$K^0$ form factor at order $p^6$ of chiral perturbation theory

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Abstract

This paper describes the calculation of the electromagnetic form factor of the $K^0$ meson at order $p^6$ of chiral perturbation theory which is the next-to-leading order correction to the well-known $p^4$ result achieved by Gasser and Leutwyler. On the one hand, at order $p^6$ the chiral expansion contains 1- and 2-loop diagrams which are discussed in detail. Especially, a numerical procedure for calculating the irreducible 2-loop graphs of the sunset topology is presented. On the other hand, the chiral Lagrangian $\mathcal{L}^{(6)}$ produces a direct coupling of the $K^0$ current with the electromagnetic field tensor. Due to this coupling one of the unknown parameters of $\mathcal{L}^{(6)}$ occurs in the contribution to the $K^0$ charge radius.

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1 Introduction

Chiral perturbation theory has established itself as a powerful effective theory of low energy strong
interactions. Based on the symmetry of the underlying QCD, chiral perturbation theory produces
a systematic low-energy expansion of the observables in this regime. Unfortunately, because of the
non-renormalizability of the effective theory, higher powers in the energy expansion require higher
loop Feynman integrals and as input an ever increasing number of renormalization constants. The
$p^4$-Lagrangian involves ten free parameters which were determined in the fundamental papers of
Gasser and Leutwyler [1]. For the $p^6$ Lagrangian there are already more than a hundred of them
[2]. Many interesting low energy hadronic amplitudes have now been calculated to order $p^4$ and to
order $p^6$ [3] in the case of $SU(2) \times SU(2)$ chiral perturbation theory which involves only one mass
scale. Recent progress in the calculation of massive two-loop integrals allows now calculations to
order $p^6$ in the full $SU(3) \times SU(3)$ chiral perturbation theory. With so many unknown parameters,
however, one may ask about the usefulness of such calculations. Here are a few arguments in favour:

- In a given class of experiments, such as the electromagnetic and weak form factors of the
  light mesons, only a limited number of renormalization constants enter and relations between
  amplitudes can be tested [4].

- The unknown constants enter only polynomially and precision experiments could separate
  the unambiguous predictions (see e.g. the subsequent discussion).

- Knowledge of the exact low-energy functional form of an amplitude may be important for the
  experimental extraction of low-energy parameters such as charge radii.

- The results may be used in model calculations which predict the polynomial terms. These
  calculations can then be compared with experiment.

- The question of convergence of the chiral perturbation theory may be addressed.

We have embarked on the program of a full $p^6$ and two-loop analysis of the various semi-leptonic
form factors of pions and kaons in full $SU(3) \times SU(3)$ chiral perturbation theory. From the point
of view of practicality it seems prudent to begin such an ambitious program with the simplest
process possible involving the fewest number of renormalization parameters. Here this would be
the electromagnetic form factor of the neutral kaon. There exists no tree level contribution to
$O(p^4)$ and therefore the one-loop contributions must be finite. At $O(p^6)$, however, a tree level term
exists which couples the $K^0$ directly to the electromagnetic field tensor of the external photon, and
a new renormalization parameter enters. In this paper we will present the results of our $O(p^6)$
calculation of the neutral kaon’s electromagnetic form factor. It is defined by the matrix element of
the electromagnetic current between neutral kaon states:

$$\langle K^0, p' | J_\mu | K^0, p \rangle = (p + p')_\mu F^{K^0}(q^2),$$

(1)

where $q = p - p'$ is the momentum of the photon and $J_\mu$ stands for the electromagnetic current
carried by the light quarks which is a linear combination of the chiral currents $V^3_\mu$ and $V^8_\mu$:

$$J_\mu = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s = V^3_\mu + \frac{1}{\sqrt{3}} V^8_\mu.$$  

(2)

We neglect the contributions of the heavy quarks.

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2 The effective Lagrangian and the diagrams

Chiral perturbation theory is formulated in terms of an effective Lagrangian involving an increasing number of covariant derivatives as well as external sources,

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \ldots \]  

where

\[ \mathcal{L}^{(2)} = \frac{F^2}{4} \text{Tr}(D_\mu U D^\mu U^\dagger) + \frac{F^2}{4} \text{Tr}(\chi U^\dagger + U \chi^\dagger) \]  

\[ \mathcal{L}^{(4)} = L_1 \{ \text{Tr}(D_\mu U D^\mu U^\dagger) \}^2 + L_2 \text{Tr}(D_\mu U D_\nu U^\dagger) \text{Tr}(D^\nu U D^\mu U^\dagger) \]  

\[ + L_3 \text{Tr}(D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger) + L_4 \text{Tr}(D_\mu U D^\mu U^\dagger) \text{Tr}(\chi U^\dagger + U \chi^\dagger) \]  

\[ + L_5 \text{Tr}(D_\mu U D^\mu U^\dagger (\chi U^\dagger + U \chi^\dagger)) + L_6 \{ \text{Tr}(\chi U^\dagger + U \chi^\dagger) \}^2 \]  

\[ + L_7 \{ \text{Tr}(\chi U - U^\dagger \chi) \}^2 + L_8 \text{Tr}(\chi U^\dagger \chi U^\dagger + U \chi^\dagger U \chi^\dagger) \]  

\[ - i L_9 \text{Tr}(L_{\mu\nu} D^\mu U D^\nu U^\dagger + R_{\mu\nu} D^\mu U D^\nu U) + L_{10} \text{Tr}(L_{\mu\nu} U R^{\mu\nu} U^\dagger) \]  

\[ \mathcal{L}^{(6)} = \frac{\beta}{F^2} F_{\mu\nu} \text{Tr}(\partial_\mu U \partial^\nu U^\dagger) + \ldots, \]  

\[ U(x) = \exp[i\phi(x)/F] \] is the unitary 3×3 matrix made up of the Goldstone fields, \( F \) is the pion decay constant in the chiral limit, \( D_\mu \) the covariant derivative involving external vector- and axial vector sources of the corresponding currents, \( F_{\mu\nu} \) is the electromagnetic field tensor, and \( \chi \) is related to the quark mass matrix. In \( \mathcal{L}^{(6)} \) we have only displayed the terms relevant to the \( K^0 \) form factor. The \( \mathcal{L}^{(6)} \) term is extracted from reference [2] where in their notation \( \beta = \frac{1}{2} F^2 (m_K^2 - m_\pi^2) (2B_{24} - B_{25}) \).

Each term in \( \mathcal{L}_{\text{eff}} \) produces vertices of 2, 4, 6, ... mesons with or without attached photon. According to Weinberg’s power counting theorem [8] the form factor at order \( p^6 \) of the chiral expansion is the sum of all 2-loop diagrams with vertices from \( \mathcal{L}^{(2)} \), all 1-loop diagrams with vertices from \( \mathcal{L}^{(2)} \) and one vertex from \( \mathcal{L}^{(4)} \), and a tree graph with two vertices from \( \mathcal{L}^{(4)} \) or one vertex from \( \mathcal{L}^{(6)} \). Each diagram has two external meson lines representing the incoming and outgoing \( K^0 \) and one external photon line. All relevant diagrams for the unrenormalized \( K^0 \) form factor are given in fig. 1. To indicate the origin of the vertices in the Feynman diagrams we use the following notation: \( \mathcal{L}^{(2)} \)-vertices are denoted by filled circles •, \( \mathcal{L}^{(4)} \)-vertices by filled squares ■ and an \( \mathcal{L}^{(6)} \)-vertex by an open square □. Only the \( \mathcal{L}^{(6)} \) vertex is new.

In each loop diagram, the internal lines symbolize an arbitrary \( \pi, K, \) or \( \eta \) meson and they must be summed over. Due to the derivative couplings of \( \mathcal{L}_{\text{eff}} \) the Feynman rules of the vertices are quite involved. For the relevant vertices they are discussed in appendix [7].

The diagrams (1a) and (1b) represent the well-known result of order \( p^4 \) which is the leading order approximation of the \( K^0 \) form factor [1]. The pure \( p^6 \) diagrams consist of two groups:

- the reducible diagrams, i.e. the 1-loop diagrams (2a)–(2e), the 2-loop diagrams (3a)–(3f), where the loops are independent of each other, and the tree graph (4)
- the irreducible 2-loop diagrams (5a)–(5c).

All diagrams of the first group can be reduced to elementary 1-loop integrals. The results are presented in section [8]. The calculation of the irreducible 2-loop diagrams is much harder to carry out: since in \( SU(3) \) chiral perturbation theory there are three different mass scales of the same order of magnitude \( (m_\pi, m_K, m_\eta) \), these 2-loop integrals can no longer be expressed by elementary
Figure 1: The diagrams for the $K^0$-form factor up to order $p^6$. $\mathcal{L}^{(2)}$-vertices are denoted by filled circles ($\bullet$), $\mathcal{L}^{(4)}$-vertices by filled squares ($\blacksquare$), and an $\mathcal{L}^{(6)}$-vertex by an open square ($\square$).
analytical functions. In section 4 we discuss their contributions to the \( K^0 \) form factor, and in appendix B we describe our numerical algorithm for the evaluation of these diagrams.

The contributions of the diagrams can be classified further according to their dependence on the momentum \( q \) of the photon. Diagrams, where the photon momentum does not flow through, are polynomials in \( q^2 \): (1b), (2e), (3d), (5a), and (5c) do not depend on \( q^2 \), (2c) is a polynomial in \( q^2 \) of degree \( \leq 1 \), and (4) has at most degree 2. Since the normalization of the \( K^0 \) form factor is fixed by the charge of the particle due to the Ward-Fradkin-Takahashi identity \[ F_{K^0}(0) = 0 \], the interesting contributions are those with genuine \( q^2 \) dependence. Nevertheless, we calculate all diagrams, because this procedure offers additional possibilities to check the calculation.

### 3 Reducible diagrams

The diagrams (1a)–(3f) are 1-loop integrals or products thereof and can be calculated analytically. Inserting the Feynman rules into the vertices yields integrands of the structure

\[
\frac{V(k^2, kp_1, kp_2, kl, l^2, lp_1, lp_2)}{P}
\]

where \( p_1 \) and \( p_2 \) are the external momenta of the incoming and the outgoing \( K^0 \), \( k \) and \( l \) are the internal loop momenta, the denominator \( P \) is a product of scalar propagator factors depending on the topology of the diagram, and the numerator \( V \) is a polynomial coming from the vertex factors. These integrals can be expressed by a set of basic 1-loop functions in the following way: First, \( p_1 \) and \( p_2 \) are transformed into \( p = p_1 + p_2 \) and the momentum \( q = p_1 - p_2 \) of the photon. Then, some factors in the numerator are canceled against some propagator factors in the denominator via

\[
k^2 = P(k, m^2) + m^2
\]

\[
kq = \frac{1}{2} \left[ P(k + q, m^2) + m^2 - k^2 - q^2 \right]
\]

\[
l^2 = P(l, m^2) + m^2
\]

\[
lq = \frac{1}{2} \left[ P(l + q, m^2) + m^2 - l^2 - q^2 \right]
\]

with \( P(k, m) = k^2 - m^2 \). What remains are 1-loop 1-point and 2-point functions with tensor numerators which are decomposable into Lorentz covariants. As our calculation shows, there are in fact only two 1-loop functions by means of which all reducible loop contributions of the \( K^0 \) form factor can be expressed: On the one hand the scalar 1-point function with its derivative,

\[
A(m^2) = \quad \quad \text{and} \quad \quad A'(m^2) = \frac{\partial A}{\partial m^2},
\]

and on the other hand one specific tensor coefficient of the 2-point function, also with derivative,

\[
B(q^2, m^2) = \quad \quad \quad \quad \text{and} \quad \quad B'(q^2, m^2) = \frac{1}{2} \frac{\partial B}{\partial m^2}.
\]
These basic 1-loop integrals are defined in detail in appendix A. In the following equations we give the contribution \( \Delta \) of each reducible diagram to the \( K^0 \) form factor in terms of the basic functions \( A \) and \( B \):

\[
\Delta^{(1a)} = \frac{1}{F^2} \left\{ B(q^2, m_{\pi}^2) - B(q^2, m_K^2) \right\}
\]

\[
\Delta^{(1b)} = -\frac{1}{2F^2} \left\{ A(m_{\pi}^2) - A(m_K^2) \right\}
\]

\[
\Delta^{(2a)} = \frac{4}{F^4} \left\{ B(q^2, m_{\pi}^2)[2(m_{\pi}^2 + 2m_K^2)L_4 + 2(m_{\pi}^2 + m_K^2)L_5 - q^2L_3] - B(q^2, m_K^2)[2(m_{\pi}^2 + 2m_K^2)L_4 + 4m_K^2L_5 - q^2L_3] \right\}
\]

\[
\Delta^{(2b)} = \frac{2}{F^4} \left\{ B(q^2, m_{\pi}^2)[4(m_{\pi}^2 + m_K^2)L_4 + 4m_K^2L_5 + q^2L_9] - B(q^2, m_K^2)[4(m_{\pi}^2 + 2m_K^2)L_4 + 4m_K^2L_5 + q^2L_9] \right\}
\]

\[
\Delta^{(2c)} = -\frac{1}{F^4} \left\{ A(m_{\pi}^2)[4(m_{\pi}^2 + m_K^2)L_4 + 4(m_{\pi}^2 + m_K^2)L_5 + q^2L_9] - A(m_K^2)[4(m_{\pi}^2 + m_K^2)L_4 + 8m_K^2L_5 + q^2L_9] \right\}
\]

\[
\Delta^{(2d)} = \frac{16}{F^4} \left\{ B(q^2, m_{\pi}^2)[L_4(2m_{\pi}^2 + m_K^2) + 5m_K^2] - B(q^2, m_K^2)[L_4(2m_{\pi}^2 + m_K^2) + 5m_{\pi}^2] + B'(q^2, m_{\pi}^2)m_{\pi}^2[L_4(2m_{\pi}^2 + m_K^2) + 5m_{\pi}^2 - 2L_6(2m_{\pi}^2 + m_K^2) - 2L_8m_K^2] - B'(q^2, m_K^2)m_K^2[L_4(2m_{\pi}^2 + m_K^2) + 5m_K^2 - 2L_6(2m_{\pi}^2 + m_K^2) - 2L_8m_{\pi}^2] \right\}
\]

\[
\Delta^{(2e)} = \frac{1}{F^4} \left\{ 4A(m_{\pi}^2)[L_4(2m_{\pi}^2 + m_K^2) + 5m_K^2] - 4A(m_K^2)[L_4(2m_{\pi}^2 + m_K^2) + 5m_{\pi}^2]ight. \\
- 4A'(m_{\pi}^2)m_{\pi}^2[L_4(2m_{\pi}^2 + m_K^2) + 5m_K^2 - 2L_6(2m_{\pi}^2 + m_K^2) - 2L_8m_K^2] + 4A'(m_K^2)m_K^2[L_4(2m_{\pi}^2 + m_K^2) + 5m_{\pi}^2 - 2L_6(2m_{\pi}^2 + m_K^2) - 2L_8m_{\pi}^2] \right\}
\]

\[
\Delta^{(3a)} = -\frac{2}{F^4} \left\{ B(q^2, m_{\pi}^2)^2 + B(q^2, m_{\pi}^2)B(q^2, m_K^2) - 2B(q^2, m_{\pi}^2)^2 \right\}
\]

\[
\Delta^{(3b)} = \frac{1}{12F^4} \left\{ 20A(m_{\pi}^2)B(q^2, m_{\pi}^2) + 10A(m_K^2)B(q^2, m_K^2) + 3A(m_{\pi}^2)B(q^2, m_K^2) - 30A(m_K^2)B(q^2, m_{\pi}^2) \right\}
\]

\[
\Delta^{(3c)} = \frac{1}{6F^4} \left\{ 3B(q^2, m_{\pi}^2)[A(m_{\pi}^2) + 2A(m_K^2) + A(m_{\pi}^2)] - 4B(q^2, m_{\pi}^2)[A(m_{\pi}^2) + 2A(m_K^2)] + 2B'(q^2, m_{\pi}^2)m_{\pi}^2[A(m_{\pi}^2) - 3A(m_K^2)] + B'(q^2, m_K^2)m_K^2[A(m_K^2)(3m_{\pi}^2 + m_{\pi}^2)] \right\}
\]

\[
\Delta^{(3d)} = -\frac{1}{24F^4} \left\{ 3A(m_{\pi}^2)A(m_{\pi}^2) + 6A(m_{\pi}^2)^2 - 6A(m_K^2)A(m_{\pi}^2) - 8A(m_{\pi}^2)^2 + 2m_{\pi}^2A'(m_{\pi}^2)A(m_{\pi}^2) - 6m_{\pi}^2A'(m_{\pi}^2)A(m_{\pi}^2) \right\}
\]
are independent.

In summary, the total contribution of diagrams (1a) and (1b) to the $\eta$ form factor vanishes, the only diagrams which are subject to renormalization are the $O(p^4)$ diagrams (1a) and (1b). Due to renormalization there are three modifications of these diagrams:

- Wave function renormalization of the outer $K^0$ lines requires an additional factor $1 + \delta Z_K$ where $\delta Z_K$ is the well-known $O(p^4)$ result (cf. [1])

$$\delta Z_K = -\frac{1}{4F^2} \left\{ A(m_{\eta}^2) + 2A(m_K^2) + A(m_\pi^2) + 32L_4(2m_K^2 + m_\pi^2) + 32L_5m_\eta^2 \right\}.$$  \hspace{1cm} (28)

- Renormalization of the pion decay constant $F_\pi$ has to be taken into account: the plain $O(p^4)$ result contains a factor $1/F^2$ where $F$ can no longer be identified with $F_\pi$ at order $p^6$. Instead, one has to consider the $O(p^4)$ correction $F_\pi = F(1 + \delta f)$ with (cf. [1])

$$\delta f = \frac{1}{2F^2} \left\{ A(m_K^2) + 2A(m_\pi^2) + 8L_4(2m_K^2 + m_\pi^2) + 8L_5m_\pi^2 \right\}.$$  \hspace{1cm} (29)

- Analogously, the mass renormalizations are to be included, i.e. the masses $m$ appearing in (15ff) are related to the physical masses $m_{ph}$ via $m_{ph}^2 = m^2 + \Sigma(m_{ph}^2)$ where $\Sigma$ stands for the $O(p^4)$ self energies (cf. [1])

$$-\Sigma_\pi(m_\pi^2) = \frac{1}{6F^2} \left\{ -A(m_{\eta}^2)m_\pi^2 + 3A(m_\pi^2)m_\pi^2 + 48L_4m_\pi^2(2m_K^2 + m_\pi^2) + 48L_5m_\eta^2 + 96L_6m_\pi^2(-2m_K^2 - m_\pi^2) + 96L_8m_\pi^4 \right\}.$$  \hspace{1cm} (30)

$$-\Sigma_K(m_K^2) = \frac{1}{12F^2} \left\{ A(m_{\eta}^2)(3m_{\eta}^2 + m_\pi^2) + 96L_4m_\pi^2(2m_K^2 + m_\pi^2) + 96L_5m_K^4 + 192L_6m_\pi^2(-2m_K^2 - m_\pi^2) - 192L_8m_K^4 \right\}.$$  \hspace{1cm} (31)

In summary, the total contribution of diagrams (1a) and (1b) to the $K^0$ form factor at order $p^6$ is given by

$$\left( 1 + \delta Z_K \right) \left( \Delta^{(1a)} + \Delta^{(1b)} \right) \bigg|_{m^2 = m_{ph}^2 - \Sigma(m_{ph}^2), F = F(1 - \delta f)}.$$  \hspace{1cm} (32)
Renormalization corrections of the other diagrams are of order $p^8$ and can be neglected.

The sum of all reducible diagrams including the above renormalization corrections takes the following compact form:

$$F^4 \Delta^{(\text{red})} = \frac{1}{24} \left[ -A(m_0^2) \left[ 4 A(m_K^2) - 8 B(q^2, m_K^2) - A(m_K^2) + 2 B(q^2, m_K^2) \right] + 16 A(m_K^2)^2 + 10 A(m_K^2) A(m_K^2) - 23 A(m_K^2)^2 - 80 A(m_K^2) B(q^2, m_K^2) + 32 A(m_K^2) B(q^2, m_K^2) - 28 A(m_K^2) B(q^2, m_K^2) + 70 A(m_K^2) B(q^2, m_K^2) + 96 B(q^2, m_K^2)^2 - 48 B(q^2, m_K^2) B(q^2, m_K^2) - 48 B(q^2, m_K^2)^2 \right] + 4 q^2 L_3 \left[ B(q^2, m_K^2) - B(q^2, m_K^2) \right] + (4 m_0^2 L_5 + q^2 L_9 \left[ A(m_K^2) - 2 B(q^2, m_K^2) - A(m_K^2) + 2 B(q^2, m_K^2) \right].$$

Note that the mass derivatives $A'$ and $B'$ and the $\mathcal{L}^{(4)}$ parameters $L_4, L_6,$ and $L_8$ which occurred in some diagrams have vanished. The only $\mathcal{L}^{(4)}$ parameters which are relevant for the $K^0$ form factor at order $p^6$ are $L_3, L_5,$ and $L_9.

### 4 Irreducible diagrams

In the irreducible diagrams (5a)–(5c) the 2-loop integrations are not independent of each other as they were in the reducible graphs. That is why genuine 2-loop functions enter the stage which cannot be expressed by 1-loop integrals only.

Inserting the Feynman rules yields integrands with a similar structure as in (3), e.g. for diagram (5b)

$$\frac{V(k^2, kp_1, kp_2, kl, l^2, lp_1, lp_2)}{P(k+p_1, m_1^2) P(k+p_2, m_2^2) P(k+l, m_3^2) P(l, m_3^2)},$$

where $V$ is a polynomial of degree equal to the number of vertices. After canceling factors via

$$kp_1 = \frac{1}{2} \left[ P(k + p_1, m_1^2) + m_0^2 - k^2 - p_1^2 \right]$$

$$kp_2 = \frac{1}{2} \left[ P(k + p_2, m_2^2) + m_0^2 - k^2 - p_2^2 \right]$$

$$kl = \frac{1}{2} \left[ P(k + l, m_3^2) + m_0^2 - k^2 - l^2 \right]$$

$$l^2 = P(l, m_3^2) + m_0^2$$

we are left with reducible integrals which can be calculated analytically, and with some genuine 2-loop integrals of the sunset-topology, i.e. the 3-point functions

$$T_{\alpha_1, \alpha_2, \beta} (q^2; p_1^2, p_2^2, m_0^2, m_1^2, m_2^2, m_3^2) = \int \frac{d^D k}{i(2\pi)^D} \frac{d^D l}{i(2\pi)^D} \frac{(l p_1)^{\alpha_1} (l p_2)^{\alpha_2} (k^2)^\beta}{P(k + p_1, m_0^2) P(k + p_2, m_1^2) P(k + l, m_2^2) P(l, m_3^2)}$$

and the 2-point functions

$$S_{\alpha, \beta} (p^2; m_1^2, m_2^2, m_3^2) = \int \frac{d^D k}{i(2\pi)^D} \frac{d^D l}{i(2\pi)^D} \frac{(lp)^\alpha (k^2)^\beta}{P(k + p, m_1^2) P(k + l, m_2^2) P(l, m_3^2)}.$$
In diagram (5b), five different mass flows of intermediate mesons must be regarded:

\[ \Delta^{(5b)} = \Delta^{(5b)}_{\pi^+} + \Delta^{(5b)}_{\pi^-} + \Delta^{(5b)}_{\pi^0} + \Delta^{(5b)}_{\pi^+} + \Delta^{(5b)}_{\pi^-}, \]  

where \( \Delta^{(5b)}_{rsl} \) means that meson \( r \) couples to the photon and the other two lines are mesons of type \( s \) and \( t \). Each mass flow is handled separately, and its contribution to the \( K^0 \) form factor can be expressed in terms of the basic 1- and 2-loop functions \( A, B, S_{\alpha,\beta}, T_{\alpha_1,\alpha_2,\beta} \), where for the latter at most the tensor indices

\[ S_{2,0}, S_{1,1}, S_{1,0}, S_{0,0}, S_{0,1}, S_{0,2}, T_{0,0,0,0}, T_{0,0,0,1}, T_{0,0,0,2}, T_{0,0,3}, T_{1,0,0}, T_{1,0,1}, T_{1,0,2}, T_{1,1,0}, T_{1,1,1} \]

are needed. In the following table we give the detailed result for each mass flow. We omit the arguments of the 2-loop functions \( S_{\alpha,\beta} \) and \( T_{\alpha_1,\alpha_2,\beta} \): for mass flow \( \Delta^{(5b)}_{rsl} \) they are understood to be \( S_{\alpha,\beta}(m_K^2; m_T^2, m_s^2, m_t^2) \) and \( T_{\alpha_1,\alpha_2,\beta}(q^2; m_K^2, m_K^2, m_s^2, m_t^2, m_s^2, m_t^2) \).

\[ \Delta^{(5b)}_{\pi^+} = \frac{1}{8 F^4 m_K^2 (4m_K^2 - q^2)} \left\{ \begin{array}{c} - T_{1,1,1} \cdot 32 m_K^2 + T_{1,1,0} \cdot 16 m_K^2 (4m_K^2 - q^2) + T_{1,0,2} \cdot 16 m_K^2 \\ + T_{1,0,1} \cdot 8 m_K^2 (-4m_K^2 + q^2) - T_{0,0,3} \cdot 2 m_K^2 + T_{0,0,2} \cdot m_K^2 (4m_K^2 - q^2) \\ + S_{2,0} \cdot 16 (2m_K^2 - q^2) + S_{1,1} \cdot 8 (-4m_K^2 + q^2) + S_{1,0} \cdot 8 m_K^2 (4m_K^2 - q^2) \\ + S_{0,2} \cdot (6m_K^2 - q^2) + S_{0,1} \cdot 2 m_K^2 (-4m_K^2 + q^2) \\ + A(m_K^2)^2 \cdot (4m_K^4 - 8m_K^2 m_T^2 - m_K^2 q^2 + 2m_T^2 q^2) \\ + A(m_T^2) B(q^2, m_K^2) \cdot 4 m_K^2 (4m_K^2 - q^2) \end{array} \right\} \]  

\[ \Delta^{(5b)}_{\pi^-} = \frac{1}{24 F^4 m_K^2 (4m_K^2 - q^2)} \left\{ \begin{array}{c} - T_{0,0,3} \cdot 18 m_K^2 + T_{0,0,2} \cdot 3 m_K^2 (28m_K^2 - 3q^2) + T_{0,0,1} \cdot 8 m_K^4 (-16m_K^2 + 3q^2) \\ + T_{0,0,0} \cdot 16 m_K^6 (4m_K^2 - q^2) + S_{0,2} \cdot 3 (10m_K^2 - q^2) + S_{0,1} \cdot 2 m_K^2 (-44m_K^2 + 5q^2) \\ + S_{0,0} \cdot 8 m_K^4 (8m_K^2 - q^2) \\ + A(m_K^2)^2 A(m_T^2) \cdot (4m_K^4 - 8m_K^2 m_T^2 - m_K^2 q^2 + 2m_T^2 q^2) \\ + A(m_T^2) B(q^2, m_K^2) \cdot 10 m_K^2 (4m_K^2 - q^2) \\ + A(m_T^2) B(q^2, m_K^2) \cdot 10 m_T^2 (4m_K^2 - q^2) \end{array} \right\} \]  

\[ \Delta^{(5b)}_{\pi^0} = \frac{1}{12 F^4 m_K^2 (4m_K^2 - q^2)} \left\{ \begin{array}{c} - T_{1,1,1} \cdot 24 m_K^2 + T_{1,1,0} \cdot 12 m_K^2 (4m_K^2 - q^2) + T_{1,0,2} \cdot 24 m_K^2 \\ + T_{1,0,1} \cdot 12 m_K^2 (-4m_K^2 + q^2) - T_{0,0,3} \cdot 24 m_K^4 + T_{0,0,1} \cdot 12 m_K^4 (6m_K^2 - q^2) \\ + T_{0,0,0} \cdot 12 m_K^6 (-4m_K^2 + q^2) + S_{2,0} \cdot 12 (2m_K^2 - q^2) + S_{1,1,1} \cdot 10 (-4m_K^2 + q^2) \\ + S_{1,0} \cdot 8 m_K^2 (4m_K^2 - q^2) + S_{0,2} \cdot (4m_K^2 - q^2) + S_{0,1} \cdot 24 m_K^4 \\ + S_{0,0} \cdot 4 m_K^4 (-10m_K^2 + q^2) \\ + A(m_K^2)^2 \cdot 3 m_K^2 (-4m_K^2 + q^2) \\ + A(m_T^2) B(q^2, m_K^2) \cdot 2 m_K^2 (-4m_K^2 + q^2) \end{array} \right\} \]  

\[ \Delta^{(5b)}_{\pi^+} = \frac{1}{24 F^4 m_K^2 (4m_K^2 - q^2)} \left\{ \begin{array}{c} - T_{1,1,1} \cdot 24 m_K^2 + T_{1,1,0} \cdot 12 m_K^2 (4m_K^2 - q^2) + T_{1,0,2} \cdot 24 m_K^2 \\ + T_{1,0,1} \cdot 12 m_K^2 (-4m_K^2 + q^2) - T_{0,0,3} \cdot 24 m_K^4 + T_{0,0,1} \cdot 12 m_K^4 (6m_K^2 - q^2) \\ + T_{0,0,0} \cdot 12 m_K^6 (-4m_K^2 + q^2) + S_{2,0} \cdot 12 (2m_K^2 - q^2) + S_{1,1,1} \cdot 10 (-4m_K^2 + q^2) \\ + S_{1,0} \cdot 8 m_K^2 (4m_K^2 - q^2) + S_{0,2} \cdot (4m_K^2 - q^2) + S_{0,1} \cdot 24 m_K^4 \\ + S_{0,0} \cdot 4 m_K^4 (-10m_K^2 + q^2) \\ + A(m_K^2)^2 \cdot 3 m_K^2 (-4m_K^2 + q^2) \\ + A(m_T^2) B(q^2, m_K^2) \cdot 2 m_K^2 (-4m_K^2 + q^2) \end{array} \right\} \]
\[ T_{1,1,1} \cdot 72 \, m_K^2 + T_{1,1,0} \cdot 36 \, m_K^2 (-2m_K^2 - 2m_\pi^2 + q^2) \\
+ T_{1,0,2} \cdot 24 \, m_K^2 + T_{1,0,1} \cdot 12 \, m_K^2 (-4m_K^2 - 4m_\pi^2 + q^2) \\
+ T_{1,0,0} \cdot 12 \, m_K^2 (2m_K^4 + 4m_K^2m_\pi^2 - m_\pi^2q^2 + 2m_\pi^2 - m_K^2q^2) \\
- T_{0,0,3} \cdot 6 \, m_K^2 + T_{0,0,2} \cdot 3 \, m_K^2 (6m_K^2 + m_\pi^2 - q^2) \\
+ T_{0,0,1} \cdot 6 \, m_K^2 (-3m_K^4 - 6m_K^2m_\pi^2 + m_\pi^2q^2 - 3m_\pi^2 + m_K^2q^2) \\
+ T_{0,0,0} \cdot 3 \, m_K^2 (2m_K^2 + 2m_\pi^2 - q^2)(m_K^2 + m_\pi^2) \\
+ S_{2,0} \cdot 36 (-2m_K^2 + q^2) + S_{1,1} \cdot 2 (-4m_K^2 + q^2) \\
+ S_{1,0} \cdot 2 (4m_K^4 + 4m_K^2m_\pi^2 - m_K^2q^2 - m_\pi^2q^2) + S_{0,2} \cdot 2 (7m_K^2 - q^2) \\
+ S_{0,1} \cdot 4 (-7m_K^4 - 7m_K^2m_\pi^2 + m_\pi^2q^2 + m_K^2q^2) \\
+ S_{0,0} \cdot 2 (7m_K^2 - q^2)(m_K^2 + m_\pi^2) \\
+ A(m_K^2) B(q^2, m_\pi^2) \cdot 22 \cdot m_K^2 (-4m_K^2 + q^2) \\
+ A(m_K^2) B(q^2, m_\pi^2) \cdot 8 \, m_K^2 (4m_K^2 - q^2) \right\} \\
\Delta^{(5b)}_{\pi K K} = \frac{1}{24F^4 m_K^2 (4m_K^2 - q^2)} \left\{ \\
T_{1,1,1} \cdot 72 \, m_K^2 + T_{1,1,0} \cdot 36 \, m_K^2 (-2m_K^2 - 2m_\pi^2 + q^2) \\
- T_{1,0,2} \cdot 72 \, m_K^2 + T_{1,0,1} \cdot 12 \, m_K^2 (4m_K^2 + 12m_\pi^2 - 3q^2) \\
+ T_{1,0,0} \cdot 12 \, m_K^2 (2m_K^4 + 4m_K^2m_\pi^2 - m_\pi^2q^2 - 6m_\pi^2 + 3m_K^2q^2) \\
- T_{0,0,3} \cdot 18 \, m_K^2 + T_{0,0,2} \cdot 3 \, m_K^2 (-2m_K^2 - 18m_\pi^2 + 3q^2) \\
+ T_{0,0,1} \cdot 2 \, m_K^2 (-5m_K^4 + 6m_K^2m_\pi^2 + 3m_\pi^2q^2 + 27m_\pi^2 - 9m_K^2q^2) \\
+ T_{0,0,0} \cdot m_K^2 (q^2 - 2m_K^2 - 2m_\pi^2)(-3m_\pi^2 + m_K^2)^2 \\
+ S_{2,0} \cdot 36 (-2m_K^2 + q^2) + S_{1,1} \cdot 30 (4m_K^2 - q^2) \\
+ S_{1,0} \cdot 6 (-4m_K^4 - 20m_K^2m_\pi^2 + m_\pi^2q^2 + 5m_K^2q^2) + S_{0,2} \cdot 6 (-7m_K^2 + q^2) \\
+ S_{0,1} \cdot 4 (m_K^4 + 21m_K^2m_\pi^2 - m_K^2q^2 - 3m_\pi^2q^2) \\
+ S_{0,0} \cdot 2 (m_K^2 - 3m_\pi^2)(3m_K^4 + 7m_K^2m_\pi^2 - m_K^2q^2 - m_\pi^2q^2) \\
+ A(m_K^2) A(m_\pi^2) \cdot 4 (12m_K^4 - 8m_K^2m_\pi^2 - 3m_K^2q^2 + 2m_\pi^2q^2) \\
+ A(m_K^2) B(q^2, m_\pi^2) \cdot 2 \, m_K^2 (4m_K^2 - q^2) \\
+ A(m_\pi^2) B(q^2, m_\pi^2) \cdot 16 \, m_K^2 (-4m_K^2 + q^2) \right\}. \\
(46)
\]

Diagrams (5a) and (5c) yield the same contribution because of time reversal invariance. In these diagrams, there are three mass flows

\[ \Delta^{(5a)} = \Delta^{(5a)}_{\pi K K} + \Delta^{(5a)}_{\pi K} , \]

which can be expressed by the basic functions \( A \) and \( S_{\alpha,\beta} \):

\[ \Delta^{(5a)}_{\pi K K} = \frac{1}{24F^4 m_K^2} \left\{ -8 \, S_{2,0} + S_{1,1} + 12 \, m_K^2 S_{1,0} + S_{0,2} - 6 \, m_K^2 S_{0,1} \right\} \]

\[ + 3 \, m_K^2 A(m_K^2) A(m_\pi^2) + 2 \, (2m_K^2 - m_\pi^2) A(m_\pi^2) \right\} \]

\[ \Delta^{(5a)}_{\pi K} = \frac{1}{48F^4 m_K^2} \left\{ 6 \, S_{1,1} - 8 \, m_K^2 S_{1,0} + 3 \, S_{0,2} + 2 \, m_K^2 S_{0,1} - 8 \, m_K^2 S_{0,0} \right\} . \]
\[ -6 m_\pi^2 A(m_K^2) A(m_\pi^2) - 4 m_K^2 A(m_\pi^2) A(m_\eta^2) \\
- 2 \left( m_K^2 + m_\pi^2 \right) A(m_\pi^2) A(m_\eta^2) \right\}
\]
\[ \Delta^{(5a)}_{KKK} = \frac{1}{24 F^4 m_K^2} \left\{ -4 S_{2,0} + 2 S_{1,1} - S_{0,2} + 3 m_K^2 S_{0,1} + m_K^2 A(m_K^2)^2 \right\}. \tag{51} \]

Except for special kinematic situations the genuine 2-loop integrals \( S_{\alpha,\beta} \) and \( T_{\alpha_1,\alpha_2,\beta} \) cannot be calculated analytically. In appendix B we describe the method how we calculated them by splitting them into one part which contains the divergence and can be evaluated analytically, and a second part which is finite and can be done numerically.

5 Evaluation of the diagrams and checks

In the previous sections all \( \mathcal{O}(p^6) \) contributions to the \( K^0 \) form factor have been written in terms of some basic 1- and 2-loop functions. Before evaluating the contributions explicitly, we must specify a renormalization scheme.

In our calculation we are using dimensional regularization and the so called GL-scheme which is defined in the following way: Each diagram of order \( \mathcal{O}(p^{2n}) \) is multiplied with a factor \( e^{(1-\varepsilon)\alpha(\varepsilon)} \) where \( D = 4 - 2\varepsilon \) is the dimension of space-time and \( \alpha(\varepsilon) \) is given by

\[ (4\pi)^\varepsilon \Gamma(-1 + \varepsilon) = - e^{\alpha(\varepsilon)} \frac{\varepsilon}{\varepsilon}, \tag{52} \]

that is

\[ \alpha(\varepsilon) = \varepsilon (1 - \gamma + \log 4\pi) + \varepsilon^2 \left( \frac{\pi^2}{12} + \frac{1}{2} \right) + \mathcal{O}(\varepsilon^3). \tag{53} \]

Because of \( \alpha(0) = 0 \) the total \( \mathcal{O}(p^6) \) result is unchanged in \( D = 4 \) dimensions. The reason for this modification of each diagram is to eliminate the geometric factor \( (4\pi)^\varepsilon \Gamma(1 + \varepsilon) \) appearing in the 1-loop integrals \( A \) and \( B \). This renormalization scheme is very similar to the well-known MS-bar scheme: the only difference is that in MS-bar the left-hand side of the defining equation (52) is \( - (4\pi)^\varepsilon \Gamma(\varepsilon) \) so that there a geometric factor of \( (4\pi)^\varepsilon \Gamma(\varepsilon) \) is eliminated from the 1-loop integrals.

The GL-scheme extends the usual 1-loop scheme introduced by Gasser and Leutwyler \cite{GL} in a natural way which can be understood from an inspection of the renormalization constants \( L_i \) of \( \mathcal{L}^{(4)} \): In \( D \)-dimensional space-time they have dimension \( D - 4 \) and their dimension is generated by the mass scale \( \mu \) of dimensional regularisation:

\[ L_i = \mu^{D-4} L_i(\mu, D). \tag{54} \]

\( L_i(\mu, D) \) has the same \( \mu \)-dependence as a 1-loop integral, because \( L_i \) itself is independent of \( \mu \). It can be expanded in a Laurent series around \( \varepsilon = 0 \) in the same way as a 1-loop integral:

\[ L_i(\mu, D) = \frac{L_i^{(-1)}}{\varepsilon} + L_i^{(0)}(\mu) + \varepsilon L_i^{(1)}(\mu) + \mathcal{O}(\varepsilon^2). \tag{55} \]

In the usual 1-loop scheme one chooses

\[ L_i^{(-1)} = - \frac{\Gamma_i}{32\pi^2} \tag{56} \]

\[ L_i^{(0)}(\mu) = L_i^{\text{ren}}(\mu) - \frac{\Gamma_i}{32\pi^2} \left[ 1 - \gamma + \log(4\pi) \right], \tag{57} \]
where $\Gamma_i$ are numbers which can be found in [1]. The second term in $L_i^{(0)}$ is constructed so that it cancels in the $\epsilon^0$-coefficient after multiplication with $e^{-\alpha(\epsilon)}$:

$$L_i^{GL}(\mu, D) := e^{-\alpha(\epsilon)} L_i(\mu, D) = \frac{L_i^{(1)}}{\epsilon} + L_i^{(0),GL}(\mu) + \epsilon L_i^{(1),GL}(\mu) + O(\epsilon^2)$$

with $L_i^{(0),GL}(\mu) = L_i^{\text{ren}}(\mu)$.

The dimension of the $L^{(6)}$-parameter $\beta$ appearing in [3] can be treated in the same way:

$$\beta = \mu^{2D-8} \beta(\mu, D),$$

where $\beta(\mu, D)$ behaves like a 2-loop integral. Its Laurent series in the above GL-scheme is given by

$$\beta^{GL}(\mu, D) := e^{-2\alpha(\epsilon)} \beta(\mu, D) = \frac{\beta^{(2),GL}}{\epsilon^2} + \frac{\beta^{(1),GL}(\mu)}{\epsilon} + \beta^{(0),GL}(\mu) + O(\epsilon).$$

Before discussing the numerical results we enumerate the checks which we performed on our calculation:

- Because of charge conservation, the on-shell current matrix element in [3] must not contain a term proportional to $(p - p')_\mu$. This is manifestly the case for each individual diagram after inserting the Feynman rules into the vertices.

- According to the Ward-Fradkin-Takahashi identity [3] the form factor must be equal to the charge of the particle for zero momentum transfer. This is not the case for each individual diagram, but for the sum of all reducible and the sum of all irreducible diagrams separately.

- The most important check of our calculation follows from an analysis of the divergent parts of the loop diagrams: their sum must be equal to the negative divergent part of the tree graph (4), so that the sum of all $p^6$ diagrams is finite. Since diagram (4) is a tree graph, it is a polynomial in masses and momenta and cannot produce logarithms thereof. Thus, all logarithmic terms must cancel in the sum of the divergent parts of the loop diagrams. For the irreducible diagrams, we find in the GL-scheme

$$\Delta_{\text{irr. div. part}} = \frac{2q^2(2m_K^2 - m_\pi^2)}{9(4\pi F)^4 \epsilon^2} + \frac{1}{432(4\pi F)^4 \epsilon} \left\{ -16m_\pi^2(10m_K^2 - 7m_\pi^2) \text{Lg}(m_K^2) 
- m_\pi^2(32m_K^2 - 80m_\pi^2 - 9q^2) \text{Lg}(m_K^2) - 3q^2(4m_K^2 - m_\pi^2) \text{Lg}(m_\eta^2) 
- b_0(q^2; m_K^2, m_\pi^2) (160m_K^4 - 112m_K^2m_\pi^2 - 76m_\pi^2q^2 + 28m_\pi^2q^2 - 9q^4) 
- b_0(q^2; m_\pi^2, m_\eta^2) (32m_K^2m_\pi^2 - 8m_\pi^2q^2 - 80m_\pi^4 + 56m_\pi^2q^2 - 9q^4) 
+ 80q^2(m_\pi^2 - m_\eta^2) \right\}$$

using $\text{Lg}(m^2) = \log(m^2/(4\pi \mu^2)) + \gamma$, and for the reducible 1- and 2-loop diagrams

$$\Delta_{\text{red. div. part}} = -\Delta_{\text{irr. div. part}} + \frac{2q^2}{(4\pi)^2 F^4 \epsilon}(m_K^2 - m_\pi^2) L_3^{\text{ren}}.$$  

In fact, the logarithmic terms vanish in the sum of all diagrams, but no longer separately for the group of reducible and irreducible diagrams.
The previous two formulae (61) and (62), together with equation (27), fix the divergent part of the relevant $\mathcal{L}^{(6)}$ constant $\beta$ which occurs in the $K^0$ form factor:

$$\beta_{\text{div}} = -\frac{m_K^2 - m_\pi^2}{8\pi^2\varepsilon} L^\text{ren}_3. \quad (63)$$

Note that there is no $1/\varepsilon^2$ contribution, and as a consequence the given $1/\varepsilon$ result is independent of the renormalization scheme and the renormalization mass $\mu$, because it is the leading order term in the Laurent expansion of $\beta$ w.r.t. $\varepsilon$.

For the numerical evaluation we choose the following input data:

$$m_K = 495\text{MeV} \quad (64)$$
$$m_\pi = 0.28m_K = 138.6\text{MeV} \quad (65)$$
$$F_\pi = 92.4\text{MeV}. \quad (66)$$

Note that we have used $m_\eta^2$ as a short-hand notation for the Gell-Mann-Okubo term $\frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2$ in all contributions listed in the previous sections. The loop integrals are calculated at mass scale $\mu = m_\rho = 770\text{MeV}$. At that scale the relevant $\mathcal{L}^{(4)}$ parameters are given by

$$L^\text{ren}_3(m_\rho) = (-35 \pm 13) \cdot 10^{-4} \quad (67)$$
$$L^\text{ren}_5(m_\rho) = (14 \pm 5) \cdot 10^{-4} \quad (68)$$
$$L^\text{ren}_9(m_\rho) = (69 \pm 7) \cdot 10^{-4} \quad (69)$$

Finally, at $\mathcal{O}(p^6)$ not only the $\varepsilon^0$-coefficient $L_i^{(0),GL} = L_i^\text{ren}$ occurs in the result, but also the coefficient $L_i^{(1),GL}$ of the subsequent order $\varepsilon^1$ of the Laurent expansion (58). It can be proved that these constants $L_i^{(1),GL}$ always enter the result in a specific combination with some $\mathcal{L}^{(6)}$ parameters, so that they are no new degrees of freedom and can be chosen arbitrarily: a change in their values would only lead to a redefinition of some $\mathcal{L}^{(6)}$ constants. For our calculation, we choose

$$L_i^{(1),GL}(m_\rho) = 0. \quad (70)$$

With these input data and definitions we obtain the contributions shown in fig. 2 for the reducible and the irreducible loop diagrams. The corrections are to be compared to the leading order result which is also shown in fig. 2.

6 Discussion and Results

Since at order $p^6$ there exists a direct coupling of the $K^0$ to the external photon, the slope of the form factor at zero momentum transfer is not predicted. Diverse models make predictions for the direct coupling $\beta$ of Eq. (6), but ultimately only a precise measurement of the $K^0$ charge radius would fix the constant $\beta$. On the other hand, to extract the neutral kaon’s charge radius from data a knowledge of it’s precise functional dependence on the momentum transfer $q^2$ is essential. Our explicit results come here extremely handy, as only one parameter (the charge radius) needs to be fitted to the data.

Alternatively one may consider the observable $F^{K^0}(q^2)/q^2$ in the comparison of theory with experiment. Here the unknown constant $\beta$ only influences the vertical position of the predicted
Figure 2: Contributions to the $K^0$ form factor: the solid line is the leading order 1-loop result, the dotted curves denote the corrections at order $p^6$ due to the reducible resp. irreducible loop diagrams in the GL-scheme (on the one hand diagrams (2a)-(3f) and the $p^6$ contributions of the renormalized leading order diagrams (1a)-(1b), on the other hand diagrams (5a)-(5c)). The contribution of diagram (4) is missing: It would add a straight line through the origin with a slope given by the $L^{(6)}$-parameter $\beta$.

form factor curve, but not its shape. The predictions are plotted in fig. 3. We do not include the errors due to the $L^{(4)}$ constants as their effect tends to be proportional to $q^2$, and therefore cancels in the plot of fig. 3.

It can be observed that the $O(p^6)$ correction to the $O(p^4)$ approximation is unexpectedly large. This result may indicate a breakdown of chiral perturbation theory or, as the $O(p^6)$ term represents only the first correction to the leading $O(p^4)$ term, it may be due to the fact that the $O(p^4)$ term is accidentally small. A detailed measurement of the form factor of the neutral kaon may provide unique information on the convergence of $SU(3) \times SU(3)$ chiral perturbation theory. This is because the prediction of the shape of $F^{K_0}(q^2)/q^2$, in contrast to most other $O(p^6)$ results, does not involve any of the unknown 143 freely adjustable parameters of [2]. There exist plans for experiments on the form factor of the neutral kaon at CEBAF and MAMI C, and we can only hope that these experiments are realized.

A The relevant 1-loop integrals

All reducible diagrams contributing to the $K^0$ form factor were reduced to two basic integrals in section 3. Here, we give the definitions and the results of these basic integrals in dimensional regularisation.

The first basic 1-loop integral is the 1-point function

$$A(m^2) = \mu^{4-D} \int \frac{d^Dk}{i(2\pi)^D} \frac{1}{k^2 - m^2}, \quad (71)$$

where $\mu$ is the scale of dimensional regularisation and $D = 4 - 2\varepsilon$ is the dimension of space-time. What is relevant in a 2-loop calculation are the coefficients of the Laurent expansion of the integral
around $\varepsilon = 0$ up to order $\varepsilon^1$. For $A$ we have

$$A(m^2) = \frac{a_{-1}}{\varepsilon} + a_0 + \varepsilon a_1 + O(\varepsilon^2)$$

(72)

with

$$a_{-1} = \frac{m^2}{(4\pi)^2}$$

(73)

$$a_0 = \frac{m^2}{(4\pi)^2} \left\{ 1 - \text{Lg}(m^2) \right\}$$

(74)

$$a_1 = \frac{m^2}{(4\pi)^2} \left\{ 1 + \frac{\pi^2}{12} - \text{Lg}(m^2) + \frac{\text{Lg}(m^2)^2}{2} \right\}$$

(75)

where we have introduced the abbreviation

$$\text{Lg}(m^2) = \log\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma_{\text{Euler}}.$$  

(76)

The second basic 1-loop integral, the 2-point function $B(q^2, m^2)$, is defined as the coefficient of the Lorentz-decomposition of the tensor integral

$$\mu^{4-D} \int \frac{d^Dk}{i(2\pi)^D} \frac{k^\mu k^\nu}{[(k + q)^2 - m^2][k^2 - m^2]}$$

(77)

coming along with the metric tensor $g_{\mu\nu}$, where the other Lorentz-covariant is $q_\mu q_\nu$. It can be expressed further by $A$ and the scalar 2-point function

$$B_0(q^2, m^2) = \mu^{4-D} \int \frac{d^Dk}{i(2\pi)^D} \frac{1}{[(k + q)^2 - m^2][k^2 - m^2]}$$

(78)
in the following way: Let

\[ B_0(q^2, m^2) = \frac{b_{-1}}{\varepsilon} + b_0 + \varepsilon b_1 + \mathcal{O}(\varepsilon^2) \]  
(79)

\[ B(q^2, m^2) = \frac{(B)_{-1}}{\varepsilon} + (B)_0 + \varepsilon (B)_1 + \mathcal{O}(\varepsilon^2) \]  
(80)

be those parts of the Laurent series which are relevant in a 2-loop calculation. \(b_{-1}, b_0\) and \(b_1\) are functions of \(m^2\) and \(q^2\) and are given below. The relationship between \(B\) on the one side and the scalar integrals \(A\) and \(B_0\) on the other side follows from a tensor decomposition:

\[ B(q^2, m^2) = \frac{1}{4(D-1)} \left\{ 2A(m^2) + (4m^2 - q^2)B_0(q^2, m^2) \right\} . \]  
(81)

The relevant Laurent coefficients of \(B\) are given in terms of those of \(A\) and \(B_0\) via the polynomial

\[ P(x_1, x_2, x_3) = \frac{1}{12} \left( x_1 + 2x_2 + \frac{4}{9}x_3 \right) \]  
(82)

in the following way:

\[ (B)_{-1} = 2P(a_{-1}, 0, 0) + (4m^2 - q^2)P(b_{-1}, 0, 0) \]  
(83)

\[ (B)_0 = 2P(a_0, a_{-1}, 0) + (4m^2 - q^2)P(b_0, b_{-1}, 0) \]  
(84)

\[ (B)_1 = 2P(a_1, a_0, a_{-1}) + (4m^2 - q^2)P(b_1, b_0, b_{-1}) . \]  
(85)

The basic Laurent coefficients \(b_{-1}, b_0\) and \(b_1\) of the scalar 2-point function \(B_0\) involve logarithms and dilogarithms:

\[ (4\pi)^2 b_{-1} = 1 \]  
(86)

\[ (4\pi)^2 b_0 = 2 - \text{Lg}(m^2) - \tau T_1 \]  
(87)

\[ (4\pi)^2 b_1 = 2 + \frac{\pi^2}{12} (1 - 2\tau) + \frac{1}{2} (\text{Lg}(m^2) - 2)^2 + \tau \left( T_1 \text{Lg}(m^2) - 2T_1 - \frac{1}{2}T_1^2 + T_2 + 2T_3 \right) \]  
(88)

where we have defined

\[ \tau = \sqrt{1 - \frac{4m^2}{q^2}} \]  
(89)

\[ T_1 = \log \left( \frac{\tau + 1}{\tau - 1} \right) \]  
(90)

\[ T_2 = \log^2 \left( \frac{\tau - 1}{2\tau} \right) \]  
(91)

\[ T_3 = \text{Li}_2 \left( \frac{\tau - 1}{2\tau} \right) . \]  
(92)

An infinitesimal negative imaginary part of all masses is understood.

The Taylor expansions of the Laurent coefficients of \(B\) w.r.t. \(q^2\) is needed if one is interested only in the small \(q^2\) behaviour of the form factor:

\[ (4\pi)^2 (B)_0 = -\frac{m^2}{2} \left\{ \text{Lg}(m^2) - 1 \right\} + \frac{q^2}{12} \text{Lg}(m^2) + \mathcal{O}(q^4) \]  
(93)

\[ (4\pi)^2 (B)_1 = \frac{m^2}{4} \left\{ (\text{Lg}(m^2) - 1)^2 + 1 + \frac{\pi^2}{6} \right\} - \frac{q^2}{24} \left\{ \text{Lg}(m^2)^2 + \frac{\pi^2}{6} \right\} + \mathcal{O}(q^4) . \]  
(94)

With the help of the above formulae the basic 1-loop functions \(A\) and \(B\) and therefore all reducible loop contributions to the \(K^0\) form factor are reduced to logarithms and dilogarithms.
B The irreducible 2-loop integrals

In section 4 the irreducible 2-loop diagrams of the $K^0$-form factor were expressed by a set of basic 2-loop integrals of the 2-point and 3-point sunset topologies $S_{\alpha,\beta}$ and $T_{\alpha_1,\alpha_2,\beta}$, cf. (39ff). Fig. 4 shows the diagrams and the flows of momenta.

In the case of three different masses ($m_\pi$, $m_K$, and $m_\eta$) it is no longer possible to reduce them to elementary analytical functions. Therefore, we chose a numerical approach for our calculation which will be outlined in the following. The procedure is explained in the example of the 3-point function $T_{\alpha_1,\alpha_2,\beta}$; the 2-point function $S_{\alpha,\beta}$ is simpler and can be treated along the same lines. Tensor integrals of the relevant topologies are decomposed into Lorentz covariants and the algorithm is applied to the coefficient functions.

The idea is to split the integral $T_{\alpha_1,\alpha_2,\beta}$ into two parts,

$$T_{\alpha_1,\alpha_2,\beta} = T_{\alpha_1,\alpha_2,\beta}^N + T_{\alpha_1,\alpha_2,\beta}^A,$$

(95)

where $T_{\alpha_1,\alpha_2,\beta}^A$ contains the divergences, but is of a simpler structure so that it can be calculated analytically, and $T_{\alpha_1,\alpha_2,\beta}^N$ is finite in $D = 4$ dimensions and can be evaluated numerically.

Our method of achieving a suitable decomposition (95) is based on the well-known BPHZ regularization procedure [7] which makes use of the following property of Feynman integrals: each Feynman diagram without subdivergences behaves asymptotically as a polynomial in its external momenta. Therefore, an UV-divergence of a subdivergence-free diagram may be extracted by subtracting a Taylor polynomial of sufficient degree w.r.t. the external momenta.

We apply this subtraction of Taylor polynomials in two steps: First, we consider the $l$-subdiagram which is a 2-point function with external momentum $k$, the loop momentum of the other loop integration. Subtraction of a Taylor polynomial w.r.t. $k$ renders the $l$-subgraph finite and yields an integral

$$\hat{T}_{\alpha_1,\alpha_2,\beta} = \int dk \frac{(k^2)^\alpha}{P_{k+p_1,m_0^2}P_{k+p_2,m_1^2}} \left(1 - Taylor_k\right) \int dl \frac{(lp_1)^{\alpha_1}(lp_2)^{\alpha_2}}{P_{l+m_2^2}P_{l,m_3^2}}$$

(96)

which has no subdivergences. The diagrammatic structure of $\hat{T}_{\alpha_1,\alpha_2,\beta}$ is shown in figure 5.

In the second step we subtract a Taylor polynomial from $\hat{T}_{\alpha_1,\alpha_2,\beta}$ w.r.t. its external momenta $p_1$ and $p_2$: since $\hat{T}_{\alpha_1,\alpha_2,\beta}$ has no subdivergences left, we end up with a finite integral which is defined to be $T_{\alpha_1,\alpha_2,\beta}^N$ in the decomposition (95).

This decomposition has the desired properties: On the one hand, the part $T_{\alpha_1,\alpha_2,\beta}^A$ containing the divergences has the diagrammatic structure shown in fig. 6 and can be reduced to well-known analytic functions. This is obvious for the first and third topology of fig. 6 which are products of
1-loop integrals. The second topology in fig. 6 has the structure of a scalar 2-loop vacuum bubble which are discussed in detail in [8].

On the other hand, the finite part $T_{\alpha_1,\alpha_2,\beta}^N$ is accessible to numeric evaluation, because seven of its eight integrations can be done analytically, and the integrand of the final integration is a smooth function consisting of logarithms and dilogarithms. This can be understood from the following observation: since the $l$-subintegral is a 2-point function with external momentum $k$, it may be represented by a dispersion integral

$$\int_0^\infty d\zeta \frac{f_{\alpha_1,\alpha_2}(\zeta, m_2^2, m_3^2, p_1, p_2)}{\zeta - k^2} ,$$

where the $k$ dependence occurs only in the dispersion denominator. Interchanging the $k$ and $\zeta$ integrations we end up with an integral

$$\int_0^\infty d\zeta \int d^4k \frac{(k^2)^\beta}{P_{k+p_1, m_0^2} P_{k+p_2, m_1^2}(\zeta - k^2)} ,$$

where the $k$ integration can be done analytically if one recognizes the dispersion denominator $\zeta - k^2$ as a propagator with mass $\zeta$ and momentum $k$. Thus, the $k$ subintegration is a 1-loop 3-point function which is given in terms of dilogarithms in [9].

This procedure of reducing the numeric part $T_{\alpha_1,\alpha_2,\beta}^N$ to a 1-dimensional integration is illustrated diagramatically in figure 7: writing the $l$-subgraph as a dispersion integral and subsequently interchanging the integrations has the effect of shrinking the $l$-subgraph to a single propagator of mass $\zeta$ which must finally be integrated over.
The above method of calculating the irreducible 2-loop diagrams is described in more detail in [10]. We implemented the algorithm in a set of REDUCE programs which can be supplied on request.

C Feynman rules

In this section we discuss the momentum space Feynman rules which are needed for the relevant vertices occurring in the diagrams of the $K^0$-form factor (cf. fig. 1). There are ten different vertex types which are to be considered: vertices from $\mathcal{L}^{(2)}$, $\mathcal{L}^{(4)}$, and $\mathcal{L}^{(6)}$ with an even number of meson legs and with or without an additional photon (fig. 8).

\begin{align}
\mathcal{L}^{(2)}_{4\text{-meson}} &= \frac{1}{24F^2} \text{Tr}([\phi, \partial_\mu \phi]\phi \partial^\mu \phi) + \frac{1}{48F^2} \text{Tr}(\chi \phi^4) \\
\mathcal{L}^{(2)}_{6\text{-meson}} &= \frac{1}{720F^4} \left\{ \text{Tr}(\partial_\mu \phi \partial^\mu \phi \phi^4) - 4 \text{Tr}(\partial_\mu \phi \phi \partial^\mu \phi \phi^3) + 3 \text{Tr}(\partial_\mu \phi \phi^2 \partial^\mu \phi \phi^2) \right\} - \frac{1}{1440F^4} \text{Tr}(\chi \phi^6). \tag{99/100}
\end{align}

In general, the Feynman rules for a vertex with $n$ meson legs is obtained from $\mathcal{L}_{\text{eff}}$ by first determining all monoms in the expansion of $\mathcal{L}_{\text{eff}}$ which contain exactly $n$ factors $\phi$ or derivatives thereof. Then, transformation into momentum space is done by replacing each derivative $\partial_\mu$ by $-ip_\mu$ where $p_\mu$ is the momentum flowing into the corresponding leg of the vertex. Finally, the vertex must be symmetrized in all its meson fields due to Bose symmetry.

For the vertices (a) and (b) of fig. 8 the relevant monoms of the expansion of $\mathcal{L}^{(2)}$ are

\begin{align}
\mathcal{L}_4 &= \frac{1}{24F^2} \text{Tr}([\phi, \partial_\mu \phi]\phi \partial^\mu \phi) + \frac{1}{48F^2} \text{Tr}(\chi \phi^4) \\
\mathcal{L}_6 &= \frac{1}{720F^4} \left\{ \text{Tr}(\partial_\mu \phi \partial^\mu \phi \phi^4) - 4 \text{Tr}(\partial_\mu \phi \phi \partial^\mu \phi \phi^3) + 3 \text{Tr}(\partial_\mu \phi \phi^2 \partial^\mu \phi \phi^2) \right\} - \frac{1}{1440F^4} \text{Tr}(\chi \phi^6). \tag{99/100}
\end{align}
In our calculation we avoided writing down the Feynman rules for a general particle flow in terms of the $SU(3)$ structure constants $f_{abc}$ and $d_{abc}$. Instead, we calculated the traces in the Feynman rules for each concrete flow of particles separately and inserted the Gell-Mann matrices

$$\phi = \sum_{a=1}^{8} \phi_a \lambda_a \frac{1}{2}$$

$$\begin{pmatrix}
\pi^0 + \frac{1}{\sqrt{3}} \eta & \sqrt{2} \pi^+ & \sqrt{2} K^+ \\
\sqrt{2} \pi^- & -\pi^0 + \frac{1}{\sqrt{3}} \eta & \sqrt{2} K^0 \\
\sqrt{2} K^- & \sqrt{2} K^0 & -\frac{2}{\sqrt{3}} \eta
\end{pmatrix}$$

(101)

directly into our REDUCE programs where also the symmetrisation is done. In this way, the Feynman rules can be read off from the relevant monoms of $L_{\text{eff}}$. The only step which requires some effort is the determination of these monoms.

The mass matrix can be expressed in terms of the unrenormalized meson masses and takes the following form in the isospin limit:

$$\chi = \text{diag}(m_{\pi}^2, m_{\pi}^2, 2m_K^2 - m_{\pi}^2).$$

(102)

For vertices (c) and (d) of fig. 8 the monoms defining the Feynman rules are

$$L_{2\text{-meson}}^{(4)} = \frac{2L_1}{F^2} \text{Tr}(\partial_\mu \phi \partial_\nu \phi) \text{Tr}(\chi) + \frac{2L_5}{F^2} \text{Tr}(\chi \partial_\mu \phi \partial_\nu \phi) - \frac{4L_6}{F^2} \text{Tr}(\chi \phi^2) \text{Tr}(\chi)$$

(103)

$$L_{4\text{-meson}}^{(4)} = \frac{L_1}{F^4} \left\{ \text{Tr}(\partial_\mu \phi \partial_\nu \phi) \right\}^2 + \frac{L_2}{F^4} \text{Tr}(\partial_\mu \phi \partial_\nu \phi) \text{Tr}(\partial_\mu \phi \partial_\nu \phi)$$

$$+ \frac{L_3}{F^4} \text{Tr}(\partial_\mu \phi \partial_\nu \phi)$$

$$\frac{-L_4}{3F^4} \left\{ \text{Tr}(\partial_\mu \phi \partial_\nu \phi \partial_\rho \phi) \text{Tr}(\chi) + 3 \text{Tr}(\partial_\mu \phi \partial_\nu \phi) \text{Tr}(\chi \phi^2) \right\}$$

$$\frac{-L_5}{6F^4} \left\{ 2 \text{Tr}(\chi \phi^2 \partial_\mu \phi \partial_\nu \phi) + 3 \text{Tr}(\chi \phi \partial_\mu \phi \partial_\nu \phi \partial_\rho \phi) - \text{Tr}(\chi \partial_\mu \phi \partial_\nu \phi \partial_\rho \phi) $$

$$+ \text{Tr}(\chi \partial_\mu \phi \phi^2 \partial_\nu \phi) - \text{Tr}(\chi \partial_\mu \phi \phi \partial_\nu \phi \partial_\rho \phi) + 2 \text{Tr}(\chi \partial_\mu \phi \partial_\nu \phi \partial_\rho \phi \partial_\sigma \phi) \right\}$$

$$\frac{L_6}{3F^4} \left\{ \text{Tr}(\chi \phi^4) \text{Tr}(\chi) + 3 \text{Tr}(\chi \phi^2)^2 \right\} + \frac{4L_7}{3F^4} \text{Tr}(\chi \phi^3) \text{Tr}(\chi \phi)$$

$$\frac{L_8}{6F^4} \left\{ \text{Tr}(\chi^2 \phi^4) + 2 \text{Tr}(\chi \phi \phi^3) + 3 \text{Tr}(\chi \phi^2 \phi^2) + 2 \text{Tr}(\chi \phi^3 \phi) \right\}.$$ 

The remaining vertices (e)–(j) contain an even number of mesons and one photon line. In the following we give the corresponding Feynman rules for an arbitrary external boson, e.g. a photon or a $W^\pm$. The external boson enters the Lagrangian $L_{\text{eff}}$ as a gauge field of the chiral symmetry group in the covariant derivative

$$D_\mu U = \partial_\mu U + iUl_\mu - ir_\mu U$$

(105)

and can be expressed by the Gell-Mann matrices $T_a = \lambda_a/2$:

$$l_\mu = \sum_{a=1}^{8} l_\mu^a T_a, \quad r_\mu = \sum_{a=1}^{8} r_\mu^a T_a.$$ 

(106)
Therefore, it suffices to specify only $J_J$ symmetric under intrinsic parity transformation, $n_J$ and $L_L$ a left-handed gauge boson the analogue is true for a right handed one. $J_J^{L/R}$ is therefore the relevant monomial part of $L$ which yields the Feynman rules for the meson vertices with one external boson. Since $L$ is symmetric under intrinsic parity transformation, $J_J^{L,R}$ and $J_J^{R,L}$ are not independent of each other:

$$J_J^{R,a}(U) = J_J^{L,a}(U^\dagger) \quad \text{resp.} \quad J_J^{R,a}(\phi) = J_J^{L,a}(-\phi).$$

Therefore, it suffices to specify only $J_J^{L,a}$ for the relevant vertices.

The Feynman rule of the vertex from $L(2)$ with $n$ mesons and one external boson is given by

$$J_J^{L,a}[L(2)]_{n\text{-mesons}} = \frac{-iF^2}{2} \frac{i^n}{n!} F^n \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \text{Tr}(T_a \phi^k \partial_{\mu, a} \phi^{n-k-1}).$$

Vertices (e)–(g) are special cases of this formula for $n = 2, 4, 6$.

The $L(4)$-vertices with one external boson, (h) and (i), follow from

$$J_J^{L,a}[L(4)]_{2\text{-mesons}} = \frac{2iL_4}{F^4} \text{Tr}(T_a[\partial_{\mu, a} \phi, \phi]) \text{Tr}(\chi)$$

$$+ \frac{iL_5}{F^2} \text{Tr}(T_a(\chi \partial_{\mu, a} \phi + \partial_{\mu, a} \phi \partial_{\mu, a} \phi - \phi \partial_{\mu, a} \phi - \phi \partial_{\mu, a} \phi)$$

$$- \frac{iL_9}{F^2} \partial'' \text{Tr}(T_a[\partial_{\mu, a} \phi, \partial_{\nu, a} \phi])$$

and

$$J_J^{L,a}[L(4)]_{4\text{-mesons}} = \frac{2iL_1}{F^4} \text{Tr}(T_a[\partial_{\mu, a} \phi, \phi]) \text{Tr}(\partial_{\mu, a} \phi \partial'' \phi)$$

$$+ \frac{2iL_2}{F^4} \text{Tr}(T_a[\partial'' \phi, \phi]) \text{Tr}(\partial_{\mu, a} \phi \partial_{\nu, a} \phi)$$

$$+ \frac{iL_3}{F^4} \text{Tr}(T_a(\{\partial_{\mu, a} \phi, \partial_{\nu, a} \phi \partial'' \phi\} \phi - \phi [\partial_{\mu, a} \phi, \partial_{\nu, a} \phi \partial'' \phi])$$

$$- \frac{iL_4}{6F^4} \{6 \text{Tr}(T_a[\partial_{\mu, a} \phi, \phi]) \text{Tr}(\chi \phi^2) + \text{Tr}(T_a(3 \phi \partial_{\mu, a} \phi, \phi) \text{Tr}(\chi)\}$$

$$- \frac{iL_5}{12F^4} \{ \text{Tr}(T_a(3 \phi \partial_{\mu, a} \phi, \phi^2 + 2 \phi \partial_{\mu, a} \phi \phi - 2 \phi \chi \partial_{\mu, a} \phi, \phi^2) + 2 \partial_{\mu, a} \phi \chi \phi^3)$$

$$+ \text{Tr}(T_a(4 \phi \chi \partial_{\mu, a} \phi + 3 [\partial_{\mu, a} \phi, \phi] \chi \phi^2 + 3 \phi^2 [\partial_{\mu, a} \phi, \phi] + 2 \{\partial_{\mu, a} \phi, \phi^2\} \chi \phi)$$

$$+ \text{Tr}(T_a(-4 \phi \partial_{\mu, a} \phi \chi \phi - 2 \phi^3 \chi \partial_{\mu, a} \phi - \phi \partial_{\mu, a} \phi \phi^2 \chi + \phi^2 \partial_{\mu, a} \phi, \phi^4) \}$$

$$+ \frac{iL_9}{12F^4} \partial'' \{ \text{Tr}(T_a(-3 \phi \partial_{\mu, a} \phi, \partial_{\nu, a} \phi) + \partial_{\mu, a} \phi \partial_{\nu, a} \phi \partial_{\mu, a} \phi, \partial_{\nu, a} \phi)$$

$$+ \text{Tr}(T_a(2 \phi^2 [\partial_{\mu, a} \phi, \partial_{\nu, a} \phi] + 2 [\partial_{\mu, a} \phi, \partial_{\nu, a} \phi] \partial_{\mu, a} \phi, \partial_{\nu, a} \phi)$$

$$[\partial_{\mu, a} \phi, \partial_{\nu, a} \phi] - \partial_{\mu, a} \phi, \partial_{\nu, a} \phi) \}.$$

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Finally, vertex (j) stems from $\mathcal{L}^{(6)}$ and takes the following form in the notation of [2]:

$$J^{L,\mu}_{\mu,a}[\mathcal{L}^{(6)}]_{\text{mesons}} = \frac{iB_9}{2F^2} \text{Tr} T_a[\chi, \{\partial_\mu \partial_\nu \phi, \partial^\nu \phi\}]$$
$$+ \frac{iB_{14}}{4F^2} \text{Tr} T_a(2\chi \phi \partial_\mu \phi \chi - 2\chi \partial_\mu \phi \phi + \chi \chi \partial_\mu \phi \phi
- \chi \phi \partial_\mu \phi + \partial_\mu \phi \chi \chi - \phi \partial_\mu \chi \chi)$$
$$+ \frac{iB_{15}}{2F^2} \left\{ \text{Tr}(\chi \phi) \text{Tr} T_a[\partial_\mu \phi, \chi] - \text{Tr}(\chi \partial_\mu \phi) \text{Tr} T_a[\phi, \chi] \right\}$$
$$+ \frac{iB_{16}}{4F^2} \text{Tr}(\chi) \text{Tr} T_a[\{\partial_\mu \phi, \phi\}, \chi]$$
$$+ \frac{iB_{17}}{2F^2} \text{Tr} T_a(\phi \partial_\mu \phi \chi - \partial_\mu \phi \chi \chi \phi + \phi \chi \partial_\mu \phi - \chi \chi \partial_\mu \phi \phi)$$
$$+ \frac{iB_{18}}{2F^2} \text{Tr}(\chi) \text{Tr} T_a(\phi \partial_\mu \phi \chi - \partial_\mu \phi \chi \phi + \phi \chi \partial_\mu \phi - \chi \partial_\mu \phi \phi)$$
$$+ \frac{iB_{19}}{F^2} \left\{ \text{Tr}(\chi \phi) \text{Tr} (T_a[\phi, \partial_\mu \phi]) + 2 \text{Tr}(\chi [\phi, \partial_\mu \phi]) \text{Tr}(T_a[\phi, \partial_\mu \phi]) \right\}$$
$$+ \frac{iB_{20}}{F^2} \text{Tr}(\chi \partial_\mu \phi) \text{Tr} (T_a[\phi, \chi])$$
$$+ \frac{iB_{21}}{F^2} \text{Tr}(\chi) \text{Tr} (T_a[\partial_\mu \phi])$$
$$+ \frac{iB_{22}}{F^2} \text{Tr} T_a(\partial_\nu \partial^\nu [\partial_\mu \partial_\beta \phi, \partial^\beta \phi] - \partial_\mu \partial^\nu [\partial_\nu \partial_\beta \phi, \partial^\beta \phi])$$
$$+ \frac{iB_{23}}{F^2} \partial^\nu \partial^\beta \text{Tr} T_a([\partial_\mu \partial_\beta \phi, \partial_\nu \phi] - [\partial_\nu \partial_\beta \phi, \partial_\mu \phi])$$
$$- \frac{iB_{24}}{F^2} \partial^\nu \text{Tr}([\partial_\mu \partial_\nu \phi, \partial_\nu \phi], \chi, T_a)$$
$$- \frac{iB_{25}}{F^2} \partial^\nu \text{Tr} T_a(\partial_\mu \phi \partial_\nu \phi - \partial_\nu \phi \chi \partial_\mu \phi)$$
$$- \frac{iB_{26}}{F^2} \partial^\nu \text{Tr} (T_a[\partial_\mu \phi, \partial_\nu \phi]) \text{Tr}(\chi)$$
$$+ \frac{iB_{27}}{2F^2} \partial^\nu \text{Tr} T_a(\{\partial_\mu \phi, \{\partial_\nu \phi, \chi\}\}, [\partial_\mu \phi, \{\partial_\nu \phi, \chi\}]).$$

Note that in (3) we have given only the part of $\mathcal{L}^{(6)}$ which is relevant for the $K^0$ form factor. We convinced ourselves by inserting the Feynman rule (113) into diagram (4) that only the terms containing $B_{24}$ and $B_{25}$ yield a nonzero contribution to the $K^0$ form factor.

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