Quantized Innovations Bayesian Filtering

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Abstract

The paper provides simple formulas of Bayesian filtering for the exact recursive computation of state conditional probability density functions given quantized innovations signal measurements of a linear stochastic system. This is a topic of current interest because the innovations signal should be white and therefore efficient in its use of channel capacity and in the design and optimization of the quantizer. Earlier approaches, which we reexamine and characterize here, have relied on assumptions concerning densities or approximations to yield recursive solutions, which include the sign-of-innovations Kalman filter and a Particle filtering technique. Our approach uses the Kalman filter innovations at the transmitter side and provides a point of comparison for the other methods, since it is based on the Bayesian filter. Computational examples are provided.

1 Introduction

Efficient transmission of signals over a limited capacity digital channel requires coding and quantization. Here we examine the recently explored problem of linear state estimation based on quantized innovations signals to reconstruct the conditional probability density function of the predicted and filtered system state given these measurements. We draw on [2, 9, 5] for algorithmic approaches to this problem, which in turn are based on Bayesian filtering formulations. The Bayesian filter is a recursive approach for the computation of predicted and filtered state conditional probability density functions (pdfs) given a sequence of measurements. It is an approach to nonlinear filtering. Our contribution is to develop for linear stochastic systems a signal transmission scheme based on the quantized Kalman filter innovations at the transmitter and an associated explicit Bayesian filtering solution at the receiver. We also draw comparisons to other works through the provision of a unifying framework from which to view the development. The advantage of this new approach is that it can benefit from the whiteness of the Kalman filter innovations and its zero-mean and known covariance properties to realize optimal coding and Lloyd-Max optimal quantization [11]. We also postulate that it has improved error recovery properties because of the reconvergent behavior of the Kalman filter.

An innovations signal, \( \epsilon_k \), is formed by subtracting from a measured signal, \( y_k \), its one-step-ahead prediction, \( \hat{y}_{k|k-1} \). In signal processing, such calculations are performed in order to whiten a signal prior to transmission over a bandlimited channel and thereby to achieve more efficient use of the channel capacity. In the formal context of Kalman filtering for linear Gaussian systems, the innovations formed from the state prediction is statistically white; a property that we shall apply. The prediction can equally be computed as a reconstruction of a signal available at the receiver, which is achieved at the transmitter by using the computed receiver signal \( \bar{r}_k \) and control \( u_k \) as the inputs to the predictor. Such a scheme forms the basis of the ITU-T G.726 adaptive differential pulse-coded modulation speech codec standard. The topic of investigation in this paper is the collection of methods using quantized innovations signals for the construction of a state state conditional pdf at the receiver. This is depicted in Figure 1, where the reconstruction is performed by a Bayesian filter, which we derive in the sequel. We presume that the input signal, \( \{ u_k \} \), is available to both ends; a property that occurs in remotely located sensing or filtering.

We use an \( m \)-level quantizer, \( Q_k(\cdot) \), consisting of a collection of \( m \) intervals \( \{ z_{k,t}, z_{k,u} \} \), which form a disjoint covering of the real line and a corresponding rule, \( Q_k^{-1}(\cdot) \), for dequantizing the received signal into a real number — for a \( p \)-vector channel quantizer, we take a scalar quantizer in each channel. When the innovations signal \( \epsilon_k \) lies within the range \( \{ z_{k,t}, z_{k,u} \} \) the output of the quantizer is transmitted as one of \( m \) symbols which is then dequantized at the receiver.

\[ y_{k+1} = A_k x_k + B_k u_k + w_k \]
\[ y_k = C_k x_k + v_k \]

Fig. 1. Block diagram of quantized innovations system.
as $i_k$ as one of $m$ distinct values, which we take also to be a value within the range $(z_{k,1}, z_{k,u})$ including for the two edge saturation levels. A 3-bit uniform quantizer-dequantizer cascade is shown in Figure 2. Evidently, $i_k = Q_k^{-1} Q_k(\zeta)$. The input is the innovations signal, $\epsilon_k$, and the output is the recovered innovations signal, $i_k$. The values $\pm \zeta$ denote the upper and lower saturation limits of the quantizer, which we take for simplicity to be symmetric. Our results apply equally to any well defined quantizer-dequantizer pair without reliance on symmetry or uniformity – this is important because it admits optimized quantizer designs.

Fig. 2. Three-bit ($m = 8$) uniform quantizer-dequantizer pair input-output relation with $\zeta = 0.6222$ and $L = 2\zeta/2^3 = 0.1555$. The methods of the paper apply equally to non-symmetric and non-uniform quantizers.

The receiver reconstructs a conditional pdf of the plant state $x_k$ given the received quantized innovations data, $\{i_k, i_{k-1}, \ldots, i_0\}$. We shall consider three distinct constructions of this measurement signal and corresponding receiver pdfs, distinguished where necessary by superscripts. The detailed algorithms will be presented in Section 3.

**Method K:** A Kalman filter is operated at the transmitter using signals $\{y_k\}$ and $\{u_k\}$ to produce Kalman output prediction $\hat{x}_{k|k-1}$, and Kalman innovations signal $\epsilon_k$, which is then quantized, transmitted and dequantized at the receiver.

$$\bar{\epsilon}_k = Q_k^{-1} Q_k(\epsilon_k) = Q_k^{-1} Q_k(y_k - \hat{x}_{k|k-1}),$$  \hspace{1cm} (1)

The receiver conducts its own processing of this data using a K-Bayesian filter developed in Section 3. This is depicted in Figure 3.

**Method R:** The transmitter computes the received signal $i_k$ and uses a copy of the receiver’s R-Bayesian filter to construct $p^R(x_k|\hat{x}_{k-1})$, its own version of receiver’s conditional state prediction pdf with conditional mean

Fig. 3. Block diagram of Method K showing calculation of the innovations signal, $\epsilon_k$. The K-Bayesian filter is detailed in Section 3. Then

$$i_k^R = Q_k^{-1} Q_k(y_k - C_k \hat{x}_{k|k-1}).$$  \hspace{1cm} (2)

This is illustrated in Figure 4.

**Method S:** A modified Kalman filter recursion replaces the R-Bayesian filter of Method R at both transmitter and receiver, yielding synchronized: predicted and filtered state estimates $\hat{x}_{k|k}$ and $\hat{x}_{k-1|k-1}$, output prediction $\hat{y}_{k|k-1} = C_k \hat{x}_{k|k-1}$, using data $\hat{x}_{k|k-1}$. At the transmitter,

$$i_k^S = Q_k^{-1} Q_k(y_k - C_k \hat{x}_{k|k-1}).$$  \hspace{1cm} (3)

The Bayesian filter and its progeny Kalman filter are presented in brief detail in Section 2.1 for reference. Section 3 is then devoted to the explication of the three methods in terms of their algorithms based on the Bayesian and Kalman filter recursions. Computational comparisons are made in Section 4 and conclusions are then provided. The central contribution of the paper is to recognize the transmitter-side signal model underpinning Method K and its corresponding complete Bayesian filter for the receiver side. The subsequent reinterpretation of Methods R and S and their comparison are separate contributions enabled by the new Bayesian computation. In the paper we denote: $p^i(\cdot)$ is a general pdf; $N(\mu, \Sigma)$ is the Gaussian pdf with mean value $\mu$ covariance matrix $\Sigma$. 

Fig. 4. Block diagram of Method R showing calculation of the innovations signal, $\epsilon_k^R$. The R-Bayesian filter is detailed in Section 3.
2 Background and other literature

2.1 The Bayesian Filter and Kalman Filter

Suppose we have a state-space system as follows.

\[
x_{k+1} = f_k(x_k, w_k), \quad z_k = h_k(x_k, v_k),
\]

(4)

where process noise \( \{w_k\} \) and measurement noise \( \{v_k\} \) are white and of known densities and joint density. The functions \( f_k(\cdot, \cdot) \) and \( h_k(\cdot, \cdot) \) are not necessarily linear time-varying system equations. Denoting \( Z_k = \{z_0, z_1 \ldots z_k\} \), the goal of the Bayesian filter is to compute the predicted state conditional probability density function, \( p(x_k|Z_{k-1}) \), and the filtered pdf, \( p(x_k|Z_k) \), from an initial density \( p(x_0) = p(x_0|Z_{-1}) \).

The Bayesian filter recursion [6] is

\[
p(x_k|Z_{k-1}) = \int_{x_{k-1}} p(x_k|x_{k-1})p(x_{k-1}|Z_{k-1})dx_{k-1},
\]

(6)

\[
p(x_k|Z_k) = \frac{p(z_k|x_k)p(x_k|Z_{k-1})}{\int_{x_k} p(z_k|x_k)p(x_k|Z_{k-1})dx_k}.
\]

(7)

is expressed in terms of \( p(x_k|x_{k-1}|Z_{k-1}) \), the filtered conditional pdf from the other half of the recursion, and \( p(x_k|x_{k-1}) \) derived from (4) using knowledge of function \( f \) and the distribution of \( w_k \).

is expressed in terms of \( p(x_k|Z_{k-1}) \), the predicted conditional pdf from the other half of the recursion, and \( p(z_k|x_k,Z_{k-1}) \) derived from (5) using knowledge of function \( h \) and the distribution of \( v_k \).

For linear Gaussian systems, the Bayesian filter yields the explicit conditional pdf calculation of the Kalman filter. The Kalman filter, however, is much more efficient in that it propagates the Gaussian conditional densities via conditional means and conditional covariances only. For the linear system,

\[
x_{k+1} = A_kx_k + B_kv_k + w_k,
\]

(8)

\[
y_k = C_kx_k + v_k,
\]

(9)

with noises \( \{w_k\} \sim \mathcal{N}(0, Q_k) \), \( \{v_k\} \sim \mathcal{N}(0, R_k) \) and \( x_0 \sim \mathcal{N}(\hat{x}_{0|0}, \Sigma_{0|0}) \) all mutually independent, the Kalman filter recursion is

\[
L_k = \Sigma_{k|k-1}^K (C_k \Sigma_{k|k-1}^K C_k^T + R_k)^{-1},
\]

\[
\epsilon_k = y_k - C_k \hat{x}_{k|k-1} = y_k - \hat{y}_{k|k-1},
\]

\[
\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k \epsilon_k,
\]

\[
\Sigma_{k|k}^K = (I - L_k C_k) \Sigma_{k|k-1}^K,
\]

\[
\Sigma_{k+1|k} = A_k \Sigma_{k|k}^K A_k + Q_k,
\]

\[
\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + B_k w_k.
\]

Therefore, we see the predicted state estimation error signal, \( \hat{x}_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k} \), follows the recursion:

\[
K_k = A_k L_k,
\]

\[
\hat{x}_{k+1|k} = (A_k - K_k C_k) \hat{x}_{k|k-1} + w_k - K_k v_k,
\]

(10)

We appeal later to the following properties of the Kalman filter.

Lemma 1 [1] The Kalman filter innovations is white and

\[
\epsilon_k \sim \mathcal{N}(0, C_k \Sigma_{k|k-1}^K C_k^T + R_k).
\]

Lemma 2 [3] Subject to uniform complete detectability of \( [A_k, C_k] \), uniform complete stabilizability of \( [A_k, Q_k] \), and the existence of \( R > 0 \) such that \( R_k \geq R \), the discrete Kalman filter is exponentially asymptotically stable.

This section has been devoted to a brief presentation of the Bayesian and Kalman filters based on signal models (4-5) and (8-9) respectively. In the following sections, algorithms will be presented in which the state process, \( \{x_k\} \), and measurement sequence, \( \{z_k\} \), will be identified with different signals on the transmitter and receiver sides respectively. In particular, we point out that the signal model for Method K possesses a state, which we denote \( Z_k \), consisting of the concatenation of the plant state and the Kalman filter state at the transmitter. It is this observation which admits the Bayesian filter derivation in this paper.

2.2 Other literature

The sign-of-innovations Kalman filter (SOI-KF) was proposed by Ribeiro et al. [2] and yields a precursor to Method S, applicable in the case of 1-bit quantization of the innovations. Indeed, [2] appears to have been the progenitor for this Bayesian approach to quantized innovations filtering, hence the nomenclature for Method R derived in Section 3. The development of the 1-bit Method S proceeds from a careful study of the Bayesian filter leading to the statement of Method R as above but not its precise implementation. We draw on the observations of [2] to make Method R explicit before proceeding with the analysis of Method S for the SOI-KF and its extension in the multi-level quantizer Kalman filter (MLQ-KF) of [10,9]. The
advantage of the SOI-KF and MLQ-KF is that a variant of the Kalman filter recursion is computed for approximate conditional means and covariances in place of calculation of the entire pdf. Such reduced calculations suffice because the state predicted conditional pdf is assumed to be Gaussian. This assumption is evaluated (and shown to be somewhat wanting) in the computational examples of Section 4.

Sukhavasi and Hassibi in [5] consider the Method K signal model (8-9) for plant state \( x_k \) and demonstrate that the conditional state density given quantized measurements at the receiver is the density of a sum of two independent random variables. The first random variable is the transmitter-side Kalman filter state conditional mean estimate and the second possesses a multivariate truncated Gaussian density involving the complete history of previous quantized measurements. Based on this decomposition, the authors provide a recursive estimation algorithm called Kalman-like particle filter. The decomposition is non-recursive and the algorithm requires approximation for implementation. The authors empirically establish its performance relative to the SOI-KF of [2].

For the Method K problem, [4] observe that, while the predicted pdf, \( p(x_k | E_{k-1}) \), can be computed from the traditional Bayesian filtering formula (6), the filtered pdf can be computed by the following formula.

\[
p(x_k | E_k) = \int_{E_k} p(x_k | E_k)p(E_k | E_k) dE_k, \quad (11)
\]

where \( p(x_k | E_k) \) is a Gaussian probability density function and \( p(E_k | E_k) \) is a truncated Gaussian function whose dimension increases with time. That is, this is not a recursive formula. The integration operation in (11) is replaced by a recursive formula based on a mid-point approximation to the integrals.

2.3 Contributions of this paper

The earlier works either do not yield recursive solutions to the state estimation problem or rely on approximation of the densities to achieve recursion. Our approach, Method K, is both recursive and exact, subject to the accuracy of computation of the integrals of Bayesian filtering. By the same token, Method K relies on the computation of these integrals, while Method S is Kalman-filter-like in its computational demands, which are significantly more modest. The properties described in Lemmata 1 and 2 imply advantages of Method K.

**Property 1**: Since \( \{ \epsilon_k \} \) and \( \{ \tilde{\epsilon}_k \} \) are both white sequences, they are well coded from the perspective that there is no predictable component correlated with past or future data.

**Property 2**: Building on the coding property, since the innovations sequence is Gaussian and of a priori known variance \( C_k \Sigma_{K_k|k-1} C_k + R_k \), the quantizer \( Q_k(\cdot) \) can be designed to optimize performance using, for example, the Lloyd-Max optimal quantization rule [11]. This yields maximum entropy coding of the transmitted data.

**Property 3**: Under the conditions of Lemma 2, the transmitter-side Kalman filter forgets exponentially its initial conditions. That is, the dependence of the Kalman filter and of the receiver-side Bayesian filter on initial conditions disappears over time. This property has been studied further for nonlinear filters by van Handel [8,7]. In the context here, this property suggests that, subject to well-posedness of the Kalman filtering signal model at the transmitter, the Method K receiver should eventually resynchronize after an isolated error in reception. Similarly, the Bayesian filter integral recursion should exhibit some robustness to inaccuracies in the evaluation of the integrals.

It is not immediately evident that Method R and its descendant Method S enjoy these same properties.

3 Analysis of techniques

In this section, we analyze each method under the following common assumptions.

A.1 The plant is linear and described by (8-9).
A.2 Process and measurement noises \( \{ w_k \} \sim \mathcal{N}(0, Q_k) \) and \( \{ v_k \} \sim \mathcal{N}(0, R_k) \) are independent and white.
A.3 Initial state \( x_0 \sim \mathcal{N}(\tilde{x}_{0|0}, \Sigma_{0|0}) \) is independent from \( \{ w_k \} \) and \( \{ v_k \} \).
A.4 The receiver knows: \( \tilde{x}_{0|0}, \Sigma_{0|0} \), and \( A_k, B_k, u_k, C_k, Q_k, R_k, Q_k(\cdot), Q_k^{-1}(\cdot) \), for every value of \( k \geq 0 \).

3.1 Method K of this paper

From (10), the receiver side of Method K possesses a state-space and measurement description

\[
Z_{k+1} = \begin{bmatrix} x_{k+1} \\ \tilde{x}_{k+1|k} \end{bmatrix} = \begin{bmatrix} A_k & 0 \\ 0 & A_k - K_k C_k \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_{k|k-1} \end{bmatrix} + \begin{bmatrix} B_k & I \\ 0 & -K_k \end{bmatrix} \begin{bmatrix} w_k \\ v_k \end{bmatrix},
\]

\[
\tilde{e}_k = Q_k^{-1} Q_k \begin{bmatrix} 0 & C_k \end{bmatrix} Z_k + \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} w_k \\ v_k \end{bmatrix}.
\]

By comparison to (4-5), this immediately invites the application of the Bayesian filter (6-7) with the substitutions...
\( z_k \mapsto x_k, \varepsilon_k \mapsto z_k, \) and \( \tilde{E}_k \mapsto Z_k. \)

\[
p^K(Z_k | \tilde{E}_{k-1}) = \int_{z_{k-1}} p(Z_k \mid Z_{k-1}) p^K(Z_{k-1} | \tilde{E}_{k-1}) dZ_{k-1},
\]

(14)

\[
p^K(Z_k | \tilde{E}_k) = \frac{p(\varepsilon_k \mid Z_k) p^K(Z_k | \tilde{E}_{k-1})}{\int_{z_k} p(\varepsilon_k \mid Z_k) p^K(Z_k | \tilde{E}_{k-1}) dZ_k},
\]

\[
= \frac{p(\varepsilon_k \mid Z_k) p^K(Z_k | \tilde{E}_{k-1})}{p(\varepsilon_k)} \cdot
\]

(15)

The above equations form a recursive algorithm to compute the predicted pdf and filtered pdf for each time. We show the detail of each term:

1. In (14),

\[
p(Z_k \mid Z_{k-1}) = p(w_{k-1} = x_k - A_{k-1}x_{k-1} - B_{k-1}u_{k-1})
\times p(K_{k-1}v_{k-1} = A_{k-1}((x_k^K)_{k-1} - x_{k-1}) - B_{k-1}u_{k-1}
- K_{k-1}C_{k-1}(x_{k-1} - x_{k-1})).
\]

(16)

This probability can be computed from the joint Gaussian distribution of \( w_{k-1} \) and \( v_{k-1} \), corresponding to Assumption A.2.

2. In (15), the whiteness of the innovations signal \( \{ \varepsilon_k \} \), and therefore of \( \{ \varepsilon_k \} \), admits

\[
p(\varepsilon_k \mid \tilde{E}_{k-1}) = p(\varepsilon_k) = \int_{z_{k-1}} p(\varepsilon_k) d\varepsilon_k,
\]

\[
= \int_{z_{k-1}} N(0, C_k \Sigma_k C_k^T + R_k) d\varepsilon_k,
\]

(17)

where \( (z_{k-1}, z_{k-1}) \) is a quantization interval at time \( k. \)

3. The term \( p(\varepsilon_k \mid Z_k) \) can be computed as follows:

\[
p(\varepsilon_k \mid Z_k) = \int_{z_{k-1}} p(\varepsilon_k \mid Z_{k-1}) d\varepsilon_k,
\]

\[
= \int_{z_{k-1}} p(\varepsilon_k) = C_k x_{k-1}^K + v_{k-1} d\varepsilon_k,
\]

\[
= \int_{z_{k-1}} N(C_k x_{k-1}^K, R_k) d\varepsilon_k.
\]

(18)

4. The initial probability density function at the receiver

\[
p(Z_0 | Z_{-1}) = p(Z_0) = p \left( \left[ \begin{array}{c} x_0 \\ \tilde{z}_{0|-1} \end{array} \right] \right),
\]

\[
= N \left( \left[ \begin{array}{c} \tilde{x}_{0|-1} \\ 0 \end{array} \right], \Sigma_{0|-1} \right).
\]

(19)

The second equality corresponds to Assumption A.4 that the transmitter and receiver are synchronized at time zero.

We summarize the quantized Kalman filter innovations Bayesian filter, Method K, as follows.

**Method K** algorithm at the receiver.

Kalman filter at the transmitter

**Required function:** \( p^K(Z_{k-1} | \tilde{E}_{k-1}). \)

1. Compute \( p(Z_k \mid Z_{k-1}) \) from (16).
2. Compute **predicted pdf** \( p^K(Z_k | \tilde{E}_{k-1}) \) from (14).
3. Receive \( \tilde{z}_k = Q_k^{-1} Q_k \left[ y_k - C_k x_{k-1}^K \right] \).
4. Compute \( p(\tilde{z}_k) \) from (17) and \( p(\tilde{z}_k \mid Z_k) \) from (18).
5. Compute **filtered pdf** \( p^K(Z_k | \tilde{E}_k) \) from (15).

### 3.2 Method R of Ribeiro et al. [2]

Method R of [2] uses the Bayesian filter (6-7) with measurement sequence \( \tilde{I}_{k-1} = \{ \tilde{I}_{k-1}, \ldots, \tilde{I}_0 \} \) followed by a conditional mean calculation.

\[
p^R(x_k \mid \tilde{I}_{k-1}) = \int_{x_k} p(x_k \mid x_{k-1}) p^R(x_{k-1} \mid \tilde{I}_{k-1}) d x_{k},
\]

(20)

\[
p^R(x_k \mid \tilde{I}_{k-1}) = \frac{p(\tilde{I}_k \mid x_k, \tilde{I}_{k-1}) p^R(x_{k-1} \mid \tilde{I}_{k-1})}{\int_{x_k} p(\tilde{I}_k \mid x_k, \tilde{I}_{k-1}) p^R(x_{k-1} \mid \tilde{I}_{k-1}) d x_k},
\]

(21)

\[
\tilde{x}_{k|k-1} = \mathbb{E}[x_k | \tilde{I}_{k-1}] = \int_{x_k} x_k p^R(x_{k-1} \mid \tilde{I}_{k-1}) d x_k.
\]

(22)

System state equation (4) admits the substitution into (20)

\[
p(x_k \mid x_{k-1} = x_k - A_{k-1}x_{k-1} - B_{k-1}u_{k-1}),
\]

(23)
which forms part of (16) in Method K. Additionally for Method R, from (2) and the quantizer intervals \( \{ z_{k,t}^n, z_{k,u}^n \} \),

\[
p(\bar{r}_k^R | x_k, \bar{r}_{k-1}^R) = p \left( Q_k^{-1} Q_k [C_k (x_k - \bar{x}_k^{R|k-1}) + v_k] | x_k, \bar{r}_{k-1}^R \right),
\]

\[
= \int_{z_{k,l}}^{z_{k,u}} p \left( C_k (x_k - \bar{x}_k^{R|k-1}) + v_k | x_k, \bar{r}_{k-1}^R \right) dv_k,
\]

\[
= \int_{z_{k,l}}^{z_{k,u}} N \left( C_k (x_k - \bar{x}_k^{R|k-1}), R_k \right) dv_k.
\]  

(24)

**Method R** of Ribeiro et al. [2] algorithm at transmitter and receiver

**Required function:** \( p^R(x_{k-1} | \bar{r}_{k-1}^R) \)

1. Compute \( p(x_k | x_{k-1}) \) from (23).
2. Compute predicted pdf \( p^R(x_k | \bar{r}_{k-1}^R) \) from (20).
3. Compute the conditional mean \( \bar{x}_k^{R|k-1} \) from (22).
4. Compute at the transmitter

\[
\bar{r}_k^R = Q_k^{-1} Q_k \left[ y_k - C_k \bar{x}_k^{R|k-1} \right],
\]

and receive \( r_k^R \) at the receiver.
5. Compute \( p^R(\bar{r}_k^R | x_k, \bar{r}_{k-1}^R) \) from (24).
6. Compute filtered pdf \( p^R(x_k | \bar{r}_k^R) \) from (21).

3.3 Method S of Ribeiro et al. [2] and You et al. [10]

In [2], Ribeiro et al. derive from their Method R a Kalman-filter-like recursion based on the one-bit quantization (signum) of the associated innovations signal, \( \{ \bar{r}_k^S \} \) depicted in Figure 4. This is called the sign of innovations Kalman filter (SOI-KF) and the algorithm operates at both the transmitter and receiver. You et al. [10] extend these ideas from one-bit quantization of SOI-KF to the multiple-level quantized innovations Kalman filter (MLQ-KF), which we refer to as Method S. The derivations of both works operate under the assumption that the predicted pdf \( p^S(x_{k+1} | \bar{r}_k^S) \) is Gaussian.

Denote the normal density function and \( \sqrt{\pi}/2 \text{erfc}(x) \) as

\[
\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}), \quad \alpha_z = \int_{z}^{\infty} \phi(x)dx.
\]

Further, denote the \( n \)th positive level of a symmetric, mid-rise, \( N \)-level, scalar quantizer by \( \{ z_{k,l}^n, z_{k,u}^n \} \). Then the Kalman-filter-like recursion of Method S/MLQ-KF for

\[
| \bar{r}_k^S | \in \{ z_{k,l}^n, z_{k,u}^n \}
\]

is

\[
\hat{x}_k^S = A_k \hat{x}_k^{S|k-1} + B_k \bar{r}_k^S, \\
\Sigma_k^S = A_k \Sigma_k^{S|k-1} A_k^T + Q_k,
\]

\[
\hat{x}_k^S = \hat{x}_k^{S|k-1} + \text{sgn}(\bar{r}_k^S) \left( \phi(z_{k,l}^n) - \phi(z_{k,u}^n) \right),
\]

\[
\times \frac{\Sigma_k^{S|k-1} C_k^T}{C_k \Sigma_k^{S|k-1} C_k^T + R_k},
\]

\[
\Sigma_k^S = \Sigma_k^{S|k-1} - \sum_{n=1}^{N} \frac{\left[ \phi(z_{k,l}^n) - \phi(z_{k,u}^n) \right]^2}{|\alpha_{z,n}^+ - \alpha_{z,n}^-|},
\]

\[
\times \frac{\Sigma_k^{S|k-1} C_k^T}{C_k \Sigma_k^{S|k-1} C_k^T + R_k}.
\]  

(25)

(26)

(27)

(28)

where \text{sgn}(\cdot) is the signum (or sign) function

\[
\text{sgn}(\bar{r}_k^S) = \begin{cases} +1 & \text{if } \bar{r}_k^S \geq 0 \\ -1 & \text{if } \bar{r}_k^S < 0 \end{cases}
\]

**Method S** of Ribeiro et al. [2] and You et al. [10]

algorithm at transmitter and receiver

**Required data:** \( \hat{x}_k^{S|k-1}, \Sigma_k^{S|k-1} \)

1. Compute predicted values \( \hat{x}_k^S \) and \( \Sigma_k^S \) from time-update (25-26).
2. Compute at the transmitter

\[
\bar{r}_k^S = Q_k^{-1} Q_k \left[ y_k - C_k \hat{x}_k^{S|k-1} \right],
\]

and receive \( r_k^S \) at the receiver.
3. Compute filtered values \( \hat{x}_k^S \) and covariance \( \Sigma_k^S \) from measurement update (27-28).

4 Comparison and examples

In this section we present a few simple simulation comparisons between the methods, noting that Method K yields the precise conditional densities.

4.1 Examples

**Case 1: Almost fixed state, Method K:** The first example has a state which is almost constant and well measured. It is provided to demonstrate that Method K is sound. The system has scalar state which obeys

\[
x_{k+1} = x_k + w_k, \\
y_k = x_k + v_k,
\]
with \( w_k \sim \mathcal{N}(0, 0.0001) \), \( v_k \sim \mathcal{N}(0, 0.00001) \), \( x_0 \sim \mathcal{N}(0, .02) \). The 3-bit quantizer is that depicted in Figure 2.

At the transmitter Kalman filter, the innovations value is \( \epsilon_0 = 0.1160 \), leading to a received quantized innovations value \( \bar{\epsilon}_0 \in (0, 0.1555] \). The reconstructed conditional pdfs are close to truncated Gaussians with slight smoothing at the edges according to the narrow densities of \( w_k \) in the predictor and \( v_k \) in the filter. Figure 5 is the result calculated in matlab using 101 sample points for the integrals. These computed pdfs make sense. The next few steps do not alter the figures significantly.

![Graph showing predicted and filtered pdfs](image)

Fig. 5. Example 1 predicted and filtered pdfs at the receiver computed using Bayesian filtering Method K with true innovations \( \epsilon_0 = 0.1160 \) and received quantized innovations \( \bar{\epsilon}_0 \in (0, 0.1555] \) via the 3-bit quantizer in Figure 2.

**Case 2: Comparison of methods:** Consider the system

\[
x_{k+1} = 0.95x_k + w_k, \\
y_k = x_k + v_k.
\]

with \( w_k \sim \mathcal{N}(0, 0.01) \), \( v_k \sim \mathcal{N}(0, 0.01) \), \( x_0 \sim \mathcal{N}(0, .02) \) and the same 3-bit quantizer shown in Figure 2. We compare the sequence of predicted and filtered pdfs computed using Method K and Method R, and the predicted Gaussian pdf only from Method S/MLQ-KF using the computed mean and covariance. These are contrasted with the transmitter-side Kalman filter innovations Gaussian pdf. The results are displayed in Figures 6-10.

The signals are as follows.

**Time 0:**

\[
x_0 = -0.0319, \quad v_0 = 0.1117, \\
w_0 = -0.1089, \quad x_1 = -0.1392, \\
\hat{x}_{0|0}^K = \hat{x}_{0|0}^R = \hat{x}_{0|0}^S = 0, \\
\epsilon_0 = \epsilon_0^R = \epsilon_0^S = 0.0798, \\
\bar{\epsilon}_0 = \bar{\epsilon}_0^R = \bar{\epsilon}_0^S \in (0, 0.1555], \\
\hat{x}_{1|0}^S = 0.0085, \quad \Sigma_{1|0}^S = 0.0181.
\]

**Time 1:**

\[
x_1 = -0.1392, \quad v_1 = 0.0033, \\
w_1 = 0.0553, \quad x_2 = -0.0770, \\
\epsilon_1 = -0.1898, \quad \epsilon_1^R = -0.1853, \\
\bar{\epsilon}_1 = \bar{\epsilon}_1^R \in (-0.3111, -0.1555], \\
\hat{x}_{2|1}^S = 0.0161, \quad \Sigma_{2|1}^S = 0.0177, \\
\hat{x}_{2|1}^K = -0.0630, \quad \Sigma_{2|1}^K = 0.0156.
\]

**Time 2:**

\[
x_2 = -0.0770, \quad v_2 = 0.1101, \\
w_2 = 0.1544, \quad x_3 = 0.0813, \\
\epsilon_2 = -0.0140, \quad \epsilon_2^R = -0.1230, \\
\bar{\epsilon}_2 = \bar{\epsilon}_2^R \in (-0.1555, 0], \\
\hat{x}_{3|2}^S = 0.0231, \quad \Sigma_{3|2}^S = 0.0176, \\
\hat{x}_{3|2}^K = -0.0679, \quad \Sigma_{3|2}^K = 0.0155.
\]

**Time 3:**

\[
x_3 = 0.0813, \quad v_3 = 0.0086, \\
w_3 = -0.1492, \quad x_4 = -0.0720, \\
\epsilon_3 = 0.1492, \quad \epsilon_3^R = 0.0352, \\
\bar{\epsilon}_3 = \bar{\epsilon}_3^R \in (0, 0.1555], \\
\hat{x}_{4|3}^S = 0.0297, \quad \Sigma_{4|3}^S = 0.0175, \\
\hat{x}_{4|3}^K = 0.0216, \quad \Sigma_{4|3}^K = 0.0155.
\]

**Time 4:**

\[
x_4 = -0.0720, \quad v_4 = -0.0742, \\
\epsilon_4 = -0.0936, \quad \epsilon_4^R = -0.1180, \\
\bar{\epsilon}_4 = \bar{\epsilon}_4^R \in (-0.1555, 0], \\
\hat{x}_{5|4}^S = 0.0360, \quad \Sigma_{5|4}^S = 0.0175, \\
\hat{x}_{5|4}^K = -0.0335, \quad \Sigma_{5|4}^K = 0.0155.
\]

This second example demonstrates that the methods are workable. Although the explicit computation of pdfs is numerically demanding compared with the moment recursions. However, we note from this sequence of figures that Methods K and R yield pdfs which circumscribe the transmitter-side Kalman filter pdfs. So the actual state pdf is feasible.
5 Conclusions

We have developed via Bayesian filtering an exact and recursive approach to the reconstruction from quantized Kalman filter innovations signals of predicted and filtered state conditional pdfs. The key observation was to recognize the role played by the Kalman filter state in describing the transmitter signal model. Comparisons were made to a candidate Bayesian approach underpinning the SOI-KF derivation and to the MLQ-KF, indicating some issues with the latter methods. Some specific advantages of the new method of this pa-
per were identified in terms of coding performance, optimal quantization, and recovery from error.

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