Cohomological Hall algebras and perverse coherent sheaves on toric Calabi-Yau 3-folds

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Abstract

To a smooth local toric Calabi-Yau 3-fold $X$ we associate the Drinfeld double of the (equivariant spherical) Cohomological Hall algebra in the sense of Kontsevich and Soibelman. This Drinfeld double is a generalization of the notion of the Cartan doubled Yangian defined earlier by Finkelberg and others. We extend this “3d Calabi-Yau perspective” on the Lie theory furthermore by associating a root system to certain families of $X$.

By general reasons, the COHA acts on the cohomology of the moduli spaces of certain perverse coherent systems on $X$ via “raising operators”. We conjecture that the Drinfeld double acts on the same cohomology via not only by raising operators but also by “lowering operators”. We also conjecture that this action factors through the shifted Yangian of the above-mentioned root system. We add toric divisors to the story and explain the shifts in the shifted Yangian in terms of the intersection numbers with the divisors. We check the conjectures in several examples.

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1 Introduction

The paper is devoted to Cohomological Hall algebras (COHA) of toric Calabi-Yau 3-folds and their representations.

More precisely, let $X$ be a smooth local toric Calabi-Yau 3-fold which is a resolution of singularities $f : X \rightarrow Y$, where $Y$ is affine. Assume that the fibers have dimension at most 1 and each component of the exceptional fiber is $\mathbb{P}^1$. Let $D^b\text{Coh}(X)$ be the bounded derived category of coherent sheaves on $X$. The map $f : X \rightarrow Y$, by general theory of Bridgeland and Van den Bergh [3, 63], provides $D^b\text{Coh}(X)$ with a $t$-structure whose heart is the module category $\text{Mod}_{\mathcal{A}_0}$. When $\mathcal{A}_0$ is the Jacobian algebra of a quiver with potential $(Q, W)$, the general construction of Kontsevich and Soibelman [35] yields a COHA of the pair $(Q, W)$. We denote it by $H_X$ and consider to be the COHA associated to this $t$-structure. By general reasons $H_X$ acts on the cohomology of the moduli spaces of certain sheaves (e.g. torsion free sheaves, or sheaves supported on a toric divisor) on $X$ via “raising operators”. The goal of the paper is to construct various “doubles” of $H_X$ as well as their representations on the cohomology of these moduli spaces of sheaves by adding the “lowering operators”.

We expect that there exists the notion of “generalized root system” associated to the above data, which characterizes the Drinfeld double of the (equivariant spherical) COHA algebraically as the Cartan doubled Yangian of this root system. The root system is a subset in the topological K-theory of the category $\text{Mod}_{\mathcal{A}_0}$ (lattice $\Gamma$ in the notation of
endowed with a bilinear form. There is a notion of simple root (in the case of quivers with potential they correspond to the vertices of the quiver), the notion of real and imaginary roots. We will discuss generalized root systems in more detail in the future publications, but we will use the above terminology without quotations in this paper. Geometrically, the Drinfeld double is expected to act on the cohomology of the moduli space of perverse coherent system of \( X \) in the sense of \([42, 43]\). These actions factor through those of the so-called shifted Yangians. We also give a prediction for the shift (see §8.1). In particular, the shift depends on the moduli problem and the choice of stability conditions. Representations similar to ours in the case of “quiver Yangian” were considered recently in \([39]\).

1.1 The Drinfeld double

Let us summarize some properties of the Drinfeld double in question. Assuming that \( H_X \) has a shuffle description we construct the Drinfeld double \( D(H_X) \) (see \([56, 40]\) for the terminology) of \( H_X \), as well as the Drinfeld double \( D(SH) \) of the equivariant spherical subalgebra \( SH := SH_X \) of \( H_X \) in §2. The Drinfeld double \( D(SH) \) generalizes the notion of the Cartan doubled Yangian in \([17]\). We prove the following properties of \( D(SH) \):

1. There is a triangular decomposition of \( D(SH) \cong SH^- \otimes SH^0 \otimes SH^+ \), where \( SH^-, SH^+ \) are isomorphic to \( SH \), and \( SH^0 = \mathbb{C}[\psi_i(s) \mid i \in I, s \in \mathbb{Z}] \) is the algebra of polynomials of infinitely many variables.

2. The generators of \( SH^+ \) are given by \( \{ e_i^{(r)} \mid i \in I, s \in \mathbb{N} \} \). The relations among the generators can be computed by the shuffle description of \( SH \).

3. Similarly, the generators of \( SH^- \) are given by \( \{ f_i^{(r)} \mid i \in I, s \in \mathbb{N} \} \). The relations among the generators can be computed by the shuffle description of \( SH \).

4. The action of \( SH^0 \) on \( SH^+ \) is determined by formula \((14)\). This gives the relations among \( \{ \psi_i^{(s)} \} \) and \( \{ e_j^{(r)} \} \). In a similar way we obtain the relations between \( \{ \psi_i^{(s)} \} \) and \( \{ f_j^{(r)} \} \).

5. The commutation relations between \( \{ e_i^{(r_1)} \} \) and \( \{ f_j^{(r_2)} \} \) are given in Proposition \(2.3.1\), which are determined by the definition of Drinfeld double \(2.2\) (by taking \( x = b = 1 \) and \( a = e_i^{(r_1)}, y = f_j^{(r_2)} \) in \(2.2\)).

1.2 Moduli space of perverse coherent systems

For each “simple root” of the above-mentioned root system there is a corresponding subalgebra of \( D(SH_X) \). This subalgebra is either isomorphic to the Yangian of \( \mathfrak{sl}_2 \) or the infinite Clifford algebra (see \([64]\)). Moreover, our root system always has an “imaginary root”, and is in a sense of affine-type.

It follows from §8.1 that, on the cohomology of moduli spaces associated to sheaves supported on the whole \( X \), or more precisely, perverse coherent systems in the sense of \([42, 43]\), the action of the Drinfeld double \( D(SH_X) \) factors through the shifted Yangian of the generalized root system where it shifts the imaginary root.

In this paper we refer to the perverse coherent system in \([42, 43]\) as to the rank-1 perverse coherent system. They are pairs consisting of a perverse coherent sheaf supported on the exceptional fiber of \( X \to Y \) and a homomorphism to it from the structure sheaf of \( X \). On
the other hand, the action of COHA on the cohomology of the moduli spaces associated
with sheaves supported on a toric divisor in \( X \) or on the cohomology of the moduli spaces
of perverse coherent systems supported on a toric divisor is expected to lead to shifts of real
roots (see [53]). The shifts match with the prediction from physics that shifts of imaginary
roots occur when considering the D6-brane wrapping \( X \) (sheaves supported on the whole \( X \))
[29, 49] and shifts of real roots occur when considering configurations of D4-branes wrapping
various four-cycles inside \( X \) (sheaves supported on a toric divisor in \( X \)) [1, 22, 47, 48].

The remaining part of the Introduction is devoted to examples in which these expec-
tations have been verified. Let \( Y = Y_{m,n} \) be an algebraic subvariety of \( \mathbb{C}^4 \) defined by the
equation \( xy = z^m w^n \), where \((x, y, z, w)\) are the standard coordinates on \( \mathbb{C}^4 \). It is known
that there is a toric Calabi-Yau resolution of singularities \( X = X_{m,n} \rightarrow Y_{m,n} \) given by
a proper morphism. By [42, §1] there is a tilting vector bundle \( P \) on \( X_{m,n} \) such that
\( A_0 = \text{End}_{X}(P) \) is isomorphic to the Jacobian algebra of the explicitly written quiver with
potential \((Q, W) = (Q_{m,n}, W_{m,n})\). The generating function of (non-equivariant) cohomol-
gy of the moduli space of rank-1 perverse coherent systems on \( X_{m,n} \) was studied in [42]
(see also [43] for \( X = X_{1,1} \)).

We consider special cases with \( Y = Y_{1,0}, Y = Y_{1,1} \) or \( Y = Y_{2,0} \) in this paper. The general
case \( f : X \rightarrow Y \) as in the second paragraph of §1 is expected to be reduced to one of these
three cases.

If \( Y = Y_{1,0} = \mathbb{C}^3 \) then, by definition, the resolution \( X \) is \( \mathbb{C}^3 \). In the two other case \( X \)
is the total space of a rank-2 vector bundle on \( \mathbb{P}^1 \). Note that the condition that \( X \) is a
resolution of singularities of its affinization implies that \( X \) is one of the following types:

a) the resolved conifold \( X_{1,1} = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \rightarrow Y_{1,1} \);
b) resolution of singularities of the orbifold \( Y_{2,0} \), \( X_{2,0} = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}) \rightarrow Y_{2,0} =
\mathbb{C}^2/(\mathbb{Z}/2) \times \mathbb{C} \).

The quiver with potential for \( X = \mathbb{C}^3 \) is

\[
\begin{array}{c}
\bullet \\
\circ \\
\bullet
\end{array}
\]

\[
X = \mathbb{C}^3 \quad W = B_3[B_1, B_2]
\]

The quivers with potential for the other two cases are:

\[
\begin{array}{c}
\bullet \\
\circ \\
\bullet
\end{array}
\]

\[
X = \text{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \quad W = a_1b_1a_2b_2 - a_1b_2a_2b_1
\]

\[
\begin{array}{c}
\bullet \\
\circ \\
\bullet
\end{array}
\]

\[
X = \text{Tot}(\mathcal{O}(-2) \oplus \mathcal{O}) \quad W = B_2B_1B_3 - B_2B_3B_1 + B_2B_1B_3 - B_2B_3B_1
\]
1.3 Imaginary roots and shifted Yangians

The moduli space of rank-1 perverse coherent systems depends on a choice of stability condition. We refer the readers to [42] for the definition of stability condition, which we briefly recall in §6 for convenience of the reader. The space of all stability conditions is a real vector space which contains a hyperplane arrangement. The latter divides the vector space into the disjoint union of open connected components called chambers. A stability condition \( \zeta \) is said to be generic if it belongs to the complement of the union of hyperplanes, i.e. to the interior of a chamber. The moduli space \( M_\zeta \) of \( \zeta \)-stable perverse coherent systems depends only on the chamber that contains \( \zeta \) but not on the particular \( \zeta \). For \( X = X_{m,n} \) the hyperplane arrangement coincides with the one given by root hyperplanes of the affine Lie algebra of type \( A_{m+n-1} \). The chamber structure of the space of stability conditions was described in [42]. Depending on the chamber the moduli space of stable perverse coherent systems can be identified with the moduli spaces which appear in Donaldson-Thomas (DT), Pandharipande-Thomas (PT) or non-commutative Donaldson-Thomas (NCDT) theories. We will sometimes refer to them as DT, PT or NCDT moduli spaces depending on the chamber under consideration.

If \( X = X_{1,1} \) or \( X = X_{2,0} \), a stability condition is given by a pair of real numbers \( \zeta = (\zeta_0, \zeta_1) \). Walls of the chambers are labeled by non-negative integers, see [43, Figure 1]:

\[
m \zeta_0 + (m + 1) \zeta_1 = 0, m \in \mathbb{Z}_{\geq 0}, \quad \text{the DT side, (3)}
\]

\[
\zeta_0 + \zeta_1 = 0, \quad \text{the imaginary root hyperplane, (4)}
\]

\[
(m + 1) \zeta_0 + m \zeta_1 = 0, m \in \mathbb{Z}_{\geq 0}, \quad \text{the PT side. (5)}
\]

The DT moduli space corresponds to the chamber which is positioned immediately below the imaginary root hyperplane and above \( m \zeta_0 + (m + 1) \zeta_1 = 0, \) for \( m \gg 0 \). For the PT moduli space we consider the chamber which is immediately above the imaginary root hyperplane and below \( (m + 1) \zeta_0 + m \zeta_1 = 0, \) for \( m \gg 0 \).

One can describe \( M_\zeta \) in terms of a quiver with potential. For \( X = \mathbb{C}^3 \) this is straightforward, see §5 for the details. The case \( X = X_{1,1} \) is discussed in §6.1.

Assume in general that \( (Q, W) \) is the quiver with potential such that \( A_0 = \text{End}_X(\mathcal{P}) \) is isomorphic to the Jacobian algebra of \( (Q, W) \) [42] §1. From \( (Q, W) \) one constructs a framed quiver \( \tilde{Q} \) with the framed potential \( \tilde{W} \). The framed quiver is obtained from \( Q \) by adding an additional framing vertex as well as an arrow from the framing vertex to one of the vertices of \( Q \). We define \( \tilde{W} = W \). The case of \( X_{1,1} \) is depicted below.

\[
\begin{align*}
V_0 & \quad \overset{a_1}{\longrightarrow} \quad V_1 \\
& \quad \overset{a_2}{\longrightarrow} \\
& \quad \overset{\iota}{\downarrow} \\
1 & \quad \overset{b_1}{\longrightarrow} \quad V_0 \\
& \quad \overset{b_2}{\longrightarrow}
\end{align*}
\]

Extended quiver \( \tilde{Q} \) for \( X_{1,1} = \text{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \) (6)

The stability condition \( \zeta \) defines the corresponding stability condition \( \tilde{\zeta} \) for the abelian category of representations of \( (Q,W) \) (i.e. the abelian category of representations of the corresponding Jacobi algebra). It is given in terms of the slope function \( \theta_\tilde{\zeta} \) (see §6). The
moduli space $\mathfrak{M}_\zeta$ is isomorphic to moduli space of $\theta_\zeta$-stable representations of $(\tilde{Q}, \tilde{W})$ with 1-dimensional framing.

Notice that $X$ admits an action of the torus $T \simeq (\mathbb{C}^*)^2$ which preserves the canonical class. Denote by $\mathcal{P}$ be the tilting vector bundle on $X$. The tilting object $\mathcal{P}$ is equivariant with respect to this $T$-action, and hence induces a $T$-action on $\mathfrak{M}_\zeta$.

The following result should be true in general, but we formulate it in the case of $X_{1,1}$.

**Proposition 1.3.1** Let $\zeta$ be a generic stability condition. Then there is an action of $\mathcal{H}_X$ on $H^*_{c,T}(\mathfrak{M}_\zeta, \varphi_{tr W} Q)^\vee$.

The $\mathcal{H}_X$-action in Proposition 1.3.1 is given by the “raising operators”. We expect the action of $\mathcal{S}\mathcal{H}_X$ in Proposition 1.3.1 extends to its Drinfeld double $D(\mathcal{S}\mathcal{H}_X)$ by adding the action of “lowering operators”. Furthermore, the action of the Drinfeld double factors through the shifted Yangian where it shifts the imaginary root.

Let $\mathcal{E} = (z_1, z_2, \cdots, z_l) (l \in \mathbb{Z})$ be complex parameters. Definition of the shifted Yangian $Y_l(\mathcal{E})$ of $\hat{\mathfrak{gl}}(1)$ as an associative algebra depending on the parameters $\mathcal{E}$ is recalled in §4. We refer to it as the Yangian which is $l$-positively ($-l$-negatively) shifted in parameters $\mathcal{E}$ if $l > 0$ (if $l < 0$). When $l = 0$ and there are no parameters, and we obtain the usual affine Yangian $Y_{h_1, h_2, h_3}(\hat{\mathfrak{gl}}(1))$. Here the parameters $h_i, 1 \leq i \leq 3$ correspond to the weights of the Calabi-Yau action of the torus $(\mathbb{C}^*)^3$, hence $h_1 + h_2 + h_3 = 0$. We hope the reader will not confuse the subscripts in the cases of shifted and usual Yangians.

The relation between shifted Yangians and COHAs can be summarized in our case in the following way.

**Theorem 1.3.2**

1. Let $X = \mathbb{C}^3$. The action of $\mathcal{H}_X$ on $H^*_{c,T}(\bigsqcup_{n \in \mathbb{N}} \text{Hilb}^n(\mathbb{C}^3), \varphi_{tr W} Q)^\vee$ extends to $Y_{-1}(z_1)$.

2. Let $X = X_{1,1}$ be the resolved conifold. Then for a generic stability condition on the PT-side of the imaginary root hyperplane, the action of $Y_{h_1, h_2, h_3}(\hat{\mathfrak{gl}}(1))^+$ on $H^*_{c,T}(\mathfrak{M}_\zeta, \varphi_{tr W} Q)^\vee$ extends to the one of $Y_{1}(z_1)$.

**Remark 1.3.3**

a) The algebra $Y_1(z_1)$ can also be realized as the deformed double current algebra for $\hat{\mathfrak{gl}}(1)$. The fact that this algebra acts on the PT-moduli space of the resolved conifold was originally conjectured by Costello from holography considerations in M-theory [6, § 14 Conjecture].

b) We expect the same calculation goes through for an arbitrary $X_{m,n}$. We expect that the action of $\mathcal{H}_{X_{m,n}}$ extends to the action of the shifted versions of affine Yangian $\hat{\mathfrak{g}}(m|n)$ [53, 54]. The above discussion suggests that the choice of stability condition leads to the shift of imaginary roots in the case when we consider the action on the cohomology of the moduli spaces of torsion free sheaves on $X_{m,n}$.

### 1.4 Real roots, toric divisors and shifted Yangians

We are going to illustrate the idea in a simple example. Consider the Calabi Yau 3-fold $X_{2,0} = T^* \mathbb{P}^1 \times \mathbb{C}$ and the effective divisor $D$ to be fiber of $T^* \mathbb{P}^1 \times \mathbb{C} \to \mathbb{P}^1$ over the

---

1 See also closely related discussion in [14, 21, 23, 35, 37, 38] and references therein.
north pole or over the south pole of $\mathbb{P}^1$. Depending on the choice of the toric divisor the corresponding geometry can be depicted as in the following toric diagram

\[
\begin{array}{c}
1 \\
\end{array}
\begin{array}{c}
\end{array}
\begin{array}{c}
\end{array}
\]

In Section 7.1 we obtain a quiver $Q$ with potential $W$ associated with the pair consisting of the Calabi Yau 3-fold $T^*\mathbb{P}^1 \times \mathbb{C}$ and one of the above toric divisors $D$. The construction uses a $\mathbb{Z}/2\mathbb{Z}$ symmetry of the quiver with potential associated with the pair consisting of the Calabi Yau 3-fold $\mathbb{C}^3$ and the effective divisor which is the coordinate plane.

By the dimension reduction, $(Q, W)$ can be described as the following “chainsaw quiver”

\[
\begin{array}{c}
B_3 \\
V_0 \\
B_1 \\
V_1 \\
\tilde{B}_3 \\
\end{array}
\begin{array}{c}
B_1 \\
\tilde{B}_1 \\
\end{array}
\begin{array}{c}
I_{13} \\
\triangle 1 \\
J_{13} \\
\end{array}
\]

with relations

\[
\tilde{B}_1 \tilde{B}_3 - B_3 \tilde{B}_1 = I_{13} J_{13}, \quad \tilde{B}_3 B_1 = B_1 B_3
\]

Analogously to the K-theory case in [46], the $(1, 0)$-shifted Yangian of $\widehat{\mathfrak{gl}}(2)$ (see Definition in §7.1) acts on the cohomology of the chainsaw quiver variety. Hence it also acts on

\[
\bigoplus_{V_0, V_1} H^*_{\epsilon, \text{GL}(V_0) \times \text{GL}(V_1) \times T}(\text{Rep}(Q, V_0, V_1)^{st}, \varphi_{trW})^\vee.
\]

Here the superscript $st$ refers to the locus of stable representations.

In physics language the above example describes a stack of $D4$-branes wrapping a union of smooth toric divisors [1, 22, 47, 48]. The corresponding moduli space is the moduli space of pure sheaves supported on the divisors, with the generic rank at each component specified by the number of the $D4$-branes. We expect that the Yangian shifted by real simple roots acts on the cohomology of the corresponding moduli space. The shift is determined by the intersection number of the $\mathbb{P}^1$ corresponding to the real simple root and the divisor, counted with multiplicity given by the number of $D4$-branes.

**Contents of the paper**

In §2 following the original approach of Kontsevich and Soibelman we recall the definition of COHA associated to a smooth local toric Calabi-Yau 3-fold. When the COHA has a shuffle description, we construct the Drinfeld double of the equivariant spherical COHA.
In the Drinfeld double of the spherical COHA, one has a commutative subalgebra which is isomorphic to the polynomial algebra with generators labelled by simple roots and \( \mathbb{Z} \). We refer this subalgebra as the Cartan doubled subalgebra. We define the shifted Yangian as the quotient of the Drinfeld double by certain relations among the generators of the Cartan doubled subalgebra, which are determined by a coweight. For a specific choice of the quiver with potential, we show in §3 that the Cartan doubled Yangian of a Kac-Moody Lie algebra defined in [17] surjects to the Drinfeld double of the equivariant spherical COHA of this quiver with potential. We also recall the description of COHA of \( \mathbb{C}^3 \) from our previous paper. Furthermore, we prove that the Drinfeld double corresponding to the resolved conifold has a quotient algebra which is isomorphic to the affine Yangian of \( \hat{\mathfrak{g}}(1) \). In §4 we discuss the shifted affine Yangians of \( \hat{\mathfrak{g}}(1) \). We also discuss the relationship of COHA to quantized Coulomb branch algebras of the Jordan quiver gauge theory. We connect this discussion with a special case of the conjecture of Costello.

In §5 we construct geometrically an action of the 1-negatively shifted affine Yangian on the cohomology of the Hilbert scheme of \( \mathbb{C}^3 \). In §6 we construct geometrically an action of the one-positively shifted affine Yangian on cohomology of the PT moduli space of the resolved conifold. In §7 we construct geometrically an action of shifted affine Yangian on the cohomology of the moduli of perverse coherent systems supported a toric divisor in the resolved \( \mathbb{Z}_2 \)-orbifold.

In §8 we make a proposal in the general case. Namely, we expect the action of the Drinfeld double of the spherical COHA on more general moduli spaces of perverse coherent sheaves on the general toric Calabi-Yau 3-fold \( X_{m,n} \). The action of the Drinfeld double is expected to factor through the shifted Yangian, with the shift determined by certain intersection numbers. We verify this proposal in the examples above.

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2 The Drinfeld double of COHA associated to a quiver with potential

2.1 The Hopf algebra structure

Let $R$ be a commutative integral domain of characteristic zero. Let $H$ be a Hopf algebra over $R$. By definition, $H$ possesses the following $R$-linear operations

1. multiplication $\mu : H \otimes H \to H$,
2. unit $\eta : R \to H$,
3. comultiplication $\Delta : H \to H \otimes H$,
4. co-unit $\epsilon : H \to R$
5. antipode $S : H \to H$

which satisfy the following conditions

(a) $(H, \mu, \eta)$ is an associative $R$-algebra.
(b) $(H, \Delta, \epsilon)$ is an associative $R$-coalgebra.
(c) $\Delta, \epsilon$ are algebra homomorphisms.
(d) $\mu(S \otimes 1)\Delta = \mu(1 \otimes S)\Delta = \eta \epsilon$.

We denote by $H$ the cohomological Hall algebra (COHA for short) associated with a quiver with potential $(Q, W)$ defined by Kontsevich and Soibelman [35]. We briefly recall the definition of $H$. We write $Q = (I, H)$ with $I$ being the set of vertices and $H$ the set of arrows. For a dimension vector $v = (v_i)_{i \in I} \in \mathbb{N}^I = \mathbb{Z}_{\geq 0}^I$, let $\text{Rep}(Q, v)$ be the algebraic variety (in fact with non-canonical structure of a vector space) parameterizing representations of $Q$ on the $I$-graded complex vector space whose degree $i \in I$-piece is $\mathbb{C}^{v_i}$. On $\text{Rep}(Q, v)$ there is an action of $\text{GL}_v \times T$ where $\text{GL}_v = \prod_{i \in I} \text{GL}_{v_i}$ acts by changing basis and the action of the torus $T = (\mathbb{C}^*)^2$ preserves $W$ (below the $T$-action is induced from its Calabi-Yau action on $X$ and hence on $\text{End}_{\text{Coh}(X)}(P)$). The trace of $W$ gives rise to the regular function $\text{tr} W$ on $\text{Rep}(Q, v)$ which is invariant under the $(\text{GL}_v \times T)$-action. In this notation the equivariant COHA of $(Q, W)$, is a $\mathbb{N}^I$-graded vector space

$$H = H^{(Q, W)} := \bigoplus_{v \in \mathbb{N}^I} H_v$$

with $H_v = H^*_{c, \text{GL}_v \times T}(\text{Rep}(Q, v), \varphi_{\text{tr} W})^\vee$, endowed with the multiplication defined in [35 § 7.6]. We also denote $\bigcup_{v \in \mathbb{N}^I} \text{Rep}(Q, v)$ by $\text{Rep}(Q)$. Let $R := \mathbb{C}[\text{Lie}(T)]$ be the ring of functions on $\text{Lie}(T)$. Then $H$ is an $R$-algebra.

We now construct the Drinfeld double of $H$. To do this, we assume $H$ has a shuffle description. Explicitly, for $v \in \mathbb{N}^I$, let $\mathfrak{g}_v := \prod_{i \in I} \mathfrak{gl}_{v_i}$ be the Lie algebra, and $\mathfrak{h}_v \subset \mathfrak{g}_v$ be its Cartan subalgebra. The Weyl group $S_v$ naturally acts on the ring of functions $\mathbb{C}[\mathfrak{h}_v]$. For each $v \in \mathbb{N}^I$, $\mathbb{C}[\mathfrak{h}_v]$ is a polynomial ring with variables $\{x_{1(i)}, x_{2(i)}, \ldots, x_{v_i(i)} | i \in I\}$. For simplicity, we will denote this set of variables by $x_{[1,v]}$. 
Let $H$ be the shuffle algebra defined as follows. As a vector space, we have $H = \oplus_{v \in \mathbb{N}^I} R[h_v]^{S_v}$. Consider the embeddings

$$R[h_v]^{S_{v_1}} \subset R[h_{v_1+v_2}]^{S_{v_1} \times S_{v_2}}$$

$$R[h_v]^{S_{v_2}} \subset R[h_{v_1+v_2}]^{S_{v_1} \times S_{v_2}}$$

For any pair $(p, q)$ of positive integers, let $Sh(p, q)$ be the subset of $S_{p+q}$ consisting of $(p, q)$-shuffles (permutations of $\{1, \cdots, p + q\}$ that preserve the relative order of $\{1, \cdots, p\}$ and $\{p+1, \cdots, p+q\}$). Set $Sh(v_1, v_2) := \prod_i Sh(v_1^{(i)}, v_2^{(i)})$. The multiplication of $H$ given as

$$* : R[h_{v_1}]^{S_{v_1}} \otimes R[h_{v_2}]^{S_{v_2}} \to R[h_{v_1+v_2}]^{S_{v_1+v_2}}$$

$$f \ast g = \sum_{\sigma \in Sh(v_1, v_2)} \sigma(f \cdot g \cdot \text{fac}(x_{[v_1]} | x_{[v_1+v_2]})),$$

where $\text{fac}(x_{[v_1]} | x_{[v_1+1,v_1+v_2]}) \in (R[h_{v_1}]^{S_{v_1}} \otimes R[h_{v_2}]^{S_{v_2}})_{\text{loc}}$ is an explicit rational function with denominator $\prod_{i,v_1} \prod_{b,v_1} (a_n^{(i)} - x_b^{(i)})$. Moreover, it has the property that

$$\text{fac}(x_{A_1 \cup A_2} | x_B) = \text{fac}(x_{A_1} | x_B) \text{fac}(x_{A_2} | x_B),$$

for any subsets $x_{A_1}, x_{A_2}, x_{A_1 \cup A_2}, x_B$ of the collection of variables $x_{[v_1]}$. In the case that $(Q, W)$ has a cut, the formula of $\text{fac}(x_A | x_B)$ can be found in [68, Appendix A]. We do not need the explicit expression in this paper. Here the subindex $\text{loc}$ stands for localization with respect to the divisor defined by $\text{fac}(x_A | x_B)$.

Let $H^0 := \mathbb{C}[\psi_{i,r} | i \in I, r \in \mathbb{N}]$ be the polynomial ring with infinitely many formal variables. Let $\psi_i(z) = 1 + \sum_{r \geq 0} \psi_{i,r} z^{-r-1} \in H^0[z^{-1}]$ be the generating series of generators $\psi_{i,r} \in H^0$. Similar to [66], we define the extended shuffle algebra $H^* := H^0 \ltimes H$ using the $H^0$-action on $H$ by

$$\psi_i(z) g |_{x_{[v_1]}} z^{-r-1} := \frac{\text{fac}(z | x_{[v_1]})}{\text{fac}(x_{[v_1]} | z)}, \text{ for any } g \in H_v$$

Following the same construction as in [66], we define a localized coproduct on $H^*$

$$\Delta : H^* \to \sum_{v_1 + v_2 = v} (H^*_{v_1} \otimes H^*_{v_2})_{\text{loc}}.$$
where $\psi_{[1,v_1]}(x_{[v_1+1,v_2]}) := \prod_{k \in I} \prod_{j=v_{e_1(k)}+1,\ldots,v_{e_2(k)}} \psi_k(x_j^{(k)})$. The subindex $[1,v_1]$ indicates that the factor $\psi_k(x_j^{(k)}) = 1 + \sum_{v=0}^\infty \psi_{k,v} \otimes (x_j^{(k)})^{-v-1}$ lies in $H_{e_1,loc} \otimes H_{e_2,loc}$. Define the co-unit

$$
\epsilon : H^e \to R, \psi_i(z) \mapsto 1, f(x_{[1,v_i]} \mapsto 0.
$$

for $v \neq 0$, $\psi_i(z) \in H^0[[z^{-1}], f(x_{[1,v_i]} \in H_v$. Define the antipode $S : H^e_{loc} \to H^e_{loc}$ as follows.

$$
\psi_i(z) \mapsto \psi_i^{-1}(z) \quad (12)
$$

$$
f(x_{[1,v_i]} \mapsto (-1)^{|v|} \psi_{[1,v]}^{-1}(x_{[1,v]} f(x_{[1,v_i]} \quad (13)
$$

for $\psi_i(z) \in H^0[[z^{-1}], f(x_{[1,v_i]} \in H_v$. Here $\psi_{[1,v]}(x_{[1,v]} = \prod_{k \in I} \Pi_{j=v_{e_1(k)}+1,\ldots,v_{e_2(k)}} \psi_k(x_j^{(k)})$. The subindex $[1,v_1]$ indicates that $\psi_{[1,v]}^{-1}(x_{[1,v]}$ lies in $H^e_v$. We extend $S$ to $H^e$ by requiring $S$ to be an anti-homomorphism. (i.e. $S(a \star b) = S(b) \star S(a)$). In particular, when $v = e_i$, we have

$$
S(E_i(u)) \mapsto -\psi_i^{-1}(x_i)E_i(u).
$$

Clearly, the above assignment defines an anti-homomorphism on $H^0_{loc}$.

**Lemma 2.1.1**

1. Choose $a = f(x_{[1,v_1]} \in H_{e_1}$, $b = g(x_{[1,v_2]} \in H_{e_2}$. We have $S(a \star b) = S(b) \star S(a)$.  

2. The map $S$ respects the action $H_{e_1}$.  

**Proof.** For (1): We have

$$
S(g(x_{[1,v_2]} \star S(f(x_{[1,v_1]} = (-1)^{|v_2|}(-1)^{|v_1|} \psi_{[1,v_2]}^{-1}(x_{[1,v_2]} g(x_{[1,v_2]} \star \psi_{[v_2+1,v]}^{-1}(x_{[v_2+1,v]}(f(x_{[v_2+1,v]})
$$

$$
= (-1)^{|v|} \psi_{[1,v_2]}^{-1}(x_{[1,v_2]} g(x_{[1,v_2]} \cdot \frac{\text{fac}(x_{[v_2+1,v]}(x_{[1,v]})}{\text{fac}(x_{[v_2+1,v]}(x_{[1,v]}))} f(x_{[v_2+1,v]})
$$

$$
= (-1)^{|v|} \psi_{[1,v]^{-1}(x_{[1,v]}) \sum_{\sigma \in S_{v_2-v_1}} \sigma \left( g(x_{[1,v_2]} \cdot f(x_{[v_2+1,v]} \cdot \frac{\text{fac}(x_{[v_2+1,v]}(x_{[1,v_2]}))}{\text{fac}(x_{[v_2+1,v]}(x_{[1,v_2]}))}
$$

$$
= (-1)^{|v|} \psi_{[1,v]^{-1}(x_{[1,v]}) \left( f(x_{[1,v_1]} \star g(x_{[1,v_2]}\right).
$$

We now prove (2). We have

$$
S(\psi_i(z) f(x_{[1,v]}) \psi_i(z)^{-1}) = \psi_i(z) S(f(x_{[1,v]})) \psi_i(z)^{-1} = (-1)^{|v|} \psi_i(z) \psi_{[1,v]}^{-1}(x_{[1,v]} S(f(x_{[1,v]})) \psi_i(z)^{-1}
$$

$$
= (-1)^{|v|} \psi_{[1,v]}^{-1}(x_{[1,v]} \psi_i(z) f(x_{[1,v]}) \psi_i(z)^{-1}
$$

$$
= (-1)^{|v|} \psi_{[1,v]}^{-1}(x_{[1,v]} f(x_{[1,v]}) \frac{\text{fac}(z|x_{[1,v]}(x_{[1,v]}))}{\text{fac}(x_{[1,v]}(x_{[1,v]}))}
$$

On the other hand, we compute

$$
S(f(x_{[1,v]}) \frac{\text{fac}(z|x_{[1,v]}(x_{[1,v]}))}{\text{fac}(x_{[1,v]}(x_{[1,v]}))} = (-1)^{|v|} \psi_{[1,v]}^{-1}(x_{[1,v]} f(x_{[1,v]}) \frac{\text{fac}(z|x_{[1,v]}(x_{[1,v]}))}{\text{fac}(x_{[1,v]}(x_{[1,v]}))}
$$

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This completes the proof. ■.

The above Lemma shows the assignments \([12, 13]\) determines a unique anti-homomorphism on \(H_{loc}^f\).

**Lemma 2.1.2** The anti-homomorphism \(S\) satisfies the axiom \(\mu(S \otimes 1) \Delta = \mu(1 \otimes S) \Delta = \eta c\).

**Proof.** We have

\[
\mu(S \otimes 1) \Delta(\psi_i(w)) = \mu(S \otimes 1)(\psi_i(w) \otimes \psi_i(w)) = \mu(\psi_i^{-1}(w) \otimes \psi_i(w)) = 1.
\]

Similarly, \(\mu(1 \otimes S) \Delta(\psi_i(w)) = 1\).

For a dimension vector \(v = (v^i)_{i \in I} \in \mathbb{N}^I\), we define \(|v| := \sum_{i \in I} v^i\) and \((\begin{smallmatrix} v \\ v_1 \end{smallmatrix}) := \prod_{i \in I} (v^i_{v_1})\).

For \(f(x[1,v]) \in H_v\), we have

\[
\mu(S \otimes 1) \Delta(f(x[1,v])) = \mu(S \otimes 1) \sum_{v_1+v_2=v} \frac{\psi[1,v_1](x_{v_1+1,v}) f(x[1,v_1] \otimes x_{v_1+1,v})}{\text{fac}(x_{v_1+1,v} | x_{[1,v_1]})} = 0.
\]

In the second equality, the denominator becomes \(\text{fac}(x_{1,v_1}|x_{v_1+1,v})\) since we switch \(S(\psi[1,v_1](x_{v_1+1,v}) = \psi^{-1}_{[1,v_1]}(x_{v_1+1,v})\) in front of \(f\) using

\[
\psi^{-1}_{[1,v_1]}(x_{v_1+1,v}) f(x[1,v_1] \otimes x_{v_1+1,v}) = f(x[1,v_1] \otimes x_{v_1+1,v}) \psi^{-1}_{[1,v_1]}(x_{v_1+1,v}) \frac{\text{fac}(x_{v_1,v} | x_{v_1+1,v})}{\text{fac}(x_{v_1+1,v} | x_{[1,v_1]})}.
\]

Similarly, we compute

\[
\mu(1 \otimes S) \Delta(f(x[1,v])) = \mu(1 \otimes S) \sum_{v_1+v_2=v} \frac{\psi[1,v_1](x_{v_1+1,v}) f(x[1,v_1] \otimes x_{v_1+1,v})}{\text{fac}(x_{v_1+1,v} | x_{[1,v_1]})} = 0.
\]

This completes the proof. ■.

### 2.2 The Drinfeld double of COHA

In this section, we follow the terminology from the book [30] Chapter 3. Let \(A\) and \(B\) be two Hopf algebras. A skew-Hopf pairing of \(A\) and \(B\) is an \(R\)-bilinear function

\[
(\cdot, \cdot) : A \times B \to R
\]

that satisfies the conditions
Let us take $\mathcal{A}$ equipped with opposed comultiplication described as follows.

We consider $H^{\text{coop}}_{\text{loc}}$ that satisfies $H^{\text{coop}}_{\text{loc}} = \mathcal{H}^0 \otimes H_{\text{loc}}$ as associative algebras, but is equipped with opposed comultiplication described as follows.

1. $\phi_i(z) := (-1)^{l_i+1}\psi_i(z)$, where $l_i$ is the number of loops at vertex $i \in I$.

2. The $\mathcal{H}^0$–action on $H$ is by

$$\phi_i(z)g\phi_i^{-1}(z) := \frac{\text{fac}(z|_{[1,v_i]})}{\text{fac}(z|_{[1,v]})}g, \text{for any } g \in H_v \quad (14)$$

3. The coproduct on $H^{\text{coop}}_{\text{loc}}$ is given by

$$\Delta_B : H^{\text{coop}}_{\text{loc}} \to H^{\text{coop}}_{\text{loc}} \otimes H^{\text{coop}}_{\text{loc}},$$

$$\phi_i(w) \mapsto \phi_i(w) \otimes \phi_i(w),$$

$$g(x_{[1,v]}) \mapsto \sum_{v_1 + v_2 = v} \frac{g(x_{[1,v_1]} \otimes x_{[v_1+1,v]})\phi_{[v_1+1,v]}(x_{[1,v_2]})}{\text{fac}(x_{[v_1+1,v]}|x_{[1,v_1]})}$$

4. The antipode $S_B$ is given by

$$\phi_i(z) \mapsto \phi_i(z)^{-1}$$

$$g(x_{[1,v]}) \mapsto (-1)^{|v|}g(x_{[1,v]})\phi_{[1,v]}^{-1}(x_{[1,v]})$$

Similar to the proofs in the case of $H^t$, one easily verify that $H^{\text{coop}}_{\text{loc}}$ is a Hopf algebra.

Let us take $A$ to be the Hopf algebra $H^{\text{loc}}_t$ constructed in 2.1 and $B$ to be the Hopf algebra $H^{\text{coop}}_{\text{loc}}$. We now construct a bilinear skew-Hopf pairing between $H^{\text{loc}}_t$ and $H^{\text{coop}}_{\text{loc}}$ as follows:

$$\langle \cdot, \cdot \rangle : H^{\text{loc}}_t \otimes H^{\text{coop}}_{\text{loc}} \to R$$

- $(f_v, g_w) = 0$ if $v \neq w$, $(f_v, \phi_i(z)) = 0$, $(\psi_i(z), g_w) = 0$,
- $(\psi_k(u), \psi_l(w)) = \frac{\text{fac}(u|_w)}{\text{fac}(w|_u)}$ for any $k, l \in I$.
- $(f_{x_i}, g_{c_i}) := \text{Res}_{x=\infty} f(x^{(i)}) \cdot g(-x^{(i)})dx$.

where $f_v \in H_v$, $g_w \in H_w$ and $\psi_i(z) \in SH[[z]]$. We extend the pairing to the entire $H_v \times H^t_{\text{coop}}$ using the property $(a, bb') = (\Delta_A(a), b \otimes b')$ and $(aa', b) = (a \otimes a', \Delta_B^t(b))$. An explicit formula for $(f_v, g_w)$ can be found in [66].

We now verify this is a skew-Hopf pairing. (a) is obvious; (b)(c) can be proved the same way as in [66].

**Lemma 2.2.1** $(S_A(a), b) = (a, S_B^{-1}(b))$
Proof. Let $a = \psi_k(z), b = \psi_l(w)$.

$$(S_A(\psi_k(z)), \psi_l(w)) = (\psi_k^{-1}(z), \psi_l(w)) = \frac{\text{fac}(w|u)}{\text{fac}(u|w)}$$

$$(\psi_k(z), S_B^{-1}(\psi_l(w))) = (\psi_k(z), \psi_l^{-1}(w)) = \frac{\text{fac}(w|u)}{\text{fac}(u|w)}.$$  

Thus, $(S_A(\psi_k(z)), \psi_l(w)) = (\psi_k(z), S_B^{-1}(\psi_l(w)))$.

Let $a = f_v, b = \psi_k(z)$.

$$(S_A(f_v), \psi_k(z)) = ((-1)^{|e|} \psi_k^{-1}(x_{[1,v]}) f(x_{[1,v]}), \psi_k(z)) = ((-1)^{|e|} \psi_k^{-1}(x_{[1,v]}) \otimes f(x_{[1,v]}), \psi_k(z) \otimes \psi_k(z)) = 0.$$  

$$(f_v, S_B^{-1}(\psi_k(z))) = (f_v, \psi_k^{-1}(z)) = 0.$$  

Thus, $(S_A(f_v), \psi_k(z)) = (f_v, S_B^{-1}(\psi_k(z)))$.

Let $a = f_v, b = g_{uw}$.

$$(S_A(f_v), g_{uw}) = ((-1)^{|e|} \psi_k^{-1}(x_{[1,v]}) f(x_{[1,v]}), g_{uw}) = \left((-1)^{|e|} \psi_k^{-1}(x_{[1,v]}) \otimes f(x_{[1,v]}), \sum_{\{w_1+w_2=w\}} \frac{\phi_{[w_1+1,w]}(x_{[1,w_1]} \otimes x_{[w_1+1,w]})}{\text{fac}(x_{[w_1+1,w]}|x_{[1,w_1]})} \right)$$

$$= \delta_{v,w} \left((-1)^{|e|} \psi_k^{-1}(x_{[1,v]}) \otimes f(x_{[1,v]}), g(x_{[1,v]}) \right)$$

$$(f_v, S_B^{-1} g_{uw}) = (f_v, (-1)^{|w|} \psi_k^{-1}(x_{[1,w]}) g(x_{[1,w]})) = \left((-1)^{|w|} \psi_k^{-1}(x_{[1,w]}) \otimes g(x_{[1,w]}), \sum_{\{v_1+v_2=v\}} \frac{\phi_{[v_1+1,v]}(x_{[v_1+1,v]} \otimes x_{[1,v_1]}))}{\text{fac}(x_{[v_1+1,v]}|x_{[1,v_1]}))} \right)$$

$$= \delta_{v,w} \left((-1)^{|w|} \psi_k^{-1}(x_{[1,w]}) \otimes g(x_{[1,w]}), g(x_{[1,w]}) \right)$$

Thus, we have $(S_A(f_v), g_{uw}) = (f_v, S_B^{-1} g_{uw})$, since $(\psi_k^{-1}(x_{[1,v]}), \phi_{[v_1+1,v]}(x_{[1,v]})) = (\psi_k^{-1}(x_{[1,v]}), \phi_{[v_1+1,v]}(x_{[1,v]}))$. This completes the proof. $lacksquare$

It is known that if $A, B$ are Hopf algebras, endowed with a skew-Hopf paring. Then, there is a unique Hopf structure on $A \otimes B$, called the Drinfeld double of $(A, B, (\cdot, \cdot))$, determined by the following properties [30] Lemma 3.2.2

(a) $(a \otimes 1)(a' \otimes 1) = aa' \otimes 1$,

(b) $(1 \otimes b)(1 \otimes b') = 1 \otimes bb'$,

(c) $(a \otimes 1)(1 \otimes b) = a \otimes b$,

(d) $(1 \otimes b)(a \otimes 1) = \sum (a_1, S_B(b_1)) a_2 \otimes b_2(a_3, b_3)$,

for all $a, a' \in A, b, b' \in B$. We use the notation that $\Delta_A^2(a) = \sum a_1 \otimes a_2 \otimes a_3$ and $\Delta_B^2(b) = \sum b_1 \otimes b_2 \otimes b_3$.

Let $Q' = (I', H') \subset Q$ be a subquiver. That is, $I' \subset I$ is a subset and $H' = \{h \in H \mid$ the incoming and outgoing vertices of $h$ are in $I'\}$. It is clear that the shuffle algebra $H'$ associated to $Q'$ is a subalgebra of the shuffle algebra $H$ associated to $I$. 

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Proposition 2.2.2 We have $D(H')$ is a subalgebra of $D(H)$.

Proof. The Hopf structure of $H$ when restricted to $H'$ gives a Hopf structure on $H'$ and the restriction of the pairing is still a skew Hopf pairing. The multiplication on $D(H')$ by \[ \psi \] is compatible with the multiplication on $D(H)$. This completes the proof. $\blacksquare$

Let $SH$ be the equivariant spherical COHA (\cite{7,35}). By definition, $SH$ is a subalgebra of $H$ generated by $H_{e_i}$, as $i$ varies in $I$. We will skip the word “equivariant” if it does not lead to a confusion. Similarly, let $SH$ be the spherical shuffle algebra, which is generated by $H_{e_i}$, as $i$ varies in $I$. Let $D(SH) \subset D(H)$ be the subalgebra generated by $H_{e_i}$, $H_{e_i}^{coop}$, as $i$ varies in $I$.

We assume that there is an algebra homomorphism

$$H \to H$$

which is an isomorphism after passing to the localization $- \otimes_{R[h_v]} R[h_v]^{S_v}$, for each $v$. We have an induced algebra epimorphism

$$SH \to SH,$$

which is an isomorphism after passing to the same localization. We define the Drinfeld double $D(SH)$ of the spherical COHA as $D(SH)$.

### 2.3 Relations in the Drinfeld double

Let $i \in I$ be a vertex of the quiver, let $e_i$ be the dimension vector which is 1 at $i \in I$ and 0 otherwise. Since any 1-by-1 matrices naturally commute and hence the potential function vanishes, we have $H_{GL_{e_i}}^{\mathbb{C}}(\text{Rep}(Q, e_i), \varphi_i w) \cong R[x^{(i)}]$. Define $H_i$ to be the subalgebra of $H$, generated by $R[x^{(i)}]$.

We compute the relations in the Drinfeld double of COHA. First note that $SH^r, SH^{r,coop}$ are two sub-algebras of $D(SH)$.

Proposition 2.3.1 Let $E_i(u), F_j(v)$ be the generating series

$$E_i(u) := \sum_{r \geq 0} (x^{(i)})^r u^{-r-1} \in H^r[[u]], \quad F_j(v) := \sum_{r \geq 0} (-x^{(j)})^r v^{-r-1} \in H^r[[v]],$$

for $i, j \in I$. Then, in $D(SH)$, we have the relation

$$\begin{cases}
[E_i(u), F_j(v)] = \delta_{ij} \left( \frac{\psi_i(u) \otimes 1 \otimes \phi_i(v)}{u-v} - \frac{\psi_i(v) \otimes 1 \otimes \phi_i(u)}{u-v} \right), & \text{if } \# \text{ edge loops at } i \text{ is odd}, \\
\{E_i(u), F_j(v)\} = \delta_{ij} \left( \frac{\psi_i(u) \otimes 1 \otimes \phi_i(v)}{u-v} - \frac{\psi_i(v) \otimes 1 \otimes \phi_i(u)}{u-v} \right), & \text{if } \# \text{ edge loops at } i \text{ is even}.
\end{cases}$$

where $[a, b] := ab - ba$ is the commutator and $\{a, b\} := ab + ba$ is the super commutator.

Proof. By the coproduct formula, we have

$$\Delta^2(E_i(u)) = \psi_i(x^{(i)}) \otimes \psi_i(x^{(i)}) \otimes E_i(u) + \psi_i(x^{(i)}) \otimes E_i(u) \otimes 1 + E_i(u) \otimes 1 \otimes 1$$

$$\Delta^2(F_j(v)) = F_j(v) \otimes \phi_j(x^{(j)}) \otimes \phi_j(x^{(j)}) + 1 \otimes F_j(v) \otimes \phi_j(x^{(j)}) + 1 \otimes 1 \otimes F_j(v).$$
In the multiplication formula \([2,2]\), we choose \(b = F_j(v), a = E_i(u)\). Using the following pairings

\[(\psi_i(x^{(i)}), 1) = 1, \quad (1, \phi_j(x^{(j)})) = (-1)^{e_j, L},\]

\((E_i(u) - F_j(v)\phi_j^{-1}(x^{(j)})) = (\psi_i(x^{(i)}) \otimes E_i(u) + E_i(u) \otimes 1, -F_j(v) \otimes \phi_j^{-1}(x^{(j)})) = -(-1)^{e_j, L}(E_i(u), F_j(v)),\]

we have

\[
(1 \otimes F_j(v))(E_i(u) \otimes 1) = \sum (a_1, S_B(b_1)) a_2 \otimes b_2 (a_3, b_3)
\]

\[
= (\psi_i(x^{(i)}), 1) \psi_i(x^{(i)}) \otimes 1 (E_i(u), F_j(v)) + (\psi_i(x^{(i)}), 1) E_i(u) \otimes F_j(v) (1, \phi_j(x^{(j)}))
\]

\[
+ (E_i(u), -F_j(v)\phi_j^{-1}(x^{(j)})) 1 \otimes \phi_j(x^{(j)})(1, \phi_j(x^{(j)}))
\]

\[
= (-1)^{e_j, L} E_i(u) \otimes F_j(v) + (\psi_i(x^{(i)}) \otimes 1 - 1 \otimes \phi_j(x^{(j)}))(E_i(u), F_j(v))
\]

\[
= (-1)^{e_j, L} E_i(u) \otimes F_j(v) + \delta_{ij} \frac{1}{u - x^{(i)}} \cdot \frac{1}{v - x^{(i)}}
\]

\[
= (-1)^{e_j, L} E_i(u) \otimes F_j(v) + \delta_{ij} \frac{\psi_i(u) \otimes 1 - 1 \otimes \phi_i(u) - (\psi_i(v) \otimes 1 - 1 \otimes \phi_i(v))}{u - v}
\]

Here the sign \((-1)^{e_i + 1}\) is 1 (resp. is -1), when \(i\) is has odd number of edge loops (resp. even number of edge loops). Thus, we have the desired relation. \(\blacksquare\).

**Proposition 2.3.2** The relation between \(E_i(u) \otimes 1\) and \(1 \otimes \phi_j(v)\) is given as

\[
(1 \otimes \phi_j(v))(E_i(u) \otimes 1)(1 \otimes \phi_j^{-1}(v)) = E_i(u) \frac{\text{fac}(v|x^{(i)})}{\text{fac}(x^{(i)}|v)} \otimes 1.
\]

**Proof.** By the coproduct formula, we have

\[
\Delta^2(E_i(u)) = \psi_i(x^{(i)}) \otimes \psi_i(x^{(i)}) \otimes E_i(u) + \psi_i(x^{(i)}) \otimes E_i(u) \otimes 1 + E_i(u) \otimes 1 \otimes 1
\]

\[
\Delta^2(\phi_j(v)) = \phi_j(v) \otimes \phi_j(v) \otimes \phi_j(v).
\]

In the multiplication formula \([2,2]\), we choose \(b = \phi_j(v), a = E_i(u)\). We have

\[
(1 \otimes \phi_j(v))(E_i(u) \otimes 1) = \sum (a_1, S_B(b_1)) a_2 \otimes b_2 (a_3, b_3)
\]

\[
= (\psi_i(x^{(i)}), \phi_j^{-1}(v)) E_i(u) \otimes \phi_j(v)(1, \phi_j(v))
\]

\[
= \frac{\text{fac}(v|x^{(i)})}{\text{fac}(x^{(i)}|v)} E_i(u) \otimes \phi_j(v)
\]

This is equivalent to the action

\[
(1 \otimes \phi_j(v))(E_i(u) \otimes 1)(1 \otimes \phi_j^{-1}(v)) = E_i(u) \frac{\text{fac}(v|x^{(i)})}{\text{fac}(x^{(i)}|v)} \otimes 1.
\]

This completes proof. \(\blacksquare\).

**Proposition 2.3.3** The relation between \(\psi_i(u) \otimes 1\) and \(1 \otimes \phi_j(v)\) is given as

\[
(1 \otimes \phi_j(v))(\psi_i(u) \otimes 1) = (\psi_i(u) \otimes 1)(1 \otimes \phi_j(v)).
\]

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Proof. By the coproduct formula, we have
\[
\Delta^2(\psi_i(u)) = \psi_i(u) \otimes \psi_i(u) \otimes \psi_i(u), \quad \Delta^2(\phi_j(v)) = \phi_j(v) \otimes \phi_j(v) \otimes \phi_j(v).
\]
In the multiplication formula (2.2), we choose \(b = \phi_j(v), a = \psi_i(u)\). We have
\[
(1 \otimes \phi_j(v)) (\psi_i(u) \otimes 1) = (\psi_i(u), \phi_j^{-1}(v)) \psi_i(u) \otimes \phi_j(v)(\psi_i(u), \phi_j(v)) = \psi_i(u) \otimes \phi_j(v).
\]
This completes proof. \(\square\)

2.4 Cartan doubled Yangian and Shifted Yangians

By Proposition 2.3.3 in the Drinfeld double \(D(SH)\), there is a commutative subalgebra which is isomorphic to \(H^0 \otimes H^0\). For fixed \(i \in I\), label the Cartan elements \(\psi_{i,r} \otimes 1 \in H^0 \otimes 1 (r \geq 0)\) and \(1 \otimes \phi_{i,r} \in 1 \otimes H^0 (r \geq 0)\) in \(D(SH)\) by \(Z\) as follows
\[
\psi^{(r)}_i := \begin{cases} 
\psi_{i,r} \otimes 1 - 1 \otimes \phi_{i,r}, & \text{if } r \geq 0, \\
\psi_{i,-r-1} \otimes 1 + 1 \otimes \phi_{i,-r-1}, & \text{if } r < 0.
\end{cases}
\]
(15)

When \(g\) is the Kac-Moody Lie algebra, the Cartan doubled Yangian of \(g\) is introduced in \([17]\) (and recalled in Definition 3.2.1). In Proposition 3.2.2, we show that the Cartan doubled Yangian of a Kac-Moody algebra surjects to the Drinfeld double of some \(SH(X_m,0)\). Note that the index set of \(\psi^{(r)}_i\) is \(I \times Z\). This explains the name “Cartan doubled”.

A coweight \(\mu\) is a \(Z\)-linear \(Z\)-valued function on \(Z^I\). In \([17]\), a shifted Yangian of a Kac-Moody algebra \([17, \text{Definition 3.5}]\) is defined to be a quotient of the Cartan doubled Yangian by a coweight \(\mu\). Motivated by the loc.cit. we propose the following definition.

**Definition 2.4.1** For any coweight \(\mu\) we define the shifted Yangian \(Y_\mu\) to be the quotient of \(D(SH)\) by the relations
\[
\psi_i^{(p)}, \text{ for all } p < -\langle \mu, \alpha_i \rangle.
\]

3 Examples: COHA and the Drinfeld double

3.1 Subalgebras associated to vertices

Let \((Q, W)\) be a quiver with potential, and let \(i \in I\) be a vertex of the quiver. Recall \(H_i\) is the subalgebra of \(H\) generated by the dimension vector \(e_i\).

**Proposition 3.1.1** In the above notation the algebra \(H_i\) is isomorphic to the positive part of the Clifford algebra if \(i\) has no edge loop, \(Y^+_\hbar(\mathfrak{sl}_2)\) if \(i\) has 1 edge loop.

**Proof.** The COHA of this subquiver in both cases was computed in \([35, \text{§ 2.5}]\). In particular it was identified with the positive part of the infinite Clifford algebra in the first case, see \([35, \text{§ 2.5}]\). In the second case it is identified with \(Y^+_\hbar(\mathfrak{sl}_2)\) (see also \([67]\)). \(\square\)

**Example 3.1.2** For the Dynkin quiver \(Q = A_1\) endowed with the potential \(W = 0\), the Drinfeld double \(D(H)\) is generated by
\[
e^{(r)}, f^{(r)}, \psi^{(s)}, r \geq 0, s \in Z,
\]
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subject to the following relations:

\[ [\psi^{(a)}, \psi^{(b)}] = 0 \]
\[ \{e^{(r_1)}, e^{(r_2)}\} = 0, \quad \{f^{(r_1)}, f^{(r_2)}\} = 0, \]
\[ \{e^{(r_1)}, \psi^{(r_2)}\} = 0, \quad \{f^{(r_1)}, \psi^{(r_2)}\} = 0, \]
\[ \{e^{(r_1)}, f^{(r_2)}\} = \psi^{(r_1+r_2)}, \]

where \(a, b := ab + ba\), and \(\psi^{(s)}\) is defined in \(15\). Since the \(A_1\) quiver has no arrows, the conjugation action \(16\) of \(\psi(z)\) on \(H\) is given as

\[ \psi(z)g(\psi(z))^{-1} = \frac{\text{fac}(z)}{\text{fac}(z)} g = \frac{z - x}{x - z} g = -g. \]

Proposition \(2.3.2\) implies \(\psi(z)g(\psi(z))^{-1} = -g\). Therefore, \(\{\psi^{(s)}, e^{(r)}\} = 0\), for \(s \in \mathbb{Z}\). Similarly, \(\{\psi^{(s)}, f^{(r)}\} = 0\), for \(s \in \mathbb{Z}\). Proposition \(2.3.1\) implies that \(\{e^{(r_1)}, f^{(r_2)}\} = \psi^{(r_1+r_2)}\).

**Example 3.1.3** Let \(Q\) be the Jordan quiver, i.e. it has one vertex and one loop. Then for the potential \(W = 0\) the Drinfeld double \(D(H)\) is generated by

\[ e^{(r)}, f^{(r)}, \psi^{(s)}, r \geq 0, s \in \mathbb{Z}, \]

subject to the Cartan doubled Yangian relations for \(\mathfrak{sl}_2\) (see Definition \(3.2.1\) taking \(I\) to be a point). In particular, we have the relation

\[ [e(u), f(v)] = \frac{\psi^+(u) - \psi^+(v)}{u - v}, \]

where \(\psi^+(u) = \psi(u) \otimes 1 - 1 \otimes \psi(u)\) which follows from Proposition \(2.3.1\).

### 3.2 The Cartan doubled Yangian of a Kac-Moody Lie algebra

Let us start with the following definition.

**Definition 3.2.1** \(7\) Define 3.1] Let \(g\) be a Kac-Moody Lie algebra. The Cartan double Yangian \(Y_{\infty}(g)\) is the \(C\)-algebra generated by \(E_i^{(q)}, F_i^{(q)}, H_i^{(p)}\), for \(i \in I, q > 0, p \in \mathbb{Z}\), subject to the relations

\[ [H_i^{(p)}, H_j^{(q)}] = 0, \quad (\text{HH}) \]
\[ [E_i^{(p)}, E_j^{(q)}] = \delta_{ij} H_i^{(p+q-1)}, \quad (\text{EF}) \]
\[ [H_i^{(p+1)}, E_j^{(q)}] - [H_i^{(p)}, E_j^{(q+1)}] = \frac{\alpha_i \cdot \alpha_j}{2} (H_i^{(p)} E_j^{(q)} + E_j^{(q)} H_i^{(p)}), \quad (\text{HE}) \]
\[ [H_i^{(p+1)}, F_j^{(q)}] - [H_i^{(p)}, F_j^{(q+1)}] = -\frac{\alpha_i \cdot \alpha_j}{2} (H_i^{(p)} F_j^{(q)} + F_j^{(q)} H_i^{(p)}), \quad (\text{HF}) \]
\[ [E_i^{(p+1)}, E_j^{(q)}] - [E_i^{(p)}, E_j^{(q+1)}] = \frac{\alpha_i \cdot \alpha_j}{2} (E_i^{(p)} E_j^{(q)} + E_j^{(q)} E_i^{(p)}), \quad (\text{EE}) \]
\[ [F_i^{(p+1)}, F_j^{(q)}] - [F_i^{(p)}, F_j^{(q+1)}] = -\frac{\alpha_i \cdot \alpha_j}{2} (F_i^{(p)} F_j^{(q)} + F_j^{(q)} F_i^{(p)}), \quad (\text{FF}) \]
\[ i \neq j, N = 1 - \alpha_i \cdot \alpha_j, \quad \text{sym}[E_i^{(p_1)}, [E_i^{(p_2)}, \ldots, [E_i^{(p_N)}, E_j^{(q)}] \ldots]] = 0, \]
\[ i \neq j, N = 1 - \alpha_i \cdot \alpha_j, \quad \text{sym}[F_i^{(p_1)}, [F_i^{(p_2)}, \ldots, [F_i^{(p_N)}, F_j^{(q)}] \ldots]] = 0, \]

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Now let $\mathfrak{g}$ be a symmetric Kac-Moody Lie algebra with Dynkin diagram $\Gamma$. Let $(Q,W)$ be the tripled quiver with potential as in [24]. Let $\mathcal{H}$ be the COHA of $(Q,W)$.

**Proposition 3.2.2** In the above notation we have an epimorphism

$$Y_\infty(\mathfrak{g}) \to D(S\mathcal{H}).$$

**Proof.** Consider the following generating series of $H_i^{(p)}(p \in \mathbb{Z})$

$$H_i^>(z) := \sum_{p \geq 0} H_i^{(p+1)} z^{-p-1}, \quad H_i^<(z) := 1 + \sum_{p \leq 0} H_i^{(p)} z^{-p-1}.$$ 

Define a map $Y_\infty(\mathfrak{g}) \to D(S\mathcal{H})$ by

$$E_i^{(0)} \mapsto (x^{(i)})^{q-1} \otimes 1 \in S\mathcal{H}^e \otimes 1,$$

$$F_i^{(0)} \mapsto 1 \otimes (-x^{(i)})^{q-1} \in 1 \otimes S\mathcal{H}^e$$

$$H_i^>(z) \mapsto \psi_i^+(z) = \psi_i(z) \otimes 1 - 1 \otimes \psi_i(z),$$

$$H_i^<(z) \mapsto \psi_i^-(z) = \psi_i(z) \otimes 1 + 1 \otimes \psi_i(z).$$

By [65] Theorem 7.1, the map respects the relations of $[\text{EE}], [\text{FF}]$ and the two Serre relations. By Proposition 2.3.3, the map preserves the relation $[\text{HE}]$. Proposition 2.3.1 implies the map respects the relations of $[\text{HE}], [\text{EF}]$. Using the action (14) and Proposition 2.3.2, the map respects the relations of $[\text{HE}], [\text{HF}]$. This completes proof. $\blacksquare$.

### 3.3 Quiver with potential coming from toric local Calabi-Yau 3-folds

With notations as in §4 let $f : X \to Y$ be a resolution of singularities, where $X$ is a smooth local toric Calabi-Yau 3-fold and $Y$ is affine. There is a tilting generator $\mathcal{P}$ of $D^b \text{Coh}(X)$ so that the functor

$$R \text{Hom}_X(\mathcal{P}, -) : D^b \text{Coh}(X) \to D^b(\text{Mod-} \mathcal{A}_0)$$

induces an equivalence of derived categories. Here $\mathcal{A}_0 := \text{End}_X(\mathcal{P})^{op}$ is a coherent sheaf of non-commutative $\mathcal{O}_Y$-algebras, and $\text{Mod-} \mathcal{A}_0$ is the abelian category of coherent sheaves of (right) modules over $\mathcal{A}_0$. Let $\mathcal{A} \subset D^b \text{Coh}(X)$ be a heart of $D^b \text{Coh}(X)$ corresponding to the heart $\text{Mod-} \mathcal{A}_0$ of the standard $t$-structure of $D^b(\text{Mod-} \mathcal{A}_0)$ under the functor $R \text{Hom}_X(\mathcal{P}, -)$. One example of $\mathcal{A}$ is $\text{Perv}^{-1}(X/Y)$, the abelian category of perverse coherent sheaves in the sense of Bridgeland and Van den Bergh [3, 63]. It is known that $\mathcal{A}_0$ can be identified with the Jacobian algebra of a quiver with potential $(Q,W)$. We denote the COHA of the pair $(Q,W)$ also be $\mathcal{H}_X$.

Note that each vertex $i \in I$ of $Q$ gives rise to a simple object $S_i$ in the heart of the induced $t$-structure on $D^b \text{Coh}(X)$. We say $S_i$ is *bosonic* if the simple object $S_i$ in $D^b \text{Coh}(X)$ has the property that $\text{Ext}^*(S_i, S_i) = H^{2*}(\mathbb{P}^3)$; we say $S_i$ is *fermionic* if the simple object $S_i$ is a spherical object, that is, $\text{Ext}^*(S_i, S_i) = H^*(S^3)$. We have the following general proposition.

**Proposition 3.3.1** For a vertex $i \in I$ of the quiver $Q$, assume the corresponding simple object $S_i$ is a line bundle on $j_* \mathcal{O}_\mathbb{P}^1(m)[n]$ with $j : \mathbb{P}^1 \hookrightarrow X$ and $m,n \in \mathbb{Z}$. Then, $S_i$ is either bosonic or fermionic.
Proof. It is not difficult to calculate the self-extension of this line bundle. Without loss of
generality, assume this line bundle is trivial. Taking the formal neighbourhood of $P_1$ in $X$,
we get a rank-2 vector bundle on $P_1$ the determinant of which is $\mathcal{O}(-2)$. By the assumption
that $X$ is birational to its affinization $Y$, we get that this rank-2 vector bundle has no ample
line sub-bundles, therefore is either $\mathcal{O} \oplus \mathcal{O}(-2)$ or $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In the first case, $\mathcal{O}_{P_1}$
has 1-dimensional self-extension. In the second case, $\mathcal{O}_{P_1}$ has no non-trivial self-extensions.
The entire $\text{Ext}^\bullet(S_i, S_i)$ is then determined by the Calabi-Yau condition $\blacksquare$.

Therefore, when $S_i$ is bosonic, it follows from Example 3.1.3 that $D(H_i)$ is the Cartan
doubled Yangian of $\mathfrak{sl}_2$. When $S_i$ is fermionic, it follows from Example 3.1.2 that $D(H_i)$ is
the infinite Clifford algebra.

We give three examples of quivers with potentials coming from this setup.

Example 3.3.2 Let $X = X_{m,0}$. Then, the defining equation $xy = z^m$ of $Y_{m,0}$ is a type
$A$-singularity. All the simple objects are bosonic. Indeed, let $S_m = \mathbb{C}^2/\mathbb{Z}_m$ be the type
$A_{m-1}$ Kleinian surface singularity, and $\tilde{S}_m$ its crepant resolution. Then, $X_{m,0} \cong \tilde{S}_m \times \mathbb{C}$.
The toric diagram of $X_{m,0}$ is depicted in figure 1. The quiver with potential is the tripled
quiver of $\tilde{A}_{m-1}$ in the sense of of [24].

Example 3.3.3 Let $X = X_{m,n}$. A toric diagram of one possible resolution of $Y_{m,n}$ is shown
in figure 2. Other possibilities can be obtained by considering different orderings of the left
and the right $(1,0)$ lines ending on a sequence of $(k,1)$ lines. All possible choices for $Y_{2,1}$
are depicted in figure 3. One can then associate a bosonic simple object to each internal
line bounded by two $(1,0)$ lines ending from the same side and a fermionic simple object to
each internal line bounded by $(1,0)$ lines ending from opposite directions. The quiver with
potential is given explicitly in [42].
Figure 3: Three possible orderings of horizontal lines for $Y_{2,1}$ associated to the two possible choices of the root system of $\mathfrak{gl}(2|1)$. The root system associated to the left figure consists of a single bosonic and a single fermionic root whereas the left figure is associated with the root system consisting of two fermionic roots. The third possible ordering is a simple reflection of the first one.

However, there are more examples which have not been well studied in mathematical literature.

**Example 3.3.4** Let $X = S \times C$, where $S$ is an elliptically fibered K3-surface, with special fiber $C$ being a collection of $\mathbb{P}^1$’s in the $\tilde{A}_n$-type configuration. That is, let $\Delta$ be the collection irreducible components of $C$ each of which is isomorphic to $\mathbb{P}^1$. There is one component of $C$, removing which results a curve $C_0$ an open neighborhood of which is isomorphic to $\tilde{S}_m \times C$ as in Example 3.3.2 above. Although there is no contraction of this collection of $\mathbb{P}^1$’s with rational singularity, hence the method of constructing a tilting bundle from [63] does not apply here, the result of [25] still suggest that there is a $t$-structure on $\mathbb{D}^b \text{Coh}(X)$ that contains $\mathcal{O}_{C_i}(1)[-1]$ with $i \in I$ and $\mathcal{O}_{C_0}$ as simple objects. It is easy to verify that all these simple objects are bosonic.

### 3.4 Shifted Yangians and perverse coherent systems

Let us explain how the notion of shifted Yangians appears later on in our paper. We will consider the moduli space of stable perverse coherent system on $X$. As to be explained in 7 and 8, the moduli space depends not only on a choice of the stability parameter $\zeta$, but also on an algebraic cycle $\chi$ of the form

$$N_0X + \sum_{i=1}^{r} N_i D_i,$$

where $N_i \in \mathbb{N}$, and each $D_i$ is a toric divisor. Here $\chi$ determines a framing of the quiver with potential $(Q, W)$. We mark the framings associated to $[X]$ by $\square$ and the ones associated to each $D_i$ by $\triangle$. For example, when $X = X_{2,0}$, and $D_1$ is the fiber of the 0 $\in \mathbb{P}^1$, the framed quiver is as follows

\[^2\text{In physics terminology } \chi \text{ should be thought of as a stack of D6-branes and D4-branes wrapping the corresponding cycles.}\]
We are going to discuss the framed quiver with potential for general $\chi$ in the future publications. In this paper we only consider some special cases. In particular, in §8 we will discuss how in general $\zeta$ and $\chi$ determine a coweight $\mu$. We also explain that we expect that the action of $D(S\mathcal{H})$ cohomology of perverse coherent systems factors through $Y_\mu$. Furthermore, we expect that the algebra $D(S\mathcal{H})$ has a coproduct, which after passing to the quotient becomes $Y_\mu \to Y_{\mu_1} \otimes Y_{\mu_2}$ if $\mu_1 + \mu_2 = \mu$. This coproduct is compatible with hyperbolic restriction to fixed points of a subtorus acting on the framings, similar to §55. Therefore, we can reduce the verification of the action of $Y_\mu$ for a general $\chi$ to the cases when the framing dimension is 1.

3.5 COHA of $C^3$

Consider the case when $X = C^3$. The quiver with potential is recalled in the Introduction. We denote the quiver by $Q_3$ and the potential by $W_3$. The algebraic properties of the COHA has been studied [55 §2.3], building up on earlier works [57, 16]. In particular, there is a subalgebra of the equivariant COHA of $(Q_3, W_3)$ which is called the equivariant spherical COHA of $(Q_3, W_3)$ (or $C^3$). Recall that in [55 Theorem 7.1.1.], the equivariant spherical COHA was identified with $Y_1, Y_2, Y_3 (\hat{\mathfrak{gl}}(1))^+$. Furthermore, using the same coproduct as in §2.2, one can show the reduced Drinfeld double of $S\mathcal{H}_{C^3}$ is isomorphic to the entire Yangian $Y_1, Y_2, Y_3 (\hat{\mathfrak{gl}}(1))^+$ [55]. Similar argument as in Proposition 3.2.2 then implies that the shifted Yangians $Y_l(z)$ from §4 are quotients of $D(S\mathcal{H}_{C^3})$.

3.6 COHA of the resolved conifold

In this section, the toric Calabi-Yau 3-fold is $X = X_{1,1}$, the resolved conifold. The corresponding quiver with potential was described in the Introduction (see (2)). Recall that we denote the quiver by $Q_{1,1}$ and the potential by $W_{1,1}$, and we denote the corresponding COHA by $\mathcal{H}_{X_{1,1}}$. By Proposition 3.3.1 there are positive parts of two Clifford subalgebras in $\mathcal{H}_{X_{1,1}}$ associated to the two vertices of the quiver $Q_{1,1}$. The goal of this section is to show that there is an algebra homomorphism $\mathcal{H}_{X_{1,1}} \to \mathcal{H}_{C^3}$, where $\mathcal{H}_{C^3}$ is the COHA for $C^3$ from §3.5.

For any $n \in \mathbb{N}$ consider the open subset $\text{Rep}(Q_{1,1}, (n, n))^0 \subset \text{Rep}(Q_{1,1}, (n, n))$ consisting of such representations that the map $b_1$ is an isomorphism. We will show below that
\( \oplus_n H^*_n, GL(n,n) \times_T (\text{Rep}(Q_{1,1}, (n,n))^0, \varphi_{\text{tr}} W_{1,1})^V \) has a natural algebra structure, so that the restriction to open subset

\( \oplus_n H^*_n, GL(n,n) \times_T (\text{Rep}(Q_{1,1}, (n,n)), \varphi_{\text{tr}} W_{1,1})^V \to \oplus_n H^*_n, GL(n,n) \times_T (\text{Rep}(Q_{1,1}, (n,n))^0, \varphi_{\text{tr}} W_{1,1})^V \)

is an algebra homomorphism.

Observe that there is a canonical isomorphism of vector spaces

\( \oplus_n H^*_n, GL(n,n) \times_T (\text{Rep}(Q_{1,1}, (n,n))^0, \varphi_{\text{tr}} W_{1,1})^V \cong \oplus_n H^*_n, GL(n,n) \times_T (\text{Rep}(Q_{3,1}, (n,n)), \varphi_{\text{tr}} W_{3,1})^V \).

Furthermore, with the algebra structures on both sides, the above isomorphism of vector spaces is an isomorphism of algebras with the latter \( \oplus_n H^*_n, GL(n,n) \times_T (\text{Rep}(Q_{3,1}, (n,n)), \varphi_{\text{tr}} W_{3,1})^V \)

\begin{proposition}
Restriction to the above-defined open subset induces the algebra homomorphism \( \mathcal{H}_{X,1} \to \mathcal{H}_{C^3} \). Its image contains \( Y_{h_1, h_2, h_3}(\text{gl}(1))^\dagger \).
\end{proposition}

\textbf{Proof.} Consider the affine space parameterizing representations of \( Q_{1,1} \) on the underlying vector space \( V = (V_1, V_2) \) with maps \( a_i \) from \( V_1 \) to \( V_2 \) and \( b_i \) back to \( V_1 \) for \( i = 1, 2 \). We denote such space by \( \text{Rep}(Q_{1,1}, V) \). We first recall the multiplication of \( \mathcal{H}_{X_{1,1}} \). For this, consider pairs of flags\

\[ 0 \to V_1 \subset V'_1 \to V''_1 \to 0 \quad \text{and} \quad 0 \to V_2 \subset V'_2 \to V''_2 \to 0. \]

Fixing one choice of such a pair, the space parameterizing all such pairs can be identified with \( \text{GL} \) where \( \text{GL}(V'_1) \times \text{GL}(V'_2) \), and \( P = P_1 \times P_2 \) is the parabolic subgroup with \( P_i = \{ x \in \text{GL}(V_i) \mid x(V_i) \subset V'_i \}, i = 1, 2 \). Consider \( \text{Rep}(Q_{1,1}, V') \) and the subspace \( Z \) consisting of representations such that \( (V_1, V_2) \) is a sub-representation. We have the following diagram of correspondences

\[ \begin{array}{ccc}
\text{G} \times_P Z^n & \xrightarrow{\phi} & \text{G} \times_P \text{Rep}(Q_{1,1}, V') \\
G \times_P \left( \text{Rep}(Q_{1,1}, V') \times \text{Rep}(Q_{1,1}, V'') \right) & \xrightarrow{\psi} & \text{Rep}(Q_{1,1}, V')
\end{array} \]

where \( \phi \) is an affine bundle. The potential \( W_{1,1} \) defines functions \( \text{tr}((W_{1,1})_V^\vee (W_{1,1})_V^\vee) \) on \( \text{Rep}(Q_{1,1}, V') \times \text{Rep}(Q_{1,1}, V'') \) and \( \text{tr}((W_{1,1})_V) \) on \( Z \) by restricting the function \( \text{tr}(W_{1,1}) \).

We have \( \text{tr}((W_{1,1})_V^\vee (W_{1,1})_V^\vee) \circ \phi = \text{tr}(W_{1,1}) \).

The map \( \eta \) is a closed embedding and \( \psi \) is a projection, both of which are compatible with the potential functions.

Let \( Z^0 \) be the intersection of \( Z \) with \( \text{Rep}(Q_{1,1}, V')^0 \). Note that \( P \) acts on \( Z^0 \), and that the natural map \( \eta^0 : Z^0 \to \text{Rep}(Q_{1,1}, V')^0 \) is a pullback

\[ \begin{array}{ccc}
Z^0 & \to & \text{Rep}(Q_{1,1}, V')^0 \\
\downarrow & & \downarrow \\
Z & \to & \text{Rep}(Q_{1,1}, V')
\end{array} \]

\[ 23 \]
In particular, restriction to the open subsets intertwines $\eta_*$ and $(\eta^0)_*$. That is, we have the commutative diagram

$$H^*_c(G \times T)(G \times P Z, \varphi_{\text{tr} \, W_{1,1}})^\vee \xrightarrow{\eta_*} H^*_c(G \times T)(G \times P \text{Rep}(Q_{1,1}, V'), \varphi_{\text{tr} \, W_{1,1}})^\vee$$

Similarly, $\psi^0 : G \times P \text{Rep}(Q_{1,1}, V')^0 \to \text{Rep}(Q_{1,1}, V')^0$ is pullback of $\psi$, and hence after taking cohomology the restrictions to the open subsets intertwines $\psi_*$ and $(\psi^0)_*$.

$$H^*_c(G \times T)(G \times P \text{Rep}(Q_{1,1}, V'), \varphi_{\text{tr} \, W_{1,1}})^\vee \xrightarrow{\psi_*} H^*_c(G \times T)(\text{Rep}(Q_{1,1}, V'), \varphi_{\text{tr} \, W_{1,1}})^\vee$$

Also, clearly the map $\phi$ induces the map $\phi^0$ by restriction to open subsets as in the following diagram. Again after taking cohomology restrictions to the open subsets intertwines $\phi^*$ and $(\phi^0)^*$.

$$H^*_c(G \times P (\text{Rep}(Q_{1,1}, V) \times \text{Rep}(Q_{1,1}, V''))) \xrightarrow{\phi^*} H^*_c(G \times P Z)^\vee$$

To summarize, we have the following commutative diagram

$$G \times P Z \xrightarrow{\phi^0} G \times P \text{Rep}(Q_{1,1}, V')^0$$

For $V' = (V'_1, V'_2)$, we now impose a condition that $b_1 : V'_2 \to V'_1$ is an isomorphism. The space of isomorphisms $\text{Isom}(V'_2, V'_1)$ is a torsor over $\text{GL}(V'_1)$. In particular, $\text{GL}(V'_1)$ acts transitively on $\text{Rep}(Q_{1,1}, V')^0$ by the change of basis on $V'_1$. The quotient

$$\text{Rep}(Q_{1,1}, V')^0 / \text{GL}(V'_1) \cong \text{Rep}(Q_3, V'_1)$$

is canonically identified with $\text{Rep}(Q_3, V'_1)$ with $B_1 = a_1 \circ b_1$, $B_2 = a_2 \circ b_1$, and $B_3 = b_1^{-1} \circ b_2$. The action of $\text{GL}(V'_2)$ on the left hand side and the action of $\text{GL}(V'_1)$ on the right hand side are compatible.
Similarly, the parabolic subgroup $P_1$ acts on $Z^0$ transitively. Thus, we have the identification

$$(G \times P Z^0) / GL(V'_1) = GL(V'_2) \times P_2 (Z^0 / P_1) \cong GL(V'_2) \times P_2 Z_{Q_3} \cong GL(V'_1) \times P_2 Z_{Q_3},$$

where $Z_{Q_3}$ is the correspondence used in the multiplication of $\mathcal{H}_{Q_3}$.

Note that the action of $P$ on $\text{Rep}(Q_1, V)^0 \times \text{Rep}(Q_1, V'')^0$ factors through the projection $P \to GL(V) \times GL(V'')$. Therefore, we have

$$G \times P (\text{Rep}(Q_1, V)^0 \times \text{Rep}(Q_1, V'')^0) \cong GL(V'_2) \times P_2 \begin{pmatrix} \text{Rep}(Q_1, V)^0 \times \text{Rep}(Q_1, V'')^0 \\ P_1 \end{pmatrix}$$

$$\cong GL(V'_2) \times P_2 (\text{Rep}(Q_1, V)^0 \times \text{Rep}(Q_1, V'')^0) \cong GL(V'_1) \times P_1 \left( \text{Rep}(Q_3, V_1) \times \text{Rep}(Q_3, V'') \right)$$

To finish the proof, it remains to notice that quotient of the top correspondence of diagram 17 by $GL(V'_1)$ becomes

$$GL(V'_1) \times P_1 \left( \begin{pmatrix} \text{Rep}(Q_3, V_1) \\ \text{Rep}(Q_3, V''') \end{pmatrix} \right) \xrightarrow{\phi_{Q_3}} GL(V'_1) \times P_1 Z_{Q_3} \xrightarrow{\psi_{Q_3} \circ \psi_{Q_3}} \text{Rep}(Q_3, V'_1)$$

This is the correspondence that defines multiplication of $\mathcal{H}_{Q_3}$. This finishes the proof. \[\square\]

4 Shifted Yangians of $\widehat{\mathfrak{gl}(1)}$ and Costello conjectures

4.1 Definition of the shifted Yangians of $\widehat{\mathfrak{gl}(1)}$

Let $h_1, h_2, h_3$ be the deformation parameters with $h_1 + h_2 + h_3 = 0$. Set $\sigma_2 := h_1 h_2 + h_1 h_3 + h_2 h_3$, and $\sigma_3 = h_1 h_2 h_3$. Let $\mathcal{C} = (z_1, z_2, \cdots, z_{m})$ be the parameters ($l \in \mathbb{Z}$).

**Definition 4.1.1** Let $Y_l(z)$ be a $\mathbb{C}[h_1, h_2]$-algebra generated by $D_{0,m}(m \geq 1)$, $e_n, f_n$ ($n \geq 0$) with the relations

$$[D_{0,m}, D_{0,n}] = 0(m, n \geq 1)$$

$$[D_{0,m}, e_n] = -e_{m+n-1}(m \geq 1, n \geq 0)$$

$$[D_{0,m}, f_n] = -f_{m+n-1}(m \geq 1, n \geq 0)$$

$$3[e_{m+2}, e_{n+1}] - 3[e_{m+1}, e_{n+2}] - [e_{m+2}, e_n] + [e_m, e_{n+3}]$$

$$\quad + \sigma_2([e_{m+1}, e_n] - [e_m, e_{n+1}]) + \sigma_3(e_m e_n + e_n e_m) = 0$$

$$3[f_{m+2}, f_{n+1}] - 3[f_{m+1}, f_{n+2}] - [f_{m+2}, f_n] + [f_m, f_{n+3}]$$

$$\quad + \sigma_2([f_{m+1}, f_n] - [f_m, f_{n+1}]) - \sigma_3(f_m f_n + f_n f_m) = 0$$

$$\text{Sym}_{\mathcal{E}_3}[e_{i_1}, e_{i_2}, e_{i_3+1}] = 0$$

$$\text{Sym}_{\mathcal{E}_3}[f_{i_1}, f_{i_2}, e_{i_3+1}] = 0$$

$$[e_m, f_n] = h_{m+n}$$

$$1 - \sigma_3 \sum_{n \geq 0} h_n z^{-(n+1)} = \begin{cases} \prod_{i=1}^l (z - z_i) \cdot \psi(z), & l \geq 0, \\ \prod_{i=1}^l \frac{1}{(z - z_i)} \cdot \psi(z), & l \leq 0. \end{cases}$$
where the equality \((26)\) means the coefficients of \(z^{-i}\) on both sides are equal for each \(i \geq 1\).

Here \(\psi(z) := \exp\left(-\sum_{n \geq 0} D_{0,n+1} \varphi_n(z)\right)\) and the function \(\varphi_n(t)\) is a formal power series in \(t\) depending on \(h_1, h_2, h_3\). It is given by the following formula.

\[
\exp\left(\sum_{n \geq 0} (-1)^{n+1} a^n \varphi_n(z)\right) = \frac{(z + a - h_1)(z + a - h_2)(z + a - h_3)}{(z + a + h_1)(z + a + h_2)(z + a + h_3)},
\]

where \(a\) is any element in \(C\).

When \(l = 0\), this is the Yangian of \(\widehat{\text{gl}(1)}\) when \(z_i = 0\). We denote it by \(Y_{h_1, h_2, h_3}(\widehat{\text{gl}(1)})\).

When \(l > 0\) (\(l < 0\)), we refer to it as the \(l\)-positively (\(-l\)-negatively) shifted Yangian in parameters \(\vec{z}\). The positively shifted Yangian \(Y_{l}(\vec{z})\) is first defined in \([34, \text{Section 6}]\).

**Remark 4.1.2** In the above presentation, we have the correspondence compared with the notation in \([34, \text{Section 6}]\).

\[
h_1 \mapsto h, h_1 \mapsto t, h_3 \mapsto -h - t, D_{0,n} \mapsto \frac{D_{0,n}}{h}.
\]

We also take the central elements \(c_0 = c_1 = \cdots = 0\) in the central extension \(\text{SHF}\) of \(Y_{h_1, h_2, h_3}(\widehat{\text{gl}(1)})\), so that the factor \(\frac{(1-(h+1)x)(1+xtz)}{1-(h+1)(1+|x|)z}\) in \([34]\) does not show up in the formula of \(\psi(z)\)

### 4.2 Relation to the Costello conjectures

We explain how Theorem 4.1.3 implies a special case of Costello conjectures \([6]\).

Let \(X_\ell = \mathbb{C}^2/\mathbb{Z}/\mathbb{Z}^\ell\) be a crepant resolution of \(\mathbb{C}^2/(\mathbb{Z}/\mathbb{Z}^\ell)\). We have the Hilbert scheme \(X_\ell^{[k]}\) of \(k\) points on \(X_\ell\), which is smooth and quasi-projective. We denote the space of quasi-maps from \(\mathbb{P}^1\) to \(X_\ell^{[k]}\) of degree \(d\) by \(\mathcal{M}_{k,d,l}\). We follow the notation of quasi-maps from \([50, \text{Section 2.3}]\).

In \([6]\) the author defined the algebra \(A_{\ell,h,\epsilon}\) which is a deformation of the enveloping algebra \(U(\text{Diff}_*(\mathbb{C}) \otimes \text{gl}(l))\). Here \(\text{Diff}_*(\mathbb{C})\) denotes the algebra of differential operators on the line \(\mathbb{C}\) with \([\partial_{\xi}, z] = \epsilon\), and \(h\) is the deformation parameter.

It is conjectured in \([6, \text{P73}]\) that \(A_{\ell,h,\epsilon}\) acts on \(\bigoplus_{j} H^*_j(\mathcal{M}_{k,d,l}, \mathcal{P})\). The space \(\mathcal{M}_{k,d,l}\) can be realized as the critical locus of a regular function (“potential”) defined similarly to the one in \([50, \text{Section 4.3}]\). The sheaf of vanishing cycles of this function is the perverse constructible sheaf \(\mathcal{P}\) on \(\mathcal{M}_{k,d,l}\) \([50, \text{P72}]\). Here \(h, \epsilon\) are equivariant parameters corresponding to the 2-dimensional torus \(T = C^*_h \times C^*_\epsilon\) action.

It is explained in \([6]\) that the algebra \(A_{\ell,h,\epsilon}\) is a certain limit of Coulomb branch algebra of the cyclic quiver gauge theory, which in turn has been algebraically described as cyclotomic rational Cherednik algebra in \([34]\). Since we need a more detailed description, we briefly recall this algebra.

First recall the spherical cyclotomic rational Cherednik algebra (see \([34, \text{Section 5}]\)). Let \(\Gamma_N := N \times (\mathbb{Z}/\mathbb{Z})^N\) be the wreath product of the symmetric group and the cyclic group of order \(l\). Denote a fixed generator of the \(i\)-th factor of \((\mathbb{Z}/\mathbb{Z})^N\) by \(\alpha_i\). The group \(\Gamma_N\) acts on \(C(\xi_1, \cdots, \xi_N, \eta_1, \cdots, \eta_N)\) by

\[
\alpha_i(\xi_i) = \epsilon \xi_i, \quad \alpha_i(\xi_j) = \xi_j, \alpha_i(\eta_i) = \epsilon^{-1} \eta_i, \alpha_i(\eta_j) = \eta_j \quad (i \neq j)
\]

with the obvious \(S_N\)-action. Here \(\epsilon\) denotes a primitive \(l\)-th root of unity.

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Definition 4.2.1 The cyclotomic rational Cherednik algebra $H^{\text{cyc}}_{N,l}$ for $\mathfrak{gl}(N)$ is the quotient of the algebra $\mathbb{C}[h_1, h_2, c_1, \cdots, c_{l-1}]$ with relations

\begin{equation}
[\xi_i, \xi_j] = 0 = [\eta_i, \eta_j], \quad (i, j = 1, \cdots, N) \tag{27}
\end{equation}

\begin{equation}
[\eta_i, \xi_j] = \begin{cases} 
-h_1 + h_2 \sum_{k \neq i} \sum_{m=1}^{l-1} s_{ik} \alpha^m_i \alpha^{-m}_{k} + \sum_{m=1}^{l-1} c_m \alpha^m_i & \text{if } i = j \\
-h_2 \sum_{m=0}^{l-1} s_{ij} \epsilon^m \alpha_i \alpha^{-m}_j & \text{if } i \neq j 
\end{cases} \tag{28}
\end{equation}

Let $e_{\Gamma_N}$ be the idempotent for the group $\Gamma_N$. The spherical cyclotomic rational Cherednik algebra is defined as

$$SH^{\text{cyc}}_{N,l} = e_{\Gamma_N} H^{\text{cyc}}_{N,l} e_{\Gamma_N}.$$  

Theorem 4.2.2 \cite[Theorem 1.1, Theorem 6.14, Theorem 1.5]{34}

1. Let $A_{h_1}$ be the quantized Coulomb branch algebra of the gauge theory $(G, N) = (\text{GL}(N), \mathfrak{gl}(N) \oplus (\mathbb{C}^N)^{\mathbb{Z}/l})$. If $l > 0$ then $A_{h_1}$ is isomorphic to $SH^{\text{cyc}}_{N,l}$ with an explicitly given correspondence between parameters in both algebras in loc.cit.

2. Let $A_{h_1}$ be the quantized Coulomb branch algebra for $\dim(V) = N$, $\dim(W) = 1$. Then, we have a surjective homomorphism of algebras $\Psi : Y_l(z) \to A_{h_1}$.

It follows from the above theorem that we have an epimorphism

$$Y_l(z) \to SH^{\text{cyc}}_{N,l},$$

for all $N$, such that $e_0 \mapsto e_{\Gamma_N} \left( \sum_{i=1}^{N} \xi_i \right) e_{\Gamma_N}$.

The relation of the shifted Yangian $Y_l(z)$ and the spherical cyclotomic rational Cherednik algebra $SH^{\text{cyc}}_{N,l}$ is

$$Y_l(z) = \lim_{N \to \infty} SH^{\text{cyc}}_{N,l}.$$  

Here according to \cite[§ 13.4.2, Remark]{6} the limit is expected to be made precise using Deligne categories, similar to the tensor categories extending those of representations of the symmetric groups $S_n$ to complex values of $n$. Then the index $N$ becomes a central element in the limit algebra. See also \cite{13} for an analogue of this limit process via Deligne categories. It is expected that the generators and relations in \cite{26} give another presentation of this algebra.

According to \cite[Theorem 1.6.1]{6} there is an epimorphism $A_{l, h, e} \to SH^{\text{cyc}}_{N,l}$, for all $N$. Thus, $A_{l, h, e}$ is also isomorphic to the limit $\lim_{N \to \infty} SH^{\text{cyc}}_{N,l}$.

This gives the isomorphisms

$$Y_l(z) \cong A_{l, h, e} \cong \lim_{N \to \infty} SH^{\text{cyc}}_{N,l}.$$  

In this paper, we focus on the case where $l = 1$, thus $X_l = \mathbb{C}^2$, and $X_l^{[k]}$ is the Hilbert scheme $\text{Hilb}^k(\mathbb{C}^2)$. The above-mentioned algebra $A_{1, h, e}$ \cite{34} is isomorphic to $Y_1(z_1)$.

Let $X_{1,1}$ be the resolved conifold. The PT moduli space of $X_{1,1}$ space is identified with the space of $\mathcal{O}(-1)$-twisted quasi-maps from $\mathbb{P}^1$ to $\text{Hilb}^k(\mathbb{C}^2)$ \cite[§ 4.3.19 Exercise 4.3.22]{51}. Recall that the PT moduli space parameterizes the complexes $\mathcal{O}_{X_{1,1}} \to \mathcal{F}$ of sheaves on $X_{1,1}$, with the condition that the sheaf $\mathcal{F}$ is pure 1-dimensional (i.e., has no 0-dimensional subsheaves), and $\text{Cokernel}(s)$ is 0-dimensional. The sheaf $\mathcal{F}$ is shown to be an extension of
the structure sheaf of a curve in $X_{1,1}$ by a zero-dimensional sheaf supported on this curve \[52 \S 1.3\]. Note that $X_{1,1}$ is the total space of a rank-2 vector bundle on $\mathbf{P}^1$, the projection to $\mathbf{P}^1$ gives a map from the curve supporting $\mathcal{F}$ to $\mathbf{P}^1$. Pushing-forward via the projection $\pi: X_{1,1} \to \mathbf{P}^1$, the sheaf $\mathcal{F}$ gives a vector bundle on $\mathbf{P}^1$ the rank of which is equal to the degree of the map from this curve to $\mathbf{P}^1$. The action of $\pi_*\mathcal{O}_{X_{1,1}}$ on the vector bundle $\pi_*\mathcal{F}$ is equivalent to a pair of commuting $\mathcal{O}(-1)$-valued Higgs fields. The section of $\mathcal{F}$ is equivalent to a section of the vector bundle $\pi_*\mathcal{F}$, which generates the bundles under the action of $\pi_*\mathcal{O}_{X_{1,1}}$ hence of the two Higgs fields. This is the same as a quasi-map from $\mathbf{P}^1$ to the Hilbert scheme of points on $\mathbf{C}^2$.

By Theorem 1.3.2, we have an action of the shifted Yangian $Y_1(z_1)$ on the cohomology of the moduli space of perverse coherent systems on $X_{1,1}$ with stability parameter on the PT-side of the imaginary root hyperplane. This can be viewed as an action of $A_{l,h,\epsilon}$ on $\bigoplus_d H^*_T(M_{k,d,l}, P)$.

5 Hilbert scheme of $\mathbf{C}^3$ and shifted Yangians

In this section we prove Theorem 1.3.2(1).

5.1 The Hilbert scheme of points on $\mathbf{C}^3$

Consider the following framed quiver $\tilde{Q}_3$ with potential

\[
\begin{tikzpicture}
  \node (B1) at (0,0) {$B_1$};
  \node (B2) at (1,1) {$B_2$};
  \node (B3) at (1,-1) {$B_3$};
  \node (I) at (0,-2) {$I$};
  \draw [->] (B1) to [out=90,in=180] (B2);
  \draw [->] (B1) to [out=90,in=0] (B3);
  \draw [->] (B2) to [out=270,in=90] (B3);
  \draw [->] (I) to [out=0,in=270] (B1);
\end{tikzpicture}
\]

The potential is $W_3 = \tilde{W}_3 = B_3([B_1, B_2])$

A representation of the quiver $Q_3$ of dimension $n$ is stable framed if the following additional property is satisfied

\[ C(B_1, B_2, B_3)I(C) = C^n. \]

Here $C(B_1, B_2, B_3)$ is the algebra of non-commutative polynomials in the variables $B_1, B_2, B_3$.

The set of stable framed representations of dimension $n$ is denoted by $\mathcal{M}(n, 1)^{st}$. It consists of triples of $n \times n$ matrices $B_1, B_2, B_3 \in \text{End}(\mathbf{C}^n)$, together with a cyclic vector $v = I(1) \in \mathbf{C}^n$. The group $\text{GL}_n$ acts by conjugation. Cyclicity of $v$ means that it generates $\mathbf{C}^n$ under the action of $B_i$'s.

Note that the critical locus $\text{Crit}(\tilde{W})$ of the potential $\text{tr} \tilde{W}$ in $\mathcal{M}(n, 1)^{st}$ consists of triples of commuting matrices $(B_1, B_2, B_3)$ satisfying the property that $\text{Im}(I)$ is a cyclic vector of $\mathbf{C}^n$ under the three matrices. Therefore, the quotient stack $\text{Crit}(\tilde{W})/\text{GL}_n$ is a scheme isomorphic to $\text{Hilb}^n(\mathbf{C}^3)$. Here $\text{Hilb}^n(\mathbf{C}^3)$ is the Hilbert scheme of $n$-points on $\mathbf{C}^3$ defined by

\[ \text{Hilb}^n(\mathbf{C}^3) = \{ J \subset C[x_1, x_2, x_3] \mid J \text{ is an ideal, and } \dim(C[x_1, x_2, x_3]/J) = n \}. \]
The isomorphism $\text{Crit}(\bar{W})/\text{GL}_n \cong \text{Hilb}^n(\mathbb{C}^3)$ is given such as follows. To an ideal $J \subset \mathbb{C}[x_1, x_2, x_3]$ let us choose an isomorphism $\mathbb{C}[x_1, x_2, x_3]/J \cong \mathbb{C}^n$. Then the linear maps $B_1, B_2, B_3 \in \text{End}(\mathbb{C}^n)$ are given by multiplications by $x_1, x_2, x_3 \mod J$. Furthermore $I \in \text{Hom}(\mathbb{C}, \mathbb{C}^n)$ is $I(1) = 1 \mod J$. Different choices of basis of $\mathbb{C}[x_1, x_2, x_3]/J$ give isomorphic representations. Conversely, if we have $(B_1, B_2, B_3, I) \in \text{Crit}(\bar{W})/\text{GL}_n$, then, the ideal $J$ is defined as the kernel of the following map

$$\phi : \mathbb{C}[x_1, x_2, x_3] \to \mathbb{C}^n, f \mapsto f(B_1, B_2, B_3)I(1).$$

Note that $\phi$ is surjective by the framed stability condition. Hence $\text{dim}(\mathbb{C}[x_1, x_2, x_3]/J) = n$.

There is an action of $(\mathbb{C}^*)^3$ on $\text{Hilb}^n(\mathbb{C}^3)$ induced by rescaling of the coordinates $x_1, x_2, x_3$. Fixed points $(\text{Hilb}^n(\mathbb{C}^3))(\mathbb{C}^*)^3$ of this action are in one-to-one correspondence with 3-dimensional partitions of $n$.

By [55, Theorem 5.1.1], we have an action of $\mathcal{SH}^{(Q_3, W_3)}$ on

$$\bigoplus_{n \geq 0} H^*_{c, \mathbb{GL}_n \times T}(\mathcal{M}(n, 1)^{st}, \varphi_{W_3} Q)^\vee \cong \bigoplus_{n \geq 0} H^*_{c, T}(\text{Hilb}^n(\mathbb{C}^3), \varphi_{W_3} Q)^\vee$$

given by natural correspondences.

### 5.2 Action of the shifted Yangian

In this section, we will construct an action of the shifted Yangian $Y_{-1}(z_1)$ on

$$\bigoplus_{n \geq 0} H^*_{c, T}(\text{Hilb}^n(\mathbb{C}^3), \varphi_{W_3} Q)^\vee.$$ 

In general the scheme $\text{Hilb}^n(\mathbb{C}^3)$ is singular which makes considerations more complicated than in the case $n = 2$. However, we have the following isomorphism of cohomology groups

$$H^*_{c, T}(\text{Hilb}^n(\mathbb{C}^3), \varphi_{W_3} Q)^\vee \cong H^*_{c, \mathbb{GL}_n \times T}(\mathcal{M}(n, 1)^{st}, \varphi_{W_3} Q)^\vee,$$

where $\mathcal{M}(n, 1)^{st}$ and $\mathcal{M}(n, 1)^{st}/\text{GL}_n$ are smooth.

Let $V$ denote the standard coordinate vector space of dimension $n + 1$. Fix $\xi \subset V$, a one dimensional subspace and let $V_2 := V/\xi$ be the quotient. Consider the following correspondence

$$\begin{array}{ccc}
\mathcal{M}(n, n + 1)^{st} & \xrightarrow{p} & \mathcal{M}(n, 1)^{st} \\
\downarrow q & & \downarrow q \\
\mathcal{M}(n + 1, 1)^{st} & \xrightarrow{p} & \mathcal{M}(n + 1, 1)^{st}
\end{array}$$

where

$$\mathcal{M}(n, n + 1)^{st} = \{(B_1, B_2, B_3, I) \in \mathcal{M}(n + 1, 1)^{st} \mid B_i(\xi) \subset \xi\}.$$ 

The parabolic subgroup $P := \{x \in \text{GL}_{n+1} = \text{End}(V) \mid x(\xi) \subset \xi\}$ acts on $\mathcal{M}(n, n + 1)^{st}$.

The map $p$ is given by $(B_1, B_2, B_3, I) \mapsto (B_1, B_2, B_3, I) \mod \xi$, where $I \mod \xi$ is the composition of $I : \mathbb{C} \to V$ with the projection $V \to V_2 = V/\xi$. The map $q$ is the natural inclusion.

It induces a correspondence

$$\begin{array}{ccc}
\mathcal{M}(n, n + 1)^{st}/P & \xrightarrow{p} & \mathcal{M}(n + 1, 1)^{st}/\text{GL}_{n+1} \\
\downarrow q & & \downarrow q \\
\mathcal{M}(n, 1)^{st}/\text{GL}_n & \xrightarrow{p} & \mathcal{M}(n + 1, 1)^{st}/\text{GL}_n
\end{array}$$
The subspace $\xi \subset V$ gives a tautological line bundle on the correspondence $\mathcal{M}(n, n + 1, 1)^{st}/P$, which will be denoted by $L$. Recall that the Levi subgroup of $P$ is $\mathbb{C}^* \times \text{GL}_n$ where $\xi$ above is the standard weight 1 representation of the $\mathbb{C}^*$-factor. Hence, $c_1(L)$ coincide with the equivariant variable of the $\mathbb{C}^*$-factor. For any rational function $G(x)$ in one variable with coefficients in $H^*_T(pt)$ we define the class $G(c_1(L))$ in a localization of $H^*_{c,T}(\mathcal{M}(n, n + 1, 1)^{st}, \varphi_{\tilde{W}_3}Q)^\vee$. Here the localization is taken with respect to $H_{c,T}(pt) \cong H_{c,T}(pt)[x]$.

Define the actions of the operators

\[ E(G) : H^*_{c,\text{GL}_n \times T}(\mathcal{M}(n, 1)^{st}, \varphi_{\tilde{W}_3}Q)^\vee_{loc} \to H^*_{c,\text{GL}_{n+1} \times T}(\mathcal{M}(n+1, 1)^{st}, \varphi_{\tilde{W}_3}Q)^\vee_{loc}, \]

(30)

\[ F(G) : H^*_{c,\text{GL}_{n+1} \times T}(\mathcal{M}(n+1, 1)^{st}, \varphi_{\tilde{W}_3}Q)^\vee_{loc} \to H^*_{c,\text{GL}_n \times T}(\mathcal{M}(n, 1)^{st}, \varphi_{\tilde{W}_3}Q)^\vee_{loc} \]

(31)

by the following convolutions:

\[ E(G) := q_\ast(G(c_1(L)) \otimes p^\ast), \quad F(G) := p_\ast(G(c_1(L)) \otimes q^\ast). \]

Here the subscript $loc$ means localization of $H^*_T(pt)$-modules. Note that the fixed point loci with respect to the $T$-actions are finite for all three spaces above. Hence the push-forward $p_\ast$ is well-defined after passing to the localization despite of the fact that $p$ is not a proper map.

Let $\lambda$ be a 3-dimensional Young diagram with $n$ boxes, which we denote as a partition $\lambda \vdash n$. The dual of the compactly supported cohomology group $H^*_{c,\text{GL}_n \times T}(\mathcal{M}(n, 1)^{st}, \varphi_{\tilde{W}_3}Q)^\vee_{loc}$ has a basis given by $\{\lambda | \lambda \vdash n\}$. Let $(\lambda + [\square]) \vdash (n + 1)$ be the Young diagram obtained by adding a box $\square$ to $\lambda$. We use the notation $(\lambda | (E(G))|\lambda + [\square])$ for the coefficient of $\lambda + [\square]$ in the expansion of $E(G)(\lambda)$. Similar convention is used for $(\lambda + [\square](F(G))|\lambda)$.

We denote by $\square_{i,j,k}$ the box in the 3-dimensional Young diagram $\lambda$ which has coordinates $(i,j,k)$. Here we follow the French convention in writing the coordinates of boxes. For example, all boxes has non-negative coordinates and the corner box has coordinates $(0,0,0)$.

Recall that $(\mathbb{C}^*)^3$ naturally acts on $\mathbb{C}^3$ and $H^*_{c,\text{GL}_n \times T}(\mathcal{M}(n, 1)^{st}, \varphi_{\tilde{W}_3}Q)^\vee_{loc}$ has a basis given by $\{\lambda | \lambda \vdash n\}$. Let $\chi$ be the $\mathbb{C}^*$-equivariant variable of the $i$th $\mathbb{C}^*$-factor of $(\mathbb{C}^*)^3$. We also have the 2-dimensional torus $T \subset (\mathbb{C}^*)^3$ preserving the canonical line bundle, and $H^*_{c,T}(pt) = Q[\hbar_1, \hbar_2, \hbar_3]/(\hbar_1 + \hbar_2 + \hbar_3 = 0)$. We also have an $\mathbb{C}^*$ acting on the framing, and let $\chi$ be the $\mathbb{C}^*$-equivariant variable. For any box $\square_{i,j,k}$, we consider the following element $x_{\square_{i,j,k}} := \chi + i\hbar_1 + j\hbar_2 + k\hbar_3 \in H^*_{c,T \times \mathbb{C}^*(pt)}$. In the proof, we show this element is equal to the equivariant Chern root of some vector bundle.

**Proposition 5.2.1** The matrix coefficients of the operators $E(G), F(G)$ in the basis of fixed points are as follows:

\[ \langle \lambda | (E(G))|\lambda + [\square] \rangle = \text{Res}_{z=x_{\square}} G(z) \frac{1}{z-x_{\text{loc}}} \prod_{\square \in \lambda} (z-x_{\square} - h_1)(z-x_{\square} - h_2)(z-x_{\square} - h_3)^{z-x_{\square}} \]

Here for a box $\square$ or $\square$ in a 3-dimensional Young diagram, $x_{\square}$ or $x_{\square}$ stands for the $T$-weight of the corresponding box, which is an element in $H^*_{T}(pt)$.

**Proof.** Both formulas can be derived from (59) and (56). Formula (56) gives

\[ \langle \lambda + [\square]|(f(G))|\lambda \rangle = G(c_1(L)) \frac{e(T_{\lambda}(\mathcal{M}(n+1, 1)^{st}/\text{GL}_{n+1}))}{e(T_{\lambda,\lambda + [\square]}(\mathcal{M}(n, n+1)^{st}/P))} \]

30
Here $e$ stands for the $T$-equivariant Euler class.

In order to use [59] we recall that the correspondence $G \times_p M(n, n+1, 1)^{st}$ has a free $G$-action. The Levi subgroup of $P$ is $\mathbf{C}^* \times \text{GL}(n)$. We denote the equivariant variable corresponding to the $\mathbf{C}^*$-action by $z$. Then at the point $(\lambda, \lambda + \mathbf{1}) \in G \times_p M(n, n+1, 1)^{st}$ the tautological bundle $L$ becomes the standard 1-dimensional representation of $\mathbf{C}^*$, and hence $c_1(L)|_{(\lambda, \lambda + \mathbf{1})} = z$. Now [59] gives

$$\langle \lambda | (e(G))| \lambda + \mathbf{1} \rangle = \text{Res}_{z=x} G(c_1(L)) \frac{e(T_\lambda(M(n, 1)^{st}/\text{GL}_n))}{e(T_{\lambda, \lambda + \mathbf{1}}(M(n, n+1, 1)^{st}/P))}$$

Here $e$ denote the $T \times \mathbf{C}^*$-equivariant Euler class, where $\mathbf{C}^*$ is the above-mentioned factor of the Levi subgroup of $P$.

Next we calculate all the tangent spaces at the fixed points. We start by calculating the tangent space for $M(n+1, 1)^{st}/\text{GL}_{n+1}$. Let $V$ be the tautological bundle of rank $n+1$. For simplicity, for any vector bundle $X$, we write the dual bundle as $X^\vee$. Hence, for two vector bundles $X, Y$ we have $\text{Hom}(X, Y) = X^\vee \otimes Y$.

The tangent complex $T(M(n+1, 1)^{st}/\text{GL}_{n+1})$ is given by

$$\text{End}(V) \to \left( \mathbf{C}^3 \otimes \text{End}(V) + \text{Hom}(W, V) \right).$$

Its class in the Grothendieck group of $M(n+1, 1)^{st}/\text{GL}_{n+1}$ is then given by

$$T(M(n+1, 1)^{st}/\text{GL}_{n+1}) = \mathbf{C}^3 \otimes V^\vee \otimes V + W^\vee \otimes V - \text{End}(V).$$

The tangent complex of the correspondence $M(n, n+1, 1)^{st}/P$ has the following class in the Grothendieck group, where we do not include the terms coming from obstruction theory.

$$T(M(n, n+1, 1)^{st}/P) = \mathbf{C}^3 \otimes P + W^\vee \otimes V - P.$$

Now formula [59] gives the following

$$\langle \lambda | (E(G))| \lambda + \mathbf{1} \rangle = \text{Res}_{z=x} G(c_1(L)) \frac{e(T_\lambda(M(n, 1)^{st}/\text{GL}_n))}{e(T_{\lambda, \lambda + \mathbf{1}}(M(n, n+1, 1)^{st}/P))}$$

$$\begin{align*}
&= \text{Res}_{z=x} G(c_1(L)) \frac{e(\mathbf{C}^3 \otimes V_2^\vee \otimes V_2 + W^\vee \otimes V - \text{End}(V_2))}{e(\mathbf{C}^3 \otimes P + W^\vee \otimes V - P)} \\
&= \text{Res}_{z=x} G(c_1(L)) \frac{e(V_2^\vee \otimes \xi)}{e(\mathbf{C}^3 \otimes V_2^\vee \otimes \xi)} \frac{1}{e(W^\vee \otimes \xi)}
\end{align*}$$

Here as before $z$ is the equivariant Chern root of $\xi$. We take the basis of $V_2$ to be $\{ \Box | \Box \in \lambda \}$. Then the equivariant Chern root of the line spanned by $\Box$ is the $T$-weight of $\Box$, which is $x_\Box$ in our notation. The three loops $B_1, B_2, B_3$ have weights $h_1, h_2, h_3$ respectively. Let $\chi$ be the Chern roots of $W$. Thus, we have

$$\langle \lambda | (E(G))| \lambda + \mathbf{1} \rangle = \text{Res}_{z=x} G(z) \prod_{\Box \in \lambda} \frac{z - x_\Box}{(z - x_\Box - h_1)(z - x_\Box - h_2)(z - x_\Box - h_3)} \frac{1}{z - \chi}$$

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Similarly, by \[56\] we have
\[
\langle \lambda + \mathbf{1} | (F(G)) | \lambda \rangle = G(c_1(L)) \frac{e(T_\lambda(M(n + 1, 1)^{st}/GL_{n+1})}{e(T_{\lambda, \mathbf{1}}(M(n, n + 1, 1)^{st}/P))} \frac{e(C^3 \otimes V^\vee \otimes V + W^\vee \otimes V - \text{End}(V))}{e(C^3 \otimes P + W^\vee \otimes V - P)} = G(c_1(L)) \frac{e(C^3 \otimes \xi \otimes V_2)}{e(\xi \otimes V_2)}
\]
\[
=G(c_1(L)) \prod_{\square \in \lambda} (x_{\square} - x_{\square} + \hbar_1)(x_{\square} - x_{\square} + \hbar_2)(x_{\square} - x_{\square} + \hbar_3)
\]
This completes the proof. ■

Now in the definition \[39\] above we take the function \(G(x)\) to be \(x^i\). For \(e_i \in Y_{-1}(z_1)\), we define the action of \(e_i\) on \(\bigoplus_{n \geq 0} H^1_{c,T}(\text{Hilb}^n(C^3), \varphi_{\tilde{W}_n}Q)^\vee\) by the operator \(E(x^i)\) above. Similarly, for \(f_i \in Y^-_{-1}(z)\), we define the action of \(f_i\) by the operator \(F(x^i)\) above. The parameter \(z_1\) acts by \(\chi\).

We define the action of \(\psi(z)\) on \(\bigoplus_{n \geq 0} H^1_{c,T}(\text{Hilb}^n(C^3), \varphi_{\tilde{W}_n}Q)^\vee\) via the Chern polynomial of the following class in the Grothendieck group
\[
(q_1q_2)V + (q_1q_3)V + (q_2q_3)V - q_1V + q_2V + q_3V.
\]
Here \(q_1, q_2, q_3\) are the three coordinate lines of \(C^3\), considered as \(T\)-representations. We refer to \[35\ § 4.1\] for the definition of Chern polynomial.

**Proposition 5.2.2** Let \(\lambda \vdash n\) be a 3d partition. The eigenvalue of \(\psi(z)\) on \([\lambda]\) equals
\[
\prod_{\square \in \lambda} \frac{z - x_{\square} + \hbar_1}{z - x_{\square} - \hbar_1} \frac{z - x_{\square} + \hbar_2}{z - x_{\square} - \hbar_2} \frac{z - x_{\square} + \hbar_3}{z - x_{\square} - \hbar_3}.
\]

**Definition 5.2.3** For a 3d Young diagram \(\lambda\), an addible box is a box in another 3d Young diagram \(\lambda'\), such that \(\lambda'\) is obtained by adding this box to \(\lambda\). A removable box in \(\lambda\) is a box such that \(\lambda\) with this box removed is still a 3d Young diagram.

**Lemma 5.2.4** Assume \(\hbar_1 + \hbar_2 + \hbar_3 = 0\). Then \(h(z)|_{\lambda} = \frac{1}{z - x_{\square}^{000}} \psi(z)|_{\lambda}\) is multiplication by a rational function, the poles of which are at the addible and removable boxes of \(\lambda\).

**Proof.** Follows from a careful examination of the cancellations. ■

### 5.3 Checking the relations of the shifted Yangian

#### 5.3.1 Checking the relations \[21\] \[22\] \[23\] \[24\]
This follows from the epimorphism
\[
Y^+_{-1}(z_1) \rightarrow SH^{(Q_3, W_3)}_1, \quad e_n \mapsto \lambda^n, \quad Y^-_{-1}(z_1) \rightarrow SH^{(Q_3, W_3)}_1, \quad (-1)^{n+t} f_{n+t} \mapsto \lambda^n
\]
and the compatibility of the COHA action and the \(Y^\pm_{-1}(z_1)\) actions on the Hilbert scheme.
5.3.2 Checking the relation \[19\]

This is exactly the same as the affine Yangian case.

Let \(e(y)\) be the generating series \(\sum_{n=0}^{\infty} e_n y^{n+1}\), where \(y\) is a formal variable. The relation \[19\] is equivalent to the following relation (see \[34\])

\[
\psi(z) e(y) = \left( e(y) \psi(z) \left( z - y^{-1} - h_1 \right) \left( z - y^{-1} - h_2 \right) \left( z - y^{-1} - h_3 \right) \right) + \quad (32)
\]

where \((\cdot)_+\) denotes the part with positive powers in \(y\). Here \(\left( z - y^{-1} - h_1 \right) \left( z - y^{-1} - h_2 \right) \left( z - y^{-1} - h_3 \right) \) is regarded as an element in \(\mathbb{C}[h_1, h_2][[z^{-1}]]\).

The action of the series \(e(y)\) on \(\bigoplus_{n \geq 0} H^*_{\mathfrak{e}, \mathfrak{f}}(\text{Hilb}^n(\mathfrak{C}^3), \varphi_{\mathfrak{W}_3} Q) \) is given by the operator \(E(\frac{1}{y - z})\). By Proposition 5.2.1 the matrix elements of \(e(y)\) are given by

\[
\langle \lambda | (e(y)) | \lambda + \square \rangle = \frac{y}{1 - x \square y} \text{Res}_{z=x} \frac{1}{z - x \square_{\text{Hilb}}} \prod_{\square \in \lambda} \frac{z - x \square}{(z - x \square - h_1)(z - x \square - h_2)(z - x \square - h_3)}.
\]

Let \(A := \text{Res}_{z=x} \frac{1}{z - x \square_{\text{Hilb}}} \prod_{\square \in \lambda} \frac{z - x \square}{(z - x \square - h_1)(z - x \square - h_2)(z - x \square - h_3)}\). Applying both sides of (32) to the partition \(\lambda\), and looking at the coefficient of \(\lambda + \square\), we obtain

\[
\langle \lambda | (\psi(z) e(y)) | \lambda + \square \rangle = \psi(z)_{|_{\lambda+\square}} \langle \lambda | (e(y)) | \lambda + \square \rangle = \psi(z)_{|_{\lambda+\square}} \frac{y}{1 - x \square y} A
\]

We have following inductive formula of \(\psi(z)\)

\[
\psi(z)_{|_{\lambda+\square}} = \psi(z)_{|_{\lambda}} \frac{z - x \square - h_1}{z - x \square + h_1} \frac{z - x \square - h_2}{z - x \square + h_2} \frac{z - x \square - h_3}{z - x \square + h_3}
\]

Therefore, the left hand side of (32) becomes

\[
\langle \lambda | (\psi(z) e(y)) | \lambda + \square \rangle = \psi(z)_{|_{\lambda}} \frac{z - x \square - h_1}{z - x \square + h_1} \frac{z - x \square - h_2}{z - x \square + h_2} \frac{z - x \square - h_3}{z - x \square + h_3} \frac{y}{1 - x \square y} A
\]

Applying the right hand side of (32) to the partition \(\lambda\), we obtain the right hand side is

\[
\frac{y}{1 - x \square y} \psi(z)_{|_{\lambda}} \left( \left( z - y^{-1} - h_1 \right) \left( z - y^{-1} - h_2 \right) \left( z - y^{-1} - h_3 \right) \right) A
\]

The equality (32) follows from applying the operator \(\text{Res}_{y=(x \square - 1) y^i}\) on both sides.

5.3.3 Checking the relation \[25\]

Let us introduce the operator \(h_{i+j} := [e_i, f_j]\) on \(\bigoplus_{n \geq 0} H^*_{\mathfrak{e}, \mathfrak{f}}(\text{Hilb}^n(\mathfrak{C}^3), \varphi_{\mathfrak{W}_3} Q)\). By the formula of Proposition 5.2.1 it is clear that \([e_i, f_j]\) is \([e_i', f_j']\), if \(i + j = i' + j'\). Define the generating series

\[
h(z) := 1 - \sigma_3 \sum_{i \geq 0} h_i z^{i-1}
\]

The relation \[25\] is equivalent to (see \[61\] Proposition 1.5)

\[
\sigma_3 (w - z)[e(z), f(w)] = h(z) - h(w).
\]
Then, the relation (25) becomes that

\[ h(z) = \frac{1}{z-x_{\square}^0} \psi(z). \]

Let \( \lambda \vdash n \). By Proposition 5.2.1, we have

\[
\langle \lambda + \square | (f_j) | \lambda \rangle \langle \lambda | (e_i) | \lambda + \square \rangle = \text{Res}_{z=x_{\square}} z^{i+j} \frac{1}{z-x_{\square}^0} \prod_{\square \in \Lambda} \frac{(z-x_{\square} + h_1)(z-x_{\square} + h_2)(z-x_{\square} + h_3)}{(z-x_{\square} - h_1)(z-x_{\square} - h_2)(z-x_{\square} - h_3)}
\]

On the other hand, using again Proposition 5.2.1, we have

\[
\langle \lambda - \square | (e_i) | \lambda \rangle \langle \lambda | (f_j) | \lambda - \square \rangle = \text{Res}_{z=x_{\square}} z^{i+j} \frac{1}{z-x_{\square}^0} \prod_{\square \in \Lambda} \frac{(z-x_{\square} + h_1)(z-x_{\square} + h_2)(z-x_{\square} + h_3)}{(z-x_{\square} - h_1)(z-x_{\square} - h_2)(z-x_{\square} - h_3)}
\]

This implies that

\[
[e_i, f_j]_{\lambda} = - \sum_{\text{addible boxes}} \text{Res}_{z=x_{\square}} z^{i+j} \frac{1}{z-x_{\square}^0} \prod_{\square \in \Lambda} \frac{(z-x_{\square} + h_1)(z-x_{\square} + h_2)(z-x_{\square} + h_3)}{(z-x_{\square} - h_1)(z-x_{\square} - h_2)(z-x_{\square} - h_3)}
\]

\[
- \sum_{\text{removable boxes}} \text{Res}_{z=x_{\square}} z^{i+j} \frac{1}{z-x_{\square}^0} \prod_{\square \in \Lambda} \frac{(z-x_{\square} + h_1)(z-x_{\square} + h_2)(z-x_{\square} + h_3)}{(z-x_{\square} - h_1)(z-x_{\square} - h_2)(z-x_{\square} - h_3)}
\]

\[
= - \sum_{\text{addible boxes} \cup \text{removable boxes}} \text{Res}_{z=x_{\square}} z^{i+j} \frac{1}{z-x_{\square}^0} \prod_{\square \in \Lambda} \frac{(z-x_{\square} + h_1)(z-x_{\square} + h_2)(z-x_{\square} + h_3)}{(z-x_{\square} - h_1)(z-x_{\square} - h_2)(z-x_{\square} - h_3)}
\]

\[
= \text{Res}_{z=\infty} z^{i+j} \frac{1}{z-x_{\square}^0} \prod_{\square \in \Lambda} \frac{(z-x_{\square} + h_1)(z-x_{\square} + h_2)(z-x_{\square} + h_3)}{(z-x_{\square} - h_1)(z-x_{\square} - h_2)(z-x_{\square} - h_3)}
\]

where the last equality follows from the residue theorem and Lemma 5.2.4.

Then we have

\[
h(z) = \frac{1}{z-x_{\square}^0} \prod_{\square \in \Lambda} \frac{(z-x_{\square} + h_1)(z-x_{\square} + h_2)(z-x_{\square} + h_3)}{(z-x_{\square} - h_1)(z-x_{\square} - h_2)(z-x_{\square} - h_3)} \tag{33}
\]

We understand the equality (33) as the equality of all coefficients for powers \( z^{-(i+1)}, i \geq 0 \) the power series expansions of rational functions.

**Remark 5.3.1** More generally, consider the quiver with potential given by 3 loops quiver, with framing vector space \( \mathbf{C}^l \).
Let $\mathcal{M}(n, l)^{st}$ be the set of stable representations of the above quiver, let $\text{Coh}_l(C^3, n) = \mathcal{M}(n, l)^{st}/ GL_n$. Let $T_1$ be the maximal torus of $GL_l$ acting by change of basis on the framing. By \cite[Theorem 5.1.1]{SS}, we have an action of $\mathcal{SH}^{(Q_3, W_3)}$ on

$$\bigoplus_{n \geq 0} H^*_c(GL_n \times T_1 \times T(\mathcal{M}(n, l)^{st}, \varphi_{W_3} Q)^{\vee} \cong \bigoplus_{n \geq 0} H^*_c(T_1 \times T(\text{Coh}_l(C^3, n), \varphi_{W_3} Q)^{\vee}$$

given by natural correspondences. The same calculation as above shows that this action extends to an action of the shifted Yangian $Y_{-1}(z_1, z_2, \cdots, z_l)$.

6 PT moduli space of the resolved conifold

In this section we prove Theorem 1.3.2(2).

6.1 The COHA action

In this section, we are going to prove that on cohomology of the moduli space associated with a stability condition chosen on the PT side of the imaginary root hyperplane, there is an action of the equivariant spherical COHA $\mathcal{H}_{C^3}$. Furthermore, the above action can be lifted to the Drinfeld double $D(\mathcal{H}_{C^3})$. The latter gives rise to the action of the 1-shifted affine Yangian of $\mathfrak{gl}(1)$.

Let $(Q, W)$ be the quiver with potential from \cite{Kontsevich}. Let $\tilde{Q}$ be the framed quiver (see \cite{Kontsevich}) obtained by adding to the quiver $Q$ a framing vertex $\infty$ and an extra edge $i_\infty$ going from $\infty$ to 0. Here 0 denotes one of the vertices of $Q$. We extend the potential for $Q$ by the formula $\tilde{W} = W$. For $\zeta = (\zeta_0, \zeta_1, \zeta_\infty)$ a triple of real numbers and a triple of vector spaces $(V_0, V_1, V_\infty)$ associated to the following 3 vertices of extended quiver $\tilde{Q}$, a representation $F = (a_1, a_2, b_1, b_2, t)$ of $\tilde{Q}$ with dimension vector $(\dim(V_0), \dim(V_1), \dim(V_\infty))$, we define the slope of $F$ to be

$$\theta_\zeta(F) := \frac{\zeta_0 \dim(V_0) + \zeta_1 \dim(V_1) + \zeta_\infty \dim(V_\infty)}{\dim(V_0) + \dim(V_1) + \dim(V_\infty)}.$$

A representation $F$ of $\tilde{Q}$ is said to be $\theta_\zeta$-(semi-)stable if

$$\theta_\zeta(F') < (\leq) \theta_\zeta(F)$$

for any nonzero proper $\tilde{Q}$-subrepresentation $F'$ of $F$.

For any pair of real numbers $\zeta = (\zeta_0, \zeta_1)$, define $\zeta_\infty := -\zeta_0 \dim(V_0) - \zeta_1 \dim(V_1)$ so that we have a triple $\tilde{\zeta} = (\zeta_0, \zeta_1, \zeta_\infty)$. We say the representation $F$ with dimension vector $(\dim(V_0), \dim(V_1), 1)$ is $\tilde{\zeta}$-stable, if it is $\theta_{\tilde{\zeta}}$-stable. Note that in the special case $\zeta_0 = \zeta_1 = -1$, a representation of $\tilde{Q}$ is $\theta_{\tilde{\zeta}}$-stable if every $\tilde{Q}$-subrepresentation of $(V_0, V_1, V_\infty)$ containing the framing $V_\infty$ is either $V_\infty$ or the entire $(V_0, V_1, V_\infty)$. For this choice of $\zeta$ we also call a $\zeta$-stable representation cyclicly stable.

Let $\mathcal{M}_\zeta(v_0, v_1)$ denote the moduli space of $\zeta$-stable $\tilde{Q}$-representations, such that $\dim(V_0) = v_0, \dim(V_1) = v_1, \dim(V_\infty) = 1$. The space $\bigsqcup_{v_0, v_1} \mathcal{M}_\zeta(v_0, v_1)$ is isomorphic to the moduli space $\mathcal{M}_{\zeta}$ of $\zeta$-stable perverse coherent systems in the sense of Nagao and Nakajima \cite[§ 1.3]{Nagao}. We refer to $\zeta = (\zeta_0, \zeta_1)$ as the stability parameter, and the 2-dimensional real vector space $\{\zeta = (\zeta_0, \zeta_1) \mid \zeta_1 \in \mathbb{R}\}$ the space of stability conditions.

Let us recall some terminologies from the Introduction. A stability parameter $\zeta$ is called generic if every $\zeta$-semi-stable point is in fact $\zeta$-stable. The locus of non-generic stability
conditions is a union of hyperplanes (which in this case is a union of lines) called \textit{walls}. The complement of the hyperplane arrangement is the union of open connected components which we call \textit{chambers}. The moduli space $\mathcal{M}_\zeta$ in the case of generic $\zeta$ depends only on the chamber that contains $\zeta$. This is the case we are mostly interested in.

The chamber structure of the space of stability conditions for $X_{1,1}$ considered here is described in [43, Figure 1]. In particular, one wall is given by the imaginary root hyperplane $\zeta_0 + \zeta_1 = 0$. We consider the chambers above this imaginary root hyperplane, i.e. those where $\zeta_0 + \zeta_1 > 0$. Those walls are given by

$$(m + 1)\zeta_0 + m\zeta_1 = 0, \quad \text{where} \quad m \in \mathbb{Z}_{\geq 0}.$$ 

Consider a generic stability condition $\zeta_m$ which is above this wall but close to it. We use a different description of the moduli space $\mathcal{M}_{\zeta_m}$ more relevant for our purpose. For this, we consider the following quiver $\tilde{Q}_\zeta$ with potential $W_\zeta$ (see [13, Figure 9]). Note that $\tilde{Q}_\zeta$ is also obtained from $Q$ by adding the vertex $\infty$ which by an abuse of terminology we still refer to as the framing, but the set of arrows depends on $m$. We consider representation of $\tilde{Q}_\zeta$ on a triple of vector spaces $(V_0, V_1, V_\infty)$ associated to the 3 vertices as before. Again a representation of $\tilde{Q}_\zeta$ is said to be \textit{cyclicly stable} if every $\tilde{Q}_\zeta$-subrepresentation of $(V_0, V_1, V_\infty)$ containing the framing $V_\infty$ is either $V_\infty$ or the entire $(V_0, V_1, V_\infty)$ [43]. The moduli space $\mathcal{M}_m$ of cyclically stable $\tilde{Q}_\zeta$-representations with $V_\infty = \mathbb{C}$ is again isomorphic to $\mathcal{M}_{\zeta_m}$.

$$\begin{array}{c}
\begin{array}{c}
V_1 \\
\bullet
\end{array}
\end{array}$$

The quiver $\tilde{Q}_{\zeta_m}$ with potential $W_\zeta$ given by

$$a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1 + p_1 b_1 q_1 + p_2 (b_1 q_2 - b_2 q_1) + \cdots + p_m (b_1 q_m - b_2 q_{m-1}) - p_{m+1} b_2 q_m$$

There is a 4-dimensional torus $(\mathbb{C}^*)^4$ acting by scaling $(a_1, a_2, b_1, b_2)$. However, the 1-dimensional subtorus $T_0 = \{(t, t, t^{-1}, t^{-1}) \in (\mathbb{C}^*)^4\}$ acts trivially. Then we have the action of the 3-dimensional subtorus of $(\mathbb{C}^*)^4$ with the first coordinate being 1 $\in \mathbb{C}^*$. Note that this 3-dimensional torus maps isomorphically to $(\mathbb{C}^*)^4/T_0$ under the quotient map. We write an element in this subtorus as $(1, t, q, h) \in (\mathbb{C}^*)^4$. We consider $T \subseteq (\mathbb{C}^*)^4$ consisting of $(1, t, q, h)$ with $tqh = 1$. The action of $T$ on $X$ preserves the canonical line bundle. In particular, the induced action on the moduli spaces preserves the potential.

The weights of $p_i$ and $q_i$ are determined by the conditions

$$\text{wt}(p_i) = \text{wt}(q_i^{-1}), \quad \text{wt}(p_{i+1}) = \text{wt}(q_i^{-1}) t^{-1}.$$ 

Therefore, we have

$$\text{wt}(b_1) = 1, \quad \text{wt}(b_2) = t, \quad \text{wt}(a_1) = q, \quad \text{wt}(a_2) = t^{-1} q^{-1}, \quad \text{wt}(p_i) = t^{1-i}, \quad \text{wt}(q_i) = t^{i-1}.$$
Theorem 6.1.1

1. There is a natural action of $\GL(\mathfrak{m}_m)$, single out by the condition that $b_1$ is surjective. The natural flat pull-back to the open subset induces the isomorphism $H^*_{c,T}(\mathfrak{m}_m, \varphi_{tr} W, \mathbb{Q})^\vee \to H^*_{c,T}(\mathfrak{m}_m, \varphi_{tr} W, \mathbb{Q})^\vee$ (up to localization).

**Theorem 6.1.1**

1. There is a natural action of $\mathcal{H}_{\mathfrak{c}^0}$ on $H^*_{c,T}(\mathfrak{m}_m, \varphi_{tr} W, \mathbb{Q})$.

2. This action extends to an action of $Y_1(z_1)$.

We prove here only part 1 of the statement. In order to prove part 2 it suffices to find the commutation relations of the raising and lowering operators acting on $H^*_{c,T}(\mathfrak{m}_m, \varphi_{tr} W, \mathbb{Q})$, which we leave until §6.4.

**Proof.** We keep the notation of Section 3.6. To avoid confusion, here in the proof, we denote by $W$ in place of $V_\infty$ the vector space at the framing vertex of the quiver. Let

$\Surj(V'_2, V'_1) = \{x \in \Hom(V'_2, V'_1) \mid x \text{ is surjective}\}$

be the set of surjective morphisms from $V'_2$ to $V'_1$. Let

$\mathcal{M}(V'_i, W)^0 := (\Surj(V'_2, V'_1) \oplus \Hom(V'_2, V'_1) \oplus \Hom(V'_1, V'_2) \oplus \Hom(V'_1, W)^{0 \oplus m+1} \oplus \Hom(W, V'_2)^{0 \oplus m})$

Note that the quotient $\Surj(V'_2, V'_1)/\GL(V'_1)$ is a Grassmannian. Consider the following space:

$\GL(V'_1) \times P_1 \mathcal{M}(V'_i, W)^0 = \{(\xi, b_1, b_2, a_1, a_2) \mid \xi \in V'_1, b_1 : V'_2 \to V'_1, b_2 : V'_2 \to V'_1, a_i : V'_1 \to V'_2, i = 1, 2\}$

Denote by $\mathcal{M}(V'_i, W)^{st,0} \subset \mathcal{M}(V'_i, W)^0$ the stable locus.

We define the following spaces

$\tilde{\mathcal{M}} = \{(\xi_1, \xi_2, b_1, b_2, a_1, a_2) \mid \xi_2 \subset V'_2, \xi_1 \subset V'_1, b_1 : V'_2 \to V'_1, b_2 : V'_2 \to V'_1, a_i : V'_1 \to V'_2, i = 1, 2\}$

$G \times P Z^0 = \{(\xi_1, \xi_2, b_1, b_2, a_1, a_2) \in \tilde{\mathcal{M}} \mid b_2(\xi_2) \subset \xi_1, a_i(\xi_1) \subset \xi_2, i = 1, 2\}$.

We have a closed embedding $i : G \times P Z^0 \to \tilde{\mathcal{M}}$ and an affine bundle projection $\pi : \tilde{\mathcal{M}} \to \GL(V'_i) \times P_1 \mathcal{M}(V'_i, W)^0$.

We have the following correspondence

$$\begin{array}{c}
G \times P \left(\Rep(Q_{1,1}, V)_0 \times \mathcal{M}(V'_i, W)^0\right) \\
\phi \downarrow \pi \\
G \times P \mathcal{M}(V'_i, W)^0
\end{array}$$

We have the following isomorphisms

$$\begin{align*}
G \times P \left(\Rep(Q_{1,1}, V)_0 \times \mathcal{M}(V'_i, W)^{st,0}\right) & \cong \GL(V'_i) \\
& \cong \GL(V'_2) \times P_1 \left(\Rep(Q_{1,1}, V)_0 \times \mathcal{M}(V''_i, W)^{st,0}\right) \\
& \cong \GL(V'_2) \times P_2 \left(\Rep(Q_{1,1}, V)_0 \times \mathcal{M}(V''_i, W)^{st,0}\right) \\
& \cong \GL(V'_2) \times P_2 \left(\Rep(Q, V_1) \times (\mathcal{M}(V''_i, W)^{st,0} \mathcal{M}(V'_i))\right)
\end{align*}$$

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Taking the quotient of the correspondence (34) by GL($V'_1$), we have the correspondence

$$GL(V'_2) \times_{P_2} (Z^0/P_1) \xleftarrow{i} \tilde{M}/GL(V'_1) \xrightarrow{\pi} \mathcal{M}(V', W)^0/P_1$$

$$GL(V'_2) \times_{P_2} \left( \text{Rep}(Q_3, V_1) \times \mathcal{M}(V'', W)^0/GL(V'') \right)$$

The action is defined by

$$\psi_* \circ (\pi^{-1})^* \circ i_* \circ \phi^*$$

on the stable locus of (35), where $\pi^{-1}$ exists since $\pi$ is an affine bundle. ■

6.2 The tangent space

In this section, we describe the tangent spaces to the torus fixed points in the open locus $\mathcal{M}^0_m \subset \mathfrak{M}_m$, as well as the correspondence used in (35). Again in this section to avoid confusion we denote by $W$ the vector space at the framing vertex of the quiver in place of $V_m$.

Recall that $\mathfrak{M}^0_m(\mathcal{V})$ is the quotient $\mathcal{M}(\mathcal{V}, W)^{st, 0}/(\text{GL}(V_1) \times \text{GL}(V_2))$, where $\mathcal{M}(\mathcal{V}, W)^{st, 0}$ is the cyclically stable locus of

$$\mathcal{M}(\mathcal{V}, W)^0 = \left( \text{Surj}(V_2, V_1) \oplus \text{Hom}(V_2, V_1) \oplus \text{Hom}(V_1, V_2)^{\oplus 2} \oplus \text{Hom}(V_1, W)^{\oplus m+1} \oplus \text{Hom}(W, V_2)^{\oplus m} \right).$$

The quotient $\text{Surj}(V_2, V_1)/\text{GL}(V_1)$ is a Grassmannian, denoted by $\text{Grass}(v_1, V_2)$, parameterizing quotients $b_1 : V_2 \rightarrow V_1$ with the dimension of $V_1$ being $v_1$. Therefore, $\mathcal{M}(\mathcal{V}, W)^0/\text{GL}(V_1)$ is an affine bundle over the Grassmannian with fiber

$$F := \text{Hom}(V_2, V_1) \oplus \text{Hom}(V_1, V_2)^{\oplus 2} \oplus \text{Hom}(V_1, W)^{\oplus m+1} \oplus \text{Hom}(W, V_2)^{\oplus m}.$$

The tangent space of $\mathfrak{M}^0_m(\mathcal{V})$ is an extension of the above vector space $F$ by the tangent space of $\text{Grass}(v_1, V_2)$, taking into account of the GL($V_2$)-action. Again we denote by

$$\text{Gr}T = \mathcal{H}om(\mathcal{V}, V_1)$$

$$\oplus (\mathcal{H}om(V_2, V_1) \oplus \mathcal{H}om(V_1, V_2)^{\oplus 2} \oplus \mathcal{H}om(V_1, W)^{\oplus (m+1)} \oplus \mathcal{H}om(W, V_2)^{\oplus m})$$

such that $b_1(\xi_2) \subset \xi_1$, $b_1$ is an isomorphism $GL(V'_1)$

$$\mathcal{F}1 := \{ (\xi_1, \xi_2, b_1) \mid \xi_2 \subset V'_2, \xi_1 \subset V'_1, b_1 : V'_2 \rightarrow V'_1$$

such that $b_1(\xi_2) \subset \xi_1$, $b_1$ is an isomorphism $GL(V'_1)$

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which parameterizes diagrams of vector spaces as follows

\[
\begin{array}{c}
0 \rightarrow \xi_1 \rightarrow V'_1 \rightarrow V_1 \rightarrow 0 \\
\uparrow b_1 \quad \uparrow b_1 \quad \uparrow b_1 \\
0 \rightarrow \xi_2 \rightarrow V'_2 \rightarrow V_2 \rightarrow 0 \\
\end{array}
\]

where \( \dim(V_i) = v_i \) and \( \dim(V'_i) = v_i + 1 \), for \( i = 1, 2 \). Note that \((G \times P Z^0)/GL(V'_1)\) is an affine bundle over \( \widetilde{Fl} \).

The fiber of the affine bundle is isomorphic to

\[
P(V'_2, V'_1) \oplus P(V'_1, V'_2)^{\oplus 2} \oplus \mathcal{H}om(V'_1, W)^{\oplus (m + 1)} \oplus \mathcal{H}om(W, V'_2)^{\oplus m}
\]

where

\[
\begin{align*}
P(V'_2, V'_1) &:= \{ x \in \mathcal{H}om(V'_2, V'_1) \mid x(\xi_2) \subset \xi_2 \}, \\
P(V'_1, V'_2) &:= \{ x \in \mathcal{H}om(V'_1, V'_2) \mid x(\xi_1) \subset \xi_1 \}.
\end{align*}
\]

Let \( Fl(v_1 + 1, 1, V'_2) \) be the flag variety of pairs consisting of a quotient \( b_1 : V'_2 \rightarrow V'_1 \) with \( \dim(V'_1) = v_1 + 1 \) and a line \( \xi_1 \subseteq V'_1 \). Then the base space \( \widetilde{Fl} \) maps to \( Fl(v_1 + 1, 1, V'_2) \), where the map is given by forgetting the subspace \( \xi_2 \). This map realizes \( \widetilde{Fl} \) as an affine bundle with each fiber isomorphic to \( \text{Hom}(\xi_2, V'_2) \).

The tangent bundle of the flag variety \( Fl(v_1 + 1, 1, V'_2) \) has the same class in the Grothendieck group as

\[
\mathcal{H}om(V'_2, V'_1) \oplus \mathcal{H}om(\xi, V'_1).
\]

Hence there is a filtration on the tangent sheaf of \((G \times P Z^0)/GL(V'_1)\) with associated graded being

\[
\begin{align*}
\mathcal{H}om(V'_2, V'_1) &\oplus \mathcal{H}om(\xi, V'_1) \\
\oplus P(V'_2, V'_1) &\oplus P(V'_1, V'_2)^{\oplus 2} \oplus \mathcal{H}om(V'_1, W)^{\oplus (m + 1)} \oplus \mathcal{H}om(W, V'_2)^{\oplus m} \\
\oplus \text{Hom}(\xi_2, V'_2).
\end{align*}
\]

Note that the affine bundle \( \pi : \widetilde{\mathcal{M}} \rightarrow GL(V'_1) \times_{p_1} \mathcal{M}(V', W)^0 \) (see (34)) has fiber isomorphic to \( \text{Hom}(\xi_2, V'_2) \).

In the action defined by (30), we only focus on the contribution of the tangent sheaf of \((G \times P Z^0)/GL(V'_1)\) without the term \( \text{Hom}(\xi_2, V'_2) \).

As before, the action of \( GL(V'_1) \) induces a Lie algebra map

\[
\text{End}(V'_2) \rightarrow \mathcal{T}_{(G \times P Z^0)/GL(V'_1)}.
\]

The tangent bundle of the quotient \( Z^{st,0}/(P_1 \times P_2) = (G \times P Z^0)/(GL(V'_1) \times GL(V'_2)) \) is again given by the cokernel of the above action map.

\[39\]
6.3 Fixed points

In this subsection we will use the terminology and some results of \[43\]. The following are the pictures for the empty room configurations (ERC) for the finite type pyramid partitions with length 3 and length 4 (\[43\ Figure 12\]).

In general, for ERC for the finite type pyramid partitions with length $m$, there are $1 \times m$ black stones on the first layer, $1 \times (m - 1)$ white stones on the second layer, $2 \times (m - 1)$ black stones on the third, $2 \times (m - 2)$ white stones on the fourth, and so on until we reach $m \times 1$ black stones.

The following definition can be found in [5].

**Definition 6.3.1** A finite type pyramid partition of length $m$ is a finite subset $\Pi$ of the ERC of length $m$ in which, for every stone in $\Pi$, the stones directly above it are also in $\Pi$.

**Proposition 6.3.2** [43 Proposition 4.14] The set of $T$-fixed points in $\mathfrak{M}_m$ is isolated and parameterized by finite type pyramid partitions of length $m$.

By our conventions on the torus action, the weights of the black stones on the top layer in the empty room partition are

$$1, t, t^2, \ldots, t^{m-1}.$$ 

The weights of the black stones on the layer 3 are

$$ qt, ht; qt^2, ht^2; \ldots, qt^{m-1}, ht^{m-1}. $$

The weights of the black stones on the layer 5 are

$$ q^2 t^2, qht^2, h^2 t^2, q^2 t^3, qht^3, h^2 t^3, \ldots, q^2 t^{m-1}, h^2 t^{m-1}. $$

The last one is the $2(m - 1) + 1 = (2m - 1)$th layer, where the weights of the black stones are

$$ q^{m-1} t^{m-1}, q^{m-2} h t^{m-1}, q^{m-3} h^2 t^{m-1} \ldots, h^{m-1} t^{m-1}. $$

By the definition of the torus weight of $b_1$, for each pair of black and white stones as in Figure (38), the weight of the white stone is the same as that of the black stone.
Let us consider a pair consisting of a black stone and a white stone such that the black stone is right above of the white stone (as in the following picture):

![Diagram](image)

We say that a pair of stones as in (38) is *removable*, if after removing the pair, we still get a pyramid partition. Similarly, we say that a pair of stones as in (38) is *addible* to a pyramid partition, if they are not in the pyramid partition but after adding the pair to the pyramid partition, we still get one.

Let us analyze what the addible and removable pairs are for a given pyramid partition. For this we use the following terminology in order to describe the relative position of one stone with respect to another one:

- **Front**
- **Back**
- **Left** (q)
- **Right** (h)

Furthermore, in this terminology the word *above* will mean *up with respect to the paper surface*, and *below* will mean *down with respect to the paper surface*.

A pair of black and white stones is addible, if the following conditions hold for the black stone in this pair:

1. (Black condition) There is a black stone in front of it in the pyramid partition.
2. (White condition) If there are white stones above it in the empty room partition (see the following picture), then these white stones have to be in the pyramid partition.

For a pair of black and white stones let us analyze the above two conditions for the black stone in this pair. In order to avoid confusion we refer to the black stone in the pair as *black-one*. If the black condition holds for the black-one, then there is another black stone, which we call *black-zero*, which is positioned in front of the black-one. Moreover, the white condition holds automatically for black-zero. Unless one of the white stones above black-zero is the end of a chain of the whites, the white condition also holds for black-one.

Similarly, a pair of black and white stones as in (38) is removable, if the pair is at the
end of a chain like this

No black stone below it

and there is no black stone below the last white stone.

**Proposition 6.3.3** Assume \( t + q + h = 0 \). Poles of the function

\[
h(z) = (-1)^{1[\text{black only stones}]}(-1)^{m+1}(z - \chi - mt) \prod_{b\in\text{black stones and white stones}} \frac{(z - x_b + t)(z - x_b + q)(z - x_b + h)}{(z - x_b - t)(z - x_b - q)(z - x_b - h)} \prod_{b\in\text{black stones only}} \frac{(z - x_b)(z - x_b + q)(z - x_b + h)}{(z - x_b - t)}
\]

are at the addible and removable pairs.

**Proof.** By definition the function \( h(z) \) is product of factors coming from each stone. Let us analyze the contribution of each chain of stones. From the above discussion of addible and removable pairs, in order to prove the proposition it suffices to show that \( h(z) \) has the following zeros and poles. Here the formula of \( h(z) \) is given as a product over all the stones in \( \lambda \). We calculate the contribution of one chain of stones in the product, by taking the product of the corresponding factors over all the stones in this chain.

**Zeros:**

1. For a chain of black stones, we want the product of factors from this chain to have two zeros above the end stone. That is, assume the end stone is \( x \), then we want two zeros at \( x - h \) and \( x - q \) respectively.

2. For a chain of white stones we want the product of factors from this chain to have two zeros, under the stones in front of the end stone of the chain. That is, let the end stone of this chain be \( x \), then the two zeros are at \( x + t + q \) and \( x + t + h \). ( The stone in front of \( x \) is \( x + t \). The two places \( x + t + q \) and \( x + t + h \), where we want the zeros to be, are the two stones underneath \( x + t \).

3. Similarly, a chain of black stones should contribute one zero at the beginning stone.

**Poles:**

1. A chain of black stones (with or without white stones) should contribute a pole at the end stone of the chain.
2. A chain of black stones and white stones should contribute a pole at the end stone.

Now we verify that $h(z)$ does have the aforementioned zeros and poles from the contribution of each chain of stones. Suppose we have a chain of black stones (with marked weights) as on the following picture

```
    x
   /\  \\
  /   \ /
 /     \ /
```

The function $h(z)$ has the following factor

$\frac{(z-x)(z-x+q)(z-x+h)(z-x-t+t)(z-x-t+q)(z-x-t+h)}{(z-x-t)(z-x-t-q)(z-x-t-h)(z-x-2t+q)(z-x-2t+h)(z-x-2t-q)(z-x-2t-h)}$

Thus, the pole of $h(z)$ is $xt^a$. It is clear that $xt^a$ is an addible place.

Suppose we have a chain of black and white stones (with marked weights) as on the following picture

```
    x
   /\  \\
  /   \ /
 /     \ /
```

To illustrate the idea of the proof, we compute this case in the explicit example as in the picture. This can be made in general. In this case, the function $h(z)$ has the following factor

$\frac{(z-x)(z-x+q)(z-x+h)(z-x-t+t)(z-x-t+q)(z-x-t+h)}{(z-x-t)(z-x-t-q)(z-x-t-h)(z-x-2t+q)(z-x-2t+h)(z-x-2t-q)(z-x-2t-h)}$

Thus, the poles of $h(z)$ are $x+2t$ and $x+3t$. It is clear that $xt^3$ is an addible place, and $xt^2$ is a removable place.
6.4 Raising and lowering operators

Taking into account weights and using the canonical isomorphism \( \text{Hom}(V_2, V_1) \cong V_2^\vee \otimes V_1 \), the matrix coefficients of raising operator can be calculated using the Appendix \( \text{A.1} \) in the following way.

\[
e^{\left( tV_2^\vee \otimes V_1 + qV_1^\vee \otimes V_2 + hV_1^\vee \otimes V_2 + V_2^\vee \otimes V_1 \right) \oplus \oplus_{i=1}^m (t^{i-1}W^\vee \otimes V_2) \oplus \oplus_{i=1}^{m+1} (t^{i-1}V_1^\vee \otimes W) - \text{End}(V_2)}
\]

\[
e^{\left( tP(V_2', V_1') + qP(V_1', V_2') + hP(V_1', V_2') + V_2'^\vee \otimes V_1' \right) \oplus \oplus_{i=1}^m (t^{i-1}W^\vee \otimes V_2') \oplus \oplus_{i=1}^{m+1} (t^{i-1}V_1'^\vee \otimes W) - \text{End}(V_2')}
\]

\[
= e^{(tV_2^\vee \otimes \xi + qV_1^\vee \otimes \xi + hV_1^\vee \otimes \xi)} e^{(V_1^\vee \otimes \xi)} e^{(\oplus_{i=1}^{m+1} (t^{i-1}W^\vee \otimes \xi))}
\]

We denote a pyramid partition by \( \lambda \), and we denote by \( \lambda + \blacksquare \) the one obtained from \( \lambda \) by adding a pair \( \{3\} \) denoted by \( \blacksquare \). Let \( \{x_w | w = \text{white stone}\} \) be the Chern roots of \( V_1 \), and let \( \{x_b | b = \text{black stone}\} \) be the Chern roots of \( V_2 \). Let \( \chi \) be the Chern root of \( W \). Then the above formula gives

\[
\langle \lambda | e_i | \lambda + \blacksquare \rangle = \text{Res}_{z=\blacksquare} (-1)^{|\{\text{black only stones}\}|} z^i \prod_{b \in \text{black stones}} \frac{z - x_b}{z - x_b - t} \prod_{w \in \text{white stones}} \frac{1}{(z - x_w - q)(z - x_w - h)} \prod_{i=1}^m \frac{1}{z - \chi - (i - 1)t}
\]

Similarly, the lowering operator is given by

\[
e^{\left( tV_2^\vee \otimes V_1 + qV_1^\vee \otimes V_2 + hV_1^\vee \otimes V_2 + V_2^\vee \otimes V_1 \right) \oplus \oplus_{i=1}^m (t^{i-1}W^\vee \otimes V_2) \oplus \oplus_{i=1}^{m+1} (t^{i-1}V_1^\vee \otimes W) - \text{End}(V_2)}
\]

\[
e^{\left( tP(V_2', V_1') + qP(V_1', V_2') + hP(V_1', V_2') + V_2'^\vee \otimes V_1' \right) \oplus \oplus_{i=1}^m (t^{i-1}W^\vee \otimes V_2') \oplus \oplus_{i=1}^{m+1} (t^{i-1}V_1'^\vee \otimes W) - \text{End}(V_2')}
\]

\[
= e^{(t\xi^\vee \otimes V_1 + q\xi^\vee \otimes V_2 + h\xi^\vee \otimes V_2)} e^{(\xi^\vee \otimes V_1)} e^{(\oplus_{i=1}^{m+1} (t^{i-1}\xi^\vee \otimes W))}
\]

The matrix coefficient is given by

\[
\langle \lambda + \blacksquare | f_j | \lambda \rangle = z^j \prod_{b \in \text{black stones}} (z - x_b + q)(z - x_b + h) \prod_{w \in \text{white stones}} \frac{z - x_w + t}{z - x_w} \prod_{i=1}^{m+1} \frac{1}{z - x_w - \chi - (1 - i)t} \big|_{z=\blacksquare}
\]
Then we have
\[ h(z) = (-1)^{|\{\text{black only stones}\}|} \prod_{b \in \text{black stones}} \frac{(z - x_b)(z - x_b + q)(z - x_b + h)}{z - x_b - t} \prod_{w \in \text{white stones}} \frac{(z - x_w)(z - x_w - q)(z - x_w - h)}{z - x_w + t} \prod_{i=1}^{m} \frac{z - \chi - (i - 1)t}{z - \chi + (1 - i)t} \prod_{i=1}^{m+1} (-1)^{(z - \chi - mt)} \prod_{b \in \text{black stones} \text{ and white stones}} \frac{(z - x_b + t)(z - x_b + q)(z - x_b + h)}{(z - x_b - t)(z - x_b - q)(z - x_b - h)} \prod_{b \in \text{black stones only}} \frac{(z - x_b)(z - x_b + q)(z - x_b + h)}{(z - x_b - t)} \]

Similar to the proof in § 5.3, the operators \( e_i, f_j \), and \( h(z) \) satisfy the relations of \( Y_1(z_1) \), where \( z_1 = \chi + mt \). This concludes the proof of Theorem 6.1.1(2).

7 Shift of the real roots for the resolved orbifold

Let us now move to the discussion of sheaves supported on toric divisors inside of the toric Calabi-Yau 3-fold \( X = X_{m,n} \). As we mentioned in the Introduction, in physics language these divisors correspond to configurations of D4-branes. We also mentioned that they are expected to give rise to shifts of real roots of the affine Yangian. The shifts are determined by the intersection numbers of the corresponding divisor with the projective lines \( P^1 \)’s inside \( X = X_{m,n} \). As we will see in the examples below this is indeed the case.

For any embedded smooth rational curve \( P^1 \subseteq X \) its formal neighbourhood is isomorphic to the one of the embedded smooth rational curve \( P^1 \) in either \( X_{2,0} \) or \( X_{1,1} \). For any toric divisor \( D \subseteq X \), the formal completion at \( P^1 \) is isomorphic to the one for the standard toric divisors in either \( X_{2,0} \) or \( X_{1,1} \). Therefore, we study these cases as the basic building blocks.

7.1 Chainsaw quiver

Consider the Calabi Yau 3-fold \( T^*P^1 \times C = \text{Tot}(\mathcal{O}_{P^1}(-2) \oplus \mathcal{O}_{P^1}) \). We take the effective divisor \( D \) to be fiber of \( T^*P^1 \times C \to P^1 \) over the north pole or over the south pole of \( P^1 \). This is illustrated by the toric diagram (7).

Let us discuss one of these diagrams (say \( D \) is the fiber of \( T^*P^1 \times C \to P^1 \) over the north pole of \( P^1 \), since the other is similar. Note that the \((x, z)\)-coordinate hyperplane divisor \( C^2_{xz} \subseteq C^3_{xyz} \) is invariant under the natural \( \mathbb{Z}/2\)-action, hence defines a divisor in \( C^3/(\mathbb{Z}/2) \). The strict transform under the resolution of singularity \( T^*P^1 \times C \to C^3/(\mathbb{Z}/2) \) is the divisor \( D \), the fiber of \( T^*P^1 \times C \to P^1 \) over the north pole of \( P^1 \). Therefore we can describe the moduli space of stable framed sheaves on \( T^*P^1 \times C \) that supported on \( D \) such as follows.

Start with the quiver with potential \((Q^+_1, W^+_1)\), which is quiver description for the moduli space of framed sheaves on \( C^3 \) that supported on the divisor \( C^2_{xz} \). Note that the
framing \((Q_3^{fr}, W_3^{fr})\) has to do with the toric divisor \(C_{xz}\), which is different from the framing \((\tilde{Q}_3, \tilde{W}_3)\) considered in §5.

The potential is \(W_3^{fr} = B_2([B_1, B_3] + I_{13}J_{13})\).

Explicitly, the group \(\mathbb{Z}/2\mathbb{Z}\) acts on \(C^3\) in such a way that its generator \(\sigma \in \mathbb{Z}/2\mathbb{Z}\) is represented by the matrix

\[
\sigma \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

The torus \((\mathbb{C}^*)^2 \subset (\mathbb{C}^*)^3\) acts on \(C^3\) by rescaling the coordinates by \(t_1, t_2, t_3\) subject to the Calabi-Yau condition \(t_1 t_2 t_3 = 1\). The above formula gives an embedding of the group \(\mathbb{Z}/2\mathbb{Z}\) into the subtorus \((\mathbb{C}^*)^2\). Under the \(\mathbb{Z}/2\mathbb{Z}\) action the weights of the arrows of \(Q^{fr}_3\) are given by

\[
B_1 \sim -1, \ B_2 \sim -1, \ B_3 \sim 1 \\
I_{13} \sim +1, \ J_{13} \sim -1.
\]

We decompose the vector spaces \(K, N_{13}\) into eigenspaces with eigenvalues 1, -1.

\[
K = K^0 \oplus K^1 \\
N_{13} = N_{13}^0 \oplus N_{13}^1
\]

Setting \(N_{13}^1\) to be zero, we obtain the framed quiver \(Q\) from (16) (with the square node removed), with potential \(W\) given by

\[
W = -(B_2 \tilde{B}_1 \tilde{B}_3 - \tilde{B}_2 \tilde{B}_3 B_1 + \tilde{B}_2 B_1 B_3 - B_2 B_3 \tilde{B}_1) + B_2 I_{13} J_{13}
\]

Framed stability condition implies that the image of \(I_{13}\) is invariant under \(B_1, B_3\) and \(\tilde{B}_1, \tilde{B}_3\) and generates the whole space.

We now use the dimensional reduction of a quiver with potential and a cut (see [35, Section 4.8], [9, Appendix]). We take the cut to be the set of arrows consisting of \(B_2, \tilde{B}_2\). Below we present the dimensionally reduced quiver with relations in our case, referring the reader to [38, Appendix] for a description of the COHA for the dimensionally reduced quiver with relations for a general cut.

Let \(Q' := Q \setminus \{B_2, \tilde{B}_2\}\) be the quiver obtained by removing the two arrows \(B_2, \tilde{B}_2\). A picture of the quiver obtained is (8). Then, \(\text{Rep}(Q, V_0, V_1) = \text{Rep}(Q', V_0, V_1) \times (\text{Hom}(V_0, V_1) \oplus \cdots)\).
Hom(V_1, V_0). We have the following diagram

$$
\begin{align*}
Z \times \mathbb{A}^n & \xrightarrow{\imath} \text{Rep}(Q, V_0, V_1) \\
\downarrow & \\
Z & \xrightarrow{\pi} \text{Rep}(Q', V_0, V_1) \rightarrow \text{pt}
\end{align*}
$$

where the affine space $\mathbb{A}^n$ is $\text{Hom}(V_0, V_1) \oplus \text{Hom}(V_1, V_0)$, and $Z$ is the algebraic variety

$$
Z = \{ z \in \text{Rep}(Q', V_0, V_1) \mid \text{tr}(W)(z, l) = 0, \text{ for all } l \in \text{Hom}(V_0, V_1) \oplus \text{Hom}(V_1, V_0) \}
$$

$$
= \{ z \in \text{Rep}(Q', V_0, V_1) \mid \frac{\partial W}{\partial B_2}(z) = 0, \frac{\partial W}{\partial \tilde{B}_2}(z) = 0 \}
$$

$$
= \text{Rep}(C[Q']/J, V_0, V_1),
$$

where $J$ is the ideal generated by

$$
\frac{\partial W}{\partial B_2} = - \tilde{B}_1 \tilde{B}_3 - B_3 \tilde{B}_1 + I_{13} J_{13}, \quad \frac{\partial W}{\partial \tilde{B}_2} = -(- \tilde{B}_3 B_1 + B_1 B_3).
$$

(39)

Then [35 Section 4.8] and [9 Appendix] give an isomorphism

$$
H^*_{c, \text{GL}(V_0) \times \text{GL}(V_1) \times T}(\text{Rep}(Q, V_0, V_1), \varphi_{trW}) \cong H^*_{c, \text{GL}(V_0) \times \text{GL}(V_1) \times T}(Z \times \mathbb{A}^n, Q).
$$

Note that the variety $Z$ is the representation variety of the quiver $Q'$ with relations on the vector spaces $(V_0, V_1)$. The quiver $Q'$ with relation (39) is known as a chainsaw quiver (defined in [20]).

The following definition of Yangian of $\mathfrak{gl}(n)$ can be found in [44] and [19].

**Definition 7.1.1** The Yangian of $\mathfrak{gl}(n)$ is generated by

$$
\{ e_i^{(r)}, f_i^{(r)}, g_j^{(r)} \mid 1 \leq i < n, 1 \leq j \leq n, r \geq 1 \}.
$$

Define the following generating series of the generators

$$
e_i(z) := \sum_{r \geq 1} e_i^{(r)} z^{-r}, f_i(z) := \sum_{r \geq 1} f_i^{(r)} z^{-r}, g_j(z) := 1 + \sum_{r \geq 1} g_j^{(r)} z^{-r}.
$$
They are subject to the following relations \cite[Lemma 2.48]{19}.

\begin{align}
[g_i(z), g_j(w)] &= 0; \\
(z - w)[g_i(z), e_j(w)] &= (\delta_{i,j} - \delta_{i,j+1})g_i(z)(e_j(z) - e_j(w)); \\
(z - w)[g_i(z), f_j(w)] &= (\delta_{i,j+1} - \delta_{i,j})(f_j(z) - f_j(w))g_i(z); \\
[e_i(z), f_j(w)] &= 0, \text{ if } i \neq j; \\
(z - w)[e_i(z), e_i(w)] &= \frac{g_{i+1}(w)}{g_i(w)} - \frac{g_{i+1}(z)}{g_i(z)}; \\
(z - w)[e_i(z), e_{i+1}(w)] &= -e_i(z)e_{i+1}(w) + e_i(w)e_{i+1}(w) - [e^{(1)}_{i+1}, e_i(w)] + [e^{(1)}_i, e_{i+1}(z)]; \\
[e_i(z), e_j(w)] &= 0, \text{ if } |i - j| > 1; \\
[e_i(z), e_j(w)] + [e_i(z), e_j(w)] &= 0, \text{ if } |i - j| = 1; \\
(z - w)[f_i(z), f_j(w)] &= (f_i(z) - f_j(w))^2; \\
(z - w)[f_i(z), f_{i+1}(w)] &= f_{i+1}(w)f_i(z) - f_{i+1}(w)f_i(w) + [f_i(w), f_{i+1}^{(1)}] - [f_i(z), f_{i+1}^{(1)}]; \\
[f_i(z), f_j(w)] &= 0, \text{ if } |i - j| > 1; \\
[f_i(z), [f_i(z), f_j(w)] + [f_i(z), f_j(w)]] &= 0, \text{ if } |i - j| = 1. 
\end{align}

Let \( r = (r_1, \cdots, r_n) \in \mathbb{N}^n \), we define the \( r \)-shifted Yangian of \( \widehat{\mathfrak{gl}(n)} \) to be the algebra generated by

\[ \{ e^{(r)}_i, f^{(r)}_i, g^{(r)}_j | i, j \in \mathbb{Z}/n\mathbb{Z}, r \geq 1 \}, \]

subject to the same relations (40)–(52), except the relation (44) is modified to be

\begin{equation}
(z - w)[e_i(z), f_i(w)] = \frac{g_{i+1}(w)}{g_i(w)} - (z^{r_{i+1} - r_i}) \frac{g_{i+1}(z)}{g_i(z)}. 
\end{equation}

We have identified the cohomology \( H^*_c(\text{GL}(V_0) \times \text{GL}(V_1)) \times T(\text{Rep}(Q, V_0, V_1), \varphi_{trW} Q) \) with the cohomology of the chainsaw quiver \( H^*_c(\text{GL}(V_0) \times \text{GL}(V_1)) \times T(Z \times A^n, Q) \) via the dimensional reduction. It is shown in \cite{16} that the shifted quantum toroidal algebra of \( \mathfrak{gl}(N) \) acts on the K-theory of the chainsaw quiver variety. Following the same calculation, with K-theory replaced by cohomology, we obtain

**Theorem 7.1.2** The \((1,0)\)-shifted Yangian of \( \widehat{\mathfrak{gl}(2)} \) acts on

\[ \bigoplus_{V_0, V_1} H^*_c(\text{GL}(V_0) \times \text{GL}(V_1)) (\text{Rep}(Q, V_0, V_1), \varphi_{trW} Q)^\vee. \]
7.2 Few more examples

7.2.1 Nakajima quiver varieties

Consider the CY 3-fold $T^* \mathbb{P}^1 \times \mathbb{C} = \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1})$, and the effective divisor $\mathcal{O}(-2)$ on $\mathbb{P}^1$. This is illustrated in the following toric diagram.

![Toric Diagram](image)

The quiver description is given by the following figure:

![Quiver Diagram](image)

The potential is given by the formula:

$$B_2 \tilde{B}_1 \tilde{B}_3 - \tilde{B}_2 \tilde{B}_3 B_1 + \tilde{B}_2 B_1 B_3 - B_2 B_3 \tilde{B}_1 + B_3 I_{12} J_{12}.$$  

The framed stability condition says that any subspace containing the image of $I_{12}$ and invariant under the action of $B_1, B_2, \tilde{B}_1, \tilde{B}_2$ coincides with the whole space (i.e. the image of 1 is a cyclic vector). Then using the dimension reduction with respect to the loops, the corresponding moduli spaces are the Nakajima quiver varieties with framing 1. It is known that there is an action of the affine Yangian of $\mathfrak{sl}_2$ (without shifting).

7.2.2 Blowup of $\mathbb{C}^2$

Let $X$ be the resolved conifold $\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. As the effective divisor we take the total space of one of the line bundle summands over $\mathbb{P}^1$. We denote them by $\mathcal{O}(-1)_1$ and $\mathcal{O}(-1)_2$ respectively. This is illustrated in the following toric diagram.

![Toric Diagram](image)
The corresponding quiver is

with the potential

\[ a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1 + b_1 i j. \]

Applying the dimensional reduction to the above quiver with potential with the cut consisting of the edge \( b_1 \), we obtain the quiver with relations from [45], which described stable framed sheaves on the blowup of \( \mathbb{C}^2 \). We postpone the study of shifted Yangian action to the future.

8 What to expect for general toric local Calabi-Yau 3-folds

8.1 Moduli spaces and shifted Yangians

In this subsection we explain how to put examples from Sections 3.3.2, 3.3.3 and 3.3.4 into a more general framework.

Fix a quiver with potential \((Q, W)\). Assume \( Q \) is symmetric. (It follows from this assumption that the quadratic form \( \chi_R \mid V \), defined in [35] is symmetric.) In [35] Section 6.2, Question 6.2, Kontsevich-Soibelman conjecture that there exits a \( \mathbb{Z} \)-graded Lie algebra \( g \) (BPS Lie algebra), such that the (non-equivariant) COHA of \((Q, W)\) is isomorphic to the universal enveloping algebra of the current algebra \( g \otimes \mathbb{C}[T] \) of \( g \). This conjecture was proved later by Davison and Meinhardt [10]. In the special case when \((Q, W)\) is the “tripled” quiver of a simply-laced Kac–Moody Lie algebra as in [24], it follows from the recent paper [11] of Davison that the zeroth piece of the perverse filtration on the COHA of \((Q, W)\) is isomorphic to the universal enveloping algebra of \( g \). The latter contains the upper-triangular subalgebra of the Kac–Moody Lie algebra. As a consequence, if the tripled quiver \( Q \) has exactly one loop at each vertex, then the \((i, j)\)-th entry of the Cartan matrix of the Kac-Moody Lie algebra is equal to the number of arrows of \( Q \) between \( i \) and \( j \) if \( i \neq j \). The diagonal entries are all equal to 2.

We do not know an explicit description of the root system of the BPS Lie algebra for general symmetric quivers with potential. Nevertheless, assuming furthermore that the potential \( W \) is homogeneous, the three examples 3.3.2, 3.3.3 and 3.3.4 suggest that the root system of the BPS Lie algebra should contain a sub root system, the Cartan matrix of
which is given as
\[
\langle \alpha_i, \alpha_j \rangle = \begin{cases} 
-\# \{ h : i \rightarrow j, h \in H \} & \text{if } i \neq j; \\
2 & \text{if } i = j, \text{ and } i \text{ has one loop}; \\
0 & \text{if } i = j, \text{ and } i \text{ has no loops.}
\end{cases} 
\tag{54}
\]

This sub root system is that of a Lie subalgebra of the BPS Lie algebra, corresponding to the spherical subalgebra of the COHA. The Drinfeld double of the entire COHA is expected to be a Cartan doubled Yangian of the root system of the entire BPS Lie algebra.

Now go back to geometry. Let \( X \) be a toric CY 3-fold as in §3. We are in the setting of §3.3 that is, \( f : X \rightarrow Y \) is a resolution with \( Y \) affine and the fibers of \( f \) are at most 1-dimensional. As mentioned in the introduction of [42, pg.2] that such affine toric CY 3-fold \( Y \) can be classified in terms of the lattice polygon in \( \mathbb{R}^2 \). The classification is given by the family \( Y = Y_{m,n} \), together with finitely many exceptional cases. When \( Y = Y_{m,n} \), the corresponding quiver is symmetric [42, Section 1.2].

Let \( \text{Mod} \_1 \mathcal{A}_0 \subseteq \text{Mod} \_2 \mathcal{A}_0 \) be the full subcategory consisting of coherent sheaves of \( \mathcal{A}_0 \)-modules whose set-theoretical supports as coherent sheaves on \( Y \) are zero-dimensional. Similarly, we have the abelian category \( \mathcal{A}_f \), which is equivalent to \( \text{Mod} \_1 \mathcal{A}_0 \) under the functor \( \text{RHom}_{\mathcal{X}}(P, -) \).

Let \( K_0(\mathcal{A})_\mathbb{Z} \) be the Grothendieck group of \( \mathcal{A} \). Let \( \{ S_i \}_{i \in I} \) be a collection of pairwise distinct simple objects in \( \mathcal{A}_f \). For each \( S_i \), let \( [S_i] \in K_0(\mathcal{A})_\mathbb{Z} \) be its class in the Grothendieck group, which we call a simple root. Using this terminology, the set \( \{ [S_i] \}_{i \in I} \) of simple roots is an integer basis of \( K_0(\mathcal{A})_\mathbb{Z} \). Thus the latter can be thought of as a root lattice. We call a simple root \( [S_i] \) bosonic if the simple object \( S_i \) in \( D^b \text{Coh}(X) \) is bosonic, that is \( \text{Ext}^*(S_i, S_i) = H^{2*}(P^3) \). We call a simple root \( [S_i] \) fermionic if the simple object \( S_i \) is fermionic, that is \( \text{Ext}^*(S_i, S_i) = H^{*}(S^3) \). This terminology is explained in §3.1.

In the three examples 3.3.2, 3.3.3 and 3.3.4, the Cartan matrix (54) can be alternatively described as the following pairing on \( K_0(\mathcal{A})_\mathbb{Z} \)
\[
\langle [S_i], [S_j] \rangle = \begin{cases} 
-\text{dim}(\text{Ext}^1(S_i, S_j)) & \text{if } i \neq j; \\
2 & \text{if } i = j, \text{ and } i \text{ is bosonic;}
\end{cases} 
\tag{54}
\]

Indeed, in all these cases, the algebra \( \mathcal{A}_0 \) has a grading coming from the \( C^* \)-action on \( X \) which contracts \( X \) to its special fiber, so that the tilting bundle is \( C^* \)-equivariant. The quiver with potential \( (Q, W) \), whose Jacobian algebra is \( \mathcal{A}_0 \), can be chosen so that the vertices are labeled by \( I \), and the number of arrows from \( i \) to \( j \) is \( \text{dim} \text{Ext}^1(S_i, S_j) \) for \( i, j \in I \) [2] Theorem 3.1 and proof. Hence, the matrix (54) can be expressed in this way. In particular, this gives rise to a free abelian group \( K_0(\mathcal{A})_\mathbb{Z} \) endowed with a basis and an integer pairing. In general, this resembles a part of some "Cartan data", possibly that of a sub root system of the BPS Lie algebra.

Example 8.1.1 In Example 3.3.3, reading off the sequence of bosonic and fermionic simple objects by following the diagram (3) from the bottom to the top (or equivalently from the top to the bottom that produces a reflected Dynkin diagram), one recovers different root systems of the \( \mathfrak{g}(m|n) \) algebra [68]. The above-defined root system thus coincides with the root system of \( \mathfrak{g}(m|n) \). This configuration has been considered by various authors including [39, 54, 62].
In Example 3.3.4, the generalized root system is expected to be of “double affine” type $A_n$. The construction of §2.2 is expected to give a definition of double affine Yangian of type $A_n$ [7].

The moduli space of perverse coherent system of $X$ in the sense of §1.2 will depend on a choice of the stability parameter $ζ$, as well as the algebraic cycle $χ$. Here the stability parameter is understood in the same sense as in §6. We denote by $X$ the space of stability parameters of the perverse coherent system. It can be identified with $K_0(D^b \text{Coh}(X))_{\mathbb{R}}$. We say that $ζ ∈ X$ is generic if it lies in the interior of a chamber. Each generic $ζ$ determines an abelian subcategory $\mathcal{A}_ζ$, which depends only on the chamber containing $ζ$. In the case when $X = X_{m,n}$ this abelian subcategory comes from a tilting bundle explicitly constructed in [2] §1.

The above-mentioned parameter $χ$ is an algebraic cycle of the form $N_0 X + \sum_{i=1}^r N_i D_i$, where $N_i ∈ \mathbb{N}$, and each $D_i$ is a toric divisor. Taking its class in the Borel-Moore homology with integral coefficients, we get $[χ] ∈ H^{BM}_6(X, \mathbb{Z})$. As we only consider algebraic cycles, we have $[χ] ∈ H^{BM}_6(X, \mathbb{Z}) ⊕ H^{BM}_4(X, \mathbb{Z})$.

We can think of $[χ]$ as a coweight (i.e. an integer functional on the root lattice) in the following way. The 0-th homology of $X$ is one dimensional, let $[pt]$ be its basis. Let \{[$C_1], \cdots, [C_{m+n-1}]$\} be the classes of curves $C_i ≃ \mathbb{P}^1$ which give a basis of $H_2(X, \mathbb{Z})$. Let

$$ch : K_0(\mathcal{A}_ζ) → H_*(X)$$

be the homological Chern character map [8] Section 5.9]. Each simple object $S_i ∈ K_0(\mathcal{A}_ζ)$ gives rise to a class $ch([S_i]) ∈ H_*(X)$. The shift is then determined by the pairing

$$⟨χ, ch[S_i]⟩ := −[χ] ∩ ch[S_i]$$

for all $i ∈ I$, (55)

where $∩ : H^{BM}_6(X) ⊕ H_*(X) → \mathbb{Z}$ is the intersection pairing [8] Proposition 2.6.18]. Since the set of simple objects $\{[S_i] | i ∈ I\}$ gives rise to the set of simple roots spanning the root lattice, the homology class $χ$ gives rise to a coweight via the pairing (55).

Let us illustrate the above discussion in the following three examples.

1. Consider the PT moduli space of $X_{1,1}$. There are two simple objects $\{S_0, S_1\} = \{O_{\mathbb{P}^1}, O_{\mathbb{P}^1}(m), O_{\mathbb{P}^1}(m+1)[−1]\}$

They generate the abelian category (heart of the corresponding t-structure) $\mathcal{A}_m^+$. Their classes in $K_0(\mathcal{A}_m^+)$ gives the two simple roots. Consider the short exact sequence $0 → O → O(m) → O_m → 0$ of sheaves on $\mathbb{P}^1$, where $O_m$ is the skyscraper sheaf at $0 ∈ \mathbb{P}^1$. Thus, in $K_0(\mathcal{A}_m^+)$, we have $[O_{\mathbb{P}^1}(m)] = [O] + m[O_m]$. We calculate the homological Chern character map using the devissage principle [8] Proposition 5.9.3], we have

$$ch[S_0] = ch[O_{\mathbb{P}^1}(m)] = [C] + m[pt],$$

$$ch[S_1] = ch[O_{\mathbb{P}^1}(m+1)[−1]] = −[C] − (m + 1)[pt]$$

where $[C]$ is the $\mathbb{P}^1$ class in $H_2(X)$. In this case $χ = [X_{1,1}]$. The intersection pairing is given by $[X_{1,1}] ∩ C = 0, [X_{1,1}] ∩ [pt] = 1$. Note that the imaginary root $δ = ch[S_0] + ch[S_1]$. This implies that

$$⟨χ, δ⟩ = −χ ∩ δ = 1.$$ 

Thus, we have the +1 shift of the imaginary root, cf. in [6].
2. Consider $X = X_{2,0}$, and $D$ be the fiber of the vector bundle $X_{2,0} \to \mathbb{P}^1$ over either the north pole or the south pole of $\mathbb{P}^1$, cf. §7.1. There are two simple objects

$$\{S_0, S_1\} = \{\mathcal{O}_{\mathbb{P}^1}(-m), \mathcal{O}_{\mathbb{P}^1}(-m - 1)[-1]\}$$

in $K_0(\mathcal{A}_m)$, which corresponds to the two simple roots. Consider the short exact sequence $0 \to \mathcal{O}(-m) \to \mathcal{O} \to \mathcal{O}^{\oplus m} \to 0$, where $\mathcal{O}_0$ is the skyscraper sheaf at $0 \in \mathbb{P}^1$. Thus, in $K_0(\mathcal{A}_m)$, we have $[\mathcal{O}_{\mathbb{P}^1}(-m)] = [\mathcal{O}] - m[\mathcal{O}_0]$. Applying the homological Chern character map, we have

$$ch[S_0] = ch[\mathcal{O}_{\mathbb{P}^1}(-m)] = [C] - m[pt],$$

$$ch[S_1] = ch[\mathcal{O}_{\mathbb{P}^1}(-m - 1)[-1]] = -[C] + (m + 1)[pt],$$

where $[C]$ is the $\mathbb{P}^1$ class in $H_2(X)$. In this case $\chi = [D]$. The intersection pairing is given by

$$[D] \cap [C] = 1, [D] \cap [pt] = 0.$$

This implies that

$$\langle \chi, ch[S_0] \rangle = -\chi \cap ch[S_0] = -1, \langle \chi, ch[S_1] \rangle = -\chi \cap ch[S_1] = 1.$$

Thus, the shifts of the two simple roots are $-1, 1$ respectively, cf. §7.1.

3. Consider the DT moduli space of $\mathbb{C}^3$. In this case $\chi = [\mathbb{C}^3]$. The imaginary root $\delta$ is given by the class $[pt]$. The intersection pairing is given by

$$\langle \chi, \delta \rangle = -[\mathbb{C}^3] \cap \delta = -1.$$

Thus we have the $-1$ shift of the imaginary root, cf. §5.

### 8.2 Cartan Doubled Yangian from geometry

Let $D^b \text{Coh}(X)$ be the bounded derived category of coherent sheaves on $X$, and $D^{Z_2} \text{Coh}(X)$ be the derived category of 2-periodic complexes of coherent sheaves on $X$ (see §4 for the definition). Then we have a functor

$$F : D^b \text{Coh}(X) \to D^{Z_2} \text{Coh}(X)$$

by taking direct sum of odd and even degree complexes. We expect that there is a geometric construction of the Cartan doubled Yangian $Y_\infty$, constructed by taking cohomology of certain moduli stack of objects in $D^{Z_2} \text{Coh}(X)$ endowed with Hall multiplication, similarly to §4.

Let $\mathcal{A} \subset D^b \text{Coh}(X)$ be the heart of $D^b \text{Coh}(X)$ corresponding to a tilting bundle as in §1. Denote by $H_\mathcal{A}$ the COHA associated to $\mathcal{A}$ constructed as in §1. We expect the choice of the t-structure defines an algebra embedding $H_\mathcal{A} \to Y_\infty$, as well as a triangular decomposition

$$Y_\infty \cong H_\mathcal{A} \otimes H_0 \otimes H_\mathcal{A}^{[1]},$$

where the subalgebra $H_0$ is a polynomial algebra $\mathbb{C}[h_1, h_2][\psi_{i,k}]_{i \in I, k \in \mathbb{Z}}$, with infinite variables labeled by $I \times \mathbb{Z}$.

For each $i \in I$, let $(S_i) \subseteq \mathcal{A}$ be the Serre subcategory of $\mathcal{A}$ generated by $S_i$. Then, the Cartan doubled Yangian $Y_\infty^{(S_i)}$, associated to $D^{Z_2}((S_i))$, is a subalgebra of $Y_\infty$. 

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Let $\delta$ be the imaginary root defined in the same way as in [31, Chapter 5]. That is, $\delta$ is a root, but $\delta$ is not in the Weyl group orbit of the simple roots. We expect that a quotient of $Y_{\infty}$ is isomorphic to $Y_{h_1, h_2, h_3}(\mathfrak{g}(1))$, which corresponds to the root $\delta$ of $Y_{\infty}$.

We are interested in representations of $Y_{\infty}$ on the cohomology of the moduli spaces of perverse coherent systems on $X$. In the case $X = X_{m,n}$, everything can be spelled out in the language of quivers with potential using [12]. Let $\mathcal{M}_{X}$ be the moduli space of stable perverse coherent systems. There is a potential function $\hat{W}$ defining the symmetric obstruction theory of $\mathcal{M}_{X}$. Let us consider the vector space $V := H^*_c(\mathcal{M}_{X}, \varphi_W C)^{\vee}$. Then we expect the following.

**Conjecture 8.2.1** The algebra $Y_{\infty}$ is isomorphic to $D(SH)$ constructed in § 2.3. There is an action of $Y_{\infty}$ on $V$ in agreement with the general philosophy of [35]. Moreover, this action factors through an action of the shifted affine Yangian, with the shift determined by $[x]$ and $\zeta$ as above.

### 8.3 A braid group action

Starting from the heart of the $t$-structure coming from a tilting bundle as above, one can construct the braid group action on $D^b \text{Coh}(X)$ induced by mutations with respect to subcategories $(S_i), i \in I$. We expect that on one hand this braid group action induces an action on $D^b \text{Coh}(X)$ and hence on the algebra $Y_{\infty}$. On the other hand it induces an action on $K_0(D^b \text{Coh}(X))$, and therefore an action on $X$. We expect that this action gives rise to an action of the Weyl group of the generalized root system described in § 8.1. The set of roots in $K_0(D^b \text{Coh}(X))$ determines a hyperplane arrangement in $X$. The braid group acts by reflections with respect to these hyperplanes.

In particular, for two adjacent chambers separated by a root hyperplane, the reflection with respect to the root hyperplane sends the heart of the $t$-structure associated to one chamber to that of the other one. These two chambers determine two Borel subalgebras of $Y_{\infty}$ which differ by the action of a Weyl group element.

As an illustration we explicitly calculate below the effect of the action of the affine Weyl group on Iwahori subgroups in the well-known case of $\hat{A}_2$. Then we discuss a similar picture in an example of a particular Calabi-Yau 3-fold in Remark [3.3.2].

**Example 8.3.1** Let $G := \text{SL}_2(C)$ and $T$ be its maximal torus of diagonal matrices. Denote by $B$ the Borel subgroup consisting of upper triangular matrices, and $B^-$ the one consisting of lower triangular matrices.

Let $G((t)) := \text{SL}_2((t))$, $U^-((t)) := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subset G((t))$ be the corresponding groups over the field of Laurent series $C((t))$. The Coxeter presentation of the affine Weyl group is $\hat{W} = \left\{ s_0, s_1 \mid s_0^2 = s_1^2 = 1 \right\}$. It is isomorphic to $Z_2 \ltimes Z$, where $Z_2$ is generated by $s_1$ and $Z$ is generated by $s_0s_1$. In particular, any element can be uniquely written as $w = (s_0s_1)^m$ or $w = s_1(s_0s_1)^m$ for some $m \in Z$. In $G((t))$, we have a subgroup, which is the Tits extension of $\hat{W}$, with generators $n_{s_0}, n_{s_1}$ defined as

$$n_{s_0} := \begin{bmatrix} 0 & -t^{-1} \\ t & 0 \end{bmatrix}, \quad n_{s_1} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$  

Thus, $(n_{s_0}n_{s_1})^m = \begin{bmatrix} t^{-m} & 0 \\ 0 & t^m \end{bmatrix}$

For any Weyl group element $w = (s_0s_1)^m$ (resp. $w = s_1(s_0s_1)^m$), we write the corresponding element in $G((t))$ as $w = (n_{s_0}n_{s_1})^m$ (resp. $w = n_{s_1}(n_{s_0}n_{s_1})^m$) slightly abusing the notation.
We have the following three subgroups of $G(t)$,

- the positive Iwahori $I^+ = \{ g \in G[[t]] \mid g(0) \in B \}$,
- the level 0 Iwahori $I^0 = T[[t]]U^-(t)$,
- the negative Iwahori $I^- = \{ g \in G[[t]] \mid g(\infty) \in B^- \}$.

Following [41], for $c \in C, k \in Z$, we define

$$X_{\alpha_1+k\delta}(c) = \begin{bmatrix} 1 & ct^k \\ 0 & 1 \end{bmatrix}, \quad X_{-\alpha_1+k\delta}(c) = \begin{bmatrix} 1 & 0 \\ ct^k & 1 \end{bmatrix}. $$

The following points are in $I^+$:

$$X_{-\alpha_1+\delta}(1) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}, \quad X_{\alpha_1}(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and they determine } I^+ \text{ in the following sense. We take } X_{-\alpha_1+\delta}(1) \text{ and } X_{\alpha_1}(1) \text{ to be the set of simple roots. Then, any positive root will be } X_{\alpha_1+k\delta}(c) \text{ if } \alpha = \alpha_1 \text{ and } k \in Z_{\geq 0} \text{ or if } \alpha = -\alpha_1 \text{ and } k \in Z_{\geq 0}. \text{ We have}

$$I^+ = \{ a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_i \in C[[t]], a_3 \in tC[[t]], det(a) = 1 \} = ( \prod_{k \geq 0, c} X_{-\alpha_1+k\delta}(c))^T([[t]]) ( \prod_{k \geq 0, c} X_{\alpha_1+k\delta}(c)).$$

Similarly we can determine $I^-$ and $I^0$ in terms of their simple roots. For any group element $g \in G((t))$, the conjugations $gI^+g^{-1}$, $gI^-g^{-1}$, $gI^0g^{-1}$ are also Iwahori subgroups. Then, we have the following 3 sets, each consisting a collection of Iwahori subgroups of $G((t))$.

- $G((t))/I^+$, the thin or positive level affine flag variety,
- $G((t))/I^0$, the semi-infinite or level 0 affine flag variety,
- $G((t))/I^-$, the thick or negative level affine flag variety.

With the structure of Weyl group above, we have the following orbit decompositions

$$G((t)) = \bigsqcup_{x \in \hat{W}} I^+xI^+, \quad G((t)) = \bigsqcup_{y \in \hat{W}} I^0yI^+, \quad G((t)) = \bigsqcup_{z \in \hat{W}} I^-zI^+, \quad \text{where in each decomposition the set of orbits are labelled by } \hat{W}. \text{ In particular, two Iwahori subgroups from two different components are not conjugate to each other. A direct computation determines the conjugations of } I^+ \text{ in terms of simple roots}

$$wX_{-\alpha_1+\delta}(1)w^{-1} = \begin{cases} X_{-\alpha_1+(2m+1)\delta}(1) & \text{if } w = (ns_0ns_1)^m; \\
X_{\alpha_1+(2m+1)\delta}(-1) & \text{if } w = n_{s_1}(ns_0ns_1)^m. \end{cases}$$

$$wX_{\alpha_1}(1)w^{-1} = \begin{cases} X_{\alpha_1-2m\delta}(1) & \text{if } w = (ns_0ns_1)^m; \\
X_{-\alpha_1-(2m)\delta}(-1) & \text{if } w = n_{s_1}(ns_0ns_1)^m. \end{cases}$$

Furthermore, $wX_{-\alpha_1+\delta}(1)w^{-1}$ and $wX_{\alpha_1}(1)w^{-1}$ determine another Iwahori subgroup $w(I^+)$ which is conjugate to $I^+$. A similar calculation can be done for $I^-$ and $I^0$. 

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Finally we make a remark about the comparison of this Weyl group action with the braid group action on the derived category. We follow the notations above.

**Remark 8.3.2** In the example $X = C \times T^*P^1$ the root system from §8.1 is that of $\widehat{\mathfrak{sl}_2}$.

We now describe a bijection between $I^+$-orbits on the thin affine flag variety $G((t))/I^+$ with the $t$-structures $\mathfrak{A}_m^+$ parametrized by the chambers on the PT side of the imaginary root hyperplane. This bijection is so that the simple roots of $A$ with the $t$-structures $\mathfrak{A}_m^+$ coincide with the simple objects in $\mathfrak{A}_m^+$. To describe this bijection, we use the description of the orbits from the decomposition $G((t)) = \bigsqcup_{x \in W} I^+ x I^+$ and the Tits extension of Weyl group elements as in Example 8.3.1. For simplicity, we can choose $m > 0$ when $w = (n_{s_0} n_{s_1})^m$, and $m < 0$, when $w = n_{s_1} (n_{s_0} n_{s_1})^m$. (The other choice corresponds to the flop of $X$.) Therefore, the above computation shows that the simple objects

$$\{ch[S_0] = [C] - m[p_0], \ ch[S_1] = -[C] + (m+1)[p_0] \}$$

in $\mathfrak{A}_m^+$ match up with the simple roots

$$\{X_{\alpha_1 - m\delta}(1), X_{-\alpha_1 + (m+1)\delta}(1) \}.$$

This bijection is equivariant with respect to the affine Weyl group actions.

Similarly, the $I^-$-orbits in $G((t))/I^-$ are in natural bijection with the $t$-structures $\mathfrak{A}_m^-$, so that the simple roots of $I \in G((t))/I^-$ coincide with the simple objects in $\mathfrak{A}_m^-$. The positive level Iwahori and the negative ones are not related by the action of the Weyl group. Instead they differ by a homological shift $[1]$, and hence are isomorphic as abstract algebras. The level zero Iwahori subgroup and its conjugations (as elements in $G$) do not occur here in the context of the COHA of perverse coherent sheaves.

In general, however, we do not expect two Iwahori subalgebras to be isomorphic in the case when they come from two $t$-structures that do not differ by the action of a braid group element. A similar idea of exploring COHA’s associated to two different $t$-structures can be found in §6.1 in an attempt to “categorify the wall-crossing”. We also mention the idea of the construction of the “derived COHA” which contains all “root COHA’s” corresponding to rays in $\mathbb{R}^2$ having the common vertex at the point $(0,0)$, which was proposed by Kontsevich and Soibelman in 2012 (unpublished).

A Residue pushforward formula in critical cohomology

A.1 Convolution operators on fixed points

Let $X$ be a complex algebraic $H$-variety, where $H$ is a complex affine algebraic group. We assume that $F := X^H$ consists of isolated fixed points $F = \{F_i\}$ and $|F| < \infty$. Let $i : F \hookrightarrow X$ be the inclusion. Denote by $\varphi_f$ the vanishing cycle complex associated to the function $f$ on $X$. Note that, by assumption, we have

$$H^*_{c,H}(F, \varphi_{i_0 f}C)^\vee = H^*_{c,H}(F, C)^\vee = \oplus_i H^*_{c,H}(F_i, C)^\vee$$

The classes $i_*(1_i)$ form a basis of $H^*_{c,H}(X, \varphi_f C)^\vee$. Let us consider the map on localizations $i_* : H^*_{c,H}(F, \varphi_{i_0 f}C)_{loc} \to H^*_{c,H}(X, \varphi_f C)_{loc}$ induced by

$$i_* : H^*_{c,H}(F, \varphi_{i_0 f}C)^\vee \to H^*_{c,H}(X, \varphi_f C)^\vee.$$
Let $N_F X$ be the normal bundle of $F$ inside $X$, and $e(N_F X)$ be its Euler characteristic. We also have
\[ i^* i_* \alpha = e(N_F X), \quad i_*^{-1} \beta = \frac{i^* \beta}{e(N_F X)} \]
for $\alpha \in H^*_{\text{c}, H}(F, \varphi_{i \circ f})^\vee$, and $\beta \in H^*_{\text{c}, H}(X, \varphi_f)^\vee$ (see, e.g., [9, Proposition 2.16(1)]).

Consider the correspondence

\[ \begin{array}{ccc}
F_W & \xrightarrow{q} & F_Y \\
\cap i_W & \xrightarrow{p} & \\
F_X & \xrightarrow{i_X \cap} & W \xrightarrow{q} \ X \\
\cup i_Y & \xrightarrow{p} & \cap i_Y \xrightarrow{p} \ Y \\
\end{array} \]

We have
\[ i_{X}^{-1} q_* p^* i_Y^*(1_{F_Y}) = q_{F_Y} i_{W_Y}^{-1} p^* i_Y^*(1_{F_Y}) \]
\[ = \sum_i \frac{i_{W_Y}^*}{e(N_{F_W Y})} p^* i_Y^*(1_{F_Y}) \]
\[ = \sum_i \frac{(p_F)^* i_Y^*}{e(N_{F_W Y})} i_Y^*(1_{F_Y}) \]
\[ = \sum_i \frac{(p_F)^* e(N_{F_Y} Y)}{e(N_{F_W Y})} \]

A.2 Jeffery-Kirwan residue formula in critical cohomology

Here we discuss the Jeffery-Kirwan residue localization formula in the setting of critical cohomology. We follow the proof of Guillemin and Kalkman [27], also taking into account the description of the vanishing cycle complex of Kapranov [32].

A.2.1 De Rham model of equivariant critical cohomology

First, we recall the de Rham model of critical cohomology following [32], taking into account the group action as in [21, § 2.10]. Let $Y$ be smooth complex algebraic variety endowed with an algebraic $C^*$-action as well as with a $C^*$-invariant regular function $f$. In order to use the $C^\infty$-de Rham complexes, we consider sheaves in the analytic topology. Recall that a fine sheaf is a sheaf with partition of unity, and that a fine sheaf is acyclic under direct image functor, and hence for any sheaf $\mathcal{F}$ the higher derived direct image can be calculated by applying the direct image functor to a resolution of $\mathcal{F}$ by fine sheaves. Recall that Poincaré Lemma implies that a resolution of the constant sheaf $C_Y$ by fine sheaves is given by the de Rham complex $(\Omega^*_{Y}, d)$. Here we use subscript notation for the sheaves, and $\Omega^*_{Y}$ for the space of global sections. The space of compactly supported sections is denoted by $\Omega^*_c(Y)$. Let $\varphi_f(C_Y)$ denote the sheaf of vanishing cycles. A complex representing $\varphi_f(C_Y)$ was given in [22]. Let $\overset{\cdot}{d} := d + df \wedge$ be the “twisted” differential on $\Omega_Y$. The complex $(\Omega_Y, \overset{\cdot}{d})$ is quasi-isomorphic to $\varphi_f(C_Y)$. Let $\pi_Y : Y \to \text{pt}$ be the structure map. Recall that $Y$ is smooth and hence the Verdier dualizing sheaf $D_Y$ is $C_Y$ homologically shifted to the dimension of
Y. The cohomology of Y valued in the vanishing cycle complex \( H^*_c(Y, \varphi_f C)^\vee := \pi_* \varphi_f D_Y \) is then calculated as the cohomology of the complex obtained by taking global sections of the de Rham complex representing \( \varphi_f C_Y [\dim Y] \).

Let \( \pi : X \to Y \) be a proper flat map of smooth complex algebraic varieties, and \( f \) a regular function on \( Y \). We have a map \( \pi_* : H^*_c(X, \varphi_f C)^\vee \to H^*_c(Y, \varphi_f C)^\vee \) induced by applying \( \pi_* \) to the map of complexes on \( Y \)

\[
\pi_*[\Omega_X, \tilde{d}] \to [\Omega_Y, \tilde{d}],
\]

where the individual map of sheaves \( \pi_* \Omega_X^* \to \Omega_Y^* \) is given by integration along fibers of \( \pi \).

The algebraic \( C^* \)-action induces an action of the maximal torus \( S^1 \subseteq C^* \) when \( Y \) is considered as a smooth manifold. In the calculations below we follow [24, § 2.10] taking into account the \( S^1 \)-action via the Cartan model. Note that the Borel construction used in [9] differs from the Cartan model by a completion. The cohomology \( H^*_c, S^1, (Y, \varphi_f C)^\vee \) can be calculated via the de Rham model \( \Omega^* S^1, (Y, \varphi_f) := \Omega^*(Y)^{S^1} \otimes C[x] \) with the differential \( \tilde{d} = d + i(v) \otimes x + \wedge df \). Here \( d \) is the de Rham differential on forms, \( v \) is the vector field induced by the \( S^1 \)-action, and \( d_{S^1} := d + i(v) \otimes x \) is the differential calculating the usual Cartan model of equivariant cohomology.

### A.2.2 An explicit formula of the Kirwan map

Let \( \overline{X} \) be a smooth complex algebraic variety with a \( C^* \)-action of dimension \( n \). In order to consider de Rham complex of the GIT quotient \( \overline{X} / \xi C^* \) for a character \( \xi \) of \( C^* \), we use Kirwan’s theorem to identify \( \overline{X} / \xi C^* \) with the Hamiltonian reduction. More precisely, the GIT quotient \( \overline{X} / \xi C^* \) is diffeomorphic to \( \mu^{-1}(\xi)/S^1 \) where \( \mu \) is the moment map of the compact Lie group \( S^1 \subseteq C^* \). Let \( X = \mu^{-1}(\xi, \infty) \), which is a smooth manifold with boundary and an \( S^1 \)-action that preserves the boundary.

Abusing the notation, for any smooth manifold \( N \) with boundary, endowed with an \( S^1 \)-action as well as with a smooth function \( f \) which preserves the boundary and \( f \), we denote the complex \( \Omega^*(N)^{S^1} \otimes C[x] \) with the differential \( \tilde{d} = d + i(v) \otimes x + \wedge df \) by \( \Omega^*_{S^1}(N, \varphi_f) \).

In the setup above, let \( X \) be a smooth manifold with boundary endowed with a smooth \( S^1 \)-action. Assume the \( S^1 \)-action on \( \partial X \) is locally free, so that \( \partial X / S^1 \cong M \) is again a smooth algebraic variety. Let \( f \) be an \( S^1 \)-invariant regular function on \( X \). Then, the restriction of \( f \) to \( \partial X \) is an \( S^1 \)-invariant smooth function, which is again obtained from pulling back a regular function on \( M \), which by an abuse of notation we still denote by \( f \).

We define the Kirwan map \( \kappa : H^*_c, S^1, (X, \varphi_f)^\vee \to H^*_c(M, \varphi_f)^\vee \) to be the following map at the level of de Rham complexes. We have the restriction of forms \( \iota^* : \Omega^*_c(X, \varphi_f) \to \Omega^*_c(\partial X, \varphi_f) \), as well as the pullback \( \pi^* : \Omega^*_c(\partial X, \varphi_f) \to \Omega^*_c(M, \varphi_f) \). Then, for any closed form \( \alpha \in \Omega^*(M, \varphi_f) \), \( \kappa(\alpha) \) is the class of any form \( \gamma \) with the property that \( \pi^*(\gamma) \) has the same class as \( \iota^* \alpha \). That is, \( \iota^* \alpha - \pi^*(\gamma) \) is a boundary cycle. The calculation of Guillemin and Kalkman [27] gives an explicit formula for \( \gamma \), which we recall here.

Let \( \theta \) be a \( S^1 \)-invariant one form such that \( \iota(v) \theta = 1 \), which is well-defined on the complement of \( X^{S^1} \). Consider the formal expression \( \nu_0 = \frac{\theta}{x + df + \theta \wedge df} \). From definition, we have \( \tilde{d}(\theta) = x + df + \theta \wedge df \), in particular \( \tilde{d}(x + df + \theta \wedge df) = 0 \). Therefore, \( \tilde{d}(\nu_0) = \frac{\tilde{d}(\theta)}{x + df + \theta \wedge df} = 1 \).

Let \( \nu = \Omega^*_c, (X, \varphi_f) \) such that \( \tilde{d}(\nu) = 0 \). Then, \( \tilde{d}(\alpha \wedge \nu) = \alpha \). We consider

\[
\nu_0 = \sum_{n \leq 0} \left( \tilde{d}^{n} \right)^{n} \text{as a Laurent power series in } x \text{ with coefficients in } \Omega(X)^{S^1}.
\]

Now let \( \{X_i\}_{i=1, \ldots, N} \) be the connected components of the fixed point set, and let \( U_i \) be pairwise disjoint tubular neighbourhoods of \( X_i \) in \( X \) such that \( U_i \cap \partial X = \emptyset \). Let \( i^*_k : X_k \to X \)
be the embedding. Assume \( \deg \alpha = \dim \partial X - 1 \). Then, by Stocks theorem, we have

\[
0 = \int_X \alpha = \int_X d\nu_0 \wedge \alpha = \sum_{k=1}^N \int_{\partial U_k} \frac{\theta_\alpha}{x + d\theta + \theta \wedge df} + \int_{\partial X} \frac{\theta_\alpha}{x + d\theta + \theta \wedge df}.
\]

We confine ourselves to the case when \( X^{S^1} = \bigcup X_k \) is a finite set. We now follow the calculation of \([28] \text{§ 10.8.6}\) to show that \( \int_{\partial U_k} \frac{\theta_\alpha}{x + d\theta + \theta \wedge df} \) converges to \( \frac{i_k^* \alpha}{e(\nu_k)} \) where \( \nu_k \) is the tangent space of \( X_k \) and \( e(\nu_k) \) is the equivariant Euler class, which can be calculated as the product of the weights with multiplicities on \( \nu_k \). Indeed, as in loc. cit., we have \( \alpha = f(x) + d_{S^1} \beta \) where \( f(x) \) is the restriction of \( \alpha \) at \( X_k \) and \( \beta \in \Omega_{S^1}(U_k) \), hence \( \nu_0 \wedge \alpha = f(x)\nu_0 + \beta + d_{S^1}(\nu_0 \wedge \beta) \). Integrating on \( \partial U_k \), the second term vanishes; the third term is of higher order in terms of the radius of \( U_k \); the first term becomes \( f(x) \int_{\partial U_k} \nu_0 \), where the calculation in local coordinate in loc. cit. shows that \( \int_{\partial U_k} \nu_0 = \frac{1}{x^{n+1}e(\nu_k)} \). To summarize, we obtain

\[
\int_{\partial X} \frac{\theta_\alpha}{x + d\theta + \theta \wedge df} = \sum_{k=1}^N \frac{i_k^* \alpha}{x^n e(\nu_k)} \quad (57)
\]

under the assumption that \( \deg \alpha = \dim \partial X - 1 \). By degree consideration \([69] \text{page 21}\), we also have \( \alpha \wedge \nu_0 = \nu + \beta x^{-1} \) with \( \beta = \text{Res}_{x=\infty}(\alpha \wedge \nu_0) \). In particular, \( \alpha = d(\nu) + i(\nu)\beta \) with \( i(\nu)\beta = \pi^* \gamma \) for some \( \gamma \in \Omega^*(M, \varphi_f) \). In other words, \( \kappa(\alpha) = \text{Res}_{x=\infty} \pi_* \frac{\theta_\alpha}{x + d\theta + \theta \wedge df} \).

A.2.3 Conclusion

Assume that \( X \) is endowed with an action of \( S \times S^1 \), where \( S \) is a compact Lie group which acts in a Hamiltonian way. Let \( \{E_m\} \) be a system of \( S \)-spaces the limit of which is the Borel construction of \( ES \). Following the consideration in \([69] \text{§ 5.2}\), for any \( \alpha \in H^*_c(S^1 \times S^1, M, \varphi) \) of a certain degree \( i \), we choose a representative in \( \Omega_{S^1}(E_m, \varphi) \) with \( i = \dim X_m - 1 \). In the argument above, replacing \( X \) by \( X_m := E_m \times S X \), taking residue from both sides of \((57)\) and taking into account \((58)\), we obtain the residue pushforward formula in \( S \)-equivariant cohomology

\[
\int_M \kappa(\alpha) = \sum_{k=1}^N \text{Res}_{x=\infty} \frac{i_k^* \alpha}{e(\nu_k)} \quad (59)
\]

in \( H^*_c(M, \varphi) \to H^*_c(M, \varphi) \).

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