The Holographic Principle and the Renormalization Group

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We prove a c-theorem for holographic theories.

1 A Proposal for a Holographic c-Function

In [1] Zamolodchikov proved that for local and unitary two-dimensional field theories there exists a function of the couplings, hereinafter called the c-function, \( c(g_i) \), such that

\[
- \beta_i \partial_i c \leq 0
\] (1)

along the renormalization group flow. For fixed points of the flow, the c-function reduces to the central extension of the Virasoro algebra. Some generalizations of the c-function to realistic four-dimensional theories have been suggested; let us mention in particular Cardy's proposal on \( S^4 \):

\[
c \equiv \int_{S^4} \sqrt{g} < T >
\] (2)

where \( T \) is the trace of the energy-momentum tensor of the theory. There is no complete agreement as to whether a convincing proof of the theorem exists for dimension higher than two (cf. [2] for a recent attempt).

It is not difficult to invent a holographic definition of the central extension for \( N=4 \) super Yang-Mills theories using the AdS/CFT map, especially in the light of the IR/UV connection pointed out by Susskind and Witten in [3].

In order to understand this construction properly, let us recall the original 't Hooft's presentation of the holographic principle, stemming from the Bekenstein-Hawking entropy formula for black holes [4]. Given a bounded region of instantaneous \( d-1 \) space \( V \) of volume \( vol_{d-1}(V) \), holography states that all the physical information on processes in \( V_{d-1} \) can be codified in terms of surface variables, living on the boundary of \( V, \partial V \). More precisely, the number of holographic degrees of freedom is given by:

\[
N_{d.o.f.} \sim \frac{vol_{(d-2)}(\partial V)}{G^{(d)}}
\] (3)

\text{\small aContribution to the Encuentros Relativistas ERE-98 (Luis Bel’s Festschrift)}
Physically this means that we have precisely one degree of freedom in each area cell of size given by the Planck length. In spite of the fact that the Bekenstein bounds would suggest the radical approach that any physics in V can be mapped into holographic degrees of freedom in ∂V , the list of theories suspected to admit holographic projection is still small, and always involves gravity. (Susskind indeed suggested from the beginning that string theory should be holographic).

Let us consider for concreteness a four dimensional CFT defined on a spacetime with topology $S^3 \times \mathcal{R}$ and with the natural metric in $S^3$. Let us also introduce an ultraviolet cutoff $\delta$, and let us correspondingly divide the sphere $S^3$ into small cells of size $\delta^3$. The number of cells is clearly of order $1/\delta^3$.

We would like now to define the number of degrees of freedom in terms of the central extension as

$$N_{dof} = \frac{c}{\delta^3} \quad (4)$$

Notice that here the parameter $c$ plays the rôle of the number of degrees of freedom in each cell. If the theory we are considering is the holographic projection of some supergravity in the bulk, it is natural to rewrite this in terms of Eq (3), but with the $\text{vol}(\partial V)$ now replaced by a section of the bulk at $\delta = \text{constant}$, with $\delta$ being now identified with the holographic parameter. We are thus led to the identification

$$c \equiv \lim_{\delta \to 0} \frac{\delta^3 \text{Vol}(\partial V_\delta)}{G_5} \quad (5)$$

In the particular example of $AdS_5 \times S_5$ this yields $c = \frac{R^4}{4\alpha'}$, with $R^4 = \alpha'^2 Ng^2$.

2 Renormalization Group Flow along Null Geodesics

We shall in this section study the renormalization group evolution of the postulated $c$-function; that is, its dependence on the holographic variable, $\rho$. In order to do that, the first point is to identify exactly what we understand by area (that is, $\text{vol}(\partial V)$). Our definition clearly involves the quotient between an area defined close to the horizon and an inertial area, so that:

$$c(\delta) \equiv \frac{1}{G_d} \frac{\text{vol}_{(d-2)}(\mathcal{J}_\delta)\text{vol}_{(d-2)}(\text{inertial})}{(d-2)(d-3)}$$

The meaning of the preceding formula is as follows (cf. 5). The first term, $\text{vol}_{(d-2)}(\mathcal{J}_\delta)$ is the volume computed on $\mathcal{J}$ regularized with an UV cutoff $\delta$ ($R^3/\delta^3$ in the familiar example of $AdS_5 \times S_5$). The other factor, $\text{vol}_{(d-2)}(\text{inertial})$ stands for the equivalent volume measured by an inertial observer which does
not feel the gravitational field (that is, \( \delta^3 \) in AdS\(_5\)). The whole thing is then divided by the d-dimensional Newton’s constant.

Our c-function will obey a renormalization group equation without anomalous dimensions:

\[
\delta \frac{\partial c}{\partial \delta} + \sum_i \beta_i(g) \frac{\partial c}{\partial g_i} = 0
\]

which means that to prove the c-theorem we just have to show that:

\[
\delta \frac{\partial c}{\partial \delta} \leq 0
\]

Now, to study the evolution in the bulk of the c-function, we need to know how this regularized definition of area evolves as we penetrate into the bulk. It is now only natural to identify the UV cutoff \( \delta \) with the affine parameter \( \hat{u} \) introduced in the previous section such that \( \hat{u} \sim 0 \).

We would like also to argue that it is quite convenient to study the evolution of the area along null geodesics entering the bulk. First of all, the whole set-up is conformally invariant (which is not the case for timelike geodesics). In addition, there is a very natural definition of transverse space there. In a Newman-Penrose orthonormal tetrad \( b^a \), which is a sort of complexified light cone, because in terms of a real orthonormal tetrad, \( e^a \),

\[
\begin{align*}
l^\mu \partial_\mu &\equiv e^+ \equiv \frac{1}{\sqrt{2}}(e^0 + e^3) \\
n^\mu \partial_\mu &\equiv e^- \equiv \frac{1}{\sqrt{2}}(e^0 - e^3) \\
m^\mu \partial_\mu &\equiv e_T \equiv \frac{1}{\sqrt{2}}(e^1 - ie^2) \\
mbar^\mu \partial_\mu &\equiv \ebar_T \equiv \frac{1}{\sqrt{2}}(e^1 + ie^2),
\end{align*}
\]

one can easily find the optical scalars\(^b\) of the geodesic congruence. One has, in particular, that

\[
\rho \equiv -\nabla_\mu l^\mu m^\nu \bar{m}^\nu = - (\theta + i\omega)
\]

where the expansion, \( \theta \), is defined by \( \theta \equiv \frac{1}{2} \nabla_\alpha l^\alpha \) and the rotation, \( \omega \), is a scalar which measures the antisymmetric part of the covariant derivative of the tangent field: \( \omega^2 \equiv \frac{1}{2} \omega_{\alpha\beta} \omega^{\alpha\beta} \), with \( \omega_{\alpha\beta} \equiv \nabla_{[\alpha} l_{\beta]} \).

\(^b\)We choose to present the formulas in the four dimensional case by simplicity, but it should be clear that no essential aspect depends on this.
Let us now consider a congruence of null geodesics. This means that we have a family $x^\mu(u,v)$, such that $v$ tells in which geodesic we are, and $u$ is an affine parameter of the type previously considered. The connecting vector (geodesic deviation) $Z^\mu \equiv x^\mu(u,v) - x^\mu(u,v + \delta v)$ connects points on neighboring geodesics, and by construction satisfies

$$\mathcal{L}(l)Z^\mu = 0$$

that is, $l^\mu \nabla_\mu Z^\alpha = Z^\mu \nabla_\mu l^\alpha$. Although the modulus of the vector $Z$ is not conserved, it is not difficult to show that its projection on $l^\mu$ is a constant of motion. Penrose and Rindler call abreast the congruences for which this projection vanishes. In this case one can show that $h = 0$, where $h$ is defined from the projection of the geodesic deviation vector on the Newman-Penrose tetrad:

$$Z^\alpha = g l^\alpha + \zeta \bar{m}^\alpha + \bar{\zeta} m^\alpha + h n^\alpha$$

Under the preceding circumstances, the triangle $(0,\zeta_1,\zeta_2)$ is contained in $\Pi$, the 2-plane spanned by the real and imaginary parts of $m^\alpha$. Now it can be proven that, calling $A_2$ the area of this elementary triangle,

$$l^\alpha \nabla_\alpha A_2 = -(\rho + \bar{\rho}) A_2 = 2 \theta A_2$$

This fact relates in a natural way areas with null geodesic congruences.

Using this information we can write at once:

$$\hat{u} \frac{dc(\hat{u})}{d\hat{u}} = (\theta \hat{u} + d - 2)c(\hat{u})$$

It is worth noting at this point that $\hat{\theta}$ is finite (it corresponds to Einstein’s static universe in the standard AdS example). The divergence in $\theta$ stems from the conformal transformation necessary to go from $\hat{g}_{\alpha\beta}$ to $g_{\alpha\beta}$, to wit:

$$\theta = \hat{\theta} + \frac{d - 2}{2} \frac{N.Z}{\Omega}$$

In order that inertial and $\mathcal{J}$ units be the same, it is natural to measure inertial areas in units of $\delta \equiv \frac{\hat{\mathcal{J}}}{(N.Z)}$, where $(N,Z)_{\mathcal{J}}$ represents the scalar product of the vector $N^\mu \equiv -\nabla^\mu \Omega$ and $Z^\mu$ computed at $\hat{u} = 0$. (This is an effect similar to the usual redshift factor). Doing that one gets that the first derivative vanishes to first order:

$$\hat{u} \frac{dc(\hat{u})}{d\hat{u}} = 0$$

\footnote{In the general case, it is plain that in this way we build a $d-2$-volume}
But we can now invoke a well known theorem by Raychaudhuri:

\[ l^\mu \nabla_\mu \theta = \omega^2 - \frac{1}{2} R_{\mu\nu} l^\mu l^\nu - \sigma \bar{\sigma} - \theta^2 \]  

(17)

(\text{where the shear} \ \sigma \bar{\sigma} \equiv \frac{1}{2} \nabla_{[\beta} l_{\alpha]} \nabla^{[\beta} l^{\alpha]} - \frac{1}{4} (\nabla_\alpha l^\alpha)^2)

The Ricci term in the above equation vanishes for Einstein spaces, and the rotation must necessarily be zero if we want the flow lines to be orthogonal to the surfaces of transitivity; that is, that there exists a family of hypersurfaces \( \Sigma \), such that \( l_\mu = \nabla_\mu \Sigma \).

This shows that under these conditions

\[ \dot{u} \frac{d\theta}{d\dot{u}} < 0 \]  

(18)

which is enough to prove the c-theorem in the holographic case of present interest.

More details on the geometrical approach to the holographic map can be found in our paper.

Acknowledgments

This work has been partially supported by the European Union TMR program FMRX-CT96-0012 Integrability, Non-perturbative Effects, and Symmetry in Quantum Field Theory and by the Spanish grant AEN96-1655. The work of E.A. has also been supported by the European Union TMR program ERBFMRX-CT96-0090 Beyond the Standard model and the Spanish grant AEN96-1664. Enrique Alvarez is grateful to Luis Bel for accepting him as his student certain day a long time ago.

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