ON BLACK HOLE SOLUTIONS
IN MODEL WITH ANISOTROPIC FLUID

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Abstract

A family of spherically symmetric solutions in the model with 1-component anisotropic fluid is considered. The metric of the solution depends on a parameter $q > 0$ relating radial pressure and the density and contains $n - 1$ parameters corresponding to Ricci-flat “internal space” metrics. For $q = 1$ and certain equations of state ($p_i = \pm \rho$) the metric coincides with the metric of black brane solutions in the model with antisymmetric form. A family of black hole solutions corresponding to natural numbers $q = 1, 2, \ldots$ is singled out. Certain examples of solutions (e.g. containing for $q = 1$ Reissner-Nordström, $M2$ and $M5$ black brane metrics) are considered. The post-Newtonian parameters $\beta$ and $\gamma$ corresponding to the 4-dimensional section of the metric are calculated.

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1 Introduction

Currently, there is a certain interest to $p$-brane solutions with horizon (see, for example, [1] and references therein) defined on product manifolds $\mathbb{R} \times M_0 \times \ldots \times M_n$. $P$-brane solutions (e.g. black brane ones) usually appear in the models with antisymmetric forms and scalar fields (see also [4]-[14]). Cosmological and spherically symmetric solutions with $p$-branes are usually obtained by the reduction of the field equations to the Lagrange equations corresponding to Toda-like systems [13]. An analogous reduction for the models with multicomponent "perfect" fluid was done earlier in [17, 18]. Earlier extensions of the Schwarzschild, Tangerlini, Reissner-Nordstrom and Majumdar-Papapetrou solutions to diverse dimensions see in [2, 3].

For cosmological models with antisymmetric forms without scalar fields any $p$-brane is equivalent to an anisotropic perfect fluid with the equations of state:

$$p_i = -\rho, \quad \text{or} \quad p_i = \rho,$$

when the manifold $M_i$ belongs or does not belong to the brane worldvolume, respectively (here $p_i$ is the pressure in $M_i$ and $\rho$ is the density, see Section 2).

In this paper we use this analogy in order to find a new family of exact spherically-symmetric solutions in the model with 1-component anisotropic fluid for more general equations of state (see Appendix for more familiar form of eqs. of state):

$$p_r = -\rho(2q - 1)^{-1}, \quad p_0 = \rho(2q - 1)^{-1},$$

and

$$p_i = \left(1 - \frac{2U_i}{d_i}\right)\rho/(2q - 1),$$

$i > 1$, where $\rho$ is a density, $p_r$ is a radial pressure, $p_i$ is a pressure in $M_i$, $i = 2, \ldots, n$. Here parameters $U_i$ ($i > 1$) are arbitrary and the parameter $q > 0$ obey $q \neq 1/2$. The manifold $M_0$ is $d_0$-dimensional sphere in our case and $p_0$ is the pressure in the tangent direction. The case $q = 1$ was considered earlier in [20].

The paper is organized as follows. In Section 2 the model is formulated. In Section 3 a subclass of spherically symmetric solutions (generalizing solutions from [20]) is presented and black hole solutions with integer $q$ are singled out.
Section 4 deals with certain examples of solutions containing for \( q = 1 \) the Reissner-Nordström metric, \( M2 \) and \( M5 \) black brane metrics. In Section 5 the post-Newtonian parameters for the 4-dimensional section of the metric are calculated.

2 The model

Here, we consider a family of spherically symmetric solutions to Einstein equations with an anisotropic fluid matter source

\[
R^M_N - \frac{1}{2} \delta^M_N R = kT^M_N
\]  

(2.1)

defined on the manifold

\[
M = \mathbb{R} \times (M_0 = S_{d_0}) \times \frac{M_1 = \mathbb{R}}{\text{time}} \times M_2 \times \ldots \times M_n,
\]

(2.2)

with the block-diagonal metrics

\[
ds^2 = e^{2\gamma(u)} du^2 + \sum_{i=0}^{n} e^{2X_i(u)} h^{(i)}_{m_i n_i} dy_{m_i} dy_{n_i}.
\]

(2.3)

Here \( \mathbb{R} = (a, b) \) is interval. The manifold \( M_i \) with the metric \( h^{(i)} \), \( i = 1, 2, \ldots, n \), is the Ricci-flat space of dimension \( d_i \):

\[
R_{m_i n_i} [h^{(i)}] = 0,
\]

(2.4)

and \( h^{(0)} \) is standard metric on the unit sphere \( S_{d_0} \)

\[
R_{m_0 n_0} [h^{(0)}] = (d_0 - 1) h^{(0)}_{m_0 n_0},
\]

(2.5)

\( u \) is radial variable, \( \kappa \) is the multidimensional gravitational constant, \( d_1 = 1 \) and \( h^{(1)} = -dt \otimes dt \).

The energy-momentum tensor is adopted in the following form

\[
(T^M_N) = \text{diag}(-(2q-1)^{-1} \rho, (2q-1)^{-1} \rho \delta^{m_0}_{k_0}, -\rho, -p_2 \delta^{m_2}_{k_2}, \ldots, -p_n \delta^{m_n}_{k_n}),
\]

(2.6)

where \( q > 0 \) and \( q \neq 1/2 \). The pressures \( p_i \) and the density \( \rho \) obeys the relations (1.3) with arbitrary constants \( U_i, i > 1 \).

In what follows we put \( \kappa = 1 \) for simplicity.
3 Exact solutions

Let us define

1. $U_0 = 0$, \hspace{1cm} (3.1)
2. $U_1 = q$, \hspace{1cm} (3.2)
3. $(U, U) = U_i G^{ij} U_j$, \hspace{1cm} (3.3)

where $U = (U_i)$ is $(n+1)$-dimensional vector and

$$G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D}$$ \hspace{1cm} (3.4)

are components of the matrix inverse to the matrix of the minisuperspace metric \[15, 16\]

$$(G_{ij}) = (d_i \delta_{ij} - d_i d_j)$$ \hspace{1cm} (3.5)

and $D = 1 + \sum_{i=0}^{n} d_i$ is the total dimension.

In our case the scalar product (3.3) reads

$$(U, U) = q^2 + \sum_{i=2}^{n} \frac{U_i^2}{d_i} + \frac{1}{2 - D} \left( q + \sum_{i=2}^{n} U_i \right)^2$$ \hspace{1cm} (3.6)

It is proved in Appendix that the relation 1° implies $(U, U) > 0$.

For the equations of state (1.2) and (1.3) we have obtained the following spherically symmetric solutions to the Einstein equations (2.1) (see Appendix)

$$ds^2 = J_0 \left( \frac{dr^2}{1 - 2\mu r^d} + r^2 d\Omega^2_{d_0} \right) - J_1 \left( 1 - \frac{2\mu}{r^d} \right) dt^2$$ \hspace{1cm} (3.7)

$$+ \sum_{i=2}^{n} J_i h_{m_i n_i}^{(i)} dy^{m_i} dy^{n_i},$$

$$\rho = \frac{(2q - 1)(dq)^2 P(P + 2\mu)(1 - 2\mu r^{-d}) q^{-1}}{2(U, U) H^2 J_0 r^{2d_0}},$$ \hspace{1cm} (3.8)

by methods similar to obtaining $p$-brane solution [13]. Here $d = d_0 - 1$, $d\Omega^2_{d_0} = h^{(0)}_{ma n_0} dy^{m_0} dy^{n_0}$ is spherical element, the metric factors
\[ J_i = H^{-2U_i/(U,U)}, \quad H = 1 + \frac{P}{2\mu} \left[ 1 - \left( 1 - \frac{2\mu}{r^d} \right)^q \right] ; \quad (3.9) \]

\[ P > 0, \mu > 0 \] are constants and

\[ U^i = G^{ij}U_j = \frac{U_i}{d_i} + \frac{1}{2 - D} \sum_{j=0}^n U_j. \quad (3.10) \]

Using (3.10) and \( U_0 = 0 \) we get

\[ U^0 = \frac{1}{2 - D} \sum_{j=0}^n U_j \quad (3.11) \]

and hence one can rewrite (3.7) as follows

\[ ds^2 = J_0 \left[ \frac{dr^2}{1 - \frac{2\mu}{r^d}} + r^2d\Omega^2 - H^{-2q/(U,U)} \left( 1 - \frac{2\mu}{r^d} \right) dt^2 + \right. \]

\[ + \sum_{i=2}^n H^{-2U_i/(d_i(U,U))} h_m^{(i)} \delta_{m_i} dy^{m_i} dy^{n_i} \left] . \quad (3.12) \]

**Remark 1.** We note that the density \( \rho \) is positive for \( 2q > 1 \) and negative for \( 2q < 1 \). For \( 2q = 1 \) the solution also exists. In this case \( \rho = 0 \) and the energy-momentum tensor should be rewritten in terms of \( p_0 \)

\[ (T^M_N) = \text{diag}(-p_0, p_0 \delta_{k_0}^{m_0}, 0, p_2 \delta_{k_2}^{m_2}, \ldots, p_n \delta_{k_n}^{m_n}), \quad (3.13) \]

where

\[ p_0 = \frac{d^2P(P + 2\mu)(1 - 2\mu r^{-d})^{-1/2}}{8(U,U)H^2J_0r^{2d_0}}. \quad (3.14) \]

**Black holes for natural** \( q \). For natural

\[ q = 1, 2, \ldots \quad (3.15) \]

the metric has a horizon at \( r^d = 2\mu = r_h^2 \). Indeed, for these values of \( q \) the function \( H(r) > 0 \) is smooth in the interval \((r_*, +\infty)\) for some \( r_* < r_h \). (For odd \( q = 2m + 1 \) one get \( r_* = 0 \). A global structure of the black hole solution corresponding to these values of \( q \) will be a subject of a separate publication.
For $2U^0 \neq -1$ and $0 < q < 1$ we get a singularity $r^d \to 2\mu$. Indeed, due to Einstein equations the scalar curvature of the metric is proportional to

$$T^M_M = [D - 2 - 2 \sum_{j=0}^n U_j] p_0 = (1 + 2U^0)(D - 2)p_0 \quad (3.16)$$

but $p_0$ (proportional to $\rho$) diverges when $r^d \to 2\mu$ for $q < 1$ (see (3.8)).

**Remark 2.** For non-integer $q > 1$ the function $H(r)$ has a non-analytical behaviour in the vicinity of $r^d = 2\mu$. In this case one may conject that the limit $r^d \to 2\mu$ also corresponds to singularity but this subject needs a separate investigation.

4 Examples

Up till now $U_i$ were arbitrary in our solution.

Here we consider certain examples of the solution with

$$U_2 = q d_2, \quad U_i = 0, \quad (4.1)$$

for $i > 2$ and the equations of state: $p_2 = -\rho$, and $p_j = \rho/(2q - 1)$ for $j > 2$. The $q = 1$ case describing the fluid analogue of $p$-brane solution with $p = d_2$ was considered in [20].

4.1 Solutions for $D = 4$

Let us consider the 4-dimensional space-time manifold $\mathbb{R} \times S^2 \times \mathbb{R}$. The metric and the density $\rho$ from (3.12) and (3.8) read

$$ds^2 = H^{2/q} \left[ \frac{dr^2}{1 - \frac{2\mu}{r}} + r^2 d\Omega^2_2 - H^{-4/q} \left( 1 - \frac{2\mu}{r} \right) dt^2 \right], \quad (4.2)$$

$$\rho = \frac{(2q - 1)P(P + 2\mu)}{H^{2+(2/q)}r^4} \left( 1 - \frac{2\mu}{r} \right)^{q-1}. \quad (4.3)$$

Here $H = 1 + \frac{P}{2\mu} \left[ 1 - (1 - \frac{2\mu}{r})^q \right]$. For $q = 1$ by changing of variable $r = r' - P$ we obtain the standard Reissner-Nordström metric.
\[
ds^2_{RN} = - \left( 1 - \frac{2GM}{r'} + \frac{Q^2}{r'^2} \right) dt^2 + \left( 1 - \frac{2GM}{r'} + \frac{Q^2}{r'^2} \right)^{-1} dr'^2 + r'^2 d\Omega^2 \quad (4.4)
\]

with the charge squared \( Q^2 = P(P+2\mu) \) and the gravitational radius \( GM = P + \mu \). Here, obviously, \( Q^2 < (GM)^2 \).

4.2 Solutions for \( D = 11 \)

Here we consider two examples of solutions for the case \( D = 11 \) and \( n = 3 \). These solutions are generalizations of solutions for \( q = 1 \) from [20].

\((M2)_q\)-solutions. For \( U_2 = qd_2 = 2q \), \( U_3 = 0 \) we get from (3.12):

\[
ds^2 = H^{1/(3q)} \left[ \frac{dr'^2}{1 - \frac{2\mu}{r'}} + r'^2 d\Omega^2_{d0} - H^{-1/q} \left( 1 - \frac{2\mu}{r'^d} \right) dt^2 \right. \\
+ H^{-1/q} h^{(2)}_{m_2n_2} dy^{m_2} dy^{n_2} + h^{(3)}_{m_3n_3} dy^{m_3} dy^{n_3} \bigg].
\]

For \( q = 1 \) this formula gives the metric of the electric \( M2 \) black brane solution in 11-dimensional supergravity [8]. The density (3.8) has the following form:

\[
\rho = \frac{(2q-1) d^2 P(P + 2\mu)(1 - 2\mu r^{-d})^{q-1}}{4H^{2+(1/3q)} r^{2d_0}}. \quad (4.6)
\]

\((M5)_q\)-solution. Now we put \( U_2 = qd_2 = 5q \). The metric reads:

\[
ds^2 = H^{2/(3q)} \left[ \frac{dr'^2}{1 - \frac{2\mu}{r'}} + r'^2 d\Omega^2_{d0} - H^{-1/q} \left( 1 - \frac{2\mu}{r'^d} \right) dt^2 + \\
H^{-1/q} h^{(2)}_{m_2n_2} dy^{m_2} dy^{n_2} + h^{(3)}_{m_3n_3} dy^{m_3} dy^{n_3} \bigg],
\]

and the density is

\[
\rho = \frac{(2q-1) d^2 P(P + 2\mu)(1 - 2\mu r^{-d})^{q-1}}{4H^{2+(2/3q)} r^{2d_0}}. \quad (4.8)
\]

For \( q = 1 \) we get the metric of the magnetic \( M5 \) black brane solution in 11-dimensional supergravity [8].
5 Physical parameters

5.1 Gravitational mass and PPN parameters

Here we put $d_0 = 2$ ($d = 1$). Let us consider the 4-dimensional space-time section of the metric (3.12). Introducing a new radial variable by the relation:

$$r = R \left(1 + \frac{\mu}{2R}\right)^2,$$

we rewrite the 4-section in the following form:

$$ds^2_{(4)} = H^{-2U/(U,U)} \left[ -H^{-2q/(U,U)} \left( \frac{1 - \frac{\mu}{2R}}{1 + \frac{\mu}{2R}} \right)^2 dt^2 + \left(1 + \frac{\mu}{2R}\right)^4 \delta_{ij} dx^i dx^j \right]$$  \hspace{1cm} (5.2)

$i, j = 1, 2, 3$. Here $R^2 = \delta_{ij} x^i x^j$.

The parametrized post-Newtonian (Eddington) parameters are defined by the well-known relations

$$g^{(4)}_{00} = -(1 - 2V + 2\beta V^2) + O(V^3), \hspace{1cm} (5.3)$$
$$g^{(4)}_{ij} = \delta_{ij}(1 + 2\gamma V) + O(V^2), \hspace{1cm} (5.4)$$

$i, j = 1, 2, 3$. Here

$$V = \frac{GM}{R}$$ \hspace{1cm} (5.5)

is the Newtonian potential, $M$ is the gravitational mass and $G$ is the gravitational constant.

From (5.2)-(5.4) we obtain:

$$GM = \mu + \frac{Pq(q + U^0)}{(U,U)}$$ \hspace{1cm} (5.6)

and

$$\beta - 1 = \frac{|A|}{(GM)^2}(q + U^0), \hspace{1cm} (5.7)$$
$$\gamma - 1 = -\frac{Pq}{(U,U)GM}(q + 2U^0), \hspace{1cm} (5.8)$$
where

\[ |A| = \frac{1}{2}q^2 P(P + 2\mu)/(U,U) \text{ (see Appendix), or, equivalently,} \]

\[ P = -\mu + \sqrt{\mu^2 + 2|A|(U,U)q^{-2}} > 0. \]

For fixed \( U \), the parameter \( \beta - 1 \) is proportional to the ratio of two physical parameters: the anisotropic fluid density parameter \( |A| \) and the gravitational radius squared \( (GM)^2 \). For compact internal spaces the parameter \( |A| \) is proportional to the effective mass of the fluid outside the external horizon (for natural \( q \)), i.e. to the integral of \( \rho \) over the region \( r^d > 2\mu \).

### 5.2 Hawking temperature

The Hawking temperature of the black hole may be calculated using the well-known relation [19]

\[ T_H = \frac{1}{4\pi \sqrt{-g_{tt}g_{rr}}} \left. \frac{d(-g_{tt})}{dr} \right|_{\text{horizon}}. \quad (5.9) \]

We get

\[ T_H = \frac{d}{4\pi(2\mu)^{1/d}} \left( 1 + \frac{P}{2\mu} \right)^{-q/(U,U)} . \quad (5.10) \]

Here \( q = 1, 2, \ldots \).

For the 4-dimensional solution (4.2) we get \( T_H = \frac{1}{8\pi \mu} \left( 1 + \frac{P}{2\mu} \right)^{-2/q} \). For \( D = 11 \) metrics (4.5) and (4.7) the Hawking temperature reads

\[ T_H = \frac{d}{4\pi(2\mu)^{1/d}} \left( 1 + \frac{P}{2\mu} \right)^{-1/(2q)} . \]

### 6 Conclusions

In this paper, using our methods developed earlier for obtaining perfect fluid and p-brane solutions, we have considered a family of spherically symmetric solutions in the model with 1-component anisotropic fluid when the equations
of state (1.2) and (1.3) are imposed. The metric of any solution contains \((n - 1)\) Ricci-flat ”internal” space metrics and depends upon arbitrary parameters \(U_i, i > 1\).

For \(q = 1\) and certain equations of state (with \(p_i = \pm \rho\)) the metric of the solution coincides with that of black brane (or black hole) solution in the model with antisymmetric forms without dilatons [20]. For natural numbers \(q = 1, 2, \ldots\) we obtained a family of black hole solutions.

Here we also considered certain examples of solutions with horizon, e.g. (fluid) generalizations of charged black hole and \(M2, M5\) black brane solutions.

We have also calculated for possible estimations of observable effects of extra dimensions the post-Newtonian parameters \(\beta\) and \(\gamma\) corresponding to the 4-dimensional section of the metric and the Hawking temperature as well. The parameter \(\beta - 1\) is written in terms of ratios of the physical parameters: the perfect fluid parameter \(|A|\) and the gravitational radius squared \((GM)^2\).

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Appendix

A Lagrange representation

It is more convenient for finding of exact solutions, to write the stress-energy tensor in cosmological-type form

\[
(T^M_N) = \text{diag}(-\hat{\rho}, \hat{\rho}_0 \delta^m_0, \hat{p}_1 \delta^m_1, \ldots, \hat{p}_n \delta^m_n),
\]  

(A.1)

where \(\hat{\rho}\) and \(\hat{p}_i\) are "effective" density and pressures, respectively, depending upon the radial variable \(u\) and the physical density \(\rho\) and pressures \(p_i\) are related to the effective ("hat") ones by formulas

\[
\rho = -\hat{p}_1, \quad p_r = -\hat{\rho}, \quad p_i = \hat{p}_i, \quad (i \neq 1).
\]  

(A.2)

The equations of state may be written in the following form

\[
\hat{p}_i = (1 - 2U_i d_i) \hat{\rho},
\]  

(A.3)

where \(U_i\) are constants, \(i = 0, 1, 2, \ldots, n\). It follows from (A.2), (A.3) and \(U_1 = q\) that

\[
\rho = (2q - 1)\hat{\rho}.
\]  

(A.4)

The conservation law equations \(\nabla_M T^M_N = 0\) (following from Einstein equations) may be written, due to relations (2.3) and (A.1) in the following form:

\[
\dot{\hat{\rho}} + \sum_{i=0}^n d_i \dot{X}^i (\hat{\rho} + \hat{p}_i) = 0.
\]  

(A.5)

Using the equation of state (A.3) we get

\[
\dot{\hat{\rho}} = -A e^{2U_i X^i - 2\gamma_0},
\]  

(A.6)

where \(\gamma_0(X) = \sum_{i=0}^n d_i X^i\) and \(A\) is constant.

The Einstein equations (2.1) with the relations (A.3) and (A.6) imposed are equivalent to the Lagrange equations for the Lagrangian

\[
L = \frac{1}{2} e^{-\gamma + \gamma_0(X)} G_{ij} \dot{X}^i \dot{X}^j - e^{\gamma - \gamma_0(X)} V,
\]  

(A.7)
where

\[ V = \frac{1}{2} d_0 (d_0 - 1) e^{2U_i^{(0)} X^i} + A e^{2u_i X^i} \]  

(A.8)
is the potential and the components of the minisupermetric \( G_{ij} \) are defined in (3.5).

\[ U_i^{(0)} X^i = -X^0 + \gamma_0(X), \quad U_i^{(0)} = -\delta_i^0 + d_i, \quad A_0 = \frac{1}{2} d_0 (d_0 - 1), \]  

(A.9)
i = 0, \ldots, n \) (for cosmological case see [17, 18]).

For \( \gamma = \gamma_0(X) \), i.e. when the harmonic time gauge is considered, we get the set of Lagrange equations for the Lagrangian

\[ L = \frac{1}{2} G_{ij} \ddot{X}^i \ddot{X}^j - V, \]  

(A.10)

with the zero-energy constraint imposed

\[ E = \frac{1}{2} G_{ij} \dot{X}^i \dot{X}^j + V = 0. \]  

(A.11)

It follows from the restriction \( U_0 = 0 \) that

\[ (U^{(0)}, U) \equiv U_i^{(0)} G^{ij} U_j = 0. \]  

(A.12)

Indeed, the contravariant components \( U_i^{(0)} = G^{ij} U_j \) are the following ones

\[ U^{(0)i} = -\frac{\delta^i_0}{d_0}. \]  

(A.13)

Then we get \( (U^{(0)}, U) = U^{(0)i} U_i = -U_0/d_0 = 0 \). In what follows we also use the formula

\[ (U^{(0)}, U^{(0)}) = \frac{1}{d_0} - 1 < 0, \]  

(A.14)

for \( d_0 > 1 \).

Now we prove that \( (U, U) > 0 \). Indeed, minisupermetric has the signature \((-\ldots, +, \ldots, +) \) [15, 16], vector \( U^{(0)} \) is time-like and orthogonal to vector \( U \neq 0 \). Hence the vector \( U \) is space-like.
B General spherically symmetric solutions

When the orthogonality relations (A.12) and 3 of (3.1) are satisfied the Euler-Lagrange equations for the Lagrangian (A.10) with the potential (A.8) have the following solutions (see relations from [18] adopted for our case):

\[ X^i(u) = -\sum_{\alpha=0}^{1} \frac{U^{(\alpha)i}}{(U^{(\alpha)\alpha})} \ln |f_{\alpha}(u - u_{\alpha})| + c^i u + \bar{c}^i, \]  \hspace{1cm} (B.15)

where \( U^{(1)} = U \), \( u_{\alpha} \) are integration constants; and vectors \( c = (c^i) \) and \( \bar{c} = (\bar{c}^i) \) are orthogonal to the \( U^{(\alpha)} = (U^{(\alpha)i}) \), i.e. they satisfy the linear constraint relations

\[ U^{(0)}(c) = U^{(0)}_i c^i = -c^0 + \sum_{j=0}^{n} d_j c^j = 0, \]  \hspace{1cm} (B.16)

\[ U^{(0)}(\bar{c}) = U^{(0)}_i \bar{c}^i = -\bar{c}^0 + \sum_{j=0}^{n} d_j \bar{c}^j = 0, \]  \hspace{1cm} (B.17)

\[ U(c) = U_i c^i = 0, \]  \hspace{1cm} (B.18)

\[ U(\bar{c}) = U_i \bar{c}^i = 0. \]  \hspace{1cm} (B.19)

Here

\[ f_{\alpha}(\tau) = R_{\alpha} \frac{\sinh(\sqrt{C_{\alpha}} \tau)}{\sqrt{C_{\alpha}}}, \quad C_{\alpha} > 0, \quad \eta_{\alpha} = +1, \]

\[ R_{\alpha} \frac{\cosh(\sqrt{C_{\alpha}} \tau)}{\sqrt{C_{\alpha}}}, \quad C_{\alpha} > 0, \quad \eta_{\alpha} = -1, \]  \hspace{1cm} (B.20)

\[ R_{\alpha} \frac{\sin(\sqrt{|C_{\alpha}|} \tau)}{\sqrt{|C_{\alpha}|}}, \quad C_{\alpha} < 0, \quad \eta_{\alpha} = +1, \]

\[ R_{\alpha} \tau, \quad C_{\alpha} = 0, \quad \eta_{\alpha} = +1, \]

\( \alpha = 0, 1; \) where \( R_0 = d_0 - 1, \eta_0 = 1, R_1 = \sqrt{2|A|}(|U, U|), \eta_1 = -\text{sign}A. \)

The zero-energy constraint, corresponding to the solution (B.15) reads

\[ E = \frac{1}{2} \sum_{\alpha=0}^{1} \frac{C_{\alpha}}{(U^{(\alpha)\alpha})} + \frac{1}{2} G_{ij} \bar{c}^i c^j = 0. \]  \hspace{1cm} (B.21)
**Special solutions.** The horizon condition (i.e. infinite time of propagation of light for $u \to +\infty$) lead us to the following integration constants

\begin{align*}
\bar{c}^i &= 0, \\
\tilde{c}^i &= \bar{\mu} \sum_{\alpha=0}^{1} \frac{U_1^{(\alpha)} U^{(\alpha)i}}{(U^{(\alpha)}, U^{(\alpha)})} - \bar{\mu} \delta^i_1, \\
C_\alpha &= (U_1^{(\alpha)})^2 \bar{\mu}^2,
\end{align*}

(B.22, B.23, B.24)

where $\bar{\mu} > 0$, $\alpha = 0, 1$.

We also introduce a new radial variable $r = r(u)$ by relations

\begin{equation}
\exp(-2\bar{\mu}u) = 1 - \frac{2\mu}{r^d}, \quad \mu = \bar{\mu} / d > 0, \quad d = d_0 - 1,
\end{equation}

(B.25)

and put $u_1 < 0$, $A < 0$, $u_0 = 0$.

The relations of the Appendix imply the formulae (3.7) and (3.8) for the solution from Section 3 with

\begin{align*}
H &= \exp(-\bar{\mu}U_1 u) f_1(u - u_1), \\
A &= -\frac{(dq)^2}{2(U, U)} P(P + 2\mu),
\end{align*}

(B.26)

$P > 0$. 

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