Charge and Spin Supercurrents in Magnetic Josephson Junctions with Spin Filters and Domain Walls

Samme M. Dahir, Anatoly F. Volkov, and Ilya M. Eremin

Institut für Theoretische Physik III, Ruhr-Universität Bochum, D-44780 Bochum, Germany

We analyze theoretically the influence of domain walls (DWs) on the DC Josephson current in magnetic superconducting $S_m/F_l/F_l/F_m$ junctions. The Josephson junction consists of two "magnetic" superconductors $S_m$ (superconducting film covered by a thin ferromagnetic layer), spin filters $F_l$ and a ferromagnetic layer $F$ with or without DW (DWs). The spin filters $F_l$ allow electrons to pass with one specific spin orientation, such that the Josephson coupling is governed by a fully polarized long-range triplet component. In the absence of DW(s), the Josephson and spin currents are nonzero when the right and left filters, $F_{l,r}$, pass electrons with equal spin orientation and differ only by a temperature-independent factor. They become zero when the spins of the triplet Cooper pairs passing through the $F_{l,r}$ have opposite directions. Furthermore, for the different chiralities of the injected triplet Cooper pairs the spontaneous currents arise in the junction yielding a diode effect. Once a DW is introduced, it reduces the critical Josephson current $I_c$ in the case of equal spin polarization and makes it finite in the case of opposite spin orientation. The critical current $I_c$ is maximal when the DW is in the center of the F film. A deviation of the DW from the center generates a force that pushes the DW to the center of the F film. In addition, we consider the case of an arbitrary number $N$ of DW’s, with the case $N = 2$ corresponding to a model system for a magnetic skyrmion.

Over the past decade, there has been a significant interest in studying the properties of superconductor/ferromagnet (S/F) hybrid structures. One of the particular aspects of these heterostructures is related to remarkable phenomena caused by the magnetic interaction of topological textures in the superconductor (Abrikosov and Pearl vortices [11,12]), and in the ferromagnet (domain walls or skyrmions [8,10]). The interaction of vortices with the magnetic field in a ferromagnet may results in a spontaneous generation of vortices in the superconductor $S$ in S/F bilayers. This effect occurs in the absence of a direct contact between the electron systems in S and F (no proximity effect) and is caused by the magnetic field generated by vortices or the magnetic textures.

At the same time, the penetration of Cooper pairs into a ferromagnet (the proximity effect) leads to a number of further interesting effects. In particular, the Josephson current in S/F/S junctions may change sign in a certain temperature interval (see [13]) and also review [14,15]. Another interesting effect is the triplet component which arises in S/F hybrid structures with a inhomogeneous magnetisation $\mathbf{M}(r)$ in F. If the magnetization is uniform, the Cooper pairs penetrating into the ferromagnet consist of singlet and short-range triplet components, respectively. Both components penetrate into the ferromagnet over a short lengthscale $\xi_J \approx \sqrt{D_F/J}$ (in the diffusive case), where $D_F$ is the diffusion coefficient and $J$ is the exchange field which, in most of ferromagnets, is much larger than the temperature $T$. If the magnetisation $\mathbf{M}(r)$ is non-homogeneous, as occurs, for example, in S/F/$F$ structure, then a long range triplet component (LRTC) may occur in the system. Here, $\mathbf{F}$ is a weak ferromagnet with magnetization magnitude $\mathbf{m}$ much less than $\mathbf{M}$ and a direction is non-collinear to $\mathbf{M}$. This component propagates into the F region over a long, compared to $\xi_J$, length of the order of $\xi_T \approx \sqrt{D_F/\pi T}$.

In this case, the superfluid component in most part of F consists solely of triplet Cooper pairs. For example, the Josephson coupling in S/$F_m/F/F_m/S$ structure can be realized through the LRTC as it was predicted (see also review [16,17] and references therein) and observed experimentally [18,19]. Interestingly, the long-range triplet Cooper pairs with spin-up and -down orientations penetrate the ferromagnet $F$ regardless of the magnetization orientation $\mathbf{M}$ so that the spin current $I_s$ in $S/F_m/F/F_m/S$ Josephson junctions is absent, whereas the charge current $I_Q$ is non-zero. Only in the presence of spin filters at the $S/F_m$ interfaces the current $I_Q$ becomes finite.

In this manuscript, we calculate the Josephson charge $I_Q$ and spin $I_{sp}$ currents in the $S_m/F_l/F_l/F_m$ Josephson junctions under various conditions, where $S_m = S/F_m$ is a conventional superconductor covered by a thin ferromagnetic layer. First we consider the system without DWs and calculate the currents $I_{Q,sp}$: a) in the absence or presence of spin filters at the S/F interfaces, b) for equal or different polarizations or chiralities of the triplet Cooper pairs injected into F from the left and right superconductors S. Most importantly, we also study the influence of the domain walls in F (DWs) on the $I_Q$ and $I_{sp}$ in the dirty case when the condensate Green’s function $\hat{f}$ obey the Usadel equation. Within this approximation the Green’s functions $f$ do not depend on the momentum direction $\mathbf{p}/|\mathbf{p}|$. Therefore, according to the Pauli principle, the functions for the LRTC $\hat{f}(t,t') \sim \langle c_\uparrow(t)c_\downarrow(t') \rangle$ are zero at coinciding times $t = t'$. In other words, these are odd functions of the Matsubara frequency $\omega$, $\hat{f}(\omega) = -\hat{f}(-\omega)$, so that summing over all $\omega$ gives zero: $\hat{f}(t,t) \sim \sum_\omega \hat{f}(\omega) = 0$. The triplet odd-frequency Cooper pairs exist in any superconducting system if there is a Zeeman interaction of electron spins and a magnetic or
exchange field. This case was studied long ago. Unlike homogeneous superconductors with the Zeeman interaction, where the triplet component coexists with the singlet one, the recently studied hybrid S/F systems allow the separation of triplet and singlet Cooper pairs. In addition, we assume a weak proximity effect allowing linearization of the necessary equations and the boundary conditions yielding simple analytical expressions for \( f(r) \) and the currents \( I_{Q,sp} \).

Although the Josephson effect has been studied for similar structures in various limiting cases (see references above as well as \( \text{[55–61]} \)), there is no systematic study of the dependence of the \( I_{Q,sp} \) on spin polarization, chiralities and the presence of the spin filters and DWs. In particular, we show that although the current \( I_Q \) is zero for opposite polarization directions and different chiralities of injected Cooper pairs in the presence of spin filters, it becomes finite in the presence of DWs. We will consider an arbitrary number of DWs and pay a special attention to the case of two DWs. The latter case may be regarded as a model of magnetic texture such as skyrmion with \( N = 1 \) winding number (like Bloch or Neel skyrmion) when the magnetization profile \( \mathbf{M} \) has the same orientation outside the DWs and the opposite orientation between DWs.\( \text{[62–63]} \)

### I. Basic Equations

We consider an \( S_m/\text{Fl}/F/\text{Fl}/S_m \) Josephson junction with one or several domain walls (DW) in the F film (wire). Schematically the considered system is shown in Fig.1. The junction consists of two "magnetic" superconductors \( S_m \) and of two filters (Fl) which allow only electrons with a single spin polarization, parallel or antiparallel to the \( z \) axis, to pass through. The "magnetic" superconductors may be made of conventional superconductors covered by ferromagnetic thin films with the magnetization aligned parallel to the \( x \)- or \( y \)-axes. The magnetization vector \( \mathbf{M} = (0, 0, M) \) is supposed to be oriented along the \( z \)-axis. The filters may be magnetic insulators selecting electrons with a spin collinear to the \( z \) axis. The Cooper pairs penetrating into the F film due to proximity effect consist of triplet long-range components only. We assume that the proximity effect is weak as it is the case in most of experimental setups. The Cooper pairs are described by a matrix quasiclassical Green’s function \( \hat{f}(x) \), which is supposed to be small \( |\hat{f}| << 1 \). The function \( \hat{f}(x) \) in the F film obeys the linearized Usadel equation.\( \text{[15–20, 63–64]} \)

\[
-\partial_x^2 \hat{f} + \kappa^2 \hat{f} + \frac{i\kappa^2}{2} \left( n_z(x) \left[ \hat{X}_{03}, \hat{f} \right] + n_k(x) \left[ \hat{X}_{0k}, \hat{f} \right] \right) = 0, \quad (1)
\]

with matching conditions at \( x = l_i \)

\[
\hat{f}|_{x=0} - \hat{f}|_{x=0} = 0, \quad (3)
\]

\[
\partial_x \hat{f}|_{x=0} - \partial_x \hat{f}|_{x=0} = i(\kappa_{DW}/2) \left[ \hat{X}_{02}, \hat{f} \right], \quad (4)
\]

where \( \kappa_{DW} = \kappa^2 d_{DW} \).

A solution of Eq.\((2)\) consists of a short-range and long-range components, respectively. The first one decays on a short distance of the order of \( \xi_J \approx \sqrt{D_F}/J \), while the second varies on a much longer characteristic length of the order of \( \xi_T \approx \sqrt{D_F}/\pi T \). Observe that the condensate matrix Green’s function \( \hat{f} \) is off-diagonal in the Gor’kov-Nambu space, \( i.e. \hat{f} \) is proportional to \( \hat{\tau}_1, \hat{\tau}_2 \) matrices. In addition, the long-range triplet component (LRTC), \( f_{lr} \), is also off-diagonal in the spin space, \( i.e. \hat{f} \) is proportional to \( \hat{\sigma}_1, \hat{\sigma}_2 \) matrices such that the third term in Eq.\((2)\) for this component vanishes. This means that in a general case the matrix LRTC \( f_{lr} \) obeying Eq.\((2)\) can be written in the form

\[
f_{lr}(x) = \sum_{i,k} a_{ik}(x) \hat{X}_{ik}, \quad (5)
\]

where \( \{i,k\} = \{1, 2; 1, 2\} \). A concrete form of the LRTC is determined by the boundary conditions at \( x = \)}
where field in the function in the weak ferromagnet \( F \). The term \( \hat{X} \) describes the tunneling of Cooper pairs through the filters and is defined as

\[
\hat{X} = \sqrt{\frac{\omega - i J_m}{\Delta^2}}
\]

where \( \Delta \) is conductivity of the F film with equal probabilities, and therefore the number of the triplet pairs with both spin orientations in the F is the same. This case has been called nematic LRTC in Ref.\(^{70}\). If \( |\Delta| = |\Delta| \), then \( s = \pm 1 \), and the triplet Cooper pairs are fully polarized with total spin parallel or antiparallel to the \( z \)-axis. Note that a magnetic half-metal can be used as a spin filter. The case of \( s = 0 \) corresponds to the absence of filters at the F/F interfaces.

Eqs. (7-11) are a generalization of the Kupriyanov-Lukichev boundary conditions\(^{73}\) which in turn were obtained from the Zaitsev’s boundary conditions\(^{71}\) (see also Ref.\(^{22}\) where the applicability of the Kupriyanov-Lukichev boundary conditions is discussed).

Till now we assumed that the phases of the order parameter in the superconductors S are chosen equal to zero. The presence of the phases \( \pm \varphi / 2 \) at \( S_{l,d} \) can be easily introduced via a gauge transformation \( \hat{S}_p = \exp \left( \pm i \hat{X}_{30} \varphi / 4 \right) \cdot \hat{g}_{s,p} = \hat{S}_p \cdot \hat{g}_S \cdot \hat{S}_p \) (see, for example\(^{22}\)) so that the boundary condition (7) can be written as

\[
\partial_x \hat{I} |_{x = \pm L} = \pm \kappa_b |\sin(\varphi / 2) \pm i \hat{X}_{30} \sin(\varphi / 2)| \hat{I}_{1,r} F_{S-}.
\]

The parameter \( s = \pm 2 \Re (\hat{T} U^*/|\hat{T}|^2 + |U|^2) \) characterizes the degree of spin-up and spin-down polarization of the triplet Cooper pairs injected into the film F. If \( U = 0 \), Cooper pairs with up and down spins penetrate into the F film with equal probabilities, and therefore the number of the triplet pairs with both spin orientations in the F is the same. This case has been called nematic LRTC in Ref.\(^{70}\). If \( |\Delta| = |\Delta| \), then \( s = \pm 1 \), and the triplet Cooper pairs are fully polarized with total spin parallel or antiparallel to the \( z \)-axis. Note that a magnetic half-metal can be used as a spin filter. The case of \( s = 0 \) corresponds to the absence of filters at the F/F interfaces.

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\[
\partial_x \hat{I} |_{x = \pm L} = \pm \kappa_b |\sin(\varphi / 2) \pm i \hat{X}_{30} \sin(\varphi / 2)| \hat{I}_{1,r} F_{S-}.
\]

The matrix condensate function \( \hat{X}_m F_{S-} \) describes a long-range triplet component in the film F, but it becomes a long-range one in the F film because of the non-collinearity of the the magnetization vectors \( \mathbf{m} \) and \( \mathbf{M} \). Note that the functions \( \hat{X}_{12} F_{S-} \) and \( \hat{X}_{21} F_{S-} \), written explicitly, consist of triplet components with up and down spins \( \hat{X}_{12} F_{S-} \sim \langle c_1(t)c_1(0) \rangle + \langle c_2(t)c_2(0) \rangle \), \( \hat{X}_{21} F_{S-} \sim \langle c_1(t)c_2(0) \rangle - \langle c_2(t)c_1(0) \rangle \) so that the function \( \hat{X} F_{S-} = \hat{X}_{12} \pm \hat{X}_{21} F_{S-} \) describes the Cooper pairs polarized in one direction, see Appendix A for further details.

Knowing the Green’s functions \( \hat{T} \), we can readily calculate the charge \( I_Q = \mathbf{I}_Q \cdot \mathbf{e}_x \) and the spin currents \( I_{sp} = \mathbf{I}_{sp}^{(z)} \cdot \mathbf{e}_x \) using the following expressions

\[
I_Q = \frac{\sigma F}{e} 2\pi T \sum_{\omega \geq 0} I_{Q,\omega}, \quad I_{sp} = \mu_B \frac{\sigma F}{e} 2\pi T \sum_{\omega \geq 0} I_{sp,\omega}
\]

where the ”spectral” currents \( I_{Q,\omega} \) and \( I_{sp,\omega} \) are defined as

\[
I_{Q,\omega} = (i/4) \text{Tr} \{ \hat{r}_3 \cdot \hat{S}_3 \hat{f}_{LR} \hat{f}_{LR} \} = i \{ \hat{f} \partial_x \hat{f} \} \quad \text{at} \quad \omega = \omega_{30} \tag{16}
\]

and further details are given in Appendix B. Similar formulas were used in Refs.\(^{16,17}\). Observe that the traces in the Nambu space for charge and spin currents are actually different, which was often overlooked previously.
In order to find the Josephson current, we need to solve Eq. [2] with the matching conditions [3] and boundary conditions [7].

We first consider the case of the F film with a uniform magnetization, $\mathbf{M} = (0, 0, M)$, without DWs. Although such magnetic Josephson junctions have been already studied previously in different limiting cases (ballistic and diffusive) using various mostly numerical techniques [21-22], we will discuss the main results in the dirty case and in the limit of the weak proximity effect. Then the formulas for currents acquire a simple analytical form, not known previously, that allows for a straightforward physical interpretation. In addition, we will focus our study on the case of fully polarized triplet Cooper pairs of different chiralities.

In particular, the solution of Eq. [2], $\hat{J}_{LR,0}$, which obeys the boundary conditions [7] has the form

$$\hat{J}_{LR,0} = \hat{C} \cosh(\kappa_\omega x) + \hat{S} \sinh(\kappa_\omega x)$$

with

$$\hat{C} = \frac{\kappa_b}{2\kappa_\omega} \left[ \hat{X}_+ \cos \left( \frac{\varphi}{2} \right) + i \hat{X}_{30} \cdot \hat{X}_{-} \sin \left( \frac{\varphi}{2} \right) \right] F_{S-}$$

$$\hat{S} = \frac{\kappa_b}{2\kappa_\omega} \left[ \hat{X}_- \cos \left( \frac{\varphi}{2} \right) + i \hat{X}_{30} \cdot \hat{X}_{+} \sin \left( \frac{\varphi}{2} \right) \right] F_{S-}$$

where we have defined $\hat{X}_+ = \hat{X}_r \pm \hat{X}_l$. Substituting $\hat{J}_{LR,0}$ from Eq. [18] into Eqs. [16,17], we obtain

$$I_{Q,\omega} = i \hat{C} \omega \left[ -i \{ \hat{X}_r \cdot \hat{X}_l \}_{30} \sin \varphi + \{ \hat{X}_r \cdot \hat{X}_l \}_{03} \sin \varphi \right] F_{S-}$$

$$I_{sp,\omega} = i \hat{S} \omega \left[ -i \{ \hat{X}_r \cdot \hat{X}_l \}_{30} \sin \varphi + \{ \hat{X}_r \cdot \hat{X}_l \}_{03} \sin \varphi \right] F_{S-}$$

$$\hat{I}_\omega = \frac{(\kappa_b F_{S-})^2}{\kappa_\omega \sinh(2\kappa_\omega)}.$$  

and $\kappa_\omega = \sqrt{2|\omega|/D_F}$. Observe that the matrices $\hat{C}, \hat{S}$ and $\hat{X}_r, \hat{X}_l$ anticommute with matrices $\hat{X}_{30}, \hat{X}_{03}$ so that the traces $\{ C^2 \}_{30}, \{ C^2 \}_{03}$ etc. are equal to zero. In the following we calculate the charge and spin currents for different cases in detail.

A. Currents in the absence of filters.

For the case of equal chiralities of the triplet Cooper pairs injected from the right (left) S/F1 interfaces $(\mathbf{m}_r|\mathbf{m}_l|\mathbf{e}_x$ or $\mathbf{m}_r|\mathbf{m}_l|\mathbf{e}_y)$ and defining $\hat{X}_r = \hat{X}_1 = \hat{X}^{(x)}$ or $\hat{X}_r = \hat{X}_1 = \hat{X}^{(y)}$, the charge and spin "spectral" currents are

$$I_{Q,(x,y)} = I_{Q,\omega} = \hat{I}_\omega \sin \varphi$$

$$I_{sp,\omega} = \hat{I}_\omega = 0$$

i.e. the charge current has the usual form $I_{Q,\omega} = \hat{I}_\omega \sin \varphi$ whereas the spin current is zero. For the case of different chiralities ($\hat{X}_r = \hat{X}^{(y)}_{12}, \hat{X}_l = \hat{X}^{(1)}_{11}$) the currents are given by

$$I_{Q,(x,y)} = 0$$

$$I_{sp,\omega} = \hat{I}_\omega \cos \varphi$$

where indices (x,y) and (x,y) refer to the chirality of the Cooper pairs penetrating the film F on the right and on the left that is, $I_{Q,(x,y)} \sim \{ \hat{X}_r^{(x)} \cdot \hat{X}_l^{(y)} \}$. We also note an important feature of the obtained currents. In particular, the critical "spectral" current $\hat{I}_{\omega}$ in the considered S/m/Fl/F/Fl/S/m junction has the sign opposite to that in S/N/S Josephson junction since in the latter case the critical current $\hat{I}_{S/N/S} \sim F_S^2 > 0$, while in the system under consideration $\hat{I}_{\omega} \sim F_{S-}^2 < 0$ (see Eq. [7]), here $F_S = \Delta/\sqrt{\omega^2 + \Delta^2}$. This is a simple representation of the fact that the LRTC leads to a $\pi$-Josephson coupling.

Observe that the Josephson current $I_{Q,(x,y)}$ is finite for collinear orientations of the magnetic moments $\mathbf{m}$ in the left and right films $F_R$ and is zero ($I_{Q,\omega} = 0$) for the orthogonal orientations of the vectors $\mathbf{m}_r,l$. The opposite is true for the spin current. It is zero in the case of vectors $\mathbf{m}_r = \mathbf{m}_l$ and is finite if $\mathbf{m}_r \cdot \mathbf{m}_l = 0$, i.e., when the vectors $\mathbf{m}_r,l$ are orthogonal. Moreover, in the second case a spontaneous spin current arises in the system even when the phase difference $\varphi$ is zero.

The formulas for the currents [24,27] are derived for the case when the vectors $\mathbf{m}_r,l$ lie in the plane perpendicular to the z-axis so that $\mathbf{m}_r,l \cdot \mathbf{e}_z = \cos \alpha_{r,l} = 0$. They can be easily generalized for the arbitrary angles $\alpha_{r,l}$. Taking into account that only the components $\mathbf{m}_{x,y} \cdot \mathbf{e}_{x,y}$ contribute to the LRTC, in a more general case the currents $\hat{I}$ are equal to

$$\hat{I}_\alpha = \hat{I} \sin \alpha_r \sin \alpha_l$$

The formulas for the charge and spin currents $I_{Q,sp,\omega}$ are represented in Table 1. The angles $\alpha_{r,l}$ are chosen to be equal to $\pi/2$ so that $\sin \alpha_r = \sin \alpha_l = 1$.

filters, the currents are spin independent. As we show below the presence of spin filters makes both currents spin dependent. For the case of equal chiralities, i.e., $\hat{X}_r = \hat{X}_l = \hat{X}^{(v)}$ ($v = x$ or $y$), the charge and spin currents can be found by using formulas for $\hat{X}^{(x)}$ and $\hat{X}^{(y)}$,
Table 1. Summary of the charge and spin supercurrents in magnetic Josephson junctions with spin filters and their modifications due to domain walls for different ferromagnetic filter orientations.

| CURRENTS         | XX(YY) no Fl | XY no Fl | XX(YY) + Fl | XY + Fl          |
|------------------|--------------|----------|-------------|------------------|
| $I_Q$            | $\tilde{I} \sin \varphi$ | 0        | $\tilde{I} (1 + s_r s_l) \sin \varphi$ | $-\tilde{I} (s_r + s_l) \cos \varphi$ |
| $I_{sp}$         | 0            | $\tilde{I} \cos \varphi$ | $\tilde{I} s_r s_l \sin \varphi$ | $-\tilde{I} (1 + s_r s_l) \cos \varphi$ |

Corrections to the Currents due to DWs

|                  | XX(YY) + Fl, P case | XX(YY) + Fl, AP case |
|------------------|---------------------|----------------------|
| $I_Q$            | $(\tilde{I} - \delta \tilde{I}) \sin \varphi$ | $\delta \tilde{I} \sin \varphi$ |

II. Modifications of the Currents due to DWS.

Next we consider the modifications of the currents, obtained above, for the case of the domain wall in the F film. We restrict our analysis to the case of equal chiralities (the generalization to the case of different chiralities is straightforward) and also assume that the spacing between the nearest DWs is much larger than the decay length of the short-range component $f_{sr}$, i.e., $|l_1 - l_2| \gg \xi \approx \sqrt{D_F J}$. The main effect of the domain wall is the creation of a short-range triplet component, which results in a correction $\delta \hat{f}_{ir}$ to the long-range component $\hat{f}_{ir,O}$ defined by Eq. [18]. While the short-range component exists only near each DW, the LRTC extends over a larger distance, which can be of the order of $L$.

In particular, the correction $\delta \hat{f}_{ir}$ arises due to matching conditions for the function $\delta \hat{f}_{ir}(x)$ at $x = l_i$, where $l_i$ is the coordinate of a DW. These conditions for $\delta \hat{f}_{ir,0}(x)$ and its partial derivative are

$$\left[\delta \hat{f}_{ir}\right]|_{l_i} = 0,$$

$$\left[\partial_x \delta \hat{f}_{ir}\right]|_{l_i} = i \kappa_{DW}/2 \left[\hat{X}_{o2}, \hat{f}_{sr}(l)\right]|_{+}.$$

As usual, these are complemented by the boundary conditions

$$\partial_x \delta \hat{f}_{ir}|_{\pm L} = 0.$$

In the presence of several DWs, the solution for $\delta \hat{f}_{ir}$ can be represented in the form

$$\delta \hat{f}_{ir}(x) = \sum_i \delta \hat{f}_{ir}^{(i)}(x).$$
where the $\delta \hat{f}_{ir}^{(i)}(x)$ is a perturbation of the LRTC generated by the $i-$th DW. In order to find this function, one needs to determine a short-range component $\hat{f}_{sr}^{(i)}$ produced by the $i-$th DW, which we do in the next subsection.

### A. Short-range Component generated by the Domain Wall

The short-range component obeys Eq. (2) and matching conditions (37,38) that can be written as

\[
|\hat{f}_{sr}| = l = 0, \quad [\partial_x \hat{f}_{sr}] = i\frac{\kappa_{DW}}{2}|\hat{X}_{02}, \hat{f}_{ir,0}|, \quad (37)
\]

where we also dropped the subindex $i$ in $l_i$ for simplicity. Taking into account Eqs. (41-42), we can rewrite Eq. (38) as follows

\[
[\partial_x \hat{f}_{sr}] = i\kappa_{DW} \left( \hat{C}_2 \cos \frac{\tilde{l}}{\sinh \tilde{L}} + \hat{S}_2 \sinh \frac{\tilde{l}}{\cosh \tilde{L}} \right) \quad (39)
\]

where $\tilde{l} = \kappa_{\omega} l$, $\tilde{L} = \kappa_{\omega} L$ and $\hat{C}_2 = |\hat{X}_{02}, \hat{C}|$, $\hat{S}_2 = |\hat{X}_{02}, \hat{S}|$. A solution for the short-range component, Eq. (2), obeying the matching conditions (37,38) in the vicinity of $i-$th DW can written in the form

\[
\hat{f}_{sr} = \hat{f}_{sr}^{(A)} \cos(\varphi/2) + \hat{f}_{sr}^{(B)} \sin(\varphi/2). \quad (40)
\]

where the matrices $\hat{f}_{sr}^{(A,B)}$ Green’s functions contain exponentially decaying functions

\[
A^{(y)}, \quad B^{(y)} \quad \text{are equal to}
\]

\[
A_p^{(y)} = A_{AP}^{(y)} = A_p^{(x)}/s, \quad (47)
\]

\[
B_p^{(y)} = B_{AP}^{(y)} = B_p^{(x)}/s. \quad (48)
\]

In the next section, we calculate the function \(\delta \hat{f}_{ir,0}(x)\).

### B. Correction to the LRTC due to a domain wall

Finally, the correction $\delta \hat{f}_{ir}(x)$ obeys the equation

\[
-\partial_x^2 \delta \hat{f}_{ir} + \kappa_{\omega}^2 \delta \hat{f}_{ir} = 0, \quad (49)
\]

complemented by the conditions (33,35). The solution of Eq. (49), which obeys the boundary conditions (35), is

\[
\delta \hat{f}_{ir}(x) = \begin{cases}
\hat{C}_< \cosh \left( \tilde{x} + \tilde{L} \right), & -L < x < L, \\
\hat{C}_< \cosh \left( \tilde{x} - \tilde{L} \right), & L < x < L,
\end{cases} \quad (50)
\]

The matrices $\hat{C}_{\leq}$ are found from the matching conditions (33,34)

\[
\hat{C}_{\leq} = \hat{C}_{\leq}^{(A)} \cos(\varphi/2) + i \hat{X}_{30} \hat{C}_{\leq}^{(B)} \sin(\varphi/2) \quad (51)
\]

and find for $\hat{C}_{\leq}^{(A,B)}$

\[
\hat{C}_{\leq}^{(A)} = \hat{a}_{\leq} = -4 \frac{\kappa_{DW}}{\kappa_{\omega}} \frac{\cosh \left( \tilde{L} + \tilde{l} \right)}{\sinh \left( 2\tilde{L} \right)} A \hat{X}_{n2}, \quad (52)
\]

\[
\hat{C}_{\leq}^{(B)} = \hat{b}_{\leq} = -4 \frac{\kappa_{DW}}{\kappa_{\omega}} \frac{\cosh \left( \tilde{L} + \tilde{l} \right)}{\sinh \left( 2\tilde{L} \right)} B \hat{X}_{n2} \quad (53)
\]
where the signs ± correspond to \( x \gtrless l \) and \( n = 1 \) for \( y \)-chirality and \( n = 2 \) for \( x \)-chirality.

Having known the long-range Green’s function \( \hat{f}_{tr} = \hat{f}_{tr,0} + \delta \hat{f}_{tr} \), we can find a change of the current in the presence of a DW.

### C. Change of the Currents due to domain wall

The corrections to the currents are

\[
\delta I_Q = (\sigma_F/e) 2\pi T \sum_{\omega \geq 0} \delta I_{Q,\omega}
\]

\[
\delta I_{sp} = \mu_B (\sigma_F/e^2) 2\pi T \sum_{\omega \geq 0} \delta I_{sp,\omega}
\]

and the partial currents \( \delta I_{Q,\omega} \) and \( \delta I_{sp,\omega} \) are given by

\[
\begin{align*}
\delta I_{Q,\omega} & = i(\delta \hat{f}_{tr,0} \partial_x \hat{f}_{tr,0} + \hat{f}_{tr,0} \partial_x \delta \hat{f}_{tr})_{30}, \\
\delta I_{sp,\omega} & = i(\delta \hat{f}_{tr,0} \partial_x \hat{f}_{tr,0} + \hat{f}_{tr,0} \partial_x \delta \hat{f}_{tr})_{33}
\end{align*}
\]

We find

\[
\begin{align*}
\delta I_{Q,\omega} & = \kappa_\omega \left((\hat{C}^{(B)} + \hat{S}^{(B)}) \cdot \hat{a} - (\hat{C}^{(A)} + \hat{S}^{(A)}) \cdot \hat{b}\right)_{00} \\
\delta I_{sp,\omega} & = \kappa_\omega \left((\hat{C}^{(B)} + \hat{S}^{(B)}) \cdot \hat{a} - (\hat{C}^{(A)} + \hat{S}^{(A)}) \cdot \hat{b}\right)_{33}
\end{align*}
\]

Here, the matrices \( \hat{C}^{(A,B)} \) and \( \hat{S}^{(A,B)} \) are presented in the Appendix C (Eqs. (C1-C4)), and the matrices \( \hat{a} \equiv \hat{a}_r \), \( \hat{b} \equiv \hat{b}_\omega \) are defined in Eqs. [52][53].

Then, we find for the currents of Cooper pairs injected from the right and left \( S_m \) reservoirs with equal chiralities and arbitrary spin polarizations

\[
\begin{align*}
\delta I_Q & = \delta I_{Q,\omega} \sin \varphi, \\
\delta I_{sp} & = \delta I_{sp,\omega} \sin \varphi.
\end{align*}
\]

The critical currents \( \delta I_{Q,\omega} \) and \( \delta I_{sp,\omega} \) depend on the chiralities and polarizations of Cooper pairs propagating from the right and from the left. For the case (a) \( (xx) - \)chiralities, \( P \)-case \((s = s_r = s_l)\)

\[
\begin{align*}
\delta I_{Q,\omega;P}^{(xx)} & = -2 \kappa_{DW}^2 \left( \frac{\kappa_b}{\kappa_\omega} \right) \frac{F_S}{2} \cosh \left( L + \frac{i}{2} \right) \cosh \left( \frac{L - i}{2} \right) \sinh^2(L) \\
\delta I_{sp,\omega;P}^{(xx)} & = \delta I_{Q,\omega;P}^{(xx)}
\end{align*}
\]

(b) \( (xx) - \)chiralities, \( AP \)-case \((s = s_r = -s_l)\) we find

\[
\begin{align*}
\delta I_{Q,\omega;AP}^{(xx)} & = - \delta I_{Q,\omega;P}^{(xx)}, \\
\delta I_{sp,\omega;AP}^{(xx)} & = - \delta I_{sp,\omega;P}^{(xx)}
\end{align*}
\]

Comparing this equation and Eqs. [62][63], we see that the signs of the currents \( \delta I_{Q,\omega;AP}^{(xx)} \) and \( \delta I_{sp,\omega;AP}^{(xx)} \) are changed. Finally, in the case (c) \( (yy) - \)chiralities, \( P(AP)-cases\), the currents are

\[
\begin{align*}
\delta I_{Q,\omega;P}^{(yy)} & = \delta I_{Q,\omega;P}^{(xx)} = \delta I_{Q,\omega;AP}^{(yy)} \\
\delta I_{sp,\omega;P}^{(yy)} & = - \delta I_{sp,\omega;P}^{(xx)} = \delta I_{sp,\omega;AP}^{(yy)}
\end{align*}
\]

In the case of the \( yy \)-chirality, the coefficients \( A^{(y,y)} \) and \( B^{(y,y)} \) do not depend on the polarization \( s \). That is, the currents are equal for different spin orientations:

\[
\begin{align*}
\delta I_{Q,\omega;P}^{(yy)} & = \delta I_{Q,\omega;AP}^{(yy)} \quad \text{and} \quad \delta I_{sp,\omega;P}^{(yy)} = \delta I_{sp,\omega;AP}^{(yy)}
\end{align*}
\]

The analysis of the obtained results shows that the DW reduces the Josephson charge and spin currents if Cooper pairs injected from the right and left superconductors have parallel spin orientation. Thus, the action of the DW on the critical current in this case is analogous to the action of paramagnetic impurities, which decrease the penetration length of the LRT [2012]. In the case of antiparallel orientations, the DW makes the Josephson critical current finite. It is interesting to note that the maximum magnitude of the total Josephson current \( I_Q = |I_{Q,0} + \delta I_Q| \) is achieved at \( l = 0 \). This means that the Josephson energy has a minimum if the DW is located in the center of the junction for the case of parallel spin polarized Cooper pairs. Note also that the correction to the current \( \delta I_{Q,0} \) is proportional to the square of \( \kappa_{DW} \): \( \delta I_{Q,0} \sim K_0 \kappa_0^2 \), thus, the contribution to the current due to DW does not depend on whether the magnetisation vector \( \mathbf{M} \) in the Bloch DW

![FIG. 2](Color online.) Temperature dependence of the normalized critical current \( I_{c,0} \) in the absence of DW (black curve) and a change of the normalized critical current \( \delta I_{c} \) due to DW (red curve), which is subtracted from \( I_{c,0} \) to obtain the total current. For the sake of the presentation the magnitude of the latter is multiplied by the factor of 5. The change \( \delta I_{c} \) decreases the Josephson critical current \( I_c \), if \( I_{c,0} \) is not zero and makes \( I_c \) finite if \( I_{c,0} = 0 \) (antiparallel spin orientations of triplet Cooper pairs injected from the left and from the right). The temperature \( T \) and the exchange \( J_m \) in F, are normalized to \( \Delta(0) \). The parameter \( J \equiv J_m/\Delta(0) \) is chosen to be equal to 3 (see Appendix D).
current decreases drastically with increasing $T$ multiplied this dependence by the factor 50 because the critical $T/\xi(\text{black})$ and $L/\xi(I)$ assumption above ($I$ the DW reduces the critical current $I_c$, $\kappa_c$, $\kappa_{DW}$, $\Delta(0) = 0$).

The long-range triplet Cooper pairs, penetrating into the magnetic layer $F_m$ from the S banks into the F film, consists of singlet and triplet Cooper pairs penetrating the superconducting condensate in a thin magnetic layer $F_m$. The spin filters $F_l$ pass only the triplet Cooper pairs which are long range in $F$ because the magnetisation vector $m$ in $F_m$ is perpendicular to the magnetisation vector $M|e_z$ in the F film. The long-range triplet Cooper pairs, penetrating into the $F$ film, differ in chiralities, i.e., by orientation of the vector $m$ ($m|e_x$ or $m|e_y$), and in polarization of the total spin of the triplet Cooper pairs. First, we considered the case of a uniform magnetisation in $F$, $M(x) = \text{const}$, and of the absence of spin filters. Then, the LRTC consists of equal numbers of fully polarized triplet pairs with opposite directions of the total spin $s_{l,r}$ (the nematic case in terminology of Ref.[12]). In this case, the spin current $I_{sp}^{(xx)} = I_{sp}^{(yy)}$ is zero and the Josephson current $I_Q^{(xx)} = I_Q^{(yy)}$ is finite.

D. Change of the Currents due to two domain walls

We assume that the spacing between the nearest DWs is much larger than $\kappa_f$. In this case each DW contributes to the Josephson current independently from others. Thus, the correction to the Josephson current due to, for example two DWs, is given by

$$\delta I_Q^{(xx)} = -2\frac{(\kappa_{DW}^2 F_S^-)^2}{K_0\kappa_{cd}^2 \sinh^2(2\bar{L})}\left[\cosh(2\bar{L}) + \cosh(2\bar{l}) + \cosh(2\bar{d}l)\right] \sin \varphi,$$

where $\bar{l} = (\bar{l}_1 + \bar{l}_2)/2$, $\delta \bar{l} = \bar{l}_1 - \bar{l}_2$. According to the assumption above $(\delta \kappa_f) \gg 1$. This formula means that the DW reduces the critical current $I_c = I_{c0} + \delta I_c$ in the $P$-case and makes it finite in the $AP$-case. The decrease of the current $I_c = I_{c0} - |\delta I_c|$ in the $P$-case would be minimal if $\bar{l} = 0$, i.e., the two DWs are located in the center of the $F$ film.

III. CONCLUSIONS

We have calculated the Josephson charge $I_Q$ and spin $I_{sp}$ currents in an $S/F_m/F_l/F/F_l/F_m/S$ Josephson junction when the Josephson coupling is realized via different types of a long-range triplet component (LRTC). The superconducting condensate in a thin magnetic layer $F_m$ consists of singlet and triplet Cooper pairs penetrating from the S banks into the $F_m$ film. The spin filters $F_l$ pass only the triplet Cooper pairs which are long range in $F$ because the magnetic field $m$ in $F_m$ is perpendicular to the magnetisation vector $M|e_z$ in the F film. The long-range triplet Cooper pairs, penetrating into the $F$ film, rotate clockwise or counterclockwise. The results of the change of the Josephson currents due to a single domain wall are also summarized in Table I.

In Fig.2 we plot the temperature dependence of the Josephson critical current $I_{Q,0}(T)$ in the absence of a DW and a correction to the current due to a DW located in the center of the F film ($l = 0$), see also Appendix D for details of the numerics. One can see that the critical current $I_{Q,0}(T)$ and the correction due to a DW decrease monotonously with increasing the temperature. For completeness we show in Fig.3 the dependence $I_{Q,0}(J)$ for two temperatures. A similar dependence shows the correction to the Josephson current due to a DW.
may occur in the absence of the phase difference and the direction of the charge current depends on spins. The spontaneous currents may be the reason for the Josephson diode effect, discussed recently. All these results are summarized in Table I.

We have studied the change of the charge and spin currents in the presence of arbitrary number of DWs in the F film. It turns out that a DW reduces the critical Josephson current if the spin directions of the Cooper pairs injected from the right and left superconductors \( S_m \) are parallel \( (s_r = s_l) \). The critical current reaches a maximum if the DW is located in the center of the F film. In the case of an antiparallel spins, \( s_r = -s_l \), the critical current \( I_{c,0}^{(AP)} \) in the absence of a DW is zero, but becomes finite in the presence of a DW.

The case of two DWs, which may be considered as a model of a skyrmion, is particularly interesting. The dependence of the change of the critical current \( \delta I_{Q,\omega}^{(x)} \) due to two DWs is given by Eq. (67). In the case of parallel spins \( (s_r = s_l) \), the critical current \( I_{Q,c} \) has a maximum if the DWs are located in the center of the F film. In the case of antiparallel spins \( (s_r = -s_l) \), the maximum \( I_{Q,c} \) corresponds to the location of two DWs at the edges of the F film.

IV. ACKNOWLEDGEMENTS

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First we calculate the exact Green’s functions $\hat{G}_{ik}$ and show that they and the quasiclassical Green’s functions $\hat{g}^{(2)}(s)$ and $\hat{g}^{(3)}(s)$ describe the fully polarized triplet Cooper pairs with spin $s = \pm 1$. We use the Nambu indices defined in Ref. \cite{14}, so that $c_{n,s} = c_s$ and $c_{n,s} = c_{s}^\dagger$ for $n = 2$; $c_s = c_1$ for $s = 1$ ($\bar{s} = 2$) and $c_s = c_{\bar{s}}$ for $s = 2$ ($\bar{s} = 1$). The Green’s function $\hat{G}_{12}$ is

$$
\hat{G}_{12}(t,t') = -i \left\langle c_{n,s}(t) \hat{X}_{12} c_{n,s}(t^\dagger) \right\rangle 
= -i \left\langle c_{n,s}(t) \tau_1 \otimes \sigma_2 c_{n,s}(t^\dagger) \right\rangle 
= -i \left\langle c_s(t) \sigma_2 c_1^\dagger(t') + c_{\bar{s}}(t) \sigma_2 c_{\bar{s}}^\dagger(t') \right\rangle 
= -i \left\langle -c_{\bar{s}}(t) c_1(t') + c_s(z) c_{\bar{s}}(t') - c_{\bar{s}}^\dagger(z) c_1^\dagger(t') + c_s^\dagger(z) c_{\bar{s}}^\dagger(t') \right\rangle
$$

(A1)

**Appendix A: Green’s Functions $\hat{G}_{ik}$**

First we calculate the exact Green’s functions $\hat{G}_{ik}$ and show that they and the quasiclassical Green’s functions $\hat{g}^{(2)}(s)$ and $\hat{g}^{(3)}(s)$ describe the fully polarized triplet Cooper pairs with spin $s = \pm 1$. We use the Nambu indices defined in Ref. \cite{14}, so that $c_{n,s} = c_s$ and $c_{n,s} = c_{s}^\dagger$ for $n = 2$; $c_s = c_1$ for $s = 1$ ($\bar{s} = 2$) and $c_s = c_{\bar{s}}$ for $s = 2$ ($\bar{s} = 1$). The Green’s function $\hat{G}_{12}$ is

$$
\hat{G}_{12}(t,t') = -i \left\langle c_{n,s}(t) \hat{X}_{12} c_{n,s}(t^\dagger) \right\rangle 
= -i \left\langle c_{n,s}(t) \tau_1 \otimes \sigma_2 c_{n,s}(t^\dagger) \right\rangle 
= -i \left\langle c_s(t) \sigma_2 c_1^\dagger(t') + c_{\bar{s}}(t) \sigma_2 c_{\bar{s}}^\dagger(t') \right\rangle 
= -i \left\langle -c_{\bar{s}}(t) c_1(t') + c_s(z) c_{\bar{s}}(t') - c_{\bar{s}}^\dagger(z) c_1^\dagger(t') + c_s^\dagger(z) c_{\bar{s}}^\dagger(t') \right\rangle
$$

(A1)
and
\[
\hat{G}_{21}(t,t') = -i \left\langle c_{ns}(t) \hat{X}_{21} c_{n's'}^\dagger(t') \right\rangle
= -i \left\langle c_\uparrow(z)c_\uparrow(t') + c_\downarrow(z)c_\downarrow(t') - c_\uparrow(z)c_\downarrow(t') - c_\downarrow(z)c_\uparrow(t') \right\rangle
\] (A2)

Analogously, we obtain for \(\hat{G}_{11}\) and \(\hat{G}_{22}\)
\[
\hat{G}_{11}(t,t') = -i \left\langle c_\uparrow(z)c_\uparrow(t') + c_\downarrow(z)c_\downarrow(t') + c_\downarrow(z)c_\uparrow(t') + c_\uparrow(z)c_\downarrow(t') \right\rangle
\] (A3)
\[
\hat{G}_{22}(t,t') = -i \left\langle c_\uparrow(z)c_\uparrow(t') - c_\downarrow(z)c_\downarrow(t') - c_\downarrow(z)c_\uparrow(t') - c_\uparrow(z)c_\downarrow(t') \right\rangle
\] (A4)

Combining Eqs. (A1-A4), one can write
\[
\hat{G}^{(x)}(t,t') \equiv \hat{G}_{11} - s\hat{G}_{22} = -2i \left\langle c_\downarrow(t)c_\uparrow(t') + c_\uparrow(t)c_\downarrow(t') \right\rangle
\] (A5)
\[
\hat{G}^{(y)}(t,t') \equiv \hat{G}_{12} + s\hat{G}_{21} = 2i(-1)^s \left\langle c_\downarrow(t)c_\uparrow(t') + c_\uparrow(t)c_\downarrow(t') \right\rangle
\] (A6)

Eqs. (A5,A6) show that both Green’s functions \(\hat{G}^{(x)}\) and \(\hat{G}^{(y)}\) are off-diagonal in the Nambu-space and define triplet Cooper pairs with spin up (\(s = 1\)) and down (\(s = -1\)), which describe a fully polarized triplet component. Since the matrix structure of the Green’s functions does not change upon going over to the quasiclassical functions,
\[
\hat{g}_{BVE} = \frac{i}{\pi \nu} \int d\xi \hat{G}
\] (A7)

the same statement is true for matrix functions \(\hat{g}_{BVE}\).

Note the transformation suggested by Ivanov-Fominov
\[
\hat{g} = \hat{U} \cdot \hat{g}_{BVE} \cdot \hat{U}^\dagger
\] (A8)
\[
\hat{U} = \frac{1}{2} \left( \hat{X}_{00} + i\hat{X}_{33} \right) \cdot \left( \hat{X}_{00} - i\hat{X}_{33} \right)
\] (A9)

transform the function \(\hat{g}_{BVE}\) introduced in Ref[23] into the functions \(\hat{g}\), employed here. It does not change the relations because the matrix \(\hat{U}\) commutes with the matrices \(\hat{X}_{00}\) and \(\hat{X}_{33}\).

These functions arise as a result of the action of the spin filters (we set \(T = 1, U = \pm 1\)).
\[
\hat{G}^{(x)} = T_{33} \cdot \hat{G}_{11} \cdot T_{33}^\dagger
\] (A10)
\[
\hat{G}^{(y)} = T_{33} \cdot \hat{G}_{12} \cdot T_{33}^\dagger
\] (A11)
\[
T_{33} = \frac{1}{\sqrt{2}} \left( T + U \hat{X}_{33} \right)
\] (A12)

**Appendix B: Charge and Spin currents**

The charge density \(\rho\) is equal to
\[
\rho(t,t') = C_Q \sum_p \left\langle c_{ns}^\dagger(t) \hat{X}_{30} c_{n's'}(t') \right\rangle
= C_Q \sum_p \left\langle c_{ns}^\dagger(t) \hat{\tau}_3 c_{n's'}(t') \right\rangle = \left\langle c_\uparrow(t)c_\downarrow(t') - c_\downarrow(t)c_\uparrow(t') \right\rangle
= C_Q \sum_p \left\langle c_\uparrow(t)c_\uparrow(t') + c_\downarrow(t)c_\downarrow(t') - c_\uparrow(t)c_\downarrow(t') - c_\downarrow(t)c_\uparrow(t') \right\rangle
\] (B1)

where \(C_Q\) is a constant which will be defined below. The operators \(c_{ns}^\dagger(t'), c_{n's'}(t)\), as before, depend on times \(t, t'\).

For equals times \(t = t'\), we obtain
\[
\rho(t) = 2C_Q \sum_p \left\langle c_\uparrow^\dagger c_\uparrow + c_\uparrow^\dagger c_\downarrow \right\rangle
= -2iC_Q \sum_p \left\{ \hat{G} \right\}_{30} = \frac{2}{\pi} C_Q \nu(0) \{\hat{g}_{BVE}\}_{00}
\] (B2)
Here, $\hat{g}_{BVE}$ is the quasiclassical Green’s function derived in [20]. The magnetic moment is

$$M = C_M \sum_p \langle c^\dagger_{ns}(t) \hat{X}_{03} c_{n's'}(t') \rangle$$

$$= C_M \sum_p \langle c^\dagger_s \sigma_3 c_{s'} + c_s \sigma_3 c^\dagger_{s'} \rangle$$

$$= C_M \sum_p \langle c^\dagger_s(t) c_3(t') + c^\dagger_s(t) c_3(t') \rangle$$

(B3)

For equal times $t = t'$, we obtain

$$M(t) = 2 C_M \sum_p \langle c^\dagger_s(t) c_3(t) + c^\dagger_s(t) c_3(t) \rangle$$

$$\Rightarrow 2i C_M \sum_p \{ \hat{G} \} = \frac{2}{\pi} C_M \{ \hat{g}_{BVE} \}_{33}$$

(B4)

To find the formula for the charge (spin) current, consider the Usadel equation for the Keldysh function

$$\dot{\tau}_3 \cdot \partial_t \hat{g} + \partial_t \hat{g} \cdot \tau_3 = D_F \partial_x (\hat{g} \cdot \partial_x \hat{g}) + i J \left[ \hat{X}_{33}, \hat{g} \right]$$

(B5)

Introducing $\tilde{t} = (t + t')/2$ and $\tau = t - t'$, Eq. (B5) can be written as

$$\frac{1}{2} \partial_{\tilde{t}}[\tilde{\tau}_3, \hat{g}] + \partial_\tau [\tilde{\tau}_3, \hat{g}] = D_F \partial_x (\hat{g} \cdot \partial_x \hat{g}) + i J \left[ \hat{X}_{33}, \hat{g} \right]$$

(B6)

We multiply Eq. (B6) first by $\hat{X}_{30}$, then by $\hat{X}_{03}$ and calculate the trace. We get the law of conservation of the charge and the magnetization

$$\frac{\partial \rho}{\partial \tilde{t}} = - \partial_\tau j_Q \quad \frac{\partial M}{\partial \tilde{t}} = - \partial_\tau j_{sp}$$

(B7)

where the charge current $j_Q$ is equal to

$$j_Q = - \sigma_n 2\pi e \sum_{\omega \geq 0} \frac{1}{4} \text{Tr} \{ \tilde{\tau}_3 \hat{g} \nabla \hat{g} \}$$

$$= - \frac{\sigma_n}{e} 2\pi T \sum_{\omega \geq 0} \{ \hat{g} \nabla \hat{g} \}_{30}$$

(B8)

and the spin current $j_{sp}$ is given by

$$j_{sp} = - \mu_B \sigma_n (i\pi T) \sum_{\omega \geq 0} \nabla \{ \hat{g} \}_{03}$$

(B9)

The charge density $\rho$ is

$$\rho = e \nu(0) (i2\pi T) \sum_{\omega \geq 0} \{ \hat{g} \}_{00}$$

(B10)

The Drude conductivity $\sigma_n$ is

$$\sigma_n = 2\nu(0) D_n e^2$$

(B11)

The magnetic moment $M_z$ is (see, for example, [20], Eq. (A28))

$$M_z = \mu_B \nu(0) (i2\pi T) \sum_{\omega \geq 0} \{ \hat{g} \}_{33}$$

(B12)
Appendix C: Coefficients in the Change of the Currents due to two DWs

The coefficients $\hat{C}^{(A,B)}$ and $\hat{S}^{(A,B)}$ in Eqs. (58-59) are determined by Eqs. (19-20). They are equal to

$$\hat{C}^{(A)} = \frac{\kappa_b}{2\kappa_\omega} \left( \bar{X}_r + \bar{X}_l \right) \cos(\varphi/2) F_{S-},$$  \hspace{1cm} (C1)
$$\hat{C}^{(B)} = \frac{\kappa_b}{2\kappa_\omega} \left( \bar{X}_r - \bar{X}_l \right) \sin(\varphi/2) F_{S-},$$  \hspace{1cm} (C2)
$$\hat{S}^{(A)} = \frac{\kappa_b}{2\kappa_\omega} \left( \bar{X}_r - \bar{X}_l \right) \cos(\varphi/2) F_{S-},$$  \hspace{1cm} (C3)
$$\hat{S}^{(B)} = \frac{\kappa_b}{2\kappa_\omega} \left( \bar{X}_r + \bar{X}_l \right) \sin(\varphi/2) F_{S-}.$$  \hspace{1cm} (C4)

The matrices $\hat{a}, \hat{b}$ equal

$$\hat{a} = -4r_\omega \frac{\cosh(\bar{L} + \bar{l})}{\sinh(2\bar{L})} A \cos(\varphi/2) \bar{X}_{n2};$$ \hspace{1cm} (C5)
$$\hat{b} = -4r_\omega \frac{\cosh(\bar{L} + \bar{l})}{\sinh(2\bar{L})} B \sin(\varphi/2) \bar{X}_{n2}.$$ \hspace{1cm} (C6)

with $n = 1, 2$ for $y$- and $x$-chiralities.

Appendix D: Details of the numerics

The critical current $\tilde{I}_0(t)$ of the considered Josephson junction without DWs is

$$\tilde{I}_0(t) = I_0 N(t),$$  \hspace{1cm} (D1)
$$I_0 = \frac{\sigma_F \Delta}{e} \xi_\Delta \kappa_b^2$$ \hspace{1cm} (D2)

$$N(t) = 2\pi t \sum_{n \geq 0} \left[ \operatorname{Im} \frac{\tilde{\Delta}(t)}{\sqrt{(t_n + i \bar{J}_m)^2 + \tilde{\Delta}(t)^2}} \right]^2 \frac{1}{\sqrt{t_n} \sinh \left( 2\bar{L} \sqrt{t_n} \right)}$$ \hspace{1cm} (D3)

where $\xi_\Delta = \sqrt{D_F/2\Delta}$, $\bar{L} = L/\xi_\Delta$, $t_n = \pi t(2n + 1)$, $t = T/\Delta(0)$.

The correction to the current due to a single DW is

$$\delta \bar{I}(t) = -I_{DW} N_{DW}(t),$$  \hspace{1cm} (D4)
$$I_{DW} = \frac{\sigma_F \Delta}{e} (\xi_\Delta \kappa_{DW} \kappa_b^2)^2 \xi_\Delta,$$  \hspace{1cm} (D5)

$$N_{DW}(t) = 2\pi t \sum_{n \geq 0} \left[ \operatorname{Im} \frac{\tilde{\Delta}(t)}{\sqrt{(t_n + i \bar{J}_m)^2 + \tilde{\Delta}(t)^2}} \right]^2 \frac{\cosh \left( \sqrt{t_n} (\bar{L} + \bar{l}) \right) \cosh \left( \sqrt{t_n} (\bar{L} - \bar{l}) \right)}{t_n \left( \sinh \left( 2\bar{L} \sqrt{t_n} \right) \right)^2}$$ \hspace{1cm} (D6)

The temperature dependence of $\tilde{\Delta}(t) \equiv \Delta(T)/\Delta(0)$ can be approximated as

$$\Delta(T) \approx \Delta(0) \tanh \left( 1.74 \sqrt{\left( T_c/T - 1 \right)} \right)$$ \hspace{1cm} (D7)

In the limits of $T = 0$ and $T \to T_c$ it reproduces the limiting expressions (see, for example $^{83}$)

$$2\Delta(0) \approx 3.5T_c$$ \hspace{1cm} (D8)
$$\Delta(T)|_{T \to T_c} \approx 3.06 \sqrt{(T_c - T)/T_c}$$ \hspace{1cm} (D9)