Modified Multi-Key Fully Homomorphic Encryption Based on NTRU Cryptosystem without Key-Switching

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Abstract: The Multi-Key Fully Homomorphic Encryption (MKFHE) based on the NTRU cryptosystem is an important alternative to the post-quantum cryptography due to its simple scheme form, high efficiency, and fewer ciphertexts and keys. In 2012, López-Alt et al. proposed the first NTRU-type MKFHE scheme, the LTV12 scheme, using the key-switching and modulus-reduction techniques, whose security relies on two assumptions: the Ring Learning With Error (RLWE) assumption and the Decisional Small Polynomial Ratio (DSPR) assumption. However, the LTV12 and subsequent NTRU-type schemes are restricted to the family of power-of-2 cyclotomic rings, which may affect the security in the case of subfield attacks. Moreover, the key-switching technique of the LTV12 scheme requires a circular application of evaluation keys, which causes rapid growth of the error and thus affects the circuit depth. In this paper, an NTRU-type MKFHE scheme over prime cyclotomic rings without key-switching is proposed, which has the potential to resist the subfield attack and decrease the error exponentially during the homomorphic evaluating process. First, based on the RLWE and DSPR assumptions over the prime cyclotomic rings, a detailed analysis of the factors affecting the error during the homomorphic evaluations in the LTV12 scheme is provided. Next, a Low Bit Discarded & Dimension Expansion of Ciphertexts (LBD&DEC) technique is proposed, and the inherent homomorphic multiplication decryption structure of the NTRU is proposed, which can eliminate the key-switching operation in the LTV12 scheme. Finally, a leveled NTRU-type MKFHE scheme is developed using the LBD&DEC and modulus-reduction techniques. The analysis shows that the proposed scheme compared to the LTV12 scheme can decrease the magnitude of the error exponentially and minimize the dimension of ciphertexts.

Key words: NTRU-type Multi-Key Fully Homomorphic Encryption (MKFHE); prime cyclotomic rings; Low Bit Discarded (LBD); homomorphic multiplication decryption structure

1 Introduction

The Multi-Key Fully Homomorphic Encryption (MKFHE) can perform arbitrary operations on encrypted data at different public keys (users), and the final ciphertext can be jointly decrypted by all the involved users. Moreover, the operation process between the ciphertexts of different users can be entrusted to the cloud offline, avoiding the interaction between users in the secure Multi-Party Computing (MPC) protocol. So, the MKFHE can be widely used in ciphertext retrieval [1], secure MPC [2–4], privacy-preserving protocol [5], etc.

In 2012, López-Alt et al. [6] proposed the concept of MKFHE for the first time and constructed the first MKFHE scheme LTV12 based on the NTRU cryptosystem (called NTRU-type MKFHE). Many studies on the MKFHE have been reported [7–14]. Similar
to the traditional single-key Fully Homomorphic Encryption (FHE), the MKFHE mainly includes NTRU-type MKFHE, GSW-type MKFHE\cite{8–11}, and BGV-type MKFHE\cite{22–25}. Among the three types of MKFHE schemes, the NTRU-type MKFHE scheme is the fastest in encryption and decryption, and has the simplest form, and uses the least ciphertexts and keys. The underlying scheme of the NTRU encryption has been used to design various cryptographic primitives, including the digital signatures\cite{15}, identity-based encryption\cite{16,17], and multi-linear maps\cite{18,19}.

The security of the NTRU-type homomorphic encryption schemes is based on the Ring Learning With Error (RLWE) assumption and Decisional Small Polynomial Ratio (DSPR) assumption. Stehlé and Steinfeld showed that the DSPR assumption could be reduced to the RLWE assumption under certain conditions (refer to Ref. \cite{20} for details). The RLWE represents an algebraic variant of the LWE\cite{21}, whose hardness can be reduced to the hardness of the worst-case problems on ideal lattices in the standard model. However, recently, it has been shown that subfield attacks\cite{22–25} affected the asymptotic security of NTRU-type schemes for large moduli \( q \). Yu et al.\cite{26,27} considered a variant of NTRU encrypt over prime cyclotomic rings and obtained the INDistinguish-ability under Chosen-Plaintext Attack (IND-CPA) secure results in the standard model assuming the hardness of the worst-case problems on ideal lattices, which was shown to be a good choice to resist the subfield attacks.

In LTV12\cite{6}, the leveled NTRU-type MKFHE scheme was adopted by using the key-switching (known as the relinearization) and modulus-reduction techniques\cite{28,29}. However, the key-switching process needed to be carried out before reducing the modulus during the homomorphic evaluations, which increased the error significantly. Hitherto, much work on the design and security research of the NTRU-type FHE scheme has been done\cite{30,31}, but there are few outstanding results of the NTRU-type MKFHE.

The main contributions of this work can be summarized as follows.

(1) The NTRU-type MKFHE (LTV12) over prime cyclotomic rings is adopted, and the parameters that affect the growth of error during the homomorphic evaluations are analyzed.

(2) A Low Bit Discarded and Dimension Expansion of Ciphertexts (LBD&DEC) technique is used to modify the inherent decryption structure of the NTRU-type MKFHE, which eliminates the key-switching in homomorphic multiplication and reduces the ciphertext dimension.

(3) A leveled NTRU-type MKFHE scheme over prime cyclotomic rings, which successfully eliminates the relinearization and greatly decreases the error magnitude, is designed.

The rest of this paper is organized as follows. In Section 2, the basic mathematical techniques used in this work, and the RLWE and DSPR assumptions are presented. In Section 3, the NTRU-type MKFHE over prime cyclotomic rings is introduced, and its cryptographic properties are analyzed. In Section 4, the inherent homomorphic decryption structure is modified by using the LBD&DEC technique, and the detailed analysis of the size of error, ciphertext, and evaluation key, etc., is provided. In Section 5, a multi-key somewhat homomorphic encryption scheme is designed by using the LBD&DEC technique, and the parameter comparison between our scheme and the LTV12 scheme is given. In Section 6, a leveled NTRU-type MKFHE scheme without key-switching is presented. The conclusion is provided in Section 7.

2 Preliminary

Assume \( \lambda \) denotes the security parameter, and \( \text{negl}(\lambda) \) denotes a negligible function of \( \lambda \); \( a \) denotes the row vector, \( a_i \) represents the \( i \)-th element of \( a \), and \( a^T \) represents the column vector. The element located in the \( i \)-th row and the \( j \)-th column of matrix \( C \) is represented as \( C[i, j] \). In general, vectors can be regarded as a row matrix. Let \( v \) and \( w \in \mathbb{R}^m \), where \( R \) denotes the cyclotomic rings, and assume the dimension of vector \( v \) and \( w \) is \( m \), so \( v \cdot w = (v, w) = v_1 w_1 + v_2 w_2 + \cdots + v_k w_k \) denotes the inner product, \( v \) and \( w \in \mathbb{R}^m \).

In this paper, the prime cyclotomic rings \( R = \mathbb{Z}[x]/\Phi_n(x) \) and \( R_q = R/qR \) are used, where \( \Phi_n(x) = x^{n−1} + x^{n−2} + \cdots + 1 \) (\( n \) denotes a prime), \( q = q(\lambda) \) denotes a prime, and it satisfies \( q = 1 \mod n \). Addition and multiplication operation on these rings are component-wise in their coefficients, and the coefficients of \( R_q \) are reduced to the range \( \left\{−q/2, q/2\right\} \), except for \( q = 2 \). We require the ability to sample from the probability distribution \( \chi, \) i.e., the truncated discrete Gaussian distribution \( D_{2^r, \sigma} \), with Gaussian parameter \( \sigma \), deviation \( r = \sigma \sqrt{2\pi} \), and Gaussian function \( e^{-\pi x^2/\sigma^2} \). Refer to Ref. \cite{32} for a detailed description of the discrete Gaussian distribution. Let \( \chi = \chi(\lambda) \) be a \( B \)-bound error distribution over \( R \) whose coefficients are
in the range $[-B, B]$. For the probability distribution $D$, $x \leftarrow D$ denotes that $x$ is sampled from $D$.

The vector length $l$ is generally measured by the Euclidean norm. For $v \in R$, we use $\|v\|_\infty = \max_{0 \leq i < n-1} |v_i|$ to denote the standard $l_\infty$-norm and use $\|v\|_1 = \sum_{i=0}^{n-1} |v_i|$ to denote the standard $l_1$-norm.

The security of our scheme is based on the RLWE and DSPR assumptions. Following Refs. [26] and [27], a brief introduction to these assumptions over the prime cyclotomic rings is provided.

**Definition 1** RLWE assumption: Let $\lambda$ be a security parameter. $q = q(\lambda) \in Z$ is a prime integer. 

$$\Phi_n(x) = x^{n-1} + x^{n-2} + \cdots + 1$$

is the sub-cyclotomic polynomial. For the polynomial ring defined by $R = Z[x]/\Phi_n(x)$ and $R_q = R/qR$, and an error distribution $\chi \sim \chi(\lambda)$ over $R$, the RLWE assumption states that the following two distributions cannot be distinguished:

1. One samples $(a_i, b_i)$ uniformly from $R_q^{n+1}$, and (2) one first draws $a_i \leftarrow R_q^n$ uniformly, and samples $(a_i, b_i) \in R_q^{n+1}$ by choosing $s_i \leftarrow R_q^n$ and $e_i \sim \chi$ uniformly, and set $b_i = (a_i, s_i) + e_i$.

**Definition 2** DSPR assumption: Let $\lambda$ be a security parameter. $q = q(\lambda) \in Z$ is a prime integer. 

$$\Phi_n(x) = x^{n-1} + x^{n-2} + \cdots + 1$$

is the sub-cyclotomic polynomial. For the polynomial ring $R = Z[x]/\Phi_n(x)$ and $R_q = R/qR$, and a $B$-bound error distribution $\chi \sim \chi(\lambda)$ over $R$, the DSPR assumption states that the following two distributions cannot be distinguished:

1. A polynomial $h = 2g/f$, where $f = 2f' + 1$, and it is reversible over $R_q$, and (2) a polynomial $h$ sampled uniformly at random over $R_q$.

Two subroutines: Here two subroutines BitDecomp() and Powersof2(), which are widely used in the FHE schemes, are introduced. Assuming that $l = \lceil\log q\rceil$, these two subroutines can be expressed as follows.

- **BitDecomp** $(x \in R_q)$: $R_q \rightarrow R_q^l$. On input $x \in R_q$, outputs $x \mapsto (x_0, x_1, \ldots, x_{l-1}) \in \{0, 1\}^l$ (For convenience, we denote BitD(x) $\in R_q^l$).

- **Powersof2** $(x \in R_q)$: $R_q \rightarrow R_q^l$. On input $x \in R_q$, outputs $x \mapsto (x, 2x, \ldots, 2^{l-1}x)$, where $2^{l-1} < q/2$ (For convenience, we denote Pof2(x) $\in R_q^l$).

It’s obviously to verify that $(\text{BitD}(x), \text{Pof2}(y)) = (x, y) \mod q$, where $(x, y)$ denotes the product of polynomials $x, y \in R_q$.

**Key-switching technique:** The relinearization technique in the LTV12 scheme is also known as the key-switching technique[28,29]. The key-switching technique can be used to reduce the dimension of expanded ciphertext to the normal level. Generally, it can be used to transform a ciphertext $c \in R_q$ (under the secret key $f$) to another ciphertext $c_{evk} \in R_q$ (under the secret key $f_{evk}$) while the corresponding message stays unchanged. Let $l = \lceil\log q\rceil$, the key-switching process mainly consists of two procedures.

1. **KeySwitch** $(c, k, q)$: Compute the ciphertext vector $c' = \text{BitD}(c) \in R_q^l$, and output $c_{evk} = c' \cdot k_t = \langle \text{BitD}(c), k_t \rangle \in R_q$.

There are some useful confusions in the following.

**Lemma 1**[26] Let $\Phi_n(x) = x^{n-1} + x^{n-2} + \cdots + 1$, and $R = Z[x]/\Phi_n(x)$, where $n$ is a prime. For any $a, b \in R$, it holds that

$$\|ab\|_\infty \leq (n-1)\|a\|_\infty \|b\|_\infty .$$

According to Lemma 1, the following Lemma 2 can be drawn.

**Lemma 2** Let $a, b \in R$ be sampled from a discrete Gaussian distribution with parameter $B\sqrt{2\pi}$ and bound $B$, under the worst-case conditions, the bound of $ab \mod \Phi_n(x)$ is $\|ab\|_\infty \leq (n-1)B^2$, for convenience, mod $\Phi_n(x)$ is omitted. In particular, when the bound of $b$ is 1, it holds that $\|ab\|_\infty \leq 2(n-1)B$.

**Remark** According to Lemma 2, if $a, b, c \in R$, we have $\|abc\|_\infty \leq 2(n-1)\|ab\|_\infty \|c\|_\infty \leq 2^2(n-1)^2B^3$. So Lemma 2 yields the following corollary.

**Corollary 1** Let $\lambda$ be a security parameter, for the polynomial ring given by $R = Z[x]/\Phi_n(x)$, $\Phi_n(x) = x^{n-1} + x^{n-2} + \cdots + 1$, where $n$ is a prime, $\chi \sim \chi(\lambda)$ is a $B$-bound error distribution, and $q = q(\lambda) \in Z$ is a prime integer, let $s_1, s_2, \ldots, s_k \leftarrow \chi$. Then, we have $\|\prod_{i=1}^{k} s_i\|_\infty \leq 2^{k-1}(n-1)(k-1)B^k$.

3 Basic NTRU-Type Multi-Key Somewhat Homomorphic Encryption (MKSHE) over Prime Cyclotomic Rings

The NTRU key pairs consist of ring elements $(h, f)$, such that $h = [2g/f]_q$, where $g$ and $f$ denote small elements sampled from a $B$-bounded distribution $\chi$ and $f$ is invertible in $R_q$, respectively. Further recall that an NTRU ciphertext has the form of $c := [m + h\hat{s} + 2\hat{e}]_q$ for small elements $\hat{s}$ and $\hat{e}$ sampled from $\chi$, and $m$ can be recovered by computing $[f\hat{c}]_q \pmod{2}$. 

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The NTRU-type homomorphic encryption naturally supports the homomorphic evaluations between ciphertexts of different secret keys, which can be easily proven. Generally, it is assumed that there are four users \(A, B, C,\) and \(D,\) corresponding to the four public keys \((pk_A, pk_B, pk_C, pk_D)\) and four secret keys \((sk_A, sk_B, sk_C, sk_D)\), respectively. Plaintexts \((m_A, m_B, m_C, m_D)\) can be encrypted as \((\hat{c}_A, \hat{c}_B, \hat{c}_C, \hat{c}_D)\), where \(\hat{c}_i = h_i \hat{s} + 2\hat{e} + m_i \in R_q, i \in \{A, B, C, D\}.\) Set \(\{pk_A, pk_B, pk_C\} \in K_1, \{pk_B, pk_C, pk_D\} \in K_2,\) so \(K_1 \cap K_2 = \{pk_B, pk_C\},\) and \(K = K_1 \cup K_2 = \{pk_A, pk_B, pk_C, pk_D\}.\) It should be noted that since the method to process the cubic (or larger order) of a ciphertext product is similar to quadratic order, only the quadratic order of the ciphertext product is considered in this paper.

In particular, the two joint ciphertexts can be denoted as \(\hat{c}_1 = \hat{c}_A \hat{c}_B \hat{c}_C\) and \(\hat{c}_2 = \hat{c}_B \hat{c}_C \hat{c}_D,\) which can be decrypted to the joint plaintexts \(m_1 = m_A m_B m_C\) and \(m_2 = m_B m_C m_D,\) respectively, by using the joint secret keys \(F_{K_1} = f_{A} f_{B} f_{C}\) and \(F_{K_2} = f_{B} f_{C} f_{D}.\) That is \(F_{K_1} \hat{c}_1 = F_{K_1} m_1 + v_1\) and \(F_{K_2} \hat{c}_2 = F_{K_2} m_2 + v_2.\) Similarly, to decrypt \(\hat{c}_1 \hat{c}_2 = \hat{c}_A^2 \hat{B} \hat{C} \hat{D}\) we need to multiply \(F_{K_1} F_{K_2} = f_{A} f_{B} f_{C} f_{D}.\) Thus, the magnitude of the coefficients of \(F_{K_1} F_{K_2}\) grows exponentially with the degree of the evaluated circuit. Namely, after \(L\) multiplications, the needed joint secret key will represent the product of \(L\) polynomials, and the magnitude of the coefficients in this product will increase exponentially with \(L.\) In order to solve these problems, the joint secret key \(F_{K} = f_{A} f_{B} f_{C} f_{D},\) which has no quadratic items, is used to complete homomorphic decryption. Since \(K_1, K_2 \in (K_1 \cup K_2),\) we have

\[
\begin{align*}
[F_{K}(\hat{c}_1 + \hat{c}_2)]_q &= [F_{K}(m_A m_B m_C + m_B m_C m_D) + f_D v_1 + f_A v_2]_q.
\end{align*}
\]

However, since there are no \(f_A^2\) and \(f_B^2\) in \(F_{K}\), the multiplication cannot be decrypted correctly. Thus, we have

\[
[F_{K} \hat{c}_1 \hat{c}_2]_q \neq [F_{K} m_A m_B m_C^2 m_D + \text{error}_{\text{mult}}]_q,
\]

where \(\text{error}_{\text{mult}}\) represents the error of homomorphic multiplication decryption.

Therefore, the key-switching technique is used in the LTV12 scheme to re-linearize \(\text{error}_{\text{mult}} = \hat{c}_1 \hat{c}_2\), and switch \([F_{K} \hat{c}_1 \hat{c}_2]_q\) to the decryption structure given by

\[
[F_{K}(\hat{c}_1 \hat{c}_2)]_q = [F_{K_1} F_{K_2}(\hat{c}_1 \hat{c}_2) + \text{error}_{\text{mult}}]_q.
\]

Although \(\hat{c}_1 \hat{c}_2\) is decrypted by \(F_{K}\), the dependence of the coefficient' magnitude of the joint secret key on the circuit degree is eliminated, the error grows rapidly during the key-switching process. In the following, the error growing trend is explained in detail.

### 3.1 Scheme

The correctness and error of the NTRU-type multi-key homomorphic encryption scheme in the LTV12 scheme are analyzed. For prime cyclotomic rings, the correctness of the LTV12 scheme does not change, but the error is different compared to the LTV12 scheme on power-of-two cyclotomic rings. The basic LTV12 scheme over prime cyclotomic rings is described in this section, and a detailed analysis of the factors affecting the error growth during the homomorphic evaluations is provided.

Let \(\lambda\) be a security parameter, \(q = q(\lambda) \in \mathbb{Z}\) is a prime integer, \(\Phi_q(x) = x^{n-1} + x^{n-2} + \cdots + 1\) is the sub-cyclotomic polynomial. For the polynomial ring \(R = \mathbb{Z}[x]/\Phi_q(x)\) and \(R_q = R/qR,\) if the error distribution \(\chi \sim \chi^q\) over \(R\) is \(B\)-bound, the \(\chi^{log q}\) is also a \(B\)-bound error distribution space.

As stated earlier in this section, we let \(K_1, K_2\) denote the two public-key sets containing \(N\) users. In the LTV12 scheme, the exponential dependence of the error on \(N\) is not eliminated, so it is assumed that there is an \(a\)-priori upper bound on \(N,\) that is \(N \approx n^2,\) with constant \(c \in (0, 1).\) Without loss of generality, we assume \(K_1 \cap K_2 = \{pk_1, \ldots, pk_j\}, K_1 \cup K_2 = \{pk_1, pk_2, \ldots, pk_j\},\) where \(j \in [0, N], r \in [N, 2N].\)

The basic NTRU-type MKSHE over prime cyclotomic rings can be expressed as follows. This expression represents the basic MKSHE (called BC-MKSHE) scheme, whose security is based on the RLWE and DRSW assumptions over prime cyclotomic rings.

1. **BC-MKSHE. KeyGen(\(1^l\)):** Sample \(f^*, g \sim \chi,\) and set \(f = 2f^* + 1,\) so that \(f = 1 \pmod{2}\). If \(f\) is not invertible in \(R_q,\) resample \(f^* \sim \chi.\) Set \(h = 2g/f \in R_q,\) so \(pk := h \in R_q,\) \(sk := f \in R.\) For all \(r \in \{1, \ldots, l\}\) (here \(l = \lceil \log q \rceil\) ), sample \(s_e, e \sim \chi,\) compute the evaluation key vector \(k_e = hs_e + 2e + \text{Po}2(f) \in R_q^l.\)

   Output: \((pk, sk, evk) = (h, f, k_e).\)

2. **BC-MKSHE. Enc(pk, m):** Sample \(\hat{\chi}, \hat{e} \sim \chi.\)

   Output: \((\hat{c}, \hat{\chi}) \sim \chi.\)

3. **BC-MKSHE. Dec(sk, \hat{c}, \hat{\chi}):** Let \(u := (sk_1sk_2\ldots sk_N\hat{\chi}) \in R_q.\)

   Output: \(m^* := u \pmod{2}.\)

4. **BC-MKSHE. KeySwitch(\(\hat{c}, k_e, q\)):** Given the ciphertext \(\hat{c}\) and the evaluation key \(k_e,\) and output \([\text{BitDec}(\hat{c}), k_e]_q\).**

5. **BC-MKSHE. Eval.Add(\(\hat{c}_1, \hat{c}_2\)):** Given two
ciphertexts \( \hat{c}_1 \) and \( \hat{c}_2 \) with the corresponding public-key sets \( K_1 \) and \( K_2 \), output the ciphertext \( \hat{c}_{\text{add}} = [\hat{c}_1 + \hat{c}_2]_q \in R_q \).

(6) BC-MKSHE. Eval. Mult(\( \hat{c}_1, \hat{c}_2 \), KeySwitch): Given two ciphertexts \( \hat{c}_1 \) and \( \hat{c}_2 \) with the corresponding public-key sets \( K_1 \) and \( K_2 \), let \( \hat{c}_0 = \hat{c}_1 \hat{c}_2 \). For \( j \in [0, N] \),

(a) If \( j \neq 0 \), output \( \hat{c}_{\text{mult}} = \hat{c}_0 \in R_q \).

(b) If \( j \neq 0 \), for \( t \in [1, j] \), compute \( \hat{c}_t = \text{KeySwitch}(\hat{c}_{t-1}, k_{t,t}, q) \in R_q \).

Let \( \hat{c}_{\text{mult}} = \hat{c}_j \) at the end of the iteration.

3.2 Correctness analysis

Multi-key homomorphism: Let \( F_{K_1} \) and \( F_{K_2} \) be the joint decryption keys for ciphertext \( \hat{c}_1 \) and \( \hat{c}_2 \), respectively. Also, let \( F_t = F_{t-1}(f_t)^{-1} \), \( t \in [1, j] \), where \( F_0 = F_{K_1}F_{K_2} \). According to the LTV12 scheme, the addition and multiplication on ciphertexts can be decrypted using the product of the users’ secret keys in set \( K = K_1 \cup K_2 \). For given two ciphertexts \( \hat{c}_1 \) and \( \hat{c}_2 \) with the corresponding public-key sets \( K_1 \) and \( K_2 \), we have \( F_{K_1}\hat{c}_1 = F_{K_1}m_1 + v_1 \), \( F_{K_2}\hat{c}_2 = F_{K_2}m_2 + v_2 \), and \( \|F_{K_1}\hat{c}_1\|_\infty = \|F_{K_2}\hat{c}_2\|_\infty < \psi \), where \( v_1 \) and \( v_2 \) denote the error. Then, in the case of addition, we have \( F_{K}\hat{c}_{\text{add}} = F_{K}(m_1 + m_2) + F_{K-K_1}v_1 + F_{K-K_2}v_2 + v_1 + v_2 \).

Therefore, \( \hat{c}_{\text{add}} \) decrypts correctly. The multiplication case is more complex, and in that case, we have

\[
F_t\hat{c}_t = F_{t-1}(f_t)^{-1}(\text{BitD}(\hat{c}_{t-1}) \cdot 2g) + 2F_{t}(\text{BitD}(\hat{c}_{t-1}) \cdot e) + F_{t-1}\hat{c}_{t-1}.
\]

For all \( t \in [1, j] \), by using the key-switching technique, we can get the decryption structure in the following form:

\[
F_K\hat{c}_{\text{mult}} = F_j\hat{c}_j = \text{error}_{\text{mult}} + F_{K_1}F_{K_2}\hat{c}_0
\]

where \( F_K = F_j = f_1f_2\cdots f_r \) has no quadratic item of \( f_t \). Equation (1) represents the inherent decryption structure of the NTRU-type homomorphic multiplication. By rounding of \([F_{K_1}F_{K_2}\hat{c}_0]_q \), we get \( F_{K_1}F_{K_2}m_1m_2 \), and thus \( \hat{c}_{\text{mult}} \) can be decrypted correctly by \( F_K \). This inherent decryption structure that is deduced by using the key-switching technique is a guarantee that we can successfully decrypt \( \hat{c}_{\text{mult}} \) by \( F_K \).

Error analysis: Assume that there is no intersection between the public keys of the \( N \) users in \( K_1 \) or \( K_2 \). For instance, consider the worst case of the ciphertext \( \hat{c}_1 \) encrypted by the public-key set \( K_1 \), and let \( \hat{c}_1 \) be the multiplication of all users’ fresh ciphertexts, i.e., \( \hat{c}_1 = \hat{c}_1^1\hat{c}_2^2\cdots \hat{c}_N^N \) denotes the ciphertext of \( m_1 = m_1^1m_2^2\cdots m_N^N \), where \( \hat{c}_i^j \) denotes a fresh ciphertext, so \( \hat{c}_1^j f_i = m_1^1 f_i + v_1^j \).

By applying Lemma 2 and Corollary 1, we easily get \( \|\hat{c}_1^j f_i\|_\infty < (2B + 1) + 3(n - 1)(2B + 1)^2 \). Let \( (2B + 1) + 3(n - 1)(2B + 1)^2 = \psi_0 \), then in the above-mentioned worst case, \( \hat{c}_1 F_{K_1} = (\hat{c}_1^1 f_1)(\hat{c}_2^2 f_2)\cdots (\hat{c}_N^N f_N) \). So \( \psi \) can be bounded by

\[
\psi \leq 2N^1(N^1)^1(\psi_0)^N < 2^{3N^1}(n - 1)^{N^1}(2B + 1)^N.
\]

Necessarily, let \( \psi < q/2 \).

Based on Corollary 1, we have \( \|F_K\|_\infty \leq 2^{r-1}(n - 1)^{r-1}(2B + 1)^r \), and for convenience, we set \( E_r \leq 2^{r-1}(n - 1)^{r-1}(2B + 1)^r \).

Since the error generated by homomorphic addition is much smaller than the multiplication, we analyze only the error generated by the homomorphic multiplication. Thus, it can be written

\[
\|F_j\hat{c}_j\|_\infty < 3l(n - 1)E_{r+1} + \|F_{j-1}\hat{c}_{j-1}\|_\infty
\]

(2) Then, we obtain

\[
\|F_{j-1}\hat{c}_{j-1}\|_\infty \leq 3l(n - 1)\sum_{i=1}^{j} E_{r+i} + 2(n - 1)\psi^2
\]

Thus, the final error is bounded by the following:

\[
\|F_j\hat{c}_j\|_\infty \leq 3l(n - 1)\sum_{i=1}^{j} E_{r+i} + 2(n - 1)\psi^2 < 3l(n - 1)E_{r+j} + (1 + \sum_{i=1}^{j} E_{r+i}) + 2(n - 1)\psi^2
\]

(3) Based on Eq. (3), it can be found that there are two main factors affecting the error growth, which are the number of union secret keys \( N \) and the length of vector BitD(\( \hat{c}_j \)). Hence, these two factors can be used to control the error growth by applying the two following methods:

1. Exponential dependence of the error on \( N \) can be reduced by eliminating the key-switching operations;
2. Error magnitude can be decreased by reducing the length of a ciphertext vector.

4 Modified Multi-Key Homomorphic Multiplication Decryption Structure

The key-switching technique causes fast error growth. The main way to decrease the error is to eliminate the key-switching in the BC-MKSHE. However, to ensure that the scheme works correctly, the decryption structure of the homomorphic multiplication has to be modified. In Refs. [31] and [33], the decryption structure of a single-key homomorphic encryption scheme was studied. Chen[33] extended the ciphertext to the vector form, and performed the homomorphic decryption in the vector space. The main disadvantage of this approach
is that the increase in the ciphertext dimension makes the homomorphic evaluations more complex. However, in our multi-key homomorphic encryption scheme, the homomorphic decryption structure is improved by expanding the ciphertext dimension, and the Low Bit Discarded (LBD) method is used to control the ciphertext space size.

4.1 LBD technique

In Ref. [34], the efficiency of the fully homomorphic encryption scheme was enhanced by discarding the lower bits. In this section, the LBD based on the functions BitDecomp() and Powersof2() is presented.

LBD: By discarding elements with small coefficients (i.e., low bits) of Powersof2() vector, a lower-dimension vector that has no influence on the final decryption is obtained. For instance, for a given ciphertext $c_i$ and a secret key $f_i$, the inner product of BitD($c_i$) and Pof2($f_i$) can be obtained,

$$
\text{BitD}(c_i) \cdot \text{Pof2}(f_i) = (c_{i,1}, c_{i,2}, \ldots, c_{i,l})(2^0 f_i, 2^1 f_i, \ldots, 2^{l-1} f_i) = 2^0 c_{i,1} f_i + c_{i,2} 2^1 f_i + \cdots + c_{i,l} 2^{l-1} f_i \tag{4}
$$

where $c_{i,\tau} \in R_2, \tau \in [1, l]$. Compared to $2^{l-1} c_{i,l} f_i$, the value of $2^{d-1} c_{i,d} f_i$ ($d \ll l$) is small. Thus, when $l$ is large, discarding the terms of $c_{i,1} f_i, 2 c_{i,2} f_i, 2^2 c_{i,3} f_i, \ldots, 2^{d-1} c_{i,d} f_i$ has little effect on the overall value of Eq. (4). Further, the LBD function can be defined as LBD$_{d_1 \rightarrow d_2}$($\text{Pof2}(f_i)$), which means that the columns from $d_1$ to $d_2$ (we denote as $(d_1 \rightarrow d_2)$) of vector Pof2($f_i$) are discarded. According to Eq. (4), assume that the $(1 \rightarrow d)$ columns of Pof2($f_i$) are discarded, to ensure the correctness of the mathematical operation, we should discard the $(1 \rightarrow d)$ columns of BitD($c_i$). Therefore, we get

$$
\text{LBD}_{d_1 \rightarrow d}(\text{BitD}(c_i)) \cdot \text{LBD}_{d_1 \rightarrow d}(\text{Pof2}(f_i)) = (c_{i,d+1}, \ldots, c_{i,l})(2^d f_i, \ldots, 2^{l-1} f_i) = c_{i,d+1} 2^d f_i + \cdots + c_{i,l} 2^{l-1} f_i \tag{5}
$$

According to Eq. (5), the dimensions of vectors BitD($c_i$) and Pof2($f_i$) are reduced after the LBD, while the operation efficiency is improved.

If we set $l = \lfloor \log q \rfloor$ and $d = \lfloor \log q \rfloor$, the LBD technique can be described,

$$
\text{LBD}_{d_1 \rightarrow d}(\text{BitD}(c_i)) \cdot \text{LBD}_{d_1 \rightarrow d}(\text{Pof2}(f_i)) = \sum_{t=1}^{l} c_{i,t} 2^{t-1} f_i - \sum_{t=1}^{d} c_{i,t} 2^{t-1} f_i = \text{BitD}(c_i) \cdot \text{Pof2}(f_i)_{l = \lfloor \log q \rfloor - d = \lfloor \log q \rfloor} \tag{6}
$$

According to Eq. (6), the LBD is based only on the functions BitDecomp() and Powersof2().

4.2 Modified method

In this section, the LBD technique is employed to improve the homomorphic decryption process.

**Method 1:** In the BC-MKSHHE scheme, the LBD technique can be used to discard redundant bits of the evaluation key vector to simply the key-switching operations. So we have Method 1 in the following.

**Step 1 (Discard low bits of the evaluation key vector):** Let $\beta$ and $d$ be positive constants, and $l = \lfloor \log q \rfloor$. Perform the LBD functions to obtain the evaluation key. Output: $\tilde{k}_r = \text{LBD}_{\beta \rightarrow d + \beta - 1}(k_r) \in R_q^{l-d}$.

**Step 2 (Simplify KeySwitch function):** Compute the ciphertext vector $\text{LBD}_{\beta \rightarrow d + \beta - 1}(\text{BitD}(	ilde{c})) \in R_q^{l-d}$.

Output is as follows:

$$
\text{KeySwitch}_{\text{LBD}}(\tilde{c}, \tilde{k}_r, q) = \left(\lfloor \text{LBD}_{\beta \rightarrow d + \beta - 1}(\text{BitD}(\tilde{c})), \text{LBD}_{\beta \rightarrow d + \beta - 1}(\tilde{k}_r)\rfloor\right)_q.
$$

**Step 3 (Compute the homomorphic multiplication of ciphertexts):** Given two ciphertexts $\tilde{c}_1$ and $\tilde{c}_2$ with the corresponding public-key sets $K_1$ and $K_2$, let $\tilde{c}_0 = \tilde{c}_1 \cdot \tilde{c}_2$. For $j \in [0, N]$, $\tilde{\tilde{c}}_j = \text{KeySwitch}_{\text{LBD}}(\tilde{c}_1, \tilde{k}_j, q) \in R_q$.

Set $c_{\text{mult}} = \tilde{\tilde{c}}_j$ at the end of the iteration.

**Correctness verification of Method 1:** Sample $\tilde{\tilde{s}}_{\text{r}}, \tilde{\tilde{E}}_r \leftarrow 1^{l-d}$, we have

$$
F_{l-1} \tilde{\tilde{c}}_{\text{r}} = F_{l-1} \left(\text{LBD}_{\beta \rightarrow d + \beta - 1}(\text{BitD}(\tilde{c}_{\text{r}}-1)) \cdot 2g \cdot \tilde{\tilde{s}}_{\text{r}}\right) + 2F_{l-1} \left(\text{LBD}_{\beta \rightarrow d + \beta - 1}(\text{BitD}(\tilde{c}_{l-1})) \cdot \tilde{\tilde{e}}_{\text{r}}\right) = F_{l-1} \left(\sum_{\xi = \beta + 1}^{d + \beta - 2} 2^{5\xi - 1} \tilde{c}_{\text{r}l-1}\xi\right) + F_{l-1} \tilde{\tilde{c}}_{\text{r}l-1} \tag{7}
$$

where $\tilde{\tilde{c}}_{\text{r}l-1}\xi \in R_2$ and $\xi$ is a constant variable. According to Eq. (6) and since $F_{l-1}(\text{mod } 2) = 1$, we set $\beta > 1$ to ensure that $F_{l-1} \left(\sum_{\xi = \beta + 1}^{d + \beta - 2} 2^{5\xi - 1} \tilde{c}_{\text{r}l-1}\xi\right)$ is an even element. Thus, if $\beta > 1$, we have $F_{l-1} \tilde{\tilde{c}}_{\text{r}l-1}(\text{mod } 2) = F_{l-1} \tilde{\tilde{c}}_{\text{r}l-1}(\text{mod } 2)$. Further, according to Eq. (7), the error magnitude is given in the following:

$$
\| F_j \tilde{\tilde{c}}_{\text{r}} \|_\infty = \| F_j f_j^{-1} (\text{LBD}_{\beta \rightarrow d + \beta - 1}(\text{BitD}(\tilde{c}_{\text{r}-1})) \cdot 2g \cdot \tilde{\tilde{s}}_{\text{r}}) + 2F_{l-1} \left(\text{LBD}_{\beta \rightarrow d + \beta - 1}(\text{BitD}(\tilde{c}_{l-1})) \cdot \tilde{\tilde{e}}_{\text{r}}\right)\|_\infty + \| F_j \tilde{\tilde{c}}_{\text{r}l-1}\|_\infty < (3l - d + (2^{d+2} - 4)) \times (n-1)E_{l+1} + F_{l-1} \tilde{\tilde{c}}_{\text{r}l-1}\|_\infty \tag{8}
$$

For any $a, b \in R_q$, it holds that $\| a - b \|_\infty \leq \| a \|_\infty + \| b \|_\infty$. Since $F_t = F_{l-1}(f_t)^{-1}$, the magnitude of $F_j f_j^{-1} g \cdot \tilde{\tilde{s}}_{\text{r}} \|_\infty$, $F_j \tilde{\tilde{e}}_{\text{r}} \|_\infty$, and $F_{l-1} \tilde{\tilde{c}}_{\text{r}l-1}\|_\infty$ in Eq.
(8) are the same. Compared to Eq. (2), it can be found that when \( d > 1 \), \( \| F_j \tilde{c}_j \| \|_\infty > \| F_j c_j \| \|_\infty \) which means that low bits are discarded, the error increases. Moreover, the value of \( \| F_j \tilde{c}_j \| \|_\infty \) increases with the use frequency \( k \). Let \( O \) be a positive constant. Compute the plaintext \( C \) to decrypt \( c \). Use public key \( pk = C \) where \( C = \text{LBD}_d \cdot c \). To keep the probability \( P_{\text{of}2} \) of change to encrypt the plaintext vector \( d \) for given \( K_1 \) and \( K_2 \). To ensure the correctness of the decryption. Accordingly, the modified decryption structure can be improved by the following steps.

**Step 1 (Discard the plaintext vector):** Let \( l = \lfloor \log q \rfloor \), \( d \) is a positive constant. Compute the plaintext vector \( \tilde{m} = \text{LBD}_{2 \rightarrow d+1} (\text{Pof}2(m)) (m \in [0, 1]) \).

Output: \( \tilde{m} = (m, 2d+1 m, \ldots, 2l-1 m) \in R^l_q \).

**Step 2 (Ciphertext expansion):** Let \( pk := h \in R_q \), \( sk := f \in R \). Sample \( s, e \rightarrow \chi \). Use public key \( pk \) to encrypt the plaintext vector \( \tilde{m} = (m, 2d+1 m, \ldots, 2l-1 m) \).

Output: \( c := \tilde{m} + hs + 2e \in R^l_q \).

**Step 3 (Set the decryption structure):** For given two ciphertext vectors \( c_1 \) and \( c_2 \) with the corresponding public-key sets \( K_1 \) and \( K_2 \), compute the matrix \( C \).

\[
C = \text{LBD}_{2 \rightarrow d+1} (\text{BitD}(c_1^t)) \in R^l_q \times [l \times d] .
\]

Output: \( e_{\text{mult}} = C \cdot c_2^t \in R^l_q \).

**Step 4 (Select decryption element):** Select the first element \( c_{\text{mult}, 1} \) from the ciphertext vector \( c_{\text{mult}} \). Here, \( F_K = F_j = f_1 f_2 \cdots f_r \) and \( K = K_1 \cup K_2 \).

Output \( \text{Dec}(F_K, c_{\text{mult}, 1})(\text{mod} 2) \).

We modify \( \tilde{c} \in R_q \) to obtain \( c := \text{Pof}2(m) + h\tilde{s} + 2\tilde{e} \in R^l_q \) instead of \( c := \text{Pof}2(m + h\tilde{s} + 2\tilde{e}) \), so that the error generated by the term of \( \text{Pof}2(h\tilde{s} + 2\tilde{e}) \) can be removed. In Section 4.3, the advantages of this change will be introduced when calculating the error magnitude.

**Correctness verification of Method 2:** According to Step 2, set \( \alpha = \{1, 2\} \), we get

\[
e_a^t := \begin{bmatrix}
2^0 m_a + h s_{a, 1} + 2 e_{a, 1} \\
2^1 m_a + h s_{a, d_2, d_2} + 2 e_{a, d_2, d_2} \\
\vdots \\
2^{l-1} m_a + h s_{a, l, l} + 2 e_{a, l, l}
\end{bmatrix} \in R^l_q .
\]

Thus, we have \( \text{BitD}^t(e_a^t) \in R^{(l \times d)} \). To keep the correctness of \( \text{BitD}^t(e_a^t) \cdot c^t_e \), we perform Step 3. After discarding the \( (2 \rightarrow d+1) \) columns of matrix \( \text{BitD}^t(e_a^t) \), an \((l - d) \times (l - d) \) matrix is obtained,

\[
C = \text{LBD}_{2 \rightarrow d+1} (\text{BitD}(e_a^t)) = \begin{bmatrix}
(c_{a, 1})_1 & (c_{a, 1})_{d+2} & \cdots & (c_{a, 1})_{l} \\
(c_{a, 1})_{d+1} & (c_{a, d+2})_{d+2} & \cdots & (c_{a, d+2})_{l} \\
\vdots & \vdots & \ddots & \vdots \\
(c_{a, l})_1 & (c_{a, l})_{d+2} & \cdots & (c_{a, l})_{l}
\end{bmatrix}.
\]

So, according to Step 4, we use the joint secret key \( F_K \) to decrypt \( e_{\text{mult}} \).

\[
F_K \cdot e^t_{\text{mult}} = 2^0 m_2 + h s_{2, 1} + 2 e_{2, 1} + 2^{d+1} m_2 + h s_{2, d_2, d_2} + 2 e_{2, d_2, d_2} + \cdots + 2^{l-1} m_2 + h s_{2, l, l} + 2 e_{2, l, l}.
\]
We set $c_1 F_{K_1} = m_1 F_{K_1} + v_1'$ and $c_2 F_{K_2} = m_2 F_{K_2} + v_2'$, let $\|v_1'\|_{\infty} = \|v_2'\|_{\infty} < \eta$. Thus, the first element is selected as decryption,

$$c_{\text{mult,1}} F_K = (c_{1,1})_1 (2^{m_2 F_K + F_K e_{\epsilon,1}^1}) +$$

$$\sum_{\xi = d+2} l (c_{1,1})_\xi (2^{m_1 F_K + F_K e_{\epsilon,2}^1}) =$$

$$\sum_{\xi = 2} l (c_{1,1})_\xi (2^{m_1 F_K + F_K e_{\epsilon,2}^1}) =$$

$$\sum_{\xi = 2} l (2^{c_{1,1}})_\xi m_2 F_K + \sum_{\xi = 1} l (c_{1,1})_\xi F_K e_{\epsilon,2}^1 =$$

$$\sum_{\xi = 2} l (c_{1,1})_\xi F_K e_{\epsilon,2}^1 + \sum_{\xi = 2} l (2^{c_{1,1}})_\xi m_2 F_K =$$

$$m_{1,1} m_2 F_K + m_{2,2} F_{K-K_1} v_{1,1}^1 + \sum_{\xi = 1} l (c_{1,1})_\xi F_K - K_1 v_{1,1}^1 -$$

$$\sum_{\xi = 2} l (c_{1,1})_\xi F_K - K_1 v_{1,1}^1 =$$

$$\sum_{\xi = 2} l (c_{1,1})_\xi F_K - K_1 v_{1,1}^1 (9)$$

where $\|v_{1,1}'\|_{\infty} = \|v_{1,1}'\|_{\infty}$ and $\|v_{2,1}'\|_{\infty} = \|v_{2,1}'\|_{\infty}$. So we can get $c_{\text{mult,1}} F_K \text{(mod 2)} = m_{1,1} m_2$. The error magnitude is bounded by $\|c_{\text{mult,1}} F_K\|_{\infty} \leq q/2$.  

4.3 Parameters analysis

As already explained, by using the LBD&DEC technique, the original, inherent decryption structure of the NTRU-type MKHE can be modified.

**Theorem 1** Let LBD constant $d$, $c_d F_{K_d} = m_d F_{K_d} + v_d'$, and $\|v_d'\|_{\infty} < \eta$. Let $c_{\text{mult,1}}$ be the first column of the ciphertext vector $c_{\text{mult}} = C \cdot c_{\text{mult}^T} \in K_q^{l-d}$, and $C = \text{LBD}_{2,d+1}(\text{BitD}(c_{\text{mult}}^1)) \in K_q^{(l-d) \times (l-d)}$. Then, we have

$$c_{\text{mult,1}} F_K = m_{1,1} m_2 F_K + v_{\text{mult,1}}$$

and

$$\|v_{\text{mult,1}}\|_{\infty} \leq \left(2 (n-1) (l - d) + 1 \right) \times$$

$$\left( \frac{3}{2} E_r + 3 N(n-1)(l-d) E_{r+1} +$$

$$1 + N(2^{d+2} - 4)(n-1)(2(n-1)(l-d) + 1) \times$$

$$2^{d+2} - 4)(n-1) E_r, \right.$$ 

where $E_r = 2^{(n-1)} (2B + 1)^r$.

**Proof** According to Eq. (9), the error of $c_{\text{mult,1}} F_K$ can be bounded as follows:

$$\|c_{\text{mult,1}} F_K\|_{\infty} = \left| m_{1,1} m_2 F_K +$$

$$\left( m_{2,2} F_{K-K_1} v_{1,1}^1 + \sum_{\xi = 1} l (c_{1,1})_\xi F_{K-K_1} v_{1,1}^1 -$$

$$\sum_{\xi = 2} l (c_{1,1})_\xi F_{K-K_1} v_{1,1}^1 \right) =$$

$$\left( \sum_{\xi = 1} l (c_{1,1})_\xi F_{K-K_1} v_{1,1}^1 -$$

$$\sum_{\xi = 2} l (c_{1,1})_\xi F_{K-K_1} v_{1,1}^1 \right) \right| \leq$$

$$\left| m_{1,1} m_2 F_K + \sum_{\xi = 1} l (c_{1,1})_\xi F_{K-K_1} v_{1,1}^1 \right|$$

$$\leq E_r + 2 (n-1) E_{r-1} (10)$$

Thus, the starting error can be easily obtained as $\eta_0 < 3 (n-1) (2B + 1)^2$. Then, the magnitude of $\eta$ is calculated. For $N$ users having a set of public-keys $K_1$, the worst case is that $c_1$ is the product of all the users' fresh ciphertexts. Specifically, for $N = 2$, we have $\{e_1^1, f_1\}$ and $\{e_1^2, f_2\}$, where $e_1^1$ and $e_1^2$ denote the fresh ciphertext vectors. The magnitude of $\eta$ is bounded,

$$\eta_N = 2 \leq (2(n-1)(2B + 1))(2(n-1)(l-d) + 1) \eta_0 +$$

$$(2^{d+2} - 4)(n-1) E_2.$$ 

Further, as $N = 3$, set $\{e_1^1, e_1^2, f_1' = f_1 f_2\}$ and $\{e_3^1, f_3\}$. So the magnitude of $\eta$ is bounded,

$$\eta_N = (2(n-1)(2B + 1))(2(n-1)(l-d) + 1) \eta_0 +$$

$$(2^{d+2} - 4)(n-1) E_3.$$ 

Accordingly, for $N$ users, set $\{e_1^1, \ldots e_{N-1}^1, f_1' = f_1 \ldots f_{N-1}\}$ and $\{e_N^1, f_N\}$. So the magnitude of $\eta$ is bounded,

$$\eta_N = 2 \leq (2(n-1)(2B + 1))(2(n-1)(l-d) + 1) \eta_0 +$$

$$(2^{d+2} - 4)(n-1) E_N.$$ 

This yields to the following relationship:

$$\eta \leq (2(n-1)(2B + 1)) N \eta_0 +$$

$$(2^{d+2} - 4)(n-1) E_N.$$ 

Further, as $N = 3$, set $\{e_1^1, e_1^2, f_1' = f_1 f_2\}$ and $\{e_3^1, f_3\}$. So the magnitude of $\eta$ is bounded,

$$\eta_N = 2 \leq (2(n-1)(2B + 1))(2(n-1)(l-d) + 1) \eta_0 +$$

$$(2^{d+2} - 4)(n-1) E_3.$$ 

Accordingly, for $N$ users, set $\{e_1^1, \ldots e_{N-1}^1, f_1' = f_1 \ldots f_{N-1}\}$ and $\{e_N^1, f_N\}$. So the magnitude of $\eta$ is bounded,
By combining Eqs. (10) and (11), we get
\[ \|c_{\text{mult},1}F_K\|_\infty \leq E_r + (2(n-1)(l-d) + 1) \times \left( \frac{3}{2}E_{r+2} + 3N(n-1)(l-d)E_{r+1} \right) + (1 + N(2^{d+2} - 4)(n-1)(2(n-1)(l-d) + 1)) \times (2^{d+2} - 4)(n-1)E_r. \]

So, the decryption structure is obtained \( c_{\text{mult},1}F_r = m_1m_2F_r + v_{\text{mult},1}. \) Since
\[ \|c_{\text{mult},1}F_r\|_\infty = \|m_1m_2F_r\|_\infty + \|v_{\text{mult},1}\|_\infty. \]

Further, we obtain
\[ \|v_{\text{mult},1}\|_\infty \leq (2(n-1)(l-d) + 1) \times \left( \frac{3}{2}E_{r+2} + 3N(n-1)(l-d)E_{r+1} \right) + (1 + N(2^{d+2} - 4)(n-1)(2(n-1)(l-d) + 1)) \times (2^{d+2} - 4)(n-1)E_r. \]

According to Theorem 1, it can be found that when \( d = 0, \|v_{\text{mult},1}\|_\infty \leq (2(n-1)(l+1)) \times \left( \frac{3}{2}E_{r+2} + 3N(n-1)(l+1)E_{r+1} \right) \). This denotes the error generated only by the ciphertext dimension extension technique.

Theorem 2  Set LBD constant \( d. \) The LBD&DEC technique can decrease both the ciphertext dimension and error magnitude, and \( d \) satisfies the following relationship:
\[ d = \{\max(d)\}(2^d - 1)/d \leq 3(n-1)(2B+1)/2, \quad d > 0. \]

Proof  Take the homomorphic multiplication as an example, the LBD technique is to decrease the ciphertext dimension and error magnitude. If the LBD technique is not used in the ciphertext vector, the decryption structure is assumed as \( c_{\text{mult}} \cdot F_K = \text{BitD}(c_1^{*}) \cdot (c_2^{*})^{T} \cdot F_K. \) The decryption can be completed by using the first column of \( c_{\text{mult}} \cdot F_K. \)
\[ c_{\text{mult},1}F_K = m_1m_2F_K + m_2F_{K-K}v_{1,1} + \sum_{s=1}^{1}(c_{1,1}^{*})c_{F-K-K_2}v_{2,1}^{*}. \]

Assume the magnitude of errors \( v_{1,1}^{*} \) and \( v_{2,1}^{*} \) is \( \eta'. \) According to Eq. (10),
\[ \|c_{\text{mult},1}F_K\|_\infty \leq E_r + (2(n-1)l + 1) 2(n-1)E_{r-N} \eta'. \]

Obviously, the LBD can reduce the ciphertext dimension from \( l \) to \( (l-d), \) but we want to decrease the error magnitude at the same time. Note that the starting error is \( \eta_0 < 3(n-1)(2B+1)^2. \) According to Theorem 1, when \( d = 0, \) we can obtain
\[ \eta' \leq 3E_{N+2}/2 + 3N(n-1)E_{N+1}. \]

Compared to Eq. (11), there is a constant \( d > 0 \) that makes \( \eta' > \eta, \) which is given by
\[ 3N(n-1)dE_{N+1} > N(2^{d+2} - 4)(n-1)E_{N} \]
\[ \Rightarrow 3(n-1)(2B+1) > \frac{(2^{d+1} - 2)}{d} \]

It can be easily found that \( (2^d - 1)/d \) is incremental of \( d (d > 0). \) So, \( d \) has an upper bound that makes \( \eta \) be the closest to \( \eta'. \) We let \( d_1 \) be the upper bound of \( d, \) if \( d = d_1, \) then \( \eta \approx \eta'. \) Therefore, we need to verify the correctness of the relationship \( \|c_{\text{mult},1}F_K\|_\infty \geq \|c_{\text{mult},1}F_K\|_\infty. \) So, we have
\[ \|c_{\text{mult},1}F_K\|_\infty > \|c_{\text{mult},1}F_K\|_\infty \]
\[ \Rightarrow d_1(2(n-1)E_{r-N} \eta') > (2^{d_1+1} - 2)E_{r} \]
\[ \Rightarrow \frac{3}{2}E_{r+2} + 3N(n-1)lE_{r+1} > \frac{(2^{d_1+1} - 2)E_{r}}{d_1} \]
\[ \Rightarrow (n-1)^2(2B+1)(Nl+2B+1) > \frac{(2^{d_1+1} - 2)}{d_1} \]

Since \( (n-1)^2(2B+1)(Nl+2B+1) > 3(n-1)(2B+1) \) is obviously satisfied, Eq. (13) holds. Thus, for any value of \( N, \) we can select \( d \) that satisfies \( d = \max(d)\)(\( 2^d - 1)/d \leq 3(n-1)(2B+1)/2, \) \( d > 0) \) to ensure \( \|c_{\text{mult},1}F_K\|_\infty \geq \|c_{\text{mult},1}F_K\|_\infty. \]

According to Theorems 1 and 2, the LBD&DEC can be used to improve the decryption structure, while decreasing the error magnitude. Consequently, Method 2 can be used to modify the NTRU-type multi-key homomorphic encryption schemes.

5 Modified NTRU-Type Multi-Key Somewhat Homomorphic Encryption

According to the analysis provided in Section 4, Method 2 has two advantages.

(1) The DEC technique improves the decryption structure of the NTRU-type scheme and eliminates the key-switching operations, which significantly decreases the dependence of the error on \( N \) (the dependence is exponentially decreasing).

(2) The LBD technique reduces the ciphertext dimension and further decreases the error magnitude.

Based on Method 2, we propose an NTRU-type MKSHE by using the LBD&DEC technique.

5.1 Modified NTRU-type MKSHE

Let \( \lambda \) be a security parameter; \( q = q(\lambda) \) is a prime integer; and \( \Phi_p(x) = x^{n-1} + x^{n-2} + \cdots + 1 (n \) is a prime) is the sub-cyclotomic polynomial. For the polynomial ring given by \( R = \mathbb{Z}[x]/\Phi_p(x) \) and \( R_q = R/qR, \) and a \( B \)-bound error distribution \( \chi = \chi(\lambda) \) over \( R, \) set \( \chi' \) as a \( B \)-bound error distribution space, where \( l = [\log q]. \) The modified NTRU-type MKSHE (denote as M-MKSHE) can be described as follows.
(1) M-MKSHE. KeyGen($1^k$): Sample $f', g \leftarrow \chi$, and set $f = 2f' + 1$, so that $f \equiv 1 \pmod{2}$. If $f$ is not invertible in $R_q$, resample $f' \leftarrow \chi$. Set $h = 2g/f \in R_q$, so $pk := h \in R_q, sk := f \in R$. Set $l = \lfloor \log q \rfloor$, and let discard constant $d$ satisfy the following relationship:

\[ d = \lfloor \max(d) \rfloor \frac{(2^d - 1)/d}{(n - 1)(2B + 1)/2, \quad d > 0}. \]

Output: $(pk, sk, int) = (h, f, d)$.

(2) M-MKSHE. Enc($pk, m$): Compute $\hat{m} = LBD_{2 \rightarrow d+1}(\text{Pof} 2(m))$, and sample $s, e \leftarrow \chi^{1 \to d}$.

Output: $c := \hat{m} + hs + 2e \in R_q^{1 \to d}$.

(3) M-MKSHE. Dec($sk, c$): Select the first element $c_1$ from the ciphertext vector $c$, let $u := (sk_1sk_2 \ldots sk_N)c_1 \in R_q$.

Output: $m' := u \pmod{2}$.

(4) M-MKSHE. Eval. Add($c_1, c_2$): Compute the addition of $c_1$ and $c_2$ as $c_{\text{add}} = [c_1 + c_2]_q$.

(5) M-MKSHE. Eval. Mult($c_1, c_2$): Compute the matrix $C = LBD_{2 \rightarrow d+1}(\text{BitD}(c_1^T)) \in R_q^{(l-1) \times (l-1)}$.

Output: $c_{\text{multi}} = C \cdot c_2^T \in R_q^{1 \to d}$. 

5.2 Analysis results

(1) Correctness of M-MKSHE scheme

**Homomorphic multiplication:** Considering the decryption of homomorphic multiplication, we can get $c_{\text{multi}} \cdot F_K = (LBD_{2 \rightarrow d+1}(\text{BitD}(c_1^T))) \cdot c_2^T \in R_q^{l \to d}$. By selecting the first column of $c_{\text{multi}} \cdot F_K$, we get

\[
\begin{align*}
&c_{\text{multi},1}F_K = m_1m_2F_K + \left( m_2F_K - K_1v'_1v'_1,1 + \sum_{s=2}^{d+1} (c_{1,1}^sF_K - K_2v'_2,1)^{-1} \right. \\
&\left. \sum_{s=2}^{d+1} m_2(c_{1,1}^sF_K - K_2v'_2,1) \right) \\
\end{align*}
\]

Note that $[c_{\text{multi},1}F_K]_q \pmod{2} = m_1m_2$.

**Remark:** It is not necessary to consider whether $K_1 \cap K_2$ is empty.

(2) Circuit depth of M-MKSHE scheme

**Theorem 3** For the parameter values provided above, the M-MKSHE can evaluate any circuit depth, $L = \frac{\log q}{(N - j + 1)\log(n - 1) + \log q + O(1)}$.

**Proof** As already mentioned, in this work we consider only the error of multiplication. Without loss of generation, we set $K_1 \cap K_2 = \{pk_1, \ldots, pk_j\}$, so $r = 2N - j$. For any level of multiplication operations, the multiplication of ciphertexts can be decrypted by $F_K$. According to Theorem 1, after the first level homomorphic multiplication evaluation, the error of $c_{\text{multi},1}F_K$ (here $c_{\text{multi},1}$ denotes the first element of vector $c_{\text{multi}}$ in the first level) is bounded,

\[
\|v_{\text{multi},1}\|_\infty \leq (2(n - 1)(l - d) + 1) \left( \frac{3}{2}E_r + 2 \right)
\]

At the second level, we set

\[
\|
\begin{align*}
&\sum_{s=2}^{d+1} (c_{1,s}^sF_K - K_2v'_2,1)^{-1} \right. \\
&\left. \sum_{s=2}^{d+1} m_2(c_{1,s}^sF_K - K_2v'_2,1) \right) \\
\end{align*}
\]

\[
\in R_q^{l \to d}.
\]

In the same way, we can get $c^{(2)}_2 \in R_q^{l \to d}$. Let $c^{(2)}_1$ and $c^{(2)}_2$ correspond to the public-key sets $K_1$ and $K_2$, respectively. It should be noted,
where $\theta$ is a constant variable. The magnitude of $m^{(L)}(t)$ is $0$, which yields to $\|v_{\text{mult,1}}^{(L)}\|_\infty + Q \times E_r < \frac{3}{2} \hat{P} \times E_{L_r-(L-1)N+2} + 3N \times P \times E_{L_r-(L-1)N+1} + (1 + N \times Q \times P) (P \times Q) E_{2R+2N} + Q \times E_r$.

After $L$ levels of homomorphic operations, the error magnitude can grow up, $\|v_{\text{mult,1}}^{(L)}\|_\infty < 2(n-1)\|E_{R-N}\|_\infty + Q \times E_r < \frac{3}{2} \hat{P} \times E_{L_r-(L-1)N+2} + 3N \times P \times E_{L_r-(L-1)N+1} + (1 + N \times Q \times P) (P \times Q) E_{2R+2N} + Q \times \sum_{\theta=2}^{L} \hat{P}^{\theta-1} \times E_{(3(n-1))/2}N$ (14)

where $\theta$ is a constant variable. The magnitude of $m^{(L)}(t)$ is ignored because it is much smaller than $\|v_{\text{mult,1}}^{(L)}\|_\infty$, and let $\|v_{\text{mult,1}}^{(L)}\|_\infty < q/2$.

According to Theorem 2, the LBD technique is not used in our scheme, the value of $d$ is 0, which yields to the following error bound of $\|v_{\text{mult,1}}^{(L)}\|_\infty$:

$$\|v_{\text{mult,1}}^{(L)}\|_\infty < \frac{3}{2} (1 + 2(n-1)\hat{P} \times E_{L_r-(L-1)N+2} + 3N(1 + 2(n-1)) \hat{P} \times E_{L_r-(L-1)N+1}$$

Thus, by selecting $d = \max(d)/\sqrt{2(n-1)}$, the magnitude of $\|v_{\text{mult,1}}^{(L)}\|_\infty$ becomes infinitely close to $\|v_{\text{mult,1}}^{(L)}\|_\infty$. The limit state is selected at each level, and the final error after the circuit depth of $L$ satisfies the following relationship:

$$\|v_{\text{mult,1}}^{(L)}\|_\infty < \frac{q}{2}$$

$$\Rightarrow L \log(1 + 2(n-1)\hat{P}) + (L - (L-1)N + 2) \times \log(2(n-1)(2B+1)^2) + \log\left(1 + \frac{N}{4(n-1)(2B+1)^2}\right) < \log q + \log 3$$

$$\Rightarrow L \approx \frac{\log q}{(N-j+1) \log(n-1) + \log \log q + O(1)} \quad (15)$$

According to Eqs. (14) and (15), with the increase of parameter $N$, the error magnitude increases, and the circuit depth decreases. However, $j$ can reduce the impact of $N$, which is contrary to the BC-MKSHE scheme.

5.3 Parameters comparison

In the BC-MKSHE scheme, after one homomorphic multiplication operation, the error satisfies the following:

$$\text{Error}_{\text{BC-MKSHE}} < 6!(n-1)E_{2N} + 2^N E_{4N}$$

However, in our M-MKSHE scheme, the upper bound of the error is given by:

$$\text{Error}_{\text{M-MKSHE}} < 3(n-1)\frac{1}{2}E_{r+2} + 3(n-1)lE_{r+1}$$

The ratio of the two previous error bounds is given, $\text{Ratio} = \frac{\text{Error}_{\text{BC-MKSHE}}}{\text{Error}_{\text{M-MKSHE}}} \approx \frac{6!(n-1)E_{2N} + 2^N E_{4N}}{(2(n-1)l + 1)\frac{1}{2}E_{r+2} + 3(n-1)lE_{r+1}} + E_r$.

$$\Rightarrow \frac{2^N E_{2N}}{N(n-1) \log q + O(1)} \quad (16)$$

According to Eq. (16), the error magnitude of our M-MKSHE scheme is decreased exponentially compared to the BC-MKSHE scheme.

In the following, the comparison of these two schemes regarding the other parameters is provided, such as the ciphertext size, secret key size, public key size, and evaluation key size.

Take one homomorphic multiplication operation as an example. In the BC-MKSHE scheme, the ciphertexts are two polynomials in $R_q$, whose degree is smaller than $(n-1)$, so the size of ciphertexts is $(2(n-1) \log q$. Also, the public keys are $2N$ polynomials in $R_q$, so the size of public keys is $2N(n-1) \log q$. Further, the joint secret keys for decrypting are $r$ polynomials in $R$, and their coefficients are smaller than $(2B + 1)$, so the size of joint secret keys is $r(n-1) \log(2B + 1)$. Furthermore, the evaluation keys are $|\log q|$-dimensional polynomials whose degree is smaller than $(n-1)$. Then, after $j$-times evaluations, the size of the evaluation keys is $j(n-1) |\log q| \log q$.

In our modified scheme, the key-switching technique is not used and none of the evaluation keys is
required. The ciphertexts are \((\log q - d)\)-dimensional polynomial vectors, so their size is 
\[2(n - 1)(\log q - d)\log q.\] The same as for the BC-MKSHE, the size of public keys is 
\[2N(n - 1)\log q,\] and the size of joint secret keys is 
\[(n - 1)\log(2B + 1).\] See Table 1 for details, the comparison of the parameters of the M-MKSHE and 
BC-MKSHE schemes is provided.

As shown in Table 1, our scheme does not require the evaluation key, and the error magnitude is reduced 
exponentially, but the ciphertext size is increased by \((\log q - d)\) times.

6 Leveled NTRU-Type Fully Homomorphic Encryption

According to Theorem 3, the circuit depth is reduced with the decrease of \(N\), so the modulus-reduction technique has to be used to decrease the error magnitude after every homomorphic evaluation.

**Modulus-reduction**\[28, 29\]: Modulus-reduction technique can change the inner modulus \(q\) of a ciphertext \(c\) to the smaller modulus \(p\) (\(p = q \mod 2\)) while roughly scaling down the error by the ratio of \(p/q\) and preserving the correctness of the decryption under the same secret key.

**ModulusSwitch\((c, q, p)\)**: For input \(c \in R_p\), and a smaller modulus \(p\), output is \(c' \in R_p\), which is the closest element to \((p/q) \cdot c\) and \(c' = c \mod 2\).

**Lemma 3**\[28\]: Let \(p\) and \(q\) be two odd modulus, let \(c \in R_q\), and define \(c' \in R_q\), whose value is the closest to \((p/q)c\), then \(c' \equiv c \mod 2\).

**Output**:\(\{p, sk, \text{int} = \{h(0^\ast), f(L)\}\}^\ast\):

(2) **M-MKFHE. Enc\((pk, m)\)**: Sample \(s(0^\ast), e(0^\ast) \leftarrow \chi^{l_0 - d}\), let \(l_0 = \log q\) and \(\hat{n} = (\text{Pof}\,2(m))^T \in R_{q_0}\). Output the ciphertext vector, \(c(0^\ast) := h(0^\ast)s(0^\ast) + 2e(0^\ast) + LBD_{2^{l_0} - d}(\hat{n}) \in R_{q_0}\).

(3) **M-MKFHE. Dec\((sk_1, sk_2, \ldots, sk_N, c(L))\)**: Select the first element \(c_1^{(L)} \in R_{q_L}\) from ciphertext vector \(c(L) \in R_{q_L}\), set \(u := (sk_1sk_2\ldots sk_N)c_1^{(L)} \in R_{q_L}\).

**Output**:\(m' := u \mod 2\).

(4) **M-MKFHE. Eval. Add\((c_1^{(t)}, c_2^{(t)})\)**: For the two ciphertexts \(c_1^{(t)}, c_2^{(t)} \in R_{q_L}\), see the \(l_t\)-th level, compute the addition of \(c_1^{(t)}\) and \(c_2^{(t)}\) as \(c_{\text{add}}^{(t)} := [c_1^{(t)} + c_2^{(t)}]_{q_L} \in R_{q_L}\). Then, reduce the modulus, so we have \(c_{\text{add}}^{(t + 1)} = (q_{t+1} / q_t) \cdot c_{\text{add}}^{(t)} \mod 2\).

**Output**:\(c_{\text{add}}^{(t+1)} = LBD_{2^{n_0} - d + 1 - l_t - d}(c_{\text{add}}^{(t+1)})\).

(5) **M-MKFHE. Eval. Mult\((c_1^{(t)}, c_2^{(t)})\)**: For the two

| Table 1 | Comparison of parameters between BC-MKSHE and M-MKSHE. |
|---------|---------------------------------------------------------|
| Parameter                          | BC-MKSHE scheme | M-MKSHE scheme |
| Maximum size of error               | \(6(n - 1)E_2N + 2^N E_4N\) | \(E_r + (2(n - 1) - l - d) + \left(\frac{3}{2}E_{r + 2} + 3^N(n - 1)(l - d)E_{r + 1}\right) + (2^{d + 2} - 4)(n - 1)E_r + (2(n - 1) - l - d) + 1(N(2^{d + 2} - 4)(n - 1))E_r\) |
| Ciphertext size                     | \(2(n - 1)\log q\) | \(2(n - 1)(\lceil \log q \rceil - d) \log q\) |
| Evaluation key size                 | \(j(n - 1)(\lceil \log q \rceil) \log q\) | \(0\) |
| Public key size                     | \(2N(n - 1)\log q\) | \(2N(n - 1)\log q\) |
| Secret key size                     | \(r(n - 1)(2B + 1)\) | \(r(n - 1)(2B + 1)\) |

Note: \(d = \lceil \max(d)\rceil(2^d + 1 - 2)/d \leq 3(n - 1)(2B + 1), d > 0\).
ciphertexts $c_1^{(l)}$ and $c_2^{(l)} \in R_q^{[\log q] - d}$ in the $i$-th level, compute the multiplication of $c_1^{(l)}$ and $c_2^{(l)}$ as $c_{\text{mult}}^{(l)} = \text{LBD}_{2^{-d} + 1} \left( \text{BitDecomp}(c_1^{(l)}) \right) \cdot c_2^{(l)} \in R_q^{l_i - d}$. Then, by reducing the modulus, we get $c_{\text{mult}}^{(l+1)} = (q_{l_i+1}/q_i) \cdot c_{\text{mult}}^{(l)} \pmod{2}$.

Output: $c_{\text{mult}}^{(l+1)} = \text{LBD}_{l_{i+1} - d + 1 \rightarrow l_i - d} (c_{\text{mult}}^{(l)})$.

6.2 Analysis

(1) Scheme framework

The process of homomorphic operation in the M-MKFHE scheme is shown in Fig. 1. The flowchart presented in Fig. 1 can be used as a model framework for algorithm design. In Fig. 1, the ciphertext is expanded to the vector starting from the plaintext vector, i.e., $c := \text{LBD}_{2^{-d} + 1} \left( \text{BitDecomp}(c_1^{(l)}) \right) \cdot c_2^{(l)} \in R_q^{l - d}$. Only the first element of the ciphertext vector is decrypted. Therefore, the correctness of the first term of $LBD_{2^{-d} + 1} \left( \text{BitDecomp}(c_1^{(l)}) \right) \cdot c_2^{(l)}$ should be ensured. In order to complete the homomorphic operation, the ciphertext has to be maintained in the vector form. Although the complexity of the ciphertext calculation is increased when the ciphertext vectors are multiplied, the key-switching technique is not used in our scheme.

(2) Correctness

For $B \ll q_L$, the selected LBD constant $d$ is suitable for all levels of homomorphic operations. Thus, to reduce the modulus every time, we need to perform the LBD to discard some rows of the ciphertext vector. For instance, at the $i$-th level, when the homomorphic operations are completed, we get the $(l_i - d)$-dimensional ciphertexts $c_{\text{add}}^{(i)}$ and $c_{\text{mult}}^{(i)} \in R_q^{l_i - d}$. After the ciphertext is decomposed by BitDecomp(): $R_{q_{l_i+1}} \rightarrow R_{q_{l_i+1}}^{[\log q_{l_i+1}] - d}$ at the $(i + 1)$-th level, $c_{\text{add}}^{(i)}$ and $c_{\text{mult}}^{(i)}$ are decomposed to $(l_i - d) \times (l_i + 1 - d)$-dimension matrices. Therefore, the following algorithm has to be performed, and the last $(l_i - l_{i+1})$-th rows of the matrices have to be discarded to keep the correctness of the next homomorphic operation. The conversion progress is provided in Algorithm 1.

The conversion algorithm is important to achieve a fully homomorphic operation in our M-MKFHE scheme. It can be seen that the ciphertext dimension is reduced with the increase in the circuit depth $L$ by using the LBD&DEC technique. So, the modulus-reduction of our M-MKFHE scheme has two main advantages: (1) Reducing the modulus can decrease the error magnitude. (2) Reducing the modulus can also decrease the ciphertext dimension at different levels. Both of these advantages can improve the efficiency of the MKFHE scheme.

(3) Security

Our leveled M-MKFHE denotes a modified MKFHE in the LTV12 scheme. The techniques of LBD& DEC are used. The security of dimension expansion depends on the RLWE and DSPR assumptions over prime cyclotomic rings. The LBD is based on functions BitDecomp() and Powersof(2). As known in Ref. [6], functions BitDecomp() and Powersof(2) have no effect on security. Thus, the LBD technique does not affect the security of our scheme. According to Refs. [6, 26], our M-MKFHE scheme is IND-CPA secured under the RLWE and DSPR assumptions over prime cyclotomic rings.

7 Conclusion

By using the LBD&DEC technique, our modified multi-key FHE improves the inherent homomorphic

![Fig. 1 Process of homomorphic operation in the M-MKFHE scheme.](image-url)
multiplication decryption structure of the NTRU in the LTV12 scheme, and successfully eliminates the key-switching operations and decreases the magnitude of error exponentially. Moreover, our scheme can more effectively process the quadratic part of a ciphertext product. The LBD technique used in our M-MKFHE can minimize the ciphertext dimension and improve the efficiency of the homomorphic operation.

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