BIFURCATIONS ANALYSIS OF LESLIE-GOWER PREDATOR-PREY MODELS WITH NONLINEAR PREDATOR-HARVESTING

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ABSTRACT. In the present paper the dynamics of a Leslie-Gower predator-prey model with Michaelis-Menten type predator harvesting is studied. We give out all the possible ranges of parameters for which the model has up to five equilibria. We prove that these equilibria can be topological saddles, nodes, foci, centers, saddle-nodes, cusps of codimension 2 or 3. Numerous kinds of bifurcations also occur, such as the transcritical bifurcation, pitchfork bifurcation, Bogdanov-Takens bifurcation and homoclinic bifurcation. Several numerical simulations are carried out to illustrate the validity of our results.

1. Introduction. In our real life, along with the increase of people’s demand for food and resources, the exploitation of some biological resources is increasing. In fact, the exploitation of biological resources and the harvesting of populations are commonly practiced in fishery, forestry, and wildlife administration [6, 8]. Hence it is important to find a sustainable development policy to protect the ecosystem. The predator-prey models are widely used to describe the interaction of two species, one species feeds on another. The dynamical properties of the predator-prey models not only can be used to analyze the relations between the prey and predator or to predict whether they can coexist, but also get insight into the optimal management of renewable resources [8, 30]. Therefore, in recent years, predator-prey model has becomes one of the most popular areas in biological systems and many classic models have been proposed [34, 24, 32, 2, 10, 21].

In this paper, we are interested in the following new predator-prey system

\[
\begin{align*}
\dot{x} &= r_1 x (1 - \frac{x}{K}) - a x y \\
\dot{y} &= r_2 y (1 - \frac{y}{b x}) - \frac{q E y}{m_1 E + m_2 y}, \quad \text{if } (x, y) \neq (0, 0) \\
\dot{y} &= 0, \quad \text{if } (x, y) = (0, 0) 
\end{align*}
\]

where \(x\) and \(y\) represent population densities of prey and predator, respectively. The parameters \(r_1, r_2, K, a, b, q, E, m_1, m_2\) are positives. And, the prey grows with intrinsic growth rate \(r_1\) and carrying capacity \(K\) in the absence of predation. \(r_2\) and \(a\) stand for the intrinsic growth rate of the predator, and the maximal predator per capita consumption rate, respectively. \(b\) is a measure of food quality that the prey provides towards the predator births, \(b x\) takes on the role of a prey-dependent

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carrying capacity for the predator. $q$ is the catchability coefficient, $E$ is the effort applied to harvest individuals and $m_1, m_2$ are suitable constants. Moreover, the ratio $r_2/q$ is known as the biotechnical productivity of the predator species.

As we know, one of important predator-prey models is the Leslie-Gower model [34], which is a modification of the Lotka-Volterra model [31]. The Leslie-Gower type model can be described by the autonomous bi-dimensional, where the prey grows logistically and the interaction between prey and predator is expressed by Holling type-I functional response [16]. While the subsistence of predator depends exclusively on prey population, hence the conventional environmental carrying capacity of predator is taken to be proportional to prey abundance. For the considerations, many researchers [17, 10, 28, 29] investigated the system

\[
\begin{aligned}
\dot{x} &= r_1x(1 - \frac{x}{K}) - axy - \frac{qEx}{m_1E + m_2x} \\
\dot{y} &= r_2y(1 - \frac{y}{bx}) \quad \text{if } (x, y) \neq (0, 0) \\
\dot{y} &= 0 \quad \text{if } (x, y) = (0, 0)
\end{aligned}
\] (2)

The stability of the interior equilibrium is studied in [34] by numerical methods. Lindstrom [28] investigated the nonexistence, existence and limit cycles. Hsu and Huang [17] prove that all the solutions are bounded and positive if their initial values are in the first quadrant, and study the globally asymptotical stability of the interior equilibrium using Liapunov function and LaSalle’s invariance principle.

For the commercial purpose and the economic interest, people introduced various harvesting in the predator-prey model [6, 27, 36]. Since the predator-prey model with harvesting has richer dynamics, then the effect of harvesting on the dynamics of predator-prey system and the role of harvesting in the management have attracted great attentions [3, 35, 25, 11]. Generally, there are three basic types of harvesting reported in the literatures: (i) constant-yield harvesting can be described by a constant, which independent of the size of the population under harvest [18, 37]; (ii) constant-effort harvesting that means the number of individuals harvested per unit of time is proportional to the current population [27, 26]; and (iii) nonlinear harvesting or called Michaelis-Menten type harvesting [17, 14], which is a Holling type-II function about the current population. Amongst the three types of harvesting, Michaelis-Menten type harvesting is more realistic from biological and economic point of view. It shows that the nonlinear harvesting function exhibits saturation effect with respect to both the stock abundance and the effort level.

Gupta et al. [13] studied following Leslie-Gower predator-prey model with Michaelis-Menten type prey harvesting

\[
\begin{aligned}
\dot{x} &= r_1x(1 - \frac{x}{K}) - axy - \frac{qEx}{m_1E + m_2x} \\
\dot{y} &= r_2y(1 - \frac{y}{bx}) \quad \text{if } (x, y) \neq (0, 0) \\
\dot{y} &= 0 \quad \text{if } (x, y) = (0, 0)
\end{aligned}
\] (3)

The system has up to five equilibrium including the origin. The origin can be attractor and saddle. Other equilibria can be saddles, nodes, focus, centers and saddle-node. Some bifurcations of codimension 1 were discussed, such as saddle-node bifurcations and Hopf bifurcations. One year later, Gupta and Chandra [15] analysed the dynamics of a modified Leslie-Gower predator-prey model with Michaelis-Menten type prey harvesting.

In this paper, we consider Leslie-Gower predator-prey model with Michaelis-Menten type predator harvesting [14]. We find that there are many interesting results such as the origin is always a saddle, some equilibria are the cusps of codimension 2 and of codimension 3, and system [14] undergoes very rich bifurcations. For
mathematical simplicity, we do some rescalings in system \([1]\). Let \(s = r_2 t, u = x/K, v = ay/r_1\). After rewriting \((u, v, s)\) back to \((x, y, t)\), system \([1]\) becomes
\[
\begin{aligned}
\dot{x} &= \gamma x (1 - x - y) \\
\dot{y} &= y (1 - \frac{au}{x}) - \frac{hy}{c + y}, \quad \text{if } (x, y) \neq (0, 0) \\
\dot{y} &= 0, \quad \text{if } (x, y) = (0, 0)
\end{aligned}
\]
where \(\gamma = r_1/r_2, \alpha = r_1/(abK), h = aqE/(m_2 r_1 r_2), c = am_1 E/(m_2 r_1)\).

We investigate the positive equilibria and their phase portraits of system \([3]\). Let \(h_1 = 1 + c + 2\alpha + \alpha c - 2\sqrt{\alpha(1 + \alpha)(1 + c)}\). When \(h > h_1\), there are no interior equilibria, which implies that the extinction of prey or predator occurs. When \(h \leq h_1\), there are one or two interior equilibria, depending on the ranges of the positive parameters. It is shown that the maximum harvesting rate \(h_{\text{MSY}}\) of system \([3]\) is depend on the values of \(1/\alpha\) i.e. the food quality that the prey provides towards the predator births. When \(\alpha < 1/c\), i.e. the food is enough for the predator births, then \(h_{\text{MSY}} = h_1\). When \(\alpha > 1/c\), i.e. the food is not enough for the predator births, then \(h_{\text{MSY}} = c < h_1\). When \(h \leq h_1\), we obtain the phase portraits of \([3]\) near each equilibrium. We shall show that they are topological saddles, nodes, foci, centers, saddle-nodes of codimension 1, nonhyperbolic stable node of codimension 2, cusp of codimension 3.

When the equilibria are nonhyperbolic, we give bifurcation analysis for system \([4]\). System \([3]\) has rich bifurcations. It is shown that system \([4]\) can exhibit numerous kinds of bifurcation phenomena in terms of the original parameters in the model, such as the transcritical bifurcation, pitchfork bifurcation, saddle-node bifurcation, Hopf bifurcations, Bogdanov-Takens bifurcation and homoclinic bifurcation. Those dynamical behaviors can be better for us to understand how the nonlinear predator harvesting affects the dynamics of system \([4]\).

The organization of this paper is as following. In Section 2, we study the various conditions for which the existence of the equilibrium of the model. The phase portraits of each equilibrium such as saddle, node, focus, saddle-node, cusp of codimension 2 and cusp of codimension 3 are given in Section 3. In section 4, we consider the Hopf bifurcations of the model depending on all parameters. By computing the first Lyapunov number, the subcritical and supercritical Hopf bifurcations are obtained. Some numerical analysis are given to illustrate these Hopf bifurcations. In section 5, the Bogdanov-Takens bifurcation of codimension 2 is studied. This paper ends by a brief discussion in section 6.

2. Equilibria of the model. In the following, we will first prove some basic property of system \([4]\), such as the positivity and boundedness of solutions as well as the permanence of system \([4]\). Recall that system \([4]\) is said to be \textit{permanent} if there exist positive constants \(\omega_1\) and \(\omega_2\), such that each positive solution \((x(t), y(t))\) of system \([4]\) with initial condition \((x_0, y_0) \in \text{Int } \Omega\) satisfies,
\[
\min \{\liminf_{t \to +\infty} x(t), \liminf_{t \to +\infty} y(t)\} \geq \omega_1 \quad \text{and} \quad \max \{\limsup_{t \to +\infty} x(t), \limsup_{t \to +\infty} y(t)\} \leq \omega_2.
\]

We will use the following Lemma which is proved in \([4]\) to get the boundedness of solutions of \([4]\).

\textbf{Lemma 2.1.} \textit{If } a, b > 0 \textit{ and } \frac{dX}{dt} \leq \left(\geq\right)X(t)(a - bX(t)) \textit{ with } X(0) > 0, \textit{ then }
\[
\limsup_{t \to +\infty} X(t) \leq \frac{a}{b} \left(\liminf_{t \to +\infty} X(t) \geq \frac{a}{b}\right).
\]
Proposition 1. (i) The first quadrant is positive invariant for system (4).

(ii) All solutions (x(t), y(t)) of system (4) with the initial condition (x_0, y_0) in the first quadrant are bounded for all t ≥ 0.

(iii) The system (4) is permanent if α > 1 and c > h.

Proof. (i) Let \( \bar{x}(t) \) be a solution of \( \dot{x} = \gamma x(1-x) \) and \( \bar{y}(t) \equiv 0 \). Then \( (\bar{x}(t), \bar{y}(t)) \) is a solution of (4). Hence the x-axis is a positive invariant of (4). By the interpretation of the Leslie-Gower terms when \( x = 0 \) described earlier, we have \( y \equiv 0 \). We see that any solution of (4) starting at the first quadrant can not cross y-axis to the second quadrant. Then the first quadrant is positive invariant.

(ii) From the first equation of system (4), we can get
\[
\dot{x} = \gamma x(1 - x - y) \leq \gamma x(1 - x).
\]
From Lemma 2.1 we have \( \limsup_{t \to +\infty} x(t) \leq 1 \). Thus there exists a positive constant \( M_1 \) such that \( x(t) \leq M_1 \) for all \( t \geq 0 \).

For the second equation of (4), we have
\[
\dot{y} = y(1 - \frac{\alpha y}{x}) - \frac{hy}{c+y} \leq y(1 - \frac{\alpha y}{M_1}).
\]
By Lemma 2.1 we have \( \limsup_{t \to +\infty} y(t) \leq M_1/\alpha \). Hence, there exist a positive constant \( M_2 \) such that \( y(t) \leq M_2 \) for all \( t \geq 0 \).

(iii) If \( \alpha > 1 \), then there exists a \( \varepsilon > 0 \) such that \( 1 + \varepsilon < \alpha \). From the proof of (ii), we get \( x(t) \leq 1 + \varepsilon \) and \( y(t) \leq (1 + \varepsilon)/\alpha \) for the sufficiently large \( t \).

Hence, for the sufficiently large \( t \), the first equation of system (4) can write
\[
\dot{x} = \gamma x(1 - x - y) \geq \gamma x(1 - x - (1 + \varepsilon)/\alpha) = \gamma x(\zeta_1 - x),
\]
where \( \zeta_1 = 1 - (1 + \varepsilon)/\alpha > 0 \). By Lemma 2.1 we have \( \liminf_{t \to +\infty} x(t) \geq \zeta_1 \). Hence, \( x(t) \geq \zeta_1/2 \) for sufficiently large \( t \).

Now using positivity of \( y \) and for sufficiently large \( t \), from the second equation of system (4) we have
\[
\dot{y} = y(1 - \frac{\alpha y}{x} - \frac{h}{c+y}) \geq y(1 - \frac{2\alpha y}{\zeta_1} - \frac{h}{c}) = y(\zeta_2 - \frac{2\alpha y}{\zeta_1}),
\]
where \( \zeta_2 = 1 - h/c \). If \( c > h \) i.e. \( \zeta_2 > 0 \), from Lemma 2.1 we have \( \liminf_{t \to +\infty} y(t) \geq \frac{\zeta_1 \zeta_2}{2\alpha} \).

Then, combine with the proof of (ii), we can choosing \( \omega_1 = \min\{\zeta_1, \zeta_1 \zeta_2/(2\alpha)\} \) and \( \omega_2 = \max\{1, M_1/\alpha\} \), we get the permanence of the system (4).

\( \square \)

Remark 1. From Proposition 1(ii), when \( \alpha > 1 \) and \( c > h \), i.e. \( r_1 > abK \) and \( 1 < m_1(r_2/q) \), system (4) is permanent. This implies that when the intrinsic growth rate \( r_1 \) of prey and the biotechnical productivity of the predator \( r_2/q \) are large, then predator and prey will persist and extinction will not occur. Biologically, the density of prey is large, then the predator has enough food to get. Hence the prey and predator can coexist.

For the practical biological meaning, we will study system (4) in \( \Omega := \{(x, y) | x \geq 0, y \geq 0\} \). In the sequel of this section, we will consider the equilibrium of system (4) in \( \Omega \). When \( y = 0 \), system (4) has equilibrium \( E_0 = (0, 0) \) and \( E_1 = (1, 0) \). Note the right side of the second equation of system (4) is singular at \( (x, y) = (0, 0) \). After redefining at \( (x, y) = (0, 0) \) as in the third equation of (4), there is no confusion.
To find the equilibria in \( \text{Int}\Omega \) suffices to solve the following system:

\[
\begin{cases}
(1 + \alpha)x^2 - (1 + c + 2\alpha + \alpha c - h)x + \alpha(1 + c) = 0 \\
y = 1 - x
\end{cases}
\]  

(5)

**Notations.**

\[ h_i = \alpha + (1 + \alpha)(1 + c) + (-1)^i2\sqrt{\alpha(1 + \alpha)(1 + c)}, i = 1, 2. \]

\[ x_2 = \sqrt{\alpha(1+c) \frac{1}{1+\alpha}}, y_2 = 1 - x_2, x_i = \frac{1+c+2\alpha+\alpha c-h}{2(1+\alpha)}y_i = 1 - x_i, i = 3, 4. \]

where \( \Delta := h^2 - 2h(1 + c + 2\alpha + \alpha c) + (1 + c + \alpha c)^2 = (h - h_1)(h - h_2). \)

Through direct calculations, we have the following results.

**Lemma 2.2.**

(1) \( 0 < h_1 < 1 + c + 2\alpha + \alpha c < h_2. \)

(2) \( h_1 \geq c. \) Equality holds if and only if \( c = 1/\alpha. \)

(3) \( h_1 > c(1 + \alpha) - 1 \) if and only if \( c < 1/\alpha. \)

The following results give all the equilibria of system (4) in \( \text{Int}\Omega. \)

**Theorem 2.3.**

(1) If \( \alpha < 1/c \) and \( h > h_1 \) or \( \alpha \geq 1/c \) and \( h \geq c, \) then (4) has no equilibria in \( \text{Int}\Omega. \)

(2) If \( \alpha < 1/c \) and \( h = h_1, \) then (4) has a unique equilibria \( E_2 = (x_2, y_2) \) in \( \text{Int}\Omega. \)

(3) If \( \alpha < 1/c \) and \( c < h < h_1, \) then (4) has two equilibria \( E_3 = (x_3, y_3) \) and \( E_4 = (x_4, y_4) \) in \( \text{Int}\Omega. \)

(4) If \( h < c \) or \( \alpha < 1/c \) and \( h = c, \) then (4) has a unique equilibria \( E_3 = (x_3, y_3) \) in \( \text{Int}\Omega. \)

**Figure 1.** The number of interior equilibriums of system (4).

Fig.1 shows the number of interior equilibriums for different values of \( h. \) The blue solid line is for prey nullcline and the red dashed cure is for predator nullcline.

From Theorem 2.3, we see that the maximum harvesting rate \( h_{\text{MSY}} \) of system (4) is depend on the values of the food quality that the prey provides towards the predator births \( 1/\alpha. \) When \( \alpha < 1/c, \) i.e., the food quality is good for the predator births, the \( h_{\text{MSY}} = h_1. \) When \( \alpha > 1/c, \) i.e., the food quality is bad for the predator births, the \( h_{\text{MSY}} = c < h_1. \)

3. **Phase portraits of the equilibria.** Now, we consider the dynamics of system (4) in the neighborhood of each equilibrium. Let \( (x, y) \) be an equilibrium of (4). The Jacobian matrix \( A(x, y) \) of system (4) at \( (x, y) \) is important in discussing the
dynamics of (4). Through direct calculations, we have the Jacobian matrix $A(x, y)$ at an equilibrium $(x, y)$ of system (4).

$$A(x, y) = \begin{bmatrix} \gamma(1 - 2x - y) & -\gamma x \\ \frac{\alpha y^2}{x^2} & 1 - \frac{2\alpha y}{x} - \frac{\beta c}{(c + y)^2} \end{bmatrix}$$ \hspace{1cm} (6)$$

We first consider the dynamical properties of the equilibria on the boundary. At the equilibria $E_0 = (0, 0)$, the Jacobian matrix cannot be calculated directly because the ratio $\frac{x}{y}$ is not defined at $E_0$. To understand the dynamical behaviors of $(0, 0)$, as in [19], we need a nonlinear transformations to remove the singularity and expand the system to the whole $x$-axis. Our goal is to introduce transformation $x = u, y = uv$ and expand the system to the whole $x$-axis.

**Theorem 3.1.** The equilibrium $E_0 = (0, 0)$ of system (4) is a saddle.

**Proof.** Since (4) is singular at $(0, 0)$, we introduce the transformation to consider the dynamics of $(0, 0)$ of (4) such that its the dynamics can be obtained from the ones of the equilibria of the transformed system on the $v$-axis.

Take the change of variable $x = u, y = uv$. Then (4) can be changed into the following system

$$\begin{cases} \dot{u} = \gamma u(1 - u - uv) \\ \dot{v} = (1 - \gamma)v + \gamma uv - \alpha v^2 + \gamma uv^2 - \frac{h v}{c + uv} \end{cases} \hspace{1cm} (7)$$

Clearly, $(0, 0)$ is an equilibrium of system (7). It has another non-negative equilibrium $(0, \frac{c(1 - \gamma) - h}{c})$ on $v$-axis if and only if $\gamma < 1$ and $h < c(1 - \gamma)$.

By calculation, the Jacobian matrix for system (7) at $(0, 0)$ is given by

$$A(0, 0) = \begin{bmatrix} \gamma & 0 \\ 0 & \frac{c - h - \gamma c}{c} \end{bmatrix}$$ \hspace{1cm} (8)$$

It has two eigenvalues $\gamma > 0$ and $\frac{c - h - \gamma c}{c}$. Thus, if $\gamma < 1$ and $h < c(1 - \gamma)$, then $(0, 0)$ is an unstable node. If $\gamma \geq 1$ or $\gamma < 1$ and $h > c(1 - \gamma)$, then $(0, 0)$ is a saddle. In addition, the Jacobian matrix for the system (7) at $(0, \frac{c(1 - \gamma) - h}{c})$ has two eigenvalues $\gamma > 0$ and $-\frac{c - h - \gamma c}{c} < 0$. Thus, $(0, \frac{c(1 - \gamma) - h}{c})$ is a saddle.

From above discussion, we conclude that the equilibrium $E_0 = (0, 0)$ of system (4) behaves like a saddle, See figure 2. The proof is completed.

Now, we consider the dynamics of $E_1 = (1, 0)$. By (3), we have

$$A(E_1) = \begin{bmatrix} -\gamma & -\gamma \\ 0 & 1 - h/c \end{bmatrix}$$ \hspace{1cm} (9)$$

**Theorem 3.2.** (1) If $h < c$, then $E_1$ is a saddle;

(2) If $h > c$, then $E_1$ is a stable node;

(3) If $h = c$ and $c \neq 1/\alpha$, then $E_1$ is a saddle-node of codimension 1; (4) If $h = c$ and $c = 1/\alpha$, then $E_1$ is a nonhyperbolic stable node of of codimension 2.

**Proof.** (1) Note that $|A(E_1)| = -\gamma(1 - h/c) < 0$ if $h < c$. Then $E_1$ is a saddle.

(2) If $h > c$, then $|A(E_1)| > 0, tr(A(E_1)) = -\gamma - \frac{h - c}{c} < 0$ and $(tr(A(E_1)))^2 - 4|A(E_1)| = (\gamma + \frac{h - c}{c})^2 - 4\gamma \frac{h - c}{c} = (\gamma - \frac{h - c}{c})^2 \geq 0$. Hence $E_1$ is a stable node.

(3) When $h = c$, we have $|A(E_1)| = 0$ and $tr(A(E_1)) = -\gamma \neq 0$. Under transformation $u = x - 1, v = y$, (4) can be changed into

$$\begin{cases} \dot{u} = -\gamma(u + v + u^2 + uv) \\ \dot{v} = \beta v^2 + o((u, v)^2) \end{cases} \hspace{1cm} (10)$$
where $\beta = \frac{1-\alpha c}{c}$. Make transformation $u = -(u_1 + v_1), v = u_1$. Then system (10) becomes

$$
\begin{cases}
\dot{u}_1 = \beta u_1^2 + o((u_1, v_1)^2) \\
\dot{v}_1 = -\gamma v_1 + o((u_1, v_1))
\end{cases}
$$

(11)

Thus, when $h = c$ and $c \neq 1/\alpha$, we have $\beta \neq 0$. Hence $E_1$ is a saddle-node of codimension 1.

(4) When $h = c$ and $c = 1/\alpha$, then $\beta = 0$. System (11) becomes

$$
\begin{cases}
\dot{u}_1 = -(\alpha + 1/c^2)u_1^3 - \alpha u_1^2v_1 + o((u_1, v_1)^3) \\
\dot{v}_1 = -\gamma v_1 + o((u_1, v_1))
\end{cases}
$$

(12)

Note that $-(\alpha + 1/c^2) < 0$ and $-\gamma < 0$. We have $E_1$ is a nonhyperbolic stable node of codimension 2.

**Remark 2.** From Theorem 3.2, when we take $h$ as a bifurcation parameter, then system (1) undergoes a transcritical bifurcation around $E_1$ if $h = c \neq 1/\alpha$ and a pitchfork bifurcation around $E_1$ if $h = c = 1/\alpha$. In fact, if we denote $X = (x, y)^T, X_0 = (1, 0)^T$ and $f(X, h) = (f_1(X, h), f_2(X, h))^T$, where $f_1(X, h) = \gamma x(1 - x - y), f_2(X, h) = y(1 - \frac{\alpha y}{x}) - \frac{hy}{c+y}$, then system (1) can be rewrite as $X = f(X, h)$ and $f(X_0, h) \equiv 0$.

From (9) we know when $h = c$, the matrix $A = Df(X_0, c)$ has a simple eigenvalue $\lambda = 0$ with eigenvector $v = (1, -1)^T$ and $A^T$ has an eigenvector $w = (0, 1)^T$ corresponding to the eigenvalue $\lambda = 0$. By computation, when $h = c \neq 1/\alpha$ we have $w^T f_h(X_0, c) = 0, w^T[Df_h(X_0, c)v] = 1/c \neq 0$ and $w^T[D^2f(X_0, c)(v, v)] = -2(\alpha - 1/c) \neq 0$. Hence, if $h = c = 1/\alpha$, then system (1) undergoes a transcritical bifurcation around $E_1$. If $h = c = 1/\alpha$, then $w^T f_h(X_0, c) = 0, w^T[Df_h(X_0, c)v] = 1/c \neq 0, w^T[D^2f(X_0, c)(v, v)] = 0$ and $w^T[D^3f(X_0, c)(v, v, v)] = 6(\alpha + 1/c^2) \neq 0$, hence system (1) undergoes a pitchfork bifurcation around $E_1$.

From the ecological view, the transcritical bifurcation and pitchfork bifurcation occurs around $E_1$ gives the maximum threshold for continuous harvesting without the extinction risk of the predator species.

In the following, we shall consider the dynamical properties of the interior equilibria. If $(x, y) \in Int\Omega$ is an interior equilibria of system (4), then (5) becomes:

$$
A(x, y) = \begin{bmatrix}
-\gamma x & -\gamma x \\
\frac{\alpha y^2}{x^2} & 1 - \frac{2\alpha y}{x} - \frac{hy}{c+y^2}
\end{bmatrix}
$$

(13)

From $1 - \frac{\alpha y}{x} - \frac{hy}{c+y} = 0$, we have

$$
|A(x, y)| = -\gamma x[1 - \frac{2\alpha y}{x} - \frac{c}{c+y}(1 - \frac{\alpha y}{x}) - \frac{y}{x}(1 - \frac{h}{c+y})]
$$

$$
= -\gamma x[\frac{1}{c+y}(y - \frac{\alpha cy}{x} - \frac{2\alpha y^2}{x}) - \frac{y(c-h+y)}{x(c+y)}]
$$

(14)

$$
= -\gamma y \left[\frac{x}{c+y}(x - (1+2\alpha)y - (c-h+\alpha c)) \right] = \frac{\gamma y}{c+y} r(x)
$$

where $r(x) := -2(1+\alpha)x - h + 1 + c + 2\alpha + \alpha c$. Thus, $|A(x, y)|$ and $r(x)$ have the same signs.
For the trace, we get
\[
tr(A(x, y)) = -\gamma x + 1 - \frac{2\alpha y}{x} - \frac{c}{c+y} (1 - \alpha y) x
\]
\[
= -\gamma x + \frac{1}{c+y}(y - \frac{\alpha cy}{x} - \frac{2\alpha y^2}{x})
\]
\[
= -\gamma x + \frac{y}{x(c+y)} [(1 + 2\alpha)x - \alpha(c + 2)]
\]
(15)

Let \( \gamma_2 = \frac{x^2}{x_2} \) and \( \alpha(c) = \frac{4 + c - \sqrt{c^2 + 4c + 12}}{2\sqrt{c + 4c + 12}} \). By direct calculations, we have \( 0 < \alpha(c) < 1/c \).

**Theorem 3.3.** (1) If \( \alpha < 1/c, h = h_1 \) and \( \gamma \neq \gamma_2 \), then \( E_2 \) is a saddle-node of codimension 1;

(2) If \( \alpha < 1/c, h = h_1, \gamma = \gamma_2 \) and \( \alpha \neq \alpha(c) \), then \( E_2 \) is a cusp of codimension 2;

(3) There is \( c_0 > 0 \) such that if \( \alpha < 1/c, h = h_1, \gamma = \gamma_2 \) and \( \alpha = \alpha(c) \), then \( E_2 \) is a cusp of codimension 3 for \( 0 < c < c_0 \).

**Proof.** From the formulas of \( x_2 \) and \( h_1 \), we have \( r(x_2) = 0 \),
which implies that \( |A(E_2)| = 0 \). Hence, we have \( 1 - \frac{2\alpha y}{x_2} - \frac{hc}{(c + y_2)^2} = \frac{\alpha y_2}{x_2} \), and then
\[
tr(A(E_2)) = -\gamma x_2 + 1 - \frac{2\alpha y_2}{x_2} - \frac{hc}{(c + y_2)^2} = -\gamma x_2 + \frac{\alpha y_2}{x_2}.
\]
(1) From above analysis, we obtain that \( tr(A(E_2)) \neq 0 \) when \( \gamma \neq \gamma_2 \).

Under the transformations \( u = x - x_2, v = y - y_2 \), then (11) can be changed into the following system
\[
\begin{cases}
\dot{u} = a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv \\
\dot{v} = b_{10}u + b_{01}v + b_{20}u^2 + b_{11}uv + b_{02}v^2 + o(|(u,v)|^2)
\end{cases}
\]
(16)
where \( a_{10} = -\gamma x_2, a_{20} = a_{11} = -\gamma, b_{10} = \frac{\alpha y_2}{x_2} \), \( b_{01} = 1 - \frac{2\alpha y}{x_2} - \frac{hc}{(c + y_2)^2} \), \( b_{20} = -\frac{\alpha y_2}{x_2} \), \( b_{11} = \frac{h_1}{x_2^2} \), \( b_{02} = h_1(c + y_2)^2 - \frac{\alpha}{x_2} \).

Making transformation \( u_1 = b_{01}u - a_{01}v, v_1 = a_{10}u + a_{01}v \), (16) becomes
\[
\begin{cases}
\dot{u}_1 = a_{10}u_1^2 + a_{20}u_1 + a_{3}v_1^2 + o(|(u_1,v_1)|^2) \\
\dot{v}_1 = \rho v_1 + o(|(u_1,v_1)|)
\end{cases}
\]
(17)

where \( \rho = a_{10} + b_{10}, a_1 = \frac{\alpha y_2}{\rho^3}(b_{20} - b_{11} + b_{02}), a_2 = \frac{1}{\rho^2}(2(a_{20}b_{10} - a_{10}b_{20} + b_{10}b_{02})
+ (b_{10} - a_{10})(a_{20}b_{10} - a_{10}b_{20})), a_3 = \frac{1}{\rho^3}[(a_{20}b_{10} - a_{10}b_{20}) - b_{10}b_{02}] \).

Clearly, \( \rho = tr(A(E_2)) \neq 0 \). Note that \( \frac{h_1c}{(c + y_2)^2} = 1 - \frac{2\alpha y}{x_2} - \frac{\alpha y_2}{x_2} = \frac{\alpha c}{x_2} \), we have
\[
b_{20} - b_{11} + b_{02}
= -\frac{\alpha y_2}{x_2} \frac{2\alpha y_2}{x_2} + \frac{h_1c}{(c + y_2)^2} - \frac{\alpha}{x_2} x_2^3 + \frac{1}{c + y_2} = \frac{\alpha x_2}{x_2^3}(c + y_2) \neq 0.
\]

Thence, \( a_1 = \frac{\alpha y_2}{\rho^3}(b_{20} - b_{11} + b_{02}) \neq 0 \). we know that \( E_2 \) is a saddle-node of codimension 1.
(2) When \( \gamma = \gamma_2 \), then \( tr(A(E_2)) = a_{10} + b_{01} = 0 \). Then system \((16)\) is
\[
\begin{align*}
\dot{u} &= a_{10} u + a_{01} v + a_{20} u^2 + a_{11} uv \\
\dot{v} &= b_{10} u + b_{01} v + b_{20} u^2 + b_{11} uv + b_{02} v^2 + o((u, v)^2)
\end{align*}
\]
where \( a_{10} = a_{01} = -\gamma_2 x_2, a_{20} = a_{11} = b_{20} = -\gamma_2, b_{10} = b_{01} = b_{21} = \frac{\gamma_2 x_2 (c + y_2)}{y_2}, b_{02} = \frac{\gamma_2 x_2 (c + y_2)}{y_2} \).

Let \( u_2 = u, v_2 = -b_{01} u + a_{01} v \). Then system \((18)\) is transformed into
\[
\begin{align*}
\dot{u}_2 &= v_2 + \kappa_{11} u_2 v_2 \\
\dot{v}_2 &= l_{20} u_2^2 + l_{11} u_1 v_2 + l_{02} v_2^2 + o((u_2, v_2)^2)
\end{align*}
\]
where \( \kappa_{11} = 1/x_2, l_{20} = -\frac{\gamma_2 x_2 (1 + c)}{y_2(c + y_2)}, l_{11} = \frac{\gamma_2 (1 + c)(a - \alpha(c))(1 + \alpha + \alpha(c))(c^2 + 4 + 12)}{y_2(c + y_2)(1 + \alpha)^2(x_2^2 + 4 + 4c)x_2 + 1 + c}, l_{02} = -\frac{c - x_2}{y_2(c + y_2)} \).

Make transformation \( u_3 = u_2, v_3 = v_2 + \kappa_{11} u_2 v_2 \). Then, system \((19)\) becomes
\[
\begin{align*}
\dot{u}_3 &= u_3 \\
\dot{v}_3 &= l_{20} u_3^2 + l_{11} u_1 v_3 + (\kappa_{11} + l_{02}) v_3^2 + o((u_3, v_3)^2)
\end{align*}
\]

Note that \( l_{20} = -\frac{\gamma_2 x_2 (1 + c)}{y_2(c + y_2)} \neq 0 \). We show that \( E_2 \) is a cusp. Letting \( dt = [1 - (\kappa_{11} + l_{02}) u_3] d\tau \) and changing back \( \tau \) to \( t \), \((20)\) is
\[
\begin{align*}
\dot{u}_4 &= u_4 \\
\dot{v}_4 &= l_{20} u_4^2 + l_{30} u_4^3 + l_{21} u_4^2 v_4 + l_{12} u_4 v_4^2 + l_{03} v_4^3 + l_{40} u_4^4 + l_{31} u_4^3 v_4 \quad (23)
\end{align*}
\]
where \( l_{20} = -\frac{\gamma_2 x_2 (1 + c)}{y_2(c + y_2)}, l_{30} = -\gamma_2 \frac{(2 + x_2 - c)^2 + 2a + 4c}{2x_2}, l_{21} = \frac{\gamma_2 x_2 (c + y_2)}{2x_2} \), \( l_{12} = \frac{4 + c + 2(x_2 - c^2)}{4x_2} \), \( l_{03} = -\frac{c(3y_2^2 + 8x_2)}{4x_2^2} \), \( l_{40} = \frac{\gamma_2 x_2 (c + y_2)^2}{2x_2} - c \gamma_2 x_2 (y_2 + 2) \), \( l_{31} = \frac{\gamma_2 x_2 (c + y_2)^2}{2x_2} - c \gamma_2 x_2 (y_2 + 2) \), \( l_{22} = \frac{2 - x_2}{y_2} + \frac{(2 + x_2 - c)^3}{4x_2} \), \( l_{13} = \frac{c(3y_2^2 + 8x_2)}{4x_2^2} \), \( l_{04} = -\frac{c y_2^2}{8x_2} \).

Let \( u_5 = u_4 - \frac{4}{2 + x_2} v_4^2 + \frac{4}{2 + x_2} v_5 \), \( v_5 = v_4 - l_{03} u_4 v_4^2 \). Then system \((22)\) becomes
\[
\begin{align*}
\dot{u}_5 &= v_5 - \frac{l_{12} u_5^2}{2 + x_2} + \frac{3}{2 + x_2} u_5 v_5^2 + o((u_5, v_5)^2) \\
\dot{v}_5 &= l_{20} u_5^2 + l_{30} u_5^3 + l_{21} u_5^2 v_5 + l_{12} u_5 v_5^2 + l_{03} u_5^3 + l_{40} u_5^4 + (l_{31} - l_{20} l_{03}) u_5^3 v_5 \quad (24)
\end{align*}
\]
The transformation \( u_6 = u_5 - \frac{l_{03} u_4 v_4^2}{2 + x_2} - \frac{l_{12} u_5^3}{2 + x_2} = u_5 - \frac{l_{03} u_4 v_4^2}{2 + x_2} - \frac{l_{12} u_5^3}{2 + x_2} \) brings system \((24)\) into
\[
\begin{align*}
\dot{u}_6 &= v_6 + o((u_6, v_6)^4) \\
\dot{v}_6 &= l_{20} u_6^2 + l_{30} u_6^3 + l_{21} u_6^2 v_6 + l_{12} u_6 v_6^2 + l_{03} u_6^3 + l_{40} u_6^4 + (l_{31} - 3l_{20} l_{03}) u_6^3 v_6 \quad (25)
\end{align*}
\]
Note that $l_{20} = \frac{2^{2}(1+c)}{2} > 0$. Thus we can make the change of variables $u_{7} = u_{6}, v_{7} = \frac{1}{\sqrt{l_{20}}}(v_{6} + o(|(u_{6}, v_{6})|^{4}))$ and $\tau = \sqrt{l_{20}}t$. System (25) becomes

$$
\begin{cases}
\dot{u}_{7} &= v_{7} \\
\dot{v}_{7} &= u_{7} + \frac{\lambda_{34}}{l_{20}}u_{7}^{3} + \frac{\lambda_{32}}{l_{20}}u_{7}^{2} + \frac{\lambda_{42}}{l_{20}}v_{7}\frac{\lambda_{32}}{l_{20}}u_{7}^{3} + \frac{\lambda_{42}}{l_{20}}v_{7}\frac{\lambda_{32}}{l_{20}}u_{7}^{2} \\
&+ v_{7}^{2}(l_{12}u_{7} + l_{22}v_{7}^{2}) + o((|u_{7}, v_{7})|^{4})
\end{cases}
$$

(26)

From proposition 5.3 in [23] (also see Lemma 3.2 in [18]), system (26) is equivalent to

$$
\begin{cases}
\dot{u}_{8} &= v_{8} \\
\dot{v}_{8} &= u_{8}^{2} + Gu_{8}v_{8} + o((|u_{8}, v_{8})|^{4})
\end{cases}
$$

(27)

where $G = \frac{l_{34} - 3l_{20}l_{31}}{\sqrt{l_{20}}} - \frac{l_{30}l_{31}}{l_{20}\sqrt{l_{20}}} = \frac{1}{l_{20}\sqrt{l_{20}}}((l_{20})_{31} - 3l_{20}l_{30} - l_{30}l_{21})$.

Through direct computations, $G = \frac{2\sqrt{2}\alpha(\gamma^{2}x_{1}^{2}(1+c))}{x_{2}^{2}}\psi(c)$, where

$$
\psi(c) = \frac{2c^{2}(2 - x_{2}) - c(4x_{2} + 13) + 2(26x_{2} - 7)}{4x_{2}} + \frac{cy_{2}^{2}(c^{2} + 4c + 84)}{16x_{2}}
$$

Notice that $x_{2}, y_{2}$ are continuous in $c$. Hence $G$ is continuous in $c$. When $c = 0$, then $x_{2} = 2 - \sqrt{3}, y_{2} = \sqrt{3} - 1, \alpha(0) = \frac{2 - \sqrt{3}}{2\sqrt{3}}$ and $\psi(0) = \frac{206(2 - \sqrt{3}) - 7}{(2\sqrt{3} - 1)^{2}} < 0$. Hence $G|_{c=0} < 0$. There is a $c_{0} > 0$ such that $G < 0$ for $c \in (0, c_{0})$. Thus, we have that $E_{2}$ is a cusp of codimension 3. Therefore, the proof is completed.

**Theorem 3.4.** If $\alpha < 1/c$ and $c < h < h_{1}$, then $E_{4}$ is a saddle.

**Proof.** From [14], we know that $|A(E_{4})| = \frac{2y_{4}}{c^{2} + y_{4}}r(x_{4})$. By the formula of $x_{4}$, we have $r(x_{4}) = -\sqrt{\Delta} < 0$. Thus $|A(E_{4})| < 0$, which implies that $E_{4}$ is a saddle.

Now, we study the phase portraits near $E_{3} = (x_{3}, y_{3})$.

**Notations.**

$$h_{0} = \frac{(1+c+\alpha c)^{2}}{(1+2\alpha)(2+c)}, \delta = (1 + 2\alpha)x_{3} - \alpha(c + 2), \omega_{0} = \frac{\delta y_{3}}{x_{3}^{2}(c+y_{3})}, \omega_{i} = \frac{y_{3}(\sqrt{\delta + r(x_{3})} + (-1)^{i}\sqrt{r(x_{3})})^{2}}{x_{3}^{2}[c+y_{3}]}$$

In fact, from the formulas of $x_{3}$ and $r(x_{3})$, we have $r(x_{3}) = \sqrt{\Delta} > 0$. And, through direct calculations, we have the following results.

**Lemma 3.5.** (1) If $\alpha < 1/c$, then $h_{0} \leq h_{1} < 1 + \frac{1+c+\alpha c}{1+2\alpha}$.

(2) If $\alpha \geq 1/c$ and $h < c$ or $\alpha < 1/c$ and $h \leq h_{0}$, then $\delta \leq 0$.

(3) If $\alpha < 1/c$ and $h_{0} < h < h_{1}$, then $\delta > 0$.

(4) If $\alpha \geq 1/c$ and $h < c$ or $\alpha < 1/c$ and $h < h_{1}$, then $\delta + r(x_{3}) > 0$.

**Proof.** (1) By the formulas of $h_{1}$, we have

$$h_{1} = \frac{2(1+2\alpha)(\sqrt{\alpha(1+\alpha)(1+c)} - 2\alpha(1+\alpha)(c+2))}{1+2\alpha} = \frac{2\alpha(1+\alpha)(1+c+\alpha)(1-\alpha)}{(1+2\alpha)(1+2\alpha)(\sqrt{\alpha(1+\alpha)(1+c) + \alpha(1+\alpha)(c+2)}}$$

which implies that $h_{1} < \frac{1+c+\alpha c}{1+2\alpha}$ if and only if $\alpha < 1/c$. 

From the formulas of $h_0$ and $h_1$, we have

$$h_1 - h_0 = \frac{1}{(1 + 2\alpha)(2 + c)} \{(1 + c + \alpha)(1 + 2\alpha)(2 + c) - (1 + c + \alpha)\} + 2(1 + 2\alpha)(2 + c)(\alpha - \sqrt{\alpha(1 + \alpha)(1 + c)})$$

$$= \frac{(1 + c + \alpha)(1 + 4\alpha + ac) + 2(1 + 2\alpha)(2 + c)(\alpha - \sqrt{\alpha(1 + \alpha)(1 + c)})}{(1 + 2\alpha)(2 + c)}$$

$$= \frac{[(1 + 2\alpha)\sqrt{1 + c - (2 + c)\sqrt{\alpha(1 + \alpha)}}]^2}{(1 + 2\alpha)(2 + c)} \geq 0$$

which implies that $h_1 \geq h_0$.

(2) From the formulas of $\delta$ and $x_3$, we have

$$\delta = \frac{1}{2(1 + \alpha)} \{(1 + 2\alpha)|1 + c + 2\alpha + ac - h - \sqrt{\Delta}| - 2\alpha(1 + \alpha)(c + 2)\}$$

$$= \frac{1 + 2\alpha}{2(1 + \alpha)} \{(1 + c + \alpha)\frac{1 + c + \alpha}{1 + 2\alpha} - h - \sqrt{\Delta}\}.$$

If $\frac{1 + c + ac}{1 + 2\alpha} \leq h$, then $\delta < 0$.

If $\frac{1 + c + ac}{1 + 2\alpha} > h$, then we have

$$\delta = \frac{2\alpha(1 + c + ac)}{(1 + 2\alpha)(\frac{1 + c + ac}{1 + 2\alpha} - h + \sqrt{\Delta})} (1 - \alpha).$$

Then we have $\delta \leq 0$ if $\alpha \geq 1/c$.

From (1), we have known that $h_0 \leq h_1 < \frac{1 + c + ac}{1 + 2\alpha}$ if $\alpha < 1/c$. Thus if $h < h_1$, then $\frac{1 + c + ac}{1 + 2\alpha} > h$. Then

$$\delta = \frac{2\alpha[h(1 + 2\alpha)(2 + c) - (1 + c + ac)^2]}{1 + c + ac - h(1 + 2\alpha) + (1 + 2\alpha)\sqrt{\Delta}}$$

$$= \frac{2\alpha(1 + 2\alpha)(2 + c)}{1 + c + ac - h(1 + 2\alpha) + (1 + 2\alpha)\sqrt{\Delta}} (h - h_0).$$

(28)

Hence $\delta \leq 0$ if $h \leq h_0$.

(3) From (28), we see that $\delta > 0$ if $\alpha < 1/c$ and $h_0 < h < h_1$.

(4) From the formulas of $r(x_3)$ and $\delta$, we have

$$\delta + r(x_3) = \frac{1 + 2\alpha}{2(1 + \alpha)} \left( \frac{1 + c + \alpha}{1 + 2\alpha} - h + \frac{\sqrt{\Delta}}{1 + 2\alpha} \right).$$

Clearly, $\delta + r(x_3) > 0$ if $h < \frac{1 + c + ac}{1 + 2\alpha}$. From (1), we have known that if $\alpha < 1/c$, then $h_1 < \frac{1 + c + ac}{1 + 2\alpha}$. Thence, when $\alpha < 1/c$ and $h < h_1$, then $\delta + r(x_3) > 0$. Thus (4)(ii) is proved.

From above analysis, we obtain that $\delta + r(x_3) > 0$ when $h \leq \frac{1 + c + ac}{1 + 2\alpha}$. If $\frac{1 + c + ac}{1 + 2\alpha} < h < c$, then we can get from (28) that

$$\delta + r(x_3) = \frac{\Delta - [(1 + 2\alpha)h - (1 + c + ac)]^2}{2(1 + \alpha)[(1 + 2\alpha)h - (1 + c + ac) + \sqrt{\Delta}]}$$

$$= \frac{2h\alpha(c - h)}{(1 + 2\alpha)[h - \frac{1 + c + ac}{1 + 2\alpha} + \sqrt{\Delta}]} > 0.$$

Hence, if $\alpha \geq 1/c$ and $h < c$, we have $\delta + r(x_3) > 0$. The proof is complete. □
Theorem 3.6. (1) If $\alpha \geq 1/c, h < c$ and $\gamma \in (0, \omega_1) \cup (\omega_2, \infty)$ or $\alpha < 1/c, h \leq h_0$ and $\gamma \leq \omega_1$ or $\alpha < 1/c, h < h_1$ and $\gamma \geq \omega_2$, then $E_3$ is a hyperbolic stable node. 

(2) If $\alpha \geq 1/c, h < c$ and $\omega_1 < \gamma < \omega_2$ or $\alpha < 1/c, h \in (0, h_0) \cup (h_0, h_1)$ and $\omega_1 < \gamma < \omega_2$, then $E_3$ is a hyperbolic stable focus. 

(3) If $\alpha < 1/c, h_0 < h < h_1$ and $\gamma = \omega_0$, then $E_3$ is a weak focus or center. 

(4) If $\alpha < 1/c, h_0 < h < h_1$ and $\gamma \leq \omega_1$, then $E_3$ is a hyperbolic unstable node. 

(5) If $\alpha < 1/c, h_0 < h < h_1$ and $\omega_1 < \gamma < \omega_0$, then $E_3$ is a hyperbolic unstable focus.

Proof. By (14), we have $|A(E_3)| = \frac{\gamma y_3}{c + y_3} r(x_3) = \frac{\gamma y_3}{c + y_3} \sqrt{\Delta} > 0$. From (15), we get $tr(A(E_3)) = -\gamma x_3 + \frac{y_3 \delta}{3(y_3 + y_1)} = x_3(-\gamma + \omega_0)$. By the formulas of $\omega_0$ and $\delta$, we have $\delta < 0$ if and only if $\omega_0 < 0$. Then we have:

(A1) If $\delta \leq 0$, then $tr(A(E_3)) < 0$. 

(A2) If $\delta > 0$, then $tr(A(E_3)) < 0$ is equivalent to $\gamma > \omega_0$. 

Moreover, we have

$$ (tr(A(E_3)))^2 - 4|A(E_3)| = \gamma^2 x_3^2 - \frac{2y_3(\delta + 2r(x_3))}{c + y_1} \gamma + \frac{y_3^2 \delta^2}{x_3^2(c + y_1)^2} = x_3^2(\gamma - \omega_1)(\gamma - \omega_2). $$

Note that $\omega_1 < \omega_2$. Then

(B) $(tr(A(E_3)))^2 - 4|A(E_3)| \geq 0$ if and only if $\gamma \leq \omega_1$ or $\gamma \geq \omega_2$. 

From the formulas of $\omega_0, \omega_1$ and $\omega_3$, we have the facts:

(C1) If $\delta > 0$, then $0 < \omega_1 < \omega_0 < \omega_2$. 

(C2) If $\delta \leq 0$ and $\delta + r(x_3) > 0$, then $\omega_0 < 0 < \omega_1 < \omega_2$. 

From (A1), (A2), (B), (C1), (C2) and Lemma 3.5, the results follow. 

Remark 3. The so-called ‘biological control paradox’ states that we cannot have a low and stable prey equilibrium density. By Theorems 3.4 and 3.6, we see that the $E_3$ is a saddle and $E_3$ may be a stable node or focus, this contradicts with the ‘biological control paradox’ because $x_3 > x_1$. 

From the formula of $h_0$, we have that $h_0 \geq c$ since $h_0 - c = \frac{(1-\alpha c)^2}{(1+2\alpha)(1+c)} \geq 0$. In fact, we always have that $h_0 \geq c$ since $h_0 - c = \frac{1-\alpha c)^2}{(1+2\alpha)(1+c)} \geq 0$. By Theorems 3.2 and 3.6 (1(ii), (iii)), we know that the system will have two stable equilibrium ($E_1$ and $E_3$) for some parameters, i.e. the bi-stability case occurs. That is there are some regions such that the predator will go extinction, but for some other regions the prey and predator can co-exist.
shown in Fig.2. Thus, it satisfies Theorems 3.1(2), 3.4 and 3.6(2)(ii). We do some numerical calculations. Take \((\gamma, \alpha, c, h) = (0.1086, 0.5354, 0.9559, 0.5675)\), then \(x_3 = 0.1688, x_4 = 0.9028, y_4 = 0.0972, \omega_1 = 6.677 \times 10^{-6}, \omega_2 = 0.1919\) and \(h_0 = 0.5677\). Thus, it satisfies Theorems 3.1(2), 3.4 and 3.6(2)(ii). Hence the system (3) has a boundary equilibrium \(E_1\) which is saddle, and has a unique interior equilibrium \(E_3 = (0.5981, 0.4019)\) is a stable focus, which is shown in Fig.4.

Set \((\gamma, \alpha, c, h) = (0.1086, 0.5354, 0.9559, 0.5675)\), then we have \(x_3 = 0.8312, y_3 = 0.1688, x_4 = 0.9028, y_4 = 0.0972, \omega_1 = 6.677 \times 10^{-6}, \omega_2 = 0.1919\) and \(h_0 = 0.5677\). Thus, it satisfies Theorems 3.1(2), 3.4 and 3.6(2)(ii). Hence the system (3) has a boundary equilibrium \(E_1\) and two interior equilibrium \(E_3 = (0.8312, 0.1688), E_4 = (0.9028, 0.0972)\), among which \(E_1\) is a stable node, \(E_3\) is a stable focus and \(E_4\) is a saddle. Thus, system has two stable equilibrium i.e. bi-stability occurred. This is shown in Fig.5.

4. Hopf bifurcation. In Theorem 3.6(3), we obtain that if \(\alpha < 1/c, h_0 < h < h_1\) and \(\gamma = \omega_0\), then \(E_3\) is a weak focus or center. The system (3) may occur hof bifurcation near the \(E_3\). The purpose of this section is to study the Hopf bifurcation of system (3) and its direction for some parameter values.

In order to study the Hopf bifurcation, we need some knowledge of the Hopf bifurcation theory of the autonomous system. To state the result, we consider the following system

\[
\begin{aligned}
\dot{x} &= a_1 x + a_0 y + p(x, y) := f(x, y) \\
\dot{y} &= b_1 x + b_0 y + q(x, y) := g(x, y)
\end{aligned}
\]

where \(p(x, y) = \sum_{i+j=2}^{\infty} a_{ij} x^i y^j\) and \(q(x, y) = \sum_{i+j=2}^{\infty} b_{ij} x^i y^j\).

If \(D = a_{10} b_0 - a_{01} b_1 > 0\), then it follows from the formula (3') of section 4.4 in [33] that the first Liapunov number, denoted by \(\sigma\) is given by

\[
\sigma = -\frac{3\pi}{2a_0 b_1 D^{1/2}} \sum_{i=1}^{\infty} \xi_i
\]
Lemma 4.1. Assume that \( D > 0 \) and \( a_{10} + b_{01} = 0 \). Then the following assertions hold.

(1) If \( \sigma < 0 \) (or \( \sigma > 0 \)), then the equilibrium \((0, 0)\) is a stable (or unstable) center or stable (or unstable) focus with multiplicity one.

(2) If \( \sigma < 0 \) (or \( \sigma > 0 \)), then a supercritical (or subcritical) Hopf bifurcation occurs at \((0, 0)\) at the bifurcation value \( \mu = a_{10} + b_{01} = 0 \).

(3) If \( \sigma < 0 \) (or \( \sigma > 0 \)), then a unique stable (or unstable) limit cycle bifurcates from \((0, 0)\) at the bifurcation value \( \mu = a_{10} + b_{01} \) increases from zero.

Now, we consider the Hopf bifurcation of system \((\text{[I]}\)). We first translate the equilibrium \(E_3 = (x_3, y_3)\) of system \((\text{[I]}\)) to the origin by using the transformation \(u = x - x_3, v = y - y_3\). Thus, \((\text{[I]}\)) can be changed into the following system

\[
\begin{align*}
\dot{u} &= a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv \\
\dot{v} &= b_{10}u + b_{01}v + a_{20}v^2 + b_{11}uv + b_{02}v^2 \\
&\quad + b_{30}u^3 + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3 + o(|(u, v)|^3)
\end{align*}
\]  

(32)

where \(a_{10} = a_{01} = -\omega_0 x_3, a_{20} = a_{11} = -\omega_0, b_{10} = \frac{\omega y_3}{x_3}, b_{01} = 1 - \frac{2\omega y_3}{x_3} - \frac{hc}{(c+yn)^2}, b_{20} = -\frac{\omega_0 y_3}{x_3}, b_{11} = \frac{2\omega y_3}{x_3}, b_{02} = \frac{hc}{(c+yn)^2} - \frac{\omega_0}{x_3}, b_{30} = \frac{\omega y_3}{x_3}, b_{21} = \frac{2\omega y_3}{x_3}, b_{12} = \frac{2\omega y_3}{x_3}, b_{03} = -\frac{\omega_0 y_3}{x_3} - \frac{hc}{(c+yn)^2}.

By the transformation, we know that \((0, 0)\) a center or weak focus of \((\text{[I]}\)). The Jacobian matrix of \((\text{[I]}\)) at \((0, 0)\) is

\[
A(0, 0) = \begin{bmatrix} 
-\omega_0 x_3 & \frac{-\omega_0 y_3}{x_3} \\
\frac{\omega y_3}{x_3} & 1 - \frac{2\omega y_3}{x_3} - \frac{hc}{(c+yn)^2}
\end{bmatrix}
\]

Through direct calculations, we have \(tr(A(0, 0)) = 0\) and

\[
D = |A(0, 0)| = \frac{\omega_0 y_3}{c + y_3} \sqrt{\Delta} > 0.
\]

By computations, we obtain

\[
\begin{align*}
\xi_1 &= \frac{\omega y_3}{x_3} (\omega_0 - b_{02}), \quad \xi_2 = \omega_0^2 x_3^2 \left( 4\omega x_3^3 - \frac{2\omega y_3}{x_3^2} - \omega_0 b_{02} \right), \quad \xi_3 = 0, \\
\xi_4 &= \frac{\omega_0 y_3}{x_3} \frac{5\omega y_3}{x_3}, \quad \xi_5 = -\frac{\omega y_3}{x_3} \left( 2\omega_0 y_3 b_{02} - \omega_0 \right), \quad \xi_6 = -\frac{\omega y_3}{x_3} \left( 2\omega_0 y_3 b_{02} - \omega_0 \right), \quad \xi_7 = -\frac{\omega y_3}{x_3} \left( 2\omega_0 y_3 b_{02} - \omega_0 \right), \quad \xi_8 = -\omega_0 x_3 (\omega_0 x_3 - \frac{\omega y_3}{x_3} (\frac{2\omega y_3}{x_3} b_{03} - \frac{\omega y_3}{x_3})),
\end{align*}
\]
Noting that \( x_3 + y_3 = 1 \). Hence,

\[
\sigma = -\frac{3\pi}{2a_1D^{3/2}} \sum_{i=1}^{8} \xi_i \\
= -\frac{3\pi}{2a_1D^{3/2}} \left[ 2\alpha \omega^3 x_3 - \frac{2\alpha \omega^3 y_3^2}{x_3} + \frac{2\alpha^2 \omega^2 y_3^2}{x_3^2} - \left( \frac{\omega^3}{x_3^3} + 4\alpha \omega^2 y_3 + \frac{\alpha \omega^2 y_3^2}{x_3^3} \right) \right] \\
+ \left( \frac{2\omega \alpha^2 y_3^3}{x_3^3} \right) b_{02} + \left( \frac{2\omega \alpha y_3^2}{x_3} \right) b_{02} - 3\alpha \omega^3 y_3^3 \left( \omega_0 - \frac{\alpha y_3^2}{x_3^3} \right) b_{03} = \frac{3\pi \Delta}{2x_3} \left( \frac{c + y_3}{\omega_0 y_3} \right)^\frac{3}{2} \sum_{i=1}^{3} \eta_i.
\]

where \( \eta_1 = \frac{\omega}{x_3^3} \left( 1 + \alpha \right) \left( 4 + x_3 \right) - \frac{2\omega}{x_3^3} \left( 2 + 9\alpha \right) x_3 - 4\alpha \right) + \frac{2\alpha^3 y_3^2}{x_3^3}, \eta_2 = -\frac{h_0}{\left( c + y_3 \right)^2} \left( \omega_0 x_3 - \alpha \omega_0 \left( 6 - c - x_3 \right) + \frac{2\omega_0 (4c + 7)}{x_3^3} - \frac{2\omega_0 (1 + c)}{x_3^3} + \frac{2\alpha^3 y_3^2 (1 + x_3)}{x_3^3} \right) \right] \eta_3 = -\frac{h_0}{\left( c + y_3 \right)^3} \left( 3\alpha \omega_0 y_3 + \frac{3\alpha^3 y_3^4}{x_3^3} - \frac{h_0 \omega_0 x_3}{c + y_3} - \frac{2h_0 \alpha y_3^2}{x_3^3 (c + y_3)^2} \right).

Thus, we have the following theorem.

**Theorem 4.2.** Assume that \( \alpha < 1/c, h_0 < h < h_1 \) and \( \gamma = \omega_0 \). Then

1. If \( \sigma < 0 \), then a supercritical Hopf bifurcation of \( \text{(1)} \) occurs at the equilibrium \( E_3 \) and an stable limit cycle will occurred near the \( E_3 \) as the bifurcation value \( \mu = a_{10} + b_{01} \) increases from zero.
2. If \( \sigma = 0 \), then system \( \text{(1)} \) has at least two limit cycles for some suitable parameter values.
3. If \( \sigma > 0 \), then a subcritical Hopf bifurcation of \( \text{(1)} \) occurs at the equilibrium \( E_3 \) and an unstable limit cycle will occurred near the \( E_3 \) as the bifurcation value \( \mu = a_{10} + b_{01} \) increases from zero.
By numerical calculations, let \((c, \alpha, \gamma) = (0.7154, 0.1123, 0.4459)\) and \(h = 1.071\). Then \(h_0 = 0.9697, h_1 = 1.0945\) and \(\omega_0 = 0.4459\). Also, we have \(D = 0.0452 > 0, \sigma = -10.566 < 0\). From Theorems 3.2(2), 3.4 and 4.2(1), we know that \(E_1\) is a stable node, \(E_2 = (0.5212, 0.4788)\) is a saddle and system has a stable limit cycle near \(E_3 = (0.3323, 0.6677)\) this means \(E_3\) is a weak focus of multiplicity 1 and unstable, which is shown in Fig.6.

Set \((c, \alpha, \gamma) = (0.7154, 0.1123, 0.4183)\) and \(h = 1.063609962\). Then \(h_0 = 0.9697, h_1 = 1.0945\) and \(\omega_0 = 0.4183\). Also, we have \(D = 0.0491 > 0, \sigma = 0.0\). From Theorems 3.2(2), 3.4 and 1.2(2), we know that \(E_1\) is a stable node, \(E_4 = (0.5386, 0.4614)\) is a saddle and system has two limit cycles near \(E_3 = (0.3216, 0.6784)\) which means \(E_3\) is a weak focus of multiplicity 2, which is shown in Fig.7.

Take \((c, \alpha, \gamma) = (0.7154, 0.1123, 0.4160)\) and \(h = 1.063\). Then \(h_0 = 0.9697, h_1 = 1.0945\) and \(\omega_0 = 0.4160\). We also have \(D = 0.0494 > 0, \sigma = 0.8181 > 0\). From Theorems 3.2(2), 3.4 and 1.2(3), we know that \(E_1\) is a stable node, \(E_4 = (0.5399, 0.4601)\) is a saddle and system has an unstable limit cycle near \(E_3 = (0.3208, 0.6792)\) this means \(E_3\) is a weak focus of multiplicity 1 and stable, which is shown in Fig.8.

5. BT bifurcation. From Theorem 5.1. When \(\alpha < 1/c, h = h_1, \gamma = \gamma_2\) and \(\alpha \neq \alpha(c)\), then \(E_2\) is a cusp of codimension 2. In this section, we will investigate the Bogdanov-Takens bifurcation (i.e. BT bifurcation). In fact, we have the following theorem.

Theorem 5.1. When \(\alpha < 1/c, h = h_1, \gamma = \gamma_2\) and \(\alpha > \alpha(c)\) \((\alpha < \alpha(c))\), then system (34) have a unique interior equilibrium \(E_2\) which is a cusp of codimension 2 i.e. BT singularity. If we choose \(h\) and \(\gamma\) as bifurcation parameters, then system (34) undergoes repelling (attracting) BT bifurcation in a small neighborhood of the interior equilibrium \(E_2\) as \((h, \gamma)\) varies near \((h_1, \gamma_2)\). Thence, there exist some parameter values such that system (34) has an unstable (stable) limit cycle, and there exist some other parameter values such that system (34) has an unstable (stable) homoclinic loop.

Proof. We choose \(h\) and \(\gamma\) as two bifurcation parameters. Consider the following perturbed system of (34).

\[
\begin{align*}
\dot{x} &= (\gamma_2 + \varepsilon_1)x(1 - x - y) \\
\dot{y} &= y(1 - \frac{\sigma}{c}) - \frac{(h_1 + \varepsilon_2)y}{c + y}
\end{align*}
\]

(33)

It is clear that when \(\varepsilon_1 = \varepsilon_2 = 0\), then system (33) has a unique interior equilibrium \(E_2\) and it is a cusp of codimension 2.

We remove \((x_2, y_2)\) to the origins. Let \(u = x - x_2\) and \(v = y - y_2\), then system (33) becomes

\[
\begin{align*}
\dot{u} &= p_{10}u + p_{01}v + p_{20}u^2 + p_{11}uv \\
\dot{v} &= q_0 + q_{10}u + q_{01}v + q_{20}u^2 + q_{11}uv + q_{02}v^2 + h.o.t.
\end{align*}
\]

(34)

where \(p_{10} = p_{01} = -\gamma_2 x_2 - x_2 \varepsilon_1, p_{20} = p_{11} = -\gamma_2 - \varepsilon_1, q_0 = -\frac{\sigma}{c + y_2}, q_{10} = \gamma_2 x_2, q_{01} = \gamma_2 x_2 - \frac{c}{c + y_2} \varepsilon_2, q_{20} = -\gamma_2, q_{11} = \frac{2 \gamma_2 x_2}{y_2}, q_{02} = \frac{\gamma_2 x_2 (c - x_2)}{y_2 (c + y_2)} + \frac{c}{c + y_2} \varepsilon_2.

Make \(u_1 = u, v_1 = p_{10}u + p_{01}v + p_{20}u^2 + p_{11}uv\). Then (33) becomes

\[
\begin{align*}
u_1 &= u_1 \\
v_1 &= k_0 + k_{10}u_1 + k_{01}v_1 + k_{20}u_1^2 + k_{11}u_1 v_1 + k_{02}v_1^2 + h.o.t.
\end{align*}
\]

(35)
where \( k_0 = p_{01}q_0, k_{10} = p_{01}q_{10} - p_{10}q_0 + p_{11}q_0, k_{01} = p_{10} + q_0, k_{20} = p_{01}q_{20} - p_{20}q_0 + p_{11}q_{10} - p_{10}q_{11}, k_{11} = 2p_{01} + q_0 - \frac{1}{p_{01}} (p_{10}p_{11} + 2p_{10}q_0), k_{02} = \frac{1}{p_{01}} (p_{11} + q_0) \).

When \( \varepsilon_1 \to 0, \varepsilon_2 \to 0 \), we can get that \( k_{20} \to \frac{\gamma_2^2 x_2 (1 + c)}{\mu_4 (c + y_2)} > 0 \). Then the following transformation makes sense for small \( \varepsilon_1, \varepsilon_2 \). With the change of variable \( u_2 = u_1 + \frac{k_{01}}{k_{20}}, v_2 = v_1 \), system (35) is

\[
\begin{cases}
  u_2 = v_2 \\
  v_2 = r_0 + r_1 v_2 + k_{20} u_2^2 + k_{11} u_2 v_2 + k_{02} v_2^2 + h.o.t.
\end{cases}
\]

where \( r_0 = k_0 - \frac{k_{10} k_{11}}{4k_{20}} \) and \( r_1 = k_{01} - \frac{k_{10} k_{11}}{2k_{20}} \).

We introduce a new time variable \( \tau \), which is satisfied \( dt = (1 - k_{02} u_2) d\tau \) and assuming that a dot over a variable now means differentiation with respect to \( \tau \). We relabel as \( t \) for simplicity. Then system (35) can be transformed into

\[
\begin{cases}
  \dot{u}_2 = (1 - k_{02} u_2) v_2 \\
  \dot{v}_2 = (1 - k_{02} u_2) (r_0 + r_1 v_2 + k_{20} u_2^2 + k_{11} u_2 v_2 + k_{02} v_2^2 + h.o.t.)
\end{cases}
\]

Set \( u_3 = u_2, v_3 = (1 - k_{02} u_2) v_2 \). Then (37) becomes

\[
\begin{cases}
  \dot{u}_3 = v_3 \\
  \dot{v}_3 = r_0 - 2r_0 k_{02} u_3 + r_1 v_3 + (k_{20} + r_0 k_{10}^2) u_3^2 + (k_{11} - r_1 k_{02}) u_3 v_3 + h.o.t.
\end{cases}
\]

Let \( u_4 = u_3 - \frac{r_0 k_{02}}{k_{20} + r_0 k_{02}^2}, v_4 = v_3 \). Then system (38) becomes

\[
\begin{cases}
  \dot{u}_4 = v_4 \\
  \dot{v}_4 = \lambda_1 + \lambda_2 v_4 + \lambda_3 u_4^2 + \lambda_4 u_4 v_4 + h.o.t.
\end{cases}
\]

where \( \lambda_1 = r_0 (1 - \frac{r_0 k_{02}}{k_{20} + r_0 k_{02}^2}), \lambda_2 = r_1 + \frac{r_0 k_{02}}{k_{20} + r_0 k_{02}^2} (k_{11} - r_1 k_{02}), \lambda_3 = k_{20} + r_0 k_{10}^2, \lambda_4 = k_{11} - r_1 k_{02} \).

Notice that \( \lambda_1 \to 0, \lambda_2 \to 0, \lambda_3 \to k_{20} > 0 \) and \( \lambda_4 \to k_{11} > 0 \), when \( \varepsilon_i \to 0, (i = 1, 2) \). Thus we have \( \lambda_1, \lambda_4 \to k_{20}, k_{11} > 0 \). Making transformation \( u_5 = \frac{\lambda_3}{\lambda_4} u_4, v_5 = \frac{\lambda_3}{\lambda_4} v_4, \tau = \frac{\lambda_3}{\lambda_4} t \), we rewrite \( \tau \) as \( t \) for simplicity and hence we can get

\[
\begin{cases}
  \dot{u}_5 = v_5 \\
  \dot{v}_5 = \mu_1 + \mu_2 v_5 + u_5^2 + u_5 v_5 + h.o.t.
\end{cases}
\]

Moreover, the Jacobian matrix is evaluated with \( (\varepsilon_1, \varepsilon_2) = (0, 0) \)

\[
\begin{vmatrix}
  \partial (u_1, u_2) \\
  \partial (\varepsilon_1, \varepsilon_2)
\end{vmatrix} = \begin{pmatrix}
  0 & \frac{(y_2^2 + c y_2 - 2 x_2) (2 y_2^2 + 2 y_2 c - c) (y_2^2 + c y_2 - 2 x_2) (y_2^2 + 2 y_2 c - c) (c + y_2)}{\gamma_2 x_2 (1 + c) (c + y_2) (c + y_2) (c + y_2)^2 (c + y_2)^2} \\
  \frac{(y_2^2 + c y_2 - 2 x_2) (2 y_2^2 + 2 y_2 c - c) (y_2^2 + c y_2 - 2 x_2) (c + y_2)}{\gamma_2 x_2 (1 + c) (c + y_2) (c + y_2) (c + y_2)^2 (c + y_2)^2} & 0
\end{pmatrix}
\]

Therefore, by the Bogdanov and Takens bifurcation theorems in [12, 22], we obtain the results. The proof is complete.
Remark 4. By Theorems 5.1 we can see that these bifurcations can lead to a potentially dramatic shift in the system dynamics, hence it’s ecologically important. Through a saddle-node bifurcation, the system can have zero, one or two interior equilibrium as the harvesting rate crosses its critical value. Thus, there are some values of parameters such that the prey and the predator co-exist in the form of a positive equilibrium for different initial values. By a Hopf bifurcation, system can have at least a limit cycle. So, there are some values of parameters such that the prey and the predator co-exist in the form of a positive equilibrium for all initial values lying inside the periodic orbit or an periodic solution with a finite period for all initial values on the periodic orbit. Through a homoclinic bifurcation, we know that the system \( (\epsilon_1,\epsilon_2) \) will have a homoclinic loop. Thence, there are some values of parameters such that the prey and the predator co-exist in the form of a positive equilibrium for all initial values lying inside the homoclinic loop or a periodic orbit with infinite period for all initial values on the homoclinic loop.

We do some numerical calculations to illustrate the theoretic results. Let \((c, \alpha, \gamma, h) = (0.605997, 0.11163, 0.6172, 1.0041)\). When \((\epsilon_1,\epsilon_2) = (0,0)\), then system have a cusp of codimension 2 \(E_2 = (0.4016, 0.5984)\), which is shown in Fig.9.

Set \((\epsilon_1,\epsilon_2) = (-0.067, -0.0167)\), then system have an unstable limit cycle appears through Hopf bifurcation in a small neighborhood of \(E_3 = (0.3311, 0.6689)\) and a saddle \(E_4 = (0.4871, 0.5129)\), which is shown in Fig.10.

Take \((\epsilon_1,\epsilon_2) = (-0.0451, -0.0119)\), then system have a stable focus \(E_3 = (0.3412, 0.6588)\), a saddle \(E_4 = (0.4721, 0.5273)\) and an unstable homoclinic loop, which is shown in Fig.11.

When we take \((\epsilon_1,\epsilon_2) = (-0.017, -0.039)\), then system have a stable focus \(E_3 = (0.2991, 0.7009)\) and a saddle \(E_4 = (0.3931, 0.4609)\), which is shown in Fig.12.

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