REFLECTIONS ON TOPOLOGICAL QUANTUM FIELD THEORY

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Abstract

The aim of this article is to introduce some basic notions of Topological Quantum Field Theory (TQFT) and to consider a modification of TQFT, applicable to embedded manifolds. After an introduction based around a simple example (Section 1) the notion of a $d$-dimensional TQFT is defined in category-theoretical terms, as a certain type of functor from a category of $d$-dimensional cobordisms to the category of vector spaces (Section 2). A construction due to Turaev, an operator-valued invariant of tangles, is discussed in Section 3. It bears a strong resemblance to 1-dimensional TQFTs, but carries much richer structure due to the fact that the 1-dimensional manifolds involved are embedded in a 3-dimensional space. This leads us, in Section 4, to propose a class of TQFT-like theories, appropriate to embedded, rather than pure, manifolds.

1. INTRODUCTION

To introduce the idea of TQFT, we start by considering the simplest compact phase space, namely $S^2$ with symplectic form $\omega$ given in terms of a local complex coordinate $z$ by $\omega = idz \wedge d\bar{z}/(1 + |z|^2)^2$. Locally $\omega$ is exact, $\omega = d\theta$, where $\theta$ is a 1-form, but globally $\omega$ is the curvature of a non-trivial complex line bundle (i.e. having fibre isomorphic to $\mathbb{C}$) with connection $\theta$. In geometric quantization, the Hilbert space corresponding to $(S^2, \omega)$ is the space of holomorphic sections of this line bundle. It can easily be shown that there are only two linearly independent holomorphic sections, and thus the Hilbert space is isomorphic to $\mathbb{C}^2$. (This illustrates the general principle that a finite phase space volume gives rise to a finite Hilbert space dimension.)

Turning to dynamics, since we are describing a topological theory the Hamiltonian operator $\hat{H}$ is taken to be simply the zero $2 \times 2$ matrix. Thus the quantum evolution operator $\exp(i\hat{H}t)$ is the $2 \times 2$ identity matrix and its trace is 2. We will now associate 1-dimensional manifolds and operators in the following way: the interval $I$ is associated with $\exp(i\hat{H}t)$ and the circle $S^1$ with $\text{Tr} \exp(i\hat{H}t)$ (regarded as an operator from $\mathbb{C}$ to $\mathbb{C}$). The idea is that the path integral representation of each operator is given by an integral over fields defined on the corresponding $(0+1)$-dimensional spacetime manifold. Incidentally, from the Legendre transformation one finds that the action is $S = \int \theta$, since $0 = H = \theta - \mathcal{L}$ where $\mathcal{L}$ is the Lagrange density. Thus the factor $\exp iS$ in the path integral may be interpreted geometrically as the holonomy of the connection $\theta$. 

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Irrespective of how the above assignment is motivated, from a mathematical standpoint it distinguishes between the two fundamental 1-dimensional manifolds $I$ and $S^1$ and thus provides a topological classification of (connected) 1-manifolds. Of course in one dimension there is very little to classify, but the same feature persists for TQFTs relating to higher-dimensional manifolds, where the classification problem can pose very great challenges.

In the next section we shall describe an elegant axiomatic approach to TQFT, which allows one to bypass the difficulties associated with path integrals and captures certain essential features common to all TQFTs.

2. THE DEFINITION OF TQFT

One of the main approaches to defining the notion of a TQFT uses the elegant language of category theory. For this reason we start with a very brief discussion of categories and functors.

A category $\mathcal{C}$ consists of (a) a class of objects, (b) for any ordered pair of objects $(V, W)$ a set $\text{Hom}(V, W)$ of morphisms $V \rightarrow W$, and (c) an operation of composition of morphisms, $(V \rightarrow W, U \rightarrow V) \mapsto U \rightarrow W$ satisfying, (C1) associativity of composition, and (C2) existence of an identity morphism for each object (i.e. for any $X$ there exists a morphism $X \rightarrow X$ such that, for any $X \rightarrow Y$, $f \circ 1_X = f$ and $1_Y \circ f = f$).

Intuitively, the objects are sets or spaces, possibly endowed with additional structures, and the morphisms are structure-preserving maps. Two examples are:

1. $\text{Vect}(k)$, the category whose objects are all finite-dimensional vector spaces over the ground ring $k$, and whose morphisms are all $k$-linear maps.

2. $d - \text{Cobord}$, the category whose objects are smooth, compact, oriented $(d - 1)$-dimensional manifolds without boundary, and whose morphisms are cobordisms i.e. smooth, compact, oriented $d$-dimensional manifolds with boundary, identified up to orientation-preserving diffeomorphisms which restrict to the identity on the boundary.

The unusual feature of this second example is that the morphisms are not maps between manifolds in the usual sense, but are themselves (equivalence classes of) manifolds of one dimension higher than the the objects. The middle surface in Figure 1 depicts a 2-cobordism from the circle $S^1$ to the disjoint union of two copies of the circle, namely the familiar “trousers” surface.

![Figure 1: Composition of 2-cobordisms](image-url)
For simplicity, in what follows we will describe cobordisms as $d$-dimensional manifolds, rather than equivalence classes thereof. For $\Sigma_1$ and $\Sigma_2$ two objects, a cobordism from $\Sigma_1$ to $\Sigma_2$, written $\Sigma_1 \xrightarrow{M} \Sigma_2$, is a manifold $M$ whose boundary is $\Sigma_1 \amalg \Sigma_2$, where $\Sigma_1'$ means $\Sigma_1$ with the opposite orientation and $\amalg$ is the disjoint union. The composition of a cobordism from $\Sigma_1$ to $\Sigma_2$ and one from $\Sigma_2$ to $\Sigma_3$ is given by gluing the two cobordisms along $\Sigma_2$. The identity morphism for an object $\Sigma$ is the cylinder cobordism $\Sigma \times I$, where $I$ is the standard interval. Figure 1 depicts the composition of the identity morphism for the circle and the “trousers” cobordism.

A fundamental concept in category theory is that of a functor between two categories: given two categories $\mathcal{C}$ and $\mathcal{C}'$, a (covariant) functor from $\mathcal{C}$ to $\mathcal{C}'$ is (a) an assignment to each object $V$ of $\mathcal{C}$ of an object $F(V)$ of $\mathcal{C}'$, and (b) an assignment to each morphism $V \xrightarrow{f} W$ of $\mathcal{C}$ of a morphism $F(V) \xrightarrow{F(f)} F(W)$ of $\mathcal{C}'$, such that (F1) $F(1_V) = 1_{F(V)}$ for every object $V$ of $\mathcal{C}$, and (F2) $F(f \circ g) = F(f) \circ F(g)$ for every pair of morphisms $f, g$ of $\mathcal{C}$ whose composition is defined.

Categories may possess various kinds of additional structures, of which we mention the following: (A) a product of objects, and corresponding product of morphisms, (B) a unit object and unit morphism with respect to this product, and (C) an involution on the class of objects. The categories $\text{Vect}(k)$ and $d - \text{Cobord}$ introduced above possess all three structures. In $\text{Vect}(k)$ (A) is the tensor product $(V, W) \mapsto V \otimes W$ together with the tensor product of two linear maps $(V \xrightarrow{f} W, V' \xrightarrow{g} W') \mapsto (V \otimes V' \xrightarrow{f \otimes g} W \otimes W')$, (B) is the ring $(k \otimes V = V$, etc.) together with 1 (regarded as a linear map $k \to k$), and finally (C) is $V \mapsto V^*$ (the dual vector space of $V$). (Here we are identifying $V$ and $V^{**}$.) In $d - \text{Cobord}$ (A) is the disjoint union $(\Sigma_1, \Sigma_2) \mapsto \Sigma_1 \amalg \Sigma_2$ and $(\Sigma_1 \xrightarrow{M} \Sigma_1', \Sigma_2 \xrightarrow{N} \Sigma_2') \mapsto \Sigma_1 \amalg \Sigma_2 \xrightarrow{M \amalg N} \Sigma_1' \amalg \Sigma_2'$, (B) is $\emptyset$, the empty $(d-1)$-dimensional manifold ($\emptyset \amalg \Sigma = \Sigma$ etc.) together with $\emptyset$, the empty $d$-dimensional manifold, and (C) is $\Sigma \mapsto \Sigma$ (the manifold $\Sigma$ with the opposite orientation).

We are now in a position to define what a TQFT is: A $(d$-dimensional) topological quantum field theory is a functor $d - \text{Cobord} \longrightarrow \text{Vect}(k)$ respecting the structures (A), (B) and (C).

A TQFT is commonly described by assignments $\Sigma \mapsto V_\Sigma$ for objects and $M \mapsto Z_M$ for morphisms. The definition implies that these obey the properties (F1) $Z_{\Sigma \times I} = 1_{V_\Sigma}$, (F2) $Z_{M \circ M'} = Z_M \circ Z_{M'}$, (AA) $V_{\Sigma \amalg \Sigma'} = V_\Sigma \otimes V_{\Sigma'}$ and $Z_{M \amalg M'} = Z_M \otimes Z_{M'}$, (BB) $V_\emptyset = k$; $Z_\emptyset = 1$, and (CC) $V_{\Sigma^*} = V_\Sigma$.

As mentioned in the previous section, the physical intuition underlying this definition is that $V_\Sigma$ is the Hilbert space associated with the spacelike manifold $\Sigma$ and $Z_M$ is the path integral associated to the spacetime $M$.

It should be mentioned at this stage that there are other approaches to defining a TQFT. In particular, the original axiomatic definition of TQFT due to Atiyah [1] has a slightly different flavour. In that approach a TQFT is an assignment $\Sigma \mapsto V_\Sigma$ of vector spaces to closed $(d-1)$-manifolds, together with an assignment $M \mapsto Z_M$, which assigns to a $d$-dimensional manifold $M$, with boundary $\partial M = \Sigma$, an element of the vector space $V_\Sigma$, both assignments obeying a number of axioms. The definition in terms of a functor from $d - \text{Cobord}$ to $\text{Vect}(k)$ has been used by various authors, e.g. [2][3][4][5].

To give an idea of how this axiomatic definition of TQFT works, we will reanalyse the TQFT discussed in the introduction from this standpoint. This is a 1-dimensional TQFT i.e. a functor from $1 - \text{Cobord}$ to $\text{Vect}(C)$. The objects of $1 - \text{Cobord}$, being 0-dimensional manifolds, are either $\emptyset$ or the disjoint union of single points. Thus, by the properties (AA) and (BB) above, it is enough to specify $V_p$, where $p$ stands for
a point, and we set $V_p = \mathbb{C}^2$. Turning to the morphisms, connected 1-manifolds are diffeomorphic to either the interval $I$ or the circle $S^1$. By (F1) above $Z_I = \text{id}_{\mathbb{C}^2}$. Thus to complete the description of this TQFT we need to obtain $Z_{S^1}$. Now $\partial S^1 = \emptyset$ so $S^1$ is a cobordism from $\emptyset$ to $\emptyset$, and under the TQFT functor is transported to a linear map $Z_{S^1} : V_\emptyset = \mathbb{C} \rightarrow V_\emptyset = \mathbb{C}$. Thus $Z_{S^1}$ may be identified with a number and it remains to show that this number is 2, i.e. the trace of the $2 \times 2$ identity matrix. This is a relatively simple exercise, but for reasons of space we will not go into details.

The discussion of TQFT in this section and the previous one were mainly intended to convey a flavour of the subject. It is clear that the example of TQFT which was studied has very little interest for the topological classification of manifolds, since in dimension 1 the only (connected) manifolds to classify are the interval and the circle. One way of achieving richer results is to go up in dimension. A notable example is of course the 3-manifold invariant arising from a 3-dimensional TQFT based on the Chern-Simons action [6]. It would go too far to discuss here the many other examples of higher dimensional TQFTs, since this is a vast and expanding subject. Thus we limit ourselves to giving some references to the literature [3][4][7][8][9][10][11]. In the next two sections however we will propose a second way to achieve richer structure for TQFTs, namely by considering a modified cobordism category whose objects and morphisms are embedded manifolds.

### 3. OPERATOR INVARIANTS OF TANGLES

In this section we will look at a motivating example of the kind of “embedded” TQFT alluded to at the end of the previous section. The construction of tangle invariants to be described is due to Turaev [12]. We will present tangles as a category, called $\mathcal{OTa}$ by Turaev (oriented tangles). The objects of this category are finite sequences of + or − signs, interpreted geometrically as oriented points lying in integer positions on the positive $x_1$ axis in $\mathbb{R}^2$ coordinatised by $(x_1, x_2)$. The morphisms of $\mathcal{OTa}$ are (oriented) tangles: consider $\mathbb{R}^3$ coordinatised by $(x_1, x_2, x_3)$ and two parallel planes in $\mathbb{R}^3$, being $x_3 = 0$ and $x_3 = 1$. Both planes are to be thought of as objects in the above sense, and thus contain a number of oriented points along the lines $x_2 = x_3 = 0$ and $x_2 = 0$, $x_3 = 1$ respectively. A tangle is a 1-dimensional oriented submanifold of $\mathbb{R}^2 \times [0, 1] \subset \mathbb{R}^3$ whose boundary consists of the oriented points in the top and bottom planes, these being the only points where the tangle intersects the top and bottom planes. Furthermore two tangles are identified if one is carried to the other by an orientation-preserving diffeomorphism of $\mathbb{R}^2 \times [0, 1]$ which is the identity restricted to the top and bottom planes (see Figure 2).

![Figure 2: A tangle](image-url)
A tangle is regarded as a morphism from the orientation-reversed object in the lower plane to the object in the upper plane, in the same way as a cobordism from \( \Sigma_1 \) to \( \Sigma_2 \) was a manifold \( M \) with boundary \( \Sigma_1 \sqcup \Sigma_2 \). Thus the tangle in Figure \([2]\) is a morphism from \((-+--+)\) to \((-+-+)\). (Here we have adopted a different convention to \([12]\), where points associated to upwards-pointing strands are labelled + and to downwards-pointing strands −. Our convention aims to make contact with the TQFT axioms from the previous section.)

If \( T_1 \) is a tangle from \( O_1 \) to \( O_2 \) and \( T_2 \) is a tangle from \( O_2 \) to \( O_3 \), we form the composition \( T_2 \circ T_1 \) by concatenation, placing the second tangle on top of the first and shrinking in the \( x_3 \) direction. The identity morphism for an object is simply the corresponding trivial tangle, which has all its strands parallel to the \( x_3 \) axis, oriented up or down as appropriate.

The category \( \mathcal{OT}_a \) also possesses extra structures analogous to those of \( d-\text{Cobord} \), namely (A) a product \( \otimes \) of objects and corresponding product of morphisms/tangles, given by juxtaposition (e.g. for objects one has \((+-)(-+-)=(+-+-+)--\) and for tangles \( T_1 \otimes T_2 \) is the tangle obtained by placing \( T_2 \) to the right of \( T_1 \)), (B) a unit object and unit morphism for this product, being the empty sequence and the empty tangle respectively, and finally (C) an involution on objects given by \( + \leftrightarrow - \) in the sequences.

Thus the category of oriented tangles is very similar to the category of 1-cobordisms. There are however some important differences arising from the fact that tangles are embedded in a 3-dimensional space, whereas the morphisms of \( 1-\text{Cobord} \) are abstract 1-dimensional manifolds. In the first place, it is easy to find inequivalent tangles which are diffeomorphic as abstract manifolds. Second, the product (A) is no longer commutative in \( \mathcal{OT}_a \).

Tangles are very pleasing objects. They are a simultaneous generalization of both braids and knots, since braids are tangles with all strands pointing downwards, say, and knots, or more generally links, are just tangles from \( \emptyset \) to \( \emptyset \). At the same time, tangles lend themselves well to algebraization. Just as in the case of braids, where any braid can be written as a composition of elementary braids, simple over- and undercrossings of two adjacent strands, so also one can express any tangle as a composition of a set of elementary tangles. The equivalence of tangles under diffeomorphisms of \( R^2 \times [0,1] \) can be expressed algebraically as a set of relations between these generators. Geometrically these correspond to a number of tangle moves, analogous to the three Reidemeister moves for knot diagrams. We refer to \([12]\) for further details.

Now the main and beautiful result of the Turaev paper \([12]\) is the construction of a functor \( F \) from the category of oriented tangles to \( \text{Vect}(k) \), where \( k \) is now \( \mathbb{Z}[q,q^{-1}] \), the ring of Laurent polynomials in \( q \) with integer coefficients. Let \( V \) be a finite-dimensional vector space over \( k \) and \( V^* \) its dual. Then the functor \( F \) assigns \( \emptyset \) to \( k \), + to \( V \), − to \( V^* \) and sequences of + and − are sent to the corresponding tensor product. Acting on morphisms, \( F \) sends a tangle \( O_1 \xrightarrow{T} O_2 \) to a linear map \( F(O_1) \xrightarrow{F(T)} F(O_2) \). For instance the tangle depicted in Figure \([2]\) corresponds under \( F \) to a linear map from \( V^* \otimes V^* \otimes V \otimes V^* \otimes V \) to \( V^* \otimes V \otimes V^* \). This is how the term “operator invariant” arises, since \( F \) assigns a linear operator to each tangle, and this assignment does not depend on the particular representative of the equivalence class.

It would take us too far here to go into the details of the construction of \( F \). The idea is to find operators corresponding to the elementary tangles referred to above, since by the functoriality of \( F \) one has for the composition of two tangles \( F(T_1 \circ T_2) = F(T_1) \circ F(T_2) \). In particular the operators for the elementary over- and undercrossings are obtained from a certain class of quantum R-matrices, i.e. matrices satisfying the quantum Yang-Baxter equation. This equation is the algebraic counterpart of the third
Reidemeister move for knot diagrams. For further details we refer, once again, to [12].

Now for the special case of tangles which are knots or links, the functor $F$ yields a linear map from $k$ to $k$, which may be identified with an element of $k$. This element is essentially the Homfly polynomial of the knot or link, a knot polynomial which generalises both the Alexander-Conway and Jones polynomials. Thus Turaev’s construction can be viewed as a generalisation to tangles of the Homfly polynomial.

The point of discussing this example is hopefully clear by now: the functor $F$ which assigns operator invariants to tangles bears a very strong resemblance to a 1-dimensional TQFT. However, by virtue of the fact that the cobordisms are now embedded manifolds the structure of the functor $F$ is far richer than that of the simple 1-dimensional TQFT discussed in Sections 1 and 2. We are thus led to seek a new class of TQFT-like theories which can encompass embedded cobordisms, a point of view which will be explored now in the final section.

4. TQFT FOR EMBEDDED MANIFOLDS

In this section we shall outline some ideas on how to modify the definition of TQFT given in Section 2 in order to describe embedded manifolds. Our first suggestion attempts to capture the features of the tangle example discussed in the previous section. A “tangle-type TQFT” will be a functor from $(d, n)$-section. A “tangle-type TQFT” will be a functor from $(d, n)$ to $\text{Vect}(k)$, where $(d, n)$-$\text{Cobord}$ is, loosely speaking, the category of $d$-dimensional cobordisms embedded in $n$ dimensions.

To define the objects of $(d, n)$-$\text{Cobord}$ we start by defining basic objects. Consider embeddings of closed $(d-1)$-dimensional manifolds in $I \times \mathbb{R}^{n-2}$ such that the image is contained in the interior of $I \times \mathbb{R}^{n-2}$. Let these be connected in the following sense: if the embedding is described by a map $f : \Sigma \to I \times \mathbb{R}^{n-2}$, there do not exist open sets $U_1, U_2 \subset I \times \mathbb{R}^{n-2}$, which are topologically open discs, such that $U_1 \cap U_2 = \emptyset$, $f(\Sigma) \subset U_1 \cup U_2$ and $f(\Sigma) \cap U_1 \neq \emptyset \neq f(\Sigma) \cap U_2$. For each isotopy class of connected embeddings we choose a standard representative. These are the basic objects. All other objects are obtained by taking products of the basic objects, where the product is juxtaposition of the embeddings in an analogous fashion to the case of tangles. This, to our mind, is the natural way to generalize the objects of $\text{OTa}$ to higher dimensions: the standard embedding of a single point in $I \times \mathbb{R}$ identified with $[1/2, 3/2] \times \mathbb{R}$ maps the point to $(1, 0)$. Of course, it may be interesting to study more general situations where the objects are unrestricted embeddings, as in the analysis of braid statistics for particles in the plane [13]. The morphisms of $(d, n)$-$\text{Cobord}$ are compact $d$-dimensional manifolds embedded in $\mathbb{R}^{n-1} \times [0, 1]$, whose boundary components lie exclusively in the top and bottom hyperplanes $\mathbb{R}^{n-1} \times \{0\}$ and $\mathbb{R}^{n-1} \times \{1\}$, and both top and bottom boundaries belong to the objects of $(d, n)$-$\text{Cobord}$. The embedded $d$-dimensional manifolds are identified up to orientation-preserving diffeomorphisms of $\mathbb{R}^{n-1} \times [0, 1]$, which restrict to the identity on the top and bottom hyperplanes, as in the case of tangles.

We have already seen an example of a tangle-type TQFT, namely the functor $F$ from $(1, 3)$-$\text{Cobord}$ to $\text{Vect}(k)$ from the previous section. Going up one dimension, another tangle-type TQFT is a functor from $(2, 4)$-$\text{Cobord}$ to $\text{Vect}(k)$. Now the objects of $(2, 4)$-$\text{Cobord}$ are standard embeddings of closed 1-dimensional manifolds in $I \times \mathbb{R}^2$, i.e. representatives of knot and link classes. The morphisms are 2-dimensional surfaces embedded in $\mathbb{R}^3 \times [0, 1]$ with objects as their top and bottom boundaries, up to identification. A special case consists of the 2-knots with empty top and bottom boundaries. Such objects have been studied by a number of authors (see for instance [14]).
A second context where an embedded TQFT structure appears is parallel transport in vector bundles. Let $M$ be a smooth manifold. We can define a category $\text{Path}(M)$ as follows: the objects of $\text{Path}(M)$ are points of $M$ and the set of morphisms from $m_1$ to $m_2$ is the set of smooth paths from $m_1$ to $m_2$, i.e. maps $\gamma : [0,1] \to M$ such that $\gamma(0) = m_1$, $\gamma(1) = m_2$, which are constant in $[0,\epsilon]$ and $[\epsilon,1]$ for some $0 < \epsilon < 1/2$, identified up to a suitable equivalence relation. The composition of two paths $m_1 \xrightarrow{\gamma} m_2$ and $m_2 \xrightarrow{\gamma'} m_3$ is $m_1 \xrightarrow{\gamma \circ \gamma'} m_3$, the obvious path which follows $\gamma$ and then $\gamma'$ at double speed.

The equivalence relation is such that this composition is well-defined and associative on path equivalence classes. We refer to [15] for a description of an appropriate equivalence relation, namely rank-1 homotopy. We remark that the extra condition requiring the paths to be constant at their endpoints guarantees the smoothness of the composition.

Now, given a vector bundle $E \xrightarrow{\pi} M$ over $M$ with connection $\nabla$, we can define a “parallel-transport-type TQFT” as a functor $F$ from $\text{Path}(M)$ to $\text{Vect}(k)$, given by $m \mapsto \pi^{-1}(m)$ (the fibre over $m$) for objects, and $(m_1 \xrightarrow{\gamma} m_2) \mapsto (\pi^{-1}(m_1) \xrightarrow{F(\gamma)} \pi^{-1}(m_2))$ for morphisms, where $F(\gamma)$ is the isomorphism between the fibres over the endpoints induced by parallel transport along $\gamma$.

A natural modification of the previous example occurs when $\nabla$ is flat. Then we can replace $\text{Path}(M)$ by the homotopy groupoid, regarded as a category whose morphisms are homotopy classes of paths on $M$.

Thus we see that a number of interesting examples have an embedded TQFT structure. A question which naturally arises is: do these theories come about from some classical theory and, if so, what is the corresponding topological action? For the lowest-dimensional tangle-type TQFT a candidate for the action is presumably some version of the Kontsevich integral [16] [17]: suppose the tangle is a braid (i.e. none of the strands doubles back on itself). Any plane $x_3 = c$ for $c \in [0,1]$ constant intersects the braid in a fixed number of points, say $n$, and thus varying the plane from $x_3 = 0$ to $x_3 = 1$ we get a film of $n$ points moving in the plane. In other words, a braid defines a path in the configuration space for $n$ identical non-coincident particles $\mathbb{C}^n \setminus \Delta$, where we have identified $\mathbb{R}^2$ with $\mathbb{C}$ and $\Delta$ is the diagonal subset of $\mathbb{C}^n$, i.e. the union of hyperplanes $\{(z_1, \ldots, z_n) | z_i = z_j \}$ for $i \neq j$. We can introduce a flat connection on $\mathbb{C}^n \setminus \Delta$, namely the Knizhnik-Zamolodchikov connection $A_{KZ}$, and the flatness implies that the parallel transport along the path in $\mathbb{C}^n \setminus \Delta$ is invariant under homotopy. In terms of braids this means that the parallel transport is the same for two braids which are related by an orientation-preserving diffeomorphism of $\mathbb{C} \times [0,1]$, which is the identity on the top and bottom planes and which preserves each horizontal plane $x_3 = c$. Now the Kontsevich integral construction can be extended to tangles, [18], [19] giving a functorial assignment from tangles to certain vector spaces generated by so-called chord diagrams. Thus the topological action for tangles should be something of the form $\text{tr} \int f_* A_{KZ}$ where $f$ maps from $[0,1]$ to some modified configuration space, appropriate to tangles rather than braids.

In conclusion, the study of tangles from a TQFT angle suggests a promising method of extending the TQFT approach to embedded manifolds. If the codimension of the embedded manifolds is not too large, one can expect a considerably richer structure for embedded TQFTs compared to pure TQFTs for the same dimension. From the physical point of view, it is interesting to note that for this class of theories, we have written down a quantum theory straight away, without having started from a classical theory, e.g. in terms of some classical Lagrangian. Indeed it seems likely that the classical action will have a complicated structure, if the tangle example is anything to go by. Thus the study of TQFTs may suggest some new understanding of the nature of quantum theories, with no dynamics to cloud the issue, at least in the first instance.
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