Quantum criticality of the imperfect Bose gas in $d$ dimensions

P Jakubczyk and M Napiórkowski

Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, Hoża 69, 00-681 Warsaw, Poland
E-mail: pjak@fuw.edu.pl and marnap@fuw.edu.pl

Received 22 August 2013
Accepted 23 September 2013
Published 24 October 2013

Abstract. We study the low-temperature limit of the $d$-dimensional imperfect Bose gas. Relying on an exact analysis of the microscopic model, we establish the existence of a second-order quantum phase transition to a phase involving the Bose–Einstein condensate. The transition is triggered by varying the chemical potential and persists at non-zero temperatures $T$ for $d > 2$. We extract the exact phase diagram and identify the scaling regimes in the vicinity of the quantum-critical point focusing on the behavior of the correlation length $\xi$. The length $\xi$ develops an essential singularity exclusively for $d = 2$. We follow the evolution of the phase diagram varying $d$. For $d > 2$ our results agree with renormalization-group-based analysis of the effective bosonic order-parameter models with the dynamical exponent $z = 2$.

Keywords: rigorous results in statistical mechanics, Bose–Einstein condensation (theory), critical exponents and amplitudes (theory), quantum phase transitions (theory)
1. Introduction

Exactly soluble models are rare in the theory of phase transitions and their prominent importance is hardly disputable. In systems displaying quantum criticality [1]–[6], properties specific to different spatial dimensionalities are entangled, which yields a rich structure of the phase diagram and leads to violations of standard paradigms. For this reason quantum-critical systems are harder to analyze, and are often also more interesting, as compared to their classical counterparts. As a matter of fact, it is hardly ever the case that a microscopic model displaying a quantum-critical point can be exactly studied both at temperature $T = 0$ and for $T > 0$.

Interacting Bose systems play a distinct role in quantum many-body physics, and the last two decades have brought tremendous progress in their understanding, both on the experiment and theory sides. In particular, a number of important rigorous results concerning Bose–Einstein condensation [7] have been established in recent years. These are however in almost all cases restricted to $T = 0$ and do not allow access to the unconventional universal features of the phase diagram at $T > 0$ and comparison to computations based on non-exact techniques. A basic aspect of Bose–Einstein condensation is the actual order of the transition. In the non-interacting case it is third order (within the Ehrenfest classification); on the other hand, order-parameter based studies of dilute Bose systems [8]–[17] as well as related effective low-energy bosonic models for underlying Fermi systems [6], [18]–[20] typically truncate the effective action at quartic order, biasing the system towards a second-order transition. It is however well known that different fluctuation-related effects tend to change the order of both thermal and quantum phase transitions [5], [21]–[29], and often destabilize them towards first order. In fact, it is hardly ever the case that the transition between the microscopic model and the effective order-parameter description can be made in a fully controlled way. It is also well known that Bose–Einstein condensation is sensitive to the boundary conditions and displays different properties at different dimensionalities $d$. For these reasons insights brought by simple interacting microscopic models soluble at $T \geq 0$ and arbitrary dimensionalities seem very valuable.

doi:10.1088/1742-5468/2013/10/P10019
In this paper we consider the quantum-critical properties of the imperfect Bose gas \[30\]–\[34\] (IBG) in \(d\) spatial dimensions. For \(d = 3\) and off the \(T \to 0\) limit this system was rigorously studied \[30, 31\] decades ago. It was however only recently that the analysis was extended to arbitrary \(d > 2\) \[35\] and it was demonstrated that in the classical regime (in the immediate vicinity of Bose–Einstein condensation at \(T > 0\) and for \(d > 2\)) this system belongs to the universality class of the \(d\)-dimensional Berlin–Kac (spherical) model \[36\]–\[38\]. It was however thus far not appreciated that the system displays a quantum-critical point (QCP) and is susceptible to exact analysis also off the thermal phase transition, and even for \(d \leq 2\).

2. The model and its solution

The Hamiltonian of the imperfect Bose gas \[30\] reads

\[
\hat{H}_{\text{IBG}} = \sum_k \frac{\hbar^2 k^2}{2m} \hat{n}_k + \frac{a}{2V} \hat{N}^2,
\]

using the standard notation. The mean-field interaction energy \(H_{\text{mf}} = (a/2V) \hat{N}^2\) \((a > 0)\) emerges from a binary repulsive interaction \(v(r)\) in the Kac limit \(\lim_{\gamma \to 0} \gamma^d v(\gamma r)\), where the interaction strength is suppressed, but its range diverges \[33\]. This fact clarifies the actual physical content of the model \(1\). The particles are spinless and are enclosed in a box of volume \(V = L^d\); the system is subject to periodic boundary conditions. We ultimately pass to the thermodynamic limit. Applying a Hubbard–Stratonovich type transformation, the grand canonical partition function of the IBG can be cast in the form \[39\]

\[
\Xi(T, L, \mu) = -ie^{i(\beta \mu^2/2a) V} \left( \frac{V}{2\pi a \beta} \right)^{1/2} \int_{\alpha \beta - i \infty}^{\alpha \beta + i \infty} ds e^{-s \hat{V} \phi_b(s)},
\]

where

\[
\phi_b(s) = -\frac{s^2}{2a \beta} + \frac{\mu s}{a} - \frac{1}{\lambda^d g_{d/2+1}(e^s)} + \frac{1}{V} \ln(1 - e^s).
\]

Here \(\beta = (k_B T)^{-1}\), \(\lambda = h/\sqrt{2\pi m k_B T}\) denotes the thermal wavelength, and the constant \(\alpha < 0\) in the integration limits is arbitrary. The Bose functions are defined via

\[
g_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}.
\]

In addition to \(\lambda\) and \(L\) we immediately identify the lengthscales \(L_T = (a/\beta)^{1/d}\) and \(L_\mu = (a|\mu|^{-1})^{1/d}\). We observe that \(L_\mu\) and \(L_T\) are not defined for the ideal Bose gas, where the limit \(a \to 0\) is taken before the thermodynamic limit. The other important observation is that all the scales, \(L_\mu, L_T, \lambda\), as well as the correlation length (see below), diverge at the QCP at \(T = 0, \mu = 0\). Also note that at arbitrary \(\mu\) the limit \(T \to 0\) involves at least two divergent lengthscales \(L_T\) and \(\lambda\). One finds \(L_T \gg \lambda\) for \(d < 2\) and \(L_T \ll \lambda\) for \(d > 2\).
The saddle-point approximation to equation (2) becomes exact in the thermodynamic limit. The saddle-point equation 0 = \phi'(s)|_{s=s_0} can be cast in the form

\[-s_0 \left( \frac{\lambda}{L_T} \right)^d + \text{sgn}(\mu) \left( \frac{\lambda}{L_\mu} \right)^d = g_{d/2}(e^{s_0}) + \left( \frac{\lambda}{L} \right)^d \frac{e^{s_0}}{1-e^{s_0}}.\]  

(5)

Its properties strongly depend on dimensionality. In particular the Bose function \(g\) is finite for \(x \rightarrow 1^-\) if \(d > 2\) and diverges otherwise. The solution \(s_0\) to equation (5) is related to the correlation length \(\xi\) via \([39]\)

\[\xi = k\lambda |s_0|^{-1/2},\]

(6)

where the constant \(k\) depends on the boundary conditions \([40, 41]\), and thus \(|s_0| \ll 1\) implies the divergence of \(\xi\). The character of this divergence at the QCP is determined by \(d\) and the way the limits \(T \rightarrow 0\) and \(\mu \rightarrow 0\) are taken. In what follows we analyze the distinct regimes of the \((\mu, T)\) phase diagram for different values of \(d\). We extract the behavior of the correlation length from equation (6).

### 2.1. \(d = 2\)

For \(d = 2\), equation (5) always admits a finite solution \(s_0 < 0\) in the limit \(L \rightarrow \infty\). This is clear after observing that the function \(g_1(e^{s_0})\) is monotonically increasing and features a logarithmic singularity for \(s_0 \rightarrow 0^-\)

\[g_1(e^{s_0}) = -\log |s_0| + \cdots \]  

for \(s_0 \rightarrow 0^-\).

(7)

We consider \(\lambda/L \ll 1\) and perform an asymptotic analysis of equation (5) in the limit of vanishing \(T\) and \(\mu\). If the obtained solution fulfills \(|s_0| \ll 1\), we impose the condition \(g_1(e^{s_0}) \gg (\lambda/L)^2 e^{s_0}/(1-e^{s_0})\) in addition to \(\lambda/L \ll 1\). This assures that \(L\) is large enough that the bulk scaling behavior is achieved. The solution to equation (5) depends on how the limits \(T \rightarrow 0\) and \(\mu \rightarrow 0\) are taken. For \(|s_0| \ll (L_T/L_\mu)^2\) the first term in equation (5) is negligible. We refer to the set of values of \(\mu\) and \(T\) assuring this condition as Regime I. Obviously, in this case \(\mu > 0\). If we additionally require that \(\lambda/L_\mu \gg 1\), we obtain \(|s_0| \ll 1\) and we may use the asymptotic expansion equation (7). The other possibility \(\lambda/L_\mu \ll 1\) leads to a contradiction within Regime I. For Regime I we find that achieving bulk scaling requires that \(L\) is large enough that

\[L/L_\mu \gg e^{\lambda^2/(2L_\mu^2)}.\]

(8)

The consistency requirements are that

\[L_T/L_\mu \gg e^{-\lambda^2/(2L_\mu^2)}, \quad \lambda \gg L_\mu, \quad \mu > 0.\]

(9)

This is straightforwardly translated to a relation between the thermodynamic parameters \((\mu, T)\), see figure 1. Under these assumptions we obtain

\[s_0 = -e^{-\lambda^2/L_\mu^2} + \cdots = -e^{-\tau \mu/(k_B T)} + \cdots\]

(10)

with \(\tau = h^2/(2\pi am)\). In consequence the correlation length \(\xi \sim T^{-1/2} e^{\tau \mu/(2k_B T)}\) shows an essential singularity in the limit \(T \rightarrow 0\) at positive \(\mu\), which follows from equation (6).

We now consider the case where \(|s_0| \gg (L_T/L_\mu)^2\) and the left-hand side of equation (5) is dominated by the first term. We refer to this case as Regime II. All \(\mu\) and \(T\) dependences
then drop out of equation (5) and we find
\[ s_0 = \tilde{s}_0(a, m) + \cdots, \tag{11} \]
where \( \tilde{s}_0(a, m) \) is a solution to the equation
\[ -\tau s_0 = g_1(e^{s_0}). \tag{12} \]
The only requirement for the system size is \( \lambda/L \ll 1 \) and the consistency requirement may be written as
\[ \beta|\mu| \ll |\tilde{s}_0(a, m)|. \tag{13} \]
It now follows from equation (6) that \( \xi \sim T^{-1/2} \). This agrees with the general prediction that \( \xi \sim T^{-(d+z-2)/(2z)} \) when we insert the presently relevant value \[[1]\] of the dynamical exponent \( z = 2 \).

The remaining scaling regime (Regime III) occurs if \( g_1(e^{s_0}) \) is negligible as compared to both the terms on the left-hand side of equation (5). In this case we find
\[ s_0 = \beta \mu. \tag{14} \]
Consistency requires that \( \mu < 0 \) and \(-\mu \gg \beta^{-1}\). The temperature dependences of \( \xi \) cancel and we find \( \xi \sim |\mu|^{-1/2} \) in agreement with the standard Landau theory.

The \( d = 2 \) phase diagram is summarized in figure 1. The crossover lines separating the three scaling regimes are straight. The crossover line slope is controlled by the dimensionless parameter \( \tau \). The quantum-critical regime (II) expands upon decreasing \( \tau \) (i.e. for strong interactions and/or large particle masses) and shrinks for increasing \( \tau \). The essential singularity of \( \xi \) in Regime I derives from the logarithmic singularity of the Bose function \( g_1 \). For \( d > 2 \) an analogous calculation yields a power-law divergence of \( \xi \) for \( \mu > 0, T \to 0 \).

2.2. \( d \in [2,4] \)

For \( d > 2 \) the Bose function \( g_{d/2}(e^{s_0}) \) is bounded and in the limit \( \lambda/L \to 0 \) one finds a non-zero solution to equation (5) only for \( \mu < \mu_c(T) \), where
\[ \mu_c(T) = a\zeta\left(\frac{d}{2}\right)\lambda^{-d}. \tag{15} \]
Here \( \zeta(a) = g_a(1) \) denotes the Riemann zeta function. For \( \mu > \mu_c \) the system is in the phase involving the Bose–Einstein condensate, where \( s_0 = 0 \) and the last term in equation (5) gives a finite contribution in the limit \( \lambda/L \to 0 \). In this phase the system hosts long-ranged excitations and \( \xi \) is infinite. Equation (15) describes the shape of the line \( \mu_c(T) \) of thermal phase transitions persisting down to \( (\mu, T) = (0, 0) \). We identify the shift exponent \( \psi = 2/d \) from \( \mu_c \sim T^{1/\psi} \). The obtained value agrees with the general prediction \( \psi = z/(d + z - 2) \) upon inserting the dynamical exponent \( z = 2 \). In \( d = 3 \) we compare the obtained phase boundary equation (15) to the results for the dilute Bose gas, calculated using leading-order perturbative expansion in the interaction coupling \( u_0 \) of the effective action \[[1]\]. Somewhat surprisingly the results coincide upon the identification \( u_0 \to ((2\pi)^3/2) (\sqrt{\hbar}B)^3 a \). Observe that in deriving equation (15) we made no assumptions concerning the magnitude of the interactions, while the results of \[[1]\]
Figure 1. The phase diagram of the imperfect Bose gas in $d = 2$. The crossover lines separating the three scaling regimes are straight and their slope is controlled by the parameter $\tau = h^2/(2\pi am)$. The correlation length $\xi$ in Regime I is given by equations (10) and (6) and displays behavior of the type $\xi \sim T^{-1/2}e^{\tau\mu/(2k_BT)}$. In Regime II we find $\xi \sim T^{-1/2}$, while in Regime III $\xi \sim |\mu|^{-1/2}$. The results in Regime II are consistent with the generic expectations for bosonic order-parameter models that $\xi \sim T^{-(d+\kappa-2)/(2\kappa)}$. In Regime III the value $\nu = 1/2$ conforms with Landau mean-field theory.

hold only for small $u_0$ and in addition should rather refer to short-ranged microscopic interactions.

We now perform an asymptotic analysis of equation (5) approaching the limit $T \to 0$, $\mu \to 0$. Similarly to the case of $d = 2$ we first consider the case of $|s_0| \ll (L_T/L_\mu)^d$, where the first term in equation (5) is negligible. With the additional condition $|s_0| \ll 1$ we may expand the Bose function for $d \in [2, 4 \, [42]$

$$g_{d/2}(e^{s_0}) = \zeta \left(\frac{d}{2}\right) + \Gamma \left(1 - \frac{d}{2}\right)|s_0|^{(d-2)/2} + \cdots.$$  

This leads to

$$|s_0| = \left[\frac{\lambda^d}{a} (\mu - \mu_c)\frac{1}{\Gamma(1-d/2)}\right]^{2/(d-2)} + \cdots,$$  

which applies for

$$k_B T \gg \frac{h^d}{(2\pi m)^{d/2}a} \frac{1}{\Gamma(1-d/2)} \frac{\mu_c - \mu}{\mu^{(d-2)/2}}$$  

and $\mu > 0$. The result (17) together with equation (6) lead to $\xi \sim (\mu_c - \mu)^{-1/(d-2)}$, which identifies the $\nu$ exponent in the vicinity of the thermal phase transition. Consistently with [35] the thermal phase transition is in the universality class of the spherical model. The relation (18) defines the region displaying thermal scaling and may be understood as a Ginzburg criterion [36]. In the present case, where $|s_0| \ll (L_T/L_\mu)^d$, considering $|s_0| \gg 1$ leads to inconsistencies. It should be appreciated that the occurrence of non-Landau critical indices at the phase transition for $T > 0$ contrasts the present system with other models which are solved exactly by a saddle-point calculation. Examples are

doi:10.1088/1742-5468/2013/10/P10019
Quantum criticality of the imperfect Bose gas in $d$ dimensions

Figure 2. The phase diagram of the imperfect Bose gas in $d \in [2, 4]$. The bold line is a locus of second-order phase transitions to a low-$T$ phase involving the Bose–Einstein condensate. Its shape $T_c(\mu) \sim \mu^\psi$ is governed by the universal exponent $\psi = 2/d$. In Regime I the system shows scaling behavior dominated by thermal fluctuations and the divergence of $\xi$ at the transition is governed by the critical exponent $\nu = 1/(d - 2)$, which is characteristic of the universality class of the spherical model. The Ginzburg line bounding Regime I from above follows the same power law as the $T_c$-line. The quantum-critical regime (II) occurs for $|\mu| \ll T^{d/2}$. The correlation length within this regime is characterized by the exponent $\nu = d/4$. Within Regime III the singularity at $(\mu, T) = (0, 0)$ is cut off by $\mu$ rather than $T$. Within this regime $\xi \sim \mu^{-1/2}$, which is typical of mean-field theory and conforms with the general expectation.

the reduced BCS model [43], and pure forward-scattering models for symmetry-breaking Fermi surface deformations [44].

In the remaining portion of the phase diagram one identifies two scaling regimes. For $|s_0| \gg (L_T/L_\mu)^d$ we find

$$|s_0| = \zeta \left( \frac{d}{2} \right) \frac{a \beta}{\lambda^d},$$

and therefore $\xi \sim T^{-d/4}$. This again agrees with the renormalization-group-based prediction $\xi \sim T^{-(d+z-2)/2z}$ with $z = 2$. The result (19) applies for $|\mu| \ll a \zeta \left( \frac{d}{2} \right) \frac{1}{\lambda^d}$,

$$|\mu| \ll a \zeta \left( \frac{d}{2} \right) \frac{1}{\lambda^d},$$

which defines Regime II in figure 2. In the opposite case ($|\mu| \gg a \zeta (d/2)(1/\lambda^d)$) and for $\mu < 0$ one finds a result analogous to Regime III in $d = 2$, where $s_0 = \beta \mu$. The results obtained for $d \in (2, 4)$ are summarized in the phase diagram, figure 2. We emphasize that despite the apparent triviality of the interaction term in the Hamiltonian equation (1), the structure of the phase diagram of figure 2 has very little in common with the one of the non-interacting gas (taking $a \to 0$ before the thermodynamic limit). Recall that the thermodynamics of the non-interacting Bose gas is defined only for $\mu \leq 0$, and condensation occurs exclusively for $\mu = 0$ at sufficiently low temperatures.
For $d = 4$ one replaces the expansion in equation (16) with $g_{d/2}(e^{s_0}) = \zeta(d/2) + |s_0| \log |s_0| + \cdots$. This leads to

$$s_0 = \zeta(2) \frac{\mu_c - \mu}{\mu_c \log ((\mu_c - \mu)/\mu_c)}$$

(21)

and yields a logarithmic correction to scaling behavior of $\xi$ in the thermal regime (I). The results obtained for $d \in (2, 4)$ in the quantum-critical regime (II) and in the ‘$T = 0$’ regime (III) apply also in $d = 4$.

2.3. $d > 4$

For $d > 4$ and $|s_0| \ll 1$ we have

$$g_{d/2}(e^{s_0}) = \zeta\left(\frac{d}{2}\right) - \zeta\left(\frac{d}{2} - 1\right) |s_0| + \cdots,$$

(22)

and the problem simplifies because the first term in equation (5) dominates over all other contributions which depend on $s_0$. One therefore obtains

$$s_0 = \beta(\mu - \mu_c) + \cdots$$

(23)

in all the regimes where $|s_0| \ll 1$.

3. Summary

In summary, we have analyzed the low-temperature limit of the imperfect Bose gas in $d$ spatial dimensions. The model corresponds to the Kac scaling limit, where repulsive two-particle interactions are scaled to become progressively weaker and long-ranged. Despite superficial similarities to the ideal Bose gas, the model features a very different structure of the phase diagram, in particular in the properties related to Bose–Einstein condensation. By an exact analysis performed at the microscopic level, we have established the existence of a quantum-critical point in the phase diagram spanned by $(\mu, T)$. Bose–Einstein condensation persists for $T > 0$ in dimensionalities $d > 2$ and the thermal transition is in the universality class of the $d$-dimensional spherical model. The system is a rare example of a microscopic model, where one can exactly establish the occurrence of a second-order quantum phase transition and analyze the quantum-critical properties without recourse to an effective model and (approximate and not always fully controllable) renormalization-group methods. Interestingly, the system is solved exactly by a saddle-point approach, but exhibits non-Landau critical indices. Instead, for $2 < d < 4$ and in the thermal regime, one finds the behavior characteristic of the Berlin–Kac universality class.

In fact, the close relation between the IBG and the spherical model may be extended beyond the thermal regime by invoking the quantum variant of the latter [38]. The quantum-critical properties of the spherical model were addressed in [45], and were later put in a much more general context in [46]. As far as universal properties are concerned, our present results match with those relevant for the spherical model and may be extracted within the framework of [46] in any region of the phase diagram.

Our expression for the finite-$T$ transition line (up to a factor defining the interaction constants) fully coincides with the result obtained from an effective order-parameter field
Quantum criticality of the imperfect Bose gas in \(d\) dimensions

theory at leading order in an expansion in interactions. This is interesting because the present calculation is valid at arbitrarily large interactions.

For \(d = 2\) the correlation length shows essentially singular behavior in one of the scaling regimes emergent in the limit \(T \to 0\). In all the other cases including \(d < 2\) we find power-law divergences. For \(d > 2\) the overall structure of the phase diagram is consistent with the predictions of the renormalization-group theory of effective order-parameter models with the dynamical exponent \(z = 2\). In particular, the vicinity of the quantum-critical point splits into the classical regime, dominated by the thermal fluctuations, the quantum-critical regime, and the quantum regime, where the essential system properties are the same as in \(T = 0\). The correlation length in the phase involving the Bose–Einstein condensate is infinite, indicating generic coherent behavior.

Acknowledgments

We would like to thank F Benitez, A Eberlein, N Hasselmann, W Metzner, P Nowakowski, B Obert, and H Yamase for discussions and a number of very useful comments. We acknowledge funding by the National Science Centre via 2011/03/B/ST3/02638.

References

[1] Sachdev S, 2011 Quantum Phase Transitions (Cambridge: Cambridge University Press)
[2] 2008 Quantum matter, Science 319 1201 (special issue)
[3] 2008 Quantum phase transitions, Nature Phys. 4 157 (special issue)
[4] 2010 Quantum criticality and novel phases, Phys. Status Solidi b 247 457 (special issue)
[5] Belitz D, Kirkpatrick T R and Vojta T. 2005 Rev. Mod. Phys. 77 579
[6] von Loehneysen H, Rosch A, Vojta M and Woelfle P, 2007 Rev. Mod. Phys. 79 1015
[7] Lieb E H, Seiringer R, Solovej J P and Yngvason J, 2005 The Mathematics of the Bose Gas and its Condensation (Basel: Birkhauser)
[8] Kolomeisky E B and Straley J P, 1992 Phys. Rev. B 46 11749
[9] Kolomeisky E B and Straley J P, 1992 Phys. Rev. B 46 13942
[10] Bijlsma M and Stoff H T, 1996 Phys. Rev. A 54 5085
[11] Andersen J O and Strickland M, 1999 Phys. Rev. A 60 1442
[12] Kolomeisky E, Newman T J, Straley J P and Qi X, 2000 Phys. Rev. Lett. 85 1146
[13] Cherny A Y and Shanenko A A, 2001 Phys. Rev. E 64 027105
[14] Crisan M, Bodea D, Groiu I and Tifrea I, 2002 J. Phys A: Math. Gen. 35 239
[15] Metikas G, Zobay O and Alber G, 2004 Phys. Rev. A 69 043614
[16] Nikolič P and Sachdev S, 2007 Phys. Rev. A 75 033608
[17] Rançon A and Dupuis N, 2012 Phys. Rev. A 85 063607
[18] Hertz J A, 1976 Phys. Rev. B 14 1165
[19] Millis A J, 1993 Phys. Rev. B 48 7183
[20] Jakubczyk P, Strack P, Katanin A A and Metzner W, 2008 Phys. Rev. B 77 195120
[21] Coleman S and Weinberg E, 1973 Phys. Rev. D 7 1888
[22] Halperin B I, Lubensky T C and Ma S, 1974 Phys. Rev. Lett. 32 292
[23] Fucito F and Parisi G, 1981 J. Phys. A: Math. Gen. 14 L499
[24] Belitz D, Kirkpatrick T R and Vojta T, 1999 Phys. Rev. Lett. 82 4707
[25] Chubukov A V, Pepin C and Rech J, 2004 Phys. Rev. Lett. 92 147003
[26] Greenblatt R L, Aizenman M and Lebowitz J L, 2009 Phys. Rev. Lett. 103 197201
[27] Jakubczyk P, 2009 Phys. Rev. B 79 125115
[28] Jakubczyk P, Metzner W and Yamase H, 2009 Phys. Rev. Lett. 103 220602
[29] Yamase H, Jakubczyk P and Metzner W, 2011 Phys. Rev. B 83 125121
[30] Davies E B, 1972 Commun. Math. Phys. 28 69
[31] Buffet E and Pulé J V, 1983 J. Math. Phys. 24 1608
[32] Zagrebnov V A and Bru J, 2011 Phys. Rep. 530 291

doi:10.1088/1742-5468/2013/10/P10019
Quantum criticality of the imperfect Bose gas in $d$ dimensions

[33] Lewis J T, 1986 *Statistical Mechanics and Field Theory: Mathematical Aspects* (Lecture Notes in Physics) vol 257 (New York: Springer)

[34] van der Berg M, Lewis J T and de Smedt P, 1984 *J. Stat. Phys.* **37** 697

[35] Napiórkowski M, Jakubczyk P and Nowak K, 2013 *J. Stat. Mech.* 06015

[36] Amit D and Martin-Mayor V, 2005 *Field Theory, the Renormalization Group and Critical Phenomena* (Singapore: World Scientific)

[37] Gunton J D and Buckingham M J, 1967 *Phys. Rev.* **166** 152

[38] Brankov J G, Dantchev D M and Tonchev N S, 2000 *The Theory of Critical Phenomena in Finite-Size Systems—Scaling and Quantum Effects* (Singapore: World Scientific)

[39] Napiórkowski M and Piasecki J, 2011 *Phys. Rev. E* **84** 061105

[40] Napiórkowski M and Piasecki J, 2012 *J. Stat. Phys.* **147** 1145

[41] Napiórkowski M and Piasecki J, 2013 unpublished

[42] Ziff R M, Uhlenbeck G and Kac M, 1977 *Phys. Rep.* **32** 169

[43] Mühlhaselel B, 1962 *J. Math. Phys.* **3** 522

[44] Yamase H, Oganesyan V and Metzner W, 2005 *Phys. Rev. B* **72** 035114

[45] Verbeure A and Zagrebnov V A, 1992 *J. Stat. Phys.* **69** 329

[46] Momont B, Verbeure A and Zagrebnov V A, 1997 *J. Stat. Phys.* **89** 633

doi:10.1088/1742-5468/2013/10/P10019