Non-linear diffusion in $R^D$ and in Hilbert Spaces, a Cylindrical/Functional Integral Study

LUIZ C.L. BOTELHO
Departamento de Matemática Aplicada,
Instituto de Matemática, Universidade Federal Fluminense,
Rua Mario Santos Braga
24220-140, Niterói, Rio de Janeiro, Brazil
e-mail: botelho.luiz@ig.com.br

Abstract

We present a proof for the existence and uniqueness of weak solutions for a cut-off and non cut-off model of non-linear diffusion equation in finite-dimensional space $R^D$ useful for modelling flows on porous medium with saturation, turbulent advection, etc. - and subject to deterministic or stochastic (white noise) stirrings. In order to achieve such goal, we use the powerful results of compacity on functional $L^p$ spaces (the Aubin-Lion Theorem). We use such results to write a path-integral solution for this problem.

Additionally, we present the rigourous functional integral solutions for the Linear Diffussion equation defined in Infinite-Dimensional Spaces (Separable Hilbert Spaces). These further results are presented in order to be useful to understand Polymer cylindrical surfaces probability distributions and functionals on String theory.
1 Introduction

The deterministic non-linear diffusion equation is one of the most important topics in the Mathematical-Physics of the non-linear evolution equation theory [1-3]. An important class of initial-value problems in turbulence has been modeled by non-linear diffusion stirred by random sources [4].

The purpose of this paper in Mathematical methods for Physics is to provide a model of non-linear diffusion were one can use and understand the compacity functional analytic arguments to produce theorems of existence and uniqueness on weak solutions for deterministic stirring in $L^\infty([0,T] \times L^2(\Omega))$. We use these results to give a first step “proof” for the famous Rosen path integral representation for the Hopf characteristic functional associated to the white-noise stirred non-linear diffusion model. These studies are presented on section II.

In section III we present a study of a Linear diffusion equation in a Hilbert Space, which is the basis of the famous Loop Wave Equations in String and Polymer surface theory ([3], [4]).

2 The Non-Linear Diffusion

Let us start our paper by considering the following non-linear diffusion equation in some strip $\Omega \times [0,T]$ with $\Omega$ denoting a $C^\infty$-compact domain of $R^D$.\footnote{See example 9.2-2 and 9.3 in "Transport Phenomena" by R. Byran Bird, Warren E. Stewart, Edwin N. Lightfoot; John Wiley & Sons, 1960, pages 272–276, 304–309.}

$$\frac{\partial U(x,t)}{\partial t} = (+\Delta U)(x,t) + \Delta(^\wedge)(F(U(x,t)) + f(x,t)) \quad (1)$$

with initial and Dirichlet boundary conditions as given below.

$$U(x,0) = g(x) \in L^2(\Omega) \quad (2)$$

$$U(x,t) \mid_{\partial \Omega} = 0 \quad (for \ t > 0) \quad (3)$$

We note that the non-linearity of the diffusion-spatial term of the parabolic problem eq(1) takes into account the physical properties of non-linear porous medium’s diffusion saturation
physical situation where this model is supposed to be applied [1] - by means of the hypothesis that the regularized Laplacean operator $\Delta^{(\wedge)}$ in the non-linear term of the governing diffusion eq.(1) has a cut-off in its spectral range. Additionally we make the hypothesis that the non-linear function $F(x)$ is a compact support real continuously differentiable function on the extended interval $(-\infty, \infty)$. The external source $f(x, t)$ is supposed to belong to the space $L^\infty([0, T] \times L^2(\Omega))$ or to be a white-noise external stirring of the form ([2] - pp. 61) when in the random case

$$F(\cdot, t) = \frac{d}{dt} \left\{ \sum_{n \in \mathbb{Z}} \sqrt{\lambda_n} \beta_n(t) \varphi_n(\cdot) \right\} = \frac{d}{dt} w(t).$$

(4)

Here $\{\varphi_n\}$ denotes a complete orthonormal set on $L^2(\Omega)$ and $\beta_n(t), n \in \mathbb{Z}$ are independent Wiener processes.

Let us show the existence and uniqueness of weak solutions for the diffusion problem above stated by means of Galerking Method for the case of deterministic $f(x, t) \in L^\infty([0, T] \times L^2(\Omega))$.

Let $\{\varphi_n(x)\}$ be spectral eigen-functions associated to the Laplacean $\Delta$. Note that each $\varphi_n(x) \in H^2(\Omega) \cap H^1_0(\Omega)$ [3]. We introduce now the (finite-dimensional) Galerkin approximants

$$U^{(n)}(x, t) = \sum_{i=1}^{n} U_i^{(n)}(t) \varphi_i(x)$$

$$f^{(n)}(x, t) = \sum_{i=1}^{n} (f(x, t), \varphi_i(x))_{L^2(\Omega)} \varphi_i(x)$$

subject to the initial-conditions

$$U^{(n)}(x, 0) = \sum_{i=1}^{n} (g(x), \varphi_i)_{L^2(\Omega)} \varphi_i(x)$$

(5)

Here $(\cdot, \cdot)_{L^2(\Omega)}$ denotes the usual inner product on $L^2(\Omega)$.

After substituting eqs.(5), (6) in eq.(1), one gets the weak form of the non-linear diffusion equation in the finite-dimension approximation as a mathematical well-defined systems of ordinary non-linear differential equations, as a result of an application of the Peano existence-solution theorem.

$$\left( \frac{\partial U^{(n)}(x, t)}{\partial t}, \varphi_j(x) \right)_{L^2(\Omega)} + \left( -\Delta U^{(n)}(x, t), \varphi_j(x) \right)_{L^2(\Omega)}$$

$$= \left( \nabla^{(\wedge)} \cdot \left[ (F'(U^{(n)}(x, t)) \nabla^{(\wedge)} U^{(n)}(x, t)], \varphi_j(x) \right) \right)_{L^2(\Omega)} + (f^{(n)}(x, t), \varphi_j(x))_{L^2(\Omega)}$$

(7)

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By multiplying the associated system eq.(7) by $U^{(n)}$ we get the diffusion equation in the finite dimensional Galerking sub-space in the integral form:

$$\frac{1}{2} \frac{d}{dt} \|U^{(n)}\|_{L^2(\Omega)}^2 + (-\Delta U^{(n)}, U^{(n)})_{L^2(\Omega)} + \int_\Omega d^3x (F'(U^{(n)})(\nabla(U^{(n)})) \cdot \nabla(U^{(n)}))(x, t) = (f, U^{(n)})_{L^2(\Omega)} \tag{8}$$

This result, by its turn, yields a prior estimate for any positive integer $p$:

$$\frac{1}{2} \frac{d}{dt} \left( \|U^{(n)}\|_{L^2(\Omega)}^2 \right) + \gamma(\Omega) \|U^{(n)}\|_{L^2(\Omega)}^2 + \|F'(U^{(n)})\frac{1}{2}(\nabla U^{(n)})\|_{L^2(\Omega)}^2 \
\leq \frac{1}{2} \left\{ p \|f(x, t)\|_{L^2(\Omega)}^2 + \frac{1}{p} \|U^{(n)}\|_{L^2(\Omega)}^2 \right\} \tag{9}$$

Here $\gamma(\Omega)$ is the Garding-Poincaré constant on the inequality of the quadratic form associated to the Laplacean operator defined on the domain $H^2(\Omega) \cap H_0^1(\Omega)$.

$$\|U^{(n)}\|_{H^1(\Omega)}^2 = (-\Delta U^{(n)}, U^{(n)})_{L^2(\Omega)} \geq \gamma(\Omega) \|U^{(n)}\|_{L^2(\Omega)}^2. \tag{10}$$

By choosing the integer $p$ big enough and applying the Gronwall lemma, we obtain that the set of function $\{U^{(n)}(x, t)\}$ forms a bounded set in $L^\infty([0, T], L^2(\Omega)) \cap L^\infty([0, T], H_0^1(\Omega))$ and in $L^2([0, T], L^2(\Omega))$. As a consequence of this boundeness property of the function set $\{U^{(n)}\}$, there is a sub-sequence weak-star convergent to a function $\overline{U}(t, x) \in L^\infty([0, T], L^2(\Omega))$, which is the candidate for our “weak” solution of eq.(1).

Another important estimate is to consider again eq.(9), but now considering the Sobolev space $H_0^1(\Omega)$ on this estimate eq.(9), namely:

$$\frac{1}{2} \left( \|U^{(n)}(T)\|_{L^2(\Omega)}^2 - \|U^{(n)}(0)\|_{L^2(\Omega)}^2 \right) + \mathcal{C}_0 \int_0^T dt \|U^{(n)}\|_{H_0^1(\Omega)}^2 \
\leq \frac{1}{2} p \left( \int_0^T \|f\|_{L^2(\Omega)}^2 dt \right) + \frac{1}{2p} \left( \int_0^T \|U^{(n)}\|_{L^2(\Omega)}^2 dt \right) < M < \infty \tag{11}$$

since we have the coerciviness condition for the Laplacean operator

$$(-\Delta U^{(n)}, U^{(n)})_{L^2(\Omega)} \geq \mathcal{C}_0(U^{(n)}, U^{(n)})_{H_0^1(\Omega)} \tag{12}$$

Note that $\|U^{(n)}(0)\|_{L^2(\Omega)}^2 \leq 2 \|g(x)\|_{L^2(\Omega)}$ (see eq.(8)) and $\{\|U^{(n)}(T)\|_{L^2(\Omega)}^2\}$ is a bounded set of real positive numbers.
As a consequence of a prior estimate of eq.(11), one obtains that the previous sequence of functions \{U^{(n)}\} ∈ \(L^\infty([0,T], H_0^1(\Omega) \cap H^2(\Omega))\) forms a bounded set on the vector valued Hilbert space \(L^2([0,T], H_0^1(\Omega))\) either.

Finally, one still has another a priori estimate after multiplying the Galerkin system eq.(7) by the time-derivatives \(\dot{U}^{(n)}\), namely (with \(f(x,t) \equiv 0\) for simplicity)

\[
\int_0^T dt \left\| \frac{dU_n(t)}{dt} \right\|^2_{L^2(\Omega)} \\
\leq \text{Real}(AU_n(T), U_n(T)) - (AU_n(0), U_n(0)) \\
+ \int_0^T dt \left\| \Delta^{(\lambda)} F(U_n(t)) \right\|_{L^2(\Omega)} \frac{dU_n}{dt} \right\|_{L^2(\Omega)} \\
\leq \frac{1}{2p} \left( \int_0^T \left\| \Delta^{(\lambda)} F(U_n(t)) \right\|_{L^2(\Omega)}^2 dt \right) \\
+ \frac{1}{2p} \left( \int_0^T \left\| \frac{dU_n}{dt} \right\|^2_{L^2(\Omega)} \right) + \gamma(\Omega) ||U_n(0)||^2
\]

(13)

By noting that

\[
\int_0^T \left\| \Delta^{(\lambda)} F(U_n(t)) \right\|_{L^2(\Omega)}^2 dt \\
\leq \left\| \Delta^{(\lambda)} \right\|_{\text{op}}^2 \times \left( \sup_{x \in [-\infty, \infty]} \{F(x)\} \right)^2 \\
\times \int_0^T dt \left\| U_n(t) \right\|^2_{L^2(\Omega)} < \infty
\]

(14)

one obtains as a further result that the set of the derivatives \(\{\frac{dU_n}{dt}\}\) is bounded in \(L^2([0,T], L^2(\Omega))\) (so in \(L^2([0,T], H^{-1}(\Omega))\)).

At this point we apply the famous Aubin-Lion theorem [3] to obtain the strong convergence on \(L^2(\Omega)\) of the set of the Galerkin approximants \(\{U_n(x,t)\}\) to our candidate \(\overline{U}(x,t)\), since this set is a compact set in \(L^2([0,T], L^2(\Omega))\) (see appendix A).

By collecting all the above results we are lead to the strong convergence of the \(L^2(\Omega)\)-sequence of functions \(F(U_n(x,t))\) to the \(L^2(\Omega)\) function \(F(\overline{U}(x,t))\).

We now assemble the above obtained rigorous mathematical results to obtain \(\overline{U}(x,t)\) as a
weak solution of eq.(1) for any test function $v(x, t) \in C_0^\infty([0, T])$, $H^2(\Omega) \cap H_0^1(\Omega))$

$$\lim_{n \to \infty} \int_0^T dt \left[ \left( U^{(n)} - \frac{dv}{dt} \right)_{L^2(\Omega)} + (-\Delta U^{(n)}, v)_{L^2(\Omega)} \right]$$

$$= \lim_{n \to \infty} \int_0^T dt (f^{(n)}, v)$$

or in the weak-generalized sense above mentioned

$$\int_0^T dt \left( \left( U(x, t), -\frac{dv}{dt} \right)_{L^2(\Omega)} 
+ (U(x, t), (-\Delta v)(x, t))_{L^2(\Omega)} 
+ (F(U(x, t), -(\Delta v)(x, t))_{L^2(\Omega)} \right)$$

$$= \int_0^T dt (f(x, t), v(x, t))_{L^2(\Omega)},$$

since $v(0, x) = v(T, x) \equiv 0$ by our proposed space of time-dependent test functions as $C_0^\infty([0, t]$, $H^2(\Omega) \cap H_0^1(\Omega))$, suitable to be used on the Rosens path integrals representations for stochastic systems (see equations (22a)-(22b) in what follows).

The uniqueness of our solution $\bar{U}(x, t)$, comes from the following lemma [4], if $F(x)$ is an injective function.

**Lemma 1.** If $\bar{U}_{(1)}$ and $\bar{U}_{(2)}$ in $L^\infty([0, T] \times L^2(\Omega))$ are two functions satisfying the weak relationship below

$$\int_0^T dt \left\{ \left( \bar{U}_{(1)} - \bar{U}_{(2)}, -\frac{dv}{dt} \right)_{L^2(\Omega)} 
+ (\bar{U}_{(1)} - \bar{U}_{(2)}, +\Delta v)_{L^2(\Omega)} 
+ (F(\bar{U}_{(1)} - F(\bar{U}_{(2)}), +\Delta v)_{L^2(\Omega)} \right\} \equiv 0$$

then $\bar{U}_{(1)} = \bar{U}_{(2)}$ a.e in $L^\infty([0, T] \times L^2(\Omega))$. The proof of eq.(17) is easily obtained by considering the family of test functions on eq.(16) of the following form $v_n(x, t) = g(t) e^{\alpha_n t} \phi_n(x)$ with $-\Delta \phi_n(x) = \alpha_n \phi_n(x)$ and $g(t) = 1$ for $(\varepsilon, T - \varepsilon)$ with $\varepsilon > 0$ arbitrary. We can see that it reduces to the obvious identity ($\alpha_n > 0$).

$$\int_{\varepsilon}^{T-\varepsilon} dt \exp(\alpha_n t) (F(\bar{U}_{(1)} - F(\bar{U}_{(2)}), \phi_n))_{L^2(\Omega)} \equiv 0,$$
which means that $F(\overline{U}_1) = F(\overline{U}_2)$ a.e on $(0,T) \times \Omega$ since $\varepsilon$ is an arbitrary number. We have thus

$$\overline{U}_1 = \overline{U}_2 \quad \text{almost everywhere} \quad (19)$$

Let us now consider a path-integral solution of eq.(1) (with $g(x) = 0$) for $f(x,t)$ denoting the white-noise stirring [4].

$$E(f(x,t)f(x',t')) = \lambda \delta^{(D)}(x-x')\delta(t-t') \quad (20)$$

where $\lambda$ is the noise-strength.

The first step is to write the generating process stochastic functional through the Rosen-Feynman path integral identities [4]

$$Z[J(x,t)] = E_f \left[ \exp \left\{ i \int_0^T dt \int_\Omega d^D x U(x,t,J(x,t)) \right\} \right] \quad (21-a)$$

$$= E_f \left[ \int D^F[U] \delta^{(F)}(\partial_t U - \Delta U - \Delta^{(\lambda)}(F(U) - f)) \right] \times \exp \left\{ i \int_0^T dt \int_\Omega d^D x U(x,t)J(x,t) \right\} \quad (21-b)$$

$$= E_f \left[ \int D^F[U] D^F[\lambda] \exp \left\{ i \int_0^T dt \int_\Omega d^D x \lambda(x,t) \times (\partial_t U - \Delta U - \Delta^{(\lambda)}(F(U) - f)) \right\} \right]$$

$$\times \exp \left\{ i \int_0^T dt \int_\Omega d^D x U(x,t)J(x,t) \right\} \quad (21-c)$$

$$= \int D^F[U] \exp \left\{ -\frac{1}{2\lambda} \int_0^T dt \int_\Omega d^D x \times (\partial_t U - \Delta U - \Delta^{(\lambda)}(F(U))^2(x,t)) \right\}$$

$$\times \exp \left\{ i \int_0^T dt \int_\Omega d^D x U(x,t)J(x,t) \right\} \quad (21-d)$$

The important step made rigorous mathematically possible on the above written (still formal) Rosen’s path integral representation by our previous rigorous mathematical analysis is the use of the delta functional identity on eq.(21-b) which is true only in the case of the existence and uniqueness of the solution of the diffusion equation in the weak sense at least for multiplier Lagrange fields $\lambda(x,t) \in C_0^\infty([0,T], H^2(\Omega) \cap H^1_0(\Omega))$.

As an important mathematical result to be pointed out is that in general case of a nonporous medium [4] in $R^3$, where one should model the diffusion non linearity by a complete Laplacean $\Delta F(U(x,t))$, one should observes that the set of (cut-off) solutions $\{U^{(\lambda)}(x,t)\}$ of
eq.(1) still remains a bounded set on $L^\infty([0,T],L^2(\Omega))$. Since we have the a priori estimate uniform bound for the $U^{(n)}$-derivatives below in $D = 3$ (with $G'(x) = F(x)$ and $F(0) = 0$). Namely:

$$
\left| \int_0^T dt \left\| \frac{dU^{(n)}}{dt} \right\|_{L^2(\Omega)}^2 \right| \leq \int_0^T dt \left( \int \Delta F(U^{(n)}(t)) \cdot \left( \frac{dU^{(n)}(t)}{dt} \right) \right) + \int \Delta G(U^{(n)}(t,x)) \cdot \left( \frac{dU^{(n)}(t)}{dt} \right) \leq \left( \int \Delta G(U^{(n)}(t,x)) \cdot \left( \frac{dU^{(n)}(t)}{dt} \right) \right) + \sup_{0 \leq t \leq T} \left\{ \frac{1}{p} \left\| f(x,t) \right\|_{L^2(\Omega)}^2 + \left\| U^{(n)}(t) \right\|_{L^2(\Omega)}^2 \right\} \leq \frac{1}{2p} \left\| f \right\|_{L^\infty((0,T),L^2(\Omega))}^2 + \frac{1}{2p} \left\| U^{(n)}(t) \right\|_{L^\infty((0,T),L^2(\Omega))}^2 < \infty $$

(22)

Where $U^{(n)}(T,x)|_{\partial \Omega} = U^{(n)}(0,x)|_{\partial \Omega} = 0$ (see eq.(3). The uniform bound for the derivatives is achieved by choosing $\frac{1}{2p} < 1$.

As another point worth to call the attention for we note that the above considered function space is the dual of the Banach space $L^1([0,T],L^2(\Omega))$. So, one can extract from the above set of cut-off solutions a candidate $\overline{U}^{(\infty)}(x,t)$, in the weak-star topology of $L^\infty([0,T],L^2(\Omega))$ for the above cited case of cut-off removing $\wedge = +\infty$ [6]. However, we will not proceed throughly in this straightforward technical question of cut-off removing in our model of non-linear diffusion in this paper for general spaces $R^D$.

Finally, we remark that in the one-dimensional case $\Omega \in R^1$, one can further show by using the same compacity methods the existence and uniqueness of the diffusion equation added with the hydrodynamic advective term $\frac{1}{2p} \partial_x (U(x,t))^2$, which turns the diffusion eq.(1) as a kind of non-linear Burger equation on a porous medium.

It appears very important to remark that Galerking methods applied directly to the finite-dimensional stochastic eq.(7) (see eq.(4)) may be saving-time computer simulation candidates for the “turbulent” path-integral eq.(22a)-eq.(22d) evaluations by approximate numerical methods ([2]-second reference).
3 The Linear Diffusion in the space $L^2(\Omega)$

Let us now present some mathematical results for the diffusion problem in Hilbert Spaces formed by square-integrable functions $L^2(\Omega)$ [5], with the domain $\Omega$ denoting a compact set of $\mathbb{R}^D$.

The diffusion equation in the infinite-dimensional space $L^2(\Omega)$ is given by the following functional differential equation (see first reference of [5] for the mathematical notation).

$$\frac{\partial \psi[f(x);t]}{\partial t} = \frac{1}{2} Tr_{L^2(\Omega)} \left( [Q D_f^2 \psi[f(x,t)]] \right), \quad \psi[f(x), t \to 0^+] = \Omega[f(x)], \quad (23)$$

Here $\psi[f(x), \cdot]$ is a time-dependent functional to be determined through the governing eq.(23) and belonging to the space $L^2(L^2(\Omega), dQ\mu(f))$ with $dQ\mu(f)$ denoting the Gaussian measure on $L^2(\Omega)$ associated to $Q$ – a fixed positive self-adjoint trace class operator $\mathbf{f}_1(L^2(\Omega))$ – and $D_f^2$ is the second – Frechet derivative of the functional $\psi[f(x), t]$ which is given by a $f(x)$-dependent linear operator on $L^2(\Omega)$ with associated quadratic form $(D_f^2 \psi[f(x), t] \cdot g(x), h(x))_{L^2(\Omega)}$.

By considering explicitly the spectral base of the operator $Q$ on $L^2(\Omega)$

$$Q \varphi_n = \lambda_n \varphi_n, \quad (24)$$

The $L^2(\Omega)$-infinite – dimensional diffusion equation takes the usual form:

$$\Psi[\sum_n f_n \varphi_n, t] = \psi^{(\infty)}[(f_n), t] \quad (25a)$$

$$\Omega[\sum_n f_n \varphi_n] = \Omega^{(\infty)}[(f_n)] \quad (25b)$$

$$\frac{\partial \psi^{(\infty)}[(f_n), t]}{\partial t} = \sum_n [(\lambda_n \Delta f_n) \psi^{(\infty)}[(f_n), t]] \quad (25c)$$

$$\psi^{(\infty)}[(f_n), 0] = \Omega^{(\infty)}[(f_n)] \quad (25d)$$

For instance

$$Q^{+1}(x, y) = I_\Omega(x)((-\Delta)^\alpha + m^2)^{-1}(x, y)I_\Omega(y) \text{ in } \mathbb{R}^n$$

with $I_\Omega(z) = \begin{cases} 1 & \text{if } z \in \Omega \\ 0 & \text{if } z \in \Omega' \end{cases}$ and $\alpha > \frac{D}{2}$. Note that $Tr_{L^2(\Omega)}(Q) = (\text{vol}(\Omega)) \int \frac{d^D p}{(2\pi)^{D/2}}$.
or in the Physicist’ functional derivative form (see ref. [5]).

\[ \frac{\partial}{\partial t} \psi[f(x), t] = \int_{\Omega} d^D x \int_{\Omega} d^D x' Q(x, x') \frac{\delta^2}{\delta f(x') \delta f(x)} \psi[f(x), t] \]  

(26a)

\[ \psi[f(x), 0] = \Omega[f(x)] \]  

(26b)

Here the integral operator Kernel of the trace class operator is explicitly given by

\[ Q(x, x') = \sum_n (\lambda_n \varphi_n(x) \varphi_n(x')) \]  

(26c)

A solution of eq.(26a) is easily written in terms of Gaussian path-integrals [5] which reads on the physicist’s notations

\[ \psi[f(x), t] = \int_{L^2(\Omega)} D^F[g(x)] \Omega[f(x) + g(x)] \times \frac{1}{t} \det \left[ \frac{1}{t} Q^{-1} \right] \]

\[ \times \exp \left\{ -\frac{1}{2t} \int_{\Omega} d^D x \int_{\Omega} d^D x' g(x) \cdot Q^{-1}(x, x') g(x') \right\} \]  

(27)

Rigorously, the correct functional measure on eq.(27) is the normalized Gaussian measure with the following Generating functional

\[ Z[j(x)] = \int_{L^2(\Omega)} d\mu Q[g(x)] \exp \left\{ \frac{i}{\Omega} \int j(x) g(x) d^D x \right\} \]

\[ = \exp \left\{ -\frac{t}{2} \int_{\Omega} d^D x \int_{\Omega} d^D x' j(x) Q^{+1}(x, x') j(x') \right\} \]  

(28)

At this point, it becomes important remark that when writting the solution as a Gaussian-path integral average as done in eq.(27), all the \( L^2(\Omega) \) functions in the functional domain of our diffusion functional field \( \psi[f(x), t] \) belongs to the functional domain of the quadratic form associated to the classe trace operator \( Q \) the so-called reproducing kernel of the operator \( Q \) which is not the whole Hilbert Space \( L^2(\Omega) \) as naively indicated on eq.(27), but the following subset of it:

\[ \text{Dom}(\psi[\cdot, t]) = \{ f(x) \in L^2(\Omega) | Q^{-\frac{1}{2}} f \in L^2(\Omega) \} \subset \neq L^2(\Omega) \]  

(29)

The above written result gives a new generalization of the famous Cameron-Martin theorem that the usual Wienner measure (defined by the one-dimensional Laplacean with Dirichlet conditions on the interval end-points) is translation invariant, i.e \( d^{\text{Wien}} \mu[f + g] = d^{\text{Wien}} \mu[f] \times \)
\[
\left(\frac{d^{\text{Wien}}[f+g]}{d^{\text{Wien}}[f]}\right), \text{ if and only if the shift function } g(x) \text{ is absolutely continuous with derivative on } L^2([a,b]). \text{ In other words } g \in H^1_0([a,b]) = \text{Dom}\left\{\sqrt{-\frac{d^2}{dx^2}}\right\}.
\]

Another point important to call the reader attention is that one can write eq.(27) in the usual form of Diffusion in finite dimensional case (see appendix B)

\[
\psi[f(x), t] = \int_{L^2(\Omega)} D^F[g(x)]\Omega[f(x) + \sqrt{t}g(x)] \times \det[Q^{-1}] \\
\exp\left\{-\frac{1}{2t} \int_{\Omega} dP x \int_{\Omega} dP x' g(x)Q^{-1}(x, x')g(x)\right\},
\]

(30)

At this point is worth call the reader attention that \(d_Q\mu\) and \(d_Q\mu\) Gaussian measures are singular to each other by a direct application of Kakutani theorem for Gaussian infinite dimensional measures for any time \(t > 0\).

\[
d_tQ\mu[g(x)]/d_Q\mu[g(x)] = +\infty
\]

(31)

Let us apply the above results for the Physical diffusion of Polymer Rings (closed strings) described by Periodic Loops \(\vec{X}(\sigma) \in \mathbb{R}^D, 0 \leq \sigma \leq T, \vec{X}(\sigma + T) = \vec{X}(\sigma)\) with a non-local diffusion coefficient \(Q(\sigma, \sigma')\) (such that \(\int_0^T d\sigma \int_0^T d\sigma' Q(\sigma, \sigma') = Tr[Q] < \infty\)). The functional governing equation in Loop Space (formed by Polymer rings) is given by

\[
\frac{\partial \psi^{(e)}[\vec{X}(\sigma); A]}{\partial A} = \int_0^T d\sigma \int_0^T d\sigma' Q_{ij}^{(e)}(\sigma, \sigma') \frac{\delta^2}{\delta \vec{X}_i(\sigma)\delta \vec{X}_j(\sigma)} \psi^{(e)}[\vec{X}(\sigma), A]
\]

(32a)

\[
\psi^{(e)}[\vec{X}(\sigma); 0] = \exp\left\{-\frac{\lambda}{2} \int_0^T d\sigma \int_0^T d\sigma' \vec{X}_i(\sigma)M_{ij}(\sigma, \sigma')\vec{X}_j(\sigma')\right\}.
\]

(32b)

Here the ring polymer surface probability distribution \(\psi^{(e)}[\vec{X}(\sigma), A]\) depends on the area parameter \(A\), the area of the cylindrical polymer surface of our surface-polymer chain. Note the presence of a parameter \(\varepsilon\) on the above written objects takes into account the local (the integral operator kernel) case \(Q(\sigma, \sigma') = \delta(\sigma - \sigma')\) as a limiting case of the rigorously mathematical well-defined (class trace) situation on the end of the observable evaluations

\[
Q_{ij}^{(e)}(\sigma, \sigma') = \frac{1}{\sqrt{\pi\varepsilon}} \exp\left(-\frac{(\sigma - \sigma')^2}{\varepsilon^2}\right) \times \delta_{ij}
\]

(33)
The solution of eq.(32a) is straightforwardly written in the case of a self-adjoint kernel $M$ on $L^2(\Omega \times \Omega)$ (with $[M, Q] = 0$).

$$\exp \left\{ -\frac{1}{2} \int_0^T d\sigma \int_0^T d\sigma' \vec{X}_i(\sigma) M_{ij}(\sigma, \sigma') \vec{X}_j(\sigma') \right\}$$

$$\times \det^{-\frac{1}{2}}[1 + A\lambda M(Q^{(e)})^{-1}]$$

$$\exp \left\{ +\frac{1}{2} \int_0^T d\sigma \int_0^T d\sigma' (\vec{M}\vec{X})_i(\sigma) \left( (\lambda M + (Q^{(e)})^{-1} \cdot \frac{1}{A}) (\vec{M}\vec{X})_j(\sigma') \right) \right\}$$

The functional determinant can be reduced to the evaluation of an integral equation

$$\det^{-\frac{1}{2}}[1 + A\lambda M(Q^{(e)})^{-1}]$$

$$= \exp \left\{ -\frac{1}{2} Tr_{L^2(\Omega)} \log(1 + \lambda AM(Q^{(e)})^{-1}) \right\}$$

$$= \exp \left\{ -\frac{A}{2} Tr_{L^2(\Omega)} \int_0^\lambda d\lambda' [(Q^{(e)})^{-1} M(1 + \lambda' A(Q^{(e)})^{-1} M)^{-1}] \right\}$$

$$= \exp \left\{ -\frac{A}{2} Tr_{L^2(\Omega)} \int_0^\lambda d\lambda' R(\lambda') \right\}$$

(35)

Here the kernel operator $R(\lambda')$ satisfies the integral equation (accessible for numerical analysis)

$$R(\lambda')(1 + \lambda' A(Q^{(e)})^{-1} M = (Q^{(e)})^{-1} M$$

(36)

Which in the local case of $\varepsilon \to 0^+$, when considered in the final result eq.(34) - eq.(35), produces the explicitly candidate solutions for our Polymer-surface probability distribution with $M$ a class trace operator on the Loop space: $L^2([0, T])$.

$$\psi[\vec{X}(\sigma), A] = \exp \left\{ -\frac{1}{2} Tr_{L^2(\Omega)} \int_0^\lambda d\lambda' [M(Q^{(e)} + \lambda' AM)^{-1}] \right\}$$

$$\times \exp \left\{ -\frac{\lambda}{2} \int_0^T d\sigma \int_0^T d\sigma' X_i(\sigma) \cdot M_{ij}(\sigma, \sigma') \vec{X}_j(\sigma') \right\}$$

$$\times \exp \left\{ +\frac{1}{2} \int_0^T d\sigma \int_0^T d\sigma' (\vec{M}\vec{X})_i(\sigma) \left( \lambda M + (Q^{(e)})^{-1} \cdot \frac{1}{A} \right) (\sigma, \sigma') (\vec{M}\vec{X})_j(\sigma') \right\}$$

(37)

It is worth calling the reader attention that if $A \in \mathfrak{f}_1$ and $B$ is a bounded operator - so $A \cdot B$ is a class trace operator-, the functional determinant $\det[1 + AB]$ is a well-defined object as a direct result of the obvious estimate, result which was used to arrive at eq.(37).

$$\lim_{N \to \infty} \prod_{n=0}^N (1 + \lambda_n) \leq \exp \left( \sum_{n=0}^N \lambda_n \right) = \exp(TrAB)$$

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As a last comment on the linear infinite-dimensional diffusion problem eq. (23), let us sketchy a (rigorous) proof that eq. (27) is the unique solution of eq. (23). Firstly, let us consider the initial condition on eq. (23) as belonging to the space of all mappings \( G : L^2(\Omega) \to \mathbb{R} \) that are twice Fréchet differentiable on \( L^2(\Omega) \) with uniformly continuous and bounded second derivative \( D^2 G \) (a bounded operator of \( \mathcal{L}(L^2(\Omega)) \) with norm \( C \)). This set of mappings will be denoted by \( uC^2[L^2(\Omega), \mathbb{R}] \). It is, thus, straightforward to see through an application of the mean value theorem that the following estimate holds true

\[
\sup_{f(x) \in Q^\frac{1}{2} L^2(\Omega)} |\psi[f(x), t] - G[f(x)]| \\
\leq \int_{L^2(\Omega)} |G(f(x) + g(x)) - G(g(x))| d_{tQ}\mu[g(x)] \\
\leq \int_{L^2(\Omega)} \left[ DG(f(x), g(x))_{L^2(\Omega)} + \int_0^1 d\sigma (1 - \sigma)(D^2 G[f(x) + \sigma g(x)]g(x), g(x)_{L^2(\Omega)}) \right] \\
\times d_{tQ}\mu[g(x)] \\
\leq 0 + C \int_0^1 d\sigma (1 - \sigma) \int_{L^2(\Omega)} \|g(x)\|^2_{L^2(\Omega)} d_{tQ}\mu[g(x)] \\
\leq C \left( \int_{L^2(\Omega)} \|g(x)\|^2_{L^2(\Omega)} d_{tQ}\mu[g(x)] \right) \\
\leq C Tr(tQ) = (C Tr(Q)t \to 0 \text{ as } t \to 0^+). \tag{38}
\]

We have thus defined a strongly continuous semi-group on the Banach Space \( UC^2[L^2(\Omega), \mathbb{R}] \) with infinitesimal generator given by the infinite-dimensional Laplacean \( Tr[QD^2] \) acting on the space \( L^2(Q^\frac{1}{2}(L^2(\Omega)), \mathbb{R}) \). By the general theory of semi-groups on Banach spaces we obtain that eq. (27) satisfies the infinite-dimensional diffusion initial value problem eq. (23), at least for initial conditions on the space \( uC^2[L^2(\Omega), \mathbb{R}] \). Since purely Gaussian functionals belong to \( uC^2[L^2(\Omega), \mathbb{R}] \) and they form a dense set on the space \( L^2(L^2(\Omega), d_{tQ}\mu) \), we get the proof of our result for general initial condition on \( L^2(L^2(\Omega), d_{tQ}\mu) \).

Finally, we point out that the general solution of the diffusion problem on Hilbert Space with sources and sinks, namely

\[
\frac{\partial}{\partial t} \psi[f(x), t] = \frac{1}{2} Tr_{L^2(\Omega)}[Q D^2 \psi[f(x), t]] - V[f(x)]\psi[f(x), t] \tag{39}
\]
with
\[
\psi[f(x), t \to 0^+] = \Omega[f(x)], \quad (40)
\]
possesses a generalized Feynman-Wiener-Kac Hilbert \( L^2(\Omega) \) space valued path integral representation, which in the Feynman Physicist formal notation reads as

\[
\psi[h(x), T] = \int_{C([0,T],L^2(\Omega))} \mathcal{D}^F[X(\sigma)] \\
\times \exp \left\{ -\frac{1}{2} \int_0^T d\sigma \left( \frac{dX}{d\sigma}, Q^{-1} \frac{dX}{d\sigma} \right)_{L^2(\Omega)} (\sigma) \right\} \\
\times \Omega \left[ \left( \int_0^T X(\sigma) d\sigma \right) + X(0) \right] \\
\times \exp \left\{ -\int_0^T d\sigma V \left( \left( \int_0^T X(\sigma') d\sigma' \right) + X(0) \right) \right\} \quad (41)
\]

Where the paths satisfy the end-point constraint \( X(T) = h(x) \in L^2(\Omega); X(0) = f(x) \in L^2(\Omega). \)
Appendix A: The Aubin-Lion Theorem

Just for completeness in this mathematical appendix for our mathematical oriented readers, we intend to give a detailed proof of the basic result on compactness of sets in function spaces of the form $L^2(\Omega)$ and throughout used on section 2. We have, thus, the Aubin-Lion Theorem\cite{3} in the Gelfand triplet $H^1_0(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) = (H^1_0(\Omega))^*$

“Aubin-Lion - If \( \{U_n(x, t)\} \) is a sequence of time-differentiable functions in a bounded set of $L^2([0, T], H^1_0(\Omega))$ such that its time derivatives forms a bounded set of $L^2([0, T], H^{-1}_0(\Omega))$, we have that $\{U_n(x, t)\}$ is a compact set on $L^2([0, T], L^2(\Omega))$”.

Proof: the basic fact we are going to use to give a mathematical proof of this theorem is the following identity (Ehrling’s lemma): For any given $\varepsilon > 0$, there is a constant $C(\varepsilon)$ such that

$$\|U_n\|_{L^2(\Omega)} \leq \varepsilon\|U_n\|_{H^1_0(\Omega)} + C(\varepsilon)\|U_n\|_{H^{-1}(\Omega)}^2$$  \(\text{(A-1)}\)

As a consequence, we have the following estimate

$$\int_0^T \|U_n - U_m\|_{L^2([0, T], L^2(\Omega))}^2 \leq \int_0^T dt(\|U_n - U_m\|_{H^1_0(\Omega)}^2 + C(\varepsilon)\|U_n - U_m\|_{H^{-1}(\Omega)}^2)^2$$

$$\leq \varepsilon^2 \left( \int_0^T dt\|U_n - U_m\|_{H^1_0(\Omega)}^2 \right) + (C(\varepsilon))^2 \left( \int_0^T dt\|U_n - U_m\|_{H^{-1}(\Omega)}^2 \right)$$

$$+ 2\varepsilon C(\varepsilon) \left( \int_0^T dt(\|U_n - U_m\|_{H^1_0(\Omega)} \times \|U_n - U_m\|_{H^{-1}(\Omega)}) \right)$$

$$\leq \varepsilon^2 \left( \int_0^T dt\|U_n - U_m\|_{H^1_0(\Omega)}^2 \right) + (C(\varepsilon))^2 \left( \int_0^T dt\|U_n - U_m\|_{H^{-1}(\Omega)}^2 \right)$$

$$+ 2\varepsilon C(\varepsilon) \left( \int_0^T dt\|U_n - U_m\|_{H^1_0(\Omega)}^2 \right)^{\frac{1}{2}} \left( \int_0^T dt\|U_n - U_m\|_{H^{-1}(\Omega)}^2 \right)^{\frac{1}{2}}$$

$$\leq 2\varepsilon^2 M + 2\varepsilon C(\varepsilon) M^{\frac{1}{2}} \left( \int_0^T dt\|U_n - U_m\|_{H^{-1}(\Omega)}^2 \right)^{\frac{1}{2}}$$

$$+ (C(\varepsilon))^2 \left( \int_0^T dt\|U_n - U_m\|_{H^{-1}(\Omega)}^2 \right)^{\frac{1}{2}}$$  \(\text{(A2)}\)

At this point, we use the Arzela-Ascoli theorem to see that $\{U_n(x, t)\}$ is a compact set on
the space $C([0, T], H^{-1}(\Omega))$ since we have the set equicontinuity:

$$
\|U_n(t) - U_m(s)\|_{H^{-1}(\Omega)} \leq \int_s^t \|U_n'(\tau)\|_{H^{-1}(\Omega)} d\tau
$$

$$
\leq |t - s|^\left(1 - \frac{1}{2}\right) \times \left(\int_0^T \|U_n'(\tau)\|^2_{H^{-1}(\Omega)} d\tau\right)^{\frac{1}{2}}
$$

$$
\leq \overline{M}|t - s|^\frac{1}{2}
$$

(A3)

It is a crucial step now by remarking that $H^1_0(\Omega)$ is compactly immerse in $L^2(\Omega)$ (Rellich Theorem). Let us not that for each $t$ (almost everywhere in $[0, T]$), $U_n(x, t)$ is a bounded set on $H^1_0(\Omega)$ since $U_n(x, t)$ belongs to a bounded set $L^2([0, T], H^1_0(\Omega))$ by hypothesis. As a consequence, $\{U_{nk}(x, t)\}$ is a compact set on $L^2(\Omega)$ (Rellich Theorem) and so in $H^{-1}(\Omega)$ almost everywhere in $[0, T]$ since $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$.

By an application of the Arzela-Ascoli theorem, there is a sub-sequence $\{U_{nk}(x, t)\}$ of $\{U_n(x, t)\}$ (and still denoted by $\{U_n(x, t)\}$) such that it converges uniformly to a given function $\overline{U}(x, t) \in C([0, T], H^{-1}(\Omega))$. As a direct result of this fact we, have that (for $T < \infty$!) for $(n, m) \to \infty$.

$$
\left(\int_0^T \|U_n - U_m\|^2_{H^{-1}(\Omega)}\right)^{\frac{1}{2}} \leq (\sup |U_n - U_m|_{C([0, T], H^{-1}(\Omega)}) \times \left(\int_0^T 1 \cdot dt\right)^{\frac{1}{2}} \to 0 \quad (A4)
$$

Returning to our estimate eq.(A2), we see that this sub-sequence is a Cauchy sequence in $L^2([0, T], L^2(\Omega))$. As a consequence, for each fixed $t \in [0, T]$ (almost everywhere), $U_n(x, t)$ converges to $\overline{U}(x, t)$ in $L^2(\Omega)$.
Appendix B: The Linear Diffusion Equation

Let us show mathematically the basic functional integral representation eq.(30) for the $L^2(\Omega)$- Space Diffusion Equation eq.(23).

As a first step for such proof, let us call the reader attention that one should consider the second order (Laplacean) $D^2U(x,t)$ as a bounded operator in $L^2(\Omega)$ in order to the operatorial composition with the positive definite class trace operator $Q$ still be a class trace operator as it is explicitly supposed in the right-hand side of eq.(23).

We thus impose as the sub-space of initial condition the Diffusion Equation eq.(23) for the (dense) vector sub-space of $C(L^2(\Omega), R)$ composed of all functionals of the form.

\[ f(x) = \int_{L^2(\Omega)} dQ\mu(p) F(p) \exp(i\langle p, x \rangle_{L^2(\Omega)}) \]  

(B1)

with $F(p) \in L^2(L^2(\Omega), dQ\mu)$.

By substituting the initial condition eq.(B1) into the integral representation eq.(30) and by using the Fubbini-Toneli Theorem to exchange the needed integrations order in the estimate below, we get:

\[ U(x,t) = \int_{L^2(\Omega)} f(x + \sqrt{t}\xi)dQ\mu(\xi) \]

\[ = \int_{L^2} dQ\mu(\xi) \left\{ \int_{L^2} dQ\mu(p) F(p) e^{i\langle p, x + \sqrt{t}\xi \rangle_{L^2}} \right\} \]

\[ = \int_{L^2} dQ\mu(p) F(p) \cdot e^{i\langle p, x \rangle_{L^2}} e^{-\frac{1}{2}t\langle p, Qp \rangle_{L^2}}. \]  

(B2)

Note that we have already proved that $U(x,t)$ is a bounded functional of $C( L^2(\Omega) \times [0, \infty]; R)$ on the basis of our hypothesis on the initial functional data eq.(B1).

At this point we observe that the second order Frechet derivatives of the Functional $\exp i\langle p, x \rangle_{L^2}$ are easily (explicitly) evaluated as \([7]\]

\[ QD^2 \left( e^{i\langle p, x \rangle_{L^2}} \right) = \left( \sum_{\ell=1}^{\infty} \lambda_\ell \frac{\partial^2}{\partial^2 x_\ell} \right) \left[ e^{i\langle \sum_{n=1}^{\infty} p_n x_n \rangle} \right] = - \langle p, Qp \rangle_{L^2} e^{i\langle p, x \rangle_{L^2}} \]  

(B3)

We have thus a straightforward proof of our claim above cited on the basis again of the
chosen initial date sub-space

\[
Tr[QD^2U(x, t)] \\
\leq \int_{L^2(\Omega)} d\mu(p) |F(p)| |\langle p, Qp \rangle| L^2 \\
\leq \left( \int_{L^2(\Omega)} d\mu(p) |F(p)|^2 \right)^{\frac{1}{2}} \left( \int_{L^2(\Omega)} d\mu(p) |\langle p, Qp \rangle| L^2(\Omega)|^2 \right)^{\frac{1}{2}} \\
\leq (TrQ)\|F\|_{L^2(L^2(\Omega), d\mu)} < \infty. \tag{B4}
\]

Now, it is a simply application to verify that eq.(B2) satisfies the Diffusion Equation in \(L^2(\Omega)\) (or in any other Separable Hilbert Space). Namely:

\[
\frac{\partial U(x, t)}{\partial t} = \int_{L^2(\Omega)} d\mu(p) F(p) e^{i\langle p, x \rangle} \left\{ -\frac{1}{2} \langle p, Qp \rangle L^2 - \frac{1}{2} \left\langle \lim_{t \to 0^+} e^{-\frac{t}{2} \langle p, Qp \rangle L^2} \right\rangle \right\} \\
\times e^{-\frac{t}{2} \langle p, Qp \rangle L^2} \tag{B5}
\]

\[
Tr_{L^2(\Omega)}[QD^2U(x, t)] = \int_{L^2(\Omega)} d\mu(p) F(p) \left\{ \frac{D^2}{2} e^{i\langle p, x \rangle} \right\} e^{-\frac{t}{2} \langle p, Qp \rangle L^2} \\
= \int_{L^2(\Omega)} d\mu(p) F(p) \left\{ -\langle p, Qp \rangle L^2 \right\} e^{i\langle p, x \rangle} e^{-\frac{t}{2} \langle p, Qp \rangle L^2} \tag{B6}
\]

with

\[
U(x, 0) = \int_{L^2(\Omega)} d\mu(p) F(p) e^{i\langle p, x \rangle} \left\{ \lim_{t \to 0^+} e^{-\frac{t}{2} \langle p, Qp \rangle L^2} \right\} = f(x). \tag{B7}
\]
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