Gravitational field around a screwed superconducting cosmic string in scalar-tensor theories

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Abstract

We obtain the solution that corresponds to a screwed superconducting cosmic string (SSCS) in the framework of a general scalar-tensor theory including torsion. We investigate the metric of the SSCS in Brans-Dicke theory with torsion and analyze the case without torsion. We show that in the case with torsion the space-time background presents other properties different from that in which torsion is absent. When the spin vanish, this torsion is a $\phi$-gradient and then it propagates outside of the string. We investigate the effect of torsion on the gravitational force and on the geodesics of a test-particle moving around the SSCS. The accretion of matter by wakes formation when a SSCS moves with speed $v$ is investigated. We compare our results with those obtained for cosmic strings in the framework of scalar-tensor theory.

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1 Introduction

Topological defects like cosmic strings\cite{1, 2, 3, 4} have been studied in different contexts\cite{5} like, for example, to understand the primordial Universe and the mechanism of structure formation \cite{6, 7, 8, 9, 10} in very early eras.

Cosmic string presents superconducting properties \cite{11} and in this case, it may behaves like both bosonic (see Ref.\cite{12} and references therein) and fermionic strings \cite{13, 14}. The superconductivity is supposed to be relevant during or very soon after the phase transition in which the string was formed.

It has been argued that gravity may be described by a scalar-tensorial gravitational field, at least at sufficiently high energy scales. From the theoretical point of view, scalar-tensor theories of gravitation, in which gravity is mediated by one or several long-range scalar field in addition to the usual tensor field present in Einstein’s theory, are the most natural alternatives to general relativity. In these theories the gravitational interaction is mediated by a (spin-2) graviton and by a (spin-0) scalar field \cite{15, 16}.

There are compelling suggestions from astrophysical observations that Einstein’s original description of gravity may require the inclusion of hitherto undetected fields of either gravitational or matter fields at least at the first moments of the Universe. As example, we can mention that the dilaton fields were, certainly, very relevant in primordial Universe in comparison with gravitational fields but nowadays the scalar contributions is small.

On the other hand, torsion fields were analyzed in the geometry of a cosmic string whose presence could have been influenced the formation and evolution of structures in the Universe \cite{17}. Other effect of torsion corresponds to the contribution to neutrino oscillations \cite{18}. Due to the role played by the torsion several authors have already discussed that torsion may have been an important element in the early Universe, when the quantum effects of gravitation were drastically important\cite{19, 20}. Also the torsion is important from the phenomenological point of view and it may be relevant in cosmology. This importance is associated with the modifications of
kinematic quantities, like shear, vorticity, acceleration, expansion and their evolution equations due to the presence of torsion \[21, 22, 23, 24, 25\].

Therefore, from the previous considerations concerning the importance of scalar-tensor theory of gravity and of the theory of gravity which takes into account the torsion, we will investigate the superconducting cosmic string in the framework of a general scalar-tensor theory with torsion in which the presence of a dilaton field and torsion are present and is supposed to persist from the period of formation of the string on. We study the formation of the cosmic string wakes in this context and we analyze the contribution of the current carried by the string. Our main purpose is to study how the cosmological effects of long strings are affected by torsion and scalar fields in the background generated by scalar-tensor gravities with torsion as compared with general relativity. We show that the cosmic string wakes moving in space-time with torsion present effects similar to the wiggly cosmic string, assuming, of course, the validity of the Pogosian and Vachaspati conclusion \[26\].

To incorporate the Pogosian and Vachaspati statement \[26\], we propose that their straight strings with small-scale structures (wiggles) may resemble the strings endowed with torsion in our picture (screwed strings). In so doing, we postulate that the small-scale structures existing in wiggly strings can be approximately scaled to the geometrical deformation that torsion produces on ordinary strings. This premise leads us to the idea that the primordial spectrum of perturbations in the CMBR, as observed by COBE, may reasonably be reproduced if one uses the freedom in the parameter-space of numerical models of structure formation based on wiggly strings.

The wakes produced by "wiggly" cosmic strings can result in an efficient process of formation of large scale structure and affect the microwave-background isotropy \[27\]. In the case of pure scalar-tensor gravity \[28\], the cosmic string wakes present very similar structure to the wiggly cosmic string in general relativity. In the framework of scalar-tensor theory which we are considering, we will show that the presence of the torsion amplify this effect.

The shape of the matter (radiation) power spectrum can be obtained by fol-
Following the evolution of a network of long ordinary straight strings interacting with the universe matter (radiation) content. A string evolves in such a way that its characteristic curvature radius at time $t$ is $\sim t$ (see Ref. [8], and references therein). Each string moves with typical speed $v \sim 1$. The translational motion of a string creates a wedge-shaped wake behind it with a deficit angle $8\pi G\mu$, assuming that $4\pi G\mu v$ is greater than the thermal velocity in the network. Under these conditions the density contrast is $\delta \varepsilon / \varepsilon \sim 1$, while the wake typical length scale and mass are $\sim t$ and $M_w \sim 8\pi G\mu t^3 \delta \varepsilon \sim \mu t$, respectively. Once the wake forms particles fall into it with transversal velocity $v_t \sim 4\pi G\mu$, and acceleration $a_w \sim 2\pi GM_w / t^2$. Collisionless particles travel a distance shorter than the wake width: $v_t^2 / 2a_w \sim 4\pi G\mu t$. In order to be trapped by the wake potential well, baryons must deposit their energy via a shock. Gravitational instability will force wakes to grow up to reach masses $\sim 10^{16} M_\odot$ and length $L_{sc} \sim 20h^{-1} \text{Mpc}$, characteristic of observed superclusters of galaxies.

A short description of scalar-tensor theories with torsion fields is presented in section 2, where the torsion is considered in a general form. In section 3 we analyze the specific case where the torsion is a gradient of the scalar field and we obtain the superconducting cosmic string solution in general scalar-tensor theories with torsion. Sections 4 and 5 are devoted to some applications and in section 6 we end up with some conclusions.

### 2 Scalar-tensor theory with torsion

The scalar-tensor theory with torsion is an extension of Einstein’s general relativity in which a scalar field is coupled minimally to the gravitational field and a dynamical torsion term is considered additionally. This coupling is referred to the Jordan-Fierz frame, where the action takes the form

$$ I = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ \tilde{\phi} \tilde{R} - \frac{\omega(\tilde{\phi})}{\tilde{\phi}} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \right] + I_m (e^\mu_a, \Omega^\mu_{ab}, \Psi), \quad (2.1) $$


where \( I_m(e_a^\mu, \Omega_{ab}^\mu, \Psi) \) is the action of the matter which in the general case takes into account all fields. The function \( \omega \) in scalar-tensor theory has a \( \tilde{\phi} \) dependence but in Brans-Dicke theory it is a constant.

The scalar curvature \( \tilde{R} \) is a function of the vierbeins \( e_a^\mu \) and spin-connections \( \Omega_{ab}^\mu \) [29]. In Einstein-Cartan (EC) theory, \( \tilde{R} \) is given by

\[
\tilde{R} = e_a^\mu e^\nu_b (\Omega_{\mu,\nu}^{ab} - \Omega_{\nu,\mu}^{ab} + \Omega_a^c \Omega_{c\nu}^{b\mu} - \Omega_a^c \Omega_{c\mu}^{b\nu})
\]

(2.2)

where the latin indices \( a, b \) are flat type and the greek ones \( \mu, \nu \) are world type.

As this point we write the torsion in general form given by

\[
S_a^{\mu\nu} = e_a^{\mu,\nu} - e_a^{\nu,\mu} + \Omega_a^c e_{\nu}^c - \Omega_a^c e_{\mu}^c,
\]

(2.3)

and the field equation for spin connection \( \Omega^a_{\mu,ab} \) as

\[
\tilde{\phi}(S_{\alpha\beta}^\mu + \delta_{\beta}^\mu S_{\lambda\alpha}^\lambda - \delta_{\alpha}^\mu S_{\lambda\beta}^\lambda) = 8\pi \sigma_{\alpha\beta}^\mu + \frac{1}{2}(\delta_{\alpha}^\mu \phi_{\beta} - \delta_{\beta}^\mu \phi_{\alpha}),
\]

(2.4)

where \( \sigma_{\alpha\beta}^\mu \) is the spin angular momentum tensor defined by

\[
\sigma_{\alpha\beta}^\mu = \frac{1}{\sqrt{-g}} (e_a^a e_{\beta}^b - e_a^a e_{\alpha}^b) \delta_{\beta}^\mu (\delta_{\alpha}^\beta \Omega_{ab}^\mu) (\sqrt{-g I_m}).
\]

(2.5)

The contraction of the \( \beta \) and \( \mu \) indices leads to

\[
\tilde{\phi} S_{\alpha\beta}^\mu = 8\pi \Sigma_{\alpha\beta}^\mu + \frac{1}{2}(\delta_{\alpha}^\mu \phi_{\beta} - \delta_{\beta}^\mu \phi_{\alpha}),
\]

(2.6)

where \( \Sigma_{\alpha\beta}^\mu \) is given by

\[
\Sigma_{\alpha\beta}^\mu = \sigma_{\alpha\beta}^\mu + \frac{1}{2}(\delta_{\alpha}^\mu \sigma_{\beta}^\lambda - \delta_{\beta}^\mu \sigma_{\alpha}^\lambda).
\]

(2.7)

It is worth to call attention to the fact that in the scalar curvature \( \tilde{R} \), the scalar function \( \tilde{\phi} \) can act as source of the torsion field. Thus, in the absence of spins, the torsion field may be generated by a non-spin term, the gradient of the scalar field and therefore, in the absence of spin the torsion does not vanish. Therefore, it can propagates with the scalar field and thus we can write the torsion as
\[ S_{\mu\nu}^\lambda = \left( \delta_\mu^\lambda \partial_\nu \tilde{\phi} - \delta_\nu^\lambda \partial_\mu \tilde{\phi} \right) / 2 \tilde{\phi}. \] (2.8)

The most general affine connection \( \Gamma_{\lambda\nu}^\alpha \) written in terms of the contortion tensor \( K_{\lambda\nu}^\alpha \) is

\[ \Gamma_{\lambda\nu}^\alpha = \{^\alpha_{\lambda\nu}\} + K_{\lambda\nu}^\alpha, \] (2.9)

where the quantity \( \{^\alpha_{\lambda\nu}\} \) is the Christoffel symbol computed from the metric tensor \( g_{\mu\nu} \) and the contortion tensor \( K_{\lambda\nu}^\alpha \) can be written in terms of the torsion field as:

\[ K_{\lambda\nu}^\alpha = -\frac{1}{2}(S_{\lambda\nu}^\alpha + S_{\nu\lambda}^\alpha - S_{\lambda\nu}^\alpha) \] (2.10)

In this case the scalar curvature \( \tilde{R} \) given by (2.1) in the Jordan-Fierz frame can be written as

\[ \tilde{R} = \tilde{R}(\{\}) + \epsilon \frac{\partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi}}{\phi^2}, \] (2.11)

where \( \tilde{R}(\{\}) \) is the Riemann scalar curvature in the Jordan-Fierz frame and \( \epsilon \) is the torsion coupling constant [17].

Although action (2.1) shows explicitly this scalar-tensor gravity’s character, we will adopt, for technical reasons, to work in the conformal (Einstein) frame in which the kinematic terms of the scalar and the tensor fields do not mix and the action is given by

\[ I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [R - 2g^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi}] + I_m[\Psi_m, \Lambda^2(\phi) g_{\mu\nu}], \] (2.12)

where \( g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab} \) is a pure rank-2 tensor in the Einstein frame and \( R \) is the curvature scalar such that

\[ R = R(\{\}) + 4\epsilon \alpha(\phi)^2 \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \] (2.13)

It is interesting to note that action (2.12) is obtained from (2.1) by a conformal transformation.
\[ \tilde{g}_{\mu\nu} = \Lambda^2(\phi)g_{\mu\nu}, \]  

and by a redefinition of the quantity

\[ G\Lambda^2(\phi) = \tilde{\phi}^{-1} \]

which shows up that any gravitational phenomena will be affected by the variation of the gravitation “constant” \( G \) in the scalar-tensorial gravity, and finally, by introducing a new parameter

\[ \alpha^2 \equiv \left( \frac{\partial \ln \Lambda(\phi)}{\partial \phi} \right)^2 = [2\omega(\tilde{\phi}) + 3]^{-1}, \]

which can be interpreted as the (field-dependent) coupling strength between matter and the scalar field.

In order to make our calculations as general as possible, we choose not to specify the factors \( \Lambda(\phi) \) and \( \alpha(\phi) \) (the field-dependent coupling strength between matter and the scalar field), leaving them as arbitrary functions of the scalar field.

In the conformal frame, the Einstein equations are modified. A straightforward calculation shows that they turn into

\[ R_{\mu\nu} = 2\kappa \partial_\mu \phi \partial_\nu \phi + 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T). \]

\[ G_{\mu\nu} = 2\kappa \partial_\mu \phi \partial_\nu \phi - \kappa g_{\mu\nu}g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + 8\pi G T_{\mu\nu}, \]

where \( \kappa \) is \( \phi \)-function defined by

\[ \kappa(\phi) = 1 - 2\epsilon\alpha(\phi)^2 \]

which has two contributions: one given by scalar-tensor term and the other by the torsion. We note that the last equation brings a new information and shows that the matter distribution behaves as a source for \( \phi \) and \( g_{\mu\nu} \) as well. The energy-momentum tensor is defined as usual.
\[
T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta I_m}{\delta g_{\mu\nu}},
\]  
(2.18)

but in the conformal frame it is no longer conserved \(\nabla_\mu T^\mu_\nu = \alpha(\phi) T_{\nu}T^\mu\). It is clear from transformation (2.14) that we can relate quantities from both frames in such a way that \(\hat{T}^{\mu\nu} = \Lambda^{-6}(\phi) T^{\mu\nu}\) and \(\hat{T}_\nu^\mu = \Lambda^{-4}(\phi) T_\nu^\mu\).

In the scalar-tensor theory, the Einstein equations are modified by the presence of the field \(\phi\) and are obtained by applying the variational principle to (2.12) with \(R\) given by (2.13). The equation for \(\phi\) reads as follows:

\[
\Box_g \phi = -4\pi G\alpha(\phi)T,
\]  
(2.19)

where

\[
\Box_g \phi = \frac{1}{\sqrt{-g}} \partial_\mu \left[ \sqrt{-g} \partial^\mu \phi \right].
\]  
(2.20)

It brings some new information because it doesn’t appear in general relativity, and shows us that a matter distribution in the space behaves like a source for \(\phi\), and, as usual, for \(g_{\mu\nu}\) as well. Up to now, we have dealt with the purely gravitational sector, but in what follows, we will introduce the action for the matter that describes a cosmic string.

### 3 Superconducting cosmic string in scalar-tensor theory

In order to describe the simplest superconducting cosmic string in a scalar-tensor theory, we require the matter action to carry a pair of complex scalar and gauge fields, in an Abelian Higgs model whose action is given by

\[
I_m = \int d^4x \sqrt{\tilde{g}} \left[ -\frac{1}{2} D_\mu \Phi(D^\mu \Phi)^* - \frac{1}{2} D_\mu \Sigma(D^\mu \Sigma)^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} H_{\mu\nu} H^{\mu\nu} - V(|\Phi|, |\Sigma|) \right],
\]  
(3.1)
where $D_\mu \Sigma = (\partial_\mu + ieA_\mu)\Sigma$ and $D_\mu \Phi = (\partial_\mu + iqC_\mu)\Phi$ are the covariant derivatives. The reason why the gauge fields do not minimally couple to torsion is well discussed in Refs.\,[16, 30]. The field strengths are defined as usually as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $H_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$, with $A_\mu$ and $C_\nu$ being the gauge fields and $V(|\Phi|, |\Sigma|)$. This action given by Eq.(3.1) has a $U(1) \times U(1)'$ symmetry, where the $U(1)'$ group, associated with the $\varphi$-field, is broken by the vacuum and gives rise to vortices of the Nielsen-Olesen type\,[31]

$$
\Phi = \varphi(r)e^{i\theta},
$$

$$
C_\mu = \frac{1}{\eta}[P(r) - 1]\delta_\mu^\theta,
$$

in which $(t, r, \theta, z)$ are usual cylindrical coordinates. The boundary conditions for the fields $\varphi(r)$ and $P(r)$ are the same as those of ordinary cosmic strings\,[31], namely

$$
\varphi(r) = \eta \quad r \to \infty \quad P(r) = 0 \quad r \to \infty
\quad \varphi(r) = 0 \quad r = 0 \quad P(r) = 1 \quad r = 0.
$$

The other $U(1)$-symmetry, that we associate with electromagnetism, acts on the $\Sigma$-field. This symmetry is not broken by the vacuum; however, it is broken in the interior of the defect. The $\Sigma$-field in the string core, where it acquires an expectation value, is responsible for a bosonic current being carried by the gauge field $A_\mu$. The only non-vanishing components of the gauge fields are $A_z(r)$ and $A_t(r)$, and the current-carrier phase may be expressed as $\zeta(z, t) = \omega_1 t - \omega_2 z$. Notwithstanding, we focus only on the magnetic case\,[12]. Their configurations are defined as:

$$
\Sigma = \sigma(r)e^{i\zeta(z, t)},
$$

$$
A_\mu = \frac{1}{\epsilon}[A(r) - \frac{\partial \zeta(z, t)}{\partial z}]\delta_\mu^z,
$$

because of the rotational symmetry of the string itself. The fields responsible for the cosmic string superconductivity have the following boundary conditions

$$
\frac{d}{dr}\sigma(r) = 0 \quad r = 0 \quad A(r) \neq 0 \quad r \to \infty
\quad \sigma(r) = 0 \quad r \to \infty \quad A(r) = 1 \quad r = 0.
$$
The potential $V(\varphi, \sigma)$ triggering the spontaneous symmetry breaking can be fixed by:

$$V(\varphi, \sigma) = \frac{\lambda_\varphi}{4} (\varphi^2 - \eta^2)^2 + f_{\varphi\sigma} \varphi^2 \sigma^2 + \frac{\lambda_\sigma}{4} \sigma^4 - \frac{m_\sigma^2}{2} \sigma^2,$$  \hspace{1cm} (3.6)

where $\lambda_\varphi$, $\lambda_\sigma$, $f_{\varphi\sigma}$ and $m_\sigma$ are coupling constants. Constructed in this way, this potential possesses all the ingredients that make it viable to generate the formation of a superconducting cosmic string, as it is well-established. In addition, it is extended to include a new term describing the interaction with the torsion field. The presence of this interaction term does not affect the occurrence of the string ground states. However, it adds with a torsion density in the string core due to the coupling with the charged particle flux.

Let us consider a cosmic string in a cylindrical coordinate system, $(t, r, \theta, z)$, $(r \geq 0$ and $0 \leq \theta < 2\pi)$ with the metric defined in Einstein frame as

$$ds^2 = e^{2(\gamma - \psi)}(-dt^2 + dr^2) + \beta^2 e^{-2\psi} d\theta^2 + e^{2\psi} dz^2,$$  \hspace{1cm} (3.7)

where $\gamma$, $\psi$ and $\beta$ depend only on $r$. We can find the relations between the parameters of the metric through Einstein equations as given by Eq (2.16). Then, in the space-time with the metric defined in Eq (3.7), these equations are

$$\beta'' = 8\pi G \beta (T_t^t + T_r^r) e^{2(\gamma - \psi)}$$

$$\left(\beta \gamma'\right)' = 8\pi G \beta (T_r^r + T_\theta^\theta) e^{2(\gamma - \psi)},$$  \hspace{1cm} (3.8)

$$\left(\beta \psi'\right)' = 4\pi G \beta (T_t^t + T_r^r + T_\theta^\theta - T_z^z) e^{2(\gamma - \psi)},$$

and the $\phi$ - equation given by (2.19) is

$$\left(\beta \kappa \phi'\right)' = 4\pi G \beta T_\alpha(\phi) e^{2(\gamma - \psi)},$$  \hspace{1cm} (3.9)
where (') denotes “derivative with respect to r”.

The non-vanishing components of the energy-momentum tensor are

\[
T^t_t = -\frac{1}{2} \Lambda^2(\phi) \{ e^{2(\Psi - \gamma)}(\varphi'^2 + \sigma'^2) + \frac{e^{2\Psi}}{\beta^2} \varphi'^2 P^2 + e^{-2\Psi} \sigma'^2 A^2 \\
+ \Lambda^{-2}(\phi)e^{-2\gamma}(\frac{A'^2}{4\pi e^2}) + \Lambda^{-2}(\phi) \frac{e^{2(\Psi - \gamma)}}{\beta^2} \frac{P'^2}{4\pi q^2} + 2\Lambda^2(\phi) V(\varphi, \sigma) \}
\]

\[
T^r_r = \frac{1}{2} \Lambda^2(\phi) \{ e^{2(\Psi - \gamma)}(\varphi'^2 + \sigma'^2) - \frac{e^{2\Psi}}{\beta^2} \varphi'^2 P^2 - e^{-2\Psi} \sigma'^2 A^2 \\
+ \Lambda^{-2}(\phi)e^{-2\gamma}(\frac{A'^2}{4\pi e^2}) - \Lambda^{-2}(\phi) \frac{e^{2(\Psi - \gamma)}}{\beta^2} \frac{P'^2}{4\pi q^2} - 2\Lambda^2(\phi) V(\varphi, \sigma) \}
\]

\[
T^\theta_\theta = -\frac{1}{2} \Lambda^2(\phi) \{ e^{2(\Psi - \gamma)}(\varphi'^2 + \sigma'^2) - \frac{e^{2\Psi}}{\beta^2} \varphi'^2 P^2 + e^{-2\Psi} \sigma'^2 A^2 \\
+ \Lambda^{-2}(\phi)e^{-2\gamma}(\frac{A'^2}{4\pi e^2}) - \Lambda^{-2}(\phi) \frac{e^{2(\Psi - \gamma)}}{\beta^2} \frac{P'^2}{4\pi q^2} + 2\Lambda^2(\phi) V(\varphi, \sigma) \}
\]

\[
T^z_z = -\frac{1}{2} \Lambda^2(\phi) \{ e^{2(\Psi - \gamma)}(\varphi'^2 + \sigma'^2) + \frac{e^{2\Psi}}{\beta^2} \varphi'^2 P^2 - e^{-2\Psi} \sigma'^2 A^2 \\
- \Lambda^{-2}(\phi)e^{-2\gamma}(\frac{A'^2}{4\pi e^2}) + \Lambda^{-2}(\phi) \frac{e^{2(\Psi - \gamma)}}{\beta^2} \frac{P'^2}{4\pi q^2} + 2\Lambda^2(\phi) V(\varphi, \sigma) \}
\]

The external solutions of the Eq.(3.8) have the same form of the scalar tensor theory [32], but the \(\phi\)-solution is different and comes from

\[
\phi' = \kappa^{-1} \lambda \frac{1}{r}.
\]

This implies that

\[
R = 2(\psi'' + \frac{1}{\beta} \psi' - \phi'^2 + \frac{m}{\beta^2}) e^{2(\Psi - \gamma)} = 2\phi'^2 e^{2(\psi - \gamma)}
\]

is different from the result obtained in the framework of pure scalar-tensor theories of gravity [32].

Let us make an estimation of the order of magnitude of the correction induced by \(\kappa^{-1} \lambda\). It is very illustrative to consider a particular form for the arbitrary function
Λ(φ), corresponding to the Brans-Dick theory, Λ = e^{αφ}, with α^2 = \frac{1}{2w+3}, (w=cte).

Thus, for this case,

$$\phi' = \lambda \frac{(w + \frac{3}{2})}{(\bar{w} + \frac{3}{2})r}$$

(3.13)

where \(\bar{w} = \omega - \epsilon\). Using the values for \(w\) such that \(w > 2500\) (consistent with solar system experiments made by Very Baseline Interferometry (VLBI) [33], we have that \(\omega >>> \epsilon\) and thus \(\lambda \frac{(w + \frac{3}{2})}{(\bar{w} + \frac{3}{2})} \sim \lambda\). Therefore, the external solution in the Brans-Dicke theory in this limit is the same as in the case of the superconducting cosmic string in scalar-tensor theory [32].

The external metric for the SSCS takes, thus, the form

$$ds^2 = \left(\frac{r}{r_0}\right)^{-2n} W^2(r) \left[ \left(\frac{r}{r_0}\right)^{2m^2} (-dt^2 + dr^2) + B^2 r^2 d\theta^2 \right] + \left(\frac{r}{r_0}\right)^{2n} \frac{1}{W^2(r)} dz^2,$$

(3.14)

with \(W(r) = [(r/r_0)^{2n} + p]/[1 + p]\) and the parameters \(n, \lambda\) and \(m\) are given by \(n^2 = \kappa^{-1}\lambda^2 + m^2\), with \(\kappa^{-1}\) constant in the Brans-Dicke theory.

Now, let us find the solutions for a SSCS by considering the weak field approximation (for a review of the procedure see Ref.[32]). To do this let us assume that the metric \(g_{\mu\nu}\) and the scalar field \(\phi\) can be written as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

$$\Lambda(\phi) = \Lambda(\phi_0) + \Lambda'(\phi_0) \phi(1),$$

$$T_{\mu\nu} = T_{(0)\mu\nu} + T_{(1)\mu\nu},$$

$$\phi = \phi_0 + \phi(1),$$

(3.15)

where \(\Lambda'(\phi_0) = \Lambda(\phi_0)\alpha(\phi_0)\), \(\eta_{\mu\nu} = diag(-, +, +, +)\) is the Minkowski metric tensor and \(\phi_0\) is a constant.

The energy per unit length \(U\), the tension per unit length \(\tau\) and the current density \(I\), are given, respectively, by
\begin{align}
U &= -2\pi \int_0^{r_0} T_1^t r dr; \\
\tau &= -2\pi \int_0^{r_0} T_2^z r dr; \\
\text{and} \\
I &= 2\pi e \int_0^{r_0} r dr \sigma^2 A
\end{align}

In the case of a space-time with torsion, we can find the matching conditions using the fact that \(\{\alpha_{\mu \nu}\}^{(+)}\rangle_{r=r_0} = \{\alpha_{\mu \nu}\}^{(-)}\rangle_{r=r_0}\), and the metricity constraint \(\nabla_\rho g_{\mu\nu} \rangle_{r=r_0} = [\nabla_\rho g_{\mu\nu}]_{r=r_0} = 0\). Thus, we find the following continuity conditions

\begin{align}
[g_{\mu\nu}]_{r=r_0}^{(-)} &= [g_{\mu\nu}]_{r=r_0}^{(+)}, \\
[\alpha_{\mu\nu}]_{r=r_0}^{(+)} + 2[g_{\alpha\rho}K_{(\mu\nu)}]_{r=r_0}^{(+)} &= [\alpha_{\mu\nu}]_{r=r_0}^{(-)} + 2[g_{\alpha\rho}K_{(\mu\nu)}]_{r=r_0}^{(-)},
\end{align}

where \((-\rangle\) represents the internal region and \((+\rangle\) corresponds to the external region around \(r = r_0\). In analyzing the junction conditions we notice that the contortion contributions appear neither in the internal nor in the external regions, differently from the results obtained in Refs. (34, 35).

In order to compare the external solutions we demand a linearisation of these ones since they are exact solutions. Then, we make a change of variable \(r \rightarrow \rho\), such that

\begin{equation}
\rho = r \left[1 + \tilde{G}_0(4U + I^2) - 4\tilde{G}_0 U \ln \frac{r}{r_0} - 2\tilde{G}_0 I^2 \ln^2 \frac{r}{r_0}\right],
\end{equation}

and, in this way, we have that at first order in \(\tilde{G}_0\), the \(\phi_{(1)}\)-solution is given by

\begin{equation}
\phi_{(1)} = 2\tilde{G}_0 \kappa^{-1} \alpha(\phi_0)(U + \tau - I^2) \ln \frac{\rho}{r_0}.
\end{equation}

where we used Eq. (2.19) and the fact that \(\tilde{G}_0 \equiv G\Lambda^2(\phi_0)\).
Doing the identification of the coefficients of both linearised metrics, we finally obtain

\[ m^2 = 4\tilde{G}_0I^2 \]
\[ B^2 = 1 - 8\tilde{G}_0(U + \frac{I^2}{2}) \]
\[ \lambda = 2\tilde{G}_0\alpha(\phi_0)(U + \tau - I^2) \]
\[ p = 1 + \tilde{G}_0^{1/2}(U - \tau - I^2). \quad (3.22) \]

Using the same procedure to obtain the superconducting cosmic string solution in the framework of the scalar-tensor theories \[\text{[32]}\], we find the metric in Einstein frame as

\[
\begin{align*}
    ds^2 &= \left\{ 1 + 4\tilde{G}_0 \left[ I^2 \ln^2 \frac{\rho}{r_0} + (U - \tau + I^2) \ln \frac{\rho}{r_0} \right] \right\} (-dt^2 + d\rho^2) \\
    &+ \left\{ 1 - 4\tilde{G}_0 \left[ I^2 \ln^2 \frac{\rho}{r_0} + (U - \tau - I^2) \ln \frac{\rho}{r_0} \right] \right\} dz^2 \\
    &+ \rho^2 \left[ 1 - 8\tilde{G}_0(U + \frac{I^2}{2}) + 4\tilde{G}_0(U - \tau - I^2) \ln \frac{\rho}{r_0} + 4\tilde{G}_0I^2 \ln^2 \frac{\rho}{r_0} \right] d\theta^2. \quad (3.23)
\end{align*}
\]

Note that this metric corresponds to the same one obtained in the case of pure scalar-tensor theories. The reason for this coincidence is that the contribution due to the torsion comes out only in the third order and therefore it does not appear in the linearized solution we have considered. Otherwise, some new physical effects appears as we shall see in the next section.

The deficit angle associated with the space-time given by metric \((3.23)\) is

\[
\Delta \theta = 2\pi \left[ 1 - \frac{1}{\sqrt{|g_{\rho \rho}|}} \frac{d}{\rho} \sqrt{|g_{\theta \theta}|} \right],
\]

which can be written as

\[
\Delta \theta = 4\pi \tilde{G}_0(U + \tau + I^2). \quad (3.24)
\]
4 Particle deflection near a SSCS

In this section we study the geodesic equation in the space-time under consideration. To do this, we have to work with the metric given by Eq.(3.23) in the Jordan-Fierz frame. Then, if we consider a section perpendicular to the string, i. e., \(dz = 0\), then we have

\[
ds_{\perp}^2 = \Lambda(\phi_0)^2(1 - h_{tt})[-dt^2 + dr^2 + (1 - b)r^2d\theta^2],
\]

(4.1)

where \(h_{tt}\) is

\[
h_{tt} = -4\tilde{G}_0\{I^2(\ln(\frac{\rho}{r_0}))^2 + [\alpha_+U - \alpha_-(\tau - I^2)]\ln(\frac{\rho}{r_0})\},
\]

(4.2)

with \(b\) and \(\alpha\) being given by

\[
b = 8\tilde{G}_0\left[U + \frac{I^2(1 + 2\ln(\frac{\rho}{r_0}))}{2}\right],
\]

(4.3)

\[
\alpha_{\pm} = [1 \pm \frac{1}{2}\kappa^{-1}\alpha^2(\phi_0)].
\]

(4.4)

We know that when the string possesses current there appear gravitational forces. We shall consider the effect that torsion plays on the gravitational force generated by a SSCS on a particle moving around the defect, assuming that the particle has no charge. Let us consider the particle with speed \(|\mathbf{v}| \leq 1\), in which case the geodesic equation becomes

\[
\frac{d^2x^i}{d\tau^2} + \Gamma^i_{tt} = 0,
\]

(4.5)

where \(i\) is the spatial coordinate index and the connection can be written as in Eq.(2.9), with the non-vanish terms given by

\[
\Gamma^i_{tt} = \left\{^i_{tt}\right\} + K^i_{tt}
\]

(4.6)
with $K_{(tt)}^{i}$ being the contortion which is symmetric in the two-first indices. The only non-vanish part is

$$K_{(tt)}^{r} = \frac{\phi'}{2\phi} \sim -\alpha(\phi_0)\phi'(1) = -\frac{1}{\rho} \tilde{G}_0 \kappa^{-1} \alpha^2(\phi_0)(U + \tau - I^2).$$

(4.7)

The gravitational acceleration around the string gets the form

$$a = \nabla h_{tt} - \frac{2 \tilde{G}_0 \kappa^{-1} \alpha^2(\phi_0)(U + \tau - I^2)}{\rho},$$

(4.8)

with $g_{tt} = -1 + h_{tt}$ in Eq.(3.23). Therefore, the torsion contribution to the force is

$$f_{\text{tors}} = -\frac{2m}{\rho} \tilde{G}_0 \kappa^{-1} \alpha^2(\phi_0)(U + \tau - I^2)$$

(4.9)

We also note that the gravitational pull is related to the $h_{tt}$ component that has explicit dependence on the torsion, as shown in Eq.(4.2). From the last equation, the force that the SSCS exerts on a test particle can be explicitly written as

$$f = -\frac{m}{\rho} \left[ 4 \tilde{G}_0 I^2 \left( 1 + \frac{(U - \tau)}{I^2} + 2 \ln(\rho/r_0) \right) + 4 \tilde{G}_0 \kappa^{-1} \alpha^2(\phi_0)(U + \tau - I^2) \right].$$

(4.10)

A quick glance at the last equation allows us to understand the essential role played by torsion in the context of the present formalism. If torsion is present, even in the case in which the string has no current, an attractive gravitational force appears. In the context of the SSCS, torsion acts such to enhance the force that a test particle feels outside the string. This peculiar fact may have meaningful astrophysical and cosmological effects, as for example, influencing the process of formation of structures.

5 Large-scale structure formation

Let us consider the deflection of particles moving past the string. Assuming for simplicity that the direction of propagation is perpendicular to the string, we can write the metric (4) in terms of Minkowskian coordinates in the form
\[ ds^2 = (1 - h_{00})[dt^2 - dx^2 - dy^2] \] (5.1)

In last section we concluded that in a space-time with torsion there is a change in the geodesics due to the presence of the symmetric part of the contortion (4.5). To study the formation of a wake behind a moving screwed cosmic string, we will first consider the rest frame of the string with a velocity \( v \) in the x direction. In this situation, all components of geodesic equations are

\[
\frac{d^2 x^i}{d \tau^2} + \Gamma^i_{(\mu \nu)} u^\mu u^\nu = 0, \quad (5.2)
\]

whose linearized forms are given by

\[
2\ddot{x} = -(1 - \dot{x}^2 - \dot{y}^2) \partial_x h_{tt} + (1 - \dot{y}^2) \alpha(\phi_0) \partial_x \phi(1), \quad (5.3)
\]

\[
2\ddot{y} = -(1 - \dot{x}^2 - \dot{y}^2) \partial_y h_{tt} + (1 - \dot{x}^2) \alpha(\phi_0) \partial_y \phi(1), \quad (5.4)
\]

where \( h_{00} \) is given by (4.2) and the overdot denotes derivative with respect to \( t \). We can analyze Eqs. (5.3) and (5.4) in which concerns the contribution coming from the last term.

We need only consider terms of first order in \( \tilde{G}_0 \), in which case (5.4) can integrated over the unperturbed trajectory \( x = vt, y = y_0 \). Then, we can transform to the frame in which the string has a velocity \( v \). Then, particles enter the wake with a transverse velocity

\[
v_t = 4\pi \tilde{G}_0(U + \tau + I^2)v\gamma + \frac{4\pi \tilde{G}_0\kappa^{-1}\alpha^2(\phi_0)(U + \tau - I^2)}{v\gamma} \quad (5.5)
\]

The first term is the usual velocity impulse of the particles due to the deficit angle. The second term contains the contribution due to the torsion.

The motion of the string creates the wakes, then we can compute the characteristic total mass of the large structures formed in this way as
\[ M_w = 2vt^3 \]  

(5.6)

The gravitational acceleration in the field of the wake is

\[ a_w = 2\pi GM_w/t^2. \]  

(5.7)

Thus the wake width, i.e., the scale of the largest structures in the universe, turns out to be

\[ \Delta l \equiv \frac{v^2}{2a_w} \sim vt. \]  

(5.8)

From these equations, and using \( t_{eq} = 4 \times 10^{10} (\Omega h)^{-2} \) s for the universe age at the radiation-matter equilibrium phase, we can determine the main characteristics of the gravitational perturbations induced by wakes formed from SSCS.

### 6 Conclusion

It is possible that torsion had a physically relevant role during the early stages of the Universe’s evolution. Along these lines, torsion fields may be potentially sources of dynamical stresses which, when coupled to other fundamental fields (i.e., the gravitational and scalar fields), might have performed an important action during the phase transitions leading to formation of topological defects such as the SSCS here we have considered. Therefore, it seems an important issue to investigate basic models and scenarios involving cosmic defects within the context of scalar-tensor theories with torsion. We showed that torsion as well as scalar fields has a small but non-negligible contribution to the geodesic equation obtained from the contortion term and from the scalar fields, respectively. From a physical point of view, these contributions, certainly, are important, and must be considered. The motivation to consider this scenario comes form the fact that scalar-tensor gravitational fields are important for a consistent description of gravity, at least at sufficiently high energy scales, and that torsion as well can induce some physical effects and could
be important at some energy scale, as for example, in the low-energy limit of string theory.

As we showed in this work, massless particles (such as photons) will be deflected by an angle $\Delta \theta = 4\pi \tilde{G}(0)(U + \tau + I^2)$. From the observational point of view, it would be impossible to distinguish a screwed string from its general relativity partner, just by considering effects based on deflection of light (i.e., double image effect, for instance). On the other hand, trajectories of massive particles will be affected by the torsion coupling (which is generated by a space-time with torsion) \[36, 37\].

If the string is moving with normal velocity, $v$, through matter, a transversal velocity appears which is given by Eq. (5.5). It is worth to call attention to the fact that there exists, in this case, a new contribution to the transversal velocity given by $v_t = \frac{4\pi \tilde{G}(0)\alpha^2(\phi_0)(U + \tau - I^2)}{\gamma}$ which is associated with scalar-tensor torsion. We showed that the propagation of photons is unaffected by a screwed superconducting cosmic string and it is only affected by the angular deficit. This result shows us that the effect of torsion on massive particles is qualitatively different from its effect on radiation; this aspect becomes especially relevant when calculating CMBR-anisotropy and the power spectrum as wiggly cosmic strings. One expects that this feature could help to partially by-pass the current difficulties in reconciling the COBE normalized matter power spectrum with the observational data in the cosmic string model.
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References

[1] A. Vilenkin, Phys. Rev. D 23, 852 (1981); W. A. Hiscock, Phys. Rev. D 31, 3288 (1985); J. R. Gott III, Astrophys. J. 288, 422 (1985); D. Garfinkel, Phys. Rev. D 32, 1323 (1985).

[2] M. B. Hindmarsh and T. W. B. Kibble, Rep. Prog. Phys. 58, 477 (1995).

[3] A. Vilenkin and E. P. S. Shellard, Cosmic Strings and other Topological Defects (Cambridge University Press, 1994).

[4] T. W. B. Kibble, Phys. Rep. 67, 183 (1980).

[5] T. W. Kibble, J. of Phys. A9, 1387 (1976).

[6] E. Bertschinger and P. N. Watts, Astrophys. J. 328, 23 (1988).

[7] R. Brandenberger, N. Kaiser, D. Schramm, and N. Turok, Phys. Rev. Lett. 59, 2371 (1987).

[8] J. Silk and A. Vilenkin, Phys. Rev. Lett. 53, 1700 (1984).

[9] N. Turok and R. Brandenberger, Phys. Rev. D 33, 2175 (1986).

[10] A. Vilenkin, Phys. Rep. 121, 263 (1985).

[11] E. Witten, Nucl. Phys. B249, 557 (1985).

[12] P. Peter and D. Puy, Phys. Rev. D 48, 5546 (1993).

[13] R. Jackiw and P. Rossi, Nucl. Phys. B 190, 681 (1981).
[14] S.C. Davis, Int. J. Theor. Phys. 38, 2889 (1999); hep-ph/9901417 (1999); S.C. Davis et al., Phys. Rev. D 62, 043503 (2000); hep-ph/9912356 (1999).

[15] I. Buchbinder, S.D. Odintsov and I. Shapiro, Effective Action in Quantum Gravity, Institute of Physics Publishing, Bristol and Philadelphia (1992).

[16] V. de Sabbata and M. Gasperini, Introduction to Gravitation, World Scientific Publishing (1985).

[17] William M. Baker, Class. Quantum Grav. 7, 717 (1990).

[18] M. Adak, T. Dureli and L. H. Ryder, Class. Quantum Grav. 18, 1503 (2001).

[19] V. De Sabbata, IL Nuovo Cimento, 107 A, 363 (1994).

[20] Y. Duan, G. Yang and Y. Jiang, Helv. Phys. Acta 70, 565 (1997).

[21] A. Trautman, Nature 242, 7 (1973).

[22] J. Stewart and P. Hajicek, Nature 244, 96 (1973).

[23] M. Demianski, R. De Ritis, G. Platania, P. Scudellaro and C. Stornaiolo, Phys. Rev D 35, 1181 (1987).

[24] G. F. Ellis and M. Bruni, Phys. Rev. D 40, 1804 (1989); G.F. Ellis and J. Hwang, Phys. Rev D 40, 1819 (1989); G. F. Ellis, M. Bruni and J. Hwang, Phys. Rev D 42, 1035 (1990).

[25] D. Palle, Nuovo Cim. B 114, 853 (1999).

[26] L. Pogosian and T. Vachaspati, Phys. Rev. D 60, 083504 (1999).

[27] T. Vashaspati and A. Vilenkin, Phys. Rev. Lett. 67, 1057 (1991).

[28] S. R. M. Masalskiene and M. E. X. Guimarães, Class. Quantum Grav. 17, 3055 (2000).
[29] S.W.Kim, Phys. Rev. D 34, 1011 (1986).

[30] F.W.Hehl et al., Rev. Mod. Phys. 48, 393 (1976).

[31] H.B Nielsen and P. Olesen, Nucl.Phys. 61, 45 (1973).

[32] C.N.Ferreira, M.E.X. Guimarães and J.A.Helayel-Neto, Nucl.Phys. B 581, 165 (2000).

[33] Eubanks T M et al. Bull.Am. Phys. Soc. Abstract # K11.05., (1997).

[34] W.Arkuszewski et al., Commun.Math.Phys. 45, 183 (1975).

[35] R.A.Puntigam and H.H.Soleng, Class. Quantum Grav. 14, 1129 (1997).

[36] C.N.Ferreira, Class. Quantum Grav., (2001), in press.

[37] H.Kleinert, Phys.Lett. B440 283 (1998); H.Kleinert, Gen. Rel. Grav. 32 769 (2000); H.Kleinert, Gen. Rel. Grav. 32 1271 (2000).