A REMARK ON VIRTUAL ORIENTATIONS FOR COMPLETE INTERSECTIONS

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The aim of this note is to give a simple definition of genus zero virtual orientation classes (or fundamental classes) for projective complete intersections or, more generally, complete intersections in convex varieties, and to prove a push forward formula (see Lemmas 1 and 2 below) for them.

1. Let $X$ be a smooth complex projective variety. Let $X_\beta$ denote the stack of stable maps $f : C \to X$ of genus $g$ curves with $n$ marked points, such that $f_*([C]) = \beta \in H_2(X)$. Here $n$ and $g$ will be fixed and suppressed from the notations. Let $\pi_\beta : \tilde{X}_\beta \to X_\beta$ be the universal curve, and $\psi_\beta : \tilde{X}_\beta \to X$ be the canonical map.

A vector bundle $N$ over $X$ is called convex if $R^1\pi_\beta^*\psi_\beta^*N = 0$ for all $\beta$. In this case we will use the notation $N_\beta$ for the vector bundle $\pi_\beta^*\psi_\beta^*N$ over $X_\beta$. The variety $X$ is called convex if the tangent bundle $T_X$ is convex.

Example. Let $X = \mathbb{P}^N$, and $g = 0$. Then $X$ is convex, and any bundle of the form $N = \oplus \mathcal{O}(l_a)$ ($l_a \geq 0$) is convex.

2. Let $X$ be a convex variety and $N$ a convex bundle over $X$. Let $s : X \to N$ be a regular (i.e. transversal to the zero section) section of $N$, $i : Y := s^{-1}(0) \hookrightarrow X$ the subscheme of its zeros.

Set $Y_\beta := \coprod Y_\gamma$, the disjoint union over all $\gamma \in H_2(Y)$ mapping to $\beta$. The section $s$ induces the section $s_\beta : Y_\beta \to N_\beta$, and we have $Y_\beta = s_\beta^{-1}(0)$. We denote by $i_\beta$ the embedding $Y_\beta \hookrightarrow X_\beta$.

Definition 1. The virtual dimension $\dim \text{virt}(Y_\beta)$ is the number $\dim(X_\beta) - \text{rk}(N_\beta)$.

The section $s$ is not necessarily regular. If it is, then the virtual dimension of $Y_\beta$ coincides with its usual dimension.

3. Construction. Our goal is to define certain class in the homology Chow group $A_{\dim \text{virt}(Y_\beta)}(Y_\beta)$. We will use the key construction from the intersection theory, [F], 6.1. We have a cartesian square

$$
\begin{array}{ccc}
Y_\beta & \xrightarrow{i_\beta} & X_\beta \\
\downarrow i_\beta & & \downarrow s_d \\
X_\beta & \xrightarrow{s_\beta} & N_\beta
\end{array}
$$

where $s_0$ denotes the zero section. Let $C$ be the normal cone of $Y_\beta$ in $X_\beta$. We have the canonical closed embedding $C \hookrightarrow i_\beta^*N_\beta$, whence the class $[C] \in A_d(i_\beta^*N)$ where $d =$
$\dim(C) = \dim(X_\beta)$. We set by definition

$$[Y_\beta]^{\text{virt}} := (p^*)^{-1}([C]).$$

Here $p^* : A_*(Y_\beta) \rightarrow A_{*+\text{rk}(N)}(i_\beta^*N)$ is the isomorphism induced by the projection $p : i_\beta^*N \rightarrow Y_\beta$.

4. Push forward formula. Lemma 1. We have

$$i_\beta^*([Y_\beta]^{\text{virt}}) = [X_\beta] \cap e(N_\beta)$$

where $e$ denotes the Euler (top Chern) class.

This formula was written down as a conjecture by Yu.I.Manin (talk at MPI, February 1997).

Proof. We can identify $s_0 : X_\beta \rightarrow N_\beta$ with the embedding of the scheme of zeros of the regular diagonal section of the vector bundle $N_\beta \times N_\beta \rightarrow N_\beta$, and then apply [F], 6.3.4. □

5. Iteration. Let $M$ be a convex vector bundle over $Y$, and $j : Z \hookrightarrow Y$ be the subscheme of zeros of a regular section $t : Y \rightarrow M$. We set by definition,

$$\dim^{\text{virt}}(Z_\beta) := \dim^{\text{virt}}(Y_\beta) - \text{rk}(M_\beta).$$

We define

$$[Z_\beta]^{\text{virt}} \in A_{\dim^{\text{virt}}(Z_\beta)}(Z_\beta)$$

by the same construction as in no. 3, with $[X_\beta]$ replaced by $[Y_\beta]^{\text{virt}}$.

If $M = i^*N'$ and $t$ is the restriction of a regular section $s' : X \rightarrow N'$ then $Z$ may be defined as the subscheme of zeros of the regular section $s \oplus s' : X \rightarrow N \oplus N'$, and the definition of $[Z_\beta]^{\text{virt}}$ at one step using $s \oplus s'$ coincides with the previous one (which used first $s$, then $t$).

We have a generalization of Lemma 1:

6. Push forward formula. II. Lemma 2. We have

$$j_\beta^*([Z_\beta]^{\text{virt}}) = [Y_\beta]^{\text{virt}} \cap e(M_\beta).$$

The proof is the same as in Lemma 1.

7. The above constructions have obvious equivariant versions.

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References

[F] W.Fulton, Intersection theory, Springer-Verlag, Berlin et al., 1984.
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