Sensor self-interaction, scale-invariant spin dynamics, and the $\hbar$ limit of field sensing

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We describe quantum limits to field sensing that relate noise, geometry and measurement duration to fundamental constants, with no reference to particle number. We cast the Tesche and Clarke (TC) bound on dc-SQUID sensitivity as such a limit, and find analogous limits for volumetric spin-precession magnetometers, e.g. alkali vapors and nitrogen-vacancy centers in diamond. The volumetric limits arise from scale-invariance of the sensors’ internal dynamics, through which particle number dependence vanishes. We find energy resolution per bandwidth $E_B \geq \alpha h$, $\alpha \sim 1$, in agreement with the TC limit, for paradigmatic spin-based measurements of static and oscillating magnetic fields.

The quantum limits of measurement is a rich topic of both fundamental and practical interest. The theory of these limits informs many other topics, including the statistics of parameter estimation [1], the geometry of quantum states [2], entanglement in many-body systems [3], quantum information processing [4,5], and quantum non-locality [6,7]. Understanding quantum measurement effects has led to improved sensitivity in gravitational wave detectors [8,9] and progress toward similar improvements in measurements of time [10–13], dc magnetic fields [14,15] and radio-frequency fields [16,17].

The vast majority of prior work on quantum sensitivity limits concerns the problem of linear interferometric parameter estimation. For example, the standard quantum limit (SQL) $\langle \delta \phi^2 \rangle \geq 1/N$ and the Heisenberg limit (HL) $\langle \delta \phi^2 \rangle \geq 1/N^2$ constrain linear estimation of a phase $\phi$ given the resource of $N$ two-level systems. These dimensionless limits acquire units, e.g. length or time, through implementation-dependent scale factors, e.g. a wavelength or a transition frequency. Because these scale factors, as well as the available $N$, can vary greatly from one implementation to another, such dimensionless limits do not by themselves provide benchmarks by which to compare different sensor implementations.

Here we study a qualitatively different kind of quantum sensitivity limit, one that contains no implementation-specific scale factors, and no makes reference to available resources, only to the quantity to be measured and to the method of measurement. To see what form such a limit could take, consider a sensor that measures the field $B$ in a volume $V$ over an observation time $T$, and gives a reading $B_{\text{obs}} = B_{\text{true}} + \delta B$, where $B_{\text{true}}$ is the true value of the field and $\delta B$ is the measurement error – a zero-mean random variable if the sensor is properly calibrated. The mean apparent magnetostatic energy in the sensor volume is $E_{\text{obs}} = \langle B^2_{\text{obs}} \rangle V/(2\mu_0) = B^2_{\text{true}} V/(2\mu_0) + \langle \delta B^2 \rangle V/(2\mu_0)$, where the second term expresses the sensor’s so-called “energy resolution,” which more properly can be identified as the bias of the apparent energy. Allowing for averaging of repeated measurements with independent noise, any general limit on sensing of static fields will have the form

$$\frac{\langle \delta B^2 \rangle VT}{2\mu_0} = \frac{S_B(0)V}{2\mu_0} \geq S$$

where $S_B(0)$ is the low-frequency limit of the power spectral density of variable $x$, and $S$ is a constant with units of action. As a purely empirical observation, several technologies to sense low-frequency magnetic fields approach $S = \hbar$ while to date none surpasses it [20]. Such a limit makes no reference to available resources, and the scale factors are the fundamental constants $\mu_0$ and $\hbar$.

We can obtain a first implementation-independent limit from a well-known analysis of dc superconducting quantum interference devices (dc SQUIDs) by Tesche and Clarke (TC) [21]. Considering a lumped-circuit model for dc SQUID magnetometers with resistively-shunted Josephson junctions, TC computed the sensitivity, i.e. power spectral density of the equivalent noise, for an optimized device. At zero temperature, the sensitivity is limited by zero-point current fluctuations in the shunt resistances, to give $S_\Phi(0)/(2L) \geq \hbar$ where $\Phi$ is the flux through the SQUID loop, and $L$ is the loop inductance [22,23]. With careful construction, small dc SQUID devices have reported $S_\Phi(0)/(2L)$ as low as $2\hbar$ [24,25].

The implementation-dependent factor $L$ can be eliminated by noting that a wire loop has $\Phi = BA$ and $L = \sqrt{A}\mu_0/\alpha$, where $A$ is the loop area and $\alpha$ is a wire-geometry factor of order unity [27]. This gives a limit that concerns only field geometry and time: When measuring the field on a patch of area $A$ in a time $T$ with a dc-SQUID, the limiting sensitivity is equivalent to minimum energy per bandwidth of $\langle \delta B^2 \rangle T A^{3/2}/(2\mu_0) \geq \alpha h$.

In what follows we illustrate and derive the analogous limit for a second important field sensing technology, the spin-precession sensor. As the name suggests, such devices detect magnetic fields by the precession induced in an ensemble of spins. Notable examples include alkali vapors [28,29], nitrogen-vacancy centers in diamond [30,31], and spinor Bose-Einstein condensates [32,33].
FIG. 1. Time evolution of spin statistics and sensitivity for fixed-spin dc magnetometry for a variety of magnetic moments and field strengths, showing the general behaviour of seemingly exponential loss of coherence through self-interaction, and sensitivity nearly independent of implementation specifics. Following an initial tip into the plane orthogonal to the magnetic field, an ensemble spin-1/2 particles, with fixed random positions, is allowed to freely evolve. Red and blue graphics show, on the left scale, the per-spin polarization $S^{(q)}/M$ and $S^{(q)}_0/M$, respectively, with curves showing mean value and shaded regions showing mean plus-minus one rms deviation, i.e. square root of the corresponding diagonal elements of the spin covariance matrix $\Gamma_S$. Green curve shows, on the right scale, $E_R/h$ with a minimum of $\approx 0.7$ at $T = T_{\text{opt}} \approx 0.5 \Upsilon_{d}^{-1}$. Computed using an ensemble of $Q = 4 \times 10^4$ clusters of $M = 2$ PPP-distributed spins.

One might expect such sensors to be described by the “quantum metrology” analysis of linear interferometry [5], which gives rise to the SQL and HL given above. If this were the case, there could be no energy resolution limit, because the spin number density $\rho = N/V$ could be taken to infinity, such that $1/N$ and thus $(\delta B^2)$ approach zero for fixed $V$, leaving vanishing $E_R$. At high densities interactions cannot be neglected, however, and the linear interferometry results are not directly applicable [34].

Prior works on quantum sensing with interacting particles have considered scaling [35, 36] and optimization [34, 37] scenarios. Here we show that self-interaction in [34] scenarios. Here we show that self-interaction in spin-precession sensors produces self-similar spin dynamics, through which the limiting sensitivity becomes independent of all implementation-dependent parameters, and indeed gives $S = \alpha \hbar$, as in the TC analysis. Prior modeling of high-density NV-center ensembles [38] and high-density alkali vapors [39, 40] have noted variance of sensitivity with respect to spin density.

We consider a generic prepare-evolve-project sensing protocol, in which an ensemble of $N$ spins $\{s_i\}$, each with spin quantum number $s$, is initialized in a product state $|\Psi_0\rangle = |\psi_0\rangle^{\otimes N}$, allowed to evolve under a Hamiltonian $H$ containing as parameters the unknown positions of the spins $\{x_i\}$ and a field component $B$ to be estimated, e.g. the dc or rf field amplitude. The spins are detected at time $T$ by projection of the total spin $S = \sum_i s_i$ onto a direction $n$. Taking $V$ as constant, $T(\delta B^2)$ determines the energy resolution. For large $N$, $S \cdot n$ is nearly gaussian, and optimal estimation [41] $B$ can be understood using propagation of error: If $\Gamma_S$ is the covariance matrix of $S$, with elements $\langle S_i S_j + S_j S_i \rangle / 2 - \langle S_i \rangle \langle S_j \rangle$, the sensitivity limit is [42]

$$T(\delta B^2) \geq \min_{n,T} \frac{\text{tr} \Gamma_S(T) \cdot n}{|\partial_T (S \cdot n)|^2}. \quad (2)$$

Here the expectation is defined as

$$\langle A \rangle = \int \langle \Psi_0 | U^\dagger(t) A U(t) | \Psi_0 \rangle P_{\rho}(\{x_i\}) \, d\{x_i\}, \quad (3)$$

and includes classical averaging over configurations $\{x_i\}$ with probability density $P_{\rho}(\{x_i\})$, parametrized by $\rho$. For example, $\{x_i\}$ could be distributed as a Poisson point process (PPP), in which an infinitesimal volume $dV$ contains a spin with probability $p dV$. $U$ is the solution to the Schrödinger equation and thus depends on $\{x_i\}$ and $B$. Due to coherent signal accumulation at short times and decoherence at long times, the minimum in Eq. (2) occurs at a finite time $T = T_{\text{opt}}$. Examples (explained in detail below) are shown in Fig. [4].

Considering a uniform magnetic field $B$ with a constant component $B_{dc}$ along the $z$-axis, and including the dipole-dipole (dd) interaction, the Hamiltonian is

$$H = -\gamma \hbar \sum_i s_i \cdot B + \sum_{i \neq j} H_{dd}^{(ij)} \quad (4)$$

$$H_{dd}^{(ij)} = \frac{\gamma^2 \hbar^2 \mu_0}{4 \pi r_{ij}^3} [s_i \cdot s_j - 3(s_i \cdot \mathbb{R}_{ij})(s_j \cdot \mathbb{R}_{ij})], \quad (5)$$

where $\gamma$ is the gyromagnetic ratio, $\mathbb{R}_{ij} = r_{ij}/r_{ij}$, $r_{ij} \equiv x_i - x_j$, and $r_{ij} \equiv |r_{ij}|$.

In Eq. (4), the first term drives precession of $S$ at the Larmor frequency $\omega_L = 2\pi/T_L = \gamma B$. If $R_{ij}(\theta)$ is the rotation about axis $z$ (parallel to $B$) by an angle $\theta$, we can define a rotating frame by writing $A(t) \equiv R_z(\omega_L t) A(t)$ where $A$ is any lab-frame vector observable. In this frame the $B_{dc}$ term vanishes, leaving the rotating-frame Hamiltonian

$$\tilde{H} = \sum_{i \neq j} \hbar \frac{\Upsilon_{dd}}{s^2 \mu_0 r_{ij}^3} [\tilde{s}_i \cdot \tilde{s}_j - 3(\tilde{s}_i \cdot \tilde{\mathbb{R}}_{ij})(\tilde{s}_j \cdot \tilde{\mathbb{R}}_{ij})]. \quad (6)$$

where $\Upsilon_{dd} \equiv \gamma^2 \hbar^2 \mu_0/\langle 4\pi \rangle$ is the strength of the dipole-dipole coupling, with units of angular frequency.
This transformation leaves unchanged the limit found in Eq. (2), because of the optimization over the read-out direction \( \mathbf{n} \). We note that the \((\bar{s}_i \cdot \mathbf{R}_{ij})(\bar{s}_j \cdot \mathbf{R}_{ij})\) term has explicit time dependence (in \( \mathbb{R} \)) and causes relaxation of the collective spin through their coupling to \( x \) direction.

We now explore the general features of this problem through numerical simulation. Due to the rapid fall-off of \( H^{(i)}_{\text{dd}} \) with \( r_{ij} \), and the fact that \( H^{(i)}_{\text{dd}} \) vanishes when averaged over a sphere of constant \( r_{ij} \), the dynamics of any given spin \( s_i \) will be determined mostly by its nearest neighbors and by \( \mathbf{B} \). This motivates the following approximation: we consider the full system of \( N \) spins as an ensemble of \( Q = N/M \) clusters of \( M \) spins each, with each cluster evolving independently under \( H \). The collective spin is then \( \mathbf{S} = \sum_{q=1}^{Q} \mathbf{S}^{(q)} \), where \( \mathbf{S}^{(q)} = \sum_{s_i \in c^{(q)}} s_i \) is the total spin of cluster \( c^{(q)} \). Within each cluster, positions \( \{ \mathbf{x}_i^{(q)} \} \) are assigned by finding the \( M - 1 \) closest, PPP-distributed points to \( \mathbf{x}_1^{(q)} \), which is taken as the origin. For a product-state initial condition, the \( \mathbf{S}^{(q)} \) are independent, so that \( \Gamma_S = \sum_q \Gamma_{S^{(q)}} \). We compute \( U(T) \) using Eq. (6) and matrix exponentiation for \( Q \sim 10^4 \) clusters, to find \( \Gamma_{S^{(q)}}(T) \), (\( \mathbf{S}^{(q)}(T) \)), and its derivatives.

We first show the case of dc magnetometry, in which \( \mathbf{B}(t) = (0, 0, B_{dc}) \), the initial state is \( |\phi_0\rangle = |+x\rangle \), and \( B = B_{dc} \). Representative results are shown in Fig. 1. The mean amplitude of oscillation shows a steady and seemingly exponential decline, while the elements of the covariance matrix saturate, with the result that the imprecision reaches a minimum at which \( \langle \delta B_{dc}^2 \rangle VT/(2\mu_0 h) \approx 0.7 \) at a finite time \( T_{\text{opt}} = \Upsilon_{\text{dd}} \approx 0.5 \). As shown in Fig. 2, the limiting sensitivity improves with increasing \( M \), but saturates at about \( M = 6 \). Similarly, \( s \) larger than \( 1/2 \) provides an advantage that appears to saturate about \( s = 1 \). Simulations (not shown) with other conditions, including different spin quantum number, gyromagnetic ratio, field strength and density, find very similar limiting sensitivities, strongly suggesting an implementation-independent limit for fixed spin-precession sensors.

To understand this limit, we first note that Eq. (9) is time-periodic. This motivates a Kapitza approach in which we divide the dynamics of \( \{s_i\} \) into a slowly-varying “secular” part and a “micro-motion” part oscillating at \( \omega_L \). The micromotion is smaller than the secular part by a factor \( \sim \Upsilon_{\text{dd}}/\omega_L \), and for sufficient \( \omega_L \) becomes negligible \(^{19}\). The secular motion is governed by the Larmor-cycle-averaged Hamiltonian

\[
\bar{H}(t) = \frac{1}{T_L} \int_0^{t+T_L} dt' \hat{H}(t')
\]

where the subscript \( z \) indicates the component along \( \hat{z} \), i.e., along the dc field. The \( x \) and \( y \) components are lost in the cycle average. We note that \( \omega_L \) no longer appears.

We can now understand the effect of density, using a strategy from renormalization group (RG) theory. We imagine dividing the sensor volume into \( \lambda \) equal sub-volumes, while also increasing the density by a factor \( \lambda \). We indicate post-transformation quantities with primes, e.g. \( \rho' = \rho \lambda \). If \( P_\lambda \) is self-similar, in the sense that the statistical distribution of \( \{\lambda^{1/3}(\mathbf{x}_i' - \mathbf{x}_i')\} \) within a sub-volume the same as that of \( \{\mathbf{x}_i - \mathbf{x}_i\} \) in the full volume, and again assuming edge effects are negligible, the sub-volumes now represent \( \lambda \) independent, reduced-scale realizations of the original sensor. A PPP for example has such self-similarity.

For a given configuration \( \{\mathbf{x}_i\} \rightarrow \{\lambda^{-1/3}\mathbf{x}_i\} \), the Hamiltonian is \( \bar{H}' = \lambda \bar{H} \), implying a speed-up of the rotating-frame dynamics by a factor \( \lambda \). When averaged over \( \{\mathbf{x}_i\} \), this produces faster evolution of spin means \( \langle \bar{s}_i'(t) \rangle = \langle \bar{s}_i(\lambda t) \rangle \) and correlators \( \langle s_i'(t) s_j'(t) \rangle = \langle s_i(\lambda t) s_j(\lambda t) \rangle \). Considering then the collective spin \( \mathbf{S} \), which sums the \( \lambda \) sub-volumes, we find \( \langle S'(t) \rangle = \lambda \langle S(\lambda t) \rangle \) and \( \Gamma_{S'}(t) = \lambda \Gamma_S(\lambda t) \). Inserting into Eq. (2), we find \( T_{\text{opt}}' = \lambda^{-1} T_{\text{opt}} \) and thus the same limiting sensitivity, independent of \( \lambda \) and thus of \( \rho \).

The specific values of the other implementation-dependent factors \( \gamma \) and \( s \) are similarly irrelevant: a change to \( \gamma' s' = \gamma s \), through dynamics and quantum noise scaling, gives \( \langle S'(t) \rangle = \lambda \langle S(\lambda t) \rangle \) and \( \Gamma_{S'}(t) = \lambda \Gamma_S(\lambda t) \), \( T_{\text{opt}}' = \lambda^{-1} T_{\text{opt}} \), and again the same limiting sensitivity. The only remaining, i.e. non-relevant, factors are fundamental constants, making the limit implementation-independent within the secular regime \( \omega_L \gg \Upsilon_{\text{dd}} \).

Again using the cluster simulation, we study a range of \( \omega_L / \Upsilon_{\text{dd}} \), see Fig. 3 and find that outside of the secular regime \( T\langle \delta B_{dc}^2 \rangle \) is about a factor of two larger than in the secular regime, confirming that the secular result is in fact the global optimum.
Using $\hat{H}$ we can also consider RF magnetometry \[19\], which requires only minor modification to the above discussion. Now $B(t) = (B_{RF} \cos \omega t, B_{RF} \sin \omega t, B_{dc})$, the initial state is $|\phi_0\rangle = |+z\rangle$, and the unknown is $B = B_{RF}$. As before, we take $\{x_i\}$ to be PPP-distributed. In the rotating frame, the RF field appears fixed, and contributes a term $-\frac{\gamma}{\hbar} B_{RF} \sum_i \hat{s_i} \cdot \hat{x}$ to both Eqs. \[6\] and \[7\]. For $\gamma |B_{RF}| \ll \gamma_{dd}$, i.e. for weak-field detection, the contribution of this term to the dynamics is small, and the scale-invariance arguments proceed as above. Using Eq. \[7\] we numerically evaluate $\Gamma_S$ and $\delta_{B_{RF}} (\mathbf{S} \cdot \mathbf{n})$ by the cluster expansion as above. Results indicate a lower-bound of $E_{RF}/\hbar \approx 1/4$, see \[32\].

Thus far, the discussion has concerned only spins with fixed positions. This describes at least one important sensor type, color centers in solids. For mobile spins, we consider a set of trajectories $\{x_i(t)\}$, and the RG argument proceeds as above if the trajectories’ distribution is self-similar in the sense that $\{\lambda^{1/3}[x_i'(t) - x_j'(t)]\}$ has the same distribution as $\{x_i(\lambda t) - x_j(\lambda t)\}$. This describes sub-diffusive $(\Delta x)^2 \propto \Delta t$ transport, as opposed to diffusive $(\Delta x)^2 \propto \Delta t$ or ballistic $\Delta x \propto \Delta t$ transport. As such, we cannot directly use the RG argument to establish an implementation-independent limit for vapor- or gas-phase spin-precession sensors. Nonetheless, we recover scale-invariance in two scenarios.

First, precisely because the decoherence rate $\gamma_{dd} \propto \rho$ grows faster with density than does the $t \propto \rho^{2/3}$ (diffusive) or $t \propto \rho^{1/3}$ (ballistic) transport time across the inter-particle spacing, for sufficiently large densities (or sufficiently slow transport) the spins will appear effectively immobile and can be treated as fixed, so that the conclusions given above for fixed spins apply. Second, in the opposite extreme of highly-mobile, weakly-coupled spins, we can expect short-range collisional processes to cause spin depolarization faster than do long-range spin-spin interactions. Such collisions produce a decoherence rate $\Gamma \propto \rho$ in both diffusive and ballistic regimes \[43\]. It has been reported \[44\] that the limiting “spin-destruction” (SD) collision rates $\Gamma_{SD}$ in alkali-alkali collisions are consistent with a molecular dipole-dipole interaction, such that $\Gamma_{SD} \propto \gamma^2$. If this is the case, the sensitivity is similarly implementation-independent. In the case of $^{87}$Rb, measured SD rates are moreover consistent with a limiting sensitivity $E_{RF}/\hbar \approx 1$ \[40\].

Conclusions - We have identified a new kind of quantum sensing limit, one that applies to dimensioned physical quantities such as length or field strength, but which makes reference neither to available quantum resources such as particle number, nor to implementation-dependent scale factors such as the sensing particles’ wavelength or transition frequency. For spin-precession sensors, the limit is a consequence of scale-invariance in the self-decoherence dynamics of spin-ensembles. For sensors employing fixed, randomly-placed spins, the limiting “energy resolution per bandwidth” is near the reduced Planck constant, a result that coincides with the limit for dc-SQUID sensors.

Why such a limit would fall so close to the quantum of action is an intriguing open question. We note that unlike many quantum limits derivable from the geometry of quantum states \[2\], this one appears to involve in a fundamental way the dynamics of open quantum systems: it describes the rate at which entropy enters the initially well-ordered spin degrees of freedom from the center-of-mass (cm) degrees of freedom. We leave for future work the interesting question of whether the limit can be “beaten” by dynamical decoupling of interacting spins \[45\], or by using spin ensembles with reduced cm entropy, e.g. spinor Bose-Einstein condensates \[33\] or microscopically-ordered spin systems.

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do not much concern us here. For this to be an implementation-independent limit, it suffices that $A/L^2$, and thus $\alpha$, is bounded from below.
Supplementary information for: Sensor self-interaction, scale-invariant spin dynamics, and the \( \hbar \) limit of field sensing

**OPTIMAL READOUT**

To efficiently evaluate Eq. (2), it is convenient to note that by parametrizing \( n = (\cos \theta, \sin \theta, 0) \), the variance of the estimate of \( B \) can be written

\[
\langle \delta B^2 \rangle = \min_n \frac{\| \Gamma \|_2}{Z} = \frac{\min_\theta \langle \partial_t (S_x \cos \theta + S_y \sin \theta) \rangle}{\langle \partial_t (\langle S_x \rangle \cos \theta + \langle S_y \rangle \sin \theta) \rangle}
\]

\[
= \frac{\| \Gamma \|_2}{Z}
\]

(8)

where

\[
Z \equiv \langle \partial_t (S_x \rangle \rangle^2 \text{var} S_y + 2 \text{cov}(S_x, S_y) \langle \partial_t (S_x \rangle \rangle \langle \partial_t (S_y \rangle \rangle + \langle \partial_t (S_y \rangle \rangle^2 \text{var} S_x
\]

(9)

and \( |\cdot| \) indicates the determinant of the upper left 2 x 2 sub-matrix.

**KAPITZA-THEORY DYNAMICS**

To understand the conditions under which spin-interaction dynamics will simplify due to Larmor precession, we adapt the classical Kapitza method to the spin problem at hand. We begin by writing the equations of motion for the spins, in the frame rotating at \( \omega_L \), and thus governed by Hamiltonian of Eq. (6). We have

\[
\frac{d}{dt} \mathbf{s}^{(i)} = \sum_{k \neq i} \frac{\gamma^2 \hbar^2 \mu_0}{4 \pi^2 \eta^2} \left[ 3(\mathbf{s}^{(i)} \times \mathbf{R}^{(ik)}) (\mathbf{R}^{(ik)} \cdot \mathbf{R}^{(ik)}) + (\mathbf{s}^{(i)} \times \mathbf{s}^{(k)}) \right]
\]

(10)

where \( \mathbf{R} \equiv R_z(\omega_L t) \mathbf{r} \) is the rotated vector joining the spins. The precession period is \( T \equiv 2 \pi / \omega_L \).

Collecting all spin components into a single vector \( \mathbf{z} = \bigoplus_i \mathbf{s}^{(i)} \), we note that Eq. (10) has the form

\[
\frac{d}{dt} z_i = \mathcal{R}_{ijk}(t) z_j z_k
\]

(11)

where the tensor of coefficients \( \mathcal{R} \) is periodic: \( \mathcal{R}(t+T) = \mathcal{R}(t) \). It is convenient to identify a time \( t_a \) as the start of a cycle, and divide \( \mathcal{R}(t) \) as \( \mathcal{R}(t) = \mathcal{R} + \hat{\mathcal{R}}(t) \) where the cycle-averaged part is \( \mathcal{R} = T^{-1} \int_{t_a}^{t_a+T} \mathcal{R}(t) dt \).

We write \( \mathbf{z} = p + q \), where \( p \) is the slowly-varying “secular motion” and \( q \) is the small and rapidly-varying “micro-motion,” defined as the solution of

\[
\frac{d}{dt} q_i = \hat{\mathcal{R}}_{ijk}(t)
\]

(12)

with zero cycle-average: \( \int_{t_a}^{t_a+T} q(t) dt = 0 \). We can write the formal solution

\[
q_i(t) = \int_{t_a}^{t} \hat{\mathcal{R}}_{ijk}(t') dt' p_j(t_a) p_k(t_a).
\]

(13)

We note that \( q \sim T \), and thus \( q \) becomes small for large \( \omega_L \).

Using the smallness of \( q \) we expand the r.h.s. of Eq. (11), as applies to the time period \( t \in [t_a, t_a+T) \) to find

\[
\frac{d}{dt} z_i = \mathcal{R}_{ijk}(t) z_j (t_a) z_k(t_a) + q_i(t) [\partial_t \mathcal{R}_{ijk}(t) z_j z_k]_{z=p(t_a)} + O(q^2).
\]

(14)

We drop the doubly-small \( O(q^2) \) term and integrate over one cycle to find the cycle-averaged rate of change

\[
\frac{dp_i}{dt} = \frac{\Delta p_i}{T} = \mathcal{R}_{ijk}(t) p_j(t_a) p_k(t_a)
\]

\[
+ \frac{1}{T} \int_{t_a}^{t_a+T} dt' \int_{t_a}^{t} dt'' \mathcal{R}_{ijk}(t') p_j(t_a) p_k(t_a)
\]

\[
\times [\partial_t \mathcal{R}_{ijk}(t) z_j z_k]_{z=p(t_a)},
\]

(15)

which now refers only to \( p \). The second term describes the effect of micromotion on the secular dynamics.

We note that the first term in Eq. (15) scales as \( \mathcal{Y}_{dd} \), and contains both the \( \mathbf{s}^{(i)} \times \mathbf{s}^{(k)} \) factor that produces spin-exchange and \( (\mathbf{s}^{(i)} \times \mathbf{zR}_{jk}^{(ik)}) (\mathbf{R}^{(ik)} \cdot \mathbf{R}^{(ik)}) \) obtained by cycle-averaging \( (\mathbf{s}^{(i)} \times \mathbf{R}^{(ik)}) (\mathbf{R}^{(ik)} \cdot \mathbf{R}^{(ik)}) \). This latter factor is responsible for loss of angular momentum to the centre of mass degrees of freedom. The second term in Eq. (15) scales as \( \mathcal{Y}_{dd}/\omega_L \), i.e., smaller than the first by a factor \( \mathcal{Y}_{dd}/\omega_L \). It is this smallness that justifies using the cycle-averaged hamiltonian of Eq. (7) for large \( \omega_L \).

**RF MAGNETOMETER RESULTS**

In Fig. 4 we show results of numerical simulations with the scenario and methods described in the text. The results appear to converge to \( E_R/\hbar \approx 1/4 \).
FIG. 4. RF sensitivity, as in Fig. 2. Convergence of numerical results with increasing number of spins $M$ and spin quantum number $s$. Vertical axis shows time-optimal energy resolution, found as the minimum of $\langle \delta B_{DC}^2 \rangle VT/(2\mu_0 \hbar)$ versus $T$ as shown in Figure 1. Red, yellow, orange, and green show $s = 1/2, 1, 3/2$ and 2, respectively. Computed with $Q = 1 \times 10^4$ except for $s = 1/2$, $M = 7$, computed with $Q = 3 \times 10^4$. 