An HLL Riemann solver for the hybridised discontinuous Galerkin formulation of compressible flows

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Abstract
This work proposes a high-order hybridised discontinuous Galerkin (HDG) formulation of the Harten-Lax-Van Leer (HLL) Riemann solver for compressible flows. A unified framework is introduced to present Lax-Friedrichs, Roe and HLL Riemann solvers via appropriate definitions of the HDG numerical fluxes. The resulting high-order HDG method with HLL Riemann solver is evaluated through a set of numerical simulations of inviscid compressible flows in different regimes, from subsonic isentropic flows to transonic and supersonic problems with shocks. The accuracy of the proposed method is comparable with the one of Lax-Friedrichs and Roe numerical fluxes in subsonic and transonic flows. The superior performance of HLL is highlighted in supersonic cases, where the method provides extra robustness, being able to produce positivity preserving approximations without the need of any user-defined entropy fix.

1 Introduction

The majority of commercial, industrial and open source unstructured computational fluid dynamics (CFD) solvers employ finite volume (FV) or low-order stabilised finite element (FE) methods. Low-order FV and FE methods are robust, easy to implement and provide a competitive alternative for the computation of steady state CFD solutions. However, for complex flow problems involving transient effects, such as the propagation of vortices over long distances, low-order methods are known to introduce high dissipation and dispersion errors.

High-order methods have become popular due to the lower dissipation and dispersion errors, when compared to low-order methods. This has prompted the extension of FV and stabilised FE schemes to high-order. Discontinuous Galerkin (DG) methods, often seen as a methodology to combine the advantages of both FV and FE schemes, have become one of the most adopted approaches within the high-order community. Contrary to FV methods, DG methods are known to introduce high accuracy and reduced dissipation and dispersion errors.
methods define high-order local approximations. In addition, in DG methods, the stabilisation term required for solving convection dominated problems is easier to define when compared to traditional stabilised FE methods [2,4,16,17]. The DG framework allows to devise high-order numerical methods, to enforce element-by-element conservation and to efficiently exploit parallel computing architectures. However, the duplication of nodes at the interface of neighbouring elements has limited its application mostly to academic problems, see the discussion in [20] and references therein. Hybrid discretisation methods, e.g. the hybridised and hybridisable discontinuous Galerkin (HDG) methods [15,20] and the hybrid high-order (HHO) method [18], obtained from the hybridisation of traditional DG schemes, provide a significantly less expensive alternative [31,65]. The HDG approach reduces the number of globally coupled degrees of freedom via the introduction of a hybrid variable, namely the trace of the unknown variable on the mesh faces, and appropriately defined inter-element numerical fluxes. Recently, special attention has been devoted to the HDG method which relies on a mixed formulation for second-order problems [12,15,28,31,43,52].

In the context of compressible flows, several HDG methods have been proposed [23,46,47,55,64], exploring different hybridisation strategies and different functional spaces for the approximation of the unknown variables. In contrast, the definition of approximate Riemann solvers for HDG methods have received considerably less attention, and only the traditional Lax-Friedrichs and Roe solvers have been considered [11,17]. It is worth noting that in the inviscid limit, i.e. for the Euler equations, HDG methods based on primal and mixed formulations are equivalent.

This work considers the numerical solution of the Euler equations of gas dynamics and formulates, for the first time in the context of HDG, the Harten-Lax-Van Leer (HLL) Riemann solver [30]. The method is presented within a unified framework for the derivation of approximate Riemann solvers for compressible flows in the context of high-order HDG methods. The framework includes the existing Lax-Friedrichs and Roe solvers and the novel HLL solver. As shown using a series of two dimensional examples, the use of HLL Riemann solver is especially important in the context of supersonic flows, where the Roe numerical flux may fail to provide physically admissible solutions because of a lack of dissipation [45,53], whereas Lax-Friedrichs produces over-dissipative approximations [37,39]. On the contrary, the HLL Riemann solver provides a robust framework to compute accurate solutions while guaranteeing positiveness of the approximate density and pressure fields [24,53].

The remainder of this paper is organised as follows. Section 2 introduces the hybridised DG formulation of the Euler equations. In section 3, a unified description of the Riemann solvers in the context of high-order HDG methods is presented and the novel HLL Riemann solver is proposed for hybrid discretisations. Section 4 discusses the solver for the resulting nonlinear problem and the numerical strategy to treat solutions with discontinuities and sharp gradients. A set of numerical examples for a variety of flow conditions, from subsonic isentropic flows to transonic and supersonic problems with shocks, is presented in section 5 to demonstrate optimal convergence properties, accuracy and robustness of the proposed HLL Riemann solver in the context of a high-order HDG discretisation. Finally, section 6 summarises the main results of this work.
2 Hybridised discontinuous Galerkin formulation of the Euler equations

2.1 Problem statement

Consider an open bounded computational domain $\Omega \subset \mathbb{R}^{n_{sd}}$, being $n_{sd}$ the number of spatial dimensions, with boundary $\partial \Omega$ and let $T_{\text{end}} > 0$ be the final time of interest.

The Euler equations, describing the conservation of mass, momentum and energy for inviscid compressible flows and assuming that there are no external volume forces, can be expressed in non-dimensional conservative form as

$$
\begin{cases}
\frac{\partial U}{\partial t} + \nabla \cdot F(U) = 0 & \text{in } \Omega \times (0, T_{\text{end}}], \\
U = U^0 & \text{in } \Omega \times \{0\}, \\
B(U, U_{\infty}) = 0 & \text{on } \partial \Omega \times (0, T_{\text{end}}].
\end{cases}
$$

where the vector of conservative variables, $U$, and the inviscid flux tensor, $F(U)$, are given by

$$
U := \begin{pmatrix} \rho \\ \rho v \\ \rho E \end{pmatrix} \quad \text{and} \quad F(U) := \begin{bmatrix} \rho v \\ \rho v T \\ \rho v \otimes v + p I_{n_{sd}} \end{bmatrix} + \begin{bmatrix} (\rho E + p)v \\ (\rho E + p) v T \end{bmatrix}.
$$

The vector $B$ in equation (1) is used to define, in a compact form, the boundary conditions that are detailed in section 2.2.1, and the subscript $\infty$ indicates free stream values. In equation (2), $\rho$ denotes the density of the fluid, $v$ the velocity vector, $p$ the pressure, $E$ the total energy per unit mass and $I_{n_{sd}}$ is the $n_{sd} \times n_{sd}$ identity matrix.

The nonlinear hyperbolic system (1) is closed by an equation of state which, for a perfect polytropic gas, is

$$
p = (\gamma - 1) \rho \left( E - \frac{1}{2} \| v \|^2 \right)
$$

where $\gamma$ denotes the ratio of the specific heat coefficients of the fluid and takes value $\gamma = 1.4$ for air.

2.2 The hybridised discontinuous Galerkin framework

Consider a partition of the domain $\Omega$ in $n_{el}$ disjoint subdomains $\Omega_e$ such that $\Omega = \bigcup_{e=1}^{n_{el}} \Omega_e$. The mesh skeleton, or internal interface, $\Gamma$ is defined as

$$
\Gamma := \bigcup_{e=1}^{n_{el}} \partial \Omega_e \setminus \partial \Omega.
$$

The strong form of the Euler equations can be written in the so-called broken
computational domain as

\[
\begin{cases}
\frac{\partial U_e}{\partial t} + \nabla \cdot F(U_e) = 0 & \text{in } \Omega_e \times (0, \text{end}], \text{ for } e = 1, \ldots, n_{el}, \\
U_e = U^0 & \text{in } \Omega_e \times \{0\}, \text{ for } e = 1, \ldots, n_{el}, \\
b(U, U_\infty) = 0 & \text{on } \partial \Omega \times (0, \text{end}], \\
[U \otimes n] = 0 & \text{on } \Gamma \times (0, \text{end}], \\
[F(U)n] = 0 & \text{on } \Gamma \times (0, \text{end}],
\end{cases}
\]

where \( U_e \) denotes the restriction of the solution \( U \) to an element \( \Omega_e \), \( n \) is the outward unit normal vector and \([\cdot]\) denotes the jump operator defined at each internal interface \( \Gamma \) as the sum

\[
[U] = U^+ + U^-
\]

of the values in the elements \( \Omega^+ \) and \( \Omega^- \) on the left and right of the interface, respectively \[36\].

### 2.2.1 Strong form of local and global problems

The HDG method solves equation (5) in two stages \[15,42–44\]. First, \( n_{el} \) local problems, given by

\[
\begin{cases}
\frac{\partial U_e}{\partial t} + \nabla \cdot F(U_e) = 0 & \text{in } \Omega_e \times (0, \text{end}], \\
U_e = U^0 & \text{in } \Omega_e \times \{0\}, \\
U_e = \hat{U} & \text{on } \partial \Omega_e \times (0, \text{end}],
\end{cases}
\]

for \( e = 1, \ldots, n_{el} \), define the solution \( U \) in each element as a function of an independent variable \( \hat{U} \), representing the trace of the solution on \( \Gamma \cup \partial \Omega \).

Then, \( \hat{U} \) is computed as the solution of a global problem imposing the boundary conditions on \( \partial \Omega \) and enforcing inter-element continuity of the solution and of the normal fluxes on \( \Gamma \) via the so-called transmission conditions, namely

\[
\begin{cases}
b(U, \hat{U}, U_\infty) = 0 & \text{on } \partial \Omega \times (0, \text{end}], \\
[U \otimes n] = 0 & \text{on } \Gamma \times (0, \text{end}], \\
[F(U)n] = 0 & \text{on } \Gamma \times (0, \text{end}],
\end{cases}
\]

where \( b(U, \hat{U}, U_\infty) = 0 \) is a compact notation to express boundary conditions and it involves the hybrid variable. Given two disjoint partitions, \( \Gamma_\infty \) and \( \Gamma_w \), of the boundary such that \( \partial \Omega = \Gamma_\infty \cup \Gamma_w \), boundary conditions are defined as

\[
\hat{B} := \begin{cases}
A^+_n(\hat{U})(U - \hat{U}) + A^-_n(\hat{U})(U_\infty - \hat{U}) & \text{on } \Gamma_\infty \times (0, \text{end}], \\
\rho - \hat{\rho}, [(I_{n_{el}} - n \otimes n)\rho v - \hat{\rho}v]^T, \rho E - \hat{\rho}E & \text{on } \Gamma_w \times (0, \text{end}],
\end{cases}
\]

where \( \Gamma_\infty \) denotes the portion of the boundary where inflow or outflow conditions are imposed and \( \Gamma_w \) denotes the portion of the boundary where wall or symmetry conditions are enforced \[23,41,47\].

To define the inflow or outflow boundary conditions, the inviscid Jacobian matrix in the normal direction to the boundary is defined as

\[
A_n(\hat{U}) := 
\]
\[ \partial F(\vec{U})/\partial \vec{U} \cdot \vec{n}. \] The spectral decomposition of the matrix is then computed, namely \( A_\lambda(\vec{U}) = L \Lambda R \), where \( L \), \( R \) and \( \Lambda \) denote the matrices of left eigenvectors, right eigenvectors and eigenvalues, respectively. The matrices \( A_n^\pm \) appearing in equation (9) are defined as \( A_n^\pm := (A_n \pm |A_n|)/2 \), where \( |A_n(\vec{U})| := L \Lambda |R \) and the matrix \( |A| \) is a diagonal matrix containing the absolute value of the eigenvalues in \( \Lambda \).

### 2.2.2 Weak form of local and global problems

According to the notation introduced in [27,60], the following discrete functional spaces are introduced for the vector of conservative variables

\[ \mathcal{W}_h := \{ W \in [\mathcal{L}_2(\Omega)]^{n+2} : W|_{\Omega_e} \in [\mathcal{P}^k(\Omega_e)]^{n+2} \forall \Omega_e, e = 1, \ldots, n_{e1} \}, \quad (10a) \]

\[ \tilde{\mathcal{W}}_h := \{ \tilde{W} \in [\mathcal{L}_2(S)]^{n+2} : \tilde{W}|_{\Gamma_i} \in [\mathcal{P}^k(\Gamma_i)]^{n+2} \forall \Gamma_i \subset S \subseteq \Omega \cup \partial \Omega \}, \quad (10b) \]

where \( \mathcal{P}^k(\Omega_e) \) and \( \mathcal{P}^k(\Gamma_i) \) denote the spaces of polynomial functions of complete degree at most \( k \) in \( \Omega_e \) and on \( \Gamma_i \), respectively.

Moreover, the classical notation for \( \mathcal{L}_2 \) inner products of vector and tensor-valued functions on a generic subdomain \( D \subset \Omega \) is considered

\[ \left< V, W \right>_D := \int_D V \cdot W \, d\Omega \quad \text{and} \quad \left< V, W \right>_D := \int_D V : W \, d\Omega. \quad (11) \]

The corresponding \( \mathcal{L}_2 \) inner products on a surface \( S \subset \Gamma \cup \partial \Omega \) are denoted by \( \left< \cdot, \cdot \right>_S \).

The following discrete weak form of the local problems is obtained: for every element \( \Omega_e \), \( e = 1, \ldots, n_{e1} \), given \( \vec{U} \in \mathcal{L}_2 \left( (0, T_{\text{end}}]; \mathcal{W}_h \right) \), find an approximation \( U_e \in \mathcal{L}_2 \left( (0, T_{\text{end}}]; \mathcal{W}_h \right) \) such that

\[ \left. \left< W, \frac{\partial U_e}{\partial t} \right>_{\Omega_e} - \left< \nabla W, F(U_e) \right>_{\Omega_e} + \left< W, \nabla(\vec{U}_e) \vec{n} \right>_{\partial \Omega_e} = 0, \quad (12) \]

for all \( W \in \mathcal{L}_2 \left( (0, T_{\text{end}}]; \mathcal{W}_h \right) \), where the functional space \( \mathcal{L}_2 \left( (0, T_{\text{end}}]; \mathcal{W}_h \right) \) features \( \mathcal{L}_2 \left( (0, T_{\text{end}}]; \mathcal{W}_h \right) \) functions in time with spatial approximation in \( \mathcal{W}_h \), being \( \mathcal{W}_h \) the restriction of \( \mathcal{W}_h \) to the element \( \Omega_e \). An analogous definition holds for \( \mathcal{L}_2 \left( (0, T_{\text{end}}]; \tilde{\mathcal{W}}_h \right) \).

**Remark 1.** In [12], the trace of the numerical flux \( \overline{F(U_e)} \vec{n} \) on the boundary \( \partial \Omega_e \) is introduced. It is worth noting that its choice affects the quality and the accuracy of the resulting HDG approximation. Different definitions of such numerical fluxes are detailed in section 3, where a unified framework for the formulation of Riemann solvers in the context of HDG methods is presented.

To derive the discrete weak form of the global problem, first observe that the first transmission condition in [8] is automatically fulfilled owing to the condition \( U_e = \vec{U} \) imposed by the local problems on the boundary \( \partial \Omega_e \) of each element and to the uniqueness of \( \vec{U} \) on the mesh skeleton. Thus, from [8], it
follows that the global problem is: find $\bar{U} \in L_2 \left((0, T_{\text{end}}); \bar{W}^b \right)$ such that
\[
\sum_{e=1}^{n_{el}} \left\{ \langle \bar{W}, F(U_e^+) n \rangle_{\partial \Omega_e, \Gamma} + \langle \bar{W}, \tilde{B}(U, \bar{U}, U_\infty) \rangle_{\partial \Omega_e, \partial \Omega} \right\} = 0, \quad (13)
\]
for all $\bar{W} \in L_2 \left((0, T_{\text{end}}); \bar{W}^b \right)$, where the boundary flux $\tilde{B}$ takes the expressions detailed in [6].

3 A unified framework for Riemann solvers in hybridised discontinuous Galerkin methods

As mentioned above, the choice of the numerical fluxes $F(U_e)n$ appearing in equations (12) and (13) is essential for the accurate solution of the Euler equations. More precisely, such numerical fluxes need to encapsulate the information on the convective nature of the flow under analysis. Their approximation has been studied using Riemann solvers in the context of DG methods [17, 37, 51] and, more recently, of HDG [41, 46, 47].

3.1 Riemann solvers in standard DG methods

Consider a pair of neighbouring elements, $\Omega^+_e$ and $\Omega^-_e$, with shared interface $\Gamma_i = \partial \Omega^+_e \cap \partial \Omega^-_e \subset \Gamma$. The solution at each side of the interface is denoted by $U^+_e$ and $U^-_e$, whereas $U^*(U^+_e, U^-_e)$ represents an intermediate state between $U^+_e$ and $U^-_e$. Following the monograph by Toro [63], the definition of Lax-Friedrichs, Roe and HLL Riemann solvers is first recalled for standard DG formulations.

3.1.1 Lax-Friedrichs Riemann solver

The first option is represented by the Lax-Friedrichs numerical flux. This Riemann solver is obtained as an extrapolation of the result for a scalar convection equation [35] and defines the numerical flux as
\[
F(U_e)n^\pm = \frac{1}{2} \left[ F(U^+_e) + F(U^-_e) \right] n^\pm + \frac{\lambda^*_{\text{max}}}{2} (U^+_e - U^-_e), \quad (14)
\]
where $\lambda^*_{\text{max}} := |v^* \cdot n| + c^*$ is the maximum eigenvalue of the matrix $A_n(U^*)$ evaluated at the intermediate state $U^*$, and $c := \sqrt{\gamma p/\rho}$ denotes the speed of sound. It is well-known that the Lax-Friedrichs numerical flux [14] is extremely robust but leads to over-diffusive solutions.

3.1.2 Roe Riemann solver

The Roe Riemann solver [54] introduces a different stabilisation in the equations for conservation of mass, momentum and energy by means of the matrix $|A_n(U^*)|$ that linearises the convective fluxes $F(U^*)$. More precisely, the Roe numerical fluxes is given by
\[
F(U_e)n^\pm = \frac{1}{2} \left[ F(U^+_e) + F(U^-_e) \right] n^\pm + |A_n(U^*)|(U^+_e - U^-_e), \quad (15)
\]
where $A_n(U^*)$ and $|A_n(U^*)|$ are the matrices introduced in section 2.2.1 and evaluated at the intermediate state $U^*$.

Although being more accurate than the Lax-Friedrichs flux, Roe Riemann solver is not positivity preserving and it may produce nonphysical solutions in transonic and supersonic cases due to the violation of entropy conditions [50,53]. In this context, the linearised Roe solver is modified via a so-called entropy fix (EF) in order to recover the entropy conditions. The entropy fix by Harten and Hyman (HH) [29] proposes the following modification of the Roe numerical flux

$$F(U_e)n^\pm = \frac{1}{2} \left[ F(U_e^+) + F(U_e^-) \right] n^\pm + |A_n(U^*)|(U_e^\pm - U_e^\mp), \quad (16)$$

where $|A_n(U^*)|$ denotes a dissipation matrix. The HH-EF dissipation matrix is defined as $|A_n(U^*)| := L\Phi R$, being $L$ and $R$ the left and right eigenvector matrices previously introduced and $\Phi$ a diagonal matrix such that

$$\Phi_{ii} := \begin{cases} |\lambda_i|, & \text{if } |\lambda_i| > \delta \\ \delta, & \text{otherwise} \end{cases}, \quad (17)$$

where $\lambda_i$ denotes the $i$-th eigenvalue of the matrix $\Lambda$ introduced above.

Remark 2. In the expression of the dissipation matrix, a user-defined threshold parameter $\delta > 0$ needs to be appropriately tuned to introduce the correct amount of extra diffusion for the problem under analysis. Note that, generally, $\delta \ll \lambda_{\max}$. Nonetheless, it is worth recalling that this value is problem-dependent and may require an empirical tuning to provide the best performance of the Roe solver.

### 3.1.3 Harten-Lax-van Leer Riemann solver

An alternative approach to remedy the entropy violation of the Roe solver is represented by the HLL Riemann solver [30]. Such approach relies on a weighted average of the information in two neighbouring elements $\Omega_e^+$ and $\Omega_e^-$ and leads to the following numerical flux

$$\overline{F(U_e)n^\pm} = \left[ s^+F(U_e^+) - s^-F(U_e^-) \right] n^\pm + s^+s^- \frac{u_\pm - u_\mp}{s^+ - s^-}(U_e^\pm - U_e^\mp), \quad (18)$$

where $s^+ := \max(0, v^\star \cdot n^+ + c^\star)$ and $s^- := \min(0, v^\star \cdot n^- - c^\star)$ denote the estimates of the smallest and largest wave speeds, respectively, with the corresponding signs.

### 3.2 Riemann solvers in hybridised DG methods

In this section, a unified framework for the formulation of the above introduced Riemann solvers in the context of HDG methods is presented. The framework includes, for the first time, the formulation of the HLL Riemann solver within an HDG formulation of the Euler equations. This derivation stems from the seminal work of Peraire and co-workers on linear and nonlinear convection-diffusion equations [42,43] and on compressible flows [41,46,47]. The topic has also been studied in [5].
The general structure of the trace of the HDG numerical flux for a nonlinear problem is
\[
F(U_e) n = F(\bar{U}) n + \tau(U_e - \bar{U}), \tag{19}
\]
where \(\tau\) is a stabilisation matrix which encapsulates the information of the Riemann solvers. It is worth noting that in (19) the hybrid variable \(\bar{U}\) defined on the interface \(\Gamma_i\) between two neighboring elements \(\Omega^+_e\) and \(\Omega^-_e\) is utilised as the intermediate state \(U^*\) introduced in section 3.1.

In order to derive the formulation of Lax-Friedrichs, Roe and the newly proposed HLL Riemann solver in the context of HDG methods, the inter-element continuity of the trace of the numerical fluxes imposed in the global problem (8) is considered, namely
\[
\int F(U_e) n = 0.
\]
It follows that the sum of the contributions \(F(U_e) n\) from two neighbouring elements is set to zero. Exploiting definition (19) and observing that \(\int F(\bar{U}) n = 0\) because of the uniqueness of \(\bar{U}\) on the internal faces, the above transmission condition reduces to
\[
(\tau^+ + \tau^-) \bar{U} = \tau^+ U^+_e + \tau^- U^-_e, \tag{20}
\]
where constant stabilisation matrices \(\tau^+\) and \(\tau^-\) have been considered along the interface seen from element \(\Omega^+_e\) and \(\Omega^-_e\), respectively. Under the assumption of \((\tau^+ + \tau^-)\) being invertible, the intermediate state \(\bar{U}\) is determined pointwise as
\[
\bar{U} = (\tau^+ + \tau^-)^{-1} [\tau^+ U^+_e + \tau^- U^-_e]. \tag{21}
\]
Hence, the numerical flux (19) is formulated as an explicit function of the left and right states \(U^+_e\) and \(U^-_e\). From the framework above, two cases are analysed hereafter. On the one hand, a stabilisation matrix continuous across the interface is obtained by setting \(\tau^+ = \tau^-\). On the other hand, a stabilisation matrix, discontinuous across the interface, is considered when \(\tau^+ \neq \tau^-\).

3.2.1 Continuous stabilisation across the interface: Lax-Friedrichs and Roe Riemann solvers

Consider a continuous definition of the stabilisation matrix across the interface, that is \(\tau^+ = \tau^- = \tau\). It follows
\[
\bar{U} = \frac{U^+_e + U^-_e}{2}, \tag{22a}
\]
\[
F(U_e)n^\pm = F \left( \frac{U^+_e + U^-_e}{2} \right) n^\pm + \frac{1}{2} \tau(U^+_e - U^-_e). \tag{22b}
\]

By considering \(\bar{U}\) as an intermediate state between \(U^+_e\) and \(U^-_e\) and under appropriate choices of the stabilisation matrix \(\tau\), a formulation mimicking Lax-Friedrichs and Roe Riemann solvers for DG methods, see [14] and [15], is retrieved for HDG methods [16][17]. More precisely, for each element \(\Omega_e, e = 1, \ldots, n_1\), setting \(\tau = \tilde{\lambda}_{\text{max}} I_{\text{max}}\), with \(\tilde{\lambda}_{\text{max}} := ||\bar{\nu} \cdot n|| + \tilde{c}\), the Lax-Friedrichs numerical flux is retrieved for the HDG method, namely
\[
\tilde{F}(U_e)n = \tilde{F}(\bar{U})n + \tilde{\lambda}_{\text{max}}(U_e - \bar{U}). \tag{23}
\]
Similarly, the intermediate state \( \tilde{U} \) and the stabilisation matrix \( \bar{\tau} = |A_n(\tilde{U})| \) lead to the formulation of the Roe Riemann solver in the context of HDG methods, that is,

\[
\bar{F}(U_e)n = F(\tilde{U})n + |A_n(\tilde{U})|(U_e - \tilde{U}).
\]

Finally, the HH-EF variant of the Roe numerical flux is given by \( \bar{\tau} = |A_n^t(\tilde{U})| \), according to the correction to matrix \( A_n(\tilde{U}) \) introduced in (17).

**Remark 3.** It is worth noting that the stabilisation matrix introduced in (23) for the Lax-Friedrichs Riemann solver is isotropic, whereas for the Roe numerical fluxes in (24), different values of the stabilisation term are introduced in the equations of conservation of mass, momentum and energy.

### 3.2.2 Discontinuous stabilisation across the interface: Harten-Lax-van Leer Riemann solver

Consider an isotropic stabilisation matrix, discontinuous across the interface, defined as \( \bar{s}^\pm = s^\pm I_{n+2} \), with \( s^+ \neq s^- \). It follows

\[
\tilde{U} = \frac{s^+ U^+_e + s^- U^-_e}{s^+ + s^-},
\]

\[
\bar{F}(U_e)n^\pm = F \left( \frac{s^+ U^+_e + s^- U^-_e}{s^+ + s^-} \right) n^\pm + \frac{s^+ s^-}{s^+ + s^-} (U^+_e - U^-_e).
\]

It is worth noting that here the intermediate state in (25a) is obtained as a weighted average of the states \( U^+_e \) and \( U^-_e \). From this framework, an HLL-type numerical flux, mimicking the behaviour of (18) for DG approaches, is devised for the first time in the context of HDG methods. More precisely, the HLL Riemann solver is

\[
\bar{F}(U_e)n = F(\tilde{U})n + s^+(U_e - \tilde{U}),
\]

where \( s^+ := \max(0, \tilde{v} \cdot n + \tilde{c}) \).

**Remark 4.** A variant of the HLL Riemann solver in (26), the so-called Harten-Lax-van Leer-Einfeldt (HLLE) numerical flux [21], can be devised by simply modifying the stabilisation parameter \( s^+ \) as

\[
s^+ := \max(0, \tilde{v} \cdot n + \tilde{c}, v^+ \cdot n + c^+, v^- \cdot n + c^-),
\]

being \( \odot^+ \) and \( \odot^- \) the variables associated with the states \( U^+_e \) and \( U^-_e \), respectively, at each side of the interface under analysis. Numerical experiments have shown that, in the context of high-order discretisations, the practical difference between HLL and HLLE numerical fluxes is not significant since the jumps across the interface are very small. Henceforth, the former choice is considered for simplicity.

### 4 Implementation details of the high-order hybridised DG solver

In this section, some details on the implementation of the nonlinear solver in the high-order HDG method and on the numerical treatment of solutions with discontinuities and sharp gradients are provided.
4.1 Solver for the nonlinear system of equations

Introducing the numerical flux in the weak forms of the local and global problems, the corresponding discrete problems are obtained. More precisely, the resulting discrete local problems are: for \( e = 1, \ldots, n_{el} \), given \( \hat{U} \in L_2 \left( (0, T_{end}); \hat{W}^h \right) \), find an approximation \( U_e \in L_2 \left( (0, T_{end}); \tilde{W}^h \right) \) such that

\[
\left( W, \frac{\partial U_e}{\partial t} \right)_{\Omega_e} - (\nabla W, F(U_e))_{\Omega_e} + \left( W, F(\hat{U}) \right)_{\partial \Omega_e} + \left( W, \tau(U_e - \hat{U}) \right)_{\partial \Omega_e} = 0,
\]

for all \( W \in L_2 \left( (0, T_{end}); \tilde{W}^h \right) \). It is worth noting that the stabilisation matrix \( \tau \) varies according to the choice of the Riemann solver, as detailed in section 3.2.

Similarly, the formulation of the discrete global problem is obtained: find an approximation \( \tilde{U} \in L_2 \left( (0, T_{end}); \hat{W}^h \right) \) such that

\[
\sum_{e=1}^{n_{el}} \left\{ \left( \tilde{W}, F(\tilde{U})n \right)_{\partial \Omega_e \cap \Gamma} + \left( \tilde{W}, \tau(U_e - \tilde{U}) \right)_{\partial \Omega_e \cap \Gamma} + \left( \tilde{W}, \hat{B}(U, \tilde{U}, U\infty) \right)_{\partial \Omega_e \cap \partial \Omega} \right\} = 0,
\]

for all \( \tilde{W} \in L_2 \left( (0, T_{end}); \hat{W}^h \right) \).

An isoparametric approximation in space is considered for the primal, \( U \), and hybrid, \( \tilde{U} \), variables. Thus, for each element \( \Omega_e, e = 1, \ldots, n_{el} \), the semi-discrete form of the local problem is

\[
M_e \frac{dU_e}{dt} + R_e(U_e, \tilde{U}) = 0,
\]

where \( U_e \) and \( \tilde{U} \) are the vectors of the nodal values of the primal and hybrid variables, respectively, \( M_e \) is the mass matrix associated with element \( \Omega_e \) and \( R_e(U_e, \tilde{U}) \) is the residual vector defined as

\[
R_e(U_e, \tilde{U}) := -C_{uu}^e(U_e) + C_{u\tilde{u}}^e(\tilde{U}) + A_{uu}^e U_e - A_{u\tilde{u}}^e \tilde{U},
\]

where the vectors \( C_{uu}^e(\cdot) \) and the matrices \( A_{uu}^e \) in (31) are obtained from the spatial discretisation of the integral terms in equation (28).

The focus of this work is the development of a robust Riemann solver guaranteeing accurate results for different flow regimes. Thus, special attention is devoted to the spatial discretisation of the Euler equations and the simulations in section 5 only consider steady state problems. In this context, time integration is employed as a relaxation method to ease the convergence process of the solution. It is worth emphasising that in presence of transient problems, the use of high-order time integrators, e.g. backward difference formula (BDF) schemes or diagonally implicit Runge-Kutta (DIRK) methods, is critical to obtain accurate results.
For the sake of simplicity and in order to present the fully discrete form of equation (30), the backward Euler method is considered hereafter for the treatment of the time derivative, leading to

\[ M_e U_{n+1}^e + \Delta t \mathbf{R}_e(U_{n+1}^e, \tilde{U}_{n+1}^e) = M_e U_n^e, \]  

(32)

where the superindex \( n \) denotes the quantities at time \( t^n \in (0, T_{\text{end}}] \). To solve equation (32), the Newton-Raphson algorithm is utilised. It is worth noting that the last two terms in (31) are linear since the stabilisation matrix \( \tau \) introduced by the different Riemann solvers of section 3.2 is evaluated at instant \( t^n \).

Analogously, from the global problem (29) it follows

\[ \mathbf{R}_e(\tilde{U}_{n+1}^e) = 0. \]  

(33)

The residual vector for the global problem is given by

\[ \tilde{\mathbf{R}}_e(\tilde{U}) := \sum_{e=1}^{n_{\text{el}}} \left\{ \mathbf{C}_e^e(\tilde{U}) + \mathbf{A}_{\tilde{e}u} U_e - \mathbf{A}_{\tilde{e}\tilde{u}} \tilde{U} + \mathbf{B}_e \right\}, \]  

(34)

where \( \tilde{\mathbf{B}}_e \) accounts for the boundary conditions and \( U_e \) is substituted by the Newton-Raphson solution of the local problem (32). As for the local residual, the stabilisation matrix \( \tau \) defined by the Riemann solvers is evaluated at time \( t^n \), whence the second and third terms in (34) are linear. The nonlinear problem (33) is solved by means of a Newton-Raphson iterative method.

4.2 Shock-capturing technique

It is known that high-order methods experience an oscillating behaviour in the vicinity of shocks and regions with sharp gradients. For this purpose, an artificial dissipation term is added to regularise the numerical approximation of the problem. More precisely, the artificial viscosity is introduced on the left-hand side of the conservation equation (28) by means of a discretised Laplace operator, following standard approaches in the context of DG and SUPG methods [1,6,11,59], namely

\[ (\nabla W, \varepsilon \nabla U)_{\Omega_e}. \]  

(35)

Remark 5. The additional dissipation introduced via the artificial viscosity term facilitates the fulfillment of entropy conditions for the Roe Riemann solver. Indeed, a transonic example with shock is presented in section 5.3, where an accurate solution is achieved using the Roe solver without the need of any entropy fix. Nonetheless, it is worth emphasising that this behaviour is problem-dependent and in other flow regimes, e.g. supersonic flows, the Roe numerical flux suffers numerical issues in providing physically admissible solutions (see section 5.4).

For the successful implementation of shock capturing techniques, two aspects are critical: the discontinuity sensor and the amount of artificial viscosity introduced.

For the sensor, the smoothness indicator \( S_e \) introduced in [49] and expressed in terms of the density field according to [48], namely

\[ S_e := \frac{\left( \rho_e - \tilde{\rho}_e, \rho_e - \tilde{\rho}_e \right)_{\Omega_e}}{(\rho_e, \rho_e)_{\Omega_e}}, \]  

(36)
is utilised to detect the regions with discontinuities. In (36), \( \rho_e \) denotes the density in the element \( \Omega_e \), computed using a polynomial approximation of degree \( k \), and \( \tilde{\rho}_e \) is its truncation of order \( k-1 \). The sensor measures the regularity of the approximate solution based on the rate of decay of its Fourier coefficients. More precisely, if \( S_e > k^{-4} \), such approximation is expected to be at most \( C^0 \), whereas smooth functions are expected to decay more rapidly \[6\].

Following \[32, 57\], the sensor (36) is implemented using nodal basis functions. It follows that

\[
S_e = \frac{\rho_e^T V^{-T} P V^{-1} \rho_e}{\rho_e^T V^{-T} V^{-1} \rho_e},
\]

(37)

where \( \rho_e \) is the vector containing the nodal values of the density field in the element \( \Omega_e \), \( V \) is the Vandermonde matrix whose inverse maps the Lagrange basis onto the orthonormal one and \( P \) is the orthogonal projection matrix onto the space of monomials of degree \( k \), namely

\[
P := \text{diag}(n_L, \ldots, n_L, n_H, \ldots, n_H),
\]

(38)

being \( n_L \) and \( n_H \) the number of degrees of freedom for monomials of degree \( k-1 \) and \( k \), respectively. In two dimensions, it holds \( n_L := k+1 \) and \( n_H := k(k+1)/2 \).

Concerning the artificial viscosity introduced, its amount in each element is determined according to

\[
\varepsilon_e = \begin{cases} 
0, & \text{if } s_e < s_0 - \xi, \\
\varepsilon_0 \left(1 + \sin \left( \frac{\pi (s_e - s_0)}{2\xi} \right) \right), & \text{if } s_0 - \xi < s_e < s_0 + \xi, \\
\varepsilon_0, & \text{if } s_e > s_0 + \xi,
\end{cases}
\]

(39)

where \( s_e := \log_{10} S_e \), \( \varepsilon_0 \sim h/k \) and \( s_0 \) and \( \xi \) are selected such that \( s_0 + \xi = -4 \log_{10} k \) and \( s_0 - \xi \) is sufficiently large to detect the regions in which mild shock waves are present \[32\]. For the transonic and supersonic cases in sections 5.3 and 5.4, \( s_0 - \xi = -11 \log_{10} k \) is considered. Finally, following \[48\], a \( C^0 \) reconstruction of the artificial viscosity is considered. For this purpose, at each vertex \( \tilde{x} \) of the mesh, the maximum value of the viscosity \( \varepsilon_e \) in the patch of elements centred in \( \tilde{x} \) is selected and a global continuous viscosity field \( \varepsilon \) is obtained by performing a linear interpolation in each element.

5 Numerical studies

In this section, a set of numerical examples is considered to test the performance and the optimal approximation properties of the HLL Riemann solver for the HDG method in different flow regimes. Qualitative and quantitative comparisons between the accuracy of the HLL Riemann solver and the traditional Lax-Friedrichs and Roe numerical fluxes are also provided. Special emphasis is devoted to show the ability of the HLL Riemann solver to provide parameter-free physically admissible solutions for supersonic cases, contrary to the Roe Riemann solver, and the lower amount of numerical dissipation it introduces with respect to the Lax-Friedrichs solver.
Figure 1: Ringleb flow - Triangular meshes of $\Omega = [0, 1]^2$ for the $h$-convergence analysis.

5.1 Convergence analysis: Ringleb flow

The Ringleb flow problem is considered to verify the optimal convergence of the HDG method with the different numerical fluxes under analysis. It consists of a smooth transonic 2D solution of the Euler equations with analytical expression obtained via the hodograph method [10]. For any given spatial coordinates $(x, y)$, the solution of the Ringleb flow can be computed by solving the following nonlinear implicit equation in terms of the speed of sound $c$,

$$
\left(x + \frac{J}{2}\right)^2 + y^2 = \frac{1}{4\rho^2V^4},
$$

where the following relationships for density $\rho$, radial velocity $V$ and $J$ hold

$$
\rho = \frac{c^{2/(\gamma-1)}}, \quad V = \sqrt{\frac{2(1-c^2)}{\gamma-1}}, \quad J = \frac{1}{c} + \frac{1}{3c^3} + \frac{1}{5c^5} - \frac{1}{2} \log \left(\frac{1+c}{1-c}\right).
$$

The exact velocity and pressure fields are

$$
v = \begin{pmatrix} -\text{sgn}(y)V \sin \theta \\ V \cos \theta \end{pmatrix} \quad \text{and} \quad p = \frac{1}{\gamma}c^{2\gamma/(\gamma-1)},
$$

where $\text{sgn}(\cdot)$ is the sign operator, $\sin \theta := \Psi V$ and

$$
\Psi := \sqrt{\frac{1}{2V^2} + \rho \left(x + \frac{J}{2}\right)}.
$$

The Ringleb flow is solved in the domain $\Omega = [0, 1]^2$ with far field conditions imposed on all the boundaries, i.e. $\Gamma_\infty = \partial\Omega$. The computational domain is discretised using uniform meshes of triangular elements. Figure 1 displays the first three levels of refinement employed.

The approximate solution of the Mach number distribution computed on the mesh in figure 1a using polynomial degree $k = 1, \ldots, 3$ is depicted in figure 2. The results clearly display the gain in accuracy obtained increasing the degree of the polynomial approximation, even in presence of extremely coarse meshes, motivating the interest in high-order discretisations.
Figure 2: Ringleb flow - Mach number distribution computed using the HLL Riemann solver on the first level of mesh refinement with polynomial degree $k = 1, \ldots, 3$.

Figure 3: Ringleb flow - Mesh convergence of the $L_2$ error of (a) density, (b) momentum and (c) energy, using Lax-Friedrichs (LF), Roe and HLL Riemann solvers and polynomial degree of approximation $k = 1, \ldots, 4$. 
An $h$-convergence study is performed using a degree of approximation ranging from $k = 1$ up to $k = 4$ and for the three Riemann solvers presented in section 3. Figure 3 displays the error for the conserved variables, i.e. $\rho$, $\rho v$ and $\rho E$, measured in the $L_2(\Omega)$ norm, as a function of the characteristic mesh size $h$. It can be observed that the three Riemann solvers lead to an optimal rate of convergence $h^{k+1}$ and a comparable accuracy in all cases.

5.2 Entropy production: subsonic flow past a circular cylinder

The subsonic flow around a circular cylinder at free stream Mach number $M_\infty = 0.3$ is considered to assess the numerical dissipation introduced by the HLL Riemann solver in the context of HDG methods, in comparison with Lax-Friedrichs and Roe Riemann solvers.

It is known that the geometrical error introduced by low-order descriptions of curved boundaries is responsible for a substantial nonphysical entropy production [3]. Possible solutions involve the modification of the wall boundary condition [34] or the incorporation of the exact boundary representation [56]. As mentioned earlier, isoparametric approximations are considered in this work. Therefore, only approximations of degree at least $k = 2$ are reported, preventing the geometrical error from dominating over the dissipative behaviour of the Riemann solvers under analysis.

Two meshes are considered for this example. The coarsest mesh consists of 1,104 elements with 32 elements to discretise the circle, whereas the finest mesh has 4,635 elements and 64 subdivisions on the circle. A detailed view of the corresponding meshes near the cylinder is depicted in figure 4.

![Mesh 1 and Mesh 2](image)

(a) Mesh 1  (b) Mesh 2

Figure 4: Subsonic flow around a cylinder - Detail of the meshes near the 2D cylinder, featuring (a) 32 and (b) 64 subdivisions on the circular boundary.

For isentropic subsonic flows, entropy production is a measure of the numerical dissipation introduced by the spatial discretisation. The nonphysical entropy production is computed via the so-called entropy error, namely

$$\varepsilon_{\text{ent}} = \frac{p}{p_\infty} \left( \frac{\rho_\infty}{\rho} \right)^{\gamma - 1}, \quad (44)$$

measuring the relative error of the total pressure with respect to the undisturbed flow in an isentropic process.
The first line of figure 5 shows the Mach number distribution and isolines of the numerical solution computed on the first mesh with $k = 1, \ldots, 3$, using the HLL Riemann solver. Although the computed distribution of the Mach number is comparable in the three settings, the superiority of high-order approximations becomes evident when the corresponding entropy errors are compared (Fig. 5, bottom). The results clearly display that, increasing the polynomial degree of discretisation, the numerical dissipation introduced by the method is localised in the vicinity of the cylinder and its overall amount is reduced.

Figure 5: Subsonic flow around a cylinder - Mach number distribution and isolines (top) and entropy error in logarithmic scale (bottom) computed on the first mesh using the HLL Riemann solver with $k = 2$ (left), $k = 3$ (middle) and $k = 4$ (right).

To quantify the differences between the three Riemann solvers, the nonphysical entropy production is compared through the $L_2$ norm of the entropy error, measured along the surface of the cylinder. Figure 6 displays the quantity (44) as a function of the number of degrees of freedom of the global problem, for the two meshes under analysis and an increasing value of the polynomial degree used to approximate the solution. The results show the entropy production of the HLL Riemann solver is almost identical when compared to the Lax-Friedrichs Riemann solver. As expected for a subsonic flow, the entropy production is slightly lower for the Roe Riemann solver.

It is worth noting that the differences between the three Riemann solvers are less important as the polynomial degree of the approximation increases. This confirms the observation above on the reduced amount of numerical dissipation introduced by the method as the degree of the discretisation increases and the consequent extra accuracy provided by high-order approximations. Henceforth, and in order to fully exploit the advantages of the presented HDG solver with
the different Riemann solvers, only high-order approximations are considered.

5.3 Shock treatment: transonic flow over a NACA 0012 aerofoil

A transonic case is considered to test the ability of the proposed HDG method with HLL Riemann solver to capture solutions with sharp gradients and discontinuities using high-order approximations. The example involves the computation of the transonic flow over a NACA 0012 aerofoil, at free stream conditions $M_\infty = 0.8$ and angle of attack $\alpha = 1.25^\circ$, see for instance \cite{59,62,66}.

Figure 7: Transonic flow over a NACA 0012 aerofoil - Mach number distribution computed using HLL Riemann solver with polynomial degree of approximation $k = 4$.

The steady state problem is solved via a relaxation approach with a time step $\Delta t = 10^{-1}$ such that the Courant number is $C = 22$. Convergence to the steady state is achieved when the residual of the steady terms of the continuity equation reaches $10^{-6}$ or is decreased by three orders of magnitude from its maximum value.
All Riemann solvers are equipped with the shock capturing technique described in section 4.2 and the value $\varepsilon_0 = 0.4$ is selected. In the case under analysis, no entropy fix is required by the Roe flux since the artificial viscosity introduced by the shock capturing strategy allows the Riemann solver to fulfill the entropy conditions. Nonetheless, it is worth remarking that the need of an entropy fix is not known \textit{a priori} and the value of the corresponding parameter $\delta$ depends upon the problem and requires to be appropriately tuned by the user, as it will be shown in section 5.4 for the case of a supersonic flow over the NACA 0012 aerofoil.

A mesh with 1,877 triangular elements, without any specific refinement in the shock region, is used and an approximation degree $k = 4$ is considered. The far field boundary is placed 10 chord units away from the aerofoil.

Figure 7 displays the Mach number distribution computed using the HLL Riemann solver. An accurate description of the flow around the aerofoil is obtained and the shock is precisely captured with a coarse mesh, owing to the high-order polynomial approximation constructed using the HDG framework and the shock capturing term introduced. The resolution of the shock is clearly related to the local mesh size and sharper representations may be obtained by performing local mesh refinement in the shock region, as described in [40]. Comparable results, not reported here for brevity, were obtained by the proposed HDG method with Lax-Friedrichs and Roe Riemann solvers.

The accuracy of the different numerical fluxes is thus evaluated comparing the pressure coefficient, given by

$$C_p = \frac{p - p_\infty}{0.5 \rho_\infty v_\infty^2},$$

over the aerofoil profile.

A well resolved solution, in agreement with experimental data from [66], is obtained using all Riemann solvers. The results in figure 8 display that the HLL Riemann solver provides an approximation without oscillations and with accuracy similar to the one of the Roe numerical fluxes near the upper, stronger shock. It is worth noting that the jumps appearing at the extrema of the shock region are due to the discontinuous nature of the HDG approximation. The lower, weaker shock, is reproduced less precisely by the three Riemann solvers and HLL presents a behaviour closer to the Lax-Friedrichs solution in this case.

Next, the entropy production is considered for this non-isentropic case. In this context, such quantity allows to estimate the numerical dissipation introduced in the upstream region before the shock and the entropy produced by the artificial viscosity.

On the one hand, the results in figure 9 show that the regions of activation of the sensor are almost identical for the three Riemann solvers. On the other hand, the different amount of numerical dissipation introduced by the numerical fluxes is responsible for the production of entropy. As observed in figure 8, the HLL Riemann solver presents a behaviour similar to the Roe one in the vicinity of the upper, stronger shock, where comparable approximations are achieved. On the contrary, the Lax-Friedrichs numerical flux introduces the largest amount of numerical dissipation in this region, as shown in figure 9d. In the vicinity of the weaker shock on the lower part of the aerofoil, the three Riemann solvers show a similar entropy production. Finally, the Roe solver provides the most
Figure 8: Transonic flow over a NACA 0012 aerofoil - Pressure coefficient around the aerofoil surface computed using different Riemann solvers with polynomial degree of approximation $k = 4$ and detailed views of the lower (left) and upper (right) shocks.
Figure 9: Transonic flow over a NACA 0012 aerofoil - Regions of activation of the shock sensor (left) and entropy production in logarithmic scale (right) for HLL (top), Lax-Friedrichs (LF, middle) and Roe (bottom) Riemann solvers using a polynomial degree of approximation $k = 4$. 
accurate results in the region near the trailing edge, where the HLL numerical flux introduces extra dissipation.

5.4 Positivity preservation: supersonic flow around a NACA 0012

The last example considers the supersonic flow around a NACA 0012 aerofoil at free stream Mach number $M_\infty = 1.5$ and zero angle of attack [49]. A time step $\Delta t = 8 \times 10^{-2}$ is considered to advance in time and the corresponding Courant number is $C = 20$. Convergence to the steady state is achieved when the residual of the steady terms of the continuity equation reaches $10^{-6}$ or is decreased by three orders of magnitude from its maximum value. The shock treatment is handled by means of the technique discussed in section 4.2, with a value of the artificial viscosity $\varepsilon_0 = 1$ and 2.5 times larger than the one considered in section 5.3 in order to account for the extra strength of the shock.

This supersonic problem is especially challenging since it features an abrupt shock in front of the aerofoil. In such case, Riemann solvers may fail to provide physically admissible solutions, leading to a violation of the positiveness of the approximate density and pressure fields [24, 45, 53]. The HLL numerical flux does not suffer from such issue and provides a robust framework for the approximation of supersonic flows without the need of an entropy fix, i.e. with no user intervention. The Mach number distribution computed using the HLL Riemann solver with a polynomial degree of approximation $k = 4$ is presented in figure 10.

As for the case of the transonic flow discussed above, the method is able to accurately capture the physics of the problem, even on a coarse mesh, owing to the high-order functional discretisation introduced by the HDG scheme.

The map of the entropy production is reported in figure 11 for the HLL and Lax-Friedrichs numerical fluxes. The results display that the HLL Riemann solver introduces a limited amount of numerical dissipation in the vicinity of the front shock. On the contrary, the Lax-Friedrichs solver is responsible for a large entropy production in the shock region, confirming its over-diffusive nature also in supersonic problems. Similarly to the transonic case, figure 11 confirms...
Figure 11: Supersonic flow over a NACA 0012 aerofoil - Regions of activation of the shock sensor (left) and entropy production in logarithmic scale (right) for HLL (top) and Lax-Friedrichs (LF, bottom) Riemann solvers using polynomial degree of approximation $k = 4$. 
that the shock-capturing sensor is activated in the same regions independently on the Riemann solver considered.

Next, the HLL and Roe Riemann solvers are compared. Figure 12 shows the minimum nodal value of the pressure computed using the Roe numerical flux with no entropy fix, with an HH entropy fix $\delta = 0.1$ and the HLL Riemann solver. In the case with no entropy fix, the Roe solver displays an insufficient numerical dissipation. After few iterations, negative values of the pressure are computed, leading to a nonphysical solution. This error is amplified from one time step to the following ones and rapidly leads to the divergence of the Newton-Raphson algorithm employed to solve the nonlinear problem.

To remedy this issue, inherent to the Roe Riemann solver, an HH entropy fix with an empirically tuned value of the threshold parameter $\delta$ is considered. It is worth emphasising that the tuning of such parameter is problem-dependent. With this setting, the HDG method with Roe Riemann solver converges to a steady state solution but small values of the pressure field are obtained. The corresponding Mach number distribution computed using the Roe numerical flux with entropy fix parameter $\delta = 0.1$ is reported in figure 13. Nonphysical overshoots of the solution are identified in the region of the front shock (Fig. 13c). Such oscillations appear despite the artificial viscosity is introduced in the corresponding elements as displayed in figure 13d. Hence, this value of the HH entropy fix parameter leads to insufficient stabilisation and a higher threshold needs to be introduced.

On the contrary, the HLL numerical flux provides a robust approximation with no oscillations (Fig. 13a) without the need of any entropy fix and introducing a lower amount of artificial viscosity as displayed in figure 13b. It is worth noting that the colour scale of figure 13 keeps the same gradation of colours of figure 10 for the interval $M \in [0, 1.8]$ but extends up to $M = 3.6$ to visualise the peak values achieved by the overshoos in the Roe solution.

Of course, the stability issue experienced by the Roe Riemann solver can
Figure 13: Supersonic flow over a NACA 0012 aerofoil - Detail of the Mach number distribution (left) and corresponding artificial viscosity (right) in the front shock near the leading edge computed using HLL (top) and Roe Riemann solver with HH entropy fix with threshold parameter $\delta = 0.1$ (bottom) with polynomial degree of approximation $k = 4$. 
be fixed by increasing the threshold value $\delta$ of the HH entropy fix. Numerical results, not reported here for brevity, showed that a value $\delta = 0.25$ or larger allows the high-order HDG solver to achieve a physically admissible solution with no overshoots. In this context, the main drawback of the Roe Riemann solver is the necessity of a problem-dependent, empirical tuning of the threshold parameter $\delta$ in the definition of the HH entropy fix as discussed above. On the contrary, HLL numerical flux provides a robust, parameter-free strategy to produce positivity preserving, thus physically admissible, solutions.

6 Concluding remarks

This paper introduces a unified framework to describe traditional Riemann solvers, namely Lax-Friedrichs, Roe and Harten-Lax-Van Leer, in the context of the high-order hybridised discontinuous Galerkin method. According to the HDG rationale, the intermediate state utilised to evaluate the numerical fluxes is constructed by means of the HDG hybrid variable and the information of the Riemann solver itself is encapsulated in the HDG stabilisation matrix. Hence, an HLL Riemann solver for compressible flows is devised for the first time for an HDG discretisation.

Optimal convergence properties of the HDG discretisation have been verified using Lax-Friedrichs, Roe and HLL Riemann solvers on a problem with analytical solution leading to similar levels of accuracy for all the approximations. Then, a set of 2D numerical examples has been presented to show the advantages of high-order approximations for compressible flow problems and the capabilities of the novel HLL numerical flux in different flow regimes, from subsonic to supersonic, with special attention to its comparison with well-established Lax-Friedrichs and Roe Riemann solvers in the context of HDG.

In isentropic subsonic cases, HLL Riemann solver achieves results comparable with both Lax-Friedrichs and Roe, the latter being slightly more accurate since it introduces the least amount of numerical dissipation. For transonic cases, a shock-capturing technique is considered for all the Riemann solvers. The overall performance of the HLL Riemann solver is again comparable with the one of the Roe method, whereas the Lax-Friedrichs numerical flux appears to be slightly more diffusive. Finally, the HLL Riemann solver exhibits a superior performance in supersonic cases. On the one hand, contrary to the Lax-Friedrichs approach, HLL numerical flux introduces a limited amount of numerical dissipation, which is comparable to the one of the Roe Riemann solver. On the other hand, contrary to the Roe approach, HLL features a robust and parameter-free solver able to produce positivity preserving solutions without the need of any entropy fix to be tuned according to the problem under analysis.

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