(3 + 1)-formulation for gravity with torsion and non-metricity: II. The hypermomentum equation

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Received 30 July 2021, revised 19 September 2021
Accepted for publication 1 October 2021
Published 26 October 2021

Abstract
In this article, we consider a particular case of metric-affine \(f(R)\)-gravity for \(f(R) = R\), i.e. the metric-affine general relativity (MAGR). As a companion to the first article in the series, we perform the (3 + 1) decomposition to the hypermomentum equation, obtained from the minimization of the MAGR action \(S[g, \omega]\) with respect to the connection \(\omega\). Moreover, we show that the hypermomentum tensor \(H\) could be entirely constructed from ten hypersurfaces variables that arise from its dilation, shear, and rotational (spin) parts. The (3 + 1) hypermomentum equations consist of one scalar equation, three vector equations, three matrix equations, and one tensor equation of order-(\(\frac{3}{2}\)). Together with the (3 + 1) decomposition of the traceless torsion constraint, which consists of a scalar and a vector equation, we obtain ten hypersurface equations that are the main result in this article. Finally, we consider some special cases of MAGR, namely, the zero hypermomentum, metric, and torsionless cases. For vanishing hypermomentum, we could retrieve the metric compatibility and torsionless condition in the (3 + 1) framework, forcing the affine connection to be Levi-Civita as in the standard general relativity.

Keywords: torsion, non-metricity, hypermomentum, 3 + 1 formulation, metric-affine gravity

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1. Introduction

Modified gravity theories have gained attention in these few decades [1–3]. Attempts to modify gravity arise from the interest to explain exotic phenomenologies such as dark matter and dark energy, as well as to solve the problem of quantum gravity. One of these modified theories branches is the metric-affine gravity (MAG) [4–8]. The theory differs from the standard general relativity (GR) following the generalization of three objects. The first generalization is on the action integral [7, 8]. While the standard GR could be derived from the Einstein–Hilbert action $S = \int \mathcal{R} \, \text{vol}$ (with $\mathcal{R}$ is the Ricci scalar as a functional of metric $g$ on the manifold $(\mathcal{M}, g)$), the MAG could use more general forms of action, including all the geometrical trinity of gravity’s contributions (curvature, torsion, and non-metricity) [8, 9]. The second generalization is applied to the covariant derivative (a special case of connection defined on the tangent bundle) $\nabla$ on $(\mathcal{M}, g)$. Standard GR adopts a special case of covariant derivative known as Levi-Civita, where the vector potential $\Gamma$ of $\nabla$ can always be written as a function of metric $g$ on $(\mathcal{M}, g)$. On the other hand, MAG uses a general affine connection $\nabla$, which is generally independent of the metric [4–8]. The vector potential $\omega$ of $\nabla$, which physicists usually label as the connection, may contain non-metricity and torsional parts. Because of the independence of the connection from the metric, the action of MAG is also a functional on $\omega$, as well as a functional of $g$. A special case of MAG with Einstein–Hilbert action $S[g, \omega] = \int \mathcal{R} \, \text{vol}$ is known as metric-affine general relativity (MAGR) or generalized Palatini gravity (however, these terminologies could be used differently, as in [10]). Meanwhile, a more general family of such theories with the $f(\mathcal{R})$ action, where $f(\mathcal{R})$ is the polynomial functions of $\mathcal{R}$, is known as metric-affine-$f(\mathcal{R})$ gravity [11, 12]. Both theories are considered as special cases of MAG. The last generalization is on the existence of coupling between matter and connection, which give rise to a new quantity known as the hypermomentum. This quantity vanishes in the standard GR.

A detailed and complete explanation of MAG could be found in [7, 8], while a detailed account of $f(\mathcal{R})$ and Metric-Affine $f(\mathcal{R})$ could be found in [1, 11, 12]. Some of the latest progress on these theories include the mathematical and geometrical analysis [13, 14], ghost-free higher-order curvature gravity [15–17], metric-affine cosmology [18–22], and the $(3 + 1)$ formulation of special cases of MAG [23–28], which become the main interest in this article.

As mentioned briefly in [32], the $(3 + 1)$ formulation of GR was introduced by Darmois [33], Lichnerowicz [34–36], and Fouré-Bruhat [37, 38], and then was popularized by Arnowitt, Deser, and Misner in their famous article [39]. In the $(3 + 1)$ formulation, the covariant fields on spacetime are split into their temporal and spatial parts to give rise to the dynamical variables on the 3D spatial hypersurface. The $(3 + 1)$ formulation of $f(\mathcal{R})$-gravity for some special cases had been done in [23–26], while some attempts on their quantization had been done in [29–31]. In our previous article [28], we apply the $(3 + 1)$ formulation for the MAGR (this includes the geometrical trinity of gravity: curvature, torsion, and non-metricity), where the focus is on the stress–energy–momentum equation (or the generalized Einstein field equation), obtained by minimizing the action with respect to the metric $g$. Here, as a companion article to [28], the focus is on the hypermomentum equation obtained from the minimization of the action with respect to the connection $\omega$. The existence of the hypermomentum $H$ in the rhs of the hypermomentum equation accommodates certain kinds of matter fields that give rise to torsion and non-metricity. Another equation of interest in this article is the traceless torsion constraint, which emerges as the consequence of a consistent application of projective invariant constraint. The origin of this constraint has been described in detail in [8, 13, 14]. Another brief explanation could be seen in [11, 12, 28].
The main result in this article is the \((3+1)\) formulation of the hypermomentum equation for the MAGR. Following our earlier work in [28], we use the additional hypersurface variables with non-Riemannian origins to write these \((3+1)\) equations. The hypermomentum \(H\), following [40–43], is decomposed into its dilation, shear, and rotational (spin) parts. Moreover, we could show that the hypermomentum could be constructed completely from 10 hypersurfaces variables. In the end, by setting the hypermomentum to be zero, we could retrieve the torsionless and metric compatibility in the \((3+1)\) framework, hence forcing the affine connection to be Levi-Civita. This is possible with the help of the traceless torsion constraint [28].

The structure of this article is organized as follows: in section 2, we briefly review the geometrical setting and some definitions in [28]; these include the introduction of additional variables on the hypersurface. Also, in this section, we perform the \((3+1)\) decomposition of the torsion \(T\) and the non-metricity factor \(\nabla Xg\), for any \(X \in T_pM\).

Section 3 contains complete derivations on the equations of motion of metric-affine \(f(R)\) gravity, i.e. the stress–energy–momentum equation, obtained by minimizing \(S\) with respect to \(g\), and the hypermomentum equation, obtained by minimizing \(\tilde{S}\) with respect to \(\omega\). The derivation of the hypermomentum equation differs from the standard derivation in [11, 12]; however, we show that both ways are equivalent. The traceless torsion constraint is also introduced in this section. At the end of this section, we consider a particular case of metric-affine \(f(R)\)-gravity, where \(f(R) = R\), i.e. the MAGR or generalized Palatini gravity. From this stage up to the rest of this article, we will only consider this case.

In section 4, we perform the \((3+1)\) decomposition to the hypermomentum equation. The first step is to split the hypermomentum \(H\) in the rhs of the hypermomentum equation into its temporal and spatial parts. The split of the hypermomentum could be simplified further by decomposing \(H\) into the dilation, shear, and rotational (spin) parts. With these, the hypermomentum could be entirely constructed from ten hypersurfaces variables. The second step is to split the lhs of the hypermomentum equation using some \((3+1)\) relations we derived in section 2. The results are \((3+1)\) hypermomentum equations, which consist of a scalar equation, three vector equations, three matrix equations, and a tensor equation of order \((21)\).

Moreover, we perform the \((3+1)\) decomposition of the traceless torsion constraint, resulting in a scalar and one vector equation; collecting these results altogether, we obtain ten hypersurface equations, which are the main results in this article.

In section 5, we consider some special cases of MAGR. For the first case, we set the hypermomentum \(H = 0\), and then show that this gives the metric compatibility and torsionless condition of the Levi-Civita connection. For the second and third cases, we apply the metric compatibility and torsionless condition separately to the \((3+1)\) hypermomentum equations; for these two cases, the hypermomentum, as well as the dynamical equations, are restricted by constraints.

Finally, we discuss some subtleties of the results and conclude our works in section 6.

2. The geometrical setting and definitions

The geometrical setting is similar with the one we used in our first article in the series [28]. Let \(M\) be a four-dimensional, globally hyperbolic, Lorentzian manifold, equipped with a Lorentzian metric \(g\) and an affine connection \(\nabla\) that is independent from \(g\). As mentioned in the introduction, \(\omega\) is the vector potential of \(\nabla\), but physicists usually labels it as the connection. Let \(\Sigma\) be the three-dimensional (Riemannian) hypersurface on \(M\), and \(\hat{n}\) be the unit vector normal to \(\Sigma\) at each point \(p \in \Sigma\) satisfying \(g_p(\hat{n}_p, \hat{n}_p) = -1\). The metric \(g\) on \(M\) induces a Riemannian metric \(\tilde{q}\) on the hypersurface \(\Sigma\):

\[
\tilde{q} = g + \hat{n}^* \otimes \hat{n}^*,
\]
with \( \hat{n}^* \in T^*_p \mathcal{M} \) is the covariant vector to \( \hat{n} \), satisfying \( \hat{n}^* = g(\hat{n}, \cdot) = g(\hat{n}) \) (the label \( p \) is omitted for simplicity).

### 2.1. The adapted coordinate, lapse function, and shift vector

Let \( x^\alpha = \{x^0, x^i\} \) be a local coordinate on \( \mathcal{M} \), with \( x^0 \) and \( x^i \) are, respectively, the temporal and spatial parts of \( x^\alpha \). The corresponding coordinate vector basis on \( T_p \mathcal{M} \) is \( \partial_\mu = \{\partial_0, \partial_i\} \). Any vector \( V \in T_p \mathcal{M} \) could be decomposed as follows:

\[
V = \langle dx^\alpha, V \rangle \partial_\mu,
\]

with \( V^\mu \) is the component of \( V \), \( dx^\mu \) is the coordinate covector basis on \( T^*_p \mathcal{M} \), and \( \langle \ldots \rangle \) is the inner-product defined on the tangent space of \( \mathcal{M} \) satisfying \( \langle dx^\mu, \partial_\nu \rangle = \partial_\nu (dx^\mu) = \delta^\mu_\nu \).

Equivalently, \( dx^\mu = g^{\mu\nu} \partial_\nu \) and \( g_{\mu\nu} = g(\partial_\mu, \partial_\nu) = \langle \partial_\mu, \partial_\nu \rangle \).

Let us consider another set of vector basis which equally spans \( T_p \mathcal{M} \), namely \( (\hat{n}, \partial_\nu) \), where \( \partial_\nu \) is simultaneously the coordinate vector basis corresponding to the local coordinate \( \{x^i\} \) of the hypersurface \( \Sigma \). As a consequence, \( g(\hat{n}, \partial_\nu) = n_\nu = 0 \). \( (\hat{n}, \partial_\nu) \) is known as the adapted vector basis to the hypersurface \( \Sigma \). In this basis, the temporal component of \( \{\partial_\nu\}, \partial_0 \), is decomposed into:

\[
\partial_0 = - (\hat{n}^*, \partial_0) \hat{n} + (3dx^i, \partial_0) \partial_i, 
\]

where \( 3dx^i = 3q^{ij}g(\partial_j) \), is the covector basis which spans \( T^*_p \Sigma \). \( N \) and \( N^i\partial_i \) are, respectively, the lapse function and the shift vector of \( \partial_0 \). By solving \( \langle dx^\nu, \partial_0 \rangle = \delta^\nu_0 \), one could obtain that:

\[
dx^0 = -\hat{n}^*N^{-1}, \quad (3)
\]

\[
dx^i = \hat{n}^*N^iN^{-1} + 3dx^i, \quad (4)
\]

Using (2)–(4), the components of metric \( g \) and its inverse \( g^{-1} = g^* \) could be written in terms of lapse and shift as follow:

\[
g(\partial_0, \partial_0) = g_{00} = N^iN_i - N^2, \quad g^*(dx^0, dx^0) = g^{00} = -N^{-2},
\]

\[
g(\partial_0, \partial_i) = g_{0i} = N_i, \quad g^*(dx^0, dx^i) = g^{0i} = N^iN^{-2},
\]

\[
g(\partial_i, \partial_j) = g_{ij} = 3q^{ij}, \quad g^*(dx^i, dx^j) = g^{ij} = 3q^{ij} - (N^iN^j)N^{-2},
\]

where \( 3q^{ij} \) satisfies \( 3q^{ij} = g^*(3dx^i, 3dx^j) \). One could derive the following relations that will be useful for the later derivations in this article [28]:

\[
\hat{n} = n^0\partial_0 + n^i\partial_i = N^{-1}(\partial_0 - N^0\partial_i), \quad N = N^0\partial_0 + N_i\partial_i = -Nn^i\partial_i,
\]

\[
\hat{n}^* = n_0dx^0 + n_i dx^i = -Nd^0, \quad N^* = N_0dx^0 + N_i dx^i = N(N^0dx^0 + q_{ij}N^jdx^i).
\]

For a complete review on \((3 + 1)\)-decomposition and the hypersurface foliation of a manifold, one could consult [32, 44, 46, 47].

### 2.2. The additional variables

Following [28], all the \((3 + 1)\) quantities will be written in terms of the additional variables, motivated by the concept of geometrodynamics introduced by Wheeler in [45]. In this concept,
a system in GR could be equally described by fields on hypersurface $\Sigma$; these additional variables are examples of them. The three-fields satisfy the dynamical equations that resulted from (3 + 1) decomposition of the (covariant) Euler–Lagrange equations. Mathematical quantities presented in the dynamical equations are usually written in the terms (or partial derivatives) of $\nabla_X Y$, where $X \in T_p\mathcal{M}$ and $Y$ is a general tensor of order $\left( \frac{n}{m} \right)$. In the (3 + 1) formulation, they are split into their temporal and spatial parts. Therefore, it is convenient to rewrite $\nabla_X Y$ in terms of the additional variables:

$$\nabla_\alpha \dot{\alpha} = \Theta (\dot{\alpha}) \hat{\alpha} + \alpha' \partial_\alpha,$$

$$\nabla_\alpha \dot{\beta} = \Theta (\dot{\beta}) \hat{\beta} + \alpha' \partial_\beta,$$

$$\nabla_\alpha \dot{\gamma} = K_{ij} \dot{\alpha} + \omega_{ij}^k \partial_\gamma,$$

$$\nabla_\alpha \dot{\delta} = \Delta_{ij} \dot{\delta} + \Delta_{ij}^\ell \partial_\delta.$$  

The additional variables have geometrical interpretation as follows [28]: $(\Theta (\dot{\alpha})$, $\omega_{ij}^k$) are the temporal and spatial components of the four-acceleration $\nabla_\alpha \dot{\alpha} := \alpha' \partial_\alpha$. Specifically, $\Theta (\dot{\alpha}) = -\langle \dot{\alpha}^*, \nabla_\alpha \dot{\alpha} \rangle$ is the angle between the four-velocity with the four-acceleration, which is zero for a metric connection, while $\alpha' = \langle \frac{d}{dt}, \nabla_\alpha \dot{\alpha} \rangle$ is the components of the (relativistic) three-acceleration. $K$ and $\mathcal{K}$ are extrinsic curvatures of the first and second kind, defined as follows:

$$K_{ij} = K (\partial_i, \partial_j) = -g (\nabla_i \partial_j, \dot{\alpha} - \langle \dot{\alpha}^*, \nabla_i \partial_j \rangle),$$

$$\mathcal{K}_{ij} = q^3 g \mathcal{K}_{ij} = q^3 g (\nabla_\alpha \dot{\alpha}, \partial_\beta) - \langle \frac{d}{dt}, \nabla_\alpha \dot{\alpha} \rangle.$$  

$\Theta_i := -\langle \dot{\alpha}^*, \nabla_\alpha \dot{\alpha} \rangle$ is the angle between the normal $\hat{\alpha}$ and $\nabla_\alpha \dot{\alpha}$, which is a part of the acceleration tensor $a := (\nabla_\alpha \dot{\alpha}, \nabla_\alpha \dot{\beta}, \nabla_\alpha \dot{\gamma})$. $\Delta_{ij} = -\langle \dot{\alpha}^*, \nabla_\alpha \dot{\alpha} \rangle$ and $\Delta_{ij}^\ell = \langle \frac{d}{dt}, \nabla_\alpha \dot{\alpha} \rangle$ are the temporal and spatial components of the evolution generator of $\partial_\delta$. Finally, $\omega_{ij}^k = \langle \frac{d}{dt}, \nabla_i \partial_j \rangle$ is the three-connection on $\Sigma$. These additional variables, in general, have non-Riemannian origins; for a detailed explanation, consult [28].

2.3. (3 + 1)-decomposition of the torsion tensor

Due to the different conventions used in the literature, we need to fix the notations used in this article. The connection (or precisely, the vector potential of the affine connection) is written as $\nabla_\alpha \partial_\beta = \omega_{\mu}^\alpha \partial_\mu$, which are different from the ones used in, for examples [11, 12]. Then one could define the torsion tensor as follows [46, 47]:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad X, Y \in T_p\mathcal{M},$$  \hspace{1cm} (9)

with $[X, Y]$ denotes the Lie bracket between $X, Y \in T_p\mathcal{M}$ as follows:

$$[X, Y] = X (Y) - Y (X) = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) \partial_\nu.$$  

(One should note that there exist different conventions in the definition of torsion, for example, the definition of torsion in [11, 12] is $T_{\alpha \beta}^\gamma = \frac{1}{2} (\omega_{\alpha \beta}^\gamma - \omega_{\beta \gamma}^\alpha - \omega_{\gamma \alpha}^\beta)$, with a factor $\frac{1}{2}$). In terms of components, the torsion tensor (9) reads as:

$$T(X, Y) = X^\alpha Y^\beta T (\partial_\alpha, \partial_\beta),$$  \hspace{1cm} (10)

$$T (\partial_\alpha, \partial_\beta) = \nabla_\alpha \partial_\beta - \nabla_\beta \partial_\alpha - [\partial_\alpha, \partial_\beta] = \left( \omega_{\alpha \beta}^\gamma - \omega_{\beta \gamma}^\alpha \right) \partial_\gamma.$$  \hspace{1cm} (11)
using the fact that the Lie bracket of coordinates basis is zero, \([\partial_i, \partial_j] = 0\).

The torsion \(T(\partial_i, \partial_j)\) is decomposed into temporal and spatial parts, namely, the quantities \(T(\dot{n}, \dot{n}), T(\dot{n}, \dot{\Theta}) = -T(\partial_i, \dot{n})\), and \(T(\partial_i, \dot{\Theta})\). By the antisymmetry of the first and third indices of \(T(\partial_i, \partial_j)\), it is clear that:

\[
T(\dot{n}, \dot{n}) = 0.
\]

From the definition (9), and then inserting the additional variables (5)–(8), one could obtain that:

\[
T(\dot{n}, \dot{\Theta}) = -T(\partial_i, \dot{n}) = (\Delta_i - \Theta_i - \langle \dot{n}^i, \partial_i \dot{n}\rangle) \dot{n} + \left(\Delta^i_j + K^i_j + 3dx^i, \partial_j \dot{n}\right) \partial_j,
\]

and:

\[
T(\partial_i, \dot{\Theta}) = 3T(\partial_i, \partial_j) + (K(\partial_i, \partial_j) - K(\partial_j, \partial_i)) \dot{n}.
\]

(12)–(14) are the \((3 + 1)\) decomposition of \(T(\partial_i, \partial_j)\). Using (10) and (11) and the fact that:

\[
\langle 3dx^i, \partial_j \dot{n}\rangle = \partial_i n^j = -\frac{1}{N}\partial_0 N^i,
\]

one could rewrite (12)–(14) in terms of their components.

Let us define the component of torsion \(T^i_{\alpha \beta}\) as follows:

\[
T^i_{\alpha \beta} \partial_\alpha = g^{\lambda \beta} T^i_{\alpha \lambda} \partial_\alpha = g^{\lambda \beta} T(\partial_\mu, \partial_\beta) \partial_\alpha + g^{\lambda \beta} T(\partial_\mu, \partial_\alpha) \partial_\beta,
\]

where the last index of \(T^i_{\alpha \lambda}\) is raised by the inverse metric \(g^*\). Some important quantities for the derivation in the next sections are several contractions of (15) with combinations of the temporal and/or spatial vector basis as follows:

\[
n^\mu n^\beta T^i_{\alpha \beta} \partial_\alpha = n^\mu n^\beta T^i_{\alpha \beta} \partial_\alpha = T(\dot{n}, \dot{n}),
\]

\[
n^\mu T^i_{\alpha \beta} \partial_\alpha = n^\mu g^{\lambda \beta} T^i_{\alpha \lambda} \partial_\alpha = g^{\lambda \beta} T(\dot{n}, \partial_\lambda) \partial_\alpha = N^i N^{-1} T(\dot{n}, \dot{n}) + 3q^i T(\dot{n}, \dot{\Theta}),
\]

\[
n^\mu T^i_{\alpha \beta} \partial_\alpha = n^\mu T^i_{\alpha \beta} \partial_\alpha = T(\dot{\Theta}, \dot{n}),
\]

\[
T^i_{\alpha \beta} \partial_\beta = g^{\mu \lambda} T^i_{\alpha \lambda} \partial_\alpha = N^i N^{-1} T(\partial_\mu, \dot{n}) + 3q^\mu T(\partial_\mu, \dot{\Theta}).
\]

Inserting (12)–(14) to these 4 equations gives:

\[
n^\mu n^\beta T^i_{\alpha \beta} \partial_\alpha = 0,
\]

\[
n^\mu T^i_{\alpha \beta} \partial_\alpha = 3q_{ij} \left(\Delta_j - \Theta_j - \langle \dot{n}^i, \partial_j \dot{n}\rangle\right) \dot{n} + 3q^j \left(\Delta^j - K^j_k + \langle 3dx^j, \partial_k \dot{n}\rangle\right) \partial_k,
\]

\[
n^\beta T^i_{\alpha \beta} \partial_\alpha = -\left(\Delta_i - \Theta_i - \langle \dot{n}^i, \partial_\Theta \dot{n}\rangle\right) \dot{n} + \left(\Delta^j - K^j_k + \langle 3dx^j, \partial_k \dot{n}\rangle\right) \partial_j,
\]

\[
T^i_{\alpha \beta} \partial_\beta = \left(-N^i N^{-1} \left(\Delta_i - \Theta_i - \langle \dot{n}^i, \partial_\Theta \dot{n}\rangle\right) + 3q^k (K^i_k - K^k_i)\right) \dot{n}
\]

\[
- N^i N^{-1} \left(\Delta^k - K^k_j + \langle 3dx^k, \partial_j \dot{n}\rangle\right) \partial_k + 3q^k T(\partial_\mu, \dot{\Theta}).
\]

We consider these four quantities because they will appear explicitly in the \((3 + 1)\) decomposition of the hypermomentum equation.

6
2.4. (3 + 1)-decomposition of the non-metricity factor

One could decompose the non-metricity factor $\nabla_X g$ into its temporal and spatial parts, for every $X \in T_p M$; however, considering the form of equations (5)–(8), it is more convenient to obtain the (3 + 1) decomposition of the quantity $\nabla_X g^*\$, where $g^* = g^{\mu\nu} \partial_\mu \partial_\nu$ is the contravariant version of $g$. Notice that $g^* = g^{-1}\$, since $g^{\mu\nu} g_{\mu\nu} = \delta^\nu_\nu$. Taking the contravariant version of the decomposition of metric in (8), one could write:

$$\nabla_X g^* = \nabla_X (3 q^* - \hat{n} \otimes \hat{n})\ .$$

Now, taking the covariant derivative of $g^*$ in the direction of $\hat{n}$ and $\partial_i$, then using (5)–(8) to the term $\nabla_{\hat{n}} (\hat{n} \otimes \hat{n})$ (or $\nabla_i (\hat{n} \otimes \hat{n})$), one could obtain:

$$\nabla_{\hat{n}} g^* = -2 \Theta (\hat{n}) \hat{n} \otimes \hat{n} + (\Delta^i - \alpha^i) (\hat{n} \otimes \partial_i + \partial_i \otimes \hat{n}) + 3 \nabla_{\hat{n}} 3 q^*, \quad (20)$$

$$\nabla_i g^* = -2 \Theta_i \hat{n} \otimes \hat{n} + (K_{ij} - K_{ij}^j) (\hat{n} \otimes \partial_j + \partial_j \otimes \hat{n}) + 3 \nabla_i 3 q^*, \quad (21)$$

where:

$$3 \nabla_{\hat{n}} 3 q^* = (3 \nabla_{\hat{n}} 3 q^{ik}) \partial_i \otimes \partial_j = (\hat{n} [3 q^{ik}] + \Delta^j k 3 q^{ki} + \Delta^i k 3 q^{jk}) \partial_i \otimes \partial_j, \quad (22)$$

$$3 \nabla_i 3 q^* = (3 \nabla_i 3 q^{ik}) \partial_i \otimes \partial_k = (\partial_i 3 q^{ik} + \omega^{ik}_j + \omega^{jk}_i) \partial_i \otimes \partial_k. \quad (23)$$

To obtain equations (22) and (23), one could simply derive $3 \nabla_{\hat{n}} 3 q^{ik}$ or $3 \nabla_i 3 q^{ik}$ in the indices of $i, j$, then write the quantities in terms of additional variables, i.e. $\Delta^i j$ and $3 \omega^{ik}_j$.

The hypermomentum equations will contain specific terms with the torsional and non-metricity contribution; hence it is convenient to derive the (3 + 1) decomposition of several non-metricity terms that will appear on the hypermomentum equation as follows. First, $\nabla_X g^*$ could be written in indices as:

$$\nabla_X g^* = X^\mu (\nabla_{\mu} g^{\alpha\beta}) \partial_\alpha \otimes \partial_\beta, \quad (24)$$

where $\nabla_{\mu} g^{\alpha\beta} = \partial_{\mu} g^{\alpha\beta} + \omega^{\alpha\beta}_\mu + \omega^{\beta\alpha}_\mu$. One could consider $\nabla_X g^*$ as a $(\frac{2}{2})$-order tensor that acts on two covectors. Therefore, acting $\nabla_X g^*$ on $Y^*, Z^* \in T_p M$ gives:

$$\left(\nabla_X g^*\right) (dx^\alpha, dx^\beta) = X^\mu \nabla_{\mu} g^{\alpha\beta}, \quad (24)$$

$$\left(\nabla_X g^*\right) (Y^*, Z^*) = X^\mu Y_\alpha Z_\beta \nabla_{\mu} g^{\alpha\beta}, \quad (25)$$

$$\left(\nabla_X g^*\right) (Y^*, \cdot) = \left(\nabla_X g^*\right) (\cdot, Y^*) = (X^\mu Y_\alpha \nabla_{\mu} g^{\alpha\beta}) \partial_\beta. \quad (26)$$

Using (24)–(26) with the help of (2) and (3), one could apply the (3 + 1) decomposition to the non-metricity terms, which will arise in the hypermomentum equation as follows:

$$\nabla_{\mu} g^{\alpha\beta} = - (\nabla_{\mu} g^*) (\hat{n}^\alpha, dx^\beta) - N^{-1} N^\alpha (\nabla_{\mu} g^*) (\hat{n}^\alpha, dx^\beta) + (\nabla_{\mu} g^*) (dx^\alpha, dx^\beta), \quad (27)$$

$$g^{\alpha\beta} \nabla_{\mu} g_{\alpha\beta} = - g_{\alpha\beta} \nabla_{\mu} g^{\alpha\beta} = - (N^{-2} N^\alpha N_i - 1) (\nabla_{\mu} g^*) (\hat{n}^\alpha, \hat{n}^\alpha) + 2 N^{-1} N_i (\nabla_{\mu} g^*) (\hat{n}^\alpha, dx^\beta) - 3 q_{ij} (\nabla_{\mu} g^*) (dx^\alpha, dx^\beta). \quad (28)$$

The last quantity is known as the Weyl vector $Q_\mu = - g^{\alpha\beta} \nabla_{\mu} g_{\alpha\beta} [46, 47]$. One needs to be careful that (27) and (28) are written in $dx^\alpha$ instead of $\delta dx^\alpha$. A complete (3 + 1) decomposition
3. The metric-affine $f(R)$-gravity

3.1. The action of metric-affine $f(R)$-gravity

In this section, we will briefly review the theory of metric-affine $f(R)$-gravity. The review of the concepts is highly based on [8, 11, 12]. There exist some minor modifications on several derivations of the results; however, these do not alter the physical interpretations of the theory. If one is restricted to work in the scope of metric-affine $f(R)$-gravity, then the most general action in the theory is [12]:

$$S[g, \omega, \chi, \zeta] = S_{GR}[g, \omega] + S_{\text{matter}}[g, \omega] + S_{\text{LM}}[\chi, g, \omega] + S_{\text{sym}}[\zeta, g].$$

(29)

The action $S$ is a functional over the metric $g$, connection $\omega$, and Lagrange multipliers $\chi$ and $\zeta$. The connection $\omega$ is a general affine connection independent of the metric $g$ [8, 11, 12].

The first term $S_{GR}[g, \omega]$ is the gravitational term:

$$S_{GR}[g, \omega] = \int_M f(R[\omega]) \text{vol};$$

$$f(R) = \ldots + \frac{c_2}{R^2} + \frac{c_1}{R} - 2\Lambda + \frac{R^2}{k_2} + \frac{R^3}{k_3} + \ldots$$

with:

$$R(\partial_{\mu}, \partial_{\nu}) \partial_{\beta} = \nabla_{\mu} \nabla_{\nu} \partial_{\beta} - \nabla_{\nu} \nabla_{\mu} \partial_{\beta} = \left(\partial_{\mu} \omega_{\nu}^{\alpha}_{\beta} - \partial_{\nu} \omega_{\mu}^{\alpha}_{\beta} + \omega_{\mu}^{\alpha}_{\sigma} \omega_{\sigma}^{\beta}_{\nu} - \omega_{\nu}^{\sigma}_{\beta} \omega_{\mu}^{\alpha}_{\sigma}\right) \partial_{\alpha}.$$

(30)

$$\text{Ric} = \left< dx^\alpha, R(\partial_{\mu}, \partial_{\nu}) \partial_{\beta} \right> dx^\nu \otimes dx^\beta = \delta^\alpha_\mu R^{\alpha}_{\nu \beta} dx^\nu \otimes dx^\beta,$$

(31)

$$\mathcal{R} = \text{tr} \text{Ric} = g^{\alpha \beta} R_{\nu \beta}.$$  

(32)

$R$, $\text{Ric}$, and $\mathcal{R}$ are the generalized Riemann curvature tensor, Ricci tensor, and (Ricci) scalar curvature. Notice that in this article we write the Riemann tensor as $R(\partial_{\mu}, \partial_{\nu}) \partial_{\beta} = R_{\mu \nu \alpha \beta}$, which is a different notation convention with [11, 12]. In the theory of $f(R)$-gravity, the main modification of the action is carried by this $f(R)$ term [12], where these curvatures are derived from a Levi-Civita connection $\tilde{\nabla}$. If one considers a more general metric-affine $f(R)$ theory, as in this article, the Levi-Civita connection is generalized to an affine connection $\nabla$, such that it contains non-Riemannian contributions, i.e. torsion and non-metricity [11, 12].

The second term on the rhs of (29), $S_{\text{matter}}[g, \omega]$, is the Lagrangian of the matter density, which is a functional of $g$ and $\omega$. In the standard GR, the connection $\omega$ is a function of $g$; hence $S_{\text{matter}}$ only depends on the metric. However, even in the Palatini formulation of gravity where $\omega$ is independent of $g$, there exists an a priori assumption that there is no coupling between the matter with connection [11]. As mentioned in the Introduction, the MAG generalizes the matter of (27) and (28) could be obtained by inserting (4), but this is not necessary for the moment. The readers interested in detailed review of the foliation method on non-Riemannian geometry could consult [47].
term to be coupled independently to the connection. As a consequence of this generalization, the variation of $S_{\text{matter}}$ with respect to $\omega$ gives a new quantity: the hypermomentum tensor $H$ that accommodates certain kinds of matter fields that give rise to non-Riemannian contributions.

The third term $S_{\text{LM}}$ is the Lagrange multiplier term, defined as follows:

$$S_{\text{LM}} = \int_{M} \chi^\mu T_\mu \text{vol},$$

with $\chi^\mu$ is the Lagrange multiplier corresponding to the traceless torsion constraint [8, 11, 12]:

$$T_\mu = T_\alpha^\alpha_\mu = 0.$$ (33)

The traceless torsion constraint is introduced to guarantee the consistency of the equations of motion under the projective invariance transformation [8, 11, 12]. Leaving the connection $\omega$ unconstrained gives rise to unspecified degrees of freedom in the equation of motion: in the Palatini formulation, the minimization of the $\int \mathcal{R}[\omega] \text{vol} + S_{\text{matter}}[g]$ with respect to $\omega$ gives the Palatini tensor $P_{\alpha\beta}^\mu = 0$ [8, 11]. However the Palatini tensor is traceless: $P_{\alpha\beta}^\mu = 0$ [8, 14]. Consequently, in four-dimension, one could only obtain 60 independent equations from $P_{\alpha\beta}^\mu = 0$, which are not enough to specify the 64 independent variables of a general affine connection [8, 14, 28]. The maximal condition on the connection one could obtain from $P_{\alpha\beta}^\mu = 0$ is [8, 14]:

$$\omega_{\mu}^{\alpha\beta} = \Gamma_{\mu}^{\alpha\beta} = \frac{1}{3} \mathcal{T}_{\lambda}^\mu T^\lambda \delta_\beta^\alpha,$$ (34)

with $\Gamma_{\mu}^{\alpha\beta}$ is the Levi-Civita connection and $T_\mu$ is the torsion vectorial degrees of freedom. These unspecified degrees of freedom could be eliminated by introducing the projective transformation on the connection [8, 12, 14]:

$$\omega_{\mu}^{\alpha\beta} \rightarrow \omega_{\mu}^{\alpha\beta} + \delta_\beta^\alpha \xi_\mu,$$ (35)

which preserves the Ricci scalar $\mathcal{R}$. By setting $\xi_\mu = \frac{1}{3} T_\mu$ and performing the transformation to (34), then:

$$\omega_{\mu}^{\alpha\beta} = \Gamma_{\mu}^{\alpha\beta},$$

is the solution to the equation $P_{\mu}^{\alpha\beta} = 0$, which matches the result of the standard GR. However, as a consequence of performing (35) to $\omega$, the connection is now constrained such that its torsional part is traceless, i.e. satisfying (33). As discovered in [13, 14], (33) is not the only possible condition to guarantee the consistency of the equations of motion under (35), one could equally choose the Weyl tensor $Q_{\mu} = 0$, or moreover, the linear combination between $Q_{\mu}$ and $T_\mu$ to be zero. In this article, we choose the constraint $T_\mu = 0$, which is frequently used in the literature. The treatment where $Q_{\mu} = 0$ could be found elsewhere. For a more detailed explanation of the projective invariant problem, one could consult [8, 11, 12, 14, 28].

The last term in (29), $S_{\text{sym}}$, is another Lagrange multiplier term related to another ‘constraint’ of the system, namely, the symmetry of the metric tensor. One should note that for a general affine connection, the Ricci tensor (31) is not symmetric by the interchange of the $(\nu\beta)$ indices; in other words, the Ricci tensor has an antisymmetric part $R_{\nu\beta} = R_{(\nu}\beta) + R_{\nu[\beta]}$ [15] (here respectively, we use $(,)$ and $[,]$ to label symmetry and antisymmetry). However, this antisymmetric part does not contribute to the Ricci scalar $\mathcal{R}$ since the contraction of $R_{(\nu}\beta)$ with $g^{\nu\beta}$ is always zero. This is due to the fact that the metric is always symmetric,
namely $g^{[\nu \beta]} = 0$. To obtain consistent equations of motion, one needs as well to consider this constraint as follows:

$$S_{\text{sym}} = \int_M \zeta_{\alpha \beta} g^{[\alpha \beta]} \text{vol},$$

with $\zeta$ is the Lagrange multiplier corresponding to the constraint $g^{[\alpha \beta]} = 0$.

### 3.2. The stress–energy–momentum equation

In this subsection, we will briefly review the derivation of the stress–energy–momentum equation which arise from the minimization of $S$ with respect to the metric $g$. As mentioned previously, there are some minor modifications on the derivations in this article; however, these do not alter the physical interpretations of the theory. Other variations in deriving the results could be found on the literature [8, 11, 12].

The variation of a functional $G[x]$ with respect to a variable $x$ is defined as [44]:

$$\delta x G = \left. \frac{d}{ds} G[x + s \delta x] \right|_{s=0}. \quad (36)$$

With this relation in mind, we will derive all the equations of motions of the action (29). The variation of the action $S$ with respect to $g$, assuming the variations of $\omega, \chi$, and $\zeta$ are zero, gives:

$$\delta g S = \int_M \left( \delta f (R[\omega]) \right) \text{vol} + f (R[\omega]) \delta g \text{vol} + \delta g S_{\text{matter}} [g, \omega] + \delta g S_{\text{LM}} + \delta g S_{\text{sym}}. \quad (37)$$

The variation of $f(R)$, with (36) and the chain rule, gives $\delta f (R) = f' (R) \delta g R$, with $f' (R) = \frac{d f}{d R}$ [12]. Next, the variation of the Ricci scalar with respect to $g$ is $\delta g R = (\delta g^{[\alpha \beta]} R_{\alpha \beta})$, where the term $\delta g R_{\alpha \beta} = 0$, since for a general affine connection, the Ricci tensor $R_{\alpha \beta}$ is only a function of $\omega$ [44]. Therefore, by the variation of $\delta g$, we have:

$$\delta g f (R) = f' (R) R_{\alpha \beta} \delta g^{[\alpha \beta]}. \quad (39)$$

The variation of $S_{\text{matter}} [g, \omega]$ is defined as [8, 11, 12]:

$$\delta g S_{\text{matter}} [g, \omega] := - \int_M \kappa T_{\alpha \beta} \delta g^{[\alpha \beta]} \text{vol}, \quad (40)$$

with $T$ (not to be confused with the torsion $T$) is the standard stress–energy–momentum tensor, symmetric in the $(\alpha, \beta)$ indices [12].

The variation of $S_{\text{LM}}$ with respect to $g$ is:

$$\delta g S_{\text{LM}} = \int_M \left( \delta g \chi^\mu \right) T_{\alpha \mu}^\alpha \text{vol} + \int_M \chi^\mu \left( \delta g T_{\alpha \mu}^\alpha \right) \text{vol} + \int_M \chi^\mu T_{\alpha \mu}^\alpha \left( \delta g \text{vol} \right).$$
Both the first and second terms are zero because the Lagrange multiplier $\chi^\mu$ and the trace of torsion $T_\mu$ are independent of the metric $g$. Therefore, using (38):

$$\delta_g S_{LM} = - \frac{1}{2} \int_M \left( \frac{1}{2} \chi^\mu T_\nu \chi^\mu \chi_{\nu\beta} \right) (\delta g^{\alpha\beta}) \text{vol}.$$  \hspace{1cm} (41)

The variation of $S_{sym}$ with respect to $g$ is:

$$\delta_g S_{sym} = \int_M \left( \delta_g \zeta_{\alpha\beta} \right) g^{\alpha\beta} \text{vol} + \int_M \left( \delta g^{\alpha\beta} \right) \zeta_{\alpha\beta} g^{\beta\gamma} \left( \delta_g g_{\gamma\delta} \right) \text{vol}.$$  \hspace{1cm} (42)

The first term is zero because the Lagrange multiplier $\zeta_{\alpha\beta}$ is independent from the metric $g$. The second term is:

$$\delta_g g^{\alpha\beta} = \frac{1}{2} \delta_g \left( g^{\alpha\beta} - g^{\beta\alpha} \right) = \frac{1}{2} \left( \delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu \right) \delta g^{\mu\nu}.$$  \hspace{1cm} (43)

Finally, inserting (38)–(42) to (37) gives:

$$\delta_g S = \int_M \left( f'(R) R_{\alpha\beta} - \frac{1}{2} f(R) g_{\alpha\beta} - \kappa T_{\alpha\beta} - \frac{1}{2} \left( \chi^\mu T_\nu \chi^\mu + \zeta_{\mu\nu} \delta^{[\mu\nu]} \right) g_{\alpha\beta} + \zeta_{\alpha\beta} \right) \delta g^{\alpha\beta} \text{vol}.$$  \hspace{1cm} (44)

$G$ is the generalized Einstein tensor, and (43) is the stress–energy–momentum equation of metric-affine $f(R)$-gravity. The stress–energy–momentum equation in [8, 11, 12] does not explicitly contain the term $- \frac{1}{2} \chi^\mu T_\nu \chi^\mu g_{\alpha\beta}$ in the generalized Einstein tensor $G_{\alpha\beta}$. Without these terms, the corresponding stress–energy–momentum equation will give rise to another (unnecessary) constraint: $G_{\alpha\beta} = 0$, since $T_{\alpha\beta}$ is always symmetric. To prevent this unnecessary situation without leaving the scope of metric-affine $f(R)$-gravity, one needs to consider $S_{sym}$ in the complete action; this will be discussed in the next sections.

3.3. The hypermomentum equation

This subsection will contain the derivation of the hypermomentum equation. This second equation of motion could be obtained from the variation of $S$ with respect to the affine connection $\omega$, $\delta_\omega S$, now assuming the variation of $g$, $\chi$, and $\zeta$ are zero:

$$\delta_\omega S = \int_M \left( \delta_\omega f(R[\omega]) \right) \text{vol} + f(R[\omega]) \delta_\omega \text{vol} + \delta_\omega S_{\text{matter}}[g, \omega] + \delta_\omega S_{LM} + \delta_\omega S_{\text{sym}}.$$  \hspace{1cm} (44)
We will carry the derivation of the hypermomentum equation differently from the standard derivation in [11, 12] to have a simpler form for the $(3 + 1)$ decomposition. As we had mentioned in the previous derivation, $\delta\omega$ vol is zero since it is only a function of $g$. Similar with (39), we have:

$$\delta\omega f(\mathcal{R}) = f'(\mathcal{R}) \delta\omega \mathcal{R}. \tag{45}$$

The quantity $\delta\omega \mathcal{R}$ could be derived as follows. With definition (30), one could show that the variation of Ricci tensor (31) with respect to $\omega$ is:

$$\delta\omega R_{\alpha\beta} = \nabla_\mu \delta\omega_\alpha^{\mu} \beta - \nabla_\alpha \delta\omega_\mu^{\mu} _\beta + T^\sigma_{\mu\sigma} \delta\omega_\sigma^{\mu} _\beta, \tag{46}$$

where $\nabla$ acts on all the indices of $\omega$. Since $\omega$ is independent of $g$, we have:

$$\delta\omega \mathcal{R} = \delta\omega g^{\alpha\beta} R_{\alpha\beta} = g^{\alpha\beta} \delta\omega R_{\alpha\beta} = g^{\alpha\beta} \left( \nabla_\mu \delta\omega_\alpha^{\mu} \beta - \nabla_\alpha \delta\omega_\mu^{\mu} _\beta + T^\sigma_{\mu\sigma} \delta\omega_\sigma^{\mu} _\beta \right). \tag{47}$$

Inserting (47) to (45) gives:

$$\delta\omega f(\mathcal{R}) = f'(\mathcal{R}) g^{\alpha\beta} \nabla_\mu \delta\omega_\alpha^{\mu} \beta - f'(\mathcal{R}) g^{\alpha\beta} \nabla_\alpha \delta\omega_\mu^{\mu} _\beta + f'(\mathcal{R}) g^{\alpha\beta} T^\sigma_{\mu\sigma} \delta\omega_\sigma^{\mu} _\beta. \tag{48}$$

Let us focus on the first term of the rhs of (48). We could write the following relation concerning the first term:

$$\nabla_\mu f'(\mathcal{R}) g^{\alpha\beta} \delta\omega_\alpha^{\mu} \beta = \left( \nabla_\mu f'(\mathcal{R}) g^{\alpha\beta} \right) \delta\omega_\alpha^{\mu} \beta + f'(\mathcal{R}) g^{\alpha\beta} \nabla_\mu \delta\omega_\alpha^{\mu} \beta, \tag{49}$$

with $X_\sigma = f'(\mathcal{R}) g^{\alpha\beta} \delta\omega_\alpha^{\mu} \beta$. Moreover, we could have:

$$\nabla_\mu X_\alpha = \partial_\mu X_\alpha + \omega_\alpha^\sigma X_\sigma = \partial_\mu X_\alpha + \Gamma^\sigma_{\mu\sigma} X_\sigma + Q_\mu^{\alpha} X_\sigma + C^\alpha_{\mu\sigma} X_\sigma, \tag{50}$$

where $\tilde{\nabla}$ is the Levi-Civita connection. $Q$ and $C$ are the non-metricity and the torsion parts of $\omega$, known respectively as the disformation/deflection and the contorsion tensor [28]:

$$Q_{\mu\nu\beta} = -\frac{1}{2} \left( \nabla_\mu g_{\nu\beta} - \nabla_\nu g_{\mu\beta} + \nabla_\beta g_{\mu\nu} \right), \tag{51}$$

$$C_{\mu\nu\beta} = \frac{1}{2} \left( T_{\mu\nu\beta} + T_{\nu\beta\mu} - T_{\beta\mu\nu} \right). \tag{52}$$

Using (50), we have:

$$\nabla_\mu X_\mu = \tilde{\nabla}_\mu X_\mu + (Q_\mu^{\mu} + C^\mu_{\mu\sigma}) X_\sigma, \tag{53}$$

where we use $\tilde{\nabla}_\mu X_\mu = \tilde{\nabla}_\mu X_\mu$ using the fact that $\tilde{\nabla}_\mu g^{\alpha\beta} = 0$ for the Levi-Civita connection. With (49) and (53), we obtain:

$$f'(\mathcal{R}) g^{\alpha\beta} \nabla_\mu \delta\omega_\alpha^{\mu} \beta = \tilde{\nabla}_\mu Y_\alpha + \left( Q_\alpha^{\alpha} + C^\alpha_{\alpha\sigma} \right) Y_\sigma - \left( \nabla_\alpha f'(\mathcal{R}) g^{\alpha\beta} \right) \delta\omega_\sigma^{\mu} _\beta. \tag{54}$$

Doing the same way with the second term in the rhs of (48), we have:

$$f'(\mathcal{R}) g^{\alpha\beta} \nabla_\alpha \delta\omega_\mu^{\mu} _\beta = \tilde{\nabla}_\alpha Y_\alpha + \left( Q_\alpha^{\alpha} + C^\alpha_{\alpha\sigma} \right) Y_\sigma - \left( \nabla_\alpha f'(\mathcal{R}) g^{\alpha\beta} \right) \delta\omega_\sigma^{\mu} _\beta. \tag{55}$$
where $X^\sigma = f'(R) g^{\alpha\beta} \delta\omega^\alpha_{\beta}$ and $Y^\sigma = f'(R) g^{\alpha\beta} \delta\omega^\mu_{\mu \beta}$. Inserting (54) and (55) to (48) gives:

\[
\begin{align*}
\delta_w f (R) &= \tilde{\nabla}^\mu X_\mu + \left( Q^\sigma_{\mu \sigma} + C_{\mu \sigma} \right) X^\sigma - \left( \nabla_\mu f'(R) g^{\alpha \beta} \right) \delta\omega^\mu_{\mu \beta} \\
&\quad - \tilde{\nabla}^\sigma Y_\sigma - \left( Q^\alpha_{\sigma \alpha} + C^\alpha_{\sigma \alpha} \right) Y^\sigma + \left( \nabla_\alpha f'(R) g^{\alpha \beta} \right) \delta\omega^\mu_{\mu \beta} \\
&\quad + f'(R) g^{\alpha \beta} T^\alpha_{\mu \alpha} \delta\omega^\mu_{\mu \beta}.
\end{align*}
\]

Changing the dummy indices and performing some tensor algebra straightforwardly, (56) could be written as:

\[
\delta_w f (R) = \tilde{\nabla}^\mu \left( X_\mu - Y_\mu \right) + \left( Q^\lambda_{\mu \lambda} + C^\lambda_{\mu \lambda} - \nabla_\nu \right) 2 \delta^\mu_{\nu \gamma} \tilde{g}^{\alpha \beta} + T^\alpha_{\mu \alpha} f'(R) \delta \omega^\mu_{\mu \beta}.
\]

Let us keep the result for the moment and focus on the remaining terms in relation (44). The third term in the rhs of (44) is the variation of the matter Lagrangian with respect to the connection:

\[
\delta_w S_{\text{matter}} [g, \omega] := -\int_M \kappa H^\alpha_{\mu \beta} \delta\omega^\mu_{\mu \beta} \text{vol}.
\]

$H$ is the hypermomentum, which vanishes for standard matter in the scope of original GR. Non-vanishing hypermomentum, which can only occur in modified GR, generates non-Riemannian spacetime containing torsion and non-metricity. Certain kinds of matter that contain hypermomentum will have internal degrees of freedom, such as spins [40–43]. In fact, a hypothetical matter carrying hypermomentum called hyperfluid could be constructed from the continuity equation of the MAG model [48].

The fourth term in the rhs of (44) is the $S_{LM}$ part:

\[
\delta_w S_{LM} = \int_M \left( \chi^\mu T^\alpha_{\alpha \mu} \text{vol} \right) = \int_M \left( \delta_w \chi^\mu \right) T^\alpha_{\alpha \mu} \text{vol} + \int_M \chi^\mu \left( \delta_w T^\alpha_{\alpha \mu} \right) \text{vol}
\]

\[
+ \int_M \chi^\mu T^\alpha_{\alpha \mu} \left( \delta_w \text{vol} \right),
\]

since $\chi$ and (vol) are independent of $\omega$. One could easily obtain $\delta_w T^\alpha_{\alpha \mu}$ from the definition of torsion tensor in (9): $\delta_w T^\alpha_{\alpha \mu} = \left( \delta^\alpha_{\mu \beta} - \delta^\beta_{\mu \alpha} \right) \delta\omega^\mu_{\mu \beta}$, which could be inserted to (59) to give:

\[
\delta_w S_{LM} = \int_M \left( \chi^\alpha \delta^\beta_{\mu \sigma} - \chi^\beta \delta^\sigma_{\mu \alpha} \right) \delta\omega^\mu_{\mu \beta} \text{vol}.
\]

The last term in the rhs of (44) is $S_{sym}$:

\[
\delta_w S_{sym} = \int_M \left( \zeta^\alpha g^{\alpha \beta} \text{vol} \right) = 0,
\]

since $\zeta$, $g$ and (vol) are independent of $\omega$.

Inserting (57), (58), (60), and (61) to (44) gives:

\[
\delta_w S = \int_M \tilde{\nabla}^\mu \left( X_\mu - Y_\mu \right) \text{vol} + \int_M \left( \nabla_\nu f'(R) g^{\alpha \beta} \right) \delta\omega^\mu_{\mu \beta} + f'(R) \delta \omega^\mu_{\mu \beta} - \kappa H^\alpha_{\mu \beta} \left( \chi^\alpha \delta^\beta_{\mu \sigma} - \chi^\beta \delta^\sigma_{\mu \alpha} \right) \delta\omega^\mu_{\mu \beta} \text{vol}.
\]
The first term in the rhs of (62) is a total divergence, which is equal to zero if $\mathcal{M}$ is compact [44]; let us apply this case for simplicity. Then one is left with the second, third, and last terms. The Euler–Lagrange equation, which minimizes the action (29) for any variation of the connection, is then:

$$
\left(\left\{Q^\lambda_\nu + C^\lambda_\nu - \nabla_\nu \right\} (2\delta^\alpha_\mu g^{\alpha\beta}) + T^\alpha_\mu \right) f'(\mathcal{R}) = \kappa H^\alpha_\mu + \chi^\alpha_\mu \delta^\alpha_\mu - \chi^\beta_\mu \delta^\alpha_\mu. \quad (63)
$$

Notice that the covariant derivative $\nabla_\nu$ acts on both $\delta^\rho_\mu g^{\alpha\beta}$ and $f'(\mathcal{R})$. Moreover, using (51) and (52), and using the symmetries of the disformation $\mathcal{Q}$ and contortion $C$ [28], (63) becomes:

$$
\left(\left\{T^\lambda_\mu - \frac{1}{2} S^{\alpha\lambda}_\mu \nabla_\nu g_{\sigma\lambda} - \nabla_\nu \right\} (2\delta^\alpha_\mu g^{\alpha\beta}) + T^\alpha_\mu \right) f'(\mathcal{R}) = \kappa H^\alpha_\mu + 2\chi^\alpha_\mu \delta^\alpha_\mu. \quad (64)
$$

$P$ is known as the Palatini tensor, and (64) is the hypermomentum equation of the metric-affine $f(\mathcal{R})$-gravity.

One could show that (64) is equivalent to the original EL equation in the standard derivation in [11, 12] as follows. Breaking the antisymmetric part explicitly into its component, (64) could be written as:

$$
\left(\left\{-\frac{1}{2} \left( g^{\sigma\lambda} \nabla_\nu g_{\sigma\lambda} \right) g^{\alpha\beta} - \nabla_\nu g^{\alpha\beta} \right\} \delta^\rho_\mu + \frac{1}{2} \left( g^{\sigma\lambda} \nabla_\nu g_{\sigma\lambda} \right) g^{\mu\nu} \nabla_\nu g^{\alpha\beta} \right) f'(\mathcal{R})
\quad + \left( T^\lambda_\mu - \frac{1}{2} \left( g^{\sigma\lambda} \nabla_\nu g_{\sigma\lambda} \right) g^{\mu\nu} \nabla_\nu g^{\alpha\beta} \right) f'(\mathcal{R}) = \kappa H^\alpha_\mu + \chi^\alpha_\mu \delta^\alpha_\mu - \chi^\beta_\mu \delta^\alpha_\mu. \quad (65)
$$

Using the fact that $\nabla_\nu g = g g^{\alpha\beta} \nabla_\nu g_{\alpha\beta}$, where $g$ is the determinant of $g_{\alpha\beta}$, one could obtain:

$$
\nabla_\nu g = \frac{1}{2} g^{\alpha\lambda} \nabla_\nu g_{\sigma\lambda} + \nabla_\mu \right) g^{\alpha\beta}. \quad (66)
$$

Inserting (66) to (65) gives:

$$
\left( -\left( \nabla_\nu \sqrt{-g} g^{\alpha\beta} \right) \delta^\rho_\mu + \frac{1}{2} \left( \nabla_\nu \sqrt{-g} g^{\alpha\beta} \right) + \sqrt{-g} T^\lambda_\mu g^{\alpha\beta} - \sqrt{-g} T^\lambda_\mu g^{\mu\nu} \nabla_\nu g^{\alpha\beta} \right) f'(\mathcal{R})
\quad = \sqrt{-g} \left( \kappa H^\alpha_\mu + 2\chi^\alpha_\mu \delta^\alpha_\mu \right). \quad (65)
$$

which is equivalent to the standard form of hypermomentum equation in [11, 12], up to the factor 2, due to the difference in the definition of torsion tensor (9).

3.4. The traceless torsion and symmetry constraints

There are two remaining equations of motion that arise from the variation of $S$ with respect to the Lagrange multipliers. The variation of $S$ with respect to $\chi$ gives the traceless torsion constraint $T^\mu_\lambda = 0$. It should be kept in mind that this equation does not emerge from the dynamics, but as we had mentioned previously, it is introduced at the kinematical level to remove the inconsistencies that arise under the projective invariant transformation [8, 12, 14].

The last equation of motion is obtained from variation of $S$ with respect to $\zeta$; this gives the symmetry of the metric: $g^{\alpha\beta} = 0$. Similar to the previous constraint, this equation does not emerge from the dynamics, but it is introduced to guarantee the consistency of the equations of motion that arise under an a priori assumption that the metric in metric-affine $f(\mathcal{R})$-gravity is always symmetric.
To conclude this section, we collect all the equations of motion we had derived previously, i.e. (43) and (64), together with the two constraints. Therefore, the variation of $\mathcal{S}$ with respect to $g$, $\omega$, $\chi$, and $\zeta$, gives 4 equations of motion:

\[
\begin{align*}
  f'(R)R_{\alpha\beta} - \frac{1}{2}(f(R) + \chi^{\mu}T^{\lambda}_{\lambda\mu} + \zeta_{\mu\nu}g^{\mu\nu})g_{\alpha\beta} + \zeta_{[\alpha\beta]} &= \kappa T_{\alpha\beta}, \tag{67} \\
  (T_{\lambda\nu}^{\lambda} + Q_{\lambda\nu}^{\lambda} - \nabla_{\nu})\left(2\delta^{[\nu}_{\mu}g^{\alpha\beta]} + T^{\alpha\beta}_{\mu}\right) f'(R) &= \kappa H^{\alpha\beta} + 2\lambda^{[\alpha\beta]}, \tag{68} \\
  T_{\lambda\mu}^{\lambda} &= 0, \tag{69} \\
  g^{[\alpha\beta]} &= 0. \tag{70}
\end{align*}
\]

Let us solve (68) for $\chi$, following [12]. Contracting the indices ($\beta$, $\mu$) gives $\chi_{\alpha} = -\frac{1}{3}\kappa H^{\alpha\beta}_{\beta}$. On the other hand, from the symmetries of (67), we could deduce that $\zeta_{[\alpha\beta]} \equiv -R_{[\alpha\beta]}$, since $g_{\alpha\beta}$ and $T_{\alpha\beta}$ are always symmetric. Inserting these results together with (69) and (70) to (67) and (68) gives the simplified version of the equations of motion that could be found in the literature, for example [8, 11, 12]:

\[
\begin{align*}
  f'(R)R_{\alpha\beta} - \frac{1}{2}f(R)g_{\alpha\beta} &= \kappa T_{\alpha\beta}, \tag{71} \\
  (Q_{\lambda\nu}^{\lambda} - \nabla_{\nu})\left(2\delta^{[\nu}_{\mu}g^{\alpha\beta]} + T^{\alpha\beta}_{\mu}\right) f'(R) &= \kappa H^{\alpha\beta} - \frac{2}{3}\kappa H^{[\alpha}_{\mu}\delta^{\beta]}_{\mu}, \tag{72} \\
  T_{\lambda\mu}^{\lambda} &= 0, \tag{73}
\end{align*}
\]

where we omit the last equation $g^{[\alpha\beta]} = 0$ since we are always working with a symmetric metric $g$.

3.5. Special case: metric-affine GR (MAGR)

The theory of MAGR (or generalized Palatini gravity) is a special case of metric-affine $f(R)$-gravity for $f(R) = R$. With this requirement, (71)–(73) becomes:

\[
\begin{align*}
  R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} &= \kappa T_{\alpha\beta}, \tag{74} \\
  (Q_{\lambda\nu}^{\lambda} - \nabla_{\nu})\left(2\delta^{[\nu}_{\mu}g^{\alpha\beta]} + T^{\alpha\beta}_{\mu}\right) &= \kappa \left(H_{\mu}^{\alpha\beta} - \frac{2}{3}H^{[\alpha}_{\mu}\delta^{\beta]}_{\mu}\right), \tag{75} \\
  T_{\lambda\mu}^{\lambda} &= 0. \tag{76}
\end{align*}
\]

As the $(3+1)$ decomposition of the stress–energy–momentum equation (74) had been done in [28], in the next part of this article, we will focus on the $(3+1)$ decomposition of the hypermomentum equation (75).

4. $(3+1)$ hypermomentum equation

In this section, we will perform the $(3+1)$ decomposition to the hypermomentum equation (75). We divide the task into two steps: first, to split the hypermomentum tensor into its temporal and spatial part, and second, to perform the ADM decomposition to the hypermomentum equation.
4.1. \((3 + 1)\) decomposition of hypermomentum tensor

4.1.1. The dilation, shear, and spin (rotation) parts of the hypermomentum. The hypermomentum tensor derived in (58), is a \( \ell^2 \)-order tensor \( \mathcal{H} = \mathcal{H}^{\mu \beta \alpha \nu} \partial_\mu \otimes d x^\alpha \otimes \partial_\nu \). Adapting the terminology of gauge theory, let us called \( \mu \) as the spacetime index, and \( (\alpha, \beta) \) as the ‘internal’ indices of \( \mathcal{H} \). One could consider \( \mathcal{H} \) as a matrix-valued vector, where their ‘internal’ part acts on a pair of \( (Y, Z^*) \in T_p \mathcal{M} \times T_p^* \mathcal{M} \) as follows [28]:

\[
\mathcal{H} (Y, Z^*) = \mathcal{H} (Y^\alpha \partial_\alpha, Z^\beta d x^\beta) = Y^\alpha Z^\beta \mathcal{H} (\partial_\alpha, d x^\beta). \tag{77}
\]

Notice that \( \mathcal{H} (\partial_\alpha, d x^\beta) \) is a vector, namely:

\[
\mathcal{H} (\partial_\alpha, d x^\beta) = H^{\mu \alpha \beta} \partial_\mu. \tag{78}
\]

In terms of free-index notations, this could be written as \( \mathcal{H}^* (Y, Z) \in T_p \mathcal{M} \):

\[
\mathcal{H} (Y, Z^*) = \mathcal{H} (Y, g (Z)) := \mathcal{H}^* (Y, Z) = Y^\alpha Z^\beta \mathcal{H}^* (\partial_\alpha, \partial_\beta), \quad Z \in T_p \mathcal{M}. \tag{79}
\]

As proposed in [4], the hypermomentum could be decomposed according to the symmetries of its ‘internal’ indices as \( \mathcal{H}^{\mu \alpha \beta} = \mathcal{H}^{\mu (\alpha \beta)} + \mathcal{H}^{\mu [\alpha \beta]} \). The antisymmetric part is defined as the spin (rotation) of the hypermomentum:

\[
\mathcal{H}^{\mu [\alpha \beta]} := \Omega^{\mu \alpha \beta},
\]

while the symmetric part could be furthermore decomposed as \( \mathcal{H}^{\mu (\alpha \beta)} = \frac{1}{2} g_{\alpha \beta} D^\mu + \Lambda^{\mu \alpha \beta} \), with \( d \) is the dimension of the manifold. \( D^\mu \) is the dilation vector:

\[
D^\mu := \text{tr} \mathcal{H}^\mu = \mathcal{H}^\mu \nu \nu,
\]

and \( \Lambda^{\mu \alpha \beta} \) as the shear of the hypermomentum:

\[
\Lambda^{\mu \alpha \beta} := \mathcal{H}^{\mu (\alpha \beta)} - \frac{1}{2} g_{\alpha \beta} \mathcal{H}^{\mu \nu \nu}.
\]

Notice that these symmetries decompositions are not defined for \( \mathcal{H}^{\mu \alpha \beta} \), but instead for \( \mathcal{H}^{\mu \alpha \beta} \) with the last index lowered in (78). Using the relation in (79), then one could obtain, for any \( X^* \in T_p^* \mathcal{M} \):

\[
\langle X^* , \mathcal{H} (Y, Z^*) \rangle = \langle X^* , \mathcal{H}^* (Y, Z) \rangle = \frac{1}{4} \langle X^* , D \rangle \ g (Y, Z) + \langle X^* , \Lambda (Y, Z) \rangle + \langle X^* , \Omega (Y, Z) \rangle, \quad \tag{80}
\]

or in indices:

\[
\langle X^\mu, \mathcal{H} (Y, Z^\nu) \rangle = Y^\alpha Z_\beta \mathcal{H}^{\mu \alpha \beta} X_\mu = X_\mu Y^\alpha Z^\beta \mathcal{H}^{\mu \alpha \beta} = X_\mu Y^\alpha Z^\beta \left( \frac{1}{4} g_{\alpha \beta} D^\mu + \Lambda^{\mu \alpha \beta} + \Omega^{\mu \alpha \beta} \right). \]

Here, we insert the dimension \( d = 4 \).
The dilation $\mathcal{D} = D^\mu \partial_\mu$ is a (scalar valued) four-vector, while the shear $\Lambda = \Lambda^\alpha_{\alpha \beta} \partial_\mu \otimes dx^\alpha \otimes dx^\beta$ and the rotation $\Omega = \Omega^\mu_{\alpha \beta} \partial_\mu \otimes dx^\alpha \otimes dx^\beta$ are matrix-valued four-vectors, satisfying the following symmetries:

$$\Lambda (Y, Z) = \Lambda (Z, Y), \quad \Lambda^\alpha_{\alpha \beta} = \Lambda^\beta_{\beta \alpha}, \quad (81)$$

$$\Omega (Y, Z) = -\Omega (Z, Y), \quad \Omega^\mu_{\alpha \beta} = -\Omega^{\mu \beta}_{\alpha \alpha}, \quad (82)$$

Applying the concept of geometrodynamics by Wheeler [45], with the help of symmetries (81) and (82), we could construct the hypermomentum $\mathcal{H}$ by using 10 hypersurface variables as follows.

**Dilation $(3 + 1)$ decomposition.** The dilation $\mathcal{D}$ is a four-vector, hence it could be decomposed into the (scalar) temporal dilation $\langle \dot{n}^* \dot{n}^* \rangle \dot{n}$ and (three-vector) spatial dilation $\langle 3d \dot{x}^j, \mathcal{D} \rangle$:

$$\mathcal{D} = -\langle \dot{n}^* \mathcal{D} \rangle \dot{n} + \langle 3d \dot{x}^j, \mathcal{D} \rangle \partial_j. \quad (83)$$

We usually write $\mathcal{D} = \partial_\mu \partial^\mu n$ as $3 \dot{n}^*$. Hence, the four-vector dilation $\mathcal{D}$ could be constructed from 2 hypersurface variables, namely, the scalar $\partial_\mu \partial^\mu n$ and three-vector $3 \dot{n}^*$.

**Shear $(3 + 1)$ decomposition.** The shear $\Lambda$, which is a traceless, symmetric matrix-valued four-vector, could be decomposed into:

$$\Lambda (\dot{n}, \dot{n}) = \text{tr} \sigma = \frac{1}{3} q^i \sigma_{ij}, \quad (84)$$

$$\Lambda (\partial_i, \partial_i) = \Lambda (\dot{\partial}_i, \dot{n}) := \tau_i, \quad (85)$$

$$\Lambda (\partial_i, \partial_j) = \Lambda (\dot{\partial}_i, \dot{\partial}_j) := \sigma_{ij}. \quad (86)$$

The condition (84) could be obtained from the tracelessness of $\Lambda$:

$$\text{tr} \Lambda = g^* \langle \dot{n}^* \dot{n}^* \rangle \Lambda (\dot{n}, \dot{n}) + 2 g^* \langle \dot{n}^*, 3d \dot{x} \rangle \Lambda (\dot{n}, \partial_i) + g^* \langle 3d \dot{x}^i, 3d \dot{x} \rangle \Lambda (\partial_i, \partial_j). \quad (87)$$

With definition (86) and using the fact that $g^* \langle \dot{n}^* \dot{n}^* \rangle = -1, g^* \langle \dot{n}^*, 3d \dot{x} \rangle = 0$, and $g^* \langle 3d \dot{x}^i, 3d \dot{x} \rangle = 3 q^i$, one could obtained (84). Note that $\tau$ and $\sigma$ are respectively, a vector-valued and matrix-valued four-vector, therefore, they could be furthermore decomposed as follows:

$$\Lambda (\dot{n}, \dot{n}) = \text{tr} \sigma = -\langle \dot{n}^* \text{tr} \sigma \rangle \dot{n} + \langle 3d \dot{x}^j, \text{tr} \sigma \rangle \partial_j, \quad (88)$$

$$\Lambda (\partial_i, \dot{n}) = \Lambda (\partial_i, \dot{n}) = \tau_i = -\langle \dot{n}^*, \tau_i \rangle \dot{n} + \langle 3d \dot{x}^i, \tau_i \rangle \partial_j, \quad (89)$$

$$\Lambda (\partial_i, \partial_j) = \Lambda (\partial_i, \partial_i) = \sigma_{ij} = -\langle \dot{n}^*, \sigma_{ij} \rangle \dot{n} + \langle 3d \dot{x}^i, \sigma_{ij} \rangle \partial_j. \quad (90)$$

$\sigma_{ij}^k$ is known as the three-dimensional spatial shear tensor. The shear tensor $\Lambda$ could be constructed from 4 hypersurface variables, namely, the three-vector $\langle \dot{n}^*, \tau_i \rangle$, the matrices $\tau_i^j$ and $\langle \dot{n}^*, \sigma_{ij} \rangle$, and the $\langle \mathbf{1} \rangle$-tensor $\sigma_{ij}^k$. Note that $\langle \dot{n}^*, \text{tr} \sigma \rangle$ and $\langle \text{tr} \sigma \rangle^k$ could be obtained by contracting $\langle \dot{n}^*, \sigma_{ij} \rangle$ and $\sigma_{ij}^k$ with $3 q^i$. 

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Spin \((3 + 1)\) decomposition. Finally, the spin \(\Omega\), which is an antisymmetric-valued four-vector, is decomposed into:

\[
\Omega (\hat{n}, \hat{n}) = 0, \tag{87}
\]
\[
\Omega (\hat{n}, \partial_i) = -\Omega (\partial_i, \hat{n}) = k_i, \tag{88}
\]
\[
\Omega (\partial_i, \partial_j) = -\Omega (\partial_j, \partial_i) = \varepsilon_{ijk} l^k. \tag{89}
\]

Condition (87) is a consequence of the antisymmetricity of \(\Omega\). \(k\), which is a vector-valued four-vector, is usually known as the ‘boost’ part of a four-dimensional rotation, while \(l^k\), also a vector valued four-vector, is usually known as the 3D rotation part. They could be furthermore decomposed into:

\[
\Omega (\hat{n}, \partial_i) = -\Omega (\partial_i, \hat{n}) = k_i = -\langle \hat{n}^*, k_i \rangle \hat{n} + \langle 3 dx^l, k_i \rangle \partial_j, \]
\[
\Omega (\partial_i, \partial_j) = -\Omega (\partial_j, \partial_i) = \varepsilon_{ijk} l^k = -\varepsilon_{ijk} \langle \hat{n}^*, l^k \rangle \hat{n} + \varepsilon_{ijk} \langle 3 dx^m, l^k \rangle \partial_m.
\]

The rotation bivector \(\Omega\) then could be constructed from 4 hypersurface variables, namely, the three-vectors \(\langle \hat{n}^*, k_i \rangle\) and \(\langle \hat{n}^*, l^k \rangle\), and the matrices \(k_i^l\) and \(l^i_{\mu\nu}\).

To conclude this subsection, the hypermomentum \(H\) could be constructed completely from these ten hypersurface variables: \(D = (D(\hat{n}), 3D)\), \(\tau_i = \left(-\langle \hat{n}^*, \tau_i \rangle, \tau_i^j \right), \sigma_{ij} = \left(-\langle \hat{n}^*, \sigma_{ij} \rangle, \sigma_{ij}^k \right), k_i = \left(-\langle \hat{n}^*, k_i \rangle, k_i^l \right),\) and \(l^k = \left(-\langle \hat{n}^*, l^k \rangle, l^k_{\mu\nu}\right)\).

4.1.2. The hypermomentum \((3 + 1)\) decomposition. Now, with the dilation, shear, and spin decomposition, we could proceed to decompose the hypermomentum \(H\) completely. Using the notation in (77), one could decompose the hypermomentum into its ‘internal’ indices into the normal and parallel parts, with respect to the hypersurface \(\Sigma\) [28]:

\[
H(\hat{n}^*, \hat{n}^*) = -\langle \hat{n}^*, H(\hat{n}^*, \hat{n}^*) \rangle \hat{n} + \langle 3 dx^l, H(\hat{n}^*, \hat{n}^*) \rangle \partial_l, \]
\[
H(\hat{n}^*, 3 dx^i) = -\langle \hat{n}^*, H(\hat{n}^*, 3 dx^i) \rangle \hat{n} + \langle 3 dx^l, H(\hat{n}^*, 3 dx^i) \rangle \partial_l, \]
\[
H(\partial_i, \hat{n}^*) = -\langle \hat{n}^*, H(\partial_i, \hat{n}^*) \rangle \hat{n} + \langle 3 dx^l, H(\partial_i, \hat{n}^*) \rangle \partial_l, \]
\[
H(\partial_i, 3 dx^j) = -\langle \hat{n}^*, H(\partial_i, 3 dx^j) \rangle \hat{n} + \langle 3 dx^l, H(\partial_i, 3 dx^j) \rangle \partial_l. \tag{90}
\]

This is similar to the decomposition of the stress–energy–momentum tensor \(T\) into its 3 components, i.e. the energy \(E\), the momentum \(p_i\), and the stress \(S_{ij}\), where the decomposition is based on the split of the ‘spacetime’ into space and time.

Let us derive an important relation using (79):

\[
\langle X^*, H (Y, 3 dx^j) \rangle = \langle X^*, q^j H^* (Y, \partial_j) \rangle, \tag{91}
\]

which will be used in the following derivation. Now using (80) and (91), together with the help of (79), (81), (82), and the dilation, shear, and rotation split (83), (84)–(86), (87) and (89), the
temporal and spatial components in (90) becomes:

\[
\langle \dot{n}^*, \mathcal{H} (\dot{n}, \dot{n}^*) \rangle = -\frac{1}{4} \langle \dot{n}^*, \mathcal{D} \rangle + \langle \dot{n}^*, \Lambda (\dot{n}, \dot{n}) \rangle = \frac{1}{4} \mathcal{D} (\dot{n}) + \langle \dot{n}^*, \text{tr} \, \sigma \rangle ,
\]

\[
\langle 3 \text{dx}^i, \mathcal{H} (\dot{n}, \dot{n}^*) \rangle = -\frac{1}{4} \langle 3 \text{dx}^i, \mathcal{D} \rangle + \langle 3 \text{dx}^i, \Lambda (\dot{n}, \dot{n}) \rangle = -\frac{1}{4} \mathcal{D} + (\text{tr} \, \sigma)^i \cdot (92) \]

\[
\langle \dot{n}^*, \mathcal{H} (\dot{n}, 3 \text{dx}^i) \rangle = 3 q^{ij} \left( \langle \dot{n}^*, \Lambda (\dot{n}, \partial_j) \rangle + \langle \dot{n}^*, \Omega (\dot{n}, \partial_j) \rangle \right) = 3 q^{ij} (\dot{n}^* \cdot \partial_j + k^j) ,
\]

\[
\langle 3 \text{dx}^i, \mathcal{H} (\dot{n}, 3 \text{dx}^i) \rangle = 3 q^{ik} \left( \langle 3 \text{dx}^i, \Lambda (\dot{n}, \partial_k) \rangle + \langle 3 \text{dx}^i, \Omega (\dot{n}, \partial_k) \rangle \right) = 3 q^{ik} (\dot{\tau}_k + k^k) .
\]

(93)

\[
\langle \dot{n}^*, \mathcal{H} (\partial_j, 3 \text{dx}^i) \rangle = \frac{1}{4} \delta^j_i \langle \dot{n}^*, \mathcal{D} \rangle + 3 q^{ik} \left( \langle \dot{n}^*, \Lambda (\partial_j, \partial_k) \rangle + \langle \dot{n}^*, \Omega (\partial_j, \partial_k) \rangle \right)
\]

\[
= \frac{1}{4} \delta^j_i \mathcal{D} (\dot{n}) + 3 q^{ik} \left( \langle \dot{n}^*, \sigma_{ik} + \varepsilon_{iklm} \mathcal{F}^m \rangle \right) ,
\]

\[
\langle 3 \text{dx}^i, \mathcal{H} (\partial_j, 3 \text{dx}^i) \rangle = \frac{1}{4} \delta^j_i \langle 3 \text{dx}^i, \mathcal{D} \rangle + 3 q^{ik} \left( \langle 3 \text{dx}^i, \Lambda (\partial_j, \partial_k) \rangle + \langle 3 \text{dx}^i, \Omega (\partial_j, \partial_k) \rangle \right)
\]

\[
= \frac{1}{4} \delta^j_i \mathcal{D} + 3 q^{ik} \left( \sigma_{ik} + \varepsilon_{iklm} \mathcal{F}^m \right) .
\]

(95)

(92)–(95) are the complete \((3 + 1)\) decomposition of the hypermomentum tensor \(\mathcal{H}\).

Another important relation is the decomposition of \(\text{tr} \, \mathcal{H} = \mathcal{H} (\partial_\mu, \text{dx}^\nu) = \mathcal{H}^{\nu}_{\mu \nu} \partial_\nu\):

\[
\text{tr} \, \mathcal{H} = - \langle \dot{n}^*, \mathcal{H} (\partial_\mu, \text{dx}^\nu) \rangle \dot{n} + \langle 3 \text{dx}^i, \mathcal{H} (\partial_\mu, \text{dx}^\nu) \rangle \partial_i .
\]

(96)

On the other hand, using (3) and (4), (96) could be written as:

\[
\text{tr} \, \mathcal{H} = - \mathcal{H} (\dot{n}, \dot{n}^*) + \mathcal{H} (\partial_i, 3 \text{dx}^i) .
\]

(97)

Hence, using (97) and (80):

\[
\langle \dot{n}^*, \text{tr} \, \mathcal{H} \rangle = - \mathcal{D} (\dot{n}) - \langle \dot{n}^*, \text{tr} \, \sigma \rangle + \langle \dot{n}^*, 3 \text{tr} \, \Lambda \rangle = - \mathcal{D} (\dot{n}) ,
\]

\[
\langle 3 \text{dx}^i, \text{tr} \, \mathcal{H} \rangle = \mathcal{D} + \langle 3 \text{dx}^i, \text{tr} \, \sigma \rangle + \langle 3 \text{dx}^i, 3 \text{tr} \, \Lambda \rangle = \mathcal{D} ,
\]

(98)

with \(3 \text{tr} \, \Lambda = 3 q^{ij} \Lambda (\partial_i, \partial_j) = \text{tr} \, \sigma\) and \(3 \text{tr} \, \Omega = 3 q^{ij} \Omega (\partial_i, \partial_j) = 0\). The quantities (92), (95) and (98) will be used for the \((3 + 1)\) decomposition of the hypermomentum equation (75).

### 4.2. \((3 + 1)\) decomposition of hypermomentum equation

In this subsection, we will derive the main results in this article, that are the \((3 + 1)\) decomposition to the hypermomentum equation (75) and the traceless torsion constraint (76). 

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4.2.1. The \((3 + 1)\) hypermomentum equation in adapted coordinate. The hypermomentum equation \( (75) \) could be decomposed into 4 equations as follows:

\[
n^n n^\alpha T^\alpha_{\mu} \partial_\alpha - n^g \left( \nabla_g g^{\sigma\rho} \right) \partial_\alpha + n_\beta \left( \nabla_\nu g^{\sigma\rho} \right) \hat{n} = \kappa n^n n^\beta \mathcal{H}^{\alpha \beta} \partial_\alpha - \frac{2}{3} n^n n^\beta \xi \mathcal{H}^{(\alpha)} \sigma \delta_\mu ^{(3)} \partial_\alpha,
\]

\[
= \kappa n^n \mathcal{H}^{,\beta} \partial_\beta - n^n \delta_\beta ^{(2) \mathcal{H}^{(\alpha)} \sigma \delta_\mu ^{(3)} \partial_\alpha,
\]

\[
n^n g^{\beta} T^\alpha_{\mu} \partial_\alpha + \frac{1}{2} (g_{\sigma\lambda} \nabla_{\nu} g^{\sigma\lambda}) (n^n g^{\beta} \partial_\alpha) - (\nabla_\delta g^{\alpha}) \partial_\alpha + (\nabla_\nu g^{\alpha}) \hat{n}
\]

\[
= \kappa n^n \mathcal{H}^{,\beta} \partial_\beta - n^n \delta_\beta ^{(2) \mathcal{H}^{(\alpha)} \sigma \delta_\mu ^{(3)} \partial_\alpha,
\]

\[
n^n T^\alpha_{\mu} \partial_\alpha + \frac{1}{2} (g_{\sigma\lambda} \nabla_{\nu} g^{\sigma\lambda}) (\delta^{\beta \gamma} \hat{n} - n^n \partial_\gamma) - n^\beta \left( \nabla_\nu g^{\alpha} \right) \partial_\alpha + (\nabla_\nu g^{\alpha}) \partial_\alpha + (\nabla_\nu g^{\alpha}) \partial_\nu + (\nabla_\nu g^{\alpha}) \partial_\nu
\]

\[
= \kappa n^n \mathcal{H}^{,\beta} \partial_\beta - n^n \delta_\beta ^{(2) \mathcal{H}^{(\alpha)} \sigma \delta_\mu ^{(3)} \partial_\alpha,
\]

by acting the \(\mu, \beta\) indices in \( (75) \) with four combinations of the temporal \( \hat{n} \) and spatial vector (and covector) basis \( \partial_i \) [28]. Note that by definition \( (77) \), one needs to be careful that the quantity \( \mathcal{H}^{,\beta} \partial_\beta \neq \mathcal{H} (\partial_{\mu}, 3 \partial') \).

Only in the Gauss (normal) coordinate, \( \mathcal{H} (\partial_{\mu}, 3 \partial') = \mathcal{H} (\partial_{\mu}, 3 \partial') \). Using the torsion decomposition \((16)\)–\((19)\) and some useful relations in \((25)\)–\((28)\) and \((99)\) becomes:

\[
T (\hat{n}, \hat{n}) - (\nabla_\delta g^{\alpha}) (\delta^{\beta \gamma} \hat{n}) + \Phi (\hat{n}) \hat{n} = \kappa \left( \mathcal{H} (\hat{n}, \hat{n}) + \frac{1}{3} (\mathcal{T} \mathcal{H} + \langle \hat{n}, \mathcal{T} \mathcal{H} \rangle) \hat{n} \right).
\]

\[
T (\hat{n}, g^{\alpha}) (\delta^{\beta \gamma} \hat{n}) + \left( \Phi' - \frac{1}{2} \Psi' \right) \hat{n} + \frac{1}{2} \Psi (\hat{n}) g^{\alpha} (\delta^{\beta \gamma}) = \kappa \mathcal{H} (\hat{n}, \delta^{\beta \gamma})
\]

\[
= -\frac{1}{3} \left( \langle \delta^{\beta \gamma}, \hat{n} \rangle \mathcal{T} \mathcal{H} - \langle \delta^{\beta \gamma}, \mathcal{T} \mathcal{H} \rangle \hat{n} \right),
\]

\[
T (\partial_{\mu}, \hat{n}) - (\nabla_\delta g^{\alpha}) (\delta^{\beta \gamma} \hat{n}) + \frac{1}{2} \Psi \hat{n} + \left( \Phi (\hat{n}) - \frac{1}{2} \Psi (\hat{n}) \right) \partial_\mu = \kappa \left( \mathcal{H} (\partial_{\mu}, \hat{n}) + \frac{1}{3} (\langle \hat{n}, \mathcal{T} \mathcal{H} \rangle) \partial_\mu \right)
\]

\[
g^{\alpha \beta} T (\partial_{\mu}, \partial_{\nu}) - (\nabla_\delta g^{\alpha}) (\delta^{\beta \gamma} \hat{n}) + \frac{1}{2} \Psi \hat{n} + \left( \Phi' - \frac{1}{2} \Psi' \right) \delta_\mu
\]

\[
= \kappa \left( \mathcal{H} (\partial_{\mu}, \partial_{\nu}) - \frac{1}{3} (\delta_\mu \mathcal{T} \mathcal{H} - \langle \delta_\mu, \mathcal{T} \mathcal{H} \rangle) \partial_\mu \right) \right) \right)
\]

with:

\[
\Phi (\hat{n}) = - (\nabla_\delta g^{\alpha}) (\delta^{\beta \gamma} \hat{n}) = N^{-1} N^{\delta} (\nabla_\delta g^{\alpha}) (\delta^{\beta \gamma} \hat{n}) + (\nabla_\delta g^{\alpha}) (\delta^{\beta \gamma} \hat{n})
\]

\[
\Phi' = - (\nabla_\delta g^{\alpha}) (\delta^{\beta \gamma} \hat{n}) = N^{-1} N^{\delta} (\nabla_\delta g^{\alpha}) (\delta^{\beta \gamma} \hat{n}) + (\nabla_\delta g^{\alpha}) (\delta^{\beta \gamma} \hat{n})
\]
\[
\Psi (\hat{n}) = \left( N^{-2}Nj - 1 \right) \left( \nabla_\text{g}^* \right) \left( \hat{n}^*, \hat{n}^* \right) - 2N^{-1}Nj \left( \nabla_\text{g}^* \right) \left( \hat{n}^*, \text{d}x^i \right)
+ 3 q_{jk} \left( \nabla_\text{g}^* \right) (\text{d}x^j, \text{d}x^k)
\]

\[
\Psi_j = \left( N^{-2}Nj - 1 \right) \left( \nabla_\text{g}^* \right) \left( \hat{n}^*, \hat{n}^* \right) - 2N^{-1}Nj \left( \nabla_\text{g}^* \right) \left( \hat{n}^*, \text{d}x^i \right)
+ 3 q_{jk} \left( \nabla_\text{g}^* \right) (\text{d}x^j, \text{d}x^k)
\]

and \( \Psi^j = 3 q^{ij} \Psi_j \).

Applying the torsion and non-metricity decomposition in (12)–(14) and (20)–(23) to the lhs of (100), and applying (92)–(95) and (98) to the rhs of (100), one could rewrite the four equations in terms of the additional variables as follows:

\[
\left( K_i^j - K_i^j \right) \hat{n} + (\Delta^i - \alpha^i) \partial_i = -\kappa \left( \frac{1}{3} \mathcal{D} (\hat{n}) + \langle \hat{n}, \text{tr} \sigma \rangle \right) \hat{n} + \kappa \left( \langle \text{tr} \sigma \rangle + \frac{1}{12} \mathcal{D} \right) \partial_i
\]

(101)

\[
\left( \Delta^j - 2\Theta^j - g (\partial^i \hat{n}, \hat{n}) + 3 \nabla_i^3 q^{ij} - \frac{1}{2} q_{jk}^3 \nabla_i^3 q^{jk} - N^{-1}N^j \left( K_i^j - K_i^j \right) \right) \hat{n}
+ \left( \Delta^j - K^j + \langle 3 \text{d}x^j, \partial^i \hat{n} \rangle + \left( \Theta (\hat{n}) + \frac{1}{2} q_{jk}^3 \nabla_i^3 q^{jk} \right) \right) 3 q^{ij}
- \nabla_i^3 q^{ij} + N^{-1}N^j (\Delta^j - \alpha^j) \partial_j = \text{rhs1},
\]

(102)

\[
\left( \frac{1}{2} q_{jk}^3 \nabla_i^3 q^{jk} - \Delta_i + g (\partial^i \hat{n}, \hat{n}) \right) \hat{n} + \left( K_i^j - K_i^j + \Theta (\hat{n}) - \frac{1}{2} q_{jk}^3 \nabla_i^3 q^{jk} \right) \partial_i
+ \left( K_i^j - \Delta_i^j - \langle 3 \text{d}x^j, \partial^i \hat{n} \rangle \right) \partial_j = \text{rhs2},
\]

(103)

\[
\left( K_i^j - K_i^j + \frac{1}{2} q_{jk}^3 \nabla_i^3 q^{jk} - \Delta_i + g (\partial^i \hat{n}, \hat{n}) \right) N^{-1}N^j \hat{n}
+ 3 q_{jk}^3 \nabla_i^3 q^{jk} + \left( \Delta_i - \alpha_i + 3 \nabla_k^3 q^{jk} - \Theta - \frac{1}{2} q_{jk}^3 \nabla_i^3 q^{jk} \right) \partial_i
+ \left( \Theta (\hat{n}) - \frac{1}{2} q_{jk}^3 \nabla_i^3 q^{jk} \right) N^{-1}N^j \partial_i
+ \left( \Theta_i + \frac{1}{2} q_{jk}^3 \nabla_i^3 q^{jk} \right) 3 q^{jk} - 3 \nabla_i^3 q^{jk}
+ N^iN^{-1} \left( K_i^k - \Delta_i^k - \langle 3 \text{d}x^j, \partial^i \hat{n} \rangle \right) \partial_i = \text{rhs3},
\]

(104)

with:

\[
\text{rhs1} = \kappa \left( \frac{1}{3} \mathcal{D} - \langle \hat{n}, \tau^i + k^i \rangle - N^iN^{-1} \left( \frac{1}{3} \mathcal{D} (\hat{n}) + \langle \hat{n}, \text{tr} \sigma \rangle \right) \right) \hat{n}
+ \kappa \left( \tau^i + k^i + N^iN^{-1} \left( \frac{1}{12} \mathcal{D} + (\text{tr} \sigma)^i \right) \right) \partial_j.
\]
\[ \text{rhs}2 = \kappa \left( -\langle \tilde{n}^*, \tau_i - k_i \rangle \tilde{n} - \frac{1}{3} D (\tilde{n}) \partial_j + \left( \tau'_j - k'_j \right) \partial_j \right), \]

\[ \text{rhs}3 = \kappa \left( -\frac{1}{12} \delta^j_i D (\tilde{n}) - \frac{3}{2} q^k \langle \tilde{n}^*, \sigma_{ik} + \varepsilon_{ilm} P^m \rangle - N^i N^{-1} \langle \tilde{n}^*, \tau_i - k_i \rangle \right) \tilde{n} \]

\[ + \kappa \left( -\frac{1}{3} (D^j - N^{-1} N^j D (\tilde{n})) \partial_i \right) \]

\[ + \kappa \left( -\frac{1}{12} \delta^j_i D^k + \frac{3}{2} q^l (\sigma^k_{il} + \varepsilon_{ilm} P^m) + N^i N^{-1} (\tau_i - k_i) \right) \partial_k \]

Contracting (101)–(104) with \( \tilde{n}^* \) and \( 3 \, dx^l \), we obtain 8 equations. (101) gives:

\[ K_i^j - K_j^i = -\kappa \left( \frac{1}{4} D (\tilde{n}) + \langle \tilde{n}^*, \tau_i - k_i \rangle \right), \]

\[ \Delta^j - \alpha^j = \kappa \left( (\tau^j + \frac{1}{12} D^j) \right), \]

(102) gives:

\[ \Delta^j - 2 \Theta^j - g (\partial^j \tilde{n}, \tilde{n}) + \nabla_i^3 q^j - \frac{3}{2} q^k \nabla_i \nabla_j \nabla_l \nabla_k = \kappa \left( \frac{1}{3} D^j - \langle \tilde{n}^*, \tau^j + k^j \rangle \right), \]

\[ - \kappa N^j N^{-1} \left( \frac{1}{4} D (\tilde{n}) + \langle \tilde{n}^*, \tau^j + k^j \rangle \right), \]

(103) gives:

\[ \frac{1}{2} 3 q_{jk} \nabla_i \nabla_j \nabla^k \Delta_i + g (\partial_i \tilde{n}, \tilde{n}) = -\kappa \left( \tilde{n}^*, \tau_i - k_i \right), \left( K_i^j - K_j^i + \Theta (\tilde{n}) - \frac{1}{2} q_{kl} \nabla_i \nabla_j \nabla^k \right) \delta^j_i \]

\[ + K_i^j - \Delta_i \langle 3 \, dx^j, \partial_i \tilde{n} \rangle = \kappa \left( \frac{1}{3} D (\tilde{n}) \delta^j_i + \tau^j_i - k^j_i \right), \]

and finally, (104) gives:

\[ K_i^j - K_j^i + \left( \frac{1}{2} 3 q_{jl} \nabla_j \nabla^l - \Delta_i + g (\partial_i \tilde{n}, \tilde{n}) \right) N^i N^{-1} \]

\[ = \kappa \left( -\frac{1}{12} \delta^j_i D (\tilde{n}) - \frac{3}{2} q^k \nabla_i \nabla^k + \varepsilon_{ilm} P^m \right) - \kappa N^i N^{-1} \langle \tilde{n}^*, \tau_i - k_i \rangle \],

\[ 3 T_{ij}^k + \left( \Theta_i + \frac{1}{2} q_{lm} \nabla_i \nabla^l \nabla^k \right) 3 q^k - 3 \nabla_i \nabla^k \]

\[ + N^i N^{-1} \left( K_i^k - \Delta_i^k - \langle 3 \, dx^k, \partial_i \tilde{n} \rangle \right) + \left( \Delta_i - \alpha_i + 3 \nabla_i \nabla^k - \Theta \right) - \frac{1}{2} q_{lm} \nabla_i \nabla^k \nabla^j \nabla^l \]
\[
\begin{align*}
+ \left( \Theta (\hat{n}) - \frac{1}{2} q_{lm} \nabla_n q^{lm} + \mathcal{K}_i^j - K_i^j \right) N^{-1} N^j \right) \delta_i^k = \text{rhs} 4, \\
\text{with:} \\
\text{rhs} 4 = \kappa \left( \frac{1}{3} (D^j - N^{-1} N^j D (\hat{n})) \delta_i^k - \frac{1}{12} \delta_i^j D^k + \frac{3}{2} q^k \left( \sigma^k_{\ j} + \varepsilon_{jlm} \hat{\sigma}^{lm} \right) \\
+ N^j N^{-1} \left( r^k_i - k_i^k \right) \right),
\end{align*}
\]

These 8 equations are the \((3 + 1)\) ADM decomposition of the hypermomentum (75).

4.2.2. The traceless torsion constraint ADM decomposition. The hypermomentum equation (75) provides 60 independent equations, 4 more equations are needed to determine uniquely the connection from the equation of motions (74) and (75). These vectorial degrees of freedom are reduced by traceless torsion constraint (76), consult [8, 14] or [28] for some detailed explanations. Let us write
\[
T = T_{\mu \nu} dx^\nu \in T_p^* M \text{ as a one-form where its components satisfy:}
\]
\[
T_{\nu} = T_{\mu \nu} = \langle dx^\mu, T (\partial_\mu, \hat{n}) \rangle = 0.
\]
Contracting \( T \) with the normal \( \hat{n} \) and \( \partial_i \), and using (3) and (4) gives:
\[
n^\nu T_{\nu} = n^\nu T_{\mu \nu} = - \langle 3 dx^i, T (\partial_i, \hat{n}) \rangle,
\]
\[
T_i = T_{\mu}^{\mu i} = - \langle \hat{n}^*, T (\partial_i, \hat{n}) \rangle + \langle 3 dx^i, T (\partial_j, \partial_i) \rangle,
\]
then using torsion decomposition (12)–(14), one obtains [28]:
\[
\langle T, \hat{n} \rangle = n^\nu T_{\nu} = n^\nu T_{\mu \nu} = \Delta_i^i - \mathcal{K}_i^i + \langle 3 dx^i, \partial_i \hat{n} \rangle = 0, \hspace{1cm} (106)
\]
\[
\langle T, \partial_i \rangle = T_i = T_{\mu}^{\mu i} = \Delta_i - \Theta_i - \langle \hat{n}^*, \partial_i \hat{n} \rangle + 3 T_j^j = 0. \hspace{1cm} (107)
\]
(106) and (107) are the \((3 + 1)\) ADM decomposition of the traceless torsion constraint (76).

4.2.3. Classification of the equations. The hypermomentum equation (75), together with the traceless torsion constraint (76), provides 10 equations. Collecting and classifying all these results, we have 2 scalar equations, resulting from the hypermomentum equation and the traceless torsion constraint:
\[
\mathcal{K}_i^j - K_i^j = - \kappa \left( \frac{1}{4} D (\hat{n}) + \langle \hat{n}^*, \text{tr} \sigma \rangle \right),
\]
\[
\Delta_i^i - \mathcal{K}_i^i + \langle 3 dx^i, \partial_i \hat{n} \rangle = 0;
\]
4 vectors equations, 3 from the hypermomentum equations:
\[
\Delta_i^i - \alpha_i^i = \kappa \left( \text{tr} \sigma \right)^i + \frac{1}{12} D^i,
\]
\[
\frac{1}{2} 3 q_{jk} \nabla_j 3 q^{jk} - \Delta_i + g (\partial_i \hat{n}, \hat{n}) = \kappa \langle \hat{n}^*, k_i - \tau_i \rangle,
\]
\[ \Delta^1 - 2 \Theta^j - g (\partial^j \hat{n}, \hat{n}) + 3 \nabla_j q_{ji} - \frac{1}{2} q_{ik} 3 \nabla^i q^{ik} - N^{-1} N^j \left( K^j_i - K_j^i \right) \]
\[ = \kappa \left( \frac{1}{2} \mathcal{D}^j - \langle \hat{n}^i, \tau^i + k^i \rangle \right) - \kappa N \left( N^i - 1 \right) \mathcal{D}^j (\hat{n}) + \langle \hat{n}^i, \tau^i, \sigma^i \rangle, \]
and 1 from the traceless torsion constraint:
\[ \Delta_i - \Theta_i - (\hat{n}^i, \partial_i \hat{n}) + 3 T_j^i = 0; \]
3 matrix equations:
\[ \Delta^i - K^j_i + \left( 3 \nabla^j x^i, \partial^i \hat{n} \right) - \nabla^i \nabla^i q_{ij} + N^{-1} N^j \left( \Delta^i - \alpha^i \right) + \Theta (\hat{n}) + \frac{1}{2} q_{ij} \nabla^i 3 q^{ij} \]
\[ = \kappa \left( \tau^i + k^i + N \left( \frac{1}{2} \mathcal{D}^i + \langle \tau^i, \sigma^i \rangle \right) \right), \]
\[ = \kappa \left( \frac{1}{2} \mathcal{D}^i (\hat{n}) \delta^i_j + \tau^i_j - k^i_j \right), \]
\[ \mathcal{K}_j^i - K_j^i + \left( \frac{1}{2} q_{ij} \nabla^i 3 q^{ij} - \Delta_i + g (\partial_i \hat{n}, \hat{n}) \right) N^{-1} \]
\[ = \kappa \left( \frac{1}{2} \delta^i_j \mathcal{D} (\hat{n}) - \frac{1}{2} q_{ij} 3 \nabla^i 3 q^{ij} \right) N^{-1} - \kappa \left( N \left( N^i - 1 \right) \langle \hat{n}^i, \tau_i, k_i \rangle \right), \]
and finally 1 tensor equation of order-(1):
\[ 3 T_j^i + \left( \Theta_j + \frac{1}{2} q_{lm} \nabla^j 3 q^{lm} - 3 q^j_i + N^i N^{-1} \left( K^i_j - \Delta^i_j - \langle 3 \nabla^i, \partial_j \rangle \right) \right) \]
\[ + \left( \Delta^i - \alpha^i + 3 \nabla^i q^{ij} - \Theta^i - \frac{1}{2} q_{lm} 3 \nabla^i 3 q^{lm} \right) \]
\[ + \left( \Theta (\hat{n}) - \frac{1}{2} q_{lm} \nabla^i 3 q^{lm} + \mathcal{K}_j^i - K_j^i \right) \left( N^{-1} \right) \delta^i_j = \text{rhs} 4, \]
with rhs4 satisfies (105). These 10 equations are the main result in this article, that are, the complete \((3 + 1)\) decomposition of (75) and (76) in the adapted (not necessarily normal) coordinate. We will discuss these main results in section 6.

5. Special cases

5.1. Special case I: zero hypermomentum

Let us take a special case where the hypermomentum \(H\) vanishes. The scalar equations are simplified as follows:
\[ \mathcal{K}_i^i - K_i^i = 0, \quad (108) \]
\[
\Delta^i - K^i + \left( 3d x^i, \partial n \right) = 0. \tag{109}
\]

With the scalar equation (108), the vector equations becomes:
\[
\Delta^i - \alpha^i = 0, \quad \Theta^i = \frac{1}{2} 3 \nabla_j 3 q_{ji}, \tag{110}
\]
\[
\Delta^i - \Theta^i - g \left( \partial^i n, n \right) = \frac{1}{2} \left( 3 q_{jk} 3 \nabla^i q^k - 3 \nabla_j 3 q_{jk} \right). \tag{111}
\]
\[
\Delta^i - \Theta^i - g \left( \partial^i n, n \right) = - 3 T^i_{ji}. \tag{112}
\]

Inserting (108) and using vector equations (111) and (112), together with the projective invariant constraint (113), the matrix equations becomes:
\[
\left( \Theta (n) + \frac{1}{2} 3 q_{kl} \left( \n \left[ 3 q^j \right] + \Delta^j + \Delta^k \right) \right) 3 q^{ij} - \n \left[ 3 q^j \right] - \Delta^{ij} - K^{ij} + \left( 3 d x^i, \partial^i \n \right) = 0, \tag{114}
\]
\[
\left( \Theta (n) - \frac{1}{2} 3 q_{kl} \left( \n \left[ 3 q^j \right] + \Delta^j + \Delta^k \right) \right) 3 q^{ij} + K^{ij} - \Delta^{ij} - \left( 3 d x^j, \partial^i \n \right) = 0, \tag{115}
\]
\[
K^{ij} - K^{ji} = 0. \tag{116}
\]

Contracting (114) and (115) with \(3 q_{ji}\), and then using both the scalar equations, gives:
\[
3 \Theta (n) = - \frac{1}{2} 3 q_{kl} \left( \n \left[ 3 q^j \right] + \Delta^j + \Delta^k \right),
\]
\[
3 \Theta (n) = \frac{3}{2} q_{kl} \left( \n \left[ 3 q^j \right] + \Delta^j + \Delta^k \right),
\]
which is only satisfied if:
\[
\Theta (n) = 0, \quad \n \left[ 3 q^j \right] + \Delta^j + \Delta^k = 0. \tag{117}
\]

Inserting (117) and (118) to (114)–(116) gives:
\[
\Delta^k - K^{ij} + \left( 3 d x^i, \partial^i \n \right) = 0, \tag{119}
\]
\[
K^{ij} - K^{ij} = 0, \quad K^{ij} - K^{ij} = 0. \tag{120}
\]

The remaining equation is the tensor equation. Raising all its indices, then using vector equations (110)–(113) and matrix equations (119)–(121), gives:
\[
\frac{1}{2} \left( 3 \nabla_i \nabla^i q_{jl} + 3 \nabla_j \nabla^j q_{il} \right) q_{jk} - 3 \nabla^i q_{jk} \\
+ \frac{1}{N} \left( - \Delta^i + \nabla^i q_{jk} - \left( 3 \nabla^j q_{ik} \right) \right) + \frac{1}{2} \left( 3 \nabla_j q_{ij} - 3 \nabla^i q_{ij} \right) q_{jk} + 3 T^{ij} = 0.
\]

(122)

Contracting (122) with \( q_{jk} \) gives:

\[
3 \nabla^i q_{ij} - 3 q_{jk} \nabla^i q_{jk} = 0,
\]

then decomposing this tensor equation into the symmetric and antisymmetric parts of the \((i, j)\)-indices, gives:

\[
3 \nabla^i q_{ij} = 0,
\]

(123)

and

\[
3 T^{ij} = 0.
\]

(124)

Let us solve the simplified equations (108), (109), (110)–(113), (119)–(121), (123) and (124). The easiest way is to start from (123), which is satisfied if:

\[
3 \nabla^i q_{jk} = 0.
\]

(125)

Inserting (125) to (122) gives:

\[
3 T^{ij} = \frac{1}{N} \left( \Delta^i - \Theta^i - \left( 3 \nabla^j q_{ik} \right) \right).
\]

(126)

Now we have the following equations for the torsionless condition:

\[
K_{ij} - K_{ji} = 0, \quad \Delta_i - \Theta_i - \left( \nabla^k q_{ij} \right) = 0,
\]

(127)

and metric compatibility:

\[
\Theta(\hat{n}) = 0, \quad K_{ij} - K_{ij} = 0,
\]

\[
\Theta^i = 0, \quad 3 \nabla^i q_{jk} = 0
\]

(128)

\[
\Delta^i - \alpha^i = 0, \quad 3 \nabla^i q_{jk} = 0.
\]

We could conclude that for zero hypermomentum, the affine connection becomes Levi-Civita, and the \((3 + 1)\) stress–energy–momentum equations for MAGR return to the original EFE, see [28]. Furthermore, without the traceless torsion constraint (106) and (107), it is not possible to retrieve the metric compatibility and torsionless condition in the absence of hypermomentum [28].
5.2. Special case II: metric connection in normal coordinate

For the next two subsections, we will use a special coordinate where the lapse \( N = 1 \) and the shift \( \mathbf{N} = 0 \), i.e. the normal (Gauss) coordinate. The choice of coordinates will not alter the physical interpretation but will greatly simplify our calculation. For the metric connection (not necessarily torsionless), we apply the metricity compatibility (128) to the 8 \((3 + 1)\) equations in the normal coordinate. Notice that the metricity condition causes the Weyl tensor \( Q_\mu = 0 \) (but not vice versa), and this guarantees the consistency of the equations of motion under the projective invariance transformation [14]. The connection could be uniquely determined only from (74) and (75), hence, one does not need to consider the traceless torsion constraint (76).

With the metricity condition (128), the 8 additional variables reduce into 4, where we choose to work with only \( \Delta^i, K^{ij}, \Delta^i, \) and \( 3 T_{i}^\cdot k \). The \((3 + 1)\) equations consist of 1 scalar equation:

\[
-\kappa \left( \frac{1}{4} \mathcal{D} (\hat{n}) + \langle \hat{n}^*, \text{tr} \sigma \rangle \right) \equiv 0, \tag{129}
\]

3 vector equations:

\[
\kappa \left( (\text{tr} \sigma)^i + \frac{1}{12} \mathcal{D}^i \right) \equiv 0, \tag{130}
\]

\[
\Delta^i = \kappa \left( \frac{1}{3} \mathcal{D}^i - \langle \hat{n}^*, \tau^i + k^i \rangle \right),
\]

\[
\Delta^i = -\kappa \langle \hat{n}^*, k^i - \tau^i \rangle,
\]

3 matrix equations:

\[
\Delta^{ij} - K^{ij} = \kappa \left( \tau^{ji} + k^{ji} \right),
\]

\[
K_{ij} - \Delta^j = \kappa \left( -\frac{1}{3} \mathcal{D} (\hat{n}) \delta^j_i + \tau^j_i - k^j_i \right),
\]

\[
K_{ij} - K^{ij} = \kappa \left( -\frac{1}{12} \delta_{ij} \mathcal{D} (\hat{n}) - \frac{3}{q} q^k \langle \hat{n}^*, \sigma_{ik} + \epsilon_{iklm} P^m \rangle \right),
\]

and 1 tensor equation of order \((i)\):

\[
3 T_{i}^\cdot k = \kappa \left( \frac{1}{3} (\mathcal{D}^i) \delta^k_i - \frac{1}{12} \delta^k_i \mathcal{D}^k + \frac{3}{q} q^k \langle \hat{n}^*, \sigma_{ik} + \epsilon_{iklm} P^m \rangle \right). \tag{132}
\]

The scalar equation and the first vector equation are constraints on the hypermomentum \( \mathcal{H} \).

The remaining two vector equations, which could be written compactly as:

\[
\Delta^i = \kappa \left( \frac{1}{3} \mathcal{D}^i - \langle \hat{n}^*, \tau^i + k^i \rangle \right),
\]

\[
\Delta^i = \kappa \langle \hat{n}^*, \tau^i - k^i \rangle, \tag{131}
\]

could be solved to obtain the additional variable \( \Delta^i \) in terms of hypermomentum in (131) and 1 constraint on \( \mathcal{H} \):

\[
\frac{1}{3} \mathcal{D}^i - 2 \langle \hat{n}^*, \tau^i \rangle = 0. \tag{132}
\]
The matrix equations could be compactly written as follows:

\[ K^{ij} - \Delta^{ij} = -\kappa \left( \tau^i + k^i \right), \]  
\[ K^{ij} - \Delta^{ij} = \kappa \left( -\frac{1}{3} D (\hat{n}) 3 q^{ij} + \tau^i - k^i \right), \]  
\[ K^{ij} - K^{ji} = \kappa \left( -\frac{1}{12} q^{ij} D (\hat{n}) - 3 q^{im} 3 q^{jk} (\hat{n}^*, \sigma_{nk} + \varepsilon_{nkml} F^m) \right). \]

Solving (133) and (134) gives another constraint on \( \mathcal{H} \):

\[ 2 \tau^{ji} - \frac{1}{3} D (\hat{n}) 3 q^{ij} = 0. \]  

Now let us focus on the last matrix equation, which could be written as:

\[ 2 K^{[ij]} = \kappa \left( -\frac{1}{12} q^{ij} D (\hat{n}) - \langle \hat{n}^*, \sigma^{(ij)} \rangle - \langle \hat{n}^*, \epsilon^{(ij) m} F^m \rangle \right), \]

notice that \( \sigma^{(ij)} = 0 \) since \( \sigma^{ij} \) is symmetric, and:

\[ 3 q^{in} 3 q^{jk} (\hat{n}^*, \sigma_{nk}) = 3 q^{im} 3 q^{jk} (\hat{n}^*, \sigma_{nk}) = \langle \hat{n}^*, \epsilon^{(ij) m} F^m \rangle. \]

We could split this equation in its symmetric and antisymmetric parts:

\[ \kappa \left( -\frac{1}{12} q^{ij} D (\hat{n}) - \langle \hat{n}^*, \sigma^{(ij)} \rangle \right) = 0, \]  
\[ -\frac{1}{2} \kappa \langle \hat{n}^*, \epsilon^{(ij) m} F^m \rangle = K^{[ij]}. \]

(136) is another constraint on \( \mathcal{H} \).

Let us solve (133) and (134), that could be written as:

\[ K^{ij} + \hat{n} \left[ 3 q^{ij} \right] + \Delta^{ij} = -\kappa \left( \tau^i + k^i \right), \]
\[ K^{ij} - \Delta^{ij} = \kappa \left( -\frac{1}{3} D (\hat{n}) 3 q^{ij} + \tau^i - k^i \right), \]

using (22) and the fact that \( 3 \nabla^a 3 q^* = 0 \) for a metric connection. Substituting and eliminating these two equations give:

\[ 2 K^{ij} + \hat{n} \left[ 3 q^{ij} \right] + 2 \Delta^{ij} = -\kappa \left( 2 \tau^i + \frac{1}{3} D (\hat{n}) 3 q^{ij} \right), \]
\[ \hat{n} \left[ 3 q^{ij} \right] + 2 \Delta^{ij} = -\kappa \left( 2 \tau^i - \frac{1}{3} D (\hat{n}) 3 q^{ij} \right). \]

If we split these two into their symmetric and antisymmetric parts, we have:

\[ K^{i[j]} = -\kappa \left( k^{[ij]} + \frac{1}{6} D (\hat{n}) 3 q^{ij} \right) - \frac{1}{2} \hat{n} \left[ 3 q^{ij} \right], \]  
\[ K^{i[j]} + \Delta^{i[j]} = -\kappa k^{[ij]}. \]
\[ \Delta^{(i)} = -\kappa \left( \tau^{(i)} - \frac{1}{6} \mathcal{D} (\hat{n}) \gamma^{ij} \right) - \frac{1}{2} \hat{n} \left[ 3 \gamma^{ij} \right]. \] (140)

\[ -\kappa T^{[i]} = 0. \] (141)

(141) is another constraint on \( \mathcal{H} \). Using (137)–(140), we could solve for \( K^{(i)}, K^{(j)}, \Delta^{(i)}, \) and \( \Delta^{(j)} \) to obtain:

\[ K^{ij} = \kappa \left( \frac{1}{6} \mathcal{D} (\hat{n}) \gamma^{ij} - \frac{1}{2} \langle \hat{n}^*, \epsilon^{ij}_m F^m \rangle - k^{(j)} \right) - \frac{1}{2} \hat{n} \left[ 3 \gamma^{ij} \right], \]

\[ \Delta^{ij} = \kappa \left( \frac{1}{6} \mathcal{D} (\hat{n}) \gamma^{ij} + \frac{1}{2} \langle \hat{n}^*, \epsilon^{ij}_m F^m \rangle - k^{(j)} - \tau^{(j)} \right) - \frac{1}{2} \hat{n} \left[ 3 \gamma^{ij} \right]. \]

The last remaining equation is the tensor equation, where we have raised all of its indices:

\[ 3T^{[kl]} = \kappa \left( \frac{1}{3} (\mathcal{D})^l \gamma^{ij} - \frac{1}{12} \gamma^{ij} D^k + (\sigma^{ij} + \epsilon^{ij}_m \rho^m) \right). \]

Splitting the index \((i, j)\) into its symmetric and antisymmetric part gives:

\[ 0 = \kappa \left( \frac{1}{3} 3q^{(ik)} D^j - \frac{1}{12} 3q^{ij} D^k + \sigma^{(ij)} \right) \] (142)

\[ 3T^{ijkl} = \kappa \left( \frac{1}{3} 3q^{(ik)} D^{jl} + \epsilon^{ij}_m \rho^m \right). \]

The symmetric part gives another constraint to \( \mathcal{H} \), while the antisymmetric part gives solution to \( 3T^{ijkl} \).

Collecting all the constraints on \( \mathcal{H} \), namely, (129), (130), (132), (135), (136), (141), and (142), and solving them nicely give 7 constraints on the hypermomentum:

\[ \langle \hat{n}^*, \tau^* \rangle = \frac{1}{6} \mathcal{D}, \quad \langle \hat{n}^*, \tau \sigma \rangle = \frac{1}{4} \mathcal{D} (\hat{n}), \quad \langle \hat{n}^*, \sigma^{ij} \rangle = \frac{1}{12} q^{ij} D \]

\[ \tau^* = \frac{1}{6} \mathcal{D} (\hat{n}) \gamma^{ij}, \quad \tau^{[i]} = 0, \quad (\tau \sigma)^{ij} = -\frac{1}{12} D, \quad \sigma^{ij} = -\frac{1}{3} \gamma^{ij} D^k + \frac{1}{12} \gamma^{ij} D^k. \] (143)

Finally, we write the additional variables in terms of \( \mathcal{D} \):

\[ \Delta^{i} = \alpha^{i} = \kappa \left( \frac{1}{6} D^i - \langle \hat{n}^*, k^i \rangle \right), \]

\[ \Delta^{ij} = \kappa \left( \frac{1}{2} \langle \hat{n}^*, \epsilon^{ij}_m F^m \rangle - k^{(j)} \right) - \frac{1}{2} \hat{n} \left[ 3 \gamma^{ij} \right]. \] (144)

\[ K^{ij} = K^{ij} = \kappa \left( \frac{1}{6} \mathcal{D} (\hat{n}) \gamma^{ij} - \frac{1}{2} \langle \hat{n}^*, \epsilon^{ij}_m F^m \rangle - k^{(j)} \right) - \frac{1}{2} \hat{n} \left[ 3 \gamma^{ij} \right], \]

\[ 3T^{ijkl} = \kappa \left( \frac{1}{3} 3q^{(ik)} D^{jl} + \epsilon^{ij}_m \rho^m \right), \]

while the remaining ones, namely \( \Theta (\hat{n}), \theta^i, 3\nabla_a \gamma^{ij}, \) and \( 3\nabla_a 3q^{ik}, \) are zero due to the metric compatibility (128).
Equations in (143) indicate that the hypermomentum $\mathcal{H}$ is constrained, one could choose $D'(\hat{n})$ and $D'$ from the dilation $D$, and $k_i, \hat{l}$ from the rotation $\Omega$ as the free quantities. The shear $\Lambda$ is determined by $D$ and $\Omega$, and could be obtained by inserting the constraint (143) to (84)–(86):

$$\Lambda(\hat{n}, \hat{n}) = \text{tr} \, \sigma = \frac{1}{4}D(\hat{n})\hat{n} - \frac{1}{12}D'\partial_i,$$

$$\Lambda(\hat{n}, \partial_i) = \tau_i = \frac{1}{6} \left( -D\hat{n} + D(\hat{n})\delta_i^j \partial_j \right), \quad (145)$$

$$\Lambda(\partial_i, \partial_j) = \sigma_{ij} = \frac{1}{12} 3 q_{ij}D(\hat{n})\hat{n} + \left( \frac{1}{3} \delta_{ij} D_H + \frac{1}{12} 3 q_{ij} D' \right) \partial_k.$$ 

Together with the $(3 + 1)$ metric-compatible stress–energy–momentum equations in normal coordinate as follows [28]:

$$\frac{1}{2} \left( \frac{3}{2} \mathcal{R} - \text{tr} \left( K^2 \right) + \left( \text{tr} K \right)^2 \right) = \kappa E, \quad (146)$$

$$\frac{1}{2} \left( 3\nabla_j \left( K^j_i + \Delta l^j_i \right) - 3\nabla_j K^j_i + 3 T^j_i K^j_k + \Delta_i \Delta^j_k - \Delta^j_k \hat{n} - \hat{n} \left[ \frac{1}{2} \Omega_{ij} + \frac{1}{12} 3 q_{ij} \mathcal{D}_H + \frac{1}{12} 3 q_{ij} D' \right] \partial_k \right) = \kappa p_i,$$

$$\hat{n} \left[ K_{ij} + \frac{1}{2} \nabla_i \left( \Delta^i + \Delta_j \right) + \frac{1}{2} \Omega_{ij} - \frac{1}{2} \nabla_i \left( \Delta^i + \Delta_j \right) \right] = \kappa S_{ij} - \frac{1}{2} q_{ij} \kappa (S - E),$$

(144) and (145) describe the metric compatible MAGR systems. These results, for examples, could be used to describe the dynamics of systems with torsion considered in [50–55].

5.3. Special case III: torsionless connection in normal coordinate

For the last case, we apply the torsionless condition (127) in the normal coordinate. With this, we have 6 independent variables: $\Theta(\hat{n}), 3 \alpha, K, 3 \omega, \Theta = \Delta$, and $\Delta^i = K^i_j$. Note that the traceless torsion constraint (76) is automatically satisfied in this case, so we have 8 equation that must be satisfied, consisting 1 scalar equation:

$$\Delta^i - K^i_j = -\kappa \left( \frac{1}{4} D(\hat{n}) + \langle \hat{n}, \text{tr} \sigma \rangle \right), \quad (147)$$

3 vectors equations:

$$\Delta^i - \alpha^i = \kappa \left( \text{tr} \sigma \right)^i + \frac{1}{12} D^i,$$

$$- \Delta^i + 3 \nabla_j \left( 3 q^j - \frac{1}{2} q_{jk} 3 \nabla^3 q^k - \kappa \left( \frac{1}{3} D^i - \langle \hat{n}, \tau^i + k^i \rangle \right) \right),$$

$$\frac{1}{2} q_{jk} 3 \nabla^3 q^k - \Delta_i = \kappa \langle \hat{n}, k_i - \tau_i \rangle.$$ 

3 matrix equations:

$$\left( \Theta(\hat{n}) + \frac{1}{2} q_{kl} \left( \hat{n} \left[ 3 q^l \right] + \Delta^l + \Delta^l \right) \right) 3 q^i_j - \hat{n} \left[ 3 q^i_j \right] - \Delta^i - \Delta^i = \kappa \left( \tau^i + k^i \right),$$

$$K^i_j - \Delta^i_j + \left( \Delta^i - K^i_j + \Theta(\hat{n}) - \frac{1}{2} q_{kl} \left( \hat{n} \left[ 3 q^l \right] + \Delta^l + \Delta^l \right) \right) \delta^i_j.$$
Here, the temporal and spatial components of the evolution generator of $\partial_i$ satisfy $\Delta_i = \Theta_i$ and $\Delta^j_i = K^j_i$ by the torsionless condition.

The three vector equations could be simplified as follows:

\[
\begin{align*}
\Delta^i - \Theta^i &= \kappa \left( (\text{tr} \sigma)^i + \frac{1}{12} D^i \right), \quad (148) \\
3 \nabla_j q^j - 2 \Delta_i &= \kappa \left( \frac{1}{3} D^j - 2 \langle \hat{n}^j, \tau^j \rangle \right), \quad (149) \\
3 \nabla_j q^j - 3 q_k \nabla^j q^k &= \kappa \left( \frac{1}{3} D^j - 2 \langle \hat{n}^j, K^j \rangle \right). \quad (150)
\end{align*}
\]

Meanwhile, raising the indices $(i, j)$ of the matrix equations, gives:

\[
\begin{align*}
\Theta (\hat{n}) + \frac{1}{2} q_{kl} (\hat{n} [^3 q^j] + \Delta^k + \Delta^l) \right) 3 q^j - \hat{n} [^3 q^j] - \Delta^j - \Delta^j = \kappa (\tau^j + k^j), \\
K^{ij} - \Delta^j + \left( \Delta^j_i - K^j_i + \Theta (\hat{n}) - \frac{1}{2} q_{kl} (\hat{n} [^3 q^j] + \Delta^k + \Delta^l) \right) 3 q^j \\
= \kappa \left( -\frac{1}{3} D (\hat{n}) 3 q^j + \tau^j - k^j \right), \\
\Delta^j - K^j = \kappa \left( -\frac{1}{12} q_{ij} D (\hat{n}) - \langle \hat{n}^j, \sigma^{ij} + \varepsilon_m F^m \rangle \right).
\end{align*}
\]

These 3 equations needs to be split into their symmetric and antisymmetric parts as follows:

\[
\begin{align*}
\left( \Theta (\hat{n}) + \frac{1}{2} q_{kl} (\hat{n} [^3 q^j] + \Delta^k + \Delta^l) \right) 3 q^j - \hat{n} [^3 q^j] - 2 \Delta^{(ij)} = \kappa (\tau^{(ij)} + k^{(ij)}), \quad (151) \\
0 &= \kappa (\tau^{(ij)} + k^{(ij)}), \quad (152) \\
K^{ij} - \Delta^{(ij)} + \left( \Delta^j_i - K^j_i + \Theta (\hat{n}) - \frac{1}{2} q_{kl} (\hat{n} [^3 q^j] + \Delta^k + \Delta^l) \right) 3 q^j
\end{align*}
\]
and:
\[ \Delta^{(\rho)} - K^\rho = \kappa \left( \frac{1}{12} 3 q^{\rho ij} + \tau^{(\rho)} - k^{(\rho)} - \langle \hat{n}^*, \sigma^{(\rho)} \rangle \right). \]  
(155)

and:
\[ \Delta^{(\mu)} = -\kappa \langle \hat{n}^*, e^{\mu}_{\phantom{\mu}m} \rangle. \]  
(156)

Notice that the extrinsic curvature of the first kind \( K^{ij} \), by the torsionless condition (127), is symmetric on its indices. Substituting (152) to (154) gives the antisymmetric part of \( \Delta^{ij} \):
\[ \Delta^{(ij)} = -2\kappa T^{(ij)}. \]  
(157)

Substituting (154) to (156) gives constraint on \( H \):
\[ 2\kappa T^{[ij]} = \kappa \langle \hat{n}^*, e^{ij}_{\phantom{ij}m} \rangle \]  
(158)

together with (152). Equations (153) and (155) gives:
\[ \left( \Delta_{ij} - K_{ij} + \Theta (\hat{n}) - \frac{1}{2} 3 q_{ij} \left( \hat{n} [\tau] + \Delta_{ij} + \Delta_{ij}^k \right) \right) 3 q_{ij} = \kappa \left( -\frac{5}{12} D (\hat{n}) 3 q^{ij} + \tau^{(i)} - k^{(i)} - \langle \hat{n}^*, \sigma^{(i)} \rangle \right). \]  
(159)

Contracting this equation with \( 3 q_{ij} \) and substituting (147), gives:
\[ \Theta (\hat{n}) - \frac{1}{2} 3 q_{ij} \left( \hat{n} [\tau] + \Delta_{ij} + \Delta_{ij}^k \right) = \kappa \left( \frac{1}{3} (\text{tr} \tau - \text{tr} k) + \frac{2}{3} \langle \hat{n}^*, \text{tr} \sigma \rangle - \frac{1}{6} D (\hat{n}) \right), \]  
(160)

with \( \text{tr} \tau = 3 q_{ij} \tau^{ij} = 3 q_{ij} T^{(ij)} \) and \( \text{tr} k = 3 q_{ij} k^{ij} = 3 q_{ij} \kappa^{(ij)} \). Contracting (151) with \( 3 q_{ij} \) gives:
\[ 3 \Theta (\hat{n}) + \frac{1}{2} 3 q_{ij} \left( \hat{n} [\tau] + \Delta_{ij} + \Delta_{ij}^k \right) = \kappa (\text{tr} \tau + \text{tr} k). \]  
(161)

(159) and (160) could be solved for \( \Theta (\hat{n}) \) and \( \nabla_{\hat{n}} 3 q^{ij} \) as follows:
\[ \Theta (\hat{n}) = \kappa \left( \frac{1}{3} (\text{tr} \tau + \frac{1}{6} \text{tr} k) - \frac{1}{24} D (\hat{n}) + \frac{1}{6} \langle \hat{n}^*, \text{tr} \sigma \rangle \right), \]  
(161)
\[ (\nabla_{\hat{n}} 3 q^{ij}) = \frac{1}{3} \kappa 3 q^{ij} \left( \text{tr} k + \frac{1}{4} D (\hat{n}) - \langle \hat{n}^*, \text{tr} \sigma \rangle \right). \]  
(162)

Inserting (161) and (162) to (151) gives:
\[ \Delta^{(ij)} = -\frac{1}{2} \kappa (\tau^{(ij)} + k^{(ij)}) - \frac{1}{2} \hat{n} [\tau] 3 q^{ij} \]  
\[ + \frac{1}{2} \kappa 3 q^{ij} \left( \frac{1}{3} \text{tr} \tau + \frac{2}{3} \text{tr} k + \frac{1}{12} D (\hat{n}) - \frac{1}{3} \langle \hat{n}^*, \text{tr} \sigma \rangle \right). \]  
(163)
while using (163) with (157) and (155) gives:

\[
\Delta^{ij} = \kappa \left( -2 \kappa \tau^{ij} - \frac{1}{2} \left( \tau^{(i \beta)} + k^{(i \beta)} \right) + \frac{3}{2} q^{ij} \left( \frac{1}{3} \text{tr} \ \tau + \frac{2}{3} \text{tr} \ k + \frac{1}{12} D(\hat{n}) - \frac{1}{3} \langle n^* , \text{tr} \ \sigma \rangle \right) \right) - \frac{1}{2} \hat{n} \ [3 q^{ij}] ,
\]

(164)

\[
K^{ij} = K^{ij} = \kappa \left( \langle \hat{n}^* , \sigma^{(i \beta)} \rangle - \frac{1}{2} \left( \tau^{(i \beta)} + k^{(i \beta)} \right) + \frac{3}{2} q^{ij} \left( \frac{1}{8} D(\hat{n}) + \frac{1}{6} \text{tr} \ \tau - \frac{1}{6} \langle n^* , \text{tr} \ \sigma \rangle \right) \right) - \frac{1}{2} \hat{n} \ [3 q^{ij}] .
\]

(165)

Finally, let us raise the index \( i \) of the tensor equation, and then contract it with \( ^3q_{ij} \) to obtain:

\[
\Theta^i + \frac{1}{2} \lambda^i \nabla_j \lambda^{3q^{ij}} = - \frac{1}{2} \left( \Delta^i - \alpha^i \right) + \kappa \left( \frac{1}{8} D^i + \frac{1}{2} q^{i m} \left( \sigma^{k m} + \varepsilon_{m n k l} \right) \right) .
\]

Inserting (148), and using torsionless condition, we obtain:

\[
\Delta^i + \frac{1}{2} \lambda^i \nabla_j \lambda^{3q^{ij}} = \kappa \left( \frac{1}{2} \left( \sigma^i + \frac{1}{12} D^i + \frac{1}{2} q^{i m} \left( \sigma^{k m} + \varepsilon_{m n k l} \right) \right) \right) .
\]

(166)

Using (166) to (149) and (150), one could solve for \( \Delta^i \) and \( ^3\nabla^3q^{ik} \):

\[
\Delta^i = \kappa \left( \frac{1}{2} \langle \hat{n}^* , \sigma^i \rangle - \frac{1}{24} D^i - \frac{1}{4} \left( \text{tr} \ \sigma^i \right) + \frac{1}{4} q^{i m} \left( \sigma^{k m} + \varepsilon_{m n k l} \right) \right) ,
\]

(167)

\[
^3\nabla^3q^{ik} = \frac{1}{3} \kappa \left( - \langle \hat{n}^* , \sigma_i \rangle + 2 \langle \hat{n}^* , k_i \rangle - \frac{1}{12} D_i - \frac{1}{2} \left( \text{tr} \ \sigma_i \right) + \frac{1}{2} \left( \sigma^i + \varepsilon_{i m n k} \right) \right) \ .
\]

(168)

And last, from (148), we could obtain \( \alpha^i \):

\[
\alpha^i = \kappa \left( \frac{1}{2} \langle \hat{n}^* , \sigma^i \rangle - \frac{5}{24} D^i + \frac{1}{2} q^{i m} \left( \sigma^{k m} + \varepsilon_{m n k l} \right) \right) .
\]

(169)

The solutions to the equation of motions in terms of the additional variables, are (161), (162), (164), (165), (167), (168), and (169), together with \( ^3T (\hat{i} \hat{j}, \hat{j} \hat{k}) = 0 \). Meanwhile, there are 2 constraints on \( \mathcal{H} \), namely, (152) and (158), that are the constraint on the antisymmetric part of \( \tau^{ij} \) dan \( k^{ij} \):

\[
\tau^{(i \beta)} = - \frac{1}{2} \varepsilon^{(i \beta) m} \langle \hat{n}^* , l^m \rangle ,
\]

(170)

\[
k^{(i \beta)} = \frac{1}{2} \varepsilon^{(i \beta) m} \langle \hat{n}^* , l^m \rangle .
\]

(171)
Note that the index $i$ on $\tau^{ij}$ dan $k^{ij}$ define the spatial indices, while the $j$ define the ‘internal’ indices. Only the antisymmetric parts of $\tau^{ij}$ dan $k^{ij}$ are constrained.

Finally, using (170) and (171), we could rewrite the equations of the additional variables and collect all the results:

$$\Theta = \kappa \left( \frac{1}{3} \text{tr} \tau + \frac{1}{6} \text{tr} k \right) - \frac{1}{24} \mathcal{D} \left( \hat{n} \right) + \frac{1}{6} \left( \hat{n}, \text{tr} \sigma \right),$$

$$\Theta' = \Delta' = \kappa \left( \frac{1}{2} \left( \hat{n}, \tau \right) - \frac{1}{24} \mathcal{D}' - \frac{1}{4} \left( \text{tr} \sigma \right)' + \frac{1}{4} \left( \sigma_{nk} + \varepsilon_{nkmj} q^m \right) \right),$$

$$\alpha' = \kappa \left( \frac{1}{2} \left( \hat{n}, \tau \right) - \frac{1}{2} \mathcal{D}' - \frac{5}{4} \left( \text{tr} \sigma \right)' + \frac{1}{4} \left( \sigma_{nk} + \varepsilon_{nkmj} q^m \right) \right),$$

$$(\nabla_{k} 3 q^{k}) = \frac{1}{3} \kappa \left( \left\langle \hat{n}, \tau \right\rangle - \frac{1}{4} \mathcal{D} \left( \hat{n} \right) - \left\langle \hat{n}, \text{tr} \sigma \right\rangle \right),$$

$$3 \nabla_{i} 3 q^{i} = \frac{\sqrt{3}}{\kappa} \left( \left\langle \hat{n}, 2k_i - \tau_i \right\rangle - \frac{1}{12} \mathcal{D}_i - \frac{1}{2} \left( \text{tr} \sigma \right) \right) \right)$$

$$(1 + \frac{1}{2} \left( \sigma_{nk} + \varepsilon_{nkmj} q^m \right) \right) 3 q^{i},$$

$$K^{ij} = K^{ij} = \kappa \left( \left( \hat{n}, \sigma^{ij} \right) \right) - \frac{1}{2} \left( \tau^{(k)} + k^{(k)} \right)$$

$$+ \frac{3}{4} q^{i} \left( \frac{1}{8} \mathcal{D} \left( \hat{n} \right) + \frac{1}{6} \text{tr} \tau + \frac{1}{3} \text{tr} k - \frac{1}{6} \left( \hat{n}, \text{tr} \sigma \right) \right) - \frac{1}{2} \hat{n} \left[ 3 q^{ij} \right],$$

$$\Delta^{ij} = \kappa \left( \left( \hat{n}, \sigma^{ij} \right) \right) - \frac{1}{2} \left( \tau^{(k)} + k^{(k)} \right)$$

$$+ \frac{1}{4} q^{i} \left( \frac{1}{3} \text{tr} \tau + \frac{2}{3} \text{tr} k + \frac{1}{12} \mathcal{D} \left( \hat{n} \right) - \frac{1}{3} \left( \hat{n}, \text{tr} \sigma \right) \right) - \frac{1}{2} \hat{n} \left[ 3 q^{ij} \right],$$

with:

$$T^{k}_{ij} = \omega_{i}^{k} - \omega_{j}^{k} = 0.$$
6. Discussions and conclusions

6.1. Discussions

6.1.1. Complete \((3 + 1)\) equations of MAGR system. For the completeness of this article, we manage to split each quantity into its temporal and spatial parts, which give rise to 10 hypersurface variables. Let us review some of the significant results obtained in [28]. First, to make sense of the \((3 + 1)\) decomposition of the hypermomentum tensor \(\mathcal{H}\), let us take an analogy of the formalism introduced in [4], and define a new terminology only based on their symmetries of its internal indices (i.e. dilation \(D\), shear \(\mathbf{\Lambda}\), rotation \(\mathbf{\Omega}\)) [4]. However, these three quantities are still covariant, and in section 4, we manage to split each quantity into its temporal and spatial parts, which give rise to 10 hypersurface variables \(\mathcal{D} = (D(n), \frac{1}{2}D), \tau_l = (\langle n^i, \tau_i \rangle, \tau^l)\), \(\sigma_{ij} = (\langle n^i, \sigma_{ij} \rangle, \sigma^k_{ij}), k_l = (\langle n^i, k_i \rangle, k^l)\), and \(\hat{t}^i = (\langle n^i, \hat{t}^i \rangle, t^l)\). To have a similar analog with the \((3 + 1)\) split of the stress–energy–momentum-tensor \(\mathcal{T}\), let us abandon for the moment the terminology introduced in [4], and define a new terminology only based on the hypersurface splitting as follows. \(\mathcal{H}\) is decomposed into 3 parts: (a) the purely-time part, which is a scalar observable:

\[
\langle \hat{n}^i, \mathcal{H} (\hat{n}, \hat{n}^+ \rangle) := \mathcal{E}, \tag{173}
\]

(b) the mixed part, which contains three three-vector observables:

\[
\langle \hat{n}^i, \mathcal{H} (\hat{n}, \hat{n}^+ \rangle := \psi, \tag{174}
\]

\[
\langle \hat{n}^i, \mathcal{H} (\hat{n}, \hat{n}^+ \rangle := \phi, \tag{175}
\]

and three \(3 \times 3\)-matrix observables:

\[
\langle \hat{n}^i, \mathcal{H} (\hat{n}, \hat{n}^+ \rangle := \mathcal{P}^{ij}, \tag{176}
\]

\[
\langle \hat{n}^i, \mathcal{H} (\hat{n}, \hat{n}^+ \rangle := \mathcal{Q}^{ij}, \tag{177}
\]

\[
\langle \hat{n}^i, \mathcal{H} (\hat{n}, \hat{n}^+ \rangle := \mathcal{R}^{i}_{j}, \tag{178}
\]

and last, (c) the purely spatial part, which is a \(\binom{1}{1}\)-order tensor, or a \(3 \times 3\)-matrix-valued three-vector observable:

\[
\langle \hat{n}^i, \mathcal{H} (\hat{n}, \hat{n}^+ \rangle := h^i_{jk}, \tag{179}
\]

In this terminology, \(\mathcal{H}\) could be constructed entirely from these 8 variables. Unlike \(E, p_s, S_{ij}\), the physical interpretation of each term is unclear and unfamiliar because the matter models that contain hypermomentum are still hypothetical. These 8 variables could be written in terms of...
10 former variables \( \mathcal{D} = \{ \mathcal{D}(\hat{\tau}), \pi^0, \pi^i, \kappa^i_j \}, \tau_i = (-\langle \hat{\pi}_i, \tau \rangle, \tau_i), \sigma_{ij} = (-\langle \hat{\pi}_i, \sigma_{ij} \rangle, \sigma_{ij}^k, k_i = (-\langle \hat{\pi}_i, k_i \rangle, k_i^j \), and \( \hat{\pi} = (-\langle \hat{\pi}_i, \hat{\pi} \rangle, \hat{\pi}^k \) using (173)–(176) and (92)–(95).

Writing the \((3 + 1)\) hypermomentum equation in this new terminology, one could obtain the complete \((3 + 1)\) equations describing MAGR systems (in normal coordinates) as follows: (a) from the stress–energy–momentum equation, containing 1 scalar (energy), 1 vector (momentum), 1 matrix (stress–energy) equations [28]:

\[
\frac{1}{2} \left( 3\mathcal{R} - \text{tr}(\mathcal{K} \mathcal{C}) + (\text{tr} \mathcal{K})(\text{tr} \mathcal{C}) - \left( \mathcal{K}_{ij}^i + \mathcal{K}_{ij}^j \right) \Delta_j - \hat{n} [\text{tr} \mathcal{K}] \right) + \frac{3}{4} q_{i}^{j} h_{i}^{j} + 3 \nabla_i \alpha_i - 3 q^i q^j \nabla_j \Delta_i - \Theta_i (\alpha_i + 3 q^i \Delta_j)
\]

\[
\left. + \Theta (\hat{n}) (\text{tr} \mathcal{K} + \text{tr} \mathcal{C}) \right) = \kappa E,
\]

\[
\frac{1}{2} \left( 3\nabla_i \mathcal{K}_{ij}^i - 3\nabla_i \mathcal{K}_{ij}^j + 3 T_i^j \mathcal{K}_{ij} - \alpha^i (K_{ij} + K_{ji}) \right)
\]

\[
+ 3 \nabla_i \Delta_j - \hat{n} \left( \frac{3}{2} \omega^j_i + (\Delta_i + \Theta_i) \mathcal{K}_{ij}^i + (\Delta_j - \Theta_j) \mathcal{K}_{ij}^j - \partial_i (\Theta_i \hat{n}) + \hat{n} [\Theta_i] \right) = \kappa p_i,
\]

\[
\hat{n} [K_{(ij)}] - \Theta_i \Delta_j - 3 \nabla_i \Delta_j + 3 R_{(ij)}
\]

\[
\left. + (\text{tr} \mathcal{K} + \Theta (\hat{n})) K_{(ij)} - K_{(ij)} \mathcal{K}_{kl} - K_{(ij)} \Delta_{(kl)} = \kappa S_{ij} - \frac{1}{2} q_{j} q_{i} (S - E) \right),
\]

(b) from the \((3 + 1)\) hypermomentum equation, containing 1 scalars, 3 vectors, 3 matrices, 1 tensor equations:

\[
\mathcal{K}_{ij}^i - K_{ij}^i = - \kappa E,
\]

\[
\Delta_i - \alpha_i = \kappa \left( \frac{2}{3} q^i + \frac{1}{2} h_i^j \right),
\]

\[
\frac{1}{2} \frac{3}{4} q_{k} q_{j} + 3 \nabla_{j} q_{k} - \Delta_i = - \kappa \varphi_i,
\]

\[
\Delta_i - 2 \Theta_i + 3 \nabla_j q^j - \frac{1}{2} q_{k} q_{j} + 3 \nabla_{i} q_{k} = \kappa \left( \frac{1}{3} (h_i^j - \Theta_i) - \varphi_i \right),
\]

\[
\Delta_{i} - \mathcal{K}_{ij} = - \nabla_i \mathcal{K}_{ij} + \left( \Theta (\hat{n}) + \frac{1}{2} q_{k} \nabla_{j} q_{k} \right) q_{i}^{j} = \kappa P_{i}^j,
\]

\[
\left( \mathcal{K}_{ij} + \Theta (\hat{n}) - \frac{1}{2} q_{k} \nabla_{j} q_{k} \right) d_{i}^{j} + K_{ij} - \Delta_i = - \kappa \left( \frac{1}{3} (\mathcal{E} - \mathcal{R}_{ij}^k) d_{i}^{j} - Q_{i}^j \right),
\]

\[
\left( \mathcal{K}_{ij}^i - K_{ij}^i + \Theta (\hat{n}) - \frac{1}{2} q_{k} \nabla_{j} q_{k} \right) d_{i}^{j} + K_{ij} - \Delta_i = - \kappa \left( \frac{1}{3} (\mathcal{E} - \mathcal{R}_{ij}^k) d_{i}^{j} - Q_{i}^j \right),
\]
\[ K^i_j - K^j_i = -\kappa \left( \frac{1}{3} \left( \mathcal{E} - \mathcal{R}^k_k \right) \delta^j_i + \mathcal{R}^j_i \right) \]  

(186)

\[ 3T_{ij}^{\;\,jk} + \left( \Theta_i + \frac{1}{2} q_{lm} 3\nabla_i 3q^{lm} \right) 3q^{ik} - 3\nabla_i 3q^{jk} \]  

(187)

and last, (c) from the traceless torsion constraint, containing 1 scalar and 1 vector equations (106) and (107) which are only additional constraints on the variables. Note that if the connection is either metric or torsionless, (106) and (107) are not needed.

Now let us analyze the dynamics of the general MAGR system. First, in the standard GR, we have 10 equations of motion, the scalar Hamiltonian constraint (1 variable), the vector momentum constraint (3 variables), and the dynamics (6 independent variables) [28, 32, 44]:

\[ \frac{1}{2} \left( 3\mathcal{R} - \text{tr} \left( K^2 \right) + (\text{tr} K)^2 \right) = \kappa E, \]  

(188)

\[ 3\nabla_j K^i_j - 3\nabla_i K^j_i = \kappa p_i, \]  

(189)

\[ \hat{n} \left[ K_{ij} \right] + \frac{3}{2} R_{ij} + (\text{tr} K) K_{ij} - 2K_i^k K_{kj} = \kappa S_{ij} - \frac{1}{2} 3 q_{ij} \kappa (S - E). \]  

(190)

Given the information of \( E, p_i, S_{ij} \) (which completely construct \( \sigma \)), (188)–(190) needs to be solved to obtain 10 unknowns variables: the lapse \( N \), and the shift \( N \), and the three-metric \( 3q_{ij} \). Obtaining these 10 variables is equivalent to obtaining the information of \( g_{\mu\nu} \) in the Lagrangian framework. On the other hand, in the MAGR case, we have 3 Euler–Lagrange equations (74)–(76); this gives 74 independent equations: 10 from the stress–energy–momentum equations (177)–(179), 60 from hypermomentum equations (180)–(187), 4 from traceless torsion constraint (106) and (107). Given the information of \( E, p_i, S_{ij} \) and \( E, \theta^i, \phi^i, \varphi^i, P^i, Q_{ij}, R_j^i, h^i_j \) (or 10 in Hehl terminology), there are 74 unknowns that need to be solved: \( N, N, 3q_{ij} \), and the 8 additional variables: (1 for \( \Theta (\hat{n}) \), 3 for each \( \Theta_i, \Delta_i, 3\alpha_i \), 9 for \( K, K, \Delta^j_i \), and 27 for \( 3\omega \)). The numbers of the independent equations and the number of the unknowns in an MAGR system are equivalent. This gives a good start for the well-posedness of the Cauchy problem, but this is not in the scope of our article.

The second difference with the standard GR is the number of the constraint and dynamics in the system. As discovered in [28], the \( (3+1) \) decomposition of the generalized stress–energy tensor (177)–(179) do not contain constraint equations; all of the 10 equations are dynamical (all of them contain derivative of time). Only when the hypermomentum is zero then the connection becomes Levi-Civita, thus the generalized energy and momentum equations return to the standard Hamiltonian and diffeomorphism constraints in GR. In the \( (3+1) \) hypermomentum equations (180)–(187), 6 of them are constraint equations, only (184) and (185) of the hypermomentum equation are dynamical, which contain time derivative terms in the form of \( \nabla^a \delta^j_i q^{ij} \). These dynamical terms on the hypermomentum equation are uniquely a consequence of the non-metricity condition. It is clear that the remaining traceless torsion (106) and (107) are also constraint equations. Hence, for a complete general MAGR system, we have 5 dynamical equations and 8 constraint equations.
6.1.2. The relations between stress–energy and hypermomentum tensor. As mentioned clearly in [8], the stress–energy tensor $T$ is not completely independent from the hypermomentum tensor $H$. This is because they originate from the same source $S_{\text{matter}}$. The relation comes from the fact that $\delta_{\omega_S} S_{\text{matter}} = \delta_S S_{\text{matter}}$. With definitions (40) and (58), one could derive [8]:

$$
\partial T_{\mu\nu} \partial \omega_{\alpha\beta} = \partial H_{\alpha\beta} \partial g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} H_{\alpha\beta} = \frac{1}{\sqrt{-g}} \partial \left( \sqrt{-g} H_{\alpha\beta} \right),
$$

(191)

which is a relation that must be satisfied by both $T$ and $H$. For the zero hypermomentum case, $H_{\alpha\beta} = 0$, and therefore, the stress–energy tensor needs to satisfy $\partial T_{\mu\nu} \partial \omega_{\alpha\beta} = 0$, i.e. $T$ is independent from the connection $\omega$. For the next two cases, i.e. the metric or torsionless cases, the hypermomentum are constrained (equation (145) for metric and (170) and (171) for torsionless case). These will restrict the degrees of freedom of the corresponding stress–energy tensor (in the set of equations (146) for metric and (172) for torsionless case), via relation (191). One could apply the $(3+1)$ formulation to relation (191), but this is out of the scope in this article.

6.1.3. Comments on the symmetricity constraint. Without the symmetricity constraint (70), the Euler–Lagrange equation (67) will become:

$$
f' (R) R_{\alpha\beta} = -\frac{1}{2} \left( f (R) + \chi^{\mu} T_{\lambda \mu} \right) g_{\alpha\beta} = \kappa T_{\alpha\beta}.
$$

(192)

Even with reducing the degrees of freedom of the general affine connection $\omega$ via the traceless torsion constraint (69), the Ricci tensor $R_{\alpha\beta}$ will still possess the antisymmetric part:

$$
R_{[\alpha\beta]} = \frac{1}{2} \left( R_{\alpha\beta} - R_{\beta\alpha} \right) = \frac{1}{2} \left( \nabla_\mu T_{\mu \alpha\beta} + \nabla_\alpha T_{\mu\beta \mu} + \nabla_\beta T_{\mu\mu \alpha} - R_{\alpha\beta} \mu_\mu - T_{\sigma\alpha\beta} T_{\mu\mu \sigma} \right).
$$

(193)

Inserting the traceless torsion constraint (69) to (193) will only give:

$$
R_{[\alpha\beta]} = \frac{1}{2} \left( \nabla_\mu T_{\mu \alpha\beta} - R_{\alpha\beta} \mu_\mu \right),
$$

(194)

where $R_{\alpha\beta} \mu_\mu$ is the homothetic curvature that arise from the non-metricity condition [13–15]. Since $g_{\alpha\beta}$ and $T_{\alpha\beta}$ is symmetric, from (192), we obtain another condition from the dynamics, relating the torsion and the non-metricity as follows:

$$
\nabla_\mu T_{\mu \alpha\beta} = R_{\alpha\beta} \mu_\mu.
$$

This unnecessary constraint could be avoided by introducing the symmetricity constraint (70) in the level of the action, such that the antisymmetric part $R_{[\alpha\beta]}$ simply does not enter the equation of motion, see section 3.

6.2. Conclusions and further remarks

Let us summarize the results achieved from our work. First, we have performed the $(3+1)$-decomposition to the torsion tensor and the non-metricity factor. These are equations (12)–(14) and (20)–(23) in section 2, which are written using the additional variables (see [28]). Second, we have given a complete derivation of the equation of motion of metric-affine $f (R)$-gravity, i.e. (71)–(73); these equations are well-known, and we do not add some new results, except a
different way to rederive the hypermomentum equation and the explicit use of the symmetricity constraint. Third, following [4], we decomposed the hypermomentum tensor into the dilatation, shear, and rotational (spin) parts and then split each part into its temporal and spatial components. Hence, we have 10 hypersurface variables that are responsible for the construction of the hypermomentum tensor. The main result of this article is the $(3+1)$-formulation of the hypermomentum equation, together with the $(3+1)$ traceless torsion constraint. They are obtained by inserting the $(3+1)$ torsion tensor, $(3+1)$ non-metricity factor, and $(3+1)$ hypermomentum tensor from the previous sections to equations (75) and (76). These resulting equations are given at the end of section 4, containing 2 scalar equations, 4 vector equations, 3 matrix equations, and 1 tensor equation of order $(\frac{3}{2})$. We provided several special cases for the $(3+1)$ hypermomentum equation: the zero hypermomentum, the metric connection, and the torsionless connection case in section 5.

Collecting the results in this article with our previous results in [28], we have obtained a complete $(3+1)$-formulation for MAGR, or generalized Palatini gravity. The first difference of this theory with the standard GR lies in the connection: in MAGR, the connection is not constrained. This difference manifests in the $(3+1)$ equation of motion. Similar to the standard GR, in the MAGR, the covariant stress--energy--momentum equation is decomposed into 3 canonical equations: the generalized energy (or Hamiltonian) equation, the generalized momentum (or the diffeomorphism) equation, and the generalized stress--energy equation. However, unlike the $(3+1)$ standard GR, none of the $(3+1)$ MAGR are constraint equations since all of them contain the derivative of quantities with respect to the time coordinate. The second main difference between MAGR and standard GR is the dependence of matter on the connection, which in MAGR will give rise to the hypermomentum tensor. In the standard Palatini gravity, this term vanished, and with the traceless torsion constraint, it gives equivalent dynamics to the standard GR. In our works, we have shown that for vanishing hypermomentum, the $(3+1)$ hypermomentum equation, together with traceless torsion constraint, are equivalent to the metric compatibility and torsionless condition, hence constraining the affine connection to be Levi-Civita. Furthermore, with the metric compatibility and torsionless condition, one could retrieve the Hamiltonian and diffeomorphism constraint from the generalized energy and momentum equation, hence obtaining the standard dynamics of GR.

As mentioned in [28], there are several subtleties we encounter along with our work. The first subtlety is technical, concerning the tracelessness of the torsion tensor, which is given by hand, or in other words, is introduced at the kinematical level via the projective transformation. First, taking the tracelessness of the torsion tensor is not the only way to solve the projective invariance problem; one could equally choose the Weyl vector to be zero, or more generally, the linear combination of both the trace of torsion and the latter, as discussed in [8, 14]. The choice will affect the form of the resulting $(3+1)$ equations. From the theoretical perspective, it is important to obtain the $(3+1)$ formulation in the most general setting. Second, it is more favorable to introduce the tracelessness of torsion (or moreover, the vanishing of Weyl vector) at the dynamical level, where for the traceless torsion case, this had been done in [49] by adding extra terms in the action, at least quadratic in torsion. Another more phenomenological subtlety is on the physical interpretation of the hypermomentum tensor. There has been ongoing research on the candidates of matter that will give a non-vanishing hypermomentum, and hence causing the torsion and non-metricity in spacetime. Some progress on this problem had been addressed in [50–62], and research in this direction is highly encouraged. We expect our works in this article could contribute to further research in the direction of the complete Hamiltonian analysis, the existence of the Cauchy problem, and some extension to $(3+1)$ metric-affine-$f(\mathcal{R})$ and other metric-affine theories.
Acknowledgment

AS was funded by Riset Institut Teknologi Bandung. FPZ was funded by Riset Institut Teknologi Bandung and RISTEKDIKTI.

Data availability statement

No new data were created or analysed in this study.

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