1. Introduction. Star formation occurs in our galaxy primarily inside cold interstellar clouds. Observations indicate that gas motion inside these clouds is highly supersonic in the broad range of scales $0.1 - 1000$ pc. At these scales the clouds lack any characteristic structure and their scaling properties are to some extent universal. This universality is the signature of the supersonic and super–Alfvénic magneto–hydrodynamic (MHD) turbulence that governs the dynamics of the clouds [1, 2, 3, 4, 5, 6].

The properties of turbulent MHD flows are controlled by the rms sonic Mach number, $M_S$, (ratio of rms flow velocity and sound velocity) and the rms Alfvénic Mach number, $M_A$, (ratio of rms flow velocity and Alfvén velocity). In interstellar clouds these numbers are large, $M_S \sim 1 - 30$, $M_A > 1$. The corresponding supersonic motion produces the observed strong density enhancements [8], some of which are Jeans–unstable and undergo gravitational collapse. This initial ‘turbulent’ stage of density fragmentation should be described statistically, in terms of velocity and density probability distribution functions or correlators; its properties are inherently related to the statistical properties of compressible MHD turbulence [8].

Early studies suggesting the importance of astrophysical supersonic turbulence were undertaken by von Weizsäcker [10] and later by Larson [11], who discovered that the spectrum of the velocity field in molecular clouds was steeper than the spectrum predicted by the standard Kolmogorov model. The discrepancy was attributed to the strong intermittency of supersonic turbulence. However, the analytical understanding of such turbulence remained rather poor.

Advances in computational and observational techniques have recently allowed measurements of power spectra and higher order correlators of velocity and density fields, reviving the interest in supersonic turbulence. Modern observations generate surveys covering a range of scales comparable or even exceeding the resolution of numerical simulations. The interstellar medium can now be viewed as a laboratory for investigating supersonic motion unachievable with Earth–based experiments.

The turbulent field can be characterized by the scaling of its structure functions [see below]. The intermittency is defined as the departure of this scaling from the Kolmogorov one. So far, only the limiting cases of $M_S \gg 1$ and $M_S = 0$ have been investigated. An analytical description of supersonic turbulence, $M_S \gg 1$, has been proposed in [12] and confirmed numerically in [13, 14, 15]. It is based on the observation that the most intense dissipative structures in supersonic flows are two–dimensional shocks. This approach makes use of the model by She and Lévéque [16] for incompressible turbulence, $M_S = 0$, where the dissipative structures are one–dimensional vortex filaments.

In this letter we report the results of a series of numerical MHD simulations where we traced the change of the velocity structure functions over the broad range of Mach numbers $0.3 \leq M_S \leq 10$. We discovered that their scalings can be described by the general model where only one parameter, the dimension of the most intense dissipative structures, needs to be varied as a function of $M_S$. Our results thus provide a method for obtaining the velocity scaling in interstellar clouds once their Mach numbers have been inferred from observations.

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The scaling of velocity structure functions in incompressible turbulence is best described by the She–Lévéque formula [16, 17]:

\[ \frac{\zeta(p)}{\zeta(3)} = \frac{p}{9} + \frac{2}{3} \left[ 1 - \left( \frac{2}{3} \right)^{p/3} \right]. \quad (2) \]

This is a specific case of the more general formula

\[ \frac{\zeta(p)}{\zeta(3)} = \gamma p + C (1 - \beta^p) \]

with the normalization condition,

\[ C = (1 - 3\gamma)/(1 - \beta^3). \quad (4) \]

Two basic assumptions are used to derive (4), as discussed in [18]. The first assumption is the existence of a symmetric relation between the intensity of fluctuations \( F_p(\ell) \) and \( F_{p+1}(\ell) \) with respect to a translation in \( p \):

\[ F_{p+1}(\ell) = A_p F_p(\ell) \beta F(\infty)(\ell)^{1-\beta}, \quad (5) \]

where the \( p \)th order intensity of fluctuations is defined as \( F_p(\ell) = S_{p+1}(\ell)/S_p(\ell) \), and \( A_p \) are constants independent of \( \ell \) (generally found to be independent of \( p \) as well).

The functions \( F_p(\ell) \) are associated with higher intensity fluctuations with increasing values of \( p \). The parameter \( \beta \) is a measure of intermittency. In Kolmogorov’s 1941 model [19], \( \beta \to 1 \) and \( \zeta(p) = p/3 \) This is the limit of no intermittency. Lower \( \beta \) corresponds to higher degree of intermittency, or to the increasing role played by strong persistent fluctuations (structures).

The second assumption is

\[ F_{\infty} \sim S_3^\gamma. \quad (6) \]

The value of the parameter \( \gamma \) is related to very high order moments and is therefore difficult to constrain with experiments. Kolmogorov’s turbulence corresponds to \( \gamma = 1/3 \).

The parameter \( C \) has been interpreted as the Hausdorff codimension of the support of the most singular dissipative structures [17]. In incompressible turbulence the most dissipative structures are along vortex filaments with Hausdorff dimension \( D = 1 \) and so \( C = 2 \). Boldyrev [12] has proposed to extend the scaling (4) to supersonic turbulence, with the assumption that the Hausdorff dimension of the support of the most singular dissipative structures is \( D = 2 \) (\( C = 1 \)), because dissipation of supersonic turbulence occurs mainly in sheet–like shocks. He used \( \gamma = 1/9 \) as in incompressible turbulence, which implies \( \beta^3 = 1/3 \), that is a higher degree of intermittency than in incompressible turbulence. With these parameters one obtains the velocity scaling proposed for supersonic turbulence in [12]:

\[ \frac{\zeta(p)}{\zeta(3)} = \frac{p}{9} + 1 - \left( \frac{1}{3} \right)^{p/3}. \quad (7) \]

This velocity scaling has been found to provide a very accurate prediction for numerical simulations of highly supersonic and super–Alfvénic turbulence [13, 14]. The same scaling has also been proposed for incompressible MHD turbulence [20], where dissipation occurs mainly in two dimensional current sheets.
In the present work we obtain the structure function scaling in simulations of compressible super-Alfvénic MHD turbulence as a function of the rms sonic Mach number of the flow, $M_S$, for constant value of the rms Alfvénic Mach number, $M_A \gg 1$ (the limiting case of strong magnetic field was investigated analytically and numerically in [21, 22, 23]). We use an isothermal equation of state and vary the value of $M_S$ in different experiments by varying the thermal energy. As a result, the initial value of $M_A=10$ remains unchanged from run to run. In our numerical method the total magnetic flux is conserved. However, the magnetic energy is amplified.

The initial value of the ratio of average magnetic and dynamic pressures is $\langle P_m \rangle_{in}/\langle P_d \rangle_{in} = 0.005$ for all runs.

The value of the same ratio, averaged over the time interval for which the structure functions are computed, is given in Table 1. Because the average dynamic pressure is always in excess of the average magnetic pressure, $\langle P_m \rangle/\langle P_d \rangle \ll 1$, the dissipation is expected to occur primarily in vortex filaments (roughly corresponding to magnetic flux tubes) in subsonic runs and in shocks in supersonic runs. In the next section we argue that when we increase the value of $M_S$, the Hausdorff dimension of the support of the most dissipative structures grows continuously from $D \approx 1$ to $D \approx 2$.

3. Results. We have solved the three dimensional compressible MHD equations, for an isothermal gas with initially uniform density and magnetic fields, in a staggered mesh with $250^3$ computational cells and periodic boundary conditions. Turbulence is set up as an initial large scale random and solenoidal velocity field (generated in Fourier space with power only in the range of wavenumber $1 \leq K \leq 2$) and maintained with an external large scale random and solenoidal force, correlated at the largest scale turn-over time. Details about the numerical method are given in [2].

We have run six experiments with different values of $M_S$, in the range $0.4 \leq M_S \leq 9.5$ (Table I), for approximately 10 dynamical times. The transverse structure functions of velocity are computed for the last four dynamical times of each experiment, up to the order $p=10$. Their power law slope, $\zeta(p)$, is obtained from a least square fit in the range $2 \leq \ell \leq 96$. The standard deviation of the slope is used as an estimate of the uncertainty of $\zeta(p)$.

Figure 4 shows the structure functions plotted versus the third order one for the $M_S=9.5$ experiment. The structure functions are compensated by their least square fit slope, $\zeta(p)$. We study the structure functions relative to the third order to exploit the concept of extended self-similarity [17, 24], that is the property of the relative scaling to extend up to a range of scales affected by dissipation (numerical dissipation in this case).

The exponents $\zeta(p)$, normalized to the third order one, are plotted in Figure 2 divided by their value predicted in [12] and given by [6]. The $M_S=0.4$ and $M_S=0.8$ models follow the She–Léveque scaling of incompressible turbulence. As the value of $M_S$ increases, the velocity scaling becomes gradually more intermittent. At $M_S=9.5$ the scaling is very close to the analytical prediction for supersonic turbulence [12], as previously shown [12, 14].

The validity of the assumption of hierarchical symmetry expressed by [5], can be tested with a log–log plot of $F_{p+1}/F_p$ versus $F_p/F_1$, obtained by varying the argument $\ell$ in [5]. If the plot is a straight line, the parameters $A_p$ are constants independent of $\ell$ and the assumed hierarchical symmetry expressed by [5] is satisfied. This
TABLE I: Values of the rms sonic Mach number, average magnetic to dynamic pressure ratio at initial time and averaged over the time interval for which the structure functions are computed, \( \beta \) parameter from fitting \( \zeta(p)/\zeta(3) \) and Hausdorff dimension of the most dissipative structures.

| Parameter | Value |
|-----------|-------|
| \( \langle M_s \rangle \) | 0.39, 0.77, 1.5, 2.6, 4.8, 9.5 |
| \( \langle P_{\text{rms}} \rangle \) | 0.005, 0.005, 0.005, 0.005, 0.005, 0.005 |
| \( \langle P_{\text{rms}} \rangle \) | 0.15, 0.14, 0.09, 0.11, 0.09, 0.08 |
| \( \beta \) | 0.87, 0.87, 0.84, 0.79, 0.73, 0.69 |
| \( D \) | 1.0, 1.0, 1.34, 1.70, 1.90, 2.01 |

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