} = 0$, we obtain

$$0 = \frac{a_3}{288} + \frac{7a_5}{480} + \frac{7a_6}{480} + \frac{7a_7}{23040} + \frac{7a_9}{11520} + \frac{7a_{10}}{46080} - \frac{7a_{13}}{46080}$$

\[\vdots\]

$$- \frac{a_{100}}{3} - \frac{2a_{101}}{3} - \frac{a_{102}}{2} + \frac{5a_{103}}{2} + 144a_{104} - \frac{1}{4608}.\]  

(99)

From the degree 1 part of $\tau_{2,0} \Phi(\tau_1,1,\tau_{1,1})|_{t=0} = 0$, we obtain

$$0 = \frac{7a_3}{480} + \frac{7a_4}{480} + \frac{7a_5}{480} + \frac{7a_6}{480} + \frac{7a_7}{23040} + \frac{7a_8}{11520} + \frac{7a_9}{23040} + \frac{7a_{10}}{11520}$$

\[\vdots\]

$$- \frac{a_{33}}{96} - \frac{a_{34}}{96} + \frac{a_{35}}{96} + \frac{a_{36}}{288} + \frac{a_{37}}{288} - \frac{a_{38}}{48}.\]  

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\[ \begin{align*} 
&+ \frac{a_{38}}{6} + \frac{a_{39}}{6} + \frac{a_{40}}{144} - \frac{a_{57}}{48} + \frac{a_{60}}{144} + \frac{a_{63}}{288} + \frac{a_{64}}{288} + \frac{a_{67}}{288} \\
&+ \frac{a_{68}}{288} + \frac{a_{69}}{288} + \frac{a_{70}}{72} - \frac{a_{72}}{13824} - \frac{a_{73}}{13824} - \frac{a_{74}}{4608} - \frac{a_{75}}{4608} - \frac{a_{76}}{6912} \\
&- \frac{a_{77}}{9} + \frac{a_{78}}{9} + \frac{a_{79}}{13824} + \frac{a_{80}}{13824} + \frac{a_{81}}{2304} + \frac{a_{82}}{2304} + \frac{a_{83}}{2304} + \frac{a_{84}}{1728} + \frac{a_{84}}{1728} \\
&+ \frac{a_{92}}{4} + \frac{a_{93}}{8} + \frac{a_{94}}{8} + \frac{a_{97}}{4} - \frac{a_{98}}{4} - \frac{a_{99}}{4} - \frac{5a_{101}}{3} + \frac{5a_{102}}{6} \\
&- \frac{7a_{103}}{3} + 192a_{104} - \frac{1}{4608}. 
\end{align*} \]

From the degree 1 part of \( \tau_{2,0}(\tau_{1,1}, \tau_{2,0})|_{t=0} = 0 \), we obtain

\[ \begin{align*} 
0 &= -\frac{a_2}{7a_{11}} - \frac{11a_5}{96} - \frac{379a_6}{2880} + \frac{7a_7}{960} + \frac{7a_8}{11520} + \frac{193a_9}{138240} + \frac{463a_{10}}{138240} \\
&+ \frac{a_{34}}{23040} + \frac{a_{36}}{46080} + \frac{a_{37}}{46080} + \frac{a_{39}}{288} + \frac{a_{40}}{96} + \frac{a_{47}}{48} + \frac{a_{53}}{288} \\
&+ \frac{a_{68}}{13a_{57}} - \frac{a_{69}}{13a_{57}} + \frac{a_{70}}{6} - \frac{a_{72}}{6} + \frac{a_{73}}{288} + \frac{a_{74}}{13824} + \frac{a_{75}}{13824} + \frac{a_{76}}{13824} \\
&+ \frac{a_{77}}{9} + \frac{a_{78}}{9} + \frac{a_{79}}{13824} + \frac{a_{80}}{13824} + \frac{a_{81}}{2304} + \frac{a_{82}}{2304} + \frac{a_{83}}{2304} + \frac{a_{84}}{1728} + \frac{a_{84}}{1728} \\
&- \frac{a_{86}}{144} - \frac{a_{94} - a_{98}}{2} - \frac{a_{99}}{2} - \frac{a_{101}}{3} + \frac{5a_{102}}{3} - \frac{10a_{103}}{3} \\
&+ 528a_{104} - \frac{59}{16128}. 
\end{align*} \]

From the degree 1 part of \( \tau_{3,1}(\tau_{2,0}, \tau_{0,0})|_{t=0} = 0 \), we obtain

\[ \begin{align*} 
0 &= \frac{31a_1}{96768} - \frac{a_2}{288} - \frac{7a_3}{2880} + \frac{7a_4}{960} - \frac{7a_7}{2880} - \frac{a_8}{1152} - \frac{7a_9}{34560} + \frac{7a_{10}}{34560} \\
&+ \frac{a_{34}}{69120} + \frac{a_{37}}{2880} + \frac{3a_{40}}{13824} + \frac{a_{41}}{13824} + \frac{a_{43}}{96} + \frac{a_{46}}{138240} + \frac{a_{45}}{288} + \frac{a_{52}}{144} \\
&+ \frac{a_{56}}{288} + \frac{a_{57}}{36} + \frac{a_{58}}{288} + \frac{a_{60}}{288} + \frac{a_{61}}{36} + \frac{a_{63}}{36} + \frac{a_{66}}{288} + \frac{a_{67}}{288} \\
&- \frac{a_{68}}{288} + \frac{a_{70}}{48} + \frac{a_{72}}{13824} + \frac{a_{75}}{13824} + \frac{a_{77}}{288} + \frac{7a_{79}}{9} + \frac{a_{80}}{1728} + \frac{7a_{9}}{13824} + \frac{7a_{9}}{13824} \\
&+ \frac{a_{82}}{6912} + \frac{a_{83}}{3} + \frac{5a_{84}}{3} + \frac{a_{85}}{6} + \frac{5a_{89}}{3} + \frac{a_{95}}{6} + \frac{12}{6} \\
&- \frac{a_{100}}{6} + \frac{a_{101}}{3} - \frac{5a_{103}}{3} + \frac{a_{105}}{13824} - \frac{1}{1536}. 
\end{align*} \]

From the degree 0 part of \( \tau_{2,1}(\tau_{2,0}, \tau_{0,0})|_{t=0} = 0 \), we obtain

\[ \begin{align*} 
0 &= \frac{31a_1}{10752} - \frac{7a_3}{4608} - \frac{7a_9}{4608} - \frac{7a_{10}}{4608} + \frac{7a_{11}}{23040} + \frac{7a_{12}}{23040} + \frac{7a_{13}}{23040} \\
&- \frac{7a_{20}}{23040} - \frac{7a_{21}}{4608} - \frac{7a_{22}}{4608} - \frac{7a_{23}}{4608} - \frac{a_{71}}{6} - \frac{a_{72}}{6} + \frac{a_{73}}{6} - \frac{a_{74}}{2304}. 
\end{align*} \]
From the degree 0 part of $\tau_2,3,0 \Phi(t_0, t_0)|_{t=0} = 0$, we obtain

$$0 = \begin{align*} &859 a_1 + 7 a_2 + 7 a_3 + 7 a_4 + 7 a_5 + 7 a_6 + 7 a_7 + 7 a_8 + 229 a_8 \\ &+ 229 a_9 + 229 a_10 + 19 a_11 + 19 a_12 + a_13 + 7 a_14 + 7 a_15 + 7 a_{16} \\ &+ 7 a_{17} + 7 a_{18} + 7 a_{19} + 19 a_{20} + 229 a_{21} + 229 a_{22} + 229 a_{23} + 7 a_{24} \\ &+ 7 a_{25} + 7 a_{26} + 7 a_{27} + 7 a_{28} + 7 a_{29} + 7 a_{30} + 31 a_{31} + a_{32} \\ &+ 288 + 2880 + 2880 + 2880 + 2880 + 2880 + 2880 + 2880 + 2880 = 1360 + 288 \end{align*}$$

From the degree 1 part of $\tau_3,1 \tau_3,0 \Phi(t_0, t_0)|_{t=0} = 0$, we obtain

$$0 = \begin{align*} &53 a_1 + a_2 + 7 a_3 + 7 a_4 + 7 a_5 + 7 a_6 + 7 a_7 + 7 a_8 + 61 a_8 \\ &+ 61 a_9 + 61 a_{10} + 43 a_{11} + 43 a_{12} + 43 a_{13} + a_{14} + a_{15} + a_{16} \\ &+ 34560 + 34560 + 34560 + 34560 + 34560 + 34560 + 34560 + 34560 + 34560 = 34560 + 34560 \end{align*}$$
\[ + \frac{a_{102}}{3} - \frac{10a_{103}}{3} + 240a_{104} + \frac{a_{105}}{768} - \frac{53}{6912}. \]  

From the degree 1 part of \( \tau_{2,1\tau_{3,1}}\Phi(\tau_{0,0}, \tau_{0,0}) \)|\(_{t=0} = 0\), we obtain

\[
0 = -\frac{5a_1}{48384} + \frac{a_2}{144} + \frac{7a_3}{720} + \frac{7a_4}{720} + \frac{7a_5}{720} + \frac{7a_6}{720} + \frac{7a_7}{960} - \frac{5a_8}{5a_9} - \frac{5a_{10}}{7a_{11}} + \frac{7a_{12}}{7a_{13}} + \frac{a_{14}}{72} + \frac{a_{15}}{48} + \frac{a_{16}}{72} + \frac{7a_{17}}{1440} + \frac{a_{18}}{72} + \frac{a_{19}}{48} + \frac{7a_{20}}{5a_{21}} - \frac{5a_{22}}{5a_{23}} + \frac{a_{24}}{a_{25}} + \frac{7a_{26}}{7a_{27}} + \frac{7a_{28}}{a_{29}} + \frac{7a_{30}}{7a_{31}} - \frac{a_{32}}{a_{33}} - \frac{a_{34}}{a_{33}} - \frac{a_{35}}{a_{36}} + \frac{a_{38}}{a_{39}} + \frac{a_{40}}{a_{41}} + \frac{a_{42}}{a_{43}} + \frac{a_{44}}{a_{45}} + \frac{a_{46}}{a_{47}} + \frac{a_{49}}{a_{50}} + \frac{a_{51}}{a_{52}} - \frac{a_{53}}{a_{54}} - \frac{a_{55}}{a_{56}} - \frac{a_{57}}{a_{58}} + \frac{a_{66}}{a_{67}} + \frac{a_{68}}{a_{69}} + \frac{a_{70}}{a_{71}} + \frac{a_{72}}{a_{73}} + \frac{a_{74}}{5a_{75}} + \frac{a_{77}}{7a_{82}} + \frac{a_{83}}{a_{84}} + \frac{2a_{84}}{a_{88}} + \frac{2a_{88}}{3} + \frac{a_{89}}{3} + \frac{a_{90}}{3} - \frac{a_{91}}{12} + \frac{a_{92}}{2} + \frac{a_{93}}{24} + \frac{a_{94}}{24} - \frac{a_{97}}{12} - \frac{a_{98}}{12} - \frac{a_{100}}{6} - \frac{a_{101}}{6} - \frac{a_{102}}{5a_{103}} - \frac{a_{104}}{3} + \frac{5a_{105}}{80a_{104}} + \frac{7}{13824} - \frac{7}{4608}. \]  

From the degree 1 part of \( \tau_{2,1\tau_{4,0}}\Phi(\tau_{0,0}, \tau_{0,0}) \)|\(_{t=0} = 0\), we obtain

\[
0 = \frac{923a_{1}}{290304} + \frac{17a_{2}}{576} - \frac{a_{3}}{30} - \frac{a_{4}}{30} - \frac{a_{5}}{30} - \frac{a_{6}}{30} - \frac{109a_{8}}{69120} - \frac{109a_{9}}{69120} - \frac{69120}{a_{10}} - \frac{7a_{11}}{6760} - \frac{7a_{12}}{6760} - \frac{a_{13}}{64} - \frac{45a_{14}}{73a_{15}} + \frac{7a_{15}}{73a_{16}} + \frac{5a_{16}}{73a_{17}} + 11a_{17} + \frac{5a_{18}}{64} + \frac{7a_{19}}{7680} - \frac{a_{20}}{69120} + \frac{120}{a_{21}} - \frac{120}{a_{22}} + \frac{a_{23}}{960} - \frac{a_{24}}{6050} + \frac{a_{25}}{144} + \frac{a_{26}}{144} + \frac{a_{27}}{144} - \frac{a_{28}}{144} - \frac{a_{29}}{144} - \frac{a_{30}}{144} - \frac{7a_{31}}{144} + \frac{a_{32}}{288} - \frac{a_{33}}{288} + \frac{a_{34}}{17a_{37}} + \frac{a_{35}}{a_{38}} + \frac{a_{39}}{a_{41}} + \frac{a_{40}}{17a_{43}} + \frac{a_{41}}{17a_{44}} + \frac{a_{42}}{a_{43}} + \frac{a_{44}}{a_{45}} + \frac{a_{46}}{a_{47}} - \frac{a_{48}}{144} - \frac{a_{49}}{144} - \frac{a_{50}}{144} - \frac{a_{51}}{144} - \frac{a_{52}}{144} - \frac{a_{53}}{144} - \frac{a_{54}}{144} - \frac{a_{55}}{144} - \frac{a_{56}}{144} - \frac{a_{57}}{144} - \frac{a_{58}}{144} - \frac{a_{59}}{144} - \frac{a_{60}}{144} - \frac{a_{61}}{144} - \frac{a_{62}}{144} - \frac{a_{63}}{144} - \frac{a_{64}}{144} - \frac{a_{65}}{144} - \frac{a_{66}}{144} - \frac{a_{67}}{144} - \frac{a_{68}}{144} - \frac{a_{69}}{144} - \frac{a_{70}}{144} - \frac{a_{71}}{144} - \frac{a_{72}}{144} - \frac{a_{73}}{144} - \frac{a_{74}}{144} - \frac{a_{75}}{144} - \frac{a_{76}}{144} - \frac{a_{77}}{144} - \frac{a_{78}}{2} + \frac{a_{79}}{2} + \frac{a_{80}}{2} + \frac{a_{81}}{2} + \frac{a_{82}}{2} + \frac{a_{83}}{2} + \frac{a_{84}}{2} - \frac{a_{85}}{2} - \frac{a_{86}}{2} - \frac{7a_{88}}{2} + \frac{a_{89}}{2} + \frac{a_{90}}{2} + \frac{a_{91}}{2} + \frac{a_{92}}{2} + \frac{a_{93}}{2} + \frac{a_{94}}{2} + \frac{a_{95}}{2} + \frac{a_{96}}{2} + \frac{a_{97}}{2} + \frac{a_{98}}{2} + \frac{a_{99}}{2} + \frac{a_{100}}{2} + \frac{a_{101}}{2} + \frac{a_{102}}{2} + \frac{a_{103}}{2} + \frac{200a_{104}}{13824} + \frac{11a_{105}}{13824} - \frac{5489}{483840}. \]
From the degree 1 part of $\tau_{2,0} \tau_{4,1} \Phi(\tau_{0,0}, \tau_{0,0})|_{t=0} = 0$, we obtain

$$0 = \frac{139a_1}{241920} + \frac{a_2}{240} + \frac{a_3}{240} - \frac{a_4}{240} + \frac{a_5}{720} - \frac{a_6}{720} + \frac{7a_7}{960} - \frac{7a_8}{4608} - \frac{7a_{25}}{69120} - \frac{2a_{26}}{4608} + \frac{a_{27}}{11520} - \frac{a_{28}}{69120} - \frac{a_{29}}{69120} - \frac{a_{30}}{240} + \frac{11a_{31}}{240} - \frac{a_{32}}{480} + \frac{7a_{33}}{480} - \frac{7a_{34}}{480} + \frac{5a_{37}}{288} - \frac{a_{38}}{288} + \frac{a_{39}}{288} + \frac{a_{40}}{288} - \frac{a_{41}}{288} - \frac{a_{42}}{288} - \frac{5a_{43}}{288} - \frac{5a_{44}}{288} - \frac{a_{46}}{288} - \frac{a_{47}}{288} - \frac{a_{49}}{288} - \frac{5a_{50}}{288} - \frac{5a_{51}}{288} - \frac{5a_{52}}{288} - \frac{5a_{53}}{288} - \frac{a_{54}}{288} - \frac{a_{55}}{288} - \frac{a_{56}}{288} - \frac{a_{57}}{288} - \frac{a_{58}}{288} - \frac{a_{59}}{288} + \frac{a_{60}}{288} - \frac{a_{62}}{288} - \frac{a_{65}}{288} - \frac{a_{66}}{288} + \frac{a_{67}}{288} + \frac{a_{68}}{288} + \frac{a_{70}}{288} + \frac{a_{71}}{288} + \frac{a_{72}}{288} + \frac{a_{73}}{288} + \frac{5a_{74}}{288} + \frac{5a_{75}}{288} + \frac{a_{77}}{288} + \frac{5a_{79}}{288} + \frac{a_{82}}{288} + \frac{a_{83}}{288} + \frac{5a_{84}}{288} + \frac{a_{85}}{288} + \frac{a_{86}}{288} + \frac{5a_{88}}{288} + \frac{5a_{89}}{288} + \frac{5a_{90}}{288} + \frac{a_{89}}{288} + \frac{a_{90}}{288} + \frac{a_{91}}{288} + \frac{a_{92}}{288} + \frac{a_{93}}{288} + \frac{a_{94}}{288} + \frac{a_{99}}{288} + \frac{a_{101}}{288} + \frac{a_{102}}{288} + \frac{13a_{103}}{288} + \frac{6}{2} + \frac{12}{4} + \frac{12}{6} + \frac{6}{6} + \frac{6}{6} + \frac{199}{13824} - \frac{53760}{13824}.$$  \hspace{1cm} (108)

From the degree 1 part of $\tau_{2,1} \tau_{2,1} \Phi(\tau_{0,0}, \tau_{0,1})|_{t=0} = 0$, we obtain

$$0 = \frac{a_1}{13824} + \frac{a_2}{13824} + \frac{7a_7}{23040} + \frac{a_8}{13824} + \frac{a_9}{23040} + \frac{5a_{10}}{13824} + \frac{7a_{11}}{23040} + \frac{7a_{12}}{23040} - \frac{23040}{13824} - \frac{5a_2}{13824} - \frac{a_4}{13824} - \frac{7a_7}{23040} + \frac{7a_{15}}{23040} - \frac{a_{16}}{13824} + \frac{a_{18}}{23040} + \frac{7a_{19}}{23040} + \frac{7a_{20}}{23040} - \frac{23040}{13824} - \frac{a_{21}}{13824} - \frac{5a_{24}}{13824} - \frac{a_{25}}{13824} - \frac{2a_{29}}{23040} + \frac{a_{30}}{23040} + \frac{a_{31}}{13824} + \frac{a_{33}}{23040} + \frac{a_{37}}{23040} + \frac{a_{40}}{13824} - \frac{a_{44}}{23040} + \frac{a_{47}}{23040} + \frac{a_{49}}{23040} + \frac{a_{50}}{23040} + \frac{a_{52}}{23040} + \frac{a_{53}}{23040} - \frac{a_{57}}{13824} + \frac{a_{59}}{13824} - \frac{a_{62}}{13824} + \frac{a_{64}}{13824} + \frac{a_{67}}{13824} + \frac{a_{70}}{13824} + \frac{a_{71}}{13824} + \frac{a_{73}}{13824} + \frac{a_{74}}{13824} + \frac{a_{75}}{13824} + \frac{a_{77}}{13824} + \frac{a_{78}}{13824} + \frac{a_{79}}{13824} + \frac{a_{79}}{13824} + \frac{a_{80}}{13824} + \frac{a_{81}}{13824} + \frac{a_{82}}{13824} + \frac{a_{83}}{23040} + \frac{a_{84}}{23040} + \frac{a_{88}}{23040} + \frac{a_{89}}{23040} + \frac{a_{90}}{23040} + \frac{a_{91}}{13824} + \frac{a_{92}}{13824} + \frac{a_{93}}{13824} + \frac{a_{94}}{13824} + \frac{a_{99}}{13824} + \frac{a_{101}}{13824} + \frac{a_{102}}{13824} + \frac{13a_{103}}{13824} + \frac{6}{2} + \frac{12}{4} + \frac{12}{6} + \frac{6}{6} + \frac{6}{6} + \frac{199}{13824} - \frac{53760}{13824}.$$  \hspace{1cm} (109)

From the degree 1 part of $\tau_{2,1} \tau_{2,1} \Phi(\tau_{0,0}, \tau_{1,0})|_{t=0} = 0$, we obtain

$$0 = \frac{a_2}{48} + \frac{a_3}{288} + \frac{a_4}{288} + \frac{7a_5}{23040} + \frac{7a_6}{23040} + \frac{7a_7}{23040} - \frac{5a_9}{23040} - \frac{5a_{10}}{23040} + \frac{a_{11}}{69120} - \frac{41a_{13}}{69120} + \frac{11a_{15}}{69120} + \frac{a_{16}}{16} + \frac{a_{17}}{69120} + \frac{5a_{24}}{288} + \frac{a_{28}}{288} + \frac{a_{30}}{288}.$$  \hspace{1cm} (1010)
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