Neutrino emissivity of anisotropic neutron superfluids

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Abstract

We examine the influence of the anisotropy of the superfluid energy gap and residual Fermi-liquid interactions in the triplet-correlated neutron liquid onto neutrino energy losses through neutral weak currents. The neutrino-pair emission caused by the pair breaking and formation processes and by the spin-wave decays is considered for the case of the $^3P_2$ pairing in the state with $m_j = 0$. The simple analytical formulae are obtained. A comparison with the previous results of the average-angle approach shows that the gap anisotropy leads to quenching of the neutrino emissivity caused by the pair recombination processes on about 15% and to substantial suppression of the spin-wave decays. Residual particle-hole interactions increase the energy losses in both the channels on about 5%.
I. INTRODUCTION

Neutrino emission from a superfluid neutron liquid is currently thought to be the dominant cooling mechanism of the baryon matter, for some ranges of the temperature and/or matter density. One of the mechanisms leading the neutron star cooling consists on the recombination of thermally excited baryon BCS pairs into the condensate with emission of neutrino pairs via neutral weak currents [1]. It is generally accepted that, for temperatures near the associated superfluid critical temperatures $T_c$, emission from pair breaking and formation (PBF) processes dominates the neutrino emissivities in many cases. This idea has become the basis of the minimal cooling paradigm for superfluid neutron stars [2, 3].

According to modern investigations [4–8], the neutrino radiation caused by the pairing of baryons into singlet state is strongly suppressed, and the dominant neutrino emission, at the long-cooling epoch, occurs from the triplet-correlated condensate of superfluid neutrons which, as expected, exists in the superdense core of the star. Neutrino energy losses owing to the triplet PBF processes have been initially derived in Ref. [9], ignoring the anomalous weak interactions. The self-consistent approach to this problem was developed in Refs. [10–12], where the anomalous effective vertices are found as the solution of the Dyson's equations in the ladder approximation. Poles of the found analytical solutions [13, 14] indicate the existence of the undamped collective oscillations of the order parameter in the superfluid $^3P_2$ condensate of neutrons with $m_j = 0$. Some of the collective excitations with the excitation energy smaller than the superfluid energy gap are able to decay into neutrino pairs. This is, for example, the "so-called" normal-flapping mode which represents the nonunitary excitation with a nonzero average spin expectation value and can be termed by spin wave. The neutrino decay of such spin waves could be important for thermally-emitting neutron stars, presumably cooling through the combination of neutrino emission from the interior and photon cooling from the surface, the latter is responsible for their observed thermal emissions [15].

Exact solutions of the vertex equations are complicated by the anisotropy of the triplet order parameter. Therefore the preliminary analysis of the collective oscillations and the neutrino energy losses from the $^3P_2$ superfluid neutron liquid has been performed in the average-angle approximation replacing the anisotropic energy gap in the quasiparticle energy by its value averaged over angles [13, 16, 17]. The spin-wave energy, as found in this
approach, is proportional to the temperature-dependent energy gap amplitude $\Delta(T)$.

However the more rigorous analysis with taking into account of the gap anisotropy has shown that the average-angle approximation is not justified for examination of the collective oscillations of the superfluid condensate. It was found [14] that the gap anisotropy leads to a strong decreasing of the level energy along with a lowering of the temperature. One can expect that this is to suppress substantially the neutrino energy losses caused by the spin wave decays because the rate of neutrino losses is strongly dependent on the wave energy.

In this paper we examine the effect of the anisotropy of the superfluid energy gap and the residual particle-hole interactions onto the neutrino energy losses from the $^3P_2$ superfluid neutron liquid with $m_j = 0$. We consider the neutrino emission caused by the PBF processes and by the spin-wave decays (SWD).

The paper is organized as follows. The next section contains some preliminary notes. We discuss the effective strong interactions in the superdense neutron matter, the order parameter for the triplet pair-correlated system with strong interactions and introduce the notation used below. In Sec. III, we present the general expression for neutrino emissivity of the medium through neutral weak currents in terms of the imaginary part of the current-current correlator. In Sec. IV, we examine the anomalous vertices with taking into account of both the Fermi-liquid effects and the anisotropy of the superfluid energy gap for the case of $^3P_2$ pairing with $m_j = 0$, as is adopted in the minimal cooling paradigm [3]. We also discuss the self-consistent response of the superfluid neutron liquid onto external neutrino field. Finally, in Sec. V, we evaluate the self-consistent neutrino energy losses from the PBF and SWD processes and compare them with the neutrino losses caused by modified urca processes and nn-bremsstrahlung. Section VI contains a short summary of our findings and the conclusion. A short Appendix contains some transformation of the previous results.

In this work we use the standard model of weak interactions, the system of units $\hbar = c = 1$, and the Boltzmann constant $k_B = 1$.

II. PRELIMINARY NOTES AND NOTATION

The order parameter, $\hat{D} = D_{\alpha\beta}$, arising owing to triplet pairing of quasiparticles in a degenerate Fermi system, represents a $2 \times 2$ symmetric matrix in spin space, $(\alpha, \beta = \uparrow, \downarrow)$. 
Near the Fermi surface this matrix can be written as (see e.g. [18])

$$\hat{D}(n) = \Delta \bar{b}(n) \cdot \hat{\sigma} \hat{g},$$  \hspace{1cm} (1)

where $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ are Pauli spin matrices, and $\hat{g} = i\hat{\sigma}_2$.

The spin-angle structure of the triplet condensate is defined by a vector $\bar{b}(n)$ in spin space which depends on the direction of the quasiparticle momentum $p$. The angular dependence of the order parameter is represented by the unit vector $n = p/p$ which defines the polar angles $(\theta, \varphi)$ on the Fermi surface,

$$n_1 = \sin \theta \cos \varphi, \quad n_2 = \sin \theta \sin \varphi, \quad n_3 = \cos \theta.$$  \hspace{1cm} (2)

We assume that the (temperature-dependent) gap amplitude $\Delta$ is a complex constant (on the Fermi surface), and $\bar{b}(n)$ is a real vector which we normalize by the condition

$$\langle \bar{b}^2(n) \rangle = 1.$$  \hspace{1cm} (3)

Hereafter we use the angle brackets to denote angle averages,

$$\langle ... \rangle \equiv \frac{1}{4\pi} \int d\mathbf{n} \cdots = \frac{1}{2} \int_{-1}^{1} dn_3 \int_{0}^{2\pi} d\varphi \cdots.$$  \hspace{1cm} (4)

The triplet pairing leads to the energy gap, $\Delta \bar{b}(n)$, in the quasiparticle spectrum which is in general anisotropic. We are mostly interested in the values of quasiparticle momenta $p$ near the Fermi surface $p \simeq p_F$, where the quasiparticle energy is given by

$$E_p = \sqrt{\varepsilon_p^2 + \Delta^2 \bar{b}^2(n)},$$  \hspace{1cm} (5)

with

$$\varepsilon_p \simeq V_F(p - p_f),$$  \hspace{1cm} (6)

and $V_F \ll 1$ is the Fermi velocity of the nonrelativistic neutrons. Here the fact is used that, in the absence of external fields, the gap amplitude $\Delta$ is real.

The spin-orbit interaction among quasiparticles is known to dominate in the nucleon matter of a high density [19, 20] with the most attractive channel of interactions in the $^3P_2$ state with $s = 1, j = 2, l = 1$. In this case the order parameter in the superfluid system can be constructed with the aid of the set of mutually orthogonal complex vectors $b_{m_j}(n)$
which generate standard spin-angle functions of the total angular momentum \( j = 2 \) and \( m_j = 0, \pm 1, \pm 2 \), so that

\[
b_{m_j}(n) \sigma \hat{g} \equiv \sum_{m_s + m_l = m_j} \left( \frac{1}{2} \right)^{1/2} \alpha \beta |1m_s \rangle (11m_s m_l |2m_j \rangle Y_{1,m_l}(n),
\]

and are normalized by the condition

\[
\langle b^*_{m'_j} b_{m_j} \rangle = \delta_{m_j m'_j}.
\]

These can be found in the form (See details in Ref. [14]):

\[
b_0 = \sqrt{1/2} (-n_1, -n_2, 2n_3), \quad b_1 = -\sqrt{3/4} (n_3, in_3, n_1 + in_2),
\]

\[
b_2 = \sqrt{3/4} (n_1 + in_2, in_1 - n_2, 0), \quad b_{-m_j} = (-)^{m_j} b^*_{m_j}.
\]

In our approach it is necessary to distinguish the interactions in the channel of two quasiparticles from the interactions in the particle-hole channel. Since we are interested in values of quasiparticle momenta near the Fermi surface, \( p \simeq p_F n \), the momentum transferred in the collision of two quasiparticles is of the order of \( 2p_F \). In this case the non-central spin-orbit and tensor interactions are most important in the superdense neutron matter. The attractive spin-orbit interaction which dominates in the channel of two quasiparticles is the basic reason of the neutron pairing [20]. The most attractive channel corresponds to spin, orbital, and total angular momenta \( s = 1, l = 1, \) and \( j = 2 \), respectively, and pairs quasiparticles into the \( ^3P_2 \) states with \( m_j = 0, \pm 1, \pm 2 \). The substantially smaller tensor interactions lift the strong paramagnetic degeneracy inherent in pure \( ^3P_2 \) pairing and mix states of different magnetic quantum numbers. It is well known that the tensor interactions generate also a small \( ^3F_2 \) admixture to the ground state and markedly modify the \( ^3P_2 \) energy gap [21–24].

The purpose of our study is however not the ground state but the linear response of the superfluid system onto external neutral weak currents. In this work we assume that the ground state of the superfluid neutron system and the magnitude of the energy gap at the Fermi surface are the known external parameters. Neutrino emissivity of the neutron system with a mixed \( ^3P_2-^3F_2 \) superfluid condensate was examined in Ref. [12]. According to this work, incorporating of the small admixture of the \( ^3F_2 \) state requires of sophisticated calculations but does not affect noticeably the excitation spectra and the neutrino emissivity
through neutral weak currents. Accordingly, throughout this paper, we neglect tensor forces. The pairing interaction, in the most attractive channel, can then be written as

\[ \phi \Gamma_{\alpha \beta, \gamma \delta} (p, p') = V(p, p') \sum_{m_j} (b_{m_j}(n) \hat{\sigma} \hat{g})_{\alpha \beta} \left( \hat{g} \sigma b_{m_j}^* (n') \right)_{\gamma \delta}, \]  

where \( \rho = p_F M^*/\pi^2 \) is the density of states near the Fermi surface; and \( V(p, p') \) is the interaction amplitude.

Consider now the interactions in the particle-hole channel. In our analysis, we shall use the fact that the Fermi-liquid interactions do not interfere with the pairing phenomenon if approximate hole-particle symmetry is maintained in the system, i.e., the Fermi-liquid interactions remain unchanged upon pairing. Near the Fermi surface, the Fermi-liquid effects are reduced to the standard particle-hole interactions:

\[ \phi \delta_{\alpha \gamma, \beta \delta} (n n') = f(n n') \delta_{\alpha \beta} \delta_{\gamma \delta} + g(n n') \sigma_{\alpha \beta} \sigma_{\gamma \delta}. \]  

For generality, Eq. (11) should be supplemented with contributions from spin-orbit and tensor interactions. However, in uniform media, the momentum of particle-hole type excitations equals the transferred momentum \( q \). We are interested in the medium response at the time-like momentum transfer, \( q \leq \omega \sim \Delta \). In this case the contribution from the noncentral interactions, which depend on the transferred momentum \( q \), would be proportional to some power of \( q/p_F \ll 1 \) and vanish in the limit, \( q \to 0 \), which we are interested in (see below).

III. NEUTRINO ENERGY LOSSES

The emission of neutrino pairs is kinematically possible thanks to the existence of a superfluid energy gap, which admits the quasiparticle transitions with time-like momentum transfer \( k = (\omega, q) \), as required by the final neutrino pair: \( k = k_1 + k_2 \). We consider the standard model of weak interactions through neutral weak currents. After integration over the phase space of escaping neutrinos and antineutrinos the total energy which is emitted into neutrino pairs per unit volume and time is given by the following formula (see details, e.g., in Ref. [26]):

\[ \epsilon = -\frac{G_F^2 N}{192 \pi^5} \int_0^\infty d\omega \int d^3 q \frac{\omega \Theta(\omega - q)}{\exp \left( \frac{\omega}{T} \right) - 1} \text{Im} \Pi^\mu\nu_{\text{weak}} (\omega, q) (k_\mu k_\nu - k^2 g_{\mu\nu}), \]

(12)
where \( \Theta(x) \) is the Heaviside step-function; \( \mu, \nu = 0, 1, 2, 3 \) are Dirac indices; \( \mathcal{N} = 3 \) is the number of neutrino flavors; \( G_F \) is the Fermi coupling constant, and \( \Pi^{\mu\nu}_{\text{weak}} \) is the retarded weak polarization tensor of the medium.

In general, the weak polarization tensor of the medium is a sum of the vector-vector, axial-axial, and mixed terms. The mixed vector-axial polarization has to be an antisymmetric tensor, and its contraction in Eq. (12) with the symmetric tensor \( k_\mu k_\nu - k^2 g_{\mu\nu} \) vanishes. Thus only the pure-vector and pure-axial polarizations should be taken into account. We then obtain \( \Pi^{\mu\nu}_{\text{weak}} = C_V^2 \Pi^{\mu\nu}_V + C_A^2 \Pi^{\mu\nu}_A \), where \( C_V \) and \( C_A \) are vector and axial-vector weak coupling constants of a neutron, respectively.

The Fermi velocity is small in the nonrelativistic system, \( V_F \ll 1 \), and we can study the neutrino energy losses in the lowest order over this small parameter. We are interested in the time-like domain of the transferred energy and momentum, \( q < \omega \), and \( \omega \gtrsim \Delta \), in accordance with the total energy and momentum of escaping neutrino pairs. Since the transferred space momentum comes in the polarization functions of the medium in a combination \( q V_F \ll \omega, \Delta \), one can evaluate the polarization functions in the limit \( q = 0 \). Conservation of the vector current requires \( \Pi^{\mu\nu}_V(\omega > 0, q = 0) = 0 \). This relation reflects the fact that the neutrino-pair emission through the vector channel of neutral weak currents is strongly suppressed in nonrelativistic systems \[4\]. Therefore we focus on the axial channel of the weak interactions, assuming \( \Pi^{\mu\nu}_{\text{weak}}(\omega, q = 0) \simeq C_A^2 \Pi^{\mu\nu}_A(\omega, q = 0) \). Further simplification is possible due to the fact that, in the lowest (zero) order over the particle velocity, only the space component of the axial-vector vertex survives. This allows to write

\[
\Pi^{\mu\nu}_{\text{weak}}(\omega, q = 0) \simeq C_A^2 \delta^{\mu i} \delta^{\nu j} \Pi^{ij}_A(\omega, q = 0)
\]

with \( i, j = 1, 2, 3 \).

The field interaction with a superfluid should be described with the aid of four effective three-point vertices. There are two ordinary effective vertices corresponding to creation of a particle and a hole by the field that differ by direction of fermion lines and – two anomalous vertices that correspond to creation of two particles or two holes. Accordingly, in graphical representation, the polarization tensor represents a superposition of loops incorporating the ordinary and anomalous vertices connected by the ordinary and anomalous Green functions, as depicted in Fig. 1. We use the adopted graphical notation for the ordinary and anomalous propagators, \( \hat{G} = \quad \quad \rightarrow \quad \rightarrow \), \( \hat{G}^{-}(p) = \quad \rightarrow \rightarrow \), \( \hat{F}^{(1)} = \quad \rightarrow \rightarrow \), and \( \hat{F}^{(2)} = \quad \rightarrow \rightarrow \). The analytic
FIG. 1. Graphs for the polarization tensor. Discarding the residual particle-hole interactions we show the ordinary vertices points.

Form of the quasiparticle propagators in the momentum representation can be written as

\[ \hat{G}(p_\kappa, p) = G(p_\kappa, p) \hat{1}, \quad \hat{F}^{(1)}(p_\kappa, p) = F(p_\kappa, p) \bar{b}\hat{\sigma}\hat{g}, \]

(14)

\[ \hat{F}^{(2)} = \hat{F}^\dagger(-p_\kappa, -p) = \hat{g}\hat{\sigma}\bar{b}F(-p_\kappa, -p). \]

(15)

We define the scalar Green functions

\[ G(p_\kappa, p) = \frac{-ip_\kappa - \varepsilon_p}{p_\kappa^2 + E_p^2}, \quad F(p_\kappa, p) = F(-p_\kappa, -p) = \frac{\Delta}{p_\kappa^2 + E_p^2}. \]

(16)

where \( p \) is the quasiparticle momentum, and \( p_\kappa = (2\kappa + 1)\pi T \) with \( \kappa = 0, \pm 1, \pm 2, \ldots \) is the fermionic Matsubara frequency which depends on the temperature \( T \). The quasiparticle energy is given by

\[ E_p = \sqrt{\varepsilon_p^2 + \Delta^2\bar{b}^2(n)} \]

(17)

with

\[ \varepsilon_p \simeq v_F(p - p_F). \]

(18)

IV. EFFECTIVE VERTICES AND POLARIZATION FUNCTIONS

The anomalous effective vertices, which we denote as \( \hat{T}^{(1)}(n, \omega) \) and \( \hat{T}^{(2)}(n, \omega) \), are given by infinite sums of the diagrams, taking into account the pairing interaction in the ladder approximation [27]. The ordinary effective vertices, \( \hat{\tau}(n, \omega) \), \( \hat{\tau}^-(n, \omega) = \hat{\tau}^T(-n, \omega) \), incorporating the particle-hole interactions can be evaluated in the random-phase approximation [28]. This can be expressed by the set of Dyson equations symbolically depicted by graphs in Fig. 2 where the particle-hole interactions (11) are shown by the shaded rectangle. Wavy lines represent the pairing interaction (10). The first diagram on the right-hand side of the first line is the three-point vertex of a free particle. In the nonrelativistic case, the bare axial-vector vertex is given by the spin matrices \( \hat{\sigma} \).
FIG. 2. Dyson equations for full ordinary and anomalous vertices. The particle-hole interaction is shown by the shaded rectangle. Wavy lines represent the pairing interaction.

The analytic form of these equations, which are to be solved simultaneously with the gap equations, was repeatedly considered before. The equations can be substantially simplified due to the fact that we are interested in the processes near the Fermi surface. After a series of algebraic manipulations (see details in Ref. [11]) the solution valid for \( q = 0 \) can be obtained in the form:

\[
\hat{T}^{(1)} = \sum_{m_j} B_{m_j} (\omega_\eta) \left( \hat{\sigma} b_{m_j} \right) \hat{g},
\]

\[
\hat{T}^{(2)} = \sum_{m_j} B_{m_j} (\omega_\eta) \hat{g} \left( \hat{\sigma} b_{m_j} \right),
\]

and

\[
\hat{\tau} = \phi (n, \omega_\eta) \hat{\sigma},
\]

where \( \omega_\eta = 2\pi \eta \) with \( \eta = 0, \pm 1, \pm 2, \ldots \) is the bosonic Matsubara frequency, and \( \phi (n, \omega_\eta) = \phi (-n, \omega_\eta) \) and \( B_{m_j} (\omega_\eta) \) are to satisfy the equations (we omit the \( \omega_\eta \) dependence of the functions):

\[
B_{m_j} = -\frac{i \omega_\eta}{\Delta} \frac{1}{\chi_{m_j}} \left\langle i \left( b_{m_j}^* \times \bar{b} \right) \mathcal{I}_0 \phi \right\rangle,
\]

\[
\phi = 1 + \frac{1}{3} \int \frac{dn'}{4\pi} g(nn') \left[ \sum_{m_j} \frac{i \omega_\eta \mathcal{I}_0 i \left( b_{m_j}^* \times \bar{b} \right) \cdot B_{m_j} - 4\bar{b}^2 \mathcal{I}_0 \phi}{\Delta} \right]_{n'}
\]

with

\[
\chi_{m_j} \equiv \left\langle \left( b_{m_j}^* b_m - \bar{b}^2 \right) A \right\rangle - \frac{\omega^2}{2\Delta^2} \left\langle \left( b_{m_j}^* b_m \right) \mathcal{I}_0 \right\rangle - 2 \left\langle \left( b_{m_j}^* \bar{b} \right) \left( \bar{b} b_{m_j} \right) \mathcal{I}_0 \right\rangle.
\]
The functions $A(n)$ and $I_0(\omega_\eta; n)$ are given by
\[ A(n) \equiv \int_{-\infty}^{\infty} d\varepsilon \left( \frac{1}{2E} \tanh \frac{E}{2T} - \frac{1}{2\varepsilon} \tanh \frac{\varepsilon}{2T} \right), \] (25)
and
\[ I_0(\omega_\eta; n) = \int_{-\infty}^{\infty} d\varepsilon \frac{\Delta^2}{E 4E^2 + \omega_\eta^2} \tanh \frac{E}{2T}. \] (26)
with
\[ E = \sqrt{\varepsilon^2 + \Delta^2 \bar{b}^2(n)}. \] (27)

To solve the set of Eqs. (22), (23) we expand $\phi(n)$ in spherical harmonics
\[ \phi(n,\omega_n) = \sum_{lm} \phi_{l,m}(\omega_n) Y_{l,m}(\theta, \varphi), \] (28)
where $\theta$ and $\varphi$ are, respectively, the polar and azimuthal angles of the vector $n$, and write
\[ g(nn') = \sum_{l=0}^{\infty} \frac{4\pi g_l}{2l + 1} \sum_{\mu} Y_{l,\mu}(\theta, \varphi) Y_{l',\mu}^*(\theta', \varphi'), \] (29)

Inserting (28) and (29) into Eqs. (22) and (23) we obtain
\[ B_{mj} = -i\omega_\eta \frac{1}{\Delta} \chi_{mj} \sum_{lm} \phi_{l,m} \langle I_0^i \left( \vec{b}_{mj} \times \bar{b} \right) Y_{l,m} \rangle \] (30)
and
\[ \phi_{l,m} = \sqrt{4\pi} \delta_{l,0} \delta_{m,0} + \frac{g_l}{2l + 1} \sum_{l',m'} \left( \frac{\omega_\eta^2}{4\Delta^2} \lambda_{l,m;l',m'} - \eta_{l,m;l',m'} \right) \phi_{l',m'}, \] (31)
where we defined dimensionless quantities
\[ \lambda_{l,m;l',m'} \equiv 4\pi \sum_{m_j} \frac{1}{\chi_{mj}} \langle Y_{l,m}^* I_0^i (\vec{b}_{mj} \times \bar{b}) \rangle \cdot \langle Y_{l',m'}^* I_0^i (\vec{b}_{mj}^* \times \bar{b}) \rangle, \] (32)
and
\[ \eta_{l,m;l',m'} \equiv 4\pi \langle Y_{l,m}^* \bar{b}^2 I_0 Y_{l',m'} \rangle. \] (33)

We will focus on the condensation into the state $^3P_2$ with $m_j = 0$ which is conventionally considered as the preferable one in the bulk matter of neutron stars. In this case
\[ \vec{b}(n) = \vec{b}_0(n), \quad \bar{b}^2 = \frac{1}{2} \left( 1 + 3n_3^2 \right) \] (34)
Since the function $I_0(\omega_n, \bar{b}^2)$ is axial symmetric the integration over the azimuthal angle can be done in Eqs. (32) and (33). By performing subsequent summation over $m_j = 0, \pm 1, \pm 2$
we find that only $m_j = \pm 1$ contribute, and $\chi_{-1} = \chi_1 \equiv \chi$. In this way we obtain (the $\omega_\eta$ dependence is omitted):

$$
\lambda_{l,m;\nu,m'} = -\delta_{m,0} \delta_{m',0} \frac{3 \beta_l \beta_{\nu}}{2} \chi,
$$

with

$$
\beta_l (\omega_\eta) \equiv \langle \bar{b}^2 (n_3^2) I_0 (\omega_\eta, n_3^2) P_l (n_3) \rangle ,
$$

where $P_l (x)$ are the Legendre polynomials, and

$$
\eta_{l,m;\nu,m'} = \delta_{m,m'} \eta_{l,m;\nu,m}.
$$

In these notation we obtain the Eq. (31) in the form

$$
\phi_{l,m} = \sqrt{4\pi} \delta_{l,0} \delta_{m,0} - \frac{g_l}{2l+1/3} \sum_{\nu} \left( \frac{3}{2} \frac{\omega_\eta^2}{4 \Delta^2} \frac{2 \beta_l \beta_{\nu}}{\chi} + \eta_{l,m;\nu,m} \right) \phi_{\nu,m},
$$

For $m \neq 0$, this equation has only trivial solution $\phi_{l,m \neq 0} = 0$. For $\phi_l \equiv \phi_{l,0}$ Eq. (38) takes the form

$$
\phi_l = \sqrt{4\pi} \delta_{l,0} - \frac{g_l}{2l+1/3} \sum_{\nu} \left( \frac{3}{2} \frac{\omega_\eta^2}{4 \Delta^2} \frac{2 \beta_l \beta_{\nu}}{\chi} + \eta_{l,\nu} \right) \phi_{\nu},
$$

with $\beta_l$, as given in Eq. (36) and

$$
\gamma_{l,\nu} (\omega_\eta) \equiv \langle P_l (n_3) \bar{b}^2 (n_3^2) I_0 (\omega_\eta, n_3^2) P_{\nu} (n_3) \rangle .
$$

Since the $\phi (n, \omega) = \phi (-n, \omega)$ only even values of $l$ contribute into the expansion (28) which takes the form

$$
\phi (n, \omega_\eta) = \frac{1}{\sqrt{4\pi}} \sum_{l=\text{even}} \phi_l (\omega_\eta) P_l (n_3).
$$

The Fermi-liquid parameters are not well known for a wide range of neutron densities we consider. Some known data allow to hope, however, that, in Eq. (39), the parameters $g_l / (2l + 1)$ decrease rapidly for $l \geq 2$. For example, according to Ref. [24], at the Fermi momentum $p_F = 1.7$ fm$^{-1}$ one has $g_0 = 0.842$ while $g_2 / 2 = 0.0438$. Therefore we take the approximation $g_l = 0$ for $l \geq 2$ thus obtaining

$$
\phi = \frac{\chi}{\left( 1 + \frac{4}{3} g_0 \langle \bar{b}^2 I_0 \rangle \chi + 2 g_0 \Omega_\eta^2 \langle \bar{b}^2 I_\nu \rangle \chi \right)}.
$$

Inserting this expression into Eq. (22) gives $B_0 = B_{\pm 2} = 0$,

$$
B_{\pm 1} (\omega_\eta) = -\frac{i \omega_\eta}{2 \Delta} \sqrt{\frac{3}{2} \left( 1 + \frac{4}{3} g_0 \langle \bar{b}^2 I_0 \rangle \chi + 2 g_0 \Omega_\eta^2 \langle \bar{b}^2 I_\nu \rangle \chi \right)} \left( \begin{array}{c} 1 \\ \mp i \end{array} \right),
$$
\( \chi \equiv \left\langle 2 \left[ -\Omega_\eta^2 b_1^* b_1 - (b_1^* b_1^* b_1^* b_1) \right] I_0 + (b_1^* b_1 - \bar{b}^2) A \right\rangle. \) \hfill (44)

In obtaining Eq. (43) we used the identities \( \beta_0 = \gamma_0, \equiv \left\langle \bar{b}^2 I_0 \right\rangle. \) After the replacement \( i\omega_n \rightarrow \omega + i0 \) we obtain the analytic continuation to the retarded vertex for particles. The replacement \( i\omega_n \rightarrow \omega - i0 \) gives the advanced vertex for holes.

From Eqs. (19), (20) and (21) we find

\[
\hat{T}^{(1)}(n, \Omega) = -f(\Omega, y) [e^* (\hat{\sigma} b_1) \hat{g} + e (\hat{\sigma} b_{-1}) \hat{g}], \tag{45}
\]

\[
\hat{T}^{(2)}(n, \Omega) = -f^*(\Omega, y) [e^* \hat{g} (\hat{\sigma} b_1) + e \hat{g} (\hat{\sigma} b_{-1})] \tag{46}
\]

and

\[
\hat{\tau} = \phi(\Omega, y) \hat{\sigma}, \tag{47}
\]

\[
\hat{\tau}^- = \phi^*(\Omega, y) \hat{\sigma}^T \tag{48}
\]

with \( e = (1, i, 0), \)

\[
\Omega = \frac{\omega}{2\Delta(T)}, \quad y = \frac{\Delta(T)}{T}. \tag{49}
\]

The functions \( f(\Omega, y) \) and \( \phi(\Omega, y) \) are given by

\[
f(\Omega, y) \equiv \sqrt{\frac{3}{2}} \frac{\Omega \left\langle \bar{b}^2 I_0 \right\rangle}{\left( 1 + \frac{4}{3} g_0 \left\langle \bar{b}^2 I_0 \right\rangle \right) \chi - 2g_0 \Omega^2 \left\langle \bar{b}^2 I_0 \right\rangle^2} \tag{50}
\]

and

\[
\phi(\Omega, y) \equiv \frac{\chi}{(1 + \frac{4}{3} g_0 \left\langle \bar{b}^2 I_0 \right\rangle) \chi - 2g_0 \Omega^2 \left\langle \bar{b}^2 I_0 \right\rangle^2}, \tag{51}
\]

where

\[
\chi(\Omega, y) = \frac{1}{4} \left[ \int_0^1 dn_3 \left[ 6\Omega^2 (1 + n_3^2) - 3n_3^2 (1 - n_3^2) \right] I_0 (n_3^2; \Omega, y) \right.
\]

\[
+ \int_0^1 dn_3 (1 - 3n_3^2) A (n_3^2, y) \right], \tag{52}
\]

and

\[
\left\langle \bar{b}^2 I_0 \right\rangle = \frac{1}{2} \int_0^1 dn_3 (1 + 3n_3^2) I_0. \tag{53}
\]

In obtaining Eq. (46) we used the identity \( I_0 (\omega - i0) = I_0' (\omega + i0). \)

Making use of the effective vertices (45)-(48) and the propagators (14)-(16) one can calculate the axial polarization tensor by the diagrams of Fig. 1 (See details in Refs. [10, 13]).
For $q = 0$ we obtain:

$$\Pi_{ij}^\Lambda (\Omega, y) = -4\rho \left[ \text{Re} \phi (\Omega, y) \left\langle \left( \delta_{ij} - \frac{\bar{b}_i \bar{b}_j}{\bar{b}^2} \right) \bar{b}^2 I_0 \right\rangle 
- (\delta_{ij} - \delta_{i3} \delta_{j3}) \sqrt{\frac{3}{2}} \Omega \text{Re} f (\Omega) \left\langle \bar{b}^2 I_0 \right\rangle \right].$$

(54)

with $\bar{b} (n) = \sqrt{1/2} (-n_1, -n_2, 2n_3)$ and $\bar{b}^2 = \frac{1}{2} (1 + 3n_3^2)$, as given by Eq. (34).

The imaginary part of $\Pi_{ij}^\Lambda$ that arises owing to the PBF processes originates from the function $I_0 (n_3; \Omega, y)$ at $\Omega > \bar{b}_{\text{min}} = 1/\sqrt{2}$. A one more contribution into the imaginary part of the axial polarization tensor (54) arises from the pole of the function $f (\Omega, y)$ at $\Omega = \Omega_s < \bar{b}_{\text{min}}$. This contribution describes the neutrino energy losses caused by the spin wave decays. The PBF and SWD processes operate in different kinematical domains, so that the imaginary part of the polarization tensor consists of two clearly distinguishable contributions, $\text{Im} \Pi_{ij}^\Lambda = \text{Im} \Pi_{ij}^{\text{PBF}} + \text{Im} \Pi_{ij}^{\text{SWD}}$, which we now consider.

V. NEUTRINO LOSSES

A. PBF channel

First we examine the PBF processes occurring at $\omega > 2\Delta_n \equiv 2 \bar{b} (n) \Delta$ or, equivalently, at $\Omega > \bar{b}_{\text{min}}$. From Eq. (54) we obtain

$$\text{Im} \Pi_{ij}^\Lambda \left( \Omega > \frac{1}{\sqrt{2}} \right) = -4\rho \left[ \text{Re} \phi \left\langle \left( \delta_{ij} - \frac{\bar{b}_i \bar{b}_j}{\bar{b}^2} \right) \bar{b}^2 \text{Im} I_0 \right\rangle 
- 4\rho (\delta_{ij} - \delta_{i3} \delta_{j3}) \sqrt{\frac{3}{2}} \Omega \text{Re} f \left\langle \bar{b}^2 \text{Im} I_0 \right\rangle \right].$$

(55)

Inserting this expression into Eqs. (13) and (12) we calculate the contraction of $\text{Im} \Pi_{\mu\nu}^{\text{weak}}$ with symmetric tensor $k_{\mu} k_{\nu} - k^2 g_{\mu\nu}$ to obtain

$$\text{Im} \Pi_{\mu\nu}^{\text{weak}} (k_{\mu} k_{\nu} - k^2 g_{\mu\nu}) \simeq -4\rho \left( \text{Re} \phi - \sqrt{\frac{3}{2}} \Omega \text{Re} f \right) \times \left\langle (2\omega^2 - 2q^2 - q_{\perp}^2) \bar{b}^2 \text{Im} I_0 \right\rangle,$$

(56)

where we use the local frame with $Oz \parallel \bar{b}$, and $q_\parallel$ and $q_\perp$ are defined as

$$q_\parallel = \frac{1}{\bar{b}^2} (q \bar{b})^2, \quad q_\perp = q^2 - q_\parallel^2.$$

(57)
The functions $I_0$, $\phi$ and $f$ are independent of the space momentum of the neutrino pair. Therefore the integral over $d^3q$ in Eq. (12) can be performed, and we obtain the neutrino emissivity in the form:

$$\epsilon = \frac{32}{15\pi^6} C_A^2 G_F^2 p_F M^* N T_c^7 \int_0^\infty d\Omega \frac{y^7 \Omega^6}{\exp(2y\Omega) - 1} \left( \text{Re} \phi - \sqrt{\frac{3}{2} \Omega^2} \text{Re} f \right) \langle \bar{b}^2 \text{Im} I_0 \rangle. \quad (58)$$

Imaginary part of the analytic continuation of the function (26) is given by

$$\text{Im} I_0 (\omega + i0) = -\text{sign} (\Omega) \frac{\pi \Theta (\Omega^2 - \bar{b}^2)}{4\Omega \sqrt{\Omega^2 - \bar{b}^2}} \tanh \left( \frac{y\Omega}{2} \right). \quad (59)$$

With $\bar{b}^2$ given in Eq. (31) one finds for $\Omega^2 > 1/2$:

$$\langle \bar{b}^2 \text{Im} I_0 \rangle = \frac{\pi^2 \sqrt{6}}{96} \frac{1}{\Omega} (1 + 2\Omega^2) \tanh \left( \frac{y\Omega}{2} \right) \Theta \left( \Omega^2 - \frac{1}{2} \right) \Theta (2 - \Omega^2)$$

$$+ \frac{\pi \sqrt{6}}{48} \frac{1}{\Omega} \left[ (2\Omega^2 + 1) \arcsin \frac{3}{2\Omega^2 - 1} - \sqrt{6}\Omega^2 - 12 \right]$$

$$\times \tanh \left( \frac{y\Omega}{2} \right) \Theta (\Omega^2 - 2) \quad (60)$$

The integration over $d\Omega$ is now parted into two intervals:

$$\epsilon = \epsilon_0 \tau^7 y^7 \left[ 4 \int_{1/\sqrt{2}}^{\sqrt{2}} d\Omega \frac{(1 + 2\Omega^2) \Omega^5}{(e^{y\Omega} + 1)^2} \left( \text{Re} \phi - \sqrt{\frac{3}{2} \Omega^2} \text{Re} f \right) \right.$$  

$$+ \frac{8}{\pi} \int_{\sqrt{2}}^\infty d\Omega \frac{\Omega^5}{(e^{y\Omega} + 1)^2} \left( \text{Re} \phi - \sqrt{\frac{3}{2} \Omega^2} \text{Im} f \right)$$

$$\times \left[ (2\Omega^2 + 1) \arcsin \frac{\sqrt{3}}{2\Omega^2 - 1} - \sqrt{6}\Omega^2 - 12 \right] \right], \quad (61)$$

where we denote $\tau = T/T_c$, and $y = y(\tau)$;

$$\epsilon_0 \equiv \frac{\sqrt{6}}{180\pi^4} C_A^2 G_F^2 p_F M^* N T_c^7$$

$$= 1.88 \times 10^{20} \left( \frac{M^*}{M} \right) \left( \frac{p_F}{M_c} \right) T_{9c} N_c C_A^2 \frac{\text{erg}}{\text{cm}^3 \text{s}}. \quad (62)$$

$M$ and $M^*$ are the effective and bare nucleon masses, respectively; $T_{9c} \equiv T_c/10^9\text{K}$. 

Equation (61) with $f (\Omega, y)$ and $\phi (\Omega, y)$ as given in Eqs. (50) and (51) improves previous results obtained in Ref. [10], where the vertex function was evaluated in the BCS and average-angle approximation yielding

$$f_{av} (\Omega) \simeq \frac{1}{2} \sqrt{\frac{3}{2}} \frac{1}{\Omega} \phi_{BCS} = 1. \quad (63)$$
Therefore before proceeding to the detailed analysis of the neutrino losses caused by the PBF processes, we examine the obtained equations for the above approximations.

Replacing $f(\Omega) \rightarrow f_{av}(\Omega)$ in Eq. (61) leads to the following result

$$
\epsilon_{av}(\tau) = \epsilon_0 \tau^7 y^7 \left[ \int_{1/\sqrt{3}}^{\sqrt{2}} d\Omega \frac{(1 + 2\Omega^2) \Omega^5}{(e^{y\Omega} + 1)^2} 
+ \frac{2}{\pi} \int_{\sqrt{2}}^{\infty} d\Omega \frac{\Omega^5}{(e^{y\Omega} + 1)^2} \left( (2\Omega^2 + 1) \arcsin \frac{\sqrt{3}}{\sqrt{2\Omega^2 - 1}} - \sqrt{6\Omega^2 - 12} \right) \right]. \quad (64)
$$

The neutrino emissivity, as found in Ref. [10] in the average-angle approximation, is written in the form of the two-fold integral over the Fermi surface and over the energy of escaping neutrino pairs. The integration over the Fermi surface in that expression can be performed analytically. In the Appendix, we demonstrate that such the integration results in Eq. (64)
For numerical evaluation of the neutrino losses, with making use of Eqs. (61) and (64) it is necessary to know the function \( y(\tau) \) which in general is to be found with the aid of the gap equation. For the case \( m_j = 0 \) the function is well investigated. We can adjust for example the simple fit to \( y(\tau) = \sqrt{2(1-\tau)}(0.7893 + 1.188/\tau) \), as suggested in Ref. [9].

In Fig. 3 we show the neutrino energy losses computed according to the exact Eqs. (61, 50, 51) in comparison with the energy losses in the average-angle approximation, as given by Eq. (64). The gap anisotropy leads to quenching of the PBF neutrino emissivity on about 15%, while the exchange particle-hole interactions with \( g_0 = 0.842 \) slightly increase the neutrino energy losses. The neutrino emissivity which incorporates both the effects is less than the result obtained in the average-angle approximation on approximately 10%.

**B. SWD channel**

The function \( \mathcal{I}_0(n_3; \Omega, y) \) is real at \( \Omega < \bar{b}_{\text{min}} \). From Eq. (55), (50) and (51) we find in this domain

\[
\Pi^{\mu\nu}_{\text{weak}} \left( k_\mu k_\nu - k^2 g_{\mu\nu} \right) \simeq -4\rho \left( \chi - \frac{3}{2} \frac{\Omega^2}{\bar{b}^2 \mathcal{I}_0} \right) \times \left< \left( 2\omega^2 - 2q_0^2 - q_\perp^2 \right) \bar{b}^2 \mathcal{I}_0 \right> \left( 1 + \frac{4}{3} g_0 \left< \bar{b}^2 \mathcal{I}_0 \right> \right) \chi - 2g_0\Omega^2 \left< \bar{b}^2 \mathcal{I}_0 \right>^2, \tag{65}
\]

This function has a pole owing to existence of eigen oscillations of the condensate at the frequency \( \omega_s \) satisfying the condition

\[
\left( 1 + \frac{4}{3} g_0 \left< \bar{b}^2 \mathcal{I}_0 \right> \right) \chi - 2g_0\Omega_s^2 \left< \bar{b}^2 \mathcal{I}_0 \right>^2 = 0, \tag{66}
\]

where all the functions are to be taken at \( \Omega = \Omega_s \), defined as

\[
\Omega_s = \frac{\omega_s}{2\Delta (T)}. \tag{67}
\]

As discussed in Ref. [14], the corresponding nonunitary oscillations look like the ”normal-flapping” mode in \(^3\text{He}-\text{A} \) [30].

Solution to Eq. (66) can by found by assuming that the oscillation frequency of this wave is small, \( \Omega_s^2 \ll 1 \). This allows to neglect \( \omega^2 \) in the integrand of Eq. (26) and write

\[
\mathcal{I}_0(n_3; \omega_s) \simeq \tilde{I}_0(n_3) \equiv \frac{\Delta^2}{4} \int_{-\infty}^{\infty} \frac{d\varepsilon}{E^3} \tanh \frac{E}{2T}, \tag{68}
\]
After this simplification the analytic solution to Eq. (66) can be written as:

\[
\Omega^2_s = \frac{\int_0^1 dn_3 \left[ 3n_3^2 (1 - n_3^2) \hat{I}_0 - (1 - 3n_3^2) A \right]}{6 \int_0^1 dn_3 (1 + n_3^2) \hat{I}_0 - 8g_0 \langle \hat{b}^2 \hat{I}_0 \rangle \left( 1 + \frac{4}{3}g_0 \langle \hat{b}^2 \hat{I}_0 \rangle \right)^{-1}},
\]

(69)

where the function \( A \) is given by Eq. (25), and

\[
\langle \hat{b}^2 \hat{I}_0 \rangle = \frac{1}{2} \int_0^1 dn_3 (1 + 3n_3^2) \hat{I}_0.
\]

(70)

In Ref. [11] the right-hand side of Eq. (69) was evaluated for \( g_0 = 0 \) in the average-angle approximation assuming that the anisotropic gap in the quasiparticle energy is replaced by its average-angle magnitude, \( \hat{b}^2 \Delta^2 (T) \to \Delta^2 (T) \). Such approach results in \( \Omega^2_s \to 1/20 \) or, equivalently, \( \omega_s^{av} \sim \Delta (T) / \sqrt{5} \) with a simple temperature dependence of the excitation energy only through the gap amplitude. However, the more accurate calculation [14] has shown that, for the eigen modes, the average-angle approximation is valid only in the limit \( T \to T_c \). The gap anisotropy leads to a strong decreasing of the energy of the flapping mode at lowering of the temperature, and \( \omega_s \) tends to zero when \( T \to 0 \), as shown in Fig. 4.

![FIG. 4. The frequency \( \omega_s (\tau) \) of the spin wave at \( q = 0 \) in units of \( 2\Delta (\tau) \) versus reduced temperature \( \tau = T/T_c \). Solid curve is calculated for \( g_0 = 0.842 \). Long-dashed curve corresponds to \( g_0 = 0 \).](image-url)
Near the pole $\Omega \simeq \Omega_s$ we can approximate

$$
\Pi^{\mu
u}_{\text{weak}} (k_\mu k_\nu - k^2 g_{\mu\nu}) 
\simeq \frac{-2\rho \left(\chi - \frac{3}{2} \Omega^2 \langle \vec{b}^2 \mathcal{I}_0 \rangle \right) \left(2\omega^2 - 2q_\parallel^2 - q_\perp^2\right) \langle \vec{b}^2 \mathcal{I}_0 \rangle}{\Omega \left|1 + \frac{4}{3} g_0 \langle \vec{b}^2 \mathcal{I}_0 \rangle\right| \partial \chi / \partial \Omega^2 - 2 g_0 \langle \vec{b}^2 \mathcal{I}_0 \rangle^2}_{\Omega = \Omega_s} (\Omega - \Omega_s + i0). \quad (71)
$$

The displacement of the pole in a complex $\omega$-plane is chosen so that to obtain the retarded polarization. Such a displacement is equivalent to the presence of a delta-function imaginary part

$$
\text{Im} \Pi^{\mu
u}_{\text{weak}} (k_\mu k_\nu - k^2 g_{\mu\nu}) 
\simeq -\frac{2\pi \rho \left(\chi - \frac{3}{2} \Omega^2 \langle \vec{b}^2 \mathcal{I}_0 \rangle \right) \left(2\omega^2 - 2q_\parallel^2 - q_\perp^2\right) \langle \vec{b}^2 \mathcal{I}_0 \rangle}{\Omega_s \left|1 + \frac{4}{3} g_0 \langle \vec{b}^2 \mathcal{I}_0 \rangle\right| \partial \chi / \partial \Omega^2 - 2 g_0 \langle \vec{b}^2 \mathcal{I}_0 \rangle^2}_{\Omega = \Omega_s} \delta (\Omega - \Omega_s). \quad (72)
$$

To evaluate this expression we expand the function $\mathcal{I}_0$ near the pole to obtain

$$
\mathcal{I}_0 \simeq \tilde{I}_0 + (\Omega^2 - \Omega_s^2) \tilde{I}_1 \quad (73)
$$

with $\tilde{I}_0$, as given in Eq. (68), and

$$
\tilde{I}_1 \equiv \frac{\Delta^4}{4} \int_{-\infty}^{\infty} \frac{d\varepsilon}{E^0} \tanh \frac{E}{2T}. \quad (74)
$$

In this way we find

$$
\partial \chi / \partial \Omega^2 \simeq \frac{3}{4} \left(2 \left(1 + n_3^2\right) \tilde{I}_0 - \left(n_3^2 \left(1 - n_3^2\right) \tilde{I}_1\right)\right) \quad (75)
$$

and

$$
\chi (\Omega_s, y) \simeq \frac{1}{4} \left[6 \Omega_s^2 \left(1 + n_3^2\right) \tilde{I}_0 - 3 \left(n_3^2 \left(1 - n_3^2\right) \tilde{I}_0\right) + \left(1 - 3n_3^2\right) A\right].
$$

The remaining calculations, similar to ones performed in the previous section, result in the following neutrino energy losses caused by the decay of spin waves:

$$
\epsilon_{\text{SWD}} = \varepsilon_0 \frac{64}{3\pi} \sqrt{6} \frac{\tau^7 \bar{y} \bar{\Omega}_s^5 \exp (2y\Omega_s) - 1}{\left(1 + \frac{4}{3} g_0 \langle \vec{b}^2 \mathcal{I}_0 \rangle\right) \partial \chi / \partial \Omega^2 - 2 g_0 \langle \vec{b}^2 \mathcal{I}_0 \rangle^2} \quad (76)
$$

In this expression the spin-wave relative frequency $\Omega_s (\tau)$ is defined by Eq. (66). The SWD neutrino emissivity, as obtained earlier in the average-angle approximation, can be obtained from this expression if to replace the relative frequency with a constant $\Omega_s \to 1/20$. 

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It was already mentioned the gap anisotropy leads to a strong decreasing of the wave energy along with lowering of the temperature. This is to suppress substantially the neutrino energy losses caused by the spin wave decays because the rate of neutrino losses is strongly dependent on the wave energy.

FIG. 5. SWD neutrino emissivity from neutron matter versus the temperature $T$ in logarithmic scale. The solid curve is calculated by Eq. (61) with taking into account the anisotropy of the energy gap and the residual particle-hole interactions in the spin-spin channel with the Landau parameter $g_0 = 0.842$. Short-dashed curve is same but with $g_0 = 0$. The long-dashed curve is calculated in the average-angle approximation and with $g_0 = 0$, as given by Eq. (64).

In Fig. 5 we show SWD neutrino emissivity from neutron matter versus the temperature $T$ in logarithmic scale. We show also the neutrino emissivity as calculated in the anisotropic BCS approximation and with taking into account of both the anisotropy of the energy gap and residual particle-hole interactions. The Fermi-liquid effects lead to a minor (about 15%)
increase of the neutrino emissivity in the SWD channel, however this emissivity is small in comparison with the result obtained in the average-angle approximation.

C. Competitive neutrino processes

![Graph of PBF and SWD neutrino emissivities versus temperature](image)

FIG. 6. PBF and SWD neutrino emissivities versus temperature $T$ in comparison with the modified urca and bremsstrahlung emissivities at $k_F = 1.7$.

The neutrino emissivity for the processes discussed above is plotted in Fig. 6 together with the modified urca and bremsstrahlung emissivities with the suppression factors resulting from superfluidity as obtained in Ref. [31].

The emissivity from the PBF dominates everywhere below the critical temperature for the $^3P_2$ superfluidity except the narrow temperature domain near the critical point, where the modified urca processes are more operative. Neutrino losses caused by the bremsstrahlung and SWD are less effective than in the PBF processes.
VI. SUMMARY AND CONCLUSION

We have calculated the neutrino energy losses from the $^3P_2(m_j = 0)$ superfluid neutron liquid with accurate taking into account of the anisotropy of the superfluid energy gap and minimal account of residual Fermi-liquid interactions. In our analysis we have used the integral expressions for anomalous three-point effective vertices derived in Refs. [10, 14]. We have examined the neutrino energy losses through neutral weak currents caused by the pair breaking and formation processes and by the spin-wave decays. The corresponding neutrino emissivity is given in Eqs. (61) and (76).

Earlier the neutrino losses have been calculated discarding the Fermi-liquid effects with using the average-angle approximation, where the anisotropic energy gap in equations for the anomalous vertices was replaced by its average-angle magnitude, $b^2 \Delta^2(T) \to \Delta^2(T)$. A comparison of the results allows to make some inferences concerning validity of the average-angle approach.

As shown above, the gap anisotropy leads to quenching of the PBF neutrino emissivity on about 15%, while the residual particle-hole interactions increase the PBF emissivity on about 5%. This difference is practically indistinguishable in the logarithmic scale in Fig. 6. Up to accuracy about ten percents, one can neglect both the anisotropy of the effective weak vertices and the Fermi-liquid effects in the PBF processes and use the simple expression in the form of one-fold integral (64) for practical estimates.

However, the exact account of the anisotropy dramatically modifies the SWD neutrino losses. This fact has a simple explanation. As found in Ref. [10], the relative spin-wave energy is constant in the average-angle approximation, $\Omega_{av} = 1/\sqrt{5}$. Taking into account of the gap anisotropy leads to a strong temperature dependence of the energy of this collective mode [14]. The wave frequency diminishes and tends to zero at the lowering of the temperature, as shown in Fig. 4. Since the rate of the neutrino losses is strongly dependent on the wave energy the anisotropy leads to a rapid decrease of the SWD energy losses along with lowering of the temperature. A comparison of different competitive processes presented in Fig. 6 allows to conclude that the dominant energy losses from the $^3P_2$ superfluid neutron liquid are generated by the PBF processes except the narrow temperature domain near the critical point, where the modified urca processes are more operative.
Appendix A: Neutrino emissivity in the average-angle approximation

The neutrino emissivity, as obtained in Ref. [10] in the average-angle approximation, is written in the form of the two-fold integral

$$\epsilon_{av} \simeq \frac{2}{15\pi^5} G_F^2 C_A^2 N_\nu p_F M^* T_e^7 \tau^7 y^2 \int \frac{dn}{4\pi} \int_{0}^{\infty} dx \frac{z^4}{(1 + e^z)^2}, \quad (A1)$$

where $z = \sqrt{x^2 + b^2 y^2}$. With the aid of the change $x = y\sqrt{\Omega^2 - b^2}$ one can recast this expression to the form

$$\epsilon_{av} \simeq \frac{2}{15\pi^5} G_F^2 C_A^2 N_\nu p_F M^* T_e^7 \tau^7 y^2 \int \frac{dn}{4\pi} \int_{0}^{\infty} dx \frac{(x^2 + b^2 y^2)^2}{(1 + e^{y\Omega})^2} \quad (A2)$$

Further simplification is possible if to change the order of integration and write the emissivity in the form

$$\epsilon_{av} \simeq \frac{2}{15\pi^5} G_F^2 C_A^2 N_\nu p_F M^* T_e^7 \tau^7 y^7 \int d\Omega \frac{\Omega^5}{(1 + e^{y\Omega})^2} \int \frac{dn}{4\pi} b^2 \Theta (\Omega - \bar{b}) \sqrt{\Omega^2 - b^2}, \quad (A3)$$

where $\bar{b}^2 = (1 + 3n_3^2)/2$ and $\Theta (x)$ is the Heavyside’s step-function. The integration over the Fermi surface gives

$$\int \frac{dn}{4\pi} b^2 \Theta (\Omega - \bar{b}) \sqrt{\Omega^2 - b^2} = \frac{\pi}{4\sqrt{6}} (1 + 2\Omega^2) \Theta (\Omega - 1/\sqrt{2}) \Theta (\sqrt{2} - \Omega)$$

$$+ \frac{\sqrt{6}}{12} \left( (2\Omega^2 + 1) \arcsin \frac{\sqrt{3}}{\sqrt{2}\Omega^2 - 1} - \sqrt{6}\Omega^2 - 12 \right) \Theta (\Omega - \sqrt{2}) \quad (A4)$$

Inserting this expression into Eq. (A3) we arrive to Eq. (64).

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