Stabilization of perturbed integrator chains using Lyapunov-Based Homogeneous Controllers

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Abstract

In this paper, we present a Lyapunov-based homogeneous controller for the stabilization of a perturbed chain of integrators of arbitrary order \( r \geq 1 \). The proposed controller is based on homogeneous controller for stabilization of pure integrator chains. The advantages to control the homogeneity degree of the controller are also discussed. A bounded-controller with minimum amplitude of discontinuous control and a controller with fixed-time convergence are synthesized, using control of homogeneity degree, and their performances are shown in simulations. It is demonstrated that the homogeneous arbitrary HOSM controller [1] is a particular case of our controller.

I. Introduction

The problem of finite-time stabilization of a perturbed integrator chain arises in many control applications. For example, electromechanical systems such as motorized actuators or robotic arms are modeled as perturbed double integrators. Another application is in Higher Order Sliding Mode Control (HOSM) [2], which consists of the stabilization of an auxiliary system arising as a perturbed integrator chain built from the output and its higher time derivatives [3]. The finite-time stability problem was addressed in relation with homogeneous systems in [4], and homogeneity concept was used for stabilization of linear systems in [5]. In [6], the link between finite-time stabilization and homogeneity of a system was established, and it was shown that a homogeneous system is finite-time stable if and only if it is asymptotically stable and has a
negative homogeneity degree. This result has, since then, been used for the development of many controllers for pure and perturbed integrator chains. A homogeneous nonsmooth proportional-derivative controller for robot manipulators (double integrator system) was developed in [7]. This work was generalized for an arbitrary-length integrator chain in [8]. In [9] and [10], negative homogeneity was used for the finite-time stabilization of a class of nonlinear systems that includes perturbations at each integrator link.

Among Sliding Mode techniques, the homogeneity approach was used in [11], [12], to demonstrate finite-time stabilization of the arbitrary order sliding mode controllers for Single Input Single Output (SISO) systems [1]. A robust Multi Input Multi Output (MIMO) HOSM controller was also presented in [13], using a constructive algorithm with geometric homogeneity based finite-time stabilization of an integrator chain. A controller, which stabilizes a perturbed integrator chain of arbitrary length using only the signs of state variables, was presented in [14]. A Lyapunov-based approach for arbitrary HOSMC controller design was first presented in [15]. In these works, it was shown that a class of homogeneous controllers that satisfies certain conditions, could be used to stabilize perturbed integrator chains.

In this paper, we present a continuation of [15], and develop a Lyapunov-based robust controller for the finite-time stabilization of a perturbed integrator chain of arbitrary order, with bounded uncertainty. The main focus of this paper is to obtain various properties in the controller by changing the degree of homogeneity. The homogeneous controller for perturbed integrator chains is developed from a discontinuous Lyapunov-based controller for pure integrator chains. It is then demonstrated that the homogeneity degree can be controlled in the neighborhood of zero, such that the amplitude of discontinuous control is kept to its minimum possible value when the states have converged. It is also shown that the recently developed “Fixed-Time” stability notion can be achieved by changing the homogeneity degree. Fixed-time stability, introduced as uniform stability of double integrator systems in [16], refers to the finite-time stability of systems, where the convergence time is bounded and independent of the system’s initial state. In [16] and [17], fixed-time convergence to a neighborhood of the origin was demonstrated for second order systems and arbitrary order respectively. In [18], fixed-time convergence controllers were developed for linear systems, insuring guaranteed convergence exactly to zero. Based on the control of homogeneity, the controller presented in this paper ensures fixed-time convergence to zero of a perturbed chain of integrators.
The paper is organized as follows: the problem formulation as well as the motivation and contributions of the paper are discussed in Section 2. The controller design is presented in Section 3 and its special cases are demonstrated in Section 4. Simulation results are shown in Section 5 and concluding remarks are given in Section 6.

II. PROBLEM FORMULATION, MOTIVATION AND CONTRIBUTION

In this section, we will present the mathematical formulation of the perturbed integrator chain problem. Then, the motivation behind using homogeneity based controllers will be discussed, and finally, the contribution of this paper will be presented in the context of the problem and motivation.

A. Problem Formulation

Let us consider an uncertain nonlinear system:

\[
\begin{aligned}
\dot{x}(t) &= f(x, t) + g(x, t)u, \\
y(t) &= s(x, t),
\end{aligned}
\]

(1)

where \( x \in \mathbb{R}^n \) is the state vector and \( u \in \mathbb{R} \) is the input control. The sliding variable \( s \) is a measured smooth output-feedback function and \( f(x, t) \) and \( g(x, t) \) are uncertain smooth functions. It is assumed that the relative degree, \( r \) of the system [19] is globally well defined, uniform and time invariant [3] and the associated zero dynamics are asymptotically stable. For autonomous systems, \( r \) is the minimum order of time derivatives of the output \( y(t) \) in which the control input \( u \) appears explicitly. This means that, for suitable functions \( \varphi(x, t) \) and \( \gamma(x, t) \), we obtain

\[
y^{(r)}(t) = \varphi(x(t), t) + \gamma(x(t), t)u(t).
\]

(2)

The functions \( \gamma(x, t) \) and \( \varphi(x, t) \) are assumed to be bounded by positive constants \( \gamma_m \leq \gamma_M \) and \( \varphi \), such that, for every \( x \in \mathbb{R}^n \) and \( t \geq 0 \)

\[
0 < \gamma_m \leq \gamma(x, t) \leq \gamma_M, \quad |\varphi(x, t)| \leq \varphi.
\]

(3)

Defining \( s^{(i)} := \frac{d^i}{dt^i}y \); the goal of \( r^{th} \) order sliding mode control is to arrive at, and keep the following manifold in finite-time:

\[
s^{(0)}(x, t) = s^{(1)}(x, t) = \cdots = s^{(r-1)}(x, t) = 0.
\]

(4)
To be more precise, let us introduce $z = [z_1 \ z_2 \ ... \ z_r]^T := [s \ s \ s^{(r-1)}]^T$. Then (4) is equivalent to $z = 0$. Since the only available information on $\tilde{\phi}(x,t)$ and $\tilde{\gamma}(x,t)$ are the bounds (3), it is natural to consider a more general control system instead of System (2), such as

$$\dot{z} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ \vdots \\ 0 \varepsilon(t) + \gamma(t)u \end{pmatrix},$$

(5)

where the new functions $\phi$ and $\gamma$ are arbitrary measurable functions that verify the condition

$$(H1) \quad \phi(t) \in [-\bar{\phi}, \bar{\phi}], \quad \gamma(t) \in [\gamma_m, \gamma_M].$$

(6)

The objective of this paper is to design controllers which stabilize System (5) to the origin in finite-time. Since these controllers are to be discontinuous feedback laws $u = U(z)$, solutions of (5) will fall in the realm of differential inclusions and need to be understood here in Filippov sense, i.e. the right hand vector set is enlarged at the discontinuity points of (5) to the convex hull of the set of velocity vectors obtained by approaching $z$ from all the directions in $\mathbb{R}^r$, while avoiding zero-measure sets [20].

B. Motivation behind Homogeneity based control

Let us consider the following one-dimensional differential equation

$$\dot{z} = \omega(z) = -c \ |z|^\alpha.$$

where $\alpha \geq 0$, $c > 0$ and $|z|^\alpha := |z|^\alpha \ sign(z)$. The term $sign(z)$ is a multi-valued function, equal to $z/|z|$ if $z \neq 0$ and $[-1, 1]$ if $z = 0$.

The degree of homogeneity of Equation (7) is $\kappa = \alpha - 1$ and this system is stable for all $c > 0$ and $\alpha \geq 0$. However, different characteristics can be obtained in the system, depending upon the value of $\alpha$:

- $\alpha = 0$: the convergence to zero occurs in finite-time. The controller $\omega(z)$ is uniformly bounded for $z \in \mathbb{R}$ but discontinuous at $z = 0$;
- $0 < \alpha < 1$: the convergence to zero occurs in finite-time. The controller $\omega(z)$ is unbounded and tends to zero as $|z| \to 0$;
- $\alpha > 1$: the convergence to zero is asymptotic, however the convergence time to the sphere $\mathbb{B}(0,1)$ is uniformly bounded by a constant. The controller $\omega(z)$ is unbounded.
Based on these observations, we can construct controllers forcing $z$ to converge to zero in finite-time by changing $\alpha$:

- an unbounded controller obtained by changing $\kappa$ from $\kappa_1 > 0$ to $\kappa_2 < 0$ when $z$ reaches the sphere $B(0,1)$. This ensures the convergence to zero in finite-time, bounded by a constant.
- a uniformly bounded controller whose amplitude tends to zero as $z$ converges to zero. This controller is obtained by changing the homogeneity degree from $\kappa_1 = -1$ to $-1 < \kappa_2 < 0$.

In the case of a single perturbed integrator, (7) is replaced by the following differential inclusion:

$$\dot{z} \in u(z) \left[ \gamma_m, \gamma_M \right] + [-\bar{\phi}, +\bar{\phi}],$$

where $\gamma_m \leq \gamma_M$ and $\bar{\phi}$ are arbitrary positive constants. In [15], it was demonstrated that the controller

$$u(z) = \frac{1}{\gamma_m} \left( \omega(z) + \bar{\phi} \text{sign}(\omega(z)) \right),$$

defined for $0 < \alpha < 1$, forces $z$ to converge to zero in finite-time. According to the standard comparison principle [21] the rate of convergence of (8) is faster than (7). It can be noted that the controller is valid for $0 \leq \alpha \leq 2$.

C. Contribution

In this work, we extend the above observations related to homogeneous controllers for a single integrator to the stabilization of perturbed integrator chains of arbitrary order $r \geq 1$, based on a controller which stabilizes pure integrator chains. The study is focused on the stabilization of a perturbed integrator chain, therefore the zero-dynamics of the nonlinear system (1) are not treated. It is shown that the well-known homogeneous robust arbitrary HOSM controller [1] is a particular case of our controller, for a particular choice of homogeneity degree. It is also demonstrated that a bounded controller is synthesized using a change of homogeneity degree, such that the controller has a reduced amplitude at $z = 0$. Then a fixed-time controller is obtained, also by controlling the homogeneity degree.

III. CONTROLLER DESIGN

We will now develop the controller in two steps. The stabilization of a pure integrator chain will be considered first. Then the study will be extended to the case of a perturbed integrator chain.
A. Useful definitions, lemmas and theorems

We need the following definitions to state our results. Consider the differential system

\[ \dot{z} = f(t, z), \quad z \in \mathbb{R}^r. \quad (9) \]

**Definition 1.** [18], [22] The equilibrium point \( z = 0 \) of System (9) is said to be locally **finite-time** stable in a neighborhood \( \hat{U} \subset \mathbb{R}^r \) if (i) it is asymptotically stable in \( \hat{U} \); (ii) it is finite-time convergent in \( \hat{U} \), i.e. for any initial condition \( z_0 \), \( z(t, z_0) = 0, \forall t \geq T(z_0) \), where \( T(z_0) \) is called the settling-time function. The equilibrium point \( z = 0 \) is globally finite-time stable if \( \hat{U} = \mathbb{R}^r \). The equilibrium point is **fixed-time** stable if (i) it is globally finite-time stable; (ii) the settling-time function is bounded by a constant \( T_{\text{max}} \), i.e. \( \exists T_{\text{max}} > 0 : \forall z_0 \in \mathbb{R}^r, T(z_0) \leq T_{\text{max}} \).

**Definition 2.** [18] The set \( S \) is said to be globally **finite-time** attractive for (9), if for any initial condition \( z_0 \), the trajectory \( z(t, z_0) \) of (9), achieves \( S \) in finite-time \( T(z_0) \). Moreover, the set \( S \) is said to be **fixed-time** attractive for (9), if (i) it is globally finite-time stable; (ii) the settling-time function is bounded by a constant \( T_{\text{max}} \).

Let us recall the following theorem:

**Theorem 1.** [6], [22] Suppose there exists a positive definite \( C^1 \) function \( V \) defined on a neighborhood \( \hat{U} \subset \mathbb{R}^r \) of the equilibrium point \( z = 0 \) and real numbers \( C > 0 \) and \( \alpha \geq 0 \), such that the following condition is true for every trajectory \( z \) of System (9),

\[ \dot{V} + CV^\alpha(z(t)) \leq 0, \quad \text{if } z(t) \in \hat{U}, \quad (10) \]

where \( \dot{V} \) is the time derivative of \( V(z(t)) \).

Then all trajectories of System (9) which stay in \( \hat{U} \) converge to zero. If \( \hat{U} = \mathbb{R}^r \) and \( V \) is radially unbounded, then System (9) is globally stable with respect to the equilibrium point \( z = 0 \).

Depending on the value \( \alpha \), we have different types of convergence: if \( 0 \leq \alpha < 1 \), the equilibrium point \( z = 0 \) is finite-time stable ([22]), if \( \alpha = 1 \), it is exponentially stable and if \( \alpha > 1 \) the equilibrium point \( z = 0 \) is asymptotically stable equilibrium and, for every \( \varepsilon > 0 \), the set \( B(0, \varepsilon) = \{z \in \hat{U} : V(z) < \varepsilon\} \) is fixed-time attractive.

We next recall the concept of homogeneity. Consider the time-invariant differential system

\[ \dot{z} = f(z), \quad z \in \mathbb{R}^r. \quad (11) \]
Definition 3. [8] The family of dilations $\zeta^p_\varepsilon$, $\varepsilon > 0$, are the linear maps defined on $\mathbb{R}^r$ given by

$$\zeta^p_\varepsilon(z_1, \ldots , z_r) = (\varepsilon^{p_1}z_1, \ldots , \varepsilon^{p_r}z_r),$$

where $p = (p_1, \ldots , p_r)$ with the dilation coefficients $p_i > 0$, for $i = 1, \ldots , r$.

Definition 4. [8] The vector field $f(z) = (f_1(z), \ldots , f_r(z))^T$ is homogeneous of degree $\kappa \in \mathbb{R}$ with respect to the family of dilation $\zeta^p_\varepsilon$ if, for every $z \in \mathbb{R}^r$ and $\varepsilon > 0$,

$$f_i(\varepsilon^{p_1}z_1, \ldots , \varepsilon^{p_r}z_r) = \varepsilon^{p_i + \kappa} f_i(z_1, \ldots , z_r), \quad i = 1, \ldots , r, \quad \varepsilon > 0.$$  

System (11) is called homogeneous, if the vector field $f(z)$ is homogeneous.

Definition 5. [8] A function $\Omega(z)$ is homogeneous of degree $a \in \mathbb{R}^+$ with respect to the family of dilation $\zeta^p_\varepsilon$ if, for every $z \in \mathbb{R}^r$ and $\varepsilon > 0$,

$$\Omega(\varepsilon^{p_1}z_1, \ldots , \varepsilon^{p_r}z_r) = \varepsilon^{a} \Omega(z_1, \ldots , z_r).$$

Definition 6. The homogeneous norm $\Gamma_i(z)$ for $z \in \mathbb{R}^i$ is defined by

$$\Gamma_i(z) \equiv \Gamma_i(z_1, \ldots , z_i) = \left( \sum_{j=1}^{i} |z_j|^{c/p_j} \right)^{1/c},$$

where $c = \max(p_1, \ldots , p_r) > 0$.

In this case, the unit sphere $S_i$ is given by

$$S_i = \{ z \in \mathbb{R}^i : \Gamma_i(z) = 1 \}.$$

Lemma 1 (Lemma 4.2 of [6]). Suppose $\Omega_1$ and $\Omega_2$ are continuous real-valued functions on $\mathbb{R}^r$, homogeneous with respect to $\zeta^p_\varepsilon$ of degrees $d_1 > 0$ and $d_2 > 0$, respectively, and $\Omega_1$ is positive definite. Then, for every $z \in \mathbb{R}^r$,

$$\left[ \min_{\{z: \Omega_1(z) = 1\}} \left[ \Omega_1(z) \right]^{\frac{d_2}{d_1}} \right] \leq \Omega_2(z) \leq \left[ \max_{\{z: \Omega_1(z) = 1\}} \left[ \Omega_1(z) \right]^{\frac{d_2}{d_1}} \right].$$

(12)

Proposition 1 (Proposition 1 of [23]). Let $\Omega$ be a positive definite $C^1$ function, homogeneous of degree $a$ with respect to $\zeta^p_\varepsilon$. Then, for all $i = 1, \ldots , r$; $\frac{\partial \Omega}{\partial z_i}$ is homogeneous of degree $(a - p_i)$. 
B. Stabilization of a pure chain of integrator

Consider the following pure integrator chain:

\[
\begin{align*}
\dot{z}_i &= z_{i+1}, \quad i = 1, \ldots, r-1, \\
\dot{z}_r &= u.
\end{align*}
\] (13)

The following result guarantees the stabilization of (13).

**Theorem 2.** Let \( r \) be the order of the pure integrator chain given in (13). For \( \kappa \in [-1/r, 1/r] \), set

\[ p_i = 1 + (i-1)\kappa, \quad i = 1, \ldots, r, \]

and finally let \( c \) be a positive constant such that \( c \geq \max(p_1, \ldots, p_r) \). Then there exist constants \( l_i > 0, \ i = 1, \ldots, r \), independent on \( \kappa \), such that the feedback control law \( u = \omega_{\kappa}(z) := v_r \) defined inductively by

\[
\begin{align*}
v_0 &= 0, \\
v_i &= -l_i N_i \text{sign}(z_i - v_{i-1}), \\
i &= 1, \ldots, r,
\end{align*}
\] (14)

stabilizes System (13), where \( N_i, i = 1, \ldots, r \) are defined by

\[ N_i = \left( \sum_{j=1}^{i} |z_j|^{c/p_j} \right)^{\frac{p_i + \kappa}{c}}. \] (15)

There also exists a homogeneous Lyapunov function \( V_{\kappa}(z) \) for the closed-loop system (13) with the state-feedback \( u \), that satisfies \( \dot{V}_{\kappa} \leq -C V_{\kappa}^{\frac{c+1+\kappa}{c+1}} \), for some positive constant \( C \).

**Proof:**

For \( 1 \leq i \leq r \), we define

\[
\begin{align*}
w_i &:= \left( \sum_{j=1}^{i} |z_j|^{c/p_j} \right) \text{sign}(z_i - v_{i-1}), \\
W_i &:= \int_{v_{i-1}(z_1, \ldots, z_{i-1})}^{v_i(z_1, \ldots, z_{i-1}, s)} w_i(z_1, \ldots, z_{i-1}, s) \, ds, \\
&= \left( |z_1|^{\frac{c}{p_1}} + \cdots + |z_{i-1}|^{\frac{c}{p_{i-1}}} \right) |z_i - v_{i-1}| + \frac{|z_i|^{\frac{c}{p_i} + 1} - |v_{i-1}|^{\frac{c}{p_i} + 1}}{p_i + 1}.
\end{align*}
\] (16)

It can be seen that \( W_i \) is positive definite function with respect to \( v_{i-1} - z_i \), homogeneous with respect to \( \xi_{p_i}^{c} \) of degree \( (c + p_i) \). We introduce \( \bar{W}_i := W_i^{\delta_i} \), where \( \delta_i = \frac{c + 1}{c + p_i} \), so that all functions \( \bar{W}_i \) are homogeneous of the same homogeneity degree \( (c + 1) \).
Lemma 2. With the notations above and \(1 \leq i \leq r\), there exist positive constants \(k_i\), such that:

\[
\dot{W}_i \leq k_i |w_i|^{c+1}/c.
\]  

(17)

Proof: We can get that \(\left(\sum_{j=1}^{i} |z_j|^{c/p_j}\right)\) is a homogeneous function with respect to \(\xi_p^c\) of degree \(c\). Then according to Lemma 1 and for a given \(\kappa\), there exists a constant \(K_i\) depending on \(\kappa\), such that

\[
\dot{W}_i \leq K_i(\kappa) \left(\sum_{j=1}^{i} |z_j|^{c/p_j}\right),
\]

where \(K_i(\kappa) = \max_{\kappa \in [-1/r,1/r]} \dot{W}_i\). Then, the choice of \(k_i\), as \(k_i = \max_{\kappa \in [-1/r,1/r]} K_i(\kappa)\), implies (17).

We proceed to prove the theorem by induction on \(r\).

Step 1: Consider \(\dot{z}_1 = u\). For any \(l_1 > 0\), taking \(u = \omega_\kappa(z_1) = -l_1 |z_1|^{(p_1+\kappa)/p_1}\) stabilizes the closed-loop system. The Lyapunov function \(V_1 = W_1 = |z_1|^{1+c}/(1+c)\) is homogeneous of degree \(c+1\) and

\[
\dot{V}_1 = -l_1 |z_1|^{c+1}/c \leq -\eta_1 V_1^{c+1}/c,
\]

(18)

for some constant \(\eta_1 > 0\).

Step i: Assume that the conclusion holds true till \(i-1\). Define the Lyapunov function \(V_i\) by \(V_i = V_{i-1} + \dot{W}_i = \sum_{j=1}^{i} \dot{W}_j\). We get

\[
\dot{V}_i = \sum_{j=1}^{i-1} \frac{\partial \dot{W}_i}{\partial z_j} z_{i+j} + w_i V_i \dot{W}_i \frac{c(i-1)k}{c+p_i} + \dot{V}_{i-1} + \frac{\partial V_{i-1}}{\partial z_{i-1}} (z_i - v_{i-1}),
\]

(19)

\[
\leq \sum_{j=1}^{i-1} \frac{\partial \dot{W}_i}{\partial z_j} z_{i+j} - l_i V_i \dot{W}_i \frac{c+p_i}{c+p_i} + \dot{V}_{i-1} + \frac{\partial V_{i-1}}{\partial z_{i-1}} (z_i - v_{i-1}),
\]

The fact that \(\dot{W}_i\) are homogeneous with respect to \(\xi_p^c\) of degree \((c+1)\) for each \(i = 1, \cdots, r\), implies that \(V_i\) are homogeneous of degree \((c+1)\) with respect to \(\xi_p^c\) as well. In addition, according to Proposition 1, \(\dot{V}_i\) are homogeneous of degree \((c+1-\kappa)\) with respect to \(\xi_p^c\). Then without loss
of generality, the study can be restricted to the unit sphere $S_i$.  

By taking $l_i$ such that

$$l_i > 2k_i \max_{z \in S_i} \left\{ \sum_{j=1}^{i-1} \frac{\partial \bar{W}_i}{\partial z_j} z_{i+1} + \frac{\partial V_{i-1}}{\partial z_{i-1}} (z_i - v_{i-1}) \right\}, \forall \kappa \in [-1/r, 1/r],$$

(20)

and setting $\eta_i := l_i / 2k_i^{1+\epsilon}$, we get

$$\dot{V}_i \leq - \sum_{j=1}^{i} \eta_j \bar{W}_j^{c+1+\epsilon}. \quad (21)$$

At the final step, all parameters $l_i$ are determined, with $V_\kappa(z) = V_r = \sum_{j=1}^{r} \bar{W}_j$ and

$$\dot{V}_\kappa(z) \leq - \sum_{j=1}^{i} \eta_j \bar{W}_j^{c+1+\epsilon} \leq - \eta \sum_{j=1}^{i} \bar{W}_j^{c+1+\epsilon},$$

where $\eta := \min \eta_i$ for $i = 1, \cdots, r$.

We get $\dot{V}_\kappa \leq -CV_\kappa^{c+1+\epsilon}$, where, according to lemma [1], $C \geq \eta 2^{r-1/c+1}$. 

C. Stabilization of an r-perturbed chain of integrator

From the result obtained in Theorem [2] we now proceed to the stabilization of the perturbed integrator chain presented in System (5). The extension of Theorem [2] to the case of System (5) is based on the following result of [15].

**Theorem 3.** [15] Let $\omega(z)$ and $V(z)$ be respectively, a state-feedback control law stabilizing System (13) and a Lyapunov function for the closed-loop system, which satisfy the hypotheses of Theorem [1] and obey the following additional conditions: for every $z \in \hat{U}$,

$$\frac{\partial V}{\partial z_r}(z) \omega(z) \leq 0, \quad \omega(z) = 0 \Rightarrow \frac{\partial V}{\partial z_r}(z) = 0.$$

Then, for arbitrary constants $m, n \geq 1$, the following control law stabilizes System (5):

$$u(z) = \frac{m}{\eta_m} (\omega(z) + n \phi \text{sign}(\omega(z))). \quad (22)$$

The function $V(z)$ remains a Lyapunov function for the closed-loop system (5) with the feedback $u(z)$, and satisfies Condition (10). If $\hat{U} = \mathbb{R}^r$ and $V(z)$ is radially unbounded, then the closed-loop system (5) with the feedback $u(z)$ is globally stable with respect to the origin.
Proof: This theorem is a generalization of Theorem 2 of [15], and can be proven in the same way.

It can be shown that the controller presented in Theorem 2 satisfies the conditions presented in Theorem 3. We calculate
\[
\frac{\partial V}{\partial z_r} \omega = \frac{\partial \bar{W}_r}{\partial z_r} v_r - \frac{l_r}{k_r^{c+1}} \tilde{W}_r^{c+1} \leq 0, \quad \text{and}
\]
\[
\omega = 0 \Rightarrow -l_r |w_r|^\frac{r_k + 1}{c} \text{sign}(z_r - v_r - 1) = 0 \Rightarrow \frac{\partial V}{\partial z_r} \equiv \frac{\partial \bar{W}_r}{\partial z_r} = 0.
\]

Remark 1. It should be noted that this controller is not unique, and all homogeneous controllers satisfying the conditions of Theorem 3 are valid (e.g. [8]).

As indicated in Section 2 of [12], in order to stabilize the uncertain System (5) by a state-feedback controller \( u = u(z) \), it is necessary that the controller be discontinuous at \( z = 0 \), and satisfy
\[
\lim_{||z|| \to 0} |u(z)| \geq \frac{\phi}{\gamma_m} =: M_{\min}.
\]

IV. DISCUSSION OF SPECIAL CASES

In this section, we consider some specific choices of the homogeneity degree, in order to obtain further results. First, it is shown that for a particular choice of homogeneity degree, the homogeneous HOSM controller presented in [1] becomes a special case of our controller. Then, a bounded controller with minimum amplitude \( M_{\min} \) of discontinuous control at \( z = 0 \) is designed. Finally, a controller with fixed-time convergence is synthesized.

A. Robust Homogeneous Arbitrary HOSM Controller

Let us consider the controller presented in [1].

Proposition 2. Provided \( l_1, \ldots, l_{r-1}, M > 0 \) are chosen sufficiently large in the listed order, the bounded controller \( u = -M \text{sign}(\phi_{r-1}) \) provides finite-time stability for System (5), where \( \phi_{r-1} \)
is defined inductively as

\[
N_1 = |z_1|^{\frac{r-1}{r}}, \\
N_i = \left( |z_1|^{\frac{d}{r}} + |z_2|^{\frac{d}{r-1}} + |z_i|^{\frac{d}{r-i+1}} \right)^{\frac{r-i}{d}}, \\
i = 2, \ldots, r-1. \\
\phi_0 = z_1, \\
\phi_i = z_{i+1} + i_i N_i \text{sign}(\phi_{i-1}), \\
i = 2, \ldots, r-1.
\]

(23)

with \(d > r\) is an arbitrary positive constant.

**Proof:** Consider the functions \(N_i\) in Equation (15). Let us fix the parameters \(\kappa\) and \(c\) as follows \(\kappa = -1/r\) and \(c = d/r\). Then, defining \(\phi_i = z_{i+1} - v_i\), we find:

\[
N_1 = |z_1|^{\frac{r-1}{r}}, \\
N_i = \left( |z_1|^{\frac{d}{r}} + |z_2|^{\frac{d}{r-1}} + |z_i|^{\frac{d}{r-i+1}} \right)^{\frac{r-i}{d}}, \\
i = 2, \ldots, r. \\
\phi_0 = z_1, \\
\phi_i = z_{i+1} + i_i N_i \text{sign}(\phi_{i-1}), \\
i = 2, \ldots, r-1.
\]

(24)

According to Theorem 3 and by taking \(m = n = 1\), the state-feedback control law for the stabilization of System (5) can be expressed as: \(u = -(l_i N_i/K_m + \phi/K_m)\text{sign}(\phi_{r-1})\). It can be seen that this particular choice of parameter \(\kappa\) gives \(N_r \equiv 1\), then for the positive constant \(M\) defined by

\[
M := \left( \frac{l_r}{\gamma_m} + \frac{\Phi}{\gamma_m} \right),
\]

the controller \(u = -M\text{sign}(\phi_{r-1})\) provides finite-time stability for System (5).

**B. Homogeneous controller with minimum amplitude of discontinuous control at \(z = 0\)**

The amplitude of discontinuous control, in the case of [1], is equal to \(M = \left( \frac{l_r}{\gamma_m} + \frac{\Phi}{\gamma_m} \right)\). We shall now see that this amplitude can be reduced to its minimum level \(M_{min}\) when the state \(z\) tends to zero, by changing the degree of homogeneity.
Proposition 3. For $k \in (-1/r, 0)$ and $A > 0$ satisfying
\[
\max_{V_k(z) \leq A} |\omega_k(z)| \leq l_r,
\]
we define the function

\[
U_{k,A}(z) := \begin{cases} 
\omega_{-1/r}(z) & \text{if } V_k(z) > A, \\
\omega_k(z) & \text{if } V_k(z) \leq A.
\end{cases}
\]

Then the closed-loop system (5) with the controller $u(z) := \frac{1}{\gamma_m} \left( U_{k,A}(z) + \phi \text{sign}(U_{k,A}(z)) \right)$ is stable in finite-time, and $u(z)$ is bounded with minimum amplitude of discontinuity $M_{\min}$ at $z = 0$.

Proof: Consider the following sets
\[
S_1 = \{ z \in \mathbb{R}^r : |\omega_k(z)| \leq l_r \}, \\
S_2 = \{ z \in \mathbb{R}^r : V_k(z) \leq A \}.
\]
According to Condition (25), we have $S_2 \subset S_1$. As $V_{-1/r}(z) < 0, \forall z \notin S_2$, $z$ will reach $S_1$ and $S_2$ successively in finite-time. Once $z \in S_2$, $U_{k,A}(z)$ is equal to $\omega_k(z)$, with $|\omega_k(z)| \leq l_r$. Therefore, $z$ will stay in $S_2$ and converges to zero in finite-time, as $V_k(z) < 0, \forall z \notin S_1, \forall z \neq 0$. Clearly $U_{k,A}(z)$ tends to zero as $z$ tends to zero. As a result
\[
\lim_{\|z\| \to 0} |u(z)| = \frac{\Phi}{\gamma_m} = M_{\min}, \\
\forall z \in \mathbb{R}^r, |u(z)| \leq M_{\min} + \frac{l_r}{\gamma_m}.
\]

C. Fixed-time Homogeneous controller

In certain cases, it is required that the controller converges within a fixed interval of time, irrespective of its initial condition. This can also be achieved by changing the homogeneity degree.

Proposition 4. For $k \in (-1/r, 0)$ and $B > 0$, define
\[
E := \min_{V_k(z) = B} V_{-k}(z) > 0,
\]
and the function

\[
U_{k,B}(z) := \begin{cases} 
\omega_{-k}(z) & \text{if } V_k(z) > B, \\
\omega_k(z) & \text{if } V_k(z) \leq B.
\end{cases}
\]
Then the controller \( u(z) := \frac{1}{\gamma_m} (U_{k,B}(z) + \bar{\phi} \text{sign}(U_{k,B}(z))) \) stabilizes System (5) in fixed-time \( T \leq T_u + T_f \). The values of \( T_u \) and \( T_f \) are given by

\[
T_u = \frac{E_{c+1}^k}{c+1} C \quad \text{and} \quad T_f = \frac{B_{c+1}^{k}}{c+1} C.
\]

**Proof:** The conclusion follows by integrating the differential equation \( \dot{V} = -CV^\alpha \) on appropriate time intervals. Consider first the following sets

\[
S_1 = \{ z \in \mathbb{R}^r : V_{-k}(z) \leq E \} \quad \text{and} \quad S_2 = \{ z \in \mathbb{R}^r : V_k(z) \leq B \}.
\]

According to Condition (27), we get that \( S_1 \subset S_2 \). Clearly, \( z \) will reach \( S_2 \) in a fixed-time, bounded by a constant \( T_u \), calculated as follows:

for \( \alpha = 1 - \frac{k}{c+1} \), \( \int_E^{+\infty} \frac{dV}{V^\alpha} = -C \int_0^{T_u} dt \), then \( T_u = \frac{E_{c+1}^k}{c+1} C \).

When \( z \) reaches \( S_2 \), i.e. \( V_k(z) = B \), \( z \) will converge to zero in a finite-time, bounded by \( T_f \), calculated as follows:

for \( \alpha = 1 + \frac{k}{c+1} \), \( \int_B^0 \frac{dV}{V^\alpha} = -C \int_{T_u}^{T_u+T_f} dt \), then \( T_f = \frac{B_{c+1}^{k}}{c+1} C \).

\[\blacksquare\]

**V. Simulation Results**

In this section, we illustrate the performance of our proposed controllers using the following perturbed triple integrator defined by:

\[
\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3, \quad \dot{z}_3 = \varphi + \gamma u,
\]

with \( \varphi = \sin(t) \) and \( \gamma = 3 + \cos(t) \). Then, we have

\[ \gamma_m = 2, \quad \gamma_M = 4, \quad \varphi = 1. \]

The parameters of the controller are chosen as follows:

\[ l_1 = 1, \quad l_2 = 4, \quad l_3 = 7. \]

We start first by fixing the parameter \( \kappa \) for different values \( \{ \frac{1}{8}, -\frac{1}{8}, -\frac{1}{3} \} \).

For \( \kappa > 0 \), Figure 1 shows a fast convergence of the states to a neighborhood of zero by an unbounded controller, otherwise the convergence to zero is asymptotic. For \( -1/3 < \kappa < 0 \), the convergence of the states to zero in finite-time is obtained by an unbounded controller with a
minimum amplitude of the discontinuous control at \( z = 0 \), as shown in Figure 2. The finite-time convergence of the states is also shown in Figure 3 for \( \kappa = -1/3 \), using a bounded controller with a large discontinuous control at \( z = 0 \).

A bounded controller which ensures a minimum discontinuous control amplitude at zero is shown in Figure 4 by switching \( \kappa \) in neighborhood of zero, from \(-1/3\) to \(-1/8\).

A fixed-time controller is shown in Figure 5. Figure 6 shows that the convergence time will not exceed 8.5 sec for any initial condition.

![Graph 1](a) control law \( u \) versus time (s).  
(b) \( z_1 \) and \( z_2 \) versus time (s).

**Fig. 1.** test for \( \kappa > 0 \)

![Graph 2](a) control law \( u \) versus time (s).  
(b) \( z_1 \) and \( z_2 \) versus time (s).

**Fig. 2.** test for \(-1/r < \kappa < 0\)

![Graph 3](a) control law \( u \) versus time (s).  
(b) \( z_1 \) and \( z_2 \) versus time (s).

**Fig. 3.** test for \( \kappa = -1/r \) (case of [I])
VI. CONCLUSIONS

In this paper, we presented a Lyapunov-based method for designing finite-time convergent controllers for stabilization of perturbed integrator chains. This work is based on a homogeneous controller which stabilize pure integrator chains. The presented controller can stabilize integrator

![Figure 6](image-url)  
**Fig. 6.** Convergence time versus initial condition.
chains of any arbitrary order. It was also shown that the properties of minimum discontinuity amplitude of the controller and fixed-time convergence can be obtained by changing the homogeneity degree of the controller. Future researches consist of studying the possibility of continuous control of homogeneity.

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