The generalization error of max-margin linear classifiers: High-dimensional asymptotics in the overparametrized regime

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Abstract

Modern machine learning models are often so complex that they achieve vanishing classification error on the training set. Max-margin linear classifiers are among the simplest classification methods that have zero training error (with linearly separable data). Despite this simplicity, their high-dimensional behavior is not yet completely understood. We assume to be given i.i.d. data \((y_i, x_i), i \leq n\) with \(x_i \sim N(0, \Sigma)\) a \(p\)-dimensional Gaussian feature vector, and \(y_i \in \{+1, -1\}\) a label whose distribution depends on a linear combination of the covariates \(\langle \theta^*, x_i \rangle\). We consider the proportional asymptotics \(n, p \to \infty\) with \(p/n \to \psi\), and derive exact expressions for the limiting prediction error. Our asymptotic results match simulations already when \(n, p\) are of the order of a few hundreds.

We explore several choices for the the pair \((\theta^*, \Sigma)\), and show that the resulting generalization curve (test error error as a function of the overparametrization ratio \(\psi = p/n\)) is qualitatively different, depending on this choice. In particular we consider a specific structure of \((\theta^*, \Sigma)\) that captures the behavior of nonlinear random feature models or, equivalently, two-layers neural networks with random first layer weights. In this case, we observe that the test error is monotone decreasing in the number of parameters. This finding agrees with the recently developed ‘double descent’ phenomenology for overparametrized models.

1 Introduction

1.1 Background

Modern machine learning methods often require minimizing a highly non-convex empirical risk, a prominent example being multilayer neural networks. It is common practice to increase the complexity of the neural network architecture until the optimization landscape simplifies, and gradient descent (GD) or stochastic gradient descent (SGD) succeed in achieving vanishing training error. Often this approach leads to good generalization properties, even when the resulting model is so overparametrized that could fit purely random labels with zero training error [ZBH+16].

How is it possible that large overparametrization does not lead to overfitting and large generalization error? A popular explanation of this behavior makes use of the notion of ‘implicit regularization’: the optimization algorithm (GD, SGD or their variants) effectively selects a specific model among all the ones with vanishing training error. The selected model minimizes a certain notion of complexity. This intuition has been made precise in certain specific examples (see, e.g., [SHN+18, GLSS18b, LMZ17, GLSS18a, ACHL19]). For instance when performing linear regression with square loss, GD converges to the minimum \(\ell_2\)-norm solution among all the ones that interpolate the data (i.e. achieve vanishing training error).

In this paper we study linear classification: we are given data \(\{(y_i, x_i)\}_{i \leq n}\) where \(y_i \in \{+1, -1\}\) are labels and \(x_i \in \mathbb{R}^p\) are feature vectors, and would like to predict the label of a fresh sample \(x_{\text{new}}\) via \(\text{sign}(\langle \hat{\theta}, x_{\text{new}} \rangle)\),
for some vector $\hat{\theta} \in \mathbb{R}^d$. A popular way to fit such a linear classifier is to use GD to minimize the logistic (a.k.a. cross-entropy) loss

$$
\hat{\theta}^{k+1} = \hat{\theta}^k - s_k \nabla \hat{L}_n(\theta^k), \quad \hat{L}_n(\theta) \equiv \frac{1}{n} \sum_{i=1}^{n} \left\{ -y_i \langle \theta, x_i \rangle + \log \left( e^{\theta^T x_i} + e^{-\theta^T x_i} \right) \right\}.
$$

(1.1)

Throughout this paper, we will focus on the overparametrized regime in which there exists a separator, i.e. a vector $\theta$ such that $y_i \langle \theta, x_i \rangle > 0$ for all $i \leq n$.

It was proven in [SHN+18] that gradient descent (with respect to logistic loss) selects a specific separator, namely the max-margin classifier:

$$
\hat{\theta}^{\text{MM}}(y, X) \in \arg \max \left\{ \min_{i \leq n} y_i \langle \theta, x_i \rangle : \| \theta \|_2 = 1 \right\}.
$$

(1.2)

More precisely, we have $\theta^k / \| \theta^k \| \to \hat{\theta}^{\text{MM}}$ as $k \to \infty$.

Over the last year, the generalization properties of overparametrized models have attracted considerable interest (we refer to Section 3 for a brief overview). A common phenomenology has been demonstrated empirically in a number of statistical models, including kernel methods, random forests, and multilayer neural networks [BHMM19]:

1. The training error decreases as a function of the number of parameters, until it vanishes at a critical value of the overparametrization ratio $\psi = p/n$ (here we denote generically by $p$ the number of parameters). This also referred to as the interpolation threshold, and we will denote it by $\psi^*$. We will refer to models above this threshold (i.e. such that $p/n > \psi^*$) as to ‘overparametrized’ models, and to models below the threshold as to ‘underparametrized’ models.

2. The test error, as a function of the overparametrization ratio, peaks at the interpolation threshold.

3. Overparametrized models outperform underparametrized ones, i.e. they have smaller test error.

4. Optimal prediction accuracy (minimum test error) is achieved at large overparametrization ratios, i.e. for $\psi \gg 1$.

5. Optimal prediction accuracy is achieved with small or sometimes vanishing regularization.

This is referred to as the ‘double descent’ scenario, to emphasize the peculiar behavior of the test error, which descends again after the interpolation threshold.

Several elements of this picture have been rigorously studied in simple models [BHX19, BHM18, HMRT19, BLLT19]. In particular, [HMRT19] used random matrix theory to derive exact predictions for the test error in the case of ridge regression and minimum norm least-squares regression. Several of the above phenomena can be reproduced in this setting. In particular, the generalization curve displays a peak at the interpolation threshold, which in this case is at $\psi^*_n = 1$. This peak becomes an actual singularity in the limit of vanishing regularization: namely the test error diverges as $n, p \to \infty$ with $p/n \to 1$. Further, for certain misspecification structures, the test error has a global minimum in the overparametrized regime.

A more complex setting was studied in [MM19], building on earlier results of [HMRT19]. These papers consider the nonparametric regression problem, namely the problem of learning a function on the $d$-dimensional sphere $S^{d-1}$: $f : S^{d-1} \to \mathbb{R}$, given data $x_i \sim \text{Unif}(S^{d-1})$, $y_i = f(x_i) + \epsilon_i$, $i \leq n$. Regression is performed using a two-layers neural network with fixed (random) first-layer weights. Second-layer weights are fitted using ridge regression. This is the random features model [Nea96, RR08], and can be viewed as a randomized approximation of kernel ridge regression. The paper [MM19] obtains an asymptotically exact characterization of the generalization error. This asymptotic prediction reproduces the above phenomena, and yields several new insights as well.

Can we extend this analysis beyond square loss and ridge regularization? Is the qualitative picture unchanged? Max-margin linear classification is an interesting setting to address these question, for a number of reasons. First of all, most applications of neural networks are to classification rather than regression. Cross-entropy is a far most popular choice of loss function than square loss, and classification error is a standard metric of accuracy. As explained above, in the overparametrized regime, gradient descent with
cross-entropy loss converges to the max-margin classifier. Hence it provides another example (as min-norm least squares) of ‘implicitly regularized’ fitting procedure.

Extending earlier results to the classification setting is not a purely technical problem. Indeed, the new setting will necessarily produce new phenomena for at least two reasons: (i) The classification error is bounded, and therefore it cannot diverge at the interpolation threshold, as is the case for min-norm linear regression. As a consequence, while for min-norm linear regression, the test error is always decreasing in an interval \( \psi \in (\psi^*, \psi^* + \Delta) \). As we will show, this is not always the case for classification. (ii) The interpolation threshold is not at \( \psi^* = 1 \). Indeed, a classical result of Cover [Cov65] yields \( \psi^* = 1/2 \) for the special case in which \( y_i \sim \text{Unif}((+1, -1)) \) independently of \( x_i \), provided the \( \{x_i\}_{i \leq n} \) are in generic positions. This result was recently generalized by Candès and Sur [CS18] for the significantly more challenging setting in which \( \mathbb{P}(y_i = +1|x_i) = (1 + e^{-\langle \theta^*, x_i \rangle})^{-1} \) and \( x_i \) is Gaussian. We also note that [SC19] studies logistic regression in the underparametrized regime \( \psi < \psi^* \) (for isotropic \( x_i \)). In contrast, here will focus on the overparametrized case \( \psi > \psi^* \).

From a technical viewpoint, while the ridge regression estimator can be written explicitly in terms of the eigenvalue decomposition of the covariance matrix \( \Sigma = \Sigma_n \) and the ‘true’ parameters vector \( \theta^* = \theta_{*,n} \). Let \( \Sigma_n = \sum_{i=1}^p \lambda_i v_i v_i^T \) be the eigenvalue decomposition of \( \Sigma \), with \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p \). Our first assumption requires that \( \Sigma \) is well conditioned.

**Assumption 1.** There exist a constant \( c, C \in (0, \infty) \) such that, letting \( \lambda_{\min}(\Sigma_n) = \lambda_1(\Sigma_n), \lambda_{\max}(\Sigma_n) = \lambda_p(\Sigma_n) \),
\[
c \leq \lambda_{\min}(\Sigma_n) \leq \lambda_{\max}(\Sigma_n) \leq C.\]

Our second assumption concerns the eigenvalue distribution of \( \Sigma_n \) as well as the decomposition of \( \theta_{*,n} \) in the basis of eigenvectors of \( \Sigma_n \).

**Assumption 2.** Let \( \rho_n = \langle \theta_{*,n}, \Sigma_n \theta_{*,n}\rangle^{1/2} \) and \( \bar{w}_i \equiv \sqrt{p} \lambda_i(v_i, \theta_{*,n})/\rho_n \). Then the empirical distribution of \( \{(\lambda_i, \bar{w}_i)\}_{1 \leq i \leq n} \) converges in Wasserstein-2 distance to a probability distribution \( \mu \) on \( \mathbb{R}_{>0} \times \mathbb{R} \):
\[
\frac{1}{p} \sum_{i=1}^p \delta(\lambda_i, \bar{w}_i) \xrightarrow{W_2} \mu.
\]
In particular, \( \int w^2 \mu(d\lambda, dw) = 1 \), and \( \rho_n \to \rho \), where \( 1/\rho^2 \equiv \int (w^2/\lambda) \mu(d\lambda, dw) \).

We refer the reader to Appendix A for a reminder of the definition of the Wasserstein distance \( W_2 \). Here, we limit ourselves to mentioning that convergence in \( W_2 \) is equivalent to weak convergence plus convergence of the second moment, see e.g. [Vil08]. In particular, the condition \( \int (w^2/\lambda) \mu(d\lambda, dw) = 1/\rho^2 \), implies \( \lim_{n \to \infty} ||\theta_{*,n}||_2 = 1 \). Notice that this choice of normalization implies no loss of generality: if \( \lim_{n \to \infty} ||\theta_{*,n}||_2 = c \neq 1 \), we can rescale \( \theta_{*,n} \) (letting \( \theta_{*,n}^{\text{new}} = \theta_{*,n}/c \)) and the function \( f \) (letting \( f^{\text{new}}(t) = f(ct) \)), as to satisfy the assumed normalization.

Finally, we state our assumptions on the function \( f \).
Assumption 3. Define $T = YG$ where $G \sim N(0, 1)$ and $\mathbb{P}(Y = 1 \mid G) = 1 - \mathbb{P}(Y = -1 \mid G) = f(\rho \cdot G)$. We assume $f : \mathbb{R} \to [0, 1]$ to be continuous, and to satisfy the following non-degeneracy condition:

$$\inf \left\{ x : \mathbb{P}(T < x) > 0 \right\} = -\infty \quad \text{and} \quad \sup \left\{ x : \mathbb{P}(T > x) > 0 \right\} = \infty.$$  

It is easy to check that the non-degeneracy condition is satisfied for most ‘reasonable’ choices of $f$. In particular, it is sufficient that $f(x) \in (0, 1)$ for all $x$.

Under these assumptions, we establish the following results.

Asymptotic characterization of the maximum margin. Define the maximum margin by

$$\kappa_n(y, X) \equiv \max \left\{ \min_{i \leq n} y_i(\theta, x_i) : \theta \in \mathbb{R}^p, \|\theta\|_2 = 1 \right\}. \quad (1.4)$$

We prove that $\kappa_n(y, X) \to \kappa^*(\mu, \psi)$ almost surely as $n \to \infty$, for some non-random asymptotic margin $\kappa^*(\mu, \psi)$. We give an explicit characterization of the limiting value $\kappa^*(\mu, \psi)$, stated in Section 4.

As a corollary, we derive the limiting value of the interpolation threshold, i.e. the minimum number of parameters per dimensions above which the data are separable with a positive margin: $\psi^*(\mu, \psi) \equiv \inf \{\psi \geq 0 : \kappa^*(\mu, \psi) > 0\}$. (This generalizes the recent result of [CS18].)

Asymptotic characterization of prediction error. Let the test error be defined by

$$\text{Err}_n(y, X) \equiv \mathbb{P}(y^{\text{new}} \langle \hat{\theta}^{\text{MM}}(y, X), x^{\text{new}} \rangle \leq 0), \quad (1.5)$$

where expectation is with respect to a fresh sample $(y^{\text{new}}, x^{\text{new}})$ independent of the data $(y, X)$. We will sometimes refer to $\text{Err}_n(y, X)$ as to the prediction error. We prove that $\text{Err}_n(y, X) \to \text{Err}^*(\mu, \psi)$ for a non-random limit $\text{Err}^*(\mu, \psi)$, which we characterize explicitly, cf. Section 4.

Special cases. We evaluate the asymptotic expressions for the maximum margin $\kappa^*(\mu, \psi)$ and the prediction error and $\text{Err}^*(\mu, \psi)$, in several special cases. Namely, we consider special sequences of the covariance $\Sigma_n$, and the true parameters’ vector $\theta^{*}$, resulting in limit distributions $\mu$.

In particular, we consider a choice of the pair $(\Sigma_n, \theta^{*})$ which captures the second order statistics of the random features model of [BBV06, RR08] or, equivalently, of two-layer neural networks with random first layer weights. Based on [MM19] we expect this Gaussian covariates model to have the same asymptotic prediction error as the actual random features model. We observe that the prediction error decreases monotonically with the number of random features, in the overparametrized regime.

This confirms the general ‘double descent’ phenomenology developed in [BHX19, BHM18, HMRT19, BLLT19], and provides the first exact asymptotics beyond simple ridge regression.

Technical innovation. Our analysis is based on Gordon’s Gaussian comparison inequality [Gor1998] and, in particular its application to convex-concave problems developed in [TOH15]. This approach allows to replace the original optimization problem by a simpler one, that is nearly separable. By studying the asymptotics of this equivalent problem, it is possible to obtain a precise characterization of the original problem in terms of the solution of a set of nonlinear equations.

However, this asymptotic characterization holds only if the set of nonlinear equations admit a unique solution. Proving uniqueness can be challenging, and is normally done on a case-by-case basis. Here, we develop a new technique to prove uniqueness. In extreme synthesis, we construct, in a natural way, an infinite-dimensional convex problem whose KKT conditions are equivalent to the same set of nonlinear equations. We exploit this underlying convex structure to prove uniqueness. We believe this technique is potentially applicable to a broad set of problems.

We will begin our exposition in Section 2 by presenting the special applications. We will then survey related work in Section 3, and state our general results in Section 4. Section 5 outlines the proof of these results, deferring most of the technical work to the appendices.
2 Special examples and numerical illustrations

In this section we illustrate our main results (to be presented in Section 4) by considering a few special cases, namely special sequences of the true parameter vector $\theta_{*,n}$, and covariance matrix $\Sigma_n$. We also discuss statistical insights that can be drawn from the analysis of these cases.

2.1 Isotropic well specified model

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Max-margin linear classification for isotropic well specified data. Left: maximum margin (scaled by $\sqrt{\psi} = \sqrt{p/n}$). Right: test error. Labels are generated using the logistic function $f(x) = (1 + e^{-\beta x})^{-1}$ with $\beta = 1, 2, 8$ (from bottom to top on the left, and from top to bottom on the right). Vertical dashed lines correspond to the interpolation threshold $\psi^*$, and continuous lines to the analytical predictions of Corollary 2.1. Symbols are empirical results for the max-margin (left) and prediction error (right). Here $p = 800$ and we vary $n = p/\psi$, averaging results over 20 instances. Error bars (barely visible) report standard errors on the empirical means of 20 instances.}
\end{figure}

We begin by considering the simplest case, namely isotropic covariates $x_i \sim N(0, I_p)$ (i.e $\Sigma_n = I_p$). In this case, by rotational invariance, the margin and prediction error do not depend on the vector $\theta_{*,n}$ which has unit norm. Figure 3 report the results of numerical experiments with $p = 800$ and various values of $n$. We observe that the classification error decreases as $n$ increase, i.e. as $\psi$ decreases, until it crosses a threshold below which the data is no longer separable.

In order to state our characterization of the maximum margin and prediction error, we introduce the function $F_\kappa : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ (for $\kappa \in \mathbb{R}$):

$$F_\kappa(c_1, c_2) = \left( \mathbb{E} \left[ (\kappa - c_1 YG - c_2 Z)^2 \right] \right)^{1/2}$$

where

\begin{align*}
Z \perp (Y,G), \\
Z \sim N(0,1), G \sim N(0,1), \\
\mathbb{P}(Y = +1 \mid G) = f(G), \\
\mathbb{P}(Y = -1 \mid G) = 1 - f(G).
\end{align*}

The next corollary is an immediate consequence of our main result, Theorem 1.

**Corollary 2.1.** Consider the isotropic model, and let $f$ satisfy Assumption 3. Define $L(c; \kappa, \psi) = F_\kappa(c, \sqrt{1 - \kappa^2}) - \sqrt{\psi(1 - \kappa^2)}$. Then the following hold:

(a) The maximum margin $\kappa_n(y, X)$ converges almost surely to a strictly positive limit if and only if $\psi > \psi_{iso}^*$, where the interpolation threshold is given by

$$\psi_{iso}^* = \min_{c \geq 0} F_0(c, 1)^2.$$
(b) For any $\psi > \psi^*_\text{iso}$ the asymptotic maximum margin is given by $\lim_{n \to \infty} \kappa^*_n(y, X) \to \kappa^*_\text{iso}(\psi)$, where

$$\kappa^*_\text{iso}(\psi) = \inf \left\{ \kappa \geq 0 : F_\kappa(c, \sqrt{1-c^2}) - \sqrt{\psi(1-c^2)} > 0 \text{ for all } c \in [0,1] \right\}. \quad (2.3)$$

(c) The asymptotic prediction error is given by $\lim_{n \to \infty} \text{Err}^*_n(y, X) = \text{Err}^*_\text{iso}(\psi)$, where

$$\text{Err}^*_\text{iso}(\psi) = \mathbb{P}\left( c^*_\text{iso}(\psi)YG + \sqrt{1-c^*_\text{iso}(\psi)^2}Z \leq 0 \right), \quad (2.4)$$

$$c^*_\text{iso}(\psi) \equiv \arg \min_{c \in [0,1]} \left\{ F_{\kappa=c^*_\text{iso}(\psi)}(c, \sqrt{1-c^2}) - \sqrt{\psi(1-c^2)} \right\}. \quad (2.5)$$

(Here expectation is taken with respect to the random variables $(G, Y, Z)$ with joint distribution defined in Eq. (2.1).)

In Figure 2 we compare the theoretical prediction for the maximum margin and test error given in the last corollary with the numerical results: the agreement is excellent. Our analytical prediction confirm the observation made above: the error is monotone increasing in $\psi$, for $\psi > \psi^*$. It is possible to show that $\text{Err}^*_\text{iso}(\psi) \to 1/2$ as $\psi \uparrow \infty$, while $\text{Err}^*_\text{iso}(\psi) \to \text{Err}^*_\text{iso}(\psi^*)$ as $\psi \downarrow \psi^*$, where (in general) $\text{Err}^*_\text{iso}(\psi^*) \in (0,1/2)$.

Notice that this behavior is different from the one observed for min-norm least squares [BHX19, HMRT19], under isotropic covariates (and for a similarly well-specified model). In that case, at small signal-to-noise ratio (SNR), the error is monotone decreasing for $\psi > \psi^*$, while at high SNR it is monotone decreasing in an interval $\psi \in (\psi^*, \psi_{\text{min}})$, and increasing for $\psi > \psi_{\text{min}}$. This different behavior can be explained, at least in part, by the observation that the square loss is unbounded and diverges (for min-norm least squares) at the interpolation threshold. Hence it is necessarily decreasing right above that threshold. In contrast, since the classification error is bounded, it can be monotone increasing with the overparametrization ratio $\psi = p/n$.

**Figure 2.** Isotropic well-specified model: comparing margin-based bounds and actual test error. In the lower part of the plot, analytical predictions (continuous lines) and numerical simulations (circles) for the test error: same data as in Figure 3. Dot-dashed lines are margin-based estimates of the test error $\sqrt{\psi}/\kappa^*(\psi)$ (see text).

The maximum margin $\kappa^*_\text{iso}(\psi)$ is also monotone increasing with $\psi$, with $\kappa^*_\text{iso}(\psi) \downarrow 0$ as $\psi \downarrow \psi^*$, and $\kappa^*_\text{iso}(\psi)/\sqrt{\psi} \to 1$ as $\psi \to \infty$. Notice that the relation between margin and classification error is somewhat counterintuitive. On the basis of classical margin theory [SSBD14], one would expect that the test error increases when the margin decreases. The opposite happens in Fig. 3: as $\psi$ decreases both the error and the margin decrease. The explanation to this counterintuitive phenomenon is related to the fact that, in the present regime, the margin does not concentrate near its population value (which is not separable in this case).
In order to further clarify this phenomenon, in Figure 2 we compare the actual test error (both numerical simulations, and the predictions of Corollary 2.1), with a margin-based bound from [SSBD14, Theorem 26.14]. The latter implies, with our notations,

$$\text{Err}_n(y, X) \leq \frac{4r\sqrt{\psi}}{\kappa^*}(\psi) + o_n(1), \quad r^2 \equiv \int x\mu(dx, dw).$$  \hspace{1cm} (2.6)

Here $r$ is the typical (normalized) radius of the feature vectors, namely the asymptotic value of $r^2_n = \mathbb{E}\|x_1\|^2/p$ (in the present case, $r = 1$). Even discarding the factor 4 (which we do in Figure 2), this upper bound has the wrong qualitative dependence on $\psi$ and is never non-trivial in the present setting (never smaller than 1).

### 2.2 Isotropic misspecified models

Assuming the model to be well specified can be unrealistic. Further, the well specified setting cannot be used to understand the beneficial effect (in terms of approximation error) of adding more covariates. In this section we keep considering isotropic covariates, but introduce a simple misspecification structure.

We assume that label $y_i$ depend on a potentially infinitely dimensional feature vector $z_i \in \mathbb{R}^\infty$, $z_i \sim N(0, I_{\infty})$, via

$$P(y_i = +1 | z_i) = f_0(\langle \beta_*, z_i \rangle).$$  \hspace{1cm} (2.7)

Note that this makes mathematical sense as long as $\beta_* \in \ell_2$. Without loss of generality, we can assume $\|\beta_*\|_2 = 1$. We learn a max-margin classifier over the first $p$ features. Namely, we write $z_i = (x_i, \tilde{z}_i)$ where $x_i \in \mathbb{R}^p$ contains the first $p$ coordinates of $z_i$, and $\tilde{z}_i$ contains the other coordinates. We then apply max-margin classification to data ${\{(y_i, x_i)\}_i \leq n}$. Given a vector $v \in \mathbb{R}^\infty$, we write $\Pi_{\leq \ell} v$ for the $\ell$-dimensional vector formed by the first $\ell$ entries of $v$.

Figure 3. Isotropic misspecified model. Left: maximum margin (scaled by $\sqrt{\psi} = \sqrt{p/n}$). Right: test error. Labels are generated according to Eq. (2.7), with $f_0(x) = (1 + e^{-\beta x})^{-1}$, with $\beta = 8$. The vertical dashed line corresponds to the interpolation threshold $\psi_{\text{inter}}$, and the continuous line to the analytical prediction from Corollary 2.2. Symbols are empirical results for the prediction error, for $n = 400$, $p_0 = 800$ and varying $p = n\psi$, averaged over 20 instances. Error bars (barely visible) report standard errors on the empirical means of 20 instances.

This setting can be reduced to the one in the previous section (isotropic well-specified model), whereby labels are assigned according to Eq. (1.3), with $f(x) = \mathbb{E}\{f_0(\sqrt{\gamma_n}x + \sqrt{1 - \gamma_n}G)\}$ with $G' \sim N(0, 1)$ and $\gamma_n = \|\Pi_{\leq p}\beta_{*,n}\|_2$. Further $\theta_{*,n} = \Pi_{\leq p}\beta_{*,n}/\gamma_n$. In words, those features that are not included in the model, and that correspond to non-zero entries in $\beta_{*,n}$, act as additional noise in the labels. As more and more features are added to the model, the corresponding noise variance $(1 - \gamma_n)$ decreases.
In order to state the asymptotic characterization of the max margin classifier, we modify the function of Eq. (2.1) as follows

\[
F_{\kappa,\gamma}(c_1, c_2) = \left( \mathbb{E} \left[ (\kappa - c_1 Y^2 G - c_2 Z)^2 \right] \right)^{1/2} \quad \text{where} \quad \begin{cases} 
Z \perp (Y, G), \\
\mathbb{P}(Y_0 = +1 | G) = \mathbb{E}[f_0(\sqrt{\gamma}G + \sqrt{1-\gamma}G')|G], \\
\mathbb{P}(Y_0 = -1 | G) = 1 - \mathbb{E}[f_0(\sqrt{\gamma}G + \sqrt{1-\gamma}G')|G], \\
Z, G, G' \sim_{iid} \mathcal{N}(0, 1). 
\end{cases}
\]

Corollary 2.2. Consider the misspecified isotropic model, and let \( f_0 \) satisfy Assumption 3. Further assume \( \beta_{*,n} \in \mathbb{R}^\infty \) to be such that \( \|\beta_{*,n}\|_2 = 1 \) and \( \|\Pi_{\leq \rho(n)}\beta_{*,n}\|_2 \to \gamma(\psi) \) as \( n \to \infty \). For any \( \psi > 0 \), define

\[
\kappa^*_\min(\psi) = \inf \left\{ \kappa \geq 0 : F_{\kappa,\gamma}(c, \sqrt{1-c^2}) - \sqrt{\psi(1-c^2)} > 0 \quad \forall c \in [0, 1] \right\}.
\]

(a) The maximum margin \( \kappa^*_n(y, X) \) converges almost surely to a strictly positive limit if and only if \( \psi > \psi^*_\min \equiv \inf \{ \psi > 0 : \kappa^*_\min(\psi) > 0 \} \).

(b) For \( \psi > \psi^*_\min \), the asymptotic max-margin is given by \( \lim_{n \to \infty} \kappa^*_n(y, X) \to \kappa^*_\min(\psi) \).

(c) The asymptotic prediction error is given by \( \lim_{n \to \infty} \text{Err}_n(y, X) = \text{Err}^*_\min(\psi) \), where

\[
\text{Err}^*_\min(\psi) = \mathbb{P} \left( c^*_\min(\psi) Y \gamma(\psi) G + \sqrt{1 - c^*_\min(\psi)^2} Z \leq 0 \right),
\]

\[
c^*_\min(\psi) \equiv \arg \min_{c \in [0, 1]} \left\{ F_{\kappa = \kappa^*_\min(\psi), \gamma(\psi)}(c, \sqrt{1-c^2}) - \sqrt{\psi(1-c^2)} \right\}.
\]

Note that the misspecified model has in important conceptual advantage over the well specified one: the data distribution (2.7) is independent of the number of features.

In Figure 3 we consider a misspecified problem in which \( \beta_{*,n} \) puts equal asymptotically weight over the first \( p_0 \) features, where \( p_0/n \to \psi_0 \in (0, \infty) \). Explicitly, we assume \( \beta_{*,i} \in \{1/\sqrt{p_0}, -1/\sqrt{p_0} \} \) for \( i \leq p_0 \), and \( \beta_{*,i} = 0 \) for \( i > p_0 \). This results in \( \gamma(\psi) = \psi/\psi_0 \) if \( \psi \leq \psi_0 \) and \( \gamma(\psi) = 1 \) for \( \psi > \psi_0 \). Note that the same limiting function \( \gamma(\psi) \) is obtained for other choices of the vector \( \beta_{*,n} \). For instance, if \( \beta_{*,i} \) is a uniformly random vector drawn independent for each \( n \), with unit norm and support on \( \{1, \ldots, p_0\} \), the assumptions of Corollary 2.2 are satisfied again, with \( \gamma(\psi) = \min(\psi/\psi_0, 1) \) as before.

In this example the test error decreases in the overparametrized regime for \( \psi^*_\min < \psi < \psi_0 \), and then increases again for \( \psi_0 < \psi \). As explained above, adding more features reduces the approximation error, and hence reduces the test error. A similar behavior was observed in [HMRT19] for the case of min-norm least squares regression. Notice that the maximum margin is monotone increasing in \( \psi \), with \( \kappa^*(\psi)/\sqrt{\psi} < 1 \). Further, since \( \kappa^*(\psi)/\sqrt{\psi} < 1 \), the classical margin-based bound of Eq. (2.6) is always larger than one. As for the well-specified model, the margin does not seem to capture the behavior of the actual test error.

### 2.3 A random features model

We next discuss a data distribution which fits within our general setting but is expected to be asymptotically equivalent to random features models, i.e. two-layers neural networks with random first layer weights. Random features methods originate in the work of Neal [Nea96], Balcan, Blum, Vempala [BBV06], and of Rahimi, Recht [RR08]. A sequence of recent papers [JGH18, DZPS18, CB18] suggests that in the so-called ‘lazy training’ regime, the behavior of multilayer networks is well approximated by certain random features model, whereby the randomness is connected with the initialization of the training process.

As mentioned in the previous section, [MM19] derived the asymptotic generalization error of random features ridge regression (under square loss). While features are highly non-Gaussian in this model, [MM19] also proved a surprising universality result: the test error is the same as for a model with Gaussian features, whose correlation structure is the same as for the random features. A related phenomenon arises for kernel inner product random matrices, see [CS13, FM19]. Here we conjecture that a similar universality phenomenon holds for classification, and derive the test error for the resulting Gaussian covariates model.
2.3.1 Motivation: random features

We first describe a model for binary classification using random features. We are given data \( \{(y_i, z_i)\}_{i \leq n} \), whereby \( y_i \in \{+1, -1\}, z_i \in \mathbb{N}(0, I_d) \) and

\[
P(y_i = +1 | z_i) = f_0((\beta, z_i)) \quad \|\beta\|_2 = 1.
\]  

(2.12)

In order to perform classification, we proceed as follows:

(i) We generate features \( \hat{x}_{ij} = \sigma(\langle w_j, z_i \rangle) \) where \( \sigma: \mathbb{R} \rightarrow \mathbb{R} \) is a non-linear function. Here \( w_j, j \leq p \) are \( d \)-dimensional vectors which we draw uniformly on the unit sphere \( S^{d-1}(1) \), \( (w_j)_{j \leq p} \sim \text{Unif}(S^{d-1}(1)) \).

(ii) We find a max-margin separating hyperplane for data \( \{(y_i, \hat{x}_i)\}_{i \leq n} \), where \( \hat{x}_i = (x_{ij})_{j \leq p} \).

Equivalently, letting \( W \in \mathbb{R}^{p \times d} \) be the matrix with rows \( w_i, 1 \leq i \leq p \), we have \( \hat{x}_i = \sigma(Wz_i) \) (where \( \sigma \) is understood to act componentwise on the vector \( Wz_i \)). We then predict via

\[
y\left( z^{\text{new}} \right) = \text{sign} \left( \hat{\theta}^{\alpha}(\theta, \sigma(Wz^{\text{new}})) \right),
\]

(2.13)

where the max-margin classifier if defined by

\[
\hat{\theta}^{\alpha}(\theta, Z) \in \arg \max \left\{ \min_{1 \leq n} y_i(\theta, \sigma(Wz_i)) : \|\theta\|_2 = 1 \right\}.
\]

(2.14)

This can be described as a two layers neural network, with random first-layer weights which are fixed to \( W \) and non-optimized. Second-layer weights are instead given by \( \theta \in \mathbb{R}^p \) and chosen as to maximize the margin.

2.3.2 Gaussian model

Notice that the random features model of Eq. (2.13) is not a linear classifier in the Gaussian vector \( z \). Hence, at first sight, the results of this paper might seem irrelevant to the analysis of the model (2.13). However –following [MM19]– we will now construct a Gaussian covariates model and conjecture that its asymptotic behavior is the same as for the random features in the regime \( p, n, d \rightarrow \infty \) with \( p/d \rightarrow \psi_1 \), \( n/d \rightarrow \psi_2 \). This conjecture will be confirmed by numerical experiments.

In order to motivate our construction, we decompose the activation function in \( \mathcal{L}^2(\mathbb{R}, \nu_G) \) (the space of square-integrable functions, with respect to \( \nu_G \) the standard Gaussian measure) as follows

\[
\sigma(u) = \gamma_0 + \gamma_1 u + \gamma_2 \sigma_{\perp}(u).
\]

(2.15)

Here \( \gamma_0 = \mathbb{E}\{\sigma(G)\}, \gamma_1 = \mathbb{E}\{G\sigma(G)\} \) and \( \gamma_2 = \mathbb{E}\{G \sigma(G)^2\} - \mathbb{E}\{G\sigma(G)\}^2 - \mathbb{E}\{\sigma(G)\}^2 \). (Expectation is over \( G \sim \mathcal{N}(0, 1) \).) We can then rewrite the random features model of the previous section as follows

\[
ex_{ij} = \gamma_0 + \gamma_1 \langle w_j, z_i \rangle + \gamma_2 \xi_{ij}, \quad \xi_{ij} = \sigma_{\perp}(\langle w_j, z_i \rangle),
\]

(2.16)

\[
g_i = \langle \beta, z_i \rangle, \quad P(y_i = +1 | g_i) = f_0(g_i).
\]

(2.17)

Note that the random variables \( \xi_{ij} \) have zero mean and unit variance, by construction. Further \( \mathbb{E}_{z_i} \{\xi_{ij}(w_j, z_i)\} = 0 \) since by construction \( \mathbb{E}\{\sigma_{\perp}(G)G\} = 0 \). This suggest to replace the \( \xi_{ij} \) by a collection of independent random variables:

\[
ex_{ij} = \gamma_1 \langle w_j, z_i \rangle + \gamma_2 \xi_{ij}, \quad \xi_{ij} \sim \mathcal{N}(0, 1),
\]

(2.18)

\[
g_i = \langle \beta, z_i \rangle, \quad P(y_i = +1 | g_i) = f_0(g_i),
\]

(2.19)

Here \( (\xi_{ij})_{i \leq n, j \leq p} \) are drawn independently of \( \{w_i\}_{i \leq p}, \{x_j\}_{j \leq p} \).

Note that under this model \( x_i \) and \( g_i \) are jointly Gaussian. Without loss of generality, we can write them as \( x_i \sim \mathcal{N}(0, \Sigma_n), g_i = \alpha_n (\theta_{n, x}, x_i) + \varepsilon_i \) where \( \varepsilon_i \sim \mathcal{N}(0, \tau_n^2) \) is independent of \( x_i \), for suitable choices of the \( \Sigma_n \) and \( \tau_n \). We state the overall conjecture explicitly below.
Conjecture 1. Let $\kappa^*_{nRF}(y, Z)$ and $\text{Err}^*_{nRF}(y, Z)$ be the maximum margin and test error of the random features model of Section 2.3.1. Further define the effective Gaussian features model following the general definition of Section 1.2, with $\psi = \psi_1/\psi_2$ and the following choice of $\Sigma_n, \theta_{*,n}$, $f_n$:

\[
\Sigma_n = \gamma_1^2 WW^T + \gamma_2^2 I_p, \quad \text{(2.20)}
\]

\[
\theta_{*,n} = \alpha_n^{-1} \gamma_1 (\gamma_1^2 WW^T + \gamma_2^2 I_p)^{-1} W \beta^*, \quad \text{(2.21)}
\]

\[
f_n(x) = E\{f_0(\alpha_n x + \tau_n G)\}, \quad \text{(2.22)}
\]

\[
\alpha_n^2 = \gamma_1^2 \beta^T (\gamma_1^2 WW^T + \gamma_2^2 I_p)^{-1} W \beta^*, \quad \text{(2.23)}
\]

\[
\tau_n^2 = 1 - \gamma_1^2 \beta^T (\gamma_1^2 WW^T + \gamma_2^2 I_p)^{-1} W \beta^*. \quad \text{(2.24)}
\]

Let $\kappa^*_{GF,n}(y, Z)$ and $\text{Err}^*_{GF,n}(y, Z)$ be the maximum margin and test error in the effective Gaussian features model. Then, in the limit $p, n, d \to \infty$ with $p/d \to \psi_1$ and $n/d \to \psi_2$, we have

\[
|\kappa^*_{RF,n}(y, Z) - \kappa^*_{GF,n}(y, Z)| \xrightarrow{P} 0, \quad |\text{Err}^*_{RF,n}(y, Z) - \text{Err}^*_{GF,n}(y, Z)| \xrightarrow{P} 0. \quad \text{(2.25)}
\]

(Here $P$ denotes convergence in probability.)

We can use the general results presented in the next section in order to computing the asymptotic margin and test error of the effective Gaussian features model. In order to do so, we need to verify Assumptions 1.2, 3. Assumption 1 immediately follows since $\lambda_{\min}(\Sigma_n) \geq \gamma_1^2 > 0$, and (for any $c > 0$) $\lambda_{\max}(\Sigma_n) \leq \gamma_1(1 + \sqrt{p/d + c})^2 + \gamma_2^2$, with probability at least $1 - \exp(-\Theta(d))$, by standard bounds on the eigenvalues of Wishart random matrices [Ver18a].

Next, we need to check Assumption 2 and determine the limit probability measure $\mu$. Fix numbers $\gamma_1, \gamma_*, \psi_1, \psi_2 > 0$, and consider the following probability measure on $(0, \infty)$:

\[
\mu_\lambda(dx) = \begin{cases} 
(1 - \psi_1^{-1}) \delta_0 + \psi_1^{-1} \nu_1(x) dx & \text{if } \psi_1 > 1, \\
\nu_\psi(x) dx & \text{if } \psi_1 \in (0, 1),
\end{cases} \quad \text{(2.26)}
\]

\[
\nu_\psi(x) = \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{2\pi \lambda x} 1_{x \in [\lambda_-, \lambda_+]}, \quad \text{(2.27)}
\]

\[
\lambda_{\pm} = (1 \pm \sqrt{\lambda})^2. \quad \text{(2.28)}
\]

By Marchenko-Pastur’s law, the empirical spectral distribution of $WW^T$ converges in $W_2$ to $\mu_\lambda$ almost surely as $p, d \to \infty$ [BS10]. Let $X \sim \mu_\lambda$ independent of $G \sim N(0, 1)$. Using Eq. (2.21), we obtain that (recalling from Assumption 2 that $\tilde{w}_i = \sqrt{\rho \lambda}(\psi_1, \theta_{*n})/\rho_n$, $\rho_n = (\theta_{*n}, \Sigma_n \theta_{*n})^{1/2}$)

\[
\frac{1}{p} \sum_{i=1}^p \delta_{(\lambda_i, \psi_1 \tilde{w}_i)} W \xrightarrow{D} \mu \equiv \text{Law}(X, W),
\]

where

\[
X = \gamma_1^2 \tilde{X} + \gamma_*^2, \quad W = \frac{\gamma_1 \sqrt{\psi_1 \tilde{X} G}}{C_0(\gamma_1^2 \tilde{X} + \gamma_*^2)^{1/2}}, \quad C_0 = E\left\{\frac{\gamma_1^2 \psi_1 \tilde{X}}{(\gamma_1^2 \tilde{X} + \gamma_*^2)^{1/2}}\right\}^{1/2}. \quad \text{(2.29)}
\]

Finally, we need to check Assumption 3. Using Eq. (2.24), we obtain $\tau_n \to \tau$ as $n \to \infty$, where:

\[
\tau^2 = 1 - \psi_1 E\left\{\frac{\gamma_1^2 \tilde{X}}{\gamma_1^2 \tilde{X} + \gamma_*^2}\right\}. \quad \text{(2.30)}
\]

Since $\tau^2 > 0$, it follows from Eq. (2.22) that Assumption 3 holds.

In Figure 4 we report the results of numerical simulations within the random features model of Section 2.3.1, with ReLU activations. We compare the outcome of these simulation with the analytical predictions for the effective Gaussian features model, which we described above. The asymptotic equivalence conjecture, stated formally above, appears to be confirmed by the numerical results: predictions are in excellent
agreement with simulations. We also note that the test error is monotonically decreasing with the over-parametrization ratio $p/n = \psi_1/\psi_2$, and its global minimum is achieved at large overparametrization: this confirms the double descent scenario described in the introduction. Finally we note that the margin is monotone increasing for $\psi > \psi_\star$. At first sight, this might seem to suggest that the decrease in classification error is related to the increase in the margin. However a closer look reveals that the normalized margin $\kappa^*(\psi)/(r\sqrt{\psi})$ is always smaller than one. Therefore the classical margin bound of Eq. (2.6) is always larger than one (even neglecting the factor 4). Once again, the behavior of the error in Figure 4 appears difficult to explain in terms of standard margin theory.

One particular prediction of our theory is that the test error and the margin should depend on the activation function only through the two coefficients $\gamma_1$ and $\gamma_\star$. We check this numerically by repeating the same simulation of Figure 4, but using a different activation function. Namely, we use activation $\sigma_2(x) = 0.5x_+ + a_0x_+^2 + a_1x_+(1 + x_+)^{-1}$, where we choose $a_0$ and $a_1$ as to obtain the same values of $\gamma_1, \gamma_\star$ as for ReLU. Figure 5 reports the outcome of numerical simulations with activation $\sigma_2$. As conjectured, the two sets of numerical results in Figures 4 and 5 are hardly distinguishable.

3 Related work

The machine learning community has devoted significant attention to the analysis of maximum margin classifiers. An incomplete selection of references include [Bar98, AB09, KP02, BM02, KST09, Kol11]. This line of work develops upper bounds on the generalization error (difference between test and training error) in terms of the complexity (e.g. the Radamacher complexity) of the underlying function class. In the case of maximum margin classification, this approach yields upper bounds that depend on the empirical margin or the empirical margin distribution. The latter, in turn, concentrates around the population margin distribution.

We do not expect such tight concentration properties in the regime $p \asymp n$, which we consider here. As a consequence, margin-based bound are unlikely to capture the detailed properties of generalization curves that we derived here.

The regime $n \ll p$ has been studied within high-dimensional statistics. In this setting, the parameters’ vector is assumed to be highly structured. For instance, in high-dimensional regression, the sparsity $s_0$ is assumed to be much smaller than the sample size $s_0 \ll n/\log p$ [BVDG11]. Concentration of measure is sufficient to prove consistency in such highly structured problems,
In contrast, a growing body of work focuses on a different ‘noisy high-dimensional regime’ in which the sample size is proportional to the number of parameters, and the estimation error (suitably rescaled) converges to a non-trivial limit [Mon18]. Asymptotically exact results have been obtained in a large array of problems including sparse regression using $\ell_1$ penalization (Lasso) [BM12, MM18], general regularized linear regression [DJM13, ALMT14, TOH15], robust regression [EKBB+13, DM16, EK18, TAH18], Bayesian estimation within generalized linear models [BKM+19], logistic regression [CS18, SC19], low rank matrix estimation [DAM16, LM19, BDM+18], and so on. Several new mathematical techniques have been developed to address this regime: constructive methods based on message passing algorithms [BM11, BLM15]; Gaussian comparison methods based on Gordon’s inequality [Gor88, TOH15]; interpolation techniques motivated from statistical physics [BM19].

The study of this ‘noisy high-dimensional’ regime has a long history in statistical physics. Non-rigorous methods from spin glass theory have been successfully used since the eighties in this context. We refer to [EVdB01] for an overview of this early work, and to [MPV87, MM09] for general introductions. An early breakthrough in this line of work was the result by Elizabeth Gardner [Gar88], who computed the maximum margin $\kappa^*(\psi)$ for the special case of isotropic features and purely random labels (i.e. $\Sigma_n = I_p$ and $f(x) = 1/2$).

Here we follow the approach based on Gordon’s inequality [Gar88], as formalized by Thrampoulidis, Oymak, Hassibi [TOH15]. The closest results in the earlier literature are the analysis of logistic regression by Sur and Candès [CS18, SC19], and the recent paper on regularized logistic regression by Salehi, Abbasi and Hassibi [SAH19]. Both of these analyses focus on the underparametrized regime, in which the maximum likelihood estimator is well defined and (with high probability) unique. By contrast, we focus on the overparametrized regime here. Further, while earlier work assumes isotropic covariates $x_i \sim N(0, I_p)$, we consider a general covariance structure, under Assumptions 1 and 2. This is crucial in order to be able to capture the behavior of overparametrized random features models. From a technical point of view, our approach is related to the one of [SAH19]. Notice however that [SAH19] does not prove uniqueness of the minimizer of Gordon’s optimization problem, while this is the main technical challenge that we address in our proof (in a more complicated setting, due to the general covariance).

Finally, as discussed in the introduction, our work is connected to a growing line of research that investigates the behavior of generalization error in overparametrized model that interpolate the data (i.e. achieve vanishing training error). This work was largely motivated by the empirical observation that deep neural networks fit perfectly the data and yet generalize well [ZBH+16]. It was noticed in [BMM18] that this behavior is significantly more general than neural networks, while the double descent scenario was put forward in [BHMM19]. Independently, [GJS+19] observed the same phenomenon in the context of multilayer networks, and connected it to phase transitions in physics.
Mathematical results about generalization behavior in the overparametrized regime have been obtained in several recent papers [BHM18, BHX19, LR18, RZ18, MVS19, HMRT19, BLLT19, MM19]. However, all of earlier work has focused on least squares (or ridge) regression with square loss. (Certain nearest-neighbor-like methods are also considered in [BHM18].) As already mentioned, maximum margin classification presents some important conceptual differences. The closest earlier results in this literature are [HMRT19, MM19] which use random matrix theory to characterize ridge regression in the proportional asymptotics \( p, n \to \infty \) with \( p/n = \psi \).

4 Main results

Our main theorem characterizes the asymptotic value of the maximum margin \( \kappa^*(\mu, \psi) \) and the asymptotic generalization error of the max-margin classifier, to be denoted by \( \text{Err}^*(\mu, \psi) \).

4.1 Introducing the asymptotic predictions

We start by defining our general analytical predictions \( \kappa^*(\mu, \psi) \) and \( \text{Err}^*(\mu, \psi) \). Recall that \( \rho \in \mathbb{R}_{>0} \) and the probability measure \( \mu \) on \( \mathbb{R}_{>0} \times \mathbb{R} \) are defined by Assumption 2. For any \( \kappa \geq 0 \), define \( F_\kappa : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) by

\[
F_\kappa(c_1, c_2) = \left( \mathbb{E} \left[ \left( \kappa - c_1 Y G - c_2 Z \right)^2 \right] \right)^{1/2}
\]

where

\[
\begin{align*}
Z &\sim \mathcal{N}(0, 1), \quad G \sim \mathcal{N}(0, 1) \\
\mathbb{P}(Y = +1 \mid G) &= f(\rho \cdot G) \\
\mathbb{P}(Y = -1 \mid G) &= 1 - f(\rho \cdot G)
\end{align*}
\]

(4.1)

Let the random variables \( X, W \) be such that \( (X, W) \sim \mu \) (where \( \mu \) is defined by Assumption 2). Introduce the constants

\[
\zeta = \left( \mathbb{E}_\mu[X^{-1}W^2] \right)^{-1/2} \quad \text{and} \quad \omega = \left( \mathbb{E}_\mu[(1 - \zeta^2 X^{-1})W^2] \right)^{1/2}.
\]

(4.2)

Define the functions \( \psi_+ : \mathbb{R}_{>0} \to \mathbb{R} \) and \( \psi_- : \mathbb{R}_{>0} \to \mathbb{R} \) by

\[
\psi_+(\kappa) = \begin{cases} 
0 & \text{if } \partial_1 F_\kappa(\zeta, 0) > 0, \\
\partial_2^2 F_\kappa(\zeta, 0) - \omega^2 \partial_1^2 F_\kappa(\zeta, 0) & \text{if otherwise},
\end{cases}
\]

\[
\psi_-(\kappa) = \begin{cases} 
0 & \text{if } \partial_1 F_\kappa(-\zeta, 0) > 0, \\
\partial_2^2 F_\kappa(-\zeta, 0) - \omega^2 \partial_1^2 F_\kappa(-\zeta, 0) & \text{if otherwise}.
\end{cases}
\]

(4.3)

Finally, we define \( \psi^*(0) \) and \( \psi^* : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0} \) by

\[
\psi^*(0) = \min_{c \in \mathbb{R}} F_0^2(c, 1),
\]

(4.4)

\[
\psi^*(\kappa) = \max\{\psi^*(0), \psi_+(\kappa), \psi_-(\kappa)\}.
\]

(4.5)

The next proposition guarantees that the definition of \( \kappa^*(\psi) \), and \( \text{Err}^*(\psi) \) given below are meaningful. Its proof is deferred to Appendix B.

**Proposition 4.1.** (a) For any \( \psi > \psi^*(\kappa) \), the following system of equations has unique solution \( (c_1, c_2, s) \in \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \) (here expectation is taken with respect to \( (X, W) \sim \mu \)):

\[
-c_1 = \mathbb{E}_\mu \left[ \frac{\partial_1 F_\kappa(c_1, c_2) - c_1^2 - \partial_2 F_\kappa(c_1, c_2)}{c_2^2 \partial_2 F_\kappa(c_1, c_2) X^{1/2} + \psi^2 X^{-1/2}} \right] W^2 X^{1/2}
\]

\[
c_1^2 + c_2^2 = \mathbb{E}_\mu \left[ \frac{\psi X + (\partial_1 F_\kappa(c_1, c_2) - c_1^2 - \partial_2 F_\kappa(c_1, c_2)) W^2 X}{(c_2^2 \partial_2 F_\kappa(c_1, c_2) X^{1/2} + \psi^2 X^{-1/2})^2} \right] W^2 X
\]

\[
1 = \mathbb{E}_\mu \left[ \frac{\psi + (\partial_1 F_\kappa(c_1, c_2) - c_1^2 - \partial_2 F_\kappa(c_1, c_2)) W^2 X}{(c_2^2 \partial_2 F_\kappa(c_1, c_2) X^{1/2} + \psi^2 X^{-1/2})^2} \right].
\]

(4.6)
(b) Define the function $T : (\psi, \kappa) \rightarrow \mathbb{R}$ (for any $\psi > \psi^*(\kappa)$) by

$$T(\psi, \kappa) = \psi^{-1/2} \left( F_\kappa(c_1, c_2) - c_1 \partial_1 F_\kappa(c_1, c_2) - c_2 \partial_2 F_\kappa(c_1, c_2) \right) - s, \quad (4.7)$$

where $(c_1(\psi, \kappa), c_2(\psi, \kappa), s(\psi, \kappa))$ is the unique solution of Eq. (4.6) in $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$. Then we have

(i) $T(\cdot, c_1(\cdot, \cdot), c_2(\cdot, \cdot), s(\cdot, \cdot))$ are continuous functions in the domain $\{(\psi, \kappa): \psi > \psi^*(\kappa)\}$.

(ii) For any $\kappa > 0$, the mapping $T(\cdot, \kappa)$ is strictly monotonically decreasing, and satisfies

$$\lim_{\psi \nearrow +\infty} T(\psi, \kappa) < 0 < \lim_{\psi \searrow \psi^*(\kappa)} T(\psi, \kappa). \quad (4.8)$$

(iii) For any $\psi > 0$, the mapping $T(\psi, \cdot)$ is strictly monotonically increasing, and satisfies

$$\lim_{\kappa \nearrow +\infty} T(\psi, \kappa) = \infty. \quad (4.9)$$

We are now in position to define $\kappa^*(\mu, \psi)$ and $\text{Err}^*(\mu, \psi)$.

**Definition 4.1.** Recall the function $T$ defined at Eq. (4.7). For any $\psi > \psi^*(0)$, we define the asymptotic max-margin as

$$\kappa^*(\mu, \psi) = \inf \{ \kappa \geq 0 : T(\psi, \kappa) = 0 \}. \quad (4.10)$$

We further define the asymptotic generalization error $\text{Err}^* : (0, \infty) \rightarrow [0, 1]$ by

$$\text{Err}^*(\mu, \psi) = \mathbb{P} \left( \nu^*(\psi) YG + \sqrt{1 - \nu^*(\psi)^2} Z \leq 0 \right), \quad (4.11)$$

where probability is over $Z \perp (Y, G)$, with $G, Z \sim \mathcal{N}(0, 1)$ and $\mathbb{P}(Y = +1 \mid G) = f(\rho \cdot G) = 1 - \mathbb{P}(Y = -1 \mid G)$. Further $c_i^*(\psi) = c_i(\psi, \kappa^*(\psi))$, $i \in \{1, 2\}$.

Proposition 4.1 shows that the mapping $\psi \rightarrow \kappa^*(\mu, \psi)$ is well-defined, strictly monotonically increasing, and satisfies $\lim_{\kappa \rightarrow \infty} \kappa^*(\mu, \psi) = \infty$. In the following, we will often omit the argument $\mu$ from $\kappa^*$, $\text{Err}^*$.

### 4.2 Main statement

Below we present the main mathematical result of this paper. Section 5 presents the proof of this theorem, with most technical legwork deferred to the appendices. The next subsection briefly describe our proof technique.

**Theorem 1.** Consider i.i.d. data $(y, X) = \{(y_i, x_i)\}_{i \leq n}$ where $x_i \sim \mathcal{N}(0, \Sigma_n)$ and $\mathbb{P}(y_i = +1 \mid x_i) = f(\theta_{s,n}, x_i)$. Assume $n, p \rightarrow \infty$ with $p/n \rightarrow \psi \in (0, \infty)$, and $\theta_{s,n}, \Sigma_n, f$ satisfying Assumptions 1, 2, 3. (In particular, $\rho, \mu$ are defined by Assumption 2.)

Let $\psi^*(0)$ be defined as per Eq. (4.4), and $\kappa^*(\mu, \psi)$, $\text{Err}^*(\mu, \psi)$ be determined as per Definition 4.1. Then the following hold:

(a) The data are linearly separable with a margin which is bounded away from 0 if and only if $\psi > \psi^*(0)$.

(b) Let $\kappa_n(y, X) \equiv \max_{\theta} \min_{i \leq n} y_i \theta_i$ be the maximum margin for data $(y, X)$. In the overparametrized regime $\psi > \psi^*(0)$ we have, almost surely,

$$\lim_{n \rightarrow \infty} \kappa_n(y, X) = \kappa^*(\mu, \psi). \quad (4.12)$$

(c) Let $\text{Err}_n(y, X) \equiv \mathbb{P}(y^{\text{new}}(\hat{\theta}^{\text{MM}}(y, X), x^{\text{new}}) \leq 0)$ be the prediction error of the maximum margin classifier. In the overparametrized regime $\psi > \psi^*(0)$ we have, almost surely,

$$\lim_{n \rightarrow \infty} \text{Err}_n(y, X) = \text{Err}^*(\mu, \psi). \quad (4.13)$$
Remark: Point (a) in Theorem 1 is a generalization of the recent result of [CS18], which concerns the case in which \( f(x) \) is a logistic function.

The main content of Theorem 1 is in parts (b) and (c). To the best of our knowledge, the only case that had been characterized before is the one of isotropic covariates and purely random labels (i.e. \( \Sigma_n = I_p(n) \) and \( f(x) = 1/2 \)). In this case the asymptotic value of the maximum margin was first determined rigorously by Shcherbina and Tirozzi [ST03], confirming the non-rigorous result by Gardner [Gar88].

4.3 Proof technique

Parts (a), (b). Consider first the problem of determining the asymptotics of the maximum margin. Recall that \( X \in \mathbb{R}^{n \times p} \) denotes the matrix with rows \( x_1, \ldots, x_n \) and, for any \( \kappa > 0 \), define the event

\[
\mathcal{E}_{n,\psi,\kappa} = \left\{ \exists \theta \in \mathbb{R}^p, \|\theta\|_2 \leq 1 \text{ such that } y_i(x_i, \theta) \geq \kappa \text{ for } i \in [n] \right\}.
\]

In order to prove Theorem 1.\( (b) \), we would like to determine for which pairs \( (\psi, \kappa) \) we have \( P(\mathcal{E}_{n,\psi,\kappa}) \rightarrow 1 \) and for which pairs instead \( P(\mathcal{E}_{n,\psi,\kappa}) \rightarrow 0 \).

To this end, we define \( \xi_{n,\psi,\kappa} \) by

\[
\xi_{n,\psi,\kappa} = \min_{\|\theta\|_2 \leq 1} \max_{\|\lambda\|_2 \leq 1, y \in \mathbb{R}^p} \frac{1}{\sqrt{p}} \lambda^T(\kappa y - X\theta).
\]

(4.14)

We then have:

\[
\{ \xi_{n,\psi,\kappa} > 0 \} \iff \mathcal{E}_{n,\psi,\kappa} \quad \text{and} \quad \{ \xi_{n,\psi,\kappa} = 0 \} \iff \mathcal{E}_{n,\psi,\kappa}^c.
\]

(4.15)

This equivalence follows immediately from the following identities

\[
\mathcal{E}_{n,\psi,\kappa} = \left\{ \exists \theta \in \mathbb{R}^p, \|\theta\|_2 \leq 1 \text{ such that } \| (\kappa 1 - (y \circ X\theta))_+ \|_2 = 0 \right\},
\]

\[
\xi_{n,\psi,\kappa} = \min_{\|\theta\|_2 \leq 1} \frac{1}{\sqrt{p}} \| (\kappa 1 - (y \circ X\theta))_+ \|_2.
\]

(4.16)

We are then reduced to study the typical value of the minimax problem (4.14). Notice that this problem is convex in \( \theta \), concave in \( \lambda \), and linear in the Gaussian random matrix \( X \). We use Gordon’s Gaussian comparison inequality [Gor88] (and in particular a refinement due to Thrampoulidis, Oymak, Hassibi [TOH15]) to study the asymptotics of \( \xi_{n,\psi,\kappa} \). From a technical viewpoint, the most challenging step amounts to proving existence and uniqueness of the solution of the equivalent Gordon’s problem.

Part (c). Let \( G, Z \) be independent \( \mathcal{N}(0,1) \). Define for \( r \in \mathbb{R}_{\geq 0} \) and \( \nu \in [-1,1] \) the error function:

\[
Q(r, \nu) = P \left( \nu YG + \sqrt{1 - \nu^2}Z \leq 0 \right) \quad \text{where} \quad \begin{cases} P(Y = +1 \mid G) = f(r \cdot G) \\ P(Y = -1 \mid G) = 1 - f(r \cdot G) \end{cases}
\]

(4.17)

The generalization error of the max-margin classifier is:

\[
\text{Err}_n(y, X) = Q \left( \|\hat{\theta}_{\sigma,n}\|_{\Sigma_n}, \frac{\hat{\theta}_{\sigma,n}^\text{MM}}{\|\hat{\theta}_{\sigma,n}\|_{\Sigma_n}} \right).
\]

(4.18)

Comparing this expression to Theorem 1.\( (c) \), and recalling that \( \|\theta_{\sigma,n}\|_{\Sigma_n} \rightarrow \rho \) by Assumption 2, we see that it is sufficient to prove that, for each \( \psi > \psi^*(0) \),

\[
\lim_{n \rightarrow \infty} \frac{\langle \hat{\theta}_{\sigma,n}^\text{MM}, \theta_{\sigma,n} \rangle_{\Sigma_n}}{\|\theta_{\sigma,n}\|_{\Sigma_n}} = \nu^*(\psi).
\]

(4.19)

To this end, we generalize the definition of \( \xi_{n,\psi,\kappa} \) as follows. For any compact set \( \Theta_p \subseteq \mathbb{R}^p \), we define the quantity \( \xi_{n,\psi,\kappa}(\Theta_p) \) by

\[
\xi_{n,\psi,\kappa}(\Theta_p) = \min_{\theta \in \Theta_p} \max_{\|\lambda\|_2 \leq 1, y \in \mathbb{R}^p} \frac{1}{\sqrt{p}} \lambda^T(\kappa y - X\theta).
\]

(4.20)
We begin by setting \( \sum R \) dependent of the choice of a basis on \( \mathbb{R}^p \). In order to control the left hand side of Eq. (4.18), we consider sets of the form
\[
\Theta_p = \left\{ \theta \in \mathbb{R}^p : \|\theta\|_2 \leq 1, \frac{(\hat{\theta}^{MM}_{\theta, n}, \Sigma_n)}{\|\theta_n\|_{\Sigma_n}} \in J_n \right\},
\]
for suitable sequences of compact sets \( J_n \subseteq \mathbb{R} \). Using Gordon’s inequality to lower bound \( \xi_{n, \psi, \kappa}(\Theta_p) \), we can guarantee that Eq. (4.18) holds.

## 5 Proof of Theorem 1

This section provides a complete outline of the proof of Theorem 1, deferring most technical steps to the appendices.

Notice that the definition of the joint distribution of \((y, x)\) and the statements in Theorem 1 are independent of the choice of a basis on \( \mathbb{R}^p \). We can therefore work in the basis in which \( \Sigma \) is diagonal. This amounts to assuming that \( \Sigma = \Lambda \) is diagonal.

The proof proceeds through a sequence of steps to progressively simplify the quantities \( \xi_{n, \psi, \kappa} \) \( \xi_{n, \psi, \kappa}(\Theta_p) \). We begin by setting \( \xi_{(0)}_{n, \psi, \kappa} = \xi_{n, \psi, \kappa} \) and \( \xi_{(0)}_{n, \psi, \kappa}(\Theta_p) = \xi_{n, \psi, \kappa}(\Theta_p) \).

### Step 1: Reduction from \( \xi_{(0)}_{n, \psi, \kappa} \) to \( \xi_{(1)}_{n, \psi, \kappa} \) via Gordon’s comparison inequality

We use Gordon’s comparison inequality to reduce the original minimax of a complicated Gaussian process to that of a much simpler Gaussian process. We state Gordon’s comparison inequality below for reader’s convenience [Gor88, TOH15].

**Theorem 2** (Theorem 3 from [TOH15]). Let \( \mathcal{C}_1 \subseteq \mathbb{R}^p \) and \( \mathcal{C}_2 \subseteq \mathbb{R}^n \) be two compact sets and let \( T : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathbb{R} \) be a continuous function. Let \( X = (X_i)_{i,j} \sim N(0, 1) \in \mathbb{R}^{p \times n} \), \( g \sim N(0, I_p) \) and \( h \sim N(0, I_n) \) be independent vectors and matrices. Define,
\[
Q_1(X) = \min_{w_1 \in \mathcal{C}_1} \max_{w_2 \in \mathcal{C}_2} w_1^T X w_2 + T(w_1, w_2)
\]
\[
Q_2(g, h) = \min_{w_1 \in \mathcal{C}_1} \max_{w_2 \in \mathcal{C}_2} \|w_2\|_2 g^T w_1 + \|w_1\|_2 h^T w_2 + T(w_1, w_2)
\]

Then the following hold:

1. For all \( t \in \mathbb{R} \)
   \[
P(Q_1(X) \leq t) \leq 2 P(Q_2(g, h) \leq t).
   \]

2. Suppose \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are both convex, and \( T \) is convex concave in \((w_1, w_2)\). Then, for all \( t \in \mathbb{R} \)
   \[
P(Q_1(X) \geq t) \leq 2 P(Q_2(g, h) \geq t).
   \]

Let \( g \sim N(0, I_p) \), \( h \sim N(0, I_n) \), \( u \sim N(0, I_n) \) be independent Gaussian vectors and \( w \) the unit vector in the direction of \( \Lambda^{1/2} \theta \), i.e. \( w = \Lambda^{1/2} \theta / \|\Lambda^{1/2} \theta\|_2 \). Further, let \( y \in \{+1, -1\}^n \) be such that \( y_i \) is conditional independent of \( g, h \) and \((u_j)_{j \neq i} \) given \( u_i \), with \( P(y_i = +1| u_i) = f(\rho_n u_i) \) and \( \rho_n = \|\Lambda^{1/2} \theta\|_2 \). Define \( \xi_{(1)}_{n, \psi, \kappa} \) and \( \xi_{(1)}_{n, \psi, \kappa}(\Theta_p) \) by letting
\[
\xi_{(1)}_{n, \psi, \kappa}(\Theta_p) := \min_{\theta \in \Theta_p} \max_{\|\lambda\|_2 \leq 1, \lambda \otimes y \geq 0} \frac{1}{\sqrt{p}} \left( \chi^T (ky - (\Lambda^{1/2} w, \theta) u - \|\Pi_w \Lambda^{1/2} \theta\|_2 h) + \|\lambda\|_2 g^T \Pi_w \Lambda^{1/2} \theta \right),
\]
and \( \xi_{(1)}_{n, \psi, \kappa} = \xi_{(1)}_{n, \psi, \kappa}(B^p(1)). \)

We can apply Gordon’s inequality (Theorem 2) to relate \( \xi_{(0)}_{n, \psi, \kappa} \) to \( \xi_{(1)}_{n, \psi, \kappa} \) : the result is given in the next lemma, whose proof can be found in Appendix D.1.
Lemma 5.1. The following inequalities hold for any $t \in \mathbb{R}$, any compact set $\Theta_p \subseteq \mathbb{R}^p$:
\[
\begin{align*}
\mathbb{P}(\xi_{n,\psi,\kappa}^{(0)} \leq t) &\leq 2 \mathbb{P}(\xi_{n,\psi,\kappa}^{(1)} \leq t) \quad \text{and} \quad \mathbb{P}(\xi_{n,\psi,\kappa}^{(0)} \geq t) \leq 2 \mathbb{P}(\xi_{n,\psi,\kappa}^{(1)} \geq t), \\
\mathbb{P}(\xi_{n,\psi,\kappa}^{(0)}(\Theta_p) \leq t) &\leq 2 \mathbb{P}(\xi_{n,\psi,\kappa}^{(1)}(\Theta_p) \leq t).
\end{align*}
\]  

Further, if $\Theta_p$ is convex, we have
\[
\mathbb{P}(\xi_{n,\psi,\kappa}^{(0)}(\Theta_p) \geq t) \leq 2 \mathbb{P}(\xi_{n,\psi,\kappa}^{(1)}(\Theta_p) \geq t).
\]  

Step 2: Reduction from $\xi_{n,\psi,\kappa}^{(1)}$ to $\xi_{n,\psi,\kappa}^{(2)}$. A simple calculation gives
\[
\xi_{n,\psi,\kappa}^{(1)}(\Theta_p) = (\xi_{n,\psi,\kappa}^{(1)}(\Theta_p))_+,
\]  
where
\[
\xi_{n,\psi,\kappa}^{(1)} := \min_{\|\theta\|_2 \leq 1} \frac{1}{\sqrt{p}} \left( \left\| (\kappa \textbf{1} - \langle \Lambda^{1/2} w, \theta \rangle) (y \otimes u) - \|\Pi_{w^\perp} \Lambda^{1/2} \theta\|_2 (y \otimes h) \right\| + g^T \Pi_{w^\perp} \Lambda^{1/2} \theta \right),
\]
\[
\xi_{n,\psi,\kappa}^{(1)}(\Theta_p) := \min_{\theta \in \Theta_p} \frac{1}{\sqrt{p}} \left( \left\| (\kappa \textbf{1} - \langle \Lambda^{1/2} w, \theta \rangle) (y \otimes u) - \|\Pi_{w^\perp} \Lambda^{1/2} \theta\|_2 (y \otimes h) \right\| + g^T \Pi_{w^\perp} \Lambda^{1/2} \theta \right).
\]

Recall the definition of $F_\kappa$. Now we define the quantity $\xi_{n,\psi,\kappa}^{(2)}$ and $\xi_{n,\psi,\kappa}^{(2)}$ by,
\[
\xi_{n,\psi,\kappa}^{(2)} := \min_{\|\theta\|_2 \leq 1} \psi^{-1/2} \cdot F_\kappa \left( \langle \Lambda^{1/2} w, \theta \rangle, \|\Pi_{w^\perp} \Lambda^{1/2} \theta\|_2 \right) + \frac{1}{\sqrt{p}} g^T \Pi_{w^\perp} \Lambda^{1/2} \theta
\]
\[
\xi_{n,\psi,\kappa}^{(2)}(\Theta_p) := \min_{\theta \in \Theta_p} \psi^{-1/2} \cdot F_\kappa \left( \langle \Lambda^{1/2} w, \theta \rangle, \|\Pi_{w^\perp} \Lambda^{1/2} \theta\|_2 \right) + \frac{1}{\sqrt{p}} g^T \Pi_{w^\perp} \Lambda^{1/2} \theta.
\]

The next lemma allows us to move from $\xi_{n,\psi,\kappa}^{(1)}$ to $\xi_{n,\psi,\kappa}^{(2)}$.

Lemma 5.2. The following convergence holds for any sequence of compact sets $\{\Theta_p\}_{p \in \mathbb{N}}$ satisfying $\Theta_p \subseteq \{\theta \in \mathbb{R}^p : \|\theta\|_2 \leq 1\}$:
\[
\left| \mathbb{E} \left[ \xi_{n,\psi,\kappa}^{(1)}(\Theta_p) - \left( \xi_{n,\psi,\kappa}^{(1)}(\Theta_p) \right)_+ \right] \right| \xrightarrow{p} 0.
\]  

In particular, we have
\[
\left| \mathbb{E} \left[ \xi_{n,\psi,\kappa}^{(1)} - \left( \xi_{n,\psi,\kappa}^{(2)} \right)_+ \right] \right| \xrightarrow{p} 0.
\]

One important benefit of this reduction is that both $\xi_{n,\psi,\kappa}^{(2)}$ and $\xi_{n,\psi,\kappa}^{(2)}(\Theta_p)$ (for $\Theta_p$ convex) are minima of convex optimization problems (in contrast, the optimization problems defining $\xi_{n,\psi,\kappa}^{(1)}$ and $\xi_{n,\psi,\kappa}^{(1)}(\Theta_p)$ are not convex). Indeed, notice that the function $F_\kappa(c_1, c_2)$ is convex in $(c_1, c_2)$. We collect all the useful properties of the function $F_\kappa(c_1, c_2)$ in the next lemma, whose proof is deferred to Appendix B.2.

Lemma 5.3. The following properties hold for $F_\kappa : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$:

(a) If $\kappa > 0$, then the function $F_\kappa : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is strictly convex.

(b) For any fixed $c_1 \in \mathbb{R}$, the function $c_2 \mapsto F_\kappa(c_1, c_2)$ is strictly increasing for $c_2 \in \mathbb{R}_{\geq 0}$.

(c) The function $F_\kappa : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable.

(d) For any $c_1, c_2 \in \mathbb{R}$, the function $\kappa \mapsto F_\kappa(c_1, c_2)$ is strictly increasing for $\kappa \in \mathbb{R}$.
Step 3: Analysis of $\xi_{n,ψ,κ}^{(2)}$

Characterizing the limit of $\xi_{n,ψ,κ}^{(2)}$ is the technically most challenging part. Our approach is to find a new representation of $\xi_{n,ψ,κ}^{(2)}$ that allows one to easily guess its asymptotic behavior. Let us define $\bar{w} = \sqrt{p}w$. We have:

$$
\xi_{n,ψ,κ}^{(2)} = \min_{\|θ\|_2 ≤ 1} \psi^{-1/2} \cdot F_k \left( \langle θ, Λ_{1/2}w \rangle, \|Π_{w^⊥} Λ_{1/2}θ \|_2 \right) + \frac{1}{\sqrt{p}} g^T Π_{w^⊥} Λ_{1/2}θ \\
= \min_{\frac{1}{p} \sum_{i=1}^p \theta_i^{1/2} \leq 1} \psi^{-1/2} \cdot F_k \left( \frac{1}{p} \sum_{i=1}^p (Π_{w^⊥}(Λ_{1/2}θ))_i, \left( \frac{1}{p} \sum_{i=1}^p (Π_{w^⊥}(Λ_{1/2}θ))_i^2 \right)^{1/2} \right) + \frac{1}{p} \sum_{i=1}^p (Π_{w^⊥}g)_i \lambda_i^{1/2}θ_i.
$$

(5.8)

Let $Q_n$ be the empirical distribution of the coordinates of $(g, Λ, \bar{w})$, i.e., the probability measure on $\mathbb{R}^3$ defined by

$$
Q_n = \frac{1}{p} \sum_{i=1}^p \delta_{(g_i, Λ_i, \bar{w}_i)}.
$$

(5.9)

Let $L^2(Q_n) = L^2(Q_n, \mathbb{R}^3)$ be the space of functions $h : \mathbb{R}^3 \to \mathbb{R}$, $(g, Λ, w) \mapsto h(g, Λ, w)$ that are square integrable with respect to $Q_n$. Notice that the $n$ points that form $Q_n$ are almost surely distinct, and therefore we can identify this space with the space of vectors $θ \in \mathbb{R}^p$. We also define three random variables in the same space by $G(g, Λ, w) = g$, $X(g, Λ, w) = Λ$, $W(g, Λ, w) = w$. With these definitions, we can rewrite the expression in Eq. (5.8) as

$$
\xi_{n,ψ,κ}^{(2)} = \min_{h \in L^2(Q_n) : \|h\|_{Q_n} \leq 1} \psi^{-1/2} \cdot F_k \left( \langle X^{1/2}h, W \rangle_{Q_n}, \|Π_{W^⊥}(X^{1/2}h)\|_{Q_n} \right) + \langle X^{1/2}Π_{W^⊥}(G), h \rangle_{Q_n}.
$$

(5.10)

Now we define $Q_∞ := N(0, 1) \otimes μ$. By Assumption 2, the following convergence holds almost surely

$$
Q_n \xrightarrow{W_2} Q_∞.
$$

(5.11)

Motivated by the representation in Eq. (5.10) and the convergence in Eq. (5.11), we define $ξ_{ψ,κ}$ by

$$
ξ_{ψ,κ} := \min_{h = h(g, X, W) ∈ L^2(∞)} \psi^{-1/2} \cdot F_k \left( \langle X^{1/2}h, W \rangle_Q, \|Π_{W^⊥}(X^{1/2}h)\|_Q \right) + \langle X^{1/2}Π_{W^⊥}(G), h \rangle_{Q_∞}.
$$

(5.12)

In other words, in defining $ξ_{ψ,κ}$, we replace $Q_n$ on the right-hand side of Eq. (5.10) by its limit $Q_∞$. Proposition 5.4 below characterizes both the asymptotic behavior of the optimal value $ξ_{n,ψ,κ}^{(2)}$ and the optimal solution $θ_{n,ψ,κ}^{(2)}$ of the problem defined in Eq. (5.8) (note that $ξ_{n,ψ,κ}^{(2)}$ and $θ_{n,ψ,κ}^{(2)}$ are random). The proof of Proposition 5.4 can be found in Appendix B.

**Proposition 5.4.** (a) If $ψ ≤ ψ^+(κ)$, then almost surely

$$
\lim_{n \to ∞, p/n \to ψ} ξ_{n,ψ,κ}^{(2)} > 0.
$$

(b) If $ψ > ψ^+(κ)$, then almost surely

$$
\lim_{n \to ∞, p/n \to ψ} ξ_{n,ψ,κ}^{(2)} = ξ_{ψ,κ} = T(ψ, κ).
$$

(5.13)

Moreover, the minimum $θ_{n,ψ,κ}^{(2)}$ (of the problem defined in Eq. (5.8)) is unique and satisfies

$$
\lim_{n \to ∞, p/n \to ψ} \langle θ_{n,ψ,κ}^{(2)}, Λ_{1/2}w \rangle = c_1(ψ, κ), \quad \text{and} \quad \lim_{n \to ∞, p/n \to ψ} \|θ_{n,ψ,κ}^{(2)}\|_{Λ_{1/2}} = (c_1^2(ψ, κ) + c_2^2(ψ, κ))^{1/2},
$$

(5.14)

where $(c_1(ψ, κ), c_2(ψ, κ))$ is defined as in Proposition 4.1.
Let us emphasize that the almost sure convergence from \( \lim_{n \to \infty} \xi^{(2)}_{n,\psi,\kappa} = \xi_{\psi,\kappa} \) (i.e., Eq. (5.13)) is not an immediate consequence of the convergence \( Q_n \xrightarrow{p/n} Q_\infty \) (Eq. (5.11)). Indeed, the optimization problem defining \( \xi^{(2)}_{n,\psi,\kappa} \) has dimension \( p \) increasing with \( n \), while the problem defining \( \xi_{\psi,\kappa} \) is infinite-dimensional (cf. Eq. (5.10) and Eq. (5.12)). As a consequence, elementary arguments from empirical process theory do not apply: we refer to Appendix B for details.

As an immediate consequence of Proposition 4.1 and Proposition 5.4, we obtain that,
\[
\liminf_{n \to \infty, p/n \to \psi} \xi^{(2)}_{n,\psi,\kappa} > 0 \quad \text{for } \kappa > \kappa^*(\psi).
\]
\[
\lim_{n \to \infty, p/n \to \psi} \xi^{(2)}_{n,\psi,\kappa} \leq 0 \quad \text{for } \kappa \leq \kappa^*(\psi).
\]
Together with Lemma 5.1 and Lemma 5.2, we can pass the above result to \( \xi_{n,\psi,\kappa} \):

- For \( \kappa > \kappa^*(\psi) \)
  \[
  \lim_{n \to \infty, p/n \to \psi} \mathbb{P} (\xi_{n,\psi,\kappa} > 0) = 1. \tag{5.15}
  \]
- For \( \kappa < \kappa^*(\psi) \)
  \[
  \lim_{n \to \infty, p/n \to \psi} \mathbb{P} (\xi_{n,\psi,\kappa} = 0) = 1. \tag{5.16}
  \]
- For \( \kappa = \kappa^*(\psi) \), we have for any \( \varepsilon > 0 \),
  \[
  \lim_{n \to \infty, p/n \to \psi} \mathbb{P} (\xi_{n,\psi,\kappa} \in [0, \varepsilon]) = 1. \tag{5.17}
  \]
This characterizes the asymptotics of \( \xi_{n,\psi,\kappa} \). We proceed analogously to characterize the behavior of \( \xi_{n,\psi,\kappa}(\Theta_p) \) and therefore determine the high-dimensional limit of \( \langle \tilde{\theta}_n^{\text{MM}}, \theta_{\ast,n} \rangle \Sigma_n / (\|\theta_{\ast,n}\|\Sigma_n, \|\tilde{\theta}_n^{\text{MM}}\|\Sigma_n) \). The main result of this analysis is presented in the next proposition, whose proof is given in appendix E.

**Proposition 5.5.** Let \( \psi > \psi^*(0) \). For the max-margin linear classifier \( \mathbf{x} \to \text{sign}(\langle \tilde{\theta}_n^{\text{MM}}, \mathbf{x} \rangle) \), we have
\[
\frac{\langle \tilde{\theta}_n^{\text{MM}}, \mathbf{x} \rangle}{\|\theta_{\ast,n}\|\Sigma_n, \|\tilde{\theta}_n^{\text{MM}}\|\Sigma_n} \xrightarrow{p} \nu^*(\psi). \tag{5.18}
\]

**Summary** We are now in a position to prove Theorem 1. Parts (a), (b) follow immediately from Eq. (4.15), Eq. (5.15) and Eq. (5.16). Part (c) follows from Eq. (4.17) and Proposition 5.5.

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**A Notations**

We typically use lower case letters to denote scalars (e.g. \( a, b, c, \cdots \in \mathbb{R} \)), boldface lower case to denote vectors (e.g. \( \mathbf{u}, \mathbf{v}, \mathbf{w}, \cdots \in \mathbb{R}^d \)), and boldface upper case to denote matrices (e.g. \( \mathbf{X}, \mathbf{Z}, \cdots \in \mathbb{R}^{d_1 \times d_2} \)). The standard scalar product of two vectors \( \mathbf{u}, \mathbf{v} \in \mathbb{R}^d \) will be denoted by \( \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\top \mathbf{v} = \sum_{i=1}^d u_i v_i \). The corresponding norm is \( \|\mathbf{v}\|_2 = (\mathbf{v}^\top \mathbf{v})^{1/2} \). We will define other norms and scalar products within the text.

We occasionally use the notation \( [a \pm b] \equiv [a - b, a + b] \equiv \{ x \in \mathbb{R} : a - b \leq x \leq a + b \} \) for intervals on the real line.

Given two probability measures \( \mu_1, \mu_2 \) on \( \mathbb{R}^d \), their Wasserstein distance \( W_2 \) is defined as
\[
W_2(\mu_1, \mu_2) \equiv \left\{ \inf_{\gamma \in C(\mu_1, \mu_2)} \int \|\mathbf{x}_1 - \mathbf{x}_2\|^2 \gamma(d\mathbf{x}_1, d\mathbf{x}_2) \right\}^{1/2}, \tag{A.1}
\]
where the infimum is taken over the set of couplings \( C(\mu_1, \mu_2) \) of \( \mu_1, \mu_2 \).

Throughout the paper, we are interested in the limit \( n, p \to \infty \), with \( p/n \to \psi \in (0, \infty) \). We do not write this explicitly each time, and often only write \( n \to \infty \) (as in, for instance, \( \lim_{n \to \infty} \)). It is understood that \( p = p_n \) is such that \( p_n/n \to \psi \).
B Properties of the asymptotic optimization problem

In this appendix we derive some important properties of the asymptotic optimization problem that determines the asymptotic maximum margin and prediction error. This has two formulations: the one given in Proposition 4.1, in terms of the three parameters \((c_1, c_2, s)\) and the infinite-dimensional optimization in Eq. (5.12).

We begin by recalling some definitions, and introducing new ones in the next subsection. We will then establish some useful properties of the function \(F\) in Section B.2, and of the asymptotic optimization problem (5.12) in Sections B.3 to B.5.

B.1 Definitions

Given a probability distribution \(\mathbb{P}\) on \(\mathbb{R}^m\), we write \(\mathcal{L}^2(\mathbb{P}) = \mathcal{L}^2(\mathbb{R}^m, \mathbb{P})\) for the Hilbert space of square integrable functions \(h : \mathbb{R}^m \to \mathbb{R}\), with scalar product

\[
\langle h_1, h_2 \rangle_\mathbb{P} = \mathbb{E}_\mathbb{P}\{h_1(Z)h_2(Z)\} = \int h_1(z)h_2(z) \mathbb{P}(dz),
\]

and corresponding norm \(\|h\|_\mathbb{P} = \langle h, h \rangle_\mathbb{P}^{1/2}\). (As usual, measurable functions are considered modulo the equivalence relation \(h_1 \sim h_2 \iff \mathbb{P}(h_1 \neq h_2) = 0\).)

We use \(W^\perp(\mathbb{P})\) to denote the subspace of \(\mathcal{L}^2(\mathbb{P})\) orthogonal to the random variable \(W \in \mathcal{L}^2(\mathbb{P})\):

\[
W^\perp(\mathbb{P}) = \{ h \in \mathcal{L}^2(\mathbb{P}) : \langle h, W \rangle_\mathbb{P} = \mathbb{E}_\mathbb{P}[hW] = 0 \}.
\]

We denote by \(\Pi_{W^\perp, \mathbb{P}}\) the orthogonal projection operator onto the orthogonal complement \(W^\perp(\mathbb{P})\), i.e., for any \(h \in \mathcal{L}^2(\mathbb{P})\), we define

\[
\Pi_{W^\perp, \mathbb{P}}(h) = h - \frac{\langle h, W \rangle_\mathbb{P}}{\|W\|_\mathbb{P}^2}W.
\]  

Notice that the projector \(\Pi_{W^\perp, \mathbb{P}}\) depends on \(\mathbb{P}\) because the scalar product \(\langle h, W \rangle_\mathbb{P}\) and the norm \(\|W\|_\mathbb{P}\) do. However, we typically will drop this dependency as it is clear from the context.

In all of our applications, we will actually consider \(m = 3\), denote by \((g, x, w)\) the coordinates in \(\mathbb{R}^3\), and by \(G(g, x, w) = g, X(g, x, w) = x, W(g, x, w) = w\) the corresponding random variables. We will be particularly interested in two cases:

1. \(\mathbb{P} = \mathbb{Q}_\infty := \mathbb{N}(0, 1) \otimes \mu\), with \(\mu\) as per Assumption 2.
2. \(\mathbb{P} = \mathbb{Q}_n = p^{-1}\sum_{i=1}^p \delta_{(g_i, \lambda_i, \bar{w}_i)}\), the empirical distribution defined in Section 5.

Define \(R_{\psi, \kappa, \mathbb{P}} : \mathcal{L}^2(\mathbb{P}) \to \mathbb{R}\) by

\[
R_{\psi, \kappa, \mathbb{P}}(h) = \psi^{-1/2} \cdot F_\kappa \left( \langle h, X^{1/2}W \rangle_\mathbb{P}, \|\Pi_{W^\perp}(X^{1/2}h)\|_\mathbb{P} \right) + \langle h, X^{1/2}\Pi_{W^\perp}(G) \rangle_\mathbb{P},
\]  

where \(F_\kappa(c_1, c_2)\) is defined as per Eq. (2.1). We consider the optimization problem:

\[
\begin{align*}
\text{minimize} & \quad R_{\psi, \kappa, \mathbb{P}}(h), \\
\text{subject to} & \quad \|h\|_\mathbb{P} \leq 1,
\end{align*}
\]  

and denote its minimum value by \(R^*_{\psi, \kappa, \mathbb{P}}\), i.e.,

\[
R^*_{\psi, \kappa, \mathbb{P}} = \min \left\{ R_{\psi, \kappa, \mathbb{P}}(h) \mid \|h\|_\mathbb{P} \leq 1 \right\}.
\]
B.2 Properties of the function $F$: Proof of Lemma 5.3

Proof of Lemma 5.3  Recall the definition of $F$
\[
F_{\kappa}(c_1,c_2) = \left( \mathbb{E} \left[ (\kappa - c_1 Y G - c_2 Z)^2 \right] \right)^{1/2},
\]
with expectation taken w.r.t $\mathcal{N}(0,1)$ and by using two facts: (i) $F(c_2) = \mathbb{E}[c_2 Z^2]$, (ii) $\mathbb{E}[c_2 Z]$ is strictly increasing for $c_2 \geq 0$. Assumption 3 implies that, when $(c_1, c_2) \neq (c'_1, c'_2)$ and $\kappa \neq 0$,
\[
\mathbb{P}(b_1 c_1 - c_2 Z) \neq b_2 (\kappa - c'_1 T - c'_2 Z) > 0.
\]
for any nonzero pair $(b_1, b_2)$. Hence, inequality (i) holds strictly for any $(c_1, c_2) \neq (c'_1, c'_2)$. This proves that the function $(c_1, c_2) \mapsto F_{\kappa}(c_1, c_2)$ is strictly convex.

(b) The function $c_2 \mapsto F_{\kappa}(c_1, c_2)$ is strictly increasing for $c_2 \geq 0$. Denote $Z_1, Z_2, Z_3$ to be mutually independent $\mathcal{N}(0,1)$ random variables. Note for any $c_2^{(1)} \geq c_2^{(2)} \geq 0$, there exist $c_3^{(2)}$ such that
\[
c_2^{(1)} Z_1 \overset{d}{=} c_2^{(2)} Z_2 + c_3^{(2)} Z_3.
\]
Thus, when $c_2^{(1)} \geq c_2^{(2)} \geq 0$, we have that
\[
F_{\kappa}(c_1, c_2^{(1)}) = \left( \mathbb{E} |\kappa - c_1 T - c_2^{(1)} Z_1|^2 \right)^{1/2} = \left( \mathbb{E} |\kappa - c_1 T - c_2^{(2)} Z_2 - c_3^{(2)} Z_3|^2 \right)^{1/2}
\]
\[
\geq \left( \mathbb{E} |\kappa - c_1 T - c_2^{(2)} Z_2 + c_3^{(2)} \mathbb{E} Z_3|^2 \right)^{1/2} = \left( \mathbb{E} |\kappa - c_1 T - c_2^{(2)} Z_2|^2 \right)^{1/2} = F_{\kappa}(c_1, c_2^{(2)})
\]
where (i) holds due to Jensen’s inequality. Note that (i) becomes a strict inequality whenever $c_2^{(1)} > c_2^{(2)}$.

(c) $F_{\kappa} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable. This follows by an application of dominated convergence, by using two facts: (i) the mapping $x \mapsto c_2^{(1)}$ is continuously differentiable, and (ii) $F_{\kappa}(c_1, c_2) > 0$ for any $(c_1, c_2) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$ (indeed $(\kappa - c_1 Y G - c_2 Z)^2 \geq 0$ and by Assumption 3 the inequality is strict with positive probability).

(d) The function $\kappa \mapsto F_{\kappa}(c_1, c_2)$ is strictly increasing. Let $\kappa_2 > \kappa_1$. Then $F_{\kappa_2}(c_1, c_2)^2 - F_{\kappa_1}(c_1, c_2)^2 = \mathbb{E}(g(Y, G, Z))$, where $g(Y, G, Z) = (\kappa_2 - c_1 Y G - c_2 Z)^2 - (\kappa_1 - c_1 Y G - c_2 Z)^2$ is non-negative, and strictly positive with positive probability (again by Assumption 3).
Lemma B.1. Suppose \((c_1, c_2) \in \mathbb{R} \times \mathbb{R}_{\geq 0}\) satisfies the condition
\[
\partial_1 F_\kappa(c_1, c_2) = 0.
\]
Then, we have the estimate:
\[
\partial_2 F_\kappa(c_1, c_2) \leq \min_{c \in \mathbb{R}} F_0(c, 1).
\]

**Proof** Since \(F_\kappa\) is convex by Lemma 5.3, we have for all \(c \in \mathbb{R}, t \in \mathbb{R}_{\geq 0},\)
\[
F_\kappa(ct, t) \geq F_\kappa(c_1, c_2) + \partial_2 F_\kappa(c_1, c_2)(t - c_2).
\]
This shows in particular that for any \(c \in \mathbb{R}\)
\[
F_0(c, 1) = \lim_{t \to \infty} \frac{F_\kappa(ct, t)}{t} \geq \partial_2 F_\kappa(c_1, c_2).
\]
Taking minimum over \(c\) on both sides gives the desired claim. \(\square\)

### B.3 Properties of \(R_{\psi, \kappa, \mathbb{P}}\)

In this section we state three lemmas establishing several properties of the variational problem (B.4). We will prove these properties in the next subsections.

**Lemma B.2.** Assume that \(\mathbb{E}[|W|^2] = 1\) and \(\mathbb{P}(X \in [x_{\min}, x_{\max}]) = 1\) for some \(x_{\min}, x_{\max} \in (0, \infty)\).

Then the function \(R_{\psi, \kappa, \mathbb{P}} : \mathcal{L}^2(\mathbb{P}) \to \mathbb{R}\) is lower semicontinuous (in the weak topology) and strictly convex.

As a consequence, the minimum of the optimization problem (B.3) is achieved at a unique function \(h^* \in \mathcal{L}^2(\mathbb{P})\). (Uniqueness holds in the sense that, any other minimizer \(h^*\) must satisfy \(\mathbb{P}(h^* \neq h^*) = 0\).)

**Lemma B.3.** Under the assumptions of Lemma B.2, define \(\zeta(\mathbb{P}), \eta > 0\) by
\[
\zeta(\mathbb{P}) = \|X^{-1/2}W\|^{-1}_{\mathbb{P}},
\]
\[
\eta = \frac{1}{6x_{\max}^{1/2}} \cdot \min \left\{ \min_{|c_1| \leq x_{\max}^{1/2}, 0 \leq c_2 \leq x_{\max}^{1/2}} \{F_\kappa(c_1, c_2) - F_0(c_1, c_2)\}, \min_{|c_1| \leq x_{\max}^{1/2}} F_\kappa(c_1, 0) \right\}.
\]
(Note \(\eta\) is independent of \(\mathbb{P}\).)

Further, the call that \(\psi^*(0) \equiv 0\), and assume one of the three conditions below to be satisfied:

- **A.** \(\psi^*(0)^{1/2} \geq \psi^{1/2}\|\Pi_{W^\perp}(G)\|_{\mathbb{P}} - \eta\).
- **A.** \(\partial_1 F_\kappa(\zeta, 0) \leq \eta\) and \(\partial_2 F_\kappa(\zeta, 0) \geq \|\partial_1 F_\kappa(\zeta, 0)(1 - \zeta^2 X^{-1})W + \psi^{1/2}\Pi_{W^\perp}(G)\|_{\mathbb{P}} - \eta\).
- **A.** \(\partial_1 F_\kappa(-\zeta, 0) \geq -\eta\) and \(\partial_2 F_\kappa(-\zeta, 0) \geq \|\partial_1 F_\kappa(-\zeta, 0)(1 - \zeta^2 X^{-1})W + \psi^{1/2}\Pi_{W^\perp}(G)\|_{\mathbb{P}} - \eta\).

Then we have
\[
R^*_{\psi, \kappa, \mathbb{P}} \geq \psi^{-1/2}x_{\max}^{1/2}\eta > 0.
\]

**Lemma B.4.** Under the assumptions of Lemma B.3, further assume all of the three conditions below are satisfied:

- **B.** \(\psi^*(0)^{1/2} < \psi^{1/2}\|\Pi_{W^\perp}(G)\|_{\mathbb{P}}\).
- **B.** Either \(\partial_1 F_\kappa(\zeta, 0) > 0\) or \(\partial_2 F_\kappa(\zeta, 0) < \|\partial_1 F_\kappa(\zeta, 0)(1 - \zeta^2 X^{-1})W + \psi^{1/2}\Pi_{W^\perp}(G)\|_{\mathbb{P}}\).
- **B.** Either \(\partial_1 F_\kappa(-\zeta, 0) < 0\) or \(\partial_2 F_\kappa(-\zeta, 0) < \|\partial_1 F_\kappa(-\zeta, 0)(1 - \zeta^2 X^{-1})W + \psi^{1/2}\Pi_{W^\perp}(G)\|_{\mathbb{P}}\).

Then the following hold
Throughout this proof, we keep $\psi, \kappa, \mathbb{P}$ fixed, and hence we drop them from the the arguments of $\mathcal{R}$ to simplify notations (hence writing $\mathcal{R}(h) = \mathcal{R}_{\psi, \kappa, \mathbb{P}}(h)$). Further, we will drop the subscripts $\mathbb{P}$ from $\langle h_1, h_2 \rangle_{\mathbb{P}}$ and $\|h\|_{\mathbb{P}}$.

We begin by noticing that $\mathcal{R} : \mathcal{L}^2(\mathbb{P}) \to \mathbb{R}$ is lower semicontinuous with respect to the weak-* topology (which coincide with the weak topology since $\mathcal{L}^2(\mathbb{P})$ is an Hilbert space). Indeed note that: (i) The mappings $h \mapsto \langle h, X^{1/2}W \rangle$ and $h \mapsto \langle h, X^{1/2} \Pi_{W^*}(G) \rangle$ are continuous; (ii) The mapping $h \mapsto \|\Pi_{W^*}(X^{1/2}h)\|$ is lower semicontinuous. Since for any $h \in \mathcal{L}^2(\mathbb{P})$, we have the variational representation

$$
\|\Pi_{W^*}(X^{1/2}h)\|^2 = \|h\|^2 - \langle h, W \rangle^2 = \sup_{h' : \|h'\| \leq 1} \langle h', h \rangle^2 - \langle h, W \rangle^2. \tag{B.17}
$$

Since $h \mapsto \|\Pi_{W^*}(X^{1/2}h)\|$ is the supremum of continuous functions, it is lower semicontinuous. (ii) $(c_1, c_2) \mapsto F_\kappa(c_1, c_2)$ is continuous (by Lemma 5.3.(e)), and monotone increasing in $c_2$ (by Lemma 5.3.(b)).

Together, (i), (ii), (iii) imply the lower semicontinuity of $\mathcal{R}$. Since the constraint set $\{h : \|h\| \leq 1\}$ is sequentially compact w.r.t the weak-* topology by the Banach-Alaoglu Theorem, this immediately implies that the minimum of the optimization problem (B.3) is achieved by some $h^* \in \mathcal{L}^2(\mathbb{P})$.

In order to prove uniqueness of the minimizer, we show that $\mathcal{R} : \mathcal{L}^2(\mathbb{P}) \to \mathbb{R}$ is strictly convex

$$
\frac{1}{2} \left( L(h_0) + L(h_1) \right) > L \left( \frac{1}{2} (h_0 + h_1) \right) \quad \text{for any } h_0, h_1 \text{ such that } P(h_0 \neq h_1) > 0. \tag{B.18}
$$

**B.3.1 Proof of Lemma B.2**

Throughout this proof, we keep $\psi, \kappa, \mathbb{P}$ fixed, and hence we drop them from the the arguments of $\mathcal{R}$ to simplify notations (hence writing $\mathcal{R}(h) = \mathcal{R}_{\psi, \kappa, \mathbb{P}}(h)$). Further, we will drop the subscripts $\mathbb{P}$ from $\langle h_1, h_2 \rangle_{\mathbb{P}}$ and $\|h\|_{\mathbb{P}}$.

We begin by noticing that $\mathcal{R} : \mathcal{L}^2(\mathbb{P}) \to \mathbb{R}$ is lower semicontinuous with respect to the weak-* topology (which coincide with the weak topology since $\mathcal{L}^2(\mathbb{P})$ is an Hilbert space). Indeed note that: (i) The mappings $h \mapsto \langle h, X^{1/2}W \rangle$ and $h \mapsto \langle h, X^{1/2} \Pi_{W^*}(G) \rangle$ are continuous; (ii) The mapping $h \mapsto \|\Pi_{W^*}(X^{1/2}h)\|$ is lower semicontinuous. Since for any $h \in \mathcal{L}^2(\mathbb{P})$, we have the variational representation

$$
\|\Pi_{W^*}(X^{1/2}h)\|^2 = \|h\|^2 - \langle h, W \rangle^2 = \sup_{h' : \|h'\| \leq 1} \langle h', h \rangle^2 - \langle h, W \rangle^2. \tag{B.17}
$$

Since $h \mapsto \|\Pi_{W^*}(X^{1/2}h)\|$ is the supremum of continuous functions, it is lower semicontinuous. (ii) $(c_1, c_2) \mapsto F_\kappa(c_1, c_2)$ is continuous (by Lemma 5.3.(e)), and monotone increasing in $c_2$ (by Lemma 5.3.(b)).

Together, (i), (ii), (iii) imply the lower semicontinuity of $\mathcal{R}$. Since the constraint set $\{h : \|h\| \leq 1\}$ is sequentially compact w.r.t the weak-* topology by the Banach-Alaoglu Theorem, this immediately implies that the minimum of the optimization problem (B.3) is achieved by some $h^* \in \mathcal{L}^2(\mathbb{P})$.

In order to prove uniqueness of the minimizer, we show that $\mathcal{R} : \mathcal{L}^2(\mathbb{P}) \to \mathbb{R}$ is strictly convex

$$
\frac{1}{2} \left( L(h_0) + L(h_1) \right) > L \left( \frac{1}{2} (h_0 + h_1) \right) \quad \text{for any } h_0, h_1 \text{ such that } P(h_0 \neq h_1) > 0. \tag{B.18}
$$

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Pick $h_0, h_1 \in \mathcal{L}^2(\mathbb{P})$ such that $\mathbb{P}(h_0 \neq h_1) > 0$. Denote $h_{1/2} = \frac{1}{2}(h_0 + h_1)$. Notice that

$$\frac{1}{2}(\mathcal{R}(h_0) + \mathcal{R}(h_1)) - \mathcal{R}(h_{1/2})$$

$$= \psi^{-1/2} \cdot \left\{ \frac{1}{2} F_\kappa \left( h_0, X^{1/2} W, \| \Pi_{W^\perp} (X^{1/2} h_0) \| \right) + \frac{1}{2} F_\kappa \left( h_1, X^{1/2} W, \| \Pi_{W^\perp} (X^{1/2} h_1) \| \right) \right. $$

$$- F_\kappa \left( h_{1/2}, X^{1/2} W, \| \Pi_{W^\perp} (X^{1/2} h_{1/2}) \| \right) \right\}$$

$$\geq \psi^{-1/2} \left\{ F_\kappa \left( h_{1/2}, X^{1/2} W, \| \Pi_{W^\perp} (X^{1/2} h_{1/2}) \| \right) \right\}$$

$$- F_\kappa \left( h_{1/2}, X^{1/2} W, \| \Pi_{W^\perp} (X^{1/2} h_{1/2}) \| \right) \right\}$$



\[(i) \geq 0.\]

where (i) follows since $F_\kappa$ is convex by Lemma 5.3.(a) and (ii) follows since $F_\kappa$ is increasing with respect to its second argument by Lemma 5.3.(b) and the triangle inequality $\frac{1}{2} \left( \| \Pi_{W^\perp} (X^{1/2} h_0) \| + \| \Pi_{W^\perp} (X^{1/2} h_1) \| \right) \geq \| \Pi_{W^\perp} (X^{1/2} h_{1/2}) \|$.

Next we prove that one of the inequalities (i) and (ii) must be strict when $\mathbb{P}(h_0 \neq h_1) > 0$. To see this, suppose both inequalities (i) and (ii) become equalities for some $h_0, h_1$. By Lemma 5.3.(a), we know that $F_\kappa$ is strictly convex, and strictly increasing its second argument. Thus, if both inequalities (i) and (ii) become equalities, $h_0, h_1$ and $h_{1/2} = \frac{1}{2}(h_0 + h_1)$ need to satisfy

$$\left\langle h_0, X^{1/2} W \right\rangle = \left\langle h_1, X^{1/2} W \right\rangle,$$

$$\| \Pi_{W^\perp} (X^{1/2} h_0) \| = \| \Pi_{W^\perp} (X^{1/2} h_1) \|,$$

$$\| \Pi_{W^\perp} (X^{1/2} h_{1/2}) \| = \frac{1}{2} \left( \| \Pi_{W^\perp} (X^{1/2} h_0) \| + \| \Pi_{W^\perp} (X^{1/2} h_1) \| \right).$$

Now, the first equality of Eq. (B.19) is equivalent to

$$\left\langle X^{1/2} (h_0 - h_1), W \right\rangle = 0,$$

and the last two equalities are equivalent to

$$\Pi_{W^\perp} (X^{1/2} (h_0 - h_1)) = 0 \quad \mathbb{P} - a.s.$$

(B.21)

Thus, if both inequalities (i) and (ii) become equalities, it must happen that

$$X^{1/2} (h_0 - h_1) = 0 \quad \mathbb{P} - a.s.$$

which implies $\mathbb{P}(h_0 \neq h_1) = 0$ since we assumed $\mathbb{P}(X > 0) = 1$. This completes the proof that $\mathcal{R} : \mathcal{L}^2(\mathbb{P}) \rightarrow \mathbb{R}$ is strictly convex in the sense of Eq. (B.18).

Strict convexity implies immediately uniqueness of the minimizer of $\mathcal{R}$. Indeed, given two minimizers $h^*$ and $\tilde{h}^*$, we must have $\mathbb{P}(\tilde{h}^* \neq h^*) = 0$, because otherwise $h_{1/2} = (h^* + \tilde{h}^*)/2$ would achieve a strictly smaller cost.

### B.3.2 Proof of Lemma B.3

Throughout this proof, we keep $\mathbb{P}$ fixed, and hence we drop it from the the arguments of $\mathcal{R}$ to simplify notations (hence writing $\mathcal{R}_{\psi, \kappa}(h) = \mathcal{R}_{\psi, \kappa, \mathbb{P}}(h)$), and from $\langle h_1, h_2 \rangle_\mathbb{P}$ and $\| h \|_\mathbb{P}$.

We organize the proof in three parts depending on which of the three conditions A1, A2 or A3 holds.

**Condition A1 holds:**

$$\psi^*(0)^{1/2} \geq \psi^{1/2} \| \Pi_{W^\perp} (G) \| - \eta. \quad (B.22)$$
Define the constant $c_\kappa$ by

$$c_\kappa = \min_{|c_1| \leq \frac{1}{2}, 0 \leq c_2 \leq \frac{1}{2}} \{ F_\kappa(c_1, c_2) - F_0(c_1, c_2) \}. \tag{B.23}$$

Note that $c_\kappa > 0$ strictly since $\kappa \mapsto F_\kappa(c_1, c_2)$ is strictly increasing and $F_\kappa(c_1, c_2) - F_0(c_1, c_2)$ is continuous in $c_1, c_2$ by Lemma 5.3.

Since for any $h$ such that $\|h\| \leq 1$, we have

$$\langle h, X^{1/2}W \rangle \leq \|X^{1/2}W\| \|h\| \leq x_{\max}^{1/2} \quad \text{and} \quad \|\Pi_{W^1}(X^{1/2}h)\| \leq \|X^{1/2}h\| \leq x_{\max}^{1/2}, \tag{B.24}$$

the definition of $c_\kappa$ at Eq. (B.24) implies for any $h$ satisfying $\|h\| \leq 1$,

$$\mathcal{R}_{\psi,\kappa}(h) \geq \mathcal{R}_{\psi,0}(h) + \psi^{-1/2}c_\kappa. \tag{B.25}$$

Now we note that, by Cauchy-Schwartz,

$$\langle X^{1/2}h, \Pi_{W^1}(g) \rangle = \langle \Pi_{W^1}(X^{1/2}h), \Pi_{W^1}(G) \rangle \leq \|\Pi_{W^1}(X^{1/2}h)\| \|\Pi_{W^1}(G)\|. \tag{B.26}$$

Thus we have that for any $h \in \mathcal{L}^2(\mathbb{P})$ satisfying $\|h\| \leq 1$,

$$\mathcal{R}_{\psi,\kappa}(h) \overset{(i)}{=} \mathcal{R}_{\psi,0}(h) + \psi^{-1/2}c_\kappa = \psi^{-1/2} \left( F_0 \left( \langle h, X^{1/2}W \rangle, \|\Pi_{W^1}(X^{1/2}h)\| + c_\kappa \right) + \langle X^{1/2}h, \Pi_{W^1}(G) \rangle \right) \overset{(ii)}{\geq} \left( \psi^{-1/2} \cdot \min_{c \in \mathbb{R}} F_0(c, 1) \right) \cdot \|\Pi_{W^1}(X^{1/2}h)\| + \psi^{-1/2}c_\kappa - \|\Pi_{W^1}(G)\| \|\Pi_{W^1}(X^{1/2}h)\| \overset{(iii)}{=} \left( \psi^*(0) / \psi \right)^{1/2} \cdot \|\Pi_{W^1}(G)\| \|\Pi_{W^1}(X^{1/2}h)\| + \psi^{-1/2}c_\kappa \overset{(iv)}{\geq} \psi^{-1/2}c_\kappa - \psi^{-1/2}x_{\max}^{1/2} \eta \overset{(iv)}{=} \psi^{-1/2}x_{\max}^{1/2} \eta,$$

where in (i), we use Eq. (B.25); in (ii), we use the bound in Eq. (B.26) and the fact that $F_0(c_1, c_2) = c_2 F_0(c_1/c_2, 1) \geq c_2 \min_{c \in \mathbb{R}} F_0(c, 1)$ for any $(c_1, c_2) \in \mathbb{R} \times \mathbb{R}_+$; in (iii), we use the assumption in Eq. (B.22) and the fact that $\|\Pi_{W^1}(X^{1/2}h)\| \leq \|X^{1/2}h\| \leq x_{\max}^{1/2}$ for all $h$ satisfying $\|h\| \leq 1$; (iv) follows from the definition of $c_\kappa$ and $\eta$. This proves that

$$\mathcal{R}_{\psi,\kappa}^* := \min_{h: \|h\| \leq 1} \mathcal{R}_{\psi,\kappa}(h) \geq \psi^{-1/2}x_{\max}^{1/2} \eta \geq 0.$$

Condition A2 holds:

$$\partial_1 F_\kappa(\zeta, 0) \leq 0 \quad \text{and} \quad \partial_2 F_\kappa(\zeta, 0) \geq \|\partial_1 F_\kappa(\zeta, 0)(1 - \zeta^2 X^{-1})W + \psi^1/2\Pi_{W^1}(G)\| - \eta. \tag{B.27}$$

To start with, by Lemma 5.3, $F_\kappa$ is convex. Hence, for any $h$,

$$\mathcal{R}_{\psi,\kappa}(h) = \psi^{-1/2} F_\kappa(h, X^{1/2}W), \Pi_{W^1}(X^{1/2}h)) + \langle \Pi_{W^1}(G), X^{1/2}h \rangle \overset{\geq}{\geq} \psi^{-1/2} \left( F_\kappa(\zeta, 0) + \partial_1 F_\kappa(\zeta, 0) \left( \langle h, X^{1/2}W \rangle - \zeta \right) + \partial_2 F_\kappa(\zeta, 0) \|\Pi_{W^1}(X^{1/2}h)\| \right) + \langle \Pi_{W^1}(G), \Pi_{W^1}(X^{1/2}h) \rangle. \tag{B.28}$$

Now by Cauchy-Schwartz inequality, the inequality below holds for any $h$ such that $\|h\| \leq 1$:

$$\zeta = \zeta^2 \|X^{-1/2}W\| \geq \zeta^2 \langle h, X^{-1/2}W \rangle = \zeta^2 \langle X^{1/2}h, X^{-1/2}W \rangle \tag{B.29}$$

This immediately implies for any $h$ such that $\|h\| \leq 1$,

$$\langle h, X^{1/2}W \rangle - \zeta \leq \langle X^{1/2}h, W \rangle - \langle X^{1/2}h, \zeta^2 X^{-1}W \rangle = \langle X^{1/2}h, (1 - \zeta^2 X^{-1})W \rangle \tag{B.30}$$
Note that \( \langle W, (1 - \zeta^2 X^{-1}) W \rangle = 0 \), since \( \|W\| = 1 \). This implies that
\[
(1 - \zeta^2 X^{-1}) W = \Pi_{W^\perp} ((1 - \zeta^2 X^{-1}) W).
\] (B.31)

The last two displays imply that, for any \( h \) such that \( \|h\| \leq 1 \),
\[
\langle h, X^{1/2} W \rangle - \zeta \leq \langle X^{1/2} h, \Pi_{W^\perp} ((1 - \zeta^2 X^{-1}) W) \rangle = \langle \Pi_{W^\perp} (X^{1/2} h), (1 - \zeta^2 X^{-1}) W \rangle
\] (B.32)

Therefore, we have for all \( h \) such that \( \|h\| \leq 1 \),
\[
\partial_1 F_\kappa(\zeta, 0) \left( \langle h, X^{1/2} W \rangle - \zeta \right) \\
= \langle \partial_1 F_\kappa(\zeta, 0) - \eta \rangle \left( \langle h, X^{1/2} W \rangle - \zeta \right) + \eta \left( \langle h, X^{1/2} W \rangle - \zeta \right)
\]
\[
\overset{(i)}{\geq} \langle \partial_1 F_\kappa(\zeta, 0) - \eta \rangle \left( \Pi_{W^\perp} (X^{1/2} h), (1 - \zeta^2 X^{-1}) W \right) + \eta \left( \langle h, X^{1/2} W \rangle - \zeta \right)
\]
\[
\overset{(ii)}{=} \partial_1 F_\kappa(\zeta, 0) \Pi_{W^\perp} (X^{1/2} h), (1 - \zeta^2 X^{-1}) W \right) + \eta \left( \langle h, X^{1/2} W \rangle - \zeta - \langle X^{1/2} h, (1 - \zeta^2 X^{-1}) W \rangle \right)
\]
\[
\overset{(iii)}{\geq} \partial_1 F_\kappa(\zeta, 0) \Pi_{W^\perp} (X^{1/2} h), (1 - \zeta^2 X^{-1}) W - 4\eta x_{\text{max}}^2.
\] (B.33)

where in (i) we use Eq. (B.32) and the assumption \( \partial_1 F_\kappa(\zeta, 0) \leq \eta \); in (ii), we note the identity in Eq. (B.31); in (iii), we use the bounds below that hold for all \( h \) with \( \|h\| \leq 1 \):
\[
\left| \langle h, X^{1/2} W \rangle \right| \leq \|X^{1/2} W\| \leq x_{\text{max}}^{1/2}, \quad \zeta \leq (\psi_{\text{max}}^{-1/2} \|W\|)^{-1} = x_{\text{max}}^{1/2},
\]
\[
\langle X^{1/2} h, (1 - \zeta^2 X^{-1}) W \rangle \leq \|X^{1/2} (1 - \zeta^2 X^{-1}) W\| \leq \|X^{1/2} W\| + \zeta^2 \|X^{-1/2} W\| = x_{\text{max}}^{1/2} + \zeta \leq 2x_{\text{max}}^{1/2}.
\]

Substituting Eq. (B.33) into Eq. (B.28), we have for all \( h \) satisfying \( \|h\| \leq 1 \),
\[
\mathcal{R}_{\psi, \kappa}(h) \geq \psi^{-1/2} \left( F_\kappa(\zeta, 0) + \partial_2 F_\kappa(\zeta, 0) \right) \Pi_{W^\perp} (X^{1/2} h) \right) + 4\psi^{-1/2} x_{\text{max}}^{1/2} \eta
\]
\[
+ \psi^{-1/2} \partial_2 F_\kappa(\zeta, 0) (1 - \zeta^2 X^{-1}) W + \Pi_{W^\perp} (G) \Pi_{W^\perp} (X^{1/2} h)
\]
\[
\overset{(i)}{\geq} \psi^{-1/2} F_\kappa(\zeta, 0) - 4\psi^{-1/2} x_{\text{max}}^{1/2} \eta
\]
\[
+ \psi^{-1/2} \left( \partial_2 F_\kappa(\zeta, 0) - \left( \partial_1 F_\kappa(\zeta, 0) (1 - \zeta^2 X^{-1}) W + \psi^{1/2} \Pi_{W^\perp} (G) \right) \right) \Pi_{W^\perp} (X^{1/2} h) \right)
\]
\[
\overset{(ii)}{\geq} \psi^{-1/2} F_\kappa(\zeta, 0) - 5\psi^{-1/2} x_{\text{max}}^{1/2} \eta \geq \psi^{-1/2} x_{\text{max}}^{1/2} \eta.
\]

where, in (i), we use the Cauchy-Schwartz inequality and in (ii), we use the assumption (B.27) and the bound \( \|\Pi_{W^\perp} (X^{1/2} h)\| \leq x_{\text{max}}^{1/2} \) that holds whenever \( \|h\| \leq 1 \). As a consequence, this proves that
\[
\mathcal{R}_{\psi, \kappa} = \min_{h \|h\| \leq 1} \mathcal{R}_{\psi, \kappa}(h) \geq \psi^{1/2} x_{\text{max}}^{1/2} \eta.
\]

**Condition A3** holds:
\[
\partial_1 F_\kappa(-\zeta, 0) \geq -\eta \quad \text{and} \quad \partial_2 F_\kappa(-\zeta, 0) \geq \left( \partial_1 F_\kappa(-\zeta, 0) (1 - \zeta^2 X^{-1}) W + \psi^{1/2} \Pi_{W^\perp} (G) \right) - \eta.
\]

In this case, the inequality \( \mathcal{R}_{\psi, \kappa} > 0 \) follows from essentially the same argument as in the previous point. We omit the details.

**B.3.3 Proof of Lemma B.4**

Throughout the proof, we will drop \( \mathbb{P} \) from subscripts in order to lighten the notations.

By Lemma 5.3, the function \( h \mapsto \mathcal{R}_{\psi, \kappa, \mathbb{P}}(h) \) is strictly convex, by Lemma B.2. Hence, the unique minimizer of problem (B.4) is determined by the Karush—Kuhn—Tucker (KKT) conditions. Namely, \( h \) is
the minimum of problem (B.4) if and only if, for some scalar \( s \) and some measurable function \( Z = Z(g, x, w) \), the following hold

\[
X^{1/2} \Pi_{W^+}(G) + \psi^{-1/2} X^{1/2} \left( \partial_1 F_\kappa \left( \langle h, X^{1/2} W \rangle, \| \Pi_{W^+}(X^{1/2} h) \| \right) W + \\
+ \partial_2 F_\kappa \left( \langle h, X^{1/2} W \rangle, \| \Pi_{W^+}(X^{1/2} h) \| \Pi_{W^+}(Z) \right) + s h = 0.
\]

(B.34)

\[
Z = \begin{cases} \| \Pi_{W^+}(X^{1/2} h) \|^{-1} \cdot \Pi_{W^+}(X^{1/2} h) & \text{if } \| \Pi_{W^+}(X^{1/2} h) \| > 0 \\
Z'(g, x, w) & \text{where } \| Z' \| \leq 1 & \text{if } \| \Pi_{W^+}(X^{1/2} h) \| = 0.
\end{cases}
\]

For completeness, we provide a derivation of the KKT conditions in Appendix F.1.

We claim that the lemma holds for any \( \Delta = \Delta_{\psi, \kappa}(P) > 0 \) that satisfies the conditions stated next. Let us first define the constants

\[
\beta = x_{\max}/x_{\min}, \quad \beta' = (1 + \beta^{1/2})(1 + x_{\min}^{-1/2}),
\]

(B.35)

\[
M = \psi^{1/2} \| \Pi_{W^+}(G) \|_p + \max_{|c| \leq x_{\min}^{1/2}} |\partial_1 F_\kappa(c, 0)| + \max_{|c| \leq x_{\min}^{1/2}} |\partial_2 F_\kappa(c, 0)|,
\]

(B.36)

the functions \( \gamma_+, \gamma_- : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) (here we use the notation \([a \pm b] \equiv [a - b, a + b] \):

\[
\gamma_+(\Delta) = \max_{\delta \in [0, \Delta], c \in [-\zeta \pm \beta^{1/2} \Delta]} \{ |\partial_1 F_\kappa(c, \delta) - \partial_1 F_\kappa(\zeta, 0)| + |\partial_2 F_\kappa(c, \delta) - \partial_2 F_\kappa(\zeta, 0)| \}
\]

\[
\gamma_-(-\Delta) = \max_{\delta \in [0, \Delta], c \in [-\zeta \pm \beta^{1/2} \Delta]} \{ |\partial_1 F_\kappa(c, \delta) - \partial_1 F_\kappa(-\zeta, 0)| + |\partial_2 F_\kappa(c, \delta) - \partial_2 F_\kappa(-\zeta, 0)| \}
\]

(B.37)

and the functions \( \tilde{\gamma}_+, \tilde{\gamma}_- : \mathbb{R}_+ \to \mathbb{R}_+ \),

\[
\tilde{\gamma}_+(\Delta) = 2\zeta x_{\min}^{-1/2} (\gamma_+(\Delta) + M \beta' \Delta) \quad \text{and} \quad \tilde{\gamma}_-(\Delta) = 2\zeta x_{\min}^{-1/2} (\gamma_-(\Delta) + M \beta' \Delta).
\]

(B.38)

Then \( \Delta = \Delta_{\psi, \kappa}(P) \) needs to satisfy all the conditions below:

**C1.** \( \Delta \) is smaller than some constant:

\[
\beta^{1/2} \Delta \leq \zeta \quad \text{and} \quad (1 + \beta^{1/2}) \zeta \Delta \leq x_{\max}^{1/2}.
\]

(B.39)

**C2.** \( \Delta \) satisfies the following strengthened version of condition B2:

\[
\begin{align*}
\text{either} & \quad \partial_1 F_\kappa(\zeta, 0) > \gamma_+(\Delta) \\
\text{or} & \quad \partial_2 F_\kappa(\zeta, 0) < \| \partial_1 F_\kappa(\zeta, 0)(1 - \zeta^2 X^{-1}) W + \psi^{1/2} \Pi_{W^+}(G) \|_p - \tilde{\gamma}_+(\Delta).
\end{align*}
\]

(B.40)

**C3.** \( \Delta \) satisfies the following strengthened version of condition B3:

\[
\begin{align*}
\text{either} & \quad \partial_1 F_\kappa(-\zeta, 0) < -\gamma_-(-\Delta) \\
\text{or} & \quad \partial_2 F_\kappa(-\zeta, 0) < \| \partial_1 F_\kappa(-\zeta, 0)(1 - \zeta^2 X^{-1}) W + \psi^{1/2} \Pi_{W^+}(G) \|_p - \tilde{\gamma}_-(\Delta).
\end{align*}
\]

(B.41)

It is clear that we can choose \( \Delta = \Delta_{\psi, \kappa}(P) > 0 \) satisfying the conditions Eq. (B.39) to Eq. (B.41), as well as the claims in the lemma.

We claim that the KKT conditions (B.34) imply that any minimizer \( h \) and its associated dual variable \( s \) must satisfy

\[
s > 0 \quad \text{and} \quad \| \Pi_{W^+}(X^{1/2} h) \| > \Delta.
\]

(B.42)

To show this, first assume by contradiction \( s = 0 \). Since \( X > 0 \) by assumption, Eq. (B.34) now implies

\[
\Pi_{W^+}(G) + \psi^{-1/2} \left( \partial_1 F_\kappa \left( \langle h, X^{1/2} W \rangle, \| \Pi_{W^+}(X^{1/2} h) \| \right) W + \partial_2 F_\kappa \left( \langle h, X^{1/2} W \rangle, \| \Pi_{W^+}(X^{1/2} h) \| \right) \Pi_{W^+}(Z) \right) = 0.
\]

(B.43)
Hence, by taking inner products with $W$ on both sides of Eq. (B.43), and using $\|W\| = 1$, we get that
\[
\psi^{-1/2} \cdot \partial_1 F_\kappa \left( \langle h, X^{1/2}W \rangle, \| \Pi_{W^\perp} (X^{1/2}h) \| \right) = 0. \tag{B.44}
\]
Plugging Eq. (B.44) into Eq. (B.43), we get that
\[
\psi^{-1/2} \cdot \partial_2 F_\kappa \left( \langle h, X^{1/2}W \rangle, \| \Pi_{W^\perp} (X^{1/2}h) \| \right) \Pi_{W^\perp} (Z) = - \Pi_{W^\perp} (G). \tag{B.45}
\]
Note that $\| \Pi_{W^\perp} (Z) \| \leq \| Z \| \leq 1$. By taking norm on both sides of Eq. (B.45), we get the bound:
\[
\psi^{-1/2} \cdot \partial_2 F_\kappa \left( \langle h, X^{1/2}W \rangle, \| \Pi_{W^\perp} (X^{1/2}h) \| \right) \geq \| \Pi_{W^\perp} (G) \|. \tag{B.46}
\]
Now, we recall Lemma B.1. By Lemma B.1, Eq. (B.44) and Eq. (B.46) imply that
\[
(\psi^*(0)/\psi)^{1/2} = \psi^{-1/2} \cdot \min_{c \in \mathbb{R}} F_0 (c, 1) \geq \| \Pi_{W^\perp} (G) \|. \tag{B.47}
\]
This contradicts our assumption on $\psi$. We therefore conclude that $s > 0$.

next, again by contradiction, assume $s > 0$ but $\| \Pi_{W^\perp} (X^{1/2}h) \| \leq \Delta$. Denote
\[
c_1 = \langle h, X^{1/2}W \rangle, \quad c_2 = \| \Pi_{W^\perp} (X^{1/2}h) \| \quad \text{and} \quad \delta_h = \Pi_{W^\perp} (X^{1/2}h).
\]
Now using the above notation, and multiplying Eq. (B.34) by $X^{-1/2}$, we reach the identity:
\[
\Pi_{W^\perp} (G) + \psi^{-1/2} (\partial_1 F_\kappa (c_1, c_2)W + \partial_2 F_\kappa (c_1, c_2)\Pi_{W^\perp} (Z)) + sX^{-1/2}h = 0. \tag{B.48}
\]
Now, the orthogonal decomposition gives $X^{1/2}h = c_1 W + \delta_h$. This gives the representation
\[
h = c_1 X^{-1/2}W + X^{-1/2}\delta_h. \tag{B.49}
\]
Now, we note the bounds below on $|c_1|$ and $c_2$ (recall $\beta = x_{\max}/x_{\min}$ from Eq. (B.35)):
\[
0 \leq c_2 \leq \Delta \quad \text{and} \quad \zeta - \beta^{1/2}\Delta \leq |c_1| \leq \zeta + \beta^{1/2}\Delta. \tag{B.50}
\]
The first part of Eq. (B.50) follows by our assumption $\| \Pi_{W^\perp} (X^{1/2}h) \| \leq \Delta$. To show the second part of Eq. (B.50), note first that $\|h\| = 1$ since $s > 0$ and the KKT condition. Note further
\[
\|X^{-1/2}\delta_h\| \leq x_{\min}^{-1/2} \|\delta_h\| \leq x_{\min}^{-1/2} \Delta \quad \text{and} \quad x_{\min}^{1/2} \leq \zeta = \|X^{-1/2}W\|^{-1} \leq x_{\max}^{1/2}. \tag{B.51}
\]
Hence, by taking norm on both sides of Eq. (B.49) and using triangle inequality, we get
\[
1 = \|h\| \in \left[ |c_1|\zeta^{-1} - x_{\min}^{-1/2}\Delta, |c_1|\zeta^{-1} + x_{\min}^{-1/2}\Delta \right]. \tag{B.52}
\]
As $\zeta \leq x_{\max}^{1/2}$ by Eq. (B.51), the above bound implies the second part of Eq. (B.50). Now, we divide our discussion into two cases, based on the sign of $c_1$:

1. $c_1 > 0$. Since $c_1 > 0$ and $\beta^{1/2}\Delta < \zeta$, we know from Eq. (B.50) that
\[
|c_1 - \zeta| \leq \beta^{1/2}\Delta. \tag{B.53}
\]
Recall our definition of $\gamma_+ (\Delta)$ at Eq. (B.37):
\[
\gamma_+ (\Delta) = \max_{c_1 \in [\zeta + \beta^{1/2}\Delta], c_2 \in [0, \Delta]} \left\{ |\partial_1 F_\kappa (c_1, c_2) - \partial_1 F_\kappa (\zeta, 0)| + |\partial_2 F_\kappa (c_1, c_2) - \partial_2 F_\kappa (\zeta, 0)| \right\}.
\]
We start from the basic fact that
\[
\|W\| = 1 \quad \text{and} \quad \| \Pi_{W^\perp} (Z) \| \leq \| Z \| \leq 1. \tag{B.54}
\]
Thus, we can use triangle inequality to see from Eq. (B.48) that
\[
\Pi_{W^\perp}(G) + \psi^{-1/2} \left( \partial_1 F_\kappa(\zeta, 0)W + \partial_2 F_\kappa(\zeta, 0)\Pi_{W^\perp}(Z) + R_+ \right) + sX^{-1/2}h = 0. 
\] (B.55)
holds for for some function $R_+ = R_+(G, X, W)$ with $\|R_+\| \leq \gamma_+(\Delta)$. Now, we take inner products with $W$ on both sides of Eq. (B.55) and we can get that
\[
\psi^{-1/2} \left( \partial_1 F_\kappa(\zeta, 0) + \langle W, R_+ \rangle \right) + s\langle W, X^{-1/2}h \rangle = 0. 
\] (B.56)
Now we can eliminate the variable $s$ from Eq. (B.55) and Eq. (B.56) and get
\[
\langle W, X^{-1/2}h \rangle \left( \psi^{1/2} \Pi_{W^\perp}(G) + \partial_1 F_\kappa(\zeta, 0)W + \partial_2 F_\kappa(\zeta, 0)\Pi_{W^\perp}(Z) + R_+ \right) = (\partial_1 F_\kappa(\zeta, 0) + \langle W, R_+ \rangle) X^{-1/2}h. 
\] (B.57)
Now, we give estimates on terms on the LHS and RHS of Eq. (B.57). First of all, by taking inner products with $X^{-1/2}W$ in Eq. (B.49), the triangle inequality gives us the inclusion (as above, $|a \pm b| \leq |a - b, a + b|$):
\[
\langle W, X^{-1/2}h \rangle \in [c_1 \zeta^{-2} \pm x_{\min}^{-1}\Delta] \subseteq [\zeta^{-1} \pm (1 + \beta^{1/2})x_{\min}^{-1}\Delta]. 
\] (B.58)
where $(i)$ uses $c_1 > 0$ and the bound at Eq. (B.53). Next, the definition of $M$ implies
\[
M \geq \psi^{1/2} \||\Pi_{W^\perp}(G)\| + |\partial_1 F_\kappa(\zeta, 0)| + |\partial_2 F_\kappa(\zeta, 0)|, 
\]
so we know from Eq. (B.54) and the triangle inequality that
\[
\|\psi^{1/2} \Pi_{W^\perp}(G) + \partial_1 F_\kappa(\zeta, 0)W + \partial_2 F_\kappa(\zeta, 0)\Pi_{W^\perp}(Z)\| \leq M. 
\] (B.59)
Lastly, we have the estimates that holds for all $h$ satisfying $\|h\| \leq 1$:
\[
\|\langle W, X^{-1/2}h \rangle\| \leq \|X^{-1/2}W\| \|h\| \leq x_{\min}^{-1/2}, \quad |\langle W, R_+ \rangle| \leq \gamma_+(\Delta), \quad \|X^{-1/2}h\| \leq x_{\min}^{-1/2} 
\] (B.60)
Thus, with Eq. (B.57), and the estimates in Eq. (B.58), Eq. (B.59) and Eq. (B.60), we get
\[
\zeta^{-1} \left( \psi^{1/2} \Pi_{W^\perp}(G) + \partial_1 F_\kappa(\zeta, 0)W + \partial_2 F_\kappa(\zeta, 0)\Pi_{W^\perp}(Z) \right) - \partial_1 F_\kappa(\zeta, 0)X^{-1/2}h = \hat{R}_+ 
\] (B.61)
for some function $\hat{R}_+ = \hat{R}_+(g, x, w)$ satisfying $\|\hat{R}_+\| \leq x_{\min}^{-1/2} \left( 2\gamma_+(\Delta) + MX_{\min}^{-1/2}(1 + \beta^{1/2})\Delta \right)$. Finally, we can use Eq. (B.49), Eq. (B.51), Eq. (B.53) to get the estimate
\[
\|h - \zeta X^{-1/2}W\| \leq |c_1 - \zeta| \|X^{-1/2}W\| + \|X^{-1/2}\delta_h\| \leq x_{\min}^{-1/2}(\beta^{1/2} + 1)\Delta. 
\]
We can substitute the above bound in Eq. (B.61) and get that
\[
\left( \psi^{1/2} \Pi_{W^\perp}(G) + (\partial_1 F_\kappa(\zeta, 0)W + \partial_2 F_\kappa(\zeta, 0)\Pi_{W^\perp}(Z)) \right) - \zeta^2 \partial_1 F_\kappa(\zeta, 0)X^{-1}W = \hat{R}_+, 
\] (B.62)
where the remainder term $\hat{R}_+ = \hat{R}_+(g, X, W)$, by triangle inequality, satisfies
\[
\|\hat{R}_+\| \leq \zeta \|\hat{R}_+\| + \zeta |\partial_1 F_\kappa(\zeta, 0)| x_{\min}^{-1/2}(\beta^{1/2} + 1)\Delta \leq \hat{\gamma}_+(\Delta), 
\] (B.63)
(we recall the definition of $\hat{\gamma}_+(\Delta)$ at Eq. (B.38)). By algebraic manipulation of Eq. (B.62), we get
\[
-\partial_2 F_\kappa(\zeta, 0)\Pi_{W^\perp}(Z) = \partial_1 F_\kappa(\zeta, 0)(1 - \zeta^2 X^{-1})W + \psi^{1/2} \Pi_{W^\perp}(G) - \hat{R}_+. 
\] (B.64)
Now that $\|\Pi_{W^\perp}(Z)\| \leq \|Z\| \leq 1$. By taking norm on both sides of Eq. (B.64), we get
\[
\partial_2 F_\kappa(\zeta, 0) \geq \|\partial_1 F_\kappa(\zeta, 0)(1 - \zeta^2 X^{-1})W + \psi^{1/2} \Pi_{W^\perp}(G)\| - \hat{\gamma}_+(\Delta). 
\] (B.65)
Moreover, since \( s > 0 \) by assumption and also we have \( \langle W, X^{-1/2} h \rangle > 0 \) by Eq. (B.58), we must have \( \partial_1 F_\kappa(\zeta, 0) + \langle W, R_+ \rangle < 0 \) due to Eq. (B.56). Triangle inequality gives

\[
\partial_1 F_\kappa(\zeta, 0) < \|R_+\| \leq \gamma_+(\Delta).
\] (B.66)

Summarizing, we see that the case where \( s > 0 \), \( \|\pi_{W^\perp}(X^{1/2}h)\| \leq \Delta \) can happen, only if

\[
\partial_1 F_\kappa(\zeta, 0) < \gamma_+(\Delta) \quad \text{and} \quad \partial_2 F_\kappa(\zeta, 0) \geq \|\partial_1 F_\kappa(\zeta, 0)(1 - \zeta^2 X^{-1})W + \psi^{1/2} \pi_{W^\perp}(G)\| - \bar{\gamma}_+(\Delta).
\]

which contradicts the assumed condition on \( \psi, \zeta, \Delta \).

2. \( c < 0 \). Similar to the previous case, one can show that, this can happen only if

\[
\partial_1 F_\kappa(-\zeta, 0) > -\gamma_-(\Delta) \quad \text{and} \quad \partial_2 F_\kappa(-\zeta, 0) \geq \|\partial_1 F_\kappa(-\zeta, 0)(1 - \zeta^2 X^{-1})W + \psi^{1/2} \pi_{W^\perp}(G)\| - \bar{\gamma}_-(\Delta).
\]

which contradicts the assumed condition on \( \psi, \zeta, \Delta \).

Summarizing the above discussion, we have shown the desired result in Eq. (B.42).

Using the fact that \( s > 0 \) and \( \|\pi_{W^\perp}(X^{1/2}h)\| > 0 \), we can simplify the KKT condition (B.34). Denote

\[
c_1 = \langle h, X^{1/2}W \rangle \quad \text{and} \quad c_2 = \|\pi_{W^\perp}(X^{1/2}h)\|.
\] (B.67)

The KKT condition (i.e., Eq. (B.34)) can be equivalently written as:

\[
X^{1/2} \pi_{W^\perp}(G) + \psi^{-1/2} X^{1/2} (\partial_1 F_\kappa(c_1, c_2) W + \partial_2 F_\kappa(c_1, c_2) \pi_{W^\perp}(Z)) + sh = 0
\]

\[
\|h\| = 1, \quad Z = c_2^{-1} \cdot \pi_{W^\perp}(X^{1/2}h)
\]

Observe that

\[
\pi_{W^\perp}(Z) = c_2^{-1} \pi_{W^\perp}(X^{1/2}h) = c_2^{-1} \left( X^{1/2}h - (X^{1/2}h, W)W \right) = c_2^{-1} \left( X^{1/2}h - c_1 W \right).
\]

The KKT condition can be equivalently represented as

\[
\pi_{W^\perp}(G) + \psi^{-1/2} \left( \partial_1 F_\kappa(c_1, c_2) W + c_2^{-1} \partial_2 F_\kappa(c_1, c_2)(X^{1/2}h - c_1 W) \right) + sX^{-1/2}h = 0 \quad \text{and} \quad \|h\| = 1. \] (B.68)

The first equation imply

\[
h = -\frac{\psi^{1/2} \pi_{W^\perp}(G) + (\partial_1 F_\kappa(c_1, c_2) - c_1 c_2^{-1} \partial_2 F_\kappa(c_1, c_2)) W}{c_2^{-1} \partial_2 F_\kappa(c_1, c_2) X^{1/2} + \psi^{1/2} sX^{-1/2}}
\] (B.69)

Now, we plug in the above expression of \( h \) into the three equations below (we note the three equations all follow easily from Eq. (B.67) and Eq. (B.68)):

\[
c_1 = E_p[WX^{1/2}h], \quad c_1^2 + c_2^2 = E_p[Xh^2] \quad \text{and} \quad 1 = E_p[h^2]
\]

we get the expressions below:

\[
-c_1 = E_p \left[ \frac{(\psi^{1/2} \pi_{W^\perp}(G) + (\partial_1 F_\kappa(c_1, c_2) - c_1 c_2^{-1} \partial_2 F_\kappa(c_1, c_2)) W)W X^{1/2}}{c_2^{-1} \partial_2 F_\kappa(c_1, c_2) X^{1/2} + \psi^{1/2} sX^{-1/2}} \right]
\]

\[
c_1^2 + c_2^2 = E_p \left[ \frac{(\psi^{1/2} \pi_{W^\perp}(G) + (\partial_1 F_\kappa(c_1, c_2) - c_1 c_2^{-1} \partial_2 F_\kappa(c_1, c_2)) W)^2 X}{(c_2^{-1} \partial_2 F_\kappa(c_1, c_2) X^{1/2} + \psi^{1/2} sX^{-1/2})^2} \right]
\]

\[
1 = E_p \left[ \frac{(\psi^{1/2} \pi_{W^\perp}(G) + (\partial_1 F_\kappa(c_1, c_2) - c_1 c_2^{-1} \partial_2 F_\kappa(c_1, c_2)) W)^2}{(c_2^{-1} \partial_2 F_\kappa(c_1, c_2) X^{1/2} + \psi^{1/2} sX^{-1/2})^2} \right]
\] (B.70)

Recall that the minimum \( h^* \) is unique. Thus the value \( c_1 = \langle h, X^{1/2}W \rangle \), \( c_2 = \|\pi_{W^\perp}(X^{1/2}h)\| \) and hence the value \( s \) that satisfy the KKT condition Eq. (B.34) is unique. Since the KKT condition, i.e., Eq. (B.34) is
equivalent to Eq. (B.70), this implies the existence and uniqueness of \((c_1, c_2, s)\) that satisfy the Eq. (B.70). Moreover, we have that solution \((c_1, c_2)\) satisfies
\[
|c_1| = |\langle h, X^{1/2}W \rangle| \leq \|X^{1/2}h\| \leq x_{\text{max}}^{1/2} \quad \text{and} \quad c_2 = \|\Pi_{W^\perp}(X^{1/2}h)\| \leq \|X^{1/2}h\| \leq x_{\text{max}}^{1/2}.
\]

Now, by taking inner products with \(X^{1/2}h\) on both sides of the first equation of Eq. (B.68), we get
\[
\langle \Pi_{W^\perp}(G), X^{1/2}h \rangle + \psi^{-1/2}(c_1 \partial_1 F_\kappa(c_1, c_2) + c_2 F_\kappa(c_1, c_2)) + s = 0,
\]  
where we use the fact that
\[
\langle X^{1/2}h, (X^{1/2}h - c_1 W) \rangle = \|\Pi_{W^\perp}(X^{1/2}h)\|^2 \quad \text{and} \quad \langle X^{-1/2}h, X^{1/2}h \rangle = \|h\|^2 = 1.
\]
Eq. (B.71) now gives that \(\mathcal{R}_{\psi, \kappa}^*\) can be characterized by
\[
\mathcal{R}_{\psi, \kappa}^* = \psi^{-1/2}(F_\kappa(c_1, c_2) - c_1 \partial_1 F_\kappa(c_1, c_2) - c_2 \partial_2 F_\kappa(c_1, c_2)) - s,
\]  
where again, \((c_1, c_2, s)\) on the right-hand side above is the unique solution of Eq. (B.70).

### B.4 Consequences for \(\mathbb{P} = \mathbb{Q}_n\) and \(\mathbb{P} = \mathbb{Q}_\infty\)

The technical lemmas in Section B.3 can be directly applied to \(\mathbb{P} = \mathbb{Q}_n\) and \(\mathbb{P} = \mathbb{Q}_\infty\), yielding some important consequences. Here \(\psi^i(\kappa)\) is defined as in the statement of Proposition 4.1, see Eq. (4.5).

**Corollary B.5.** If \(\psi \leq \psi^i(\kappa)\), then, almost surely,
\[
\mathcal{R}_{\psi, \kappa, \mathbb{Q}_n}^* > 0 \quad \text{and} \quad \liminf_n \mathcal{R}_{\psi, \kappa, \mathbb{Q}_n}^* > 0.
\]

**Proof** Define the quantities
\[
\zeta_n = \zeta(\mathbb{Q}_n) = \|X^{-1/2}W\|_{\mathbb{Q}_n}^{-1} \quad \text{and} \quad \zeta_\infty = \zeta(\mathbb{Q}_\infty) = \|X^{-1/2}W\|_{\mathbb{Q}_\infty}^{-1}.
\]

Then, by definition of \(\psi^i(\kappa)\), one the following conditions hold:

1. \(\psi^i(0)^{1/2} \geq \psi^{1/2} \geq \psi^{1/2} \cdot \|\Pi_{W^\perp}(G)\|_{\mathbb{Q}_\infty}\)
2. \(\partial_1 F_\kappa(\zeta_\infty, 0) \leq 0 \quad \text{and} \quad \partial_2 F_\kappa(\zeta_\infty, 0) \geq \|\partial_1 F_\kappa(\zeta_\infty, 0)(1 - \zeta_\infty^2 X^{-1})W + \psi^{1/2} \Pi_{W^\perp}(G)\|_{\mathbb{Q}_\infty}\)
3. \(\partial_1 F_\kappa(-\zeta_\infty, 0) \geq 0 \quad \text{and} \quad \partial_2 F_\kappa(-\zeta_\infty, 0) \geq \|\partial_1 F_\kappa(-\zeta_\infty, 0)(1 - \zeta_\infty^2 X^{-1})W + \psi^{1/2} \Pi_{W^\perp}(G)\|_{\mathbb{Q}_\infty}\)

Therefore, the assumptions of Lemma B.3 are satisfied for \(\mathbb{P} = \mathbb{Q}_\infty\). Further recall that, by Eq. (5.11),
\[
W_2(\mathbb{Q}_n, \mathbb{Q}_\infty) \to 0,
\]
and therefore
\[
\|\Pi_{W^\perp}(G)\|_{\mathbb{Q}_n} \to \|\Pi_{W^\perp}(G)\|_{\mathbb{Q}_\infty}, \quad \zeta_n = \zeta(\mathbb{Q}_n) \to \zeta(\mathbb{Q}_\infty) = \zeta_\infty, \quad \|\partial_1 F_\kappa(\zeta_n, 0)(1 - \zeta_n^2 X^{-1})W + \psi^{1/2} \Pi_{W^\perp}(G)\|_{\mathbb{Q}_n} \to \|\partial_1 F_\kappa(\zeta_\infty, 0)(1 - \zeta_\infty^2 X^{-1})W + \psi^{1/2} \Pi_{W^\perp}(G)\|_{\mathbb{Q}_\infty}.
\]

Therefore, the three conditions stated in Lemma B.3 are also satisfied when \(\mathbb{P} = \mathbb{Q}_n\) for sufficiently large \(n\). The result follows by applying the Lemma B.3.

**Corollary B.6.** If \(\psi > \psi^i(\kappa)\), then, almost surely:
(a) For both $\mathbb{P} = \mathbb{Q}_\infty$ and $\mathbb{P} = \mathbb{Q}_n$ (and $n$ sufficiently large), the system of equations (B.70) has unique solutions $(c_1, \psi, \kappa)(\mathbb{P}), c_{2, \psi, \kappa}(\mathbb{P}), s_{\psi, \kappa}(\mathbb{P})) \in \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. 

(b) For $\mathbb{P} = \mathbb{Q}_\infty$, the system of equations (B.70) is equivalent to the system (4.6), and therefore we can identify 

$$c_{1, \psi, \kappa}(\mathbb{Q}_\infty) = c_1(\psi, \kappa), \quad c_{2, \psi, \kappa}(\mathbb{Q}_\infty) = c_2(\psi, \kappa), \quad s_{\psi, \kappa}(\mathbb{Q}_\infty) = s(\psi, \kappa),$$

where $(c_1(\psi, \kappa), c_2(\psi, \kappa), s(\psi, \kappa))$ is the unique solution of Eq. (4.6) in $\mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. 

(c) For all $n$ sufficiently large, the minimizer $\hat{\theta}_{n, \psi, \kappa}^{(2)}$ of the problem (5.8) satisfies 

$$\left(\hat{\theta}_{n, \psi, \kappa}^{(2)}, \Lambda_n^{1/2}w\right) = c_{1, \psi, \kappa}(\mathbb{Q}_n) \quad \text{and} \quad \left\|\Pi_{W}^{\perp} \Lambda_n^{1/2} \hat{\theta}_{n, \psi, \kappa}^{(2)}\right\| = c_{2, \psi, \kappa}(\mathbb{Q}_n).$$

(d) The following representation holds for both $\mathbb{P} = \mathbb{Q}_\infty$ and $\mathbb{P} = \mathbb{Q}_n$ (and $n$ sufficiently large) 

$$\mathcal{R}_{\psi, \kappa, \mathbb{P}} = \psi^{-1/2} \left( F_{\kappa}(c_{1, \psi, \kappa}(\mathbb{P}), c_{2, \psi, \kappa}(\mathbb{P})), c_{2, \psi, \kappa}(\mathbb{P}) - \left( c_{2, \psi, \kappa}(\mathbb{P}) \cdot \delta_{1} F_{\kappa}(c_{1, \psi, \kappa}(\mathbb{P}), c_{2, \psi, \kappa}(\mathbb{P})), c_{2, \psi, \kappa}(\mathbb{P})) \right) - s_{\psi, \kappa}(\mathbb{P}). \right. \quad \text{(B.77)}$$

(e) The following bounds hold for both $\mathbb{P} = \mathbb{Q}_\infty$ and $\mathbb{P} = \mathbb{Q}_n$ (and $n$ sufficiently large) 

$$|c_{1, \psi, \kappa}(\mathbb{P})| \leq x_{\max}^{1/2} \text{ and } \Delta_{\psi, \kappa}(\mathbb{Q}_\infty) \leq c_{2, \psi, \kappa}(\mathbb{P}) \leq x_{\max}^{1/2}, \quad \text{(B.78)}$$

$$s_{\psi, \kappa}(\mathbb{Q}_n) \leq \Delta_{\psi, \kappa}(\mathbb{Q}_\infty)^{-1}(1 + \psi^{-1} C^2)^{1/2} x_{\max}, \quad \text{(B.79)}$$

where $\Delta_{\psi, \kappa}(\mathbb{Q}_\infty) > 0$ is jointly continuous with respect to $\psi, \kappa$ and with respect to $\mathbb{Q}_\infty$ (in the $W_2$ topology).

**Proof** By definition of $\psi^+(\kappa)$, all the following conditions are satisfied (with $\zeta_\infty$ given by Eq. (B.74))

1. $\psi^+(0)^{1/2} < \psi^{1/2} \cdot \|\Pi_{W}^{\perp}(G)\|_{\mathbb{Q}_\infty}$

2. Either $\partial_{1} F_{\kappa}(\zeta_\infty, 0) > 0$ or $\partial_{2} F_{\kappa}(\zeta_\infty, 0) < \|\partial_{1} F_{\kappa}(\zeta_\infty, 0)(1 - \zeta_\infty^2 X^{-1})W + \psi^{1/2} \Pi_{W}^{\perp}(G)\|_{\mathbb{Q}_\infty}$. 

3. Either $\partial_{1} F_{\kappa}(\zeta_\infty, 0) < 0$ or $\partial_{2} F_{\kappa}(\zeta_\infty, 0) < \|\partial_{1} F_{\kappa}(\zeta_\infty, 0)(1 - \zeta_\infty^2 X^{-1})W + \psi^{1/2} \Pi_{W}^{\perp}(G)\|_{\mathbb{Q}_\infty}$.

Hence, the assumptions of Lemma B.4 are satisfied for $\mathbb{P} = \mathbb{Q}_\infty$. Since, by Eq. (5.11), $W(\mathbb{Q}_n, \mathbb{Q}_\infty) \to 0$, the assumptions of Lemma B.4 are also satisfied for $\mathbb{P} = \mathbb{Q}_n$ for all sufficiently large $n$.

Then the claims (a)-(d) immediately follow by applying Lemma B.4.

Equation B.78 follows immediately from Lemma B.4(d) for $\mathbb{P} = \mathbb{Q}_\infty$. For $\mathbb{P} = \mathbb{Q}_n$ notice that, by the continuity of $\mathbb{P} \mapsto \Delta_{\psi, \kappa}^{(2)}(\mathbb{P})$ with respect to the $W_2$ topology, we have $\Delta_{\psi, \kappa}(\mathbb{Q}_n) \geq \Delta_{\psi, \kappa}(\mathbb{Q}_\infty)/2$ for all $n$ large enough, whence the claim follows by eventually redefining $\Delta_{\psi, \kappa}(\mathbb{Q}_\infty)$.

In order to prove the bound (B.79) let us define the functions $V(c_1, c_2, s)$ and $V^\perp(c_1, c_2, s)$ by

$$V(c_1, c_2, s) := \left[ \psi^{1/2} \Pi_{W}^{\perp}(G) + \left( \partial_{1} F_{\kappa}(c_1, c_2) - c_1 c_2^{-1} \partial_{2} F_{\kappa}(c_1, c_2) \right)W \right]^2 - 1,$$

$$V^\perp(c_1, c_2, s) := \psi^{-1} s^{-2} X \left[ \psi^{1/2} \Pi_{W}^{\perp}(G) + \left( \partial_{1} F_{\kappa}(c_1, c_2) - c_1 c_2^{-1} \partial_{2} F_{\kappa}(c_1, c_2) \right)W \right]^2 - 1.$$

Recall that $(c_1, c_2, s) = (c_{1, \psi, \kappa}(\mathbb{P}), c_{2, \psi, \kappa}(\mathbb{P}), s_{\psi, \kappa}(\mathbb{P}))$ satisfies the system of equations (4.6) for either $\mathbb{P} = \mathbb{Q}_\infty$ or $\mathbb{P} = \mathbb{Q}_n$ (and $n$ is sufficiently large), whence

$$E_{\mathbb{P}} \left[ V(c_1(\mathbb{P}), c_2(\mathbb{P}), s(\mathbb{P})) \right] = 0. \quad \text{(B.80)}$$

Lemma 5.3 implies $\partial_{2} F_{\kappa}(c_1, c_2) \geq 0$ for all $(c_1, c_2) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$. Thus $V(c_1, c_2, s) \leq V^\perp(c_1, c_2, s)$ for all $(c_1, c_2, s) \in \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Thus, for either $\mathbb{P} = \mathbb{Q}_\infty$ or $\mathbb{P} = \mathbb{Q}_n$ (and $n$ is sufficiently large)

$$E_{\mathbb{P}} \left[ V^\perp(c_1(\mathbb{P}), c_2(\mathbb{P}), s(\mathbb{P})) \right] \geq 0,$$
which by an algebraic manipulation is equivalent to (writing for simplicity $c_1 = c_1(P)$, $c_2 = c_2(P)$, $s = s(P)$)

\[(sc_2)^2 \leq \mathbb{E}_P \left\{ X \left[ c_2 \Pi_{W^\perp}^*(G) + \psi^{-1/2} (c_2 \cdot \partial_1 F_\kappa(c_1, c_2) - c_1 \cdot \partial_2 F_\kappa(c_1, c_2)) W \right] \right\}^2 \]  

(B.81)

Note that we have $X \leq x_{\text{max}}$ by assumption, and $|c_1(P)| \leq x_{\text{max}}^{1/2}$ and $|c_2(P)| \leq x_{\text{max}}^{1/2}$ by Eq. (B.78). Therefore

\[(sc_2)^2 \leq x_{\text{max}}^2 \mathbb{E}_P \left[ \left( |\Pi_{W^\perp}^*(G)| + \psi^{-1/2} (|\partial_1 F_\kappa(c_1, c_2)| + |\partial_2 F_\kappa(c_1, c_2)|) W \right)^2 \right] \]

\[= x_{\text{max}}^2 \left[ \mathbb{E}_P [\Pi_{W^\perp}^*(G)]^2 + \psi^{-1} (|\partial_1 F_\kappa(c_1, c_2)| + |\partial_2 F_\kappa(c_1, c_2)|)^2 \right]. \]

where we use the fact that $\mathbb{E}_P [W^2] = 1$ for both $P = Q_\infty$ and $P = Q_n$. The claimed bound in Eq. (B.79) follows (eventually redefining $\Delta_\psi, \kappa(P)$) by setting $C = \max_{|c_1| \leq x_{\text{max}}^{1/2}, c_2 \in [0, x_{\text{max}}^{1/2}]} |\partial_1 F_\kappa(c_1, c_2)|$, and using the fact that $W_2(Q_n, Q_\infty) \to 0$ by Eq. (5.11), which implies

\[\lim_{n \to \infty} \mathbb{E}_{Q_n} [(\Pi_{W^\perp}^*(G))^2] = \mathbb{E}_{Q_\infty} [(\Pi_{W^\perp}^*(G))^2] = 1. \quad \text{(B.82)}\]

\[\square\]

### B.5 Proof of Proposition 4.1

Point (a) is follows immediately from Corollary B.6(a). Indeed, the system of equations (B.70) coincides with the system (4.6) for $P = Q_\infty$. By Corollary B.6(d), we also have for $\psi > \psi^i(\kappa)$,

\[T(\psi, \kappa) = R^*_{\psi, \kappa, Q_\infty}. \quad \text{(B.83)}\]

First of all, we claim that $(\psi, \kappa) \to T(\psi, \kappa)$ is continuous and strictly increasing with respect to $\kappa$, and strictly decreasing with respect to $\psi$.

In order to prove this claim, recall that, by Eq. (B.4),

\[R^*_{\psi, \kappa, P} = \min \left\{ R_{\psi, \kappa, P}(h) \mid \|h\|_P \leq 1 \right\}, \quad \text{(B.84)}\]

\[R_{\psi, \kappa, P}(h) = \psi^{-1/2} \cdot F_\kappa (\langle h, X^{1/2} W_P \rangle_P, \|\Pi_{W^\perp} (X^{1/2} h)\|_P) + \langle h, X^{1/2} \Pi_{W^\perp}^*(G) \rangle_P. \quad \text{(B.85)}\]

Notice that: (i) for any $c_1, c_2 \in \mathbb{R}$, $\kappa \to F_\kappa(c_1, c_2)$ is strictly increasing and strictly increasing; (ii) as a consequence, for any fixed $h$, $R_{\psi, \kappa, P}(h)$ is strictly increasing with respect to $\kappa$ and decreasing with respect to $\psi$; (iii) the minimum $R^*_{\psi, \kappa, Q_\infty}$ in the above optimization problem is achieved by some $h^*_{\psi, \kappa} \in L^2(Q_\infty)$ with $\|h^*_{\psi, \kappa}\|_{Q_\infty} \leq 1$ (see Lemma B.2). The claim that $(\psi, \kappa) \to T(\psi, \kappa)$ is continuous and strictly increasing with respect to $\kappa$ then follows by a standard argument. This proves point the continuity of $T$ in point (b.i) and the monotonicity properties in points (b.ii) and (b.iii).

Next, we claim that

\[\lim_{\psi \to \infty} T(\psi, \kappa) = \lim_{\psi \to \infty} R^*_{\psi, \kappa, Q_\infty} < 0. \]

Indeed, let $\bar{h} = X^{-1/2} G / \|X^{-1/2} G\|_{Q_\infty}$. Then $\|\bar{h}\|_{Q_\infty} \leq 1$ and $\langle \bar{h}, X^{1/2} \Pi_{W^\perp}^*(G) \rangle_{Q_\infty} = 1 / \|X^{-1/2} G\|_{Q_\infty} > 0$. By definition, $R^*_{\psi, \kappa, Q_\infty} \leq L_{\psi, \kappa, Q_\infty}(\bar{h})$. This implies

\[R^*_{\psi, \kappa, Q_\infty} \leq \psi^{-1/2} \cdot \max_{|c_1| \leq x_{\text{max}}^{1/2}, c_2 \in [0, x_{\text{max}}^{1/2}]} F_\kappa(c_1, c_2) - 1 / \|X^{-1/2} G\|_{Q_\infty}. \]

Now the desired result follows by taking $\psi \to \infty$. This proves the first bound in point (b.ii). The second bound in point (b.ii) follows because

\[\lim_{\psi \to \psi^i(\kappa)} T(\psi, \kappa) \overset{(i)}{=} \lim_{\psi \to \psi^i(\kappa)} R^*_{\psi, \kappa, Q_\infty} \geq R^*_{\psi^i(\kappa), \kappa, Q_\infty} \overset{(ii)}{> 0}. \]

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where (i) is due to Eq (B.83) and (ii) holds because of Corollary B.5.

Third, we show that
\[
\lim_{\kappa \to \infty} T(\psi, \kappa) = \lim_{\kappa \to \infty} R_{\kappa, Q, \infty}^* = \infty.
\]
This is due to the following bound on \( R_{\kappa, Q, \infty}^* \):
\[
R_{\kappa, Q, \infty}^* \geq \psi^{-1/2} \min_{|c_1| \leq x_{\min}/x_{\max}^2, c_2 \in [0, x_{\max}^2]} F_\kappa(c_1, c_2) - x_{\max}^{1/2},
\]
and the fact that \( \lim_{\kappa \to \infty} \min_{|c_1| \leq x_{\min}/x_{\max}^2, c_2 \in [0, x_{\max}^2]} F_\kappa(c_1, c_2) = \infty \). This concludes the proof of point (b.iii).

Last, we show that \( c_1(\cdot, \cdot), c_2(\cdot, \cdot), s(\cdot, \cdot) \) are continuous function on the domain \( \{(\psi, \kappa) : \psi > \psi^*(\kappa)\} \). Pick any point \((\psi_0, \kappa_0)\) such that \( \psi_0 > \psi^*(\kappa_0) \). Let \( \{\psi_l\}_{l \in \mathbb{N}} \) and \( \{\kappa_l\}_{l \in \mathbb{N}} \) be two sequences such that \( \psi_l \to \psi_0 \) and \( \kappa_l \to \kappa_0 \). It suffices to show that
\[
\lim_{l \to \infty} (c_1(\psi_l, \kappa_l), c_2(\psi_l, \kappa_l), s(\psi_l, \kappa_l)) = (c_1(\psi_0, \kappa_0), c_2(\psi_0, \kappa_0), s(\psi_0, \kappa_0)). \tag{B.86}
\]
Corollary B.6 implies that for all \( l \in \mathbb{N} \) and \( l = 0, \)
\begin{align*}
    c_1(\psi_l, \kappa_l) &= c_1(\psi_l, \kappa_l)(Q_\infty) \in [-x_{\max}^{1/2}, x_{\max}^{1/2}], \\
    c_2(\psi_l, \kappa_l) &= c_2(\psi_l, \kappa_l)(Q_\infty) \in [\Delta_{\psi_l}(Q_\infty), x_{\max}^{1/2}], \\
    s(\psi_l, \kappa_l) &= s(\psi_l, \kappa_l)(Q_\infty) \in [0, \Delta_{\psi_l}(Q_\infty)]^{-1}(1 + \psi^{-1} C^2)^{1/2} x_{\max}^{1/2}. \tag{B.87}
\end{align*}
Now, by definition, \( \lim_{m \to \infty} \Delta_{\psi_l}(Q_\infty) = \Delta_{\psi, \kappa_0}(Q_\infty) > 0 \). Thus, we can pick \( M > m > 0 \) such that \( (c_1(\psi_l, \kappa_l), c_2(\psi_l, \kappa_l), s(\psi_l, \kappa_l)) \in \mathcal{S} := [-M, M] \times [m, M] \times [0, M] \) for all large enough \( l \). Now, we show that any limit point \((c_1(\psi_l, \kappa_l), c_2(\psi_l, \kappa_l), s(\psi_l, \kappa_l))\) must be \((c_1(\psi_0, \kappa_0), c_2(\psi_0, \kappa_0), s(\psi_0, \kappa_0))\). To do this, first take any limit point of \((c_1(\psi_l, \kappa_l), c_2(\psi_l, \kappa_l), s(\psi_l, \kappa_l))\), and denote it by \((\bar{c}_1, \bar{c}_2, \bar{s})\). It is clear that \((\bar{c}_1, \bar{c}_2, \bar{s}) \in \mathcal{S}\), and moreover, \((\bar{c}_1, \bar{c}_2, \bar{s})\) must satisfy the system of equations (B.70) for \( \psi = \psi_0, \kappa = \kappa_0, \mathbb{P} = Q_\infty \). The next lemma (proved in the next section) now shows that \( \bar{s} \) must be non-negative.

**Lemma B.7.** Assume \( \psi > \psi^*(0) \). Suppose that \((c_1, c_2, s)\) satisfies the system of equations (B.70) (with \( \mathbb{P} = Q_\infty \)) for some \( c_2 \neq 0 \). Then we must have that \( s \neq 0 \).

We therefore know that \((\bar{c}_1, \bar{c}_2, \bar{s})\) satisfies both \( \bar{c}_2 > 0 \) and \( \bar{s} > 0 \) and the system of equations (B.70) corresponding to \( \psi = \psi_0, \kappa = \kappa_0, \mathbb{P} = Q_\infty \). Now since that solution is known to be unique, by Corollary B.6.(a), we conclude that \((\bar{c}_1, \bar{c}_2, \bar{s}) = (c_1(\psi_0, \kappa_0), c_2(\psi_0, \kappa_0), s(\psi_0, \kappa_0))\). This proves the convergence statement (B.86), and concludes the proof of point (b.i).

**B.5.1 Proof of Lemma B.7**

Assume by contradiction that \((c_1, c_2, 0)\) is a solution of Eq. (B.70) (with \( \mathbb{P} = Q_\infty \)) for some \( c_1 \) and some \( c_2 \neq 0 \). Denote \( h = h(G, X, W) \) to be the function
\[
h = \frac{c_2 \Pi_{W^1}(G) + \psi^{1/2}(c_2 \partial_1 F_\kappa(c_1, c_2) - c_1 \partial_2 F_\kappa(c_1, c_2)) W}{\psi^{1/2} \partial_2 F_\kappa(c_1, c_2) X^{1/2}}. \tag{B.88}
\]
The condition that \((c_1, c_2, 0)\) is a solution of the system of equations (B.70) is then equivalent to
\[
E_{Q_\infty}[h W X^{1/2}] = -c_1, \quad E_{Q_\infty}[h^2 X] = c_1^2 + c_2^2, \quad E_{Q_\infty}[h^2] = 1. \tag{B.89}
\]
Since \( E_{Q_\infty}[W^2] = 1 \) and \( E_{Q_\infty}[\Pi_{W^1}(G) W] = 0 \), the first of these equations is equivalent to
\[
\psi^{1/2} c_2 \partial_1 F_\kappa(c_1, c_2) = 0,
\]
and since \( c_2 \neq 0 \), we conclude that
\[
\partial_1 F_\kappa(c_1, c_2) = 0. \tag{B.90}
\]
Using Eq. (B.90), and the fact that $W$ is independent of $G$ under $Q_\infty$, the second equation of (B.89) is equivalent to

$$c_2^2((\psi^{1/2} \partial_2 F_n(c_1, c_2))^2 - 1) = 0.$$ 

Again, since $c_2 \neq 0$ and $\partial_2 F_n(c_1, c_2) > 0$ by Lemma 5.3, we conclude that

$$\psi^{1/2} \partial_2 F_n(c_1, c_2) = 1. \quad (B.91)$$

Now Lemma B.1 implies that

$$(\psi/\psi^*(0))^{1/2} = \psi^{1/2} \cdot \min_{c \in \mathbb{R}} F_0(c, 1) \geq 1,$$ \hspace{1cm} (B.92)

which contradicts our assumed condition on $\psi$.

C Analysis of Gordon’s optimization problem: Proof of Proposition 5.4

Notice that $\xi_{n,\psi,\kappa}^{(2)} = \mathcal{R}_{\psi,\kappa,Q_\infty}^*$. Therefore, point (a) follows by Corollary B.5 and we can assume hereafter $\psi > \psi^+(\kappa)$.

We claim that the following holds for any $\psi > \psi^+(\kappa)$:

$$\lim_{n \to \infty, p, n \to \psi} (c_1, \psi, \kappa(Q_n), c_2, \psi, \kappa(Q_n)) = (c_1, \psi, \kappa(Q_\infty), c_2, \psi, \kappa(Q_\infty)) \quad (C.1)$$

Before proving this claim, let us show that it implies point (b):

- Equation (5.13) follows from the Corollary B.6.(d) and Eq. (C.1).
- The first limit in Eq. (5.14) follows from

$$\lim_{n \to \infty} \langle \xi_{n,\psi,\kappa}^{(2)}, \mathbf{A}^{1/2} w \rangle \overset{(a)}{\to} \lim_{n \to \infty} c_1, \psi, \kappa(Q_n) \overset{(b)}{=} c_1, \psi, \kappa(Q_\infty) = c_1(\psi, \kappa),$$

where (a) is a consequence of Corollary B.6.(c) and (b) follows from Eq. (C.1).
- Finally, the second limit in Eq. (5.14) follows from the same argument

$$\lim_{n \to \infty} \| \hat{\theta}_{n,\psi,\kappa}^{(2)} \|_{\mathbf{A}_n} = \left( c_1, \psi, \kappa(Q_\infty)^2 + c_2, \psi, \kappa(Q_\infty)^2 \right)^{1/2} = \left( c_1(\psi, \kappa)^2 + c_2(\psi, \kappa)^2 \right)^{1/2}.$$

We are now left with the task of proving Eq. (C.1). We fix $\psi, \kappa$ in the rest of the proof. For notational convenience, we drop $\psi, \kappa$ from the arguments in what follows. Define the functions

$$V_{1,n}(c_1, c_2, s) = \frac{(c_2\psi^{1/2} \Pi_{W^+ Q_n}(G) + (c_2 \partial_1 F_n(c_1, c_2) - c_1 \partial_2 F_n(c_1, c_2))) W X^{1/2}}{\partial_2 F_n(c_1, c_2) X^{1/2} + c_2 \psi^{1/2} X^{1/2}} + c_1,$$

$$V_{2,n}(c_1, c_2, s) = \frac{(c_2\psi^{1/2} \Pi_{W^+ Q_n}(G) + (c_2 \partial_1 F_n(c_1, c_2) - c_1 \partial_2 F_n(c_1, c_2))) W^2 X}{(\partial_2 F_n(c_1, c_2) X^{1/2} + c_2 \psi^{1/2} X^{1/2})^2} - (c_1^2 + c_2^2), \quad (C.2)$$

$$V_{3,n}(c_1, c_2, s) = \frac{(c_2\psi^{1/2} \Pi_{W^+ Q_n}(G) + (c_2 \partial_1 F_n(c_1, c_2) - c_1 \partial_2 F_n(c_1, c_2))) W^2}{(\partial_2 F_n(c_1, c_2) X^{1/2} + c_2 \psi^{1/2} X^{1/2})^2} - 1.$$ 

Notice that these functions depend on $n$ because $\Pi_{W^+ Q_n}$ does. We introduce the shorthands

$$(c_1, c_2, s) = (c_1(Q_n), c_2(Q_n), s(Q_n)) \quad \text{and} \quad (c_1, c_\infty, c_2, c_\infty, s) = (c_1(Q_\infty), c_2(Q_\infty), s(Q_\infty))$$

For sufficiently large $n$, $(c_1, c_2, s)$ is the solution of the system of equations

$$\mathbb{E}_{Q_n}[V_{1,n}(c_1, c_2, s)] = 0, \quad \mathbb{E}_{Q_n}[V_{2,n}(c_1, c_2, s)] = 0, \quad \mathbb{E}_{Q_n}[V_{3,n}(c_1, c_2, s)] = 0. \quad (C.3)$$

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Corollary B.6 implies the existence of $M > m > 0$ such that
\[
\limsup_{n \to \infty} |c_{1,n}| < M, \quad m < \liminf_{n \to \infty} c_{2,n} \leq \limsup_{n \to \infty} c_{2,n} < M, \quad \limsup_{n \to \infty} s_n < M. \tag{C.4}
\]
Define the compact set $\mathcal{S} := [-M, M] \times [m, M] \times [0, M]$. The next lemma establishes for each $i = 1, 2, 3$ the uniform convergence result of $E_{Q_n}(V_{i,n}(c_1, c_2, s))$ to $E_{Q_\infty}(V_{i,\infty}(c_1, c_2, s))$ on the compact set $\mathcal{S}$. To avoid interrupting the flow, we defer its proof to the next subsection.

**Lemma C.1.** For $i = 1, 2, 3$, we have almost surely
\[
\lim_{n \to \infty} \sup_{(c_1, c_2, s) \in \mathcal{S}} |E_{Q_n}(V_{i,n}(c_1, c_2, s)) - E_{Q_\infty}(V_{i,\infty}(c_1, c_2, s))| = 0 \tag{C.5}
\]

Now, we are ready to show the desired convergence result in Eq. (C.1). We prove that any limit point of $(c_{1,n}, c_{2,n}, s_n)$ must be $(c_{1,\infty}, c_{2,\infty}, s_\infty)$. To do this, first take any limit point of $(c_{1,n}, c_{2,n}, s_n)$, and denote it to be $(\tilde{c}_1, \tilde{c}_2, \tilde{s}^*)$. Since by definition we have for $i = 1, 2, 3$,
\[
E_{Q_n}[V_{i,n}(c_{1,n}, c_{2,n}, s_n)] = 0,
\]
the triangle inequality immediately implies that, for $i = 1, 2, 3$,
\[
|E_{Q_\infty}[V_{i,\infty}(\tilde{c}_1^*, \tilde{c}_2^*, \tilde{s}^*)]| \leq |E_{Q_\infty}[V_{i,\infty}(\tilde{c}_1^*, \tilde{c}_2^*, \tilde{s}^*) - E_{Q_\infty}[V_{i,\infty}(c_{1,n}, c_{2,n}, s_n)]| + |E_{Q_\infty}[V_{i,\infty}(c_{1,n}, c_{2,n}, s_n)] - E_{Q_n}[V_{i,n}(c_{1,n}, c_{2,n}, s_n)]|
\tag{C.6}
\]
for all $n \in \mathbb{N}$. Now, by definition of $\mathcal{S}$, $(c_{1,n}, c_{2,n}, s_n) \in \mathcal{S}$ for large enough $n$. Hence,
\[
|E_{Q_\infty}[V_{i,\infty}(\tilde{c}_1^*, \tilde{c}_2^*, \tilde{s}^*)]| \leq \limsup_n |E_{Q_\infty}[V_{i,\infty}(\tilde{c}_1^*, \tilde{c}_2^*, \tilde{s}^*) - E_{Q_\infty}[V_{i,\infty}(c_{1,n}, c_{2,n}, s_n)]| + \limsup_n \sup_{(c_1, c_2, s) \in \mathcal{S}} |E_{Q_\infty}[V_{i,\infty}(c_1, c_2, s) - E_{Q_n}[V_{i,n}(c_{1,n}, c_{2,n}, s_n)]| = 0,
\tag{C.7}
\]
where the last identity uses the fact that the mapping $(c_{1,n}, c_{2,n}, s_n) \to E_{Q_n}[V_{i,\infty}(c_{1,n}, c_{2,n}, s_n)]$ is continuous on $\mathcal{S}$, and the uniform convergence result by Lemma C.1. This shows that any limit point $(\tilde{c}_1^*, \tilde{c}_2^*, \tilde{s}^*) \in \mathcal{S}$ must satisfy the system of equations below
\[
E_{Q_\infty}[V_{1,\infty}(c_1, c_2, s)] = 0, E_{Q_\infty}[V_{2,\infty}(c_1, c_2, s)] = 0, E_{Q_\infty}[V_{3,\infty}(c_1, c_2, s)] = 0. \tag{C.8}
\]
Now we recall Lemma B.7. By Lemma B.7, we know $(\tilde{c}_1^*, \tilde{c}_2^*, \tilde{s}^*)$ must satisfy both $\tilde{c}_2^* > 0$ and $\tilde{s}^* > 0$ and the system of equations (C.8). Now since the solution of the system of equations (C.8) is unique in $\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ (by Corollary B.6), this shows that $(\tilde{c}_1^*, \tilde{c}_2^*, \tilde{s}^*) = (c_{1,\infty}, c_{2,\infty}, s_\infty)$. Thus, we proved that any limit point of \{(c_{1,n}, c_{2,n}, s_n)\}$_{n \in \mathbb{N}}$ must be $(c_{1,\infty}, c_{2,\infty}, s_\infty)$. This implies the desired convergence (C.1).

**C.1 Proof of Lemma C.1**

By Eq. (5.11), we know that $Q_n \xrightarrow{\text{w}} Q_\infty$ almost surely. Therefore, it is sufficient to prove that $Q_n \xrightarrow{\text{w}} Q_\infty$ implies
\[
\limsup_{n \to \infty} \sup_{(c_1, c_2, s) \in \mathcal{S}} |E_{Q_n}(V_{i,n}(c_1, c_2, s)) - E_{Q_\infty}(V_{i,\infty}(c_1, c_2, s))| = 0.
\]

Let us denote for $i \in \{1, 2, 3\}$
\[
\bar{V}_{i,n}(c_1, c_2, s) = E_{Q_n}[V_{i,n}(c_1, c_2, s)] \quad \text{and} \quad \bar{V}_{i,\infty}(c_1, c_2, s) = E_{Q_\infty}[V_{i,\infty}(c_1, c_2, s)].
\]

By Arzelà-Ascoli theorem, it suffices to show that
(a) For each $i \in \{1, 2, 3\}$, we have for any fixed $(c_1, c_2, s) \in \mathcal{S}$
\[
\lim_{n \to \infty} |\bar{V}_{i,n}(c_1, c_2, s) - \bar{V}_{i,\infty}(c_1, c_2, s)| = 0.
\]
(b) For each $i \in \{1, 2, 3\}$, the functions \{\bar{V}_{i,n}(c_1, c_2, s)\}$_{n \in \mathbb{N}}$ is equicontinuous on $\mathcal{S}$, i.e., for any $\varepsilon > 0$, there exists some $\delta > 0$, such that for any $(c_1, c_2, s), (c_1', c_2', s') \in \mathcal{S}$ satisfying $\|(c_1, c_2, s) - (c_1', c_2', s')\|_2 < \delta,
\sup_{n \in \mathbb{N}} |\bar{V}_{i,n}(c_1, c_2, s) - \bar{V}_{i,n}(c_1', c_2', s')| \leq \varepsilon.$

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Proof of Point (a) Recall that \( Q_n \xrightarrow{w} Q_\infty \) implies

\[
\lim_{n \to \infty} \| E_{Q_n}[f(G, X, W)] - E_{Q_\infty}[f(G, X, W)] \| = 0,
\]

for any continuous function \( f : \mathbb{R}^3 \to \mathbb{R} \) satisfying \( \sup_{(G, X, W) \in \mathbb{R}^3} \frac{f(G, X, W)}{\| (G, X, W) \|_2} < \infty. \)

We will use this fact to prove the statement of point (a). For notational simplicity, we introduce

\[
\begin{align*}
V_{1,n}(c_1, c_2, s) &= (c_2 \psi^{1/2} W_{1,n}(G) + (c_2 \partial_1 F_n(c_1, c_2) - c_1 \partial_2 F_n(c_1, c_2)) W) \cdot X^{1/2}, \\
V_{2,n}(c_1, c_2, s) &= (c_2 \psi^{1/2} W_{2,n}(G) + (c_2 \partial_1 F_n(c_1, c_2) - c_1 \partial_2 F_n(c_1, c_2)) W)^2 X, \\
V_{3,n}(c_1, c_2, s) &= (c_2 \psi^{1/2} W_{3,n}(G) + (c_2 \partial_1 F_n(c_1, c_2) - c_1 \partial_2 F_n(c_1, c_2)) W)^3 \\
V_1(c_1, c_2, s) &= \partial_1 F_n(c_1, c_2) X^{1/2} + c_2 \psi^{1/2} s X^{-1/2}, \\
V_2(c_1, c_2, s) &= (\partial_2 F_n(c_1, c_2) X^{1/2} + c_2 \psi^{1/2} s X^{-1/2})^2, \\
V_3(c_1, c_2, s) &= (\partial_2 F_n(c_1, c_2) X^{1/2} + c_2 \psi^{1/2} s X^{-1/2})^3.
\end{align*}
\]

Note that, by definition, we have

\[
\begin{align*}
V_{1,n}(c_1, c_2, s) &= V_{1,n}(c_1, c_2, s) + c_1 \equiv U_{1,n}(c_1, c_2, s) + c_1, \\
V_{2,n}(c_1, c_2, s) &= V_{2,n}(c_1, c_2, s) - (c_1^2 + c_2^2) \equiv U_{2,n}(c_1, c_2, s) - (c_1^2 + c_2^2), \\
V_{3,n}(c_1, c_2, s) &= V_{3,n}(c_1, c_2, s) - 1 \equiv U_{3,n}(c_1, c_2, s) - 1.
\end{align*}
\]

so we have for any \( i \in \{1, 2, 3\}, (c_1, c_2, s) \in S, \)

\[
\begin{align*}
\bar{V}_{i,n}(c_1, c_2, s) - \bar{V}_{i,\infty}(c_1, c_2, s) &= \mathbb{E}_{Q_n} \left[ \frac{V_{i,n}(c_1, c_2, s)}{V_i(c_1, c_2, s)} \right] - \mathbb{E}_{Q_\infty} \left[ \frac{V_{i,\infty}(c_1, c_2, s)}{V_i(c_1, c_2, s)} \right] \\
&= \left( \mathbb{E}_{Q_n} \left[ \frac{V_{i,n}(c_1, c_2, s)}{V_i(c_1, c_2, s)} \right] - \mathbb{E}_{Q_\infty} \left[ \frac{V_{i,\infty}(c_1, c_2, s)}{V_i(c_1, c_2, s)} \right] \right) + \mathbb{E}_{Q_n} \left[ \frac{(V_{i,n}(c_1, c_2, s) - V_{i,\infty}(c_1, c_2, s))}{V_i(c_1, c_2, s)} \right].
\end{align*}
\]

To prove point (a), it is sufficient to prove for any \( (c_1, c_2, s) \in S: \)

\[
\lim_{n \to \infty} \text{Err}_{n,1,i}(c_1, c_2, s) = 0 \quad \text{and} \quad \lim_{n \to \infty} \text{Err}_{n,2,i}(c_1, c_2, s) = 0.
\]

We begin with the first limit in Eq. (C.13). The idea is to apply the convergence statement in Eq (C.9). We claim that

\[
\text{ess sup}_{(G, X, W) \in \mathbb{R}^3} \frac{V_{i,\infty}(c_1, c_2, s)}{V_i(c_1, c_2, s)} \|(G, X, W)\|_2 < \infty
\]

(where the essential sup holds both under \( Q_n \) and under \( Q_\infty \)). This follows by the below two observations.

- There exists \( C_{\text{up}} = C_{\text{up}}(S, \psi, \kappa) > 0 \) such that for any \( i \in \{1, 2, 3\}, \)

\[
\text{sup}_{(c_1, c_2, s) \in S (G, X, W) \in \mathbb{R}^3} \frac{V_{i,\infty}(c_1, c_2, s)}{V_i(c_1, c_2, s)} \|(G, X, W)\|_2 \leq C_{\text{up}}.
\]
For convenience of the reader, we write explicitly $V_{i,\infty}^\uparrow(c_1, c_2, s)$:

\[
\begin{align*}
V_{1,\infty}^\uparrow(c_1, c_2, s) &= (c_2 \psi^{1/2} G + (c_2 \partial_1 F_\kappa(c_1, c_2) - c_1 \partial_2 F_\kappa(c_1, c_2)) W)WX^{1/2}, \\
V_{2,\infty}^\uparrow(c_1, c_2, s) &= (c_2 \psi^{1/2} G + (c_2 \partial_1 F_\kappa(c_1, c_2) - c_1 \partial_2 F_\kappa(c_1, c_2)) W^2 X, \\
V_{3,\infty}^\uparrow(c_1, c_2, s) &= (c_2 \psi^{1/2} G + (c_2 \partial_1 F_\kappa(c_1, c_2) - c_1 \partial_2 F_\kappa(c_1, c_2)) W^2).
\end{align*}
\]

Equation (C.15) holds because (i) $\partial_1 F_\kappa(c_1, c_2)$ and $\partial_2 F_\kappa(c_1, c_2)$ are continuous functions of $(c_1, c_2)$ (by Lemma 5.3) and thus are uniformly bounded on the compact set $S$ (ii) by Assumption 1, there exists a constant $C < \infty$ such that, almost surely, $X < C$. Therefore, there exists a quadratic polynomial of $(G, W)$ such that $|V_{i,\infty}^\uparrow(c_1, c_2, s)|$ is bounded by that quadratic polynomial for any $(c_1, c_2, s)$.

- There exists $c_{\text{low}} = c_{\text{low}}(S, \psi, \kappa) > 0$ such that for any $i \in \{1, 2, 3\}$,

\[
\min_{(c_1, c_2, s) \in S} V_{i,\infty}^\uparrow(c_1, c_2, s) > c_{\text{low}}. \tag{C.17}
\]

This is because (i) we have the lower bound $V_{i,\infty}^\uparrow(c_1, c_2, s) > \partial_2 F_\kappa(c_1, c_2)X^{1/2}$ and $V_{i,\infty}^\uparrow(c_1, c_2, s) > (\partial_2 F_\kappa(c_1, c_2)X^{1/2})^2$ for $i \in \{2, 3\}$; (ii) for some $c > 0$ we have $X > c$ almost surely (by Assumption 1; (iii) for some constant $'c' > 0$, we have $\partial_2 F_\kappa(c_1, c_2) > c'$ for all $(c_1, c_2, s) \in S$ since $\partial_2 F_\kappa(c_1, c_2)$ is a positive continuous function (by Lemma 5.3) and $S$ is compact.

Next, we establish the second limit in Eq. (C.13). In light of Eq. (C.17), it is sufficient to prove for any $(c_1, c_2, s) \in S$,

\[
\lim_{n \to \infty} \mathbb{E}_{Q_n} \left[ V_{i,n}^\uparrow(c_1, c_2, s) - V_{i,\infty}^\uparrow(c_1, c_2, s) \right] = 0. \tag{C.18}
\]

For future use, let’s prove a strengthened version of Eq (C.18), i.e.

\[
\lim_{n \to \infty} \sup_{(c_1, c_2, s) \in S} \mathbb{E}_{Q_n} \left[ V_{i,n}^\uparrow(c_1, c_2, s) - V_{i,\infty}^\uparrow(c_1, c_2, s) \right] = 0. \tag{C.19}
\]

The difference between $V_{i,n}^\uparrow(c_1, c_2, s)$ and $V_{i,\infty}^\uparrow(c_1, c_2, s)$ is that the term $\Pi_{W^+, Q_n}(G)$ in $V_{i,n}^\uparrow(c_1, c_2, s)$ differs from the term $\Pi_{W^+, Q_\infty}(G) = G$ in $V_{i,\infty}^\uparrow(c_1, c_2, s)$. More concretely, let us define the differences

\[
\begin{align*}
\Delta_1(G) &= G - \Pi_{W^+, Q_n}(G) = a_nW \\
\Delta_2(G) &= G^2 - \Pi_{W^+, Q_n}(G) = 2a_nGW - a_n^2W^2,
\end{align*}
\]

where $a_n = \mathbb{E}_{Q_n}[GW]$ (recall that $\mathbb{E}_{Q_n}[W^2] = 1$). Since $\mathbb{E}_{Q_n}[GW] \to \mathbb{E}_{Q_\infty}[GW] = 0$ (recall the key fact $Q_n \xrightarrow{W} Q_\infty$), this shows that

\[
\lim_{n \to \infty} a_n = 0.
\]

Now we compare $V_{i,n}^\uparrow(c_1, c_2, s)$ and $V_{i,\infty}^\uparrow(c_1, c_2, s)$ (cf. Eq (C.10) and Eq (C.16)). The difference of $V_{i,n}^\uparrow$ and $V_{i,\infty}^\uparrow$ can be characterized by the difference between $G$ and $\Pi_{W^+, Q_n}(G)$:

\[
\begin{align*}
V_{1,\infty}^\uparrow(c_1, c_2, s) - V_{1,n}^\uparrow(c_1, c_2, s) &= a_n f_1(c_2, \psi) W^{1/2}X \\
V_{2,\infty}^\uparrow(c_1, c_2, s) - V_{2,n}^\uparrow(c_1, c_2, s) &= 2a_n f_1(c_2, \psi) f_2(c_1, c_2, \psi, \kappa) W^2X + f_1^2(c_2, \psi)(2a_nGW - a_n^2W^2)X \\
V_{3,\infty}^\uparrow(c_1, c_2, s) - V_{3,n}^\uparrow(c_1, c_2, s) &= 2a_n f_1(c_2, \psi) f_2(c_1, c_2, \psi, \kappa) W^2 + f_1^2(c_2, \psi)(2a_nGW - a_n^2W^2).
\end{align*}
\]

where $f_1, f_2$ are functions depending only on $c_1, c_2, \psi, \kappa$ and independent of $G, W, X$:

\[
f_1(c_2, \psi) = c_2 \psi^{1/2} \text{ and } f_2(c_1, c_2, \psi, \kappa) = c_2 \partial_1 F_\kappa(c_1, c_2) - c_1 \partial_2 F_\kappa(c_1, c_2). \tag{C.22}
\]

Now that Eq (C.19) follows since (i) we have the convergence $a_n \to 0$, (ii) each function $f_1$ and $f_2$ are uniformly bounded on $S$, and (iii) each individual term involving $(G, W, X)$ on the RHS of Eq (C.21) i.e., $W^2X^{1/2}, GWX, W^2X, GW, W$, are upper and lower bounded by some quadratic function of $(G, W)$ (since $c < X < C$ for some constant $0 < c < C < \infty$ by Assumption 1), and thus the expectation $\mathbb{E}_{Q_n}[|GW^2X^{1/2}|], \mathbb{E}_{Q_n}[|GWX|], \mathbb{E}_{Q_n}[|W^2X|], \mathbb{E}_{Q_n}[GW], \mathbb{E}_{Q_n}[W^2]$ are uniformly bounded over $n \in \mathbb{N}$ thanks to Eq (C.9).
Proof of Point (b) Recall Eq. (C.11), and define \( \bar{U}_{i,n}(c_1, c_2, s) = E_{Q_n} U_{i,n}(c_1, c_2, s) \), notice that it is sufficient to prove equicontinuity of \( \bar{U}_{i,n} \). Pick any \((c_1, c_2, s), (c_1', c_2', s') \in \mathcal{S}\),

\[
\bar{U}_{i,n}(c_1, c_2, s) - \bar{U}_{i,n}(c_1', c_2', s') = E_{Q_n} \left[ \frac{V_{i,n}^+(c_1, c_2, s)}{V_i^+(c_1, c_2, s)} - \frac{V_{i,n}^+(c_1', c_2', s')}{V_i^+(c_1', c_2', s')} \right] \tag{C.23}
\]

In light of Eq. (C.17) and Eq. (C.23), it is sufficient to prove that, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \((c_1, c_2, s), (c_1', c_2', s') \in \mathcal{S}\) satisfying \( \| (c_1, c_2, s) - (c_1', c_2', s') \|_2 < \delta \),

\[
\sup_{n \in \mathbb{N}} E_{Q_n} \left[ V_{i,n}^+(c_1, c_2, s) V_i^+(c_1', c_2', s') - V_{i,n}^+(c_1', c_2', s') V_i^+(c_1, c_2, s) \right] \leq \epsilon. \tag{C.24}
\]

By triangle inequality

\[
\left| V_{i,n}^+(c_1, c_2, s) V_i^+(c_1', c_2', s') - V_{i,n}^+(c_1', c_2', s') V_i^+(c_1, c_2, s) \right| \leq V_{i,n}^+(c_1, c_2, s) \left( V_i^+(c_1, c_2, s) - V_i^+(c_1', c_2', s') \right) + V_{i,n}^+(c_1', c_2', s') \left( V_i^+(c_1', c_2', s') - V_i^+(c_1, c_2, s) \right) \tag{C.25}
\]

Now, the desired goal in Eq. (C.24) follows by the above triangle inequality and the following four observations.

- There exists some \( C_{up} = C_{up}(\mathcal{S}, \psi, \kappa) > 0 \) such that for any \( i \in \{1, 2, 3\} \),

\[
\sup_{(c_1, c_2, s) \in \mathcal{S}} V_{i,n}^+(c_1, c_2, s) < C_{up}. \tag{C.26}
\]

This is because (i) for some universal \( c, C > 0 \) we have \( C > X > c \) by Assumption 1 and (ii) the continuous function \( \partial_2 F_{\kappa}(c_1, c_2) \) is bounded on the compact set \( \mathcal{S} \).

- For any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that as long as we have \( \| (c_1, c_2, s) - (c_1', c_2', s') \|_2 < \delta \),

\[
\sup_{n \in \mathbb{N}} E_{Q_n} \left| V_{i,n}^+(c_1, c_2, s) - V_{i,n}^+(c_1', c_2', s') \right| \leq \epsilon. \tag{C.27}
\]

An inspection of \( V_{i,n}^+(c_1, c_2, s) \) (recall Eq. (C.10)) implies that each \( V_{i,n}^+(c_1, c_2, s) \) takes the form of

\[
V_{i,n}^+(c_1, c_2, s) = \sum_j f_{j,i}(c_1, c_2, s, \psi, \kappa) \cdot g_{j,i} \left( \Pi_{W^+, Q_n}(G), X, W \right) \tag{C.28}
\]

where each \( f_{j,i} \) is a continuous function of \((c_1, c_2, s, \psi, \kappa)\) independent of \((W, G, X)\) and each \( g_{j,i} \) is a polynomial of \( \left( \Pi_{W^+, Q_n}(G), X, W \right) \) independent of \((c_1, c_2, s, \psi, \kappa)\), and each \( g_{j,i} \) is at most a quadratic function of \( \left( \Pi_{W^+, Q_n}(G), W \right) \). Now, we note the following three facts.

1. The triangle inequality bound on the LHS of Eq. (C.27) in terms of \( f_{j,i} \) and \( g_{j,i} \)

\[
\sup_{n \in \mathbb{N}} E_{Q_n} \left| V_{i,n}^+(c_1, c_2, s) - V_{i,n}^+(c_1', c_2', s') \right| \leq \sum_j \left| f_{j,i}(c_1, c_2, s, \psi, \kappa) - f_{j,i}(c_1', c_2', s', \psi, \kappa) \right| \cdot \sup_{n \in \mathbb{N}} E_{Q_n} \left[ \| g_{j,i} \left( \Pi_{W^+, Q_n}(G), X, W \right) \|_2 \right]
\]

2. The fact that the function \( f_{j,i}(c_1, c_2, s, \psi, \kappa) \) is uniformly continuous w.r.t \((c_1, c_2, s) \in \mathcal{S}\) for any fixed \( \psi, \kappa \), i.e., for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that as long as \( \| (c_1, c_2, s) - (c_1', c_2', s') \|_2 < \delta \),

\[
| f_{j,i}(c_1, c_2, s, \psi, \kappa) - f_{j,i}(c_1', c_2', s', \psi, \kappa) | \leq \epsilon.
\]
3. There exists some \( C_{\text{up}} < \infty \) such that for all \( i,j \)
\[
\sup_{n \in \mathbb{N}} \mathbb{E}_{Q_n} \left[ |g_{j,i}(\Pi_{W^+,Q_n}(G), X, W)| \right] \leq C_{\text{up}} < \infty. \tag{C.29}
\]
This is due to the fact that (i) \( |X| < C \) for some universal \( C > 0 \) by Assumption 1, (ii) \( g_{j,i} \) is at most a quadratic function of \( (\Pi_{W^+,Q_n}(G), W) \) if holding \( X \) constant, and thus for some \( C_{\text{up}} < \infty \)
\[
\sup_{n \in \mathbb{N}} |g_{j,i}(\Pi_{W^+,Q_n}(G), W)| \leq C_{\text{up}}(\Pi_{W^+,Q_n}(G)^2 + W^2) \tag{C.30}
\]
(iii) since \( Q_n \xrightarrow{W_2} Q_\infty \), we have that
\[
\sup_{n \in \mathbb{N}} \mathbb{E}_{Q_n} \left[ (\Pi_{W^+,Q_n}(G)^2 + W^2) \right] < \infty. \tag{C.31}
\]
Now the desired claim in Eq. (C.27) follows immediately from the above observations.

- There exists some \( C_{\text{up}} = C_{\text{up}}(S, \psi, \kappa) > 0 \) such that for any \( i \in \{1,2,3\} \),
\[
\sup_{n \in \mathbb{N}} \sup_{(c_1,c_2,s) \in S} \mathbb{E}_{Q_n} \left[ V_{i,n}(c_1, c_2, s) \right] < C_{\text{up}}. \tag{C.32}
\]
In fact, by Eq. (C.28), we have that
\[
\sup_{n \in \mathbb{N}} \sup_{(c_1,c_2,s) \in S} \mathbb{E}_{Q_n} \left[ V_{i,n}(c_1, c_2, s) \right] \leq \sum_j \sup_{(c_1,c_2,s)} |f_{j,i}(c_1, c_2, s, \psi, \kappa)| \sup_{n \in \mathbb{N}} \mathbb{E}_{Q_n} \left[ |g_{j,i}(\Pi_{W^+,Q_n}(G), X, W)| \right].
\]
Now that Eq. (C.32) follows since (i) the continuous functions \( |f_{j,i}(c_1, c_2, s, \psi, \kappa)| \) are uniformly bounded on the compact set \( (c_1,c_2,s) \in S \) and (ii) the uniform upper bound of \( \mathbb{E}_{Q_n}[|g_{j,i}|] \) in Eq. (C.29).

- For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that as long as we have \( \|(c_1, c_2, s) - (c_1', c_2', s')\|_2 < \delta \),
\[
\left| V_i^+(c_1, c_2, s) - V_i^+(c_1', c_2', s') \right| \leq \varepsilon. \tag{C.33}
\]
An inspection of \( V_i^+(c_1, c_2, s) \) (recall Eq. (C.10)) implies that each \( V_i^+(c_1, c_2, s) \) takes the form of
\[
V_i^+(c_1, c_2, s) = \sum_j \tilde{f}_{j,i}(c_1, c_2, s, \psi, \kappa) \cdot \tilde{g}_{j,i} \left( X^{1/2}, X^{-1/2} \right) \tag{C.34}
\]
where each \( \tilde{f}_{j,i} \) is a continuous function of \( (c_1,c_2,s,\psi,\kappa) \) independent of \( (W,G,X) \) and each \( \tilde{g}_{j,i} \) is a quadratic polynomial of \( (X^{1/2}, X^{-1/2}) \) independent of \( c_1, c_2, s, \psi, \kappa, W, G \). Thus, we have
\[
\left| V_i^+(c_1, c_2, s) - V_i^+(c_1', c_2', s') \right| \leq \sum_j \left| f_{j,i}(c_1, c_2, s, \psi, \kappa) - f_{j,i}(c_1', c_2', s', \psi, \kappa) \right| \cdot \left| \tilde{g}_{j,i} \left( X^{1/2}, X^{-1/2} \right) \right|
\]
Now that the desired Eq. (C.33) follows because (i) the functions \( f_{j,i}(c_1, c_2, s, \psi, \kappa) \) is uniformly continuous w.r.t \( (c_1, c_2, s) \in S \) for any fixed \( \psi, \kappa \) and (ii) for some \( C_{\text{up}} < \infty \), we have \( \left| \tilde{g}_{j,i}(X^{1/2}, X^{-1/2}) \right| \leq C_{\text{up}} \) almost surely under \( Q_n \) for all \( n \in \mathbb{N} \) because \( |X| < C \) for some universal \( C > 0 \) by Assumption 1.

D Reduction to Gordon’s optimization problem

D.1 Proof of Lemma 5.1

Recall that \( X \) has i.i.d. rows \( x_i \sim \mathcal{N}(0, A) \), with \( A \) a diagonal matrix. We rewrite \( X = \bar{X} \Lambda^{1/2} \), where \( (X_{ij})_{i \leq n, j \leq p} \sim \mathcal{N}(0, 1) \). Therefore, Eq. (4.19) yields
\[
\xi_{n,\psi,\kappa}^{(0)}(\Theta_p) = \min_{\|\theta\|_2 \leq 1, \theta \in \Theta_p} \max_{\|\lambda\|_2 \leq 1, y \in \Lambda \geq 0} \frac{1}{\sqrt{p}} \lambda^\top (\kappa y - \bar{X} \Lambda^{1/2} \theta). \tag{D.1}
\]
We need to be cautious when we apply Theorem 2 to \( \xi^{(0)}_{w,\psi,\kappa} \); \( y \) is not independent of the Gaussian random matrix \( \bar{X} \). To circumvent this technical difficulty, recall that \( w = \theta_*/\|\theta_*\| \) (\( w \) can be chosen arbitrarily if \( \theta_* = 0 \)). We decompose \( \bar{X} \) into orthogonal components as follows:
\[
\bar{X} = uw^T + X\Pi_{w^\perp} \quad \text{where} \quad u = Xw \sim N(0, I_n).
\] (D.2)
(recall the unit vector \( w \) that parallels \( \bar{\theta}_* \)) Since \( \bar{X} \) is isotropic Gaussian, \( \bar{X}\Pi_{w^\perp} \) is independent of \((u,y)\).
Substituting in Eq. (D.1), we get
\[
\xi^{(0)}_{n,\psi,\kappa}(\Theta_p) = \min_{\|\theta\| \leq 1, \theta \in \Theta_p} \max_{\lambda \in \lambda_p, \lambda \geq 0} \frac{1}{\sqrt{p}} \Lambda^T(\kappa y - \langle \Lambda^{1/2} w, \theta \rangle u - \bar{X}\Pi_{w^\perp} \Lambda^{1/2} \theta). \tag{D.3}
\]
Consider, to be definite, the case \( \Theta_p = B^p(1) \) (corresponding to \( \xi^{(0)}_{n,\psi,\kappa} = \xi^{(0)}_{n,\psi,\kappa}(B^p(1)) \)). By conditioning on \((u,y)\), we can apply Theorem 2 to get for any \( t \in \mathbb{R} \):
\[
\mathbb{P}(\xi^{(0)}_{n,\psi,\kappa} \leq t | u, y) \leq 2 \mathbb{P}(\xi^{(1)}_{n,\psi,\kappa} \leq t | u, y) \quad \text{and} \quad \mathbb{P}(\xi^{(0)}_{n,\psi,\kappa} \geq t | u, y) \leq 2 \mathbb{P}(\xi^{(1)}_{n,\psi,\kappa} \geq t | u, y).
\]
Taking expectation over \( u, y \) on both sides of the equation gives for any \( t \in \mathbb{R} \),
\[
\mathbb{P}(\xi^{(0)}_{n,\psi,\kappa} \leq t) \leq 2 \mathbb{P}(\xi^{(1)}_{n,\psi,\kappa} \leq t) \quad \text{and} \quad \mathbb{P}(\xi^{(0)}_{n,\psi,\kappa} \geq t) \leq 2 \mathbb{P}(\xi^{(1)}_{n,\psi,\kappa} \geq t). \tag{D.4}
\]
The claim for \( \xi^{(0)}_{n,\psi,\kappa}(\Theta_p), \xi^{(1)}_{n,\psi,\kappa}(\Theta_p) \) follows by the same argument.

**D.2 Proof of Lemma 5.2**

Let us introduce the notation
\[
f^{(1)}_{n,\psi,\kappa}(\theta) = \frac{1}{\sqrt{p}} \left\| \left( \kappa 1 - \langle \Lambda^{1/2} w, \theta \rangle (y \ominus u) - \|\Pi_{w^\perp} \Lambda^{1/2} \theta\| \right) \right\|_2 + \frac{1}{\sqrt{p}} g^T \Pi_{w^\perp} \Lambda^{1/2} \theta,
\]
\[
f^{(2)}_{n,\psi,\kappa}(\theta) = \psi^{-1/2} \cdot F_\kappa \left( \langle \Lambda^{1/2} w, \theta \rangle, \|\Pi_{w^\perp} \Lambda^{1/2} \theta\| \right) + \frac{1}{\sqrt{p}} g^T \Pi_{w^\perp} \Lambda^{1/2} \theta.
\]
We claim that
\[
\sup_{\theta : \|\theta\| \leq 1} \left| f^{(1)}_{n,\psi,\kappa}(\theta) - f^{(2)}_{n,\psi,\kappa}(\theta) \right| \xrightarrow{\mathbb{P}} 0. \tag{D.5}
\]
Let us show that this claim implies the statement of Lemma 5.2. By definition, Eq. (5.4), Eq. (5.5) and Eq. (5.6) gives for any compact \( \Theta_p \),
\[
\xi^{(1)}_{n,\psi,\kappa}(\Theta_p) = \xi^{(1)}_{n,\psi,\kappa}(\Theta_p)_+ \quad \text{and} \quad \xi^{(2)}_{n,\psi,\kappa}(\Theta_p) = \min_{\theta \in \Theta_p} \left( f^{(2)}_{n,\psi,\kappa}(\theta) \right)_+ \tag{D.6}
\]
Note that the mapping \( x \rightarrow (x)_+ \) is Lipschitz. Thereby, we have when \( \Theta_p \subseteq \{ \theta : \|\theta\| \leq 1 \} \),
\[
\left| \xi^{(1)}_{n,\psi,\kappa}(\Theta_p) - \xi^{(2)}_{n,\psi,\kappa}(\Theta_p) \right| \leq \sup_{\theta : \|\theta\| \leq 1} \left| \left( f^{(1)}_{n,\psi,\kappa}(\theta) \right)_+ - \left( f^{(2)}_{n,\psi,\kappa}(\theta) \right)_+ \right| \leq \sup_{\theta : \|\theta\| \leq 1} \left| f^{(1)}_{n,\psi,\kappa}(\theta) - f^{(2)}_{n,\psi,\kappa}(\theta) \right|. \tag{D.7}
\]
With the uniform convergence result at Eq. (D.5), this immediately implies the desired Lemma 5.2.

In the rest of the proof, we prove the claim (D.5). We introduce the two functions
\[
g^{(1)}_{n,\psi,\kappa}(\nu, q) = \frac{1}{\sqrt{p}} \left\| \kappa 1 - \nu (y \ominus u) - q (y \ominus h) \right\|_2 \quad \text{and} \quad g^{(2)}_{n,\psi,\kappa}(\nu, q) = \psi^{-1/2} \cdot F_\kappa(\nu, q), \tag{D.8}
\]
and we denote for each \( \theta \in \mathbb{R}^p \),
\[
\nu(\theta) = \langle \Lambda^{1/2} w, \theta \rangle \quad \text{and} \quad q(\theta) = \|\Pi_{w^\perp} \Lambda^{1/2} \theta\|.
\]
By definition, we have that for all $\boldsymbol{\theta} \in \mathbb{R}^p$,
\[
\left| f_{n,\psi,\kappa}^{(1)}(\boldsymbol{\theta}) - f_{n,\psi,\kappa}^{(2)}(\boldsymbol{\theta}) \right| = \left| g_{n,\psi,\kappa}^{(1)}(\nu(\boldsymbol{\theta}), \varphi(\boldsymbol{\theta})) - g_{n,\psi,\kappa}^{(2)}(\nu(\boldsymbol{\theta}), \varphi(\boldsymbol{\theta})) \right|
\tag{D.9}
\]

Note $\|w\|_2 \leq 1$ and $\|A\|_{op} \leq C$ by Assumption 1. Thus, for all $\boldsymbol{\theta} \in \{\theta : \|\theta\|_2 \leq 1\}$,
\[
|\nu(\boldsymbol{\theta})| \leq \|\Lambda^{1/2}w\|_2 \leq C^{1/2} \quad \text{and} \quad q(\boldsymbol{\theta}) \leq \|\Lambda^{1/2}\theta\|_2 \leq C^{1/2}.
\tag{D.10}
\]

Therefore, Eq. (D.9) and Eq. (D.10) immediately implies that
\[
\sup_{\|\theta\|_2 \leq 1} \left| f_{n,\psi,\kappa}^{(1)}(\boldsymbol{\theta}) - f_{n,\psi,\kappa}^{(2)}(\boldsymbol{\theta}) \right| \leq \sup_{|\nu| \leq C^{1/2}, q \leq C^{1/2}} \left| g_{n,\psi,\kappa}^{(1)}(\nu, q) - g_{n,\psi,\kappa}^{(2)}(\nu, q) \right|
\tag{D.11}
\]

By Eq. (D.11), we can establish the desired uniform convergence result (D.5), by proving
\[
\sup_{|\nu| \leq C^{1/2}, q \leq C^{1/2}} \left| g_{n,\psi,\kappa}^{(1)}(\nu, q) - g_{n,\psi,\kappa}^{(2)}(\nu, q) \right| \xrightarrow{p} 0.
\tag{D.12}
\]

The proof of Eq. (D.12) is based on standard uniform convergence argument from empirical process theory. To start with, let us introduce the i.i.d random variables $\{Z_i(\nu, q)\}_{i=1}^n$ by
\[
Z_i(\nu, q) = (\kappa - \nu y_i u_i - q y_i h_i)_+.
\tag{D.13}
\]

so we can have by definition,
\[
\left( g_{n,\psi,\kappa}^{(1)}(\nu, q) \right)^2 = \frac{1}{p} \sum_{i=1}^n Z_i(\nu, q)^2 = \psi^{-1} \cdot \frac{1}{n} \sum_{i=1}^n Z_i(\nu, q)^2.
\]

Now, it is natural to introduce the quantity $\tilde{g}_{n,\psi,\kappa}^{(1)}(\nu, q)$ such that
\[
\left( \tilde{g}_{n,\psi,\kappa}^{(1)}(\nu, q) \right)^2 = \psi^{-1} \cdot \mathbb{E} \left[ Z(\nu, q)^2 \right] = \psi^{-1} \cdot \mathbb{E} \left[ (\kappa - \nu y - q y h)^2_+ \right],
\tag{D.14}
\]

where $Z(\nu, q) = (\kappa - \nu y - q y h)_+$ and the expectation on the RHS is taken w.r.t the random variables $(y, u, h)$, whose joint distribution is specified by
\[
(h, u) \perp y, \quad h, u \sim N(0, 1), \quad \mathbb{P}(y = 1 \mid u) = f(\|\theta\_n^*\| \Sigma \cdot u)
\tag{D.15}
\]

Recall the definition of $g_{n,\psi,\kappa}^{(2)}(\nu, q)$ in Eq. (D.8). We know that
\[
\left( \tilde{g}_{n,\psi,\kappa}^{(2)}(\nu, q) \right)^2 = \psi^{-1} \cdot F_\kappa(\nu, q) = \psi^{-1} \cdot \mathbb{E}[(\kappa - \nu Y U - q Y H)^2_+] \tag{D.16}
\]

where the expectation on the RHS is taken w.r.t the random variables $(Y, U, H)$, whose joint distribution is specified by
\[
(H, U) \perp Y, \quad H, U \sim N(0, 1), \quad \mathbb{P}(Y = 1 \mid U) = f(\rho \cdot U).
\tag{D.17}
\]

We can prove the desired Eq. (D.12) by showing that

- The uniform convergence from $g_{n,\psi,\kappa}^{(1)}(\nu, q)$ to $\tilde{g}_{n,\psi,\kappa}^{(1)}(\nu, q)$:
\[
\sup_{|\nu| \leq C^{1/2}, 0 \leq q \leq C^{1/2}} \left| g_{n,\psi,\kappa}^{(1)}(\nu, q) - \tilde{g}_{n,\psi,\kappa}^{(1)}(\nu, q) \right| \overset{a.s.}{\rightarrow} 0 \tag{D.18}
\]

- The uniform convergence from $\tilde{g}_{n,\psi,\kappa}^{(1)}(\nu, q)$ to $\tilde{g}_{n,\psi,\kappa}^{(2)}(\nu, q)$:
\[
\sup_{|\nu| \leq C^{1/2}, 0 \leq q \leq C^{1/2}} \left| \tilde{g}_{n,\psi,\kappa}^{(1)}(\nu, q) - \tilde{g}_{n,\psi,\kappa}^{(2)}(\nu, q) \right| \overset{a.s.}{\rightarrow} 0 \tag{D.19}
\]
In the rest of the proof, for notational simplicity, we introduce the compact set \( S_C \subseteq \mathbb{R}^2 \)

\[
S_C = \left\{ (\nu, q) : |\nu| \leq C^{1/2}, 0 \leq q \leq C^{1/2} \right\}.
\]

Now, we establish the below three important facts.

(a) There exists some constant \( c_0 > 0 \) independent of \( n, \nu, q \), such that for all \( t > 0 \), and \((\nu, q) \in S_C \)

\[
\mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n} Z_i(\nu, q)^2 - \mathbb{E}[Z(\nu, q)^2] \right| > t \right) \leq 2 \exp \left( -c_0 n \min \{ t, t^2 \} \right),
\]

where \( Z(\nu, q) = (\kappa - \nu y u - q y h)_+ \) and \( y, u, h \) is distributed according to Eq. (D.15). Indeed, it is not hard to show for some \( M < \infty \) independent of \( \nu, q, n \), we have for all \((\nu, q) \in S_C \), the i.i.d random variables \( \{ Z_i(\nu, q)^2 \}_{1 \leq i \leq n} \) are subgaussian with parameter at most \( M \), and thereby \( \{ Z_i(\nu, q)^2 \}_{1 \leq i \leq n} \) are subexponential with parameter at most \( M^2 \). Thus, the desired concentration inequality at Eq. (D.20) follows by the standard Bernstein inequality [Ver18b, Thm 2.8].

(b) There exists some numerical constant \( c_1 > 0 \) so that with probability at least \( 1 - \exp(-c_1 n) \), the mapping \((\nu, q) \mapsto \bar{g}_{n,\psi,\kappa}^{(1)}(\nu, q) \) is \( 3\psi^{-1/2} \)-Lipschitz continuous. Indeed, it is not hard to show for some \( M < \infty \) independent of \( \nu, q, n \), we have for all \((\nu, q) \in S_C \), the i.i.d random variables \( \{ Z_i(\nu, q)^2 \}_{1 \leq i \leq n} \) are subgaussian with parameter at most \( M \), and thereby \( \{ Z_i(\nu, q)^2 \}_{1 \leq i \leq n} \) are subexponential with parameter at most \( M^2 \). Thus, the desired concentration inequality at Eq. (D.20) follows by the standard Bernstein inequality [Ver18b, Thm 2.8].

(c) Define \( \Delta(\nu, q) = -\nu y u - q y h \). By the elementary inequality \((a + b)_+ \leq a_+ + b_+ \), we have

\[
Z(\nu, q)_+ \leq Z(\nu', q')_+ + \Delta(\nu - \nu', q - q')_+.
\]

Therefore, Minkowski’s inequality implies that for any pairs \((\nu, q)\) and \((\nu', q')\),

\[
(\mathbb{E}[Z(\nu, q)_+]^2)^{1/2} \leq (\mathbb{E}[Z(\nu', q')_+]^2)^{1/2} + \mathbb{E}[\Delta(\nu - \nu', q - q')_+^2]^{1/2},
\]

where we are taking expectation over \( y, u, h \) whose distribution is specified by Eq. (D.15). Now that

\[
\mathbb{E}[\Delta(\nu, q)_+^2] \leq \mathbb{E}[\Delta(\nu', q')_+^2] \leq 2(\nu^2 \mathbb{E}[|y u|^2] + q^2 \mathbb{E}[|y h|^2]) = 2(\nu^2 + q^2).
\]

Thereby, Eq. (D.22) implies for any pairs \((\nu, q)\) and \((\nu', q')\),

\[
\left( \mathbb{E}[Z(\nu, q)_+]^2 \right)^{1/2} - \left( \mathbb{E}[Z(\nu', q')_+]^2 \right)^{1/2} \leq 2 \| (\nu - \nu', q - q') \|_2.
\]

This proves the result as \( \bar{g}_{n,\psi,\kappa}^{(1)}(\nu, q) = \psi^{-1/2} \cdot (\mathbb{E}[Z(\nu, q)_+]^2)^{1/2} \).

**Proof of (D.18)** Now, we can prove Eq. (D.18) via standard covering argument. Take the minimal \( \epsilon_n = n^{-1/4} \) covering of the set \( S_C \), and we denote them by \( \{(\nu_i, q_i)\}_{i \in T_n} \). Standard volume argument shows that \( T_n \leq 2C^2 \epsilon_n^{-2} \). Denote the event \( \mathcal{E}_n \) to be

\[
\mathcal{E}_n = \left\{ \max_{i \in [T_n]} \left| \bar{g}_{n,\psi,\kappa}^{(1)}(\nu_i, q_i) - \bar{g}_{n,\psi,\kappa}^{(1)}(\nu, q) \right| \leq \epsilon_n \right\} \cap \left\{ (\nu, q) \mapsto \bar{g}_{n,\psi,\kappa}^{(1)}(\nu, q) \right. \left. \text{is } 3\psi^{-1/2}-\text{Lipschitz} \right\}.
\]

It is clear that on the event \( \mathcal{E}_n \), we have

\[
\sup_{(\nu, q) \in S_C} \left| \bar{g}_{n,\psi,\kappa}^{(1)}(\nu, q) - \bar{g}_{n,\psi,\kappa}^{(1)}(\nu, q) \right| \leq (3\psi^{-1/2} + 1)\epsilon_n.
\]
Point (a), (b) and the union bound above shows that \( \mathbb{P}(\mathcal{S}_n) \geq 1 - (T_n + 1) \exp(-c_2 n \varepsilon_n^2) \) for some numerical constant \( c_2 > 0 \). By Borel-Cantelli lemma, (almost surely) there exists some \( N \in \mathbb{N} \) such that \( \mathcal{S}_n \) happens for all \( n > N \). Thus, for some \( N \in \mathbb{N} \), we have Eq. (D.26) holds for all \( n > N \), and since \( \varepsilon_n \to 0 \), this shows the desired almost sure convergence:

\[
\lim_{n \to \infty} \sup_{(\nu, q) \in \mathcal{S}} \left| \tilde{g}^{(1)}_{n, \psi, \kappa}(\nu, q) - \tilde{g}^{(1)}_{n, \psi, \kappa}(\nu, q) \right| = 0.
\]

**Proof of (D.19)** Note that \( \|\theta_{*, n}\|_{\Sigma} \to \rho \). By dominated convergence theorem, we have for \((\nu, q) \in \mathcal{S}\)

\[
\lim_{n \to \infty} \left| \tilde{g}^{(1)}_{n, \psi, \kappa}(\nu, q) - \tilde{g}^{(2)}_{\psi, \kappa}(\nu, q) \right| = 0.
\]

Point (c) above implies that the class of functions \( \{\tilde{g}^{(1)}_{n, \psi, \kappa}(\nu, q)\}_{n \in \mathbb{N}} \) is equicontinuous. Thus, Arzelà-Ascoli theorem implies the desired uniform convergence:

\[
\lim_{n \to \infty} \sup_{(\nu, q) \in \mathcal{S}} \left| \tilde{g}^{(1)}_{n, \psi, \kappa}(\nu, q) - \tilde{g}^{(2)}_{\psi, \kappa}(\nu, q) \right| = 0. \tag{D.27}
\]

### E Asymptotics of the prediction error: Proof of Proposition 5.5

We introduce the notation:

\[
\tilde{\kappa}_n = \min_{i \in [n]} \mu_i(\hat{\Theta}_{n, x_i}^\text{MM}), \quad \nu_n = \frac{\langle \hat{\Theta}_{n}^\text{MM}, \theta_{*, n} \rangle_{\Sigma_n}}{\| \hat{\Theta}_{n}^\text{MM} \|_{\Sigma_n} \| \theta_{*, n} \|_{\Sigma_n}}, \tag{E.1}
\]

We define auxiliary functions \( \xi^{(i)}_{n, \alpha, \kappa}(\nu) \), \( \xi^{(i)}_{n, \alpha, \kappa}(\nu_1, \nu_2) \) as follows:

- We set for \( i \in \{0, 1, 2\} \), \( \nu \in [-1, 1] \),

\[
\xi^{(i)}_{n, \psi, \kappa}(\nu) = \xi^{(i)}_{n, \psi, \kappa}(\Theta_{p}(\nu)), \tag{E.2}
\]

where

\[
\Theta_{p}(\nu) = \left\{ \theta \in \mathbb{R}^p : \| \theta \|_2 \leq 1, \frac{\langle \theta_{*, n}, \theta \rangle_{\Sigma_n}}{\| \theta_{*, n} \|_{\Sigma_n} \| \theta \|_{\Sigma_n}} = \nu \right\}. \tag{E.3}
\]

So in particular, we have

\[
\xi^{(2)}_{n, \psi, \kappa}(\nu) = \min_{\| \theta \|_2 \leq 1, \langle \theta_{*, n}, \Lambda^{1/2} w \rangle_{\Sigma_n} / \| \theta \|_{\Lambda^{1/2}} = \nu} \psi^{-1/2} F_{\kappa} \left( \langle \Lambda^{1/2} w, \theta \rangle, \| \Pi_{w} \Lambda^{1/2} \theta \|_2 \right) + \frac{1}{\sqrt{p}} g^T \Pi_{w} \Lambda^{1/2} \theta. \tag{E.4}
\]

Note by definition

\[
\xi^{(2)}_{n, \psi, \kappa} = \min_{\nu \in [-1, 1]} \xi^{(2)}_{n, \psi, \kappa}(\nu) = \min_{\| \theta \|_2 \leq 1} \frac{1}{\sqrt{p}} \| (\kappa 1 - y \odot X \theta) + \|_2
\]

\[
\xi^{(2)}_{n, \psi, \kappa} = \min_{\nu \in [-1, 1]} \xi^{(2)}_{n, \psi, \kappa}(\nu) = \min_{\| \theta \|_2 \leq 1} \psi^{-1/2} F_{\kappa} \left( \langle \Lambda^{1/2} w, \theta \rangle, \| \Pi_{w} \Lambda^{1/2} \theta \|_2 \right) + \frac{1}{\sqrt{p}} g^T \Pi_{w} \Lambda^{1/2} \theta, \tag{E.5}
\]

- We set for any \( i \in \{0, 1, 2\} \), \( \nu_1, \nu_2 \in [-1, 1] \),

\[
\tilde{\xi}^{(i)}_{n, \alpha, \kappa}(\nu_1, \nu_2) = \min \left\{ \min_{\nu \leq \nu_1} \xi^{(i)}_{n, \alpha, \kappa}(\nu), \min_{\nu \geq \nu_2} \xi^{(i)}_{n, \alpha, \kappa}(\nu) \right\}. \tag{E.6}
\]
The above definitions imply
\[ n_k = \sup_{\nu} \{ \xi_{n,\psi,\kappa} = 0 \}, \quad \hat{\nu}_n \in \{ \nu : \xi_{n,\psi,\hat{\kappa}_n}(\nu) = 0 \}. \tag{E.7} \]

Now, we prove our desired goal of the proposition, i.e.,
\[ \hat{\nu}_n \overset{P}{\to} \nu^*(\psi). \tag{E.8} \]

For any \( \varepsilon > 0 \), denote \( \nu^*_\varepsilon(\psi; \varepsilon) = \nu^*(\psi) + \varepsilon \) and \( \nu^*_\varepsilon(\psi; \varepsilon) = \nu^*(\psi) - \varepsilon \). By Eq. (E.7), it suffices to show that
\[ \lim_{n \to \infty, p/n \to \psi} \mathbb{P} \left( \xi_{n,\psi,\hat{\kappa}_n}(\nu^*_\varepsilon(\psi; \varepsilon), \nu^*_\varepsilon(\psi; \varepsilon)) > 0 \right) = 1 \text{ for any } \varepsilon > 0. \tag{E.9} \]

The rest of the proof will establish Eq. (E.9). Define for any \( \nu_0, \nu_1 \in [-1, 1], \)
\[ \text{gap}(\nu_0, \nu_1) = \min_{c_1,0/\sqrt{c_1,0 + c_2,0} = \nu_0, \ c_1,1/\sqrt{c_1,1 + c_2,1} = \nu_1, \ \max\{c_1,0 + c_2,0, \ c_1,1 + c_2,1\} \leq \max, \ c_{1,1/2} = (c_0 + c_1,1)/2, \ c_{2,1/2} = (c_0 + c_2,1)/2} \left\{ \frac{1}{2} (F_\kappa(c_1,0, c_2,0) + F_\kappa(c_1,1, c_2,1)) - F_\kappa(c_1,1/2, c_2,1/2) \right\}. \tag{E.10} \]

By Lemma 5.3, \( F_\kappa \) is strictly convex and continuous, and thus we know that (i) \( (\nu_0, \nu_1) \to \text{gap}(\nu_0, \nu_1) \) is lower-semicontinuous and (ii) \( \text{gap}(\nu_0, \nu_1) > 0 \) when \( \nu_0 \neq \nu_1 \). The crucial observation is Lemma E.1 below, whose proof we defer into Section E.1.

**Lemma E.1.** For any \( \nu_0, \nu_1 \in [-1, 1] \), we have
\[ \frac{1}{2} \left( \xi^{(2)}_{n,\psi,\kappa}(\nu_0) + \xi^{(2)}_{n,\psi,\kappa}(\nu_1) \right) - \xi^{(2)}_{n,\psi,\kappa} \geq \psi^{-1/2} \cdot \text{gap}(\nu_0, \nu_1). \tag{E.11} \]

Now we are ready to prove Eq. (E.9). Recall our notation that \( \hat{\theta}^{(2)}_{n,\psi,\kappa} \in \mathbb{R}^p \) is the optimal solution of the optimization problem defining \( \xi^{(2)}_{n,\psi,\kappa} \) (see Eq. (E.5)). Let us denote \( \hat{\nu}^{(2)}_{n,\psi,\kappa} \) to be
\[ \hat{\nu}^{(2)}_{n,\psi,\kappa} = \left( \xi^{(2)}_{n,\psi,\kappa}, \Lambda^{1/2} w \right) / \| \xi^{(2)}_{n,\psi,\kappa} \| \Lambda. \]

By definition, it is clear for all \( n, \psi, \kappa, \)
\[ \xi^{(2)}_{n,\psi,\kappa} = \xi^{(2)}_{n,\psi,\kappa} \left( \hat{\nu}^{(2)}_{n,\psi,\kappa} \right). \tag{E.12} \]

Now, we use Lemma E.1. Plugging \( \nu_0 = \hat{\nu}^{(2)}_{n,\psi,\kappa}, \kappa = \kappa^*(\psi) \) into Eq. (E.11), and using Eq. (E.12), we get
\[ \xi^{(2)}_{n,\psi,\kappa^*(\psi)}(\nu_1) - \xi^{(2)}_{n,\psi,\kappa^*(\psi)}(\nu_1) \geq \psi^{-1/2} \cdot \text{gap}(\hat{\nu}^{(2)}_{n,\psi,\kappa^*(\psi)}, \nu_1). \tag{E.13} \]

holds for all \( \nu_1 \in [-1, 1] \). Now, we define for any \( \nu_0, \nu_1, \nu_2 \in [-1, 1], \kappa > 0 \)
\[ \text{gap}(\nu_0, \nu_1, \nu_2) = \min \left\{ \min_{\nu \leq \nu_1} \text{gap}(\nu_0, \nu), \ min_{\nu \geq \nu_2} \text{gap}(\nu_0, \nu) \right\}. \tag{E.14} \]

Note that (i) \( (\nu_0, \nu_1, \nu_2) \to \text{gap}(\nu_0, \nu_1, \nu_2) \) is lower-semicontinuous (because \( \text{gap}(\cdot, \cdot) \) is lower semicontinuous), and (ii) for any \( \nu_1 < \nu_0 < \nu_2 \) \( \text{gap}(\nu_0, \nu_1, \nu_2) > 0 \) (because \( \text{gap}(\kappa)(\nu_0, \nu) > 0 \) for any \( \nu \neq \nu_0 \)). Now, by Eq. (E.13), Eq. (E.6) and Eq. (E.14), we know for all \( \nu_1, \nu_2 \in [-1, 1], \)
\[ \xi^{(2)}_{n,\psi,\kappa^*(\psi)}(\nu_1) - \xi^{(2)}_{n,\psi,\kappa^*(\psi)}(\nu_1) \geq 2\psi^{-1/2} \cdot \text{gap}(\hat{\nu}^{(2)}_{n,\psi,\kappa^*(\psi)}, \nu_1, \nu_2). \tag{E.15} \]

Recall Proposition 5.4. We have almost surely (recall that \( c_i^*(\psi) = c_i(\psi, \kappa^*(\psi)) \) for \( i \in \{1, 2\} \)),
\[ \lim_{n \to \infty, p/n \to \psi} \xi^{(2)}_{n,\psi,\kappa^*(\psi)} = T(\psi, \kappa^*(\psi)) = 0 \text{ and } \lim_{n \to \infty, p/n \to \psi} \hat{\nu}^{(2)}_{n,\psi,\kappa^*(\psi)} = \frac{c_1^*(\psi)}{\sqrt{(c_1^*(\psi))^2 + (c_2^*(\psi))^2}} = \nu^*(\psi). \tag{E.16} \]
Therefore, for any fixed $\nu_1, \nu_2$, by Eq. (E.15) and Eq. (E.16), we have almost surely
\[
\liminf_{n \to \infty, p/n \to \psi} \xi_{n, \psi, \kappa^*}(\nu_1, \nu_2) \geq 2\psi^{-1/2} \cdot \|n\|_{\psi}(\nu_1, \nu_2). \tag{E.17}
\]
Now, for any fixed $\varepsilon > 0$, we define $\eta(\psi; \varepsilon) > 0$ by
\[
\eta(\psi; \varepsilon) = 2\psi^{-1} \cdot \|n\|_{\psi}(\nu_1^*, \nu_1^*(\psi; \varepsilon), \nu_2^*(\psi; \varepsilon)). \tag{E.18}
\]
then Eq. (E.17) implies in particular that
\[
\liminf_{n \to \infty, p/n \to \psi} \mathbb{P} \left( \xi_{n, \psi, \kappa^*}(\nu_1, \nu_2) \geq \eta(\psi; \varepsilon) \right) = 1. \tag{E.19}
\]
Notice that $\xi_{n, \psi, \kappa^*}(\nu_1, \nu_2) = \xi_{n, \psi, \kappa^*}(\psi)(\theta_p(\nu_1, \nu_2))$ where the set $\theta_p(\nu_1, \nu_2)$ is defined by
\[
\theta_p(\nu_1, \nu_2) = \left\{ \theta : \|\theta\|_2 \leq 1, \frac{(\theta_{\kappa, \nu_1, \nu_2})_{\Sigma_n}}{\|\theta_{\kappa, \nu_1, \nu_2}\|_{\Sigma_n}} \leq \nu_1 \right\} \cup \left\{ \theta : \|\theta\|_2 \leq 1, \frac{(\theta_{\kappa, \nu_1, \nu_2})_{\Sigma_n}}{\|\theta_{\kappa, \nu_1, \nu_2}\|_{\Sigma_n}} \geq \nu_2 \right\}. \tag{E.20}
\]
Thus, Eq. (E.19), Lemma 5.1 and Lemma 5.2 imply that
\[
\lim_{n \to \infty} \mathbb{P} \left( \xi_{n, \psi, \kappa^*}(\nu_1, \nu_2) > \eta(\psi; \varepsilon) \right) = 1. \tag{E.21}
\]
Finally, we notice that

- The function $\kappa \to \xi_{n, \psi, \kappa}(\nu_1, \nu_2)$ is $((\psi)^{-1/2})$-Lipschitz for all $n, \psi, \nu_1, \nu_2$. This is due to (i) the mapping $\kappa \to \frac{1}{\sqrt{p}} \|(\kappa 1 - y \circ X_\theta)\|_2$ is $((\psi)^{-1/2})$ Lipschitz, and (ii) the variational characterization below:
\[
\xi_{n, \psi, \kappa^*}(\nu_1, \nu_2) = \min_{\theta, \|\theta\| \leq 1, \theta \in \theta_p(\nu_1, \nu_2)}\frac{1}{\sqrt{p}} \|(\kappa 1 - y \circ X_\theta)\|_2.
\]
- the convergence $\hat{\kappa}_n \xrightarrow{P} \kappa^*(\psi)$ which is implied by Eq. (E.7), Eq. (5.15) and Eq. (5.16).

Using the above two facts, and the high probability bound at Eq. (E.21), we get
\[
\lim_{n \to \infty} \mathbb{P} \left( \xi_{n, \psi, \hat{\kappa}_n}(\nu_1, \nu_2) > \eta(\psi; \varepsilon) \right) = 1. \tag{E.22}
\]
This gives the desired claim at Eq. (E.9), and thus the proposition.

### E.1 Proof of Lemma E.1

Let $\theta_0 \in \mathbb{R}^p, \theta_1 \in \mathbb{R}^p$ be such that $\langle A^{1/2} w, \theta_0 \rangle / \|\theta_0\|_A = \nu_0, \langle A^{1/2} w, \theta_1 \rangle / \|\theta_1\|_A = \nu_1$ and
\[
\xi_{n, \psi, \kappa^*}(\nu_0) = \psi^{-1/2} \cdot F_n \left( \langle A^{1/2} w, \theta_0 \rangle, \|\Pi_{w^1} A^{1/2} \theta_0\|_2 \right) + \frac{1}{\sqrt{p}} g^T \Pi_{w^1} A^{1/2} \theta_0
\]
\[
\xi_{n, \psi, \kappa^*}(\nu_1) = \psi^{-1/2} \cdot F_n \left( \langle A^{1/2} w, \theta_1 \rangle, \|\Pi_{w^1} A^{1/2} \theta_1\|_2 \right) + \frac{1}{\sqrt{p}} g^T \Pi_{w^1} A^{1/2} \theta_1. \tag{E.23}
\]
Denote $\theta_{1/2} = \frac{1}{2}(\theta_0 + \theta_1)$. By definition of $\xi_{n, \psi, \kappa^*}$, we know that
\[
\xi_{n, \psi, \kappa^*} \leq \psi^{-1/2} \cdot F_n \left( \langle A^{1/2} w, \theta_{1/2} \rangle, \|\Pi_{w^1} A^{1/2} \theta_{1/2}\|_2 \right) + \frac{1}{\sqrt{p}} g^T \Pi_{w^1} A^{1/2} \theta_{1/2}
\]
\[
\leq \psi^{-1/2} \cdot F_n \left( \langle A^{1/2} w, \theta_{1/2} \rangle, \frac{1}{2} \left( \|\Pi_{w^1} A^{1/2} \theta_0\|_2 + \|\Pi_{w^1} A^{1/2} \theta_1\|_2 \right) \right) + \frac{1}{\sqrt{p}} g^T \Pi_{w^1} A^{1/2} \theta_{1/2}. \tag{E.24}
\]
where the second line follows since $F_\kappa$ is increasing w.r.t its second argument (see Lemma 5.3). Denote
\[ c_{1,0} = \langle \Lambda^{1/2} w, \theta_0 \rangle, \quad c_{1,1} = \langle \Lambda^{1/2} w, \theta_1 \rangle, \quad c_{2,0} = \| \Pi_{w^*} \Lambda^{1/2} \theta_0 \|_2, \quad c_{2,1} = \| \Pi_{w^*} \Lambda^{1/2} \theta_1 \|_2. \]
Let $c_{1,1/2} = \frac{1}{2}(c_{1,0} + c_{1,1})$ and $c_{2,1/2} = \frac{1}{2}(c_{2,0} + c_{2,1})$. From Eq. (E.23) and Eq. (E.24), we have
\[ \frac{1}{2} \left( \epsilon_n^{(2)}(\nu_0) + \epsilon_n^{(2)}(\nu_1) \right) - \epsilon_n^{(2)}(\xi_n) \geq \psi^{-1/2} \cdot \frac{1}{2} \left( F_\kappa(c_{1,0}, c_{2,0}) + F_\kappa(c_{1,1}, c_{2,0}) \right) - F_\kappa(c_{1,1/2}, c_{2,1/2}). \]
Note that $\nu_0 = c_{1,0}/\sqrt{c_{1,0}^2 + c_{2,0}^2}$ and $\nu_1 = c_{1,1}/\sqrt{c_{1,1}^2 + c_{2,1}^2}$. This gives the desired Lemma E.1.

\section*{F Some technical results}

\subsection*{F.1 Proof of KKT conditions (B.34)}

In this appendix we prove the KKT conditions (B.34) for the optimization problem (B.4). The argument is quite standard and we present it mainly for completeness.

We begin by a simple variant of the Hahn-Banach theorem, which deals with the case of two closed convex sets $C_1, C_2$ of which only one is bounded. While this is a minor technical difference from the standard setting, we do not know a good reference for this result.

\textbf{Lemma F.1.} Let $C_1$ and $C_2$ be closed convex sets in a Hilbert space $\mathcal{H}$. Assume $C_1$ is bounded, and $C_1 \cap C_2 = \emptyset$. Then, there exists some $h \in \mathcal{H}$ such that
\[ \inf_{g_1 \in C_1} \langle g_1, h \rangle > \sup_{g_2 \in C_2} \langle g_2, h \rangle. \] (F.1)

\textbf{Proof} Define the distance $\text{dist}(C_1, C_2)$ by
\[ \text{dist}(C_1, C_2) = \inf_{g_1, g_2 \in C_1, C_2} \| g_1 - g_2 \|. \] (F.2)

We claim that $\text{dist}(C_1, C_2) = \| g_1^* - g_2^* \|$ for some $g_1^* \in C_1$ and $g_2^* \in C_2$. To show this, let $g_{1,n} \in C_1$ and $g_{2,n} \in C_2$ be such that $\| g_{1,n} - g_{2,n} \| \to \text{dist}(C_1, C_2)$. Since $C_1$ is bounded, we know that $\{g_{1,n}\}_{n \in \mathbb{N}}$ is bounded and therefore $\{g_{2,n}\}_{n \in \mathbb{N}}$ is also bounded. Hence, for some $M > 0$, we have $g_{2,n} \in C_2 \cap B(M)$ for all $n \in \mathbb{N}$ (B(r) denotes the closed ball with radius $r$ in $\mathcal{H}$). Note that: (i) by Banach-Alaoglu Theorem, since $C_1$ and $C_2 \cap B(M)$ are bounded and closed, they are compact they are compact with respect to the weak-topology; (ii) the mapping $h \to \| h \|$ is lower semicontinuous with respect to the weak-topology. By (i), we can choose a weak limit point $\{g_{1,n}^*, g_{2,n}^*\} \in C_1 \times C_2$ of the sequence $\{(g_{1,n}, g_{2,n})\}_{n \in \mathbb{N}}$. By (ii), we have $\text{dist}(C_1, C_2) \geq \| g_1^* - g_2^* \|$ and therefore $\text{dist}(C_1, C_2) = \| g_1^* - g_2^* \| > 0$.

Now, we denote for each $\delta > 0$ the set
\[ C_1^\delta = \{ x : \| x - x_1 \| \leq \delta \text{ for some } x_1 \in C \}. \]

Then $C_1^\delta$ is bounded, closed and convex. Moreover, if we let $\delta = \text{dist}(C_1, C_2)/2$, then $C_1^\delta \cap C_2 = \emptyset$. By the Hahn-Banach theorem shows there exists $h \in \mathcal{H} \setminus \{0\}$ such that
\[ \inf_{g_1 \in C_1^\delta} \langle g_1, h \rangle \geq \sup_{g_2 \in C_2} \langle g_2, h \rangle. \] (F.3)

Notice that, since $h \neq 0$, we have
\[ \inf_{g_1 \in C_1} \langle g_1, h \rangle = \inf_{g_1 \in C_1^\delta} \langle g_1, h \rangle. \] (F.4)

The desired result follows by Eq. (F.3) and Eq. (F.4).

\textbf{Lemma F.2.} A point $h \in \mathcal{L}^2(\mathcal{P})$ is a minimizer of the optimization problem (B.4) if and only if it satisfies the KKT conditions (B.34). Further, by Lemma B.2, such a minimizer is unique.
Proof The probability measure $\mathbb{P}$ will be fixed throughout the proof, and hence we will drop it from the subscripts in $\mathcal{R}_{\psi,\kappa}(h)$ and $\|h\|_{\mathbb{P}}$, $\langle h, h_2 \rangle_{\mathbb{P}}$.

We define $\partial \mathcal{R}_{\psi,\kappa}(h)$ to be the following subset of $L^2(\mathbb{P})$:

$$
\partial \mathcal{R}_{\psi,\kappa}(h) := \left\{ X^{1/2} \Pi_{W^\perp} (G) + \psi^{-1/2} : X^{1/2} \left( \partial_t F_\kappa \left( \langle h, X^{1/2}W \rangle, \| \Pi_{W^\perp} (X^{1/2}h) \| \right) \right) W^\perp \right\}
$$

$$
\partial_2 F_\kappa \left( \langle h, X^{1/2}W \rangle, \| \Pi_{W^\perp} (X^{1/2}h) \| \right) \Pi_{W^\perp} (Z) : Z \in S_h \right\}
$$

$$(F.5)$$

$$
S_h = \left\{ \left\{ \| \Pi_{W^\perp} (X^{1/2}h) \|^{-1} \right\} \Pi_{W^\perp} (X^{1/2}h) \right\} \text{ if } \| \Pi_{W^\perp} (X^{1/2}h) \| \neq 0
$$

$$
\{ Z : \| Z \| \leq 1 \} \text{ if } \| \Pi_{W^\perp} (X^{1/2}h) \| = 0.
$$

A standard calculation yields the following (see [HUL13, Chapter VI]):

- For any $\Delta h \in L^2(\mathbb{P})$ and any $t \in \mathbb{R}$, we have

$$
\mathcal{R}_{\psi,\kappa}(h + t\Delta h) \geq \mathcal{R}_{\psi,\kappa}(h) + t \sup_{g \in \partial \mathcal{R}_{\psi,\kappa}(h)} \langle g, \Delta h \rangle.
$$

$$(F.6)$$

- For any $\Delta h \in L^2(\mathbb{P})$, we have

$$
\mathcal{R}_{\psi,\kappa}(h + t\Delta h) \leq \mathcal{R}_{\psi,\kappa}(h) + t \sup_{g \in \partial \mathcal{R}_{\psi,\kappa}(h)} \langle g, \Delta h \rangle + o(t) \text{ (} t \to 0)\).$$

$$(F.7)$$

We will next show that the KKT conditions (B.34)) are sufficient and necessary for $h$ to be optimal.

**Sufficiency** Suppose the KKT conditions (B.34)) hold for some primal variable $h \in L^2(\mathbb{P})$ and dual variable $s$. This implies $-sh \in \partial L_{\psi,\kappa}(h)$ for some $s \geq 0$. Now we divide our discussion into two cases.

- If $\|h\| < 1$, then $s = 0$ by the KKT conditions (B.34)). Thus, $0 \in \partial \mathcal{R}_{\psi,\kappa}(h)$. Hence, Eq. (F.6) immediately implies that $h$ is a minimizer of the optimization problem.

- If $\|h\| = 1$, then for some $s \geq 0$, we have $-sh \in \partial \mathcal{R}_{\psi,\kappa}(h)$. Now, since any feasible direction $\Delta h$ (i.e., $h + t\Delta h \in \{ h : \| h \| \leq 1 \}$ for some $t \geq 0$) must satisfy $(h, \Delta h) \leq 0$, again by Eq. (F.6) implies that $h$ is a minimizer of the optimization problem.

**Necessity** Let $h^*$ be a minimizer. Then we know that $\mathcal{R}_{\psi,\kappa}(h^* + t\Delta h) \geq \mathcal{R}_{\psi,\kappa}(h^*)$ for any $\Delta h, t$ such that $h^* + t\Delta h \in \{ h : \| h \| \leq 1 \}$. Again, we divide our discussion into two cases.

- Suppose the minimizer $h^*$ also satisfies $\|h^*\| < 1$. By Eq. (F.7), we know that, for any $\Delta h \in L^2(\mathbb{P})$,

$$
\sup_{g \in \partial \mathcal{R}_{\psi,\kappa}(h^*)} \langle g, \Delta h \rangle \geq 0.
$$

$$(F.8)$$

Thus we must have $\partial \mathcal{R}_{\psi,\kappa}(h^*) = \{ 0 \}$. Hence, the pair $(h, s) = (h^*, 0)$ satisfies the KKT conditions (B.34)).

- Suppose the minimum $h^*$ also satisfies $\|h_{\min}\| = 1$. By Eq. (F.7), we know for any $\Delta h \in L^2(\mathbb{P})$ such that $(h^*, \Delta h) < 0$ (such $\Delta h$ is a feasible direction, i.e, $h + t\Delta h \in \{ h : \| h \| \leq 1 \}$ for $t > 0$ small enough), we must have

$$
\sup_{g \in \partial \mathcal{R}_{\psi,\kappa}(h^*)} \langle g, \Delta h \rangle \geq 0.
$$

$$(F.9)$$

Since $\partial \mathcal{R}_{\psi,\kappa}(h)$ is bounded, a standard perturbation argument implies that Eq. (F.9) continues to hold for any $\Delta h \in L^2(\mathbb{P})$ such that $(h^*, \Delta h) \leq 0$. Thus, if we define the convex set $C = \{ -sh^* : s \geq 0 \}$, we have

$$
\inf_{h' \in C} (h', \Delta h) \geq 0 \text{ for some } \Delta h \in L^2(\mathbb{P}) \implies \sup_{g \in \partial \mathcal{R}_{\psi,\kappa}(h)} \langle g, \Delta h \rangle \geq 0.
$$

$$(F.10)$$
Assume $C \cap \partial \mathcal{R}_{\psi, \kappa}(h^*) = \emptyset$. Since $C, \partial \mathcal{R}_{\psi, \kappa}(h^*)$ are closed and convex, and moreover $\partial \mathcal{R}_{\psi, \kappa}(h^*)$ is bounded, Lemma F.1 implies the existence of $\Delta_h$ such that
\[
\inf_{h' \in C} \langle h', \Delta_h \rangle > \sup_{g \in \partial \mathcal{R}_{\psi, \kappa}(h)} \langle g, \Delta_h \rangle. \tag{F.11}
\]
Since $C$ is a cone, this implies $\inf_{h' \in C} \langle h', \Delta_h \rangle = 0$. This shows the existence of $\Delta_h$ such that
\[
\inf_{h' \in C} \langle h', \Delta_h \rangle = 0 \quad \text{and} \quad \sup_{g_h \in \partial \mathcal{R}_{\psi, \kappa}(h)} \langle g_h, \Delta_h \rangle < 0, \tag{F.12}
\]
which thus contradicts Eq. (F.10). Therefore, $C \cap \partial \mathcal{R}_{\psi, \kappa}(h^*) \neq \emptyset$. This means that there exists $s$ such that the pair $(h^*, s)$ satisfies the KKT conditions (B.34)).

This concludes the proof. □

References

[AB09] Martin Anthony and Peter L Bartlett, Neural network learning: Theoretical foundations, Cambridge University Press, 2009. 11

[ACHL19] Sanjeev Arora, Nadav Cohen, Wei Hu, and Yuping Luo, Implicit regularization in deep matrix factorization, arXiv:1905.13655 (2019). 1

[ALMT14] Dennis Amelunxen, Martin Lotz, Michael B McCoy, and Joel A Tropp, Living on the edge: Phase transitions in convex programs with random data, Information and Inference: A Journal of the IMA 3 (2014), no. 3, 224–294. 12

[Bar98] Peter L Bartlett, The sample complexity of pattern classification with neural networks: the size of the weights is more important than the size of the network, IEEE Transactions on Information Theory 44 (1998), no. 2, 525–536. 11

[BBV06] Maria-Florina Balcan, Avrim Blum, and Santosh Vempala, Kernels as features: On kernels, margins, and low-dimensional mappings, Machine Learning 65 (2006), no. 1, 79–94. 4, 8

[BDM+18] Jean Barbier, Mohamad Dia, Nicolas Macris, Florent Krzakala, and Lenka Zdeborová, Rank-one matrix estimation: analysis of algorithmic and information theoretic limits by the spatial coupling method, arXiv:1812.02537 (2018). 12

[BHM18] Mikhail Belkin, Daniel J Hsu, and Partha Mitra, Overfitting or perfect fitting? risk bounds for classification and regression rules that interpolate, Advances in Neural Information Processing Systems, 2018, pp. 2300–2311. 2, 4, 13

[BHMM19] Mikhail Belkin, Daniel Hsu, Siyuan Ma, and Soumik Mandal, Reconciling modern machine-learning practice and the classical bias–variance trade-off, Proceedings of the National Academy of Sciences 116 (2019), no. 32, 15849–15854. 2, 12

[BHX19] Mikhail Belkin, Daniel Hsu, and Ji Xu, Two models of double descent for weak features, arXiv:1903.07571, 2019. 2, 4, 6, 13

[BKM+19] Jean Barbier, Florent Krzakala, Nicolas Macris, Léo Miolane, and Lenka Zdeborová, Optimal errors and phase transitions in high-dimensional generalized linear models, Proceedings of the National Academy of Sciences 116 (2019), no. 12, 5451–5460. 12

[BLLT19] Peter L Bartlett, Philip M Long, Gábor Lugosi, and Alexander Tsigler, Benign overfitting in linear regression, arXiv:1906.11300 (2019). 2, 4, 13
Mohsen Bayati, Marc Lelarge, and Andrea Montanari, *Universality in polytope phase transitions and message passing algorithms*, The Annals of Applied Probability **25** (2015), no. 2, 753–822.

Peter L Bartlett and Shahar Mendelson, *Rademacher and gaussian complexities: Risk bounds and structural results*, Journal of Machine Learning Research **3** (2002), no. Nov, 463–482.

Mohsen Bayati and Andrea Montanari, *The dynamics of message passing on dense graphs, with applications to compressed sensing*, IEEE Trans. on Inform. Theory **57** (2011), 764–785.

Mohsen Bayati and Andrea Montanari, *The LASSO risk for gaussian matrices*, IEEE Trans. on Inform. Theory **58** (2012), 1997–2017.

Jean Barbier and Nicolas Macris, *The adaptive interpolation method: a simple scheme to prove replica formulas in bayesian inference*, Probability Theory and Related Fields **174** (2019), no. 3-4, 1133–1185.

Mikhail Belkin, Siyuan Ma, and Soumik Mandal, *To understand deep learning we need to understand kernel learning*, arXiv:1802.01396, 2018.

Z. Bai and J. Silverstein, *Spectral Analysis of Large Dimensional Random Matrices*, Springer, 2010.

Peter Bühlmann and Sara Van De Geer, *Statistics for high-dimensional data: methods, theory and applications*, Springer Science & Business Media, 2011.

Lenaic Chizat and Francis Bach, *A note on lazy training in supervised differentiable programming*, arXiv:1812.07956 (2018).

Thomas M Cover, *Geometrical and statistical properties of systems of linear inequalities with applications in pattern recognition*, IEEE Transactions on Electronic Computers (1965), no. 3, 326–334.

Xiuyuan Cheng and Amit Singer, *The spectrum of random inner-product kernel matrices*, Random Matrices: Theory and Applications **2** (2013), no. 04, 1350010.

Emmanuel J Candès and Pragya Sur, *The phase transition for the existence of the maximum likelihood estimate in high-dimensional logistic regression*, arXiv:1804.09753 (2018).

Yash Deshpande, Emmanuel Abbe, and Andrea Montanari, *Asymptotic mutual information for the balanced binary stochastic block model*, Information and Inference: A Journal of the IMA **6** (2016), no. 2, 125–170.

David L Donoho, Iain Johnstone, and Andrea Montanari, *Accurate prediction of phase transitions in compressed sensing via a connection to minimax denoising*, IEEE transactions on information theory **59** (2013), no. 6, 3396–3433.

David Donoho and Andrea Montanari, *High dimensional robust m-estimation: Asymptotic variance via approximate message passing*, Probability Theory and Related Fields **166** (2016), no. 3-4, 935–969.

Simon S Du, Xiyu Zhai, Barnabas Poczos, and Aarti Singh, *Gradient descent provably optimizes over-parameterized neural networks*, arXiv:1810.02054 (2018).

Noureddine El Karoui, *On the impact of predictor geometry on the performance on high-dimensional ridge-regularized generalized robust regression estimators*, Probability Theory and Related Fields **170** (2018), no. 1-2, 95–175.

Noureddine El Karoui, Derek Bean, Peter J Bickel, Chinghway Lim, and Bin Yu, *On robust regression with high-dimensional predictors*, Proceedings of the National Academy of Sciences **110** (2013), no. 36, 14557–14562.
[Mon18] Andrea Montanari, Mean field asymptotics in high-dimensional statistics: From exact results to efficient algorithms, Proceedings of the International Congress of Mathematicians, World Scientific, 2018, pp. 2957–2980. 12

[MPV87] Marc Mézard, Giorgio Parisi, and Miguel A. Virasoro, Spin glass theory and beyond, World Scientific, 1987. 12

[MVS19] Vidya Muthukumar, Kailas Vodrahalli, and Anant Sahai, Harmless interpolation of noisy data in regression, arXiv:1903.09139 (2019). 13

[Nea96] Radford M Neal, Priors for infinite networks, Bayesian Learning for Neural Networks, Springer, 1996, pp. 29–53. 2, 8

[RR08] Ali Rahimi and Benjamin Recht, Random features for large-scale kernel machines, Advances in Neural Information Processing Systems, 2008, pp. 1177–1184. 2, 4, 8

[RZ18] Alexander Rakhlin and Xiyu Zhai, Consistency of interpolation with laplace kernels is a high-dimensional phenomenon, arXiv:1812.11167 (2018). 13

[SAH19] Fariborz Salehi, Ehsan Abbasi, and Babak Hassibi, The impact of regularization on high-dimensional logistic regression, arXiv preprint arXiv:1906.03761 (2019). 12

[SC19] Pragya Sur and Emmanuel J Candès, A modern maximum-likelihood theory for high-dimensional logistic regression, Proceedings of the National Academy of Sciences 116 (2019), no. 29, 14516–14525. 3, 12

[SHN+18] Daniel Soudry, Elad Hoffer, Mor Shpigel Nacson, Suriya Gunasekar, and Nathan Srebro, The implicit bias of gradient descent on separable data, The Journal of Machine Learning Research 19 (2018), no. 1, 2822–2878. 1, 2

[SSBD14] Shai Shalev-Shwartz and Shai Ben-David, Understanding machine learning: From theory to algorithms, Cambridge University Press, 2014. 6

[ST03] Mariya Shcherbina and Brunello Tirozzi, Rigorous solution of the Gardner problem, Communications in Mathematical Physics 234 (2003), no. 3, 383–422. 15

[TAH18] Christos Thrampoulidis, Ehsan Abbasi, and Babak Hassibi, Precise error analysis of regularized m-estimators in high dimensions, IEEE Transactions on Information Theory 64 (2018), no. 8, 5592–5628. 12

[TOH15] Christos Thrampoulidis, Samet Oymak, and Babak Hassibi, Regularized linear regression: A precise analysis of the estimation error, Conference on Learning Theory, 2015, pp. 1683–1709. 4, 12, 15, 16

[Ver18a] Roman Vershynin, High-dimensional probability: An introduction with applications in data science, vol. 47, Cambridge University Press, 2018. 10

[Ver18b] ______, High-dimensional probability: An introduction with applications in data science, vol. 47, Cambridge University Press, 2018. 43

[Vil08] Cédric Villani, Optimal transport: old and new, vol. 338, Springer Science & Business Media, 2008. 3

[ZBH+16] Chiyuan Zhang, Samy Bengio, Moritz Hardt, Benjamin Recht, and Oriol Vinyals, Understanding deep learning requires rethinking generalization, arXiv:1611.03530 (2016). 1, 12