Suppression of superfluid stiffness near Lifshitz-point instability to finite momentum superconductivity

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We derive the Ginzburg-Landau theory for finite momentum superconductivity from a model with local attraction and repulsive pair-hopping and show that it must include up to sixth order derivatives of the order parameter. For weak pair-hopping there is a line of Lifshitz points with continuous change from zero to finite momentum which for larger pair-hopping is replaced by a bicritical region where pair momentum changes discontinuously. Proximity to a Lifshitz point provides a mechanism for reduced superfluid stiffness and we discuss implications of the model for the cuprate superconductors.

Periodically modulated superconductivity is a common theme in several fields that deal with quantum many-body physics; ranging from cold atoms and solid-state systems, to dense nuclear matter. Such a state was first considered in systems with a Zeeman split population of spins, referred to as a Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state. A similar state, but without a symmetry breaking field, is discussed in the context of cuprate superconductors and referred to as a pair-density wave (PDW). For one, the PDW state is suggested to account for the suppression of superconductivity in LBCO at 1/8 doping. Moreover, observations in the pseudogap phase, such as a prevalence of diamagnetic response, arcs in the Fermi surface, and anomalous quantum oscillations at large magnetic fields, have been put forward as evidence for a more ubiquitous PDW state. Further, PDW-like states breaking time-reversal symmetry have been discussed to account for the apparent finite Kerr-angle. A modulated superconducting state, $\Delta_Q$, has been observed in STM measurements consistent with coexisting superconductivity (SC), $\Delta_0$, and charge-density wave (CDW) $\rho_Q$. Recently a more direct signature was reported in terms of a double period, $\rho_Q$, CDW that would follow from coexisting SC and PDW order, thus indicating that the PDW is an intrinsic order in the cuprate superconductors.

Lacking a clear microscopic origin of the PDW state, effective Ginzburg-Landau (GL) theories have been utilized to explore the implications of such a state. In this article instead, we start with an effective microscopic model with repulsive (“$\pi$-phase”) pair-hopping interactions known to generate PDW states even in the absence of spin population imbalance. We derive an effective GL theory, that necessarily includes up to sixth order derivatives of the order parameter, and explore it in the context of the BCS to BEC crossover. As seen in Figure 1, the homogeneous state becomes unstable to a finite momentum time-reversal breaking Fulde Ferrell (FF) type state $\Delta(r) = \Delta_Q e^{iQ\cdot r}$. This happens at arbitrarily small pair-hopping, $\alpha > 0$, for a sufficiently large attraction, $g$, and occurs through a Lifshitz point where the stiffness to deformations of the order parameter vanishes to lowest order. For $\alpha > \alpha_{SL}$ there is instead a line of bicritical points at finite temperatures, where the FF state is near degenerate to a translational symmetry breaking Larkin-Ovchinnikov (LO) PDW state $\Delta(r) = \Delta_Q \cos(Q\cdot r)$. At
the intersection of these transitions $\alpha = \alpha_{SL}$, $g = g_{SL}$, $T = 0$ there is a special multicritical “super-Lifshitz” (SL) point with extra soft fluctuations, $\omega \sim \tilde{q}^6$, and distinct mean-field exponents.

Using a unified description of the full momentum dependence of the GL theory clarifies the interdependence of zero and finite momentum states. In particular, we find that proximity to a Lifshitz point generically decreases the zero momentum superfluid stiffness leading to a suppression of $T_c$.[30] It also shows that superconducting states with dominant uniform order and subdominant PDW order and states where the roles are reversed, both suggested to exist in the cuprate superconductors[6, 21], are closely related within the same model.

Model — To keep the discussion as general as possible we consider a 2D continuous field theory with on-site s-wave pairing and pair-hopping[41]

$$H = \int \frac{\psi^\dagger_{\sigma}(-\nabla^2 + m^2)\psi_{\sigma}(r)}{2m} \, dr - \frac{g_0}{2} \int_{r_1, r_2} T(r_1-r_2) \psi^\dagger_{\sigma_1} r_1 \psi^\dagger_{\sigma_2} r_2 \psi_{\sigma_1} r_2 \psi_{\sigma_2} r_2$$

(1)

with $\hbar = 1, k_B = 1$, and summation over repeated indices, and

$$T(r_1-r_2) = \delta(r_1-r_2) - \alpha \delta \left( r_1-r_2 \pm \frac{\vec{x}}{\hat{y}} \lambda \right).$$

(2)

We define $g_{\text{pair}} = \alpha g_0$ as the strength of the pair-hopping interaction and $g = g_0(1-4\alpha)$ as the strength of zero momentum attraction. We consider $\alpha > 0$, i.e. repulsive pair-hopping. Further, we consider a type II superconductor and ignore fluctuations of the gauge field. The interaction breaks rotational symmetry to $C_4$, and introduces a periodic modulation scale $\Lambda$, but preserves translational symmetry. We use $\lambda = 2\pi k_F^{-1}$, that is, comparable to the inter-particle distance.

We consider the model (1, 2) as an effective model for spontaneous emergence of PDW superconductivity. However, this sort of pair-hopping interactions have been suggested both as off-diagonal terms of the microscopic Coulomb interaction[30, 32, 42] and as effective interactions in stripe ordered systems[31, 43].

Method — We address the model using a Hubbard-Stratonovich transformation for a bosonic finite momentum pair field $\Delta(p, i\Omega_n)$ with $\Omega_n = \frac{2\pi n}{N}$, a bosonic Matsubara frequency, which couples bi-linearly to the electronic field[44]. Integrating out the fermions, and expanding to fourth order in the pair field, we write the partition function $Z = Z_0 \text{Tr} e^{-\beta F(\Delta)}$ in terms of the GL free energy functional

$$F = \frac{1}{\beta} \left( \int_{\Delta, i\Omega} \Gamma^{-1}_{\Delta, i\Omega} |\Delta(p, i\Omega)|^2 + \frac{u}{2} \int |\Delta|^4 \right)$$

(3)

with

$$\Gamma^{-1}_{\Delta, i\Omega} = \frac{T^{-1}_{\Delta, i\Omega}}{g_0} \int_{k, \omega} G(k, i\omega_n) G(-k, i\omega_n)$$

(4)

(where the quartic term is written schematically, for details see [44]). Here $T^{-1}_{\Delta, i\Omega} = -2 \alpha \cos(\frac{2\pi p_x}{d_x}) - 2 \alpha \cos(\frac{2\pi p_y}{d_y})$ and $G(k, i\omega_n) = \frac{1}{i\omega_n - \xi(k)}$, with fermionic Matsubara frequency $\omega_n = \frac{2\pi n+1}{2} \pi$, and dispersion $\xi(k) = k^2/2m - \mu$. We expand around the normal state of the dominant mode $Q, \Delta(Q) = 0$, thus our theory will hold near $T_c$. At this mode, the quartic term takes the form $u(Q) = \int G^2(k+Q, i\omega) G^2(-k, -i\omega)$. (In the case of expansion around two simultaneous modes additional interactions should also be included). Further, $\Gamma^{-1}_{\Delta, i\Omega}$ has a logarithmic UV-divergence which we regularize by introducing a high energy cut-off $\epsilon_L = \frac{A^2}{2m}$.

![FIG. 2. Mean-field phase diagram of the free energy $E_\text{F}$ (for $Q_ \neq 0$) with SC, $Q_+ = 0$, and FF, $Q_+ \neq 0$, phases. (a) Phases as a function of $a$ ($Q^2$) and $c$ ($Q^4$), and (inset) the corresponding functional form of $A(Q_\pm)$ for $r = 0$. For $c > 0$, $a = 0$ there is a line of Lifshitz points with a continuous transition in $Q_\pm$, as shown in (b). (N.B. when including fluctuations the Lifshitz point $r = a = 0$ is pushed to $T = 0$, see Figure 1.) Solid black line $c < 0$, $a = \frac{c^2}{2\pi}$ indicates a line of critical points where $Q_+$ jumps, as shown in (c). Black dashed in (c) marks a 1st order transition (vertical assuming $u(Q) = u$, whereas other dashed lines in (a) and (c) are boundaries of meta-stable subleading phases (for $0 < a < c^2/4$) that are not thermodynamic transitions. The black dot in (a) marks the “super-Lifshitz” point discussed in the text.

Ginzburg-Landau free energy — To characterize the phase diagram of the model we first study the GL free energy, $F_M$, which is the static field (mean-field) version of
The interaction $T(p)$ is minimized along the diagonals, thus using rotated coordinates $p_x = \frac{p_x + p_y}{\sqrt{2}}$ is convenient. We find that for given $(T, \mu, g, \alpha)$, $\Gamma^{-1}(Q, 0)$ can be characterized by a sixth order polynomial in $Q_+, Q_-:

\begin{align*}
F_M &= A(Q)|\Delta Q|^2 + \frac{u(Q)}{2}|\Delta Q|^4, \\
A(Q) &= r + \sqrt{2}Q^2 + bQ_+^2Q_-^2 + c(Q_+^4 + Q_-^4) + \frac{1}{6}Q^6.
\end{align*}

(For convenience we have fixed the magnitude of the $Q^6$ term.)

The theory ensures that $u > 0$, $b > 0$, but $r$, $a$, and $c$ can have either sign. In minimizing the energy in terms of momentum, $Q$, and order parameter, $\Delta Q$, we can pick $Q_+$ $(Q_+ = 0)$ w.l.o.g. There are in general three possible minima given by $Q_+ = 0$, $Q_+ = \pm Q_0$ where $Q_0 = -\frac{c}{2} + \sqrt{\frac{c^2}{4} - a}$. The phase diagram in the $(r, a, c)$ space is outlined in Figure 2. Here we recognize a continuous evolution of $Q_+$ from a SC to an FF state through a Lifshitz point (for a vector order parameter [35]) when $a$ changes sign and $c > 0$. For $c < 0$ there is a region of coexisting local minima at $Q_+ = 0$ and $Q_+ = \pm Q_0$. The critical surfaces meet at a line of bicritical points given by $r = A(Q_0) = 0$ (solid black line in Figure 2) where $Q_+$ jumps. That it is a bicritical transition (1st order), rather than tetracritical transition (coexisting order), is due to the competition of SC and FF which we will discuss below. At the super-Lifshitz point, $a = c = 0$, $A(Q) \sim r + \frac{1}{2}Q^6$ along $Q_+$. The mean-field correlation length exponent along the soft directions will change from $\nu = 1/4$ at a Lifshitz point, to $\nu = 1/6$ at the super-Lifshitz point. Similarly, approaching along $T = T_c, c = 0$ in the FF state by tuning $a$, the exponent $Q \sim |a|^{\beta_k}$ is given (in mean-field) by $\beta_k = 1/2$ for a Lifshitz point, and $\beta_k = 1/4$ for the SL point [39].

BCS to BEC crossover — To find the phase-diagram within the BCS to BEC crossover we go beyond mean-field theory by not only considering the Thouless criterion

$$\min_{p=Q} \Gamma^{-1}(p, 0) = 0,$$

but also the condition for a fixed particle number [35] [37]

$$n = \frac{1}{\beta \text{Vol.}} \frac{\partial \ln Z}{\partial \mu} = n_F + n_B,$$

where $n_F$ is the free fermion density and $n_B$ the contribution from pre-formed pairs (neglecting $O(\Delta^4)$). Thus, we solve for $T_c, \mu_c$ for a given $g, \alpha$. In the weak coupling limit, $g \rightarrow 0$, the pairs are loosely bound with $n_F \gg n_B$, yielding the BCS expression $\mu = \varepsilon_F$, with $\varepsilon_F$ the bare Fermi energy. As the interaction increases, $\mu$ will become negative, with all fermions bound up in pairs ($n_F \ll n_B$), which in turn will condense, see Figure 3.

The details of the analysis concern the general form of $n_B = -\text{Tr} \Gamma \frac{\partial \Gamma^{-1}}{\partial \mu}$, which is determined by the analytical structure of the pair-propagator, $\Gamma(p, z)$, representing the two-particle spectrum [35]. In strong coupling $\Gamma(p, z)$ is well approximated by a simple pole structure yielding (for details see [44])

$$n_B = 2 \int \frac{1}{q \left( \beta \left( \frac{\mu}{\kappa^2} + \frac{\mu^2}{m_B^2} \right) \right)} \exp \left( \frac{3(3/2)}{\text{reg.}} \frac{\Delta}{\pi} \right) \sqrt{\pi} T \sqrt{m_B m_B}$$

where we have expanded $\Gamma^{-1}(p, z) \approx r_Q - \kappa Q z + \frac{\kappa^2}{2} q_i^2 q_i, i = \pm$ around the saddle point $(p = Q + q)$, before performing the Matsubara sum over $z = i\Omega$. $m_B = \frac{n_B}{\alpha_Q}$ is the boson mass, reflecting the curvature at the saddle point. For the $Q = 0$ saddle point the mass is isotropic $m_B^2 \propto a^{-1}$ from [51]; while for the $Q_+ > 0$ FF state the mass is anisotropic, with $m_B^2 \propto (bQ_0^2)^{-1}$ and $m_B^2 \propto (-4a - 2cQ_0^2)^{-1}$. To regularize the IR-divergence of the 2D bosonic occupation and capture the Kosterlitz-Thouless transition, we follow Stintzing et al. [37] (see also [49] and introduce a third dimension (whose energy-scale equals the thermal expectation value) indicated in [S10]. Using [S10] will provide a valid description for the strong and weak coupling limit. For intermediate coupling, $\mu \sim 0$ and $n_B \sim n_F$, this analysis could be improved upon by including interaction with scattering.
states \cite{17}.

The solution to (1) and (7) was studied for \( n = \frac{k_F^2}{2\pi} = 0.2 \) (in inverse length units), \( \Lambda = 10k_F \) and a range of values of \( \alpha \). The phase diagram is shown in Figure 1 with \( Q_+ \), \( m_B \), \( \mu \) and \( n_B \) presented in Figure 3. For \( \alpha < \alpha_{SL} \), with \( \alpha_{SL} \approx 0.02 \), a transition through a Lifshitz point is realized, and we are moving along a path equivalent to the green arrow in Figure 2 as \( g \) increases. For \( \alpha > \alpha_{SL} \), \( c \) changes sign giving rise to coexisting saddle points along the red arrow as \( g \) increases.

For small \( g \) we see the expected BCS behavior \( T_c \approx 1.13\sqrt{\varepsilon_\Lambda \varepsilon_{FF}} \) (note \( \varepsilon_\Lambda >> \varepsilon_{FF} \)). We note that \( T_c \) in the FF phase do not saturate due to continued decrease of effective mass from pair-hopping.

**Weak pair-hopping instability**— We now show that any finite \( \alpha \) leads to an instability towards FF (\( a < 0 \)). First, note that the existence of a Lifshitz point is equivalent to a diverging bosonic mass, or vanishing phase-stiffness (assuming finite \( \kappa_Q \)), which leads to increased fluctuation, and according to (S10) \( T_c \to 0 \), which is clearly seen in Figure 1. (Strictly speaking (S10) is not valid for \( m_B = \infty \), and higher order terms in \( q \) need to be included \cite{14}.) \( T_c = 0 \) ensures that the Lifshitz point will be in the deep BEC limit, \( \mu/T \to -\infty \). (Because of perfect nesting \( \Gamma^{-1}(p=0,0) \) can only be finite for \( T \to 0 \) and \( q > 0 \) if \( \mu < 0 \).) The effect of this can be seen in Figure 3(c), (d), where increasing \( \mu \) moves the BEC phase to smaller \( g \). An exact relation for \( m_B \) can be derived in the deep BEC limit

\[
m_B^2|_{p=0} = 2m \left( 1 - \frac{\pi \alpha \Lambda^2 \lambda^2}{mg(1 - 4\alpha) \sinh^2 \left( \frac{2\pi m}{mg} \right)} \right)^{-1}.
\]

With \( \alpha = 0 \) we find the expected \( m_B \to 2m \). Instead, for \( \alpha > 0 \) there is always a \( g = g_0 \), with a divergent mass, proving the existence of the Lifshitz point. \( g_0(\alpha) \) is shown as a blue line in Figure 1. We also note that this relation is independent of density, i.e. the line of Lifshitz points is stationary with regard to \( n \).

For larger \( \alpha \), \( \alpha > \alpha_{SL} \), the situation is quite different. Here the zero and finite momentum branches coexist, leading to a bicritical point at which the momentum jumps and almost immediately attains the maximum value \( Q_{+\text{max}} = 2\sqrt{2}/\lambda \). We also see that the bicritical point is on the weak coupling side with renormalized, yet positive \( \mu \). The behavior near the formation of the \( Q_+ \neq 0 \) branch is somewhat intricate\cite{14}. However, these are meta-stable points without any corresponding thermodynamic transition; thus we marked this part with a dashed line in Figure 1 extrapolated to \( T = 0 \).

**Coexisting orders**— In the FF regime there is a 4-fold degeneracy between states at \( Q = \pm Q_0, \hat{Q}_+ \), and \( Q = \pm Q_0 \hat{Q}_- \), also, at the proposed bicritical point there is degeneracy between SC and FF. With interactions of the form \( \gamma(Q, \hat{Q})|\Delta_Q|^2|\Delta_{\hat{Q}}|^2 \) the criterion for coexistence reads \( \gamma(Q, \hat{Q}) \leq \sqrt{\mu(Q)\mu(\hat{Q})} \) with \( \gamma(Q, \hat{Q}) = 2f/G(k + Q, \omega)G(k + \hat{Q}, \omega)G^2(-k, -\omega) \). We will not discuss this in detail but only infer three important regimes \cite{14}:

(i) For \( Q \to 0 \) we find \( \gamma(0, 0) = 2u(0) \), which implies that FF is stable for small \( Q \), i.e. around the Lifshitz point. This also shows that near the super-Lifshitz point the transition is bicritical, as opposed to tetracritical. We have also checked that this holds for larger \( Q \) near \( T_c \).

(ii) At strong coupling, \( \mu < 0 \), the FF state is stable for small enough \( Q \), determined by the binding energy of the pairs.

(iii) Then we are only left with the possibility of forming an LO state in weak-coupling, \( \mu > 0 \), for larger \( Q \). In this case, the FF state depends on parameters in greater details. Extrapolating our model to \( T < T_c \) we find an instability towards the diagonal LO state (PDW) \( \Delta_Q = \Delta_{-Q} \), with \( Q = Q_0\hat{Q}_+ \) (or equivalent). Also the LO-FF hybrid type state, \( \Delta_{Q_0} = \Delta_{-Q_0} \), such that \( \Delta(\bar{r}, \omega) \sim e^{iQ_0\bar{r}\pm \sqrt{\lambda}} \cos(Q_0y/\sqrt{\lambda}) \), which breaks both time reversal and translational invariance, is stable (but sub-leading to LO) at low temperature. Additional states, e.g. checkerboard containing all four degenerate FF states are also possible \cite{29, 43}.

For the LO state one should also consider the interaction \( \gamma(0, Q - Q)\Delta_Q^2\Delta_{-Q}^2\pm c.c. \) that may turn the 1st order SC-LO transition to a coexistence phase.

**Conclusions**— We have seen how an arbitrarily weak repulsive pair-hopping for sufficiently strong local attraction, \( g \), leads to an instability from zero to finite momentum SC. At weak pair-hopping, this is manifested as an SC dome of \( T_c \) versus \( g \) ending at a strong coupling quantum Lifshitz point with a transition into an FF state. At larger pair-hopping, the dome is hidden under a bicritical transition where the system changes from SC with sub-dominant FFLO order, to a state where the roles are reversed. At the intersection of Lifshitz and bicritical behavior, there is a “super-Lifshitz” point with extra soft fluctuations and distinct critical exponents. The Lifshitz transition forces the system to the strong coupling regime, \( \mu < 0 \), with pre-formed pairs in the normal state, whereas the bicritical transition occurs in the weak to intermediate coupling regime, \( \mu > 0 \).

Making connections to the cuprate superconductors it is natural to speculate that the low \( T_c \) of the underdoped materials may be due to suppressed phase-stiffness\cite{10} caused by proximity to a Lifshitz instability to finite momentum superconductivity. In addition there is evidence in the cuprates for both of the regimes with subdominant order: Recent evidence for PDW order near vortex cores in BSCCO suggest that suppression of SC leads to enhancement of PDW order, consistent with subdominant PDW order \cite{19, 21}. In LBCO, at 1/8 doping, there is evidence of 2D superconductivity, that has been attributed to interlayer frustrated PDW, which only at lower temperatures gives way to a 3D Meissner state and homogeneous SC, consistent with subdominant SC order\cite{6}. \cite{9}.

Several features of the model remain to be explored...
further, including a more detailed study of the relative prevalence of the various FF/LO type states, and to include charge order that may additionally favor LO over FF.

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Supplementary Material for
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EFFECTIVE THEORY - HUBBARD STRATONOVICH TRANSFORMATION

Rewriting equation (1) in [49], by going to reciprocal space, and introducing the Nambu-spinor, \( \Psi_k = [\psi_{\uparrow, k} \psi_{\downarrow, k}] \), yields

\[
H = \int \frac{d^2k}{(2\pi)^2} \tau_+ \Psi_k \Psi_{k'} + \frac{1}{g_0} \int \Delta^*(p) \Delta(p) \int \frac{d^2k}{(2\pi)^2} \tau_+ \Psi_k \Psi_{k'} + \frac{1}{g_0} \int \Delta^*(p) \Delta(p)
\]

where \( \tau_{\pm} = \frac{\tau_{\uparrow} \pm \tau_{\downarrow}}{2} \) with \( \tau_i \) being the Pauli-matrices. Further, \( \int \frac{d^2k}{(2\pi)^2} \), \( \varepsilon(k) = \frac{k^2}{2m} \) and \( T(p) = 1 - 2\alpha \cos(\frac{Q}{2} p_x) - 2\alpha \cos(\frac{Q}{2} p_y) \). We measure lengths and energies in \( l \) and \( \frac{1}{m_T} \), where \( l \) is some microscopic length scale and \( m \) the band mass.

The partition function \( Z = \text{Tr} e^{-\beta (H - \mu N)} \) can be expressed as a coherent state path integral. We utilize the Hubbard-Stratonovich transformation[50], which replaces the interaction with a fluctuating bosonic field coupling bilinearly to the electronic field. Thus

\[
Z = \int D \Psi^* D \Psi D \Delta^* D \Delta e^{-S(\Psi, \Delta)}, \quad S(\Psi, \Delta) = \int \frac{d^2k}{(2\pi)^2} \frac{1}{g_0} \int \Delta^*(p) \Delta(p) \int \frac{d^2k}{(2\pi)^2} \tau_+ \Psi_k \Psi_{k'} + \frac{1}{g_0} \int \Delta^*(p) \Delta(p)
\]

where \( \xi(k) = \varepsilon(k) - \mu \) and \( k = (k, i\omega) \), \( p = (p, i\Omega) \) for the bosonic and fermionic modes respectively. The total action can be written \( S = S_{\text{eff}}^c + S^\Delta \) where \( S^\Delta \) is the second term in (S2) and

\[
S_{\text{eff}}^c(\Delta) = -\text{Tr} \ln \beta G^{-1} = -\text{Tr} \ln \beta G_0^{-1} + \sum_n \frac{1}{n} \text{Tr} (G_0 \Sigma)^n
\]

which is obtained by integrating out the electronic degrees of freedom and expanding around the normal state \( \Delta = 0 \). Thus, our theory will hold for small \( \Delta \), that is, near \( T_c \). Keeping only forth order terms of \( \Delta \) (all odd terms vanish in the Pauli-matrix space) in \( S \) we write the partition function as \( Z = Z_0 \text{Tr} e^{-\beta F(\Delta)} \) where

\[
F = \frac{1}{\beta} \left( \int_{p, i\Omega} \Gamma^{-1}(p, i\Omega) |\Delta(p, i\Omega)|^2 + \frac{1}{2} \sum_{p_1, p_2, p_3} u(p_1, p_2, p_3) \Delta(p_1) \Delta^*(p_2) \Delta(p_3) \Delta^*(p_1 - p_2 + p_3) \right),
\]

\[
\Gamma^{-1}(p, i\Omega) = \frac{T^{-1}(p)}{g_0} - \int_k G(k + \frac{p}{2}, i\omega + i\Omega) G(-k + \frac{p}{2}, -i\omega), \quad G(k, i\omega) = \frac{1}{i\omega - \xi(k)}.
\]

We anticipate that the onset of instability, in general, will occur at finite momenta \( Q \), i.e.

\[
\min_{p-Q} \Gamma^{-1}(p, 0) = 0.
\]

In Figure S1 we show a few realizations of \( \Gamma^{-1}(p, 0) \), as a function of \( p = p_x \), for parameter values presented in Figure 3 in [49]. We clearly see the continuous development of a finite momentum, \( p_+ = Q_+ \), minimum for small \( \alpha \), and a discrete jump in momentum for bigger \( \alpha \). We also see that the structure of \( \Gamma^{-1}(p, 0) \) is well captured by the characteristic sixth order polynomial presented in equation (5) in [49], even for large \( Q_+ \) (\( Q_+ \sim 2\pi/\lambda \)). We evaluate the repulsive forth order term at the dominating mode \( Q \), \( u(p_1, p_2, p_3) \to u(Q) \) where

\[
uu(Q) = \int_k G^2(k + \frac{Q}{2}, i\omega) G^2(-k + \frac{Q}{2}, -i\omega).
\]
\( \Gamma^{-1}(p, i\Omega) \) shows a logarithmic UV-divergence which we regularize by introducing a high energy cut-off \( \epsilon = \frac{\Lambda^2}{2m} \). However, in 2D the instability to a superconducting state is in fact equivalent to the existence of a bound state\[34, 51\]. This means that we can express the bare interaction strength, \( g \), in terms of the bound state energy in 2D, \( E_b \), through the relation
\[
\frac{\Gamma^{-1}(0)}{g} = \frac{m}{4\pi} \ln \left( \frac{2\epsilon \Lambda}{E_b} \right).
\]
Here \( \epsilon \Lambda \) cancels exactly in \( \Gamma^{-1}(p, i\omega) \). In this work we keep the explicit cut-off, however we note that \( \langle S5 \rangle \) yields \( \mu \to -E_b/2 \), in the strong coupling limit (\( \mu/T \to -\infty \)), i.e. we have to overcome the binding energy in order to break the pair.

![FIG. S1. \( \Gamma^{-1}(p, 0) \) for \( \alpha = 0.01 \) (dashed green) and \( \alpha = 0.05 \) (solid red) for three different interaction strengths (other variables are the same as in Figure (3) in [49]). For \( \alpha = 0.01 \) we observe a transition through a Lifshitz point where the minimum shifts continuously from \( Q = 0 \) to \( |Q| > 0 \) roughly at \( \frac{mg}{2\pi} \simeq 0.4 \). For \( \alpha = 0.05 \) we observe a transition through a bicritical point where the momentum changes discontinuously for \( \frac{mg}{2\pi} \simeq 0.2488 \).](image)

**COEXISTING ORDERS**

To investigate the possibility of other composite orders we include all anticipated modes, \(-Q, 0, Q, -\hat{Q}, \hat{Q}\) (where \( Q = Q_0 \hat{Q}_+ \) and \( \hat{Q} = Q_0 \hat{Q}_- \)), simultaneously. We find the following additional terms of the free energy (arising from \( u(p_1, p_2, p_3) \))
\[
\begin{align*}
\gamma(0, Q)|\Delta_0|^2|\Delta_Q|^2, \\
\gamma(Q, Q')|\Delta_Q|^2|\Delta_{Q'}|^2, \\
\gamma(0, Q, -Q)|\Delta_0|\Delta_Q^* \Delta_{-Q} + c.c.
\end{align*}
\]
represented as Feynman in Figure S2. The combinatorial factors arise from cyclic permutation and charge conjugation. However, for the second diagram the charge conjugation is included explicitly in \( \langle S7 \rangle \).

From \( \langle S6 \rangle \) and \( \langle S8 \rangle \) we see that that \( \gamma(0, 0) = 2u(0) \), and correspondingly \( \gamma(Q, Q') > \sqrt{u(Q)u(Q')} \) holds for small \( Q \). This implies that the FF state is stable for small \( Q \). We can understand this from the proximity in parameter space.
to the uniform state. Whereas FF corresponds to a small deformation of the SC, with locally preserved superfluid density, any linear combination of the four FF states will have nodes in the real space pair wave-function and fails to take advantage locally of the full condensation energy.

At strong coupling, \( \mu < 0 \), where the binding energy of the pair, \( E_b \sim |\mu| \), is much bigger than the modulation energy, \( \sim Q^2 \), we anticipate the integrand of \( u \) and \( \gamma \) to be similar (i.e. for \( Q < Q_c \) with \( Q_c \propto \sqrt{|\mu|} \)) . This was checked numerically and indeed we find \( \gamma(Q, Q') > u(Q) \), thus FF is stable against forming nodes even in this case.

We are then left with the possibility of forming a LO state in a weak-coupling system, \( \mu > 0 \), at larger \( Q \). We do not find instability towards LO at \( T_c \) in the presented parameter regime. But, if we extrapolate below the critical temperature \( T < T_c \) we find an instability towards the diagonal LO state (PDW) \( \Delta_Q = \Delta_{-Q} \), with \( Q = Q_0 Q_x \), or equivalent. However, we are departing from the validity regime of this analysis by extrapolating below \( T_c \). One simple improvement would be to consider expanding around a saddle-point in the ordered phase. This is left for future works.

**BOSONIC OCCUPATION NUMBER**

The contribution to the density of fermions from pre-formed pairs is given by

\[
n_B = -\text{Tr} \frac{\partial \Gamma^{-1}}{\partial \mu}, \tag{S9}
\]

which is determined by the analytical structure of the pair-propagator \( \Gamma(p, \Omega) \), describing a two-particle spectrum\[36]. In 2D there exists a bound state, \( \Omega(p) = \frac{p^2}{4m_B} \), at all interaction energies, as well as a two particle continuum represented by a branch cut for \( \Omega(p) > -2\mu + \frac{p^2}{4m_B} \). In the weak-coupling limit, where \( \mu = \epsilon_F \), this leads to relaxation dynamics due to decay of weakly bound pairs. Nevertheless, \( n_B \) turns out to be negligible because of the high phase stiffness, or small bosonic mass, in accordance with the BCS results. However, in strong coupling limit \( \mu \to -E_b/2 \), and the branch cut and pole becomes increasingly separated. Thus, the low energy physics is well described by only keeping the, now freely propagating, bound state and \( (S9) \) takes the form

\[
n_B = \int_q \left( \exp \left( \frac{a_{Q,i} q_i^2}{\kappa_Q} \right) - 1 \right)^{-1} \left[ -\frac{1}{\kappa_Q} \frac{\partial r_Q}{\partial \mu} - \frac{\partial}{\partial \mu} \left( \frac{a_{Q,i} q_i^2}{\kappa_Q} \right) \right]. \tag{S10}
\]

Here we expanded around a saddle point \( Q_x = Q_0 \) at \( p = Q + q \) as \( \Gamma^{-1}(p, z) \approx r_Q - \kappa_Q z + \frac{a_{Q,i} q_i^2}{\kappa_Q} \), \( i = \pm \) before performing the Matsubara sum over \( z = i \Omega \). \( m_B = \frac{\kappa_0}{a_{Q,i}} \) is the boson mass, reflecting the curvature at the saddle point. In the strong coupling limit one finds \(-\frac{1}{\kappa_Q} \frac{\partial r_Q}{\partial \mu} = 2 \) (for \( Q^2/(2m) \ll |\mu| \)).

At this point we run in to an expected problem, that there is no long range order superconductivity in 2D. This becomes apparent since the Bose-integral in \( (S10) \) diverges in 2D. There is however a transition in the Kosterlitz-Thouless (KT) sense, where the low energy state is one with quasi-long range order and a finite super-fluid density.
However the physics of the KT-transition is lost by resorting to the Gaussian approximation \cite{37}. Instead we choose to regularize the divergent integral by allowing the bosons to move out in the third dimension, the $z-$direction. A way to do this is by substituting \( \frac{q^2}{2m_B^0} \rightarrow \frac{q^2}{2m_B^0} + \frac{q^2}{2m_B^0} \int \frac{d^2 q}{(2\pi)^2} \rightarrow \frac{2\pi}{\sqrt{(q^2_0)}} \int \frac{d^3 q}{(2\pi)^3} \), where \( (q^2_0) = 2m_B^0 T \) is the thermal expectation value of the momenta in the $z-$direction \cite{37,40}. With these considerations we can express the first and second term in \( (S10) \) as

\[
n^{(1)}_B = 8\pi T \sqrt{m_B^0 m_B} \int \frac{d^3 \tilde{q}}{(2\pi)^3} \frac{1}{e^{\frac{\tilde{q}^2}{m_B^0}} - 1}, \quad n^{(2)}_B = \frac{16\pi^2 \alpha^2}{mg^{-2}} T^2 (m_B^0 + m_B^{-}) \sqrt{m_B^0 m_B} \int \frac{d^3 \tilde{q}}{(2\pi)^3} \frac{\tilde{q}_+^2 + \tilde{q}_-^2}{e^{\frac{\tilde{q}^2}{m_B^0}} - 1} \quad (S11)
\]

where we introduced the dimensionless momenta \( \tilde{q}_i = q_i/\sqrt{2m_B^0 T} \) and the integration is done over a sphere of infinite radius. The second term, \( n^{(2)}_B \), vanishes for \( \alpha = 0 \). But, even for finite \( \alpha \), this term can be neglected for small enough densities, \( \frac{mg}{2\pi n_B^0 x} \gg \alpha \) (where \( \frac{mg}{2\pi n_B^0 x} \gtrsim 0.1 \) in this work). Thus, we find

\[
n_B = \frac{\zeta(3/2)}{\sqrt{\pi}} T \sqrt{m_B^0 m_B} \quad (S12)
\]

presented in equation (8) in \cite{49}. From the above discussion we understand that using \( (S12) \) will provide a valid description for the strong and weak coupling limit (where it vanishes due to small effective mass). However, at intermediate coupling, when \( \mu \sim 0 \) and \( n_B \sim n_F \), both free and bound fermions coexists and there is a contribution from scattering states which is not accounted for properly here\cite{37}. Nevertheless, since the bound state exist for all interaction in 2D (in contrast to the 3D case) there is reason to believe that \( (S12) \) might still qualitatively give the right description.

**Corrections to \( n_B \)**

From the divergence of the bosonic mass we anticipate a suppression of \( T_c \) due to the increase of fluctuation. Indeed, from \( (S12) \) the divergence of the mass is accompanied by a suppression of \( T_c \) to zero. However, strictly speaking, for \( m_B \rightarrow \infty \) the derivation of \( (S12) \) breaks down since the inclusion of higher order kinetic terms necessarily become of importance and will regulate the divergence of \( n_B \) as \( m_B \rightarrow \infty \) (for fixed \( T \)). We employ the expansion \( \frac{q^2}{2m_B^0} + cq^4 \), where \( m_B, c \) are evaluated at the stable point \( Q \), and study corrections when \( m_B \rightarrow \infty \) for two cases. (Here we neglect cross terms like \( bq^2 q^2 \) which do not change the result presented below. Also, note that \( b, c \) is not the same as in equation (5) in \cite{30}.)

(i) We start by studying the case when only one mass diverge, say \( m_B^{-} \rightarrow \infty \) and \( m_B^{+} \) finite. This could happen when the finite \( Q \) solution loses its support. We note that the divergence of \( (S10) \) lies in the IR. Thus, introducing \( \tilde{q}_{-} = q_{-}/\sqrt{2m_B^{-} T} \) and \( \tilde{q}_{+} = q_{+}/\sqrt{c T} \) we can write \( n_B \) for small momenta as

\[
n^{(1)}_B \sim \sqrt{T m_B^{-}} \left( \frac{T}{c} \right)^{1/4} \int \tilde{q}^{-1/2} d\tilde{q}, \quad n^{(2)}_B \sim \sqrt{T m_B^{-}} \left( \frac{T}{c} \right)^{3/4} \frac{\alpha \lambda^2}{mg} \int \tilde{q}^{1/2} d\tilde{q}. \quad (S13)
\]

(We left out the angular part of the integral, as well as numerical constants.) Both integrals are convergent meaning that \( c \) will determine temperature at this point.

(ii) At the Lifshitz point the mass diverge in two direction, thus for \( m_B^{\pm} \rightarrow \infty \) we consider \( \tilde{q}_z = q_z/\sqrt{2m_B^{\pm} T} \) and \( \tilde{q}_{\pm} = q_{\pm}/\sqrt{c T} \) yielding

\[
n^{(1)}_B \sim \left( \frac{T}{c} \right)^{1/2} \int \tilde{q}^{-1} d\tilde{q}, \quad n^{(2)}_B \sim \frac{\alpha \lambda^2}{mg} \left( \frac{T}{c} \right) \int d\tilde{q} \quad (S14)
\]

Here, the first term is divergent for finite \( T \), thus it forces \( T \rightarrow 0 \). This means that \( (S12) \) correctly captures the vanishing of \( T_c \) at the Lifshitz point, this is why we keep \( (S12) \) even in this case. The second term is finite and negligible also in this case. Nevertheless, \( T_c \) will suffer from corrections near the Lifshitz point.

Further, note that \( (S10) \) only considers one order at a time. At points of coexistence, like \( \frac{mg}{2\pi n_B^0 x} = 0.25 \) for \( \alpha = 0.05 \) where SC and FF are degenerate, one should consider contribution from both orders to \( n_B \). However in weak coupling this is not expected to be important since \( n_F \) either way dominates. In strong coupling though, it is likely to be important since it redistributes the dominating pre-formed pairs into the different orders.
Corrections from time dependent part

There is an anomalous time-dependent term $\eta_Q zq_+, i = \pm$, which arise in the expansion of $\Gamma^{-1}$ for $Q_0 > 0$, $\Gamma^{-1}(p, z) \approx r_Q - \kappa_Q z + \frac{\eta_Q}{2} q_i^2 + \eta_Q z q_+, i = \pm$. The inclusion of this term shifts the location of the poles for finite $q_+$ and $(S10)$ would take the same form, but with $\kappa_Q \to \kappa_Q(1 + q_+ \eta_Q/\kappa_Q)$. We note that, approximately, the highest momenta of relevance in $(S10)$ is $q_{\text{max}}^2 \approx T \kappa_Q/a_Q$. From the simulations we have found $q_{\text{max}} \eta_Q/\kappa_Q \lesssim 0.1$ and this term was excluded.

One interesting feature that can be seen from Figure (3) in [49] is that the mass, $m_B = \sqrt{m_B^+, m_B^-}$, attains a finite value at the end of the $Q_0 > 0$ branch for $\alpha = 0.05$. Because of the vanishing curvature, $a_{Q_0, \pm} = 0$, the mass is expected to diverge, since $m_B^+ = \frac{\kappa_Q}{a_{Q_0, \pm}}$. However, it turns out that $\kappa_Q \to 0$ in this limit as well, yielding a finite mass. The vanishing of $\kappa_Q$ means that we need to consider higher order terms in frequency, but this was not considered in this work since the ending of this branch corresponds to meta-stable points without any true thermodynamical transition.