CHAPTER 1

One-Dimensional Quantum Spin Liquids

P. Lecheminant

Laboratoire de Physique Théorique et Modélisation, CNRS UMR 8089,
Université de Cergy-Pontoise, 5 mail Gay-Lussac, Neuville sur Oise,
95301 Cergy-Pontoise Cedex, France

This chapter is intended as a brief overview of some of the quantum spin
liquid phases with unbroken SU(2) spin symmetry available in one di-
mension. The main characteristics of these phases are discussed by means
of the bosonization approach. A special emphasis is laid on the interplay
between frustration and quantum fluctuations in one dimension.

1. Introduction

A central issue in the study of strongly correlated systems is the classifica-
tion of all possible Mott insulating phases at zero temperature. The general
strategy for describing the possible phases associated with the spin degrees
of freedom (the so-called “quantum magnetism”) of Mott insulators con-
sists of analysing localized spin models such as the antiferromagnetic (AF)
Heisenberg model:

\[ \mathcal{H} = J \sum_{<i,j>} \vec{S}_i \cdot \vec{S}_j + .., \]  

(1)

where \( \vec{S}_i \) are quantum spin-\( S \) operators on sites \( i \) of a lattice, and the sum
\( < i,j > \) is over the nearest neighbor sites with an antiferromagnetic ex-
change (\( J > 0 \)); the ellipsis represents additional terms like second-neighbor
competing exchange interaction or ring exchange that can be eventually
added to the Heisenberg Hamiltonian \( \mathcal{H} \). Generically, the ground state
of the model \( \mathcal{H} \) without these additional terms displays long-range Néel or-
dering (\( < \vec{S}_i > \neq 0 \) which breaks spontaneously the SU(2) symmetry of the
lattice Hamiltonian \( \mathcal{H} \). The low-energy excitations are gapless spin-waves
(i.e. magnons) as expected when a continuous symmetry is spontaneously broken.

A central focus of quantum magnetism over the years, starting from the proposal made by Anderson, has been the search for a spin liquid behavior i.e. a phase with no magnetic long-range Néel order. Spin liquids phases are expected to be stabilized in low dimensions or in presence of frustration, i.e. in situations where quantum fluctuations can strongly suppress magnetism. It gives rise to rich physics with exotic low-energy excitations which require in most cases the use of non-perturbative techniques to fully determine its properties. In the two-dimensional case, different spin liquid phases have been found and display bond ordering or topological ordering (for a review see for instance Refs. 2, 3 and references therein). An interesting attempt to develop a systematic classification of two-dimensional spin liquids has also recently been explored by Wen.

In one dimension, the quantum analog of the Mermin-Wagner theorem shows that quantum fluctuations always disorder the ground state in systems with a continuous symmetry. The generic situation in one dimension is thus to have a spin liquid phase. The one dimensional case is also extremely favorable since several non-perturbative techniques are available to fully characterize the physical properties of different spin liquids. These powerful techniques include integrability, conformal field theory (CFT), the bosonization approach, and numerical calculations such as the density matrix renormalization group (DMRG) approach. The determination of the properties of spin liquids is not a purely academic problem since many physical realizations of spin chains have been synthesized over the years. The reader may consult for instance the recent reviews on experiments in quasi-one dimensional spin systems.

The main questions in the study of one-dimensional spin liquids are in order: What are the different spin liquid phases available at zero-temperature? What are the different physical properties of these spin liquids? What is the nature of the quantum phase transition between two different spin liquids? Last but not least, the problem of dimensional crossover, i.e. the fate of the spin liquid as the two dimensional case is reached is also a central topic. In particular, is it possible to stabilize a spin liquid phase in $d = 2$ starting from the $d = 1$ case?

In this work, we shall mainly be concerned with the two first questions for spin liquids with unbroken SU(2) spin symmetry. Using the bosonization approach, we shall discuss some of the physical properties of spin liquids that have been found in spin chains, spin ladders with or without frustra-
An important question related to this problem is how can one classify all these 1D spin liquid phases at $T = 0$? A first distinction concerns the existence or absence of a spectral gap in the model. In a gapless system (critical spin liquid), one has quasi-long range Néel order. The leading asymptotics of spin-spin correlation functions display a power law behavior with an exponent characterized by a well defined underlying CFT $^{15}$. In contrast, in a spin liquid phase with a spectral gap, spin-spin correlation functions have an exponential decay with the distance due to the existence of a finite correlation length $\xi$: $\langle \vec{S}_i \cdot \vec{S}_j \rangle \sim \exp(-|i - j|/\xi)$. At this point, it is worth stressing that two gapful spin liquid phases sharing the same thermodynamic properties (typically a thermal activation law at low T) may display very different behaviors in other physical quantities of interest such as, for instance, dynamical or optical properties. It is thus necessary to make a scrutiny analysis of the properties of the phase before elaborating a complete classification.

Let us first consider the ground state of a gapful spin liquid phase. A (discrete) symmetry might be spontaneously broken in the ground state resulting on a ground-state degeneracy. The spin liquid phase may display also an hidden topological order $^{16}$ which manifests itself in a ground-state degeneracy which depends on the nature of boundary conditions (BC) used i.e. periodic or open BC. The resulting chain-end (edge) excitations in the open BC case can be observed in the NMR profile of a spin chain compound doped with non-magnetic impurities like Zn or Mg. Another important distinction between two spin liquids relies on the quantum number carried by elementary excitations. The fractionalized or integer nature of this quantum number has crucial consequences in the dynamical structure factor of the system which directly probes the elementary nature of the spin-flip. A sharp spectral peak, characteristic of a well defined $S = 1$ mode, or a broad (incoherent background) feature in the dynamical structure factor can be seen in inelastic neutron scattering experiments. Finally, an interesting question is the existence or not of bound states in the energy spectrum below the two-particle continuum. Spin liquid phases may then be distinguished at this level which is also an important fact from the experimental point of view since observation of magnetic singlet bound states can be realized through light scattering experiments $^{13}$. This classification is certainly far from being complete but it enables us to distinguish between several spin liquid phases and investigate the nature of the quantum phase transition between them.

This chapter is organized as follows: We present, in Section 2, the different spin liquid phases that occur in unfrustrated spin chains and spin
ladders. The main effects of frustration in one-dimensional spin liquids are described in Section 3. In particular, we shall observe that frustration plays its trick by allowing deconfined spinons (carrying fractional $S = 1/2$ quantum number) as elementary excitations and it provides a non-trivial source of incommensurability. Finally, our concluding remarks are given in Section 4.

2. Unfrustrated spin chains

The paradigmatic model to investigate the properties of 1D quantum magnets in absence of frustration is the AF Heisenberg spin chain given by the Hamiltonian:

$$\mathcal{H} = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1},$$

where $\vec{S}_i$ is a spin-$S$ operator at the ith site of the chain and the exchange interaction is antiferromagnetic: $J > 0$. In the classical limit, the model displays Néel long-range order: $\vec{S}_i \sim S(-1)^i \vec{n}$, $\vec{n}$ being an unit vector with an arbitrary fixed orientation in spin space. This solution breaks the SU(2) symmetry of the model (2) down to U(1). Corresponding to this symmetry breaking scheme, there are two Goldstone modes which propagate with the same velocity (Lorentz invariance in the low-energy limit). In the language of CFT, it corresponds to a field theory with central charge $c = 2$ ($c = 1$ being the central charge of a free massless boson field). Physically, these Goldstone modes are nothing but a doublet of gapless spin waves modes associated with slow modulations in the orientation of the vector $\vec{n}$ which represents the order parameter of the Néel magnetic structure. In the quantum case, this Néel solution in one dimension is destabilized by strong quantum fluctuations no matter how large $S$ is: a spin liquid phase is formed.

2.1. Spin-1/2 Heisenberg chain

In the ultra quantum limit i.e. $S = 1/2$, the nature of the spin liquid phase can be fully determined since the model is exactly solvable for $S = 1/2$ by means of the Bethe ansatz approach. In particular, starting with Bethe’s seminal work, a host of exact results have been obtained over the years for ground state properties, magnetic susceptibility, thermodynamics, excitation spectrum, and correlation functions.

The model displays quantum criticality properties which belong to the Wess-Zumino-Novikov-Witten (WZNW) $su(2)_1$ universality class. The
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low-energy limit of the spin-1/2 AF Heisenberg chain \(2\) is described by the \(\text{su}(2)_1\) WZNW CFT with central charge \(c = 1\) perturbed by marginally irrelevant current-current interaction \(15, 24\). The resulting Hamiltonian density reads as follows:

\[
\mathcal{H}_{\text{eff}} = \frac{2\pi v}{3} \left( J_R^2 + J_L^2 \right) + \lambda \vec{J}_R \cdot \vec{J}_L, \tag{3}
\]

where \(J_R\) and \(J_L\) are respectively the right and left \(\text{su}(2)_1\) currents which generate the \(\text{su}(2)_1\) WZNW CFT \(25\); these currents satisfy the \(\text{su}(2)_1\) Kac-Moody commutation relations:

\[
\left[ J^a_R, L(x), J^b_R, L(y) \right] = \mp \frac{i\delta^{ab}}{4\pi} \delta'(x - y) + i\epsilon^{abc} J^c_R, L(x) \delta(x - y). \tag{4}
\]

In Eq. (3), \(v\) is the spin velocity and \(\lambda < 0\) so that the last contribution is a marginally irrelevant term that renormalizes to zero in the far infrared (IR) limit. This perturbation accounts for logarithmic corrections in the spin-spin correlation \(26\) which is exactly known \(27\) in the long-distance limit:

\[
\langle \vec{S}_0 \cdot \vec{S}_r \rangle \approx \frac{(-1)^r}{(2\pi)^{3/2}} \left( \frac{\ln r}{r} \right)^{1/2}. \tag{5}
\]

The most striking feature of this spin liquid phase stems from the nature of its elementary excitations. They have been elucidated by Faddeev and Takhtajan \(22\) within the Bethe ansatz approach and consist of fractional \(S = 1/2\) massless excitations called spinons. The lowest excitations are fourthfold degenerate and correspond to a triplet \((S = 1)\) and a singlet \((S = 0)\). The resulting energy spectrum is a continuum in \((k, \omega)\) space between a lower boundary (the des Cloizeaux-Pearson dispersion relation \(21\))

\[
\omega_{dcp} = \pi J \left| \sin k/2 \right| (-\pi < k \leq \pi)
\]

and an upper boundary \(\omega_u = \pi J |\sin(k/2)|\) (see Fig. 1). The central point of the analysis is that this continuum can be interpreted as being made up of two spin-1/2 excitations (spinons) with the dispersion:

\[
\omega_{\text{spinon}} = \frac{\pi J}{2} \sin k, \tag{6}
\]

with \(0 < k < \pi\). A spinon has thus a wave-vector restricted to only half of the Brillouin zone. A triplet (or singlet) excitation with momentum \(k\) is then described by two spinons with momenta \(k_1\) and \(k_2\) \((0 < k_1 \leq k_2 < \pi)\) such that:

\[
\omega(k) = \omega_{\text{spinon}}(k_1) + \omega_{\text{spinon}}(k_2)
\]

with \(k = k_1 + k_2 - 2\pi)\) if \(0 < k \leq \pi\) (respectively \(-\pi < k < 0\)). It corresponds to a two-parameter continuum \(\omega(k) = \pi J \sin(k/2) \cos(k/2 - q_1)\) with \(0 < \)
$q_1 < k/2$ for $0 < k < \pi$ and $\pi + k < q_1 < \pi + k/2$ for $-\pi < k < 0$ which identifies with the continuum of the spin-1/2 AF Heisenberg chain.

![Graph](image)

Fig. 1. Two-spinon continuum of the spin-1/2 AF Heisenberg chain.

A magnon excitation carrying $S = 1$ quantum number is thus no longer an elementary excitation in this model but is fractionalized into two spinons. One important experimental signature of these spinons is the absence of any sharp peak in the dynamical susceptibility structure factor since a single spin-flip generates a triplet excitation which is not elementary here but made up of two spinons. The spectral density is then a convolution of the spectral densities of the individual spinons. These spinons have fascinating properties in particular they obey semion statistics intermediate between bosons and fermions, as it can be seen by the study of the elementary excitations of the Haldane-Shastry model which belongs to the same universality class as the spin-1/2 AF Heisenberg chain. An heuristic way to describe the spinons is to consider an anisotropic XXZ version of the Heisenberg model:

$$\mathcal{H}_{XXZ} = J_z \sum_i S_i^z S_{i+1}^z + \frac{J_\perp}{2} \sum_i (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) .$$

This model is still integrable and the ground state is two-fold degenerate for $J_z > J_\perp$ as the result of the spontaneous breaking of a discrete $Z_2$ symmetry. In this regime, a spectral gap is opened and the elementary excitations are massive spinons. A simple way to understand these properties is to consider the Ising limit when $J_z \gg J_\perp$ where the ground state reduces to the two Néel ordered states: $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\ldots$ and $\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\ldots$. The elementary excitations are massive kinks which carry $\Delta S^z = \pm 1/2$ quantum number and interpolate between these two ground states: $\uparrow\uparrow\uparrow\downarrow\downarrow\times\downarrow\uparrow\ldots$ The composite nature of a magnon excitation can then be readily seen by starting from a ground state and apply a single spin-flip on it to obtain the configuration:
Applying the exchange $J_\perp$ interaction of the model (7) on this state, this magnon excitation with $\Delta S^z = -1$ is in fact made up of two kinks of the Néel order i.e. two spinons: $\uparrow \downarrow \times \uparrow \downarrow \downarrow \uparrow \downarrow \uparrow \downarrow \downarrow$. At the transition $J_z = J_\perp$ (isotropic limit), the ground state is disordered by “condensation” of these kinks and the low-lying excitations carrying $\Delta S^z = \pm 1/2$ are transmuted from gapful kinks to gapless spinons. Magnons are still built as pairs of these spinons and have a gapless energy spectrum.

This remarkable production of two particles from a single spin-flip has been explored in inelastic neutron scattering measurements on quasi-1D compounds that are good experimental realizations of the model (2). The first clear evidence of spinons comes from neutron experiments on KCuF$_3$ that found a continuum of magnetic excitations consistent with that expected from unbound spinons pair. The spinon continuum has also been observed in several other realizations of the spin-1/2 AF Heisenberg chain as the copper benzoate Cu(C$_6$D$_5$COO)$_2$.3D$_2$O$_3$, BaCu$_2$Si$_2$O$_7$, and Cu(C$_4$H$_4$N$_2$)(NO$_3$)$_2$.3.

### 2.2. Haldane’s conjecture

After the elucidation of the elementary excitations of the spin-1/2 AF Heisenberg chain by Faddeev and Takhtajan in the early eighties, a central question of 1D quantum magnetism concerns the stability of the spinons. What is the fate of these exotic excitations upon switching on small deviations from the model (2)? Do these additional terms lead to a confinement of the spinons or do they still remain deconfined?

A simple modification of the spin-1/2 AF Heisenberg chain (2) is not to add perturbations but to change the Hilbert space namely to consider the general spin-$S$ case. The resulting model is no longer integrable and one has to resort to approximate methods to describe its physical properties. This question has lead to one of the most profound result in the field of 1D quantum magnetism: the difference between integer and half-integer spins AF Heisenberg chains predicted by Haldane. Integer spin chains are incompressible spin liquids with a finite gap (the so-called Haldane gap) in the energy spectrum. In that case, the spinons are confined and the elementary excitations of the model are massive (optical) $S = 1$ magnons.

In contrast, for half-integer spins, the model displays quantum criticality with similar universal properties as the $S = 1/2$ case that are described by the $su(2)_1$ WZNW universality class. The elementary excitations are still, in this case, gapless spinons.
This remarkable distinction between integer and half-integer spins, historically called Haldane’s conjecture, has been obtained by means of a semiclassical analysis of the model with a special emphasis on the topological nature of the order parameter fluctuations. For a review of this approach, the reader may consult the reviews 35, 24 and the books 36, 37, 14. In the large $S$ limit, the Euclidean action that describes the low-energy properties of the spin-$S$ AF Heisenberg chain reads as follows

$$S_{\text{eff}} = \frac{v}{2g} \int d\tau dx \left[ \frac{1}{v^2} (\partial_\tau \vec{n})^2 + (\partial_x \vec{n})^2 \right] + i 2\pi S Q(\vec{n}) ,$$

(8)

where $v = 2J S$, $g = 2/S$, and $\vec{n}$ is the order parameter of the Néel collinear state. The second contribution in Eq. (8) is a topological term since $Q(\vec{n}) = \int d\tau dx \vec{n} \cdot (\partial_\tau \vec{n} \wedge \partial_x \vec{n})/4\pi$ is an integer, called the Pontryagin index, which measures the number of times the spin configuration $\vec{n}(x, \tau)$ covers the surface of the unit sphere $S^2$. In more mathematical terms, the configurations $\vec{n}(x, \tau)$, with fixed boundary conditions at infinity, are mappings of the sphere $S^2$ onto $S^2$ with homotopy classes classified by an integer $\Pi_2(S^2) = \mathbb{Z}$, which is nothing but the Pontryagin index $Q$. For integer spin chains, the term $2\pi S Q(\vec{n})$ in Eq. (8) has no effect on the path integral of the model and can be discarded. The effective action (8) reduces then to the one of the two-dimensional O(3) non-linear sigma model which is a massive integrable field theory 38, 39. The exact low-energy spectrum of this field theory consists of a massive bosonic triplet with mass $m \sim e^{-2\pi/g}$ and it exhibits no bound states. The AF Heisenberg chain with integer spin is thus expected to have low-lying triplet excitations with a gap that scales as $\Delta \sim ve^{-\pi S}$ when $S$ is large.

In the case of half-integer spins, the topological term in Eq. (8) manifests itself in the path integral through a phase factor $(-1)^Q$ which gives rise to quantum interference between topologically distinct paths in space-time of the order parameter field $\vec{n}(x, \tau)$. It turns out that this process protects the model from a dynamically generated mass gap by quantum fluctuations and, in contrast, it leads to a non-perturbative massless flow towards an IR conformally invariant fixed point 40, 41, 42 which belongs to the $su(2)_1$ WZNW universality class. Since this fixed point describes also the universal properties of the spin-1/2 AF Heisenberg chain, all half-integer spins chains should exhibit the same type of emerging quantum criticality with for instance spin-spin correlations that decay according to the power law behavior 43.
2.3. Haldane spin liquid: spin-1 Heisenberg chain

The simplest incompressible one-dimensional spin liquid phase corresponds to the spin-1 AF Heisenberg chain. The existence of a Haldane gap in this model has generated an intense activity over the years after its prediction. It has been confirmed numerically from exact diagonalizations on finite samples, quantum Monte-Carlo methods, transfer matrix computations, and finally from DMRG calculations. In particular, this latter technique predicts a gap $\Delta = 0.41050(2)J$ and a correlation length $\xi \simeq 6.03(1)$ lattice spacings. From the experimental point of view, several Haldane compounds have been synthesized over the years (see for instance Ref. [12] for a recent review). The two most studied compounds are CsNiCl$_3$ and Ni(C$_2$H$_8$N$_2$)$_2$NO$_2$(ClO$_4$) (NENP) where spin-1 local moments are provided by Ni$^{2+}$ ions. In particular, inelastic neutron scattering experiments on NENP confirm the existence of the Haldane gap. This energy gap has also been observed in several other quasi-1D spin-1 materials as (CH$_3$)$_4$NNi(NO$_2$)$_3$, AgVP$_2$S$_6$, and Y$_2$BaNiO$_5$, which is probably the best realization of the spin-1 AF Heisenberg chain. Finally, it is worth noting that the recent compounds Ni(C$_5$D$_{14}$N$_2$)$_2$N$_3$(PF$_6$) (NDMAP) and Ni(C$_5$H$_{14}$N$_2$)$_2$N$_3$(ClO$_4$) (NDMAZ) enable to investigate experimentally the high magnetic field properties of the spin-1 Heisenberg chain. From the theoretical point of view, several methods have been introduced to shed light on the nature of the mechanism of the Haldane-gap phenomena and to determine the main characteristics of this incompressible spin liquid phase. These approaches are the non-linear sigma model field theory obtained in the large spin limit, the valence bond state (VBS) description, the Majorana fermions method, and the restricted Hilbert space approach.

Probably, the simplest and most appealing approach to study the physical properties of the spin-1 AF Heisenberg chain consists to add a bi-quadratic interaction to the model (2):

$$H_\beta = J \sum_i \left[ \vec{S}_i \cdot \vec{S}_{i+1} + \beta \left( \vec{S}_i \cdot \vec{S}_{i+1} \right)^2 \right].$$

One of the main interest of this extended Heisenberg model is that for $\beta = 1/3$, the so-called AKLT point, the model has an exactly solvable ground state which captures the main characteristics of the Haldane spin liquid phase. At this special point, the Hamiltonian (3) is equivalent to the sum of projection operators $P_2(i,i+1)$ that project onto the spin-2
contribution for every pair of nearest-neighbor spins:

\[ \mathcal{H}_{AKLT} = \mathcal{H}_{\beta=1/3} = 2J \sum_i \left[ P_2(i, i+1) - \frac{1}{3} \right]. \]  

(10)

From this structure, the ground state of this model can be constructed exactly using nearest-neighbor valence bonds. Following Affleck and coworkers pioneer work\(^{57}\), each original $S = 1$ spin is written as two spin-1/2 variables in a triplet state. The ground state is then obtained by coupling into a singlet state all nearest-neighbor spin-1/2, thus forming a crystalline pattern of valence bonds (see Fig. 2). This state is called the valence-bond-solid (VBS) state. For periodic BC and an even number of sites, this VBS state is the unique ground state and does not break the translation symmetry of the Hamiltonian\(^{10}\). It can also be shown rigorously\(^{57}\) that the AKLT model has a spectral gap whose value has been computed numerically: $\Delta \simeq 0.7J$\(^{61}\). In addition, the authors of Ref.\(^{57}\) have also been able to show that the spin-spin correlation function decays exponentially with the distance with a finite correlation length equals to $\xi = 1/\ln 3$ lattice spacings. The VBS state is therefore the simplest incompressible spin liquid phase with a unique ground state and a gap to all excitations.

Despite the exponential decay of correlations in this spin liquid phase, there is a subtle form of hidden AF ordering. This hidden order was first discovered by den Nijs and Rommelse\(^{62}\) using an equivalence of the spin-1 chain to a two-dimensional restricted solid-on-solid model and also later by Tasaki\(^{63}\) by means of a geometrical approach. A simple way to exhibit this hidden AF ordering is to look at the VBS state written in the conventional $S_z$ representation. A typical configuration has the following structure: $..0 \uparrow 0..0 \downarrow \uparrow 0 \downarrow 0..0 \uparrow 0..0$, i.e. each $\uparrow (S_n^z = +1$ state) is followed by $\downarrow (S_n^z = -1$ state) with an arbitrary number of 0 states ($S_n^z = 0$ state) between and vice versa. If we ignore the spins with $S_n^z = 0$ state then the remaining spins display a perfect long-range spin-1/2 Néel ordering: $\uparrow \downarrow \uparrow \downarrow \downarrow \downarrow \ldots$ The VBS state has thus a perfect dilute AF Néel order. Because of the arbitrary
number of $S^n_z = 0$ sites inserted between $S^p_z = \pm 1$ sites, the long-range AF order is invisible in the spin-spin correlation function which has an exponential decay. However, this hidden order becomes manifest in the non-local string order parameter introduced by den Nijs and Rommelse and defined by

$$\mathcal{O}_\text{string}^\alpha = -\lim_{|i-j| \to \infty} |i-j| \langle S^\alpha_i \exp \left( i \pi \sum_{k=j+1}^{i-1} S^\alpha_k \right) S^\alpha_j \rangle,$$

(11)

with $\alpha = x, y, z$. In the VBS ground-state, this order parameter can be computed exactly: $\mathcal{O}_\text{string}^\alpha = 4/9$ i.e. this object exhibits long-range order in the VBS phase. The existence of such a hidden order is essential to the basic mechanism of the Haldane gap: breaking this topological order to create an excitation costs a finite energy gap. The properties of this dilute AF order was further elucidated by Kennedy and Tasaki. They were able to show, using a non-local unitary transformation, that the long-range order $\mathcal{O}_\text{string}^\alpha \neq 0$ and the Haldane gap are related to a spontaneous breaking of a hidden $Z_2 \times Z_2$ symmetry. In particular, the string order parameter becomes the usual local ferromagnetic order parameter under this non-local unitary transformation. Another important consequence of this broken $Z_2 \times Z_2$ symmetry is the existence of a quasi-degeneracy of four lowest energy levels for a finite-size chain. This can also be understood within the VSB state since for open BC the AKLT model has exactly four ground states with the existence of two spin-1/2 degrees of freedom that are unpaired at each end of the chain (see Fig. 3). Three of these states constitute a spin triplet whereas the last one is a spin singlet.

Beside the fact that only the ground state of the Hamiltonian is known exactly, the AKLT approach is also very useful to describe low-lying excitations of the model through a variational approach starting from the VBS state. A first approach consists to create a “magnon” excitation by

![Fig. 3. VBS state with open boundary conditions; unpaired bonds are left at the boundaries resulting into two free spin-1/2 objects at the edges.](image-url)
applying a spin-flip on the VBS ground state ($|VBS\rangle$):

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \exp(ikj) S_j^+|VBS\rangle,$$

(12)

$N$ being the system size. It is very tempting to interpret this excitation as a standard $S = 1$ magnon i.e. an excitation which modifies only locally the AF Néel order. In fact, the trial wave function (12) does not describe a magnon excitation of the perfect hidden Néel order of the VBS state but it has instead a solitonic nature. Indeed, in the $S_z$ representation, a typical configuration that appears in $S_j^+|VBS\rangle$ has the following structure:

..0↑0..0↓↑0↑0↑..0↑0..

Removing the 0 states, this configuration becomes ↑↓↑↑↓↑.. i.e. a soliton or domain wall which destroys non-locally the AF Néel long-range order. An alternative approach to describe the elementary excitations of the AKLT (10) model is to start from the VBS state and to promote the spins of the link $(j, j + 1)$ into a triplet state $|\Phi^a_j\rangle$.

This low-lying excitation carrying momenta $k$ describes a moving triplet bond, usually called crackion, and it is defined by

$$|\Phi^a(k)\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \exp(ikj) |\Phi^a_j\rangle.$$

(13)

In fact, the two approaches (12) and (13) are related and describe the same dispersion relation for the triplet excitation. The solitonic nature of the excitation becomes manifest by considering open boundary conditions and the non-local unitary transformation introduced by Kennedy and Tasaki. In that case, the trial excited wave function corresponds to a domain wall that interpolates between two of the four possible ground states.

One of the main success of this VBS approach stems from the fact that the properties of the VBS state are not special to the AKLT point but have a larger extent. This model turns out to be smoothly connected to the spin-1 Heisenberg chain in the sense that they share the same physical properties. In this respect, numerical investigations of the phase diagram of the bilinear-biquadratic spin-1 Hamiltonian have revealed that this model with $|\beta| < 1$ belongs to a same phase, the Haldane phase, whose main characteristics are well described by the VBS state. In particular, the origin of the Haldane gap in the spin-1 Heisenberg chain is a consequence of a hidden $Z_2 \times Z_2$ broken symmetry as for the AKLT model. At the Heisenberg point $\beta = 0$, the hidden AF Néel ordering is no
longer perfect as at the AKLT point. It is only locally destroyed so that there is still long-range order at $\beta = 0$ with a non-zero string order parameter $O_{\text{string}}^a \simeq 0.374$ \cite{76,77,46}. A second consequence of the spontaneous breaking of the $Z_2 \times Z_2$ symmetry is the two almost free $S = 1/2$ degrees of freedom at the ends of a finite chain with open BC. At the Heisenberg point, exact diagonalizations \cite{68} of finite open samples with an even number of sites have shown that the ground state is a singlet and the existence of an exponentially low-lying triplet state (the so-called Kennedy triplet) in the Haldane gap. This leads to a fourfold ground-state degeneracy in the thermodynamic limit which can be simply understood by means of the VBS description of Fig. \ref{fig:3} as described above. The existence of these two $S = 1/2$ edge states is a robust and striking property of the Haldane spin liquid phase and they are present in the whole region $|\beta| < 1$ of the bilinear-biquadratic spin-1 model \cite{73,74}. The physical properties of these $S = 1/2$ chain-boundary excitations in the open spin-1 chain have been investigated in detail numerically \cite{46,75,76,77,78,79,73,74,80} and also analytically \cite{81,82,83}. Remarkably enough, this liberation of fractional spin-1/2 degrees of freedom in a spin-1 Heisenberg chain has been observed experimentally in the spin-1 compound NENP cut by non-magnetic impurities ($\text{Zn}^{2+}$ or $\text{Mg}^{2+}$) \cite{84,85} and also doped with magnetic ions ($\text{Cu}^{2+}$) \cite{86}. In particular, these spin-1/2 chain-boundary excitations reveal itself as satellite peaks in the NMR profile of the Mg-doped Y$_2$BaNiO$_5$ \cite{87}. A very recent inelastic neutron scattering experiment \cite{88} further probe the microscopic structure of these edge states through the wave-vector dependence of the Zeeman resonance.

### 2.4. General spin-$S$ case

For higher spin-$S$ value ($S > 1$), the difference between integer-spin and half-integer AF Heisenberg chains has been explained within the semiclassical analysis pioneered by Haldane \cite{34,35} and also by means of the Abelian \cite{89} and non-Abelian \cite{40,90} bosonization approaches. The Haldane’s conjecture for $S > 1$ has also been investigated numerically and experimentally over the years.

The numerical studies \cite{91,92,93,94} of the $S = 3/2$ AF Heisenberg chain report that the model displays a critical behavior that belongs to the same universality class as the $S = 1/2$ Heisenberg chain. Several quasi-1D AF compounds with spin $3/2$ have been found like CsVBr$_3$, CsVCl$_3$ \cite{95,96}, and AgCrP$_2$S$_6$ \cite{97}. In particular, the uniform susceptibility
of the spin-3/2 AgCrP$_2$S$_6$ material measured experimentally is consistent with the result for spin-1/2, i.e. confirming the universal behavior of half-integer spins. In the $S = 2$ case, the existence of a Haldane gap has been established in various quantum Monte-Carlo approaches and in DMRG calculations. In the thermodynamic limit, a finite gap of $\Delta = 0.085(5)J$ and a finite correlation length $\xi = 49(1)$ lattice spacings have been found in DMRG. Several Haldane spin-2 compounds have been discovered over the years (see Ref. for a review) like CsCrCl$_3$ and (2,2'-bipyridine)trichloromanganase(III) (MnCl$_3$(bipy)). Experiments on this last compound report the existence of a gap $\Delta/J = 0.07 \pm 0.02$ which is in good agreement with the numerical results.

Integer $S > 1$ spin chains share similar properties as the $S = 1$ case. In particular, there is still a hidden AF order in the integer $S > 1$ case. A non-local string order parameter that measures this topological order has been proposed by Oshikawa:

$$ O^\alpha_{\text{string}} \left( \frac{\pi}{S} \right) = - \lim_{|i-j| \to +\infty} \langle S^\alpha_i \exp \left( i\pi/S \sum_{k=j+1}^{i-1} S^\alpha_k \right) S^\alpha_j \rangle, \quad (14) $$

where $\alpha = x, y, z$. This order parameter can be exactly computed in spin-$S$ generalization of the VBS state and numerically for the Heisenberg chain. For instance, in the spin-2 case, DMRG calculations found $O^\alpha_{\text{string}} (\pi/2) = 0.726(2)$. However, the full characterization of the hidden AF order for integer spin $S > 1$ is less clear than for $S = 1$ where, as seen above, it is related to a broken $Z_2 \times Z_2$ symmetry. In fact, the nature of the hidden symmetry breaking is still an open problem for $S > 1$. Insight on this issue might be gained by studying the structure of edge states of open integer spin-$S$ chains. Higher-$S$ VBS states with open BC have $(S + 1)^2$-fold degenerate ground states with the presence of two free $S/2$ degrees of freedom at each end of the chain in close parallel to the $S = 1$ case (see Fig. 3). A non-linear sigma model approach of the open spin-$S$ Heisenberg chain reveals the existence of spin-$S/2$ edge states in the integer spin case. These results lead us to expect that $Z_{2S} \times Z_{2S}$ might be a good candidate to describe the hidden symmetry breaking scheme in the integer spin-$S$ case. In the $S = 2$ case, the approximate nine-fold degeneracy of the ground state of finite open chains have been observed numerically by DMRG. The presence of these spin-1 chain-end excitations has also been confirmed experimentally in the ESR study of the spin-2 Haldane compound CsCrCl$_3$ doped with non-magnetic Mg$^{2+}$ ions.
The liberation of these fractional spin-$S/2$ degrees of freedom when non-magnetic impurities are introduced in integer spin-$S$ AF Heisenberg chain is a remarkable fact. It stems from the presence of a non-trivial topological ordering in the ground state of the spin-$S$ chain. In this respect, the existence of these edge states opens the possibility to make a distinction between spin-$S$ Haldane liquid phases. These massive spin liquid phases share the same thermodynamics properties but have for instance different chain-end ESR response due to the special nature of their edge states. In fact, this topological distinction between gapped spin liquid phases can also be applied to the massless case. All half-integer AF Heisenberg spin chains have the same bulk universal properties described by an IR fixed point which belongs to the $su(2)_1$ WZNW universality class. However, the ground state of half-integer spin chains still sustains a topological ordering. Indeed, by means of a non-linear sigma model approach, Ng [106] has given evidence that edge states with fractionalized quantum number $(S - 1/2)/2$ exist in open half-integer spin-$S$ AF Heisenberg chain. This interesting result has been confirmed numerically by examining open spin chains with spins up to $S = 9/2$ by DMRG [77],[108]. Recently, a numerical investigation [109] of the non-local string order parameter (14) directly reveals the presence of a topological order in half-integer spin-$S$ AF Heisenberg chains. The existence of this hidden topological order (or the related chain-end excitations) leads to a topological distinction between gapless spin liquid phases with the same emerging quantum criticality.

2.5. Two-leg spin ladder

A further impetus for the study of low-dimensional spin systems was given by the discovery of spin-ladder materials one decade ago [111]. These systems consist of a finite number ($n_{\text{leg}}$) of spin-1/2 AF Heisenberg chains coupled by a transverse exchange interaction $J_\perp$. The Hamiltonian of this model reads as follows

$$H_{n_{\text{leg}}} = J_{\parallel} \sum_{i}^{n_{\text{leg}}} \sum_{a=1}^{n_{\text{leg}}} \vec{S}_{a,i} \cdot \vec{S}_{a,i+1} + J_{\perp} \sum_{i}^{n_{\text{leg}}} \sum_{a=1}^{n_{\text{leg}}-1} \vec{S}_{a,i} \cdot \vec{S}_{a+1,i},$$

where $\vec{S}_{a,i}$ is a spin-1/2 operator at site $i$ on the $a$-th chain and the in-chain exchange $J_{\parallel}$ is antiferromagnetic ($J_{\parallel} > 0$). One of the main interest of these spin ladders stems from the fact that they are intermediate objects between 1D and 2D systems. Moreover, these quasi-1D systems share many properties with the cuprates and, because of their relative simplicity, may
provide some insights on mechanisms behind high-$T_c$ superconductivity.

Despite this original motivation to study spin ladders, these systems display striking properties which make them interesting by themselves in the field of 1D quantum magnetism. In particular, in close parallel to the qualitative difference between integer and half-integer AF Heisenberg spin chains, the universal properties of spin ladders with open boundary conditions in the transverse direction strongly depend on the parity of the number of legs $n_{\text{leg}}$. Ladders with even number of legs are spin liquids with a finite gap in the excitation spectrum and exponentially decaying spin-spin correlations. The original massless spinons of the spin-1/2 chains get confined by the interchain coupling to form coherent optical spin-1 magnon excitations. Upon doping, these ladders display quasi-long-range superconducting pairing correlations with an approximate d-wave symmetry. In contrast, odd-legged ladders have a gapless spectrum which is characterized by a central charge $c = 1$ corresponding to an effective $S = 1/2$ AF Heisenberg spin chain. Upon doping, odd-legged ladders exhibit a metallic behavior typical of a Luttinger liquid. This difference on the energy spectrum of the spin ladders depending on the parity of $n_{\text{leg}}$ is strongly reminiscent of the Haldane conjecture for AF Heisenberg spin chains with $S = n_{\text{leg}}/2$. It can be simply understood from the fact that in the strong ferromagnetic rung coupling limit the spin ladder is indeed equivalent to a single $S = n_{\text{leg}}/2$ AF Heisenberg chain.

Several spin ladders materials have been discovered over the years. The family of compounds Sr$_{n-1}$Cu$_{n+1}$O$_{2n}$ approximately realize ladders with $n_{\text{leg}} = (n + 1)/2$, $n = 3, 5, 7...$ legs. The even-odd scenario in ladders was confirmed experimentally from susceptibility and muons resonance measurements on this compound with $n = 3$ and $n = 5$. The existence of a spin gap in the two-leg case ($n_{\text{leg}} = 2$) was found in experiments on the two-leg compounds $A_{14}\text{Cu}_{24}O_{41}$ ($A_{14} \equiv \text{La}_6\text{Ca}_8$, (Sr,La,Ca)$_{14}$), Cu$_2$(C$_5$H$_{12}$N$_2$)$_2$Cl$_4$, Cu$_2$(Cu$_2$(C$_5$H$_{12}$N)$_2$)$_2$Cl$_4$, and the organic compound BIP-BNO. However, it is worth noting that additional interactions are required to fully describe these materials such as for instance a non-negligible ring exchange for the compound $A_{14}\text{Cu}_{24}O_{41}$ or a small diagonal interchain exchange interaction for Cu$_2$(C$_5$H$_{12}$N$_2$)$_2$Cl$_4$.

The existence of a spin gap in the two-leg case can be anticipated by considering a strong-coupling analysis of the model with $n_{\text{leg}} = 2$. In the limit of strong AF interchain coupling $J_\perp \gg J_{||}$, the ground state consists of singlet bonds formed across the rungs of the ladder, with triplet
excitations separated by a large energy gap of order $J_\perp$. Perturbations in small $J_\parallel$ will cause these singlet bonds to resonate and the triplet excitations form a band with bandwidth $\sim J_\parallel$ but the spectral gap survives $J_\perp$ \cite{126}. In the strong ferromagnetic interchain $-J_\perp \gg J_\parallel$ coupling case, local spins $S = 1$ associated with each rung of the ladder are formed leading thus to a spin-1 AF Heisenberg chain with a non-zero Haldane gap in the energy spectrum. The cross-over between strong and weak coupling limits has been carefully analysed in numerical calculations \cite{127,128,129,130}. It turns out that the ground state in the two strong-coupling limits $|J_\perp|/J_\parallel \gg 1$ evolves adiabatically with increasing $J_\parallel$. The spin gap survives for arbitrarily large $J_\parallel/|J_\perp|$ and finally vanishes for $J_\perp = 0$, i.e. when the two spin-1/2 AF Heisenberg chains are decoupled. The critical point $J_\perp = 0$ separates thus two strong-coupling massive phases: a rung-singlet phase for $J_\perp > 0$ and a phase with $J_\perp < 0$ which is smoothly connected to the Haldane phase of the spin-1 chain.

The opening of the spin gap upon switching on the interchain coupling can be investigated by means of the bosonization approach \cite{131,132,89}. In particular, the field theory that accounts for the massless spinon-optical magnon transmutation, when $J_\perp$ is small, corresponds to an $SO(3) \times Z_2$ symmetric model of four massive Majorana (real) fermions \cite{132,133}, i.e. four off-critical 2D Ising models (for a review see for instance the book \cite{10}). The resulting low-energy field theory is described by the following Hamiltonian density

$$H_{\text{bos} = 2} = \frac{-iv}{2} \left( \tilde{\xi}_R \cdot \partial_x \xi_L - \xi_L \cdot \partial_x \tilde{\xi}_R \right) - i m_t \tilde{\xi}_R \cdot \tilde{\xi}_L - \frac{iv_s}{2} \left( \xi_0^R \partial_x \xi_0^L - \xi_0^L \partial_x \xi_0^R \right) - im_s \xi_0^R \xi_0^L + H_{\text{marg}}, \quad (16)$$

where $\tilde{\xi}_{R,L}$ are a triplet of right and left-moving Majorana fermions that describe the $S = 1$ low-lying excitations of the two-leg spin ladder and the Majorana fermion $\xi_0^{R,L}$ accounts for the singlet excitation. The masses of these real fermions are given by in the weak coupling limit: $m_t = J_\perp \lambda^2 / 2\pi$ and $m_s = -3J_\perp \lambda^2 / 2\pi$, $\lambda$ being a non-universal constant. These triplet and singlet massive Majorana fermions are weakly coupled by a marginal perturbation associated to the last term of Eq. (16):

$$H_{\text{marg}} = \frac{g_1}{2} \left( \tilde{\xi}_R \cdot \tilde{\xi}_L \right)^2 + g_2 \tilde{\xi}_R \cdot \tilde{\xi}_L \xi_0^R \xi_0^L, \quad (17)$$

with $g_1 = -g_2 = \pi a_0 J_\perp / 2$, $a_0$ being the lattice spacing. The model \cite{10}, derived in the weak coupling limit $|J_\perp| \ll J_\parallel$, is expected to have a larger
extent and to capture in fact the low-energy properties of the two-leg spin ladder for arbitrary $J_\perp$ with a suitable redefinition of the masses $m_{s,t}$, velocities $v_{s,t}$, and coupling constants $g_{1,2}$. This stems from the continuity between the weak- and strong-coupling limits in the two-leg spin ladder observed numerically. In addition, in the strong ferromagnetic rung limit $-J_\perp \gg J_\parallel$, the singlet excitation described by the Majorana fermion $\tilde{\xi}_{R,L}^0$ are frozen ($|m_s| \to \infty$) so that the low-energy properties of the model (16) are governed by the triplet magnetic excitations corresponding to the fermions $\tilde{\xi}_{R,L}$. The resulting model coincides with the low-energy field theory of the spin-1 AF Heisenberg chain obtained by Tsvelik by perturbing around the Babujian-Takhtajan integrable point of the bilinear-biquadratic spin-1 chain with $\beta = -1$. The Hamiltonian correctly thus captures the low-energy properties of the two-leg spin ladder in the limit $-J_\perp \gg J_\parallel$ where it reduces to the spin-1 AF Heisenberg chain.

Despite of its apparent simplicity, the model (16) is not an integrable field theory and takes the form a massive $SO(3) \times Z_2$ Gross-Neveu model. However, the leading behavior of the physical quantities of the two-leg spin ladder can be determined by treating the marginal contribution (17) perturbatively. As shown in Ref. 132, this term leads to a renormalization of the masses and velocities so that the low-energy description (16) simply reduces to four independent massive Majorana fermions, i.e. four decoupled off-critical 2D Ising models. This mapping onto off-critical 2D Ising models can then be exploited to derive the low-energy properties of the two-leg spin ladder such as, for instance, the spectrum of elementary excitations, low-T thermodynamics and the leading asymptotics of spin-spin correlation functions. To this end, one needs to express the spin operators $\vec{S}_{a,i}$ in terms of the Ising fields. In the continuum limit, the spin densities separate into the smooth and staggered parts, $\vec{S}_a(x) = \vec{J}_{aL}(x) + \vec{J}_{aR}(x) + (-1)^{x/a} \vec{n}_a(x)$. The chiral su(2)$_1$ currents, uniform parts of the spin densities, can be written locally in terms of the Majorana fermions:

$$\vec{J}_{aR,L} = -\frac{i}{4} \tilde{\xi}_{R,L} \wedge \tilde{\xi}_{R,L} + i \frac{\tau_a}{2} \tilde{\xi}_{R,L} \tilde{\xi}_{R,L}^0,$$

with $\tau_1 = +1$ and $\tau_2 = -1$. In contrast, the staggered magnetizations $\vec{n}_a$ are non-local in terms of the underlying fermions and express in terms of the order $\langle \sigma_a \rangle$ and disorder $\langle \mu_a \rangle$ of the Ising models:

$$\vec{n}_+ \sim (\mu_0 \mu_1 \sigma_2 \sigma_3, \mu_0 \sigma_1 \mu_2 \sigma_3, \mu_0 \sigma_1 \sigma_2 \sigma_3),$$

$$\vec{n}_- \sim (\sigma_0 \sigma_1 \mu_2 \sigma_3, \sigma_0 \mu_1 \sigma_2 \sigma_3, \sigma_0 \mu_1 \mu_2 \sigma_3),$$

(18)
where \( \vec{n}_\pm = \vec{n}_1 \pm \vec{n}_2 \). Due to the existence of non-zero masses \( m_{s,t} \) for the fermions in Eq. (16), all related Ising models are non-critical. A spectral gap is thus present in the two-leg spin ladder for all signs of the interchain interaction \( J_\perp \) and it opens linearly with \( J_\perp \) in the weak coupling limit. In addition, the Ising description (18,19) of the spin densities allows the calculation of the leading asymptotics of spin-spin correlations using exact results of the two-point function of a non-critical Ising model. Since the signs of the triplet and singlet masses are always opposite in Eq. (16), it can be shown that the dynamical spin susceptibility displays a sharp single-magnon peak near \( q = \pi/a_0 \) and \( \omega = |m_t| \) which reflects the elementary nature of the triplet excitations for all signs of \( J_\perp \).

In this respect, the spin liquid phase of the two-leg spin ladder is very similar to the Haldane phase of the spin-1 chain. However, it has been stressed recently that the phase with \( J_\perp < 0 \), smoothly connected to the Haldane phase of the spin-1 chain, and the rung-singlet phase for \( J_\perp > 0 \) are, in fact, topologically distinct gapped phases (137). The distinction is intimately related to the short-range valence bond structure of their ground states. In close parallel to the classification of short-range valence bond configurations on a two-dimensional square lattice (138), two different topological classes can be defined in the one-dimensional case by counting the number \( Q_y \) of valence bonds crossing an arbitrary vertical line (137). In the case of the rung-singlet phase, \( Q_y \) is always even while it is odd for the Haldane phase with \( J_\perp < 0 \). Two different non-local string order parameters can then be defined in connection to this topological distinction (129, 130, 137):

\[
O_{\text{even}}^\alpha = -\lim_{|i-j| \to +\infty} \langle (S_{1,i+1}^\alpha + S_{2,i}^\alpha) e^{i\pi \sum_{k=j+1}^{i-1} (S_{1,k+1}^\alpha + S_{2,k}^\alpha)} (S_{1,j+1}^\alpha + S_{2,j}^\alpha) \rangle
\]

\[
O_{\text{odd}}^\alpha = -\lim_{|i-j| \to +\infty} \langle (S_{1,i}^\alpha + S_{2,i}^\alpha) e^{i\pi \sum_{k=j+1}^{i-1} (S_{1,k}^\alpha + S_{2,k}^\alpha)} (S_{1,j}^\alpha + S_{2,j}^\alpha) \rangle,
\]

with \( \alpha = x, y, z \). The Haldane and rung-singlet phases are then characterized by \( O_{\text{odd}}^\alpha \neq 0, O_{\text{even}}^\alpha = 0 \), and by \( O_{\text{even}}^\alpha \neq 0, O_{\text{odd}}^\alpha = 0 \) respectively. The order parameters (20) reveal also the difference nature of the hidden AF Néel order in the Haldane phase (non-zero triplet states along the rung) and in the rung-singlet phase (non-zero triplet states along the diagonal) (129, 130, 137). This topological difference can be discussed in light of the Ising description (19) of the two-leg spin ladder. The string order parameters (20) can be expressed in terms of the order and disorder Ising operators (139) \( O_{\text{odd}}^\alpha \sim \langle \sigma_1^\alpha \rangle^2 \langle \sigma_2^\alpha \rangle^2 \) and \( O_{\text{even}}^\alpha \sim \langle \mu_1 \rangle^2 \langle \mu_2 \rangle^2 \). For a ferromagnetic (respectively antiferromagnetic) interchain coupling, the Ising models in the triplet sector are in their ordered (respectively disordered) phases so that \( O_{\text{odd}}^z \neq 0 \).
and $O^x_{\text{even}} = 0$ (respectively $O^x_{\text{odd}} = 0$ and $O^x_{\text{even}} \neq 0$). In this Ising description, the phases with $J_\perp > 0$ and $J_\perp < 0$ are thus simply related by a Kramers-Wannier duality transformation on the underlying Ising models.

The topological distinction between these two phases becomes manifest when analysing the ground-state degeneracy depending on the nature of BC used. In the open BC case, as noted by the authors of Ref. [137], ground states of gapped spin liquid states characterized by an odd value of $Q_y$ have spin-1/2 edge states, while these end states disappear when $Q_y$ is even. The existence of these $S = 1/2$ chain-end degrees of freedom leads to a ground-state degeneracy in a two-leg spin ladder with open BC. In the Haldane phase of the ladder with $J_\perp < 0$, finite open chains with an even number of sites have a singlet ground-state with an exponentially low-lying triplet in the spin gap resulting on a fourfold ground-state degeneracy in the thermodynamic limit. In contrast, the ground state in the rung-singlet phase is always unique whether open or periodic BC are used: no low-lying triplet states are found inside the spin gap in open ladder with $J_\perp > 0$ and an even number of sites. In this respect, the rung-singlet phase with $J_\perp > 0$, in contrast to the $J_\perp < 0$ case, is not equivalent to the Haldane phase characterized by $S = 1/2$ chain-end excitations even though they share similar properties such as the presence of a spin gap, and a non-zero string order parameter. The existence or absence of spin-1/2 edge states in the open two-leg spin ladder can also be discussed within the Ising model description [16]. In particular, the fourfold ground-state degeneracy of the open two-leg ladder with a ferromagnetic interchain coupling can be obtained within this approach. Indeed, for $J_\perp < 0$, the three Ising models for the triplet sector are all in their ordered phases while the Ising model for the singlet degrees of freedom belongs to its disorder phase so that $\langle \sigma_i \rangle \neq 0$ ($i = 1, 2, 3$) and $\langle \sigma_0 \rangle = 0$. In that case, each Ising model in the triplet sector has a doubly degenerate ground state which gives thus an eightfold degeneracy. However, there is a redundancy in the Ising description since the triplet Hamiltonian in Eq. (16), the total uniform and staggered magnetizations in Eqs. (18,19) are all invariant under the transformation: $\xi_{R,L}^i \rightarrow -\xi_{R,L}^i, \mu_i \rightarrow \mu_i, \sigma_i \rightarrow -\sigma_i, i = 1, 2, 3$. This leads to a physical fourfold ground-state degeneracy as it should be in the Haldane phase. Another advantage of this Ising model description is to make explicit the spontaneous breaking of a hidden $Z_2 \times Z_2$ symmetry associated to the ground-state degeneracy for $J_\perp < 0$. The existence of this hidden symmetry can also be revealed with help of a non-local unitary transformation on the lattice spins of the two-leg ladder [140] as in the spin-1 AF Heisenberg chain [63].
Finally, the Haldane and rung-singlet phases of the two-leg spin ladder can also be distinguished by their optical properties. In the rung-singlet phase, the existence of singlet or triplet two-magnon bound states below the two-magnon continuum has been predicted theoretically by a number of groups. Recently, the singlet two-magnon bound state has been observed in the optical conductivity spectrum of the compound (La,Ca)$_{14}$Cu$_{24}$O$_{41}$ which contains layers with Cu$_2$O$_3$ two-leg spin ladder with $J_\perp > 0$. On the contrary, it is expected that for a ferromagnetic interchain coupling no bound states are present below the two-particle continuum as in the spin-1 AF Heisenberg chain. The existence of bound states in the two-leg spin ladder can be discussed in the context of the field theory approach. Indeed, the marginal term plays its trick by giving rise to an effective interaction between the magnon excitations. In the AF interchain case, the contribution is in fact marginal relevant and the interaction between magnons are attractive. The presence of bound states in this case can be argued qualitatively as a result of this attractive interaction. The Hamiltonian takes the form of a SO(4) Gross-Neveu model up to irrelevant contributions and a duality transformation $\xi_R^0 \rightarrow -\xi_R^0$ in the singlet sector. The latter model is integrable and, for $J_\perp > 0$, the low-lying excitations are massive kinks and anti-kinks interpolating between the two degenerate ground states resulting from the spontaneous symmetry breaking of the $Z_2$ symmetry: $\xi_R^a \xi_L^a \rightarrow -\xi_R^a \xi_L^a$, $a = 0, ..., 3$. The mass terms in the full Hamiltonian break explicitly this $Z_2$ symmetry and the ground-state degeneracy is lifted so that kink configurations are no longer asymptotic states of the field theory. The situation is in close parallel to the two-dimensional Ising model in its low-temperature phase upon switching on a magnetic field. The mass terms are then expected to induce a linear confining potential between the kinks giving rise to a sequence of bound states. In contrast, when $J_\perp < 0$, the perturbation is a marginal irrelevant contribution and the spectrum of the model consists of massive fermions and their multiparticle excitations.

2.6. Non-Haldane spin liquid

The Haldane and rung-singlet phases of the two-leg spin ladder are not the only possible gapped spin-liquid states available in 1D unfrustrated quantum magnets. In this respect, Nersesyan and Tsvelik have discussed the example of a gapped spin liquid phase without any coherent magnon excitations. The spectral function of this state displays a broad feature rather
than a single sharp magnon peak. Such a spin liquid can be stabilized by the introduction of a four-spin interchain interaction that couples two spin-1/2 AF Heisenberg chains:

\[
\mathcal{H}_{so} = J \sum_i \sum_{a=1}^2 \vec{S}_{a,i} \cdot \vec{S}_{a,i+1} + K \sum_i \left( \vec{S}_{1,i} \cdot \vec{S}_{1,i+1} \right) \left( \vec{S}_{2,i} \cdot \vec{S}_{2,i+1} \right). \tag{21}
\]

The biquadratic interaction represents an interchain coupling for the spin-dimerization operators \(\epsilon_{a,i} \sim (-1)^i \vec{S}_{a,i+1} \cdot \vec{S}_{a,i}\) of each chain which can be effectively generated by spin-phonon interaction. A second motivation to investigate the effect of this four-spin interaction stems from orbital degeneracy. In most of transition metal compounds, in addition to the usual spin degeneracy, the low-lying electron states are also characterized by orbital degeneracy. A starting point to study magnetic properties of magnetic insulators with Jahn-Teller ions is the two-band Hubbard-like models. At quarter-filling (one electron per atom), this system is a Mott insulator in the limit of strong Coulomb interaction and the state of each ion can be characterized by a spin degrees of freedom \(\vec{S}_{1,i}\) and an orbital state described by a pseudo-spin-1/2 \(\vec{S}_{2,i}\). In the large Coulomb repulsion limit, the simplest Hamiltonian that describes the competition between spin and orbital degrees of freedom in one dimension reduces to the spin-orbital model (21). In particular, this model has been introduced by Pati et al. to explain the unusual magnetic properties of the quasi-one-dimensional spin gapped material Na\(_2\)Ti\(_2\)Sb\(_2\)O\(_4\)\(^\text{148}\).

The spin-orbital model (21) is unfrustrated for \(K < 0\) whereas the frustration manifests itself in the antiferromagnetic case \(K > 0\) only in the intermediate regime \(K \simeq J\) as we shall see later. The Hamiltonian (21) is invariant under independent SU(2) rotations in the spin (\(\vec{S}_1\)) and orbital (\(\vec{S}_2\)) spaces. For generic couplings, the model (21) is thus SU(2) \(\times\) SU(2) \(\times\) Z\(_2\) symmetric, the additional Ising symmetry being the exchange between the spins \(\vec{S}_1\) and \(\vec{S}_2\). In the weak coupling limit \(|K| \ll J\), this underlying SO(4) symmetry of the spin-orbital model (21) is reflected in the form of its low-energy Hamiltonian density which is described in terms of four massive Majorana fermions:\(^\text{133,149}\)

\[
\mathcal{H}_{so} \simeq -i \frac{v}{2} \sum_{a=0}^3 \left( \xi_R^a \partial_x \xi_L^a - \xi_L^a \partial_x \xi_R^a \right) - im \sum_{a=0}^3 \xi_R^a \xi_L^a, \tag{22}\]

with \(m = \alpha K\), \(\alpha\) being a positive non-universal constant. In Eq. (22), a marginal contribution, similar to Eq. (17), has been neglected. In contrast to the low-energy description\(^\text{136}\) of the standard two-leg spin ladder, the
triplet and singlet Majorana modes become equally important here as a consequence of the SO(4) symmetry of the spin-orbital model \[21\]. The resulting low-energy properties of the model can then be determined by this mapping onto four off-critical 2D Ising models. As the staggered magnetizations \[19\], the spin dimerization fields \(\epsilon_{1,2}\) can be expressed in terms of the order and disorder operators of the underlying Ising models:

\[
\epsilon_a \sim \mu_1\mu_2\mu_3\mu_4 \pm \tau_a\sigma_1\sigma_2\sigma_3\sigma_4.
\]

For an AF biquadratic coupling (\(K > 0\)), the four Ising models belong to their ordered phases so that the model \[22\] enters a spontaneously dimerized phase with a finite gap and \(\langle \epsilon_1 \rangle = -\langle \epsilon_2 \rangle = \pm |\epsilon_0|\). The ground state is thus two-fold degenerate and dimerizes with an alternating pattern as shown in Fig. 4 (a). In contrast, in the ferromagnetic \(K < 0\) case, the dimerization is now in-phase between the two chains: \(\langle \epsilon_1 \rangle = \langle \epsilon_2 \rangle = \pm |\epsilon_0|\) as depicted by Fig. 4 (b). A first distinction between this gapped spin liquid state and the Haldane and rung-singlet phases of the two-leg spin ladder stems from the ground-state degeneracy and the fact that the lattice translation symmetry is spontaneously broken in the ground states.

Fig. 4. Ground states of the spin-orbital model in the weak coupling limit \(|K| \ll J\). The bonds indicate a singlet pairing between the spins. (a) Alternating dimerization for \(K > 0\); (b) in-phase dimerization for \(K < 0\). In each case, a second ground state is obtained by applying the one-step translation symmetry on each chain to the states depicted here.
of Fig. 4. A more drastic difference appears at the level of the low-lying excitations. Indeed, the elementary excitations of the spin-orbital model are neither optical magnons nor massive spinons but a pair of propagating massive triplet or singlet kinks connecting two spontaneously dimerized ground states. An example of such a triplet excitation is described in Fig. 5 for the staggered dimerization phase with $K > 0$. The composite nature of the $S = 1$ excitation reveals itself in the dynamical structure factor of the model which can be determined using the Ising description of the total and relative staggered magnetizations. The dynamical magnetic susceptibility displays a two-particle threshold instead of a sharp magnon peak near $q = \pi/a_0$ as in the two-leg spin ladder or the spin-1 chain. This incoherent background in the dynamical structure factor leads to a new gapped spin liquid phase in unfrustrated quantum magnetism without any coherent magnon excitations. In this respect, this state has been called non-Haldane spin liquid by Nersesyan and Tsvelik. The existence of this spin liquid phase has been confirmed non-perturbatively at two special points in the phase diagram of the spin-orbital model. At $J = 3K/4$, the ground state is exactly known and is a product of checkerboard-ordered spin and orbital singlets as in Fig. 4 (a) obtained in the weak coupling limit. At $K = -4J$, the model is exactly solvable and the ground-state energy and triplet energy gap have been determined exactly. The system has two spontaneously dimerized ground states and belongs to the class of non-Haldane spin liquid.

Finally, it is worth noting that the alternating and in-phase dimerization phases can be distinguished in close parallel to the topological distinction between the two gapped phases of the two-leg spin ladder. Using the topological criterion of Kim et al., one observes from Fig. 4 that the number $Q_y$ of valence bonds crossing an vertical line is odd (respectively even) for

![Fig. 5. Triplet excitation of the alternating dimerization described in terms of a pair of dimerization kinks of each chain.](image-url)
the staggered (respectively in-phase) dimerization. The two non-Haldane spin liquid phases with $K > 0$ and $K < 0$ belong thus to two different topological classes. This difference becomes manifest when investigating the structure of the edge states of the two phases with open BC. The physical properties of these boundary excitations of the semi-infinite spin-orbital model (21) can be determined by means of the Ising mapping (22) of the weak coupling limit similarly to the cut two-leg ladder (22). For $K < 0$, such edge states are absent whereas, in the $K > 0$ case, two spin-1/2 chain-end excitations are expected (22).

3. Frustration effects

In this section, we shall review some of the main aspects of the interplay between frustration and quantum fluctuations in AF spin chains and spin ladders. The natural question is whether frustration can stabilize new types of spin liquid phases with exotic spin excitations not encountered in Section 2. The paradigmatic model to analyse the effect of frustration in spin chains is the $J_1 - J_2$ spin-1/2 Heisenberg chain with Hamiltonian:

$$\mathcal{H} = J_1 \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \sum_i \vec{S}_i \cdot \vec{S}_{i+2},$$

(24)

where the next-nearest neighbor coupling $J_2 > 0$ is a competing AF interaction which introduces frustration. This model can also be viewed as a frustrated two-leg spin ladder where the spin chains are coupled in a zigzag way as shown in Fig. 6. Let us first discuss some of the main characteristics of frustration in 1D spin systems by means of a semiclassical approach.

**Fig. 6.** Two-leg zigzag ladder.

3.1. Semiclassical analysis

Classically, one of the main effect of frustration is to favor non-collinear magnetic ordering where the spins lie in a plane rather than along a single
direction as in unfrustrated spin systems:

\[ \vec{S}_i = S \cos (\vec{Q} \cdot \vec{r}_i) \vec{n}_1 + S \sin (\vec{Q} \cdot \vec{r}_i) \vec{n}_2, \]  

(25)

where \( \vec{r}_i \) is the spatial location of the site \( i \) and \( \vec{n}_{1,2} \) are two mutually orthogonal fixed vectors in spin space with unit length. The magnetic structure corresponding to the ordering (25) is a circular spiral with a pitch angle related to the wave-vector \( \vec{Q} \) which is, in general, incommensurate. In the case of the \( J_1 - J_2 \) chain, the spiral ground state is stabilized when \( J_2/J_1 > 1/4 \). The spins are arranged in a canted configuration in which each spin makes an angle \( \alpha \) with its predecessor such that \( \cos \alpha = -J_1/4J_2 \).

The classical ground state of the model is doubly degenerate since the spin configurations (25) can turn clockwise and counterclockwise with the same energy along the spiral axis \( \vec{n}_3 = \vec{n}_1 \wedge \vec{n}_2 \). A \( \mathbb{Z}_2 \) discrete symmetry characterized by this right- and left-handed chirality is thus spontaneously broken in this helical structure. A corresponding chiral ordering can be defined and detected by the chiral order parameter:

\[ \langle (\vec{S}_i \wedge \vec{S}_{i+1}) \cdot \vec{n}_3 \rangle \neq 0. \]

In contrast to the Néel state of unfrustrated magnets, the SU(2) spin rotation of the Heisenberg Hamiltonian is completely broken in the non-collinear state (25). Three Goldstone or spin-wave modes are thus expected here from this spontaneous symmetry breaking scheme. It leads to a CFT with central charge \( c = 3 \) with extended criticality in comparison to the unfrustrated case (\( c = 2 \), see Section 2). A second distinction stems from the spin-wave calculation for non-collinear ordered states which reports the presence of two different spin-wave velocities. Therefore, the low-energy field theory describing canted magnets is expected to be non-Lorentz invariant. To identify this effective field theory, one needs to define a suitable order parameter for canted magnets. The long-wavelength fluctuations of the spin configurations (26) can be captured by introducing a two-component complex field \( z_a, a = 1, 2 \) of unit modulus (\( |z_1|^2 + |z_2|^2 = 1 \)) such that

\[ \vec{n}_1 + i\vec{n}_2 = \epsilon_{ac} z_c \vec{\sigma}_{ab} z_b, \]  

(26)

where \( \epsilon_{ab} \) is the standard antisymmetric tensor and \( \vec{\sigma} \) is a vector formed by the Pauli matrices. It is indeed straightforward to see that this identification correctly reproduces the constraints: \( \vec{n}_1^2 = \vec{n}_2^2 = 1 \) and \( \vec{n}_1 \cdot \vec{n}_2 = 0 \) if the two-component complex field \( z_a \) belongs to the three-dimensional surface of the unit sphere in four dimensions denoted by \( \mathbb{S}^3 \). Moreover, it is worth noting that the field \( z = (z_1, z_2) \) transforms like a \( S = 1/2 \) spinor under spin rotations. The representation (26) is double-valued since \( z \) and \(-z\)
describe the same non-collinearly ordered state. The order parameter space corresponding to canted magnets is thus $S^3/Z_2$.

A crucial consequence of this identification is that the order parameter allows topologically nontrivial vortices having a $2\pi$ circulation since $\pi_1(S^3/Z_2) = Z_2$ [15]. Upon encircling such a $Z_2$ vortex, called now a vison [15], by a closed loop, the value of the spinor complex field $z = (z_1, z_2)$ changes smoothly from $z$ to $-z$, i.e. two points diametrically opposed on the surface of a sphere $S^3$. The existence of these stable vison defects reveals itself in the semiclassical description of frustrated spin chains like the $J_1 - J_2$ Heisenberg chain [24]. In particular, in sharp contrast to unfrustrated spin chains, no Pontryagin topological term has been found in the effective action that describes the low-energy properties of the $J_1 - J_2$ Heisenberg chain [157,158] and two-dimensional frustrated Heisenberg antiferromagnets [159] in the large spin limit. This stems from the very special topological nature of the order parameter for canted magnets which has a trivial second homotopy group: $\pi_2(S^3/Z_2) = 0$ [155]. However, there is still a subtle distinction between half-integer and integer spins in the $J_1 - J_2$ Heisenberg chain related to the existence of stable visons. A Berry phase calculation shows that tunneling between sectors with different $Z_2$ topological number is possible for integer spins but not for half-integer spins due to cancellation between pairs of paths. The ground state of the $J_1 - J_2$ model is thus non-degenerate for integer spins and two-fold degenerate (two different sectors non-coupled by tunneling effect) for half-integer spins. The quantum number carried by elementary excitations depends also strongly on the nature of the spin. In the half-integer case, there is an energy gap towards the creation of $Z_2$ visons so that these topological defects are strongly suppressed. The low-energy excitations are then described by the spinor field $z = (z_1, z_2)$ of Eq. [20] which is free to propagate in absence of visons and carries a $S = 1/2$ quantum number under spin rotations. Neglecting anisotropy, the Euclidean effective action that governs the leading low-energy properties of the $J_1 - J_2$ chain for half-integer spins is the SU(2) × SU(2) principal chiral model defined by

$$S_G = \frac{v}{2g} \int dx d\tau \text{Tr} \left( \partial_\mu G^\dagger \partial_\mu G \right), \quad (27)$$

where $G$ is a SU(2) matrix formed by the two complex fields $z_{1,2}$. This field theory is integrable and its low-energy spectrum consists of massive excitation with $S = 1/2$ quantum number i.e. massive deconfined spinons. In contrast, in the integer spin case, the visons now proliferate in
the ground state so that the spinor field \( z = (z_1, z_2) \) cannot any longer be defined as single-valued configurations. The suitable fields that describe the low-energy excitations of the \( J_1 - J_2 \) chain with integer spins are bilinear of \( z \) and correspond to the \( \vec{n}_{1,2} \) fields of Eq. (26). In that case, neglecting anisotropy, the effective field theory that captures the leading low-energy properties of the frustrated spin chain with integer spins is the non-linear sigma model \( \text{SO}(3) \times \text{SO}(3) \) with action:

\[
S_R = \frac{v}{2g} \int \! dx d\tau \, \text{Tr} \left( \partial_\mu R^{-1} \partial_\mu R \right),
\]

where \( R \) is a rotation matrix made of the triplet vectors \( \vec{n}_{1,2} \) and \( \vec{n}_3 = \vec{n}_1 \wedge \vec{n}_2 \). The low-lying spectrum of this field theory can be determined by a large \( N \) approach or by means of a strong-coupling analysis: it consists of a triplet of \( S = 1 \) massive magnons. In the integer spin case, the spinons are thus expected to be confined into optical magnons.

An important prediction, obtained in this semiclassical description, is thus the existence of a gapped spin liquid phase for half-integer spins with a two-fold degenerate ground state and massive deconfined spinons. Such a state is stabilized by frustration and represents a spin liquid phase not encountered in Section 2. We shall now consider the ultra-quantum case with \( S = 1/2 \) where a combination of numerical and field theoretical techniques can be used to fully determine the main characteristics of this spin liquid phase.

### 3.2. Spin liquid phase with massive deconfined spinons

The phase diagram of the spin-1/2 \( J_1 - J_2 \) Heisenberg chain has been studied extensively over the years after the bosonization analysis of Haldane. This problem is not also a purely academic question since inorganic compounds such as \( \text{CuGeO}_3 \) or \( \text{LiV}_2\text{O}_5 \) can be considered as prototypes of the spin-1/2 \( J_1 - J_2 \) chain. In particular, values such as \( J_1 \simeq 160 \text{ K} \) and \( J_2/J_1 \simeq 0.36 \) have been proposed for \( \text{CuGeO}_3 \). In addition, the quasi-1D compound \( \text{SrCuO}_2 \) contains a collection of spin-1/2 Heisenberg chains assembled pairwise in an array of weakly coupled zigzag ladders. These zigzag ladders are built from corner-sharing Cu-O chains with an exchange \( J_2 \) staked pairwise in edge-sharing geometry. The frustrating interaction \( J_1 \) between the chains (see Fig. 6) stems from the nearly 90° Cu-O-Cu bonds and is expected to be weak. Finally, two other possible realizations of the spin-1/2 \( J_1 - J_2 \) chain have been recently proposed: the compound \( (\text{N}_2\text{H}_5)\text{CuCl}_3 \) which can be
described as a two-leg zigzag spin ladder with $J_1/J_2 \simeq 0.25$ and Cu[2-(2-aminomethyl)pyridine]Br$_2$ with a ratio $J_2/J_1 \simeq 0.2$.

A starting point for investigating the phase diagram of the spin-1/2 $J_1 - J_2$ chain (24) is to consider the weak coupling limit when $J_2 \ll J_1$. This enables us to study the stability of the massless spinons of the spin-1/2 AF Heisenberg chain upon switching on a small next-nearest-neighbor frustrating interaction $J_2$. In this weak-coupling regime, the low-energy effective Hamiltonian density of the model reads as follows:

$$H_{\text{eff}} = \frac{2\pi v}{3} (\vec{J}_L^2 + \vec{J}_R^2) + \gamma \vec{J}_L \cdot \vec{J}_R,$$

(29)

where $\vec{J}_{L,R}$ are the left and right su(2) currents that generate the su(2) quantum criticality of the spin-1/2 AF Heisenberg chain. The low-energy physics of the model are thus mainly determined by the marginal current-current interaction of Eq. (29) with coupling constant $\gamma \simeq J_2 - J_{2c}$, $J_{2c}$ being a non-universal positive constant. For a small value of $J_2$, one has $\gamma < 0$ so that the interaction in Eq. (29) is a marginal irrelevant contribution. The low-energy physics of the $J_1 - J_2$ chain is thus identical to that of the spin-1/2 AF Heisenberg chain which is governed by the su(2) fixed point with additional logarithmic corrections introduced by the current-current interaction. In this respect, frustration plays no important role in the regime $J_2 < J_{2c}$ and the spin-1/2 Heisenberg phase with massless spinons is stable upon switching on a small value of $J_2$. However, for $J_2 > J_{2c}$ ($\gamma > 0$), the current-current interaction becomes marginal relevant and a strong coupling regime develops with a dynamically generated spectral gap. A phase transition of Berezinskii-Kosterlitz-Thouless (BKT) type occurs at $J_2 = J_{2c}$ which separates the gapless spin-1/2 Heisenberg phase from a fully massive region. The actual value of the transition has been determined numerically by different groups to be: $J_{2c} \simeq 0.2411 J_1$.

At this point, it is worth noting that the quasi-1D material Cu[2-(2-aminomethyl)pyridine]Br$_2$ can be described by a spin-1/2 $J_1 - J_2$ Heisenberg chain with $J_2/J_1 \simeq 0.2$ and should thus belong to the critical Heisenberg phase with $J_2 < J_{2c}$. The absence of a spin gap for this compound has been reported experimentally.

The main characteristics of the strong coupling massive phase with $J_2 > J_{2c}$ can be determined from Eq. (29). The field theory is indeed integrable and corresponds to a chiral Gross-Neveu model or a non-Abelian version of the Thirring model with a SU(2) symmetry. The low-energy excitations of the model are a massive doublet (massive spinons) with
A striking effect of frustration is thus the formation of a mass for the spinons of the spin-1/2 AF Heisenberg chain without confining them into $S = 1$ excitations as in the two-leg spin ladder. A second consequence of frustration is the presence of spontaneously dimerization in the model. A simple way to exhibit this dimerization is to use the bosonization approach to express the Hamiltonian (29) in terms of a $\beta^2 = 8\pi$ sine-Gordon model [165]:

$$
H_{\text{eff}} = \frac{v}{2} \left[ (\partial_x \Phi)^2 + (\partial_x \Theta)^2 \right] + \frac{\gamma}{2\pi} \partial_x \Phi \partial_x \Phi - \frac{\gamma a_0^2}{4\pi^2} \cos \sqrt{8\pi} \Phi,
$$

(30)

where $\Phi_{R,L}$ are the chiral components of the bosonic field $\Phi$ ($\Phi = \Phi_R + \Phi_L$) and $\Theta$ is its dual field ($\Theta = \Phi_L - \Phi_R$). The SU(2) invariance of the model (30) is hidden in the structure of the interaction with a single coupling constant and the fact that the bosonic field $\Phi$ is compactified on a circle with a special radius $R = 1/\sqrt{2\pi}$ consistent with the SU(2) symmetry. This compactification leads to the following identification:

$$
\Phi \sim \Phi + 2\pi R n = \Phi + n\sqrt{2\pi},
$$

(31)

$n$ being integer. In the phase with $J_2 > J_{2c}$, one has $\gamma > 0$ so that the bosonic field is pinned at one of its minima: $\langle \Phi \rangle = p\sqrt{\pi/2}$, $p$ being integer. However, by taking into account the identification (31), the $\beta^2 = 8\pi$ sine-Gordon model (30) has only two inequivalent ground states with $\langle \Phi \rangle = 0$ and $\langle \Phi \rangle = \sqrt{\pi/2}$. This two-fold degeneracy can be interpreted as resulting from the spontaneous breaking of a discrete $Z_2$ symmetry. This symmetry identifies with the one-step translation symmetry ($T_a$) which is described by the following shift on the bosonic field [24]: $\Phi \rightarrow \Phi + \sqrt{\pi/2}$. This symmetry is spontaneously broken in the phase $J_2 > J_{2c}$ and the two ground-state field configurations are connected by this translation symmetry. An order parameter designed to characterize this phase is the spin dimerization operator $\epsilon_n = (-1)^n \vec{S}_n \cdot \vec{S}_{n+1}$ which admits the following bosonic representation in the continuum limit $\epsilon \sim \cos \sqrt{2\pi} \Phi$ [24,10]. This operator changes sign under the lattice translation symmetry and has a non-zero expectation value $\langle \epsilon \rangle \neq 0$ in the two ground states of the $\beta^2 = 8\pi$ sine-Gordon model [30].

For $J_2 > J_{2c}$, frustration stabilizes thus a gapful spontaneously dimerized phase which is characterized by a two-fold degenerate ground state and a spontaneous breaking of the lattice translation symmetry. The elementary excitations of this phase are massive spinons that carry $S = 1/2$ quantum number and identify with the kinks of the underlying dimerization. A simple way to understand the emergence of these deconfined spinons
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is to consider the Majumdar-Ghosh (MG) point at $J_2 = J_1/2$ where the ground state of the lattice model (24) is exactly known. This MG point for the spontaneously dimerized phase plays a similar role than the AKLT point (10) for describing the main properties of the Haldane phase in the phase diagram of the spin-1 bilinear-biquadratic chain (9). For $J_2 = J_1/2$, the Hamiltonian (24) takes the following form up to a constant:

$$H_{MG} = \frac{3J_1}{4} \sum_i P_{3/2}(i-1, i, i+1),$$

(32)

where $P_{3/2}(i-1, i, i+1)$ projects the total spin of the three spins located at sites $i-1, i, i+1$ onto the $S = 3/2$ subspace. For an even number of sites $N$ and periodic BC, the ground state of the MG model (32) is two-fold degenerate and corresponds to the two singlet states:

$$|\Phi_1\rangle = [1, 2][3, 4]...[N−1, N]$$

$$|\Phi_2\rangle = [2, 3][4, 5]...[N, 1],$$

(33)

where $[i, i+1]$ denotes a singlet bond between the spins at the $i$ and $i+1$ sites. The two ground states (33) are represented in Fig. 7. The lattice translation symmetry $T_{a_0}$ is broken in the two ground states (33) and it exchanges them: $T_{a_0}|\Phi_{1,2}\rangle = |\Phi_{2,1}\rangle$. The existence of gap in the MG model (32) has been shown rigorously by Affleck et al. (57) and the spin-spin correlation function can be determined exactly at the MG point (182):

$$\langle \vec{S}_i \cdot \vec{S}_j \rangle = \frac{3}{4} \delta_{i,j} - \frac{3}{8} \delta_{|i−j|,1}.$$  

(34)
This correlation function is thus zero for distance larger than one lattice spacing. The ground-state degeneracy and the spontaneous breaking of a discrete $Z_2$ symmetry suggest the existence of topological excitations which interpolate between the two ground states (33). These kink excitations can be viewed as the insertion of a spin-1/2 in a sea of singlet valence bond states as shown in Fig. 8. This spin-1/2 excitation (spinon) is nothing but a domain wall between the two states (33). For periodic BC, these kinks

Fig. 8. Dimerization kink (spinon) with $S = 1/2$ quantum number.

states appear always in pairs (see Fig. 9) and the low-lying excitations of the MG model can be built starting from the state (35)

$$ |egin{array}{c} p, m \end{array} \rangle = [1, 2] .. [2p - 3, 2p - 2] \alpha_{2p-1} [2p, 2p + 1] .. [2m - 2, 2m - 1] \alpha_{2m} [2m + 1, 2m + 2] .. [N - 1, N], \quad (35) $$

where $\alpha_{2p-1}$ and $\alpha_{2m}$ denote spin-1/2 states located at the sites $2p-1$ and $2m$ respectively. A variational approach of the low-lying excitations can then be done by considering a linear combination of states (35) and it leads to a triplet and singlet continuum spin excitations. In particular,

Fig. 9. Spin-1 excitation built from two spinons.

the dispersion relation of a massive spinon, obtained within this approach, takes the form (36)

$$ \epsilon(k) = \frac{5J_1}{8} + \frac{J_1}{2} \cos(2ka_0). \quad (36) $$
This dispersion relation has been verified numerically by exact diagonalizations. Finally, as shown by Caspers and Magnus, there are additional exact singlet and triplet bound-states at momentum $k = \pi/2a_0$ which are degenerate with energy $E = J_1$. Exact diagonalization and DMRG calculations have confirmed the existence of these bound-states for a small range of momenta close to $k = \pi/2a_0$ in the region $J_2 \geq J_1/2$.

In summary, the spontaneously dimerized phase for $J_2 > J_{2c}$ represents a distinct spin liquid phase stabilized by frustration. The main distinction, from the spin liquid phases of the two-leg spin ladder and spin-orbital model, originates in the fractionalized nature of the quantum number carried by elementary excitations. In particular, instead of a sharp $S = 1$ magnon peak as in the two-leg spin ladder, the dynamical structure factor of the model in this dimerized phase displays an incoherent background with additional bound-states features. The main difference between the dimerized phase of Fig. 7 and the staggered dimerization phase of the spin-orbital model stems from the nature of the low-lying excitations. In the latter phase, the excitations are a pair of propagating massive triplet kinks, as described above, whereas here the elementary excitations are massive spinons.

This dimerized phase, with deconfined massive spinons excitations, extends in the entire region with $J_2 > J_{2c}$ as it has been shown numerically. Fig. 10 represents the evolution of the spin gap $\Delta$, computed by DMRG, as function of the next-nearest neighbor interaction $J_2$. As depicted

![Figure 10](image-url)

Fig. 10. Evolution of the spin gap, computed by DMRG, as a function of $J_2$ for $J_2 > J_{2c}$; taken from Ref. 178.
by Fig. 10, the spin gap is maximum for $J_2/J_1 \approx 0.6$ and decreases to zero in the large $J_2$ limit. The absence of a spin gap in this regime can be easily understood since for $J_2 \gg J_1$, the $J_1 - J_2$ spin chain can be better viewed as a two-leg zigzag ladder (see Fig. 6). In the limit $J_2 \to +\infty$, the two spin-$1/2$ Heisenberg chains are decoupled so that the model is critical: $\Delta = 0$. From the results of Fig. 6, it is very tempting to conclude that the whole region with $J_2 > J_{2c}$ describes a single phase. However, a subtle qualitative change occurs in the model at the level of spin correlation functions after the MG point in close parallel to the Haldane phase of the spin-1 bilinear-biquadratic chain [17,18] just after the AKLT point [10]. Indeed, an incommensurate behavior develops in the real-space spin-spin correlation function for $J_2 > J_1/2$. This incommensurability can be interpreted as a quantum signature of the spiral structure of the classical ground state of the $J_1 - J_2$ spin chain for $J_2 > J_1/4$. After the MG point, the leading asymptotic of the spin-spin correlation function behaves as

$$\langle \vec{S}_i \cdot \vec{S}_{i+r} \rangle \sim \frac{1}{r^{1/2}} \cos(q r a_0) \exp(-r a_0/\xi),$$

(37)

where the oscillation factor depends on $J_2/J_1$. In parallel to the classification of commensurate-incommensurate transitions in Ising spin systems [19], the MG point, as the AKLT point [10], is a disorder point of the first kind [22]. The momentum $q_{\text{max}}$ which maximizes the static structure factor (Fourier transform of the spin-spin correlation function) becomes incommensurate not at the MG point but slightly after $J_{2L} \approx 0.5206 J_1$ at a Lifshitz point [22]. For $J_2 > J_{2L}$, the static structure factor displays a double peak structure rather than a single peak. The low-lying excitations are then characterized by an incommensurate momenta $q_{\text{max}} \neq \pi/a_0$. In contrast to the classical case, it is worth noting that the BKT phase transition between the critical and dimerized phases and the onset of incommensurability at the Lifshitz point occur at different points of the phase diagram, $J_{2c} \approx 0.2411 J_1$ and $J_{2L} \approx 0.5206 J_1$ respectively. Though no real phase transition occurs for $J_2 > J_{2c}$ in the model, the dimerized phase in the region $J_2 > J_{2L}$ can be distinguished from that at $J_{2c} < J_2 < J_{2L}$ due to the existence of this incommensurability. In this respect, the spontaneously dimerized phase for $J_2 > J_{2L}$ with deconfined massive spinons and incommensurate correlations represents a remarkable 1D spin liquid phase stabilized by frustration.

This incommensurability induced by frustration is, in fact, quite general in 1D and not restricted to the spin-$1/2$ case. In particular, this phenomenon appears also in the spin-$1$ case. The zero-temperature phase diagram of the
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spin-1 $J_1 - J_2$ Heisenberg chain has been investigated numerically and a spectral gap exists for all finite value of $J_2$. A first-order transition occurs at $(J_2/J_1)_T \approx 0.744$ separating a Haldane phase from a double Haldane phase. The quantum number carried by elementary excitations is $S = 1$ in full agreement with the semi-classical approach described above. Frustration plays its trick here by giving rise to a similar onset of incommensurability as in the spin-$1/2$ case. A disorder point of the first kind occurs at $(J_2/J_1)_D \approx 0.284$ together with a Lifshitz point $(J_2/J_1)_L \approx 0.3725$ after which the static structure factor develops a two-peak structure.

### 3.3. Field theory of spin liquid with incommensurate correlations

Frustration represents thus a novel mechanism in one dimension for generating incommensurability as external magnetic fields or Dzyaloshinskii-Moriya interaction. The zigzag ladder representation (Fig. 6) of the $J_1 - J_2$ model enables one to investigate, by a weak coupling approach $J_1 \ll J_2$, the main characteristics of frustration and to shed light on the mechanism that gives rise to incommensurate correlations in the large $J_2$ regime. In the spin-$1/2$ case, the onset of incommensurability and the presence of deconfined massive spinons might be understood starting from the limit where the two spin-$1/2$ AF Heisenberg chains are decoupled.

The continuum limit of the $S = 1/2$ two-leg zigzag ladder has been analysed by several groups. The interacting part of the Hamiltonian density of the low-energy field theory reads as follows:

$$\mathcal{H}_{int} \approx g_1 \left( \vec{J}_{1L} \cdot \vec{J}_{2R} + \vec{J}_{1L} \cdot \vec{J}_{1R} \right) + g_2 \vec{n}_1 \cdot \partial_x \vec{n}_2,$$

where $\vec{J}_{a,L,R}$ ($a = 1, 2$) are the left-right su(2)$_1$ currents corresponding to the continuum description of the $a$th spin-$1/2$ AF Heisenberg chain. In Eq. $\vec{n}_a$ denotes the staggered magnetization of the spin density of the chain with index $a = 1, 2$. In the continuum limit, this $\vec{n}_a$ field identifies with the vector part of the primary field, with scaling dimension $\Delta_n = 1/2$, of the su(2)$_1$ WZNW CFT transforming according to the fundamental representation of SU(2). In this continuum description, frustration suppresses geometrically the standard backscattering contribution $\vec{n}_1 \cdot \vec{n}_2$ which governs the low-energy physics of the two-leg spin ladder. This term represents a strongly relevant perturbation of scaling dimension 1. It corresponds to the energy operators of the underlying Ising models of the low-
energy description of the two-leg spin ladder. As seen in Section 2, this backscattering contribution leads to the formation of a spin gap in this model as well as the confinement of the original massless spinons into optical magnons. In presence of frustration, the interacting part is now only marginal relevant and consists of two terms of different nature. The first one, with coupling constant $g_1$, is a current-current interaction similar to that which appears in the effective field theory of the spin-1/2 Heisenberg chain in the weak coupling limit $J_2 \ll J_1$. The second marginal contribution in Eq. (38), called twist term, $O_{\text{twist}} = \vec{n}_1 \cdot \partial_x \vec{n}_2$ is a novel parity-breaking perturbation which contains the staggered magnetizations of each chain and also a spatial derivative. In contrast to the current-current perturbation, this twist term is not a scalar under the Lorentz transformation in 1 + 1 dimensions but behaves as a vector. In the CFT jargon, this kind of perturbation is characterized by a non-zero conformal spin $S = \pm 1$ together with its scaling dimension $\Delta = 2$. In fact, this twist perturbation has been forgotten in the first bosonization analysis of the two-leg zigzag spin ladder. It has been discovered by Nersesyan et al. and independently by Allen. It is interesting to observe that a similar operator appears in the effective field theory approach to the spin-1 two-leg zigzag ladder in the large $J_2$ limit. This twist perturbation is thus the hallmark of frustration in the low-energy description of frustrated spin ladders with zigzag interchain interaction.

The effect of such a non-zero conformal spin perturbation is non-trivial since the usual irrelevant versus relevant criterion of perturbative field theory does not hold for such a non-scalar contribution (see for instance the discussion in the book). In the case of a $S = \pm 1$ perturbation, it is expected that the generic effect of this contribution is the onset of incommensurability. In this respect, a simple example of incommensurability arising from such a perturbation is the spin-1/2 XXZ Heisenberg chain in a magnetic field along the z-axis. The Hamiltonian density of this model, obtained within the bosonization approach, reads as follows:

$$\mathcal{H} = \frac{v}{2} \left[ (\partial_x \Phi)^2 + (\partial_x \Theta)^2 \right] - g \cos \beta \Phi - h \partial_x \Phi. \quad (39)$$

Here the $S = \pm 1$ perturbation is described by the uniform part of the spin density $\partial_x \Phi$. It is well known that in this model, when the magnetic field $h$ is increased, a commensurate-incommensurate phase transition takes place with the appearance of an incommensurate phase with critical correlation functions. This transition occurs for a finite or vanishing magnetic field depending on the relevance or irrelevance of the
cos $\beta$ term. A similar incommensurability generated by a $S = \pm 1$ conformal spin perturbation occurs in the spin-1/2 XXZ Heisenberg chain in a transverse magnetic field, in a model of two spinless Luttinger chains weakly-coupled by a single-particle interchain hopping and in the quantum axial next-to-nearest neighbor Ising (ANNNI) chain in a transverse magnetic field. Finally, it is important to note that there exists some exactly solvable CFT models perturbed by $S = \pm 1$ conformal spin term for which the presence of incommensurability can be shown non-perturbatively.

All these results suggest that the twist term $O_{\text{twist}} = \vec{n}_1 \cdot \partial_x \vec{n}_2$, as proposed by Nersesy $\ddots$ et al. should be at the origin of the incommensurability found numerically in the spin-1/2 $J_1 - J_2$ Heisenberg chain when $J_2 > 0.52 J_1$. To this end, this twist term and the current-current interaction of Eq. (38) can be expressed in terms of the four Majorana fermions $\xi^a_{R,L}, a = 0, 1, 2, 3$ of the continuum limit of the standard two-leg spin ladder. In particular, the current-current interaction is built from of the following tensor: $O^{ab}_{cc} = \xi^a_R \xi^b_L \xi^c_R \xi^d_L$ with $a, b, c, d = 0, 1, 2, 3$ and $a \neq b$. The twist perturbation is also local in terms of these Majorana fermions but with a different structure. A typical term that enters its expression is $O^{abcd}_{\text{twist}} = \xi^a_R \xi^b_L \xi^c_L \xi^d_L$ with $a, b, c, d = 0, 1, 2, 3$ and $a \neq b \neq c \neq d$. The non-zero conformal spin of this twist term is reflected here by the different number of right and left fermions in this expression. The field theory expressed in terms of these four Majorana fermions, turns out not to be integrable. The renormalization group (RG) flow analysis reveals that the current-current interaction and the twist term are equally important in the IR limit. They reach a strong coupling regime simultaneously with a fixed ratio. The nature of this strong coupling regime is still an open problem since the emerging IR field theory is not integrable and one cannot disentangle the effect of the two interactions of Eq. (38). However, it is very tempting to explain the main characteristics of the spontaneously dimerized phase with incommensurate correlation of the spin-1/2 $J_1 - J_2$ Heisenberg chain in the large $J_2$ limit from these two contributions. On one hand, the current-current contribution of Eq. (38), equivalent to two decoupled SU(2) Thirring model, has similar properties as the field theory obtained in the weak coupling regime $J_2 \ll J_1$. The spontaneously dimerization and the existence of massive deconfined spinons should result from this interaction. In fact, this current-current interaction appears alone, without a twist term, in the continuum limit of a frustrated two-leg spin ladder with crossings along a special line of the couplings.
been shown that this model, in the weak coupling limit, is characterized by a weak spontaneously dimerization with massive spinons as elementary excitations. Finally, as already stressed, the twist perturbation of Eq. (38) should be at the origin of the incommensurability in the large $J_2$ limit. In this respect, the effect of the twist term of the two-leg zigzag ladder has been analysed within a RPA approach and it leads indeed to some incommensurability behavior in the spin-spin correlation function.

A simple way to disentangle the effects of the current-current interaction and the twist perturbation is to introduce an exchange anisotropy which makes the twist contribution more relevant in the RG sense. The extreme case is the XY version of the spin-1/2 $J_1 - J_2$ Heisenberg chain with Hamiltonian:

$$H_{XY} = J_1 \sum_n \left( S_n^x S_{n+1}^x + S_n^y S_{n+1}^y \right) + J_2 \sum_n \left( S_n^x S_{n+2}^x + S_n^y S_{n+2}^y \right). \quad (40)$$

In the ladder limit when $J_2 \gg J_1$, the low-energy physics of this model can be analysed by means of the bosonization approach. Introducing two bosonic fields $\Phi_\pm$, the resulting bosonic Hamiltonian density of the model reads as follows:

$$H_{XY} \simeq \frac{v}{2} \sum_{a=\pm} \left( (\partial_x \phi_a)^2 + (\partial_x \theta_a)^2 \right) + g \partial_x \theta_+ \sin \left( \sqrt{2\pi} \theta_- \right), \quad (41)$$

where $\Theta_\pm$ are the dual fields associated to the bosonic fields $\Phi_\pm$. The Hamiltonian describes a nontrivial field theory with a relevant $S = \pm 1$ conformal spin twist perturbation with scaling dimension $\Delta_g = 3/2$. In this strong anisotropic XY case, the current-current perturbation is less dominant and can be safely neglected to derive the nature of the phase of the model when $J_2 \gg J_1$. The presence of incommensurability in the system can then be found using a mean-field analysis of the model by decoupling the two pieces of the twist term of Eq. (41). In particular, the leading asymptotic behavior of the transverse spin-spin correlation functions of the model obtained by this mean-field approach is given by:

$$\langle S_1^a(x) S_n^a(0) \rangle \sim \frac{\exp(iqx)}{|x|^{1/4}}, \quad a = 1, 2, \quad (42)$$

which displays an incommensurate critical behavior with an oscillating factor: $q - \pi/a_0 \sim (J_1/J_2)^2$. The physical picture that emerges from this mean-field analysis is the existence of a critical spin nematic phase that preserves the U(1) and time-reversal symmetries and displays long-range chiral ordering in its ground state: $\langle (\vec{S}_{n+1} \wedge \vec{S}_n)_z \rangle \neq 0$. It is important to
note that this spin nematic phase does not break the time-reversal symmetry but spontaneously breaks a $Z_2$ symmetry of the model which is a tensor product of a site-parity and link-parity symmetries on the two chains: $P_L^{(1)} \times P_S^{(2)}$. The $z$-component of the spin current $J_{zs}^a$ associated to the $a$th spin-1/2 XY chain ($a = 1, 2$) takes a non-zero expectation value in the ground state of the mean field Hamiltonian $199$:

$$\langle J_{zs}^a \rangle = \langle J_{zs}^b \rangle = -\frac{v}{\sqrt{2\pi}} \langle \partial_x \Theta_+ \rangle \neq 0.$$ (43)

As a result, this produces a picture of local nonzero spin currents polarized along the $z$-anisotropy axis circulating around the triangular plaquettes of the two-leg zigzag spin ladder. This phase has been found numerically by Nishiyama $218$, who has investigated the existence of chiral order of the Josephson ladder with half a flux quantum per plaquette, and also by Hikihara et al. $219$ by means of the DMRG approach. This critical incommensurate phase is the quantum analogue of the classical spiral phase with chiral ordering $153$. It is thus natural to expect that this phase is not restricted to the spin-1/2 case but should exist in the range of parameters of the model $199$ in the general spin case. Indeed, the existence of this critical spin nematic phase in the spin-$S$ case has been shown numerically for $S = 1, 3/2, 2$ $219, 220$ by means of a semiclassical method $221$, and finally by the Abelian bosonization approach $222$ of a general spin $S$ introduced by Schulz $89$.

3.4. Extended criticality stabilized by frustration

In addition to incommensurability, a second characteristic of classical canted magnets is the presence of three gapless spin-wave modes instead of two as in the Néel colinear state. An interesting question is whether frustration can lead to new type of emerging quantum criticality not encountered in unfrustrated spin chains or ladders. In this last case, the critical behavior of half-integer AF Heisenberg spin chains or odd-legged spin ladders is characterized by one gapless bosonic mode i.e. by a CFT with central charge $c = 1$. As it has already been pointed out in the previous section, a striking effect of frustration in the continuum limit stems from the fact that the low-energy effective field theory is mainly governed by marginal interactions. New type of IR critical behaviors may result from the delicate balance between these marginal perturbations. In the following, we shall give some examples of critical phases and quantum critical points with extended criticality stabilized by frustration in 1D spin systems.
3.4.1. Critical phases with SU(N) quantum criticality

A first example of a system with extended criticality is the 1D spin-orbital model \( (21) \) in a regime where frustration reveals itself \( K \sim J \). At the special point \( J = K/4 \), the Hamiltonian \( (21) \) can be expressed in terms of a product of two-body permutation operator in \( S_1 \) and \( S_2 \) subspaces:

\[
H_{so} = J \sum_i \left( 2 \vec{S}_{1,i} \cdot \vec{S}_{1,i+1} + \frac{1}{2} \right) \left( 2 \vec{S}_{2,i} \cdot \vec{S}_{2,i+1} + \frac{1}{2} \right)
= J \sum_i P_{i,i+1}^{(S_1=1/2)} P_{i,i+1}^{(S_2=1/2)}.
\] (44)

Since this Hamiltonian exchanges both \( \vec{S}_{1,i} \) and \( \vec{S}_{2,i} \) spins at the same time, the spin-orbital model at \( J = K/4 \) is not only SU(2) \( \times \) SU(2) symmetric but actually has an enlarged SU(4) symmetry which unifies the spin and orbital degrees of freedom \( \text{[223,224,225]} \). More precisely, the Hamiltonian \( (21) \) can be recasted as an AF Heisenberg spin chain with SU(4) spins up to a constant:

\[
H_{SU(4)} = J \sum_i \sum_{A=1}^{15} T_i^A T_{i+1}^A,
\] (45)

where \( T^A \) are the 15 generators belonging to the fundamental representation of SU(4). This model is exactly solvable by means of the Bethe ansatz \( \text{[220]} \) and its low-energy spectrum consists of three gapless spinons with wave vectors \( \pm \pi/2a_0 \) and \( \pi/a_0 \) \( \text{[220,227]} \). As shown by Affleck \( \text{[228]} \), the critical theory corresponds to the \( \text{su}(4)_1 \) WZNW model with central charge \( c = 3 \) (three massless bosonic modes). The existence of this quantum critical point with a SU(4) symmetry allows us to study the spin-orbital model \( (21) \) by a continuum description in an intermediate coupling regime \( J \sim K \) where frustration shows off. In this respect, it is important to notice that, according to the Zamolodchikov’s c theorem \( \text{[229]} \), the SU(4) critical point with central charge \( c = 3 \) cannot be reached by a RG trajectory starting from the decoupling limit \( (K = 0) \) of two spin-1/2 AF Heisenberg chains with total central charge \( c = 2 \). Stated differently, the physics in the neighborhood of the SU(4) point cannot be understood in terms of weakly coupled \( S = 1/2 \) Heisenberg chains. The strategy to tackle with this intermediate coupling regime is to start from the SU(4) Hubbard chain at quarter-filling and apply the bosonization approach to obtain the continuum description of the spin densities \( \vec{S}_{1,2}(x) \) at the SU(4) point in the large Coulomb repulsion limit. The low-energy field theory which describes small
deviations from the SU(4) symmetric point can then be derived \(^ {230,231}\). The resulting effective Hamiltonian density associated to the symmetry breaking scheme SU(4) → SU(2) × SU(2) of the spin-orbital model \(^ {24}\) reads as follows \(^ {230}\):

\[
H_{\text{eff}} = -\frac{iv_s}{2} \left( \tilde{\xi}_{sR} \cdot \partial_x \tilde{\xi}_{sR} - \tilde{\xi}_{sL} \cdot \partial_x \tilde{\xi}_{sL} \right) - \frac{iv_o}{2} \left( \tilde{\xi}_{oR} \cdot \partial_x \tilde{\xi}_{oR} - \tilde{\xi}_{oL} \cdot \partial_x \tilde{\xi}_{oL} \right) + g_1 \left( \tilde{\xi}_{sR} \cdot \tilde{\xi}_{sL} \right)^2 + g_2 \left( \tilde{\xi}_{oR} \cdot \tilde{\xi}_{oL} \right)^2 + g_3 \left( \tilde{\xi}_{sR} \cdot \tilde{\xi}_{sL} \right) \left( \tilde{\xi}_{oR} \cdot \tilde{\xi}_{oL} \right),
\]

(46)

where \(\tilde{\xi}_{sR,L}\) and \(\tilde{\xi}_{oR,L}\) are two triplet of Majorana fermions which act respectively in the spin and orbital sectors. In Eq. (46), we have considered a more general situation than the model \(^ {24}\) by allowing the exchange in spin \((J_1)\) and orbital \((J_2)\) channels to be different: the three coupling constants \(g_i\) are independent. The interaction of the low-energy field theory \(^ {14}\) is marginal and describes two SO(3) Gross-Neveu models marginally coupled. This field theory is not integrable so that one has to recourse to perturbation theory to elucidate the phase diagram of the spin-orbital model in the vicinity of the SU(4) symmetric point. In fact, the all-order beta functions of the field theory \(^ {14}\) can be determined using the approach of Gerganov et al. \(^ {232}\). These authors have computed the all-order beta functions of a general model with anisotropic current-current interactions in a special minimal scheme prescription. The model \(^ {14}\) can be viewed as a current-current interaction corresponding to the symmetry breaking scheme SO(6) → SO(3) × SO(3). The application of the general formula given in Ref. \(^ {232}\) confirms the conclusions derived from the one-loop calculation \(^ {230,231}\) which reveals the existence of two phases with different remarkable properties.

A first phase, for \(J_1 \approx J_2 > K/4\), has a spectral gap and the ground state has a similar staggered dimerization as the weak-coupling phase of Fig. \(^ {41}\). Provided the anisotropy is not too large \(J_1 \neq J_2\) in the vicinity of the SU(4) symmetric point, the massive phase displays an approximate SO(6) ∼ SU(4) enlarged symmetry. Indeed, by neglecting the velocity anisotropy in Eq. \(^ {14}\), the RG equations reveal a flow to strong coupling with an attraction along a special direction with enlarged symmetry described by the following Hamiltonian density \(^ {230}\):

\[
H_{\text{IR}} \approx -\frac{iv_s}{2} \sum_{a=1}^{6} \left( \xi^a_R \partial_x \xi^a_R - \xi^a_L \partial_x \xi^a_L \right) + g_* \left( \sum_{a=1}^{6} \xi^a_R \xi^a_L \right)^2,
\]

(47)

with \(g_* > 0\). This Hamiltonian is identified as the SO(6) Gross-Neveu model.
which is a massive integrable field theory. Its spectrum is known and consists of the fundamental fermion, with mass $M$, together with a kink and anti-kink with mass $m_{\text{kink}} = M/\sqrt{2}$. The initial model, SU(2) x SU(2) symmetric, acquires thus in the IR limit an enlarged SO(6) symmetry. In more physical terms, the spin and orbital degrees of freedom are unified and described by a same multiplet with six components. A similar example of symmetry restoration by interactions is the emergence of a SO(8) symmetry in weakly-coupled two-leg Hubbard ladder at half-filling and in the SU(4) Hubbard chain at half-filling.

The second phase with $J_1 \approx J_2 < K/4$ has striking properties. All couplings in Eq. (46) flow to zero in the IR limit and the interaction is marginal irrelevant. The six Majorana fermions are thus massless and the phase displays extended quantum criticality characterized by a central charge $c = 3$. Spin-spin correlation functions decay algebraically with exponent $3/2$ and exhibit a four-site periodicity ($2k_F = \pi/2a_0$). The critical behavior at the

![Phase diagram](image)

Fig. 11. Phase diagram of the spin-orbital model obtained by DMRG; $x = J_2/K$ and $y = J_1/K$; taken from Ref. 231.

SU(4) point extends to a finite region of the phase diagram of the spin-orbital model. In this respect, this remarkable gapless phase with extended quantum criticality $c = 3$ represents a new universality class, stabilized by frustration, in spin chains and spin ladders. It is worth noting that this phase has a maximum of gapless modes allowed by the classical structure of a spiral (three Goldstone modes).

The phase diagram of the anisotropic spin-orbital model with
$J_1 \neq J_2$ has been investigated numerically by means of the DMRG technique. Figure 11 represents the resulting $T = 0$ phase diagram obtained in Refs. 235, 231. In phase I, both spin and orbital degrees of freedom are in fully polarized ferromagnetic states. In phase II, the orbital degrees of freedom are still in a ferromagnetic state whereas the spin degrees of freedom are now critical (criticality of a spin-$1/2$ AF Heisenberg chain) and vice versa in phase III. Phase IV corresponds to the staggered dimerized phase of Fig. 4 while phase V is the $c = 3$ gapless phase with extended quantum criticality. In fact, this critical phase has been first discovered only along a special line $-K/4 < J_1 = J_2 < K/4$ in the DMRG calculation of Pati et al. 147. Additional DMRG works 235, 231 have shown the existence of this gapless phase in an extended region (Phase V) of the phase diagram (Fig. 11).

A second model which displays extended quantum criticality induced by frustration is the spin-1 bilinear-biquadratic chain (9) with a biquadratic coupling constant $\beta > 1$. In this regime, frustration manifests itself and the classical ground state is an incommensurate spiral. The Haldane phase of the spin-1 bilinear-biquadratic chain (9) with $|\beta| < 1$ ends at the integrable point with $\beta = 1$, the so-called Uimin-Lai-Sutherland 236 point. This model can be expressed in terms of SU(3) spins and takes the form of a SU(3) AF Heisenberg spin chain:

$$\mathcal{H}_{SU(3)} = J \sum_i \sum_{A=1}^{8} T_i^AT_{i+1}^A,$$

(48)

where $T^A_i$ are the 8 generators belonging to the fundamental representation of SU(3). This model is exactly solvable by means of the Bethe ansatz 230, 220 and its low-energy spectrum consists of two gapless spinons with wave vectors $\pm 2\pi/3a_0$. As shown by Affleck 228, the critical theory corresponds to the $su(3)_1$ WZNW model with central charge $c = 2$ (two massless bosonic modes). The existence of this quantum critical point with a SU(3) symmetry allows to study the spin-1 bilinear-biquadratic chain (9) in the vicinity of $\beta \simeq 1$ by means of a field theory approach 237. The low-energy effective field theory is described by marginal current-current interactions associated to the symmetry breaking scheme SU(3) $\rightarrow$ SU(2) 237. The one-loop RG flow near $\beta \simeq 1$ reveals the existence of two distinct phases. On one hand, a phase with a dynamical mass generation when $\beta < 1$ which signals the onset of the Haldane phase. On the other hand, a massless phase for $\beta > 1$ where interactions are marginal irrelevant. The SU(3) quantum criticality of the Uimin-Lai-Sutherland point, with two gap-
less bosonic modes, extends to a finite region of the phase diagram of the generalized spin-1 Heisenberg chain \( [1] \). This massless phase has been first predicted in a numerical investigation of this model by Fáth and Sólyom \([238]\). This massless phase with approximate SU(3) symmetry, stabilized by frustration, is similar in spirit to the previous SU(4) massless phase of the spin-orbital model \([21]\). Finally, it is worth noting that this \( c = 2 \) massless phase appears also in the phase diagram of the so-called spin tube model, a three-leg spin ladder with frustrated periodic interchain interaction, in a magnetic field \([239]\).

3.4.2. Chirally stabilized critical spin liquid

Frustration can also induce an exotic quantum critical point in frustrated spin ladders. A possible lattice realization of this phenomenon is a model (see Fig. 12) of three \( S = 1/2 \) AF Heisenberg spin chains weakly coupled by on-rung \( J_\perp \) and plaquette-diagonal \( J_\times \) interchain interactions \([240]\). The Hamiltonian of this model reads as follows:

\[
\mathcal{H}_x = J_\parallel \sum_i \sum_{a=1}^3 \vec{S}_{a,i} \cdot \vec{S}_{a,i+1} + J_\perp \sum_i \vec{S}_{2,i} \cdot (\vec{S}_{1,i} + \vec{S}_{3,i}) + J_\times \sum_i \left[ (\vec{S}_{1,i} + \vec{S}_{3,i}) \cdot \vec{S}_{2,i+1} + (\vec{S}_{1,i+1} + \vec{S}_{3,i+1}) \cdot \vec{S}_{2,i} \right].
\]

\( (49) \)

In the continuum limit with \( J_\perp, J_\times \ll J_\parallel \), the Hamiltonian of the lattice

Fig. 12. Three-leg spin ladder with crossings.
model\textsuperscript{[49]} takes the form:
\begin{equation}
\mathcal{H}_x = \frac{2\pi v}{3} \sum_{a=1}^{3} \left( J_{aR}^2 + J_{aL}^2 \right) + \tilde{g} \vec{n}_2 \cdot (\vec{n}_1 + \vec{n}_3) + g \left[ J_{2R} \cdot \left( J_{1L} + J_{3L} \right) + J_{2L} \cdot \left( J_{1R} + J_{3R} \right) \right],
\end{equation}
with \( g = a_0 (J_- + 2J_x) \) and \( \tilde{g} = a_0 (J_- - 2J_x) \). In Eq. \textsuperscript{(50)}, \( \vec{J}_{aR,L} \) \((a = 1, 2, 3)\) are the right and left chiral \( su(2) \) currents which accounts for the low-energy description of the uniform part of the spin density of the \( a \)-th spin-1/2 chain. It is interesting to note that in Eq. \textsuperscript{(50)}, there is no marginally relevant twist perturbation \( \vec{n}_1 \partial_x \vec{n}_2 \) which appears in the continuum description of the two-leg spin ladder with a small zigzag interchain coupling \textsuperscript{[85]}. The two interaction terms in Eq. \textsuperscript{(50)} are of different nature. On one hand, the first contribution, with coupling constant \( g \), is a strongly relevant perturbation with scaling dimension \( \Delta_g = 1 \). On the other hand, the second term is a current-current interaction which is only marginal and, as long as \( \tilde{g} \) is not too small, can be discarded. As a result, for generic values of \( g \) and \( \tilde{g} \), the low-energy physics of the model \textsuperscript{(49)} will be essentially that of the standard three-leg ladder, and frustration will play no role (except for renormalization of mass gaps and velocities). The important point here is that in contrast with non-frustrated ladders, the two coupling constants \( g, \tilde{g} \) can vary independently, and there exists a vicinity of the line \( J_- = 2J_x \) \((\tilde{g} = 0)\) where the low-energy properties of the model are mainly determined by current-current interchain interaction.

Along the special line \( J_- = 2J_x \), the low-energy effective Hamiltonian \textsuperscript{(50)} simplifies as follows\textsuperscript{[241]}
\begin{equation}
\mathcal{H}_x = -\frac{i\nu}{2} \left( \xi_{R}^0 \partial_x \xi_{R}^0 - \xi_{L}^0 \partial_x \xi_{L}^0 \right) + \frac{2\pi v}{3} \left( J_{2R}^2 + J_{2L}^2 \right) + \frac{\pi v}{2} \left( I_{R}^2 + I_{L}^2 \right) + g \left[ J_{2R} \cdot I_{L} + J_{2L} \cdot I_{R} \right],
\end{equation}
where \( \xi_{R,L}^0 \) are Majorana fermions which are associated to the \( Z_2 \) \((1 \to 3)\) discrete interchange symmetry between the surface chains labelled \( a = 1, 3 \) in the lattice model \textsuperscript{(49)}; the total chiral current of the surface chains is noted \( \vec{I}_{R,L} = \vec{J}_{R,L} + \vec{J}_{3R,L} \) and corresponds to a \( su(2) \) \( WZNW \) current. The Hamiltonian \textsuperscript{(51)} has an interesting structure. First, the Majorana fermions \( \xi_{R,L}^0 \) do not participate in the interaction and remain thus critical in the IR limit. These massless degrees of freedom can be interpreted as an effective two-dimensional Ising model at \( T = T_c \) which accounts for singlet excitations between the surface chains. All non-trivial physics of the model
is incorporated in the current-current dependent part of the Hamiltonian \( \mathcal{H}_{cc} \), denoted by \( \mathcal{H}_{cc} \), which describes marginally coupled \( \text{su}(2)_1 \) and \( \text{su}(2)_2 \) WZNW models. Moreover, this Hamiltonian separates into two commuting and *chirally asymmetric* parts: \( \mathcal{H}_{cc} = \mathcal{H}_1 + \mathcal{H}_2 \), \( ([\mathcal{H}_1, \mathcal{H}_2] = 0) \), where

\[
\mathcal{H}_1 = \frac{\pi v}{2} \vec{I}_R^2 + \frac{2\pi v}{3} \vec{J}_{2L}^2 + g \vec{I}_R \cdot \vec{J}_{2L}, \\
(52)
\]

and \( \mathcal{H}_2 \) is obtained from \( \mathcal{H}_1 \) by inverting chiralities of all the currents. The model (52) is integrable by means of the Bethe-ansatz approach \( [\text{[163,244]}] \). A simple RG analysis shows that, at \( g > 0 \), the interaction is marginally relevant. Usually the development of a strong coupling regime is accompanied by a dynamical mass generation and the loss of conformal invariance in the strong coupling limit. However, here due to the chiral asymmetry of \( \mathcal{H}_1 \), it turns out that the effective interaction flows towards an intermediate fixed point where conformal invariance is recovered with a smaller central charge. This critical behavior has been identified as the universality class of chirally stabilized fluids, introduced by Andrei, Douglas, and Jerez \( [\text{241}] \). In the case of the model (52), the symmetry of the IR fixed point, obtained from the Bethe-ansatz analysis, \( [\text{241}] \) turns out to be \( \text{su}(2)_1|_R \times \mathbb{Z}_2|_L \). This result can also be derived using a Toulouse point approach \( [\text{240}] \). As a whole, taking into account of the contribution of the Majorana fermions \( \xi^0_{R,L} \), the model \( [\text{51}] \) displays critical properties characterized by a fixed point with a \( \text{su}(2)_1 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry and a \( c = 2 \) central charge. This emerging quantum criticality can be interpreted as the criticality resulting from an effective \( S = 1/2 \) AF Heisenberg chain and two decoupled critical Ising models. Frustration, introduced by the diagonal exchange interaction \( J_x \), shows up in a non-trivial way by stabilizing a quantum critical point with central charge \( c = 2 \). This IR behavior differs from the standard \( c = 1 \) quantum criticality in unfrustrated three-leg spin ladder by the two gapless non-magnetic, singlet, degrees of freedom described by the Ising models. The physical properties of the model at this \( c = 2 \) critical point can be determined by a Toulouse point approach of the model \( [\text{51},240] \). The slowest spin-spin correlation functions of the model at the IR critical point correspond to the staggered correlations between the spins of the surface chains which decay with a universal exponent \( 3/2 \). As a consequence, the low-temperature dependence of the NMR relaxation rate scales as \( 1/T \sim \sqrt{T} \) in contrast to \( 1/T \sim \text{const} \) for the spin-1/2 AF Heisenberg chain \( [\text{10}] \). The \( c = 2 \) quantum critical point, induced by frustration, describes thus a new universality class in spin ladders. In addition, the Toulouse point analysis enables us to investigate the effect of the neglected backscattering term.
\vec{n}_2 \cdot (\vec{n}_1 + \vec{n}_3) of the model (50) in the vicinity of the line $\tilde{g} = 0$. The main effect of this operator is to open a spectral gap in the Ising degrees of freedom of the $c = 2$ fixed point but has no dramatic effect on the effective $S = 1/2$ AF Heisenberg chain. The $c = 2$ IR fixed point is thus unstable with respect to the interchain backscattering term and the model (50) will display IR critical properties governed by the $c = 1$ fixed point of the three-leg spin ladder or the $S = 1/2$ AF Heisenberg spin chain.

Finally, it is worth noting that the zero-temperature phase diagram of the three-leg spin ladder with crossings (49) has been recently investigated by means of the DMRG approach. Two critical phases have been found corresponding to the $S = 3/2$ AF Heisenberg chain and the three-leg spin ladder. Frustration, introduced through the diagonal interchain interaction $J_\perp$, induces a quantum phase transition between these two critical phases belonging to the standard $c = 1$ universality class. At large $J_\perp$, the transition has been found to be of first order. In the weak coupling regime, the transition occurs at $J_\perp = 2J_\times$ when $J_\times < 0.8J_\parallel$ in full agreement with the field theoretical description (50) of the model. Extended DMRG calculations are required to fully characterize the nature of the phase boundary at $J_\perp = 2J_\times$ in the weak coupling regime in particular to verify the chiral spin liquid behavior with $c = 2$ criticality proposed in Ref. 240.

4. Concluding remarks

The suppression of magnetism by quantum fluctuations in one dimension gives rise to a large variety of zero-temperature spin liquid phases with striking different physical properties. The canonical examples are the emerging quantum criticality of the spin-1/2 AF Heisenberg chain, controlled by gapless elementary excitations with fractional quantum numbers (spinons), and the formation of an incompressible Haldane spin liquid phase in the spin-1 case with optical $S = 1$ magnon excitations. Several other types of incompressible spin liquid phases can be stabilized with different dynamical or optical properties. A distinct spin liquid behavior may be revealed in the form of the dynamical structure factor with the presence or not of a sharp spectral peak which directly probes the elementary nature of the triplet excitation of the phase. In this respect, the staggered dimerized phase of the two-leg spin ladder with a biquadratic interchain exchange represents a non-Haldane spin liquid phase even though a spectral gap is formed by quantum fluctuations. The main difference between this phase and the Haldane phase of the spin-1 Heisenberg chain stems from the com-

\[ \vec{n}_2 \cdot (\vec{n}_1 + \vec{n}_3) \]
posite nature of the spin-flip $S = 1$ excitation due to the presence of a two-particle threshold in the dynamical structure factor instead of a sharp magnon peak. Two gapped spin liquid phases can also be distinguished at the level of the topology of the short-range valence bond description of their ground states. A spin liquid phase may exhibit a ground-state degeneracy depending on the nature of boundary conditions used. In particular, for open boundary conditions, chain-end spin excitations can result from this ground-state degeneracy leading to well defined satellite peaks in the NMR profile of the system doped with non-magnetic impurities. An example of this topological distinction between two gapped phases is provided by the two-leg spin ladder. In the Haldane phase of the open two-leg ladder with a ferromagnetic rung interchain $J_\perp < 0$, $S = 1/2$ edge states are formed whereas these chain-end excitations disappear in the rung singlet phase of the ladder with $J_\perp > 0$.

Frustration, i.e. the impossibility to satisfy simultaneously every pairwise interaction, can induce additional types of spin liquid phases with exotic properties. In the spin-1/2 case, one of the most striking effect of frustration is the stabilization of a spontaneously dimerized phase with massive deconfined spinon excitations. Frustration represents thus a direct route to fractionalization. The existence of these deconfined spinon excitations has important consequences, as for instance, in the study of doping effects of such a spin liquid phase with non-magnetic impurities. On general grounds, it is expected that no free spin degrees of freedom are generated around non-magnetic impurities due to the presence of deconfined spinons

In the case of the spontaneously dimerized phase of the spin-1/2 $J_1$-$J_2$ Heisenberg chain, the absence of induced free spin degrees of freedom or edge states has been shown recently. A second remarkable effect of frustration is the onset of incommensurability in spin chains whose classical ground state has a spiral structure. In this respect, frustration is a novel mechanism in one dimension for generating incommensurate behavior as external magnetic fields or Dzyaloshinskii-Moriya interaction. An example of such a spin liquid phase with incommensurate correlation is the quantum $J_1 - J_2$ Heisenberg chain in the large next-nearest neighbor limit. When the SU(2) symmetry of this model is explicitly broken, an incommensurate phase is produced by frustration with local non-zero spin currents, polarized along the anisotropic axis, circulating around triangular plaquettes. This phase is the quantum signature of the classical spiral phase induced by frustration. Finally, a last effect of frustration, discussed in this review, is the possible realization of a new type of emerging quantum criticality in
spin chains and ladders. In the continuum description, the low-energy effective field theory of a frustrated spin chain is mainly governed by marginal interactions. An extended IR critical behavior characterized by a CFT with central charge $c > 1$ may result from the delicate balance between these marginal contributions.

Regarding perspectives, a natural question to raise is the existence of a novel 1D spin liquid phase with unbroken SU(2) spin symmetry which displays physical properties not described in this review. In particular, an interesting possibility is the realization of a 1D version of the chiral spin liquid phase \cite{245} which breaks spontaneously the time-reversal symmetry. In fact, such a phase has been identified recently in the phase diagram of the two-leg spin ladder with a four-spin cyclic exchange \cite{246,247,246}. In the large ring-exchange limit, a spin liquid phase characterized by a non-zero scalar-chirality operator $\langle \vec{S}_{1,i} \cdot (\vec{S}_{2,i} \wedge \vec{S}_{1,i+1}) \rangle$, breaking both parity and time-reversal symmetries, has been found in DMRG calculations \cite{246,247}. Using duality arguments, the authors of Ref. \cite{247} have also exhibited several spin ladder models with exact ground-state which display spontaneous breaking of the time-reversal symmetry. Finally, a central issue is the possibility to stabilize a two-dimensional SU(2)-invariant spin liquid phase with exotic properties starting from the one-dimensional limit. Recently, a model of two-dimensional frustrated spin system with SU(2) spin symmetry and strong spatially anisotropy has been introduced by Nersesyan and Tsvelik \cite{248}. This model can also be interpreted as a collection of weakly coupled spin chains. It has been shown that, in the limit of infinite number of chains, this system displays a spin liquid phase with massive deconfined fractional spin-1/2 excitations (spinons) \cite{249}. The existence of spinons in two-dimensional spin systems has also been found in a crossed-chain model \cite{249} which is a 2D anisotropic version of the pyrochlore lattice. Remarkably enough, for a range of coupling constants, \cite{249,250} this system is a non-dimerized spin liquid with deconfined gapless spinons as elementary excitations. This spin liquid phase, stabilized by frustration, is an example of a SU(2) version of a sliding Luttinger liquid phase \cite{251}. We hope that other two-dimensional spin liquid phases with exotic properties will be reported in the near future.

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References

1. P. W. Anderson, Mater. Res. Bull. 8, 153 (1973); Science 235, 1196 (1987).
2. S. Sachdev and K. Park, Ann. Phys. 298, 58 (2002); S. Sachdev, ibid. 303, 226 (2003).
3. C. Lhuillier and G. Misguich, Frustrated quantum magnets, in High Magnetic fields, edited by C. Berthier, L. P. Lévy, and G. Martinez, p 161 (Lecture Notes in Physics Vol 595, Springer, 2002); arXiv: cond-mat/0109146.
4. X. G. Wen, Phys. Rev. B 65, 165113 (2002).
5. N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1307 (1968).
6. A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B 241, 333 (1984).
7. P. Di Francesco, P. Mathieu, and D. Sénéchal, Conformal Field Theory (Springer-Verlag, New York, 1997).
8. A. Luther and I. Peschel, Phys. Rev. B 9, 2911 (1974).
9. S. Coleman, Phys. Rev. B 11, 2088 (1975).
10. A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, Bosonization and Strongly Correlated Systems (Cambridge University Press, Cambridge, 1998).
11. S. R. White, Phys. Rev. Lett. 69, 2863 (1992); Phys. Rev. B 48, 10345 (1993).
12. M. Yamashita, T. Ishii, and H. Matsuzaka, Coord. Chem. Rev. 198, 347 (2000).
13. P. Lemmens, G. Güntherodt, and C. Gros, Phys. Rep. 375, 1 (2003).
14. S. Sachdev, Quantum Phase Transitions (Cambridge University Press, Cambridge, 1999).
15. I. Affleck, Phys. Rev. Lett. 55, 1355 (1985); Nucl. Phys. B 265, 409 (1986).
16. X. G. Wen, Adv. in Phys. 44, 405 (1995).
17. H. Bethe, Z. Phys. 71, 205 (1931).
18. L. Hulthén, Ark. Mat., Astron. Fys. 26A, 1 (1938).
19. R. B. Griffiths, Phys. Rev. 133, A768 (1964); C. N. Yang and C. P. Yang, Phys. Rev. 150, 327 (1966); S. Eggert, I. Affleck, and M. Takahashi, Phys. Rev. Lett. 73, 332 (1994); S. Lukyanov, Nucl. Phys. B 522, 533 (1998); A. Klümper and D. C. Johnston, Phys. Rev. Lett. 84, 4701 (2000).
20. M. Takahashi, Thermodynamics of One-Dimensional Solvable Models (Cambridge University Press, Cambridge, 1999).
21. J. des Cloizeaux and J. J. Pearson, Phys. Rev. 128, 2131 (1962).
22. L. D. Faddeev and L. A. Takhtajan, Phys. Lett. A 85, 375 (1981).
23. A. Luther and I. Peschel, Phys. Rev. B 9, 2911 (1974); A. H. Bougourzi, M. Couture, and M. Kacir, ibid. 54, 12869 (1996); A. Abada, A. H. Bougourzi,
and B. Si-Lakhal, *Nucl. Phys. B* **497**, 733 (1997); N. Kitanine, J. M. Maillet, and V. Terras, *ibid.* **554**, 647 (1999).

24. I. Affleck, *Field theory methods and quantum critical phenomena*, in Les Houches, session XLIX, *Champs, Cordes et Phénomènes critiques* (Elsevier, New-York, 1989).

25. V. G. Knizhnik and A. B. Zamolodchikov, *Nucl. Phys. B* **247**, 83 (1984).

26. I. Affleck, D. Gepner, H. J. Schulz, and T. Ziman, *J. Phys. A* **22**, 511 (1989); R. R. P. Singh, M. E. Fisher, and R. Shankar, *Phys. Rev. B* **39**, 2562 (1989); T. Giamarchi and H. J. Schulz, *ibid.*, 4620 (1989).

27. I. Affleck, *J. Phys. A* **31**, 4573 (1998); T. Hikihara and A. Furusaki, *Phys. Rev. B* **58**, R583 (1998); S. Lukyanov and V. Terras, *Nucl. Phys. B* **654**, 325 (2003).

28. F. D. M. Haldane, *Phys. Rev. Lett.* **66**, 1529 (1991).

29. F. D. M. Haldane, *Phys. Rev. Lett.* **60**, 635 (1988); B. S. Shastry, *ibid.*, 639 (1988).

30. S. E. Nagler, D. A. Tennant, R. A. Cowley, T. G. Perring, and S. K. Satija, *Phys. Rev. B* **44**, 12361 (1991); D. A. Tennant, T. G. Perring, R. A. Cowley, and S. E. Nagler, *Phys. Rev. Lett.* **70**, 4003 (1993); D. A. Tennant, R. A. Cowley, S. E. Nagler, and A. M. Tsvelik, *Phys. Rev. B* **52**, 13368 (1995).

31. D. C. Dender, D. Davidović, D. H. Reich, C. Broholm, K. Lefmann, and G. Aeppli, *Phys. Rev. B* **53**, 2583 (1996).

32. I. Tsukada, Y. Sasago, K. Uchinokura, A. Zheludev, S. Maslov, G. Shirane, K. Kakurai, and E. Ressouche, *Phys. Rev. B* **60**, 6601 (1999).

33. P. R. Hammar, M. B. Stone, D. H. Reich, C. Broholm, P. J. Gibson, M. M. Turnbull, C. P. Landee, and M. Oshikawa, *Phys. Rev. B* **59**, 1008 (1999).

34. F. D. M. Haldane, *Phys. Rev. Lett.* **50**, 1153 (1983); *Phys. Lett. A* **93**, 464 (1983).

35. I. Affleck, *J. Phys. Condens. Matter* **1**, 3047 (1989).

36. E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, 1991).

37. A. Auerbach, *Interacting Electrons and Quantum Magnetism*, (Springer-Verlag, New York, 1994).

38. A. B. Zamolodchikov and Al. B. Zamolodchikov, *Ann. Phys.* **120**, 253 (1979).

39. P. B. Wiegmann, *Phys. Lett. B* **152**, 209 (1985).

40. I. Affleck and F. D. M. Haldane, *Phys. Rev. B* **36**, 5291 (1987).

41. R. Shankar and N. Read, *Nucl. Phys. B* **336**, 457 (1990).

42. A. B. Zamolodchikov and Al. B. Zamolodchikov, *Nucl. Phys. B* **379**, 602 (1992).

43. R. Botet and R. Jullien, *Phys. Rev. B* **27**, 613 (1983); R. Botet, R. Jullien, and M. Kolb, *ibid.* **28**, 3914 (1983); M. Kolb, R. Botet, and R. Jullien, *J. Phys. A* **16**, L673 (1983); U. Glaus and T. Schneider, *Phys. Rev. B* **30**, 215 (1984); J. B. Parkinson and J. C. Bonner, *ibid.* **32**, 4703 (1985); H. J. Schulz and T. A. L. Ziman, *ibid.* **33**, 6545 (1986); A. Moreo, *ibid.* **35**, 8562 (1987); T. Sakai and M. Takahashi, *ibid.* **42**, 1090, 4537 (1990); O. Golinelli, T. Jolicoeur, and R. Lacaze, *ibid.* **46**, 10854 (1992); S. Haas, J. Riera, and E.
Dagotto, *ibid.* 48, 3281 (1993); M. Kaburagi, I. Harada, and T. Tonegawa, *J. Phys. Soc. Jpn.* 62, 1848 (1993); T. Sakai and M. Takahashi, *ibid.* 63, 755 (1994).

44. M. P. Nightingale and H. W. Blöte, *Phys. Rev. B* 33, 659 (1986); M. Marcu and J. Müller, *Phys. Lett. A* 119, 469 (1987); M. Takahashi, *Phys. Rev. B* 38, 5188 (1988); K. Nomura, *Phys. Rev. B* 40, 2421 (1989); S. Liang, *Phys. Rev. Lett.* 64, 1597 (1990); J. Deisz, M. Jarrell, and D. L. Cox, *Phys. Rev. B* 42, 4869 (1990); S. V. Meshkov, *ibid.* 48, 6167 (1993); S. Miyashita and S. Yamamoto, *ibid.*, 913 (1993); J. Deisz, M. Jarrell, and D. L. Cox, *ibid.*, 10227 (1993).

45. K. Kubo and S. Takada, *J. Phys. Soc. Jpn.* 55, 438 (1986); K. Betsuyaku, *Phys. Rev. B* 34, 8125 (1986); K. Kubo, *ibid.* 46, 866 (1992).

46. S. R. White and D. A. Huse, *Phys. Rev. B* 48, 3844 (1993); E. S. Sorensen and I. Affleck, *Phys. Rev. Lett.* 71, 1633 (1993); *Phys. Rev. B* 49, 15771 (1994).

47. W. J. L. Buyers, R. M. Morra, R. L. Armstrong, M. J. Hogan, P. Gerlach, and K. Hirakawa, *Phys. Rev. Lett.* 56, 371 (1986); M. Steiner, K. Kakurai, J. K. Kjems, D. Petitgrand, and R. Pynn, *J. Appl. Phys.* 61, 3953 (1987); R. M. Morra, W. J. L. Buyers, R. L. Armstrong, and K. Hirakawa, *Phys. Rev. B* 38, 543 (1988); Z. Tun, W. J. L. Buyers, R. L. Armstrong, K. Hirakawa, and B. Briat, *ibid.* 42, 4677 (1990); I. A. Zaliznyak, L. P. Regnault, and D. Petitgrand, *ibid.* 50, 15824 (1994).

48. J. P. Renard, M. Verdaguer, L. P. Regnault, W. A. C. Erkelens, J. Rossat-Mignod, and W. G. Stirling, *Europhys. Lett.* 3, 949 (1987).

49. J. P. Renard, M. M. Verdaguer, L. P. Regnault, W. A. C. Erkelens, J. Rossat-Mignod, J. Ribas, W. G. Stirling, and C. Vettier, *J. Appl. Phys.* 63, 3538 (1988); L. P. Regnault, C. Vettier, J. Rossat-Mignod, and J. P. Renard, *Physica B* 156-157, 247 (1989); *ibid.* 180-181, 188 (1992); S. Ma, C. Broholm, D. H. Reich, B. J. Sternlieb, and R. W. Erwin, *Phys. Rev. Lett.* 69, 3571 (1992); L. P. Regnault, I. Zaliznyak, J. P. Renard, and C. Vettier, *Phys. Rev. B* 50, 9174 (1994); S. Ma, D. H. Reich, C. Broholm, B. J. Sternlieb, and R. W. Erwin, *ibid.* 51, 3289 (1995).

50. V. Gadet, M. Verdaguer, V. Briois, A. Gleizes, J. P. Renard, P. Beauvillain, C. Chappert, T. Goto, K. Le Dang, and P. Veillet, *Phys. Rev. B* 44, 705 (1991).

51. H. Mutka, C. Payen, P. Molini, J. L. Soubeyroux, P. Colombet, and A. D. Taylor, *Phys. Rev. Lett.* 67, 497 (1991).

52. J. Darriet and J. P. Renard, *Solid State Commun.* 86, 409 (1993); T. Sakaguchi, K. Kakurai, T. Yokoo, and J. Akimitsu, *J. Phys. Soc. Jpn.* 65, 3025 (1996).

53. G. Xu, J. F. Ditus, T. Ito, K. Oka, H. Takagi, C. Broholm, and G. Aeppli, *Phys. Rev. B* 54, R6827 (1996).

54. Z. Honda, H. Askawa, and K. Katsumata, *Phys. Rev. Lett.* 81, 2566 (1998); Y. Chen, Z. Honda, A. Zheludev, C. Broholm, K. Katsumata, and S. M. Shapiro, *ibid.* 86, 1618 (2001).

55. Z. Honda, K. Katsumata, H. Aruga Katori, K. Yamada, T. Ohishi, T. Man-
53

abe, and M. Yamashita, *J. Phys. Condens. Matter* **9**, L83 (1997).
56. A. Zheludev, Z. Honda, K. Katsumata, R. Feyerherm, and K. Prokes, *Europhys. Lett.* **55**, 868 (2001); A. Zheludev, Z. Honda, Y. Chen, C. L. Broholm, K. Katsumata, and S. M. Shapiro, *Phys. Rev. Lett.* **88**, 077206 (2002).
57. I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, *Phys. Rev. Lett.* **59**, 799 (1987); *Commun. Math. Phys.* **115**, 477 (1988).
58. A. M. Tsvelik, *Phys. Rev. B* **42**, 10499 (1990).
59. G. Gómez-Santos, *Phys. Rev. Lett.* **63**, 790 (1989).
60. H.-J. Mikeska, *Europhys. Lett.* **19**, 39 (1992).
61. G. Fáth and J. Sólyom, *J. Phys. Condens. Matter* **5**, 8983 (1993).
62. M. den Nijs and K. Rommelse, *Phys. Rev. B* **40**, 4709 (1989).
63. H. Tasaki, *Phys. Rev. Lett.* **66**, 798 (1991).
64. T. Kennedy and H. Tasaki, *Phys. Rev. B* **45**, 304 (1992); *Commun. Math. Phys.* **147**, 431 (1992).
65. D. P. Arovas, A. Auerbach, and F. D. M. Haldane, *Phys. Rev. Lett.* **60**, 531 (1988).
66. S. Knabe, *J. Stat. Phys.* **52**, 627 (1988).
67. R. Scharf and H.-J. Mikeska, *J. Phys. Condens. Matter* **7**, 5083 (1995).
68. T. Kennedy, *J. Phys. Condens. Matter* **2**, 5737 (1990).
69. K. Chang, I. Affleck, G. W. Hayden, and Z. G. Soos, *J. Phys. Condens. Matter* **1**, 153 (1998); E. S. Sorensen and A. P. Young, *Phys. Rev. B* **42**, 754 (1990); J. Deisz, *ibid.* **46**, 2885 (1992).
70. S. M. Girvin and D. P. Arovas, *Phys. Scr. T* **27**, 156 (1989); D. P. Arovas and S. M. Girvin, *Recent Progress in Many-Body Theories*, Vol 3, 315 (Plenum Press, New York, 1992).
71. Y. Hatsugai and M. Kohmoto, *Phys. Rev. B* **44**, 11789 (1991); K. Kubo, *ibid.* **46**, 866 (1992); F. Alcaraz and Y. Hatsugai, *ibid.*, 13914 (1992).
72. U. Schollwöck, T. Jolicoeur, and T. Garel, *Phys. Rev. B* **53**, 3304 (1996); O. Golinelli, T. Jolicoeur, and E. S. Sorensen, *Eur. Phys. J B* **11**, 199 (1999).
73. E. Polizzi, F. Mila, and E. S. Sorensen, *Phys. Rev. B* **58**, 2407 (1998).
74. S.-W. Tsai and J. B. Marston, *Phys. Rev. B* **62**, 5546 (2000).
75. S. Miyashita and S. Yamamoto, *Phys. Rev. B* **48**, 913, 9528 (1993); *ibid.* **51**, 3649 (1995).
76. M. Kaburagi, I. Harada, and T. Tonegawa, *J. Phys. Soc. Jpn.* **62**, 1848 (1993).
77. S. Qin, T. K. Ng, and Z. B. Su, *Phys. Rev. B* **52**, 12844 (1995).
78. Ö. Legeza, G. Fáth, and J. Sólyom, *Phys. Rev. B* **55**, 291 (1997); Ö. Legeza and J. Sólyom, *ibid.* **59**, 3606 (1999).
79. C. D. Batista, K. Hallberg, and A. A. Aligia, *Phys. Rev. B* **58**, 9248 (1998); *ibid.* **60**, R12553 (1999).
80. F. Alet and E. S. Sorensen, *Phys. Rev. B* **62**, 14116 (2000).
81. P. Mitra, B. Halperin, and I. Affleck, *Phys. Rev. B* **45**, 5299 (1992).
82. E. S. Sorensen and I. Affleck, *Phys. Rev. B* **49**, 15771 (1994).
83. P. Lecheminant and E. Orignac, *Phys. Rev. B* **65**, 174406 (2002).
84. S. H. Glarum, S. Geschwind, K. M. Lee, M. L. Kaplan, and J. Michel, *Phys. Rev. Lett.* **67**, 1614 (1991).
85. T. Goto, S. Satoh, Y. Matsumata, and M. Hagiwara, *Phys. Rev. B* **55**, 2709 (1997).
86. M. Hagiwara, K. Katsumata, I. Affleck, B. I. Halperin, and J. P. Renard, *Phys. Rev. Lett.* **65**, 3181 (1990).
87. F. Tedoldi, R. Santachiara, and M. Horvatić, *Phys. Rev. Lett.* **83**, 412 (1999).
88. M. Kenzelmann, G. Xu, I. A. Zaliznyak, C. Broholm, J. F. DiTusa, G. Aeppli, T. Ito, K. Oka, and H. Takagi, *Phys. Rev. Lett.* **90**, 087202 (2003).
89. H. J. Schulz, *Phys. Rev. B* **34**, 6372 (1986).
90. D. C. Cabra, P. Pujol, and C. von Reichenbach, *Phys. Rev. B* **58**, 65 (1998).
91. A. Moreo, *Phys. Rev. B* **35**, 8562 (1987); C. C. Alcaraz and A. Moreo, *ibid.* **46**, 2896 (1992).
92. T. Ziman and H. J. Schulz, *Phys. Rev. Lett.* **59**, 140 (1987).
93. K. Hallberg, X. Q. G. Wang, P. Horsch, and A. Moreo, *Phys. Rev. Lett.* **76**, 4955 (1996).
94. J. Lou, J. Dai, S. Qin, Z. Su, and L. Yu, *Phys. Rev. B* **62**, 8600 (2000).
95. M. Niel, C. Cros, G. Le Flem, M. Pouchard, and P. Hagenmüller, *Physica B* **86-88**, 702 (1977).
96. S. Itoh, K. Kakurai, Y. Endoh, and H. Tanaka, *Physica B* **213-214**, 161 (1995).
97. H. Mutka, C. Payen, and P. Molinié, *Europhys. Lett.* **21**, 623 (1993).
98. N. Hatano and M. Suzuki, *J. Phys. Soc. Jpn.* **62**, 1346 (1993); S. V. Meshkov, *Phys. Rev. B* **48**, 6167 (1993); J. Deisz, M. Jarell, and D. L. Cox, *ibid.* **48**, 10227 (1993); G. Sun, *ibid.* **51**, 8370 (1995); S. Yamamoto, *Phys. Rev. Lett.* **75**, 3348 (1995).
99. Y. Nishiyama, K. Totsuka, N. Hatano, and M. Suzuki, *J. Phys. Soc. Jpn.* **64**, 414 (1995).
100. U. Schollwöck and T. Jolicoeur, *Europhys. Lett.* **30**, 493 (1995); U. Schollwöck, O. Golinelli, and T. Jolicoeur, *Phys. Rev. B* **54**, 4038 (1996).
101. S. Qin, Y.-L. Liu, and L. Yu, *Phys. Rev. B* **55**, 2721 (1997); S. Qin, X. Wang, and L. Yu, *ibid.* **56**, R14251 (1997).
102. G. E. Granroth, M. W. Meisel, M. Chaparala, T. Jolicoeur, B. H. Ward, and D. R. Talham, *Phys. Rev. Lett.* **77**, 1616 (1996).
103. S. Gopolan, T. M. Rice, and M. Sigrist, *Phys. Rev. B* **49**, 8901 (1994).
104. S. R. White, R. M. Noack, and D. J. Scalapino, *Phys. Rev. Lett.* **73**, 886 (1994).
105. B. Frischmuth, B. Ammon, and M. Troyer, *Phys. Rev. B* **54**, R3714 (1996);
B. Frischmuth, S. Haas, G. Sierra, and T. M. Rice, *ibid.* 55, R3340 (1997).
114. A. G. Rojo, *Phys. Rev. B* 53, 9172 (1996).
115. G. Sierra, *J. Phys. A* 29, 3299 (1996).
116. For a review, see for instance: J. Voit, *Rep. Prog. Phys.* 58, 977 (1995); H. J. Schulz, G. Cuniberti, and P. Pieri, *Lecture notes of the Chia Laguna summer school* (Springer Verlag, 1997); arXiv: cond-mat/9807366.
117. M. Azuma, Z. Hiroi, M. Takano, K. Ishida, and Y. Kitaoka, *Phys. Rev. Lett.* 73, 3463 (1994).
118. K. Kojima, A. Keren, G. M. Luke, B. Nachumi, W. D. Wu, Y. J. Uemura, M. Azuma, and M. Takano, *Phys. Rev. Lett.* 74, 2812 (1995).
119. S. A. Carter, B. Batlogg, R. J. Carla, J. J. Krajewski, W. F. Peck Jr., and T. M. Rice, *Phys. Rev. Lett.* 77, 1378 (1996); R. S. Eccleston, M. Uehara, J. Akimitsu, H. Eisaki, N. Motoyama, and S.-I. Uchida, *ibid.* 81, 1702 (1998); T. Imai, K. R. Thurber, K. M. Shen, A. W. Hunt, and F. C. Chou, *ibid.*, 220 (1998); M. Takigawa, N. Motoyama, H. Eisaki, and S. Uchida, *Phys. Rev. B* 57, 1124 (1998); L. P. Regnault, A. H. Moudden, J. P. Boucher, E. Lorenzo, A. Hiess, A. Vietkin, and A. Revcolevschi, *Physica B* 259-261, 1038 (1999); A. Glozar, G. Blumberg, B. S. Dennis, B. S. Shastry, N. Motoyama, H. Eisaki, and S. Uchida, *Phys. Rev. Lett.* 87, 197202 (2001).
120. C. A. Hayward, D. Poilblanc, and L. P. Lévy, *Phys. Rev. B* 54, R12649 (1996); G. Chaboussant, P. A. Crowell, L. P. Lévy, O. Piovesana, A. Madouri, and D. Mailly, *ibid.* 55, 3046 (1997); G. Chaboussant, M.-H. Julien, Y. Fagot-Reviron, L. P. Lévy, C. Berthier, M. Horvatić, and O. Piovesana, *Phys. Rev. Lett.* 79, 925 (1997); G. Chaboussant, Y. Fagot-Reviron, M.-H. Julien, M. E. Hansson, C. Berthier, M. Horvatić, L. P. Lévy, and O. Piovesana, *ibid.* 80, 2713 (1998).
121. P. R. Hammar, D. H. Reich, C. Broholm, and F. Trouw, *Phys. Rev. B* 57, 7846 (1998).
122. B. C. Watson et al., *Phys. Rev. Lett.* 86, 5168 (2001).
123. K. Katoh, Y. Hosokoshi, K. Inoue, and T. Goto, *J. Phys. Soc. Jpn.* 69, 1008 (2000); K. Katoh, Y. Hosokoshi, K. Inoue, M. I. Bartashevich, H. Nakano, and T. Goto, *J. Phys. Chem. Sol.* 63, 1277 (2002).
124. S. Brehmer, H.-J. Mikeska, M. Müller, N. Nagaosa, and S. Uchida, *Phys. Rev. B* 60, 329 (1999); M. Matsuda, K. Katsumata, R. S. Eccleston, S. Brehmer, and H.-J. Mikeska, *ibid.* 62, 8903 (2000).
125. T. S. Nunner, P. Brune, T. Kopp, M. Windt, and M. Grüninger, *Phys. Rev. B* 66, 180404 (2002).
126. T. Barnes, E. Dagotto, J. Riera, and E. S. Swanson, *Phys. Rev. B* 47, 3196 (1993).
127. K. Hida, *J. Phys. Soc. Jpn.* 60, 1347 (1991); *ibid.* 64, 4896 (1995).
128. H. Watanabe, *Phys. Rev. B* 50, 13442 (1994); *ibid.* 52, 12508 (1995).
129. Y. Nishiyama, N. Hatano, and M. Suzuki, *J. Phys. Soc. Jpn.* 64, 1967 (1995).
130. S. R. White, *Phys. Rev. B* 53, 52 (1996).
131. S. P. Strong and A. J. Millis, *Phys. Rev. Lett.* 69, 2419 (1992); *Phys. Rev. B* 50, 9911 (1994).
132. D. G. Shelton, A. A. Nersesyan, and A. M. Tsvelik, Phys. Rev. B 53, 8521 (1996).
133. A. A. Nersesyan and A. M. Tsvelik, Phys. Rev. Lett. 78, 3969 (1997); Errata 79, 1171 (1997).
134. L. Takhtajan, Phys. Lett. A 87, 479 (1982); J. Babujian, ibid. 90, 479 (1982).
135. D. J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).
136. D. Allen and D. Sénéchal, Phys. Rev. B 55, 299 (1997).
137. E. H. Kim, G. Fáth, J. Sólyom, and D. J. Scalapino, Phys. Rev. B 62, 14965 (2000); G. Fáth, Ó. Legeza, and J. Sólyom, ibid. 63, 134403 (2001).
138. N. Read and B. Chakraborty, Phys. Rev. B 40, 7133 (1989).
140. S. Takada and H. Watanabe, J. Phys. Soc. Jpn. 61, 39 (1992).
141. K. Damle and S. Sachdev, Phys. Rev. B 57, 8307 (1998).
142. O. P. Sushkov and V. N. Kotov, Phys. Rev. Lett. 81, 1941 (1998); V. N. Kotov, O. P. Sushkov and R. Eder, Phys. Rev. B 59, 6266 (1999).
143. C. Jurecka and W. Brenig, Phys. Rev. B 61, 14307 (2000).
144. S. Trebst, H. Monien, C. J. Hamer, Z. Weihong, and R. R. P. Singh, Phys. Rev. Lett. 85, 4373 (2000); W. Zheng, C. J. Hamer, R. R. P. Singh, S. Trebst, and H. Monien, Phys. Rev. B 63, 144410 (2001).
145. M. Windt, M. Grüninger, T. Nunner, C. Knetter, K. P. Schmidt, G. S. Uhrig, T. Kopp, A. Freimuth, U. Ammerahl, B. Büchner, and A. Revcolevschi, Phys. Rev. Lett. 87, 127002 (2001).
146. K. I. Kugel and D. I. Khomskii, Sov. Phys. Usp. 25, 231 (1982).
147. S. K. Pati, R. R. P. Singh, and D. I. Khomskii, Phys. Rev. Lett. 81, 5406 (1998).
148. E. Axtell, T. Ozawa, S. Kauzlarich, and R. R. P. Singh, J. Solid State Chem. 134, 423 (1997).
149. E. Orignac, R. Citro, and N. Andrei, Phys. Rev. B 61, 11533 (2000).
150. A. K. Kolezhuk and H.-J. Mikeska, Phys. Rev. Lett. 80, 2709 (1998); A. K. Kolezhuk, H.-J. Mikeska, and U. Schollwöck, Phys. Rev. B 63, 064418 (2000).
151. M. J. Martins and B. Nienhuis, Phys. Rev. Lett. 85, 4956 (2000).
152. P. Lecheminant and E. Orignac, unpublished.
153. J. Villain, J. Phys. Fr 38, 385 (1977); Chiral order in Helimagnets, proceedings of the 13th IUPAP Conference on Statistical Physics, edited by C. Weil, D. Cabid, C. G. Kuper, and I. Riess (1978).
154. T. Jolicoeur and J. C. Le Guillou, Phys. Rev. B 40, 2727 (1989).
155. See for instance, N. D. Mermin, Rev. Mod. Phys. 51, 591 (1979).
156. T. Senthil and M. P. A. Fisher, Phys. Rev. B 62, 7850 (2000).
157. S. Rao and D. Sen, Nucl. Phys. B 424, 547 (1994); J. Phys. Condens. Matter 9, 1831 (1997).
158. D. Allen and D. Sénéchal, Phys. Rev. B 51, 6394 (1995).
159. T. Dombre and N. Read, Phys. Rev. B 39, 6797 (1989).
160. F. D. M. Haldane, talk at the workshop New Theoretical Approaches to Strongly Correlated Systems, Cambridge (2000); F. D. M. Haldane, Phys. Rev. Lett. 61, 1029 (1988).
161. J. von Delft and C. L. Henley, *Phys. Rev. Lett.* 69, 3236 (1992); *Phys. Rev. B* 48, 965 (1992).
162. D. Loss, D. P. DiVincenzo, and G. Grinstein, *Phys. Rev. Lett.* 69, 3232 (1992).
163. A. M. Polyakov and P. B. Wiegmann, *Phys. Lett.* 131 B, 121 (1983); *ibid.* 141, 223 (1984).
164. E. Ogievetsky, N. Reshetikhin, and P. B. Wiegmann, *Nucl. Phys. B* 280, 45 (1987).
165. F. D. M. Haldane, *Phys. Rev. B* 25, 4925 (1982); Errata *ibid.* 26, 5257 (1982).
166. M. Hase, I. Terasaki, and K. Uchinokura, *Phys. Rev. Lett.* 70, 3651 (1993).
167. For a review, see for instance: J. P. Boucher and L. P. Regnault, *J. Phys. I* 6, 1939 (1996).
168. M. Isobe and Y. Ueda, *J. Phys. Soc. Jpn.* 65, 3142 (1996); N. Fujiwara, H. Yasuoka, M. Isobe, Y. Ueda, and S. Maegawa, *Phys. Rev. B* 55, R11945 (1997).
169. G. Castilla, S. Chakravarty, and V. J. Emery, *Phys. Rev. Lett.* 75, 1823 (1995); J. Riera and A. Dobry, *Phys. Rev. B* 51, 16098 (1995).
170. M. Matsuda and K. Katsumata, *J. Mag. Mag. Mat.* 140-144, 1671 (1995); M. Matsuda, K. Katsumata, K. M. Kojima, M. Larkin, G. M. Luke, J. Merrin, B. Nachumi, Y. J. Uemura, H. Eisaki, N. Motoyama, S. Uchida, and G. Shirane, *Phys. Rev. B* 55, R11953 (1997).
171. N. Motoyama, H. Eisaki, and S. Uchida, *Phys. Rev. Lett.* 76, 3212 (1996).
172. N. Maeshima, M. Hagiwara, Y. Narumi, K. Kindo, T. Kobayashi, and K. Okunishi, *J. Phys. Condens. Matter* 15, 3607 (2003).
173. H. Kikuchi, H. Nagasawa, Y. Ajiro, T. Asano, and T. Goto, *Physica B* 284-288, 1631 (2000).
174. V. L. Berezinskii, *Sov. Phys. JETP* 34, 610 (1972); J. M. Kosterlitz and D. J. Thouless, *J. Phys. C* 6, 1181 (1973); J. M. Kosterlitz, *ibid.* 7, 1046 (1974).
175. K. Okamoto and K. Nomura, *Phys. Lett. A* 169, 433 (1992); K. Nomura and K. Okamoto, *J. Phys. A* 27, 5773 (1994).
176. R. Chitra, S. Pati, H. R. Krishnamurthy, D. Sen, and S. Ramasesha, *Phys. Rev. B* 52, 6581 (1995).
177. S. Eggert, *Phys. Rev. B* 54, 9612 (1996).
178. S. White and I. Affleck, *Phys. Rev. B* 54, 9862 (1996).
179. N. Andrei and J. H. Lowenstein, *Phys. Rev. Lett.* 43, 1698 (1979); *Phys. Lett. B* 90, 106 (1980).
180. A. A. Belavin, *Phys. Lett. B* 87, 117 (1979).
181. C. K. Majumdar and D. K. Ghosh, *J. Math. Phys.* 10, 1388, 1399 (1969).
182. B. S. Shastry and B. Sutherland, *Phys. Rev. Lett.* 47, 964 (1981).
183. W. J. Caspers, K. M. Emmett, and W. Magnus, *J. Phys. A* 17, 2687 (1984).
184. E. Sorensen, I. Affleck, D. Augier, and D. Poilblanc, *Phys. Rev. B* 58, R14701 (1998).
185. W. J. Caspers and W. Magnus, *Phys. Lett. A* 88, 103 (1982).
186. G. Fáth and A. Süto, *Phys. Rev. B* 62, 3778 (2000).
187. K. Nomura, *J. Phys. Soc. Jpn.* **72**, 476 (2003).
188. R. Bursill, G. A. Gehring, D. J. J. Furnell, J. B. Parkinson, T. Xiang, and C. Zeng, *J. Phys. Condens. Matter* **7**, 8605 (1995).
189. S. Watanabe and H. Yokoyama, *J. Phys. Soc. Jpn.* **68**, 2073 (1999).
190. A. A. Aligia, C. D. Batista, and F. H. L. Essler, *Phys. Rev. B* **62**, 3259 (2000).
191. J. Stephenson, *Can. J. Phys.* **47**, 2621 (1969); *ibid.* **48**, 1724, 2118 (1970); *J. Math. Phys.* **12**, 420 (1970).
192. T. Tonegawa, M. Kaburagi, N. Ichikawa, and I. Harada, *J. Phys. Soc. Jpn.* **61**, 2890 (1992).
193. S. Pati, R. Chitra, D. Sen, H. R. Krishnamurthy, and S. Ramasesha, *Europhys. Lett.* **33**, 707 (1996).
194. A. Kolezhuk, R. Roth, and U. Schollwöck, *Phys. Rev. Lett.* **77**, 5142 (1996); *Phys. Rev. B* **55**, 8928 (1997).
195. R. Roth and U. Schollwöck, *Phys. Rev. B* **58**, 9264 (1998).
196. A. Kolezhuk and U. Schollwöck, *Phys. Rev. B* **65**, 100401 (2002).
197. I. Dzyaloshinskii, *J. Phys. Chem. Solids* **4**, 241 (1958); T. Moriya, *Phys. Rev. Lett.* **4**, 228 (1960); *Phys. Rev. 120*, 91 (1960).
198. D. Allen, PhD thesis, Sherbrooke university, 1998.
199. A. A. Nersesyan, A. O. Gogolin, and F. H. L. Essler, *Phys. Rev. Lett.* **81**, 910 (1998).
200. D. C. Cabra, A. Honecker, and P. Pujol, *Eur. Phys. J. B* **13**, 55 (2000).
201. D. Allen, F. H. L. Essler, and A. A. Nersesyan, *Phys. Rev. B* **61**, 8871 (2000).
202. C. Itoi and S. Qin, *Phys. Rev. B* **63**, 224423 (2001).
203. D. Allen and D. Sénéchal, *Phys. Rev. B* **61**, 12134 (2000).
204. R. Chitra and T. Giamarchi, *Phys. Rev. B* **55**, 5816 (1997).
205. G. I. Dzhaparidze and A. A. Nersesyan, *JETP Lett.* **27**, 334 (1978); *J. Low. Temp. Phys.* **37**, 95 (1979).
206. V. L. Pokrovsky and A. L. Talapov, *Phys. Rev. Lett.* **42**, 65 (1979); *Sov. Phys. JETP 51*, 134 (1980).
207. H. J. Schulz, *Phys. Rev. B* **22**, 5274 (1980).
208. Y. Okwamoto, *J. Phys. Soc. Jpn.* **49**, 8 (1980).
209. A. A. Nersesyan, A. Luther, and F. V. Kusmartsev, *Phys. Lett. A* **176**, 363 (1993).
210. D. V. Dmitriev, V. Ya. Krivnov, and A. A. Ovchinnikov, *Phys. Rev. B* **65**, 172409 (2002); D. V. Dmitriev, V. Ya. Krivnov, A. A. Ovchinnikov, and A. Langari, *JETP 95*, 538 (2002).
211. A. Dutta and D. Sen, *Phys. Rev. B* **67**, 094435 (2003).
212. A. A. Nersesyan, *Phys. Lett. A* **153**, 49 (1991); A. A. Nersesyan, A. Luther, and F. V. Kusmartsev, *JETP Lett.* **55**, 692 (1992).
213. P. Fendley and C. Nayak, *Phys. Rev. B* **63**, 115102 (2001).
214. D. Allen, P. Azaria, and P. Lecheminant, *J. Phys. A* **34**, L305 (2001).
215. J. L. Cardy, *Nucl. Phys. B* **389**, 577 (1993).
216. A. M. Tsvelik, *Nucl. Phys. B* **612**, 479 (2001).
217. D. Allen, private communication and unpublished.
218. Y. Nishiyama, *Eur. Phys. J. B* **17**, 295 (2000).
219. T. Hikihara, M. Kaburagi, and H. Kawamura, *Phys. Rev. B* **63**, 174430 (2001); *Prog. Theor. Phys. Suppl.* **145**, 58 (2002).
220. M. Kaburagi, H. Kawamura, and T. Hikihara, *J. Phys. Soc. Jpn.* **68**, 3185 (1999); T. Hikihara, M. Kaburagi, H. Kawamura, and T. Tonegawa, *ibid.* **69**, 259 (2000).
221. A. K. Kolezhuk, *Phys. Rev. B* **62**, R6057 (2000); *Prog. Theor. Phys. Suppl.* **145**, 29 (2002).
222. P. Lecheminant, T. Jolicoeur, and P. Azaria, *Phys. Rev. B* **63**, 174426 (2001); T. Jolicoeur and P. Lecheminant, *Prog. Theor. Phys. Suppl.* **145**, 23 (2002).
223. Y. Q. Li, M. Ma, D. N. Shi, and F. C. Zhang, *Phys. Rev. Lett.* **81**, 3527 (1998).
224. Y. Yamashita, N. Shibata, and K. Ueda, *Phys. Rev. B* **58**, 9114 (1998).
225. B. Frischmuth, F. Mila, and M. Troyer, *Phys. Rev. Lett.* **82**, 835 (1999).
226. B. Sutherland, *Phys. Rev. B* **12**, 3795 (1975).
227. Y.-Q. Li, M. Ma, D.-N. Shi, and F.-C. Zhang, *Phys. Rev. B* **60**, 12781 (1999).
228. A. B. Zamolodchikov, JETP Lett. **43**, 730 (1986).
229. P. Azaria, A. O. Gogolin, P. Lecheminant, and A. A. Nersesyan, *Phys. Rev. Lett.* **83**, 624 (1999); P. Azaria, E. Boulat, and P. Lecheminant, *Phys. Rev. B* **61**, 12112 (2000).
230. C. Itoi, S. Qin, and I. Affleck, *Phys. Rev. B* **61**, 6747 (2000).
231. G. V. Uimin, *JETP Lett.* **12**, 225 (1970); C. K. Lai, *J. Math. Phys.* **15**, 1675 (1974).
232. B. Frischmuth, F. Mila, and M. Troyer, *Phys. Rev. Lett.* **86**, 4753 (2001); A. Leclair, *Phys. Rev. B* **64**, 045329 (2001).
233. H.-H. Lin, L. Balents, and M. P. A. Fisher, *Phys. Rev. B* **58**, 1794 (1998).
234. C. Itoi and M.-H. Kato, *Phys. Rev. B* **55**, 8295 (1997).
235. G. Fáth and J. Sólyom, *Phys. Rev. B* **44**, 11836 (1991); *ibid.* **47**, 872 (1993).
236. R. Citro, E. Orignac, N. Andrei, C. Itoi, and S. Qin, *J. Phys. Condens. Matter* **12**, 3041 (2000).
237. P. Azaria, P. Lecheminant, and A. A. Nersesyan, *Phys. Rev. B* **58**, R8881 (1998); P. Azaria and P. Lecheminant, *Nucl. Phys. B* **575**, 439 (2000).
238. N. Andrei, M. Douglas, and A. Jerez, *Phys. Rev. B* **58**, 7619 (1998).
239. X. Wang, N. Zhu, and C. Chen, *Phys. Rev. B* **66**, 172405 (2002).
240. S. Sachdev and M. Vojta, in *Proceedings of the XIII International Congress on Mathematical Physics*, edited by A. Fokas, A. Grigoryan, T. Kibble, and B. Zegarlinski, (International Press, Boston, 2001); arXiv: cond-mat/0009202.
241. B. Normand and F. Mila, *Phys. Rev. B* **65**, 104411 (2002).
242. X. G. Wen, F. Wilczek, and A. Zee, *Phys. Rev. B* **39**, 11413 (1989).
243. A. Läuchli, G. Schmid, and M. Troyer, *Phys. Rev. B* **67**, 100409(R) (2003).
247. T. Hikihara, T. Momoi, and X. Hu, *Phys. Rev. Lett.* 90, 087204 (2003); T. Momoi, T. Hikihara, M. Nakamura, and X. Hu, *Phys. Rev. B* 67, 174410 (2003).

248. A. A. Nersesyan and A. M. Tsvelik, *Phys. Rev. B* 67, 024422 (2003); F. A. Smirnov and A. M. Tsvelik, arXiv: cond-mat/0304634; M. J. Bhaseen and A. M. Tsvelik, arXiv: cond-mat/0305194.

249. O. A. Starykh, R. R. P. Singh, and G. C. Levine, *Phys. Rev. Lett.* 88, 167203 (2002); R. R. Singh, O. A. Starykh, and P. J. Freitas, *J. Appl. Phys.* 83, 7387 (1998).

250. P. Sindzingre, J.-B. Fouet, and C. Lhuillier, *Phys. Rev. B* 66, 174424 (2002).

251. V. J. Emery, E. Fradkin, S. A. Kivelson, and T. C. Lubensky, *Phys. Rev. Lett.* 85, 2160 (2000); A. Vishwanath and D. Carpentier, *Phys. Rev. Lett.* 86, 676 (2001); R. Mukhopadhyay, C. L. Kane, and T. C. Lubensky, *Phys. Rev. B* 63, 081103 (2001); *ibid.* 64, 045120 (2001); S. L. Sondhi and K. Yang, *ibid.* 63, 054430 (2001).