Equivariant Intersection Theory

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1 Introduction

The purpose of this paper is to develop an equivariant intersection theory for actions of linear algebraic groups on algebraic schemes. The theory is based on our construction of equivariant Chow groups. They are algebraic analogues of equivariant cohomology groups which satisfy all the functorial properties of ordinary Chow groups. In addition, they enjoy many of the properties of equivariant cohomology. The principal results of this paper are:

(1) If a group $G$ acts with finite stabilizers on a scheme $X$, then rational equivariant Chow groups can be identified with the rational Chow groups of a quotient. As a result, we show that the rational Chow groups of quotients of smooth varieties by group actions have a canonical ring structure. This extends and simplifies previous work of Mumford ([Mu]), Gillet ([Gi]) and Vistoli ([Vi]). In addition the integral Chow groups are an invariant of the quotient stack $[X/G]$, so we can associate an integral Chow ring to smooth

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quotient stacks.

(2) There is a Riemann-Roch isomorphism between a completion of equivariant $K$-theory of coherent sheaves and a completion of equivariant Chow groups. This extends the Riemann-Roch theorem of Baum, Fulton and MacPherson to the equivariant case.

(3) There is a localization for torus actions relating the the equivariant Chow groups of a scheme to the Chow groups of the fixed locus. Such a theorem is a hallmark of other equivariant theories such as cohomology and $K$-theory. The localization theorems in equivariant cohomology and $K$-theory imply residue formulas such as Bott’s ([B-V], [A-B], [I-N]), which can now be proved using intersection theory.

Previous work on equivariant intersection theory ([B], [G], [V]) defined equivariant Chow groups using invariant cycles on $X$. The definition we give of equivariant Chow groups, in contrast, is modeled on Borel’s definition of equivariant cohomology. Borel’s insight was to replace the original topological space $X$ by a homotopic space $X \times EG$, where $EG$ is a contractible space on which $G$ acts freely. Since $G$ acts freely, there is a nice quotient $X_G$ of $X \times EG$ by $G$ and equivariant cohomology is defined as the cohomology of $X_G$. To define equivariant Chow groups one needs an appropriate algebraic replacement for $EG$. This was supplied by Totaro [T], who used finite dimensional representations to approximate the infinite dimensional space $EG$. In particular if $V$ is a representation of $G$, let $U$ denote an open set on which $G$ acts freely and has a a quotient $U \to U/G$ which is a principal bundle. For any linear algebraic group, the representation can be chosen so that $V - U$ has arbitrarily large codimension. If $X$ is a $G$-scheme then, under mild hypotheses on $G$ or $X$ (see below), $X \times U$ has a quotient $X \times^G U$ so that $X \times U \to X \times^G U$ is a principal $G$-bundle. The group $A_{\dim V + i - \dim G}(X \times^G U)$ is independent of $V$ as long as the codimension of $V - U$ is sufficiently large. This defines the $i$-th equivariant Chow group $A^G_i(X)$.

Because $X \times U \to X \times^G U$ is a principal $G$-bundle, cycles on $X \times^G U$ exactly correspond to $G$-invariant cycles on $X \times U$. Since we only consider cycles of codimension smaller than the dimension of $X \times (V - U)$, we may in fact view these as $G$-invariant cycles on $X \times V$. In other words, instead of considering only $G$-invariant cycles on $X$ we consider $G$-invariant cycles on
for any sufficiently big representation of $G$. By enlarging the class of cycles we allow, we obtain a theory with many good properties.

By construction, the equivariant Chow groups $A^*_G(X)$ inherit most of the properties of ordinary Chow groups. In particular if $X$ is smooth, then there is an intersection product on the equivariant Chow groups $A^*_G(X)$, no matter how badly $G$ acts on $X$. When $G$ acts properly and a quotient $X/G$ exists we prove (Theorem 2) that $A^*_G(X) \otimes \mathbb{Q} = A^*_G(X/G) \otimes \mathbb{Q}$. As a result, this proves that if $G$ acts properly on a smooth variety $X$, then the rational Chow groups of a quotient $X/G$ have a canonical intersection product (Corollary 2). This extends the results of Vistoli, who proved such a theorem when $G$ acts with finite, reduced stabilizers. This theorem should be useful for doing intersection theory on moduli spaces of objects which possess infinitesimal automorphisms. It can also be used to do intersection theory on toric varieties in arbitrary characteristic. Furthermore, by avoiding the use of algebraic stacks, our proof is much simpler.

Another interesting aspect of the theory is that the groups $A^*_G(X)$ are actually an invariant of the quotient stack $[X/G]$ (Proposition 3). Thus if $X$ is smooth, then there is an integral intersection ring associated to the quotient stack $[X/G]$. When $[X/G]$ is Deligne-Mumford (i.e. $G$ acts with finite, reduced stabilizers) then our ring tensored with $\mathbb{Q}$ agrees with the rings of Gillet and Vistoli. It would be interesting to compute the torsion in the equivariant Chow ring in examples of moduli stacks such as curves of low genus.

Our results on quotient stacks also suggest that there should be integral Chow rings associated to arbitrary smooth stacks. Motivated by the equivariant Chow ring, we expect that this ring would have torsion in arbitrarily high degree. However, we do not know how to construct such a ring in general.

The connection between equivariant $K$-theory and equivariant Chow groups is given by our equivariant Riemann-Roch theorem (Theorem 4). We prove that there is an isomorphism $\tau_X : K^*_0(X) \otimes \mathbb{Q} \to A^*_G(X) \otimes \mathbb{Q}$ between the completion of $K$-theory along the augmentation ideal of the representation ring $R(G)$, and the completion of the $A^*_G(X)$ along the augmentation ideal of the equivariant Chow ring of a point.

Along the way we prove a theorem (Theorem 3) which shows that the completion of $K^*_G(X)$ along the augmentation ideal of $R(G)$ is the same as the completion along the augmentation ideal of $K^*_G(X)$.
to results of \cite{CEPT} and answers a special case of a conjecture of \cite{Ko}.

In the last part of the paper we prove a localization theorem for torus actions. If a torus $T$ acts on $X$ with fixed locus $X^T \subset X$, then $A^*_T(X) \otimes_{R_T} (R^+_T)^{-1} = A^*_T(X^T) \otimes (R^+_T)^{-1} R_T$, where $R^+_T$ denotes the multiplicative set of homogeneous elements of positive degree in $R_T = Sym(T)$.

Finally, using the localization theorem, we prove the Bott residue formula for actions of (split) tori on smooth complete varieties. This formula has been recently applied in enumerative geometry (cf. \cite{E-S}) so we include an intersection theoretic proof. Our line of argument follows that of \cite{A-B} using equivariant intersection theory in place of equivariant cohomology. (Note that Iversen and Nielsen \cite{I-N} gave an algebraic proof of this formula - for smooth projective varieties - using equivariant $K$-theory. Also, using techniques of algebraic deRahm homology, Hübli and Yekutieli \cite{H-Y} proved a version - in characteristic 0 - for the action of an algebraic vector field with isolated fixed points.)

In Section 6 we discuss extensions of the theory to group schemes over a regular base scheme, and in the appendix we prove a number of technical results about group actions and quotients in arbitrary characteristic.

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2 Definitions and basic properties

2.1 Conventions and Notation

Except in Section 6, all schemes are assumed to be of finite type over a field of arbitrary characteristic. A variety is a reduced and irreducible scheme. An algebraic group is always assumed to be linear.

If an algebraic group $G$ acts on a scheme $X$ then the action is said to be closed if the orbits of geometric points are closed in $X$. It is proper if the action map $G \times X \to X \times X$ is proper. Finally, we say that it is free if the action map is a closed embedding. By \cite[Prop. 0.9]{GIT} if the action is free and a geometric quotient scheme $X/G$ exists, then $X$ is a principal $G$ bundle over $X/G$. 

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Throughout the paper we will assume that at least one of the following hypotheses on \(X\) or \(G\) is satisfied.

1. \(X_{\text{red}}\) is quasi-projective and the action is linearized with respect to some projective embedding.

2. \(G\) is connected and \(X_{\text{red}}\) equivariantly embeds as a closed subscheme in a normal variety.

3. \(G\) is special in the sense of \(\text{Sem-Chev}\); i.e. all principal \(G\)-bundles are locally trivial in the Zariski topology. (Examples of special groups are tori, solvable and unipotent groups as well as \(GL(n), SL(n),\) and \(Sp(2n)\). Finite groups are not special, nor or the orthogonal groups \(SO(2n)\) and \(SO(2n+1)\).)

For simplicity of exposition, we will usually assume that \(X\) is equidimensional.

### 2.2 Equivariant Chow groups

Let \(X\) be an \(n\)-dimensional scheme. We will denote the \(i\)-th equivariant Chow group of \(X\) by \(A_i^G(X)\). It is defined as follows.

Let \(G\) be a \(g\)-dimensional algebraic group. Choose an \(l\)-dimensional representation \(V\) of \(G\) such that \(V\) has an open set \(U\) on which \(G\) acts freely and whose complement has codimension more than \(n - i\). Assume that a quotient \(U \to U/G\) (necessarily a principal bundle) exists. (Such representations exist for any group; see Lemma \(\text{[2]}\) of the Appendix.) The principal bundle \(U \to U/G\) is Totaro’s finite dimensional approximation of the classifying bundle \(EG \to BG\) (see \(\text{[16]}\) and \(\text{[E-G]}\)). The diagonal action on \(X \times U\) is also free, and since one hypothesis (1)-(3) holds, there is a quotient \(X_{\text{red}} \times U \to (X_{\text{red}} \times U)/G\) which is a principal \(G\) bundle\(^1\) (Prop \(\text{[22]}\)). We will usually denote this quotient by \((X_{\text{red}} \times^G U)\) or \(X_G\).

**Definition-Proposition 1** Set \(A_i^G(X)\) (the \(i\)-th equivariant Chow group) to be \(A_{i+l-g}(X_G)\), where \(A_*\) is the usual Chow group. This group is independent of the representation as long as \(V - U\) has sufficiently high codimension.

\(^1\)Without any hypothesis on \(X\) or \(G\), we only know that the quotient exists as an algebraic space.
Remark. In the sequel, the notation $U \subset V$ will refer to an open set in a representation on which the action is free. Because we are working with Chow groups, we will, when no confusion can arise, abuse notation and act as if all schemes are reduced.

Proof of Definition-Proposition 1. We will use Bogomolov’s double fibration argument. Let $V_1$ be another representation of dimension $k$ such that there is an open $U_1$ with a principal bundle quotient $U_1 \to U_1/G$ and whose complement has codimension at least $n - i$. Let $G$ act diagonally on $V \oplus V_1$. Then $V \oplus V_1$ contains an open set $W$ which has a principal bundle quotient $W/G$ and contains both $U \oplus V_1$ and $V \oplus U_1$. Thus, $A_{i+k+l-g}(X \times^G W) = A_{i+k+l-g}(X \times^G (U \oplus V_1))$ since $(X \times^G W) - (X \times^G (U \oplus V_1))$ has dimension smaller than $i + k + l - g$. On the other hand, the projection $V \oplus V_1 \to V$ makes $X \times^G (U \oplus V_1)$ a vector bundle over $X \times^G U$ with fiber $V_1$ and structure group $G$. Thus, $A_{i+k+l-g}(X \times^G (U \oplus V_1)) = A_{i+l-g}(X \times^G U)$. Likewise, $A_{i+k+l-g}(X \times^G W) = A_{i+k-g}(X \times^G U_1)$, as desired. □

Example For the classical groups, the representations and subsets can be constructed explicitly. In the simplest case, if $G = \mathbb{G}_m$ then we can take $V$ to an $l$-dimensional representation with weights one, $U = V \setminus \{0\}$, and $U/G = \mathbb{P}^{l-1}$. If $G = GL_n$, take $V$ to be the vector space of $n \times p$ matrices ($p > n$), with $GL_n$ acting by left multiplication, and let $U$ be the subset of matrices of maximal rank. Then $U/G$ is the Grassmannian $Gr(n,p)$.

Remarks Now that we have defined equivariant Chow groups, we will use the notation $X_G$ to mean a mixed quotient $X \times^G U$ for any representation $V$ of $G$. If we write $A_{i+l-g}(X_G)$ then $V - U$ is assumed to have codimension more than $n - i$ in $V$. (As above $n = \dim X$, $l = \dim V$ and $g = \dim G$.)

If $Y \subset X$ is an $m$-dimensional $G$-invariant subvariety, then it has a $G$-equivariant fundamental class $[Y]_G \in A^*_m(X)$. However, unlike ordinary Chow groups, $A^*_G(X)$ can be non-zero for any $i \leq n$, including negative $i$. The projection $X \times U \to U$ induces a map $X_G \to U$ with fiber $X$. Restriction to a fiber gives a map $i^* : A^*_G(X) \to A^*_u(X)$ from equivariant Chow groups to ordinary Chow groups. The map is independent of the choice of fiber because any two points of $U/G$ are rationally equivalent. For any $G$-invariant subvariety $Y \subset X$, $i^*([Y]_G) = [Y]$.
2.3 Functorial properties

In this section all maps $f : X \rightarrow Y$ are assumed to be $G$-equivariant.

If $f : X \rightarrow Y$ is proper, then by descent, the induced map $f_G : X_G \rightarrow Y_G$ is also proper. Likewise, if $f : X \rightarrow Y$ is flat of relative dimension $k$ then $f_G : X_G \rightarrow Y_G$ is flat of dimension $k$.

**Definition 1** Define proper pushforward $f_* : A^G_i(X) \rightarrow A^G_i(Y)$, and flat pullback $f^* : A^G_i(Y) \rightarrow A^G_{i-k}(X)$ as $f_G^* : A_{i+t-g}(X_G) \rightarrow A_{i+t-g}(Y_G)$ and $f_G^* : A_{i+t-g}(Y_G) \rightarrow A_{i+t-g-k}(X_G)$ respectively.

If $f : X \rightarrow Y$ is smooth, then $f : X_G \rightarrow Y_G$ is also smooth. Furthermore, if $f : X \rightarrow Y$ is a regular embedding, then $f \times id : X \times U \rightarrow Y \times U$ is a regular embedding. In the cartesian diagram

$$
\begin{array}{ccc}
X \times U & \rightarrow & Y \times U \\
\downarrow & & \downarrow \\
X_G & \rightarrow & Y_G \\
\end{array}
$$

the vertical arrows are flat and surjective so by [F-L, Prop. IV 3.5] the map $X_G \rightarrow Y_G$ is also a regular embedding. In particular, if $f : X \rightarrow Y$ is an l.c.i morphism in the sense of [Fu, Section 6.6], then $f_G : X_G \rightarrow Y_G$ is also l.c.i.

**Definition 2** If $f : X \rightarrow Y$ is l.c.i. of codimension $d$ then define $f^* : A^G_i(Y) \rightarrow A^G_{i-d}(X)$ as $f_G^* : A_{i+t-g}(Y_G) \rightarrow A_{i+t-g-d}(X_G)$.

**Proposition 1** The maps $f_*$ and $f^*$ above are well defined.

Proof: We will use the double fibration argument. Let $V_1$ be another representation. Then we have a cartesian diagram

$$
\begin{array}{ccc}
X \times^G (U \oplus V_1) & \rightarrow & Y \times^G (U \oplus V_1) \\
\downarrow & & \downarrow \\
X \times^G U & \rightarrow & Y \times^G U \\
\end{array}
$$

The vertical maps are flat, and their pullbacks are the isomorphisms which allowed us to define $A^G_i$. Since flat pullback is compatible with proper pushforward, the equivariant pushforward $f_*$ is well defined. Likewise the flat pullback is compatible with flat and l.c.i pullback, so $f^*$ is also well defined. $\Box$
2.4 Chern classes

Let $X$ be a scheme with a $G$ action, and let $E$ be an equivariant vector bundle. For each $i, j$ define a map $c^G_j(E) : \Lambda^G_i(X) \to \Lambda^G_{i-j}(X)$ as follows. Let $V$ be an $l$-dimensional representation such that $V - U$ has high codimension. By hypothesis, there is a principal bundle $X \times U \to X_G$. Thus by [GIT, Prop. 7.1] there is a quotient $E_G$ of $E \times U$.

**Lemma 1** $E_G \to X_G$ is a vector bundle.

Proof. The bundle $E_G \to X_G$ is an affine bundle which locally trivial in the étale topology since it becomes locally trivial after the smooth base change $X \times U \to X_G$. Also, the transition functions are affine since they are affine when pulled back to $X \times U$. Hence, by descent, $E_G \to X_G$ is locally trivial in the Zariski topology and has affine transition functions; i.e., $E_G$ is a vector bundle over $X_G$. □

Identify $\Lambda^G_i(X)$ and $\Lambda^G_{i-j}(X)$ with $\Lambda^{i+l-g}(X_G)$ and $\Lambda^{i-j+l-g}(X_G)$ respectively.

**Definition-Proposition 2** Define equivariant Chern classes $c^G_j(E) : \Lambda^G_i(X) \to \Lambda^G_{i-j}(X)$ by $c^G_j(E) \cap \alpha = c_j(E_G) \cap \alpha \in \Lambda^{i-j+l-g}(X_G)$. This definition does not depend on the choice of representation.

Proof: Let $V_1$ be another representation. Then the pullback of $E \times U$ to $X \times (U \oplus V_1)$ is isomorphic to the quotient $E \times (U \oplus V_1)$. □

Given the above propositions, equivariant Chow groups satisfy all the formal properties of ordinary Chow groups ([Fu, Chapters 1-6]). In particular, if $X$ is smooth, there is an intersection product on the the equivariant Chow groups $\Lambda^G_*(X)$ which makes $\oplus \Lambda^G_*(X)$ into a graded ring.

2.5 Operational Chow groups

Define equivariant operational Chow groups $\Lambda^G_*(X)$ as operations $c(Y \to X) : \Lambda^G_*(Y) \to \Lambda^G_{*}(Y)$ for every $G$-map $Y \to X$. As for ordinary operational Chow groups ([Fu, Chapter 17]), these operations should be compatible with the operations on equivariant Chow groups defined above (pullback for l.c.i. morphisms, proper pushforward, etc.) From this definition it is clear
that for any $X$, $A^*_G(X)$ has a ring structure. The ring $A^*_G(X)$ is positively graded, and $A^*_G(X)$ can be non-zero for any $i \geq 0$.

Note that by construction, the equivariant Chern classes defined above are elements of the equivariant operational Chow ring.

**Proposition 2** If $X$ is smooth of dimension $n$, then $A^i_G(X) \simeq A^{n-i}_G(X)$.

**Proof.** Define a map $A^i_G(X) \to A^{n-i}_G(X)$ by the formula $c \mapsto c \cap [X]_G$. Define a map $A^{n-i}_G(X) \to A^i_G(X)$, $\alpha \mapsto c_\alpha$ as follows. If $Y \to X$ is a $G$-map, then since $X$ is smooth, the graph $\gamma_f : Y \to Y \times X$ is a $G$-map which is a regular embedding. If $\beta \in A^*_G(Y)$ set $c_\alpha \cap \beta = \gamma_f^*(\beta \times \alpha)$ (note that the cartesian product of equivariant classes is well defined).

**Claim (cf. [Fu, Proposition 17.3.1]):** $\beta \times (c \cap [X]_G) = c \cap (\beta \times [X]_G)$.

Given the claim, the formal arguments of [Fu, Proposition 17.4.2] show that the two maps are inverses.

**Proof of Claim:** The equivariant class $\beta$ is represented by a cycle on some $Y_G$, which we can assume to be the fundamental class of a subvariety $W \subset Y_G$. Let $\tilde{W}$ be the inverse image of $W$ in $Y \times U$. Then $\beta$ pulls back to $[\tilde{W}]_G$ by the equivariant projection map $Y \times U \to Y$. By requiring $V - U$ to have sufficiently high codimension, we may assume that the pullback on Chow groups is an isomorphism in the appropriate degrees. Replacing $Y$ by $Y_G$, we may assume $\beta = [\tilde{W}]_G$. Since $\tilde{W}$ is $G$-invariant, the projection $p : \tilde{W} \times X \to X$ is equivariant. Thus,

$$(c \cap [X]_G) \times [\tilde{W}]_G = p^*(c \cap [X]_G) = c \cap p^*([X]_G) = c \cap ([X_G] \times [\tilde{W}]_G).$$

$\square$

Let $V$ be a representation such that $V - U$ has codimension more than $k$, and set $X_G = X \times^G U$. In the remainder of the subsection we will discuss the relation between $A^k_G(X)$ and $A^k(X_G)$ (ordinary operational Chow groups).

Recall [Fu, Definition 18.3] that an envelope $\pi : \tilde{X} \to X$ is a proper map such that for any subvariety $W \subset X$ there is a subvariety $\tilde{W}$ mapping birationally to $W$ via $\pi$. In the case of group actions, we will say that $\pi : \tilde{X} \to X$ is an *equivariant* Chow envelope, if $\pi$ is $G$-equivariant, and if we can take $\tilde{V}$ to be $G$-invariant for $G$-invariant $V$. If there is an open set $X^0 \subset X$ over which $\pi$ is an isomorphism, then we say $\pi : \tilde{X} \to X$ is a *birational* envelope.
Lemma 2 If $\pi : \tilde{X} \to X$ is an equivariant (birational) envelope, then $p : \tilde{X}_G \to X_G$ is a (birational) envelope ($\tilde{X}_G$ and $X_G$ are constructed with respect to a fixed representation $V$). Furthermore, if $X^0$ is the open set over which $\pi$ is an isomorphism (necessarily $G$-invariant), then $p$ is an isomorphism over $X^0_G = X^0 \times^G U$.

Proof: Fulton [Fu, Lemma 18.3] proves that the base extension of an envelope is an envelope. Thus $\tilde{X} \times U \xrightarrow{\pi \times \text{id}} X \times U$ is an envelope. Since the projection $X \times U \to X$ is equivariant, this envelope is also equivariant. If $W \subset X_G$ is a subvariety, let $W'$ be its inverse image (via the quotient map) in $X \times U$. Let $W'$ be an invariant subvariety of $\tilde{X} \times U$ mapping birationally to $W'$. Since $G$ acts freely on $\tilde{X} \times U$ it acts freely on $\tilde{W}'$, and $\tilde{W} = \tilde{W}'/G$ is a subvariety of $\tilde{X}_G$ mapping birationally to $W$. This shows that $\tilde{X}_G \to X_G$ is an envelope; it is clear that the induced map $\tilde{X}_G \to \tilde{X}$ is an isomorphism over $X^0_G$. $\blacksquare$

Suppose $\tilde{X} \xrightarrow{\pi} X$ is an equivariant envelope which is an isomorphism over $U$. Let $\{S_i\}$ be the irreducible components of $S = X - X^0$, and let $E_i = \pi^{-1}(S_i)$. Then $\{S_iG\}$ are the irreducible components of $X_G - X^0_G$ and $E_iG = p^{-1}(S_iG)$.

Theorem 1 If $X$ has an equivariant non-singular envelope $\pi : \tilde{X} \to X$ such that there is an open $X^0 \subset X$ over which $\pi$ is an isomorphism, then $A^k_G(X) = A^k(X_G)$.

Proof: If $\pi : \tilde{X} \to X$ is an equivariant non-singular envelope, then $p : \tilde{X}_G \to X_G$ is also an envelope and $\tilde{X}_G$ is non-singular. Thus, by [K], Lemma 1.2] $p^* : A^*(X_G) \to A^*(\tilde{X}_G)$ is injective. The image of $p^*$ is described inductively in [K], Theorem 3.1]. A class $\tilde{c} \in A^*(\tilde{X}_G)$ equals $p^*c$ if and only if for each $E_iG$, $\tilde{c}|_{E_iG} = p^*c_i$ where $c_i \in A^*(E_i)$. This description follows from formal properties of operational Chow groups, and the exact sequence [K], Theorem 2.3]

$$A^*(X_G) \xrightarrow{p} A^*(\tilde{X}_G) \xrightarrow{p_1 - p_2} A^*(\tilde{X}_G \times_{X_G} \tilde{X}_G)$$

where $p_1$ and $p_2$ are the two projections from $\tilde{X}_G \times_{X_G} \tilde{X}_G$.

By Proposition 2 above, we know that $A^k_G(X) = A^k(\tilde{X}_G)$. We will show that $A^k_G(X)$ and $A^k(X_G)$ have the same image in $A^k(\tilde{X}_G)$. By Noetherian induction we may assume that $A^k(S_i) = A^k((S_i)_G)$. To prove the theorem, it
suffices to show that there is also an exact sequence of equivariant operational Chow groups

\[ 0 \to A^*_G(X) \xrightarrow{\pi_*} A^*_G(\tilde{X}) \xrightarrow{p_1^* - p_2^*} A^*(\tilde{X} \times_X \tilde{X}) \]

This can be checked by working with the action of \( A^*_G(X) \) on a fixed Chow group \( A_i(X_G) \) and arguing as in Kimura’s paper. □

**Corollary 1** If equivariant resolution of singularities holds (in particular if the characteristic is 0), and \( V - U \) has codimension more than \( k \), then \( A^k_G(X) = A^k(X_G) \).

Proof (c.f. [Ki, Remark 3.2]). We must show the existence of an equivariant envelope \( \pi : \tilde{X} \to X \). By equivariant resolution of singularities, there is a resolution \( \pi_1 : \tilde{X}_1 \to X \) such that \( \pi_1 \) is an isomorphism outside some invariant subscheme \( S \subset X \). By Noetherian induction, we may assume that we have constructed an equivariant envelope \( \tilde{S} \to S \). Now set \( \tilde{X} = \tilde{X}_1 \cup \tilde{S} \). □

### 2.6 Equivariant higher Chow groups

In this section assume that \( X \) is quasi-projective. Bloch ([B]) defined higher Chow groups \( \tilde{A}^i(X,m) \) as \( H_m(Z^i(X,\cdot)) \) where \( Z^i(X,\cdot) \) is a complex whose \( k \)-th term is the group of cycles of codimension \( i \) in \( X \times \Delta^k \) which intersect the faces properly. Since we prefer to think in terms of dimension rather than codimension we will define \( A^p(X,m) \) as \( H_m(Z_p(X,\cdot)) \), where \( Z_p(X,k) \) is the group of cycles of dimension \( p+k \) in \( X \times \Delta^k \) intersecting the faces properly. When \( X \) is equidimensional of dimension \( n \), then \( A^p(X,m) = A^{n-p}(X,m) \).

If \( Y \subset X \) is closed, there is a localization long exact sequence. The advantage of indexing by dimension rather than codimension is that the sequence exists without assuming that \( Y \) is equidimensional.

**Lemma 3** Let \( X \) be equidimensional, and let \( Y \subset X \) be closed, then there is a long exact sequence of higher Chow groups

\[ \ldots \to A_p(Y,k) \to A_p(X,k) \to A_p(X-Y,k) \to \ldots \to A_p(Y) \to A_p(X) \to A_p(X-Y) \to 0 \]

(there is no requirement that \( Y \) be equidimensional).
Proof. This is a simple consequence of the localization theorem of [3]. By induction it suffices to prove the lemma when $Y$ is the union of two irreducible components $Y_1, Y_2$. In particular, we will prove that the complexes $Z_p(X - (Y_1 \cup Y_2), \cdot)$ and $\frac{Z_p(X, \cdot)}{Z_p(Y_1 \cup Y_2, \cdot)}$ are quasi-isomorphic.

By the original localization theorem, $Z_p(X - (Y_1 \cup Y_2), \cdot) \simeq \frac{Z_p(X - Y_1, \cdot)}{Z_p(Y_2 - (Y_1 \cap Y_2), \cdot)}$ and $Z_p(X - Y_1, \cdot) \simeq \frac{Z_p(X, \cdot)}{Z_p(Y_2, \cdot)}$. By induction on dimension, we can assume that the lemma holds for schemes of smaller dimension, so $Z_p((Y_2 - (Y_1 \cap Y_2), \cdot) \simeq \frac{Z_p(Y_2, \cdot)}{Z_p(Y_1 \cap Y_2, \cdot)}$. Finally note that $\frac{Z_p(Y_2, \cdot)}{Z_p(Y_1 \cap Y_2, \cdot)} = \frac{Z_p(Y_2 - (Y_1 \cap Y_2), \cdot)}{Z_p(Y_1 \cap Y_2, \cdot)}$. Combining all our quasi-isomorphisms we have

$Z_p(X - (Y_1 \cup Y_2), \cdot) \simeq \frac{Z_p(X, \cdot)}{Z_p(Y_1 \cup Y_2, \cdot)} \simeq \frac{Z_p(X, \cdot)}{Z_p(Y_1 \cup Y_2, \cdot)}$ as desired. ⊘

If $X$ is quasi-projective with a $G$-action, we can define equivariant higher Chow groups $A^G_i(X, m)$ as $A^{l+2g}_i(X_G, m)$, where $X_G$ is formed from an $l$-dimensional representation $V$ such that $V - U$ has high codimension. The homotopy lemma for higher Chow groups shows that $A^G_i(X, m)$ is well defined.

Our reason for constructing equivariant higher Chow groups is to obtain a long exact sequence for a $G$-invariant subscheme $Y$ of a quasi-projective scheme $X$ with a $G$-action.

**Proposition 3** Let $X$ be an equidimensional $G$-scheme, and let $Y \subset X$ be an invariant subscheme. There is a long exact sequence of higher equivariant Chow groups

$$
\ldots \to A_p^G(Y, k) \to A_p^G(X, k) \to A_p^G(X - Y, k) \to \ldots \to A_p^G(Y) \to A_p^G(X) \to A_p^G(X - Y) \to 0.
$$

□

### 2.7 Cycle Maps

If $X$ is a complex algebraic variety with the action of a complex algebraic group, then we can define equivariant Borel-Moore homology $H^G_{BM,i}(X)$ as $H^G_{BM,i+2l-2g}(X_G)$ for $X_G = X \times^G U$. As for Chow groups, the definition is
independent of the representation, as long as $V - U$ has sufficiently high codimension, and we obtain a cycle map

$$cl : A^G_i(X) \to H^G_{BM,2i}(X)$$

compatible with the usual operations on equivariant Chow groups (\cite[cf. Chapter 19]{F}).

Note that $H^G_{BM,i}(X)$ is not the same as $H_i(X \times^G EG)$, where $EG \to BG$ is the topological classifying bundle. However, if $X$ is smooth, then $X_G$ is also smooth, and $H_{BM,i}(X_G)$ is dual to $H^{2n-i}(X_G) = H^{2n-i}(X \times^G EG) = H_G^{2n-i}(X)$, where $n$ is the complex dimension of $X$. In this case we can interpret the cycle map as giving a map

$$cl : A^G_i(X) \to H^{2i}_G(X)$$

If $X$ is compact, and the open sets $U \subset V$ can be chosen so that $U/G$ is projective, then Borel-Moore homology of $X_G$ coincides with ordinary homology, so $H^G_{BM,*}(X)$ can be calculated with a compact model. However In general, however, $U/G$ is never projective. If $G$ is a torus, then $U/G$ can be taken to be a product of projective spaces. If $G = GL_n$, then $U/G$ can be taken to be a Grassmannian

(see the example in Section 2.1)

If $G$ is semisimple, then $U/G$ cannot be chosen projective, for then the hyperplane class would be a nontorsion element in $A^1_G$, but by \cite{E-G} $A^*_G \otimes \mathbb{Q} \cong \mathbb{S}(\hat{T})^W \otimes \mathbb{Q}$, which has no elements of degree 1. Nevertheless for semisimple (or reductive) groups we can obtain a cycle map

$$cl : A^G_*(X) \otimes \mathbb{Q} \to H^T_{BM,*}(X; \mathbb{Q})^W$$

by identifying $A^*_G(X) \otimes \mathbb{Q}$ with $A^T_*(X)^W \otimes \mathbb{Q}$ and $H^G_{BM,*}(X; \mathbb{Q})$ with $H^T_{BM,*}(X; \mathbb{Q})^W$; if $X$ is compact then the last group can be calculated with a compact model.

### 3 Intersection theory on quotients

One of the uses of equivariant intersection theory is to study intersection theory on quotient stacks and their moduli. In particular, we show below that the rational Chow groups of moduli spaces which are group quotients of a smooth variety have an intersection product – even when there are infinitesimal automorphisms.
3.1 Chow groups of quotients

Let $G$ act on a scheme $X$, and assume that a geometric quotient $X \to X/G$ exists.

**Proposition 4** If $G$ acts freely, then $A^G_*(X, m) = A_*(X/G, m)$ (the isomorphism of higher Chow groups requires $X$ to be quasi-projective).

Proof. If the action is free, then $(V \times X)/G$ is a vector bundle over $X/G$. Thus $X_G$ is an open set in this bundle with arbitrarily high codimension, and the proposition follows from homotopy properties of (higher) Chow groups. $\square$

**Theorem 2** If $G$ acts properly on a quasi-projective variety $X$, so that $X/G$ is quasi-projective, then

1. $A^G_*(X, m) \otimes \mathbb{Q} = A_*(X/G, m) \otimes \mathbb{Q}$ for all $m \geq 0$.

2. There is an isomorphism of operational Chow rings
   
   $$p^* : A^*(X/G) \otimes \mathbb{Q} \to A^*_G(X) \otimes \mathbb{Q}.$$

**Remarks.** (1) If the action is proper, then the stabilizers are complete. Since $G$ is affine they must in fact be finite. We will sometimes mention that the stabilizers are finite for emphasis.

(2) When $X$ is the set of stable points for some linearized action of $G$ and $X/G$ is quasi-projective, then the action is proper.

(3) The condition that $X/G$ is quasi-projective is only required because the localization long exact sequence for higher Chow groups has only been proved for quasi-projective schemes.

In practice, many interesting varieties arise as quotients of a smooth variety by a connected algebraic group which acts with finite stabilizers. Examples include simplicial toric varieties and various moduli spaces such as curves, vector bundles, stable maps, etc. There is a long history of work on the problem of constructing an intersection product on the rational Chow groups.\footnote{In characteristic $p$, the definition of geometric quotient used here is slightly weaker than the one given in [GIT]. See the appendix.}
group of quotients of smooth varieties. In characteristic 0, Mumford (\cite{Mu}) proved the existence of an intersection product on the rational Chow groups of $\overline{\mathcal{M}}_g$, the moduli space of stable curves. Gillet (\cite{Gi}) and Vistoli (\cite{Vi}) constructed intersection products on quotients in arbitrary characteristic – provided that the stabilizers of geometric points are reduced. In characteristic 0, Gillet (\cite{Gi}, Thm 9.3) showed that his product on $\mathcal{M}_g$ agreed with Mumford’s, and in \cite{Ed}, Lemma 1.1] it was shown that Vistoli’s product also agreed with Mumford’s.

As a corollary to Theorem 2 we obtain a simple proof of the existence of intersection products on the rational Chow groups of quotients for a group acting with finite but possibly non-reduced stabilizers. Furthermore, when the stabilizers are reduced, our product agrees with Gillet’s and Vistoli’s (Proposition 11). In particular, this answers \cite{Vi}, Conjecture 6.6] affirmatively for moduli spaces of quotient stacks.

**Corollary 2** Let $Y$ be a quasi-projective variety which is isomorphic to a geometric quotient $X/G$, where $X$ is smooth and $G$ acts properly (hence with finite stabilizers) on $X$. Then the rational Chow groups $A_*(Y)_\mathbb{Q}$ have an intersection product. This product is independent of the presentation of $Y$ as a quotient.

Proof of Corollary 2. Since $X$ is smooth, the equivariant Chow groups $A^G_*(X)$ have an intersection product induced by the isomorphism $A^G_*(X) \to A^G_*(X)$ (with $\mathbb{Z}$ coefficients). By Theorem 2 $A^G_*(X)_\mathbb{Q} = A_*(Y)_\mathbb{Q}$ so the groups $A_*(Y)_\mathbb{Q}$ inherit a ring structure. Since the intersection product on $A_*(Y)$ is induced by the multiplication in $A^*(Y)$ it depends only on $Y$. □

Proof of part (1) of Theorem 2. For simplicity of exposition we give the proof assuming that the group $G$ is connected of dimension $g$. (This way we can assume that the set-theoretic inverse image in $X$ of a subvariety of $X/G$ is a single variety rather than a possible disjoint union of varieties.) All coefficients – including those of cycle groups – are assumed to be rational. If $G$ acts properly on $X$, then $G$ acts properly on $X \times \Delta^m$ by acting trivially on the second factor. In this case, the boundary map of the higher Chow group complex preserves invariant cycles, so there is a subcomplex of invariant cycles $Z_*(X, \cdot)^G$. Set

$$A_*( [X/G], m) = H_m (Z_* (X, \cdot)^G, \partial).$$
Now if $X \to X/G$ is a geometric quotient, then so is $X \times \Delta^m \to X \times \Delta^m$. Define a map $\pi^*: Z_k(X, m) \otimes \mathbb{Q} \to Z_{k+g}(X, m)^G \otimes \mathbb{Q}$ for all $m$ as follows. Let $F \subset X/G \times \Delta^m$ be a $k+m$-dimensional subvariety intersecting the faces properly, then $H = (\pi^{-1}F)_{\text{red}}$ is a $G$-invariant $(k+m+g)$-dimensional subvariety of $X \times \Delta^m$ which intersects the faces properly. Thus, $[H] \in Z_{k+g}(X, m)^G$. Let $e_H$ be the order of the stabilizer at a general point of $H$, and let $i_H$ be the degree of the purely inseparable extension $K(F) \subset K(H)^G$. Set $\pi^*[F] = \alpha_H[H] \in Z_{k+g}(X, m)$, where $\alpha_H = \frac{e_H}{i_H}$. Since $G$-invariant subvarieties of $X \times \Delta^m$ exactly correspond to subvarieties of $X/G \times \Delta^m$, $\pi^*$ is an isomorphism of cycles for all $m$.

**Proposition 5** Let

$$
\begin{array}{ccc}
Z & \xrightarrow{g} & X \\
p & \downarrow & \pi & \downarrow \\
Q & \xrightarrow{f} & Y \\
\end{array}
$$

be a commutative diagram of quotients with $f$ and $g$ proper. Then $p^* \circ g_* = f_* \circ \pi^*$ as maps $Z_*(Q) \to Z_*(Z, m)^G$.

Proof of Proposition 5. The proposition is an immediate consequence of the following lemma.

**Lemma 4** Suppose $G$ acts properly (hence with finite stabilizers) on varieties $Z$ and $X$. Let

$$
\begin{array}{ccc}
Z & \xrightarrow{g} & X \\
p & \downarrow & \pi & \downarrow \\
Q & \xrightarrow{f} & Y \\
\end{array}
$$

be a commutative diagram of geometric quotients with $f$ and $g$ finite and surjective. Then

$\frac{e_Z}{i_Z}[K(Q) : K(Y)] = \frac{e_X}{i_X}[K(Z) : K(Z)]$.

Proof. Since we are checking degrees, we may replace $Y$ and $Q$ by $K(Y)$ and $K(Q)$, and $X$ and $Z$ by their generic fibers over $Y$ and $Q$ respectively. Then
we have a commutative diagram of varieties.

\[
\begin{array}{ccc}
Z & \to & X \\
\downarrow & & \downarrow \\
\text{spec}(K(Z)^G) & \to & \text{spec}(K(X)^G) \\
\downarrow & & \downarrow \\
\text{spec}(K(Q)) & \to & \text{spec}(K(Y))
\end{array}
\]

Since \(i_Z := [K(Z)^G : K(Q)]\) and \(i_X := [K(X)^G : K(Y)]\), it suffices to prove

\[
e_Z[K(Z)^G : K(X)^G] = e_X[K(Z) : K(X)].
\]

By [Bo, Paragraph 6.5] the extensions \(K(Z)^G \subset K(Z)\) and \(K(X)^G \subset K(X)\) are separable (transcendental). Thus, after finite separable base extensions, we may assume that there are sections \(s : \text{spec}(K(X)^G) \to U\) and \(t : \text{spec}(K(Z)^G) \to W\). In this case the stabilizer group schemes over \(W\) and \(U\) are isomorphic to \(K(Z)^G \times G\) and \(K(X)^G \times G\) respectively ([GIT, Proof of Prop 0.7]). Thus, \(e_Z = [K(Z)^G \times K(G) : K(Z)]\) and \(e_X = [K(X)^G \times K(G) : K(X)]\). The lemma follows. \(\square\)

**Proposition 6** The map \(\pi^*\) commutes with the boundary operator of the higher Chow groups. In particular, there is an induced isomorphism of Chow groups

\[
A_k(X/G, m) \simeq A_{k+g}([X/G], m).
\]

Proof of Proposition 6. If

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & X \\
p \downarrow & & \pi \downarrow \\
Q & \xrightarrow{f} & X/G
\end{array}
\]

is a commutative diagram of quotients with \(f\) and \(g\) finite and surjective, then \(f_*\) and \(g_*\) are surjective as maps of cycles. Thus, by Proposition 6 it suffices to prove \(p^* : Z_*(Q) \to Z_*(X)^G\) commutes with \(\partial\).

By Proposition 23 of the appendix, there is a commutative diagram of quotients such that \(p : Z \to Q\) is a principal bundle. Since \(p\) is flat, \(p^*\) commutes with \(\partial\) and Proposition 6 follows. \(\square\)
Suppose $T \subset X$ is a $G$-invariant subvariety. Let $S \subset X/G$ be its image under the quotient map. Set $U = X - T$ and $V = X/G - U$. Then we have two commutative diagrams of geometric quotients:

$$
\begin{array}{ccc}
T & \xrightarrow{i} & X \\
\pi \downarrow & & \pi \downarrow \\
S & \xrightarrow{j} & X/G
\end{array}
\quad
\begin{array}{ccc}
U & \xrightarrow{j} & X \\
\pi \downarrow & & \pi \downarrow \\
V & \xrightarrow{j} & X/G
\end{array}
$$

Lemma 5 Let $\alpha \in Z_k(X/G, m)$ and $\beta \in Z_k(S, m)$.

1. $\pi^* j^* \alpha = j^* \pi^* \alpha$ in $Z^G_{k+g}(U, m)$.
2. $\pi^* i_s \beta = i_s \pi^* \beta$ in $Z^G_{k+g}(X, m)$.

Proof of Lemma 5. If $\alpha = [F]$ and $H = \pi^{-1}(F)_{\text{red}}$, then $\pi^* j^* \alpha$ and $j^* \pi^* \alpha$ are both multiples of $[H \cap U]$. Since $e_{[H \cap U]} = e_{[H]}$, and $i_{[H \cap U]} = i_{[H]}$, the multiplicities are the same. This proves (1).

Part (2) was proved in Proposition 3. □

As a consequence of Proposition 3 and Lemma 5, we obtain the following proposition.

Proposition 7 Let $T \subset X$ be an invariant subvariety. If $S$, $U$, and $V$ are as above, then there is a commutative diagram of isomorphisms

$$
\begin{array}{c}
\ldots \to A_*(T/G, m) \to A_*([X/G], m) \to A_*([U/G], m) \to \ldots \\
\cong \uparrow & \cong \uparrow & \cong \uparrow \\
\ldots \to A_*(S, m) \to A_*(X/G, m) \to A_*(V, m) \to \ldots
\end{array}
$$

□

Next, note that there is a map

$$
\alpha : A_*([X/G], m) \to A^G_*(X, m)
$$

defined by the formula

$$
[F] \in Z^G_*(X, m) \mapsto [F]_G
$$

which commutes with equivariant proper pushforward and equivariant flat pullback.

Proposition 8 If $G$ acts properly on $X$, and a quasi-projective geometric quotient $X \to X/G$ exists, then $\alpha$ is an isomorphism.
Proof of Proposition \[\text{III}.\] By the Nullstellensatz there is a point of \(X\) which is finite over the generic point of \(X/G\). Thus, by generic flatness, there is a locally closed subvariety \(Z \subset X\), and an open set \(W \subset X/G\) such that the projection \(Z \to W\) is finite and flat. Let \(U = \pi^{-1}(W)\). Since \(G\) acts properly, the map \(G \times Z \to U\) is finite. Shrinking \(Z\) (and thus \(U\)) we may assume that \(G \times Z \to U\) is also flat. By Noetherian induction and the localization long exact sequence (which exists by Proposition \[\text{III}.\]) it suffices to prove that \(\alpha : A^*_G([U/G], m) \to A^*_G(U, m)\) is an isomorphism.

Taking Chow groups, we obtain a commutative diagram where all maps commute.

\[
\begin{array}{ccc}
A^*_G(G \times Z, m) & \xrightarrow{\alpha} & A^*_G(U, m) \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
A_*(G \times Z/G, m) & \xrightarrow{\alpha} & A_*([U/G])
\end{array}
\]

(The right horizontal arrows are proper pushforward divided by the degree, and the left horizontal arrows are flat pullback.) Chasing the diagram shows that if the left vertical arrow is an isomorphism then so is the right vertical arrow. Since \(G\) acts freely, \(\alpha : A_*(G \times Z/G, m) \to A_*(G \times Z, m)^G\) is an isomorphism by Proposition \[\text{III}.\] \(\Box\).

We have now proved part (1) of Theorem \[\text{II}.\]

Proof of part (2) of Theorem \[\text{II}.\] (cf. [Vi, Proposition 6.1]). Suppose \(c \in A^*(X/G)_\mathbb{Q}\), \(Z \to X\) is a \(G\)-equivariant morphism, and \(\alpha \in A^*_G(Z)\). For any representation \(V\), there are maps \(Z_G \to X_G \to X/G\). If \(V\) is chosen so that \(\alpha\) is represented by a class \(\alpha_V \in A_*(Z_G)\) we can define

\(p^* c \cap \alpha = c \cap \alpha_V\).

As usual, this definition is independent of the representation, so \(p^* c \cap \alpha \in A^*_G(Z)\).

(1) \(p^*\) is injective.

Proof of (1). Suppose \(p^* \cap \alpha = 0\) for all \(G\)-maps \(Z \to X\) and all \(\alpha \in A^*_G(Z)\). By base change, it suffices to show \(c \cap x = 0\) for all \(x \in A_*(X/G)_\mathbb{Q}\).

By Proposition \[\text{III}.\] there is a finite map \(Y \to X/G\), and a principal bundle \(Z \to Y\) together with a finite \(G\)-map \(Z \to X\). Thus we obtain a commutative diagram

\[
\begin{array}{ccc}
Z_G & \xrightarrow{g} & X_G \\
q \downarrow & & \downarrow p \\
Y & \xrightarrow{f} & X/G
\end{array}
\]
where the horizontal maps are proper and surjective. Choose \( y \in A_*(Y) \) so that \( f_*(y) = x \). Since \( q \) is flat

\[ 0 = c \cap q^*y = q^*(c \cap y). \]

Since \( q^* \) is an isomorphism in the appropriate degrees, \( c \cap y = 0 \). Thus

\[ 0 = f_*(c \cap y) = c \cap f_*y = c \cap x \]

as desired.

(2) \( p^* \) is surjective.

Proof of (2). Suppose \( d \in A^*_G(X) \). Define \( c \in A^*(X/G) \) as follows: If \( Y \to X/G \) and \( y \in A_*(Y) \), set \( c \cap y = d \cap \pi^*y \) where \( \pi : X \times_{X/G} Y \to Y \) is the quotient map, and \( \pi^*y \in A_*(X \times_{X/G} Y/G) \). Let \( \widetilde{B}_G \to B \) be the principal \( G \)-bundle \( B \times_{[X/G]} X \), (The fiber product is a scheme, although the product is taken over the quotient stack \([X/G]\)). Typically, \( \widetilde{B}_G \) is the structure bundle of a projective bundle \( \mathbb{P}(p_*L) \) for a relatively very ample line bundle \( L \) on \( Y \).

Since \( X \) is smooth, there is an equivariant pullback \( f^*: A^*_G(X) \to A^*_G(B_G) \) of the induced map \( B_G \to B \). Given a family of curves \( Y \to B \) there is a map

3.2 Intersection products on moduli

Equivariant intersection theory gives a nice way of working with cycles on a singular moduli space \( M \) which is a quotient \( X/G \) of a smooth variety by a group acting with finite stabilizers. Given a subvariety \( W \subset M \) and a family \( Y \to B \) of schemes parametrized by \( M \) there is a map \( B \to M \). We wish to define a class \( f^*([W]) \in A_*B \) corresponding to how the image of \( B \) intersects \( W \). This can be done (after tensoring with \( \mathbb{Q} \)) using equivariant theory.

By Theorem 2, there is an isomorphism \( A_*(M) \mathbb{Q} = A^*_G(X) \) which takes \([W]\) to the equivariant class \( w = \int_w [f^1W]_G \). Let \( B_G \to B \) be the principal \( G \)-bundle of a projective bundle \( \mathbb{P}(p_*L) \) for a relatively very ample line bundle \( L \) on \( Y \). Since \( X \) is smooth, there is an equivariant pullback \( f^*: A^*_G(X) \to A^*_G(B_G) \) of the induced map \( B_G \to B \). Identifying \( A^*_G(B) \) with \( A_*B \) we obtain our class \( f^*([W]) \).

Example (Moduli of stable curves) Let \( \mathcal{M}_g \) be the moduli space of curves. Let \( \Delta \subset \mathcal{M}_g \) be the Weil divisor corresponding to (stable) nodal curves which are formed by identifying a curve of genus \( i \) to a curve of genus \( g - i \) at a point. Given a family of curves \( Y \to B \) there is a map
We wish to define a cycle $\delta_B$ corresponding to the intersection of the image of $B$ with $\Delta_i$. Such a class can be defined using Vistoli’s intersection theory on the Deligne-Mumford stack $\overline{\mathcal{M}}_g$, since there is a Gysin pullback $A_*(B) \otimes \mathbb{Q} \overset{i^*}{\to} A_* B \otimes \mathbb{Q}$ corresponding to the inclusion $\delta_i \hookrightarrow \overline{\mathcal{M}}_g$. Let $\delta_i$ be the inverse image of $\Delta_i$ in $\overline{\mathcal{M}}_g$. If the image of $B$ is completely contained in $\Delta_i$, then, to calculate $\delta_B$ we need to use an excess intersection formula

$$\delta_B = i^*(\delta_i) = c_1(O_{\delta_i}((\delta_i)) \cap [B].$$

Similarly, if $\Theta \subset \overline{\mathcal{M}}_g$ is a subvariety of codimension $d$ corresponding to a smooth substack $\theta \subset \overline{\mathcal{M}}_g$, then we would like to assert that

$$\theta_B = c_d(N_{\theta} \overline{\mathcal{M}}_g) \cap [B]$$

where $N_{\theta} \overline{\mathcal{M}}_g$ is the normal bundle to $\theta$ in $\overline{\mathcal{M}}_g$. Unfortunately, such formulas were not fully developed in [Vi], so their use can not be completely justified. Consider the equivariant situation. The moduli space $\overline{\mathcal{M}}_g$ is a geometric quotient $H_g/G$ where $H_g$ is the (smooth) Hilbert scheme of pluricanonically embedded stable curves and $G = PGL(N)$ for some $N$. Given an abstract family $Y \to B$ let $Y_H \to B_H$ be the corresponding family of pluricanonically embedded curves. Let $\Delta_H \overset{i_H}{\hookrightarrow} H_g$ be the corresponding $G$ invariant divisor in $A^*_H(H_g)$. Since $H_g$ is smooth, we obtain an equivariant line bundle $O(\Delta_H)$. Let $B_H \to H_g$ be the corresponding equivariant map. Then by equivariant excess intersection (which follows from ordinary excess intersection on schemes $(H_g \times U)/G)$

$$i_H^*(\Delta_H) = c_1(O_{\Delta_H}(\Delta_H)) \cap [B].$$

The corresponding formula in $A_* B \otimes \mathbb{Q}$ follows from the identification of $A_*(B)$ with $A^*_H(B_H)$.

Similar formulas (involving Chern classes of the equivariant normal bundle) hold for cycles of higher codimension. In [Ed, Section 3] explicit excess intersection formulas were given in codimension 1 and 2, for various nodal loci. The approach there is similar to the discussion above, although equivariant Chow groups were not used. Instead, a result of Vistoli (proved in characteristic 0, although true in arbitrary characteristic) was used to identify $A^1(B)$ (codimension-one cycles) with $A^1(B_H)$ because $B_H \to B$ is a principal $PGL(N)$ bundle. The statements of [Ed, Section 3] can proved in in arbitrary characteristic using the methods outlined above.
3.3 Chow groups of quotient stacks

If $G$ acts on $X$ we let $[X/G]$ denote the quotient stack. This is a stack in the sense of Artin, and exists without any assumptions on the $G$-action. By the next proposition, the equivariant Chow groups do not depend on the presentation as a quotient, so they are an invariant of the stack.

**Proposition 9** Suppose that $[X/G] \simeq [Y/H]$ as quotient stacks. Then $A^i_G(X) \simeq A^i_H(Y)$ for all $i$.

Proof: Suppose $\dim G = g$ and $\dim H = h$. Let $V_1$ be an $l$-dimensional representation of $G$, and $V_2$ an $M$ dimensional representation of $H$. Let $X_G = X \times^G U_1$ and $Y_H = X \times^H U_2$, where $U_1$ (resp. $U_2$) is an open set on which $G$ (resp. $H$) acts freely. Since the diagonal of a quotient stack is representable, the fiber product $Z = X_G \times [X/G] Y_H$ is a scheme. This scheme is a bundle over $X_G$ and $Y_H$ with fiber $U_2$ and $U_1$ respectively. Thus, $A_{i+l-g}(X_G) = A_{i+l+m-h}(Z) = A_{i+m-h}(Y_H)$ and the proposition follows. \(\square\)

**Remark.** Proposition 9 suggests that there should be a notion of Chow groups of an arbitrary algebraic stack which can have non-zero torsion in arbitrarily high degree. This situation would be analogous to the cohomology of quasi-coherent sheaves on the étale (or flat) site (cf. [D-M, p. 101]).

If $G$ acts properly with finite, reduced stabilizers, then $[X/G]$ is a separated Deligne-Mumford stack. The rational Chow groups $A_*([X/G]) \otimes \mathbb{Q}$ were first defined by Gillet and coincide with the groups $A_*([X/G]) \otimes \mathbb{Q}$ defined above. More generally, if $G$ acts with finite stabilizers which are not reduced, then then Gillet’s definition can be extended and we can define the “naive” Chow groups $A_k([X/G])_{\mathbb{Q}}$ as the group generated by $k$-dimensional integral substacks modulo rational equivalences. With this definition we expect that $A^G_*([X/G])_{\mathbb{Q}} = A_*([X/G])_{\mathbb{Q}}$. To prove such an isomorphism in general requires that the naive Chow groups of the stack satisfy the homotopy property (i.e. if $F \to G$ is a vector bundle in the category of stacks, then $A_*(F)_{\mathbb{Q}} = A_*(G)_{\mathbb{Q}}$). However, if a quasi-projective quotient exists, then Proposition 9 can be restated in the language of stacks as

**Proposition 10** Let $G$ be a $g$-dimensional group which acts properly on a scheme $X$ (so the quotient $[X/G]$ is a separated Artin stack). Assume that a quasi-projective moduli scheme $X/G$ exists for $[X/G]$. Then $A^G_*([X/G]) \otimes \mathbb{Q} = A_{i-g}([X/G]) \otimes \mathbb{Q}$. \(\square\)
Remarks. (1) Although $A^\ast_G(X) \otimes \mathbb{Q} = A_\ast([X/G]) \otimes \mathbb{Q}$, the integral Chow groups may have non-zero torsion for all $i < \dim X$. It would be interesting to compute this torsion in examples such as moduli spaces of curves of low genus.

(2) In general, a separated quotient stack should always have an algebraic space as coarse moduli space. Thus, an extension of the present theory to algebraic spaces would eliminate the need for any assumptions in the proposition.

With the identification of $A^\ast_G(X) \otimes \mathbb{Q}$ and $A_\ast([X/G]) \otimes \mathbb{Q}$ there are three intersection products on the rational Chow groups of a smooth Deligne-Mumford quotient stack with a moduli space – the equivariant product, Vistoli’s product defined using a via a gysin pullback for regular embeddings of stacks, and Gillet’s product defined using the product in higher $K$-theory. The next proposition shows that they are identical.

**Proposition 11** If $X$ is smooth and $[X/G]$ is a separated Deligne-Mumford stack (so $G$ acts with finite, reduced stabilizers) with a quasi-projective moduli space $X/G$, then the intersection products on $A_\ast([X/G]) \otimes \mathbb{Q}$ defined by Vistoli and Gillet are the same as the equivariant product on $A^\ast_G(X) \otimes \mathbb{Q}$.

Proof. If $V$ is an $l$-dimensional representation, then all three products agree on the smooth quotient scheme $(\mathbb{V}, \mathbb{G}) (X \times U)/G$. Since the flat pullback of stacks $f : A^\ast([X/G]) \otimes \mathbb{Q} \to A^\ast((X \times U)/G) \otimes \mathbb{Q}$ commutes with all 3 products, and is an isomorphism to arbitrarily high codimension, the proposition follows. □

4 **Equivariant Riemann-Roch**

In this section we construct an equivariant Todd class map and prove an equivariant Riemann-Roch theorem for $G$-schemes. The theorem involves completions of equivariant $K$-groups and Chow groups because the groups $A^\ast_G(X)$ (resp. $A_\ast^G(X)$) can have terms of arbitrarily large negative (resp. positive) degree. Thus, the Todd class and Chern character map to completions of these groups. The map $\tau_X^G : K^G_0(X) \to A^G_\ast(X) \otimes \mathbb{Q}$ factors through a completion of $K^G_0(X)$ and we obtain an isomorphism $\tau_X^G : K^G_0(X) \to A^G_\ast(X) \otimes \mathbb{Q}$.
This section has two parts. In the first, we define $\hat{A}_G(X,i)$ and $\hat{K}'_G(X)$ as completions of $A^*_G(X,i)$ and $K_i^G(X)$ along certain ideals. We then prove an analogue of a theorem of Atiyah and Segal which gives a more geometric description of these completions. In the second part we construct the Todd class map $\tau^n_G : K'_0^G(X) \to A^*_G(X) \mathbb{Q}$ and show that it induces an isomorphism $\tau^n_G : K'_i^G(X) \cong A^*_G(X) \mathbb{Q}$. The construction is an easy consequence of the nonequivariant Riemann-Roch theorem and our geometric description of the completions. Finally, we discuss a conjecture of Vistoli.

4.1 Completions of equivariant K-groups and Chow groups

Let $R(G)$ denote the representation ring of $G$. Let $K^G_0(X)$ denote the Grothendieck group of $G$-equivariant vector bundles on $X$, and let $K^G_i(X)$ denote the $i$-th higher $K$-group of the category of $G$-equivariant coherent sheaves ([Th]). As in the non-equivariant case, $K^G_0(X)$ is a ring under tensor product, and $K^G_i(X)$ is a module for that ring. Also, $K^G_0(X)$ and $K^G_i(X)$ are $R(G)$ modules via the isomorphism $R(G) \cong K^G_0(pt) = K^G_0$.

Let $P \subset K_0^G = R(G)$ denote the ideal of virtual representations of dimension 0, and let $K_i^G(X)$ be the completion of $K_i^G(X)$ along $P$. Let $Q = A^G_+ \subset A^*_G$ be the augmentation ideal, and let $A^*_G(X,i)$ be the completion of $A^*_G(X,i)$ along $Q$.

Let $\tilde{Q} = A^*_G(X) \subset A^*_G$ denote the augmentation ideal, then $QA^*_G(X) \subset \tilde{Q}$. Let $\tilde{A}^*_G(X)$ denote the completion of $A^*_G(X)$ along $\tilde{Q}$. Likewise, let $\tilde{P} \subset K_0^G(X)$ denote the ideal of virtual bundles of rank 0 (the kernel of the rank map), then $PK_0^G(X) \subset \tilde{P}$. Let $K_i^{\tilde{G}}(X)$ denote the completion of $K_i^G(X)$ along $\tilde{P}$.

We will show below that there are isomorphisms

$$K_i^{\tilde{G}}(X) \cong K_i^G(X)$$

$$A^*_G(X,i) \cong A^*_G(X,i).$$

To show this, we will compare these completions with more geometrically defined ones.

Partially order the set $\mathcal{V}$ of representations of $G$ by the rule $W < V$ if $W$ is a summand in $V$. For each representation, let $V^f$ be the open set of
points whose orbits are closed in $V$ and which have trivial stabilizer. The collections \( \{ K_i^G(X \times V) \} \) and \( \{ A_i^G((X \times V),i) \} \) are inverse systems since the inclusion $V^f \oplus W \hookrightarrow (V \oplus W)^f$ induces restriction maps

\[
K_i^G(X \times (V \oplus W)^f) \rightarrow K_i^G(X \times V^f \oplus W) \simeq K_i^G(X \times V^f)
\]

\[
A_i^G((X \times (V \oplus W)^f),i) \rightarrow A_i^G((X \times V^f \oplus W),i) \simeq A_i^G((X \times V^f),i).
\]

By identifying $K_i^G(X \times V)$ with $K_i^G(X)$ and $A_i^G((X \times V),i)$ with $A_i^G(X,i)$ we obtain restriction maps

\[
r_V' : K_i^G(X) \rightarrow K_i^G(X \times V^f)
\]

\[
r_V : A_i^G(X,i) \rightarrow A_i^G((X \times V^f),i)
\]

and thus inverse systems \( \{ r_V(K_i^G) \} \) and \( \{ r_V(A_i^G(X,i)) \} \).

**Theorem 3** There are isomorphisms of completions

\[
\lim_{\leftarrow V} r_V(K_i^G(X)) \simeq K_i^G(X)
\]

and if $X$ is quasi-projective we also have

\[
\lim_{\leftarrow V} r_V(K_i^G(X)) \simeq \hat{K}_i^G(X)
\]

\[
\lim_{\leftarrow V} r_V(A_i^G(X,i)) \simeq \hat{A}_i^G(X,i) \simeq \tilde{A}_i^G(X,i).
\]

**Remark:** A similar equality of completions was proved (for $K_0^G$) in [CEPT] for actions of finite groups of projective varieties defined over rings of integers of number fields.

As a result of this identification of completions, we can prove a particular case of a conjecture of Köck ([Kö]) for arbitrary reductive groups acting on regular schemes of finite type over a field. Set $K(X,G) = \oplus K_i^G(X) = \oplus K_i^G(X)$, and let $\tilde{K}(X,G)$ be the completion along the augmentation ideal $\tilde{P}$ of $K_0^G(X)$.

**Corollary 3** Let $X \xrightarrow{f} Y$ be a proper equivariant morphism of quasi-projective, regular schemes. Then $f_* : K(X,G) \rightarrow K(Y,G)$ is continuous with respect to the $\tilde{P}$-adic topologies.
Proof. The pushforward $f_*$ induces a map of inverse systems

$$\lim_{\leftarrow V} r'_V(K'^G_i(X)) \to \lim_{\leftarrow V} R'_V(K'^G_i(Y))$$

The corollary follows from the identification of completions in Theorem 3. □

Proof of Theorem 3. The two statements have essentially identical proofs, so we will only prove the isomorphisms in $K$-theory. Furthermore, the proof that

$$\lim_{\leftarrow V} r'_V(K'^G_i(X)) \simeq \widetilde{K}'^G_i(X)$$

is virtually identical to the proof that

$$\lim_{\leftarrow V} r'_V(K'^G_i(X)) \simeq \widehat{K}'^G_i(X)$$

so we will only prove the latter. (The only difference in the proof of the former is that we need to assume $X$ is quasi-projective so we can compare the $\gamma$ filtration and the topological filtration on $K^G_0(X).$) This statement is the analogue of [A-S, Theorem 2.1], except that we do not need the hypothesis that $\widehat{K}'^G_i(X)$ is finite over $R(G).$ As in [A-S] we first prove the result for a torus and from this deduce the general case. Our proof of the torus case is somewhat different from that of [A-S], but the passage to the general case uses their arguments, which we have repeated for completeness.

Step 1. We first prove the result if $G = T$ is a torus. We have filtrations of $K'^G_i(X)$ by the ideals $\ker r'_V$ and by powers of the ideal $P = P_T.$ It suffices to show that the filtrations have bounded difference. This is a consequence of the next two lemmas.

**Lemma 6** Let $V$ be a representation of $T$ and $W \subset V$ a subrepresentation of codimension $l.$ Let $i : X \times W \to X \times V$ be the inclusion. Then $i_*(K'^T_i(X \times W)) \in PK'^T_i(X \times V).$

Proof of Lemma 6. We can find a chain of $T$-invariant subspaces $W = W_l \subset W_{l-1} \subset \ldots \subset W_0 = V$ where the codimension of $W_j$ in $V$ is $j.$ By induction on codimension, it suffices to consider the case where the codimension of $W$ is 1. By the projection formula for equivariant $K$-theory, $i_*(K'^T_i(X \times W)) = i_*(\mathcal{O}_{X \times W})K'^T_i(X).$ However,

$$i_*(\mathcal{O}_{X \times W}) = [\mathcal{O}_{X \times V} - [(V/W) \otimes_k \mathcal{O}_{X \times V}]],$$

which is in $PK'^T_i(X \times V).$ □
Lemma 7
\[ P^sK_0^T \subset \ker r'_V \subset P^lK_0^T(X) \]
for any \( s > d + \dim X - \dim T \).

Proof of Lemma 7. If \( V \) is a representation of a torus then \( V^u = V - V^f \) is a finite union of linear subspaces (Appendix, Proposition [21]) which by assumption have codimension at least \( l \) in \( V \). From the localization long exact sequence

\[ \ldots \to K_i^T(X \times V^u) \xrightarrow{i_*} K_i^T(X \times V) \xrightarrow{r'_V} K_i^T(X \times V^f) \to \ldots \]

we know that \( \ker r'_V = i_*(K_i^T(X \times V^u)) \). The image of \( K_i^T(X \times V^u) \) is generated by the images of \( K_i^T(X \times W) \) for each linear space \( W \subset V^u \). By Lemma [2], these images are contained in \( P^lK_i^T(X \times V) = K_i^T(X) \). Hence \( \ker r'_V \subset P^lK_i^T(X) \).

For the other inclusion, note that \( K_i^T(X \times V^f) = K_i^T(X_T) \), and \( P^sK_0^T(X \times V^f) \subset F^sK_i^T(X_T) \), where \( F \) denotes the \( \gamma \)-filtration on \( R(G) \). Since a point is projective, \( F^sK_i^T(X_T) \subset F_{\text{top}}^sK_i^T(X_T) \). Thus, if \( s > \dim X_T \), then \( F^sK_i^T(X_T) = 0 \). Hence \( P^sK_i^T(X_T) \subset K_i^T(X) \), as desired. \( \square \)

This lemma implies the desired equality of completions for the case of a torus.

Step 2. We prove the result for \( G = GL_n \). Let \( j : T \hookrightarrow G \) be the inclusion and let \( j^* : K_i^G(X) \to K_i^T(X) \) be the induced restriction map.

Lemma 8 There is a functorial map \( j_* : K_i^T(X) \to K_i^G(X) \) such that \( j_* j^* \) is the identity. Hence \( K_i^G(X) \) is a direct summand in \( K_i^T(X) \).

Proof of Lemma 8. This is proved in [Alt, Prop. 4.9] for topological K-theory. The same proof works in the algebraic setting: the main ingredient is the projective bundle theorem, which was proved in this setting by Thomason [Th2, Theorem 3.1]. \( \square \)

From the proof of Step 1, in computing \( \lim_{\leftarrow V} K_i^T(X \times V^f) \) we need not consider all representations of \( T \): it suffices to consider the subsystem of representations of \( T \) which are restrictions of representations of \( G \), then \( \ker r^G_V = \ker r^T_V \cap K_i^G(X) \).

The submodule \( K_i^G(X) \) of \( K_i^T(X) \) inherits two topologies from \( K_i^T(X) \): the topology induced by the ideals \( \ker r^T_V \cap K_i^G(X) = \ker r^G_V \), and the
topology induced by powers of the ideal $P_T$. Because $K_i'^G(X)$ is a direct summand in $K_i'^T(X)$, by Lemma 7 these topologies coincide. On the other hand, as noted in [A-S], the ideals $P_T$ and $P_G R(T)$ have bounded difference, so they induce the same topology on $K_i'^T(X)$. The restriction of this topology to $K_i'^G(X)$ is the topology induced by powers of the ideal $P_G$. Putting these facts together, we conclude that $\lim_{\leftarrow} V K_i'^G(X \times V') \simeq \hat{K}_i'^G(X)$ for $G = GL_n$, as desired.

**Step 3.** We now deduce the result for general $G$. Embed $G$ into $H = GL_n$. Then $K_i'^G(X) = K_i'^H(X \times^G H)$ (Proposition 6.2).

As above, we may restrict our attention to representations of $G$ which are restrictions of representations of $H$, then

$$\lim_{\leftarrow} V r'_V(K_i'^G(X)) \simeq \lim_{\leftarrow} V r'_V(K_i'^H((X \times^G H))).$$

As noted in [A-S], the $P_H$-adic and $P_G$-adic topologies coincide on any $R(G)$-module, and hence, by the result for $H = GL_n$, we have

$$\lim_{\leftarrow} V r'_V(K_i'^G(X)) \simeq K_i'^G(X),$$

as desired. □

If $X$ is any $G$-scheme, let $K_i'^G(X)Q = K_i'^G(X) \otimes Q$, and $A^G_i(X, i)Q = A^G_i(X, i) \otimes Q$.

**Corollary 4** There are isomorphisms of completions

$$\lim_{\leftarrow} V \{r'_V(K_i'^G(X))Q\} \simeq K_i'^G(X)Q \simeq \hat{K}_i'^G(X)Q$$

and if $X$ is quasi-projective

$$\lim_{\leftarrow} V \{r_V(A^G_i(X, i))\} \simeq \hat{A}^G_i(X, i)Q.$$

**Proof:** The proof is the same as above, except we do not need $X$ to be quasi-projective to show that $\hat{P}^s K_i'^G(X)Q \subseteq \ker r'_V$ for $s >> 0$. Instead, we can apply the non-equivariant Riemann-Roch isomorphism of [Fu, Chapter 18]

□

\footnote{Note that $X \times^G H$ is a scheme, because of our hypothesis on $X$ or $G$.}
Example. Since inverse limits do not commute with tensoring with $\mathbb{Q}$, $\lim_{\leftarrow V} \{ r'_V(K'_G(X))_\mathbb{Q} \}$ need not be equal to $(\lim_{\leftarrow V} r'_V(K'_G(X))) \otimes \mathbb{Q}$. For example if $G = \mathbb{Z}/2\mathbb{Z}$ and $X = pt$ then

$$K'_0(X) = K'_0(X) = R(G) = \mathbb{Z}[u]/(u^2 - 1).$$

In this case $\lim_{\leftarrow V} r'_V(K'_0(X)) \simeq \mathbb{Z}_2$ (the 2-adic integers). Thus, $(\lim_{\leftarrow V} r'_V(K'_0(X)) \otimes \mathbb{Q}) = \mathbb{Q}_2$. On the other hand, $\lim_{\leftarrow V}(r'_V(K'_0(X))_\mathbb{Q}) = \mathbb{Q}$.

However, there is a map

$$\lim_{\leftarrow V} r'_V(K'_i(X)) \rightarrow \lim_{\leftarrow V} \{ r'_V(K'_i(X))_\mathbb{Q} \}$$

which induces a map

$$(\lim_{\leftarrow V} r'_V(K'_i(X))) \otimes \mathbb{Q} \rightarrow \lim_{\leftarrow V} \{ r'_V(K'_i(X))_\mathbb{Q} \}.$$  

Likewise there is a map

$$(K'_i(X)) \otimes \mathbb{Q} \rightarrow K'_i(X)_\mathbb{Q}$$

which need not be an isomorphism.

4.2 The equivariant Riemann-Roch isomorphism

Before stating the Riemann-Roch theorem we need to define the equivariant Chern character. For this we need a suitable completion of $A^*_G(X)$. Elements of this completion should operate on $\hat{A}^*_G(X)$. There are three candidates for this completion:

1. $\lim_{\leftarrow V} A^*_G(X \times V^f)$;
2. $\hat{A}^*_G(X)$, the completion with respect to $\hat{Q}$;
3. $\hat{A}^*_G(X)$, the completion with respect to $Q$.

If $X$ is smooth, then all three completions are equal. In general, we do not know whether they are equal because of the lack of a suitable exact sequence for operational Chow groups. However, all of these operate on $A^*_G(X)$ by virtue of the isomorphisms proved in the last subsection. There are maps

$$A^*_G(X) \rightarrow A^*_G(X) \rightarrow \lim_{\leftarrow V} A^*_G(X \times V^f).$$
The first map is because $Q \subset \tilde{Q}$. The second is because the map $A^*_G(X) \to A^*_G(X \times V^f)$ induces a map $A^*_G(X) \to A^*_G(X \times V^f)$ (because high powers of $\tilde{Q}$ map to zero in $A^*_G(X \times V^f)$). These maps are compatible with restrictions and hence induce a map to the inverse limit. We will define a Chern character map with image in $A^*_G(X)$.

**Definition 3** Define the equivariant Chern character

$$ch_G : K^*_G(X) \to A^*_G(X)_{\tilde{Q}}$$

by the formula

$$ch_G(E) = r + c^G_1(E) + \frac{1}{2}(c^G_1(E)^2 - 2c^G_2(E)) + \ldots.$$ 

Let $V$ be a representation of $G$ such that $V - V^f$ has codimension more than $i$. If $E \to X$ is an equivariant vector bundle, then $c^G_i(E)$ restricts to $c_i(E \times^G V^f)$ under the restriction map $A^*_G(X) \to A^*_G(X \times^G V^f)$. Thus, the equivariant Chern character $ch_G : K^*_G(X) \to A^*_G(X)$ restricts to the ordinary Chern character $ch_{X \times^G V^f} : K(X \times^G V^f) \to A^*(X \times^G V^f)$.

**Proposition 12** There is a factorization

$$ch_G : K^*_G(X) \to \hat{K}^*_G(X) \to \tilde{K}^*_G(X) \tilde{A}^*_G(X)_{\tilde{Q}}.$$ 

Proof. The proof follows from the fact that $ch(P^n)$ and $ch(\tilde{P}^n)$ are contained in $\tilde{Q}^n$ for any $n > 0$. □

We will also denote the map $\tilde{K}^*_G(X) \to A^*_G(X)_{\tilde{Q}}$ by $ch_G$.

**Theorem 4** (Equivariant Riemann-Roch)

There are maps

$$\tau^G_X : K^*_G(X) \to A^*_G(X)_{\tilde{Q}}$$

with the following properties (cf. [F1, Chapter 18]):

1. $\tau^G_X$ is covariant for equivariant proper morphisms.
2. If $\epsilon \in K^*_G(X)$ and $\alpha \in \tilde{K}^*_G(X)$ then $\tau^G_X(\epsilon \alpha) = ch^G_X(\epsilon) \cap \tau^G_X(\alpha)$. (Recall that $A^*_G(X)$ operates on $A^*_G(X)$ because of the isomorphisms of completions.)
If $f : X \to Y$ is a $G$-equivariant l.c.i. morphism, then
\[ \text{ch}^G f_*(\epsilon) = f_*((Td^G(T_f)\text{ch}^G(\epsilon)) \]
and
\[ \tau^G_X(f^*\alpha) = Td^G(T_f)f^*\tau^G_X(\alpha). \]

If $V \subset X$ is a $G$-invariant subvariety of dimension $k$, then
\[ \tau^G_X(O_V - [V]_G) \in F_{k-1}(\hat{A}_G^*(X)_Q) \]
where $F_{k-1}(\hat{A}_G^*(X)_Q)$ denotes the subgroup of cycles of “dimension” strictly less than $k$.

(5) $\tau^G_X$ factors through the map $K_0^G(X) \to K_0^G(X)_Q$ and induces an isomorphism between $K_0^G(X)_Q$ and $A^*_G(X)_Q$.

(6) If $X$ is quasi-projective, and $i > 0$, then there is an isomorphism
\[ \tau^G_X : \lim_{\leftarrow V} K_i^G(X \times V^f)_Q \cong \lim_{\leftarrow V} A^*_G((X \times V^f), i)_Q. \]

Proof. When $i = 0$, the restriction maps $r_V$ and $r'_V$ are surjective. Thus by Theorem 3
\[ K_0^G(X) = \lim_{\leftarrow V} K_0^G(X \times V^f) \]
and
\[ A^*_G(X)_Q = \lim_{\leftarrow V} (A^*_G(X \times V^f) \otimes Q). \]
By non-equivariant Riemann-Roch ([Fu, Chapter 18]), for each representation $V$ of $G$ there is a map
\[ \tau_{X \times V^f} : K_0^G(X \times V^f) \to A^*_G(X \times V^f) \otimes Q \]
satisfying the analogues of (1) - (5). To prove the theorem it suffices to show that maps $\{\tau_{X \times V^f}\}$ are compatible with the inverse system maps. This is quite straightforward. Let $V$ and $W$ be representations. Let $\pi : (X \times V^f \oplus W)/G \to (X \times V^f)$. Then $\pi$ is smooth and $Td_{T_\pi} = 1$. Thus $\pi^* \cdot (\tau_{X \times V^f}) = \tau_{X \times V^f \oplus W} \cdot \cdot \cdot$. Likewise, if $i : (X \times V^f \oplus W) \to (X \times (V \oplus W)^f)$ is the inclusion map, then $i^* \cdot \tau_{X \times (V \oplus W)^f} = \tau_{X \times V^f \oplus W} \cdot i^*$.

To prove (6) we argue as above, using Bloch’s Riemann-Roch isomorphism
\[ K_i(X \times G V^f) \cong A_*(X \times G V^f), i). \]

\[ \square \]
Vistoli's conjecture  By composing the map above with the natural map $K_0^G(X) \rightarrow K_0^{G'}(X)$ we get a map $\tau_X^G : K_0^G(X) \rightarrow \hat{A}_*(X)_Q$. When $G$ acts on $X$ with finite reduced stabilizers then Vistoli [Vi1] stated a theorem which asserted the existence of a map

$$\tau : K_0^{G'}(X) \otimes Q \rightarrow A_*([X/G] \otimes Q)$$

satisfying properties (1)-(4) above (here $[X/G]$ is the Deligne-Mumford quotient stack). By Theorem 2 $A_*([X/G]) \otimes Q = A^G_*(X) \otimes Q$. Thus $I^d(A^G_*(X) \otimes Q) = 0$ for $d \gg 0$, so $A^G_*(X)_Q = A^G_*(X) \otimes Q$. Thus Vistoli’s map is a special case of our map $\tau^G_X$, since it is uniquely determined by properties (1)-(4). Vistoli noted that this map need not be an isomorphism and made the following conjecture about its kernel.

Conjecture 1 ([Vi1, Conjecture 2.4]) Suppose that $G$ acts on $X$ with finite reduced stabilizers. If $\alpha \in \ker(\tau^G_X : K_0^G(X) \rightarrow A_*([X/G]_Q))$ then there exists an element $\epsilon \in K_0^G(X)$ with every non-zero rank (meaning $\epsilon$ is represented by a complex of locally free sheaves whose homology is non-zero at the generic point of every subvariety) such that $\epsilon \alpha = 0$.

The results of this section identify the kernel: it is exactly the kernel of the completion map $K_0^G(X) \otimes Q \rightarrow K_0^{G'}(X)_Q$.

Proposition 13 Suppose $K_0^G(X)$ is Noetherian and $K_0^{G'}(X)$ is finitely generated over $K_0^G(X)$. Then $\alpha \in \ker(\tau^G_X)$ if and only if $(1 + \delta)\alpha = 0$ for some $\delta \in K_0^G(X)$ of (virtual) rank 0.

Proof. The proof follows immediately from Krull’s theorem. $\square$

5 Localization

In this section we discuss properties of equivariant Chow groups that are similar to properties of equivariant cohomology. In the first part, we give the relationship between $A^G_*(X)$ and $A^T_*(X)$ when $G$ is a connected reductive group with maximal torus $T$. The remainder of the section is devoted to actions of (split) tori. In particular, we prove two localization theorems (Theorems 5, 6). Following ideas of [A-B] they yield a characteristic free proof of the Bott residue formula for split torus actions on complete varieties over a field of arbitrary characteristic (Theorem 7).
5.1 Connected reductive groups

Denote by $A^\ast_G$ or $R_G$ the equivariant Chow ring of a point (the equivariant Chow groups of a point have a ring structure since a point is smooth). If $G$ is a connected reductive group then by \cite{EG}, $R_G \otimes \mathbb{Q} = \text{Sym}(\hat{T})^W \otimes \mathbb{Q}$, where $\hat{T}$ is the group of characters of the maximal torus and $W$ is the Weyl group. When $G$ is special in the sense of \cite{Sem-Chev} then $R_G = \text{Sym}(\hat{T})^W$ exactly (\cite{EG}). Under this identification we will write $R_d^G = A^d_G = A_G - d$. Via pullback from a point, $A_G^\ast(X)$ has the structure of an $R_G$-module.

If $G = T$ is a split torus, then $W$ is trivial, and the identification $R_T = \text{Sym}(\hat{T})$ is given explicitly as follows. If $\lambda \in \hat{T}$, let $k_\lambda$ denote the corresponding 1-dimensional representation of $T$, and let $L_\lambda$ denote the line bundle $U \times^T k_\lambda \to U/T$. The map $\hat{T} \to A^1_T$ given by $\lambda \mapsto c_1(L_\lambda)$ extends to a ring isomorphism $\text{Sym}(\hat{T}) \to R_T$. If $f : T \to S$ is a homomorphism of tori, then there is a pullback map $f^* : \hat{S} \to \hat{T}$. This extends to a ring homomorphism $f^* : \text{Sym}(\hat{S}) \to \text{Sym}(\hat{T})$, or in other words, a map $f^* : R_S \to R_T$.

**Proposition 14** Let $G$ be a connected reductive group with split maximal torus $T$ and Weyl group $W$. Then $A^G_\ast(X) \otimes \mathbb{Q} = A_T^\ast(X)^W \otimes \mathbb{Q}$. If $G$ is special the isomorphism holds with integer coefficients.

Proof: If $G$ acts freely on $U$, then so does $T$. Thus for a sufficiently large representation $V$, $A^T_\ast(X) = A_{i+l-t}((X \times U)/T)$ and $A^G_\ast(X) = A_{i+l-g}((X \times U)/G)$ (here $l$ is the dimension of $V$, $t$ the dimension of $T$ and $g$ the dimension of $G$). On the other hand, $(X \times U)/T$ is $G/T$ bundle over $(X \times U)/G$. Thus $A_k((X \times U/T)) \otimes \mathbb{Q} = A_{k+g-t}((X \times U)/G)^W \otimes \mathbb{Q}$ and if $G$ is special, then the equality holds integrally (\cite{EG}) and the proposition follows. □

Thus, for connected reductive groups, to compute equivariant Chow groups (at least with rational coefficients), it suffices to understand equivariant Chow groups for tori. We begin with the following proposition.

**Proposition 15** If $T$ acts trivially on $X$, then $A^T_\ast(X) = A_\ast(X) \otimes R_T$.

Proof. If the action is trivial then $(U \times X)/T = U/T \times X$. The spaces $U/T$ can be taken to be products of projective spaces, so $A_\ast(U/T \times X) = A_\ast(X) \otimes A_\ast(U/T)$. □
Remark. If the action is trivial, the pullbacks $A^*X_T \to A^*X$ and $A^*(U/T) \to A^*X_T$ induce an inclusion of $A^*X \otimes R_T \subset A^*_T(X)$ as a subring. If $X$ is smooth, then the inclusion is an isomorphism by Proposition 2.

5.2 Fixed loci and the localization theorem

For the remainder of this section, all Chow groups have rational coefficients, and for simplicity of exposition, we assume that tori are split.

If $X$ is a scheme with a $T$-action, we may put a closed subscheme structure on the locus $X^T$ of points fixed by $T$.

Now $R_T = \text{Sym}(\hat{T})$ is a polynomial ring. Set $Q = (R_T^+)^{-1} \cdot R_T$, where $R_T^+$ is the multiplicative system of homogeneous elements of positive degree.

**Theorem 5** (localization) The map $i_*^T : A_*(X^T) \otimes Q \to A_*(X) \otimes Q$ is surjective, and if $X$ is quasi-projective it is an isomorphism.

**Remark.** The quasi-projectivity assumption is needed to apply the long exact sequence for higher Chow groups. The strategy of the proof is similar to [Th2, Theorem 5.3].

Proof. Applying the localization exact sequence for higher equivariant Chow groups

$$\ldots \to A_*^T(X^T) \to A_*^T(X) \to A_*^T(X - X^T) \to 0$$

the theorem follows from the following proposition.

**Proposition 16** If $T$ acts on $X$ without fixed points, then there exists $r \in R_T^+$ such that $r \cdot A_*^T(X, m) = 0$. (Recall that $A_*^T(X, m)$ refers to $T$-equivariant higher Chow groups.)

Suppose $f : T \to S$ is a homomorphism of tori. As discussed above, there is a pullback map $f^* : A_*^S \to A_*^T$.

**Lemma 9** (cf. [A-B]) Suppose there is a $T$-map $X \overset{\phi}{\to} S$. Then $t \cdot A_*^T(X) = 0$ for any $t = f^*s$ with $s \in R_S^+$. 34
Proof of Lemma 9. Since $A^*_S$ is generated in degree 1, we may assume that $s$ has degree 1. After clearing denominators we may assume that $s = c_1(L_s)$ for some line bundle on a space $U/S$. The action of $t = f^*s$ on $A_*(X_T)$ is just given by $c_1(\pi^T_t f^*L_s)$ where $\pi_T$ is the map $U \times^T X \to U/T$. To prove the lemma we will show that this bundle is trivial.

First note that $L_s = U \times^S k$ for some action of $S$ on the one-dimensional vector space $k$. The pullback bundle on $X_T$ is the line bundle

$$U \times^T (X \times k) \to X_T$$

where $T$ acts on $k$ by the composition of $f : T \to S$ with the original $S$-action. Now define a map

$$\Phi : X_T \times k \to U \times^T (X \times k)$$

by the formula

$$\Phi(e, x, v) = (e, x, \phi(x) \cdot v)$$

(where $\phi(x) \cdot v$ indicates the original $S$ action). This map is well defined since

$$\Phi(et, t^{-1}x, v) = (et, t^{-1}x, \phi(t^{-1}x) \cdot v)$$

as required. This map is easily seen to be an isomorphism with inverse $(e, x, v) \mapsto (e, x, \phi(x)^{-1} \cdot v)$. □

Proof of Proposition 16. Since $A^*_G(X) = A^*_G(X_{red})$ we may assume $X$ is reduced. Working with each component separately, we may assume $X$ is a variety. Let $X^0 \subset X$ be the $(G$-invariant) locus of smooth points. By Sumhiro’s theorem [Su], the action of a torus on a normal variety is locally linearizable (i.e. every point has an affine invariant neighborhood). Using this theorem it is easy to see that the set $X(T_1) \subset X^0$ of points with stabilizer $T_1$ can be given the structure of a locally closed subscheme of $X$. Furthermore, only finitely many subgroups can occur as stabilizers (Appendix, Lemma [11]), so there is some $T_1$ such that $U = X(T_1)$ is open in $X^0$, and thus in $X$.

The torus $T' = T/T_1$ acts without stabilizers, but the action of $T'$ on $U$ is not a priori proper. However, by [Th1, Proposition 4.10], we can replace $U$ by a sufficiently small open set so that $T'$ acts freely on $U$ and a principal bundle quotient $U \to U/T$ exists. Shrinking $U$ further, we can assume that
this bundle is trivial, so there is a $T$ map $U \to T'$. Hence, by the lemma, $t \cdot A^T_*(U) = 0$ for any $t \in A^*_T$ which is pulled back from $A^*_{T'}$.

Let $Z = X - U$. By induction on dimension, we may assume $p \cdot A^*_T Z = 0$ for some homogeneous polynomial $p \in R_T$. From the long exact sequence of higher Chow groups,

$$\ldots A^*_T(Z,m) \to A^*_T(X,m) \to A^*_T(U,m) \to \ldots$$

it follows that $tp$ annihilates $A^*_T(X)$ where $t$ is the pullback of a homogeneous element of degree 1 in $R_S$.

Remark: Using only the short exact localization sequence for ordinary equivariant Chow groups (which does not require an assumption of quasi-projectivity) shows that $i_*$ is surjective. □

5.3 Explicit localization and the integration formula

The localization theorem in equivariant cohomology has a more explicit version for manifolds. This yields an integration formula from which the Bott residue formula is easily deduced ([A-B], [B-V]). In this section we prove the analogous results for equivariant Chow groups of smooth varieties. Because equivariant Chow theory has formal properties similar to equivariant cohomology, the arguments are almost the same as in [A-B]. As before we assume that all tori are split.

Let $F$ be a scheme with a trivial $T$-action. If $E \to F$ is a $T$-equivariant vector bundle on $F$, then $E$ splits canonically into a direct sum of vector subbundles $\bigoplus_{\lambda \in \hat{T}} E_\lambda$, where $E_\lambda$ consists of the subbundle of vectors in $E$ on which $T$ acts by the character $\lambda$. The equivariant Chern classes of an eigenbundle $E_\lambda$ are given by the following lemma.

**Lemma 10** Let $F$ be a scheme with a trivial $T$-action, and let $E_\lambda \to F$ be a $T$-equivariant vector bundle of rank $r$ such that the action of $T$ on each vector in $E_\lambda$ is given by the character $\lambda$. Then for any $i$,

$$c^T_i(E_\lambda) = \sum_{j \leq i} \binom{r-j}{i-j} c_j(E_\lambda) \lambda^{i-j}.$$

In particular the component of $c^T_r(E_\lambda)$ in $R^*_T$ is given by $\lambda^r$. □
As noted above, $A^*_T(F) \supset A^*F \otimes R_T$. The lemma implies that $c^*_T(E)$ lies in the subring $A^*F \otimes R_T$. Because $A^N F = 0$ for $N > \dim F$, elements of $A^i F$, for $i > 0$, are nilpotent elements in the ring $A^*_T(F)$. Hence an element $\alpha \in A^dF \otimes R_T$ is invertible in $A^*_T(F)$ if its component in $A^0F \otimes R^d_T \cong R^d_T$ is nonzero.

For the remainder of this section $X$ will denote a smooth variety with a $T$ action. If $X$ is smooth then by [Iv] the fixed locus $X^T$ is also smooth. For each component $F$ of the fixed locus $X^T$ the normal bundle $N_F X$ is a $T$-equivariant vector bundle over $F$. Note that the action of $T$ on $N_F X$ is non-trivial.

**Proposition 17** If $F$ is a component of $X^T$ with codimension $d$ then $c^*_d(N_F X)$ is invertible in $A^*_T(F) \otimes \mathbb{Q}$.

Proof: By ([Iv], Proof of Proposition 1.3]), for each closed point $f \in F$, the tangent space $T_f F$ is equal to $(T_f X)^T$, so $T$ acts with non-zero weights on the normal space $N_f = T_f X/T_f F$. Hence the characters $\lambda_i$ occurring in the eigenbundle decomposition of $N_F X$ are all non-zero. By the preceding lemma, the component of $c^*_d(N_F X)$ in $R^d_T$ is nonzero. Hence $c^*_d(N_F X)$ is invertible in $A^*_T(F) \otimes \mathbb{Q}$, as desired. $\square$

Using this result we can get, for $X$ smooth, the following more explicit version of the localization theorem.

**Theorem 6** (Explicit localization) Let $X$ be a smooth (not necessarily quasi-projective) variety with a torus action. Let $\alpha \in A^*_T(X) \otimes \mathbb{Q}$. Then

$$\alpha = \sum_F i_{F*} \frac{i^*_F \alpha}{c^*_{d_F}(N_F X)},$$

where the sum is over the components $F$ of $X^T$ and $d_F$ is the codimension of $F$ in $X$.

Proof: By the surjectivity part of the localization theorem, we can write $\alpha = \sum_F i_{F*}(\beta_F)$. Therefore, $i^*_F \alpha = i^*_F i_{F*}(\beta_F)$ (the other components of $X^T$ do not contribute); by the self-intersection formula, this is equal to $c^*_{d_F}(N_F X) \cdot \beta_F$. Hence $\beta_F = \frac{i^*_F \alpha}{c^*_{d_F}(N_F X)}$ as desired. $\square$
If \( X \) is complete, then the projection \( \pi_X : X \to \text{pt} \) induces push-forward maps \( \pi^T_X : A^*_T X \to R_T \) and \( \pi^T_X : A^*_T X \otimes \mathbb{Q} \to \mathbb{Q} \). There are similar maps with \( X \) replaced with any component \( F \) of \( X_T \). Applying \( \pi^T_X \) to both sides of the explicit localization theorem, and noting that \( \pi^T_X i^*_F = \pi^T_F \), we deduce the “integration formula” (cf. [A-B, Equation (3.8)]).

**Corollary 5 (Integration formula)** Let \( X \) be smooth and complete, and let \( \alpha \in A^*_T (X) \otimes \mathbb{Q} \). Then

\[
\pi_X^* (\alpha) = \sum_{F \subset X_T} \pi^*_F \left\{ \frac{i^*_F \alpha}{c^T_{d_F} (N_F X)} \right\}
\]

as elements of \( \mathbb{Q} \). \( \square \)

**Remark.** If \( \alpha \) is in the image of the natural map \( A^*_T (X) \to A^*_T (X) \otimes \mathbb{Q} \) (which need not be injective), then the equation above holds in the subring \( R_T \) of \( \mathbb{Q} \). The reason is that the left side actually lies in the subring \( R_T \); hence so does the right side. In the results that follow, we will have expressions of the form \( z = \sum z_j \), where the \( z_j \) are degree zero elements of \( \mathbb{Q} \) whose sum lies in the subring \( R_T \). The pullback map from equivariant to ordinary Chow groups gives a map \( i^* : R_T = A^*_T (\text{pt}) \to \mathbb{Q} = A_* (\text{pt}) \), which identifies the degree 0 part of \( R_T \) with \( \mathbb{Q} \). Since \( \sum z_j \) is a degree 0 element of \( R_T \), it is identified via \( i^* \) with a rational number. Note that \( i^* \) cannot be applied to each \( z_j \) separately, but only to their sum. In the integration and residue formulas below we will identify the degree 0 part of \( R_T \) with \( \mathbb{Q} \) and suppress the map \( i^* \).

The preceding corollary yields an integration formula for an element \( a \) of the ordinary Chow group \( A_0 X \), provided that \( a \) is the pullback of an element \( \alpha \in A^*_T X \).

**Proposition 18** Let \( a \in A_0 X \), and suppose that \( a = i^* \alpha \) for \( \alpha \in A^*_T X \). Then

\[
\deg (a) = \sum_F \pi^*_F \left\{ \frac{i^*_F \alpha}{c^T_{d_F} (N_F X)} \right\}
\]

**Proof:** Consider the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i} & X_T \\
\downarrow \pi_X & & \downarrow \pi^T_X \\
\text{pt} & \xrightarrow{i} & U/T.
\end{array}
\]

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We have \( \pi_X (a) = \pi_X i^*(\alpha) = i^* \pi^T_X(\alpha) \). Applying the integration formula gives the result. \( \square \)

### 5.4 The residue formula

Let \( E \to X \) be a \( T \)-equivariant vector bundle of rank \( r \) on a complete, smooth \( n \)-dimensional variety. Let \( p(x_1, x_2, \ldots, x_r) \) be a polynomial of weighted degree \( n \), where the degree of \( x_i \) is \( i \). The integration formula above will allow us to compute \( \deg (p(c_1(E), \ldots, c_r(E)) \cap [X]) \) in terms of the restriction of \( E \) to \( X^T \).

As a notational shorthand, write \( p(E) \) for \( p(c_1(E), \ldots, c_r(E)) \) and \( p^T(E) \) for \( p(c^T_1(E), \ldots, c^T_r(E)) \). Write \( p^T(E|_F) \) for \( p(c^T_1(E|_F), \ldots, c^T_r(E|_F)) \) and notice that \( p(E) \cap [X] = i^*(p^T(E) \cap [X^T]) \). We can therefore apply the preceding proposition to get the Bott residue formula.

**Theorem 7 (Bott residue formula)** Let \( E \to X \) be a \( T \)-equivariant vector bundle of rank \( r \) on a complete, smooth \( n \)-dimensional variety, and let \( p(x_1, x_2, \ldots, x_r) \) be a polynomial of weighted degree \( n \). Then

\[
\deg (p(E) \cap [X]) = \sum_{F \subseteq X^T} \pi^T_{F^*} \left\{ \frac{p^T(E|_F) \cap [F]^T}{c^T_{d_F}(N_{F^*}X)} \right\}.
\]

\( \square \)

By Lemma [11] the equivariant Chern classes \( c^T_i(E|_F) \) and \( c^T_{d_F}(N_{F^*}X) \) can be computed in terms of the characters of the torus occurring in the eigenbundle decompositions of \( E|_F \) and \( N_{F^*}X \) and the Chern classes of the eigenbundles. The above formula can then be readily converted (cf. [A-B]) to more familiar forms of the Bott residue formula not involving equivariant cohomology. We omit the details. If the torus \( T \) is 1-dimensional, then degree zero elements of \( Q \) are rational numbers, and the right hand side of the formula is just a sum of rational numbers. This is the form of the Bott residue formula which is most familiar in practice.
6 Actions of group schemes over an arbitrary base

Let $S$ be a regular scheme, and let $X/S$ be an $S$-scheme with an action of a connected reductive group scheme $G/S$. With appropriate assumptions on $G/S$ (see below), it is possible to use Seshadri’s results on geometric reductivity over an arbitrary base to extend much of our theory.

If $S = \text{Spec}(\mathbb{Z})$ and $G/S$ is reductive, then the theory goes through more or less intact. In particular, if $X/\mathbb{Z}$ is a smooth scheme acted on by a reductive group scheme $G/\mathbb{Z}$, then there is an equivariant Chow ring $A^*_G(X)$. Such a ring should be useful for studying intersection theory on moduli in mixed characteristic.

6.1 Definitions

**Definition 4** Let $G/S$ be a smooth group scheme. Let $E/S$ be a vector bundle (i.e., $\text{Spec}(\text{Sym}(E^*))$ where $E/S$ is a locally free $G$-module). The bundle $E/S$ is said to be a representation of $G/S$ if there is an action $G \times E \rightarrow E$ which is linear on each fiber.

We assume the following condition on $G/S$:

(*) There exist representations $E/S$ with a non-empty open set $U/S$ such that $G/S$ acts freely on $U$.

**Proposition 19** If $G/S$ is a smooth group scheme, then condition (*) above is satisfied if either

1. $G/S$ is the pullback of a group scheme $G_R/\text{Spec}(R)$ where $R$ is a Dedekind domain.
2. The geometric fibers of $G/S$ are all semisimple with trivial center.

Proof. By [Se, Lemma 1, Proposition 3], $R(G_R)$ is a projective $R$-module which contains a finitely generated $G$-invariant $R$-module. Since $R$ is a Dedekind domain, this module is projective. Pulling back to $S$ gives the desired representation.

By [SGA3, Expose II], the Lie algebra $\text{Lie}(G/S)$ is a vector bundle over $S$. Since $G$ has trivial center, the adjoint action of $G$ on the vector bundle $\text{Lie}(G/S) \times_S \text{Lie}(G/S)$ is generically free. □
Henceforth we will assume that $G/S$ is reductive. If condition (*) holds, we can find representations $E/S$ and open set $U/S$ so that $E - U$ has arbitrarily high codimension. By Seshadri’s theorem ([Sc]) there is a principal bundle quotient $U \to U/G$.

The arguments of Proposition 22 yield

**Proposition 20** Assume one of the following:

1. $X/S$ equivariantly embeds in a projective bundle over $S$.
2. $X/S$ is normal.

Then a principal bundle quotient $X \times U \to X \times^G U$ exists. $\square$

As a consequence of Proposition 20 we can define equivariant Chow groups.

**Definition 5** Assume that condition (*) on $G/S$ holds, as well as one of the hypotheses (1) or (2) of Proposition 20. Define the $i$-th equivariant Chow group as $A_i^{G}(X \times^G U)$ where $l = \dim (U/S)$ and $g = \dim (G/S)$. As for algebraic schemes, the definition is independent of the representation.

### 6.2 Results over an arbitrary base

Since most of the results of intersection theory hold for schemes over a regular base ([Fu, Chapter 19]), most of the results on equivariant Chow groups also hold.

In particular, the functorial properties with respect to proper, flat and l.c.i maps hold.

If $S = \text{Spec}(R)$ where $R$ is a Dedekind domain, then there is an intersection product on $A^G_*(X)$ for $X/S$ smooth.

If $G$ acts freely on $X$ and a quotient $X/G$ exists, then $A^G_*(X) = A_*(X/G)$.

If the stabilizer group scheme for the action of $G$ on $X$ is finite over $S$, and a quotient $X/G$ exists, then we expect that $A^G_*(X)_Q = A_*(X/G)_Q$. However, to prove such a statement using the techniques of this paper would require a localization long exact sequence for higher Chow groups over an arbitrary
If $S$ is regular and $T/S$ is a split torus, then the equality of completions of $K^T_i(X)$ with respect to either the augmentation ideal of $K^T_0(S)$ or the augmentation ideal of $K^T_0(X)$ holds (cf. Theorem 3). The analogous equality of completions for (higher) Chow groups also holds. From the torus case, we can deduce the corresponding equality of completions for the totally split (i.e. pulled back from the split groups over $Spec(\mathbb{Z})$) classical groups $G = SL(n,S), Sp(2n,S)$. The argument is the same as in Section 4.2. The key point is that for these groups $K^G_i(X)$ is a direct summand in $K^T_i(X)$; this can be proved (as in Lemma 8) by realizing $G/B$ as a sequence of projective bundles and applying Thomason’s projective bundle theorem. (Note that for group schemes $G$, as for groups, there is a scheme $G/B$.) For $G = SO(n,S), G/B$ can be realized as a sequence of quadric bundles, and the analysis of [E-G1] applied to deduce the result with rational coefficients. Once the analogue of Theorem 3 is proved, the corresponding Riemann-Roch statements follow.

A form of the localization theorem for split torus actions also holds over an arbitrary base. However, we can only prove a localization isomorphism if the fixed locus is regularly embedded in $X$. Again, the obstruction is the lack of a long exact sequence for higher Chow groups over an arbitrary base.

Finally, if $G/S$ is smooth but not reductive and $G$ embeds as a closed subgroup of $GL(n,S)$, then a quotient $X \times^G U$ exists as an algebraic space. To develop an equivariant intersection theory in this case would require further facts about Chow groups of algebraic spaces.

7 Appendix

Here, we collect some useful results about actions of algebraic groups acting on algebraic schemes in arbitrary characteristic.

7.1 Torus actions
Lemma 11  If $X$ is a variety with an action of a torus $T$, then there is an open $U \subset X$ so that the stabilizer is constant for all points of $U$. \\

Proof: It suffices to prove the lemma after finite base change, so we may assume that $T$ is split. Let $\tilde{X} \to X$ be the normalization map. This map is $T$-equivariant and is an isomorphism over an open set. Thus we may assume $X$ is normal. By Sumihiro’s theorem, the $T$ action on $X$ is locally linearizable, so it suffices to prove the lemma when $X = V$ is a vector space and the action is diagonal.

If $V = k^n$, then let $U = (k^*)^n$. The $n$-dimensional torus $G^m_n$ acts transitively on $U$ in the obvious way. This action commutes with the given action of $T$. Thus the stabilizer at each closed point of $U$ is the same. ✷

Proposition 21  Let $V$ be a vector space with a linear action of a torus $T$. Let $V^f$ be the set of points with closed orbits and trivial stabilizers. Then the set $V^u = V - V^f$ is a finite union of linear subspaces.

Proof: Again, after finite base change we may assume that $T$ is split. Let $V^c$ denote the set of points in $V$ whose $T$-orbits are closed in $V$. We first prove that $V - V^c$ is a finite union of linear subspaces. Choose a basis $\{v_i\}$ on which $T$ acts diagonally. If $\dim T = 1$, then the $T$-orbit of $v = \sum a_i v_i$ is not closed if and only if the weights of the non-zero coordinates are either all non-negative or all non-positive. Thus, $V - V^c$ is defined by the vanishing of various subsets of coordinate hyperplanes, hence is a finite union of linear subspaces. For $T$ of arbitrary dimension, $T \cdot v$ is closed iff for all 1-dimensional subtori $S \subset T$, $S \cdot v$ is closed (this follows from [GIT, Prop. 2.4]). This in turn holds if $S \cdot v$ is closed for a sufficiently general $S \subset T$, so the result follows from the case $\dim T = 1$.

To complete the proof we must show that the complement of the set of points with trivial stabilizer is a union of linear subspaces. This follows from two facts.

(1) If $G \subset T$ is a subgroup, then $L_G = \{v \in V|G \subset Stab(v)\}$ is a linear subspace.

(2) $V$ can be covered by a finite number of $L_G$’s by Lemma [11]. ✷
7.2 Principal bundles

Lemma 12 ([E-G]) Let $G$ be an algebraic group. For any $i > 0$, there is a representation $V$ of $G$ and an open set $U \subset V$ such that $V - U$ has codimension more than $i$ and such that a principal bundle quotient $U \to U/G$ exists.

Proof. Embed $G$ into $GL(n)$ for some $n$. Assume that $V$ is a representation of $GL(n)$ and $U \subset V$ is an open set such that a principal bundle quotient $U \to U/GL(n)$ exists. Since $GL(n)$ is special, this principal bundle is locally trivial in the Zariski topology. Thus $U$ is locally isomorphic to $W \times GL(n)$ for some open $W \subset U/GL(n)$. A quotient $U/G$ can be constructed by patching the quotients $W \times GL(n) \to W \times (GL(n)/G)$.

We have thus reduced to the case $G = GL(n)$. Since the action of an affine group is locally finite, there as an equivariant closed embedding of $G \hookrightarrow V$ into a sufficiently large vector space $V/k$. Consider the open set $U \subset V$ of points with trivial stabilizers which are stable for the $G$ action on $V$. Since $G$ acts freely on itself, $G \subset U$; hence $U$ is non-empty. Since the stabilizers are trivial, the action on $U$ is free, and the GIT quotient $U \to U/G$ is a principal bundle. Now if $V_1 = V \oplus V$, then ([GIT, Proposition 1.18]) $U_1 = (U \oplus V) \cup (V \oplus U) \subset V_1^*$. Thus a principal bundle quotient $U_1 \to U_1/G$ exists, and the codimension of $V_1 - U_1$ is strictly smaller than the codimension of $V - U$. Thus, by taking the direct sum of a sufficiently large number of copies $V$, we may assume that $V - U$ has arbitrarily high codimension. □

Let $G$ be an algebraic group, let $U$ be a scheme on which $G$ acts freely, and suppose that a principal bundle quotient $U \to U/G$ exists.

Proposition 22 Let $X$ be an algebraic scheme with a $G$ action. Assume that at least one of the following hypotheses holds.

(1) $X$ is quasi-projective with a linearized $G$-action.

(2) $G$ is connected and $X$ is equivariantly embedded as a closed subscheme of a normal variety.

(3) $G$ is special.

Then a principal bundle quotient $X \times U \to (X \times^G U)$ exists.
Proof. If $X$ is quasi-projective with a linearized action, then there is an 
equivariant line bundle on $X \times U$ which is relatively ample for the projection $X \times U \to U$. By [GIT, Prop 7.1] a principal bundle quotient $X \times^G U$ exists.

Now suppose that $X$ is normal and $G$ is connected. By Sumhiro’s theorem [Su], $X$ can be covered by invariant quasi-projective open sets which have a linearized $G$ action. Thus, by [GIT, Prop 7.1] we can construct a quotient $X_G = X \times^G U$ by patching the quotients of the quasi-projective open sets in the cover.

If $X$ equivariantly embeds in a normal variety $Y$, then by the above paragraph a principal bundle quotient $Y \times U \to Y \times^G U$ exists. Since $G$ is affine, the quotient map is affine, and $Y \times U$ can be covered by affine invariant open sets. Since $X \times U$ is an invariant closed subscheme of $Y \times U$, $X \times U$ can also be covered by invariant affines. A quotient $X \times^G U$ can then be constructed by patching the quotients of the invariant affines.

Finally, if $G$ is special, then $U \to U/G$ is a locally trivial bundle in the Zariski topology. Thus $U = \bigcup \{U_\alpha\}$ where $\phi_\alpha : U_\alpha \simeq G \times W_\alpha$ for some open $W_\alpha \subset U/G$. Then $\psi_\alpha : X \times U_\alpha \to X \times W_\alpha$ is a quotient, where $\psi_\alpha$ is defined by the formula $(x, w, g) \mapsto (g^{-1}x, w)$ (Here we assume that $G$ acts on the left on both factors of $X \times G$).

### 7.3 Quotients

Following Vistoli, we define a geometric quotient $X \xrightarrow{\pi} Y$ to be a map which satisfies properties i)-iii) of [GIT, Definition 0.6]. In particular, we do not require that $\mathcal{O}_Y = \pi_* (\mathcal{O}_X)^G$. The advantage of this definition is that it is preserved under base change. In characteristic 0 there are no inseparable extensions, so our definition agrees with Mumford’s ([GIT, Prop 0.2]).

The following proposition is an analogue of [V1, Prop 2.6]. The proof is similar.

**Proposition 23** Let $G$ act properly on a variety $X$ (hence with finite, but possibly non-reduced stabilizers), so that a geometric quotient $X \to Y$ exists. Then there is a commutative diagram of quotients

$$
\begin{array}{ccc}
Z & \rightarrow & X \\
\downarrow & & \downarrow \\
Q & \rightarrow & Y
\end{array}
$$
where $Z \to Q$ is a principal $G$-bundle and the horizontal maps are finite and surjective.

Proof. By [GIT, Lemma p. 14], there is a finite map $Q \to Y$, with $Q$ normal, so that the pullback $X_1 \to Q$ has a section in the neighborhood of every point. Cover $Q$ by a finite number of open sets $\{U_\alpha\}$ so that $X_1 \to Q$ has a section $U_\alpha \to V_\alpha$ where $V_\alpha = \pi^{-1}(U_\alpha)$.

Define a $G$-map

$$\phi_\alpha : G \times U_\alpha \to V_\alpha$$

by the formula

$$(g, y) \mapsto gs_\alpha(y).$$

The action is proper so each $\phi_\alpha$ is proper. Since the stabilizers are finite, $\phi_\alpha$ is in fact finite.

To construct a principal bundle $Z \to Q$ we must glue the $G \times U_\alpha$’s along their intersection. To do this we will find isomorphisms $\phi_{\alpha\beta} : s_\alpha(U_{\alpha\beta}) \to s_\beta(U_{\alpha\beta})$ which satisfy the cocycle condition.

For each $\alpha, \beta$, let $I_{\alpha\beta}$ be the scheme which parametrizes isomorphisms of $s_\alpha$ and $s_\beta$ over $U_{\alpha\beta}$ (i.e. a section $U_{\alpha\beta} \to I_{\alpha\beta}$ corresponds to a global isomorphism $s_\alpha(U_{\alpha\beta}) \to s_\beta(U_{\alpha\beta})$). The scheme $I_{\alpha\beta}$ is finite over $U_{\alpha\beta}$ (but possibly totally ramified in characteristic $p$) since it is defined by the cartesian diagram

$$\begin{array}{ccc}
I_{\alpha\beta} & \to & U_{\alpha\beta} \\
\downarrow & & \downarrow 1 \times s_\beta \\
G \times U_{\alpha\beta} & \xrightarrow{1 \times \phi_\alpha} & U_{\alpha\beta} \times V_{\alpha\beta}
\end{array}$$

(Note that $I_{\alpha\alpha}$ is the stabilizer of $s_\alpha(U_\alpha)$.)

Over $U_{\alpha\beta\gamma}$ there is a composition giving multiplication morphisms which are surjective when $\gamma = \beta$.

$$I_{\alpha\beta} \times U_{\alpha\beta\gamma} I_{\beta\gamma} \to I_{\alpha\gamma}$$

which gives multiplication morphisms which are surjective when $\gamma = \beta$.

After a suitable finite (but possibly inseparable) base change, we may assume that there is a section $U_{\alpha\beta} \to I_{\alpha\beta}$ for every irreducible component of $I_{\alpha\beta}$. (Note that $I_{\alpha\beta}$ need not be reduced.) Fix an open set $U_\alpha$. For $\beta \neq \alpha$ choose a section $\phi_{\alpha\beta} : U_{\alpha\beta} \to I_{\alpha\beta}$. Since the $I_{\alpha\beta}$’s split completely and $I_{\alpha\alpha}$ is a group scheme, there are sections $\phi_{\beta\alpha} : U_{\alpha\beta} \to I_{\beta\alpha}$ so that $\phi_{\alpha\beta} \cdot \phi_{\beta\alpha}$ is the
identity section of $U_{\alpha \alpha}$. For any $\beta, \gamma$ we can define a section of $I_{\beta \gamma}$ over $U_{\alpha \beta \gamma}$ as the composition $\phi_{\beta \alpha} \cdot \phi_{\alpha \gamma}$. Because $I_{\beta \gamma}$ splits, the $\phi_{\beta \alpha}$’s extend to sections over $U_{\beta \gamma}$.

By construction, the $\phi_{\beta \gamma}$’s satisfy the cocycle condition. We can now define $Z$ by gluing the sets $G \times U_{\beta}$ along the $\phi_{\beta \gamma}$’s. □

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