Some generalizations and unifications of $C_K(X)$, $C_\psi(X)$ and $C_\infty(X)$

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Dedicated to Professor Azarpanah

Abstract. Let $\mathcal{P}$ be an open filter base for a filter $\mathcal{F}$ on $X$. We denote by $C^P(X)$ ($C_\infty^P(X)$) the set of all functions $f \in C(X)$ where $Z(f) = \{x : |f(x)| < 1\}$ contains an element of $\mathcal{P}$. First, we observe that every proper subrings in the sense of Acharyya and Ghosh (Topology Proc. 2010) has such form and vice versa. Afterwards, we generalize some well known theorems about $C_K(X), C_\psi(X)$ and $C_\infty(X)$ for $C^P(X)$ and $C_\infty^P(X)$. We observe that $C_\infty^P(X)$ may not be an ideal of $C(X)$. It is shown that $C_\infty^P(X)$ is an ideal of $C(X)$ and for each $F \in \mathcal{F}$, $X \setminus F$ is bounded if and only if the set of non-cluster points of the filter $\mathcal{F}$ is bounded. By this result, we investigate topological spaces for which $C_\infty^P(X)$ is an ideal of $C(X)$ whenever $\mathcal{P} = \{A \subseteq X : A$ is open and $X \setminus A$ is bounded $\}$ (resp., $\mathcal{P} = \{A \subseteq X : X \setminus A$ is finite $\}$). Moreover, we prove that $C^P(X)$ is an essential (resp., free) ideal if and only if the set $\{V : V$ is open and $X \setminus V \in \mathcal{F}\}$ is a $\pi$-base for $X$ (resp., $\mathcal{F}$ has no cluster point). Finally, the filter $\mathcal{F}$ for which $C_\infty^P(X)$ is a regular ring (resp., $z$-ideal) is characterized.

Keywords: local space, bounded subset, $z$-ideal, regular ring, essential ideal, $\mathcal{F}$-$CG_\delta$ subset.

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1 Introduction

In this paper, $X$ assumed to be a completely regular Hausdorff space. $C(X)$($C^*(X)$) stands for the ring of all real valued (bounded) continuous functions on $X$. A subcollection $\mathcal{B}$ of a filter $\mathcal{F}$ is a filter base for $\mathcal{F}$ if and only if each element of
Our aim of this paper is to reveal some important properties of a special kind of generalized form of $C_K(X)$ and $C_{∞}(X)$, which is denoted by $C^P(X)$ and $C_{∞P}(X)$. In section 2, some examples of these subrings are given and we prove that for any open filter base $P$, there is an ideal $P'$ of closed sets such that $C^P(X) = C_{P'}(X)$ and $C_{∞P}(X) = C_{∞P'}(X)$ and vice versa whenever $C_{∞P}(X)$ is a proper subring of $C(X)$ (see [1]). It is shown that $C^P(X)$ is a free ideal if and only if $F$ has no cluster point. Consequently, we observe that $X$ is a local space (i.e., there is an open filter base $P$ for a filter $F$ which $F$ has no cluster point) if and only if there is an open filter base $P$ such that $C^P(X)$ is a free ideal. In this section, we show that $C^P(X)$ (resp., $C_{∞P}(X)$) is a zero ideal if and only if any element of $P$ ($F$) is dense in $X$. A subset $A$ of $X$ is called an $F$-$CGδ$, if $A = \bigcap_{i=1}^{∞} A_i$, where each $A_i$ is an open subset, $X \setminus A_i$ and $A_{i+1}$ are completely separated and each $A_i \in F$. We prove that that $C^P(X) = C_{∞P}(X)$ if and only if every closed $F$-$CGδ$ is an element of $F$. We give an example of an open filter base $P$ for a filter $F$ such that $C_{∞P}(X)$ is not an ideal of $C(X)$. It is also shown that $C_{∞P}(X)$ is an ideal of $C(X)$ and for any $F \in F$, $X \setminus F$ is bounded if and only if the set of non-cluster points of the filter $F$ is bounded which is a generalization of Theorem 1.3 in [6]. Consequently, if $X$ is a pseudocompact space, then for any open filter base $P$, $C_{∞P}(X)$ is an ideal of $C(X)$.

Section 3 is devoted to the essentiality of $C^P(X)$ and ideals in $C_{∞P}(X)$. In [5], it was proved that an ideal $I$ of $C(X)$ is an essential ideal if and only if $\bigcap Z[I]$ does not contain an open subset (i.e., $\text{int} \bigcap Z[I] = \emptyset$). Azarpah in [4], proved that $C_{K}(X)$ is an essential ideal if and only if $X$ is an almost locally compact (i.e., every non-empty open set of $X$ contains a non-empty open set with compact closure). We generalize these results for $C^P(X)$ and
2 $C_\mathcal{P}^\mathcal{P}(X)$ and $C_{\infty,\mathcal{P}}(X)$

Let $\mathcal{P}$ be an open filter base for a filter $\mathcal{F}$ on topological space $X$, we denote by $C_\mathcal{P}^\mathcal{P}(X)$ the set of all functions $f$ in $C(X)$ for which $Z(f)$ contains an element of $\mathcal{P}$. Also $C_{\infty,\mathcal{P}}(X)$ denotes the family of all functions $f \in C(X)$ for which the set $\{x : |f(x)| < \frac{1}{n}\}$ contains an element of $\mathcal{P}$, for each $n \in \mathbb{N}$.

Recall that, for a subset $A$ of $X$, $O_A = \{f : A \subseteq \text{int}Z(f)\}$.

**Lemma 2.1.** The following statements hold.

1. $C_\mathcal{P}^\mathcal{P}(X)$ is a $z$-ideal of $C(X)$ contained in $C_{\infty,\mathcal{P}}(X)$.
2. $C_\mathcal{P}^\mathcal{P}(X) = \sum_{A \in \mathcal{P}} O_A = \bigcup_{A \in \mathcal{P}} O_A$
3. $C_{\infty,\mathcal{P}}(X)$ is a proper subring of $C(X)$.

**Proof.** (1) By definition of $C_\mathcal{P}^\mathcal{P}(X)$ and since $\mathcal{P}$ is a base filter, it is easy to see that $C_\mathcal{P}^\mathcal{P}(X)$ is a $z$-ideal and contained in $C_{\infty,\mathcal{P}}(X)$.

(2) Let $f \in C_\mathcal{P}^\mathcal{P}(X)$. Then there exists $A \in \mathcal{P}$ such that $A \subseteq Z(f)$. Hence $A \subseteq \text{int}Z(f)$, i.e., $f \in O_A \subseteq \sum_{A \in \mathcal{P}} O_A$. If $f \in \sum_{A \in \mathcal{P}} O_A$, then there are $1 \leq i \leq n$ and $f_i \in O_{A_i}$ such that $f = f_1 + \ldots + f_n$, thus $\bigcap_{i=1}^n A_i \subseteq \bigcap_{i=1}^n \text{int}Z(f_i) \subseteq Z(f)$. But $\bigcap_{i=1}^n A_i$ contains an element of $\mathcal{P}$, so $f \in C_\mathcal{P}^\mathcal{P}(X)$. The proof of the second equality is obvious.

(3) First, we observe that $C_{\infty,\mathcal{P}}(X)$ is a proper subset of $C(X)$. For, if $C_{\infty,\mathcal{P}}(X) = C(X)$, then $\emptyset \in \mathcal{P}$, which is a contradiction. On the other hand, we have

\[
\{x : |f(x) - g(x)| < \frac{1}{n}\} \supseteq \{x : |f(x)| < \frac{1}{2n}\} \cap \{x : |g(x)| < \frac{1}{2n}\} \text{ and } \{x : |f(x)g(x)| < \frac{1}{n}\} \supseteq \{x : |f(x)| < \frac{1}{\sqrt{n}}\} \cap \{x : |g(x)| < \frac{1}{\sqrt{n}}\}.
\]
Some generalizations and unifications of $C_K(X)$, $C_\psi(X)$ and $C_\infty(X)$

Therefore, $C_{\infty,\mathcal{P}}(X)$ is a proper subring of $C(X)$.

Recall that a family $\mathcal{P}$ of closed subsets of $X$ is called an ideal of closed sets in $X$, if it satisfies in the following conditions.

1. If $A, B \in \mathcal{P}$, then $A \cup B \in \mathcal{P}$.
2. If $A \in \mathcal{P}$ and $B \subseteq A$ with $B$ closed in $X$, then $B \in \mathcal{P}$.

In [1], Acharyya and Ghosh for ideal $\mathcal{P}$ of closed subsets of $X$ defined $C_\mathcal{P}(X)$ and $C_{\infty,\mathcal{P}}(X)$ as follows:

$$C_\mathcal{P}(X) = \{ f \in C(X) : d(X \setminus Z(f)) \in \mathcal{P} \} ;$$

$$C_{\infty,\mathcal{P}}(X) = \{ f \in C(X) : \{ x : |f(x)| \geq \frac{1}{n} \} \in \mathcal{P}, \text{ for each, } n \in \mathbb{N} \}.$$

In the next result we give a new presentation of these subrings. We note that for ideal $\mathcal{P}$ of closed sets, $C_{\infty,\mathcal{P}}(X)$ may be $C(X)$ but by Proposition 2.1 $C_{\infty,\mathcal{P}}(X)$ for each open filter $\mathcal{P}$ is a proper subring.

**Proposition 2.2.** The following statements hold.

1. For every open filter base $\mathcal{P}$, there exists an ideal $\mathcal{P}'$ of closed sets such that $C_\mathcal{P}(X) = C_{\mathcal{P}'}(X)$ and $C_{\infty,\mathcal{P}}(X) = C_{\infty,\mathcal{P}'}(X)$.

2. If $C_{\infty,\mathcal{P}}(X)$ is a proper subring of $C(X)$, then there is an open filter base $\mathcal{Q}$ such that $C_\mathcal{P}(X) = C_{\mathcal{Q}}(X)$ and $C_{\infty,\mathcal{P}}(X) = C_{\infty,\mathcal{Q}}(X)$.

**Proof.** (1) Let $\mathcal{P}$ be an open filter base. Consider $\mathcal{P}'$ as follows:

$$\mathcal{P}' = \{ A : A \text{ is closed and } A \subseteq X \setminus B \text{ for some } B \in \mathcal{P} \}.$$

Then, it is easy to see that $\mathcal{P}'$ is an ideal of closed sets in $X$, $C_\mathcal{P}(X) = C_{\mathcal{P}'}(X)$ and $C_{\infty,\mathcal{P}}(X) = C_{\infty,\mathcal{P}'}(X)$.

(2) Assume that $C_{\infty,\mathcal{P}}(X)$ is a proper subring of $C(X)$ and $\mathcal{Q} = \{ A \subseteq X : X \setminus A \in \mathcal{P} \}$. Then we can see that $\mathcal{Q}$ is an open filter, $C_\mathcal{P}(X) = C_{\mathcal{Q}}(X)$ and $C_{\infty,\mathcal{P}}(X) = C_{\infty,\mathcal{Q}}(X)$. □

**Example 2.3.** Let $X$ be a non-compact Hausdorff space and $\mathcal{P} = \{ A \subseteq X : X \setminus A \text{ is compact} \}$. Then $\mathcal{P}$ is an open filter base, $C_\mathcal{P}(X) = C_K(X)$ and $C_{\infty,\mathcal{P}}(X) = C_\infty(X)$.

**Example 2.4.** Let $\mathcal{P} = \{ A : A \text{ is open and } X \setminus A \text{ is Lindelöf} \}$ and $X$ be a non-Lindelöf space. Then $\mathcal{P}$ is an open filter and we have

$$C_{\infty,\mathcal{P}}(X) = \{ f : X \setminus Z(f) \text{ is a Lindelöf subset of } X \} \text{ and}$$

$$C_\mathcal{P}(X) = \{ f : X \setminus Z(f) \text{ is a Lindelöf subset of } X \}.$$
To see this, let \( f \in C_\infty \mathcal{P}(X) \). Then for each \( n \in \mathbb{N} \), \( \{ x : |f(x)| \geq \frac{1}{n} \} \subseteq X \setminus A \) for some \( A \in \mathcal{P} \), so it is a Lindelöf subset of \( X \). On the other hand we have \( X \setminus Z(f) = \bigcup_{n=1}^{\infty} \{ x : |f(x)| \geq \frac{1}{n} \} \), hence \( X \setminus Z(f) \) is a Lindelöf subset of \( X \), by Theorem 3.8.5 in [10]. If \( X \setminus Z(f) \) be a Lindelöf subset of \( X \), then \( \{ x : |f(x)| \geq \frac{1}{n} \} \) is a Lindelöf subset of \( X \) so \( \{ x : |f(x)| < \frac{1}{n} \} \) contains an element of \( \mathcal{P} \), i.e., \( f \in C_\infty \mathcal{P}(X) \). Similarly we may prove that \( C_\infty \mathcal{P}(X) = \{ f : X \setminus Z(f) \) is a Lindelöf subset of \( X \}. \)

In the sequel, we assume that \( \mathcal{P} \) is an open filter base for a filter \( \mathcal{F} \).

**Proposition 2.5.** If the complement of any element of \( \mathcal{P} \) is Lindelöf, then \( C_\infty \mathcal{P}(X) \subseteq \bigcap_{p \in \nu X \setminus X} M^p \). Where by \( \nu X \) we mean the real compactification of \( X \) (see [11]).

**Proof.** Let \( f \in C_\infty \mathcal{P}(X) \) and \( p \in \nu X \setminus X \) (i.e., \( M^p \) is a free real maximal ideal). Then for any \( x \in X \setminus Z(f) \) there exist \( f_x \in M^p \) such that \( x \in X \setminus Z(f_x) \). Hence \( X \setminus Z(f) \subseteq \bigcup_{x \in X} X \setminus Z(f_x) \). Now, by Example 2.4 and hypothesis, \( X \setminus Z(f) \) is Lindelöf, so there is a countable subset \( S \) of \( X \) such that \( X \setminus Z(f) \subseteq \bigcup_{x \in S} X \setminus Z(f_x) \) and each \( f_x \in M^p \). On the other hand, there exists \( h \in C(X) \) such that \( Z(h) = \bigcap_{x \in S} Z(f_x) \). Therefore \( h \in M^p \) and \( Z(h) \subseteq Z(f) \). But \( M^p \) is a z-ideal, so \( f \in M^p \), i.e., \( C_\infty \mathcal{P}(X) \subseteq \bigcap_{p \in \nu X \setminus X} M^p \). □

Recall that, a subset \( A \) of \( X \) is called bounded (relative pseudocompact) subset, if for every function \( f \in C(X) \), \( f(A) \) is a bounded subset of \( \mathbb{R} \), see [14].

**Example 2.6.** \( \mathcal{P} = \{ A : A \) is open and \( X \setminus A \) is pseudocompact \} \) and \( X \) be a non-pseudocompact space. Then \( \mathcal{P} \) is an open filter base and

\[ C_\mathcal{P}(X) = C_\psi(X) = \{ f : X \setminus Z(f) \) is pseudocompact \}, \]

\[ C_\infty \mathcal{P}(X) = \{ f : \{ x : |f(x)| \geq \frac{1}{n} \} \) is pseudocompact \} \].

For, suppose that \( f \in C_\mathcal{P}(X) \), then \( Z(f) \supseteq A \) for some \( A \in \mathcal{P} \). Hence \( X \setminus Z(f) \subseteq X \setminus A \). This implies that \( X \setminus Z(f) \) is a bounded subset. Now by [14] Theorem 2.1, \( X \setminus Z(f) \) is pseudocompact, i.e., \( f \in C_\psi(X) \). If \( f \in C_\psi(X) \), then \( X \setminus Z(f) \) is an element of \( \mathcal{P} \) and \( Z(f) \supseteq X \setminus Z(f) \), i.e., \( f \in C_\mathcal{P}(X) \). For more details about \( C_\psi(X) \), the reader referred to [14]. Similarly we may prove the second equality.

**Remark 2.7.** Let \( \mathcal{P} = \{ A : X \setminus A \) is finite \} \) and \( X \) is an infinite space. Then \( C_\mathcal{P}(X) = C_F(X) \), and \( C_\infty \mathcal{P}(X) = \{ f : \{ x : |f(x)| \geq \frac{1}{n} \} \) is finite for each \( n \in \mathbb{N} \}. \) In this case, \( C_\infty \mathcal{P}(X) = C_F(X) \) if and only if the set of isolated points of \( X \) is finite. To see this, let \( \{ x_1, x_2, ..., x_n, ... \} \) be a subset of isolated points in \( X \). Define \( f_n(x) = \begin{cases} \frac{1}{n} & x = x_n \\ 0 & x \neq x_n \end{cases} \) and \( f(x) = \sum_{n=0}^{\infty} f_n(x) \). Then
there exists $F \in C_{\infty}(X)$ and $f \notin C_{\infty}(X)$. On the other hand, for points $x \in X \setminus Z(f)$, then there exists a compact subset $K \subseteq X$ such that $x \in \text{int}(K)$, but $x \notin K \in \mathcal{P}$, hence $x \notin X \setminus \overline{K}$, i.e., $\text{int}(K) \cap (X \setminus K) \neq \emptyset$, this is a contradiction, thus $\bigcap_{A \in \mathcal{P}} \overline{A} = \emptyset$.

**Example 2.10.** [11, 4. M] Let $X$ be an uncountable space in which all points are isolated points except for a distinguished point $s$. A neighborhood of $s$ is any set containing $s$ which complement is countable. Then $X$ is a local space. To see this, let $Y = \{x_1, x_2, \ldots\}$ be a countable subset of $X$, where $s \notin Y$. Put $A_n = \{x_n, x_{n+1}, \ldots\}$ and $\mathcal{P} = \{A_n : n \in \mathbb{N}\}$. Then $\mathcal{P}$ is an open filter base on $X$. Now, for any $n \in \mathbb{N}$, the set $X \setminus A_n$ is a neighborhood of $s$, so $s \notin \bigcap_{A \in \mathcal{P}} \overline{A_n}$, thus $\bigcap_{A \in \mathcal{P}} \overline{A} = \emptyset$.

In the following we see an example of a topological space which is not a local space.

**Example 2.11.** [11, 4. N] For each $n \in \mathbb{N}$, let $A_n = \{n, n+1, \ldots\}$ and $E = \{A_n : n \in \mathbb{N}\}$ which points in $\mathbb{N}$ are isolated point and a neighborhood of $\sigma$ is of the form $U \cup \{\sigma\}$ which $U \in \mathcal{E}$. Note that any set contains $\sigma$ is closed. Now if there is an open base $\mathcal{P}$ for some filter $\mathcal{F}$ on $X$ such that $\mathcal{F}$ has no cluster point, then there exists $F \in \mathcal{F}$ such that $\sigma \notin \overline{F}$, but $\sigma$ has a neighborhood say $U \cup \{\sigma\}$ such that $U \in \mathcal{E}$ and $U \cup \{\sigma\} \subseteq X \setminus \overline{F}$. Since $\mathcal{E}$ is a base for $\mathcal{E}$, then there exists $n \in \mathbb{N}$ and $A_n \in E$ such that $A_n \cup \{\sigma\} = \{n, n+1, \ldots\} \cup \{\sigma\} \subseteq U \cup \{\sigma\} \subseteq X \setminus \overline{F}$. On the other hand, for points $x = 1, 2, \ldots, n-1$ there exists $F_1, F_2, \ldots, F_{n-1}$ in $\mathcal{F}$, such that $i \in X \setminus F_i$ for $1 \leq i \leq n$, so $X \subseteq (X \setminus \overline{F}) \cup \bigcup_{i=1}^{n-1} (X \setminus F_i)$, therefore $\emptyset \in \mathcal{F}$. This a contradiction, i.e., $X$ is not a local space.

We have already observe that $C^{\mathcal{F}}(X)$ is a $z$-ideal in $C(X)$. In the following proposition we find a condition over which $C^{\mathcal{F}}(X)$ is a free ideal.

**Proposition 2.12.** Let $\mathcal{P}$ be an open filter base for a filter $\mathcal{F}$. Then $C^{\mathcal{F}}(X)$ is a free ideal if and only if $\mathcal{F}$ has no cluster point (i.e., $\bigcap_{A \in \mathcal{P}} \overline{A} = \emptyset$).
Some generalizations and unifications of $C_K(X), C_\varphi(X)$ and $C_\infty(X)$

**Proof.** Let $\bigcap_{F \in \mathcal{F}} \overline{F} \neq \emptyset$. Then there exists $x \in \bigcap_{F \in \mathcal{F}} \overline{F}$. By hypothesis, there is $f \in C^p(X)$ such that $x \notin X \setminus Z(f)$. On the other hand there is $A \in \mathcal{F}$ such that $(X \setminus Z(f)) \cap A \neq \emptyset$. But $x \notin A$ implies that $(X \setminus Z(f)) \cap A = \emptyset$. Thus this is a contradiction. Conversely, let $x \in X$. We have $\bigcap_{F \in \mathcal{F}} F = \emptyset$, so there exists $F \in \mathcal{F}$ such that $x \notin F$. By completely regularity of $X$, there exists $f \in C(X)$ such that $f(x) = 1$ and $f(F) = 0$. Hence $f \in C^p(X)$ and $x \notin Z(f)$, i.e., $C^p(X)$ is a free ideal.

It is easy to see that $X$ is a locally compact non-compact space if and only if $\mathcal{P} = \{A \subset X : X \setminus A$ is compact$\}$ is an open filter base for some filter $\mathcal{F}$ with no cluster point. So, by the above proposition we have the following corollaries.

**Corollary 2.13.** $C_K(X)$ is a free ideal if and only if $X$ is a locally compact non-compact space.

**Corollary 2.14.** A space $X$ is a local space if and only if $C_P(X)$ is a free ideal for some open filter base $\mathcal{P}$ on $X$.

**Proof.** By Proposition 2.12 the verification is immediate.

**Proposition 2.15.** Let $\mathcal{P}$ be an open filter base. The following statements are equivalent.

1. Every element of $\mathcal{P}$ is dense in $X$.
2. $C_\infty(\mathcal{P})(X) = (0)$.
3. $C^p(X) = (0)$.

**Proof.** (1)$\Rightarrow$(2) Let for every $A \in \mathcal{P}$, $\overline{A} = X$ and $f \in C_\infty(\mathcal{P})(X)$. Then the set $\{x : |f| \leq \frac{1}{n+1}\}$ is $X$, so for any $n > 1$ we have $\{x : |f| < \frac{1}{n-1}\} \supseteq \{x : |f| \leq \frac{1}{n}\} = X$, i.e., $f = 0$.

(2)$\Rightarrow$(3) This is evident.

(3)$\Rightarrow$(1) Suppose that $C^p(X) = 0$ and $A \in \mathcal{P}$. If $\overline{A} \neq X$, then there exists $x \in X \setminus \overline{A}$, hence we define $f \in C(X)$ such that $f(x) = 1$ and $f(\overline{A}) = 0$, i.e., $f \in C^p(X) = 0$, which implies that $f = 0$, this is a contradiction.

**Corollary 2.16.** Let $X = \mathbb{Q}$ with usual topology and $\mathcal{P} = \{A \subset \mathbb{Q} : \mathbb{Q} \setminus A$ is compact$\}$. Then $C_\infty(\mathcal{P})(X) = C_\infty(X) = (0)$.

**Proof.** Every element of $\mathcal{P}$ is dense in $X$, so by Proposition 2.13 $C_\infty(\mathcal{P})(X) = C_\infty(X) = (0)$.

**Definition 2.17.** Let $\mathcal{F}$ be a filter on $X$. A subset $A$ of $X$ is an $\mathcal{F}$-$\text{CG}_\delta$ if $A = \bigcap_{i=1}^{\infty} A_i$, where each $A_i \in \mathcal{F}$ and is open and for each $i$, $X \setminus A_i$ and $\overline{A_{i+1}}$ are completely separated (see [11]).
Example 2.18. Let \( F = \{ F \subseteq X : X \setminus F \text{ is compact} \} \) and \( X \) is a non-compact Hausdorff space. Then for every open locally compact \( \sigma \)-compact subspace \( A \), \( X \setminus A \) is an \( FCG_\delta \). Since, by [10, p. 250], \( A = \bigcup_{i=1}^{\infty} A_i \) where \( A_i \subseteq int A_{i+1} \) and each \( A_i \) is compact so \( X \setminus A \) is an \( FCG_\delta \) set.

In the following lemma, we give a characterization of a closed \( FCG_\delta \) subset.

Lemma 2.19. Let \( A \) be a closed subset of space \( X \). Then \( A \) is an \( FCG_\delta \) set if and only if \( A = Z(f) \) for some \( f \in C_\infty P(X) \).

Proof. Let \( A \) be an \( FCG_\delta \). Then \( A = \bigcap_{n=1}^{\infty} A_n \), where each \( A_n \) is an element of \( F \), \( X \setminus A_n \) and \( A_{n+1} \) are completely separated. Now, for each \( n \in \mathbb{N} \), there exists \( f_n \in C(X) \) such that \( f_n(\overline{A_{n+1}}) = 0 \), \( f_n(X \setminus A_n) = 1 \), then \( f = \sum \frac{1}{n} f_n \) is an element of \( C(X) \). By Weierstrass M-test, \( A = Z(f) \). We claim that \( f \in C_\infty P(X) \). Let \( x_0 \in A_{n+1} \). Then \( f_1(x_0) = f_2(x_0) = ... f_n(x_0) = 0 \) and so \( f(x_0) \leq \frac{1}{2n+1} + \frac{1}{2n} + ... \leq \frac{1}{2^n} < \frac{1}{n} \). Therefore \( x_0 \in \{ x : |f(x)| < \frac{1}{n} \} \), and hence \( A_{n+1} \subseteq \{ x : |f(x)| < \frac{1}{n} \} \), i.e., \( f \in C_\infty P(X) \). Conversely, suppose that \( A = Z(f) \) for some \( f \in C_\infty P(X) \). Then \( A = \bigcap_{n=1}^{\infty} A_n \), where \( A_n = \{ x : |f(x)| < \frac{1}{n} \} \in F \) for each \( n \in \mathbb{N} \), \( X \setminus A_n \) and \( A_{n+1} \) are disjoint zero-sets, and hence completely separated, i.e., \( A \) is an \( FCG_\delta \).

Proposition 2.20. \( C_\infty P(X) = C^P(X) \) if and only if every closed \( FCG_\delta \) is an element of \( F \).

Proof. Suppose that condition holds. We know that \( C^P(X) \) is a subset of \( C_\infty P(X) \). It is enough to prove that \( C_\infty P(X) \subseteq C^P(X) \). Let \( f \in C_\infty P(X) \), then by Lemma 2.12 \( Z(f) \) is a closed \( FCG_\delta \). Hence \( Z(f) \) contains an element of \( P \), i.e., \( f \in C^P(X) \). Conversely, suppose that \( C_\infty P(X) = C^P(X) \) and \( A \) is a closed \( FCG_\delta \). By lemma 2.12 \( A = Z(f) \) for some \( f \in C_\infty P(X) \), now \( f \in C^P(X) \) implies that \( A = Z(f) \) contains an element of \( P \), i.e., \( A \in F \).

In the above proposition we have seen that if every closed \( FCG_\delta \) is an element of \( F \), then \( C_\infty P(X) \) is an ideal of \( C(X) \). But in general, \( C_\infty P(X) \) may not be an ideal of \( C(X) \) as we will see in the sequel.

Example 2.21. Let \( P = \{ \mathbb{R} \setminus \{ \frac{k}{n} : n \in \mathbb{N} \} \} \). Then it is easy to see that \( P \) is an open filter base on \( \mathbb{R} \). Now, we show that \( C_\infty P(\mathbb{R}) \subseteq C_\infty (\mathbb{R}) \) is not an ideal of \( C(\mathbb{R}) \). For see this, Consider

\[
f(x) = \begin{cases} 
0 & x \leq 0 \\
1 & 1 \leq x 
\end{cases} \quad \text{and} \quad g(x) = \begin{cases} 
0 & x \leq 0 \\
x^2 & 1 \leq x 
\end{cases}.
\]

Then \( f \in C_\infty P(\mathbb{R}) \), \( g \in C(\mathbb{R}) \) and we have
is not bounded, then there exist

\( A \) such that for each \( n \in \mathbb{N} \), \( h(x_n) \geq n \). \( A \) is open so any \( \{x_n\} \) is an isolated point, therefore we can define \( f_n(x) = \begin{cases} \frac{1}{n} & x = n \\ 0 & x \neq n \end{cases} \)

and \( f(x) = \sum_{n=0}^{\infty} f_n(x) \) such that \( f_n \in C(X) \) and so \( f \in C(X) \). We have \( \{x : |f| < \frac{1}{n}\} = \{x_{n+1}, x_{n+2}, \ldots\} \). Now any \( x_n \in X \setminus F_n \) for some \( F_n \in \mathcal{F} \). This implies that \( X \setminus \{x_1, \ldots, x_n\} \) contains an element of \( \mathcal{F} \) hence contains an element of \( \mathcal{P} \), i.e., \( f \in C_{\infty}(X) \). But we have \( \{x : |f| < \frac{1}{n}\} = X \setminus \{x_1, x_2, \ldots\} \), if there is \( P \in \mathcal{P} \) such that \( X \setminus \{x_1, x_2, \ldots\} \supseteq P \), then \( \{x_1, x_2, \ldots\} \subseteq X \setminus F_n \), which contradicts by hypothesis, so \( f \notin C_{\infty}(X) \), i.e., \( C_{\infty}(X) \) is not an ideal of \( C(X) \), this is a contradiction.

(2)\(\Rightarrow\)(3). It is easily seen that the complement of every closed \( \mathcal{F}-CG_\delta \) is a subset of non-cluster points of the filter \( \mathcal{F} \) so is bounded.
Corollary 2.24. If $X$ is a pseudocompact space, then for any open filter base $\mathcal{P}$, $C_{\infty}\mathcal{P}(X)$ is an ideal of $C(X)$.

Proof. If $X$ is a completely regular pseudocompact Hausdorff space and $\mathcal{P}$ be an open filter base for filter $\mathcal{F}$, then any subset of $X$ is bounded so the set of non-cluster point of $\mathcal{F}$ is bounded, thus by Theorem 2.23 $C_{\infty}\mathcal{P}(X)$ is an ideal of $C(X)$. □

Corollary 2.25. Let $X$ be a local space. Then for any open filter base $\mathcal{P}$, $C_{\infty}\mathcal{P}(X)$ is an ideal of $C(X)$ and for any $A \in \mathcal{P}$, $X \setminus \overline{A}$ is bounded if and only if $X$ is a pseudocompact non-compact space.

Proof. If $X$ is pseudocompact, then by Corollary 2.24, for any open filter base $\mathcal{P}$, $C_{\infty}\mathcal{P}(X)$ is an ideal of $C(X)$ and for any $A \in \mathcal{P}$, $X \setminus \overline{A}$ is bounded. Now let $X$ be a local space, then there exist an open filter base $\mathcal{P}$ for some filter $\mathcal{F}$ on $X$ such that $\mathcal{F}$ has no cluster point so $X$ is the set of non-cluster point of filter $\mathcal{F}$. Hence by Theorem 2.23 $X$ is bounded, i.e., $X$ is pseudocompact. □

Corollary 2.26. Let $X$ be a non-pseudocompact space and $\mathcal{P} = \{A: A$ is open and $X \setminus A$ is bounded $\}$. Then $C_{\infty}\mathcal{P}(X)$ is an ideal of $C(X)$ if and only if any union of the interior of closed bounded subsets is a bounded subset.

Proof. If any union of the interior of closed bounded subsets is a bounded subset, then the set of non-cluster points of open filter $\mathcal{P}$ is bounded so by Theorem 2.24 $C_{\infty}\mathcal{P}(X)$ is an ideal of $C(X)$. Conversely, If $A = \bigcup_{\alpha \in S} \text{int} A_\alpha$ where for each $\alpha \in S$, $A_\alpha$ is a closed bounded set, then we have $X \setminus A_\alpha \in \mathcal{P}$ and $\text{int} A_\alpha = X \setminus (X \setminus A_\alpha)$, so $A$ is contained in the set of non-cluster points of open filter $\mathcal{P}$, i.e., $A$ is bounded. □
Corollary 2.27. Let $X$ be an an infinite space and $\mathcal{P} = \{A \subseteq X : X \setminus A$ is finite $\}$. Then $C_{\infty}\mathcal{P}(X)$ is an ideal of $C(X)$ if and only if the set of isolated points of $X$ is bounded.

Proof. Let $A \in \mathcal{P}$. We have $X \setminus \overline{A}$ is an open finite subset, thus the set of non-cluster points of $\mathcal{P}$ is contained in the set of isolated points of $X$, so is bounded, hence by Theorem 2.23 $C_{\infty}\mathcal{P}(X)$ is an ideal of $C(X)$. Conversely, let $A$ be the set of isolated points of $X$. Then $A = \bigcup_{x \in A} \{x\}$, each $\{x\}$ is a clopen subset, $X \setminus \{x\} \in \mathcal{P}$ and $\{x\} = \text{int}\{x\} = X \setminus (X \setminus \{x\})$, so $A$ is contained in the set of non-cluster points of open filter $\mathcal{P}$, i.e., $A$ is bounded. \qed

Example 2.28. (a). If $\mathcal{P} = \{A \subseteq \mathbb{R} : \mathbb{R} \setminus A$ is bounded $\}$. Then $C_{\infty}\mathcal{P}(\mathbb{R})$ is not an ideal of $C(\mathbb{R})$. Because $\bigcup_{n=1}^\infty (0, n)$ is not bounded.

(b). If $\mathcal{P} = \{A \subseteq \mathbb{R} : \mathbb{R} \setminus A$ is finite $\}$. Then by Corollary 2.27 $C_{\infty}\mathcal{P}(\mathbb{R})$ is an ideal of $C(\mathbb{R})$.

Remark 2.29. Any closed bounded in a normal space is a pseudocompact and any pseudocompact Lindelöf space is compact, so if $X$ is a realcompact normal space and $\mathcal{P}$ equal be the set of all subsets whose complements are bounded subsets of $X$, then $C_{\infty}\mathcal{P}(X) = C_\infty(X)$, for example in $\mathbb{R}$ if $\mathcal{P}$ equal be the set of all subsets of $\mathbb{R}$, whose complements are bounded, then $C_{\infty}\mathcal{P}(\mathbb{R}) = C_\infty(\mathbb{R})$. In particularly, if $X$ is a Lindelöf space and $\mathcal{P}$ equal be the set of all subsets of $X$, whose complements are bounded, then $C_{\infty}\mathcal{P}(X) = C_\infty(X)$.

3 $C^\mathcal{P}(X)$ as an essential ideal.

Topological spaces $X$ for which $C_\infty(X)$ (resp., $C_K(X)$) is an essential ideal was characterized by Azarpanah, in [4]. In this section we characterize topological spaces $X$ for which $C^\mathcal{P}(X)$ is an essential ideal.

Proposition 3.1. An ideal $E$ in $C_{\infty}\mathcal{P}(X)$ is an essential ideal if and only if $\bigcap Z[E]$ does not contain a subset $V$ where $X \setminus V$ $\in \mathcal{F}$ and $\text{int}\, V \neq \emptyset$.

Proof. Let $X \setminus V = F \in \mathcal{F}$, $\bigcup_{f \in E} \text{coz}(f) \subseteq F$ and $\text{int}\, V \neq \emptyset$, i.e., $\overline{F} \neq X$. Then there exist $x \in X$ such that $x \notin \overline{F}$, it follows that there is $f \in C(X)$ such that $f(x) = 1$, $f(F) = 0$, i.e., $f \in C_{\infty}\mathcal{P}(X)$. Now for any $g \in E$ we have $X \setminus Z(f) \subseteq X \setminus F \subseteq Z(g)$, i.e., $fg = 0$ so $(f) \cap E = 0$, which contradicts the essentiality of $E$. Conversely, let $0 \neq f \in C_{\infty}\mathcal{P}(X)$. Then there is $a \in X$ such that $|f(a)| > \frac{1}{n}$ for some $n \in \mathbb{N}$, hence $a \in X \setminus \{x : |f| \leq \frac{1}{n}\}$, i.e., $\{x : |f| \leq \frac{1}{n}\} \neq X$. We know that $\{x : |f| < \frac{1}{n}\} \in \mathcal{F}$. By hypothesis, $X \setminus \{x : |f| < \frac{1}{n}\} \notin \bigcap Z[E]$. Therefore, there exists $b \in X \setminus \{x : |f| \leq \frac{1}{n}\}$ and $g \in E$ such that $g(b) \neq 0$, i.e, $fg \neq 0$ thus $E$ is an essential ideal in $C_{\infty}\mathcal{P}(X)$.

\qed
Recall that a collection \( \mathcal{B} \) of open sets in a topological space \( X \) is called a \( \pi \)-base if every open set of \( X \) contains a member of \( \mathcal{B} \). The reader is referred to \([8], [9], [10], [12], \) and \([15]\). The next result is a generalization of Theorem 3.2 in \([\text{3}]\).

**Theorem 3.2.** \( C^P(X) \) is an essential ideal if and only if \( \{ V : V \) is open and \( X \setminus V \in \mathcal{F} \} \) is a \( \pi \)-base for \( X \).

**Proof.** Let \( U \) be a proper open set in \( X \). By regularity of \( X \), there exist a non-empty open set \( V \) such that \( V \subseteq \text{cl}V \subseteq U \). Now find \( f \in C(X) \) where \( f(cV) = \{1\}, f(x) = 0 \) for some \( x \notin U \). If \( X \setminus V \in \mathcal{F} \), there is noting to proved. Suppose \( X \setminus V \notin \mathcal{F} \). If \( V \subseteq Z(h) \) for every \( h \in C^P(X) \), then \( V \subseteq \bigcap Z(C^P(X)) \), which implies that \( C^P(X) \) is not an essential ideal, by \([\text{3}]\). Theorem 3.1. Therefor there is some \( h \in C^P(X) \) such that \( V \setminus (X \setminus Z(h)) \neq \emptyset \), i.e., there is some \( x_0 \in V \) for which \( h(x_0) \neq 0 \). Clearly \( fh \in C^P(X) \). So \( W = X \setminus Z(fh) \) is contained in \( X \setminus F \) for some \( F \in \mathcal{F} \). If \( W' = W \cap V \), then \( W' \) is a non-empty open set in \( U \) and \( X \setminus W' \in \mathcal{F} \).

Conversely, We will prove that for every non-unit \( g \in C(X) \), \( C^P(X) \cap (g) \neq 0 \). Since \( X \setminus Z(g) \) is an open set, then there is an open set \( U \) where \( U \subseteq \text{cl}U \subseteq X \setminus Z(g) \), and there is an open set \( V \subseteq U \) such that \( X \setminus V \in \mathcal{F} \). Then \( V \subseteq U \subseteq X \setminus Z(g) \). Define \( f \in C(X) \) such that \( f(X \setminus V) = 0, f(x) = 1 \) for some \( x \in V \). Since \( X \setminus V \subseteq Z(f) \) so \( f \in C^P(X) \). On the other hand \( Z(g) \subseteq X \setminus V \subseteq Z(f) \) so \( fg \neq 0 \) and \( fg \in C_P(X) \cap (g) \). \( \square \)

### 4 \( C_{\infty P}(X) \) as a \( z \)-ideal and a regular ring.

We know that \( C_{\infty P}(X) \) is a subring of \( C(X) \), in this section, we see that \( C_{\infty P}(X) \) is a \( z \)-ideal if and only if every cozero-set containing a \( \mathcal{F} \)-\( CG_\delta \) is an element of \( \mathcal{F} \). Also we prove that, \( C_{\infty P}(X) \) is a regular ring (i.e., for each \( f \in C_{\infty P}(X) \) there exists \( g \in C_{\infty P}(X) \) such that \( f = f^2g \) if and only if every closed \( \mathcal{F} \)-\( CG_\delta \) is an open subset and belong to \( \mathcal{F} \).

**Proposition 4.1.** The subring \( C_{\infty P}(X) \) is a \( z \)-ideal of \( C(X) \) and if and only if every cozero-set containing a closed \( \mathcal{F} \)-\( CG_\delta \) is an element of \( \mathcal{F} \).

**Proof.** First, we prove that \( Z(f) \subseteq Z(g) \) and \( f \in C_{\infty P}(X) \), implies that \( g \in C_{\infty P}(X) \). To see this, we know that \( Z(f) \subseteq Z(g) \subseteq \{ x : |g(x)| < \frac{1}{n} \} \), for all \( n \in \mathbb{N} \). But \( \{ x : |g(x)| < \frac{1}{n} \} \) is a cozero-set and \( Z(f) \) is a closed \( \mathcal{F} \)-\( CG_\delta \). So, by hypothesis, \( \{ x : |g(x)| < \frac{1}{n} \} \) is an element of \( \mathcal{F} \), i.e., \( g \in C_{\infty P}(X) \). Now, suppose that \( f \in C_{\infty P}(X) \) and \( g \in C(X) \). Then \( Z(f) \subseteq Z(fg) \), shows that \( fg \in C_{\infty P}(X) \). Thus \( C_{\infty P}(X) \) is a \( z \)-ideal of \( C(X) \). Conversely, Suppose that \( X \setminus Z(f) \) is a cozero-set contains a closed \( \mathcal{F} \)-\( CG_\delta \) subset \( A \). By Lemma \([2, 19]\) there exist \( g \in C_{\infty P}(X) \), such that \( A = Z(g) \), so \( Z(g) \subseteq X \setminus Z(f) \). Now we...
define \( h = \frac{g^2}{f^2 + g^2} \). Then \( h \in C(X) \) and \( Z(g) \subseteq Z(h) \), therefore \( h \in C_{\mathcal{P}}(X) \).

On the other hand, for each \( n \in \mathbb{N} \), \( \{ x : |h(x)| < \frac{1}{n} \} \subseteq X \setminus Z(f) \), hence \( X \setminus Z(f) \in \mathcal{F} \). This completes the proof. \( \Box \)

**Theorem 4.2.** \( C_{\mathcal{P}}(X) \) is a regular ring if and only if every closed \( \mathcal{F}-\mathcal{CG}_\delta \) is an open subset and belongs to \( \mathcal{F} \).

**Proof.** First, we prove that every closed \( \mathcal{F}-\mathcal{CG}_\delta \) is an open subset. By Lemma 2.19 every closed \( \mathcal{F}-\mathcal{CG}_\delta \) is of the form \( Z(f) \) for some \( f \in C_{\mathcal{P}}(X) \). But \( Z(f) = Z(f \land n) \), for each \( n \in \mathbb{N} \) and \( \{ x : |f \land n| < \frac{1}{m} \} = \{ x : |f| < \frac{1}{m} \} \). So we can let \( f \) be bounded. Regularity of \( C_{\mathcal{P}}(X) \) implies that, there exists \( g \in C_{\mathcal{P}}(X) \) such that \( f = f^2 g \). Then \( X \setminus Z(1 - f g) \subseteq \text{int}Z(f) \).

If \( x \in X \setminus Z(f) \), then \( x \in Z(1 - f g) \), which contradict \( x \in Z(f) \), i.e., \( Z(f) \) is an open subset. On the other hand for every \( x \in X \setminus Z(f) \), \( g(x) = \frac{1}{f(x)} \) and hence \( g(x) \geq \frac{1}{n} \), where \( n \) is an upper bounded for \( |f| \). Therefore \( X \setminus Z(f) \subseteq \{ x : |g| \geq \frac{1}{n} \} \), i.e., \( Z(f) \supseteq \{ x : |g| < \frac{1}{n} \} \). But \( \{ x : |g| < \frac{1}{n} \} \) contains an element of \( \mathcal{P} \) so \( Z(f) \in \mathcal{F} \). Conversely, Suppose \( f \in C_{\mathcal{P}}(X) \). \( Z(f) \) is a closed \( \mathcal{F}-\mathcal{CG}_\delta \) so by hypothesis, is an open subset which belong to \( \mathcal{F} \). We define \( g(x) = 0 \) for \( x \in Z(f) \) and \( g(x) = \frac{1}{f(x)} \) for \( x \in X \setminus Z(f) \). Then \( g \in C(X) \), \( f = f^2 g \) and \( \{ x : |g| < \frac{1}{n} \} \supseteq Z(f) \), i.e., \( g \in C_{\mathcal{P}}(X) \). \( \Box \)

**Corollary 4.3.** (a) Let \( \mathcal{P} = \{ A : A \) is open and \( X \setminus A \) is Lindelöf \} \) and \( X \) is a non-Lindelöf space. Then \( \mathcal{P} \) is an open filter and \( C_{\mathcal{P}}(X) \) is a regular ring if and only if every closed \( \mathcal{P}-\mathcal{CG}_\delta \) is an open subset.

(b) \( C_{\infty}(X) \) is a regular ring if and only if every open locally compact \( \sigma \)-compact subset is compact.

**Proof.** (a) It is easily seen that \( \mathcal{P} \) is an open filter. If \( C_{\mathcal{P}}(X) \) is a regular ring, then by Theorem 4.2 every closed \( \mathcal{P}-\mathcal{CG}_\delta \) is an open subset. Now let \( A \) be a closed \( \mathcal{P}-\mathcal{CG}_\delta \) which is an open subset. Then by Lemma 2.19, \( A = Z(f) \) for some \( f \in C_{\mathcal{P}}(X) \). But \( X \setminus A = X \setminus Z(f) = \bigcup_{n=1}^{\infty} \{ x : |f(x)| \geq \frac{1}{n} \} \), hence by [10] Theorem 3.8.5, \( X \setminus A \) is a Lindelöf subset of \( X \), i.e., \( A \in \mathcal{P} \). By Theorem 4.2, \( C_{\mathcal{P}}(X) \) is a regular ring.

(b) If \( A \) is an open locally compact \( \sigma \)-compact subset, then \( X \setminus A \) is a closed \( \mathcal{P}-\mathcal{CG}_\delta \), where \( \mathcal{P} = \{ A \subseteq X : X \setminus A \) is compact \} and \( C_{\infty}(X) = C_{\mathcal{P}}(X) \), so by Theorem 4.2 we are done. \( \Box \)

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Some generalizations and unifications of $C_K(X)$, $C_\psi(X)$ and $C_\infty(X)$

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