Matrix Completion with Prior Subspace Information via Maximizing Correlation

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Abstract—This paper studies the problem of completing a low-rank matrix from a few random entries with the help of prior subspace information. Assuming that we have access to the column and row subspaces of the desired matrix, a new approach is proposed to incorporate the prior subspace information into the vanilla matrix completion program. Apart from promoting low-rank property, the proposed approach maximizes the correlation between the desired matrix and the matrix constructed by prior subspace information. The theoretical guarantee is established for the proposed method, which shows that with suitable prior information, the proposed approach can reduce the sample size of matrix completion by a logarithmic factor.

I. INTRODUCTION

Matrix completion refers to recovering a low-rank matrix from a small number of random entries

\[ Y = \mathcal{R}_p(X^*) , \]

where \( Y \in \mathbb{R}^{n \times n} \) denotes the matrix of measurements, \( X^* \in \mathbb{R}^{n \times n} \) denotes the original rank-\( r \) matrix to be estimated and \( \mathcal{R}_p(\cdot) \) denotes the Bernoulli sampling operator [1].

To complete the low-rank matrix, the nuclear norm minimization program is proposed to promote low-rank property

\[
\min_{X} \|X\|_* \quad \text{s.t.} \quad Y = \mathcal{R}_p(X),
\]

where \( \|\cdot\|_* \) denotes the nuclear norm. The related performance guarantees and algorithms have been given in a large number of literature; see e.g. [4]–[14] and references therein. The theoretical results indicate that \( O(rn \log^2 n) \) samples are sufficient to accurately complete the matrix for an incoherent rank-\( r \) matrix.

In many applied problems, however, the prior information about the \( r \)-dimensional column and row subspaces of the desired matrix \( X^* \) is available to us, which are denoted by \( \mathcal{U}_r \) and \( \mathcal{V}_r \), respectively. One promising example in collaborative filtering [15]–[17] is to predict new ratings based on the ratings for different movies from a large number of users, where \( \mathcal{U}_r \) might denote the similarities among users and \( \mathcal{V}_r \) might illustrate the similarities among movies. Another important instance in link prediction [18], [19] is to complete missing links among users in a social network by using the given links, where \( \mathcal{U}_r = \mathcal{V}_r \) represents the friendship among users. An extra significant example in dynamic sensor network location is to locate battery-powered sensors by using previous distance matrix [20], where \( \mathcal{U}_r \) (resp. \( \mathcal{V}_r \)) denotes the \( r \)-dimensional column (resp. row) space of the previous distance matrix.

From these applications, a natural question to ask is whether we can leverage the prior information to improve the performance of standard matrix completion; if so, how and how well.

There exists a rich literature on matrix completion under different prior subspace information. Higher dimensional subspace information was studied in [21]–[23]. A scheme with perfect \( s \)-dimensional subspace information \( (s \geq r) \) was studied in [21], [22] to obtain a better improvement than standard matrix completion. A following approach with imperfect \( s \)-dimensional subspace information was proposed in [23], where Chiang et al. completed the original matrix by splitting it into a low-rank estimate in the subspace and a low-rank perturbation outside the subspace.

Different from the above works, other researchers considered improving the performance of matrix completion by using \( r \)-dimensional imperfect subspace information. In [8], [24]–[26], the authors studied a diagonal weighted nuclear norm minimization approach for matrix completion. The theoretical results showed that by choosing proper weights, the approach could outperform the standard low-rank matrix completion.

In [27], Eftekhari et al. proposed another weighted matrix completion method by projecting the target matrix on the constructed subspaces. Their results showed that with suitable side information, the approach can decrease the sample size by a logarithmic factor compared with standard procedure. In [28], the authors gave an analysis for a similar projection scheme, which obtained unique optimal weights that minimize the sample size.

In this paper, we propose a novel approach to integrate \( r \)-dimensional prior subspace information into matrix completion problem by maximizing the correlation between \( X \) and \( \Phi \)

\[
\min_{X} \|X\|_* - \lambda \langle \Phi, X \rangle \quad \text{s.t.} \quad Y = \mathcal{R}_p(X)
\]

where \( \Phi \) is constructed from the prior subspace information and \( \lambda \in [0, 1] \) is a tradeoff parameter. Previous work [29] proposed a similar approach but the measurements are sub-Gaussian, which is far from practical applications in matrix completion problem since the measurement operator in matrix
where the completion are highly structured, i.e., partial unit orthogonal matrices. This paper employs totally different tools to establish the theoretical guarantees for the highly structured measurement operator. In particular, with reliable prior information, our approach can theoretically reduce the number of measurements by a logarithmic factor compared to standard matrix completion, i.e., \( O(\sqrt{n} \log n) \) samples are the optimal bound for an incoherent rank-\( r \) matrix in this case.

The paper is organized as follows. We introduce some useful preliminaries in Section \( \text{II} \) Performance guarantees for matrix completion with prior subspace information are presented in Section \( \text{III} \) Simulations are included in Section \( \text{IV} \) and the conclusion is drawn in Section \( \text{V} \).

II. PRELIMINARIES

In this section, we provide some helpful notation, definitions and propositions, which will be used in the sequel. Throughout the paper, \( I_n \in \mathbb{R}^{n \times n} \) denotes identity matrix and \( 0_n \in \mathbb{R}^{n \times n} \) denotes all-zero matrix.

The sampling operator is defined as follows

\[
R_p(X) = \sum_{i,j=1}^{n} \delta_{ij} \langle e_i e_j^T, X \rangle e_i e_j^T,
\]

where \( e_i \) denotes the \( i \)-th canonical basis in \( \mathbb{R}^n \), \( \{\delta_{ij}\}_{i,j=1}^{n} \) are independent Bernoulli random variables and \( \delta_{ij} \) takes 1 with probability \( p_{ij} \) and 0 with probability \( 1 - p_{ij} \). Here, \( p_{ij} \) is the observation probability for the \( (i,j) \)-th entry and \( \{p_{ij}\}_{i,j=1}^{n} \) are independent with each other. It means that we can observe \( m = \sum_{i,j=1}^{n} p_{ij} \) samples in expectation.

Next, we review an important definition named leverage scores for matrix completion [8].

**Definition 1** (Leverage scores). Let the thin SVD for a rank-\( r \) matrix \( X \in \mathbb{R}^{n \times n} \) be \( U_r \Sigma_r V_r^T \). Define \( \mathcal{U}_r = \text{span}\{U_r\} \) and \( \mathcal{V}_r = \text{span}\{V_r\} \). Then the leverage scores \( \mu_i(\mathcal{U}_r) \) with respect to the \( i \)-th row of \( X \), and \( \nu_j(\mathcal{V}_r) \) with respect to the \( j \)-th column of \( X \) are defined as

\[
\mu_i = \mu_i(\mathcal{U}_r) = \frac{n}{r} \|U_r^T e_i\|_2^2, \quad i = 1, 2, \ldots, n,
\]

\[
\nu_j = \nu_j(\mathcal{V}_r) = \frac{n}{r} \|V_r^T e_j\|_2^2, \quad j = 1, 2, \ldots, n.
\]

Then the coherence parameter [4] of \( X \) can be expressed as

\[
\eta(X) = \max \{\mu_i(\mathcal{U}_r), \nu_j(\mathcal{V}_r)\}.
\]

It’s easy to verify that \( \eta(X) \in [1, \frac{\sqrt{r}}{2}] \). Moreover, when \( \eta(X) \) is small, i.e., \( \mathcal{U}_r \) and \( \mathcal{V}_r \) are spanned by vectors with nearly equal entries in magnitude, we call that \( X \) is *incoherent*; when \( \eta(X) \) is large, i.e., \( \mathcal{U}_r \) or \( \mathcal{V}_r \) contains a “spiky” basis, we call that \( X \) is *coherent*.

Then we introduce an important result in matrix analysis [27], [30]. A simple extension of Lemma 3 from [27] achieves the following general result.

**Lemma 1.** Consider a rank-\( r \) matrix \( X \in \mathbb{R}^{n \times n} \). Let \( U_r \) and \( \bar{U}_r \) in \( \mathbb{R}^{n \times r} \) be orthonormal bases for \( r \)-dimensional subspaces \( \mathcal{U}_r = \text{span}(X) \) and \( \bar{\mathcal{U}}_r \), respectively. And let the SVD of

\[
U_r^T \bar{U}_r = L_L \cos(\Gamma) R_L^T, \quad \text{where} \quad L_L \in \mathbb{R}^{r \times r} \text{ and } R_L \in \mathbb{R}^{r \times r}
\]

are orthogonal matrices, \( \Gamma = \text{diag}\{\gamma_1, \ldots, \gamma_r\} \in \mathbb{R}^{r \times r} \) is a diagonal matrix, which contains the principal angles between \( \mathcal{U}_r \) and \( \bar{\mathcal{U}}_r \), with \( \pi/2 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_r \geq 0 \). The diagonal matrix \( \cos(\Gamma) \) is defined as

\[
\cos(\Gamma) \triangleq \text{diag}\{\cos \gamma_1, \cos \gamma_2, \ldots, \cos \gamma_r\} \in \mathbb{R}^{r \times r},
\]

and \( \sin(\Gamma) \in \mathbb{R}^{r \times r} \) is defined likewise. Then, there exist \( U'_r, U''_r \in \mathbb{R}^{n \times r} \), and \( U''_{n-2r} \in \mathbb{R}^{n \times (n-2r)} \) such that

\[
B_L = \begin{bmatrix} U_r & U'_r & U''_{n-2r} \end{bmatrix} \begin{bmatrix} L_L & L_L & I_{n-2r} \end{bmatrix} \in \mathbb{R}^{n \times n},
\]

\[
\bar{B}_L = \begin{bmatrix} \bar{U}_r & \bar{U}'_r & \bar{U}''_{n-2r} \end{bmatrix} \begin{bmatrix} R_L & R_L & I_{n-2r} \end{bmatrix} \in \mathbb{R}^{n \times n},
\]

are orthonormal bases for \( \mathbb{R}^n \). Furthermore, we have

\[
B_L^T \bar{B}_L = \begin{bmatrix} \cos(\Gamma) & \sin(\Gamma) \\ -\sin(\Gamma) & \cos(\Gamma) \end{bmatrix} I_{n-2r}.
\]

For \( r \)-dimensional subspaces \( \mathcal{V}_r = \text{span}(X^T) \) and \( \bar{\mathcal{V}}_r \), let \( V_r \) and \( \bar{V}_r \in \mathbb{R}^{r \times r} \) be orthonormal bases for \( \mathcal{V}_r \) and \( \bar{\mathcal{V}}_r \), respectively. Let the SVD of \( V_r^T \bar{V}_r = L_R \cos(\Psi) \bar{R}_R^T \) with orthogonal matrices \( L_R, \bar{R}_R \in \mathbb{R}^{r \times r} \) and diagonal matrix \( \Psi \in \mathbb{R}^{r \times r} \), we use the same way to construct the orthonormal bases

\[
B_R = \begin{bmatrix} V_r & V'_r & V''_{n-2r} \end{bmatrix} \begin{bmatrix} L_R & L_R & I_{n-2r} \end{bmatrix} \in \mathbb{R}^{n \times n},
\]

\[
\bar{B}_R = \begin{bmatrix} \bar{V}_r & \bar{V}'_r & \bar{V}''_{n-2r} \end{bmatrix} \begin{bmatrix} R_R & R_R & I_{n-2r} \end{bmatrix} \in \mathbb{R}^{n \times n},
\]

such that

\[
B_R^T \bar{B}_R = \begin{bmatrix} \cos(\Psi) & \sin(\Psi) \\ -\sin(\Psi) & \cos(\Psi) \end{bmatrix} I_{n-2r}.
\]

Similarly, the diagonal matrix \( \Psi = \text{diag}\{\eta_1, \ldots, \eta_r\} \in \mathbb{R}^{r \times r} \) contains the principal angles between \( \mathcal{V}_r \) and \( \bar{\mathcal{V}}_r \) in a non-increasing order.

Define the leverage scores of subspace \( \mathcal{U}_r = \text{span}(\{U_r, \bar{U}_r\}) \) and \( \bar{\mathcal{V}}_r = \text{span}(\{V_r, \bar{V}_r\}) \) as follows

\[
\mu_i \triangleq \mu_i(\mathcal{U}_r), \quad i = 1, 2, \ldots, n,
\]

\[
\nu_j \triangleq \nu_j(\bar{\mathcal{V}}_r), \quad j = 1, 2, \ldots, n.
\]

Under the assumptions of Lemma 1, we introduce some notations for conciseness

\[
A_{cc} = L_L \cos(\Gamma) \bar{R}_L^T \bar{R}_R \cos(\Psi) L_R^T,
\]

\[
A_{cs} = L_L \cos(\Gamma) \bar{R}_L^T \bar{R}_R \sin(\Psi) L_R^T,
\]

\[
A_{sc} = L_L \sin(\Gamma) \bar{R}_L^T \bar{R}_R \cos(\Psi) L_R^T,
\]

\[
A_{ss} = L_L \sin(\Gamma) \bar{R}_L^T \bar{R}_R \sin(\Psi) L_R^T.
\]
By slightly modifying Theorem 2 in [8] with sampling operator \( R_p(\cdot) \), we have the following result for standard matrix completion [1].

**Proposition 1 (Theorem 2, [3]).** Let \( X^* \in \mathbb{R}^{n \times n} \) be a rank-\( r \) matrix, and \( Y = R_p(X^*) \in \mathbb{R}^{n \times n} \) denote the matrix of measurements. Let \( \mu_i \) and \( \nu_j \) be the leverage scores as defined in Definition[7]. If

\[
1 \geq p_{ij} \gtrsim \frac{(\mu_i + \nu_j) r \log^2 n}{n},
\]

for all \( i, j = 1, \ldots, n \), then with high probability, \( X^* \) is the unique solution for program (2).

From the results of Proposition[1] we can conclude that we need \( O(r n \log^2 n) \) samples to accurately complete the matrix for an incoherent rank-\( r \) matrix.

### III. Performance Guarantees

In this section, we give the theoretical analysis for the proposed procedure (2). Assume that the prior subspace information of the \( r \)-dimensional column and row spaces of \( X^* \), denoted by \( U_r \) and \( V_r \), respectively, is available to us. By leveraging the prior subspace information, we construct \( \Phi = U_r V_r^T \), where \( U_r \in \mathbb{R}^{n \times r} \) and \( V_r \in \mathbb{R}^{n \times r} \) are the orthonormal bases for subspaces \( U_r \) and \( V_r \), respectively. By using the golfing scheme to construct the dual certificate [5], [8], we give the main theoretical results as follows.

**Theorem 1.** Let \( X^* \in \mathbb{R}^{n \times n} \) be a rank-\( r \) matrix with thin SVD \( X^* = U_r \Sigma_r V_r^T \) for \( U_r \in \mathbb{R}^{n \times r}, V_r \in \mathbb{R}^{n \times r} \) and \( \Sigma_r \in \mathbb{R}^{r \times r} \). Let the column and row subspaces of \( X^* \) be \( U_r = \text{span}(U_r) \) and \( V_r = \text{span}(V_r) \), respectively. Assume that the \( r \)-dimensional prior subspace information \( U_r \) about \( U_r \) and \( V_r \) about \( V_r \) is known beforehand. Let \( \Gamma \in \mathbb{R}^{r \times r} \) be diagonal whose entries are the principal angles between \( U_r \) and \( U_r \) and \( V_r \) about \( V_r \). Let \( \mu_i, \nu_j, \mu_i, \text{and} \nu_j \) be the leverage scores defined as before. Suppose \( \min_{i,j} p_{ij} \geq l^2 \), where \( l^{-1} \) is polynomial in \( n \).

\[
1 \geq p_{ij} \gtrsim \max\{\log(\alpha_1 n), 1\} \cdot \frac{(\mu_i + \nu_j) r \log n}{n} \cdot \max\left\{\left(\frac{\alpha_3 \beta}{1 - 2 \alpha_2}\right)^2, 1\right\},
\]

for all \( i, j = 1, \ldots, n \), and

\[
\alpha_2 < \frac{1}{2},
\]

then with high probability, we can achieve exact recovery of \( X^* \) by solving the program (2). The parameters are defined as

\[
\alpha_1 = \left\| I_r - \lambda A_{cc} \right\|_F, \quad \alpha_3 = \left\| I_r - \lambda A_{cc} \right\|_F + \left\| \lambda A_{cc} \right\| + \left\| \lambda A_{cs} \right\|,
\]

and

\[
\beta = 1 \lor \sqrt{2 \max_i \frac{\mu_i}{\mu_i} \lor \sqrt{2 \max_j \frac{\nu_j}{\nu_j}},
\]

where \( a \lor b = \max(a, b) \).

**Proof:** See Appendix A.  

**Remark 1.** If there is no prior information, i.e., \( \lambda = 0 \), then the proposed procedure (2) reduces to standard procedure (1).

In this case, we have \( \alpha_1 = \sqrt{r}, \alpha_2 = 0, \alpha_3 = 1 \) and \( \beta = \sqrt{2} \) by simple calculation [8]. According to Theorem[1] the bound of sample probability becomes

\[
1 \geq p_{ij} \gtrsim \frac{(\mu_i + \nu_j) r \log n \cdot \log(\sqrt{n})}{n}.
\]

This result means that for incoherent matrices, \( O(r n \log^2 n) \) samples are needed to complete the matrix correctly, which agrees with the result of standard matrix completion in Proposition[1].

**Remark 2 (Good prior subspace information).** When the prior subspace information is exactly the same as the original one, i.e., \( U_r = U_r \) and \( V_r = V_r \), we get \( \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0 \) and \( \beta = \sqrt{2} \) by setting \( \lambda = 1 \). In this case, the sample probability bound for our procedure (2) reduces to

\[
1 \geq p_{ij} \gtrsim \frac{(\mu_i + \nu_j) r \log n}{n}.
\]

It means that the proposed method can decrease the sample size of standard matrix completion (1) by a logarithmic factor. Besides, we come to a similar conclusion when the prior subspace information is very close to the original one.

**Remark 3 (Bad prior subspace information).** When the prior subspace information is different from the original subspace information, the largest principal angle between subspaces doesn’t approach to 0, which will lead to \( \alpha_1 = c > 0 \). In this case, the proposed procedure will have similar or even worse performance compared with standard procedure. So a better choice is to set \( \lambda = 0 \).

**Remark 4.** By following the technique in [6, Theorem 7], it’s straightforward to extend the proposed approach to matrix completion with bounded noise.

### A. On the choice of tradeoff parameter

The bound of sample size is influenced by parameters \( \alpha_1, \alpha_2, \alpha_3 \) and \( \beta \), but \( \alpha_1 \) is the deciding factor. From the definition, we have

\[
= \lambda^2 \left( \left\| A_{cc} \right\|_F^2 + \left\| A_{cc} \right\|_F^2 + \left\| A_{sc} \right\|_F^2 \right) - 2 \lambda \text{tr}(A_{cc}) + r.
\]

Choosing the optimal

\[
\lambda^* = \left\| A_{cc} \right\|_F^2 + \left\| A_{cc} \right\|_F + \left\| A_{sc} \right\|_F^2.
\]

We achieve the minimum of \( \alpha_1 \)

\[
\alpha_1^* = \sqrt{r \frac{\text{tr}(A_{cc})}{\left\| A_{cc} \right\|_F^2 + \left\| A_{cc} \right\|_F + \left\| A_{sc} \right\|_F^2}}.
\]
Accordingly, the optimal choice of weights $\lambda$ becomes

$$\lambda^* = \frac{\sum_{i=1}^r \cos^2 \gamma_i}{r - \sum_{i=1}^r \sin \gamma_i} \approx \frac{\sum_{i=1}^r \cos^2 \gamma_i}{r},$$

and we achieve the minimum

$$\alpha_1^* \approx \sqrt{r - \left(\sum_{i=1}^r \cos^2 \gamma_i\right)^2}.$$  

In this case, when the prior subspace information is very close to the original subspace, the choice of $\lambda^*$ is $\lambda^* \approx 1$ and then $\alpha_1^* \approx 0$. At the opposite extreme, when the subspace information is very poor, the best choice is $\lambda^* \approx 0$.

IV. NUMERICAL SIMULATIONS

In this section, we carry out simulations to verify the correctness of the theoretical guarantees.

Let rank-$r$ matrix $X^* \in \mathbb{R}^{n \times n}$ be the original matrix. We construct $X^*$ as follows: generate two independent Gaussian matrix $G_1$, $G_2 \in \mathbb{R}^{n \times r}$; let $\mathcal{U}_r$ (resp. $\mathcal{V}_r$) be the $r$-dimensional column orthogonal matrix of spaces $\mathcal{U}_r = \text{span}\{G_1\}$ (resp. $\mathcal{V}_r = \text{span}\{G_2\}$); then let $X^* = U_r \Sigma_r V_r^T$, where $\Sigma_r = \text{diag}\{1, \ldots, r, 0, \ldots, 0\}$. Next, we generate the perturbed matrix $\hat{X} = X^* + \sigma Z \in \mathbb{R}^{n \times n}$, where the entries of $Z$ obey a zero-mean Gaussian distribution with standard deviation 1 and $\sigma > 0$ is a constant. By taking truncated rank-$r$ SVD for $\hat{X}$, i.e., $\hat{X} = \hat{U}_r \hat{\Sigma}_r \hat{V}_r^T$, we set $\Phi = \hat{U}_r \hat{V}_r^T$.

We set $n = 32$, $r = 4$ and $\text{tol} = 10^{-3}$ for the simulation. For a specific sampling probability $p$, we give successful probability by averaging 50 trials. A trial is a successful one as long as the solution $\hat{X}$ satisfies

$$\|X^* - \hat{X}\|_F < \text{tol}.$$ 

Let $p$ increase from 0 to 1 with step $1/n$, then we can get the simulation results.

We consider two kinds of prior information: good prior information ($\sigma = 0.01$) and bad prior information ($\sigma = 0.1$). For any kind of prior information, we compare the performance of four methods: standard matrix completion (1), the proposed matrix completion via maximizing correlation (2), diagonal weighted matrix completion from (8) and weighted matrix completion from (27). For the weighted matrix completion approach, we set the optimal weights according to (27). Besides, we make simulations for (2) with different weights $\lambda$ to check the choice of $\lambda$.

The results of matrix completion under standard prior information ($\sigma = 0.01$) are shown in Fig. 1(a). Fig. 1(b) gives the comparisons for the four methods. Here, we set $\lambda = 1$. The results illustrate that the proposed approach has the best performance, which performs a little better than weighted matrix completion. Although diagonal weighted matrix completion has worse performance than the proposed approach and the weighted approach, it performs much better than standard matrix completion. In Fig. 1(b), we show the performance of the proposed approach with different weights $\lambda$. The results indicate that the integration of side information reduces the sampling probability of standard matrix completion ($\lambda = 0$). Furthermore, the larger the parameter $\lambda$, the better the perfor-
performance. The optimal $\lambda$ calculated by (8) is $\lambda^* = 0.9895$, which is very close to 1 and coincides with the simulation results.

In Fig. 2 we repeat the experiments under weaker prior information ($\sigma = 0.1$). In Fig. 2(a), the performance of the proposed, weighted and diagonal method deteriorates sharply compared with the plots in Fig. 1(a). From the result, we see that the proposed method slightly outperform standard matrix completion while the other two methods underperform the standard one. Fig. 2(b) also experiences a sharp degradation in terms of performance. In Fig. 2(b), all the results for different $\lambda$ coincide together, showing a slightly improvement than standard matrix completion. The optimal $\lambda$ calculated by (8) is $\lambda^* = 0.5750$, which is consistent with simulation results.

V. CONCLUSION

In this paper, we have proposed a novel method to complete a low-rank matrix from a small collections of its entries with the aid of prior subspace information. We have given the theoretical results for the proposed method. The results demonstrate that with reliable side information, the proposed method can decrease the number of measurements for standard matrix completion by a logarithmic factor. Numerical results have been provided to verify the theoretical results.

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For all $i,j$, let $\hat{\mu}_i$ and $\hat{\nu}_j$ be the leverage scores defined as before. Suppose that $\min_{i,j} p_{ij} \geq l^2$, where $l^{-1}$ is polynomial in $n$. If

$$1 \geq p_{ij} \geq \max\{\log (\alpha_1 n), 1\} \cdot \frac{\hat{\mu}_i + \hat{\nu}_j}{n} \log n \\
\cdot \max \left\{ \left( \frac{\alpha_3 \beta}{1 - 2\alpha_2} \right)^2, 1 \right\}$$

for all $i,j = 1, \ldots, n$, and

$$\alpha_2 < \frac{1}{2},$$

then with high probability, there exists $Y \in \text{range}(\mathcal{R}_p)$ satisfying

$$\|\mathcal{P}_\mathcal{T}\perp(\Phi) + \mathcal{P}_\mathcal{T}\perp(Y)\| < \frac{1}{2}.$$
and 
\[ \|U_r V_r^T - \lambda P_T(\Phi) - P_T(Y)\|_F \leq \frac{l}{2\sqrt{2}} \]
where 
\[ \alpha_1 = \left\| \begin{bmatrix} I_r - \lambda A_{cc} + \lambda A_{cs} & \lambda A_{cs} \\ 0 & 0 \end{bmatrix} \right\|_F, \]
\[ \alpha_2 = \|\lambda A_{ss}\|, \quad \alpha_3 = \|I_r - \lambda A_{cc}\| + \|\lambda A_{cc}\| + \|\lambda A_{cs}\|, \]
and 
\[ \beta = 1 \vee \sqrt{2 \max_i \frac{\mu_i}{\mu_i}} \vee \sqrt{2 \max_j \frac{\nu_j}{\nu_j}}. \]

Proof: See Appendix B

Now we are ready to prove Theorem 1. Consider any feasible solution \((X^* + Z)\) to problem (2) for non-zero matrix \(Z \in \{Z : \mathcal{R}_P(Z) = 0\}\). Let \(W \in \mathbb{R}^{n \times n}\) be a matrix satisfying \(W \in \{W : W^T U_r = 0, W V_r = 0, \|W\| \leq 1\}\) and \((\mathcal{I}, P_{T^+}(Z)) = \|P_{T^+}(Z)\|_s\). Then we have \(W = P_{T^+}(W)\) and \(U_r V_r^T + W \in \partial \|X^*\|_s\). According to the definition of subdifferential, for any non-zero matrix \(Z \in \ker(\mathcal{R}_P)\), we have

\[ \|X^* + Z\|_s - \lambda \langle \Phi, X^* + Z \rangle \geq \|X^*\|_s - \lambda \langle \Phi, X^* \rangle + \langle U_r V_r^T + W - \lambda \Phi, Z \rangle \] (12)

Let \(Y \in \text{range}(\mathcal{R}_P)\), then we have \((Y, Z) = 0\) and

\[ \|X^* + Z\|_s - \lambda \langle \Phi, X^* + Z \rangle \geq \|X^*\|_s - \lambda \langle \Phi, X^* \rangle + \langle U_r V_r^T + W - \lambda \Phi - Y, Z \rangle \]

Using Holder’s inequality and the properties of \(W\) yields

\[ \langle U_r V_r^T + W - \lambda \Phi - Y, Z \rangle = \langle U_r V_r^T - \lambda P_T(\Phi) - P_T(Y), P_T(Z) \rangle + \langle W - \lambda P_{T^+}(\Phi) - P_{T^+}(Y), P_{T^+}(Z) \rangle \]
\[ \geq - \|U_r V_r^T - \lambda P_T(\Phi) - P_T(Y)\|_F \|P_{T^+}(Z)\|_F + (1 - \|\lambda P_{T^+}(\Phi) + P_{T^+}(Y)\|_s) \|P_{T^+}(Z)\|_s \]
\[ \geq - \|U_r V_r^T - \lambda P_T(\Phi) - P_T(Y)\|_F \|P_{T^+}(Z)\|_F + (1 - \|\lambda P_{T^+}(\Phi) + P_{T^+}(Y)\|_s) \|P_{T^+}(Z)\|_F \]

The second inequality follows from \(\|P_{T^+}(Z)\|_s \geq \|P_{T^+}(Z)\|_F\).

Suppose the assumptions of Lemma 3 is satisfied, using Lemma 3 yields

\[ \langle U_r V_r^T + W - \lambda \Phi - Y, Z \rangle \geq \frac{l}{2\sqrt{2}} \|P_{T^+}(Z)\|_F + \frac{l}{2} \|P_{T^+}(Z)\|_F \geq 0, \]

where the last inequality applies Lemma 2 and \(\min_{i,j} p_{ij} \geq l^2\).

By incorporating (13) into (12), we have

\[ \|X^* + Z\|_s - \lambda \langle \Phi, X^* + Z \rangle \geq \|X^*\|_s - \lambda \langle \Phi, X^* \rangle \]

for any non-zero matrix \(Z \in \{Z : \mathcal{R}_P(Z) = 0\}\), which completes the proof.

APPENDIX B

PROOF OF LEMMA 3

In this section, we use golfing scheme to construct the dual certificate by following [5, 8].

We define the diagonal matrices \(M = \text{diag}\{\mu_1, \ldots, \mu_n\}\) and \(N = \text{diag}\{\nu_1, \ldots, \nu_n\}\). Then let us introduce the following norms which, respectively, measure the (weighted) largest entry and largest \(\ell_2\) norm of the rows or columns of a matrix [8].

Definition 2 (\(\mu(\infty)\) norm and \(\mu(\infty, 2)\) norm). For a rank-\(r\) matrix \(X \in \mathbb{R}^{n \times n}\), we set

\[ \|X\|_{\mu(\infty)} = \sqrt{\sup_{i,j} \left| \frac{X_{ij}}{\nu_i \nu_j} \right|} \]

where \(|A|_{\infty}\) returns the largest entry of matrix \(A\) in magnitude. Moreover, for a rank-\(r\) matrix \(X \in \mathbb{R}^{n \times n}\), we define

\[ \|X\|_{\mu(\infty, 2)} = \sqrt{\max_{i,j} \left| \frac{X_{ij}}{\nu_i \nu_j} \right|} \]

where \(a \vee b = \max\{a, b\}\) and \(\|X\|_{\mu(\infty, 2)}\) denotes the largest \(\ell_2\) norm of the rows of \(X\).

Before proving Lemma 3, let’s give some useful lemmas which will be used in the proof.

Lemma 4 (Lemma 9, [8]). For probabilities \(\{p_{ij}\} \subset (0, 1)\), consider the measurement operator \(\mathcal{R}_P(\cdot)\) defined in Eq. 3 and projection operator \(P_T\) defined in Eq. 11. Then, except with a probability of at most \(n^{-20}\),

\[ \|P_T - P_T R P_T(\cdot)\|_{F \to F} \leq \frac{1}{2}, \]

provided that

\[ \frac{(\mu_i + \nu_j) r \log n}{n} \leq p_{ij} \leq 1, \quad \forall i, j \in [1, n], \quad (14) \]

where \(\|A(\cdot)\|_{F \to F} = \sup\|X\|_{F \leq 1} \|A(X)\|_F\), and \((AB)(\cdot) = A(B(\cdot))\).

Lemma 5 (Lemma 10, [8]). Consider a fixed \(X \in \mathbb{R}^{n \times n}\). For some universality constant \(\Delta \geq 1\), if

\[ \frac{\Delta^2 (\mu_i + \nu_j) r \log n}{n} \leq p_{ij} \leq 1, \quad \forall i, j \in [1, n], \]

holds, then

\[ \|R_P - I\|_{\mu(\infty)} \leq \frac{\lambda}{\Delta} \left( \|X\|_{\mu(\infty)} + \|X\|_{\mu(\infty, 2)} \right), \]

except with a probability of at most \(n^{-20}\). Here, \(I(\cdot)\) is the identity operator.
Lemma 6 (Lemma 12, [3]). Consider a fixed matrix $X \in T \subset \mathbb{R}^{n \times n}$. Then except with a probability of at most $n^{-20}$, it holds that
\[
\| (P_T - P_T R_p P_T) (X)\|_{\mu(\infty)} \leq \frac{1}{2} \| X\|_{\mu(\infty)},
\]
as long as (14) holds.

Lemma 7 (Lemma 11, [3]). Consider a fixed matrix $X \in T \subset \mathbb{R}^{n \times n}$ (i.e., $P_T(X) = X$). Then except with a probability of at most $n^{-20}$, it holds that
\[
\| (P_T - P_T R_p P_T) (X)\|_{\mu(\infty,2)} \leq \frac{1}{2} \|X\|_{\mu(\infty)} + \frac{1}{2} \|X\|_{\mu(\infty,2)},
\]
as long as (14) holds.

Lemma 8. For $W_0 = U_r V_r^T - \lambda P_T (U_r V_r^T)$ and $P_T \perp (U_r V_r^T)$, it holds that
\[
W_0 \|_F = \alpha_1, \quad \| \lambda P_T \perp (U_r V_r^T) \| = \alpha_2,
\]
\[
\| W_0 \|_{\mu(\infty,2)} \leq \alpha_3 \beta, \quad \| W_0 \|_{\mu(\infty)} \leq \alpha_3 \beta,
\]
where
\[
\alpha_1 = \left\| \begin{bmatrix} I_r - \lambda A_{cc} & \lambda A_{cs} \\ \lambda A_{sc} & 0 \end{bmatrix} \right\|_F,
\]
\[
\alpha_2 = \| \lambda A_{sc} \|, \quad \alpha_3 = \| I_r - \lambda A_{cc} \| + \| \lambda A_{sc} \| + \| \lambda A_{cs} \|,
\]
and
\[
\beta = 1 \vee \sqrt{2 \max_i \frac{\mu_i}{\mu_j} \vee \sqrt{2 \max_j \frac{\mu_i}{\mu_j}}},
\]

**Proof:** See Appendix C.

In order to measure $X^*$, we use $K$ independent measurement operators $R_q(\cdot)$ instead of $R_p(\cdot)$, which means the probability $p_{ij}$ and $q_{ij}$ satisfies
\[
(1 - q_{ij})^K = 1 - p_{ij}, \quad i, j = 1, \ldots, n, \tag{15}
\]
for a given number $K$. Let $Y = Y_K$, then
\[
\| W_K \|_F = \| U_r V_r^T - \lambda P_T (\Phi) - P_T (Y) \|_F \leq 2^{-K} \| W_0 \|_F,
\]
except with a probability of at most $K n^{-20}$. Let $K \geq \max \left\{ \log \left( \frac{2\sqrt{q}}{l} \right), 1 \right\}$, where $l^{-1}$ is polynomial in $n$. Then except with a probability of at most
\[
K n^{-20} = O(\log(\alpha_1 n)) \cdot n^{-20} = o(n^{-19}).
\]
From the triangle inequality, we have
\[
\| \lambda P_T \perp (\Phi) + P_T \perp (Y) \| \leq \| P_T \perp (Y) \| + \| \lambda P_T \perp (\Phi) \|.
\]
From Lemma 5 except with a probability of at most $K n^{-20} = O(\log(\alpha_1 n))$, we have
\[
\| P_T \perp (Y) \| \leq \sum_{j=1}^K \| P_T \perp R_q(W_{j-1}) \|
\]
\[
= \sum_{j=1}^K \| P_T \perp (R_q(W_{j-1}) - W_{j-1}) \|
\]
\[
\leq \sum_{j=1}^K \| (R_q - \mathbb{I}) W_{j-1} \|
\]
\[
\leq \frac{1}{\Delta} \sum_{j=1}^K (\| W_{j-1} \|_{\mu(\infty)} + \| W_{j-1} \|_{\mu(\infty,2)}),
\]
except as long as
\[
\Delta^2 (\mu_i + \nu_j) r \log n \leq q_{ij} \leq 1, \quad \forall i, j \in [1, n],
\]
and
\[
\Delta \geq 1.
\]
The second line holds since $W_{j-1} = P_T (W_{j-1})$ and the third line holds since $\| P_T (X) \| \leq \| X \|$ for any $X \in \mathbb{R}^{n \times n}$. Using Lemma 4 leads to
\[
\| W_{j-1} \|_{\mu(\infty)} = \| (P_T - P_T R_q P_T) (W_{j-2}) \|_{\mu(\infty)}
\]
\[
\leq \frac{1}{2} \| W_{j-2} \|_{\mu(\infty)},
\]
except with a probability of at most $n^{-20}$, as long as
\[
(\mu_i + \nu_j) r \log n \leq q_{ij} \leq 1, \quad \forall i, j \in [1, n].
\]
By iteration, we obtain
\[
\| W_{K} \|_F \leq 2^{-K} \| W_0 \|_F,
\]
except with a probability of at most $K n^{-20}$.
except with a probability of at most \( o(n^{-19}) \), since \( j \leq K \). By using Lemma \( \ref{lem:prob_bound} \), we obtain

\[
\| W_{j-1} \|_{\mu(\infty,2)} = \| (P_T - P_T R_q P_T) (W_{j-2}) \|_{\mu(\infty,2)} \\
\leq \frac{1}{2} \| W_{j-2} \|_{\mu(\infty)} + \frac{1}{2} \| W_{j-2} \|_{\mu(\infty,2)},
\]

\[
\leq \frac{j-1}{2j-1} \| W_0 \|_{\mu(\infty)} + \frac{1}{2j-1} \| W_0 \|_{\mu(\infty,2)},
\]

except with a probability of at most \( o(n^{-19}) \) due to the fact \( j \leq K \). It follows that

\[
\| P_T(Y) \| \leq \frac{1}{\Delta} \sum_{j=1}^{K} \frac{j}{2j-1} \| W_0 \|_{\mu(\infty)} \\
+ \frac{1}{\Delta} \sum_{j=1}^{K} \frac{1}{2j-1} \| W_0 \|_{\mu(\infty,2)} \\
< \frac{1}{\Delta} \left( 4 \| W_0 \|_{\mu(\infty)} + 2 \| W_0 \|_{\mu(\infty,2)} \right),
\]

where \( \sum_{j=1}^{K} \frac{j}{2j-1} < 4 \) and \( \sum_{j=1}^{K} \frac{1}{2j-1} < 2 \) for finite \( K \).

By using Lemma \( \ref{lem:prob_bound} \) except with a probability of at most \( o(n^{-19}) \), we have

\[
\| P_T(Y) \| + \| \lambda P_T(\Phi) \| < \frac{4 \xi_1 + 2 \xi_2}{\Delta} + \alpha_2 \\
\leq \frac{6 \alpha_3 \beta}{\Delta} + \alpha_2 = \frac{4}{2},
\]

provided that

\[
\Delta^2 \left( \mu_i + \nu_j \right) r \log n \leq q_{ij} \leq 1, \quad \forall i, j \in [1, n].
\]

where we set

\[
\Delta = \max \left\{ \left( \frac{12 \alpha_3 \beta}{1 - 2 \alpha_2} \right)^2, 1 \right\}.
\]

So we conclude that if

\[
\max \left\{ \left( \frac{\alpha_3 \beta}{1 - 2 \alpha_2} \right)^2, 1 \right\} \cdot \frac{(\mu_i + \nu_j) r \log n}{n} \leq q_{ij} \leq 1,
\]

we have

\[
\| \lambda P_T(\Phi) + P_T(Y) \| < \frac{1}{2}
\]

with high probability. Finally, according to \( \ref{eq:prob_bound} \), if \( \{q_{ij}\} \) are small enough, which means \( n \) is large enough, we have

\[
p_{ij} = 1 - (1 - q_{ij})^K \geq K q_{ij} \\
\geq \max \left\{ \log \left( \frac{2 \sqrt{2 \alpha_1}}{\lambda} \right), 1 \right\} \cdot \frac{(\mu_i + \nu_j) r \log n}{n} \\
\geq \max \left\{ \left( \frac{\alpha_3 \beta}{1 - 2 \alpha_2} \right)^2, 1 \right\} \\
\geq \max \left\{ \log (\alpha_1 n), 1 \right\} \cdot \frac{(\mu_i + \nu_j) r \log n}{n}.
\]

\[
\begin{align*}
\text{APPENDIX C} & \quad \text{PROOF OF LEMMA} \ref{lem:prob_bound} \\
\text{In this section, we will use principal angles between subspaces to bound } & \| W_0 \|_{\mu(\infty)} \text{ and } \| W_0 \|_{\mu(\infty,2)} \text{ and } \| P_T(\tilde{U}_r \tilde{V}_r^T) \|, \text{ where } W_0 = U_r V_r^T - \lambda P_T(\tilde{U}_r \tilde{V}_r^T). \text{ Before that, we give an alternative expression of } W_0 \text{ first. For } \text{convenience, we review the definition}\n\end{align*}
\]

\[
A_{cc} = L_L \cos(\Gamma) R_L^T R_R \cos(H) L_R^T, \\
A_{cs} = L_L \cos(\Gamma) R_L^T R_R \sin(H) L_R^T, \\
A_{sc} = L_L \sin(\Gamma) R_L^T R_R \cos(H) L_R^T, \\
A_{ss} = L_L \sin(\Gamma) R_L^T R_R \sin(H) L_R^T,
\]

and define the following matrices

\[
C_L \triangleq \begin{bmatrix} L_L & L_L & I_{n-2r} \end{bmatrix}, \quad D_L \triangleq \begin{bmatrix} R_L & R_L & I_{n-2r} \end{bmatrix}, \\
C_R \triangleq \begin{bmatrix} L_R & L_R & I_{n-2r} \end{bmatrix}, \quad D_R \triangleq \begin{bmatrix} R_R & R_R & I_{n-2r} \end{bmatrix}.
\]

We know \( U_r^T \tilde{U}_r = L_L \cos(\Gamma) R_L^T \) and \( V_r^T \tilde{V}_r = L_R \cos(H) R_R^T \). Besides, Lemma \( \ref{lem:prob_bound} \) immediately implies that

\[
U_r = B_L \begin{bmatrix} L_L^T & 0 \end{bmatrix}, \quad \tilde{U}_r = B_L \begin{bmatrix} \cos(\Gamma) R_L^T & - \sin(\Gamma) R_R^T \end{bmatrix}, \\
V_r = B_R \begin{bmatrix} L_R^T & 0 \end{bmatrix}, \quad \tilde{V}_r = B_R \begin{bmatrix} \cos(H) R_R^T & - \sin(H) R_R^T \end{bmatrix}.
\]

By incorporating the above expressions, we have

\[
\begin{align*}
\mathcal{P}_T(\tilde{U}_r \tilde{V}_r^T) &= U_r U_r^T \tilde{U}_r \tilde{V}_r^T + \tilde{U}_r \tilde{V}_r^T V_r^T \\
&= B_C \begin{bmatrix} A_{cc} & -A_{cs} \end{bmatrix} C_B^T R_B^T, \\
&= B_C \begin{bmatrix} I_r & -\lambda A_{cc} \end{bmatrix} C_B^T R_B^T.
\end{align*}
\]

So we have

\[
\begin{align*}
W_0 &= U_r V_r^T - \lambda \mathcal{P}_T(\tilde{U}_r \tilde{V}_r^T) \\
&= B_C \begin{bmatrix} I_r & -\lambda A_{cc} \end{bmatrix} C_B^T R_B^T.
\end{align*}
\]
1) The new expression of $\|W_0\|_F$. Expressing $W_0$ by the principal angles \([17]\) yields

$$
\|W_0\|_F = \left\| U_r V_r^T - \lambda P_T(U_r V_r^T) \right\|_F = \left\| B_L C_L^T \begin{bmatrix} I_r - \lambda A_{cc} & \lambda A_{cs} \\ \lambda A_{cs} & 0_r \end{bmatrix} 0_{n-2r} \right\|_F \right\|_F 
$$

where the third equality holds due to the rotational invariance.

2) The bound of $|W_0|_{\mu(\infty)}$. By using \([17]\), the definition of $B_L$ and $B_R$ and the triangle inequality, we have

$$
\|W_0\|_{\mu(\infty)} \leq \left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} U_r \cdot (I_r - \lambda A_{cc}) \cdot V_r^T \left( \frac{r N}{n} \right)^{-\frac{1}{2}} \right\|_\infty + \left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} U_r' \cdot \lambda A_{cs} \cdot V_r^T \left( \frac{r N}{n} \right)^{-\frac{1}{2}} \right\|_\infty + \left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} U_r \cdot \lambda A_{cs} \cdot V_r^T \left( \frac{r N}{n} \right)^{-\frac{1}{2}} \right\|_\infty.
$$

By using

$$
\|XY\|_\infty \leq \|X\|_{(\infty,2)} \|Y\|_{(\infty,2)}
$$

and

$$
\|XY\|_{(\infty,2)} \leq \|X\|_{(\infty,2)} \|Y\|, \tag{19}
$$

we have

$$
\|W_0\|_{\mu(\infty)} \leq \left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} U_r \right\|_{(\infty,2)} \left\| I_r - \lambda A_{cc} \right\| + \left\| \left( \frac{r N}{n} \right)^{-\frac{1}{2}} V_r \right\|_{(\infty,2)} + \left\| \lambda A_{cs} \cdot V_r \right\|_{(\infty,2)}.
$$

Then we can obtain

$$
\|W_0\|_{\mu(\infty)} \leq \left\| I_r - \lambda A_{cc} \right\| + \left\| \lambda A_{cs} \right\| \sqrt{2 \max_i \frac{\hat{\mu}_i}{\mu_i}} + \left\| \lambda A_{cs} \right\| \sqrt{2 \max_j \frac{\hat{\nu}_j}{\nu_j}}.
$$

where the first inequality applies the following properties

$$
\left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} U_r \right\|_{(\infty,2)} = 1, \left\| \left( \frac{r N}{n} \right)^{-\frac{1}{2}} V_r \right\|_{(\infty,2)} = 1,
$$

$$
\left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} U_r' \right\|_{(\infty,2)} \leq 2 \max_i \frac{\hat{\mu}_i}{\mu_i}, \left\| \left( \frac{r N}{n} \right)^{-\frac{1}{2}} V_r' \right\|_{(\infty,2)} \leq 2 \max_j \frac{\hat{\nu}_j}{\nu_j}, \tag{20}
$$

which are obtained by standard calculation \([27]\).  

3) The bound of $|W_0|_{\mu(\infty,2)}$. We recall the definition of $W_0|_{\mu(\infty,2)}$ as follows

$$
\|W_0\|_{\mu(\infty,2)} = \left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} W_0 \right\|_{(\infty,2)} \left\| \left( \frac{r N}{n} \right)^{-\frac{1}{2}} W_0^T \right\|_{(\infty,2)}.
$$

Now we bound $\left\| \left( \frac{r M}{n} \right)^{-1/2} W_0 \right\|_{(\infty,2)}$ first

$$
\left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} W_0 \right\|_{(\infty,2)} = \left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} \left[ B_L C_L^T \begin{bmatrix} I_r - \lambda A_{cc} & \lambda A_{cs} \\ \lambda A_{cs} & 0_r \end{bmatrix} 0_{n-2r} \right] \right\|_{(\infty,2)},
$$

where the above equality uses the rotational invariance. Using triangle inequality yields

$$
\left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} W_0 \right\|_{(\infty,2)} \leq \left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} U_r \cdot (I_r - \lambda A_{cc}) \right\|_{(\infty,2)} + \left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} U_r' \cdot \lambda A_{cs} \right\|_{(\infty,2)} + \left\| \left( \frac{r M}{n} \right)^{-\frac{1}{2}} U_r \cdot \lambda A_{cs} \right\|_{(\infty,2)}.
$$
By using (19) and (20), we obtain
\[
\left\| \left( \frac{rM}{n} \right)^{-\frac{1}{2}} W_0 \right\|_{(\infty,2)} \leq \left\| \left( \frac{rM}{n} \right)^{-\frac{1}{2}} U_r \right\|_{(\infty,2)} \left\| I_r - \lambda A_{cc} \right\| \\
+ \left\| \left( \frac{rM}{n} \right)^{-\frac{1}{2}} U'_r \right\|_{(\infty,2)} \left\| \lambda A_{sc} \right\| \\
+ \left\| \left( \frac{rM}{n} \right)^{-\frac{1}{2}} U_r \right\|_{(\infty,2)} \left\| \lambda A_{cs} \right\|
\]
\[
\leq \left\| I_r - \lambda A_{cc} \right\| + \lambda A_{sc} \sqrt{\max_i \frac{\hat{\mu}_i}{\mu_i}} + \lambda A_{cs} .
\]
Similarly, we have the other bound
\[
\left\| \left( \frac{rN}{n} \right)^{-\frac{1}{2}} W_0^T \right\|_{(\infty,2)} \leq \left\| I_r - \lambda A_{cc} \right\| + \left\| \lambda A_{sc} \right\| \\
+ \left\| \lambda A_{cs} \right\| \sqrt{\max_i \frac{\bar{\nu}_i}{\nu_i}}
\]
Therefore, we obtain
\[
\left\| W_0 \right\|_{(\infty,2)} \leq \left( \left\| I_r - \lambda A_{cc} \right\| + \left\| \lambda A_{sc} \right\| + \left\| \lambda A_{cs} \right\| \right) \cdot
\left( 1 \lor \sqrt{\max_i \frac{\hat{\mu}_i}{\mu_i}} \lor \sqrt{\max_j \frac{\bar{\nu}_j}{\nu_j}} \right).
\]
4) The new expression of \( \left\| \lambda P_{T^\perp}(\Phi) \right\| \). By applying rotational invariance, we obtain
\[
\left\| P_{T^\perp}(\vec{U}_r \vec{V}_r^T) \right\| = \left\| \vec{U}_r \vec{V}_r^T - P_{T}(\vec{U}_r \vec{V}_r^T) \right\|
\]
\[
= B_L C_L^T \begin{bmatrix} A_{cc} & -A_{cs} \\ -A_{sc} & A_{ss} \end{bmatrix}_{0_{n-2r}} B_R C_L^T \\
- B_L C_L^T \begin{bmatrix} A_{cc} & -A_{cs} \\ -A_{sc} & A_{ss} \end{bmatrix}_{0_{n-2r}} B_R C_L^T \\
= B_L C_L^T \begin{bmatrix} 0_r & 0_r \\ 0_r & A_{ss} \end{bmatrix}_{0_{n-2r}} B_R C_L^T \\
= \left\| A_{ss} \right\| .
\]