TORSION ORDER OF SMOOTH PROJECTIVE SURFACES

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Abstract. To a smooth projective variety $X$ whose Chow group of 0-cycles is $\mathbb{Q}$-universally trivial one can associate its torsion order $\text{Tor}(X)$, the smallest multiple of the diagonal appearing in a cycle-theoretic decomposition à la Bloch-Srinivas. We show that $\text{Tor}(X)$ is the exponent of the torsion in the Néron-Severi group of $X$ when $X$ is a surface over an algebraically closed field $k$, up to a power of the exponential characteristic of $k$.

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Introduction. Let $X$ be a smooth projective irreducible variety over a field $k$. Assume that $CH_0(X_{k(X)}) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}$: this is the strongest case of “decomposition of the diagonal” à la Bloch-Srinivas [5]. To $X$ is associated its torsion order $\text{Tor}(X)$, the smallest multiple of the diagonal of $X$ appearing in such a decomposition (Definition 1.5). This number is also studied by Chatzistamatiou and Levine in [6].

The integer $\text{Tor}(X)$ kills all normalised motivic birational invariants of smooth projective varieties in the sense of Definition 1.1 (Lemma 1.6). In particular, away from char $k$, the exponent of the torsion subgroup of the geometric Néron-Severi group of $X$ divides $\text{Tor}(X)$ (Corollary 5.4); the main result of this paper is that we have equality when $X$ is a surface and $k$ is algebraically closed: this result was announced in [12, Remark 3.1.5 3)]. In the special case where $\text{Tor}(X) = 1$, it
was obtained previously in [17] and [1] (see also Theorem A.1 in the appendix).

The equality follows from a short exact sequence (Corollary 5.4 a)):

\[(0.1)\quad 0 \to CH^2(X_{k(X)}) \to Tor(H^2(X), H^3(X))^2 \to H^{nr}_3(X \times X, \mathbb{Q}/\mathbb{Z}(2)) \to 0\]

where \(H^s(X)\) is Betti cohomology of \(X\) with integer coefficients in characteristic 0 (for simplicity; in positive characteristic, use \(l\)-adic cohomology). It also shows that \(CH^2(X_{k(X)})\) is finite (away from the characteristic of \(k\)), with a very explicit bound.\(^1\)

The exact sequence \((0.1)\) is a special case of a more general one appearing in Theorem 5.3, which implies in particular the finiteness of \(CH^2(X_{k(Y)})\) for any other smooth projective \(Y\), and an explicit bound on its order. See Theorem A.6 for another proof of this finiteness, and a different bound.

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1. Basic properties of the torsion order

1.1. Review of birational motives. We fix a base field \(k\), and write \(\text{Sm}^\text{proj} = \text{Sm}^\text{proj}(k)\) for the category of smooth projective \(k\)-varieties. Recall from [12] the category \(\text{Chow}^\alpha(k, A)\) of birational Chow motives with coefficients in a commutative ring \(A\): there is a commutative diagram of functors

\[\text{Sm}(k) \xrightarrow{h} \text{Chow}^\text{eff}(k, A) \xrightarrow{h^\circ} \text{Chow}^\alpha(k, A)\]

where \(\text{Chow}^\text{eff}(k, A)\) is the covariant category of effective Chow motives with coefficients in \(A\) (opposite to that of [16]), and Hom groups in \(\text{Chow}^\alpha(k, A)\) are characterized by the formula

\[\text{Chow}^\alpha(k, A)(h^\alpha(Y), h^\alpha(X)) = CH_0(X_{k(Y)}) \otimes A\]

for \(X, Y \in \text{Sm}^\text{proj}(k)\) (with \(Y\) irreducible). When \(A = \mathbb{Z}\), we simplify the notation to \(\text{Chow}^\text{eff}(k), \text{Chow}^\alpha(k), \text{Chow}^\text{eff}, \text{Chow}^\alpha\).

\(^1\)It would be interesting to completely determine \(CH^2(X_{k(X)})\): for example, when \(X\) is an Enriques surface and \(\text{char } k = 0\), is it \(\mathbb{Z}/2\) or \((\mathbb{Z}/2)^2\)?
1.2. Motivic birational invariants. Let $X \in \text{Sm}^{\text{proj}}(k)$ be irreducible, with $CH_0(X_{k(X)}) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}$: this condition is equivalent to Bloch-Srinivas’ decomposition of the diagonal relative to a closed subset of dimension 0 [5]. By [12, Prop. 3.1.1], this means that the birational motive $h^o(X)$ of $X$ in the category $\text{Chow}^o(k, \mathbb{Q})$ is trivial, i.e., that the projection map $h^o(X) \to h^o(\text{Spec } k) =: 1$ is an isomorphism in $\text{Chow}^o(k, \mathbb{Q})$. Then $CH_0(X_K) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}$ for any field extension $K$ of $k$ (loc. cit., Condition (vi)).

To such an $X$, we want to associate a numerical invariant. To motivate it, let us introduce a definition:

Definition 1.1. A motivic invariant of smooth projective varieties with values in an additive category $A$ is a functor $F: \text{Sm}^{\text{proj}} \to A$ which factors through an additive functor $\text{Chow}^{\text{eff}} \to A$. We say that $F$ is birational if it further factors through $\text{Chow}^o$. The invariant $F$ is normalised if $F(\text{Spec } k) = 0$.

Remark 1.2. If $X,Y \in \text{Sm}^{\text{proj}}$ are (stably) birationally equivalent, then $h^o(X) \simeq h^o(Y)$ in $\text{Chow}^o$ [12, Prop. 2.3.8]. Hence, to be a motivic birational invariant is stronger than to be a (stable) birational invariant. It is much stronger: $h^o(S) \xrightarrow{\sim} 1$ for $S$ the Barlow surface [2], a complex surface of general type.

Examples 1.3. a) For any cycle module $M_*$ in the sense of Rost [15], any $K \supseteq k$ and any $n \in \mathbb{Z}$, $X \mapsto A^0(X_K, M_n)$ (resp. $X \mapsto A^0(X_K, M_n)$) defines a contravariant (resp. covariant) motivic birational invariant with values in $\text{Ab}$, the category of abelian groups [12, Cor. 6.1.3].

b) In particular, for $M_* = K_*^M$ (Milnor $K$-theory), the functor $X \mapsto A^0(X_K, M_0) = CH_0(X_K)$ is a motivic birational invariant. When $K = k(Y)$ for some $Y \in \text{Sm}^{\text{proj}}$, this is also obvious by the interpretation of $CH_0(X_K)$ as $\text{Chow}^o(h^o(Y), h^o(X))$.

c) Given a contravariant motivic invariant $F$, we get two (contravariant) normalised invariants by the formulas

$$\overline{F}(X) = \text{Ker}(F(k) \to F(X)),$$

$$\overline{F}(X) = \text{Coker}(F(k) \to F(X))$$

and similarly for covariant motivic invariants:

$$\overline{F}(X) = \text{Coker}(F(X) \to F(k)),$$

$$\overline{F}(X) = \text{Ker}(F(X) \to F(k)).$$

They are birational if $F$ is birational.

d) Suppose that $F$ is a motivic invariant with values in the category of $\mathbb{Z}[1/p]$-modules, where $p$ is the exponential characteristic of $k$ (or, more generally, in a $\mathbb{Z}[1/p]$-linear additive category); assume $F$ contravariant
to fix ideas. Then $F$ is birational if and only if, for any $Y \in \text{Sm}^{\text{proj}}$, the map $F(Y) \to F(Y \times \mathbb{P}^1)$ is an isomorphism. This follows from [12, Th. 2.4.2].

**Definition 1.4.** The category $\text{Chow}^o_{\text{norm}}$ is the quotient of $\text{Chow}^o$ by the ideal generated by $1$.

Thus a motivic birational invariant is normalised if and only if it factors through $\text{Chow}^o_{\text{norm}}$.

Let $M, N \in \text{Chow}^o$. By definition, $\text{Chow}^o_{\text{norm}}(M, N)$ is the quotient of $\text{Chow}^o(M, N)$ by the group of morphisms $f : M \to N$ which factor through $1$. If $M = h^o(Y)$ and $N = h^o(X)$, this gives

\[ \text{Chow}^o_{\text{norm}}(h^o(Y), h^o(X)) \cong \text{Coker}(CH_0(X) \to CH_0(X_{k(Y)})). \]

**1.3. The torsion order.** If now the birational motive of $X$ is trivial in $\text{Chow}^o(k, \mathbb{Q})$, then the image of $h^o(X)$ in $\text{Chow}^o_{\text{norm}}$ is torsion; in other words, there is an integer $n > 0$ such that $n1_{h^o(X)} = 0$ in $\text{Chow}^o_{\text{norm}}(h^o(X), h^o(X))$.

**Definition 1.5.** The smallest such integer $n$ is called the **torsion order** of $X$, and denoted by $\text{Tor}(X)$. We extend this to arbitrary (connected) $X$ by setting $\text{Tor}(X) = 0$ if $h^o(X)$ is not trivial in $\text{Chow}^o(k, \mathbb{Q})$.

If $p$ is the exponential characteristic of $k$, we write $\text{Tor}^p(X)$ for the part of $\text{Tor}(X)$ which is prime to $p$ (so $\text{Tor}^p(X) = \text{Tor}(X)$ if $\text{char} k = 0$).

In $\text{Chow}^o$, the identity morphism $1_{h^o(X)}$ is given by $\eta_X \in CH_0(X_{k(X)})$, where $\eta_X$ is the generic point viewed as a closed point of $X_{k(X)}$. This gives a concrete description of the torsion order:

**Lemma 1.6.** Suppose that $CH_0(X_{k(X)}) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}$. Then the torsion order of $X$ is the order $n$ of $\eta_X$ in the group $CH_0(X_{k(X)})/CH_0(X)$ (it is 0 if and only if $\eta_X$ has infinite order). Moreover, we have $nF(X) = 0$ for any normalised motivic birational invariant $F$. In particular,

\[ nCH_0(X_K) = n\frac{CH_0(X_K)}{CH_0(X)} = 0 \quad \text{for any } K \geq k \]

where $CH_0(X_K) = \text{Ker}(CH_0(X_K) \xrightarrow{\text{deg}} \mathbb{Z})$.

**Proof.** The first and second statements are tautological; the third follows as a special case of the second. \qed

**1.4. Torsion order and index.** Another important invariant is

**Definition 1.7.** The **index** of an irreducible $X \in \text{Sm}^{\text{proj}}$ is the positive generator of $\text{Im}(\text{deg} : CH_0(X) \to \mathbb{Z})$. We denote it by $I(X)$. 
Proposition 1.8. Let $X \in \text{Sm}^{\text{proj}}$, irreducible. Write $n$ for its torsion order and $d$ for its index.

a) If $F$ is a motivic invariant and $\underline{F}$ is as in Example 1.3 c), then we have $d\underline{F}(X) = 0$.

b) $n$ is divisible by $d$.

c) Suppose $CH_0(X_k(X)) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}$. If $x \in CH_0(X)$ is an element of degree $d$, then $m(x_{k(X)} - d\eta_X) = 0$ in $CH_0(X_{k(X)})$ for some $m > 0$, and $n \mid md$.

d) If $d = 1$ and $m$ is minimal in c), then $n = m$.

Proof. a) Suppose $F$ is contravariant. Let $\alpha \in F(k)$ be such that $\pi^*\alpha = 0$, where $\pi : X \to \text{Spec } k$ is the structural morphism. If $x \in CH_0(X)$ is an element of degree $d$, it defines a morphism $x : 1 \to h^\alpha(X)$ such that $\pi \circ x = d$. Hence $d\alpha = 0$.

b) A diagram chase yields an exact sequence

\[
CH_0(X_0) \rightarrow CH_0(X_{k(X)})_0 \rightarrow \frac{CH_0(X_{k(X)})}{CH_0(X)} \rightarrow \mathbb{Z}/d \rightarrow 0
\]

where $CH_0(X_K)_0$ was defined in Lemma 1.6 and the last map sends the class of $\eta_X$ to 1.

c) The first claim follows from (1.1), and the second follows from pushing this identity in $CH_0(X_{k(X)})/CH_0(X)$.

d) If $d = 1$, (1.1) yields a surjection

\[
CH_0(X_{k(X)})_0 \twoheadrightarrow \frac{CH_0(X_{k(X)})}{CH_0(X)}.
\]

Let $y \in CH_0(X_{k(X)})_0$ mapping to the class of $\eta_X$. This means that $\eta_X - y = x_{k(X)}$ for some $x \in CH_0(X)$, and necessarily $\deg(x) = 1$. By Lemma 1.6, we have $ny = 0$ so the conclusion is true for this choice of $x$. But if $x' \in CH_0(X)$ is of degree 1, then $n(x' - x) = 0$ hence the conclusion remains true when replacing $x$ by $x'$.

\[\square\]

Remark 1.9. When $d = 1$, we can avoid the recourse to the category $\text{Chow}^\text{norm}_0$: in this case, the morphism $h^\alpha(X) \to 1$ is (noncanonically) split, hence we may consider its kernel $h^\alpha(X)_0 \in \text{Chow}^\alpha$. The endomorphism ring of this birational motive is canonically isomorphic to $CH_0(X_{k(X)})/CH_0(X)$.

1.5. Change of base field and products.

Proposition 1.10. Let $K/k$ be a field extension. Then:

a) $\text{Tor}(X_K) \mid \text{Tor}(X)$.

b) If $k$ and $K$ are algebraically closed, then $\text{Tor}(X_K) = \text{Tor}(X)$. 


Proof. a) is obvious, and b) follows from the rigidity theorem for torsion in Chow groups [13]. \[\square\]

**Proposition 1.11.** For any connected \(X, Y \in \text{Sm}^{\text{proj}}\), \(\text{Tor}(X \times Y) | \text{Tor}(X) \text{Tor}(Y)\).

Proof. If \(\text{Tor}(X) = 0\) or \(\text{Tor}(Y) = 0\), this is obvious. Otherwise, let \(m > 0\) (resp. \(n > 0\)) be such that \(m1_{h^o(X)}\) (resp. \(n1_{h^o(Y)}\)) factors through \(1\). Then \(mn1_{h^o(X \times Y)} = m1_{h^o(X)} \otimes n1_{h^o(Y)}\) factors through \(1 \otimes 1 = 1\). \[\square\]

2. TORSION ORDER FOR CYCLE MODULES

For any abelian group \(A\), write

\[
\exp(A) = \inf\{m > 0 \mid mA = 0\}
\]

and, by convention, \(\exp(A) = 0\) if no such integer \(m\) exists. Also write

\[
\exp^p(A) = \exp(A[1/p]).
\]

2.1. General case. We refine the notion of torsion order as follows:

**Definition 2.1.** Let \(M\) be a cycle module. For \(X \in \text{Sm}^{\text{proj}}\), \(K \supseteq k\) and \(n \in \mathbb{Z}\), write \(F_n(X_K) = \text{Coker}(M_n(K) \to A^0(X_K, M_n))\): then \(X \mapsto F_n(X_K)\) is a normalised motivic birational invariant in the sense of Definition 1.1. We set

\[
\text{Tor}_K(X, M_n) = \exp(F_n(X_K)),
\]

\[
\text{Tor}(X, M_n) = \text{lcm}_{K \supseteq k} \text{Tor}_K(X, M_n),
\]

\[
\text{Tor}(X, M) = \text{lcm}_n \text{Tor}(X, M_n).
\]

where \(\text{lcm}\) means lower common multiple.

By Lemma 1.6, \(\text{Tor}_K(X, M_n) | \text{Tor}(X, M_n) | \text{Tor}(X)\). Moreover,

**Lemma 2.2.** \(\text{Tor}(X, M_{n-1}) | \text{Tor}(X, M_n)\).

Proof. Let \(K/k\) be an extension. We have a naturally split exact sequence ([15, Prop. 2.2] and its proof):

\[
0 \to A^0(X_K, M_n) \to A^0(X_{K(t)}, M_n) \to \bigoplus_{x \in (\text{A}^1_K)^{(1)}} A^0(X_{K(x)}, M_{n-1}) \to 0.
\]

Indeed, \(K \mapsto A^0(X_K, M_n)\) defines a cycle module. Comparing with the same exact sequence for \(X = \text{Spec} \, k\), we get the conclusion. \[\square\]
2.2. **Unramified cohomology of degree \(\leq 2\).** For \(K \supseteq k\), we write \(K\) for an algebraic closure of \(K\) and \(G_K = \text{Gal}(K/K)\). Let \(p\) be the exponential characteristic of \(k\). We compute \(\text{Tor}(X, \mathcal{H}_n)\) for low values of \(n\), where \(\mathcal{H}_n\) is the cycle module \(K \mapsto H^n_{\text{ét}}(K, (\mathbb{Q}/\mathbb{Z})^i(n-1))\) with \((\mathbb{Q}/\mathbb{Z})^i(n-1) := \lim_{\longrightarrow (m,p)} \mu_m^\otimes n^{-1}\). As is well-known, we have

\[
A^0(X_K, \mathcal{H}_n) = \begin{cases} 
H^0(K, (\mathbb{Q}/\mathbb{Z})^i(-1)) & \text{for } n = 0 \\
H^1(X_K, (\mathbb{Q}/\mathbb{Z})^i) & \text{for } n = 1 \\
\text{Br}(X_K)[1/p] & \text{for } n = 2.
\end{cases}
\]

Let \(X\) be such that \(CH_0(X_{k(X)}) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}\); then \(b^1(X) = 0\) and \(b^2(X) = \rho(X)\) where \(b^i(X)\) (resp. \(\rho(X)\)) denotes the \(i\)-th Betti number (resp. the Picard number) of \(X\) [12, Prop. 3.1.4 3)]. In particular, we have \(\text{Pic}_{X/k}^0 = 0\) and for any \(K \supseteq k\), \(H^1(X_K, (\mathbb{Q}/\mathbb{Z})^i) \xrightarrow{\sim} \text{NS}(X_K)_{\text{tors}}[1/p]\) and similarly \(\text{Br}(X_K)[1]/p \xrightarrow{\sim} H^2_{\text{ét}}(X_K, \mathbb{Z}_l)_{\text{tors}}\) for \(l \neq p\), so \(\text{Br}(X_K)[1/p] \xrightarrow{\sim} \text{Br}(X_K)[1/p]\). (We neglected Tate twists in these computations.)

In the sequel, we abbreviate \(X_{\bar{k}}\) to \(\bar{X}\); for simplicity, we assume \(I(X) = 1\) so that \(H^i(K, (\mathbb{Q}/\mathbb{Z})^j) \rightarrow H^i(X_K, (\mathbb{Q}/\mathbb{Z})^j)\) is split injective for any \(K, i, j\). The Hochschild-Serre spectral sequence then gives isomorphisms (see Definition 2.1 for the notation \(F_n\)):

\[
F_0(X_K) = 0, \quad F_1(X_K) = (\text{NS}(\bar{X})_{\text{tors}}[1/p])^{G_K}
\]

and an exact sequence

\[
0 \rightarrow H^1(K, \text{NS}(\bar{X}))[1/p] \rightarrow F_2(X_K) \rightarrow (\text{Br}(\bar{X})[1/p])^{G_K}.
\](2.1)

For \(K \supseteq \bar{k}\), \(G_K\) acts trivially on \(\text{NS}(\bar{X})\) and \(\text{Br}(\bar{X})\). Then \(H^1(K, \text{NS}(\bar{X})) = \text{Hom}(G_K, \text{NS}(\bar{X})_{\text{tors}})\) and the last map in (2.1) is split surjective: indeed, \(\text{Br}(\bar{X})[1/p]\) maps to \(F_2(X_K)\) by functoriality. This yields:

**Proposition 2.3.** Let \(X\) be such that \(I(X) = 1\) and \(CH_0(X_{k(X)}) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}\). Then

\[
\text{Tor}(X, \mathcal{H}_0) = 1
\]

\[
\text{Tor}(X, \mathcal{H}_1) = \exp^p(\text{NS}(\bar{X})_{\text{tors}})
\]

\[
\text{Tor}(X, \mathcal{H}_2) = \text{lcm}(\exp^p(\text{NS}(\bar{X})_{\text{tors}}), \exp^p(\text{Br}(\bar{X})))
\]

In particular, \(\text{Tor}(X)\) is divisible by \(\exp^p(\text{NS}(\bar{X})_{\text{tors}})\) and \(\exp^p(\text{Br}(\bar{X}))\).

\(\square\)

(Of course, one could recover this conclusion directly by considering the normalised motivic birational functors \(X \mapsto \text{NS}(\bar{X})_{\text{tors}}[1/p]\) and \(X \mapsto \text{Br}(\bar{X})[1/p]\).)
Remark 2.4. When $k$ is algebraically closed, the above computation yields $\text{Tor}_k(X, \mathcal{H}_1) = \exp^p(\text{NS}(X)_{\text{tors}})$ and $\text{Tor}_k(X, \mathcal{H}_2) = \exp^p(\text{Br}(X))$.

When $\dim X = 2$, $\exp^p(\text{NS}(\overline{X})_{\text{tors}}) = \exp^p(\text{Br}(\overline{X}))$ by Poincaré duality. We shall see in Corollary 5.4 that, then, $\text{Tor}^p(\overline{X}) = \exp^p(\text{NS}(\overline{X})_{\text{tors}}) = \exp^p(\text{Br}(\overline{X}))$. In view of Proposition 2.3, this also yields

$$\text{Tor}^p(X) = \text{Tor}(X, \mathcal{H}) \quad \text{if} \quad \dim X \leq 2.$$ 

Question 2.5. Is the equality (2.2) true in general? In other words, does the cycle module $\mathcal{H}_*$ always compute the torsion index?

3. Extension of functors

Definition 3.1. If $F$ is a contravariant functor from smooth $k$-schemes of finite type to abelian groups, we extend it to smooth $k$-schemes essentially of finite type by the formula

$$\tilde{F}(X) = \lim_{\rightarrow} F(X')$$

where $X'$ runs through the smooth models of finite type of $X/k$.

Note that if $F(X) = A^n_{\text{alg}}(X)$, then $F$ is defined on all smooth $k$-schemes (not necessarily of finite type), but does not commute with filtering colimits; so the natural map

$$\tilde{A}^n_{\text{alg}}(X) \to A^n_{\text{alg}}(X)$$

is not an isomorphism in general, see [12, Rk. 2.3.10 2)]. By contrast, we have:

Lemma 3.2. For any cycle module $M$, the functors $A^p(-, M_q)$ of §4 below commute with filtering colimits of smooth schemes.

Proof. This is obvious, since the same is true for the cycle complexes of [15].

As a special case, one recovers the commutation of Chow groups with filtering colimits [3, Lemma 1A.1].

4. The Rost spectral sequence

Let $M$ be a cycle module. For any smooth $X/k$, recall its cycle cohomology with coefficients in $M$:

$$A^p(X, M_q) = H^p(\cdots \to \bigoplus_{x \in X^{(p)}} M_{q-p}(k(x)) \to \cdots)$$

where the differentials are defined through Rost’s axioms. We assume:
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(i) $M_n = 0$ for $n < 0$;

(ii) $M_0(K) = A$ for any $K/k$, where $A$ is a torsion-free abelian group.

By Rost’s axioms [15], there is then a canonical homomorphism of cycle modules

$$K^M \otimes A \to M$$

where $K^M$ is the cycle module given by Milnor’s $K$-theory. For any $n \geq 0$, this yields a surjective homomorphism

$$CH^n(X) \otimes A = A^n(X, M_n) \to A^n(X, M_n) =: A^n_M(X).$$

We may thus think of $A^n_M(X)$ as the group of cycles of codimension $n$ modulo “$M$-equivalence”.

**Examples 4.1.**

1) For $M = K^M \otimes A$, we get $A^n_M(X) = CH^n(X) \otimes A$.

2) Let $H$ be Betti cohomology (in characteristic 0) or $l$-adic cohomology (in characteristic $\neq l$): in the first case, let $A = \mathbb{Z}$ and in the second case let $A = \mathbb{Z}_l$. For a function field $K/k$, set

$$H_n(K) := \bar{H}^n(Spec K, A(n))$$

see Definition 3.1. (This is not the cycle module $\mathcal{H}$ considered in Subsection 2.2.) By [4, Th. 7.3] and [11, proof of Prop. 4.5], one has

$$A^n_{\mathcal{H}}(X) = A^n_{alg}(X) \otimes A$$

where $A^n_{alg}(X)$ is the group of cycles of codimension $n$ on $X$, modulo algebraic equivalence.

We now take two smooth $k$-varieties $X, Y$, and study the Rost spectral sequence [15, Cor. 8.2] attached to the first projection $\pi : Y \times X \to Y$:

$$E_1^{p,q}(r) = \bigoplus_{y \in Y^{(p)}} A^q(X_{k(y)}, M_{r-p}) \Rightarrow A^{p+q}(Y \times X, M_r)$$

abutting to the coniveau filtration on $A^{p+q}(Y \times X, M_r)$ with respect to $Y$. Note that $A^q(X_{k(y)}, M_{r-p}) = 0$ for $p + q > r$ by Condition (i) on $M$, hence $E_1^{p,q}(r) = 0$ in that range.

Take $r = 2$: we only have to consider $p + q \leq 2$. By definition, we have for a function field $K/k$ (see (4.1) for the notation $A^n_M$):

$$A^q(X_K, M_q) = \lim_{U^q} A^n_M(X \times U) =: \bar{A}^n_M(X_K)$$
where \( U \) runs through smooth models of \( K \) as above (see Lemma 3.2). This yields

\[
E_2^{0,2}(2) = \tilde{A}_M^2(X_k(Y)) \\
E_2^{1,1}(2) = \text{Coker}\left( A^1(X_k(Y), M_2) \to \bigoplus_{y \in Y^{(1)}} \tilde{A}_M^1(X_k(y)) \right) \\
E_2^{2,0}(2) = \text{Coker}\left( \bigoplus_{y \in Y^{(1)}} A^0(X_k(y), M_1) \to Z^2(Y) \otimes A \right).
\]

The latter group is a quotient of \( A^2_M(Y) \) (consider the maps \( M_1(k(y)) \to A^0(X_k(y), M_1) \)). If \( X \) has a 0-cycle of degree 1, the map \( A^2_M(Y) \to A^2_M(Y \times X) \) is split, hence \( \pi^*: A^3_M(Y) \to E_2^{2,0}(2) \) is an isomorphism. Thus \( E_2 = E_\infty \) in the Rost spectral sequence. We summarise this discussion:

**Proposition 4.2.** Let \( \text{gr}_Y A^2_M(X \times Y) \) be the associated graded to the coniveau filtration relative to \( Y \). Assume that \( X \) has a 0-cycle of degree 1. Then we have isomorphisms

\[
\text{gr}_Y^0 A^2_M(X \times Y) = \tilde{A}_M^2(X_k(Y)) \\
\text{gr}_Y^1 A^2_M(X \times Y) = \text{Coker}(A^1(X_k(Y), M_2) \to \bigoplus_{y \in Y^{(1)}} \tilde{A}_M^1(X_k(y))) \\
\text{gr}_Y^2 A^2_M(X \times Y) = A^2_M(Y).
\]

Moreover, we have an exact sequence:

\[
(4.3) \quad 0 \to A^1(Y, M_2) \to A^1(Y \times X, M_2) \to A^1(X_k(Y), M_2) \\
\quad \to \bigoplus_{y \in Y^{(1)}} \tilde{A}_M^1(X_k(y)) \to A^2_M(Y \times X)/A^2_M(Y).
\]

5. **Trivial birational motives of surfaces**

We start with a special case of Proposition 4.2:

**Theorem 5.1.** Suppose \( k \) algebraically closed, and let \( X/k \) be a smooth projective variety such that \( \text{Pic}^0_{X/k} = 0 \). Then for any smooth \( Y \), there is an exact sequence

\[
(5.1) \quad CH^2(Y) \oplus \text{Pic}(Y) \otimes \text{NS}(X) \oplus CH^2(X) \\
\quad \to CH^2(Y \times X) \to CH^2(X_k(Y))/CH^2(X) \to 0
\]

where the maps \( CH^2(X), CH^2(Y) \to CH^2(Y \times X) \) are induced by the two projections, and the map \( \text{Pic}(Y) \otimes \text{NS}(X) \to CH^2(Y \times X) \) is given by the cross-product of cycles.
A version of this theorem is found in Merkurjev’s appendix [14]; I thank J.-L. Colliot-Thélène for pointing out this reference.

Proof. Consider the Rost spectral sequence (4.2) for the cycle module $M = K^M$. Since $\text{Pic}^0_{X/k} = 0$, we have $\text{NS}(X) \xrightarrow{\sim} \text{Pic}(X_{k(y)})$ for any $y \in Y^{(1)}$, hence

$$E_2^{1,1} = \text{Coker}(A^1(X_{k(Y)}, K_2) \to Z^1(Y) \otimes \text{NS}(X)).$$

Then the natural map $k(Y)^* \otimes \text{NS}(X) \to A^1(X_{k(Y)}, K_2)$ realises $E_2^{1,1}$ as a quotient of $\text{Pic}(Y) \otimes \text{NS}(X)$. We conclude by applying Proposition 4.2. 

\[\square\]

Theorem 5.1 may be compared with a computation of the cohomology of $Y \times X$. We use $l$-adic cohomology, neglecting Tate twists: so $H^i(X) := \prod_{l \neq p} H^i_{\text{ét}}(X, \mathbb{Z}_l)$, where $p$ is the exponential characteristic of $k$ ($p = 1$ if $\text{char } k = 0$). If $k = \mathbb{C}$, we have $H^i(X) \simeq H^i_{\text{ét}}(X) \otimes \prod_l \mathbb{Z}_l$, by M. Artin’s comparison theorem. We note that the choice of a rational point of $X$ gives a retraction of the map $F(Y) \to F(Y \times X)$ for any contravariant functor $F : \text{Sm}^{\text{proj}} \to \text{Ab}$; the quotient $F(Y \times X, Y)$ is therefore a direct summand of $F(Y \times X)$. Then the Künneth formula gives split exact sequences

\[\text{(5.2)} \quad 0 \to H^3(X) \to H^3(Y \times X, Y) \to \text{Tor}(H^2(Y), H^2(X)) \to 0 \]

and

\[\text{(5.3)} \quad 0 \to H^2(Y) \otimes H^2(X) \oplus H^1(Y) \otimes H^3(X) \oplus H^4(X) \]
\[\to H^4(Y \times X, Y) \]
\[\to \text{Tor}(H^2(Y), H^3(X)) \oplus \text{Tor}(H^3(Y), H^2(X)) \to 0.\]

We now make the following

**Assumption 5.2.** $k$ is algebraically closed, $Y$ is projective and $X$ is a surface such that $CH_0(X_{k(X)}) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}$.

Recall that, then, $\text{Alb}(X) = \text{Pic}_{X/k}^0 = 0$ and $CH^2(X) = \mathbb{Z}$ (Roîtman’s theorem), so that Theorem 5.1 applies. Recall also that

$$H^1(X) = 0$$
$$\text{NS}(X) \otimes \hat{\mathbb{Z}}' \xrightarrow{\sim} H^2(X)$$
$$H^3(X) \simeq \text{Hom}(\text{NS}(X)_{\text{tors}}, (\mathbb{Q}/\mathbb{Z})^\vee)$$
$$H^4(X) = \hat{\mathbb{Z}}'$$
where \( \hat{Z}' = \prod_{l \neq p} Z_l \). Thus (5.1) and (5.3) yield a commutative diagram

\[
\begin{array}{ccc}
\text{Pic}(Y) \otimes \text{NS}(X) \otimes \hat{Z}' & \rightarrow & CH^2(Y \times X, Y) \otimes \hat{Z}' \\
\psi \downarrow & & \downarrow \varphi \\
\text{cl}_{Y \times X, Y}^2 & \rightarrow & \text{Coker cl}_{Y \times X, Y} \\
\end{array}
\]

(5.4)

\[0 \rightarrow H^2(Y) \otimes H^2(X) \otimes H^1(Y) \otimes H^3(X) \otimes \hat{Z}' \rightarrow H^4(Y \times X, Y) \rightarrow \text{Tor}(H^2(Y), H^3(X)) \otimes \text{Tor}(H^3(Y), H^2(X)) \rightarrow 0.
\]

An obvious generalisation of the exact sequence (1.1) boils down to an isomorphism

\[CH_0(X_{k(Y)})_0 \cong CH_0(X_{k(Y)})/CH_0(X).
\]

In (5.4), the left vertical map \( \psi \) is diagonal; its cokernel is

\[\text{Coker } \psi = H^2_{tr}(Y) \otimes \text{NS}(X) \oplus H^1(Y) \otimes H^3(X)
\]

where \( H^2_{tr}(Y) \) := Coker \( \text{cl}_Y^1 \), and its kernel is \( \text{Pic}^0(Y) \otimes \text{NS}(X) \otimes \hat{Z}' \) (we use here that \( H^2_{tr}(Y) \) is torsion-free). The snake lemma thus yields an exact sequence

\[(5.5) \quad \text{Pic}^0(Y) \otimes \text{NS}(X) \otimes \hat{Z}' \xrightarrow{\alpha} \text{Ker } \text{cl}_{Y \times X, Y}^2 \xrightarrow{\beta} \text{Ker } \varphi \xrightarrow{\gamma} H^4_{tr}(Y \times X, Y) \otimes \hat{Z}' \rightarrow H^4(Y \times X, Y).
\]

To go further, we use étale motivic cohomology as in [11]; the cycle class map \( \text{cl}_{Y \times X, Y}^2 \) extends to an étale cycle class map [11, (3-1)]:

\[\tilde{\text{cl}}_{Y \times X, Y}^2 : H^4_{\text{ét}}(Y \times X, Y, \mathbb{Z}(2)) \otimes \hat{Z}' \rightarrow H^4(Y \times X, Y).
\]

**Theorem 5.3.** *Under Assumption 5.2, \( \text{Ker } \text{cl}_{Y \times X, Y}^2 \) and \( \text{Ker } \text{cl}_{Y \times X, Y}^2 \) are torsion-free; the exact sequence (5.5) yields a surjection

\[\text{Pic}^0(Y) \otimes \text{NS}(X) \otimes \hat{Z}' \rightarrow \text{Ker } \text{cl}_{Y \times X, Y}^2
\]

and an exact sequence of finite groups

\[(5.6) \quad 0 \rightarrow \text{Ker } \varphi \rightarrow H^2_{tr}(Y) \otimes \text{NS}(X)_{\text{tors}} \oplus H^1(Y) \otimes H^3(X) \rightarrow H^3_{\text{int}}(Y \times X, Y; (\mathbb{Q}/\mathbb{Z})'/(2)) \rightarrow \text{Coker } \varphi \rightarrow 0
\]

where \( H^3_{\text{int}}(Y \times X, Y; (\mathbb{Q}/\mathbb{Z})'/(2)) := \lim_{(m, p) = 1} H^3_{\text{int}}(Y \times X, Y; \mu_m^{\otimes 2}). In particular, \( CH_0(X_{k(Y)})/CH_0(X)[1/p] \simeq CH_0(X_{k(Y)})_{\text{tors}}[1/p] \) is finite.*

**Proof.** This proof is ugly, mainly because the Leray spectral sequence for étale motivic cohomology relative to the projection \( (Y \times X, Y) \rightarrow Y \) does not behave as well as the spectral sequence (4.2). So, instead of comparing directly étale motivic and \( l \)-adic cohomology, we have to wiggle through.
We have a commutative diagram
\[(5.7)\]
\[
\begin{array}{ccc}
H^3_{\text{ét}}(Y \times X, Y; (\mathbb{Q}/\mathbb{Z})'(2)) & \xrightarrow{\sim} & \lim_{\substack{\longrightarrow \ \ \\ m \to \infty}} H^3_{\text{ét}}(Y \times X, Y, \mu_m^{\otimes 2}) \\
\downarrow & & \downarrow \\
H^4_{\text{ét}}(Y \times X, Y, \mathbb{Z}(2)) \otimes \mathbb{Z}' & \xrightarrow{\text{cl}_Y^2 \times \text{cl}_X^2} & H^4(Y \times X, Y)
\end{array}
\]
in which the right vertical map is injective, because \(H^3(Y \times X, Y)\) is torsion by (5.2). Thus \(\ker \text{cl}_Y^2 \times \text{cl}_X^2\) is torsion-free, and so is its subgroup \(\ker \text{cl}_Y^2 \otimes \text{cl}_X^2\). But the image of \(\alpha\) in (5.5) is divisible, hence a direct summand. Therefore the image of \(\beta\) is torsion-free, hence 0. So we get the surjection promised in the theorem, and an exact sequence
\[(5.8)\]
\[0 \to \ker \varphi \to H^2_{\text{tr}}(Y) \otimes \text{NS}(X) \oplus H^1(Y) \otimes H^3(X) \to \text{coker cl}_Y^2 \otimes \text{coker cl}_X^2 \to 0.\]

As a consequence, \(\ker \varphi\) is finitely generated; since it is torsion it must be finite, hence \(\text{CH}_0(X_k(Y))/\text{CH}_0(X)[1/p]\) is finite.

We now deduce from [11, Th. 1.1] the following surjection:
\[(5.9)\]
\[H^3_{\text{nr}}(Y \times X, Y; (\mathbb{Q}/\mathbb{Z})'(2)) \to (\text{coker cl}_Y^2 \otimes \text{coker cl}_X^2)_{\text{tors}}\]
(if \(k = \mathbb{C}\), this is due to Colliot-Thélène–Voisin [10, Th. 3.7], with Betti cohomology instead of \(l\)-adic cohomology). This map has divisible kernel; however, \(Z \mapsto H^3_{\text{nr}}(Y \times Z, Y; (\mathbb{Q}/\mathbb{Z})'(2))\) is a normalised motivic birational invariant, hence \(H^3_{\text{nr}}(Y \times X, Y; (\mathbb{Q}/\mathbb{Z})'(2))\) is killed by \(\text{Tor}(X)\) and therefore finite; so (5.9) is an isomorphism.

Let \(M = \text{coker cl}_Y^2 \otimes \text{coker cl}_X^2\) tors; by [11, Cor. 3.5], this is actually \(\text{coker cl}_Y^2 \otimes \text{coker cl}_X^2\) tors, although we won’t use it. The composition of the map \(\gamma\) of (5.5) with the projection \(p : \text{coker cl}_Y^2 \otimes \text{coker cl}_X^2 \to M\) has image isomorphic to \(H^2_{\text{tr}}(Y) \otimes (\text{NS}(X)/\text{tors})\).

I claim that \(p \circ \gamma\) is surjective. To see this, choose a retraction \(\rho\) of the map \(\theta\) in Diagram (5.4); composing \(\rho \circ \text{cl}_Y^2 \otimes \text{cl}_X^2\) with the projection to \(\text{coker} \psi\), we get an induced map
\[
\text{CH}^2(X_k(Y))/\text{CH}^2(X) \otimes \mathbb{Z}' \to \text{coker} \psi
\]
whose composition with \(\text{coker} \psi \to H^2_{\text{tr}}(Y) \otimes (\text{NS}(X)/\text{tors})\) is 0 since \(\text{CH}^2(X_k(Y))/\text{CH}^2(X)\) is torsion. This shows that \(\rho\) induces a map
\[
\tilde{\rho} : \text{coker cl}_Y^2 \otimes \text{coker cl}_X^2 \to H^2_{\text{tr}}(Y) \otimes (\text{NS}(X)/\text{tors})
\]
factoring through a left inverse of the inclusion \(H^2_{\text{tr}}(Y) \otimes (\text{NS}(X)/\text{tors}) \hookrightarrow M\) induced by \(\gamma\). But \(\gamma \otimes \mathbb{Q}\) is an isomorphism, since \(\ker \varphi\) and \(\text{coker} \varphi\) are torsion; therefore \(p \circ \gamma\) is surjective as claimed.
Chasing in (5.8) with this information and using the isomorphism (5.9) now yields the exact sequence (5.6). □

**Corollary 5.4.** a) Suppose $Y = X$. Then we have a commutative diagram of short exact sequences:

\[
\begin{array}{cccc}
0 & \rightarrow & CH^2(X \times X) \otimes \hat{\mathbb{Z}} & \rightarrow & H^4(X \times X) & \rightarrow & H^3_{nr}(X \times X, (\mathbb{Q}/\mathbb{Z})'(2)) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & CH^2(X'_{k(X)})/CH^2(X)[1/p] & \rightarrow & \text{Tor}(H^2(\mathbb{X}), H^3(\mathbb{X}))^2 & \rightarrow & H^3_{nr}(X \times X, (\mathbb{Q}/\mathbb{Z})'(2)) & \rightarrow & 0.
\end{array}
\]

b) In particular, the first map of (5.1) (for $Y = X$) has $p$-primary torsion kernel, and $\text{Tor}^p(\mathbb{X}) = \exp^p(\text{NS}(\mathbb{X})_{\text{tors}})$.

**Proof.** Indeed, we have $\text{Pic}^0(\mathbb{X}) = H^1(\mathbb{X}) = H^2_{\text{nr}}(\mathbb{X}) = 0$, and Theorem 5.3 boils down to the injectivity of $\text{cl}^2_{X \times X, X}$ and $\varphi$, plus an isomorphism $H^3_{\text{nr}}(X \times X, (\mathbb{Q}/\mathbb{Z})'(2)) \xrightarrow{\sim} \text{Coker } \varphi$. But $H^3_{\text{nr}}(X, (\mathbb{Q}/\mathbb{Z})'(2)) = 0$, hence $H^3_{\text{nr}}(X \times X, (\mathbb{Q}/\mathbb{Z})'(2)) \xrightarrow{\sim} H^3_{\text{nr}}(X \times X, (\mathbb{Q}/\mathbb{Z})'(2))$. □

**Corollary 5.5.** If $Y$ is a curve, we have a short exact sequence

\[0 \rightarrow CH^2(X'_{k(Y)})_{\text{tors}}[1/p] \rightarrow H^1(Y) \otimes H^3(X) \rightarrow \text{Coker } \text{cl}^2_{Y \times X} \rightarrow 0.\]

**Proof.** In this case, $H^3_{\text{nr}}(Y) = 0$ and the target of $\varphi$ is 0. □

**Remarks 5.6.** a) The special case $\text{NS}(\mathbb{X})_{\text{tors}} = 0$ and $\text{char } k = 0$ of Corollary 5.4 b) was proven in [1, Cor. 1.10] and [17, Prop. 2.2]. As Colliot-Thélène points out, the methods of [8] imply that for any smooth projective $k$-variety $X$ with $b^1 = 0$ and $b^2 = \rho$, $\text{Ker}(CH^2(X'_k) \rightarrow CH^2(X'_K))$ is killed by $\exp(\text{NS}(\mathbb{X})_{\text{tors}}) \cdot \exp(\text{Br} (X))$ (see Theorem A.1).

b) In the first version of this paper, I had proven Corollaries 5.4 and 5.5 but had doubts on the finiteness of $CH_0(X'_{k(Y)})_{\text{tors}}$ in general. Colliot-Thélène provided a proof based on his 1991 CIME course [9], see Theorem A.6. This encouraged me to find a proof in the spirit of this note, and Theorem 5.3 is the result. Note that the group $\Theta$ appearing in [9, Th. 7.1] coincides with $H^4_{\text{et}}(X, \mathbb{Z}(2))_{\text{tors}}$. In this spirit, a weaker analogue of [9, Th. 7.3] is the following fact: for any field $F$, the functor $\mathbf{Sm}^{\text{proj}}(F) \ni Z \mapsto \text{Ker}(H^4_{\text{et}}(Z, \mathbb{Z}(2)) \rightarrow H^3_{\text{et}}(\mathbb{Z}(1)) \rightarrow H^2_{\text{et}}(Y, \mathbb{Z}(1))$ is injective for any smooth projective $Y$). As a consequence, $\text{Ker}(H^4_{\text{et}}(X, \mathbb{Z}(2)) \rightarrow H^3_{\text{et}}(X'_F, \mathbb{Z}(2))$ is killed by $\text{Tor}(X)$ if $X$ has a trivial birational motive.
A.1. **Introduction.** On donne des conséquences faciles de résultats établis dans [8] (avec W. Raskind) et dans le rapport de synthèse [9], en particulier dans une section où je développais des arguments de S. Saito et de P. Salberger.

A.2. **Notations et rappels.** Pour simplifier les énoncés, on se limite ici aux variétés définies sur un corps de caractéristique nulle. On note $k$ une clôture algébrique de $k$. Pour une telle $k$-variété $X$, supposée projective, lisse, géométriquement connexe sur le corps $k$, on note $\overline{X} = X \times_k \overline{k}$. On note $b_i$ le $i$-ième nombre de Betti $l$-adique de $\overline{X}$. On sait qu’il est indépendant du nombre premier $l$. On note $\rho$ le rang du groupe de Néron-Severi géométrique $\text{NS}(\overline{X})$. Pour tout entier $i$, on note ici $H^i(\overline{X}, \mathbb{Z}(j)) := \prod_l H^i_{\text{ét}}(\overline{X}, \mathbb{Z}(j))$. Le sous-groupe de torsion $H^i(\overline{X}, \mathbb{Z}(j))_{\text{tors}}$ est fini. On note $e_i$ son exposant. Pour $k = \mathbb{C}$ le corps des complexes, $H^i_{\text{Betti}}(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Z}_l \simeq H^i_{\text{ét}}(X, \mathbb{Z}_l)$. On sait que l’on a un isomorphisme de groupes finis $\text{NS}(\overline{X})_{\text{tors}} = H^3(\overline{X}, \mathbb{Z}(1))_{\text{tors}}$. Le groupe de Brauer $\text{Br}(\overline{X})$ de $\overline{X}$ est extension du groupe fini $H^3(\overline{X}, \mathbb{Z}(1))_{\text{tors}}$ par $(\mathbb{Q}/\mathbb{Z})^{b_2 - \rho}$. La condition $H^1(X, \mathbb{O}_X) = 0$ équivaut à $b_1 = 0$. La condition $H^2(X, \mathbb{O}_X) = 0$ équivaut (théorie de Hodge) à $\rho = b_2$, c’est-à-dire à la finitude du groupe de Brauer de $\overline{X}$. Pour $X$ une variété lisse, on note $CH^i(X)$ le groupe de Chow des cycles de codimension $i$ de $X$. Pour $X$ une variété projective, on note $CH_i(X)$ le groupe de Chow des cycles de dimension $i$ de $X$.

A.3. **Exposant de torsion.** L’énoncé suivant aurait pu être inclus dans [8]. Comme indiqué formellement ci-dessus, l’entier $e_i$ est l’annulateur de la torsion du $i$-ème groupe de cohomologie entière.

**Théorème A.1.** Soit $k$ un corps de caractéristique zéro. Soit $X$ une $k$-variété projective, lisse, connexe, satisfaisant $X(k) \neq \emptyset$. Supposons que le réseau $\text{NS}(\overline{X})_{\text{tors}}$ admet une base globalement respectée par le groupe de Galois absolu de $k$.

(a) Supposons $b_1 = 0$ et $\rho = b_2$. Alors le groupe de torsion $\text{Ker}[CH^2(X) \to CH^2(\overline{X})]$ est annulé par le produit $e_2.e_3$, qui est aussi le produit de l’exposant de $\text{NS}(\overline{X})_{\text{tors}}$ et de l’exposant du groupe $\text{Br}(\overline{X})$.

(b) Si de plus $b_3 = 0$, alors $CH^2(X)_{\text{tors}}$ est annulé par $e_2.e_3.e_4$.

**Démonstration.** Il suffit de suivre les démonstrations du §3 de [8]. On note $H^i(k, \bullet)$ les groupes de cohomologie galoisienne.
Sous l’hypothèse $H^1(X, O_X) = 0$, le théorème 1.8 de [8] donne une suite exacte de modules galoisiens

$$0 \to D_0 \to H^0(X, \mathcal{K}_2) \to H^2(X, \hat{\mathbb{Z}}(1))_{\text{tors}} \to 0$$

où $D_0$ est uniquement divisible. Le groupe $K_2\mathbb{K}$ est uniquement divisible. On a la suite exacte

$$0 \to H^0(X, \mathcal{K}_2)/K_2\mathbb{K} \to K_2\mathbb{K}(X)/K_2\mathbb{K} \to K_2\mathbb{K}(X)/H^0(X, \mathcal{K}_2) \to 0.$$

Comme on a supposé $X(k) \neq \emptyset$, on a $H^1(k, K_2\mathbb{K}(X)/K_2\mathbb{K}) = 0$ [7, Theorem 1]. On voit alors que le groupe $H^1(k, K_2\mathbb{K}(X)/H^0(X, \mathcal{K}_2))$ est un sous-groupe de $H^2(k, H^2(X, \hat{\mathbb{Z}}(1))_{\text{tors}})$ et donc est annulé par $e_2$.

Sous les deux hypothèses $H^2(X, O_X) = 0$ et $H^1(X, O_X) = 0$ (cette dernière garantissant $Pic(X) = \text{NS}(X)$), le théorème 2.12 de [8] donne une suite exacte de modules galoisiens

$$0 \to D_1 \to \text{NS}(X) \otimes \mathbb{K} \to H^1(X, \mathcal{K}_2) \to [D_2 \oplus H^3(X, \hat{\mathbb{Z}}(2))_{\text{tors}}] \to 0,$$

où $D_1$ et $D_2$ sont uniquement divisibles. L’hypothèse que l’action du groupe de Galois sur $\text{NS}(X)/\text{tors}$ est triviale assure via le théorème 90 de Hilbert que l’on a $H^1(k, \text{NS}(X) \otimes \mathbb{K}) = 0$. De la suite exacte ci-dessus on déduit que $H^1(k, H^1(X, \mathcal{K}_2))$ est un sous-groupe de $H^1(k, H^3(X, \hat{\mathbb{Z}}(2))_{\text{tors}})$ et donc est annulé par $e_3$.

La proposition 3.6 de [8] fournit une suite exacte

$$H^1(k, K_2\mathbb{K}(X)/H^0(X, \mathcal{K}_2)) \to \text{Ker}[CH^2(X) \to CH^2(X)]$$

$$\to H^1(k, H^1(X, \mathcal{K}_2)).$$

On voit donc que $\text{Ker}[CH^2(X) \to CH^2(X)]$ est annulé par le produit $e_2.e_3$.

Par Bloch et Merkurjev-Suslin, $CH^2(X)_{\text{tors}}$ est un sous-quotient de $H^3_{et}(X, \mathbb{Q}/\mathbb{Z}(2))$ [9, Théorème 3.3.2]. Si $b_3 = 0$, alors $CH^2(X)_{\text{tors}}$ est un sous-quotient de $H^4(X, \hat{\mathbb{Z}}(2))_{\text{tors}}$, d’exposant $e_4$. Sous les hypothèses du théorème, on obtient alors que $CH^2(X)_{\text{tors}}$ est annulé par $e_2.e_3.e_4$.

\[ \square \]

**Remarques** A.2. 1) Soit $Y$ une variété projective et lisse sur le corps des complexes $\mathbb{C}$ satisfaisant les hypothèses du théorème. Pour tout corps $k$ contenant $\mathbb{C}$, le théorème s’applique à la $k$-variété $X = Y \times_k k$.

L’hypothèse sur l’action galoisienne est alors automatiquement satisfaite pour la $k$-variété $X$, car on a $\text{NS}(Y) = \text{NS}(X)$.

2) Lorsque $e_2 = 1 = e_3$, l’énoncé (a) est le théorème 3.10 b) de [8].

3) Si $X$ est une surface, $e_4 = 1$, et $b_1 = b_3$. En outre, $e_2 = e_3$. Sous les hypothèses du théorème, on trouve que le groupe $CH^2(X)_{\text{tors}} = $
CH_0(X)_{tors} est annulé par le carré de l’exposant de la torsion de NS(\(\overline{X}\)).

A.4. Finitude. On utilise ici les notations et résultats du §7 de [9].

Théorème A.3. Soient \(k\) un corps de caractéristique zéro et \(\overline{\mathbb{F}}\) une clôture algébrique. Soit \(X\) une \(k\)-variété projective et lisse, géométriquement intègre. Notons \(\overline{X} = X \times_k \overline{\mathbb{F}}\). Notons \(b_i \in \mathbb{N}\) les nombres de Betti \(l\)-adiques de \(\overline{X}\) et \(\rho = \text{rang}(\text{NS}(\overline{X}))\). Supposons \(H^1(X, O_X) = 0\), ce qui équivaut à \(b_1 = 0\). Supposons aussi \(H^2(X, O_X) = 0\), ce qui équivaut à \(\rho = b_2\). Supposons \(b_3 = 0\). Alors le conoyau de l’application

\[
H^3_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}(2)) \oplus [H^1(X, \mathbb{K}_2) \otimes \mathbb{Q}/\mathbb{Z}] \to H^3_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}(2))
\]

est d’exposant fini.

Démonstration. L’hypothèse \(b_3 = 0\) implique que le groupe \(H^3_{\text{ét}}(\overline{X}, \mathbb{Q}/\mathbb{Z}(2))\) s’identifie au groupe fini \(H^4_{\text{ét}}(\bar{X}, \mathbb{Z}(2))_{\text{tors}}\). L’énoncé est alors une conséquence immédiate du Théorème 7.3 de [9], auquel je renvoie pour les notations.

\(\square\)

Théorème A.4. Soient \(k\) un corps de caractéristique zéro et \(\overline{\mathbb{F}}\) une clôture algébrique. Soit \(X\) une \(k\)-variété projective et lisse, géométriquement intègre. Notons \(\overline{X} = X \times_k \overline{\mathbb{F}}\). Supposons que chacun des entiers \(b_1\), \(b_2 - \rho\) et \(b_3\) associés à \(\overline{X}\) est nul. Supposons \(X(k) \neq \emptyset\). Alors il existe un entier \(N > 0\) annulant le groupe \(CH^2(X)_{\text{tors}}\) et tel que pour tout entier \(n > 0\) multiple de \(N\), l’application

\[
CH^2(X)_{\text{tors}} \to CH^2(X)/n \to H^4_{\text{ét}}(X, \mu_n^{\otimes 2})
\]

composée de la projection naturelle et de l’application classe de cycle en cohomologie étale est injective.

Démonstration. Il suffit de combiner le théorème A.3 avec le théorème 7.2 de [9].

\(\square\)

Remarque A.5. Si \(X\) est une surface, l’hypothèse \(b_3 = 0\) est impliquée par \(b_1 = 0\).

On dit qu’un corps \(k\) de caractéristique zéro est à cohomologie galoisienne finie si pour tout module fini galoisien \(M\) sur \(k\), tous les groupes de cohomologie galoisienne \(H^i(k, M)\) sont finis. Parmi les corps de caractéristique zéro satisfaisant cette propriété, on trouve : les corps algébriquement clos, les corps réels clos, les corps \(p\)-adiques, les corps de séries formelles itérées sur un des corps précédents.

Théorème A.6. Soit \(k\) un corps de caractéristique zéro à cohomologie galoisienne finie. Soit \(K\) un corps de type fini sur \(k\). Soit \(X\) une \(K\)-variété projective et lisse satisfaisant \(X(K) \neq \emptyset\). Notons \(\overline{X} = X \times_K \overline{K}\).
Supposons que chacun des entiers \( b_1, b_2 - \rho \) et \( b_3 \) associés à \( \overline{X} \) est nul. Alors le groupe \( CH^2(X)_{\text{tors}} \) est fini.

**Démonstration.** D’après le théorème A.4, il existe un entier \( n > 0 \) tel que le groupe \( CH^2(X)_{\text{tors}} \) s’identifie à un sous-groupe de l’image de l’application classe de cycle

\[
CH^2(X)/n \to H^4_{\text{ét}}(X, \mu_n^{\text{\otimes 2}}).
\]

Soit \( Y \) une \( k \)-variété intègre de corps des fonctions \( K \). Quitte à restreindre la \( k \)-variété \( Y \) à un ouvert non vide convenable, il existe un \( Y \)-schéma intègre, projectif et lisse \( \mathcal{X} \to Y \) dont la fibre générique est la \( K \)-variété \( X \). L’application de restriction \( CH^2(\mathcal{X}) \to CH^2(X) \) est surjective, et les applications classe de cycle \( CH^2(X)/n \to H^4_{\text{ét}}(X, \mu_n^{\text{\otimes 2}}) \) et \( CH^2(\mathcal{X})/n \to H^4(\mathcal{X}, \mu_n^{\text{\otimes 2}}) \) sont compatibles. L’image de \( CH^2(X)/n \to H^4_{\text{ét}}(X, \mu_n^{\text{\otimes 2}}) \) est donc dans l’image de la restriction \( H^4(\mathcal{X}, \mu_n^{\text{\otimes 2}}) \to H^4_{\text{ét}}(X, \mu_n^{\text{\otimes 2}}) \). Sous les hypothèses du théorème, les groupes \( H^i(W, \mu_n^{\text{\otimes j}}) \) sont finis pour toute variété \( W \) de type fini sur \( k \), en particulier \( H^4(X, \mu_n^{\text{\otimes 2}}) \) est fini. On conclut que \( CH^2(X)_{\text{tors}} \) est fini. □

**Remarque A.7.** Si \( X \) est une \( K \)-surface, \( b_1 = b_3 \) et l’hypothèse est simplement que \( b_1 = 0 \) et \( b_2 - \rho = 0 \), et la conclusion est que \( CH_0(X)_{\text{tors}} \) est fini.

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