THEOREM OF MATHER ON GENERIC PROJECTIONS FOR SINGULAR VARIETIES

A. ALZATI - E. BALLICO - G. OTTAVIANI

Abstract. The theorem of Mather on generic projections of smooth algebraic varieties is also proved for the singular ones. jet spaces, generic projections.
primary 32C40; secondary 14B05, 14N05.

1. INTRODUCTION

In [1] it appeared a self-contained proof of the following transversality theorem of Mather on generic projections (see [2]) in the setting of Algebraic Geometry:
Theorem (1,1): let $X$ be a smooth subvariety of codimension $c$ of the complex projective space $\mathbb{P}^n$. Let $T$ be any linear subspace of $\mathbb{P}^n$ of dimension $t$ such that $T \cap X = \emptyset$ (so $t \leq c - 1$). For any $i_1 \leq t + 1$ let $X_{i_1} = \{x \in X | \dim((TX)_x \cap T) = i_1 - 1\}$ (the dimension of $\emptyset$ is $-1$). When $X_{i_1}$ is smooth, for any $i_2 \leq i_1$ define $X_{i_1,i_2} = \{x \in X_{i_1} | \dim((TX_{i_1})_x \cap T) = i_2 - 1\}$ and so on; for $i_k \leq \ldots \leq i_2 \leq i_1$ define (when possible) $X_{i_1,\ldots,i_k}$. For $T$ in a Zariski open set of the Grassmanian $Gr(\mathbb{P}^t,\mathbb{P}^n)$, each $X_{i_1,\ldots,i_k}$ is smooth (and so the above definitions are possible) until (increasing $k$) it becomes empty and its codimension $\nu_I$ in $X$ can be calculated (where $I = (i_1, i_2, \ldots, i_k)$).

We refer to [1] for the calculation of $\nu_I$ and for comments and remarks about the theorem.

This theorem was stated for smooth subvarieties of $\mathbb{P}^n$ but the same proof can also be used for the smooth open set $X$ of a singular algebraic variety $Y$ except for the crucial th. 3.15, (p. 409 of [1]), in which the compactness of $X$ is needed.

In this short note we want to replace the proof in [1] with a little longer proof which works also in the case under examination. We obtain the following:
Theorem (1,2): theorem (1,1) still holds if $X$ is replaced with the smooth open subvariety of a possibly singular projective variety $Y$.

2. BACKGROUND

Let $Y$ be a singular algebraic subvariety of the n-dimensional projective space $\mathbb{P}^n$ over the complex numbers. Let $X$ be the smooth open set of $Y$. First of all we outline the proof of Mather’s theorem given in [1] and we introduce some notation.

Fix an integer $t$ with $0 \leq t \leq c - 1$. Let $L$ be a $(n-t-1)$-dimensional linear subspace of $\mathbb{P}^n$.

Let $F = \{P^t \in Gr(\mathbb{P}^t,\mathbb{P}^n) | P^t \cap X = \emptyset \text{ and } P^t \cap L = \emptyset\}$. For any $f \in F$ let $p_f : X \to L$ be the linear projection centered in $f$ and let $j^k p_f$ be its k-jet $(j^k p_f : X \to J^k(X,L))$ sends every $x \in X$ into the k-jet of $p_f$ in $x$, see [1] for the
definition of $J^k(X, L)$). Let $I = (i_1, i_2, ..., i_k)$ be any sequence of integers with $(i_1 \geq i_2 \geq \ldots \geq i_k \geq 0)$.

Let $g : X \times F \to J^k(X, L)$ be given by: $g(x, f) = (j^k p_f)_{x, f}$.

The proof of Mather’s theorem is divided into two steps:

1) define in $J^k(X, L)$ some submanifolds $\Sigma^I$ with the property that $j^k p_f^{-1}(\Sigma^I) = X_f$ (when $X_f$ are defined), this definition is not trivial and it is due to Boardman: $\Sigma^I$ are the so called Thom-Boardman singularities, they are smooth, locally closed and of codimension $\nu_I$;

2) show that there exists a Zariski open set $U \subset F$ such that for any $f \in U$, $j^k p_f : X \to J^k(X, L)$ is transversal to $\Sigma^I$.

The proof of step 1) runs exactly as in [1].

To prove step 2) firstly we remark, (see [1], prop. 3.13), that for any smooth subvariety $W \subset J^k(X, L)$ there exists a Zariski open set $U \subset F$ such that for any $f \in U$, $j^k p_f : X \to J^k(X, L)$ is transversal to $W$ if $g$ is transversal to $W$. Secondly we give the following definition. Let $\varphi : X \to J^k(X, L)$ be a holomorphic map and let $W \subset J^k(X, L)$ be a smooth subvariety, then define:

\[ \delta(\varphi, W, x) = 0 \text{ if } \varphi(x) \notin W \]

\[ \delta(\varphi, W, x) = \dim [J^k(X, L)] - \dim [TW_{\varphi(x)} + d\varphi (TX)_x] \text{ if } \varphi(x) \in W \text{ where } TW \text{ and } TX \text{ are the tangent spaces and } d \text{ stands for the usual differential.} \]

Note that $\delta(\varphi, W, x) \geq 0$ and that $\varphi$ is transversal to $W$ at $x$ if and only if $\delta(\varphi, W, x) = 0$.

As in [1], th. 3.10 and 3.11, it can be shown that for $W = \Sigma^I \subset J^k(X, L)$ the following condition $(*)$ is satisfied:

\[ (*) \delta(g, W, (x, f)) \leq \delta(j^k p_f, W, x) \text{ for any } (x, f) \in X \times F \text{ and equality holds if and only if } \delta(j^k p_f, W, x) = 0. \]

Therefore to prove step 2) all that we need is the following:

**Theorem (2.1):** with the previous notation, assume that condition $(*)$ is satisfied for some smooth subvariety $W \subset J^k(X, L)$; then there exists a Zariski open set $U \subset F$ such that for any $f \in U$, $j^k p_f : X \to J^k(X, L)$ is transversal to $W$.

The proof of this theorem (th. 3.15 in [1]) must be rewritten in our case. In §3 we will give this proof and so we will also prove theorem (1.2).

### 3. Proof of theorem (2.1)

Let us define $\delta_g = \text{Sup}_{(x, f) \in X \times F} \{ \delta(g, W, (x, f)) \}$, moreover let us define $A = \{(x, f) \in X \times F | \delta(g, W, (x, f)) = \delta_g \} \subset X \times F$, $A$ is a Zariski closed set in $X \times F$. Note that if $\delta_g = 0$ theorem (2.1) is true (see th. 3.13 in [1]), so we can assume $\delta_g \neq 0$ and $A \neq \emptyset$.

Let $\pi_2 : X \times F \to F$ be the natural projection. $X \times F$ is equipped with the induced Zariski topology from $Y \times F$. Let $\overline{A}$ be the Zariski closure of $A$ in $Y \times F$; let $\pi_3 : Y \times F \to F$ be the natural projection, $\pi_3(\overline{A})$ is a Zariski closed set of $F$. If $\pi_3(\overline{A})$ is a proper subset of $F$, we can consider $F' = F \setminus \pi_3(\overline{A})$ and $g' = g_{|X \times F'}$. The assumptions of the theorem are true for $F'$ and $g'$ and $\delta_g' < \delta_g$. If the corresponding $\pi_3(\overline{A})$ were a proper subset of $F'$ we would get $F''$ and $g''$ and so on. After a finite number of steps we would get $F'$ and $g'$, for which the assumptions would be still true, with $\delta_g' = 0$, so the theorem would be proved.

Hence we have only to prove that $\pi_3(\overline{A})$ is a proper subset of $F$.

By contradiction let us assume that $\pi_3(\overline{A}) = F$, then $F = \overline{\pi_2(A)}$. 
We can choose \((x_0, f_0) \in A \) and \(z_0 = (j^k g)_{(x_0, f_0)} \in W\). As \(\delta(g, W, (x_0, f_0))\) is strictly positive, by assumption we get that \(\delta(j^k p_{f_0}, W, x_0)\) is strictly positive too, hence \(j^k p_{f_0}\) is not transversal to \(W\) at \(x_0\).

\(W\) is smooth at \(x_0\) so it is a local complete intersection, then it is possible (see [1], proof of th. 3.15) to get a smooth subvariety \(W' \subset J^k(X, L)\) and a smooth dense open Zariski set \(Z \subset X \times F\) such that: \(W' \subset W\), \(\dim(W') - \dim(W) = \delta_g, g\) is transversal to \(W'\) at \((x, f)\) for any \((x, f) \in Z\).

The holomorphic map \(g_{|Z} : Z \to J^k(X, L)\) is transversal to \(W'\) so \(g_{|Z}^{-1}(W') = g^{-1}(W') \cap Z\) is smooth in \(X \times F\).

Let us consider \(\pi = \pi_{(y^*^{-1}(W'), x^*_{|z})} = \pi_{3(y^*^{-1}(W'), x^*_{|z})} : g^{-1}(W') \cap Z \to F\).

It is easy to see that:

1. \(\pi_2(A \cap Z) = F\)

2. \(\pi_2(g^{-1}(W') \cap Z) = F\).

Moreover \(F = \pi_2(g^{-1}(W'))\) and otherwise there would exist a Zariski open set \(B \subset F\) such that \(B \cup \pi_2(g^{-1}(W')) = \emptyset\), hence for any \(f \in B\) and for any \(x \in X, (x, f) \notin g^{-1}(W')\), i.e. \(g(x, f) \notin W'\), i.e. \(g(x, f) \notin W\), i.e. for any \(f \in B\) and for any \(x \in X, \delta(g, W, (x, f)) = 0\) and the theorem would be immediately proved (see th. 3.13 of [1]).

It follows: \(\pi_2(g^{-1}(W') \cap Z) \subseteq \pi_2(g^{-1}(W')) \cap \pi_2(Z) \subseteq \pi_2(g^{-1}(W')) \cap \pi_2(Z) = F\), therefore:

By (1) \([\pi_2(A \cap Z)] \cap D \neq \emptyset\), then we can choose \(f_1 \in [\pi_2(A \cap Z)] \cap D\) such that \(\pi^{-1}(f_1)\) is smooth, of the expected codimension and biholomorphic to a Zariski open set of \(J^k_{f_1} = J^k(p_{f_1})^{-1}(W') \subset X\). We can also choose \(x_1 \in X\) such that \((j^k p_{f_1})^{-1}(W')\) is smooth, of the expected codimension and smooth at \(x_1\), i.e. \(\delta(j^k p_{f_1}, W', x_1) = 0\).

On the other hand \(f_1 \in \pi_2(A \cap Z),\) hence it is possible to choose \(x_1 \in X\) such that \((x_1, f_1) \in A\), i.e. \(\delta(g, W, (x_1, f_1)) = \delta_g\).

Let \(z_1 = (j^k p_{f_1})_{x_1}\) then:

\[\delta(j^k p_{f_1}, W', x_1) = \dim[j^k(X, L)] - \dim(TW')_{z_1} + d(j^k p_{f_1}, TX)_{x_1}\]
\[\delta(j^k p_{f_1}, W, x_1) = \dim[j^k(X, L)] - \dim(TW)_{z_1} + d(j^k p_{f_1}, TX)_{x_1}\]

and \(0 = \delta(j^k p_{f_1}, W', x_1) \geq \delta(j^k p_{f_1}, W, x_1) - \delta_g\).

But assumption (*) and the fact that \((x_1, f_1) \in A\) imply:

\[0 \geq \delta(j^k p_{f_1}, W, x_1) - \delta_g > \delta(g, W, (x_1, f_1)) - \delta_g = \delta_g - \delta_g = 0,\] contradiction!

4. Cones

In this brief section we want to remark that when \(Y\) is a cone it is possible to use Mather’s theorem (1.1).

For instance let us assume that \(Y\) is a cone in \(\mathbb{P}^n\) of vertex \(V\) on a smooth subvariety \(B\) of \(\mathbb{P}^n\) whose span is \(\mathbb{P}^s\) with \(\dim(Y) = \gamma = b + v + 1, \dim(B) = b, \dim(V) = v, s = v + 1\).

Let \(T\) be a generic \(t\)-dimensional subspace of \(\mathbb{P}^n\) with: \(T \cap Y = \emptyset, t \leq -1, \gamma \geq (t + 1)(n - \gamma)\). Let \(Y_{t+1} = \{ y \in Y \} y\) is a smooth point, \((TY)_{y} > T\).

If \(Y\) were smooth Mather’s theorem (1.1) would say that, for generic \(T, Y_{t+1}\) is a smooth subvariety of \(Y\) and \(\dim(Y_{t+1}) = \gamma - (t + 1)(n - \gamma)\), in our case we have:
Proposition: the closure of $Y_{t+1}$ is a cone of dimension $\gamma - (t+1)(n-\gamma)$ with vertex $V$ over a smooth variety.

As $Y$ is a cone we remark that $t \leq s - 1$ ($t \leq n - \gamma - 1 = s - b - 1$), hence there exists a linear subspace $H \cong \mathbb{P}^s$ in $\mathbb{P}^n$ such that $H \supset T$ and $H \cap V = \emptyset$. We can assume that $B = H \cap Y$ and we can apply th. (1.1) to $\mathbb{P}^s$, $T$ and $B$ as $B$ is smooth, $T \cap B = \emptyset$ and $T$ is generic in $\mathbb{P}^s$ with respect to $B$. If $t \leq b - 1$ and $b \geq (t+1)(s-b)$ (for instance when $t = 0$ and $2b \geq s$) then $B_{t+1} = \{y \in B | (TB)_y \supset T\}$ is a smooth subvariety of $B$ and $\dim(B_{t+1}) = b - (t+1)(s-b)$. On the other hand $(TB)_y \supset T$ if and only if $(TY)_y \supset T$ as $(TY)_y = \langle V, (TB)_y \rangle$ i.e. $(TB)_y = (TY)_y \cap H$, hence $Y_{t+1} \cap H = B_{t+1}$ and the closure in $Y$ of $Y_{t+1}$ is another cone of vertex $V$ over $B_{t+1}$. This cone has dimension $b - (t+1)(s-b) + v + 1 = \gamma - (t+1)(n-\gamma)$ which is exactly the expected dimension when $Y$ is smooth.

5. References

[1] A. Alzati, G. Ottaviani: "The theorem of Mather on generic projections in the setting of Algebraic Geometry" Manuscr. Math. 74 391-412 (1992).
[2] J. N. Mather: "Generic projections" Ann. of Math. 98 226-245 (1973).

All authors are members of Italian GNSAGA. Work supported by Murst funds.

Addresses:
A. Alzati: Dip. di Matematica Univ. di Milano, via C. Saldini 50 20133-Milano (Italy).
E-mail: alzati@mat.unimi.it

E. Ballico: Dip. di Matematica Univ. di Trento, via Sommarive 14 38050-Povo Trento (Italy).
E-mail: ballico@science.unitn.it

G. Ottaviani: Dip di Matematica Univ. di Firenze, viale Morgagni 67/A 50134-Firenze (Italy).
E-mail: ottavian@udini.math.unifi.it