LIMIT CYCLES FOR REGULARIZED PIECEWISE SMOOTH SYSTEMS WITH A SWITCHING MANIFOLD OF CODIMENSION TWO

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ABSTRACT. In this paper we consider an $n$ dimensional piecewise smooth dynamical system. This system has a co-dimension 2 switching manifold $\Sigma$ which is an intersection of two co-dimension one switching manifolds $\Sigma_1$ and $\Sigma_2$. We investigate the relation of periodic orbit of PWS between periodic orbit of its regularized system. If this PWS system has an asymptotically stable crossing periodic orbit $\gamma$ or sliding periodic orbit, we establish conditions to ensure that also a regularization of the given system has a unique, asymptotically stable, limit cycle in a neighbourhood of $\gamma$, converging to $\gamma$ as the regularization parameter goes to 0.

1. Introduction. Piecewise smooth (PWS) systems have been studied by many authors during the past few years, because they have many applications in economy, control theory, mechanical systems with dry frictions, impact oscillators, pest control models and so on. See [2], [3], [15], [16], [25] and the references therein.

Among many ways in which one can study PWS systems, the regularization method is very useful, because it replaces the discontinuous system with a smooth system. This is helpful because we are more familiar with smooth systems and we have more tools to investigate smooth systems. The first authors to formally introduce this technique were Sotomayor and Teixeira, see [24], and recently this method has been applied to study singularities, Filippov sliding vector fields and dynamical behavior near sliding regions of the PWS vector field. See [17], [18], [19] and [21].

Bifurcation of periodic orbits of planar Filippov systems with one switching line and the unperturbed periodic orbits cross the switching line transversally multiple times has been studied in [6] and [14]. When the unperturbed system has a limit cycle, they give some conditions for its persistence. Limit cycle bifurcations for a class of perturbed planar piecewise smooth systems with 4 switching lines are investigated in [26]. Many authors have studied the persistence of limit cycles for

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regularized planar vector fields, see [5] and [23]. However, these results rely heavily on the planar nature of the problem. In particular, they apply Poincaré-Bendixson Theorem to obtain the existence of limit cycles. In our recent work [11], we apply Brouwer fixed Theorem to obtain the existence of limit cycles and get similar results in more general n-dimensional PWS system with a discontinuity hyperplane.

Here we recall some results in [11]. Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( h(x) : \mathbb{R}^n \to \mathbb{R} \). Define the switching manifold as \( R_+ = \{ q \in \mathbb{R}^n : H(q) > 0 \} \), \( R_- = \{ q \in \mathbb{R}^n : H(q) < 0 \} \) and \( S = \{ q \in \mathbb{R}^n : H(q) = 0 \} \). The gradient \( \nabla H(q) \neq 0 \). Consider the following system:

\[
\dot{x} = F_0(x) = \begin{cases} 
F_+(x), & \text{if } x \in R_+ \\
F_-(x), & \text{if } x \in R_- .
\end{cases}
\]  

Here, \( F_- \) and \( F_+ \) are \( C^r \) functions, where \( r \geq 1 \), which we assume to be well defined in \( R_\mp \), on \( S \), and in a neighborhood of \( S \). Write \( F_0 = (F_-, F_+) \). A smooth function \( \phi : \mathbb{R} \to \mathbb{R} \) is a transition function if \( \phi(x) = -1 \) for \( x \leq -1 \), \( \phi(x) = 1 \) for \( x \geq 1 \) and \( \phi'(x) > 0 \) if \( x \in (-1, 1) \).

The \( \phi \)-regularization of \( F_0 = (F_-, F_+) \) is a 1-parameter family of vector fields \( F_\epsilon \), connecting \( F_- \) and \( F_+ \), and giving the following regularized system for (1):

\[
\dot{x} = F_\epsilon(x) = \frac{1}{2} \left( 1 - \phi \left( \frac{H(x)}{\epsilon} \right) \right) F_- (x) + \frac{1}{2} \left( 1 + \phi \left( \frac{H(x)}{\epsilon} \right) \right) F_+ (x).
\]  

For brevity, we will use the notation \( \phi_\epsilon(z) = \phi(\frac{z}{\epsilon}) \). Here \( \epsilon \) is a small positive parameter. The vector field \( F_\epsilon \) is an average of \( F_- \) and \( F_+ \) inside the boundary layer \( \{ x \in \mathbb{R}^n | -\epsilon < H(x) < \epsilon \} \), while it is equal to either \( F_- \) or \( F_+ \) outside the boundary layer.

In [5] and [23], the authors consider discontinuous planar systems. They show that if \( \gamma \) is a hyperbolic periodic orbit of (1) in \( \mathbb{R}^2 \), then, under suitable assumptions, the regularized vector field (2) has a hyperbolic limit cycle \( \gamma_\epsilon \), converging to \( \gamma \) as \( \epsilon \to 0 \).

In [11], we consider vector fields in \( \mathbb{R}^n \), with a co-dimension 1 hyperplane of discontinuity, for any \( n \geq 2 \). Under appropriate assumptions, we have proved that, if the original PWS system has an asymptotically stable (crossing or sliding) periodic orbit, then so will the regularized system. We mention that this is not a trivial generalization. There are two main difficulties. The first difficulty is that for higher dimensional piecewise smooth systems we do not have a Poincaré-Bendixson Theorem to help us in establishing existence of the limit cycle (e.g. [4], [5], [6], [23]); extensions of the Poincaré-Bendixson Theorem for systems in \( \mathbb{R}^n \), see [22], [27], require special type of systems (competitive or monotone systems), which do not fit our type of problem. The second difficulty is to establish the stability of the limit cycle of the regularized problem. We have done this by using the monodromy matrices of the discontinuous and regularized problems, and showing that the latter converges (as \( \epsilon \to 0 \)) to the former.

PWS system with a switching manifold of co-dimension 2 has been studied by Alexander and Seidman in [1]. Recently, Dieci and Gugliemi consider the regularization of PWS systems with a switching manifold of co-dimension 2, see [13]. However, the relation of periodic orbit of PWS between periodic orbit of its regularized system is not clear.

In this paper, we consider PWS systems of the following type:

\[
\dot{x} = f(x), \quad f(x) = f_i(x), \quad x \in R_i, \quad i = 1, 2, 3, 4. 
\]
with initial condition \( x(0) = x_0 \in R_i \) for some \( i \). \( R_i \in \mathbb{R}^n \) are open, disjoint and connected sets, and we may as well think that \( \mathbb{R}^n = \bigcup_i R_i \). Each \( f_i \) is smooth on \( R_i, i = 1,2,3,4 \). We will assume that each \( f_i \) is actually smooth in an open neighborhood of each \( R_i, i = 1,2,3,4 \).

We now consider equation (3), where \( R_i \)'s are locally separated by two intersecting smooth surfaces of codimension 1, \( \Sigma_1 = \{ x \in \mathbb{R}^n : h_1(x) = 0 \}, \nabla h_1(x) \neq 0, x \in \Sigma_1 \) and \( \Sigma_2 = \{ x \in \mathbb{R}^n : h_2(x) = 0 \}, \Sigma = \Sigma_1 \cap \Sigma_2. \Sigma = \{ x \in \mathbb{R}^n : h(x) = 0, h : \mathbb{R}^n \rightarrow \mathbb{R}^2 \}, h(x) = \left( \begin{array}{c} h_1(x) \\ h_2(x) \end{array} \right), \nabla h_1(x) \neq 0, x \in \Sigma_2. \nabla h_1(x) \) and \( \nabla h_2(x) \) are linearly dependent in a neighborhood of \( \Sigma. \) Without loss of generality, we label the four regions \( R_i \)'s and related vector fields as follows:

\[
R_1 : f_1 \quad \text{when} \quad h_1 < 0, h_2 < 0, \quad R_2 : f_2 \quad \text{when} \quad h_1 < 0, h_2 > 0, \\
R_3 : f_3 \quad \text{when} \quad h_1 > 0, h_2 > 0, \quad R_4 : f_4 \quad \text{when} \quad h_1 > 0, h_2 < 0. \tag{4}
\]

Take \( \epsilon > 0 \) small, the regularization of (3) is the following differential equation

\[
\dot{x} = f_\epsilon(x) = \frac{1 - \phi_\epsilon(h_1(x))}{2} f_1(x) + \frac{1}{2} \phi_\epsilon(h_1(x)) f_2(x) + \frac{1 + \phi_\epsilon(h_1(x))}{2} f_3(x) + \frac{1}{2} \phi_\epsilon(h_1(x)) f_4(x). \tag{5}
\]

Under suitable assumptions, we will prove if the original PWS system (3) has an asymptotically stable periodic orbit, then so will the regularized system (5). This work can be seen as a follow-up of [11]. The remainder of this paper is organized as follows. We give some definitions and state our main result in section 2. In section 3, we will introduce Filippov sliding vector fields. In section 4, we will prove our main result. Our conclusions will be given in section 5.

2. Basic definitions and main result. In this section, we will give definitions and assumptions.

For system (1), if a solution intersects \( S \) at an attractive sliding point \( x \) then it must remain on \( S. \) However the vector field \( F_0 \) is not defined on \( S \) and a sliding vector field needs to be defined. We follow Filippov (see [15]) and for each \( x \in S \) that verifies the first condition in (8) we define the sliding vector field as

\[
F_S(x) = \frac{1}{2} [(1 - \phi^*)F_- + (1 + \phi^*)F_+] (x), \quad \phi^*(x) = \frac{\nabla H^T(F_- + F_+)}{\nabla H^T(F_- - F_+)}(x), \tag{6}
\]

where the value of \( \phi^*(x) \) in (6) is such that \( \nabla H^T F_S(x) = 0. \)

We assume that \( f_i \) are \( C^r, i = 1,2,3,4, \) \( r \geq 1, \) and in a neighborhood of \( \Sigma_1, \Sigma_2 \) and \( \Sigma \). We denote the flow of (3) as \( \varphi^i_\epsilon(x) \) and the flow of (5) as \( \varphi^\epsilon(x). \)

**Definition 1.** Crossing conditions at \( x \in \Sigma = \Sigma_1 \cap \Sigma_2 \) are given by

\[
\begin{align*}
n_1^T f_1(x) > 0, \quad n_2^T f_1(x) > 0, \\
n_1^T f_2(x) > 0, \quad n_2^T f_2(x) > 0, \\
n_1^T f_3(x) > 0, \quad n_2^T f_3(x) > 0, \\
n_1^T f_4(x) > 0, \quad n_2^T f_4(x) > 0.
\end{align*} \tag{7}
\]

or similar conditions with opposite signs, where \( n_1(x) \) and \( n_2(x) \) are unit normals to the tangent planes \( T_x(\Sigma_1) \) and \( T_x(\Sigma_2) \), respectively.
Definition 2. Conditions that guarantee attractive sliding at \( x \in \Sigma \) are given by
\[
\begin{align*}
& n_1^T f_1(x) > 0, n_2^T f_1(x) > 0, \\
& n_1^T f_2(x) > 0, n_2^T f_2(x) < 0, \\
& n_1^T f_3(x) < 0, n_2^T f_3(x) > 0, \\
& n_1^T f_4(x) < 0, n_2^T f_4(x) < 0.
\end{align*}
\] (8)

3. Filippov sliding vector fields.

3.1. Codimension one case. The case when \( \Sigma \) is of codimension one discontinuity surface is well understood. For the completeness, we recall it briefly. Let
\[
\begin{align*}
& \Sigma_1^+ = \{ x \in \Sigma_1 : h_2(x) > 0 \}, \quad \Sigma_1^- = \{ x \in \Sigma_1 : h_2(x) < 0 \}; \\
& \Sigma_2^+ = \{ x \in \Sigma_2 : h_1(x) > 0 \}, \quad \Sigma_2^- = \{ x \in \Sigma_2 : h_1(x) < 0 \}.
\end{align*}
\]
Note that the sliding vector filed is actually defined on the codimension one switching manifold and it is tangent to this switching manifold. i.e.
\[
\begin{align*}
& x \in \Sigma_1^- : f_{\Sigma_1} = (1 - \alpha^-) f_1 + \alpha^- f_4, 0 \leq \alpha^- \leq 1; \nabla h_1^T f_{\Sigma_1} = 0; \\
& x \in \Sigma_1^+ : f_{\Sigma_1} = (1 - \alpha^+) f_1 + \alpha^+ f_3, 0 \leq \alpha^+ \leq 1; \nabla h_1^T f_{\Sigma_1} = 0; \\
& x \in \Sigma_2^- : f_{\Sigma_2} = (1 - \beta^-) f_1 + \beta^- f_2, 0 \leq \beta^- \leq 1; \nabla h_2^T f_{\Sigma_2} = 0; \\
& x \in \Sigma_2^+ : f_{\Sigma_2} = (1 - \beta^+) f_1 + \beta^+ f_3, 0 \leq \beta^+ \leq 1; \nabla h_2^T f_{\Sigma_2} = 0.
\end{align*}
\] (9) (10) (11) (12)

Definition 3. If a solution of (3) reaches the switching manifold \( \Sigma_1 \) or \( \Sigma_2 \) or \( \Sigma \) at an attractive sliding point \( x \). Then \( x \) is said to be a transversal entry point.

3.2. Codimension 2 case: Algebraic ambiguity. Following [9] and [15], we call Filippov sliding vector filed (on \( \Sigma \)) any vector field of the form
\[
f_\Sigma(x) = \sum_{i=1}^{4} \lambda_i(x) f_i(x), \quad \text{where} \quad \lambda_i(x) \geq 0 \quad \text{and} \quad \sum_{i=1}^{4} \lambda_i(x) = 1, x \in \Sigma. \tag{13}
\]
subject to the constraint that \( f_\Sigma(x) \) lies in the tangent plane to \( \Sigma \) at \( x \), i.e.
\[
(\nabla h_j(x))^T f_\Sigma(x) = 0, \quad \text{for} \quad j = 1, 2. \tag{14}
\]
Since we have 4 unknowns in 3 equations, it’s easy to know we will fail to select uniquely the coefficients \( \lambda_i, i = 1, 2, 3, 4 \).

There are many methods to choose Filippov sliding vector filed. For instance, bilinear interpolation method and moments method. However, we are interested in the former one. To our knowledge, it was first seen in [1] and then it has been discussed thoroughly in [7], [9] and so on. This method selects the Filippov sliding vector field as follows.
\[
f_\Sigma := (1 - \alpha)(1 - \beta)f_1 + (1 - \alpha)\beta f_2 + \alpha(1 - \beta)f_3 + \alpha\beta f_4, \tag{15}
\]
where \( \alpha, \beta \in (0, 1) \), \( \lambda_B = \begin{pmatrix} (1 - \alpha)(1 - \beta) \\ (1 - \alpha)\beta \\ \alpha(1 - \beta) \\ \alpha\beta \end{pmatrix} \), \( W = \begin{pmatrix} w_1^1 & w_1^2 & w_1^3 & w_1^4 \\ w_2^1 & w_2^2 & w_2^3 & w_2^4 \\ w_3^1 & w_3^2 & w_3^3 & w_3^4 \\ w_4^1 & w_4^2 & w_4^3 & w_4^4 \end{pmatrix} \) such that \( W \lambda_B = 0 \). \( w_i^j = \nabla h_i^T f_j, i = 1, 2, j = 1, 2, 3, 4 \).

Assume that trajectories of the PWS system (3) exist in a neighborhood \( U \) of \( \Sigma \). This must be understood to imply that in case the value of \( x_0 \) is on either \( \Sigma_1 \) or \( \Sigma_2 \) (but not on \( \Sigma \)), there may be sliding motion on \( \Sigma_1 \) or \( \Sigma_2 \) according to Filippov’s first order theory. However, we also remark that motion in a neighborhood of \( \Sigma \)
may not be uniquely defined, such as when \( x_0 \in \Sigma_1 \) but sliding motion on \( \Sigma_1 \) is repulsive. With this in mind, we will still write \( \phi^t(x_0) \) to indicate a continuous Filippov trajectory of the system. The general characterization of attractive \( \Sigma \) will require that \( \Sigma \) be stable (with respect to the initial conditions) and approached by trajectories of the system. We say that \( \Sigma \) is stable if for any \( x_0 \in \Sigma \), and for any \( \epsilon > 0 \) sufficiently small, there is \( \delta > 0 \) such that if \( u \in B_\delta(x_0)/(\Sigma \cap B_\delta(x_0)) \), then the distance between \( \phi^t(u) \) and \( \Sigma \) satisfies \( d(\phi^t(u) - \Sigma) \leq \epsilon \) for all \( t \geq 0 \). However, in many situations of interest, we will want a more restrictive condition requiring not only that \( \Sigma \) is attractive, but it is also approached in finite time. We will assume that \( \Sigma \) is attractive in finite time upon sliding, see the following definition in [9].

**Definition 4.** \( \Sigma \) attracts in finite time trajectories of (3), if: (i) \( \Sigma \) is stable; (ii) for any \( x_0 \in U/(\Sigma \cap U) \), where \( U \) is a neighborhood of \( \Sigma \), there exists a first (finite) time \( \tau(x_0) \geq 0 \) such that \( \phi^\tau(x_0) \in \Sigma \).

**Definition 5.** Assume that the trajectory of system (3) slides on \( \Sigma \). When the trajectory reaches a point \( x_1 \in \Sigma \), where one-and-only-one of the following four conditions satisfied.

(i) Exiting on \( \Sigma_2^- \) or \( \Sigma_2^+ \):
- (a) \( \nabla h_1(x)^T f_{\Sigma_2^-}(x) = 0 \), or
- (b) \( \nabla h_1(x)^T f_{\Sigma_2^+}(x) = 0 \).

(ii) Exiting on \( \Sigma_1^- \) or \( \Sigma_1^+ \):
- (a) \( \nabla h_2(x)^T f_{\Sigma_1^-}(x) = 0 \), or
- (b) \( \nabla h_2(x)^T f_{\Sigma_1^+}(x) = 0 \).

Then \( \tilde{x} \) is called a (first order) generic tangential exit point. Here \( f_{\Sigma_2^-}, f_{\Sigma_2^+}, f_{\Sigma_1^-}, f_{\Sigma_1^+} \) are called exit vector fields, respectively. See [10].

In Section 4, the jumps in the derivatives of the solution will be considered, and the monodromy matrix along \( \gamma \) is defined with the aid of suitable saltation matrices. As for smooth dynamical systems, we also study stability properties of a periodic orbit \( \gamma \) of PWS system via the eigenvalues of the monodromy matrix, i.e. the Floquet multipliers.

**Definition 6.** Let \( \gamma \) be a periodic orbit of (3). Let \( \mu_1, \mu_2, \ldots, \mu_n \) be the corresponding Floquet multipliers. We say that \( \gamma \) is asymptotically stable if one of the multipliers is 1, say \( \mu_1 = 1 \), and all other \( \mu_i \)'s are less than 1 in modulus.

For the crossing periodic orbit case, our main result is the following

**Theorem 7.** Let \( \gamma \) be an asymptotically stable crossing periodic orbit of (3). Then, for \( \epsilon > 0 \) sufficiently small, system (5) has a unique limit cycle \( \gamma_\epsilon \) in a neighborhood of \( \gamma \). Moreover \( \gamma_\epsilon \) is asymptotically stable and \( \lim_{\epsilon \to 0} \gamma_\epsilon = \gamma \).

If \( \gamma \) is a sliding periodic orbit, without loss of generality, we assume that it looks like Figure 2. This kind of sliding periodic orbit has been studied in [10] for 3D case. We assume that \( \Sigma \) is the intersection of two hyperplanes \( \Sigma_1 = \{x \in \mathbb{R}^n : h_1(x) = x_1 = 0\} \) and \( \Sigma_2 = \{x \in \mathbb{R}^n : h_2(x) = x_2 = 0\} \). Let \( \bar{x} \in \Sigma \) be a tangential exit point that satisfies condition (i)(a) in definition 5. Take the initial condition as \( x(0) = \bar{x} \). This periodic orbit starts sliding on \( \Sigma_1^+ \) with the sliding vector filed \( f_{\Sigma_1^+} \). Then at \( x = x_1 \in \Sigma_2^+ \), the trajectory exits \( \Sigma_2^+ \) smoothly and enters in \( R_3 \). Then at \( x = x_2 \in \Sigma_1^+ \), the trajectory reaches \( \Sigma_1^+ \) transversally and slides on it. At
$x = x_3 \in \Sigma$, the trajectory reaches $\Sigma$ transversally and slides on it with the sliding vector field $f_\Sigma$ until it reaches the exit point $x_4 = \bar{x}$. Assume that system (3) has an asymptotically stable sliding periodic orbit $\gamma$ given as the following Figure 2.

For the sliding periodic orbit case, we have the following

**Figure 1.** Crossing periodic orbit

**Theorem 8.** Assume that system (3) has an asymptotically stable sliding periodic orbit $\gamma$ and $\Sigma$ is attractive in finite time upon sliding. Assume moreover that the entrance point $x_2$ in $\Sigma^+_1$ is a transversal entry point and that both the exit points $\bar{x}$ from $\Sigma$ and $x_1$ from $\Sigma^+_2$ are first order tangential exit points. Then, for $\epsilon > 0$ sufficiently small, there exists a unique limit cycle $\gamma_\epsilon$ of system (5) in a neighborhood of $\gamma$. Moreover $\gamma_\epsilon$ is asymptotically stable and $\lim_{\epsilon \to 0} \gamma_\epsilon = \gamma$.

**Figure 2.** Sliding periodic orbit
4. Proof of main result. In this section, we prove Theorem 7. We will treat the crossing periodic orbit when it crosses only codimension one switching manifold and it crosses co-dimension two switching manifold separately.

4.1. Crossing co-dimension one switching manifold. In this part, we consider the case in which $\gamma$ meets $\Sigma_1$ and $\Sigma_2$ in just four crossing points, denote them with $\tilde{x}_i$, $i = 1, 2, 3, 4$. See Figure 1. We can write Theorem 7 as the following

**Theorem 9.** Assume that (3) has an asymptotically stable periodic orbit $\gamma$ with totally four transversal crossing points with $\Sigma_1$ and $\Sigma_2$. Then, for $\epsilon > 0$ sufficiently small, there exists one and only one periodic orbit $\gamma_\epsilon$ of (5) in a neighborhood of $\gamma$. Moreover $\gamma_\epsilon$ is asymptotically stable and $\lim_{\epsilon \to 0} \gamma_\epsilon = \gamma$.

We will prove Theorem 9 according to the following steps.

1. Prove that (5) has at least one limit cycle. To do this, we will define a Poincaré map $P_\epsilon$, and use Brouwer’s fixed point Theorem to show that it has a fixed point. This will give at least one limit cycle $\gamma_\epsilon$ of (5), and we will show that $\gamma_\epsilon \to \gamma$ when $\epsilon \to 0$.

2. Then, we will show that $\gamma_\epsilon$ is asymptotically stable, so $\gamma_\epsilon$ is the unique limit cycle of (5), for $\epsilon$ sufficiently small.

Let us define a Poincaré map associated to $\gamma$. Without loss of generality, we will assume that the periodic solution associated to $\gamma$ crosses $\Sigma_2$ at $\tilde{x}_1$ coming from $\Sigma^-_2$ and entering in $\Sigma^+_1$ at $\tilde{x}_2$, and then again (at a later time) crosses $\Sigma^-_2$ at $\tilde{x}_2$ and $\Sigma^+_1$ at $\tilde{x}_3$.

Let $\varphi^t_i$ denote the flows of $f_i$. Let $B_\delta(\tilde{x}_1)$ be a neighborhood of $\tilde{x}_1$ in $\mathbb{R}^n$ and denote with $B_\delta(\tilde{x}_1, \Sigma_2)$ its intersection with $\Sigma_2$ and with $\overline{B_\delta(\tilde{x}_1, \Sigma_2)}$ its closure. Then for $\delta$ sufficiently small the Poincaré map $P_\epsilon(x) = \varphi^t_{\epsilon, t_\epsilon}(x)$, where $t_{\epsilon}(x)$ is the first return time to $\Sigma_2$, is well defined and smooth in $x$ and it takes a point $x$ in $B_\delta(\tilde{x}_1, \Sigma_2)$ into a neighborhood of $\tilde{x}_2$. Similarly, we can define a Poincaré map $P_\epsilon(x)$ that, due to the asymptotic stability of $\gamma$, takes a point $x$ in a neighborhood of $\tilde{x}_2$ into $B_\delta(\tilde{x}_1, \Sigma_2)$. Let $P = P_\epsilon \circ P_{\epsilon, t}$ : $B_\delta(\tilde{x}_1, \Sigma_2) \to B_\delta(\tilde{x}_1, \Sigma_2)$ be the Poincaré map of system (3). Then $P$ is well defined and smooth with its inverse in $B_\delta(\tilde{x}_1, \Sigma_2)$ and since $\gamma$ is asymptotically stable, $P$ satisfies $P(B_\delta(\tilde{x}_1, \Sigma_2)) \subset B_\delta(\tilde{x}_1, \Sigma_2)$ for $\delta$ sufficiently small. Let $\psi_\delta$ be the boundary of $B_\delta(\tilde{x}_1, \Sigma_2)$, then $\psi_\delta$ is the intersection of the $(n-1)$-sphere of center $\tilde{x}_1$ and radius $\delta$ with $\Sigma_2$. The set $P(\psi_\delta)$ is a diffeomorphic image of $\psi_\delta$. Let $V$ be the union of all trajectories of (3) with initial point on $\psi_\delta$ and endpoint on $P(\psi_\delta)$ together with $B_\delta = B_\delta(\tilde{x}_1, \Sigma_2) \setminus P(B_\delta(\tilde{x}_1, \Sigma_2))$ and let $\hat{V}$ be the compact subset of $\mathbb{R}^n$ whose boundary is $V$. Then all solution trajectories of (3) that intersect $B_\delta$ will do so transversally, will enter $\hat{V}$ and will remain inside it. The periodic orbit $\gamma$ attracts all trajectories inside $\hat{V}$.

Stability of $\gamma$ can be studied via the monodromy matrix $X(T)$ at $\gamma$. Let $T$ be the period of $\gamma$ and assume that $\varphi^t_1(\tilde{x}_1) = \tilde{x}_4$, $\varphi^t_1(\tilde{x}_1) = \tilde{x}_2$, $\varphi^t_3(\tilde{x}_2) = \tilde{x}_3$, $\varphi^t_2(\tilde{x}_3) = \tilde{x}_1$. Then, $X(T)$ can be written as the composition of the following matrices (e.g., see [12, 20]):

$$X(T) = X(T, t_2)S_{12}(\tilde{x}_3)X(t_2, t_4)S_{43}(\tilde{x}_2)X(t_4, t_1)S_{14}(\tilde{x}_4)X(t_1, 0)S_{21}(\tilde{x}_1)$$

where $S_{21}(\tilde{x}_1) = I + \frac{(f_2-f_1)}{h_1}T h_1^T(\tilde{x}_1)$ and $S_{43}(\tilde{x}_2) = I + \frac{(f_4-f_3)}{h_1}T h_1^T(\tilde{x}_2)$, $S_{14}(\tilde{x}_4) = I + \frac{(f_1-f_3)}{h_1}T h_1^T(\tilde{x}_4)$, $S_{32}(\tilde{x}_3) = I + \frac{(f_2-f_4)}{h_1}T h_1^T(\tilde{x}_3)$ are so-called saltation matrices,
while the fundamental matrix solutions $X(t, 0)$, $X(t, 1)$ and $X(t, 4)$ satisfy
\[
\begin{align*}
\dot{X}(t, 0) &= Df_1(\varphi_0^t(\bar{x}_1))X(t, 0), & X(0, 0) &= I; \\
\dot{X}(t, 1) &= Df_4(\varphi_0^t(\bar{x}_2))X(t, 1), & X(0, 1) &= I, \\
\dot{X}(t, 4) &= Df_2(\varphi_0^t(\bar{x}_2))X(t, 4), & X(0, 4) &= I.
\end{align*}
\]

The four salutation matrices in (16) are nonsingular and hence $X(T)$ has an eigenvalue at 1 and all the other eigenvalues are less than 1 in modulus because of asymptotic stability of $\gamma$.

To prove the existence of a periodic orbit of (5) in a neighborhood of $\gamma$ we employ the Poincaré map of (5). In a neighborhood of $\bar{x}_1$ and $\bar{x}_2$ solutions of (5) intersect $\Sigma_2$ transversally and hence we can consider the following Poincaré map $P_\gamma : B_\delta(\bar{x}_1, \Sigma_2) \to B_\delta(\bar{x}_1, \Sigma_2)$ that associates to a point in $B_\delta(\bar{x}_1, \Sigma_2)$ its first return to $B_\delta(\bar{x}_1, \Sigma_2)$. The following proposition establishes the existence of at least one periodic orbit of (5).

**Proposition 10.** The map $P_\gamma$ has at least one fixed point in $B_\delta(\bar{x}_1, \Sigma_2)$.

In order to prove this proposition we will need the following Lemma.

**Lemma 11.** For each $x_0 \in B_\delta(\bar{x}_1)$ the following is satisfied
\[
\lim_{\varepsilon \to 0} \varphi_{\varepsilon}^t(x_0) = \varphi_0^t(x_0),
\]
uniformly for $t$ in a compact interval.

**Proof.** Denote by $\varphi_0^t$ and $\varphi_{\varepsilon}^t$ the flows of (3) and (5) respectively. Together with $\Sigma_2$, consider also the hyperplanes $\Sigma_{2, \varepsilon} = \{ x \in \mathbb{R}^n | h_2(x) = \varepsilon \}$ and $\Sigma_{2, -\varepsilon} = \{ x \in \mathbb{R}^n | h_2(x) = -\varepsilon \}$. In what follows, for $x_0 \in B_{\delta}(\bar{x}_1)$, we want to estimate the distance between $\varphi_0^t(x_0)$ and $\varphi_{\varepsilon}^t(x_0)$ at their intersection points with $\Sigma_2$, $\Sigma_{2, \varepsilon}$ and $\Sigma_{2, -\varepsilon}$.

Without loss of generality assume that $x_0 \in B_{\delta}(\bar{x}_1, \Sigma_2)$. Then for $\delta$ and $\varepsilon$ sufficiently small $\nabla h_2 f_1(x_0) < 0$, $\nabla h_2 f_2(x_0) < 0$. Let $\bar{t}_1$ be such that $x_1 = \varphi_0^{\bar{t}_1}(x_0) \in \Sigma_{2, -\varepsilon}$ and similarly, let $x_1' = \varphi_{\varepsilon}^{\bar{t}_1}(x_0) \in \Sigma_{2, -\varepsilon}$, with $x_1, x_1'$ in a neighborhood of $\bar{x}_1$. We want to bound $\| \varphi_0^t(x_0) - \varphi_{\varepsilon}^t(x_0) \|$ and show that it goes to zero when $\varepsilon \to 0$. To fix ideas, assume $\bar{t}_1 > \bar{t}_1$. Let $L_1 = \max_{x_0 \in [0, \bar{t}_1]} |Df_1(\varphi_0^t(x_0)), \max_{x_0 \in [0, \bar{t}_1]} |Df_2(\varphi_{\varepsilon}^t(x_0))|$ and $M_1 = \max_{t \in [0, \bar{t}_1]} |f_1(\varphi_{\varepsilon}^t(x_0))|$, $i = 1, 2$. Then the following inequality holds
\[
\begin{align*}
\| \varphi_0^{\bar{t}_1}(x_0) - \varphi_{\varepsilon}^{\bar{t}_1}(x_0) \| &\leq \| \int_{0}^{\bar{t}_1} f_1(\varphi_0^s(x_0))ds - f_1(\varphi_{\varepsilon}^s(x_0))ds \| + \\
&\quad \| \int_{0}^{\bar{t}_1} \left( \frac{1}{2} \phi_{\varepsilon}^s(s) \right)(f_1 - f_2)(\varphi_{\varepsilon}^s(x_0))ds \| + \| \int_{0}^{\bar{t}_1} f_2(\varphi_{\varepsilon}^s(x_0))ds \| \\
&\leq L_1 \int_{0}^{\bar{t}_1} \| \varphi_0^s(x_0) - \varphi_{\varepsilon}^s(x_0) \|ds + \bar{t}_1(M_1 + M_2) + (\bar{t}_1 - \bar{t}_1)(M_1 + M_2) \\
&\leq \bar{t}_1(M_2 + M_1)e^{\bar{t}_1 L_1},
\end{align*}
\]
where the last inequality follows from Gronwall’s Lemma. Moreover using $h_2(\varphi_0^{\bar{t}_1}(x_0)) = -\varepsilon$, if we consider the Taylor polynomial in Lagrange form of $\varphi_0^{\bar{t}_1}(x_0)$ at the point $\bar{t}_1 = 0$, we obtain
\[
\bar{t}_1 = \frac{-\varepsilon}{\nabla h_2(\varphi_0^0(x_0))1^{T} f_1(\varphi_0^0(x_0))}, \quad \eta \in (0, \bar{t}_1).
\]
In particular \( \lim_{t \to 0} \overline{t}_1 = 0 \) and, in a similar way, \( \lim_{t \to 0} t'_1 \to 0 \). This, together with the bound for \( \| \varphi^{t_1}(x_0) - \varphi^{t'_1}(x_0) \| \), implies \( \lim_{t \to 0} \varphi^{t}_1(x_0) = \varphi^{t'_1}(x_0) \).

For \( t > t_1 \), \( \varphi^{t}_0(x_0) \) moves away from \( \Sigma_{2,-} \) in direction opposite to \( \nabla h_1(x) \). But it will meet \( \Sigma_{1,-} \) in a neighborhood of \( \bar{x}_1 \). Note that now we have \( f_1 = f_\epsilon \). Let \( \bar{t}_2 \) be such that \( x_2 = \varphi^{t_1}_0(x_1) \in \Sigma_{1,-} \) and \( \bar{t}'_2 \) be such that \( x'_2 = \varphi^{t'_1}_0(x'_1) \in \Sigma_{1,-} \).

To fix ideas, again we assume \( t'_2 > \bar{t}_2 \). Let \( M_- = \max_{t \in [\bar{t}_2, t'_2]} \| f_1(\varphi^{t}_0(x_1)) \|, \max_{t \in [0, t_2]} Df_1(\varphi^{t}_0(x_1)) \) be the local Lipschitz constant for \( f_1 \), then the following bound holds

\[
\| \varphi^{t}_0(x_1) - \varphi^{t'_1}(x'_1) \|
\leq \| x_1 - x'_1 \| + \int_{\bar{t}_2}^{t'_2} \| f_1(\varphi^{t}_0(x_1)) - f_1(\varphi^{t}_0(x'_1)) \| ds + (t'_2 - \bar{t}_2)M_-,
\]

where the last inequality follows from Gronwall’s Lemma. Notice that \( \lim_{t \to 0} t'_2 = \bar{t}_2 \), since \( x'_2 \to x_1 \) and \( f_\epsilon = f_1 \) for \( h_1(x) < -\epsilon \) and \( h_2(x) < -\epsilon \). Then we have \( \lim_{t \to 0} \varphi^{t}_0(x'_1) = \varphi^{t}_0(x_1) \). In a similar way we can show that \( \| \varphi^{t}_0(x_0) - \varphi^{t}_0(x_0) \| \to 0 \) up to their first return time to \( B_\delta(\bar{x}_1, \Sigma_2) \). This proves the Lemma.

As a consequence of Lemma 11, this Corollary holds.

Corollary 12. As \( \epsilon \to 0 \), \( P_\epsilon \) converges pointwise to \( P \) in \( \overline{B}_\delta(\bar{x}_1, \Sigma_2) \).

Proof of Proposition 10. We will prove that \( P_\epsilon(\overline{B}_\delta(\bar{x}_1, \Sigma_2)) \subset \overline{B}_\delta(\bar{x}_1, \Sigma_2) \). Then the statement will follow from Brouwer’s fixed point Theorem. Consider the following relation between \( \psi_\delta \) and \( P(\psi_\delta) \): \( \beta(\psi_\delta, P(\psi_\delta)) = \min_{a \in \psi_\delta, b \in P(\psi_\delta)} \| a - b \| \).

Then \( \beta(\psi_\delta, P(\psi_\delta)) > \overline{\eta} > 0 \) and \( \overline{\eta} \) is bounded away from 0 since \( P(\overline{B}_\delta(\bar{x}_1, \Sigma_2)) \subset B_\delta(\bar{x}_1, \Sigma_2) \). Then \( \eta, 0 < \eta < \overline{\eta} \) be fixed, and for every \( x \in B_\delta(\bar{x}_1, \Sigma_2) \) denote with \( B_\eta(x, \Sigma_2) \) the intersection of the \( \eta \)-ball centered at \( P(x) \) with \( \Sigma_2 \). Then \( B_\eta(x, \Sigma_2) \subset B_\delta(\bar{x}_1, \Sigma_2) \). Corollary 12 implies that for every \( x \in B_\delta(\bar{x}_1, \Sigma_2) \) there exists \( \epsilon_{\eta,x} > 0 \) such that, for \( \epsilon \in (0, \epsilon_{\eta,x}) \), \( P_\epsilon(x) \in B_\eta(P(x), \Sigma_2) \subset B_\delta(\bar{x}_1, \Sigma_2) \). The proof follows upon noticing that for every \( x, \epsilon_{\eta,x} \) is bounded away from 0.

Proposition 13. System (5) has at least a periodic orbit.

Proof. Proposition 10 ensures existence of a fixed point of \( P_\epsilon \) in \( B_\delta(\bar{x}_1, \Sigma_2) \). We need to exclude the possibility that the fixed point is an equilibrium point for \( f_\epsilon \). Assume by contradiction that there exists \( \bar{x} \in B_\delta(\bar{x}_1, \Sigma_2) \) such that \( f_\epsilon(\bar{x}) = 0 \). Then it follows that

\[
\nabla h_2(\bar{x})^T f_\epsilon(\bar{x}) = \nabla h_2(\bar{x})^T \left( \frac{1}{2}(1 - \phi_\epsilon) f_1 + \frac{1}{2}(1 + \phi_\epsilon) f_2 \right)(\bar{x}) = 0.
\]

Note that \( \nabla h_2^T f_1(\bar{x}) > 0, \nabla h_2^T f_2(\bar{x}) > 0 \) and \( \frac{1}{2}(1 - \phi_\epsilon), \frac{1}{2}(1 - \phi_\epsilon) > 0 \), hence we reach a contradiction. 

An invariant region \( V_\epsilon \) for \( f_\epsilon \) can be built by considering the union of all trajectories with initial points on \( \psi_\delta \) and endpoints on \( P_\epsilon(\psi_\delta) \) together with \( B_\delta(\bar{x}_1, \Sigma_2) \). Then \( (B_\delta(\bar{x}_1, \Sigma_2)) = B_\delta^\circ \). Let \( \tilde{V}_\epsilon \) be the compact subset of \( \mathbb{R}^n \) with boundary \( V_\epsilon \). Then all trajectories of (1) that cross \( B_\delta^\circ \) will do so transversally, will enter \( \tilde{V}_\epsilon \) and will
remain inside it. We will show that (5) has a unique periodic orbit \( \gamma_\epsilon \) in \( \bar{V}_\epsilon \) and \( \gamma_\epsilon \) attracts all the solutions inside \( \bar{V}_\epsilon \).

Next, for each \( \epsilon \), select a fixed point of \( P \) and denote it with \( x_\epsilon \). Let \( \gamma_\epsilon = \{ x \in \mathbb{R}^n : x = \varphi^\epsilon_t(x_\epsilon), \ t \in \mathbb{R} \} \), be the corresponding periodic orbit. In order to complete the proof of Theorem 9 we will show

i) as \( \epsilon \to 0 \), \( x_\epsilon \to \bar{x}_1 \), which in turn will imply \( \gamma_\epsilon \to \gamma \);

ii) for \( \epsilon \) sufficiently small, \( \gamma_\epsilon \) is asymptotically stable and this allows us to exclude that (5) has a family of periodic orbits that converges to \( \gamma \).

Let \( x \in B_\delta(\bar{x}_1, \Sigma_2) \) and let \( \varphi^\epsilon_0(x) \) be the solution of (3) with initial condition \( x \). Denote by \( x_2 \) its intersection with \( \Sigma_2 \) in a neighborhood of \( \bar{x}_2 \) and let \( x_3 = P(x) \). Notice that \( x_3 \neq x \) unless \( x = \bar{x}_1 \). Let \( t_2 \) be such that \( \varphi^\epsilon_0(t_2)(x) = x_2 \) and \( T(x) \) be such that \( \varphi^T(x)(x) = x_3 \). The following Lemma holds.

**Lemma 14.** Denote with \( X(t, 0, x) \) and \( X_\epsilon(t, 0, x) \) respectively the fundamental matrix solution of (3) along the solution \( \varphi^\epsilon_0(x) \) and the fundamental matrix solution of (5) along \( \varphi^\epsilon_\epsilon(x) \), \( x \in B_\delta(\bar{x}_1, \Sigma_2) \) Then

\[
\lim_{\epsilon \to 0} X_\epsilon(T_\epsilon(x), 0, x) = X(T(x), 0, x),
\]

where with \( T_\epsilon(x) \) we denote the first return time of \( \varphi^\epsilon_\epsilon(x) \) to \( B_\delta(\bar{x}_1, \Sigma_2) \).

**Proof.** In what follows we will omit the explicit dependence of \( X(t, 0, x) \) and \( X_\epsilon(t, 0, x) \) on \( x \). The principal matrix solution of (5) along \( \varphi^\epsilon_\epsilon(x) \) satisfies

\[
\dot{X}_\epsilon = Df_\epsilon(\varphi^\epsilon_\epsilon(x))X_\epsilon, \quad X_\epsilon(0) = I,
\]

where \( I \) is the identity matrix and \( Df_\epsilon(x) \) is the Jacobian of \( f_\epsilon \).

Similarly to what we have done in the proof of Lemma 11, we consider the intersection of \( \varphi^\epsilon_\epsilon(x) \) with \( \Sigma_2, \Sigma_{2, \epsilon} \) and \( \Sigma_{2, -\epsilon} \), and use these intersection points to rewrite \( X_\epsilon(T_\epsilon(x), 0) \) as product of transition matrices. Let \( x_0^\epsilon \in \Sigma_{2, \epsilon} \) be such that \( \varphi^\epsilon_t(x_0^\epsilon) \) meets \( \Sigma_2 \) at time \( t_1^\epsilon \) at the point \( x \), i.e. \( \varphi^\epsilon_{t_1^\epsilon}(x_0^\epsilon) = x \), and \( \nabla h^T_{\epsilon} f_\epsilon(x_0^\epsilon) < 0 \). We evaluate the monodromy matrix \( X_\epsilon(T_\epsilon(x), 0) \) along the shifted solution \( \varphi^\epsilon_{t_1^\epsilon}(x_0^\epsilon) \). Notice that in a whole neighborhood of \( x \) the following inequality is satisfied \( \nabla h^T_{\epsilon} f_\epsilon(x) < 0 \) so that \( \varphi^\epsilon(x_0^\epsilon) \) intersects \( \Sigma_{2, -\epsilon} \) and then \( \Sigma_{1, \epsilon} \) at two isolated points: \( x_2^\epsilon = \varphi^\epsilon_{t_2^\epsilon}(x_0^\epsilon) \) and \( x_3^\epsilon = \varphi^\epsilon_{t_3^\epsilon}(x_0^\epsilon) \). Then the trajectory enters the set \( \Sigma_{1, \epsilon} \) at time \( t_2^\epsilon \). Then the trajectory enters the set \( \Sigma_{2, -\epsilon} \) again in \( x_2^\epsilon \) and then \( \Sigma_{1, -\epsilon} \) again in \( x_3^\epsilon \) at time \( t_1^\epsilon \). See the following Figure 3. At \( t = T_\epsilon(x) \), \( \varphi^\epsilon_{t_1^\epsilon}(x_0^\epsilon) \) returns to \( \Sigma_{2, \epsilon} \). With these notations, we can rewrite \( X_\epsilon \) as follows

\[
X_\epsilon(T_\epsilon(x), 0) = X_\epsilon(T_\epsilon(x), t_{11}^\epsilon)X_\epsilon(t_{11}^\epsilon, t_{0}^\epsilon)X_\epsilon(t_{0}^\epsilon, t_{5}^\epsilon)X_\epsilon(t_{5}^\epsilon, t_{6}^\epsilon)X_\epsilon(t_{4}^\epsilon, t_{1}^\epsilon)X_\epsilon(t_{1}^\epsilon, t_{21}^\epsilon)X_\epsilon(t_{21}^\epsilon, 0).
\]

We will show that \( X_\epsilon \) in (19) converge to the corresponding factors in \( X(T, 0, x) \), where

\[
X(T, 0, x) = X(T, t_2)S_{32}(\bar{x}_3)X(t_2, t_4)S_{43}(\bar{x}_2)X(t_4, t_1)S_{14}(\bar{x}_4)X(t_1, 0)S_{21}(\bar{x}_1).
\]
For the factor $X_\epsilon(t_2',0)$, we rewrite it as $X_\epsilon(t_2',0) = X_\epsilon(t_2',t_1')X_\epsilon(t_1',0)$. The limit of $X_\epsilon(t_2',0)$, as $\epsilon \to 0$, will exist if the limits of the other two factors do. We have

$$X_\epsilon(t_1',0)$$

$$= I + \int_{t_0}^{t_1'} Df_\epsilon(x(t))X_\epsilon(t,0)dt$$

$$= I + \frac{1}{2} \int_{t_0}^{t_1'} \left[ (1 - \phi_\epsilon(\frac{h_2(x(t))}{\epsilon}))Df_1 + (1 + \phi_\epsilon(\frac{h_2(x(t))}{\epsilon}))Df_2 \right] (x(t))X_\epsilon(t,0)dt$$

$$+ \frac{1}{\epsilon} \int_{t_0}^{t_1'} \frac{1}{2} \phi_\epsilon' \left( \frac{h_2(x(t))}{\epsilon} \right) (-f_1 + f_2)\nabla h_2(x(t))^T X_\epsilon(t,0)dt.$$

The first integral goes to 0 as $\epsilon \to 0$, since $t_1' \to 0$ and the integrand is bounded. The second integral is dealt with by noticing that $\phi_\epsilon'(\frac{h_2}{\epsilon})$ is positive and thus we can consider the change of variable $t \to x_2$, along with $\frac{1}{\epsilon} \int_{t_0}^{t_1'} \phi_\epsilon' \frac{h_2}{\epsilon} dx_2 = -\frac{1}{2}$, and the mean value Theorem for integrals to obtain

$$\frac{1}{2\epsilon} \int_{t_0}^{t_1'} \frac{1}{\nabla h_2^T f_\epsilon(x)} \phi_\epsilon' \left( \frac{x_2}{\epsilon} \right) (f_2 - f_1)(x)\nabla h_2^T X_\epsilon dx_2 = -\frac{1}{2} \frac{(f_1 - f_2)\nabla h_2^T X_\epsilon}{\nabla h_2^T f_\epsilon},$$

where the entries of the matrix on the right-hand side are evaluated at some value $t$, $0 \leq t < t_1'$, possibly different for each matrix entry. Since all objects on the right-hand side are smooth in $[0,t_1']$, $t_1' \to 0$, $f_\epsilon$ at $t = 0$ is $f_1$, and $X_\epsilon(t_1',0) = I + \mathcal{O}(t_1')$, by expanding around $t = 0$ (or $x_1 = \epsilon$), we conclude that the limit as $\epsilon \to 0$ of $X_\epsilon(t_1',0)$ exists and it is:

$$\lim_{\epsilon \to 0} X_\epsilon(t_1',0) = I + \frac{1}{2} \frac{(f_1 - f_2)\nabla h_2^T X_\epsilon}{\nabla h_2^T f_1}(x).$$

For $X_\epsilon(t_2',t_1')$, we have

$$X_\epsilon(t_2',t_1') = I + \int_{t_1'}^{t_2'} Df_\epsilon(x(t))X_\epsilon(t,t_1')dt.$$
implies in particular that there exists \( \epsilon > 0 \) such that for each \( \epsilon \), \( \lim_{\epsilon \to 0} X_\epsilon(t',0) \) exists, and it is given by

\[
\lim_{\epsilon \to 0} X_\epsilon(t',0) = I + \left[ \frac{f_1 - f_2}{\nabla h_2^T f_2} \right]_{x} = S_{21}(\bar{x}_1).
\] (25)

Similarly to (25), we can prove \( \lim_{\epsilon \to 0} X_\epsilon(t'_3,t'_2) = S_{14}(\bar{x}_4) \).

Finally, \( f_\epsilon = f_1 \) in \( [t'_2, t'_3] \) and \( f_\epsilon = f_2 \) in \( [t'_1, T_\epsilon] \). This, together with the following limits: \( x'_3, x'_4, x'_5 \to x_2 \), and \( t'_3, t'_4, t'_5 \to t_2 \), and \( T_\epsilon(x) \to T(x) \), imply:

\[
\lim_{\epsilon \to 0} X_\epsilon(t'_3,t'_2) = X(t_2,0), \quad \text{and} \quad \lim_{\epsilon \to 0} X_\epsilon(T_\epsilon(x),0) = X(T(x),0).
\]

Lemma 14 implies in particular that \( X_\epsilon(T_\epsilon(x),0,x) \) is bounded for all \( \epsilon \). At first, this might come as a surprise, since the derivative with respect to \( x_1 \) of \( f_\epsilon \) is multiplied by the factor \( \frac{1}{\epsilon} \) in the boundary layer, (see also the first column of \( DF_\epsilon \)). However, the boundary layer and the time spent inside the boundary layer go to zero linearly with \( \epsilon \), so that the derivative of the vector field in the boundary layer remains finite in the limit. Moreover, boundedness of \( X_\epsilon(T_\epsilon(x),0,x) \) implies Lipschitzianity of the solution for each \( \epsilon > 0 \), as the following corollary states.

**Corollary 15.** There exists \( \beta > 0 \) such that for each \( t \geq 0 \) and for each \( \epsilon > 0 \)

\[
\| \phi_\epsilon'(x) - \phi_\epsilon'(y) \| \leq \beta \| x - y \|, \quad x, y \in B_\delta(\bar{x}_1).
\]
Proof. The proof follows from the mean value Theorem for integrals applied to

\[ \| \varphi'_x(x) - \varphi'_y(y) \| = \| \int_0^1 X_e(t, 0, s \epsilon x + (1 - s)y)(x - y) \, ds \|. \]

\[ \square \]

**Corollary 16.** There exists \( \alpha > 0 \) such that for each \( \epsilon > 0 \)

\[ \| P_\epsilon(x) - P_\epsilon(y) \| \leq \alpha \| x - y \|, \quad x, y \in B_\delta(\bar{x}_1, \Sigma_2) \]

Proof. The result follows from the equality

\[ DP_\epsilon(x) = E^T X_e(T_\epsilon(x), 0, x) E, \]

where \( x \in B_\delta(\bar{x}_1, \Sigma_2) \), \( E = (e_2, \ldots, e_n) \), \( e_j \) is the \( j \)-th canonical vector and \( T_\epsilon(x) \) is the first return time of \( \varphi'_e(x) \) to \( B_\delta(\bar{x}_1, \Sigma_2) \). Then \( \| DP_\epsilon(x) \| \leq \| X_e(T_\epsilon(x), 0, x) \| \) and the norm on the right is bounded as \( \epsilon \to 0 \) because of Lemma 14. \[ \square \]

The following Lemma is the first part of Main step i) in the proof of Theorem 9.

**Lemma 17.**

\[ \lim_{\epsilon \to 0} x_\epsilon = \bar{x}_1. \]

Proof. Let us denote with \( x_\epsilon^k \) and \( \bar{x}_1^k \) the \( k \)-th component of \( x_\epsilon \) and \( \bar{x}_1 \), \( k \geq 2 \). (The first components are 0.) Let

\[ \lim inf_{\epsilon \to 0} x_\epsilon^2 = \bar{x}_1^2, \quad \lim sup_{\epsilon \to 0} x_\epsilon^2 = \bar{x}_1^2 \]

and let \( x_{\epsilon_1} \) and \( x_{\epsilon_2} \) be two sequences such that \( \lim_{\epsilon_1 \to 0} x_{\epsilon_1}^2 = \bar{x}_1^2 \) and \( \lim_{\epsilon_2 \to 0} x_{\epsilon_2}^2 = \bar{x}_1^2 \). From \( x_{\epsilon_1} \) and \( x_{\epsilon_2} \) we can extract convergent subsequences which, with abuse of notation, we still use \( x_{\epsilon_1} \) and \( x_{\epsilon_2} \). Let

\[ \lim_{\epsilon_1 \to 0} x_{\epsilon_1} = \bar{x}_1, \quad \lim_{\epsilon_2 \to 0} x_{\epsilon_2} = \bar{x}_1. \]  

Notice that \( \lim_{\epsilon_1 \to 0} P_\epsilon(x_{\epsilon_1}) = \lim_{\epsilon_2 \to 0} x_{\epsilon_2} = \bar{x}_1 \) and \( \lim_{\epsilon_1 \to 0} P_\epsilon(x_{\epsilon_1}) = \lim_{\epsilon_2 \to 0} x_{\epsilon_2} = \bar{x}_1 \).

Then

\[ \| P_{\epsilon}(x) - \bar{x}_1 \| \leq \| P_{\epsilon}(x) - P_{\epsilon}(\bar{x}_1) \| + \| P_{\epsilon}(\bar{x}_1) - P_{\epsilon}(x_{\epsilon_1}) \| + \| P_{\epsilon}(x_{\epsilon_1}) - \bar{x}_1 \|. \]

The three terms to the right of the inequality go to zero for \( \epsilon_1 \to 0 \). Indeed we have \( P_{\epsilon}(x) \to P_{\epsilon}(x_{\epsilon_1}) \), \( P_{\epsilon}(x_{\epsilon_1}) \to \bar{x}_1 \) and \( \| P_{\epsilon}(\bar{x}_1) - P_{\epsilon}(x_{\epsilon_1}) \| \leq \alpha \| \bar{x}_1 - x_{\epsilon_1} \| \) because of Corollary 16. It then follows that \( \bar{x}_1 = P_{\epsilon}(x_{\epsilon_1}) \), so that \( \bar{x}_1 = \bar{x}_1 \). Similarly, \( \bar{x}_1 = \bar{x}_1 \).

As a consequence of this reasoning, \( \lim_{\epsilon \to 0} x_{\epsilon k} = \bar{x}_1 k \). To show convergence of \( x_{\epsilon k} \), \( k = 3, \ldots, n \), we use reasonings analogous to the ones used for (26) together with the reasonings in this proof. \[ \square \]

Lemma 17 and Corollary 15 imply the following inequality

\[ \| \varphi'_e(x_\epsilon) - \varphi'_e(\bar{x}_1) \| \leq \| \varphi'_e(x_\epsilon) - \varphi'_e(\bar{x}_1) \| + \| \varphi'_e(\bar{x}_1) - \varphi'_e(\bar{x}_1) \| \]

\[ \leq \beta \| x_\epsilon - \bar{x}_1 \| + \| \varphi'_e(\bar{x}_1) - \varphi'_e(\bar{x}_1) \|. \]  

(27)

In particular, (27) and Lemma 11 ensure \( \gamma_\epsilon \to \gamma \) and this proves Main step i) above.

Lemma 14 and Lemma 17 together with continuity of \( X_e \) with respect to \( x \in B_\delta(\bar{x}_1, \Sigma_2) \), imply that for all \( \mu > 0 \) there exists \( \epsilon_\mu \) sufficiently small so that for \( \epsilon < \epsilon_\mu \) the following inequality is satisfied

\[ \| X_e(T_\epsilon, 0, x_\epsilon) - X(T, 0, \bar{x}_1) \| \leq \| X_e(T_\epsilon, 0, x_\epsilon) - X_e(T_\epsilon(\bar{x}_1), 0, \bar{x}_1) \| + \| X_e(T_\epsilon(\bar{x}_1), 0, \bar{x}_1) - X(T, 0, \bar{x}_1) \| < \mu. \]
As a consequence, for $\epsilon$ sufficiently small, $X_\epsilon(T_\epsilon, 0, x_\epsilon)$ has all eigenvalues less than 1 and one equal to 1 (since $\gamma_\epsilon$ is periodic). This implies Main step ii), i.e., $\gamma_\epsilon$ is asymptotically stable and hence isolated. This completes the proof of Theorem 9.

4.2. Crossing co-dimension two switching manifold. In this part, we will consider the case when system (3) crosses not only the co-dimension one switching manifold $\Sigma_1$ or $\Sigma_2$, but also the co-dimension two switching manifold $\Sigma$, see definition 1. We assume that system (3) has a crossing periodic orbit as the following Figure 4 shows.

![Figure 4](image)

**Theorem 18.** Assume that system (3) has an asymptotically stable crossing periodic orbit $\gamma$ with one transversal crossing point with $\Sigma$ as given in Figure 4. Then, for $\epsilon > 0$ sufficiently small, there exists one and only one periodic orbit $\gamma_\epsilon$ of (5) in a neighborhood of $\gamma$. Moreover $\gamma_\epsilon$ is asymptotically stable and $\lim_{\epsilon \to 0} \gamma_\epsilon = \gamma$.

We will prove Theorem 18 according to the following steps.

1. Prove that (5) has at least one limit cycle. To do this, we will define a Poincaré map $P_\epsilon$, and use Brouwer’s fixed point Theorem to show that it has a fixed point. This will give at least one limit cycle $\gamma_\epsilon$ of (5), and we will show that $\gamma_\epsilon \to \gamma_0$ when $\epsilon \to 0$.

2. Then, we will show that $\gamma_\epsilon$ is asymptotically stable, so $\gamma_\epsilon$ is the unique limit cycle of (5), for $\epsilon$ sufficiently small.

We define a Poincaré map associated to $\gamma$. Without loss of generality, we will assume that the periodic solution associated to $\gamma$ crosses $\Sigma_2$ at $\bar{x}_1$ coming from $\Sigma_2^-$ and entering in $\Sigma_1^-$ at $\bar{x}_4$, and then again (at a later time) crosses $\Sigma_2^+$ at $\bar{x}_2$ and $\Sigma_1^+$ at $\bar{x}_3$.

Let $\varphi_t^\epsilon$ denote the flows of $f_\epsilon$. Let $B_\delta(\bar{x}_1)$ be a neighborhood of $\bar{x}_1$ in $\mathbb{R}^n$ and denote with $B_\delta(\bar{x}_1, \Sigma_2)$ its intersection with $\Sigma_2$ and with $\overline{B_\delta(\bar{x}_1, \Sigma_2)}$ its closure. Then for $\delta$ sufficiently small the Poincaré map $P_\epsilon(x) = \varphi_{t_-}(x)$, where $t_-(x)$ is the first return time to $S$, is well defined and smooth in $x$ and it takes a point $x$
in $B_\delta(x_2, \Sigma_2)$ into a neighborhood of $x_2$. Similarly, we can define a Poincaré map $P_+ (x)$ that, due to the asymptotic stability of $\gamma$, takes a point $x$ in a neighborhood of $x_2$ into $B_\delta(x_1, \Sigma_2)$. Let $P = P_+ \circ P_-$ : $B_\delta(x_1, \Sigma_2) \to B_\delta(x_1, \Sigma_2)$ be the Poincaré map of system (3). Then $P$ is well defined and smooth with its inverse in $B_\delta(x_1, \Sigma_2)$ and since $\gamma$ is asymptotically stable, $P$ satisfies $P(B_\delta(x_1, \Sigma_2)) \subset B_\delta(x_1, \Sigma_2)$ for $\delta$ sufficiently small. Let $\psi_d$ be the boundary of $B_\delta(x_1, \Sigma_2)$, then $\psi_d$ is the intersection of the $(n - 1)$-sphere of center $x_1$ and radius $\delta$ with $\Sigma_2$. The set $P(\psi_d)$ is a diffeomorphic image of $\psi_d$. Let $V$ be the union of all trajectories of (3) with initial point on $\psi_d$ and endpoint on $P(\psi_d)$ together with $B_\delta = B_\delta(x_1, \Sigma_2) \setminus P(B_\delta(x_1, \Sigma_2))$ and let $\tilde{V}$ be the compact subset of $\mathbb{R}^n$ whose boundary is $V$. Then all solution trajectories of (3) that intersect $B_\delta$ will do so transversally, will enter $\tilde{V}$ and will remain inside it. The periodic orbit $\gamma$ attracts all trajectories inside $\tilde{V}$.

Stability of $\gamma$ can be studied via the monodromy matrix $X(T)$ at $\gamma$. Let $T$ be the period of $\gamma$ and assume that $\varphi^{\delta}_{1}(x_1) = x_0$, $\varphi^{\delta}_{2}(x_0) = x_2$, $\varphi^{\delta}_{3}(x_2) = x_3$, $\varphi^{\delta}_{4}(x_3) = x_1$. Then, $X(T)$ can be written as the composition of the following matrices (e.g., see [12, 20]):

$$X(T) = X(T, t_2)S_{22}(\bar{x}_3)X(t_2, t_4)S_{13}(\bar{x}_2)X(t_4, t_1)S_{21}(\bar{x}_0)X(t_1, 0)S_{21}(\bar{x}_1)$$

(28)

where $S_{21}(\bar{x}_1) = I + \left(\frac{\partial f - f_h}{\partial h} h_T^2(\bar{x}_1)\right)$ and $S_{13}(\bar{x}_2) = I + \left(\frac{\partial f - f_h}{\partial h} h_T^2(\bar{x}_2)\right)$. $S_{22}(\bar{x}_3)$ is the so-called saltation matrices, while the fundamental matrix solutions $X(t, 0)$, $X(t, t_1)$ and $X(t, t_4)$ satisfy

$$\begin{align*}
\dot{X}(t, 0) &= Df_1(\varphi^0_0(\bar{x}_1))X(t, 0), \\
\dot{X}(t, t_1) &= Df_1(\varphi^0_1(\bar{x}_1))X(t, t_1), \\
\dot{X}(t, t_4) &= Df_2(\varphi^0_2(\bar{x}_2))X(t, t_4),
\end{align*}$$

Moreover, according to [12] and considering the orientation of our crossing periodic orbit, we have two different cases for $S_{21}(\bar{x}_0)$:

$$S_{21}(\bar{x}_0) = \begin{bmatrix}
I & (\frac{f_3 - f_4}{\partial h} h_T^2(\bar{x}_0)) \\
\frac{\partial f - f_h}{\partial h} h_T^2 f_4 & I + \frac{\partial f - f_h}{\partial h} h_T^2 f_4
\end{bmatrix},$$

(29)

or

$$S_{21}(\bar{x}_0) = I + \frac{(f_3 - f_4)}{\partial h} h_T^2(\bar{x}_0).$$

(30)

All the saltation matrices in (16) are nonsingular and hence $X(T)$ has an eigenvalue at 1 and all the other eigenvalues are less than 1 in modulus because of asymptotic stability of $\gamma$.

To prove the existence of a periodic orbit of (5) in a neighborhood of $\gamma$ we employ the Poincaré map of (5). In a neighborhood of $\bar{x}_1$ and $\bar{x}_2$ solutions of (5) intersect $\Sigma_2$ transversally and hence we can consider the following Poincaré map $P_c : B_\delta(x_1, \Sigma_2) \to B_\delta(x_1, \Sigma_2)$ that associates to a point in $B_\delta(x_1, \Sigma_2)$ its first return to $B_\delta(x_1, \Sigma_2)$. The following proposition establishes the existence of at least one periodic orbit of (5).

**Proposition 19.** The map $P_c$ has at least one fixed point in $B_\delta(x_1, \Sigma_2)$. 

Let $x \in B_\delta(x_1, \Sigma_2)$ and let $\varphi^0_0(x)$ be the solution of (3) with initial condition $x$. Denote with $x_2$ its intersection with $\Sigma_2$ in a neighborhood of $\bar{x}_2$ and let $x_3 = P(x)$. Notice that $x_3 \neq x$ unless $x = \bar{x}_1$. Let $t_2$ be such that $\varphi^0_2(x) = x_2$ and $T(x)$ be such that $\varphi^{T(x)}(x) = x_3$. The following Lemma holds.
Lemma 20. Denote with $X(t, 0, x)$ and $X_\epsilon(t, 0, x)$ respectively the fundamental matrix solution of (3) along the solution $\varphi_0^t(x)$ and the fundamental matrix solution of (5) along $\varphi_\epsilon^t(x)$, $x \in B_\delta(\bar{x}_1, \Sigma_2)$ Then
\[
\lim_{\epsilon \to 0} X_\epsilon(T_\epsilon(x), 0, x) = X(T(x), 0, x),
\]
where with $T_\epsilon(x)$ we denote the first return time of $\varphi_\epsilon^t(x)$ to $B_\delta(\bar{x}_1, \Sigma_2)$.

Proof. In what follows we will omit the explicit dependence of $X(t, 0, x)$ and $X_\epsilon(t, 0, x)$ on $x$. The principal matrix solution of (5) along $\varphi_\epsilon^t(x)$ satisfies
\[
\dot{X}_\epsilon = Df_\epsilon(\varphi_\epsilon^t(x))X_\epsilon, \quad X_\epsilon(0) = I,
\]
where $I$ is the identity matrix and $Df_\epsilon(x)$ is the Jacobian of $f_\epsilon$.

The remaining proof of this theorem are similar to what we have done in the theorem 9. Hence we omit the details. $\square$

4.3. Sliding periodic orbit. We will prove Theorem 8 according to the following steps.

(1) Prove that (5) has at least one limit cycle. To do this, we will define a Poincaré map $P_\epsilon$, and use Brouwer’s fixed point Theorem to show that it has a fixed point. This will give at least one limit cycle $\gamma_\epsilon$ of (5), and we will show that $\gamma_\epsilon \to \gamma_0$ when $\epsilon \to 0$.

(2) Then, we will show that $\gamma_\epsilon$ is asymptotically stable, so $\gamma_\epsilon$ is the unique limit cycle of (5), for $\epsilon$ sufficiently small.

Although the time arrow is for convenience only, we will work with this figure in mind when defining the Poincaré map. We first define a Poincaré map for the nonsmooth system in a neighborhood of $\gamma$. To do so, let $\bar{x}_1 \in \gamma \cap R_3$ and take a cross section $\Sigma_0$ to $\gamma$ at $\bar{x}_1$. For $\delta$ sufficiently small, let $B_\delta(\bar{x}_1)$ be the ball centered at $\bar{x}_1$ with radius $\delta$. Denote by $B_\delta(\bar{x}_1, \Sigma_0) = B_\delta(\bar{x}_1) \cap \Sigma_0$, then all solutions of system (3) with initial condition on $B_\delta(\bar{x}_1, \Sigma_0)$ reach $\Sigma_1^+$ in a neighborhood of $x_2$ and they will start sliding on $\Sigma_1^+$ since $x_2$ is a transversal entry point.

Let $n(x) = \nabla h_1(x) \times \nabla h_2(x)$, since the solution trajectories will arrive $\Sigma$ in finite time, then all solutions that slide on $\Sigma_1^+$ in a neighborhood of $\gamma$ will enter $\Sigma$ and slide on $\Sigma$ until they reach $\Pi_{2out} = \{ x \in B_\eta(x_4) \mid g_2(x_4) = 0 \}$, $g_2(x) = \nabla h_2^T f_{\Sigma_1^+} (x)$. Then $\nabla g_2^T f_{\Sigma_1^+}(x_4) > 0$ implies that all solutions that slide on $\Sigma$ in a neighborhood of $\gamma$. Denote by $\Pi_{3out} = \{ x \in B_\eta(x_1) \mid g_3(x_1) = 0 \}$, $g_3(x) = \nabla h_3^T f_3(x)$. Then $\nabla g_3^T f_3(x_1) > 0$ implies that all solutions that slide on $\Sigma_2^+$ in a neighborhood of $\gamma$ will cross $\Pi_{3out}$ transversally and enter $R_3$ and finally they will reach $\Sigma_0$ again.

We define the map $P : B_\delta(\bar{x}_1, \Sigma_0) \to B_\delta(\bar{x}_1, \Sigma_0)$ as the composition of the following five smooth maps:

1) $P_1 : \Sigma_0 \to \Sigma_1^+, P_1(x) = \varphi_{t_3}^1(x)$, where $t_3(x)$ is the first time at which $\varphi_{t_3}^1(x)$ meets $\Sigma_1^+$;

2) $P_{\Sigma_1^+} : \Sigma_1^+ \to \Sigma, P_{\Sigma_1^+}(x) = \varphi_{t_{\Sigma_1^+}}^{t_{\Sigma_1^+}}(x)$, where with $\varphi_{t_{\Sigma_1^+}}^{t_{\Sigma_1^+}}$ we denote the flow of the sliding Filippov vector field $f_{\Sigma_1^+}(x)$ and $t_{\Sigma_1^+}(x)$ is the time at which $\varphi_{t_{\Sigma_1^+}}^{t_{\Sigma_1^+}}(x)$ meets $\Sigma$;

3) $P_2 : \Sigma \to \Sigma_2^+ \cap \Pi_{2out}, P_2(x) = \varphi_{t_{\Sigma}}^{t_{\Sigma}}(x)$, where $t_{\Sigma}(x)$ is the time at which $\varphi_{t_{\Sigma}}^{t_{\Sigma}}(x)$ reaches $\Pi_{2out}$ at $\bar{x} = x_4$. 


4) \( P_2 : \Sigma_2^+ \cap \Pi_{2\text{out}} \rightarrow \Sigma_2^+ \cap \Pi_{3\text{out}}, \) \( P_2(x) = \varphi_{\Sigma_2^+}^{t}(x), \) where \( t_{\Sigma_2^+}(x) \) is the time at which \( \varphi_{\Sigma_2^+}^{t}(x) \) reaches \( \Sigma_0. \)

5) \( P_3 : \Sigma_2^+ \cap \Pi_{3\text{out}} \rightarrow \Sigma_0, \) \( P_3(x) = \varphi_{\Sigma}^{\tilde{t}_3}(x), \) where \( \tilde{t}_3(x) \) is the first time at which \( \varphi_{\Sigma}^{\tilde{t}_3}(x) \) reaches \( \Sigma_0 \) from \( \Sigma_2^+ \cap \Pi_{3\text{out}}. \)

Since \( P \) is a contraction, \( P(B_3(\bar{x}_1, \Sigma_0)) \) is a proper subset of \( B_3(\bar{x}_1, \Sigma_0). \) We study the stability of \( \gamma \) using the monodromy matrix of (3) along \( \gamma. \) Then the monodromy matrix \( X(T, 0) \) along \( \gamma \) is given by the following expression (e.g., see [12] and [10].)

\[
X(T, 0) = X_3(T, t_{\Sigma_1}^+)X_{\Sigma_2}^+(t_{\Sigma_2}^+, t_\Sigma)S(\bar{x})X_\Sigma(t_\Sigma, t_{\Sigma_1}^+)S_\Sigma(x_3) \\
\cdot X_{\Sigma_1}^+(t_{\Sigma_1}^+, t_3)S_{\Sigma_1}(x_2)X_3(t_3, 0), \tag{32}
\]

where

\[
S_\Sigma(x_3) = I + (f\Sigma_2(x_3) - f_{\Sigma_1^+}(x_3))\frac{\nabla h_2(x_3)^T}{\nabla h_2(x_3)^T f_{\Sigma_1^+}(x_3)},
\]

\[
S_{\Sigma_1}(x_2) = I + (f_{\Sigma_1^+}(x_2) - f_3(x_2))\frac{\nabla h_1(x_2)^T}{\nabla h_1(x_2)^T f_3(x_2)}.\]

and \( f_3(\bar{x}) = \omega(\bar{x})f_{\Sigma_2}^{\bar{x}}(\bar{x}), \) \( S(\bar{x}) = \frac{1}{\omega(\bar{x})}I \) is the saltation matrix, \( I \) is the \( n \times n \) identity matrix, see the proposition 12 of [10]. If \( f_3(\bar{x}) = f_{\Sigma_2}^{\bar{x}}(\bar{x}), \) the corresponding solution will exit \( \Sigma \) with continuous vector field, then \( \omega(\bar{x}) = I, \) hence the saltation matrix 
\( S(\bar{x}) = I. \) Under our assumptions, \( S(\bar{x}) = I. \) This implies (32) is the following

\[
X(T, 0) = X_3(T, t_{\Sigma_1}^+)X_{\Sigma_2}^+(t_{\Sigma_2}^+, t_\Sigma)X_\Sigma(t_\Sigma, t_{\Sigma_1}^+)S_\Sigma(x_3) \\
\cdot X_{\Sigma_1}^+(t_{\Sigma_1}^+, t_3)S_{\Sigma_1}(x_2)X_3(t_3, 0). \tag{33}
\]

The five fundamental matrices in (32) solve the following Cauchy problems:

\[
\begin{align*}
\dot{X}_3(t, 0) &= Df_3(x(t))X_3(t, 0), \quad X_3(0, 0) = I, \\
\dot{X}_{\Sigma_1^+}(t, t_3) &= Df_{\Sigma_1^+}(x(t))X_{\Sigma_1^+}(t, t_3), \quad X_{\Sigma_2}^+(t_3, t_3) = I, \\
\dot{X}_\Sigma(t, t_{\Sigma_1^+}) &= Df_\Sigma(x(t))X_\Sigma(t, t_{\Sigma_1^+}), \quad X_\Sigma(t_{\Sigma_1^+}, t_{\Sigma_1^+}) = I, \\
\dot{X}_{\Sigma_2}^+(t, t_\Sigma) &= Df_{\Sigma_2}^{\bar{x}}(x(t))X_{\Sigma_2}^+(t, t_\Sigma), \quad X_{\Sigma_2}^+(t_\Sigma, t_\Sigma) = I, \\
\dot{X}_3(t, t_{\Sigma_1^+}) &= Df_3(x(t))X_{\Sigma_2}^+(t, t_{\Sigma_1^+}), \quad X_3(t_{\Sigma_1^+}, t_{\Sigma_1^+}) = I.
\end{align*}
\]

The monodromy matrix associated with \( \gamma \) has two Floquet multipliers equal to 0 and one equal to 1, all the other eigenvalues are less than 1 in modulus because of the asymptotical stability.

Recall that in [11], we have proved when system (1) has an asymptotically stable sliding periodic orbit, the flow \( \phi_\epsilon \) of its regularization system will uniformly converge to the flow \( \phi_0 \) of system (1). See Lemma 18. This lemma actually tells us the uniform convergence of the flow for regularization PWS to the flow of original PWS system when it has a codimension one switching manifold. However, this result is not sufficient to deal with our case, because our PWS system not only has codimension one switching manifolds \( \Sigma_1 \) and \( \Sigma_2, \) but also has a codimension two switching manifold \( \Sigma. \) Fortunately, we have similar results for the codimension two case, the similar result has been proved in [8] for a given cubic transition function. We state this result as follows.
Consider the $\epsilon$-neighborhood $S$ of $\Sigma$: $S = \{-\epsilon \leq x_1, x_2 \leq \epsilon\}$ and choose two specific transition functions as follows:

$$
\alpha(x_1) = \begin{cases} 
1, & \text{if } x_1 > \epsilon, \\
\frac{1}{2} + \frac{x_1}{\epsilon}(3 - (\frac{x_1}{\epsilon})^2), & \text{if } x_1 \in [-\epsilon, \epsilon], \\
0, & \text{if } x_1 < -\epsilon.
\end{cases}
$$

$$
\beta(x_2) = \begin{cases} 
1, & \text{if } x_2 > \epsilon, \\
\frac{1}{2} + \frac{x_2}{\epsilon}(3 - (\frac{x_2}{\epsilon})^2), & \text{if } x_2 \in [-\epsilon, \epsilon], \\
0, & \text{if } x_2 < -\epsilon.
\end{cases}
$$

The so-called bilinear regularization is the following one parameter family of vector fields

$$
f^\epsilon_B(x) = (1 - \alpha(x_1))[(1 - \beta(x_2))f_1(x) + \beta(x_2)f_2(x)] + \alpha(x_1)\frac{1}{2}[(1 - \beta(x_2))f_3(x) + \beta(x_2)f_4(x)].
$$

(34)

In [8], authors have proved that if $\Sigma$ is attractive in finite time, the solution of regularized system (34) converges uniformly to the sliding solution with bilinear vector field (15) on intervals of the form $[0, T]$. Moreover, this result holds for general transition function.

We now define the Poincaré map $P_\epsilon$ for (5) as the map $P_\epsilon : B_\delta(\bar{x}_1, \Sigma_0) \to \Sigma_0$ so that for every point $x \in B_\delta(\bar{x}_1, \Sigma_0)$, $P_\epsilon(x)$ is the first return point of $\varphi^{\epsilon}_{t_\epsilon}(x)$ to $\Sigma_0$.

Lemma 18 in [11] and the results in [8] insures that $P_\epsilon$ is well defined and it implies the following Proposition.

**Proposition 21.** $P_\epsilon$ converges pointwise to $P$.

Propositions 22 and 23 below will allow us to view the Poincaré map as the composition of some maps, similarly to what we did for $P$. This will lead to a decomposition of the fundamental matrix solution of (5) into several different factors, as for the discontinuous case (see (33)).

**Proposition 22.** Orbits of (5) corresponding to solutions with initial conditions in a neighborhood of $\gamma$ must intersect each $\Pi^\epsilon_{out}$ transversally, $i=2,3$.

**Proof.** Since we have proved the case when the orbits slides on the co-dimension one switching manifold in [11]. We only need to prove the case if the orbits slide on the co-dimension two switching manifold $\Sigma$. So it is sufficient for us to prove the case of orbits of (5) corresponding to solutions with initial conditions in a neighborhood of $\gamma$ must intersect $\Pi^\epsilon_{out}$ transversally.

Let $S^\epsilon_{out} = \Pi^\epsilon_{out} \cap \Sigma_{2,\epsilon}$, and note that for all $x \in S^\epsilon_{out}$, $\nabla n^T f_\epsilon(x) = \nabla n^T f^\epsilon_{\Sigma_{2,\epsilon}}(x) = 0$. The set $S^\epsilon_{out}$ divides $\Sigma_{2,\epsilon}$ in two regions, denote them as $S^-_{\epsilon}$ and $S^+_{\epsilon}$, such that for all $x \in S^\pm_{\epsilon}$, $\nabla n^T f_\epsilon(x) = \nabla n^T f^\epsilon_{\Sigma_{2,\epsilon}}(x) \geq 0$. Lemma 18 in [11] implies that an orbit $\psi_\epsilon$ of (5) in a neighborhood of $\gamma$ must intersect $S^-_{\epsilon}$ in an isolated point $x^\epsilon$ in a neighborhood of $x_4$ and moreover it must be $\nabla n^T f^\epsilon_{\Sigma_{2,\epsilon}}(x^\epsilon_4) < 0$. Along the sliding arc of $\gamma$, and in a neighborhood of it, the following inequality must be satisfied: $\nabla n^T f^\epsilon_{\Sigma_{2,\epsilon}}(x_4) > 0$. It follows that in a neighborhood of $x_4$ there are points of $\psi_\epsilon$ that satisfy $\nabla n^T f^\epsilon_{\Sigma_{2,\epsilon}}(x) > 0$, while $\nabla n^T f^\epsilon_{\Sigma_{2,\epsilon}}(x_4) < 0$. Continuity of solutions with respect to $x$ imply that there must be a point $\bar{x}_\epsilon$ of $\psi_\epsilon$ so that
Proof. Let $g(x) = \nabla n^T f_{\Sigma^+}(x) = 0$ and $\nabla g^T f_\epsilon(x) < 0$. The statement of the proposition follows. \hfill \square

**Proposition 23.** Solution trajectories of (5) corresponding to solutions with initial conditions in $B_3(\bar{x}_1, \Sigma_0)$ must intersect each $\Pi_{\text{out}}$ only in one point, before returning to $\Sigma_0$, $i = 2, 3.$

**Proof.** We will prove this proposition by contradiction. We just prove that solution trajectories of (5) corresponding to solutions with initial conditions in $B_3(\bar{x}_1, \Sigma_0)$ will intersect $\Pi_{\text{out}}$ only in one point, before returning to $\Sigma_0$. For $\Pi_{\text{out}}$, we just need to do similar work as we have done in [11]. Let $x_0 \in B_3(\bar{x}_1, \Sigma_0)$ and consider the two flows $\varphi^t_\epsilon(x_0)$ and $\varphi^t_0(x_0)$. Let $t_1$ be such that $\varphi^t_0(x_0)$ meets $\Sigma$ in $x_3$ in a neighborhood of $x_3$, starts sliding on $\Sigma$ and at $t_2 > t_1$ intersects $\Pi_{\text{out}}$ transversally at a point $\hat{x}_4$ in a neighborhood of $x_4$: $\varphi^t_0(x_0) = \hat{x}_4 \in \Pi_{\text{out}} \cap \Sigma$, and $\nabla g^T f_{\Sigma^+}(\hat{x}_4) < 0$, with $g(x) = \nabla n^T f_{\Sigma^+}(x)$.

Similarly, $\varphi^t_\epsilon(x_0)$ intersects $\Sigma_{\text{out}}$ at an isolated point $x^*_1$. $\nabla n^T f_\epsilon(x^*_1) = \nabla n^T f_{\Sigma^+}(x^*_1) > 0$. Let $t'_1$ be such that $x^*_1 = \varphi^{t'_1}_\epsilon(x_0)$. Then the solution remains inside the boundary layer and for $t > t'_1$ it crosses $\Pi_{\text{out}}$ transversally at a point $\hat{x}$ such that $\nabla n^T f_\epsilon(\hat{x}) < 0$, see Proposition 22. At time $t'_2 > t$, the solution crosses $\Sigma_{\text{out}}$ and leaves the boundary layer. Assume by contradiction that there are $t_1, t_2 \in (t, t'_2)$, with $t_1 < t_2$, so that $\hat{x}_{1,2} = \varphi^{t_1, t_2}_\epsilon(x_0)$ are two other intersection point with $\Pi_{\text{out}}$. Then it must be

$$g(\hat{x}_{1,2}) = 0, \quad \nabla g^T f_\epsilon(\hat{x}_{1,2}) > 0, \quad \nabla g^T f_\epsilon(\hat{x}_{2}) < 0.$$ (35)

Because of uniform convergence of $\varphi^t_\epsilon(x_0)$ to $\varphi^t_0(x_0)$, we know that $t'_1 \to t_1$ and $t, t_1, t_2, t'_2 \to t_2$. Then $f_\epsilon(\hat{x}_1)$ is arbitrarily close to $f_\epsilon(\hat{x}_4)$. But at the exit point $x_4$ it must be $\nabla g^T f_{\Sigma^+}(x_4) < 0$ and this is in contradiction with (35). \hfill \square

The following is a consequence of Brouwer fixed point Theorem and of Lemma 21.

**Proposition 24.** $P_\epsilon$ has at least a fixed point in $B_3(\bar{x}_1, \Sigma_0)$.

**Proof.** See the proof of Proposition 10. \hfill \square

Moreover, (5) can not have equilibria in a neighborhood of $\gamma$. Indeed outside the boundary layer $f_\epsilon = f_i$, $i = 1, 2, 3, 4$. If there is an equilibrium $\bar{x}$ inside the boundary layer, then $f_\epsilon(\bar{x}) = 0$ and $\phi_\epsilon$ is such that

$$0 = \sum_{i=1}^{2} \left( \frac{1}{2} - \phi_\epsilon(h_i(\bar{x})) \right) f_i(\bar{x}) + \sum_{i=1}^{4} \left( \frac{1}{2} + \phi_\epsilon(h_i(\bar{x})) \right) f_i(\bar{x}),$$

in particular $\nabla h_i^T f_\epsilon(\bar{x}) = 0, \ i = 1, 2$ so that $\alpha = \frac{1+\phi_\epsilon(h_1(\bar{x}))}{2}$ and $\beta = \frac{1+\phi_\epsilon(h_2(\bar{x}))}{2}$. In particular

$$\nabla h_i^T f_\epsilon(\bar{x}) = 0, \ i = 1, 2$$

in (15) and hence $f_\epsilon(\bar{x}) = f_i(\bar{x}) = 0$, a contradiction. Then, to each fixed point of $P_\epsilon$ there corresponds a periodic orbit of (5).

For each $\epsilon$, we select an arbitrary fixed point of $P_\epsilon$ and we denote it as $x_{\epsilon}$. Let $\gamma_{\epsilon}$ be the corresponding periodic orbit. What follows mimics the reasonings employed for the case of a crossing periodic orbit. We will show the following two points hold.

i) $x_{\epsilon} \to \bar{x}_1$, which in turn will imply $\gamma_{\epsilon} \to \gamma$;
ii) $\gamma_\epsilon$ is asymptotically stable so that it is the unique periodic orbit of (2) in a neighborhood of $\gamma$.

Let $x \in B_\delta(\bar{x}_1, \Sigma_0)$ and let $\varphi_0^\epsilon(x)$ be the corresponding solution of (3). At time $T(x)$ the solution reaches $\Sigma$ again at a point different from $x$, unless $x = \bar{x}_1$. The following Lemma holds.

**Lemma 25.** Let $X(t,0,x)$ be the fundamental matrix solution of (3) along $\varphi_0^\epsilon(x)$ and $X_\epsilon(t,0,x)$ be the fundamental matrix solution of (5) along $\varphi_\epsilon^\epsilon(x)$. Then

$$\lim_{\epsilon \to 0} X_\epsilon(T_\epsilon(x),0,x) = X(T(x),0,x),$$

where $T_\epsilon(x)$ denotes the first return time of $\varphi_\epsilon^\epsilon(x)$ to $\Sigma_0$.

**Proof.** Below, we omit the dependence of $X$ and $X_\epsilon$ upon $x$. We adopt the notation used in Appendix A of [11].

Together with $\Sigma_1$, consider the hyperplane $\Sigma_{1,-\epsilon} = \{ x \in \mathbb{R}^n | h_1(x) = -\epsilon \}$. For $\epsilon$ sufficiently small, uniform convergence of $\varphi_\epsilon^\epsilon(x)$ to $\varphi_0^\epsilon(x)$ implies that $\varphi_\epsilon^\epsilon(x)$ meets $\Sigma_{1,-\epsilon}$ in two isolated points: $x_2^\epsilon$ in a neighborhood of $x_2$ and $x_3^\epsilon$ in a neighborhood of $x_3$. Let $t_3^\epsilon$ be such that $\varphi_\epsilon^\epsilon(x_3^\epsilon) = x_3^\epsilon = \varphi_0^\epsilon(x_3)$. Then $\lim_{\epsilon \to 0} x_3^\epsilon = x_2$ and $\lim_{\epsilon \to 0} t_3^\epsilon = t_3^\epsilon$.

Let $\phi^\star(x_2)$ be as in (6) and let $\tau = \frac{t_3^\epsilon}{\epsilon}$. For $0 < \mu < \epsilon$, let $\tau_\mu$ be such that, for $\tau > \tau_\mu$, $|\tilde{\phi}(\tau) - \phi^\star(x_2)| < \frac{\mu}{2\epsilon}$, see equation (23) in [11]. Then $\tau_\mu$ satisfies equation (27) in [11] and $t_\mu = \epsilon \tau_\mu \to 0^+$ as $\epsilon \to 0$. Let $x_1^\mu,\epsilon = \epsilon t_1 \varphi_\epsilon^\epsilon + \epsilon \tau_\mu$, $y_\mu,\epsilon = E^T \varphi_\epsilon^\epsilon (t_3^\epsilon + \epsilon \tau_\mu)$ and denote with $\phi^\mu,\epsilon$ the corresponding value of $\phi^\star$ evaluated at the point $\frac{x_1^\mu,\epsilon}{\epsilon}$. Then $|\phi^\mu,\epsilon - \phi^\star((0,y_\mu,\epsilon)^T)| < \mu < \epsilon$, see equation (28) of [11].

We write the fundamental matrix solution $X_\epsilon(T_\epsilon(x),0)$ as product of different transition matrices

$$X_\epsilon(T_\epsilon(x),0) = X_\epsilon(T_\epsilon(x),t_3^\epsilon + t_\mu) X_\epsilon(t_3^\epsilon + t_\mu, t_3) X_\epsilon(t_3^\epsilon, t_3^\epsilon + t_\mu) X_\epsilon(t_3^\epsilon, t_3).$$

We want to show that each transition matrix in (36) converges to the corresponding transition matrix in (33).

Lemma 18 in [11] and Proposition 23 imply $\lim_{\epsilon \to 0} x_3^\epsilon = x_3$ and $\lim_{\epsilon \to 0} t_{\Sigma_1}^\epsilon = t_{\Sigma_1}^\epsilon$. Moreover $\lim_{\epsilon \to 0} \phi^\mu,\epsilon = \phi^\star(x_2)$ since $\phi_\epsilon(x(t))$ converges uniformly to $\phi^\star(x(t))$ in $[t_3 + t_\mu, t_{\Sigma_1}^\epsilon]$ (see the last steps of the proof of Lemma 18 in Appendix A in [11]).

From the reasoning above it follows that

$$\lim_{\epsilon \to 0} X_\epsilon(t_3^\epsilon, 0) = X(t_3, 0).$$

For the second piece, we write

$$X_\epsilon(t_3^\epsilon + t_\mu, t_3^\epsilon) = I + \int_{t_3^\epsilon}^{t_3^\epsilon + t_\mu} Df_\epsilon(\varphi_\epsilon(t)) X_\epsilon(t, t_3^\epsilon) dt$$

$$= I + \frac{1}{2} \int_{t_3^\epsilon}^{t_3^\epsilon + t_\mu} ((1 - \phi_\epsilon) Df_2(\varphi_\epsilon(t)) + (1 + \phi_\epsilon) Df_3(\varphi_\epsilon(t))) X_\epsilon(t, t_3^\epsilon) dt$$

$$+ \frac{1}{2 \epsilon} \int_{t_3^\epsilon}^{t_3^\epsilon + t_\mu} \phi_\epsilon'(\frac{x_1}{\epsilon}) (f_3 - f_2) \nabla h_1(x)^T X_\epsilon(t, t_3^\epsilon) dt.$$
In what follows we reason similarly to the proof of Lemma 14. The fundamental matrix solution $X_\epsilon$ must satisfy $X_\epsilon(t_3^\epsilon + t_\mu, t_3^\epsilon) = f_\epsilon(\varphi^{t_3^\epsilon + t_\mu}_\epsilon(x))$. When we take the limit as $\epsilon \to 0$, then $\varphi^{t_3^\epsilon + t_\mu}_\epsilon(x) = x_2$, $f_\epsilon(x_2^\epsilon) \to f_3(x_2)$, and $\lim_{\epsilon \to 0} f_\epsilon(\varphi^{t_3^\epsilon + t_\mu}_\epsilon(x)) = f_{\Sigma_3^+}(x_2)$. In particular, we have that $\lim_{t \to t_3^\epsilon} \lim_{\epsilon \to 0} f_\epsilon(\varphi^{t}_\epsilon(x)) = \lim_{t \to t_3^\epsilon} \lim_{\epsilon \to 0} f_\epsilon(\varphi^{t}_\epsilon(x))$ and this discontinuity is reflected also in the limit of the fundamental matrix solution. Indeed, if $L = \lim_{\epsilon \to 0} X_\epsilon(t_3^\epsilon + t_\mu, t_3^\epsilon)$ exists, then it must satisfy

$$L f_3(x_2) = f_{\Sigma_3^+}(x_2).$$

(38)

Now, the first integral in (37) goes to zero as $\epsilon \to 0$ since $t_\mu = \epsilon \tau_\mu \to 0$ and the integrand is bounded.

For the second integral, inside the interval $(t_3^\epsilon, t_3^\epsilon + t_\mu)$, we can write

$$X_\epsilon(t, t_3^\epsilon) = I + R_\epsilon(x_2^\epsilon, t),$$

$$f_\epsilon(x) = f_1(x_2^\epsilon) + (t - t_3^\epsilon) Df_1(x_2^\epsilon) f_\epsilon(x_2^\epsilon) + \text{h.o.t., } i = 1, 2$$

where $R_\epsilon$ is bounded, and $\|R_\epsilon\| \to 0$ as $t \to t_3^\epsilon$.

Using this in the integral above, since $\phi^{\epsilon}_\epsilon > 0$, we get

$$\frac{1}{2\epsilon} \int_{t_3^\epsilon}^{t_3^\epsilon + t_\mu} \phi^{\epsilon}_\epsilon \left( \frac{x_1}{\epsilon} \right) (f_3 - f_2) \nabla h_1(x) X_\epsilon(t, t_3^\epsilon) dt$$

$$= \frac{1}{2\epsilon} \int_{t_3^\epsilon}^{t_3^\epsilon + t_\mu} \phi^{\epsilon}_\epsilon \left( \frac{x_1}{\epsilon} \right) dt \left[ (f_3 - f_2) \nabla h_1^T(x_2) + \bar{E} \right]$$

$$= \left[ (f_3 - f_2) \nabla h_1^T(x_2) + \bar{E} \right] c(\epsilon),$$

where $c(\epsilon) = \frac{1}{2\epsilon} \int_{t_3^\epsilon}^{t_3^\epsilon + t_\mu} \phi^{\epsilon}_\epsilon \left( \frac{x_1}{\epsilon} \right) dt$, and $\bar{E}$ is the error matrix whose components are each computed at possibly different values of $\epsilon \in (t_3^\epsilon, t_3^\epsilon + t_\mu)$.

Now, for $\epsilon \to 0$, $\bar{E} \to 0$ and $x_2^\epsilon \to x_2$, and $t_3^\epsilon \to t_3$ from the right. Thus, for $\epsilon \to 0$, the second integral gives

$$\left[ (f_3 - f_2) \nabla h_1^T(x_2) \right] \lim_{\epsilon \to 0} c(\epsilon),$$

and so would have

$$\lim_{\epsilon \to 0} X_\epsilon(t_3^\epsilon + t_\mu, t_3^\epsilon) = I + \left[ (f_3 - f_2) \nabla h_1^T(x_2) \right] \lim_{\epsilon \to 0} c(\epsilon).$$

and since (38) holds, we must have that

$$\lim_{\epsilon \to 0} c(\epsilon)(f_3 - f_2) \nabla h_1^T f_3(x_2) = (f_{\Sigma_3^+} - f_3)(x_2) = (\alpha^+ - 1)(f_3 - f_2)(x_2),$$

and hence $\lim_{\epsilon \to 0} c(\epsilon) = \frac{\alpha^+ - 1}{\nabla h_1^T f_3(x_2)}$. Thus, finally we get

$$\lim_{\epsilon \to 0} X_\epsilon(t_3^\epsilon + t_\mu, t_3^\epsilon) = S_{\Sigma_3^+}(x_2),$$

where $S_{\Sigma_3^+}$ is the saltation matrix from $R_3$ to $\Sigma_3^+$. For the third transition matrix in (36), we have

$$X_\epsilon(t_3^\epsilon + t_\mu, t_3^\epsilon + t_\mu) = I + \int_{t_3^\epsilon + t_\mu}^{t_3^\epsilon + t_\mu} Df_\epsilon(\varphi_\epsilon) X(t, t_3^\epsilon + t_\mu) dt.$$
\( f_\epsilon (\varphi_\epsilon^t(x)) \) converges to \( f_{\Sigma_1^+} (\varphi_0^t(x)) \) for \( t \in [t_\xi + t_\mu, t_\Sigma^+] \). Hence

\[
\lim_{\epsilon \to 0} Df_\epsilon (\varphi_\epsilon(t)) = \lim_{\epsilon \to 0} (1 - \varphi_\epsilon(t)) Df_2 (\varphi_\epsilon(t)) + \varphi_\epsilon(t) Df_3 (\varphi_\epsilon(t)) + \frac{1}{\epsilon} \varphi_\epsilon^t \left( \frac{1}{\epsilon} \right) ((f_3 - f_2)(\varphi_\epsilon(t)) = Df_{\Sigma_1^+}(t).
\]

It follows that

\[
\lim_{\epsilon \to 0} X(t_{\Sigma_1^+}^t, t_\mu, t_\Sigma^+) = X(t_{\Sigma_1^+}^t, t_\mu).
\]

For the fourth transition matrix in (36), we have

\[
X_\epsilon (t_{\Sigma_1^+}^t + t_\mu) = I + \int_{t_{\Sigma_1^+}^t}^{t_{\Sigma_1^+}^t + t_\mu} Df_\epsilon (\varphi_\epsilon(x)) X(t, t_{\Sigma_1^+}^t) dt
\]

where

I_1 = \int_{t_{\Sigma_1^+}^t}^{t_{\Sigma_1^+}^t + t_\mu} \left[ 1 - \varphi_\epsilon(t_\mu) \right] Df_1 (x) + \left[ 1 - \varphi_\epsilon(t_\mu) \right] Df_3 (x)
\]

\[
I_2 = \int_{t_{\Sigma_1^+}^t}^{t_{\Sigma_1^+}^t + t_\mu} \left\{ \varphi_\epsilon (t_\mu) \left[ 1 - \varphi_\epsilon(t_\mu) \right] Df_1 (x) + \left[ 1 - \varphi_\epsilon(t_\mu) \right] Df_3 (x) \right\} \nabla h_1 (x) X(t, t_{\Sigma_1^+}^t) dt.
\]

I_3 = \int_{t_{\Sigma_1^+}^t}^{t_{\Sigma_1^+}^t + t_\mu} \left\{ \varphi_\epsilon (t_\mu) \right\} \left[ 1 - \varphi_\epsilon(t_\mu) \right] Df_1 (x) + \left[ 1 - \varphi_\epsilon(t_\mu) \right] Df_3 (x)
\]

The fundamental matrix solution \( X_\epsilon \) satisfies the following inequality

\[
X_\epsilon (t_{\Sigma_1^+}^t + t_\mu) f_\epsilon (x_3) = f_\epsilon (\varphi_\epsilon^t (x)).
\]

Let \( \epsilon \to 0 \), we have that \( \varphi_\epsilon^t (x) = x_3 \), \( f_\epsilon (x_3) \to f_{\Sigma_1^+} (x_3) \), and \( \lim_{\epsilon \to 0} f_\epsilon (\varphi_\epsilon^t (x)) = f_{\Sigma_1^+} (x_3) \). In particular, we have that \( \lim_{\epsilon \to 0} f_\epsilon (\varphi_\epsilon^t (x)) = f_{\Sigma_1^+} (x_3) \) and this discontinuity is reflected also in the limit of the fundamental matrix solution. Indeed, if \( L_1 = \lim_{\epsilon \to 0} X_\epsilon (t_{\Sigma_1^+}^t + t_\mu, t_{\Sigma_1^+}^t) \) exists, then it must satisfy

\[
L_1 f_{\Sigma_1^+} (x_3) = f_{\Sigma_1^+} (x_3).
\]

Now, \( I_1 \) goes to zero as \( \epsilon \to 0 \) since \( t_\mu = \epsilon t_\mu \to 0 \) and the integrand is bounded. Since \( \varphi_\epsilon (x) \to \varphi_0 (x) \) uniformly and \( t_{\Sigma_1^+}^t \to t_{\Sigma_1^+}^t \). Moreover, \( \phi_\epsilon (t) \to \phi(y(t)) \) uniformly for \( t \in [t_{\Sigma_1^+}^t, t_\mu] \), with \( y(t) = E T \varphi (t) \) so that \( f_\epsilon (\varphi_\epsilon (x)) \to f_{\Sigma_1^+} (\varphi_0 (x)) \) for \( t \in [t_{\Sigma_1^+}^t, t_{\Sigma_1^+}^t + t_\mu] \).

For \( I_2 \), inside the interval \( (t_{\Sigma_1^+}^t, t_{\Sigma_1^+}^t + t_\mu) \), we can write

\[
X_\epsilon (t, t_{\Sigma_1^+}^t) = I + R_\epsilon (x_2', t),
\]

\[
f_\epsilon (x) = f_\epsilon (x_3') + (t - t_{\Sigma_1^+}^t) Df_\epsilon (x') f_\epsilon (x_3') + \text{h.o.t., } i = 2, 3
\]
where $R_\epsilon$ is bounded, and $\|R_\epsilon\| \to 0$ as $t \to t^\epsilon_{\Sigma^1_+}$.

Using this in the integral above, since $\phi'_\epsilon > 0$, we get

$$\int_{t^\epsilon_{\Sigma^1_1}}^{t^\epsilon_{\Sigma^1_1} + t_{\epsilon,\mu}} \frac{\phi'_\epsilon(h_2(x))}{2\epsilon} \left[ - \frac{1 - \phi_s(h_2(x))}{2} f_1(x) - \frac{1 + \phi_s(h_2(x))}{2} f_2(x) \right] \nabla h_1(x) X(t, t^\epsilon_{\Sigma^1_1}) dt$$

$$= \frac{1}{2\epsilon} \int_{t^\epsilon_{\Sigma^1_1}}^{t^\epsilon_{\Sigma^1_1} + t_{\epsilon,\mu}} \phi'_\epsilon\left(\frac{x_1}{\epsilon}\right) dt \left[ - \frac{1 - \phi_s(h_2(x))}{2} f_1(x) - \frac{1 + \phi_s(h_2(x))}{2} f_2(x) \right] \bigg|_{t = t^\epsilon_{\Sigma^1_1}}.$$

Denote by $c_1(\epsilon) = \frac{1}{2\epsilon} \int_{t^\epsilon_{\Sigma^1_1}}^{t^\epsilon_{\Sigma^1_1} + t_{\epsilon,\mu}} \phi'(\frac{x_1}{\epsilon}) dt$.

Now, for $\epsilon \to 0$, $x_3^\epsilon \to x_3$, and $t^\epsilon_{\Sigma^1_1} \to t_{\Sigma^1_1}$ from the right. Thus, for $\epsilon \to 0$, we have

$$- f_{\Sigma^1_2} \nabla h_1^T(x_3) \lim_{\epsilon \to 0} c_1(\epsilon) f_{\Sigma^1_1}(x_3) = 0.$$

Analogously, we can prove that the other part of $I_2$ goes to 0. Hence for $\epsilon \to 0$, $I_2 \to 0$.

So we have

$$\lim_{\epsilon \to 0} X_\epsilon(t^\epsilon_{\Sigma^1_1} + t_{\epsilon,\mu}, t^\epsilon_{\Sigma^1_1}) = I + [(f_{\Sigma^1_1}^+ - f_{\Sigma^1_1}^-) \nabla h_2^T(x_3)] f_{\Sigma^1_1}^+(x_3) \lim_{\epsilon \to 0} c_2(\epsilon),$$

where $c_2(\epsilon) = \frac{1}{2\epsilon} \int_{t^\epsilon_{\Sigma^1_1}}^{t^\epsilon_{\Sigma^1_1} + t_{\epsilon,\mu}} \phi'(\frac{x_1}{\epsilon}) dt$. Notice that $f_{\Sigma^1}(x_3) = [(1 - \alpha^*) f_{\Sigma^1_1}^+ + \alpha^* f_{\Sigma^1_1}^-](x_3)$, we have

$$[(f_{\Sigma^1_1}^+ - f_{\Sigma^1_1}^-) \nabla h_2^T(x_3)] f_{\Sigma^1_1}^+(x_3) \lim_{\epsilon \to 0} c_2(\epsilon) = (f_{\Sigma^1} - f_{\Sigma^1_1}^-)(x_3),$$

and hence $\lim_{\epsilon \to 0} c_2(\epsilon) = \frac{\alpha^*-1}{\nabla h_2^T f_{\Sigma^1_1}(x_3)}$. Thus, finally we get

$$\lim_{\epsilon \to 0} X_\epsilon(t^\epsilon_{\Sigma^1_1} + t_{\epsilon,\mu}, t^\epsilon_{\Sigma^1_1}) = S_{\Sigma^1}(x_3).$$

Finally, we get the convergence of the transition matrix in (36), $X_\epsilon(T_\epsilon(x), t^\epsilon_1)$, to the corresponding term of $X(T(x), 0)$ (see also (32)) as $\epsilon \to 0$.

As in Section 4.1, Lemma 25 implies that $\varphi^\epsilon_i$ and $P_\epsilon$ are Lipschitz for all $\epsilon$. Then the following result follows.

**Lemma 26.**

$$\lim_{\epsilon \to 0} x^\epsilon = \bar{x}_1.$$

**Proof.** Let us denote with $x^\epsilon_k$ the $k$-th component of $x_\epsilon$. Let

$$\liminf_{\epsilon \to 0} x^\epsilon_1 = \underline{x}_1, \quad \limsup_{\epsilon \to 0} x^\epsilon_1 = \overline{x}_1$$

and let $x_{\epsilon_1}$ and $x_{\epsilon_2}$ be two sequences such that $\lim_{\epsilon \to 0} x_{\epsilon_1}^1 = \underline{x}_1$ and $\lim_{\epsilon \to 0} x_{\epsilon_2}^1 = \overline{x}_1$. From $x_{\epsilon_1}$ and $x_{\epsilon_2}$, we can extract convergent subsequences which we still denote as $x_{\epsilon_1}$ and $x_{\epsilon_2}$. Let

$$\lim_{\epsilon \to 0} x_{\epsilon_1} = \underline{x}, \quad \lim_{\epsilon \to 0} x_{\epsilon_2} = \overline{x}.$$  \hspace{1cm} (40)

As in Lemma 17, using the fact that $P_\epsilon$ is Lipschitz for all $\epsilon$, we obtain that $P_\epsilon(\underline{x}) = \underline{x}$ and $P_\epsilon(\overline{x}) = \overline{x}$, so that $\underline{x} = \overline{x} = \bar{x}_1$. This proves convergence of the first component of $x_\epsilon$ to the first component of $\bar{x}_1$. The proof for the other components is done in a similar way. \hfill $\square$
Note that Lemma 25 and Lemma 26 together with continuity of $X_\epsilon$ with respect to $x$ imply that $X_\epsilon(T_\epsilon, 0, x_\epsilon)$ has all eigenvalues less than 1 and one equal to 1 (since $\gamma_\epsilon$ is periodic). This implies that $\gamma_\epsilon$ is asymptotically stable and hence isolated, and Theorem 8 follows.

5. **Conclusions.** In this paper we have considered an $n$ dimensional PWS systems with a co-dimension two switching manifold, i.e an intersection of two codimension one switching manifolds. This work is a continuation of [11]. We have further assumed that this PWS system has an asymptotically stable periodic orbit $\gamma$, including crossing periodic or sliding periodic orbit. Our main results show that if $\gamma$ does not lie entirely on the switching manifold $\Sigma_1$ or $\Sigma_2$ nor $\Sigma$, then a regularization of the PWS system also has a unique asymptotically stable limit cycle in a neighborhood of $\gamma$, converging to $\gamma$ as the regularization parameter goes to 0. The case of $\gamma$ lying entirely on one of $\Sigma_1$ or $\Sigma_2$ or $\Sigma$ remains to be considered.

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