NOTES ON COARSE GRAininGS AND FUNCTIONS OF OBSERVABLES

A. DVUREČENSKIJ, P. LAHTI, S. PULMANNOVÁ, AND K. YLINEN

Abstract. Using the Naimark dilation theory we investigate the question under what conditions an observable which is a coarse graining of another observable is a function of it. To this end, conditions for the separability and for the Boolean structure of an observable are given.

Keywords: semispectral measure, Naimark dilation, coarse graining, separable observable, Boolean observable.

1. Introduction

Let \((\Omega, \mathcal{A})\) be a measurable space, \(\mathcal{H}\) a complex Hilbert space, \(\mathcal{L}(\mathcal{H})\) the set of bounded operators on \(\mathcal{H}\), and \(E : \mathcal{A} \to \mathcal{L}(\mathcal{H})\) a normalized positive operator measure, that is, a semispectral measure. We call such measures observables of a physical system described by \(\mathcal{H}\).

Let \((\mathcal{K}, \widetilde{E}, V)\) be a Naimark dilation of \(E\) into a spectral measure \(\widetilde{E}\), that is, \(\widetilde{E} : \mathcal{A} \to \mathcal{L}(\mathcal{K})\) is a projection measure acting on a Hilbert space \(\mathcal{K}\) and \(V : \mathcal{H} \to \mathcal{K}\) an isometric linear map such that \(E(X) = V^* \widetilde{E}(X)V\) for all \(X \in \mathcal{A}\). We say that an observable \(E : \mathcal{A} \to \mathcal{L}(\mathcal{H})\) is separable if it has a Naimark dilation \((\mathcal{K}, \widetilde{E}, V)\) which is separable, that is, the range \(\widetilde{E}(\mathcal{A})\) of \(\widetilde{E}\) is a separable Boolean sub-\(\sigma\)-algebra in the projection lattice \(\mathcal{P}(\mathcal{K})\) of the Hilbert space \(\mathcal{K}\). (We use the lattice theoretical terminology as introduced in \([13]\).)

We recall that a Boolean sub-\(\sigma\)-algebra \(\mathcal{B}\) of \(\mathcal{P}(\mathcal{K})\) is separable, if there exists a countable subset \(B\) such that the smallest Boolean sub-\(\sigma\)-algebra of \(\mathcal{B}\) containing \(B\) is \(\mathcal{B}\). The importance of such sub-\(\sigma\)-algebras of \(\mathcal{P}(\mathcal{K})\) lies in the following fact: a Boolean sub-\(\sigma\)-algebra \(\mathcal{R}\) of \(\mathcal{P}(\mathcal{K})\) is the range of a real projection measure \(F : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{K})\), that is, \(\mathcal{R} = F(\mathcal{B}(\mathbb{R}))\) if and only if \(\mathcal{R}\) is separable \([13]\) Lemma 3.16]. Furthermore, in that case, if \(\mathcal{R}_1\) is a Boolean sub-\(\sigma\)-algebra contained in \(\mathcal{R}\), then there is a real Borel function \(f\) such that \(\mathcal{R}_1 = F^f(\mathcal{B}(\mathbb{R}))\), where \(F^f(X) = F(f^{-1}(X))\) \([13]\) Theorem 3.9], see also \([2]\) Lemma 4.11].

Consider now two observables \(E_1\) and \(E\) defined on the \(\sigma\)-algebras \(\mathcal{A}_1\) and \(\mathcal{A}\) of the measurable spaces \((\Omega_1, \mathcal{A}_1)\) and \((\Omega, \mathcal{A})\), respectively,
and taking values in $\mathcal{L}(\mathcal{H})$. We say that $E_1$ is a \textit{function} of $E$ if there is a measurable function $f : \Omega \to \Omega_1$ such that $E_1 = E^f$, that is, $E_1(X) = E(f^{-1}(X))$ for all $X \in \mathcal{A}_1$. Clearly, if $E_1$ is a function of $E$, then the range of $E_1$ is contained in the range of $E$. In general, for any two observables $E_1$ and $E$, if $E_1(\mathcal{A}_1) \subset E(\mathcal{A})$ we say that $E_1$ is a \textit{coarse graining} of $E$.

Assume that $E_1$ is a coarse graining of $E$. If $(\mathcal{K}, \tilde{E}, V)$ is a Naimark dilation of $E$, we let $\mathcal{R}_1$ be the set of all projections $P \in \tilde{E}(\mathcal{A})$ such that $V^*PV \in E_1(\mathcal{A}_1)$. Then

$$E_1(\mathcal{A}_1) = V^*\mathcal{R}_1V \subset E(\mathcal{A}) = V^*\tilde{E}(\mathcal{A})V.$$ 

Calling two observables \textit{equivalent} if their ranges are the same we observe that if $\tilde{E}(\mathcal{A})$ is a separable Boolean sub-$\sigma$-algebra of $\mathcal{P}(\mathcal{K})$, then $\tilde{E}$ is equivalent to a real projection measure $\mathcal{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{K})$. If, in addition, $\mathcal{R}_1$ is a Boolean sub-$\sigma$-algebra of $\tilde{E}(\mathcal{A})$ then it can be expressed as $\mathcal{R}_1 = F^f(\mathcal{B}(\mathbb{R}))$ for some Borel function $f$. In this case observables $E_1$ and $E$ are equivalent to the two real functionally related semispectral measures $E_1^r$ and $E^r$, where $E_1^r(X) = V^*F^f(X)V$ and $E^r(X) = V^*F(X)V$ for all $X \in \mathcal{B}(\mathbb{R})$.

The questions of interest for this study are the following. First, under what conditions is an observable separable? Secondly, if an observable is a coarse graining of another observable, when is it a function of the latter? Sections $2$ and $3$ are devoted to the separability questions whereas in Section $4$ we study the question of functional relations between observables.

\textbf{Remark 1.} For positive operator measures $E_1$ and $E$, the condition $E_1(\mathcal{A}_1) \subset E(\mathcal{A})$ need not imply that $E_1$ is a function of $E$. However, $E_1$ and $E$ may still be functionally related (functionally coexistent) so that there is a positive operator measure $F$ with measurable functions $f$ and $g$ such that $E_1 = F \circ f^{-1}$ and $E = F \circ g^{-1}$. Indeed, as an illustration of this phenomenon, consider the real scalar measures $E$ and $E_1$ concentrated, respectively, on the sets $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$ such that $E(\{x_1\}) = E(\{x_2\}) = 1/8, E(\{x_3\}) = E(\{x_4\}) = 3/8$, and $E_1(\{y_1\}) = E_1(\{y_2\}) = E_1(\{y_3\}) = 1/8, E_1(\{y_4\}) = 5/8$. Clearly, the range of $E_1$ is contained in that of $E$, but there is no function $f : \{x_1, x_2, x_3, x_4\} \to \{y_1, y_2, y_3, y_4\}$ such that $E_1(Y) = E(f^{-1}(Y))$. Indeed, if such a function exists, we must have $E_1(\{y_1\}) = E(f^{-1}(\{y_1\})) = 1/8$, which gives $f^{-1}(\{y_1\}) = \{x_1\}$, or $f^{-1}(\{y_1\}) = \{x_2\}$, and $E_1(\{y_1\}) = E(f^{-1}(\{y_1\}))$, which yields $f^{-1}(\{y_1\}) = \{x_1, x_2, x_3\}$ or $f^{-1}(\{y_1\}) = \{x_1, x_2, x_4\}$. Both $E$ and $E_1$ are, however, functions of the observable $\{z_i\} \mapsto F(\{z_i\}) = 1/8, i = 1, \ldots, 8$. 
2. Separable Boolean $\sigma$-algebras

In this section we collect, for the reader’s convenience, some basic observations in the context of separable Boolean sub-$\sigma$-algebras of the projection lattice of a Hilbert space. The proofs follow readily from known facts and the results themselves may be part of the folklore of the subject though hard to find in the literature.

Let $\mathcal{B}$ be a Boolean algebra. An atom of $\mathcal{B}$ is any non-zero element $a$ of $\mathcal{B}$ such that $b \leq a$ for $b \in \mathcal{B}$ implies $b = 0$ or $b = a$. Let $\text{At}(\mathcal{B})$ be the set of all atoms of $\mathcal{B}$. If $\text{At}(\mathcal{B}) = \emptyset$, $\mathcal{B}$ is said to be atomless. If $a$ and $b$ are two different atoms of $\mathcal{B}$, then they are disjoint, $a \land b = 0$.

If $\mathcal{B}_i = (\mathcal{B}_i; 0_i, 1_i; \prime_i)$, $i = 1, 2$, are Boolean $\sigma$-algebras, then their Cartesian product $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$ is again a Boolean $\sigma$-algebra with operations defined coordinatewise, the least and the greatest elements being $0 = (0_1, 0_2)$ and $1 = (1_1, 1_2)$, respectively.

**Proposition 2.** Let $\mathcal{B}$ be a Boolean $\sigma$-algebra such that every system of mutually orthogonal non-zero elements of $\mathcal{B}$ is at most countable. Then $\mathcal{B}$ can be decomposed in the form $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, where $\mathcal{B}_1$ is a Boolean $\sigma$-algebra isomorphic with the power set $2^N$, where $N$ is a finite or countable cardinal, and $\mathcal{B}_2$ is an atomless Boolean $\sigma$-algebra.

**Proof.** Let $\text{At}(\mathcal{B})$ be the set of all atoms of $\mathcal{B}$. Since any two different atoms $a$ and $b$ of $\mathcal{B}$ are mutually orthogonal, $a \leq b'$, $0 \leq |\text{At}(\mathcal{B})| \leq \aleph_0$.

Define $a_0 := \forall \{a : a \in \text{At}(\mathcal{B})\}$; if $\text{At}(\mathcal{B}) = \emptyset$, we put $a_0 := 0$. For any element $a \in \mathcal{B}$, we have the decomposition

$$a = (a \land a_0) \lor (a \land a'_0). \quad (1.1)$$

Define $\mathcal{B}_1 := \{a \in \mathcal{B} : a \leq a_0\}$ and $\mathcal{B}_2 := \{a \in \mathcal{B} : a \leq a'_0\}$. Then $\mathcal{B}_1 = (\mathcal{B}_1; 0, a_0, a'_0)$, where $x' a_0 := x' \land a_0$ for $x \in \mathcal{B}_1$, and $\mathcal{B}_2 = (\mathcal{B}_2; 0, a'_0, a'_0)$, where $x' a'_0 := x' \land a'_0$ for $x \in \mathcal{B}_2$, are Boolean $\sigma$-algebras such that $\mathcal{B}_1$ is isomorphic with the $\sigma$-algebra $2^N$, where $N = |\text{At}(\mathcal{B})|$, and $\mathcal{B}_2$ is atomless. In view of (1.1) we have the decomposition $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$. \qed

The set $\mathcal{P}(\mathcal{H})$ of all projections on $\mathcal{H}$ forms a complete orthomodular lattice with respect to the operator order and orthocomplementation $P \mapsto P^\perp := I_\mathcal{H} - P$, with $I_\mathcal{H} = I$ and $O_\mathcal{H} = O$ being the identity and zero operators on $\mathcal{H}$.

**Theorem 3.** Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{B}$ be a Boolean sub-$\sigma$-algebra of $\mathcal{P}(\mathcal{H})$. Then $\mathcal{B}$ is separable. In particular, if $\mathcal{H}$ is finite dimensional, then $\mathcal{B} = 2^N$, where $N$ is an integer such that $1 \leq N \leq \dim \mathcal{H}$. 


Proof. Using Proposition 2 we decompose the $\sigma$-algebra $\mathcal{B}$ in the form $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, where $\mathcal{B}_1$ is isomorphic with $2^N$, $N = |\text{At}(\mathcal{B})|$, and $\mathcal{B}_2$ is atomless. Let $P_0 = \bigvee\{P : P \in \text{At}(\mathcal{B})\}$ and denote $\mathcal{H}_0 = P_0(\mathcal{H})$.

Assume $\dim \mathcal{H} = \aleph_0$. If $P_0 = I_\mathcal{H}$, then $\mathcal{B} = \mathcal{B}_1$, and $\mathcal{B}$ is separable. If $P_0 \neq I_\mathcal{H}$, then $I_\mathcal{H} - P_0 \neq O$, and since $\mathcal{B}_2$ is atomless, we have $\dim(\mathcal{H}_0) = \aleph_0$. In addition, $\mathcal{B}_2$ is a Boolean $\sigma$-algebra which is a subalgebra of $\mathcal{P}(\mathcal{H}_0)$. Let $\mathbb{B}_2$ be the von Neumann algebra generated by $\mathcal{B}_2$. Then $\mathbb{B}_2$ is a commutative von Neumann algebra acting in the infinite-dimensional complex separable Hilbert space $\mathcal{H}_0$, and the projection lattice of $\mathbb{B}_2$ coincides with $\mathcal{B}_2$ which is atomless. Therefore, by [12, Theorem III.1.22], $\mathbb{B}_2$ is isomorphic with the von Neumann algebra $L^\infty(0, 1)$ (the space of all essentially bounded functions on the unit interval $(0, 1)$ with respect to the Lebesgue measure). Since the projections from $L^\infty(0, 1)$ are only characteristic functions, they have a countable generator, consequently, $\mathbb{B}_2$ has a countable generator. Because $\mathcal{B}_1$ is generated by the countable set of atoms, in view of $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, $\mathcal{B}$ is separable.

Assume now $\dim \mathcal{H} < \infty$. Then $P_0 = I_\mathcal{H}$ and therefore, $\mathcal{B} = 2^N$. □

3. Separable observables

A Naimark dilation $(\mathcal{K}, \tilde{E}, V)$ of a semispectral measure $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is minimal if $\mathcal{K}$ is the closed linear span of $\{\tilde{E}(X) | X \in \mathcal{A}\}$. As is well known, a minimal dilation always exists and it is unique up to an isometric isomorphism [10].

Lemma 4. Let $(\Omega, \mathcal{A})$ be a measurable space with a separable $\sigma$-algebra $\mathcal{A}$ and let $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a normalized positive operator measure acting on a complex separable Hilbert space $\mathcal{H}$. If $(\mathcal{K}, \tilde{E}, V)$ is a minimal Naimark dilation of $E$, then $\mathcal{K}$ is separable.

Proof. Let $\mathcal{F}$ be a countable collection of subsets of $\Omega$ which generates the $\sigma$-algebra $\mathcal{A}$, and let $\mathcal{R}$ be the ring generated by $\mathcal{F}$. Since $\mathcal{F}$ is countable, the ring $\mathcal{R}$ is countable [3, Theorem I.5.C]. Let $\mathcal{C}$ be the complex linear span of the characteristic functions $\chi_X$ of the sets $X \in \mathcal{R}$, and let $\tilde{\mathcal{C}}$ be its closure in the set of bounded functions $\Omega \to \mathbb{C}$ (with respect to the sup-norm). $\tilde{\mathcal{C}}$ is a separable commutative $C^*$-algebra. Let $\Phi : \tilde{\mathcal{C}} \to \mathcal{L}(\mathcal{H})$ be the positive linear map corresponding to the normalized positive operator measure $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, $\Phi(f) = \int f \, dE$. Then $\Phi$ is completely positive [10, Theorem 3.10]. Let $(\mathcal{K}, \pi, V)$ be its minimal Stinespring dilation. The Hilbert space $\mathcal{K}$ is separable [10, p. 46]. Let $P_o : \mathcal{R} \to \mathcal{L}(\mathcal{K})$ be the projection-valued set function
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defined by $P_o(X) = \pi(\chi_X)$ for all $X \in \mathcal{R}$. Then $V^* P_o(X)V = E(X)$ for all $X \in \mathcal{R}$. From its construction it easily follows that $P_o$ is weakly $\sigma$-additive on $\mathcal{R}$.

For any $\varphi \in \mathcal{K}$ and $X \in \mathcal{R}$ denote $\mu_{\varphi,\varphi}^o(X) = \langle \varphi | P_o(X)\varphi \rangle$. Since $\mu_{\varphi,\varphi}^o$ is $\sigma$-additive on $\mathcal{R}$, it has a unique extension to a (positive) measure $\mu_{\varphi,\varphi}$ on $\mathcal{A}$. For any $\varphi, \psi \in \mathcal{K}$ and $X \in \mathcal{A}$ denote

$$\mu_{\varphi,\psi} = \frac{1}{4} \sum_{k=1}^{4} i^k \mu_{\varphi+i^k\psi,\varphi+i^k\psi}.$$ 

Elementary estimates show that the map $(\varphi, \psi) \mapsto \mu_{\varphi,\psi}(X)$ is a bounded sesquilinear form for each $X \in \mathcal{A}$, and we get a positive operator measure $\tilde{P} : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ which extends $P_o$.

It remains to be shown that the map $\tilde{P}$ is a projection measure. We denote by $M(\mathcal{R})$ the monotone class generated by $\mathcal{R}$. The class $\{X \in \mathcal{A} | \tilde{P}(X)^2 = \tilde{P}(X)\}$ contains $\mathcal{R}$ and is easily seen to be a monotone class and so it equals $\mathcal{A}$ [3, Theorem I.6.B]. Clearly, $V^* \tilde{P}(X)V = E(X)$ for all $X \in \mathcal{A}$ and $(\mathcal{K}, \tilde{P}, V)$ constitutes a minimal dilation of $E$ and $\mathcal{K}$ is separable.

An alternative approach would be to use in the above proof Naimark’s dilation theory [11, Appendix, Theorem 1] instead of Stinespring’s.

Remark 5. A physically relevant dilation $(\mathcal{K}, \tilde{E}, V)$ of a quantum observable $E$ is typically not minimal, see e.g. [8]. An interesting example of a dilation acting on a nonseparable Hilbert space appears in [9] for the canonical phase observable.

Corollary 6. Let $(\Omega, \mathcal{A})$ be a measurable space with a separable $\sigma$-algebra $\mathcal{A}$ and let $\mathcal{H}$ be a complex separable Hilbert space. Any normalized positive operator measure $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is separable.

Proof. Let $(\mathcal{K}, \tilde{E}, V)$ constitute a minimal Naimark dilation of $E$. The set $\{\tilde{E}(X)\varphi | X \in \mathcal{A}, \varphi \in \mathcal{H}\}$ is dense in $\mathcal{K}$. By Lemma [3] $\mathcal{K}$ is separable. Therefore, by Theorem [3] $\tilde{E}(\mathcal{A})$ is a separable Boolean sub-$\sigma$-algebra of $\mathcal{P}(\mathcal{K})$.

4. BOOLEAN OBSERVABLES

The Boolean structure of the range of an observable plays an important role in the functional calculus of observables. We therefore recall the following results. Here $\mathcal{E}(\mathcal{H})$ denotes the set of effect operators on $\mathcal{H}$, i.e., $\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{L}(\mathcal{H}) : O \leq A \leq I\}$.

Proposition 7. The range $E(\mathcal{A})$ of an observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is a Boolean subalgebra of the set $\mathcal{E}(\mathcal{H})$ of effects if and only if $E$ is projection valued.
Proof. For any $X \in \mathcal{A}$ the product $E(X)E(X')$ is a positive lower bound of $E(X)$ and $E(X')$. If $E(\mathcal{A})$ is Boolean then $E(X) \wedge E(X') = O$, and thus $E(X)E(X') = O$, that is, $E(X)^2 = E(X)$. On the other hand, if $E$ is projection valued, then the claim follows from the multiplicativity of the spectral measure and from the fact that for any two projections $P$ and $R$ their greatest lower bound and smallest upper bound in $\mathcal{E}(\mathcal{H})$ are the same as in $\mathcal{P}(\mathcal{H})$, that is, $P \wedge R$ and $P \vee R$, respectively. 

The order structure of the set of effects $\mathcal{E}(\mathcal{H})$ is highly complicated. For instance, if $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is an observable, then for any $X, Y \in \mathcal{A}$, the effect $E(X \cap Y)$ is a lower bound of the effects $E(X)$ and $E(Y)$, but these effects need not have the greatest lower bound $E(X) \wedge_{\mathcal{E}(\mathcal{H})} E(Y)$ and even if $E(X) \wedge_{\mathcal{E}(\mathcal{H})} E(Y)$ exists it need not coincide with $E(X \cap Y)$. When the order and the complement of $\mathcal{E}(\mathcal{H})$ are restricted to the range $E(\mathcal{A})$ of $E$ it is possible that the system $(E(\mathcal{A}), \leq, \not \gtrless)$ is a Boolean $\sigma$-algebra without $E$ being projection valued. To express that option it is useful to introduce two further concepts. We say that an observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is **regular** if for any $O \neq E(X) \neq I$, neither $E(X) \leq E(X')$ nor $E(X') \leq E(X)$, and it is **$\Delta$-closed** if for any triple of pairwise orthogonal elements $A, B, C \in E(\mathcal{A})$, the sum $A + B + C$ is in $E(\mathcal{A})$.

From [33][7] the following results are then obtained.

**Proposition 8.** For any observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ the following three conditions are equivalent.

a) $(E(\mathcal{A}), \leq, \not \gtrless)$ is a Boolean $\sigma$-algebra.

b) $E$ is regular.

c) $E$ is $\Delta$-closed.

Consider now two observables $E_1$ and $E$ defined on the $\sigma$-algebras $\mathcal{A}_1$ and $\mathcal{A}$ of the measurable spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega, \mathcal{A})$, respectively, and taking values in $\mathcal{L}(\mathcal{H})$, with $\mathcal{H}$ being complex and separable. Assume that $E_1$ is a coarse graining of $E$, that is, $E_1(\mathcal{A}_1) \subset E(\mathcal{A})$. Let $(\mathcal{K}, \tilde{E}, V)$ be a Naimark dilation of $E$, with separable $\mathcal{K}$, and let $\mathcal{R}_1$ be again the set of projections $P \in \tilde{E}(\mathcal{A})$ such that $V^*PV \in E_1(\mathcal{A}_1)$.

**Proposition 9.** With the above notations, $\mathcal{R}_1$ is a Boolean sub-$\sigma$-algebra of $\mathcal{P}(\mathcal{K})$ if and only if there is a real Borel function $f$ and a real semispectral measure $E_r$ such that $E$ is equivalent with $E_r$ and $E_1$ is equivalent with $E_r f$.

**Proof.** If $\mathcal{R}_1$ is a Boolean sub-$\sigma$-algebra of $\mathcal{P}(\mathcal{K})$ then, as a subset of $\tilde{E}(\mathcal{A})$, it is also separable. Thus by the results [13][3, Theorem 3.9] there is a real projection measure $F_r$ and a real Borel
function \( f \) such that \( \tilde{E}(A) = F_r(B(R)) \) and \( R_1 = F^f_r(B(R)) \). The semispectral measures \( E_r := V^*F_rV \) and \( E^f_r := V^*F^f_rV \) are now as required. The other direction is immediate.

We say that an observable \( E : A \to \mathcal{L}(\mathcal{H}) \) has the \textit{V-property} with respect to a subset \( Q \) of \( E(A) \) if for each \( X,Y \in A \) and \( C \in Q \) the inequality \( E(X) \leq C \leq E(Y) \) implies that there is a \( Z \in A \) such that \( X \subset Z \subset Y \) and \( C = E(Z) \). The importance of this property is in the fact that for any two (real) observables \( E_1 \) and \( E \), if \( E_1(A) \subset E(A) \) and if \( E \) has the \textit{V-property} on \( E_1(A) \), then \( E_1 \) is a function of \( E \) [6].

Lemma 10. With the above notations, \( O_K, I_K \in R_1 \), and if \( P \in R_1 \) then also \( P^\perp \in R_1 \). Moreover, for any \( P,R \in R_1 \), if \( P \leq R \), then \( V^*PV \leq V^*RV \). In addition, the observable \( E \) has the \textit{V-propety} on \( R_1 \).

Proof. If \( P \in R_1 \), then \( V^*PV = E_1(X) \) for some \( X \in A_1 \) and thus \( E_1(X') = I_{\mathcal{H}} - E_1(X) = V^*V - V^*PV = V^*(I_K - P)V \), so that \( P^\perp \in R_1 \). If \( P \leq R \), then for any \( \psi \in \mathcal{K} \), \( \langle \psi | P\psi \rangle \leq \langle \psi | R\psi \rangle \), and thus, in particular, for any \( \varphi \in \mathcal{H} \), \( \langle \varphi | E_1(X)\varphi \rangle = \langle \varphi | V^*PV\varphi \rangle = \langle V^*E_1(X)V\varphi | V\varphi \rangle \leq \langle V^*RV\varphi | V^*\varphi \rangle = \langle V^*RV\varphi | E_1(Y)\varphi \rangle \). To demonstrate the \textit{V-property}, let \( X,Y \in A \), \( X \subseteq Y \), so that \( E(X) \leq \tilde{E}(X) \). Assume that \( P \in R_1 \) is such that \( \tilde{E}(X) \leq P \leq \tilde{E}(Y) \). Let \( Z \in A \) be such that \( \tilde{E}(Z) = P \). Then for \( Z_1 = X \cup (Y \cap Z) \) we have \( X \subseteq Z_1 \subseteq Y \), and \( \tilde{E}(Z_1) = \tilde{E}(X) \lor (\tilde{E}(Y) \land \tilde{E}(Z)) = (\tilde{E}(X) \lor \tilde{E}(Y)) \land (\tilde{E}(X) \lor P) = \tilde{E}(Y) \land P = P \).

Remark 11. The assumption that \( \tilde{E} \) has the \textit{V-property} on \( R_1 \) does not imply that \( E \) has the \textit{V-property} on \( E_1(A) \). For an illustration, see Remark [1].

Proposition 12. With the above notations, if \( E_1 \) is projection valued, then \( R_1 \) is a Boolean sub-\( \sigma \)-algebra of \( \tilde{E}(A) \).

Proof. For any \( P \in \mathcal{P}(K) \), \( V^*PV \in \mathcal{P}(\mathcal{H}) \) if and only if \( VV^*P = PVV^* \). Let \( P,R \in R_1 \) so that there are \( X,Y \in A_1 \) such that \( V^*PV = E_1(X) \) and \( V^*RV = E_1(Y) \). Then

\[
V^*P \land RV = V^*PRV = V^*VV^*PRV = V^*PVV^*RV = E_1(X)E_1(Y) = E_1(X \cap Y)
\]

showing that \( R_1 \) is closed under \( \land \). By the de Morgan laws, the same is true for \( \lor \). If \( (P_n)_{n=1}^{\infty} \) is a sequence of mutually orthogonal projections.
of $\mathcal{R}_1$, that is, $P_n \leq P_m^\perp$ for all $n \neq m$, then also $E_1(X_n) \leq E_1(X_m)^\perp = E_1(X_m')$. Therefore,

$$V^*(\sqrt{P_n})V = V^*(\sum P_n)V = \sum V^*P_nV = \sum E_1(X_n) = E_1(\bigcup X_n)$$

(where the series converge weakly) which shows the $\sigma$-property of $\mathcal{R}_1$.

□

**Corollary 13.** Let $\Omega_1$ and $\Omega$ be complete separable metric spaces and let $\mathcal{B}(\Omega_1)$ and $\mathcal{B}(\Omega)$ be their respective Borel $\sigma$-algebras. Assume that $\Omega_1$ and $\Omega$ have the cardinality of $\mathbb{R}$. Consider the observables $E_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}(\mathcal{H})$ and $E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ such that $E_1$ is a coarse graining of $E$. If $E_1$ is projection valued, then $E_1 = E^f$ for some Borel function $f : \Omega \to \Omega_1$.

**Proof.** Since $\Omega_1$ and $\Omega$ are complete separable metric spaces with the cardinality of $\mathbb{R}$, according to [4, Remark (ii), p. 451], there are bijections $\alpha : \Omega \to \mathbb{R}$ and $\beta : \Omega_1 \to \mathbb{R}$ which are such that $\alpha, \alpha^{-1}, \beta$, and $\beta^{-1}$ are Borel measurable. Now $E^\alpha$ and $E^\beta_1$ are real observables with the same ranges as $E_1$ and $E$, respectively. By [13, Theorem 3.9] there is a measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $E^\alpha_1(X) = E^\alpha(g^{-1}(X)), X \in \mathcal{B}(\mathbb{R})$. Putting $X = \beta(Z), Z \in \mathcal{B}(\Omega)$, we obtain $E_1(Z) = E^\alpha_1(\beta(Z)) = E^\alpha(g^{-1}(\beta(Z))) = E^f(Z)$, where $f = \beta^{-1} \circ g \circ \alpha : \Omega \to \Omega_1$.

□

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Anatolij Dvurečenskij, Mathematical Institute, Slovak Academy of Sciences, SK-81473 Bratislava, Slovakia
E-mail address: dvurecen@mat.savba.sk

Pekka Lahti, Department of Physics, University of Turku, FIN-20014 Turku, Finland
E-mail address: pekka.lahti@utu.fi

Sylvia Pulmannová, Mathematical Institute, Slovak Academy of Sciences, SK-81473 Bratislava, Slovakia
E-mail address: pulmann@mat.savba.sk

Kari Ylinen, Department of Mathematics, University of Turku, FIN-20014 Turku, Finland
E-mail address: kari.ylinen@utu.fi