Hydrological system modeling: Approach for analysis with dynamical systems

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Abstract. Recent methods for sustainability analysis have involved the methodology of system dynamics for the representation of different landscape components that are analyzed. This document proposes a mathematical model based on system dynamics for the hydrological component in the sustainability analysis that includes the processes between precipitation and runoff. Additionally, the stability of the equilibrium point found with the theory of dynamic systems is studied. The modeling is carried out from a diagram of levels and flows, in order to obtain a system of ordinary differential equations of first order, linear, non smooth, that takes as state variables static and capillary storage, superficial storage, higher gravitational storage, lower gravitational storage (aquifer) and storage in the channel. The model was validated with time series of the Chinchina basin (Caldas, Colombia). From dynamical systems point of view, the system behavior has an equilibria point that depends on the saturation value in the static and capillary storage.

1. Introduction

Recent methods for sustainability analysis have involved the methodology of system dynamics for the representation of different landscape components that are analyzed. The purpose of these analyzes is to represent the relationships between the different attributes of the landscape, exhibiting the feedback between the social, environmental and economic landscape components. This document proposes a mathematical model based on system dynamics for the hydrological component that includes the processes between precipitation and runoff from these sustainability analyzes. In this way, an analysis of the stability of equilibrium points with the theory of dynamic systems is performed and data from a hydrological station are used to validate with the model a time series of runoff from a basin.

2. The run-off modeling

A wide variety of hydrological models have been developed for the study of site-specific hydrological phenomena and hydrological cycles [1] estimating, predicting and managing water distribution and fluxes at the soil-atmosphere interface, as a function of various parameters that are used for describing soil and watershed characteristics [2].

The proposed method for the run-off modeling in this article is the system dynamics, which is a method for studying complex and dynamical systems providing a powerful contextual tool to
decision-makers [3–5]. The objectives reported in the state of the art for modeling hydrological systems with system dynamics include sustainable water planning and management, global environmental analysis, performance analysis, risk assessment, water quality monitoring and analysis, reservoir system management, and developing a learning tool for decision-makers [4, 6–8]. The objective of this paper is obtaining a hydrological model that can be included in the sustainability analysis proposed in [9].

The interest of the dynamical systems analysis is to find the equilibria points that allows to obtain interesting conclusions about the behavior of the model in common and extreme situations, for example, [10] and [11]. For understanding the dynamical systems approach of the precipitation-runoff relationship presented in this paper is recommended to check the following references: [12–15].

3. Methodology

The system that we will model corresponds to a portion of the hydrological cycle in which the precipitates run down the basin until reaching a control point or exit point of the basin, as shown in Figure 1.

![Figure 1](image_url)

**Figure 1.** Hydrological system studied: precipitation and runoff relationship.

Figure 1, begins by showing how precipitation falls on vegetation and soil, in what is called capillary storage, which corresponds to water retained by capillary forces between soil particles, and storage in vegetation (static storage). For simplicity these two storages have been considered as one: the static and capillary storage $S_1$, given in millimeters (mm). Static and capillary storage have a limited storage capacity that is known as available water or maximum static
storage capacity $H_u$, given in mm. When the storage $S_1$ exceeds its storage capacity $H_u$, the water flows in excesses that conduct the water towards what is called the surface storage $S_2$ given in mm.

Surface storage can be released by direct runoff, by evaporation, by transpiration and by infiltration into the soil. Direct runoff ends in discharge at the exit of the basin. The evaporation in this article has been considered with the transpiration that occurs in the static and capillary storage, therefore, it will not be considered evapotranspiration in surface storage, nor in any other storage in our representation. Infiltration enters water into the upper subsurface storage of the soil or gravitational storage $S_3$, given in mm, which distributes the flow by percolation or by what is called interflow or subsurface flow. Percolation favors the accumulation of water in the lower subsurface layer of the soil, also called aquifer $S_4$ and given in mm, while the interflow flows into the stream at the outlet of the basin. The aquifer is emptied through the base flow, which carries water to the point of exit from the basin, and through the underground water loss, which are part of the balance of matter of our model. The sum of the runoff, the interflow and the base flow, constitutes the flow at the exit point of the basin, given in millimeters per day (mm/day). These flows represent what in hydrology is known as rapid response, intermediate response and slow response, components of the hydrological response of a basin. The residence times $\tau_i$, $i = 1, \ldots, 3$, given in day, of each storage are different. This residence time is understood as the maximum residence time of the water in each of these storage areas.

![Figure 2. Stocks and flows diagram of the precipitation and runoff relationship.](image-url)
In the model presented in this document have been considered so that the residence time of the surface storage $\tau_1$ is greater than the residence time of the gravitational storage $\tau_2$, which in turn is greater than the time of residence of aquifer storage $\tau_3$. In summary $\tau_3 < \tau_2 < \tau_1$. This formula corresponds to the aggregate version of the open hydrologic simulation model (SHIA) [8]. Additionally, the infiltration rate $\delta_1$ is higher than the percolation rate $\delta_2$, which in turn is higher than the underground loss rate $\delta_3$. In summary $\delta_3 < \delta_2 < \delta_1$. All rates are given in $(\text{day}^{-1})$.

The variables $H_u$, $\delta_1$, $\delta_2$, $\delta_3$, $\tau_1$, $\tau_2$ and $\tau_3$ are soil properties that can be obtained from direct measurement in a timely manner and it is not possible to obtain it for the whole basin; these parameters can be estimated by using indirect information. For this reason and due to the high variability of these values in the natural basins, it is not possible to make a reliable field estimation of these parameters, which is why a calibration and validation process of the model proposed for the system is required that you want to implement.

The interpretation by means of a diagram of levels and flows of Figure 1 that outlines the system that we will model mathematically, is presented in Figure 2. From which the equations that appear in the following lines are obtained.

Will be called the mathematical model of the precipitation and run-off relationship, the next system of ordinary differential equations of the first order (Equation (1)).

\[
\begin{align*}
\dot{S}_1 &= \begin{cases} 
R(t) - \min\{\lambda \text{PET}(t), S_1\} & S_1 \leq H_u \\
-\min\{\lambda \text{PET}(t), S_1\} & S_1 > H_u 
\end{cases} \\
\dot{S}_2 &= \begin{cases} 
-k_1 S_2 & S_1 \leq H_u \\
R(t) - k_1 S_2 & S_1 > H_u 
\end{cases} \\
\dot{S}_3 &= \delta_1 S_2 - k_2 S_3 \\
\dot{S}_4 &= \delta_2 S_3 - k_3 S_4 \\
\dot{S}_5 &= 0
\end{align*}
\]

(1)

Where we have taken for simplicity $k_i = 1/\tau_i$, $i = 1, \ldots, 3$. Note that the Equation (1) for the change of channel storage is zero, which means that in the channel what enters (volumes of water per unit time of runoff, interflow and base flow) is exactly the same to what comes out (the output flow).

4. Results and discussion

The analytical solution is presented in Equation (2). Given that the system of Equation (2) includes two-time series that are the result of field measurements and that the mathematical model is linear, we find the analytical solution of the system (Equation (2)) and calculate the limit when time tends to infinity, to establish the equilibrium points of the system (Equation (2)).
not receive the excess saturation, the other storages can only lose their volumes until empty.

storage in the channel, tend to empty, because indeed if the static and capillary storage does the static and capillary storage tends to be the initial storage plus the accumulation of the saturation value of $H$ (Equation (3)) depends on whether the static and capillary storage tends to infinity, we find the stable equilibrium point (Equation (3)).

When the static and capillary storage $S_1$ does not exceed the saturation value $H_u$ ($S_1 \leq H_u$), the static and capillary storage tends to be the initial storage plus the accumulation of the precipitation $\Omega$ minus the evapotranspiration volume $\lambda \Lambda$. Meanwhile, other storage, except for storage in the channel, tend to empty, because indeed if the static and capillary storage does not receive the excess saturation, the other storages can only lose their volumes until empty.

In the case where static and capillary storage $S_1$ exceeds the saturation value $H_u$ ($S_1 > H_u$); 1) static and capillary storage tends to be the initial volume minus the volume that evapotranspires, 2) Surface storage tends to be the volume accumulated by precipitation, 3) gravitational storage tends to be $\delta_1/k_2$ times the volume accumulated in precipitation, that is, depends on the quotient between the infiltration rate $\delta_1$ and the interflow rate $k_2$, 4) in the equilibrium, the aquifer accumulates $\delta_2/k_3$ times the accumulated in the equilibrium value of the gravitational storage, that is, the accumulation depends on the percolation rate $\delta_2$ and the base flow rate, and 5) the accumulation in the channel in the balance is the initial value of the storage.

From the equilibrium value, an interpretation could also be made of what would happen if the precipitation were to become zero in a basin, that is, in the face of a critical drought. The
precipitation accumulation value $\Omega$ would be null and the equilibrium value of the system would be $(S_1, S_2, S_3, S_4, S_5)_c = (S_1(0) - \lambda A, 0, 0, 0, S_5(0))$, which is interpreted as evapotranspiration of the water content of the static and capillary storage, in addition to the total disappearance of water volumes in the surface, gravitational and aquifer.

5. Evaluation of the model
For the evaluation of the model, we have carried out two different activities: calibration using values in Table 1, and model validation on the Chinchina river basin (Department of Caldas, Colombia), Figure 3.

Table 1. Values of the parameters and initial conditions that were used in the simulations presented in this article.

| Parameter                              | Symbol | Unit | Value     |
|----------------------------------------|--------|------|-----------|
| Potential evapotranspiration factor    | $\lambda$ |      | 0.91      |
| Useful water or maximum capillary storage capacity | $H_u$ | mm   | 350.20    |
| Soil infiltration rate                 | $\delta_1$ | day$^{-1}$ | 0.20     |
| Subsurface percolation rate            | $\delta_2$ | day$^{-1}$ | 0.003   |
| Rate of underground losses in the subsoil | $\delta_3$ | day$^{-1}$ | 0.00002 |
| Residence time on surface             | $\tau_1$ | day | 3.79      |
| Residence time in upper subsurface    | $\tau_2$ | day | 8.28      |
| Residence time in subsurface below    | $\tau_3$ | day | 192.58    |
| Capillary and static storage          | $S_1(0)$ | mm | 317.60    |
| Surface storage                       | $S_2(0)$ | mm | 8.68      |
| Gravitational storage                 | $S_3(0)$ | mm | 25.55     |
| Storage in aquifer                    | $S_4(0)$ | mm | 265.70    |
| Stream storage                        | $S_5(0)$ | mm | 10.00     |

The simulations were obtained by means of the freeware Vensim PLE, using the Runge-Kutta automatic integration method.

![Simulated and real time series of the output flow for the month of December 2010 in the Chinchina river basin.](image-url)
6. Conclusions

An aggregate hydrological mathematical model of precipitation-runoff type, constructed from the interpretation of the hydrological cycle through a diagram of levels and flows, is proposed. The proposed model is a system of differential equations of the first order, linear, non smooth, with five state variables: static and capillary storage, surface storage, upper gravitational storage, lower gravitational storage (aquifer) and water storage. storage in the channel.

The system of differential equations presents a single stable equilibrium point that corresponds to the channel at the outlet of the basin. The point of equilibrium must have been solving the system analytically, which is simple because the system is linear. The procedure consists of applying the limit when time tends to infinite (or less infinite) on the solution of the system and is used when the algebraic system generated by vector fields cannot be solved. In the case of the study, the algebraic system that emerges from the equalization of the vectorial field with zero for the obtaining of equilibrium points, cannot be solved analytically because the equation depends on the precipitation time series \( R(t) \) and the evapotranspiration \( EVT(t) \) with which the model is fed.

The results of the analysis performed on the mathematical model of the Chinchina river basin through the use of dynamical systems, is not conclusive to establish any dependence on the behavior of the system with respect to a particular parameter, although sensitivity is observed in all the parameters. This means that changes in the stability or in the number of equilibria of the system are not expected due to the variation of parameters, that is to say, bifurcations are not obtained for this representation of the hydrological system.

Although the model is linear and does not reproduce exactly the time series of the output flow, it approaches very well, which is very good, given the simplicity of the model, which also allows some level of mathematical treatment.

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