Fixed-parameter Approximability of Boolean MinCSPs

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Abstract. The minimum unsatisfiability version of a constraint satisfaction problem (MinCSP) asks for an assignment where the number of unsatisfied constraints is minimum possible, or equivalently, asks for a minimum-size set of constraints whose deletion makes the instance satisfiable. For a finite set $\Gamma$ of constraints, we denote by $\text{MinCSP}(\Gamma)$ the restriction of the problem where each constraint is from $\Gamma$. The polynomial-time solvability and the polynomial-time approximability of $\text{MinCSP}(\Gamma)$ were fully characterized by Khanna et al. [32]. Here we study the fixed-parameter (FP-) approximability of the problem: given an instance and an integer $k$, one has to find a solution of size at most $g(k)$ in time $f(k) \cdot n^{O(1)}$ if a solution of size at most $k$ exists. We especially focus on the case of constant-factor FP-approximability. Our main result classifies each finite constraint language $\Gamma$ into one of three classes:

1. $\text{MinCSP}(\Gamma)$ has a constant-factor FP-approximation.
2. $\text{MinCSP}(\Gamma)$ has a (constant-factor) FP-approximation if and only if Nearest Codeword has a (constant-factor) FP-approximation.
3. $\text{MinCSP}(\Gamma)$ has no FP-approximation, unless FPT = W[1].

We give some evidence that problems in the second class do not have constant-factor FP-approximations: we show that there is no such approximation if both the Exponential-Time Hypothesis (ETH) and the Linear PCP Conjecture (LPC) are true, and we also show that such an approximation would imply the existence of an FP-approximation for the $k$-Densest Subgraph problem with ratio $1 - \epsilon$ for any $\epsilon > 0$.

1 Introduction

Satisfiability problems and, more generally, Boolean constraint satisfaction problems (CSPs) are basic algorithmic problems arising in various theoretical and applied contexts. An instance of a Boolean CSP consists of a set of Boolean variables and a set of constraints; each constraint restricts the allowed combination of values that can appear on a certain subset of variables. In the decision version of the problem, the goal is to find an assignment that simultaneously satisfies every constraint. One can also define optimization versions of CSPs: the goal can be to find an assignment that maximizes the number of satisfied constraints, minimizes the number of unsatisfied constraints, maximizes/minimizes the weight (number of 1s) of the assignment, etc. [19].

Since these problems are usually NP-hard in their full generality, a well-established line of research is to investigate how the complexity of the problem changes for restricted versions of the problem. A large body of research deals with language-based restrictions: given any finite set $\Gamma$ of Boolean constraints, one can consider the special case where each constraint is restricted to be a member of $\Gamma$. The ultimate research goal of this approach is to prove a dichotomy theorem: a complete classification result that specifies for each fine constraint set $\Gamma$ whether the restriction to $\Gamma$ yields an easy or hard problem. Numerous classification theorems of this form have been proved for various decision and optimization versions for Boolean and non-Boolean CSPs [16,13,10,11,9,12,8,25,31,33,47,37]. In particular, for $\text{MinCSP}(\Gamma)$, which is the optimization problem asking for an assignment minimizing the number of unsatisfied constraints, Creignou et al. [19] obtained a classification of the approximability for every finite Boolean constraint language $\Gamma$. The goal of this paper is to characterize the approximability of Boolean $\text{MinCSP}(\Gamma)$ with respect to the more relaxed notion of fixed-parameter approximability.

Parameterized complexity [26,28,22] analyzes the running time of a computational problem not as a univariate function of the input size $n$, but as a function of both the input size $n$ and a relevant parameter $k$ of the input. For example, given a $\text{MinCSP}$ instance of size $n$ where we are looking for a

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solution satisfying all but $k$ of the constraints, it is natural to analyze the running time of the problem as a function of both $n$ and $k$. We say that a problem with parameter $k$ is fixed-parameter tractable (FPT) if it can be solved in time $f(k) \cdot n^{O(1)}$ for some computable function $f$ depending only on $k$. Intuitively, even if $f$ is, say, an exponential function, this means that problem instances with "small" $k$ can be solved efficiently, as the combinatorial explosion can be confined to the parameter $k$. This can be contrasted with algorithms with running time of the form $n^{O(k)}$ that are highly inefficient even for small values of $k$. There are hundreds of parameterized problems where brute force gives trivial $n^{O(k)}$ algorithms, but the problem can be shown to be FPT using nontrivial techniques; see the recent textbooks by Downey and Fellows [26] and by Cygan et al. [22]. In particular, there are fixed-parameter tractability results and characterization theorems for various CSPs [37, 13, 34, 35].

The notion of fixed-parameter tractability has been combined with the notion of approximability [15, 17, 27, 14, 18]. Following [16, 38], we say that a minimization problem is fixed-parameter approximable (FPA) if there is an algorithm that, given an instance and an integer $k$, in time $f_1(k) \cdot n^{O(1)}$ either returns a solution of cost at most $f_2(k) \cdot k$, or correctly states that there is no solution of cost at most $k$. The two crucial differences compared to the usual setup of polynomial-time approximation is that (1) the running time is not polynomial, but can have an arbitrary factor $f(k)$ depending only on $k$ and (2) the approximation ratio is defined as a function of $k$ (as opposed to being a function of the input size $n$). In this paper, we mostly focus on the case of constant-factor FPA, that is, when $f_2(k) = c$ for some constant $c$.

Schaefer’s Dichotomy Theorem [40] identified six classes of finite Boolean constraint languages (0-valid, 1-valid, Horn, dual-Horn, bijunctive, affine) for which the decision CSP is polynomial-time solvable, and shows that every language $\Gamma$ outside these classes yields NP-hard problems. Therefore, it makes sense to study MinCSP only within these six classes, as it is otherwise already NP-hard to decide if the optimum is 0 or not, making approximation or fixed-parameter tractability irrelevant. Within these classes, polynomial-time approximability and fixed-parameter tractability seem to appear in orthogonal ways: the classes where we have positive results for one approach is very different from the classes where the other approach helps. For example, 2CNF DELETION (also called ALMOST 2SAT) is fixed-parameter tractable [45, 30], but has no polynomial-time approximation algorithm with constant approximation ratio, assuming the Unique Games Conjecture [15]. On the other hand, if $\Gamma$ consists of the three constraints $(x, \bar{x})$, and $(a \rightarrow b) \land (c \rightarrow d)$, then the problem is W[1]-hard [40], but belongs to the class IHS-1 and hence admits a constant-factor approximation in polynomial time [32].

By investigating constant-factor FP-approximation, we are identifying a class of tractable constraints that unifies and generalizes the polynomial-time constant-factor approximable and fixed-parameter tractable cases. We observe that if each constraint in $\Gamma$ can be expressed by a 2SAT formula (i.e., $\Gamma$ is bijunctive), then we can treat the MinCSP instance as an instance of 2SAT DELETION, at the cost of a constant-factor loss in the approximation ratio. Thus the fixed-parameter tractability of 2SAT DELETION implies MinCSP has a constant-factor FP-approximation if the finite set $\Gamma$ is bijunctive. If $\Gamma$ is in IHS-B for some constant $B$, then MinCSP is known to have a constant-factor approximation in polynomial time, which clearly gives another class of constant-factor FP-approximable constraints. Our main results show that probably these two classes cover all the easy cases with respect to FP-approximation (see Section 2 for the definitions involving properties of constraints).

**Theorem 1.** Let $\Gamma$ be a finite Boolean constraint language.

1. If $\Gamma$ is bijunctive or IHS-B for some constant $B \geq 1$, then MinCSP($\Gamma$) has a constant-factor FP-approximation.
2. Otherwise, if $\Gamma$ is affine, then MinCSP($\Gamma$) has an FP-approximation (resp., constant-factor FP-approximation) if and only if NEAREST CODEWORD has an FP-approximation (resp., constant-factor FP-approximation).
3. Otherwise, MinCSP($\Gamma$) has no fixed-parameter approximation, unless FPT = W[P].

Given a linear code over $GF[2]$ and a vector, the NEAREST CODEWORD problem asks for a codeword in the code that has minimum Hamming distance to the given vector. There are various equivalent formulations of this problem: ODD SET is a variant of HITTING SET where one has to select at most $k$ elements to hit each set exactly an odd number of times, and it is also possible to express the problem as finding a solution to a system of linear equations over $GF[2]$ that minimizes the number of unsatisfied equations. Arora et al. [3] showed that, assuming NP $\not\subseteq$ DTIME($n^{\text{poly} (\log n)}$), it is not possible to approximate NEAREST CODEWORD within ratio $2^{\log^{1-\epsilon} n}$ for any $\epsilon > 0$. In particular, this implies that a constant-factor polynomial-time approximation is unlikely. We give some evidence that even constant-factor FP-
approximation is unlikely. First, we rule out this possibility under the assumption that the Linear PCP Conjecture (LPC) and the Exponential-Time Hypothesis (ETH) both hold.

**Theorem 2.** Assuming LPC and ETH, Nearest Codeword has no constant-factor FP-approximation for any ratio \( r \).

Second, we connect the FP-approximability of Nearest Codeword with the \( k \)-Densest Subgraph problem, where the task is to find \( k \) vertices that induce the maximum number of edges.

**Theorem 3.** If Nearest Codeword has a constant-factor FP-approximation for some ratio \( r \), then for every \( \epsilon > 0 \), there is a factor-\((1-\epsilon)\) FP-approximation for \( k \)-Densest Subgraph.

Thus a constant-factor FP-approximation for Nearest Codeword implies that \( k \)-Densest Subgraph can be approximated arbitrarily well, which seems unlikely. Note that Theorems 2 and 3 remain valid for the other equivalent versions of Nearest Codeword, such as Odd Set.

Post’s lattice is a very useful tool for classifying the complexity of Boolean CSPs (see e.g., [1, 20, 4]). A (possibly infinite) set \( \Gamma \) of constraints is a co-clone if it is closed under pp-definitions, that is, whenever a relation \( R \) can be expressed by relations in \( \Gamma \) using only equality, conjunctions, and projections, then relation \( R \) is already in \( \Gamma \). Post’s co-clone lattice characterizes every possible co-clone of Boolean constraints (see Table 1 and Figure 1). From the complexity-theoretic point of view, Post’s lattice becomes very relevant if the complexity of the CSP problem under study does not change by adding new pp-definable relations to the set \( \Gamma \) of allowed relations. For example, this is true for the decision version of Boolean CSP. In this case, it is sufficient to determine the complexity for each co-clone in the lattice, and a complete classification for every finite set \( \Gamma \) of constraints follows. For MinCSP, neither the polynomial-time solvability nor the fixed-parameter tractability of the problem is closed under pp-definitions, hence Post’s lattice cannot be used directly for a complexity classifications. However, as observed by Khanna et al. [32] and subsequently exploited by Dalmau et al. [23, 24], the constant-factor approximability of MinCSP is closed under pp-definitions (modulo a small technicality related to equality constraints). We observe that the same holds for constant-factor FP-approximability and hence Post’s lattice can be used for our purposes. Therefore, the classification results amounts to identifying the maximal easy and the minimal hard co-clones.

The paper is organized as follows. Sections 2 and 3 contain preliminaries on CSPs, approximability, Post’s lattices and reductions. A more technical restatement of Theorem 1 in terms of co-clones is stated at the end of Section 3. Section 4 gives FPA algorithms, Section 5 establishes the equivalence of some CSPs with Odd Set, and Section 6 proves inapproximability results for CSPs. Section 7 proves Theorems 2 and 3 the conditional hardness results for Odd Set.

## 2 Preliminaries

A subset of \( \{0, 1\}^n \) is called an \( n \)-ary Boolean relation. In this paper, a constraint language \( \Gamma \) is a finite collection of finitary Boolean relations. When a constraint language \( \Gamma \) contains only a single relation \( R \), i.e., \( \Gamma = \{R\} \), we usually abuse notation and we write \( R \) instead of \( \{R\} \). The decision version of CSP, restricted to finite constraint language \( \Gamma \) is defined the following way.

**CSP(\( \Gamma \))**

\[ \text{Input:} \text{ A pair } \langle V, C \rangle, \text{ where} \]
\[ \text{\quad} V \text{ is a set of variables,} \]
\[ \text{\quad} C \text{ is a set of constraints } \{C_1, \ldots, C_q\}, \text{ i.e., } C_i = (s_i, R_i), \text{ where } s_i \text{ is a tuple of variables of length } n_i, \text{ and } R_i \in \Gamma \text{ is an } n_i\text{-ary relation.} \]

**Question:** Does there exist a solution, that is, a function \( \varphi : V \rightarrow \{0, 1\} \) such that for each constraint \( (s, R) \in C \), with \( s = (v_1, \ldots, v_n) \), the tuple \( \varphi(v_1), \ldots, \varphi(v_n) \) belongs to \( R \)?

Note that we can alternatively look at a constraint as a Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \), where \( n \) is a non-negative integer called the arity of \( f \). We say that \( f \) is satisfied by an assignment \( s \in \{0, 1\}^n \) if \( f(s) = 1 \). For example, if \( f(x, y) = x + y \mod 2 \), then the corresponding relation is \( \{(0, 1), (1, 0)\} \); we also denote addition modulo 2 with \( x \oplus y \).

We recall the definition of a few well-known classes of constraint languages. A Boolean constraint language \( \Gamma \) is
- \( \theta \)-valid, if each \( R \in \Gamma \) contains a tuple in which all entries are 0;
- \( I \)-valid, if each \( R \in \Gamma \) contains a tuple in which all entries are 1;
- \( \text{IHS-B+} \) (for implicative hitting set-bounded+) if each \( R \in \Gamma \) can be expressed by a conjunction of clauses of the form \( \neg x \lor \neg y \lor z \lor \neg x_k \);
- \( \text{IHS-B} \) if each \( R \in \Gamma \) can be expressed by a conjunction of clauses of the form \( x \lor \neg y \lor \neg x_k \);
- \( \text{IHS-B-} \) if it is an \( \text{IHS-B+} \) or and \( \text{IHS-B-} \) language;
- \( \text{Horn} \), if each \( R \in \Gamma \) can be expressed by a conjunction of Horn clauses, i.e., clauses that have at most one positive literal;
- \( \text{dual-Horn} \), if each relation \( R \in \Gamma \) can be expressed by a conjunction of dual-Horn clauses, i.e., clauses that have at most one negative literal;
- \( \text{affine} \), if each relation \( R \in \Gamma \) can be expressed by a conjunction of relations defined by equations of the form \( x_1 \oplus \cdots \oplus x_n = c \), where \( c \in \{0, 1\} \);
- \( \text{self-dual} \) if for each relation \( R \in \Gamma \), if \( (a_1, \ldots, a_n) \in R \), then \( (\neg a_1, \ldots, \neg a_n) \in R \)

For every finite constraint language \( \Gamma \), we consider the following problem.

\[
\text{MinCSP}(\Gamma)
\]

**Input:** A pair \( \langle V, C \rangle \), where \( V \) is a set of variables, \( C \) is a set of constraints, and an integer \( k \).

**Question:** Does there exist a deletion set, that is, a set \( W \subseteq C \) such that \( |W| \leq k \), and there exists a solution for the CSP(\( \Gamma \))-instance \( \langle V, C \setminus W \rangle \)?

For technical reasons, it will be convenient to work with a slight generalization of the problem where we can specify that certain constraints are “undeletable.”

\[
\text{MinCSP}^+(\Gamma)
\]

**Input:** A pair \( \langle V, C \rangle \), where \( V \) is a set of variables and \( C \) is a set of constraints, a subset \( C^* \subseteq C \) of undeletable constraints, and an integer \( k \).

**Question:** Does there exist a deletion set, that is, a set \( W \subseteq C \setminus C^* \) such that \( |W| \leq k \), and there exists a solution for the CSP(\( \Gamma \))-instance \( \langle V, C \setminus W \rangle \)?

For these two problems, we call feasible solution a set of potentially more than \( k \) constraints whose removal yields a satisfiable instance. Note that, contrary to MinCSP for which removing all the constraints constitute a trivially feasible solution, it is possible that an instance of MinCSP\(^+\) has no feasible solution.

We will use two types of reductions to connect the approximability of optimization problems. The first type perfectly preserves the optimum value (or cost) of instances.

**Definition 1.** An optimization problem \( A \) has a cost-preserving reduction to problem \( B \) if there are two polynomial-time computable functions \( F \) and \( G \) such that

1. For any feasible instance \( I \) of \( A \), \( F(I) \) is a feasible instance of \( B \) having the same optimum cost as \( I \).
2. For any feasible instance \( I \) of \( A \), if \( S' \) is a solution for \( F(I) \), then \( G(I, S') \) is an instance of \( I \) having cost at most the cost of \( F(I) \).

It is easy to show that MinCSP\(^+\) can be reduced to MinCSP, showing that the existence of undeletable constraints does not make the problem significantly more general. Note that, in the previous definition, if instance \( I \) has no feasible solution, then the behavior of \( F \) on \( I \) is not defined.

**Lemma 1.** There is a cost-preserving reduction from MinCSP\(^+\) to MinCSP.

**Proof.** The function \( F \) on a feasible instance \( I \) of MinCSP\(^+\) is defined the following way. Let \( m \) be the number of constraints. We construct \( F(I) \) by replacing each undeletable constraint with \( m + 1 \) copies. If \( I \) is a feasible instance of MinCSP\(^+\), then \( I \) has a solution with at most \( m \) deletions, which gives a solution of \( F(I) \) as well, showing that \( OPT(F(I)) \leq OPT(I) \leq m \). Conversely, \( OPT(F(I)) \leq m \) implies that an optimum solution of \( F(I) \) uses only the deletable constraints of \( I \), otherwise it would need to delete all \( m + 1 \) copies of an undeletable constraints. Thus \( OPT(I) \leq OPT(F(I)) \) and hence \( OPT(I) = OPT(F(I)) \) follows.

The function \( G(I, S') \) on a feasible instance \( I \) if MinCSP\(^+\) and a feasible solution \( S' \) of \( F(I) \) is defined the following way. If \( S' \) deletes only the deletable constraints of \( I \), then \( G(I, S') = S' \) also a
feasible solution of $I$ with the same cost. Otherwise, if $S'$ deletes at least one undeletable constraint, then it has cost at least $m + 1$, as it has to delete all $m + 1$ copies of the constraint. Now we define $G(I, S')$ to be the set of all (at most $m$) deletable constraints; by assumption, $I$ is a feasible instance of $\text{MINCSP}^+$, hence $G(I, S')$ is a feasible solution of cost at most $m + 1$

The second type of reduction that we use is the standard notion of $A$-reductions\footnote{If the name of a clone is $L_3$, for example, then the corresponding co-clone is $\text{Inv}(L_3)$ ($\text{Inv}$ is defined, for example, \textit{in} [6], which is denoted by $IL_3$.
}, which preserve approximation ratios up to constant factors. Note that we slightly deviate from the standard definition by not requiring any specific behavior of $F$ when $I$ has no feasible solution.

**Definition 2.** A minimization problem $A$ is $A$-reducible to problem $B$ if there are two polynomial-time computable functions $F$ and $G$ and a constant $\alpha$ such that

1. For any feasible instance $I$ of $A$, $F(I)$ is a feasible instance of $B$.
2. For any feasible instance $I$ of $A$, and any feasible solution $S'$ of $F(I)$, $G(I, S')$ is a feasible solution for $I$.
3. For any feasible instance $I$ of $A$, and any $r \geq 1$, if $S'$ is an $r$-approximate solution for $F(I)$, then $G(I, S')$ is an $(\alpha r)$-approximate feasible solution for $I$.

**Proposition 1.** If optimization problem $A$ is $A$-reducible to optimization problem $B$ and $B$ admits a constant-factor FPA algorithm, then $A$ also has a constant-factor FPA algorithm.

### 3 Post’s lattice, co-clone lattice, and a simple reduction

A clone is a set of Boolean functions that contains all projections (that is, the functions $f(a_1, \ldots, a_n) = a_k$ for $1 \leq k \leq n$) and is closed under arbitrary composition. All clones of Boolean functions were identified by Post\footnote{Use of this lattice for CSPs, Post’s lattice can be transformed to another lattice whose elements are not co-clones. \textit{in} [44], and he also described their inclusion structure, hence the name of Post’s lattice. To make for $1$ Post’s lattice, co-clone lattice, and a simple reduction

For brevity, we often write “$\exists \land$-definable” instead of “pp-definable without equality”. For brevity, we often write “$\exists \land$-definable” instead of “pp-definable without equality”. If $S$ is a set of relations, $S$ is pp-definable ($\exists \land$-definable) from $\Gamma$ if every relation is $S$ is pp-definable ($\exists \land$-definable) from $\Gamma$.

For a set of relations $\Gamma$, we denote by $\langle \Gamma \rangle$ the set of all relations that can be pp-defined over $\Gamma$. We refer to $\langle \Gamma \rangle$ as the co-clone generated by $\Gamma$. The set of all co-clones forms a lattice. To give an idea about the connection between Post’s lattice and the co-clone lattice, we briefly mention the following theorem, and refer the reader to, for example,\footnote{For a set of relations $\Gamma$, we denote by $\langle \Gamma \rangle$ the set of all relations that can be pp-defined over $\Gamma$. We refer to $\langle \Gamma \rangle$ as the co-clone generated by $\Gamma$. The set of all co-clones forms a lattice. To give an idea about the connection between Post’s lattice and the co-clone lattice, we briefly mention the following theorem, and refer the reader to, for example, [21], which is denoted by $IL_3$.

**Theorem 3.** The lattices of Boolean clones and Boolean co-clones are anti-isomorphic.

Using the above comments, it can be seen (and it is well known) that the lattice of Boolean co-clones has the structure shown in Figure 1. In the figure, if co-clone $C_2$ is above co-clone $C_1$, then $C_2 \supset C_1$. The names of the co-clones are indicated in the nodes, and where we follow the notation of Böhler et al\footnote{Use of this lattice for CSPs, Post’s lattice can be transformed to another lattice whose elements are not co-clones. In [44], and he also described their inclusion structure, hence the name of Post’s lattice. To make}.\footnote{Use of this lattice for CSPs, Post’s lattice can be transformed to another lattice whose elements are not co-clones. In [44], and he also described their inclusion structure, hence the name of Post’s lattice. To make an idea about the connection between Post’s lattice and the co-clone lattice, we briefly mention the following theorem, and refer the reader to, for example, [21], which is denoted by $IL_3$.}
| Co-clone Order Base | Base |
|---------------------|------|
| IBF 0              | \{=\}, \{\emptyset\} |
| IR_0 1             | \{\bar{x}\} |
| IR_1 1             | \{x\} |
| IR_2 1             | \{x, \bar{x}\}, \{xx\} |
| IM 2               | \{x \rightarrow y\} |
| IM_1 2             | \{x \rightarrow y, x\}, \{x \wedge (y \rightarrow z)\} |
| IM_0 2             | \{x \rightarrow y, \bar{x}\}, \{\bar{x} \wedge (y \rightarrow z)\} |
| IM_2 2             | \{x \rightarrow y, x, \bar{x}\}, \{x \rightarrow y, \bar{x} \rightarrow y\}, \{x \bar{y} \wedge (u \rightarrow v)\} |
| IS_0^m             | m \{\text{OR}^m\} |
| IS'_{0}^m          | m \{\text{NAND}^m\} |
| IS_0 1             | m \{\text{OR}^m|m \geq 2\} |
| IS_1 1             | m \{\text{NAND}^m|m \geq 2\} |
| IS_0 2             | m \{\text{OR}^m, x, \bar{x}\} |
| IS'_{0} 2          | m \{\text{OR}^m|m \geq 2\} \cup \{x, \bar{x}\} |
| IS_0 1             | m \{\text{OR}^m, x \rightarrow y\} |
| IS_0 1             | m \{\text{OR}^m|m \geq 2\} \cup \{x \rightarrow y\} |
| IS'_{0} 1          | m \{\text{OR}^m, x, \bar{x}, x \rightarrow y\} |
| IS_0 2             | m \{\text{NAND}^m|m \geq 2\} \cup \{x, \bar{x}, x \rightarrow y\} |
| IS'_{1} 2          | m \{\text{NAND}^m, x, \bar{x}\} |
| IS_0 1             | m \{\text{NAND}^m, x \rightarrow y\} |
| IS_0 1             | m \{\text{NAND}^m|m \geq 2\} \cup \{x \rightarrow y\} |
| IS'_{0} 1          | m \{\text{NAND}^m, x, \bar{x}, x \rightarrow y\} |
| IS_0 2             | m \{\text{NAND}^m|m \geq 2\} \cup \{x, \bar{x}, x \rightarrow y\} |
| IS_1 2             | 2 \{x \oplus y\} |
| IS_2 2             | 2 \{x \oplus y, x \rightarrow y\}, \{xy, \bar{xy}z\} |
| IS_3 4             | 4 \{\text{EVEN}^4\} |
| IS_0 3             | 3 \{\text{EVEN}^3, \bar{x}\}, \{\text{EVEN}^3\} |
| IS_1 3             | 3 \{\text{EVEN}^3, x\}, \{\text{ODD}^3\} |
| IS_2 3             | 3 \{\text{EVEN}^3, x, \bar{x}\}, \text{every \{EVEN}^m, x\} \text{ where } n \geq 3 \text{ is odd} |
| IS_3 4             | 4 \{\text{EVEN}^4, x \oplus y\}, \{\text{ODD}^4\} |
| IV 3               | 3 \{x \lor y \lor \bar{z}\} |
| IV_0 3             | 3 \{x \lor y \lor \bar{z}, x\} |
| IV_1 3             | 3 \{x \lor y \lor \bar{z}, x\} |
| IV_2 3             | 3 \{x \lor y \lor \bar{z}, x\} |
| IV 3               | 3 \{x \lor y \lor \bar{z}\} |
| IV_0 3             | 3 \{x \lor y \lor \bar{z}, x\} |
| IV_1 3             | 3 \{x \lor y \lor \bar{z}, x\} |
| IV_2 3             | 3 \{x \lor y \lor \bar{z}, x\} |
| IV 3               | 3 \{x \lor y \lor \bar{z}\} |
| IV_0 3             | 3 \{x \lor y \lor \bar{z}, x\} |
| IV_1 3             | 3 \{x \lor y \lor \bar{z}, x\} |
| IV_2 3             | 3 \{x \lor y \lor \bar{z}, x\} |
| IV 3               | 3 \{x \lor y \lor \bar{z}\} |
| IV_0 3             | 3 \{x \lor y \lor \bar{z}, x\} |
| IV_1 3             | 3 \{x \lor y \lor \bar{z}, x\} |
| IV_2 3             | 3 \{x \lor y \lor \bar{z}, x\} |

Table 1. Bases for all Boolean co-clones. (See [a] for a complete definition of relations that appear.) The order of a co-clone is the minimum over all bases of the maximum arity of a relation in the base. The order is defined to be infinite if there is no finite base for that co-clone.
Fig. 1. Classification of Boolean CSPs according to constant ratio fixed-parameter approximability.
For a co-clone $C$ we say that a set of relations $\Gamma$ is a base for $C$ if $C = \langle \Gamma \rangle$, that is, any relation in $C$ can be pp-defined using relations in $\Gamma$. Böhlér et al. give bases for all co-clones in \[1\], and the reader can consult this paper for details. We reproduce this list in Table 1

It is well-known that pp-definitions preserve the complexity of the decision version of CSP: if $\Gamma_2 \subseteq \langle \Gamma_1 \rangle$ for two finite languages $\Gamma_1$ and $\Gamma_2$, then there is a natural polynomial-time reduction from CSP($\Gamma_2$) to CSP($\Gamma_1$). The same is not true for MinCSP: the approximation ratio can change in the reduction. However, it has been observed that this change of the approximation ratio is at most a constant (depending on $\Gamma_1$ and $\Gamma_2$)\[2,3,4\]: we show the same here in the context of parameterized reductions.

**Lemma 2.** Let $\Gamma$ be a constraint language, and $R$ be a relation that is pp-definable over $\Gamma$ without equality. Then there is an $A$-reduction from MinCSP($\Gamma \cup \{ R \}$) to MinCSP($\Gamma$).

**Proof.** One direction is trivial. Let $I$ be an instance of MinCSP($\Gamma \cup \{ R \}$). Let $\varphi(x_1, \ldots, x_k)$ be a primitive positive formula defining $R$ from $\Gamma$. Then, $\varphi$ is of the form $\exists y_1, \ldots, y_l \psi(x_1, \ldots, x_k, y_1, \ldots, y_l)$, where $\psi$ is the quantifier-free part of $\varphi$. The key and well-known (in similar contexts) observation is that $\psi$ can be alternatively seen as an instance of MinCSP($\Gamma$). More precisely, we define the instance associated to $\psi, I_{\psi}$, as the instance that has variables $x_1, \ldots, x_k, y_1, \ldots, y_l$ and contains for every atomic formula $S(v_1, \ldots, v_r)$ in $\psi$, the constraint $((v_1, \ldots, v_r), S)$. It follows that for any assignment $s : x_1, \ldots, x_k, y_1, \ldots, y_l \to A$, $s$ is a solution of $I_{\psi}$ if and only if $\psi(s(x_1), \ldots, s(x_k), s(y_1), \ldots, s(y_l))$ holds.

We obtain an instance $I'$ of MinCSP($\Gamma$) from $I$ using the following replacement. For each constraint $C = ((u_1, \ldots, u_q), R)$ in $I$, we identify the quantifier-free part $\psi_C(u_1, \ldots, u_q, y_{C,1}, \ldots, y_{C, l})$ of the formula corresponding to $C$, and then replace $C$ with the set of constraints of the instance $I_{\psi_C}$, where $y_{C,1}, \ldots, y_{C, l}$ are newly introduced variables. We leave the rest of the constraints intact.

Any deletion set $X_I$ for $I$ is translated to a deletion set $X_{I'}$ of $I'$ as follows. If $C = ((u_1, \ldots, u_q), P) \in X_I$ and $P \neq R$, then we place $((u_1, \ldots, u_q), P) \in X_{I'}$. If $P = R$, then we place all the constraints that replaced $C$ into $X_{I'}$. Since the number of these constraints is bounded by a constant, we obtain only a constant blow-up in the solution size. The converse can be shown similarly.

By repeated applications of Lemma \[3\] the following corollary establishes that we need to provide approximation algorithms only for a few MinCSPs, and these algorithms can be used for other MinCSPs associated with the same co-clone.

**Corollary 1.** Let $C$ be a co-clone and $B$ be a base for $C$. If the equality relation can be $\exists \forall \land$-defined from $B$, then for any finite $\Gamma \subseteq C$, there is an $A$-reduction from MinCSP($\Gamma$) to MinCSP($B$).

For hardness results, we would like to argue that if a co-clone $C$ is hard, then any constraint language $\Gamma$ generating the co-clone is hard. However, there are two technical issues here. First, co-clones are infinite and we defined the problems for finite constraint languages. Therefore, we formulate this requirement instead by saying that a finite base $B$ of the co-clone $C$ is hard. Second, pp-definitions require equality relations, which may not be expressible by $\Gamma$. However, as the following theorem shows, this is an issue only if $B$ contains relations where the coordinates are always equal (which will not be the case in our proofs). A $k$-ary relation $R$ is irredundant if for every two different coordinates $1 \leq i < j \leq k$, $R$ contains a tuple $(a_1, \ldots, a_k)$ with $a_i \neq a_j$. A set of relations $S$ is irredundant if any relation in $S$ is irredundant.

**Theorem 5 \[2,5\].** If $S \subseteq \langle \Gamma \rangle$ and $S$ is irredundant, then $S$ is $\exists \forall \land$-definable from $\Gamma$.

Therefore, if we consider an irredundant base $B$ of co-clone $C$, then we can formulate the following result.

**Corollary 2.** Let $B$ be an irredundant base for some co-clone $C$. If $\Gamma$ is a finite constraint language with $C \subseteq \langle \Gamma \rangle$, then there is an $A$-reduction from MinCSP($B$) to MinCSP($\Gamma$).

**Proof.** Since $B \subseteq \langle \Gamma \rangle$ and $B$ is irredundant, $B$ can be $\exists \forall \land$-defined over $\Gamma$ using Theorem \[2\]. Then repeated applications of Lemma \[3\] shows the existence of the reduction from MinCSP($B$) to MinCSP($\Gamma$).

By the following lemma, if the constraint language is self-dual, then we can assume that it also contains the constant relations.

**Lemma 3.** Let $\Gamma$ be a self-dual constraint language. Assume that $x \oplus y \in \Gamma$. Then there is a cost-preserving reduction from MinCSP($\Gamma \cup \{ x, \bar{x} \}$) to MinCSP($\Gamma$).

\[4\] We note that EVEN$^4$ can be pp-defined using DUP$^3$. Therefore the base $\{ \text{DUP}^3, \text{EVEN}^4, x \oplus y \}$ given by Böhlér et al. \[1\] for IN$_2$ can be actually simplified to $\{ \text{DUP}^3, x \oplus y \}$. 
Proof. Let \( I \) be an instance of \( \text{MINCSP}(\Gamma \cup \{ x, \bar{x} \}) \). We construct an instance \( J \) of \( \text{MINCSP}^*(\Gamma') \) such that a deletion set of size \( k \) of \( I \) corresponds to a deletion set of size \( k \) of \( J \); then the result for \( \text{MINCSP} \) follow from the fact that there is a cost-preserving reduction from \( \text{MINCSP}^* \) to \( \text{MINCSP} \) (Lemma 1). Every constraint of the form \( R(x_1, \ldots, x_r) \) in \( I \) is placed into \( J \). We introduce an undeletable constraint \( x \oplus y \) to \( J \), where \( x \) and \( y \) are new variables. For any constraint \( v = 0 \) in \( I \), we add a constraint \( x = v \) (i.e., constraints \( x \oplus u \) and \( u \oplus v \)) to \( J \). For any constraint \( v = 1 \) in \( I \), we add a constraint \( x = v \) in \( J \).

Let \( W_I \) be a deletion set for \( I \). To obtain a deletion \( W_J \) set of the same size for \( J \), any constraint that is not of the form \( v \) constraint \( x \) is expressed as \( x \oplus u \) and \( u \oplus v \), then we place \( u \oplus v \) into \( W_J \), and for any constraint \( w = 1 \) in \( W_I \), we place the constraint \( y = w \) into \( W_J \). Then assigning 0 to \( x \) and 1 to \( y \), and for the remaining variables of \( J \) using the assignment for the variables of \( I \), we obtain a satisfying assignment for \( J \).

The converse can be done by essentially reversing the argument, except that we might need to use the complement of the satisfying assignment for \( J \) to satisfy constraints of the form \( x = 0 \) and \( x = 1 \).

The following theorem states our trichotomy classification in terms of co-clones.

**Theorem 6.** Let \( \Gamma \) be a finite set of Boolean relations.

1. If \( \langle \Gamma \rangle \subseteq C \) (equivalently, if \( \Gamma \subseteq C \)), where \( C \in \{ \Pi_0, \Pi_1, \text{IS}_{00}, \text{IS}_{10}, \text{ID}_2 \} \), then \( \text{MINCSP}(\Gamma) \) has a constant-factor \( \text{FPA} \) algorithm. (Note in these cases \( \Gamma \) is 0-valid, 1-valid, \( \text{IHS-B}^+ \), \( \text{IHS-B}^- \), or bijunctive, respectively.)
2. If \( \langle \Gamma \rangle \in \{ \Pi_2, \Lambda_4 \} \), then \( \text{MINCSP}(\Gamma) \) is equivalent to \( \text{NEAREST CODEWORD} \) and to \( \text{ODD SET under A-reductions} \) (Note that these constraint languages are affine.)
3. If \( C \subseteq \langle \Gamma \rangle \), where \( C \in \{ \text{IE}_2, \text{IV}_2, \text{IN}_2 \} \), then \( \text{MINCSP}(\Gamma) \) does not have a constant-factor \( \text{FPA} \) algorithm unless \( \text{FPT} = \text{W}[P] \). (Note that in these cases \( \Gamma \) can \( \exists \forall \)-define either arbitrary Horn relations, or arbitrary dual Horn relations, or the \( \text{NAE}^3 \) relation, defined in Lemma 7.)

Looking at the co-clone lattice, it is easy to see that Theorem 6 covers all cases. It is also easy to check that Theorem 6 formulated in the introduction follows from Theorem 6.

Theorem 6 is proved the following way:

- Statement 1 is proved in Sections 4.1, 4.3 (Corollaries 3, 4, and 5).
- Statement 2 is proved in Section 5 (Theorem 9).
- Statement 3 is proved in Sections 6.1 (Corollary 7 and Lemma 7).

This completes the proof of Theorem 6.

## 4 CSPs with FPA algorithms

In this section, we prove the first statement of Theorem 6 by going through co-clones one by one.

### 4.1 Co-clones \( \Pi_0 \) (0-valid) and \( \Pi_1 \) (1-valid)

As every relation of a 0-valid \( \text{MINCSP} \) is always satisfied by the all 0 assignment, and every relation of a 1-valid \( \text{MINCSP} \) is always satisfied by the all 1 assignment, so we have a trivial algorithm for these problems.

**Corollary 3.** If \( \langle \Gamma \rangle \subseteq \Pi_0 \) or \( \langle \Gamma \rangle \subseteq \Pi_1 \), then \( \text{MINCSP}(\Gamma) \) is polynomial-time solvable.

### 4.2 Co-clone \( \text{ID}_2 \) (bijunctive)

The problem \( \text{ALMOST 2-SAT} \) is defined as \( \text{MINCSP}(\Gamma(2\text{-SAT})) \), where \( \Gamma(2\text{-SAT}) = \{ x \lor y, x \lor \neg y, \neg x \lor \neg y \} \).

**Theorem 7** ([45]). \( \text{ALMOST 2-SAT} \) is fixed-parameter tractable.

Every bijunctive relation can be pp-defined by 2-SAT, thus the constant-factor approximability for bijunctive languages follow easily from the FPT algorithm for \( \text{ALMOST 2-SAT} \) and from Corollary 4.

**Corollary 4.** If \( \langle \Gamma \rangle \subseteq \text{ID}_2 \), then \( \text{MINCSP}(\Gamma) \) has a constant-factor \( \text{FPA} \) algorithm.

**Proof.** We check in Table 1 that \( B = \{ x \oplus y, x \rightarrow y \} \) is a base for the co-clone \( \text{ID}_2 \). Relations in \( B \) (and equality) can be \( \exists \forall \)-defined over \( \Gamma(2\text{-SAT}) \), so the result follows from Corollary 4.
4.3 Co-clones IS_{00} (IHS-B+) and IS_{10} (IHS-B-)

We first note that if \( \langle I \rangle \) is in IS_{00} or IS_{10}, then the language is IHS-B for some \( B \geq 2 \).

**Lemma 4.** If \( \langle I \rangle \subseteq IS_{00} \), then there is an integer \( B \geq 2 \) such that \( \langle I \rangle \) is IHS-B+. If \( \langle I \rangle \subseteq IS_{10} \), then there is an integer \( B \geq 2 \) such that \( \langle I \rangle \) is IHS-B-.

**Proof.** We note that there is no finite base for IS_{00}, thus \( \langle I \rangle \) is a proper subset of IS_{00} (see Table 3). There is no proper base for IS_{01}, IS_{02}, or IS_{03} either. It follows from the structure of the co-clone lattice (Figure 3) that there exists a finite \( B \geq 2 \) such that \( \langle I \rangle \subseteq IS_{10}^B \), which implies that \( \langle I \rangle \) is IHS-B+. The proof of the second statement is analogous.

By Lemma 4, if \( \langle I \rangle \subseteq IS_{00} \), then \( \langle I \rangle \) is generated by the relations \( \neg x, x \rightarrow y, x_1 \lor \cdots \lor x_k \) for some \( k \geq 2 \). The MinCSP problem for this set of relations is known to admit a constant-factor approximation.

**Theorem 8** ([19], Lemma 7.29). MinCSP(\( \neg x, x \rightarrow y, x_1 \lor \cdots \lor x_k \)) has a \((k+1)\)-factor approximation algorithm (and hence has a constant-factor FPA algorithm).

Now Theorem 8 and Corollary 1 imply that there is a constant-factor FPA algorithm for MinCSP(\( \langle I \rangle \)) whenever \( \langle I \rangle \) is in the co-clone IS_{00} or IS_{10} (note that equality can be \( \exists y \)-defined using \( x \rightarrow y \)). In fact, one can observe that the resulting algorithm is a polynomial-time approximation algorithm: Theorem 8 gives a polynomial-time algorithm and this is preserved by Corollary 1.

**Corollary 5.** If \( \langle I \rangle \subseteq IS_{00} \) or \( \langle I \rangle \subseteq IS_{10} \), then MinCSP(\( \langle I \rangle \)) has a constant-factor FPA algorithm.

Note that Theorem 7.25 in [19] gives a complete classification of Boolean MinCSPs with respect to constant-factor approximability. As mentioned in the proof above, these MinCSPs also admit a constant-factor approximation algorithm. The reason we need Corollary 5 is to have the characterization in terms of the co-clone lattice.

5 CSPs equivalent to Odd Set: co-clones IL_{2} (affine) and IL_{3} (affine & self-dual)

In this section we show the equivalence of several problems under \( \Lambda \)-reductions. We identify CSPs that are equivalent to the following well-known combinatorial problems.

| Nearest Codeword |
|-------------------|
| **Input:** An \( m \times n \) matrix \( A \), and an \( m \)-dimensional vector \( b \). |
| **Output:** An \( n \) dimensional vector \( x \) that minimizes the Hamming distance between \( Ax \) and \( b \). |

| Odd Set |
|---------|
| **Input:** A set-system \( S = \{ S_1, S_2, \ldots, S_m \} \) over universe \( U \). |
| **Output:** A subset \( T \subseteq U \) of minimum size such that every set of \( S \) is hit an odd number of times by \( T \), that is, \( \forall i \in [m], |S_i \cap T| \) is odd. |

**Even/Odd Set** is the same problem as **Odd Set**, except that we can specify whether a set should be hit an even or odd number of times (the objective is the same as in **Odd Set:** find a subset of minimum size satisfying the requirements). We call a set that must be hit an odd (even) number of times **odd (even)** set. We show that this does not make the problem much more general: there is a parameter preserving reduction from **Even/Odd Set** to **Odd Set**.

**Lemma 5.** There is a cost-preserving reduction from **Even/Odd Set** to **Odd Set**.

**Proof.** Let \( I \) be the instance of **Even/Odd Set**. If all sets in \( I \) are even sets, then the empty set is an optimal solution. Otherwise, fix an arbitrary odd set \( S_o \) in \( I \). We obtain an instance \( I' \) of **Odd Set** by introducing every odd set of \( I \) into \( I' \), and for each even set \( S_e \) of \( I \), we introduce the set \( S_e \triangle S_o \) into \( I' \), where \( \triangle \) denotes the symmetric difference of two sets. This completes the reduction.

Let \( W \) be a solution of size \( k \) for \( I \). We claim that \( W \) is also a solution of \( I' \). Then those sets of \( I' \) that correspond to odd sets of \( I \) are obviously hit an odd number of times by \( W \). We have to show that the remaining sets of \( I' \) are also hit an odd number of times. Let \( T = S_e \triangle S_o \) be such a set for some even
Theorem 9. The following problems are equivalent under cost-preserving reductions:

(1) Nearest Codeword, (2) Odd Set, (3) MinCSP(B₂), and (4) MinCSP(B₃).

Proof. (1) ⇒ (2): Let A be an $m \times n$ generator matrix for the Nearest Codeword problem, and c be an $m$-dimensional vector such that we want to find a codeword of Hamming distance at most $k$ from c. Let C be the set of all codewords generated by A, i.e., vectors in the column space of A. Let $A^\perp$ be the $\ell \times m$ matrix whose rows form a basis for the subspace perpendicular to the column space of A. Then $w \in C$ if and only if $A^\perp w = 0$. Assume now that $z$ is a vector that differs from $c$ at most in $k$ positions. Then we can write $z = z' + c$, where the weight of $z'$ is the distance between $z$ and $z'$. To find such a $z$, we write $A^\perp (z' + c) = 0$, and now we wish to find a solution that minimizes the weight of $z'$. Observe that $A^\perp z' = A^\perp c$ (since we are working in GF(2)). This can be encoded as a problem where we have a ground set $U = \{1, \ldots, m\}$, and sets $S_i, 1 \leq i \leq \ell$, defined as follows. Element $j \in S_i$ if $A^\perp(i, j) = 1$. We want to find a subset $W \subseteq U$ of size at most $k$ such that $S_i$ is hit an even number of times if the $i$-th element of the vector $A^\perp$ is 0, and an odd number of times if it is 1. This is an instance of the EVEN/ODD Set problem. Using Lemma [5], we can further reduce this problem to Odd Set, and we are done.

(2) ⇒ (3): We show that there is a cost-preserving reduction from Odd Set to MinCSP*(B₂) (and hence to MinCSP(B₂) by Lemma [6]). First we 3-way-express the relation ODDⁿ using B₂. We induct on $n$. For $n = 2$, we have that ODD²(x₁, x₂) = EVEN³(x₁, x₂, 0, 1). Assume we have a formula that defines ODDⁿ. Then observe that

$$\text{ODD}^{n+1}(x₁, \ldots, x_{n+1}) = \exists u \text{ODD}^n(x₁, \ldots, x_n, u) \land \text{EVEN}^3(u, x_n, x_{n+1}).$$

The variables of the MinCSP instance J are the elements of the ground set of the Odd Set instance I. For each set $\{y₁, \ldots, yₖ\}$ of I, we add an undeletable constraint ODD*(y₁, \ldots, yₖ) to J. Finally, for each variable $y$ that appears in a constraint, we add the constrain $\overline{y}$. It is easy to see that a hitting set of size $k$ for I (consisting of constraints of the form $\overline{y}$) corresponds to a deletion set of size $k$ for I.

(3) ⇒ (4): This follows from Lemma [6].

(4) ⇒ (1): Let I be the MinCSP(B₂) instance, and assume it has $n$ variables and $m$ constraints. We define a Nearest Codeword instance J as follows. The matrix A has dimension $m \times n$, and columns are indexed by the variables of I. If the $i$-th constraint of I is EVEN¹(x₁, x₂, x₃, x₄), then the $i$-th row of A has 1-s in positions $j₁, j₂, j₃, j₄$, and the $i$-th entry of vector b is 0. If the $i$-th constraint of I is $x_k₁ \oplus x_k₂$, then the $i$-th row of A has 1-s in positions $k₁$ and $k₂$, and the $i$-th entry of b is 1. Clearly, a deletion set of size $k$ for I corresponds to a solution of J having distance $k$ from vector b.
6 Hard CSPs: Horn, dual-Horn and IN₂

In this section, we prove statement 3 of Theorem 6: we prove the inapproximability of MinCSP \( FPT = W[P] \) Theorem 10 \([39]\). Monotone Circuit Satisfiability are hard.

6.1 Co-clones IV₂ (Horn), IE₂ (dual-Horn), and IN₂ (self-dual)

We use Corollary \([\text{C}]\) to establish the inapproximability of Horn-SAT and dual-Horn-SAT, assuming that FPT \( \neq W[P] \). Using the co-clone lattice, this will show hardness of approximability of MinCSP(\( \Gamma \)) if \( \langle \Gamma \rangle \in \{IV₂,IE₂\} \).

**Lemma 6.** If there is an FPA algorithm for MinCSP(\( \{x \lor y \lor \bar{z},x,\bar{x}\} \)) or MinCSP(\( \{\bar{x} \lor \bar{y} \lor z,x,\bar{x}\} \)) with constant approximation ratio, then FPT = W[P].

**Proof.** We show that there is a parameter preserving polynomial-time reduction from Monotone Circuit Satisfiability to MinCSP(\( \{x \lor y \lor \bar{z},x,\bar{x}\} \)). This is sufficient by Corollary \([\text{C}]\). Let C be the MCS instance. We produce an instance I of MinCSP(\( \text{C} \)) as follows. We think of inputs of C as gates, and we refer to these as “input gates”. This will simplify the discussion. For each gate of C, we introduce a new variable into I, and we let f denote the natural bijection from the gates and inputs of C to the variables of the instance I.

We add constraints to simulate each AND gate of C as follows. Observe first that the implication relation \( x \rightarrow y \) can be expressed as \( y \lor x \lor \bar{x} \). For each AND gate \( G \) such that \( G_1 \) and \( G_2 \) are the gates feeding into \( G \) (note that \( G_1 \) and \( G_2 \) are allowed to be input gates), we add two constraints to I as follows. Let \( y = f(G), x_1 = f(G_1), \) and \( x_2 = f(G_2) \). We place the constraints \( y \rightarrow x_1, y \rightarrow x_2 \). We observe that the only way variable y could take on value 1 is if both \( x_1 \) and \( x_2 \) are assigned 1. (In this case, note that y could also be assigned 0 but that will be easy to fix.)

Similarly, we add constraints to simulate each OR gate of C as follows. For each OR gate \( G \) such that \( G_1 \) and \( G_2 \) are the gates feeding into \( G \), we add a constraint to I, we add the constraint \( x_1 \lor x_2 \lor y \) to I, where \( y = f(G), x_1 = f(G_1), \) and \( x_2 = f(G_2) \). Note that if both \( x_1 \) and \( x_2 \) are 0, than \( y \) is forced to have value 0. (Otherwise \( y \) can take on either value 0 or 1, but again, this difference between an OR gate and our gadget will be easy to handle.)

In addition, we add a constraint \( x_o = 1 \), where \( x_o \) is the variable such that \( x_o = f(G) \), where G is the output gate. We define all constraints that appeared until now to be undeletable, so that they cannot be removed in solution of the MinCSP(\( \text{C} \)) instance. To finish the construction, for each variable \( x \) such that \( x = f(G) \) where G is an input gate, we add a constraint \( x = 0 \) to I. We call these constraints input constraints. Note that only input constraints can be removed.
If there is a satisfying assignment \( \varphi_C \) of \( C \) (from gates of \( C \) to \( \{0, 1\} \)) of weight \( k \), then we remove the input constraints \( x = 0 \) of \( I \) such that \( \varphi_C(G) = 1 \), where \( f(G) = x \). Clearly, the map \( \varphi_C \circ f^{-1} \) is a satisfying assignment for \( I \), where we needed \( k \) deletions.

For the other direction, assume that we have a satisfying assignment \( \varphi_I \) for \( I \) after removing some \( k \) input constraints (note that if any other constraints are removed, we can simply ignore those deletions). We repeatedly change \( \varphi_I \) as long as either of the following conditions apply. If \( x_1, x_2 \) and \( y \) are such that \( f^{-1}(x_1) \) and \( f^{-1}(x_2) \) are gates feeding into gate \( f^{-1}(y) \) where \( f^{-1}(y) \) is an AND gate, and \( \varphi_I(x_1) = 1, \varphi_I(x_2) = 1, \varphi_I(y) = 0 \), then we change \( \varphi_I(y) \) to 1. Similarly, if \( f^{-1}(y) \) is an OR gate, \( 1 \in \{ \varphi_I(x_1), \varphi_I(x_2) \} \), \( \varphi_I(y) = 0 \), then we change \( \varphi_I(y) \) to 1. It follows form the definition of the constraints we introduced for AND and OR gates that once we finished modifying \( \varphi_I \), the resulting assignment \( \varphi \) is still a satisfying assignment. Now it follows that \( \varphi \circ f \) is a weight \( k \) satisfying assignment for \( I \).

To show the inapproximability of \( \text{MinCSP}((x \lor y \lor z, x, \bar{x}) \} \), we note that there is a parameter preserving bijection between instances of \( \text{MinCSP}((x \lor y \lor z, x, \bar{x}) \} \) and \( \text{MinCSP}((x \lor y \lor z, x, \bar{x}) \}) \); given an instance \( I \) of either problem, we obtain an equivalent instance of the other problem by replacing every literal \( \ell \) with \( \bar{\ell} \). Satisfying assignments are converted by replacing 0-s with 1-s and vice versa.

As \( \{ x \lor y \lor z, x, \bar{x} \} \) (resp., \( \{ x \lor y \lor z, x, \bar{x} \} \)) is an irredundant base of \( IV_2 \) (resp., \( IE_2 \), Corollary 2 implies hardness if \( \langle I \rangle \) contains \( IV_2 \) or \( IE_2 \).

**Corollary 7.** If \( I \) is a finite constraint language with \( IV_2 \subseteq \langle I \rangle \) or \( IE_2 \subseteq \langle I \rangle \), then \( \text{MinCSP}(I) \) is not \( FP \)-approximable, unless \( \text{FPT} = \text{W}[P] \).

Finally, we consider the co-clone \( IN_2 \).

**Lemma 7.** If \( I \) is a finite constraint language with \( IN_2 \subseteq \langle I \rangle \) then \( \text{MinCSP}(I) \) is not \( FP \)-approximable, unless \( P = \text{NP} \).

**Proof.** Let \( \text{NAE}^3 = \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\} \). From Table 1, we see that \( \text{NAE}^3 \) is a base for the co-clone \( IN_2 \) of all self-dual languages. If there was a constant-factor \( FP \) algorithm for \( \text{MinCSP} \) \( \text{NAE}^3 \), then setting the parameter to 0 would give a polynomial time decision algorithm for \( \text{CSP} \) \( \text{NAE}^3 \). But \( \text{CSP} \) \( \text{NAE}^3 \) is \( \text{NP} \)-complete \[49\], so there is no constant-factor \( FP \) algorithm for \( \text{MinCSP} \) \( \text{NAE}^3 \) unless \( \text{NP} = \text{P} \).

7 Odd Set is probably hard

In this section we provide evidence that problems equivalent to \( \text{Nearest Codeword} \) and \( \text{Odd Set} \) (in particular, problems in Theorem 6(2)) are hard, i.e., they are unlikely to have a constant-factor \( FP \) algorithm.

In the \( k \)-\text{DENSEST SUBGRAPH} problem, we are given a graph \( G = (V, E) \) and integer \( k \); the task is to find a set \( S \) of \( k \) vertices that maximizes the number of edges in the induced subgraph \( G[S] \). Note that an exact algorithm for \( k \)-\text{DENSEST SUBGRAPH} would imply an exact algorithm for \( \text{CLIQUE} \). Due to its similarity to \( \text{CLIQUE} \), it is reasonable to assume that \( k \)-\text{DENSEST SUBGRAPH} is hard even to approximate.

We formulate the following specific hardness assumption.

**Assumption 11** There is an \( \varepsilon > 0 \) such that for any function \( f \), one cannot approximate \( k \)-\text{DENSEST SUBGRAPH} within ratio \( 1 - \varepsilon \) in time \( f(k) \cdot n^{O(1)} \).

It will be more convenient to work with slightly different version of \( k \)-\text{DENSEST SUBGRAPH}. In the \( \text{MULTICOLORED} \) \( k \)-\text{DENSEST SUBGRAPH} problem, we are given a graph \( G = (V, E) \) whose vertex-set \( V \) is partitioned into \( k \) classes \( C_1, \ldots, C_k \), and the goal is to find a set \( S = \{ v_1, \ldots, v_k \} \) of \( k \) vertices satisfying \( v_i \in C_i \) for each \( i \in [k] \), and maximizing the number of edges in the induced subgraph \( G[S] \).

Let us observe that Assumption 11 implies the same inapproximability for \( \text{MULTICOLORED} \) \( k \)-\text{DENSEST SUBGRAPH} as well. Indeed, for the sake of contradiction, let us assume that for any \( \varepsilon > 0 \), \( \text{MULTICOLORED} \) \( k \)-\text{DENSEST SUBGRAPH} admits an \( (1 - \varepsilon) \)-approximation running in time \( f(k) \cdot n^{O(1)} \) for some function \( f \). Then, by the technique of \text{color coding} \[2\], one can also approximate \( k \)-\text{DENSEST SUBGRAPH}. Each vertex of the (non-colored) instance graph is given a color in \( [k] \) uniformly at random. With probability at least \( \frac{1}{k^4} \), this coloring has given pairwise distinct colors to the \( k \) vertices of an optimal solution to \( k \)-\text{DENSEST SUBGRAPH}, and therefore those \( k \) vertices also constitute an optimal solution to \( \text{MULTICOLORED} \) \( k \)-\text{DENSEST SUBGRAPH} in the colored graph. Then, we run our approximation on the
Multicolored $k$-Densest Subgraph instance. If we repeat that process $\alpha k^k$ times for some large $\alpha$, we get, with probability at least $1 - 2^{-\alpha}$ an $(1 - \varepsilon)$-approximation for $k$-Densest Subgraph running in time $g(k) \cdot n^{O(1)}$, where $g(k) = \alpha k^k f(k)$. This algorithm can be derandomized with a $k$-perfect hash family, that is a set $\mathcal{C}$ of $k$-colorings (not necessarily proper) of the graph such that for each subset $S$ of $k$ vertices, and for each coloring $\mathcal{P}$ of $S$, one global coloring of the family $\mathcal{C}$ has a projection to $S$ equals to $\mathcal{P}$. A $k$-perfect hash family with $e^k k^{O(\log k)} \log n$ colorings can be computed in time $e^k k^{O(\log k)} \cdot n^{O(1)}$, which would give a deterministic $(1 - \varepsilon)$-approximation running in time $e^k k^{O(\log k)}(f(k) + 1) \cdot n^{O(1)}$.

Odd Set has the so-called self-improvement property. Informally, a polynomial time (resp. fixed-parameter time) approximation within some ratio $r$ can be turned into a polynomial time (resp. fixed-parameter time) approximation within some ratio close to $\sqrt{r}$.

Lemma 8. If there is an $r$-approximation for Odd Set running in time $f(n, m, k)$ where $n$ is the size of the universe, $m$ the number of sets, and $k$ the size of an optimal solution, then for any $\varepsilon > 0$, there is an $(1 + \varepsilon)\sqrt{r}$-approximation running in time $max(f(1 + n + n^2, 1 + m + nm, 1 + k + k^2), O(n^{1 + \frac{\varepsilon}{2}}))$.

Proof. The following reduction is inspired by the one showing the self-improvement property of Nearest Codeword [3]. Let $S = \{S_1, \ldots, S_m\}$ be any instance over universe $U = \{x_1, \ldots, x_n\}$. Let $\varepsilon > 0$ be any real positive value and $k$ be the size of an optimal solution. We can assume that $k \geq \frac{\varepsilon}{2}$ since one can find an optimal solution by exhaustive search in time $O(n^{1 + \frac{\varepsilon}{2}})$. We build the set-system $S' = S \cup \bigcup_{i \in [n]} S'_i \cup \{\{e\}\}$ over universe $U' = U \cup \bigcup_{i \in [n]} \{x'_i\} \cup \{e\}$ such that $S'_i = \{x_i\} \cup \{x'_j | x_j \in S_i\}$. Note that the size of the new family is squared. We show that there is a solution of size at most $k$, and if there is a solution of size at most $1 + k + k^2$ to instance $S'$.

If $T$ is a solution to $S$, then $T' = \{e\} \cup T \cup \{x'_i | x_i \in T\}$ is a solution to $S'$. Indeed, sets in $S \cup \{\{e\}\}$ are obviously hit an odd number of times. And, for any $i \in [n]$ and $j \in [m]$, set $S'_i$ is hit exactly once (by $e$) if $x_i \notin T$, and is hit by $e$, $x_i$, plus as many elements as $S_i$ is hit by $T$; so again an odd number of times. Finally, $[T'] = |T| + |T|^2$.

Conversely, any solution to $S'$ should contain element $e$ (to hit $\{e\}$), and should intersect $U$ in a subset $T$ hitting an odd number of times each set $S_i$ $(\forall i \in [n])$. Then, for each $x_i \in T$, each set $S'_i$ with $j \in [m]$ is hit exactly twice by $e$ and $x_i$. Thus, one has to select a subset of $\{x'_1, \ldots, x'_n\}$ to hit each set of the family $\{S'_1, \ldots, S'_n\}$ an odd number of times. Again, this needs as many elements as a solution to $S$ needs. So, if there is a solution to $S'$ of size at most $1 + k + k^2$, then there is a solution to $S$ of size at most $k$. In fact, we will only use the weaker property that if there is a solution to $S'$ of size at most $k$, then there is a solution to $S$ of size at most $\sqrt{k}$.

Now, assuming there is an $r$-approximation for Odd Set running in time $f(n, m, k)$, we run that algorithm on the instance $S'$ produced from $S$. This takes time $f(1 + n + n^2, 1 + m + nm, 1 + k + k^2)$ and produces a solution of size $r(1 + k + k^2)$. From that solution, we can extract a solution $T$ to $S$ by taking its intersection with $U$. And $T$ has size smaller than $\sqrt{r(1 + k + k^2)} \leq \sqrt{r}(k + 1) = (1 + \frac{\varepsilon}{2})\sqrt{r}k \leq (1 + \varepsilon)\sqrt{r}k$.

Repeated application of the self-improvement in Lemma 8 shows that any constant-ratio approximation implies the existence of $(1 + \varepsilon)$-approximation for arbitrary small $\varepsilon > 0$.

Corollary 8. If Odd Set admits an FPA algorithm with some ratio $r$, then, for any $\varepsilon > 0$, it also admits an FPA algorithm with ratio $1 + \varepsilon$.

Proof. We observe that for any $r' > 1$, there exists an $\varepsilon > 0$ such that $(1 + \varepsilon)\sqrt{r'} \leq \sqrt{r'}$. Thus, applying twice the reduction of Lemma 8, we can improve any fixed-parameter $r'$-approximation to a fixed-parameter $\sqrt{r'}$-approximation. Therefore, applying repeatedly the self-improvement property, we would obtain an FPA algorithm with ratio arbitrarily close to 1.

Now we show that an approximation for Odd Set with ratio $1 + \frac{\varepsilon}{2}$ implies the existence of a $(1 - \varepsilon)$-approximation for $k$-Densest Subgraph. In light of Corollary 8, this means that any constant-factor approximation for Odd Set would violate Assumption 11.

Theorem 12. For any ratio $r$, Odd Set does not have an FPA algorithm with ratio $r$, unless Assumption 11 fails.

Proof. Let $\varepsilon > 0$ be such that $k$-Densest Subgraph and therefore Multicolored $k$-Densest Subgraph do not admit a fixed-parameter $(1 - \varepsilon)$-approximation. We show that an FPA algorithm with ratio $1 + \frac{\varepsilon}{2}$ for Odd Set would contradict Assumption 11. Let $G = (V = C_1 \cup \ldots \cup C_k, E)$ be an instance of Multicolored $k$-Densest Subgraph, and let $X$ be an
optimal solution inducing $m$ edges. For any $\{i, j\} \in {\binom{[k]}{2}}$, we let $E_{\{i, j\}}$ to be the set of edges between $C_i$ and $C_j$.

We build $2(\frac{k}{2})$ instances of Odd Set: one for each subset of $\{1, \ldots, k\}$. One such subset is $\mathcal{P} := \{\{i, j\} \mid E_{\{i, j\}} \cap E(X) \neq \emptyset\}$. In words, $\mathcal{P}$ is a correct guess of which $E_{\{i, j\}}$ are inhabited by the edges induced by the optimal solution $X$. Let $\mathcal{V}$ be the set of indices $i \in [k]$ such that $i$ appears in at least one pair of $\mathcal{P}$, and let $k' = |\mathcal{V}|$. Informally, $\mathcal{V}$ corresponds to the color classes of the vertices which are not isolated in the subgraph induced by $X$.

The universe $U$ consists of an element $x_v$ per vertex $v$ of $C_i$ such that $i \in \mathcal{V}$ and an element $x_e$ per edge $e$ in $E(X)$ such that $\{i, j\} \in \mathcal{P}$. For any vertex $u \in C_i$ and any $j \in [k]$ such that $\{i, j\} \in \mathcal{P}$, we set $S_{u,j} = \{x_v \mid v \in C_i \text{ and } v \neq i \} \cup x_{uv} \mid uv \in E_{\{i, j\}}$ and $u \in \{v, w\}$, and for each $i \in \mathcal{V}$, $S_i = \{x_v \mid v \in C_i\}$. The set-system is $\mathcal{I} = (U, \mathcal{S} = \bigcup_{i \in C_i} \bigcup_{j \in \mathcal{P}} S_{u,j} \cup \bigcup_{v \in C_i} S_v)$.

First, we show that the instance of Odd Set built for subset $\mathcal{P}$ admits a solution of size $k' + m$. Let $X' = \{a_1, \ldots, a_{k'}\} \subseteq X$ be the $k'$ vertices which are not isolated in $G[X]$. We claim that $Z = \{x_{a_i} \mid i \in \mathcal{P}\}$ is an odd set of $\mathcal{I}$. Each $S_i$ with $i \in \mathcal{V}$ is hit by exactly one element of $Z$ since no two $a_p$'s can come from the same color class. Each $S_{u,j}$ with $u \in C_i$, $\{i, j\} \in \mathcal{P}$, and $u \not\in X'$ is hit exactly once by $x_{a_p}$ where the color class of $a_p$ is $C_i$. Each $S_{u,j}$ with $u = a_p \in C_i \cap X'$, $\{i, j\} \in \mathcal{P}$ is hit exactly once by $x_{a_p}$ where the color class of $a_p$ is $C_j$. Finally, $Z$ has the desired size $|X' + |E(X')| = k' + |E(X)| = k' + m$.

Since we have established that $\mathcal{I}$ has an odd set of size $k' + m$, our supposed $1 + \frac{\epsilon}{3}$-approximation would return a solution $Z$ of at most $(1 + \frac{\epsilon}{3})(k' + m)$ elements. We now show how to obtain a good approximation for Multicolored $k$-Densest Subgraph from such a solution to Odd Set. By construction, for each $i \in \mathcal{V}$, the set $S_i$ should be hit an odd number of times, that is $|Z \cap S_i|$ is odd. In particular, $Z \cap S_i$ is non-empty. So, we can build a set $\{x_{u,i} \mid i \in \mathcal{V}\}$ where $x_{u,i}$ is an arbitrary element of $Z \cup S_i$.

Let $\mathcal{E} = \{S_{u,j} \mid \{i, j\} \in \mathcal{P}\}$. Each of the $2|\mathcal{P}|$ sets of $\mathcal{E}$ (note that if, say, $\{1, 2\} \in \mathcal{P}$, then both $S_{1,2}$ and $S_{2,1}$ become a member of $\mathcal{E}$) are hit an even number of times by $Z \cap \bigcup_{i \in \mathcal{V}} S_i$. Indeed, $|(Z \cap \bigcup_{i \in \mathcal{V}} S_i) \cap S_{u,j}| = |Z \cap S_i \setminus \{x_{u,i}\}| = |Z \cap S_i| - 1$ which is even. We observe that each $S_{u,j} \in \mathcal{E}$ intersects with only one other set of $\mathcal{E}$, namely, $S_{u,j}$. So, we need at least $|\mathcal{P}|$ elements to hit the sets in $\mathcal{E}$. If there is an edge between $u_i$ and $u_j$, both $S_{u,i}$ and $S_{u,j}$ can be hit at the same time by including element $x_{u,i,j}$ into the solution. Otherwise $S_{u,i}$ and $S_{u,j}$ are disjoint and at least two elements are necessary to hit them. As there are at least $\frac{k'}{2}$ edges on $k'$ non isolated vertices, we have $y \geq \frac{k'}{2}$. The set $Z \setminus (S_1 \cup \cdots \cup S_{k'})$ contains at most $|Z| - k \leq (1 + \frac{\epsilon}{2})m + \frac{\epsilon}{2}k' \leq (1 + \frac{\epsilon}{2})m + \frac{\epsilon}{2}m = (1 + \epsilon)m$ elements and these elements hit ever set in $\mathcal{E}$. Thus, it can be true only for at most $\epsilon m$ of the pairs in $\mathcal{P}$ that the two sets $S_{u,i}, S_{u,j} \in \mathcal{E}$ cannot be hit by a single element of $Z$. Equivalently, it is true for at least $(1 - \epsilon)m$ of the $m$ pairs in $\mathcal{P}$ that the two sets $S_{u,i}, S_{u,j} \in \mathcal{E}$ are hit by a single element of $Z$. As mentioned previously, that element can only be $x_{u,i,j}$. The fact that such an element actually exists means that there is an edge between $u_i$ and $u_j$. Therefore, $\{u_i\}_{i \in \mathcal{V}}$ induces at least $(1 - \epsilon)m$ edges. It follows that $Z$ is an $(1 - \epsilon)$-approximation for the instance of Multicolored $k$-Densest Subgraph; a contradiction to Assumption [1].

The next evidence against an FPA algorithm for Odd Set is conditional upon the so-called Linear PCP conjecture (LPC, for short).

**Assumption 13 (Linear PCP Conjecture)** There exist constants $0 < \alpha < 1$, $A, B > 0$, such that Max 3-SAT on $n$ variables can be decided with completeness $1$ and error $\alpha$ by a verifier using log $n + A$ random bits and reading $B$ bits out of the proof.

LPC is more an open question than a conjecture. Though, for the moment, LPC has almost always proved to be a necessary hypothesis in showing that a specific problem cannot admit an FPA algorithm [7]. If LPC turns out to be true, the consequence for approximation is that there is a linear reduction introducing a constant gap from 3-SAT to Max 3-SAT. Thus, if we combine this fact with the sparsification lemma of Impagliazzo et al. [30], we may observe the following result:

**Lemma 9 (Lemma 2, [7]).** Under LPC and ETH, there are two constants $r < 1$ and $\delta > 0$ such that one cannot distinguish satisfiable instances of Max 3-SAT with $m$ clauses from instances where at most $rm$ clauses are satisfied in time $2^{m^r}$.

The previous result was in fact stated slightly more generally allowing a weaker form of LPC where the completeness is not 1 but $1 - \epsilon$. We re-stated the lemma this way since we will need perfect completeness. The state-of-the-art PCP concerning the inapproximability of Max 3-SAT, only implies the following:

**Theorem 14 ([31]).** Under ETH, one cannot distinguish satisfiable instances of Max 3-SAT from instances where at most $(\frac{2}{3} + \alpha(1))m$ clauses are satisfied in time $2^{m^{1-o(1)}}$. 
Now, we are set for the following result:

**Theorem 15.** Under LPC and ETH, for any ratio $r$, Odd Set does not have an FPA algorithm with ratio $r$.

**Proof.** Again, the idea is to assume an FPA algorithm with ratio $1 + \varepsilon$ for Odd Set, show that it would imply a too good approximation for MAX 3-SAT in subexponential time, therefore contradicting Lemma [9] and then conclude with Lemma [8].

Let $\phi = \bigwedge_{i \leq k} C_i$ be any instance of MAX 3-SAT, where the $C_i$s are 3-clauses over the set of $n$ variables $V$. We partition the clauses arbitrarily into $k$ sets $A_1, A_2, \ldots, A_k$ of size roughly $\frac{n}{k}$. We denote by $V_i$ the set of all the variables appearing in at least one clause of $A_i$; each $V_i$ has size at most $\frac{3n}{k}$. Of course, while the $A_i$’s are a partition of the clauses, the $V_i$’s can intersect with each other. We build an instance $I = (U, S)$ of Odd Set the following way. For each $i \in [k]$, set $U_i$ contains one element $x(A, i)$ per assignment $A$ of $V_i$ that satisfies all the clauses inside $A_i$. The universe $U$ is $\bigcup_i U_i$. For each $i \neq j \in [k]$, for each variable $x \in V_i \cap V_j$, we set $S_{y,i,j} = \{x(A,i) \mid y$ is set to true by $A\} \cup \{x(A,j) \mid y$ is set to false by $A\}$. Observe that $S_{y,i,j}$ is a solution of size $k$.

If $\phi$ is satisfiable, we fix a (global) satisfying assignment $A_g$. We claim that $S = \{x(A,i) \mid A$ agrees with $A_g$ in the entire $V_i\}$ is a solution of size $k$ to the Odd Set instance. Set $S$ is of size $k$ since for each $i \in [k]$ exactly one element $x(A,i)$ can be such $A$ agrees with $A_g$. This also shows that each set $U_i$ is hit exactly once by $S$. Finally, for each $i \neq j \in [k]$ and for each variable $x \in V_i \cap V_j$, we know that an optimal odd set has more than $\frac{n}{k}$ times (0 or 2). By construction, $y$ is set to true by $A$ and $x(A,j) \mid y$ is set to false by $A$ can be hit at most once. Besides, $y$ is set to true by $A$ is hit exactly once by $S$ if $x(A,j) \mid y$ is set to false by $A$ is not hit by $S$, since the partial assignments mapped to the elements in $S$ necessarily agree. Therefore, $S_{y,i,j}$ is hit exactly once by $S$ and $S$ is a solution.

Now, we assume that Odd Set admits an FPA algorithm with ratio $1 + \varepsilon$ for a small $\varepsilon$ that we will fix later. If the solution $S$ returned by this algorithm on instance $I$ is of size greater than $(1 + \varepsilon)k$, then we know that an optimal odd set has more than $k$ elements, so we know that $\phi$ is not satisfiable. So, we can assume that $|S| \leq (1 + \varepsilon)k$. Each $U_i$ has to be hit at least once and $U_i$s are pairwise disjoint, so we can arbitrarily decompose $S$ into $P \sqcup R$, where $P$ is of size $k$ and hits each $U_i$ exactly once, and therefore $|R| \leq \varepsilon k$. Thus, at least $(1 - \varepsilon)k$ sets $V_i$s are hit exactly once by $S$. We denote by $U$ the set of such sets $U_i$s. Let $A_g$ be the assignment of $V$ agreeing on each assignment $A$ of $V_i$ such that $x(A,i)$ is the only element hitting $U_i \in U$ (and setting the potential remaining variable arbitrarily). Assignment $A_g$ is well defined since if $x(A,i)$ is the only element hitting $U_i \in U$ and $x(A',j)$ is the only element hitting $U_j \in U$, and assignments $A$ and $A'$ disagree on a variable $y$, then $S_{y,i,j}$ would be hit an even number of times (0 or 2) by $A$. By construction, $A_g$ satisfies all the clauses in the $A_i$s such that $U_i \in U$, that is at least $(1 - \varepsilon)k \times \frac{n}{k} = (1 - \varepsilon)n \times |A|$ clauses. Let $r$ and $\delta$ be two constants satisfying Lemma [8]. If we choose $\varepsilon = \frac{1 - \varepsilon}{2}$, this number of clauses exceed $rn$, so we would know that the instance is satisfiable.

Say, the running time of the FPA algorithm is $f(k)(|U| + |S|)^{2c}$ for some constant $c$. We may observe that $|U| \leq k2^{\frac{m}{2}}$ and $|S| \leq k + 2(\frac{k}{m})n$. Thus, the running time is $g(k)n^c2^{\frac{m}{2}}$. Setting $k = \frac{m}{2}$, this running time would be better than $2^{\delta m}$, contradicting LPC or ETH.

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