Foldability of simplicial surfaces onto a triangle

Markus Baumeister*

April 30, 2019

In recent years, complicated folding structures have been employed more often, with applications ranging from architecture ([BW08]) to robotics ([SMH16]). Concurrently, mathematical aspects of paper folding have been developed, spanning a diverse range of topics (contrast [JL19], [LO09], [Hul94], [Izm14], [SC31], [Sta10], [Tac09], [WY10], and [DO10]). One interesting question is how a given surface built up from triangles can be folded. There has been some progress towards answering this question, mostly with explicit angles ([BH96], [ABD01]). Since this problem is very hard in principle, we try to simplify it by considering a purely combinatorial model of folding, independent of any angles.

In this paper, we analyse which surfaces can be folded onto a triangle in this combinatorial model. Considering only combinatorial folding alters the situation drastically. For example, the octahedron can combinatorially be folded onto a triangle, although it is rigid under regular folding ([Deh16]). In contrast, a tetrahedron cannot combinatorially be folded onto a triangle. In particular, the notion of combinatorial folding focuses on intrinsic properties of the surface instead of the embedding.

In order to describe combinatorial folding properly, we have to choose an appropriate model. Although our model could be described as a triangulation of a two–dimensional manifold ([GY03], [DLRS10]) or as a combinatorial manifold ([IKN17]), we prefer a more intuitive, combinatorial description, which focuses on the incidence relations between vertices, edges, and faces directly. We eschew the additional manifold structure since our analysis does not depend on it – only the incidence relations between vertices, edges, and faces are strictly necessary.

We will not describe the complete combinatorial folding theory in this paper, and will restrict our definition in Section 2 to the specific case of “folding onto a triangle”. Our main result is that any combinatorial simplicial surface that can be folded onto a triangle is orientable and admits a vertex–3–colouring. With these properties given, we can reformulate the triangle–folding–problem as the search for a cyclic permutation whose products with certain involutions have a specified number of cycles (compare Corollary 4.13 for the precise formulation).

*Lehr- und Forschungsgebiet Algebra, RWTH Aachen University
1 Covering a Triangle

This paper is concerned with the question “When can a simplicial surface be folded onto a single triangle?”.

Our definition of simplicial surfaces differs from that in the literature. It is important to note that a simplicial surface (in our sense) is not a simplicial complex, since different simplices may have the same vertices. The closest definition can be found in [Pol].

Definition 1.1. A simplicial surface is a quadruple \((V, E, F, \prec)\) such that

1. \(V, E, F\) are finite sets (called vertices, edges, and faces) and \(\prec \subseteq (V \times E) \cup (V \times F) \cup (E \times F)\) is a transitive relation, called incidence.

2. For every edge \(e \in E\), there are exactly two vertices incident to \(e\).

3. For every face \(f \in F\) there are three incident vertices \(v_1, v_2, v_3\) and three incident edges \(e_1, e_2, e_3\) such that \(v_i\) and \(v_{i+1}\) are incident to \(e_i\) for \(1 \leq i \leq 3\) (where \(v_4 := v_1\)).

4. For every edge \(e \in E\), there are at most two faces incident to \(e\).

5. For every vertex \(v \in V\) there is a finite sequence \((e_1, f_1, e_2, f_2, \ldots)\) such that
   • The \(e_i\) are pairwise distinct and exactly those edges incident to \(v\).
   • The \(f_i\) are pairwise distinct and exactly those faces incident to \(v\).
   • \(e_i\) and \(e_{i+1}\) are incident to \(f_i\).
   • If the final element of the sequence is a face, then \(e_1\) is incident to that face.

6. For every vertex \(v \in V\), there is an edge \(e \in E\) such that \(v \prec e\).

7. For every edge \(e \in E\), there is a face \(f \in F\) with \(e \prec f\).

The simplicial surface is closed if there are exactly two faces incident to every edge.

While we allow our simplicial surfaces to be disconnected, it is often convenient to restrict to connected simplicial surfaces.

Example 1.2. For any set \(M\) with three elements we can define a triangle as the simplicial surface \((\text{Pot}_1(M), \text{Pot}_2(M), \text{Pot}_3(M), \varnothing)\).

Example 1.3. We can depict the incidence structure of a simplicial surface graphically. The following surface is a torus.
Here, we have $V = \{A, B, C, D\}$, $E = \{a, b, \ldots, l\}$ and $F = \{1, \ldots, 8\}$.

The process of “folding onto a triangle” can be separated into two conditions that need to be fulfilled. The first one is a surjective map from the surface to the triangle, and the second one consists of restrictions imposed by folding. We start with the characterisation of the surfaces that can be mapped to a triangle.

**Definition 1.4.** Let $S_1 = (V_1, E_1, F_1, \prec_1)$ and $S_2 = (V_2, E_2, F_2, \prec_2)$ be two simplicial surfaces. A **simplicial map** $\varphi$ is a map

$$\varphi : V_1 \uplus E_1 \uplus F_1 \to V_2 \uplus E_2 \uplus F_2,$$

such that $\varphi(V_1) \subseteq V_2$, $\varphi(E_1) \subseteq E_2$, $\varphi(F_1) \subseteq F_2$ and $x \prec y$ implies $\varphi(x) \prec \varphi(y)$.

If there is a simplicial map that is an inverse of $\varphi$, then $\varphi$ is called **simplicial isomorphism**.

It is important to note that the vertices of a triangle can be coloured with three distinct colours. The same colouring gives necessary and sufficient conditions for the existence of a map from the surface to the triangle.

**Definition 1.5.** Let $S = (V, E, F, \prec)$ be a simplicial surface. A map $c_V : V \to \{1, 2, 3\}$ is called **vertex–3–colouring** if, for every edge $e \in E$ and distinct vertices $v_1, v_2 \in V$ with $v_1 \prec e$ and $v_2 \prec e$, we have $c_V(v_1) \neq c_V(v_2)$.

**Remark 1.6.** A simplicial surface admits a vertex–3–colouring if and only if there exists a simplicial map from the surface to the triangle.

### 2 Folding restrictions

After classifying all maps from a simplicial surface onto a triangle, we incorporate the folding restrictions. To do so, we utilise the edge–colouring which is induced by the vertex–colouring.¹

¹This colouring can be interpreted as proper colouring of the face–edge–graph of the simplicial surface.
Definition 2.1. Let $S = (V, E, F, \prec)$ be a simplicial surface. A map $c_E : E \to \{1, 2, 3\}$ is called edge–3–colouring if, for every face $f \in F$ and distinct edges $e_1, e_2 \in E$ with $e_1 \prec f$ and $e_2 \prec f$, we have $c_E(e_1) \neq c_E(e_2)$.

Every vertex–3–colouring $c_V$ of $S$ defines an edge–3–colouring of $S$ via $c_E(e) = c_V(v_1) + c_V(v_2) - 2$, where $v_1, v_2$ are the vertices incident to $e$. This is called the induced edge–3–colouring.

We note in passing that our induced edge–3–colourings coincide with the mmm–structures considered in [BNPS17].

To formulate the folding restrictions, we need to be more specific about the concept “folding onto a triangle”. Here, we restrict our attention to closed simplicial surfaces.

Intuitively, we want to formalise this picture:

Clearly, the faces are ordered by the folding. We model this as a linear order on the faces of the simplicial surface.

Additionally, real materials usually do not self–intersect. As those intersections can only become relevant at the edges, we focus our attention there. Clearly, if edges of the simplicial surface are mapped to different edges of the triangle, they do not come into conflict with each other. Therefore, we only have to consider whether two edges that lie in the same colour–class of the induced edge–3–colouring, are intersection–free.

Since we only consider closed simplicial surfaces, each of these edges is adjacent to exactly two faces. Since they come from an edge–3–colouring, these are four different faces. If these four faces are ordered by the folding, one of the following cases has to manifest (we depict faces as vertical lines and edges as half–circles):

The edges only intersect in the middle case. Therefore, we can define folding as follows:

Definition 2.2. Let $(V, E, F, \prec)$ be a closed simplicial surface with vertex–3–colouring $c_V$ and induced edge–3–colouring $c_E$. A triangle–folding is a total ordering $<$ on $F$ such that:

If there are two edges with faces $\{f_1, f_2\}$ and $\{g_1, g_2\}$, such that $f_1 < g_1 < f_2 < g_2$ holds, then these edges have different edge colours.

3 Relation between linear and cyclic orders

Folding onto a triangle requires a linear order to be intersection–free. Since it is more convenient to work with a cyclic order, we give a proof of their equivalence.
Definition 3.1. Let $M$ be a finite set. A cyclic order on $M$ is a cycle $\sigma \in \text{Sym}(M)$ of length $|M|$.

From a linear order we can construct a cyclic order by making the smallest element follow the largest one. Conversely, we can cut a cyclic order at any point to obtain a linear order.

Definition 3.2. Let $M$ be a finite set.

1. Let $m_1 < m_2 < \cdots < m_{|M|}$ be a total order on $M$. The induced cyclic order is the cyclic order $\sigma_\prec: M \to M$, $x \mapsto \begin{cases} m_{i+1} & x = m_i \text{ for } i < |M|, \\ m_1 & x = m_{|M|}. \end{cases}$

2. Let $\sigma$ be a cyclic order on $M$. Given $m \in M$, the induced linear order $<_\sigma$ is defined as $m <_\sigma \sigma(m) <_\sigma \sigma^2(m) <_\sigma \cdots <_\sigma \sigma^{|M|-1}(m)$.

A partition can be intersection–free for both linear and cyclic orders.

Definition 3.3. Let $M$ be a finite set and $P$ a partition of two–element–subsets of $M$.

1. Let $\prec$ be a linear order on $M$. We call $P$ intersection–free with respect to $\prec$ if $\{m_1, m_2\}, \{n_1, n_2\} \in P$ implies that $m_1 < n_1 < m_2 < n_2$ is impossible.

2. Let $\sigma$ be a cyclic order on $M$. We call $P$ intersection–free with respect to $\sigma$ if there are no $m \in M$ and $1 \leq i < j < k < |M|$ such that $\{m, \sigma^i(m)\} \in P$ and $\{\sigma^j(m), \sigma^k(m)\} \in P$.

The conversion between linear and cyclic orders does not change the intersections.

Remark 3.4. Let $M$ be a finite set and $P$ a partition of two–element–subsets of $M$.

1. If $\prec$ is a linear order on $M$ and $P$ is intersection–free with respect to $\prec$, then $P$ is also intersection–free with respect to the induced cyclic order $\sigma_\prec$.

2. If $\sigma$ is a cyclic order on $M$ and $P$ is intersection–free with respect to $\sigma$, then $P$ is also intersection–free with respect to any induced linear order $<_\sigma$.

Proof. 1. Assume there are $1 \leq i < j < k$ and $m \in M$ such that $\{m, \sigma_\prec^s(m)\} \in P$ and $\{\sigma_\prec^i(m), \sigma_\prec^k(m)\} \in P$.

Let $n$ be the $\prec$–minimum of $\{m, \sigma_\prec^i(m), \sigma_\prec^j(m), \sigma_\prec^k(m)\}$. Then $n = \sigma_\prec^s(m)$ with $s \in \{0, i, j, k\}$. We consider the case $s = i$ in detail, the other ones are analogous.

We have $\{m, \sigma_\prec^j(m)\} = \{\sigma_\prec^{|M|-i}(n), \sigma_\prec^{j-i}(n)\}$ and $\{\sigma_\prec^i(m), \sigma_\prec^k(m)\} = \{n, \sigma_\prec^{k-i}(n)\}$. Since $n < \sigma_\prec^{j-i}(n) < \sigma_\prec^{k-i}(n) < \sigma_\prec^{|M|-i}(n)$, this contradicts $P$ being intersection–free with respect to $\prec$.

2. This is obvious. □
4 Circle representations

We went to the effort of converting linear and cyclic orders into each other to describe them as permutations. For \( n \in \mathbb{N} \) we use \( n \) to denote the set \( \{1, 2, \ldots, n\} \).

Given a closed simplicial surface with \( 2n \) faces\(^2\) and an edge–3–colouring, each colour class defines a partition of the faces into two–element–subsets (each edge is mapped to the set of its two adjacent faces). This partition can also be represented by a fix–point–free involution in \( \text{Sym}(2n) \).

Example 4.1. Up to renaming the colours, the torus from Example 1.3 has exactly one vertex–3–colouring with colour classes \( \{D\} \), \( \{B, C\} \), and \( \{A\} \).

This induces the edge–3–colouring with colour classes \( \{a, b, c, d\} \), \( \{e, g, i, k\} \), and \( \{f, h, j, l\} \). Each colour class defines a partition of the faces, which can be interpreted as a fix–point–free involution.

| colour class | colour partition | colour involution |
|--------------|------------------|------------------|
| \{a, b, c, d\} | \{2, 7\}, \{3, 6\}, \{1, 4\}, \{5, 8\} | (1,4)(2,7)(3,6)(5,8) |
| \{e, g, i, k\} | \{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\} | (1,2)(3,4)(5,6)(7,8) |
| \{f, h, j, l\} | \{2, 3\}, \{4, 5\}, \{6, 7\}, \{1, 8\} | (1,8)(2,3)(4,5)(6,7) |

Definition 4.2. Let \( (V, E, F, \preceq) \) be a closed simplicial surface with edge–3–colouring \( c_E \).

For each colour \( C \) the colour partition is \( \{\{f \in F \mid e \preceq f\} \mid e \in E, c_E(e) = C\} \) and the colour involution is a map \( \rho : F \to F \) that assigns to each face \( f \) the unique face \( g \) sharing an edge of colour \( C \) with it.

With Definition 3.3 and Definition 4.2 we can reformulate the definition of triangle–folding.

Remark 4.3. Let \( (V, E, F, \prec) \) be a closed simplicial surface with vertex–3–colouring \( c_V \) and induced edge–3–colouring \( c_E \). A total order \( \prec \) on \( F \) is a triangle–folding if and only if all colour partitions of \( c_E \) are intersection–free with respect to \( \prec \).

We want to restate the folding restrictions in terms of a cyclic permutation on the faces and the fix–point–free involutions defined by the edge–3–colouring. Let \( \sigma \in \text{Sym}(2n) \) denote a cyclic permutation of the \( 2n \) faces and suppose that \( \rho \in \text{Sym}(2n) \) is an involution representing an edge colour class.

We will show that \( \rho \) is intersection–free with respect to \( \sigma \) if and only if their product \( \sigma \rho \) has exactly \( n + 1 \) cycles. To prove this claim, we translate the permutations into a geometric setting. We arrange the elements of \( 2n \) into a circle, with the order defined by \( \sigma \). The involution is depicted by connecting two points in the same orbit.

Consider the involutions \( (1, 4)(2, 7)(3, 6)(5, 8) \) and \( (1, 2)(3, 4)(5, 6)(7, 8) \) from the previous example, together with the cyclic order \( (1, 4, 3, 8, 5, 2, 7, 6) \). The first one is intersection–free, the second one is not.

\(^2\)In a closed simplicial surface, the number of faces is always even: By counting the number of edge–face–pairs in two different ways, we obtain that \( 2E = 3F \), where \( E \) is the number of edges and \( F \) is the number of faces.
Definition 4.4. Let \( n \in \mathbb{N} \) and \( \sigma \) be a cyclic order on \( 2n \). Let \( \{ M_j \}_{1 \leq j \leq n} \) be a partition of \( 2n \) into two–element–subsets. A circle representation \( C \) of \( (\sigma, \{ M_j \}_{1 \leq j \leq n}) \) is a triple \( (\iota, \{ Z_k \}_{1 \leq k \leq 2n}, \{ S_j \}_{1 \leq j \leq n}) \) with

1. A map \( \iota : 2n \to \{ x \in \mathbb{C} \mid \| x \| = 1 \} \) with \( \iota(\sigma(m)) = e^{\frac{2\pi i}{2n}} \iota(m) \) for all \( m \in 2n \).
2. The arcs \( Z_k := \{ \iota(k) \cdot e^{\frac{2\pi i x}{2n}} \mid x \in [0, 1] \} \) with origin \( \iota(k) \) and target \( \iota(\sigma(k)) \).
3. The line segments \( S_j := \{ \alpha(x_1) + (1 - \alpha)x_2 \mid M_j = \{ x_1, x_2 \}, \alpha \in [0, 1] \} \).

The circle representation is intersection–free if the line segments are pairwise disjoint.

If \( \{ M_j \}_{1 \leq j \leq n} \) is the set of orbits of a fix–point–free involution \( \rho \), we call \( C \) a circle representation of \( (\sigma, \rho) \).

This notion of intersection–free is connected with the concepts from Definition 3.3.

Lemma 4.5. Let \( \sigma \) be a cyclic order on \( 2n \) and \( \{ M_j \}_{1 \leq j \leq n} \) a partition of \( 2n \) into two–element–subsets. Then the following statements are equivalent:

1. \( \{ M_j \}_{1 \leq j \leq n} \) is intersection–free with respect to \( \sigma \).
2. Any circle representation of \( (\sigma, \{ M_j \}_{1 \leq j \leq n}) \) is intersection–free.

Proof. Consider two line segments \( \overline{\iota(a_1)\iota(a_2)} \) and \( \overline{\iota(a_3)\iota(a_4)} \). Since rotations and reflections do not change the intersection of line segments, we may assume \( \iota(a_1) = 1 \). For \( j \in \{1, 2, 3\} \) we write \( \iota(a_j) = e^{i\varphi_j} \) for some \( 0 < \varphi_j < 2\pi \).

\( \{ a_1, a_2 \} \) and \( \{ a_3, a_4 \} \) intersect with respect to \( \sigma \) if and only if \( 0 < \varphi_3 < \varphi_2 < \varphi_4 \) or \( 0 < \varphi_4 < \varphi_2 < \varphi_3 \) holds.

The line segment \( \overline{\iota(a_1)\iota(a_2)} \) divides the circle \( \{ x \in \mathbb{C} \mid \| x \| = 1 \} \) into two connected components. The two boundary components are parametrized by the intervals \( [0, \varphi_2] \) and \( [\varphi_2, 2\pi] \). We distinguish two cases:

1. If \( \{ \varphi_3, \varphi_4 \} \cap [0, \varphi_2] \) contains only one element, \( \iota(a_3) \) and \( \iota(a_4) \) lie in different connected components. Since the circle is convex, the line segment \( \overline{\iota(a_3)\iota(a_4)} \) lies within the circle, so it has to intersect the line segment \( \overline{\iota(a_1)\iota(a_2)} \).
2. If \( \{ \varphi_3, \varphi_4 \} \cap [0, \varphi_2] \) contains zero or two elements, \( \iota(a_3) \) and \( \iota(a_4) \) lie in the same connected component. Since both connected components are convex, the line segment \( \iota(a_3)\iota(a_4) \) is completely contained in one of them. In particular, it does not intersect the line segment \( \iota(a_1)\iota(a_2) \).

If we consider an intersection–free circle representation of \((\sigma, \rho)\), we observe a correspondence between bounded connected components and orbits of the product \( \rho\sigma \).

It connects a bounded connected component with the origins of the arcs contained in its boundary, which are in turn given as the orbits of \( \rho\sigma \). In the illustration above, the orbits are \( \{1, 3\} \) and \( \{2, 6, 8\} \).

To prove the correspondence, we have to analyse the bounded connected components of a circle representation in detail. This will take the remainder of this section and culminate in theorem 4.12.

**Lemma 4.6.** Let \((\iota, \{Z_k\}_{1 \leq k \leq 2n}, \{S_j\}_{1 \leq j \leq n})\) be an intersection–free circle representation and \( U := \bigcup_{k=1}^{2n} Z_k \cup \bigcup_{j=1}^{n} S_j \). Then \( \mathbb{C} \setminus U \) has exactly \( n+1 \) bounded connected components.

**Proof.** We interpret the sets \( Z_k \) and \( S_j \) as the edges of a planar graph with vertices \( \{\iota(k) | 1 \leq k \leq 2n\} \). This graph has \( 2n \) vertices and \( 2n + n \) edges. Since its \textit{Euler}–characteristic is 1, there are exactly \( n + 1 \) bounded connected components. 

The main technical analysis is contained in the following lemma, which describes the specific form of the bounded connected components explicitly.

**Lemma 4.7.** Let \((\iota, \{Z_k\}_{1 \leq k \leq 2n}, \{S_j\}_{1 \leq j \leq n})\) be an intersection–free circle representation and \( U := \bigcup_{k=1}^{2n} Z_k \cup \bigcup_{j=1}^{n} S_j \). The connected components of the complement \( \mathbb{C} \setminus U \) have the properties:

- There is exactly one unbounded connected component \( \{ x \in \mathbb{C} | ||x|| > 1 \} \). Its boundary is \( \bigcup_{k=1}^{2n} Z_k \).

- There are exactly \( n + 1 \) bounded connected components. The boundary of each bounded connected component has the form \( \bigcup_{i=1}^{m} \left( Z_{k_i} \cup S_{j_i} \right) \) for some \( m \in \mathbb{N} \) such that
  1. The intersections \( Z_{k_t} \cap S_{j_t} \) and \( S_{j_t} \cap Z_{k_{t+1}} \) contain exactly one element for each \( 1 \leq t \leq m \) (we set \( j_{m+1} := j_1 \) for this purpose).

8
2. If $\iota(a_t)$ is the unique element in $S_{j_{t-1}} \cap Z_{k_t}$ and $\iota(b_t)$ is the unique element in $Z_{k_t} \cap S_{j_t}$, then $\sigma(a_t) = b_t$.

This lemma characterises the boundary of the bounded connected components. The first condition states that each boundary consists of an alternating sequence of complete arcs and line segments. The second one defines the order in which this sequence should be traversed.

**Proof.** Since all line segments lie in $\{x \in \mathbb{C} | |x| \leq 1\}$ and $\bigcup_{j=1}^{2n} Z_j = \{x \in \mathbb{C} | |x| = 1\}$, the claim concerning the unbounded connected component is clear.

Let $\kappa$ be a bounded connected component. Since the circle representation is intersection–free, the boundary of $\kappa$ is the union of some $Z_k$ and $S_t$.

1. Let $\iota(k)$ be in the boundary of $\kappa$. This lies in $Z_k$, $Z_{k+1}$ and a line segment. Since the line segment divides the circle $\{x \in \mathbb{C} | |x| \leq 1\}$ into two connected components, it lies in the boundary of exactly two connected components (since they are intersection–free). As $\iota(k)$ is an endpoint for the segment, any connected component which has the line segment as a boundary will also have one of the arcs as boundary. Since the two connected components adjacent to the segment contain different arcs, $Z_k$ and $Z_{k+1}$ cannot be both contained in $\kappa$.

Therefore, for each point $\iota(k)$ in the boundary of $\kappa$, there is exactly one arc and exactly one line segment in the boundary. It follows that arcs and line segments alternate along the boundary of $\kappa$, which shows the first property.

2. We have either $\sigma(a_1) = b_1$ or $\sigma(b_1) = a_1$. If the second equality holds, we invert the ordering of the sets in the boundary. Hence, suppose $\sigma(a_1) = b_1$.

Then $\iota(b_1)$ is the target of an arc. The other point in $S_{j_1}$ is $\iota(a_2)$. This point is the source of the arc $Z_{k_2}$ (otherwise the two arcs would be separated by the line segment). In particular, we have $\sigma(a_2) = b_2$. The full claim follows by induction.

This technical characterisation allows the construction of the bijection between connected components and orbits of a group.

**Lemma 4.8.** Let $\sigma$ be a cyclic order on $2n$ and $\rho$ a fix–point–free involution on $2n$. Let $(\iota, \{Z_k\}_{1 \leq k \leq 2n}, \{S_j\}_{1 \leq j \leq n})$ be an intersection–free circle representation of $(\sigma, \rho)$ and use the notation from Lemma 3.4.

Then we have a bijection between the bounded connected components of $\mathbb{C}\setminus U$ and the orbits of $\rho \sigma$: a connected component with boundary $\bigcup_{k=1}^{m} (Z_{k_t} \cup S_{j_t})$ is mapped to $\bigcup_{t=1}^{m} (S_{j_{t-1}} \cap Z_{k_t})$ (with $j_0 := j_m$), which consists of the origins of the arcs contained in the boundary of the connected component.

**Proof.** The map is well–defined: Let $\iota(a_t)$ be the unique element of $S_{j_{t-1}} \cap Z_{k_t}$, then $\sigma(\iota(a_t))$ is the unique element in $Z_{k_t} \cap S_{j_t}$. In particular, it lies in $S_{j_t}$. Then $\rho \sigma(\iota(a_t))$ is the other endpoint of this line segment, and therefore the unique element in the intersection $S_{j_t} \cap Z_{k_{t+1}}$. Therefore, this image is an orbit of $\rho \sigma$.
To show injectivity, we consider an element $\iota(t)$. This is the source of one arc and the target of another. Since only sources are mapped by our construction, the images of different bounded connected components are disjoint.

For surjectivity, consider an orbit $B$ of $\rho\sigma$. This orbit is not empty, so there is a $b \in B$. This $b$ is the source of one arc $Z$. Then the unique bounded connected component with $Z$ in its boundary is mapped to $B$.

By combining Lemma 4.8 and Lemma 4.9 we have shown that a circle representation of $(\sigma, \rho)$ being intersection–free implies that $\rho\sigma$ has exactly $n+1$ cycles.

We now proceed in the opposite direction: If $\sigma\rho$ has exactly $n+1$ cycles, we construct an intersection–free circle representation of $(\sigma, \rho)$.

To do so, we need some general properties of permutations.

**Lemma 4.9.** Let $\pi \in \text{Sym}(n)$ be a permutation with more than $n/2$ cycles. Then $\pi$ has at least one fixed point.

**Proof.** Suppose $\pi$ has no fixed point. Then every orbit of $\langle \pi \rangle$ contains at least two elements. Since orbits are disjoint, $\pi$ has to contain more than $n/2 \cdot 2$ elements, which is a contradiction.

If $\rho\sigma$ has exactly $n+1$ cycles, it needs to have a fixed point. This is significant: In a circle representation, a fixed point corresponds to a line segment whose end–points have minimal distance. In particular, it can be removed from a circle representation without changing whether it is intersection–free.

To prove this reduction, we first show that after the removal of this line segment there is another fixed point. Then we use an inductive argument to prove the complete claim.

**Lemma 4.10.** Let $\sigma$ be a $2n$–cycle and $\rho$ a fix–point–free involution in $\text{Sym}(2n)$, such that $\rho\sigma$ has exactly $n+1$ cycles ($n > 1$). Let $f \in 2n$ be a fixed point of $\rho\sigma$.

We define permutations on $2n \setminus \{f, \sigma(f)\}$ (remember that $\{f, \sigma(f)\}$ is an orbit of $\rho$):

\[
\hat{\rho}(k) := \rho(k) \\
\hat{\sigma}(k) := \begin{cases} 
\sigma^3(k) & \text{if } \sigma(k) = f \\
\sigma(k) & \text{otherwise}
\end{cases}
\]

Then $\hat{\rho}\hat{\sigma}$ has exactly $n$ cycles.

**Proof.** To show the claim, we analyse how the orbits $B$ of $\rho\sigma$ change.

1. $B = \{f\}$: This orbit is removed.

2. $\sigma(f) \in B$: The only element on which the orbit might change is the precursor of $\sigma(f)$, namely $(\rho\sigma)^{-1}\sigma(f) \in B$. Since $\rho = \rho^{-1}$, we have

\[
(\rho\sigma)^{-1}\sigma(f) = \sigma^{-1}\rho^{-1}\sigma(f) = \sigma^{-1}(\rho\sigma)(f) = \sigma^{-1}(f)
\]

Now we compute its image under the modified permutations:

\[
\hat{\rho}\hat{\sigma}\sigma^{-1}(f) = \rho\sigma^3\sigma^{-1}(f) = (\rho\sigma)(\sigma(f))
\]
In other words, the image of $\sigma^{-1}(f)$ under $\hat{\rho}\hat{\sigma}$ is the same as the image of $\sigma(f)$ under $\rho\sigma$. The new orbit is $B\setminus\{\sigma(f)\}$.

We have to show that this new orbit is not empty. For it to be empty, we would need $\sigma^2(f) = f$, i.e. $n = 1$.

3. $f, \sigma(f) \notin B$: In this case, nothing changes. The only difference can appear if there is an element $k$ of the orbit such that $\sigma(k) = f$. In this case $\rho\sigma(k) = \sigma(f)$.

In total, the number of orbits is reduced by 1.

We now proceed with formulating the induction.

**Lemma 4.11.** Let $\sigma$ be a $2n$–cycle and $\rho$ a fix–point–free involution in $\text{Sym}(2n)$, such that $\rho\sigma$ has exactly $n+1$ cycles. Then every circle representation of $(\sigma, \rho)$ is intersection–free.

**Proof.** We prove the claim by induction on $n$. For $n = 1$, we have $\sigma = \rho = (1,2)$. Therefore, there is only one line segment, which cannot intersect with any other one.

Now assume $n > 1$. By Lemma 4.9 there is a fixed point $f$ of $\rho\sigma$, i.e. $\rho(\sigma(f)) = f$. Denote $g := \sigma(f)$, then we have $\rho(g) = f$.

The line segment between $f$ and $g$ cannot intersect any other line segment, since $f$ and $g$ are neighbours on the circle. We only need to show that the other line segments do not intersect.

To do so, we define $\hat{\rho}$ and $\hat{\sigma}$ as in Lemma 4.10. Their product has exactly $n$ cycles. By the induction hypothesis, their circle representation is intersection–free.

We can now formulate the main theorem.

**Theorem 4.12.** Let $\sigma$ be a $2n$–cycle and $\rho$ a fix–point–free involution on $2n$. Then the following are equivalent:

- The orbits of $\rho$ are intersection–free with regards to $\sigma$.
- $\rho\sigma$ has exactly $n + 1$ orbits on $2n$.

**Proof.** By Lemma 4.5, the orbits of $\rho$ are intersection–free if and only if a circle representation is intersection–free. Combining Lemma 4.6 and Lemma 4.8 then gives one direction of the proof. The other direction follows from Lemma 4.11.

**Corollary 4.13.** Let $(V, E, F, \prec)$ be a closed simplicial surface with vertex–3–colouring $c_V$ and induced edge–3–colouring $c_E$.

There is a triangle–folding of the surface if and only if there exists an $|F|$–cycle $\sigma \in \text{Sym}(F)$ such that its product with all colour involutions has exactly $\frac{|F|}{2} + 1$ cycles.

**Proof.** By Remark 4.3, $\prec$ is a triangle–folding if and only if all colour partitions are intersection–free with respect to $\prec$. By remark 3.4 we can replace $\prec$ by a cyclic order $\sigma$. Since the colour partitions are the orbits of the colour involutions, the claim follows from theorem 4.12.
5 Orientability

In this section, we show that only orientable simplicial surfaces can be folded onto a triangle. This is based on the following observation:

**Remark 5.1.** Let $\overline{xy}$ be a line segment in an intersection–free circle representation. Then $y = e^{2\pi i \frac{k}{2n}} x$ with $k$ odd.

**Proof.** By the definition of circle representations, there is a $k \in \mathbb{Z}$ with $y = e^{2\pi i \frac{k}{2n}} x$.

This line segment divides the circle into two connected components. The boundary of one of them consists precisely of the points $\{e^{2\pi i \frac{l}{2n}} | 0 \leq l \leq k\}$. These points have to be connected in pairs of two, therefore the number of elements in this set has to be even. This is only possible if $k$ is odd.

Since a circle representation subdivides the circle into an even number of points, the concept of even and odd distances makes sense.

**Definition 5.2.** Let $C$ be a circle representation with map $\iota : 2n \to C$. For two elements $a, b \in 2n$ there is a $k \in \mathbb{Z}$ with

$$\iota(y) = e^{2\pi i \frac{k}{2n}} \iota(x).$$

If $k$ is even, $a$ and $b$ have **even distance**. Otherwise, they have **odd distance**.

**Remark 5.3.** All points with even distance from each other form an equivalence class.

Combining Remark 5.1 and Remark 5.3, we get a necessary criterion for foldability:

**Lemma 5.4.** Let $\rho_1, \rho_2, \rho_3$ be three fix–point–free involutions on $2n$ such that there is a cyclic order $\sigma$ that is intersection–free with respect to the orbit partition of each of them.

If $\langle \rho_1, \rho_2, \rho_3 \rangle$ is transitive on $2n$, the even–distance equivalence relation is only dependent on the involutions.

**Proof.** By Remark 5.1, two numbers in an orbit of any $\rho_i$ have to lie in different equivalence classes. Since there are only two classes and $\langle \rho_1, \rho_2, \rho_3 \rangle$ is transitive on $2n$, this determines the class membership of every number.

A more pedestrian formulation would be, whether it is possible to relabel $2n$ in such a way, that all involutions always swap even and odd numbers.

Our next step is the geometric interpretation of this equivalence relation. An orientation maps each face to a cyclic order of its vertices such that the orders of adjacent faces are compatible. For convenience, we only define this for simplicial surfaces with vertex–3–colourings.
Definition 5.5. Let \((V, E, F, \prec)\) be a simplicial surface with vertex-3-colouring \(c_V\). A simplicial orientation is a map \(z : F \to \{(1, 2, 3), (1, 3, 2)\}\), such that, for two faces \(f_1\) and \(f_2\) with a common edge, \(z(f_1) = z(f_2)^{-1}\) holds.

Theorem 5.6. Let \(S\) be a simplicial surface that can be folded onto a triangle. Then \(S\) has a simplicial orientation.

Proof. By Remark 1.6, \(S\) has a vertex-3-colouring and an induced edge-3-colouring with involutions \(\rho_1, \rho_2,\) and \(\rho_3\). We argue for each orbit of \(\langle \rho_1, \rho_2, \rho_3 \rangle\) separately.

By Remark 5.3, the faces fall in two equivalence classes. Mapping one of the classes to \((1, 2, 3)\) and the other one to \((1, 3, 2)\) defines a simplicial orientation.

6 Acknowledgements

I am grateful for the support of my supervisor, Alice Niemeyer, in writing this paper. I am also grateful for the funding by the “Graduiertenkolleg Experimentelle konstruktive Algebra” during the writing of this paper.
References

[ABD+01] Esther M. Arkin, Michael A. Bender, Erik D. Demaine, Martin L. Demaine, Joseph S. B. Mitchell, Saurabh Sethia, and Steven S. Skiena, *When can you fold a map?*, Algorithms and Data Structures: 7th International Workshop, WADS 2001 Providence, RI, USA, August 8–10, 2001 Proceedings (Berlin, Heidelberg) (Frank Dehne, Jörg-Rüdiger Sack, and Roberto Tamassia, eds.), 2001, pp. 401–413.

[BH96] Marshall Bern and Barry Hayes, *The complexity of flat origami*, Proceedings of the Seventh Annual ACM-SIAM Symposium on Discrete Algorithms (Philadelphia, PA, USA), SODA ’96, Society for Industrial and Applied Mathematics, 1996, pp. 175–183.

[BNPS17] Karl-Heinz Brakhage, Alice C. Niemeyer, Wilhelm Plesken, and Ansgar Strzelačzyk, *Simplicial surfaces controlled by one triangle*, 2017. MR 3747985

[BW08] Hani Buri and Yves Weinand, *Origami - folded plate structures, architecture*, 10th WCTE (2008), 1–8.

[Deh16] Max Dehn, *Über die starreheit konvexer polyeder*, Mathematische Annalen 77 (1916), 466–473 (Deutsch).

[DLRS10] Jesus A. De Loera, Jorg Rambau, and Francisco Santos, *Triangulations: Structures for algorithms and applications*, 1st ed., Springer Publishing Company, Incorporated, 2010.

[DO10] Erik D. Demaine and Joseph O’Rouke, *Geometric Folding Algorithms, Linkages, Origami, Polyhedra*, Cambridge University Press, 32 Avenue of the Americas, New York, 2010.

[GY03] Jonathan L. Gross and Jay Yellen, *Handbook of graph theory*, Discrete Mathematics and its Applications, CRC Press, 2003.

[Hul94] Thomas Hull, *On the mathematics of flat origamis*, Congressus Numerantium 100 (1994), 215–224.

[IKN17] Ivan Izmestiev, Steven Klee, and Isabella Novik, *Simplicial moves on balanced complexes*, Advances in Mathematics 320 (2017), 82 – 114.

[Izm14] Ivan Izmestiev, *Classification of flexible kokotsakis polyhedra with quadrangular base*, 2014.

[JL19] Robert J. Lang, *Origami and geometric constructions*, published on http://www.langorigami.com 03 2019.

[LO09] R.J. Lang and OrigamiUSA., *Origami 4*, Origami (AK Peters), Taylor & Francis, 2009.
[Pol]  Konrad Polthier, *Polyhedral surfaces of constant mean curvature*, Habilitationsschrift, TU-Berlin (Febr. 2002), 1-212.

[SG31]  R. Sauer and H. Graf, *Über flächenverbiegung in analogie zur verknickung offener facettenfläche*, Mathematische Annalen 105 (1931), no. 1, 499–535.

[SMH+16]  H. Shigemune, S. Maeda, Y. Hara, N. Hosoya, and S. Hashimoto, *Origami robot: A self-folding paper robot with an electrothermal actuator created by printing*, IEEE/ASME Transactions on Mechatronics 21 (2016), no. 6, 2746–2754.

[Sta10]  Hellmuth Stachel, *A kinematic approach to kokotsakis meshes*, Computer Aided Geometric Design 27 (2010), no. 6, 428 – 437.

[Tac09]  Tomohiro Tachi, *Generalization of rigid foldable quadrilateral mesh origami*, Proc. Int. Assoc. Shell and Spatial Struct.(Valencia, Spain), 2009, pp. 2287–2294.

[WY10]  Weina Wu and Zhong You, *Modelling rigid origami with quaternions and dual quaternions*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 466 (2010), no. 2119, 2155–2174.