UNBOUNDED GENERALIZATIONS OF THE FUGLEDE-PUTNAM THEOREM AND APPLICATIONS TO THE COMMUTATIVITY OF SELF-ADJOINT OPERATORS

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Abstract. In this article, we prove and disprove several generalizations of unbounded versions of the Fuglede-Putnam theorem. As applications, we give conditions guaranteeing the commutativity of a bounded self-adjoint operator with an unbounded closed symmetric operator.

1. Essential background

All operators considered here are linear but not necessarily bounded. If an operator is bounded and everywhere defined, then it belongs to $B(H)$ which is the algebra of all bounded linear operators on $H$ (see [22] for its fundamental properties).

Most unbounded operators that we encounter are defined on a subspace (called domain) of a Hilbert space. If the domain is dense, then we say that the operator is densely defined. In such case, the adjoint exists and is unique.

Let us recall a few basic definitions about non-necessarily bounded operators. If $S$ and $T$ are two linear operators with domains $D(S)$ and $D(T)$ respectively, then $T$ is said to be an extension of $S$, written as $S \subset T$, if $D(S) \subset D(T)$ and $S$ and $T$ coincide on $D(S)$.

An operator $T$ is called closed if its graph is closed in $H \oplus H$. It is called closable if it has a closed extension. The smallest closed extension of it is called its closure and it is denoted by $\overline{T}$ (a standard result states that a densely defined $T$ is closable iff $T^*$ has a dense domain,

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and in which case $T = T^{**}$. If $T$ is closable, then

$$S \subset T \Rightarrow \overline{S} \subset \overline{T}.$$ 

If $T$ is densely defined, we say that $T$ is self-adjoint when $T = T^*$; symmetric if $T \subset T^*$; normal if $T$ is closed and $TT^* = T^*T$.

The product $ST$ and the sum $S + T$ of two operators $S$ and $T$ are defined in the usual fashion on the natural domains:

$$D(ST) = \{x \in D(T) : Tx \in D(S)\}$$

and

$$D(S + T) = D(S) \cap D(T).$$

In the event that $S$, $T$ and $ST$ are densely defined, then

$$T^*S^* \subset (ST)^*,$$

with the equality occurring when $S \in B(H)$. If $S + T$ is densely defined, then

$$S^* + T^* \subset (S + T)^*$$

with the equality occurring when $S \in B(H)$.

Let $T$ be a linear operator (possibly unbounded) with domain $D(T)$ and let $B \in B(H)$. Say that $B$ commutes with $T$ if

$$BT \subset TB.$$ 

In other words, this means that $D(T) \subset D(TB)$ and

$$BTx = TBx, \forall x \in D(T).$$

Let $A$ be an injective operator (not necessarily bounded) from $D(A)$ into $H$. Then $A^{-1} : \text{ran}(A) \to H$ is called the inverse of $A$, with $D(A^{-1}) = \text{ran}(A)$.

If the inverse of an unbounded operator is bounded and everywhere defined (e.g. if $A : D(A) \to H$ is closed and bijective), then $A$ is said to be boundedly invertible. In other words, such is the case if there is a $B \in B(H)$ such that

$$AB = I \text{ and } BA \subset I.$$ 

If $A$ is boundedly invertible, then it is closed.

The resolvent set of $A$, denoted by $\rho(A)$, is defined by

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is bijective and } (\lambda I - A)^{-1} \in B(H)\}.$$ 

The complement of $\rho(A)$, denoted by $\sigma(A)$,

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$

is called the spectrum of $A$. 
Recall also that the product of two closed operators need not be closed (see [24]). However, and it is known (among other results), that $TS$ is closed if $T$ is closed and $S \in B(H)$ or if $T^{-1}$ is in $B(H)$ and $S$ is closed.

If a symmetric operator $T$ is such that $\langle Tx, x \rangle \geq 0$ for all $x \in D(T)$, then we say that $T$ is positive, and we write $T \geq 0$. When $T$ is self-adjoint and $T \geq 0$, then we can define its unique positive self-adjoint square root, which we denote by $\sqrt{T}$.

If $T$ is densely defined and closed, then $T^*T$ (and $TT^*$) is self-adjoint and positive (a celebrated result due to von-Neumann, see e.g. [30]). So, when $T$ is closed then $T^*T$ is self-adjoint and positive whereby it is legitimate to define its square root. The unique positive self-adjoint square root of $T^*T$ is denoted by $|T|$. It is customary to call it the absolute value or modulus of $T$. If $T$ is closed, then (see e.g. Lemma 7.1 in [30])

$$D(T) = D(|T|) \text{ and } \|Tx\| = ||T|x||, \forall x \in D(T).$$

Next, we recall some definitions of unbounded non-normal operators. A densely defined operator $A$ with domain $D(A)$ is called hyponormal if

$$D(A) \subset D(A^*) \text{ and } \|A^*x\| \leq \|Ax\|, \forall x \in D(A).$$

A densely defined linear operator $A$ with domain $D(A) \subset H$, is said to be subnormal when there are a Hilbert space $K$ with $H \subset K$, and a normal operator $N$ with $D(N) \subset K$ such that

$$D(A) \subset D(N) \text{ and } Ax = Nx \text{ for all } x \in D(A).$$

In the end, we recall some basic facts about matrices of non-necessarily bounded operators. Let $H$ and $K$ be two Hilbert spaces and let $A : H \oplus K \to H \oplus K$ (we may also use $H \times K$ instead of $H \oplus K$) be defined by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where $A_{11} \in L(H)$, $A_{12} \in L(K,H)$, $A_{21} \in L(H,K)$ and $A_{22} \in L(K)$ are not necessarily bounded operators. If $A_{ij}$ has a domain $D(A_{ij})$ with $i,j = 1,2$, then

$$D(A) = (D(A_{11}) \cap D(A_{21})) \times (D(A_{12}) \cap D(A_{22}))$$

is the natural domain of $A$. So if $(x_1, x_2) \in D(A)$, then

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{pmatrix}. $$
Also, recall that the adjoint of \[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\] is not always \[
\begin{pmatrix}
A_{11}^* & A_{21}^* \\
A_{12}^* & A_{22}^*
\end{pmatrix}
\] (even when all domains are dense including the main domain \(D(A)\)) as known counterexamples show. Nonetheless, e.g.,
\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}^* = \begin{pmatrix}
A^* & 0 \\
0 & B^*
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
0 & C \\
D & 0
\end{pmatrix}^* = \begin{pmatrix}
0 & D^* \\
C^* & 0
\end{pmatrix}
\]

if \(A, B, C\) and \(D\) are all densely defined.

2. Introduction

The aim of this paper is twofold. In the first part, we obtain some generalizations of the Fuglede-Putnam theorem involving unbounded operators. In the second part, we apply the Fuglede-Putnam theorem to obtain conditions guaranteeing the commutativity of self-adjoint operators, one of them is bounded.

Recall that the original version of the Fuglede-Putnam theorem reads:

**Theorem 2.1.** ([10], [28]) If \(A \in B(H)\) and if \(M\) and \(N\) are normal (non necessarily bounded) operators, then
\[
AN \subset MA \implies AN^* \subset M^* A.
\]

There have been many generalizations of the Fuglede-Putnam theorem since Fuglede’s paper. However, most generalizations were devoted to relaxing the normality assumption. Apparently, the first generalization of the Fuglede theorem to an unbounded \(A\) was established in [27]. Then the first generalization involving unbounded operators of the Fuglede-Putnam theorem is:

**Theorem 2.2.** If \(A\) is a closed and symmetric operator and if \(N\) is an unbounded normal operator, then
\[
AN \subset N^* A \implies AN^* \subset NA
\]
whenever \(D(N) \subset D(A)\).

In fact, the previous result was established in [18] under the assumption of the self-adjointness of \(A\). However, and by scrutinizing the proof in [18] or [19], it is seen that only the closedness and the symmetricity of \(A\) were needed. Other unbounded generalizations may be consulted in [21] and [3] and some of the references therein. In the end, readers may wish to consult the survey [25] exclusively devoted to the Fuglede-Putnam theorem and its applications.

In the second part of this manuscript, we continue the investigations initiated in the thesis [19], and then in [14] inter alia. More precisely, we show that if \(B \in B(H)\) is self-adjoint and \(A\) is densely defined,
closed and symmetric, then $BA \subset AB$ given that $AB$ or $BA$ is e.g. normal.

3. Generalizations of the Fuglede-Putnam theorem

If a densely defined operator $N$ is normal, then so is its adjoint. However, if $N^*$ is normal, then $N^{**}$ does not have to be normal (unless $N$ itself is closed). A simple counterexample is to take the identity operator $I_D$ restricted to some unclosed dense domain $D \subset H$. Then $I_D$ cannot be normal for it is not closed. But, $(I_D)^* = I$ which is the full identity on the entire $H$, is obviously normal. Notice in the end that if $N$ is a densely defined closable operator, then $N^*$ is normal if and only if $N$ is.

The first improvement is that in the very first version by B. Fuglede, the normality of the operator is not needed as only the normality of its closure will do. This observation has already appeared in [5], but we reproduce the proof here.

**Theorem 3.1.** Let $B \in B(H)$ and let $A$ be a densely defined and closable operator such that $\overline{A}$ is normal. If $BA \subset AB$, then

$$BA^* \subset A^*B.$$  

*Proof.* Since $\overline{A}$ is normal, $A^* = \overline{A^*}$ remains normal. Now,

$$BA \subset AB \implies B^*A^* \subset A^*B^* \quad \text{(by taking adjoints)}$$

$$\implies B^*\overline{A} \subset \overline{A}B^* \quad \text{(by using the classical Fuglede theorem)}$$

$$\implies BA^* \subset A^*B \quad \text{(by taking adjoints again),}$$

establishing the result. $\square$

*Remark.* Notice that $BA^* \subset A^*B$ does not yield $BA \subset AB$ even in the event of the normality of $A^*$ (see [24]).

Let us now turn to the extension of the Fuglede-Putnam version. A similar argument to the above one could be applied.

**Theorem 3.2.** Let $B \in B(H)$ and let $N, M$ be densely defined closable operators such that $\overline{N}$ and $\overline{M}$ are normal. If $BN \subset MB$, then

$$BN^* \subset M^*B.$$  

*Proof.* Since $BN \subset MB$, it ensues that $B^*M^* \subset N^*B^*$. Taking adjoints again gives $B\overline{N} \subset \overline{M}B$. Now, apply the Fuglede-Putnam theorem to the normal $\overline{N}$ and $\overline{M}$ to get the desired conclusion

$$BN^* \subset M^*B.$$  

$\square$
Jabłoński et al. obtained in [15] the following version.

**Theorem 3.3.** If \( N \) is a normal (bounded) operator and if \( A \) is a closed densely defined operator with \( \sigma(A) \neq \mathbb{C} \), then:

\[
NA \subset AN \iff g(N)A \subset Ag(N)
\]

for any bounded complex Borel function \( g \) on \( \sigma(N) \). In particular, we have \( N^*A \subset AN^* \).

**Remark.** It is worth noticing that B. Fuglede obtained, long ago, in [11] a unitary \( U \in B(H) \) and a closed and symmetric \( T \) with domain \( D(T) \subset H \) such that \( UT \subset TU \) but \( U^*T \not\subset TU^* \).

Next, we give a generalization of Theorem 3.3 to an unbounded \( N \), and as above, only the normality of \( N \) is needed.

**Theorem 3.4.** Let \( p \) be a one variable complex polynomial. If \( N \) is a densely defined closable operator such that \( \sigma[p(A)] \neq \mathbb{C} \), then

\[
NA \subset AN \iff N^*A \subset AN^*
\]

whenever \( D(A) \subset D(N) \).

**Remark.** This is indeed a generalization of the bounded version of the Fuglede theorem. Observe that when \( A, N \in B(H) \), then \( \overline{N} = N \), \( D(A) = D(N) = H \), and \( \sigma[p(A)] \) is a compact set.

**Proof.** First, we claim that \( \sigma(A) \neq \mathbb{C} \), whereby \( A \) is closed. Let \( \lambda \) be in \( \mathbb{C} \setminus \sigma[p(A)] \). Then, and as in [8], we obtain

\[
p(A) - \lambda I = (A - \mu_1 I)(A - \mu_2 I) \cdots (A - \mu_n I)
\]

for some complex numbers \( \mu_1, \mu_2, \cdots, \mu_n \). By consulting again [8], readers see that \( \sigma(A) \neq \mathbb{C} \).

Now, let \( \lambda \in \rho(A) \). Then

\[
NA \subset AN \iff NA - \lambda N \subset AN - \lambda N = (A - \lambda I)N.
\]

Since \( D(A) \subset D(N) \), it is seen that \( NA - \lambda N = N(A - \lambda I) \). So

\[
N(A - \lambda I) \subset (A - \lambda I)N \iff (A - \lambda I)^{-1}N \subset N(A - \lambda I)^{-1}.
\]

Since \( \overline{N} \) is normal, we may now apply Theorem 3.1 to get

\[
(A - \lambda I)^{-1}N^* \subset N^*(A - \lambda I)^{-1}
\]

because \( (A - \lambda I)^{-1} \in B(H) \). Hence

\[
N^*A - \lambda N^* \subset N^*(A - \lambda I) \subset (A - \lambda I)N^* = AN^* - \lambda N^*.
\]

But

\[
D(AN^*) \subset D(N^*) \quad \text{and} \quad D(N^*A) \subset D(A) \subset D(N) \subset D(\overline{N}) = D(N^*).
\]
Thus, $D(N^*A) \subset D(AN^*)$, and so
\[ N^*A \subset AN^* , \]
as needed. \hfill \square

Now, we present a few consequences of the preceding result. The first one is given without proof.

**Corollary 3.5.** If $N$ is a densely defined closable operator such that $N$ is normal and if $A$ is an unbounded self-adjoint operator with $D(A) \subset D(N)$, then
\[ NA \subset AN \implies N^*A \subset AN^* . \]

**Corollary 3.6.** If $N$ is a densely defined closable operator such that $N$ is normal and if $A$ is a boundedly invertible operator, then
\[ NA \subset AN \implies N^*A \subset AN^* . \]

**Proof.** We may write
\[ NA \subset AN \implies NAA^{-1} \subset ANA^{-1} \implies A^{-1}N \subset NA^{-1} . \]
Since $A^{-1} \in B(H)$ and $N$ is normal, Theorem 3.1 gives
\[ A^{-1}N^* \subset N^*A^{-1} \text{ and so } N^*A \subset AN^* , \]
as needed. \hfill \square

A Putnam’s version seems impossible to obtain unless strong conditions are imposed. However, the following special case of a possible Putnam’s version is worth stating and proving. Besides, it is somewhat linked to the important notion of anti-commutativity (cf. [32]).

**Proposition 3.7.** If $N$ is a densely defined closable operator such that $N$ is normal and if $A$ is a densely defined operator with $\sigma(A) \neq \mathbb{C}$, then
\[ NA \subset -AN \implies N^*A \subset -AN^* \]
whenever $D(A) \subset D(N)$.

**Proof.** Consider
\[ \tilde{N} = \begin{pmatrix} N & 0 \\ 0 & -N \end{pmatrix} \text{ and } \tilde{A} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \]
where $D(\tilde{N}) = D(N) \oplus D(N)$ and $D(\tilde{A}) = D(A) \oplus D(A)$. Then $\tilde{N}$ is normal and $\tilde{A}$ is closed. Besides $\sigma(\tilde{A}) \neq \mathbb{C}$. Now
\[ \tilde{N}\tilde{A} = \begin{pmatrix} 0 & NA \\ -NA & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & -AN \\ AN & 0 \end{pmatrix} = \tilde{A}\tilde{N} , \]
for $NA \subset -AN$. Since $D(\tilde{A}) \subset D(\tilde{N})$, Theorem 3.4 applies, i.e. it gives $\tilde{N}^{*}\tilde{A} \subset \tilde{A}\tilde{N}^{*}$ which, upon examining their entries, yields the required result. 

We finish this section by giving counterexamples to some "generalizations".

**Example 3.8.** ([21]) Consider the unbounded linear operators $A$ and $N$ which are defined by

$$Af(x) = (1 + |x|)f(x) \text{ and } Nf(x) = -i(1 + |x|)f'(x)$$

(with $i^2 = -1$) on the domains

$$D(A) = \{f \in L^2(\mathbb{R}) : (1 + |x|)f \in L^2(\mathbb{R})\}$$

and

$$D(N) = \{f \in L^2(\mathbb{R}) : (1 + |x|)f' \in L^2(\mathbb{R})\}$$

respectively, and where the derivative is taken in the distributional sense. Then $A$ is a boundedly invertible, positive, self-adjoint unbounded operator. As for $N$, it is an unbounded normal operator $N$ (details may consulted in [21]). It was shown that such that

$$AN^{*} = NA \text{ but } AN \not\subset N^{*}A \text{ and } N^{*}A \not\subset AN$$

(in fact $ANf \neq N^{*}Af$ for all $f \neq 0$).

So, what this example is telling us is that $NA = AN^{*}$ (and not just an "inclusion"), that $N$ and $N^{*}$ are both normal, $\sigma(A) \neq \mathbb{C}$ (as $A$ is self-adjoint), but $NA \not\subset AN^{*}$.

This example can further be beefed up to refute certain possible generalizations.

**Example 3.9.** (Cf. [26]) There exist a closed operator $T$ and a normal $M$ such that $TM \subset MT$ but $TM^{*} \not\subset M^{*}T$ and $M^{*}T \not\subset TM^{*}$. Indeed, consider

$$M = \begin{pmatrix} N^{*} & 0 \\ 0 & N \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$$

where $N$ is normal with domain $D(N)$ and $A$ is closed with domain $D(A)$ and such that $AN^{*} = NA$ but $AN \not\subset N^{*}A$ and $N^{*}A \not\subset AN$ (as defined above). Clearly, $M$ is normal and $T$ is closed. Observe that $D(M) = D(N^{*}) \oplus D(N)$ and $D(T) = D(A) \oplus L^2(\mathbb{R})$. Now,

$$TM = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} N^{*} & 0 \\ 0 & N \end{pmatrix} = \begin{pmatrix} 0_{D(N^{*})} & 0_{D(N)} \\ AN^{*} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0_{D(N)} \\ AN^{*} & 0 \end{pmatrix}$$
where e.g. $0_{D(N)}$ is the zero operator restricted to $D(N)$. Likewise

$$MT = \begin{pmatrix} N^* & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ NA & 0 \end{pmatrix}. $$

Since $D(TM) = D(AN^*) \oplus D(N) \subset D(NA) \oplus L^2(\mathbb{R}) = D(MT)$, it ensues that $TM \subset MT$. Now, it is seen that

$$TM^* = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & N^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ AN & 0 \end{pmatrix}$$

and

$$M^*T = \begin{pmatrix} N & 0 \\ 0 & N^* \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ N^*A & 0 \end{pmatrix}.$$ 

Since $ANf \neq N^*Af$ for any $f \neq 0$, we infer that $TM^* \not\subset M^*T$ and $M^*T \not\subset TM^*$.

4. SOME APPLICATIONS TO THE COMMUTATIVITY OF SELF-ADJOINT OPERATORS

In [1], [7], [13], [14], [16], [18], [19], [20], [25], and [29], the question of the self-adjointness of the normal product of two self-adjoint operators was tackled in different settings (cf. [2]). In all cases, the commutativity of the operators was reached.

Here, we deal with the similar question where the unbounded (operator) factor is closed and symmetric which, and it is known, differs from self-adjointness (the two classes can behave quite differently, cf. [24]).

First, we give a perhaps known lemma (cf. Lemmata 2.1 & 2.2 in [12]). See also [17] for the case of normality.

**Lemma 4.1.** ([16]) Let $A$ and $B$ be self-adjoint operators. Assume that $B \in B(H)$ and $BA \subseteq AB$. Then the following assertions hold:

(i) $AB$ is a self-adjoint operator and $AB = BA$,

(ii) if $A$ and $B$ are positive so is $AB$.

We shall also have need for the following result:

**Lemma 4.2.** Let $B \in B(H)$ be self-adjoint. If $BA \subseteq AB$ where $A$ is closed, then $f(B)A \subseteq Af(B)$ for any continuous function $f$ on $\sigma(B)$. In particular, and if $B$ is positive, then $\sqrt{BA} \subseteq A\sqrt{B}$.

**Remark.** In fact, the previous lemma was shown in ([18], Proposition 1) under the assumption "$A$ being unbounded and self-adjoint", but by looking closely at its proof, we see that only the closedness of $A$ was needed (cf. [4] and [15]).
We are now ready to state and prove the first result of this section.

**Theorem 4.3.** Let $A$ be an unbounded closed and symmetric operator with domain $D(A)$, and let $B \in B(H)$ be positive. If $AB$ is normal, then $BA \subset AB$, and so $AB$ is self-adjoint. Also, $BA$ is self-adjoint.

Besides, $|A|B \subset |B|A$, and so $|A|B$ is self-adjoint and positive. Moreover, $|A|B = |B|A$.

**Proof.** Since $B \in B(H)$ is self-adjoint, we have $(BA)^* = A^*B$ and $BA^* \subset (AB)^*$. Now, write

$$B(AB) = BAB \subset BA^*B \subset (AB)^*B.$$  

Since $AB$ and $(AB)^*$ are both normal, the Fuglede-Putnam theorem applies and gives

$$B(AB)^* \subset (AB)^{**}B = ABB = AB^2,$$

i.e.

$$B^2A \subset B^2A^* \subset B(AB)^* \subset AB^2.$$  

Since $A$ is closed and $B \in B(H)$ is positive, Lemma 4.2 gives $BA \subset AB$.

To show that $AB$ is self-adjoint, we proceed as follows: Observe that

$$BA \subset BA^* \subset (AB)^*.$$  

Since we also have $BA \subset AB$, we now know that

$$BAx = ABx = (AB)^*x$$

for all $x \in D(A)$. This says that $AB$ and $(AB)^*$ coincide on $D(A)$. Denoting the restrictions of the latter operators to $D(A)$ by $T$ and $S$ respectively, it is seen that

$$T - S \subset 0, T \subset AB, \text{ and } S \subset (AB)^*.$$  

Hence

$$(AB)^* - AB \subset T^* - S^* \subset (T - S)^* = 0.$$  

Since $D(AB) = D[(AB)^*]$ thanks to the normality of $AB$, it ensues that $AB = (AB)^*$, that is, $AB$ is self-adjoint.

Now, we show that $BA$ is self-adjoint. First, we show that $BA$ is normal. Clearly $BA^* \subset A^*B$ for we already know that $BA \subset AB$.

Hence

$$BA^*A \subset A^*BA \subset A^*AB.$$  

Therefore

$$BABA = ABA^*B \subset A^*B^2$$

and

$$(BA)^*BA = A^*BAB \subset A^*AB^2.$$
By Lemma 4.1, it is seen that both of $AA^*B^2$ and $AA^*B^2$ are self-adjoint. By the maximality of self-adjoint operators, it ensues that

$$BA(BA)^* = AA^*B^2$$

and

$$(BA)^*BA = A*AB^2.$$ 

Since $AB$ is self-adjoint, $(AB)^2$ is self-adjoint. But

$$(AB)^2 = ABAB \subset AA^*B^2$$

and so $(AB)^2 = AA^*B^2$. Similarly,

$$ABAB \subset A^*BAB \subset A^*AB^2$$

or $(AB)^2 = A^*AB^2$. Therefore, we have shown that

$$(BA)^*BA = BA(BA)^*.$$ 

In other words, $BA$ is normal.

To infer that $BA$ is self-adjoint, observe that $BA \subset AB$ gives $BA \subset AB$, but because normal operators are maximally normal, we obtain $BA = AB$, from which we derive the self-adjointness of $BA$.

To show the last claim of the theorem, consider again $BA^*A \subset A^*AB$. So, $B|A| \subset |A|B$ by the spectral theorem say. Since $B \geq 0$, Lemma 4.1 gives the self-adjointness and the positivity of $|A|B$, as well as $|A|B = B|A|$. This completes the proof.  

Remark. Under the assumptions of the preceding theorem (by consulting [5]), we have:

$$|AB| = |BA| = |A|B = |B|A.$$ 

Corollary 4.4. Let $A$ be an unbounded closed and symmetric operator and let $B \in B(H)$ be positive. Suppose that $AB$ is normal. Then

$BA$ is closed $\implies$ $A$ is self-adjoint.

In particular, if $B$ is invertible, then $A$ is self-adjoint.

Proof. By Theorem 4.3, $BA$ is self-adjoint and $BA = AB$. Hence

$$BA^* \subset (AB)^* = (BA)^* = BA.$$ 

So, when $BA$ is closed, $BA^* \subset BA$. Therefore, $D(A^*) \subset D(A)$, and so $D(A) = D(A^*)$. Thus, $A$ is self-adjoint, as required.

Corollary 4.5. Let $A$ be an unbounded closed and symmetric operator with domain $D(A)$, and let $B \in B(H)$ be positive. If $BA^*$ is normal, then $BA \subset AB$, and so $BA^*$ is self-adjoint.

Proof. Since $BA^*$ is normal, so is $(BA^*)^* = AB$. To obtain the desired conclusion, one just need to apply Theorem 4.3. 

□
The case of the normality of $BA$ was unexpectedly trickier. After a few attempts, we have been able to show the result.

**Theorem 4.6.** Let $A$ be an unbounded closed and symmetric operator with domain $D(A)$, and let $B \in B(H)$ be self-adjoint. Assume $BA$ is normal. Then $A$ is necessarily self-adjoint.

If we further assume that $B$ is positive, then $BA$ becomes self-adjoint and $BA = AB$.

We are now ready to show Theorem 4.6.

**Proof.** First, recall that since $BA$ is normal, $BA$ is closed and $D(BA) = D((BA)^*)$.

Write

$$A(BA) \subset A^*BA = (BA)^*A.$$  

Since $BA$ is normal and $D(BA) = D(A)$, Theorem 2.2 is applicable and it gives

$$A(BA)^* \subset (BA)^{**}A = BAA = BA^2,$$

i.e. $AA^*B \subset BA^2$. Since $A$ is symmetric, we may push the previous inclusion to further obtain $AA^*B \subset BAA^*$, that is, $|A^*|^2B \subset B|A^*|^2$.

Next, we claim that $B|A^*|$ is closed too. To see that, observe that as $B \in B(H)$, then $(BA)^* = A^*B$. Hence $BAA = (A^*B)^*$ or $BA = (A^*B)^*$ because $BA$ is already closed. By Lemma 11 in [6], the last equation is equivalent to $(|A^*|^B)^* = B|A^*|$ which gives the closedness of $B|A^*|$ as needed.

Now, we have

$$B|A^*|(|A^*|)^* = B|A^*|^2B \subset B^2|A^*|^2.$$  

It then follows by Corollary 1 in [9] that

$$B|A^*|(|B|A^*|)^* = B^2|A^*|^2$$

for $B|A^*|(|B|A^*|)^*$, $B^2$, and $|A^*|^2$ are all self-adjoint. The self-adjointness of $B|A^*|(|B|A^*|)^*$ also implies that $B^2|A^*|^2$ is self-adjoint as well, i.e.

$$B^2|A^*|^2 = (B^2|A^*|^2)^* = |A^*|^2B^2.$$  

In particular, $B^2|A^*|^2$ is closed. So, Proposition 3.7 in [7] implies that $B|A^*|^2$ is closed.

The next step is to show that $B|A^*|^2$ is normal. As $|A^*|^2B \subset B|A^*|^2$, it ensues that

$$B|A^*|^2(B|A^*|^2)^* = B|A^*|^4B \subset B^2|A^*|^4$$

and

$$(B|A^*|^2)^*B|A^*|^2 = |A^*|^2B^2|A^*|^2 \subset B^2|A^*|^4.$$
Since $B|A^*|^2(B|A^*|^2)^* = (B|A^*|^2)^*B|A^*|^2$, and $|A^*|^2$ are all self-adjoint, Corollary 1 in [9] yields

$$B|A^*|^2(B|A^*|^2)^* = (B|A^*|^2)^*B|A^*|^2 = (B^2|A^*|^4).$$

Therefore, $B|A^*|^2$ is normal. So, since $B \in B(H)$ is self-adjoint and $|A^*|^2$ is self-adjoint and positive, it follows by Theorem 1.1 in [14] that $B|A^*|^2$ is self-adjoint and $B|A^*|^2 = |A^*|^2B$.

By applying Theorem 10 in [4], it is seen that $B|A^*| = |A^*|B$

due to the self-adjointness and the positivity of $|A^*|$.

We now have all the necessary tools to establish the self-adjointness of $A$. Indeed,

$$D(A^*) = D(|A^*|) = D(B|A^*|) = D(|A^*|B) = D(A^*B) = D((BA)^*) = D(BA) = D(A).$$

Thus, $A$ is self-adjoint as it is already symmetric.

Finally, when $B \in B(H)$ is positive and since $A$ is self-adjoint, $(BA)^* = AB$ is normal. By Theorem [13], $AB$ is self-adjoint or $(BA)^*$ is self-adjoint. In other words,

$$BA = (BA)^* = AB,$$

and this marks the end of the proof. $\square$

Generalizations to weaker classes than normality vary. Notice in passing that in [7], the self-adjointness of $BA$ was established for a positive $B \in B(H)$ and an unbounded self-adjoint $A$ such that $BA$ is hyponormal and $\sigma(BA) \neq \mathbb{C}$. The next result is of the same kind.

**Proposition 4.7.** Let $B \in B(H)$ be positive and let $A$ be a densely defined closed symmetric operator. If $(AB)^*$ is subnormal or if $BA^*$ is closed and subnormal, then $BA \subset AB$.

Moreover, if $A$ is self-adjoint, then $AB$ is self-adjoint. Besides, $AB = BA$.

**Proof.** The proof relies on a version of the Fuglede-Putnam theorem obtained by J. Stochel in [31]. Write

$$B[(AB)^*]^* = B(AB)^{**} = BAB \subset BA^*B \subset (AB)^*B.$$

Since $(AB)^*$ is subnormal, Theorem 4.2 in [31] yields

$$B^2A \subset B^2A^* \subset B(AB)^* \subset (AB)^{**}B = AB^2.$$
The same inclusion is obtained in the event of the subnormality of $BA^*$. Indeed, write

$$B(BA^*)^* = BAB \subset BA^*B.$$ 

Applying again Theorem 4.2 in [31] gives

$$B(BA^*) \subset (BA^*)^*B = AB^2.$$ 

Therefore, and as above, we obtain $B^2 A \subset AB^2$.

Now, since $B \geq 0$ and $A$ is closed, it follows that $BA \subset AB$.

Finally, when $A$ is self-adjoint, Lemma 4.1 implies that $AB$ is self-adjoint and $AB = BA$, as needed.

There are still more cases to investigate. As is known, if $N \in B(H)$ is such that $N^2$ is normal, then $N$ need not be normal (cf. [23]). The same applies for the class of self-adjoint operators.

The first attempted generalization is the following: Let $A, B$ be two self-adjoint operators, where $B$ is positive, and such that $(AB)^n$ is normal for some $n \in \mathbb{N}$ such that $n \geq 2$. Does it follow that $AB$ is self-adjoint?

The answer is negative even when $A$ and $B$ are $2 \times 2$ matrices. This is seen next:

**Example 4.8.** Take

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Then $A$ is self-adjoint and $B$ is positive (it is even an orthogonal projection). Also,

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ whilst } (AB)^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for all $n \geq 2$. In other words, $AB$ is not self-adjoint while all $(AB)^n$, $n \geq 2$, are patently self-adjoint.

Let us pass to other possible generalizations.

**Proposition 4.9.** Let $B \in B(H)$ be positive and let $A$ be a closed and symmetric operator. Assume $AB^n$ is normal for a certain positive integer $n \in \mathbb{N}$. Then

1. $BA \subset AB$ (hence $BA$ is symmetric).
2. If it is further assumed that $B$ is invertible, then $A$ is self-adjoint. Besides, all of $AB^{1/n}$ and $B^{1/n}A$ are self-adjoint for all $n \geq 1$.

**Proof.**
(1) Since $B^n$ is positive for all $n$ and $AB^n$ is normal, it follows by Theorem 4.3 that $AB^n$ is self-adjoint and $B^nA \subset AB^n$. By Lemma 4.2 it is seen that $BA \subset AB$.

(2) Since $AB^n$ is normal and $B^nA$ is closed (as $B^n$ is invertible), Corollary 4.4 yields the self-adjointness of $A$.

Finally, since $BA \subset AB$ and $B$ is positive, it follows that $B^{1/n}A \subset AB^{1/n}$, from which we derive the self-adjointness of $AB^{1/n}$ and $B^{1/n}A = B^{1/n}A$, as suggested.

□

Similarly, we have:

Proposition 4.10. Let $B \in B(H)$ be positive and let $A$ be a closed and symmetric operator. Assume that $B^nA$ is normal for some positive integer $n \in \mathbb{N}$. Then $A$ and $BA$ are self-adjoint, and $BA = AB$.

One of the tools to prove this result is:

Lemma 4.11. (Cf. Proposition 3.7 in [7]) Let $B \in B(H)$ and let $A$ be an arbitrary operator such that $B^nA$ is closed for some integer $n \geq 2$. Suppose further that $BA$ is closable. Then $BA$ is closed.

Proof. Let $(x_p)$ be in $D(B^nA)$ and such that $x_p \to x$ and $BAx_p \to y$.

Since $B^{n-1} \in B(H)$, $B^nAx_p \to B^{n-1}y$. Since $B^nA$ is closed, we obtain $x \in D(B^nA) = D(A)$. Since $BA \subset BA$ and $x \in D(BA)$, we have

$$BAx = \overline{BAx} = \lim_{p \to \infty} BAx_p = y$$

by the definition of the closure of an operator. We have therefore shown that $BA$ is closed, as wished. □

Now, we show Proposition 4.10.

Proof. Since $B^n$ is positive, Theorem 4.6 gives both the self-adjointness of $A$ and $B^nA$. Moreover, $B^nA = AB^n$. Using Lemma 4.2 or else, we get $BA \subset AB$ (only the inclusion suffices to finish the proof). The equation $B^nA = AB^n$ contains the closedness of $B^nA$ which, by a glance at Lemma 4.11 yields $BA = AB$ by consulting Lemma 4.1. □

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