Slowing down of so-called chaotic states: “Freezing” the initial state

M. Belger¹, S. De Nigris†², X. Leoncini‡¹,³

¹ Aix-Marseille Université, Université de Toulon, CNRS, CPT UMR 7332, 13288 Marseille, France
² Department of Mathematics and Namur Center for Complex Systems-naXys, University of Namur, 8 rempart de la Vierge, 5000 Namur, Belgium
³ Center for Nonlinear Theory and Applications, Shenyang Aerospace University, Shenyang 110136, China

Abstract
The so-called chaotic states that emerge on the model of XY interacting on regular critical range networks are analyzed. Typical time scales are extracted from the time series analysis of the global magnetization. The large spectrum confirms the chaotic nature of the observable, anyhow different peaks in the spectrum allows for typical characteristic time-scales to emerge. We find that these time scales $\tau(N)$ display a critical slowing down, i.e they diverge as $N \to \infty$. The scaling law is analyzed for different energy densities and the behavior $\tau(N) \sim \sqrt{N}$ is exhibited. This behavior is furthermore explained analytically using the formalism of thermodynamic-equations of the motion and analyzing the eigenvalues of the adjacency matrix.

1 Introduction
Macroscopic chaotic behavior is often linked to out-of-equilibrium states, one of the most prominent display of such phenomenon is most certainly turbulence. The resulting chaotic or turbulent states result from various macroscopic instabilities and bifurcations, and their persistence is usually driven by strong gradients or energy fluxes. When considering isolated systems with many degrees of freedom, some similar behavior can be found, but typically it is a transient during which, starting from a given initial condition, the system relaxes to some thermodynamical equilibrium [1]. Microscopic “molecular” chaos plays there an important role for relaxation; however, in the equilibrium state, macroscopic variables are at rest, despite the microscopic chaos. It is nevertheless possible to extend this transient state: indeed in recent years there has been an extensive study of the so-called quasi-stationary states (QSSs), that emerge after a violent relaxation in systems with long-range interactions [2–5]. These states have the peculiarity that their lifetime diverges with the number of constituents, so that the limits $N \to \infty$ and $t \to \infty$ do not commute. In fact it has been shown that some of these states are non-stationary but can display regular oscillations and, therefore, they represent a different kind of steady state [6–9]. Moreover, as can be observed in [10], both the lifetime of the state and the “transient” relaxation time from the

¹ email address: denigris.sarah@gmail.com
² Corresponding author, email address: xavier.leoncini@cpt.univ-mrs.fr

ISSN 2164-6376, eISSN 2164-6414/S-see front materials © 2014 L&H Scientific Publishing, LLC. All rights reserved.
QSS to the equilibrium diverge with system size. During these relaxation periods we can expect to observe some long lived but transient chaotic-like features in these isolated states [11]. As mentioned, these transient periods correspond to some kind of relaxation, nevertheless, more recently, persistent chaotic macroscopic behavior in an isolated system has been exhibited. These states occur over a wide range of energy. They were first spotted on systems of rotators evolving on a regular lattice, with a critical range of interaction and number of neighbors [12]. Further studies have shown that this behavior occurred as well on so-called lace networks, when the effective network dimension was around \(d = 2\) [13]. Studying these systems for different number of constituents \(N\) and a fixed density of energy \(\epsilon\), the chaotic behavior of the order parameter was persistent, and it appeared that the width of the fluctuations around its mean value was not changing with \(N\), implying an infinite susceptibility over a given range of values of \(\epsilon\). However it was evident, at least qualitatively, that some changes in the characteristic time scales of the fluctuations were present and depended on the system's sizes.

In this paper we focus on this dependence of the fluctuations time scales with system size, we shall show that the observed scaling \(\tau(N) \sim \sqrt{N}\) is different than the typical relaxation time scales observed in QSS, and provide a theoretical explanation of these time scales in the low energy range. The paper is organized as follows: in the first part we describe the considered model and remind the reader of the previously obtained results. We then move on to a numerical study of the characteristic time scales of the fluctuations by analyzing the frequency spectrum of the measured order parameter, where a scaling behavior \(\tau(N) \sim \sqrt{N}\) is clearly exhibited. The presence of a large and broad spectrum allows us to infer that the signal is indeed chaotic. We then perform an analytical study of the thermodynamical wave spectrum at low energies and we indeed confirm the numerically exhibited scaling. This evidence confirms that these chaotic states are not QSS's and that the chaotic behavior can be expected to be an actual permanent feature or characteristic of these “equilibrium” states.

2 Description of the model

Originally the model we shall consider was tailored in order to uncover the threshold of a long range interacting system. As such it was inspired from the fact that the so-called \(\alpha\)-HMF model (see [14, 15]) displayed similar thermodynamical properties as the mean field model (for \(\alpha < 1\)), also dubbed the HMF model, which over the years has become de facto the paradigmatic model to study and test new ideas when studying long range system. In the \(\alpha\)-HMF rotators are located on a one-dimensional lattice, and the coupling constant \(J_{ij}\) between the spins decreases according to a power-law with the distance between the rotators \(J_{ij} \sim |i - j|^{-\alpha}\), so that all rotators are coupled. The initial idea of the proposed model was to consider a range \(r\) of neighbors who equally interact with a sharp edge, meaning that \(J_{ij} \sim \text{Cst}\) if \(|i - j| < r\) and 0 if \(|i - j| \geq r\). We set up a window function, but what is important here is that we allow \(r\) to be a function of the total number of spins \(N\). The range is parametrized using a characteristic exponent \(1 \leq \gamma \leq 2\), which measures as well the total number of links (interactions) being present in the system. When \(\gamma = 1\), we are on a one-dimensional chain with a short range interactions (in our case with just nearest neighbors interactions), while when \(\gamma = 2\), we retrieve the mean field model, with all rotators equally interacting with each other. To get more specific we now present the details of the rotators model placed on a one-dimensional lattice with periodic boundary conditions. The Hamiltonian of the considered system writes

\[
H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2k} \sum_{i,j}^{N} \epsilon_{i,j}(1 - \cos(q_i - q_j)),
\]

where \(k\) is the constant number of links (connections) per rotator which scales with \(\gamma\) as

\[
k \equiv \frac{1}{N} \sum_{i,j} \epsilon_{i,j} = \frac{2^{2-\gamma}(N-1)\gamma}{N},
\]

and is related to the range by the simple relation \(k = 2r\). The matrix \(\epsilon_{i,j}\) is the adjacency matrix, defined as
Fig. 1 Magnetization versus time, for a fixed density of energy $\epsilon \approx 0.4$ and different values of $\gamma$. The size of the system is $N = 2^{13}$. For $\gamma = 1.25$ there is no magnetization (the residual magnetization is due to finite size effects see for instance ), for $\gamma = 1.75$ we observed a finite almost constant magnetization, while for $\gamma = 1.5$ large fluctuations of order one are observed. Simulations have been performed using a time step $\delta t = 10^{-3}$.

$$\epsilon_{i,j} = \begin{cases} 1 & \text{if } \|i-j\| \leq r \\ 0 & \text{otherwise} \end{cases},$$

(3)

where $\|i-j\|$ stands for the smallest distance between two site on the one dimensional lattice with periodic boundary conditions. From the Hamiltonian we directly get the equations of the motion of the rotators.

$$\dot{q}_i = p_i$$

(4)

$$\dot{p}_i = -\frac{1}{k} \sum_{j=1}^{N} \epsilon_{i,j} \sin(q_i - q_j).$$

(5)

A full study of the equilibrium properties of this model has been made in [12,16]. The order parameter that we monitored is the total magnetization of the system $M$, defined as

$$M = \begin{cases} M_x = \frac{1}{N} \sum_{i=1}^{N} \cos q_i \\ M_y = \frac{1}{N} \sum_{i=1}^{N} \sin q_i \end{cases} = M \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}$$

The results are as follows:

- For $\gamma < 1.5$ the system behaves as a short range model, meaning that no order parameter emerges in the thermodynamic limit and no phase transition exists. For the short range case ($\gamma = 1$), this result is consistent with the predictions of the Mermin-Wagner theorem, which predicts that no order parameter can exist for systems with dimensions $d \leq 2$, due to the existence of a continuous symmetry group (here the global translation/rotation symmetry $q_i \to q_i + \theta$).

- For $\gamma > 1.5$ the system behaves like the mean field model, meaning that a second order transition at a critical density of energy of $\epsilon_c = 0.75$, is observed. All curves $M_x(\epsilon)$ appear as independent of $\gamma$ and fall on the mean field one.

- For $\gamma = 1.5$ for a range of temperatures below the critical energy one, a chaotic state is observed. The magnetization displays steady and large incoherent fluctuations, which do not appear to be dependent on
The system size is here \( N = 2^9 \). The final time and data sampling of the simulation is identical to the one performed in Fig. 1. We notice that the fluctuations are indeed of the same order, however we can notice that the typical time scale of the fluctuations appear to be faster than in Fig. 1.

system size, implying an infinite susceptibility. The time dependence of these fluctuations is the subject of this paper. Note also that the transition of the Berezinsky-Kosterlitz-Thouless type has not been detected (see for details [12]).

To illustrate the phenomena described, we have plotted in Fig. 1 the evolution of the order parameter at a fixed density of energy \( \epsilon \) for three different values of \( \gamma \) and a system size of \( N = 2^{13} \). Simulations have been performed using the optimal fifth order symplectic integrator described in [17], and the fast-Fourier transform made use of the FFTW package. We can notice the peculiar regime that appears for \( \gamma = 1.5 \) where the magnetization displays what looks like a macroscopic chaotic behavior.

In the next section we shall study in more detail the temporal behavior of the order parameter in these chaotic states.

3 Critical slowing down

3.1 Numerical study

In this section we study numerically the behavior of the order parameter for different values of \( \epsilon, \gamma = 1.5 \) and different system sizes with the aim of uncovering the timescales characterizing the fluctuations. Indeed we can notice in Fig. 2 that the typical time scale of the fluctuations appears to depend on the system size, as the magnetization fluctuations are much faster for \( N = 512 \) (Fig. 2) than for \( N = 8192 \) (Fig. 1). Also, even though the signal plotted in Fig. 2 looks turbulent, it may just be the consequence of the presence of a few unrelated modes. In order to confirm the chaotic nature of the signal, we decided to analyze its Fourier spectrum. An example of such spectrum is displayed in Fig. 3. We can notice that the spectrum is continuous, differently from the one given by a quasi-periodic signal, so it is definitely of the chaotic (turbulent) type. However we can notice as well some broad peaks which are associated to the decreasing harmonics in this signal. Indeed, we can relate these peaks to the typical scale of fluctuations that visually appeared in the figures 2 and 1. In order to determine the scaling with system size, we performed a sequence of numerical simulations, with a fixed density of energy, fixed total time and different system sizes. In these simulations, the initial condition is extracted from a Gaussian distribution for both the \( p_i \)'s and \( q_i \)'s. The signal analysis is performed over the data that has been averaged during the second half of the total simulation's time. The results are displayed in Fig. 4, where the
Fig. 3 Fourier spectrum of a “chaotic” signal of the order parameter. The considered system size is $N = 2^{14}$. We can notice that we obtain a continuous spectrum with some broad peaks, and associated harmonics. The time dependence is typically chaotic, and does not correspond to a quasi-periodic signal.

Fig. 4 Scaling of the localization of the frequency peaks versus system size. The locations of the first three peaks (harmonics) are represented. One notices a global uniform scaling of the different peaks, and a slowing down of the typical fluctuation time. The scaling law shows a decrease of the peak frequencies as $f \sim 1/\sqrt{N}$. 
locations of the three first peaks displayed in Fig. [3] are represented versus system size in a log-log plot. One notices a universal scaling of the typical fluctuation time scale, with all peaks having a frequency that decreases as $f \sim N^{-1/2}$. This scaling was initially not anticipated as one would naturally expect a behavior similar to what has been observed for QSS’s, with a typical lifetime scaling $\tau \sim N^{\alpha}$, with $\alpha = 1$ or higher values. The observed scaling $\tau \sim N^{1/2}$ is another confirmation that these chaotic states do not correspond to transient regimes, but are “steady”. We had already run very large time simulations in without noticing any visible change in the dynamics of the order parameter, but a transient with a large value of $\alpha$ could still have been possible.

We now move on to a theoretical hint at the observed scaling law, and the confirmation as well that these are not transient states.

3.2 Theoretical analysis

In order to perform our analysis we carry out a similar calculation as the one performed in [12], that had allowed us to prove that $\gamma = 1.5$ was a threshold between the short range and the long range behavior. The method was proposed in a general context and explicitly developed for lattice’s system in [18]. In order to be more self-consistent we review the method from the start, and apply it to the considered system (1).

As already stated we consider a lattice (in dimension $D = 1$ for our system) of $N$ sites with coordinates $x_i = 1, \cdots, N$. At each site $i$ we have a particle, coupled to some neighbors, each having a momentum $p_i$ and conjugate coordinate $q_i$. We recall that we shall consider thermodynamical equilibrium properties (even though we are looking at some dynamical properties) so the units are such that the lattice spacing, the Boltzmann constant, and the mass are equal to one. Also from the form of the Hamiltonian (1), a calculation within the canonical ensemble will imply that the $p_i$ are distributed according to a Gaussian distribution. Since we are working on a lattice, with periodic boundary conditions, we can represent the momentum as a superposition of Fourier modes:

$$p_i = \sum_{k=0}^{Nk_0} \alpha_k \cos(kx_i + \phi_k),$$  \hspace{1cm} (6)

where the wavenumber $k$ is in the reciprocal lattice (an integer multiple of $k_0 = 2\pi/N^{(1/D)}$), the wave amplitude is $\alpha_k$, and since we want the momenta to be Gaussian distributed variables in the thermodynamic limit, we consider that the random phase $\phi_k$ is uniformly distributed on the circle. Therefore, we should, given some conditions on the amplitude, be able to apply the central limit theorem. The momentum set is labeled, using (5), with the set of phases $\ell \equiv \{\phi_k\}$. Note also that this equation can also be interpreted as a change of variables, from $p$ to $\alpha$, with constant Jacobian (the change is linear and we chose an equal number of modes and particles).

Before proceeding, we would like to make some remarks. First, it is clear from the Hamiltonian (1) that we have a translational invariance, which implies that the total momentum of the system is conserved. Since physics should not change we make a simple Galilean transform in order to choose a reference frame where the total momentum is zero. The total momentum is directly linked to the zero mode, so this choice implies thus that we have to take $\alpha_0 = 0$. Second, since we know that in the canonical ensemble the variance of $p_i$ is fixed and equal to the temperature of the system, we shall assume that the $\alpha_k$ are all of the same order (we need a large number of relevant modes for the center-limit theorem to apply). Given these assumptions and using the relation $\langle p_i^2 \rangle = \sum \alpha_k^2 / 2$ (we average over the random phases), we write $\langle p_i^2 \rangle \approx T$ and obtain $\alpha_k^2 \approx O(|T/N|)$ (we call this relation the Jeans condition [19]).

We now move on to the associated conjugated variable of $p_i$, since we have $q_i = p_i$, we write it as

$$q_i = \alpha_0 + \sum_{k=k_0}^{Nk_0} \alpha_k \cos(kx_i + \phi_k),$$  \hspace{1cm} (7)

where $\alpha_0$ is a constant since $\alpha_0 = 0$, corresponding to the constant average of the $q_i$’s. In order to proceed, since we are below the mean-field critical temperature, we make a low temperature hypothesis: thus, we can assume that neighboring $q_i$’s are not too different although no global long range order exists. Assuming that
the difference \( q_i - q_j \) when the rotators interact is small, we expand the Hamiltonian and obtain:

\[
H = \sum_i \frac{p_i^2}{2} + \frac{1}{4k} \sum_{i,j} \epsilon_{ij} (q_i - q_j)^2.
\]

Using the previous expressions derived for \( q_i \) and \( p_i \) and averaging over the random phases we end up with an effective Hamiltonian

\[
\frac{\langle H \rangle}{N} = \frac{1}{2} \sum_{l=1}^{N} \alpha_l^2 + \alpha_l^2 (1 - \lambda_l),
\]

where

\[
\lambda_l = \frac{2}{k} \sum_{m=1}^{k/2} \cos \left( \frac{2\pi ml}{N} \right)
\]

are the eigenvalues of the adjacency matrix.

We can extract from this a dispersion relation, indeed we have

\[
\frac{d}{dt} \left( \frac{\partial \langle H \rangle}{\partial \dot{\alpha}_l} \right) = -\frac{\partial \langle H \rangle}{\partial \alpha_l} \ddot{\alpha}_l = -\omega_l^2 \alpha_l
\]

(10)

As mentioned this computation was already used in in order to show that the critical threshold between short range and long range behavior was \( \gamma = 1.5 \); we used this formalism in order to compute analytically the value of the magnetization in the thermodynamic limit. In the present case, we stress the fact that the dispersion relation (10) embeds also some dynamical informations since we have access to the typical frequencies that we can expect to find in the system. This dynamical information was not used in previous papers leveraging this formalism, but nevertheless the understanding of the observed scaling law could provide new avenues for this approach.

We can now use this dynamical feature in order to explain the critical slowing down by monitoring how the spectrum behaves as we change the size of the system, for the specific situation with \( \gamma = 1.5 \), i.e. \( k \sim \sqrt{N} \). For this purpose, we consider a specific mode \( l \); we have

\[
\omega_l^2 = (1 - \lambda_l)
\]

(11)

\[
= 1 - \frac{1}{k} \left[ \sin \left( \frac{(k+1)l\pi}{N} \right) \sin \left( \frac{1}{N} \pi \right) \right] - 1.
\]

(12)

In order to proceed we shall consider that \( N \to \infty \), thus \( N \gg \sqrt{N} \), i.e \( N \gg k \) and that \( l \) is fixed, we can then perform an expansion of the expression (11), and in order to avoid the first order \( \omega_l^2 = 0 \) result, we shall expand it to third order using \( \sin(x) = x - x^3/6 + o(x^3) \). We then obtain (omitting the \( o(x^3) \) notation)

\[
\omega_l^2 \approx k + 1 - \frac{1}{k} \left[ \frac{(k+1)l\pi}{N} - \frac{(k+1)l^3\pi^3}{6N^3} \right] \approx k + 1 - \frac{l^3\pi^2}{6N^2}
\]

\[
\approx k + 1 \left[ 1 - \frac{l^3\pi^2}{6N^2} \right] \approx k \left[ \frac{(k+1)^2l^2\pi^2}{6N^2} - \frac{l^3\pi^2}{6N^2} \right]
\]

\[
\approx \frac{k^2}{N^2} \sim \frac{1}{N}.
\]

We recover analytically the critical slowing down exhibited numerically in Fig. 4 and confirm that the scaling law leads to \( \omega \sim 1/\sqrt{N} \), and thus characteristic time scales of order \( \sqrt{N} \).
4 Concluding remarks

In this paper we have analyzed the typical time scale $\tau(N)$ over which the chaotic behavior (fluctuations) of the order parameter evolves as a function of system size. First after a numerical study, we have exhibited that $\tau(N) \sim \sqrt{N}$. Then this behavior has been afterwards confirmed theoretically, by showing that each of the frequencies, associated to modes of the dual lattice, scaled as $\omega_k \sim 1/\sqrt{N}$ with system size. The direct consequences of these results go in two directions. First we confirmed the chaotic states observed and discussed in [12,13,16] indeed are not a transient state like a QSS and, because of the presence of a large continuous spectrum, we can as well confirm the chaotic nature of the macroscopic behavior in these states, much like a turbulent one. Second, when performing our theoretical analysis using the formalism developed in [18], we were able to show for the first time that it is possible to uncover some dynamical information from this formalism, and the successful prediction of the scaling law shows that the formalism is adequate to predict some finite size dynamical features of systems with many degrees of freedom with underlying Hamiltonian microscopic dynamics.

As a whole the typical decay of the characteristic time scale has another important consequence: indeed should we consider an $N \to \infty$ limit, the fluctuations should stop and the system will end up frozen in its initial magnetic state. It is important to comment that still the infinite susceptibility would remain, so the system should remain extremely sensitive to any external perturbation. This critical slowing down with system size has been observed in other types of networks with different structure. Thus, beside confirming the same behavior arises considering lace networks as a substrate, an interesting perspective would be to check if there are any similarities to what has been already reported, and if this phenomenon could be of practical use, like for instance to slow down the waves propagation in some localized regions.

acknowledgements

S.D.N and X.L. would like to thank S. Ogawa for fruitful discussions and remarks during the preparation of this manuscript.

References

[1] E. Fermi, J. Pasta, and S. Ulam. Los Alamos Reports, (LA-1940), 1955.
[2] T. Dauxois, S. Ruffo, E. Arimondo, and M. Wilkens, editors. Dynamics and Thermodynamics of Systems with Long Range Interactions, volume 602 of Lect. Not. Phys. Springer-Verlag, Berlin, 2002.
[3] Alessandro Campa, Thierry Dauxois, and Stefano Ruffo. Statistical mechanics and dynamics of solvable models with long-range interactions. Phys. Rep., 480:57–159, 2009.
[4] A. Campa, T. Dauxois, D. Fanelli, and S. Ruffo. Physics of Long-Range Interacting Systems. Oxford University Press, 2014.
[5] Y. Levin, R. Pakter, F. B. Rizzato, T. N. Teles, and F. P. C. Benetti. Nonequilibrium statistical mechanics of systems with long-range interactions. Phys. Rep., 535:1–60, 2014.
[6] J. P. Holloway and J. J. Dorning. Phys. Rev. A, 44:3856, 1991.
[7] T. L. Van den Berg, D. Fanelli, and X. Leoncini. Stationary states and fractional dynamics in systems with long range interactions. EPL, 89:50010, 2010.
[8] Y. Y. Yamaguchi. Phys. Rev. E, 84:016211, 2011.
[9] S. Ogawa, J. Barre, H. Morita, and Y. Y. Yamaguchi. Phys Rev. E, 89:063007, 2014.
[10] Alessio Turchi, Duccio Fanelli, and Xavier Leoncini. Existence of Quasi-stationary states at the Long Range threshold. Commun. Nonlinear. Sci. Numer. Simulat., 16(12):4718–4724, December 2011.
[11] F. L. Antunes, F. P. C. Benetti, R. Pakter, and Y. Levin. Chaos and relaxation to equilibrium in systems with long-range interactions. Phys. Rev. E, 92:052123, 2015.
[12] Sarah De Nigris and Xavier Leoncini. Emergence of a non trivial fluctuating phase in the XY model on regular networks. EPL, 101:10002, 2013.
[13] Sarah De Nigris and X. Leoncini. Crafting networks to achieve, or not achieve, chaotic states. Phys. Rev. E, 91:042809,
2015.

[14] Celia Anteneodo and Constantino Tsallis. Breakdown of exponential sensitivity to initial conditions: Role of the range of interactions. *Phys. Rev. Lett.*, 80:5313–5316, 1998.

[15] Alessandro Campa, Andrea Giansanti, Daniele Moroni, and Constantino Tsallis. Long-range interacting classical systems: universality in mixing weakening. *Phys. Lett. A*, 286:251, 2001.

[16] Sarah De Nigris and Xavier Leoncini. Critical behaviour of the XY -rotors model on regular and small world networks. *Phys. Rev. E*, 88(1-2):012131, 2013.

[17] R. I. McLachlan and P. Atela. The accuracy of symplectic integrators. *Nonlinearity*, 5:541–562, 1992.

[18] Xavier Leoncini and Alberto Verga. Dynamical approach to the microcanonical ensemble. *Phys. Rev. E*, 64(6):066101, 2001.

[19] J. H. Jeans. *The dynamical theory of gases*. Cambridge Univ. Press, 1916.