Application of the black hole-fluid analogy: identification of a vortex flow through its characteristic waves

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Black holes are like bells; once perturbed they will relax through the emission of characteristic waves. The frequency spectrum of these waves is independent of the initial perturbation and, hence, can be thought of as a ‘fingerprint’ of the black hole [1–3]. Since the 1970s scientists have considered the possibility of using these characteristic modes of oscillation to identify astrophysical black holes [4–6]. With the recent detection of gravitational waves, this idea has started to turn into reality [7, 8]. Inspired by the black hole-fluid analogy [9–12], we demonstrate the universality of the black-hole relaxation process through the observation of characteristic modes emitted by a hydrodynamical vortex flow. The characteristic frequency spectrum is measured and agrees with theoretical predictions obtained using techniques developed for astrophysical black holes [13, 14]. Our findings allow for the first identification of a hydrodynamical vortex flow through its characteristic waves. The flow velocities inferred from the observed spectrum agree with a direct flow measurement. Our approach establishes a non-invasive method, applicable to vortex flows in fluids and superfluids alike, to identify the wave-current interactions and hence the effective field theories describing such systems.
According to General Relativity, the late stage of the relaxation process of an astrophysical black hole is expected to depend only on its mass and angular momentum, and not on the details of its formation process [5]. This opens up the possibility of Black Hole Spectroscopy: the identification of the spacetime geometry through the measurement of the frequency spectrum of gravitational waves emitted by a newly formed black hole. Motivated by the black hole-fluid analogy, which implies that universal processes (e.g. Hawking radiation [15–17] and superradiance [18]) manifest themselves similarly in black holes and condensed matter systems, we apply the black hole spectroscopy idea to a vortex fluid flow.

The analogy, built on the works of Unruh [9] and Visser [10], relies on the fact that perturbations of some condensed matter systems, e.g. shallow water waves propagating on the free surface of an irrotational vortex flow [11], are mathematically equivalent to scalar waves propagating around a rotating black hole (see [12] for a review). Such a vortex flow, commonly called a draining bathtub (DBT) flow, is uniquely described by the velocity field $v(t, r, \theta) = v(r) = -\frac{D}{r} \hat{r} + \frac{C}{r} \hat{\theta}$. The two parameters, denoted C (for circulation) and D (for drain), are analogous to the angular momentum and mass of a rotating black hole.

Once perturbed, a vortex flow will relax through the emission of surface waves that propagate on the air-water interface. Such waves correspond to small deformations, $\delta h(t, x)$, of the unperturbed surface elevation, where $t$ denotes time and $x$ represents the spatial coordinates on the two-dimensional free surface. For axisymmetric free surface vortex, it is convenient to adopt polar coordinates $x = (r, \theta)$ and decompose the perturbations in terms of its azimuthal components,

$$\delta h(t, r, \theta) = \text{Re} \left[ \sum_{m \in \mathbb{Z}} \delta h_m(t, r) e^{im\theta} \right], \quad (1)$$

where the azimuthal number $m \in \mathbb{Z}$ indicates an $m$-fold symmetry with respect to the polar angle $\theta$.

Towards the end of the relaxation process, each azimuthal component is well-approximated (in an open system) by a superposition of time-decaying modes, called quasinormal modes (QNMs) [13]. Each QNM oscillates at the characteristic frequency $f_{mn}$ and has an amplitude that decays exponentially in time with a characteristic timescale of $1/\Gamma_{mn}$. The overtone number $n \in \mathbb{N}$ classifies the QNMs according to their decay times. The set of complex frequencies $\omega_{\text{QNM}}(m, n) = 2\pi f_{mn} + i\Gamma_{mn}$ is called the QNM spectrum. For the DBT flow, the QNMs have been extensively studied and their spectrum was calculated
using various methods [19, 20]. The QNM spectra of more realistic vortex flows, either due to the presence of vorticity [21] or dispersion [22], have also been investigated.

One of the techniques available to estimate the QNM spectrum of a black hole is based on the properties of light-rings (LRs) [13, 14]. The LRs of a black hole are the orbits (i.e. the equilibrium points in the radial direction) of massless particles. The relation between QNMs and LR modes comes from the fact that, in many (but not all [23]) spacetimes, QNMs can be seen as waves travelling on the unstable orbits and slowly leaking out [2]. For a fluid flow, these modes can also be identified as the lowest energy modes capable of propagating across the entire flow (see Methods). As such, they constitute the most favourable channel to transfer energy in and out of the system. While QNMs strongly rely on the openness of the system, the LR modes, being independent of the boundary conditions, do not. In particular, the presence of a non-open boundary condition, either at infinity or at the horizon, will modify the late-time behaviour of the relaxation process. More precisely, the decay times of characteristic waves will be altered by reflections from the boundaries. Additionally, their non-oscillatory behaviour will be further modified by damping in the system and recurring perturbations. The oscillatory part of the LR spectrum, $f_\star(m)$, therefore provides a more robust quantity to characterise the fluid flow in finite size experiments. By proving the existence of LR modes in fluid flows $^1$ and by using their oscillation frequencies to extract flow parameters, our experiment provides a practical realisation of the black hole spectroscopy idea.

We set up a vortex flow out of equilibrium to observe the emission of characteristic modes during its relaxation. We call such a restless fluid flow an Unruh vortex $^2$. Our experiment was conducted in a 3 m long and 1.5 m wide rectangular tank with a 2 cm-radius sink hole at the centre. Water is pumped continuously from one corner at a flow rate of 15±1 ℓ/min. The sink-hole is covered until the water raises to a height of 10.00±0.05 cm. Water is then allowed to drain, leading to the formation of an Unruh vortex. We recorded the perturbations of the free surface when the flow was in a quasi-stationary state at a water depth of 5.55±0.05 cm. The water elevation was recorded using the Fast-Chequerboard Demodulation method [24] and the entire procedure was repeated 25 times.

1 Even though these are surface water waves and not electromagnetic waves, we shall still refer to them as LR modes due to the fluid-gravity analogy.

2 This name comes from the German word “Unruhe” which means restlessness and was chosen in acknowledgment of W. G. Unruh, the founder of analogue gravity.
Fig. 1. **Normalised power spectral densities.** The power spectral density is compared with the minimum energy curve $f_{\text{min}}^+(m, r)$, plotted in red, for various $m$. The maxima of $f_{\text{min}}^+(m, r)$ indicate the location of the light-rings, $r_{LR}(m)$, which are shown in dashed white lines. For $m < 0$, the spectral density peaks and the minimum energy line are distinguishable for small radii. For $m > 0$, we observe two signals whose peaks are radius-dependent. The upper one follows the minimum energy line and corresponds, most probably, to random noise generated locally. The lower one follows the angular velocity of the fluid flow according to $f_\alpha(m, r) = mv\theta(r)/(2\pi r)$ (orange curve) and is likely sourced by potential vorticity perturbations.
The resulting Unruh vortex is axisymmetric to a good approximation, allowing us to perform an azimuthal decomposition, as in (1), to study its characteristic modes. We select specific azimuthal modes by performing a polar Fourier transform and we extract the associated radial profiles $\delta h_m(t, r)$. Azimuthal modes with $m > 0$ are co-rotating with the flow while modes with $m < 0$ are counter-rotating with the flow. By calculating the time Fourier transform of $\delta h_m(t, r)$, we estimate the Power Spectral Density (PSD) of each $m$-mode for $r \in [7.4 \text{ cm}, 25 \text{ cm}]$. In Fig. 1 we present the PSDs of a single experiment for a range of co- and counter-rotating modes (see Methods for data analysis and flow characterisation).

We can identify two different behaviours depending on the sign of $m$. For negative $m$'s, the PSDs are approximately constant over the window of observation. The spectral density is peaked around a single frequency, which allows us to define the position-independent spectrum $f_{\text{peak}}(m)$ shown in Fig. 2. This spectrum can be used twofold. First, by employing a standard Particle Imaging Velocimetry (PIV) technique, we estimate the circulation parameter to be $C \approx 151 \text{ cm}^2/\text{s}$ and the drain parameter to be negligible, i.e. $D \approx 0 \text{ cm}^2/\text{s}$ (see Methods). Using these values we can predict the characteristic mode spectrum $f_{\text{PIV}}^*(m)$ using the LR properties, as shown by the dashed orange curve in Fig. 2. We observe that the model describing the characteristic oscillations of an Unruh vortex as LR modes is consistent with the data. This is the first experimental observation of the oscillatory part of the LR spectrum.

Second, after having validated our approach, we perform analogue black hole spectroscopy to characterise the fluid flow (as an alternative to PIV). By leaving the flow parameters $C$ and $D$ unspecified, we look for the best match (in terms of non-linear regression analysis) between the experimental spectrum $f_{\text{peak}}(m)$ of counter-rotating modes and the corresponding theoretical predictions for the LR spectrum. This reduces the DBT parameter space from two dimensions to one, constraining the flow parameters $C$ and $D$ to lie on the red curve shown in Fig. 3. Any pair of points along this curve will give the same spectrum, $f_{\text{BM}}^*(m)$, represented by the solid black curve in Fig. 2. The region between the dashed orange curves shown in Fig. 3 represents the 95% confidence intervals for the values of $C$ and $D$. This region overlaps the yellow rectangle which represents the possible flow parameters found using PIV. Note that, in this case, the black hole spectroscopy method imposes a slightly stronger constraint on the circulation parameter than PIV.

We highlight that in order to uniquely determine $C$ and $D$ the positive $m$ part of the LR
Fig. 2. **Characteristic spectrum of the Unruh vortex.** The frequency spectrum $f_{\text{peak}}(m)$, extracted from the experimental data and represented by green dots, is compared with the theoretical prediction for the light-ring frequencies, $f_*(m)$. The error bars indicate the standard deviation over 25 experiments. The dashed orange curve is the predicted spectrum, $f_\text{PIV}^*(m)$, computed for $C = 151 \text{ cm}^2/\text{s}$ and $D = 0 \text{ cm}^2/\text{s}$. These flow parameters were obtained via an independent flow measurement, in our case Particle Imaging Velocimetry (PIV). The two spectra agree, confirming the detection of light-ring mode oscillations. The solid black curve, $f_\text{BM}^*(m)$, is the non-linear regression of the experimental data to the draining bathtub vortex model, and provides the values for C and D presented in the red curve of Fig. 3.

Although the LR modes are absent in the PSDs of the co-rotating modes shown in Fig. 1, two distinct, radius-dependent, signals are present. We can understand their origin using the flow parameters previously obtained. By computing the minimum energy line, $f_{\text{min}}^+(m, r)$ (red curve), we observe that one of the signals corresponds to random noise generated
Fig. 3. **Flow characterisation.** The intensity of the background image represents the normalised weighted sum of squared residuals between the experimental spectrum, $f_{\text{peak}}(m)$, and the theoretical prediction for the light-ring frequencies, $f^{\star}(m)$, as a function of the flow parameters. The red curve represents the family of possible values for C and D that best match the experimental data (using the method of weighted least squares). The area delimited by the dashed orange curves represents the 95% bootstrap confidence interval (see Methods). It overlaps with the yellow rectangle on the bottom right corner, which corresponds to the flow parameters obtained using Particle Imaging Velocimetry. The spread along the C-direction represents the 95% confidence interval estimated via the likelihood function. The spread along the D-direction represents the extracted upper bound for D. In the top right corner we present a detailed view of the parameter space where the two flow measurements overlap.

locally. The other signal is related to the angular velocity of the fluid flow and can be matched with the curve $f_{\alpha}(m, r) = mv^\theta(r)/(2\pi r)$, shown in orange. This peak lies below the minimum energy and, hence, corresponds to evanescent modes. A possible explanation for their appearance is that potential vorticity (PV) perturbations act as a source for them \cite{25}. In irrotational flows (which is the regime in which our observations are made), PV is carried by the flow as a passive tracer. Various $m$ components of PV will therefore source free-surface deformations which are transported at frequencies $f_{\alpha}(m, r)$. These observations
strengthen our confidence in the flow parameters obtained.

Our experiment exhibits a new facet of the fluid-gravity analogy [9–11] which has led to a better understanding of fundamental phenomena such as Hawking radiation [26, 27] and superradiance [28, 29]. Besides providing the first observation of light-ring mode oscillations, this successful demonstration of the principle behind black hole spectroscopy paves the way for real-life applications of the fluid-gravity analogy. This method can be used as an alternative to the standard fluid flow visualisation techniques, such as particle imaging velocimetry, that require tracer particles. In particular, when suitable tracer particles are hardly found or do not exist, like in superfluids [30], this is a promising non-invasive method to characterise fluid flows.

Methods

A. Data analysis.

In the following we outline the details of the detection method used and the various steps behind the data analysis leading to the PSDs shown in Fig. 1 and to the LR spectrum presented in Fig. 2. The free surface of the water is obtained using the Fast-Chequerboard Demodulation method [24]. A periodic pattern is placed at the bottom of the tank in a region of 59 cm $\times$ 84 cm. The pattern is composed of two orthogonal sinusoidal waves with wavelengths of 6.5 mm each. Deformations of this pattern due to free surface fluctuations are recorded in a region of 58 cm $\times$ 58 cm over the vortex using a Phantom Camera Miro Lab 340 high speed camera at a frame rate of 40 fps over 16.3 s with an exposure time of 24000 $\mu$s. For each of the 651 pictures of the deformed pattern, we reconstruct the free surface in the form of an array $h(t_k, x_i, y_j)$ giving the height of the water at the 1600 $\times$ 1600 points on the free surface $(x_i, y_j)$ at every time step $t_k$. The typical amplitude of the free surface deformations corresponds, at most, to 2% of the unperturbed water height. This justifies the linear treatment employed.

To select specific azimuthal numbers, we choose the centre of our coordinate system to be the centre of the hole and convert the signal from cartesian to polar coordinates. In addition to this change of coordinates, we discard all data points within a minimal radius

3 The MATLAB code used for this is available at: https://github.com/swildeman/fcd
$r_{\text{min}} \approx 7.4$ cm. This cropping is necessary as there is no clear pattern in the centre. This is either due to the hole at the bottom of the tank or due to the curvature of the vortex itself deforming the pattern too much to be detectable. We also discard points above a radius $r_{\text{max}} \approx 25$ cm, to avoid errors coming from the edge of the images. After this step, our data is in the form $h(t_k, r_i, \theta_j)$ with $r_{\text{min}} < r_i < r_{\text{max}}$.

Before selecting azimuthal components, we construct the analytic representation of the water elevation by adding to the real signal $h$ an imaginary part constructed from its Hilbert transform: $h_\mathcal{C} = h + i\mathcal{H}(h)$. The Hilbert transform $\mathcal{H}(h)$ is computed by means of a discrete Fourier transform and by removing the redundant negative frequency components of the time-spectrum. We have verified that other methods of computing the analytic representation, e.g. wavelet transforms, give identical results. This step is crucial as it will allow us to distinguish between $m > 0$ and $m < 0$, i.e. between waves that are co- and counter rotating with respect to the flow. At this stage we are left with a complex valued array $h_\mathcal{C}(t_k, r_i, \theta_j)$, such that $\text{Re}(h_\mathcal{C}) = h$. In the following we will discard the index $\mathcal{C}$ and keep in mind that we are now dealing with a complex array.

We then perform a discrete Fourier transform in the angular direction to separate the various azimuthal components $h(t_k, r_i, m) = \sum_j h(t_k, r_i, \theta_j) e^{-im\theta_j} \Delta \theta$. From this, it is possible to estimate the Power Spectral Density $PSD(f, r_i, m)$ of the waves emitted by the vortex for every azimuthal number $m$ and every radius $r_i$,

$$PSD(f, r_i, m) \propto |\tilde{h}(f, r_i, m)|^2,$$

where $\tilde{h}$ is the time Fourier transform of the height field at fixed $(m, r_i)$. In Fig. 1, we show the normalised PSDs for a range of $m$ values. The PSDs are finally averaged over the radius in order to look at the r-independent frequency content, i.e. the oscillation frequency of the LR modes.

Various averaged PSDs are presented in Supplementary Fig. 1. For each one of them, corresponding to a different $m$, the location of the peak, $f_{\text{peak}}(m)$, is obtained by a parabolic interpolation of the maximum of the Power Spectrum $\mathcal{P}_m$ and its nearest neighbouring points. This is the spectrum presented in Fig. 2. By repeating the procedure using a different choice for the center, located 10 pixels ($\approx 4$mm) away from the original, we observed that the maximum deviation in the location of a frequency peak is approximately 2%, attesting the robustness of the procedure against an inaccurate choice for the centre.
Supplementary Fig. 1. **Typical radius-averaged power spectral densities.** The location of peaks, denoted by $f_{\text{peak}}(m)$, correspond to the oscillation frequencies of the light-ring modes. The values of $f_{\text{peak}}(m)$ extracted from the data are plotted in Fig. 2.

### B. Light-ring properties and minimum energy

It is known that the QNM spectrum can be approximated using the properties of the LRs via [14]:

$$\omega_{\text{QNM}}(m) \approx \omega_\star(m) - i\Lambda(m) \left( n + \frac{1}{2} \right),$$  \hspace{1cm} (3)

where $\omega_\star(m) = 2\pi f_\star(m)$ is the angular frequency of an m-mode orbiting on the LR, $\Lambda(m)$ is the Lyapunov exponent of the orbit for this specific m-mode, and $n$ is the overtone number. In 2D systems where dispersion is absent, the evolution of perturbations can be described by means of a potential function $V(\omega, m, r)$ (see for e.g. [21, 31, 32]). In this context, the properties of the LR can be found in the high-$m$ limit of the potential: $V(\omega, m \gg 1, r)$. When dispersion is present, one does not have access to such a description.

In order to evaluate the LR spectrum, we look at the trajectories of rays. Such trajectories are found by solving Hamilton’s equation with the Hamiltonian $\mathcal{H}(\omega, m, r, k_r)$ given through
the dispersion relation \[22\]. In the case of surface waves propagating on a background flow, the dispersion relation reads

\[
\omega = \omega_d(k) = \pm F(k) + v \cdot k, \quad \text{with} \quad F(k) = \sqrt{\left(gk + \frac{\sigma}{\rho}k^3\right) \tanh(hk)}, \quad (4)
\]

where \(k = |k|\), \(g = 9.81 \text{ m/s}^2\) is the gravitational acceleration on Earth, \(\sigma = 0.0728 \text{ N/m}\) is the surface tension of water, \(\rho = 997 \text{ kg/m}^3\) is the density of water, \(h\) is the water depth, and \(v\) is the velocity of the background flow. This amounts to a WKB approximation and the wave vector in polar coordinates is given by \(k = (\frac{m}{r}, k_r)\). The LR oscillation frequency, \(f_*(m)\), is then calculated by looking for the critical point of the Hamiltonian. More precisely, for a given azimuthal number \(m\) and for fixed flow parameters, we solve numerically the following system of equations:

\[
\mathcal{H} = 0, \quad \frac{\partial \mathcal{H}}{\partial r} = 0 \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial k_r} = 0. \quad (5)
\]

The spectrum \(f_{\text{PIV}}^*(m)\) plotted in Fig. 2 is obtained by solving the above system for \(m \in [-25, -1]\) with the flow parameters \(C = 151 \text{ cm}^2/\text{s}\) and \(D = 0 \text{ cm}^2/\text{s}\) given by PIV. The black hole spectroscopy idea suggests that one can revert the process and extract the flow parameters from the spectrum instead. By comparing the experimental data \(f_{\text{peak}}(m)\) with the theoretical prediction \(f_*(m)\) through a non-linear regression analysis, we find the light-ring spectrum that best fits the data (i.e. that minimizes the weighted sum of squared residuals). This spectrum \(f_{\text{BM}}^*(m)\), also plotted in Fig.2, provides the values for \(C\) and \(D\) presented in the red curve of Fig. 3. The 95% confidence interval, delimited by the dashed orange curves in Fig.3, is obtained by using the bias-corrected and accelerated bootstrap method for 5000 samples \[33\].

We can also give a new interpretation of the LR conditions by noticing that

\[
\frac{\partial \mathcal{H}}{\partial k_r} = \pm F(k) \frac{\partial \omega_d}{\partial k_r} \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial r} = \pm F(k) \frac{\partial \omega_d}{\partial r}. \quad (6)
\]

This means that a critical point of the Hamiltonian is also a critical point of the dispersion relation. In particular, the LR modes will satisfy the condition \(\frac{\partial \omega_d}{\partial k_r} = 0\). This condition defines two curves, \(\omega_{\text{min}}^\pm = 2\pi f_{\text{min}}^\pm(m, r)\) (one for each branch of the dispersion relation), representing the minimum frequency required by a specific \(m\)-mode to be able to propagate at a radius \(r\). In Fig. 1, the \(f_{\text{min}}^\pm\) curves are plotted in red as a function of \(r\) for various values of \(m\). They separate the \((f, r)\)-plane in two regions. On the one hand, modes with
a frequency $f$ above the minimum energy curve, $f > f_{\text{min}}^+$, have a real-valued $k_r$, and are therefore able to propagate. On the other hand, modes with a frequency $f$ below the minimal energy threshold, $f < f_{\text{min}}^+$, having imaginary $k_r$ values, correspond to evanescent modes.

The second LR condition imposes that $\partial \omega_d/\partial r = 0$, implying that LR modes correspond to the lowest energy modes that are able to propagate across the entire fluid. By sitting at the top of the minimum energy curve, the LR modes possess a real valued $k_r$ everywhere, allowing them to transfer energy across the entire system. This resembles the standard procedure used in the shallow water regime. Indeed, we note that in this regime the minimum energy curves $\omega_{\text{min}}^\pm$ correspond to $\omega_\pm$ of [21], and therefore the eikonal potential $V_{\text{eik}}$ can be reconstructed from the minimum energy curves, namely $V_{\text{eik}} = -(\omega - \omega_{\text{min}}^+)(\omega - \omega_{\text{min}}^-)$.

As a last remark, we note that one can also use the dispersion relation $\omega_d$ to determine the imaginary part of the LR spectrum, which is related to the Lyapunov exponent $\Lambda$. Substituting equation (6) into the formula to compute the Lyapunov exponent (see Eq. (5.7) in [22]), one finds that

$$\Lambda = \text{det}([d^2 \omega_d]),$$

where $[d^2 \omega_d]$ is the Hessian matrix of $\omega_d$ evaluated on the LR.

C. Particle Imaging Velocimetry

As an independent method to characterise the background flow (and therefore validate the analogue black hole spectroscopy approach), we use Particle Imaging Velocimetry (PIV). PIV is a technique frequently used in engineering to measure fluid configurations. It relies on seeding the flow with small tracer particles and recording them using a high-speed camera as they follow the fluid streamlines. Each image thus obtained is split into smaller windows, which are then compared with similar windows in the consecutive image. For each window in a given image we compute the correlation table with respect to windows in the next image. By finding the maxima of the correlation tables, one is able to identify the displacement of each window between images. Repeating the procedure for all pairs of consecutive images, one can reconstruct the evolution of the velocity field in the region captured by the images (see e.g. [34] for a thorough review).

In our experiment, the flow was seeded with plyolite particles with an average diameter of 1 mm. The particles were illuminated with a 2 mm-thick light sheet provided by a Yb-
doped laser of characteristic wavelength 523 nm. The laser was aligned in all cases to be pointing directly at the vortex and the height of the sheet was adjusted to 5.4 cm, i.e. just below the water’s surface. Due to the presence of a shadow in the camera’s field of view (located where the laser sheet intersects the vortex core), the laser source was placed at three different positions: half way along the length of the tank and offset by 1.2 m to either side. The camera was positioned such that the pixel size was 0.32 mm. Two measurements were taken for each laser position, giving a total of 6 experimental runs. Since PIV requires the presence of tracer particles, it was not possible to run PIV simultaneously with the Fast-Chequerboard Demodulation method.

In each experimental run the particles were recorded for 5.445s using a Phantom Miro Lab 340 camera, with a Sigma 24-70mm F2.8 lens attached, at a resolution of 1600x1600 pixels and a frame rate of 200 frames per second (corresponding to 1090 images). The images were analysed using the MATLAB extension PIVlab, developed in [34]. We used PIVlab’s window deformation tool, with spline deformation, to reduce the number of erroneous vector identifications. The analysis was performed over three iterations using an interrogation area of 256 x 256, 128 x 128 and 64 x 64 pixels respectively, each with 50% overlapping.

The velocity field \( v(t_k, r_i, \theta_j) \) thus obtained is first averaged in time and then decomposed into an angular part, \( v^\theta(r_i, \theta_j) \), and a radial one, \( v^r(r_i, \theta_j) \), where \( r_i \in [7.4 \text{ cm}, 20 \text{ cm}] \). We note that \( v^r \ll v^\theta \) in the region of observation. The maximum value of the radial velocity is approximately a tenth of the angular one. More importantly, the bias in our PIV method (estimated by applying the method to simulated data) has the same order of magnitude as the measured radial velocities. This implies that we cannot extract a reliable value for the drain parameter D in the analysed region. In fact, it is expected that most of the draining in free surface vortices occurs through the bulk and through the boundary layer at the bottom of the tank, with a negligible radial flow at the surface, especially far from the vortex core [35–37]. Nevertheless, we can determine an upper-bound for D, namely \( D_{\text{max}} = \max_{i,j} |r_i v^r(r_i, \theta_j)| = 39 \text{ cm}^2/\text{s} \). This is the upper-bound used in Fig. 3.

In the window of observation, \( v^\theta(r_i, \theta_k) \) is found to be \( \theta \)-independent to a good approximation. This allows us to average the angular profile over the angles to obtain \( v^\theta(r_i) \). To justify this procedure we make the azimuthal decomposition \( v^\theta(r_i, \theta_j) = v^\theta(r_i) + U_{\text{asym}}(r_i, \theta_j) \), where \( U_{\text{asym}}(r_i, \theta_j) = \sum_{m \neq 0} v^{\theta,m}(r_i) e^{-im\theta_j} \) is the asymmetric part of the flow field. We then compare
Supplementary Fig. 2. **Angular component of the velocity profile.** Green dots correspond to the averaged (over time, angle and experiments) angular velocity. The error bars indicate the standard deviation. The orange curve is the weighted least-squares fit to the angular velocity $v^\theta = C/r$ of the Draining Bathtub model, corresponding to $C = 151 \text{ cm}^2/\text{s}$.

The energy of the asymmetric and axisymmetric parts, finding that $\sum_{m \neq 0} \mathcal{E}_m / \mathcal{E}_0 \sim 1.3\%$, where $\mathcal{E}_m = \sum_i |v^\theta,m(r_i)|^2$ is the energy of each azimuthal component. This indicates that our flow is highly symmetric.

The averaged (over time, angle and experiments) angular velocity profile is displayed in Supplementary Fig. 2. The error associated with the averaging is given by the standard deviation and indicates the spread in the data about the mean value. To check the stationarity of the flow, we compare the angular velocity vector field in the first and final seconds of our experimental runs, finding that the maximum difference at any point is smaller than the uncertainty due to the vector identification inherent in the PIV. Hence, we deem the velocity field to be stationary within the error of our method. Since a draining vortex is expected to be irrotational sufficiently far from the drain hole, we fit the mean angular velocity with the function $v^\theta(r) = C/r$, as shown in Supplementary Fig. 2. The extracted value for $C$ is $151 \text{ cm}^2/\text{s}$. Along with this value, we also compute the 95% confidence interval via the
likelihood function, which is used in conjunction with $D_{\text{max}}$, to plot the yellow box in Fig. 3.

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