Fully Distribution-free Center-outward Rank Tests for Multiple-output Regression and Manova

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FULLY DISTRIBUTION-FREE CENTER-OUTWARD RANK TESTS FOR MULTIPLE-OUTPUT REGRESSION AND MANOVA

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Extending rank-based inference to a multivariate setting such as multiple-output regression or MANOVA with unspecified \( d \)-dimensional error density has remained an open problem for more than half a century. None of the many solutions proposed so far is enjoying the combination of distribution-freeness and efficiency that makes rank-based inference a successful tool in the univariate setting. A concept of center-outward multivariate ranks and signs based on measure transportation ideas has been introduced recently. Center-outward ranks and signs are not only distribution-free but achieve in dimension \( d > 1 \) the (essential) maximal ancillarity property of traditional univariate ranks, hence carry all the “distribution-free information” available in the sample. We derive here the Hájek representation and asymptotic normality results required in the construction of center-outward rank tests for multiple-output regression and MANOVA. When based on appropriate spherical scores, these fully distribution-free tests achieve parametric efficiency in the corresponding models.

1. Introduction. Rank-based testing methods have been quite successful in testing problems for single-output regression and linear models such as ANOVA (see the classical monographs by Hájek and Šidák (1967), Randles and Wolfe (1979) or Puri and Sen (1985)) and univariate linear time series (Hallin et al. (1985), Koul and Saleh (1993), Hallin and Puri (1994)). Being distribution-free, they remain valid over the full class of absolutely continuous distributions. In linear models (this includes testing for single-output regression slopes, testing for treatment effects in analysis of variance, testing against location shifts in two-sample problems) and ARMA time series, they do reach parametric or semiparametric efficiency bounds at given reference densities, thus reconciling the objectives of robustness and efficiency.

Extending those attractive features to a multivariate context has been a long-standing open problem, for which many solutions have been proposed in the literature. Puri and Sen (1971)
for a variety of problems in multivariate analysis (including multiple-output regression and MANOVA) and Hallin et al. (1989) for VARMA time series models construct tests based on componentwise ranks which, however, fail to be distribution-free. Building upon an ingenious multivariate extension of the L₁ definition of quantiles, Oja (1999, 2010) defines the so-called spatial ranks; the resulting tests are neither distribution-free nor efficient. Tests based on the ranks of various concepts of statistical depth also have been proposed (Liu (1992), Liu and Singh (1993), He and Wang (1997), Zuo and He (2006)). While distribution-free, the ranks of statistical depth, however, are failing to exploit any directional information, and hence typically do not allow for any type of asymptotic efficiency. As for the tests based on the Mahalanobis ranks and signs proposed by Hallin and Paindaveine (2002a,b, 2004, 2005), they do achieve, within the class of linear models and linear time series with elliptical densities, parametric or semiparametric efficiency at correctly specified elliptical reference densities; their distribution-freeness, hence their validity, unfortunately, is limited to the class of elliptical distributions.

Inspired by measure transportation ideas, a new concept of ranks and signs for multivariate observations has been introduced recently under the name of Monge-Kantorovich ranks and signs in Chernozhukov et al. (2017), under the name of center-outward ranks and signs in Hallin (2017) and Hallin et al. (2020a), along with the related concepts of center-outward distribution and quantile functions. Contrary to earlier concepts, those ranks and signs are extending to dimension d > 1 the essential maximal ancillarity property of univariate ranks that can be interpreted as a finite-sample form of semiparametric efficiency; the corresponding empirical center-outward distribution functions, moreover, satisfy a Glivenko-Cantelli result.

Concepts of center-outward ranks and signs have been successfully applied (Boeckel et al. (2018); Deb and Sen (2019); Ghosal and Sen (2019); Shi et al. (2019, 2020)) in the construction of distribution-free tests of independence between random vectors and multivariate goodness-of-fit; applications to the study of tail behavior and extremes can be found in De Valk and Segers (2018); Beirlant et al. (2019) are using the related center-outward empirical quantiles in the analysis of multivariate risk; Hallin et al. (2019) are proposing center-outward R-estimators for VARMA time series models with unspecified innovation densities. The present paper goes one step further in the direction of a toolkit of distribution-free tests for multiple-output multivariate analysis, by deriving a Hájek-type asymptotic representation result for linear center-outward rank statistics. Asymptotic normality follows as a corollary, from which center-outward rank tests are constructed for multiple-output regression models (including, as special cases, MANOVA and two-sample location models). Those tests are distribution-free, hence valid, over the entire family of absolutely continuous distributions; for adequate choice of the scores, parametric efficiency is attained at chosen densities.

Outline of the paper

The paper is organized as follows. Section 2 briefly describes the main tools to be used: center-outward distribution and quantile functions (Section 2.1) and their empirical counterparts, the center-outward ranks and signs (Section 2.2). The main
properties of these concepts are summarized in Section 2.3 (Proposition 2.1); their invariance/equivariance properties are established in Proposition 2.2. Section 3 is entirely devoted to the key results of this paper, which extend and generalize the classical approach by Hájek and Šidák (1967): a Hájek type asymptotic representation for multivariate center-outward linear rank statistics (Section 3.2) and the resulting asymptotic normality result (Section 3.3). Section 4.1 describes the multiple-output regression model to be considered throughout, which contains, as particular cases, the two-sample location and MANOVA models, of obvious practical importance. Local asymptotic normality is established in Section 4.2 for this model under general error densities (Proposition 4.1) and, for the purpose of future comparisons, for the particular case of elliptical distributions (Proposition 4.2). The center-outward tests we are proposing are described in Section 5.2, along with (Corollary 5.2) their local asymptotic optimality properties. Due to their importance in applications, the particular cases of the hypothesis of equal locations in the two-sample problem and the hypothesis of no treatment effect in MANOVA are considered in Section 5.3. Sections 6.1 and 6.2 propose some simple choices of score functions, extending the classical median-test-score (based on center-outward signs only), Wilcoxon, and van der Waerden (normal-score) tests. Section 6.3 discusses affine invariance issues. Section 7 is devoted to a numerical exploration of the finite-sample performance of our rank tests which appear to outperform their competitors in non-elliptical situations while performing equally well under ellipticity. Section 7.3 presents an application to a four-dimensional real dataset of (highly non-elliptical) archeological observations from three excavation sites in present-day Israel. While traditional MANOVA methods cannot reject the hypothesis of no treatment effect, our fully distribution-free center-outward rank-based test rejects it quite significantly.

2. Center-outward distribution functions, ranks, and signs in $\mathbb{R}^d$.

2.1. Center-outward distribution functions. Throughout, denote by $Z^{(n)}$ a triangular array $(Z_1^{(n)}, \ldots, Z_n^{(n)})$, $n \in \mathbb{N}$ of i.i.d. $d$-dimensional random vectors with distribution $P$ in the family $\mathcal{P}_d$ of absolutely continuous distributions on $\mathbb{R}^d$. The notation $\text{spt}(P)$ is used for the support of $P$, $\text{spt}(P)$ for its interior. The open (resp. closed) unit ball and the unit hypersphere in $\mathbb{R}^d$ are denoted by $S_d$ (resp. $\bar{S}_d$) and $S_{d-1}$, respectively; $U_d$ stands for the spherical uniform distribution over $S_d$, $\mu_d$ for the Lebesgue measure over $\mathbb{R}^d$; $I_d$ is the $d \times d$ unit matrix, $I_A$ the indicator of the Borel set $A$.

The definition of the center-outward distribution function of $P$ is particularly simple for $P$ in the so-called class $\mathcal{P}_d^+$ of distributions with nonvanishing densities—namely, the class of all distributions with density $f := dP/d\mu_d$ such that, for all $D \in \mathbb{R}^+$, there exist constants $\lambda_{D;P}^-$ and $\lambda_{D;P}^+$ satisfying

$$0 < \lambda_{D;P}^- \leq f(z) \leq \lambda_{D;P}^+ < \infty \quad (2.1)$$

for all $z$ with $\|z\| \leq D$ (so that $\text{spt}(P) = \mathbb{R}^d$ and $P$-a.s. is equivalent to $\mu_d$-a.e.). The main result in McCann (1995) then implies the existence of an a.e. unique convex lower semi-continuous function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ with gradient $\nabla \varphi$ such that $\nabla \varphi \# P = U_d$. Call $F_z := \nabla \varphi$ the center-outward distribution function of $P$. It follows from Figalli (2018) that $F_z$ defines a homeomorphism between the punctured unit ball $S_d \setminus \{0\}$ and its image $\mathbb{R}^d \setminus F_z^{-1}(\{0\})$.

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$^4$Namely, the spherical distribution with uniform (over $[0, 1]$) radial density—equivalently, the product of a uniform over the distances to the origin and a uniform over the unit sphere $S_{d-1}$. For $d = 1$, it coincides with the Lebesgue uniform; for $d \geq 2$, it has unbounded density at the origin.

$^5$We borrow from measure transportation the convenient notation $T \# P$ (with the terminology $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ pushes $P$ forward to $T \# P$) for the distribution under $Z \sim P$ of $T(Z)$.
call $Q_{\pm} : u \mapsto Q_{\pm}(u) := F_{\pm}^{-1}(u)$, $u \neq 0$ the *center-outward quantile function*. Figalli also shows that, defining $Q_{\pm}(0) := F_{\pm}^{-1}(\{0\})$ yields a convex and compact subset with Lebesgue measure zero in $\mathbb{R}^d$, the *center-outward median set* of $P$.

All the intuition and all the properties of center-outward distribution and quantile functions hold for this special case (considered in Hallin (2017)), and the reader may like to restrict to it. The general case $P \in \mathcal{P}_d$ (considered in Hallin et al. (2020a)), however, requires the slightly more technical definitions below, due to the fact that $P$-a.s., in general, is no longer equivalent to $\mu_d$-a.e. and the possibility of a more or less regular boundary to spt$(P)$.

For $P$ in $\mathcal{P}_d$ but not necessarily in $\mathcal{P}_d^+$, McCann (1995) implies the existence of an a.e. unique real-valued convex lower semi-continuous function $\psi$ with domain $S_d$ such that $\nabla\psi \& U_d = P$. Without loss of generality, we further can extend $\psi$ to $\mathbb{R}^d$ by defining $\psi(u) := \liminf_{\|u\| \to 1} \psi(v)$ for $\|u\| = 1$ and $\psi(u) := \infty$ for $\|u\| > 1$. The center-outward quantile function $Q_{\pm}$ then is defined as the gradient $\nabla\psi$, with domain $S_d$, of that extended $\psi$. Considering the Legendre transform $\varphi(z) := \sup_{u \in S_d} \langle \psi(u), u \rangle$ of $\psi$, define center-outward distribution function of $P$ as $\mathbf{F}_{\pm} := \nabla\varphi$, with domain $\mathbb{R}^d$ and range in $S_d$.

See Section 2.3 for the main properties of $\mathbf{F}_{\pm}$ and $Q_{\pm}$ and Section 2 of Hallin et al. (2020a) for further details.

Some properties of the center-outward ranks, such as distribution-freeness or the independence between ranks and signs, hold for all $P \in \mathcal{P}_d$. Some others—essentially, asymptotic results—require slightly more regular distributions. Following Hallin et al. (2020a), define

$$\mathcal{P}_d^+ := \{P = \nabla\Upsilon \# U_d \mid \Upsilon \text{ is convex, } \nabla\Upsilon \text{ a homeomorphism over } S_d \setminus \{0\} \text{ with } \nabla\Upsilon(\{0\}) \text{ compact, convex, and } \mu_d(\nabla\Upsilon(\{0\})) = 0\}.$$

Obviously, $\mathcal{P}_d^+ \subset \mathcal{P}_d$; it follows from Proposition 2.3 in Hallin et al. (2020a) that $\mathcal{P}_d^+$ also contains the class $\mathcal{P}_d^{\text{conv}}$ of all distributions $P \in \mathcal{P}_d$ with (bounded or unbounded) convex support satisfying (2.1) for all $z \in \text{spt}(P)$ with $\|z\| \leq D$, which in turn contains $\mathcal{P}_d^+$: hence, $\mathcal{P}_d \subset \mathcal{P}_d^{\text{conv}} \subset \mathcal{P}_d^+ \subset \mathcal{P}_d$.

2.2. *Center-outward ranks and signs.* Except for a few particular cases such as spherical distributions, the above definitions are not meant for the analytical derivation of $\mathbf{F}_{\pm}$ and $Q_{\pm}$ which typically involves Monge-Ampère partial differential equations; estimation is possible, though, via their empirical counterparts $\mathbf{F}_{\pm}^{(n)}$ and $Q_{\pm}^{(n)}$, based on center-outward ranks and signs, which we now describe.

Associated with the $n$-tuple $Z_1^{(n)}, \ldots, Z_n^{(n)}$, the *empirical center-outward distribution function* $\mathbf{F}_i^{(n)}$ is mapping $Z_1^{(n)}, \ldots, Z_n^{(n)}$ to a “regular” grid $\mathcal{G}_n$ of the unit ball $S_d$. That grid $\mathcal{G}_n$ is obtained as follows:

(a) first factorize $n$ into $n = n_R n_S + n_0$, with $0 \leq n_0 < \min(n_R, n_S)$;
(b) next consider a “regular array” $\mathcal{G}_{n_S} := \{s_1^{n_S}, \ldots, s_{n_S}^{n_S}\}$ of $n_S$ points on the sphere $S_{d-1}$ (see the comment below);
(c) finally, the grid consists in the collection $\mathcal{G}_n$ of the $n_R n_S$ points $g$ of the form

$$(r/(n_R + 1)) s_s^{n_S}, \quad r = 1, \ldots, n_R, \quad s = 1, \ldots, n_S,$$

along with $(n_0$ copies of) the origin in case $n_0 \neq 0$: a total number $n - (n_0 - 1)$ or $n$ of distinct points, thus, according as $n_0 > 0$ or $n_0 = 0$.

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6 Regularity problems in the area are notoriously tricky: see, e.g., Caffarelli (1996).

7 In particular, no closed forms of $\mathbf{F}_{\pm}$ and $Q_{\pm}$ are known for non-spherical elliptical distributions.
By “regular” we mean “as uniform as possible”, in the sense, for example, of the low-discrepancy sequences of the type considered in numerical integration and Monte-Carlo methods (see, e.g., Niederreiter (1992), Judd (1998), Dick and Pillichshammer (2014), or Santner et al. (2003)). The only mathematical requirement needed for Proposition 2.1 below is the weak convergence, as \( n_S \to \infty \), of the uniform discrete distribution over \( \mathcal{S}_{n_S} \) to the
uniform distribution over \( \mathcal{S}_{d-1} \). A uniform i.i.d. sample of points over \( \mathcal{S}_{d-1} \) satisfies such a requirement but fails to produce as neat a concept of ranks and signs; moreover, one easily can construct arrays that are “more regular” than an i.i.d. one. For instance, one could select \( n_S \) i.i.d. random points \( \mathcal{S}_{n_S} \) in \( \mathbb{R}^d \); the optimal pairing between the sample and the grid is bijective; the number of tied values involved is \( 0 \) or 1 according as \( n_S \) is even or odd. One also could consider factorizations of the form \( n = n_R n_S + n_0 \) with \( n_S \) even and \( 0 \leq n_0 < \min(2n_R, n_S) \), then require \( \mathcal{S}_n \) to be symmetric with respect to the origin, automatically yielding \( \sum_{s=1}^{n_S} s_{i,s} = 0 \).

The empirical counterpart \( F_{i,s}^{(n)} \) of \( F_{i,s} \) is defined as the (bijective, once the origin is given) mapping from \( Z_1^{(n)}, \ldots, Z_n^{(n)} \) to the grid \( \mathcal{S}_n \) that minimizes the sum of squared Euclidean distances \( \sum_{i=1}^n \| F_{i,s}^{(n)}(Z_i^{(n)}) - Z_i^{(n)} \|^2 \). That mapping is unique with probability one; in practice, it is obtained via a simple optimal assignment (pairing) algorithm (a linear program; see Section 4 of Hallin (2017) for details).

Call center-outward rank of \( Z_i^{(n)} \) the integer \( i = n_R + 1 )\| F_{i,s}^{(n)}(Z_i^{(n)}) \| \quad i = 1, \ldots, n \)
and center-outward sign of \( Z_i^{(n)} \) the unit vector
\[
S_{i,s}^{(n)} := \frac{F_{i,s}^{(n)}(Z_i^{(n)})}{\| F_{i,s}^{(n)}(Z_i^{(n)}) \|} \quad \text{for } F_{i,s}^{(n)}(Z_i^{(n)}) \neq 0;
\]
for \( F_{i,s}^{(n)}(Z_i^{(n)}) = 0 \), put \( S_{i,s}^{(n)} = 0 \).

Some desirable finite-sample properties, such as strict independence between the ranks and the signs, only hold for \( n_0 = 0 \) or 1, due to the fact that the mapping from the sample to the grid is no longer injective for \( n_0 \geq 2 \). This, which has no asymptotic consequences (since the number \( n_0 \) of tied values involved is \( o(n) \) as \( n \to \infty \)), is easily taken care of by the following tie-breaking device:

(i) randomly select \( n_0 \) i.i.d. random directions \( s_1^{(0)}, \ldots, s_{n_0}^{(0)} \) in \( \mathcal{S}_{n_S} \), then

(ii) replace the \( n_0 \) copies of the origin with the new gridpoints
\[
[1/2(n_R + 1)] s_1^{(0)}, \ldots, [1/2(n_R + 1)] s_{n_0}^{(0)}.
\]
The resulting grid (for simplicity, the same notation \( \mathcal{S}_n \) is used) no longer has multiple points, and the optimal pairing between the sample and the grid is bijective; the \( n_0 \) smallest ranks, however, take the non-integer value 1/2.

2.3. Main properties. This section summarizes some of the main properties of the concepts defined in Sections 2.1 and 2.2; further properties and a proof for Proposition 2.1 can be found in Hallin et al. (2020a).

**Proposition 2.1.** Let \( F_{i,s} \) denote the center-outward distribution function of \( P \in \mathcal{P}^d \). Then,

(i) \( F_{i,s} \) is a probability integral transformation of \( \mathbb{R}^d \); namely, \( Z \sim P \iff F_{i,s}(Z) \sim U_d \); by construction, \( \| F_{i,s}(Z) \| \) is uniform over the interval \( [0, 1] \), \( F_{i,s}(Z)/\| F_{i,s}(Z) \| \) uniform over the sphere \( S_{d-1} \), and they are mutually independent.
Let \( Z_{i}^{(n)} \), \( Z_{i}^{(n)} \) be i.i.d. with distribution \( P \in P_{d} \) and center-outward distribution function \( F_{\pm} \). Then,

(ii) \( \left( F_{\pm}^{(n)}(Z_{1}^{(n)}), \ldots, F_{\pm}^{(n)}(Z_{n}^{(n)}) \right) \) is uniformly distributed over the \( n!/n_{0}! \) permutations with repetitions of the gridpoints in \( G_{n} \) with the origin counted as \( n_{0} \) indistinguishable points (resp. the \( n! \) permutations of \( G_{n} \) if either \( n_{0} \leq 1 \) or the tie-breaking device described in Section 2.2 is adopted);

(iii) if either \( n_{0} = 0 \) or the tie-breaking device described in Section 2.2 is adopted, the \( n \)-tuple of center-outward ranks \( R_{1:n;\pm}^{(n)} \) and the \( n \)-tuple of center-outward signs \( S_{1:n;\pm}^{(n)} \) are mutually independent;

(iv) if either \( n_{0} \leq 1 \) or the tie-breaking device described in Section 2.2 is adopted, the \( n \)-tuple \( \left( F_{\pm}^{(n)}(Z_{1}^{(n)}), \ldots, F_{\pm}^{(n)}(Z_{n}^{(n)}) \right) \) is strongly essentially maximal ancillary.\(^8\)

Assuming, moreover, that \( P \in P_{d} \),

(v) (Glivenko-Cantelli) \( \max_{1 \leq i \leq n} \left| F_{\pm}^{(n)}(Z_{i}^{(n)}) - F_{\pm}^{(n)}(Z_{i}) \right| \to 0 \) a.s. as \( n \to \infty \).

Center-outward distribution functions, ranks, and signs also inherit, from the invariance features of Euclidean distances, elementary but quite remarkable invariance and equivariance properties under orthogonal transformations. Denote by \( F_{Z_{\pm}}^{Z} \) the center-outward distribution function of \( Z \) and by \( F_{Z_{\pm}}^{Z(n)} \) the empirical distribution function of a sample \( Z_{1}, \ldots, Z_{n} \) associated with a grid \( G_{n} \).

**Proposition 2.2.** Let \( \mu \in \mathbb{R}^{d} \) and denote by \( O \) a \( d \times d \) orthogonal matrix. Then,

(i) \( F_{\pm}^{\mu+OZ}(\mu + OZ) = OF_{Z}^{Z}(z), \) \( z \in \mathbb{R}^{d} \);

(ii) denoting by \( F_{\pm}^{\mu+OZ(n)} \) the empirical distribution function of the sample \( \mu + OZ_{1}, \ldots, \mu + OZ_{n} \) associated with the grid \( O\bar{G}_{n} \) (hence, by \( F_{Z_{\pm}}^{Z(n)} \) the empirical distribution function of the sample \( Z_{1}, \ldots, Z_{n} \) associated with the grid \( G_{n} \)),

\[
(2.3) \quad F_{\pm}^{\mu+OZ(n)}(\mu + OZ) = OF_{Z(n)}^{Z}(Z), \quad i = 1, \ldots, n;
\]

(iii) the center-outward ranks \( R_{1:n;\pm}^{(n)} \) and the cosines \( S_{1:n;\pm}^{(n)} \) computed from the sample \( Z_{1}, \ldots, Z_{n} \) and the grid \( G_{n} \) are the same as those computed from the sample \( \mu + OZ_{1}, \ldots, \mu + OZ_{n} \) and the grid \( O\bar{G}_{n} \).

**Proof.** Starting with (ii), note that, for any \( (z_{1}, \ldots, z_{n}) \in \mathbb{R}^{nd}, (u_{1}, \ldots, u_{n}) \in \mathbb{R}^{nd} \), and \( \mu \in \mathbb{R}^{d} \), denoting by \( \pi \) a permutation of \( \{1, \ldots, n\} \),

\[
\sum_{i=1}^{n} \| \mu + z_{i} - u_{\pi(i)} \|^{2} - \sum_{i=1}^{n} \| z_{i} - u_{\pi(i)} \|^{2} = n \mu^{T} \mu + 2 \mu^{T} \sum_{i=1}^{n} z_{i} - 2 \mu^{T} \sum_{i=1}^{n} u_{i}
\]

does not depend on \( \pi \); the optimal pairing between the \( \mu + z_{i} \)'s and the \( u_{i} \)'s thus does not depend on \( \mu \), so that \( F_{\pm}^{\mu+Z(n)}(\mu + Z_{i}) = F_{Z_{\pm}}^{Z(n)}(Z_{i}) \) for all \( i \) (with \( F_{\pm}^{\mu+Z(n)} \) and \( F_{Z_{\pm}}^{Z(n)} \) constructed from the same grid \( G_{n} \)). As for \( F_{\pm}^{OZ(n)}(OZ) \) computed from \( O\bar{G}_{n} \)

\(^8\)See Section 2.4 and Appendices D1 and D2 of Hallin et al. (2020a) for a precise definition of this crucial property and a proof.
and \( \text{OF}^{Z_{(n)}}(Z_i) \) computed from \( \mathfrak{G}_n \), they obviously coincide since the Euclidean distances on which they are based coincide. Part (iii) of the proposition is an immediate consequence.

Turning to (i), note that \( F_{\pm} \), as the gradient of a convex function, enjoys (see, e.g., Rockafellar (1966)) cyclical monotonicity: for any finite collection of points \( z_1, \ldots, z_k \in \mathbb{R}^{nd} \), it holds that

\[
\langle F_{\pm}(z_1), z_2 - z_1 \rangle + \langle F_{\pm}(z_2), z_3 - z_2 \rangle + \ldots + \langle F_{\pm}(z_k), z_1 - z_k \rangle \leq 0.
\]

Equivalently, considering the grid \( \mathfrak{G}_k := \{ F_{\pm}(z_1), \ldots, F_{\pm}(z_k) \} \), any \( k \)-tuple of the form \( (z_i, F_{\pm}(z_i)) \), \( i = 1, \ldots, k \) constitutes an optimal coupling minimizing

\[
S^{(k)}_z := \sum_{i=1}^{k} \| F_{\pm}(z_i) - z_{\pi(i)} \|^2
\]

over the \( k! \) permutations \( \pi \) of \( \{1, \ldots, k\} \): denoting by \( F_{\pm}^{z_{(k)}} \) the minimizer of \( S^{(k)}_z \), thus,

\[
F_{\pm}^{z_{(k)}}(z_i) = F_{\pm}(z_i), \quad i = 1, \ldots, k \text{ for any } k.
\]

Now, for fixed \( k \), (ii) applies, so that, similar to (2.3),

\[
F^{\mu \cdot \mbox{OZ}_{(k)}}_{\pm}(\mu + \mbox{OZ}_n) = \text{OF}^{Z_{(k)}}_{\pm}(z_i).
\]

In view of (2.4) (for \( F_{\pm} = F_Z^{(k)} \)), however,

\[
F^{\mu \cdot \mbox{OZ}_{(k)}}_{\pm}(\mu + \mbox{OZ}_n) = F^{\mu \cdot \mbox{OZ}}_{\pm}(\mu + \mbox{OZ}_n) \quad \text{and} \quad F^{Z_{(k)}}_{\pm}(z_i) = F^{Z}_{\pm}(z_i).
\]

The result follows from piecing together (2.5) and (2.6).

These orthogonal equivariance and invariance properties, however, do not extend to non-orthogonal affine transformations.

3. Linear center-outward rank statistics: Hájek representation and asymptotic normality.

3.1. Linear center-outward rank statistics. Linear rank statistics in this context depend on a score function \( J : \mathbb{S}_d \rightarrow \mathbb{R}^d \) and are indexed by triangular arrays \( \{c^{(n)}_1, \ldots, c^{(n)}_n\} \) of real numbers (regression constants). On those score functions and regression constants we are making the following assumptions.

**Assumption 3.1.**

(i) \( J : \mathbb{S}_d \rightarrow \mathbb{R}^d \) is continuous over \( \mathbb{S}_d \);

(ii) for any sequence \( s^{(n)} = \{s^{(n)}_1, \ldots, s^{(n)}_n\} \) of \( n \)-tuples in \( \mathbb{S}_d \) such that the uniform discrete distribution over \( s^{(n)} \) converges weakly to \( U_d \) as \( n \rightarrow \infty \),

\[
\lim_{n \rightarrow \infty} n^{-1} \text{tr} \sum_{r=1}^{n} J(s^{(n)}_r)J'(s^{(n)}_r) = \text{tr} \int_{\mathbb{S}_d} J(u)J'(u) \, dU_d
\]

where \( \int_{\mathbb{S}_d} J(u)J'(u) \, dU_d < \infty \) has full rank.

As we shall see, a special role is played, in relation with spherical distributions, by score functions of the form

\[
J(u) := J(\|u\|) \frac{u}{\|u\|} 1_{\{\|u\| \neq 0\}} \quad u \in \mathbb{S}_d
\]
for some function $J : [0, 1) \to \mathbb{R}$. Assumption 3.1 then holds if (i) $J$ is continuous and (ii)

$$(3.9) \quad 0 < \lim_{n \to \infty} n^{-1} \sum_{r=1}^{n} J^2 \left( \frac{r}{n+1} \right) = \int_{0}^{1} J^2(u) \, du < \infty$$

(a sufficient condition for (3.9) is the traditional assumption that $J$ has bounded variation, i.e. is the difference of two nondecreasing functions).

As for the regression constants, we assume that the classical Noether conditions hold.

ASSUMPTION 3.2. The $c_i^{(n)}$’s are not all equal (for given $n$) and satisfy

$$(3.10) \quad \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n))^2 / \max_{1 \leq i \leq n} (c_i^{(n)} - \bar{c}(n))^2 \to \infty \quad \text{as} \quad n \to \infty$$

where $\bar{c}(n) := n^{-1} \sum_{i=1}^{n} c_i^{(n)}$.

Associated with the score functions $J$, consider the $d$-dimensional statistics

$$(3.11) \quad T_a^{(n)} = \left( \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n))^2 \right)^{-1/2} \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n)) \mathbf{J}(F_{\pm}(Z_i^{(n)})),$$

$$T_e^{(n)} := \left( \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n))^2 \right)^{-1/2} \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n)) \mathbb{E} \left[ \mathbf{J}(F_{\pm}(Z_i^{(n)})) \left| \frac{F_{\pm}(Z_i^{(n)})}{\| F_{\pm}(Z_i^{(n)}) \|} \right. \right],$$

and

$$T^{(n)} := \left( \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n))^2 \right)^{-1/2} \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n)) \mathbf{J}(F_{\pm}(Z_i^{(n)})).$$

Adopting Hájek’s terminology, call $T_a^{(n)}$ an approximate-score linear rank statistic and $T_e^{(n)}$ an exact-score linear rank statistic. As we shall see, both $T_a^{(n)}$ and $T_e^{(n)}$ admit the same asymptotic representation $T^{(n)}$, hence are asymptotically equivalent. For score functions of the form (3.8), we have

$$T_a^{(n)} = \left( \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n))^2 \right)^{-1/2} \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n)) \mathbf{J}(\frac{R_{i,\pm}^{(n)}}{nR + 1}) S_{i,\pm}^{(n)},$$

$$T_e^{(n)} = \left( \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n))^2 \right)^{-1/2} \times \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n)) \mathbb{E} \left[ \mathbf{J}(\| F_{\pm}(Z_i^{(n)}) \|) \left| \frac{F_{\pm}(Z_i^{(n)})}{\| F_{\pm}(Z_i^{(n)}) \|} \right. \right] \mathbf{F}_{\pm}(Z_i^{(n)}),$$

and

$$T^{(n)} = \left( \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n))^2 \right)^{-1/2} \sum_{i=1}^{n} (c_i^{(n)} - \bar{c}(n)) \mathbf{J}(\| F_{\pm}(Z_i^{(n)}) \|) \frac{F_{\pm}(Z_i^{(n)})}{\| F_{\pm}(Z_i^{(n)}) \|}.$$
3.2. **Asymptotic representation.** The following proposition is a center-outward multivariate counterpart of the asymptotic results in Section V.1.6 of Hájek and Šidák (1967). Throughout this section, we assume that the empirical distribution function $F_{\pm}^{(n)}$ is computed from a triangular array $(Z_1^{(n)}, \ldots, Z_n^{(n)})$, $n \in \mathbb{N}$ of i.i.d. $d$-dimensional random vectors with distribution $P \in \mathcal{P}_d$ and center-outward distribution function $F_{\pm}$.

**Proposition 3.1** (Hájek representation). Let Assumptions 3.1 and 3.2 hold and $Z_1^{(n)}, \ldots, Z_n^{(n)}$ be i.i.d. with distribution $P \in \mathcal{P}_d$. Then,

$$(i) \quad T_{a}^{(n)} - T^{(n)} = o_{q.m.}(1), \quad (ii) \quad T_{e}^{(n)} - T^{(n)} = o_{q.m.}(1),$$

and

$$(iii) \quad T_{a}^{(n)} - T_{e}^{(n)} = o_{q.m.}(1)$$

as $n \to \infty$ in such a way that $n_R \to \infty$ and $n_S \to \infty$.

**Proof.** We throughout write $Z_i$ for $Z_i^{(n)}$. First consider Part (i) of the proposition. We have

$$T_{a}^{(n)} - T^{(n)} = \left(\sum_{i=1}^{n} (c_i^{(n)} - \bar{c}^{(n)})^2\right)^{-1/2} \left[\sum_{i=1}^{n} (c_i^{(n)} - \bar{c}^{(n)}) [J(F_{\pm}^{(n)}(Z_i)) - J(F_{\pm}(Z_i))].$$

Let $a_i^{(n)} := a(F_{\pm}^{(n)}(Z_i), F_{\pm}(Z_i)) := J(F_{\pm}^{(n)}(Z_i)) - J(F_{\pm}(Z_i))$. Then,

$$\|T_{a}^{(n)} - T^{(n)}\|^2 = \left(\sum_{i=1}^{n} (c_i^{(n)} - \bar{c}^{(n)})^2\right)^{-1} \left[\sum_{i=1}^{n} (c_i^{(n)} - \bar{c}^{(n)})^2 \|a_i^{(n)}\|^2 + \sum_{i \neq j} (c_i^{(n)} - \bar{c}^{(n)}) (c_j^{(n)} - \bar{c}^{(n)}) a_i^{(n)'} a_j^{(n)}\right].$$

Since $E\|a_i^{(n)}\|^2 = E\|a_i^{(n)}\|^2$ and $Ea_i^{(n)'} a_j^{(n)} = E a_i^{(n)'} a_j^{(n)}$, we get

$$E\|T_{a}^{(n)} - T^{(n)}\|^2 = E\|a_1^{(n)}\|^2 - E a_1^{(n)'} a_2^{(n)} \leq 2E\|a_1^{(n)}\|^2.$$

Hence, it only remains to show that

$$E\|a_1^{(n)}\|^2 = E\left\|J(F_{\pm}^{(n)}(Z_1)) - J(F_{\pm}(Z_1))\right\|^2$$

$$= E\|\zeta_n - \zeta\|^2 \to 0$$

with $\zeta_n := J(F_{\pm}^{(n)}(Z_1))$ and $\zeta := J(F_{\pm}(Z_1))$. It follows from the Glivenko-Cantelli theorem in Hallin et al. (2020a), the continuity of $J$, and the continuity over $\mathbb{R} \setminus \{0\}$ of $x \mapsto x/\|x\|$ that $\zeta_n \to \zeta$ a.s. Furthermore, Assumption 3.1 implies that

$$E\|\zeta_n - \zeta\|^2 = tr E(J(F_{\pm}^{(n)}(Z_1)) J'(F_{\pm}^{(n)}(Z_1)) \to tr E(J(F_{\pm}(Z_1)) J'(F_{\pm}(Z_1)) = E\|\zeta\|^2.$$

It follows (see, for instance, part (iv) of Theorem 5.7 in (Shorack, 2000, Chapter 3)) that $E\|\zeta_n - \zeta\|^2 \to 0$. This concludes the proof for Part (i) of the proposition.

---

9 The notation $o_{q.m.}(1)$ stands for a sequence of random vectors tending to zero in quadratic mean (hence also in probability).
Turning to Part (ii), put

$$b_1^{(n)} := J(F_{\pm}(Z_{i})) - E \left[ J(F_{\pm}(Z_{i})) \left| F_{\pm}^{(n)}(Z_{i}) \right. \right],$$

and let us show that $E(b_1^{(n)})^2 = o(1)$. Since $\|\zeta_n - \zeta\|$ tends a.s. to zero, it follows from the Egorov theorem (see, e.g., Theorem 7.5.1 in Dudley (1989)) that, for any $\varepsilon > 0$, there is a set $A \subset \Omega$ such that

$$P(A) > 1 - \varepsilon \quad \text{and} \quad \sup_{\omega \in A} \|\zeta_n(\omega) - \zeta(\omega)\| \to 0.$$

Denoting by $A^c$ the complement of $A$ in $\Omega$, we have

$$E\|b_1^{(n)}\|^2 = E\|\zeta - E[\zeta F_{\pm}^{(n)}(Z_1)]\|^2$$

$$= E\|\zeta 1_A + \zeta 1_{A^c} - E[\zeta 1_A + \zeta 1_{A^c} F_{\pm}^{(n)}(Z_1)]\|^2$$

$$\leq 3E\|\zeta 1_A - E[\zeta 1_A F_{\pm}^{(n)}(Z_1)]\|^2 + 3E\|\zeta 1_{A^c}\|^2$$

$$+ 3E\|E[\zeta 1_{A^c} F_{\pm}^{(n)}(Z_1)]\|^2$$

$$=: 3 \left( I_1^{(n)} + I_2 + I_3^{(n)} \right), \quad \text{say.}$$

In view of the square-integrability of $\zeta$, $I_2$ can be made arbitrarily small as $\varepsilon \to 0$. As for $I_3^{(n)}$, we have

$$I_3^{(n)} = E \left[ \|E[\zeta 1_A F_{\pm}^{(n)}(Z_1)]\|^2 \right] \leq E \left[ E[\|\zeta 1_A\|^2 F_{\pm}^{(n)}(Z_1)] \right] = E\|\zeta 1_A\|^2$$

where $P(A) \leq \varepsilon$, so that $I_3^{(n)}$ also is arbitrarily small as $\varepsilon \to 0$.

It remains to prove that $I_1^{(n)} \to 0$ as $n \to \infty$. Recall that $F_{\pm}^{(n)}(Z_1)$, with probability $(1 - n_0/n)$ tending to one, is a point of the regular grid $\mathcal{G}_n$ of $n - (n_0 - 1)1_{n_0>0}$ points in the unit ball used in the construction of $F_{\pm}^{(n)}$. Moreover, for any $g \in \mathcal{G}_n \setminus \{0\}$, we have $P[F_{\pm}^{(n)}(Z_1) = g] = 1/n$. Define

$$B_0^{(n)} := \{\omega : F_{\pm}^{(n)}(Z_1)(\omega) = g\}.$$

Clearly, $\{B_0^{(n)}, g \in \mathcal{G}_n\}$ constitutes a disjoint partition of $\Omega$ and $P(B_0^{(n)}) \to 0$ uniformly in $g \in \mathcal{G}_n$ as $n \to \infty$. Then,

$$I_1^{(n)} = E \left[ \sum_{g \in \mathcal{G}_n} \left( \zeta 1_A 1_{B_0^{(n)}} - E[\zeta 1_A F_{\pm}^{(n)}(Z_1)] 1_{B_0^{(n)}} \right) \right]^2$$

$$= E \sum_{g \in \mathcal{G}_n} \left( \zeta 1_A 1_{B_0^{(n)}} - E[\zeta 1_A F_{\pm}^{(n)}(Z_1)] 1_{B_0^{(n)}} \right)^2$$

where the latter equality follows form the fact that $1_{B_0^{(n)}} 1_{B_0^{(n)}} = 1_{g=h}$. Since $B_0^{(n)}$ is an atom of $\sigma(F_{\pm}^{(n)}(Z_1))$, the latter conditional expectation is a constant on $B_0^{(n)}$, namely

$$E[\zeta 1_A F_{\pm}^{(n)}(Z_1)] 1_{B_0^{(n)}} = \frac{1_{B_0^{(n)}}}{P(B_0^{(n)})} \int_{\eta \in B_0^{(n)}} \zeta(\eta) 1_A(\eta) dP(\eta).$$
Hence,

\[ I_1^{(n)} = \sum_{g \in \Phi_n} \int_{\Omega} \left\| 1_{B_g^{(n)}}(\omega) \int_{\eta \in B_g^{(n)}} [\zeta(\omega)1_A(\omega) - \zeta(\eta)1_A(\eta)] \frac{dP(\eta)}{P(B_g^{(n)})} \right\|^2 dP(\omega) \]

\[ = \sum_{g \in \Phi_n} \int_{\Omega} \left\| 1_{B_g^{(n)}}(\omega) \int_{\eta \in B_g^{(n)}} \left[ (\zeta(\omega) - \zeta_n(\omega))1_A(\omega) + (\zeta_n(\eta) - \zeta(\eta))1_A(\eta) \right] \frac{dP(\eta)}{P(B_g^{(n)})} \right\|^2 dP(\omega) \]

since \( \zeta_n(\omega)1_A(\omega)1_{B_g^{(n)}}(\omega) = J(g) = \zeta_n(\eta)1_A(\eta)1_{B_g^{(n)}}(\eta) \) on \( A \cap B_g^{(n)} \). Now, we are almost done. Since, for \( \omega \in A \), we have the uniform convergence of \( \| \zeta_n(\omega) - \zeta(\omega) \| \) to zero, we may bound the integrand uniformly. More precisely, for any \( \varepsilon > 0 \) there exists \( n_\varepsilon \) such that \( \| \zeta(\omega) - \zeta_n(\omega) \| < \varepsilon \) for all \( n \geq n_\varepsilon \) and all \( \omega \in A \), so that, from Jensen’s inequality,

\[ I_1^{(n)} \leq \sum_{g \in \Phi_n} \int_{\Omega} 1_{B_g^{(n)}}(\omega) \int_{B_g^{(n)}} \left[ 2\| \zeta(\omega) - \zeta_n(\omega) \|^2 1_A(\omega) + 2\| \zeta(\eta) - \zeta_n(\eta) \|^2 1_A(\eta) \right] \frac{dP(\eta)}{P(B_g^{(n)})} dP(\omega) \]

\[ \leq \sum_{g \in \Phi_n} \int_{\Omega} 1_{B_g^{(n)}}(\omega) 4\varepsilon^2 \frac{P(B_g^{(n)})}{P(B_g^{(n)})} dP(\omega) = 4\varepsilon^2 E \sum_{g \in \Phi_n} 1_{B_g^{(n)}} = 4\varepsilon^2. \]

Part (ii) of the proposition follows. Part (iii) is an immediate consequence of Parts (i) and (ii).

\[ \square \]

3.3. Asymptotic normality. The asymptotic normality of \( T_a^{(n)} \) and \( T_e^{(n)} \) follows from Proposition 3.1 and the asymptotic normality of \( T^{(n)} \), along with the distribution-freeness of \( T_a^{(n)} \) and \( T_e^{(n)} \).

**Proposition 3.2.** Let Assumptions 3.1 and 3.2 hold and \( Z_1^{(n)}, \ldots, Z_n^{(n)} \) be i.i.d. with distribution \( P \in \mathcal{P}_d \). Then, \( T_a^{(n)} \), \( T_e^{(n)} \), and \( T^{(n)} \) are asymptotically normal as \( n \to \infty \) (in such a way that \( n_R \to \infty \) and \( n_S \to \infty \)), with mean 0 and covariance \( \int_{S_d} J(u)J'(u) dU_d \) reducing, for \( J \) of the form (3.8), to \( d^{-1} \int_0^1 J^2(u) du I_d \).

**Proof.** First assume that \( P \in \mathcal{P}_d^\pm \), with center-outward distribution function \( F_\pm \). In view of Proposition 3.1, establishing the result for \( T^{(n)} \) is sufficient. Put \( V := F_\pm(Z_1^{(n)}) \). Then \( V \overset{D}{=} U W \), where \( U \) and \( W \) are mutually independent, \( U \) is uniform over \([0, 1]\), and \( W \) is uniform over the unit sphere \( S_{d-1} \). Clearly,

\[ E T^{(n)} = 0 \quad \text{and} \quad \text{Var} \left( T^{(n)} \right) = \text{Var} J(V) = \int_{S_d} J(u)J'(u) dU_d \]

so that, for \( J \) of the form (3.8),

\[ \text{Var} \left( T^{(n)} \right) = E J^2(U) \text{Var} W = \frac{1}{d} \int_0^1 J^2(u) du I_d \]

since \( \text{Var} W = \frac{1}{d} I_d \) (see, e.g. page 34 of Fang et al. (2017)). Now, \( T^{(n)} \) is a sum of independent variables, and the Noether condition (3.10) ensures that the Feller-Lindenberg condition.
holds. The desired asymptotic normality result (for $T_a^{(n)}$ and $T_e^{(n)}$, under $P \in \mathcal{P}_d^+$) thus follows from the central limit theorem. Finally, consider the general case $P \in \mathcal{P}_d$. Distribution-freeness implies that the finite-$n$ distributions of $T_a^{(n)}$ and $T_e^{(n)}$ are the same under $P \in \mathcal{P}_d$ as under $P' \in \mathcal{P}_d^+$. Hence, their asymptotic distributions under $P \in \mathcal{P}_d$ and $P' \in \mathcal{P}_d^+$ also coincide. This completes the proof. \hfill \Box

4. Multiple-output linear models. The objective of this section is to construct, based on the center-outward ranks and signs of Section 3, rank tests for the slopes of multiple-output linear models, extending to a multivariate setting the methods developed, e.g. in Puri and Sen (1985) for the single-output case.

4.1. The model. Consider the multiple-output linear (or multiple-output regression) model under which an observed $Y^{(n)}$ satisfies

$$Y^{(n)} = 1_n \beta'_0 + C^{(n)} \beta + \varepsilon^{(n)},$$

where $1_n := (1, \ldots, 1)'$, $Y^{(n)} = \begin{pmatrix} Y^{(n)}_{11} & Y^{(n)}_{12} & \cdots & Y^{(n)}_{1d} \\ \vdots & \vdots & & \vdots \\ Y^{(n)}_{n1} & Y^{(n)}_{n2} & \cdots & Y^{(n)}_{nd} \end{pmatrix} \in \mathbb{R}^{n \times d}$ is an $n \times d$ matrix of $n$ observed $d$-dimensional outputs,

$$C^{(n)} = \begin{pmatrix} c^{(n)}_{11} & c^{(n)}_{12} & \cdots & c^{(n)}_{1m} \\ \vdots & \vdots & & \vdots \\ c^{(n)}_{n1} & c^{(n)}_{n2} & \cdots & c^{(n)}_{nm} \end{pmatrix} \in \mathbb{R}^{n \times m}$$

an $n \times m$ matrix of (specified) deterministic covariates,

$$\beta'_0 = (\beta_{01}, \ldots, \beta_{0d}) \quad \text{and} \quad \beta = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1d} \\ \vdots & \vdots & & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{md} \end{pmatrix} \in \mathbb{R}^{m \times d}$$

a $d$-dimensional intercept and an $m \times d$ matrix of regression coefficients, and

$$\varepsilon^{(n)} = \begin{pmatrix} \varepsilon^{(n)}_{11} & \varepsilon^{(n)}_{12} & \cdots & \varepsilon^{(n)}_{1d} \\ \vdots & \vdots & & \vdots \\ \varepsilon^{(n)}_{n1} & \varepsilon^{(n)}_{n2} & \cdots & \varepsilon^{(n)}_{nd} \end{pmatrix} \in \mathbb{R}^{n \times d}$$

an $n \times d$ matrix of nonobserved i.i.d. $d$-dimensional errors $\varepsilon_i^{(n)}$, $i = 1, \ldots, n$ with density $f^\varepsilon$. If $\beta_0$ is to be identified, a location constraint has to be imposed on $f^\varepsilon$. One could think of the classical constraint $\mathbb{E}\varepsilon_i^{(n)} = 0$ (requiring the existence of a finite mean): $\beta_0 + \beta' \varepsilon_i^{(n)}$ then is to be interpreted as the expected value of $Y_i^{(n)}$ for covariate values $c_i^{(n)}$. In the context of this paper, however, a more natural location constraint (which moreover does not require any integrability condition) is $F_\varepsilon^\mp(0) = 0$, where $F_\varepsilon^\mp$ stands for the center-outward distribution function of the $\varepsilon_i^{(n)}$: $0$ and $\beta_0 + \beta' \varepsilon_i^{(n)}$ then are center-outward medians for $\varepsilon$ and $Y_i^{(n)}$, respectively.

In most applications, however, one is interested mainly in the impact of the input covariates $c_i^{(n)}$ on the output $Y_i^{(n)}$: the matrix $\beta$ is the parameter of interest, and $\beta_0$ is a nuisance.
There is no need, then, for identifying \( \beta_0 \) nor qualifying \( \beta_0 + \beta' c_i^{(n)} \) as a mean or a center-outward median for \( Y_i^{(n)} ; \beta \) is to be interpreted as a matrix of treatment effects governing the shift \( \delta \beta \) in the distribution of the \( d \)-dimensional output produced by a variation \( \delta \) in the \( m \)-dimensional covariate. Center-outward ranks and signs being insensitive to shifts, there is even no need to specify, nor to estimate \( \beta_0 \).

4.2. Local Asymptotic Normality (LAN). The model (4.1) is easily seen to be locally asymptotically normal (LAN) under the following two classical assumptions.

**Assumption 4.1.** The square root \( z \mapsto (f\varepsilon)^{1/2}(z) \) of the error density is differentiable in quadratic mean,\(^{10}\) with quadratic mean gradient \( \nabla (f\varepsilon)^{1/2} \). Letting

\[
\varphi_{f\varepsilon} := -2\nabla (f\varepsilon)^{1/2} / (f\varepsilon)^{1/2},
\]

assume moreover that the information matrix \( \mathcal{I}_{f\varepsilon} := \mathbb{E} \left[ \varphi_{f\varepsilon}(\varepsilon) \varphi_{f\varepsilon}'(\varepsilon) \right] \) has full rank \( d \).

On the regression constants \( C_i^{(n)} \), we borrow from Hallin and Paindaveine (2005) the following assumptions; note that Part (iii) requires that each of the \( m \) triangular arrays of constants \( c_{ij}^{(n)}, i \in \mathbb{N}, j = 1, \ldots, m \) satisfies Assumption 3.2.

**Assumption 4.2.** Letting \( \mathbf{V}_c^{(n)} := n^{-1} \sum_{i=1}^{n} (\mathbf{e}_i^{(n)} - \bar{c}^{(n)})(\mathbf{e}_i^{(n)} - \bar{c}^{(n)})' \), with \( \bar{c}^{(n)} := n^{-1} \sum_{i=1}^{n} c_i^{(n)} \), denote by \( \mathbf{D}_c^{(n)} \) the diagonal matrix with elements \( (\mathbf{V}_c^{(n)})_{jj} \), \( j = 1, \ldots, m \);

(i) \( (\mathbf{V}_c^{(n)})_{jj} > 0 \) for \( j = 1, \ldots, m \);

(ii) defining \( \mathbf{R}_c^{(n)} := \mathbf{D}_c^{(n)-1/2} \mathbf{V}_c^{(n)} \mathbf{D}_c^{(n)-1/2} \), the limit \( \mathbf{R}_c := \lim_{n \to \infty} \mathbf{R}_c^{(n)} \) exists, is positive definite, and factorizes into \( \mathbf{R}_c = (\mathbf{K}_c \mathbf{K}_c')^{-1} \) for some full-rank \( m \times m \) matrix \( \mathbf{K}_c \);

(iii) the following Noether conditions hold: letting \( c_{ij}^{(n)} := n^{-1} \sum_{i=1}^{n} c_{ij}^{(n)} \),

\[
\lim_{n \to \infty} \sum_{i=1}^{n} (c_{ij}^{(n)} - c_{ij}')^2 / \max_{1 \leq i \leq n} (c_{ij}^{(n)} - c_{ij}')^2 = \infty, \quad j = 1, \ldots, m.
\]

Letting \( \mathbf{Z}_i^{(n)} = \mathbf{Z}_i^{(n)}(\beta) := Y_i^{(n)} - \mathbf{1}_n \beta_0 - \beta' c_i^{(n)} \), the following result readily follows from, e.g., (Lehmann and Romano, 2005, Theorem 12.2.3). In order to simplify the notation, we throughout adopt the same contiguity rates as in Hallin and Paindaveine (2005). Namely, we consider local perturbations of the parameter \( \beta \) of the form \( \beta + \nu(n) \tau \) where \( \tau \) is an \( m \times d \) matrix and \( \nu(n) := n^{-1/2} \mathbf{K}_c^{(n)} \), with \( \mathbf{K}_c^{(n)} := (\mathbf{D}_c^{(n)})^{-1/2} \mathbf{K}_c \). This is a notational convenience and has no impact on the form of locally asymptotically optimal test statistics.

**Proposition 4.1.** Under Assumptions 4.1 and 4.2, the model (4.1) is LAN (with respect to \( \beta \)), with central sequence \( \Delta_{\beta_0;f\varepsilon}^{(n)}(\beta) := n^{1/2} \text{vec} \Lambda_{\beta_0;f\varepsilon}^{(n)} \) where

\[
\Lambda_{\beta_0;f\varepsilon}^{(n)} := \frac{1}{n} \sum_{i=1}^{n} \mathbf{K}_c^{(n)}(\mathbf{e}_i^{(n)} - \bar{c}^{(n)}) \varphi_{f\varepsilon}'(\mathbf{Z}_i^{(n)})
\]

and Fisher information \( \mathcal{I}_{f\varepsilon} \otimes \mathbf{I}_m \)

\(^{10}\)It follows from a result by Lind and Roussas (1972) independently rediscovered by Garel and Hallin (1995) that quadratic mean differentiability is equivalent to partial quadratic mean derivability with respect to all variables.
LAN for the same linear model (4.1) has been established (in the broader context of regression with VARMA errors in Hallin and Paindaveine (2005) under the assumption that the error density \( f^e \) is centered elliptical,\(^{11}\) that is, has the form

\[
(4.3) \quad f^e (z) = \kappa_d \mu_{d, f}^{-1/2} \left( (z/\Sigma^{-1/2}z)^1/2 \right)
\]

with \( \kappa_d := (2\pi^{d/2}/\Gamma(d/2)) \int_{r>0} r^{d-1} F^e_r (r) dr \) for some symmetric positive definite shape matrix \( \Sigma \) and some radial density \( f \) (over \( \mathbb{R}_0^+ \)) such that \( f(z) > 0 \) Lebesgue-a.e. in \( \mathbb{R}_0^+ \) and \( \int_{r>0} r^{d-1} f(r) dr < \infty \). When \( \varepsilon \) is elliptical with shape matrix \( \Sigma \) and radial density \( f \), the modulus \( \| \Sigma^{-1/2} \varepsilon \| \) has density

\[
\int_{r>0} r^{d-1} f(r) dr < \infty.
\]

Assumption 4.1 then is equivalent to the mean square differentiability, with quadratic mean derivative \((\hat{f}^{1/2})'\)

A traditional choice which, however, rules out heavy-tailed radial densities with infinite second-order moments, is the empirical covariance matrix of the \( Z_i^{(n)} \)'s. An alternative, which satisfies Assumption 4.3 without any moment assumptions, is Tyler’s estimator of scatter, see Theorem 4.1 in Tyler (1987) for strong consistency, Theorem 4.2 for asymptotic normality.

Under Assumption 4.3, which entails the affine invariance of \( Z_i^{(n)} \)\(^{\text{ell}} \). Proposition 4.1 takes the following form.

**Proposition 4.2.** Under Assumptions 4.2 and 4.3, the model (4.1) with error density \( f^e \) of the elliptical type (4.3) and quadratic mean differentiable \( f^{1/2} \) is LAN (with respect to \( \beta \)), with central sequence

\[
(4.5) \quad \Delta^{(n)}_{\Sigma^{(n)}; \beta_0; f} (\beta) := n^{1/2} \left( \left( \Sigma^{(n)} \right)^{-1/2} \otimes I_m \right) \text{vec} \left( \Delta^{(n)}_{\Sigma^{(n)}; \beta_0; f} (\beta) \right) + o_P(1)
\]

\(^{11}\)For simplicity, we henceforth are dropping the word “centered.”
where

\[ \Delta^{(n)_{\text{ell}}}_{\Sigma; \beta; i; j}(\beta) := \frac{1}{n} \sum_{i=1}^{n} \varphi_i(\|Z_{i}^{(n)_{\text{ell}}}\|)K_{c}^{(n)_{\text{ell}}}(c_{i}^{(n)} - \bar{c}^{(n)}) \left( \frac{Z_{i}^{(n)_{\text{ell}}}}{\|Z_{i}^{(n)_{\text{ell}}}\|} \right)' \]

leading to a Fisher information matrix \( \frac{1}{n} I_{d; i} \Sigma^{-1} \otimes I_m \).

This LAN result, where the residuals are subjected to preliminary (empirical) sphericization via \((\tilde{\Sigma}^{(n)})^{-1/2}\), highlights the fact that elliptical families with given \( f \) are parametrized by the form of the central sequence \((4.2)\), and hence distribution-free, of the central sequence \(n\) and \(\bar{c}\) leading to a Fisher information matrix \((4.6)\) where \(\Delta_{\Sigma; \beta; i; j}(\beta) = \left( I_d \otimes \Sigma^{-1/2} \right) \Delta_{I_d; \beta; i; j}(\beta) \), the limiting Gaussian shift experiments associated with elliptical and spherical errors coincide (with the perturbation \(\text{vec}(\tau)\) of \(\text{vec}(\beta)\) in the elliptical case corresponding to a perturbation \(\text{vec}(\varsigma) = \left( I_d \otimes \Sigma^{-1/2} \right) \text{vec}(\tau)\) in the spherical case). That invariance under linear sphericization of local limiting Gaussian shifts, however, does not extend to the general case of Proposition 4.1.

5. Rank tests for multiple-output linear models.

5.1. Elliptical or Mahalanobis rank tests. Rank-based inference for elliptical multiple-output linear models was developed in Hallin and Paindaveine (2005). The ranks and the signs there are the elliptical or Mahalanobis ranks \(R_{i; n_{\text{ell}}}^{(n)_{\text{ell}}}\) and signs \(S_{i; n_{\text{ell}}}^{(n)_{\text{ell}}}\) —namely, the ranks of the moduli \(\|Z_{i}^{(n)_{\text{ell}}}\|\) and the signs \(S_{i; n_{\text{ell}}}^{(n)_{\text{ell}}} := \frac{Z_{i}^{(n)_{\text{ell}}}}{\|Z_{i}^{(n)_{\text{ell}}}\|}\), both computed, in agreement with the above remark on the spherical nature of elliptical families, after the empirical sphericization \((4.4)\).

The validity of tests based on those elliptical ranks and signs, unfortunately, requires elliptical \(f\). A welcome relaxation of stricter Gaussianity assumptions, ellipticity remains an extremely strong symmetry requirement; it is made, essentially, for lack of anything better but is unlikely to hold in practice. If the assumption of ellipticity is to be waived, elliptical ranks and signs are losing their distribution-freeness for the benefit of the center-outward ranks and signs. And, since center-outward ranks and signs, in view of Proposition 2.2, are invariant under location shift, center-outward rank tests can address the (more realistic) unspecified intercept case without any additional estimation step.

5.2. Center-outward rank tests. Denote by \(F_{i; n}^{(n)_{\text{ell}}}\) the empirical center-outward distribution associated with the \(n\)-tuple \((Z_{1}^{(n)_{\text{ell}}}, \ldots, Z_{n}^{(n)_{\text{ell}}}\) where \(Z_{i}^{(n)_{\text{ell}}}\) now is defined as \(Y_{i; n}^{(n)_{\text{ell}}} = \beta'c_{i}^{(n)}\), by \(R_{i; n}^{(n)_{\text{ell}}}\) and \(S_{i; n}^{(n)_{\text{ell}}}\), respectively, the corresponding center-outward ranks and signs. In line with the form of the central sequence \((4.2)\), consider

\[ \Delta_{J; n}^{(n)_{\text{ell}}} := n^{-1} \sum_{i=1}^{n} K_{c}^{(n)_{\text{ell}}}(c_{i}^{(n)} - \bar{c}^{(n)})J'( \frac{R_{i; n}^{(n)_{\text{ell}}}}{n_{R} + 1} S_{i; n}^{(n)_{\text{ell}}} ) . \]

It follows from the asymptotic representation result of Proposition 3.1 that, when the actual density is \(f\), for the scores \(J = \varphi_{f_{\beta}} \circ F_{\varphi}^{-1}\), with \(\varphi_{f_{\beta}}\) defined in Assumption 4.1

\[ \Delta_{\beta; f_{\beta}}^{(n)_{\text{ell}}}(\beta) := n^{1/2} \text{vec} \Delta_{J; n}^{(n)_{\text{ell}}} = \Delta_{\beta; f_{\beta}}^{(n)_{\text{ell}}}(\beta) + o_{P}(1) \]

and \(\Delta_{\beta; f_{\beta}}^{(n)_{\text{ell}}}(\beta)\) thus constitutes a version, based on the center-outward ranks and signs and hence distribution-free, of the central sequence \(\Delta_{\beta; f_{\beta}}^{(n)_{\text{ell}}}(\beta)\) in \((4.2)\). The following asymptotic normality result then holds.
PROPOSITION 5.1. Assume that $Y_{i}^{(n)}$ satisfies (4.1) and let Assumptions 3.1 and 4.2 hold. Then,

(i) $n^{1/2}\text{vec} \Lambda_{J}^{(n)\pm}$ is asymptotically normal, with mean 0 and covariance

$$\mathcal{I}_{J} \otimes I_{m} \quad \text{where} \quad \mathcal{I}_{J} := \int_{S_{d}} J(u)J'(u)dU_{d},$$

under the null hypothesis $H_{0}^{(n)}(\beta^{0})$ that $\beta = \beta^{0}$ while the intercept $\beta_{0}$ and the distribution $P \in \mathcal{P}_{d}$ of the $\varepsilon$'s remain unspecified;

(ii) the test rejecting $H_{0}^{(n)}(\beta^{0})$ whenever the test statistic

$$Q_{J}^{(n)\pm} := n\left(\text{vec} \Lambda_{J}^{(n)\pm}\right)^{'} \mathcal{I}_{J}^{-1} \otimes I_{m} \left(\text{vec} \Lambda_{J}^{(n)\pm}\right)$$

exceeds the $(1 - \alpha)$ quantile of a chi-square distribution with md degrees of freedom has asymptotic level $\alpha$ as $n \to \infty$;\(^{12}\)

(iii) for $J = \varphi_{f} \circ F_{\pm}$ where $F_{\pm}$ denotes the center-outward distribution function associated with $f^{c}$, the covariance $\mathcal{I}_{J} \otimes I_{m}$ coincides with $\mathcal{I}_{J^{c}} \otimes I_{m}$ and the test based on $Q_{J}^{(n)\pm}$ is, under error density $f^{c}$, locally asymptotically maximin, at asymptotic level $\alpha$, for the null hypothesis $H_{0}^{(n)}(\beta^{0})$.

PROOF. First assume that the error distribution $P$, with center-outward distribution function $F_{\pm}$, is in $\mathcal{P}_{d}$. Noting that, for column vectors $a$ and $b$, we have $\text{vec}(ab') = b \otimes a$,

$$n^{1/2}\text{vec} \Lambda_{J}^{(n)\pm} = n^{-1/2} \sum_{i=1}^{n} \text{vec}\left[K_{c}^{(n)'}(c_{i}^{(n)} - \bar{c}(n))J'(\frac{R_{i}^{(n)}}{nR + 1}S_{i}^{(n)})\right]$$

$$= n^{-1/2} \sum_{i=1}^{n} J\left(\frac{R_{i}^{(n)}}{nR + 1}S_{i}^{(n)}\right) \otimes \left(K_{c}^{(n)'}(c_{i}^{(n)} - \bar{c}(n))\right).$$

It follows from Proposition 3.1 that this latter statistic is asymptotically equivalent to

$$T_{J} = n^{-1/2} \sum_{i=1}^{n} J\left(F_{\pm}(Z_{i}^{(n)})\right) \otimes \left(K_{c}^{(n)'}(c_{i}^{(n)} - \bar{c}(n))\right)$$

which is a sum of independent variables such that $ET_{J} = 0$ and

$$\text{Var} T_{J} = n^{-1} \sum_{i=1}^{n} \text{Var}\left[J\left(F_{\pm}(Z_{i}^{(n)})\right) \otimes \left(K_{c}^{(n)'}(c_{i}^{(n)} - \bar{c}(n))\right)\right]$$

$$= n^{-1} \sum_{i=1}^{n} \text{E}\left[J\left(F_{\pm}(Z_{i}^{(n)})\right) \otimes \left(K_{c}^{(n)'}(c_{i}^{(n)} - \bar{c}(n))\right) \times J'(F_{\pm}(Z_{i}^{(n)}) \otimes (c_{i}^{(n)} - \bar{c}(n))K_{c}^{(n)}\right]$$

$$= n^{-1} \sum_{i=1}^{n} \text{E}\left[J\left(F_{\pm}(Z_{i}^{(n)})\right)J'(F_{\pm}(Z_{i}^{(n)})\right)$$

\(^{12}\)Since $Q_{J}^{(n)\pm}$ is distribution-free under the null hypothesis $H_{0}^{(n)}(\beta^{0})$, the finite-$n$ size of this test is uniform over $H_{0}^{(n)}(\beta^{0})$, hence uniformly close to $\alpha$ for $n$ large enough. This is in sharp contrast with daily practice pseudo-Gaussian tests, which remain asymptotically valid under a broad range of distributions, albeit not uniformly so.

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\[ \otimes \left( K_c^{(n)r} \langle c_i^{(n)} - \bar{c}^{(n)} \rangle (c_i^{(n)} - \bar{c}^{(n)})' K_c^{(n)r} \right) \]

\[ = \int_{\mathcal{S}_d} J(u) J'(u) \, dU_d \otimes n^{-1} K_c^{(n)r} \sum_{i=1}^n \left[ (c_i^{(n)} - \bar{c}^{(n)}) (c_i^{(n)} - \bar{c}^{(n)})' \right] K_c^{(n)r} \]

which tends to \( \mathcal{I}_J \otimes I_m \) as \( n \to \infty \). The Lindeberg condition is satisfied, so that \( T_J \), hence also \( n^{1/2} \vec{\Lambda}_J^{(n)\pm} \), has the announced asymptotic normal distribution.

Finally, consider the general case of an absolutely continuous \( P \in \mathcal{P}_d \); as in the proof of Proposition 3.2, distribution-freeness implies that the asymptotic distribution of \( n^{1/2} \vec{\Lambda}_J^{(n)\pm} \) is the same under \( P \in \mathcal{P}_d \) as under \( P \in \mathcal{P}_d^+ \). This completes the proof of Part (i). In view of (5.2); Parts (ii) and (iii) readily follow. \( \square \)

**COROLLARY 5.2.** (i) In the particular case of a spherical score of the form (3.8), the test statistic \( Q_J^{(n)\pm} \) simplifies into

\[ Q_J^{(n)\pm} = \frac{nd}{\int_0^1 J^2(u) \, du} \left( \vec{\Lambda}_J^{(n)\pm} \right)' \left( \vec{\Lambda}_J^{(n)\pm} \right) \]

where

\[ \vec{\Lambda}_J^{(n)\pm} := n^{-1} \sum_{i=1}^n J \left( \frac{R_{i;\pm}}{nR + 1} \right) K_c^{(n)r} (c_i^{(n)} - \bar{c}^{(n)}) S_{i;\pm}^{(n)r} \]

and \( n^{1/2} \vec{\Lambda}_J^{(n)\pm} \) is asymptotically normal with mean \( 0 \) and variance \( d^{-1} \int_0^1 J^2(u) \, du \mathbf{I}_{md} \).

(ii) The test statistic \( Q_J^{(n)\pm} \) with spherical score

\[ J_i := \varphi_{f_i} \circ (F_{d,i})^{-1} \]

yields locally asymptotically optimal tests under the spherical density with radial density \( f \).

5.3. Some particular cases. In this section, we provide explicit forms of the test statistic for the two-sample and MANOVA problems. Because of their simplicity and practical value (see Section 6.1), we concentrate on the case (5.4) of spherical scores, from which the general case (5.3) is easily deduced (essentially, by substituting \( J \left( \frac{R_{i;\pm}}{nR + 1} S_{i;\pm}^{(n)} \right) \chi_i \) for \( J \left( \frac{R_{i;\pm}}{nR + 1} S_{i;\pm}^{(n)} \right) S_{i;\pm}^{(n)} \)).

5.3.1. Center-outward rank tests for two-sample location. An important particular case is the two-sample location model, where \( n = n_1 + n_2 \) and (4.1) holds with covariates of the form \( \mathbf{C}^{(n)} = (1'_{n_1}, 0'_{n_2})' \) (with \( 1_{n_1} \) an \( n_1 \)-dimensional column vector of ones, \( 0_{n_2} \) an \( n_2 \)-dimensional column vector of zeros); the parameter \( \beta = (\beta_{11}, \ldots, \beta_{1d})' \) here is a \( d \)-dimensional row vector.

The objective is to test the null hypothesis \( H_0 : \beta = 0_d \) under which the distributions of \( Y_1^{(n)}, \ldots, Y_{n_1}^{(n)} \) and \( Y_{n_1+1}^{(n)}, \ldots, Y_n^{(n)} \) coincide. Elementary computation yields

\[ \bar{c}^{(n)} = n_1/n, \quad V_c^{(n)} = n_1 n_2 / n^2, \quad \text{and} \quad K_c = 1. \]

If the regular grid \( \mathcal{G}_n \) is chosen such that \( \| \sum_{s=1}^{n_s} S_s^{(n)} \| = 0 \) (which is always possible, see Section 2.2), \( \sum_{i=1}^n J \left( \frac{R_{i;\pm}}{nR + 1} \right) S_{i;\pm}^{(n)} = 0 \) and the test statistic (5.4) takes the simple form

\[ Q_J^{(n)\pm} = \left( \frac{nd}{n_1 n_2} \int_0^1 J^2(u) \, du \right) \left\| \sum_{i=1}^{n_1} J \left( \frac{R_{i;\pm}}{nR + 1} \right) S_{i;\pm}^{(n)} \right\|^2. \]
Assumption 4.2 (iii) yields \( \lim_{n \to \infty} n \min \{n_1, n_2\} / \max \{n_1, n_2\} = \infty \), which holds whenever
\[
\lim_{n \to \infty} \min \{n_1, n_2\} = \infty.
\]

Under Assumptions 3.1 and (5.7), with \( P \in \mathcal{P}_d \), \( Q_J^{(n)} \) is, under \( H_0 \), asymptotically \( \chi^2 \) with \( d \) degrees of freedom and the null hypothesis can be rejected at asymptotic level \( \alpha \) whenever \( Q_J^{(n)} \) exceeds the \( (1 - \alpha) \) quantile of a \( \chi^2 \) distribution.

5.3.2. Center-outward rank tests for MANOVA. Another important special case of model (4.1) is the multivariate \( K \)-sample location or MANOVA model. The observation here decomposes into \( K \) samples, with respective sizes \( n_1, \ldots, n_K \) and \( n = \sum_{k=1}^K n_k \). Precisely, \( Y^{(n)} = (Y^{(n;1)}, \ldots, Y^{(n;k)}, \ldots, Y^{(n;K)}) \)
with
\[
Y^{(n;k)} = \begin{pmatrix}
Y_{k;11}^{(n)} & Y_{k;12}^{(n)} & \cdots & Y_{k;1d}^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{k;n_k1}^{(n)} & Y_{k;n_k2}^{(n)} & \cdots & Y_{k;n_kd}^{(n)}
\end{pmatrix}
\]
and (4.1) holds with the matrix of covariates
\[
C^{(n)} = \begin{pmatrix}
d_{11}^{(n)} & d_{12}^{(n)} & \cdots & d_{1K-1}^{(n)} \\
d_{21}^{(n)} & d_{22}^{(n)} & \cdots & d_{2K-1}^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
d_{K1}^{(n)} & d_{K2}^{(n)} & \cdots & d_{KK-1}^{(n)}
\end{pmatrix}
\]
where \( d_{ij}^{(n)} = I_{ij}, i, j = 1, \ldots, K \) and \( i = 1, \ldots, K \) and \( j = 1, \ldots, K - 1 \). The null hypothesis is the hypothesis of no treatment effect \( H_0 : \beta = 0_{(K-1) \times d} \).

Letting \( V^{(n)} := (n_1/n, \ldots, n_{K-1}/n)' \), the matrix \( V_c^{(n)} \) in Assumption 4.2 takes the form \( V_c^{(n)} = \text{diag}\{V^{(n)}\} - V^{(n)}V^{(n)'} \), where \( \text{diag}\{V^{(n)}\} \) stands for the diagonal matrix with diagonal entries \( V^{(n)} \). If the regular grid \( \mathcal{S}_n \) is chosen such that \( \| \sum_{s=1}^{n_s} s_{s}^{n_s} \| = 0 \), the test statistic (5.4) simplifies into
\[
Q_J^{(n)} = \frac{d}{\int_0^1 J^2(u) \, du} \sum_{k=1}^K \left\| \sum_{i=n_k+1}^{n_{k+1}} J \left( \frac{R_{i;\pm}^{(n)}}{n_{R} + 1} \right) S_{i;\pm}^{(n)} \right\|^2
\]
where \( (V_c^{(n)})^{-1/2} \) is substituted for its limit \( K_c^{(n)} \).

Assumption 4.2(iii) is satisfied as soon as \( \lim_{n \to \infty} \min \{n_1, \ldots, n_K\} \to \infty \). Assuming moreover that \( \liminf_{n \to \infty} n_k/n > 0 \) and \( \limsup_{n \to \infty} n_k/n < \infty \) for \( 1 \leq k \leq K \), the limit matrix \( R_c \) is positive definite\(^{13}\) and Assumption 4.2(ii) is satisfied as well. Then, under the null hypothesis of no treatment effect, \( Q_J^{(n)} \) is asymptotically chi-square with \( (K - 1)d \) degrees of freedom and the test rejecting \( H_0 \) whenever \( Q_J^{(n)} \) exceeds the corresponding \( (1 - \alpha) \) quantile has asymptotic level \( \alpha \) irrespective of the actual error distribution \( P \in \mathcal{P}_d \). This test is a multivariate generalization of the well-known univariate rank test for \( K \)-sample equality of location (the univariate one-way ANOVA hypothesis of no treatment effect), see (Hájek and Šidák, 1967, p.170). Note that, for \( K = 2 \), \( Q_J^{(n)} \) coincides with the two-sample test statistic obtained Section 5.3.1.

\(^{13}\)This limit possibly can exist along subsequences, with asymptotic statements modified accordingly. For the sake of simplicity, we do not include this in subsequent results.
6. Choosing a score function. Section 5 allows us to construct, based on any score function (\(J\) or \(J_s\)) satisfying Assumption 3.1 (either with (3.7) or (3.9)), strictly distribution-free center-outward rank tests of the null hypothesis \(H_0^{(n)}(\beta^0)\) under which \(\beta = \beta^0\) while the intercept \(\beta_0\) and the error distribution \(P \in P_d\) remain unspecified. All these tests, however, depend on a score function to be selected by the practitioner. Some will favor simple scores of the spherical type (see Section 6.1); others may want to base their choice on efficiency considerations (see Section 6.2).

6.1. Standard score functions. Popular choices are the spherical sign test, Wilcoxon and van der Waerden scores. Let us describe them, in more details, in the particular case of the two-sample problem.

The two-sample sign test is based on the degenerate score \(J_{\text{sign}}(r) := 1\) for \(r \in (0, 1)\); using the fact that \(\sum_{i=1}^n S^{(n)}_{i; \pm} = 0\), one gets for (5.6), with the notation of Section 5.3.1, the very simple test statistic
\[
Q^{(n)}_{\text{sign}} = \frac{nd}{n_1 n_2} \left\| \sum_{i=1}^{n_1} S^{(n)}_{i; \pm} \right\|^2.
\]

The choice \(J_{\text{wilcoxon}}(r) := r\) similarly yields the Wilcoxon two-sample test: noting that \(\sum_{i=1}^n R^{(n)}_{i; \pm} S^{(n)}_{i; \pm} = 0\) holds if \(\sum_{i=1}^n S^{(n)}_{i; \pm} = 0\) and that \(\int_0^1 r^2 \, dr = 1/3\), this yields the test statistic
\[
Q^{(n)}_{\text{wilcoxon}} = \frac{3nd}{n_1 n_2 (n_R + 1)} \left\| \sum_{i=1}^{n_1} R^{(n)}_{i; \pm} S^{(n)}_{i; \pm} \right\|^2.
\]

As for the two-sample van der Waerden test, it is based on the Gaussian or van der Waerden scores \(J_{\text{vdw}}(r) := (\psi_d^{-1}(r))^{1/2}\), where \(\psi_d\) denotes the cumulative distribution function of a chi-square variable with \(d\) degrees of freedom. Clearly \(\int_0^1 J_{\text{vdw}}^2(r) \, dr = \int_0^\infty x \psi_d(x) = d\) and, provided that \(\sum_{i=1}^n S^{(n)}_{i; \pm} = 0\),
\[
\sum_{i=1}^n \psi_d^{-1}\left(\frac{R^{(n)}_{i; \pm}}{n_R + 1}\right)^{1/2} S^{(n)}_{i; \pm} = 0.
\]

Hence, the van der Waerden center-outward rank test statistics takes the form
\[
Q^{(n)}_{\text{vdw}} = \frac{n}{n_1 n_2} \left\| \sum_{i=1}^{n_1} \psi_d^{-1}\left(\frac{R^{(n)}_{i; \pm}}{n_R + 1}\right)^{1/2} S^{(n)}_{i; \pm} \right\|^2.
\]

6.2. Score functions and efficiency. The tests statistics in Section 6.1 offer the advantage of a structure paralleling the structure of the numerator of the classical Gaussian \(F\) test—basically substituting, in the latter, \(S^{(n)}_{i; \pm}\) (sign test scores), \(R^{(n)}_{i; \pm} S^{(n)}_{i; \pm}\) (Wilcoxon scores), or \(\left(\psi_d^{-1}\left(\frac{R^{(n)}_{i; \pm}}{n_R + 1}\right)\right)^{1/2} S^{(n)}_{i; \pm}\) (van der Waerden scores) for the sphericized residuals (4.4)\(^{14}\) and adopting the adequate standardization.

The choice of a score function also can be guided by efficiency considerations, selecting \(J\) in relation to some reference distribution under which efficiency is to be attained. This, in the

\(^{14}\) The computation of which, moreover, requires the specification of \(\beta_0\) or its consistent estimation—something center-outward ranks and signs do not need in view of their shift-invariance.
univariate case, yields the normal (van der Waerden), Wilcoxon or sign test scores, achieving efficiency under Gaussian, logistic, or double exponential reference densities; as we shall see, $Q_{\text{sign}}^{(n)\pm}$ and $Q_{\text{sw}}^{(n)\pm}$ similarly achieve efficiency at spherical exponential and Gaussian reference distributions.\footnote{Due to the fact that the density $f_{d,|\cdot|}$ of the modulus of a spherical logistic fails to be logistic for $d > 1$, \(Q_{\text{Wilcoxon}}^{(n)\pm}\), however, does not enjoy efficiency under spherical logistic; this is also the case of the elliptical rank tests based on Wilcoxon scores in Hallin and Paindaveine (2002a,b, 2005).}

In the same spirit, one could contemplate the idea of achieving, based on center-outward rank tests, efficiency at some selected reference distribution $P_d^0$ in $P_d$ (with density $f^0_d$ and center-outward distribution function $F^0_{0;\pm}$ satisfying the adequate regularity assumptions). Indeed, it follows from Proposition 5.1 that efficiency under $P_d^0$ can be achieved by a test based on the test statistic $Q_j^{(n)\pm}$ given in (5.3) with score $J = \varphi_{f^0_\cdot} \circ (F^0_{0;\pm})^{-1}$.

This, however, raises two problems. First, in order for $\varphi_{f^0_\cdot}$ to be analytically computable, the distribution $P_d^0$ has to be fully specified (up to location and a global scaling parameter), with closed-form density function $f^0_d$. Second, the corresponding score $J = \varphi_{f^0_\cdot} \circ (F^0_{0;\pm})^{-1}$ also involves the center-outward quantile function $(F^0_{0;\pm})^{-1}$ for which, except for a few particular cases (spherical distributions), no explicit form is available in the literature.

Once $P_d^0$ is fully specified, in principle, it can be simulated, and an arbitrarily precise numerical evaluation of $(F^0_{0;\pm})^{-1}$ can be obtained, to be plugged into $J$. This may be computationally heavy, but increasingly efficient algorithms are available in the very active domain of numerical measure transportation: see, e.g., Mérigot (2011) or Peyré and Cuturi (2019).

Now, choosing a fully specified reference $P_d^0$ may be embarrassing—this means, for instance, a skew-$t$ distribution with specified degrees of freedom, shape matrix, and skewness parameter (without loss of generality, location can be taken as $0$), a multinormal or elliptical distribution with specified radial density and specified (up to a positive global factor) covariance (again, the mean can be taken as $0$), ... Fortunately, a full specification of $P_d^0$ can be relaxed to the specification of a parametric family with parameter $\vartheta$, say, such as the family $P_{\text{skew}}$ of all skew-$t$ distributions with location $0$ (parameters: a shape matrix and a $d$-tuple of skewness parameters) or the family $P_{\text{ell}}$ of all elliptical distributions (4.3) with radial density $f$ (parameter: a scatter matrix). The unspecified parameter $\vartheta$ of $P_d^0$ indeed can be replaced, in the numerical evaluation of $F^0_{0;\pm}$, with consistent estimated values provided that the estimator $\hat{\vartheta}$ are measurable with respect to the order statistic\footnote{The order statistic of the $n$-tuple $Z_1, \ldots, Z_n$ of $d$-dimensional ($d > 1$) random vectors can be defined as any reordering $Z_{(1)}, \ldots, Z_{(n)}$ generating the $\sigma$-field of permutation-invariant Borel sets of $\sigma(Z_1, \ldots, Z_n)$; for instance, the one resulting from ordering the observations $Z_i$ from smallest to largest first component.} of the residuals $Z_i^{(n)}$.

Plugging these estimators into the score $J$—this include the standardization factor and the numerical evaluation of $F^0_{0;\pm}$—yields data-driven (order-statistic-driven) scores $J_i^{(n)}$. Conditionally on the order statistic, the corresponding test statistic is still distribution-free and its (conditional) critical values remain unconditionally correct. However, these critical values involve the order statistic: the resulting tests therefore no longer are ranks tests but permutation tests.\footnote{Similar data-driven score ideas have been proposed in the univariate context by Dodge and Jurečková (2000).}

The theoretical properties, feasibility, and finite-sample performance of this data-driven approach should be explored and numerically assessed—this is, however, beyond the scope of this paper and we leave it for future research.

In view of this, no obvious non-spherical candidate emerges, in dimension $d > 1$, as a reference density. The center-outward test statistic achieving optimality at the spherical distributions with radial density $f$ is $Q_j^{(n)\pm}$ with $J_f$ as in (5.5).
6.3. Affine invariance and sphericization. Affine invariance (testing) or equivariance (estimation), in “classical multivariate analysis,” is generally considered an essential and inescapable property. Closer examination, however, reveals that this particular role of affine transformations is intimately related to the affine invariance of Gaussian and elliptical families of distributions. When Gaussian or elliptical assumptions are relaxed, affine transformations are losing this privileged role and the relevance of affine invariance/equivariance properties is much less obvious.

When $Y^{(n)}A'$ (where $A$ is an arbitrary full-rank $d \times d$ matrix), is observed instead of $Y^{(n)}$, $\hat{\Sigma}^{(n)}$ is replaced with $\hat{\Sigma}^{(n)}_A = A\hat{\Sigma}^{(n)}A'$, yielding sphericized residuals of the form $Z^{(n)}_{A;i} = (A\hat{\Sigma}^{(n)}A')^{-1/2}AZ_i^{(n)}$ instead of $Z_i^{(n)}$. It follows from elementary calculation that $Z^{(n)}_{A;i} = PZ_i^{(n)}$ with $P = (A\hat{\Sigma}^{(n)}A')^{-1/2}A(\hat{\Sigma}^{(n)}A')^{1/2}$ orthogonal. Strictly speaking, sphericized residuals, thus, are not affine-invariant. This possible discrepancy between sphericized residuals is due to the fact that square roots such as $\hat{\Sigma}^{(n)}^{-1/2}$ are only defined up to an orthogonal transformation; choosing the symmetric root is a convenient choice, but does not yield $P = I_d$.

However, the moduli $\|Z^{(n)}_{A;i}\|$ and $\|Z_i^{(n)}\|$ coincide, irrespective of $P$, and so do the cosines
\[
\frac{\langle Z^{(n)}_{A;i}, Z^{(n)}_{A;j} \rangle}{\|Z^{(n)}_{A;i}\| \|Z^{(n)}_{A;j}\|} \quad \text{and} \quad \frac{\langle Z_i^{(n)}, Z_j^{(n)} \rangle}{\|Z_i^{(n)}\| \|Z_j^{(n)}\|}, \quad i, j = 1, \ldots, n.
\]

The affine-invariance of typical elliptical-rank-based test statistics, which are quadratic forms involving those moduli and cosines, follows.

Being measurable with respect to the ranks of the moduli $\|Z_i^{(n)}\|$ and the scalar products $\langle Z_i^{(n)}, Z_j^{(n)} \rangle / \|Z_i^{(n)}\| \|Z_j^{(n)}\|$, the elliptical rank statistics developed in Hallin and Paindaveine (2005) are affine-invariant; this is in full agreement with our previous remark that the limiting local Gaussian shifts in elliptical experiments are unaffected under affine transformations.

The center-outward distribution functions, ranks and signs cannot be expected to enjoy similar affine-invariance properties—actually, it has been proved (Proposition 3.14 in Cuesta-Albertos et al. (1993)) that they do not. If, however, affine invariance is considered an indispensible property, it is easily restored: choosing your favorite (consistent under ellipticity) estimator of scatter $\hat{\Sigma}^{(n)}$ (which also requires, in case $\beta_0$ is not specified, an estimator of location $\hat{\mu}^{(n)}$), just compute the sphericized residuals $Z_i^{(n)}$ defined in (4.4) prior to computing the center-outward ranks and signs and performing the tests: in view of Proposition 2.2, the resulting center-outward ranks and signs enjoy the same affine-invariance properties (invariance of the ranks and the cosines of signs) as the elliptical ones.

If the actual density $f^\varepsilon$ is elliptical, this linear sphericization does not modify the local experiment, hence local asymptotic powers and, in case the scores themselves are spherical, efficiency properties, are preserved. If the actual density $f^\varepsilon$ is not elliptical, however, such linear sphericization\footnote{The Cholesky square root does: see Proposition 2 in Hallin et al. (2020b).} has a nonlinear impact\footnote{Actually, only a “second-order sphericization,” as the distribution of $Z_i^{(n)}$ still fails to be spherical unless the error distribution itself was.} on $f^\varepsilon$-based central sequences: the corresponding Gaussian shift experiments are not preserved and, irrespective of the scores they are based on, the local asymptotic powers of center-outward rank tests are affected. Summing up, preliminary sphericization does restore affine-invariance of center-outward rank tests.
tests while preserving their local powers under ellipticity, but distorts those local powers under non-elliptical error densities. Figures 4 and 5 below provide examples where that distortion significantly deteriorates the power.

Whether affine-invariance is desirable or not is open to discussion. In “classical multivariate analysis,” that is, under Gaussian or elliptical densities, linear sphericization preserves local experiments, making affine invariance a natural requirement. When considering more general error distributions $P$, linear transformations are losing their privileged status: they no longer sphericize the distribution $P$ of a typical $Z \sim P$ and no longer preserve local experiments. Moreover, while all consistent-under-ellipticity estimators $\Sigma^{(n)}$ and $\hat{\mu}^{(n)}$ yield, under ellipticity, the same limiting location and scatter values, distinct estimators, under non-elliptical densities, will converge to distinct and sometimes hardly interpretable limits: $\hat{\mu}^{(n)} = \overline{X}^{(n)}$ (the arithmetic mean) and $\hat{\mu}^{(n)} = \text{Oja median}$, Oja (1983), which asymptotically coincide under ellipticity, may yield completely distinct locations; what is the relevance of Tyler’s scatter matrix in a distribution where sign curves are not straight lines and decorrelation of radii (through which center $\hat{\mu}^{(n)}$?) makes little sense? etc. The distortion of local powers under non-elliptical error densities thus depends on the choice of $\Sigma^{(n)}$ and $\hat{\mu}^{(n)}$, which is hard to justify. While easily implementable, affine invariance/equivariance, in such a context, is thus a disputable requirement.

7. Some numerical results. A small Monte Carlo simulation study is conducted (Sections 7.1–7.2) in order to explore the finite-sample performance of our tests. Results are presented for two-sample location and MANOVA models, and limited to Wilcoxon score function $J(r) = r$; other choices for $J$ lead to very similar figures, which we therefore do not report.

7.1. Two-sample location test. Consider first a two-sample location problem in dimension $d = 2$. Two independent random samples of size $n_1 = n_2 = n/2$ were generated and the two test statistics $Q^{(n)\text{ell}}_{\text{Wilcoxon}}$ and $Q^{(n)\pm}_{\text{Wilcoxon}}$ (see Section 5.3.1) were computed. The sample covariance matrix $\hat{\Sigma}$ was used for the computation of $A^{(n)\text{ell}}_{\text{Wilcoxon}}$ and the elliptical or Mahalanobis ranks and signs.

Rejection frequencies were computed for the following error densities:

(a) a centered bivariate normal distribution with unit variances and correlation $\rho = 1/4$;
(b) a centered bivariate $t$-distribution with the same scaling matrix as in (a) and $\nu$ degree of freedom, $\nu = 1$ (Cauchy) and $\nu = 3$;\footnote{The bivariate $t$-distribution with $m$ degrees of freedom and scaling matrix $A'A$ is the one defined in Example 2.5 of Fang et al. (1990) as the distribution of a random vector $\xi := \mu + A'\zeta \sqrt{m}/\sqrt{3}$ where $\zeta \sim N_{2}(0, I_{2})$ and $s \sim \chi_{m}^{2}$, independent of $\zeta$—not to be confused with the elliptical distribution with Student radial density $f$.}
(c) a mixture, with weights $w_1 = 1/4$ and $w_2 = 3/4$, of two bivariate normal distributions with means $\mu_1 = (3/4, 0)'$ and $\mu_2 = (-1/4, 0)'$ and covariance matrices

\[
\Sigma_1 = \begin{pmatrix} 1 & 2/3 \\ 2/3 & 1 \end{pmatrix} \quad \text{and} \quad \Sigma_2 = \begin{pmatrix} 1 & -2/3 \\ -2/3 & 1 \end{pmatrix},
\]

respectively;
(d) a mixture, with weights $w_1 = 1/4$ and $w_2 = 3/4$, of two bivariate $t_1$ (Cauchy) distributions centered at $\mu_1 = (3/4, 0)'$ and $\mu_2 = (-1/4, 0)'$, with the same scaling matrices $\Sigma_1$ and $\Sigma_2$ as in (c);
(e) a “U-shaped” mixture, with weights \( w_1 = 1/2, w_2 = 1/4, \) and \( w_3 = 1/4, \) of three bivariate normal distributions, \( \mathcal{N}_2(\mu_1, \Sigma_1), \mathcal{N}_2(\mu_2, \Sigma_2), \) and \( \mathcal{N}_2(\mu_3, \Sigma_3) \) where

\[
\begin{align*}
\mu_1 &= (0, 0)', \\
\mu_2 &= (-3, 1)', \\
\mu_3 &= (3, 1)', \\
\end{align*}
\]

and

\[
\begin{align*}
\Sigma_1 &= \begin{pmatrix} 2 & 0 \\ 0 & 1/8 \end{pmatrix}, \\
\Sigma_2 &= \begin{pmatrix} 1/2 & -1/3 \\ -1/3 & 1/2 \end{pmatrix}, \\
\Sigma_3 &= \begin{pmatrix} 1/2 & 1/3 \\ 1/3 & 1/2 \end{pmatrix};
\end{align*}
\]

(f) an “S-shaped” mixture, with equal weights \( w = 1/3, \) of three bivariate normal distributions, \( \mathcal{N}_2(\mu_4, \Sigma_4), \mathcal{N}_2(\mu_5, \Sigma_5), \) and \( \mathcal{N}_2(\mu_6, \Sigma_4) \) where

\[
\begin{align*}
\mu_4 &= (-9/2, -1/2)', \\
\mu_5 &= (0, -1/2)', \\
\mu_6 &= (9/2, 1)', \\
\end{align*}
\]

and

\[
\begin{align*}
\Sigma_4 &= \begin{pmatrix} 3/2 & -\sqrt{3/8} \\ -\sqrt{3/8} & 1 \end{pmatrix}, \\
\Sigma_5 &= \begin{pmatrix} 3/2 & \sqrt{3/8} \\ \sqrt{3/8} & 1 \end{pmatrix}, \\
\Sigma_6 &= \begin{pmatrix} 3/2 & -\sqrt{3/8} \\ -\sqrt{3/8} & 1 \end{pmatrix};
\end{align*}
\]

(g) a skew \( t \)-distribution with \( \nu \) degrees of freedom, \( \nu = 1 \) and 3, with skewness parameter \( \alpha = (5, -3)', \) scaling matrix \( \Sigma_7 = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}, \) and location parameter \( \xi = 0. \)
For the interpretation of the parameters of the skewed $t$-distribution see (Azzalini and Capitanio, 2014, Chapter 6).

Mixture error densities quite naturally appear in the context of hidden heterogeneities due, for instance, to omitted covariates; as for asymmetries, they are likely to be the rule rather than the exception. Samples of size 200 from the Gaussian mixtures $(c)$, $(e)$, and $(f)$ and from the skew $t$-distribution with 3 degrees of freedom $(g)$ are shown in Figure 1.

![Normal distribution](image1.png) ![t-distribution, 1 df](image2.png) ![t-distribution, 3 df](image3.png)

**Fig 2.** The empirical powers of two-sample location tests based on the Wilcoxon center-outward rank statistic (solid line), the elliptical rank statistic (dashed line), and the two-sample Hotelling test (dotted line) as functions of the shift $\delta$ under normal and Student $t$ error densities; sample sizes $n_1 = n_2 = 50$, 200, and 450.

The first sample was generated from one of the distributions $(a)$–$(g)$, and the second sample was drawn from the same distribution shifted by the vector $(\delta, \delta)'$ for $\delta \in [0.00, 0.24]$. Each simulation was replicated $N = 1000$ times and the empirical size and power of the test were computed for $\alpha = 0.05$. The resulting rejection frequencies illustrate the dependence of the power on the parameter $\delta$; they are provided in Figures 2, 3, 4, and 5.

The power of the center-outward rank test statistic $Q^{(n)}_c$ is plotted as a solid line, the power of the elliptical rank test statistic $Q^{(n)}_{ell}$ as a dashed line. For the sake of comparison,
The empirical powers of two-sample location tests based on the Wilcoxon center-outward rank statistic (solid line), the Wilcoxon elliptical rank test statistic (dashed line), and the two-sample Hotelling test (dotted line), as functions of the shift $\delta$, for the mixtures of two normal (left panel) and two $t_1$ error densities (right panel), respectively; sample sizes $n_1 = n_2 = 50, 200, \text{and } 450$.

The empirical powers of two-sample location tests based on the Wilcoxon center-outward rank statistic (solid line), the Wilcoxon elliptical rank test statistic (dashed line), the two-sample Hotelling test (dotted line), and the center-outward rank statistic computed from linearly sphericized residuals (dot-dashed line), as functions of the shift $\delta$, for the "U-shaped" (left panel) and the "S-shaped" mixtures of three normal error densities (right panel), respectively; sample sizes $n_1 = n_2 = 50, 200, \text{and } 450$.

we also provide the power of the classical two-sample Hotelling test, shown as a dotted line. Three different sample sizes $n_1 = n_2 = 50, 200, \text{and } 450$ (hence, $n = 100, 400, \text{and } 900$) are considered, yielding three groups of curves (from light gray to black, colors in the online version). The regular grids $G_n$ for computation of the center-outward ranks and signs are constructed with $n_S = n_R = 10$ for $n = 100, n_S = n_R = 20$ for $n = 400, \text{and } n_S = n_R = 30$ for $n = 900$. 
Figure 5. The empirical powers of two-sample location tests based on the Wilcoxon center-outward rank statistic (solid line), the Wilcoxon elliptical rank test statistic (dashed line), the two-sample Hotelling test (dotted line), and the center-outward rank statistic computed from linearly sphericized residuals (dot-dashed line), as functions of the shift $\delta$ for skew $t$ error densities with $\nu = 1.1$ (left panel) and $\nu = 3$ (right panel), respectively; sample sizes $n_1 = n_2 = 50, 200, \text{and} 450$.

Figure 2 displays the empirical power curves for the elliptical distributions (a) and (b). The results for the normal distribution are very similar for the three tests: rank-based tests (Wilcoxon scores), thus, are no less powerful than the optimal Hotelling test. As expected, the Hotelling test fails for the $t_1$ distribution, while the elliptical test based on the sample covariance matrix performs surprisingly well here (the robustness benefits of ranks). The tests based on $Q^{(n)}_{\text{all}}$ and $Q^{(n)}_{\pm \text{Wilcoxon}}$ both outperform the Hotelling test also for the $t$-distribution with 3 degrees of freedom. The conclusion is that center-outward rank tests perform equally well as elliptical rank tests under elliptical densities.

The remaining distributions (c)–(g) are non-elliptical ones. Results for the mixtures (c) and (d) are shown in Figure 3. For the mixture (c) of two normals, the results obtained for the three tests are still quite similar, but the center-outward rank test based on $Q^{(n)}_{\pm \text{Wilcoxon}}$, in general, yields the largest power. For the mixture (d) of two $t_1$ (Cauchy) distributions, the Hotelling test fails miserably and the center-outward rank test very clearly outperforms the elliptical rank test for all sample sizes.

Figures 4 and 5 provide the results for the mixtures (e)–(f) and the skew $t$-distribution (g), respectively. The power curve for the test statistic $Q^{(n)}_{\pm \text{Wilcoxon}}$ computed from the linearly sphericized residuals (using the sample mean and the sample covariance matrix as estimators of location and scatter) is added as a dot-dashed line. In all these plots, the center-outward rank test statistic leads to the largest power. Note that the linear sphericization of the residuals, which makes the test affine-invariant, may noticeably deteriorate the power (see the discussion in Section 6.3).

7.2. One-way MANOVA. The performance of center-outward rank tests is very briefly studied here for one-way MANOVA with $K = 3$ groups, still for $d = 2$. Two random samples were generated from the distribution (a) (Gaussian) or (e) (U-shaped mixture of three Gaussians), as described in Section 7.1, and the third sample was drawn from the same distribution shifted by the vector $(\delta, \delta)'$ for $\delta \in [0.00, 0.24]$. A balanced design with groups of size $n_1 = n_2 = n_3 = 75$ (hence $n = 225$) and $n_1 = n_2 = n_3 = 300$ (hence $n = 900$) was
considered. For \( n = 225 \), the grid \( \mathcal{G}_n \) is constructed with \( n_R = n_S = 15 \); for \( n = 900 \), we set \( n_R = n_S = 30 \). As in Section 7.1, the results are presented for the Wilcoxon scores \( J(r) = r \) only—other choices lead to very similar conclusions.

The empirical power curves are plotted in Figure 6 for the center-outward rank test based on \( Q_n^{(\text{Wilcoxon})} \) (solid line), the elliptical rank test statistic \( Q_n^{(\text{ell, Wilcoxon})} \) (dashed line), and the classical Pillai’s test based on an approximate F-distribution (dotted line). For the normal distribution, all the three tests perform very similarly. For the non-elliptical mixture distribution, however, the center-outward rank test provides a noticeably larger power compared to the other two methods.

![Graph showing power curves for normal and U-shaped mixture distributions.](image)

**Figure 6.** The empirical powers of MANOVA tests based on the Wilcoxon center-outward rank statistic (solid line), the Wilcoxon elliptical rank test statistic (dashed line), and Pillai’s classical test (dotted line), as functions of the shift \( \delta \), for the normal distribution (left panel) and the U-shaped mixture of three normals (right panel); sample sizes \( n_1 = n_2 = n_3 = 75 \) and \( 300 \).

### 7.3. An empirical illustration

The practical value of the center-outward rank tests is illustrated by the following archeological application where classical methods fail to detect any treatment effect. The data consist of \( n = 126 \) measurements of \( \text{MgO}, \text{P}_2\text{O}_5, \text{CoO}, \) and \( \text{Sb}_2\text{O}_3 \) in natron glass vessels excavated from three Syro-Palestinian sites in present-day Israel: Apollonia (\( n_1 = 54 \) observations), Bet Eli’ezer (\( n_2 = 17 \) observations), and Egypt (\( n_3 = 55 \) observations); a fourth site only has two observations, and therefore was dropped from the analysis. This dataset is has been originally analyzed by Phelps et al. (2016) with the objective of detecting possible differences among the three sites. Bivariate plots of these four variables are shown in Figure 7, where one can observe that the marginal distributions of CoO, and Sb\(_2\)O\(_3\) exhibit heavy tails and are very far from normal, while the joint distribution seems far from elliptical.

First, all the two-dimensional data subsets corresponding to the bivariate plots in Figure 7 were analyzed (six bivariate MANOVA models, thus). Pillai’s classical test yields non-significant \( p \)-values for all combinations, see Table 1. On the other hand, the center-outward tests are able to detect differences between the three groups whenever the variable CoO is included in the analysis. The center-outward ranks and signs can be computed from a grid \( \mathcal{G}_n \) with either \( n_S = 7 \) and \( n_R = 18 \) or \( n_S = 18 \) and \( n_R = 7 \); both choices have been implemented in Table 1 below (c-o tests I and II, respectively).

Table 1 reveals some differences in the \( p \)-values for c-o tests I and II which, however, yield the same conclusions at significance level \( \alpha = 0.05 \). The tests based on elliptical ranks (based
on the sample covariance function) lead to highly non-significant $p$-values for all couples of variables; consequently, the corresponding results are not presented here.

|          | Pillai’s test | c-o test I | c-o test II |
|----------|---------------|------------|-------------|
| MgO      | 0.3547        | 0.0946     |             |
| P$_2$O$_5$ | 0.3817        | 0.0000     |             |
| MgO      | 0.1217        | 0.0000     | 0.0000      |
| MgO      | 0.2268        | 0.1865     | 0.3236      |
| P$_2$O$_5$ | 0.1491        | 0.0000     | 0.0000      |
| P$_2$O$_5$ | 0.1957        | 0.0561     | 0.3004      |
| CoO      | 0.1453        | 0.0000     | 0.0000      |

TABLE 1

$p$-values for the bivariate Pillai MANOVA and the Wilcoxon center-outward rank tests based on $n_R = 7$, $n_S = 18$ (c-o test I) and $n_R = 18$, $n_S = 7$ (c-o test II), respectively.

Next, the MANOVA comparison is conducted for the full 4-dimensional dataset. Pillai’s test leads to a $p$-value 0.1553, which is far from significant at level $\alpha = 0.05$: no difference gets detected among the three groups, thus, with this classical method. In sharp contrast, the Wilcoxon center-outward rank test (with $n_R = 7$ and $n_S = 18$) yields a $p$-value $10^{-15}$, which is highly significant. The elliptical Wilcoxon rank test (based on sample covariance matrix), on the other hand, with $p$-value 0.5827, also fails to detect anything at level $\alpha = 0.05$.

**FIG 7.** The content of MgO, P$_2$O$_5$, CoO, and Sb$_2$O$_3$ in natron glass vessels from Appolonia (circles), Bet Eli‘ezer (triangles), and Egypt (squares).
8. Conclusion and perspectives. Classical multivariate analysis methods, which are
daily practice in a number of applied domains, remain deeply marked by Gaussian and elliptical
assumptions. In particular, no distribution-free approach is available so far for hypothesis
testing in multiple-output regression models, which include the fundamental two-sample and
MANOVA models—except for the elliptical or Mahalanobis rank tests developed in Hallin and Paindaveine (2004) which, unfortunately, require the strong assumption of elliptic sym-
metry, an assumption which is unlikely to hold in most applications. Based on the recent
concept of center-outward ranks and signs, this paper proposes the first fully distribution-
free tests of the hypothesis of no treatment effect in the context, thereby extending to the
multivariate case the classical Hájek approach to univariate rank-based inference (Hájek and Šidák, 1967). Simulations and an empirical example demonstrate the excellent performance
of the method. This is only a first step into the direction of complete toolkit of distribution-
free methods for multiple-output analysis of variance problems, but it lays the theoretical
bases (asymptotic representation and asymptotic normality results for linear center-outward
rank statistics) and theoretical guidelines (Hájek projection of LAN central sequences) for
further developments.

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