**Abstract**

The present work is the first of a series of two papers, in which we analyse the higher variational equations associated to natural Hamiltonian systems, in their attempt to give Galois obstruction to their integrability. We show that the higher variational equations $\mathcal{V}E_p$ for $p \geq 2$, although complicated they are, have very particular algebraic structure. Precisely they are solvable if $\mathcal{V}E_1$ is virtually Abelian since they are solvable inductively by what we call the second level integrals. We then give necessary and sufficient conditions in terms of these second level integrals for $\mathcal{V}E_p$ to be virtually Abelian (see Theorem 3.1). Then, we apply the above to potentials of degree $k = \pm 2$ by considering their $\mathcal{V}E_p$ along Darboux points. And this because their $\mathcal{V}E_1$ does not give any obstruction to the integrability. In Theorem 1.2, we show that under non-resonance conditions, the only degree two integrable potential is the harmonic oscillator. In contrast for degree $-2$ potentials, all the $\mathcal{V}E_p$ along Darboux points are virtually Abelian (see Theorem 1.3).

**key words:** Hamiltonian systems; integrability; differential Galois theory;

**MSC2000 numbers:** 37J30, 70H07, 37J35, 34M35.

1 Introduction

Our aim in this paper is to developed methods and tools which allow effective investigation of the integrability of complex Hamiltonian through the analysis of the differential Galois group of their higher order variational equations along a particular solution. This approach is described in, e.g., [2, 1]. The most general result was obtained in [10]. It gives necessary conditions for the complete meromorphic integrability of a meromorphic Hamiltonian system defined on a complex analytic symplectic manifold $M^{2n}$. These conditions are expressed in the following result.

**Theorem 1.1** (Morales-Ramis-Simó). *Assume that a meromorphic Hamiltonian system is integrable in the Liouville sense with first integrals which are meromorphic in a connected neighbourhood*
U of the phase curve $\Gamma$ corresponding to a non-equilibrium solution, and are functionally independent in $U \setminus \Gamma$. Then, for each $p \in \mathbb{N}$, the identity component $(G_p)^c$ of the differential Galois group $G_p$ of $p$-th order variational equations $VE_p$ along $\Gamma$ is Abelian.

For background material, detailed exposition and proof of the above theorem we refer the reader to [10], and [3]. Numerous successful applications of this theorem were obtained just by dealing with the first variational equations $VE_1$. For an overview of these results see, e.g., [9, 8]. The $VE_p$ with $p \geq 2$ are much more complicated systems than $VE_1$. This is why, at present time no systematic studies of the higher variational equations have been made. The aim of these two papers is, among other things, to extract some general structure of the $VE_p$ for a certain wide class of systems.

In order to apply the above theorem one has to know a particular solution of the considered system. Generally it is not easy to find such a particular solution of a given system of nonlinear differential equations. This is the reason why so many efforts have been devoted to natural Hamiltonian systems with homogeneous potentials, for which we can find particular solutions in a systematic way.

Indeed, let us consider a class of Hamiltonian systems with $n$ degrees of freedom generated by a natural Hamiltonian function of the form

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V(q), \quad q = (q_1, \ldots, q_n),$$

where $V$ is a homogeneous function of degree $k \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$. For such systems, the corresponding Hamilton’s equations have the canonical form

$$\frac{d}{dt} q = p, \quad \frac{d}{dt} p = -V'(q), \quad (1.1)$$

where $V'(q) := \text{grad} V(q)$.

Generally equations (1.1) admit particular solutions of the following form. Let a non-zero vector $d \in \mathbb{C}^n$ satisfy

$$V'(d) = \gamma d, \quad \text{where } \gamma \in \mathbb{C}^*. $$

Such a vector is called a proper Darboux point of the potential $V$. It defines a two dimensional plane in the phase space $\mathbb{C}^{2n}$, given by

$$\Pi(d) := \{(q, p) \in \mathbb{C}^{2n} \mid q = \varphi d, \ p = \psi d, \ (\varphi, \psi) \in \mathbb{C}^2 \}. $$

This plane is invariant with respect to system (1.1). Equations (1.1) restricted to $\Pi(d)$ have the form of one degree of freedom Hamilton’s equations

$$\frac{d}{dt} \varphi = \psi, \quad \frac{d}{dt} \psi = -\gamma \varphi^{k-1}, \quad (1.2)$$

with the following phase curves

$$\Gamma_{k,e} := \left\{ (\varphi, \psi) \in \mathbb{C}^2 \mid \frac{1}{2} \psi^2 + \frac{\gamma}{k} \varphi^k = e \right\} \subset \mathbb{C}^2, \quad e \in \mathbb{C}. $$

In this way, a solution $(\varphi, \psi) = (\varphi(t), \psi(t))$ of (1.2) gives rise to solution $(q(t), p(t)) := (\varphi d, \psi d)$ of equations (1.1) with the corresponding phase curve

$$\Gamma_{k,e} := \{(q, p) \in \mathbb{C}^{2n} \mid (q, p) = (\varphi d, \psi d), \ (\varphi, \psi) \in \Gamma_{k,e} \} \subset \Pi(d). \quad (1.3)$$
In this context significant obstructions to the integrability were obtained just by dealing with the first variational equations $\text{VE}_1$. However, as we proved in [4], for $k = \pm 2$, no obstruction can be found on the level of $\text{VE}_1$, since the differential Galois group of $\text{VE}_1$ is virtually Abelian.

For $k = 2$ our main result in is the following theorem.

**Theorem 1.2.** Let $V$ be a homogeneous potential of degree $k = 2$ satisfying the following assumptions:

1. it has a proper Darboux point, i.e., there exits a non-zero vector $d$, such that $V'(d) = \gamma d$ with $\gamma \in \mathbb{C}^*$;

2. the Hessian matrix $\gamma^{-1}V''(d)$ is diagonalisable with eigenvalues

\[ \lambda_1 = \omega_1^2, \ldots, \lambda_n = \omega_n^2, \]

such that $\omega_1, \ldots, \omega_n$ are $\mathbb{Z}$-linearly independent;

3. $V$ is integrable in the Liouville sense.

Then

\[ V(q) = \frac{1}{2}q^T V''(d)q. \]

In other words, in the eigenbasis of $V''(d)$, the Hamiltonian has the following form

\[ H = \frac{1}{2} \sum_{i=1}^{n} (p_i^2 + \omega_i^2 q_i^2). \]

In other words, this result shows that, under the non resonant conditions given by point 2, a homogeneous potential of degree two is integrable if and only if it is a harmonic oscillator.

In contrast, for applications it is important to have a result which gives necessary conditions for the integrability without non-resonance assumptions. We formulate such results in Section 4.3.

As we already mentioned, for case $k = -2$, the second order variational equations do not give any obstacles to the integrability. Truly amazing is the fact that the variational equations of any arbitrary order do not give any obstruction to the integrability. More precisely, we show the following.

**Theorem 1.3.** Let $V$ be a homogeneous potential of degree $k = -2$ which has a proper Darboux point $d$. Then, for each $p \in \mathbb{N}$, the differential Galois group of the higher variational equations $\text{VE}_p$ along the phase curve associated with $d$ is Abelian.

In comparison with the previous result, this theorem for $k = -2$ is quite anecdotic. Nevertheless, it shows the local nature of the Galois obstruction along a particular solution. Indeed, the fact that we do not get obstruction along a Darboux point, in the present case is certainly intimately related with the integrability of homogeneous potentials of degree $k = -2$ with two degrees of freedom.

The paper is organised as follows. In Section 2 we analyse the general structure of the $\text{VE}_p$, for $p \geq 2$. This is presented in the companion a potential of arbitrary degree $k \in \mathbb{Z}$. The general $\text{VE}_p$ are complicated linear system with second member. The main goal of this Section is to show how some significant sub-systems of $\text{VE}_p$ can be extracted in order to apply the Morales-Ramis-Simó Theorem. This will be done explicitly for $\text{VE}_2$, since the
study of these equations for general degrees \( k \) will be the main goal of the second of this paper.

From Section \([2]\) it will be clear that the solutions of \( \text{VE}_{p+1} \) are obtained by adding to the solutions of \( \text{VE}_p \) a certain number of primitive integrals. Formally it means the following.

Let \( F_i/K \) be the Picard-Vessiot extension of \( \text{VE}_i \). Then we have the following tower of inclusions

\[
K \subset F_1 \subset F_2 \subset \cdots \subset F_p \subset F_{p+1},
\]

and \( F_{p+1}/F_p \) is generated by certain number of elements \( \Phi \), such that \( \Phi' \in F_p \). In Section \([3]\) we show that if the differential Galois group \( F_{p+1}/K \) is virtually Abelian, then we have a strong restrictions on the form of integrals \( \Phi \), see Theorem \([3.1]\).

Section \([4]\) contains a proof of Theorem \([1.2]\) and Section \([5]\) contains a proof of Theorem \([1.3]\).

2 General structure of higher order variational equation

2.1 Solvability of higher order variational equations and second level integrals

Let us consider a system of differential equations

\[
\frac{d}{dt} x = v(x), \quad x \in U \subset \mathbb{C}^m, \quad t \in \mathbb{C},
\]  

(2.1)

where \( U \) is an open set, and the right hand sides \( v(x) = (v_1(x), \ldots, v_m(x)) \) are holomorphic. Let \( x_0(t) \) be a particular solution of this system. In a neighborhood of \( x_0(t) \) we represent \( x \) in the following form

\[
x = x_0(t) + \varepsilon x_1 + \frac{1}{2!} \varepsilon^2 x_2 + \frac{1}{3!} \varepsilon^3 x_3 + \cdots,
\]

where \( \varepsilon \) is a formal small parameter. Inserting the above expansion into both sides of equation (2.1), and equaling terms of the same order with respect to \( \varepsilon \) we obtain a chain of equations of the form

\[
\frac{d}{dt} x_p = Ax_p + f_p(x_1, \ldots, x_{p-1}), \quad p = 1, 2, \ldots,
\]  

(2.2)

where

\[
A = \frac{\partial v}{\partial x}(x_0(t)),
\]  

(2.3)

and \( f_1 = 0 \). In this settings, the \( p \)-th equation in the chain is a linear non-homogeneous equation. Its non-homogeneous term depends on general solution of first \( p - 1 \) equations in the chain. We called it the variational equation of order \( p \), and denote it by \( \text{VE}_p \).

In this paper we work with Hamiltonian systems, so in particular \( m = 2n \) and, system (2.1) is the canonical Hamilton equations. Moreover, the matrix \( A \) in (2.3) is an element of \( \mathfrak{sp}(2n, K) \), where \( K \) is the differential ground field.

We assume that all the differential fields which appear in this paper are of characteristic zero and their fields of constants is \( \mathbb{C} = \mathbb{C} \).

Let \( K \) be a differential field. We denote \( a' \), the derivative of \( a \in K \). Recall that the Picard-Vessiot ring of a Picard-Vessiot extension \( F/K \) is the set of elements of \( F \) which are “holonomic” over \( K \). They form the set of elements of \( F \) which are solutions of some non-trivial linear differential equation with coefficients in \( K \). We denote the Picard-Vessiot ring of a Picard-Vessiot extension \( F/K \) by \( T(F/K) \).

In our considerations an important role is played by the notion of “levels of integrals”. 
Definition 2.1. Let \( K \subset F_1 \subset F_2 \) be a “tower of Picard-Vessiot extensions of \( K \)”. By this we mean that \( F_1 / K \) and \( F_2 / K \) are Picard-Vessiot extensions with \( F_1 \subset F_2 \). An element \( \Phi \in F_2 \) is called an integral of second level with respect to \( K \) iff \( \Phi' \in T(F_1 / K) \). If moreover \( \Phi' \in K \), i.e., if \( \Phi \) is a primitive integral over \( K \), we say that \( \Phi \) is an integral of the first level.

Obviously, a first level integral is an integral of second level. Moreover, let us observe that any second level integral is holonomic over \( K \), hence it belongs to \( T(F_2 / K) \).

Let us consider the following system of differential equations

\[
\begin{align*}
    x_1' &= Ax_1, \\
    x_2' &= Ax_2 + B,
\end{align*}
\]

where \( A \in \text{sp}(2n, K) \subset \text{sl}(2n, K) \). Let us assume that the elements of the one column matrix \( B \) belongs to the Picard-Vessiot ring \( T(F_1 / K) \) of equation (2.4a). According to equations (2.2) and (2.3), the second order variational equations of a general Hamiltonian system have such a form. Let \( F_2 \supset K \) be the Picard-Vessiot extension of the whole system (2.4). We have that \( K \subset F_1 \subset F_2 \), and the extension \( F_2 \supset F_1 \) is generated by a certain number of second level integrals. To see this, let us take a fundamental matrix \( X_1 \in \text{Sp}(2n, F_1) \) of equation (2.4a). In order to solve equation (2.4b) we apply the classical variation of constants method. That is, we look for a particular solution \( X_2 \) of the form \( X_2 = X_1 C \), where the column vector \( C \) satisfies

\[
C' = X_1^{-1} B.
\]

Notice that the right hand sides of the above equation belong to \( T(F_1 / K)^{2n} \) since the fundamental matrix \( X_1 \) is unimodular. Hence, we have

\[
X_2 = X_1 \int X_1^{-1} B \iff C = \int X_1^{-1} B.
\]

Thus, as claimed, the field \( F_2 \) is generated over \( F_1 \) by a certain number of elements \( \Phi \) such that \( \Phi' \in T(F_1 / K) \). Precisely, \( F_2 / F_1 \) is generated by the \( 2n \) entries of \( C \).

Taking into account the above facts, and thanks to equation (2.2), we have proved the following.

Lemma 2.2. Let us assume that \( \text{VE}_1 \) has virtually Abelian differential Galois group \( \text{Gal}(\text{VE}_1) \), and let us consider the following tower of PV extensions

\[
K \subset \text{PV}(\text{VE}_1) \subset \text{PV}(\text{VE}_2) \subset \cdots \subset \text{PV}(\text{VE}_p) \subset \text{PV}(\text{VE}_{p+1}) \subset \cdots
\]

Then the following statements hold true for arbitrary \( p \in \mathbb{N} \).

1. In each tower \( K \subset \text{PV}(\text{VE}_p) \subset \text{PV}(\text{VE}_{p+1}) \), the extension \( \text{PV}(\text{VE}_p) \subset \text{PV}(\text{VE}_{p+1}) \) is generated by the second level integrals.

2. Each \( \text{PV}(\text{VE}_p) \) is a solvable Picard-Vessiot extension.

The above lemma shows the particular structure of the \( \text{VE}_p \). Although they are big complicated systems but, nevertheless, they are solvable. As a consequence, our main goal is going to find tractable conditions that distinguish the virtually Abelian ones between all these solvable systems.
Remark 2.3 In our further considerations we will use the following superposition principle. Let us consider a linear non-homogeneous system
\[ \dot{x} = Ax + B_1 + \cdots + B_s, \quad x \in K^m, \] (2.7)
where \( K \) is a differential field, and \( A \in \mathbb{M}(m, K), B_i \in K^m \) for \( i = 1, \ldots, s \). Let \( x_1, \ldots, x_s \in K^m \), satisfy
\[ \dot{x}_i = Ax_i + B_i \quad \text{for} \quad i = 1, \ldots, i. \] (2.8)

Then
\[ \hat{x} = x_1 + \cdots + x_s, \]
is a particular solution of (2.7).

Let \( \hat{L}/K \) and \( L_i/K \) denote the Picard-Vessiot extensions of (2.7), and (2.8), respectively. By \( \hat{G} \) and \( G_i \) we denote the corresponding differential Galois groups. By the above observation we have the following inclusion
\[ \hat{L} \subset L_1 \cdots L_s, \] (2.9)
where the product denotes the composition of fields. Hence, \( \hat{G} \) is an algebraic subgroup of \( G_1 \times \cdots \times G_s \). Thus if \( G_1, \ldots, G_s \) are virtually Abelian, then \( \hat{G} \) is virtually Abelian. Moreover, if in (2.9) we have the equality, then we have also the inverse implication.

2.2 Higher order variational equations along a Darboux point

In this section we show the general structure of the second order variational equations for the Hamiltonian system (1.1) and for a particular solution \( (q(t), p(t)) = (\varphi d, \dot{\varphi} d) \) associated to a proper Darboux point \( d \).

We can rewrite equations (1.1) into Newton form
\[ \ddot{q} = F(q) \] (2.10)
where \( F(q) = -V'(q) \). We put
\[ q = q_0 + \varepsilon q_1 + \frac{1}{2!} \varepsilon^2 q_2 + \frac{1}{3!} \varepsilon^3 q_3 + \cdots \]
where \( q_0 = \varphi(t)d \) is the chosen particular solution, and \( \varepsilon \) is a formal small parameter.

Inserting the above expansion into equation (2.10) and comparing terms of the same order with respect to \( \varepsilon \) we obtain an infinite sequence of equation. The first of them \( \ddot{q}_0 = F(q_0) \), is identically satisfied by assumptions. For further purposes we need the next three equations which are the following
\[ \ddot{q}_1 = F'(q_0)q_1, \] (2.11)
\[ \ddot{q}_2 = F'(q_0)q_2 + F''(q_0)(q_1, q_1), \] (2.12)
\[ \ddot{q}_3 = F'(q_0)q_3 + 3F''(q_0)(q_1, q_2) + F^{(3)}(q_0)(q_1, q_1, q_1), \] (2.13)

From this we see that: \( \text{VE}_1 \) is a linear homogeneous equation given by (2.11). In contrast, the \( \text{VE}_p \) for \( p \geq 2 \) are non-homogeneous linear systems. But their linear part is the same as the one of \( \text{VE}_1 \). Moreover, the second term in \( \text{VE}_2 \) is a quadratic form in the solutions of \( \text{VE}_1 \). A bigger complexity appear in \( \text{VE}_3 \) given by (2.13). The second term in the right hand side of this equation is a bilinear form in solution of \( \text{VE}_1 \) and \( \text{VE}_2 \), while the third term is a cubic form in the solutions of \( \text{VE}_1 \).
Notice that in the considered case $F$ is homogeneous of degree $(k - 1)$. This is why we have
\[ F(q_0) = F(\varphi(t)d) = \varphi(t)^{k-1}F(d) = -\varphi(t)^{k-1}d, \]
and
\[ F^{(i)}(q_0) = \varphi(t)^{k-1-i}F^{(i)}(d) \]

It seems that a global investigation of the whole system (2.11)-(2.13) is too difficult. However, we are going to simplify its study by considering subsystems of them. This will be done at the level of $\text{VE}_2$ and $\text{VE}_3$.

**Remark 2.4** In the above calculations we implicitly assumed that the Darboux point $d$ satisfies $V'(d) = d$. If we have a Darboux point $c$ satisfying $V'(c) = \gamma c$, then $d = \alpha c$ satisfies $V'(d) = \alpha^{k-2}\gamma d$. Hence, if $k \neq 2$ we can choose $\alpha$ in such a way that $\alpha^{k-2}\gamma = 1$, and we do not lose the generality. For $k = 2$ we have to rescale the potential. If $V$ has a Darboux point $c$ satisfying $V'(c) = \gamma c$, then potential $\tilde{V} := \gamma^{-1}V$ has the same integrability properties as $V$, and $\tilde{V}' = c$.

**2.3 Reduction procedure for $\text{VE}_2$ and $\text{VE}_3$**

In order to simplify notations we fix the following conventions. To a differential equation over a ground field $K$ we attach a certain name, e.g., $\text{VE}_2$. The ground field over which we consider this equation is always clearly known from the context. Then, the corresponding Picard-Vessiot extension and its differential Galois group will be denoted by $\text{PV}(\text{VE}_2)$, and by $\text{Gal}(\text{VE}_2)$, respectively.

The main goal of this section is to prove that the differential Galois group $\text{Gal}(\text{VE}_2)$ is virtually Abelian if and only if the differential Galois groups of a certain number of systems extracted from $\text{VE}_2$ are virtually Abelian. Applications of these reductions are given in Section 4.3 below, they will be of crucial importance in the second part of the paper. But since the proofs of Theorems 1.2 and 1.3 are independent of these considerations. As a consequence, these section is not of crucial importance for their understanding.

Let us assume that the Hessian matrix $V''(d)$ is diagonalisable. Then, without loss of generality we can assume that it is diagonal, and we put $V''(d) = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Let us denote also
\[ q_j = (q_{1,j}, \ldots, q_{n,j}) \text{ for } j \in \mathbb{N}. \] (2.14)

Then, the system of equations (2.11), (2.12) and (2.13) reads
\begin{align*}
\dot{q}_{i,1} &= -\lambda_i \varphi(t)^{k-2}q_{i,1}, \quad (2.15) \\
\dot{q}_{i,2} &= -\lambda_i \varphi(t)^{k-2}q_{i,2} + \varphi(t)^{k-3}\Theta^i(q_1, q_1), \quad (2.16) \\
\dot{q}_{i,3} &= -\lambda_i \varphi(t)^{k-2}q_{i,3} + 3\varphi(t)^{k-3}\Theta^i(q_1, q_2) + \varphi(t)^{k-4}\Xi^i(q_1) \quad (2.17)
\end{align*}

where $1 \leq i \leq n$, and $\Theta^i$, and $\Xi^i$ are polynomials of their arguments. Notation!

\[ \Theta^i(q_1, q_2) = \sum_{a, \beta = 1}^{n} \theta_{a, \beta}^{i} q_{a, \beta} q_{\beta, 2}, \quad \text{where } \theta_{a, \beta}^{i} = D_{a, \beta}F_i(d), \] (2.18)

and
\[ \Xi^i(q_1) := \sum_{a, \beta, \gamma = 1}^{n} \xi_{a, \beta, \gamma}^{i} q_{a, \beta} q_{\beta, 1} q_{\gamma, 1}, \quad \text{where } \xi_{a, \beta, \gamma}^{i} = D_{a, \beta, \gamma}F_i(d). \] (2.19)

The first order variational equations $\text{VE}_1$ is given by (2.15). It has the form of a direct product of independent equations. Thus we have a perfect splitting of the problem at this
level. In order to perform effectively an analysis of VE\(_2\) and VE\(_3\) we have to split the problem into smaller subsystems. We can do this in the following way. We set to zero all the variables \(q_{i,1}\) except variable \(q_{a,1}\) in the system (2.15)–(2.16). We get a system of \(n\) independent subsystems of VE\(_2\) which we denote VE\(_{2,a'}\) for \(1 \leq \gamma \leq n\). Such a system has the following form

\[
\begin{align*}
\ddot{q}_{a,1} &= -\lambda_a \varphi(t)q_{a,1}^k, \\
\ddot{q}_{\gamma,2} &= -\lambda_\gamma \varphi(t)q_{\gamma,2}^k + \varphi(t)q_{a,1}^k q_{\gamma,2}^2,
\end{align*}
\] (2.20)

In a similar way, for two fixed indices \(a \neq \beta\), we distinguish \(n\) other subsystems VE\(_{2,(a,\beta)}\) of VE\(_2\). They are subsystems of (2.15)–(2.16) obtained by putting \(q_{i,1} = 0\) except for \(i \in \{a, \beta\}\). They are of the following form

\[
\begin{align*}
\ddot{q}_{a,1} &= -\lambda_a \varphi(t)q_{a,1}^k, \\
\ddot{q}_{\beta,1} &= -\lambda_\beta \varphi(t)q_{\beta,1}^k, \\
\ddot{q}_{\gamma,2} &= -\lambda_\gamma \varphi(t)q_{\gamma,2}^k + \varphi(t)q_{a,1}^k q_{\gamma,2}^2 + 2\lambda_\beta q_{a,1}^k q_{\gamma,2}^2 + 2\lambda_\alpha q_{a,1}^k q_{\gamma,2}^2. 
\end{align*}
\] (2.21)

We fix three indices \(a, \beta, \gamma \in \{1, \ldots, n\}\) such that \(a \neq \beta\). From VE\(_{2,(a,\beta)}\) we extract a system EX\(_{2,(a,\beta)}\) of the following form

\[
\begin{align*}
\ddot{q}_{a,1} &= -\lambda_a \varphi(t)q_{a,1}^k, \\
\ddot{q}_{\beta,1} &= -\lambda_\beta \varphi(t)q_{\beta,1}^k, \\
\ddot{q}_{\gamma,2} &= -\lambda_\gamma \varphi(t)q_{\gamma,2}^k + 2\varphi(t)q_{a,1}^k q_{\gamma,2}^2 + 2\lambda_\beta q_{a,1}^k q_{\gamma,2}^2.
\end{align*}
\] (2.22)

Note that this system is not a subsystem of VE\(_{2,(a,\beta)}\). According to Remark 2.3 we have

\[
\text{PV}(\text{VE}_{2,(a,\beta)}) = \text{PV}(\text{VE}_{2,a})\text{PV}(\text{VE}_{2,\beta})\text{PV}(\text{EX}_{2,(a,\beta)}).
\] (2.23)

The inclusion \(\subset\) is evident. And the reverse inclusion follows from

\[
\text{PV}(\text{EX}_{2,(a,\beta)}) \subset \text{PV}(\text{VE}_{2,a})\text{PV}(\text{VE}_{2,\beta})\text{PV}(\text{VE}_{2,(a,\beta)})
\]
which corresponds to the subtractions of particular solutions. Hence again, by Remark 2.3 we have that Gal(VE\(_{2,(a,\beta)}\)) is virtually Abelian if and only if the groups Gal(VE\(_{2,a}\)), Gal(VE\(_{2,\beta}\)), and Gal(EX\(_{2,(a,\beta)}\)) are virtually Abelian.

By the above facts and again by Remark 2.3 we have proved the following.

**Proposition 2.5.** The differential Galois group Gal(VE\(_2\)) is virtually Abelian if and only if the groups Gal(VE\(_{2,a}\)) and Gal(EX\(_{2,(a,\beta)}\)) with \(a, \beta, \gamma \in \{1, \ldots, n\}\) and \(a \neq \beta\), are virtually Abelian.

**3 Second Level Integrals and Virtually Abelian Galois Groups**

According to Lemma 2.2 we know that if Gal(VE\(_1\)) is virtually Abelian, then Gal(VE\(_p\)) is solvable for an arbitrary \(p \in \mathbb{N}\). Therefore, our main goal in this section is to find a necessary and sufficient condition which guarantee that Gal(VE\(_p\)) is virtually Abelian for \(p \in \mathbb{N}\).

From what follows, all the PV extensions \(F/K\) will have the same algebraically closed field of constants \(C = \mathbb{C}\).
We have to analyse the following structure

\[ K \subset F_1 \subset F, \]

where \( F/K \) and \( F_1/K \) are PV extensions, \( \text{Gal}(F_1/K) \) is virtually Abelian, and \( F/F_1 \) is generated by second level integrals \( \Phi_1, \ldots, \Phi_q \). We can assume that these integrals are independent over \( F_1 \). Hence, \( H := \text{Gal}(F/F_1) \) is a vector group isomorphic to \( C^q \). Therefore, we get the following exact sequence of algebraic groups

\[ 0 \rightarrow H = C^q \rightarrow \text{Gal}(F/K) \rightarrow \text{Gal}(F_1/K) \rightarrow 0. \]

As a consequence, the algebraic closure \( \bar{K} \) of \( K \) in \( F_1 \) coincides with the algebraic closure of \( K \) in \( F \).

In order to decide whether or not, \( F/K \) is virtually Abelian, we shall use the following result.

**Theorem 3.1.** Let \( K \subset F_1 \subset F \) be a tower of Picard-Vessiot extensions such that \( F/F_1 \) is generated by integral of second level over \( K \). Then \( G := \text{Gal}(F/K) \) is virtually Abelian if and only if \( G_1 := \text{Gal}(F_1/K) \) is virtually Abelian and any second level integrals \( \Phi \in F \) can be expanded into the form

\[ \Phi = R_1 + J, \quad \text{with} \quad R_1 \in T(F_1/K) \quad \text{and} \quad J' \in \bar{K}, \]

where \( \bar{K} \) is the algebraic closure of \( K \) in \( F_1 \). Moreover, a necessary condition for the virtual Abelianity of \( G \) is that we get: \( \sigma(\Phi) - \Phi \in T(F_1/K) \) for all \( \sigma \in G^\circ \).

We can interpret this result is the following way. The fact that \( \text{Gal}^\circ(F/K) \) is Abelian implies that any given second level integral can be computed thanks to first level integral and exponential of integrals over \( \bar{K} \).

### 3.1 Proof of Theorem 3.1

For sake of clarity, here we recall some classical facts about Abelian algebraic groups and Abelian PV extensions. Most of these results are contained in [5].

In a linear algebraic group \( G \) any element \( x \in G \) has a Jordan-Chevalley decomposition \( x = x_sx_u = x_u x_s \) with \( x_u \) and \( x_s \) belonging to \( G \). We shall denote by \( G_s \) and by \( G_u \) the semi-simple and the unipotent parts of \( G \).

**Proposition 3.2.**

1. If \( G \) is connected and Abelian then \( G_s \) and \( G_u \) are connected algebraic subgroups of \( G \) and \( G = G_s \times G_u \).

2. In the previous context, \( G_s \) is a torus, i.e., is isomorphic to some \((G_m)^p\).

3. A unipotent Abelian algebraic group is a vector group, i.e., is isomorphic to some \((G_a)^q\). Moreover, any algebraic group morphism between two of them is linear.

4. If \( 0 \rightarrow (G_a)^q \rightarrow G 
\rightarrow G_1 \rightarrow 0 \) is an exact sequence of connected Abelian algebraic group, then this sequence splits. That is, \( G \) contains a copy of \( G_1 \), and \( G \cong G_1 \times (G_a)^q \).

5. A connected and Abelian linear algebraic group is isomorphic to some \((G_m)^p \times (G_a)^q\).

**Proof.** Point 1 is the theorem from [6] page 100. Point 2 follows from the theorem on page 104 in [6], and point 5 follows from 1, 2, and 3.

We prove point 3. Let \( G \) be unipotent and algebraic. It can be viewed as a closed subgroup of some \( U(n, C) \) the group of the \( n \times n \) unipotent upper triangular matrices. Hence, \( g = \text{Lie}(G) \) is a subalgebra of \( \text{gl}(n, C) = \text{Lie}(U(n, C)) \), the Lie algebra of upper...
triangular nilpotent matrices. The exponential mapping \( \exp : \mathfrak{n}(n, C) \rightarrow U(n, C) \) is one to one with inverse the classical Log mapping. Therefore, \( \exp \) is also a one-to-one mapping from \( \mathfrak{g} \) to \( G \). Since \( \mathfrak{g} \) and \( G \) are Abelian, \( \exp \) is a group morphism, hence an isomorphism. More precisely, let \( (N_1, \ldots, N_d) \) be a \( C \)-basis of \( \mathfrak{g} \). The mapping \( f : C^d \cong (G_a)^d \rightarrow G \) given by

\[
f(t_1, \ldots, t_d) := \exp(t_1 N_1 + \cdots + t_d N_d) = \exp(t_1 N_1) \times \cdots \times \exp(t_d N_d),
\]

is an isomorphism of algebraic groups.

Now let \( \varphi : G \cong (G_a)^d \rightarrow G' \cong (G_a)^q \) be a morphism of algebraic groups. From a \( C \)-basis \( (e_1, \ldots, e_d) \) of \( (G_a)^d \), we get \( \forall t = (t_1, \ldots, t_d) \in \mathbb{Z}^d, \)

\[
\varphi(t_1 e_1 + \cdots + t_d e_d) = t_1 \varphi(e_1) + \cdots + t_d \varphi(e_d).
\]

But since both terms of this formula are polynomial in \( t \), (3.1) holds for all \( t \in C^d \) since \( \mathbb{Z}^d \) is Zarisky dense in \( C^d \). Hence \( \varphi \) is linear.

It remains to prove point 4. Let \( f \) be the algebraic group morphism corresponding to the arrow \( G \rightarrow G_1 \). According to [6, Th. 15.3, p. 99], for all \( x \in G \), \( f(x) = f(x_0) \) and \( f(x) = f(x_u) \). Hence, the restriction of \( f \) to \( (G_a)^q \) must be surjective and possessing a trivial kernel since the semi-simple part of \( (G_a)^q \) is trivial. As a consequence, \( (G_a)^q \cong (G_1)_q \). Similarly, we get an exact sequence for the unipotent parts \( 0 \rightarrow (G_a)^q \rightarrow (G_u)_q \rightarrow (G_1)_u \rightarrow 0 \). Thanks to point 1, we are reduced to prove that this latter sequence splits. But this is obviously true since by point 3 this sequence reduces to a sequence of linear spaces whose arrow are linear mappings.

Now we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1** Let us assume that \( G := \text{Gal}(F/K) \) is virtually Abelian. Let us denote by \( G_1 := \text{Gal}(F_1/K) \). The proof will be the consequence of the two following steps.

**First Step.** The proof reduces to the case where \( G \) is connected. In this case \( K = K_1 \) and \( G_1 \) is also connected and Abelian. According to the exact sequence (3) and Proposition 3.2 (4),

\[
G \cong G_1 \times H \cong G_1 \times (G_a)^q \text{ for some } q \in \mathbb{N}.
\]

If \( q = 0 \) then \( F = F_1 \), and the Theorem follows. Now let us assume that \( q \geq 1 \). Let us set

\[
M := F^{G_1}.
\]

Since \( G_1 \triangleleft G \), the extension \( M/K \) is Picard-Vessiot with Galois group \( \text{Gal}(M/K) \cong H \cong (G_a)^q \). From [11, Example 1.141, p.32], there exist \( J_1, \ldots, J_q \in M \) with \( J_i^H \in K \) such that \( M = K(J_1, \ldots, J_q) \). The composition field \( F_1 M \) is a differential sub-field of \( F \) which has trivial stabiliser in \( G \). Indeed, if \( \sigma \in G \) fixes point-wise all the elements of \( F_1 M \), then \( \sigma \in H \) since it fixes all the elements belonging to \( F_1 \), and it also belongs to \( G_1 \) since it fixes all the elements belong to \( M \). Therefore, \( \sigma = 1 \) and

\[
F = F_1 M = F_1(J_1, \ldots, J_q).
\]

Since \( \text{tr. deg}(F/F_1) = \dim_C(H) = q \), we deduce that \( J_1, \ldots, J_q \) are algebraically independent over \( F_1 \).

**Second Step.** Let \( \Phi \) be a second level integral. Since \( \Phi' \in F_1 \) and \( \{J_1, \ldots, J_q\} \) is a transcendental basis of \( F/F_1 \), \( \{\Phi, J_1, \ldots, J_q\} \) are \( q + 1 \) algebraically dependant first level
integrals over $F_1$. Therefore, according to the Ostrowski-Kolchin Theorem, see \[7\], there exist $(c_1, \ldots, c_q) \in \mathbb{C}^q$ such that

$$
\Phi - \sum_{i=1}^q c_i J_i = R_1 \in F_1.
$$

But $\Phi \in T(F/K)$ as well as $J := \sum_{i=1}^q c_i J_i$. Therefore, $R_1 \in T(F_1/K)$ and the first implication of the theorem is proved.

**Conversely** If $G^\circ_i$ is Abelian and each second level integral has the form $\Phi = R_1 + J$, since $F/F_1$ is generated by those $\Phi_i$; it is also generated by the corresponding $J_i$ which are integral of first level w.r.t to $\tilde{K}$. But according to Exercise 1.41 on page 32 in \[11\], $F_1/\tilde{K}$ is generated by some integrals of first level w.r.t to $\tilde{K}$ and exponentials of integrals. Therefore, the same happens for $F/\tilde{K}$ and $G^\circ$ is Abelian.

**Finally**, if $\Phi$ can be written $\Phi = R_1 + J$. Since $F/F_1$ is generated by those $\Phi_i$; it is also generated by the corresponding $J_i$ which are integral of first level w.r.t to $\tilde{K}$. But according to Exercise 1.41 on page 32 in \[11\], $F_1/\tilde{K}$ is generated by some integrals of first level w.r.t to $\tilde{K}$ and exponentials of integrals. Therefore, the same happens for $F/\tilde{K}$ and $G^\circ$ is Abelian.

This proves the necessary condition.

\[11\]

### 3.2 Additive and multiplicative versions of Theorem 3.1

Here, we apply this result in two special cases summarized in the following two lemmas.

**Lemma 3.3.** Let $L/K$ and $L_1/K$ be Picard-Vessiot extensions with $L_1 = K(I_1, \ldots, I_s)$, where $I_i' \in K$, for $i = 1, \ldots, s$. Assume that there exists a $C$-linear combination $I$ of these integrals $I_i$ which is transcendental over $K$. Assume further that there exist $\Phi \in L$, and $w \in K^*$ such that

$$
\Phi' = wI, \quad \text{that is} \quad \Phi = \int wI.
$$

Then, the Galois group $\text{Gal}(L/K)$ is virtually Abelian, implies that there exists a constant $c$ such that

$$
cI - \int w \in K.
$$

Equivalently, $\Phi$ can be computed thanks to a closed formula of the form

$$
\Phi = P(I) + J \quad \text{with} \quad P(I) := \frac{c}{2}I^2 + gI \in K[I], \quad \text{and} \quad J' \in K.
$$

**Proof.** Since a $C$-linear combination of primitive integrals over $K$ still is a primitive integral over $K$, without loss of generality, we may prove the result in the restricted case where:

$$
L_1 = K(I), \quad \text{with} \quad I' \in K, I \notin K.
$$

For all $\sigma \in G := \text{Gal}(L/K)$, we have an additive formula of the form

$$
\sigma(I) = I + c(\sigma), \quad \text{with} \quad c(\sigma) \in \mathbb{C}.
$$

Hence,

$$
\sigma(\Phi') = \sigma(wI) = w\sigma(I) = w(I + c(\sigma)) = \Phi' + c(\sigma)w.
$$

So, there exists some constant $d(\sigma)$ such that

$$
\sigma(\Phi) - \Phi = c(\sigma) \int w + d(\sigma)
$$

(3.2)
Now let us assume that $G$ is virtually Abelian. According to Theorem 3.1, for all $\sigma \in G^\circ$,

$$\sigma(\Phi) - \Phi = c(\sigma) \int w + d(\sigma) \in K[I] = T(L_1/K).$$

Hence, if we choose $\sigma_0 \in G^\circ$ such that $c(\sigma_0) = 1$, we get that

$$\int w \in K[I].$$

Therefore, the two primitive integrals $\int w$ and $I$ are algebraically dependant over $K$. In such a case, by the Ostrowski-Kolchin theorem, there are two constants $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{0\}$ such that

$$\alpha \int w + \beta I \in K.$$

But in this relation $\alpha$ cannot be zero since $I$ is transcendental, and the result follows. The converse implication follows by integration by part. \hfill \Box

**Lemma 3.4.** Let $L/K$ and $L_1/K$ be Picard-Vessiot extensions with $L_1 = K(E_1, \ldots, E_s)$, where $E_i'/E_i \in K$, for $i = 1, \ldots, s$. Assume that $L$ contain one element $\Phi$ of the following form

$$\Phi := \int \sum_{i=1}^{r} w_i M_i$$

(3.3)

where $w_i \in K^*$, and each

$$M_i = M_i(E_1, \ldots, E_s) \in \mathbb{C}[E_1, E_1^{-1}, \ldots, E_s, E_s^{-1}],$$

is a monomial, for all $1 \leq i \leq r$. Moreover, $M_1, \ldots, M_r$ are not mutually proportional, i.e.,

$$\frac{M_i}{M_j} \notin K \quad \text{for} \quad i \neq j.$$

Then, we have:

1. Each separated integral $\Phi_i := \int w_i M_i \in L$.

2. If the extension $L/L_1$ is generated by $\Phi$, then $L/K$ is virtually Abelian implies that for each $1 \leq i \leq r$, there exists $c_i \in \mathbb{C}$, such that

$$\frac{\Phi_i + c_i}{M_i} \in K.$$

**Proof.** First we prove point 1. Let us take $M = E_1^{n_1} \cdots E_s^{n_s}$ with $n_1, \ldots, n_s \in \mathbb{Z}$. If for $\sigma \in \text{Gal}(L/K)$ we have

$$\sigma(E_i) = \rho_i(\sigma)E_i, \quad \text{for} \quad 1 \leq i \leq s,$$

then

$$\sigma(M) = \chi_M(\sigma)M,$$

where $\chi_M$ is a character of $\text{Gal}(L/K)$ given by

$$\chi_M(\sigma) := \rho_1(\sigma)^{n_1} \cdots \rho_s(\sigma)^{n_s}.$$

We denote $\chi_i = \chi_{M_i}$ for $1 \leq i \leq r$. As by assumption $M_i/M_j \notin K$ for $i \neq j$, we have

$$\chi_i \neq \chi_j, \quad \text{for} \quad i \neq j,$$
Now, for each $\sigma \in \text{Gal}(L/K)$ we have

$$\sigma(\Phi') = \sum_{j=1}^{r} \sigma(w_j)M_j = \sum_{j=1}^{r} \chi_j(\sigma)w_jM_j = \sum_{j=1}^{r} \chi_j(\sigma)\Phi'_j.$$  

Hence, we get that for each $\sigma \in \text{Gal}(L/K)$

$$\sigma(\Phi) - \sum_{j=1}^{r} \chi_j(\sigma)\Phi_j \in \mathbb{C} \implies \sum_{j=1}^{r} \chi_j(\sigma)\Phi_j \in L. \quad (3.4)$$

Now, since $\chi_i \neq \chi_j$ for $i \neq j$, by the Artin-Dedekind lemma, the characters $\chi_1, \ldots, \chi_r$ are $\mathbb{C}$-linearly independent. As a consequence there exist $r$ elements $\sigma_1, \ldots, \sigma_r$ of $\text{Gal}(L/K)$ such that $r \times r$ matrix $[\chi_j(\sigma_i)]$ is invertible. If we write $r$ corresponding equations $(3.4)$ and invert this system we obtain that each $\Phi_j \in L$, for $1 \leq j \leq r$.

Now, we prove point 2. As in the proof of the previous Lemma, we can assume without loss of the generality that here, $\Phi = \int wE$, where

$$L_1 = K(E), \quad \text{with} \quad E'/E \in K, \quad \text{and} \quad \text{tr.deg}(L_1/K) = 1.$$ 

For each $\sigma \in G := \text{Gal}(K/L)$, we have a multiplicative formula of the form

$$\sigma(E) = \lambda(\sigma)E, \quad \text{with} \quad \lambda(\sigma) \in \mathbb{C}^*.$$ 

Hence,

$$\sigma(\Phi') = \sigma(wE) = w\sigma(E) = \lambda(\sigma)\Phi'.$$

So, there exists some constant $d(\sigma)$ such that

$$\sigma(\Phi) = \lambda(\sigma)\Phi + d(\sigma) \quad (3.5)$$

From $(3.5)$, the linear representation of $G^o$ in $V := \text{span}_\mathbb{C}\{1, \Phi\}$, gives a morphism of algebraic groups

$$\rho : G^o \to G_{\text{aff}}, \quad \sigma \mapsto \rho(\sigma) := \begin{bmatrix} 1 & d(\sigma) \\ 0 & \lambda(\sigma) \end{bmatrix}.$$ 

The affine group $G_{\text{aff}} \simeq G_m \ltimes G_a$ is a solvable, non Abelian group of dimension two. Since $\lambda(\sigma)$ cover all non zero constant values, the image $\rho(G^o)$ contains a maximal torus which is therefore isomorphic to $G_m$.

So, if we assume that $G^o$ is Abelian, then the image $\rho(G^o)$ must be a maximal torus. As a consequence, all the matrices $\rho(\sigma)$ with $\sigma \in G^o$ can be simultaneously diagonalisable in a fixed basis

$$\left\{ [1,0]^T, [c,1]^T \right\}, \quad \text{for some} \quad c \in \mathbb{C}.$$ 

Hence, by direct computation we get that the $\rho(\sigma)$ are of the form

$$\rho(\sigma) = \begin{bmatrix} 1 & d(\sigma) \\ 0 & \lambda(\sigma) \end{bmatrix} \quad \text{with} \quad d(\sigma) = c[\lambda(\sigma) - 1].$$

Taking this into account, we see that for all $\sigma \in G^o$, formula $(3.5)$ reads

$$\sigma(\Phi) = \lambda(\sigma)\Phi + c[\lambda(\sigma) - 1].$$

So

$$\sigma(\Phi + c) = \lambda(\sigma)[\Phi + c].$$
an this gives
\[ \sigma \left( \frac{\Phi + c}{E} \right) = \frac{\Phi + c}{E}. \]

This means that \((\Phi + c)/E\) is algebraic over \(K\). But now \(\Phi + c\) is a first level integral with respect to \(K(E)\). That it is algebraic over this field implies that it belongs to \(K(E)\). But now, the restriction morphism \(G^o \to \text{Gal}(K(E)/K) \simeq G_m\) is surjective. Therefore, \((\Phi + c)/E\) is fixed by \(G^o\) implies that it belongs to \(K\).

Let us observe that, curiously, the last lemma was proved without any reference to Theorem 3.1.

4 Potential of degree \(k = 2\) and their VE

In this section we prove Theorem 1.2 and also give some results concerning resonant cases.

4.1 The VE for \(k = 2\)

According to Proposition 2.5 to find the necessary and sufficient conditions which guarantee that \(\text{Gal}(\text{VE}_2)\) is virtually Abelian, we have to find such conditions for \(\text{VE}_{2,\alpha}^\gamma\) and \(\text{EX}_{2,(\alpha,\beta)^\gamma}\) with \(\alpha, \beta, \gamma \in \{1, \ldots, n\}\) such that \(\alpha \neq \beta\).

Since we consider the case \(k = 2\), a particular solution \(\phi(t)\) with energy \(e = 1/2\) satisfies \(\dot{\phi}^2 + \phi^2 = 1\). Thus, we can take, e.g.,
\[ \phi(t) = \sin t. \]

We denote by \(\Gamma\) the corresponding phase curve in \(\mathbb{C}^{2n}\). The first order variational equation
\[ \ddot{q}_1 := -\phi^{k-2}V''(d)q_1 = -V''(d)q_1 \]
is a matrix second order equation with constant coefficients over the differential ground field
\[ K = \mathbb{C}(\phi, \dot{\phi}) = \mathbb{C}(e^{it}). \]

Hence, we can rewrite it in the following form
\[ x_1 = Ax_1, \quad \text{where} \quad A := \begin{bmatrix} 0_n & E_n \\ -V''(d) & 0_n \end{bmatrix} \in \text{sp}(2n, \mathbb{C}). \quad (4.1) \]

It is easy to show that the matrix \(A\) has eigenvalues \(\pm i\omega\), where \(\omega^2 = \lambda\), when \(\lambda\) span the eigenvalues of \(V''(d)\). Thus, the entries of the fundamental matrix \(X_1\) of equation (4.1) belong to a ring of the form
\[ R_1 := \mathbb{C}(e^{it}) \left[ e^{\pm i\omega_1 t}; \ldots; e^{\pm i\omega_n t}; t \right], \]
where \(\omega_1, \ldots, \omega_n\) are the eigenvalues of \(V''(d)\).

With these notations and assumptions, \(\text{VE}_{2,\alpha}^\gamma\) can be rewritten into the following form
\[
\begin{align*}
\dot{x} &= -\omega_\alpha^2 x, \\
\dot{z} &= -\omega_\gamma^2 z + \frac{\theta_{\alpha,\gamma}}{\sin t} x^2.
\end{align*}
\]

(4.2)
where, to simplify notations, instead of blind variables $q_{\alpha,1}$ and $q_{\gamma,2}$, we introduce $x$ and $z$.

In a similar way we rewrite $\text{EX}^{\gamma}_{2,\alpha}$ into the form

$$
\begin{align*}
\ddot{x} &= -\omega_{\alpha}x, \\
\ddot{y} &= -\omega_{\beta}y, \\
\ddot{z} &= -\omega_{\gamma}z + 2\frac{\theta_{\epsilon,\beta}}{\sin t}xy.
\end{align*}
$$

(4.3)

The last equations in (4.2) and (4.3) have the same form

$$
\ddot{z} = -\omega^{2}z + b(t)
$$

(4.4)

with $e^{i\omega t}$ and $b(t)$ belonging to $R_{1}$.

Now, we consider equation (4.4) over $K = \mathbb{C}(e^{it})$. Our aim is to compute its Picard-Vessiot extension $L/K$. Let $L_{1}$ be the Picard-Vessiot extension of $K$ containing: $b(t), \dot{b}(t)$, and all the solutions of $\ddot{z} = -\omega^{2}z$. Since $b(t)$ is holonomic over $K$, it belongs to the Picard-Vessiot ring $T(L_{1}/K)$. Moreover, the extension $L/L_{1}$ is generated by the second level integrals over $K$. The form of these integrals is described into the following property.

**Lemma 4.1.** With the above notations the following statements hold true.

1. If $\omega = 0$, then $L/L_{1}$ is generated by

$$
\int \frac{b(t)}{\sin t} dt, \quad \text{and} \quad \int \frac{tb(t)}{\sin t} dt.
$$

2. If $\omega \neq 0$, then $L/L_{1}$ is generated by

$$
\int \frac{e^{i\omega t}b(t)}{\sin t} dt, \quad \text{and} \quad \int \frac{e^{-i\omega t}b(t)}{\sin t} dt.
$$

**Proof.** We rewrite equation (4.4) as the following non-homogeneous linear system

$$
\begin{pmatrix}
\ddot{z}_1 \\
\ddot{z}_2
\end{pmatrix} = A \begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} + b(t)
$$

(4.5)

where

$$
A = \begin{bmatrix}
0 & 1 \\
-\omega^{2} & 0
\end{bmatrix}, \quad \text{and} \quad b = \frac{1}{\sin t} \begin{bmatrix}
0 \\
b(t)
\end{bmatrix}
$$

Let us denote by $Z$ a fundamental matrix of solutions of the homogeneous part of (4.5).

The variation of constants gives a particular solution of the form $\ddot{z} = Zc$ with $c = [c_{1}, c_{2}]^{T}$ satisfying

$$
\frac{d}{dt}c = Z^{-1}b.
$$

Since $Z \in \text{GL}(2,L_{1})$, and $L/L_{1}$ is generated by the two entries of $\ddot{z}$, it is also generated by $c_{1}$ and $c_{2}$.

Now, if $\omega = 0$, then

$$
Z = \begin{bmatrix}
1 & t \\
0 & 1
\end{bmatrix}, \quad \text{and} \quad \dot{c} = Z^{-1}b = \frac{b(t)}{\sin t} \begin{bmatrix}
-t \\
1
\end{bmatrix}
$$

If $\omega \neq 0$, then

$$
Z:= \begin{bmatrix}
e^{i\omega t} & e^{-i\omega t} \\
i\omega e^{i\omega t} & -i\omega e^{-i\omega t}
\end{bmatrix}, \quad \text{and} \quad \dot{c} = Z^{-1}b = \frac{ib(t)}{2\omega \sin t} \begin{bmatrix}
e^{-i\omega t} \\
e^{i\omega t}
\end{bmatrix}
$$

Now, in both cases, the claim follows easily. \qed
4.2 Proof of Theorem 1.2

4.2.1 Taylor expansion of $V$ around the Darboux points.

We can assumed that the considered Darboux point $d$ satisfies $V'(d) = d$. Moreover, we also assume that $V''(d) = \text{diag}(\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2)$. From the Euler identity we easily deduce that $V''(d)d = d$. Hence, 1 is an eigenvalue of $V''(d)$ with the corresponding eigenvector $d$.

This implies that one of $\omega_i^2$ is one, and we can assume that $\omega_n = 1$. Thus, $d = [0, \ldots, 0, 1]^T$.

We also assumed that $V$ is analytic around this Darboux point. So, we have the following Taylor expansion

$$V(q) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \partial^\alpha V(d) \cdot \tilde{q}^\alpha$$

where $\tilde{q} = q - d = (q_1, \ldots, q_{n-1}, q_n - 1)$. In the above we use the standard multi-index notations. That is,

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \quad |\alpha| := \sum_{i=1}^n \alpha_i, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad q^\alpha := q_1^{\alpha_1} \cdots q_n^{\alpha_n},$$

and

$$\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad \text{where} \quad \partial_i^{\alpha_i} := \frac{\partial^{\alpha_i}}{\partial q_i^{\alpha_i}}.$$ 

Taking into account that $V'(d) = d$, we also have by the Euler identity $V(d) = 1/2$, and the above expansion can be written in the form

$$V(q) = \frac{1}{2} (\omega_1^2 q_1^2 + \cdots + \omega_{n-1}^2 q_{n-1}^2 + q_n^2) + \sum_{|\alpha| \geq 3} \frac{1}{\alpha!} \partial^\alpha V(d) \cdot \tilde{q}^\alpha. \quad (4.6)$$

Thus, our aim is to prove that

$$\partial^\alpha V(d) = 0, \quad \text{for all} \quad |\alpha| = m \geq 3,$$

when the potential is integrable. This will be done by induction with respect to $m$.

Let us return to the equations of motion written in Newton form given by equation (2.10), i.e.,

$$\ddot{q} = F(q),$$

where $F(q) = -V'(q)$, i.e.,

$$F_i(q) = -\frac{\partial V}{\partial q_i}(q) = -\partial_{\xi_i} V(q).$$

In the above, the $i$-th component of multi-index $\xi_i$ is equal one, and all remaining vanish. Thus, we have

$$\partial^\alpha F_i(d) = -\partial^{\alpha_\xi_i} V(d), \quad (4.7)$$

or equivalently

$$\partial^\alpha V(d) = -\partial^{\alpha_\xi_i} F_i(d).$$
4.2.2 The induction procedure.

Let \( p \geq 2 \). We assume that \( \text{VE}_p \) has the following simple form

\[
\begin{align*}
\dot{x}_i &= -\omega_i^2 x_i, \quad 1 \leq i \leq n, \\
\dot{y}_j &= -\omega_j^2 y_j + \sum_{|\alpha|=p} \frac{\zeta_j^{|\alpha|} x^\alpha}{\sin^{p-1}(t)} , \quad 1 \leq j \leq n,
\end{align*}
\]

where \( x = (x_1, \ldots, x_n) \), and

\[ \zeta_j^{|\alpha|} := \frac{1}{|\alpha|!} \partial^{|\alpha|} F_j(d). \]

According to (4.2) and (4.3), we notice that \( \text{VE}_2 \) has such form. For \( p \geq 3 \), according to Section 2.3, \( \text{VE}_p \) will have such a simple form iff

\[ \partial^{|\gamma|} F_j(d) = 0, \quad \text{with} \quad 1 \leq j \leq n \]

whenever,

\[ 2 \leq |\gamma| < p. \]

Our aim is therefore to prove that \( \zeta_j^{|\alpha|} = 0 \), for all \( 1 \leq j \leq n \), and all \( \alpha \) such that \( |\alpha| = p \).

**First Step.** Let \( K = \mathbb{C}(e^{it}) \) be our ground field, and \( L := \text{PV}(\text{VE}_p) \) be the Picard-Vessiot extension of \( \text{VE}_p \) over \( K \). The field \( L \) contains

\[ L_1 := \text{PV}(\text{VE}_1) = K(e^{i\omega_1 t}, \ldots, e^{i\omega_{n-1} t}). \]

According to Lemma 4.1, \( L \) contains second level integrals of the form

\[ \Phi_j = \int \sum_{|\alpha|=p} \frac{\zeta_j^{|\alpha|} x^\alpha}{\sin^{p-1}(t)} e^{i\omega_j t}, \quad \text{for all} \quad 1 \leq j \leq n. \]

In particular, if we choose

\[ x = (x_1, \ldots, x_n) = (e^{i\omega_1 t}, \ldots, e^{i\omega_n t}), \quad \text{with} \quad \omega_n = 1, \]

then

\[ x^\alpha = e^{i(\alpha_1 \omega_1 + \cdots + \alpha_n \omega_n) t} = e^{i(\alpha \cdot \omega)t}, \]

and \( L \) contains integrals of the form

\[ \Phi_j = \sum_{|\alpha|=p} \zeta_j^{|\alpha|} \int \frac{1}{\sin^{p-1}(t)} e^{i(\alpha \cdot \omega + \omega_j) t} = \sum_{|\alpha|=p} \zeta_j^{|\alpha|} T_p^{(\alpha \cdot \omega + \omega_j)}, \]

where the “trigonometric integrals” \( T_p^{(\omega)} \) are defined and studied in the Appendix.

In accordance with Lemma 3.4 let us consider the following monomials

\[ M_\alpha := e^{i(\alpha \cdot \omega_j) t} \]

with fixed \( j \). We claim that they are not mutually proportional. We prove this claim by contradiction. Thus, let \( \alpha \) and \( \alpha' \) be two different multi-indices, and assume that

\[ \frac{M_{\alpha'}}{M_\alpha} = e^{i(\alpha' - \alpha \cdot \omega) t} \in K = \mathbb{C}(e^{it}). \]
This implies that
\[(\alpha' - \alpha) \cdot \omega = (\alpha'_1 - \alpha_1)\omega_1 + \cdots + (\alpha'_{n-1} - \alpha_{n-1})\omega_{n-1} + (\alpha'_n - \alpha_n) = m \in \mathbb{Z}.
\]

But, since \(\omega_1, \ldots, \omega_{n-1}\) are \(\mathbb{Z}\)-linearly independent, this implies that
\[a'_i = a_i \quad \text{for} \quad 1 \leq i < n.
\]
Since \(|\alpha| = |\alpha'|\), we also have \(a'_n = a_n\), and in effect \(\alpha' = \alpha\). This is a contradictory with the assumption that \(\alpha' \neq \alpha\). In this way we proved our claim.

Now, according to the first point of Lemma 3.4, we have that each term in the sum \((4.9)\) is an element of \(L\), i.e.,
\[c^j \xi^j T^{(\alpha \cdot \omega + \omega_j)}_{p-1} \in L.
\]
Moreover, according to the second point of Lemma 3.4, we know that if \(\text{Gal}(L/K)\) is virtually Abelian, then there exists a constant \(c\) such that
\[c^j \xi^j T^{(\alpha \cdot \omega + \omega_j)}_{p-1} + c \in K = \mathbb{C}(e^{it}).
\]
It follows that \(c^j \xi^j T^{(\alpha \cdot \omega + \omega_j)}_{p-1}\) is meromorphic on \(\mathbb{C}\). When \(p = 2\), by Lemma A1 \(T^{(\alpha \cdot \omega + \omega_j)}_{p-1}\) is never meromorphic, and hence \(c^j = 0\). When \(p \geq 3\), by Lemma A4, \(T^{(\alpha \cdot \omega + \omega_j)}_{p-1}\) is not meromorphic unless that
\[\alpha \cdot \omega + \omega_j \in \mathbb{Z}.
\]
Since \(\alpha \in \mathbb{N}^n\), the condition \(\alpha \cdot \omega + \omega_j \in \mathbb{Z}\) is satisfied only when \(j = n\) and \(\alpha = (0, \ldots, 0, p)\).

Summarizing, we proved that if \(\alpha \neq (0, \ldots, 0, p)\), or \(j \neq n\), then \(c^j = 0\).

**SECOND STEP.** It still has to be shown that \(c^j = 0\), for \(j = n\) and \(\alpha = \alpha_0 = (0, \ldots, 0, p)\). We know that
\[c^j_{\alpha_0} = 1 \cdot \partial^n F_j(d) = -\frac{1}{(\alpha + \varepsilon_j)!} \partial^n + \varepsilon_j V(d),
\]
and so, \(c^n_{\alpha_0}\) is proportional to \(\partial^n + 1 V(d)\). Notice that \(\partial^n V(q)\) is homogeneous of degree \((2 - p)\). Thus, we have thanks to Euler’s identity
\[\sum_{i=1}^{n} q_i \partial_i \partial^n V(q) = (2 - p) \partial^n V(q).
\]
Evaluating both sides of the above identity at \(q = d = (0, \ldots, 0, 1)\) we obtain
\[\partial^n + 1 V(d) = (2 - p) \partial^n V(d). \quad (4.10)
\]
By induction we assume that \(V_{d-1}\) reduces to \(V_1\), so \(c^n_{(0, \ldots, 0, p-1)} \simeq \partial^n V(d) = 0\). Hence we also have \(c^n_{(0, \ldots, 0, p)} = 0\).

Summarizing, we prove that \(c^j_{\alpha} = 0\) for all \(1 \leq j \leq n\), and all \(\alpha\) such that \(|\alpha| = p\).
**Third Step.** Finally, notice that we show that

$$\partial^n V(d) = 0, \quad \text{with} \quad 3 \leq |a| \leq p + 1.$$  

Thus $VE_{p+1}$ has the same form as the assumed form of $VE_p$. Hence the induction hypothesis go on.

We can conclude the proof of the theorem observing that the first step of the induction procedure applies to $VE_2$, that is when $p = 2$. Indeed the first step goes without any change. Moreover, in the second step with $p = 2$, the identity (4.10) gives

$$\partial^n V(d) = 0 \quad \text{for} \quad n \geq 1,$$

So, we have the correct initial step of induction. This ends the proof.

### 4.3 The Group $\text{Gal}(VE_2)$ in Resonant Cases when $k = 2$

The following two lemmas give the necessary and sufficient conditions which guarantee that $\text{Gal}(VE_{2,a})$ and $\text{Gal}(EX_{2,(a,\beta)})$ are virtually Abelian.

**Lemma 4.2.** The group $\text{Gal}(VE_{2,a})$ is virtually Abelian iff either $\theta_{a,\alpha}^\gamma \neq 0$, or $\theta_{a,\alpha}^\gamma \neq 0$, and $\omega_\alpha, \omega_\gamma \in \mathbb{Q}^*.$

**Lemma 4.3.** The group $\text{Gal}(EX_{2,(a,\beta)})$ is virtually Abelian iff either $\theta_{a,\beta}^\gamma \neq 0$, or $\theta_{a,\beta}^\gamma \neq 0$, and $\omega_\alpha, \omega_\beta, \omega_\gamma \in \mathbb{Q}^*.$

#### 4.3.1 Proof of Lemma 4.2

In this subsection we denote by $L/K$ the Picard-Vessiot extension of $VE_{2,a}$. The field $L$ contains the field $L_1$, where $L_1/K$ is the Picard-Vessiot extension associated with the system of homogeneous equations $\dot{x} + \omega_\alpha^2 x = 0$ and $\dot{z} + \omega_\gamma^2 z = 0.$

If $\theta_{a,\alpha}^\gamma = 0$, then $VE_{2,a}$ is a subsystem of $VE_2$, therefore $\text{Gal}(L/K)$ is virtually Abelian.

Let us observe also that if both $\omega_\alpha$ and $\omega_\gamma$ belongs to $\mathbb{Q}^*$, then the field $L_1 = K(e^{i\omega_\alpha t}, e^{i\omega_\gamma t})$ is a finite extension of $K$. Moreover, the field $L/L_1$ is generated by a finite set of integrals of first level with respect to $L_1$. As a consequence, we have

$$\text{Gal}^p(L/K) = \text{Gal}(L/L_1) \simeq G_{\alpha}^p,$$

for a certain positive integer $p$. Hence, $\text{Gal}(L/K)$ is virtually Abelian.

The above observations show that we have to consider the case $\theta_{a,\alpha}^\gamma \neq 0$. But in this case, without loss of the generality, we can assume that $\theta_{a,\alpha}^\gamma = 1$. To investigate whether or not $\text{Gal} L/K$ could be virtually Abelian, we need a description of the field $L_1$, and of the extension $L/L_1$. These informations are collected in the following table.

| case $(\omega_\alpha, \omega_\gamma)$ | $L_1$ | $\text{span}_C(x^2)$ | $\sin(t)\Phi$ |
|-----------------------------------|-------|---------------------|----------------|
| 1 $(0,0)$ | $\{t\}$ | $\{1,t,t^2\}$ | $\{1,t,t^3\}$ |
| 2 $(0,*)$ | $\{t,e^{i\omega_\gamma t}\}$ | $\{1,t,t^2\}$ | $\{t^d e^{\pm i\omega_\gamma t}, 0 \leq d \leq 2\}$ |
| 3 $(*,0)$ | $\{t,e^{i\omega_\gamma t}\}$ | $\{1,e^{\pm 2i\omega_\gamma t}\}$ | $(1,e^{\pm 2i\omega_\gamma t}, t,e^{\pm 2i\omega_\gamma t})$ |
| 4 $(*,*)$ | $\{e^{i\omega_\gamma t}, e^{i\omega_\gamma t}\}$ | $\{1,e^{\pm 2i\omega_\gamma t}\}$ | $\{e^{\pm i\omega_\gamma t}, e^{i(\pm \omega_\gamma \pm 2\omega_\gamma)t}\}$ |

Let us explain the origin of this table. The exact form of the field $L_1$ depends on whether $\omega_\alpha$ or $\omega_\gamma$ vanish. This gives us four non-equivalent cases. The symbol $*$ in the second
column means that the corresponding \( \omega \) is not zero. The column marked by \( L_1 \) contains
generators of the extension \( L_1/K \).

Now, in the formulae of Lemma 4.1

\[
\frac{b(t)}{\sin(t)} = \frac{x^2}{\sin(t)},
\]

where \( x \) belongs to the solution space \( \text{span}_C(x) := \text{Sol}\{\ddot{x} + \omega^2 x = 0\} \). This is why, \( b(t) \) spans the second symmetric power of this vector space. That is:

\[
\text{span}_C(b) := \text{span}_C(x)^{\mathbb{Z}^2},
\]
is a three dimensional vector space over \( \mathbb{C} \). Now, the second level integrals \( \Phi \) given by

\[
L := \int b(t) \sin(t) \, dt \quad \text{or} \quad \Phi = \int \frac{tb(t)}{\sin(t)} \, dt
\]

Hence, \( \sin(t)\Phi \), spans the vector space

\[
\text{span}_C(b) + t \cdot \text{span}_C(b).
\]

Similarly, from Lemma 4.1 again, if \( \omega_\gamma \neq 0 \), we get that \( \sin(t)\Phi \) spans

\[
e^{i\omega_t} \cdot \text{span}_C(b) + e^{-i\omega_t} \cdot \text{span}_C(b).
\]

These considerations immediately give the last column of the table.

As a consequence, the second level integrals \( \Phi \) involved in the problem are of three types

\[
T_\omega := \int \frac{e^{i\omega t}}{\sin(t)} \, dt, \quad P_d := \int \frac{td}{\sin(t)} \, dt, \quad \text{or} \quad M_{d\omega} := \int \frac{tde^{i\omega t}}{\sin(t)} \, dt,
\]
where \( \omega \in \mathbb{C} \) and \( n \in \mathbb{N} \). These are the “trigonometric”, “polynomial” and the “mixed”
integrals. Observe further that \( T_0 = P_0 = M_{00} \). According to Lemma A.1 of the Appendix, all these integrals are not meromorphic on \( \mathbb{C} \) and are therefore transcendental over \( K = \mathbb{C}(e^{it}) \).

Now, in the three first cases of table (4.11),

\[
I := t \in L_1.
\]

We shall therefore apply Lemma 3.3 to show that in these three cases, \( \text{Gal}(L/K) \) is not
virtually Abelian.

**Case 1.** According to the Table, we can write \( L_1 = K(I) \). Moreover, we also have

\[
P_1 = \int \frac{t}{\sin(t)} \, dt = \int \dot{P}_0 I \in L, \quad \text{with} \quad \dot{P}_0 = \frac{1}{\sin(t)} \in K.
\]

So, \( \Phi = P_1 \) is a second level integral with respect to \( K \) of the form appearing in Lemma 3.3.

Hence, according to it, we get the implication

\[
\text{Gal}(L/K) \text{ is virtually Abelian} \implies cI - \int \dot{P}_0 = ct - P_0 \in K,
\]
for a certain \( c \in \mathbb{C} \). In particular this implies that \( P_0 \) is meromorphic on \( \mathbb{C} \) since \( K = \mathbb{C}(e^{it}) \) is contained into \( \mathcal{M}(\mathbb{C}) \) the field of meromorphic functions on \( \mathbb{C} \). But this is not true thanks to Lemma A.1, so \( \text{Gal}(L/K) \) is not virtually Abelian.
Case 2. Here $\omega_\alpha = 0$ but $\omega_\gamma \neq 0$. Hence, we can write $L_1 = K(t, e^{i\omega_\gamma t}) = \hat{K}(I)$ with $\hat{K} := K(e^{i\omega_\gamma t})$. We therefore get the implication

$$\text{Gal}(L/K) \text{ is virtually Abelian } \implies \text{Gal}(L/\hat{K}) \text{ is virtually Abelian},$$

and we are reduced to proving that Gal$(L/\hat{K})$ is not virtually Abelian.

By setting $I = t$ again, according to table (4.11),

$$M_{1,\omega_\gamma} = \int \frac{te^{i\omega_\gamma t}}{\sin(t)} dt = \int \hat{T}_{\omega_\gamma} I \in L, \quad \text{with} \quad \hat{T}_{\omega_\gamma} = \frac{e^{i\omega_\gamma t}}{\sin(t)} \in \hat{K},$$

is a second level integral with respect to $\hat{K}$ of the form appearing into Lemma 3.3. So we can apply this result to the field extension $L/\hat{K}$. We therefore get the implication

$$\text{Gal}(L/\hat{K}) \text{ is virtually Abelian } \implies cI - \int \hat{T}_{\omega_\gamma} = ct - T_{\omega_\gamma} \in \hat{K},$$

for a certain $c \in \mathbb{C}$. But again this is not true, since $\hat{K} \subset \mathcal{M}(\mathbb{C})$ and $T_{\omega_\gamma}$ is not meromorphic on $\mathbb{C}$.

Case 3. Is very similar to the previous one. Here we set $\hat{K} := K(e^{i\omega_\gamma t})$, and now

$$\Phi = M_{1,2\omega_\alpha} \in L.$$

Therefore, if Gal$(L/\hat{K})$ is virtually Abelian, then $T_{2\omega_\alpha}$ is meromorphic, which again is not true.

Case 4. Here we shall apply Lemma 3.4. In this case, $L_1 = K(\exp(i\omega_\alpha t), \exp(i\omega_\gamma t))$ and

$$L/L_1 = L_1(T_{\pm2\omega_\alpha \pm \omega_\gamma}, T_{\pm \omega_\gamma}),$$

is generated by eight integrals of first level with respect to $K$. Our task is now to prove the implication

$$\text{Gal}(L/K) \text{ virtually Abelian } \Rightarrow \omega_\alpha, \omega_\gamma \in \mathbb{Q}^*.$$

Let us therefore assume that there exist $\omega_0 \in \{\omega_\alpha, \omega_\gamma\}$, with $\omega_0 \not\in \mathbb{Q}$. The corresponding exponential $\exp(i\omega_0 t)$ is not algebraic over $K$, and the quasi-toroidal extension $L_1/K$ has the transcendental degree $\geq 1$. Moreover, there exists $\omega \in \{\pm2\omega_\alpha \pm \omega_\gamma, \pm \omega_\gamma\}$ which is not rational. So we can apply Lemma 3.4 with $\Phi := T_\omega$ in order to get the implication

$$\text{Gal}(L/K) \text{ virtually Abelian } \Rightarrow \frac{T_\omega + c}{\exp(i\omega t)} \in K.$$

But this further implies that $T_\omega$ is meromorphic on $\mathbb{C}$. Since it is not true, the claim follows.

These considerations finishes the proof of Lemma 4.2.

4.3.2 Proof of Lemma 4.3

Here, as in the proof of Lemma 4.2 we denote by $L/K$ the PV extension of $\text{EX}_{2,\alpha,\beta}^\gamma$. We denote also by $L_1/K$ the PV extension obtained by adding to $K$ the three solutions spaces

$$\text{span}_C(x) := \text{Sol}\{x + \omega_\alpha^2 x = 0\},$$
$$\text{span}_C(y) := \text{Sol}\{y + \omega_\beta^2 y = 0\},$$
$$\text{span}_C(z) := \text{Sol}\{z + \omega_\gamma^2 z = 0\}.$$
As before, \( L_1 \subset L \).

When \( \omega_\alpha, \omega_\beta, \omega_\gamma \in \mathbb{Q}^* \), \( L_1 = K(\exp(i\omega_\alpha t), \exp(i\omega_\beta t), \exp(i\omega_\gamma t)) \) is a finite extension of \( K \), and \( L/L_1 \) is generated by a finite set of first level integrals with respect to \( L_1 \). Hence, \( \text{Gal}(L/K) \) is virtually Abelian since its connected component is a vector group.

When \( \theta_{\alpha,\beta}^\gamma = 0 \), then \( \text{EX}_{2,\alpha,\beta}^\gamma \) is a subsystem of \( \text{VE}_1 \), hence is virtually Abelian. As a consequence, it is enough to prove that away from these two cases, \( \text{Gal}(L/K) \) is not virtually Abelian. We may therefore assume without loss of generality that \( \theta_{\alpha,\beta}^\gamma = 1 \). The collected information about \( L_1/K \) and \( L/L_1 \) are contained in the following table.

| \( (\omega_\alpha, \omega_\beta, \omega_\gamma) \) | \( L_1/K \) | \( \text{span}_C(xy) \) | \( \sin(t)\Phi \) |
|-----------------------------------------|-------------|----------------|--------------|
| 1                                      | \{t\}       | \{1, t, t^2\} | \( 1, t, t^2, t^3 \) |
| 2                                      | \{t, e^{i\omega_\gamma t}\} | \{1, t, t^2\} | \( t^d e^{\pm i\omega_\gamma t}, d = 0, 1, 2 \) |
| 3                                      | \{e^{i\omega_\beta t}, t\} | \{e^{\pm i\omega_\beta t}, te^{\pm i\omega_\gamma t}\} | \( t^d e^{\pm i\omega_\gamma t}, d = 0, 1, 2 \) |
| 4                                      | \{e^{i\omega_\beta t}, te^{i\omega_\gamma t}\} | \{e^{\pm i\omega_\beta t}, te^{\pm i\omega_\gamma t}\} | \( t^d e^{\pm i\omega_\beta t}, d = 0, 1 \) |
| 5                                      | \{e^{i\omega_\gamma t}, ei\omega_\beta t, t\} | \{e^{\pm i\omega_\gamma t}, t^{i\omega_\beta t}\} | \( t^d e^{\pm i\omega_\gamma t}, d = 0, 1 \) |
| 6                                      | \{e^{i\omega_\gamma t}, e^{i\omega_\beta t}, e^{i\omega_\gamma t}\} | \{e^{\pm i\omega_\gamma t}, t^{i\omega_\beta t}\} | \( e^{\pm i\omega_\gamma t}, d = 0, 1 \) |

As in Table (4.11), the cases are distinguished by the vanishing of at least one component in the triple \( (\omega_\alpha, \omega_\beta, \omega_\gamma) \). Thus, in general we have eight possibilities. However, due to the form \( \text{EX}_{2,\alpha,\beta}^\gamma \) and the fact that \( \theta_{\alpha,\beta}^\gamma = \theta_{\beta,\alpha}^\gamma \), the indices \( \alpha \) and \( \beta \) play a symmetric role. This is why we have to deal with only six distinct cases. The Column \( L_1/K \) gives a set of generators of this extension. The column \( \text{span}_C(xy) \) gives a basis of the vector space

\[
\text{span}_C(b) = \text{span}_C(xy) = \text{span}_C(x) \oplus \text{span}_C(y).
\]

Observe that in contrast with Table (4.11), this \( \mathbb{C} \) vector space can be of dimension 3, or 4. Now the elements of the last column are computed thanks to the same rule as in Table (4.11).

From the last column, one can easily see that all the second level integrals which appear in this context are of the type described in the proof of Lemma 4.2. We can therefore perform the proof in a similar way as before. Precisely, for Cases 1 to 5, \( \text{Gal}(L/K) \) is not virtually Abelian thanks to Lemma 3.3. In Case 6, the use of Lemma 3.4 gives the implication

\[
\text{Gal}(L/K) \text{ virtually Abelian} \Rightarrow \omega_\alpha, \omega_\beta, \omega_\gamma \in \mathbb{Q}^*.
\]

This ends the proof of this Lemma.

5 Potentials of degree \( k = -2 \) and, Proof of Theorem 1.3

In our paper [4], we found a correspondence between the first variational equations \( \text{VE}_1 \) of potentials of degree \( k \) and of degree \(-k\). Moreover, we used it to get the implication

\[
\text{Gal}(\text{VE}_1) \text{ virtually Abelian for } k = 2 \Rightarrow \text{Gal}(\text{VE}_1) \text{ virtually Abelian for } k = -2.
\]

The present theorem shows that the latter correspondence cannot hold at the level of the \( \text{VE}_p \) for \( p \geq 2 \). And because of this we get Galois obstruction along the Darboux points for \( k = 2 \), although there is no such obstruction for potentials of degree \( k = -2 \).
5.1 Ingredients for the proof

Let $k = -2$, and $d \in \mathbb{C}^n \setminus \{0\}$ be a proper Darboux point satisfying $V'(d) = d$. The corresponding phase curve is $\Gamma_e = \{(\phi d, \phi d) \in \mathbb{C}^{2n}\}$, where

$$\phi^2 = e + \frac{1}{q^2},$$

and $e \in \mathbb{C}$ is a fixed value of the energy. At first glance we analyse the solutions of the equation

$$\ddot{x} = -\frac{\lambda}{q^2} x, \quad \lambda \in \mathbb{C},$$

(5.1)

which is a sub-equation of $\text{VE}_1$. The explicit form $\phi$ depends on $e$. To investigate solutions of equation (5.1) we introduce a parameter $\omega$ defined by $\lambda = 1 - 4\omega^2$, and two functions of time $I$ and $E_\omega$. They are defined in the table below.

|   | $e = 0$ | $e \neq 0$ |
|---|---------|------------|
| $\phi$ | $\phi := \sqrt{2t}$ | $\phi := \sqrt{et^2 - \frac{1}{e}}$ |
| $E_\omega$ | $E_\omega := t^\omega$ | $E_\omega := \left(\frac{et - 1}{et + 1}\right)^\omega$ |
| $I$ | $I := \log(t)$ | $I := \log\left(\frac{et - 1}{et + 1}\right)$ |

(5.2)

Proposition 5.1. Let us consider equation (5.1) with $\lambda = 1 - 4\omega^2$. If $\omega = 0$, then its solution space is generated by $\phi$ and $\phi I$; otherwise it is generated by $\phi E_\omega$ and $\phi E_{-\omega}$.

The functions $I$ and $E_\omega$ satisfy the following relations

$$\dot{E}_\omega = \frac{2\omega}{q^2}E_\omega, \quad \dot{I} = \frac{2}{q^2}.$$

The proof of this proposition follows by direct computation and will be left to the reader.

In [4] we have shown that, for $k = -2$, the first variational equations $\text{VE}_1$ are solvable. Hence, equation (5.1) is solvable because it is a sub-equation of $\text{VE}_1$.

Let us fix the value of $e$, and denote by $K := \mathbb{C}(\phi, \dot{\phi})$. We consider the infinite dimensional Picard-Vessiot extension $F/K$ generated by the integral $I$ and the exponentials $E_\omega$ with $\omega \in \mathbb{C}$. By Proposition 5.1, $\text{Gal}(F/K)$ is virtually Abelian.

Let us consider the following vector sub-space of $\mathcal{B}$ of $F$:

$$\mathcal{B} := \text{Vect}_\mathbb{C}\{I^m E_\omega | m \in \mathbb{N}, \ \omega \in \mathbb{C}\}.$$

We note here that this vector space contains $\mathbb{C}$ and remains stable by multiplication since

$$(I^m E_\omega) \cdot (I^m' E_{\omega'}) = I^{m+m'} E_{\omega+\omega'}.$$

The basic idea of our proof of Theorem 1.3 is to show that, with the use of vectorial notations, that we get the following inclusion

$$\text{Sol}(\text{VE}_p) \subset \phi \mathcal{B}^n,$$

holds true for $p \in \mathbb{N}$. Let us observe that such an inclusion will imply that $\text{PV}(\text{VE}_p) \subset F$, and the result will follow. In order to get this inclusion we shall need the two following lemmas.
Lemma 5.2. For $b(t) \in \mathcal{B}$, we have

$$\Phi := \int \frac{b(t)}{q^2} \, dt \in \mathcal{B}.$$ 

Proof. Clearly, it is enough to show this for the integrals

$$\Phi^{(m)} := \int \frac{1}{q^2} \, dt \quad \text{for} \quad m \in \mathbb{N}, \quad \text{and} \quad \omega \in \mathbb{C}.$$ 

According to Proposition 5.1 we have:

$$\Phi^{(m)} := \int \frac{1}{q^2} \, dt = \int \frac{1}{2} \, dt = \frac{I^{(m+1)}}{2(m+1)} \in \mathcal{B}.$$ 

Moreover, for $\omega \neq 0$, we also have

$$\Phi^{(0)} := \int \frac{E \omega}{q^2} \, dt = \frac{E \omega}{2\omega} \in \mathcal{B}.$$ 

Now, if $m \geq 1$, we obtain

$$2\omega \Phi^{(m)} = \int \frac{2\omega E \omega}{q^2} \, dt = \int \frac{I^{(1)}}{q^2} \, dt \quad \text{for} \quad m \in \mathbb{N}, \quad \omega \in \mathbb{C}.$$ 

As a result we get

$$2\omega \Phi^{(m)} = I^{(1)} - 2m \Phi^{(m-1)},$$

and the result follows by induction with respect to $m$. ∎

The following result extends Proposition 5.1.

Lemma 5.3. The solution space of the equation

$$\ddot{x} = -\frac{\lambda}{q^4} x + \frac{b(t)}{q^3}, \quad \lambda \in \mathbb{C}, \quad b(t) \in \mathcal{B},$$

is contained in the vector space $q\mathcal{B}$.

Proof. For $b(t) = 0$, the thesis of the lemma is a direct consequence of Proposition 5.1. Hence, we assume that $b(t) \neq 0$, and we rewrite equation (5.3) as a system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\lambda q^{-4} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \cdot \begin{bmatrix} q^{-3} \end{bmatrix}, \quad b = b(t).$$

A particular solution of this system is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = X \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

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where $X$ is a fundamental matrix of the homogeneous part of equation (5.3), and $[c_1, c_2]^T$ is a certain solution of the following equation

$$\begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} = X^{-1} \begin{bmatrix} 0 \\ b\varphi^{-3} \end{bmatrix}.$$  

Let us write $\lambda = 1 - 4\omega^2$. In the case $\omega = 0$, by Proposition 5.1 a fundamental matrix $X$ and its inverse are given by

$$X = \begin{bmatrix} \varphi & \varphi I \\ \dot{\varphi} & \varphi I + \varphi I \end{bmatrix}, \quad X^{-1} = \frac{1}{2} \begin{bmatrix} \varphi I + \varphi I & -\varphi I \\ -\varphi & \varphi \end{bmatrix}.$$  

Hence, we have

$$\begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} bI\varphi^{-2} \\ b\varphi^{-2} \end{bmatrix},$$

so, by Lemma 5.2 $[c_1, c_2] \in B \times B$. Hence, $[x, y] \in \varphi B \times \varphi B$ and this finishes the proof for $\omega = 0$.

For $\omega \neq 0$, the fundamental matrix $X$ and its inverse for homogeneous part of equation (5.4), are given by

$$X = \begin{bmatrix} \varphi E_{\omega} & \varphi E_{-\omega} \\ \dot{\varphi} E_{\omega} + \varphi \dot{E}_{\omega} & \dot{\varphi} E_{-\omega} + \varphi \dot{E}_{-\omega} \end{bmatrix}, \quad X^{-1} = \frac{1}{2\omega} \begin{bmatrix} \varphi E_{-\omega} + \varphi \dot{E}_{-\omega} & -\varphi E_{-\omega} \\ -\varphi E_{\omega} \varphi \dot{E}_{\omega} & \varphi E_{\omega} \end{bmatrix},$$

so

$$\begin{bmatrix} \dot{c}_1 \\ \dot{c}_2 \end{bmatrix} = \frac{1}{2\omega} \begin{bmatrix} b\varphi^{-2} E_{-\omega} \\ b\varphi^{-2} E_{\omega} \end{bmatrix}. \quad (5.5)$$

Hence, also in this case, by Lemma 5.2 $[c_1, c_2] \in B \times B$. Hence, $[x, y] \in \varphi B \times \varphi B$ and this finishes the proof.

5.2 Proof of Theorem 1.3

The first step. With the use of vectorial notations, we show the following result.

**Proposition 5.4.** For $VE_1$ we have $\text{Sol}(VE_1) \subset \varphi B^n$.

**Proof.** The first order variational equation has the form

$$\ddot{q}_1 = -\varphi^{-4}V''(d)q_1. \quad (5.6)$$

We consider two possibilities: either $V''(d)$ is diagonalisable, or it is not diagonalisable.

If $V''(d)$ is diagonalisable, then $VE_1$ splits into a direct sum of equations which have the form

$$\ddot{x} = -\frac{\lambda}{\varphi^4}x, \quad (5.7)$$

where $\lambda$ is an eigenvalue of $V''(d)$. And the result follows from Lemma 5.3.

If $V''(d)$ is not diagonalisable, $VE_1$ splits into subsystems corresponding to Jordan block of the form

$$\begin{cases} 
\ddot{x} = -\lambda \varphi^{-4}x, \\
\ddot{y} = -\lambda \varphi^{-4}y + \varphi^{-4}x, \\
\ddot{z} = -\lambda \varphi^{-4}z + \varphi^{-4}y, \\
\ddots \\
\ddot{w} = -\lambda \varphi^{-4}w. 
\end{cases} \quad (5.8)$$
By Lemma 5.3, an arbitrary solution $x$ of the first equation can be written in the form $x = \phi b_0$, where $b_0 \in B$. Hence, for this solution, the second equation in (5.8) can be written in the form

$$\ddot{y} = -\frac{\lambda}{\phi^2} y + \frac{b_0}{\phi^3}.$$ 

Thus, by Lemma 5.3, $y \in \phi B$. Inserting this solution into the third equation (5.8) we also get that $z \in \phi B$. If the size of the Jordan block is greater than three we proceed inductively. As a consequence we obtain that an arbitrary solution $q_1$ belongs to $\phi B^n$, and this finishes our proof.

**The induction procedure.** Let us write the higher variational equation in Newton form, that is again, we set $F(q) := -V'(q)$. Here for convenience, we write explicitly the first three variational equations:

$$\begin{align*}
\ddot{q}_1 &= \frac{F'(d)(q_1)}{\phi^4}, \\
\ddot{q}_2 &= \frac{F'(d)(q_2)}{\phi^4} + \frac{F''(d)(q_1, q_1)}{\phi^5}, \\
\ddot{q}_3 &= \frac{F'(d)(q_1)}{\phi^4} + \frac{3F''(d)(q_1, q_2)}{\phi^5} + \frac{F^{(3)}(d)(q_1, q_1)}{\phi^6}.
\end{align*}$$

As a consequence, in general, we may write the VE $\ddot{q}_p$ in the form

$$\ddot{q}_p = \frac{F'(d)(q_p)}{\phi^4} + \frac{R_p}{\phi^5},$$

where $R_p$ is a $\mathbb{C}$-linear combination of terms of the form

$$\frac{F''(d)(q_{i_1}, q_{i_2})}{\phi^2}, \ldots, \frac{F^{(s)}(d)(q_{i_1}, \ldots, q_{i_s})}{\phi^s},$$

with $2 \leq s \leq p$, and $1 \leq i_k \leq p - 1$.

Now, let assume that by solving the VE $\ddot{q}_m$ for $1 \leq m \leq p - 1$ we found that $q_m \in \phi B^n$.

In other words $q_m = \phi B_m$ with some $B_m \in B^n$. Since $F^{(s)}(d)$ is a vectorial $s$-form with constant coefficients we get

$$\frac{F^{(s)}(d)(q_{i_1}, \ldots, q_{i_s})}{\phi^s} = \frac{F^{(s)}(d)(\phi B_{i_1}, \ldots, \phi B_{i_s})}{\phi^s} = F^{(s)}(d)(B_{i_1}, \ldots, B_{i_s}).$$

But now, since $B$ is stable by multiplication, it follows that

$$R_p \in B^n.$$

According to Lemma 5.3, we can deduce that all the solutions $q_p$ of the equations

$$\ddot{q}_p = \frac{F'(d)(q_p)}{\phi^4} + \frac{R_p}{\phi^5},$$

belong to $\phi B^n$.

This ends the proof of the theorem.
A About the analytic behaviour of some trigonometric integrals

We show the following.

**Lemma A.1.** Let \( f \in \mathcal{O}(\mathbb{C}) \) be a holomorphic function such that \( f(n\pi) \neq 0 \), for a certain \( n \in \mathbb{Z} \). Then, the function

\[
F(t) := \int \frac{f(t)}{\sin t} \, dt,
\]

(A.1)

is not meromorphic on \( \mathbb{C} \).

Indeed, if \( M_{n\pi} \) denotes the monodromy operator around \( t = n\pi \), we get

\[
M_{n\pi}(F) = F + 2\pi i f(n\pi).
\]

For \( \omega \in \mathbb{C} \) and \( n \in \mathbb{N} \), we define the following functions

\[
T_n(\omega)(t) := \int \frac{e^{i\omega t}}{\sin^n t} \, dt.
\]

(A.2)

**Lemma A.2.** The following statements holds true.

1. The function \( T_1(\omega)(t) \) is not meromorphic.
2. The function \( T_2(\omega)(t) \) is meromorphic iff \( \omega = 0 \).

**Proof.** The first statement immediately follows from Lemma 1. In order to prove the second one, we notice that in a neighborhood of \( t = 0 \) we have

\[
\sin^2 t = t^2(1 + \cdots), \quad \text{and} \quad e^{i\omega t} = 1 + i\omega t + \cdots,
\]

and thus

\[
\frac{e^{i\omega t}}{\sin^2 t} = \frac{1}{t^2} + \frac{i\omega}{t} + \cdots.
\]

Hence if \( \omega \neq 0 \) the function \( T_2(\omega)(t) \) is not meromorphic. For \( \omega = 0 \), we have \( T_2^{(0)}(t) = -\cot t \), is a meromorphic function on \( \mathbb{C} \).

**Proposition A.3.** For \( n > 2 \) the following relation holds true

\[
T_n(\omega)(t) = S_{n-2}(\omega)(t) + c_{n-2}^{(\omega)} T_{n-2}(\omega)(t),
\]

(A.3)

where

\[
S_{n-2}(\omega)(t) := -e^{i\omega t} \frac{i\omega \sin t + (n-2) \cos t}{(n-1)(n-2) \sin^{n-1} t}, \quad \text{and} \quad c_{n-2}^{(\omega)} := \frac{(n-2)^2 - \omega^2}{(n-1)(n-2)}.
\]

**Proof.** Differentiating both sides of (A.3) and making some simplifications we obtain the identity.

Now, we are ready to characterize all the cases when \( T_n^{(\omega)} \) is meromorphic.

**Lemma A.4.** The function \( T_n^{(\omega)} \) is meromorphic if and only if we either get

1. \( \omega \in \{ \pm 2k \mid 0 \leq k \leq (n-2)/2 \} \) for even \( n \) and \( n \geq 2 \),
2. \( \omega \in \{ \pm(1+2k) \mid 0 \leq k \leq (n-1)/2 \} \) for odd \( n \) and \( n \geq 3 \).
Proof. Assume that \( n \) is even. Then from relation (A.3) it follows that
\[
T_n^{(\omega)}(t) = f_n^{(\omega)}(t) + p_n(\omega) T_2^{(\omega)}(t),
\]
where \( f_n^{(\omega)} \) is a meromorphic function, and
\[
p_n(\omega) = \prod_{k=0}^{(n-2)/2} \epsilon_{2k}^{(\omega)} = a_n \prod_{k=0}^{(n-2)/2} [(2k)^2 - \omega^2], \quad a_n \neq 0.
\]
Thus, by Lemma [A.2] \( T_n^{(\omega)} \) is meromorphic iff \( p_n(\omega) = 0 \), and this occurs iff \( \omega \) belongs to the set given in the first statement.

With the case of odd \( n \) we proceed in a similar way. \( \square \)

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