5-dimensional geometries I: the general classification

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Abstract

This paper is the first of a 3-part series that classifies the 5-dimensional Thurston geometries. The present paper (part 1 of 3) summarizes the general classification, giving the full list, an outline of the method, and some illustrative examples. This includes phenomena that have not appeared in lower dimensional geometries, such as an uncountable family of geometries $\tilde{\text{SL}}_2 \times_\alpha S^3$.

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1 Introduction

By the classification of closed surfaces (see e.g. [Mat02, Thm. 5.11]), every closed surface is diffeomorphic to a quotient of $\mathbb{E}^2$, $S^2$, or $\mathbb{H}^2$ by a discrete group of isometries. It is a classical result that in dimension 2, these three are the only connected, simply-connected, complete Riemannian manifolds with transitive isometry group (see e.g. [Thu97, Thm. 3.8.2]).

The quest for the 3-dimensional generalization that became Thurston’s Geometrization Conjecture led to a version of the following definition. (The equivalence to older definitions is outlined in Part II, [Gen16a, Prop. 2.5].)

Definition 1.1 (Geometries, following [Thu97, Defn. 3.8.1] and [Fil83, §1.1]).

(i) A geometry is a connected, simply-connected homogeneous space $M = G/G_p$ where $G$ is a connected Lie group acting faithfully with compact point stabilizers $G_p$.

(ii) $M$ is a model geometry if there is some lattice $\Gamma \subset G$ that acts freely on $M$. Then the manifold $\Gamma \backslash G/G_p$ is said to be modeled on $M$. 

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(iii) $M$ is maximal if it is not $G$-equivariantly diffeomorphic to any other geometry $G'/G'_p$ with $G \subsetneq G'$. Any such $G'/G'_p$ is said to subsume $G/G_p$.

Then a closed 3-manifold is a quotient of at most one maximal model geometry, which can be determined from the fundamental group [Thu97, Thm. 4.7.8] or from the existence of certain bundle structures (usually Seifert bundles) and some topological data (usually two Euler numbers) [Sco83, Thm. 5.3]. Thurston classified the 3-dimensional maximal model geometries and found eight (see [Thu97, Thm. 3.8.4]).

In 4 dimensions, Filipkiewicz classified the maximal model geometries in [Fil83]. Though 4-manifolds without geometric decompositions [Hil02, §13.3 #3] indicate there is less hope for a straightforward generalization of geometrization, Filipkiewicz’s classification highlights a few interesting firsts. The list comprises 18 geometries and—for the first time—a countably infinite family, named $\text{Sol}^4_{m,n}$. (See e.g. [Hil02, §7.1] or [Wal86, §1, Table 1] for the names currently in use.) One of the eighteen is $\mathbb{F}^4 = \mathbb{R}^2 \ltimes \text{SL}(2, \mathbb{R})/\text{SO}(2)$, the first geometry to admit finite-volume quotients but no compact quotients.

The direction to take should now seem straightforward. One seeks a classification of maximal model geometries in all dimensions; but a handful of obstacles stand in the way of such a classification:

1. Existing classifications, including now the present paper, rely on tools that may become unusable with increasing dimension. For example, the case of discrete point stabilizers ([Thu97, Thm. 3.8.4(c)] in dimension 3, [Fil83, Ch. 6] in dimension 4, and [Gen16a, Thm. 1.1(ii)] in Part II) relies on a classification of solvable Lie algebras over $\mathbb{R}$, which is incomplete in dimensions 7 and up. (See e.g. [SW12, Introduction] for a summary of known progress, and [BFNT13] for a wider survey.)

2. The aforementioned aspects of the 4-dimensional classification suggest that new phenomena may continue to appear for a few more dimensions. A workable approach to a general classification may not be evident without knowledge of such features.

An optimistic interpretation of these obstacles is that the 5-dimensional case is both tractable and potentially illustrative. Having carried out the classification, the new phenomena are summarized in Section 2; the main result is the following list.

**Theorem 1.2 (Classification of 5-dimensional geometries).** The maximal model geometries of dimension 5 are:

1. The geometries with constant curvature:
   
   $$
   \mathbb{E}^5 = \mathbb{R}^5 \ltimes \text{SO}(5)/\text{SO}(5), \quad S^5 = \text{SO}(6)/\text{SO}(5), \quad \mathbb{H}^5 = \text{SO}(5,1)/\text{SO}(5);
   $$

2. The other irreducible Riemannian symmetric spaces $\text{SL}(3,\mathbb{R})/\text{SO}(3)$ and $\text{SU}(3)/\text{SO}(3)$;

3. The unit tangent bundles or universal covers of circle bundles:

   $$
   T^1(\mathbb{H}^3) = \text{PSL}(2,\mathbb{C})/\text{SO}(2), \quad T^1(\mathbb{E}^{1,2}) = \mathbb{R}^3 \ltimes \text{SO}(1,2)^0/\text{SO}(2), \quad U(2,1)/U(2);
   $$

   \[\text{2}\]
4. The associated bundles (see e.g. [Sha00, §1.3 Vector Bundles] for the notation):

$$\text{Heis}_3 \times \mathbb{R}^3 = (\text{Heis}_3 \times \mathbb{R}^3(2)) \times \mathbb{R}^3/(\{(0,0,s), \gamma(t), e^{\pi is}\}, s,t \in \mathbb{R})$$

$$\text{Heis}_3 \times \mathbb{R} \tilde{\text{SL}}_2 = (\text{Heis}_3 \times \mathbb{R}^3(2)) \times \tilde{\text{SL}}_2/(\{(0,0,s), \gamma(t), \gamma(s)\}, s,t \in \mathbb{R})$$

$$\tilde{\text{SL}}_2 \times_\alpha S^3 = \tilde{\text{SL}}_2 \times \mathbb{R}^3(2) \times \mathbb{R}/\{\gamma(s), e^{\pi it}, \alpha s + t\}, 0 < \alpha < \infty$$

$$\tilde{\text{SL}}_2 \times_\alpha \tilde{\text{SL}}_2 = \tilde{\text{SL}}_2 \times \tilde{\text{SL}}_2 \times \mathbb{R}/\{\gamma(s), \gamma(t), \alpha s + t\}, 0 < \alpha \leq 1$$

$L(a;1) \times S^1 L(b;1) = S^3 \times S^3 \times \mathbb{R}/\{e^{\pi is}, e^{\pi it}, \alpha s + bt\}, 0 < a \leq b$ coprime in $\mathbb{Z}$,

where the Heisenberg group Heis$_3$ is $\mathbb{R}^3$ with the multiplication law

$$(x,y,z)(x',y',z') = (x + x', y + y', z + z' + xy' - x'y),$$

on which SO(2) acts through the action of SL(2,$\mathbb{R}$) on the $x,y$ plane, and $t \mapsto e^{\pi it} \in S^1$ and $\gamma : \mathbb{R} \to SO(2) \subseteq \tilde{\text{SL}}_2$ are 1-parameter subgroups sending $\mathbb{Z}$ to the center;

5. The three principal $\mathbb{R}$-bundles with non-flat connections over the $\mathbb{F}^4$ geometry, distinguished from each other by their curvatures:

$$\mathbb{F}^5_a = \text{Heis}_3 \times \tilde{\text{SL}}_2/(\{0,0,at\}, \gamma(t)\}, a = 0 \text{ or } 1;$$

6. The six simply-connected indecomposable nilpotent Lie groups, named by their Lie algebras as in [PSWZ76, Table II], in which the point stabilizer of the identity element is a maximal compact group of automorphisms (specified in Table 3.2):

$$A_{5,1} = \mathbb{R}^4 \times \mathbb{R}_{x^2,x^2} \quad A_{5,2} = \mathbb{R}^4 \times \mathbb{R}_{x^4} \quad A_{5,3} = (\mathbb{R} \times \text{Heis}_3) \times \mathbb{R}_{x^3,y^2 \to y}$$

$$A_{5,4} = \text{Heis}_5 \quad A_{5,5} = \text{Nil}^4 \times \mathbb{R}_{3 \to 1} \quad A_{5,6} = \text{Nil}^4 \times \mathbb{R}_{4 \to 3 \to 1};$$

7. The simply-connected indecomposable non-nilpotent solvable Lie groups, specified the same way:

$$A_{5,7}^{1,1,-1,-1} = \mathbb{R}^4 \times \mathbb{R}_{x^2,x^1} \quad A_{5,9}^{1,1,-1,-1} = \mathbb{R}^4 \times \mathbb{R}_{x^1,x^2}$$

$$A_{5,7}^{-1,-1,-1} = \mathbb{R}^4 \times \mathbb{R}_{x^1,x^2} \quad A_{5,8}^{-1,-1,-1} = \mathbb{R}^4 \times \mathbb{R}_{x^2,x^1}$$

$$A_{5,10}^{-1,-1,-1} = (x-1)^2, x^1 \quad A_{5,15}^{-1,-1,-1} = (x-1)^2, (x+1)^2$$

$$A_{5,20}^0 = (\mathbb{R} \times \text{Heis}_3) \times \mathbb{R}_{(x-y)z = 1}^0, \quad A_{5,33}^{-1,-1,-1} = \mathbb{R}^3 \times \{xyz = 1\};$$

8. and all twenty-nine products of lower-dimensional geometries involving no more than one Euclidean factor, named as in [Wal86, Table I].

(a) 4-by-1:

$$S^4 \times \mathbb{E} \quad \mathbb{H}^4 \times \mathbb{E} \quad \mathbb{CP}^2 \times \mathbb{E} \quad \mathbb{CH}^2 \times \mathbb{E} \quad \mathbb{F}^4 \times \mathbb{E}$$

$$\text{Nil}^4 \times \mathbb{E} \quad \text{Sol}^4 \times \mathbb{E} \quad \text{Sol}^4 \times \mathbb{E} \quad \text{Sol}^4_{m,n} \times \mathbb{E}$$
(b) 3-by-2:

\[
\begin{array}{ccc}
S^3 \times \mathbb{E}^2 & \mathbb{E}^3 \times S^2 & \mathbb{E}^3 \times \mathbb{H}^2 \\
\mathbb{H}^3 \times \mathbb{E}^2 & S^3 \times S^2 & S^3 \times \mathbb{H}^2 \\
\text{Heis}_3 \times \mathbb{E}^2 & \text{Heis}_3 \times S^2 & \text{Heis}_3 \times \mathbb{H}^2 \\
\text{Sol}_3 \times \mathbb{E}^2 & \text{Sol}_3 \times S^2 & \text{Sol}_3 \times \mathbb{H}^2 \\
\widetilde{\text{SL}}_2 \times \mathbb{E}^2 & \widetilde{\text{SL}}_2 \times S^2 & \widetilde{\text{SL}}_2 \times \mathbb{H}^2 \\
\end{array}
\]

(c) 2-by-2-by-1:

\[
\begin{array}{ccc}
S^2 \times S^2 \times \mathbb{E} & S^2 \times \mathbb{H}^2 \times \mathbb{E} & \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{E}
\end{array}
\]

More explicit instructions for constructing these geometries—such as the solvable Lie groups and their automorphism groups—are delegated to where they occur in the classification in Parts II and III [Gen16a, Gen16b].

Roadmap. The present paper (Part I) summarizes the classification; Section 2 picks out illustrative examples, Section 3 outlines the strategy, and Section 4 briefly surveys related classifications. Part II [Gen16a] classifies the point stabilizer subgroups \( G_p \) and classifies the geometries where \( G_p \) acts irreducibly or trivially on tangent spaces. Part III [Gen16a] classifies the remaining geometries after showing that they all admit invariant fiber bundle structures (hence the name “fibering geometries”).

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2 Salient examples

2.1 New phenomena

Much of our interest in this classification is in the search for phenomena that occur for the first time in dimension 5, in hopes of finding a pattern that continues in higher dimensions. See also Section 3 for a discussion of new tools.

An uncountable family of geometries. The associated bundles \( \widetilde{\text{SL}}_2 \times_\alpha S^3 \) \((0 < \alpha < \infty)\) form an uncountable family of maximal model geometries. (Taking \( \widetilde{\text{SL}}_2 \times_\alpha S^3 \) as a circle bundle over \( S^2 \times \mathbb{H}^2 \) with an invariant connection, the parameter \( \alpha \) is a ratio of curvatures in the \( S^2 \) and \( \mathbb{H}^2 \) directions.) This and \( \widetilde{\text{SL}}_2 \times_\alpha \text{SL}_2 \) are the first occurrences of uncountable families. Since every lattice in a Lie group is finitely presented [OV00, Thm. I.1.3.1], \( \pi_1 \) of the quotient manifolds will not determine the geometries. Details are in Part III, [Gen16b, Prop. 6.35].
An infinite family without compact quotients. In fact, \( \widetilde{SL}_2 \times_\alpha S^3 \) admits compact quotients if and only if \( \alpha \) is rational [Gen16b, Prop. 6.36]. (Recall that beginning with \( \mathbb{H}^2 \), geometries can have noncompact quotients of finite volume; and beginning with \( \mathbb{F}^4 = \mathbb{R}^2 \times \text{SL}(2, \mathbb{R})/\text{SO}(2) \), model geometries might have no compact quotients.)

Non-unique maximality. The geometry \( T^1 S^3 = \text{SO}(4)/\text{SO}(2) \) is a non-maximal form of \( S^3 \times S^2 \) and \( L(1; 1) \times_{S^1} L(1; 1) \)—both of which are maximal [Gen16b, Rmk. 6.40]. This contrasts with the positive results for unique maximality listed in the discussion after [Fil83, Prop. 1.1.2].

Inequivalent compact geometries with the same diffeomorphism type. Using Barden’s diffeomorphism classification [Bar65] of simply-connected 5-manifolds by second homology and second Stiefel-Whitney class, one can prove that the associated bundles of lens spaces \( L(a; 1) \times_{S^1} L(b; 1) \) are all diffeomorphic to \( S^3 \times S^2 \) [Ott09, Cor. 3.3.2].

More broadly one can attempt to give the classification up to diffeomorphism, following previous results such as [Mos50, Cor. p. 624], [Gor77], [Ish55], and [Ott09, Thm. 1.0.3]. Most of the geometries are products of \( \mathbb{R}^k \) and some spheres; the two exceptions are \( \mathbb{C}P^2 \times \mathbb{E} \) and the rational homology sphere \( \text{SU}(3)/\text{SO}(3) \), named \( X_{-1} \) in Barden’s classification [BG02, Introduction].

Note that while the correct diffeomorphism type may be obvious enough to guess, it is not as obviously correct. The problem is proving that the space is a direct product of \( \mathbb{R}^k \) and the product of spheres onto which it deformation retracts—such a claim is false for any nontrivial vector bundle (such as \( TS^2 \)) and for a homogeneous example by Samelson discussed in [Mos55, §5 Example 4]. Instead one has to use either an explicit description of the diffeomorphism type from [Mos62, Thm. A] or the fact that sufficiently nice bundles over contractible spaces are trivial [Hus94, Cor. 10.3].

Isotropy irreducible spaces. The geometries \( \text{SL}(3, \mathbb{R})/\text{SO}(3) \) and \( \text{SU}(3)/\text{SO}(3) \) have point stabilizers \( \text{SO}(3) \) acting irreducibly on the (5-dimensional) tangent spaces. This is the first occurrence of such an action being irreducible and not the standard representation for a group of the same isomorphism type. These two geometries are still symmetric spaces, but in higher dimensions there exist homogeneous spaces with irreducibly-acting point stabilizers that are not symmetric spaces (See e.g. [HZ96, Introduction]).

2.2 Examples that highlight tools

The classification of geometries requires an increasingly wide range of tools as the dimension increases. These are a handful of examples where either unexpected tools appeared, or familiar tools exhibit behavior that is not completely obvious at first glance.

Model geometries via Galois theory and Dirichlet’s unit theorem. Some Galois theory is needed to answer questions of lattice existence, such as to prove that \( \mathbb{R} \times \text{Conf}^+ \mathbb{E}^3/\text{SO}(3) \) (where the action of \( \text{Conf}^+ \mathbb{E}^3 \) on \( \mathbb{R} \) is chosen to make the semidirect product unimodular) is not a model geometry [Gen16b, Prop. 5.1(iv)].

Dirichlet’s unit theorem makes an appearance when we construct a lattice in \( \mathbb{R}^3 \times \{xyz = 1\}^0 \) by taking a finite index subgroup of \( \mathcal{O}_K \times \mathcal{O}_K^\times \) where \( K \) is a totally real cubic field extension of \( \mathbb{Q} \) [Gen16a, Prop. 5.16].
Point stabilizers not realized. The classification of geometries starts by classifying subgroups of \(SO(5)\) in order to classify point stabilizers—but not every subgroup is realized by a maximal model geometry. For example, \(SO(3)\) in its standard representation is one such subgroup, though the non-model geometry \(\mathbb{R} \rtimes \text{Conf}^+ \mathbb{E}^3 / SO(3)\) mentioned above suggests this can be thought of as a near miss. The non-occurrence of \(SU(2)\) is another example, a feature shared by the 4-dimensional classification of geometries. Other subgroups—namely \(SO(4)\) and \(SO(3) \times SO(2)\)—are point stabilizers only of product geometries. A listing of (non-product) geometries by point stabilizer is given below in Table 3.2 after Figure 3.1 names the subgroups.

Geometries in higher dimensions with reducible isotropy and no fibering. When point stabilizers act reducibly on tangent spaces, our strategy breaks down the problem by showing the existence of an invariant fiber bundle structure. That this is possible is a convenient accident of low dimensions; higher dimensions introduce isotropy-reducible geometries that admit no fibering.

For example, in dimension 18, there is \(\text{Sp}(3)/\text{Sp}(1)\), where the embedding \(\text{Sp}(1) \hookrightarrow \text{Sp}(3)\) is given by the irreducible representation of \(\text{Sp}(1) \cong SU(2)\) on \(\mathbb{C}^6\). This has two isotropy summands but admits no nontrivial fibering since \(\text{Sp}(1)\) is maximal (so no larger group can be a point stabilizer of the base space) [DK08, Example V.10]. A strategy that continues to break the problem down using invariant fiber bundle structures may have to account for these exceptions, likely using Dynkin’s work on classifying maximal subgroups of semisimple Lie groups in [Dyn00a, Dyn00b].

Non-geometries as base spaces of fiber bundles. Even when invariant fiber bundle structures exist, a number of complications prevent the classification from having a straightforward recursive solution. Filipkiewicz warns in [Fil83, Prop. 2.1.3] that the base space of an invariant fiber bundle structure may fail to be a geometry due to noncompact point stabilizers—e.g. the action of \(\text{PSL}(2, \mathbb{C})\) on \(S^2 \cong \mathbb{CP}^1\) makes \(T^1 \mathbb{H}^3 = \text{PSL}(2, \mathbb{C})/\text{PSO}(2)\) a fiber bundle over \(S^2\). Even when point stabilizers are compact, the base may fail to be maximal (e.g. \(\text{Heis}_5\) fibers over \(\mathbb{E}^4\) with \(U(2)\) point stabilizers) or a model geometry (e.g. \(\text{Sol}^3\) over \(\text{Aff}^+ \mathbb{R}\), which cannot admit a lattice since it is not unimodular).

2.3 Examples that clarify how the classification is organized

The eight categories of Theorem 1.2 and the grouping of geometries into parametrized families involved some arbitrary choices. This section discusses the chosen method of organization and some variations.

The omission of some spaces that one might have guessed. Some of the categories in Thm. 1.2 are conspicuously missing geometries that happen to be non-model or non-maximal.

3. The tautological unit circle bundle \(U(3)/U(2)\) over \(\mathbb{CP}^2\) is non-maximal, being equivariantly diffeomorphic to \(S^5\). The two other unit tangent bundles of 3-dimensional spaces of constant curvature are also non-maximal.

\[
T^1(S^3) = SO(4)/SO(2) \cong S^2 \times S^3 \quad T^1(\mathbb{E}^3) = \mathbb{R}^3 \times SO(3)/SO(2) \cong S^2 \times \mathbb{E}^3
\]

7. Many of the solvable Lie groups arising from the list in [PSWZ76, Table II] are not unimodular and hence do not admit lattices.

8. Every product geometry with multiple Euclidean factors is non-maximal—but all other products are maximal, usually as a consequence of the de Rham decomposition theorem. (See [Gen16b, Prop. 3.12] in Part III.)
Counting the geometries and families. The list given in Thm. 1.2 includes 53 individual geometries and the following 6 infinite families of geometries.

\[ L(a; 1) \times S^1 L(b; 1), \quad a \leq b \text{ coprime positive integers} \]
\[ \widetilde{SL}_2 \times_\alpha S^3, \quad 0 < \alpha < \infty \]
\[ \widetilde{SL}_2 \times_\alpha \widetilde{SL}_2, \quad 0 < \alpha \leq 1 \]
\[ \mathbb{R}^4 \rtimes \mathbb{R}, \quad e^{tA} \text{ semisimple integer matrix with 4 real eigenvalues} \]
\[ \mathbb{R}^4 \rtimes \mathbb{R}, \quad e^{tA} \text{ semisimple integer matrix with 2 real eigenvalues} \]
\[ \text{Sol}^4_{m,n} \times \mathbb{E}, \quad m, n \in \mathbb{Z} \]

To some extent, this count depends on interpretation. The first three families could be expanded to include products of spheres and hyperbolic spaces, while the last three could be unified with \( \mathbb{R}^4 \rtimes \mathbb{R} \) to form one large family with a name like \( \text{Sol}^5_{m,n,p} \) (where \( m, n, \) and \( p \) are the middle coefficients of the characteristic polynomial of \( e^A \)). Indeed, [PSWZ76, Table II] suggests this latter unification by listing all semidirect products \( \mathbb{R}^4 \rtimes \mathbb{R} \) with diagonalizable action under the family \( A_{5,7} \). We keep the subfamilies of \( \text{Sol}^5_{m,n,p} \) separate since their point stabilizers have different dimensions.

3 Overview of method

The classification of 5-dimensional geometries \( M = G/G_p \) begins, following Thurston [Thu97, §3.8] and [Fil83, §1.2], by using the representation theory of compact groups to list the subgroups \( G_p \subseteq \text{SO}(T_p M) \) that could be point stabilizers (Figure 3.1).

Figure 3.1: Closed connected subgroups of \( \text{SO}(5) \), with inclusions. \( \text{SO}(3)_5 \) denotes \( \text{SO}(3) \) acting on its 5-dimensional irreducible representation; and \( S^1_{m/n} \) acts as on the direct sum \( V_m \oplus V_n \oplus \mathbb{R} \) where \( S^1 \) acts irreducibly on \( V_m \) with kernel of order \( m \). See Part II, [Gen16a, Prop. 3.1] for the proof.

\[
\begin{align*}
\text{SO}(5) & \quad \text{SO}(4) \quad \text{SO}(3) \times \text{SO}(2) \quad \text{SO}(3)_5 \\
\text{SO}(3) & \quad \text{SO}(2) \times \text{SO}(2) \\
\text{U}(2) & \quad \text{SU}(2) \\
\text{S}^1 & \quad \text{S}^1_{m/n} \quad \text{S}^1_{0} = \text{SO}(2) \quad \text{S}^1_{1/2} \\
\{1\} & \quad \{1\}
\end{align*}
\]

The problem divides into cases by the action of \( G_p \) on the tangent space \( T_p M \) (the “linear isotropy representation”)—more specifically, by the highest dimension of an irreducible subrepresentation \( V \).
Table 3.2: Using the classification, non-product geometries can be listed by point stabilizer.

| Stabilizer | Geometries |
|------------|------------|
| SO(5)      | $E^5$, $S^5$, $H^5$ |
| U(2)       | Heis$_5$ and $U(2,1)/U(2)$ |
| SO(3)$_5$  | $SL(3,\mathbb{R})/SO(3)$ and $SU(3)/SO(3)$ |
| SO(2) x SO(2) | $\mathbb{R}^4 \rtimes \mathbb{R}_{x-1,x+1,x+1}$ and the associated bundles (Thm. 1.2(4)) |
| SO(2)      | $\mathbb{R}^4 \rtimes \mathbb{R}$ and $\mathbb{R}^4 \rtimes \mathbb{R}$ with $2$ real roots $\mathbb{R}^4 \rtimes \mathbb{R}$ with $2$ real roots $(x-1)^2,x+1,x+1$ |
| $S^1_{1/2}$ | All line bundles over $\mathbb{F}^4$ (Thm. 1.2(5)) |
| $S^1_1$    | The two unit tangent bundles (Thm. 1.2(3)), $\mathbb{R}^4 \rtimes \mathbb{R}$, and $(\mathbb{R} \times \text{Heis}_3) \rtimes \mathbb{R}$ $x^2,x^2$ and $(x \rightarrow x_2 \rightarrow y)$ |
| $\{1\}$   | The remaining solvable Lie groups |

Figure 3.3: Flowchart of the classification. Let $V$ be an irrep in $G_p \rtimes T_pM$ of maximal dimension.

At the extremes, one can appeal to existing classifications—the classification of strongly isotropy irreducible homogeneous spaces by Manturov [Man61a, Man61b, Man66, Man98], Wolf [Wol68, Wol84], and Krämer [Krä75] when $G_p \rtimes T_pM$ is irreducible ($\dim V = 5$); and the classification of low-dimensional solvable real Lie algebras by Mubarakzyanov [Mub63] and Dozias [Doz63] if $G_p \rtimes T_pM$ is trivial ($\dim V = 1$). These cases are handled in Part II [Gen16a], including the production of an
identification key for trivial-isotropy geometries (Fig. 3.4).

Figure 3.4: Identification key for solvable geometries $G = G/\{1\}$.

Lie algebra $\mathfrak{g}$ is solvable

- nilpotent
  - 4-D abelian ideal
    - $\mathfrak{g}^4 \neq 0$ $\Rightarrow$ $\mathbb{R}^4 \times \mathbb{R}$
    - $\mathfrak{g}^4 = 0$ $\Rightarrow$ $\mathbb{R}^4 \times \mathbb{R} \cong \text{Nil}^4 \times \mathbb{E}$
  - no 4-D abelian ideal
    - $\mathfrak{g}^4 \neq 0$ $\Rightarrow$ $\text{Nil}^4 \times \mathbb{R}$
    - $\mathfrak{g}^4 = 0$ $\Rightarrow$ $\text{Nil}^4 \times \mathbb{R}$

- non-nilpotent
  - nilradical $\mathbb{R}^3$
    - $\mathbb{R}^3 \times \{xyz = 1\}^0$
  - nilradical $\mathbb{R}^4$
    - 2 Jordan blocks $\Rightarrow$ $\mathbb{R}^4 \times \mathbb{R}$
    - 3 Jordan blocks $\Rightarrow$ $\mathbb{R}^4 \times \mathbb{R}$
    - 4 Jordan blocks $\Rightarrow$ $\mathbb{R}^4 \times \mathbb{R}$

- nilradical $\mathbb{R} \oplus \mathfrak{sl}_3$
  - 1-D center $\Rightarrow$ $\langle \mathbb{R} \times \text{Heis}_3 \rangle \times \mathbb{R}$
  - 2-D center $\Rightarrow$ $\text{Sol}_1^4 \times \mathbb{E}$

Otherwise $G_p \acts T_pM$ is nontrivial and reducible ($2 \leq \dim V \leq 4$). We classify these—the “fibering geometries”—in Part III, starting by proving the existence of a $G$-invariant fiber bundle structure on $M$ [Gen16b, Prop. 3.3]. The Uniformization Theorem and version of a theorem by Obata and Lelong-Ferrand [Oba73, Lemma 1] imply the base space has an invariant conformal structure. Beyond this common behavior, the properties of the fibering and the relevant tools vary with the dimension of the subrepresentation $V$, naturally suggesting the cases in Figure 3.3.

When $\dim V = 4$, the geometries are determined by curvature and base, in a fashion closely resembling Thurston’s treatment of $\dim V = 2$ and $\dim M = 3$ in [Thu97, Thm. 3.8.4(b)]; Table 3.5 lists the results.

Otherwise, we work systematically with $G$-invariant fiber bundle structures by recasting the problem as an extension problem for the Lie algebra of $G$ and solving it with the help of Lie algebra cohomology. Over 3-dimensional base spaces there happen to be only products; but over 2-dimensional base spaces a daunting array of possibilities requires some attempt to organize the problem, summarized in Figure 3.6.
### Table 3.5: Geometries with irreducible 4-dimensional isotropy summand

| Base | Flat (product) | Curved |
|------|----------------|--------|
| $S^4$ | $S^4 \times E$ |        |
| $E^4$ | non-maximal $E^5$ |        |
| $H^4$ | $H^4 \times E$ |        |
| $\mathbb{C}P^2$ | $\mathbb{C}P^2 \times E$ | non-maximal $S^5$ |
| $\mathbb{C}H^2$ | $\mathbb{C}H^2 \times E$ | $\widetilde{U(2,1)}/U(2)$ |

### Figure 3.6: Classification strategy for geometries $M = G/G_P$ fibering over 2-D spaces $B$.

- **B has invariant metric?**
  - yes — $\tilde{G}$ is an extension of $\text{Isom}_0 B$
  - no
- **Extension of $\text{Isom}_0 E^2$ by $\mathbb{R}^3$?**
  - yes — Some nilpotent Lie groups.
  - no — then $\tilde{G}$ is a split extension
- **Levi action nontrivial?**
  - yes — $T^1\mathbb{H}^{1,2}$ and the line bundles over $\mathbb{F}^4$.
  - no — $\tilde{G}$ is a direct product

### 4 Related work

**Classification of compact homogeneous spaces.** Gorbatsevich has produced classification results for compact homogeneous spaces $M$ by using a fiber bundle described in [GOV93, §II.5.3.2] whose fibers have compact transformation group, whose base is aspherical, and whose total space is a finite cover of $M$. The classification is complete in dimension up to 5 in general, in dimension 6 up to finite covers, and in dimension 7 in the aspherical case [Gor12].

Another approach would be to group the problem by the number of isotropy summands. The Riemannian homogeneous spaces with irreducible isotropy were classified by Manturov [Man61a, Man61b, Man66, Man98], Wolf [Wol68, Wol84], and Krämer [Krä75]; and the compact Riemannian
homogeneous spaces with two isotropy summands were classified by Dickinson and Kerr in [DK08].

**Classification of naturally reductive spaces.** The naturally reductive Riemannian homogeneous spaces \( G/G_p \) — those whose geodesics through \( p \) are orbits of 1-parameter subgroups tangent to the representation complementary to \( T_1G_p \subset G_p \sa T_1G \) (see e.g. [KN69, §X.3]) — have been classified in dimension 6 by Agricola, Ferreira, and Friedrich [AFF15]; and in lower dimensions by work of Kowalski and Vanhecke (see [KPV96, §6] for a summary).

The case of dimension 5, in [KV85, Thm. 2.1],\(^1\) shares the following features with the classification of geometries.

1. Everything with SU(2) isotropy is realized by a homogeneous space with U(2) isotropy [KV91, main result (b)].

2. The associated bundle geometries appear as indecomposable naturally reductive spaces.

The differences between geometries and naturally reductive spaces bear mentioning as well:

1. Naturally reductive spaces need not be maximal as geometries, as demonstrated by non-maximal realizations of \( S^3 \cong S^3 \times S^1/S^1 \) and \( S^5 \cong \text{SU}(3)/\text{SU}(2) \).

2. Some geometries — particularly those with trivial isotropy — may not be realizable by naturally reductive spaces. In 3 dimensions, there is just \( \text{Sol}^3 \); and in 4 dimensions, there are \( \mathbb{F}^4 \) and the four solvable Lie group geometries other than \( \text{Heis}_3 \times \mathbb{E} \) and \( \mathbb{E}^4 \). In 5 dimensions, there are the unit tangent bundles \( T^1\mathbb{H}^3 \) and \( T^1\mathbb{E}^{1,2} \), the line bundles over \( \mathbb{F}^4 \), the products involving \( \text{Sol}^3 \), and any solvable Lie group geometries that are not \( \mathbb{E}^5 \), \( \text{Heis}_5 \), or a product involving \( \text{Heis}_3 \).

A chart in [KPV96, 5.1] summarizes the relations between several other classes of spaces.

**Other geometric structures.** One can replace the assumption of an invariant Riemannian metric (compact point stabilizers) with other structures. A number of difficulties may result from this: geodesic completeness may no longer coincide with other notions of completeness (e.g. in [DZ10, Thm. 2.1]); isotropy representations may fail to be semisimple or faithful; and point stabilizers may fail to act faithfully on tangent spaces, necessitating techniques like those of [vM13]. In spite of these challenges, some results are known, such as:

- Using conformal structures without relaxing other assumptions yields only \( S^n \) and \( \mathbb{E}^n \): a manifold whose conformal automorphism group acts transitively with the identity component preserving no Riemannian metric is conformally equivalent to one of the two, by theorems of Obata [Oba73, Lemma 1] and Lafontaine [Laf88, Thm. D.1].

- The interaction of complex structures and 4-dimensional geometries was investigated by Wall in [Wal86]; and almost-complex structures on homogeneous spaces up to dimension 6 are classified by Alekseevsky, Kruglikov, and Winther in [AKW14] with the additional assumption that point stabilizers are semisimple.

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1 The changes in the corrected version [KPV96, 6.4] appear to amount to (1) changing “symmetric” and “decomposable” to “locally symmetric” and “locally decomposable” and (2) changing the rational parameter to a real parameter in the Type II family (Heisenberg-group-like bundles).
Loosely analogous to almost-complex structures are 5-manifolds whose structure group can be reduced to $SO(3)_5$. The classification of such structures in the integrable case by Bobieński and Nurowski [BN07, Thm. 4.7] offers an alternative path to the classification of isotropy-irreducible geometries, and there are some further classification results in the non-integrable case by Chiossi and Fino in [FC07] and by Agricola, Becker-Bender, and Friedrich in [ABBF11].

The pseudo-Riemannian geometries were classified in dimension 3 by Dumitrescu and Zeghib in [DZ10]; and the pseudo-Riemannian naturally reductive spaces were classified in dimension 4 by Batat, López, and María [BLM15].

5 References

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