Quantum Deformation of the Poincare Supergroup 
and $\kappa$-deformed Superspace

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Abstract

The classical $r$-matrix for $N = 1$ superPoincaré algebra, given by Lukierski, Nowicki and Sobczyk is used to describe the graded Poisson structure on the $N = 1$ Poincaré supergroup. The standard correspondence principle between the even (odd) Poisson brackets and (anti)commutators leads to the consistent quantum deformation of the superPoincaré group with the deformation parameter $q$ described by fundamental mass parameter $\kappa$ ($\kappa^{-1} = \ln q$). The $\kappa$-deformation of $N = 1$ superspace as dual to the $\kappa$-deformed supersymmetry algebra is discussed.

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1 Introduction

Recently in several papers ([1] - [8]) there were considered quantum deformations of \( D = 4 \) Poincaré algebra which describes the relativistic symmetries. Subsequently we would like to stress here that during the last twenty years the supersymmetric extensions of the relativistic symmetries were one of the most studied ideas in the theory of fundamental interactions. We conclude therefore that it is natural to ask how do look the quantum deformations of superalgebras or supergroups which describe the supersymmetric extensions of the four-dimensional space-time symmetries.

The deformation of \( N = 1 \) superPoincaré algebra with fourteen generators \( I_A = (M_i, L_i, P_\mu, Q_i, \overline{Q}_\alpha) \), \( (A = 1, \ldots, 14) \) can be studied at least in two different ways:

a) By considering the Hopf subalgebras of quantum superconformal algebra \( U_q(SU(2, 2; 1)) \).

The complete description of this approach should take all possible quantum deformations of \( SU(2, 2; 1) \). In the case studied so far (see [11]) the minimal Hopf subalgebra of \( U_q(SU(2, 2; 1)) \) containing deformed \( N = 1 \) superPoincaré generators has 16 generators; 14 generators of superPoincaré algebra \( \mathcal{P}_{4;1} \) as well as the dillatation generator \( D \) and the chiral generator \( A \). We therefore have
\[
U_q(SU(2, 2; 1)) \supset U_q(\mathcal{P}_{4;1} \oplus (D \oplus A)) \tag{1.1}
\]
i. e. we obtain in such a way the quantum deformation of \( D = 1 \) super-Weyl algebra.

b) By considering the contraction of quantum super-de Sitter algebra \( U_q(OSp(1; 4)) \).

It appears that such a method provides a genuine 14-generator quantum deformation of \( N = 1 \) Poincaré superalgebra, the \( \kappa \)-deformed super-Poincaré algebra given firstly in [12], and described briefly in Sect. 2.

In this paper we shall study further the quantum deformation of \( N = 1 \) super-Poincaré group given in [12]. From the \( \kappa \)-deformed super-Poincaré algebra, which is a non-commutative Hopf algebra, there can be extracted the non-trivial classical \( r \)-matrix. Indeed, in [12] it has been shown that the graded-antisymmetric part of the coproducts in first order in deformation parameter \( h \equiv \frac{1}{\kappa} \) is given by
\[
\delta(X) = \frac{1}{\kappa}[X \otimes 1 + 1 \otimes X, r] \tag{1.2}
\]
\[
r = L_i \wedge P_i - \frac{i}{4} Q_\alpha \wedge \overline{Q}_\dot{\alpha} \equiv r^{AB} I_A \wedge I_B \tag{1.3}
\]

\(^1\)We would like to recall here that for the complexified conformal algebra one can introduce the \( R \)-matrix with 7 parameters [8]. The analogous general multiparameter deformations of quantum superalgebras were not studied in the literature (see however the partial results in [10]).
where \( A \wedge B \equiv A \otimes B - (-1)^{\eta(A)\eta(B)} B \otimes A; \ i = 1, 2, 3; \ \alpha = 1, 2. \)

The bitensor \( r \in \hat{g} \otimes \hat{g} \) given by (1.3) describes the classical \( r \)-matrix for the \( N = 1 \) Poincaré superalgebra, where \( L_i \) denotes the boost generators, \( P_i \) - the three-momenta, and \( Q_\alpha, Q_{\dot{\alpha}} \) describe the supercharges written as Weyl two-spinors. It appears that the classical \( r \)-matrix (1.3) satisfies the graded \textit{modified} classical Yang-Baxter equation \( ^2 \), which permits to introduce consistently on the space \( g^* \) dual to \( g \) the non-trivial multiplication structure, determined by the cobracket (1.2). Introducing the generators \( Z_A \in \hat{g} \) representing the supergroup parameters, one can define on the functions \( f(Z_A) \) the graded Poisson \( r \)-bracket

\[
\{f, g\} = \{f, g\}_R - \{f, g\}_L
\]

where \( (a = R, L)^3 \)

\[
\{f, g\}_a = (-1)^{\eta(A)\eta(B)} (\bar{D}_A f) r^{AB}(\bar{D}_B g)
\]

and

- \( \bar{D}_A \) denotes left derivative which is for \( a = R \) (\( a = L \)) right-invariant (left-invariant) under supergroup transformations,

- \( \bar{D}_A \) respectively denotes right derivative which is right-invariant (left-invariant)

\( \text{for} \ a = R(a = L). \)

In Sect.3 we shall consider more in detail the Poisson-Lie supergroup structure on \( N = 1 \) Poincaré supergroup. It appears that for the choice of the \( r \)-matrix given by (1.3) the Poisson bracket (1.4) can be consistently quantized in a standard way, by the substitution of (graded) Poisson brackets by (anti-)commutators. In such a way the supergroup parameters are promoted to the noncommuting generators of quantum \( N = 1 \) Poincaré supergroup, with the coproduct rules, described by the composition law of two \( N = 1 \) supersymmetry transformations.

\( ^2 \) For non-supersymmetric case see [13] - [15]

\( ^3 \) In supersymmetric case one can introduce the left- and right-side derivatives

\[
\bar{d} f = \bar{d} Z_A \frac{\partial f}{\partial Z_A} \quad \bar{d} a = \frac{\partial f}{\partial Z_A} \bar{d} A
\]

where \( \bar{d}^2 = \frac{\partial}{\partial} = 0, \) satisfying different Leibnitz rules

\[
\bar{d}(fg) = \bar{d} f g + (-1)^{\eta(f)} f \bar{d} g \quad \bar{d} (fg) = (-1)^{\eta(g)} \bar{d} f g + f \bar{d} g
\]

One gets that

\[
\frac{\bar{d} f}{\partial Z_A} = (-1)^{\eta(f)\eta(Z_A)} \frac{\partial f}{\partial Z_A}
\]

Using the relations (A.3) one can write the Poisson \( r \)-bracket on a supergroup in four different ways, which differ by suitable \textit{sign} factors. The choice (1.5) is the standard one.
It appears that after this quantization procedure the Lorentz sector of the quantum $N = 1$ Poincaré supergroup is classical - in analogy with the case of quantum Poincaré group, considered previously by Zakrzewski [16]. The deformation of the remaining generators of quantum $N = 1$ Poincaré supergroup, describing translations and supertranslations, provides the $\kappa$-deformed $N = 1$ superspace, which is discussed in Sect.4. Finally in Sect.5 we present an outlook and some unsolved problems.

2 $D = 4$ Quantum superPoincaré Algebra

The $\kappa$-deformed $D = 4$ Poincaré superalgebra given in [12] has the structure of noncommutative and noncocommutative Hopf superalgebra. It is described by the following set of relations:

a) Lorentz sector $(M_{\mu\nu} = (M_i, N_i)$, where $M_i = \frac{1}{2} \epsilon_{ijk} M_{jk}$ describe the non-relativistic $O(3)$ rotations, and $N_i$ describe boosts).

i) algebra

$$[M_i, M_j] = i \epsilon_{ijk} M_k \quad [M_i, L_j] = i \epsilon_{ijk} L_k$$

(2.1a)

$$[L_i, L_j] = -i \epsilon_{ijk} (M_k \cosh \frac{P_0}{\kappa} - \frac{1}{8\kappa} T_k \sinh \frac{P_0}{2\kappa} + \frac{1}{16\kappa^2} P_k (T_0 - 4M))$$

(2.1b)

where $(\mu = 0, 1, 2, 3)$

$$T_\mu = Q^A (\sigma_\mu)_{AB} Q^B$$

(2.2)

ii) coalgebra

$$\Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i$$

(2.3a)

$$\Delta(L_i) = L_i \otimes e^{\frac{P_0}{4\kappa}} + e^{\frac{P_0}{4\kappa}} \otimes L_i + \frac{i}{2\kappa} \epsilon_{ijk} (P_j \otimes$$

$$\otimes M_k e^{\frac{P_0}{4\kappa}} + M_j e^{-\frac{P_0}{4\kappa}} \otimes P_k) +$$

$$+ \frac{i}{8\kappa} (\sigma_i)_{\dot{\alpha}\dot{\beta}} (\bar{Q}_\alpha e^{-\frac{P_0}{4\kappa}} \otimes Q_\beta e^{\frac{P_0}{4\kappa}} + Q_\beta e^{-\frac{P_0}{4\kappa}} \otimes \bar{Q}_\alpha e^{\frac{P_0}{4\kappa}})$$

(2.3b)

iii) antipodes

$$S(M_i) = -M_i$$

$$S(N_i) = -N_i + \frac{3i}{2\kappa} P_i - \frac{i}{8\kappa} (Q \sigma_i \bar{Q} + \bar{Q} \sigma_i Q)$$

(2.4)

b) Fourmomenta sector $P_\mu = (P_i, P_0)$
i) algebra

\[[M_i, P_j] = i\epsilon_{ijk} P_k \quad [M_j, P_0] = 0 \quad (2.5a)\]

\[[N_i, P_j] = i\kappa \delta_{ij} \sinh \frac{P_0}{\kappa} \quad [N_i, P_0] = iP_i \quad (2.5b)\]

\[[P_\mu, P_\nu] = 0 (\mu, \nu = 0, 1, 2, 3) \quad (2.5c)\]

ii) coalgebra

\[\Delta(P_i) = P_i \otimes e^{\frac{P_\mu}{\kappa}} + e^{-\frac{P_\mu}{\kappa}} \otimes P_i \quad (2.6a)\]

\[\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \quad (2.6b)\]

The antipode is given by the relation \(S(P_\mu) = -P_\mu\).

c) Supercharges sector [12]

i) algebra

\[\{Q_\alpha, Q_\beta\} = 4\kappa \delta_{\alpha\beta} \sin \frac{P_0}{2\kappa} - 2P_i (\sigma_i)^{\alpha\beta} \quad (2.7a)\]

\[\{Q_\alpha, Q_\beta\} = \{Q_\dot{\alpha}, Q_\dot{\beta}\} = 0 \quad (2.7b)\]

\[[M_i, Q_\alpha] = -\frac{i}{2} (\sigma_i)^{\dot{\alpha}\beta} Q_\beta \quad [M_i, Q_\dot{\alpha}] = -\frac{i}{2} (\sigma_i)^{\alpha\dot{\beta}} Q_\beta \quad (2.7b)\]

\[[N_i, Q_\alpha] = -\frac{i}{2} \cosh \frac{P_0}{2\kappa} (\sigma_i)^{\dot{\alpha}\beta} Q_\beta \quad [N_i, Q_\dot{\alpha}] = \frac{i}{2} \cosh \frac{P_0}{2\kappa} (\sigma_i)^{\alpha\dot{\beta}} Q_\beta \quad (2.7c)\]

\[[P_\mu, Q_\alpha] = [P_\mu, Q_\dot{\alpha}] = 0 \quad (2.7d)\]

ii) coalgebra

\[\Delta(Q_\alpha) = Q_\alpha \otimes e^{\frac{P_\mu}{\kappa}} + e^{-\frac{P_\mu}{\kappa}} \otimes Q_\alpha \quad (2.8)\]

\[\Delta(Q_\dot{\alpha}) = Q_\dot{\alpha} \otimes e^{\frac{P_\mu}{\kappa}} + e^{-\frac{P_\mu}{\kappa}} \otimes Q_\dot{\alpha} \quad (2.8)\]

iii) antipodes

\[S(Q_\alpha) = -Q_\alpha \quad S(Q_\dot{\alpha}) = -Q_\dot{\alpha} \quad (2.9)\]

On the basis of the relations (2.3) - (2.7) one can single out the following features of the quantum superalgebra \(U_\kappa(\mathcal{P}_{4;1})\) :

i) The algebra coproducts and antipodes of Lorentz boosts \(N_i\) do depend on \(Q_\alpha, Q_\dot{\alpha}\) i.e. the \(\kappa\)-deformed Poincaré algebra as well as Lorentz sectors do not form the Hopf subalgebras.

ii) Putting in the formulae (2.1) - (2.6) \(Q_\alpha = Q_\dot{\alpha} = 0\) one obtains the \(\kappa\)-deformed Poincaré algebra considered in [5], i.e.

\[U_\kappa(\mathcal{P}_{4;1})|_{Q_\alpha=Q_\dot{\alpha}=0} = U_\kappa(\mathcal{P}_4)\]
From (2.5 c) we see that the fourmomenta commute. This property implies by duality the standard addition formula for the space-time fourvectors (see Sect. 4).

3 Poisson $r$-brackets For $N = 1$ Poincaré Super-group And Their Quantization

The classical $N = 1$ Poincaré Lie superalgebra with the cobracket (1.2) describes the $N = 1$ Poincaré Lie super-bialgebra $(\hat{g}, \hat{\delta})$, which is called coboundary [15] due to the relation (1.3) between the cobracket $\hat{\delta}$ and the $r$-matrix.

The coboundary super-bialgebras with the $r$-matrix satisfying the modified classical Yang-Baxter equation describe infinitesimally Poisson-Lie supergroups, with the supergroup action $(Z_A, Z_B) \rightarrow Z_A \circ Z_B$ consistent with the Poisson structure given by the $r$-Poisson bracket (1.5). These brackets satisfy the following properties:

1. Graded antisymmetry

$$\{ f, g \} = -(-1)^{(f)(g)} \{ g, f \}$$  (3.1)

2. Graded Jacobi identity

$$(-1)^{(f)(h)} \{ f, \{ g, h \} \} + (-1)^{(g)(h)} \cdot \{ h, \{ f, g \} \} + (-1)^{(f)(g)} \{ g, \{ h, f \} \} = 0$$  (3.2)

3. Graded Leibnitz rules

$$\{ f, gh \} = \{ f, g \} h + (-1)^{(f)(g)} g \{ f, h \}$$

$$\{ fg, h \} = f \{ g, h \} + (-1)^{(g)(h)} \{ f, h \} g$$  (3.3)

4. Lie – Poisson property

Let us write the coproduct induced by the composition law of two supergroup transformations

$$\Delta(Z) = Z \hat{\otimes} Z$$  (3.4)

where ”$\hat{\otimes}$” denotes that we take the composition rule described by ”$\circ$” and replace the product by the tensor product. The Lie-Poisson property takes the form

$$\Delta \{ f, g \} = \{ \Delta(f), \Delta(g) \}$$  (3.5)

where the following rule for the multiplication of graded tensor products should be used:

$$(f_1 \otimes f_2)(g_1 \otimes g_2) = (-1)^{(f_2)(g_1)} f_1 g_1 \otimes f_2 g_2$$  (3.6)
In order to calculate explicitly the Poisson bracket (1.4) one can express the right- and left-invariant derivatives in terms of the ordinary ones, i.e. rewrite (1.4) as follows

\[ \{ f, g \} = f \frac{\partial}{\partial Z^A} \omega^{AB}(z) \frac{\partial}{\partial Z^B} g \]  

(3.7)

If we observe that right

\[ \tilde{D}^{(a)}_A = \frac{\partial}{\partial Z^B} \tilde{\mu}^{(a)B}_A(Z) \quad \tilde{D}^{(a)}_A = \tilde{\mu}^{(a)B}_A(Z) \frac{\partial}{\partial Z^B} \]  

(3.8)

where \( \tilde{\mu}^{(a)}_A, \tilde{\mu}^{(a)}_A \) can be calculated by the differentiation of the composition formulae of the supergroup parameters \( Z^A \), one obtains that (\( L = +, \ R = - \)):

\[ \omega^{AB}(Z) = \tilde{\mu}^{(+)A}_C(Z) r^{CD} \tilde{\mu}^{(+)B}_D(Z) - \tilde{\mu}^{(-)A}_C(Z) r^{CD} \tilde{\mu}^{(-)B}_D(Z) \]  

(3.9)

where the leading term at \( Z = 0 \) is linear, and describes the cobracket of the \( N = 1 \) Poincaré bi-superalgebra (\( \hat{g}, \hat{\delta} \)), in accordance with the relation (1.2).

The quantization of the \( N = 1 \) superPoincaré algebra consists in two steps:

1. Write (3.9) for the independent parameters \( Z^A \) (the generators of the algebra of functions on the supergroup \( P_{4;1} \))

\[ \{ Z^A, Z^B \} = \omega^{AB}(Z) \]  

(3.10)

and calculate \( \omega^{AB} \) by choosing the functions \( \tilde{\mu}^{(a)}_A, \tilde{\mu}^{(a)}_A \) in (3.8), depending on the parametrization of the supergroup.

2. Quantize the Poisson bracket by the substitution

\[ \{ Z^A, Z^B \} \longrightarrow \begin{cases} \frac{1}{m} [\hat{Z}^A, \hat{Z}^B]_- & \text{if} \quad \eta(A) \cdot \eta(B) = 0 \\ \frac{1}{m} [\hat{Z}^A, \hat{Z}^B]_+ & \text{if} \quad \eta(A) \cdot \eta(B) = 1 \end{cases} \]  

(3.11)

where \([\hat{A}, \hat{B}]_\pm = \hat{A} \hat{B} \pm \hat{B} \hat{A}\), and choose the ordering of the \( \hat{Z} \)-variables in \( \omega^{AB} \) in such a way that the Jacobi identities are satisfied, and the coproduct (3.4) is a homomorphism of the quantized superalgebra.

Let us recall the supergroup composition law (\( A \) is a \( 2 \times 2 \) \( SL(2; \mathbb{C}) \) matrix).

\[
(X_{\mu}, \theta_{\alpha}, A^{\beta}_\alpha) \circ (X'_{\mu}, \theta'_{\alpha}, A'^{\beta}_\alpha) =
\]

\[
= (X_{\mu} + \Lambda^\nu_{\mu}(A)X'_{\nu} + \frac{i}{2}(\theta'^T A^{-1} \sigma^\mu \theta' - \theta'^T \sigma^\mu (A^+)^{-1} \theta'),
\]

\[
\theta_{\alpha} + \theta'_{\beta} (A^{-1})^\beta_{\alpha}, \quad A'^{\gamma}_\alpha A^{\beta}_{\gamma} \]

(3.12)
The formulae (3.12) permits to calculate the functions \( \dot{\alpha}^{(+)} \), \( \ddot{\alpha}^{(+)} \) in the formula (3.9). We obtain for example the following formulae for left-sided left-invariant super-derivatives:

\[
\begin{align*}
\tilde{D}^{(+)}_\alpha & = \frac{\partial}{\partial A_\alpha} - \frac{i}{2\kappa} \epsilon^{\beta\gamma}(1_2 - (AA^+)^{-1})_{\gamma\beta} - \frac{i}{2\kappa} \epsilon^{\beta\gamma}(1_2 - (AA^+)^{-1})_{\gamma\beta} \\
\tilde{D}^{(+)}_\beta & = \frac{\partial}{\partial A_\beta} \frac{i}{2\kappa} (\epsilon^{\beta\gamma}(1_2 - (AA^+)^{-1})_{\gamma\beta} - \frac{i}{2\kappa} \epsilon^{\beta\gamma}(1_2 - (AA^+)^{-1})_{\gamma\beta}) \tag{3.13}
\end{align*}
\]

and by conjugation

\[
\begin{align*}
\tilde{D}^{(+)}_\alpha & = (A^{-1})_\alpha^\beta \frac{\partial}{\partial A_\beta} + \frac{i}{2\kappa} \epsilon^{\beta\gamma}(A^+)^{-1}_{\gamma\beta} - \frac{i}{2\kappa} \epsilon^{\beta\gamma}(A^+)^{-1}_{\gamma\beta} \\
\tilde{D}^{(+)}_\beta & = (A^{-1})_\beta^\gamma \frac{\partial}{\partial A_\gamma} \frac{i}{2\kappa} (\epsilon^{\beta\gamma}(A^+)^{-1}_{\gamma\beta} - \frac{i}{2\kappa} \epsilon^{\beta\gamma}(A^+)^{-1}_{\gamma\beta}) \tag{3.14}
\end{align*}
\]

Calculating the remaining invariant derivatives on the bosonic Poincaré subgroup and inserting in the formula (3.9) the \( r \)-matrix (1.3) we obtain the following fundamental \( r \)-Poisson brackets for the coordinates \( (X_\mu, A_\alpha^\beta, A_\alpha^\dot{\beta}, \theta_\alpha, \theta_\dot{\alpha}) \) on \( N = 1 \) Poincaré supergroup [5]:

a) Lorentz subgroup \( (A_\alpha^\beta, A_\alpha^\dot{\beta}) \)

The Lorentz subgroup parameters are classical, i.e.

\[
\{ A_\alpha^\beta, A_\gamma^\delta \} = \{ A_\alpha^\beta, A_\beta^\delta \} = \{ A_\alpha^\dot{\beta}, A_\beta^\dot{\beta} \} = 0 \tag{3.15}
\]

b) Translations \( (X_\mu) \) (we denote \( \theta = (\theta_1 \theta_2), \tilde{\theta} = (\tilde{\theta}_1 \tilde{\theta}_2) \))

\[
\begin{align*}
\{ X^i, X^j \} & = \frac{i}{8\kappa} \theta^T \sigma^i (1_2 - (AA^+)^{-1}) \sigma^j \tilde{\theta} - \frac{i}{8\kappa} \theta^T \sigma^j (1_2 - (AA^+)^{-1}) \sigma^i \tilde{\theta} \\
\{ X^0, X^j \} & = -\frac{i}{\kappa} X^j + \frac{i}{8\kappa} \theta^T [\sigma^j, (AA^+)^{-1}] \tilde{\theta} \\
\{ A_\alpha^\beta, X^i \} & = \frac{1}{2\kappa} (A \sigma_i)_\alpha^\beta \Lambda^i_\alpha (A) - (\sigma^i \cdot A)_\beta \\
\{ A_\alpha^\beta, X^0 \} & = \frac{1}{2\kappa} (A \sigma_i)_\alpha^\beta \Lambda^0_\beta (A) \tag{3.16}
\end{align*}
\]

c) Supertranslations

\[
\begin{align*}
\{ \theta^\alpha, \theta^\beta \} & = \{ \theta^\dot{\alpha}, \theta^\dot{\beta} \} = 0 \quad \{ \theta^\alpha, \theta^\dot{\beta} \} = \frac{i}{2\kappa} (1_2 - (AA^+)^{-1}) \theta^\beta \\
\{ X^i, \theta_\alpha \} & = \frac{1}{4\kappa} (\theta^T \sigma^i) \gamma (1_2 - (AA^+)^{-1})^\gamma_\alpha \\
\{ X^0, \theta_\alpha \} & = -\frac{1}{4\kappa} \theta^T (1_2 + (AA^+)^{-1})^\gamma_\alpha \\
\{ A_\alpha^\beta, \theta^\gamma \} & = \{ A_\alpha^\dot{\beta}, \theta^\gamma \} = 0 \tag{3.18}
\end{align*}
\]

\footnote{We use the spinorial representation of the Lorentz generators, e.g. \( L_i = \frac{1}{4}(\sigma_i)_\alpha^\beta L^\alpha_\beta + (\tilde{\sigma}_i)_\alpha^\dot{\beta} L^\dot{\beta}_\beta \).}
In order to quantize the Poisson brackets (3.15 - 3.20) we perform the substitution (3.11). It appears that this substitution is consistent with Jacobi identities if we keep the order of the coordinate generators on rhs of (3.16) also in quantized case. Furthermore, rewriting the composition law (3.12) as the coproduct rule for the coordinate generators, i.e.

$$\Delta(X_\mu) = X_\mu \otimes 1 + \Lambda_\mu^\nu(A) \otimes X_\nu - \frac{i}{2}(A^{-1}_\alpha A^{-1}_\beta \sigma^\mu_{\alpha\beta}, \theta^\gamma \otimes \theta^\alpha + \theta^\alpha \sigma^\mu_{\alpha\beta} A^{-1}_\gamma \otimes \theta^\gamma)$$

$$\Delta(\theta_\alpha) = \theta_\alpha \otimes 1 + (A^{-1})_\alpha^\beta \otimes \theta_\beta$$

$$\Delta(A^\alpha_\beta) = A^\alpha_\beta \otimes A^\beta_\gamma$$

(3.21)

One can show that the formulae (3.21) describe the homomorphism of the quantized superalgebra given in Sect. 2. Adding the formulae for the antipodes

$$S(X^\mu) = -\Lambda_\mu^\nu(A^{-1})X^\nu \quad S(A^\alpha_\beta) = (A^{-1})_\beta^\alpha$$

$$S(\theta^\alpha) = -A^\gamma_\beta \theta^\beta$$

we see that we have obtained the complete set of relations describing the $\kappa$-deformation of $N = 1$ Poincaré supergroup.

Let us observe that

a) If we put $A^+ A = 1$, i.e. we consider the semidirect product $T_{4,4} \oplus SU(2)$ of the quantum subgroup $T_{4,4}$ (quantum fourtranslations + quantum supertranslations) and $SU(2)$ describing the space rotations, only in two relations (first relation (3.17) and second relation (3.19)) the nontrivial $\kappa$-deformation occurs.

b) If we put $A = 1$, i.e. we consider the quantum subgroup $T_{4,4}$, we obtain the $\kappa$-deformed $N = 1$ superspace. It appears that only the commutator $[X^0, \theta_\alpha]$ is $\kappa$-deformed.

c) Putting in (3.15) - (3.17) $\theta^\alpha = \theta'^\alpha = 0$ one recovers the $\kappa$-deformed inhomogeneous $ISl(2; \mathbb{C})$ group, given in [17].

4 $\kappa$-deformed $N = 1$ Superspace

Let us recall firstly that for $\kappa$-deformed relativistic theory, with infinitesimal symmetries described by the $\kappa$-deformed Poincaré algebra [1, 2, 4, 6, 7] there are two different ways of introducing the Poincaré group and space-time coordinates:

a) Using the formula (2.5 c) one can consider the space-time coordinates by considering ordinary Fourier transforms of the functions depending on the commuting fourmomenta [1, 6, 18]. In such an approach the space-time coordinate operators $\hat{X}_\mu$ commute and are introduced as the operators satisfying the relations

$$[\hat{X}_\mu, \hat{P}_\nu] = i\eta_{\mu\nu}$$

(4.1)
b) Using the duality relation for Hopf algebras described by the scalar product on quantum double with the following properties

\[ \langle \Delta(\hat{z}) | \hat{g}_1 \otimes \hat{g}_2 \rangle = \langle \hat{z} | \hat{g}_1 \hat{g}_2 \rangle \]  
\[ \langle \hat{z}_1 \otimes \hat{z}_2 | \Delta(\hat{g}) \rangle = \langle \hat{z}_1 \hat{z}_2 | \hat{g} \rangle \]  

we easily see that for standard duality relation between \( \hat{X}_\mu \) and \( \hat{P}_\mu \) generators

- non-cocommutative fourmomenta (see (2.6)) imply the non-commutativity of the coordinates [10]:

\[ [\hat{X}^i, \hat{X}^j] = 0 \quad [\hat{X}^0, \hat{X}^j] = \frac{1}{\kappa} \hat{X}^j \]  

- commutativity of the fourmomenta imply that

\[ \Delta(\hat{X}^\mu) = \hat{X}^\mu \oplus 1 + 1 \oplus \hat{X}^\mu \]  

One can rewrite the coproduct formulae (2.6) and (4.4) as the addition formulae for the fourmomenta

\[ p^{(1+2)}_i = p^{(1)}_i e^{p^{(2)}_0 \frac{\kappa}{2}} + p^{(2)}_i e^{-p^{(1)}_0 \frac{\kappa}{2}} \quad p^{(1+2)}_0 = p^{(1)}_0 + p^{(2)}_0 \]  

and for the space-time coordinates

\[ x^{\mu}_{(1+2)} = x^{\mu}_{(1)} + x^{\mu}_{(2)} \]  

If we introduce the following element of the quantum double describing the translation sector of \( \kappa \)-Poincaré (\( \hat{X}^0 = -\hat{X}_0 \), \( \hat{X}^i = -\hat{X}_i \))[6]

\[ G(\hat{X}^\mu; \hat{P}_\mu) = e^{-\frac{i}{\kappa} \hat{X}_0 \otimes \hat{P}_0} e^{i \hat{X}_i \otimes \hat{P}_i} e^{-\frac{i}{\kappa} \hat{X}_0 \otimes \hat{P}_0} \]  

one can encode the additional formulae (4.5a-b) into the following multiplication rules

\[ G(\hat{X}^\mu; p^{(1)}_\mu) G(\hat{X}^\mu; p^{(2)}_\mu) = G(\hat{X}^\mu; p^{(1+2)}_\mu) \]  
\[ G(x^{\mu}_{(1)}; \hat{P}_\mu) G(x^{\mu}_{(2)}; \hat{P}_\mu) = G(x^{\mu}_{(1+2)}; \hat{P}_\mu) \]  

We see therefore that the relations (4.6) describe the generalization of Fourier transform kernels to the case of the translation sector of \( \kappa \)-Poincaré group, with the coproducts determining their multiplication rule.

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[6] For the concepts of exponentiation of the generators of quantum double, consisting of quantum Lie algebra and dual quantum Lie group see [20] - [22], where the exponentials (4.6) are called quantum \( T \)-matrices. The notion of quantum \( T \)-matrix is related to the notion of the universal bicharacter of Woronowicz (see e.g. [19]).
Let us extend such a scheme to $N = 1$ superPoincaré case. The non-commutative Hopf algebra, describing $\kappa$-deformed superspace, is obtained by the quantization of the relations (3.16)-(3.19) with $A = 1$. One obtains

\[
[X^i, X^j] = 0 \quad [\hat{X}^0, \hat{X}^j] = \frac{1}{\kappa} \hat{X}^j
\]

\[
\{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = \{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = \{\hat{\theta}^\dot{\alpha}, \hat{\theta}^\dot{\beta}\} = 0
\] (4.8)

\[
[X^i, \hat{\theta}^\alpha] = [\hat{X}^i, \hat{\theta}^\dot{\alpha}] = 0
\]

and the coproducts (3.26) implying the following composition law in superspace:

\[
\hat{\theta}^{\alpha(1+2)} = \hat{\theta}^{\alpha(1)} + \hat{\theta}^{\alpha(2)} \quad \hat{\theta}^{\dot{\alpha}(1+2)} = \hat{\theta}^{\dot{\alpha}(1)} + \hat{\theta}^{\dot{\alpha}(2)}
\] (4.9)

\[
\hat{X}^\mu_{(1+2)} = X^\mu_{(1)} + X^\mu_{(2)} + \frac{i}{2}(\sigma^\mu)_{\alpha\dot{\beta}}(\theta^\dot{\beta}(1) \theta^\alpha(2) - \theta^\alpha(1) \theta^\dot{\beta}(2))
\]

We recall that $\kappa$-deformed $N = 1$ superalgebra is described by the relations (2.7) and the coproducts (2.8). The addition formula of the Grassmann-algebra-valued eigenvalues $q_\alpha, q_{\dot{\alpha}}$ of the supercharges, induced by (2.12), is the following

\[
q^{(1+2)}_\alpha = q^{(1)}_\alpha e^{\frac{p^{(1)}_\alpha}{2\kappa}} + q^{(2)}_\alpha e^{-\frac{p^{(1)}_\alpha}{2\kappa}}
\]

\[
q^{(1+2)}_{\dot{\alpha}} = q^{(1)}_{\dot{\alpha}} e^{\frac{p^{(1)}_{\dot{\alpha}}}{2\kappa}} + q^{(2)}_{\dot{\alpha}} e^{-\frac{p^{(1)}_{\dot{\alpha}}}{2\kappa}}
\] (4.10)

If we introduce the following quantum counterpart of the finite supertranslation group elements in momentum as well as coordinate superspace

\[
G(p_\mu, q_\alpha, q_{\dot{\alpha}}) = e^{-\frac{i}{2} \hat{X}_0 p_0} e^{i(X^i p_i + \hat{\theta}^\alpha q_\alpha + \hat{\theta}^\dot{\alpha} q_{\dot{\alpha}})} e^{-\frac{i}{2} \hat{X}_0 \tilde{p}_0}
\] (4.11a)

\[
\tilde{G}(x^\mu, \theta_\alpha, \theta_{\dot{\alpha}}) = e^{i(x^\alpha \tilde{p}_\alpha + \theta^\alpha Q_\alpha + \theta^\dot{\alpha} Q_{\dot{\alpha}})}
\] (4.11b)

where $\tilde{P}_0 = 2\kappa \sinh \frac{P_0}{2\kappa}$ and $\tilde{P}_i = P_i$, we obtain the following multiplication laws:

\[
G(p^{(1)}_\mu, q^{(1)}_\alpha, q^{(1)}_{\dot{\alpha}}) G(p^{(2)}_\mu, q^{(2)}_\alpha, q^{(2)}_{\dot{\alpha}}) = G(p^{(1+2)}_\mu, q^{(1+2)}_\alpha, q^{(1+2)}_{\dot{\alpha}})
\] (4.12a)

\[
\tilde{G}(x^{(1)}_\mu, \theta^{(1)}_\alpha, \theta^{(1)}_{\dot{\alpha}}) \tilde{G}(x^{(2)}_\mu, \theta^{(2)}_\alpha, \theta^{(2)}_{\dot{\alpha}}) = \tilde{G}(x^{(1+2)}_\mu, \theta^{(1+2)}_\alpha, \theta^{(1+2)}_{\dot{\alpha}})
\] (4.12b)

Following the discussion for ordinary supersymmetry (see e.g. [23]) one can consider the objects (4.11a) and (4.11b) as describing respectively the superfields in momentum superspace and in the usual (coordinate) superspace.

It should be mentioned that the algebra (4.8) describes the superspace coordinates in the particular Lorentz frame ($A = 1$). If we allow nontrivial Lorentz transformations, the algebra of superspace coordinates is no longer closed, and one should consider the full algebra given by (3.15-20).
5 Outlook

In this paper we presented quantum $\kappa$-deformation of $N = 1$ Poincaré supergroup, which is a non-commutative and non-co-commutative Hopf superalgebra. We would like to mention the following problems which should be further studied:

i) It appears that for the non-semisimple Lie (super)algebras the ”naive” quantization (see (3.11)) of the $r$-Poisson bracket may be very useful as a consistent quantization scheme. In [16] as well as in the case presented in this paper the ambiguities related to the ordering of the $rhs$ of the quantized $r$-Poisson brackets are resolved in the unique way. It is interesting to classify for non-semisimple Lie (super)algebras the classical $r$-matrices and find for which cases the ”naive” quantization of the $r$-Poisson bracket leads to a consistent quantization.

ii) One can show that the $\kappa$-deformed $N = 1$ supersymmetry algebra $(Q_\alpha, Q_\dot{\alpha}, P_\mu)$ as a Hopf superalgebra (see Sect.2) is dual to the Hopf superalgebra describing the $N = 1$ $\kappa$-deformed superspace (see Sect.4). It would be important to show that the whole $N = 1$ $\kappa$-deformed supergroup is dual (possibly modulo some nonlinear transformations of the generators) to the $N = 1$ $\kappa$-Poincaré superalgebra, given in [12]. We would like to stress that such duality for $D = 4$ $\kappa$-deformed Poincaré group given in [13] is not known.

iii) It would be interesting to generalize the results of [12] and of this paper to $N > 1$. We would like to mention that complete $N$-extended Poincaré superalgebra, with $N(N - 1)$ central charges, can be obtained by the construction of the superalgebra $OSp(2N; 4)$ [24]. Replacing the classical superalgebra $OSp(2N; 4)$ by its $q$-analogue $U_q(OSp(2N; 4))$ and performing quantum de-Sitter construction limit with the rescaling (2.3) one should obtain the quantum deformation of $N$-extended superPoincaré algebra. For obtaining $N$-extended $\kappa$-deformed Poincaré supergroup it is sufficient to extend the classical $r$-matrix (1.3) to $N > 1$ and follow the method presented in this paper.

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7 This programme is now under consideration, where also the classical $r$-matrices for simple quantum Lie-algebras and the ”naive” quantization of corresponding quadratic $r$-Poisson brackets are studied.
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