Error estimates for interpolation of rough data using the scattered shifts of a radial basis function

R. A. Brownlee
Department of Mathematics, University of Leicester, Leicester LE1 7RH, England

Abstract

The error between appropriately smooth functions and their radial basis function interpolants, as the interpolation points fill out a bounded domain in $\mathbb{R}^d$, is a well studied artifact. In all of these cases, the analysis takes place in a natural function space dictated by the choice of radial basis function—the native space. The native space contains functions possessing a certain amount of smoothness. This paper establishes error estimates when the function being interpolated is conspicuously rough.

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1 Introduction

In this paper we are interested in interpolation of a finite scattered data set \( A \subset \mathbb{R}^d \) by translates of a single basis function. Of the differing set ups to this problem, the one preferred in this paper is the following variational formul ation. Firstly , we require a space of continuous functions \( Z \) which carries a seminorm. The minimal norm interpolant to \( f : A \to \mathbb{R} \) on \( A \) from \( Z \) is the function \( Sf \in Z \) which agrees with \( f \) on \( A \) and has smallest seminorm amongst all other interpolants to \( f \) on \( A \) from \( Z \). The particular space we shall be concerned with is

\[
Z_m(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \hat{D}^\alpha f \in L^1_{\text{loc}}(\mathbb{R}^d), \int_{\mathbb{R}^d} w(x) |(\hat{D}^\alpha f)(x)|^2 \, dx < \infty, |\alpha| = m \right\},
\]

which carries the seminorm

\[
|f|_m := \left( \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} w(x) |(\hat{D}^\alpha f)(x)|^2 \, dx \right)^{1/2}, \quad f \in Z^m(\mathbb{R}^d).
\]

The constants \( c_\alpha \) are chosen so that \( \sum_{|\alpha|=m} c_\alpha x^{2\alpha} = |x|^{2m} \), for all \( x \in \mathbb{R}^d \). The notation \( \mathcal{S}' \) is used to denote the usual Schwartz space of distributions. The space \( Z^m(\mathbb{R}^d) \) is christened the native space. The weight function \( w : \mathbb{R}^d \to \mathbb{R} \) is initially chosen to satisfy

(W0) \( w \in C(\mathbb{R}^d \setminus \{0\}) \);
(W1) \( w(x) > 0 \) if \( x \neq 0 \);
(W2) \( 1/w \in L^1_{\text{loc}}(\mathbb{R}^d) \);
(W3) there is a positive \( \mu \in \mathbb{R} \) such that \( (w(x))^{-1} = O(|x|^{-2\mu}) \) as \( |x| \to \infty \).

A consequence of (W0)–(W3) is that \( Z^m(\mathbb{R}^d) \) is complete with respect to \( | \cdot |_m \), and if \( m + \mu - d/2 > 0 \) then \( Z^m(\mathbb{R}^d) \) is embedded in the continuous functions (see [6]). As the title of this work suggests, we expect this set up to admit minimal norm interpolants of the form

\[
(Sf)(x) := \sum_{a \in A} b_a \psi(x - a), \quad \text{for } x \in \mathbb{R}^d,
\]

for an appropriate basis function \( \psi \). We are not disappointed, but for brevity we omit the details which are well presented in [6]. The coefficients \( b_a \) in (1.1) are determined by the interpolation equations \( (Sf)(a) = f(a), a \in A \). In some situations it may be necessary to append a polynomial \( p \) onto (1.1) and take up the ensuing extra degrees of freedom by satisfying the side conditions:

\[
\sum_{a \in A} b_a q(a) = 0,
\]

whenever \( q \) is a polynomial of the same degree (or less) as \( p \). The archetypal scenario the author has in mind is \( w(x) = |x|^{2\mu} \) for \( x \in \mathbb{R}^d \), where \( \mu < d/2 \). This leads to minimal norm interpolants of the form (1.1) modulo a polynomial of degree \( m \). Here, the
radial basis function is $\psi : x \mapsto |x|^{2m+2\mu-d} \log |x|$ if $2m+2\mu-d$ is an even integer or $\psi : x \mapsto |x|^{2m+2\mu-d}$ otherwise.

It is of central importance to understand the behaviour of the error between a function $f : \Omega \to \mathbb{R}$ and its interpolant as the set $\mathcal{A}$ becomes dense in a bounded domain $\Omega$. The measure of density we employ is the fill-distance $h := \sup_{x \in \Omega} \min_{a \in \mathcal{A}} |x-a|$. One finds that there is a positive constant $\gamma(m)$, independent of $h$, such that for all $f \in \mathcal{Z}^m(\mathbb{R}^d)$,

$$
\|f - Sf\|_{L^2(\Omega)} = O(h^{\gamma(m)}), \quad \text{as } h \to 0.
$$

It is natural to ask: what happens if the function being approximated does not lie in $\mathcal{Z}^m(\mathbb{R}^d)$? It may well be that $f$ lies in $\mathcal{Z}^k(\mathbb{R}^d)$, where $k < m$ and $k + \mu - d/2 > 0$. The condition $k + \mu - d/2 > 0$ ensures that $f(a)$ exists for each $a \in \mathcal{A}$, so $Sf$ certainly exists. It is tempting to conjecture that the new error estimate should be

$$
\|f - Sf\|_{L^2(\Omega)} = O(h^{\gamma(k)}), \quad \text{as } h \to 0.
$$

We are conjecturing the same approximation order as if we had instead approximated $f$ with the minimal norm interpolant to $f$ on $\mathcal{A}$ from $\mathcal{Z}^k(\mathbb{R}^d)$. This is precisely what happens in the case $w = 1$, which was considered by Brownlee & Light in [2]. In this work, with the aid of a recent result from [1] (Lemma 2.5), we employ the technique used by Brownlee & Light to extend their work to more general weight functions. Theorem 3.5 is the definitive result we obtain. The interested reader may enjoy consulting the related papers [7, 8, 9, 10].

To close this section we introduce some notation that will be employed throughout the paper. A domain is understood to be a connected open set. The support of a function $\phi : \mathbb{R}^d \to \mathbb{R}$, denoted by $\text{supp} (\phi)$, is defined to be the closure of the set $\{x \in \mathbb{R}^d : \phi(x) \neq 0\}$. We make much use of the linear space $\Pi_m(\mathbb{R}^d)$ which consists of all polynomials of degree at most $m$ in $d$ variables. We fix $\ell$ as the dimension of this space. Finally, when we write $\hat{f}$ we mean the Fourier transform of $f$. The context will clarify whether the Fourier transform is the natural one on $L^1(\mathbb{R}^d)$, $\hat{f}(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(t)e^{-ixt} dt$, or one of its several extensions to $L^2(\mathbb{R}^d)$ or $\mathcal{S}'$.

2 Extension theorems

In this section we gather a number of useful results, chiefly about the sorts of extensions which can be carried out on our native spaces. This will first require us to establish the notion of local native spaces. To do this, we rewrite the seminorm $|f|_m$ in its direct form—that is, without the Fourier transform of $f$ appearing explicitly. Let us demand that $w$ satisfies the following additional axioms:

(W4) $w(y) = w(-y)$ for all $y \in \mathbb{R}^d$;

(W5) $w(0) = 0$ and $\hat{w}(x) \leq 0$ for almost all $x \in \mathbb{R}^d$;

(W6) $\hat{w}$ is a measurable function and for any neighbourhood $N$ of the origin, $\hat{w} \in L^1(\mathbb{R}^d \setminus N)$;
Armed with axioms (W1) and (W4)–W7 it follows from [4] that
\[ |f|_{m}^{2} = \frac{1}{2} \sum_{|\alpha|=m} c_{\alpha} \int_{\Omega} \int_{\Omega} \hat{w}(x-y)|(D^{\alpha} f)(x)-(D^{\alpha} f)(y)|^{2} \, dx \, dy, \quad f \in Z^{m}(\mathbb{R}^{d}). \] (2.1)

The notation \( C^{m}_{0}(\mathbb{R}^{d}) \) is used for the space of compactly supported \( m \)-times continuously differentiable functions on \( \mathbb{R}^{d} \). Now, let us define the following space for a domain \( \Omega \subset \mathbb{R}^{d} \),
\[ X^{m}(\Omega) := \left\{ f_{|\Omega}: f \in C^{m}_{0}(\mathbb{R}^{d}), |f|_{m,\Omega} < \infty \right\} , \]
where
\[ |f|_{m,\Omega} := \left( \frac{1}{2} \sum_{|\alpha|=m} c_{\alpha} \int_{\Omega} \int_{\Omega} \hat{w}(x-y)|(D^{\alpha} f)(x)-(D^{\alpha} f)(y)|^{2} \, dx \, dy \right)^{1/2}, \quad f \in X^{m}(\Omega). \]

A norm is placed on \( X^{m}(\Omega) \) via
\[ \|f\|_{m,\Omega} := \left( \|f\|_{W^{2m}(\Omega)}^{2} + |f|_{m,\Omega}^{2} \right)^{1/2}, \quad f \in X^{m}(\Omega). \]

The notation \( X^{m}(\Omega) \) denotes the completion of \( X^{m}(\Omega) \) with respect to \( \| \cdot \|_{m,\Omega} \), while \( Y^{m}(\Omega) \) denotes the completion of \( X^{m}(\Omega) \) with respect to \( | \cdot |_{m,\Omega} \). It is these spaces that we call the local native spaces.

We are nearly ready to state our first extension theorem, but first it is necessary to take on board four additional axioms and introduce an important type of bounded domain:

**Definition 2.1.** Let \( \Omega_{1} \) and \( \Omega_{2} \) be domains in \( \mathbb{R}^{d} \), and \( \Phi \) a bijection from \( \Omega_{1} \) to \( \Omega_{2} \). We say that \( \Phi \) is \( m \)-smooth if, writing \( \Phi(x) = (\phi_{1}(x_{1},...,x_{d}),...,\phi_{d}(x_{1},...,x_{d})) \) and \( \Phi^{-1}(x) = \Psi(x) = (\psi_{1}(x_{1},...,x_{d}),...,\psi_{d}(x_{1},...,x_{d})) \), then the functions \( \phi_{1},...,\phi_{d} \) belong to \( C^{m}(\Omega_{1}) \) and \( \psi_{1},...,\psi_{d} \) belong to \( C^{m}(\Omega_{2}) \). Let \( \Phi \) be a bijection from \( \mathbb{R}^{d} \) to \( \mathbb{R}^{d} \). We say \( \Phi \) is locally \( m \)-smooth if \( \Phi \) is \( m \)-smooth on every bounded domain in \( \mathbb{R}^{d} \).

(W8) for every locally \((m+1)\)-smooth map \( \phi \) on \( \mathbb{R}^{d} \), and every bounded subset \( \Omega \) of \( \mathbb{R}^{d} \), there is a \( C_{1} > 0 \) such that \( \hat{w}(\phi(x) - \phi(y)) \leq C_{1} \hat{w}(x-y) \), for all \( x, y \in \Omega \);
(W9) there exists a constant \( C_{2} > 0 \) such that if \( x = (x',x_{d}) \in \mathbb{R}^{d} \) and \( y = (x',y_{d}) \in \mathbb{R}^{d} \) with \( |x_{d}| \geq |y_{d}| \), then \( \hat{w}(x) \leq C_{2} \hat{w}(y) \);
(W10) \( \int_{A} \hat{w} < 0 \) whenever \( A \) has positive measure;
(W11) \( \hat{w}(y) = \hat{w}(-y) \) for all \( y \in \mathbb{R}^{d} \).

**Definition 2.2.** Let \( B = \{(y_{1},y_{2},...,y_{d}) \in \mathbb{R}^{d} : |y_{j}| < 1, 1 \leq j \leq d \} \), and set \( B_{+} = \{ y \in B : y = (y',y_{d}) \text{ and } y_{d} > 0 \} \) and \( B_{0} = \{ y \in B : y = (y',y_{n}) \text{ and } y_{n} = 0 \} \). A bounded convex domain \( \Omega \) in \( \mathbb{R}^{d} \) with boundary \( \partial \Omega \) will be called a \( V \)-domain if the following all hold:
(A1) there exist open sets \( G_1, \ldots, G_N \subset \mathbb{R}^d \) such that \( \partial \Omega \subset \bigcup_{j=1}^N G_j \);

(A2) there exist locally \((m+1)\)-smooth maps \( \phi_j : \mathbb{R}^d \to \mathbb{R}^d \) such that \( \phi_j(B) = G_j \), \( \phi_j(B_+) = G_j \cap \Omega \) and \( \phi_j(B_0) = G_j \cap \partial \Omega \), \( j = 1, \ldots, N \);

(A3) let \( \Omega_\delta \) be the set of all points in \( \Omega \) whose distance from \( \partial \Omega \) is less than \( \delta \). Then for some \( \delta > 0 \),

\[ \Omega_\delta \subset \bigcup_{j=1}^N \phi_j \left( \left\{ (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d : |y_j| < \frac{1}{m+1}, 1 \leq j \leq d \right\} \right). \]

The definition of a V-domain is taken from a paper by Light & Vail [5] in which extension theorems for our local native spaces are considered.

**Theorem 2.3** (Light & Vail [5]). Let \( \Omega \subset \mathbb{R}^d \) be a V-domain. Let \( \hat{w} : \mathbb{R}^d \to \mathbb{R} \) satisfy (W6)–(W11). Then there exists a continuous linear operator \( L : \mathcal{X}^m(\Omega) \to \mathcal{X}^m(\mathbb{R}^d) \) such that for all \( f \in \mathcal{X}^m(\Omega) \),

1. \( Lf = f \) on \( \Omega \);
2. \( \text{supp}(Lf) \) is compact and independent of \( f \);
3. \( \|Lf\|_{m, \mathbb{R}^d} \leq K\|f\|_{m, \Omega} \), for some positive constant \( K = K(\Omega) \) independent of \( f \).

A feature of the construction of the extension operator in Theorem 2.3 is that \( Lf \) can be chosen to be supported on any compact subset of \( \mathbb{R}^d \) containing \( \Omega \). For details of the construction, the reader should consult [5]. Also at our disposal is a seminorm version of Theorem 2.3.

**Theorem 2.4** (Light & Vail [5]). Let \( \Omega \subset \mathbb{R}^d \) be a V-domain. Let \( \hat{w} : \mathbb{R}^d \to \mathbb{R} \) satisfy (W6)–(W11). Given \( f \in \mathcal{Y}^m(\Omega) \), there exists a function \( f^\Omega \in \mathcal{Y}^m(\mathbb{R}^d) \) such that:

1. \( f^\Omega = f \) on \( \Omega \);
2. \( |f^\Omega|_{m, \mathbb{R}^d} \leq C|f|_{m, \Omega} \), for some positive constant \( C = C(\Omega) \) independent of \( f \).

It is convenient for us to be able to work with a norm on \( \mathcal{X}^m(\Omega) \) that is equivalent to \( \| \cdot \|_{m, \Omega} \).

**Lemma 2.5** (Brownlee & Levesley [1]). Let \( \Omega \subset \mathbb{R}^d \) be a V-domain. Let \( w : \mathbb{R}^d \to \mathbb{R} \) satisfy (W0)–(W12) and let \( m + \mu - d/2 > 0 \). Let \( b_1, \ldots, b_\ell \in \Omega \) be unisolvent with respect to \( \Pi_m(\mathbb{R}^d) \). Define a norm on \( \mathcal{X}^m(\Omega) \) via

\[ \|f\|_\Omega := \left( |f|_{m, \Omega}^2 + \sum_{i=1}^\ell |f(b_i)|^2 \right)^{1/2}, \quad f \in \mathcal{X}^m(\Omega). \]

There are positive constants \( K_1 \) and \( K_2 \) such that for all \( f \in \mathcal{X}^m(\Omega) \),

\[ K_1\|f\|_{m, \Omega} \leq \|f\|_\Omega \leq K_2\|f\|_{m, \Omega}. \]
The behaviour of the constant $K(\Omega)$ in the statement of Theorem\textsuperscript{2.3} can be understood for simple choices of $\Omega$. To realise this, we require that the weight function satisfies one further and final axiom:

(W12) there exists $C_1, C_2 > 0$ such that $C_1 h^\lambda \hat{w}(x) \leq \hat{w}(hx) \leq C_2 h^\lambda \hat{w}(x)$, for all $h > 0$, $x \in \mathbb{R}^d$.

Now, an elementary change of variables gives us:

Lemma 2.6. Let $\Omega$ be a measurable subset of $\mathbb{R}^d$. Let $w : \mathbb{R}^d \to \mathbb{R}$ be a measurable function that is nonpositive almost everywhere and satisfies (W11). Define the mapping

\[
\Pi \ni x \mapsto f(x) = a + h(x - t), \text{ where } h > 0, \text{ and } a, t, x \in \mathbb{R}^d.
\]

Then there exists a constant $K_1, K_2 > 0$, independent of $\Omega$, such that for all $f \in \mathcal{X}^m(\sigma(\Omega))$, \[K_1 \leq \frac{|f \circ \sigma|_{m, \Omega}}{h^{m-\lambda/2-d} |f|_{m, \sigma(\Omega)}} \leq K_2.\]

We are now ready to state the key result of this section, but before doing this let us make a simple observation. Look at the unisolvent points $b_1, \ldots, b_\ell$ in the statement of Lemma\textsuperscript{2.5} Since $X^m(\Omega)$ can be embedded in $C(\Omega)$, it makes sense to talk about the interpolation operator $P : X^m(\Omega) \to \Pi_m(\mathbb{R}^d)$ based on these points.

Lemma 2.7. Let $w : \mathbb{R}^d \to \mathbb{R}$ satisfy (W0)-(W12). Let $B$ be any ball of radius $h$ and centre $a \in \mathbb{R}^d$, and let $f \in X^m(B)$. Whenever $b_1, \ldots, b_\ell \in \mathbb{R}^d$ are unisolvent with respect to $\Pi_m(\mathbb{R}^d)$ let $P_\ell : C(\mathbb{R}^d) \to \Pi_m(\mathbb{R}^d)$ be the Lagrange interpolation operator on $b_1, \ldots, b_\ell$. Then there exists $c = (c_1, \ldots, c_\ell) \in B^\ell$ and $g \in X^m(\mathbb{R}^d)$ such that

1. $g(x) = (f - P_\ell f)(x)$ for all $x \in B$;
2. $g(x) = 0$ for all $|x - a| > 2h$;
3. there exists a $C > 0$, independent of $f$ and $B$, such that $|g|_{m, \mathbb{R}^d} \leq C |f|_{m, B}$.

Furthermore, $c_1, \ldots, c_\ell$ can be arranged so that $c_1 = a$.

Proof. Let $B_1$ be the unit ball in $\mathbb{R}^d$ and let $B_2 = 2B_1$. Let $b_1, \ldots, b_\ell \in B_1$ be unisolvent with respect to $\Pi_m(\mathbb{R}^d)$. Define $\sigma(x) = h^{-1}(x - a)$ for all $x \in \mathbb{R}^d$. Set $c_i = \sigma^{-1}(b_i)$ for $i = 1, \ldots, \ell$ so that $c_1, \ldots, c_\ell \in B$ are unisolvent with respect to $\Pi_m(\mathbb{R}^d)$. Take $f \in X^m(B)$. Then $(f - P_\ell f) \circ \sigma^{-1} \in X^m(B_1)$. Set $F = (f - P_\ell f) \circ \sigma^{-1}$. Let $F^{B_1}$ be constructed as an extension to $F$ on $B_1$. By Theorem\textsuperscript{2.3} and the remark following it, we can assume $F^{B_1}$ is supported on $B_2$. Define $g = F^{B_1} \circ \sigma \in X^m(\mathbb{R}^d)$. Let $x \in B$. Since $\sigma(B) = B_1$ there is a $y \in B_1$ such that $x = \sigma^{-1}(y)$. Then,

\[g(x) = (F^{B_1} \circ \sigma)(x) = F^{B_1}(y) = ((f - P_\ell f) \circ \sigma^{-1})(y) = (f - P_\ell f)(x).\]

Also, for $x \in \mathbb{R}^d$ with $|x - a| > 2h$, we have $|\sigma(x)| > 2$. Since $F^{B_1}$ is supported on $B_2$, $g(x) = 0$ for $|x - a| > 2h$. Hence, $g$ satisfies properties 1 and 2. By Theorem\textsuperscript{2.3} there is a $K_1$, independent of $f$ and $B$, such that

\[\|F^{B_1}\|_{m, B_2} \leq \|F^{B_1}\|_{m, \mathbb{R}^d} \leq K_1 \|F\|_{m, B_1}.\]
We have seen in Lemma 2.5 that if we endow \(X^m(B_1)\) and \(X^m(B_2)\) with the norms

\[
\|v\|_{B_i} = \left( |v|^2_{m,B_i} + \sum_{i=1}^\ell |v(b_i)|^2 \right)^{1/2}, \quad i = 1, 2,
\]

then \(\| \cdot \|_{B_1}\) and \(\| \cdot \|_{m,B_2}\) are equivalent for \(i = 1, 2\). Thus, there are constants \(K_2\) and \(K_3\), independent of \(f\) and \(B\), such that

\[
\|F^{B_1}\|_{B_2} \leq K_2\|F^{B_1}\|_{m,B_2} \leq K_1K_2\|F\|_{m,B_1} \leq K_1K_2K_3\|F\|_{B_1}.
\]

Set \(C_1 = K_1K_2K_3\). Since \(F^{B_1}(b_i) = F(b_i) = (f - P_c f)(\sigma^{-1}(b_i)) = (f - P_c f)(c_i) = 0\) for \(i = 1, \ldots, \ell\), it follows that \(\|F^{B_1}\|_{m,B_2} \leq C_1\|F\|_{m,B_1}\). Thus, \(|g \circ \sigma^{-1}|_{m,\mathbb{R}^d} \leq C_1(|f - P_c f) \circ \sigma^{-1}|_{m,B_1}\). Now, Lemma 2.6 can be employed twice to provide us with constants \(C_2\) and \(C_3 > 0\), independent of \(f\) and \(B\), such that

\[
|g|_{m,\mathbb{R}^d} \leq C_2h^{d+\lambda/2-m}|g \circ \sigma^{-1}|_{m,\mathbb{R}^d} \leq C_1C_2h^{d+\lambda/2-m}|f - P_c f|_{m,B_1} \leq C_1C_2C_3|f - P_c f|_{m,B}.
\]

Finally, we observe that \(|f - P_c f|_{m,B} = |f|_{m,B}\) to complete the first part of the proof. The remaining part follows by selecting \(b_1 = 0\) and choosing \(b_2, \ldots, b_\ell\) accordingly in the above construction.

\[\square\]

### 3 Error estimates

In this section we establish the error estimate conjectured in the introduction. We begin with a function \(f\) in \(Z^k(\mathbb{R}^d)\). We want to estimate

\[
\|f - S_m f\|_{L_2(\Omega)}, \quad (3.1)
\]

where \(S_m\) is the minimal norm interpolation operator from \(Z^m(\mathbb{R}^d)\) on \(A\) and \(m > k\).

The essence of the proof is as follows. Firstly, by adjusting \(f\), we obtain a function \(\tilde{f}\), still in \(Z^k(\mathbb{R}^d)\), with seminorm in \(Z^k(\mathbb{R}^d)\) not too far from that of \(f\). We then smooth \(\tilde{f}\) by convolving it with a function \(\phi \in C_0^\infty(\mathbb{R}^d)\). The key feature of the adjustment of \(f\) to \(F := \phi * \tilde{f}\) is that \(F(a) = f(a)\) for every point \(a \in A\) (Theorem 3.4). This enables us to replace \(S_m f\) with \(S_m F\) in (3.1). Furthermore, it follows that \(F \in Z^m(\mathbb{R}^d)\) so we can employ an existing \(L_2\)-error estimate to \(F - S_m F\). The remaining part of the error, \(f - F\), is easily dealt with as it vanishes on \(A\). Finally, Lemma 3.1 takes us back to an error estimate in \(Z^k(\mathbb{R}^d)\).

**Lemma 3.1.** Let \(w : \mathbb{R}^d \to \mathbb{R}\) satisfy (W0) and (W1). Let \(k \leq m\) and \(\phi \in C_0^\infty(\mathbb{R}^d)\). For each \(h > 0\) let \(\phi_h(x) = h^{-d} \phi(x/h)\) for \(x \in \mathbb{R}^d\). Then there exists a constant \(C > 0\), independent of \(h\), such that for all \(f \in Z^k(\mathbb{R}^d)\), \(|\phi_h * f|_{m,\mathbb{R}^d} \leq Ch^{k-m}|f|_{k,\mathbb{R}^d}\).

**Proof.** The case \(w = 1\) is established in [2]. The proof for this more general set up does not differ substantially so is omitted. \[\square\]
Lemma 3.2 (Brownlee & Light [2]). Suppose \( \phi \in C^\infty_0(\mathbb{R}^d) \) is supported on the unit ball and satisfies

\[
\int_{\mathbb{R}^d} \phi(x) \, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} \phi(x) x^\alpha \, dx = 0, \quad \text{for all } 0 < |\alpha| \leq k.
\]

For each \( \varepsilon > 0 \) and \( x \in \mathbb{R}^d \), let \( \phi_\varepsilon(x) = \varepsilon^{-d} \phi(x/\varepsilon) \). Let \( B \) be any ball of radius \( h \) and centre \( a \in \mathbb{R}^d \). For a fixed \( p \in \Pi_k(\mathbb{R}^d) \) let \( f \) be a mapping from \( \mathbb{R}^d \) to \( \mathbb{R} \) such that \( f(x) = p(x) \) for all \( x \in B \). Then \( (\phi_\varepsilon * f)(a) = p(a) \) for all \( \varepsilon \leq h \).

Definition 3.3. Let \( \Omega \) be an open, bounded subset of \( \mathbb{R}^d \). Let \( A \) be a set of points in \( \Omega \). The quantity \( h := \sup_{x \in \Omega} \inf_{a \in A} |x - a| \) is called the fill-distance of \( A \) in \( \Omega \). The separation of \( A \) is given by the quantity \( q := \min_{a, b \in A} \frac{|a - b|}{2} \). The quantity \( h/q \) will be called the mesh-ratio of \( A \).

Theorem 3.4. Let \( w : \mathbb{R}^d \to \mathbb{R} \) satisfy (W0)–(W12). Let \( k + m - d/2 > 0 \) and \( m \geq k \). Let \( A \) be a finite subset of \( \mathbb{R}^d \) of separation \( q > 0 \). Then for all \( f \in \mathcal{X}^k(\mathbb{R}^d) \) there exists an \( F \in \mathcal{X}^m(\mathbb{R}^d) \) such that

1. \( F(a) = f(a) \) for all \( a \in A \);
2. there exists a \( C > 0 \), independent of \( f \) and \( q \), with \( |F|_{k, \mathbb{R}^d} \leq C |f|_{k, \mathbb{R}^d} \) and \( |F|_{m, \mathbb{R}^d} \leq C q^{k-m} |f|_{k, \mathbb{R}^d} \).

Proof. Take \( f \in \mathcal{X}^k(\mathbb{R}^d) \). For each \( a \in A \) let \( B_a \subset \mathbb{R}^d \) denote the ball of radius \( \delta = q/4 \) centred at \( a \). For each \( B_a \) let \( g_a \) be constructed in accordance with Lemma 2.7. That is, for each \( a \in A \) take \( c' = (c_2, \ldots, c_\ell) \in B_a^{\ell-1} \) and \( g_a \in \mathcal{X}^k(\mathbb{R}^d) \) such that

1. \( a, c_2, \ldots, c_\ell \) are unisolvent with respect to \( \Pi_k(\mathbb{R}^d) \)
2. \( g_a(x) = (f - P_{(a,c')})f(x) \) for all \( x \in B_a \);
3. \( P_{(a,c')}f \in \Pi_k(\mathbb{R}^d) \) and \( (P_{(a,c')}f)(a) = f(a) \);
4. \( g_a(x) = 0 \) for all \( |x - a| > 2\delta \);
5. there exists a \( C_1 > 0 \), independent of \( f \) and \( B_a \), such that \( |g_a|_{k, \mathbb{R}^d} \leq C_1 |f|_{k, B_a} \).

Note that if \( a \neq b \), then \( \text{supp} \, (g_a) \) does not intersect \( \text{supp} \, (g_b) \), because if \( x \in \text{supp} \, (g_a) \) then

\[
|x - b| > |b - a| - |x - a| \geq 2q - 2\delta = 6\delta.
\]
Let \( U = \bigcup_{b \in A} \text{supp}(g_b) \), then writing \( \mathbb{R}^d = (\mathbb{R}^d \setminus U) \cup U \) we obtain
\[
\left| \sum_{a \in A} g_{a,k,R^d} \right|^2 = \frac{1}{2} \sum_{|a|=k} c_a \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{w}(x-y) \left| \sum_{a \in A} ((D^a g_a)(x) - (D^a g_a)(y)) \right|^2 \, dx \, dy
\]
\[
= \frac{1}{2} \sum_{|a|=k} c_a \left( \int_{\mathbb{R}^d \setminus U} \int_{\mathbb{R}^d \setminus U} \hat{w}(x-y) \left| \sum_{a \in A} ((D^a g_a)(x) - (D^a g_a)(y)) \right|^2 \, dx \, dy 
+ 2 \int_{\mathbb{R}^d \setminus U} \int_U \hat{w}(x-y) \left| \sum_{a \in A} ((D^a g_a)(x) - (D^a g_a)(y)) \right|^2 \, dx \, dy
\right) + \int_U \int_U \hat{w}(x-y) \left| \sum_{a \in A} ((D^a g_a)(x) - (D^a g_a)(y)) \right|^2 \, dx \, dy.
\]
(3.2)

We shall now consider each of the double integrals in (3.2) separately. Firstly, the integral over \((\mathbb{R}^d \setminus U) \times (\mathbb{R}^d \setminus U)\) is zero because \( \sum_{a \in A} g_a \) is supported on \( U \). Next, using the observation above regarding the support of \( g_a \), \( a \in A \), it follows that
\[
- \frac{1}{2} \sum_{|a|=k} c_a \int_{\mathbb{R}^d \setminus U} \int_U \hat{w}(x-y) \left| \sum_{a \in A} ((D^a g_a)(x) - (D^a g_a)(y)) \right|^2 \, dx \, dy
\]
\[
= \sum_{b \in A} \sum_{|a|=k} c_a \int_{\mathbb{R}^d \setminus U} \int_{\text{supp}(g_b)} \hat{w}(x-y) \left| \sum_{a \in A} (D^a g_a)(x) \right|^2 \, dx \, dy
\]
\[
= \sum_{b \in A} \sum_{|a|=k} c_a \int_{\mathbb{R}^d \setminus U} \int_{\text{supp}(g_b)} \hat{w}(x-y) \left| (D^a g_b)(x) \right|^2 \, dx \, dy
\]
\[
= \sum_{b \in A} \sum_{|a|=k} c_a \int_{\mathbb{R}^d \setminus U} \int_{\text{supp}(g_b)} \hat{w}(x-y) \left| (D^a g_b)(x) - (D^a g_b)(y) \right|^2 \, dx \, dy
\]
\[
\leq \sum_{b \in A} g_b^2_{k,R^d},
\]
(3.3)

Before calculating the final integral let us examine the following expression for \( b \in A \) and \( \alpha \in \mathbb{Z}^d_+ \) with \( |\alpha| = m \),
\[
\sum_{c \in A} \int_{\text{supp}(g_c)} \int_{\text{supp}(g_b)} \hat{w}(x-y) \left| (D^{\alpha} g_b)(x) - (D^{\alpha} g_c)(y) \right|^2 \, dx \, dy
\]
\[
= \sum_{c \in A} \int_{\text{supp}(g_c)} \int_{\text{supp}(g_b)} \hat{w}(x-y) \left| (D^{\alpha} g_b)(x) - (D^{\alpha} g_b)(y) + (D^{\alpha} g_c)(x) - (D^{\alpha} g_c)(y) \right|^2 \, dx \, dy
\]
\[
\leq 2 \sum_{c \in A} \int_{\text{supp}(g_c)} \int_{\text{supp}(g_b)} \hat{w}(x-y) \left| (D^{\alpha} g_b)(x) - (D^{\alpha} g_b)(y) \right|^2 \, dx \, dy
\]
\[
+ 2 \sum_{c \in A} \int_{\text{supp}(g_c)} \int_{\text{supp}(g_b)} \hat{w}(x-y) \left| (D^{\alpha} g_c)(x) - (D^{\alpha} g_c)(y) \right|^2 \, dx \, dy
\]
9
Finally, using the observation regarding the support of $g$, one again and (3.4) it follows that

$$-\frac{1}{2} \sum_{b \in A} \sum_{|\alpha| = k} c_\alpha \int_U \int_U \hat{w}(x-y) \left| \sum_{a \in A} ((D^\alpha g_a)(x) - (D^\alpha g_a)(y)) \right|^2 \, dx \, dy$$

$$= \sum_{b \in A} \sum_{|\alpha| = k} c_\alpha \int_{\text{supp}(g_a)} \int_{\text{supp}(g_b)} \hat{w}(x-y) \left| \sum_{a \in A} ((D^\alpha g_a)(x) - (D^\alpha g_a)(y)) \right|^2 \, dx \, dy$$

$$= \sum_{b \in A} \sum_{|\alpha| = k} c_\alpha \int_{\text{supp}(g_a)} \int_{\text{supp}(g_b)} \hat{w}(x-y)((D^\alpha g_b)(x) - (D^\alpha g_b)(y))^2 \, dx \, dy$$

$$+ \sum_{b \in A} \sum_{|\alpha| = k} \frac{1}{2} \sum_{c \in A} c_\alpha \int_{\text{supp}(g_c)} \int_{\text{supp}(g_b)} \hat{w}(x-y)((D^\alpha g_b)(x) - (D^\alpha g_c)(y))^2 \, dx \, dy$$

$$\leq \sum_{b \in A} |g_b|^2_{k, \mathbb{R}^d} + 2 \sum_{b \in A} |g_b|^2_{k, \mathbb{R}^d} + 2 \sum_{c \in A} |g_c|^2_{k, \mathbb{R}^d}$$

$$\leq 5 \sum_{b \in A} |g_b|^2_{k, \mathbb{R}^d}. \quad (3.5)$$

Substituting (3.3) and (3.5) into (3.2) we find

$$\left| \sum_{a \in A} g_a \right|_{k, \mathbb{R}^d}^2 \leq \sum_{a \in A} |g_a|_{k, \mathbb{R}^d}^2.$$ 

Hence, applying Condition 5 to the above inequality we have

$$\left| \sum_{a \in A} g_a \right|_{k, \mathbb{R}^d}^2 \leq 7 \sum_{a \in A} |g_a|_{k, \mathbb{R}^d}^2.$$ 

Now set $H = f - \sum_{a \in A} g_a$. It then follows from Condition 1 that $H(x) = (P_{a,c}) f(x)$ for all $x \in B_a$, and from Condition 3 that $H(a) = f(a)$ for all $a \in A$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be supported on the unit ball and enjoy the properties

$$\int_{\mathbb{R}^d} \phi(x) \, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} \phi(x) x^\alpha \, dx = 0, \quad \text{for all } 0 < |\alpha| \leq k.$$ 

Now set $F = \phi \ast H$. Using Lemma 3.1 there is a constant $C_2 > 0$, independent of $q$ and
for some appropriate constant $C > 0$, independent of $q$ and $f$, such that
\[
|F|_{m, \mathbb{R}^d}^2 \leq C_2 \delta^{2(k-m)} \left| f - \sum_{a \in A} g_a \right|_{k, \mathbb{R}^d}^2 \leq 2C_2 \delta^{2(k-m)} \left( |f|_{k, \mathbb{R}^d}^2 + \left| \sum_{a \in A} g_a \right|_{k, \mathbb{R}^d}^2 \right) 
\leq 2C_2 (1 + 7C_1^2) \delta^{2(k-m)} |f|_{k, \mathbb{R}^d}^2.
\]

Similarly, there is a constant $C_3 > 0$, independent of $q$ and $f$, such that
\[
|F|_{k, \mathbb{R}^d}^2 \leq C_3 \left| f - \sum_{a \in A} g_a \right|_{k, \mathbb{R}^d}^2 \leq 2C_3 (1 + 7C_1^2) |f|_{k, \mathbb{R}^d}^2.
\]

Thus $|F|_{m, \mathbb{R}^d} \leq Cq^{k-m} |f|_{k, \mathbb{R}^d}$ and $|F|_{k, \mathbb{R}^d} \leq C|f|_{k, \mathbb{R}^d}$ for some appropriate constant $C > 0$. Since $F = \phi \ast H$ and $H|_{B_\rho} \in \Pi_k(\mathbb{R}^d)$ for each $a \in A$, it follows from Lemma 3.2 that $F(a) = H(a) = f(a)$ for all $a \in A$.

**Theorem 3.5.** Let $\Omega \subset \mathbb{R}^d$ be a $V$-domain and let $w : \mathbb{R}^d \to \mathbb{R}$ be a measurable function satisfying (W0)–(W12). Let $k + \mu - d/2 > 0$ and $m \geq k$. For each $h > 0$, let $A_h$ be a finite, $\Pi_m(\mathbb{R}^d)$-unisolvent subset of $\Omega$ with fill-distance $h$. Assume also that there is a quantity $\rho > 0$ such that the mesh-ratio of each $A_h$ is bounded by $\rho$ for all $h > 0$. For each mapping $f : A_h \to \mathbb{R}$, let $S^h_m f$ be the minimal norm interpolant to $f$ on $A_h$ from $Z^m(\mathbb{R}^d)$. Then there exists a constant $C > 0$, independent of $h$, such that for all $f \in \mathcal{Y}^k(\Omega)$,
\[
\|f - S^h_m f\|_{L_2(\Omega)} \leq C h^{k-\lambda/2-d/2} \|f\|_{k, \Omega}, \quad \text{as } h \to 0.
\]

**Proof.** Take $f \in \mathcal{X}^k(\mathbb{R}^d)$. Construct $F$ in accordance with Theorem 3.4 and set $G = f - F$. Then $F(a) = f(a)$ and $G(a) = 0$ for all $a \in A_h$. Furthermore, there is a constant $C_1 > 0$, independent of $f$ and $h$, such that
\[
|F|_{m, \mathbb{R}^d} \leq C_1 \left( \frac{h}{\rho} \right)^k |f|_{k, \mathbb{R}^d}, \quad |G|_{k, \mathbb{R}^d} \leq |f|_{k, \mathbb{R}^d} + |F|_{k, \mathbb{R}^d} \leq (1 + C_1) |f|_{k, \mathbb{R}^d}. \quad (3.6)
\]

Thus $S^h_m f = S^h_m F$ and $S^h_k G = 0$, where we have adopted the obvious notation for $S^h_k$. Hence,
\[
\|f - S^h_m f\|_{L_2(\Omega)} = \|(F + G) - S^h_m F\|_{L_2(\Omega)} \leq \|F - S^h_m F\|_{L_2(\Omega)} + \|G - S^h_k G\|_{L_2(\Omega)}.
\]

Now, employing the error estimate in [1], there are positive constants $C_2 > 0$ and $C_3 > 0$, independent of $h$ and $f$, such that
\[
\|f - S^h_m f\|_{L_2(\Omega)} \leq C_2 h^{m-\lambda/2-d/2} |F|_{m, \Omega} + C_3 h^{k-\lambda/2-d/2} |G|_{k, \Omega}, \quad \text{as } h \to 0.
\]

Finally, using the bounds in (3.6) we have
\[
\|f - S^h_m f\|_{L_2(\Omega)} \leq C_4 h^{k-\lambda/2-d/2} |f|_{k, \mathbb{R}^d}, \quad \text{as } h \to 0, \quad (3.7)
\]
for some appropriate $C_4 > 0$. In particular, (3.7) holds for all $f \in \mathcal{X}^k(\mathbb{R}^d)$. As $\mathcal{Y}^k(\mathbb{R}^d)$ is a dense linear subspace of $\mathcal{X}^k(\mathbb{R}^d)$ then (3.7) extends to hold for all $f \in \mathcal{Y}^k(\mathbb{R}^d)$ using a standard normed space argument [3, Page 180]. To complete the proof we now let $f \in \mathcal{Y}^k(\Omega)$ and define $f^\Omega$ in accordance with Theorem 2.4. It follows that there is a $C_5 > 0$ such that
\[
\|f - S^h_m f\|_{L_2(\Omega)} \leq C_4 h^{k-\lambda/2-d/2} |f^\Omega|_{k, \mathbb{R}^d} \leq C_4 C_5 h^{k-\lambda/2-d/2} |f|_{k, \Omega}, \quad \text{as } h \to 0. \quad \Box
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References

[1] R. Brownlee and J. Levesley. A scale of improved error estimates for radial approximation in euclidean space and on spheres. Submitted to *Math. Comp.*, 2004.

[2] R. Brownlee and W. Light. Approximation orders for interpolation by surface splines to rough functions. *IMA J. Numer. Anal.*, 24(2):179–192, 2004.

[3] G. J. O. Jameson. *Topology and Normed Spaces*. Chapman and Hall, London, UK, 1974.

[4] J. Levesley and W. A. Light. Direct form seminorms arising in the theory of interpolation by translates of a basis function. *Adv. Comput. Math.*, 11(2-3):161–182, 1999.

[5] W. A. Light and M. L. Vail. Extension theorems for spaces arising from approximation by translates of a basic function. *J. Approx. Theory*, 114(2):164–200, 2002.

[6] W. A. Light and H. Wayne. Spaces of distributions, interpolation by translates of a basis function and error estimates. *Numer. Math.*, 81(3):415–450, 1999.

[7] F. J. Narcowich and J. D. Ward. Scattered-data interpolation on $\mathbb{R}^n$: Error estimates for radial basis and band-limited functions. Preprint, 2002.

[8] F. J. Narcowich, J. D. Ward, and H. Wendland. Sobolev bounds on functions with scattered zeros with application to radial basis function surface fitting. Preprint, 2003.

[9] J. Yoon. Interpolation by radial basis functions on Sobolev space. *J. Approx. Theory*, 112(1):1–15, 2001.

[10] J. Yoon. $L_p$-error estimates for “shifted” surface spline interpolation on Sobolev space. *Math. Comp.*, 72(243):1349–1367 (electronic), 2003.