Characterization and Computation of Feasible Trajectories for an Articulated Probe with a Variable-Length End Segment

Ovidiu Daescu*  Ka Yaw Teo*

Abstract

We consider an extension of the articulated probe trajectory planning problem introduced in [11], where the length $r$ of the end segment can be customized. We prove that, for $n$ line segment obstacles, the smallest length $r$ for which there exists a feasible probe trajectory can be found in $O(n^{2+\epsilon})$ time using $O(n^{2+\epsilon})$ space, for any constant $\epsilon > 0$. Furthermore, we prove that all values $r$ for which a feasible probe trajectory exists form $O(n^2)$ intervals, and can be computed in $O(n^{5/2})$ time using $O(n^{2+\epsilon})$ space. We also show that, for a given $r$, the feasible trajectory space of the articulated probe can be characterized by a simple arrangement of complexity $O(n^2)$, which can be constructed in $O(n^2)$ time.

To obtain our solutions, we design efficient data structures for a number of interesting variants of geometric intersection and emptiness query problems.

1 Introduction

The articulated probe trajectory planning problem was introduced in [11] with the following setup. We are given a two-dimensional workspace containing a set $P$ of simple polygonal obstacles with a total of $n$ vertices, and a target point $t$ in the free space, all enclosed by a circle $S$ of radius $R$ centered at $t$. An articulated probe is modeled in $\mathbb{R}^2$ as two line segments, $ab$ and $bc$, connected at point $b$. The length of $ab$ is greater than or equal to $R$, whereas $bc$ is of some small length $r \in (0, R]$. The probe is initially located outside $S$, assuming an unarticulated configuration, in which $ab$ and $bc$ are collinear, and $b \in ac$. A feasible probe trajectory consists of an initial insertion (sliding) of straight line segment $abc$ into $S$, possibly followed by a rotation of $bc$ around $b$ up to $\pi/2$ radians in either direction after the initial insertion of segment $abc$, $ab$ intersects $C$ only once and at $b$ (i.e., $b \in C$). When $bc$ rotates around $b$, the area swept by $bc$ is a sector of a circle of radius $r$ centered at $b$. For conciseness, the center of the circle on which a circular sector is based is called the center of the circular sector.

In this paper, we develop efficient algorithms for computing i) the minimum value $r > 0$ for which a feasible articulated trajectory exists, including reporting at least one such trajectory, ii) all values $r > 0$ for which a feasible articulated trajectory exists (i.e., feasible domain of $r$), and iii) the feasible trajectory space (i.e., set of all feasible trajectories) for a given value $r$.

Related work. The two-dimensional articulated probe trajectory planning problem (with a constant length $r$) was originally introduced by Teo, Daescu, and Fox [11], who presented a geometric-combinatorial algorithm for computing so-called extremal feasible probe trajectories in $O(n^2 \log n)$ time using $O(n \log n)$ space. In an extremal probe trajectory, one or two obstacle endpoints always lie tangent to the probe. The solution approach proposed in [11] can be extended to the case of polyg-
onal obstacles. For \( h \) polygonal obstacles with a total of \( n \) vertices, an extremal feasible probe trajectory can be determined in \( O(n^2 + h^2 \log h) \) time using \( O(n \log n) \) space. When a clearance \( \delta \) from the polygonal obstacles is required, a feasible probe trajectory can be obtained in \( O(n^2 + h^2 \log h) \) time using \( O(n^2) \) space.

In addition, Daescu and Teo [4] developed an algorithm for solving the articulated probe trajectory planning problem in three dimensions for a given \( r \). It was shown that a feasible probe trajectory among \( n \) triangular obstacles can be found in \( O(n^{4+\epsilon}) \) time using \( O(n^{2+\epsilon}) \) space, for any constant \( \epsilon > 0 \).

**Motivation.** Besides its general relevance in robotics, the proposed problem arises specifically in some medical applications. In minimally invasive surgeries, a rigid needle-like instrument is typically inserted through a small incision to reach a given target, after which it may perform operations such as tissue resection and biopsy. Some newer designs allow for a joint to be incorporated for moving the acting end (tip); after inserting the instrument in a straight path, the surgeon may rotate the tip around the joint to reach the target [9].

Due to the rapid advances in three dimensional printing techniques, such robotic probes can be even be customized for a given patient [3]. Rather than using a one-size-fits-all instrument, based on the patient-specific requirement and constraints, a robotic probe with a tailored-sized tip can be customarily built on-demand using three dimensional printing.

Despite its importance and relevance, as well as its rich combinatorial and geometric properties, only a handful of results have been reported [4, 11] for this trajectory planning problem.

**Results and contributions.** Recall our assumption that there is no feasible unarticulated probe trajectory (i.e., \( t \) cannot see to infinity). We begin in Section 2 by addressing our first problem of interest:

**Problem 1** Find the minimum length \( r > 0 \) of segment \( be \) such that a feasible articulated probe trajectory exists, if any, and report (at least) one such trajectory.

For brevity, a feasible articulated trajectory with the minimum length \( r \) is referred to as a feasible min-\( r \) articulated trajectory. Our approach to solving Problem 1 is as follows: i) We show that a feasible min-\( r \) articulated trajectory, if one exists, can always be perturbed, while remaining feasible, into one of a finite number of “extremal” feasible trajectories, which can be enumerated using an algebraic-geometric method (see Lemma 1 for a detailed definition of the extremal trajectories). This leads to a simple \( O(n^3 \log n) \) time, \( O(n^{2+\epsilon}) \) space algorithm, for any constant \( \epsilon > 0 \), based on enumerating and verifying the extremal trajectories for feasibility (see the full version of the paper for details). ii) We then derive an \( O(n^{2+\epsilon}) \) time and space algorithm by partially waiving the notion of computing and checking the extremal trajectories for feasibility. Specifically, the algorithm searches for a feasible min-\( r \) articulated trajectory, if any, by performing a finite sequence of perturbations and feasibility tests on certain \( O(n^2) \) extremal trajectories (whose segment \( ab \) is tangent to two obstacle endpoints). With a proper algorithmic extension to our process of finding the minimum feasible \( r \), we can compute, in \( O(n^{3/2}) \) time using \( O(n^{2+\epsilon}) \) space, the set of \( r \)-intervals for which feasible articulated trajectories exist, together with an implicit representation of feasible solutions for those values of \( r \).

In the process of deriving our solution to Problem 1, we encounter and solve a number of fundamental problems (or their special cases) that could be of theoretical interest in computational geometry. For instance, we provide an efficient data structure with logarithmic query time for solving a special instance of the circular sector emptiness query problem (i.e., for a query circular sector with a fixed arc endpoint \( t \)).

In Section 3, we address our second problem:

**Problem 2** For a given length \( r \) of segment \( bc \), compute the feasible trajectory space (i.e., set of all feasible trajectories) of the articulated probe.

We describe a geometric combinatorial approach for characterizing and computing the feasible trajectory space of the articulated probe. The feasible configuration space has a worst-case complexity of \( O(n^2) \) and can be described by an arrangement of simple curves. Using topological sweep [2], the arrangement can be constructed in \( O(n \log n + k) \) time using \( O(n + k) \) working storage, where \( k = O(n^2) \) is the number of vertices of the arrangement. By simply traversing the cells of the arrangement, we can find a feasible probe trajectory in \( O(n^2) \) time – a logarithmic factor improvement compared to the algorithm in [11].

## 2 Computing feasible min-\( r \) articulated trajectories

Recall that, for a given \( r \), \( C \) is the circle of radius \( r \) centered at \( t \). Using the rationale of [11, Lemma 2.1], we can immediately claim the following observation.

**Observation 1** Given a feasible min-\( r \) articulated trajectory, there exists an extremal feasible min-\( r \) articulated trajectory such that the probe assumes an articulated final configuration that passes through an obstacle endpoint outside \( C \) and another obstacle endpoint inside or outside \( C \).
We will later show in Lemma 1 that an extremal feasible min-\(r\) articulated trajectory is always tangent to two obstacle endpoints outside \(C\).

For ease of discussion, unless noted otherwise, we use \(bc\) and \(bt\) to denote line segment \(bc\) of the probe in its intermediate (right after the initial insertion of segment \(abc\)) and final configurations, respectively. Let \(\angle cbt\) be the angle of rotation of segment \(bc\) to reach \(t\), and let \(\sigma_{bc}\) be the circular sector swept by segment \(bc\) in order to reach \(t\). Let \(\gamma_{ct}\) denote the circular arc of \(\sigma_{bc}\). Let \(V\) denote the set of endpoints of the line segments of \(P\).

**Lemma 1** Given a feasible min-\(r\) articulated trajectory, there exists an extremal feasible min-\(r\) articulated trajectory such that, in its final configuration, \(ab\) passes through two obstacle endpoints and at least one of the following holds: I) \(\angle cbt = \pi/2\) radians, II) \(bc\) intersects an obstacle line segment at \(c\), III) \(\gamma_{ct}\) intersects an obstacle endpoint or is tangent to an obstacle line segment, IV) one of the obstacle endpoints intersected by \(ab\) coincides with \(b\), and \(\angle cbt \leq \pi/2\) radians, or V) \(bt\) passes through an obstacle endpoint.

The full proof of Lemma 1 is given in Appendix A.

**Solution approach.** We begin by emphasizing that, as stated in Lemma 1, an extremal feasible min-\(r\) articulated trajectory passes through two obstacle endpoints, neither of which is inside \(C\).

Consider the following solution approach. For each point \(v \in V\), compute the set \(R_v\) of rays with the following properties: i) Each ray originates at \(v\) and passes through a point \(u \in V \setminus \{v\}\). ii) Segment \(vb_0\) does not intersect any line segment of \(P\), where \(b_0\) is the point of tangency between the supporting line of the ray and the circle \(C\) centered at \(t\) (Figure 2A). iii) If the ray passes through \(b_0\), then the reversal of the ray does not intersect any line segment of \(P\); otherwise, the ray itself does not intersect any line segment of \(P\). \(R_v\) can be obtained in \(O(n \log n)\) time by computing the visibility polygon from \(v\) [1, 7, 10]. Since \(|V| = O(n)\), the worst-case running time for finding the set of rays \(R = \bigcup_{v \in V} R_v\) is \(O(n^2 \log n)\).

Note that each ray of \(R\) is associated with a trajectory \(T\) that has an obstacle-free segment \(ab\) passing through two obstacle endpoints. Without loss of generality, assume that \(ab\) of \(T\) passes through a pair of obstacle endpoints \(u, v \in V\), where \(u \neq v\), in the way depicted in Figure 2A. Assume that \(bc\) of \(T\) rotates clockwise to reach \(t\) (the other case is symmetrical). Let \(b_0\) be the position of \(b\) when \(\angle cbt = \pi/2\) radians, and \(c_0\) be the position of \(c\) when \(b = b_0\). In order to find a feasible min-\(r\) articulated trajectory, we perform the following sequence of steps.

**A1.** Check if the articulated trajectory \(T\) with \(\angle cbt = \pi/2\) radians is feasible. Specifically, check if the quarter circular sector bounded by \(b_0c_0\), \(b_0t\), and circular arc \(\gamma_{ct}\) (centered at \(b_0\) and emanating counter-clockwise from \(t\) to \(c_0\)) is free of obstacles (Figure 2A). If it is, then \(T\) is a feasible min-\(r\) articulated trajectory whose \(ab\) passes through \(u\) and \(v\). Otherwise, proceed with step A2.

**A2.** Check if \(b_0t\) is intersected by any obstacle (Figure 2A). If it is, then a feasible min-\(r\) articulated trajectory whose \(ab\) passes through \(u\) and \(v\) does not exist. Otherwise, proceed with steps A3 and A4.
A3. Find the closest point \( c' \in b_0c_0 \) to \( c_0 \) such that \( b_0c' \) does not intersect any obstacle (Figure 2B). Compute the center \( b' \) of the circular arc \( \gamma_{c't} \) emanating counter-clockwise from \( t \) to \( c' \), where \( b' \in vb_0 \).

A4. Check if \( b't \) is intersected by any obstacle (Figure 2B). If it is, then a feasible min-\( r \) articulated trajectory whose \( ab \) passes through \( u \) and \( v \) does not exist. Otherwise, proceed with steps A5 and A6.

A5. Find the closest point \( b'' \in vb' \) to \( b' \) such that \( b''t \) intersects an obstacle endpoint (Figure 2C). Compute the corresponding point \( c'' \) (i.e., the intersection between \( b_0c' \) and the circle of radius \( |b''t| \) centered at \( b'' \)). Note that the triangle bounded by \( b', b''t \), and \( b'b'' \) is free of obstacles.

A6. Check if the “sector” bounded by \( b'c'', b't \), and circular arc \( \gamma_{c't} \) (centered at \( b'' \) and emanating counter-clockwise from \( t \) to \( c'' \)) intersects any obstacle (Figure 2C). Note that it is equivalent to checking if the circular sector bounded by \( b'c'', b''t \), and \( \gamma_{c't} \) intersects any obstacle. If it does, then a feasible min-\( r \) articulated trajectory whose \( ab \) passes through \( u \) and \( v \) does not exist. Otherwise, proceed with step A7.

A7. At this point, observe that the articulated trajectory with the intermediate configuration represented by \( ab''c'' \) is feasible. Find the closest point \( b''' \in b'bx \) to \( b' \) such that circular arc \( \gamma_{c'''t} \) (centered at \( b''' \) and emanating counter-clockwise from \( t \) to \( c''' \)) intersects an obstacle endpoint or is tangent to an obstacle line segment (Figure 2D). Note that the “sector” bounded by \( b'c''' \), \( b't \), and circular arc \( \gamma_{c't} \) is free of obstacles. The articulated trajectory with the intermediate configuration indicated by \( ab'''c''' \) is a feasible min-\( r \) articulated trajectory whose \( ab \) passes through \( u \) and \( v \).

By simply performing an \( O(n) \)-check (i.e., check against each of the \( O(n) \) obstacles) in each of the steps above, we can obtain an \( O(n^2) \)-time “brute-force” method to find a feasible min-\( r \) articulated trajectory, if one exists. Alternatively, we can address these steps using efficient data structures, which require geometric constructs such as lower envelopes and half-space decomposition schemes. Refer to Table 1 for a summary of the query data structures, whose details are deferred to the full version of the paper. \( O(n^2) \) queries are to be processed in the worst case, resulting in a total query time bounded by \( O(n^2 \log^2 n) \). Since the preprocessing time of the query data structures is dominant overall, we have the following final result.

**Theorem 2** A feasible min-\( r \) articulated probe trajectory, if one exists, can be determined in \( O(n^{2+\epsilon}) \) time using \( O(n^{2+\epsilon}) \) space, for any constant \( \epsilon > 0 \).

The solution approach just described can be extended to find all feasible values of \( r \). The details of the algorithmic extension will be presented in the full publication, and the corresponding result is summarized in the following theorem.

**Theorem 3** All values of \( r \) for which at least one feasible trajectory exists can be determined in \( O(n^{5/2}) \) time using \( O(n^{5/2}) \) space, for any constant \( \epsilon > 0 \).

### 3 Characterizing feasible trajectory space

In this section, we describe our solution to Problem 2. We begin by explicitly characterizing the following for a given length \( r \): i) the final configuration space, ii) the forbidden final configuration space, and iii) the infeasible final configuration space.

#### 3.1 Final configuration space

In a final configuration of the articulated probe, point \( a \) can be assumed to be on \( S \), and point \( b \) lies on the circle \( C \) of radius \( r \) centered at \( t \) (Figure 1). Let \( \theta_S \) and \( \theta_C \) be the angles of line segments \( ta \) and \( tb \) measured counter-clockwise from the \( x \)-axis, where \( \theta_S, \theta_C \in [0, 2\pi) \). Since \( bc \) may rotate around \( b \) as far as \( \pi/2 \) radians in either direction, for any given \( \theta_S \), we have \( \theta_C \in [\theta_S - \cos^{-1} r/R, \theta_S + \cos^{-1} r/R] \). We call this the unforbidden range of \( \theta_C \). A final configuration of the articulated probe can be specified by \((\theta_S, \theta_C)\), depending on the locations of points \( a \) and \( b \) on circles \( S \) and \( C \),
respectively (Figure 3). The final configuration space \( \Sigma_{\text{fin}} \) of the probe can be computed in \( O(1) \) time.

![Figure 3: Final configurations of the articulated probe.](image)

(A) Each value of \( \theta_S \) is associated with an unforbidden range of \( \theta_C \) spanning from \( \theta_S - \cos^{-1}(r/R) \) to \( \theta_S + \cos^{-1}(r/R) \). (B) The unshaded region of the \((\theta_S, \theta_C)\)-plot represents the unforbidden final configuration space when \( S \) is obstacle-free.

### 3.2 Forbidden final configuration space

A final configuration is called forbidden if the final configuration (represented by \( ab \) and \( bt \)) intersects one or more of the obstacle line segments. Let \( s \) be an obstacle line segment of \( P \). We have two different cases, depending on whether \( s \) is located 1) outside or 2) inside \( C \).

**Case 1. Obstacle line segment \( s \) outside \( C \).** Let angles \( \theta_i \), where \( i = 1, \ldots, 6 \), be defined in the manner depicted in Figure 4A. Briefly, each \( \theta_i \) corresponds to an angle \( \theta_S \) at which a tangent line i) between \( C \) and \( s \) or ii) from \( t \) to \( s \), intersects \( S \). As \( \theta_S \) increases from \( \theta_1 \) to \( \theta_3 \), the upper bound of the unforbidden range of \( \theta_C \) decreases as a continuous function of \( \theta_S \). Similarly, when \( \theta_S \) varies from \( \theta_4 \) to \( \theta_6 \), the lower bound of the unforbidden range of \( \theta_C \) decreases as a continuous function of \( \theta_S \). For \( \theta_3 \leq \theta_S \leq \theta_4 \), there exists no unforbidden final configuration at any \( \theta_C \) (Figure 4B). For conciseness, the upper (resp. lower) bound of the unforbidden range of \( \theta_C \) is referred to as the upper (resp. lower) bound of \( \theta_C \) hereafter.

![Figure 4: Forbidden final configurations due to an obstacle line segment \( s \) outside \( C \).](image)

(B) For \( \theta_3 \leq \theta_S \leq \theta_4 \), there exists no unforbidden final configuration at any \( \theta_C \) (Figure 4B). For conciseness, the upper (resp. lower) bound of the unforbidden range of \( \theta_C \) is referred to as the upper (resp. lower) bound of \( \theta_C \) hereafter.

**Case 2. Obstacle line segment \( s \) inside \( C \).** We can similarly compute the forbidden final configuration space for an obstacle line segment \( s \) inside \( C \). Note in Figure 5A that angles \( \theta_i \), where \( i = 1, \ldots, 6 \), are defined differently from case 1. For \( \theta_1 \leq \theta_S \leq \theta_4 \), the upper bound of \( \theta_C \) is equivalent to \( \theta_2 \). For \( \theta_3 \leq \theta_S \leq \theta_6 \), the lower bound of \( \theta_C \) equals to \( \theta_5 \) (Figure 5B).

We can find the forbidden final configuration space for an obstacle line segment in \( O(1) \) time. Thus, for \( n \) obstacle line segments, it takes \( O(n) \) time to compute the corresponding set of forbidden final configurations. The union of these configurations forms the forbidden final configuration space \( \Sigma_{\text{fin,forb}} \) of the articulated probe. The free final configuration space of the articulated probe is \( \Sigma_{\text{fin,free}} = \Sigma_{\text{fin}} \setminus \Sigma_{\text{fin,forb}} \).
The feasible trajectory space of the articulated probe can be characterized as a subset of \( \Sigma_{fin,free} \). A final configuration is called infeasible if the circular sector associated with the final configuration (i.e., the area swept by segment \( bc \) to reach \( t \)) intersects any obstacle line segment. We denote the infeasible final configuration space as \( \Sigma_{fin,inf} \). The analytical details of the characterization of \( \Sigma_{fin,inf} \) are presented in Appendix B. Based on the analysis, we conclude that the infeasible final configuration space associated with any obstacle line segment can be found in \( O(1) \) time. As a result, it takes \( O(n) \) time to determine the infeasible final configuration space for \( n \) obstacle line segments.

### 3.4 Complexity and construction of feasible trajectory space

The feasible trajectory space of the articulated probe is represented by \( \Sigma_{fin} \setminus (\Sigma_{fin,forb} \cup \Sigma_{fin,inf}) \). Three sets of lower- and upper-bound curves, denoted as \( \sigma_{fin} \), \( \sigma_{fin,forb} \), and \( \sigma_{fin,inf} \), were obtained from characterizing the final, forbidden final, and infeasible final configuration spaces, respectively. Each of these curves is a function of \( \theta_S \) – that is, \( \theta_C(\theta_S) \).

As illustrated in Figure 3, \( \sigma_{fin} \) contains two linearly increasing curves, \( \theta_C = \theta_S - \cos^{-1} \frac{r}{R} \) and \( \theta_C = \theta_S + \cos^{-1} \frac{r}{R} \), which are defined over \( \theta_S \in [0, 2\pi) \). Each curve in \( \sigma_{fin,forb} \) is partially defined, continuous, and monotone in \( \theta_S \). Specifically, as shown in Figures 4 & 5, the curves in case 1 are monotonically decreasing with respect to \( \theta_S \), and the curves in case 2 are horizontal lines parallel to the \( \theta_S \)-axis (i.e., of some constant values of \( \theta_C \)). Furthermore, any two curves in case 1 can intersect at most once. Likewise, a curve in \( \sigma_{fin,inf} \) is bounded and monotonically increasing with respect to \( \theta_S \) (Figures 9 & 11 in Appendix B), and can intersect with another at most once.

From the observations above, it can be easily deduced that the number of intersections between any two curves in \( \sigma = \sigma_{fin} \cup \sigma_{fin,forb} \cup \sigma_{fin,inf} \) is at most one. For a set \( \sigma \) of \( O(n) \) \( x \)-monotone Jordan arcs, with at most \( c \) intersections per pair of arcs, where \( c \) is a constant, the maximum combinatorial complexity of the arrangement \( A(\sigma) \) is \( O(n^2) \) [6].

An incremental construction approach, as detailed in [5], can be used to construct the arrangement \( A(\sigma) \) in \( O(n^2 \alpha(n)) \) time using \( O(n^2) \) space, where \( \alpha(n) \) is the inverse Ackermann function. By using topological sweep [2] in computing the intersections for a collection of well-behaved curves such as those described above, the time and space complexities can be improved to \( O(n \log n + k) \) and \( O(n + k) \), respectively. Note that we can find a feasible probe trajectory by simply traversing the cells of the arrangement \( A(\sigma) \) in \( O(n^2) \) time. This implies an \( O(\log n) \) improvement over the previous result reported in [11]. We thus conclude with the following theorem.

**Theorem 4** For a positive value \( r \), the feasible trajectory space of the corresponding articulated probe can be represented as a simple arrangement of maximum combinatorial complexity \( k = O(n^2) \), which can be constructed in \( O(n \log n + k) \) time using \( O(n + k) \) space. A feasible probe trajectory, if one exists, can be determined in \( O(n^2) \) time using \( O(n^2) \) space.

### 4 Open questions

1) Our solution to Problem 1 relies on efficient data structures to address some rather specific geometric intersection and emptiness query problems. Can we improve upon those query data structures? 2) Do our techniques extend well to the variant in which a clearance is required from the obstacles? 3) Can we generalize our solution approaches to three dimensions?
Appendices

A Proof of Lemma 1

We proceed by considering the two possible scenarios implied by Observation 1.

Scenario A. A feasible min-\(r\) articulated probe trajectory exists such that \(ab\) of the trajectory passes through two obstacles endpoints \(u, v \in V\), where \(u \neq v\). Obviously, \(ab\) does not intersect the interior of any line segment of \(P\). Without loss of generality, assume that segment \(bc\) of the probe is rotated clockwise around \(b\) to reach \(t\) (the other case can be handled symmetrically), and \(ab\) passes through \(u\) and \(v\) in the way depicted in Figure 6.

Let \(h_{ab}\) denote the supporting line of \(ab\). Let \(b_0t\) be the perpendicular line segment dropped from \(t\) to line \(h_{ab}\). It is easy to observe that the minimum possible value of \(r\) for an articulated trajectory is given by the length of \(b_0t\) – that is, when \(b = b_0\) and \(\angle cbt\) is equal to \(\pi/2\) radians. Let \(T\) denote the corresponding trajectory. If \(T\) is free of obstacles, then \(T\) is a feasible min-\(r\) articulated trajectory (case I of the lemma).

Otherwise, the minimum feasible value of \(r\) is attained at some point \(b^*\) on line segment \(vb_0\), where \(b^*\) is the closest point to \(b_0\) on \(vb_0\) for which the corresponding articulated trajectory is feasible. In order to find \(b^*\), we increase \(r\) by moving \(b\) away from \(b_0\) on \(vb_0\) until the trajectory becomes feasible. Observe that, if \(bt\) intersects an obstacle line segment at any given time during the process of increasing \(r\), then the trajectory would never become feasible thereafter (illustrated by the blue and green trajectories in Figure 6).

The observations above imply that, if \(b = b_0\) is not feasible, then \(bc\) or \(\gamma_{ct}\) of \(T\) must be intersected by an obstacle line segment, or \(\sigma_{ct}\) of \(T\) must contain an obstacle line segment. By moving \(b\) away from \(b_0\) on \(vb_0\), we may rid the trajectory of obstacle line segments that intersect \(bc\), \(\gamma_{ct}\), or are contained within \(\sigma_{ct}\). Suppose that we increase \(r\) until either \(bt\) becomes tangent to an obstacle line segment or \(b\) reaches \(v\). Let \(b_1\) denote the final position of \(b\). Observe that \(b^*\) must lie somewhere between \(b_0\) and \(b_1\). In fact, as we increase \(r\), \(b = b^*\) when \(bc\) intersects an obstacle line segment at \(c\), or \(\gamma_{ct}\) intersects an obstacle endpoint or is tangent to an obstacle line segment (cases II and III of the lemma).

Figure 6: Finding the extremal feasible min-\(r\) articulated probe trajectory in Scenario A.

References

[1] E. Arkin and J. Mitchell. An optimal visibility algorithm for a simple polygon with star-shaped holes. Technical report, Cornell University Operations Research and Industrial Engineering, 1987.

[2] I. J. Balaban. An optimal algorithm for finding segments intersections. In Proceedings of the eleventh annual symposium on Computational geometry, pages 211–219, 1995.

[3] C. Culmone, G. Smit, and P. Breedveld. Additive manufacturing of medical instruments: A state-of-the-art review. Additive Manufacturing, 2019.

[4] O. Daescu and K. Teo. Computing feasible trajectories for an articulated probe in three dimensions. In 31st Annual Canadian Conference on Computational Geometry, pages 59–70, 2019.

[5] H. Edelsbrunner, L. Guibas, J. Pach, R. Pollack, R. Seidel, and M. Sharir. Arrangements. Theoretical Computer Science, 92(2):319–336, 1992.

[6] D. Halperin and M. Sharir. Arrangements. Handbook of Discrete and Computational Geometry, pages 723–762, 2017.

[7] P. J. Heffernan and J. S. Mitchell. An optimal algorithm for computing visibility in the plane. SIAM Journal on Computing, 24(1):184–201, 1995.

[8] M. Pocchiola. Graphics in flatland revisited. In Scandinavian Workshop on Algorithm Theory, pages 85–96, 1990.

[9] N. S. M. Yasin, and L. Wang. Medical technologies and challenges of robot-assisted minimally invasive intervention and diagnostics. Annual Review of Control, Robotics, and Autonomous Systems, 1:465–490, 2018.

[10] S. Suri and J. O’Rourke. Worst-case optimal algorithms for constructing visibility polygons with holes. In Proceedings of the Second Annual Symposium on Computational Geometry, pages 14–23, ACM, 1986.

[11] K. Teo, O. Daescu, and K. Fox. Trajectory planning for an articulated probe. Computational Geometry, page 101655, 2020.
Remark. Let $r_{b_0}$, $r_{b^*}$, and $r_{b_1}$ be the lengths of $bc$ when $b = b_0$, $b = b^*$, and $b = b_1$, respectively, where $r_{b_0} \leq r_{b^*} \leq r_{b_1}$. Observe that $[r_{b^*}, r_{b_1}]$ is a feasible contiguous subset of $[r_{b_0}, r_{b_1}]$. Indeed, based on the observations made thus far, it is easy to figure that, in Scenario A, there exists at most one contiguous feasible subset of $[r_{b_0}, r_{b_1}]$.

Scenario B. A feasible min-$r$ articulated probe trajectory exists such that $ab$ of the trajectory passes through an obstacle endpoint $u$, and $bt$ of the trajectory passes through an obstacle endpoint $v$, where $u, v \in V$ and $u \neq v$. Recall that $\angle cbt$ of the trajectory is less than or equal to $\pi/2$ radians. Without loss of generality, assume that segment $bc$ of the probe is rotated clockwise around $b$ to reach $t$, as in Figure 7 (the other case is symmetrical).

Figure 7: Finding the extremal feasible min-$r$ articulated probe trajectory in Scenario B.

In this case, the minimum value of $r$ for a feasible trajectory occurs when $b = v$. Let $b_0$ denote that location of $b$, and $T$ be the corresponding trajectory. If $T$ is free of obstacles, then $T$ is a feasible min-$r$ articulated trajectory (case IV of the lemma).

We now assume otherwise. Let $\rho_{b_0}$ denote the reversal (i.e., opposite in direction) of the ray emanating from $b_0$ passing through $t$. We increase $r$ by moving $b$ away from $b_0$ along $\rho_{b_0}$, while maintaining the intersection of $ab$ with $u$ and that of $bt$ with $v$, until the trajectory becomes feasible. Observe the following: i) If $\sigma_{bct}$ of $T$ intersects any obstacle line segment, then for certain there is no feasible articulated trajectory that intersects $u$ outside $C$ and $v$ inside $C$. So, $\sigma_{bct}$ of $T$ must be empty of obstacle line segments. ii) If $bt$, $bc$, or $\gamma_{ct}$ intersects an obstacle line segment at any given moment during the process of increasing $r$, then the trajectory would never become feasible thereafter.

These observations imply that, when $b = b_0$, $ab$ of $T$ must be intersected by some obstacle line segment. By increasing $r$, we may rid the trajectory of obstacle line segments that intersect $ab$. Let $b^*$ denote the closest point to $b_0$ on $\rho_{b_0}$ for which the corresponding articulated trajectory is feasible. Note that $ab$, at the moment, intersects an obstacle endpoint (case V of the lemma), as illustrated by the red trajectory in Figure 7.

Remark. Observe that we can continue to increase $r$, while still having a feasible articulated trajectory, until $b$ reaches some point $b_1$, at which either i) $ab$, $bc$, or $\gamma_{ct}$ collides with an obstacle line segment, or ii) $\angle cbt = \pi/2$. Let $r_{b_0}$, $r_{b^*}$, and $r_{b_1}$ be the lengths of $bc$ when $b = b_0$, $b = b^*$, and $b = b_1$, respectively, where $r_{b_0} \leq r_{b^*} \leq r_{b_1}$. In addition, let $r_{\pi/2}$ be the length of $bc$ when $\angle cbt = \pi/2$. According to our earlier arguments, $[r_{b^*}, r_{b_1}]$ is a feasible contiguous subset of $[r_{b_0}, r_{\pi/2}]$. In fact, there could exist multiple (disjoint) contiguous feasible subsets of $[r_{b_0}, r_{\pi/2}]$, given that $ab$ may enter and leave intersections with multiple obstacle line segments during the process of increasing $r$, while $\sigma_{bct}$ remains free of obstacle line segments (refer to the blue and green trajectories in Figure 7 for an instance).

This concludes the proof of Lemma 1.

B Characterizing infeasible final configuration space

Let $C'$ be the circle centered at $t$ and of radius $\sqrt{2r}$. A circular sector associated with a final configuration can only intersect an obstacle line segment lying inside $C'$. Instead of characterizing the lower and upper bounds of $\theta_C$ as $\theta_S$ varies from 0 to $2\pi$ (as in Section 3.2), here we perform the characterization the other way around. For conciseness, we only present arguments for the negative half of the $\theta_S$-range, which is $[\theta_C - \cos^{-1} r/R, \theta_C]$; similar arguments apply to the other half due to symmetry. We have two cases, depending on whether an obstacle line segment $s$ lies 1) inside $C$ or 2) outside $C$ and inside $C'$.

Case 1. Obstacle line segment $s$ inside $C$. For brevity, the quarter circular sector associated with a point $b$ (i.e., the maximum possible area swept by segment $bc$ to reach $t$), where the angle of $tb$ (relative to the $x$-axis) is $\theta_C$, is referred to as the quart-sector of $\theta_C$.

We define $\phi_1$, $\phi_2$, and $\phi_3$ as follows (Figure 8A). $\phi_1$ is the smallest angle $\theta_C$ at which the circular arc of the quart-sector of $\theta_C$ intersects $s$ (at one of its endpoints or interior points). $\phi_2$ is the smallest angle $\theta_C$ at which $bt$ of the quart-sector of $\theta_C$ intersects $s$ (at one of its endpoints). $\phi_3$ is the largest angle $\theta_C$ at which $bt$ of the quart-sector of $\theta_C$ intersects $s$ (at one of its endpoints). Observe that, as $\theta_C$ varies from 0 to $2\pi$, $\phi_1$ and $\phi_2$ are the angles $\theta_C$ at which the quart-sector of $\theta_C$ first and last intersects $s$, respectively.

We are only concerned with finding the lower bound of $\theta_S$ for $\theta_C \in [\phi_1, \phi_2]$, since the entire negative half of the $\theta_S$-range (i.e., $[\theta_C - \cos^{-1} r/R, \theta_C]$) is feasible for $\theta_C \in [0, \phi_1] \cup [\phi_3, 2\pi)$, and is infeasible for $\theta_C \in [\phi_2, \phi_3]$ due to intersection of $bt$ with $s$ (Figure 8A).

For $\theta_C \in [\phi_1, \phi_2]$, the lower bound of $\theta_S$ can be represented by a piecewise continuous curve, which consists of at most two pieces, corresponding to two intervals $[\phi_1, \alpha]$ and $[\alpha, \phi_2]$, where $\alpha$ is the angle $\theta_C$ of the intersection point between $C$ and the supporting line of $s$. If $\phi_1 \leq \alpha$, then the curve has two pieces; otherwise, the curve is of one single piece.

For $\theta_C \in [\phi_1, \alpha]$, the lower bound of $\theta_S$ is indicated by the endpoint $a$ of line segment $abc'$, where $c'$ is the intersection point between $s$ and the circular arc centered at $b$ (Figure 8B). If no intersection occurs between $s$ and the circular arc,
Figure 8: Infeasible final configurations due to an obstacle line segment $s$ inside $C$. Illustrations of $\theta_S$-lower bounds for (A) $\theta_C \in [\phi_1, \phi_2]$, (B) $\phi_1 < \theta_C < \alpha$, (C) $\theta_C = \alpha$, and (D) $\alpha < \theta_C < \phi_2$.

Figure 9: Infeasible final configuration space due to an obstacle line segment $s$ inside $C$.

then the lower bound of $\theta_S$ is given by the endpoint $a$ of line segment $abc'$, where $bc'$ intersects an endpoint of $s$.

For $\theta_C \in [\alpha, \phi_2]$, the lower bound of $\theta_S$ is indicated by the endpoint $a$ of line segment $abc'$, where $bc'$ intersects an endpoint of $s$ (Figure 8D). The lower bound of $\theta_S$ is equal to $\theta_C$ when $\theta_C = \phi_2$. See Figure 9 for a sketch of the infeasible final configuration space.

**Case 2. Obstacle line segment $s$ outside $C$ and inside $C'$**. As depicted in Figure 10, we only need to worry about computing the lower bound of $\theta_S$ for $\theta_C \in [\phi_1, \phi_2]$, given that the entire negative half of the $\theta_S$-range (i.e., $[\theta_C - \cos^{-1} \frac{r}{R}, \theta_C]$) is feasible for $\theta_C \in [0, \phi_1] \cup [\phi_2, 2\pi)$. The analysis is similar to case 1 and thus omitted herein. A sketch of the corresponding infeasible final configuration space is shown in Figure 11.

Observe that any of the curves just described for characterizing the lower or upper bound of $\theta_S$ can be computed in constant time. Thus, given an obstacle line segment $s$, the associated infeasible final configuration space can be found in $O(1)$ time. As a result, it takes $O(n)$ time to determine the infeasible final configuration space for $n$ obstacle line segments.
Figure 10: Infeasible final configurations due to an obstacle line segment $s$ outside $C$ and inside $C'$. Illustrations of $\theta_S$-lower bounds for (A) $\theta_C \in [\phi_1, \phi_2]$, (B) $\phi_1 < \theta_C < \alpha$, (C) $\theta_C = \alpha$, and (D) $\alpha < \theta_C < \phi_2$.

Figure 11: Infeasible final configuration space due to a line segment $s$ outside $C$ and inside $C'$. 
