The complexity of UNO

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Abstract. UNO® is one of the world-wide well-known and popular card games. We investigate
UNO from the viewpoint of combinatorial algorithmic game theory by giving some simple and
concise mathematical models for it. They include cooperative and uncooperative versions of
UNO, for example. As a result of analyzing their computational complexities, we prove that even
a single-player version of UNO is NP-complete, while it becomes in P in some restricted cases.
We also show that uncooperative two-player’s version is PSPACE-complete.

1 Introduction

Playing games and puzzles is a lot of fun for everybody, and analyzing games and puzzles
has long been attracted much interests of both mathematicians and computer scientists [4, 7].
Among various interests and directions of researchers in mathematics and computer science,
one of the central issues is their computational complexities, that is, how hard or easy to get
an answer of puzzles or to decide the winner (loser) of games [2, 3, 9]. Such games and puzzles
of interests include Nim, Hex, Peg Solitaire, Tetris, Geography, Amazons, Chess, Othello, Go,
Poker, and so on. Recently, this field is sometimes called ‘algorithmic combinatorial game
theory’ [2] to distinguish it from games arising from the other field, especially the classical
economic game theory.

In this paper, we focus on one of the well-known and popular card games called UNO†
and investigate it from the viewpoint of algorithmic combinatorial game theory to add it to
the research list. More specifically, we propose mathematical models of UNO, which is one of
the main purposes of this paper, and then examine their computational complexities. As a
result, even a single-player version of UNO is computationally intractable, while we can show
that the problem becomes rather easy under a certain restriction.

We organize this paper as follows: Section 2 introduces two mathematical models of UNO
and their variants, and also defines UNO graphs. Among those models, Section 3 focuses
on a single-player version of UNO, and investigates its complexities. In Section 4, we argue
with two-players’ version of UNO, and show that it is PSPACE-complete. Finally, Section 5
concludes the paper.

2 Preliminaries

Games are often categorized from several aspects of properties that they have when we research
it theoretically. Typical classifications are, for example, if it is multi-player or single-player,
imperfect-information or perfect-information, cooperative or uncooperative, and so on [2, 7].

† UNO® is a registered trademark of Mattel Corporation.
A single-player game is automatically perfect-information and cooperative, and is sometimes called a puzzle.

### 2.1 Game settings

UNO is one of the world-wide well-known and popular card games. It can be played by 2–10 players. Each player is dealt equal number of cards at the beginning of the game, where each (normal) card has its color and number (except for some special ones called ‘action cards’). The basic rule is that each player plays in turn, and one can discard exactly one of his/her cards at hand in one’s turn by matching the card with its color or number to the one discarded immediately before one. The objective of a single game is to be the first player to discard all the cards in one’s hand before one’s opponents. Thus, UNO is a (i) multi-player, (ii) imperfect-information, and (iii) uncooperative combinatorial game (see Appendix A for detailed rule of UNO).

In the real game setting of UNO, it is quite true that action cards play important roles to make this game complicated and interesting. However, in this paper, when we model the game mathematically, we concentrate on the most important aspect of the rules of UNO that a card has a color and a number and that one can discard a card only if its color or number match the card discarded immediately before one’s turn. In addition to obeying this fundamental property, for theoretical simplicity, we set following assumptions on our mathematical models: (a) we do not take into account either action cards nor draw pile, (b) all the cards dealt to and at hand of any player are open during the game, i.e., perfect-information, (c) we do not necessarily assume that all the players have a same number of cards at the beginning of a game (unless otherwise stated), (d) any player acts rationally, e.g., any player is not allowed to skip one’s turn intentionally, and (e) the first player can start a game by discarding any card he/she likes at hand.

### 2.2 Definitions and Notations

An UNO card has two attributes called color and number, and in general, we define a card to be a tuple \( (x, y) \in X \times Y \), where \( X = \{1, \ldots, c\} \) is a set of colors and \( Y = \{1, \ldots, b\} \) is a set of numbers. Finite number of players \( 1, 2, \ldots, p \geq 1 \) can join an UNO game. At the beginning of a single game of UNO, each card of a set of \( n \) cards \( C \) is dealt to one player among \( p \) players, i.e., each player \( i \) is initially given a set \( C_i \) of cards; \( C_i = \{t_{i,1}, \ldots, t_{i,n_i}\} \) \((i = 1, \ldots, p)\). By definition, \( \sum_{i=1}^{p} n_i = n \). Here, we assume that \( C \) is a multiple set, that is, there may be more than one card with the same color and the same number. We denote a card \( (x, y) \) dealt to player \( i \) by \( (x, y)_i \). When the number of players is one, we omit the subscript without any confusion. Throughout the paper, we assume without loss of generality that player 1 is the first to play, and players 1, 2, \ldots, \( p \) play in turn in this order.

Player \( i \) can discard (or play) exactly one card currently at hand in his/her turn if the color or the number of the card is equal to each of the card discarded immediately before player \( i \). In other words, we say that a card \( t' = (x', y') \) can be discarded immediately after a card \( t = (x, y)_i \) if and only if \( (x' = x \lor y' = y) \land i' = i + 1 \pmod{p} \). We also say that a card \( t' \) matches a card \( t \) when \( t' \) can be discarded after \( t \). A discarded card is removed from a set of cards at hand of the player. A discarding (or playing) sequence (of cards) of a card set \( C \) is a sequence of cards \( (t_{s_1}, \ldots, t_{s_k}) \) such that \( t_{s_1} \in C \) and \( t_{s_i} \neq t_{s_j} \) \((i \neq j)\). A discarding sequence \( (t_{s_1}, \ldots, t_{s_k}) \) is feasible if \( t_{s_{j+1}} \) matches \( t_{s_j} \) for \( j = 1, \ldots, k - 1 \).
In our mathematical models of UNO, we specify the problems by four parameters: number of players \( p \), number of total cards \( n \), number of colors \( c \) and the number of numbers \( b \). Two values \( c \) and \( b \) are assumed to be unbounded unless otherwise stated.

### 2.3 Models

We now define two different versions of UNO, one is cooperative and the other is uncooperative.

**Uncooperative Uno**

Instance: the number of players \( p \), and player \( i \)'s card set \( C_i \) with \( c \) colors and \( b \) numbers.

Question: determine the first player that cannot discard one’s card any more.

We refer to this **Uncooperative UNO** with \( p \) players as **Uncooperative UNO-\( p \)**. This problem setting makes sense only if \( p \geq 2 \) since UNO played by a single player becomes automatically cooperative.

**Cooperative Uno**

Instance: the number of players \( p \), player \( i \)'s card set \( C_i \) with \( c \) colors and \( b \) numbers.

Question: can all the players make player 1 win, i.e., make player 1’s card set empty before any of the other players become finished.

We abbreviate **Cooperative Uno** played by \( p \) players as **Cooperative UNO-\( p \)**, or simply as **Uno-\( p \)**. This problem setting makes sense if the number of players \( p \) is greater than or equal to 1. In **Uncooperative/Cooperative UNO**, when the number of players is given by a constant, such as UNO-2, it implies that \( p \) is no longer a part of the input of the problem. In addition to the assumptions (a)–(e) on game settings described in Subsec. 2.1, we set one additional assumption which changes depending on whether the game is cooperative or uncooperative: any player that cannot discard any card at hand (f1) skips one’s turn but still remains in the game and waits for the next turn in cooperative games, and (f2) is a loser in uncooperative games.

We define **UNO-\( p \)** graph as a directed graph to represent ‘match’ relationship between two cards in the entire card set. More precisely, a vertex corresponds to a card, and there is a directed arc from vertex \( u \) to \( v \) if and only if their corresponding cards \( t_u \) matches (can be discarded immediately after) \( t_v \). Let us consider UNO-1 graph, i.e., UNO-\( p \) graph in case that the number of players \( p = 1 \). In this case, a card \( t' \) matches \( t \) if and only if \( t \) matches \( t' \), that is, the ‘match’ relation is symmetric. This implies that UNO-1 graph becomes undirected. For UNO-2 graph, a card \( t' = (x', y')_2 \) matches \( t = (x, y)_1 \) if and only if \( t \) matches \( t' \), and therefore, UNO-2 graph also becomes undirected. Furthermore, since a player cannot play consecutively when the number of players \( p \geq 2 \), UNO-2 graph becomes bipartite. In general, since \( n \) cards of a card set \( C \) is dealt to \( p \) players at the beginning of a single UNO game, i.e., \( C \) is partitioned into \( C_i = \{ (x, y)_i \} \), UNO-\( p \) graph becomes a (restricted) \( p \)-partite graph whose partite sets correspond to \( C_i \).

### 3 Cooperative UNO

In this section, we focus on the cooperative version of UNO, and discuss its complexity when the number of players is two or one.
3.1 Two-players’ case

We first show that Uno-2 is intractable.

**Theorem 1.** Uno-2 is NP-complete.

**Proof.** Reduction from Hamiltonian Path (HP).

An instance of HP is given by an undirected graph \( G \). The problem asks if there is a Hamiltonian path in \( G \), and it is known to be NP-complete [6]. Here, we assume without loss of generality that \( G \) is connected and is not a tree, and hence that \( |V(G)| \leq |E(G)| \). We transform an instance of HP into an instance of Uno-2 as follows. Let \( C_1 \) and \( C_2 \) be the card set of players 1 and 2, respectively. We define \( C_1 = \{(i, i) \mid v_i \in V(G)\} \) and \( C_2 = \{(i, j) \mid \{v_i, v_j\} \in E(G)\} \).

Then, notice that the resulting UNO-2 graph \( G' \), which is bipartite, has partite sets \( X \) and \( Y \) (\( X \cup Y = V(G') \)) corresponding to \( V(G) \) and \( E(G) \), respectively, and represents vertex-edge incidence relationship of \( G \) (Fig. 1).

Now we show that the answer of an instance of HP is yes if and only if the answer of an instance of Uno-2 is yes. If there is a Hamiltonian path, say \( P = (v_{i_1}, v_{i_2}, \ldots, v_{i_n}) \), in the instance graph of HP, then there is a feasible discarding sequence alternatively by player 1’s and 2’s as \(((i_1, i_1)_1, (i_1, i_2)_2, (i_2, i_2)_1, \ldots, (i_{n-1}, i_{n-1})_1, (i_{n-1}, i_n)_2, (i_n, i_n)_1)\), which ends up player 1’s card before player 2’s. Conversely, if there is a feasible discarding sequence \(((i_1, i_1)_1, (i_1, i_2)_2, (i_2, i_2)_1, \ldots, (i_{n-1}, i_{n-1})_1, (i_{n-1}, i_n)_2, (i_n, i_n)_1)\), it visits all the vertices in \( X \) of \( G' \) exactly once, and thus the corresponding sequence of vertices \((v_{i_1}, v_{i_2}, \ldots, v_{i_n})\) is a simple path visiting all the vertices in \( V(G) \) exactly once, that is, a Hamiltonian path in \( G \).

The size of an instance of Uno-2 is proportional to \(|C_1| + |C_2|\). Since \(|C_1| = |V(G)|\) and \(|C_2| = |E(G)|\), the reduction is done in polynomial size in \(|V(G)| + |E(G)|\), which is the input size of an instance of HP. This completes the proof. \(\Box\)

**Corollary 1.** Uno-2 is NP-complete even when the number of cards of two players are equal.

**Proof.** Reduction from Hamiltonian Path with specified starting vertex, which is known to be NP-complete [6].

We consider the same reduction in the proof of Theorem 1. As in that proof, we can assume \(|C_1| \leq |C_2|\) without loss of generality. When \(|C_1| = |C_2|\), we are done. If \(|C_1| < |C_2|\), add \(|C_2| - |C_1|\) cards \((n + 2, n + 2)\) and a single card \((n + 2, n + 1)\) to \(C_1\), a single card \((i, n + 1)\) \((i \in \{1, \ldots, n\})\) to \(C_2\), and player 1 starts with card \((n + 2, n + 2)\). This forces the original graph \(G\) to specify a starting (or an ending) vertex of a Hamiltonian path to be \(v_i\). \(\Box\)
3.2 Single-player’s intractable case

In single-player’s case, two different versions of UNO, cooperative and uncooperative ones, become equivalent. We redefine this setting as the following:

**Uno-1 (Solitaire Uno)**

Instance: a set $C$ of $n$ cards $(x_i, y_i)$ ($i = 1, \ldots, n$), where $x_i \in \{1, \ldots, b\}$ and $y_i \in \{1, \ldots, c\}$.

Question: determine if the player can discard all the cards.

**Example 1.** Let the card set $C$ for player 1 is given by $C = \{(1, 3), (2, 2), (2, 3), (2, 3), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$. Then, a feasible discarding sequence using all the cards is $((1, 3), (2, 3), (2, 4), (3, 4), (3, 2), (2, 2), (2, 3), (4, 3), (4, 1))$ in this order, for example, and the answer is yes. The corresponding UNO-1 graph is depicted in Fig. 2.

![Fig. 2. An example of UNO-1 graph.](image)

We here investigate some basic properties of UNO-1 graphs. In UNO-1 graphs, all the vertices whose corresponding cards have either the same color or number form a clique. A line graph $L(G)$ of a given graph $G$ is a graph whose vertices are edges of $G$ and $\{(e, e') \in E(L(G)) \mid e, e' \in V(L(G)) \text{ if and only if } e \text{ and } e' \text{ share endpoints in } G\}$. A graph that contains no induced $K_{1,3}$ is called *claw-free*, and line graphs are claw-free. It is not so difficult to see that UNO-1 graphs are claw-free since at least two of the three cards that match a card must have the same color or number. Furthermore, we can observe the following fact.

**Observation 1.** A graph is UNO-1 if and only if it is a line graph of a bipartite graph.

Now we can easily understand that Uno-1 is essentially equivalent to finding a Hamiltonian path in UNO-1 graph. However, the following fact is known.

**Theorem 2.** [8] **Hamiltonian Path for line graphs of bipartite graphs is NP-complete.**

Therefore, as a corollary of this theorem, we unfortunately know that UNO is hard even for a single player.

**Theorem 3.** Uno-1 is NP-complete.

Here, we give a direct and concise proof of Theorem 3 for self-containedness and completeness instead of the one in [8], which further depends on [1].

**Proof.** A cubic graph is a graph each of whose vertex has degree 3. We reduce Hamiltonian Path for cubic graphs (HP-C), which is known to be NP-complete [5], to Uno-1.

Let an instance of HP-C be $G$. We transform $G$ into a graph $G'$, where

$$V(G') = \{(x, e) \mid x \in V(G), e = \{x, y\} \in E(G)\},$$

$$E(G') = \{((x, e), (y, e)) \mid e = \{x, y\} \in E(G)\} \cup \{(x, e_i), (x, e_j)\} \mid e_i \neq e_j\}.$$
This transformation implies that any vertex \( x \in V(G) \) is split into three new vertices \((x, e_i)\) \((i = 1, 2, 3)\) to form a clique (triangle), while each incident edge \(e_i\) \((i = 1, 2, 3)\) to \(x\) becomes incident to a new vertex \((x, e_i)\). (We call it a “node gadget” as shown in Fig. 3.) Then we prepare the card set \(C\) of the player of UNO-1 to be the set \(V(G')\), where the color and the number of \((x, e)\) are \(x\) and \(e\), respectively. We can easily confirm that there is an edge \(e = (t, t')\) in \(G'\) if and only if \(t\) and \(t'\) match, i.e., \(G'\) is the corresponding UNO-1 graph for card set \(C\).

Now it suffices to show that there is a Hamiltonian path in \(G\) of an instance of HP-C if and only if there is a Hamiltonian path in \(G'\).

Suppose there is a Hamiltonian path, say \(P = (v_{i_1}, \ldots, v_{i_n})\), in \(G\). We construct a Hamiltonian path \(P'\) in \(G'\) from \(P\) as follows. Let \(v_{i_{j-1}}, v_{i_j}, v_{i_{j+1}}\) be three consecutive vertices in \(P\) in this order, and let \(e_1 = \{v_{i_{j-1}}, v_{i_j}\}, e_2 = \{v_{i_j}, v_{i_{j+1}}\}\) and \(e_3 = \{v_{i_{j+1}}, v_{i_k}\}\) \((k \neq j-1, j+1)\). Then we replace these three vertices by the sequence of vertices \((v_{i_{j-1}}, e_1), (v_{i_j}, e_1), (v_{i_j}, e_3), (v_{i_j}, e_2), (v_{i_{j+1}}, e_2)\) in \(G'\) to form a subpath in \(P'\). For the starting two vertices \(v_{i_1}\) and \(v_{i_2}\), we replace them by the sequence of vertices \((v_{i_1}, e_1)\) \((e_1 \neq \{v_{i_1}, v_{i_2}\})\), \((v_{i_1}, e_2)\) \((e_2 \neq \{v_{i_1}, v_{i_2}\})\), \((v_{i_1}, \{v_{i_1}, v_{i_2}\})\), \((v_{i_2}, \{v_{i_1}, v_{i_2}\})\) (same for the ending two vertices). We can now confirm that the resulting sequence of vertices \(P'\) in \(G'\) form a Hamiltonian path.

For the converse, we have to show that if there is a Hamiltonian path \(P'\) in UNO-1 graph \(G'\), then there is in \(G\). If \(P'\) visits \((v, e_i)\) \((i = 1, 2, 3)\) consecutively in any order (call it “consecutiveness”) for any \(v\) (as shown in Fig. 4 (a1) or (a2)), then \(P'\) can be transformed into a Hamiltonian path \(P\) in \(G\) in an obvious way. Suppose not, that is, a Hamiltonian path \(P'\) in \(G'\) does not visit \((v, e_i)\) \((i = 1, 2, 3)\) consecutively. It suffices to show that such \(P'\) can be transformed into another path to satisfy the consecutiveness. There are two possible cases as shown in Fig. 4 (b') and (c'), both of which contain at least one end point of \(P'\) in \((v, e_i)\). In case (b'), we can resolve this inconsecutiveness in \((v, e_i)\) as shown in (b), which may result in case (c') in adjacent set of three vertices. In case (c'), in order to resolve it, we can transform it into (c), which does not contain inconsecutiveness any more.
The reduction can be done in the size proportional to the size of an instance of HP-C. Thus, the proof is completed. 

\[\square\]

### 3.3 Single-player’s tractable case

In the remaining part of this section, we will show that such an intractable problem \textsc{Uno-1} becomes tractable if the number of colors \(c\) is bounded by a constant. It will be solved by dynamic programming (DP) approach. To illustrate the DP for \textsc{Uno-1}, we first introduce a geometric view of \textsc{Uno-1} graphs.

Since an UNO card \((x, y)\) is an ordered pair of integer values standing for its color and number, it can be viewed as a (integer) lattice point in the 2-dimensional lattice plane. Then an UNO-1 graph is a set of points in that plane, where all the points with the same \(x\)- or \(y\)-coordinate form a clique. We call this way of interpretation a geometric view of UNO-1 graphs. The geometric view of an instance in Example 1 is shown in Fig. 5 (a). Now the problem \textsc{Uno-1}, which is equivalent to finding a Hamiltonian path in UNO-1 graphs, asks if, for a given set of points in the plane and starting and ending at appropriate different points, one can visit all the points exactly once under the condition that only axis-parallel moves are allowed at each point (Fig. 5 (b)).

![Fig. 5. (a) Geometric view of a UNO-1 graph, where all the edges are omitted, (b) a Hamiltonian path in the UNO-1 graph, and (c) a set of subpaths in the subgraph of the UNO-1 graph induced by the first 6 points; it shows \(h_{(1,2)} = 1, v_{(2,3)} = 1\) and \(d_{(4,4)} = 1\).](image)

**Strategy.** Let \(C\) be a set of \(n\) points and \(G\) be an UNO-1 graph defined by \(C\). Then a subgraph \(P\) forms a Hamiltonian path if and only if it is a single path that spans \(G\). Suppose a subgraph \(P\) is a spanning path of \(G\). If we consider a subset \(C'\) of the point set \(C\), then \(P[C']\) (the subgraph of \(P\) induced by \(C'\)) is a set of subpaths that spans \(G[C']\) (Fig. 5 (c)). We count and maintain the number of sets of subpaths by classifying subpaths into three disjoint subsets according to the types of their two endpoints.

Starting with the empty set of points, the DP proceeds by adding a new point according to a fixed order by updating the number of sets of subpaths iteratively. Finally when the set of points grows to \(C\), we can confirm the existence of a Hamiltonian path in \(G\) by checking the number of sets of subpaths consisting of a single subpath (without isolated vertices). Remark that, throughout this DP, we regard for convenience that an isolated vertex by itself contains a (virtual) path starting and ending at itself that spans it.

**Mechanism.** To specify a point to be added in an iteration of the DP, we define a relation \(<\) on the point set \(C\), where \(x(t)\) and \(y(t)\) are \(x\)- and \(y\)-coordinates of a point \(t\), respectively: Let \(t\) and \(t'\) be two points in \(C\), then \(t < t' \iff y(t) < y(t')\) or \((y(t) = y(t') \land x(t) < x(t'))\). When
\[ t = t', \text{ a tie breaks arbitrary. This relation } \prec \text{ defines a total order on } C, \text{ and we refer } n \text{ points in } C \text{ to } t_1, \ldots, t_n \text{ according to the increasing order of } \prec. \text{ We also define } C_\ell = \{ t_i \mid 1 \leq i \leq \ell \}. \]

Now points are added from \( t_1 \) to \( t_n \), and consider when a new point \( t_\ell = (x(t_\ell), y(t_\ell)) \) is added to \( C_{\ell-1} \). It must be added either to two, one or zero endpoints of different subpaths to form a new set of subpaths.

Now let \( P(\ell) \) be a family of sets of subpaths spanning \( G[C_\ell] \). (Recall that we regard that an isolated vertex contains a path spanning itself.) Then we classify subpaths in a set of subpaths \( P \in P(\ell) \) in the following manner: for any subpath \( P = (y(t_1), y(t_\ell)) \) and the \( y \)-coordinates of its two endpoints, either (i) both equal \( y(t_i) \) (type-h), (ii) exactly one of two equals \( y(t_i) \) (type-v), or (iii) none equals \( y(t_i) \) (type-d) holds. We count the number of such three types of subpaths in \( P \) further by classifying them by the \( x \)-coordinates of their endpoints. (Notice that types-h, -d are symmetric but type-v is not with respect to \( x \)-coordinate.) For this purpose, we prepare some subscript sets: a set of substrings \( K = \{ 1, \ldots, c \} \), sets of unordered pair of substrs \( I = \binom{K}{2} \) and \( I^+ = I \cup \{ \{i, i\} \mid i \in K \} \), and sets of ordered pair of substrs \( J = K \times K \) and \( J^- = J - \{ \{i, i\} \mid i \in K \} \).

We now introduce the following parameters \( h, v \) and \( d \) to count the number of subpaths in \( P = \{ \ell \} \) (see Fig. 5 (c)).

\[
\begin{align*}
  h_{\{i, i'\}} & : \text{#subpaths in } \mathcal{P} \text{ with endpoints } (x_i, y(t_\ell)) \text{ and } (x_{i'}, y(t_\ell)) \text{ for } \{i, i'\} \in I^+, \\
  v_{\{i, i'\}} & : \text{#subpaths in } \mathcal{P} \text{ with endpoints } (x_i, y(t_\ell)) \text{ and } (x_{i'}, y') \text{ for } \{i, i'\} \in J \text{ and } y' < y(t_\ell), \\
  d_{\{i, i'\}} & : \text{#subpaths in } \mathcal{P} \text{ with endpoints } (x_i, y') \text{ and } (x_{i'}, y'') \text{ for } \{i, i'\} \in I^+ \text{ and } y', y'' < y(t_\ell).
\end{align*}
\]

Then we define a \((2|I^| + |J|)|\)-dimensional vector \( z(\mathcal{P}) \) for a set of subpaths \( \mathcal{P} = \{ \ell \} \) as \( z(\mathcal{P}) = (h; v; d) = (h_{\{1, 1\}}, \ldots, h_{\{1, c\}}, h_{\{2, 2\}}, \ldots, h_{\{2, c\}}, h_{\{3, 3\}}, \ldots, h_{\{c, c\}}); (v_{\{1, 1\}}, \ldots, v_{\{1, c\}}, v_{\{2, 2\}}, \ldots, v_{\{2, c\}}, v_{\{3, 1\}}, \ldots, v_{\{3, c\}}); (d_{\{1, 1\}}, \ldots, d_{\{1, c\}}, d_{\{2, 2\}}, \ldots, d_{\{2, c\}}, d_{\{3, 3\}}, \ldots, d_{\{c, c\}})\). Finally, for a given vector \((h; v; d)\), we define the number of sets \( \mathcal{P} \) satisfying \( z(\mathcal{P}) = (h; v; d) \) in a family \( \mathcal{P}(\ell) \) by \( f(\ell; (h; v; d)) \), i.e., \( f(\ell; (h; v; d)) = \{|\mathcal{P} \mid \mathcal{P} \in \mathcal{P}(\ell), z(\mathcal{P}) = (h; v; d)\}|\).

Now the objective of the DP is to determine if there exists a vector \((h; v; d)\) such that \( f(n, (h; v; d)) \geq 1 \), where all the elements in \( h, v \) and \( d \) are 0 except for exactly one element is 1.

**Recursion.** As we explained, the DP proceeds by adding a new point \( t_\ell \) to \( C_{\ell-1} \). When \( t_\ell \) is added, it is connected to either 0, 1 or 2 endpoints of existing different paths, where each endpoint has \( y(t_\ell) \) or \( x(t_\ell) \) in its coordinate. The recursion of the DP is described just by summing up all possible combinations of these patterns. We treat it by dividing them into three cases, one of which has two subcases: (a) a set of base cases; (b) a case in which \( t_\ell \) is added as the first point whose \( y \)-coordinate is \( y(t_\ell) \), and (b1) as an isolated vertex, or (b2) as to be connected to an existing path; (c) all the other cases.

Now we give the DP formula in the following, in which \( h < 0 \) implies that \( h_{\{i, i'\}} < 0 \) holds for some subscript \( \{i, i'\} \) (same for \( v \) and \( d \)), and we let \( k = x(t_\ell) \) and \( \delta_{ij} \) to be a Kronecker’s delta:

\[
\begin{align*}
  f(\ell; (h; v; d)) = \begin{cases} 
    0, & \text{if } \ell = 0 \text{ or } h < 0 \text{ or } v < 0 \text{ or } d < 0, \\
    \text{(described below)}, & \text{(b1) if } \ell > 0, \text{ } y(t_\ell) > y(t_{\ell-1}), \text{ } h = 0 \ldots h_{\{k, k\}} = 1 \ldots 0, \text{ } v = 0, \\
    \text{(described below)}, & \text{(b2) if } \ell > 0, \text{ } y(t_\ell) > y(t_{\ell-1}), \text{ } h = 0, \text{ } v = 0 \ldots v_{\{k, j\}} = 1 \ldots 0, \\
    \text{(described below)}, & \text{ (c) otherwise,}
  \end{cases}
\end{align*}
\]
where the formula for case (b1) is

$$f(\ell; (h'; v; d)) = \sum_{h', v', d'} \left\{ f(\ell-1; (h'; v'; d') \mid d_{i,i'} = h'_{i,i'} + d'_{i,i'} + v'_{i,i'} + (1-\delta_{ii'})v'_{i,i'} \ (\{i,i'\} \neq \{k,j\}) \right\}$$

and the formula for case (b2) is

$$f(\ell; (h'; v; d)) = \sum_{h', v', d'} \left\{ f(\ell-1; (h'; v'; d') \mid d_{i,i'} = h'_{i,i'} + d'_{i,i'} + v'_{i,i'} + (1-\delta_{ii'})v'_{i,i'} \ (\{i,i'\} \neq \{k,j\}) \right\}$$

and the formula for case (c) becomes

$$f(\ell; (h; v; d)) = \sum_{i,i' \in I^+} f(\ell; (\ldots h_{i,i}+1\ldots)) = \sum_{i,i' \in I^+} f(\ell; (\ldots h_{i,i}+1\ldots))$$

**Timing analysis.** We first count possible combinations of arguments for $f$. Since $\ell$ varies from 0 to $n$, there are $O(n)$ possible values. All of $h$, $v$ and $d$ have $O(c^2)$ elements, each of which can have $O(n)$ possible values, and therefore $O(nc^2)$ possible values in all. To compute a single
value of \( f \), it requires \( O(n^4) \) lookups of previously computed values of \( f \) in case (c), while \( O(n^{3c^2}) \times O(n^2) \) lookups and check-sums in cases (b1) and (b2), which is greater than \( O(n^4) \). Therefore, the total running time for this DP is \( \Theta(n) \times O(n^{3c^2}) \times O(n^{3c^2+2}) = O(n^{6c^2+3}) = n^{O(c^2)}, \) which is polynomial in \( n \) when \( c \) is a constant.

Since the role of colors and numbers are symmetric in UNO games, we have the following results.

**Theorem 4.** Uno-1 is in P if \( b \) (the number of numbers) or \( c \) (the number of colors) is a constant.

### 4 Uncooperative UNO

In this section, we deal with the uncooperative version of UNO. Especially, we show that it is intractable even for two player’s case. For this purpose, we consider the following version of Generalized Geography, which is played by two players.

**Generalized Geography**

- **Instance:** a directed graph, and a token placed on an initial vertex.
- **Question:** a turn is to move the token to an adjacent vertex, and then to remove the vertex moved from from the graph. Player 1 and 2 take turns, and the first player unable to move loses. Determine the loser.

It is well-known that Generalized Geography is PSPACE-complete [9], and a stronger result is presented.

**Theorem 5.** [9] Generalized Geography for bipartite graphs is PSPACE-complete.

Now we show the hardness result for Uncooperative Uno-2.

**Theorem 6.** Uncooperative Uno-2 is PSPACE-complete.

**Proof.** Reduction from Generalized Geography for bipartite graphs (GG-B).

Let (directed) bipartite graph \( G \) with \( V(G) = X \cup Y \) be an instance of GG-B, where \( X \) and \( Y \) are two partite sets, and let \( r \in X \) be an initial vertex. To construct a corresponding Uncooperative Uno-2 instance, we first transform \( G \) into another graph \( G' \) where

\[
V(G') = \{u_s, u_t, u_c \mid u \in V(G)\},
\]

\[
E(G') = \{(u_t, u_c), (u_c, u_s) \mid u \in V(G)\} \cup \{(u_s, v_t) \mid (u, v) \in E(G)\}
\]

(Fig. 7). By construction, we can confirm that \( G' \) is a bipartite graph with \( V(G') = X' \cup Y' \), where \( X' = \{u_s, u_t \mid u \in X\} \cup \{u_c \mid u \in Y\} \) and \( Y' = \{u_s, u_t \mid u \in Y\} \cup \{u_c \mid u \in X\} \). We
let \( r' = r_t (\in X') \) be an initial vertex. It is easy to confirm that player 1 can win the game GG-B on \( G \) if and only if the player wins on \( G' \). Then we prepare card sets \( C_i \) for players \( i \) \((= 1, 2)\) by

\[
C_1 = \{(x, e), (e, y) \mid e = (x, y) \in E(G'), x \in X', y \in Y'\}
\cup\{(e, e) \mid e = (y, x) \in E(G'), x \in X', y \in Y'\},
\]

\[
C_2 = \{(y, e), (e, x) \mid e = (y, x) \in E(G'), x \in X', y \in Y'\}
\cup\{(e, e) \mid e = (x, y) \in E(G'), x \in X', y \in Y'\}.
\]

This means that we prepare three cards for each arc \( e \in E(G') \), one for player \( i \) and two for player \( 3 - i \) (Fig. 8).

![Fig. 7. Split a vertex into two edges so that edges correspond to cards.](image)

![Fig. 8. Prepare three cards \((x, e)_1\), \((e, e)_2\) and \((e, y)_1\) for an arc \( e = (x, y) \), and three cards \((e, y)_2\), \((e, e)_1\) and \((x, e)_2\) for an arc \( e = (y, x) \).](image)

Now we show that player 1 can win in an Uncooperative Uno-2 instance if and only if player 1 can win in an GG-B instance \( G' \) and \( s' \). To show this, it suffices to show that any feasible playing sequence by players 1 and 2 in an GG-B instance corresponds to a feasible discarding sequence alternatively by players 1 and 2 in the corresponding Uncooperative Uno-2 instance, and vice versa.

Suppose a situation that player 2 has just discarded a card. The discarded card belongs to either one of the following five cases: (i) \((e, x)\) for \( e = (y, x) \), (ii) \((y, e)\) for \( e = (y, x) \), (iii) \((e, e)\) for \( e = (x, y) \). Among those, for cases (ii) and (iii), since player 1 starts the game (player 1 always played before player 2’s turn), there exists exactly one card (outgoing arc) that matches the one discarded by player 2 from the end vertex of the arc corresponding to the card. This forces to traverse \( G' \) along the directed arc (in forward direction), which implies to remove corresponding end vertex from \( G' \). The only case we have to care about is case (i), where there may be multiple choices for player 1. In this case, once player 1 discarded one of match cards, the player will never play another match card afterwards, since the only card that can be discarded immediately before it has played and used up. This implies that vertex \( x \) is removed from \( G' \). (The argument is symmetric for player 1 except that the initial card is specified.)

Now we verify that Uncooperative Uno-2 is in PSPACE. For this, consider a search tree for Uncooperative Uno-2, whose root is for player 1 and every node has outgoing arcs corresponding to each player’s possible choices. Since the number of total cards for the two players is \( n \), the number of choices at any turn is \( O(n) \) and since at least one card is removed from either of the player’s card set, the number of depth of the search tree is bounded by \( O(n) \). Therefore, it requires polynomial space with respect to the input size. Thus the proof is completed. \( \square \)
5 Concluding Remarks

In this paper, we focused on UNO, the well-known card game, and gave two mathematical models for it; one is cooperative (to make a specified player win), and the other is uncooperative (to decide the player not to be able to play). As a result of analyzing their complexities, we showed that these problems are difficult in many cases, however, we also showed that a single-player’s version is solvable in polynomial time under a certain restriction.

As for an obvious future work, we can try gaining speedup in dynamic programming for UNO-1 with constant number of colors by better utilizing its geometric properties. In this direction, it may be quite natural to ask if UNO-1 is fixed-parameter tractable. Another probable direction is to investigate UNO-1 graphs from the structural point of view, since they form a subclass of claw-free graphs and seem to have interesting properties by themselves. It is also quite probable to modify our models more realistic, e.g., to take draw pile into account (as an additional player), to make all players’ cards not open, and so on.

Based on our mathematical models, it is not so difficult to invent several variations or generalizations of UNO games, even for UNO-1 (single-player’s version). Among them, we can generalize an UNO card from 2-tuple (2-dimensional) to d-tuple, that is, D-DIMENSIONAL UNO-1 with appropriate modifications to ‘match’ relation of cards. Another one is MINIMUM CARD FILL-IN, that is, given a no instance for UNO-1, find a minimum number of cards to be added to make it to be a yes instance.

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APPENDIX

A  The Rule of UNO®

UNO® is a registered trademark owned by Mattel, Inc. We introduce an excerpt of the original rules of UNO.

Contents. 108 cards as follows:

– 19 Blue cards, 0 to 9
– 19 Green cards, 0 to 9
– 19 Red cards, 0 to 9
– 19 Yellow cards, 0 to 9
– 8 Draw two cards, 2 each in blue, green, red and yellow
– 8 Reverse cards, 2 each in blue, green, red and yellow
– 8 Skip cards, 2 each in blue, green, red and yellow
– 4 Wild cards
– 4 Wild Draw Four cards

Objective of the game. To be the first player to score 500 points. Points are scored by getting rid of all the cards in your hand before your opponent(s). You score points for cards left in your opponents’ hands.

How to play. Every player picks a card. The person who picks the highest number deals. Action Cards count as zero for this part of the game.

Once the cards are shuffled each player is dealt 7 cards.

The remainder of the deck is placed face down to form a DRAW pile. The top card of the DRAW pile is turned over to begin a DISCARD pile. If an Action Card is the first one turned up from the DRAW pile, certain rules apply (see Functions of Action Cards).

The person to the left of the dealer starts play. He/she has to match the card on the DISCARD pile, either by number, color or symbol. For example, if the card is a red 7, the player must put down a red card or any color 7. Alternatively, the player can put down a Wild card (see Functions of Action Cards).

If the player doesn’t have a card to match the one on the DISCARD pile, he/she must take a card from the DRAW pile. If the card picked up can be played, the player is free to put it down in the same turn. Otherwise, play moves on to the next person in turn.

Players may choose not to play a playable card from their hand. If so, the player must draw a card from the DRAW pile. If playable, that card can be put down in the same turn, but the player may not use a card from the hand after the draw.

Functions of the Action Cards. The functions of the Action Cards, and when they may be played, are set out below.

– Draw Two Card  When this card is played, the next person to play must draw 2 cards and miss his/her turn. This card can only be played on matching colors and other Draw Two cards. If turned up at the beginning of play, the same rule applies.
- Reverse Card  This simply reverses direction of play. Play to changes direction to the right, and vice versa. The card may only be played on a matching color or on another reverse card. If this card is turned up at the beginning of play, the dealer goes first, then play moves to the right instead of the left.

- Skip Card  The next player to play after this card has been laid loses his/her turn and is “skipped”. The card may only be played on a matching color or on another Skip card. If a Skip card is turned up at the beginning of play, the player to the left of the dealer is skipped, hence the player to the left of that player commences play.

- Wild Card  The person playing this card calls for any color to continue the play, including the one currently being played, if so desired. A Wild card can be played at any time—even if the player has another playable card in the hand. If a Wild card is turned up at the beginning of play, the person to the left of the dealer determines the color, which continues play.

- Wild Draw Four Card  This is the best card to have. The person who plays it calls the color that continues play. Also, the next player has to pick up 4 cards from the DRAW pile and miss his/her turn. Unfortunately, the card can only be played when the player holding it does not have a card in his/her hand to match the color on the DISCARD pile. If the player holds matching numbers or Action Cards, however, the Wild Draw Four card may be played. A player holding a Wild Draw Four may choose to bluff and play the card illegally, but if he/she is caught certain rules apply. If this card is turned up at the beginning of play, it is returned to the deck and another card is picked.

Going out.  When a player has only one card left, he/she must yell “UNO”. Failure to do this results in having to pick up 2 cards from the DRAW pile. This is only necessary, however, if he/she is caught by one of the other players.

Once a player has no cards left, the hand is over. Points are scored and play starts over again.

If the last card played in a hand is a Draw Two of Wild Draw Four card, the next player must draw the 2 or 4 cards respectively. These cards are counted when the points are totalled.

If no player is out of cards by the time the DRAW pile is depleted, the deck is reshuffled and play continues.