DISCRETE REPRODUCING KERNEL HILBERT SPACES: SAMPLING AND DISTRIBUTION OF DIRAC-MASSES

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Abstract. We study reproducing kernels, and associated reproducing kernel Hilbert spaces (RKHSs) \( \mathcal{H} \) over infinite, discrete and countable sets \( V \). In this setting we analyze in detail the distributions of the corresponding Dirac point-masses of \( V \). Illustrations include certain models from neural networks: An Extreme Learning Machine (ELM) is a neural network-configuration in which a hidden layer of weights are randomly sampled, and where the object is then to compute resulting output. For RKHSs \( \mathcal{H} \) of functions defined on a prescribed countable infinite discrete set \( V \), we characterize those which contain the Dirac masses \( \delta_x \) for all points \( x \) in \( V \). Further examples and applications where this question plays an important role are: (i) discrete Brownian motion-Hilbert spaces, i.e., discrete versions of the Cameron-Martin Hilbert space; (ii) energy-Hilbert spaces corresponding to graph-Laplacians where the set \( V \) of vertices is then equipped with a resistance metric; and finally (iii) the study of Gaussian free fields.

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2000 Mathematics Subject Classification. Primary 47L60, 46N30, 46N50, 42C15, 65R10, 05C50, 05C75, 31C20; Secondary 46N20, 22E70, 31A15, 58J65, 81S25.

Key words and phrases. Unbounded operators, harmonic analysis, Hilbert space, reproducing kernel Hilbert space, discrete analysis, infinite matrices, binomial coefficients, Gaussian free fields, graph Laplacians, distribution of point-masses, Green’s function (graph Laplacians), orthogonal systems, bi-orthogonal systems, transforms, (discrete) Ito-isometries, optimization, determinants.
1. Introduction

A reproducing kernel Hilbert space (RKHS) is a Hilbert space $\mathcal{H}$ of functions on a prescribed set, say $V$, with the property that point-evaluation for functions $f \in \mathcal{H}$ is continuous with respect to the $\mathcal{H}$-norm. They are called kernel spaces, because, for every $x \in V$, the point-evaluation for functions $f \in \mathcal{H}$, $f(x)$ must then be given as a $\mathcal{H}$-inner product of $f$ and a vector $k_x$, in $\mathcal{H}$; called the kernel.

The RKHSs have been studied extensively since the pioneering papers by Aronszajn in the 1940ties, see e.g., [Aro43, Aro48]. They further play an important role in the theory of partial differential operators (PDO); for example as Green’s functions of second order elliptic PDOs; see e.g., [Nel57, IKL+14]. Other applications include engineering, physics, machine-learning theory (see [KH11, SZ09, CS02]), stochastic processes (e.g., Gaussian free fields), numerical analysis, and more. See, e.g., [AD93, ABDdS93, AD92, AJSV13, AJV14]. Also, see [LB04, HQKL10, ZXZ12, LP11, Vul13, SS13, HN14]. But the literature so far has focused on the theory of kernel functions defined on continuous domains, either domains in Euclidean space, or complex domains in one or more variables. For these cases, the Dirac $\delta_x$ distributions do not have finite $\mathcal{H}$-norm. But for RKHSs over discrete point distributions, it is reasonable to expect that the Dirac $\delta_x$ functions will in fact have finite $\mathcal{H}$-norm.

An illustration from neural networks: An Extreme Learning Machine (ELM) is a neural network configuration in which a hidden layer of weights are randomly sampled (see e.g., [RW06]), and the object is then to determine analytically resulting output layer weights. Hence ELM may be thought of as an approximation to a network with infinite number of hidden units.

Here we consider the discrete case, i.e., RKHSs of functions defined on a prescribed countable infinite discrete set $V$. We are concerned with a characterization of those RKHSs $\mathcal{H}$ which contain the Dirac masses $\delta_x$ for all points $x \in V$. Of the examples and applications where this question plays an important role, we emphasize three: (i) discrete Brownian motion-Hilbert spaces, i.e., discrete versions of the Cameron-Martin Hilbert space; (ii) energy-Hilbert spaces corresponding to graph-Laplacians; and finally
(iii) RKHSs generated by binomial coefficients. We show that the point-masses have finite $\mathcal{H}$-norm in cases (i) and (ii), but not in case (iii).

Our setting is a given positive definite function $k$ on $V \times V$, where $V$ is discrete (see above). We study the corresponding RKHS $\mathcal{H} (= \mathcal{H}(k))$ in detail. Our main results are Theorems 2.12, 3.18, and 4.13 which give explicit answers to the question of which point-masses from $V$ are in $\mathcal{H}$. Applications include Corollaries 3.5, 3.20, 4.5, 4.7, 4.11, and 4.12.

The paper is organized as follows: Section 2 leads up to our characterization (Theorem 2.12) of point-masses which have finite $\mathcal{H}$-norm. It is applied in sections 3 and 4 to a variety of classes of discrete RKHSs. Section 3 deals with samples from Brownian motion, and from the Brownian bridge process, and binomial kernels, and with kernels on sets $V \times V$ which arise as restrictions to sample-points. Section 4 covers the case of infinite network of resistors. By this we mean an infinite graph with assigned resistors on its edges. In this family of examples, the associated RKHSs vary with the assignment of resistors on the edges in $G$, and are computed explicitly from a resulting energy form. Our result Corollary 4.5 states that, for the network models, all point-masses have finite energy. Furthermore, we compute the value, and we study $V$ as a metric space w.r.t. the corresponding resistance metric. These results, in turn, have direct implications (Corollaries 4.7, 4.11 and 4.15) for the family of Gaussian free fields associated with our infinite network models.

A positive definite kernel $k$ is said to be universal [CMPY08] if, every continuous function, on a compact subset of the input space, can be uniformly approximated by sections of the kernel, i.e., by continuous functions in the RKHS. In Theorem 4.13 we show that for the RKHSs from kernels $k_c$ in electrical network $G$ of resistors, this universality holds. The metric in this case is the resistance metric on the vertices of $G$, determined by the assignment of a conductance function $c$ on the edges in $G$.

### 2. Discrete RKHSs

**Definition 2.1.** Let $V$ be a countable and infinite set, and $\mathcal{F}(V)$ the set of all finite subsets of $V$. A function $k : V \times V \to \mathbb{C}$ is said to be positive definite, if

$$\sum_{(x,y) \in F \times F} k(x, y) c_x c_y \geq 0 \quad (2.1)$$

holds for all coefficients $\{c_x\}_{x \in F} \subset \mathbb{C}$, and all $F \in \mathcal{F}(V)$.

**Definition 2.2.** Fix a set $V$, countable infinite.
(1) For all $x \in V$, set
\[ k_x := k (\cdot, x) : V \to \mathbb{C} \quad (2.2) \]
as a function on $V$.

(2) Let $\mathcal{H} := \mathcal{H} (k)$ be the Hilbert-completion of the span $\{k_x : x \in V\}$, with respect to the inner product
\[ \langle \sum c_x k_x, \sum d_y k_y \rangle_{\mathcal{H}} := \sum \sum c_x \overline{d_y} k (x, y) \quad (2.3) \]
modulo the subspace of functions of zero $\mathcal{H}$-norm. $\mathcal{H}$ is then a reproducing kernel Hilbert space (HKRS), with the reproducing property:
\[ \langle k_x, \varphi \rangle_{\mathcal{H}} = \varphi (x), \forall x \in V, \forall \varphi \in \mathcal{H}. \quad (2.4) \]

**Note.** The summations in (2.3) are all finite. Starting with finitely supported summations in (2.3), the RKHS $\mathcal{H} = \mathcal{H} (k)$ is then obtained by Hilbert space completion. We use physicists’ convention, so that the inner product is conjugate linear in the first variable, and linear in the second variable.

(3) If $F \in \mathcal{F} (V)$, set $\mathcal{H}_F = \text{closed span} \{k_x : x \in F\} \subset \mathcal{H}$, (closed is automatic if $F$ is finite.) And set
\[ P_F := \text{the orthogonal projection onto } \mathcal{H}_F. \quad (2.5) \]

(4) For $F \in \mathcal{F} (V)$, set
\[ K_F := (k (x, y))_{(x, y) \in F \times F} \quad (2.6) \]
as a $\#F \times \#F$ matrix.

**Remark 2.3.** It follows from the above that reproducing kernel Hilbert spaces (RKHS) arise from a given positive definite kernel $k$, a corresponding pre-Hilbert form; and then a Hilbert-completion. The question arises: “What are the functions in the completion?” Now, before completion, the functions are as specified in Definition 2.2, but the Hilbert space completions are subtle; they are classical Hilbert spaces of functions, not always transparent from the naked kernel $k$ itself. Examples of classical RKHSs: Hardy spaces or Bergman spaces (for complex domains), Sobolev spaces and Dirichlet spaces (for real domains, or for fractals [OST13, ST12, Str10]), band-limited $L^2$ functions (from signal analysis), and Cameron-Martin Hilbert spaces from Gaussian processes (in continuous time domain).
Our focus here is on discrete analogues of the classical RKHSs from real or complex analysis. These discrete RKHSs in turn are dictated by applications, and their features are quite different from those of their continuous counterparts.

**Definition 2.4.** The RKHS $\mathcal{H} = \mathcal{H}(k)$ is said to have the *discrete mass* property ($\mathcal{H}$ is called a *discrete RKHS*), if $\delta_x \in \mathcal{H}$, for all $x \in V$. Here, $\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$, i.e., the Dirac mass at $x \in V$.

**Lemma 2.5.** Let $F \in \mathcal{F}(V)$, $x_1 \in F$. Assume $\delta_{x_1} \in \mathcal{H}$. Then

$$P_F(\delta_{x_1})(\cdot) = \sum_{y \in F} (K_F^{-1} \delta_{x_1})(y) k_y(\cdot).$$

(2.7)

**Proof.** Show that

$$\delta_x - \sum_{y \in F} (K_F^{-1} \delta_{x_1})(y) k_y(\cdot) \in \mathcal{H}^\perp_F.$$  

(2.8)

The remaining part follows easily from this.

(The notation $(\mathcal{H}_F)^\perp$ stands for orthogonal complement, also denoted $\mathcal{H} \ominus \mathcal{H}_F = \{ \varphi \in \mathcal{H} \mid \langle f, \varphi \rangle_{\mathcal{H}} = 0, \forall f \in \mathcal{H}_F \}$.)

**Lemma 2.6.** Using Dirac’s bra-ket, and ket-bra notation (for rank-one operators), the orthogonal projection onto $\mathcal{H}_F$ is

$$P_F = \sum_{y \in F} |k_y \rangle \langle k_y^*|;$$

(2.9)

where

$$k_x^* := \sum_{y \in F} (K_F^{-1})_{yx} k_y$$

(2.10)

is the dual vector to $k_x$, for all $x \in V$.

**Proof.** Let $k_x^*$ be specified as in (2.10), then

$$\langle k_x^*, k_z \rangle_{\mathcal{H}} = \sum_{y \in F} \langle (K_F^{-1})_{yx} k_y, k_z \rangle_{\mathcal{H}}$$

$$= \sum_{y \in F} (K_F^{-1})_{yx} \langle k_y, k_z \rangle_{\mathcal{H}}$$

$$= \sum_{y \in F} (K_F^{-1})_{yx} (K_F)_{yz} = \delta_{x,z},$$

i.e., $k_x^*$ is the dual vector to $k_x$, for all $x \in V$. 

For \( f \in \mathcal{H} \), and \( F \in \mathcal{F}(V) \), we have
\[
\sum_{y \in F} |k_y \rangle \langle k_y^*| f = \sum_{y \in F} \langle k_y^*, f \rangle \langle \mathcal{H} \rangle k_y
\]
\[
= \sum_{(y,z) \in F \times F} (K^{-1}_F)_{z,y} \langle k_z, f \rangle \langle \mathcal{H} \rangle
\]
\[
= P_F f.
\]
This yields the orthogonal projection realized as stated in (2.9).

Now, applying (2.9) to \( \delta_{x_1} \), we get
\[
P_F (\delta_{x_1}) = \sum_{y \in F} \langle k_y^*, \delta_{x_1} \rangle \langle \mathcal{H} \rangle k_y
\]
\[
= \sum_{y \in F} \left( \sum_{z \in F} (K^{-1}_F)_{yz} \langle k_z, \delta_{x_1} \rangle \langle \mathcal{H} \rangle \right) k_y
\]
\[
= \sum_{y \in F} \left( \sum_{z \in F} (K^{-1}_F)_{yz} \delta_{x_1} (z) \right) k_y
\]
\[
= \sum_{y \in F} (K^{-1}_F \delta_{x_1}) (y) k_y,
\]
where
\[
(K^{-1}_F \delta_{x_1}) (y) := \sum_{z \in F} (K^{-1}_F)_{yz} \delta_{x_1} (z).
\]
This verifies (2.7). \qed

Remark 2.7. Note a slight abuse of notations: We make formally sense of the expressions for \( P_F (\delta_x) \) in (2.7) even in the case when \( \delta_x \) might not be in \( \mathcal{H} \). For all finite \( F \), we showed that \( P_F (\delta_x) \in \mathcal{H} \). But for \( \delta_x \) be in \( \mathcal{H} \), we must have the additional boundedness assumption (2.18) satisfied; see Theorem 2.12.

Lemma 2.8. Let \( F \in \mathcal{F}(V) \), \( x_1 \in F \), then
\[
(K^{-1}_F \delta_{x_1}) (x_1) = \| P_F (\delta_{x_1}) \|_{\mathcal{H}}^2.
\]
(2.11)

Proof. Setting \( \zeta^{(F)} := K^{-1}_F (\delta_{x_1}) \), we have
\[
P_F (\delta_{x_1}) = \sum_{y \in F} \zeta^{(F)} (y) k_F (\cdot, y)
\]
and for all $z \in F$,
\[
\sum_{z \in F} \zeta^{(F)}(z) P_F(\delta_{x_1})(z) = \sum_{F} \sum_{z} \zeta^{(F)}(z) \zeta^{(F)}(y) k_F(z, y) = \|P_F \delta_{x_1}\|_H^2.
\]
(2.12)

Note the LHS of (2.12) is given by (see Lemma 2.6)
\[
\|P_F \delta_{x_1}\|_H^2 = \langle P_F \delta_{x_1}, \delta_{x_1} \rangle_H = \sum_{y \in F} (K^{-1}_F \delta_{x_1})(y) \langle k_y, \delta_{x_1} \rangle_H = (K^{-1}_F \delta_{x_1})(x_1) = K^{-1}_F(x_1, x_1).
\]

\[\square\]

**Corollary 2.9.** If $\delta_{x_1} \in \mathcal{H}$ (see Theorem 2.12), then
\[
\sup_{F \in \mathcal{F}(V)} \langle K^{-1}_F \delta_{x_1}, \delta_{x_1} \rangle_H = \|\delta_{x_1}\|_H^2.
\]
(2.13)

The following condition is satisfied in some examples, but not all:

**Corollary 2.10.** \(\exists F \in \mathcal{F}(V)\) s.t. $\delta_{x_1} \in \mathcal{H}_F \iff K^{-1}_F \delta_{x_1} = K^{-1}_F(\delta_{x_1})(x_1)$ for all $F' \supset F$.

**Corollary 2.11 (Monotonicity).** If $F$ and $F'$ are in $\mathcal{F}(V)$ and $F \subset F'$, then
\[
(K^{-1}_F \delta_{x_1})(x_1) \leq (K^{-1}_{F'} \delta_{x_1})(x_1)
\]
(2.14)

and
\[
\lim_{F' / \mathcal{F}(V)} (K^{-1}_F \delta_{x_1})(x_1) = \|\delta_{x_1}\|_H^2.
\]
(2.15)

**Proof.** By (2.11),
\[
(K^{-1}_F \delta_{x_1})(x_1) = \|P_F \delta_{x_1}\|_H^2.
\]

Since $\mathcal{H}_F \subset \mathcal{H}_{F'}$, we have $P_F P_{F'} = P_F$, so
\[
\|P_F \delta_{x_1}\|_H^2 = \|P_F P_{F'} \delta_{x_1}\|_H^2 \leq \|P_{F'} \delta_{x_1}\|_H^2
\]
i.e.,
\[
(K^{-1}_F \delta_{x_1})(x_1) \leq (K^{-1}_{F'} \delta_{x_1})(x_1).
\]
So (2.14) follows; and the limit in (2.15) is monotone.

Theorem 2.12. Given \( V, k : V \times V \to \mathbb{R} \) positive definite (p.d.). Let \( \mathcal{H} = \mathcal{H}(k) \) be the corresponding RKHS. Assume \( V \) is countable and infinite. Then the following three conditions (i)-(iii) are equivalent; \( x_1 \in V \) is fixed:

(i) \( \delta_{x_1} \in \mathcal{H} \);
(ii) \( \exists C_{x_1} < \infty \) such that for all \( F \in \mathcal{F}(V) \), the following estimate holds:

\[
|\xi(x_1)|^2 \leq C_{x_1} \sum_{F \times F} \xi(x)\xi(y) k(x, y)
\]

(2.16)

(iii) For \( F \in \mathcal{F}(V) \), set

\[
K_F = (k(x, y))_{(x,y)\in F \times F}
\]

as a \( \#F \times \#F \) matrix. See Definition 2.2, eq. (2.6). Then

\[
\sup_{F \in \mathcal{F}(V)} (K_F^{-1} \delta_{x_1}) (x_1) < \infty.
\]

(2.18)

Proof. (i)\( \Rightarrow \) (ii) For \( \xi \in \ell^2(F) \), set

\[
h_{\xi} = \sum_{y \in F} \xi(y) k_y (\cdot) \in \mathcal{H}_F.
\]

Then \( \langle \delta_{x_1}, h_{\xi} \rangle_{\mathcal{H}} = \xi(x_1) \) for all \( \xi \).

Since \( \delta_{x_1} \in \mathcal{H} \), then by Schwarz:

\[
|\langle \delta_{x_1}, h_{\xi} \rangle_{\mathcal{H}}|^2 \leq \| \delta_{x_1} \|^2_{\mathcal{H}} \sum_{F \times F} \xi(x)\xi(y) k(x, y).
\]

(2.19)

But \( \langle \delta_{x_1}, k_y \rangle_{\mathcal{H}} = \delta_{x_1,y} = \begin{cases} 1 & y = x_1 \\ 0 & y \neq x_1 \end{cases} \); hence \( \langle \delta_{x_1}, h_{\xi} \rangle_{\mathcal{H}} = \xi(x_1) \), and so (2.19) implies (2.16).

(ii)\( \Rightarrow \) (iii) Recall the matrix

\[
K_F := (k(x, y))_{(x,y)\in F \times F}
\]

as a linear operator \( \ell^2(F) \to \ell^2(F) \), where

\[
(K_F \varphi) (x) = \sum_{y \in F} K_F (x, y) \varphi(y), \ \varphi \in \ell^2(F).
\]

(2.20)

By (2.16), we have

\[
\ker(K_F) \subset \{ \varphi \in \ell^2(F) : \varphi(x_1) = 0 \}.
\]

(2.21)
Equivalently,
\[
\ker (K_F) \subset \{ \delta_{x_1} \}^\perp \tag{2.22}
\]
and so \( \delta_{x_1} \bigg|_F \in \ker (K_F)^\perp = \text{ran} (K_F) \), and \( \exists \ \zeta^{(F)} \in l^2 (F) \) s.t.
\[
\delta_{x_1} \bigg|_F = \sum_{y \in F} \zeta^{(F)} (y) k (\cdot,y). \tag{2.23}
\]

Claim. \( P_F (\delta_{x_1}) = h_F \), where \( P_F = \text{projection onto } \mathcal{H}_F \); see (2.5) and Lemma 2.5. (See Fig 2.1.)

Proof of the claim. We only need to prove that \( \delta_{x_1} - h_F \in \mathcal{H} \ominus \mathcal{H}_F \), i.e.,
\[
\langle \delta_{x_1} - h_F, k_z \rangle_{\mathcal{H}} = 0, \ \forall z \in F. \tag{2.24}
\]
But, by (2.23),
\[
\text{LHS}(2.24) = \delta_{x_1,z} - \sum_{y \in F} k (z,y) \zeta^{(F)} (y) = 0.
\]
\( \square \)

If \( F \subset F' \), \( F,F' \in \mathcal{F} (V) \), then \( \mathcal{H}_F \subset \mathcal{H}_{F'} \), and \( P_F P_{F'} = P_F \) by easy facts for projections. Hence
\[
\| P_F \delta_{x_1} \|^2_{\mathcal{H}} \leq \| P_{F'} \delta_{x_1} \|^2_{\mathcal{H}}, \quad h_F := P_F (\delta_{x_1})
\]
and
\[
\lim_{F \nearrow V} \| \delta_{x_1} - h_F \|^2_{\mathcal{H}} = 0.
\]

\( \text{(iii)⇒(i)} \) Follows from Lemma 2.8 and Corollary 2.9. \( \square \)

Corollary 2.13. The numbers \( \left( \zeta^{(F)} (y) \right)_{y \in F} \) in (2.23) satisfies
\[
\zeta^{(F)} (x_1) = \sum_{(y,z) \in F \times F} \zeta^{(F)} (y) \zeta^{(F)} (z) k (y,z). \quad (2.25)
\]
Proof. Multiply (2.23) by $\zeta^{(F)}(z)$ and carry out the summation. □

Remark 2.14. To see that (2.23) is a solution to a linear algebra problem, with $F = \{x_i\}_{i=1}^n$, note that (2.23) $\iff$

$$
\begin{bmatrix}
  k(x_1, x_1) & k(x_1, x_2) & \cdots & k(x_1, x_n) \\
  k(x_2, x_1) & k(x_2, x_2) & \cdots & k(x_2, x_n) \\
  \vdots & \ddots & \ddots & \vdots \\
  k(x_n, x_1) & k(x_n, x_2) & \cdots & k(x_n, x_n)
\end{bmatrix}
\begin{bmatrix}
  \zeta^{(F)}(x_1) \\
  \zeta^{(F)}(x_2) \\
  \vdots \\
  \zeta^{(F)}(x_{n-1}) \\
  \zeta^{(F)}(x_n)
\end{bmatrix}
= 
\begin{bmatrix}
  1 \\
  0 \\
  \vdots \\
  0
\end{bmatrix}
$$

(2.26)

We now resume the general case of $k$ given and positive definite on $V \times V$.

Corollary 2.15. We have

$$
\zeta^{(F)}(x_1) = \|P_F(\delta x_1)\|_H^2
$$

(2.27)

where

$$
P_F(\delta x_1) = \sum_{y \in F} \zeta^{(F)}(y) k_y(\cdot)
$$

(2.28)

and

$$
\zeta^{(F)} = K_{N}^{-1}(\delta x_1), \quad N := \#F.
$$

(2.29)

Proof. It follows from (2.26) that

$$
\sum_{j} k(x_i, x_j) \zeta^{(F)}(x_j) = \delta_{1,i}
$$

and so multiplying by $\zeta^{(F)}(i)$, and summing over $i$, gives

$$
\sum_{i} \sum_{j} k(x_i, x_j) \zeta^{(F)}(x_i) \zeta^{(F)}(x_j) = \zeta^{(F)}(x_1).
$$

$$
\sum_{i} \sum_{j} k(x_i, x_j) \zeta^{(F)}(x_i) \zeta^{(F)}(x_j) = \|P_F(\delta x_1)\|_H^2
$$

□

Corollary 2.16. We have

(i)

$$
P_F(\delta x_1) = \zeta^{(F)}(x_1) k_{x_1} + \sum_{y \in F \setminus \{x_1\}} \zeta^{(F)}(y) k_y
$$

(2.30)

where $\zeta_F$ solves (2.26), for all $F \in \mathcal{F}(V)$;

(ii)

$$
\|P_F(\delta x_1)\|_H^2 = \zeta^{(F)}(x_1)
$$

(2.31)
and so in particular:

(iii) \[ 0 < \zeta^{(F)} (x_1) \leq \| \delta x_1 \|_{\mathcal{H}}^2 \] \hspace{1cm} (2.32)

Proof. Formula (2.31) follows from the definition of \( \zeta^{(F)} \) as a solution to the matrix problem \( K_N \zeta^{(F)} = \delta x_1 \), but we may also prove (2.31) directly from

\[ P_F (\delta x_1) = \sum_y \zeta^{(F)} (y) k_y. \] \hspace{1cm} (2.33)

Apply \( \langle \cdot, \delta x_1 \rangle_{\mathcal{H}} \) to both sides in (2.33), we get

\[ \langle \delta x_1, P_F (\delta x_1) \rangle_{\mathcal{H}} = \zeta^{(F)} (x_1) \]

since \( P_F = P_F^* = P_F^2 \); i.e., a projection in the RKHS \( \mathcal{H} = \mathcal{H}_V \) of \( k \).

Example 2.17 \((\#F = 2)\). Let \( F = \{ x_1, x_2 \} \), \( K_F = (k_{ij})_{i,j=1}^2 \), where \( k_{ij} := k (x_i, x_j) \). Then (2.26) reads

\[ \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} \zeta^{(F)} (x_1) \\ \zeta^{(F)} (x_2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \] \hspace{1cm} (2.34)

Set \( D := \det (K_F) = k_{11}k_{22} - k_{12}k_{21} \), then:

\( \zeta^{(F)} (x_1) = \frac{k_{22}}{D} \), \quad \zeta^{(F)} (x_2) = -\frac{k_{21}}{D} \).

Example 2.18. Let \( V = \{ x_1, x_2, \ldots \} \) be an ordered set. Set \( F_n := \{ x_1, \ldots, x_n \} \). Note that with

\[ D_n = \det (K_{F_n}) = \det \left( (k (x_i, x_j))_{i,j=1}^n \right), \] \hspace{1cm} (2.35)

\[ D_{n-1}' = (1, 1) \text{ minor of } K_{F_n} = \det \left( (k (x_i, x_j))_{i,j=2}^n \right); \] \hspace{1cm} (2.36)

then

\[ \zeta^{(F_n)} (x_1) = \frac{D_{n-1}'}{D_n} = (K_{F_n}^{-1} \delta x_1) (x_1). \] \hspace{1cm} (2.37)

Corollary 2.19. We have

\[ \frac{1}{k (x_1, x_1)} \leq \frac{k (x_2, x_2)}{D_2} \leq \cdots \leq \frac{D_{n-1}'}{D_n} \leq \cdots \leq \| \delta x_1 \|_{\mathcal{H}}^2. \]

Proof. Follows from (2.37), and if \( F \subseteq F' \) are two finite subsets, then

\[ \| P_F (\delta x_1) \|_{\mathcal{H}}^2 \leq \| P_{F'} (\delta x_1) \|_{\mathcal{H}}^2 \leq \| \delta x_1 \|_{\mathcal{H}}^2. \]
Let $k : V \times V \to \mathbb{R}$ be as specified above. Let $\mathcal{H} = \mathcal{H}(k)$ be the RKHS. We set $\mathcal{F}(V) := \text{all finite subsets of } V$; and if $x \in V$ is fixed, $\mathcal{F}_x(V) := \{F \in \mathcal{F}(V) \mid x \in F\}$.

For $F \in \mathcal{F}(V)$, let $K_F$ be the $\#F \times \#F$ matrix given by $(k(x, y))_{(x, y) \in F \times F}$. Following [KZ96], we say that $k$ is strictly positive iff (Def.) $\det K_F > 0$ for all $F \in \mathcal{F}(V)$.

Set $D_F := \det K_F$. If $x \in V$, and $F \in \mathcal{F}_x(V)$, set $K'_F :=$ the minor in $K_F$ obtained by omitting row $x$ and column $x$, see Fig 2.2.

**Corollary 2.20.** Suppose $k : V \times V \to \mathbb{R}$ is strictly positive. Let $x \in V$. Then

$$\delta_x \in \mathcal{H} \iff \sup_{F \in \mathcal{F}_x(V)} \frac{D'_F}{D_F} < \infty.$$  \ \ (2.38)

2.1. **Unbounded containment in RKHSs**

**Definition 2.21.** Let $\mathcal{K}$ and $\mathcal{H}$ be two Hilbert spaces. We say that $\mathcal{K}$ is unboundedly contained in $\mathcal{H}$ if there is a dense subspace $\mathcal{K}_0 \subset \mathcal{K}$ such that $\mathcal{K}_0 \subset \mathcal{H}$; and the inclusion operator, with $\mathcal{K}_0$ as its dense domain, is closed, i.e.,

$$\mathcal{K} \overset{\text{incl}}{\hookrightarrow} \mathcal{H}, \quad \text{dom (incl)} = \mathcal{K}_0.$$  \ \ (2.39)

Let $k : V \times V \to \mathbb{R}$ be a p.d. kernel, and let $\mathcal{H}$ be the corresponding RKHS. Set $\mathcal{H} = l^2(V)$, and

$$\mathcal{H}_0 = \text{span} \{\delta_x \mid x \in V\}.$$  \ \ (2.39)

**Proposition 2.22.** If $\delta_x \in \mathcal{H}$ for $\forall x \in V$, then $l^2(V)$ is unboundedly contained in $\mathcal{H}$.
Proof. Recall that $\mathcal{H}$ is the RKHS defined for a fixed p.d. kernel $k : V \times V \to \mathbb{R}$. Let $k_x$ be the vector in $\mathcal{H}$, given by $k_x (y) = k (x, y)$, s.t.

$$f (x) = \langle k_x, f \rangle_{\mathcal{H}}, \quad \forall f \in \mathcal{H}. \quad (2.40)$$

To finish the proof we will need:

**Lemma 2.23.** The following equation

$$\langle \delta_x, k_y \rangle_{\mathcal{H}} = \delta_{x,y} \quad (2.41)$$

holds if $\delta_x \in \mathcal{H}$ for $\forall x \in V$.

*Proof.* (2.41) is immediate from (2.40). □

**Lemma 2.24.** On

$$\text{span} \{ k_x \mid x \in V \} \subset \mathcal{H} \quad (2.42)$$

define $M k_x := \delta_x$, then by Lemma 2.23, $M$ extends to be a well defined operator $M : \mathcal{H} \to l^2 (V)$ with dense domain (2.42). We have

$$\langle k, M f \rangle_{l^2(V)} = \langle k, f \rangle_{\mathcal{H}}, \quad \forall k \in \text{span} \{ \delta_x \}, \forall f \in \text{dom} (M). \quad (2.43)$$

*Proof.* By linearity, it is enough to prove that

$$\langle \delta_x, \delta_y \rangle_{l^2} = \langle \delta_x, k_y \rangle_{\mathcal{H}} \quad (2.44)$$

holds for $\forall x, y \in V$. But (2.44) follows immediate from Lemma 2.23. □

**Corollary 2.25.** If $L : l^2 (V) \to \mathcal{H}$ denotes the inclusion mapping with $\text{dom} (L) = \text{span} \{ \delta_x : x \in V \}$, then we conclude that

$$L \subset M^*, \text{ and } M \subset L^*.$$  \hspace{1cm} (2.45)

Since $\text{dom} (M)$ is dense in $\mathcal{H}$, it follows that $L^*$ has dense domain; and that therefore $L$ is closable.

**Remark 2.26.** This also completes the proof of Proposition 2.22.

**Corollary 2.27.** Suppose $k : V \times V \to \mathbb{R}$ is as given, and that $\mathcal{H} = \text{RKHS} (k)$. Let $L$ be the densely defined inclusion mapping $l^2 (V) \to \mathcal{H}$. Then $L^* L$ is selfadjoint with dense domain in $l^2 (V)$; and $L L^*$ is selfadjoint with dense domain in $\mathcal{H}$. Moreover, the following polar decomposition holds:

$$L = U (L^* L)^{1/2} = (L L^*)^{1/2} U \quad (2.46)$$

where $U$ is a partial isometry $l^2 (V) \to \mathcal{H}$. 

3. Point-masses in concrete models

Suppose \( V \subset D \subset \mathbb{R}^d \) where \( V \) is countable and discrete, but \( D \) is open. In this case, we get two kernels: \( k \) on \( D \times D \), and \( k_V := k|_{V \times V} \) on \( V \times V \) by restriction. If \( x \in V \), then \( k_x^{(V)}(\cdot) = k(\cdot, x) \) is a function on \( V \), while \( k_x(\cdot) = k(\cdot, x) \) is a function on \( D \).

This means that the corresponding RKHSs are different, \( \mathcal{H}_V \) vs \( \mathcal{H} \), where \( \mathcal{H}_V = \text{a RKHS of functions on } V \), and \( \mathcal{H} = \text{a RKHS of functions on } D \).

**Lemma 3.1.** \( \mathcal{H}_V \) is isometrically contained in \( \mathcal{H} \) via \( \delta_x \mapsto k_x, x \in V \).

**Proof.** If \( F \subset V \) is a finite subset, and \( \xi = \xi_F \) is a function on \( F \), then

\[
\left\| \sum_{x \in F} \xi(x) k_x^{(V)} \right\|_{\mathcal{H}_V} = \left\| \sum_{x \in F} \xi(x) k_x \right\|_{\mathcal{H}}.
\]

The desired result follows from this. \( \square \)

We are concerned with cases of kernels \( k : D \times D \to \mathbb{R} \) with restriction \( k_V : V \times V \to \mathbb{R} \), where \( V \) is a countable discrete subset of \( D \). Typically, for \( x \in V \), we may have (restriction) \( \delta_x \mid_V \in \mathcal{H}_V \), but \( \delta_x \notin \mathcal{H} \); indeed this happens for the kernel \( k \) of standard Brownian motion:

\( D = \mathbb{R}_+; \)
\( V \) an ordered subset \( 0 < x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots, V = \{x_i\}^\infty_{i=1}. \)

In this case, we compute \( \mathcal{H}_V \), and we show that \( \delta_{x_i} \mid_V \in \mathcal{H}_V \); while for \( \mathcal{H}_m \) = the Cameron-Martin Hilbert space, we have \( \delta_{x_i} \notin \mathcal{H}_m \).

Also note that \( \delta_{x_1} \) has a different meaning with reference to \( \mathcal{H}_V \) vs \( \mathcal{H}_m \). In the first case, it is simply

\[
\delta_{x_1}(y) = \begin{cases} 
1 & y = x_1 \\
0 & y \in V \setminus \{x_1\}
\end{cases}.
\]

In the second case, \( \delta_{x_1} \) is a Schwartz distribution. We shall abuse notation, writing \( \delta_x \) in both cases.

In the following, we will consider restriction to \( V \times V \) of a special continuous p.d. kernel \( k \) on \( \mathbb{R}_+ \times \mathbb{R}_+ \). It is \( k(s,t) = s \wedge t = \min(s,t) \). Before we restrict, note that the RKHS of this \( k \) is the Cameron-Martin Hilbert space of function \( f \) on \( \mathbb{R}_+ \) with distribution derivative \( f' \in L^2(\mathbb{R}_+) \), and

\[
\|f\|_{\mathcal{H}}^2 := \int_0^\infty |f'(t)|^2 \, dt < \infty. \quad (3.1)
\]

For details, see below.

**Application.** The Hilbert space given by \( \|\cdot\|^2_{\mathcal{H}} \) in (3.1) is called the Cameron-Martin Hilbert space, and, as noted, it is the RKHS of \( k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} : k(s,t) := s \wedge t. \)
Now pick a discrete subset \( V \subset \mathbb{R}_+ \); then Lemma 3.1 states that the RKHS of the \( V \times V \) restricted kernel, \( k^{(V)} \) is isometrically embedded into \( \mathcal{H} \), i.e., setting
\[
J^{(V)} \left( k^{(V)}_x \right) = k_x, \quad \forall x \in V; \tag{3.2}
\]
\( J^{(V)} \) extends by “closed span” to an isometry \( \mathcal{H}_V \xrightarrow{J^{(V)}} \mathcal{H} \). It further follows from the lemma, that the range of \( J^{(V)} \) may have infinite co-dimension.

Note that \( P_V := J^{(V)} (J^{(V)})^* \) is the projection onto the range of \( J^{(V)} \). The orthocomplement is as follow:
\[
\mathcal{H} \ominus \mathcal{H}_V = \left\{ \psi \in \mathcal{H} \mid \psi(x) = 0, \ \forall x \in V \right\}. \tag{3.3}
\]

**Example 3.2.** Let \( k \) and \( k^{(V)} \) be as in (3.2), and set \( V := \pi\mathbb{Z}_+ \), i.e., integer multiples of \( \pi \). Then easy generators of wavelet functions (see e.g., [BJ02]) yield non-zero functions \( \psi \) on \( \mathbb{R}_+ \) such that
\[
\psi \in \mathcal{H} \ominus \mathcal{H}_V. \tag{3.4}
\]
More precisely,
\[
0 < \int_0^\infty |\psi'(t)|^2 \, dt < \infty, \tag{3.5}
\]
where \( \psi' \) is the distribution (weak) derivative; and
\[
\psi(n\pi) = 0, \quad \forall n \in \mathbb{Z}_+. \tag{3.6}
\]
An explicit solution to (3.4)-(3.6) is
\[
\psi(t) = \prod_{n=1}^\infty \cos\left(\frac{t}{2^n}\right) = \frac{\sin t}{t}, \quad \forall t \in \mathbb{R}. \tag{3.7}
\]
From this, one easily generates an infinite-dimensional set of solutions.

3.1. **Brownian motion**

Consider the covariance function of standard Brownian motion \( B_t, t \in [0, \infty) \), i.e., a Gaussian process \( \{B_t\} \) with mean zero and covariance function
\[
\mathbb{E}(B_sB_t) = s \wedge t = \min(s, t). \tag{3.8}
\]
We now show that the restriction of (3.8) to \( V \times V \) for an ordered subset (we fix such a set \( V \)):
\[
V : 0 < x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots \tag{3.9}
\]
has the discrete mass property (Def. 2.4).
Set \( \mathcal{H}_V = RKHS(k|_{V \times V}) \),
\[ k_V (x_i, x_j) = x_i \wedge x_j. \] (3.10)

We consider the set \( F_n = \{ x_1, x_2, \ldots, x_n \} \) of finite subsets of \( V \), and
\[ K_n = k(F_n) = \begin{bmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ x_1 & x_2 & x_2 & \cdots & x_2 \\ x_1 & x_2 & x_3 & \cdots & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} = (x_i \wedge x_j)_{i,j=1}^n. \] (3.11)

We will show that condition (iii) in Theorem 2.12 holds for \( k_V \). For this, we must compute all the determinants, \( D_n = \det (K_F) \) etc. \( n = \#F \), see Corollary 2.20.

**Lemma 3.3.**
\[ D_n = \det \left( (x_i \wedge x_j)_{i,j=1}^n \right) = x_1 (x_2 - x_1) (x_3 - x_2) \cdots (x_n - x_{n-1}). \] (3.12)

**Proof.** Induction. In fact,
\[
\begin{bmatrix} x_1 & x_1 & x_1 & \cdots & x_1 \\ x_1 & x_2 & x_2 & \cdots & x_2 \\ x_1 & x_2 & x_3 & \cdots & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} \sim \begin{bmatrix} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 - x_1 & 0 & \cdots & 0 \\ 0 & 0 & x_3 - x_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & x_n - x_{n-1} \end{bmatrix},
\]
unitary equivalence in finite dimensions. \( \square \)

**Lemma 3.4.** Let
\[ \zeta_{(n)} := K_n^{-1} (\delta_{x_1}) (\cdot) \] (3.13)
be as in eq. (2.11), so that
\[ \|P_{F_n} (\delta_{x_1})\|_{\mathcal{H}_V}^2 = \zeta_{(n)} (x_1). \] (3.14)

Then,
\[ \zeta_{(1)} (x_1) = \frac{1}{x_1} \]
\[ \zeta_{(n)} (x_1) = \frac{x_2}{x_1 (x_2 - x_1)}, \quad \text{for } n = 2, 3, \ldots, \]
and
\[ \|\delta_{x_1}\|_{\mathcal{H}_V}^2 = \frac{x_2}{x_1 (x_2 - x_1)}. \]
Proof. A direct computation shows the \((1,1)\) minor of the matrix \(K_{n}^{-1}\) is

\[
D'_{n-1} = \det \left( (x_{i} \wedge x_{j})_{i,j=2}^{n} \right) = x_{2} (x_{3} - x_{2}) (x_{4} - x_{3}) \cdots (x_{n} - x_{n-1}) \quad (3.15)
\]

and so

\[
\zeta_{(1)}(x_{1}) = \frac{1}{x_{1}}, \quad \text{and}
\]
\[
\zeta_{(2)}(x_{1}) = \frac{x_{2}}{x_{1}(x_{2} - x_{1})}
\]
\[
\zeta_{(3)}(x_{1}) = \frac{x_{2}(x_{3} - x_{2})}{x_{1}(x_{2} - x_{1})(x_{3} - x_{2})} = \frac{x_{2}}{x_{1}(x_{2} - x_{1})}
\]
\[
\zeta_{(4)}(x_{1}) = \frac{x_{2}(x_{3} - x_{2})(x_{4} - x_{3})}{x_{1}(x_{2} - x_{1})(x_{3} - x_{2})(x_{4} - x_{3})} = \frac{x_{2}}{x_{1}(x_{2} - x_{1})}
\]

\[\vdots\]

The result follows from this, and from Corollary 2.9. \(\Box\)

Corollary 3.5. \(P_{F_{n}}(\delta_{x_{1}}) = P_{F_{2}}(\delta_{x_{1}}), \forall n \geq 2.\) Therefore,

\[
\delta_{x_{1}} \in \mathcal{H}_{V}^{(F_{2})} := \text{span}\{k_{x_{1}}^{(V)}, k_{x_{2}}^{(V)}\} \quad (3.16)
\]

and

\[
\delta_{x_{1}} = \zeta_{(2)}(x_{1}) k_{x_{1}}^{(V)} + \zeta_{(2)}(x_{2}) k_{x_{2}}^{(V)} \quad (3.17)
\]

where

\[
\zeta_{(2)}(x_{i}) = K_{2}^{-1}(\delta_{x_{1}})(x_{i}), \ i = 1, 2.
\]

Specifically,

\[
\zeta_{(2)}(x_{1}) = \frac{x_{2}}{x_{1}(x_{2} - x_{1})} \quad (3.18)
\]
\[
\zeta_{(2)}(x_{2}) = \frac{-1}{x_{2} - x_{1}} ; \quad (3.19)
\]

and

\[
\|\delta_{x_{1}}\|_{\mathcal{H}_{V}}^{2} = \frac{x_{2}}{x_{1}(x_{2} - x_{1})}. \quad (3.20)
\]

Proof. Follows from the lemma. Note that

\[
\zeta_{n}(x_{1}) = \|P_{F_{n}}(\delta_{x_{1}})\|_{\mathcal{H}}^{2}
\]

and \(\zeta_{(1)}(x_{1}) \leq \zeta_{(2)}(x_{1}) \leq \cdots\), since \(F_{n} = \{x_{1}, x_{2}, \ldots, x_{n}\}\). In particular, \(\frac{1}{x_{1}} \leq \frac{x_{2}}{x_{1}(x_{2} - x_{1})}\), which yields (3.20). \(\Box\)
Remark 3.6. We showed that $\delta_{x_1} \in \mathcal{H}_V, V = \{x_1 < x_2 < \cdots\} \subset \mathbb{R}_+$, with the restriction of $s \land t$ = the covariance kernel of Brownian motion.

The same argument also shows that $\delta_{x_i} \in \mathcal{H}_V$ when $i > 1$. We only need to modify the index notation from the case of the proof for $\delta_{x_1} \in \mathcal{H}_V$. The details are sketched below.

Fix $V = \{x_i\}_{i=1}^{\infty}, \ x_1 < x_2 < \cdots$, then

$$P_{F_n}(\delta_{x_1}) = \begin{cases} 0 & \text{if } n < i - 1 \\ \sum_{s=1}^{n} (K_{F_n}^{-1}\delta_{x_1}) (x_s) k_{x_s} & \text{if } n \geq i \end{cases}$$

and

$$\|P_{F_n}(\delta_{x_1})\|_{\mathcal{H}}^2 = \begin{cases} 0 & \text{if } n < i - 1 \\ \frac{1}{x_i - x_{i-1}} & \text{if } n = i \\ \frac{x_{i+1} - x_{i-1}}{(x_i - x_{i-1})(x_{i+1} - x_i)} & \text{if } n > i \end{cases}$$

Conclusion.

$$\delta_{x_i} \in \text{span}\left\{k_{x_{i-1}}, k_{x_i}, k_{x_{i+1}}\right\}, \text{ and } \|\delta_{x_i}\|_{\mathcal{H}}^2 = \frac{x_{i+1} - x_{i-1}}{(x_i - x_{i-1})(x_{i+1} - x_i)}.$$ (3.21)

(3.22)

Corollary 3.7. Let $V \subset \mathbb{R}_+$ be countable. If $x_a \in V$ is an accumulation point (from $V$), then $\|\delta_a\|_{\mathcal{H}_V} = \infty$.

Remark 3.8. This computation will be revisited in sect. 4, in a much wider context.

Example 3.9. An illustration for $0 < x_1 < x_2 < x_3 < x_4$:

$$P_{F}(\delta_{x_3}) = \sum_{y \in F} \zeta^{(F)}(y) k_{y}(\cdot)$$

$$\zeta^{(F)} = K_{F}^{-1}\delta_{x_3}.$$  

That is,

$$\begin{bmatrix} x_1 & x_1 & x_1 & x_1 \\ x_1 & x_2 & x_2 & x_2 \\ x_1 & x_2 & x_3 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} \zeta^{(F)}(x_1) \\ \zeta^{(F)}(x_2) \\ \zeta^{(F)}(x_3) \\ \zeta^{(F)}(x_4) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Under $(K_F(x_i,x_j))_{i,j=1}^{4}$.
and

$$\zeta^{(F)}(x_3) = \frac{x_1 (x_2 - x_1) (x_4 - x_2)}{x_1 (x_2 - x_1) (x_3 - x_2) (x_4 - x_3)} = \frac{x_4 - x_2}{(x_3 - x_2) (x_4 - x_3)} = \|\delta_{x_3}\|_2^2.$$  

**Example 3.10** (Sparse sample-points). Let $V = \{x_i\}_{i=1}^\infty$, where

$$x_i = \frac{i (i - 1)}{2}, \quad i \in \mathbb{N}.$$  

It follows that $x_{i+1} - x_i = i$, and so

$$\|\delta_{x_i}\|_2^2 = \frac{x_{i+1} - x_i}{(x_i - x_{i-1}) (x_{i+1} - x_i)} = \frac{2i - 1}{(i - 1) i} \xrightarrow{i \to \infty} 0.$$  

**Conclusion.** $\|\delta_{x_i}\|_2^2 \xrightarrow{i \to \infty} 0$ if the set $V = \{x_i\}_{i=1}^\infty \subset \mathbb{R}_+$ is sparse.

Now, some general facts:

**Lemma 3.11.** Let $k : V \times V \to \mathbb{C}$ be p.d., and let $\mathcal{H}$ be the corresponding RKHS. If $x_1 \in V$, and if $\delta_{x_1}$ has a representation as follows:

$$\delta_{x_1} = \sum_{y \in V} \zeta^{(x_1)}(y) k_y,$$  

then

$$\|\delta_{x_1}\|_2^2 = \zeta^{(x_1)}(x_1).$$  

**Proof.** Substitute both sides of (3.23) into $\langle \delta_{x_1}, \cdot \rangle_\mathcal{H}$ where $\langle \cdot, \cdot \rangle_\mathcal{H}$ denotes the inner product in $\mathcal{H}$. \qed

**Application.** Suppose $V = \bigcup_n F_n$, $F_n \subset F_{n+1}$, where each $F_n \in \mathcal{F}(V)$, then if $x_1 \in F_n$, we have

$$P_{F_n}(\delta_{x_1}) = \sum_{y \in F_n} \langle x_1, K_{F_n}^{-1}y \rangle_2 k_y$$  

and

$$\|P_{F_n}(\delta_{x_1})\|_\mathcal{H} = \langle x_1, K_{F_n}^{-1}x_1 \rangle_2 = (K_{F_n}^{-1}\delta_{x_1})(x_1)$$  

and the expression $\|P_{F_n}(\delta_{x_1})\|_\mathcal{H}^2$ is monotone in $n$, i.e.,

$$\|P_{F_n}(\delta_{x_1})\|_\mathcal{H}^2 \leq \|P_{F_{n+1}}(\delta_{x_1})\|_\mathcal{H}^2 \leq \cdots \leq \|\delta_{x_1}\|_\mathcal{H}^2$$  

with

$$\sup_{n \in \mathbb{N}} \|P_{F_n}(\delta_{x_1})\|_\mathcal{H}^2 = \lim_{n \to \infty} \|P_{F_n}(\delta_{x_1})\|_\mathcal{H}^2 = \|\delta_{x_1}\|_\mathcal{H}^2.$$
Question 3.12. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be positive definite, and let $V \subset \mathbb{R}^d$ be a countable discrete subset, e.g., $V = \mathbb{Z}^d$. When does $k|_{V \times V}$ have the discrete mass property?

Examples of the affirmative, or not, will be discussed below.

3.2. Discrete RKHS from restrictions

Let $D := [0, \infty)$, and $k : D \times D \to \mathbb{R}$, with

$$k(x, y) = x \wedge y = \min(x, y).$$

Restrict to $V := \{0\} \cup \mathbb{Z}_+ \subset D$, i.e., consider $k^{(V)} = k|_{V \times V}$.

$\mathcal{H}(k)$: Cameron-Martin Hilbert space, consisting of functions $f \in L^2(\mathbb{R})$ s.t.

$$\int_0^\infty |f'(x)|^2 \, dx < \infty, \quad f(0) = 0.$$

$\mathcal{H}_V := \mathcal{H}(k^V)$. Note that

$$f \in \mathcal{H}(k^V) \iff \sum_n |f(n) - f(n+1)|^2 < \infty.$$

Lemma 3.13. We have $\delta_n = 2k_n - k_{n+1} - k_{n-1}$.

Proof. Introduce the discrete Laplacian $\Delta$, where

$$(\Delta f)(n) = 2f(n) - f(n-1) - f(n+1),$$

then $\Delta k_n = \delta_n$, and

$$\langle 2k_n - k_{n+1} - k_{n-1}, k_m \rangle_{\mathcal{H}_V} = \langle \delta_n, \delta_m \rangle_{\mathcal{H}_V} = \delta_{n,m}.$$

Remark 3.14. The same argument as in the proof of the lemma shows (mutatis mutandis) that any ordered discrete countable infinite subset $V \subset [0, \infty)$ yields

$$\mathcal{H}_V := \mathcal{H}(k|_{V \times V})$$

as a RKHS which is discrete in that (Def. 2.4) if $V = \{x_i\}_{i=1}^\infty$, $x_i \in \mathbb{R}_+$, then $\delta_{x_i} \in \mathcal{H}_V$, $\forall i \in \mathbb{N}$.

Proof. Fix vertices $V = \{x_i\}_{i=1}^\infty$,

$$0 < x_1 < x_2 < \cdots < x_i < x_{i+1} < \infty, \quad x_i \to \infty. \quad (3.27)$$
Assign conductance

\[ c_{i,i+1} = c_{i+1,i} = \frac{1}{x_{i+1} - x_i} \left( = \frac{1}{\text{dist}} \right) \]  

(3.28)

Let

\[
(\Delta f) (x_i) = \left( \frac{1}{x_{i+1} - x_i} + \frac{1}{x_i - x_{i-1}} \right) f (x_i) \\
- \frac{1}{x_i - x_{i-1}} f (x_{i-1}) - \frac{1}{x_{i+1} - x_i} f (x_{i+1}) 
\]

(3.29)

Equivalently,

\[
(\Delta f) (x_i) = (c_{i,i+1} + c_{i,i-1}) f (x_i) - c_{i,i-1} f (x_{i-1}) - c_{i,i+1} f (x_{i+1}) .
\]

(3.30)

Remark 3.15. The most general graph-Laplacians will be discussed in detail in sect. 4 below.

Then, with (3.30) we have:

\[ \Delta k_{x_i} = \delta_{x_i} \]

where \( k (\cdot, \cdot) \) = restriction of \( s \wedge t \) from \([0, \infty) \times [0, \infty) \) to \( V \times V \); and therefore

\[ \delta_{x_i} = (c_{i,i+1} + c_{i,i-1}) k_{x_i} - c_{i,i-1} k_{x_{i+1}} - c_{i,i+1} k_{x_{i-1}} \in \mathcal{H}_V \]  

(3.31)

as the RHS in the last equation is a finite sum. Note that now the RKHS is

\[ \mathcal{H}_V = \left\{ f : V \to \mathbb{C} \mid \sum_{i=1}^{\infty} c_{i,i+1} |f (x_{i+1}) - f (x_i)|^2 < \infty \right\} . \]

\[ \square \]

3.3. The Brownian bridge

Let \( D := (0, 1) = \) the open interval \( 0 < t < 1 \), and set

\[ k_{\text{bridge}} (s, t) := s \wedge t - st ; \]

(3.32)

then (3.32) is the covariance function for the Brownian bridge \( B_{\text{br}i} (t) \), i.e.,

\[ B_{\text{br}i} (0) = B_{\text{br}i} (1) = 0 \]

(3.33)
Figure 3.1. Brownian bridge $B_{\text{bri}}(t)$, a simulation of three sample paths of the Brownian bridge.

$$B_{\text{bri}}(t) = (1 - t) B \left( \frac{t}{1 - t} \right), \quad 0 < t < 1;$$  

(3.34)

where $B(t)$ is Brownian motion; see Lemma 3.1.

The corresponding Cameron-Martin space is now

$$\mathcal{H}_{\text{bri}} = \{ f \text{ on } [0,1] : f' \in L^2(0,1), f(0) = f(1) = 0 \}$$  

(3.35)

with

$$\| f \|^2_{\mathcal{H}_{\text{bri}}} := \int_0^1 |f'(s)|^2 \, ds < \infty.$$  

(3.36)

If $V = \{ x_i \}_{i=1}^{\infty}$, $x_1 < x_2 < \cdots < 1$, is the discrete subset of $D$, then we have for

$F_n \in \mathcal{F}(V), F_n = \{ x_1, x_2, \cdots, x_n \}$,

$$K_{F_n} = (k_{\text{bridge}}(x_i, x_j))_{i,j=1}^{n},$$  

(3.37)

see (3.32), and

$$\det K_{F_n} = x_1 (x_2 - x_1) \cdots (x_n - x_{n-1}) (1 - x_n).$$  

(3.38)

As a result, we get $\delta_{x_i} \in \mathcal{H}_V^{(\text{bri})}$ for all $i$, and

$$\| \delta_{x_i} \|^2_{\mathcal{H}_V^{(\text{bri})}} = \frac{x_{i+1} - x_{i-1}}{(x_{i+1} - x_i)(x_i - x_{i-1})}.$$  

Note $\lim_{x_i \to 1} \| \delta_{x_i} \|^2_{\mathcal{H}_V^{(\text{bri})}} = \infty.$
3.4. Binomial RKHS

**Definition 3.16.** Let $V = \mathbb{Z}_+ \cup \{0\}$; and

$$k_b(x, y) := \sum_{n=0}^{x \wedge y} \binom{x}{n} \binom{y}{n}, \quad (x, y) \in V \times V.$$  

where $\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$ denotes the standard binomial coefficient from the binomial expansion.

Let $\mathcal{H} = \mathcal{H}(k_b)$ be the corresponding RKHS. Set

$$e_n(x) = \begin{cases} \binom{x}{n} & \text{if } n \leq x \\ 0 & \text{if } n > x. \end{cases} \quad (3.39)$$

**Lemma 3.17** (see [AJ15]).

(i) $e_n(\cdot) \in \mathcal{H}$, $n \in V$;

(ii) $\{e_n\}_{n \in V}$ is an orthonormal basis (ONB) in the Hilbert space $\mathcal{H}$.

(iii) Set $F_n = \{0, 1, 2, \ldots, n\}$, and

$$P_{F_n} = \sum_{k=0}^{n} |\langle e_k, e_k \rangle| \quad (3.40)$$

or equivalently

$$P_{F_n} f = \sum_{k=0}^{n} \langle e_k, f \rangle \langle e_k \rangle e_k. \quad (3.41)$$

then,

(iv) Formula (3.41) is well defined for all functions $f : V \to \mathbb{C}$, $f \in \mathcal{F}\text{unc}(V)$.

(v) Given $f \in \mathcal{F}\text{unc}(V)$; then

$$f \in \mathcal{H} \iff \sum_{k=0}^{\infty} |\langle e_k, f \rangle_{\mathcal{H}}|^2 < \infty; \quad (3.42)$$

and, in this case,

$$\|f\|_{\mathcal{H}}^2 = \sum_{k=0}^{\infty} |\langle e_k, f \rangle_{\mathcal{H}}|^2.$$  

Fix $x_1 \in V$, then we shall apply Lemma 3.17 to the function $f_1 = \delta_{x_1}$ (in $\mathcal{F}\text{unc}(V)$),

$$f_1(y) = \begin{cases} 1 & \text{if } y = x_1 \\ 0 & \text{if } y \neq x_1. \end{cases}$$
Theorem 3.18. We have
\[ \|P_{F_n}(\delta_{x_1})\|_{\mathcal{F}}^2 = \sum_{k=x_1}^{n} \binom{k}{x_1}^2. \]

The proof of the theorem will be subdivided in steps; see below.

Lemma 3.19 (see [AJ15]).

(i) For all \( m, n \in V \), such that \( m \leq n \), we have
\[ \delta_{m,n} = \sum_{j=m}^{n} (-1)^{m+j} \binom{n}{j} \binom{j}{m}. \]  
(3.43)

(ii) For all \( n \in \mathbb{Z}_+ \), the inverse of the following lower triangle matrix is this: With (see Fig 3.2)
\[ L_{x,y}^{(n)} = \begin{cases} \binom{x}{y} & \text{if } y \leq x \leq n \\ 0 & \text{if } x < y \end{cases} \]  
(3.44)
we have:
\[ (L^{(n)})^{-1}_{x,y} = \begin{cases} (-1)^{x-y} \binom{x}{y} & \text{if } y \leq x \leq n \\ 0 & \text{if } x < y. \end{cases} \]  
(3.45)

Notation: The numbers in (3.45) are the entries of the matrix \( (L^{(n)})^{-1} \).

Proof. We refer to [AJ15]. In rough outline, (ii) follows from (i). \( \square \)

\[ L^{(n)} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 3 & 3 & 1 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 1 & 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & (x) & (x+1) & \cdots & * & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & 1 & \cdots & 0 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 1 & 0 & \vdots & \vdots \\ 1 & \cdots & (n) & (n+1) & \cdots & \cdots & \cdots & \cdots & n & 1 \end{bmatrix} \]

Figure 3.2. The matrix \( L_n \) is simply a truncated Pascal triangle, arranged to fit into a lower triangular matrix.
Corollary 3.20. Let $k_b$, $\mathcal{H}$, and $n \in \mathbb{Z}_+$ be as above with the lower triangle matrix $L_n$. Set

$$K_n(x, y) = k_b(x, y), \quad (x, y) \in F_n \times F_n,$$

i.e., an $(n + 1) \times (n + 1)$ matrix.

(i) Then $K_n$ is invertible with

$$K_n^{-1} = (L_n^{tr})^{-1} (L_n)^{-1};$$

an (upper triangle) \times (lower triangle) factorization.

(ii) For the diagonal entries in the $(n + 1) \times (n + 1)$ matrix $K_n^{-1}$, we have:

$$\langle x, K_n^{-1} x \rangle_{L^2} = \sum_{k=x}^{n} \binom{k}{x}^2$$

Conclusion. Since

$$\|P_{F_n}(\delta x_1)\|^2_{\mathcal{H}} = \langle x_1, K_n^{-1} x_1 \rangle_{\mathcal{H}}$$

for all $x_1 \in F_n$, we get

$$\|P_{F_n}(\delta x_1)\|^2_{\mathcal{H}} = \sum_{k=x}^{n} \binom{k}{x_1}^2$$

$$= 1 + \binom{x_1 + 1}{x_1}^2 + \binom{x_1 + 2}{x_1}^2 + \cdots + \binom{n}{x_1}^2;$$

and therefore,

$$\|\delta x_1\|^2_{\mathcal{H}} = \sum_{k=x_1}^{\infty} \binom{k}{x_1}^2 = \infty.$$

In other words, no $\delta x$ is in $\mathcal{H}$.

4. INFINITE NETWORK OF RESISTORS

Here we introduce a family of positive definite kernels $k : V \times V \to \mathbb{R}$, defined on infinite sets $V$ of vertices for a given graph $G = (V, E)$ with edges $E \subset V \times V \setminus$ (diagonal).

There is a large literature dealing with analysis on infinite graphs; see e.g., [JP10, JP11, JP13]; see also [OS05, BCF+07, CJ11].

Our main purpose here is to point out that every assignment of resistors on the edges $E$ in $G$ yields a p.d. kernel $k$, and an associated RKHS $\mathcal{H} = \mathcal{H}(k)$ such that

$$\delta x \in \mathcal{H}, \quad \text{for all } x \in V.$$
Definition 4.1. Let $G = (V, E)$ be as above. Assume

1. $(x, y) \in E \iff (y, x) \in E$;
2. $\exists c : E \to \mathbb{R}_+$ (a conductance function = resistance) such that
   (i) $c(xy) = c(yx)$, $\forall (xy) \in E$;
   (ii) for all $x \in V$, $\# \{ y \in V \mid c(xy) > 0 \} < \infty$; and
   (iii) $\exists o \in V$ s.t. for $\forall x \in V \setminus \{ o \}$, $\exists$ edges $(x_i, x_{i+1})_{n-1} \in E$ s.t. $x_0 = 0$, and
   $x_n = x$; called connectedness.

Given $G = (V, E)$, and a fixed conductance function $c : E \to \mathbb{R}_+$ as specified above, we now define a corresponding Laplace operator $\Delta = \Delta^{(c)}$ acting on functions on $V$, i.e., on $\mathcal{F}_{\text{unc}}(V)$ by

\[(\Delta f)(x) = \sum_{y \sim x} c_{xy} (f(x) - f(y)). \quad (4.2)\]

Let $\mathcal{H}$ be the Hilbert space defined as follows: A function $f$ on $V$ is in $\mathcal{H}$ iff $f(o) = 0$, and

\[\|f\|_{\mathcal{H}}^2 := \frac{1}{2} \sum_{(x,y) \in E} c_{xy} |f(x) - f(y)|^2 < \infty. \quad (4.3)\]

Lemma 4.2 ([JP10]). For all $x \in V \setminus \{ o \}$, $\exists v_x \in \mathcal{H}$ s.t.

\[f(x) - f(o) = \langle v_x, f \rangle_{\mathcal{H}}, \quad \forall f \in \mathcal{H}. \quad (4.4)\]

where

\[\langle h, f \rangle_{\mathcal{H}} = \frac{1}{2} \sum_{(x,y) \in E} c_{xy} \left( h(x) - h(y) \right) (f(x) - f(y)), \quad \forall h, f \in \mathcal{H}. \quad (4.5)\]

(The system $\{v_x\}$ is called a system of dipoles.)

Proof. Let $x \in V \setminus \{ o \}$, and use (4.2) together with the Schwarz-inequality to show that

\[|f(x) - f(o)|^2 \leq \sum_i \frac{1}{c_{x_i, x_{i+1}}} \sum_{i} c_{x_i, x_{i+1}} |f(x_i) - f(x_{i+1})|^2.\]

An application of Riesz' lemma then yields the desired conclusion.

Note that $v_x = v_x^{(c)}$ depends on the choice of base point $o \in V$, and on conductance function $c$; see (i)-(ii) and (4.3).

Now set

\[k^{(c)}(x, y) = \langle v_x, v_y \rangle_{\mathcal{H}}, \quad \forall (xy) \in (V \setminus \{ o \}) \times (V \setminus \{ o \}). \quad (4.6)\]
It follows from a theorem that $k^c$ is a Green’s function for the Laplacian $\Delta^c$ in the sense that

$$\Delta^c k^c (x, \cdot) = \delta_x$$  

where the dot in (4.7) is the dummy-variable in the action. (Note that the solution to (4.7) is not unique.)

Lemma 4.3 ([JP11]). Let $G = (V, E)$, and conductance function $c : E \to \mathbb{R}_+$ be as specified above; then $k^c$ in (4.6) is positive definite, and the corresponding RKHS $\mathcal{H} (k^c)$ is the Hilbert space introduced in (4.3) and (4.5), called the energy-Hilbert space.

Proof. See [JP10, JP11, JP13]. \hfill \square

Proposition 4.4. Let $x \in V \setminus \{o\}$, and let $c : E \to \mathbb{R}_+$ be specified as above. Let $\mathcal{H} = \mathcal{H} (k^c)$ be the corresponding RKHS. Then $\delta_x \in \mathcal{H}$, and

$$\|\delta_x\|^2_{\mathcal{H}} = \sum_{y \sim x} c_{(xy)} =: c (x).$$  

(4.8)

Proof. We study the finite matrices, defined for $\forall F \in \mathcal{F} (V)$, by

$$K_F (x, y) = k^c (x, y), \quad (x, y) \in F \times F.$$  

(4.9)

Fix $x \in V \setminus \{o\}$, and pick $F \in \mathcal{F} (V)$ such that

$$\{x\} \cup \{y \in V \mid y \sim x\} \subset F,$$  

(4.10)

see Fig 4.1; an interior point:

Figure 4.1. Neighborhood of $x$, see Def. 4.1 (ii). An interior point $x$. 
Let $F \in \mathcal{F}(V)$ be as in (4.9) and in Fig 4.1, and let $\Delta = \Delta^{(c)}$ be the Laplace operator (4.2), then for all $(x, y) \in F \times F$, we have:

$$
\langle x, K^{-1}_F y \rangle_{l^2} = \langle \delta_x, \Delta \delta y \rangle_{l^2} = (\Delta \delta_y)(x) = \begin{cases} 
c(x) & \text{if } y = x; \text{ see (4.8)} 
-c_{xy} & \text{if } y \sim x 
0 & \text{for all other values of } y
\end{cases}
$$

(4.11)

In particular,

$$
\sup_{F \in \mathcal{F}(V)} (K_F \delta_x)(x) < \infty;
$$

and in fact,

$$
\|\delta_x\|^2_{\mathcal{F}} = c(x), \text{ for all } x \in V \setminus \{o\},
$$

as claimed in the Proposition.

The last step in the present proof uses the equivalence (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) from Theorem 2.12 above.

Finally, we note that the assertion in (4.11) follows from

$$
\Delta v_x = \delta_x - \delta_o, \quad \forall x \in V \setminus \{o\}.
$$

(4.12)

And (4.12) in turn follows from (4.4), (4.2) and a straightforward computation. □

**Corollary 4.5.** Let $G = (V, E)$ and conductance $c : E \to \mathbb{R}_+$ be as specified above. Let $\Delta = \Delta^{(c)}$ be the corresponding Laplace operator. Let $\mathcal{H} = \mathcal{H}(k^{c})$ be the RKHS. Then

$$
\langle \delta_x, f \rangle_{\mathcal{H}} = (\Delta f)(x)
$$

(4.13)

and

$$
\delta_x = c(x) v_x - \sum_{y \sim x} c_{xy} v_y
$$

(4.14)

holds for all $x \in V$.

**Proof.** Since the system $\{v_x\}$ of dipoles (see (4.4)) span a dense subspace in $\mathcal{H}$, it is enough to verify (4.13) when $f = v_y$ for $y \in V \setminus \{o\}$. But in this case, (4.13) follows from (4.7) and (4.11). □

**Corollary 4.6.** Let $G = (V, E)$, and conductance $c : E \to \mathbb{R}_+$ be as before; let $\Delta^{(c)}$ be the Laplace operator, and $\mathcal{H}_E^{(c)}$ the energy-Hilbert space in Definition 4.1 (see (4.3)). Let $k^{(c)}(x,y) = \langle v_x, v_y \rangle_{\mathcal{H}_E^{(c)}}$ be the kernel from (4.6), i.e., the Green’s function
of $\Delta^{(c)}$. Then the two Hilbert spaces $\mathcal{H}_E$, and $\mathcal{H} (k^{(c)} ) = \text{RKHS} (k^{(c)} )$, are naturally isometrically isomorphic via $v_x \mapsto k_x^{(c)}$ where $k_x^{(c)} = k^{(c)} (x, \cdot)$ for all $x \in V$.

**Proof.** Let $F \in \mathcal{F} (V)$, and let $\xi$ be a function on $F$; then

$$\left\| \sum_{x \in F} \xi (x) k_x^{(c)} \right\|^2_{\mathcal{H} (k^{(c)} )} = \sum_{F \times F} \sum_{x} \xi (x) \xi (y) k^{(c)} (x, y)$$

$$= (4.6) \sum_{F \times F} \sum_{x} \xi (x) \xi (y) \langle v_x, v_y \rangle_{\mathcal{H}_E}$$

$$= \left\| \sum_{x \in F} \xi (x) v_x \right\|^2_{\mathcal{H}_E}.$$  

The remaining steps in the proof of the Corollary now follows from the standard completion from dense subspaces in the respective two Hilbert spaces $\mathcal{H}_E$ and $\mathcal{H} (k^{(c)} )$. □

In the following we show how the kernels $k^{(c)} : V \times V \to \mathbb{R}$ from (4.6) in Lemma 4.2 are related to metrics on $V$; so called *resistance metrics* (see, e.g., [JP10, AJSV13].)

**Corollary 4.7.** Let $G = (V, E)$, and conductance $c : E \to \mathbb{R}_+$ be as above; and let $k^{(c)} (x, y) := \langle v_x, v_y \rangle_{\mathcal{H}_E}$ be the corresponding Green’s function for the graph Laplacian $\Delta^{(c)}$.

Then there is a metric $R (\cdot) = R^{(c)}$ = the resistance metric, such that

$$k^{(c)} (x, y) = \frac{R^{(c)} (o, x) + R^{(c)} (o, y) - R^{(c)} (x, y)}{2}$$  

(4.15)

holds on $V \times V$. Here the base-point $o \in V$ is chosen and fixed s.t.

$$\langle V_x, f \rangle_{\mathcal{H}_E} = f (x) - f (o), \quad \forall f \in \mathcal{H}_E, \forall x \in V.$$  

(4.16)

**Proof.** See [JP10]. Set

$$R^{(c)} (x, y) = \| v_x - v_y \|^2_{\mathcal{H}_E}.  

(4.17)$$

We proved in [JP10] that $R^{(c)} (x, y)$ in (4.17) indeed defines a metric on $V$; the so called *resistance metric*. It represents the voltage-drop from $x$ to $y$ when 1 Amp is fed into $(G, c)$ at the point $x$, and then extracted at $y$.

The verification of (4.15) is now an easy computation, as follows:

$$\frac{R^{(c)} (o, x) + R^{(c)} (o, y) - R^{(c)} (x, y)}{2} = \frac{\| v_x \|^2_{\mathcal{H}_E} + \| v_y \|^2_{\mathcal{H}_E} - \| v_x - v_y \|^2_{\mathcal{H}_E}}{2}$$
Proposition 4.8. In the two cases: (i) \(B(t)\), Brownian motion on \(0 < t < \infty\); and (ii) the Brownian bridge \(B_{\text{bri}}(t)\), \(0 < t < 1\), from sect. 3, the corresponding resistance metric \(R\) is as follows:

(i) If \(V = \{x_i\}_{i=1}^{\infty} \subset (0, \infty)\), \(x_1 < x_2 < \cdots\), then

\[
R_B^{(V)}(x_i, x_j) = |x_i - x_j|.
\]

(ii) If \(W = \{x_i\}_{i=1}^{\infty} \subset (0, 1)\), \(0 < x_1 < x_2 < \cdots < 1\), then

\[
R_{\text{bridge}}^{(W)}(x_i, x_j) = |x_i - x_j| \cdot (1 - |x_i - x_j|).
\]

In the completion w.r.t. the resistance metric \(R_{\text{bridge}}^{(W)}\), the two endpoints \(x = 0\) and \(x = 1\) are identified; see also Fig 3.1.

4.1. Gaussian Processes

Definition 4.9. A Gaussian realization of an infinite graph-network \(G = (V, E)\), with prescribed conductance function \(c : E \to \mathbb{R}_+\), and dipoles \((\nu_x^c)_{x \in V \setminus \{o\}}\), is a Gaussian process \((X_x)_{x \in V}\) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\Omega\) is a sample space; \(\mathcal{F}\) a sigma-algebra of events, and \(\mathbb{P}\) a probability measure s.t., for \(\forall F \in \mathcal{F}(V)\), the random variables \((X_x)_{x \in F}\), are jointly Gaussian with

\[
\mathbb{E}(X_x) = \int_{\Omega} X_x d\mathbb{P} = 0
\]

and covariance

\[
\mathbb{E}(X_x X_y) = k^{(c)}(x, y) = \left\langle \nu_x^c, \nu_y^c \right\rangle_{\mathcal{H}_E};
\]

i.e., the covariance matrix \((\mathbb{E}(X_x X_y))_{(x,y) \in F \times F}\) is

\[
K_F(x, y) := k^{(c)}(x, y) \text{ on } F \times F.
\]

Lemma 4.10 ([JP10]). For all \(G = (V, E)\), and \(c : E \to \mathbb{R}_+\), as specified, Gaussian realizations exist; they are called Gaussian free fields.

Corollary 4.11. Let \(G = (V, E)\), \(c : E \to \mathbb{R}_+\) be as above; and let \((X_x)_{x \in V}\) be an associated Gaussian free field. Then the point Dirac-masses \((\delta_x)_{x \in V}\) have Gaussian
realizations
\[ \tilde{\delta}_x = c(x) X_x - \sum_{y \sim x} c_{xy} X_y, \quad \forall x \in V. \] (4.23)

**Corollary 4.12.** Let \( G = (V, E) \), and \( c : E \to \mathbb{R}_+ \) be as above. Let \( \{X_x\}_{x \in V} \) be the corresponding Gaussian free field, i.e., with correlation
\[ \mathbb{E}(X_x X_y) = k^{(c)}(x, y) = \left\langle v^{(c)}_x, v^{(c)}_y \right\rangle_{\mathcal{H}_E} \] (4.24)
where the dipoles \( \{v^{(c)}_x\} \subset \mathcal{H}_E \) are computed w.r.t. a chosen (and fixed) based-point \( o \in V \), i.e.,
\[ \left\langle v^{(c)}_x, f \right\rangle_{\mathcal{H}_E} = f(x) - f(o), \quad \forall f \in \mathcal{H}_E, \ x \in V. \] (4.25)
Finally, let \( R^{(c)}(x, y) \) be the corresponding resistance metric on \( V \). Then
\[ \mathbb{E}(X_x X_z) + \mathbb{E}(X_z X_y) \leq \mathbb{E}(X_x X_y) + R^{(c)}(o, z) \] (4.26)
holds for all vertices \( x, y, z \in V \); see Fig 4.2.

**Proof.** Use Corollary 4.7, and (4.17). We have
\[ \|v_x - v_y\|_{\mathcal{H}}^2 \leq \|v_x - v_z\|_{\mathcal{H}}^2 + \|v_z - v_y\|_{\mathcal{H}}^2, \]
and (4.26) now follows from (4.21).

**Figure 4.2.** Covariance vs resistance distance \( R^{(c)}(o, z) \) for three vertices \( x, y, z \in V \).

### 4.2. Metric Completion

The next theorem illustrates a connection between the universal property of a kernel in a RKHS \( \mathcal{H} \), on the one hand, and the distribution of the Dirac point-masses \( \delta_x \), on
the other. We make “distribution” precise by the quantity $E(x) := \|\delta_x\|_H^2$, the energy of the point-mass at the vertex point $x$. We introduce a metric completion $M$, and the universal property of the RKHS $H$ asserts that the functions from $H$ are continuous and $1/2$-Lipschitz on $M$, and that they approximate every continuous function on $M$ in the uniform norm. Recall, the vertex set $V$ is equipped with its resistance metric. The universal property here refers to the corresponding metric completion $M$ of the discrete vertex set. In the interesting cases (see e.g., Example 4.14), $M$ is a continuum; – in the case of the example below, the boundary of $V$ is a Cantor set. One expects the value of $E(x)$ to go to infinity as $x$ approaches the boundary $M$, and this is illustrated in the example; with an explicit formula for $E(x)$.

Of special interest is the class of networks $(V,E)$ where the resistance metric $R$ (on the given vertex vertex-set $V$) is bounded; see (ii) in Theorem 4.13 below. This class of networks, for which the diameter of $V$ measured in the resistance metric $R$ is bounded, includes networks having lots of edges with resistors occurring in parallel; see e.g., (JP11).

**Theorem 4.13.** Let $G = (V,E)$, $c : E \to \mathbb{R}_+$ be as above, and let $R^{(c)} : V \times V \to \mathbb{R}_+$ be the resistance-metric (see (4.17)). Let $M$ be the metric completion of $(V,R^{(c)})$. Then:

(i) For every $f \in H$, the function

$$V \ni x \mapsto f(x) \in \mathbb{C}$$

extends by closure to a uniformly continuous function $\tilde{f} : M \to \mathbb{C}$.

(ii) If $R^{(c)}$ is assumed bounded, then the RKHS $H$ is an algebra under point-wise product:

$$(f_1 f_2)(x) = f_1(x) f_2(x), \quad f_i \in H, \ i = 1, 2, \ x \in V.$$ (4.28)

(iii) If $M$ is compact, then $\{\tilde{f} \mid f \in H\}$ is dense in $C(M)$ in the uniform norm.

**Proof.** The assertions in (i) follow from the following two estimates:

Let $f \in H$, then

$$|f(x) - f(y)|^2 \leq \|f\|_H^2 R^{(c)}(x,y), \quad \forall x, y \in V;$$ (4.29)

and

$$|f(x)| \leq |f(o)| + R^{(c)}(o,x)^{\frac{1}{2}}.$$ (4.30)

The estimates in (4.29)-(4.30), in turn, follow from Corollaries 4.6 and 4.7.
To prove (ii), we compute the energy-norm of the product \( f_1 \cdot f_2 \) where \( f_i \in \mathcal{H} \), \( i = 1, 2 \); and we use Corollary 4.6:

\[
\sum_x \sum_y c_{xy} |f_1(x)f_2(x) - f_1(y)f_2(y)|^2
\]

\[
= \sum_x \sum_y c_{xy} |(f_1(x) - f_1(y))f_2(x) + f_1(y)(f_2(x) - f_2(y))|^2
\]

\[
\leq \sum_x \sum_y c_{xy} \left( |f_1(x) - f_1(y)|^2 + |f_2(x) - f_2(y)|^2 \right) \cdot \left( |f_2(x)|^2 + |f_1(y)|^2 \right)
\]

(by Schwarz inside)

\[
\leq \left( \|f_1\|_\infty^2 + \|f_2\|_\infty^2 \right) \cdot \left( \|f_1\|_{\mathcal{H}}^2 + \|f_2\|_{\mathcal{H}}^2 \right);
\]

and we note that the RHS is finite subject to the assumption in (ii).

Proof of (iii): We are assuming here that \( M \) is compact, and we shall apply the Stone-Weierstrass theorem to the subalgebra

\[
\left\{ \tilde{f} \mid f \in \mathcal{H} \right\} \subset C(M) .
\]

(4.31)

Indeed, the conditions for Stone-Weierstrass are satisfied: The functions on LHS in (4.31) form an algebra, by (ii), closed under complex conjugation; and it separates points in \( M \) by Corollary 4.7.

\[ \square \]

**Example 4.14** (The binary tree). Let \( A = \{0, 1\} \), and \( M := \prod_N A \) the infinite Cartesian product, as a Cantor space. Set \( V := \) all finite words:

\[
V = \bigcup_{n \in \mathbb{N}} \left\{ (\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \alpha_i \in \{0, 1\} \right\} ;
\]

(4.32)

and set \( l((\alpha_1, \alpha_2, \ldots, \alpha_n)) =: n \).

For \( \omega = (\omega_k)^{\infty}_1 \in M \), set

\[
\omega|_n := (\omega_1, \omega_2, \ldots, \omega_n) \in V.
\]

(4.33)

For two points \( \omega, \omega' \in M \), we shall need the number

\[
l(\omega \cap \omega') = \sup \{ n : \omega|_n = \omega'|_n \} .
\]

(4.34)

Let \( r : \mathbb{N} \to \mathbb{R}_+ \) be given such that

\[
r(\emptyset) = 0, \quad \sum_{n \in \mathbb{N}} r(n) < \infty .
\]

(4.35)
For conductance function \( c : E \rightarrow \mathbb{R}_+ \), set
\[
c_{\alpha, (at)} = \frac{1}{r(l(\alpha))}, \quad \forall \alpha \in V, \, t \in \{0, 1\}.
\] (4.36)

One checks that, when (4.35) holds, then
\[
\lim_{n,m \to \infty} R^{(c)}(\omega |_n, \omega |_m) = 0.
\]

Consider the graph \( G_2 = (V,E) \) where the edges are “lines” between \( \alpha \) and \( (\alpha t) \), where \( t \in \{0, 1\} \). See Fig 4.3.

**Fact.** With the settings above, the metric completion \( \widetilde{\mathcal{R}}^{(c)} \) w.r.t. the resistance metric on \( V \) is as follows: For \( \omega, \omega' \in M \) (see Fig 4.5),
\[
\widetilde{\mathcal{R}}^{(c)}(\omega, \omega') = 2 \sum_{n=l(\omega \cap \omega')}^{\infty} r(n).
\] (4.37)

Let \( \mathcal{H} \) be the corresponding energy-Hilbert space \( \simeq \) the RKHS of \( k_c \). For \( \alpha \in V \), let \( \delta_\alpha \) be the Dirac-mass at the vertex point \( \alpha \). Then
\[
\|\delta_\alpha\|^2_{\mathcal{H}} = \frac{2}{r(l(\alpha))} + \frac{1}{r(l(\alpha) - 1)}.
\] (4.38)

**Proof.** To see this, note that \( \alpha \) has the three neighbors sketched in Fig 4.3, i.e., \( \alpha^* \), \( (\alpha 0) \), and \( (\alpha 1) \), where \( \alpha^* \) is the one-truncated word,
\[
\widetilde{\mathcal{R}}^{(c)}(\omega, \omega') = 2 \sum_{n=l(\omega \cap \omega')}^{\infty} r(n).
\] (4.39)

One checks that when (4.35) is assumed, then the conditions in point (iii) of the theorem are satisfied. \( \square \)

**Corollary 4.15.** Now return to the discrete restriction of Brownian motion in sect. 3.1. Set \( V = \{x_1, x_2, x_3, \ldots\} \) where the points \( \{x_i\}^{\infty}_{i=1} \) are prescribed such that \( x_1 < \)
Figure 4.4. Histogram for $\|\delta_\alpha\|^2_{\mathcal{H}}$ as vertices $\alpha \in V$ approach the boundary. See (4.38), and note $\|\delta_\alpha\|^2_{\mathcal{H}} \to \infty$ as $\alpha \to M$.

Figure 4.5. The binary tree and its boundary, the Cantor-set.
We turn $V$ into a weighted graph $G$ as follows: The edges $E$ in $G$ are nearest neighbors; and we define a conductance function $c : E \to \mathbb{R}_+$ by setting
\[ c_{x_i x_{i+1}} := \frac{1}{x_{i+1} - x_i}, \tag{4.40} \]
and Laplace operator,
\[ (\Delta f)(x_i) = \frac{1}{x_{i+1} - x_i} (f(x_i) - f(x_{i+1})) + \frac{1}{x_i - x_{i-1}} (f(x_i) - f(x_{i-1})). \tag{4.41} \]
Then the RKHS associated with the Green’s function of $\Delta$ in (4.41) agrees with that from the kernel construction in sect. 3.1, i.e., the discrete Cameron-Martin Hilbert space.

**Proof.** Immediate from the previous Proposition and its corollaries. \(\square\)

**Acknowledgement.** The co-authors thank the following colleagues for helpful and enlightening discussions: Professors Daniel Alpay, Sergii Bezuglyi, Ilwoo Cho, Ka Sing Lau, Paul Muhly, Myung-Sin Song, Wayne Polyzou, Gestur Olafsson, Keri Kornelson, and members in the Math Physics seminar at the University of Iowa.

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