CONSISTENT SPIN-TWO COUPLING AND QUADRATIC GRAVITATION

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Abstract. A discussion of the field content of quadratic higher-derivative gravitation is presented, together with a new example of a massless spin-two field consistently coupled to gravity. The full quadratic gravity theory is shown to be equivalent to a canonical second-order theory of a massive scalar field, a massive spin-two symmetric tensor field and gravity. The conditions showing that the tensor field describes only spin-two degrees of freedom are derived. A limit of the second-order theory provides a new example of massless spin-two field consistently coupled to gravity. A restricted set of vacua of the second-order theory is also discussed. It is shown that flat-space is the only stable vacuum of this type, and that the spin-two field around flat space is unfortunately always ghost-like.

1. Introduction

One of the simplest and most enduring ideas for extending general relativity is to include in the gravitational action terms which involve higher powers of the curvature tensor. For instance, if we require the theory to be CP-even and include only terms up to quadratic in the curvature, the most general such action is [1]

\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} + \gamma R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho} \right], \tag{1.1} \]

where the parameters \( \alpha, \beta \) and \( \gamma \) have dimensions of inverse mass squared and \( \kappa \) is the gravitational coupling constant. Since the theory is purely gravitational, the parameters are generically of the order of the inverse Planck mass squared. Consequently, if we expand around flat space, the quadratic terms represent tiny, unobservable corrections to Einstein’s theory. It is of course also possible to drop the term linear in \( R \) completely. However, the pure quadratic theory, when coupled to localized positive-definite matter, fails to give a solution which is asymptotically flat [2-4]. Thus to agree with Newtonian gravity we must include the linear term.

The quadratic theory described by (1.1) gained importance when Stelle showed that, with \( \gamma = 0 \), the theory is completely renormalizable, including its coupling to matter [5]. The advent of string theory as a consistent theory of gravity, provided a further motivation for studying such theories, since quadratic corrections are explicitly present in the string effective

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action. In a different direction, the quadratic theory with $\beta = \gamma = 0$ has been used to provide a purely gravitational theory of inflation [6-8].

Here we shall be concerned with only the classical content of the quadratic theory. The most important feature is that the equations of motion include fourth-order derivatives of the metric, rather than only second-order derivatives, as in Einstein’s equations. The consequence is that there are more degrees of freedom in the theory than just the two graviton states. This can be understood by noting that, in solving the Cauchy problem, one must now specify more initial conditions, giving the higher-order time derivatives of $g_{\mu\nu}$ on the initial time-slice. These new degrees of freedom were first identified by Stelle [1]. If we follow his analysis, the first observation is that classically the action (1.1) really only describes a two parameter class of theories. The Gauss-Bonnet combination $\sqrt{-g} \left( R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right)$ is a total divergence and so does not contribute to the classical equations of motion. Consequently, we can rewrite the action in terms of, for instance, only $R^2$ and the Weyl tensor squared, $C^2 = R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2$, so that

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R + \frac{1}{6m_0^2} R^2 - \frac{1}{2m_2^2} C^2 \right], \quad (1.2)$$

where $m_0^{-2} = 6\alpha + 2\beta + 2\gamma$ and $m_2^{-2} = -\beta - 4\gamma$, and neither is restricted to be positive. Keeping only the lowest order in the fluctuations of the metric around flat space, Stelle showed that the theory contained, in addition to the usual graviton, two new fields. The $R^2$ term introduced a new scalar field with a squared-mass of $m_0^2$. The $C^2$ term introduced a new spin-two field with a squared-mass of $m_2^2$. However the latter field was ghost-like. This suggested that, while the quadratic theory is renormalizable, it unfortunately appears to be non-unitary.

Stelle’s results raise two sets of questions. The first centers around extending his results to the complete non-linear theory. Can we identify the new degrees of freedom in the full theory? If so, can the theory be rewritten in a form where the new fields are made explicit and satisfy canonical second-order equations of motion? Such a reformulation removes the obscurity of the original higher-order field equations, allowing for a more intuitive analysis of the theory. For instance, we can look for the vacuum states of the theory. We can also see if, in the full theory, the spin-two field remains ghost-like. We note that we expect some solution of the ghost problem to exist, since string theory, while giving an effective action which includes the problematic quadratic terms, is nonetheless generally accepted as unitary.

The second set of questions centers around the issue of coupling spin-two fields to gravity. It has long been known that there are problems of consistency when coupling massless fields of spin higher than one either to themselves or to other fields [9-13]. Even in the massive case we may have problems with preserving unitarity [14]. Yet quadratic gravity promises to be re-writable as a theory of a spin-two field consistently coupled to a scalar field and
gravity. What is the form of the coupling? How are the conditions required to ensure the field is pure spin-two realized? In particular, is there a massless limit? If so, how does the theory evade the problems of consistency which usually plague attempts to couple a massless spin-two field to gravity?

In this paper we will address both sets of questions. We shall give a procedure for rewriting the full quadratic theory in a canonical second-order form. Reducing the full action to a second-order form is not new, though the result does not seem to be widely known. It has been discovered and rediscovered in various places. The equivalence of full $R + R^2$ gravity to a second-order theory of a scalar field coupled dilatonically to gravity was first shown by Teyssandier and Tourrenc [15]. Whitt reproduced and extended these results, rewriting the second-order theory in canonical form by making a conformal rescaling of the metric [16]. The full quadratic action, including a $C^2$ term, was first written in second-order form by Magnano et al. [17-19] and Jakubiec and Kijowski [20]. Some discussion of the particle content of the reduced theory, though only at the linearized level, was given by Alonso et al. [21]. What is new here, however, is to put the second-order action in a truly canonical form, separating the spin-two and scalar degrees of freedom. This will be the main thrust of this paper. It will provide us with a consistent theory of a massive spin-two field coupled to gravity. Further we shall find that the theory has a sensible massless limit, providing an explicit example of a completely consistent coupling of a massless higher-spin field to gravity. We shall show how this new theory in fact falls under the class of coupled spin-two fields discussed by Wald and Cutler [22].

The paper is organized as follows. In Section 2 we discuss the problem of coupling spin-two fields, either through self-interaction terms or to other fields. In the massive case, the field must satisfy a set of generalized conditions to ensure that only the spin-two degrees of freedom propagate. In the massless case, the problems are more extreme and imply that coupled theories are generically inconsistent. We discuss the nature of the inconsistency and, following [13] and [10], show that consistency is equivalent to the presence of a new local symmetry. In Section 3 we discuss $R + R^2$ theory, which introduces a new pure scalar degree of freedom. This provides us with an example of how to rewrite quadratic gravity in a second-order form and extract the canonical degrees of freedom. We discuss the form and vacuum structure of the reduced theory. In general, by a vacuum solution we mean a stable solution of the second-order theory. However, in this paper we will only consider a restricted set of vacua in which the new scalar or spin-two degrees of freedom are covariantly constant. Within this context, we show that the only stable vacuum state corresponds to flat space. In Section 4 we turn to the $R + C^2$ theory, which has a new pure spin-two degree of freedom. Again rewriting the theory in a canonical second-order form, we show that suitable conditions can be derived to ensure that the new field is indeed pure spin-two. We investigate the vacuum structure of the reduced theory within the restricted set, and find that flat space
is the only stable vacuum solution and that the spin-two field fluctuations around the vacuum remain ghost-like. In Section 5 we describe the massless limit of the second-order theory given in Section 4. The theory is shown to be consistent and the associated local symmetry is identified. We find that the theory is related to a class of theories discussed by Cutler and Wald [22, 23]. In Section 6, the general quadratic action is discussed. Reducing to a second-order form, we make the complete canonical separation of the new scalar and spin-two degrees of freedom. Again we discuss the vacuum structure of the theory, finding that flat space is the only stable vacuum solution and that the spin-two field remains ghost-like. In the final section we briefly present our conclusions.

Throughout the paper our conventions are to use a metric of signature \((- + + +)\) and to define the Ricci tensor as

\[
R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\lambda\nu} + \Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\mu\nu} - \Gamma^\lambda_{\rho\mu} \Gamma^\rho_{\lambda\nu}.
\]

### 2. Interacting Spin-Two Theories and Consistency

In this section we first describe the free equations of motion for a spin-two field and then discuss the problems of introducing interactions in both the massive and, more critically, massless cases. Starting in flat space, we recall that any field theory must represent the symmetries of the spacetime, namely the Poincaré group. Wigner showed that the unitary irreducible representations are classified by the Casimir operators \(M^2\) and \(S^2\), which physically represent the mass squared and the spin of the field quanta [24]. One form of the irreducible representation with spin two and mass \(m\) is in terms of a symmetric tensor \(\phi_{\mu\nu}\), satisfying

\[
(\partial^2 - m^2) \phi_{\mu\nu} = 0, \\
\partial^\mu \phi_{\mu\nu} = 0, \quad \phi = 0,
\]

where \(\phi = \eta^{\mu\nu} \phi_{\mu\nu}\). The first equation (2.1) simply states that the field is an eigenstate of the operator \(M^2\) with eigenvalue \(m^2\), while the second two conditions (2.2) ensure that it is also an eigenstate of \(S^2\), with eigenvalue two. In fact these conditions can be derived from an action, first given by Fierz and Pauli [25, 26, 9]. We write

\[
S = \int d^4x \left[ \frac{1}{4} (\partial_\mu \phi \partial^\mu \phi - \partial_\mu \phi_{\nu\rho} \partial^\mu \phi^{\nu\rho} + 2 \partial^\mu \phi_{\mu\nu} \partial^\nu \phi - 2 \partial_\mu \phi_{\nu\rho} \partial^\rho \phi^{\nu\mu}) - \frac{1}{4} m^2 (\phi_{\mu\nu} \phi^{\mu\nu} - \phi^2) \right].
\]

(2.3)

To understand how this action leads to the spin-two conditions (2.1) and (2.2), we must consider the massive and massless cases separately. In the massive case, we start from the corresponding equations of motion for \(\phi_{\mu\nu}\), which read

\[
\partial^2 \phi_{\mu\nu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma \phi^{\rho\sigma} = 0, \quad \partial_\mu \phi_{\rho\sigma} = \partial_\mu \phi_{\lambda\nu} - \partial_\nu \phi_{\lambda\mu} + \partial_\lambda \phi_{\mu\nu} - \eta_{\mu\nu} \partial^2 \phi = 0.
\]

(2.4)
Taking the divergence and the trace of these equations gives a pair of conditions,
\[ \partial^\mu (\phi_{\mu\nu} - \eta_{\mu\nu}\phi) = 0, \]
\[ 2\partial^\mu \partial^\nu (\phi_{\mu\nu} - \eta_{\mu\nu}\phi) = -3m^2\phi, \]
respectively, which imply that
\[ \partial^\mu \phi_{\mu\nu} = 0, \quad \phi = 0. \]
Here the reader should note that the divergence of the left-hand side of the Pauli-Fierz equations (2.4) vanishes identically, a fact that will be important when we discuss interacting massless spin-two fields. Substituting the conditions (2.6) back into the original equation of motion then gives
\[ (\partial^2 - m^2) \phi_{\mu\nu} = 0, \]
and we find that we have succeeded in deriving the three conditions (2.1) and (2.2), as required.

These conditions act to constrain the number of propagating degrees of freedom. We start with a symmetric tensor \( \phi_{\mu\nu} \) satisfying the massive Klein-Gordon equation (2.1), so that we might expect all the ten components of \( \phi_{\mu\nu} \) to propagate. However, consider the role of the spin conditions (2.2) in solving the Cauchy problem. To solve the Klein-Gordon equations, we are required to specify twenty initial conditions, namely each component and its time derivative on some initial surface. The \( \phi = 0 \) condition implies two constraints on the initial conditions, since \( \phi \) and \( \partial_t \phi \) are initially both zero. The divergence condition implies a further eight constraints with \( \partial^\mu \phi_{\mu\nu} \) and \( \partial_t \partial^\mu \phi_{\mu\nu} \) initially both set to zero. It is not clear that the latter expression is really a constraint since it includes second-order time derivatives. However these can be removed, since we must still satisfy the Klein-Gordon field equations. The constraint can then be rewritten as \( \partial_i \partial^i \phi_{\mu\nu} - \partial_t \partial^\mu \phi_{\mu\nu} - m^2 \phi_{\mu\nu} = 0, \) where \( i \) runs over the spatial indices. Thus we are left with ten independent initial conditions, and so five degrees of freedom. Put another way, ignoring any equations of motion, a general symmetric tensor \( \phi_{\mu\nu} \) can be decomposed into one spin-two field, one spin-one and two scalar fields, a total of ten degrees of freedom. The equations of motion derived from the Pauli-Fierz action then imply two conditions (2.2), which are just sufficient to set all but the five massive spin-two degrees of freedom to zero.

It is worth noting that the divergence condition on \( \phi_{\mu\nu} \) can be promoted to the status of a conserved current. A local symmetry can be introduced in to the Pauli-Fierz action by writing it in a form somewhat akin to the Stückleberg action used for quantizing massive
gauge fields. We introduce a new vector field $\zeta_\mu$ and write

$$S = \int d^4x \left[ \frac{1}{4} \left( \partial_\mu \phi \partial^\mu \phi - \partial_\mu \phi_{\nu\rho} \partial^\mu \phi^{\nu\rho} + 2 \partial^\mu \phi_{\mu\nu} \partial^\nu \phi - 2 \partial_\mu \phi_{\nu\rho} \partial^\rho \phi^{\mu\nu} \right) + \frac{1}{4} m^2 \left( \phi_{\mu\nu} + \partial_{(\mu} \zeta_{\nu)} \right) \left[ \phi^{\mu\nu} + \partial^{(\mu} \zeta^{\nu)} \right] - \left[ \phi + \partial^\mu \zeta_\mu \right]^2 \right].$$

(2.8)

The corresponding equations of motion are

$$\partial^2 \phi_{\mu\nu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma \phi_{\rho\sigma} - \partial_\mu \partial^\lambda \phi_{\lambda\nu} - \partial_\nu \partial^\lambda \phi_{\lambda\mu} + \partial_\mu \partial_\nu \phi - \eta_{\mu\nu} \partial^2 \phi = m^2 \left[ \phi_{\mu\nu} + \partial_{(\mu} \zeta_{\nu)} \right] - \eta_{\mu\nu} \left( \phi + \partial^\lambda \zeta_{\lambda} \right),$$

(2.9)

$$\partial^\mu \left[ \phi_{\mu\nu} + \partial_{(\mu} \zeta_{\nu)} \right] - \eta_{\mu\nu} \left( \phi + \partial^\lambda \zeta_{\lambda} \right) = 0.$$  

(2.10)

The presence of the $\zeta_\mu$ field has no effect on the dynamic content of the theory since its field equation (2.10) is already implied by the divergence of the $\phi_{\mu\nu}$ equation of motion (2.9). However, it does introduce a new gauge symmetry. The action (2.8) is invariant under the combined transformation

$$\phi_{\mu\nu} \rightarrow \phi_{\mu\nu} + \partial_{(\mu} \zeta_{\nu)},$$

$$\zeta_\mu \rightarrow \zeta_\mu - \xi_\mu.$$  

(2.11)

We note that the original Pauli-Fierz action (2.3) corresponds to the modified action (2.8) with the gauge choice $\zeta_\mu = 0$. Further, we find that the conserved current for the gauge symmetry (2.11), in the gauge $\zeta_\mu = 0$, is none other than $\phi_{\mu\nu} - \eta_{\mu\nu} \phi$, and thus the divergence condition becomes a consequence of a local symmetry as promised. We shall use this Stückleberg formulation in separating the spin-two and scalar degrees of freedom in the full quadratic gravity theory as described in Section 6.

Finally we turn to the massless spin-two theory. The consequence of setting $m = 0$ is that we can no longer derive the conditions (2.2) from the field equations (2.4) following from the Pauli-Fierz action (2.3). However, we also find that the action has a new local gauge symmetry. It is now invariant under the combined transformation

$$\phi_{\mu\nu} \rightarrow \phi_{\mu\nu} + \partial_{(\mu} \xi_{\nu)},$$

(2.12)

This immediately allows four of the ten components of $\phi_{\mu\nu}$ to be transformed away, simply by choosing a gauge. In fact, we must choose the gauge $\partial^\mu \phi_{\mu\nu} = 0$ in order to satisfy one of the spin-two conditions. A residual gauge freedom, namely transformations of the form (2.12) with $\partial^\mu \xi_\mu = 0$, remains. As is the case of a massless gauge field, since the gauge $\partial^\mu \phi_{\mu\nu} = 0$ implies that the remaining components of $\phi_{\mu\nu}$ satisfy the massless Klein-Gordon equation, we find that the residual gauge symmetry is sufficient to set $\phi$ to zero together with three other linear combinations of the components of $\phi_{\mu\nu}$. Thus we are left with only two propagating degrees of freedom, the two different helicity states of a spinning, massless field.
Thus far we have described the content of the massive and massless free field equations for spin-two. We would now like to consider interactions, either through self-coupling terms or couplings to other fields. To provide an example of the issues which arise, consider coupling a spin-two field to gravity. The most natural approach is to take the Pauli-Fierz action (2.3), but to promote the ordinary derivatives to full covariant derivatives, so that

\[ S = \int d^4x\sqrt{-g} \left[ \frac{1}{4} (\nabla_\mu \phi \nabla^\mu \phi - \nabla_\mu \phi_{\nu\rho} \nabla^\mu \phi^{\rho\nu} + 2\nabla^\mu \phi_{\mu\nu} \nabla^\nu \phi - 2\nabla_\mu \phi_{\nu\rho} \nabla^\rho \phi^{\mu\nu}) \right. \\
\left. - \frac{1}{4} m^2 \left( \phi_{\mu\nu} \phi^{\mu\nu} - \phi^2 \right) \right]. \quad (2.13) \]

The corresponding field equations are then

\[ \nabla^2 \phi_{\mu\nu} + g_{\mu\nu} \nabla^\rho \nabla_\rho \phi - \nabla_\mu \nabla^\lambda \phi_{\lambda\nu} - \nabla_\nu \nabla^\lambda \phi_{\lambda\mu} + \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi = m^2 (\phi_{\mu\nu} - g_{\mu\nu} \phi), \quad (2.14) \]

again with the same form as the flat-space equations, but with ordinary derivatives replaced with covariant derivatives. Let us now try and derive the trace and divergence conditions (2.2) which in flat space restricted \( \phi_{\mu\nu} \) to describe only spin-two. Taking a trace of the field equations (2.14), as before, we obtain the condition

\[ 2 \nabla^\mu \nabla_\mu \left( \phi_{\mu\nu} - g_{\mu\nu} \phi \right) = -3m^2 \phi. \quad (2.15) \]

However, when we take the divergence of the field equations new terms arise because the covariant derivatives do not commute. We find that

\[ \nabla_\nu \left( \nabla^\lambda \phi - 2\nabla_\rho \phi^{\rho\lambda} \right) + \left( \nabla_\nu R_{\lambda\rho} - 2\nabla_\lambda R_{\nu\rho} \right) \phi^{\lambda\rho} = m^2 \left( \phi_{\mu\nu} - g_{\mu\nu} \phi \right). \quad (2.16) \]

Consider first the massive case. In flat space, the left-hand side of (2.16) was zero, so that substituting this equation into (2.15) immediately gave \( \phi = 0 \), and then hence, substituting back into (2.16), the divergence condition \( \partial^\mu \phi_{\mu\nu} = 0 \). Here this is no longer true. We note however that a generalized form of the divergence condition does still hold. The expressions (2.16) involve only first-order time derivatives of \( \phi_{\mu\nu} \), and, as such, represent four constraint equations for \( \phi_{\mu\nu} \), the same way \( \partial^\mu \phi_{\mu\nu} = 0 \) did in the flat-space case. We might also hope to derive a generalized form of \( \phi = 0 \), for instance by taking the divergence of (2.16) and substituting into (2.15). However, we find that no such condition, involving only first-order time derivatives of \( \phi_{\mu\nu} \), can be found. We are left to conclude that, though not inconsistent, the gravitationally coupled theory no longer describes pure spin-two. In terms of the Cauchy problem, the generalized divergence condition implies that only six components of \( \phi_{\mu\nu} \) propagate independently, but, without a generalized trace condition, we cannot further reduce the number to the five degrees of freedom of a massive spin-two field.

This discussion provides us with a general prescription for determining if an interacting theory describes pure spin-two. There must be five conditions derivable from the equations.
of motion. If $\Phi_i$ are the other fields in a second-order theory these conditions must be of the form

$$
 f_\mu(\phi_{\lambda\rho}, \partial_{\lambda}\phi_{\rho\sigma}; \Phi_i, \partial_{\lambda}\Phi_i, \partial_{\lambda}\partial_{\rho}\Phi_i) = 0,
$$

$$
 g(\phi_{\lambda\rho}, \partial_{\lambda}\phi_{\rho\sigma}; \Phi_i, \partial_{\lambda}\Phi_i, \partial_{\lambda}\partial_{\rho}\Phi_i) = 0.
$$

The first four expressions are the generalized divergence conditions. Linearizing in $\phi_{\mu\nu}$ and $\Phi_i$, they must agree with the free conditions, $\partial^\mu \phi_{\mu\nu} = 0$. The final expression is the generalized trace condition, and it must read $\phi = 0$ in the linearized limit. In general, for the conditions to be constraints on the initial conditions when solving the Cauchy problem for the theory, they must not involve the second-order time-derivative of $\phi_{\mu\nu}$. Here, for simplicity, we shall assume they involve none of the second-order derivatives.

In the massless case, the problems are more extreme. We note first that with $m = 0$ the action (2.13) no longer has a local symmetry. Transformations of the form (2.12), with ordinary derivatives replaced with covariant derivatives, fail because the covariant derivatives do not commute. Critically, we also find that, with $m = 0$, the only way to satisfy the condition (2.16), for any curvature $R_{\mu\nu}$, is to set $\phi_{\mu\nu} = 0$. The only alternative is to consider (2.16) not as a condition on $\phi_{\mu\nu}$, but as a condition on the curvature. This can in some cases be a satisfactory resolution. For instance, the spin-$\frac{3}{2}$ gravitino equation in supergravity gives a similar condition. However, in this case, the condition is implied by the Einstein equations and so is automatically satisfied on shell. Aragone and Deser have investigated the possibility of a similar solution to the problem of coupling a massless spin-two field to gravity [11, 12]. However, for a large class of couplings, they showed no such mechanism is possible.

In fact, these problems of consistency and the loss of local gauge invariance are related. Wald [13], and earlier Ogievetsky and Polubarinov [10], have used this connection as a way of deriving the possible consistent spin-two theories. To understand this relationship, let us first, following Wald [13], define the general consistency problem. For any massless, interacting, spin-two theory, we can separate the action into a free part, which is the massless Pauli-Fierz action, and a part describing the interactions,

$$
 S = S_{PF,m=0}[\phi_{\mu\nu}] + S_1[\phi_{\mu\nu}, \Phi_i],
$$

where $\Phi_i$ are the other fields in the theory. If, for example, we couple to gravity, the correction terms to make the ordinary derivatives in the Pauli-Fierz action into covariant derivatives will be included in $S_1$. The equations of motion following from the general action can be written as,

$$
 T_{\mu\nu}(\phi_{\rho\sigma}, \Phi_i) = \frac{\delta S}{\delta \phi_{\mu\nu}} = T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu} + \cdots = 0,
$$

where we make an expansion such that $T^{(k)}_{\mu\nu}$ is $k$-th order in the fields $\phi_{\mu\nu}$ and $\Phi_i$. We know that the first-order expression $T^{(1)}_{\mu\nu}$ must come from the free part of the action and
so is none other than the left-hand side of the massless Pauli-Fierz equations (2.4). Suppose we now look for a perturbative solution of (2.19). We denote the leading-order, linearized solution by $\phi^{(1)\mu\nu}$. By definition,

$$\mathcal{T}^{(1)\mu\nu}(\phi^{(1)\rho\sigma}) = 0. \quad (2.20)$$

We can similarly solve for $\Phi^{(1)i}$, the linearized solution of the $\Phi^i$ equations of motion. Now consider solving for the quadratic corrections to $\phi^{(1)\mu\nu}$, denoted $\phi^{(2)\mu\nu}$. They must satisfy

$$\mathcal{T}^{(1)\mu\nu}(\phi^{(2)\rho\sigma}) + \mathcal{T}^{(2)\mu\nu}(\phi^{(1)\rho\sigma}, \Phi^{(1)i}) = 0. \quad (2.21)$$

As we noted earlier $\partial^\mu \mathcal{T}^{(1)\mu\nu}(\phi_{\rho\sigma})$ is identically zero for any $\phi_{\rho\sigma}$. Thus, if we take the divergence of equation (2.21), we are left with the relation

$$\partial^\mu \mathcal{T}^{(2)\mu\nu}(\phi^{(1)\rho\sigma}, \Phi^{(1)i}) = 0. \quad (2.22)$$

This implies a condition on $\phi^{(1)\mu\nu}$ and $\Phi^{(1)i}$. The result is that not all the linearized solutions are necessarily consistent with the equation for the solution to the next order in the perturbation expansion. As we go to higher orders in the expansion, we shall find more and more conditions. Indeed it may be that none of the linearized solutions are compatible with these conditions, or as we found in the case of the naive coupling of a spin-two field to gravity only the trivial solution $\phi_{\mu\nu} = 0$ survives. This lack of a sensible procedure for perturbatively solving the field equations is one way of defining the problem of inconsistency.

To avoid inconsistency, we require (2.22) to be satisfied by any solution of the linearized equations, and that the similar conditions, arising at higher-orders in the perturbation expansion, are also automatically satisfied by solving the field equations to one order lower in the expansion. One way of ensuring this is if the full interacting action has a local symmetry of a particular form. For a second-order theory, we require the action to be invariant under infinitesimal transformations of the form [10, 13, 22]

$$\delta \phi_{\mu\nu} = \partial_{(\mu} \xi_{\nu)} + \alpha_{\mu\nu\rho\sigma} \partial_{(\rho} \xi_{\sigma)} + \beta_{\mu\nu\rho} \xi_{\rho},$$

$$\delta \Phi_i = \gamma_{i\rho\sigma} \partial_{(\rho} \xi_{\sigma)} + \epsilon_{i\rho} \xi_{\rho}. \quad (2.23)$$

Here the coefficients $\alpha_{\mu\nu\rho\sigma}$ and $\gamma_{i\rho\sigma}$ are taken to be general functions of $\phi_{\mu\nu}$ and $\Phi_i$, while $\beta_{\mu\nu\rho}$ and $\epsilon_{i\rho}$ may also depend on $\partial_{\lambda} \phi_{\mu\nu}$ and $\partial_{\lambda} \Phi_i$. When linearizing the transformation in $\phi_{\mu\nu}$ and $\Phi_i$, all the coefficients must vanish, so that we return to the old symmetry of the massless Pauli-Fierz action. If one calculates the corresponding conserved current, as is done in Wald [13] and Ogievetsky and Polubarinov [10], one finds that, order by order in the perturbation expansion, the divergence conditions, such as (2.22), are satisfied and there is no consistency problem. We also note that, since the symmetry has the same form as the gauge symmetry of the free action, we expect to be able to eliminate all but two propagating components of $\phi_{\mu\nu}$, so that it truly describes a massless spin-two field.
Both Ogievetsky and Polubarinov [10] and Wald [13] have used this formalism to prove that a generally covariant theory, such as Einstein gravity, where the $\phi_{\mu\nu}$ field really describes the metric of a curved spacetime, is the only consistent theory of a single spin-two field. Cutler and Wald [22] also considered consistent collections of spin-two fields. This will include any consistent theory of a massless spin-two field coupled to gravity. Wald [23] obtained the very interesting result that such theories correspond to theories of a metric on algebra-valued manifolds, where the coordinates are elements of an associative, commutative algebra. Here, we shall not be concerned with this geometrical interpretation, but it will be important to note Wald’s result that at least one of the spin-two fields in a consistent theory must be ghost-like. We shall return to this issue in Section 5.

In conclusion, for a massive, interacting, symmetric tensor field, we require a set of conditions (2.17) to hold if the field is to describe only spin-two. In the massless case, we require the action to have a local symmetry of the form (2.23) in order for the theory to be consistent and describe spin-two. These are the conditions we must show apply when we attempt to extract the spin-two degree of freedom from quadratic gravity.

3. The $R + R^2$ Theory

Before turning to general quadratic gravity, let us first consider the special case where we include only the scalar curvature quadratic correction, so that

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\mathcal{g}} \left( R + \frac{1}{6m^2} R^2 \right). \quad (3.1)$$

From Stelle’s linearized calculations, we expect the $R^2$ term to introduce a new scalar degree of freedom into the theory. Extracting this degree of freedom, by rewriting the theory in a canonical second-order form, will provide us with a simple example of a procedure we shall use throughout this paper. Having obtained the reduced, second-order theory, we shall briefly discuss its physical content, in particular the vacuum solutions.

In rewriting the theory in a canonical second-order form, we will follow Whitt [16]. The procedure is first to introduce an auxiliary field for the scalar curvature $R$, thus giving an action with second-order equations of motion. This is the analog of forming the Helmholtz Lagrangian, which reduces a second-order theory to a first-order theory in terms of field variables and conjugate momenta. Here we instead reduce a fourth-order theory to second-order, with the auxiliary field playing the role of the conjugate momentum. Then, by redefining the fields to grow an explicit kinetic energy term for the new field, the action is transformed to the canonical form for a scalar field coupled to Einstein gravity.
Thus, introducing a dimensionless auxiliary field $\lambda$, the action (3.1) can be rewritten as

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R + \frac{1}{6m^2} R^2 - \frac{1}{6m^2} \left( R - 3m^2\lambda \right)^2 \right] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ (1 + \lambda) R - \frac{3}{2}m^2\lambda^2 \right].$$

(3.2)

The $\lambda$ equation of motion is algebraic, giving $R = 3m^2\lambda$. Substituting this solution back into the action clearly returns one to the original higher-derivative form and so the theories are equivalent. The equations of motion for $g_{\mu\nu}$ and $\lambda$ are now second-order, but the action is not yet in a canonical form. With this in mind, consider the transformation of the Ricci scalar under a conformal rescaling of the metric. If $\bar{g}_{\mu\nu} = e^{\chi} g_{\mu\nu}$, in terms of the new metric and the covariant derivative $\nabla_{\lambda} g_{\mu\nu} = 0$, we have

$$\sqrt{-\bar{g}}R = \sqrt{-g} e^{-\chi} \left( \bar{R} + 3 \nabla^2 \chi - \frac{3}{2} \left( \nabla \chi \right)^2 \right).$$

(3.3)

Comparing with (3.2) we see that by choosing $\chi = \log (1 + \lambda)$, and conformally rescaling the metric we can produce a canonical Einstein-Hilbert term. Furthermore we also generate a canonical kinetic energy term. Thus, transforming from $(g_{\mu\nu}, \lambda)$ to $(\bar{g}_{\mu\nu}, \chi)$ and dropping total derivative terms, the action becomes

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\bar{g}} \left[ \bar{R} - \frac{3}{2} \left( \nabla \chi \right)^2 - \frac{3}{2} m^2 \left( 1 - e^{-\chi} \right)^2 \right].$$

(3.4)

This is the canonical form for Einstein gravity coupled to a new scalar field $\chi$ and reproduces the result derived by Whitt [16]. One subtlety here is that $\chi$ is only defined as a real field for $\lambda > -1$. Note, however, that despite this restriction on $\lambda$, the range of $\chi$ is unrestricted with $-\infty < \chi < \infty$. For $\lambda < -1$ we must introduce, instead, the real field $\chi = \log (-1 - \lambda)$, the effect of which is to change the sign of the overall normalization of the action. At the one special point $\lambda = -1$, the conformal rescaling becomes degenerate. Thus at this point the action cannot be put in canonical form, although there is no problem with the non-canonical formulation in terms of $\lambda$. Here, and throughout this paper, we will restrict our attention to the range of the auxiliary field(s) for which the transformed action has, when possible, the usual normalization. Finally, we also note that strictly the action (3.4) is not quite in canonical form since the scalar field $\chi$ is dimensionless and has a non-canonical normalization of its kinetic energy. This is, of course, easily remedied by introducing a dimensionful rescaled field $\chi' = (3/2\kappa^2)^{1/2}\chi$. For simplicity, throughout this paper, we will not make this final trivial rescaling.

Clearly the scalar field kinetic energy in action (3.4) has the usual sign, and, therefore, is not a ghost. The $\bar{g}_{\mu\nu}$ and $\chi$ equations of motion following from (3.4) have the conventional
forms
\[
\overline{R}_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu} \overline{R} = \frac{3}{2} \left[ \nabla_\mu \chi \nabla_\nu \chi - \frac{1}{2} \overline{g}_{\mu\nu} (\nabla \chi)^2 \right] - \frac{3}{2} \overline{g}_{\mu\nu} V(\chi),
\]
(3.5)
\[
\nabla^2 \chi = \frac{dV}{d\chi},
\]
(3.6)
respectively, where here the potential is given by
\[
V(\chi) = \frac{1}{2} m^2 (1 - e^{-\chi})^2.
\]
(3.7)
Consider the vacuum solutions of these equations. If we restrict ourselves to solutions where
the scalar field is covariantly constant,
\[
\nabla_\mu \chi = \partial_\mu \chi = 0,
\]
(3.8)
it follows from (3.6), that, \(\chi\) must extremize that potential. The \(\overline{g}_{\mu\nu}\) equation of motion (3.5)
then implies that the spacetime is a space of constant curvature, a deSitter or anti-deSitter
solution, with
\[
\overline{R} = 6V(\chi).
\]
(3.9)
Since the original metric \(g_{\mu\nu}\) is related to \(\overline{g}_{\mu\nu}\) by a conformal rescaling, \(g_{\mu\nu} = e^{\chi} \overline{g}_{\mu\nu}\)
and \(\chi\) is constant, it follows that
\[
R = e^{\chi} \overline{R}.
\]
(3.10)
Thus these vacuum states are also spaces of constant curvature with respect to the original
metric, although the radius of curvature is in general different.

The potential energy (3.7) is plotted in Figure 1. For \(m^2 > 0\) the potential has a single
stable minimum at \(\chi = 0\), with scalar excitations of mass \(m\), which we recall is generically

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{potential.pdf}
\caption{\(V(\chi)\) for quadratic higher-derivative gravity}
\end{figure}
of order of the Planck mass. For $m^2 < 0$ we have no stable minimum. Therefore, the only stable vacuum solution is for $m^2 > 0$ and $\chi = 0$. Since $V(0) = 0$, it follows that $\bar{R} = R = 0$ and, hence, that $\bar{g}_{\mu\nu} = g_{\mu\nu} = \eta_{\mu\nu}$. That is, the only stable vacuum solution where the scalar field is constant is trivial flat space with vanishing scalar vacuum expectation value.

In closing this section it is worth emphasizing two facts. The first is that it is clear from this discussion that the quadratic gravity action (3.1) which involves only the metric field $g_{\mu\nu}$, in the range of the scalar curvature given by $R > -3m^2$, is completely equivalent to the action (3.4) which involves not only a metric field $g_{\mu\nu}$ described by the usual Einstein theory, but also a massive, non-ghost scalar field $\chi$. The conclusion is that the modification of Einstein’s theory obtained by adding the $R^2$ term in (3.1), increases the number of propagating degrees of freedom contained in $g_{\mu\nu}$ from the two helicity states of the usual graviton to three. This can easily be seen by considering the fourth-order $g_{\mu\nu}$ equations of motion associated with action (3.1). The formalism we have used here, introducing the auxiliary field $\lambda$ and transforming to the $(\bar{g}_{\mu\nu}, \chi)$ variables, simply makes this fact manifest. We want to emphasize that the new degree of freedom is physical and cannot be removed from the theory by, for example, a field redefinition or gauge transformation.

The second fact we would like to point out is that the new scalar degree of freedom has a potential energy that is uniquely determined by the original action (3.1). This opens the possibility that $\chi$ or, equivalently, the original metric $g_{\mu\nu}$ might have a non-trivial vacuum state. We see from Figure 1 that this is not the case. However, it is possible that more general higher-derivative gravity theories do have non-trivial gravitational vacua. Indeed, if one considers actions which are general functions of the curvature, we have shown that this is the case [27].

4. The $R + C^2$ Theory

We now turn to the $R + C^2$ quadratic theory, which from Stelle’s linearized analysis we expect to introduce new pure spin-two degrees of freedom. The action is given by

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2m^2} C^2 \right].$$

(4.1)

We would like to introduce an auxiliary field to reduce the theory to a second-order form as we did in the case of the $R + R^2$ theory. First though, it is useful to rewrite the action in terms of only the Ricci tensor $R_{\mu\nu}$. This can be done by extracting a total divergence, a
Gauss-Bonnet term, which will not contribute to the classical equations of motion. We have,

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2m^2} C^2 \right]
\]

\[
= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2m^2} \left( R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \right) \right]
\]

\[
= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R - \frac{1}{m^2} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) - \frac{1}{2m^2} \left( R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \right) \right]
\]

\[
= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R - \frac{1}{m^2} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) \right].
\]

Written in the form given in the last line, we can now reduce the action to a second-order, but not canonical, form by introducing a symmetric tensor auxiliary field \(\pi_{\mu\nu}\). Such a procedure was first given by Magnano et al. [17-19] and Jakubiec and Kijowski [20]. We shall use a slightly different formulation in which it is easier to demonstrate that the new field \(\pi_{\mu\nu}\) is pure spin-two. We write

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2m^2} C^2 \right]
\]

\[
= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R - \frac{1}{m^2} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) \right]
\]

\[
= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R - G_{\mu\nu} \pi^{\mu\nu} + \frac{1}{4} m^2 \left( \pi_{\mu\nu} \pi^{\mu\nu} - \pi^2 \right) \right],
\]

where \(\pi = \pi_{\mu\nu} g^{\mu\nu}\) and \(G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\). The auxiliary field equation of motion then is

\[
G_{\mu\nu} = \frac{1}{2} m^2 \left( \pi_{\mu\nu} - g_{\mu\nu} \pi \right),
\]

which gives \(\pi_{\mu\nu} = 2m^{-2} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right)\). Substituting this expression back into the action yields the original fourth-order theory.

The metric equation of motion for the reduced action given in the last line of (4.3) reads

\[
\nabla^2 \pi_{\mu\nu} + g_{\mu\nu} \nabla^\rho \nabla^\sigma \pi_{\rho\sigma} - \nabla_\mu \nabla^\lambda \pi_{\lambda\nu} - \nabla_\nu \nabla^\lambda \pi_{\lambda\mu} + \nabla_\mu \nabla_\nu \pi - g_{\mu\nu} \nabla^2 \pi
\]

\[
+ R_\mu^\lambda \left( \pi_{\lambda\nu} - \frac{1}{2} g_{\lambda\nu} \pi \right) + R_\nu^\lambda \left( \pi_{\lambda\mu} - \frac{1}{2} g_{\lambda\mu} \pi \right) - \frac{1}{2} g_{\mu\nu} R^{\rho\sigma} \left( \pi_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} \pi \right) = m^2 \left( \pi_{\mu\nu} - g_{\mu\nu} \pi \right),
\]

where we have used the \(\pi_{\mu\nu}\) equation of motion (4.4) to simplify some terms. From the two equations of motion (4.4) and (4.5) it is clear that, although the action is not in canonical form, we now have two propagating fields, the metric \(g_{\mu\nu}\) and the new auxiliary field \(\pi_{\mu\nu}\). We expect the particle content of the metric, which now satisfies a second-order equation, to be the usual two helicity states. But does \(\pi_{\mu\nu}\) really describe the degrees of freedom of a massive spin-two field?
From our discussion in Section 2, we require generalized divergence and trace conditions of the form (2.17) to hold if this is the case. This is, in fact, so. Taking a trace of the $g_{\mu\nu}$ field equation (4.5) and the divergence of the $\pi_{\mu\nu}$ equation of motion (4.4), recalling that we have the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$, gives the pair of conditions,

$$\nabla^\mu (\pi_{\mu\nu} - g_{\mu\nu}\pi) = 0,$$

$$2\nabla^\mu \nabla^\nu (\pi_{\mu\nu} - g_{\mu\nu}\pi) = -3m^2\pi. \tag{4.6}$$

These imply that

$$\nabla^\mu \pi_{\mu\nu} = 0, \quad \pi = 0. \tag{4.7}$$

These conditions have precisely the required form. In fact, they are just the simplest curved-space generalization of the free Pauli-Fierz conditions (2.2). The ordinary derivatives have simply been replaced by covariant derivatives. More importantly they demonstrate that the full non-linear $R + C^2$ theory is equivalent to a pure spin-two field coupled to gravity.

However, the action (4.3) is still not in a canonical form. We would like both to grow an explicit kinetic term for the auxiliary field and to reduce the curvature terms to the canonical Einstein-Hilbert form. This should allow us to identify the mass of the spin-two field, and to determine if it is indeed ghost-like. To do this, we need to generalize the conformal transformation used in the previous section for a scalar auxiliary field. We start by rewriting the second-order action (4.3) in the suggestive form,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ \left( 1 + \frac{1}{2}\pi \right) g^{\mu\nu} - \pi^{\mu\nu} \right] R_{\mu\nu} + \frac{1}{4}m^2 \left( \pi_{\mu\nu}\pi^{\mu\nu} - \pi^2 \right). \tag{4.8}$$

It appears that to obtain a canonical Einstein-Hilbert term we need to define a new metric $\overline{g}_{\mu\nu}$ such that

$$\sqrt{-\overline{g}}g^{\mu\nu} = \sqrt{-g} \left[ \left( 1 + \frac{1}{2}\pi \right) g^{\mu\nu} - \pi^{\mu\nu} \right]. \tag{4.9}$$

The necessary transformation can be written as

$$\overline{g}^{\mu\nu} = (\det A)^{-1/2} g^{\mu\lambda} A_\lambda^\nu,$$

$$A_\lambda^\nu = A_\lambda^\nu(\phi_{\sigma\tau}) = \left( 1 + \frac{1}{2}\phi \right) \delta_\lambda^\nu - \phi_\lambda^\nu. \tag{4.10}$$

Here we have introduced the new field

$$\phi_\mu^\nu = \pi_\mu^\nu, \tag{4.11}$$

where it is understood that the indices of $\phi_\mu^\nu$ are raised and lowered using the metric $\overline{g}_{\mu\nu}$, while the indices of $\pi_{\mu\nu}$ where raised and lowered using $g_{\mu\nu}$. Although the identification is via the mixed index objects, it can be shown that $\phi_\mu^\nu$ is nonetheless symmetric. Note also that if $\pi_{\mu\nu}$ is traceless, so is $\phi_\mu^\nu$. 
The transformation can be inverted to give

\[ g_{\mu\nu} = g_{\mu\nu}(\phi_{\rho\sigma}) = (\det A)^{-1/2} A_\mu{}^\lambda g_{\lambda\nu}^\rho. \] (4.12)

so that the transformation of the Ricci tensor is given by

\[ R_{\mu\nu} = \overline{R}_{\mu\nu} - \nabla_\mu C_{\lambda}{}^{\alpha} + \nabla_\alpha C_\lambda{}^{\mu\nu} - C_\lambda{}^{\mu\rho} C^\rho{}_{\nu\lambda} + C_\lambda{}_{\mu\nu} C^\rho{}_{\rho\lambda}, \] (4.13)

where \( \nabla_\lambda g_{\mu\nu} = 0, \)

\[ C_\lambda{}_{\mu\nu} = C_\lambda{}^{\lambda}{}_{\mu\nu}(\phi_{\rho\sigma}) = \frac{1}{2} \left( g^{-1} \right)^{\lambda\rho} \left( \nabla_\mu g_{\nu\rho} + \nabla_\nu g_{\mu\rho} - \nabla_\rho g_{\mu\nu} \right). \] (4.14)

and \( g_{\mu\nu} = g_{\mu\nu}(\phi_{\rho\sigma}) \) as given in (4.12). Thus in terms of the new variables \((\overline{g}_{\mu\nu}, \phi_{\mu\nu})\), dropping a total divergence, the action becomes

\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\overline{g}} \left[ \overline{R} - \overline{g}^{\mu\nu} \left( C_\mu{}_{\nu\rho}(\phi_{\sigma\tau}) C^\rho{}_{\sigma\lambda}(\phi_{\sigma\tau}) - C_\mu{}_{\nu\rho}(\phi_{\sigma\tau}) C^\rho{}_{\rho\lambda}(\phi_{\sigma\tau}) \right) \right. \]

\[ \left. + \frac{1}{2} m^2 (\det \phi_{\sigma\tau})^{-1/2} \left( \phi_{\mu\nu} \phi_{\mu\nu} - \phi^2 \right) \right]. \] (4.15)

**Figure 2.** Schematic representation of the region in \( \phi_{\mu\nu} \) space giving conventional signature for the metric \( \overline{g}_{\mu\nu} \) compared with the same region for the field \( \lambda \)

We recall that in the last section there was a subtlety in making the final transformation to the canonical form of the action. For the range of the auxiliary field \( \lambda < -1 \), we had to define the field \( \chi = \log(-1 - \lambda) \) which led to an action with the canonical form but the wrong overall normalization. Further at the special point \( \lambda = -1 \) we could not transform the action into a canonical form. A similar subtlety appears here. From the form of the definition of \( \overline{g}_{\mu\nu} \) (4.10), we see that the new metric \( \overline{g}_{\mu\nu} \) need not have the same signature
as the original metric $g_{\mu\nu}$. The definition also becomes undefined at the points in $\phi_{\mu\nu}$-space where $\det A = 0$. We would like to be able to restrict ourselves to a range of $\phi_{\mu\nu}$ where the signature of the transformed metric is the conventional $(-,+,+,+)$, in agreement with the signature of the original metric. One way to do this is to note that, as we range in $\phi_{\mu\nu}$-space, a point where the $\overline{g}_{\mu\nu}$ metric changes sign is indicated by $\det \overline{g}_{\mu\nu}$ going to zero. From the definition (4.10), we see that

$$\det \overline{g}_{\mu\nu} = \det A \det g_{\mu\nu}. \quad (4.16)$$

Thus since we assume $\det g_{\mu\nu} \neq 0$, we find that the signature-changing points are indicated by $\det A = 0$. We also see from the definition (4.10) that when $\phi_{\mu\nu} = 0$, we have $\overline{g}_{\mu\nu} = g_{\mu\nu}$ and so the signatures necessarily agree. These facts allow us to define a region of the $\phi_{\mu\nu}$-space in which the $\overline{g}_{\mu\nu}$ signature is conventional. It is the region around the point $\phi_{\mu\nu} = 0$ bounded by the surface $\det A = 0$. This region is shown schematically in Figure 2. The unshaded region where the signature is conventional is the analog of the $\lambda > -1$ region in the scalar field case, the boundary $\det A = 0$ is the analog of $\lambda = -1$, while the excluded region is the analog of $\lambda < -1$. There may, of course, be other disconnected regions where the signatures also agree, but here we shall not consider them. It may also be possible to find a suitable transformation in the regions where the signatures do not agree, just as we defined $\chi = \log(-1 - \lambda)$ in the region $\lambda < -1$ in the last section. However, again as in the scalar field case, the resulting action will not describe Einstein gravity; it will have either the wrong normalization or a metric with the wrong signature. Here, and throughout the paper, we will restrict our attention to that region of the auxiliary field around $\phi_{\mu\nu} = 0$ for which we know the transformed action (4.15) describes conventional Einstein gravity.

The equations of motion for the action (4.15) are extremely involved. However, the spin-two conditions on $\phi_{\mu\nu}$ can be obtained directly from equations (4.7). In terms of the new variables these become,

$$\overline{\nabla}^\mu \phi_{\mu\nu} - \overline{g}^{\lambda\mu} \left(C^\rho_\lambda (\phi_{\sigma\tau}) \phi_{\rho\nu} + C^\rho_\nu (\phi_{\sigma\tau}) \phi_{\rho\mu} \right) = 0, \quad \phi = 0, \quad (4.17)$$

giving generalized divergence and trace conditions for $\phi_{\mu\nu}$ of the required form (2.17), so that we may again conclude that $\phi_{\mu\nu}$ describes only spin-two degrees of freedom. One can also, of course, obtain these conditions directly from the equations of motion of action (4.15).

Thus, we have succeeded in rewriting the original higher-derivative theory as a canonical theory of a spin-two field coupled to Einstein gravity. The spin-two field has a generalized sigma-model kinetic energy given by the non-linear $C^\lambda_{\mu\nu}$ terms, and a particular potential given by the term proportional to $m^2$. 
To make the structure of the sigma model explicit, we expand the kinetic energy and potential terms as a power series in $\phi_{\mu\nu}$ around $\phi_{\mu\nu} = 0$. We find that

$$L_{\phi_{\mu\nu}} = -\frac{1}{4} (1 - \phi + \ldots) \left[ \nabla_\mu \phi \nabla^\nu \phi - \nabla_\mu \phi_{\nu\rho} \nabla^\nu \phi_{\rho} + 2 \nabla_\mu \phi_{\nu\rho} \nabla^\nu \phi_{\rho} - 2 \nabla_\mu \phi_{\nu\rho} \nabla^\nu \phi_{\nu\rho} + \nabla_\mu \phi \nabla^\nu \phi - 2 \nabla_\mu \phi_{\nu\rho} \nabla^\nu \phi_{\nu\rho} + \ldots \right]$$

$$+ \left[ \frac{1}{2} \phi_{\mu\nu} \left( \nabla_\mu \phi \nabla^\nu \phi_{\nu\rho} + 2 \nabla_\mu \phi_{\nu\rho} \nabla^\nu \phi_{\nu\rho} + \nabla_\mu \phi_{\nu\rho} \nabla^\nu \phi_{\rho} - 2 \nabla_\mu \phi_{\nu\rho} \nabla^\nu \phi \nabla_{\nu\rho} \phi - 2 \nabla_\mu \phi_{\nu\rho} \nabla^\nu \phi_{\nu\rho} \right) + \ldots \right]$$

$$+ \frac{1}{4} m^2 (1 + \phi + \ldots) \left( \phi_{\mu\nu} \phi_{\mu\nu} - \phi^2 \right).$$

(4.18)

We see that, to lowest order, we have the curved-space version of the Pauli-Fierz action, with ordinary derivatives replaced with covariant derivatives. In the flat-space limit this reproduces Stelle’s linearized result. As in that case, the kinetic term clearly has the wrong sign, so that the spin-two field is ghost-like. For $m^2 > 0$ the ghost has Planck-scale mass $m$, while for $m^2 < 0$ it is tachyonic. We note that higher-order corrections in $\phi_{\mu\nu}$ modify both the potential and kinetic energy terms.

This result only applies, of course, in an expansion around $\phi_{\mu\nu} = 0$. A relevant question is whether the sigma-model corrections could change the spin-two field from ghost to non-ghost if we expand around a different vacuum, one where $\phi_{\mu\nu}$ takes on a non-zero expectation value. To answer this question we must first find the vacuum solutions of the reduced theory. In this paper, we shall only consider vacua in which the $\phi_{\mu\nu}$ field is covariantly constant. As stated above, the equations of motion for $g_{\mu\nu}$ and $\phi_{\mu\nu}$ are extremely involved, and, because of the tensor structure of $\phi_{\mu\nu}$, the covariantly constant constraint is insufficient to render the vacuum equations tractable. However, these equations simplify dramatically if we only consider $\phi_{\mu\nu}$ of the form $\phi_{\mu\nu} = \frac{1}{4} \phi \tilde{g}_{\mu\nu}$. The $g_{\mu\nu}$ and $\phi_{\mu\nu}$ equations of motion then become

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = \frac{3}{32} (1 + \frac{1}{4} \phi)^{-2} \left[ \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} \tilde{g}_{\mu\nu} \left( \nabla \phi \right)^2 \right] - \frac{3}{2} \tilde{g}_{\mu\nu} V(\phi),$$

(4.19)

$$\left( 1 + \frac{1}{4} \phi \right)^{-2} \left[ \nabla_\mu \nabla_\nu \phi - \tilde{g}_{\mu\nu} \nabla^2 \phi \right] + \frac{1}{4} \left( 1 + \frac{1}{4} \phi \right)^{-3} \left[ \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} \tilde{g}_{\mu\nu} \left( \nabla \phi \right)^2 \right] = -12 \tilde{g}_{\mu\nu} \frac{dV}{d\phi},$$

(4.20)

respectively, where we have the potential function

$$V(\phi) = \frac{m^2 \phi^2}{16 \left( 1 + \frac{1}{4} \phi \right)^2}.$$

(4.21)

The covariantly constant condition implies that

$$\nabla_\mu \phi = \partial_\mu \phi = 0.$$

(4.22)

It follows that $\phi$ is a constant and, from equation (4.20), that it must extremize the potential. The $g_{\mu\nu}$ equation of motion implies, as in equation (3.9) that the vacuum is a space of constant curvature, a deSitter or anti-deSitter solution with

$$\tilde{R} = 6V(\phi).$$

(4.23)
Thus these vacua are spaces of constant curvature. The original metric, $g_{\mu\nu}$ is related to $\mathcal{g}_{\mu\nu}$ through equation (4.12). For $\phi_{\mu\nu} = \frac{1}{4}\phi\mathcal{g}_{\mu\nu}$, it follows that

$$g_{\mu\nu} = \left(1 + \frac{1}{4}\phi\right)^{-1}\mathcal{g}_{\mu\nu},$$

(4.24)

and, hence, since $\phi_{\mu\nu}$ is constant, that

$$R = \left(1 + \frac{1}{4}\phi\right)\mathcal{R}.$$  

(4.25)

Thus these vacua correspond to spaces of constant curvature with respect to the original metric as well, although the radius of curvature is in general different. We should note that equation (4.24) provides us with a specific example of the problem of signature-changing transformations. When $\phi < -4$ the signature of $\mathcal{g}_{\mu\nu}$ changes sign with respect to the signature of $g_{\mu\nu}$. This means we are moving outside the conventional-signature region around $\phi_{\mu\nu} = 0$ we defined above. Note that the boundary $\phi = -4$, where the transformation becomes undefined, does indeed correspond to $\det A = 0$. Thus throughout this discussion of vacua of the form $\phi_{\mu\nu} = \frac{1}{4}\phi\mathcal{g}_{\mu\nu}$, we must assume $\phi > -4$.

![Figure 3. $V(\phi)$ for quadratic higher-derivative gravity](image)

The potential $V(\phi)$ is plotted in Figure 3. We see that the only stationary point is at $\phi = 0$, so that $\phi_{\mu\nu} = 0$. This is in fact simply a confirmation of our general result that the equations of motion imply $\phi = 0$. For $m^2 > 0$ this is a stable minimum, whereas for $m^2 < 0$ it is unstable. Since $V(0) = 0$, it follows that $\mathcal{R} = R = 0$ and, hence, that $\mathcal{g}_{\mu\nu} = g_{\mu\nu} = \eta_{\mu\nu}$.

We are led to conclude that trivial flat space with $\phi_{\mu\nu} = 0$ is the only possible stable vacuum state of the type we are considering. Therefore the spin-two excitations around the vacuum are necessarily ghost-like.

In this section we have made the five new spin-two degrees of freedom in $R + C^2$ gravity explicit by rewriting the theory in a truly canonical second-order form. It is worth stressing
that these degrees of freedom are physical and cannot be removed by a field redefinition or a gauge transformation. We have shown that the new field $\phi_{\mu\nu}$ satisfies generalized spin-two conditions of the required form. The second-order theory has an interesting and complicated structure with a sigma-model kinetic energy and a fixed potential. We have shown that the only symmetric vacuum state where $\phi_{\mu\nu}$ is covariantly constant is flat space with $\phi_{\mu\nu} = 0$. The fluctuations of $\phi_{\mu\nu}$ around this vacuum, always have the wrong-sign kinetic energy and so the ghost problem persists in the full non-linear theory.

5. A Massless Spin-Two Theory Consistently Coupled to Gravity

We showed in the previous section that the $R + C^2$ action is equivalent to an action describing a massive spin-two field coupled to gravity. Given the problem of writing a consistent theory of a spin-two field coupled to gravity, it is natural to ask if the second-order theory has a sensible massless limit. Consider the non-canonical form given in the last line of (4.3). Setting $m = 0$, we have the action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [R - G_{\mu\nu} \pi^{\mu\nu}],$$

(5.1)

From the discussion in Section 2, if this is to describe a consistent theory of spin-two, there must be a local symmetry of the form (2.23). In fact, a very simple symmetry does exist. Consider the transformation

$$\pi_{\mu\nu} \rightarrow \pi_{\mu\nu} + \nabla(\mu \xi_{\nu}).$$

(5.2)

Because of the Bianchi identity $\nabla^{\mu} G_{\mu\nu} = 0$, integrating by parts, we find that this is a symmetry of the action (5.1). Further it has the required form (2.23). It is just the simple curved-space generalization of the free Pauli-Fierz symmetry, with ordinary derivatives replaced by covariant derivatives. We conclude that we have a consistent theory of a massless spin-two field coupled to gravity.

To understand how this consistency works, consider the equations of motion following from (5.1). The $\pi_{\mu\nu}$ field equations are

$$G_{\mu\nu} = 0,$$

(5.3)

which are just the Einstein equations for empty space, implying that $R_{\mu\nu} = 0$. The $g_{\mu\nu}$ field equations then reads

$$\nabla^2 \pi_{\mu\nu} + g_{\mu\nu} \nabla^{\rho} \nabla^{\sigma} \pi_{\rho\sigma} - \nabla_{\mu} \nabla^{\lambda} \pi_{\lambda\nu} - \nabla_{\nu} \nabla^{\lambda} \pi_{\lambda\mu} + \nabla_{\mu} \nabla_{\nu} \pi - g_{\mu\nu} \nabla^2 \pi = 0,$$

(5.4)

where we have substituted by $R_{\mu\nu}$ throughout. But these are simply the equations one would derive for the fluctuations of the metric about a fixed empty-space background. That is, for a background spacetime given by $g_{\mu\nu}$ satisfying (5.3), linear perturbations of the metric satisfy the equations (5.4). Such fluctuations are known to be consistent; they represent the graviton degrees of freedom in empty space. This is the origin of the consistency of $\pi_{\mu\nu}$.
Note, however, the form of the action (5.1) implies that unlike the metric perturbation the $\pi_{\mu\nu}$ field is ghost-like. It is worth noting that since gravity is also a consistent theory of a massless spin-two field, the action (5.1) must have two local symmetries. Indeed it does. One is the $\pi_{\mu\nu}$ symmetry given above. The other is, of course, diffeomorphism invariance. We should also be able to make contact with Wald’s work on consistent theories of collections of spin-two fields [23]. Wald identified consistent theories with geometrical theories of a metric on an algebra-valued manifold, where the algebra must be commutative and associative. For an algebra with $n$ elements, the metric on the algebra-valued manifold describes $n$ different spin-two fields. In the simple case of an algebra consisting of only the identity element $e$ and a single nil-potent element $v$, Wald showed that the generalized algebra-valued Einstein-Hilbert action is really a pair of actions. Expanding in the basis $(e, v)$, we have

\begin{align}
S_e &= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R,
S_v &= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} G_{\mu\nu} \pi^{\mu\nu},
\end{align}

(5.5)

where $g_{\mu\nu}$ and $\pi_{\mu\nu}$ are the two components of the algebra-valued metric. Thus we see that the action (5.1) is the sum of the two actions of Wald’s theory. Note that the two actions really give three equations of motion for two fields, since we can vary $g_{\mu\nu}$ in $S_e$ and $g_{\mu\nu}$ and $\pi_{\mu\nu}$ in $S_v$. However, the $\pi_{\mu\nu}$ equation and the $g_{\mu\nu}$ equation from $S_e$, both give the empty space Einstein equations $G_{\mu\nu} = 0$. Thus the system is not over-constrained. Including the $g_{\mu\nu}$ equation from $S_v$, exactly the same equations of motion arise from the single action (5.1) as from the pair of Wald actions.

Finally, we point out that the massless theory can still be rewritten in the canonical form (4.15). The local symmetry (5.2) is of course preserved. However, it now has a more complicated realization, involving a simultaneous transformation of the metric and the spin-two field,

\begin{align}
\phi_{\mu\nu} &\rightarrow \phi'_{\mu\nu} = \phi_{\mu\nu} + \nabla_{(\mu} \xi_{\nu)} - C^\rho_{\mu\nu}(\phi_{\sigma\tau}) \xi_{\rho}, \\
g_{\mu\nu} &\rightarrow g'_{\mu\nu} = (1 + \frac{1}{2} \phi')^{-1} \left[ \frac{\det^{1/2} A(\phi'_{\sigma\tau})}{\det^{1/2} A(\phi_{\sigma\tau})} \left( 1 + \frac{1}{2} \phi \right) g_{\mu\nu} + (\phi'_{\mu\nu} - \phi_{\mu\nu}) \right].
\end{align}

(5.6)

Nonetheless, it is still in the required form (2.23). In particular, it collapses to the old massless free Pauli-Fierz symmetry in the linearized limit. In the $\phi_{\mu\nu}$-formulation, it is clear that the problem remains that the spin-two field is ghost-like. Wald has shown that all the algebra-valued theories he derived include at least one ghost-field. Thus, although quadratic gravity provides an interesting example of a massless spin-two field consistently coupled to gravity, it does so at the expense of an apparent loss of unitarity.
6. The General Quadratic Theory

Having dealt separately with the two special cases of pure scalar-curvature and pure Weyl-tensor corrections to gravity, we now turn to extracting the new degrees of freedom for a general quadratic action. The goal is to separate both a new spin-two and a new scalar field from the full non-linear action. As we have already discussed, using the Gauss-Bonnet theorem, the general action can be written either in terms of scalar-curvature and Weyl-tensor terms, or equivalently in terms of scalar-curvature and Ricci-tensor terms. We have

\[
S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ R + \frac{1}{6m_0^2} R^2 - \frac{1}{2m_2^2} C^2 \right]
= \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ R + \frac{1}{6m_0^2} R^2 - \frac{1}{m_2^2} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) \right].
\]  

(6.1)

Having seen how to introduce auxiliary fields for these two types of term in the last two subsections, it is very natural simply to repeat the procedure here with a pair of auxiliary fields, a scalar for the scalar-curvature correction term and a tensor for the Weyl term. Thus combining the rewritings (3.2) and (4.3) we have

\[
S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ R + \frac{1}{6m_0^2} R^2 - \frac{1}{m_2^2} \left( R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) \right]
= \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ R + \lambda R - \frac{3}{2} m_0^2 \lambda^2 - G_{\mu\nu} \pi^{\mu\nu} + \frac{1}{4} m_2^2 \left( \pi_{\mu\nu} \pi^{\mu\nu} - \pi^2 \right) \right]
= \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ e^\chi R - \frac{3}{2} m_0^2 (e^\chi - 1)^2 - G_{\mu\nu} \pi^{\mu\nu} + \frac{1}{4} m_2^2 \left( \pi_{\mu\nu} \pi^{\mu\nu} - \pi^2 \right) \right],
\]  

(6.2)

where in the last line, as before, we have defined a new variable \( \chi = \log (1 + \lambda) \). For the time being we have not made the usual accompanying conformal rescaling of the metric. The equations of motion for \( \chi \) and \( \pi_{\mu\nu} \) are, exactly as before,

\[
R = 3m_0^2 (e^\chi - 1),
G_{\mu\nu} = \frac{1}{2} m_2^2 \left( \pi_{\mu\nu} - g_{\mu\nu} \pi \right),
\]  

(6.3)

so that substituting by the \( \pi_{\mu\nu} \) equation reproduces the Weyl-tensor term, while substituting by the \( \chi \) equation reproduces the squared curvature-scalar term.

The \( g_{\mu\nu} \) equation of motion is

\[
2 e^\chi G_{\mu\nu} - 2 \left( \nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 \right) e^\chi + \frac{3}{2} g_{\mu\nu} m_0^2 (e^\chi - 1)^2
+ \nabla^2 \pi_{\mu\nu} + g_{\mu\nu} \nabla^\sigma \pi_{\rho\sigma} - \nabla_\mu \nabla^\lambda \pi_{\lambda\nu} - \nabla_\nu \nabla^\lambda \pi_{\lambda\mu} + \nabla_\mu \nabla_\nu \pi - g_{\mu\nu} \nabla^2 \pi
+ R_\mu^\lambda \left( \pi_{\lambda\nu} - \frac{1}{2} g_{\lambda\nu} \pi \right) + R_\nu^\lambda \left( \pi_{\lambda\mu} - \frac{1}{2} g_{\lambda\mu} \pi \right) - \frac{1}{2} g_{\mu\nu} R^{\rho\sigma} \left( \pi_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} \pi \right) = 0.
\]  

(6.4)
Taking a trace of this equation and a divergence of the $\pi_{\mu\nu}$ equation, after using the $\chi$ equation to remove all $R$ dependence, gives the conditions

$$\nabla^\mu (\pi_{\mu\nu} - g_{\mu\nu}\pi) = 0, \quad \pi - 2m_2^2 \nabla^2 e^\chi = 0. \quad (6.5)$$

It appears that we have derived a set of spin-two conditions for $\pi_{\mu\nu}$. However, they do not have the correct linearized form. Rather than collapsing to the free conditions $\partial^\mu \pi_{\mu\nu} = \pi = 0$, we find there is a linear dependence on the scalar field $\chi$.

This linear dependence is a consequence of the fact that it is not possible, when the auxiliary fields are introduced in this way, to separate the scalar and spin-two kinetic energy terms at the quadratic level. To make this coupling explicit, consider making the usual conformal transformation, $\tilde{g}_{\mu\nu} = e^\chi g_{\mu\nu}$, to grow a kinetic term for the scalar field. The Einstein tensor is not invariant but grows terms depending on the derivative of $\chi$, giving

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} - \frac{3}{2} \left( \tilde{\nabla} \chi \right)^2 - \frac{3}{2} m_0^2 (1 - e^{-\chi})^2 - \tilde{G}_{\mu\nu} \tilde{\pi}^{\mu\nu} \
- \left( \tilde{\nabla}_\mu \tilde{\nabla}_\nu \chi - \tilde{g}_{\mu\nu} \tilde{\nabla}^2 \chi + \tilde{\nabla}_\mu \chi \tilde{\nabla}_\nu \chi - \tilde{g}_{\mu\nu} \left( \tilde{\nabla} \chi \right)^2 \right) \tilde{\pi}^{\mu\nu} + \frac{1}{2} m_2^2 (\tilde{\pi}_{\mu\nu} \tilde{\pi}^{\mu\nu} - \pi^2) \right], \quad (6.6)$$

where now the indices of $\pi_{\mu\nu}$ are raised and lowered with the metric $\tilde{g}_{\mu\nu}$, and we take the object with both indices lowered as invariant when we change metrics. We see that because of the transformation of the Einstein tensor, there are couplings between $\tilde{\nabla}_\mu \tilde{\nabla}_\nu \chi$ and $\pi_{\mu\nu}$ at the quadratic level. Clearly if we now try and grow kinetic terms for the spin-two field by moving to a new metric $\tilde{g}_{\mu\nu}$, just as in the case of pure Weyl-squared corrections, this quadratic coupling between the kinetic energy terms for $\pi_{\mu\nu}$ and $\chi$ will persist. Further this coupling cannot be removed by simple field redefinition. Thus the spin-two part of this action cannot reduce to the Pauli-Fierz action in the quadratic limit.

Although there is nothing wrong with the previous formulation, we would prefer to have a more canonical form giving the ordinary Pauli-Fierz limit. We notice that the quadratic kinetic coupling arose because of the inhomogeneous pieces in the transformation of the Einstein tensor under a conformal rescaling. One way to circumvent such inhomogeneous terms is to recall that the Weyl-squared term is invariant under conformal rescalings. Thus an alternative procedure is to introduce only the scalar auxiliary field, leaving the Weyl-squared terms untouched, make the conformal rescaling to grow kinetic terms for the scalar
field, and only then to introduce the spin-two auxiliary field. That is,

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R + \frac{1}{6m_0^2} R^2 - \frac{1}{2m_2^2} \nabla_\mu \nabla_\nu \phi^\lambda \chi_{\lambda \mu \nu} C^\alpha_{\lambda \mu \nu} \right]
\]

\[
= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ e^\chi R - \frac{3}{2} m_0^2 \left( e^\chi - 1 \right)^2 - \frac{1}{2m_2^2} \nabla_\mu \nabla_\nu \phi^\lambda \chi_{\lambda \mu \nu} C^\alpha_{\lambda \mu \nu} \right]
\]

\[
= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ \tilde{R} - \frac{3}{2} \left( \tilde{\nabla} \chi \right)^2 - \frac{3}{2} m_0^2 \left( 1 - e^{-\chi} \right)^2 - \frac{1}{2m_2^2} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi^\lambda \chi_{\lambda \mu \nu} \right]
\]

\[
= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ \tilde{R} - \frac{3}{2} \left( \tilde{\nabla} \chi \right)^2 - \frac{3}{2} m_0^2 \left( 1 - e^{-\chi} \right)^2 - \tilde{G}_{\mu \nu} \tilde{\pi}^{\mu \nu} + \frac{1}{4} m_2^2 \left( \tilde{\pi}_{\mu \nu} \tilde{\pi}^{\mu \nu} - \tilde{\pi}^2 \right) \right],
\]

(6.7)

where, in the third line, we have made a conformal transformation to the metric \( \tilde{g}_{\mu \nu} = e^\chi g_{\mu \nu} \), and as usual dropped a Gauss-Bonnet term to rewrite the Weyl-squared term in the third line as \( R_{\mu \nu} R^{\mu \nu} - \frac{1}{3} R^2 \) before introducing the auxiliary field \( \tilde{\pi}_{\mu \nu} \). Clearly there is now no quadratic coupling between \( \chi \) and \( \pi_{\mu \nu} \). We can again derive, from the trace of the \( \tilde{g}_{\mu \nu} \) equations of motion and the divergence of the \( \tilde{\pi}_{\mu \nu} \) equations of motion, the generalized divergence and trace conditions on \( \tilde{\pi}_{\mu \nu} \). They are given by

\[
\tilde{\nabla}^\mu \left( \tilde{\pi}^{\mu \nu} - g_{\mu \nu} \tilde{\pi} \right) = 0,
\]

\[
\tilde{\pi} - m_2^{-2} \left[ \left( \tilde{\nabla} \chi \right)^2 + 2m_0^2 \left( 1 - e^{-\chi} \right)^2 \right] = 0.
\]

(6.8)

Linearizing, we now find that the conditions have no dependence on the scalar field, collapsing to the free Pauli-Fierz conditions. We can conclude that they provide generalized spin-two conditions of the correct form (2.17), and that this parametrization provides a true separation of the scalar and spin-two degree of freedom.

To complete the transformation to canonical form, we define a new metric, exactly as in (4.10), but with \( g_{\mu \nu} \) replaced with \( \tilde{g}_{\mu \nu} \), and obtain the action

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} - \frac{3}{2} \left( A^{-1}(\phi_{\sigma \tau}) \right)_\mu \tilde{\nabla}_\mu \chi \tilde{\nabla}_\nu \chi - \frac{3}{2} (\det A(\phi_{\sigma \tau}))^{-1/2} \left( 1 - e^{-\chi} \right)^2 \right.
\]

\[
- \tilde{g}^{\mu \nu} \left( \tilde{C}_{\mu \rho}^{\lambda}(\phi_{\sigma \tau}) \tilde{C}^{\rho \nu \lambda}(\phi_{\sigma \tau}) - \tilde{C}_{\mu \nu}^{\lambda}(\phi_{\sigma \tau}) \tilde{C}^{\rho \lambda \nu}(\phi_{\sigma \tau}) \right)
\]

\[
+ \frac{1}{4} m_2^2 \left( \det A(\phi_{\sigma \tau}) \right)^{-1/2} \left( \phi_{\mu \nu} \phi^{\mu \nu} - \phi^2 \right) \right].
\]

(6.9)

We have a canonical form for a spin-two field and a scalar field coupled to gravity. The spin-two field now couples to the scalar kinetic energy. It also enters the scalar potential. In both cases, however, the coupling is at the cubic level, so that the linearized field equations decouple, and, exactly as for the pure Weyl-squared case (4.18) expanding to quadratic order about \( \phi_{\mu \nu} = 0 \) in flat space, gives the Pauli-Fierz action. Furthermore, the spin-two
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conditions (6.8), in the new $\tilde{g}_{\mu\nu}$ frame become

$$
\nabla^\mu \left( \phi_{\mu\nu} - \tilde{g}_{\mu\nu} \phi \right) - \tilde{g}^{\mu\nu} C^\rho_{\lambda\mu} \phi_{\sigma\tau} \left( \phi_{\rho\nu} - \tilde{g}_{\rho\nu} \phi \right) - \tilde{g}^{\lambda\mu} C^\rho_{\lambda\nu} \phi_{\sigma\tau} \left( \phi_{\rho\mu} - \tilde{g}_{\rho\mu} \phi \right) = 0,
$$

(6.10)

and, although complicated, are still of the correct form (2.17) to imply that $\phi_{\mu\nu}$ is pure spin-two.

We can again ask about the vacuum state of the theory. As before we look for stable solutions of the equations of motion, where the auxiliary fields are covariantly constant. To render the problem tractable we again impose the additional condition that

$$
\phi_{\mu\nu} = \frac{1}{4} \phi \tilde{g}_{\mu\nu}.
$$

The equations of motion then greatly simplify to give

$$
\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = \frac{3}{2} \left( 1 + \frac{1}{4} \phi \right)^{-1} \left[ \nabla_{\mu} \chi \nabla_{\nu} \chi - \frac{1}{2} \tilde{g}_{\mu\nu} \left( \nabla \chi \right)^2 \right] \\
+ \frac{3}{32} \left( 1 + \frac{1}{4} \phi \right)^{-2} \left[ \nabla_{\mu} \phi \nabla_{\nu} \phi - \frac{1}{2} \tilde{g}_{\mu\nu} \left( \nabla \phi \right)^2 \right] - \frac{3}{2} \tilde{g}_{\mu\nu} V(\chi, \phi),
$$

(6.11)

$$
\left( 1 + \frac{1}{4} \phi \right)^{-1} \nabla^2 \chi - \frac{1}{4} \left( 1 + \frac{1}{4} \phi \right)^{-2} \nabla_{\mu} \phi \nabla_{\mu} \chi = \frac{dV}{d\chi},
$$

(6.12)

respectively, where we have introduced the potential

$$
V(\chi, \phi) = \frac{m_0^2 (1 - e^{-\chi})^2}{2 \left( 1 + \frac{1}{4} \phi \right)^2} + \frac{m_2^2 \phi^2}{16 \left( 1 + \frac{1}{4} \phi \right)^2}.
$$

(6.14)

The covariantly constant conditions imply that

$$
\nabla_{\mu} \chi = \partial_{\mu} \chi = 0,
$$

$$
\nabla_{\mu} \phi = \partial_{\mu} \phi = 0,
$$

(6.15)

and from the equations of motion we see that the vacuum must be an extremum of the potential. It is easy to show that the only solution is again trivial flat-space with $\chi = \phi_{\mu\nu} = 0$. Following exactly the discussion for the pure Weyl-squared theory, we find that the spin-two excitations around this vacuum are ghost-like, while it is clear from the form of the action (6.9) that the graviton and scalar excitations are normal. Thus, the problem of the ghost-like spin-two field persists even for general quadratic actions.

Finally we would like to understand what gave us the freedom to introduce two different auxiliary spin-two fields, $\tilde{\pi}_{\mu\nu}$ which gave the canonical Pauli-Fierz quadratic limit and $\pi_{\mu\nu}$ which did not. The two fields are not related by a simple field redefinition, but rather, given their respective trace conditions, appear to differ by terms which are derivatives of
χ. The solution is that there is an extra gauge freedom present when introducing the spin-two auxiliary field, the generalization of the Stückleberg symmetry discussed in Section 2. Specifically, instead of the last line of (6.2) we can introduce a new field ζμ and write,

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ e^x R - \frac{3}{2} \left( e^x \right)^2 - G_{\mu\nu} \pi^{\mu\nu} \right. \\
\left. + \frac{1}{4} m^2 \left( \left[ \pi_{\mu\nu} + \nabla_{(\mu} \zeta_{\nu)} \right] \left[ \pi^{\mu\nu} + \nabla^{(\mu} \zeta^{\nu)} \right] - \left[ \pi + \nabla^{\mu} \zeta_{\mu} \right]^2 \right) \right].
\] (6.16)

The presence of the field ζμ has no effect on eliminating \( \pi_{\mu\nu} \) through its own equation of motion, since using the Bianchi identity \( \nabla^{\mu} G_{\mu\nu} = 0 \), its only contribution, on substituting for \( \pi_{\mu\nu} \), is a total divergence. Further, its field equation, \( \nabla^{\mu} (\pi_{\mu\nu} - g_{\mu\nu} \pi) = 0 \), is already implied by the \( \pi_{\mu\nu} \) equation of motion. The action (6.16) is thus completely equivalent to the action in the last line of (6.2). More significantly, by including the field \( \zeta_{\mu} \), we introduce a new gauge symmetry. Namely the action is invariant under the combined transformations

\[
\begin{align*}
\pi_{\mu\nu} &\rightarrow \pi_{\mu\nu} + \nabla_{(\mu} \xi_{\nu)} , \\
\zeta_{\mu} &\rightarrow \zeta_{\mu} - \xi_{\mu}. 
\end{align*}
\] (6.17)

In this language, the original expression (6.2) corresponds to the gauge choice \( \zeta_{\mu} = 0 \). As in the free case, the divergence condition (6.5) becomes the conserved current condition (in the \( \zeta_{\mu} = 0 \) gauge) for the symmetry (6.17).

We can now understand the connection between the two spin-two fields \( \pi_{\mu\nu} \) and \( \tilde{\pi}_{\mu\nu} \). They are related by gauge transformations, though in a rather subtle way since they are defined with respect to two different metrics. \( \pi_{\mu\nu} \) is defined in the \( g_{\mu\nu} \) frame while \( \tilde{\pi}_{\mu\nu} \) is defined in the conformally rescaled \( \tilde{g}_{\mu\nu} \) frame. To see the relationship explicitly, we start by choosing the gauge \( \zeta_{\mu} = m_2^{-2} \nabla_{\mu} \chi \) in the action (6.16). As already stated, this is the same action as given in (6.2), just expressed in a different gauge. If we now make the conformal rescaling to \( \tilde{g}_{\mu\nu} = e^x g_{\mu\nu} \), as before, the \( G_{\mu\nu} \tilde{\pi}^{\mu\nu} \) term generates derivative terms coupling \( \pi_{\mu\nu} \) with \( \chi \). However, with this particular gauge choice, we find that such terms so arrange themselves to give, after some algebra and dropping a total derivative term,

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} - \frac{3}{2} \left( \tilde{\nabla} \chi \right)^2 - \frac{3}{2} m_0^2 \left( 1 - e^{-\chi} \right)^2 - \tilde{G}_{\mu\nu} \tilde{\pi}^{\mu\nu} \right. \\
\left. + \frac{1}{4} m^2 \left( \left[ \pi_{\mu\nu} - m_2^{-2} \tilde{\nabla}_{\mu} \tilde{\nabla}_{\nu} \chi \right] \left[ \pi^{\mu\nu} - m_2^{-2} \tilde{\nabla}^{\mu} \tilde{\nabla}^{\nu} \chi \right] - \left[ \pi - m_2^{-2} \tilde{\nabla}^2 \chi \right]^2 \right) \right].
\] (6.18)

We recognize this as none other than the action in the last line of (6.7), but instead of the gauge \( \tilde{\zeta}_{\mu} = 0 \), we have \( \tilde{\zeta}_{\mu} = -m_2^{-2} \tilde{\nabla}_{\mu} \chi \). The subtlety is that the new action is gauge-fixed with respect to the new metric \( \tilde{g}_{\mu\nu} \), that is, we have \( \tilde{\nabla}_{\mu} \tilde{\zeta}_{\nu} \) not \( \nabla_{\mu} \tilde{\zeta}_{\nu} \) entering the action. Thus, the procedure of introducing the auxiliary fields in two stages using the conformal invariance of the Weyl terms, is equivalent to first choosing a particular gauge in the \( g_{\mu\nu} \) frame. Then, with this choice, the transformed action arranges into a (different) gauge-fixed form in the
new $\tilde{g}_{\mu\nu}$ frame, thereby removing the coupling between the spin-two and scalar kinetic terms. This decomposition is not unique, but has advantage of also largely decoupling the kinetic energies for the two fields in the final nonlinear expression.

In summary, in this section we have been able to rewrite the general action quadratic in the curvature tensors in a canonical form (6.9), which makes the new scalar and spin-two degrees explicit at the non-linear level. The action has the correct quadratic limit, and we have shown that the spin-two field satisfies a pair of generalized conditions which constrain the number of independent components to the five degrees of freedom of a massive spin-two field. In the linearized limit these conditions become the trace and divergence conditions of the free Pauli-Fierz equation of motion. We found that the theory has a single vacuum state, when the fields are covariantly constant, at $\chi = \phi_{\mu\nu} = 0$, with the mass of the scalar and spin-two excitations generically of the order of the Planck-scale. Further, we made explicit the ghost nature of the spin-two field about this vacuum, demonstrating that this problem persists in the full non-linear theory.

7. Conclusion

As we discussed in the introduction, this paper has centered on two questions. One is how to rewrite quadratic gravity in a canonical second-order form, extracting the new scalar and spin-two degrees of freedom. The other is how such a canonical theory provides a consistent description of a coupled spin-two field. Two interesting questions arise. The first is to understand the connection between Wald’s geometrical theories of multiple spin-two fields and higher-derivative gravitation. In this context, one might consider actions which involve higher derivatives of the curvature tensors. This would imply more degrees of freedom, either new scalar or spin-two fields. If such theories can be rewritten in a canonical second-order form, they should provide examples of consistently coupled theories with multiple spin-two fields. It would be interesting to see how Wald’s algebras are realized in such theories, and what additional structure they imply for the higher-derivative theory.

The second question is whether non-trivial vacua exist in more general theories of higher-derivative gravity. Naively, all such theories represent Planck-scale corrections to Einstein gravity and so apparently have no effect on low-energy physics. However, this assumes that flat space is the only vacuum state in the theory. Generally one must always use the full theory to find the vacuum states and only then, having chosen a particular vacuum, make the low-energy expansion. Thus higher-derivative terms cannot be neglected when determining the correct vacuum state for the low-energy physics. For quadratic theories we found that the only stable vacuum was trivial flat space. To investigate more complicated theories, we can essentially use the same techniques we have developed in this paper. (Some discussion of reducing more general higher-derivative theories of gravity to a second-order form has already been given in the literature [17-20, 28-31].) Recently, this has allowed us to show
that non-trivial vacua seem to be a generic feature of more general theories [27], indicating that higher-derivative corrections to gravity may have a role to play in low-energy physics.

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