Duality and Hidden Symmetry in 2D Hinge Structures

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Recently, a new type of duality was found in some deformable mechanical networks, which induces a hidden symmetry when the structures taking the self-dual configuration. However, these structures are usually found accidentally and the origin of this duality is still unclear. In this work, we clarify the mechanism of this duality and propose a design principle of 2D periodic self-dual structures with arbitrary complexity. We find that this duality originates from the partial center inversion (PCI) symmetry of the hinge, which gives the structure an extra freedom degree without modifying its dynamics. For 2D mechanical hinge chains, this PCI symmetry results in dynamic isomers, i.e., dissimilar chain configurations, either periodic or aperiodic, having identical dynamic modes. More importantly, it also enables us to design various 2D periodic isostatic networks with this hinge duality. At last, by further studying a 2D non-mechanical (magnonic) system, we show that the duality and the associated hidden symmetry should exist in a broad range of Hamiltonian systems.

Introduction

Space group symmetries are cornerstones of condensed matter physics [1–4]. Nevertheless, physical systems can also have hidden symmetries which cannot be captured by space group theory [5–11]. Some of these hidden symmetries were previously reported at some specific points in the Brillouin zone (BZ) [6,7]. Recently, a hidden symmetry in the full BZ induced by self-duality was discovered in 2D mechanical isostatic networks [12–13]. These structures have a low coordination number, which creates many local hinges with unconstrained freedom degrees [14–18]. By tuning the open angle of hinges $\vartheta$, one can change these structures continuously from an open to a folded state [19–23]. Interestingly, these structures with small $\vartheta$ are dual to the structures with large $\vartheta$, at which the dual counterpart of the structure is itself. At this self-dual point, hidden symmetry emerges, resulting in Kramers-like double degeneracy [24], and many other interesting phenomena, like mechanical non-abelian spintronics [12], the degeneracy of elastic modulus [25], the critically-tilted Dirac cone [13], topological corner states [26] and the symmetric boundary effect [27] etc. Nevertheless, so far, only a few meticulous structures are found to be self-dual, and the origin of the duality and the corresponding hidden symmetry remain mysterious [13,28].

In this work, we illuminate the origin of this duality and propose a design principle of 2D structures with such duality. We find that this duality, which we rephrase as ‘hinge duality’, originates from a special partial center inversion (PCI) symmetry of the hinge, which gives the structure an extra freedom degree without changing the Hamiltonian. When multiple hinges are connected into a hinge chain, the combination of local PCI generates dynamic isomerism of the chain, whose dynamic modes are exactly the same. Based on the PCI and resulting hinge duality, we also propose a design principle to generate arbitrarily complicated 2D periodic mechanical structures with self-duality. At last, we also demonstrate the existence of hinge duality in a non-mechanical magnonic system, which suggests this duality should be a generic property of hinge structure, existing in a broad range of Hamiltonian systems.

FIG. 1: (a-c): Dual transformation for a single hinge, which is composed by PCI (a $\rightarrow$ b) and a global 90° rotation (b $\rightarrow$ c), with respect to the hinge point. The arrows indicate the coordinations of vibrational freedom degrees. (d-g) Dynamic isomers of hinge chain with different hinge sequences.
spring network with \( n \) nodes. The total freedom degrees in the system thus is \((2n+1)d\), with the hinge node shared by two arms. Under the harmonic approximation, all nodes do small vibrations around their equilibrated positions, and the time-dependent position of node \( i \) in arm \( a \), \( b \) can be written as

\[
\mathbf{r}^a_i(t) = \mathbf{R}^a_i + \mathbf{u}^a_i(t); \quad \mathbf{r}^b_i(t) = \mathbf{R}^b_i + \mathbf{u}^b_i(t),
\]

(1)

respectively, where \( \mathbf{R}^a_i \) and \( \mathbf{R}^b_i \) are the equilibrated positions, \( \mathbf{u}^a_i(t) \) and \( \mathbf{u}^b_i(t) \) are the time-dependent vibrational displacement for the node \( i \in \{0,1,2,\ldots,n\} \) in arms \( a \) and \( b \), respectively. The position of the hinge node is \( \mathbf{r}_0(t) = \mathbf{r}^a_0 = \mathbf{r}^b_0 = \mathbf{R}_0 + \mathbf{u}_0(t) \). Thus \( \mathbf{R}_0^a = \mathbf{R}_0^b = \mathbf{R}_0 \) and \( \mathbf{u}_0^a = \mathbf{u}_0^b = \mathbf{u}_0 \). Since two arms are structurally identical, they are connected by the rotational operation \( \hat{U}(\theta) \) with respect to the hinge point \( \mathbf{R}_0 \), i.e., \( \mathbf{R}_a^i = \hat{R}(\theta) \cdot \mathbf{R}_b^i \) with \( \theta \) the open angle between two arms. The Hamiltonian of the system is

\[
H = H_k + \sum_{i,j} \lambda \left\{ \left( \mathbf{u}^a_i - \mathbf{u}^a_j \right) \cdot \mathbf{e}^a_{ij} \right\}^2 + \left\{ \left( \mathbf{u}^b_i - \mathbf{u}^b_j \right) \cdot \mathbf{e}^b_{ij} \right\}^2
\]

(2)

where \( H_k \) represents the kinetic energy. \( \lambda \) is the effective spring constant. \( \mathbf{e}^a_{ij} = \mathbf{R}^a_i - \mathbf{R}^a_j \) and \( \mathbf{e}^b_{ij} = \mathbf{R}^b_i - \mathbf{R}^b_j \) with \( i,j \) running over all pairwise spring (bond) connections in each arm. The vibrational displacement vector \( \mathbf{X} = \{\mathbf{u}_0^a, \mathbf{u}_1^a, \mathbf{u}_1^b, \ldots, \mathbf{u}_n^a, \mathbf{u}_n^b\} \) satisfies the dynamic equation \( m \ddot{\mathbf{X}} = \mathbf{D} \cdot \mathbf{X} \) with \( \mathbf{D} \) the dynamic matrix and \( m \) the mass of each node [31,33].

Based on general symmetry consideration, when an isolated mechanical system under the central inversion with respect to arbitrary fixed point, the Hamiltonian is invariant. However, this is usually not true for a part of the system. Nevertheless, one can prove that Hamiltonian Eq. (2) is invariant when center inversion is conducted only for one arm with respect to the hinge point, e.g.,

\[
\mathbf{R}^\sigma_i = 2\mathbf{R}_0 - \mathbf{R}^a_i, \quad i \in \{0,1,2,\ldots,n\}
\]

(3)

We call this transformation partial center inversion (PCI), as depicted in Fig. 1b. This transformation leaves the vibrational freedom degrees \( \mathbf{u}^a_i \) intact. In addition, rotation symmetry guarantees that the system is invariant after a 90° counter-clockwise rotation around the hinge point (Fig. 1c). Therefore, by defining these two consecutive transformations as \( \hat{V}_0 \), we have the commutation relationship \([\hat{V}_0, H] = 0 \) or \([\hat{V}_0, D] = 0 \). Equivalently, \( \hat{V}_0 \) can also be interpreted as the combination of operator \( \hat{K} \) that changes the open angle of the hinge from \( \theta \) to \( \theta^* = \pi - \theta \), i.e., \( \mathbf{K} \mathbf{D} \cdot \hat{\theta} = \mathbf{D} \cdot \hat{\theta}^* \), and operator \( \hat{U}_0 \) that switches the corresponding nodes in two arms \( \mathbf{R}^a_i \mapsto \mathbf{R}^b_i \) and rotates all vibrational freedom degrees by 90° at the same time, namely, \( \hat{V}_0 = \hat{K} \hat{U}_0 \). In the case of \( n = 2 \), which is shown in Fig. 1, \( \hat{U}_0 \) can be written as

\[
\hat{U}_0 = \begin{pmatrix}
\hat{r} & 0 & 0 & 0 & 0 \\
0 & \hat{r} & 0 & 0 & 0 \\
0 & 0 & \hat{r} & 0 & 0 \\
0 & 0 & 0 & \hat{r} & 0 \\
0 & 0 & 0 & 0 & \hat{r}
\end{pmatrix}
\]

(4)

where \( \hat{r} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) is the 90° rotation operator. Here, the first to fifth rows in Eq. (4) correspond to \( \mathbf{u}_0^a, \mathbf{u}_1^a, \mathbf{u}_2^a, \mathbf{u}_0^b, \) respectively. As can be seen, \( \hat{U}_0 \) performs the node switching \((a,1) \equiv (b,1) \) and \((a,2) \equiv (b,2) \), while leaves the position of hinge node unchanged. Since \( \hat{K} \) commutes with \( \hat{U}_0 \), from \([\hat{V}_0, D] = 0 \) we have

\[
\hat{U}_0 \mathbf{D} \cdot \hat{\theta} \hat{U}_0^{-1} = \mathbf{D} \cdot \hat{\theta}^*.
\]

(5)

Eq. (5) expresses the dual relationship between two hinge configurations at open angles \( \theta \) and \( \theta^* \). We call this dual relationship as “hinge duality”. Especially, \( \theta = \theta^* \) corresponds to the self-dual point at which the hinge structure remains intact under \( \hat{K} \), but the dynamic modes are transformed by \( \hat{U}_0 \). One can prove that the vibrational modes before and after this transformation are orthogonal to each other, i.e., \( \mathbf{X} \cdot \hat{U}_0 \cdot \mathbf{X} = 0 \). This guarantees that at the self-dual point, an arbitrary vibration energy (frequency) level of the system is at least double degenerated. Importantly, this hinge duality is independent of the number of freedom degrees in the arm. This means that even in the continuous limit of the arm \( (n \to \infty) \), the hinge duality is still preserved.

**Dynamic isomerism of hinge chains** From the above analysis, one can see that the PCI gives the hinge an extra freedom degree which preserves the dynamic modes of the system. When multiple hinges are inter-connected to form a hinge chain, we find these extra freedom degrees are additive, i.e., a hinge chain with \( N \) hinges has \( 2^N \) dissimilar configurations whose dynamic modes are exactly the same. We call these configurations dynamic isomers, which can be labeled by a binary sequence “10110011...” of length \( N \). Here, 1 and 0 indicates two dual states with open angle \( \theta \) and \( \theta^* \) for a single hinge. In Fig. 1d-f, we show several configurations of dynamic isomers with different sequences, where sequence “000000...” or “111111...” represents two simplest periodic hinge chains (Fig. 1g, f). We also show an intermediate configuration between these two states in Fig. 1c, and a more disordered chain configuration in Fig. 1e. It’s surprising that such a disordered chain has the same vibrational eigen-modes as that of the periodic chains. It should be noticed that mode propagation in “111111...” chain configuration is rotated by 90° compared with the original configuration with sequence “000000...”. In fact, one can also construct other kinds of periodic hinge chain, e.g., that with sequence “101010...”, in which the vibration mode propagates in a different direction. These intriguing properties
can be utilized to build a new type of flexible wave-guides without loss.

Duality in 2D periodic hinge chains  From the above analysis, one can see that the two minimal periodic hinge chain with sequence ‘000000...’ and ‘111111...’ are connected by a sequence of PCI transformations and a global 90° rotation of the whole chain. These two steps actually define the dual transformation for the periodic hinge chain, which is equivalent to directly applying the dual transformation \( V_0 \) to the unit cell of the periodic structure. To see this more clearly, we use \( l \) to mark different cells and \((l,i)\) represents the \(i\) node in cell \(l\). As shown in Fig. 2a nodes \((l,1)\) and \((l,2)\) are on arm \(a\), while nodes \((l,3)\) and \((l+1,1)\) are on arm \(b\) in cell \(l\). When operator \( V_0 \) acts on the unit cell, one has the node switching: \((l,2) \Leftrightarrow (l,3)\) and \((l,1) \Leftrightarrow (l+1,1)\). Importantly, \((l,1) \Leftrightarrow (l+1,1)\) reverses the propagation direction of vibrational modes. Thus, this transformation for periodic hinge chain should be written as \( \hat{V}_1 = \hat{K} \hat{U}_1 \) with

\[
\hat{U}_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
r & 0 & 0 & 0 & 0 & 0 \\
r & 0 & 0 & 0 & 0 & 0 \\
r & 0 & 0 & 0 & 0 & 0 
\end{pmatrix} \mathcal{I}
\]

(6)

Here, the switching between \((l,1)\) and \((l+1,1)\) is expressed as the combination of operator \(\hat{T}_{a1}\) which shifts node \((l,1)\) one period, and complex conjugation \(\mathcal{I}\) which reverses the sign of wave vector \(\mathbf{q}\). Therefore, \(\hat{U}_1\) is an anti-unitary matrix satisfying \(\hat{U}_1^2 = -1\). Since \(\hat{V}_1\) commutes with \(\mathcal{D}\), we have the dual transformation relationship for the periodic chain

\[
\hat{U}_1 \mathcal{D}(\vartheta,\mathbf{q}) \hat{U}_1^{-1} = \mathcal{D}(\vartheta^*, \mathbf{q}).
\]

(7)

Similar to that in single hinge, the above dual relationship is also independent of the number of freedom degree of the arms. In Fig. 2b, we show the phononic spectrum for the periodic hinge chains shown in Fig. 2a. We find the identical spectrum for systems at \(\vartheta = 60^\circ\) and \(\vartheta = 120^\circ\), as well as the double degeneracy in the whole BZ zone at \(\vartheta = 90^\circ\), a hallmark of self-dual point [12].

Duality in 2D periodic hinge networks  From the above analysis, one can see that the existence of a hinge and corresponding dual transformation in the unit cell is the requisite for periodic chains to achieve hinge duality. This inspires us to propose a design principle of 2D periodic networks with hinge duality: i) constructing a single hinge in which each arm of the hinge should has more than three nodes; ii) making the corresponding nodes in each arms as a pair; iii) choosing two pairs of corresponding nodes and construct two vectors that connect the nodes in each pair. These two vectors define the two lattice vectors \(\mathbf{a}_1\) and \(\mathbf{a}_2\) of 2D periodic networks. Following the above procedure, one can generate arbitrary 2D periodic structures with hinge duality by design the structure of the hinge arm. In Fig. 4a-c, we show three simplest cases of 2D periodic networks, in which the arm in the hinge is composed by three nodes, with Fig. 3: the p31m twisted Kagome structure studied in Ref. [12]. These structures have the same dual transformation \( \hat{V}_2 = \hat{K} \hat{U}_2 \) with

\[
\hat{U}_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
r & 0 & 0 & 0 & 0 & 0 \\
r & 0 & 0 & 0 & 0 & 0 \\
r & 0 & 0 & 0 & 0 & 0 
\end{pmatrix} \mathcal{I}
\]

(8)

In this dual transformation, the node switchings \((l_1,l_2,1) \rightarrow (l_1+1,l_2,1)\) and \((l_1,l_2,2) \rightarrow (l_1,l_2+1,2)\) in unit cell \((l_1,l_2)\) are related with the shifting operator \(\hat{T}_{a_1}\) and \(\hat{T}_{a_2}\) along the lattice vectors \(\mathbf{a}_1\) and \(\mathbf{a}_2\), respectively. If the hinge arm is a hourglass made up by two equilateral triangles (Fig. 3i), the corresponding 2D periodic structure is a network with p2g symmetry, which can also be obtained by twisting the standard Kagome lattice. When the hinge arm is a perfect square (Fig. 3j),
one obtains a snub square crystal with p4g symmetry. The phononic spectrums for structures in Fig. 3a,b,d,e are given in Figs. S1-S4 of Supporting Information (SI). A general proof of the dual relationship in 2D periodic hinge structures is also provided in SI based on the dynamic matrix in q space, where we find all these hinge structures have the same self-dual point at $\vartheta = 90^\circ$.

The above design principle can also have more complicated variation. In Fig. 3f, we show the network with pmg symmetry first reported in [13]. The unit cell of this structure is a hinge made up by two rhombus arms with a ‘dangling’ bond on each arm. Nevertheless, the two ‘dangling’ bonds are arranged in an opposite direction, making two arms unable to map to each other by rotation. In fact, the dual transformation in this structure involves additional PCI transformation for the dangling bond (see the dashed blue bond in Fig. 3f). This example suggests there are many other design rules based on multiple PCIs for self-dual structures with different symmetries.

Structures in Fig. 3a-f are all deformable networks, some of which are isostatic Maxwell structures. In fact, the existence of free hinge in the unit cell guarantees that the structure must be deformable. Since the dual transformation only require this single freedom degree, there must be an isostatic structure associated with this dual transformation. Furthermore, one can also see that changing the mass of the corresponding node pair, or the elastic constant for corresponding bond pair in the hinge, does not affect the hinge duality. This provides additional freedom degree to tune the dynamic properties of the hinge-dual structures.

At last, we also consider a non-mechanical system with magnonic dynamics. The magnon is the collective magnetic excitation associated with the precession of the spin moments [34]. Magnon obeys the first-order dynamic equation, which is different from the second-order one for mechanical systems (see SI for the details). In Fig. S5, we show the magnonic spectrums for the p31m twisted Kagome networks under three different open angles. Similar to the phononic systems, we observe the identical magnonic spectrums for systems at dual open angles $\vartheta = 60^\circ$ and $\vartheta^* = 120^\circ$, as well as the Kramers-like degeneracy at the self-dual point $\vartheta = 90^\circ$. These results suggest that the hinge duality could be a generic property of hinge structures, insensitive to specific Hamiltonian or dynamic of the system.

**Discussion and Conclusion** In conclusion, we unveil the mechanism of hinge duality and the corresponding duality-induced hidden symmetry in mechanical and non-mechanical hinge systems. We find the hinge structure has a unique PCI symmetry, based on which the hinge can have a Hamiltonian-invariant dual transformation between two different configurations. This intriguing...
ing property leads to the dynamic isomerism of hinge chains, which can be either periodic or non-periodic. This property can be utilized to build a new type of flexible wave-guides. It is also very interesting to explore whether the dynamic isomerism exists at the molecular scale, since many small and macro- molecules are either single hinge or hinge chain structure [35]. Furthermore, we prove that PCI symmetry also exists in 2D periodic structures, and propose simple rules to design self-dual 2D periodic structure with arbitrary complexity. This provides a guideline to fabricate self-dual metamaterials with unconventional mechanical or acoustic properties [12, 13, 25–27]. At last, we show that the hinge duality also exists in non-mechanical magnonic system. Thus we expect it to be a generic property that exists a broad range of Hamiltonian or dynamic systems. In fact, during the preparing of this work, we noticed Ref. [28], which discussed the generality of duality-induced hidden symmetry from a different point of view. We expect further breakthrough in discovering other kinds of duality-induced hidden symmetry, especially in aperiodic 2D [35] and periodic 3D isostatic systems [36].

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