The Mass of the Neutrinos

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Abstract

In the theory of the Dirac equation and in the standard model, the neutrino is massless. Both these theories use Lorentz invariance. In modern approaches however, spacetime is no longer smooth, and this modifies special relativity. We show how such a modification throws up the mass of the neutrino.

1 Introduction

Today we know that neutrinos are described by the two component equation. Such an equation has a long history. It was proposed by Hermann Weyl as long ago as 1929. He argued that it would represent a massless Fermion. However the suggestion was soon rejected because such a particle would not be invariant under the parity transformation. Later experiments showed the non conservation of parity in beta decay, as suggested by Yang and Lee. Salam and Landau then proposed that neutrinos obey the Weyl equation, discarded nearly thirty years earlier. The Weyl equation itself is given by

\[ i\hbar \partial_t \phi = c\vec{\sigma} \cdot \vec{p}\phi \]  
(1)

This equation brings out particles with definite helicity states, and satisfies the condition for neutrinos. Though the Weyl equation differs from the four component Dirac equation, it is well known that the massless Dirac equation too can represent a neutrino, provided an extra constraint is satisfied. As we will briefly see below, the solutions of the Dirac equation preserve parity,
while the constraint removes two of the four components of the Dirac solution, which thus makes the solution non invariant under parity. We can examine this a little more carefully, by starting with the Dirac equation for a massless particle

$$\gamma_\mu \partial^\mu \psi(x) = 0$$

(2)

which in Hamiltonian form reads

$$i \partial_t \psi(x) = -i \gamma^0 \vec{\gamma} \cdot \nabla \psi(x)$$

(3)

In the usual representation

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma_5 = -i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

(4)

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

(5)

We also need

$$\sum = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = i \gamma_5 \gamma^0 \vec{\gamma}$$

(6)

The Hamiltonian can now be written in the form

$$H = i \gamma_5 \sum \cdot \vec{p} = i \gamma_5 [\vec{p} | s(p)]$$

(7)

We can see from (7) that the eigenfunctions of $H$ and $s(p)$ are eigenfunctions of $i \gamma_5$. The four linearly independent solutions of

$$Hu = p_0 u$$

with the $z$ axis as the direction of $\vec{p}$ are given by:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(8)

The eigenvalue of $i \gamma_5$ for these solutions are summarized in Table 1.

The first two spinors in (8) represent positive energy solutions while the last two represent negative energy solutions. Their helicities are given by respectively $+1, -1, +1$ and $-1$. 
As can be seen, the eigenvalue of \( \gamma_5 \) for a positive energy solution is the same as that of the helicity operator. For a negative energy solution on the other hand, the eigenvalue of \( \gamma_5 \) is opposite that of the helicity operator.

As is well known a neutrino is described by a two-component equation, the plane wave solutions of which have the property that for \( p_0 = +|p| \) the helicity is \(-1\), and for \( p_0 = -|p| \) the helicity is \(+1\). For this we require that the plane wave solutions of (3) need also to satisfy:

\[
\psi = -\gamma_5 \psi
\]

This constraint is invariant for proper Lorentz transformations. To put it another way, if \( \psi \) is a four-component spinor satisfying (3), the spinor \( \psi_n \) defined by

\[
\phi = \frac{1}{2} (1 - \gamma_5) \psi
\]

satisfies the condition (9). If

\[
\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}
\]

\[
\gamma_5 = i \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
\]

then the spinor \( \phi \) is essentially a two-component quantity since the projection operator \( \frac{1}{2} (1 - \gamma_5) \) annihilates the two lower components. The two-component spinor \( \phi \) satisfies the equation

\[
-\sigma \cdot \mathbf{p} \phi = p_0 \phi
\]

As we can see if we multiply (3) by \( \frac{1}{2} (1 - \gamma_5) \).

However the SuperKamiokande experiment [2] clearly showed that the neutrino has a small mass. On the other hand according to the standard model
the neutrino should be massless [3]. So in recent years there has been much work on going beyond the standard model in order to explain amongst other things, the neutrino mass. Currently, the dominant view is that the neutrino mass oscillation arises from the MSW mechanism.

We would now like to deduce the neutrino mass from a slightly different perspective, and argue that it is a result of a fuzzy spacetime structure, resulting in a modified Dirac equation.

## 2 Modified Dirac Equation

Our starting point is the fact that, if there is a minimum fundamental length $l$, the usual Quantum Mechanical commutation relations get modified as shown by Snyder a long time ago [4, 5, 6, 7]. These relations are now replaced by

$$[x, p] = \hbar' = \hbar[1 + \left(\frac{l}{\hbar}\right)^2 p^2] \text{ etc}$$

(Cf. also ref.[8]). (14) shows that effectively $\hbar$ is replaced by $\hbar'$. So,

$$E = \left[m^2 + p^2(1 + l^2 p^2)^{-2}\right]^{\frac{1}{2}}$$

or, the energy-momentum relation leading to the Klein-Gordon Hamiltonian being modified to the so called Snyder-Sidharth Hamiltonian [9],

$$E^2 = m^2 + p^2 + \propto l^2 p^4,$$

(15)

neglecting higher order terms and using natural units, $c = 1 = \hbar$.

For Fermions the analysis can be more detailed, in terms of Wilson lattices [10]. The free Hamiltonian now describes a collection of harmonic fermionic oscillators in momentum space. Assuming periodic boundary conditions in all three directions of a cube of dimension $L^3$, the allowed momentum components are

$$q \equiv \left\{q_k = \frac{2\pi}{L} v_k; k = 1, 2, 3 \right\}, \quad 0 \leq v_k \leq L - 1$$

(16)

finally leads to

$$E_q = \pm \left(m^2 + \sum_{k=1}^{3} a^{-2} \sin^2 q_k\right)^{1/2}$$

(17)
where \( a = l \) is the length of the lattice, this being the desired result. (17) shows that \( \alpha \) in (15) is positive. We have used the above analysis more to indicate that in the Fermionic case, the sign of \( \alpha \) is positive. A rigid lattice structure imposes restrictions on the spacetime - for example homogeneity and isotropy. Such restrictions are not demanded by the author’s model of fuzzy spacetime, and we use the lattice model more as a computational device (Cf. ref.[6]). This leads to a modification of the Dirac and Klein-Gordon equations at ultra high energies (Cf.ref.[6,11,12]). It may be remarked that proposals like equation (15) have been considered by several authors based on phenomenological considerations (Cf. refs.[13]-[22]). Our approach however, has been fundamental rather than phenomenological.

Once we consider a discrete spacetime structure, the energy momentum relation, gets modified [5,10] and we have,

\[
E^2 - p^2 - m^2 - \propto l^2 p^4 = 0 \tag{18}
\]

\( l \) being the minimum length interval, which could be the Planck length or more generally the Compton length. Let us now consider the Dirac equation

\[
\{\gamma^\mu p_\mu - m\} \psi \equiv \{\gamma^\circ p^\circ + \Gamma\} \psi = 0 \tag{19}
\]

If we include the extra effect shown in (18) we get

\[
\left(\gamma^\circ p^\circ + \Gamma + \beta l p^2\right) \psi = 0 \tag{20}
\]

\( \beta \) being a suitable matrix.

Multiplying (20) by the operator

\[
\left(\gamma^\circ p^\circ - \Gamma - \beta l p^2\right)
\]

on the left we get

\[
p_0^2 - \left(\Gamma \Gamma + \{\Gamma \beta + \beta \Gamma\} + \beta^2 l^2 p^4\right) \psi = 0 \tag{21}
\]

If (21), as in the usual theory, has to represent (18), then we require that the matrix \( \beta \) satisfy

\[
\Gamma \beta + \beta \Gamma = 0, \quad \beta^2 = 1 \tag{22}
\]

From the properties of the Dirac matrices [23] it follows that (22) is satisfied if

\[
\beta = \gamma^5 \tag{23}
\]
Using (23) in (20), the modified Dirac equation finally becomes

\[ \left\{ \gamma^0 p^0 + \Gamma + i\omega \gamma^5 l p^2 \right\} \psi = 0 \quad (24) \]

owing to the fact that we have (25)

\[ P\gamma^5 = -\gamma^5 P \quad (25) \]

It follows that the so called Dirac-Sidharth equation (24) is not invariant under reflections. This is a result which is to be expected because the correction to the usual energy momentum relation, as shown in (18) arises when \( l \) is of the order of the Compton wavelength. The usual Dirac four spinor \( \left( \begin{array}{c} \phi \\ \chi \end{array} \right) \) has the so called positive energy (or large) components \( \phi \) and the negative energy (or small) components \( \chi \). However, when we approach the Compton wavelength, that is as

\[ p \rightarrow mc \]

the roles are reversed and it is the \( \chi \) components which predominate. Moreover the \( \chi \) two spinor behaves under reflection as (23)

\[ \chi \rightarrow -\chi \]

In any case, this too provides an experimental test. We can also see that due to the modified Dirac equation (24), there is no additional effect on the anomalous gyromagnetic ratio. This is because, in the usual equation from which the magnetic moment is determined (24) viz.,

\[ \frac{d\vec{S}}{dt} = -\frac{e}{\mu c} \vec{B} \times \vec{S}, \]

where \( \vec{S} = \hbar \sum /2 \) is the electron spin operator, there is now an extra term

\[ \left[ \gamma^5, \sum \right] \quad (26) \]

However the expression (26) vanishes by the property of the Dirac matrices.
3  Massive Neutrinos

Taking units $c = 1 = \hbar$ again, the so called Dirac-Sidharth equation can be written as

$$\psi = (D + \nu \gamma^2 \gamma_5) \psi = (27)$$

In (27) $D$ represents the usual Dirac operator given in (24) and there is the extra term following it. Equation (27) is valid both for a massive and a massless Dirac particle [25]. We can see that, as the Hamiltonian is given by (Cf. Section 1).

$$H = \gamma^5 \sum \frac{\vec{p}}{|\vec{p}| s(\vec{p})} = (28)$$

the extra term in (27) represents a mass term. In other words due to the Snyder-Sidharth-Hamiltonian (15), the Dirac particle acquires an additional mass. However what is very interesting is that the extra term is not invariant under parity owing to the presence of $\gamma_5$. Indeed as we know from the theory of Dirac matrices [23]

$$P \gamma_5 = -P \gamma_5$$

(29)

Let us now consider the case of a massless Dirac particle. We can see that in this case (27) represents the neutrino with a mass, there now being no need for the extra constraint (9) required for massless neutrinos. Equation (27) automatically gives a parity non-conserving particle. In other words a massless particle, satisfying the Dirac equation in the usual theory now acquires a mass.

4  Remarks

1. We can now ask what does (27) represent if to start with the particle has a mass $m$? As can be seen it now acquires an extra mass, at ultra high energies. However this extra mass does not conserve parity. So, usual particles at very high energies become unstable, due to this additional contribution: the mass now has two parts, the usual mass $m$ that is invariant under reflection, but also a parity non-conserving part. We could also think of it as follows: a usual spin half particle of mass $m$, at ultra high energies shows up as two other particles with slightly different masses. It appears that this could be related to particles like the $K_0$ meson.

2. The standard model or the Dirac equation are strictly in accordance with Special Relativity and the neutrino mass is accordingly zero. Now however,
we have the SS-Hamiltonian (15) which is a very high energy correction to the usual relativistic dispersion relation. It is this modification or extra term that throws up the massive neutrino breaking the Lorentz symmetry.

3. We can look upon this in a different way [26]. The usual Dirac equation which is invariant under the Lorentz transformation including the improper parity operation is given by the representation

\[ D^{(\frac{3}{2})} \oplus D^{(0\frac{1}{2})} \]  

(30)

The solutions which are according to the two-component representation \( D^{(\frac{3}{2})} \) or \( D^{(0\frac{1}{2})} \) are not invariant under parity. However the combined four-component solution Dirac spinor in (30) is invariant under a Lorentz transformation plus the parity transformation. When we introduce the extra term in the modified Dirac equation, this term spoils the invariance under parity. If we write the usual Dirac spinor as

\[ \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \]  

(31)

then it is known as we saw [27] that at very high energies the upper or positive two spinor \( \phi \) is invariant under parity, but not the lower. That is under the parity operator \( P \)

\[ \phi \rightarrow \phi, \chi \rightarrow \chi \]  

(32)

Now \( \phi \) and \( \chi \) each are given by the \( D^{(\frac{3}{2})} \) and \( D^{(0\frac{1}{2})} \) representations. Under space reflections, they go into each other and it is for this reason that \( \psi \) given by (30) is invariant.

If we still consider this solution as an approximation to (27) also, the result of the parity operator \( P \) would now be to interchange the behavior \( \phi \) and \( \chi \) under \( P \).

5 Appendix

It is well known that at ultra high energies the massive Dirac equation goes over the massless Dirac equation because of the dominance of the kinetic energy term. In this case there is the well known CINI transformation which reduces the Dirac Hamiltonian to the form

\[ H = \frac{\vec{\chi} \cdot \vec{p}}{|\vec{p}|} E(\vec{p}) \]
where

\[ E(\vec{p}) = c \left[ \vec{p}^2 + m^2 c^2 \right]^{\frac{1}{2}} \]  

(33)

The Dirac equation now throws up the positive and negative helicity states which are described by two separate two component equations. The implication of the extra term in the SS-Hamiltonian can be seen from (33). For example if we take the massless case \( m = 0 \), there is now a new mass term in (33). However, because of the presence of \( \gamma^5 \) in the extra term, the helicity states now have two different masses indicating that the righthanded anti neutrino would have a slightly different mass compared to the lefthanded neutrino.

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