Hyperbolic geometry, continued fractions and classification of the finitely generated totally ordered simple dimension groups

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Abstract

We classify the polycyclic totally ordered simple dimension groups, i.e. dimension groups given by a dense embedding of lattice $\mathbb{Z}^n$ into the real line. Our method is based on the geometry of simple geodesics on the hyperbolic surface of genus $g \geq 2$. The main theorem says that isomorphism classes of the polycyclic totally ordered dimension groups are bijective with a generic subset of reals $\alpha$ modulo the action of group $GL(2, \mathbb{Z})$. The result is an extension of the Effros-Shen classification of the dicyclic dimension groups.

Key words and phrases: dimension group, geodesic lamination, Jacobi-Perron continued fraction

MSC.: 46L85 (noncommutative topology); 57M50 (geometric structures on low-dimensional manifolds)

1 Introduction

The dimension of a subspace of the Euclidean space $\mathbb{R}^n$ can be 1, 2, 3 or more. It is no longer true that dimension of any non-trivial subspace of the infinite-dimensional Euclidean space $\mathbb{R}^\infty$ is a positive integer. It was

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discovered by von Neumann that there exist linear subspaces in $\mathbb{R}^\infty$ of a non-integer (continuous) dimension. Namely, there exists a dimension function on the projections in a von Neumann algebra ranging in the unit interval.

Unlike the von Neumann algebras, the dimension function on the projections of a $C^*$-algebra, $A$, takes value in an abelian group, $K_0(A)$, rather than in $\mathbb{R}$, see e.g. [Effros 1981] [3], Chapter 1. For instance, if $A$ is an AF-algebra *ibid.*, then the range of the dimension function in $K_0(A)$ is known as a dimension group of $A$, see appendix for the definition. The classification of such groups is a difficult open problem.

In the remarkable paper [Effros & Shen 1980] [7], the authors classified the dicyclic dimension groups, i.e. dimension groups inside the abelian group $\mathbb{Z}^2$. Their main result says that each simple dicyclic group can be assigned a positive irrational number $\alpha$, defined modulo the action of matrix group $GL(2,\mathbb{Z})$, and such that if $\alpha = [a_0, a_1, a_2, \ldots]$ is a regular continued fraction of $\alpha$, then one gets a representation of the dicyclic group via the simplicial dimension groups:

\[
\mathbb{Z}^2 \left( \begin{array}{cc} 0 & 1 \\ 1 & a_0 \end{array} \right) \mathbb{Z}^2 \left( \begin{array}{cc} 0 & 1 \\ 1 & a_1 \end{array} \right) \mathbb{Z}^2 \left( \begin{array}{cc} 0 & 1 \\ 1 & a_2 \end{array} \right) \ldots
\]

see [Effros & Shen 1980] [7], Theorem 3.2. The irrational $\alpha$ is a slope of the straight line in the plane, which is the universal cover of two-dimensional torus, see [Effros & Shen 1980] [7] Theorem 2.1. The range of the dimension function (a positive cone) in $\mathbb{Z}^2$ is the half-plane $P_\alpha = \{(k, l) \in \mathbb{Z}^2 : \alpha k + l \geq 0\}$, and it is shown that each simple dicyclic dimension group arises in this way.

The objective of our note is similar classification of the polycyclic dimension groups, i.e. dimension groups inside the abelian group $\mathbb{Z}^n$. Recall that the universal cover for surfaces of genus $g \geq 2$ is no longer the Euclidean plane but the hyperbolic (Lobachevsky) half-plane $\mathbb{H}$. It is well known, that the role of straight lines in $\mathbb{H}$ is played by the simple geodesics, i.e. geodesics with no self-crossing points; we refer the reader to appendix for an exact definition. Thus, to classify the polycyclic dimension groups, one needs:

(i) to define a dimension group, $G$, coming from the simple geodesic, $\gamma$, on a hyperbolic surface of genus $g \geq 2$; 
(ii) to define a slope $\alpha \in \mathbb{R}$ of $\gamma$ on the surface; 
(iii) to construct a simplicial approximation of dimension group $G$ in terms of the slope $\alpha$ of geodesic $\gamma$.
The realization of (i) – (iii) is as follows. The closure, \( \bar{\gamma} \), of a simple non-periodic geodesic \( \gamma \) consists of the continuum of disjoint non-periodic geodesics known as a geodesic lamination \( \lambda \) [Casson & Bleiler 1988] [4]. Let \( |\lambda| \) be the total number of the principal regions of \( \lambda \) (ibid., p.60) and \( n = 2g + |\lambda| - 1 \), where \( g \) is the genus of the hyperbolic surface, \( X \), carrying the lamination \( \lambda \). It is known, that the set of invariant transversal measures of \( \lambda \) is a convex compact set \( \Delta_{k-1} \) of dimension \( 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \), where \( \left\lfloor \cdot \right\rfloor \) is the integer part of a number, see [Sataev 1975] [14]. By \( G_\lambda \) we understand the dimension group inside the abelian group \( \mathbb{Z}^n \), whose state space \( S(G_\lambda) \) is isomorphic to \( \Delta_{k-1} \). The assignment of \( G_\lambda \) via the invariant measures on \( \lambda \) is unique [Goodearl 1986] [9], Chapter 4. To implement (ii), we shall use the Jacobi-Perron continued fraction attached to the geodesic lamination \( \lambda \). Namely, a standard bijection (a blow-up, see the appendix) between geodesic laminations and foliations gives rise to a measured foliation \( F_\lambda \) on the surface \( X \). For simplicity, we let \( F_\lambda \) be given by trajectories of a closed 1-form \( \omega_\lambda \) on \( X \). Let \( \lambda_i = \int_{\gamma_i} \omega_\lambda \) be the periods of \( \omega_\lambda \) in a basis \( \{\gamma_1, \ldots, \gamma_n\} \) of the relative homology \( H_1(X, \text{Sing } \omega_\lambda; \mathbb{Z}) \); choosing \( \lambda_1 \neq 0 \), we let \( \theta_{i-1} = \lambda_i/\lambda_1 \) for \( i \geq 2 \) and \( \theta = (\theta_1, \ldots, \theta_{n-1}) \). Consider the Jacobi-Perron continued fraction of \( \theta \) [Bernstein 1971] [3]:

\[
\begin{pmatrix}
1 \\
\theta
\end{pmatrix} = \lim_{k \to \infty} \begin{pmatrix}
0 & 1 \\
I & b_0
\end{pmatrix} \cdots \begin{pmatrix}
0 & 1 \\
I & b_k
\end{pmatrix} \begin{pmatrix}
0 \\
1
\end{pmatrix},
\]

(2)

where \( I \) is the unit matrix, \( b_i = (b^{(i)}_1, \ldots, b^{(i)}_{n-1})^T \) a vector of the non-negative integers and \( I = (0, \ldots, 0, 1)^T \). By a slope of the geodesic \( \gamma \), we shall understand an irrational number \( \alpha \) given by the regular continued fraction:

\[
\alpha = [b^{(0)}_1 + 1, \ldots, b^{(0)}_{n-1} + 1, b^{(1)}_1 + 1, \ldots, b^{(1)}_{n-1} + 1, \ldots].
\]

(3)

(In other words, \( \alpha \) is a concatenation from zero to infinity of the vectors \( b_i \) with the entries added by 1 and, thus, all strictly positive.) Finally, to realize item (iii), one needs to restrict to the generic case \( k = 1 \), i.e. the totally ordered dimension groups. Such a restriction secures convergence of the Jacobi-Perron continued fraction [2] and one can apply the known theorem of [Effros & Shen 1979] [6] on approximation of the unimodular dimension groups by the simplicial dimension groups. A summary of our results can be formulated as follows.
Theorem 1 Each finitely generated totally ordered simple dimension group $G$ can be indexed by a positive irrational number $\alpha \in U$, where $U$ is a generic subset of $\mathbb{R}$. The parametrization, $G_\alpha$, has the following properties:

(i) if $G_\alpha$ and $G_{\alpha'}$ are order-isomorphic, then:

$$\alpha' = \frac{a\alpha + b}{c\alpha + d} \quad \text{for a matrix } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z});$$

(ii) $G_\alpha$ is the limit of the following simplicial dimension groups:

$$\mathbb{Z}^n \left( \begin{array}{cc} 0 & 1 \\ I & b_0 \end{array} \right) \mathbb{Z}^n \left( \begin{array}{cc} 0 & 1 \\ I & b_1 \end{array} \right) \mathbb{Z}^n \ldots$$

The article is organized as follows. The dimension group $G_\lambda$ is introduced in section 2. Theorem 1 is proved in section 3. Finally, in section 4 a brief account of geodesic laminations, measured foliations and the Jacobi-Perron fractions is given; it is supplemented by basic facts on the dimension groups, whose complete coverage can be found in [Rørdam, Larsen & Laustsen 2000] [13], Chapter 5.

2 Dimension groups generated by foliations

For notation, we refer the reader to appendix; let $\gamma$ be a simple non-periodic geodesic on a hyperbolic surface $X$ of genus $g \geq 2$. The closure $\bar{\gamma}$ contains a continuum of non-periodic pairwise disjoint simple geodesics, which form a perfect (Cantor) set on $X$. Such a closure is a geodesic lamination, which we shall denote by $\lambda$. The number of principal regions of $\lambda$ will be denoted by $|\lambda|$.

It is well known that $\lambda$ corresponds to a foliation $\mathcal{F}_\lambda$ on $X$, see e.g. [Thurston 1997] [15], Chapter 8.5. The foliation $\mathcal{F}_\lambda$ has $|\lambda|$ singular points of saddle type; it is obtained from the geodesic lamination $\lambda$ by a blow-down homotopy (see the appendix). Denote by $\Phi_X$ the space of foliations on $X$, whose singularity set coincides with that of $\mathcal{F}_\lambda$. The coordinates $(\lambda_1, \ldots, \lambda_n)$ of $\mathcal{F}_\lambda$ in $\Phi_X$ are given by the formula:

$$\lambda_i = \int_{\gamma_i} \omega_\lambda \, d\mu,$$
where $\omega_\lambda$ is a closed 1-form tangent to the leaves of foliation $\mathcal{F}_\lambda$, $\{\gamma_1, \ldots, \gamma_n\}$ is a basis in the relative homology $H_1(X, \text{Sing} \mathcal{F}_\lambda; \mathbb{Z})$ and $\mu$ an invariant transversal measure on the leaves of foliation $\mathcal{F}_\lambda$, see Section 4.2 for the details. It follows from the formulas for the relative homology that:

$$n = 2g + |\lambda| - 1. \quad (7)$$

On the other hand, it is known that the total number of independent invariant measures of foliation $\mathcal{F}_\lambda$ is equal to $k \leq \left\lfloor \frac{n}{2} \right\rfloor$, see [Sataev 1975] [14]. If we denote by $(\mu_1, \ldots, \mu_k)$ all such measures, then parameter $\mu$ in formula (6) can be written as $\mu = \sum_{j=1}^k \alpha_j \mu_j$, where $\alpha_j \geq 0$ are some real numbers. Thus, one gets from formula (6) an $n \times k$ matrix $(\lambda_{ij})$, whose entry $\lambda_{ij}$ is obtained by integration of $\omega$ along the contour $\gamma_i$ with $\mu = \mu_j$; each $j$-row of the matrix defines a homomorphism $h_j : \mathbb{Z}^n \to \mathbb{R}$, such that $h_j(e_i) = \lambda_{ij}$, where $e_i$ is a base element in $\mathbb{Z}^n$. It is easy to see, that the kernel of $h_j$ is a hyperplane in $\mathbb{R}^n$. Thus, one obtains a dimension group, $G_\lambda$, inside the abelian group $\mathbb{Z}^n$, which is bounded by the $k$ hyperplanes corresponding to the measures $(\mu_1, \ldots, \mu_k)$.

**Definition 1** The dimension group $G_\lambda$ is called associated to the geodesic lamination $\lambda$ (equivalently, foliation $\mathcal{F}_\lambda$).

In the sequel, our main case will be $k = 1$ (i.e. the totally ordered dimension group $G_\lambda$). Note that the foliations $\mathcal{F}_\lambda$ with a unique invariant ergodic measure are generic in the space $\Phi_x$ [Masur 1982] [10]. In this generic case, the dimension group $G_\lambda$ can be identified with the projective class of a $\mathbb{Z}$-module $\mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n \subset \mathbb{R}$, i.e. an equivalence class of the modules $\mu(\mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n)$, where $\mu > 0$ is a real number.

### 3 Proof of theorem 1

To prove theorem 1 we have to show that parametrization $G_\alpha$ of the polycyclic totally ordered simple dimension groups described in introduction has the following properties:

(i) if $G_\alpha$ and $G_{\alpha'}$ are order-isomorphic, then

$$\alpha' = \frac{a\alpha + b}{c\alpha + d} \quad \text{for a matrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z});$$
(ii) $G_\alpha$ is the limit of the following simplicial dimension groups:

\[
\begin{pmatrix} 0 & 1 \\ I & b_0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \quad \ldots
\]

We shall proceed stepwise; we refer the reader to the appendix for all preliminary facts and notation used below.

(i) Let $G$ be a finitely generated totally ordered simple dimension group. In this case $G \cong \mathbb{Z}^n$ is a dense subgroup of the real line $\mathbb{R}$. Let $\lambda_1, \ldots, \lambda_n$ be a set of generators of the subgroup; one can always choose $\lambda_i$ to be positive.

We wish to construct a measured foliation $F$ from the set of reals $\lambda_1, \ldots, \lambda_n$. The foliation is carried by a surface $X$, whose genus we shall further specify. Our goal can be achieved with the help of the method of zippered rectangles (see the appendix). First, we define the bottom of the rectangle to be $\lambda_1 + \ldots + \lambda_n$; so far the construction is not unique pending a choice of a permutation $\pi$. Next, one takes an irreducible permutation $\pi$ on the $n$ symbols; the top of the rectangle will be $\lambda_{\pi(1)} + \ldots + \lambda_{\pi(n)}$. (If one takes a different $\pi$, one would get another parameterization of dimension groups by measured foliations; note, however, that the number of such choices is always finite.) Instead of $\pi$, one can specify the number and types of the singular points of the foliation $F$; there exists a one-to-one correspondence between these two sets of data [Masur 1982] [10], [Veech 1982] [16]. Now, the method of zippered rectangles produces a unique measured foliation $F$ on a surface $X$, whose genus is given by the formula:

\[
g = \frac{1}{2} (n + 1 - |\text{Sing } F|),
\]

where $|\text{Sing } F|$ is equal to the number of cyclic permutations in the prime decomposition of $\pi$.

Using the integration formula (16), it is immediate that dimension group, generated by the foliation $F$, is order-isomorphic to $G$. Note that since $G$ is totally ordered, the foliation $F$ is uniquely ergodic.

We can now index the group $G$ by a slope $\alpha$ of the foliation $F$; we shall write the corresponding dimension group as $G_\alpha$. Denote by $i$ an (index) map which assigns to every totally ordered simple dimension group a slope $\alpha$ of the corresponding measured foliation $F$.

Let us prove the following series of lemmas, which reflect the properties of the index map.
Lemma 1  The index map $i$ is an injection.

Proof. Indeed, let to the contrary, $G \neq G'$ and $i(G) = i(G')$. By the uniqueness theorem ([Perron 1907] [12], Section 5, Theorem 4), the Jacobi-Perron fractions for the vectors $\lambda, \lambda'$ must be different in at least one term. So does the regular continued fractions, which define the slopes $\alpha, \alpha'$. Thus, by a main property of the continued fractions, we have $\alpha \neq \alpha'$. One runs into a contradiction with the initial assumption. □

Lemma 2  If the totally ordered dimension groups $G_{\alpha}$ and $G_{\beta}$ are order-isomorphic, then

$$\beta = \frac{a\alpha + b}{c\alpha + d},$$

where $a, b, c, d$ are integers such that $ad - bc = \pm 1$.

Proof. Let $G_{\alpha}$ and $G_{\beta}$ be given by the $\mathbb{Z}$-modules $\sum \mathbb{Z}\lambda_i$ and $\sum \mathbb{Z}\lambda'_i$, respectively. Since $G_{\alpha}$ and $G_{\beta}$ are order-isomorphic, after a scaling:

$$\mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n = \mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n,$$

as the subsets of the real line; moreover, there exists a positive isomorphism $G_{\alpha} \to G_{\beta}$ given by the formula $\lambda'_j = \sum_{i=1}^n a_{ij} \lambda_i$, where $A = (a_{ij})$ is invertible matrix with the non-negative integer entries $a_{ij} \geq 0$, see e.g. [Effros 1981] [5], p.10.

According to [Bauer 1996] [2], Proposition 3, matrix $A$ can be uniquely factorized as:

$$A = \left( \begin{array}{cc} 0 & 1 \\ \mathbf{I} & b'_1 \end{array} \right) \cdots \left( \begin{array}{cc} 0 & 1 \\ \mathbf{I} & b'_s \end{array} \right) := B'_1 \ldots B'_s,$$

where $b'_i = (\tilde{b}^{(i)}_1, \ldots, \tilde{b}^{(i)}_{n-1})$ are vectors of the non-negative integers. Thus, $\lambda' = B'_1 \ldots B'_s \lambda$ and by the definition of a slope:

$$\beta = [\tilde{b}^{(0)}_1 + 1, \ldots, \tilde{b}^{(s)}_{n-1} + 1, \alpha] = \frac{a\alpha + b}{c\alpha + d},$$

for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$. □

Remark 1  Note that the converse of lemma 2 is true only in the case of the dicyclic dimension groups. The effect is due to the fact that the Jacobi-Perron algorithm converges only on a generic subset of the slopes $\alpha \in \mathbb{R}$. 7
The item (i) of theorem 1 is proved.

(ii) Let \( \mathbb{Z} \lambda_1 + \ldots + \mathbb{Z} \lambda_n \) be a \( \mathbb{Z} \)-module in \( \mathbb{R} \), corresponding to the totally ordered dimension group \( G_\alpha \).

**Remark 2** The \( \lambda_i \) can be obtained from the continued fraction of \( \alpha \) using an inverse of formula (3) and the Jacobi-Perron continued fraction:

\[
\left( \frac{1}{\theta} \right) = \lim_{k \to \infty} \left( \begin{array}{cc} 0 & 1 \\ I & b_0 \end{array} \right) \cdots \left( \begin{array}{cc} 0 & 1 \\ I & b_k \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right),
\]

where \( \theta = (\lambda_2/\lambda_1, \ldots, \lambda_n/\lambda_1) \) and \( \lambda_1 \neq 0 \). It is known [Bauer 1996][2], that the fraction is convergent on a generic set \( V \subset \mathbb{R}^n \) of vectors \( \lambda \) corresponding to the foliation \( \mathcal{F}_\alpha \) with the unique ergodic measure. In particular, the totally ordered dimension groups are in bijection with such foliations and therefore our fraction is always convergent.

One can now apply a result of Effros and Shen: the dimension group \( G_\alpha \) is a unimodular dimension group given by the following limit of simplicial dimension groups:

\[
\mathbb{Z}^n \xrightarrow{\left( \begin{array}{cc} 0 & 1 \\ I & b_0 \end{array} \right)} \mathbb{Z}^n \xrightarrow{\left( \begin{array}{cc} 0 & 1 \\ I & b_1 \end{array} \right)} \mathbb{Z}^n \ldots
\]

see [Effros & Shen 1979][6], Corollary 3.3. The item (ii) follows. Theorem 1 is proved. □

### 4 Appendix

The present section contains a brief account on the geodesic laminations, measured foliations, method of zippered rectangles and the Jacobi-Perron continued fractions. The corresponding topics are covered in [Casson & Bleiler 1988][4], [Masur 1982][10] and [Bernstein 1971][3], respectively. We also add a paragraph on the dimension groups; we refer the reader to [Rørdam, Larsen & Laustsen 2000][13] for a complete account.
4.1 Geodesics laminations

Let $S$ be a finite genus surface of constant negative curvature (a hyperbolic surface). By a geodesic, one understands a maximal arc on $S$ consisting of the locally shortest sub-arcs (in the given metric on $S$). Each geodesic is the image of the open real interval $I$ under a continuous map $I \to S$.

Lemma 3 (Topological classification of geodesics) Let $p \in S$ be a point with an attached unit vector $t \in S^1$ on the surface $S$. Then:

(a) for almost all points $t \in S^1$ (w.r.t. to the Lebesgue measure on $S^1$), the geodesic line through $p$ in direction $t$ is an immersion $I \to S$, i.e. a finite or infinite curve with self-intersections;

(b) the remaining set $K \subset S^1$ has the cardinality of continuum and geodesic lines through $p$ in the direction $t \in K$ are embedded curves of one of the three types:

(i) periodic;

(ii) spiraling towards a periodic geodesic;

(iii) non-periodic, whose closure is a perfect (Cantor) subset of $S$.

Proof. See [Artin 1924] [1] and [Myrberg 1931] [11]. □

The geodesics of type (b) are called simple, since they have no self-crossing points. Every geodesic $\gamma : I \to S$ of type (iii) is recurrent, i.e. for any $t_0 \in I$ and $\varepsilon > 0$ the $\varepsilon$-neighbourhood of $p(t_0)$ has infinitely many intersections with $\gamma(t)$ provided $t > N$, where $N = N(\varepsilon)$ is sufficiently large. The topological closure of recurrent geodesic on $S$ contains a continuum of the disjoint recurrent geodesics, and called a geodesic lamination $\lambda$. The intersection of $\lambda$ with any closed curve on $S$ is a Cantor set. The set $S - \lambda$ is called a principal region of $\lambda$. The principal region can have up to $4g - 4$ connected components on the surface of genus $g \geq 2$ [Casson & Bleiler 1988] [4]. We denote by $|\lambda|$ the total number of such components.

A foliation $\mathcal{F}$ on a surface $X$ is a partition of $X$ into a disjoint union of 1-dimensional and, possibly, a finite number of 0-dimensional leaves denoted by $\text{Sing } \mathcal{F}$. The immediate examples of foliations are orbits of the flows and trajectories of the quadratic differentials $f(z)dz^2$ on $X$ [Fathi, Laudenbach & Poénaru 1979] [8]. The foliation $\mathcal{F}$ is called measured if it supports an invariant transversal measure on the leaves [Fathi, Laudenbach & Poénaru 1979] [9]. In other words:
(i) \( \text{Sing } \mathcal{F} \) consists of the \( n \)-prong saddles, where \( n \geq 3 \);

(ii) each 1-leaf is everywhere dense in \( X \).

The geodesic lamination \( \lambda \) can be obtained from \( \mathcal{F} \) by a blow-up homotopy. Namely, a separatrix of \( \mathcal{F} \) is a 1-leaf one of whose ends lie in \( \text{Sing } \mathcal{F} \). The blow-up is a replacement of the separatrix by a narrow strip \([−ε, ε] \times \mathbb{R}\) using a homotopy surgery, which does not affect the nearby leaves. The complement of the blown-up separatrices consists of leaves of \( \mathcal{F} \) that make up a perfect (Cantor) set on \( X \). It is not hard to prove (e.g. [Thurston 1997] [15]), that the above complement is homeomorphic to \( \lambda \). Moreover, \( |\lambda| \) is equal to the number of singular points of the foliation \( \mathcal{F} \). Note that each measured foliation can be given by the orbits of a closed one-form \( \omega \) on \( X \), passing, if necessary, to a double cover of \( X \) [Masur 1982] [10].

### 4.2 Method of zippered rectangles

There exists a remarkable construction, which allows to produce a measured foliation from a given set of positive reals \((\lambda_1, \ldots, \lambda_n)\). Let \( \pi \) be a permutation of \( n \) symbols. Consider a rectangle with the base \( \lambda_1 + \ldots + \lambda_n \) and the top \( \lambda_{\pi(1)} + \ldots + \lambda_{\pi(n)} \). We shall identify the open interval \((\lambda_{i-1}, \lambda_i)\) in the base with the open interval \((\lambda_{\pi(i)-1}, \lambda_{\pi(i)})\) at the top for all \( i = 1, \ldots, n \). The resulting object will be a \( k \)-holed topological surface, \( X \), of genus \( g = \frac{1}{2}(n - N(\pi) + 1) \), where \( N(\pi) \) is the number of cyclic permutations in the prime decomposition of \( \pi \) [Veech 1982] [16]. A foliation \( \mathcal{F} \) on \( X \) is defined by vertical lines given by the closed 1-form \( \omega = dx \). The order of the singular points of \( \mathcal{F} \) depends on the length of the elementary cyclic permutations and the total number of the singular points equals \( k = N(\pi) \). The singular points are located at the holes of surface \( X \). To recover \( \lambda_i \) from the 1-form \( \omega \), notice that

\[
n = 2g + N(\pi) - 1 = \dim H_1(X, \text{Sing } \mathcal{F}; \mathbb{Z}),
\]

where the last symbol stays for the relative homology of \( X \) with respect to the set of singular points of \( \mathcal{F} \). Since \( \omega = dx \), one arrives at the elementary, but important formula:

\[
\lambda_i = \int_{\gamma_i} \omega,
\]

where \( \gamma_i \) are the elements of a basis in \( H_1(X, \text{Sing } \mathcal{F}; \mathbb{Z}) \).
### 4.3 Jacobi-Perron fractions

The Jacobi-Perron algorithm and connected (multidimensional) continued fraction generalizes the euclidean algorithm (regular continued fraction) of an irrational number. Namely, let $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{R} - \mathbb{Q}$ and $\theta_{i-1} = \frac{\lambda_i}{\lambda_1}$, where $1 \leq i \leq n$. The continued fraction

$$
\begin{pmatrix}
1 \\
\theta_1 \\
\vdots \\
\theta_{n-1}
\end{pmatrix} = \lim_{k \to \infty} \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_1^{(1)} \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(1)}
\end{pmatrix} \cdots \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_1^{(k)} \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(k)}
\end{pmatrix}
$$

where $b_i^{(j)} \in \mathbb{N} \cup \{0\}$, is called a **Jacobi-Perron fraction**, see [Perron 1907][12]. To recover the integers $b_i^{(k)}$ from the vector $(\theta_1, \ldots, \theta_{n-1})$, one has to repeatedly solve the following system of equations:

$$
\begin{align*}
\theta_1 &= b_1^{(1)} + \frac{1}{\sigma_{n-1}} \\
\theta_2 &= b_2^{(1)} + \frac{\sigma_1^{(1)}}{\sigma_{n-1}} \\
&\vdots \\
\theta_{n-1} &= b_{n-1}^{(1)} + \frac{\sigma_{n-2}^{(1)}}{\sigma_{n-1}};
\end{align*}
$$

(17)

where $(\theta_1', \ldots, \theta_{n-1}')$ is the next input vector. Thus, each vector $(\theta_1, \ldots, \theta_{n-1})$ gives rise to a formal Jacobi-Perron continued fraction. Whether the fraction is convergent or not, is yet to be determined.

Let us introduce the following notation. We let $A^{(0)} = \delta_{ij}$ (the Kronecker delta) and $A_i^{(k+n)} = \sum_{j=0}^{n-1} b_i^{(k)} A_i^{(n+j)}$, $b_0^{(k)} = 1$, where $i = 0, \ldots, n-1$ and $k = 0, 1, \ldots, \infty$. The Jacobi-Perron continued fraction of the vector $(\theta_1, \ldots, \theta_{n-1})$ is said to be **convergent**, if $\theta_i = \lim_{k \to \infty} \frac{A_i^{(k)}}{A_0^{(k)}}$ for all $i = 1, \ldots, n-1$. Unless $n = 2$, convergence of the Jacobi-Perron fractions is a delicate question. To the best of our knowledge, there exists no intrinsic necessary and sufficient conditions for such a convergence. However, the Bauer criterion and the Masur-Veech theorem imply that the Jacobi-Perron fractions converge for the generic vectors $(\theta_1, \ldots, \theta_{n-1})$. Namely, let $\mathcal{F}$ be a measured foliation on the surface $X$ of genus $g \geq 1$. Recall that the foliation $\mathcal{F}$ is called uniquely ergodic if every invariant measure of $\mathcal{F}$ is a multiple of the Lebesgue measure. By the Masur-Veech theorem, there exists a generic subset $V$ in the space of all measured foliations, such that each $\mathcal{F} \in V$ is a
uniquely ergodic measured foliation. We let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be the coordinate vector of the foliation \( \mathcal{F} \). Then the following (Bauer’s) criterion is true: the Jacobi-Perron continued fraction of \( \lambda \) converges if and only if \( \lambda \in V \subset \mathbb{R}^n \) [Bauer 1996] [2].

4.4 Dimension groups

We use \( \mathbb{Z}, \mathbb{Z}^+, \mathbb{Q} \) and \( \mathbb{R} \) for the integers, positive integers, rationals and reals, respectively and \( GL(n, \mathbb{Z}) \) for the group of \( n \times n \) matrices with entries in \( \mathbb{Z} \) and determinant \( \pm 1 \).

By an ordered group we shall mean an abelian group \( G \) together with a subset \( P = G^+ \) such that \( P + P \subseteq P, P \cap (-P) = \{0\} \), and \( P - P = G \). We call \( P \) the positive cone on \( G \). We write \( a \leq b \) (or \( a < b \)) if \( b - a \in P \) (or \( b - a \in P \setminus \{0\} \)).

G is said to be a Riesz group if:

(i) \( g \in G \) and \( ng \geq 0, n \in \mathbb{Z}^+ \) implies \( g \geq 0 \);

(ii) \( u, v \leq x, y \) in \( G \) implies existence of \( w \in G \) such that \( u, v \leq w \leq x, y \).

Given ordered groups \( G \) and \( H \), we say that a homomorphism \( \varphi : G \to H \) is positive if \( \varphi(G^+) \subseteq H^+ \), and that \( \varphi : G \to H \) is an order isomorphism if \( \varphi(G^+) = H^+ \).

A positive homomorphism \( f : G \to \mathbb{R} \) is called a state if \( f(u) = 1 \), where \( u \in G^+ \) is an order unit of \( G \). We let \( S(G) \) be the state space of \( G \), i.e. the set of states on \( G \) endowed with the natural topology.

\( S(G) \) is a compact convex subset of the vector space \( \text{Hom} (G, \mathbb{R}) \). By the Krein-Milman theorem, \( S(G) \) is the closed convex hull of its extreme points, which are called pure states.

An ordered abelian group is a dimension group if it is order isomorphic to \( \lim_{m,n \to \infty} (\mathbb{Z}r(m), \varphi_{mn}) \), where the \( \mathbb{Z}r(m) \) are simplicially ordered groups (i.e. \( (\mathbb{Z}r(m))^+ \cong \mathbb{Z}^+ \oplus \ldots \oplus \mathbb{Z}^+ \)), and the \( \varphi_{mn} \) are positive homomorphisms. The dimension group \( G \) is said to be unimodular if \( r(m) = \text{Const} = r \) and \( \varphi_{mn} \) are positive isomorphisms of \( \mathbb{Z}^r \). In other words, \( G \) is the limit

\[
\mathbb{Z}^r \xrightarrow{\varphi_0} \mathbb{Z}^r \xrightarrow{\varphi_1} \mathbb{Z}^r \xrightarrow{\varphi_2} \ldots,
\]

of matrices \( \varphi_k \in GL(r, \mathbb{Z}^+) \).

The Riesz groups are dimension groups and vice versa. The Riesz groups can be viewed as the abstract dimension groups, while dimension groups
as a representation of the Riesz groups by the infinite sequences of positive homomorphisms.

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