BEREZIN NUMBER INEQUALITIES FOR HILBERT SPACE OPERATORS

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Abstract. In this paper, by using of the definition Berezin symbol, we show some Berezin number inequalities. Among other inequalities, it is shown that if $A, B, X \in \mathbb{B}(\mathcal{H})$, then
\[
\text{ber}(AX \pmXA) \leq \text{ber}^{\frac{1}{2}} (A^*A + AA^*) \text{ber}^{\frac{1}{2}} (X^*X + XX^*)
\]
and
\[
\text{ber}^2(A^*XB) \leq \|X\|^2 \text{ber}(A^*A)\text{ber}(B^*B).
\]

1. Introduction

Let $\mathbb{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$ with the identity $I$. A functional Hilbert space is a Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on a set $\Omega$, which has the property that point evaluations are continuous i.e., for each $\lambda \in \Omega$ the map $f \mapsto f(\lambda)$ is a continuous linear functional on $\mathcal{H}$.

Berezin set and Berezin number of the operator $A$ are defined by $\text{Ber}(A) := \{\tilde{A}(\lambda) : \lambda \in \Omega\}$ and $\text{ber}(A) := \sup \{|\tilde{A}(\lambda)| : \lambda \in \Omega\}$, respectively. It is clear that the Berezin symbol $\tilde{A}$ is the bounded function on $\Omega$ whose values lies in the numerical range of the operator $A$ and hence $\text{Ber}(A) \subseteq W(A)$ (numerical radius) and $\text{ber}(A) \leq w(A)$ (numerical range) for all $A \in \mathbb{B}(\mathcal{H})$. The Berezin number of an operator $A$ satisfies the following properties:

(i) $\text{ber}(A) \leq \|A\|$.

(ii) $\text{ber}(\alpha A) = |\alpha|\text{ber}(A)$ for all $\alpha \in \mathbb{C}$.

(iii) $\text{ber}(A + B) \leq \text{ber}(A) + \text{ber}(B)$.

The Berezin symbol is widely applied in the various questions of uniquely determines the operator and analysis. For further information about Berezin symbol we refer the reader to [4, 8, 9, 15] and references therein.

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In this paper, by using some ideas of [1, 16], we present several Berezin number inequalities. In particular, we obtain the inequalities

(i) \( \text{ber}(AX \pm XA) \leq \text{ber}^{\frac{1}{2}} (A^*A + AA^*) \text{ber}^{\frac{1}{2}} (X^*X + XX^*) \); 

(ii) \( \text{ber}(A^*XB + B^*YA) \leq 2\sqrt{\|X\|\|Y\|}\text{ber}^{\frac{1}{2}} (B^*B) \text{ber}^{\frac{1}{2}} (AA^*) \), 

where \( A, B, X, Y \in \mathbb{B} (\mathcal{H}(\Omega)) \).

2. The results

To prove our first result, we need the following lemma.

**Lemma 2.1.** Let \( X \in \mathbb{B} (\mathcal{H}(\Omega)) \). Then

\[
\text{ber}(X) = \sup_{\theta \in \mathbb{R}} \text{ber} \left( \text{Re}(e^{i\theta}X) \right) = \sup_{\theta \in \mathbb{R}} \text{ber} \left( \text{Im}(e^{i\theta}X) \right),
\]

where \( \text{Re}(X) = \frac{X + X^*}{2} \) and \( \text{Im}(X) = \frac{X - X^*}{2i} \).

**Proof.** Let \( \hat{k}_\lambda \) be the normalized reproducing kernel of \( \mathcal{H}(\Omega) \). It follows from

\[
\sup_{\theta \in \mathbb{R}} \left\langle \text{Re}(e^{i\theta}X) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle = \left| \left\langle X \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|
\]

that

\[
\sup_{\theta \in \mathbb{R}} \text{ber} \left( \text{Re}(e^{i\theta}X) \right) = \sup_{\theta \in \mathbb{R}} \sup_{\lambda \in \Omega} \left| \left\langle \text{Re}(e^{i\theta}X) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|
\]

\[
= \sup_{\lambda \in \Omega} \left| \left\langle X \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|
\]

\[
= \text{ber}(X).
\]

The proof of the second equation is similar. \( \square \)

**Remark 2.2.** If \( X = H + iK \) be the certain decomposition of the operator \( X \), then by using this fact

\[
|\langle Hx, x \rangle| \leq |\langle Xx, x \rangle| = |\langle Hx, x \rangle + i\langle Kx, x \rangle| = \sqrt{|\langle Hx, x \rangle|^2 + |\langle Kx, x \rangle|^2} \quad (x \in \mathcal{H})
\]

and Lemma 2.1, we have

\[
\text{ber}(H) = \text{ber} \left( \text{Re}(X) \right) \leq \text{ber}(X) \leq \sqrt{\text{ber}^2(H) + \text{ber}^2(K)}.
\]

Now, by applying Lemma 2.1, we show an upper bound for \( \text{ber}(AX \pm XA^*) \).
Theorem 2.3. Let $A, X \in B(\mathcal{H}(\Omega))$. Then

$$ber^2(AX \pmXA^*) \leq 2\|A\|^2 \left( \ber(H^2) + \ber(K^2) + \sqrt{(\ber(H^2) - \ber(K^2))^2 + \ber^2(HK + KH)} \right),$$

where $X = H + iK$ is the certain decomposition of the operator $X$.

Proof. Suppose that $\hat{k}_\lambda$ is the normalized reproducing kernel of $\mathcal{H}(\Omega)$. Then

$$\left| \langle \Re(e^{i\theta}(AX +XA^*)) \hat{k}_\lambda, \hat{k}_\lambda \rangle \right|^2 = \left| \langle \Re \left( (A\Re(e^{i\theta}X) + \Re(e^{i\theta}X)A^*) \right) \hat{k}_\lambda, \hat{k}_\lambda \rangle \right|^2$$

(since $\Re(T) = \Re(T^*)$)

$$\leq \left| \langle (A\Re(e^{i\theta}X) + \Re(e^{i\theta}X)A^*) \hat{k}_\lambda, \hat{k}_\lambda \rangle \right|^2$$

(since $|\langle \Re(T)x, x \rangle| \leq |\langle Tx, x \rangle|$)

$$\leq 2 \left( \left| \langle A\Re(e^{i\theta}X) \hat{k}_\lambda, \hat{k}_\lambda \rangle \right|^2 + \left| \langle \Re(e^{i\theta}X)A^* \hat{k}_\lambda, \hat{k}_\lambda \rangle \right|^2 \right)$$

(by the triangular inequality and the convexity $f(t) = t^2$)

$$= 2 \left( \left| \langle \Re(e^{i\theta}X) \hat{k}_\lambda, A^* \hat{k}_\lambda \rangle \right|^2 + \left| \langle A^* \hat{k}_\lambda, \Re(e^{i\theta}X) \hat{k}_\lambda \rangle \right|^2 \right)$$

$$\leq 2 \left( \|A^*\|^2 \left\| \Re(e^{i\theta}X) \hat{k}_\lambda \right\|^2 + \|A\|^2 \left\| \Re(e^{i\theta}X) \hat{k}_\lambda \right\|^2 \right)$$

$$= 4\|A\|^2 \left( \langle \Re(e^{i\theta}X) \hat{k}_\lambda, \Re(e^{i\theta}X) \hat{k}_\lambda \rangle \right)$$

$$= 4\|A\|^2 \left( \langle (\Re(e^{i\theta}X))^2 \hat{k}_\lambda, \hat{k}_\lambda \rangle \right).$$

It follows from

$$(\Re(e^{i\theta}X))^2 = (\Re(e^{i\theta}(H + iK)))^2$$

$$= (\cos \theta H - \sin \theta K)^2$$

$$= \cos^2 \theta H^2 + \sin^2 \theta K^2 - \cos \theta \sin \theta (HK + KH)$$
that
\[
\sup_{\theta \in \mathbb{R}} \left( \Re(e^{i\theta} X) \right)^2 \hat{k}_\lambda, \hat{k}_\lambda \nabla \nabla
\]
\[
= \sup_{\theta \in \mathbb{R}} \left( \cos^2 \theta \left\langle H^2 \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \sin^2 \theta \left\langle K^2 \hat{k}_\lambda, \hat{k}_\lambda \right\rangle - \cos \theta \sin \theta \left( \left\langle (HK + KH) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right) \right)
\]
\[
\leq \sup_{\theta \in \mathbb{R}} \left( \cos^2 \theta \text{ber}(H^2) + \sin^2 \theta \text{ber}(K^2) - \cos \theta \sin \theta \left( \left\langle (HK + KH) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right) \right)
\]
\[
\leq \frac{1}{2} \left( \text{ber}(H^2) + \text{ber}(K^2) + \sqrt{(\text{ber}(H^2) - \text{ber}(K^2))^2 + \left( \left\langle (HK + KH) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right)^2} \right),
\]
whence
\[
\sup_{\theta \in \mathbb{R}} \left| \Re \left( e^{i\theta} (AX + XA^*) \right) \hat{k}_\lambda, \hat{k}_\lambda \right| \leq 2\|A\|^2 \left( \text{ber}(H^2) + \text{ber}(K^2) \right.
\]
\[
+ \sqrt{(\text{ber}(H^2) - \text{ber}(K^2))^2 + \left( \left\langle (HK + KH) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right)^2} \right).
\]
Taking the supremum over all \( \lambda \in \Omega \) and using Lemma 2.1, we get
\[
\text{ber}^2(AX + XA^*)
\]
\[
= \sup_{\theta \in \mathbb{R}} \text{ber}(\Re(e^{i\theta} (AX + XA^*)))
\]
\[
\leq 2\|A\|^2 \left( \text{ber}(H^2) + \text{ber}(K^2) \right. + \sqrt{(\text{ber}(H^2) - \text{ber}(K^2))^2 + \text{ber}^2(HK + KH)} \right). \tag{2.1}
\]
Replacing \( A \) by \( iA \) in (2.1), we have
\[
\text{ber}^2(AX - XA^*)
\]
\[
\leq 2\|A\|^2 \left( \text{ber}(H^2) + \text{ber}(K^2) \right. + \sqrt{(\text{ber}(H^2) - \text{ber}(K^2))^2 + \text{ber}^2(HK + KH)} \right). \tag{2.2}
\]
Hence
\[
\text{ber}^2(AX \pm XA^*)
\]
\[
\leq 2\|A\|^2 \left( \text{ber}(H^2) + \text{ber}(K^2) \right. + \sqrt{(\text{ber}(H^2) - \text{ber}(K^2))^2 + \text{ber}^2(HK + KH)} \right)
\]
as required. \( \square \)

Theorem 2.3 includes a special case as follows.

**Corollary 2.4.** Let \( A, X \in \mathbb{B}(\mathcal{H}(\Omega)) \) Then
(i) If \( HK + KH = 0 \), then \( \text{ber}(AX \pm XA^*) \leq 2\|A\| \max \left( \text{ber}^\frac{1}{2}(H^2), \text{ber}^\frac{1}{2}(K^2) \right) \).
(ii) If \( X \) is self-adjoint, then \( \text{ber}(AX \pm XA^*) \leq 2\|A\| \text{ber}^\frac{1}{2}(X^2). \)
(iii) If \( X \) is self-adjoint, then \( \text{ber}(AX) \leq \|A\| \text{ber}^\frac{1}{2}(X^2) \).
where \( X = H + iK \) is the certain decomposition of the operator \( X \).
Proof. The first inequality follows from Theorem 2.3 and the inequality
\[
\text{ber}^2(AX \pm XA^*) \\
\leq 2\|A\|^2 \left( \text{ber}(H^2) + \text{ber}(K^2) + \sqrt{\text{ber}(H^2) - \text{ber}(K^2)} \right) \\
= 2\|A\|^2 (\text{ber}(H^2) + \text{ber}(K^2) + |\text{ber}(H^2) - \text{ber}(K^2)|) \\
= 4\|A\|^2 \max (\text{ber}(H^2), \text{ber}(K^2)).
\]
The second inequality follows from Theorem 2.3 and the hypotheses \(X = H + 0i\). For the third inequality we have
\[
\text{ber}(AX) = \sup_{\theta \in \mathbb{R}} \text{ber} \left( \Re(e^{i\theta} AX) \right) \\
= \frac{1}{2} \sup_{\theta \in \mathbb{R}} \text{ber} \left( e^{i\theta} AX + e^{-i\theta} XA^* \right) \\
\leq \|A\| \text{ber}^{\frac{1}{2}}(X^2) \quad \text{(by part (ii))}
\]
as required. \(\square\)

The following theorem gives some upper bounds for \(\text{ber}(AX \pm XA)\).

**Theorem 2.5.** Let \(A, X \in \mathbb{B}(\mathcal{H}(\Omega))\). Then

(i) \(\text{ber}(AX \pm XA) \leq \text{ber}^{\frac{1}{2}}(A^*A + AA^*) \text{ber}^{\frac{1}{2}}(X^*X + XX^*)\).

(ii) \(\text{ber}(AX \pm XA) \leq \text{ber}^{\frac{1}{2}}(A^*A + X^*X) \text{ber}^{\frac{1}{2}}(AA^* + XX^*)\).

**Proof.** Let \(\hat{k}_\lambda\) be the normalized reproducing kernel of \(\mathcal{H}(\Omega)\). Then
\[
\left| \left\langle (AX \pm XA) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| \leq \left| \left\langle AX\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| + \left| \left\langle XA\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| \\
= \left| \left\langle X\hat{k}_\lambda, A^*\hat{k}_\lambda \right\rangle \right| + \left| \left\langle A\hat{k}_\lambda, X^*\hat{k}_\lambda \right\rangle \right| \\
\leq \|X\hat{k}_\lambda\| \|A^*\hat{k}_\lambda\| + \|A\hat{k}_\lambda\| \|X^*\hat{k}_\lambda\| \\
\leq \left( \|A\hat{k}_\lambda\|^2 + \|A^*\hat{k}_\lambda\|^2 \right)^{\frac{1}{2}} \left( \|X\hat{k}_\lambda\|^2 + \|X^*\hat{k}_\lambda\|^2 \right)^{\frac{1}{2}} \\
\quad \text{(by the Cauchy-Schwartz inequality)} \\
= \left| \left\langle (A^*A + AA^*) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^{\frac{1}{2}} \left| \left\langle (X^*X + XX^*) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^{\frac{1}{2}} \\
\leq \text{ber}^{\frac{1}{2}}(A^*A + AA^*) \text{ber}^{\frac{1}{2}}(X^*X + XX^*).
Hence

\[ \text{ber}(AX \pm XA) = \sup_{\lambda \in \Omega} \left| \langle (AX \pm XA)\hat{k}_\lambda, \hat{k}_\lambda \rangle \right| \leq \text{ber}^\frac{1}{2}(A^*A + AA^*) \text{ber}^\frac{1}{2}(X^*X + XX^*). \]

Now, according to the inequality

\[ \|X\hat{k}_\lambda\| \|A^*\hat{k}_\lambda\| + \|A\hat{k}_\lambda\| \|X^*\hat{k}_\lambda\| \leq \left( \|A\hat{k}_\lambda\| + \|X\hat{k}_\lambda\| \right)^\frac{1}{2} \left( \|A^*\hat{k}_\lambda\| + \|X^*\hat{k}_\lambda\| \right)^\frac{1}{2} \]

(by the Cauchy-Schwartz inequality)

and a similar argument of the proof of part (i) we get the second inequality. \qed

For the special case \( A = I \), we have the next result.

**Corollary 2.6.** Let \( X \in \mathbb{B}(\mathcal{H}(\Omega)) \). Then

(i) \( \text{ber}^2(X) \leq \text{ber}(I + X^*X) \text{ber}(I + XX^*). \)

(ii) \( \text{ber}^2(X) \leq \frac{1}{2} \text{ber}(X^*X + XX^*). \)

**Remark 2.7.** Corollary 2.6(ii) is an improvement of (1.1). To see this, note that

\[ \text{ber}^2(X) \leq \frac{1}{2} \text{ber}(X^*X + XX^*) \]

\[ \leq \frac{\text{ber}(X^*X) + \text{ber}(XX^*)}{2} \]

\[ \leq \frac{\|X^*X\| + \|XX^*\|}{2} \]

\[ = \|X\|^2. \]

In the following theorem, we present some upper bounds of \( \text{ber}(A^*XB) \). To achieve this propose, we need the next lemma; see [14].

**Lemma 2.8.** If \( X \in \mathbb{B}(\mathcal{H}) \) and \( x, y \in \mathcal{H} \), then \( |\langle Xx, y \rangle|^2 \leq \|X|x, x\rangle \langle |X^*y, y\rangle \), in which \( |X| = (X^*X)^{\frac{1}{2}} \).

**Theorem 2.9.** Let \( A, B, X \in \mathbb{B}(\mathcal{H}(\Omega)) \). Then

(i) \( \text{ber}^2(A^*XB) \leq \|X\|^2\text{ber}(A^*A)\text{ber}(B^*B). \)

(ii) \( \text{ber}(A^*XB) \leq \frac{1}{2} \text{ber}(B^*|X|B + A^*|X^*|A). \)
Proof. If \( \hat{k}_\lambda \) is the normalized reproducing kernel of \( \mathcal{H}(\Omega) \), then

\[
\left| \left\langle A^*XB\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 = \left| \left\langle XB\hat{k}_\lambda, A\hat{k}_\lambda \right\rangle \right|^2 \leq \left\| XB\hat{k}_\lambda \right\|^2 \left\| A\hat{k}_\lambda \right\|^2 \leq \left\| X \right\|^2 \left\| B\hat{k}_\lambda \right\|^2 \left\| A\hat{k}_\lambda \right\|^2 \leq \left\| X \right\|^2 \left\langle B\hat{k}_\lambda, B\hat{k}_\lambda \right\rangle \left\langle A\hat{k}_\lambda, A\hat{k}_\lambda \right\rangle = \left\| X \right\|^2 \left\langle B\hat{k}_\lambda, A\hat{k}_\lambda \right\rangle \left\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \leq \left\| X \right\|^2 \text{ber}(A^*A) \text{ber}(B^*B),
\]

whence \( \text{ber}^2(A^*XB) = \sup_{\lambda \in \Omega} \left| \left\langle A^*XB\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right|^2 \leq \left\| X \right\|^2 \text{ber}(A^*A) \text{ber}(B^*B) \), and so we get the first inequality. Also, we have

\[
\left| \left\langle A^*XB\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| = \left| \left\langle XB\hat{k}_\lambda, A\hat{k}_\lambda \right\rangle \right| \leq \left\langle |X|B\hat{k}_\lambda, B\hat{k}_\lambda \right\rangle ^{\frac{1}{2}} \left\langle |X^*|A\hat{k}_\lambda, A\hat{k}_\lambda \right\rangle ^{\frac{1}{2}} \text{ (by Lemma 2.8)} \leq \frac{1}{2} \left( \left\langle (B^*|X|B)\hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \left\langle (A^*|X^*|A)\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right) \text{ (by the convexity } f(t) = t^2) = \frac{1}{2} \left( \left\langle (B^*|X|B + A^*|X^*|A)\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right) \leq \frac{1}{2} \text{ber} (B^*|X|B + A^*|X^*|A).
\]

Hence

\[
\text{ber}(A^*XB) = \sup_{\lambda \in \Omega} \left| \left\langle A^*XB\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| \leq \frac{1}{2} \text{ber} (B^*|X|B + A^*|X^*|A).
\]

\[ \square \]

In the special case of Theorem 2.9, for \( X = I \) we obtain the next result.

**Corollary 2.10.** Let \( A, B, X \in \mathcal{B}(\mathcal{H}(\Omega)) \). Then

(i) \( \text{ber}^2(A^*B) \leq \text{ber}(A^*A) \text{ber}(B^*B) \).

(ii) \( \text{ber}(A^*B) \leq \frac{1}{2} \text{ber}(A^*A + B^*B) \).
Corollary 2.11. Let $A, B, X \in \mathcal{B}(\mathcal{H}(\Omega))$. Then

(i) $\text{ber}(A^*XB) \leq \text{ber}^{\frac{1}{2}}(B^*|X|B) \text{ber}^{\frac{1}{2}}(A^*|X^*|A)$.

(ii) $\text{ber}(A^*XB) \leq \frac{1}{2} \text{ber} \left( \frac{\|B\|}{\|A\|} B^*|X|B + \frac{\|A\|}{\|B\|} A^*|X^*|A \right)$.

Proof. By Theorem 2.9(ii), we have

$$\text{ber}(A^*XB) \leq \frac{1}{2} \text{ber} \left( B^*|X|B + A^*|X^*|A \right) \leq \frac{1}{2} \left( \text{ber} (B^*|X|B) + \text{ber} (A^*|X^*|A) \right).$$

Now, if we replace $A$ and $B$ by $tA$ and $\frac{1}{t}B$ ($t > 0$) in inequality (2.3), respectively, then we get

$$\text{ber}(A^*XB) \leq \frac{1}{2} \left( \frac{1}{t^2} \text{ber} (B^*|X|B) + t^2 \text{ber} (A^*|X^*|A) \right).$$

It follows from

$$\min_{t>0} \left( \frac{1}{t^2} \text{ber} (B^*|X|B) + t^2 \text{ber} (A^*|X^*|A) \right) = 2\text{ber}^{\frac{1}{2}} (B^*|X|B) \text{ber}^{\frac{1}{2}} (A^*|X^*|A)$$

that we get the first inequality. Moreover, if we replace $A$ and $B$ by $\sqrt{\frac{\|A\|}{\|B\|}} A$ and $\sqrt{\frac{\|B\|}{\|A\|}} B$ Theorem 2.9(ii), respectively, we reach the second inequality. □

Using Theorem 2.9, we demonstrate some upper bounds for $\text{ber}(A^*XB + B^*YA)$.

Theorem 2.12. Let $A, B, X, Y \in \mathcal{B}(\mathcal{H}(\Omega))$. Then

(i) $\text{ber}(A^*XB + B^*YA) \leq \sqrt{2} \|X\| + |Y^*| \| \text{ber}^{\frac{1}{2}} (B^*B) \text{ber}^{\frac{1}{2}} (AA^*)$.

(ii) $\text{ber}(A^*XB + B^*YA) \leq 2\sqrt{\|X\|\|Y\|} \text{ber}^{\frac{1}{2}} (B^*B) \text{ber}^{\frac{1}{2}} (AA^*)$.

Proof. Applying Lemma 2.1 and Theorem 2.9(i), we have

$$\text{ber} (\Re(e^{i\alpha}(A^*XB \pm B^*YA))) = \text{ber} (\Re(e^{i\alpha}X \pm e^{-i\alpha}Y^*)B))$$

(since $\Re(T) = \Re(T^*)$)

$$\leq \text{ber} (A^*(e^{i\alpha}X \pm e^{-i\alpha}Y^*)B))$$

(by Lemma 2.1 for $\theta = 0$)

$$\leq \|e^{i\alpha}X \pm e^{-i\alpha}Y^*\| \text{ber}^{\frac{1}{2}} (B^*B) \text{ber}^{\frac{1}{2}} (AA^*)$$

(by Theorem 2.9(i)). (2.4)
It follows from the inequalities
\[
\| e^{i\alpha} X \pm e^{-i\alpha} Y^* \| = \left\| \begin{bmatrix} e^{i\alpha} X \pm e^{-i\alpha} Y^* & 0 \\ 0 & 0 \end{bmatrix} \right\| \\
= \left\| \begin{bmatrix} e^{i\alpha} & e^{-i\alpha} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \pm Y^* & 0 \\ X & 0 \end{bmatrix} \right\| \\
\leq \sqrt{2} \left\| \begin{bmatrix} X & 0 \\ \pm Y^* & 0 \end{bmatrix} \right\| \\
= \sqrt{2} \left| |X|^2 + |Y^*|^2 \right|^{\frac{1}{2}} \\
\leq \sqrt{2} \| X \| + |Y^*| \|
\]
(\text{applying [2, p. 775] to the function } h(t) = t^{\frac{1}{2}}),
\]
\[(2.4) \text{ and Lemma 2.1 that}
\]
\[
\ber (A^* XB \pm B^* Y A) = \sup_{\alpha \in \mathbb{R}} \ber (\Re(e^{i\alpha}(A^* XB \pm B^* Y A))) \\
\leq \sqrt{2} \| X \| + |Y^*| \| \ber^{\frac{1}{2}} (B^* B) \ber^{\frac{1}{2}} (AA^*) .
\]
Thus, we get the first inequality. Moreover, Using inequality (2.4) we have
\[
\ber (\Re(e^{i\alpha}(A^* XB \pm B^* Y A))) \leq \| e^{i\alpha} X \pm e^{-i\alpha} Y^*\| \ber^{\frac{1}{2}} (B^* B) \ber^{\frac{1}{2}} (AA^*) \\
\leq (\| X \| + \| Y \|) \ber^{\frac{1}{2}} (B^* B) \ber^{\frac{1}{2}} (AA^*) .
\]
\[(2.5)\]
Now, if we replace \( A \) by \( \sqrt{t} A \), \( B \) by \( \sqrt{t} B \), \( X \) by \( tX \) and \( Y \) by \( \frac{1}{t} Y \) (\( t > 0 \)) in inequality (2.5), then we get
\[
\ber (\Re(e^{i\alpha}(A^* XB \pm B^* Y A))) \leq \left( t \| X \| + \frac{1}{t} \| Y \| \right) \ber^{\frac{1}{2}} (B^* B) \ber^{\frac{1}{2}} (AA^*) .
\]
\[(2.6)\]
It follows from \( \min_{t>0} (t \| X \| + \frac{1}{t} \| Y \|) = 2\sqrt{\| X \| \| Y \|} \) and inequality (2.6) that
\[
\ber (A^* XB \pm B^* Y A) = \sup_{\alpha \in \mathbb{R}} \ber (\Re(e^{i\alpha}(A^* XB \pm B^* Y A))) \\
\leq 2\sqrt{\| X \| \| Y \|} \ber^{\frac{1}{2}} (B^* B) \ber^{\frac{1}{2}} (AA^*) .
\]
Hence, we get the second inequality. \( \square \)
Corollary 2.13. If $A, B, X \in \mathcal{B}(\mathcal{H}(\Omega))$, then

(i) $\ber(A^*X \pm XA) \leq 2\|X\|\ber^{1/2}(AA^*)$.

(ii) $\ber(A^*B \pm B^*A) \leq 2\ber^{1/2}(B^*B)\ber^{1/2}(AA^*)$.

Proof. If we put $B = I$ and $X = Y$ in Theorem 2.12(ii), then we reach the first inequality and if we take $X = Y = I$ in Theorem 2.12(ii), then we get the second inequality. □

It is well known that $w(A^n) \leq w^n(A)$ (2.7) for any $A \in \mathcal{B}(\mathcal{H})$ and $n \geq 1$. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Note that for any Toeplitz operator $T_\phi$ with $\phi \in L^\infty(\partial\mathbb{D})$ we have $\tilde{T}_\phi(\lambda) = \tilde{\phi}(\lambda) (\lambda \in \mathbb{D})$, where $\tilde{\phi}$ is the harmonic extension of $\phi$ into $\mathbb{D}$ (see, for instance Engliš [4]). Therefore, it is easy to see that

$$\ber(T_\phi) = \|\phi\|_\infty = \|T_\phi\| = w(T_\phi),$$

which implies that $\ber((T_\phi)^n) \leq \ber^n(T_\phi)$ for any positive integer $n$. In general, inequality (2.7) and the trivial inequality $\ber(A) \leq w(A)$ imply that

$$\ber(A^n) \leq \ber^n(A) \left(\frac{w(A)}{\ber(A)}\right)^n$$

for any $A \in \mathcal{B}(\mathcal{H}(\Omega))$ and $n \geq 1$.

It is natural to ask: does the same property holds true for the Berezin number of $A$, i.e. is it true that $\ber(A^n) \leq \ber^n(A)$?

Here we give some partial answers to this question.

Theorem 2.14. Let $A \in \mathcal{B}(\mathcal{H}(\Omega))$ be an operator such that

(i) $\lim_{\lambda \to \partial\mathbb{D}} \tilde{A}^n(\lambda) \neq 0$ for any integer $n \geq 1$;

(ii) $\lim_{\lambda \to \partial\mathbb{D}} \|(A^* - \tilde{A}^n(\lambda))k_\lambda\| = 0$.

Then $\ber(A^n) \leq \ber^n(A)$ for any integer $n \geq 1$.

Proof. First, let us prove by induction that if $\ber(A) \leq 1$, then $\ber(A^n) \leq 1$ for any integer $n \geq 1$. In fact, for $n = 1$ it is trivial. For $n = k$ we assume that $\ber(A^k) \leq 1$, and we prove that $\ber(A^{k+1}) \leq 1$. We set $L := \ber(A^{k+1})$. Then $\tilde{A}^{k+1}(\lambda) \leq L$ for all $\lambda \in \Omega$, and by virtue of condition (i), for any sequence $(\epsilon_n) \subset (0, 1)$ such that
\[ \lim_{n \to \infty} \epsilon_n = 0 \text{ there exists the sequence } (\lambda_n) \subset \Omega \text{ with } \lim_{n \to \infty} \lambda_n = \xi_0 \in \partial \mathbb{D} \text{ such that } |A^{k+1}(\lambda_n)| > L - \epsilon_n \quad (n \geq 1). \]

Then, by using that \( \text{ber}(A^k) \leq 1 \), we have

\[
L - \epsilon_n < |\tilde{A}^{k+1}(\lambda_n)| = |\left\langle A^{k+1}\hat{k}_{\lambda_n}, \hat{k}_{\lambda_n} \right\rangle| = |\left\langle A^k\hat{k}_{\lambda_n}, A^*\hat{k}_{\lambda_n} \right\rangle| = |\left\langle A^k\hat{k}_{\lambda_n}, A^*(\lambda_n)\hat{k}_{\lambda_n} \right\rangle + \hat{A}(\lambda_n) \left\langle A^k\hat{k}_{\lambda_n}, \hat{k}_{\lambda_n} \right\rangle|
\leq \|A^k\|\|A^*\hat{k}_{\lambda_n} - \hat{A}^*(\lambda_n)\hat{k}_{\lambda_n}\| + |\hat{A}(\lambda_n)||\hat{A}^k(\lambda_n)|
\leq \|A^k\|\|A^* - \hat{A}^*(\lambda_n)\|\hat{k}_{\lambda_n} + \text{ber}(A)\text{ber}(A^k),
\]

whence

\[
L \leq \|A^k\|\|A^*\hat{k}_{\lambda_n} - \hat{A}^*(\lambda_n)\hat{k}_{\lambda_n}\| + 1 + \epsilon_n.
\]

Using condition (ii) we get \( L \leq 1 \) whenever \( n \) tends to infinity. Now, since the operator \( \frac{A}{\text{ber}(A)} \) also satisfies conditions (i) and (ii), and \( \text{ber} \left( \frac{A}{\text{ber}(A)} \right) = 1 \) we obtain \( \text{ber} \left( \left( \frac{A}{\text{ber}(A)} \right)^n \right) \leq 1 \), which implies \( \text{ber}(A^n) \leq \text{ber}^n(A) \), as required and this completes the proof. \( \square \)

**Remark 2.15.** Every Toeplitz operator on the Hardy space \( \mathcal{H}^2(\mathbb{D}) \) satisfies condition (ii) of Theorem 2.14 (see Englisch [4] and Karaev [10]), and there are many Toeplitz operators satisfying conditions of Theorem 2.14 (see Axler and Zheng [3], Englisch [4] and Karaev et al. [12]).

Note that Berezin symbol has not in general multiplicative property \( \tilde{A}\tilde{B} = \tilde{AB} \) (for more information, see Kilič [13]). Our next result proves the inequality \( \text{ber}(AB) \leq \text{ber}(A)\text{ber}(B) \) for some operators.

**Proposition 2.16.** Let \( A, B \in \mathbb{B}(\mathcal{H}(\Omega)) \). If \( \lim_{\lambda \to \xi_0}\|(A - \tilde{A}(\lambda))^*\hat{k}_{\lambda}\| = 0 \) for some \( \xi_0 \in \partial \mathbb{D} \), then

\[
\lim_{\lambda \to \xi_0}|\tilde{A}\tilde{B}(\lambda)| \leq \text{ber}(A)\text{ber}(B).
\]

In particular, if \( \lim_{\lambda \to \xi_0}|\tilde{A}\tilde{B}(\lambda)| = \text{ber}(AB) \), then \( \text{ber}(AB) \leq \text{ber}(A)\text{ber}(B) \).
Proof. It follows from \( \widetilde{A}^* = \overline{A} \) that for all \( \lambda \in \mathbb{D} \) we have
\[
\widetilde{AB}(\lambda) = \left| \langle AB\hat{k}_\lambda, \hat{k}_\lambda \rangle \right|
= \left| \langle B\hat{k}_\lambda, A^*\hat{k}_\lambda \rangle \right|
\leq \|B\|\left\| A^*\hat{k}_\lambda - \widetilde{A}^*(\lambda)\hat{k}_\lambda \right\| + \text{ber}(A)\text{ber}(B),
\]
from which by using the hypotheses of the theorem, we have that there exists a point \( \xi_0 \in \partial \mathbb{D} \) such that \( \lim_{\lambda \to \xi_0} \widetilde{AB}(\lambda) \leq \text{ber}(A)\text{ber}(B) \), as desired. The second assertion of the theorem is immediate from the first one. The proposition is proved. \( \square \)

**Proposition 2.17.** If \( A, B \in \mathbb{B}(\mathcal{H}(\Omega)) \) and \( \widetilde{AB}(\lambda) \to 0 \) whenever \( \lambda \to \partial \Omega \), then there exists a point \( \lambda_0 \in \Omega \) such that
\[
\text{ber}(AB) - \text{ber}(A)\text{ber}(B) \leq \sqrt{B^*B(\lambda_0)(\widetilde{AA}^*(\lambda_0) - |\widetilde{A}(\lambda_0)|^2)}.
\]

Proof. By the same argument as in the proof of Proposition 2.16, we have
\[
\widetilde{AB}(\lambda) \leq \text{ber}(A)\text{ber}(B) + \|B\|\left\| A^*\hat{k}_\lambda - \widetilde{A}^*(\lambda)\hat{k}_\lambda \right\|
= \text{ber}(A)\text{ber}(B) + \sqrt{B^*B(\lambda)(\widetilde{AA}^*(\lambda) - |\widetilde{A}(\lambda)|^2)}
\]
for \( \lambda \in \Omega \). Since the set \( \left\{ \langle AB\hat{k}_\lambda, \hat{k}_\lambda \rangle : \lambda \in \Omega \right\} \) is bounded, there exists a sequence \( (\lambda_n) \subset \Omega \) such that \( \text{ber}(AB) = \sup_{\lambda \in \Omega} \left| \widetilde{AB}(\lambda) \right| = \lim_{n \to \infty} \left| \langle AB\hat{k}_{\lambda_n}, \hat{k}_{\lambda_n} \rangle \right| \). On the other hand, by the hypotheses \( \widetilde{AB}(\lambda) \to 0 \) whenever \( \lambda \to \partial \Omega \), and hence the sequence \( (\lambda_n) \) can not approach to the boundary \( \partial \Omega \). This shows there exists \( \lambda_0 \in \Omega \) such that \( \lim_{n \to \infty} \lambda_n = \lambda_0 \). Then we obtain from the last inequality that
\[
\text{ber}(AB) - \text{ber}(A)\text{ber}(B) \leq \sqrt{B^*B(\lambda_0)(\widetilde{AA}^*(\lambda_0) - |\widetilde{A}(\lambda_0)|^2)}.
\]
\( \square \)

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