Some New Developments of
Realization of Surfaces into $R^3$

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Abstract

This paper intends to give a brief survey of the developments on realization of surfaces into $R^3$ in the last decade. As far as the local isometric embedding is concerned, some results related to the Schlaffli-Yau conjecture are reviewed. As for the realization of surfaces in the large, some developments on Weyl problem for positive curvature and an existence result for realization of complete negatively curved surfaces into $R^3$, closely related to Hilbert-Efimov theorem, are mentioned. Besides, a few results for two kind of boundary value problems for realization of positive disks into $R^3$ are introduced.

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Given a smooth n-dimensional Riemannian manifold $(M^n,g)$, can we find a map

$$\phi : M^n \rightarrow R^p$$

such that $\phi^* h = g$

where $h$ is the standard metric in $R^p$? This is a long standing problem in Differential Geometry. The map $\phi$ is called isometric embedding or isometric immersion if $\phi$ is embedding or immersion. There are several very nice surveys. For example, for known results before 1970, particularly obtained by Russian mathematicians, see [GR], [G] and for results of $n = 2$, see [Y3] [Y4]. Whereas, what development has been made during the passed decades? In higher dimensional cases, the most important one is to improve the Nash’s theorem so that $(M^n,g)$ has isometric embedding in a Euclidean space of much lower dimension than that given by Nash and meanwhile, to use the contraction mapping principle in place of the theorem of the complicated hard implicit functions, see [H] or [GUN]. In contrast to higher dimensional cases it seems that problems of two dimensional Riemannian manifolds embedded into $R^3$ has attracted much more attention of mathematicians,

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particularly in the last decade. The present paper is devoted to a survey on the developments of isometric embedding (or immersion) of surfaces into $\mathbb{R}^3$ in the last decade. Of course the results mentioned in this survey are by no means exhaustive and depend a lot on the author’s taste.

Smoothly and isometrically immersing a surface $(M, g)$ into $\mathbb{R}^3$ is equivalent to finding three smooth ($C^s, s \geq 1$) functions $x^\alpha : M \rightarrow \mathbb{R}^3, \alpha = 1, 2, 3$ such that

$$g = dx_1^2 + dx_2^2 + dx_3^2.$$  \hfill (1)

Although as far as the formulation of (1) be concerned, $C^1$ regularity is enough and [K] gives a very nice result in this category. In order to see the role of the curvature of surfaces in the problem considered here we prefer to assume $s \geq 2$ throughout the present paper. In the sequel, sometimes we use $x, y, z$ to denote $x_1, x_2, x_3$. In a local coordinates near a point $p \in M$, the metric $g$ is of the form $g = g_{ij} du^i du^j$. Then (1) can be written as follows

$$\frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\alpha}{\partial u^j} = g_{ij}, i, j = 1, 2,$$ \hfill (2)

(2) is a system composed of three differential equations of first order and hence, this is a determine system. We say that $\vec{r} = (x_1, x_2, x_3)$ is a local smooth ($C^s$) isometric embedding in $\mathbb{R}^3$ of the given surface if $\vec{r} = (x_1, x_2, x_3)$ is a smooth ($C^s$) solution to (2) in a neighbourhood of the point $p$ and that $\vec{r}$ is a global smooth ($C^s$) isometric embedding (immersion) in $\mathbb{R}^3$ if $\vec{r} = (x_1, x_2, x_3)$ is a smooth ($C^s$) solution to (1) on $M$ and, meanwhile, is an embedding (immersion) into $\mathbb{R}^3$. This survey consists of three parts. The first and the second part include some recent developments in local isometric embedding and global isometric embedding respectively. The third part contains some developments on boundary value problems for realization of positive disks into $\mathbb{R}^3$.

**Local isometric embedding.** With the aid of the Cauchy-Kowalevsky theorem Cartan and Janet proved that any n-dimensional analytic metric always admits a local analytic isometric embedding in $\mathbb{R}^{s_n}$ with $s_n = n(n + 1)/2$. In the smooth category Gromov in [GR] proved that any n-dimensional $C^\infty$ metric always admits a local smooth isometric embedding in $\mathbb{R}^{s_n+n}$. As $n = 2$, $s_n = 5$ and from (2) the present result looks far away from the optimal in the smooth category. On the other hand, [P] proves that any smooth surface always has a local smooth isometric embedding in $\mathbb{R}^4$. In [Y2, No.22] and also in [Y1, No. 54] Yau posed to prove that any smooth surface always has a local smooth isometric embedding in $\mathbb{R}^3$. In this direction, it is Lin who first made important breakthrough and his results in [LC1] and [LC2] state

**Theorem 1.** (C. S. Lin) (1) Any $C^s, s > 10$ nonnegatively curved metric always admits a local $C^{s-6}$ isometric embedding in $\mathbb{R}^3$.

(2) If $g$ is a $C^s, s > 6$ metric and if its curvature $K$ satisfies $K(p) = 0$ and $dK(p) \neq 0$
then it admits a $C^{s-3}$ isometric embedding in $R^3$ near $p$.

By means of Lin’s technique the problem for local isometric embedding related to nonpositive curvature metric are also solvable in the following cases.

(3) In [IW], $K$ nonpositive in a neighbourhood of a point $p$ and $d^2 K(p) \neq 0$, $K_{ij} - 3$ isometric embedding in $R^3$ near $p$.

By means of Lin’s technique the problem for local isometric embedding related to nonpositive curvature metric are also solvable in the following case s.

(3) In [IW], $K$ nonpositive in a neighbourhood of a point $p$ and $d^2 K(p) \neq 0$, $dh(p) \neq 0$ and $q$ is an integer.

In what follows let us simply explain what the technique ones use while attacking the problem of local isometric embedding. Suppose that $\vec{r} = (x,y,z)$ is a smooth solution to (2) in a neighbourhood of the point in question previously. By the Gauss equations we have, in a local coordinate system,

$$\vec{r}_{ij} = \Gamma^k_{ij} \vec{r}_k + \Omega_{ij} \vec{n}$$

where subscripts $ij$ and $\nabla_{ij}$ denote Euclidean and covariant derivatives respectively and $\Omega_{ij}$ the coefficients of the second fundamental form, $\Gamma^k_{ij}$ the Christoffel symbols with respect to the metric and $\vec{n}$ the unit normal to $\vec{r}$. For each unit constant vector, for instance, the unit vector $\vec{k}$ of the z axis, taking the scale product of $\vec{k}$ and (3) and using the Gauss equations one can get

$$\det(\nabla_{ij} z) = K \det(g_{ij})(\vec{n},\vec{k})^2.$$  \hspace{1cm} (4)

Notice that

$$\det(\vec{n},\vec{k})^2 = 1 - \left(\frac{(\vec{r}_1 \times \vec{r}_2) \times \vec{k}}{|\vec{r}_1 \times \vec{r}_2|}\right)^2$$

$$= 1 - g^{ij} z_i z_j = 1 - |\nabla z|^2.$$  \hspace{1cm} (5)

Inserting the last expression into (4) we deduce the Darboux equation

$$F(z) = \det(\nabla_{ij} z) - K \det(g_{ij})(1 - |\nabla z|^2) = 0.$$  \hspace{1cm} (5)

Obviously each component of $\vec{r}$ does satisfy the Darboux equation. Conversely, for each smooth solution $z$ to (5) satisfying $|\nabla z| < 1$, $\tilde{g} = g - dz^2$ is a smooth flat metric. Therefore in simply connected domain $\Omega$ we can always find a smooth mapping $(x,y): \Omega \rightarrow R^2$ such that $dx^2 + dy^2 = g - dz^2$. So the realization of a given metric into $R^3$ is equivalent to finding a smooth (or $C^s$) solution to the Darboux equation with a subsidiary condition $|\nabla z| < 1$. If $K$ is positive or negative at the point considered, then (5) is elliptic or hyperbolic Monge-Ampere equation and the local solvability is well known for both of them. But if $K$ vanishes at this point, the situation is very complicated and so far there has been no standard way to deal with such kind of Monge-Ampere equation. Indeed, its linearized operator

$$L_z \xi = \lim_{t \rightarrow 0} \frac{F(z + t\xi) - F(z)}{t} = F^{ij} \nabla_{ij} \xi + 2K(\nabla z, \nabla \xi)$$ \hspace{1cm} (6)
where $F^{ij} = \partial \det(\nabla_{ik} z) / \partial \nabla_{ij} z$. It is easy to see that the type of this linear differential operator completely depends on $K$. When $K$ vanishes at the point considered, (6) may be degenerate elliptic, hyperbolic or mixed type and its local solvability is not clear. Using a regularized operator instead of (6) and the Nash-Moser procedure, Lin succeeded in proving Theorem 1. But it is still not clear whether there is obstruction for the local isometric embedding in smooth category even for the nonpositive curvature metric. Some results [EG] on linear degenerate hyperbolic operators of second order which have no local solvability should be noticed. Anyway, to author’s knowledge the problem for local isometric embedding of surfaces into $R^3$ is still open!!

Global isometric embedding. The first result on global isometric embedding of complete surfaces in $R^3$ is due to Weyl and Lewy for analytic metric and to Nirenberg and Pogorelov for smooth metric.

**Theorem.** (Weyl-Lewy Nirenberg-Pogorelov) Any analytic (smooth ) positive curvature metric defined on $S^2$ always admits an analytic (a smooth ) isometric embedding in $R^3$.

For noncompact case, for example, a complete smooth positive curvature metric defined on $R^2$, this problem was solved by two Russian mathematicians. Olovjanisnikov first found the weak isometric embedding based on the Aleksandrov’s theory on convex surfaces and Pogorelov proved the weak solution smooth if the metric smooth.

**Theorem.** (Olovjanisnikov-Pogorelov) Any smooth complete positive curvature metric defined on $R^2$ admits a smooth isometric embedding in $R^3$.

The next natural development is to consider the realization in $R^3$ of nonnegatively curved surfaces. Recently [GL] and [HZ] independently obtained the following result.

**Theorem 2.** (Guan-Li, Hong-Zuily) Any $C^4$ nonnegative curvature metric defined on $S^2$ always admits a $C^{1,1}$ isometric embedding in $R^3$.

Olovjanisnikov-Pogorelov’s result on complete positively curved plane is also extended to the nonnegatively curved case, see [HO3].

**Theorem 3.** Any complete $C^4$ nonnegative curvature metric defined on $R^2$ always admits a $C^{1,1}$ isometric embedding in $R^3$. Moreover it is smooth where the metric is smooth and the curvature positive.

In this direction a special case is also obtained in [AM].

As far as the regularity of isometric embedding be concerned, ones are interested in the following question. Can we improve the regularity of the isometric embedding obtained in Theorem 2 and Theorem 3 if the metric is smooth? It is very interesting that [IA] gives a $C^{2,1}$ convex surface which is not $C^3$ continuous but realizes analytic metric on $S^2$ with positive curvature except one point. On
the other hand, Pogorelov gave a $C^{2,1}$ geodesic disk with nonnegative curvature not even admitting a $C^2$ local isometric embedding in $\mathbb{R}^3$ at the center of this disk. Therefore a natural open question is: Does there exist a $C^{2,\alpha}$ ($0 < \alpha < 1$) isometric embedding in $\mathbb{R}^3$ for any sufficiently smooth (even analytic) nonnegatively curved sphere or plane?

The Hilbert theorem is one of the most important theorems in 3-Euclidean space. This theorem as well as Efimov's generalization in [EF1] provide a negative answer for the problem of realization of complete negatively curved surfaces into $\mathbb{R}^3$.

**Theorem.** (Hilbert-Efimov) Any complete surface with negative constant curvature (with curvature bounded above by a negative constant) has no $C^2$ isometric immersion in $\mathbb{R}^3$.

Another result [EF2] also due to Efimov should be mentioned.

**Theorem.** (Efimov) Let $M$ be a smooth complete negatively curved surface with curvature $K$ subject to

$$\sup_M |K|, \sup_M \text{grad} \left( \frac{1}{\sqrt{|K|}} \right) \leq C$$

for some constant $C$. Then $M$ has no $C^2$ isometric immersion in $\mathbb{R}^3$.

Evidently, Efimov's second result yields a necessary condition for a complete negatively curved surface to embed isometrically in $\mathbb{R}^3$,

$$\sup_M |\nabla \frac{1}{\sqrt{|K|}}| = \infty \text{ if } \sup_M |K| < \infty.$$  \hfill (8)

Yau posed the following question [Y1, No.57]. Find a nontrivial sufficient condition for a complete negatively curved surface to embed isometrically in $\mathbb{R}^3$.

He also pointed out that such a nontrivial condition might be the rate of decay of the curvature at infinity. Recently some development in this direction has been made in [HO2]. Let $M$ be a simply connected noncompact complete surface of negative curvature $K$. By the Hadamard theorem $\exp_p(M) \to M$ is a global diffeomorphism for each point $p \in M$ which induces a global geodesic polar coordinates $(\rho, \theta)$ on $M$ centered at $p$.

**Theorem 4.** Suppose that

(a) for some $\delta > 0$, $\rho^{2+\delta}|K|$ is decreasing in $\rho$ outside a compact set and that

(b) $\partial_\rho \ln |K|$, $i = 1, 2$ and $\rho \partial_\rho \partial_\theta \ln |K|$ bounded on $M$.

Then $M$ admits a smooth isometric immersion in $\mathbb{R}^3$.

**Remark 1.** If $M \in C^{s,1}(s \geq 4)$ and other assumptions in Theorem 4 are fulfilled, then it admits a $C^{s-1,1}$ isometric immersion in $\mathbb{R}^3$. 
Remark 2. The assumption (a) implies the rate of decay of the curvature at the infinity
\[ |K| \leq \frac{A}{\rho^{2+\delta}}, \] for a positive constant $A$. \hfill (9)

Such a condition on the decay of the curvature at the infinity is nearly sharp for the existence since if $\delta = 0$, there might be no existence. Consider a radius symmetric surface $(R^2, g)$ with the Gaussian curvature
\[ K = -\frac{A}{1 + \rho^2} \] for some positive constant $A$. \hfill (10)

where $\rho$ is the distance function from some point. Evidently
\[ \sup_{\mathbb{R}^2} \frac{1}{\sqrt{|K|}} = \frac{1}{\sqrt{A}} < \infty. \]

Therefore Efimov’s second theorem tells us that such a complete negatively curved surface $(R^2, g)$ has no any $C^2$ isometric immersion in $\mathbb{R}^3$ for arbitrary positive constant $A$. The arguments of [EF1] and [EF2] are very genuine but ones expect a more analysis proof for Efimov’s results in [EF1]. Anyway, a result in [EF3] also by Efimov which is much easier understood, shows that the negatively curved surface mentioned above has no $C^2$ isometric immersion if $A > 3$ in (10).

**Boundary value problems for isometric embedding in $\mathbb{R}^3$.** Recently, the study on boundary value problems for isometric embedding of surface in $\mathbb{R}^3$ has attracted much attention of mathematicians. Various formulations of such questions can be found in [Y4]. To author’s knowledge this field has not been extensively studied. Let $g$ be a smooth positive curvature metric defined on the closed unit disk $\bar{D}$. Throughout the present paper we always call such surfaces $(\bar{D}, g)$ positive disk. According to the classification in [Y4] there are two kinds of boundary value problems: one is Dirichlet problem and another is Neumann problem. Let us first consider the Dirichlet problem.

Given a smooth positive disk $(\bar{D}, g)$ and a complete smooth surface $\Sigma \subset \mathbb{R}^3$, can we find an isometric embedding $\vec{r}$ of $(\bar{D}, g)$ in $\mathbb{R}^3$ such that $\vec{r}(\partial D) \subset \Sigma$?

This problem is also raised in [PO1]. As a first step, one can consider a simple case. Assume that $\Sigma$ is a plane. Give a complete description of all isometric embedding of $(\bar{D}, g)$ satisfying $\vec{r}(\partial D) \subset \Sigma$.

Assume that $\Sigma : \{z = 0\}$. Then we are faced with the following boundary value problem.

**D:** To find an isometric embedding $\vec{r} = (x, y, z)$ of the given positive disk $(\bar{D}, g)$ such that $z(\partial D) = 0$.

It is easy to see that there is some obstruction for the existence of solutions to the above boundary value problem. Suppose that $\vec{r} = (x, y, z) \in C^2(\bar{D})$ is a solution to
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the above boundary value problem. Obviously the intersection $\vec{r}(\partial D)$ of $\vec{r}$ and the plane $\{z = 0\}$ is a $C^2$ planar convex curve. Denoting its curvature by $\hat{k}$ we have

$$\hat{k}^2 = k_g^2 + k_n^2$$  \hspace{1cm} (11)

where $k_g$ and $k_n$ are respectively the geodesic curvature and normal curvature of $\vec{r}(\partial D)$. $k_n$ is positive everywhere since $(\bar{D}, g)$ has positive curvature. Notice that the total curvature of $\vec{r}(\partial D)$, as a planar curve, equals $2\pi$. Hence

$$\int_{\vec{r}(\partial D)} |k_g|ds < 2\pi.$$  \hspace{1cm} (12)

This is a necessary condition in order that the above boundary value problem $D$ can be solvable. Indeed, this necessary condition is not sufficient for the solvability. In [HO4] there is a smooth positive disk satisfying (12) but not admitting any $C^2$ solution to the problem $D$. Furthermore this counter example also shows that too many changes of the sign of the geodesic curvature of the boundary will make the problem $D$ unsolvable. So we distinguish two cases

Case a: $k_g > 0$ on $\partial D$ and Case b: $k_g < 0$ on $\partial D$.  \hspace{1cm} (13)

It should be pointed out that Pogorelov gave the first solution to the problem $D$ in [PO1] which states that

**Theorem.** (Pogorelov) Let $(\bar{D}, g)$ be a smooth positive disk. Then the boundary value problem admits a solution $\vec{r} \in C^\infty(D) \cap C^{0,1}(\bar{D})$ provided that the geodesic curvature of $\partial D$ with respect to the metric $g$ is nonnegative.

Pogorelov only obtained a local smooth solution, namely, a solution smooth inside. One wonder that under what conditions the problem $D$ always admits a global smooth solution, namely, a solution smooth up to the boundary. Such global smooth solutions are obtained by [DE] for Case a if there exists a global $C^2$ subsolution $\psi$ for the Darboux equation (5) in the unit disk $D$, vanishing on $\partial D$ and with $|\nabla \psi|$ strictly less than 1 on $\bar{D}$. Recently, [HO5] removes this technique requirement.

**Theorem 5.** For Case a the problem $D$ always admits a unique solution in $C^\infty(\bar{D})$ if any one of the following assumptions is satisfied (1) $K > 0$ on $\bar{D}$, (2) $K > 0$ in $D$ and $K = 0 \neq |dK|$ on $\partial D$.

Obviously, the necessary condition (12) is always satisfied for Case a. As for Case b it looks rather complicated since there are some smooth convex surfaces in $R^3$ which are of no infinitesimal rigidity. The presence of such convex surfaces makes us fail to prove the existence of the problem $D$ of Case b by means of the standard method of continuity. Thus the solvability of the problem $D$ for Case b is still open!
Let us consider a spherical crown, in the spherical coordinates,

$$\Sigma_{\theta_*} = \{(\sin \theta \cos \phi, \sin \theta \sin \phi, -\cos \theta) | 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \theta_*\}$$

and $\theta = 0$ stands for the South pole. $\Sigma_{\theta_*}$ is the isometric embedding of the metric $g = d\theta^2 + \sin^2 \theta d\phi^2, 0 \leq \theta \leq \theta_*$. If $\theta_* > \frac{\pi}{2}$, then $\Sigma_{\theta_*}$ contains the below hemisphere and the geodesic curvature of its boundary is negative. We have in [HO4] Theorem 6. There is a countable set $\Lambda = \{\theta_1, \theta_2, ..., \theta_n, ...\} \subset (\pi/2, \pi)$ with a limit point $\pi/2$ such that $\Sigma_{\theta_*}$ is not infinitesimally rigid if $\theta_* \in \Lambda$.

In what follows we proceed to discuss the Neumann problem for realization of surfaces into $R^3$. The formulation is as follows. Give a smooth positive disk $(\bar{D}, g)$ and a positive function $h \in C^\infty(\partial D)$, The Neumann problem (later, called the problem $N$) is as follows.

$$N: \text{Find a surface } \vec{r}: \bar{D} \mapsto R^3 \text{ such that } d\vec{r}^2 = g$$

with the prescribed mean curvature $h$ on $\vec{r}(\partial D)$. (14)

Let us first introduce an invariant related to umbilical points of surfaces in $R^3$. Suppose that the given metric is of the form

$$g = E dx^2 + 2F dx dy + G dy^2, (x, y) \in \bar{D}. \quad (15)$$

Let $\vec{r}$ be a smooth isometric immersion of $(\bar{D}, g)$ with the second fundamental form

$$II = L dx^2 + 2M dx dy + N dy^2 \text{ for } (x, y) \in \bar{D}. \quad (16)$$

Definition. If $\vec{r}$ is of no umbilical points on $\partial D$, with

$$\sigma = (EM - FL) + \sqrt{-1}(GL - EN)$$

the winding number of $\sigma$ on $\partial D$ is called the index of the umbilical points of the surface $\vec{r}$ and denoted by $\text{Index}(\vec{r})$.

Obviously, this definition makes sense since $p$ is an umbilical point if and only if $\sigma(p) = 0$. Moreover, the definition of the index of the umbilical points is coordinate-free and hence, an invariant of describing umbilical points of surfaces. Such invariance comes from that of a differential form. Indeed, assume that in some orthonormal frame, the induced metric and the second fundamental form of a given surface $\vec{r}$ in $R^3$ are of the form

$$g = \omega_1^2 + \omega_2^2 \text{ and } II = h_{ij} \omega_i \omega_j$$

respectively. As is well known,

$$[(h_{11} - h_{22}) + 2ih_{12}] (\omega_1^2 + \omega_2^2)$$
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is an invariant differential form. So the index of umbilical points is nothing else but

$$\text{Index}(\vec{r}) = \text{Index} \{2h_{12} + i(h_{11} - h_{22})\}$$

if no umbilical point on $\partial D$ occurs.

The boundary value problem for realization of positive disks into $R^3$ seems to have some obstruction. Indeed, even if no imposing any restriction on the boundary the problem of realization of positive disks into $R^3$ is not always solvable. For details, refer to Gromov’s counter example [GR] which contains an analytic positive disk not admitting any $C^2$ isometric immersion in $R^3$. Therefore for the Neumann boundary value problem the following hypothesis is natural. Assume that

$$\text{(D},g) \text{ admits a } C^2 \text{ isometric immersion } \vec{r}_0 \text{ in } R^3.$$

We have in [HO6]

**Theorem 7.** If $(\bar{D},g)$ is a smooth positive disk satisfying (17) then for any nonnegative integer $n$ and arbitrary $(n + 1)$ distinct points $p_0 \in \partial D, p_1, ..., p_n \in D$, the problem $N$ admits two and only two solutions $\vec{r}$ in $C^\infty(\bar{D},R^3)$ with prescribed mean curvature $h$ on $\partial D$ and moreover,

one principal direction at $p_0$ is tangent to $\partial D$,

$$\text{Index}(\vec{r}) = n \text{ and } H(p_k) = H_0(p_k), k = 1, ..., n$$

where $H$ and $H_0$ are respectively the mean curvature of $\vec{r}$ and $\vec{r}_0$ provided that

$$\frac{h}{\sqrt{K}} - 1 > 4 \max_{\partial D} \left[ \frac{H_0}{\sqrt{K}} - 1 \right] \text{ on } \partial D.$$  \hspace{1cm} (18)

It is worth pointing out two extreme cases. The first one involves the existence. Suppose that the given positive disk $(\bar{D},g)$ is of positive constant curvature. Then it is easy to see that this positive disk admits a priori smooth isometric embedding $\vec{r}_0$ in $R^3$ which is a simply connected region of the sphere. Under the present circumstance $\vec{r}_0$ is totally umbilical and hence, the right hand side of (18) vanishes. Therefore if $(\bar{D},g)$ is of constant curvature and $\sqrt{K} < h \in C^\infty(\partial D)$, then the problem $N$ is always solvable for each nonnegative integer $n$ and arbitrary $(n + 1)$ distinct points $p_0 \in \partial D, p_1, ..., p_n \in D$.

The second extreme case involves the nonexistence. If the given positive disk is radius symmetric, i.e., $g = dr^2 + G^2(r)d\theta^2 \ 0 \leq r \leq 1$ where $G \in C^\infty([0,1])$ and $G(0) = 0, G'(0) = 1, G > 0$ as $r > 0$. Then if $G_r > -1, (\bar{D},g)$ has such a priori smooth isometric embedding in $R^3$,

$$\vec{r}_0 : x = G(r) \cos \theta, y = G(r) \sin \theta, z = - \int^1_r \sqrt{1 - G^2}\,dr.$$  \hspace{1cm} (19)

With its mean curvature $H_0 = H_0(r)$, if $H_0(1) > \sqrt{K(1)}$, then [HO6] proves that for arbitrary $h \in C^\infty(\partial D)$ satisfying $\sqrt{K(1)} \leq h < H_0(1)$ the problem $N$ has no any $C^2$ solution. Of course, if $h > 4H_0(1) - 3\sqrt{K(1)}$, by Theorem 7 the problem $N$ always admits two and only two smooth solutions for any nonnegative integer $n$ and arbitrary $n + 1$ points $p_0 \in \partial D, p_1, ..., p_j \in D$. 
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