EVOLUTION, ITS FRACTIONAL EXTENSION AND GENERALIZATION

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Abstract

The evolution of a quantity, described by a function of space and time, relates the first derivative in time of this function to a spatial operator applied to the function. The initial value of the function at time $t = 0$ is given.

The fractional extension of this evolution consists of replacing the first derivative in time by a fractional derivative of order $\alpha$, $0 < \alpha \leq 1$.

We give a relationship between the solution of the equation of evolution and the solution of the equation belonging to its fractional extension.

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1. Introduction

Let $u(x,t)$ be a function of space and time; $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$. The equation, with initial condition,

$$\frac{\partial u(x,t)}{\partial t} = [L(x)u](x,t) \quad (1)$$

$$u(x,0) = f(x) \quad (2)$$

is called an equation of evolution. The linear spatial operator $L(x)$ depends on the problem at hand. We assume that the solution $u(x,t)$ of this problem has been found.
The fractional extension of the above equation of evolution is given by the integral equation
\[ u_\alpha(x, t) = u(x, 0) + \frac{1}{\Gamma(\alpha)} \int_0^t d\tau (t - \tau)^{\alpha-1} [L(x)u_\alpha](x, \tau) \] (3)

For \( \alpha = 1 \) we recover the solution of the equation of evolution.

In this paper we find a relationship between the solutions \( u_\alpha(x, t) \) and \( u(x, t) \). A generalization of this relationship is also proposed.

2. Evolution and its fractional extension I

The equation of evolution Eqs. (1), (2) can be written as
\[ u(x, t) = f(x) + \int_0^t d\tau [L(x)u](x, \tau) \] (5)
\[ u(x, 0) = f(x). \] (6)

The Laplace transform \( \tilde{u}(x, p) \) in the variable \( t \) is given by
\[ \tilde{u}(x, p) = \int_0^\infty dt e^{-pt} u(x, t) \] (7)

and leads to
\[ [L(x)\tilde{u}](x, p) = p\tilde{u}(x, p) - f(x). \] (8)

The fractional extension of the above equation of evolution reads
\[ u_\alpha(x, t) = f(x) + \frac{1}{\Gamma(\alpha)} \int_0^t d\tau (t - \tau)^{\alpha-1} [L(x)u_\alpha](x, \tau). \] (9)

The Laplace transform \( \tilde{u}_\alpha(x, p) \) in the variable \( t \) is given by
\[ \tilde{u}_\alpha(x, p) = \int_0^\infty dt e^{-pt} u_\alpha(x, t) \] (10)

and leads to
\[ [L(x)\tilde{u}_\alpha](x, p) = p^\alpha \tilde{u}_\alpha(x, p) - f(x)p^{\alpha-1}. \] (11)

**Lemma 1.** The solutions \( \tilde{u}(x, p) \) and \( \tilde{u}_\alpha(x, p) \) are related by
\[ \tilde{u}_\alpha(x, p) = p^{\alpha-1} \tilde{u}(x, p^\alpha). \] (12)
Proof. 

\[ [L(x)\tilde{u}_\alpha](x,p) = p^{\alpha-1} [L(x)\tilde{u}](x,p) \]  
\[ = p^{\alpha-1} \{ p^{\alpha} \tilde{u}(x,p) - f(x) \} \]  
\[ = p^{\alpha} \tilde{u}_\alpha(x,p) - f(x)p^{\alpha-1} \]  

which is Eq. (11).

We now look at the Mellin transforms of \( u(x,t) \) and \( u_\alpha(x,t) \) in the variable \( t \).

\[ \hat{u}(x,s) = \int_0^\infty dt \ t^{s-1} u(x,t) \]  
\[ \hat{u}_\alpha(x,s) = \int_0^\infty dt \ t^{s-1} u_\alpha(x,t) . \]  

The relationship between the Mellin and Laplace transform of a function \( \phi(t) \) is

\[ \hat{\phi}(s) = \frac{1}{\Gamma(1-s)} \int_0^\infty dp \ p^{-s} \tilde{\phi}(p) . \]  

Lemma 2. \( \hat{u}(x,s) \) and \( \hat{u}_\alpha(x,s) \) are related by

\[ \hat{u}_\alpha(x,s) = \frac{1}{\alpha} \frac{\Gamma \left( 1 - \frac{s}{\alpha} \right)}{\Gamma(1-s)} \hat{u}(x, \frac{s}{\alpha}) . \]  

Proof. From Eqs. (15), (12) we get

\[ \hat{u}_\alpha(x,s) = \frac{1}{\Gamma(1-s)} \int_0^\infty dp \ p^{-s} p^{\alpha-1} \tilde{u}(x,p) \]  

The change of variable

\[ p^\alpha = q \]  

leads to

\[ \hat{u}_\alpha(x,s) = \frac{1}{\alpha} \frac{1}{\Gamma(1-s)} \int_0^\infty dq \ q^{-\frac{s}{\alpha}} \tilde{u}(x,q) \]  

or

\[ \hat{u}_\alpha(x,s) = \frac{1}{\alpha} \frac{1}{\Gamma(1-s)} \Gamma \left( 1 - \frac{s}{\alpha} \right) \hat{u}(x, \frac{s}{\alpha}) . \]  

This is Eq. (14).
3. Generalized Mittag-Leffler Functions

The generalized Mittag-Leffler functions \( F_{\alpha\beta}(z) \) are given by

\[
F_{\alpha\beta}(z) = \Gamma(\beta) \sum_{k=0}^{\infty} (-1)^k \frac{1}{\Gamma(\beta + \alpha k)} z^k
\]

with \( z \geq 0 \), \( 0 < \alpha \leq 1 \), \( \beta \geq \alpha \).

\( \beta = 1 \) give the usual Mittag-Leffler functions which we denote by \( F_\alpha(z) \).

The \( H \)-function representation of \( F_{\alpha\beta}(z) \) \([2, 3, 4]\) is given by

\[
F_{\alpha\beta}(z) = \Gamma(\beta) H_{12}^{11} \left\{ \frac{(0, 1)}{(0, 1)(1 - \beta, \alpha)} \right\}.
\]

The generalized Mittag-Leffler functions are the Laplace transform of probability measures on \( \mathbb{R}_+ \). The corresponding probability densities are given by

\[
f_{\alpha\beta}(z) = \Gamma(\beta) H_{11}^{10} \left( z \left| \frac{\beta - \alpha, \alpha}{(0, 1)} \right. \right).
\]

\( \beta = 1 \) gives the probability density of the usual Mittag-Leffler functions which is denoted by \( f_\alpha(z) \). It has the \( H \)-function representation

\[
f_\alpha(z) = H_{11}^{10} \left( z \left| (1 - \alpha, \alpha) \right. \right), \ 0 < \alpha < 1
\]

and

\[
f_1(z) = \delta(z - 1).
\]

For \( 0 < \alpha < 1 \), \( f_\alpha(z) \) is an entire function and vanishes exponentially for large positive \( z \). It has the power series representation

\[
f_\alpha(z) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{\Gamma(1 - \alpha - \alpha k) k!} z^k
\]

\( 0 < \alpha < 1 \), \( z \in \mathbb{R}_+ \).

4. Evolution and its Fractional Extension II
The Mellin transform of the probability density \( f_\alpha(z) \) is given through Eq. (28) as
\[
\hat{f}_\alpha(s) = \frac{\Gamma(s)}{\Gamma(1 - \alpha + \alpha s)}.
\]  

**Lemma 3.** The inverse Mellin transforms of \( \hat{u}(x, s) \) and \( \hat{u}_\alpha(x, s) \) are related by
\[
u_\alpha(x, t) = \int_0^\infty dz f_\alpha(z) \nu(x, t^\alpha z) .
\]

**Proof.** Let \( M \) denote the performance of the Mellin transform. From [5] we have
\[
\left[M \int_0^\infty dz f(tz)g(z)\right](s) = \hat{f}(s)\hat{g}(1 - s)
\]  
and thus from Eq. (19)
\[
\hat{u}_\alpha(x, s) = \frac{1}{\alpha} \hat{f} \left( 1 - \frac{s}{\alpha} \right) \hat{u}(x, \frac{s}{\alpha}) .
\]

This leads to
\[
\hat{u}_\alpha(x, s) = \frac{1}{\alpha} \left[M \int_0^\infty dz f_\alpha(z) \nu(x, tz)\right] \left( \frac{s}{\alpha} \right)
\]  
and thus from Eq. (38)
\[
u_\alpha(x, t) = \int_0^\infty dz f_\alpha(z) \nu(x, t^\alpha z) .
\]

**Theorem 1.** The solution \( u(x, t) \) of the equation of evolution Eq. (5) and the solution \( u_\alpha(x, t) \) of its fractional extension Eq. (9) are related by
\[
u_\alpha(x, t) = t^{-\alpha} \int_0^\infty dz f_\alpha \left( t^{-\alpha} z \right) \nu(x, z) .
\]

**Proof.** Make the variable transform \( t^\alpha z = w \) in Eq. (33).
5. The Green’s functions

The solution of the equation of evolution Eq. (3) can be written as

\[ u(x, t) = \int dy G(x, y; t) f(y) \]  \hspace{1cm} (43)

where we assume that the Green’s function \( G(x, y; t) \) is known.

The solution of the equation of its fractional extension Eq. (9) can be written as

\[ u_\alpha(x, t) = \int dy G_\alpha(x, y; t) f(y). \]  \hspace{1cm} (44)

According to Theorem 1, Eq. (42) the two Green’s functions are related by

\[ G_\alpha(x, y; t) = t^{-\alpha} \int_0^\infty dz f_\alpha(t^{-\alpha}z) G(x, y; t). \]  \hspace{1cm} (45)

6. Applications

1. The fractional diffusion equation [6]. The equation of evolution is given by

\[ \frac{\partial u(x, t)}{\partial t} = \Delta u(x, t) \]  \hspace{1cm} (46)

where

\[ \Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} \]  \hspace{1cm} (47)

and the initial condition

\[ u(x, 0) = f(x). \]  \hspace{1cm} (48)

The Green’s function for this problem is

\[ G(x, y; t) = G(|x - y|, t) \]  \hspace{1cm} (49)

where with

\[ r = |x - y| \]  \hspace{1cm} (50)

\[ G(r, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{r^2}{4t}}. \]  \hspace{1cm} (51)
The Green’s function for the fractional extension of Eq. (46) is then
\[ G_\alpha(r, t) = t^{-\alpha} \int_0^\infty dz \, f_\alpha(t^{-\alpha} z) G(r, z). \] (52)

We compute this integral by looking at its Mellin transform in the variable \( t \).
\[ \hat{G}_\alpha(r, s) = \frac{1}{\alpha} \hat{f}_\alpha \left( 1 - \frac{s}{\alpha} \right) \hat{G} \left( r, \frac{s}{\alpha} \right) = \frac{1}{\alpha} \Gamma \left( 1 - \frac{s}{\alpha} \right) \hat{G} \left( r, \frac{s}{\alpha} \right). \] (53) (54)

From [5] we get
\[ \hat{G}(r, s) = \pi^{-2s} 2^{-2s-\frac{n}{2}} \Gamma \left( \frac{n}{2} \right) \] (55)
and thus
\[ \hat{G}_\alpha(r, s) = \frac{1}{\alpha} \pi^{-\frac{n}{2} - 1} 2^{-\frac{n}{2}} r^{\frac{n}{2} - \frac{1}{2}} \Gamma \left( \frac{n}{2} - \frac{s}{\alpha} \right) \Gamma \left( 1 - \frac{s}{\alpha} \right) \Gamma \left( \frac{1}{2} - \frac{s}{\alpha} \right). \] (56)

Inverting the Mellin transform leads to
\[ G_\alpha(r, t) = \pi^{-\frac{n}{2} - 1} r^{-\frac{n}{2}} H_{12} \left( \frac{1}{2} r t^{-\frac{1}{2}} \left( \frac{1}{\alpha} \right) \right). \] (57)
This is indeed the Green’s function as given in [6].

2. Fractional Black-Scholes equation [7]

The equation of evolution (Black-Scholes equation) is given by
\[ \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C = 0 \] (58)
For \( S \geq 0, \ t \leq T \) (59)
and boundary condition
\[ C(S, T) = \max\{S - E, 0\}. \] (60)

With the transformation
\[ t = T - \frac{2}{\sigma^2} \tau, \ C(S, t) = A(S, \tau) \] (61)
\[ \lambda_0 = \frac{2r}{\sigma^2} \] (62)
we have the equation of evolution
\[ \frac{\partial A}{\partial \tau} = S^2 \frac{\partial^2 A}{\partial S^2} + \lambda_0 S \frac{\partial A}{\partial S} - \lambda_0 A \] (63)
with the initial value
\[ A(S, 0) = \max\{S - E, 0\} . \] (64)
The solution is given by
\[ A(S, \tau) = SN(d_1) - Ee^{-\lambda_0 \tau}N(d_2) \] (65)
where
\[ d_1 = (2\tau)^{-\frac{1}{2}} \left[ \ln \left( \frac{S}{E} \right) + (\lambda_0 + 1)\tau \right] \] (66)
\[ d_2 = (2\tau)^{-\frac{1}{2}} \left[ \ln \left( \frac{S}{E} \right) + (\lambda_0 - 1)\tau \right] \] (67)
and
\[ N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} dx e^{-\frac{1}{2}x^2} . \] (68)
According to Theorem 1 Eq. (42) the solution of the fractional extension of the Black-Scholes equation
\[ A_\alpha(S, \tau) = A(S, 0) + \frac{1}{\Gamma(\alpha)} \int_{0}^{\tau} dz (\tau - z)^{\alpha - 1} \times \left[ S^2 \frac{\partial^2 A_\alpha(S, z)}{\partial S^2} + \lambda_0 S \frac{\partial A_\alpha(S, z)}{\partial S} - \lambda_0 A_\alpha(S, z) \right] \] (69)
is given by
\[ A_\alpha(S, \tau) = \tau^{-\alpha} \int_{0}^{\infty} dz f_\alpha(\tau^{-\alpha}z) A(S, z) . \] (70)

7. Generalization

According to Eq. (33) the solution of the equation for the fractional extension of an equation of evolution is given by Eq. (33)
\[ u_\alpha(x, t) = \int_{0}^{\infty} dz f_\alpha(z) u(x, t^\alpha z) \] (71)
$f_\alpha(z)$ is a measure on $\mathbb{R}_+$ and $f_1(z) = \delta(z - 1)$.

The Laplace transform of $f_\alpha$ is the usual Mittag-Leffler function.

The generalized Mittag-Leffler function Eq. (24) is the Laplace transform of the measure $f_{\alpha\beta}(z)$ on $\mathbb{R}_+$, Eq. (27). We have

$$f_{\alpha\beta}(x) = \Gamma(\beta) \sum_{k=0}^{\infty} (-1)^k \frac{1}{\Gamma(\beta - \alpha - \alpha k)} \frac{x^k}{k!}, \; a < 1, \; \beta \geq \alpha \quad (72)$$

$$f_{1\beta}(x) = \begin{cases} (\beta - 1)(1 - x)^{\beta - 2}, & 0 \leq x \leq 1 \\ 0, & 1 < x, \; \beta > 1 \end{cases} \quad (73)$$

$$f_{11}(x) = \delta(x - 1). \quad (74)$$

It is thus tempting to generalize from the solution $u_\alpha(x, t)$ in Eq. (71) to

$$u_{\alpha\beta}(x, t) = \int_0^\infty dz f_{\alpha\beta}(z) u_{\alpha\beta}(x, t^{\alpha}z). \quad (75)$$

This expression uses the form factor $f_{\alpha\beta}$ and will be studied elsewhere. Other form factors might also be considered.

8. Summary

We assume that we know the solution of a general equation of evolution. The solution of the equation of its fractional extension is then related to the above solution by what we call a form factor. This form factor belongs to the Mittag-Leffler function. A fractional generalization is proposed by using the form factor belonging to the generalized Mittag-Leffler function.

References

[1] W. R. Schneider. Completely monotone generalized Mittag-Leffler functions. *Exposition. Math.*, 14(1):3–16, 1996.

[2] Charles Fox. The $G$ and $H$ functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, 98:395–429, 1961.

[3] B. L. J. Braaksma. Asymptotic expansions and analytic continuations for a class of Barnes-integrals. *Compositio Math.*, 15:239–341 (1964), 1964.

[4] H. M. Srivastava, K. C. Gupta, and S. P. Goyal. The $H$-functions of one and two variables. pages x+306, 1982. With applications.
[5] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Tables of integral transforms. Vol. I*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. Based, in part, on notes left by Harry Bateman.

[6] W. R. Schneider and W. Wyss. Fractional diffusion and wave equations. *J. Math. Phys.*, 30(1):134–144, 1989.

[7] Walter Wyss. The fractional Black-Scholes equation.

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