An Application of Okada’s Minor Summation Formula
to the Evaluation of a Multiple Integral
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1. INTRODUCTION

Noam Elkies and Everett Howe [1] independently noticed a certain elegant product
formula for the multiple integral
\[
\int_R \prod_{1 \leq i < j \leq k} (x_j - x_i) dx_1 \cdots dx_k,
\] (1)
where the region \( R \) is the set of \( k \)-tuples \((x_1, \ldots, x_k)\) satisfying \( 0 < x_1 < \cdots < x_k < 1 \). Later on Howe discovered that the formula was a special case of a formula for Selberg’s
integral described for example in [2], page 339.

Here we prove an apparently different generalization
\[
\int_R \det \left( x_i^{a_j - 1} \right) dx_1 \cdots dx_k = \frac{\prod_{1 \leq i < j \leq k} (a_j - a_i)}{\prod_{1 \leq i \leq k} a_i \prod_{1 \leq i < j \leq k} (a_j + a_i)}.
\] (2)
The original observation is the case \( a_j = j \).

The most interesting aspect of our proof is that we apply a limiting form of a remark-
able identity of Okada [3] for summing the \( k \) by \( k \) minors of an \( n \) by \( k \) matrix.

First we describe Okada’s formula.

Suppose that \( C \) is an \( n \) by \( k \) matrix. For \( 1 \leq i, j \leq k \) define
\[
S_{ij} = \sum_{1 \leq t < u \leq n} (C_{ti}C_{uj} - C_{tj}C_{ui}).
\]
Note that \( S \) is a skew-symmetric matrix. \( S_{ij} \) is the sum of all two-by-two minors of the
submatrix of \( C \) formed from columns \( i \) and \( j \) (in that order).

If \( k \) is even, then we can form the Pfaffian of \( S \), \( \text{Pf}(S) \), given as
\[
\sum_{\pi} \text{sg}(\pi) S_{\pi(1), \pi(2)} S_{\pi(3), \pi(4)} \cdots S_{\pi(k-1), \pi(k)},
\]
where the sum is over all permutations of \( \{1, \ldots, k\} \) satisfying
\[
\pi(1) < \pi(2), \quad \pi(3) < \pi(4), \quad \ldots, \quad \pi(k-1) < \pi(k)
\]
and
\[
\pi(1) < \pi(3) < \cdots < \pi(k-1).
\]
It is known that the square of the Pfaffian is the determinant of \( S \).

Okada’s formula states that the sum of the \( k \) by \( k \) minors of \( C \) is \( \text{Pf}(S) \). It is easily
proved. Indeed it suffices to prove the formula in the easy case of matrices \( C \) with a single
If \( k \) is odd, we can form \( S \) but not its Pfaffian. However, Okada observes that we can still sum the \( k \) by \( k \) minors of \( C \). We simply adjoin a zeroth row and zeroth column to \( C \) which are both all zero except for a single 1 at their intersection. The sum of the \( k+1 \) by \( k+1 \) minors of this augmented matrix is the same as the sum of the \( k \) by \( k \) minors of \( C \). So we can apply the preceding case to the augmented matrix. The effect on \( S \) is to add a zeroth row and a zeroth column with \( S_{0j} = -S_{j0} = \sum_{1 \leq t \leq n} C_{tj} \), the \( j \)th column sum of \( C \). We can then sum the minors of \( C \) by computing the Pfaffian of the augmented \( S \).

Now let us return to the multiple integral. Notice that, for \( x_1 < \cdots < x_k \), \( \det \left( \frac{x_i^{a_j-1}}{x_i^{a_j-1}} \right) \) can be regarded as a minor of the \( \infty \) by \( k \) matrix whose rows are indexed by \( x \) with \( 0 < x < 1 \) with the \( x \)th row equal to

\[
x^{a_1-1} \quad x^{a_2-1} \quad \ldots \quad x^{a_k-1}.
\]

Thus the integral can be regarded as a sort of sum of minors to which Okada’s formula can be applied.

It is not difficult to make this idea more precise. In fact there is an integral form of Okada’s identity.

**THEOREM.** Let \( f_1, \ldots, f_k \) be continuous functions on the interval \([a, b]\). Let

\[
I = \int_{a < x_1 < \cdots < x_k < b} \det(f_i(x_j)) \, dx_1 \cdots dx_k,
\]

\[
I_{ij} = \int_{a < x < y < b} (f_i(x)f_j(y) - f_j(x)f_i(y)) \, dxdy,
\]

and

\[
I_i = \int_{a < x < b} f_i(x) \, dx.
\]

If \( k \) is even, then \( I \) is the Pfaffian of the \( k \) by \( k \) matrix \( I_{ij} \), \( 1 \leq i, j \leq k \). If \( k \) is odd, then \( I \) is the Pfaffian of the \( k+1 \) by \( k+1 \) matrix which is \( I_{ij} \) augmented with a zeroth row and column defined by \( I_{00} = 0 \) and \( I_{0i} = -I_{i0} = I_i \) for \( 1 \leq i \leq k \).

**Proof.** Let \( n \) be an integer, \( h = (b-a)/n \) and \( x_i = a + hi, i = 0, \ldots, n \). Consider the matrix

\[
C = hf_j(x_i)_{0 \leq i \leq n, 1 \leq j \leq k}.
\]

Let \( S \) be the sum of the \( k \) by \( k \) minors of \( C \), let \( S_{ij} \) be the sum of the two by two minors of the matrix formed from columns \( i \) and \( j \) of \( C \) (in that order) and let \( S_i \) be the sum of the entries of column \( i \) of \( C \). \( S, S_i, \) and \( S_{ij} \) all depend on \( n \). Standard properties of integrals yield

\[
I = \lim_{n \to \infty} S, \quad I_i = \lim_{n \to \infty} S_i, \quad I_{ij} = \lim_{n \to \infty} S_{ij}.
\]

The rest then follows from Okada’s formula.
We can now apply Theorem 1 to prove (2). Let

\[ I = \int_R \det \left( x_i^{a_{j-1}} \right) dx_1 \cdots dx_k. \]

Then routine integration shows that

\[ I_i = \frac{1}{a_i}, \quad I_{ij} = \frac{(a_j - a_i)}{a_i a_j (a_i + a_j)}. \]  \hspace{1cm} (3)

From Theorem 1, if \( k \) is even, \( I \) is the Pfaffian of the \( k \) by \( k \) matrix

\[
\begin{bmatrix}
0 & \frac{a_2 - a_1}{a_1 a_2 (a_1 + a_2)} & \frac{a_3 - a_1}{a_1 a_3 (a_1 + a_3)} & \cdots & \frac{a_k - a_1}{a_1 a_k (a_1 + a_k)} \\
\frac{a_1 - a_2}{a_1 a_2 (a_1 + a_2)} & 0 & \frac{a_3 - a_2}{a_2 a_3 (a_2 + a_3)} & \cdots & \frac{a_k - a_2}{a_2 a_k (a_2 + a_k)} \\
\frac{a_1 - a_3}{a_1 a_3 (a_1 + a_3)} & \frac{a_2 - a_3}{a_2 a_3 (a_2 + a_3)} & 0 & \cdots & \frac{a_k - a_3}{a_3 a_k (a_3 + a_k)} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{a_1 - a_k}{a_1 a_k (a_1 + a_k)} & \frac{a_2 - a_k}{a_2 a_k (a_2 + a_k)} & \cdots & \frac{a_k - a_k}{a_k a_k (a_k + a_k)} & 0
\end{bmatrix}
\]

while if \( k \) is odd, \( I \) is the Pfaffian of the \( k + 1 \) by \( k + 1 \) matrix

\[
\begin{bmatrix}
0 & \frac{1}{a_1} & \frac{1}{a_2} & \cdots & \frac{1}{a_k} \\
-\frac{1}{a_1} & 0 & \frac{a_2 - a_1}{a_1 a_2 (a_1 + a_2)} & \cdots & \frac{a_k - a_1}{a_1 a_k (a_1 + a_k)} \\
-\frac{1}{a_2} & \frac{1}{a_1 a_2 (a_1 + a_2)} & 0 & \cdots & \frac{a_k - a_2}{a_2 a_k (a_2 + a_k)} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{a_1}{a_k} & \frac{a_1 - a_k}{a_k a_k (a_1 + a_k)} & \cdots & \frac{a_k - a_k}{a_k a_k (a_k + a_k)} & 0
\end{bmatrix}
\]

In each case the determinant is a homogeneous rational function of degree \(-2k\) so the Pfaffian is a homogeneous rational function of degree \(-k\). However, since \( I \) vanishes when any two of the \( a \)'s are equal, we know that the numerator of \( I \) must contain a factor of \( N = \prod_{1 \leq i < j \leq k} (a_j - a_i) \). Hence the denominator is of degree \( k + k(k - 1)/2 + e \), where \( e \) is the degree of the remaining factor of the numerator. On the other hand one also sees directly from the Pfaffian expressions that the denominator must divide \( D = \prod_{1 \leq i < k} a_i \prod_{1 \leq i < j \leq k} (a_i + a_j) \). It follows that \( e \) is zero and that the numerator of \( I \) is \( N \) to within a constant factor and the denominator \( D \) to within a constant factor. Thus the left side of (2) is equal to the right to within a constant factor.

We now need to show that this constant factor is 1. This is certainly true for \( k = 1 \) and \( k = 2 \) by (3). For larger \( k \), when \( k \) is even, if we multiply the first row and column of the Pfaffian matrix by \( a_1 \) and then set \( a_1 = 0 \) we obtain the matrix for the case \( k - 1 \) (applied to \( a_2, \ldots, a_k \)). Similarly when \( k \) is odd, if we multiply the second row and second column of the Pfaffian by \( a_1 \), we can easily reduce to the case \( k - 1 \). On the other hand the right side of (2) also has the property that when we multiply it by \( a_1 \) and then set \( a_1 = 0 \) we obtain the right side of (2) for \( k - 1 \) (applied to \( a_2, \ldots, a_k \)). These observations yield an immediate inductive proof that the constant is 1.
REFERENCES

1. Everett Howe, Oral communication.
2. Madan Lal Mehta, Random Matrices, Academic Press, 1991.
3. Soichi Okada, On the Generating Functions for Certain Classes of Plane Partitions, Journal of Combinatorial Theory, Volume 1, Series A, pages 1–23, 1989.