A NUNKE TYPE CLASSIFICATION IN THE LOCALLY COMPACT SETTING

SAMUEL M. CORSON AND OLGA VARGHESE

Abstract. In this short note we prove that a group $G$ is lcH-slender- that is, every abstract group homomorphism from a locally compact Hausdorff topological group to $G$ has an open kernel- if and only if $G$ is torsion-free and does not include $\mathbb{Q}$ or the $p$-adic integers $\mathbb{Z}_p$ for any prime $p$. This mirrors a classical characterization given by Nunke for slender abelian groups.

1. Introduction

R. Nunke produced in 1961 a remarkable theorem which comprehensively characterizes a special class of abelian groups via their subgroups [11]. An abelian group $A$ is slender if for every abstract group homomorphism $\phi$ whose domain is the countably infinite product $\prod_{\omega} \mathbb{Z}$ and whose codomain is $A$ there exists an $m \in \omega$ such that $\phi = \phi \circ p_m$, where $p_m : \prod_{\omega} \mathbb{Z} \to \prod_{k=0}^m \mathbb{Z} \times (0)_{k=m+1}$ is the obvious retraction map [7]. Thus $A$ is slender provided each homomorphism $\phi : \prod_{\omega} \mathbb{Z} \to A$ depends on only finitely many coordinates, or equivalently, provided any such $\phi$ has open kernel (endowing $\prod_{\omega} \mathbb{Z}$ with the Tychonov topology where each coordinate is discrete). Nunke’s theorem is that an abelian group $A$ is slender if and only if $A$ is torsion-free and does not include $\prod_{\omega} \mathbb{Z}$ or $\mathbb{Q}$ or a $p$-adic integer group $\mathbb{Z}_p$ for any prime $p$.

Slenderness can be seen as an automatic continuity condition: endowing an abelian slender group $A$ a discrete topology one sees that any homomorphism from $\prod_{\omega} \mathbb{Z}$ to $A$ is continuous. By analogy, one defines a (not necessarily abelian) group $G$ to be locally compact Hausdorff slender (abbrev. lcH-slender) provided every abstract group homomorphism from a locally compact Hausdorff topological group to $G$ has open kernel [2]. An early result of Dudley shows that, for example, free (abelian) groups are lcH-slender [5]. More recent results extend the known lcH-slender groups to include each group whose abelian subgroups are free [9].

We deduce the following complete classification of lcH-slender groups via their subgroups:

Theorem 1. A group $G$ is lcH-slender if and only if $G$ is torsion-free and does not include $\mathbb{Q}$ or any $p$-adic integer group $\mathbb{Z}_p$ as a subgroup.

This classification was recently shown in the case where $G$ is abelian [3]. Theorem 1 provides the following further examples of lcH-slender groups (using [4, Theorem I.4.1 (vii)] and [10, Theorem 1], respectively):

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Corollary 2. Torsion-free CAT(0) groups and torsion-free one-relator groups are lcH-slender.

An analogous notion of slenderness is defined by replacing the locally compact Hausdorff groups with the completely metrizable groups: $G$ is completely metrizable slender (abbrev. cm-slender) if every homomorphism from a completely metrizable topological group to $G$ has open kernel. Similarly $G$ is $n$-slender if every abstract group homomorphism from the fundamental group of a Peano continuum to $G$ has an open kernel. As the proscribed subgroups in Theorem 1 are neither cm-slender nor $n$-slender (see [3, Theorem C] and [6, Theorem 3.3]) we immediately obtain:

Corollary 3. If a group $G$ is either $n$-slender or cm-slender then $G$ is lcH-slender.

2. Proof of Theorem 1

Towards the proof of Theorem 1 we recall some notions from abelian group theory (see [7]). We use convenient alternative characterizations which suit our purposes, rather than the historical definitions. Let $A$ be an abelian group. We say $A$ is algebraically compact if $A$ is an abstract direct summand of a Hausdorff compact abelian group. We say $A$ is cotorsion if it is a homomorphic image of an algebraically compact group. Also, $A$ is cotorsion-free if the only cotorsion subgroup of $A$ is the trivial one. Importantly $A$ is cotorsion-free if and only if $A$ is torsion-free and contains no copy of $\mathbb{Q}$ or the $p$-adic integers $\mathbb{Z}_p$ for any prime $p$ [7, Theorem 13.3.8].

Lemma 4. Suppose $\phi : K \to G$ is an abstract group homomorphism with $K$ a compact topological group and $G$ torsion-free and not including $\mathbb{Q}$ or any $\mathbb{Z}_p$ as a subgroup. Then $\phi$ is trivial.

Proof. Let $k \in K$ be given. It is easy to verify that the closure $\overline{\langle k \rangle} \leq K$ of the cyclic subgroup generated by $k$ is compact abelian. Then $\phi(\overline{\langle k \rangle})$ is abelian and cotorsion, on the one hand, but cotorsion-free on the other hand by our assumptions on $G$. Therefore $\phi(\overline{\langle k \rangle})$ is trivial and the lemma is proved. \qed

Next we state some classical results regarding locally compact groups.

Proposition 5. Let $L$ be a locally compact Hausdorff group.

1. (Iwasawa’s structure Theorem, [8, Theorem 13]) If $L$ is connected then we can write $L = H_0\cdots H_jK$ where each $H_i$ is a subgroup of $L$ isomorphic to $\mathbb{R}$ and $K$ is a compact subgroup of $L$.

2. (van Dantzig’s Theorem, [1, III §4, No. 6]) If $L$ is locally compact Hausdorff and totally disconnected then $L$ has a compact open subgroup.

Proof of Theorem 1. If $G$ is lcH-slender it is certainly necessary that $G$ is torsion-free and not include $\mathbb{Q}$ or any $\mathbb{Z}_p$. If, for example, $G$ were to include torsion then $G$ would contain a subgroup of prime order $p$ and one can construct a discontinuous homomorphism $\phi : \prod_p \mathbb{Z}/p\mathbb{Z} \to G$ using a vector space argument. Similarly if $G$ includes $\mathbb{Q}$ as a subgroup one produces a discontinuous homomorphism $\phi : \mathbb{R} \to G$ by selecting a Hamel basis for $\mathbb{R}$. Were $G$ to include $\mathbb{Z}_p$ as a subgroup then the inclusion map $\mathbb{Z}_p \to G$ witnesses that $G$ is not lcH-slender.

For the other implication we suppose that $G$ is torsion-free and does not include $\mathbb{Q}$ or any $\mathbb{Z}_p$ and let $\phi : L \to G$ be an abstract group homomorphism with $L$ a locally compact Hausdorff group. Let $L^o$ denote the connected component of the identity.
element. Since $L^o$ is closed, it is itself locally compact Hausdorff, and as $L^o$ is also connected we know by Iwasawa’s structure Theorem that $L^o = H_0\cdots H_j K$ with each $H_i$ a subgroup isomorphic to $\mathbb{R}$ and $K$ a compact subgroup. We know that $\phi \upharpoonright K$ is trivial by Lemma 4 and since $G$ is torsion-free and includes no subgroup isomorphic to $\mathbb{Q}$ we know that $\phi \upharpoonright H_i$ is trivial for each $i$. Thus $\phi(L^o)$ is trivial.

Now the homomorphism $\phi : L \to G$ passes to a homomorphism $\overline{\phi} : L/L^o \to G$. By van Dantzig’s Theorem we have a compact open subgroup $K' \leq L/L^o$. Again by Lemma 4 we know that $\phi \upharpoonright K'$ is trivial, and thus $\pi^{-1}(K') \leq \ker(\phi)$ witnesses that $\ker(\phi)$ is open, where $\pi : L \to L/L^o$ is the continuous projection. \hfill \Box

**Remark 6.** A nice subgroup characterization for n-slender and cm-slender groups is still apparently beyond reach. Thus far a classification exists for such groups only in the abelian case [3].

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Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, 28049 Madrid, Spain.

*E-mail address: sammyc973@gmail.com*

Department of Mathematics, Münster University, Einsteinstraße 62, 48149, Münster, Germany

*E-mail address: olga.varghese@uni-muenster.de*