On static $n$-body configurations in relativity

Robert Beig$^1$ and Richard M Schoen$^2$

$^1$ Faculty of Physics, Gravity Group, University of Vienna, Vienna, Austria
$^2$ Department of Mathematics, Stanford University, Stanford, CA 94305, USA

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Abstract

The static $n$-body problem of general relativity states that there are, under a reasonable energy condition, no static $n$-body configurations for $n > 1$, provided the configuration of the bodies satisfies a suitable separation condition. In this paper we solve this problem in the case that there exists a closed, noncompact, totally geodesic surface disjoint from the bodies. This covers the situation where the configuration has a reflection symmetry across a noncompact surface disjoint from the bodies.

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1. Introduction and background

A classical result in Newtonian gravity is that there can be no static $n$-body configuration for which the bodies are separated by a plane disjoint from the bodies. On the other hand, one can concoct static 2-body configurations in Newtonian theory [BS] with both bodies being contractible and one body sufficiently non-convex so that the convex hulls of the bodies intersect. Analogous configurations exist for relativistic bodies (work in progress by Andersson, the first author, and Schmidt). For $n > 1$ and assuming a suitable energy condition, it is reasonable to conjecture a relativistic analogue of the Newtonian result stated above; that is, $n$-body static configurations should be impossible provided some separation condition for the bodies is satisfied. The work [Mu] has some results on the static $n$-body conjecture, but no theorem under easily stated conditions. In the present paper we show (see theorem 2.2) that an asymptotically flat triple $(M, V, g)$ with nonnegative scalar curvature which is static vacuum outside a compact set and in a neighborhood of a closed, embedded, noncompact, totally geodesic surface is trivial. This solves the static $n$-body problem in the case that the configuration has a reflection symmetry across a noncompact surface which is disjoint from the matter regions (see theorem 2.3).

Recall that static spacetimes are 4-manifolds with a metric of Lorentz signature which have a Killing vector field that is complete, everywhere timelike and hypersurface orthogonal. General relativity studies such spacetimes subject to the Einstein equations $G_{\mu\nu} = 8\pi GT_{\mu\nu}$ (see [W]). Such solutions describe the gravitational fields of time independent, non-rotating...
sources. Static spacetimes can be written as warped products $\mathbb{R} \times M$ with metric $ds^2$ of the form

$$ds^2 = -V^2(x) \, dt^2 + g_{ij}(x) \, dx^i \, dx^j$$  \hspace{1cm} (1.1)

with $V$ being a positive function and $g$ a Riemannian metric on the 3-manifold $M$. The Einstein equations then take the form

$$\Delta V = 4\pi G V (\rho + \tau)$$  \hspace{1cm} (1.2)

and

$$VR_{ij} - D_i D_j V = 4\pi G V [(\rho - \tau)g_{ij} + 2\tau_{ij}],$$  \hspace{1cm} (1.3)

where $\rho$ and $\tau_{ij} = \tau(ij)$ are respectively the energy density and the stress tensor in the rest system of the matter and $\tau = \tau_i^i$ is the trace. We are interested in solutions to these equations corresponding to $n$ isolated bodies. By this we mean the following: first the 3-manifold $(M, g)$ is asymptotically flat with $V$ tending to 1 at infinity. Second, we assume that the support of the matter fields $\rho, \tau_{ij}$ is contained in $n$ disjoint compact connected sets $\Omega_r$, with $\Omega_r$ open with smooth boundary $\partial \Omega_r$ for $r = 1, \ldots, n$. Finally we assume that all fields are sufficiently smooth (even analytic) except across $\partial \Omega_r$ where $\rho, \tau_{ij}$ and the normal components of $\partial^2 g_{ij}, \partial^2 V$ will in general have jump discontinuities. We require also that $g$ and $V$ be $C^1$ across the boundaries. Let us remark that taking the trace of (1.3) and using (1.2) we recover the time symmetric initial value constraint

$$R = 16\pi G \rho$$  \hspace{1cm} (1.4)

and taking a divergence of (1.3), using (1.2) and the contracted Bianchi identity, we find that

$$D_i (\tau_i^j) + \rho D_i V = 0,$$  \hspace{1cm} (1.5)

which plays the role of equilibrium condition for the matter variables. In order for this condition to hold distributionally across the boundaries we require the additional boundary condition

$$\tau_i^j n_j \big|_{\partial \Omega_r} = 0$$  \hspace{1cm} (1.6)

that is, the stress should have zero normal components to the boundary of the bodies. In many models of continuum mechanics the stress tensor is a functional of a collection of matter fields and their first derivatives, which renders equation (1.5) a quasilinear second order PDE with Neumann-type boundary conditions (1.6). For perfect fluids one has that $\tau_{ij} = \rho g_{ij}$ with $\rho > 0$ in $\Omega_r$ and $\rho$ a given positive non-decreasing function of $\rho$ in $\mathbb{R}^+$. There are different energy conditions which one might impose on the matter variables (see [HE]), the weakest one being that $\rho \geq 0$, which is sufficient for the positive mass theorem [SY] to be valid. Finally one might mention here the case of black holes, in which the regions $\cup_{r=1}^n \Omega_r$ are missing, but instead at the boundaries $V \big|_{\partial \Omega_r} = 0$ with $\partial \Omega_r$ being totally geodesic surfaces.

Historically, the ‘no-body situation’, i.e. $n = 0$, implies that $(M, V, g)$ is trivial (Minkowski) in the sense that $V = 1$ and $(M, g)$ is flat $\mathbb{R}^3$ was the first to be classified. This is the content of a classical result in [L] if $M$ is assumed to be diffeomorphic to $\mathbb{R}^3$ (the proof extends easily to all topologies). After many partial results it was recently shown by Masood-ul-Alam [Ma] that when matter is composed of a perfect fluid we must have $n = 1$ and the spacetime is spherically symmetric; in particular, Schwarzschild in the vacuum region. These spherical models have been studied extensively [HRU]. Solutions for $n = 1$ without (spatial) symmetries, for sources composed of ideally elastic material have been constructed in [ABS]. For black holes it is known that $n$ has to be 1 and the solution is isometric to the exterior of a Schwarzschild black hole. This has been shown in [BM] in the nondegenerate case and in [C] generally.
2. Separating surfaces

Let \((M, g)\) be an asymptotically flat Riemannian three manifold. We allow the possibility that \(M\) has a finite number \(q \geq 1\) of ends \(M_\alpha\), \(1 \leq \alpha \leq q\), each being asymptotically flat. Recall that the static vacuum equations are given by \(VR_{ij} - V_i j = 0\) and \(\Delta V = 0\) for a positive function \(V\) where \(R_{ij}\) denotes the Ricci tensor of \(g\) and \(V_{ij}\) the covariant hessian of \(V\) taken with respect to \(g\). We will be interested here in metrics which are static vacuum solutions outside a compact set, and at the very least have nonnegative scalar curvature everywhere.

We will consider a surface \(S\) which is noncompact, connected and properly embedded in \(M\). We first show that if such a surface is totally geodesic, then it has a finite number of ends each of which is asymptotic to a plane in one of the asymptotically flat ends of \(M\) at infinity.

Precisely, we show that there is a compact subset \(\bigcap (M_\alpha \cap (S \setminus K))\) that the static vacuum equations are given by \(VR_{ij} - V_i j = 0\) and \(\Delta V = 0\) for a positive function \(V\) where \(R_{ij}\) denotes the Ricci tensor of \(g\) and \(V_{ij}\) the covariant hessian of \(V\) taken with respect to \(g\). We will be interested here in metrics which are static vacuum solutions outside a compact set, and at the very least have nonnegative scalar curvature everywhere.

We will first show that if such a surface is totally geodesic, then it has a finite number of ends each of which is asymptotic to a plane in one of the asymptotically flat ends of \(M\) at infinity.

Proposition 2.1. Let \(S\) be a noncompact, connected, totally geodesic surface properly embedded in \(M\). Assume that \(S \cap M_\alpha\) is unbounded in the end \(M_\alpha\). There exist asymptotically flat coordinates defined on \(M_\alpha\) such that outside a compact set \(K\) the surface \(S \cap (M_\alpha \setminus K)\) is the union of \(k_\alpha \geq 1\) graphs of functions \(x^3 = f_p(x^1, x^2)\) for \(1 \leq p \leq k_\alpha\) such that there are constants \(a_p\) so that \(f_p - a_p\) decays like \(1/r'\) and the derivatives of the \(f_p\) decay correspondingly faster, where \(r' = r_\alpha' = \sqrt{(x^1)^2 + (x^2)^2}\). Note that this description holds for each of the ends \(M_\alpha\) for which \(S \cap M_\alpha\) is unbounded and the number \(k_\alpha\) depends on \(\alpha\) as do the coordinates and the functions \(f_p\). (We take \(k_\alpha = 0\) if \(S \cap M_\alpha\) is bounded.) We omit the dependence of the coordinates and the \(f_p\) on \(\alpha\) for notational convenience.

Moreover, for \(\alpha\) sufficiently large the compact subset of \(S\) given by \(S_\alpha = S \cap (K \cup \bigcup_{\alpha=1}^{q} \{r_\alpha' < \alpha\})\) is a compact surface with boundary curve \(C_\alpha\) (having \(k = \sum_{\alpha=1}^{q} k_\alpha\) components) such that the Euler characteristic \(\chi(S_\alpha)\) is equal to \(\chi(S)\) and \(\lim_{r_\alpha' \to \infty} \int_{C_\alpha} \kappa \ ds = 2\pi \kappa\) where \(\kappa\) is the geodesic curvature of the oriented curve \(C_\alpha\) in \(S\).

Proof. Our argument will work separately on each end, so throughout we focus attention on one end \(M_\alpha\) such that \(S \cap M_\alpha\) is unbounded and we omit explicit reference to \(\alpha\) unless needed for clarity. From the work of [B2] there exist coordinates on \(M_\alpha\) defined outside a compact set \(K\) such that \(g\) is equal to a Schwarzschild metric up to order \(r^{-2}\), that is

\[
g_{\alpha ij} = (1 + 2m/r)\delta_{ij} + O(r^{-2}),
\]

where \(m\) is the ADM mass. (We use the notation \(O(r^{-k})\) to denote a term which is bounded by a constant times \(r^{-k}\) and whose derivatives up to a fixed order decay correspondingly faster.) Since \(S\) is embedded and the manifold \(M_\alpha \setminus K\) may be chosen to be simply connected (for example, we can take it to be diffeomorphic to the exterior of a ball in \(\mathbb{R}^3\)) it follows that \(S\) is orientable. We choose the orientation for \(M\) and hence for \(S\) determined by the coordinates \(x^1, x^2, x^3\), and let \(e_1\) and \(e_2\) be an oriented local orthonormal basis for \(S\) relative to the metric \(g\). It then follows that the length \(N\) of the 2-vector \(e_1 \wedge e_2\) with respect to the Euclidean metric is \(1 + O(r^{-1})\). Therefore, using the fact that \(S\) is totally geodesic with respect to \(g\) we have \(D_{e_2}[e_1 \wedge e_2] = 0\) for \(\alpha = 1, 2\). Letting \(\nabla\) denote the Euclidean connection, observe that the difference tensor \(T = D - \nabla\) is of order \(r^{-2}\) since it is given in Euclidean coordinates by the Christoffel symbols of \(g\), so we have

\[
0 = \nabla_{e_2}(e_1 \wedge e_2) + T_{e_2}(e_1 \wedge e_2).
\]

From this we see that \(\nabla_{e_2}(e_1 \wedge e_2) = O(r^{-2})\) and therefore

\[
\nabla_{e_2}N = N^{-1}(\nabla_{e_2}(e_1 \wedge e_2)) \cdot (e_1 \wedge e_2) = O(r^{-2}).
\]
Now the length of the second fundamental form of $S$ with respect to the Euclidean metric is the Euclidean magnitude of $\nabla (N^{-1} e_1 \wedge e_2)$ taken along $S$ (since $N^{-1} e_1 \wedge e_2$ is the Euclidean unit tangent plane), and therefore the length of the Euclidean second fundamental form is $O(r^{-2})$.

Note: The argument above shows that if $\hat{g} = \delta + O(r^{-2})$, then the magnitudes of the second fundamental form of $S$ taken with respect to the indicated metrics satisfy the inequality $|A_\varphi| \leq c|A_\varphi| + cr^{-3}$ since in this case the difference tensor is $O(r^{-3})$.

Let $\sigma_0$ be a radius to be chosen large, and let $M_{a,\sigma}$ denote the part of $M_a$ exterior to the open ball of radius $\sigma \geq \sigma_0$. Let $\epsilon_0 > 0$ and consider the rescaled surface $S(\sigma_0) = \epsilon_0/\sigma_0(S \cap M_{a,\epsilon_0}) \subset \mathbb{R}^3 \setminus B_{\epsilon_0}(0)$. The length of the second fundamental form of $S(\sigma_0)$ is then equal to $\sigma_0/\epsilon_0$ times that of $S$ at corresponding points, and distances are changed by a factor of $\epsilon_0/\sigma_0$, so we see that the second fundamental form of $S(\sigma_0)$ at a point $x$ is bounded by $c(\epsilon_0/\sigma_0)|x|^2$. Since $S$ is connected, we see that either $S(\sigma_0)$ has a single component without boundary or it has $k_\varphi \geq 1$ components $S_\varphi(\sigma_0)$, $1 \leq p \leq k_\varphi$, each with boundary on $\partial B_{\epsilon_0}(0)$. In the former case it follows from proposition 3.1 (following section) that for $\sigma_0$ sufficiently large (hence the second fundamental form small with quadratic decay), $S$ is the graph of a function $f$ over a plane which we may take to be the $x^1x^2$-plane, and that the second derivatives of $f$ decay like $O((r')^{-2})$, and the first derivatives like $O((r')^{-1})$. In the second case proposition 3.1 implies that each of the $S_\varphi(\sigma_0)$ may be so described as the graph of a function $f_p$ with the same decay conditions. Note that since $S$ is embedded each of the $S_\varphi(\sigma_0)$ is a graph over the same plane.

Scaling back to the original surface $S$ we obtain the description of $S \cap (M_a \setminus K)$ as a union of graphs. To get the required decay, we use the Schwarzschild form of the $1/r$ term in the metric expansion. We observe that the metric $\hat{g}$ defined by $\hat{g} = (1 + m/r)^{-2}g$ has the property that $\hat{g} = \delta + O(r^{-2})$. Using the well-known relation for second fundamental forms of conformally related metrics we see

$$A_\varphi = A_\varphi + (1 + m/r)^{-2}\hat{\nu}(1 + m/r)\hat{g},$$

where $\hat{\nu}$ denotes the unit normal of $S$ with respect to $\hat{g}$ and for a function $\varphi$, we use $\hat{\nu}(\varphi)$ to denote the derivative of $\varphi$ in the direction $\hat{\nu}$. Since $A_\varphi = 0$ and from the asymptotic behavior of the $f_p$ we see that on the graph of $f_p$ we have $\hat{\nu}$ is plus or minus $\frac{\partial f_p}{\partial x^3}$, so we have $|A_\varphi| \leq O((r^{-2}))$, and from the fact that first derivatives of $f$ decay like $O((r')^{-1})$ it follows that $f_p$ is bounded by $O(\log r')$. Putting $x^3 = f_p$ in the bound on the second fundamental form, we see that $|A_\varphi| = O((\log r)r^{-3})$. Since the metric $\hat{g}$ is Euclidean up to terms of order $r^{-2}$, we use the note above to improve the decay on the Euclidean second fundamental form to $O((\log r)r^{-3})$. This can then be used to show that $f_p$ is bounded and has a limit ap, at infinity. Putting this information back into the second fundamental form bound tells us finally that the second derivatives of $f_p$ decay like $O((r')^{-3})$, and this implies the desired asymptotic decay.

The final statement on the behavior of the total geodesic curvature follows from the easily checked fact that the geodesic curvature of $C_{\sigma}$ is equal to $1/\sigma + O(\sigma^{-2})$ while the length of each component of $C_{\sigma}$ is equal to $2\pi \sigma + O(1)$. □

**Theorem 2.2.** Assume that $M$ is static vacuum outside a compact set and has $R \geq 0$ everywhere. Suppose there is a closed, noncompact, totally geodesic surface $S$ such that $g$ is static vacuum in a neighborhood of $S$. It follows that $M$ is isometric to the Euclidean space $\mathbb{R}^3$.

**Proof.** Let $V$ be the static potential defined in a neighborhood of $S$ and outside a compact set of $M$. We first show that $V$ is identically 1 on $S$ and that $S$ is flat (zero Gauss curvature). To
see this, we choose a local orthonormal frame so that the $e_\alpha$ are tangential for $\alpha = 1, 2$ and $e_3$ is normal to $S$. We then take the tangential trace of \((1.3)\) to obtain

$$V_{R_{\alpha\alpha}} = V_{\alpha\alpha} = \Delta S V,$$

where we have used the fact that $S$ is totally geodesic to write the trace of the covariant derivatives on $M$ in terms of the intrinsic Laplace operator on $S$. (It would be sufficient here that $S$ be minimal.) Now the Gauss equation tells us that since $S$ is totally geodesic we have

$$R_{\alpha\alpha} = R_{\alpha\beta\alpha\beta} + R_{\alpha\alpha3} = 2K + R_{33},$$

where $K$ is the intrinsic Gauss curvature of the surface $S$. Since $R = 0$ in the static vacuum region due to \((1.2)\), this implies that $R_{33} = -R_{\alpha\alpha}$, and therefore $R_{\alpha\alpha} = K$. Thus we see that the restriction of $V$ to $S$ satisfies the equation

$$\frac{1}{\Delta S} V_{\alpha\alpha} = -K V_{\alpha\alpha} = 0.$$ 

Now we let $S_\sigma$ be as in proposition 2.1, and apply the Gauss–Bonnet theorem to obtain

$$\int_{S_\sigma} K da = 2\pi \chi(S) - \int_{C_\sigma} \kappa ds.$$ 

The totally geodesic condition implies that $K = R_{1212}$ is bounded by a constant times $r^{-3}$, and thus by proposition 2.1, $K$ is an integrable function on $S$. Thus we may let $\sigma$ tend to infinity to conclude $\int_{S_\sigma} K da = 2\pi \chi(S) - 2\pi k \leq 0$ since $k \geq 1$ and the Euler characteristic of any connected noncompact surface is at most 1. On the other hand we have $K = V^{-1} \Delta S V$, so we may also write

$$\int_{S_\sigma} K da = \int_{S_\sigma} V^{-2} |\nabla S V|^2 da + \int_{C_\sigma} V^{-1} \frac{\partial V}{\partial v} ds,$$

where $v$ is the outer unit normal along $C_\sigma$. Since $V$ tends to 1 and the derivatives of $V$ decay at least as fast as $r^{-2}$ it follows that the boundary term goes to 0 as $p$ goes to infinity and we have

$$\int_{S} K da = \int_{S} V^{-2} |\nabla S V|^2 da.$$ 

We therefore conclude that the integral on the right is 0 and hence $V$ is constant on $S$. It follows that $V = 1$ on $S$, and from the equation satisfied for $V$ that $K = 0$ on $S$. It follows moreover that $\chi(S) = 1$, and hence $S$ is isometric to the Euclidean $\mathbb{R}^2$.

Now it is a known asymptotic property of the static equations ([B1, B2]), that there is a constant $m$ so that

$$V = 1 - \frac{m}{r} + o\left(\frac{1}{r^2}\right)$$

and that $m$ is equal to the ADM mass. Thus we have shown that $m$ is zero, so it follows from the positive mass theorem [SY] that $M$ is isometric to the Euclidean $\mathbb{R}^3$. This completes the proof.  

The following result is a consequence of theorem 2.2.

**Theorem 2.3.** A nontrivial relativistic static n-body configuration cannot have a reflection symmetry across a noncompact surface which is disjoint from the bodies.

**Proof.** Assume we had such a configuration with $S$ being the surface fixed by the symmetry $F$. It would then follow that $S$ is totally geodesic since a geodesic $\sigma$ beginning at a point of $S$ and initially tangent to $S$ must remain in $S$ since $F \circ \sigma$ is a geodesic with the same initial conditions and is therefore identical to $\sigma$. The result now follows from theorem 2.2.  

□
3. A technical result for surfaces in $\mathbb{R}^3$

In this section we prove the technical result used in the proof of proposition 2.1. The result is as follows:

**Proposition 3.1.** Assume that $S$ is a closed, connected, noncompact, embedded surface in $\mathbb{R}^3 \setminus B_\alpha$ where $B_r$ denotes the closed ball of radius $r$ centered at the origin. Assume also that for any point $x \in S$ we have $|A(x)| \leq c_\delta |x|^{-2}$ where $A$ denotes the second fundamental form of $S$. If $\varepsilon_0$ and $\delta_0$ are sufficiently small, then there exist Euclidean coordinates $x^1, x^2, x^3$ so that any connected component of $S \cap (\mathbb{R}^3 \setminus B_1)$ is contained in the graph of a function $x^3 = f(x^1, x^2)$ defined for $r' = \sqrt{(x^1)^2 + (x^2)^2} \geq 1/2$ such that the first and second derivatives of $f$ satisfy $|\partial f| \leq c(r')^{-1}$ and $|\partial^2 f| \leq c(r')^{-2}$.

**Proof.** We first consider the case in which $S \cap \partial B_\alpha = \emptyset$. In this case, $S$ is a closed embedded surface in $\mathbb{R}^3$. Let $P \in S$ be a point nearest the origin and note that $|P| > \varepsilon_0$. We choose Euclidean coordinates $y^1, y^2, y^3$ so that $P$ is at the origin and so that $\nu(P) = \frac{\partial}{\partial y^3}$ where $\nu$ denotes the unit normal vector field to $S$. There is a neighborhood of $0$ in $S$ which is the graph of a function $y^3 = f_1(y^1, y^2)$ defined for $\rho' = \sqrt{(y^1)^2 + (y^2)^2} \leq R$ so that $|\partial f_1| \leq 1$. We show that the set of $R$ with this property consists of all positive real numbers, and thus the entire surface $S$ may be so represented. To see this, let $R$ be the largest radius for which such a representation is possible, and use the fundamental theorem of calculus along the ray $\gamma(t) = (ty^1, ty^2, f_1(ty^1, ty^2))$ to write

$$v(y^1, y^2, f_1(y^1, y^2)) - \frac{\partial}{\partial y^3} = \int_0^1 \frac{d}{dt} v(\gamma(t)) \, dt.$$ 

Since $|\partial f_1| \leq 1$ it follows that $|\gamma'(t)| \leq \sqrt{2}\rho'$, and thus we have

$$\left|v(y^1, y^2, f_1(y^1, y^2)) - \frac{\partial}{\partial y^3}\right| \leq \sqrt{2}\rho' \int_0^1 |A(ty^1, ty^2, f_1(ty^1, ty^2))| \, dt.$$ 

Now $|ty^1, ty^2, f_1(ty^1, ty^2)| \geq \rho'$, and thus from the second fundamental form bound we have $|v(y^1, y^2, f_1(y^1, y^2)) - \frac{1}{\sqrt{2}}| \leq c\delta_0(\rho')^{-1}$. It follows that if $\delta_0$ is chosen sufficiently small we have $|\partial f(y^1, y^2)| \leq 1/2$ for $\rho' \leq R$. This contradicts the choice of $R$ as the largest radius for which $|\partial f| \leq 1$. This shows that $S$ is globally given as the graph of a function with gradient bounded by $1$. Therefore from the second fundamental form bound we have $|\partial^2 f_1| \leq c\delta_0(\rho')^{-2}$. It follows by integration as above that the first partials of $f_1$ converge to constants at infinity, and thus we may change coordinates to $x^1, x^2, x^3$ so that $S$ is given as $x^3 = f(x^1, x^2)$ and so that the first derivatives decay like $(r')^{-1}$. This gives the desired conclusion under the assumption that $S \cap \partial B_\alpha = \emptyset$.

Let us now assume that $S \cap \partial B_\alpha \neq \emptyset$. We first analyze the points of $S$ which lie on the unit sphere. Let $P \in S \cap \partial B_1$ and suppose that the tangent plane of $S$ at $P$ does not intersect $B_{2\varepsilon_0}$. If $\delta_0$ is sufficiently small this implies that a large neighborhood of $P$ on $S$ lies arbitrarily close to the tangent plane, and hence does not intersect $B_{\varepsilon_0}$. In this case the argument above implies that a connected component of $S$ is a global graph and hence we must have been in the first case. Therefore it follows that the tangent plane to $S$ at $P$ intersects $B_{2\varepsilon_0}$, and therefore since $\varepsilon_0$ is arbitrarily small, $\nu(P)$ is arbitrarily close to being tangent to the unit sphere. It follows from this that $S$ intersects $\partial B_1$ transversally, and that the curves of intersection have a small geodesic curvature. Since the curve of intersection is embedded, we can see by elementary geometry that it must consist of a finite number of curves all of which lie in a small neighborhood of a great circle with each curve being $C^2$ close to the great circle.
Now if we consider a point $P$ on one of these curves $\gamma$, then we choose coordinates $y_1, y_2, y_3$ so that the point $P$ is $(1, 0, 0)$ and that $v(P) = \frac{\partial}{\partial y_3}$. A neighborhood of $P$ in $S$ may then be represented by the graph $y_3 = f_1(y_1, y_2)$ with $f_1$ of small $C^2$ norm defined over a disk of radius $7/8$ centered at $(1, 0)$. This representation then extends to cover a neighborhood of the curve $\gamma$ by the graph $y_3 = f_1(y_1, y_2)$ defined for $1/4 \leq \rho' \leq 3/2$. If we now consider the largest value of $R$ for which this representation extends to the set $1/4 \leq \rho' \leq R$ with $|\partial f_1| \leq 1$, then we may repeat the argument above to show that $R = \infty$, and thus each of the intersection curves lies on a connected component of $S \cap (R^3 \setminus B_1)$ which has the required description as a graph of a function over the region $r' \geq 1/2$ in the plane. Note that the $1/4$ is replaced by $1/2$ since we need to do a slight rotation of coordinates to make the tangent plane at infinity to be the $x_1 x_2$-plane. We could replace $1/2$ by any fixed small radius $r_0$ by taking $\varepsilon_0$ and $\delta_0$ sufficiently small. Since $S$ is embedded, these planes must be parallel, so the description holds simultaneously for all components in a fixed system of Euclidean coordinates. This completes the proof. □

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References

[ABS] Andersson L, Beig R and Schmidt B G 2008 Static self-gravitating elastic bodies in Einstein gravity Commun. Pure Appl. Math. 61 988–1023
[B1] Beig R 1978/79 Arnowitt–Deser–Misner energy and $g_{00}$ Phys. Lett. A 69 153–5
[B2] Beig R 1980 The static gravitational field near spatial infinity Gen. Rel. Grav. 12 439–51
[BS] Beig R and Schmidt B G 2008 Celestial mechanics of elastic bodies Math. Z. 258 381–94
[BM] Bunting G L and Masood-ul-Alam A K M 1987 Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time Gen. Rel. Grav. 19 147–54
[C] Chruściel P T 1999 The classification of static vacuum spacetimes containing an asymptotically flat spacelike hypersurface with compact interior Class. Quantum Grav. 16 661–87
[HE] Hawking S W and Ellis G F R 1973 The Large Scale Structure of SpaceTime (Cambridge Monographs on Mathematical Physics) vol 1) (London: Cambridge University Press)
[HRU] Heinzle J M, Röhr N and Uggla C 2003 Dynamical systems approach to relativistic spherically symmetric static perfect fluid models Class. Quantum Grav. 20 4567–86
[L] Lichnerowicz A 1955 Théories Relativistes de la Gravitation et de l’électromagnétisme. Relativité générale et théories Unitaires (Paris: Masson et Cie)
[Ma] Masood-ul-Alam A K M 2007 Proof that stellar models are spherical Gen. Rel. Grav. 39 55–85
[Mu] Müller zum Hagen H 1974 The static two body problem Proc. Camb. Phil. Soc. 75 249–60
[SY] Schoen R and Yau S T 1979 On the proof of the positive mass conjecture in general relativity Commun. Math. Phys. 65 45–76
[W] Wald R 1984 General relativity (Chicago, IL: University of Chicago Press)