THE HYDROGEN LAMB SHIFT AND THE PROTON RADIUS

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ABSTRACT

The Lamb shift measurement and theory are now both a dynamically developing field and we give a review of the current data. Critical comparison of theory and experiment can be done using a value of the proton charge radius and we pay attention to several results of its determination.

The talk is devoted to the Lamb shift in the hydrogen atom. The workshop is for exotic atoms and first of all I would like to say that to my mind the hydrogen is one of them. The possibility both to calculate and to measure different energy intervals with extremely high accuracy make hydrogen a quite exotic system. Next, one has to remember that there is actually no special separate theory of the positronium or muonium atom, investigations of which were discussed here in detail. The same expression can be of use in a number of calculations for several atomic systems (see e. g. Refs. [1, 2]. One more point which is quite similar to the pionium and pionic hydrogen is that we investigate an atomic system but after all the result is important for particle physics, namely for the proton charge radius. All of these reasons show that hydrogen atomic properties are to be discussed among exotic ones. The workshop talk is based on a Max-Planck-Institut für Quantenoptik report [3] and all references can be found there.

The hydrogen atom is one of the most important QED systems. In contrast to muonium and positronium the nucleus is a proton and hence the energy of the level is influenced by the proton structure, i. e. by the strong interaction. The knowledge of the proton radius leads now to a limit of the theoretical value for the Lamb shift. We discuss here the problem of the radius determination, the most popular values of which are summarized in Table 1. We give a critical review of all of these values, as well as of the theory and the experiment on the Lamb shift.

Let us briefly describe all items of the Table 1. The elastic electron-proton scattering data were obtained with small momentum transfer and were extrapolated to zero momentum. The other way to extract the radius from such data is based on a dispersion relation approach which allows to involve data from other kinematic areas into fitting. The numerical calculation within the chiral limit of the lattice QCD can be corrected due to the chiral perturbative theory. And indeed one of the ways to determine the radius is based on the hydrogen Lamb shift investigation. First we mention that we expect that the uncertainties presented in Table 1 accordingly to the original works [4, 6, 7] are significant underestimations and we will not consider those anymore here. The details can be found in our review [3].

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| Value       | Reference                  | Method                                      |
|-------------|----------------------------|---------------------------------------------|
| 0.809(11) fm | Stanford, 1963             | scattering & empirical fitting              |
| 0.862(12) fm | Mainz, 1980                | scattering & empirical fitting              |
| 0.64(8) fm   | Draper et al., 1990        | lattice QCD in chiral limit                 |
| 0.88(3) fm   | Leinweber, Cohen, 1993     | lattice QCD & chiral perturbation           |
| 0.847(9) fm  | Mainz, 1996                | dispersion relation fitting                 |
| 0.890(14) fm | Garching, 1997             | hydrogen Lamb shift measurements            |

Table 1: Proton charge radius

We start our consideration with the Lamb shift. Several results obtained within different approaches are summarized in Table 2 (see Fig. 1 for more detail). The results presented are found within different experiments:

- the *LS* result is an average value of the best *direct* measurement of the Lamb splitting $S$ by Lundeen and Pipkin and older works. We also take into account a recent paper by van Wijngaaden, Holuj and Drake, published after our work on review was completed;
- the $LS/Γ$ result of an indirect measurement of the splitting $S$. The measurement was performed by Sokolov and Yakovlev for the ratio of $S$ and the $2p_{1/2}$ radiative width. The width was recalculated by Karshenboim recently including a leading radiative correction of relative order $α(Zα)^2 \ln(Zα)$ (see Ref. for detail)

$$\Gamma(2p_{1/2}) = \frac{2^{10}π}{3^8} α^3 Ry \frac{m_R}{m} \left\{ 1 + \ln \left( \frac{9}{8} \right) (Zα)^2 + \frac{16}{3π} α(Zα)^2 (1.5084...)^2 \ln \frac{1}{(Zα)^2} \right\};$$

- in case of $FS$ the measured value is the *fine structure* splitting $2s_{1/2} − 2p_{3/2}$, the most recent result for which was found by Hagley and Pipkin;
- the $OBF$ abbreviation means *optical beat frequency*. The value is an average one from recent Garching, Yale and Paris data;
- the $CAF$ result is obtained by *comparison* of two optical frequencies measured separately, namely the $1s − 2s$ interval from Garching and $2s − 8s/d$ from Paris.

To recalculate the data obtained by $FS$, $OBF$ and $CAF$ methods one has to use a significant piece of theory. In case of the fine structure that is the theory of the $2p$ states and in case of the optical measurement that is theory of a specific difference

$$\Delta(n) = E_{L}(1s) − n^3 E_{L}(ns).$$

First we explain the importance of this difference and next we consider the status of its calculation as well as of $E_{L}(2p_j)$. The progress in measurement of the either the Lamb splitting or the fine structure has been relatively slow. In contrast to that great
| Method | Lamb splitting |
|--------|----------------|
| LS     | 1057850(7) kHz |
| LS/Γ   | 1057858(2) kHz |
| FS     | 1057840(11) kHz |
| OBF    | 1057843(7) kHz |
| CAF    | 1057853(4) kHz |

Table 2: Experimental result for the Lamb splitting $S = E(2s_{1/2} - 2p_{1/2})$

development was obtained in optical measurement of the transition frequency between levels with different value of the principal quantum number $n$ (gross structure). The highest precision was reached in two-photon Doppler-free transitions like $1s \rightarrow 2s$. The problem of utilizing those results was due to the Rydberg constant determination. There is no way to find it except by investigation of the gross structure. Hence to find anything for the Lamb shift one has to measure two optical transitions and to construct some difference in which the Rydberg contribution is canceled. It is possible to instrumentally extract a beat frequency directly or indirectly within an experiment. It is also possible to do that with data obtained in two independent determinations of different intervals. But still one has to solve one more problem: the Lamb shift result is after all a combination of the Lamb shift of the $1s$ and $2s$ states and also a portion of a higher excited levels contribution is included. In order to manage that a specific difference $\Delta(n)$ was introduced by us some time ago (see Ref. 20 for detail). This difference as well as the $2p$ state Lamb shift has a much better theoretical status than the ground state Lamb shift (or $S$). A number of contributions like e. g. a three-loop term which have not been known up to date for $S$ are known for the difference and for the $2p$ states.

The theoretical expression for the difference of Eq. (1) is of the form

$$\Delta(n) = \frac{\alpha(Z\alpha)^4}{\pi} \frac{m_R^3}{m^2} \times \left\{- \frac{4}{3} \ln \frac{k_0(1s)}{k_0(ns)} \left(1 + \frac{Z}{M} \right)^2 \right.$$ 
$$+ (Z\alpha)^2 \times \left[ \left(4(\ln(n) - \psi(n + 1) + \psi(2)) - \frac{77(n^2 - 1)}{45n^2} \right) \ln \frac{1}{(Z\alpha)^2} + A_{60}^{VP}(n) + G_n^{SE}(Z\alpha) \right]$$
$$- \frac{14}{3} Zm \left(\psi(n + 1) - \psi(2) - \ln(n) + \frac{n - 1}{2n} \right) \right\} + \frac{\alpha^2(Z\alpha)^6m}{\pi^2} \ln^2 \frac{1}{(Z\alpha)^2} B_{62}. \quad (2)$$

The results of the $2p$ states are similar to that

$$\Delta E_L(2p_{1/2}) = \frac{\alpha(Z\alpha)^4}{8\pi} \frac{1}{m} \left(\frac{m_R}{m}\right)^3 \left\{- \frac{4}{3} \ln k_0(2p) \times \left(1 + 2Z \frac{m}{M} + Z^2 \frac{m}{M} \right)^2 \right.$$ 
$$+ (Z\alpha)^2 \left(\frac{103}{180} \ln \frac{1}{(Z\alpha)^2} - \frac{9}{140} + G_{2p_{1/2}}(Z\alpha) \right) \right\} - \frac{(Z\alpha)^4m}{24} \left(\frac{m_R}{m}\right)^2 \frac{g_e - 2}{2}$$

3
The one-loop contribution due to the vacuum polarization ($A^{VP}_{60}$) was found for an arbitrary $ns$ state by Karshenboim and Ivanov and for an $np_j$ state by Manakov, Nekipelov and Feinstein. The self-energy term ($G_{n}^{SE}(Z\alpha)$) is mainly determined by a value of $A^{SE}_{60}$. The last was found by Pachucki for $1s$ and $2s$ and by Jentschura and Pachucki for $2p$. The result has to be found extrapolating numerical data calculated by Mohr and by Kim and Mohr for higher nuclear charge $Z$. It is simpler to extrapolate the data for the difference of Eq. (2) and for the $p$ state in comparison to extrapolation for the ground state. The leading two-loop logarithmic term was found by Karshenboim. Pachucki and Grotch found that the recoil term of the order $(Z\alpha)^6m^2/M$ for an $ns$ state is scaling by $1/n^3$. 

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Figure 1: Different values of the Lamb splitting $S$
and so it cannot contribute to the difference. In case of the 2p states this correction was evaluated by Golosov, Yelkhovsky, Milstein and Khriplovich. The theoretical progress mentioned for the $\Delta(n)$ and the 2p Lamb shift allows to obtain results in Table 2 with pretty small theoretical uncertainty. The QED results for 4p states are also needed for the evaluation of some experimental data. The value of $A_{60}^{(SE)}$ was found by Jentschura, Mohr and Soff and the recoil effects were investigated by Yelkhovsky and Pachucki independently. That is, however, only a not significant part of the QED calculations which have to be done on a way to reach a value of the proton radius from hydrogen atom spectroscopy. It is necessary to calculate the Lamb shift of the 2s state (as we can see from the discussion above the significant part of the QED theory of 1s and 2s states is the same and we speak here about the ground state).

The expression for the ground state Lamb shift is much more complicated. The term most important for the further discussion is of the well known form

$$\delta E_{\text{nucl}}(ns) = \frac{2}{3} \frac{(Z\alpha)^4}{n^3} m_R^2 R_p^2. \quad (3)$$

That includes the proton charge radius and subtracting the QED result (Eq. (3) excl.) from the experimental value we can determine the proton charge radius. A detailed review of the ground state Lamb shift was presented by Pachucki et al. [21]. Here we mention some important features. The most recent corrections considered there are the one-loop self-energy (Pachucki; Mohr), the two-loop contribution of the order $\alpha^2(\alpha Z)^5 m$ (Pachucki and Eides, Grotch and Shelyuto), the leading two-loop logarithm of a higher order (Karshenboim), a pure recoil term (Fell et al.; Grotch and Pachucki and Shabaev and coworkers) and the radiative-recoil correction (Bhatt and Grotch contradicting to Pachucki).

We mainly agree with a consideration in review [21]. However, our result is shifted by -3.6 kHz because the $\alpha^2(\alpha Z)^6 m \log^2 Z\alpha$-correction has not been included there. We would also like to present here all sources of the theoretical uncertainty for the Lamb splitting $S$:

- unknown $\alpha(\alpha Z)^7 m$ and higher order one-loop corrections are estimated to 1 kHz;
- $\alpha^2(\alpha Z)^6 m \log^2 Z\alpha$ and higher order two-loop terms can give up to 2 kHz;
- the three-loop $\alpha^3(\alpha Z)^5 m$ contribution is here estimated preliminary to 2 kHz and needs more understanding.

The one-loop term is going to be calculated exactly and so the uncertainty is being removed [22].

Now let us turn to the discussion of the scattering results. We start from simple estimates. The value extracted from elastic electron-proton scattering cross sections is the electric form factor of the proton $G(q^2)$. We need to investigate it at low momentum transfer where the ‘signal’ which is $G - 1$ is mainly determined by the radius term $((q^2 R_p^2)/6)$. The $G - 1$ term lies at the condition of experiment in Ref. [5] between 1% and 15% (see Fig. 2). The scattering radius in Table 1 is claimed [5] to be with uncertainty within about 1%. That means that all shifts on the level of a few percents are important. We discussed in our review [5] some QED corrections to the cross section which are expected to be on a few percent level and which are beyond evaluation of Ref. [5]. In this paper we concentrate our attention to more important problem. That is due to the
normalization of the data. It is clear that the value of the electric form factor at zero momentum transfer is equal to one

\[ G_{th}(0) = 1. \]

That is absolutely correct theoretically, but the form factor cannot be measured straightforwardly. It is possible to measure only some cross section. Evaluating the data one can extract form factor from the Rosenbluth formulae. On this way the extracted experimental value which is expected to be the form factor is only consistent with the true form factor within some uncertainty. As one of results the value of \( G_{exp}(0) \) is consistent with one and it has to be found within some fitting procedure. In other words we have to write a trivial equation

\[ G_{exp}(q^2) = a_0 G_{th}(q^2), \]

where the value of \( a_0 \) is consistent with one, but not equal to that a priori. For application to the low momentum transfer electron-proton scattering we can expect that \( a_0 \) is a constant (but it really depends on experimental setup).

In both Mainz papers some special a priori prescriptions for \( a_0 \) were used. The problem of normalization was investigated by Wong prior to publication of the dispersion paper. He found that the result strongly depends on an assumption on the value of the constant \( a_0 \). The correct value of the radius corresponds to the free normalization (i.e. \( a_0 \) is one of the fitting parameters). The constant

\[ a_0 = 1.0028(22) \]
Momentum transfer $q^2$ [fm$^{-2}$]

Form factor $G(q^2) + (0.863$ fm$)^2 q^2/6$

Exp. point $a_0 = 1.0014; R_p = 0.863$ fm

$a_0 = 1.0000; R_p = 0.849$ fm

$a_0 = 1.0028; R_p = 0.877$ fm

Figure 3: Some fitting of the Mainz data with different normalization constant found from the Mainz scattering data a posteriori (see Fig. 3) is consistent with one (the prescription of Ref. is $a_0 = 1$) and with the value of $a_0 = 1.0014$ prescribed in Ref. However, on this way the uncertainty of the proton radius is significantly larger than in both Mainz papers (see Ref. for more details).

We have discussed all items of Table 1 and we present now some conclusions. The scattering value has an uncertainty of about 0.24 fm. We cannot present here any eventual figure because Wong performed his evaluation with the most important part of data but not with all of them. Not all corrections to the data were included. The result should be close to the Wong value (0.877(24) fm) but it needs more analysis. However data are quite old and practically it is not possible to reevaluate them. Hopefully, a new measurement in Mainz is going to be done. The Lamb shift examination can give a better value. We have to mention that the experiment claimed as the most accurate one (uncertainty is 2 kHz) is under some criticism and the average value of all other measurements gives now some larger uncertainty (about 3 kHz). The uncertainty for the proton radius extracted is 0.15 fm (the equal portions arise from the ground state Lamb shift QED theory and from the optical experiments). We also would not like to give any value because it is a running value now: some data are still coming. The most precise value which is included into our analysis is due to the Garching and Paris absolute measurements. Some parts of the Paris data are changing now due to recalibration of the standards. The preliminary value from the Lamb shift is in fair agreement with the Wong value but with twice larger accuracy.

The situation now, when the uncertainties from spectroscopy measurement, QED calculation and the scattering data are on the same level is quite challenging and we believe that new results are coming in all these fields. Recently a new wave of activity projecting of a muonic hydrogen experiment for the Lamb shift has been arisen at the PSI and we also hope that will give one more value for the proton radius. We also would like to attract attention to the Lamb shift measurement in hydrogen-like system with a
moderate value of the nuclear charge \( Z \simeq 5 - 20 \). It seems it is necessary to encourage such investigations because they can lead to experimental estimation of the higher order \( QED \) terms. That is to be helpful for the hydrogen Lamb shift theory.

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