A Formula for the Energy of Circulant Graphs with Two Generators

Justine Louis

Section de Mathématiques, Université de Genève, 1211 Geneva, Switzerland

Correspondence should be addressed to Justine Louis; justine.louis@unige.ch

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We derive closed formulas for the energy of circulant graphs generated by 1 and γ, where γ ≥ 2 is an integer. We also find a formula for the energy of the complete graph without a Hamilton cycle.

Let 1 ≤ γ₁ ≤ · · · ≤ γ_d be integers. The circulant graph C_{γ₁,...,γ_d}^𝑛 on n vertices labelled 0, 1, . . . , n − 1 is the 2D-regular graph such that, for all v ∈ Z/nZ, v is connected to v + γ_i mod n and to v − γ_i mod n, for all i = 1, . . . , d. The adjacency matrix A = (A_ij) of a graph on n vertices is the n×n matrix with rows and columns indexed by the vertices such that A_ij is the number of edges connecting vertices i and j. Let λ_k, k = 1, . . . , n, denote the eigenvalues of the adjacency matrix. The energy of a graph G on n vertices is defined by the sum of the absolute values of the eigenvalues of A; that is,

\[ E(G) = \sum_{k=1}^{n} |\lambda_k|. \]  

(1)

For γ ≥ 3, the energy of the circulant graph C_{1,γ}^𝑛 is given by

\[ E(C_{1,γ}^n) = 4 \sum_{m \in \{1, γ\}} \left( \sum_{k=0}^{\lfloor γ/2 \rfloor - 1} D_{\lfloor (2k+1)n/(2(γ+1)) \rfloor} \left( \frac{2\pi m}{n} \right) - \sum_{k=0}^{\lfloor γ/2 \rfloor - 2} D_{\lfloor (2k+1)n/(2(γ-1)) \rfloor} \left( \frac{2\pi m}{n} \right) \right), \]

(3)

where [x] denotes the greatest integer smaller than or equal to x and \([x]\) denotes the smallest integer greater than or equal to x.

Proof. The adjacency matrix of a circulant graph is circulant; it follows that the eigenvalues of C_{1,γ}^n are given by λ_k = 2 cos(2nk/n) + 2 cos(2γyk/n), k = 0, . . . , n − 1 (see [6]). The energy of C_{1,γ}^n is then given by

\[ E(C_{1,γ}^n) = 2 \sum_{k=0}^{n-1} \left| \cos \left( \frac{2nk}{n} \right) + \cos \left( \frac{2γyk}{n} \right) \right|. \]  

(4)

Let γ = 2. The two roots of the equation \( \cos x + \cos(2x) = 0 \) for \( x \in [0, \pi] \) are π/3 and π. We write the energy as

\[ E(C_{1,2}^n) = 4 + 4 \sum_{k=1}^{\lfloor n/2 \rfloor - 1} \left| \cos \left( \frac{2nk}{n} \right) + \cos \left( \frac{4nk}{n} \right) \right|. \]
\[ E(C^1, \gamma_n) = 4 + 4 \sum_{k=1}^{\lfloor n/6 \rfloor} \left( \cos \left( \frac{2\pi k}{n} \right) + \cos \left( \frac{4\pi k}{n} \right) \right) - 4 \sum_{k=\lceil n/6 \rceil+1}^{n/2} \left( \cos \left( \frac{2\pi k}{n} \right) + \cos \left( \frac{4\pi k}{n} \right) \right). \] (5)

The sum of \( \cos(kx) \) over consecutive \( k \)'s can be expressed in terms of the Dirichlet kernel; namely,
\[ D_n(x) = 1 + 2 \sum_{k=1}^{n} \cos(kx) = \frac{\sin \left( \left( \frac{n+1}{2} \right) x \right)}{\sin \left( \frac{x}{2} \right)}. \] (6)

As a consequence,
\[ 2 \sum_{k=n+1}^{m} \cos(kx) = D_m(x) - D_n(x). \] (7)

The energy of \( C^{1,2}_n \) is thus given by
\[ E(C^{1,2}_n) = 4D_{\lfloor n/6 \rfloor} \left( \frac{2\pi n}{n} \right) + 4D_{\lfloor n/6 \rfloor} \left( \frac{4\pi n}{n} \right) - 2D_{\lfloor n/2 \rfloor-1} \left( \frac{2\pi n}{n} \right) - 2D_{\lfloor n/2 \rfloor-1} \left( \frac{4\pi n}{n} \right). \] (8)

The formula then follows from the fact that, for odd \( n \), \( D_{(n-1)/2}(2\pi m/n) = 0 \) for \( m = 1, 2 \), and, for even \( n \), \( D_{n/2-1}(2\pi m/n) = 1 \) and \( D_{n/2-1}(4\pi m/n) = -1 \).

Hence,
\[ E(C^{1,\gamma}_n) = 2 \sum_{m \in \{1, \gamma\}} \left( D_{\lfloor n/(2(\gamma+1)) \rfloor} \left( \frac{2\pi m n}{n} \right) \right) + \sum_{\gamma < \gamma \leq n} \left( D_{\lfloor (2\gamma+3)n/(\gamma+1) \rfloor} \left( \frac{2\pi m n}{n} \right) - D_{\lfloor (2\gamma+1)n/(\gamma+1) \rfloor} \left( \frac{2\pi m n}{n} \right) \right) \] (9)

\[ - \sum_{\gamma < \gamma \leq n} \left( D_{\lfloor (2\gamma+1)n/(\gamma+1) \rfloor} \left( \frac{2\pi m n}{n} \right) - D_{\lfloor (2\gamma+1)n/(\gamma+1) \rfloor} \left( \frac{2\pi m n}{n} \right) \right). \]

Let \( \gamma \geq 3 \). For odd \( \gamma \), the \( \gamma \) solutions of the equation \( \cos x + \cos \gamma x = 0 \) for \( x \in [0, \pi] \) are given in the increasing order by \( \pi/(\gamma+1), \pi/(\gamma-1), 3\pi/(\gamma+1), 3\pi/(\gamma-1), \ldots, (\gamma-2)\pi/(\gamma-1), \gamma\pi/(\gamma-1) \). For even \( \gamma \), they are given by \( \pi/(\gamma+1), \pi/(\gamma-1), 3\pi/(\gamma+1), 3\pi/(\gamma-1), \ldots, (\gamma-3)\pi/(\gamma-1), (\gamma-1)\pi/(\gamma+1), \pi \).

Let \( n \) be odd. We split the sum over \( k \) of cosines to group the positive terms together and the negative terms together. The energy is given by
\[ E(C^{1,\gamma}_n) = 4 + 4 \sum_{k=1}^{\lfloor n/(2(\gamma+1)) \rfloor} \left| \cos \left( \frac{2\pi k n}{n} \right) + \cos \left( \frac{2\gamma k n}{n} \right) \right| \]
\[ = 4 + 4 \sum_{k=1}^{\lfloor n/(2(\gamma+1)) \rfloor} \left( \cos \left( \frac{2\pi k n}{n} \right) + \cos \left( \frac{2\gamma k n}{n} \right) \right) \]
\[ + 4 \sum_{l=0}^{\lfloor (2\gamma+3)n/(\gamma+1) \rfloor} \left( \cos \left( \frac{2\pi m n}{n} \right) \right) \]
\[ - 4 \sum_{l=0}^{\lfloor (2\gamma+1)n/(\gamma+1) \rfloor} \left( \cos \left( \frac{2\pi m n}{n} \right) \right) \]
\[ + \cos \left( \frac{2\gamma k n}{n} \right) \right). \] (10)

Writing the above relation in terms of Dirichlet kernels, we have
\[ E(C^{1,\gamma}_n) = 2 \sum_{m \in \{1, \gamma\}} \left( D_{\lfloor n/(2(\gamma+1)) \rfloor} \left( \frac{2\pi m n}{n} \right) \right) + \sum_{\gamma < \gamma \leq n} \left( D_{\lfloor (2\gamma+3)n/(\gamma+1) \rfloor} \left( \frac{2\pi m n}{n} \right) - D_{\lfloor (2\gamma+1)n/(\gamma+1) \rfloor} \left( \frac{2\pi m n}{n} \right) \right) \]
\[ - \sum_{\gamma < \gamma \leq n} \left( D_{\lfloor (2\gamma+1)n/(\gamma+1) \rfloor} \left( \frac{2\pi m n}{n} \right) - D_{\lfloor (2\gamma+1)n/(\gamma+1) \rfloor} \left( \frac{2\pi m n}{n} \right) \right). \]

Hence,
\[ E(C^{1,\gamma}_n) = \sum_{m \in \{1, \gamma\}} \left( D_{\lfloor n/(2(\gamma+1)) \rfloor} \left( \frac{2\pi m n}{n} \right) \right) \]
\[ - 4 \sum_{l=0}^{\lfloor (2\gamma+1)n/(\gamma+1) \rfloor} \left( \cos \left( \frac{2\pi m n}{n} \right) \right) \]
\[ - 2D_{\lfloor n/2 \rfloor} \left( \frac{2\pi m n}{n} \right). \] (11)

The formula follows from the fact that \( D_{\lfloor n/2 \rfloor}(2\pi m/n) = 0 \) for \( m = 1, \gamma \).
Expressing it in terms of Dirichlet kernels, we have

$$E(C_{1,\gamma}^n) = 4 + 2 \sum_{m \in \{1, \gamma\}} \left( D_{\lfloor (2y+3)n/(2(\gamma+1)) \rfloor} \left( \frac{2\pi m n}{n} \right) + \sum_{l=0}^{[y/2]-2} D_{\lfloor (2l+3)n/(2(\gamma+1)) \rfloor} \left( \frac{2\pi m n}{n} \right) - D_{\lfloor (2l+1)n/(2(\gamma-1)) \rfloor} \left( \frac{2\pi m n}{n} \right) \right)$$

$$+ \sum_{l=0}^{[y/2]-2} D_{\lfloor (2l+1)n/(2(\gamma-1)) \rfloor} \left( \frac{2\pi m n}{n} \right) - D_{\lfloor (2l+1)n/(2(\gamma+1)) \rfloor} \left( \frac{2\pi m n}{n} \right)$$

$$- 4 \sum_{k=\lfloor (2\gamma/2-1)n/(2(\gamma+1)) \rfloor+1}^{n/2-1} \left( \cos \left( \frac{2\pi k n}{n} \right) + \cos \left( \frac{2\pi \gamma k n}{n} \right) \right)$$

$$+ \cos \left( \frac{2\pi \gamma k n}{n} \right) \right) .

(14)$$

The theorem follows from the fact that $D_{n/2-1}(2\pi m/n) = 1$ for $m = 1, \gamma$. □

A graph is called hyperenergetic if its energy is greater than the one of the complete graph $K_n$. The eigenvalues of the adjacency matrix of $K_n$ are given by $n-1$ and $-1$ with multiplicity $n-1$, so that its energy is given by $E(K_n) = 2(n-1)$.

Figure 1(a) shows how the energy of $C_{1,\gamma}^n$ grows with respect to $n$ for $\gamma = 8$. We see that it is not hyperenergetic and that the energy grows more or less linearly with respect to $n$. Figure 1(b) shows the energy of $C_{n}^{3,\gamma}$ with fixed $n$ as $\gamma$ varies. We observe that the energy stays more or less constant independently of $\gamma$.

As a consequence of the theorem, we can carry out the sum of the Dirichlet kernels when the number of vertices is proportional to $2(\gamma - 1)(\gamma + 1)$.

**Corollary 2.** Given integers $\gamma \geq 3$ and $\alpha \geq 1$, the energy of the circulant graph $C_{2\alpha(n-1)(\gamma+1)}^{1,y}$ is given by

$$E\left(C_{2\alpha(n-1)(\gamma+1)}^{1,y}\right) = 4 \sum_{m \in \{1, \gamma\}} \left( \frac{\sin \left( \pi m \left( \left\lceil y/2 \right\rceil + 1 \right) / \left( 2\alpha \left( y - 1 \right) \right) \right)}{\sin \left( \pi m / (2\alpha \left( y - 1 \right) \left( y + 1 \right)) \right)} \sin \left( \left\lceil y/2 \right\rceil \pi m / (y + 1) \right) \right)$$

$$- \frac{\sin \left( \pi m \left( \left\lceil y/2 \right\rceil - 1 + 1 / (2\alpha \left( y + 1 \right)) \right) / \left( y - 1 \right) \right)}{\sin \left( \pi m / (2\alpha \left( y - 1 \right) \left( y + 1 \right)) \right)} \sin \left( \pi m / (y - 1) \right) .$$

(15)
Proof. Let $a \geq 1$ and $K \geq 0$ be integers. The sum over $k$ of Dirichlet kernels of index $(2k + 1)\alpha$ is given by
\[
\sum_{k=0}^{K} D_{(2k+1)\alpha}(x) = \sum_{k=0}^{K} \frac{\sin(((2k + 1)\alpha + 1/2)x)}{\sin(x/2)}.
\] (16)

By multiplying the summation by $\sin(ax)/\sin(ax)$ and using the trigonometric identity $2\sin\theta\sin\phi = \cos(\theta - \phi) - \cos(\theta + \phi)$, we have
\[
\sum_{k=0}^{K} D_{(2k+1)\alpha}(x) = \frac{\cos(x/2) - \cos(((2K + 2)\alpha + 1/2)x)}{2\sin(x/2)\sin(ax)}.
\] (17)

The corollary then follows by applying the above relation first with $a = \alpha(y - 1)$, $K = \lfloor y/2 \rfloor - 1$ and second with $a = \alpha(y + 1)$, $K = \lfloor y/2 \rfloor - 2$, and $x = 2\pi m/n$, $m \in \{1, y\}$. $\Box$

In [7], the author considered the graphs $K_n - H$, where $K_n$ is the complete graph on $n$ vertices and $H$ is a Hamilton cycle of $K_n$, and asked whether these graphs are hyperenergetic. In [4], the authors showed that the energy of $K_n - H$ is given by
\[
E(K_n - H) = n - 3 + \sum_{k=1}^{n-1} \frac{1 + 2\cos\left(\frac{2\pi k}{n}\right)}{\sin(\pi/n)}.
\] (18)

and that as $n$ goes to infinity, it is hyperenergetic. In the following proposition, we give a formula for it for all $n \geq 3$.

Proposition 3. For all $n \geq 3$, the energy of $K_n - H$ is given by
\[
E(K_n - H) = 2\left(n - 3 - \left(\left\lfloor \frac{2n}{3} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor\right)\right) + 2\sin((\left\lfloor n/3 \right\rfloor + 1/2)2\pi/n) - \sin((\left\lfloor 2n/3 \right\rfloor + 1/2)2\pi/n).
\] (19)

Proof. We have
\[
\sum_{k=1}^{n-1} \frac{1 + 2\cos\left(\frac{2\pi k}{n}\right)}{\sin(\pi/n)} = \sum_{k=1}^{\left\lfloor n/3 \right\rfloor} \left(1 + 2\cos\left(\frac{2\pi k}{n}\right)\right)
- \sum_{k=\left\lfloor n/3 \right\rfloor + 1}^{\left\lfloor 2n/3 \right\rfloor + 1} \left(1 + 2\cos\left(\frac{2\pi k}{n}\right)\right)
- \sum_{k=\left\lfloor 2n/3 \right\rfloor + 1}^{n-1} \left(1 + 2\cos\left(\frac{2\pi k}{n}\right)\right)
= n - 2\left(\left\lfloor \frac{2n}{3} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor\right) + 2D_{\left\lfloor n/3 \right\rfloor}\left(\frac{2\pi}{n}\right)
- 2D_{\left\lfloor 2n/3 \right\rfloor}\left(\frac{2\pi}{n}\right) + D_{n-1}\left(\frac{2\pi}{n}\right).
\] (20)

Since $D_{n-1}(2\pi/n) = -1$, the proposition follows. $\Box$

By elementary analysis, one can show that $E(K_n - H) - 2(n - 1)$ is increasing in $n$. As a consequence, we find that $K_n - H$ are hyperenergetic for all $n \geq 10$. This has been previously found in [4].

Competing Interests
The author declares that there are no competing interests regarding the publication of this paper.

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