GEOMETRIC CYCLES AND CHARACTERISTIC CLASSES OF
MANIFOLD BUNDLES

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Abstract. We produce new cohomology for non-uniform arithmetic lattices $\Gamma < \text{SO}(p, q)$ using a technique of Millson–Raghunathan. From this, we obtain new characteristic classes of manifold bundles with fiber a closed $4k$-dimensional manifold $M$ with indefinite intersection form of signature $(p, q)$. These classes are defined on a finite cover of $B \text{Diff}(M)$ and are shown to be nontrivial for $M = \#_g(S^{2k} \times S^{2k})$. In this case, the classes produced live in degree $g$ and are independent from the algebra generated by the stable (i.e. MMM) classes. We also give an application to bundles with fiber a K3 surface.

1. Introduction

The starting point of this paper is the following new result about the cohomology of certain arithmetic groups $\Gamma < \text{SO}(p, q)$.

Theorem 1. Fix $1 \leq p \leq q$ such that $p + q \geq 3$. Let $\Lambda \subset \mathbb{R}^{p+q}$ be a lattice with an integral, unimodular bilinear form of signature $(p, q)$. Consider the group $\text{SO}(\Lambda)$ of automorphisms of $\Lambda$ with determinant 1. If $p$ is odd, then for every $N \geq 1$, there is a finite-index subgroup $\Gamma < \text{SO}(\Lambda)$ so that $\dim H^p(\Gamma; \mathbb{Q}) \geq N$.

A lattice $\Lambda$ as in Theorem 1 is determined up to isomorphism by its signature $(p, q)$ and its parity (even or odd) [MH73, Ch. II, §4]. The group $\text{SO}(\Lambda)$ is a nonuniform lattice in $\text{SO}(p, q) = \text{SO}(\Lambda \otimes \mathbb{R})$.

Constructing nonzero elements of $H^*\big(\Gamma; \mathbb{Q}\big)$ is a classical important problem in the theory of arithmetic groups. Our primary interest is to use Theorem 1 to produce new characteristic classes for certain fiber bundles. Recall that a characteristic class for fiber bundles with structure group $G$ is an element of $H^*(BG)$, where $BG$ is the classifying space of $G$. Our first application is as follows.

Corollary 2. Fix $k, g \in \mathbb{N}$ such that $g \geq 3$ is odd and $k \geq g/2$. Let $W_g^{4k} = \#_g(S^{2k} \times S^{2k})$, and denote the group of orientation-preserving diffeomorphisms of $W_g$ by $\text{Diff}(W_g)$. For every $N \geq 1$, there is a finite-index subgroup $\text{Diff}^T(W_g) < \text{Diff}(W_g)$ so that $\dim H^9\big(B\text{Diff}^T(W_g); \mathbb{Q}\big) \geq N$.

Before discussing further applications, we make several remarks.

The particular finite-index subgroup $\Gamma < \text{SO}(\Lambda)$ in Theorem 1 can be made more precise. Fixing a prime $\ell$, denote the congruence subgroups $\Gamma(\ell^n) = \text{SO}(\Lambda) \cap \ker \big[ \text{SL}_{p+q}(\mathbb{Z}) \rightarrow \text{SL}_{p+q}(\mathbb{Z}/\ell^n\mathbb{Z}) \big]$. Then for any $\ell$, given $N \geq 1$, the group $\Gamma(\ell^n)$ satisfies the conclusion of Theorem 1 for $n \gg 0$.

The subgroup $\text{Diff}^T(W_g) < \text{Diff}(W_g)$ appearing in Corollary 2 is defined as the preimage of a subgroup $\Gamma < \text{O}_{g, g}(\mathbb{Z})$ under a homomorphism $\text{Diff}(W_g) \rightarrow \text{O}_{g, g}(\mathbb{Z})$; this is discussed further below.

The cohomology produced in Theorem 1 is new. Millson–Raghunathan [MR80] give uniform $\Gamma < \text{SO}(p, q)$ with $1 \leq p \leq q$ and $p$ even such that $H^p(\Gamma; \mathbb{Q}) \neq 0$. Note in particular that [MR80, pg.

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locally symmetric space \( Y \) goes back to Millson [Mil76]; see also [MR80, RS93, LS86, ANP15]. In each of these works, the spaces \( \Gamma \) are produced using geometric cycles in locally symmetric spaces. Let \( X \) be the locally symmetric space associated to \( \text{SO}(\Lambda) \) and let \( Y = \Gamma \backslash X \) be the locally symmetric space for \( \Gamma < \text{SO}(\Lambda) \). There is an isomorphism \( H^* \Gamma; \mathbb{Q} \simeq H^* \Gamma; \mathbb{Q} \). Each class \( c \in H^p \Gamma; \mathbb{Q} \) we produce is Poincaré dual to a cycle \([Z]\) \( \in H^p_{\partial} \Gamma; \mathbb{Q} \) in Borel–Moore homology with closed supports, and \([Z]\) is represented by a totally-geodesic, properly-embedded oriented submanifold \( Z \subseteq Y \). To show \([Z]\) \( \neq 0 \), we find a compact, totally-geodesic oriented submanifold \( Z' \subset Y \) of dimension \( p \) so that the intersection number \( [Z] \cdot [Z'] \in H_0(\Gamma; \mathbb{Q}) \simeq \mathbb{Q} \) is nonzero.

The cycles \([Z]\) and \([Z']\) are often called geometric cycles. The idea of finding nontrivial homology of a locally symmetric space/arithmetic group by finding a pair of geometric cycles with \([Z] \cdot [Z'] \neq 0 \) goes back to Millson [Mil76]; see also [MR80, RS93, LS86, ANP15]. In each of these works, the locally symmetric space \( Y = \Gamma \backslash X \) is either compact, or the lattice \( \Gamma \) is commensurable to \( \text{SL}_n(\mathbb{Z}) \). The spaces \( \Gamma \backslash \text{SO}(p,q)/K \) we are interested in do not fall into either of these categories. Theorem 1 extends the known results to this case.

In our argument \( Z' \subset Y \) is the quotient of a maximal periodic flat in \( X \). Theorem 1 gives a partial answer to a question of Avramidi–Nguyen-Phan [ANP15, §9].
Characteristic class interpretation. For the application to manifold bundles, the element $[Z] \in H^3_{pq}(Y; \mathbb{Q}) \simeq H^p(B\Gamma; \mathbb{Q})$ described above is not of any particular use as an abstract cohomology class. For this reason, one wants a bundle-theoretic construction of $[Z]$ as a characteristic class.

Fix a lattice $\Lambda \subset \mathbb{R}^{p+q}$ as in Theorem 1. For a CW complex $B$, a map $B \to BSO(\Lambda)$ defines a vector bundle $\mathbb{R}^{p+q} \to W \to B$ with a fiberwise lattice $\Lambda$ and a fiberwise bilinear form $\beta$ of signature $(p, q)$. We extend the structure group to $SO(p, q) > SO(\Lambda)$ (this amounts to forgetting $\Lambda$ but remembering $\beta$), and then consider the different ways to reduce the structure group from $SO(p, q)$ to its maximal compact subgroup. Each choice of reduction corresponds to a choice of a rank-$p$ subbundle $U \subset W$ on which $\beta$ is positive definite. From this setup, we build a characteristic class $c$ that measures the difficulty of choosing $U \subset W$ in a way that is “compatible with $\Lambda$.” We make this precise in [4] using classical obstruction theory, and we show that $c \in H^*(B\Gamma; \mathbb{Q})$ is dual to a geometric cycle $[Z]$.

Applications to manifold bundles. Let $M^{4k}$ be a manifold, and let $\Lambda_M$ denote the lattice $H_{2k}(M; \mathbb{Z})/\text{torsion}$ with its intersection form. Given an $M$-bundle $\pi : E \to B$, one can build a vector bundle $W \to B$ by replacing each fiber $M_b := \pi^{-1}(b)$ with its homology $H_{2k}(M_b; \mathbb{R})$. On the level of classifying spaces, this corresponds to the map

$$\alpha : B\text{Diff}(M) \to BO(\Lambda_M)$$

induced by the action $\alpha : \text{Diff}(M) \to O(\Lambda_M)$ of the group of orientation-preserving diffeomorphisms $\text{Diff}(M)$ on $\Lambda_M$ by automorphisms with determinant $\pm 1$. For $\Gamma < SO(\Lambda_M)$, we define $\text{Diff}^\Gamma(M) = \alpha^{-1}(\Gamma)$. If $\Gamma < SO(\Lambda_M)$ is finite index, then $\text{Diff}^\Gamma(M) < \text{Diff}(M)$ is also finite index. In this case, note that any $M$ bundle $E \to B$ has structure group reducing to $B\text{Diff}^\Gamma(M)$ after passing to a finite cover of $B$.

To apply Theorem 1 to manifold bundles, we are interested in the homomorphism

$$\alpha^* : H^*(B\Gamma; \mathbb{Q}) \to H^*(B\text{Diff}^\Gamma(M); \mathbb{Q}).$$

Application to $W_g^{4k} = \#_g(S^{2k} \times S^{2k})$. When $M = W_g^{4k}$, information about $\alpha^*$ can be obtained using work of Berglund–Madsen [BM17]. This is explained in the appendix, written by Manuel Kramnich, which studies $\alpha^*$ for the more general class of manifolds $W_g^{2n} = \#_g(S^n \times S^n)$ with $n \geq 3$. It is shown that $\alpha^*$ is injective in degrees $* \leq n$; see Theorem 21. Corollary 2 follows immediately from Theorems 1 and 21. As a further consequence of Theorem 21, the appendix produces the first unstable classes in the rational homology of $B\text{Diff}(W_g^{2n})$ when $n \geq 3$ is odd; see Corollary 22.

We remark that the homomorphism $H^*(BO(\Lambda_M); \mathbb{Q}) \to H^*(B\text{Diff}(M); \mathbb{Q})$ can be completely understood in the stable range using index theory. Morita [Mor87] showed this for $M$ a surface; see also [ERW15]. The techniques used to study $\alpha^*$ outside the stable range rely on surgery theory, Morlet’s lemma of disjunction, and rational homotopy theory; see the appendix.

Application to K3 surfaces. Let $M^4$ be a manifold diffeomorphic to a K3 surface. In this case $SO(\Lambda_M)$ is a lattice in $SO(3, 19)$, and by Theorem 1 we can find finite-index $\Gamma < SO(\Lambda_M)$ and a nonzero cycle $z \in H_3(\text{BG}; \mathbb{Q})$. Using the global Torelli theorem, we conclude that $z$ is in the image of $H_3(B\pi_0\text{Diff}^\Gamma(M); \mathbb{Q}) \to H_3(\text{BG}; \mathbb{Q})$. We are not able to determine if $z$ is in the image of $\alpha_* : H_*(B\text{Diff}^\Gamma(M); \mathbb{Q}) \to H_*(\text{BG}; \mathbb{Q})$, but we relate this problem to two other problems of interest. Specifically, we give an example $z \neq 0 \in H_3(\text{BG}; \mathbb{Q})$ so that

- if $z$ is in the image $\alpha_*$, then there exists a K3-surface bundle over a 3-manifold that does not admit a fiberwise Einstein metric;
• if \( z \) is not in the image of \( \alpha \), then the surjection \( \text{Diff}(M) \to \pi_0 \text{Diff}(M) \) does not split.

The first statement should be contrasted with a theorem of Donaldson \([\text{Don90}, \text{Cor. 6.3}]\) that says that every K3 bundle over \( S^1 \) admits a fiberwise Einstein metric; when the base has dimension at least 2, the corresponding statement is unknown. The question of whether \( \text{Diff}(M) \to \pi_0 \text{Diff}(M) \) splits is often referred to as the \textit{generalized Nielsen realization problem}. This problem is solved when \( M \) is a surface, and there are many different proofs \([\text{Mor87}, \text{Mar07}, \text{FH09}, \text{BCS13}, \text{ST16}]\). Morita’s solution generalizes to higher dimensions for the manifolds \( W^{2n}_g = \#_g(S^n \times S^n) \) assuming \( g \) is sufficiently large. The paper \([\text{Gia09}]\) attempts to prove that \( \text{Diff}(M) \to \pi_0 \text{Diff}(M) \) does not split in the case \( M \) is a K3 surface, but the argument has an error (see updated version on arXiv).

\textit{Odd-dimensional manifolds.} With the methods of this paper, we can also produce nontrivial characteristic classes for \( M \) bundles when \( \dim M \) is odd. A sample application to \( M = \#_3(S^d \times S^{d+1}) \) is discussed in \S 5.

\textbf{Section outline.} In \S 2 we recall the general method of constructing homology of arithmetic groups using geometric cycles. In \S 3 we apply that method to \( \text{SO}(\Lambda) \) and prove Theorem 1. In \S 4 we explain how to view geometric cycles as characteristic classes. Finally, \S 5 and the Appendix contain the applications to manifold bundles.

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2. \textbf{Homology of arithmetic groups: geometric cycles}

This section provides the setup for proving Theorem 1. We summarize the general strategy to produce geometric cycles in the homology arithmetic groups/locally symmetric manifolds. We follow \([\text{MR80}]\) and \([\text{Sch10}]\) and refer the reader to these sources for further details. In \S 3 we will apply the material of this section to the specific case of interest \( \Gamma < \text{SO}(\Lambda) \).

2.1. \textbf{Geometric cycles: the general strategy.} Fix an algebraic \( \mathbb{Q} \)-group \( G \) such that \( G(\mathbb{R}) \) is a semisimple Lie group without compact factors. We are interested in finding some nontrivial homology of a finite-index subgroup \( \Gamma < G(\mathbb{Z}) \). The particular subgroup \( \Gamma \) will not be important to us, and at several points we will replace \( \Gamma \) with a further finite-index subgroup (without changing the notation) to ensure that some geometric fact is true.

We begin by describing the locally symmetric model for \( B\Gamma \). Choose a maximal compact subgroup \( K < G(\mathbb{R}) \), and define \( X = G(\mathbb{R})/K \). The manifold \( X \) is contractible and admits a \( G(\mathbb{R}) \)-invariant Riemannian metric of nonpositive curvature. Since \( \Gamma < G(\mathbb{R}) \) is discrete, it acts properly discontinuously on \( X \) and each point-stabilizer in \( \Gamma \) is finite. We can replace \( \Gamma \) by a torsion-free, finite-index subgroup (without changing the notation) to ensure that some geometric fact is true.

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The manifold \( Y \) has an abundance of totally geodesic immersed submanifolds. Let \( G_1 < G \) be a subgroup and take \( h \in G(\mathbb{R}) \) so that \( K_1 := G_1(\mathbb{R}) \cap (hK\mathbb{h}^{-1}) \) is a maximal compact subgroup of \( G_1(\mathbb{R}) \). The image of the orbit map \( G_1(\mathbb{R}) \ni g \mapsto ghK \in X \) is totally geodesic submanifold \( G_1(\mathbb{R})/K_1 \simeq X_1 \subset X \). If \( G_1 \) is a \( Q \)-subgroup, then \( \Gamma_1 = G_1 \cap \Gamma \) is finite index in \( G_1(\mathbb{Z}) \), and the natural map \( j_1 : Y_1 = \Gamma_1 \backslash X_1 \to Y \) is a proper, totally geodesic immersion \[ \text{Sch10} \) §6].

With this setup, we are ready to discuss the general strategy for producing “geometric cycles” in the homology of \( Y \). Let \( o = eK \) be the basepoint of \( X \). First choose \( G_1, G_2 < G \) so that

\[
\text{(†) } X_1 \text{ and } X_2 \text{ have complementary dimension } d_1 + d_2 = \dim X, \text{ the intersection } X_1 \cap X_2 = \{ o \} \text{ is transverse, and } Y_1 \text{ is compact.}
\]

Then choose \( \Gamma < G(\mathbb{Z}) \) so that

\[
\text{(‡) the quotients } Y, Y_1, Y_2 \text{ are oriented manifolds, the maps } j_1, j_2 \text{ are embeddings, and the intersection } Y_1 \cap Y_2 \text{ is transverse and every intersection has a positive sign.}
\]

Given (†) and (‡), the submanifolds \( Y_1 \subset Y \) determine classes \( [Y_1] \in H_{d_1}(Y; \mathbb{Q}) \) and \( [Y_2] \in H_{d_2}^1(Y; \mathbb{Q}) \) in homology and homology with closed supports, and the algebraic intersection \( [Y_1] \cdot [Y_2] \) is nonzero, so \( [Y_1] \neq 0 \) in \( H_{d_1}(Y; \mathbb{Q}) \simeq H_{d_1}(\Gamma; \mathbb{Q}) \), and \( [Y_2] \neq 0 \) in \( H_{d_2}^1(Y; \mathbb{Q}) \simeq H_{d_2}^1(\Gamma; \mathbb{Q}) \). This is explained in more detail in \[ \text{Sch10} \).

The general strategy does not always work. Indeed, it is not always possible to achieve (†). One problem is that a totally geodesic subspace \( X_1 \subset X \) need not admit a subspace of complementary dimension. (It is shown in \[ \text{MR80} \) Theorem 1.1] that such a complement exists if \( X_1 = X^\sigma \) is the fixed set of an involutive isometry.) Another problem is that if \( Y \) is non-compact, then there is no reason \( Y_1 \) or \( Y_2 \) need be compact in general. Nevertheless, in some special cases, one can find \( G_1, G_2 \) so that (‡) is satisfied. We will see this when \( G \) is an indefinite orthogonal group in the next section.

Now we address the difficulty with (‡). There is a general theorem that ensures the first two clauses of (‡). (Ensuring that \( Y \) is oriented is easy, but ensuring \( Y_1 \) and \( Y_2 \) are oriented is already nontrivial.)

**Theorem 3** (\[ \text{Sch10} \), Theorem D). Let \( G \) be a connected semisimple algebraic \( Q \)-group, let \( G_1 < G \) be a connected reductive \( Q \)-subgroup, and let \( \Gamma < G(\mathbb{Q}) \) be an arithmetic subgroup. Then after replacing \( \Gamma \) by a finite-index subgroup, the map \( j_1 : Y_1 \to Y \) is a proper, injective, closed embedding, and each component of the image is an orientable, totally geodesic submanifold of \( Y \).

Given Theorem 3, the remaining difficulty is showing that, after replacing \( \Gamma \) with a subgroup of large index, \( Y_1 \cap Y_2 \) is a finite set of points and the intersection number at each point is +1. We explain how to approach this problem in the next subsection.

### 2.2. Intersections and double cosets.

Assume that \( G_1, G_2 < G \) satisfy (†) and that \( \Gamma < G(\mathbb{Z}) \) is torsion-free and the associated manifolds \( Y_1 \) and \( Y_2 \) are oriented, embedded submanifolds of \( Y \).

As explained in \[ \text{MR80} \) (to be reviewed below), the components of \( Y_1 \cap Y_2 \) can be identified with a certain subset \( \Omega \) of the double coset space \( \Gamma_2 \backslash \Gamma / \Gamma_1 \). Choosing coset representatives \( I(\Gamma) \subset \Gamma \) for \( \Omega \), the sign of the intersection corresponding to \( \gamma \in I(\Gamma) \) is determined by the double coset of \( \gamma \) in \( G_2^+(\mathbb{R}) \backslash G(\mathbb{R}) / G_1^+(\mathbb{R}) \), where \( G_i^+(\mathbb{R}) < G_i(\mathbb{R}) \) is the subgroup that preserves orientation on \( X_i \). In particular, if \( \gamma \) can be written \( \gamma = g_2g_1 \) with \( g_1 \in G_1^+(\mathbb{R}) \), then the corresponding intersection is positive \[ \text{MR80} \) Cor. to Lem. 2.5]. One wants to show that if \( \Gamma < G(\mathbb{Z}) \) is a congruence subgroup of large index, then every \( \gamma \in I(\Gamma) \) belongs to one of the double cosets in \( G_2^+(\mathbb{R}) \backslash G(\mathbb{R}) / G_1^+(\mathbb{R}) \) that correspond to a positive intersection number.
First we describe the components of $Y_1 \cap Y_2$ in terms of the double coset space $\Gamma_2 \backslash \Gamma / \Gamma_1$. Denote the projection $\Pi : X \to Y$. There is a bijection between $\Pi^{-1}(Y_1 \cap Y_2)$ and

$$T := \{ (\gamma, x_1, x_2) : \gamma x_1 = x_2 \} \subset \Gamma \times X_1 \times X_2$$

(an intersection downstairs is covered by an intersection upstairs, and we can translate by $\gamma$ so that the intersection happens on $X_2$). The set $T$ has an action of $\Gamma_2 \times \Gamma_1$ given by

$$(\alpha_2, \alpha_1).(\gamma, x_1, x_2) = (\alpha_2 \gamma \alpha_1^{-1}, \alpha_1 x_1, \alpha_2 x_2).$$

**Claim.** $Y_1 \cap Y_2 \simeq T / (\Gamma_2 \times \Gamma_1)$.

To prove the claim, one shows that if $(\gamma, x_1, x_2)$ and $(\gamma', x_1', x_2')$ are in $T$, then $\Pi(x_1) = \Pi(x_1')$ if and only if there exists $\gamma_i \in \Gamma_1$ so that $(\gamma', x_1', x_2') = (\alpha_2, \alpha_1)(\gamma, x_1, x_2)$. The “if” direction is obvious. For the “only if” direction, one uses the fact that $Y_1$ is embedded in $Y$, which implies that if $\gamma \in \Gamma$ and $\gamma X_1 \cap X_2 \neq \emptyset$, then $\gamma \in \Gamma_1$. More details can be found in [MR80, §2].

A similar argument shows that if $(\gamma, x_1, x_2)$ and $(\gamma', x_1', x_2')$ are in $T$, then $\Pi(x_1)$ and $\Pi(x_1')$ are in the same component of $Y_1 \cap Y_2$ if and only if $\gamma$ and $\gamma'$ lie in the same double coset $\Gamma_2 \backslash \Gamma / \Gamma_1$. See [MR80, Lem 2.3 and Prop 2.3]. In other words, $\pi_0(Y_1 \cap Y_2)$ is in bijection with

$$\Omega := \{ \Gamma_2 \gamma \Gamma_1 : \gamma X_1 \cap X_2 \neq \emptyset \} \subset \Gamma_2 \backslash \Gamma / \Gamma_1.$$

Note that $\Omega \simeq \pi_0(Y_1 \cap Y_2)$ is finite because $Y_1 \cap Y_2 \subset Y_1$ is a submanifold and $Y_1$ is compact.

Next we explain, for each $y \in \pi_0(Y_1 \cap Y_2)$, whether the intersection is positive, negative, or degenerate. Fix a set of coset representatives $I(\Gamma) \subset \Gamma$ for elements of $\Omega$, and assume that $\gamma_1 = \Id$ represents $\Gamma_2 \Gamma_1$.

Note that for $\gamma \in I(\Gamma)$ we can write $\gamma = a_2 k a_1^{-1}$, where $a_i \in G_1^+(\mathbb{R})$ and $k \in K$. This is because $\gamma X_1 \cap X_2 \neq \emptyset$ implies that there exists $x_i \in X_i$ so that $\gamma x_i = x_2$. Since $G_1^+(\mathbb{R})$ acts transitively on $X_i$, we can choose $a_i \in G_1^+(\mathbb{R})$ so that $a_i(0) = x_i$, where $0 = eK$ is the basepoint of $X = G(\mathbb{R})/K$ (and is also the intersection of $X_1$ and $X_2$). Then $a_2^{-1} \gamma a_1(0) = o$, which means $a_2^{-1} \gamma a_1 = k$ for some $k \in K$.

Since $a_i$ preserves orientation on $X_i$, the sign of the intersection $\gamma X_1 \cap X_2$ is determined by the action of $k$ on $T_0 X$.

**Claim.** If $e_1, \ldots, e_p \in T_0 X_1$ and $e_{p+1}, \ldots, e_n \in T_0 X_2$ are positively oriented bases, and if $\epsilon(\gamma)$ is defined by

$$(2) \quad k(e_1) \wedge \cdots \wedge k(e_p) \wedge e_{p+1} \wedge \cdots \wedge e_n = \epsilon(\gamma) e_1 \wedge \cdots \wedge e_n,$$

then the intersection corresponding to $\gamma \in I(\Gamma)$ is positive, negative, or degenerate according to whether $\epsilon(\gamma)$ is positive, negative, or zero.

**Proof.** The basis $(a_1 e_1, \ldots, a_p e_p)$ is positively oriented in $T_{x_1} X_1$, and the basis $(a_2^{-1} e_{p+1}, \ldots, a_2^{-1} e_n)$ is positively oriented in $T_{x_2} X_2$. We want to know if $(\gamma(a_1 e_1), \ldots, \gamma(a_p e_p), a_2^{-1} e_{p+1}, \ldots, a_2^{-1} e_n)$ is positively oriented in $T_{x_2} X$. Since $a_2$ preserves orientation on $X$, the orientation of this $n$-tuple is the same as the orientation of $(k e_1, \ldots, k e_p, e_{p+1}, \ldots, e_n)$. See also [MR80, Prop 2.3].

Next one shows that if $\gamma, \gamma' \in I(\Gamma)$ lie in the same coset $G_1^+(\mathbb{R}) \backslash G(\mathbb{R})/G_1^+(\mathbb{R})$, then $\epsilon(\gamma) = \epsilon(\gamma')$.

Assume $\gamma, \gamma' \in I(\Gamma)$ and $\gamma' = h_2 \gamma h_1$ for some $h_i \in G_2^+(\mathbb{R})$. There are two cases: the intersection $\gamma X_1 \cap X_2$ is either degenerate or not. If $\gamma X_1 \cap X_2$ is degenerate (i.e. has dimension at least 1), then the same is true for $h_2 \gamma h_1 = \gamma'$, so $\epsilon(\gamma) = 0 = \epsilon(\gamma')$. If $\gamma X_1 \cap X_2$ and hence also $\gamma' X_1 \cap X_2$ are non-degenerate, then there exists a unique $x_1, x_1' \in X_1$ and $x_2, x_2' \in X_2$ so that $\gamma x_1 = x_2$ and $\gamma' x_1' = x_2'$. Since $\gamma' = h_2 \gamma h_1$, it follows that $x_1 = h_1 x_1$ and $x_2 = h_2^{-1} x_2'$. Then if $\gamma = a_2 k a_1^{-1}$ where $a_i(0) = x_i$, then we have $\gamma' = h_2 a_2 k a_1^{-1} h_1$, and $h_1^{-1} a_1(0) = h_1^{-1}(x_1) = x_1'$ and $h_2 a_2(0) = h_2(x_2) = x_2'$, which
implies that both \( \epsilon(\gamma) \) and \( \epsilon(\gamma') \) is computed from the action of \( k \) as in \(^2\), so \( \epsilon(\gamma) = \epsilon(\gamma') \). See also [MR80] Lem. 2.5.

As a consequence, if \( \gamma \in G_2^+(\mathbb{R})G_1^+(\mathbb{R}) \), then \( \epsilon(\gamma) = +1 \). We can see this latter fact directly as follows: if \( \gamma = g_2g_1 \in G_2^+(\mathbb{R})G_1^+(\mathbb{R}) \), then

\[
(3) \quad \gamma X_1 \cap X_2 = (g_2g_1X_1) \cap X_2 = (g_2X_1) \cap X_2 = g_2(X_1 \cap g_2^{-1}X_2) = g_2(X_1 \cap X_2),
\]
and since \( g_2 \) preserves orientation on \( X \) and \( X_2 \), this implies that the sign of the intersection \( \gamma X_1 \cap X_2 \) is equal to the sign of the intersection \( X_1 \cap X_2 \), which is positive by assumption.

### 3. Geometric cycles for \( \Gamma < SO(p,q) \)

In this section we prove Theorem 1. To start set \( \Gamma = \text{SO}(\Lambda) \) with \( \Lambda \) as in the statement. We split the proof of the theorem into proving two statements:

(a) Up to replacing \( \Gamma \) by a finite-index subgroup, \( H_p(\Gamma;\mathbb{Q}) \) is nonzero.

(b) Given \( N \geq 1 \), we can replace \( \Gamma \) by a finite-index subgroup so that \( \dim H_p(\Gamma;\mathbb{Q}) \geq N \).

In §3.1 we define groups \( G_1, G_2 \) and verify that the conditions of (1) from §2 can be satisfied for a good choice of \( G_1, G_2 < G \). In §3.2 we show that we can choose \( \Gamma < G(\mathbb{Z}) \) so that (1) is also satisfied. Together these prove (a). In §3.3 we prove (b) by showing how to produce many linearly independent flat cycles.

#### 3.1. The \( SO(p,q) \) case

Fix \( 1 \leq p \leq q \). Let \( V = \Lambda \otimes \mathbb{R} \). Let \( B \) be the matrix for bilinear form on \( V \) with respect to some basis for \( \Lambda \). Consider the algebraic \( \mathbb{Q} \)-group

\[
(4) \quad G = \text{SO}(B) = \{ g \in \text{SL}_{p+q}(\mathbb{C}) : g^*Bg = B \}.
\]

Then \( \text{SO}(\Lambda) \) is the group of integer points \( G(\mathbb{Z}) \). We will choose a maximal \( \mathbb{Q} \)-split torus \( G_1 \simeq (G_m)^p \) in \( G \) (here \( G_m \) denotes the multiplicative group). We will also fix \( \lambda \in \Lambda \) with \( \lambda \cdot \lambda < 0 \) and define a \( \mathbb{Q} \)-subgroup \( G_2 = \text{SO}(B') \) in \( G \), where \( B' \) is the restriction of \( B \) to \( \lambda^\perp \). These groups have real points \( G(\mathbb{R}) \simeq \text{SO}(p,q) \), \( G_1(\mathbb{R}) \simeq \text{SO}(1,1)^p \simeq (\mathbb{R}^\times)^p \), and \( G_2(\mathbb{R}) \simeq \text{SO}(p,q-1) \). The associated symmetric spaces \( X, X_1, X_2 \) have dimensions \( pq, p(q-1) \), respectively.

For a generic choice of \( G_1, G_2 \), the intersection \( X_1 \cap X_2 \) will be transverse. To make this precise, view \( G(\mathbb{R}) \) as a group of automorphisms of \( V \simeq \mathbb{R}^{p,q} \). Consider orthogonal decompositions

\[
(5) \quad V = U_1 \oplus \cdots \oplus U_p \oplus N \quad \text{and} \quad V = P \oplus L,
\]

where \( U_i \simeq \mathbb{R}^{1,1}, N \simeq \mathbb{R}^{0,q-p}, P \simeq \mathbb{R}^{p,q-1}, \) and \( L \simeq \mathbb{R}^{0,1} \). From these decompositions, one can define groups \( G_1, G_2 \) so that \( G_1(\mathbb{R}) \) is the group that preserves the first decomposition and acts as the identity on \( N \), and \( G_2(\mathbb{R}) \) preserves the second decomposition.

**Definition.** We say the pair \( (G_1, G_2) \) is in general position (or alternatively, the corresponding decompositions \( V = U_1 \oplus \cdots \oplus U_p \oplus N \) and \( V = P \oplus L \) are in general position) if \( P \cap U_i \) is a positive line for each \( i \) and \( N \perp \cap L = \{0\} \).

**Claim.** If \( G_1, G_2 \) are in general position, then \( G_1 \cap G_2 \) is trivial.

**Proof of Claim.** Fix \( g \in G_1 \cap G_2 \). Write \( L = \langle \lambda \rangle \). Since \( (G_1, G_2) \) are in general position, \( P \cap U_i \) is a positive line for each \( 1 \leq i \leq p \), and for each \( i \) we can choose a nonzero vector \( w_i \in U_i \cap (P \cap U_i) \perp \) (\( w_i \) is unique up to scaling). Choose a basis \( v_1, \ldots, v_{q-p} \) for \( N \). Since \( \lambda \in \left[ (P \cap U_1) \oplus \cdots \oplus (P \cap U_p) \right] \perp = \langle w_1, \ldots, w_p \rangle \oplus N \), we can write \( \lambda = \sum a_i w_i + \sum b_j v_j \). Note that \( N \perp \cap L = \{0\} \) and \( \dim P \cap U_i = 1 \) imply that that \( b_j \neq 0 \) for some \( 1 \leq j \leq q-p \) and \( a_i \neq 0 \) for each \( 1 \leq i \leq p \).
The matrix $g$ acts on the complexification $V^\mathbb{C} = V \otimes \mathbb{C}$. On the one hand, $g$ preserves each $U_i^\mathbb{C}$ and also $P^\mathbb{C} \cap U_i^\mathbb{C}$. Since $P \cap U_i$ is not isotropic, $g$ acts on $U_i^\mathbb{C}$ by $\pm \text{Id}$. Also $g$ acts as the identity on $N$ since $g \in G_1$. Therefore,

$$g(\lambda) = \sum \pm a_i w_i + \sum b_j v_j.$$ 

On the other hand, $g \in G_2$ implies that $g(\lambda) = \epsilon \lambda$ for some $\epsilon \in \mathbb{C}^\times$. By the preceding equation, this is only possible if $\epsilon = 1$ and $g$ acts by the identity on each $U_i$. This implies that $g = \text{Id}$. \hfill \square

Finally, we remark that we can choose $G_1$ so that $Y_1$ is compact: Prasad–Raghunathan [PR72] prove the existence of $\tau \in G(\mathbb{Q})$ whose centralizer $C_\tau(\mathbb{R})$ is a Cartan subgroup and $C_\tau(\mathbb{R})/(\Gamma \cap C_\tau(\mathbb{R}))$ is compact.

### 3.2. Proof of Theorem 1(a).

We proceed in two steps:

- **Step 1.** After replacing $\Gamma$ by a finite-index subgroup, we can write each $\gamma \in I(\Gamma)$ as $\gamma = g_2g_1$ with $g_1 \in G_1(\mathbb{R}) = G_1^+(\mathbb{R})$, and $g_2 \in G_2(\mathbb{R})$.

- **Step 2.** When $p$ is odd, we can ensure that if $\gamma = g_2g_1$ and $g_2 \notin G_2^+(\mathbb{R})$, then $\epsilon(\gamma) = +1$.

**Step 1 (the double coset of $\gamma \in I(\Gamma)$).** The following proposition follows from the argument of [MR80] Theorem 3.1. See also [FOR00] Lem. 2.6.

**Proposition 4.** Let $G$ be an connected, reductive algebraic $\mathbb{Q}$-group with $K < G(\mathbb{R})$ a maximal compact subgroup. Fix an arithmetic subgroup $\Gamma < G(\mathbb{Z})$, and let $G_1, G_2 < G$ be connected, reductive $\mathbb{Q}$-subgroups. There exists a finite-index subgroup $\Gamma' < \Gamma$ so that if $\gamma \in \Gamma'$ and $G_2(\mathbb{R})\gamma \cap KG_1(\mathbb{R}) \neq \emptyset$, then $\gamma \in G_2(\mathbb{C})G_1(\mathbb{C})$.

In [22], we saw that if $\gamma X_1 \cap X_2 \neq \emptyset$, then $G_2(\mathbb{R})\gamma \cap KG_1(\mathbb{R}) \neq \emptyset$. Then by Proposition 4, after replacing $\Gamma$ by a finite-index subgroup, we can ensure that if $\gamma \in I(\Gamma)$, then $\gamma \in G_2(\mathbb{C})G_1(\mathbb{C})$. Our next step is to show that if $\gamma \in G_2(\mathbb{C})G_1(\mathbb{C})$, then also $\gamma \in G_2(\mathbb{R})G_1(\mathbb{R})$.

**Proposition 5.** Fix $G, G_1, G_2$ and $\Gamma$ as above. Assume the pair $(G_1, G_2)$ is in general position. For $\gamma \in \Gamma$, if $\gamma \in G_2(\mathbb{C})G_1(\mathbb{C})$, then $\gamma \in G_2(\mathbb{R})G_1(\mathbb{R})$.

**Remark 6.** As a consequence of Proposition 5 if $X_1 \cap X_2 = \emptyset$, then $\gamma X_1 \cap X_2 = \emptyset$. This follows from the computation in [3]. Thus if $X_1$ and $X_2$ are disjoint, then there exists finite index $\Gamma < G(\mathbb{Z})$ so that the quotient submanifolds $Y_1, Y_2$ are disjoint in $Y$. This observation will be used in §3.3 to produce linearly independent cycles.

**Proof of Proposition 5.** We are given $\gamma = h_2h_1$ with $h_i \in G_i(\mathbb{C})$. We claim that in fact $h_i \in G_i(\mathbb{R})$. We will show that $h_1\bar{h}_1^{-1} = e = h_2^{-1}\bar{h}_2$ (bar denotes complex conjugation). Observe that $h_1\bar{h}_1^{-1} = h_2^{-1}\bar{h}_2$ because $\bar{\gamma} = \gamma$. Then $h_1\bar{h}_1^{-1} \in G_1(\mathbb{C}) \cap G_2(\mathbb{C})$, which is trivial because $G_1, G_2$ are in general position. \hfill \square

At this point, we have shown $\gamma \in I(\Gamma)$ can be expressed as $\gamma = g_2g_1$ with $g_i \in G_i(\mathbb{R})$. If $g_i$ preserves orientation on $X_i$, then we are done, as remarked above. It is easy to show that $G_1(\mathbb{R})$ preserves orientation on $X_1$ (orientation is determined only by the component, so it suffices to consider the action of elements of $G_1(\mathbb{R}) \cap K$; then you can reduce to tangent space at basepoint, where the action of $K$ is the adjoint action; the computation is straightforward).

**Step 2 (computing $\epsilon(\gamma)$ for $\gamma \in I(\Gamma)$).** Fix a pair $(G_1, G_2)$ in general position; fix $\Gamma < G(\mathbb{Z})$ as in Proposition 4 and fix $\gamma \in I(\Gamma)$. By Propositions 4 and 5 $\gamma = g_2g_1$, where $g_i \in G_i(\mathbb{R})$. In this step we will show that $\epsilon(\gamma)$ is positive when $p$ is odd, where $\epsilon(\gamma)$ is defined in [2]. For this computation, we will make a further assumption on the pair $(G_1, G_2)$.
We want to express \( \gamma = a_2ka_1 \) with \( a_i \in G^+_1(\mathbb{R}) \) and \( k \in K \). Since \( G_1(\mathbb{R}) = G^+_1(\mathbb{R}) \), we can choose \( a_1 = g_1 \). Since \( G_2(\mathbb{R}) \cong \text{SO}(p, q - 1) \), the element \( g_2 \) belongs to \( G^+_2(\mathbb{R}) \) if and only if \( \theta(g_2) = 1 \), where \( \theta : G_2(\mathbb{R}) \to \mathbb{R}^\times/(\mathbb{R}^\times)^2 \cong \{ \pm 1 \} \) is the spinor norm, defined as follows. Any \( g \in \text{SO}(p, q) \) can be expressed as a product of reflections \( g = R^{x_1} \cdots R^{x_k} \) about \( x_i \in \mathbb{R}^{p,q} \). Then \( \theta(g) = \prod_{j=1}^k x_i^2 \mod(\mathbb{R}^\times)^2 \), which is well-defined independent of the choice of reflections. (For \( x \in \mathbb{R}^{p,q} \) the reflection \( R^x \) is defined as \( z \mapsto z - 2 \frac{x \cdot z}{x^2} x \).)

If \( \theta(g_2) = 1 \), then (taking \( a_2 = g_2 \)) and \( k = e \) shows \( \epsilon(\gamma) = +1 \), so in the remainder of this subsection we assume that \( \theta(g_2) = -1 \). We will define \( k \) as a composition of commuting reflections \( k = R^{x} R^{-x} \) about \( x_+ \in P \) and \( x_- \in L \), where \( x_+^2 = \pm 1 \). Any such \( k \) belongs to \( G_2(\mathbb{R}) \cap K \) and has \( \theta(k) = -1 \), so \( \gamma = a_2ka_1 \), where \( a_2 = g_2k \).

**Remark.** As explained in [2], the sign of \( \epsilon(\gamma) \) is determined by the action of \( k \) on \( T_oX = T_oX_1 \oplus T_oX_2 \). Note that the definition \( k = R^{x} R^{-x} \) does not use anything about \( \gamma \in I(\Gamma) \). In particular, the sign of \( \epsilon(\gamma) \) when \( \gamma = g_2g_1 \) with \( \theta(g_2) = -1 \) depends only on the choice of \( (G_1, G_2) \). In the special case \( p = q \), the argument below will show that \( \epsilon(\gamma) \) actually only depends on \( p \).

Now we choose a specific pair \((G_1, G_2)\) and define \( k \) precisely. Let \( e_1, \ldots, e_p, f_1, \ldots, f_q \in \mathbb{Q} \)-basis consisting orthogonal vectors in \( \Lambda \), where \( e_i^2 = a_i > 0 \) and \( f_j^2 = b_j < 0 \) and distinct basis vectors are orthogonal (this is possible because the form is diagonalizable over \( \mathbb{Q} \)).

Let \( G_1(\mathbb{R}) \) be the group that preserves the composition \( V = U_1 \oplus \cdots \oplus U_p \oplus N \), where \( U_i = \mathbb{R}\{e_i, f_i\} \) and \( N = \mathbb{R}\{f_{p+1}, \ldots, f_q\} \). This choice of \( G_1 \) is defined over \( \mathbb{Q} \), but \( G_1(\mathbb{Z}) < G_1(\mathbb{R}) \) is not cocompact (if \( H(\mathbb{R}) \) is semisimple and defined over \( \mathbb{Q} \), then \( H(\mathbb{Z}) \) is a lattice in \( H(\mathbb{R}) \), but here \( G_1(\mathbb{R}) \) is not semisimple). However, the decomposition \( V = U_1 \oplus \cdots \oplus U_p \oplus N \) defining \( G_1 \) can be perturbed so that the resulting group \( G'_1 \) is still defined over \( \mathbb{Q} \) and \( G'_1(\mathbb{Z}) \) is cocompact in \( G'_1(\mathbb{R}) \). This is discussed more in [3,3].

Next we specify \( G_2(\mathbb{R}) \). Fix \( \lambda \in \Lambda \cap \mathbb{R}\{f_1, \ldots, f_q\} \), set \( L = \mathbb{R}\{\lambda\} \) and \( P = L^\perp \), and define \( G_2(\mathbb{R}) \) as the group that preserves \( V = P \oplus L \). Take \( v \in L \) with \( v \cdot v = -1 \), and write \( v = \sum_{j=1}^q v_j f_j \). Note that \( v_j = 0 \) for \( p + 1 \leq j \leq q \) by our choice of \( \lambda \). We also choose \( \lambda \) so that \( v_j \neq 0 \) for \( 1 \leq j \leq p \), since this implies that \( (G_1, G_2) \) is in general position.

Define \( k = R^{e_1} \circ R^u \). As a matrix, we write (abusing notation for \( R^{e_1}, R^u \)) \( k = \left( \begin{array}{cc} R^{e_1} & 0 \\ 0 & R^u \end{array} \right) \), where \( R^{e_1} \) is the \( p \times p \) diagonal matrix with last entry \(-1\) and all other diagonal entries \(1\), and \( R^u = (R_{ij}^u) \) is the \( q \times q \) matrix with \( R_{ii}^u = 1 - 2v_i^2 \) for \( 1 \leq i \leq q \) and \( R_{ij}^u = -2v_i v_j \) for \( i \neq j \).

Via the Cartan decomposition \( g = t \oplus p \), we identify \( T_oX \cong p \cong M_{p \times q}(\mathbb{R}) \) with the vector space of \( p \times q \) matrices. Under this identification \( T_oX, T_oX_1 \oplus T_oX_2 \) corresponds to \( p = p_1 \oplus p_2 \), where \( p_1 \) consists of diagonal matrices (i.e. \( C_{ij} = 0 \) if \( i \neq j \)), and \( p_2 \) consists of \( C \) such that \( Cv = 0 \), i.e. the entries of \( C \) satisfy the equations \( 0 = \sum_{j=1}^q v_j C_{ij} \) for each \( 1 \leq i \leq p \). The action of \( (^\circ \gamma) \in K = S(\text{O}(p) \times \text{O}(q)) \) on \( p \) is given by \( C \mapsto C^{\gamma - 1} \).

According to [2], the sign of \( \epsilon(\gamma) \) is the sign of the determinant of \( p_1 \xrightarrow{k} p \xrightarrow{\pi} p_1 \), where \( \pi : p \to p_1 \) is the projection with respect to \( p = p_1 \oplus p_2 \).

**Claim.** \( \det(\pi \circ k) = (-1)^{p-1} \).

Thus if \( p \) is odd, then \( \epsilon(\gamma) > 0 \) for every \( \gamma \in I(\Gamma) \).
Proof of claim. The action of \( k \) sends the diagonal matrix \((d_1, \ldots, d_p) \in p_1 \) to

\[
X := R^{p} \cdot \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_p \end{pmatrix} \cdot R^{p}
\]

The coefficients of \( X \) are given by

\[
X_{ij} = \begin{cases} 
(-1)^{d_p}(1 - 2v_i^2) d_j & i = j \\
(-1)^{d_p+1}(2v_i v_j) d_i & i \neq j, j \leq p \\
0 & j > p,
\end{cases}
\]

where \( \delta_{ip} \) is the Kronecker delta. To compute \( \pi(X) \), we write \( X = A + C \), where \( A = \text{diag}(A_1, \ldots, A_p) \in p_1 \) and \( C \in p_2 \). This reduces to solving the system of equations for \( A_i \) for \( 1 \leq i \leq p \)

\[
X_{ii} = A_i + C_{ii} \\
X_{ij} = C_{ij} & i \neq j \\
0 = \sum_{j=1}^{q} v_j C_{ij}
\]

One finds \( A_i = -d_i \) for \( 1 \leq i \leq p - 1 \) and \( A_p = d_p \). Thus \( (\pi \circ k)(d_1, \ldots, d_p) = (-d_1, \ldots, -d_{p-1}, d_p) \). This completes the proof of the claim, and also finishes Step 2.

At this point, we have shown how to obtain \( G_1, G_2 < G \) and finite-index subgroup \( \Gamma < G(\mathbb{Z}) \) so that conditions (†) and (‡) are satisfied. This proves part (a) of Theorem 1.

Remark. Note that if we perturb \((G_1, G_2)\) then the same computation of the Claim holds. The same is true if we conjugate \((G_1, G_2)\) by \( g \in G(\mathbb{Q}) \). This observation will be important in the next subsection.

3.3. Arrangements of flats and proof of Theorem 1(b). In this section we denote flats (previously \( X_1 \)) by \( F \subset X \) and “hyperplanes” (previously \( X_2 \)) by \( H \subset X \). (Calling \( H \) an hyperplane is misleading since its codimension is \( p \). However, \( H \) is the group preserving a hyperplane \( P \subset V \), so in that sense the name is perhaps reasonable.) When we write \( V = P \oplus L \) and \( V = L_1 \oplus \cdots \oplus L_p \oplus N \), we always assume the subspaces are as in (5).

We will assume \( 2 \leq p \leq q \). The case \( p = 1 \) (i.e. \( X \) is hyperbolic space) is easy. In this section we have three goals: (i) explain when a hyperplane and a flat intersect by interpreting \( X \) as a Grassmannian; (ii) for each \( N \geq 1 \), find collections \( \{F_i\}^N_1 \) and \( \{H_i\}^N_1 \) of flats and hyperplanes defined over \( \mathbb{R} \) so that the intersection matrix \((F_i \cdot H_j)\) is invertible; (iii) explain why we can perturb the flats (resp. hyperplanes) so that they descend to compact (resp. properly embedded) submanifolds of \( Y = \Gamma \backslash X \).

Intersecting flats and hyperplanes. We begin by interpreting flats and hyperplanes under the identification \( X \simeq \text{Gr}_p(V) \), where \( \text{Gr}_p(V) \) is the Grassmannian of positive-definite \( p \)-planes in \( V \). With this interpretation, if \( H \subset X \) is a hyperplane corresponding to a decomposition \( \mathbb{R}^{p,q} = P \oplus L \), then in \( \text{Gr}_p(V) \),

\[
H \simeq \{V' \in \text{Gr}_p(V) : V' \subset P\}.
\]

Similarly, for each flat \( F \subset X \) there is a decomposition \( V = U_1 \oplus \cdots \oplus U_p \oplus N \), and in \( \text{Gr}_p(V) \)

\[
F \simeq \{V' \in \text{Gr}_p(\mathbb{R}^{p,q}) : V' = \oplus_{i=1}^p V' \cap U_i\}
\]

One can see that \( F \simeq \mathbb{R}^p \) as follows. By the assumption, \( V' \cap U_i \) is a positive line for each \( i \). The space of positive lines in \( \mathbb{R}^{1,1} \) is homeomorphic to \( \mathbb{R} \). As one varies the choice of \( V' \cap U_i \) for each \( i \), one gets a subspace of \( \text{Gr}_p(V) \) homeomorphic to \( \mathbb{R}^p \).
Lemma 7 (Intersecting flats and hyperplanes). Let $F, H \subset X$ be a flat and a hyperplane, corresponding to decompositions $V = P \oplus L$ and $V = U_1 \oplus \cdots \oplus U_p \oplus N$ that are in general position. Then $H \cap F \neq \emptyset$ if and only if $P \cap U_i$ is a positive line for each $1 \leq i \leq p$.

Proof. Suppose $V' \subset H \cap F$. Then $V' \subset P$ and $V' \cap U_i$ is a positive line for each $i$. Since the decompositions are in general position, $\dim P \cap U_i = 1$, and so $P \cap U_i = V' \cap U_i$. Thus $V'$ exists if and only if $P \cap U_i$ is a positive line for each $i$. □

A good arrangement. For each $n \geq 1$, we construct a sequence of hyperplanes $\{H_i\}_{i=1}^n$ and flats $\{F_k\}_{k=1}^n$ defined over $\mathbb{R}$ so that the intersection matrix $(H \cdot F_k)$ is invertible. The integral structure $\Lambda \subset V$ will not play a role, so we will identify $V \cong \mathbb{R}^{p,q}$ with standard basis of orthogonal vectors $\mathbb{R}^{p,q} = \langle e_1, \ldots, e_p, f_1, \ldots, f_q \rangle$ with $e_i^2 = 1$ and $f_j^2 = -1$.

To begin, let $F_0$ be the flat corresponding to $\mathbb{R}^{p,q} = \langle e_1, f_1 \rangle \oplus \cdots \oplus \langle e_p, f_p \rangle \oplus \langle f_{p+1}, \ldots, f_q \rangle =: U_1 \oplus \cdots \oplus U_p \oplus N$. Next we define a hyperplane $H_0$. First let $\phi : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ be an automorphism that acts by the identity on $\langle e_1, f_1 \rangle^\perp$, and whose restriction to $\langle e_1, f_1 \rangle$ expands $\langle e_1 + f_1 \rangle$ and contracts $\langle e_1 - f_1 \rangle$. For each $m \geq 0$, define $a_m, b_m$ by $\phi^m(e_1) = a_m e_1 + b_m f_1$. Then $\phi^m(f_1) = b_m e_1 + a_m f_1$. We will also use the shorthand $e_1^m := \phi^m(e_1)$ and $f_1^m := \phi^m(f_1)$. By definition, $a_m^2 - b_m^2 = 1$ for each $m$ (hence $a_m > b_m$), and $a_m, b_m \to \infty$ and $\frac{a_m}{b_m} \to 1$ as $m \to \infty$. Fix $m \gg 0$ (to be chosen later, depending on $n$). Let $H_0$ be the hyperplane defined by the decomposition $\mathbb{R}^{p,q} = P \oplus L$, where

$$L = \langle f_1^m + f_2 + \cdots + f_p \rangle,$$

and

$$P = L^\perp = \langle e_1^m, e_2, \ldots, e_p, f_1^m - f_2, f_3 - f_p, \ldots, f_{p-1} - f_p, f_{p+1}, \ldots, f_q \rangle.$$}

Define flats $F_k$ for $k \geq 1$ by rotating $F_0$ as follows. Fix $-1 \leq \theta < 0$ (to be chosen later, depending on $n$). Let $r : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ be the rotation that is the identity on $\langle e_1, e_2, f_1, f_2 \rangle^\perp$ and restricts to each of $\langle e_1, e_2 \rangle$ and $\langle f_1, f_2 \rangle$ as a counter-clockwise rotation of angle $\theta$. Note $r \in \text{SO}(p) \times \text{SO}(q)$ (note also that to define $r$ we have used $p, q \geq 2$). For each $k \geq 1$, define $F_k = r^k(F_0)$.

Lemma 8. The intersection $H_0 \cap F_0$ is nonempty. For $k \geq 1$, if

$$(6) \quad -(a_m + b_m) \leq \tan(k\theta) \leq -(a_m - b_m),$$

then $H_0 \cap F_k = \emptyset$.

Before we prove Lemma 8 we show that it allows us to find a desired arrangement of $H_\ell, F_k$.

Observe that for each $n \geq 1$, we can choose $m \gg 0$ and $-\frac{\pi}{4} \leq \theta < 0$ so that (6) is true for $k = 1, \ldots, n$. Thus for $k = 0, \ldots, n$, we have $H_0 \cap F_k \neq \emptyset$ if and only if $k = 0$. Now define $H_\ell = r^\ell(H_0)$. If $k \geq \ell$, then

$$H_\ell \cap F_k \neq \emptyset \iff r^\ell(H_0) \cap r^k(F_0) \neq \emptyset \iff H_0 \cap r^{\ell-k}(F_0) \neq \emptyset \iff k = \ell.$$}

Consequently, the intersection matrix $(H_\ell \cdot F_k)$ is lower triangular with 1’s on the diagonal. This matrix is invertible, as desired.

Proof of Lemma 8. The first statement is easy: $H_0$ and $F_0$ intersect in the $p$-plane $V = \langle e_1, \ldots, e_p \rangle$.

Now we prove the second statement. The flat $F_k$ corresponds to the decomposition $\mathbb{R}^{p,q} = U_1^{k} \oplus U_2^{k} \oplus U_3 \oplus \cdots \oplus U_p \oplus N$.
where \( U_i^k = r^k(U_i) \subset \mathbb{R}^{p,q} \) for \( i = 1, 2 \). Note for \( i = 1, 2 \) that \( U_i^k \) is spanned by \( r^k(e_i), r^k(f_i) \), and 
\[
r^k(e_1) = \cos(k\theta)e_1 + \sin(k\theta)e_2 \quad \text{and} \quad r^k(e_2) = -\sin(k\theta)e_1 + \cos(k\theta)e_2,
\]
and the same formulas hold when \( e_1, e_2 \) are replaced by \( f_1, f_2 \).

We will compute \( P \cap U_1^k \) and see under what conditions the intersection is a positive line. If \( v \in P \cap U_1^k \), then we can write
\[
(7) \quad v = A_1(a_m e_1 + b_m f_1) + A_2 e_2 + B_1(b_m e_1 + a_m f_1 - f_2) + A_3 e_3 + \cdots + A_p e_p + B_2(f_2 - f_3) + \cdots + B_{p-1}(f_{p-1} - f_p) + B_{p+1}f_{p+1} + \cdots + B_q f_q
\]
and also
\[
(8) \quad v = X(\cos(k\theta)e_1 + \sin(k\theta)e_2) + Y(\cos(k\theta)f_1 + \sin(k\theta)f_2).
\]

Since the coefficients on \( e_3, \ldots, e_p \) and \( f_3, \ldots, f_q \) are zero in \( (8) \). \( A_i \neq 0 \) for \( i \geq 3 \) and \( B_j \neq 0 \) for \( j \geq 2 \). Then setting equations \((7)\) and \((8)\) equal (and changing notation on the coefficients slightly),
\[
(A a_m + B b_m)e_1 + (A b_m + B a_m)f_1 + C e_2 - B f_2 = X \cos(k\theta)e_1 + Y \cos(k\theta)f_1 + X \sin(k\theta)e_2 + Y \sin(k\theta)f_2.
\]

We can simplify the corresponding system of equations to
\[
X \cos(k\theta)b_m + Y \sin(k\theta)t_m^2 = Y \cos(k\theta)a_m + Y \sin(k\theta)a_m^2,
\]
so that
\[
X = \left[ \frac{a_m}{b_m} + \frac{1}{b_m} \frac{\tan(k\theta)}{\sin(k\theta)} \right] Y.
\]

We want to know if \( X^2 - Y^2 \) is positive or negative. Since \( X^2 - Y^2 = \left( \left[ \frac{a_m}{b_m} + \frac{\tan(k\theta)}{\sin(k\theta)} \right]^2 - 1 \right) Y^2 \),
\[
\text{this is nonpositive if and only if }
\frac{-1}{b_m} \leq \frac{a_m}{b_m} + \frac{\tan(k\theta)}{\sin(k\theta)} \leq 1.
\]

This inequality is equivalent to \((6)\). If it holds, then \( H_0 \cap F_k = \emptyset \) by Lemma 7. This completes the proof.

**Cocompact flats and rational hyperplanes.** Now we explain how any flat/hyperplane in \( X \) can be perturbed to one that descends to a properly immersed submanifold of \( Y = \Gamma \backslash X \). This will allow us to perturb the arrangement constructed above to an arrangement that descends to \( Y \).

**Rational hyperplanes.** We say a hyperplane \( H \subset X \) is **rational** or **defined over** \( \mathbb{Q} \) if the line \( L \) in the corresponding decomposition \( V = P \oplus L \) is defined over \( \mathbb{Q} \) (equivalently, \( L \) is spanned by an integral vector \( \lambda \in \Lambda \)). In this case, the subgroup of \( G \) that preserves the decomposition \( P \oplus L \) is defined over \( \mathbb{Q} \). Furthermore, since the \( G(\mathbb{Q}) \) orbit of a negative rational line is dense in the space of all negative lines in \( V \), any hyperplane \( H \subset X \) can be approximated by a rational hyperplane (one way to say this: for any neighborhood \( \Omega \) of \( \partial H \) in the visual boundary \( \partial X \), there exists a rational hyperplane \( H' \) so that \( \partial H' \subset \Omega \)).

**Rational flats.** We say a flat \( F \subset X \) is **rational** if its stabilizer is defined over \( \mathbb{Q} \). In this case, it descends to a properly embedded submanifold of \( Y = X/\Gamma \). The condition that \( F \) is rational is not enough for the quotient in \( Y \) to be compact. However, by \([PR72]\), there exists \( \tau \in G(\mathbb{Q}) \) whose centralizer \( C_\tau(\mathbb{R}) \) is a Cartan subgroup and \( C_\tau(\mathbb{R})/(\Gamma \cap C_\tau(\mathbb{R})) \) is compact. The element \( \tau \) will preserve some decomposition \( \mathbb{R}^{p,q} = U_{\tau,1} \oplus \cdots \oplus U_{\tau,p} \oplus N_\tau \). The \( G(\mathbb{Q}) \) orbit of \( (U_{\tau,1}, \ldots, U_{\tau,p}) \) in the space of all \( p \)-tuples \((U_1, \ldots, U_p)\) of orthogonal subspaces \( U_i \cong \mathbb{R}^{1,1} \hookrightarrow \mathbb{R}^{p,q} \) is dense (because \( G(\mathbb{R}) \) acts transitively on such tuples and \( G(\mathbb{Q}) \subset G(\mathbb{R}) \) is dense). Thus any flat \( F \subset X \) can be approximated by a rational flat \( F' \) that is compact in the quotient \( X/\Gamma \).
In summary, to prove Theorem 1(b), given $N \geq 1$, we start with the arrangement $\{F_k\}_1^N$ and $\{H_k\}_1^N$ of flats and hyperplanes in $X$ with the lower-triangular intersection pattern. Let $\bar{F}_k$ and $\bar{H}_k$ be the images of these submanifolds in $Y = X/\Gamma$. First we perturb to get a new arrangement of rational flats and hyperplanes with the same intersection pattern so that each $\bar{F}_k$ is compact and each $\bar{H}_k$ is properly immersed in $Y$. By replacing $\Gamma$ by a finite-index subgroup, we can ensure that $\bar{F}_k$ and $\bar{H}_k$ are oriented, embedded submanifolds (Theorem 3). Next we apply Proposition 5 to each pair $(\bar{F}_k, \bar{H}_k)$ (and the corresponding subgroups $G_{\bar{F}_k}, G_{\bar{H}_k} < G$) to conclude that after replacing $\Gamma$ by yet another finite-index subgroup, we can ensure that every $\gamma \in I(\Gamma)$ belongs to $G_{\bar{F}_k}(\mathbb{R})G_{\bar{H}_k}(\mathbb{R})$. Then by Remark 6, $\bar{H}_k$ and $\bar{F}_k$ intersect if and only if $H_k$ and $F_k$ intersect, i.e. the intersection matrix $(\bar{H}_k \cdot \bar{F}_k)$ is also lower-triangular. Finally, we can pass to a further finite-index subgroup so that the diagonal entries in the intersection matrix are all positive (this is true by the computation at the end of 3.2, note that the way $H_0, F_0$ are chosen is compatible with the restriction on $(G_1, G_2)$ imposed in 3.2). Therefore, $(\bar{H}_k \cdot \bar{F}_k)$ is invertible, which implies that the homology classes $[\bar{F}_1], \ldots, [\bar{F}_N]$ are linearly independent in $H_p(Y; \mathbb{Q})$. This proves Theorem 1(b).

3.4. Remarks. We finish with some remarks and comments about further directions. We will use the notation of 2.1

1) In Millson–Raghunathan (and in many of the subsequent papers, e.g. [RS03, LSB6]) the subspaces $X_1, X_2 \subset X$ are chosen to intersect orthogonally. Millson–Raghunathan motivate this choice by observing that for a totally geodesic submanifolds $X_2 \subset X$, there exists a totally geodesic $X_1 \subset X$ of complementary dimension that meets $X_2$ orthogonally if and only if $X_2 = X^g$ for some (rational) involution on $\sigma : X \to X$.

In our case $G = SO(p,q)$ with $G_2 \simeq SO(p,q-1)$, where

$$B_{p,q} = \begin{pmatrix} I_p & -I_q \\ \end{pmatrix}$$

the space $X_2 \subset X$ is the fixed set of an involutive isometry of $X$, and the orthogonal complement $X_1$ is isometric to hyperbolic space $\mathbb{H}^p$. In this case, both $Y_1, Y_2$ are noncompact in $Y$, so $Y_1$ is unsuitable for showing that $Y_2$ gives a nontrivial homology cycle. However, one can instead choose $X_1 \simeq \mathbb{H}^p$ a maximal flat, so that the quotient $Y_1$ is compact. This was first observed in the case of $SL_n(\mathbb{Z})$ by Avramidi–Nguyen-Phan [ANP15].

2) Even if one allows the subspaces $X_1, X_2 \subset X$ to meet non-orthogonally, one still runs into the same orientability issues that are present in [MRS03]. For $p \leq q$ [MRS03] finds cocompact, arithmetic $\Gamma < SO(p,q)$ with $H^p(\Gamma; \mathbb{Q}) \neq 0$ when $p$ is even. In their case $Y_1$ and $Y_2$ are locally symmetric spaces for $SO(p,1)$ and $SO(p,q-1)$, respectively. The fact that their result works for $p$ even, while our result works for $p$ even can be traced to the difference in the adjoint action of $K = SO(p) \times O(q)$ on the Lie algebra for $SO(p,1) < SO(p,q)$ versus the Lie algebra for a maximal torus in $SO(p,q)$. It may be possible to improve the aforementioned result of [MRS03] to the case $p$ is odd by using maximal flats as in our argument above.

In a similar vein, it would be nice to improve Theorem 1 to the case $p$ is even. In the case of $SO(2,2)$ one is aided by an accidental isomorphism $SO(2,2) \simeq SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

4) For $G = Sp_{2g}$ one can try to produce nontrivial flat cycles in the homology of $\Gamma < Sp_{2g}(\mathbb{Z})$. A maximal $\mathbb{Q}$-split torus $G_1 = (G_m)^g$ gives a subspace $X_1 \subset X$ of dimension $g$, and can be chosen so that $X_1$ descends to a compact oriented submanifold $Y_1 \subset Y$ (covered by a torus). It is not clear from the Millson–Raghunathan method if $[Y_1] \neq 0$ in $H_q(Y; \mathbb{Q})$. The first problem is finding a submanifold of $X$ of complementary dimension – the dimension of $X$ is $g^2 + g$ and there is no obvious sub-symmetric space of dimension $g^2$. Instead of flat cycles, one could consider $G_2 < G$ with $G_2(\mathbb{R}) \simeq Sp_{2(g-1)}(\mathbb{R}) \times Sp_2(\mathbb{R})$ (this is the analogue of $SO(p, q-1) < SO(p, q)$). The corresponding
symmetric subspace $X_2 \subset X$ has dimension $g^2 - g + 2$. A good candidate for a complementary subspace is a symmetric space for $SU(g-1,1) < Sp_{2g}(\mathbb{R})$. Unfortunately, in this case it seems both $Y_1$ and $Y_2$ will be noncompact.

4. Vector bundles with arithmetic structure group

By Corollary [2] the classes produced in Theorem [1] give rise to characteristic classes of manifold bundles $W_g \to E \to B$ with fiber $W_g = \#_g(S^{2k} \times S^{2k})$. In this section we explain what these characteristic classes measure. This gives a new perspective on the Millson–Raghunathan construction. This will play a role in §4.

Before we begin, we recall the classification of lattices $\Lambda$ bundles for some $\ell \Bbb{R}$ but it is helpful to have these examples in mind. If the form is odd, then there exists a basis for $\Lambda$, $\text{bilinear form};$ see e.g. [MH73, Ch. II, §2] as in [3]. If the form is even, then $q = p + 8\ell$ for some $\ell \geq 0$ and $\Lambda$ is isomorphic to $H^{\oplus p} \oplus (-E_8)^{\oplus \ell}$, where

$$H = (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$$

and $E_8$ is the unique positive-definite, even, unimodular lattice of rank 8.

4.1. Vector bundles with structure group $SO(\Lambda) < SO(p,q)$. Fix $1 \leq p \leq q$ and set $n = p + q$. Fix a lattice $\Lambda \simeq \mathbb{Z}^n$ with an integral, unimodular, indefinite bilinear form $\beta$. Fix a primitive vector $\lambda \in \Lambda$ such that $\lambda \cdot \lambda < 0$. Set $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. The goal of this section is to construct a characteristic class $c_\lambda \in H^p(B\Gamma; \mathbb{Q})$ for certain $\Gamma < SO(\Lambda)$ and show that $c_\lambda$ is dual to a geometric cycle $[Y_2]$ as in $[3]$.

Let $W \to B$ be a oriented, real vector bundle with rank $n$. Let $W_b$ denote the fiber over $b \in B$. Assume that the structure group reduces from $GL^+(V)$ to $SO(V)$. This is equivalent to the existence of a fiberwise bilinear form $\beta = \{\beta_b\}_{b \in B}$ of signature $(p,q)$. We can always reduce the structure group from $SO(V)$ to its maximal compact subgroup $K \simeq SO(p) \times SO(q)$ (because they are homotopy equivalent and so are their classifying spaces). Such a reduction defines a decomposition $W \simeq U \oplus U^\perp$, where $U = \bigcup_{b \in B} U_b$ is a rank-$p$ subbundle and $\beta_b : U_b \times U_b \to \mathbb{R}$ is positive definite for each $b$. Conversely, any positive rank-$p$ subbundle $U \subset W$ defines a reduction of the structure group to $SO(p) \times SO(q)$. The structure group of $W \to B$ reduces to $SO(\Lambda)$ if and only if if there exists a fiberwise lattice $\Delta = \bigcup_{b \in B} \Lambda_b \subset W$ where $\mathbb{Z}^n \simeq \Lambda_b \subset W_b \simeq \mathbb{R}^n$ is a lattice for each $b \in B$ and the restriction of $\beta_b$ to $\Lambda_b$ takes values in $\mathbb{Z}$.

**Definition.** Fix $\Delta, V$ and $\lambda \in \Lambda$ as above. We say that a positive rank-$p$ subbundle $U \subset W$ is orthogonal to $\lambda$ at $b \in B$ if there exists an isometry $\phi : \Lambda \to \Lambda_b$ so that $U_b \subset \phi(\lambda)^\perp$. If $U \subset W$ is not orthogonal to $\lambda$ at any $b \in B$, then we say $U$ is nowhere orthogonal to $\lambda$.

The characteristic class we define will be an obstruction to finding $U \subset W$ that is nowhere orthogonal to $\lambda$. We want to translate the problem of finding $U$ to a problem about finding a section of an associated bundle.

Set $\pi = \pi_1(B)$. Let $\rho : \pi \to SO(\Lambda)$ be the monodromy of $W \to B$. Consider the symmetric space $X = K \setminus SO(V)$, which is homeomorphic to the Grassmannian

$$\text{Gr}_p(V) = \{ V' \subset V : V' \text{ is positive definite and } \dim V' = p \}.$$

To see this, note that $SO(V)$ acts transitively on $\text{Gr}_p(V)$ and the stabilizer of a point is isomorphic to $K$. 

The group $\pi$ acts on $X \cong \text{Gr}_p(V)$ via the monodromy $\rho$. For a space $Z$ with a $\pi$-action, we denote the Borel construction $Z \// \pi := \tilde{B} \times Z_\pi$, where $\pi$ acts on the universal cover $\tilde{B}$ by deck transformations and $\pi$ acts on $Z$ by the given $\pi$-action, and the quotient is by the diagonal action. For any such $Z$, there is a fibration $Z \// \pi \to * \to \pi = B$ with fiber $Z$.

Observe that for $W \to B$ with monodromy $\rho : \pi \to \text{SO}(\Lambda)$, a section of the associated bundle $X \// \pi \to B$ is equivalent to a positive rank-$p$ subbundle $U \subset W$.

Let $H_\lambda = \{ V' \in \text{Gr}_p(V) : V' \subset \lambda^+ \} \subset X$. This is the sub-symmetric space corresponding to the subgroup $\text{SO}(\lambda^+) < \text{SO}(V)$. The codimension of $H_\lambda$ in $X$ is $p$.

By [Sch10, Thm. D], there exists a torsion-free, finite-index subgroup $\Gamma_\lambda < \text{SO}(\Lambda)$ so that the $\Gamma_\lambda$-orbit of $H$ is embedded and admits a $\Gamma_\lambda$-invariant orientation. (The group $\Gamma_\lambda$ is not uniquely defined by these properties, e.g. for every prime $\ell$, there exists $m > 0$ so that the congruence subgroup $\ker[\text{SO}(\Lambda) \to \text{SO}(\Lambda/\ell^m\Lambda)]$ satisfies these properties. The construction below works for any choice of $\Gamma_\lambda$.)

Fix a finite index subgroup $\Gamma < \Gamma_\lambda$, and let $H_{\lambda, \Gamma}$ be the $\Gamma$-orbit of $H_\lambda$ in $X$. By replacing $B$ by a finite cover, we can ensure that $\rho(\pi) < \Gamma$. Set $X_0 = X \setminus H_{\lambda, \Gamma}$ and consider the bundle $X_0 \// \pi \to B$. If $W \to B$ has a positive rank-$p$ subbundle $U \subset W$ that is nowhere orthogonal to $\lambda$, then $X_0 \// \pi \to B$ has a continuous section. Now we can use obstruction theory to extract a characteristic class from this situation. For this, we need to know the first nontrivial homotopy group of $X_0$.

**Lemma 9.** Fix $k \geq 0$. If $k \leq p - 2$, then $\pi_k(X_0) = 0$. Furthermore, $\pi_{p-1}(X_0) \cong \bigoplus_{\pi_0(H_{\lambda, \Gamma})} \mathbb{Z}$.

**Proof.** First assume $k \leq p - 2$. We show any map $S^k \to X_0$ is homotopically trivial. Since $X \cong \mathbb{R}^m$ is contractible, we obtain a diagram

$$
\begin{array}{ccc}
S^k & \overset{i}{\longrightarrow} & X_0 \\
\downarrow & & \downarrow \\
D^{k+1} & \overset{j}{\longrightarrow} & X
\end{array}
$$

(11)

Without loss of generality we may assume that $i$ and $j$ are smooth and $j$ is transverse to $H_{\lambda, \Gamma}$. Since $k + 1 \leq p - 1$ and the codimension of $H_{\lambda, \Gamma}$ is $p$, if $D$ is transverse to $H_{\lambda, \Gamma}$, then $D \cap H_{\lambda, \Gamma} = \emptyset$, which shows $i$ is homotopically trivial in $X_0$.

By the Hurewicz theorem, $\pi_{p-1}(X_0) \cong H_{p-1}(X_0)$. Define a homomorphism $\phi : \pi_{p-1}(X_0) \cong H_{p-1}(X_0) \to \bigoplus_{\pi_0(H_{\lambda, \Gamma})} \mathbb{Z}$ as follows. Choose an orientation on each component of $H_{\lambda, \Gamma}$. Given $i : S^{p-1} \to X_0$, extend to $D^p \to X$ transverse to $H_{\lambda, \Gamma}$, and compute the algebraic intersection of $D^p$ with each component of $H_{\lambda, \Gamma}$.

The map $\phi$ is obviously surjective: for each component of $H_{\lambda, \Gamma}$, one can choose a $(p-1)$-sphere in its link, and the image of these generate $\bigoplus_{\pi_0(H_{\lambda, \Gamma})} \mathbb{Z}$. For injectivity, it is well-known that if $D, H$ are oriented submanifolds of an oriented manifold $X$ that intersect transversely in a finite collection of points and their algebraic intersection number is 0, then $D$ can be replaced by a homologous submanifold $D'$ with $\partial D = \partial D'$ so that $D' \cap H = \emptyset$. This shows that if $[S^{p-1} \to X_0]$ is in the kernel of $\phi$, then $[S^{p-1} \to X_0] = 0$ in $H_{p-1}(X_0)$. \hfill \Box

Applying obstruction theory (see e.g. [Sco05, pg. 197]), we can try to build a section of $X_0 \// \pi \to B$. Assume that $B$ is a CW complex. We start by choosing a section over the 0-skeleton of $B$ and work our way up inductively defining a section on the $k$-skeleton for $k \leq p - 1$ using the fact that $\pi_{k-1}(X_0) = 0$ for $k \leq p - 1$. Once we reach the $p$-skeleton we meet the first measurable obstruction, which takes the form of a cocycle $C_{\lambda, \Gamma}(W) \in H^p(B; \pi_{p-1}(X_0))$. If $C_{\lambda, \Gamma}(W) \neq 0$, then $X_0 \// \pi \to B$ has no continuous section, and so $W \to B$ does not have a positive rank-$p$ subbundle.
U that is nowhere orthogonal to \( \lambda \). This is useful, but we are interested in a less-refined, \( \mathbb{Z} \)-valued obstruction.

Since \( H_{\lambda, \Gamma} \) has a \( \Gamma \)-invariant orientation, there is a preferred generator of each coordinate of \( \bigoplus_{\rho_0(H_{\lambda, \Gamma})} \mathbb{Z} \). We use this to define an augmentation map \( \bigoplus_{\rho_0(H_{\lambda, \Gamma})} \mathbb{Z} \to \mathbb{Z} \). The augmentation map induces a map \( H^p(B; \pi_{p-1}(X_0)) \to H^p(B; \mathbb{Z}) \), which sends \( c_{\lambda, \Gamma}(W) \) to a class \( c_{\lambda, \Gamma}(W) \in H^p(B; \mathbb{Z}) \).

**Proposition 10.** Fix \( \Lambda, V, \lambda \in \Lambda \), and \( \Gamma_\lambda < \text{SO}(\Lambda) \) as above. Let \( B \) be a CW complex and let \( W \to B \) be a vector bundle with structure group \( \Gamma < \Gamma_\lambda \). If \( c_{\lambda, \Gamma}(W) \neq 0 \) in \( H^p(B; \mathbb{Z}) \), then \( W \to B \) has no positive, rank-\( p \) subbundle \( U \subset W \) that is nowhere orthogonal to \( \lambda \). Equivalently, for every positive, rank-\( p \) subbundle \( U \subset W \), there exists \( b \in B \) so that \( U \) is orthogonal to \( \lambda \) at \( b \).

If \( B \) is a closed, oriented \( p \)-manifold, then we can evaluate \( c_{\lambda, \Gamma}(W) \in H^p(B; \mathbb{Z}) \) on the fundamental class to get an integer \( \langle c_{\lambda, \Gamma}(W), [B] \rangle \in \mathbb{Z} \), which is computed as follows. We have a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X/\Gamma \\
\downarrow^{u} & & \downarrow_{\rho} \\
B & \xrightarrow{\delta} & \bar{X}/\Gamma
\end{array}
\]

Here \( u \) is a section corresponding to a positive, rank-\( p \) subbundle \( U \subset W \), and the map \( p \) is the composition \( X/\pi = \bar{B} \times X \to X/\rho(\pi) \to X/\Gamma \) (the first map collapses \( \bar{B} \) to a point). Let \( \bar{H}_{\lambda, \Gamma} \) be the image of \( H_\lambda \) in \( X/\Gamma \). By our choice of \( \Gamma_\lambda \) and the assumption \( \Gamma < \Gamma_\lambda \), the inclusion \( \bar{H}_{\lambda, \Gamma} \hookrightarrow X/\Gamma \) is a proper embedding, c.f. [Sch10] Thm. D]. Now tracing through the definitions, one finds that \( \langle c_{\lambda, \Gamma}(W), [B] \rangle \) is equal to the algebraic intersection number of \( p \circ u(B) \) with \( \bar{H}_{\lambda, \Gamma} \) in \( X/\Gamma \).

Applying the above construction to the universal bundle over \( B\Gamma \), we see that \( c_{\lambda, \Gamma} \in H^p(B\Gamma) \simeq H^p(X/\Gamma) \) is dual to the cycle \( \bar{H}_{\lambda, \Gamma} \), which is a locally symmetric space for a nonuniform lattice in \( \text{SO}(\lambda^\perp) \simeq \text{SO}(p, q-1) \). In §3, we showed that there exists \( \Gamma < \Gamma_\lambda \) so that \( [\bar{H}_{\lambda, \Gamma}] \in H^p_{pq-p}(X/\Gamma) \) is nonzero. Then \( c_{\lambda, \Gamma} \) is also nontrivial.

4.2. **Vector bundles with structure group** \( \text{SL}_n(\mathbb{Z}) \). The construction of the previous section can be repeated in other situations. Here we remark on a version for vector bundles with structure group \( \text{SL}_n(\mathbb{Z}) \). We will use this in §5 to give an application similar to Corollary 2 to odd-dimensional manifolds.

Fix the standard lattice \( \mathbb{Z}^n < \mathbb{R}^n \). Let \( \delta = (P, L) \) denote be a pair of subspaces of \( \mathbb{R}^n \) defined over \( \mathbb{Q} \) such that \( \mathbb{R}^n = P \oplus L \) and \( \dim L = 1 \). For every such \( \delta \), we will associate a finite index subgroup \( \Gamma_\delta < \text{SL}_n(\mathbb{Z}) \) and for every \( \Gamma \subset \Gamma_\delta \) we will define a characteristic class \( c_{\delta, \Gamma} \in H^{n-1}(\Gamma ; \mathbb{Z}) \) for real vector bundles \( W \to B \) with structure group in \( \Gamma \).

Suppose \( W \to B \) is a real oriented vector bundle of rank \( n \). The structure group reduces from \( \text{GL}_n^+(\mathbb{R}) \) to \( \text{SL}_n(\mathbb{Z}) \) if and only if \( W \) admits a fiberwise lattice \( \Lambda \). A reduction of the structure group from \( \text{GL}_n^+(\mathbb{R}) \) to its maximal compact \( \text{SO}(n) \) corresponds to a fiberwise inner product \( \beta \) on \( W \).

**Definition.** Fix \( \delta = (P, L) \) and \( W \to B \) and \( \Lambda \subset W \) as above. For a fiberwise inner product \( \beta \), we say that a \( (P, L) \) is \( \beta \)-orthogonal at \( b \in B \) if there exists an isomorphism \( \phi : (\mathbb{R}^n, \mathbb{Z}^n) \to (W_b, \Lambda_b) \) so that \( \phi(P) \) and \( \phi(L) \) are orthogonal with respect to \( \beta_b \). If \( (P, L) \) is not \( \beta \)-orthogonal at any \( b \in B \), we say \( (P, L) \) is nowhere \( \beta \)-orthogonal.

We can translate the problem of finding an inner product \( \beta \) so that \( (P, L) \) is nowhere \( \beta \)-orthogonal to a problem of finding a section of an associated bundle. Let \( X = \text{SO}(n) \setminus \text{SL}_n(\mathbb{R}) \). This symmetric
space can be identified with the space of unit volume inner products on \( \mathbb{R}^n \). There is a bijective correspondence between fiberwise inner products \( \beta \) on \( W \rightarrow B \) and sections of \( X \parallel \pi \rightarrow B \), where \( \pi = \pi_1(B) \) acts on \( X \) via the monodromy \( \rho : \pi \rightarrow \text{SL}_n(\mathbb{Z}) \).

Consider the submanifold \( H_\delta = \{ \text{inner products such that } \mathbb{R}^n = P \oplus L \text{ is orthogonal} \} \subset X \), which is a sub-symmetric space for \( \text{SL}_{n-1}(\mathbb{R}) \times \mathbb{R} \). By [Sch10, Thm. D], we can find a torsion-free subgroup \( \Gamma_\delta < \text{SL}_n(\mathbb{Z}) \) so that the \( \Gamma_\delta \)-orbit of \( H_\delta \) in \( X \) is embedded and has a \( \Gamma_\delta \)-invariant orientation. Fix a finite-index subgroup \( \Gamma < \Gamma_\delta \). Denote the \( \Gamma \) orbit of \( H_\delta \) in \( X \) by \( H_{\delta, \Gamma} \), and set \( X_0 = X \setminus H_{\delta, \Gamma} \).

We replace \( \pi \) with \( \pi_n \) and denote the \( \pi \)-fiber above \( X \). Assume that \( \pi_0 \) is nowhere \( \beta \)-orthogonal. We replace \( B \) with a finite cover so that the monodromy \( \rho : \pi \rightarrow \text{SL}_n(\mathbb{Z}) \) factors through \( \Gamma \).

If \( W \rightarrow B \) admits an inner product \( \beta \) so that \( (P, L) \) is nowhere \( \beta \)-orthogonal, then \( X_0 \parallel \pi \rightarrow B \) admits a continuous section. Similar to Lemma 9, we compute \( \pi_k(X_0) = 0 \) for \( k \leq n - 3 \) and \( \pi_{n-2}(X_0) \cong \bigoplus \pi_0(H_{\delta, \Gamma}) \mathbb{Z} \). Then there is an obstruction class \( c_{\delta, \Gamma}(W) \in H^{n-1}(B; \pi_{n-2}(X_0)) \), which maps to a class \( c_{\delta, \Gamma}(W) \in H^{n-1}(B; \mathbb{Z}) \) under the map induced by the augmentation \( \bigoplus \pi_0(H_{\delta, \Gamma}) \mathbb{Z} \rightarrow \mathbb{Z} \).

We summarize the above discussion with the following proposition.

**Proposition 11.** Fix \( \delta = (P, L) \) and \( \Gamma_\delta < \text{SL}_n(\mathbb{Z}) \) as above. Let \( B \) be a CW complex and let \( W \rightarrow B \) be a vector bundle with structure group \( \Gamma < \Gamma_\delta \). If \( c_{\delta, \Gamma}(W) \neq 0 \) in \( H^{n-1}(B; \mathbb{Z}) \), then \( W \rightarrow B \) does not admit an inner product \( \beta \) so that \( (P, L) \) is nowhere \( \beta \)-orthogonal. Equivalently, for every inner product \( \beta \) on \( W \) there exists \( b \in B \) so that \( (P, L) \) is \( \beta \)-orthogonal at \( b \).

The class \( c_{\delta, \Gamma} \in H^{n-1}(B; \mathbb{Z}) \cong H^{n-1}(X/\Gamma) \cong H^{\delta}_{(n^2-n)/2}(X/\Gamma) \) is dual to the cycle \( [\tilde{H}_{\delta, \Gamma}] \in H^{\delta}_{(n^2-n)/2}(X/\Gamma) \) represented by the image of \( H_\delta \) in \( X/\Gamma \). Compare with the discussion following (12). By a theorem of Avramidi–Nguyen-Phan [ANP15] for a subgroup \( \Gamma < \Gamma_\delta \) of sufficiently large index, the homology class \( [\tilde{H}_{\delta, \Gamma}] \in H^{\delta}_{(n^2-n)/2}(X/\Gamma; \mathbb{Q}) \) is nontrivial.

5. **Applications to manifold bundles**

In this section and the appendix, we give applications of Theorem 1.

### 5.1. 4-manifolds, K3 surfaces bundles, and the global Torelli theorem.

Let \( M \) be a closed oriented 4-manifold. As in the introduction, we use \( \Lambda_M \) to denote \( H_2(M; \mathbb{Z})/\text{torsion} \) with its intersection form. Assume that \( \Lambda_M \) is indefinite, and let \( (p, q) \) be the signature. Up to switching the orientation, we may assume \( p \leq q \). By Theorem 1 when \( p \) is odd, there exists a finite-index subgroup \( \Gamma < \text{SO}(\Lambda_M) \) so that \( H_p(\Gamma; \mathbb{Q}) \neq 0 \).

**Question 12.** Does the image of \( H_p(B \text{Diff}^+(M); \mathbb{Q}) \rightarrow H_p(B\Gamma; \mathbb{Q}) \) intersect the subspace spanned by flat cycles nontrivially?

One could ask a similar question for homeomorphisms or homotopy automorphisms. There does not seem to be a good reason for the answer to Question 12 to be “Yes”, other than the evidence provided by Corollary 18 and Theorem 21 below.

**Example.** Fix \( 1 \leq p \leq q \) and let \( M = M_{p,q} := (\#_p \mathbb{CP}^2) \# (\#_q \overline{\mathbb{CP}^2}) \). Then the form on \( \Lambda_M \) has matrix \( B_{p,q} \) (defined in [9]) and \( \alpha : \text{Diff}(M) \rightarrow O(\Lambda_M) \) is surjective if \( p \geq 2 \). This follows from [Wal64, Thm. 2], which shows that \( \alpha \) is surjective when \( M = N \# (S^2 \times S^2) \) is simply connected and \( Q_N \) is indefinite. Since \( \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \# \overline{\mathbb{CP}^2} \simeq (S^2 \times S^2) \# \overline{\mathbb{CP}^2} \) (see e.g. [Sco05, pgs. 124,151]), the hypotheses of Wall’s theorem are true for \( M_{p,q} \) when \( p \geq 2 \). This gives many concrete examples to study Question 12.
In the remainder of this section we study Question 12 in the case $M$ is a K3 surface (i.e. a smooth 4-manifold diffeomorphic to a K3 surface). Here the form on $\Lambda_M$ has matrix $H^{\oplus 3} \oplus (-E_8)^{\oplus 2}$ (the notation is explained in [10]). Then $SO(\Lambda_M)$ is a lattice in $SO(3, 19)$. We will be interested in the maps

$$H_i(B\operatorname{Diff}^\Gamma(M); \mathbb{Q}) \xrightarrow{\alpha_1} H_i(B\pi_0\operatorname{Diff}^\Gamma(M); \mathbb{Q}) \xrightarrow{\alpha_2} H_i(B\Gamma; \mathbb{Q}),$$

for $\Gamma < SO(\Lambda_M)$. By Theorem 11 we can find $\Gamma$ and $z \neq 0 \in H_3(B\Gamma; \mathbb{Q})$. We will study whether or not $z$ is in the image of $\alpha_2$ and $\alpha_1 \circ \alpha_2$.

**Theorem 13.** Let $M$ be a smooth oriented 4-manifold diffeomorphic to a K3 surface. There exists a finite-index subgroup $\Gamma'_M < SO(\Lambda_M)$ so that for each finite-index subgroup $\Gamma < \Gamma'_M$ and for each $i \geq 0$, the map $\alpha_2 : H_i(B\pi_0\operatorname{Diff}^\Gamma(M); \mathbb{Q}) \to H_i(B\Gamma; \mathbb{Q})$ is surjective.

Consequently, each flat cycle $z \neq 0 \in H_3(B\Gamma; \mathbb{Q})$ is in the image of $\alpha_2$. Theorem 13 is a corollary of the global Torelli theorem ([Loo81] and [Bes08, §12.K]) and can be deduced from the discussion in [Gia09].

**Proof of Theorem 13.** Let $\Gamma_M$ be the image of $\operatorname{Diff}(M) \to O(\Lambda_M)$. It is known [Mat86] that $\Gamma_M$ is finite index in $O(\Lambda_M)$. To prove the theorem, it suffices to show that the surjection $\operatorname{Diff}(M) \to \Gamma_M$ splits over a finite-index subgroup $\Gamma'_M$.

Let $\operatorname{Ein}(M)$ denote the space of unit-volume Einstein metrics on $M$, topologized as a subspace of all Riemannian metrics on $M$. One defines the **homotopy moduli space**

$$\mathcal{M}_{\operatorname{Ein}}(M) := \frac{\operatorname{Ein}(M) \times E\operatorname{Diff}(M)}{\operatorname{Diff}(M)},$$

where $E\operatorname{Diff}(M)$ is the total space of the universal principal $\operatorname{Diff}(M)$ bundle over $B\operatorname{Diff}(M)$. There is a composition of maps

$$\phi : \mathcal{M}_{\operatorname{Ein}}(M) \to B\operatorname{Diff}(M) \to B\Gamma_M.$$

As explained in [Gia09] §4-5), the group $\pi_1(\mathcal{M}_{\operatorname{Ein}}(M))$ is isomorphic to a finite-index subgroup $\Gamma'_M < \Gamma_M$, and $\phi$ induces the inclusion on $\pi_1$. \hfill $\square$

The last assertion in the proof will be further explained below (as part of the proof of Proposition 14).

**Remark.** Since $\alpha_2$ is surjective, any homology of a lattice $\Gamma < SO(3, 19)$ in the **stable range** is also in the image of $\alpha_2$. Switching to cohomology, the stable cohomology can be described as the cohomology that is pulled back along the map

$$f : B\Gamma \to BSO(3, 19) \sim BS(O(3) \times O(19)) \to BSO(3).$$

Compare with [Gia09] §3. Recall $H^*(BSO(3); \mathbb{Q}) \simeq \mathbb{Q}[p_1]$, where $p_1 \in H^4$ is the first Pontryagin class. According to the ranges in [Bor81], $f$ induces an $H_i(\_; \mathbb{Q})$-isomorphism for $i < 1$. Unfortunately, this does not provide nontrivial elements of $H^*(B\Gamma_M; \mathbb{Q})$. (This is incorrectly quoted in [Gia09] Prop. 3.6.)

Theorem 13 reduces Question 12 to studying the image of $\alpha_1$. The author does not know of a single nontrivial class in the image of this map (or a single class that is not in the image of this map). In studying $\alpha_1$, we will focus on a particular type of flat cycle $z$.

Set $V = \Lambda_M \otimes \mathbb{R} \simeq H_2(M; \mathbb{R})$, and let $X = SO(V)/K$ be the symmetric space for $G = SO(V)$. As discussed in [4] there is a homeomorphism $X \simeq \Gr_3(V)$. A vector $\delta \in \Lambda$ is called a **root vector** if
\[ \delta \cdot \delta = -2. \] As in [3], consider
\[ H_\delta = \{ V' \in \text{Gr}_3(V) : V' \subset \delta^1 \} \subset X. \]

Fix a root vector \( \delta \), and choose a rational flat \( F \subset X \) that intersects \( H_\delta \) transversely. (This can be done using the arguments of [3]) By the construction of Theorem [13] there exists \( \Gamma < \Gamma'_M \) so that \( F \) and \( H_\delta \) descend to homology cycles in \( Y = \Gamma \setminus X \) that pair nontrivially. In particular, we have a nonzero class \( z_0 \in H_3(\Gamma \Gamma'; \mathbb{Q}) \).

Now we discuss whether or not \( z_0 \in \text{Im}(\alpha_2 \circ \alpha_1) \). One approach to this question is to consider the map [14] from the proof of Theorem [13] For each finite-index subgroup \( \Gamma < \Gamma'_M \), define \( M^{T_\text{Ein}}(M) = \frac{T_{\text{Ein}}^0(M) \times E \Gamma}{\Gamma_M} \), where \( T_{\text{Ein}}^0(M) \) is one of the two path components of \( T_{\text{Ein}}^0(M) \) (these components are preserved by \( \Gamma'_M \)). If \( z_0 \) is in the image of \( \phi_\ast : H_3(M^{T_\text{Ein}}(M)) \to H_3(\Gamma \Gamma'; \mathbb{Q}) \), then \( z_0 \in \text{Im}(\alpha_2 \circ \alpha_1) \). Unfortunately, the following proposition shows that this approach does not work. Nevertheless, we have an interesting Corollary [15]

**Proposition 14.** Let \( M \) be a K3 surface. Fix \( \Gamma < \Gamma'_M \) and \( z_0 \in H_3(\Gamma \Gamma'; \mathbb{Q}) \) as above. The class \( z_0 \) is not in the image of \( \phi_\ast : H_3(M^{T_\text{Ein}}(M)) \to H_3(\Gamma \Gamma'; \mathbb{Q}) \).

**Corollary 15.** If \( z_0 \in H_3(\Gamma \Gamma'; \mathbb{Q}) \) is in the image of \( H_3(B \text{Diff}^G(M); \mathbb{Q}) \to H_3(\Gamma \Gamma'; \mathbb{Q}) \), then there exists a K3 bundle over a 3-manifold that does not admit any fiberwise Einstein metric.

Of course it may be the case that \( z_0 \) is not in the image of \( H_3(B \text{Diff}^G(M); \mathbb{Q}) \to H_3(\Gamma \Gamma'; \mathbb{Q}) \), in which case the corollary is vacuously true. In this situation, there is a different interesting corollary.

**Corollary 16.** Let \( M \) be a K3 surface. If there exists any flat cycle \( z \in H_3(\Gamma \Gamma'; \mathbb{Q}) \) is not in the image of \( H_3(B \text{Diff}^G(M); \mathbb{Q}) \to H_3(\Gamma \Gamma'; \mathbb{Q}) \), then the surjection \( \text{Diff}(M) \to \pi_0 \text{Diff}(M) \) is not split.

**Proof of Corollary 16** If a splitting exists, then \( H_\ast(B \text{Diff}^G(M); \mathbb{Q}) \to H_\ast(B \pi_0 \text{Diff}^G(M); \mathbb{Q}) \) would be surjective for every \( \Gamma < \Gamma_M \). Combining this with Theorem [13] then \( H_\ast(B \text{Diff}^G(M); \mathbb{Q}) \to H_\ast(B \Gamma \Gamma'; \mathbb{Q}) \) is also surjective for every \( \Gamma < \Gamma_M \). This contradictions the assumption that some flat cycle is not in the image of \( H_3(B \text{Diff}^G(M); \mathbb{Q}) \to H_3(\Gamma \Gamma'; \mathbb{Q}) \).

**Proof of Corollary 15** Suppose there is a class \( w \in H_3(B \text{Diff}^G(M); \mathbb{Q}) \) whose image in \( H_3(\Gamma \Gamma'; \mathbb{Q}) \) is \( z_0 \). Up to scaling, we can represent \( w \) by a map of a manifold \( h : B^3 \to B \text{Diff}^G(M) \). The pullback of the universal bundle by \( h \) is a K3 bundle \( M \to E \to B \). We claim it has no fiberwise Einstein metric.

The homotopy moduli space \( M_{\text{Ein}}(M) \) is a classifying space for K3 bundles with a fiberwise Einstein metric, so \( E \to B \) admits a fiberwise Einstein metric if and only if \( h \) lifts to a map \( \tilde{h} : B \to M_{\text{Ein}}(M) \). No such lift can exist by Proposition [14] Thus \( E \to B \) has no fiberwise Einstein metric.

As remarked in Theorem [13] Giansiracusa [Gia09] [4-5] proves that \( \pi_1(M_{\text{Ein}}(M)) \simeq \Gamma'_M \). We begin by explaining the proof of this fact, since it will be used to prove Proposition [14] For this, we give a fuller description of the topology of \( M_{\text{Ein}}(M) \), which is illuminated by the global Torelli theorem. For details see [Loo81], [Bes08], [12.K], and [Gia09] [4-5]. In [Gia09] [4.2], Giansiracusa shows that

\[ (15) \]
\[ M_{\text{Ein}}(M) \simeq \frac{T_{\text{Ein}}(M) \times E \Gamma_M}{\Gamma_M}, \]

where \( T_{\text{Ein}}(M) \) is the *Teichmüller space*. By definition \( T_{\text{Ein}}(M) \) is the quotient \( \text{Ein}(M)/\text{Diff}(M) \), where \( \text{Diff}(M) = \ker [ \text{Diff}(M) \to \Gamma_M ] \). Here the action of \( \Gamma_M \) on \( T_{\text{Ein}}(M) \) is induced from the action of \( \text{Diff}(M) \) on \( \text{Ein}(M) \) (by pulling back metrics). The global Torelli theorem determines \( \pi_0(T_{\text{Ein}}(M)) \) and the topology of each component:
The space $\mathcal{T}_{\text{Ein}}(M)$ has two homeomorphic components $\mathcal{T}_{\text{Ein}}(M) \simeq \mathcal{T}_{\text{Ein}}^0(M) \sqcup \mathcal{T}_{\text{Ein}}^0(M)$, and they are permuted by the action of $\Gamma_M$.

Let $\Gamma'_M < \Gamma_M$ be the index-2 subgroup that preserves the components of $\mathcal{T}_{\text{Ein}}(M)$. There is a $\Gamma'_M$-equivariant homeomorphism between $\mathcal{T}_{\text{Ein}}^0(M)$ and a dense subspace of $X = \text{SO}(V)/K$:

$$(16) \quad \mathcal{T}_{\text{Ein}}^0(M) \simeq X \setminus \bigcup_{\delta \in \Delta} H_\delta,$$

where $\Delta \subset \Lambda$ is the set of roots.

Using this description of $\mathcal{T}_{\text{Ein}}(M)$, it follows that $\pi_1(\mathcal{M}_{\text{Ein}}(M)) \simeq \Gamma'_M$ by the long exact sequence in homotopy associated to (15) together with the fact that the subspaces $H_\delta \subset X$ have codimension-3, so $\mathcal{T}_{\text{Ein}}(M)$ is simply connected.

**Proof of Proposition 14.** We have fixed a particular torsion-free subgroup $\Gamma < \Gamma'_M$ and a flat cycle $z_0 \in H_3(B\Gamma; \mathbb{Q})$, and we wish to show $z_0$ is not in the image of $H_3(\mathcal{M}_{\text{Ein}}^\Gamma(M); \mathbb{Q}) \to H_3(B\Gamma; \mathbb{Q})$. Recall that $z_0$ has the special property that it pairs nontrivially with the image of a root hyperplane $H_\delta$ in $Y = \Gamma \setminus X$.

Suppose for a contradiction that there exists $w \in H_3(\mathcal{M}_{\text{Ein}}^\Gamma(M); \mathbb{Q})$ whose image in $H_3(B\Gamma; \mathbb{Q})$ is $z_0$. Since $\Gamma$ is torsion free, $\mathcal{M}_{\text{Ein}}^\Gamma(M) \simeq \mathcal{T}_{\text{Ein}}^0(M) \times X$. There is a diagram that commutes up to homotopy:

$$\begin{array}{ccc}
\mathcal{M}_{\text{Ein}}^\Gamma(M) & \longrightarrow & B\text{Diff}^\Gamma(M) \\
f_1 & \downarrow & \downarrow \\
\Gamma \setminus \mathcal{T}_{\text{Ein}}^0(M) & \longrightarrow & \Gamma \setminus X
\end{array}$$

The map $f_1$ is a homotopy equivalence because $\Gamma$ acts freely on $\mathcal{T}_{\text{Ein}}(M)$ so $f_1$ is a fibration with contractible fiber $\simeq X$. The map $f_2$ is the inclusion induced by (16). The diagram commutes up to homotopy because the two compositions induce the same map on $\pi_1$ and $B\Gamma$ is Eilenberg–MacLane space.

Let $\tilde{H}_\delta$ be the image of $H_\delta$ in $\Gamma \setminus X$, and let $\bar{F}$ be a totally geodesic submanifold representing the flat cycle $z_0$. By our choice of $H_\delta$ and $F$, the algebraic intersection $\bar{F} \cdot \tilde{H}_\delta$ is nonzero. On the other hand, the existence of $w$ implies, by the diagram above, that there is a cycle $Z \to \Gamma \setminus \mathcal{T}_{\text{Ein}}^0(M) \subset \Gamma \setminus X$ that is homologous to $\bar{F}$. Since the $Z \to \Gamma \setminus X$ factors through $\Gamma \setminus \mathcal{T}_{\text{Ein}}^0(M)$, the image of $Z$ is disjoint from $\tilde{H}_\delta$, which implies that $\bar{F} \cdot \tilde{H}_\delta = Z \cdot H_\delta = 0$. This is a contradiction, so the class $w$ does not exist.

**Remark 17.** In the Teichmüller space $\mathcal{T}_{\text{Ein}}^0(M) \subset X$, as one approaches one of the subsets $H_\delta \subset X$, topologically there is an embedded sphere $f : S^2 \to M$ with $f_*[S^2] = \delta$ that is being collapsed to a point [And92]. The flat cycle $\bar{F} \subset \Gamma \setminus X$ in the proof of Proposition 14 only lifts to $\Gamma \setminus \mathcal{T}_{\text{Ein}}^0(M)$ after finitely many points are removed. From this, one obtains a K3-surface bundle over a 3-torus with finitely many punctures (note $\bar{F}$ is finitely covered by $\mathbb{T}^3$). One cannot extend the bundle over the punctures without introducing singularities. The natural object that exists over the torus with the punctures filled is a “singular” K3 bundle, i.e. it is a fiber bundle away from finitely many points in the base, and at each point in this finite collection, the fiber is the space obtained from a K3 surface by collapsing some embedded 2-sphere (with self-intersection $-2$) a point.

5.2. 2-dimensional cycles and the Mather–Thurston theorem. To end this section, we mention another example/application of our ideas.
Corollary 18. Let $M^n$ be smooth manifold. Suppose that $\pi_0 \text{Diff}(M)$ is commensurable with $\text{SL}_3(\mathbb{Z})$. Then for each $N \geq 1$ there exists a finite-index subgroup $\Gamma < \text{SL}_3(\mathbb{Z})$ so that

$$\dim H_2(B\text{Diff}^\Gamma(M); \mathbb{Q}) \geq N.$$ 

In fact, as we will see in the proof, Corollary 18 remains true if $B\text{Diff}^\Gamma(M)$ is replaced by the classifying space $B\text{Diff}^\Gamma(M)^0$ of $\text{Diff}^\Gamma(M)$ with the discrete topology. This is a stronger conclusion than in Corollary 2.

There are many manifolds that satisfy the hypothesis of Corollary 18, e.g. $M = \#_3(S^k \times S^{k+1})$ whenever $k \geq 4$, c.f. [Sn77, Thm. 13.3].

Corollary 18 follows by applying the following two theorems. The first theorem, due to Avramidi–Nguyen-Phan [ANP15], is analogous to Theorem 1. For a prime $p$ and $\ell \geq 1$, denote the congruence subgroup $\ker \left[ \text{SL}_n(\mathbb{Z}) \rightarrow \text{SL}_n(\mathbb{Z}/p^\ell\mathbb{Z}) \right]$ by $\Gamma_n(p^\ell)$.

**Theorem 19** (Avramidi–Nguyen-Phan). Given a prime $p$ and an integer $N \geq 1$, there exists $\ell_0 > 0$ so that if $\ell > \ell_0$, then $\dim H_{n-1}(\Gamma_n(p^\ell); \mathbb{Q}) \geq N$.

Like in Theorem 1 their homology comes from maximal flats in the associated symmetric space. We will focus on the case $n = 3$, which is special because we can use the following theorem.

**Theorem 20** (Mather, Thurston [Thu74]). Let $M$ be a smooth closed manifold. The group $\text{Diff}_0(M)$ of diffeomorphisms isotopic to the identity is a simple group.

**Proof of Corollary 18** Denote $\text{Mod}(M) := \pi_0 \text{Diff}(M)$. The exact sequence

$$1 \rightarrow \text{Diff}_0(M) \rightarrow \text{Diff}(M) \rightarrow \text{Mod}(M) \rightarrow 1$$

gives a 5-term exact sequence in homology

$$H_2(\text{Diff}(M)) \xrightarrow{\phi} H_2(\text{Mod}(M)) \rightarrow H_1(\text{Diff}_0(M))_{\text{Mod}(M)} \rightarrow H_1(\text{Diff}(M)) \rightarrow H_1(\text{Mod}(M)) \rightarrow 0$$

By the Mather–Thurston theorem, $H_1(\text{Diff}_0(M)) = \text{Diff}_0(M)^{ab} = 0$ so $\phi$ is surjective. Since $B\text{Diff}(M)^0 \rightarrow B\text{Mod}(M)$ factors through $B\text{Diff}(M)$, the map $H_2(B\text{Diff}(M)) \rightarrow H_2(B\text{Mod}(M))$ is also surjective. The same argument applies to $\text{Diff}^\Gamma(M)$. 

One could go further and try to extend the computations of [BM13] to the manifolds $M = \#_g(S^k \times S^{k+1})$ to show that the classes in Theorem 19 are in the image of $H_*(B\text{Diff}^\Gamma(M); \mathbb{Q}) \rightarrow H_*(B\Gamma; \mathbb{Q})$ as in Theorem 21. For another example, one could look at the diffeomorphism groups of handlebodies, i.e. boundary-connected-sums of $D^{k+1} \times S^k$ for $k \geq 4$, c.f. [BP17].

**APPENDIX A. LIFTING CYCLES FROM $B\Gamma$ TO $B\text{Diff}^\Gamma(W_g)$**

BY MANUEL KRANNICH

We denote by $\text{Diff}(W_g)$ the topological group of orientation-preserving diffeomorphisms of the iterated connected sum $W_g = \#^g(S^n \times S^n)$ in the smooth topology. Fixing an embedded disc $D^{2n} \subset W_g$, we shall also consider the manifold $W_{g,1} = W_g \setminus \text{int}(D^{2n})$ and its group of diffeomorphisms $\text{Diff}_\partial(W_{g,1})$ fixing a neighborhood of the boundary pointwise, related to $\text{Diff}(W_g)$ by a map $\text{Diff}_\partial(W_{g,1}) \rightarrow \text{Diff}(W_g)$ given by extending diffeomorphisms of $W_{g,1} \subset W_g$ via the identity.
A.1. **The action on homology.** The action of the group of diffeomorphisms $\text{Diff}(W_g)$ on the middle homology $H_n(W_g) \cong \mathbb{Z}^{2g}$ preserves the nondegenerate $(-1)^n$-symmetric intersection form $\lambda : H_n(W_g) \otimes H_n(W_g) \to \mathbb{Z}$, giving rise to a map

$$\pi_0 \text{Diff}(W_g) \longrightarrow \begin{cases} \text{Sp}_{2g}(\mathbb{Z}) & n \text{ odd} \\ \text{O}_{g,g}(\mathbb{Z}) & n \text{ even}, \end{cases}$$

whose image we denote by $G_g \leq \text{GL}_{2g}(\mathbb{Z})$. If $n$ is even or $n = 1, 3, 7$, this map is surjective and $G_g$ coincides with $\text{Sp}_{2g}(\mathbb{Z})$ or $\text{O}_{g,g}(\mathbb{Z})$ depending on the parity of $n$, whereas the image $G_g \leq \text{Sp}_{2g}(\mathbb{Z})$ for $n \neq 1, 3, 7$ odd is the finite index subgroup $\text{Sp}_{2g}^1(\mathbb{Z}) \leq \text{Sp}_{2g}(\mathbb{Z})$ of matrices preserving the standard theta characteristic (see e.g. [BM17, Ex. 5.4]). As the space of orientation-preserving embeddings $\text{Emb}(D^{2n}, W_g)$ is connected, the map $\pi_0 \text{Diff}_\partial(W_{g,1}) \to \pi_0 \text{Diff}(W_g)$ is surjective, so the images of the two groups in $\text{GL}_{2g}(\mathbb{Z})$ agree. Given a subgroup $\Gamma \leq G_g$, we denote by $\text{Diff}^\Gamma(W_g) \leq \text{Diff}(W_g)$ and $\text{Diff}^\Gamma_\partial(W_{g,1}) \leq \text{Diff}_\partial(W_{g,1})$ the preimages of $\Gamma$ with respect to the canonical maps to $G_g$.

The primary goal of this appendix is to present a proof of the following result.

**Theorem 21.** For $2n \geq 6$ and a subgroup $\Gamma \leq G_g$, the natural map

$$H^*(B\Gamma; \mathbb{Q}) \to H^*(B\text{Diff}^\partial_\Gamma(W_{g,1}); \mathbb{Q})$$

is injective in degrees $* \leq n$.

**Remark.** Since the action of $\text{Diff}^\Gamma_\partial(W_{g,1})$ on $H_n(W_g)$ factors through $\text{Diff}^\Gamma(W_g)$, the same conclusion holds for $\text{Diff}^\Gamma(W_g)$ instead of $\text{Diff}^\Gamma_\partial(W_{g,1})$.

A.2. **Stable and unstable cohomology.** As the usual inclusion $\text{GL}_{2g}(\mathbb{Z}) \subset \text{GL}_{2g+2}(\mathbb{Z})$ is covered by the map $\text{Diff}_\partial(W_{g,1}) \to \text{Diff}_\partial(W_{g+1,1})$ given by extending diffeomorphisms via the identity, it restricts to an inclusion of the form $G_g \subset G_{g+1}$, so we obtain a map $B\text{Diff}_\partial(W_{\infty,1}) \to B\text{G}_{\infty}$ by taking (homotopy) colimits. By work of Borel, Galatius–Randal-Williams, and Madsen–Weiss [Bor74b, Bor81, GRW14, MW07], the cohomology ring of both the source and the target of this map is a polynomial algebra concentrated in even degrees. Moreover, there are natural choices of polynomial generators for these rings with respect to which the induced map $H^*(B\text{G}_{\infty}; \mathbb{Q}) \to H^*(B\text{Diff}_\partial(W_{\infty,1}); \mathbb{Q})$ corresponds to an inclusion of a subset of generators; this can for instance be seen by an index-theoretic argument (see e.g. [ERW15, Sect. 2.4]). Given a subgroup $\Gamma \leq G_g$ of finite index, we have a commutative square

$$\begin{array}{ccc}
H^*(B\text{G}_{\infty}; \mathbb{Q}) & \longrightarrow & H^*(B\Gamma; \mathbb{Q}) \\
\downarrow & & \downarrow \\
H^*(B\text{Diff}_\partial(W_{\infty,1}); \mathbb{Q}) & \longrightarrow & H^*(B\text{Diff}^\partial_\Gamma(W_{g,1}); \mathbb{Q}),
\end{array}$$

whose upper horizontal arrow is an isomorphism in a range of degrees growing with $g$ by a result of Borel [Bor74b, Bor81]. In light of work of Harer and Galatius–Randal-Williams [Har85, GRW18], the same holds for the lower horizontal morphism, at least if $\Gamma = G_g$ and $2n \neq 4$. Moreover, the proof of Theorem [21] will make apparent that the vertical arrows in the diagram are (compatibly) split injective for $* < n$ and any $\Gamma \leq G_g$ if $2n \neq 4$. As a result, the cokernel of the upper horizontal map—the so-called unstable cohomology of $B\Gamma$—injects in this range of degrees into the cokernel of the lower horizontal map and thus provides a source for unstable cohomology of $B\text{Diff}^\Gamma_\partial(W_{g,1})$.

When varying $\Gamma \leq G_g$ over finite index subgroups, the rational cohomology of $B\Gamma$ in degree $g$ is arbitrarily large for $n$ even and $g$ odd; this is the main result of the body of this paper (see Theorem [1]). For the full group $\Gamma = G_g$ on the other hand, very little is known about the unstable cohomology, aside from some scattered classes: computations of Hain [Hai02] and Hulek–Tomassi.
Corollary 22. For $n$ odd, the cokernel of the natural morphism
\[ H^i(B\Diff_\partial(W_{\infty,1}); \mathbb{Q}) \to H^i(B\Diff_\partial(W_{g,1}); \mathbb{Q}) \]
is nontrivial for $(i,g) = (6,3)$ as long as $n > 5$, and for $(i,g) = (12,4)$ if $n > 11$.

Remark. To the knowledge of the author, these classes are the first unstable rational cohomology classes of $B\Diff_\partial(W_{g,1})$ known, aside from the case $2n = 2$ of surfaces.

A.3. The work of Berglund–Madsen. The proof of Theorem 21 relies on work of Berglund and Madsen [BM17], who used a combination of classical surgery theory and rational homotopy theory to construct rational models for the classifying spaces $B\Autid_\partial(W_{g,1})$ and $B\Diff_\partial(W_{g,1})$ of homotopy automorphisms and block diffeomorphisms homotopic to the identity. Using these models, they proved that the rational cohomology ring of the classifying spaces of the full automorphism spaces $\hAut_\partial(W_{g,1})$ and $\Diff_\partial(W_{g,1})$ is independent of $g$ in a range of degrees and studied the rational cohomology in this stable range. As explained above, Theorem 21 yields some information on $H^*(B\Diff_\partial(W_{g,1}); \mathbb{Q})$ in the unstable range. Its proof involves relating $H^*(B\Diff_\partial(W_{g,1}); \mathbb{Q})$ to $H^*(B\Diff_\partial(W_{g,1}); \mathbb{Q})$ by combining [BM17] with Morlet’s lemma of disjunction as in [RW17] and extending some arguments in [BM17] for spaces of automorphisms homotopic to the identity to the full automorphism spaces, in particular to show that the cohomology ring $H^*(B\Aut_\partial(W_{g,1}); \mathbb{Q})$ is a retract of $H^*(B\Diff_\partial(W_{g,1}); \mathbb{Q})$, even in the unstable range, which has not been observed so far.

A.4. The proof of Theorem 21. To simplify the exposition, we restrict to the case $\Gamma = G_g$ first. The proof for general $\Gamma \leq G_g$ proceeds almost identically and we explain the necessary adjustments at the end. We divide our arguments into four steps corresponding to maps in a factorisation
\[ B\Diff_\partial(W_{g,1}) \overset{1}{\rightarrow} B\Diff_\partial(W_{g,1}) \overset{2}{\rightarrow} B\Autid_\partial(W_{g,1}) \overset{3}{\rightarrow} B\Aut_\partial(W_{g,1}) \overset{4}{\rightarrow} Bg, \]
which we explain in the following. Up to canonical block equivalences, the topological group of block diffeomorphisms fixing a neighborhood of the boundary $\Diff_\partial(W_{g,1})$ fits between $\Diff_\partial(W_{g,1})$ and the topological monoid of homotopy automorphisms, so there are natural maps $\Diff_\partial(W_{g,1}) \to \Diff_\partial(W_{g,1}) \to \hAut_\partial(W_{g,1})$ (see e.g. [BM17 Sect. 4]), which explain the maps 1–3 in the composition above, denoting by $\hAutid_\partial(W_{g,1}) \subset \hAut_\partial(W_{g,1})$ the union of components hit by the map $\Diff_\partial(W_{g,1}) \to \hAut_\partial(W_{g,1})$. It follows from Cerf’s “concordance implies isotopy” that the map $\Diff_\partial(W_{g,1}) \to \Diff_\partial(W_{g,1})$ is an isomorphism on path components, so in particular the image of $\Diff_\partial(W_{g,1})$ in $GL_{2g}(\mathbb{Z})$ coincides with the image $G_g$ in $\Diff_\partial(W_{g,1})$. The same conclusion applies more generally to any simply-connected manifold of high dimension, but the fact that also the image of $\hAut_\partial(W_{g,1})$ in $GL_{2g}(\mathbb{Z})$ agrees with $G_g$ is more specific to the manifolds $W_{g,1}$ (see e.g. [BM17 Ex. 5.4]) and explains the final map 4 in the above composition. In what follows, we examine the quality of each of the maps 1–3 in rational cohomology.

\[ \text{Q). Extending (block) diffeomorphisms of an embedded disc } D^{2n} \subset W_{g,1} \text{ to all of } W_{g,1} \text{ by the identity induces a commutative square} \]
whose induced map on vertical homotopy fibres
\[ \tilde{\text{Diff}}_\partial(D^{2n})/\text{Diff}(D^{2n}) \to \tilde{\text{Diff}}_\partial(W_{g,1})/\text{Diff}(W_{g,1}) \]
is \((2n - 4)\)-connected by an application of Morlet’s lemma of disjunction. For \( g \gg 0 \), this can be combined with Berglund–Madsen’s work [BM17] to conclude that \( \tilde{\text{Diff}}_\partial(D^{2n})/\text{Diff}(D^{2n}) \) is \((2n - 5)\)-connected, as observed by Randal-Williams [RW17, Sect. 4]. This in turn implies that \( \tilde{\text{Diff}}_\partial(W_{g,1})/\text{Diff}(W_{g,1}) \) has no rational cohomology in degrees \( * \leq 2n - 5 \) for all \( g \geq 0 \) and thus has the following as a consequence.

**Proposition 23.** For \( 2n \geq 6 \), the induced map
\[ H^*(\tilde{\text{Diff}}_\partial(W_{g,1}), Q) \to H^*(\text{Diff}_\partial(W_{g,1}); Q) \]
is an isomorphism for \( * \leq 2n - 5 \) and a monomorphism for \( * \leq 2n - 4 \).

\( \Box \). As our result on the second map in the composition [17] is independent of the specific manifold \( W_{g,1} \), we phrase it in greater generality than needed.

**Proposition 24.** For a compact, simply-connected, stably parallelisable manifold \( M \) of dimension \( d \geq 5 \) with sphere boundary \( \partial M \cong S^{d-1} \), the natural map
\[ \text{BDiff}_\partial(M) \to \text{BhAut}_{\partial}^\infty(M) \]
induces a split epimorphism on rational cohomology rings.

**Proof.** Following [BM17, Ch. 4], we pick a base point \( * \in \partial M \) in the boundary and denote by \( \text{hAut}_\partial^\text{id}(\partial) \) the topological monoid of homotopy automorphisms \( f : M \to M \), relative to the boundary, together with a bundle automorphism of the stable normal bundle of \( M \)

\[ \vartheta \xrightarrow{f} \vartheta \]
\[ M \xrightarrow{f} M \]

that covers \( f \) and restricts to the identity over the basepoint (cf. [BM17, p. 24]). We denote by \( \text{hAut}_\partial^\text{id}(\partial) \) the kernel of the map \( \text{hAut}_\partial^\text{h}(\partial) \to \pi_0 \text{hAut}_\partial(M) \) given by taking homotopy classes and forgetting the bundle map and by \( \text{hAut}_\partial^\text{h}\hat{\subseteq}(\partial) \) the preimage of the subgroup \( \pi_0 \text{hAut}_\partial^\text{h}(\partial) \subset \pi_0 \text{hAut}_\partial(N) \). Up to replacing \( \text{hAut}_\partial^\text{id}(\partial) \) with its equivalent block-version \( \tilde{\text{hAut}}^\infty_\partial(\partial) \), there is a natural map \( \text{BDiff}_\partial(M) \to \text{BhAut}_\partial^\infty(\partial) \) by taking normal derivatives (cf. [BM17, p. 25]), which fits into a morphism of homotopy fibre sequences

\[ \text{BDiff}_\partial(M) \xrightarrow{f} \text{BDiff}_\partial(M) \to B \pi_0 \text{hAut}_\partial^\infty(M) \]
\[ \text{BhAut}_\partial^\text{id}(\partial) \to \text{BhAut}_\partial^\infty(\partial) \to B \pi_0 \text{hAut}_\partial^\infty(M). \]
Berglund–Madsen [BM17 Cor. 4.13] showed that the map between fibres is a rational homology equivalence, so by an application of the Serre spectral sequence the middle vertical map has this property as well. The proof finishes by arguing that $\text{BhAut}_\partial^\infty (\partial) \to \text{BhAut}_\partial^\infty (N)$ admits a section: after choosing a trivialisation of $\partial$, every homotopy automorphism of $M$ is canonically covered by the bundle automorphism that is the identity on each fibre with respect to the chosen trivialisation, so there is a section before taking classifying spaces $\text{hAut}_\partial^\infty (N) \to \text{hAut}_\partial^* (\partial)$. This section respects the monoid structure and hence deloops to a section as wished. □

5. We now turn towards the third map $\text{BhAut}_\partial^\infty (W_{g,1}) \to \text{BhAut}_\partial (W_{g,1})$ in the composition $\textbf{(17)}$.

**Proposition 25.** For $2n \geq 4$, the map

$$\text{BhAut}_\partial^\infty (W_{g,1}) \to \text{BhAut}_\partial (W_{g,1})$$

is a rational homology equivalence.

**Proof.** By [BM17 Prop. 7.9], the action of $\pi_0 \text{hAut}_\partial (W_{g,1})$ on the rational homology of the fibre in the map of fibration sequences

$$\text{BhAut}_\partial^\infty (W_{g,1}) \to \text{BhAut}_\partial (W_{g,1}) \to B \pi_0 \text{hAut}_\partial^\infty (W_{g,1})$$

factors through the action on homology $\pi_0 \text{hAut}_\partial (W_{g,1}) \to G_g$. Since the kernel of this map is finite (see e.g. [BM17 Prop. 5.3]), the homology of $\pi_0 \text{hAut}_\partial (W_{g,1})$ and $G_g$ with coefficients in the rational homology of $\text{BhAut}_\partial^\infty (W_{g,1})$ are naturally isomorphic. As the images of $\pi_0 \text{hAut}_\partial (W_{g,1})$ and $\pi_0 \text{hAut}_\partial^\infty (W_{g,1})$ in $G_g$ coincide, the same applies to the subgroup $\pi_0 \text{hAut}_\partial^\infty (W_{g,1})$, so the map of fibration sequences above induces an isomorphism on the $E_2$-pages of the induced rational Serre spectral sequences, which implies the assertion. □

6. Combining the previous three propositions, we conclude that the map

$$H^* (\text{BhAut}_\partial (W_{g,1}); \mathbb{Q}) \to H^* (\text{BDiff}_\partial (W_{g,1}); \mathbb{Q})$$

is injective in degrees $* \leq 2n - 4$, which will finish the proof of Theorem $\textbf{21}$ for $\Gamma = G_g$ when combined with the following proposition.

**Proposition 26.** For $2n \geq 4$, the induced map

$$H^* (BG_g; \mathbb{Q}) \to H^* (\text{BhAut}_\partial (W_{g,1}); \mathbb{Q})$$

is an isomorphism in degrees $* \leq n - 1$ and a monomorphism in degrees $* \leq n$.

**Proof.** As the morphism $\pi_0 \text{hAut}_\partial (W_{g,1}) \to G_g$ has finite kernel, it is enough to show the claimed property for the map $\text{BhAut}_\partial (W_{g,1}) \to B \pi_0 \text{hAut}_\partial (W_{g,1})$ instead. Its homotopy fibre $\text{BhAut}_\partial^\infty (W_{g,1})$ is the classifying space of the component of the identity, whose Lie algebra of rational homotopy groups $\pi_{*+1} \text{BhAut}_\partial^\infty (W_{g,1}) \otimes \mathbb{Q}$ was identified by Berglund–Madsen [BM17 Thm 5.1] as a certain graded sub Lie algebra of the Lie algebra $\text{Der}^+ L(V)$ of positive degree derivations of the free graded Lie algebra $L(V)$ on the graded vector space $V = H_n (W_g; \mathbb{Q})$ concentrated in degree $(n-1)$. Any such derivation has to raise the degree by at least $(n-1)$, so $\text{Der}^+ L(V)$ is trivial in degrees $* \leq (n-2)$. Consequently, $\text{BhAut}_\partial^\infty (W_{g,1})$ is rationally $(n-1)$-connected and the proof is finished by an application of the Serre spectral sequence. □
**Restricted actions.** For an arbitrary subgroup \( \Gamma \leq G_g \), there is a commutative ladder

\[
\begin{array}{ccccccc}
\text{BDiff}^\Gamma_\partial(W_{g,1}) & \to & \text{BDiff}^\Gamma_\partial(W_{g,1}) & \to & \text{BhAut}^\infty_\partial \Gamma(W_{g,1}) & \to & \text{BhAut}^\Gamma_\partial(W_{g,1}) & \to & \text{BG}_g \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{BDiff}_\partial(W_{g,1}) & \to & \text{BDiff}_\partial(W_{g,1}) & \to & \text{BhAut}^\infty_\partial W_{g,1} & \to & \text{BhAut}_\partial(W_{g,1}) & \to & \text{BG}_g \\
\end{array}
\]

in which each square is a homotopy pullback. We remark that the proofs for Propositions 23 and 26 presented above show something stronger than stated, namely that the respective maps are rationally acyclic in a certain range of degrees, i.e. their homotopy fibres have trivial rational homology in these ranges. This property is preserved under taking homotopy pullbacks, so the statements for the variants for subgroups \( \Gamma \leq G_g \) follow from the case \( \Gamma = G_g \). This argument does not apply to Propositions 24 and 25, but their proofs for \( \Gamma = G_g \) generalise to general subgroups \( \Gamma \leq G_g \) with only marginal changes.

**References**

[And92] M. T. Anderson. The \( L^2 \) structure of moduli spaces of Einstein metrics on 4-manifolds. *Geom. Funct. Anal.*, 2(1):29–89, 1992.

[ANP15] G. Avramidi and T. T. Am Nguyen-Phan. Flat cycles in the homology of \( \Gamma \backslash \text{SL}(m, \mathbb{R})/\text{SO}(m) \). *Comment. Math. Helv.*, 90(3):645–666, 2015.

[BCS13] M. Bestvina, T. Church, and J. Souto. Some groups of mapping classes not realized by diffeomorphisms. *Comment. Math. Helv.*, 88(1):205–220, 2013.

[Bes08] A. Besse. *Einstein manifolds*. Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition.

[BM13] A. Berglund and I. Madsen. Homological stability of diffeomorphism groups. *Pure Appl. Math. Q.*, 9(1):1–48, 2013.

[BM17] A. Berglund and I. Madsen. Rational homotopy theory of automorphisms of manifolds. https://arxiv.org/abs/1401.4096, March 2017.

[Bor74a] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[Bor74b] A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[Bor81] A. Borel. Stable real cohomology of arithmetic groups. II. In *Manifolds and Lie groups (Notre Dame, Ind., 1980)*, volume 14 of *Progr. Math.*, pages 21–55. Birkhäuser, Boston, Mass., 1981.

[BP17] B. Botvinnik and N. Perlmutter. Stable moduli spaces of high-dimensional handlebodies. *J. Topol.*, 10(1):101–163, 2017.

[Don90] S. K. Donaldson. Polynomial invariants for smooth four-manifolds. *Topology*, 29(3):257–315, 1990.

[ERW15] J. Ebert and O. Randal-Williams. Torelli spaces of high-dimensional manifolds. *J. Topol.*, 8(1):38–64, 2015.

[FH09] J. Franks and M. Handel. Global fixed points for centralizers and Morita’s theorem. *Geom. Topol.*, 13(1):87–98, 2009.

[FOR00] F. T. Farrell, P. Ontaneda, and M. S. Raghunathan. Non-univalent harmonic maps homotopic to diffeomorphisms. *J. Differential Geom.*, 54(2):227–253, 2000.

[Gia09] J. Giansiracusa. The diffeomorphism group of a K3 surface and Nielsen realization. *J. Lond. Math. Soc. (2)*, 79(3):701–718, 2009.

[GRW14] S. Galatius and O. Randal-Williams. Stable moduli spaces of high-dimensional manifolds. *Acta Math.*, 212(2):257–377, 2014.

[GRW17] S. Galatius and O. Randal-Williams. Homological stability for moduli spaces of high dimensional manifolds. II. *Ann. of Math. (2)*, 186(1):127–204, 2017.

[GRW18] S. Galatius and O. Randal-Williams. Homological stability for moduli spaces of high dimensional manifolds. I. *J. Amer. Math. Soc.*, 31(1):215–264, 2018.

[Hai02] R. Hain. The rational cohomology ring of the moduli space of abelian 3-folds. *Math. Res. Lett.*, 9(4):473–491, 2002.

[Har85] J. L. Harer. Stability of the homology of the mapping class groups of orientable surfaces. *Ann. of Math. (2)*, 121(2):215–249, 1985.
References

[HT12] K. Hulek and O. Tommasi. Cohomology of the second Voronoi compactification of $\mathcal{A}_4$. Doc. Math., 17:195–244, 2012.

[Loo81] E. Looijenga. A Torelli theorem for Kähler-Einstein $K^3$ surfaces. volume 894 of Lecture Notes in Math., pages 107–112. Springer, Berlin-New York, 1981.

[LS86] R. Lee and J. Schwermer. Geometry and arithmetic cycles attached to $SL_3(\mathbb{Z})$. I. Topology, 25(2):159–174, 1986.

[Mar07] V. Markovic. Realization of the mapping class group by homeomorphisms. Invent. Math., 168(3):523–566, 2007.

[Mat86] T. Matumoto. On diffeomorphisms of a $K^3$ surface. In Algebraic and topological theories (Kinosaki, 1984), pages 616–621. Kinokuniya, Tokyo, 1986.

[MH73] J. Milnor and D. Husemoller. Symmetric bilinear forms. Springer-Verlag, New York-Heidelberg, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.

[Mil76] J. Millson. On the first Betti number of a constant negatively curved manifold. Ann. of Math. (2), 104(2):235–247, 1976.

[Mor87] S. Morita. Characteristic classes of surface bundles. Invent. Math., 90(3):551–577, 1987.

[Mor15] D. Witte Morris. Introduction to arithmetic groups. Deductive Press, Place of publication not identified, 2015.

[MR80] J. Millson and M. S. Raghunathan. Geometric construction of cohomology for arithmetic groups. I. In Geometry and analysis, pages 103–123. Indian Acad. Sci., Bangalore, 1980.

[MW07] I. Madsen and M. Weiss. The stable moduli space of Riemann surfaces: Mumford’s conjecture. Ann. of Math. (2), 165(3):843–941, 2007.

[PR72] G. Prasad and M. S. Raghunathan. Cartan subgroups and lattices in semi-simple groups. Ann. of Math. (2), 96:296–317, 1972.

[RS93] J. Rohlf and J. Schwermer. Intersection numbers of special cycles. J. Amer. Math. Soc., 6(3):755–778, 1993.

[RW17] O. Randal-Williams. An upper bound for the pseudoisotopy stable range. Math. Ann., 368(3-4):1081–1094, 2017.

[Sch10] J. Schwermer. Geometric cycles, arithmetic groups and their cohomology. Bull. Amer. Math. Soc. (N.S.), 47(2):187–279, 2010.

[Sco05] A. Scorpan. The wild world of 4-manifolds. American Mathematical Society, Providence, RI, 2005.

[ST16] N. Salter and B. Tshishiku. On the non-realizability of braid groups by diffeomorphisms. Bull. Lond. Math. Soc., 48(3):457–471, 2016.

[Sul77] D. Sullivan. Infinitesimal computations in topology. Inst. Hautes Études Sci. Publ. Math., (47):269–331 (1978), 1977.

[Thu74] W. Thurston. Foliations and groups of diffeomorphisms. Bull. Amer. Math. Soc., 80:304–307, 1974.

[Tsh19] B. Tshishiku. Borel’s stable range for the cohomology of arithmetic groups. In preparation, February 2019.

[Ven08] T. N. Venkataramana. Virtual Betti numbers of compact locally symmetric spaces. Israel J. Math., 166:235–238, 2008.

[Wal64] C. T. C. Wall. Diffeomorphisms of 4-manifolds. J. London Math. Soc., 39:131–140, 1964.

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