Double complex SUSY-transformations: deformations of real potentials and their spectral characteristics

Sergei P. Maydanyuk *

Institute for Nuclear Research, National Academy of Sciences of Ukraine,
47, prospekt Nauki, Kiev-28, 03680, Ukraine

February 2, 2008

Abstract

In paper approach of double complex SUSY-transformations with not coincident complex energies of transformation is developed, allowing to deform given real potential $V_1$ (with keeping of its shape real) with obtaining exact solutions. The explicit solutions of the deformation of shape of the potential, its wave function at arbitrary energy, not coincident with energies of transformation, wave functions at the energies of transformation are obtained, condition of keeping of continuity of the solutions and isospectral condition are determined. Using a rectangular well of finite width with infinitely high walls as the starting $V_1$ with discrete energy spectrum, by the proposed approach new types of deformation of this potential with deformation of the energy spectrum as a whole have been obtained. The new potential contains the rectangular well as own partial case (with simultaneous transformation of the shape of this new potential, energy spectrum, wave functions of all bound states, wave function at arbitrary energy into corresponding characteristics of the rectangular well at needed choice of parameters). Using null potential as the starting $V_1$ with continuous energy spectrum, new form of reflectionless real potential has been constructed. This potential generalizes well-known reflectionless potential of the type $V_{ref}(x) = A^2(1 - 2\text{sech}^2 Ax)$, allowing: to pull down tails of the potential $V_{ref}$ in the asymptotic regions up to zero (with keeping of nonzero depth); to pull down continuously the depth of the hole; to displace arbitrary along axis $x$ the hole with its passing through zero; to create and to increase the second hole, transforming $V_{ref}$ into double-well potential; to control continuously and simply the asymmetry of the shape of such reflectionless potential. Note relative simplicity of the found potential in a comparison with variety of the reflectionless shape invariant potentials.

PACS numbers: 11.30.Pb, 03.65.-w, 12.60.Jv, 03.65.Xp, 03.65.Fd

Keywords: supersymmetric quantum mechanics, exactly solvable models, Riccati equation, complex Darboux transformations, method of factorization, reflectionless, tunneling, rectangular well.

*E-mail: maidan@kinr.kiev.ua
Designations

In paper we shall use the following designations.

- The first potential $V_1$, on the basis of which we shall construct new potentials using SUSY-transformations, we shall name as the starting potential with number “1”.

- $\phi_i^{(k)}$ and $\phi_i^{(k,1)}$, $\phi_i^{(k,2)}$ — general and two partial solutions for wave function (WF or WFs in the plural case) of the potential $V_k$ with the number $k$, where instead of index $i$ at the bottom we shall use one index from the following:
  - $n$ — the number of the level of discrete energy spectrum of the potential $V_k$ with number $k$, if this level does not coincide with energy of transformation $E$ (i.e. the energy, at which the superpotential is defined);
  - $m$ — the number of the level of the discrete energy spectrum of the potential $V_k$ with the number $k$, if this level coincides with the energy of transformation $E$;
  - $w$ — the index at energy, if it coincides with the energy of transformation $E$ and does not coincide with any level of the energy spectrum of the starting potential $V_k$.

- $\bar{\phi}_i^{(k)}$ — WF of discontinuous state, which we shall denote by stroke above.

- $E_i^{(k)}$ — the level with the number $i$ (corresponding to WF $\varphi_i^{(k)}$) of the energy spectrum for the potential $V_k$. We shall number the levels of the discrete spectrum so:
  - the lowest level, concerned with the ground bound state, by number “1”;
  - the next levels, located higher and described excited bound states, by numbers “2” and larger.

- $\phi_i^{(k)}$ — function of transformation (function of factorization), defined as partial solution of $\varphi_i^{(k)}$ (for the bound or unbound or discontinuous state).

- $E_i^{(k)}$ — energy of transformation (energy of factorization), defined at potential $V_k$ with the number $k$, coinciding with $E_i^{(k)}$.

- $W_i^{(k)}$ — superpotential with the energy of transformation $E_i^{(k)}$ and function of transformation $\phi_i^{(k)}$ concerning potential $V_k$.

- Operators $A_i$ and $A_i^+$ we define so:
  \[
  A_i = \frac{d}{dx} + W_i(x), \quad A_i^+ = -\frac{d}{dx} + W_i(x) .
  \] (1)

Sometime, in text upper or lower index at such notations will be omitted.
1 Introduction

At present time an essential progress has been achieved in study of properties of quantum systems. Here, a great importance is directed to development of methods for control (management) of spectral characteristics of these systems, realized by inverse problem approach [1–4], and by approach of supersymmetric quantum mechanics (SUSY QM) [5–8, 9, 10, 11].

In the approach of SUSY QM, ways of construction of new isospectral potentials (with obtaining of exact solutions for wave function and energy spectrum) on the basis of the given one have been studied the most deeply [12]. Darboux transformations of the first order give this naturally, allowing to come from the given potential with discrete energy spectrum to new one with the energy spectrum without the lowest level (or with its little displacement for some potentials) [10, 11, 13–14, 16, 17, 18] (see also reviews about these transformations [19], about factorization method [20, 21] and historical paper [22]). Transition to the Darboux transformations of the second order gives a possibility to delete not only the lowest but arbitrary level in the energy spectrum, with obtaining explicit form for the new potential (without divergencies) and its wave functions [23, 24, 25, 26] (see also [27, 28]). In definition of function of factorization a transition from wave function of bound state to its general solution gives a possibility to select energy of factorization arbitrary. Such a way allows to analyze what happens with the potential if to change continuously a slope of wave function at selected coordinate at arbitrary selected level or to shift continuously this level in the spectrum. The constructed by such a way new potentials can be considered as the continuous deformations of the given potential. Transition to the Darboux transformations of the higher order [29, 30] and also development of methods of nonlinear supersymmetry [30, 31] give else richer algorithms of construction of new potentials with obtaining exact solutions of their spectral characteristics. Another way to construct more expasive types of transformations with obtaining new exactly solvable potentials can be reduced to introduction of more complicated representations of the superpotential (with keeping of the order of the differential transformations). Here, we note method of extended supersymmetry [32, 33, 34, 35], matrix variants of transformations of polylinear (nonlinear) supersymmetry [36], methods of solution of system of coupled discrete Schrödinger equations [37, 38], generalizations of the formalism of SUSY-transformations into 3 and multi-dimensional spaces [10, 11, 39, 40]. Else one way to construct the new exactly solvable potentials (which can be considered as independent) consists in the development of new types of the differential transformations and their deformations (with keeping of the starting potential and obtaining of new hamiltonians of the Schrödinger type), where the standard differential operators (for example, the standard Darboux operators) appear as a partial case only. One can refer to the methods of nonlinear supersymmetry, the methods of hidden supersymmetry (hidden SUSY), developed in the approaches [41, 42, 43, 44, 45, 46]. Note papers [47, 48] with a possible last and full version of the method, unified these approaches.

However, inclusion of more complicated representations for the superpotential leads to more complicated representations for the potentials, obtained by the approach of SUSY QM. So, use of the complex superpotential gives new varieties of the potentials, which turn out complex inevitably [49, 50, 51, 52]. By use of the SUSY QM methods the exact solutions for new classes of potentials with PT-symmetry have been opened [53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70]. In research of new types of the potentials with real values, a difficulty in analysis of their bound states, in determination of divergencies which can cross out found new solution, increases essentially. How do SUSY-transformations with the complex superpotential or its more complicated representation expand a set of the early known real potentials, how difficult can bound states be calculated, how is the energy spectrum deformed? Answer on these questions will point to effectiveness of inclusion of the complex superpotential or its more complicated representations into the SUSY-transformations in search of new forms of deformations of the real potentials.

The given paper is directed to study of these questions. We restrict ourselves by consideration of one-dimensional real potentials only. Using an approach of double Darboux transformations with complex superpotentials and not coincident energies of transformations as the SUSY-transformations, we shall analyze how they allow to construct new potentials on the basis of given one. The paper is organized so. In sec. 2 a variant of scheme of determination of partial and general solutions for the superpotential, assuming it and the function of factorization to be complex and the given potential — real, is presented. In sec. 3 a formalism of the double transformations with not coincident complex energies of transformations is developed. It turns out difficult to make conclusion about effectiveness of these transformations in construction of new forms of the real potentials in a general case. Therefore, for analysis as the starting potential we shall use rectangular well of finite width with infinitely high walls in sec. 4 as example of system with completely discrete energy spectrum, and null potential in sec. 5 as example of system with completely continuous energy spectrum. In sec. 6 conclusions are presented.
2 Superpotential as a solution of Riccati equation

To clarify, which the most general form an unknown superpotential has, if we know only the potential, concerning with this superpotential in the most prevailing formalism (for example, as in [5] p. 275–277, [13] p. 2922-2923, 2925–2927, [10] p. 19–20, at an energy of factorization, located not higher then the lowest level of energy spectrum of the given potential), we inevitably come to a problem of a solution of Riccati equation, where as unknown function the superpotential is used.

Let’s consider a quantum system with one-dimensional real potential \( V_1(x) \) and discrete energy spectrum completely. One can find spectral characteristics (energy spectrum, wave functions) of this system, solving the Schrödinger equation:

\[
\hat{h}_1 \varphi_m^{(1)}(x) = \left(-\frac{d^2}{dx^2} + V_1(x)\right) \varphi_m^{(1)}(x) = E_m^{(1)} \varphi_m^{(1)}(x),
\]

where \( \hat{h}_1 \) is hamiltonian of the system, \( \varphi_m^{(1)}(x) \) is wave function (WF) of a state with a number \( m \), \( E_m^{(1)} \) is energy level, corresponding to the state with the number \( m \) and having real value, \( E_1^{(1)} \) is the lowest level of the energy spectrum with the number 1. For the given potential \( V_1 \) we introduce a superpotential \( W(x) \), defining it by such a condition (for example, according to [5], p. 287–289, if \( E = E_1^{(1)} \)):

\[
V_1(x) = W^2(x) - \frac{dW(x)}{dx} + E,
\]

where \( E \) is constant, which can have arbitrary value and we name it as energy of transformation (or energy of factorization). In this paper \( W(x) \) and \( E \) are complex. Let the potential \( V_1(x) \) be exactly solvable, i. e. one can write WFs for all states and the energy spectrum of the system with such a potential in the explicit analytical form. Let’s assume, that we know a form of the potential \( V_1(x) \) and a general solution of wave function \( \varphi^{(1)}(x) \) (before imposing boundary conditions) at arbitrary energy \( E^{(1)} \). Also we assume, that we do not know the superpotential \( W(x) \) and we shall find it. Condition [3], defining the unknown \( W(x) \), is the Riccati equation. Let’s find the superpotential, solving this equation.

At first, let’s find a partial solution of \( W(x) \). Fulfilling substitution of variable:

\[
x \to z = z(x) = u(x) + i v(x),
\]

with such condition:

\[
f(x) = \frac{dz(x)}{dx},
\]

where \( u(x), v(x) \) are real functions and \( f(x) \) is arbitrary else. Write:

\[
\frac{d}{dx} = \frac{du}{dx} \frac{d}{du} + \frac{dv}{dx} \frac{d}{dv} = u_x \frac{d}{du} + v_x \frac{d}{dv}, \quad u_x = \frac{du}{dx}, \quad v_x = \frac{dv}{dx}.
\]

With new variables the equation [3] has such a form:

\[
V_1(u) - E = W^2(u) - \frac{dW(u)}{dx} = W^2(u) - u_x \frac{dW(u)}{du} - v_x \frac{dW(u)}{dv}.
\]

Let’s introduce new function \( W_R(z) \):

\[
W_R(z) = -\frac{1}{z + z_0}, \quad z_0 = \begin{cases} -u_0 - iv_0, & \text{at } u \leq 0, \ v \leq 0; \\ -u_0 + iv_0, & \text{at } u \leq 0, \ v \geq 0; \\ u_0 - iv_0, & \text{at } u \geq 0, \ v \leq 0; \\ u_0 + iv_0, & \text{at } u \geq 0, \ v \geq 0; \end{cases}
\]

where \( u, v \) are real positive constants, \( z_0 \) is parameter, used for exclusion of divergence of \( W_R(z) \) at zero \( z = 0 \). We obtain:

\[
\frac{dW_R(z)}{du} = \frac{d}{du} \left(-\frac{1}{u + iv + z_0}\right) = \frac{1}{(u + iv + z_0)^2} = \frac{1}{(z + z_0)^2},
\]

\[
\frac{dW_R(z)}{dv} = \frac{d}{dv} \left(-\frac{1}{u + iv + z_0}\right) = \frac{i}{(u + iv + z_0)^2} = \frac{i}{(z + z_0)^2},
\]

\[
W_R^2(z) = \frac{1}{(z + z_0)^2}.
\]
One can write:
\[
\alpha \frac{dW_R}{du} + \beta \frac{dW_R}{dv} = (\alpha + i\beta) W_R^2 = \frac{\alpha + i\beta}{(z + z_0)^2},
\]
where \(\alpha\) and \(\beta\) are unknown arbitrary constants. Taking into account of this, rewrite equation (10) so:
\[
V_1(u) - \mathcal{E} = W^2(u) - u_x \frac{dW(u)}{du} - v_x \frac{dW(u)}{dv} - f^2 \left\{ (\alpha + i\beta) W_R^2 - \alpha \frac{dW_R}{du} - \beta \frac{dW_R}{dv} \right\}.
\]
Fixing \(f(u)\) by condition:
\[
W^2(x) = (\alpha + i\beta) f^2(u) W_R^2(u),
\]
from (11) we find:
\[
V_1(u) - \mathcal{E} = -u_x \frac{dW(u)}{du} - v_x \frac{dW(u)}{dv} + \alpha f^2 \frac{dW_R}{du} + \beta f^2 \frac{dW_R}{dv} =
= -u_x \sqrt{\alpha + i\beta} \frac{dW_R}{du} - v_x \sqrt{\alpha + i\beta} \frac{dW_R}{dv} + \alpha f^2 \frac{dW_R}{du} + \beta f^2 \frac{dW_R}{dv}.
\]
Now we use \(\alpha\) and \(\beta\) so:
\[
\alpha f = u_x \sqrt{\alpha + i\beta}, \quad \beta f = v_x \sqrt{\alpha + i\beta}.
\]
Then from (13) we obtain:
\[
V_1(x) - \mathcal{E} = -u_x W_R \sqrt{\alpha + i\beta} \frac{df}{du} - v_x W_R \sqrt{\alpha + i\beta} \frac{df}{dv} = -W_R \sqrt{\alpha + i\beta} \left( u_x \frac{df}{du} + v_x \frac{df}{dv} \right) =
= -W_R \sqrt{\alpha + i\beta} \left( \frac{df}{du} dx + \frac{df}{dv} dx \right) = -W_R \sqrt{\alpha + i\beta} \frac{df}{dx} = -W_R \sqrt{\alpha + i\beta} \frac{d^2(x)}{dx^2}.
\]
Taking into account explicit form of (8) for \(W_R\), we find:
\[
\left( V_1(x) - \mathcal{E} \right) \left( z(x) + z_0 \right) = \sqrt{\alpha + i\beta} \frac{d^2(x)}{dx^2} \left( z(x) + z_0 \right).
\]
One can fulfill the condition (14), using \(\alpha\) and \(\beta\) so:
\[
\alpha + i\beta = 1.
\]
In this case, equation (16) transforms into the following:
\[
- \frac{d^2(x)}{dx^2} + V_1(x) \left( z(x) + z_0 \right) = \mathcal{E} \left( z(x) + z_0 \right).
\]
Now we have obtained the Schrödinger equation with the given real potential \(V_1(x)\), where \(z(x) + z_0\) is a complex wave function (we denote it as \(\varphi^{(1)}(x)\)) for selected complex value \(\mathcal{E}\). So, let’s write the partial solution \(\tilde{W}(x)\) for the superpotential:
\[
\begin{align*}
z(x) &= C_0 \varphi^{(1)}_E(x) - z_0, \\
f(x) &= d\frac{z(x)}{dx} = C_0 \frac{d\varphi^{(1)}_E(x)}{dx}, \\
W_R(x) &= -\frac{1}{z(x) + z_0} = -\frac{1}{C_0 \varphi^{(1)}_E(x)}, \\
\tilde{W}(x) &= f(x) W_R(z(x)) = -\frac{1}{\varphi^{(1)}_E(x)} \frac{d\varphi^{(1)}_E(x)}{dx}, \\
C_0 &= \text{const.}
\end{align*}
\]
Further in this paper, we shall name function \(\varphi^{(1)}_E(x)\) (its two partial solutions \(\varphi^{(1,1)}_E(x)\) and \(\varphi^{(1,2)}_E(x)\)), defining superpotential by such a way, as function of transformation (according to logics in [23, 29], one can name it also as function of factorization) and denote it \(\phi(x)\) (the partial solutions as \(\phi^{(1,1)}(x)\) and \(\phi^{(1,2)}(x)\)).

Analyzing logic of obtaining solution of Riccati equation and using the found partial solution \(\tilde{W}(x)\) for the superpotential, we find its general solution:
\[
W(c_{new}) = \tilde{W} + \Delta W(c_{new}), \quad \Delta W(c_{new}; x) = \frac{e \int \tilde{W}(x) dx}{c_{new} - \int e \frac{\tilde{W}(x) dx}{dx}},
\]
where \( c_{\text{new}} \) is new complex constant of integration.

If we know the partial solution for wave function \( \varphi_2^{(1)}(x) \) at any complex energy \( \mathcal{E} \), then (19) and (20) determine the partial and general solutions for superpotential \( W(x) \) at complex energy of transformation \( \mathcal{E} \). A possibility of the energy of transformation to be complex introduces one new degree of freedom in definition of the superpotential \( W(x) \) on the basis of (3), keeping the given real potential \( V_1 \). Therefore, a set of all functions of transformation \( \varphi_2(x) \) is essentially larger then a set of all eigenfunctions \( \varphi_n^{(1)}(x) \), defined concerning the real potential \( V_1 \) and the discrete energy spectrum \( E_n \).

In construction of new potentials-partners on the basis of the general solution \( W(c_{\text{new}}) \) the constant \( c_{\text{new}} \) allows to deform their shapes. Another possibility to deform the new potential the imaginary part of \( \mathcal{E} \) gives. Therefore, \( c_{\text{new}} \) and \( \mathcal{E} \) can be considered as independent parameters of deformation of the new potentials.

**Analysis:**

- The solutions for superpotential are obtained at arbitrariness in a choice of boundary conditions, which can be imposed on the function of transformation. Therefore, definition of superpotential only on the basis of bound states is partial one.
- The function of transformation can be not only WF of the bound state.
- The energy of transformation \( \mathcal{E} \) is arbitrary constant (it can be real, not coinciding with the ground or excited level \( E_n^{(1)} \), or complex).
- Solution (14) proves a possibility to construct the continuous potential \( V_1 \) with own bound states and without divergencies on the basis of discontinuous superpotential \( W(x) \) with divergencies. Therefore, investigations of complex superpotentials with divergencies have a sense.
- The found general solution (20) for superpotential, defined concerning one given real \( V_1 \), has two additional independent free parameters of deformation — imaginary parts of \( c_{\text{new}} \) and \( \mathcal{E} \) (in a comparison with real superpotential for the same \( V_1 \)).

3 Double SUSY-transformations with not coincident energies of transformation: \( \mathcal{E}^{(1)} \neq \mathcal{E}^{(2)} \)

3.1 Deformation of wave function at arbitrary energy \( E_k \), not coincident with the energies of transformation: \( E_k \neq \mathcal{E}^{(1)} \) and \( E_k \neq \mathcal{E}^{(2)} \)

Let’s we know one partial solution \( \varphi_k^{(1,1)}(x) \) for wave function at energy \( E_k^{(1)} \) (not obviously coincident with the levels \( E_n^{(1)} \) of the discrete energy spectrum) of the potential \( V_1 \). On its basis one can find the second partial solution \( \varphi_k^{(1,2)}(x) \) for wave function at this energy so:

\[
\varphi_k^{(1,2)}(x) = \varphi_k^{(1,1)}(x) \int_{x_0}^{x} \frac{dx'}{(\varphi_k^{(1,1)}(x'))^2},
\]

and we construct the general solution:

\[
\varphi_k^{(1)}(c_k, x) = \varphi_k^{(1,1)}(x) + c_k \cdot \varphi_k^{(1,2)}(x) = \varphi_k^{(1,1)}(x) \left( 1 + c_k \cdot \int_{x_0}^{x} \frac{dx'}{(\varphi_k^{(1,1)}(x'))^2} \right),
\]

where \( c_k \) is arbitrary constant, which introduces one degree of freedom in deformation of the total wave function \( \varphi_k^{(1)} \) (we add index os state \( n \) to \( c \)). The value of \( c_k \) one can fix by use of boundary conditions for \( \varphi_k^{(1)} \) (we use this formalism from [33], p. 103–104; see also [33], p. 526). For arbitrary level \( E_n^{(1)} \) sometimes it is convenient as the first solution \( \varphi_n^{(1,1)} \) to use WF of the bound state. Then \( c_n \) can be considered as parameter of deformation, varying of which, we transform this state into unbound one at level \( E_n^{(1)} \), deforming the shape of \( \varphi_n^{(1)} \).

In the first transition \( V_1 \rightarrow V_2 \) we find a general solution of WF for \( V_2 \) at the arbitrary energy \( E_k \) (where \( E_k \neq \mathcal{E}^{(1)} \) and \( E_k \neq \mathcal{E}^{(2)} \)) so:

\[
\varphi_k^{(2)} = N_k^{(1)} A_{w_1}(c_{w_1}) \varphi_k^{(1)}(c_k), \quad A_{w_1}(c_{w_1}) = \frac{d}{dx} + W^{(1)}(c_{w_1}), \quad W^{(1)}(c_{w_1}) = -\frac{d}{dx} \ln \phi_{w_1}^{(1)}(c_{w_1}),
\]

(23)
where \( \phi_k^{(1)} \) is the general solution of WF with parameter \( c_k \), \( \phi_{w_1}^{(1)} \) is the first function of transformation at energy \( \mathcal{E}^{(1)} \) with parameter \( c_{w_1} \). In contrast with construction of formalism for real Darboux transformations, we use in formula (23) a logarithm with complex argument. Using the formula for the logarithm with the complex argument:

\[
\log z = \log |z| + i \arg z, 
\]

we rewrite the superpotential in (23) so:

\[
W^{(1)}(c_{w_1}) = -\frac{d}{dx} \ln \left| \phi_{w_1}^{(1)}(c_{w_1}) \right| - i \frac{d}{dx} \arg \left( \phi_{w_1}^{(1)}(c_{w_1}) \right). 
\]

One can see that complex definition of the superpotential is possible and it expands essentially possibilities in search of new types of real potentials (in a comparison with using only real superpotentials). This formula can be useful in separation of imaginary part from real one in the new potentials.

In inverse transition \( V_2 \to V_1^{(\text{def})} \), we find the general solution for WF for \( V_1^{(\text{def})} \) at the same energy \( E_k \) so:

\[
\varphi_k^{(1, \text{def})} = N_k^{(2)} A_{w_2} \varphi_k^{(2)}(c_k), \quad A_{w_2} (c_{w_2}) = -\frac{d}{dx} + W^{(2)}(c_{w_2}), \quad W^{(2)}(c_{w_2}) = \frac{d}{dx} \ln \phi_{w_2}^{(2)}(c_{w_2}),
\]

where \( \phi_{w_2}^{(2)} \) is the second function of transformation at energy \( \mathcal{E}^{(2)} \) with parameter \( c_{w_2} \).

From (23) and (26) we obtain WF at the energy \( E_k \) for the deformed \( V_1^{(\text{def})} \):

\[
\varphi_k^{(1, \text{def})} (c_k, c_{w_1}, c_{w_2}) = N_k^{(1)} N_k^{(2)} (E_k - \mathcal{E}^{(1)}) \varphi_k^{(1)} + N_k^{(1)} N_k^{(2)} \Delta W (c_{w_1}, c_{w_2}) A_{w_1} \varphi_k^{(1)},
\]

where \( \Delta W (c_{w_1}, c_{w_2}) = W^{(2)}(c_{w_2}) - W^{(1)}(c_{w_1}) \). We find \( \Delta W (c_{w_1}, c_{w_2}) \):

\[
\Delta W (w_1, w_2) = \frac{\varphi_{w_1}^{(1)} \left( \varphi_{w_2}^{(1)} \right)'' - \left( \varphi_{w_1}^{(1)} \right)'' \varphi_{w_2}^{(1)}}{\varphi_{w_1}^{(1)} \left( \varphi_{w_2}^{(1)} \right)' - \left( \varphi_{w_1}^{(1)} \right)' \varphi_{w_2}^{(1)}}. 
\]

Taking into account:

\[
-\left( \varphi_{w_1}^{(1)} \right)'' = (\mathcal{E}^{(1)} - V_1) \varphi_{w_1}^{(1)}, \quad -\left( \varphi_{w_2}^{(1)} \right)'' = (\mathcal{E}^{(2)} - V_1) \varphi_{w_2}^{(1)},
\]

we obtain:

\[
\Delta W (w_1, w_2) = \frac{(\mathcal{E}^{(1)} - \mathcal{E}^{(2)}) \varphi_{w_1}^{(1)} \varphi_{w_2}^{(1)}}{\varphi_{w_1}^{(1)} \left( \varphi_{w_2}^{(1)} \right)' - \left( \varphi_{w_1}^{(1)} \right)' \varphi_{w_2}^{(1)}}. 
\]

Using this, from (27) we find the final form of WF at the energy \( E_k \):

\[
\varphi_k^{(1, \text{def})} (c_k, c_{w_1}, c_{w_2}) = N_k^{(1)} N_k^{(2)} (E_k - \mathcal{E}^{(1)}) \varphi_k^{(1)} + N_k^{(1)} N_k^{(2)} (\mathcal{E}^{(1)} - \mathcal{E}^{(2)}) \varphi_{w_2}^{(1)} \left( \varphi_{w_2}^{(1)} \right)' - \left( \varphi_{w_1}^{(1)} \right)' \varphi_{w_2}^{(1)}.
\]

If as the first or the second function of transformation, WF of the bound state at level \( \mathcal{E}^{(1)} \) or \( \mathcal{E}^{(2)} \) is used, then one can normalize this function of transformation and obtain normalizing \( N_k^{(1)} \) and \( N_k^{(2)} \) from this:

\[
N_k^{(1)} = \begin{cases} 
( E_k - \mathcal{E}^{(1)} )^{-1/2}, & \text{at } E_k > \mathcal{E}^{(1)}, \\
i | E_k - \mathcal{E}^{(1)} |^{-1/2}, & \text{at } E_k < \mathcal{E}^{(1)};
\end{cases}
\]

\[
N_k^{(2)} = \begin{cases} 
( E_k - \mathcal{E}^{(2)} )^{-1/2}, & \text{at } E_k > \mathcal{E}^{(2)}, \\
i | E_k - \mathcal{E}^{(2)} |^{-1/2}, & \text{at } E_k < \mathcal{E}^{(2)}.
\end{cases}
\]

**Analysis:**

- Let’s formulate a sufficient condition from (31), when WF of the arbitrary state at arbitrary energy \( E_k \), not coincident with both energies of transformation \( \mathcal{E}^{(1)} \) and \( \mathcal{E}^{(2)} \), remains continuous after deformation:

\[
\phi_{w_1}^{(1)} (x) \left( \phi_{w_2}^{(1)} (x) \right)' - \left( \phi_{w_1}^{(1)} (x) \right)' \frac{1}{\phi_{w_2}^{(1)} (x)} \neq 0.
\]

This condition agrees with the requirement of inequality of Wronskian of two transformation functions to zero to provide the absence of singularities in the potential \( V_2 \), obtained on the basis of the given potential \( V_0 \) by use of Darboux transformations of the second order in [23].
3.2 Deformation of the potential

The shape of the potential $V_1$ is deformed so:

$$V_1^{\text{def}}(c_{w_1}, c_{w_2}) = V_1 + \Delta V_1(c_{w_1}, c_{w_2}),$$

$$\Delta V_1(c_{w_1}, c_{w_2}) = 2(E^{(2)} - E^{(1)}) \frac{d}{dx} \left( \frac{\varphi^{(1)}_{w_1} \varphi^{(1)}_{w_2}}{\varphi^{(1)}_{w_1} (\varphi^{(1)}_{w_2} - \varphi^{(1)}_{w_1})} \right).$$

To be the new potential $V_1^{\text{def}}$ real, we introduce the following condition:

$$\Im \left( \Delta V_1(c_{w_1}, c_{w_2, x}) \right) = 0.$$ (37)

Analysis:

- The potential $V_1^{\text{def}}$ has 4 independent complex parameters of deformation: $E_1, E_2, c_{w_1}, c_{w_2}$.
- Continuity of the potential $V_1$, continuity of all WFs at all levels and energies (with possible exclusion of $E^{(1)}$ and $E^{(2)}$) after the deformation is defined by condition (33).

3.3 Deformation of wave function at the energy of transformation $E^{(1)}$

Let's find how WF of arbitrary state at the energy of transformation $E^{(1)}$ is deformed. According to (23) and (26), one can find the general solution of the deformed WF so:

$$\varphi^{(1, \text{def})}_{k_1} = N^{(2)}_{w_1} A_{w_1} \varphi^{(2)}_{k_1} = N^{(2)}_{w_1} \left( -\frac{d}{dx} + W^{(2)} \right) \varphi^{(2)}_{k_1} = N^{(2)}_{w_1} \varphi^{(2)}_{k_1} = N^{(2)}_{w_1} \frac{-\varphi^{(2)}_{w_1} \phi^{(2)}_{w_1} + \varphi^{(2)}_{k_1} \phi^{(2)}_{w_2}}{\phi^{(2)}_{w_2}}.$$ (38)

We write the general solution for $\phi^{(2)}_{w_2}$:

$$\phi^{(2)}_{w_2} = N^{(2)}_{w_2} A_{w_2} \varphi^{(2)}_{w_2} = N^{(2)}_{w_2} \left( -\frac{d}{dx} + W^{(1)} \right) \varphi^{(2)}_{w_2} = N^{(2)}_{w_2} \frac{-\varphi^{(1)}_{w_1} \phi^{(1)}_{w_1} \phi^{(1)}_{w_2} + \varphi^{(1)}_{w_1} \phi^{(1)}_{w_2}}{\phi^{(1)}_{w_2}}.$$ (39)

and its derivative:

$$\phi^{(2, \prime)}_{w_2} = N^{(2)}_{w_2} \left( E^{(1)} - E^{(2)} \right) \varphi^{(1)}_{w_1} \varphi^{(1)}_{w_2} = N^{(2)}_{w_2} \frac{\varphi^{(1)}_{w_1} \phi^{(1)}_{w_1} \phi^{(1)}_{w_2} - \varphi^{(1)}_{w_1} \phi^{(1)}_{w_2}}{\phi^{(1)}_{w_2}}.$$ (40)
It needs to take into account, that the wave function $\varphi^{(1)}_{w_1}$ before its deformation can differ from the used function of transformation $\phi^{(1)}_{w_1}$ at energy $\mathcal{E}^{(1)}$. If such wave function coincides with the function of transformation $\phi^{(1)}_{w_1}$, then the general solution for $\varphi^{(2)}_{k_1}$ can be found so:

$$\varphi^{(2)}_{k_1} = \frac{2}{\phi^{(1)}_{w_1}} \cdot I^{(1)}_{w_1} + \lambda_1, \quad I^{(1)}_{w_1} = \int_{x_0}^{x} \left( \phi^{(1)}_{w_1}(x') \right)^2 dx'.$$

Finding the derivative of this function:

$$\varphi^{(2)}_{k_1} = \left( \frac{I^{(1)}_{w_1} + \lambda_1}{\phi^{(1)}_{w_1}} \right) = \frac{\phi^{(1)}_{w_1} - \phi^{(1)}_{w_1}'}{\phi^{(1)}_{w_1}} \cdot \left( \frac{I^{(1)}_{w_1} + \lambda_1}{\phi^{(1)}_{w_1}} \right)^2,$$

we obtain:

$$\varphi^{(1, \text{def})}_{w_1} = -N^{(2)}_{w_1} \varphi^{(1)}_{w_1} + N^{(2)}_{w_1} \left( \mathcal{E}^{(1)} - \mathcal{E}^{(2)} \right) \frac{\left( I^{(1)}_{w_1} + \lambda_1 \right)}{\phi^{(1)}_{w_1}} \frac{\phi^{(1)}_{w_2}}{\phi^{(1)}_{w_1}}.$$

If wave function $\varphi^{(1)}_{k_1}$ before its deformation does not coincide with the function of transformation $\phi^{(1)}_{w_1}$ at energy $\mathcal{E}^{(1)}$, then the function:

$$\varphi^{(2, \text{def})}_{k_1} = \frac{\lambda_1}{\varphi^{(1)}_{w_1}},$$

is not a solution of the Schrödinger equation with the new potential $V_2$ at such energy. But in such a case, non-zero function $\varphi^{(2)}_{k_1}$ can be found so:

$$\varphi^{(2)}_{k_1} = N^{(1)}_{w_1} A_{w_1} \varphi^{(1)}_{k_1} = N^{(1)}_{w_1} \left( \frac{d}{dx} + W^{(1)} \right) \varphi^{(1)}_{k_1} = N^{(1)}_{w_1} \frac{\phi^{(1)}_{w_1} \varphi^{(1, \text{def})}_{k_1} - \phi^{(1)}_{w_1} \varphi^{(1, \text{def})}_{k_1}'}{\phi^{(1)}_{w_1}}.$$

where $N^{(2)}_{w_1}$ is new normalizing factor. Calculating the derivative of this function:

$$\varphi^{(2, \text{def})}_{k_1} = N^{(1)}_{w_1} \left( \mathcal{E}^{(1)} - \mathcal{E}^{(2)} \right) \frac{\phi^{(1)}_{w_1} \varphi^{(1, \text{def})}_{k_1} - \phi^{(1)}_{w_1} \varphi^{(1, \text{def})}_{k_1}'}{\phi^{(1)}_{w_1}} \cdot \varphi^{(1)}_{k_1}.$$

we find:

$$\varphi^{(1, \text{def})}_{k_1} = N^{(1)}_{w_1} N^{(2)}_{w_1} \left( \mathcal{E}^{(1)} - \mathcal{E}^{(2)} \right) \frac{\phi^{(1)}_{w_1} \varphi^{(1, \text{def})}_{k_1} - \phi^{(1)}_{w_1} \varphi^{(1, \text{def})}_{k_1}'}{\phi^{(1)}_{w_1}} \cdot \varphi^{(1)}_{k_1}.$$

Here, one can separate the deformation of $\varphi^{(1)}_{k_1}$ explicitly:

$$\varphi^{(1, \text{def})}_{k_1} = N^{(1)}_{w_1} N^{(2)}_{w_1} \left( \mathcal{E}^{(1)} - \mathcal{E}^{(2)} \right) \varphi^{(1)}_{k_1} + N^{(1)}_{w_1} N^{(2)}_{w_1} \left( \mathcal{E}^{(1)} - \mathcal{E}^{(2)} \right) \frac{\phi^{(1)}_{w_1} \varphi^{(1, \text{def})}_{k_1} - \phi^{(1)}_{w_1} \varphi^{(1, \text{def})}_{k_1}'}{\phi^{(1)}_{w_1}} \cdot \varphi^{(1)}_{k_1}. \quad (48)$$

### 3.4 Deformation of wave function at the energy of transformation $\mathcal{E}^{(2)}$

Now we shall find how WF at the energy of transformation $\mathcal{E}^{(2)}$ is deformed. If to use the partial solution of $\varphi^{(2)}_{w_2}$ at $\varphi^{(2)}_{w_2} = \phi^{(2)}_{w_2}$, then according to (23) and (26), the general solution for the deformed WF for potential $V^{(1, \text{def})}_{w_2}$ at such energy can be found so:

$$\varphi^{(2, \text{def})}_{w_2} = \frac{I^{(2)}_{w_2} + \lambda_2}{\phi^{(2)}_{w_2}}, \quad I^{(2)}_{w_2} = \int_{x_0}^{x} \left( \phi^{(2)}_{w_2}(x') \right)^2 dx'. \quad (49)$$

where, according to (30), we have:

$$\phi^{(2)}_{w_2} = N^{(1)}_{w_2} \frac{\phi^{(1)}_{w_2} \varphi^{(1, \text{def})}_{w_2} - \phi^{(1)}_{w_2} \varphi^{(1, \text{def})}_{w_2}'}{\phi^{(1)}_{w_1}}. \quad (50)$$
Calculating the integral:

\[
I_w^{(2)}(x) = \int_{x_0}^{x} \left( \phi_w^{(2)}(x') \right)^2 dx' = \left( N_w^{(1)} \right)^2 \int_{x_0}^{x} \left( \frac{\phi_{w_1}' \phi_{w_2}^{(1)} - \phi_{w_1} \phi_{w_2}^{(1)}}{\phi_{w_1}^{(1)}} \right)^2 dx' =
\]

we obtain the deformed WF:

\[
\varphi_{w_2}^{(1, \text{def})}(x) = N_{w_2}^{(1)} \phi_{w_2}^{(1)}(x) - N_{w_2}^{(1)} \phi_{w_1}' \phi_{w_2}^{(1)}(x_0) \phi_{w_2}^{(1)}(x) - \phi_{w_1} \phi_{w_2}^{(1)}(x_0) \phi_{w_2}^{(1)}(x) + \]

\[
+ N_{w_2}^{(1)} \left( \mathcal{E}^{(2)} - \mathcal{E}^{(1)} \right) \phi_{w_1}'(x) \phi_{w_2}^{(1)}(x) - \phi_{w_1} \phi_{w_2}^{(1)}(x) \phi_{w_2}^{(1)}(x) \int_{x_0}^{x} \left( \phi_{w_2}^{(1)}(x') \right)^2 dx' +
\]

\[
+ \frac{\lambda_2 \phi_{w_1}^{(1)}(x)}{N_{w_2}^{(1)} \left( \phi_{w_1}^{(1)} \phi_{w_2}^{(1)}(x) - \phi_{w_1} \phi_{w_2}^{(1)}(x) \right)}.
\]

Now we take into account, that the solution of \( \varphi_{w_2}^{(2)} \) can differ from the function of transformation \( \varphi_{w_2}^{(2)} \). In such a case, one can find non-zero function \( \varphi_{w_2}^{(1, \text{def})} \) so:

\[
\varphi_{w_2}^{(1, \text{def})} = N_{w_2}^{(2)} A_w^{+} \varphi_{w_2}^{(2)} = N_{w_2}^{(2)} \left( -\frac{d}{dx} + W^{(2)} \right) \varphi_{w_2}^{(2)} = N_{w_2}^{(2)} \frac{\phi_{w_2}^{(2)} \phi_{w_2}^{(2)} + \phi_{w_2}^{(2)} \phi_{w_2}^{(2)}}{\phi_{w_2}^{(2)}}.
\]

Taking into account \( \text{(9)} \) and \( \text{(10)} \), we calculate the function \( \varphi_{w_2}^{(2)} \) and its derivative:

\[
\varphi_{w_2}^{(2)} = N_{w_2}^{(1)} \phi_{w_1}^{(1)} \varphi_{w_2}^{(1)} - \phi_{w_1} \varphi_{w_2}^{(1)},
\]

\[
\varphi_{w_2}^{(2),'} = N_{w_2}^{(1)} \left( \mathcal{E}^{(1)} - \mathcal{E}^{(2)} \right) \phi_{w_1}^{(1)} \phi_{w_2}^{(1)} - \phi_{w_1} \phi_{w_2}^{(1)} + \phi_{w_1} \phi_{w_2}^{(1)} \phi_{w_2}^{(1)} \phi_{w_2}^{(1)}
\]

and obtain the deformed WF \( \varphi_{w_2}^{(1, \text{def})} \):

\[
\varphi_{w_2}^{(1, \text{def})} = N_{w_2}^{(1)} N_{w_2}^{(2)} \left( \mathcal{E}^{(1)} - \mathcal{E}^{(2)} \right) \frac{\phi_{w_1}^{(1)} \phi_{w_2}^{(1)} - \phi_{w_1} \phi_{w_2}^{(1)}}{\phi_{w_1}^{(1)} \phi_{w_2}^{(1)} - \phi_{w_1} \phi_{w_2}^{(1)}}.
\]

In the obtained solution one can separate the deformation explicitly:

\[
\varphi_{w_2}^{(1, \text{def})} = N_{w_2}^{(1)} N_{w_2}^{(2)} \left( \mathcal{E}^{(2)} - \mathcal{E}^{(1)} \right) \varphi_{w_2}^{(2)} + N_{w_2}^{(1)} N_{w_2}^{(2)} \left( \mathcal{E}^{(1)} - \mathcal{E}^{(2)} \right) \frac{\phi_{w_1}^{(1)} \phi_{w_2}^{(1)} - \phi_{w_1} \phi_{w_2}^{(1)}}{\phi_{w_1}^{(1)} \phi_{w_2}^{(1)} - \phi_{w_1} \phi_{w_2}^{(1)}} \varphi_{w_2}^{(1)}.
\]

4 Example — deformation of rectangular well with finite width and infinitely high walls

Let’s consider, how the double complex SUSY-transformations are effective in construction of new real potentials on the basis of the given one. We shall analyze two cases: potential with discrete and potential with continuous energy spectra.

As the starting potential \( V_1 \) with the discrete energy spectrum, we shall use the rectangular well with finite width and infinitely high walls:

\[
V_1(x) = \begin{cases} 
  0, & \text{at } 0 < x < a, \\
  +\infty, & \text{at } x < 0 \text{ or } x > a.
\end{cases}
\]
If to restrict ourselves for this potential only the discrete values of energy higher than the wall bottom, then we have a general solution for WF for arbitrary state at arbitrary energy $E_k$:

unbound state with index $k$: $\varphi_k^{(1)}(x) = A \sin(kx + \theta)$,

bound state with number $n$: $\varphi_n^{(1)}(x) = \sqrt{\frac{2}{a}} \sin(k_n x)$ (58)

and the discrete energy spectrum:

$$E_n = E_1 n^2 = k_n^2 = \frac{\pi^2}{a^2} n^2, \quad k_n = k_1 n = \frac{\pi}{a} n, \quad n = 1, 2 \ldots$$ (59)

where $A$ and $\theta$ are arbitrary (complex) constants, WF of the bound state is normalized (is real, with accuracy to constant phase factor). Coordinates of nodes for WF of the bound state with number $n$ equal to:

$$x_l = \frac{\pi l}{k_n} = \frac{a l}{n}, \quad l = 0, 1, 2 \ldots$$ (60)

and are located at the same distances, which are decreased with increasing of $n$.

Now if to include into analysis the complex values of the energies for the potential (57), then we obtain the general solution of the wave function concerning arbitrary selected complex energy $E$:

$$\varphi_2(x) = \alpha e^{c x} + \beta e^{-c x} = \alpha e^{\chi x} e^{ik x} + \beta e^{-\chi x} e^{-ik x}, \quad c = \chi + ik,$$ (61)

where $\alpha$ and $\beta$ are arbitrary complex constants, $c$ is “complex wave vector”, $\chi = \Re(c)$ and $k = \Im(c)$ are real and imaginary parts of constant $c$. Substituting this solution into the Schrödinger equation, we find the coefficients $\chi$ and $k$:

$$(-c^2 = \mathcal{E}) \quad \rightarrow \quad \left\{ \begin{array}{l} k^2 - \chi^2 = \Re(\mathcal{E}), \\
-2 \chi k = \Im(\mathcal{E}). \end{array} \right.$$ (62)

If $\Im(\mathcal{E}) \neq 0$, then from (62) we obtain:

$$\Re(c)^2 = \frac{1}{2} \left( -\Re(\mathcal{E}) + \sqrt{\Re(\mathcal{E})^2 + \Im(\mathcal{E})^2} \right), \quad \Im(c)^2 = \frac{1}{2} \left( \Re(\mathcal{E}) + \sqrt{\Re(\mathcal{E})^2 + \Im(\mathcal{E})^2} \right).$$ (63)

If $\Im(\mathcal{E}) = 0$, then:

$$\chi = \Re(c) = 0, \quad \text{at} \quad \mathcal{E} = \Re(\mathcal{E}) > 0;$$
$$k = \Im(c) = 0, \quad \text{at} \quad \mathcal{E} = \Re(\mathcal{E}) < 0.$$ (64)

Further, as the method of the deformation we shall use the SUSY-approach with not coincident energies of transformation $\mathcal{E}^{(1)} \neq \mathcal{E}^{(2)}$. As two functions of transformation, we select the following functions:

$$\phi_1(x) = e^{c_1 x + \delta};$$
$$\phi_2(x) = \alpha e^{c_2 x} + \beta e^{-c_2 x},$$ (65)

where $\delta$ is arbitrary complex constant and $c_1$, $c_2$ are complex constants, defined by the complex functions of transformation $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$.

### 4.1 Deformation of the potential

Let’s find, how the potential $V_1$ is deformed. According to (36), the potential after deformation has a form:

$$V_1^{\text{def}}(c_{w_1}, c_{w_2}) = V_1 + \Delta V_1(c_{w_1}, c_{w_2}),$$

$$\Delta V_1(c_{w_1}, c_{w_2}) = 2 \left( \mathcal{E}^{(2)} - \mathcal{E}^{(1)} \right) \frac{d}{dx} \frac{\phi_{w_1}^{(1)}(\phi_{w_2})^{(1)} - (\phi_{w_1})^{(1)}(\phi_{w_2})^{(1)}}{\phi_{w_1}^{(1)}(\phi_{w_2})^{(1)}}.$$ (66)

Substituting here solutions (65) for the functions of transformation, we obtain:

$$\Delta V_1(c_{w_1}, c_{w_2}) = 2 \left( \mathcal{E}^{(2)} - \mathcal{E}^{(1)} \right) \frac{d}{dx} \frac{\alpha e^{c_2 x} + \beta e^{-c_2 x}}{\alpha (c_2 - c_1) e^{c_2 x} - \beta (c_2 + c_1) e^{-c_2 x}}.$$ (67)
The new potential $V_1^{(d)}$ is complex. We shall be interesting in a question, whether the complex double SUSY-transformations proposed in this paper can give new real deformations of the potential $V_1$. The deformation $\Delta V_1$ to be real, it is enough, that the function is real (we shall find only nonconstant (and nonzero) deformations $\Delta V$):

$$ f(x) = \left( E^{(2)} - E^{(1)} \right) \left( \alpha \ e^{c_2 x} + \beta \ e^{-c_2 x} \right) \left( \alpha^* \ (c_2 - c_1^*) \ e^{c_2^* x} - \beta^* \ (c_2^* + c_1^*) \ e^{-c_2^* x} \right). $$ (68)

Rewrite the function $f(x)$ explicitly:

$$ f(x) = \left( E^{(2)} - E^{(1)} \right) \left( |\alpha|^2 (c_2 - c_1^*) \ e^{(c_2 + c_2^*) x} - |\beta|^2 (c_2^* + c_1^*) e^{-(c_2 + c_2^*) x} \right) + \alpha^* \beta \ (c_2^* - c_1^*) \ e^{(c_2^* - c_2)x} - \alpha \beta^* \ (c_2 + c_1^*) \ e^{(c_2 - c_2^*) x}. $$ (69)

Taking into account:

$$ c_2 + c_2^* = 2 \Re(c_2), \quad c_2 - c_2^* = 2i \Im(c_2), $$

we obtain:

$$ f(x) = \left( E^{(2)} - E^{(1)} \right) \left( |\alpha|^2 (c_2 - c_1^*) \ e^{2 \Re(c_2)x} - |\beta|^2 (c_2^* + c_1^*) e^{-2 \Re(c_2)x} \right) + \alpha^* \beta \ (c_2^* - c_1^*) \ e^{-2i \Im(c_2)x} - \alpha \beta^* \ (c_2 + c_1^*) \ e^{2i \Im(c_2)x}. $$ (70)

On the basis of exponents $\exp \pm 2 \Re(c_2)x$ and $\exp \pm 2i \Im(c_2)x$ one can combine functions $\sin (2 \Im(c_2)x)$, $\cos (2 \Im(c_2)x)$, $\cosh (2 \Re(c_2)x)$ and $\sinh (2 \Re(c_2)x)$. Write:

$$ \left( E^{(2)} - E^{(1)} \right) \cdot |\alpha|^2 (c_2 - c_1^*) e^{2 \Re(c_2)x} - |\beta|^2 (c_2^* + c_1^*) e^{-2 \Re(c_2)x} = \right.$$ $$ = N_1 \cos (2 \Re(c_2)x) + N_2 \sinh (2 \Re(c_2)x) = e^{2 \Re(c_2)x} \frac{N_1 + N_2}{2} + e^{-2 \Re(c_2)x} \frac{N_1 - N_2}{2}, $$

$$ \left( E^{(2)} - E^{(1)} \right) \cdot \alpha^* \beta \ (c_2^* - c_1^*) \ e^{-2i \Im(c_2)x} - \alpha \beta^* \ (c_2 + c_1^*) \ e^{2i \Im(c_2)x} = $$

$$ = M_1 \cos (2 \Im(c_2)x) + M_2 \sin (2 \Im(c_2)x) = e^{2i \Im(c_2)x} \left( \frac{M_1}{2} + \frac{M_2}{2i} \right) + e^{-2i \Im(c_2)x} \left( \frac{M_1}{2} - \frac{M_2}{2i} \right), $$ (71)

where $N_1$, $N_2$, $M_1$ and $M_2$ are new real constants. At the same exponents the coefficients must be equal:

$$ \left( E^{(2)} - E^{(1)} \right) \cdot |\alpha|^2 (c_2 - c_1^*) = \frac{N_1 + N_2}{2}, \quad \left( E^{(2)} - E^{(1)} \right) \cdot \alpha^* \beta \ (c_2^* - c_1^*) = \frac{M_1}{2} - \frac{M_2}{2i}, $$

$$ -\left( E^{(2)} - E^{(1)} \right) \cdot |\beta|^2 (c_2^* + c_1^*) = \frac{N_1 - N_2}{2}, \quad -\left( E^{(2)} - E^{(1)} \right) \cdot \alpha \beta^* \ (c_2 + c_1^*) = \frac{M_1}{2} + \frac{M_2}{2i}. $$ (72)

From here we find $N_1$, $N_2$, $M_1$ and $M_2$, expressing them through $\alpha$, $\beta$, $c_1$ and $c_2$:

$$ N_1 = \left( E^{(2)} - E^{(1)} \right) \left( |\alpha|^2 (c_2 - c_1^*) - |\beta|^2 (c_2^* + c_1^*) \right), $$

$$ N_2 = \left( E^{(2)} - E^{(1)} \right) \left( |\alpha|^2 (c_2 - c_1^*) + |\beta|^2 (c_2^* + c_1^*) \right), $$

$$ M_1 = \left( E^{(2)} - E^{(1)} \right) \left( \alpha^* \beta \ (c_2^* - c_1^*) - \alpha \beta^* \ (c_2 + c_1^*) \right), $$

$$ M_2 = i \left( E^{(2)} - E^{(1)} \right) \left( -\alpha^* \beta \ (c_2^* - c_1^*) - \alpha \beta^* \ (c_2 + c_1^*) \right). $$ (73)

Taking into account only real values for $N_1$, $N_2$, $M_1$, $M_2$, we obtain condition of the real deformation of the potential:

$$ \Im \left( \left( E^{(2)} - E^{(1)} \right) \left( |\alpha|^2 (c_2 - c_1^*) - |\beta|^2 (c_2^* + c_1^*) \right) \right) = 0, $$

$$ \Im \left( \left( E^{(2)} - E^{(1)} \right) \left( |\alpha|^2 (c_2 - c_1^*) + |\beta|^2 (c_2^* + c_1^*) \right) \right) = 0, $$

$$ \Re \left( \left( E^{(2)} - E^{(1)} \right) \left( \alpha^* \beta \ (c_2^* - c_1^*) - \alpha \beta^* \ (c_2 + c_1^*) \right) \right) = 0, $$

$$ \Re \left( \left( E^{(2)} - E^{(1)} \right) \left( \alpha^* \beta \ (c_2^* - c_1^*) + \alpha \beta^* \ (c_2 + c_1^*) \right) \right) = 0. $$ (74)

Write:

$$ E^{(2)} - E^{(1)} = -(c_2)^2 + (c_1)^2 = -\Re(c_2)^2 + \Im(c_2)^2 + \Re(c_1)^2 - \Im(c_1)^2 + 2i \cdot \left( -\Re(c_2) \Im(c_2) + \Re(c_1) \Im(c_1) \right), $$ (75)
Taking into account of this, we rewrite the first two equations of (75), separating real and imaginary parts:

\[
\Re(E^{(2)} - E^{(1)}) \cdot \left\{ (\Re(\alpha)^2 + \Im(\alpha)^2) \left( -\Im(c_2) + \Im(c_1) \right) + (\Re(\beta)^2 + \Im(\beta)^2) \left( -\Im(c_2) - \Im(c_1) \right) \right\} + \\
+ \Im(E^{(2)} - E^{(1)}) \cdot \left\{ (\Re(\alpha)^2 + \Im(\alpha)^2) \left( \Re(c_2) - \Re(c_1) \right) - (\Re(\beta)^2 + \Im(\beta)^2) \left( \Re(c_2) + \Re(c_1) \right) \right\} = 0,
\]

\[
\Re(E^{(2)} - E^{(1)}) \cdot \left\{ (\Re(\alpha)^2 + \Im(\alpha)^2) \left( -\Im(c_2) + \Im(c_1) \right) + (\Re(\beta)^2 + \Im(\beta)^2) \left( -\Im(c_2) - \Im(c_1) \right) \right\} + \\
+ \Im(E^{(2)} - E^{(1)}) \cdot \left\{ (\Re(\alpha)^2 + \Im(\alpha)^2) \left( \Re(c_2) - \Re(c_1) \right) + (\Re(\beta)^2 + \Im(\beta)^2) \left( \Re(c_2) + \Re(c_1) \right) \right\} = 0.
\]

Add and subtract one equation concerning the other one:

\[
\Re(\alpha)^2 + \Im(\alpha)^2 \cdot \left\{ \Re(E^{(2)} - E^{(1)}) \left( -\Im(c_2) + \Im(c_1) \right) + \Im(E^{(2)} - E^{(1)}) \left( \Re(c_2) - \Re(c_1) \right) \right\} = 0,
\]

\[
\Re(\beta)^2 + \Im(\beta)^2 \cdot \left\{ \Re(E^{(2)} - E^{(1)}) \left( -\Im(c_2) - \Im(c_1) \right) + \Im(E^{(2)} - E^{(1)}) \left( \Re(c_2) + \Re(c_1) \right) \right\} = 0. \tag{77}
\]

And obtain the first two conditions:

1) \[
\Re(E^{(2)} - E^{(1)}) \left( -\Im(c_2) + \Im(c_1) \right) + \Im(E^{(2)} - E^{(1)}) \left( \Re(c_2) - \Re(c_1) \right) = 0,
\]

2) \[
\Re(E^{(2)} - E^{(1)}) \left( -\Im(c_2) - \Im(c_1) \right) + \Im(E^{(2)} - E^{(1)}) \left( \Re(c_2) + \Re(c_1) \right) = 0. \tag{78}
\]

If \(\Im(E_2 - E_1) = 0\), then from (78) we obtain:

\[
\Im(c_1) = 0, \quad \Im(c_2) = 0, \quad \Im(E_1) = 0, \quad \Im(E_2) = 0. \tag{79}
\]

At \(\Im(E_1) = 0 \text{ and } \Im(E_2) = 0\) the exponents \(\exp \pm 2\pi \Im(c_2) x\) in (71) and (72) equal to 1. Therefore, instead of the third and the forth conditions of (77) we must use the following one:

\[
\Im(\alpha^* \beta (c_2 - c_1^*) - \alpha \beta^* (c_2^* + c_1)) = 0. \tag{80}
\]

We obtain:

\[
\Re(\alpha) \Im(\beta) = \Im(\alpha) \Re(\beta). \tag{81}
\]

One can see, that the coefficients \(\alpha\) and \(\beta\) must be real only (with accuracy to the same complex factor), that the deformation of the potential \(\Delta V\) to be real.

If \(\Im(E_2 - E_1) \neq 0\), then adding and subtracting one expression concerning the other one in (78), we obtain:

\[
\Re(E^{(2)} - E^{(1)}) \Im(c_2) - \Im(E^{(2)} - E^{(1)}) \Re(c_2) = 0,
\]

\[
\Re(E^{(2)} - E^{(1)}) \Im(c_1) - \Im(E^{(2)} - E^{(1)}) \Re(c_1) = 0. \tag{82}
\]

Substituting the real and imaginary parts of \(E_2 - E_1\) from (76), we find:

\[
\left[ -\Re(c_2)^2 + \Im(c_2)^2 + \Re(c_1)^2 - \Im(c_1)^2 \right] \Im(c_2) - 2 \cdot \left( -\Re(c_2) \Im(c_2) + \Re(c_1) \Im(c_1) \right) \Re(c_2) = \\
= \left[ \Re(c_2)^2 + \Im(c_2)^2 + \Re(c_1)^2 - \Im(c_1)^2 \right] \Im(c_2) - 2 \cdot \left( -\Re(c_2) \Im(c_2) + \Re(c_1) \Im(c_1) \right) \Re(c_1) = 0,
\]

\[
\left[ -\Re(c_2)^2 + \Im(c_2)^2 + \Re(c_1)^2 - \Im(c_1)^2 \right] \Im(c_1) - 2 \cdot \left( -\Re(c_2) \Im(c_2) + \Re(c_1) \Im(c_1) \right) \Re(c_1) = \\
= \left[ -\Re(c_2)^2 + \Im(c_2)^2 - \Re(c_1)^2 + \Im(c_1)^2 \right] \Im(c_1) + 2 \cdot \Re(c_1) \Re(c_2) \Im(c_2) = 0. \tag{83}
\]

Multiply the first equation on \(\Im(c_2)\), and the second one — on \(\Im(c_1)\):

\[
\left[ \Re(c_2)^2 + \Im(c_2)^2 + \Re(c_1)^2 - \Im(c_1)^2 \right] \Im(c_2)^2 - 2 \Re(c_1) \Re(c_2) \Im(c_1) \Im(c_2) = 0,
\]

\[
\left[ -\Re(c_2)^2 + \Im(c_2)^2 - \Re(c_1)^2 + \Im(c_1)^2 \right] \Im(c_1)^2 + 2 \Re(c_1) \Re(c_2) \Im(c_1) \Im(c_2) = 0. \tag{84}
\]

Adding one equation with the second, we obtain:

\[
\left[ \Re(c_1)^2 + \Re(c_2)^2 \right] \left( \Im(c_2)^2 - \Im(c_1)^2 \right) + \left[ \Im(c_2)^2 - \Im(c_1)^2 \right] \left( \Re(c_2)^2 + \Re(c_1)^2 \right) = \\
= \left[ \Re(c_1)^2 + \Re(c_2)^2 + \Im(c_2)^2 + \Im(c_1)^2 \right] \cdot \left( \Im(c_2)^2 - \Im(c_1)^2 \right) = 0,
\]
At point $x$ from here we find:

$$\Im(c_1) = \pm \Im(c_2).$$  \hspace{1cm} (85)

Now subtract the second equation from the first one in (54):

$$\left(\Re(c_1)^2 + \Re(c_2)^2\right) \left(\Im(c_2)^2 + \Im(c_1)^2\right) - 4 \Re(c_1) \Re(c_1) \Im(c_1) \Im(c_2) = 0.$$ \hspace{1cm} (86)

Taking into account (85), we obtain:

1) $\Im(c_1) = +\Im(c_2) : \left(\Re(c_1)^2 + \Re(c_2)^2 - 2 \Re(c_1) \Re(c_1) = (\Re(c_1) - \Re(c_2))^2 = 0\right) \rightarrow (\Re(c_1) = +\Re(c_2)),$

2) $\Im(c_1) = -\Im(c_2) : \left(\Re(c_1)^2 + \Re(c_2)^2 + 2 \Re(c_1) \Re(c_1) = (\Re(c_1) + \Re(c_2))^2 = 0\right) \rightarrow (\Re(c_1) = -\Re(c_2)).$ \hspace{1cm} (87)

And we come to absolute coincidence between two energies of transformation. Therefore, we cannot use such solutions and the case $\Im(E_2 - E_1) \neq 0$.

### 4.2 Deformation of wave function at arbitrary energy $E_k$, not coincident with the energies of transformation: $E_k \neq E^{(1)}$ and $E_k \neq E^{(2)}$

Let’s the general solution for WF $\varphi_k^{(1)}$ at the energy $E_k$ (higher then the bottom of the well) before the deformation has a form:

$$\varphi_k^{(1)}(x) = \alpha_k \sin kx + \beta_k \cos kx,$$

where $\alpha_k$ and $\beta_k$ are arbitrary complex constants. According to (31), we find the form of this WF after the deformation:

$$\varphi_k^{(1,\text{def})}(x) = N_k^{(1)} N_k^{(2)} \cdot \left\{ \left( E_k^{(1)} - E^{(1)} \right) \left( \alpha_k \sin kx + \beta_k \cos kx \right) + \left( E^{(1)} - E^{(2)} \right) \left( \alpha e^{c_2 x} + \beta e^{-c_2 x} \right) \frac{(c_2 - c_1) \alpha e^{c_2 x} - (c_2 + c_1) \beta e^{-c_2 x}}{c_2 \left( \alpha - \beta \right) - c_1 \left( \alpha + \beta \right)} \right\}.$$ \hspace{1cm} (89)

### 4.3 Bound states and the energy spectrum

Now we consider the bound state with WF:

$$\varphi_k^{(1,\text{def})}(x = 0) = 0, \quad \varphi_k^{(1,\text{def})}(x = a) = 0.$$ \hspace{1cm} (90)

At point $x = 0$ we have:

$$\varphi_k^{(1,\text{def})}(x = 0) = N_k^{(1)} N_k^{(2)} \cdot \left\{ \left( E_k^{(1)} - E^{(1)} \right) \beta_k + \left( E^{(1)} - E^{(2)} \right) \left( \alpha + \beta \right) \frac{k \alpha_k - c_1 \beta_k}{c_2 \left( \alpha - \beta \right) - c_1 \left( \alpha + \beta \right)} \right\} = 0.$$ \hspace{1cm} (91)

From here we find:

$$\alpha_k \cdot k \left( E^{(1)} - E^{(2)} \right) \left( \alpha + \beta \right) = \beta_k \cdot \left[ E^{(1)} - E_k^{(1)} + \frac{\left( E^{(1)} - E^{(2)} \right) c_1 \left( \alpha + \beta \right)}{c_2 \left( \alpha - \beta \right) - c_1 \left( \alpha + \beta \right)} \right].$$ \hspace{1cm} (92)

At $\alpha + \beta = 0$ we obtain:

$$\beta_k = 0, \quad \varphi_k^{(1)}(x) = \alpha_k \sin kx.$$ \hspace{1cm} (93)

At $\alpha + \beta \neq 0$ we obtain:

$$\alpha_k = \beta_k \cdot T_k,$$ \hspace{1cm} (94)

where

$$T_k = \frac{c_1}{k} + \frac{\left( E^{(1)} - E_k^{(1)} \right) \left( \alpha - \beta \right) - c_1 \left( \alpha + \beta \right)}{k \left( E^{(1)} - E^{(2)} \right) \left( \alpha + \beta \right)}.$$ \hspace{1cm} (95)

Taking into account $E_k = k^2$, we rewrite (95) so:

$$T_k = \frac{c_1 + E^{(1)} \left( \alpha - \beta \right) - c_1 \left( \alpha + \beta \right)}{k \left( E^{(1)} - E^{(2)} \right) \left( \alpha + \beta \right)} - k \cdot \frac{c_2 \left( \alpha - \beta \right) - c_1 \left( \alpha + \beta \right)}{k \left( E^{(1)} - E^{(2)} \right) \left( \alpha + \beta \right)}.$$ \hspace{1cm} (96)
According to the previous section, if we consider only the real deformations of the rectangular well, then the function:

\[ S(x) = \left( \mathcal{E}^{(1)} - \mathcal{E}^{(2)} \right) \frac{\alpha e^{c_2 x} + \beta e^{-c_2 x}}{\alpha (c_2 - c_1) e^{c_2 x} - \beta (c_2 + c_1) e^{-c_2 x}} \]  

becomes real (at \( x \in [0, a] \)). Taking into account, that it does not depend on \( k \), we rewrite (96) so:

\[ T_k = \frac{c_1 + \mathcal{E}^{(1)}}{S(0)} k - \frac{k}{S(0)}. \]  

Now consider the deformed WF at the second point \( x = a \):

\[ \varphi_{k, \text{def}}^{(1)}(x = a) = N_k^{(1)} N_k^{(2)} \left[ \left( E_k^{(1)} - \mathcal{E}^{(1)} \right) \left( a_k \sin ka + \beta_k \cos ka \right) + \left( \mathcal{E}^{(1)} - \mathcal{E}^{(2)} \right) \frac{(k \alpha_k - c_1 \beta_k) \cos ka - (k \beta_k + c_1 \alpha_k) \sin ka}{(c_2 - c_1) \alpha e^{c_2 a} - (c_2 + c_1) \beta e^{-c_2 a}} \right] = 0. \]  

From here we find:

\[ \sin ka \cdot \left\{ -\alpha_k \left( E_k^{(1)} - \mathcal{E}^{(1)} \right) + S(a) (k \beta_k + c_1 \alpha_k) \right\} = \cos ka \cdot \left\{ \beta_k \left( E_k^{(1)} - \mathcal{E}^{(1)} \right) + S(a) (k \alpha_k - c_1 \beta_k) \right\}. \]  

Taking into account (94), we obtain:

\[ \sin ka \cdot \left\{ T_k \left( E_k^{(1)} - k^2 \right) + S(a) (k + c_1 T_k) \right\} = \cos ka \cdot \left\{ k^2 - \mathcal{E}^{(1)} + S(a) (k T_k - c_1) \right\}. \]  

Substituting \( T_k \) from (98), we find:

\[ \sin ka \cdot \left\{ \left[ c_1 + \mathcal{E}^{(1)} - k^2 \right] \left( \mathcal{E}^{(1)} - k^2 \right) + S(0) S(a) k^2 + S(a) c_1 \left[ c_1 + \mathcal{E}^{(1)} - k^2 \right] \right\} = \cos ka \cdot k \cdot \left\{ S(0) (k^2 - \mathcal{E}^{(1)}) + S(a) (c_1 + \mathcal{E}^{(1)} - k^2) - S(0) S(a) c_1 \right\}. \]  

Now we have obtained equation of the energy spectrum for the deformed potential. At \( \cos ka \neq 0 \) one can write:

\[ \tan ka = \frac{k (A + B k^2)}{C + D k^2 + k^4}, \]  

where

\[ \begin{align*}
A &= -S(0) \mathcal{E}^{(1)} + S(a) (c_1 + \mathcal{E}^{(1)}) - S(0) S(a) c_1, \\
B &= S(0) - S(a), \\
C &= (c_1 + \mathcal{E}^{(1)}) \mathcal{E}^{(1)} + S(a) c_1 (c_1 + \mathcal{E}^{(1)}), \\
D &= -c_1 - 2 \mathcal{E}^{(1)} - S(a) c_1 + S(0) S(a). 
\end{align*} \]  

At

\[ A = 0, \quad B = 0 \]  

we come to the energy spectrum for the rectangular well — before its deformation. For this, the following condition must be fulfilled:

\[ S(a) \cdot (1 - S(0)) \cdot c_1 = 0, \quad S(0) = S(a). \]  

### 5 Deformation of potentials with continuous energy spectra and new reflectionless potentials

Now we shall analyze whether the double complex SUSY-transformations can give some-thing interesting in construction of new real potentials with continuous energy spectra. Let’s consider the simplest case — deformation of null potential. Omitting external walls of the rectangular well (57), we transform it into such null potential. Here, one can use formulas, obtained in the previous section.
As before, we use the functions of transformation in the form \(\text{(65)}\). The deformed null potential \(V_{1}^{(\text{def})}\) has the form:

\[
V_{2}(x) = 2\left(\mathcal{E}^{(2)} - \mathcal{E}^{(1)}\right) \frac{d}{dx} \alpha e^{c_{x}x} + \beta e^{-c_{x}x}{\alpha (c_{2} - c_{1}) e^{c_{x}x} - \beta (c_{2} + c_{1}) e^{-c_{x}x}}}.
\]  

This potential is complex. To be it real, we use real negative \(\mathcal{E}^{(1)}\) and \(\mathcal{E}^{(2)}\), and we use the conditions \(\text{(79)}, \text{(81)}\):

\[
\Im(c_{1}) = 0, \quad \Im(c_{2}) = 0, \quad \Re(\alpha) \Im(\beta) = \Im(\alpha) \Re(\beta).
\]  

Lets’ analyze behavior of the potential \(V_{2}\) in asymptotic regions. At limits \(x \to \pm \infty\) we obtain (in numerator and denominator we write only maximal components):

\[
V_{2}(x \to \pm \infty) = \frac{g_{1} e^{2R(c_{2})x} + g_{2} e^{-2R(c_{2})x}}{|\alpha|^{4} c_{2} - c_{1}|^{4} e^{4R(c_{2})x} + |\beta|^{4} c_{2} + c_{1}|^{4} e^{-4R(c_{2})x}|} = 0,
\]  

where \(g_{1}\) and \(g_{2}\) are constants. So, in both asymptotic regions \(V_{2}\) tends to zero. Therefore, consideration of propagation of wave in this potential is a natural problem, and general solution for wave function for such wave in both asymptotic regions is linear combination of two plane waves \(\exp(\pm ikx)\), with opposite directions of propagation.

Now we clarify whether \(V_{2}\) has divergencies and discontinuities. From \(\text{(107)}\) we see that this potential is continuous if denominator under differential in \(\text{(107)}\) is not equal to zero. So, the condition of existence of divergency at point \(x_{d}\) is the following:

\[
\alpha (c_{2} - c_{1}) e^{c_{x}x_{d}} - \beta (c_{2} + c_{1}) e^{-c_{x}x_{d}} = 0.
\]  

We find coordinate of divergency:

\[
x_{d} = \frac{1}{2c_{2}} \log \left(\frac{\beta (c_{2} + c_{1})}{\alpha (c_{2} - c_{1})}\right).
\]  

Using formula for logarithm with complex argument:

\[
\log z = \log |z| + i \arg z,
\]  

rewrite coordinate \(\text{(111)}\) so:

\[
x_{d} = \frac{1}{2c_{2}} \log \left|\frac{\beta (c_{2} + c_{1})}{\alpha (c_{2} - c_{1})}\right| + \frac{i}{2c_{2}} \arg \left(\frac{\beta (c_{2} + c_{1})}{\alpha (c_{2} - c_{1})}\right).
\]  

Taking into account that \(x_{d}\) can be real only, we see that at arbitrary values of \(c_{1}, c_{2}, \alpha, \beta\) the expression \(\text{(113)}\) is not correct. For example, if to use:

\[
\frac{\beta (c_{2} + c_{1})}{\alpha (c_{2} - c_{1})} \in \Re, \quad \frac{\beta (c_{2} + c_{1})}{\alpha (c_{2} - c_{1})} < 0,
\]  

then the potential \(\text{(107)}\) has not divergencies and is continuous in the whole axis \(x\). So, we have proved that with exception of some values of \(c_{1}, c_{2}, \alpha, \beta\), the potential \(V_{2}(x)\) is continuous function in the whole axis \(x\).

The found potential \(\text{(110)}\) is reflectionless: if to start a plane wave from its left asymptotic region to the right, then it passes along the potential without any reflections, reaching the right asymptotic region. This is followed, for example, from Rule of construction of new reflectionless potentials on the basis of one given one, formulated in \(\text{(69)}\) (see p. 447; see some corrections in \(\text{(70)}\); one can make sure in this also, using analysis in p. 280 \(\text{(5)}\)), taking into account of continuity of both superpotentials on the whole axis. This potential is shown in Fig. \(\text{I}\).

Let’s consider a well-known reflectionless potential of such form (for example, which is presented by \(\text{(33)}\) in p. 280 \(\text{(5)}\)):

\[
V_{\text{refl}} = A_{\text{refl}}^{2} \cdot \left\{1 - \frac{2}{\cosh^{2}(\alpha_{\text{refl}} x)}\right\},
\]  

where \(A_{\text{refl}}\) and \(\alpha_{\text{refl}}\) are real constants (connected by condition: \(A_{\text{refl}} = -\alpha_{\text{refl}}\)). This potential belongs to the class of Pöschl-Teller potentials (see papers \(\text{(50)}\) — in approaches of nonlinear supersymmetry with complex potentials, \(\text{(41)} - \text{(42)}, \text{(43)} - \text{(44)}\) — in the approach of hidden nonlinear supersymmetry, \(\text{(65)}, \text{(64)}, \text{(56)}\) — in the approach with PT-symmetry, \(\text{(71)}\) — a generalization of Morse potential in two-dimensional space). One can see that in
Figure 1: Reflectionless potential: (a) — displacement of the hole with its passing through zero (the values are used: $\mathcal{E}^{(1)} = -7, -20, -1000$; $\mathcal{E}^{(2)} = -5$; $a = 10 + i100$, $\Re(b) = 3$); (b) — continuous pulling down of the hole close zero (the values are used: $\mathcal{E}^{(1)} = -20$; $\mathcal{E}^{(2)} = -1, -3, -5$; $a = 10 + i100$, $\Re(b) = 3$).

both asymptotic regions it tends to nonzero limit $A^2 \neq 0$ (direct limit $A_{\text{refl}} \to 0$ does not give zero asymptotic limits for this potential, with keeping nonzero hole), its depth at $x = 0$ is fixed by constant $A$. Our found reflectionless potential has also hole of finite depth, but one can pull down this hole continuously or displace it along axis $x$, passing zero; in both asymptotic regions the potential tends to null values. On the other side, if as the starting potential $V_1$ to use nonzero constant potential with high $A^2$, then on the basis of the double SUSY-transformations one can obtain the reflectionless potential of the form (107), but with tails in the asymptotic regions as for the potential (115). So, use of negative energies of transformation with arbitrary values in the double SUSY-transformations allows to generalize the reflectionless potential (115).

If as the first function of transformation $\phi_1$ instead of (65) to use the general its solution:

$$
\begin{align*}
\phi_1(x) &= \alpha_1 e^{c_1 x} + \beta_1 e^{-c_1 x}, \\
\phi_2(x) &= \alpha_2 e^{c_2 x} + \beta_2 e^{-c_2 x},
\end{align*}
$$

(116)

then we obtain else larger possibilities in deformation of the null potential. We shall show shape of such deformed potential without intermediate calculations. In Fig. 2 one can see, how by shifting only the second energy of transformation $\mathcal{E}_2$ from zero to the value of the first energy of transformation $\mathcal{E}_1$, one can transform smoothly (continuously) the symmetric one-well potential into symmetric double-well one. So, we have obtained the simple tools for control of asymmetry of the shape of such potentials. Note, that all these potentials satisfy to Rule of construction of new reflectionless potentials on the basis of the given one in [69] with taking into account of continuity of both superpotentials, and therefore they are reflectionless (such potentials look like the reflectionless shape invariant potentials with scaling of parameters [72, 73, 74, 75, 76, 77] and self-similar potentials [78, 79, 80] (which can be symmetric in many cases) but here formulas are simpler essentially, and we do not find else in other papers such simple way for control of asymmetry of their shape).

It is interesting to analyze whether a class of the reflectionless potentials, obtained on the basis of group approach developed in [81, 82], can be expanded after taking into account of the approach from sec. 3 for construction of the reflectionless potential (107).

6 Conclusions

In the paper the approach of double complex SUSY-transformations with not coincident complex energies of transformation is developed (see sec. 3, allowing to deform the given real potential $V_1$ (with keeping of its shape real) and its spectral characteristics with obtaining exact solutions. Note the following.

- Appropriateness of such approach has been shown with obtaining of new forms of real potentials.
• Explicit solutions of the deformation of the shape of the potential, its wave function at arbitrary energy, not coincident with the energies of transformation, wave functions at the energies of transformation are obtained, condition of keeping of continuity of the solutions (i.e. without appearance of divergencies and discontinuities), isospectral condition with keeping of energy spectra are determined (for potentials with discrete energy spectra).

• Using the rectangular well of finite width with infinitely high walls as the starting \( V_1 \) with discrete energy spectrum, by the proposed approach new types of deformation of this potential with deformation of the energy spectrum as a whole have been obtained (see sec. [4]) in particularly, we do not find such deformations in variety of deformations of the well, presented in reviews [1, 2, 3, 4]). The new potential contains the rectangular well as own partial case (with simultaneous transformation of the shape of this new potential, energy spectrum, wave functions of all bound states, wave function at arbitrary energy into corresponding characteristics of the rectangular well at needed choice of the parameters of the deformation).

• Using the null potential as the starting \( V_1 \) with continuous energy spectrum, new form of the reflectionless real potential has been constructed (see sec. [5]). This potential generalizes the well-known reflectionless potential of the type \( V_{ref}(x) = A^2(1 - 2 \text{sech}^2 Ax) \), allowing:
  - to pull down tails of the potential \( V_{ref} \) in the asymptotic regions up to zero exactly (with keeping of nonzero depth);
  - to pull down continuously the finite depth of the hole;
  - to displace arbitrary along axis \( x \) the hole with its passing through zero;
  - to create and to increase the second hole, transforming \( V_{ref} \) into double-well potential;
  - to control continuously and simply the asymmetry of the shape of such reflectionless potential.

Note relative simplicity of the found potential in a comparison with variety of the reflectionless shape invariant potentials (with scaling of parameters; for example, see [5]).

**Acknowledgements**

The author is appreciated to Prof. Mikhail S. Plyushchay for useful comments concerning existence of the hidden nonlinear supersymmetry in the Pöschl-Teller potentials and variety of forms of these potentials.
References

[1] B. N. Zakhariev, N. A. Kostov and E. B. Plehanov, *Exactly solvable one- and manychannel models (Quantum intuition lessons)*, Physics of elementary particles and atomic nuclei **21** (Iss. 4), 914–962 (1990) — [in Russian].

[2] B. N. Zakhariev and V. M. Chabanov, *Qualitative theory of control of spectra, scattering, decays (Quantum intuition lessons)*, Physics of elementary particles and atomic nuclei **25** (Iss. 6), 1561–1597 (1994) — [in Russian].

[3] Zakhariev, B. N. and Chabanov, V. M., *Physics of elementary particles and atomic nuclei* **30** (Iss. 2), 277–320 (1999).

[4] B. N. Zakhariev and V. M. Chabanov, *Physics of elementary particles and atomic nuclei* **33** (Iss. 2), 348–392 (2002).

[5] F. Cooper, A. Khare and U. Sukhatme, *Supersymmetry and quantum mechanics*, Physics Reports **251** (5–6), 267–385 (January, 1995); arXiv:hep-th/9405029.

[6] A. Lahiri, P. K. Roy, and B. Bagchi, *Supersymmetry in quantum mechanics*, International Journal of Modern Physics A **5** (8), 1383 – 1456 (April, 1990).

[7] E. Witten, *Dynamical breaking of supersymmetry*, Nuclear Physics **B188** (3), 513–554 (October, 1981).

[8] L. Gendenshtein, *Derivation of exact spectra of the Schrödinger equation by means of supersymmetry*, JETP Letters **38**, 356 (September, 1983).

[9] C. V. Sukumar, *Supersymmetry, factorisation of the Schrödinger equation and a Hamiltonian hierarchy*, Journal of Physics A: Mathematical and General **18** (2), L57–L61 (February, 1985).

[10] A. A. Andrianov, N. V. Borisov and M. V. Ioffe, *The factorization method and quantum systems with equivalent energy spectra*, Physics Letters **A105** (1–2), 19–22 (October, 1984).

[11] A. A. Andrianov, N. V. Borisov and M. V. Ioffe, *Factorization method and Darboux transformation for multidimensional Hamiltonians*, Theoretical Mathematical Physics **61** (2), 1078–1088 (November, 1984).

[12] D. L. Pursey, *New families of isospectral Hamiltonians*, Physical Review **D33** (4–15), 1048–1055 (February, 1986).

[13] C. V. Sukumar, *Supersymmetric quantum mechanics of one-dimensional systems*, Journal of Physics A: Mathematical and General **18** (15), 2917–2936 (October, 1985).

[14] C. V. Sukumar, *Supersymmetric quantum mechanics and the inverse scattering method*, Journal of Physics A: Mathematical and General **18** (15), 2937–2955 (October, 1985).

[15] C. V. Sukumar, *Potentials generated by SU (1,1)*, Journal of Physics A: Mathematical and General **19** (11), 2229–2328 (August, 1986).

[16] C. V. Sukumar, *Supersymmetry, potentials with bound states at arbitrary energies and multi-soliton configurations*, Journal of Physics A: Mathematical and General **19** (12), 2297–2316 (August, 1986).

[17] C. V. Sukumar, *Supersymmetry and potentials with bound states at arbitrary energies. II*, Journal of Physics A: Mathematical and General **20** (9), 2461–2481 (June, 1987).

[18] C. V. Sukumar, *Supersymmetrical transformations and Hamiltonians generated by the Marchenko equations*, Journal of Physics A: Mathematical and General **21** (8), L455–L458 (April, 1988).

[19] H. C. Rosu, *Short survey of Darboux transformations* (Talk given at the 1-st Burgos International Workshop on Symmetries in quantum mechanics and quantum optics, Burgos, Spain, September 21–24, 1998); quant-ph/9809056.

[20] B. Mielnik, and O. Rosas-Ortiz, *Factorization: little or great algorithm?* Journal of Physics A: Mathematical and General **37** (43), 10007–10035 (October, 2004).
[21] L. Infeld and T. E. Hull, *The factorization method*, Review of Modern Physics 23 (1), 21–68 (January, 1951).

[22] G. Darboux, C. R. Acad. Sci. 94, 1456 (1882).

[23] V. G. Bagrov and B. F. Samsonov, *Darboux transformation of the Schrödinger equation*, Physics of elementary particles and atomic nuclei 28 (4), 374–397 (July, 1997).

[24] B. F. Samsonov, *New possibilities for supersymmetry breakdown in quantum mechanics and second-order irreducible Darboux transformations*, Physics Letters A263 (4–6), 274–280 (December, 1999); quant-ph/9904009.

[25] B. F. Samsonov and F. Stancu, *Phase equivalent chains of Darboux transformations in scattering theory*, Physical Review C66 (3), 034001 [12 pages] (September, 2002); quant-ph/0204112.

[26] D. J. Fernandez, B. Mielnik, O. Rosas-Ortiz and B. F. Samsonov, *The phenomenon of Darboux displacements*, Physics Letters A294 (3–4), 168–174 (February, 2002); quant-ph/0302204.

[27] A. A. Andrianov, F. Cannata, J.-P. Dedonder, and M. V. Ioffe, *Second order derivative supersymmetry and scattering problem*, International Journal of Modern Physics A 10 (18), 2683–2702 (July, 1995); hep-th/9404061.

[28] A. A. Andrianov, M. V. Ioffe and D. N. Nishnianidze, *Polynomial SUSY in quantum mechanics and second derivative Darboux transformations*, Physics Letters A201 (2–3), 103 (May, 1995); hep-th/9404120.

[29] Bagrov, V. G., Samsonov, B. F. and Shekoyan, L. A. *N-order Darboux transformation and a spectral problem on semiaxis*, quant-ph/9804032.

[30] A. A. Andrianov and A. V. Sokolov, *Nonlinear supersymmetry in quantum mechanics: algebraic properties and differential representation*, Nuclear Physics B660 (1–2), 25–50 (June, 2003); hep-th/0301062.

[31] A. A. Andrianov and F. Cannata, *Nonlinear supersymmetry for spectral design in quantum mechanics*, Journal of Physics A: Mathematical and General 37 (43), 10297–10323 (October, 2004); hep-th/0407077.

[32] A. I. Pashnev, *One-dimensional supersymmetric quantum mechanics with N ≥ 2*, Theoretical and Mathematical Physics, 69 (2), p. 1172–1175 (November, 1986); (translated from Teoreticheskaya i Matematicheskaya Fizika, 69 (2), p. 311–315 (November, 1986)).

[33] V. N. Berezovoi and A. I. Pashnev, *Supersymmetric quantum mechanics and rearrangement of the spectra of hamiltonians*, Theoretical and Mathematical Physics, 70 (1), p. 102–107 (January, 1987); (translated from Teoreticheskaya i Matematicheskaya Fizika, 70 (1), p. 146–153 (January, 1987)).

[34] V. N. Berezovoi and A. I. Pashnev, *N=2 Supersymmetric quantum mechanics and the inverse scattering problem*, Theoretical and Mathematical Physics, 74 (3), p. 264–268 (March, 1988); (translated from Teoreticheskaya i Matematicheskaya Fizika, 74 (3), p. 392–398 (March, 1988)).

[35] V. P. Berezovoj and A. I. Pashnev, *Extended N = 2 supersymmetric quantum mechanics and isospectral Hamiltonians*, Zeitschrift für Physik C: Particles and Fields, 51, p. 525–529 (1991).

[36] B. F. Samsonov, A. A. Pecheritshin, *Chains of Darboux transformations for the matrix Schrödinger equation*, Journal of Physics A: Mathematical and General 37 (1), 239–250 (January, 2004).

[37] F. Cannata and M. V. Ioffe, *Coupled-channel scattering and separation of coupled differential equations by generalized Darboux transformations*, Journal of Physics A: Mathematical and General 26 (3), L89–L92 (February, 1993).

[38] A. A. Suzko, *Darboux transformations for a system of coupled discrete Schrödinger equations*, Physics of Atomic Nuclei 65 (8), 1553–1559 (August, 2002).

[39] M. Humi, *Darboux transformations for the Schrödinger equation in 3 dimensions*, Journal of Physics A: Mathematical and General 21 (9), 2075–2084 (May, 1988).

[40] A. Gonzalez-Lopez and N. Kamran, *The multidimensional Darboux transformations*, Journal of Geometry and Physics 26 (3–4), 202–226 (July, 1998); hep-th/9612100.
[41] F. Correa, and M. S. Plyushchay, *Hidden supersymmetry in quantum bosonic systems*, Annals of Physics **322** (10), 2493–2500 (October, 2007); hep-th/0605104.

[42] F. Correa, and M. S. Plyushchay, *Peculiarities of the hidden nonlinear supersymmetry of Poschl-Teller system in the light of Lame equation*, Journal of Physics A: Mathematical and General, — in press; arXiv:0706.1114.

[43] F. Correa, L.-M. Nieto, and M. S. Plyushchay, *Hidden nonlinear supersymmetry of finite-gap Lame equation*, Physics Letters B**644** (1), 94–98 (January, 2007); hep-th/0605104.

[44] S. M. Klishchevich, and M. S. Plyushchay, *Nonlinear supersymmetry, quantum anomaly and quasieaxtly solvable systems*, Nuclear Physics B**606** (3), 583–612 (July, 2001); hep-th/0012023.

[45] A. A. Andrianov, F. Cannata and A. V. Sokolov, *Non-linear Supersymmetry for non-Hermitian, non-diagonalizable Hamiltonians. I. General properties*, Nuclear Physics B**773** (3), 107–136 (July, 2007); math-ph/0610024.

[46] A. V. Sokolov, *Non-linear supersymmetry for non-Hermitian, non-diagonalizable Hamiltonians: II. Rigorous results*, Nuclear Physics B**773** (3), 137–171 (July, 2007); math-ph/0610022.

[47] D. Baye, G. Levai and J. M. Sparenberg, *Phase-equivalent complex potentials*, Nuclear Physics A**599** (3–4), 435–456 (March, 1996).

[48] A. A. Andrianov, M. V. Ioffe, F. Cannata and J.-P. Dedonder, *SUSY quantum mechanics with complex superpotentials and real energy*, International Journal of Modern Physics A**14** (17), 2675–2688 (1999).

[49] V. M. Chabanov and B. N. Zakhariev, *Unusual (non-Gamov) decay states*, Inverse problems **17** (4), 683–693 (August, 2001).

[50] R. N. Deb, A. Khare and B. D. Roy, *Complex optical potentials and pseudoHermitian Hamiltonians*, Physics Letters A**307** (4), 215–221 (February, 2003).

[51] C. M. Bender and S. Boettcher, *Real spectra in non-hermitian hamiltonians having PT-symmetry*, Physical Review Letters **80** (24), 5243–5246 (June, 1998).

[52] C. M. Bender, S. Boettcher and P. Meisinger, *PT symmetric quantum mechanics*, Journal of Mathematical Physics **40** (5), 2201–2229 (May, 1999); quant-ph/9809072.

[53] M. Znojil, *PT-symmetric harmonic oscillator*, Physics Letters A**259** (3–4), 220–223 (August, 1999); quant-ph/9905020.

[54] M. Znojil, *PT-symmetrically regularized Eckart, Poeschl-Teller and Hulthen potentials*, Journal of Physics A: Mathematical and General **33** (24), 4561–4572 (June, 2000); quant-ph/0101131.

[55] M. Znojil, *PT symmetric square well*, Physics Letters A**285** (1–2), 7–10 (June, 2001); quant-ph/0101131.

[56] M. Znojil and G. Levai, *The interplay of supersymmetry and PT symmetry in quantum mechanics: a case study for the Scarf II potential*, Journal of Physics A: Mathematical and General **35** (41), 8793–8804 (October, 2002).

[57] M. Znojil, *Matching method and exact solvability of discrete PT-symmetric square wells*, Journal of Physics A: Mathematical and General **39** (32), 10247–10261 (August, 2006).

[58] M. Znojil, F. Cannata, B. Bagchi and R. Roychoudhury, *Supersymmetry without hermiticity within PT symmetric quantum mechanics*, Physics Letters B**483** (1–3), 284–289 (June, 2000); hep-th/0003277.

[59] G. Levai, F. Cannata and A. Ventura, *Algebraic and scattering aspects of a PT-symmetric solvable potential*, Journal of Physics A: Mathematical and General **34** (4), 839–844 (October, 2001).
[62] G. Levai, F. Cannata and A. Ventura, *PT-symmetric potentials and the so(2,2) algebra*, Journal of Physics A: Mathematical and General **35** (24), 5041–5057 (June, 2002).

[63] M. Znojil, *PT-symmetric regularizations in supersymmetric quantum mechanics*, Journal of Physics A: Mathematical and General **37** (43), 10209–10222 (October, 2004); hep-th/0404145.

[64] G. Levai, *Exact analytic study of the PT-symmetry-breaking mechanism*, Czechoslovak Journal of Physics **54** (1), 77–84 (2004).

[65] G. Levai, *Supersymmetry without hermiticity*, Czechoslovak Journal of Physics **54** (10), 1121–1124 (2004).

[66] M. Znojil, V. Jakubsky, *Solvability and PT-symmetry in a double-well model with point interactions*, Journal of Physics A: Mathematical and General **38** (22), 5041–5056 (June, 2005).

[67] M. Znojil, *Exactly solvable models with PT-symmetry and with an asymmetric coupling of channels*, Journal of Physics A: Mathematical and General **39** (15), 4047-4061 (April, 2006).

[68] F. Cannata, J.-P. Dedonder and A. Ventura, *Scattering in PT-symmetric quantum mechanics*, Annals of Physics **322** (2), 397–433 (February, 2007); quant-ph/0606129.

[69] S. P. Maydanyuk, *SUSY-hierarchy of one-dimensional reflectionless potentials*, Annals of Physics **316** (2), 440–465 (April, 2005); hep-th/0407237.

[70] S. P. Maydanyuk, *New exactly solvable reflectionless potentials of Gamov’s type* (talk on the XXXII Winter School of Physics, February 22 - March 2, 2005, ITEP, Moscow), Surveys in HEP, **19** (3–4), 175–192 (September–December, 2004); nucl-th/0504077.

[71] F. Cannata, M. V. Ioffe, and D. N. Nishnianidze, *Pseudo-hermiticity of an exactly solvable two-dimensional model*, Physics Letters A **369**, 9–15 (April, 2007); quant-ph/0606129.

[72] R. Dutt, A. Khare and U. Sukhatme, *Supersymmetry, shape invariance, and exactly solvable potentials*, American Journal of Physics **56** (2), 163–168 (February, 1988).

[73] D. T. Barclay and C. J. Maxwell, *Shape invariance and the SWKB series*, Physics Letters A **157** (6–7), 357–360 (August, 1991).

[74] D. T. Barclay, R. Dutt, A. Gangopadhyaya, A. Khare, A. Pagnamenta and U. Sukhatme, *New exactly solvable Hamiltonians: shape invariance and self-similarity*, Physical Review A **48** (4), 2786–2797 (October, 1993); arxiv:hep-ph/9304313.

[75] A. Khare and U. P. Sukhatme, *New shape invariant potentials in supersymmetric quantum mechanics*, Journal of Physics A: Mathematical and General **26** (18), L901–L904 (September, 1993); arxiv:hep-th/9212147.

[76] D. Gomez-Ullate, N. Kamran and R. Milson, *The Darboux transformation and algebraic deformations of shape-invariant potentials*, Journal of Physics A: Mathematical and General **37** (5), 1789–1804 (February, 2004).

[77] S. P. Maydanyuk and L. M. Saryan, *Space asymmetric deformation of SIP-potentials and self-similar potentials of Shabat and Spiridonov* (talk on the 4-th International Symposium “Quantum Theory and Symmetries” and the 6-th International Workshop “Lie Theory and Its Applications in Physics”, Varna, Bulgaria, August 15-21, 2005), arXiv:hep-th/0512034, — in press.

[78] A. Shabat, *The infinite-dimensional dressing dynamical system*, Inverse Problems **8** (2), 303–308 (April, 1992).

[79] V. P. Spiridonov, *Exactly solvable potentials and quantum algebras*, Physical Review Letters **69** (3), 398–401 (July, 1992); hep-th/9112075.

[80] V. P. Spiridonov, *The factorization method: selfsimilar potentials and quantum algebras*, Lecture given at Advanced Study Institute on Special Functions 2000: “Current Perspective and Future Directions”, (Tempe, Arizona, 29 May – 9 Jun, 2000), 39 pp.; hep-th/0302046.
[81] G. A. Kerimov, A. Ventura, *Group-theoretical approach to reflectionless potentials*, Journal of Mathematical Physics 47 (8), 082108–082108-16 (2006).

[82] G. A. Kerimov, *Non-spherical symmetric transparent potentials for the three-dimensional Schrödinger equation*, Journal of Physics A: Mathematical and General 40, 11607–11615 (2007).