Gaps in full homomorphism order

Jiří Fiala\textsuperscript{1,4}

Department of Applied Mathematics  
Charles University  
Prague, Czech Republic

Jan Hubička\textsuperscript{2,5}

Computer Science Institute of Charles University (IUUK)  
Charles University  
Prague, Czech Republic

Yangjing Long\textsuperscript{3,6}

School of Mathematical Sciences  
Shanghai Jiao Tong University  
Shanghai, China

Abstract

We characterise gaps in the full homomorphism order of graphs.

Keywords: graph homomorphism, full homomorphism, homomorphism order, gap, full homomorphism duality

\textsuperscript{1} Supported by MŠMT ČR grant LH12095 and GAČR grant P202/12/G061.
\textsuperscript{2} Supported by grant ERC-CZ LL-1201 of the Czech Ministry of Education and CE-ITI P202/12/G061 of GAČR.
\textsuperscript{3} Supported by National Natural Science Foundation of China (No. 11271255)
\textsuperscript{4} Email: fiala@kam.mff.cuni.cz
\textsuperscript{5} Email: hubicka@iuuk.mff.cuni.cz
\textsuperscript{6} Email: yjlong@sjtu.edu.cn
1 Introduction

For given graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ a homomorphism $f : G \to H$ is a mapping $f : V_G \to V_H$ such that $\{u, v\} \in E_G$ implies $\{f(u), f(v)\} \in E_H$. (Thus it is an edge preserving mapping.) The existence of a homomorphism $f : G \to H$ is traditionally denoted by $G \to H$. This allows us to consider the existence of a homomorphism, $\to$, to be a (binary) relation on the class of graphs. A homomorphism $f$ is full if $\{u, v\} \notin E_G$ implies $\{f(u), f(v)\} \notin E_H$. (Thus it is an edge and non-edge preserving mapping). Similarly we will denote by $G \overset{F}{\to} H$ the existence of a full homomorphism $f : G \to H$.

As it is well known, the relations $\to$ and $\overset{F}{\to}$ are reflexive (the identity is a homomorphism) and transitive (a composition of two homomorphisms is still a homomorphism). Thus the existence of a homomorphism as well as the existence of full homomorphisms induces a quasi-order on the class of all finite graphs. We denote the quasi-order induced by the existence of homomorphisms and the existence of full homomorphism on finite graphs by $(\text{Graphs}, \leq)$ and $(\text{Graphs}, \leq^F)$ respectively. (Thus when speaking of orders, we use $G \leq H$ in the same sense as $G \to H$ and $G \leq^F H$ in the sense $G \overset{F}{\to} H$.)

These quasi-orders can be easily transformed into partial orders by choosing a particular representative for each equivalence class. In the case of graph homomorphism such representative is up to isomorphism unique vertex minimal element of each class, the (graph) core. In the case of full homomorphisms we will speak of $F$-core.

The study of homomorphism order is a well established discipline and one of main topics of nowadays classical monograph of Hell and Nešetřil [5]. The order $(\text{Graphs}, \leq^F)$ is a topic of several publications [9,2,4,1,3] which are primarily concerned about the full homomorphism equivalent of the homomorphism duality [7].

In this work we further contribute to this line of research by characterising $F$-gaps in $(\text{Graphs}, \leq^F)$. That is pairs of non-isomorphic F-cores $G \leq^F H$ such that every F-core $H'$, $G \leq^F H' \leq^F H$, is isomorphic either to $G$ or $H$. We will show:

**Theorem 1.1** If $G$ and $H$ are F-cores and $(G, H)$ is an F-gap, then $G$ can be obtained from $H$ by removal of one vertex.

First we show a known fact that F-cores correspond to point-determining graphs which have been studied in 70’s by Sumner [8] (c.f. Feder and Hell [2]). We also show that there is a full homomorphism between two F-cores if and only if there is an embedding from one to another (see [2, Section 3]). These
two observations shed a lot of light into the nature of full homomorphism order and makes the characterisation of F-gaps look particularly innocent (clearly gaps in embedding order are characterised by an equivalent of Theorem 1.1). The arguments in this area are however surprisingly subtle. This becomes even more apparent when one generalise the question to classes of graphs as done by Hell and Hernández-Cruz [4] where both results of Sumner [8] and Feder and Hell [2] are given for digraphs by new arguments using what one could consider to be surprisingly elaborate (and interesting) machinery needed to carry out the analysis.

We focus on minimising arguments about the actual structure of graphs and use approach which generalises easily to digraphs and binary relational structures in general (see Section 5). In Section 2 we outline the connection of point determining graphs and F-cores. In Section 3 we show proof of the main result. In Section 4 we show how the existence of gaps leads to a particularly easy proof of the existence of generalised dualities (main results of [9,2,4,1]).

2 F-cores are point-determining

In a graph $G$, the neighbourhood of a vertex $v \in V_G$, denoted by $N_G(v)$, is the set of all vertices $v'$ of $G$ such that $v$ is adjacent to $v'$ in $G$. Point-determining graphs are graphs in which no two vertices have the same neighbourhoods. If we start with any graph $G$, and gradually merge vertices with the same neighbourhoods, we obtain a point-determining graph, denoted by $G_{pd}$.

We write $G \sim^F H$ for any pairs of graphs such that $G \xrightarrow{F} H$ and $H \xrightarrow{F} G$. It is easy to observe that $G_{pd}$ is always an induced subgraph of $G$. Moreover, for every graph $G$ it holds that $G_{pd} \xrightarrow{F} G \xrightarrow{F} G_{pd}$ and thus $G \sim^F G_{pd}$. This motivates the following proposition:

**Proposition 2.1 ([2])** A finite graph $G$ is an F-core if and only if it is point-determining.

**Proof.** Recall that $G$ is an F-core if it is minimal (in the number of vertices) within its equivalence class of $\sim^F$. If $G$ is an F-core, $G_{pd}$ can not be smaller than $G$ and thus $G = G_{pd}$.

It remains to show that every point-determining graph is an F-core. Consider two point-determining graphs $G \sim^F H$ that are not isomorphic. There are full homomorphisms $f : G \xrightarrow{F} H$ and $g : H \xrightarrow{F} G$. Because injective full homomorphisms are embeddings, it follows that either $f$ or $g$ is not injective. Without loss of generality, assume that $f$ is not injective. Consider $u, v \in V_G$, $u \neq v$, such that $f(u) = f(v)$. Because full homomorphisms
preserve both edges and non-edges, the preimage of any edge is a complete bipartite graph. If we apply this fact on edges incident with \( f(u) \), we derive that \( N_G(u) = N_G(v) \).

**Proposition 2.2 ([2,6])** For F-cores \( G \) and \( H \) we have \( G \rightarrow H \) if and only if \( G \) is an induced subgraph of \( H \).

**Proof.** Embedding is a special case of a full homomorphisms. In the opposite direction consider a full homomorphism \( f : G \rightarrow H \). By the same argument as in the proof of Proposition 2.1 we get that \( f \) is injective, as otherwise \( G \) would not be point-determining.

\[
\begin{align*}
\text{Proposition 3.2: Characterisation of F-gaps}
\end{align*}
\]

Given a graph \( G \) and a vertex \( v \in V_G \) we denote by \( G \setminus v \) the graph created from \( G \) by removing vertex \( v \). We say that vertex \( v \) determines a pair of vertices \( u \) and \( u' \) if \( N_{G \setminus v}(u) = N_{G \setminus v}(u') \). This relation (pioneered in [2] and used in [2,4,9]) will play key role in our analysis. We make use of the following Lemma:

**Lemma 3.1** Given a graph \( G \) and a subset \( A \) of the set of vertices of \( G \) denote by \( L \) a graph on the vertices of \( G \), where \( u \) and \( u' \) are adjacent if and only if there is \( v \in A \) that determines \( u \) and \( u' \). Let \( S \) be any spanning tree of \( L \). Denote by \( B \subseteq A \) the set of vertices that determine some pair of vertices connected by an edge of \( L \) and by \( C \subseteq B \) set of vertices that determine some pair of vertices connected by an edge of \( S \). Then \( B = C \).

**Proof.** Because for every pair of vertices there is at most one vertex determining them clearly \( C \subseteq B \subseteq A \).

Assume to the contrary that there is vertex \( v \in B \setminus C \) and thus every pair determined by \( v \) is an edge of \( L \) but not an edge of \( S \). Denote by \( \{u, u'\} \) some such edge of \( L \) determined by \( v \in B \). Adding this edge to \( S \) closes a cycle. Denote by \( u = v_1, v_2, \ldots v_n = u' \) the vertices of \( G \) such that every consecutive pair is an edge of \( S \). Without loss of generality, we can assume that \( v \in N_G(v_1) \) and \( v \notin N_G(v_n) \). Because \( v \in N_G(v_i) \) implies \( v \in N_G(v_{i+1}) \) unless \( v \) determines pair \( \{v_i, v_{i+1}\} \) we also know that there is \( 1 \leq i < n \) such that \( v \) determines \( v_i \) and \( v_{i+1} \). A contradiction with the fact that \( v_i, v_{i+1} \) forms an edge of \( S \).

As a warmup we show the following theorem which also follows by [8] (also shown as Corollary 3.2 in [2] for graphs and [4] for digraphs):

\[
\begin{align*}
\end{align*}
\]
Theorem 3.2 ([8,2,4]) Every F-core $G$ with at least 2 vertices contains an $F$-core with $|V_G| - 1$ vertices as an induced subgraph.

Proof. Denote by $n$ number of vertices of $G$. If there is a vertex $v$ of $G$ such that the graph $G \setminus v$ is point-determining, it is the desired $F$-core. Consider graph $S$ as in Lemma 3.1 where $A$ is the vertex set of $G$. Because $S$ has at most $n - 1$ edges and every edge of $S$ is determined by at most one vertex, we know that there is vertex $v$ which does not determine any pair of vertices and thus $G \setminus v$ is point-determining. 

In fact both [8,4] shows that every F-core $G$ with at least 2 vertices contains vertices $v_1 \neq v_2$ such that both $G \setminus v_1$ and $G \setminus v_2$ are F-cores. This follows by our argument, too but needs bit more detailed analysis. The main idea of the following proof of Theorem 1.1 can also be adapted to show this.

Proof. (of Theorem 1.1) Assume to the contrary that there are F-cores $G$ and $H$ such that $(G, H)$ is an F-gap, but $G$ differs from $H$ by more than one vertex. By induction we construct two infinite sequences of vertices of $H$ denoted by $u_0, u_1, \ldots$ and $v_0, v_1, \ldots$ along with two infinite sequences of induced subgraphs of $H$ denoted by $G_0, G_1, \ldots$ and $G'_0, G'_1, \ldots$ such that for every $i \geq 0$ it holds that:

(i) $G_i$ and $G'_i$ are isomorphic to $G$,
(ii) $G_i$ does not contain $u_i$ and $v_i$,
(iii) $G'_i$ does not contain $u_i$ and $v_{i+1}$,
(iv) $u_i$ and $u_{i+1}$ is determined by $v_i$, and,
(v) $v_i$ and $v_{i+1}$ is determined by $u_i$.

Put $G_0 = G$ and $A = V_H \setminus V_G$. Consider the spanning tree $S$ given by Lemma 3.1. Because no vertex of $A$ can be removed to obtain an induced point-determining subgraph, it follows that every vertex must have a corresponding edge in $S$. Consequently the number of edges of $S$ is at least $|A|$. Because $G$ itself is point-determining, it follows that every edge of $S$ must contain at least one vertex of $A$. These two conditions yields to the pair of vertices $v_0 \in A = V_H \setminus V_G$ and $v_1 \in V_G$ connected by an edge in $S$ and consequently we have a vertex $u_0 \in A$ which determines them. We have obtained $G_0, u_0, v_0, v_1$ with the desired properties. This finishes the initial step of the induction.

At the induction step assume we have constructed $G_i, u_i, v_i, v_{i+1}$. We show the construction of $G'_i$ and $u_{i+1}$. We consider two cases. If $v_{i+1} \notin V_G$, we put $G'_i = G_i$. If $v_{i+1} \in V_G$, we let $G'_i$ to be the graph induced by $H$ on
Because the neighbourhood of \( v_i \) and \( v_{i+1} \) differs only by a vertex \( u_i \notin G_i \) which determines them we know that \( G'_i \) is isomorphic to \( G_i \) (and thus also to \( G \)) and moreover that \( u_i \) is not a vertex of \( G'_i \) (because \( u_i \notin V_{G_i} \) can not determine itself and thus \( u_i \neq v_i \)). If \( H \) was point-determining after removal of \( v_{i+1} \) we would obtain a contradiction similarly as before. We can thus assume that \( v_{i+1} \) determines at least one pair of vertices. Because neighbourhood \( v_{i+1} \) and \( v_i \) differs only by \( u_i \) we know that one vertex of this pair is \( u_i \). Denote by \( u_{i+1} \) the second vertex.

Given \( G'_i, u_i, u_{i+1}, v_{i+1} \) we proceed similarly. If \( u_{i+1} \notin V_{G'_i} \) we put \( G_{i+1} = G'_i \). If \( u_{i+1} \in V_{G'_i} \) we let \( G_{i+1} \) to be the graph induced by \( H \) on \( (V_{G'_i} \setminus \{ u_{i+1} \}) \cup \{ u_i \} \). Again \( G_{i+1} \) is isomorphic to \( G \) and does not contain \( u_{i+1} \) nor \( v_{i+1} \). Denote by \( v_{i+2} \) a vertex determined by \( u_{i+1} \) from \( v_{i+1} \) (which again must exist by our assumption) and we have obtained \( G_{i+1}, u_{i+1}, v_{i+1}, v_{i+2} \) with the desired properties. This finishes the inductive step of the construction.

Because \( H \) is finite, we know that both sequences \( u_0, u_1, \ldots \) and \( v_0, v_1, \ldots \) contains repeated vertices. Without loss of generality we can assume that repeated vertex with lowest index \( j \) appears in the first sequence. We thus have \( u_j = u_i \) for some \( i < j \). By minimality of \( j \) we can assume that \( v_i, v_{i+1}, \ldots v_{j-1} \) are all unique. Assume that \( v_i \) is in the neighbourhood of \( u_i \), then \( v_i \) is not in the neighbourhood of \( u_{i+1} \) (because it determines this pair) and consequently also \( u_{i+1}, u_{i+2}, \ldots , u_j \). A contradiction with \( u_j = u_i \). If \( v_i \) is not in the neighbourhood of \( u_i \) we proceed analogously. \( \Box \)

## 4 Generalised dualities always exist

To demonstrate the usefulness of Theorem 1.1 and Propositions 2.1 and 2.2 give a simple proof of the existence of generalised dualities in full homomorphism order. For two finite sets of graphs \( \mathcal{F} \) and \( \mathcal{D} \) we say that \( (\mathcal{F}, \mathcal{D}) \) is a generalised finite \( F \)-duality pair (sometimes also \( D \)-obstruction) if for any graph \( G \) there exists \( F \in \mathcal{F} \) such that \( F \not\rightarrow G \) if and only if \( G \not\rightarrow D \) for no \( D \in \mathcal{D} \).

Existence of (generalised) dualities have several consequences. To mention one, it implies that the decision problem “given graph \( G \) is there \( D \in \mathcal{D} \) and full homomorphism \( G \rightarrow D ? \)” is polynomial time solvable for every fixed finite family \( \mathcal{D} \) of finite graphs. In the graph homomorphism order the dualities (characterised in [7]) are rare. In the case of full homomorphisms they are however always guaranteed to exist.

**Theorem 4.1** ([9,2,4,1]) For every finite set of graphs \( \mathcal{D} \) there is a finite set
of graphs $\mathcal{F}$ such that $(\mathcal{F}, \mathcal{D})$ is a generalised finite $F$-duality pair.

**Proof.** Without loss of generality assume that $\mathcal{D}$ is a non-empty set of $F$-cores. Consider set $\mathcal{X}$ of all $F$-cores $G$ such that there is $D \in \mathcal{D}$, $G \to D$. Because, by Proposition 2.2, the number of vertices of every such $G$ is bounded from above by the number of vertices of $D$ and because $\mathcal{D}$ is finite, we know that $\mathcal{X}$ is finite.

Now denote by $\mathcal{F}$ the set of all $F$-cores $H$ such that $H \notin \mathcal{X}$ and there is $G \in \mathcal{X}$ such that $(G, H)$ is a gap. By Theorem 1.1 this set is finite. We show that $(\mathcal{F}, \mathcal{D})$ is a duality pair.

Consider an $F$-core $G$, either $G \in \mathcal{X}$ and thus there is $D \in \mathcal{D}$, $G \to D$ or $G \notin \mathcal{X}$ and then consider a sequence of $F$-cores $G_1, G_2, \ldots, G_{|G|} = G$ such that $G_i \in \mathcal{X}$ consists of single vertex, $G_{i+1}$ is created from $G_i$ by adding a single vertex for every $1 \leq i < |G|$ (such sequence exists by Theorem 3.2). Clearly there is $1 \leq j < |G|$ such that $G_j \in \mathcal{X}$ and $G_{j+1} \notin \mathcal{X}$. Because $(G_j, G_{j+1})$ forms a gap, we know that $G_{j+1} \in \mathcal{F}$. \qed

**Remark 4.2** A stronger result is shown by Feder and Hell [2, Theorem 3.1] who shows that if $\mathcal{D}$ consists of single graph $G$ with $k$ vertices, then $\mathcal{F}$ can be chosen in a way so it contains graphs with at most $k + 1$ vertices and there are at most two graphs having precisely $k + 1$ vertices. While, by Theorem 1.1, we can also give the same upper bound on number of vertices of graphs in $\mathcal{F}$, it does not really follow that there are at most two graphs needed. It appears that the full machinery of [2] is necessary to prove this result.

In the opposite direction it does not seem to be possible to derive Theorem 1.1 from this characterisation of dualities, because given pair of non-isomorphic $F$-cores $G \xrightarrow{F} H$ and $\mathcal{D}$ a full homomorphism dual of $\{G\}$ it does not hold that for a graph $F \in \mathcal{D}$ such that $D \xrightarrow{F} H$ there is also full homomorphism $G \xrightarrow{F} H$.

## 5 Full homomorphisms of relational structures

To the date, the full homomorphism order has been analysed in the context of graphs and digraphs only. Let us introduce generalised setting of relational structures:

A language $L$ is a set of relational symbols $R \in L$, each associated with natural number $a(R)$ called *arity*. A (relational) $L$-structure $A$ is a pair $(A, (R_A; R \in L))$ where $R_A \subseteq A^{a(R)}$ (i.e. $R_A$ is a $a(R)$-ary relation on $A$). The set $A$ is called the vertex set of $A$ and elements of $A$ are vertices. The language is usually fixed and understood from the context. If the set $A$ is
finite we call \( A \) finite structure. The class of all finite relational \( L \)-structures will be denoted by \( \text{Rel}(L) \).

A homomorphism \( f : A \to B = (B, (R_B; R \in L)) \) is a mapping \( f : A \to B \) satisfying for every \( R \in L \) the implication \((x_1, x_2, \ldots, x_{a(R)}) \in R_A \implies (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R_B \). A homomorphism is full if the above implication is equivalence, i.e. if for every \( R \in L \) we have \((x_1, x_2, \ldots, x_{a(R)}) \in R_A \iff (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R_B \).

Given structure \( A \) its vertex \( v \) is contained in a loop if there exists \((v, v, \ldots, v) \in R_A \) for some \( R \in L \) of arity at least 2. Given relation \( R_A \) we denote by \( \overline{R_A} \) its complement, that is the set of all \( a(R) \)-tuples \( t \) of vertices of \( A \) that are not in \( R_A \).

When considering full homomorphism order in this context, the first problem is what should be considered to be the neighbourhood of a vertex. This can be described as follows: Given \( L \)-structure \( A \), relation \( R \in L \) and vertex \( v \in A \) such that \((v, v, \ldots, v) \notin R_A \) the \( R \)-neighbourhood of \( v \) in \( A \), denoted by \( N^R_A(v) \) is the set of all tuples \( t \setminus v \) created from \( t \in R_A \) containing \( v \). Here by \( t \setminus v \) we denote tuple created from \( t \) by replacing all occurrences of vertex \( v \) by a special symbol \( \bullet \) which is not part of any vertex set. If \((v, v, \ldots, v) \in R_A \) then the \( R \)-neighbourhood \( N^R_A(v) \) is the set of all tuples \( t \setminus v \) created from \( t \in \overline{R_A} \cup \{(v, v, \ldots, v)\} \). The neighbourhood of \( v \) in \( A \) is a function assigning every relational symbol its neighbourhood: \( N_A(v)(R) = N^R_A(v) \).

We say that \( L \)-structure \( A \) is point-determining if there are no two vertices with same neighbourhood. With these definitions direct analogies of Proposition 2.1 and 2.2 for \( \text{Rel}(L) \) follows.

Analogies of Lemma 3.1, Theorem 3.2 and Theorem 1.1 do not follow for relational structures in general. Consider, for example, a relational structure with three vertices \( \{a, b, c\} \) and a single ternary relation \( R \) containing one tuple \((a, b, c)\). Such structure is point-determining, but the only point-determining substructures consist of single vertex. There is however deeper problem with carrying Lemma 3.1 to relational structures: if a pair of vertices \( u, u' \) is determined by vertex \( v \) their neighbourhood may differ by tuples containing additional vertices. Thus the basic argument about cycles can not be directly applied here. We consequently formulate results for relational language consisting of unary and binary relations only (and, as a special case, to digraphs):

**Theorem 5.1** Let \( L \) be a language containing relational symbols of arity at most 2. If \( A \) and \( B \) are (relational) \( F \)-cores and \((A, B)\) is an \( F \)-gap, then \( A \) can be obtained from \( B \) by removal of one vertex.
The example above shows that the limit on arity of relational symbols is actu-
tually necessary. This may be seen as a surprise, because the results about
digraph homomorphism orders tend to generalise naturally to relational struc-
tures and we thus close this paper by an open problem of characterising gaps
in full homomorphism order of relational structures in general.

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