First law of compact binary mechanics with gravitational-wave tails

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Abstract

We derive the first law of binary point-particle mechanics for generic bound (i.e. eccentric) orbits at the fourth post-Newtonian (4PN) order, accounting for the non-locality in time of the dynamics due to the occurrence of a gravitational-wave tail effect at that order. Using this first law, we show how the periastron advance of the binary system can be related to the averaged redshift of one of the two bodies for a slightly non-circular orbit, in the limit where the eccentricity vanishes. Combining this expression with existing analytical self-force results for the averaged redshift, we recover the known 4PN expression for the circular-orbit periastron advance, to linear order in the mass ratio.

Keywords: gravitational waves, compact binary system, post-Newtonian theory, Hamiltonian mechanics

1. Introduction

Analytic approximation methods in General Relativity, such as the post-Newtonian (PN) approximation \cite{1-4}, gravitational self-force (GSF) theory \cite{5-7}, and the effective one-body (EOB) model \cite{8}, play an important role both in the data analysis of gravitational waves, and for comparisons with the results from numerical relativity (NR) simulations \cite{9, 10}. Recently, significant progress has been achieved on the derivation of the equations of motion of binary systems of compact objects at the fourth post-Newtonian (4PN) order, using the Arnowitt–Deser–Misner (ADM) canonical Hamiltonian formalism in ADM-TT coordinates \cite{11-15}, the Fokker action approach in harmonic coordinates \cite{16, 17}, and effective field theory (EFT) methods \cite{18-20}. The next objective is to compute the gravitational radiation field at the 4PN
order (beyond the lowest order quadrupole radiation). So far, specific high-order tail effects in the waveform and energy flux have already been computed [21].

Gravitational wave tails also form an integral part of the conservative dynamics starting at the 4PN order (beyond the Newtonian motion) [22–24, 19]. They bring on an interesting new feature of the conservative two-body dynamics at this order of approximation, namely the non-locality in time. This has been shown in the canonical ADM Hamiltonian [11–15], the harmonic-coordinates Fokker Lagrangian [16, 17] and the EFT [24, 19] approaches. The 4PN tail term is related to the appearance of infra-red divergencies in the ADM and Fokker formalisms, and such divergencies have entailed the presence of ‘ambiguity parameters’ that have plagued—for the moment—the derivations of the 4PN dynamics [11–17, 25].

On the other hand, the conservative dynamics of binary systems of compact objects enjoys a fundamental property now known as the first law of binary mechanics. For circular orbits, this law is a particular case of a more general variational relationship, valid for systems of black holes and extended matter sources [26]. The first law for non-spinning point-particle binaries on circular orbits was established in [27]. It was later generalized to spinning binaries [28], and more recently extended to generic bound (eccentric) orbits [29]. These laws have been derived on general grounds, assuming that the conservative dynamics of the binary derives from an autonomous canonical Hamiltonian. Moreover, they have been explicitly checked to hold true up to 3PN order, and even up to 5PN order for some logarithmic terms [27, 29]. First laws of binary mechanics have also been established in the framework of black hole perturbation theory and the GSF, first in the case of corotating binaries [30], later for a test mass on generic bound orbits around a Kerr black hole [31], and more recently for a massive point particle in Kerr spacetime, including all conservative GSF effects [32].

The first law of compact binary mechanics involves the so-called ‘redshift’ factor of each point particle, first introduced in [33–36] for circular orbits, and later generalized to eccentric orbits [37, 38]. Remarkably, the law can be used to relate the redshift of one of the bodies to the binary’s binding energy and angular momentum, as well as to the relativistic periastron advance for circular orbits. Since GSF calculations can now compute the redshift, either numerically with high accuracy [39–42], or analytically to high PN orders [43–50], this translates into new information about the binary’s binding energy and angular momentum, and about the circular-orbit periastron advance, which can also be computed directly within the GSF framework [51–53]. Summarizing, thanks to the latter properties, the first laws of [27–29, 32] have already been applied to:

- Determine the numerical values of the ‘ambiguity parameters’ that appeared in the derivations of the 4PN two-body equations of motion [11–17];
- Calculate the exact GSF contributions to the binding energy and angular momentum for circular orbits, thus allowing a coordinate-invariant comparison to NR results [54];
- Compute the shift in frequencies of Schwarzschild and Kerr innermost stable circular orbits induced by the conservative part of the GSF [53–59];
- Test the cosmic censorship conjecture in a particular scenario where a massive particle subject to the GSF falls into a Schwarzschild black hole along unbound orbits [60, 61];
- Calibrate the effective potentials that enter the EOB model for circular orbits [58, 62] and mildly eccentric orbits [63–65], and spin-orbit couplings for spinning binaries [66];
- Define the analogue of the redshift of a particle for black holes in NR simulations, thus allowing further comparisons to PN and GSF calculations [67].

Given the relevance of the first laws to explore the dynamics of binary systems of compact objects, it is important to address the following question: do these relations still hold when non-local effects are accounted for, i.e. when the two-body Hamiltonian becomes a functional
(and not merely a function) of the canonical variables? In the present paper, we extend the
derivation of the first law of \cite{29} to 4PN order, for non-spinning binaries, by taking into
account the non-locality of the action due to the tail effect \cite{14, 16}. In particular, we shall
prove that the first law still holds and takes the standard form, equation (3.16) below, but with
a radial action integral that gets corrected by 4PN terms related to gravitational-wave tails, as
given in equation (3.17).

As an application of the first law, we derive the periastron advance for a slightly non-
circular orbit, in the limit where the eccentricity goes to zero, as a function of the averaged
redshift, at 4PN order and to linear order in the mass ratio. Indeed, \cite{29} showed earlier how the
first law can be used to relate the EOB potentials to the averaged redshift for slightly eccentric
orbits. Since the periastron advance for circular orbits is related to a linear combination of
two of these EOB potentials \cite{56}, this suggests that the eccentric-orbit first law can be used to
relate directly the periastron advance to the averaged redshift, in the limit of a circular orbit.

In this paper we establish such a relation, equation (5.12) below, by using our first law valid
for the non-local 4PN dynamics, and check that it is indeed fully consistent with all known
results at 4PN order.

The remainder of this paper is organized as follows. In section 2 we provide a summary of
the binary’s non-local dynamics at the 4PN order, as formulated in \cite{16, 17}. The first law with
non-local tail effects is derived in section 3, and a key formula relating the particles’
redshifts to the Hamiltonian is established in section 4. Finally, in section 5 we use the first law to relate
the GSF contribution to the periastron advance to that of the averaged redshift in the circular-
orbit limit. Three appendices give some further technical details. Throughout this paper we
use geometrized units where $G = c = 1$.

2. Summary of the 4PN non-local dynamics

In this section and the next one, we employ the canonical Hamiltonian formalism applied
to a binary system of non-spinning point masses $m_a$, with $a = 1, 2$. In an arbitrary frame
of reference, the two-body dynamics is described by canonical variables $y_a$ and $p_a$. In the
center-of-mass frame, the canonical variables are the relative position $x \equiv y_1 - y_2$ and linear
momentum $p = p_1 = -p_2$. Furthermore, introducing polar coordinates in the orbital plane,
the conjugate canonical variables read $(r, \rho, p_r, p_\rho)$.

At the 4PN order, the Hamiltonian encoding the dynamics of the binary system is of the form \cite{14–17}
\begin{equation}
H = H_0(r, p_r, p_\rho, m_a) + H_{\text{tail}}[r, \rho, p_r, p_\rho; m_a].
\end{equation}

Here, $H_0$ is a local-in-time piece, the sum of many local (or ‘instantaneous’) post-Newtonian
terms up to 4PN order. This part does not depend on the coordinate $\rho$, so that the conjugate
momentum $p_\rho$ is conserved for the local dynamics. The tail term represents a 4PN correction
which is a non-local functional of the canonical variables (hence the bracket notation used in
equation (2.1)), given by
\begin{equation}
H_{\text{tail}} = -\frac{M_{\text{ADM}}}{5} \tilde{I}_y^{(3)} \tilde{I}_y^{(3)}.
\end{equation}

Because this contribution is a small 4PN correction, thereafter we will always approximate the
ADM mass by the total mass, i.e. $M_{\text{ADM}} = m \equiv m_1 + m_2$. In the tail term (2.2), $\tilde{I}_y^{(3)}(t)$ denotes
the third time derivative of the quadrupole moment $I_y(t)$, but with accelerations order reduced

by means of the (Newtonian) equations of motion, which we indicate with the hat notation. We explicitly have
\[
\dot{I}^{(3)}_{ij}(t) = -\frac{2m}{r^3} \left( p_i n_i n_j + \frac{4p_\varphi}{r} n_i \lambda_j \right),
\]
(2.3)
where the two unit vectors that span the orbital plane are \( \mathbf{n} \equiv \mathbf{x}/r = (\cos \varphi, \sin \varphi, 0) \) and \( \lambda \equiv (-\sin \varphi, \cos \varphi, 0) \), the angular brackets surrounding indices denoting the symmetric and trace-free (STF) projection. The non-local tail factor in equation (2.2) is given by [16, 17]
\[
\hat{T}^{(s)}_{ij}(t) = \text{Pf} \int_{-\infty}^{+\infty} \frac{dt'}{|t-t'|} \hat{I}^{(s)}_{ij}(t'),
\]
(2.4)
with \( s = 3 \) or 4 in this paper. It involves Hadamard’s partie finie prescription (denoted Pf), which depends on some cut-off scale, chosen to be the coordinate separation at the current time, \( r = r(t) \). More explicitly, we have
\[
\hat{T}^{(s)}_{ij}(t) = -2 \dot{I}^{(3)}_{ij}(t) \ln r(t) + \int_{0}^{+\infty} dr \ln \left( \frac{T}{2} \right) \left[ \hat{I}^{(s+1)}_{ij}(t-t) - \hat{I}^{(s+1)}_{ij}(t+t) \right].
\]
(2.5)
The tail term (2.2) depends on the orbital phase \( \varphi \), so that \( p_\varphi \) is no longer conserved for the non-local dynamics. The dependence on the masses \( m_a \) is explicit through equation (2.3) and the ADM mass, which reduces to \( m = m_1 + m_2 \) at this order of approximation.

The non-local in time dynamics of the binary system of point masses follows from varying the non-local action
\[
S = \int dt \left[ \dot{r} p_r + \dot{\varphi} p_\varphi - H \right],
\]
(2.6)
where the overdot stands for the derivative with respect to the coordinate time \( t \). This yields ordinary looking Hamiltonian equations,
\[
\dot{r} = \frac{\delta H}{\delta p_r}, \quad \varphi = \frac{\delta H}{\delta p_\varphi}, \quad \dot{p}_r = -\frac{\delta H}{\delta r}, \quad \dot{p}_\varphi = -\frac{\delta H}{\delta \varphi},
\]
(2.7)
extcept that the partial derivatives of the Hamiltonian with respect to the canonical variables are properly replaced by functional derivatives, in order to account for the non-locality. The functional derivative of the tail term (2.2) with respect to \( r \) reads as
\[
\frac{\delta H_{\text{tail}}}{\delta r} = -\frac{2m}{5} \left[ \frac{\partial \hat{I}^{(3)}_{ij}}{\partial r} \hat{I}^{(3)}_{ij} - \frac{1}{r} \hat{I}^{(3)}_{ij} \hat{I}^{(3)}_{ij} \right].
\]
(2.8)
It involves the partial derivative of the third (order reduced) time derivative of the quadrupole moment (2.3). The second term in the right-hand side of equation (2.8) comes from the derivative acting on the Hadamard partie finie scale \( r \). Similarly, for the other variables we have
\[
\frac{\delta H_{\text{tail}}}{\delta \varphi} = -\frac{2m}{5} \frac{\partial \hat{I}^{(3)}_{ij}}{\partial \varphi} \hat{I}^{(3)}_{ij},
\]
(2.9)
while the ‘functional’ derivative with respect to the masses obviously reduces to an ordinary derivative, simply given by
\[
\frac{\delta H_{\text{tail}}}{\delta m_a} = \frac{\partial H_{\text{tail}}}{\partial m_a} = -\frac{3}{5} \lambda_i \hat{I}^{(3)}_{ij} \hat{I}^{(3)}_{ij},
\]
(2.10)
Next, we compute the time derivative of the non-local Hamiltonian (2.1) ‘on-shell,’ i.e. when the field equation (2.7) are satisfied, and obtain [17]
\[ \dot{H} = \frac{m}{5} \left[ \hat{i}_{ij}^{(4)} \hat{j}_{ij}^{(3)} - \hat{i}_{ij}^{(3)} \hat{j}_{ij}^{(4)} \right]. \] (2.11)

Hence, for the dynamics deriving from the non-local Hamiltonian (2.1), the conserved energy \( E \), such that \( dE/dt = 0 \), differs from the on-shell value of \( H \), and we have instead [17]
\[ E = H + \Delta H^{DC} + \Delta H^{AC}, \] (2.12)
where the first correction is a constant (DC) contribution, while the second correction is an oscillatory (AC) contribution. The constant piece turns out to be proportional to the total averaged gravitational-wave energy flux \( \mathcal{F} \),
\[ \Delta H^{DC} = -\frac{2m}{5} \langle \hat{i}_{ij}^{(3)} \hat{j}_{ij}^{(3)} \rangle = -2m \mathcal{F}. \] (2.13)

The AC piece, on the other hand, is defined to have zero average, \( \langle \Delta H^{AC} \rangle = 0 \), and it must necessarily satisfy \( d(\Delta H^{AC})/dt = -Q_H \), where \( Q_H \) denotes the right-hand side of equation (2.11). From these two requirements, it follows that
\[ \Delta H^{AC}(t) = \left\langle \int_{t}^{u} ds Q_H(s) \right\rangle_u, \] (2.14)
where \( \langle \rangle_u \) denotes the average with respect to the variable \( u \), as defined by equation (3.7) below. In [17], an explicit expression for the AC term is given by means of a discrete Fourier series, using the known Fourier coefficients of the quadrupole moment as a function of the orbit’s eccentricity \( e \) (to Newtonian order). The Fourier series of the AC term is also provided in equation (A.6) of appendix A below, together with further details.

Similar results hold for the angular momentum. The Hamilton equation for \( p_{\phi} \) reads
\[ \dot{p}_{\phi} = \frac{2m}{5} \frac{\partial \hat{i}_{ij}^{(3)}}{\partial \phi} \hat{j}_{ij}^{(3)}, \] (2.15)
showing that \( p_{\phi} \) is not conserved because of the non-local tail term. The conserved angular momentum \( L \), such that \( dL/dt = 0 \), is then obtained in the form
\[ L = p_{\phi} + \Delta p_{\phi}^{DC} + \Delta p_{\phi}^{AC}. \] (2.16)

The constant DC part is related to the averaged gravitational-wave flux of angular momentum, \( \mathcal{G} \), while the oscillating AC part is determined by the requirements that \( \langle \Delta p_{\phi}^{AC} \rangle = 0 \), and that it must satisfy \( d(\Delta p_{\phi}^{AC})/dt = -Q_{p_{\phi}} \), where \( Q_{p_{\phi}} \) is the right-hand side of equation (2.15). More explicitly, we have
\[ \Delta p_{\phi}^{DC} = \frac{2m}{5} \left\langle \frac{\partial \hat{i}_{ij}^{(3)}}{\partial \phi} \hat{j}_{ij}^{(3)} \right\rangle = -2m \mathcal{G}, \] (2.17a)
\[ \Delta p_{\phi}^{AC}(t) = \left\langle \int_{t}^{u} ds Q_{p_{\phi}}(s) \right\rangle_u. \] (2.17b)

See appendix A for the Fourier decomposition of \( \Delta p_{\phi}^{AC} \).
3. Derivation of the first law

In this section, starting from the non-local Hamiltonian (2.1), we shall derive a first law of compact binary mechanics that accounts for the effects of the non-local tail term (2.2) at 4PN order. To do so, we start by considering the unconstrained variation of the Hamiltonian (2.1) induced by infinitesimal changes $\delta r$, $\delta \varphi$, $\delta p_r$, $\delta p_\varphi$ and $\delta m_a$ of the canonical variables and component masses, namely

$$\delta H = \frac{\partial H_0}{\partial r} \delta r + \frac{\partial H_0}{\partial p_r} \delta p_r + \frac{\partial H_0}{\partial p_\varphi} \delta p_\varphi + \sum_a \frac{\partial H_0}{\partial m_a} \delta m_a + \delta H_{\text{tail}}.$$  (3.1)

Here, we separated out the variation of the local instantaneous piece $H_0$ from that of the non-local tail part. Next, we consider the case where the changes $(\delta H, \delta r, \delta \varphi, \delta p_r, \delta p_\varphi, \delta m_a, \delta H_{\text{tail}})$ correspond to two neighbouring solutions of the binary’s Hamiltonian dynamics. In this case, one must be careful to perform the variation of the tail term (2.2) on-shell, i.e. after having replaced into it the motion by a solution of the Hamiltonian equation (2.7). That variation is then given by

$$\delta H_{\text{tail}} = -\frac{m}{5} \left[ \delta \dot{I}_y^{(3)} I_y^{(3)} + \dot{I}_y^{(3)} \delta T_y^{(3)} + \frac{\delta m}{m} I_y^{(3)} \delta T_y^{(3)} \right].$$  (3.2)

where $\delta \dot{I}_y^{(3)}$ is the variation of the (order reduced) third time derivative of the quadrupole moment (2.3) with respect to the independent variables and masses, while $\delta T_y^{(3)}$ denotes the variation of the onshell value of the tail factor (2.4) and (2.5). While comparing two neighbouring solutions of the dynamics, we can also substitute Hamilton’s equation (2.7) into equations (3.1) and (3.2), together with the explicit expressions (2.8)–(2.10) for the tail term. A straightforward calculation then yields

$$\delta H = \dot{\varphi} \delta p_\varphi - \dot{p}_\varphi \delta \varphi + \ddot{r} \delta p_r - \dot{p}_r \delta r + \sum_a z_a \delta m_a - \frac{2m}{5} I_y^{(3)} \delta \dot{r} + \frac{m}{5} \left[ \delta \dot{I}_y^{(3)} I_y^{(3)} - \dot{I}_y^{(3)} \delta T_y^{(3)} \right].$$  (3.3)

Notice the last term in square brackets, which is similar to the first two terms in the right-hand side of equation (3.2), but with a crucial minus sign difference. Finally, in equation (3.3) we have defined the ‘redshift’ factor $z_a$ to be the derivative of the Hamiltonian with respect to the mass $m_a$, namely

$$z_a \equiv \frac{\partial H}{\partial m_a} = \frac{\partial H_0}{\partial m_a} - \frac{3}{5} \delta \dot{I}_y^{(3)} I_y^{(3)},$$  (3.4)

where we used equation (2.10). The fact that the quantity (3.4) is indeed the redshift factor of the particle $a$, namely $z_a = d\tau_a/dt$, is not trivial and will be proven in section 4 below.

Next, to simplify the tail terms in square brackets in the right-hand side of equation (3.3), we make use of the explicit Fourier series representations of the quadrupole moment and of the tail factor (2.4), which are given by the formulas (A.1) and (A.4) in appendix A. Of course, this is allowed since we are considering the on-shell variation of the tail term. It can then easily be shown that

$$\delta \dot{I}_y^{(3)} I_y^{(3)} - \dot{I}_y^{(3)} \delta T_y^{(3)} = 2I_y^{(3)} \dot{I}_y^{(3)} \left( \frac{\delta \dot{r}}{r} + \frac{\delta n}{n} \right) + \Delta,$$  (3.5)

where $\Delta \equiv 2\pi/P$ denotes the frequency associated with the period $P$ of the radial motion, while the extra piece $\Delta$ represents a more complicated expression, involving a double Fourier series over the Fourier components of the quadrupole moment and their variations.
\[
\Delta = 2 \sum_{p,q} \mathcal{T}_p \mathcal{T}_q \delta \left( \mathcal{T}_q \mathcal{T}_p \right) p^q q^3 \ln \left| \frac{p}{q} \right|.
\]

(3.6)

To be clear, we are considering the difference between two infinitesimally close configurations associated with quadrupole moments \( \mathcal{I}_q(t) \) and \( \mathcal{I}_q'(t) \). These configurations have different radial frequencies \( n \) and \( n' \), semi-major axes \( a \) and \( a' \), and eccentricities \( e \) and \( e' \), as well as different masses. The Fourier decomposition (3.6) involves the Fourier coefficients \( p \mathcal{I}_p \) and \( q \mathcal{I}_q \) (see appendix A for definitions) and different mean anomalies \( \ell = n(t - t_0) \) and \( \ell' = n'(t - t_0) \). We denote \( \mathcal{I}_q = p \mathcal{I}_q - p \mathcal{I}_q, \) \( \mathcal{D} = n - n' \), and so on, e.g. \( \delta \mathcal{I}_n q^3 = \mathcal{I}_n q^3 - \mathcal{I}_n q^3 \).

Focusing on the most general case, the time average \( \langle f \rangle \) of a given function \( f(t) \) will be defined as

\[
\langle f \rangle \equiv \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} df(t).
\]

(3.7)

But for periodic functions with period \( P \), this reduces to the usual average \( \langle f \rangle = \frac{1}{P} \int_{0}^{P} df(t) \) over one radial period. Let us first check that the time average of the quantity (3.6) is zero. Indeed, expanding the variational operation, it is clear that all the terms proportional to the logarithmic factor \( \ln \left| \frac{p}{q} \right| \) but we also have terms proportional to \( \mathcal{I}_p \mathcal{I}_q \delta \mathcal{I}_n q^3 = e^{\mathcal{I}_n q^3} - e^{\mathcal{I}_n q^3} \).

However, recall that the two configurations we consider are infinitesimally close, so we have \( p n + q n' \neq 0 \) in this case. Then, by applying the long-time average (3.7) we readily obtain \( \langle e^{\mathcal{I}_n q^3} \rangle = 0 \).

Finally, we conclude that the quantity (3.6) has, indeed, zero average \( \langle \Delta \rangle = 0 \). Therefore, substituting equation (3.5) into the variational formula (3.3) and averaging, we obtain

\[
\langle \delta H \rangle = \langle \delta \delta p_\varphi - \dot{p}_\varphi \delta \varphi \rangle + \langle \dot{r} \delta p_r - \dot{p}_r \delta r \rangle + \sum_a \langle \delta \phi_a \rangle \delta m_a + 2m \delta n \mathcal{F}.
\]

(3.8)

where we used the fact that \( n \) and \( m_a \) are constant, while the last term contains the averaged gravitational-wave flux of energy \( \mathcal{F} = \frac{1}{2} \left( I^{(3)}_q I^{(3)}_q \right) \).

To evaluate the radial contribution, we proceed as in [29]. Since the average of the time derivative of a periodic function vanishes, the radial contribution to equation (3.8) can be written as

\[
\langle \dot{r} \delta p_r - \dot{p}_r \delta r \rangle = \langle \delta \left( \mathcal{F}_p \right) \rangle = \frac{1}{P} \int_{0}^{P} \delta (\mathcal{F}_p) = \frac{2}{P} \int_{-r_{-}}^{r_{+}} \delta (p, dr).
\]

(3.9)

where \( r_{-} \) and \( r_{+} \) denote the orbit’s periastron and apastron, at which \( \dot{r} = 0 \). Next, we can pull out the variation \( \delta \) from the integral. To see this, it is convenient to write equation (3.9) as an integral over the complex plane, initially along the segment \([r_{-}, r_{+}]\) along the real axis, but then deformed into an integral over a given closed contour \( C \) surrounding \( r_{-} \) and \( r_{+} \) in the complex plane, say \( \frac{1}{P} \int_{C} \delta (p, dr) \). When doing so, since the contour is fixed, one can ignore the variation of \( r_{-} \) and \( r_{+} \) in the process. This is Sommerfeld’s well-known method of contour integrals; see e.g. [68] or appendix C in [17]. Finally, we get

\[
\langle \dot{r} \delta p_r - \dot{p}_r \delta r \rangle = \frac{1}{P} \delta \oint_{C} p_r \delta r = n \delta R.
\]

(3.10)
where we recall that \( n = 2\pi/P \) is the radial frequency, or mean motion, and where \( R \) is the radial action integral, defined by

\[
R \equiv \frac{1}{2\pi} \oint p_r \, dr = \frac{1}{\pi} \int_{r_-}^{r_+} p_r \, dr. \tag{3.11}
\]

Now, to evaluate the azimuthal contribution to (3.8), we recall that \( \dot{\phi} \) is not conserved in the non-local case (see equation (2.15)), such that the result of the calculation will not reduce to the usual \( \omega \delta L \) term. Instead, we write

\[
\langle \dot{\phi} \delta p_\phi - \dot{p}_\phi \delta \phi \rangle = \omega \delta L - \omega \delta \Delta p_{\phi}^{DC} - \langle \delta (\dot{p}_\phi^{AC} \delta \phi) \rangle. \tag{3.12}
\]

in which we used equation (2.16) as well as the fact that \( L \) and \( \Delta p_{\phi}^{DC} \) are both constant, and we introduced the orbital-averaged azimuthal frequency

\[
\omega \equiv \langle \dot{\phi} \rangle = \frac{1}{P} \int_0^P \! dt \, \dot{\phi} = n \oint \frac{d\varphi}{2\pi} = Kn, \tag{3.13}
\]

where \( 2\pi K \equiv \oint d\varphi = 2\pi + \Delta \Phi \) is the accumulated azimuthal angle per radial period, with \( \Delta \Phi \) the relativistic periastron advance. In the second line of equation (3.12), we may then use (2.17) in the second term and handle the last term just like the radial contribution (3.10), such that finally

\[
\langle \dot{\phi} \delta p_\phi - \dot{p}_\phi \delta \phi \rangle = \omega \delta L + \omega \delta (2m \mathcal{F}) - n \delta \left( \frac{1}{2\pi} \oint \Delta p_{\phi}^{AC} \, d\varphi \right). \tag{3.14}
\]

At last, we have to take into account the relationship (2.12), which implies that the term \( \langle \delta H \rangle \) in equation (3.8) is not simply equal to \( \delta E \). Instead, the conserved energy \( E \) (which includes the total rest mass \( m = m_1 + m_2 \)) gets shifted by the DC correction (2.13), while the AC correction (2.14) does not contribute since it has zero time average:

\[
\langle \delta H \rangle = \delta \langle H \rangle = \delta E + \delta (2m \mathcal{F}). \tag{3.15}
\]

Finally, collecting the intermediate results (3.8), (3.10), (3.14) and (3.15), and combining the 4PN contributions that involve the gravitational-waves fluxes \( \mathcal{F} \) and \( \mathcal{G} \), where at that order of approximation one may replace \( n \) by \( \omega \) and \( \delta n \) by \( \delta \omega \) if needed, we obtain a first law of binary mechanics that takes the standard form, as established in [29], namely

\[
\delta E = \omega \delta L + n \delta \mathcal{R} + \sum_a \langle z_a \rangle \delta m_a, \tag{3.16}
\]

but where, as anticipated above, the radial action integral (3.11) gets corrected at 4PN order by terms originating from the non-local tail:

\[
\mathcal{R} = R + 2m \left( \mathcal{F} - \frac{\mathcal{F}}{\omega} \right) = \frac{1}{2\pi} \oint \Delta p_{\phi}^{AC} \, d\varphi. \tag{3.17}
\]

Heuristically, one may interpret the additional contributions proportional to the gravitational-wave fluxes as being related to the energy and angular momentum content in gravitational waves in the far zone\(^3\). Moreover, we recall that \( 2m \mathcal{F} = -\Delta H^{DC} \) and \( 2m \mathcal{G} = -\Delta p_{\phi}^{DC} \).

\(^3\)Note that the gravitational-wave fluxes are themselves related by a first law in the adiabatic approximation, namely \( \mathcal{F} = \omega \mathcal{G} - n (R - \sum_a z_a H_a) \); see section VA in [29].
The Fourier decomposition of the last term in the right-hand side of equation (3.17) is investigated in appendix A. Importantly, we note that the correction terms in (3.17) vanish for circular orbits, because for such orbits the Newtonian gravitational-wave fluxes obey $F = \omega \mathcal{G}$, while $\dot{\varphi}$ is constant and $\langle \Delta p^{\text{NC}} \rangle = 0$. Hence, the circular-orbit condition $R = 0$ implies $\mathcal{R} = 0$.

The authors of [15, 69] discussed how the non-local Hamiltonian (2.1) and (2.2) can formally be reduced to an ordinary local Hamiltonian by means of a suitable transformation $(r, \varphi, p_r, p_\varphi) \rightarrow (r^{\text{loc}}, \varphi^{\text{loc}}, p_r^{\text{loc}}, p_\varphi^{\text{loc}})$ of the phase-space variables. Having performed such a ‘localization’ of the Hamiltonian, one could then follow [29] to derive an ordinary first law of binary mechanics. That ‘local’ law would be identical to our equation (3.16), except that the radial action integral therein, say $R^{\text{loc}}$, would be given by the usual expression defined in terms of the shifted variable $p_r^{\text{loc}}$. Of course, our modified radial action integral $R$ obtained in equation (3.17) should be identical to the local radial action integral $R^{\text{loc}}$ when it is expressed in terms of the natural invariants $E$ and $L$ (and masses $m_a$), namely

$$R(E, L) = R^{\text{loc}}(E, L) \equiv \frac{1}{2\pi} \oint \! dp_r^{\text{loc}}(r^{\text{loc}}, E, L). \quad (3.18)$$

Before closing this section, we note that one can easily derive a ‘first integral’ relationship associated with the variational first law (3.16), namely

$$E = 2\omega L + 2nR + \sum_a m_a(z_a). \quad (3.19)$$

This can be proven in various ways. For instance, one might notice that $E$ is an homogeneous function of degree one in the variables $\sqrt{L}$, $\sqrt{R}$ and $m_a$, such that (3.19) comes from applying Euler’s theorem for homogeneous functions; see [27–29].

4. Derivation of the redshift factor

In this section we shall prove that the quantity $z_a$ defined by equation (3.4) actually coincides with the redshift $d\tau_a/dt$ of the particle $a$. Our proof will be based on the use of the Fokker Lagrangian, and is a minor adaptation of the proof already given in [28], with the simplification that we consider here only non-spinning particles, but with the slight complication that the dynamics is non-local because of the 4PN tail effect.

The Fokker Lagrangian of a system of point particles was defined, e.g. in [16]. We start from the gravitation-plus-matter Lagrangian of general relativity,

$$L = L_g[g_{\mu\nu}, y_a, v_a; m_a] + L_m[g_{\mu\nu}, y_a, v_a; m_a]. \quad (4.1)$$

The gravitational part $L_g$ is the usual Einstein–Hilbert term, written in the Landau–Lifshitz form, with the harmonic gauge-fixing term; see equation (2.1) in [16]. The matter Lagrangian for the system of point particles is given by

$$L_m[g_{\mu\nu}, y_a, v_a; m_a] = -\sum_a m_a \sqrt{-g_{\mu\nu}(y_a)v^\mu_a v^\nu_a}, \quad (4.2)$$

where $y_a^\mu = (t, y_a)$ and $v^\mu_a = (1, v_a)$ denote the trajectories and ordinary coordinate velocities, with $v_a(t) \equiv y_a(t)$, and $g_{\mu\nu}(y_a)$ stands for the metric evaluated at the location of the particle $a$, following some regularization scheme, in principle dimensional regularization [16].
The Einstein field equations in harmonic coordinates follow from varying the Lagrangian (4.1) with respect to the metric. These equations are then solved perturbatively, yielding an explicit PN-iterated harmonic-coordinates solution, say
\[ g_{\mu\nu}(x) \equiv g_{\mu\nu}(x; y_b, v_b, a_b; m_b). \] (4.3)

This solution depends on the positions \( y_b \) and velocities \( v_b \) of all of the particles, but also on their accelerations and any possible derivatives of accelerations that can get generated at high PN orders, and are here symbolized by \( a_b \equiv (\dot{v}_b, \ddot{v}_b, \cdots) \). Of course, the solution (4.3) depends also on all the masses \( m_b \). The Fokker Lagrangian is then defined by inserting the explicit PN solution (4.3) back into the Lagrangian (4.1), thus obtaining
\[ L_F[y_a, v_a, a_a; m_a] \equiv L_g[g_{\mu\nu}(x; y_a, v_a, a_a; m_a); v_a, m_a]. \] (4.4)

This Lagrangian is a generalized Lagrangian, depending not only on positions and velocities, but also on accelerations and derivatives of accelerations. Taking the functional derivative with respect to the position of the particle \( a \) yields
\[ \frac{\delta L_F}{\delta y_a} = \frac{\delta L}{\delta y_a} \bigg|_{\tau_{\mu\nu}} + \frac{\delta L_m}{\delta y_a} \bigg|_{\tau_{\mu\nu}}. \] (4.5)

But since \( \delta L/\delta g_{\mu\nu} = 0 \) holds for the actual PN solution \( \bar{g}_{\mu\nu} \) of the Einstein field equations, the basic property of the Fokker Lagrangian follows, namely that its functional derivative with respect to one of the particle’s position reduces to that of the matter Lagrangian while holding the metric fixed in equation (4.2):
\[ \frac{\delta L_F}{\delta y_a} = \frac{\delta L_m}{\delta y_a} \bigg|_{\tau_{\mu\nu}}. \] (4.6)

Therefore, \( \delta L_F/\delta y_a = 0 \) yields the correct equations of motion for the system of point masses in the metric generated by the particles themselves.

Next, we can apply the very same argument for the variation of the Fokker Lagrangian with respect to the mass \( m_a \), holding \( y_b, v_b, a_b \) fixed. We find that the dependence over the mass that is hidden into the PN solution \( \bar{g}_{\mu\nu} \) gets cancelled by the fact that \( \delta L/\delta g_{\mu\nu} \big|_{\tau_{\mu\nu}} = 0 \). Hence we obtain the important result
\[ \frac{\delta L_F}{\delta m_a} = \frac{\delta L_m}{\delta m_a} \bigg|_{\tau_{\mu\nu}}. \] (4.7)

As is clear from equation (4.2), the functional derivative of the matter Lagrangian at fixed \( \bar{g}_{\mu\nu} \) in the right-hand side of (4.7) reduces to an ordinary derivative, and we get
\[ \frac{\delta L_F}{\delta m_a} = -\sqrt{-g_{\mu\nu}(y_a)} \frac{\partial \ell}{\partial \tau_{\mu\nu}}. \] (4.8)

Finally, it remains to go from the Fokker Lagrangian \( L_F \) to the corresponding Hamiltonian \( H_F \). The only subtlety is that the harmonic-coordinates Fokker Lagrangian is a generalized Lagrangian. Hence we must first get rid of the accelerations by performing suitable shifts of the trajectories, so as to obtain an ordinary Lagrangian, depending only on the positions and velocities. Such shifts have recently been performed in [16], and discussed in a more general context in [70]; notice that the 4PN tail term is also transformed into an ordinary—although still non-local—term by applying suitable shifts. Now, the new metric expressed in the new, shifted variables will take the same form as in (4.3), but without accelerations,
because the redefinition of the trajectories can be seen as being induced by a coordinate transformation of the ‘bulk’ metric. Hence the derivation given above applies to the new Lagrangian with shifted variables, and the relationship (4.7) still holds. Furthermore, that Lagrangian being ordinary, a usual Legendre transformation can be performed to define the Hamiltonian as \( H_F = \sum_a p_a^\nu \dot{y}_a^\nu - L_F \), where \( p_a^\nu = \delta L_F / \delta \dot{y}_a^\nu \) gives \( \nu_a \) as a functional of the canonical positions \( y_a \) and momenta \( p_a \). From the properties of the Legendre transformation, we readily find that the derivative of the Hamiltonian with respect to the mass \( m_a \), while holding \( y_a \) and \( p_a \) fixed, is simply

\[
\frac{\delta H_F}{\delta m_a} = -\frac{\delta L_F}{\delta m_a} = \sqrt{-g_{\mu\nu}(y_a)\dot{y}_a^\mu \dot{y}_a^\nu}.
\]  

(4.9)

Here, the velocities are to be considered as functionals of the canonical variables, \( \nu_a[y_a, p_a] \).

Since the Fokker Hamiltonian \( H_F \) that we have just introduced is precisely the Hamiltonian (2.1) that we considered in sections 2 and 3, we have proven that the quantity \( \langle z_a \rangle \) defined in equation (3.4) is indeed the redshift associated with the particle \( a \), namely that

\[
\langle z_a \rangle = \frac{d \tau_a}{dt} = \sqrt{-g_{\mu\nu}(y_a)\dot{y}_a^\mu \dot{y}_a^\nu}.
\]  

(5.10)

5. Periastron advance and averaged redshift

Throughout this section we assume that one of the two compact objects, say body 1, is much less massive than the other, and we work to linear order in the mass ratio \( q \equiv m_1/m_2 \ll 1 \), or equivalently to linear order in the symmetric mass ratio \( \nu \equiv m_1m_2/m^2 = q + \mathcal{O}(q^2) \). Our objective is to relate, in the circular-orbit limit, the \( \mathcal{O}(\nu) \) contributions to the periastron advance and to the averaged redshift \( \langle z \rangle \equiv \langle z_1 \rangle \) associated with the lighter body.

A generic bound (eccentric) orbit can be parameterized using the two orbital frequencies \( n \) and \( \omega \), or equivalently using \( \omega \) and the periastron advance \( K = \omega/n \). Hence, to first order in the symmetric mass ratio \( \nu \), we may consider the following expansions of the modified radial action variable (3.17) and the averaged redshift of the lighter body:

\[
\mathcal{R}(K, \omega) = \mathcal{R}_0(K, \omega) + \nu \mathcal{R}_1(K, \omega) + \mathcal{O}(\nu^2),
\]

(5.1a)

\[
\langle z \rangle(n, \omega) = \langle z \rangle_0(n, \omega) + \nu \langle z \rangle_1(n, \omega) + \mathcal{O}(\nu^2),
\]

(5.1b)

where \( \mathcal{R}_0 \) and \( \langle z \rangle_0 \) denote the values of those quantities in the (Schwarzschild) background, while \( \mathcal{R}_1 \) and \( \langle z \rangle_1 \) represent first-order GSF corrections.

A circular orbit is defined by the condition of a vanishing radial action: \( R = 0 \); see (3.11). Crucially, as mentioned earlier, the corrective terms in the right-hand side of (3.17) vanish in the circular-orbit limit, such that \( R = 0 \) implies \( \mathcal{R} = 0 \). For the one-parameter family of circular orbits, the frequencies \( n \) and \( \omega \) are no longer independent, i.e. \( n = n^{\text{circ}}(\omega) \), or equivalently

\[
K = K^{\text{circ}}(\omega) = K_0(\omega) + \nu K_1(\omega) + \mathcal{O}(\nu^2).
\]

(5.2)

Our goal here is to relate the \( \mathcal{O}(\nu) \) contribution to \( K^{\text{circ}}(\omega) \), namely \( K_1(\omega) \), to the \( \mathcal{O}(\nu) \) contribution \( \langle z \rangle_1(n, \omega) \) to the redshift (5.1b) in the circular-orbit limit.

Expanding the circular-orbit condition \( \mathcal{R} = 0 \) to first order in the symmetric mass ratio, while using equations (5.1a) and (5.2), we get
\[ 0 = R_0(K_0(\omega), \omega) + \nu K_1(\omega) \left( \frac{\partial R_0}{\partial K} \right)_\omega (K_0(\omega), \omega) + R_1(K_0(\omega), \omega) + \mathcal{O}(\nu^2). \] (5.3)

The first term in the right-hand side of (5.3) vanishes identically. Because the contribution \( \mathcal{O}(\nu) \) must also vanish identically, we obtain

\[ K_1(\omega) = -\frac{R_1(K_0(\omega), \omega)}{\left( \frac{\partial R_0}{\partial K} \right)_\omega (K_0(\omega), \omega)}, \] (5.4)

At this stage, it gets convenient to treat \( K \) as a function of \( \omega \) and \( R_0 \), defined by inverting \( R_0 = R_0(K, \omega) \). Since \( R_0(0) = 0 \) defines circular orbits in the background (i.e. when the mass ratio is \( \nu = 0 \)), we can rewrite equation (5.4) as

\[ K_1(\omega) = -R_1(K(0), \omega) \left( \frac{\partial K}{\partial R_0} \right)_\omega(\omega, R_0 = 0). \] (5.5)

A simple change of variables from \( (\omega, R_0) \) to the frequencies \( (\omega, n) \) yields \( (\partial K/\partial R_0) \omega = (\partial K/\partial n) \omega (\partial n/\partial R_0) \omega = -(K^2/\omega)(\partial n/\partial R_0) \omega \), and here we can replace \( K \) by the background value \( K(0) \). Therefore, equation (5.5) can be written in the convenient form

\[ K_1(\omega) = -\frac{K_0^2(\omega)}{\omega} \left[ R_1 \left( \frac{\partial R_0}{\partial n} \right)_\omega \right]^{-1} (\omega, n(\omega, R_0 = 0)), \] (5.6)

where the right-hand side is computed for \( \omega \) and \( n(\omega, R_0 = 0) \), which is the radial frequency as a function of \( \omega \) for circular orbits in the Schwarzschild background, say \( n^\text{Sch}(\omega) \).

Next, we need to relate \( R_1(\omega, n) \) to the GSF contribution \( \langle z \rangle_1(n, \omega) \) to the averaged redshift (5.1b), in the circular-orbit limit. But from equations (5.8b) and (5.9c) of [29], we know that, for any dimensionless frequencies \( (\tilde{\omega}, \tilde{n}) \equiv (m\omega, mn) \), and up to an irrelevant overall scaling of \( R_0 \) and \( R_1 \),

\[ R_0(\tilde{\omega}, \tilde{n}) = -\frac{\partial \langle z \rangle_0}{\partial \tilde{n}}, \] (5.7a)

\[ R_1(\tilde{\omega}, \tilde{n}) = -\frac{1}{2} \left( \frac{\partial \langle z \rangle_1}{\partial \tilde{n}} + \frac{\partial \langle z \rangle_0}{\partial \tilde{n}} - \tilde{n} \frac{\partial^2 \langle z \rangle_0}{\partial \tilde{n}^2} - \tilde{\omega} \frac{\partial^2 \langle z \rangle_0}{\partial \tilde{n} \partial \tilde{\omega}} \right). \] (5.7b)

These expressions were established from a first law derived starting from a local Hamiltonian. However, since we proved in section 3 that a similar first law relation holds for the non-local Hamiltonian (2.1), as long as the radial action \( R \) is replaced by \( R_0 \), we conclude that (5.7) hold when expressed in terms of the corrected radial action \( R \) (recall also equation (3.18)). Inserting these expressions into equation (5.6) yields

\[ K_1(\tilde{\omega}) = \frac{K_0^2(\tilde{\omega})}{2\tilde{\omega}} \left( \frac{\partial \langle z \rangle_0}{\partial \tilde{n}^2} \right)^{-1} \left( \frac{\partial \langle z \rangle_1}{\partial \tilde{n}} + \frac{\partial \langle z \rangle_0}{\partial \tilde{n}} - \tilde{n} \frac{\partial^2 \langle z \rangle_0}{\partial \tilde{n}^2} - \tilde{\omega} \frac{\partial^2 \langle z \rangle_0}{\partial \tilde{n} \partial \tilde{\omega}} \right), \] (5.8)

where the right-hand side is still computed at \( \omega \) and \( n^\text{Sch}(\omega) = n(\omega, R_0 = 0) \).

To evaluate more explicitly the latter expression in the circular-orbit limit, it is especially convenient to parametrize the orbit in terms of the usual Schwarzschild 'semi-latus rectum' \( p \).
and ‘eccentricity’ $e$, instead of the frequencies $\dot{\omega}$ and $\ddot{\omega}$, and to perform a Taylor expansion in the limit where $e \to 0$ (see appendix B for more details). For instance, we write

$$
\left( \frac{\partial \langle z \rangle_{(1)} }{\partial n} \right)_{\omega} = \left( \frac{\partial \langle z \rangle_{(1)} }{\partial p} \right)_{e} \left( \frac{\partial p}{\partial n} \right)_{\omega} + \left( \frac{\partial \langle z \rangle_{(1)} }{\partial e} \right)_{p} \left( \frac{\partial e}{\partial n} \right)_{\omega}.
$$

(5.9)

Adapting notations, the expressions for $\bar{n}(p,e)$, $\ddot{\omega}(p,e)$ and $\langle z \rangle_{(0)}(p,e)$ are given, for instance, in equations (2.4)–(2.10) of [38]. These relationships can be computed analytically, as Taylor expansions in the eccentricity $e$. We collect all the required results in appendix B; in particular, the coefficients appearing in (5.9) are given in equation (B.8) of [38]. These relationships can be computed analytically, as Taylor expansions in the eccentricity $e$. We collect all the required results in appendix B; in particular, the coefficients appearing in (5.9) are given in equation (B.8a) there. Moreover in the small-$e$ limit, the $\mathcal{O}(e^2)$ contribution to the averaged redshift can be expanded as

$$
\langle z \rangle_{(1)}(p,e) = \langle z \rangle_{(1)}(p) + \frac{e^2}{2} \langle z \rangle_{(1)}^2(p) + o(e^2),
$$

(5.10)

where we used the notations $\langle z \rangle_{(1)}(p) \equiv \langle z \rangle_{(1)}(p,0)$ and $\langle z \rangle_{(1)}^2(p) \equiv (\partial^2 \langle z \rangle_{(1)}/\partial e^2)(p,0)$. Accurate GSF data for $\langle z \rangle_{(1)}(p)$ were computed for separations $6 < p \leq 1200$ in [38, 63].

Note that a contribution linear in the eccentricity cannot appear in equation (5.10), otherwise the expression (5.8) for $K_{(1)}$ would be singular in the circular-orbit limit $e \to 0$, as can be seen from equations (5.9) and (B.8b). Substituting (B.8) and (5.10) into (5.9), we find that both the circular-orbit contribution $\langle z \rangle_{(1)}(p)$ and the leading finite-eccentricity contribution $\langle z \rangle_{(1)}^2(p)$ appear in the final expression for $(\partial \langle z \rangle_{(1)}/\partial n)_{\omega}$ (and hence will appear in that for $K_{(1)}(\omega)$), namely

$$
\left. \frac{\partial \langle z \rangle_{(1)} }{\partial n} \right|_{e=0} = \frac{4}{3} \frac{p^2 \sqrt{p-6}}{4p^2 - 39p + 86} \left[ p \left( p^2 - 10p + 22 \right) \frac{d\langle z \rangle_{(1)} }{dp} - (p - 2)(p - 6) \frac{\langle z \rangle_{(1)}^2 }{2} \right].
$$

(5.11)

Finally we need the closed-form expressions of the frequency derivatives of the background averaged redshift $\langle z \rangle_{(0)}(\omega,\bar{n})$ that appear in equation (5.8). In the small-$e$ limit, these are given by equations (B.9) in appendix B. Then, combining equations (5.8), (5.11) and (B.9), our final expression for the $\mathcal{O}(e^2)$ contribution to the periastron advance simply reads

$$
K_{(1)}(p) = -\sqrt{p} \frac{\sqrt{p - 3}}{(p - 6)^{3/2}} + \frac{p \sqrt{p - 3}}{(p - 6)^{3/2}} \left[ p \left( p^2 - 10p + 22 \right) \frac{d\langle z \rangle_{(1)} }{dp} - (p - 2)(p - 6) \frac{\langle z \rangle_{(1)}^2 }{2} \right].
$$

(5.12)

Equivalently, in terms of the quantity $W \equiv 1/K^2$ that was introduced in [56], namely $W(x) = 1 - 6x + \nu x^2 + \mathcal{O}(x^3)$, where $x \equiv \omega^2/3 = p^{-1} + \mathcal{O}(\nu)$, we readily find for the GSF contribution

$$
\rho(x) = 2x + 2 \sqrt{1 - 3x} \left[ \frac{1 - 10x + 22x^2}{1 - 6x} \frac{d\langle z \rangle_{(1)} }{dx} + (1 - 2x) \frac{\langle z \rangle_{(1)} x }{2x} \right].
$$

(5.13)

As an important check of these results, we verified that the formula (5.13) is recovered when combining the relationship of [56] between $\rho(x)$ and the EOB potentials $a(x)$ and $\bar{a}(x)$ on the one hand, with the expressions of [29] for $a(x)$ and $\bar{a}(x)$ in terms of $\langle z \rangle_{(1)}(x)$ and

---

4 We employ the Landau symbol $o$ for remainders with its usual meaning.
\( \langle z \rangle (\nu) \) on the other hand. As an additional check of equation (5.12), we shall also consider the behaviour of the functions \( z_{(1)}(p) \) and \( \langle z \rangle_{(1)}^2(p) \) in the weak-field limit \( p \to +\infty \), and verify that we recover the known large-\( p \) behaviour for \( K_{(1)}(p) \), known from the 4PN calculations of [15].

The gauge-invariant relation \( \langle z \rangle (n, \dot{\omega}) \) has been computed for generic orbits, up to 3PN order, for any mass ratio [38]. From this it is straightforward to derive the 3PN expansions of the \( O(\nu) \) contributions \( z_{(1)}(p) \) and \( \langle z \rangle_{(1)}^2(p) \) to \( \langle z \rangle (n, \dot{\omega}) \). On the other hand, the application of analytical techniques for linear black hole perturbations has given access to high-order PN expansions for these functions. In particular, the contribution \( \langle z \rangle_{(1)}^2(p) \) has been computed up to 4PN order in [49], and up to 9.5PN order in [65]. Combining those results, we find the 4PN-accurate formulas

\[
\frac{\langle z \rangle_{(1)}^2(p)}{2} = -\frac{1}{p} + \frac{2}{p^3} \frac{p + 5}{p^3} - \frac{3}{p^3} \left( \frac{23}{3} + \frac{41}{32} \pi^2 \right) \frac{1}{p^4} + \left( \frac{1291}{512} \pi^2 + \frac{128}{5} \gamma_E + \frac{256}{5} \ln 2 - \frac{64}{5} \ln p \right) \frac{1}{p^3} + o(p^{-5}),
\]

(5.14a)

where \( \gamma_E \) is Euler’s constant. Notice the logarithmic running appearing at 4PN order, related to the occurrence of gravitational-wave tails.

However, the expansions (5.14) cannot be right away substituted into the formulas (5.12) and (5.13). Indeed, the former results were derived while normalizing the frequencies using the black hole mass \( m_2 \), but the latter results were derived while normalizing the frequencies using the total mass \( m \). Hence, we first need to account for the correction originating from the substitutions \( \dot{\omega} = m_2 \omega + \nu m_2 \dot{\omega} + O(\nu^2) \) and \( \dot{n} = m_2 n + \nu m_2 \dot{n} + O(\nu^2) \) in \( \langle z \rangle_{(0)}(\dot{\omega}, \dot{n}) \), which is simply given by

\[
-\dot{\omega} \frac{\partial \langle z \rangle_{(0)}}{\partial \dot{\omega}} - \dot{n} \frac{\partial \langle z \rangle_{(0)}}{\partial \dot{n}} = \frac{1}{\sqrt{p(p-3)}} \left( 1 - \frac{2p^3 - 25p^2 + 92p - 102}{2(p-2)(p-3)(p-6)} \bar{e}^2 + O(e^4) \right),
\]

(5.15)

where we used equations (B.5a), (B.5b), (B.9a) and (B.9b) to evaluate this expression in the small-eccentricity limit. Then, adding the 4PN expansion of the correction term (5.15) to the 4PN expansions (5.14), and substituting the results in equations (5.12) and (5.13), we obtain the 4PN expansions of the \( O(\nu) \) contributions to \( K \) and \( W = 1/K^2 \) as

\[
K_{(1)}(p) = -\frac{7}{p^3} + \left( \frac{640}{4} + \frac{123}{32} \pi^2 \right) \frac{1}{p^4} + \left( \frac{275941}{560} + \frac{48007}{3072} \pi^2 - \frac{592}{15} \ln 2 - \frac{1458}{5} \ln 3 - \frac{2512}{15} \gamma_E + \frac{1256}{15} \ln p \right) \frac{1}{p^3} + o(p^{-4}),
\]

(5.16a)

In appendix C we use the first law (3.16) to compute the redshift up to 4PN order, for circular orbits only.
\[ \rho(x) = 14x^2 + \left( \frac{397}{2} - \frac{123}{16} \pi^2 \right) x^3 + \left( -\frac{215729}{180} + \frac{58265}{1536} \pi^2 + \frac{1184}{15} \ln 2 + \frac{2916}{5} \ln 3 + \frac{5024}{15} \ln e + \frac{2512}{15} \ln x \right) x^4 + o(x^4). \]

This last result is in full agreement with the 4PN expansion of the function \( \rho(x) \), as computed up to 9.5PN order using analytic GSF methods [65].

In order to compare the formula (5.16a) to the known 4PN result for \( K(\tilde{\omega}) \), one needs to add the contribution from the zero-th order term in equation (5.2), which can easily be computed by taking the ratio of equations (B.5a) and (B.5b) in the zero-eccentricity limit, namely

\[ K_{(0)}(p) = \sqrt{\frac{p}{p - 6}} = 1 + \frac{3}{p} + \frac{27}{2p^2} + \frac{135}{2p^3} + \frac{2835}{8p^4} + o(p^{-4}). \]  

Expressing the total periastron advance (5.2) in terms of the frequency-related PN parameter \( x \equiv \tilde{\omega}^{2/3} = p^{-1} + \mathcal{O}(\nu) \), rather than the semi-latus rectum \( p \), we find that (5.16a) and (5.17) combine to give

\[ K(x) = 1 + 3x + \left( \frac{27}{2} - 7 \nu \right) x^2 + \left( \frac{135}{2} + \left[ \frac{649}{4} + \frac{123}{32} \pi^2 \right] \nu \right) x^3 + \left( \frac{2835}{8} + \left[ -\frac{275941}{360} + \frac{48007}{3072} \pi^2 - \frac{592}{15} \ln 2 - \frac{1458}{5} \ln 3 - \frac{2512}{15} \ln e - \frac{1256}{15} \ln 1 \right] \nu \right) x^4 + o(\nu, x^4). \]  

Up to uncontrolled terms \( \mathcal{O}(\nu^2) \) and \( \mathcal{O}(\nu^3) \), this result is in entire agreement with the known 4PN result, as derived for any mass ratio in the canonical ADM framework [15] and in the harmonic-coordinates Fokker Lagrangian approach [17, 71].

Finally, let us check that the binary’s binding energy for circular orbits at the 4PN order is correctly recovered by the same method. For general orbits, the rescaled binding energy \( \tilde{E} \equiv (E - m)/(\nu \nu) \) is expressed as a function of the dimensionless frequencies \( \tilde{\omega} \) and \( \tilde{n} \). In the small mass-ratio limit we have \( \tilde{E} = E_{(0)} + \nu E_{(1)} + \mathcal{O}(\nu^2) \), where, as a consequence of the first law (see equations (5.8a) and (5.9a) in [29]),

\[ E_{(0)}(\tilde{\omega}, \tilde{n}) = \langle z \rangle_{(0)} - \tilde{\omega} \frac{\partial \langle z \rangle_{(0)}}{\partial \tilde{\omega}} - \tilde{n} \frac{\partial \langle z \rangle_{(0)}}{\partial \tilde{n}} - 1, \]  

\[ E_{(1)}(\tilde{\omega}, \tilde{n}) = \frac{1}{2} \left( \langle z \rangle_{(1)} + 2 \langle z \rangle_{(0)} - \tilde{\omega} \frac{\partial \langle z \rangle_{(1)}}{\partial \tilde{\omega}} - \tilde{n} \frac{\partial \langle z \rangle_{(1)}}{\partial \tilde{n}} + \tilde{\omega}^2 \frac{\partial^2 \langle z \rangle_{(0)}}{\partial \tilde{\omega}^2} + 2 \tilde{n} \frac{\partial^2 \langle z \rangle_{(0)}}{\partial \tilde{\omega} \partial \tilde{n}} + \tilde{n}^2 \frac{\partial^2 \langle z \rangle_{(0)}}{\partial \tilde{n}^2} \right). \]  

As before we parameterize each of these quantities by means of the Schwarzschild semi-latus rectum \( p \) and eccentricity \( e \), rather than by \( \tilde{\omega} \) and \( \tilde{n} \). Thanks to our previous computation of the periastron advance for circular orbits, it is simple to deduce from (5.19) the circular-orbit limit of the energy, say \( \tilde{E}_{\text{circ}} = E_{(0)}(\tilde{\omega}) + \nu E_{(1)}(\tilde{\omega}) + \mathcal{O}(\nu^2) \). Indeed, while \( E_{(0)}^{\text{circ}} \) is obviously given by \( E_{(0)} \) for circular orbits (i.e. by taking \( e \to 0 \) and then changing \( p^{-1} = \chi \)), the GSF
contribution $E_{\text{circ}}^{(1)}$ is not directly given by the circular limit of (5.19b). Rather, it receives an additional contribution, explicitly reading

$$E_{\text{circ}}^{(1)} = E^{(1)} - \frac{\omega}{K_{(0)}^2} \frac{\partial E^{(0)}}{\partial n},$$

(5.20)

where the right-hand side is evaluated for $e = 0$ and $p = x^{-1}$. By this method we recover the known 4PN results for the GSF limit of the circular binding energy, namely \([27, 43, 14, 17]\)

$$E(x) = m - \frac{m x}{2} \left\{ 1 + \left( -\frac{3}{4} - \frac{\nu}{12} \right) x + \left( -\frac{27}{8} + \frac{19}{8} \nu \right) x^2 + \left( \frac{675}{64} + \frac{34445}{576} - \frac{205}{96} \pi^2 \nu \right) x^3 \right. \right.$$  

$$+ \left. \left( \frac{2835}{128} + \frac{98869}{5760} - \frac{6455}{1536} \pi^2 \nu - \frac{64}{3} \ln(16x) \right) x^4 \right\}.$$  

(5.21)

The angular momentum $L(x)$ can be computed in the same way. In that case, the relevant formulas for the rescaled momentum $\hat{L} \equiv L/(m^2 \nu)$ are equations (5.8b) and (5.9b) in [29], and we add a correction term similar to the one in (5.20). The result reads

$$L(x) = \frac{m^2 \nu}{\sqrt{x}} \left\{ 1 + \left( \frac{3}{2} + \frac{\nu}{6} \right) x + \left( \frac{27}{8} - \frac{19}{8} \nu \right) x^2 + \left( \frac{135}{16} + \left[ -\frac{6889}{144} + \frac{41}{24} \pi^2 \right] \nu \right) x^3 \right.$$  

$$+ \left. \left( \frac{2835}{128} + \frac{98869}{5760} - \frac{6455}{1536} \pi^2 \nu - \frac{64}{3} \ln(16x) \right) \nu x^4 + o(\nu, x^4) \right\}.$$  

(5.22)

Of course, we may explicitly check that $dE/dx = \omega dL/dx$ at fixed masses.

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Appendix A. Fourier series and long-time average

The components of the mass quadrupole moment $I_{ij}$ of generic elliptic orbits at Newtonian order, in the center-of-mass frame, are decomposed into the discrete Fourier series

$$I_{ij}(t) = \sum_{p=-\infty}^{+\infty} I_{ij} p e^{i p t},$$

(5.1)

where $\ell = n(t - t_0)$ is the mean anomaly, with $n = 2\pi/P$ the frequency associated to the period $P$ of the orbital motion, and $t_0$ is some instant of passage at periastron. The Fourier coefficients $p I_{ij}$ depend on $n$ and the orbit’s eccentricity $e$, and are fully available as closed-form combinations of Bessel functions in appendix B of [17] and appendix A of [72]. Averaging over one orbital period, we get

$$\langle I_{ij} \rangle = \int_{0}^{2\pi} \frac{d\ell}{2\pi} I_{ij}(\ell) = I_{0} I_{ij}.$$

(A.2)

However, in this paper it is important to define the time average of a function $f(t)$ in a more general manner, when the function is not necessarily periodic, by
\( \langle f \rangle \equiv \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} df(t). \)  
\hfill (A.3)

Such a long-time average coincides with the usual average for periodic functions. An important property of the long-time average (A.3) is that it implies \( \langle \dot{f} \rangle = 0 \) for any function \( f \) that remains bounded when \( t \to \pm \infty \).

Most relevant quantities can be evaluated explicitly by inserting the Fourier series (A.1). For instance, the tail factor (2.4) reads as

\[
T^{(s)}_{ij} = -2 \sum_{p+q \neq 0} \left( i p n \right)^s I_p I_q p^3 q^3 (p - q) \ln \left| \frac{p}{q} \right| e^{i(p+q)\ell},
\]

\hfill (A.4)

where we recall that the separation \( r \) between the particles has been used as the Hadamard Pf scale. The quantity \( \mathcal{Q}_H \) that was defined in section 2 to be the right-hand side of equation (2.11), and which is such that \( \dot{H} = \mathcal{Q}_H \), can be obtained by a straightforward computation as the following (double) Fourier series\(^6\)

\[
\mathcal{Q}_H = -\frac{m}{5} n^7 \sum_{p+q \neq 0} I_p I_q p^3 q^3 (p + q) \ln \left| \frac{p}{q} \right| e^{i(p+q)\ell}.
\]

\hfill (A.5)

Since \( \mathcal{Q}_H \) contains only modes with \( p + q \neq 0 \), it averages to zero: \( \langle \mathcal{Q}_H \rangle = 0 \). The oscillatory correction term \( \Delta H^{AC} \) in the conserved energy, as defined by equations (2.12) and (2.14), can be obtained directly by integrating term by term equation (A.5). Indeed, it is necessary and sufficient to discard any integration constant so that \( \langle \Delta H^{AC} \rangle = 0 \), and we obtain

\[
\Delta H^{AC} = -\frac{m}{5} n^6 \sum_{p+q \neq 0} I_p I_q p^3 q^3 (p - q) \ln \left| \frac{p}{q} \right| e^{i(p+q)\ell}.
\]

\hfill (A.6)

On the other hand, the actual integration constant which is to be added to get the conserved energy \( E \) requires a separate analysis, which was performed in [17]. The result is the DC term given in equation (2.13), which is proportional to the total averaged gravitational-wave energy flux.

Next, we present some formulas concerning the angular momentum, and notably the AC correction term \( \Delta P^{AC}_\varphi \) therein, which as we have seen enters into the modified radial action integral intervening into the first law; see equation (3.17). The Hamiltonian equation for \( p_\varphi \) was given in equation (2.15). With spatial coordinates \((x, y, z)\) adapted to the orbital motion into the plane \((x, y)\), i.e. such that the moving triad in the orbital plane reads \( \mathbf{n} = (\cos \varphi, \sin \varphi, 0) \), \( \lambda = (- \sin \varphi, \cos \varphi, 0) \) and \( \ell = n \times \lambda = (0, 0, 1) \), we have

\[
\frac{1}{2} \frac{\partial \hat{I}^{(3)}_{ij}}{\partial \varphi} = \ell^k \epsilon_{kli} \hat{I}^{(3)}_{ij},
\]

\hfill (A.7)

where the brackets around indices denote the STF projection. This equation can be checked for instance using the explicit expression (2.3). Hence we can readily express the right-hand side of the angular momentum equation (2.15) as the following double Fourier series,

\[
Q_{p_\varphi} = -\frac{4m}{5} n^6 \sum_{p+q \neq 0} \mathcal{K} p^3 q^3 \ln \left| \frac{p}{q} \right| e^{i(p+q)\ell}.
\]

\hfill (A.8)

\(^6\) We observe that here the Hadamard partie finie scale \( r \) has cancelled out.
It involves only non-zero modes \( p + q \neq 0 \), and we have defined

\[
K_{p,q} \equiv \ell \epsilon_{ijh} T_{j}^{p} T_{h}^{q} = (T_{p}^{xx} - T_{p}^{yy}) T_{q}^{xy} - T_{q}^{yx} (T_{q}^{xx} - T_{q}^{yy}),
\]

(A.9)

By integrating term by term the Fourier series (A.8), and ignoring any additive integration constant, we obtain directly the AC correction piece in the conserved angular momentum as defined by (2.17b):

\[
\Delta p_{AC}^{\phi} = -\frac{4m}{5} n^{5} \sum_{p+q \neq 0} i K_{p,q} p^{3} q^{3} \ln \left| \frac{p}{q} \right| \epsilon_{p+q} \ell, \quad \text{(A.10)}
\]

which is such that \( \langle \Delta p_{AC}^{\phi} \rangle = 0 \). On the other hand, the obtention of the constant DC piece is less trivial \([17]\) and the result has been given in equation (2.17a).

Finally, we want to control the extra term that was found in the effective action integral appearing into the first law. According to (3.17) we have

\[
R = R + 2m(\mathcal{F} - \mathcal{P}/\omega) - I
\]

with

\[
I = \frac{1}{2\pi} \oint \Delta p_{AC}^{\phi} d\phi = \frac{1}{n} \left\langle \dot{\phi} \Delta p_{AC}^{\phi} \right\rangle.
\]

(A.11)

The Fourier transform of the ‘instantaneous’ frequency \( \dot{\phi} \) is known to the Newtonian order, which is sufficient here since (A.11) is a small 4PN quantity. We have (see e.g. \([73]\])

\[
\dot{\phi} = n \left[ 1 + 2 \sum_{k=1}^{+\infty} \alpha_{k} \cos(k\ell) \right],
\]

(A.12)

where the coefficients read [with \( f \equiv (1 - \sqrt{1 - e^{2}}) / e \) and \( J_{k} \) being the usual Bessel function]

\[
\alpha_{k} = J_{k}(ke) + \sum_{r=1}^{+\infty} f^{r} \left[ J_{k-1}(ke) + J_{k+1}(ke) \right].
\]

(A.13)

Therefore by inserting into (A.11) both the Fourier series for the instantaneous frequency (A.12) and that for the AC correction term in the angular momentum (A.10), we obtain the result

\[
I = -\frac{4m}{5} n^{5} \sum_{p+q \neq 0} i K_{p,q} \alpha_{p+q} \left( \frac{p^{3} q^{3}}{p+q} \right) \ln \left| \frac{p}{q} \right|.
\]

(A.14)

(Since \( K_{p,q} = K_{-p,-q} \) we can check that \( I \) is real.) For circular orbits one must have \( p = \pm 2 \) and \( q = \pm 2 \), such that \( I = 0 \) in that case.

### Appendix B. Small-eccentricity limit

In this appendix, we collect some results that were used in section 5, for a test mass orbiting around a Schwarzschild black hole of mass \( M \), for nearly circular orbits. Hereafter, we omit the subscript \((0)\), but all of the formulas below hold only in the test-mass limit. Recall that the frequencies \( n \) and \( \omega \) are normalized using the total mass, which here reduces to the black hole mass, i.e. \((\hat{\omega}, \hat{n}) = (M\omega, Mn)\) in this appendix.

Instead of parameterizing the bound timelike geodesic of the test particle by means of the frequencies \( \hat{\omega} \) and \( \hat{n} \), or alternatively by means of the conserved specific energy \( \mathcal{E} \) and specific angular momentum \( \mathcal{L} \), we shall use the convenient ‘semi-latus rectum’ \( p \) and ‘eccentricity’ \( e \), defined such that \([74]\)
\[ E = \left[ \frac{(p - 2 - 2e)(p + 2 + 2e)}{p(p - 3 - e^2)} \right]^{1/2}, \quad L = \frac{pM}{\sqrt{p - 3 - e^2}}. \tag{B.1} \]

Following [75], we parameterize the particle’s radial motion (in Schwarzschild coordinates) using the ‘relativistic anomaly’ \( \chi \) as

\[ r(\chi) = \frac{pM}{1 + e \cos \chi}, \tag{B.2} \]

where \( \chi = 0 \) and \( \chi = \pi \) correspond to the periastron and the apastron passages, respectively. In terms of the orbital parameters \( p \) and \( e \), we have the usual Newtonian-looking expressions \( p = 2r_+r_-/[M(r_+ + r_-)] \) and \( e = (r_+ - r_-)/(r_+ + r_-). \)

Combining equations (B.1) and (B.2) with the well-known (first integral form of the) geodesic equations of motion for a test particle in Schwarzschild spacetime, the coordinate time period of the radial motion, \( P \), the corresponding proper time period, \( T \), as well as the accumulated azimuthal angle per radial period, \( \Phi \), are given by the definite integrals [74, 57, 38]

\[
P(p, e) = \int_0^{2\pi} \frac{d\rho}{d\chi} \, d\chi = \int_0^{2\pi} \frac{Mp^2 \sqrt{(p - 2 - 2e)(p - 2 + 2e)}}{(p - 2 - 2e \cos \chi)(1 + e \cos \chi)^2 \sqrt{p - 6 - 2e \cos \chi}} \, d\chi, \tag{B.3a}
\]

\[
T(p, e) = \int_0^{2\pi} \frac{d\tau}{d\chi} \, d\chi = \int_0^{2\pi} \frac{Mp^{3/2}}{(1 + e \cos \chi)^2} \sqrt{\frac{p - 3 - e^2}{p - 6 - 2e \cos \chi}} \, d\chi, \tag{B.3b}
\]

\[
\Phi(p, e) = \int_0^{2\pi} \frac{d\varphi}{d\chi} \, d\chi = 4 \sqrt{\frac{p}{p - 6 + 2e}} \, \text{ellipK} \left( \frac{4e}{p - 6 + 2e} \right), \tag{B.3c}
\]

where \( \text{ellipK}(k) \equiv \int_0^{\pi/2} (1 - k \sin^2 \theta)^{-1/2} \, d\theta \) is the complete elliptic integral of the first kind. Then, the radial frequency \( n \), the averaged azimuthal frequency \( \omega \), and the averaged redshift variable \( \langle \zeta \rangle \) are defined as

\[
n \equiv \frac{2\pi}{P}, \quad \omega \equiv \Phi \equiv \frac{\Phi}{P}, \quad \langle \zeta \rangle \equiv \frac{T}{P}. \tag{B.4}
\]

No closed form expressions for \( n(p, e) \), \( \omega(p, e) \) and \( \langle \zeta \rangle(p, e) \) are known. Still, the definite integrals (B.3) can be computed in the small-eccentricity limit \( e \ll 1 \), yielding the following Taylor series expansions:

\[
\dot{\omega}(p, e) = \frac{1}{p^{3/2}} - \frac{3}{2} \frac{p^2 - 10p + 22}{p^3/2(p - 2)(p - 6)} \, e^2 + \mathcal{O}(e^4), \tag{B.5a}
\]

\[
\dot{n}(p, e) = \frac{\sqrt{p - 6}}{p^3/2} - \frac{3}{4} \frac{2p^3 - 32p^2 + 165p - 266}{p^2(p - 2)(p - 6)^{3/2}} \, e^2 + \mathcal{O}(e^4), \tag{B.5b}
\]

\[
\langle \zeta \rangle(p, e) = \frac{\sqrt{p - 3}}{p} - \frac{3}{2} \frac{p^2 - 10p + 22}{\sqrt{p - 3}(p - 2)(p - 6)} \, e^2 + \mathcal{O}(e^4). \tag{B.5c}
\]

Here, we gave the results up to \( \mathcal{O}(e^4) \) only, because the formulas become too cumbersome at higher orders. However the expansions (B.5) can in principle be computed up to arbitrarily high orders in powers of \( e^2 \). Then, the partial derivatives of the dimensionless frequencies \( \dot{\omega} \), \( \dot{n} \) and \( \langle \zeta \rangle \) with respect to the orbital parameters \( p \) and \( e \) read as
\[
\left( \frac{\partial \tilde{\omega}}{\partial p} \right)_e = -\frac{3}{2p^{3/2}} + \mathcal{O}(e^3), \tag{B.6a}
\]
\[
\left( \frac{\partial \tilde{\omega}}{\partial e} \right)_p = -\frac{3(p^2 - 10p + 22)}{p^{3/2}(p - 2)(p - 6)} e + \mathcal{O}(e^3), \tag{B.6b}
\]
\[
\left( \frac{\partial \tilde{n}}{\partial p} \right)_e = -\frac{3}{2} \frac{p - 8}{p^2\sqrt{p - 6}} + \mathcal{O}(e^2), \tag{B.6c}
\]
\[
\left( \frac{\partial \tilde{n}}{\partial e} \right)_p = -\frac{3}{2} \frac{2p^3 - 32p^2 + 165p - 266}{p^2(p - 2)(p - 6)^{3/2}} e + \mathcal{O}(e^3), \tag{B.6d}
\]
\[
\left( \frac{\partial \langle z \rangle}{\partial p} \right)_e = \frac{3}{2p^{3/2}\sqrt{p - 3}} + \mathcal{O}(e^2), \tag{B.6e}
\]
\[
\left( \frac{\partial \langle z \rangle}{\partial e} \right)_p = \frac{3(p^2 - 10p + 22)}{\sqrt{p(p - 3)(p - 2)(p - 6)}} e + \mathcal{O}(e^3). \tag{B.6f}
\]

From these expressions, one can easily compute the determinant of the matrix transformation from \((p, e)\) to \((\tilde{\omega}, \tilde{n})\), namely
\[
D \equiv \left| \begin{array}{c}
\frac{\partial (\tilde{\omega}, \tilde{n})}{\partial (p, e)} \\
\frac{\partial (p, e)}{\partial (\tilde{\omega}, \tilde{n})}
\end{array} \right| = \frac{9}{4} \frac{4p^2 - 39p + 86}{p^{3/2}(p - 2)(p - 6)^{3/2}} e + \mathcal{O}(e^3). \tag{B.7}
\]

Combining the expansions (B.6a)–(B.6d) and (B.7), we get the following expressions for partial derivatives that appear, among others, in equation (5.9):
\[
\left( \frac{\partial p}{\partial \tilde{n}} \right)_{\tilde{\omega}} = -\frac{1}{D} \left( \frac{\partial \tilde{\omega}}{\partial e} \right)_p = \frac{4}{3} \frac{p^3 \sqrt{p - 6}}{p^2 - 10p + 22} \frac{p^2 - 10p + 22}{4p^2 - 39p + 86} + \mathcal{O}(e^2), \tag{B.8a}
\]
\[
\left( \frac{\partial e}{\partial \tilde{n}} \right)_{\tilde{\omega}} = +\frac{1}{D} \left( \frac{\partial \tilde{\omega}}{\partial p} \right)_e = -2 \frac{p^2(p - 2)(p - 6)^{3/2} \frac{1}{e} + \mathcal{O}(e)}{3 \frac{4p^2 - 39p + 86}{e}}, \tag{B.8b}
\]
\[
\left( \frac{\partial p}{\partial \tilde{\omega}} \right)_{\tilde{n}} = +\frac{1}{D} \left( \frac{\partial \tilde{n}}{\partial e} \right)_p = \frac{2}{3} \frac{p^{5/2}}{p^{3/2}} \frac{2p^3 - 32p^2 + 165p - 266}{4p^2 - 39p + 86} + \mathcal{O}(e^2), \tag{B.8c}
\]
\[
\left( \frac{\partial e}{\partial \tilde{\omega}} \right)_{\tilde{n}} = -\frac{1}{D} \left( \frac{\partial \tilde{n}}{\partial p} \right)_e = \frac{2}{3} \frac{p^{3/2}}{p^{5/2}} \frac{(p - 2)(p - 6)(p - 8)}{4p^2 - 39p + 86} + \mathcal{O}(e). \tag{B.8d}
\]

Finally, combining equations (B.6e), (B.6f) and (B.8a), and using the chain rule from \((p, e)\) to \((\tilde{\omega}, \tilde{n})\), we obtain the following expressions for the frequency derivatives of the average redshift that appear in equations (5.8), (5.15), (5.19) and (5.20):
\[
\left( \frac{\partial \langle z \rangle}{\partial \tilde{n}} \right) = -\frac{1}{2} \frac{p - 6}{p - 3 \frac{p^{3/2}}{p - 2}} e^3 + \mathcal{O}(e^4), \tag{B.9a}
\]
\[
\left( \frac{\partial \langle z \rangle}{\partial \tilde{\omega}} \right) = -\frac{p}{\sqrt{p - 3}} \left( 1 + \frac{e^2}{2(p - 3)} + \mathcal{O}(e^4) \right). \tag{B.9b}
\]
\[
\frac{\partial^2 \langle z \rangle}{\partial \eta^2} = \frac{2}{3} \frac{p^{3/2} (p - 6)^2}{\sqrt{p - 3(4p^2 - 39p + 86)}} + O(e^2), \tag{B.9c}
\]

\[
\frac{\partial^2 \langle z \rangle}{\partial \omega \partial n} = -\frac{2}{3} \frac{p^3 (p - 6)^{3/2} (p - 8)}{\sqrt{p - 3(4p^2 - 39p + 86)}} + O(e^2), \tag{B.9d}
\]

\[
\frac{\partial^2 \langle z \rangle}{\partial \omega^2} = \frac{p^{5/2} (p - 6)(2p^3 - 34p^2 + 185p - 298)}{3(4p^2 - 39p + 86)} + O(e^2). \tag{B.9e}
\]

The calculation of these partial derivatives requires the control of \( \langle z \rangle (p, e) \) to \( O(e^2) \), and that of all derived quantities at the same relative order in \( e^2 \). Note that the first derivative \( (B.9a) \), which is \( O(e^2) \), does not contribute to the final circular-orbit result in equation \( (5.12) \).

### Appendix C. Redshift for circular orbits

In this section, we derive the 4PN expressions for the particles’ redshifts in the particular case of circular orbits. For such orbits, \( \mathcal{R} = 0 \) and the first law (3.16) implies

\[
\frac{\partial E}{\partial \omega} \bigg|_{n_a} = \omega \frac{\partial L}{\partial \omega} \bigg|_{n_a}. \tag{C.1}
\]

Moreover, by considering variations with respect to the particles’ masses \( m_a \) at fixed circular-orbit frequency \( \omega \), the first law (3.16) yields the following expression for the constant redshift \( z_a \equiv \langle z_a \rangle \) of each particle:

\[
z_a = \frac{\partial E}{\partial m_a} \omega - \omega \frac{\partial L}{\partial m_a} = \frac{\partial \mathcal{M}}{\partial m_a} \omega, \tag{C.2}
\]

where we introduced \( \mathcal{M} \equiv E - \omega L \), heuristically the binary’s energy in a co-rotating frame. Now, the expressions for the conserved circular-orbit energy \( E(\omega) \) and the angular momentum \( L(\omega) \) were recently derived up to 4PN order [14, 16]. By substituting for equations (5.4b) and (5.5) of [14] into equation (C.2), we obtain the 4PN-accurate expression for the constant redshift of particle 1 as

\[
z_1 = 1 + \left( \frac{3}{4} \frac{3}{4} \Delta + \nu \right) x + \left( -\frac{9}{16} - \frac{9}{16} \Delta - \frac{\nu}{2} - \frac{1}{8} \Delta \nu + \frac{5}{24} \nu^2 \right) x^2
\]

\[
\quad + \left( -\frac{27}{32} \frac{27}{32} \Delta - \frac{\nu}{2} + \frac{19}{16} \Delta \nu - \frac{39}{32} \nu^2 - \frac{1}{32} \Delta \nu^2 + \frac{\nu^3}{16} \right) x^3
\]

\[
\quad + \left( -\frac{405}{256} \frac{405}{256} \Delta + \left[ \frac{38}{3} \frac{41}{64} \pi^2 \nu + \frac{6889}{384} \nu^3 + \frac{41}{64} \pi^2 \Delta \nu + \left( \frac{3863}{576} + \frac{41}{192} \pi^2 \right) \nu^2 - \frac{93}{128} \Delta \nu^2 + \frac{973}{864} \nu^3 - \frac{7}{1728} \Delta \nu^3 + \frac{91}{10368} \nu^4 \right) x^4
\]

\[
\quad + \left( -\frac{1701}{512} \frac{1701}{512} \Delta + \left[ \frac{329}{15} \frac{1291}{1024} \pi^2 + \frac{64}{5} \gamma_e + \frac{32}{5} \ln (16 \nu) \right] \nu
\]

\[
\quad + \left[ -\frac{24689}{3840} - \frac{1291}{1024} \pi^2 + \frac{64}{5} \gamma_e + \frac{32}{5} \ln (16 \nu) \right] \Delta \nu + \left[ -\frac{71207}{1536} + \frac{451}{256} \pi^2 \right] \Delta \nu^2
\]

\[
\quad + \left[ -\frac{1019179}{23040} + \frac{6703}{3072} \pi^2 + \frac{64}{5} \gamma_e + \frac{32}{5} \ln (16 \nu) \right] \nu^2 + \left[ \frac{356551}{6912} - \frac{2255}{1152} \pi^2 \right] \nu^3
\]

\[
+ \left( \frac{43}{576} \Delta \nu^3 - \frac{5621}{41472} \nu^4 + \frac{55}{41472} \Delta \nu^4 - \frac{187}{6208} \nu^5 \right) x^5 + o(x^6), \tag{C.3}
\]
where \( x \equiv (m \omega)^{2/3} \) is the frequency-related PN parameter and \( \Delta \equiv (m_2 - m_1)/m = \sqrt{1 - 4\nu} \) the reduced mass difference. (We assume \( m_1 \leq m_2 \). The expression for \( z_2 \) is easily deduced by setting \( \Delta \rightarrow -\Delta \) in equation (C.3). The expression (C.3) is valid for comparable masses, and in the small mass-ratio limit \( \nu \rightarrow 0 \) we obtain

\[
\begin{align*}
z_1 &= 1 + \left( -\frac{3}{2} + 2\nu \right) x + \left( -\frac{9}{8} + \frac{\nu}{2} \right) x^2 + \left( 1 - \frac{27}{16} + \frac{19}{8} \nu \right) x^3 + \left( -\frac{405}{128} + \frac{1621}{48} - \frac{41}{32} \pi^2 \right) \nu x^4 + o(\nu, x^4).
\end{align*}
\]

(C.4)

References

[1] Schäfer G 2011 Fundam. Theor. Phys. 162 167
[2] Blanchet L 2014 Living Rev. Relativ. 17 2
[3] Foffa S and Sturani R 2014 Class. Quantum Grav. 31 043001
[4] Porto R A 2016 Phys. Rep. 663 1
[5] Poisson E, Pound A and Vega I 2011 Phys. Rev. D 84 084037
[6] Blanchet L 2014 Fundam. Theor. Phys. 177 147
[7] Pound A 2015 Fundam. Theor. Phys. 179 399
[8] Damour T 2014 Fundam. Theor. Phys. 177 111
[9] Le Tiec A 2014 Int. J. Mod. Phys. D 23 1430022
[10] Buonanno A and Sathyaprakash B S 2015 General Relativity, Gravitation: a Centennial Perspective ed A Ashtekar et al (Cambridge: Cambridge University Press) p 287
[11] Jaranowski P and Schäfer G 2012 Phys. Rev. D 86 064053
[12] Jaranowski P and Schäfer G 2013 Phys. Rev. D 87 081503
[13] Jaranowski P and Schäfer G 2015 Phys. Rev. D 92 124043
[14] Damour T, Jaranowski P and Schäfer G 2014 Phys. Rev. D 89 064058
[15] Damour T, Jaranowski P and Schäfer G 2016 Phys. Rev. D 93 084014
[16] Bernard L, Blanchet L, Bohé A, Faye G and Marsat S 2015 Phys. Rev. D 93 084037
[17] Bernard L, Blanchet L, Bohé A, Faye G and Marsat S 2017 Phys. Rev. D 95 044026
[18] Foffa S and Sturani R 2013 Phys. Rev. D 87 064011
[19] Galley C R, Leibovich A K, Porto R A and Ross A 2016 Phys. Rev. D 93 124010
[20] Foffa S, Mastrolia P, Sturani R and Sturm C 2017 Phys. Rev. D 95 104009
[21] Marchand T, Blanchet L and Faye G 2016 Class. Quantum Grav. 33 244003
[22] Blanchet L and Damour T 1988 Phys. Rev. D 37 1410
[23] Blanchet L 1993 Phys. Rev. D 47 4392
[24] Foffa S and Sturani R 2013 Phys. Rev. D 87 044056
[25] Porto R A and Rothstein I Z 2017 arXiv:1703.06433 [gr-qc]
[26] Friedman J L, Uryû K and Shibata M 2002 Phys. Rev. D 65 064035
[27] Friedman J L, Uryû K and Shibata M 2002 Phys. Rev. D 70 129904 (erratum)
[28] Le Tiec A, Blanchet L and Whiting B F 2012 Phys. Rev. D 85 064039
[29] Blanchet L, Buonanno A and Le Tiec A 2013 Phys. Rev. D 87 024030
[30] Le Tiec A 2015 Phys. Rev. D 92 084021
[31] Gralla S E and Le Tiec A 2013 Phys. Rev. D 88 044021
[32] Le Tiec A 2014 Class. Quantum Grav. 31 097001
[33] Fujita R, Isoyama S, Le Tiec A, Nakano H, Sago N and Tanaka T 2017 Class. Quantum Grav. 34 134001
[34] Detweiler S 2008 Phys. Rev. D 77 124026
[35] Sago N, Barack L and Detweiler S 2008 Phys. Rev. D 78 124024
[36] Blanchet L, Detweiler S, Le Tiec A and Whiting B F 2010 Phys. Rev. D 81 064004
[37] Blanchet L, Detweiler S, Le Tiec A and Whiting B F 2010 Phys. Rev. D 81 084033
[38] Barack L and Sago N 2011 Phys. Rev. D 83 084023
[39] Akcay S, Le Tiec A, Barack L, Sago N and Warburton N 2015 Phys. Rev. D 91 124014
[40] Shah A G, Keidl T S, Friedman J L, Kim D-H and Price L R 2011 Phys. Rev. D 83 064018
[40] Shah A G, Friedman J L and Keidl T S 2012 Phys. Rev. D 86 084059
[41] Shah A G, Friedman J L and Whiting B F 2014 Phys. Rev. D 89 064042
[42] van de Meent M and Shah A G 2015 Phys. Rev. D 92 064025
[43] Bini D and Damour T 2013 Phys. Rev. D 87 121501
[44] Bini D and Damour T 2014 Phys. Rev. D 89 064063
[45] Bini D and Damour T 2014 Phys. Rev. D 89 104047
[46] Bini D and Damour T 2015 Phys. Rev. D 91 064050
[47] Johnson-McDaniel N K, Shah A G and Whiting B F 2014 Phys. Rev. D 89 064042
[48] van de Meent M and Shah A G 2015 Phys. Rev. D 92 064025