Precision QED⊗QCD Resummation Theory for LHC Physics: Status and Update†

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Abstract

We present the elements of the IR-improved DGLAP-CS theory as it relates to the new MC friendly exponentiated scheme for precision calculation of higher order corrections to LHC physics in which IR singularities from both QED and QCD are canceled to all orders in \(\alpha\) and in \(\alpha_s\) simultaneously in the presence of rigorous shower/ME matching. We present the first MC data comparing the implied new showers themselves with the standard ones using the HERWIG6.5 MC event generator as a test case at LHC energies. As expected, the IR-improved shower re-populates lower values of the energy fraction \(z\) and lower values of the attendant \(p_T\) for the standard HERWIG6.5 input parameters. Possible phenomenological implications are discussed.

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1 Introduction

With the advent of the LHC, we enter the era of precision QCD, by which we mean predictions for QCD processes at the total theoretical precision tag of 1% or better. The attendant requirement for this theoretical precision is control of the $O(\alpha_s^2 L^{n_1}, \alpha_s \alpha L^{n_2}, \alpha^2 L^{n_3})$, $n_1 = 0, 1, 2$, $n_2 = 1, 2$, $n_3 = 2$ corrections in the presence of realistic parton showers, on an event-by-event basis – here, $L$ is a generic big log. This is the objective of our approach to precision QCD theory, which for example will be needed for the expected 2% experimental precision [1] at the LHC for processes such as

$$pp \rightarrow V + m(\gamma) + n(G) + X \rightarrow \ell\ell' + m'(\gamma) + n(G) + X, \ V = W^\pm, Z, \text{ and } \ell = e, \mu, \ell' = \nu_e, \nu_\mu.$$ 

Here, we present the elements of our approach and its recent applications in MC event generator studies, which are still preliminary.

At such a precision as we have as our goal, issues such as the role of QED are an integral part of the discussion and we deal with this by the simultaneous resummation of QED and QCD large infrared effects, $QED \otimes QCD$ resummation [2] in the presence of parton showers, to be realized on an event-by-event basis by Monte Carlo methods. This is reviewed in the next Section. Let us note already that in Refs. [3] it has been shown that QED evolution enters at the $\sim 0.3\%$ level for parton distributions and that in Refs. [4] it has been shown that EW (large Sudakov logs, etc.) effects at LHC energies, as W’s and Z’s are almost massless on the TeV scale, can enter at the several % level – such corrections must treated systematically before any claim of 1% precision can be taken seriously. We are presenting a framework in which this can be done. The new amplitude-based resummation algebra then leads to a new scheme for calculating hard hadron-hadron scattering processes, IR-improved DGLAP-CS theory [5] for parton distributions, kernels, reduced cross sections with the appropriate shower/ME matching. This is summarized in Section III. In this latter Section, with an eye toward technical precision cross checks plus possible physical effects of heavy quark masses, we also deal with the issue of quark masses as collinear regulators [6–8] as an alternative [9] to the usual practice of setting all initial state quark masses to zero in calculating ISR (initial state radiation) effects in higher order QCD corrections. We also discuss in Section III the relationship between our resummation algebra and that of Refs. [10,11], as again such comparisons will be necessary in assessing the ultimate theoretical precision tag. In Section IV, we illustrate recent results we have obtained for the effects of our new approach on the parton showers as they are generated with the HERWIG6.5 MC [12]. Extensions of such studies to PYTHIA [13] and MC@NLO [14] are in progress. Section V contains summary remarks.

As a point of reference, in Ref. [15] it has been argued that the current state-of-the-art theoretical precision tag on single Z production at the LHC is $(4.1 \pm 0.3)\% = (1.51 \pm 0.75)\%(QCD) \oplus 3.79(PDF) \oplus 0.38 \pm 0.26(EW)\%$, where the results of Refs. [14,16–21] have been used in this precision tag determination.

2 $QED \otimes QCD$ Resummation

In refs. [2], we have extended the YFS theory to the simultaneous exponentiation of the large IR terms in QCD and the exact IR divergent terms in QED, so that for the prototypical subprocesses $\bar{q}q \rightarrow$
\[ Q''\nu + m(G) + n(\gamma) \] we arrive at the new result

\[
d\tilde{\sigma}_{\text{exp}} = e^{\text{SUM}_{\text{IR}}(\text{QCD})} \sum_{m,n=0}^{\infty} \frac{1}{mn!} \int \prod_{j_1=1}^{m} \frac{d^3k_{j_1}}{k_{j_1}} \prod_{j_2=1}^{n} \frac{d^3k'_{j_2}}{k'_{j_2}} \int \frac{d^4y}{(2\pi)^4} e^{i\nu(p_1 + q_1 - p_2 - q_2 - \sum k_{j_1} - \sum k'_{j_2}) + D_{\text{QCD}}} \]

where the new YFS [23, 24] residuals, defined in Ref. [2], \( \tilde{\beta}_{m,n}(k_1, \ldots, k_m; k'_1, \ldots, k'_n) \), with \( m \) hard gluons and \( n \) hard photons, represent the successive application of the YFS expansion first for QCD and subsequently for QED. The functions \( \text{SUM}_{\text{IR}}(\text{QCD}), D_{\text{QCD}} \) are determined from their analogs \( \text{SUM}_{\text{IR}}(\text{QCD}), D_{\text{QCD}} \) in Ref. [25] via the substitutions

\[
B_{\text{QCD}}^{\text{nls}} \rightarrow B_{\text{QCD}}^{\text{nls}} + D_{\text{QCD}},
\]

\[
\tilde{S}_{\text{QCD}} \rightarrow \tilde{S}_{\text{QCD}} + \tilde{S}_{\text{QED}} \equiv \tilde{S}_{\text{QED}}
\]

everywhere in expressions for the latter functions given in Refs. [25] – see Ref. [2] for the details of this substitution. It can be readily established [2] that the QCD dominant corrections happen an order of magnitude earlier in time compared to those of QED so that the leading term \( \tilde{\beta}_{0,0} \) already gives us a good estimate of the size of the effects we study.

Important in any total theoretical precision is knowledge of possible systematic issues associated with one methods. This entails the relationship between different approaches to the same classes of corrections and moves us to the relationship between our approach to QCD resummation and the more familiar approach in Refs. [10]. It has been shown in Ref. [26] that the latter approach is entirely equivalent to the approach in Refs. [11]. Establishing the relationship between our approach and that in Refs. [10] will then suffice to relate all three approaches.

In Ref. [27] the more familiar resummation for soft gluons in Refs. [10] is applied to a general \( 2 \rightarrow n \) parton process \( \{f\} \) at hard scale \( Q \), \( f_1(p_1, r_1) + f_2(p_2, r_2) \rightarrow f_3(p_3, r_3) + f_4(p_4, r_4) + \cdots + f_{n+2}(p_{n+2}, r_{n+2}) \), where the \( p_i, r_i \) label 4-momenta and color indices respectively, with all parton masses set to zero to get

\[
\mathcal{M}_{\{r_i\}}^{[f]} = \sum_{L} \mathcal{M}_{L}^{[f]}(c_L)_{\{r_i\}}
\]

where repeated indices are summed, \( J^{[f]} \) is the jet function, \( S_{L} \) is the soft function which describes the exchange of soft gluons between the external lines, and \( H_L^{[f]} \) is the hard coefficient function. The attendant infrared and collinear poles are calculated to 2-loop order. To make contact with our approach, identify in \( Q'Q \rightarrow Q''Q'' + m(G) \) in \( \{f\} \) \( f_1 = Q, Q', f_2 = Q', f_3 = Q'', f_4 = Q'', \{f_5, \ldots, f_{n+2}\} = \{G_1, \ldots, G_m\} \) so that \( n = m + 2 \) here. Observe the following:

- By its definition in eq.(2.23) of Ref. [27], the anomalous dimension of the matrix \( S_{L} \) does not contain any of the diagonal effects described by our infrared functions \( \Sigma_{IR}(QCD) \) and \( D_{QCD} \).
- By its definition in eqs.(2.5) and (2.7) of Ref. [27], the jet function \( J^{[f]} \) contains the exponential of the virtual infrared function \( \alpha_s B_{QCD} \), so that we have to take care that we do not double count when we use \( B \) in \( \{f\} \) and the equations that lead thereto.
It follows that, referring to our analysis in Ref. [28], we identify \( \tilde{\rho}^{(m)} \) in eq.(73) in this latter reference in our theory as

\[
\tilde{\rho}^{(m)}(p_1, q_1, p_2, q_2, k_1, \ldots, k_m) = \sum_{\text{colors}, \text{spins}} |\mathcal{M}_{\{r_i\}}^{(f)|}|^2 \\
\equiv \sum_{\text{spins}, \{r_i\}} b'_{\{r_i\}} |\tilde{J}^{(f)}|^2 \sum_c \sum_{L=1}^c S_{L/P}^{(f)} H_{L/P}^{(f)} (CL_{\{r_i\}}) \left( S_{L/P}^{(f)} H_{L/P}^{(f)} (CL'_{\{r_i\}}) \right)^\dagger,
\]

where here we defined \( \tilde{J}^{(f)} = e^{-\alpha_s R_{QCD}} J^{(f)} \), and we introduced the color-spin density matrix for the initial state, \( b'_{\{r_i\}} \). Here, we recall (see Refs. [5, 28], for example) that in our theory, we have

\[
d\tilde{\sigma} = \frac{e^{2\alpha_s R_{QCD}}}{n!} \int \prod_{m=1}^n \frac{d^3k_m}{(k_m^2 + \lambda^2)^{1/2}} \delta(p_1 + q_1 - p_2 - q_2 - \sum_{i=1}^n k_i) \\
\tilde{\rho}^{(n)}(p_1, q_1, p_2, q_2, k_1, \ldots, k_n) \frac{d^4p_3 d^4q_2}{p_3^2 q_2^2},
\]

for n-gluon emission. It follows that we can repeat thus our usual steps (see Ref. [5, 28]) to get the QCD corrections in our formula (11), without any double counting of effects. This use of the results in Ref. [27] is in progress.

## 3 IR-Improved DGLAP-CS Theory: Applications

In Refs. [5, 28] it has been shown that application of the result (11) to all aspects of the standard formula for hard hadron-hadron scattering processes,

\[
\sigma = \sum_{i,j} \int dx_1 dx_2 F_i(x_1) F_j(x_2) \tilde{\sigma}(x_1 x_2 s)
\]

where we the \( \{F_i(x)\} \) and \( \tilde{\sigma} \) denote the parton densities and reduced cross section respectively, leads one to its application to the DGLAP-CS theory itself for the kernels which govern the evolution of the parton densities in addition to the the implied application to the respective hard scattering reduced cross section. The result is a new set of IR-improved kernels [5],

\[
P_{qq}(z) = C_F F_{YS}(\gamma_q) e^{\frac{1}{2} \delta_q} \frac{1+z^2}{1-z} (1-z)^{\gamma_q} - f_q(\gamma_q) \delta(1-z),
\]

\[
P_{Gq}(z) = C_F F_{YS}(\gamma_q) e^{\frac{1}{2} \delta_q} \frac{1+(1-z)^2}{z} z^{\gamma_q},
\]

\[
P_{GG}(z) = 2C_F F_{YS}(\gamma_g) e^{\frac{1}{2} \delta_g} \frac{1}{z} \left\{ \frac{1-z}{z} z^{\gamma_g} + \frac{z}{1-z} (1-z)^{\gamma_g} \\
+ \frac{1}{2} \left( z^{1+\gamma_g} (1-z) + z (1-z)^{1+\gamma_g} \right) - f_G(\gamma_G) \delta(1-z) \right\},
\]

\[
P_{gG}(z) = F_{YS}(\gamma_G) e^{\frac{1}{2} \delta_q} \frac{1}{2} \left\{ z^2 (1-z)^{\gamma_G} + (1-z)^2 z^{\gamma_G} \right\}.
\]

in the standard notation, where

\[
\gamma_q = C_F \frac{\alpha_s}{\pi} = \frac{4 C_F}{\beta_0}
\]

\[
\delta_q = \frac{\gamma_q}{2} + \frac{\alpha_s C_F}{\pi} (\frac{\pi^2}{3} - \frac{1}{2})
\]

\[
\gamma_G = C_G \frac{\alpha_s}{\pi} = \frac{4 C_G}{\beta_0}
\]

\[
\delta_G = \frac{\gamma_G}{2} + \frac{\alpha_s C_G}{\pi} (\frac{\pi^2}{3} - \frac{1}{2})
\]
and

$$F_{YFS}(\gamma_q) = \frac{e^{-C_E \gamma_q}}{\Gamma(1 + \gamma_q)},$$

so that

$$f_q(\gamma_q) = \frac{2}{\gamma_q} - \frac{2}{\gamma_q + 1} + \frac{1}{\gamma_q + 2}$$

$$f_G(\gamma_G) = \frac{n_f}{C_G (1 + \gamma_G)(2 + \gamma_G)(3 + \gamma_G) + \frac{1}{\gamma_G(1 + \gamma_G)(2 + \gamma_G)}}$$

$$+ \frac{1}{(1 + \gamma_G)(2 + \gamma_G)} + \frac{2}{1}/(3 + \gamma_G)(4 + \gamma_G)$$

$$+ \frac{1}{(2 + \gamma_G)(3 + \gamma_G)(4 + \gamma_G)}.$$ (19)

Here, $C_E = 0.5772...$ is Euler’s constant and $\Gamma(w)$ is the Euler Gamma function. We see that the kernels are integrable at the IR endpoints and this admits a more friendly MC implementation, which is in progress.

Some observations are in order. First, we note that the connection of (10) with the higher order kernel results in Refs. [29] is immediate and has been shown in Refs. [5, 28]. Second, there is no contradiction with the standard Wilson expansion, as the terms we resum are not in that expansion by its usual definition. Third, we do not change the predicted cross section: we have a new scheme such that the cross section in (6) becomes

$$\sigma = \sum_{i,j} \int dx_1 dx_2 F'_i(x_1) F'_j(x_2) \hat{\sigma}(x_1 x_2 s)$$ (20)

order by order in perturbation theory, where $\{P^{\text{exp}}\}$ factorize $\hat{\sigma}_{\text{unfactorized}}$ to yield $\hat{\sigma}'$ and its attendant parton densities $\{F'_i\}$. Fourth, when one solves for the effects of the exponentiation in (10) on the actual evolution of the parton densities from the typical reference scale of $Q_0 \sim 2\text{GeV}$ to $Q = 100\text{ GeV}$ one finds [5, 28] shifts of $\sim 5\%$ for the NS $n=2$ moment for example, which is thus of some phenomenological interest– see for example Ref. [30]. Finally, we note that we have used [2] the result [11] for single Z production with leptonic decay at the LHC (and at FNAL) to focus on the ISR alone, for definiteness and we find agreement with the literature in Refs. [31–33] for exact $O(\alpha)$ results and Refs. [34–36] for exact $O(\alpha_s^2)$ results, with a threshold QED effect of 0.3%, similar to that found for the parton evolution itself from QED in Refs. [3]. Evidently, any 1% precision tag must account for all such effects.

3.1 Shower/ME Matching

In using (11) in (20) for $\hat{\sigma}'(x,x_j)$, we intend to combine our exact extended YFS calculus with HERWIG [12] and PYTHIA [13] as follows: they generate a parton shower starting from $(x_1, x_2)$ at the factorization scale $\mu$ after this point is provided by the $\{F'_i\}$ and we may use [2] either a $p_T$-matching scheme or a shower-subtracted residual scheme where the respective new residuals $\{\hat{\beta}_{n,m}(k_1, \ldots, k_n; k'_1, \ldots, k'_m)\}$ are obtained by expanding the shower formula and the result in (11) on product and requiring the agreement with exact results to the specified order. This combination of theoretical constructs can be systematically improved with exact results order-by-order in $\alpha_s, \alpha_s$ with exact phase space. The recently developed new parton evolution algorithms in Refs. [38] may also be used here.

The issue of the non-zero quark masses in the initial state radiation is present when one wants 1% precision, as we know that the parton densities for the heavy quarks are all different and the generic size

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2 See Ref. [37] for a realization of the shower subtracted residual scheme in the context of QED parton showers.

3 The current state of the art for such shower/ME matching is given in Refs. [14], which realizes exactness at $O(\alpha_s)$. 

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of mass corrections for bremsstrahlung is $\alpha_s/\pi$ for cross sections [39], so that one would like to know whether regularizing a zero-mass ISR radiation result with dimensional methods, carrying through the factorization procedure gives the same result as doing the same calculation with the physical, non-zero mass of the quark and again carrying through the factorization procedure to the accuracy $\alpha_s^2/\pi^2$, for example. Until the analysis in Ref. [9], this cross check was not possible because in Refs. [6, 7] it was shown that there is a lack of Bloch-Nordsieck cancellation in the ISR at $O(\alpha_s^2)$ unless the radiating quarks are massless. The QCD resummation algebra, as used in [11], allows us to obviate [9] this theorem, so that now such cross checks are possible and they are in progress.

3.2 Sample MC data: IR-Improved Kernels in HERWIG6.5

We have preliminary results on IR-improved showers in HERWIG6.5: we compare the z-distributions and the $p_T$ of the IR-improved and usual DGLAP-CS showers in the Figs. 1, 2, 3. As we would expect, the IR-improved shower re-populates the soft region in both variables. The details of the implementation procedure and the respective new version of HERWIG6.5, HERWIG6.5-YFS, will appear elsewhere [40]. The analogous implementations in PYTHIA and MC@NLO are in progress, as are comparisons with IR safe observables.
4 Conclusions

The theory of Ref. [23] extends to the joint resummation of QED and QCD with proper shower/ME matching built-in. For the simultaneous QED⊗QCD resummed theory, full MC event generator realization is open: a firm basis for the complete $O(\alpha_s^2, \alpha_s, \alpha^2)$ MC results needed for precision LHC physics has been demonstrated and all the latter are in progress – see Refs. [41] for new results on $\epsilon$ expansions for the higher order Feynman integrals needed to isolate the residuals in our approach for example. This allows cross check between residuals isolated with the quark masses as regulators, something now allowed by the result in Ref. [9], and those isolated in dimensional regularization for the massless quark limit. Such cross checks are relevant for precision QCD theory. The first MC data have been shown with IR-improved showers in HERWIG6.5. The spectra are softer as expected. We look forward to the detailed comparison with IR safe observables as generated with IR-improved and with the usual showers – this will appear elsewhere. [40]. Already, semi-analytical results at the $\beta^{0.0}_{0.0}$ are consistent with the literature on single Z production, while a cross check for the analogous W production is near. As the QED is at 0.3% at threshold, it is needed for 1% precision.

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