Compact fields and mass generation

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Abstract

It is shown that the free propagator of an angular, i.e. compact, field with zero lagrangian mass acquires a nonzero propagator mass \( \omega \) ('kinematical' mass generation). To demonstrate this effect the free propagator of the goldstone boson in an \( O(2) \) model with spontaneous symmetry breaking is calculated. It is shown that this propagator is massive, the mass \( \omega \) being a function of the scalar 'condensate' \( \bar{\phi} \).

1 Introduction

Compact functional integrals appear in a number of field theoretical and statistical models. For instance, all integrals in lattice gauge theories [1] are compact as well as integrals in nonlinear sigma-models, etc. One interesting example is connected with the propagator of the goldstone boson in a theory with spontaneously symmetry breaking.

Let us consider a simple \( d \)-dimensional field theoretical model with \( O(2) \sim U(1) \) global symmetry [2] (see also [3],[4]). Throughout this paper a lattice regularization will be used and \( d \geq 3 \). Let \( \phi_x = \phi_x^{(1)} + i\phi_x^{(2)} \) be a complex scalar field defined on the infinite lattice where components \( \phi_x^{(i)} \) \( (i = 1,2) \) are noncompact, i.e. \( -\infty \leq \phi_x^{(i)} \leq \infty \). The action \( S \) is given by

\[
S = \sum_x \left[ \frac{1}{2} \sum_{\mu} \partial_\mu \phi_x^* \cdot \partial_\mu \phi_x - \frac{m^2}{2} \phi_x^* \phi_x + \lambda (\phi_x^* \phi_x)^2 \right],
\]

where \( m^2 > 0 \), \( \partial_\mu \phi_x = \phi_{x+\mu} - \phi_x \) and the lattice spacing is chosen to be unity. Evidently, action \( S \) is invariant with respect to the transformations \( \phi_x \to e^{i\alpha} \phi_x \) where \( \alpha \) is some constant. This action has an infinite number of minima ('vacua') at \( \phi_x = \bar{\phi} \cdot e^{i\theta} \) where \( \bar{\phi}^2 = m^2/4\lambda \) and \( -\pi < \theta \leq \pi \). The average of any functional \( \mathcal{O}(\phi) \) is

\[
\langle \mathcal{O}(\phi) \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} \prod_z d\phi_z^{(1)} \phi_z^{(2)} \mathcal{O}(\phi) \cdot e^{-S(\phi)},
\]

where \( Z \) is the normalization constant.

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and the partition function $Z$ is defined by $\langle 1 \rangle = 1$. To trace the appearance of the goldstone boson it is convenient to make a change of variables $\phi_x = \rho_x \cdot e^{\theta_x}$ where $0 \leq \rho_x \leq \infty$ and $-\pi < \theta_x \leq \pi$. It is important to note that ranges of the variables $\rho_x$ and $\theta_x$ do not extend over the whole real axis. Evidently, the main contribution to the integral in eq.(1.2) comes from the region $\rho_x \simeq \bar{\phi}$, at least, for large enough $m^4/\lambda$. Making the change of variables $\rho_x = \rho'_x + \bar{\phi}$ one observes that new radial field $\rho'_x$ has a mass term $m^2_x \sum_x \rho'_x^2$, and angle field $\theta_x$ has no mass term (goldstone boson). Rescaling $\theta$-field ($\theta_x \equiv \varphi_x/\bar{\phi}$) one finds the free action of the (massless) $\varphi$-field:

$$S_0(\varphi) = \frac{1}{2} \sum_{x\mu} (\partial_\mu \varphi_x)^2 \ ; \ |\varphi_x| \leq M \equiv \pi \bar{\phi} . \quad (1.3)$$

This field is compact and the corresponding free propagator $\tilde{G}_{xy}$ is given by

$$\tilde{G}_{xy} = \langle \varphi_x \varphi_y \rangle_0 = \frac{1}{Z_0} \int_{-M}^{M} \prod_z d\varphi_z \varphi_x \varphi_y \cdot e^{-S_0(\varphi)} , \quad (1.4)$$

where $Z_0$ is defined by $\langle 1 \rangle_0 = 1$. The problem is to calculate this integral at finite $M$. The same problem arises in perturbative expansion in lattice gauge theories, nonlinear sigma–models, XY–model and others.

It is the aim of this paper to calculate the free propagator defined in eq.(1.4) at $M < \infty$. The next section is dedicated to the integration identities method which permits to calculate compact functional integrals. In the third section the analytical expression for the free correlator $\tilde{G}_{xy}$ at large $M$ is given. Fourth section is dedicated to the numerical calculation of the propagator $\tilde{G}(p)$ at nonzero momenta and the comparison with analytical results. The last section is reserved for summary and discussions.

## 2 Integration identities

Let $O(\varphi)$ be any functional of the compact field $\varphi_x$ and

$$\langle O \rangle_0 = \frac{1}{Z_0} \int_{-M}^{M} \prod_z d\varphi_z \ O(\varphi) \cdot e^{-S_0(\varphi)} . \quad (2.1)$$

Let us make an infinitesimal change of variables

$$\varphi_x \rightarrow \varphi'_x = \varphi_x - \delta f_x , \quad (2.2)$$

where $\delta f_x$ could be any infinitesimal parameters and

$$-M - \delta f_x \leq \varphi'_x \leq M - \delta f_x . \quad (2.3)$$

Then
\[ \mathcal{O}(\varphi) = \mathcal{O}(\varphi') + \delta \mathcal{O}(\varphi') \equiv \mathcal{O}(\varphi') + \sum_x \delta f_x \cdot \delta \mathcal{O}_x(\varphi') + \ldots , \quad (2.4) \]

and

\[ S_0(\varphi) = \frac{1}{2} \sum_{x\mu} (\partial_x \varphi_x)^2 \equiv S_0(\varphi') + \delta S_0(\varphi') ; \]

\[ \delta S_0(\varphi') = \sum_x \delta f_x \cdot \Delta \varphi_x' + \ldots , \quad (2.5) \]

where \( \Delta = -\sum_\mu \partial_\mu \bar{\partial}_\mu \) and \( \bar{\partial}_\mu \varphi_x = \varphi_x - \varphi_x - \mu \).

In the case of finite \( M \) the variation of the limits of integration must be taken into account. For any functional \( \Phi(\varphi) \) one obtains

\[ \int_{-M}^M d\varphi_x \Phi(\varphi) = \int_{-M}^{M-\delta f_x} d\varphi_x' \Phi'(\varphi') = \int_{-M}^M d\varphi_x' \Phi' + \delta' \int_{-M}^M d\varphi_x \Phi'(\varphi) , \quad (2.6) \]

where

\[ \delta' \int_{-M}^M d\varphi_x \Phi'(\varphi) = -\delta f_x \int_{-M}^M d\varphi_x' \Phi' \cdot \left[ \delta(\varphi_x' - M) - \delta(\varphi_x' = -M) \right] + \ldots . \quad (2.7) \]

Therefore,

\[ \delta' \langle \mathcal{O} \rangle_0 \equiv -\sum_x \delta f_x \cdot \left[ \delta(\varphi_x - M) - \delta(\varphi_x + M) \right] \mathcal{O} \rangle_0 + \ldots . \quad (2.8) \]

Finally, one obtains the integration identities

\[ \delta \langle \mathcal{O} \rangle_0 \equiv \langle \delta \mathcal{O} \rangle_0 - \langle \mathcal{O} \cdot \delta S \rangle_0 + \delta' \langle \mathcal{O} \rangle_0 \Rightarrow 0 \quad (2.9) \]

or

\[ \langle \delta \mathcal{O} \rangle_0 - \sum_x \delta f_x \left\{ \Delta_x \langle \varphi_x \mathcal{O} \rangle_0 + \left[ \delta(\varphi_x - M) - \delta(\varphi_x + M) \right] \mathcal{O} \rangle_0 \right\} = 0 . \quad (2.10) \]

In the case of noncompact integrals, i.e. \( M = \infty \), one obtains standard (Ward–Takahashi) identities

\[ \langle \delta \mathcal{O} \rangle_0 - \sum_x \delta f_x \cdot \Delta_x \langle \varphi_x \mathcal{O} \rangle_0 = 0 . \quad (2.11) \]
3 Free propagator

Let us choose \( O = \varphi_y \). Then \( \delta O = \delta f_y = \sum_x \delta f_x \cdot \delta_{xy} \) and from eq.(2.10) one obtains

\[
\Delta_x \tilde{G}_{xy} + \left\langle \left[ \delta(\varphi_x - M) - \delta(\varphi_x + M) \right] \varphi_y \right\rangle_0 = \delta_{xy}, \quad (3.1)
\]

where

\[
\tilde{G}_{xy} \equiv \left\langle \varphi_x \varphi_y \right\rangle_0 = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} d^d p \ e^{i p \cdot (x-y)} \cdot \tilde{G}(p). \quad (3.2)
\]

If \( M = \infty \) then from eq.(2.11) follows

\[
\Delta_x G_{xy} = \delta_{xy}, \quad (3.3)
\]

and the solution is

\[
G_{xy} = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} d^d p \ \frac{e^{i p \cdot (x-y)}}{K^2(p)}; \quad K^2(p) = \sum_{\mu} 4 \sin^2 \frac{p_{\mu}}{2}. \quad (3.4)
\]

This propagator is well-defined at \( d \geq 3 \). Let \( J_x \) be some current. Then

\[
\left\langle e^{i J \varphi} \right\rangle_0 = \exp \left\{ -\frac{1}{2} J \tilde{G} J + \delta F(J; M) \right\}; \quad (3.5)
\]

\[
\delta F(J; M) = \sum_{n \geq 2} \delta F^{(n)}(J; M); \quad \delta F^{(n)}(J; M) = \sum_{\{x^{(i)}\}} \delta F^{(n)}_{x^{(1)}...x^{(2n)}}(M) \prod_{i=1}^{2n} J_{x^{(i)}}. \quad (3.6)
\]

At \( d \geq 3 \)

\[
\tilde{G}_{xy} \to G_{xy}; \quad F^{(n)}_{x^{(1)}...x^{(2n)}}(M) \to 0 \quad (3.7)
\]

when \( M \to \infty \). Therefore, \( \delta F(J; M) \) is a small correction at large \( M \).

Let us calculate the matrix element in eq.(2.10).

\[
\left\langle \left[ \delta(\varphi_x - M) - \delta(\varphi_x + M) \right] \varphi_y \right\rangle_0 = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\xi \ \sin(\xi M) \cdot \frac{\partial}{\partial J_y} \left\langle e^{i J \varphi} \right\rangle_0 \bigg|_{J_x = \xi \delta_{xx}}. \quad (3.7)
\]

At large enough \( M \) one can discard \( \delta F \) in eq.(3.5) and

\[
\left\langle \left[ \delta(\varphi_x - M) - \delta(\varphi_x + M) \right] \varphi_y \right\rangle_0 = \omega^2 \cdot \tilde{G}_{xy}, \quad (3.8)
\]

where
\[ \omega^2 = \frac{M \zeta^3 \sqrt{2}}{\sqrt{\pi}} \cdot e^{-\frac{1}{2} \zeta^2 M^2}, \]  

(3.9)

and

\[ \tilde{\zeta}^2 = \frac{1}{G_0}; \quad \tilde{G}_0 \equiv \tilde{G}_{xx}. \]  

(3.10)

Finally, one obtains

\[ (\Delta_x + \omega^2) \tilde{G}_{xy} = \delta_{xy}, \]  

(3.11)

and the propagator \( \tilde{G} \) is given by

\[ \tilde{G}_{xy} = \frac{1}{(2\pi)^d} \int_{\pi}^{\pi} d^dp e^{ip(x-y)} \frac{e^{-\frac{1}{2}K^2(p) + \omega^2}}{K^2(p)}, \]  

(3.12)

where \( \tilde{\zeta} \) should satisfy

\[ \frac{1}{\zeta^2} = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} d^dp \frac{1}{K^2(p) + \omega^2}. \]  

(3.13)

It is easy to see that in the limit \( M \to \infty \) \( \tilde{\zeta} \to \zeta \) where \( \zeta \) is given by

\[ \frac{1}{\zeta^2} = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} d^dp \frac{1}{K^2(p)}. \]  

(3.14)

4 Numerical calculation of \( \tilde{G}(p) \)

Results of the analytical calculation of the propagator \( \tilde{G} \) defined in eq.(1.4) can be confronted with the results of the numerical study of this propagator on a final lattice. In the previous section this propagator has been analytically studied at large values of \( M \), so that the mass squared \( \omega^2(M) \ll 1 \). On the other side, method Monte Carlo gives a possibility to study this propagator also at small values of \( M \).

The Fourier transform of the propagator

\[ \langle \varphi(p)\varphi(-p) \rangle_0 = \frac{1}{Z_0} \int_{-M}^{M} d\varphi \varphi(p)\varphi(-p) \cdot e^{-\sum_{x\mu}(\partial_\mu \varphi_x)^2} = V \tilde{G}(p) \]  

(4.1)

was calculated numerically on the finite lattice at different values of \( M \) and momenta \( p \), \( V \) being a number of sites. It is convinient to define the effective mass \( \omega_{\text{eff}}(p) \) :

\[ \omega^2_{\text{eff}}(p) = \tilde{G}^{-1}(p) - K^2(p). \]  

(4.2)
Evidently, the propagator $\tilde{G}(p)$ is massive if $\omega^2_{\text{eff}}(p)$ does not depend on $p$.

In Figure 1 the dependence of the effective mass $\omega^2_{\text{eff}}(p)$ on $K^2(p)$ for different $M$ is shown. Solid lines correspond to averages of $\omega^2_{\text{eff}}(p)$ for every $M$. One can see that the dependence on $p$ is practically absent as it was expected.

It is interesting to compare numerically calculated $\omega^2 = \omega^2(M)$ with that given in eq.(3.9) (taking into account $M \rightarrow M\sqrt{2}$ because of the different normalization of the action in eq.(4.1)). In Figure 2 one can see that at $M > 1$ the results of analytical calculation (full circles) converge to $\omega^2$ calculated numerically (open circles).

5 Summary and discussions

It is shown that the free propagator of an angular, i.e. compact, field with zero lagrangian mass acquires at $d \geq 3$ a nonzero dimensionless propagator mass $\omega$ (kinematical mass generation).

To demonstrate this effect the free propagator of the goldstone boson in an $O(2)$ lattice model with spontaneous symmetry breaking is calculated. It is shown that this propagator is massive, the mass $\omega$ being a function of the scalar ‘condensate’ $\phi$. The results of numerical simulations on the finite lattice are in good agreement with the results of analytical calculations.

In the case of spin systems lattice has a physical meaning and no continuum limit is needed. In the case of continuum theory the question of interest is the dependence of the dimensional mass $m_{\text{dim}}$

$$m_{\text{dim}} = \frac{\omega(M)}{a}$$

in the limit $a \rightarrow 0$ where $a$ is lattice spacing.

The appearance of a nonzero (propagator) mass $\omega$, i.e. the effect of kinematical mass generation, could help to resolve the problem of the strong infrared divergencies in 4d QCD at finite temperature. Indeed, in QCD with lattice regularization all integrals are compact. Therefore, $m_{\text{dim}}(a) = \omega/a$ is nonzero and provides a natural regularization of the infrared divergencies at finite spacing $a$. Finally, it can give rise to the infrared cutoff discussed in [6].

In principle, one can speculate that the mechanism of the kinematical mass generation gives a possibility to obtain massive gauge fields without the introduction of higgs bosons.

\footnote{It is worthwhile to note that $\lambda\phi^4$ theory in four dimensions has no nontrivial continuum limit [5].}
References

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Figure 1: Dependence of $\omega_{\text{eff}}^2(M;p)$ on $K^2(p)$ for different $M$. Solid lines correspond to averages $\omega^2(M)$. 
Figure 2: Dependence of $\omega^2(M)$ on $M$. Lines are to guide the eye.