Convergence of a finite element method on a Bakhvalov-type mesh for a singularly perturbed convection–diffusion equation in 2D

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Abstract
A finite element method of any order is applied on a Bakhvalov-type mesh to solve a singularly perturbed convection–diffusion equation in 2D, whose solution exhibits exponential boundary layers. A uniform convergence of (almost) optimal order is proved by means of a carefully defined interpolant.

KEYWORDS
Bakhvalov-type mesh, convection–diffusion equation, finite element method, singular perturbation, uniform convergence

1 | INTRODUCTION

Consider the elliptic boundary value problem
\[-\varepsilon \Delta u - \mathbf{b} \cdot \nabla u + cu = f \quad \text{in} \quad \Omega = (0, 1)^2, \]
\[u = 0 \quad \text{on} \quad \partial \Omega, \quad (1.1)\]
where \(\varepsilon \ll 1\) is a small positive parameter and \(\mathbf{b}(x, y) = (b_1(x, y), b_2(x, y))^T\). The functions \(b_1, b_2, c\) and \(f\) are assumed to be smooth on \(\overline{\Omega}\). We also assume that for any \((x, y) \in \Omega\),
\[b_1(x, y) \geq \beta_1 > 0, b_2(x, y) \geq \beta_2 > 0, \quad (1.2a)\]
\[c(x, y) + \frac{1}{2} \nabla \cdot \mathbf{b}(x, y) \geq \gamma > 0, \quad (1.2b)\]
where \(\beta_1, \beta_2\) and \(\gamma\) are some constants. The condition (1.2a) excludes the case of turning points ([1]). The condition (1.2b) ensures that (1.1) has a unique solution in \(H_0^1(\Omega) \cap H^2(\Omega)\) for all \(f \in L^2(\Omega)\) (see,
e.g., [1]). Because $\varepsilon$ is small, the problem is in general singularly perturbed and its solution typically has exponential boundary layers at $x = 0$ and $y = 0$ and a corner layer at $(0, 0)$.

Layer phenomena appear in different kinds of problems, for example singularly perturbed problems, which are one of the important topics in scientific computation. If the priori knowledge of layers is obtained from asymptotic analysis, different types of meshes can be designed for uniformly convergent numerical methods (see [1, 2]). Here “uniformly” means that the convergence is independent of the singular perturbation parameter. Bakhvalov-type meshes are one of the most popular layer-adapted meshes and usually have better numerical performances than Shishkin-type meshes—another popular layer-adapted meshes (see [3]). The uniform convergence theory of finite element methods on Shishkin-type meshes has been studied deeply; see [1, 4–12] and references therein.

However, it is far from mature for convergence theories of finite element methods on Bakhvalov-type meshes. One of the main reasons is that the usual Lagrange interpolant does not work for Bakhvalov-type meshes (see [13]). In [14–17], we proposed a new idea for convergence analysis on Bakhvalov-type meshes in 1D. Here we extend the analysis to two dimensions. It is not trivial to extend the idea of [14] to 2D case. In one dimension, we arbitrarily modify the definition of the usual Lagrange interpolation on the last mesh in the layer to satisfy the need of convergence analysis. In two dimensions, we cannot do the same thing because boundary conditions must be taken into account. This interpolant is carefully defined according to the characteristics of layer functions and the structures of Bakhvalov-type meshes. The interpolation errors are derived in a delicate way. Then almost uniform convergence of optimal order is proved for finite element methods.

The rest of the paper is organized as follows. In Section 2 we describe the assumptions on the regularity of the solution, introduce a Bakhvalov-type mesh and define a finite element method of any order. Some preliminary results for the subsequent analysis are also given in this section. In Section 3 we construct and analyze an interpolant to the solution for uniform convergence on the Bakhvalov-type mesh. In Section 4 almost uniform convergence of optimal order is obtained by means of the interpolant and careful analysis of the convective term in the bilinear form. In Section 5, numerical results illustrate our theoretical bounds.

Denote by $\| \cdot \|_{\infty, D}$ the norm in the Lebesgue space $L^\infty(D)$. In $L^2(D)$, the inner product and the $L^2(D)$-norm are denoted by $(\cdot, \cdot)_D$ and $\| \cdot \|_D$, respectively. In $H^1(D)$, the seminorm is denoted by $| \cdot |_1$. Here $D$ is any measurable subset of $\Omega$. When $D = \Omega$ we drop the subscript $D$ from the notation for simplicity. Throughout the article, all constants $C$ are independent of $\varepsilon$ and the mesh parameter $N$ and may take different values in different formulas.

2 | DECOMPOSITION OF THE SOLUTION, BAKHVALOV-TYPE MESH AND FINITE ELEMENT METHOD

In this section we present a decomposition of the solution to (1.1), introduce a Bakhvalov-type mesh and define a finite element method. Some preliminary inequalities are also presented.

2.1 | Regularity of the solution

We make the following assumption about the solution $u$ to (1.1), which describes the structure of $u$. This assumption is also used in [18].

**Assumption 1** Let $k \geq 1$ be a fixed integer. The solution $u$ of (1.1) can be decomposed as

$$u = S + E_1 + E_2 + E_{12},$$  
(2.1a)
where $S$ is the smooth part of $u$, $E_1$ and $E_2$ are exponential layers along the sides $x = 0$ and $y = 0$ of $\Omega$ respectively, while $E_{12}$ is an exponential corner layer at $(0, 0)$. Moreover, there exists a constant $C$ such that for all $(x, y) \in \bar{\Omega}$ and $0 \leq i + j \leq k + 1$ one has

$$\left| \frac{\partial^{i+j}S}{\partial x^i \partial y^j}(x, y) \right| \leq C,$$

(2.1b)

$$\left| \frac{\partial^{i+j}E_1}{\partial x^i \partial y^j}(x, y) \right| \leq C \varepsilon^{-i} e^{-\beta_1 x/\varepsilon},$$

(2.1c)

$$\left| \frac{\partial^{i+j}E_2}{\partial x^i \partial y^j}(x, y) \right| \leq C \varepsilon^{-j} e^{-\beta_2 y/\varepsilon},$$

(2.1d)

$$\left| \frac{\partial^{i+j}E_{12}}{\partial x^i \partial y^j}(x, y) \right| \leq C \varepsilon^{-(i+j)} e^{-(\beta_1 x + \beta_2 y)/\varepsilon}. $$

**Remark 2.1** For the case $k = 1$, that is, $0 \leq i + j \leq 2$, the existence of this decomposition of $u$ with the bounds on derivatives can be guaranteed by conditions on the data of the problem (1.1) (see [19]). The arguments in [19] make this assumption with $0 \leq i + j \leq k + 1$, credible if we impose sufficient compatibility conditions on $f$ (see some explanations in [20], sect. 7).

### 2.2 Bakhvalov-type mesh

Bakhvalov mesh first appeared in [21] and is graded in the layer. Its applications require the solution of a nonlinear equation. To avoid this difficulty, Bakhvalov-type meshes are proposed as approximations of Bakhvalov mesh (see [3]).

Let $N \geq 4$ be an even positive integer. We introduce a Bakhvalov-type mesh in the $x$-direction

$$0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1.$$

To resolve the layer along $x = 0$, the mesh is graded in $[x_0, x_{N/2}]$ and equidistant in $[x_{N/2}, 1]$. The mesh points $x_i$ is defined by

$$x_i = \begin{cases} \frac{\alpha \varepsilon}{\beta_i} \varphi(t_i) & \text{with } t_i = i/N \quad \text{for } i = 0, 1, \ldots, N/2, \\ 1 - (1 - x_{N/2}) 2(N - i)/N & \text{for } i = N/2 + 1, \ldots, N, \end{cases}$$

(2.2)

with $\sigma \geq k + 1$ and $\varphi(t) := -\ln(1 - 2(1 - \varepsilon)t)$. The parameter $\sigma$ determines the smallness of the layer terms in $x_{N/2}$. This Bakhvalov-type mesh is also analyzed in [13]. In a similar way we define the mesh $\{y_j\}_{j=0}^N$ along the $y$-direction except that we replace $\beta_1$ by $\beta_2$ in (2.4). Then we obtain a tensor-product rectangular mesh $T_N$ with mesh points $(x_i, y_j)$ (see Figure 1).

**Assumption 2** Assume that $\varepsilon \leq N^{-1}$ in our analysis, as is not a restriction in practice.

Set $h_{i,x} := x_{i+1} - x_i$ and $h_{j,y} := y_{j+1} - y_j$ for all $i, j$. A mesh rectangle is often written as $\tau_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ for a specific element and more simply as $\tau_i$ for a generic mesh rectangle.

According to [14] (lemma 3), we have the following lemma.

**Lemma 2.1** For the Bakhvalov-type mesh (2.2), one has

$$h_{0,x} \leq h_{1,x} \leq \ldots \leq h_{N/2-2,x},$$

(2.3)
\begin{equation}
C \varepsilon N^{-1} \leq h_{0,i} \leq C \varepsilon N^{-1},
\end{equation}

\begin{equation}
\frac{1}{4} \sigma \varepsilon \leq h_{N/2-2,x} \leq \sigma \varepsilon,
\end{equation}

\begin{equation}
\frac{1}{2} \sigma \varepsilon \leq h_{N/2-1,x} \leq 2 \sigma N^{-1},
\end{equation}

\begin{equation}
N^{-1} \leq h_{i,x} \leq 2 N^{-1} \quad N/2 \leq i \leq N - 1,
\end{equation}

\begin{equation}
x_{N/2-1} \geq C \sigma \varepsilon \ln N, \quad x_{N/2} \geq C \sigma \varepsilon |\ln \varepsilon|,
\end{equation}

\begin{equation}
h_{i,x}^\mu e^{-\beta i/\varepsilon} \leq C \varepsilon^\mu N^{-\mu} \quad \text{for } 0 \leq i \leq N/2 - 2 \quad \text{and} \quad 0 \leq \mu \leq \sigma.
\end{equation}

For $h_{j,y}$, $0 \leq j < N$, bounds analogous to (2.3)–(2.9) also hold.

2.3 Finite element method

On the above Bakhvalov-type mesh, define the finite element space by

\[ V_N := \left\{ v_N \in C(\overline{\Omega}) : v_N \bigg|_{\partial \Omega} = 0 \text{ and } v_N \bigg|_{\Gamma} \in Q_k(\tau) \forall \tau \in \mathcal{T}_N \right\}, \]

where $Q_k(\tau) = \text{span} \{ x^i y^j : 0 \leq i, j \leq k \}$.

The finite element method is defined as follows: Find $u_N \in V_N$ such that

\begin{equation}
a \left( u_N, v_N \right) = (f, v_N) \quad \forall v_N \in V_N,
\end{equation}

with $a(u_N, v_N) := \varepsilon (\nabla u_N, \nabla v_N) + (-b \cdot \nabla u_N + c u_N, v_N)$. Condition (1.2b) implies the coercivity

\begin{equation}
a \left( v_N, v_N \right) \geq a \left\| v_N \right\|_\varepsilon^2 \quad \text{for all } v_N \in V_N,
\end{equation}

where $a = \min\{1, \gamma\}$ and

\[ \left\| v \right\|_\varepsilon := \left\{ \varepsilon \left| v \right|_1^2 + \left\| v \right\|_2^2 \right\}^{1/2} \quad \forall v \in H^1(\Omega). \]
It follows that there exists a unique solution \( u^N \) for problem (2.10) from Lax-Milgram lemma. Clearly, (1.1) and (2.10) imply the Galerkin orthogonality property

\[
a(u - u^N, v^N) = 0 \quad \text{for all } v^N \in V^N. \tag{2.12}
\]

## 3 | INTERPOLATION AND ITS ERRORS

A new interpolation operator is introduced for uniform convergence on the Bakhvalov-type mesh. Set \( x_i^t := x_i + (s/k)h_{i,x} \) and \( y_j^t := y_j + (t/k)h_{j,y} \) for \( i, j = 0, 1, \ldots, N-1 \) and \( s, t = 0, \ldots, k-1 \). For the consistency of notation, set \( x_N^0 = x_N \) and \( y_N^0 = y_N \). For any \( v \in C^0(\Omega) \) its standard Lagrange interpolant \( v^I \in V^N \) on the Bakhvalov-type mesh can be written in the following form

\[
v^I(x, y) = \sum_{j=0}^{N-1} \sum_{s=0}^{k-1} \left( \sum_{i=0}^{N-1} \sum_{t=0}^{k-1} v(x_i^t, y_j^t) \theta_{i,j}^{s,t}(x, y) \right)
\]

\[
+ \sum_{j=0}^{N-1} \sum_{s=0}^{k-1} v(x_0^s, y_j^t) \theta_{0,j}^{s,t}(x, y) + v(x_N^0, y_N^0) \theta_{0,0}^{0,0}(x, y),
\]

where \( \theta_{i,j}^{s,t}(x, y) \in V^N \) is the piecewise \( k \)-th order hat function associated with the point \( (x_i^t, y_j^t) \). For the solution \( u \) to (1.1), recall (2.1a) in Assumption 1 and define the interpolant \( \Pi u \) by

\[
\Pi u = S' + \pi_1 E_1 + \pi_2 E_2 + \pi_{12} E_{12}. \tag{3.1}
\]

Here \( S' \) is the Lagrange interpolant to \( S \) and

\[
(\pi_i E_i)(x, y) = E_i^I - P_i E_i + B_i E_i \quad \text{for } i = 1, 2,
\]

\[
(\pi_{12} E_{12})(x, y) = E_{12}^I - P_{12} E_{12}
\]

where

\[
P_1 E_1 = \sum_{i=N/2-1}^{N/2} \sum_{s=0}^{k-1} \left( \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} E_1(x_i^t, y_j^t) \theta_{i,j}^{s,t}(x, y) + E_1(x_0^s, y_j^t) \theta_{0,j}^{s,t}(x, y) \right), \tag{3.3}
\]

\[
P_2 E_2 = \sum_{j=N/2-1}^{N/2} \sum_{s=0}^{k-1} \left( \sum_{i=0}^{N-1} \sum_{t=0}^{k-1} E_2(x_i^t, y_j^t) \theta_{i,j}^{s,t}(x, y) + E_2(x_0^s, y_j^t) \theta_{0,j}^{s,t}(x, y) \right), \tag{3.4}
\]

\[
P_{12} E_{12} = \sum_{i=N/2-1}^{N/2} \sum_{j=N/2-1}^{N/2} \left( \sum_{s=0}^{k-1} \sum_{t=0}^{k-1} E_{12}(x_i^t, y_j^t) \theta_{i,j}^{s,t}(x, y) \right) \tag{3.5}
\]

and

\[
B_1 E_1 = \sum_{i=0}^{k-1} \sum_{j=0}^{N} \sum_{s=0}^{k-1} E_1(x_i^t, y_j^0) \theta_{i,j}^{s,0}(x, y),
\]

\[
B_2 E_2 = \sum_{i=0}^{k-1} \sum_{j=0}^{N} \sum_{t=0}^{k-1} E_2(x_i^0, y_j^t) \theta_{i,j}^{0,t}(x, y). \tag{3.6}
\]

Clearly we have

\[
\Pi u \in V^N, \quad \Pi u = u' - \sum_{i=1,2} P_i E_i + \sum_{i=1,2} B_i E_i. \tag{3.7}
\]
From [22] (theorem 2.7), we have the following anisotropic interpolation results.

**Lemma 3.1** Let \( \tau \in T_N \) and \( v \in H^{k+1}(\tau) \). Then there exists a constant \( C \) such that Lagrange interpolation \( v^I \) satisfies

\[
\|v - v^I\|_\tau \leq C \sum_{i+j=k+1} h_{x,i}^k h_{y,j}^l \left\| \frac{\partial^{k+1} v}{\partial x^i \partial y^j} \right\|_\tau
\]

\[
\left\| (v-v^I)_x \right\|_\tau \leq C \sum_{i+j=k} h_{x,i}^l h_{y,j}^r \left\| \frac{\partial^{k+1} v}{\partial x^i \partial y^{j+1}} \right\|_\tau
\]

and

\[
\left\| (v-v^I)_y \right\|_\tau \leq C \sum_{i+j=k} h_{x,i}^l h_{y,j}^r \left\| \frac{\partial^{k+1} v}{\partial x^{i+1} \partial y^j} \right\|_\tau,
\]

where \( h_{x,r} \) and \( h_{y,t} \) denote the lengths along \( x \)-axis and \( y \)-axis of the rectangle \( \tau \), respectively.

Set

\[
\Omega_{11} := [x_0, x_{N/2-1}] \times [y_0, y_{N/2-1}], \quad \Omega_{12} := [x_{N/2-1}, x_N] \times [y_0, y_{N/2-1}],
\]

\[
\Omega_{21} := [x_0, x_{N/2-1}] \times [y_{N/2-1}, y_N], \quad \Omega_{22} := [x_{N/2-1}, x_N] \times [y_{N/2-1}, y_N].
\]

**Lemma 3.2** Let Assumptions 1 and 2 hold true. Let \( E^I_i, i=1,2 \), denote the Lagrange interpolants of \( E_i \), \( i=1,2 \), respectively, on the Bakhvalov-type mesh \( T_N \). Then there exists a constant \( C \) such that the following interpolation error estimates hold true:

\[
\|E_1 - E^I_1\|_{\Omega((x_{N/2-1},x_{N/2}) \times [0,1])} + \|E_2 - E^I_2\|_{\Omega((0,1) \times (y_{N/2-1},y_{N/2}))} \leq C \varepsilon^{1/2} N^{-(k+1)/2},
\]

\[
\|E_1 - E^I_1\|_e + \|E_2 - E^I_2\|_e \leq C N^{-(k+1)},
\]

\[
\|E_1 - E^I_1\|_e + \|E_2 - E^I_2\|_e \leq C N^{-k},
\]

\[
\|P_1 E_1\|_e + \|P_2 E_2\|_e + \|B_1 E_1\|_e + \|B_2 E_2\|_e \leq C N^{1/2 - \sigma},
\]

where \( P_I E_i, i=1,2 \), are defined in (3.3) and (3.4), respectively.

**Proof.** We just consider \( E_1 \), since \( E_2 \) can be analyzed in a similar way. Decompose \( \|E_1 - E^I_1\|_e \) as follows

\[
\|E_1 - E^I_1\|^2 = \|E_1 - E^I_1\|_{[x_0,x_{N/2-1}] \times [0,1]}^2
\]

\[
+ \|E_1 - E^I_1\|_{[x_{N/2-1},x_N] \times [0,1]}^2 + \|E_1 - E^I_1\|_{[y_{N/2-1},y_N] \times [0,1]}^2
\]

\[
= A_1 + A_2 + A_3.
\]
Collecting (3.8), (3.11) and (3.12), we prove the first and the second bounds. From (3.9) and (3.10) we obtain
\[
|E_1(\frac{x_i}{N/2-1}, y_j^f)| \leq CN^{-\sigma}.
\]
(3.10)

Now we consider the term $A_2$. Set $D_0 = [x_{N/2-1}, x_{N/2}] \times [0, 1]$. Recall $|E_1(\frac{x_i}{N/2-1}, y_j^f)| \leq CN^{-\sigma}$. Then we have
\[
\|E_i\|_{D_0} \leq \frac{CN^{-\sigma}}{\sqrt{2}} \left( \sum_{i=N/2-1}^{N-1} \sum_{j=0}^{k-1} \left( \sum_{l=0}^{N-1} \sum_{m=0}^{k-1} \frac{1}{\|E_i\|_{D_0}^2} + \frac{1}{\|E_i\|_{D_0}^2} \right) \right)
\]
\[
+ \frac{CN^{-2\sigma}}{\sqrt{2}} \left( \sum_{i=N/2}^{N-1} \sum_{j=0}^{k-1} \left( \sum_{l=0}^{N-1} \sum_{m=0}^{k-1} \frac{1}{\|E_i\|_{D_0}^2} + \frac{1}{\|E_i\|_{D_0}^2} \right) \right)
\]
\[
\leq \frac{CN^{-2\sigma}}{\sqrt{2}} \sum_{j=0}^{N-1} \|h_{N/2-1,x} h_{j, y}\} \leq CN^{-(2\sigma+1)},
\]
(3.9)

where we have used Lemma 2.1. Direct calculations yield
\[
\|E_1\|_{D_0} \leq C\epsilon^{1/2}N^{-\sigma}.
\]
(3.10)

From (3.9) and (3.10) we obtain
\[
A_2 \leq C (\epsilon + N^{-1}) N^{-2\sigma}.
\]
(3.11)

Note $|E_1(\frac{x_i}{N/2-1}, y_j)| \leq C\epsilon^\sigma$ for $i \geq N/2$. Then the triangle inequality and Hölder inequalities yield
\[
A_3 \leq C \left( \|E_1\|_{L^2[\frac{x_i}{N/2-1}, x_{N/2}] \times [0, 1]} + \|E_i\|_{L^2[\frac{x_i}{N/2-1}, x_{N/2}] \times [0, 1]} \right)
\]
\[
\leq C \left( \|E_1\|_{L^2[\frac{x_i}{N/2-1}, x_{N/2}] \times [0, 1]} + \|E_i\|_{L^2[\frac{x_i}{N/2-1}, x_{N/2}] \times [0, 1]} \right)
\]
\[
\leq C\epsilon^{2\sigma}.
\]
(3.12)

Collecting (3.8), (3.11) and (3.12), we prove the first and the second bounds.
where inverse inequalities \(23\) (Theorem 3.2.6) and Lemma 2.1 yield

\[
\| E - E_i \|_{\Omega_{j+1}} \leq C \varepsilon^{-1/2} N^{-\sigma}.
\]

Note \(\| E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}} \leq C \varepsilon^{-1/2} N^{-\sigma}\). Then from the triangle inequality one has

\[
\| E - E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}} \leq 2 \| E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}} + 2 \| E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}}
\]

\[
\leq 2 \| E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}} + 2 \sum_{j=0}^{N-1} \sum_{i=N/2}^{N-1} \sum_{j=0}^{N-1} \| E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}}
\]

\[
\leq C \varepsilon^{-1} N^{-2\sigma} + C \varepsilon^{2\sigma} N^2,
\]

where inverse inequalities \(23\) (Theorem 3.2.6) and Lemma 2.1 yield

\[
\sum_{j=0}^{N-1} \sum_{i=N/2}^{N-1} \| E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}} \leq C \varepsilon^{-1/2} N^{-\sigma} \leq C \varepsilon N^{-2k}.
\]

Note \(\| E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}} \leq C \varepsilon^{1/2} N^{-\sigma}\) and \(h_{j+1} \geq C \varepsilon N^{-1}\) for \(j = 0, \ldots, N - 1\). Then one has

\[
\| E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}} \leq C \sum_{i=N/2}^{N-1} \sum_{j=0}^{N-1} \| E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}}
\]

\[
\leq C \varepsilon^{-1} N^{-2\sigma} + C \varepsilon^{2\sigma} N^2.
\]

Similar to the derivations of (3.13), we have

\[
\| (E - E_i) \|_{\Omega_{j+1} \cup \Omega_{j+2}} \leq C \varepsilon N^{-2k}.
\]

Note \(\| E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}} \leq C \varepsilon^{1/2} N^{-\sigma}\) and \(h_{j+1} \geq C \varepsilon N^{-1}\) for \(j = 0, \ldots, N - 1\). Then one has

\[
\| (E - E_i) \|_{\Omega_{j+1} \cup \Omega_{j+2}} \leq 2 \| (E_i) \|_{\Omega_{j+1} \cup \Omega_{j+2}} + 2 \| (E_i) \|_{\Omega_{j+1} \cup \Omega_{j+2}}
\]

\[
\leq C \varepsilon^{-1} N^{-2\sigma} + C \varepsilon^{-1} N^{-2\sigma},
\]

where

\[
\| (E_i) \|_{\Omega_{j+1} \cup \Omega_{j+2}} = \sum_{i=N/2}^{N-1} \sum_{j=0}^{N-1} \| E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}}
\]

\[
\leq C \sum_{i=N/2}^{N-1} \sum_{j=0}^{N-1} h_{j+1} \| E_i \|_{\Omega_{j+1} \cup \Omega_{j+2}}
\]
Collecting (3.13)–(3.16) and considering \( \| E_1 - E_1^f \| \leq CN^{-(k+1)} \), we prove the third bound.

Now we consider \( \| P_1 E_1 \|_\varepsilon \). From (3.17) and \( | E(x_{N/2-1}, y_j^f) | \leq CN^{-\sigma} \), we can easily obtain

\[
\| P_1 E_1 \|_\varepsilon^2 \leq CN^{-2\sigma} \sum_{s=0}^{k-1} \left( \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} \left( \| \theta_{N/2-1,j}^t \|_\varepsilon^2 + \| \theta_{N/2-1,N}^t \|_\varepsilon^2 \right) \right) 
\leq CN^{-2\sigma} \sum_{j=0}^{N-1} \left( \varepsilon h_{N/2-1,j,y}^{-1} + \varepsilon h_{N/2-1,x,y}^{-1} + h_{N/2-1,x,y}^{-1} \right) 
\leq CN^{1-2\sigma},
\]

where we have used \( h_{N/2-1,j,y}^{-1} \leq C\varepsilon^{-1}N \) from Lemma 2.1. The terms \( P_2 E_2, B_1 E_1 \) and \( B_2 E_2 \) can be analyzed in a similar way.

**Lemma 3.3** Let Assumptions 1 and 2 hold true. Let \( E_{12}^f \) denote the Lagrange interpolant of \( E_{12} \) on the Bakhvalov-type mesh \( T_N \). Then there exists a constant \( C \) such that the following interpolation error estimates hold true:

\[
\| E_{12} - E_{12}^f \|_{\Omega_{N/2-1,N/2-1}} \leq C\varepsilon N^{-k} + Ce^{1/2}N^{-(k+1)},
\]

\[
\| E_{12} - E_{12}^f \|_\varepsilon \leq C\varepsilon N^{-k} + Ce^{1/2}N^{-(k+1)} + CN^{-1-2\sigma},
\]

\[
\| E_{12} - E_{12}^f \|_\varepsilon \leq CN^{-k},
\]

\[
\| P_1 E_{12}^f \|_\varepsilon \leq CN^{1/2-2\sigma},
\]

where \( P_1 E_{12} \) is defined in (3.5).

**Proof.** Consider

\[
\| E_{12} - E_{12}^f \|_\varepsilon^2 = \| E_{12} - E_{12}^f \|_{\Omega_{i1}}^2 + \| E_{12} - E_{12}^f \|_{\Omega_{N/2-1,N/2-1}}^2 + \| E_{12} - E_{12}^f \|_{\Omega_{i1} \cup \Omega_{i2} \cup \Omega_{i2} \setminus \Omega_{N/2-1,N/2-1}}^2.
\]

Similar to (3.8), we obtain

\[
\| E_{12} - E_{12}^f \|_{\Omega_{i1}}^2 \leq C\varepsilon^2 N^{-2k}.
\]

To analyze the second term in (3.18), we frequently use the following estimation

\[
\| E_{12}^f \|_{\Omega_{ij}}^2 \leq \| E_{12}^f \|_{\Omega_{ij}}^2 \leq Ce^{-2(\beta_1 x_i + \beta_2 y_j)} h_{i,x} h_{j,y}.
\]

From (2.1d), direct calculations yield

\[
\| E_{12} \|_{\Omega \setminus \Omega_{i1}} \leq Ce^{-\sigma}.
\]
From (3.20) and Lemma 3.1, one has
\[
\left\| E_{12}^l \right\|^2_{\Omega_{12}} = \sum_{i=N/2-1}^{N-1} \sum_{j=0}^{N/2-2} \left\| E_{12}^l \right\|^2_{\tau_{ij}} \\
\leq C \sum_{i=N/2-1}^{N-1} \sum_{j=0}^{N/2-2} e^{-2(\beta_1 y_j + \beta_2 y_j)/\varepsilon} h_{1x} h_{jy} \\
\leq C \sum_{i=N/2-1}^{N-1} \sum_{j=0}^{N/2-2} e^{-2\beta_1 y_j/\varepsilon} h_{1x} h_{jy} \leq C\varepsilon N^{-2\sigma},
\]
(3.22)
where Lemma 2.1 has been used. Similarly, we have
\[
\left\| E_{12}^l \right\|^2_{\Omega_{21}} \leq C\varepsilon N^{-2\sigma}.
\]
Using (3.20) again, we have
\[
\left\| E_{12}^l \right\|^2_{(\Omega_{22} \setminus \tau_{N/2-1,N/2-1})} = \sum_{j=N/2}^{N-1} \left\| E_{12}^l \right\|^2_{\tau_{N/2-1,j}} + \sum_{i=N/2}^{N-1} \sum_{j=N/2}^{N-1} \left\| E_{12}^l \right\|^2_{\tau_{ij}} \\
\leq \sum_{j=N/2}^{N-1} e^{-2(\beta_1 y_j + \beta_2 y_j)/\varepsilon} h_{N/2-1,x} h_{jy} + \sum_{i=N/2}^{N-1} \sum_{j=N/2}^{N-1} e^{-2(\beta_1 y_j + \beta_2 y_j)/\varepsilon} h_{1x} h_{jy} \\
\leq CN^{-1-2\sigma} \varepsilon^{2\sigma} + CN^{-2\sigma} \varepsilon^{2\sigma},
\]
(3.24)
and
\[
\left\| E_{12}^l \right\|^2_{\tau_{N/2-1,N/2-1}} \leq CN^{-2-4\sigma}.
\]
(3.25)
Then the triangle inequality and (3.19), (3.21)–(3.25) yield the first and the second bounds.

Now we consider \( \left\| (E_{12} - E_{12}^l)_x \right\| \). By similar derivations for (3.13), we have
\[
\left\| (E_{12} - E_{12}^l)_x \right\|_{\Omega_{11}} \leq CN^{-k}.
\]
(3.26)
Similar to (3.14), the following bound can be obtained
\[
\left\| (E_{12} - E_{12}^l)_x \right\|_{\Omega_{11}} \leq C\varepsilon^{-1/2} N^{1-\sigma}.
\]
(3.27)
Combing (3.26) and (3.27), we prove
\[
\left\| (E_{12} - E_{12}^l)_x \right\| \leq C\varepsilon^{-1/2} N^{1-\sigma}
\]
and the same bound for \( \left\| (E_{12} - E_{12}^l)_y \right\| \). Thus we prove the third bound. Similar to (3.17),
the final bound can be proved.}\]

Lemma 3.4 Let Assumptions 1 and 2 hold true. Let \( S^l \) and \( u^l \) denote the Lagrange interpolants of \( S \) and \( u \) on the Bakhvalov-type mesh \( \tau_N \), respectively. Let \( \Pi u \) and \( \pi_i E_i, \)
\( i = 1, 2, 12, \) be defined in (3.1) and (3.2), respectively. Then there exists a constant \( C \) such that the following interpolation error estimates hold true:
\[
\sum_{i=1,2,12} \left\| \pi_i E_i - E_i \right\| \leq CN^{-(k+1)},
\]
\[
\left\| \nabla (S - S^l) \right\| + \left\| u - u^l \right\| + \left\| u - \Pi u \right\| \leq CN^{-k}.
\]
Proof. Check the derivations in (3.17) and one finds that \( \|P_i E_i\| \leq CN^{-1/2-\sigma} \). Thus
\[
\|\pi_i E_i - E_i\| \leq \|E_i' - E_i\| + \|P_i E_i\| + \|B_i E_i\| \leq CN^{-(k+1)}.
\]
Similarly, we can prove the bounds for \( E_2 \) and \( E_{12} \).

From Lemma 3.1 and (2.1b), we prove \( \|\nabla(S - S')\| \leq CN^{-k} \) and \( \|S - S'\|_\varepsilon \leq C(e^{1/2} + N^{-1})N^{-k} \) easily. Then (2.1a), the triangle inequality, Lemmas 3.2 and 3.3 yield \( \|u - u'\|_\varepsilon \leq CN^{-k} \). Besides, from (3.7), Lemmas 3.2 and 3.3 one has \( \|u - \Pi u\|_\varepsilon \leq CN^{-k} \).  

4 | UNIFORM CONVERGENCE

Set \( \chi := \Pi u - u^N \). From (2.11), (2.12), (2.1a), (2.3) and integration by parts, one has
\[
a\|\chi\|_\varepsilon^2 \leq a(\chi, \chi) = a(\Pi u - u, \chi)
= \varepsilon \int_\Omega \nabla((\Pi u - u) \chi) \, dx \, dy + \sum_{i=1,2} \int_\Omega (E_i' - P_i E_i - E_i) \cdot \nabla \chi \, dx \, dy
+ \int_\Omega (E_{12}' - P_{12} E_{12} - E_{12}) \cdot \nabla \chi \, dx \, dy - \int_\Omega \nabla \chi \, dx \, dy
+ \sum_{i=1,2} \int_\Omega (\nabla \cdot b) (\pi_i E_i - E_i) \chi \, dx \, dy + \int_\Omega c(\Pi u - u) \chi \, dx \, dy
+ \sum_{i=1,2} \int_\Omega (B_i E_i) \cdot \nabla \chi \, dx \, dy
=: I + II + III + IV + V + VI + VII. \tag{4.1}
\]
Now we analyze the terms on the right-hand side of (4.1). The Cauchy–Schwarz inequality and Lemma 5 yield
\[
(I + VI) + (IV + V)
\leq C\|\Pi u - u\|_\varepsilon \|\chi\|_\varepsilon + C \left( \|\nabla(S' - S)\| + \sum_{i=1,2,12} \|\pi_i E_i - E_i\| \right) \|\chi\|
\leq CN^{-k} \|\chi\|_\varepsilon. \tag{4.2}
\]
We put the arguments for II, III and VII in the following three lemmas.

**Lemma 4.1** Let Assumptions 1 and 2 hold true. Let \( P_i E_i \) with \( i = 1, 2 \) be defined in (3.3) and (3.4), respectively. Then one has
\[
|II| = \left| \sum_{i=1,2} \int_\Omega (E_i' - P_i E_i - E_i) \cdot \nabla \chi \, dx \, dy \right| \leq CN^{-(k+1/2)} \|\chi\|_\varepsilon. \tag{4.3}
\]

**Proof.** For the term II, we just consider \( E_1 \) since we can analyze \( E_2 \) in a similar way. Recall \( D_0 := [x_{N/2-1}, x_{N/2}] \times [0, 1] \). According to (3.2) and (3.3), one has
\[
\int_\Omega (E_1' - P_1 E_1 - E_1) \cdot \nabla \chi \, dx \, dy
= (E_1' - P_1 E_1 - E_1, \nabla \chi)_{\Omega \backslash D_0} + (E_1' - P_1 E_1 - E_1, \nabla \chi)_{D_0}
\]
\[ (E_1^t - E_1, b \cdot \nabla \chi)_{\Omega \setminus D_0} + \left( -F_1, b \cdot \nabla \chi \right)_{\Omega \setminus D_0} \]
\[ + (F_2 - E_1, b \cdot \nabla \chi)_{D_0} \]
\[ =: T_1 + T_2 + T_3. \quad (4.4) \]

where \( (E_1^t - P_1 E_1) \mid_{\Omega \setminus D_0} = (E_1^t - E_1) \mid_{\Omega \setminus D_0} \cdot (E_1^t - P_1 E_1) \mid_{D_0} = F_2 \mid_{D_0} \) and
\[
F_1 := \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} E_1 \left( x_{N/2-1,j}, y_j^t \right) \theta_{N/2-1,1}^{0,t} + E_1 \left( x_{N/2-1,1}, y_N \right) \theta_{N/2-1,1,N}^{0,0}, \\
F_2 := \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} E_1 \left( x_{N/2,j}, y_j^t \right) \theta_{N/2,j}^{0,t} + E_1 \left( x_{N/2,j}, y_N \right) \theta_{N/2,j}^{0,0}.
\]

From Lemma 3.2 and the Cauchy–Schwarz inequality, one has
\[
|T_1| \leq C \left\| E_1 - E_1^t \right\|_{\Omega \setminus D_0} \left\| \nabla \chi \right\|_{\Omega \setminus D_0} \leq C \epsilon^{1/2} N^{-(k+1/2)} \left\| \nabla \chi \right\| \leq CN^{-(k+1/2)} \left\| \chi \right\|. \quad (4.5)
\]

Note \( |E_1 \left( x_{N/2-1,j}, y_j^t \right) | \leq CN^{-\sigma} \) for any \( j, t \) and \( h_{N/2-2,x} \leq C \epsilon \). The Cauchy–Schwarz inequality yields
\[
|T_2| \leq C \| F_1 \|_{D_1} \left\| \nabla \chi \right\|_{D_1} \leq CN^{-\sigma} \left\| \nabla \chi \right\|_{D_1} \left( \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} \left\| \theta_{N/2-1,1}^{0,t} \right\|_{D_1} + \left\| \theta_{N/2-1,1,N}^{0,0} \right\|_{D_1} \right) \leq CN^{1/2-\sigma} \left\| \chi \right\|, \quad (4.6)
\]

where \( D_1 := [x_{N/2-2}, x_{N/2-1}] \times [0, 1] \) and we have made use of the supports of hat functions \( \theta_{ij}^{0,t} \).

Now we deal with the term \( T_3 \). Note \( |E_1 \left( x_{N/2,j}, y_j^t \right) | \leq C \epsilon^\sigma \) for any \( j, t \) and \( h_{N/2-1,x} \leq CN^{-1} \). The Cauchy–Schwarz inequality yields
\[
|T_3| \leq C \left( \| E_1 \|_{D_0} + \| F_2 \|_{D_0} \right) \left\| \nabla \chi \right\|_{D_0} \leq C \left( \epsilon^{1/2} N^{-\sigma} + \epsilon^\sigma \sum_{j=0}^{N-1} h_{N/2-1,x}^{1/2} h_{j,y}^{1/2} \right) \left\| \nabla \chi \right\|_{D_0} \leq C \left( N^{-\sigma} + \epsilon^{\sigma-1/2} \right) \left\| \chi \right\|, \quad (4.7)
\]

where direct calculations and \( (2.1c) \) yield \( \| E_1 \|_{D_0} \leq C \epsilon^{1/2} N^{-\sigma} \) and \( \| F_2 \|_{D_0} \) is analyzed in a similar way to \( \| F_1 \|_{D_1} \) in (4.6).

Substituting (4.5)–(4.7) into (4.4), we are done.

Lemma 4.2 Let Assumptions 1 and 2 hold true. Let \( \pi_{12} E_{12} \) be defined in (3.2). Then one has
\[
|\text{III}| = \left| \int_{\Omega} \left( \pi_{12} E_{12} - E_{12} \right) b \cdot \nabla \chi \, dx dy \right| \leq CN^{-(k+1/2)} \left\| \chi \right\|, \quad (4.8)
\]
Proof. Set $D_2 := \Omega \backslash \tau_{N/2-1,N/2-1}$. According to (3.2), one has

$$
\int_{\Omega} (\pi_1 E_{12} - E_{12}) \cdot b \cdot \nabla \chi \, dx dy \\
= (\pi_1 E_{12} - E_{12}, b \cdot \nabla \chi)_{D_2} + (\pi_1 E_{12} - E_{12}, b \cdot \nabla \chi)_{\tau_{N/2-1,N/2-1}} \\
= (E_{12}' - E_{12}, b \cdot \nabla \chi)_{D_2} - (F_3, b \cdot \nabla \chi)_{D_2} + (F_4 - E_{12}, b \cdot \nabla \chi)_{\tau_{N/2-1,N/2-1}} \\
= Q_1 + Q_2 + Q_3,
$$

(4.9)

where $(\pi_1 E_{12})_{D_2} = (E_{12}' - F_3)_{D_2}$, $(\pi_1 E_{12})_{\tau_{N/2-1,N/2-1}} = F_4_{\tau_{N/2-1,N/2-1}}$

and

$$
P_3 := \sum_{i=1}^{k-1} E_{12} \left( x_{N/2-1}^t, y_{N/2-1}^t \right) \theta_{N/2-1,N/2-1}^{0,j} + E_{12} \left( x_{N/2-1}, y_{N/2-1} \right) \theta_{N/2-1,N/2-1}^{0,0} \\
+ \sum_{i=1}^{k-1} E_{12} \left( x_{N/2-1}^t, y_{N/2-1}^t \right) \theta_{N/2-1,N/2-1}^{0,j} \\
N_2,N/2-1 \\
N_2,N/2-1
$$

From Lemma 3.3 and the Cauchy–Schwarz inequality, one has

$$
|Q_1| \leq C \left\| E_{12}' - E_{12} \right\|_{D_2} \left\| \nabla \chi \right\|_{D_2} \leq C \varepsilon^{1/2} N^{-k+1/2} \left\| \nabla \chi \right\| \leq C N^{-k+1/2} \left\| \chi \right\| \varepsilon.
$$

(4.10)

Note $|E_{12} \left( x_{N/2-1}^t, y_{N/2-1}^t \right)| \leq CN^{-2\sigma}$ for any $0 \leq s, t \leq k-1$ and $h_{N/2-2,s}, h_{N/2-2,y} \leq C \varepsilon$. Then one has

$$
\left\| P_3 \right\|_{D_2} \leq CN^{-2\sigma} \left( \sum_{i=1}^{k-1} \left\| \theta_{N/2-1,N/2-1}^{0,j} \right\|_{D_2} + \left\| \theta_{N/2-1,N/2-1}^{0,0} \right\|_{D_2} + \sum_{i=1}^{k-1} \left\| \theta_{N/2-1,N/2-1}^{0,j} \right\|_{D_2} \right) \\
\leq CN^{-2\sigma} \left( h_{N/2-2-1,N/2-2,s}^{1/2} h_{N/2-2-1,y}^{1/2} + h_{N/2-2-1,N/2-2,y}^{1/2} + h_{N/2-2-1,N/2-1}^{1/2} \right) \\
\leq C \varepsilon^{1/2} N^{-1/2-2\sigma}.
$$

Thus the Cauchy–Schwarz inequality yields

$$
|Q_2| \leq C \left\| P_3 \right\|_{D_2} \left\| \nabla \chi \right\|_{D_2} \leq CN^{-1/2-2\sigma} \left\| \chi \right\| \varepsilon.
$$

(4.11)

Note $|E_{12} \left( x_{N/2}, y_{N/2} \right)| \leq |E_{12} \left( x_{N/2-1}^t, y_{N/2-1}^t \right)| + |E_{12} \left( x_{N/2}, y_{N/2} \right)| \leq C \varepsilon^{\sigma} N^{-\sigma}$ for any $0 \leq s, t \leq k-1$. The Cauchy–Schwarz inequality yields

$$
|Q_3| \leq C \left( \left\| P_4 \right\|_{\tau_{N/2-1,N/2-1}} + \left\| E_{12} \right\|_{\tau_{N/2-1,N/2-1}} \right) \left\| \nabla \chi \right\|_{\tau_{N/2-1,N/2-1}} \\
\leq C \left( \varepsilon^{\sigma} N^{-\sigma} h_{N/2-1,s}^{1/2} h_{N/2-1,y}^{1/2} + \varepsilon N^{-2\sigma} \right) \left\| \nabla \chi \right\|_{\tau_{N/2-1,N/2-1}} \\
\leq C \varepsilon^{1/2} N^{-2\sigma} \left\| \chi \right\| \varepsilon.
$$

(4.12)

Substituting (4.10)–(4.12) into (4.9), we are done.
Lemma 4.3  Let Assumptions 1 and 2 hold true. Let $B_i E_i$ with $i = 1, 2$ be defined in (3.6). Then one has

$$|VII| = \left| \sum_{i=1,2} \int_{\Omega} (B_i E_i) \cdot \nabla \chi \, dx dy \right| \leq CN^{-k} R(N, \varepsilon) \| \chi \|_\varepsilon, \quad (4.13)$$

where $R(N, \varepsilon) = N^{-3/2} \left| \ln \frac{3}{\varepsilon N} \right| - 0.0001 pt^{1/2}$.

Proof.  We just present the analysis for the term involving $B_1 E_1$ and the other could be analyzed in a similar way. Hölder inequalities

$$\left| \int_{\Omega} (B_1 E_1) \cdot \nabla \chi \, dx dy \right|$$

$$\leq \sum_{j=0}^{N-1} \| B_1 E_1 \|_{\infty, t_x/2 - 1, j} \| b \cdot \nabla \chi \|_{1, t_x/2 - 1, j}$$

$$\leq C \sum_{j=0}^{N-1} \left( N^{-\sigma} h_{t_x/2 - 1, x}^{1/2} h_{t_x/2 - 1, y}^{1/2} \| b \cdot \nabla \chi \|_{t_x/2 - 1, j} \right)$$

$$\leq C \left( N^{-(1+\sigma)} + N^{-(1/2+\sigma)} \left| \ln \frac{3}{\varepsilon N} \right|^{1/2} \right) \| \chi \|_\varepsilon$$

$$\leq CN^{-(1/2+\sigma)} \left| \ln \frac{3}{\varepsilon N} \right|^{1/2} \| \chi \|_\varepsilon,$$

where we have used $h_{0, y} \leq C\varepsilon N^{-1}$ in Lemma 2.1 and $h_{t_x/2 - 1, x} \leq \frac{\sigma_{h}}{\rho_{i}} \ln \frac{3}{\varepsilon N}$ from (2.2).

Remark 4.1 In practice $R(N, \varepsilon)$ is bounded: If we assume $N \geq 10$ and $\varepsilon \geq 10^{-1001}$, then $R(N, \varepsilon) \leq 1.6$.

Now we are in a position to present the main result.

Theorem 4.1  Let Assumptions 1 and 2 hold true. Let $u$ and $u^N$ be the solutions of (1.1) and (2.10), respectively. Then one has

$$\| u - u^N \|_\varepsilon \leq CN^{-k} R(N, \varepsilon),$$

where $R(N, \varepsilon) = N^{-3/2} \left| \ln \frac{3}{\varepsilon N} \right| - 0.0001 pt^{1/2}$.

Proof.  Substituting (4.2), (4.3), (4.8) and (4.13) into (4.1), we obtain $\| \Pi u - u^N \|_\varepsilon \leq CN^{-k} R(N, \varepsilon)$. From a triangle inequality and Lemma 3.4, one has

$$\| u - u^N \|_\varepsilon \leq \| u - \Pi u \|_\varepsilon + \| \Pi u - u^N \|_\varepsilon \leq CN^{-k} R(N, \varepsilon).$$

Thus we are done.

Remark 4.2  In Theorem 4.1, we show the almost optimal estimation, which is slightly better than the bound in [24], theorem 1.

Remark 4.3  We cannot obtain uniform convergence of optimal order similar to the case of 1D, that is, [14], because the boundary correction terms $B_1 E_1$ and $B_2 E_2$ cannot be estimated properly. In the future, we will continue to work on this issue.
5 | NUMERICAL EXPERIMENTS

In this section we present numerical experiments that support our theoretical results. All calculations were carried out using Intel Visual Fortran 2020 and the discrete problems were solved by the nonsymmetric iterative solver GMRES; see, for example, [25].

**Problem I**

\[-\varepsilon \Delta u - (2 + 2x - y)u_x - (3 - x + 2y)u_y + u = f(x, y) \quad \text{in } \Omega = (0, 1)^2,
\]
\[u = 0 \quad \text{on } \partial \Omega,
\]

where the right-hand side \(f\) is chosen such that

\[u(x, y) = 2 \sin(\pi x) \left(1 - e^{-\frac{x}{\varepsilon}}\right) \left(1 - y\right)^2 \left(1 - e^{-\frac{y}{\varepsilon}}\right)
\]

is the exact solution. This solution exhibits typical exponential layer behavior as described in Assumption 1. The errors \(e^{\varepsilon,N} = \|u - u^N\|_\varepsilon\) and convergence rates \(\tilde{p}^{\varepsilon,N} = \frac{\ln e^{\varepsilon,N} - \ln e^{\varepsilon,2N}}{\ln 2}\) will be analyzed so that the main result, that is, Theorem 1, can be verified.

**Problem II**

\[-\varepsilon \Delta u - b_1(x, y)u_x - b_2(x, y)u_y + u = f(x, y) \quad \text{in } \Omega = (0, 1)^2,
\]
\[u = 0 \quad \text{on } \Gamma = \partial \Omega,
\]

where

\[b_1(x, y) = \left(3 + 2x - \cos\left(\frac{\pi}{2}y\right)\right), \quad b_2(x, y) = \left(2 - \sin\left(\frac{\pi}{2}x\right) + 2y\right),
\]
\[f(x, y) = x(1-x) + y(1-y).
\]

It is unlikely to obtain the exact solution of this problem. Then it is impossible to measure the errors between the true solution \(u\) and the numerical solution \(u^N\). Instead, we therefore use the double mesh principle to estimate the errors; see [26] for a theoretical justification. That is, we compare each computed solution with the computed solution on a mesh that is twice as fine. The error estimates obtained in this way are denoted by

\[\tilde{u}^{\varepsilon,N} = \left\|u^{\varepsilon,N} - \tilde{u}^{\varepsilon,2N}\right\|_\varepsilon,
\]

where \(u^{\varepsilon,N}\) is the computed solution for the particular \(\varepsilon\) and \(N\), while \(\tilde{u}^{\varepsilon,2N}\) is the approximate solution on the mesh that is twice as fine. The error estimates obtained in this way are denoted by

\[\tilde{p}^{\varepsilon,N} = \frac{\ln \tilde{u}^{\varepsilon,N} - \ln \tilde{u}^{\varepsilon,2N}}{\ln 2}.
\]

For problems I and II, we set \(\beta_1 = 2\) and \(\beta_2 = 1\) in the definitions of Bakhvalov-type meshes, for example (2.4). For both problems, we will present errors for \(\varepsilon = 10^{-4}, 10^{-5}, \ldots, 10^{-8}\) and \(N = 8, 16, 32, 64, 128, 256\). With such \(N\) and \(\varepsilon\), Assumption 2 holds true.
### Table 1: Problem I and \( k = 1 \)

| \( \varepsilon \) | \( N \) | \( 8 \) | \( 16 \) | \( 32 \) | \( 64 \) | \( 128 \) | \( 256 \) |
|----------------|-------|------|------|------|------|------|------|
| \( 10^{-4} \)  | 0.227E+0 | 0.109E+0 | 0.540E-1 | 0.269E-1 | 0.135E-1 | 0.673E-2 |
| \( 10^{-4} \)  | 1.06   | 1.02  | 1.00  | 1.00  | 1.00  | –    |
| \( 10^{-5} \)  | 0.228E+0 | 0.109E+0 | 0.540E-1 | 0.269E-1 | 0.135E-1 | 0.673E-2 |
| \( 10^{-5} \)  | 1.06   | 1.02  | 1.00  | 1.00  | 1.00  | –    |
| \( 10^{-6} \)  | 0.229E+0 | 0.109E+0 | 0.540E-1 | 0.269E-1 | 0.135E-1 | 0.673E-2 |
| \( 10^{-6} \)  | 1.07   | 1.02  | 1.00  | 1.00  | 1.00  | –    |
| \( 10^{-7} \)  | 0.231E+0 | 0.110E+0 | 0.541E-1 | 0.269E-1 | 0.135E-1 | 0.673E-2 |
| \( 10^{-7} \)  | 1.08   | 1.02  | 1.00  | 1.00  | 1.00  | –    |
| \( 10^{-8} \)  | 0.234E+0 | 0.110E+0 | 0.541E-1 | 0.269E-1 | 0.135E-1 | 0.673E-2 |
| \( 10^{-8} \)  | 1.09   | 1.02  | 1.01  | 1.00  | 1.00  | –    |

### Table 2: Problem I and \( k = 2 \)

| \( \varepsilon \) | \( N \) | \( 8 \) | \( 16 \) | \( 32 \) | \( 64 \) | \( 128 \) |
|----------------|-------|------|------|------|------|------|
| \( 10^{-4} \)  | 0.502E-1 | 0.105E-1 | 0.247E-2 | 0.603E-3 | 0.150E-3 |
| \( 10^{-4} \)  | 2.25   | 2.10  | 2.03  | 2.01  | –    |
| \( 10^{-5} \)  | 0.501E-1 | 0.105E-1 | 0.247E-2 | 0.604E-3 | 0.150E-3 |
| \( 10^{-5} \)  | 2.25   | 2.09  | 2.03  | 2.01  | –    |
| \( 10^{-6} \)  | 0.500E-1 | 0.105E-1 | 0.246E-2 | 0.604E-3 | 0.150E-3 |
| \( 10^{-6} \)  | 2.25   | 2.09  | 2.03  | 2.01  | –    |
| \( 10^{-7} \)  | 0.500E-1 | 0.105E-1 | 0.246E-2 | 0.604E-3 | 0.150E-3 |
| \( 10^{-7} \)  | 2.25   | 2.09  | 2.03  | 2.01  | –    |
| \( 10^{-8} \)  | 0.502E-1 | 0.105E-1 | 0.246E-2 | 0.604E-3 | 0.150E-3 |
| \( 10^{-8} \)  | 2.25   | 2.10  | 2.03  | 2.01  | –    |

### Table 3: Problem II and \( k = 1 \)

| \( \varepsilon \) | \( N \) | \( 4 \) | \( 8 \) | \( 16 \) | \( 32 \) | \( 64 \) | \( 128 \) | \( 256 \) |
|----------------|-------|------|------|------|------|------|------|------|
| \( 10^{-4} \)  | 0.874E-1 | 0.417E-1 | 0.207E-1 | 0.103E-1 | 0.515E-2 | 0.257E-2 | 0.129E-2 |
| \( 10^{-4} \)  | 1.07   | 1.01  | 1.00  | 1.00  | 1.00  | –    |
| \( 10^{-5} \)  | 0.876E-1 | 0.417E-1 | 0.207E-1 | 0.103E-1 | 0.515E-2 | 0.257E-2 | 0.129E-2 |
| \( 10^{-5} \)  | 1.07   | 1.01  | 1.00  | 1.00  | 1.00  | –    |
| \( 10^{-6} \)  | 0.877E-1 | 0.417E-1 | 0.207E-1 | 0.103E-1 | 0.515E-2 | 0.257E-2 | 0.129E-2 |
| \( 10^{-6} \)  | 1.07   | 1.01  | 1.00  | 1.00  | 1.00  | –    |
| \( 10^{-7} \)  | 0.877E-1 | 0.417E-1 | 0.207E-1 | 0.103E-1 | 0.515E-2 | 0.257E-2 | 0.129E-2 |
| \( 10^{-7} \)  | 1.07   | 1.01  | 1.00  | 1.00  | 1.00  | –    |
| \( 10^{-8} \)  | 0.877E-1 | 0.417E-1 | 0.207E-1 | 0.103E-1 | 0.515E-2 | 0.257E-2 | 0.129E-2 |
| \( 10^{-8} \)  | 1.07   | 1.01  | 1.00  | 1.00  | 1.00  | –    |
Numerical results are presented in Tables 1–4 and Figures 2 and 3, which support our main result. Tables 1–4 list errors in the energy norm for Problems I and II, that is, $e^{\epsilon^N}$ and $e^{2^{k}N}$, for $\epsilon = 10^{-4}, 10^{-5}, \ldots, 10^{-8}$ and $N = 8, 16, 32, 64, 128, 256$, in the cases of $k = 1, 2$. These data show uniform convergence with respect to the singular perturbation parameter $\epsilon$. In the cases of $k = 3, 4$, errors and convergence orders also show uniform convergence, which are plotted in Figures 2 and 3.

Besides, from numerical experiments we find that the linear systems become harder to be solved by iterative solvers when $k$ and $N$ become bigger and $\epsilon$ becomes smaller.
FIGURE 3  Problem II and $\epsilon = 10^{-8}$

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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