STABLE MODELS OF HECKE OPERATORS

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Abstract: We investigate the geometry of correspondences between curves, and prove that correspondences over a non-Archimedean valued field have potential semi-stable reduction. Such models allow for an explicit description of the action of the correspondence on the cohomology of the generic fibre. Finally, we explicitly determine the skeleton of the stable model \( U_p \) of the Atkin operator on \( X_0(Np) \) at \( p \), and recover results on the action of \( U_p \) on classical and overconvergent modular forms.

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1. Introduction

1.1. Semi-stable models of correspondences. The theory of semi-stable reduction for curves has found many powerful applications, and provides an invaluable tool for understanding Galois representations arising in the cohomology of curves via Picard-Lefschetz theory. Coleman \([Col03]\) and Liu \([Liu06]\) prove potential semi-stable reduction theorems for finite morphisms between curves over a general class of fields. We generalise their results to correspondences between curves. More precisely, let \( C : Y_1 \leftarrow X \rightarrow Y_2 \) be a correspondence between smooth, proper, geometrically connected curves \( X, Y_1 \) and \( Y_2 \) over a non-Archimedean field \( K \), where the maps are finite. We prove:

Theorem A. After a finite separable field extension of \( K \), we can find a correspondence \( \mathcal{E} \):

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\pi_2} & \mathcal{Y}_2 \\
\mathcal{Y}_1 & \xleftarrow{\pi_1} & X
\end{array}
\]

where \( \mathcal{X}, \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) are semi-stable models for \( X, Y_1 \) and \( Y_2 \) and \( \pi_1, \pi_2 \) are finite morphisms, such that \( \mathcal{E} \) restricts to \( C \) on the generic fibres.

In fact, we prove something stronger. We assure that we can always find a skeletal semi-stable model \( \mathcal{E} \), which imposes the additional constraint that the singular loci of the special fibres of \( \mathcal{Y}_1, \mathcal{Y}_2 \) pull back to the singular locus of the special fibre of \( \mathcal{X} \). This condition allows us to define the skeleton associated to \( \mathcal{E} \), which is a crucial notion for applications. Theorem A therefore generalises and strengthens the
results of Coleman [Col03] and Liu [Liu06] on simultaneous semi-stable reduction for finite morphisms between curves, and should be considered as a strong analogue of the potential semi-stable reduction theorem of Deligne-Mumford [DM69].

1.2. The action on cohomology. Let $l$ be a prime different from the residue characteristic of $K$, and $C$ a correspondence between smooth, proper, geometrically connected curves over $K$ as above. We the linear map

$$C^* : H^i_{\text{ét}}(Y_1, \mathbb{Q}_l) \to H^i_{\text{ét}}(Y_2, \mathbb{Q}_l)$$

by $\pi_2^* \circ \pi_1^*$. This morphism is strict in the sense that it respects the weight-monodromy filtrations

$$C^* : M_j H^i_{\text{ét}}(Y_1, \mathbb{Q}_l) = M_j H^i_{\text{ét}}(Y_2, \mathbb{Q}_l)$$

for all $i$. Now suppose we have a skeletal semi-stable model $\mathcal{C}$ of $C$. This makes the computation of the action of $C$ on $H^1_{\text{ét}}$ completely combinatorial, as we can compute information about the map on cohomology induced by $C$ by restricting to the graded pieces of different weights. We may equally describe this as the étale realisation of the induced map

$$C^* : \mathcal{J}^3(Y_2) \to \mathcal{J}^3(Y_1)$$

between the connected components of the Néron models of the Jacobian. The computation of the map on the toric parts can be described in the same way from the skeleton of $\mathcal{C}$ by pulling back and pushing forward cycles. The Néron component group $\Phi$ can be characterised in terms of the monodromy pairing on the toric part, and we may consequently compute the induced map $C^* : \Phi_1 \to \Phi_2$. All of this is discussed in 3.10 and 3.11.

1.3. Hecke operators. Theorem A guarantees in particular that Hecke operators, considered as correspondences between modular curves, obtain semi-stable reduction after a finite separable base change. We explicitly deduce the stable model of $U_p$ on $X_0(Np)$ from the work of Edixhoven [Edi90], and determine its associated skeleton $\Sigma_{U_p}$.

**Theorem B.** Let $p \geq 5$ prime to $N$, then the correspondence $U_p$ on $X_0(Np)$ obtains semi-stable reduction over $\mathbb{Q}_{\wp}(\sigma)$ where $\sigma^{(p^2-1)/2} = p$. The special fibre and skeleton are described in Figure 1.

The stable model of Hecke operators $T_l$ away from the level is implicit in the work of Katz-Mazur [KM85] at primes dividing the level at most once, which lies at the heart of the graph algorithm of Mestre-Oesterlé [Mes86]. As an application of Theorem B we obtain a graph algorithm at $p$, and recover certain results from [Rib90] on the action of $U_p$ on various parts of the Jacobian $\mathcal{J}^{0,N}(Np)$.

1.4. Too supersingular elliptic curves. The components of the stable model $U_p$ we determine in Theorem B have a clear moduli interpretation. We say an elliptic curve over $\mathcal{C}_p$ is too supersingular if it does not have a canonical subgroup as in [Kat73] and [Lub79], and nearly too supersingular if it is $p$-isogenous to such a curve. The following is a corollary of Theorem B and generalises the work of Coleman [Col05] on components of $X_0(p^2)$. For the definitions of $Z$, $Z_1$ and $Z_2$, see Figure 1 and Section 4.7.

**Theorem C.** The points on $X_0(Np)$ reducing to a smooth point of some $Z_1$ (resp. $Z_2$) parametrise too supersingular (resp. nearly too supersingular) curves with $\Gamma_0(N)$-level structure.
1.5. **Overconvergent modular forms.** Our initial motivation was a geometric study of the Atkin operator $U_p$ on spaces of overconvergent modular forms. The components of $U_p$ classifying too supersingular curves and nearly too supersingular curves mark the boundary of the regions $X_r$ from $[\text{Col}97]$ for $r = p/(p + 1)$ and $r = 1/(p + 1)$ respectively. We obtain an explicit description of the “boundary correspondence” induced by $U_p$, and its tame ramification. The $U_p$-action on distributions on the boundary might be related to overconvergent modular functions through an integral transform.

1.6. **Guide to the literature.** Simultaneous stable reduction for finite maps is investigated in $[\text{LL}99]$. Existence of simultaneous semi-stable models for finite maps was obtained over $\mathbb{C}_p$ by Coleman $[\text{Col}03]$, and for curves over function fields of Dedekind schemes by Liu $[\text{Liu}06]$. Stronger skeletal versions were proved by Cornelissen-Kato-Kool $[\text{CKK14}]$, and using techniques of Berkovich geometry in $[\text{ABBR14}]$. There has been recent interest in the geometry of correspondences motivated by a study of their dynamical systems, see $[\text{DS}06]$. A theory of canonical heights for correspondences analogous to $[\text{CS}93]$ can be found in $[\text{Ing}14]$.

Semi-stable models of curves were first used by Deligne and Mumford $[\text{DM}69]$ to prove the irreducibility of the moduli space $\mathcal{M}_g$ of curves of genus $g$. The work of Raynaud $[\text{Ray}70]$ and Grothendieck $[\text{Gro}72]$ establishes a strong connection between semi-stable models of curves and Galois representations arising from their cohomology, which put the notion of semi-stability at the centre of arithmetic geometry. Applications that are close in spirit and flavour to some of the calculations at the end of this paper, can be found in the work of Ribet $[\text{Rib}90]$ on level lowering for modular forms. We use techniques from $p$-adic geometry that allow us to keep track of different semi-stable models, using the combinatorial notion of semi-stable vertex sets recalled below. These techniques have their origins in the work of Bosch and Lütkebohmert $[\text{BL}85]$ and Berkovich $[\text{Ber}90]$. We will make extensive use of the powerful theorems in $[\text{BR}13]$, and especially the recent work of Amini-Baker-Brugallé-Rabinoff $[\text{ABBR14}]$.

When constructing the stable model of $U_p$, we will make essential use of the literature on semi-stable models of modular curves. A semi-stable model for $X_0(2p)$ was constructed in $[\text{DR}73]$. The seminal work of Katz-Mazur $[\text{KM}85]$ constructs regular models for modular curves with a clear moduli interpretation, which are the starting point of the construction of a semi-stable model for $X_0(Np^2)$ by Edixhoven $[\text{Edi}90]$. Recently, Weinstein $[\text{Wei}12]$ has constructed a semi-stable covering for the tower of modular curves, which yields semi-stable models for modular curves in great generality.

1.7. **Outline.** We recall results on the specialisation map of adic spaces and functoriality of semi-stable models of curves in section 2 and define the important notion of the skeleton attached to a semi-stable model of a punctured curve. In section 3 we use these techniques to prove that correspondences of curves over non-Archimedean valued fields are potentially skeletally semi-stable, which generalises and strengthens the results of Coleman $[\text{Col}03]$ and Liu $[\text{Liu}06]$, and provides a strong analogue of $[\text{DM}69]$. This gives rise to the important notion of skeletons of correspondences, and aids us in computing the action of correspondences on étale cohomology groups. We then explicitly determine the stable model $U_p$ for $U_p$ on $X_0(Np)$ in section 4 and its corresponding skeleton. As a consequence of our analysis, we obtain a moduli interpretation of the components appearing in the special fibre of $U_p$ in terms of canonical subgroups.

1.8. **Acknowledgements.** The author is very grateful to Netan Dogra for countless enlightening discussions. We warmly thank Minhyong Kim for proposing a study of the geometry of correspondences, as well as many useful suggestions. We are grateful for several comments and suggestions made by Christian Johansson during the early stages of this project. Finally, we thank the European Physical Sciences Research Council and the Leatherseller’s Company for their financial support.
2. Semi-stable vertex sets and skeleta

We start by recalling the notion of semi-stable vertex sets of smooth quasi-projective curves. They supply us with a combinatorial tool to understand semi-stable models, and finite maps between them. We then introduce the skeleton of a semi-stable model, and discuss the important notion of harmonic morphisms between them. We hope to generalise this material in the future to more general types of adic spaces, so as to include such perfectoid spaces as Lubin-Tate towers [Wei12]. For now, we restrict ourselves to the algebraic situation of smooth projective curves. This theory was worked out in [ABBR14], where everything is phrased in the language of Berkovich spaces.

2.1. Notation and terminology. For a curve \( \mathcal{X} \) over a discrete valuation ring, we write \( \mathcal{X} \| \) for its special fibre, and \( \mathcal{X} \) for its generic fibre. Let \( K \) be an algebraically closed, complete, non-Archimedean field with topology induced by a non-trivial valuation \( \cdot \) of rank 1. Let \( R \) be its valuation ring with maximal ideal \( m \) and residue field \( k \). We let \( f : X \to Y \) be a finite morphism between smooth proper connected curves over \( K \), and \( X^{\text{ad}}, Y^{\text{ad}} \) the adic spaces associated to \( X, Y \). We let \( D_X \subset X(K) \) and \( D_Y \subset Y(K) \) be finite sets of type-I points, which we call punctures, such that \( f^{-1}(D_Y) = D_X \). Given a point \( x \in X^{\text{ad}}(K) \), we write \( K(x) := \mathcal{O}_{X^{\text{ad}}, x}^+ \) for its completed residue field, and \( k(x) \) for the residue field of \( K(x) \). A pointed ball is an adic space which is isomorphic to the complement of the open set \( \{ \cdot \} = \text{Spa}(K(\{ \cdot \}), R(\{ \cdot \})) \), and a pointed annulus is the complement of the open set \( \{ t \} \leq p^w \) for some \( w \in \mathbb{R}_{>0} \) which we call the width of the annulus. We note that a pointed ball has exactly one Type-V point which is not the specialisation of any Type-II point. This Type-V point is called the apex point of the pointed ball. We similarly define apex points of pointed annuli, of which there are always exactly 2. We note that our notion of pointed balls or annuli corresponds to the notion of wide open balls or annuli that was popularised by Coleman and collaborators, see for instance [Col89], [Col03], [CM10].

2.2. Definition. A semi-stable formal model of the punctured curve \((X, D_X)\) is an integral proper admissible formal \( R \)-scheme \( X \) such that its generic fibre as an adic space is isomorphic to \( X^{\text{ad}} \), and moreover

- its special fibre \( \mathcal{X} \) is a reduced connected algebraic curve over \( k \) with at most ordinary double points for singularities,
- all points in \( D_X \) reduce to distinct smooth points on \( \mathcal{X} \).

The category \( \text{FMod}^{\text{ss}}_X \) consists of semi-stable formal models of \((X, D_X)\), where a morphism between two such models is a morphism of formal \( R \)-schemes that induces the identity on \( X^{\text{ad}} \).

2.3. Relating \( \text{FMod}^{\text{ss}}_{X^\|} \) and \( \text{FMod}^{\text{ss}}_X \). Choose semi-stable formal models \( X, \mathcal{Y} \) of \((X, D_X)\) and \((Y, D_Y)\) respectively. A priori, we only have a rational map \( X \to \mathcal{Y} \) induced by \( f \). We will investigate the existence of a map between \( X \to \mathcal{Y} \), which restricts to \( f \) on the generic fibre. If we can show existence, we want to be able to determine whether we can arrange for such an extension to still be a finite map. Rather than attacking this question directly on the semi-stable models, we recall the notion of semi-stable vertex sets as in [ABBR14] Section 5). They will turn out to form a category which is equivalent to \( \text{FMod}^{\text{ss}}_{X^\|} \), with the advantage that maps between semi-stable vertex sets are simply given by inclusion. This provides us with a very convenient combinatorial way of understanding morphisms between different semi-stable formal models of \( X \).

2.4. Definition. A semi-stable vertex set of \((X, D_X)\) is a finite set \( V \) of type-II points of \( X^{\text{ad}} \) such that

- the space \( X^{\text{ad}} \setminus V \) is a disjoint union of pointed balls and finitely many pointed annuli,
- the points in \( D_X \) belong to distinct pointed balls in \( X^{\text{ad}} \setminus V \).
The category $\text{Vert}^{ss}_X$ consists of semi-stable vertex sets of $(X, D_X)$, where a morphism between two such sets is given by inclusion.

2.5. **Equivalence of $\text{FMod}^{ss}_X$ and $\text{Vert}^{ss}_X$.** We now recall the equivalence of the two categories we just defined. This equivalence is what enables us to keep track of which semi-stable formal models $\xi, \eta$ admit a finite map that extends $f$. The main ideas, which we now recall, appear in [BL85] and [Ber90].

Let $\xi$ be an object in $\text{FMod}^{ss}_X$. As in [Sch12] Theorem 2.22] we can define a specialisation map, which is a morphism of locally ringed topological spaces

$$\text{sp}_X : (X^{ad}, \mathcal{O}^{ad}_X) \rightarrow (\xi, \mathcal{O}_X),$$

whose fibres are called *formal fibres*. It is possible to determine the nature of the formal fibres of $\text{sp}_X$ case by case. By combining [BL85] Propositions 2.2 and 2.3 and [Ber90] Proposition 2.4.4, we obtain the following Theorem. See also [BPR11] Theorem 4.6.

**Theorem 1** (Bosch–Lütkebohmert, Berkovich). Let $\xi$ be a point of $\mathcal{X}_s$, then

- $\xi$ is a generic point if and only if $\text{sp}^{-1}_X(\xi)$ consists of a single type-II point of $X^{ad}$,
- $\xi$ is a smooth closed point if and only if $\text{sp}^{-1}_X(\xi)$ is a pointed ball,
- $\xi$ is an ordinary double point if and only if $\text{sp}^{-1}_X(\xi)$ is a pointed annulus.

This theorem allows us to attach to $X$ the finite set $V_X := \{\text{sp}^{-1}_X(\xi)\}_\xi$, where $\xi$ ranges over the generic points of the irreducible components of $X_s$. It follows immediately from the above theorem that $V_X$ is a semi-stable vertex set for $(X, D_X)$. This defines a functor between $\text{FMod}^{ss}_X$ and $\text{Vert}^{ss}_X$. It turns out that it is in fact an anti-equivalence of categories, which is [BPR11] Theorem 4.11.

**Theorem 2** (Baker–Payne–Rabinoff). The functor

$$\text{FMod}^{ss}_X \rightarrow \text{Vert}^{ss}_X : \xi \mapsto V_X$$

induces an anti-equivalence of categories.

2.6. **Relating $\text{Vert}^{ss}_X$ and $\text{Vert}^{ss}_Y$.** We now turn back to the question of the existence of an extension of $f$ to a finite map between two semi-stable formal models $\xi, \eta$ for $(X, D_X)$ and $(Y, D_Y)$. The equivalence we just established gives us a direct way to investigate this problem, as semi-stable vertex sets naturally live in $X^{ad}$, where we know what $f$ does. More precisely, the map $f$ induces a map $X^{ad} \rightarrow Y^{ad}$, which sends type-II points to type-II points. We will abuse notation and call this map $f$. This allows for a direct comparison of the semi-stable vertex sets attached to our semi-stable formal models. The following theorem is [ABBRT14] Theorem 5.13.

**Theorem 3** (Amini–Baker–Brugallé–Rabinoff). Let $\xi, \eta$ be semi-stable formal models of $(X, D_X)$ and $(Y, D_Y)$, then $f$ extends to a unique morphism $\xi \rightarrow \eta$ if and only if $f^{-1}(V_\eta) \subseteq V_X$. Moreover, this extension is finite if and only if $f^{-1}(V_\eta) = V_X$.

2.7. **The skeleton of a semi-stable vertex set.** To a semi-stable vertex set $V$ for $(X, D_X)$, we can associate a combinatorial structure which we call the *skeleton* $\Sigma_V$ of $X$ with respect to $V$.

First, we define $\Gamma_V$ as the set of points of $X^{ad}$ that are not contained in a pointed ball which is disjoint from $V \cup D_X$, endowed with the subspace topology. Recall that the maximal Hausdorff quotient $\overline{\Gamma}_V$, often referred to as the *Berkovich skeleton* of $X$ with respect to $V$, can naturally be viewed as a finite metric graph, as we now recall. The vertex set is $V \cup D_X$, and edges come in two flavours: There is an edge $e_p$ between two vertices in $V$ for every intersection point $p$ of the corresponding components in the special...
1 and 4 respectively. Both points are of genus 0.

Define the skeleton of \( X^\text{ad} \) with respect to \( V \), or \( \Sigma_x \), to be the pair \( \Sigma_x = (\Gamma_x, \mathcal{L}_x) \), where \( \mathcal{L}_x = \{ l(e) : e \text{ edge of } \Gamma_x \} \) consists of the set of edge lengths appearing in the maximal Hausdorff quotient \( \Gamma_x \), viewed as a metric graph. We define the valency of a Type-II point \( x \) in \( \Gamma_x \) to be the number of Type-V points in \( \{ x \} \cap \Gamma_x \), and its genus to be the genus of the residue field \( k(x) \) as defined in [2.1]. The skeleton is nothing more than a slightly enhanced version of the classical concept of the dual graph of a semi-stable formal model \( \Sigma_x \).

### 2.8. Example

We now illustrate these definitions with an example. Consider the projective closure of \( y^2 = x^3 + x^2 + p^3 \), over \( \mathcal{O}_C \), and let \( \mathcal{X} \) be the blow-up at a smooth point of the special fibre. Then \( \mathcal{X} \) is a normal semi-stable model whose special fibre consists of a nodal curve \( C_1 \) and a projective line \( C_2 \), crossing transversally. Now set \( X \) to be the generic fibre of \( \mathcal{X} \), and \( D = \{ (0, 1, 0) \} \) the point at infinity.

The formal completion \( \mathcal{X} \) of \( \mathcal{X} \) is a semi-stable formal model for \( (X,D) \), and its corresponding skeleton \( \Sigma_x \) can be visualised as

![Diagram](image)

Note that every Type-II point in \( \Sigma_x \) has valency 2 except for the two points in \( V_X \), which have valencies 1 and 4 respectively. Both points are of genus 0.

### 2.9. Hyperbolic curves and stable reduction

Recall that a smooth, proper, punctured curve \( (X,D_X) \) is said to be hyperbolic if \( \chi(X,D_X) < 0 \), where \( \chi(X,D_X) = 2 - 2g(X) - |D_X| \) is the Euler characteristic. When \( (X,D_X) \) is hyperbolic, there is a unique stable model \( \mathcal{X} \) in \( \text{FMod}_C^{\text{ss}} \), which means that every vertex of genus 0 in \( V_X \) is of valency at least three. Equivalently, it corresponds to the unique semi-stable vertex set that is minimal under inclusion, and is hence the initial object in the category \( \text{Vert}_C^{\text{ss}} \).

### 2.10. Harmonic morphisms

Given a finite morphism \( f : \mathcal{X} \to \mathcal{Y} \) of semi-stable formal models for \( (X,D_X) \) and \( (Y,D_Y) \), we know that \( f^{-1}(V_Y) = V_X \). In general, it is not true that \( f^{-1}(\Gamma_Y) = \Gamma_X \), see [ABBR14, Remark 5.23]. When this holds, we say that \( f \) is a skeletal finite morphism. For maps \( f \), we can always modify \( f \) so as to make it skeletal, see [ABBR14, Corollary 4.18]. We will not recall the procedure of this modification here, as we will describe a more general argument in detail for correspondences below, where we need to make two maps simultaneously in order to assure that they are both skeletal. This is contained in the proof of Theorem [A].

A skeletal finite morphism \( f : \mathcal{X} \to \mathcal{Y} \) gives rise to a harmonic morphism on skeleta, which was introduced for Berkovich skeleta in [ABBR14, Section 4.27]. For a Type-V point \( x \in \Gamma_x \), we write \( l_x \) to mean the width of the unique pointed annulus in \( X^\text{ad} \), and \( d_x \) to be the ramification index for the map \( f \) of that point. The
integer $d_x$, now defined for any Type-V point in $\Gamma_X$, is often referred to as the \textit{expansion factor} of $f$ at $x$. It is not difficult to show that $d_x$ equals the ramification index at the point $\text{sp}_X(x)$ of the map $X_s \to Y_s$.

The condition that $f$ is skeletal ensures that for any Type-V point $y \in \Gamma_Y$, the preimage $f^{-1}(y)$ consists entirely of Type-V points contained in $\Gamma_X$. It can be shown that the morphism $f : \Gamma_X \to \Gamma_Y$ is \textit{harmonic} in the sense that the sum

$$\sum_{x \to y} d_x,$$

is independent of the Type-V point $y \in \Gamma_Y$. In the case of tame ramification, which means that $d_x$ is coprime to the residue characteristic of $K$ for all $x$ above $y$, it is equal to the degree of $f$, see [ABBR14, Proposition 2.22].

\textbf{2.11. Remark.} Note that these conditions give rise to a notion similar to that of harmonic morphisms of metrised complexes of curves in [ABBR14, Section 2.16]. The list of properties included in their definition is considerably longer. We shall however not concern ourselves with this abstract category, as our metrised complexes of curves will always arise from skeletal finite morphisms. We therefore chose to only include the above two properties in the definition of a harmonic morphism of skeleton, as they are the only ones we shall use.

\textbf{2.12. Faithfully flat descent.} In the next section, we will be interested in more general base fields $K$ that are not necessarily algebraically closed or complete. We now recall how to descend some results to base fields $K$ equipped with a non-trivial non-Archimedean valuation $v$ of rank 1, with valuation ring $R$. A nice exposition of the theory can be found in [ABBR14, Section 5], to which we refer the reader for more details. We now sketch the results we need.

Let $(X, D)$ be a smooth, projective, geometrically connected curve over a field $K_0$ with a non-trivial non-Archimedean valuation of rank 1 and valuation ring $R_0$. Let $K = \overline{K_0}$ be the completion of the algebraic closure of $K_0$, which is itself algebraically closed, and let $R$ be the valuation ring of $K$. The tools outlined above can be used to analyse semi-stable models of $(X_K, D_K)$, after which we pass back to $(X, D)$ by the theory of faithfully flat descent. More precisely, any semi-stable model of $X_K$ is isomorphic to $\mathcal{X}_R$, where $\mathcal{X}$ is a semi-stable model of $X_{K_1}$ over some finite separable extension $K_1$ of $K_0$. Moreover, the following Lemma is proved in [ABBR14, Lemma 5.5].

\textbf{Lemma 1.} Let $f : X \to Y$ be a finite morphism between smooth, projective, geometrically connected curves over $K_0$, with semi-stable models $\mathcal{X}$ and $\mathcal{Y}$ respectively. Suppose that $f$ extends to a finite morphism $\mathcal{X}_R \to \mathcal{Y}_R$, then $f$ also extends uniquely to a finite morphism $\mathcal{X} \to \mathcal{Y}$ defined over $R_0$.

\textbf{Proof.} Amini-Baker-Brugallé-Rabinoff observe that the statement is equivalent to the saying that the induced map

$$\phi : X \to X \times_{K_0} Y \hookrightarrow \mathcal{X} \times_R \mathcal{Y}$$

makes the projection $\mathcal{X} \to \mathcal{X}$ an isomorphism, where $\mathcal{X}$ is the schematic closure of the image of $\phi$. This isomorphism can now be transferred from $R$ to $R_0$ by a standard application of faithfully flat descent. \qed

\section{Semi-stable reduction of correspondences}

In this section, we prove an analogue of the semi-stable reduction theorem of Deligne-Mumford [DM69] for correspondences. This recovers the simultaneous semi-stable reduction theorems proved by Coleman [Col03] and Liu [Liu06]. The powerful techniques from [ABBR14] outlined in the previous
section make it possible to keep track of two defining maps simultaneously, and give rise to a suitable notion of \textit{skeleta} of correspondences.

3.1. \textbf{Definitions.} Let $K$ be a field equipped with a non-trivial non-Archimedean valuation $|\cdot|$ of rank 1, whose valuation ring $R$ has maximal ideal $m$ and residue field $k$. A \textit{punctured correspondence}

$$C : (X, D_1) \rightarrow (Y_1, D_1) \leftarrow (Y_2, D_2)$$

consists of a triple $X, Y_1, Y_2$ of smooth, projective, geometrically connected curves over $K$, together with finite sets of points $D_X \subset X(K), D_1 \subset Y_1(K)$ and $D_2 \subset Y_2(K)$, and finite maps $\pi_1, \pi_2$ defined over $K$, such that $\pi_1^{-1}(D_1) = D_X = \pi_2^{-1}(D_2)$. We will often refer to $(X, D_X), (Y_1, D_1)$ and $(Y_2, D_2)$ as the \textit{constituents} of $C$, and to $\pi_1, \pi_2$ as the \textit{constituent maps} of $C$. A punctured correspondence $C$ is said to be \textit{hyperbolic} if its constituents are hyperbolic in the sense of [2.9]

3.2. \textbf{Semi-stable models of correspondences.} A \textit{semi-stable} $R$-model for a punctured correspondence $C$ with constituents as above is a diagram

$$\mathcal{C} : \mathcal{X} \rightarrow \mathcal{Y}_1 \leftarrow \mathcal{Y}_2$$

where $\mathcal{X}, \mathcal{Y}_1$ and $\mathcal{Y}_2$ are integral flat proper semi-stable $R$-curves, together with isomorphisms $\mathcal{X}_K \simeq X$ as well as $\mathcal{Y}_1,K \simeq Y_1$ and $\mathcal{Y}_2,K \simeq Y_2$, so that their formal completions along the special fibre are semi-stable formal $R$-models for $(X, D_X), (Y_1, D_1)$ and $(Y_2, D_2)$ in the sense of [2.2]. We require that the constituent maps are \textit{finite}, and that their restrictions to the generic fibres constitute $C$ via the given isomorphisms. We say that a semi-stable $R$-model $\mathcal{C}_1$ \textit{dominates} another semi-stable $R$-model $\mathcal{C}_2$, if the constituents of $\mathcal{C}_1$ dominate the constituents of $\mathcal{C}_2$ pairwise.

3.3. \textbf{Skeletal semi-stable models.} Pick a semi-stable model $\mathcal{C}$ for $C$, with constituents $\mathcal{X}, \mathcal{Y}_1$ and $\mathcal{Y}_2$ and constituents maps $\pi_1, \pi_2$. The semi-stable model $\mathcal{C}$ is said to be \textit{skeletal} if both $\pi_1$ and $\pi_2$ are skeletal in the sense of [2.10]. It is not true that any semi-stable model $\mathcal{C}$ of $C$ is skeletal, already when $\mathcal{C}$ degenerates to a map, see [ABBR14, Remark 5.23]. Below, we prove potential \textit{skeletal} semi-stability for correspondences, which is therefore stronger than potential semi-stability.

3.4. \textbf{Semi-stable models for Galois morphisms.} Before coming to a proof of potential semi-stability of correspondences, which is the content of Theorem [A], we first prove two lemma’s about maps $f : X \rightarrow Y$ that are Galois. The general case will be reduced to this in the proof of Theorem [A]. First, we prove a lemma on the transitivity of the Galois action on fibres, which we reduce to the corresponding classical statement in commutative algebra.

\textbf{Lemma 2.} Let $f : X \rightarrow Y$ be a finite morphism between smooth, projective, geometrically connected curves over $K$. If $f$ is Galois with group $G$ and $y \in Y^{\text{red}}$, then $G$ acts transitively on the finite set $f^{-1}(y) \subset X^{\text{red}}$.

\textbf{Proof.} By passing to an open affinoid neighbourhood, we reduce the question to one about affinoid $K$-algebra’s $(\mathcal{A}, \mathcal{A}^{+})$ with an action of $G$. Let $|\cdot|_y$ be a continuous valuation of $(\mathcal{A}^{+}, (\mathcal{A}^{+})^G)$, then the set of valuations in Spa$(\mathcal{A}, \mathcal{A}^{+})$ extending $|\cdot|_y$ is non-empty [Bou89, Proposition 2.4.4] and
in bijection with the set of prime ideals \( \mathfrak{P} \) above \( p := \text{Supp } |_y \). Indeed, the extension to a continuous valuation on \( \mathcal{O}/\mathfrak{P} \) is uniquely determined up to equivalence by the requirement that

\[
a \mapsto \left| \text{Norm}_G(a)^{1/|G|} \right|_y, \quad \text{where } \text{Norm}_G(a) = \prod_{g \in G} a^g
\]

which is clearly \( \leq 1 \) on \( \mathcal{O}^+ \) and hence determines an element in \( \text{Spa}(\mathcal{O}, \mathcal{O}^+) \) above \( y \). The lemma now follows from the transitivity of the Galois action on the primes above \( p \) in a finite Galois extension of Dedekind rings. \( \square \)

One final ingredient is an analysis of simultaneous semi-stable reduction of Galois maps. If we drop the condition that the extension of \( f \) in the statement should be finite, this was already proved for general finite maps by Liu-Lorenzini [LL99, Proposition 4.4]. The argument where \( K = \mathbb{C}_p \) is given in Coleman [Col03], whereas the proof we include here could be extracted from various proofs in [ABBR14].

**Lemma 3.** Let \( f : (X, D_X) \to (Y, D_Y) \) be a finite Galois morphism between smooth, projective, geometrically connected, hyperbolic curves over \( K \). Then for some semi-stable model \( \mathcal{Y} \) of \( (Y, D_Y) \), the map \( f \) extends to a finite morphism \( f : \mathcal{X} \to \mathcal{Y} \) where \( \mathcal{X} \) is the stable model of \( (X, D_X) \).

**Proof.** By the results in 2.12 we may assume that \( K \) is algebraically closed and complete. Let \( V \) be the stable vertex set of \( (X, D_X) \), and set \( W = f(V) \). By [ABBR14] Lemma 3.15, there is a minimal semi-stable vertex set \( W' \) for \( (Y, D_Y) \), containing \( W \). We will prove that \( W' = W \). Pick any element \( y \in W' \setminus W \), and any element \( x \in f^{-1}(y) \). Clearly, \( y \) must be of valency at least 3 in \( \Gamma_W \), whereas \( x \) must be contained in \( \Gamma_V \) and of valency 2.

Because \( f \) is Galois, we may use Lemma 2 to show that there is an injection from the set of Type-V points in \( \mathcal{Y} \cap \Gamma_{W'} \) to the set of Type-V points in \( \mathcal{X} \cap \Gamma_V \), which is a contradiction. Therefore \( W = W' \) and hence \( W \) is semi-stable. \( \square \)

### 3.5. Potential semi-stable reduction for correspondences

We now come to the main result of this section, which should be viewed as a strong analogue of the Deligne-Mumford theorem [DM69, Corollary 2.7] on potential semi-stable reduction of smooth projective curves. We note that we already made essential use of their theorem in Lemma 3, so we do not recover it in the degenerate case where our correspondence is a single curve with identity constituent maps.

**Theorem A.** Let \( C \) be a hyperbolic punctured correspondence. After passing to some finite separable extension of \( K \), we can ensure that \( C \) has a skeletal semi-stable \( R \)-model. Moreover, there is a unique minimal such model for the relation of domination, which is necessarily stable.

**Proof.** First we assume that \( K \) is algebraically closed and complete. Consider the Galois closure \( g : (\overline{X}, \overline{D}) \to (X, D_X) \) of both \( \pi_1 \) and \( \pi_2 \). This yields a diagram
where we set \( \tilde{D} := g^{-1}(D_X) \). This makes \((\tilde{X}, \tilde{D})\) hyperbolic, and as such there is a unique stable vertex set \( \tilde{V} \) by Deligne-Mumford. Set \( V := g(\tilde{V}) \), and \( W_i := \pi_i(V) \), which are semi-stable vertex sets by Lemma 3. By Lemma 2 we must have \( V = \pi_i^{-1}(W_i) \), and hence \( \pi_i \) extends to a finite morphism between the corresponding semi-stable formal \( R \)-models.

To ensure that these are skeletal morphisms, we might need to enlarge \( W_i \) and \( \tilde{V} \). From [ABBR14, Remark 4.19], we know that \( \pi_i(\Gamma_V) \) is the union of \( \Gamma_{W_i} \) and a finite set of edges. Now enlarge \( W_i \) to contain the endpoints of those edges, and enlarge \( V \) to contain their inverse images under \( \pi_i \) for \( i = 1, 2 \). Now run through the following procedure:

1. Redefine \( W_1 \) as \( \pi_1(V) \).
2. Redefine \( V \) as \( \pi_1^{-1}(W_1) \).
3. Redefine \( W_2 \) as \( g_2(V) \).
4. Redefine \( V \) as \( g_2^{-1}(W_2) \), and return to step (1).

Note that every time a new vertex \( v \in V \) is introduced, it appears on the stable skeleton of \((X, D_X)\), and the set of edge lengths \( \mathcal{L}_V \) of the skeleton \( \Sigma_V = (\Gamma_V, \mathcal{L}_V) \) remains constant by Lemma 2. This shows that the procedure terminates. We obtain from Theorem 3 semi-stable formal \( R \)-models corresponding to \( V, W_1, W_2 \) and finite skeletal maps between them. These can be made into algebraic semi-stable \( R \)-models by the algebraisation theorem [Abb10, Corollaire 2.3.19].

Now let \( K \) be a general non-Archimedean valued field, and let \( \overline{K} \) be the completion of an algebraic closure of \( K \) with respect to its induced valuation. Over \( \overline{K} \), we can find a semi-stable formal model \( C \), whose constituents descend to a finite separable extension by Deligne-Mumford [DM69, Corollary 2.7]. To descend the maps, we can use faithfully flat descent, as in 2.12. □

3.6. Remark. We note that in general, the stable model of \( C \) does not consist of the stable models of its constituents. As can be seen below for the case of Hecke operators, typically some extra components appear. We also note that it is still possible to find skeletal semi-stable models for correspondences that fail to be hyperbolic, but in general there will not be a minimal such model. To overcome this problem, we can fix semi-stable models for all the constituents of \( C \), and run through the above argument to construct their stable hull \( \mathcal{C} \), which is the minimal semi-stable model such that all the constituents dominate their corresponding fixed semi-stable models.

3.7. Definition. The above theorem assures that after a finite separable base change, we can always find a skeletal semi-stable model \( \mathcal{C} \) for any hyperbolic correspondence \( C \) over \( K \), which gives rise to a diagram
where $\sigma_1, \sigma_2$ are finite harmonic morphisms as defined in [2,10]. We call this diagram the **skeleton** of the skeletal semi-stable model $\mathcal{C}$ of $C$. Theorem A shows that when such a skeletal semi-stable model exists, there is a unique minimal one with respect to inclusion. We refer to this skeleton as the **stable skeleton** of $C$. The associated semi-stable model of $C$ is minimal with respect to the relation of domination, and is called the **stable model** of $C$.

3.8. **The weight-monodromy filtration.** We now recall the definition of the monodromy filtration on the $l$-adic étale cohomology of a proper smooth separated scheme $X$ of finite type over $K$, and the identification of the graded pieces with quantities that can be computed from a semi-stable model in the case of curves. By the monodromy theorem of Grothendieck [Gro72], there exists a nilpotent operator

$$N \in \text{End}(H^1_{\text{ét}}(X_{\kappa}, \mathbb{Q}_l))$$

such that every $\sigma$ in a sufficiently small open subgroup of the inertia group $I \subset \text{Gal}(K'/K)$ acts as $\exp(t_i(\sigma)N)$, where $t_i : I \to \mathbb{Z}_l(1)$ is the maximal pro-$l$ quotient map. We obtain an ascending monodromy filtration $M_*$ on $H^1_{\text{ét}}(X_{\kappa}, \mathbb{Q}_l)$, characterised by $NM_i \subseteq M_{i-1}(-1)$ and

$$N^i : \text{Gr}^M_i \sim \text{Gr}^M_{i+1}(-i)$$

From the description of the action of inertia in terms of the operator $N$, we obtain a well-defined action of the geometric Frobenius $\text{Frob}_q$ on the graded pieces, and in particular a notion of weights.

**Conjecture 1** (Weight-monodromy conjecture). The eigenvalues of $\text{Frob}_q$ on $\text{Gr}^M_i H^1_{\text{ét}}(X_{\kappa}, \mathbb{Q}_l)$ are algebraic numbers whose conjugates all have complex absolute values equal to $q^{(i+j)/2}$.

This conjecture is known in many cases. We will only need the case of curves, where everything was proved by Grothendieck [Gro72]. The case of semi-stable surfaces was settled by Rapoport-Zink [RZ82], using properties of the weight spectral sequence which we discuss below. The analogous conjecture over function fields was proved by Deligne [Del71], and many subsequent developments reduce the case of mixed characteristic to Deligne’s setting, see for instance [Sch12].

3.9. **The weight spectral sequence.** Let $\mathcal{X}$ be a semi-stable model for $X$ with components $Y_i$ in the special fibre, then the analysis of the vanishing and nearby cycles functor in [RZ82] yields the construction of the weight spectral sequence, with first page

$$E_1^{p,q} = \bigoplus_{i \geq \max(0, -p)} H^{q-2i}_{\text{et}}(Y_{[p+2i+1]}^m, \mathbb{Q}_l(-i)),$$

where $Y_{[m]}$ is the disjoint union of the $m$-fold intersections of the $Y_i$. The weight spectral sequence converges to $H^p_{\text{et}}(X_{\kappa}, \mathbb{Q}_l)$ and degenerates at $E_2$. Now let $X$ be a smooth proper curve over $K$, then the knowledge of an explicit semi-stable model $\mathcal{X}$ for $X$ helps us compute the graded pieces of the weight filtration. Indeed, the weight spectral sequence [RZ82] gives us the short exact sequence

$$0 \to H^1_{\text{ét}}(\mathcal{X}_{\kappa}, \mathbb{Q}_l) \to H^1_{\text{ét}}(X_{\kappa}, \mathbb{Q}_l) \to H^1(\Gamma_{\mathcal{X}}, \mathbb{Q}_l)(-1) \to 0.$$

We may for instance describe the weight 2 graded piece using a combinatorial analysis of the dual graph of the special fibre of our semi-stable model $\mathcal{X}$. 
3.10. **Spectral properties of correspondences.** We now recall how a correspondence $C$ gives rise to strict morphisms on cohomology groups, and we show how the knowledge of a skeletal semi-stable model for $C$ allows for a combinatorial description of this morphism. In Section 3.11 discuss the link with Néron models, and the induced action on Néron component groups.

We identify a correspondence $C : Y_1 \rightarrow X \rightarrow Y_2$ between $K$-varieties of dimension $n$ with an element $C \in \text{CH}^n(Y_1 \times Y_2)$, and define the morphism $C^* : H^i_{\text{et}}(Y_1, \mathbb{Q}_l) \rightarrow H^i_{\text{et}}(Y_2, \mathbb{Q}_l)$ by $\pi_2 \circ (C \cup -) \circ \pi_1^*$. By Frobenius equivariance at every step, this morphism is strict in the sense that it respects the weight-monodromy filtrations

$$C^*(M_j H^i_{\text{et}}(Y_1, \mathbb{Q}_l)) = M_j H^i_{\text{et}}(Y_2, \mathbb{Q}_l)$$

for all $i$. In this context, we mention a result of Saito [Sai03] that assures that the expression

$$\sum_{i=0}^{2n} \text{Tr} \left( C^* \circ \sigma_s : H^i_{\text{et}}(X, \mathbb{Q}_l) \right)$$

is independent of the prime $l$ not equal to the residue characteristic of $K$. Here, $\sigma$ is an element of the Weil group $W_K$ of $K$.

Now suppose we have a skeletal semi-stable model $\mathcal{C}$ of a correspondence $C$ between smooth, proper, geometrically connected curves over $K$. This makes the computation of the action of $C$ on $H^i_{\text{et}}$ completely combinatorial. It is proved in [Sai03] that the weight spectral sequence is equivariant under the actions of $\text{Gal}(K^s / K)$ and correspondences $C$. Therefore we can compute

$$C^* : H^i_{\text{et}}(Y_1, \mathbb{Q}_l) \rightarrow H^i_{\text{et}}(Y_2, \mathbb{Q}_l)$$

by simply restricting to the graded pieces of different weights. The map on the weight 1 part decomposes into the maps induced by the correspondences on the various components of $\mathcal{C}$. The map on the graded piece of weight 2 can be read off from the skeleton of $\mathcal{C}$, by pulling back and pushing forward cycles. We mention the work of Yoshida [Yos11] in this context, which describes a different algorithm to compute this action via intersection theory on regular models of the constituents of $C$, in arbitrary dimensions.

3.11. **Néron models.** Let $\mathcal{X}$ be a semi-stable curve over $R$, and let $\mathcal{J}$ be the Néron model of its Jacobian. The quotient of $\mathcal{J}$ by its identity connected component $\mathcal{J}^0_s$ defines the Néron component group $\Phi$, so we get the exact sequence

$$0 \rightarrow \mathcal{J}^0_s \rightarrow \mathcal{J}_s \rightarrow \Phi \rightarrow 0.$$

Recall that the results of [Ray70] imply that $\mathcal{J}^0_s \cong \text{Pic}^0(\mathcal{X}_s)$, see [Gro72 Exposé IX, 12.1.11]. This means we can further decompose $\mathcal{J}^0_s$ as follows

$$0 \rightarrow H^1(\Gamma_{\mathcal{X}_s}, \mathbb{Z}) \otimes \mathbb{G}_m \rightarrow \mathcal{J}^0_s \rightarrow \text{Pic}^0(\mathcal{X}_s) \rightarrow 0,$$

where $\mathcal{X}_s$ is the normalisation of $\mathcal{X}_s$. Note that it is not necessary to assume that $\mathcal{X}$ is regular, see [Rib90 Section 2]. The algebraic group $H^1(\Gamma_{\mathcal{X}_s}, \mathbb{Z}) \otimes \mathbb{G}_m$ is called the toric part of $\mathcal{J}^0_s$. The Néron component group $\Phi$ may be computed as the cokernel of the map from the toric part to its dual, given by the monodromy pairing [Gro72 Théorème 11.5]. If $\mathcal{C}$ is a skeletal semi-stable model of $C$, then we already described the action of $C$ on the toric part $H^1(\Gamma_{\mathcal{X}_s}, \mathbb{Z}) \otimes \mathbb{G}_m$ so we may equally read off the action of $C$ on the Néron component group $\Phi$ using Raynaud’s theorem and the monodromy pairing.
4. HECKE OPERATORS ON MODULAR CURVES

In this section, we explicitly determine stable models \( \mathcal{U}_p \) at \( p \) of Hecke operators on the elliptic modular curves \( X_0(Np) \). This is done in Theorem [\textsc{E}]. As an immediate corollary of its proof, we give a moduli interpretation of the components occurring in \( \mathcal{U}_p \) in Theorem [\textsc{C}]. This recovers and generalises a result of Coleman [\textsc{C}0\textsc{I}5].

4.1. Notation. We let \( p \geq 5 \) be a prime not dividing \( N \), and \( \mathbb{Q}_p^{nr} \) the maximal unramified extension of \( \mathbb{Q}_p \). Let \( W \) be its ring of integers, which is the ring of Witt vectors of \( \mathbb{F}_p \). The modular curve with \( \Gamma_0(N) \)-level structure over \( \mathbb{Q}_p \) will be denoted by \( X_0(N) \). Unless indicated otherwise, we will use calligraphic notation \( \mathcal{X} \) for the regular models over \( W \) constructed by Katz-Mazur [\textsc{K}M85]. The moduli problem on \((\text{Ell}/\mathbb{F}_p)\) assigning to \( E/S/\mathbb{F}_p \) the set of \( \Gamma_0(N) \)-level structures together with Igusa structures of level \( p \) on \( E/S \) as in [\textsc{K}M85] Section 12.3 has a coarse moduli scheme denoted by \( \mathcal{X}_0(N, \text{Ig}(p)) \).

4.2. Stable models of modular curves. Before we construct stable models of Hecke operators, we recall the results we need on integral models of modular curves. It was shown by Igusa [\textsc{I}gu59] that \( \mathcal{X}_0(N) \) is smooth, and by Deligne-Rapoport [\textsc{D}R73] that \( \mathcal{X}_0(Np) \) is semi-stable, with special fibre consisting of two copies of \( \mathcal{X}_0(N)_0 \), intersecting transversely at the supersingular points. The formal neighbourhood at such a point \( \sigma \) is

\[
W[[x,y]]/(xy - p^{nr}),
\]

where \( a_{\sigma} := |\text{Aut}(E_\sigma)|/2 \) for \( E_\sigma \) the supersingular elliptic curve with auxiliary \( \Gamma_0(N) \)-structure corresponding to \( \sigma \).

The curve \( \mathcal{X}_0(Np^n) \) for \( n \geq 2 \) is not semi-stable, and a description of the local ring at the supersingular points was only recently found by Weinstein [\textsc{W}a12]. Whereas it would be interesting to investigate the stable model of \( \mathcal{U}_p \) at infinite level, at present we only need the case \( n = 2 \) as worked out by Edixhoven [\textsc{Edi}90]. He described the stable model for \( X_0(Np^2) \) over \( W[\sigma] \) where \( \sigma^{(p^2-1)/2} = p \). The special fibre contains four ordinary components. Two of these, which we call the inner ordinary components, are isomorphic to \( \mathcal{X}_0(N)_0 \). The other two, \( X^+ \) and \( X^- \), are called the outer ordinary components, and are isomorphic to the Igusa curves \( \mathcal{I}_0(N, \text{Ig}(p)/\pm) \). There is one supersingular component \( Z^\sigma \) for every supersingular point \( \sigma \) on \( \mathcal{X}_0(Np^2)_0 \). Edixhoven [\textsc{Edi}90] Theorem 2.1.1] proves that \( Z^\sigma \) has affine equation

\[
Z^\sigma : y^{(p+1)/a_\sigma} = x(x - 1)^{p-1}.
\]

Every supersingular component intersects every ordinary component, and there are no other intersection points. The formal neighbourhoods at these intersection points are all regular, except where the supersingular components meet the two inner ordinary components. The singularity at those points is described by

\[
W[\sigma][[x,y]]/(xy - \sigma^{(p-1)a_\sigma}/2).
\]

4.3. The stable model of \( T_l \). Let \( l \) be a prime not dividing \( Np \). The existence of a semi-stable model for \( T_l \) on \( X_0(Np) \), even over \( \mathbb{Z}_p \), is immediate from semi-stability at \( p \) of \( \mathcal{X}_0(Np) \) and \( \mathcal{X}_0(Npl) \). This canonical Katz-Mazur model \( \mathcal{T}_l \) has skeleton:
Let $p$ be a prime number, and all the information on the action of $T_1$ on the weight 2 graded piece of $H^1_{\text{et}}(X_0(N), Q_p)$ is therefore encoded in the mixing of the edges under $\Sigma_1$. A computation with classical modular polynomials reveals the action of $T_1$, which is precisely the graph method developed by Mestre and Oesterlé [Mes86]. This can be viewed as an incarnation of the Jacquet-Langlands correspondence. Calculations can be done using the results in [Ma77] Appendix I, but we warn the reader that Proposition (1.2) in loc. cit. contains errors. See [Edi91] Section 4.4.1 for more details.

4.4. The stable model of $U_p$. Recall that $U_p$ is defined as the correspondence

$$\begin{array}{ccc}
\pi_1 & X_0(Np;p) & \pi_2 \\
\downarrow & \downarrow & \downarrow \\
X_0(Np) & X_0(Np)
\end{array}$$

where $X_0(Np;p)$ classifies triples $(E_\nu, G_1, G_2)$ of elliptic curves with $\Gamma^0(N)$-level structure, together with two distinct subgroups $G_1, G_2$ of order $p$, and the constituent maps are defined by $\pi_1 : (E_\nu, G_1, G_2) \rightarrow (E_\nu, G_1)$ and $\pi_2 : (E_\nu, G_1, G_2) \rightarrow (E_\nu/G_2, E[p]/G_2)$. Note that $X_0(Np;p) \cong X_0(p^2N)$, and hence our description of the stable model of $U_p$ is aided by the analysis in [Edi89]. We will henceforth assume that $U_p$ is hyperbolic, which is the case when $p \geq 23$. For small primes, hyperbolicity can be ensured by puncturing $X_0(Np)$ sufficiently often.

**Theorem B.** Let $p \geq 5$ coprime to $N$, such that $U_p$ on $(X_0(Np), D)$ is hyperbolic. Then $U_p$ has stable reduction over $\mathbb{Q}_p^\sigma(\sigma)$, where $\sigma = (p^2 - 1)/2 = p$. The skeleton $\Sigma_{U_p}$ is as depicted in Figure 1.

**Proof.** Blow-up $X_0(Np;p)$ at the supersingular points, and base change to $L = \mathbb{Q}_p(\sigma)$, where $\sigma = (p^2 - 1)/2 = p$. By [Liu02 Proposition 10.4.6] and the analysis of the multiplicities in [Edi89], the normalisation $X^{ss}$ of this scheme is stable. The special fibre is as described in 4.2. Now extend scalars to $\mathbb{C}_p$. As the singular locus of the special fibre is supersingular, we have $\pi_1(\Gamma^{ss}) \subseteq \Gamma_{X_0(Np)}$. The set $\pi_1^{-1}(V_{X_0(Np)})$ is equal to the four Type-II points corresponding to the ordinary components, and hence if we enlarge $V_{X_0(Np)}$ to contain the images $v_1$ and $v_2$ of the Type-II points under $\pi_1$ and $\pi_2$, whose corresponding components we denote $Z_1$ and $Z_2$ respectively, the maps will extend to finite maps between the associated semi-stable models.

It remains to identify these images. First note that $Z^{(\sigma)}$ maps to $Z^{(\sigma)}_1$ under $\pi_1$ and $Z^{(\text{Prob},\sigma)}_2$ under $\pi_2$, so that remains is to determine the distance of the Type-II points $v_1$ and $v_2$ to the Type-II point $v_{\text{ord}}$ corresponding to the ordinary component containing the cusp $\infty$. The expansion factors equal the ramification indices of the corresponding points on the ordinary components of $\pi_1$ and $\pi_2$. As $\pi_1$ and $\pi_2$ restrict to identity and Frobenius maps on the inner ordinary components, and are totally ramified above supersingular points on $X^+$, we obtain the expansion factors 1 and $(p - 1)/2$ as in Figure 1. This shows that the distances of $v^{(\sigma)}_1$ and $v^{(\sigma)}_2$ to $v_{\text{ord}}$ are $\alpha_p/(p + 1)$ and $\alpha_p/(p + 1)$ respectively, which determines them uniquely as Type-II points in $X_0(Np)$.

Finally, we descend the stable model $U_p$ back down to $\mathbb{Q}_p^\sigma(\sigma)$ and its ring of integers $W[\sigma]$ using faithfully flat descent as in 2.12. □
The skeleton of the stable model $\mathcal{U}_p$ described in this theorem is perhaps most clearly described with a picture, see Figure 1. The numbers marking the edges are the expansion factors under the maps involved, which we omitted whenever it is 1. The various components involved are described by the work of Edixhoven [Edi89], which we recalled in 4.2. We use the same notation ($X^+, X^-$, etc.) justified by the isomorphism $X_0(Np;p) \cong X_0(p^2N)$. The only mixing of edges occurring in $U_p$ is accounted for by Frobenius, in contrast with the situation of $T_l$ where edges mix in a less transparent way. This is reflected by the fact that weight $k$ cusp forms which are new at $p$ have $U_p$-eigenvalue $\pm p^{(k-2)/2}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{skeleton.png}
\caption{The skeleton of $\mathcal{U}_p$}
\end{figure}

4.5. Remark. The above two descriptions on the skeleta allow us to determine the action of $T_l$ and $U_p$ on the toric part of the Néron model of the Jacobian of $X_0(Np)$, as well as its Néron component group. We recover the description of the action of $U_p$ given by Ribet [Rib90, Proposition 3.8], as well as the graph algorithm of Mestre-Oesterlé [Mes86] as remarked above.

4.6. Example. The stable model of $T_2$ on $X_0(11)$ over $\mathbb{Z}_{11}$ is given by the Katz-Mazur model. As in the Mestre-Oesterlé graph algorithm, we determine all the 2-isogenies of supersingular $\overline{F}_{11}$-curves using the modular polynomials, and weigh them appropriately according to the number of automorphisms. We obtain that the toric part of $\mathfrak{M}^{11}_{\text{supp}}$ is 1-dimensional, and $T_2$ acts on it as multiplication by $-2$. By Picard-Lefschetz theory, the Néron component group $\Phi$ is isomorphic to $\mathbb{Z}/5\mathbb{Z}$, and $T_2$ must necessarily act on it as multiplication by $-2$. This is an instance of the Hecke action on $\Phi$ away from the level and $p$ being Eisenstein, proved by Ribet [Rib90, Theorem 3.12] in this setting, and more generally in [Edi91].

We now find the stable model of $U_{11}$ on $X_0(11)$, punctured at the cusps, over $W[\sigma]$, where $\sigma^{60} = 11$. Running through the above proof, we obtain the following picture for the skeleton of $\mathcal{U}_{11}$:
Here, the numbers denote the expansion factors of the edges, which we omit whenever it is 1. The unique cycle in the skeleton of $X_0(11)$ pulls back to the sum of 11 simple cycles, 5 passing through $X^+$, 5 passing through $X^-$, and 1 through the ordinary component containing the cusp 0. Because $\text{Frob}_{11}$ acts trivially on the set of supersingular $j$-invariants, there is no mixing of edges, and the cycles through $X^\pm$ get annihilated by $\pi_2$. The action of $U_{11}$ on the toric part of $\mathcal{J}_N$ is therefore also trivial.

4.7. Moduli interpretation of $U_p$. As a consequence of our analysis above, we obtain a moduli interpretation of the components of the special fibre of $U_p$. The ordinary components arise as strict transforms of components described in the work of Katz-Mazur [KM85], and retain their moduli interpretation. The supersingular components $Z$ arise from blowing up the supersingular points of the Katz-Mazur model over an extension of $W$, and a moduli interpretation was given by [Col05, Theorem 3.1] in level $N = 1$. The following theorem recovers this for arbitrary levels.

**Terminology.** Let $Z^{(\sigma)}$ be a supersingular component of the top constituent of $U_p$, then as above we set $Z^{(\sigma)} = \pi_1(Z^{(\sigma)})$ the supersingular components of the semi-stable models of $X_0(Np)$ appearing in $U_p$. An elliptic curve over $\mathcal{O}_C$ is said to be too supersingular if it does not have a canonical subgroup as in [Kat73] and [Lub79]. It is nearly too supersingular if it is $p$-isogenous to a too supersingular curve.

**Theorem C.** The points on $X_0(Np)$ reducing to a smooth point of some $Z_1$ (resp. $Z_2$) parametrise too supersingular (resp. nearly too supersingular) curves with level structure.

**Proof.** This follows from the above analysis, and the fact that for an open annulus $A$ with outer radius 1 and inner radius $r$, the open subannulus $A_s := \{|x| : 1 \geq |T|_x \geq s\}$ for $s \geq r$ is independent of the chosen parameter $T$ for $A$. Indeed, letting $E_{p-1}$ be a global parameter for the supersingular annuli in $X_0(Np)$, lifting the Hasse invariant, we see that $Z_1$, resp. $Z_2$, corresponds to those elliptic curves over $\mathcal{O}_C$ with Hasse invariant $p/(p+1)$, resp. $1/(p+1)$. By the work of Katz [Kat73] and Lubin [Lub79], these are exactly the too supersingular, resp. nearly too supersingular, elliptic curves.

4.8. Remark. This proof is close in spirit to the moduli interpretation of Edixhoven’s components of $X_0(Np^2)$ in tame level $N = 1$ given by Coleman [Col05]. The necessary machinery on finite maps between curves developed there is contained in our notion of expansion factors on the edges of adic skeleta and harmonicity of maps, as recalled in Section 2.10. The approach taken by Coleman therefore should generalise verbatim to any tame level $N$, and should be roughly equivalent to ours. We also note that these results recover the contractiveness of $U_p$ near the ordinary locus containing $\infty$, and as a consequence we recover compactness of the operator $U_p$ on spaces of overconvergent modular forms.
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