STATISTICAL PROPERTIES OF STOCHASTIC 2D NAVIER-STOKES EQUATIONS FROM LINEAR MODELS

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Abstract. A new approach to the old-standing problem of the anomaly of the scaling exponents of nonlinear models of turbulence has been proposed and tested through numerical simulations. This is achieved by constructing, for any given nonlinear model, a linear model of passive advection of an auxiliary field whose anomalous scaling exponents are the same as the scaling exponents of the nonlinear problem.

In this paper, we investigate this conjecture for the 2D Navier-Stokes equations driven by an additive noise. In order to check this conjecture, we analyze the coupled system Navier-Stokes/linear advection system in the unknowns \( (u, w) \). We introduce a parameter \( \lambda \) which gives a system \( (u^\lambda, w^\lambda) \); this system is studied for any \( \lambda \) proving its well posedness and the uniqueness of its invariant measure \( \mu^\lambda \).

The key point is that for any \( \lambda \neq 0 \) the fields \( u^\lambda \) and \( w^\lambda \) have the same scaling exponents, by assuming universality of the scaling exponents to the force. In order to prove the same for the original fields \( u \) and \( w \), we investigate the limit as \( \lambda \to 0 \), proving that \( \mu^\lambda \) weakly converges to \( \mu^0 \), where \( \mu^0 \) is the only invariant measure for the joint system for \( (u, w) \) when \( \lambda = 0 \).

1. Introduction. We consider the stochastic Navier Stokes equations

\[
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u = -\nabla p_1 + \frac{\partial \beta_1}{\partial t}, \quad \nabla \cdot u = 0
\]  

(1)

describing the motion of a fluid in a bounded domain \( D \) of \( \mathbb{R}^d \) \( (d = 2, 3) \). Here, \( u = u(t, \xi) \) is the velocity vector field, \( p_1 = p_1(t, \xi) \) is the scalar pressure field and \( \nu > 0 \) is the viscosity. \( \beta_1 = \beta_1(t, \xi) \) is a Gaussian random field white in time, subject to the restrictions imposed below on the space correlation. The velocity field \( u \) is subject to some boundary condition and initial condition.

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Recently (see [1] and the references therein), it has been proposed that the Navier-Stokes equation and a relevant linear advection model

\[
\frac{\partial w}{\partial t} - \nu \Delta w + (u \cdot \nabla)w = -\nabla p_2 + \frac{\partial \beta_2}{\partial t}, \quad \nabla \cdot w = 0
\]  

(2)

have the same scaling exponents of their structure functions, even if their statistics are different (i.e., in the “jargon” of stochastic PDE’s their invariant measures are different). Here \(u\) is solution of equation 1; the noises \(\beta_1\) and \(\beta_2\) are independent and identically distributed.

We recall the definition of structure function (see [18]): it is an ensemble average of power of velocity differences across a length scale. If the velocity field \(u\) is stationary in time, homogeneous and isotropic in space, the longitudinal structure function is

\[
S_p^u(l) = \langle \left[ (u(t, l\xi) - u(t, 0)) \cdot \hat{\xi} \right]^p \rangle
\]  

(3)

with \(\xi, \xi + l\hat{\xi} \in D, \ p \in \mathbb{N}\) and \(l \in \mathbb{R}; \ \hat{\xi}\) is a versor and we take the scalar product of vectors in \(\mathbb{R}^d\).

For a turbulent field it is important to know how \(S_p^u(l)\) depends on \(l\) for small \(l\). The scaling exponents \(\zeta_p\) are defined by

\[
S_p^u(l) \propto l^{\zeta_p}
\]  

(4)

and are assumed to be universal in the limit \(\nu \to 0\) when \(l\) lies in the inertial range \(\eta_0^{3/4} \leq l \leq l_0\) (see e.g. [18]); an equivalent definition is given by

\[
\zeta_p = \lim_{l \to 0} \frac{\log S_p^u(l)}{\log l}.
\]  

(5)

Notice that \(l \to 0\) implies also \(\nu \to 0\).

[1] provides numerical evidence that both the 2D Navier-Stokes equations and the Sabra shell model have the same scaling exponents of the corresponding linear advection models. If this statement were true, then it would allow to reduce the establishment of the scaling exponents for the Navier-Stokes equations 1 with \(d = 2\) to the easier problem of the establishment of the scaling exponents for the linear advection model 2. As far as the linear advection problem is concerned, its scaling exponents show anomalous behavior (see, among the others, [1, 2, 19] and the references therein).

It is then of interest to understand rigorously the properties of the joint system

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u &= -\nabla p_1 + \frac{\partial \beta_1}{\partial t}, \\
\frac{\partial w}{\partial t} - \nu \Delta w + (u \cdot \nabla)w &= -\nabla p_2 + \frac{\partial \beta_2}{\partial t}
\end{aligned}
\]  

(6)

with the divergence free condition and the boundary condition. If it has a unique invariant measure \(\mu\), then the ensemble averages are computed with respect to this measure, i.e. for the linear advection model 2

\[
S_p^w(l) = \iint \{ [w(l\hat{\xi}) - w(0)] \cdot \hat{\xi} \}^p \mu(du, dw)
\]  

(7)

and for the Navier-Stokes equation 1

\[
S_p^u(l) = \iint \{ [u(l\hat{\xi}) - u(0)] \cdot \hat{\xi} \}^p \mu(du, dw) \equiv \iint \{ [u(l\hat{\xi}) - u(0)] \cdot \hat{\xi} \}^p m(du)
\]  

(8)

being \(m(du) = \int \mu(du, dw)\) the unique invariant measure for 1.
The analysis of system 6 has been performed by adding two terms (see [3] and the references therein):

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \lambda (w \cdot \nabla)u &= -\nabla p_1 + \frac{\partial \beta_1}{\partial t} \\
\frac{\partial w}{\partial t} - \nu \Delta w + (u \cdot \nabla)w + \lambda (w \cdot \nabla)w &= -\nabla p_2 + \frac{\partial \beta_2}{\partial t}
\end{aligned}
\]

(9)

where \( \lambda \in \mathbb{R} \) is a parameter. We denote by \((u^{\lambda}, w^{\lambda})\) its solution. For \( \lambda = 0 \) we recover 6 and for \( \lambda = 1 \) system 9 is symmetric.

On the other hand, for any \( \lambda \neq 0 \), system 9 enjoys the following property: setting \( v^{\lambda} = \lambda w^{\lambda} \) and multiplying the second equation by \( \lambda \), we have a perfectly symmetric system for the pair \((u^{\lambda}, v^{\lambda})\), except for the force and initial conditions:

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + (v \cdot \nabla)u &= -\nabla p_1 + \frac{\partial \beta_1}{\partial t} \\
\frac{\partial v}{\partial t} - \nu \Delta v + (u \cdot \nabla)v + (v \cdot \nabla)v &= -\lambda \nabla p_2 + \lambda \frac{\partial \beta_2}{\partial t}
\end{aligned}
\]

(10)

with \( u^{\lambda}(0) = u_0 \) and \( v^{\lambda}(0) = \lambda w_0 \). Thus, assuming\(^1\) the universality of the scaling exponents to the force, it follows that \( u^{\lambda} \) and \( v^{\lambda} = \lambda w^{\lambda} \) have the same scaling exponents for any \( \lambda \neq 0 \). Then the same consideration holds for the couple \((u^{\lambda}, w^{\lambda})\) for any \( \lambda \neq 0 \); indeed, \( S^{p}_{\lambda v^{\lambda}}(l) = \lambda^p S^{p}_{w^{\lambda}}(l) \) and 5 gives the same \( \zeta_\lambda \) for \( v^{\lambda} \) and \( w^{\lambda} \). The crucial point is to see if this holds also in the limit as \( \lambda \to 0 \).

To this end, we shall investigate when there exists a unique invariant measure \( \mu^{\lambda} \) for 9 (for any \( \lambda \in \mathbb{R} \)) and we shall prove that there exists a subsequence \( \mu^{\lambda_n} \) which converges to the unique invariant measure \( \mu \) for system 6, as \( \lambda_n \to 0 \). We recall that the continuous dependence of the solutions to system 9 as \( \lambda \to 0 \) has been investigated rigorously in the context of certain nonlinear phenomenological shell model (see [3] for the deterministic case and [5] for the stochastic case). Moreover, [5] considers the asymptotic dynamics (for large time) on the attractor, proving the continuous dependence on \( \lambda \) of the attractor. Here we address the continuous dependence of the invariant measure with respect to the parameter \( \lambda \) for the stochastic 2D Navier-Stokes equations. We shall exploit the fact that system 9 enjoys the same properties as the stochastic 2D Navier-Stokes equations, in order to prove existence and uniqueness of an invariant measure \( \mu^{\lambda} \).

The paper is organized as follows. Section 2 is devoted to introducing the functional setting. It is split into subsections. We first introduce the functional spaces and operators. Then, the stochastic external forces and assumptions on the covariances of the noise are defined. Some properties of the Ornstein-Uhlenbeck process are introduced. These auxiliary and known results will be used in Section 3 to prove the well posedness of 9 and the uniqueness of its invariant measure for any \( \lambda \in \mathbb{R} \); the proofs are given along the same lines as for the Navier-Stokes equations 1. In Section 4, we prove our main result, i.e. the continuous dependence of the invariant measure \( \mu^{\lambda} \) with respect to the parameter \( \lambda \) is proved. We first prove that the unique invariant measure \( \mu^{\lambda} \) is tight; hence it has a limit \( m \) when \( \lambda \to 0 \). Now, in order to prove that the limit \( m \) is an invariant measure, we consider the stationary solutions of 9 whose time marginals are \( \mu^{\lambda} \). Thanks to uniform estimates computed on these stationary solutions, we prove their convergence when \( \lambda \to 0 \) and that

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\(^1\)The linear advection equation 2 has scaling exponents universal to the forcing, when the forces act only on large scales (this has been investigated by physicists, see e.g. [2, 7, 11]). Some numerical evidence shows the same for the nonlinear equation 1 (see [1]).
their limit are stationary and their marginals are given by $m$. As a consequence, we conclude that $m$ is the unique invariant measure for $\theta$ when $\lambda = 0$.

2. Notations and hypothesis.

2.1. Functional setting. We define the functional setting to study the Navier-Stokes equations. From now on, the spatial domain is the square $D = [0, 2\pi]^2$ with periodic boundary conditions.

As usual, in the periodic case we assume that the spatial mean of the vectors we are dealing with is zero. This gives a simplification in the mathematical treatment, but it does not prevent to consider non zero mean value vectors. Actually, if we can analyze the problem for zero mean vectors then the problem without this assumption can be dealt with in a similar way (see [25]).

Let $H^s$ be the space of divergence free and periodic vector fields with mean zero that belong to the Sobolev space $[H^s(D)]^2$.

The Stokes operator is defined as

$$Au = -\Delta u, \quad \forall u \in D(A) = H^2.$$  

It is a closed positive unbounded self-adjoint operator in $H$ with the inverse $A^{-1}$ which is a self-adjoint compact operator in $H^0$; by the classical spectral theorems there exists a sequence $\{\gamma_j\}_{j=1}^\infty$ of eigenvalues of the Stokes operator with $0 < \gamma_1 \leq \gamma_2 \leq \ldots$, corresponding to the eigenvectors $e_j \in D(A)$; $\{e_j\}_{j \in \mathbb{N}}$ form an orthonormal basis in $H^0$. We have that $\gamma_j$ behaves like $\frac{j}{2}$ for $j \to \infty$. The Stokes operator generates an analytic semigroup $e^{-tA}$ in $H^0$ and for each $s > 0$ there exists a constant $M > 0$ (depending on $s$ and $\nu$) such that

$$\|e^{-tA}u\|_{H^s} \leq \frac{M}{t^{s/2}} \|u\|_{H^0}, \quad u \in H^0$$

for all $t > 0$.

We can define the fractional powers $A^p$ ($p > 0$) as linear unbounded operators in $H^0$ with

$$A^p e_j = \gamma_j^p e_j, \quad D(A^p) = H^{2p}.$$  

Therefore we can characterize the Hilbert spaces $H^s$ as

$$H^s = \{u = \sum_{j=1}^{\infty} u_j e_j : \sum_{j=1}^{\infty} \gamma_j^s u_j^2 < \infty\}$$

and we set

$$\|u\|_{H^s}^2 = \sum_{j=1}^{\infty} \gamma_j^s u_j^2.$$  

Moreover, $H^{s_1}$ is densely and compactly embedded in $H^{s_2}$ for $s_1 > s_2$. Finally, $H^{-s}$ denotes the dual of $H^s$, with duality bracket $\langle \cdot, \cdot \rangle$. For $s = 0$ this is the scalar product $\langle u, v \rangle = \sum_j u_j v_j$ in $H^0$.

Let $b(\cdot, \cdot, \cdot)$ be the trilinear form defined as

$$b(u, v, z) = \int_D \left( |u(\xi) \cdot \nabla|v(\xi)) \cdot z(\xi) \right) d\xi.$$  

It is well known that there exists a continuous bilinear operator $B(\cdot, \cdot) : H^1 \times H^1 \longrightarrow H^{-1}$ such that $\langle B(u, v), z \rangle = b(u, v, z)$ for all $z \in H^1$. By the incompressibility condition, we have

$$\langle B(u, v), z \rangle = -\langle B(u, z), v \rangle$$ and $\langle B(u, v), v \rangle = 0$  

(11)
for \( u, v, z \in H^1 \). Furthermore, there exist constants \( C \) and \( C_r \) such that

\[
\begin{cases}
|B(u,v)|_{H^{-1}} \leq C|u|_{H^{1/2}}|v|_{H^{1/2}} \\
|B(u,v)|_{H^0} \leq C|u|_{H^{1/2}}|v|_{H^{3/2}} \\
|B(u,v)|_{H^{-1+r}} \leq C_r|u|_{H^{-1+r}}|v|_{H^r}
\end{cases}
\]

for \( r \geq 2 \). A more refined inequality holds: for \( r > 2 \)

\[
|B(u,v)|_{H^{-1+r}} \leq C|u|_{H^{-1+r}}|\nabla v|_{H^{-1+r}} \leq C|u|_{H^{-1+r}}|v|_{H^r}
\]

since \( H^{-1+r} \) is a multiplicative algebra for \( r > 2 \). Since

\[
|B(u,v)|_{H^1} = |(u \cdot \nabla)v|_{H^1} \leq |Du|_{L^4}|\nabla v|_{L^4} + |u|_{L^\infty}|\nabla v|_{H^1}
\]

\[
\leq C|u|_{H^{1/2}}|v|_{H^{1/2}} + C|u|_{H^{3/2}}|v|_{H^2}
\]

\[
\leq C|u|_{H^{1/2}}|v|_{H^2}
\]

these two latter inequalities are summarized in the last line of (12).

Finally, let us point out that we will use the same symbol \( C \) for different constants, if they do not play an important role.

2.2. **Stochastic driving force.** As far as the stochastic forcing terms are concerned, we refer to [9] for the basic results. Here we recall the main definitions and properties.

We introduce two independent \( H^0 \)-cylindrical Wiener processes \( \beta_1 \) and \( \beta_2 \) as follows:

\[
\beta_i(t) = \sum_{j=1}^{\infty} \omega_j^{(i)}(t)e_j,
\]

with \( \omega_j^{(i)} \) mutually independent standard (scalar) Wiener processes defined on a filtered complete probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\).

Let \( G \) be a Hilbert-Schmidt operator in \( H^0 \) and denote by \( \mathcal{R}(G) \) its range and by \( \|G\|_{HS} \) its Hilbert-Schmidt norm \( \|G\|_{HS}^2 = \sum_j \|Ge_j\|_{H^0}^2 \). If we assume that

\[
\mathcal{R}(G) \subseteq H^p
\]

for some \( p \geq 0 \), then each process \( G\beta_t \) takes values in \( C(\mathbb{R}_+; H^s) \) if

\[
p > s + 1.
\]

Indeed this is equivalent to \( A^{\frac{p}{2}}G \beta_t \) taking values in \( C(\mathbb{R}_+; H^0) \). Notice that \( A^{\frac{p}{2}}G = A^{\frac{p}{2}}(A^{\frac{p}{2}}G) \); therefore, by assuming that \( A^{\frac{p}{2}}G \) is a bounded operator in \( H^0 \), we get that \( A^{\frac{p}{2}}G \) is a Hilbert-Schmidt operator if \( A^{\frac{p}{2}}G \) is so. This happens under assumption (14), since

\[
\|A^{\frac{p}{2}}\|_{HS}^2 = \sum_j \|A^{\frac{p}{2}}e_j\|_{H^0}^2 = \sum_j \gamma_j^{s-p} \sim \sum_j j^{s-p}
\]

and the latter series is convergent if (14) is fulfilled.

From now on, we assume that

\[
\exists \varepsilon > 0 : \mathcal{R}(G) \subseteq H^{1+\varepsilon}.
\]

This implies that \( G\beta_t \in C(\mathbb{R}_+; H^0) \), \( \mathbb{P} \)-a.s.
Projecting the equations of 9 onto $H^0$, we get rid of the pressure terms and the following abstract formulation is obtained

\[
\begin{aligned}
\left\{ \begin{array}{l}
du^\lambda(t) + [\nu A u^\lambda(t) + B(u^\lambda(t), u^\lambda(t))] dt = Gd\beta_1(t), \quad t > 0 \\
dw^\lambda(t) + [\nu Aw^\lambda(t) + B(w^\lambda(t), w^\lambda(t))] dt = Gd\beta_2(t), \quad t > 0 \\
u^\lambda(0) = u_0 \\
w^\lambda(0) = w_0
\end{array} \right.
\]

with initial conditions $u_0, w_0 \in H^0$.

The Cauchy problem is studied on any finite time interval $[0, T]$.

2.3. **Compact form of the system.** Now we write this system in a compact form. Define $\tilde{H}^s = H^s \times H^s$. If $x = (x_1, x_2) \in \tilde{H}^0$ and $y = (y_1, y_2) \in \tilde{H}^0$, we define the scalar product in $\tilde{H}^0$ as

\[
\langle x, y \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle
\]

and the norms in $\tilde{H}^s$ as

\[
\|x\|_{\tilde{H}^s}^2 = \|x_1\|_{H^s}^2 + \|x_2\|_{H^s}^2, \quad x = (x_1, x_2) \in \tilde{H}^s.
\]

We define the linear operator $A : \tilde{H}^2 \subset \tilde{H}^0 \rightarrow \tilde{H}^0$ as $Ax = (Ax_1, Ax_2)$. As a consequence, $A$ is nonnegative and selfadjoint.

For every $\lambda \in \mathbb{R}$, we define the bilinear operator $\tilde{B}^\lambda$ as

\[
\tilde{B}^\lambda(x, y) = \left( B(x_1, y_1) + \lambda B(x_2, y_1), B(x_1, y_2) + \lambda B(x_2, y_2) \right),
\]

By 11, 12 and 16, we have

**Lemma 2.1.** For any $x, y, z \in \tilde{H}^1$ we have

\[
\langle \tilde{B}^\lambda(x, y), z \rangle = -\langle \tilde{B}^\lambda(x, z), y \rangle, \quad \langle \tilde{B}^\lambda(x, y), y \rangle = 0.
\]

Moreover

i) there is a constant $C > 0$ such that for any $\lambda \in \mathbb{R}$

\[
\|\tilde{B}^\lambda(x, y)\|_{\tilde{H}^{-1}} \leq C(1 + |\lambda|) \|x\|_{\tilde{H}^{1/2}} \|y\|_{\tilde{H}^{1/2}};
\]

ii) there is a constant $C > 0$ such that for any $\lambda \in \mathbb{R}$

\[
\|\tilde{B}^\lambda(x, y)\|_{\tilde{H}^0} \leq C(1 + |\lambda|) \|x\|_{\tilde{H}^{1/2}} \|y\|_{\tilde{H}^{1/2}};
\]

iii) for any $r \geq 2$ there is a constant $C_r > 0$ such that for any $\lambda \in \mathbb{R}$

\[
\|\tilde{B}^\lambda(x, y)\|_{\tilde{H}^{-r+1}} \leq C_r (1 + |\lambda|) \|x\|_{\tilde{H}^r} \|y\|_{\tilde{H}^r}.
\]

Actually the latter estimate comes from

\[
\|\tilde{B}^\lambda(x, y)\|_{\tilde{H}^{-1}} \leq C(1 + |\lambda|) \|x\|_{\tilde{H}^{1/2}} \|y\|_{\tilde{H}^2}
\]

and, for $r \geq 3, r \in \mathbb{N}$

\[
\|\tilde{B}^\lambda(x, y)\|_{\tilde{H}^{-r+1}} \leq C_r (1 + |\lambda|) \|x\|_{\tilde{H}^{r-1}} \|y\|_{\tilde{H}^r}.
\]

Hence, we write 15 in more compact form as

\[
\left\{ \begin{array}{l}
d\tilde{u}^\lambda(t) + [\nu \tilde{A}\tilde{u}^\lambda(t) + \tilde{B}\tilde{(u}^\lambda(t), \tilde{u}^\lambda(t))] dt = \tilde{G}d\tilde{\beta}(t), \quad t > 0 \\
\tilde{u}^\lambda(0) = \tilde{x}
\end{array} \right.
\]

where $\tilde{u}^\lambda = (u^\lambda, w^\lambda)$ and $\tilde{G}\tilde{\beta} = (G\beta_1, G\beta_2)$. 
The following definition of solution for 19 is given

**Definition 2.2.** A stochastic process \( \tilde{u}^\lambda \) is a generalized solution in \([0,T]\) of system 19 if

\[
\tilde{u}^\lambda \in C([0,T]; \tilde{H}^0) \cap L^2(0,T; \tilde{H}^1)
\]

\( \mathbb{P} \)-a.s. and equation 19 is satisfied \( \mathbb{P} \)-a.s. in the integral sense

\[
\langle \tilde{u}^\lambda(t), \phi \rangle + \nu \int_0^t \langle \tilde{A}^2 \tilde{u}^\lambda(s), \tilde{A} \phi \rangle ds + \int_0^t \langle \tilde{B}^\lambda(\tilde{u}^\lambda(s)), \phi \rangle ds = \langle \tilde{x}, \phi \rangle + \langle \tilde{G} \tilde{\beta}(t), \phi \rangle
\]

for all \( t \in [0,T] \) and all \( \phi \in \tilde{H}^1 \).

This definition involves stronger regularity than in [14]-[15], but in those papers one looked for the minimal assumption on the noise to define a solution. But here we are interested in more regular solutions; for this reason we shall assume that \( G \) is an Hilbert-Schmidt operator in \( H^0 \) and so we avoid some technicalities of [14]-[15], required to deal with solutions of low space regularity.

2.4. **Ornstein-Uhlenbeck process.** Let us define the Ornstein-Uhlenbeck equation

\[
dz_i(t) + \nu A z_i(t) dt = G d \beta_i(t). \tag{20}
\]

We consider its mild solution

\[
z_i(t) = e^{-\nu At} z_i(0) + \int_0^t e^{-\nu A(t-s)} G d \beta_i(s) \tag{21}
\]

whose regularity is given in the following proposition (see [8]).

**Proposition 1.** Let \( G : H^0 \to H^0 \) be a linear bounded operator such that \( \mathcal{R}(G) \subseteq H^{\alpha+\varepsilon} \) for some \( \alpha, \varepsilon > 0 \). Then, given any initial value \( z_i(0) \in H^\alpha \), there exists a version of the Ornstein-Uhlenbeck process 21 whose paths are in \( C([0,\infty); H^\alpha) \), \( \mathbb{P} \)-a.s.

Therefore, from now on our assumption on the covariance of the noises is given by

\[ [H0] : G : H^0 \to H^0 \text{ is a linear bounded operator such that } \mathcal{R}(G) \subseteq H^{1+\varepsilon} \text{ for some } \varepsilon > 0 \]

This implies that \( G \) is an Hilbert-Schmidt operator in \( H^0 \).

Notice here that the mild solution of the Ornstein-Uhlenbeck equation is also a weak solution (in the PDE sense). We refer to [9] (Theorem 5.4).

3. **Ergodicity.** In this section we deal with fixed \( \lambda \), arbitrary chosen in \( \mathbb{R} \). For the evolution problem 19, we will prove first that if the initial data \( \tilde{x} \in \tilde{H}^0 \) and \( H0 \) is fulfilled, then there exists a unique generalized solution, which is a strong solution in the probabilistic sense. This unique solution is a time-homogeneous Markov process in \( \tilde{H}^0 \). Then, we define the transition functions \( P^\lambda(t, \tilde{x}, \Gamma) := \mathbb{P}\{\tilde{u}^\lambda(t; \tilde{u}^\lambda(0) = \tilde{x}) \in \Gamma\} \) for any \( t \geq 0, \tilde{x} \in \tilde{H}^0 \) and Borel subset \( \Gamma \) of \( \tilde{H}^0 \); the associated Markov semigroup is

\[
(P^\lambda_t \psi)(\tilde{x}) = \mathbb{E}[\psi(\tilde{u}^\lambda(t; \tilde{u}^\lambda(0) = \tilde{x}))].
\]

For each \( t \), \( P^\lambda_t \) maps Borelian bounded functions \( B_b(\tilde{H}^0) \) in themselves; it is said to be Feller if it maps continuous bounded functions in themselves, i.e. \( P^\lambda_t : C_b(\tilde{H}^0) \to \).
Going further, invariant measures will be investigated, that is measures \( \mu^\lambda \) such that
\[
\int \psi \, d\mu = \int P_1^\lambda \psi \, d\mu^\lambda
\]
for any \( t > 0 \) and \( \psi \in \mathcal{C}_b(\tilde{H}^0) \). We will prove that there exists a unique invariant measure for equation 19.

3.1. Well posedness. In this section we prove existence, uniqueness and continuous dependence of the solution on the initial data for system 9. The result is a classical one, since we are in a two dimensional spatial domain. We shall first prove existence of a martingale solution. Then path-wise uniqueness and continuous dependence of the solution on the initial data. Path-wise uniqueness implies that we have actually a strong solution in the probabilistic sense.

Our procedure is to work pathwise, as in [15] and [14], whereas [6] and [26] (Ch X) look for mean square estimates. In this way we get better regularity for the paths, i.e. \( \tilde{u}^\lambda \in C([0, T]; \tilde{H}^0) \) and not only \( \tilde{u}^\lambda \in L_\infty(0, T; \tilde{H}^0) \).

**Proposition 2.** Assume \( H_0 \).

Then, for any time interval \([0, T]\) and any initial condition \( x \in \tilde{H}^0 \) there exists a unique strong generalized solution \( \tilde{u}^\lambda \) of equation 19 over \([0, T]\) with the initial condition \( \tilde{u}^\lambda(0) = \tilde{x} \), satisfying \( \mathbb{P}\text{-}a.s. \)

\[
\tilde{u}^\lambda \in C([0, T]; \tilde{H}^0) \cap L^2(0, T; \tilde{H}^1).
\]

The process \( \tilde{u}^\lambda \) is a Markov and Feller process in \( \tilde{H}^0 \).

**Proof.** The proof relies on a priori estimates for the Ornstein-Uhlenbeck process \( \tilde{z} \), solving the linear equation
\[
d\tilde{z}(t) + \nu \tilde{A} \tilde{z}(t) \, dt = \tilde{G}d\tilde{w}(t); \quad \tilde{z}(0) = 0
\]
(so \( \tilde{z} = (z_1, z_2) \) with \( z_i \in C([0, \infty; H^1]) \) a.s., see 20) and the auxiliary process \( \tilde{v}^\lambda = \tilde{u}^\lambda - \tilde{z} \), solving the deterministic equation
\[
\frac{d\tilde{v}^\lambda}{dt}(t) + \nu \tilde{A} \tilde{v}^\lambda(t) + \tilde{B}^\lambda(\tilde{v}^\lambda(t), \tilde{v}^\lambda(t), \tilde{z}(t)) = 0; \quad \tilde{v}^\lambda(0) = \tilde{x}
\]
with random coefficients. The equation for \( \tilde{v}^\lambda \) is equivalent to
\[
\frac{d\tilde{v}^\lambda}{dt}(t) + \nu \tilde{A} \tilde{v}^\lambda(t) + \tilde{B}^\lambda(\tilde{v}^\lambda(t), \tilde{v}^\lambda(t), \tilde{z}(t)) = -\tilde{B}^\lambda(\tilde{z}(t), \tilde{z}(t)). \tag{22}
\]
Local existence is easy to prove, since the nonlinearity is locally Lipschitz. Therefore it is enough to get apriori estimates to show that this solution exists on the whole time interval \([0, T]\). Actually, one should first work with the finite-dimensional (Galerkin) approximation and then pass to the limit; we omit this details since they are classical (see, e.g., [24], [25]).

We take the \( \tilde{H}^0\)-scalar product of equation 22 with \( \tilde{v}^\lambda \) and use that \( \tilde{H}^{1/2} \subset [L^4(D)]^2 \) and by interpolation \( \|v\|_{\tilde{H}^{1/2}} \leq C\|v\|_{\tilde{H}^0}^{1/2}\|v\|_{\tilde{H}^1}^{1/2} \). Therefore, by taking into
account Lemma 2.1 we get
\[
\frac{1}{2} \frac{d}{dt} \|\tilde{v}(t)\|^2_{\tilde{H}^1} + \nu \|\tilde{v}(t)\|^2_{\tilde{H}^1} = -\langle \tilde{B}^\lambda(\tilde{v}(t), \tilde{z}(t)), \tilde{v}(t) \rangle = \|\tilde{B}^\lambda(\tilde{z}(t), \tilde{z}(t)), \tilde{v}(t)\|
\]
\[
\leq (1 + |\lambda|) \|\tilde{v}(t)\|_{\tilde{H}^1}^2 + (1 + |\lambda|) \|\tilde{z}(t)\|_{\tilde{H}^1} \|\tilde{v}(t)\|_{L^4_{t}}
\]
\[
\leq C(1 + |\lambda|) \|\tilde{v}(t)\|_{\tilde{H}^0} \|\tilde{v}(t)\|_{\tilde{H}^1} + C(1 + |\lambda|) \|\tilde{v}(t)\|_{\tilde{H}^1} \|\tilde{z}(t)\|_{\tilde{H}^1}^2
\]
\[
\leq \nu \|\tilde{v}(t)\|^2_{\tilde{H}^1} + C_{\nu}(1 + |\lambda|) \|\tilde{z}(t)\|^2_{\tilde{H}^1} \|\tilde{v}(t)\|^2_{\tilde{H}^0} + C_{\nu}(1 + |\lambda|) \|\tilde{z}(t)\|^4_{\tilde{H}^1},
\]
where we have used Young inequality in the latter step. Thus
\[
\frac{d}{dt} \|\tilde{v}(t)\|^2_{\tilde{H}^0} + \nu \|\tilde{v}(t)\|^2_{\tilde{H}^1} \leq C \|\tilde{z}(t)\|^2_{\tilde{H}^1} \|\tilde{v}(t)\|^2_{\tilde{H}^0} + C \|\tilde{z}(t)\|^4_{\tilde{H}^1}
\]
where the constant $C$ depends on $\nu$ and $\lambda$ and is uniformly bounded for $\lambda$ in a bounded set. Gronwall lemma applied to
\[
\frac{d}{dt} \|\tilde{v}(t)\|^2_{\tilde{H}^0} \leq C \|\tilde{z}(t)\|^2_{\tilde{H}^1} \|\tilde{v}(t)\|^2_{\tilde{H}^0} + C \|\tilde{z}(t)\|^4_{\tilde{H}^1},
\]
gives
\[
\sup_{0 \leq t \leq T} \|\tilde{v}(t)\|^2_{\tilde{H}^0} \leq \|\tilde{z}(0)\|^2_{\tilde{H}^1} e^{C \|\tilde{z}(0)\|^2_{\tilde{H}^1} dt} + C \int_0^T e^{\int_0^r C \|\tilde{z}(s)\|^4_{\tilde{H}^1} ds} \|\tilde{z}(t)\|^4_{\tilde{H}^1} dt
\]
where the r.h.s. is finite, since $H0$ grants that the paths of $\tilde{z}$ are in $C([0, T] ; \tilde{H}^1)$, $F$-a.s.. Integrating in time $23$ we get
\[
\nu \int_0^T \|\tilde{v}(t)\|^2_{\tilde{H}^1} dt \leq \|\tilde{z}(0)\|^2_{\tilde{H}^1} + C \left( \sup_{0 \leq t \leq T} \|\tilde{v}(t)\|^2_{\tilde{H}^0} \right) \|\tilde{z}(t)\|^2_{\tilde{H}^1} dt + C \int_0^T \|\tilde{z}(t)\|^4_{\tilde{H}^1} dt
\]
where the r.h.s. is finite.

In this way one gets $\tilde{v}^\lambda \in L^\infty(0, T ; \tilde{H}^0) \cap L^2(0, T ; \tilde{H}^1)$ and $\tilde{\nu}^\lambda = \tilde{\nu} + \tilde{z} \in L^\infty(0, T ; \tilde{H}^0) \cap L^2(0, T ; \tilde{H}^1)$. The continuity in time comes from the time regularity of $\tilde{v}^\lambda$. First we notice that
\[
\|\hat{B}(u, u)\|_{\tilde{H}^{-1}} = \sup_{|\phi|_{\tilde{H}^1} \leq 1} \|\langle \hat{B}(u, u), \phi \rangle\| = \sup_{|\phi|_{\tilde{H}^1} \leq 1} \|\langle \hat{B}(u, \phi), u \rangle\|
\]
\[
\leq C(1 + |\lambda|) \|u\|_{\tilde{H}^0} \|u\|_{\tilde{H}^1},
\]
and therefore $\hat{B}(\tilde{u}^\lambda, \tilde{u}^\lambda) \in L^2(0, T ; \tilde{H}^{-1})$. Thus for the time derivative we get $\frac{d}{dt} \tilde{v}^\lambda(t) = -\hat{A} \tilde{v}^\lambda(t) - \hat{B}(\tilde{u}^\lambda(t), \tilde{u}^\lambda(t)) \in L^2(0, T ; \tilde{H}^{-1})$. This implies (see [24]) that $\tilde{v}^\lambda \in C([0, T] ; \tilde{H}^0)$. Given the existence of a process $\tilde{u}^\lambda$ whose paths are in $C([0, T] ; \tilde{H}^0) \cap L^2(0, T ; \tilde{H}^1)$, $F$-a.s., we easily get pathwise uniqueness and continuous dependence on the initial data. Let $\tilde{u}^\lambda(1)$ and $\tilde{u}^\lambda(2)$ be two solutions of 19 with initial data $\tilde{z}^{(1)}$ and $\tilde{z}^{(2)}$ respectively. Denote their difference by $U^\lambda = \tilde{u}^\lambda(1) - \tilde{u}^\lambda(2)$. Since the noise is additive, the unknown $U^\lambda$ satisfies a deterministic equation:
\[
\frac{dU^\lambda}{dt}(t) + \nu \tilde{A} U^\lambda(t) + \tilde{B}(\tilde{u}^\lambda(1)(t), \tilde{u}^\lambda(1)(t)) - \tilde{B}(\tilde{u}^\lambda(2)(t), \tilde{u}^\lambda(2)(t)) = 0
\]
i.e.
\[
\frac{dU^\lambda}{dt}(t) + \nu \tilde{A} U^\lambda(t) + \tilde{B}(U^\lambda(t), \tilde{u}^\lambda(1)(t)) + \tilde{B}(\tilde{u}^\lambda(2)(t), U^\lambda(t)) = 0.
\]
With estimates as before, we get

$$\frac{1}{2} \frac{d}{dt} \|U^\lambda(t)\|^2_{\tilde{H}^0} + \nu \|U^\lambda(t)\|^2_{\tilde{H}^1} = -(\tilde{B}^\lambda(U^\lambda(t), \tilde{u}^\lambda(1)(t)), U^\lambda(t))$$

$$\leq (1 + |\lambda|) \|U^\lambda(t)\|^2_{L^4} \|\tilde{u}^\lambda(1)(t)\|_{\tilde{H}^1}$$

$$\leq C(1 + |\lambda|) \|U^\lambda(t)\|_{\tilde{H}^0} \|U^\lambda(t)\|_{\tilde{H}^1} \|\tilde{u}^\lambda(1)(t)\|_{\tilde{H}^1}$$

$$\leq \frac{\nu}{2} \|U^\lambda(t)\|^2_{\tilde{H}^1} + C\nu(1 + \lambda^2) \|\tilde{u}^\lambda(1)(t)\|^2_{\tilde{H}^1} \|U^\lambda(t)\|^2_{\tilde{H}^0}$$

From Gronwall’s lemma we get

$$\sup_{t \in [0, T]} \|U^\lambda(t)\|^2_{\tilde{H}^0} \leq \|U^\lambda(0)\|^2_{\tilde{H}^0} e^{C \nu (1 + \lambda^2) \int_0^T \|\tilde{u}^\lambda(1)(s)\|^2_{\tilde{H}^1} ds}$$

giving continuous dependence on the initial data. When \(U^\lambda(0) = 0\), this gives \(\|U^\lambda(t)\|_{\tilde{H}^0} = 0\) for all \(t > 0\) and proves path-wise uniqueness.

The path-wise continuous dependence on the initial data

$$\tilde{u}^\lambda(2) \to \tilde{u}^\lambda(1)$$

in \(C([0, T]; \tilde{H}^0)\) if \(x(2) \to x(1)\) in \(\tilde{H}^0\)

gives the Feller property in \(\tilde{H}^0\). Indeed, given a continuous function \(\psi : \tilde{H}^0 \to \mathbb{R}\) we have \(\mathbb{P}\text{-a.s. that } \psi(\tilde{u}^\lambda(2)(t)) \to \psi(\tilde{u}^\lambda(1)(t))\) as \(x(2) \to x(1)\) in \(\tilde{H}^0\); since \(\psi\) is bounded, by dominated convergence we get the convergence also when we take the mathematical expectation, i.e.

\[P_t^\lambda \psi(x(2)) = \mathbb{E}[\psi(\tilde{u}^\lambda(2)(t))] \to \mathbb{E}[\psi(\tilde{u}^\lambda(1)(t))] = P_t^\lambda \psi(x(1)).\]

We have a regularity result.

**Proposition 3.** Assume \(R(G) \subseteq H^{2+\varepsilon}\) for some \(\varepsilon > 0\). Then for arbitrary \(\tilde{x} \in \tilde{H}^1\), there exists a unique process \(\tilde{u}^\lambda\) solution to system 19 over \([0, T]\) with the initial condition \(\tilde{u}^\lambda(0) = \tilde{x}\), such that

$$\tilde{u}^\lambda \in C([0, T]; \tilde{H}^1) \cap L^4(0, T; \tilde{H}^2)$$

\(\mathbb{P}\text{-a.s.; it is a Markov and Feller process in } \tilde{H}^1.\)

**Proof.** By assumption we have \(\tilde{x} \in C([0, T]; \tilde{H}^2), \mathbb{P}\text{-a.s.}\). We study the regularity of \(\tilde{v}^\lambda\). We take the \(\tilde{H}^0\)-scalar product of equation 22 with \(\tilde{A}\tilde{v}^\lambda\). Then

$$\frac{1}{2} \frac{d}{dt} \|\tilde{v}^\lambda(t)\|^2_{\tilde{H}^1} + \nu \|\tilde{v}^\lambda(t)\|^2_{\tilde{H}^2} = -(\tilde{B}^\lambda(\tilde{v}^\lambda(t) + \tilde{z}(t), \tilde{v}^\lambda(t) + \tilde{z}(t), \tilde{A}\tilde{v}^\lambda(t))$$

$$\leq \|\tilde{B}^\lambda(\tilde{v}^\lambda(t) + \tilde{z}(t), \tilde{v}^\lambda(t) + \tilde{z}(t))\|_{\tilde{H}^0} \|\tilde{v}^\lambda(t)\|_{\tilde{H}^2}$$

We use the bilinearity, the estimates of Lemma 2.1 and the interpolation estimates:

$$\|\tilde{B}^\lambda(\tilde{v}^\lambda + \tilde{z}, \tilde{v}^\lambda + \tilde{z})\|_{\tilde{H}^0}$$

$$\leq C(1 + |\lambda|) (\|\tilde{v}^\lambda\|_{\tilde{H}^{1/2}} \|\tilde{v}^\lambda\|_{\tilde{H}^{3/2}} + \|\tilde{v}^\lambda\|_{\tilde{H}^{1/2}} \|\tilde{z}\|_{\tilde{H}^{3/2}} + \|\tilde{z}\|_{\tilde{H}^{3/2}})$$

$$\leq C(1 + |\lambda|) (\|\tilde{v}^\lambda\|_{\tilde{H}^{1/2}} \|\tilde{v}^\lambda\|_{\tilde{H}^{3/2}} + \|\tilde{v}^\lambda\|_{\tilde{H}^{1/2}} \|\tilde{v}^\lambda\|_{\tilde{H}^{1/2}} \|\tilde{z}\|_{\tilde{H}^{3/2}} + \|\tilde{z}\|_{\tilde{H}^{3/2}})$$

So, by Young inequality

$$\|\tilde{B}^\lambda(\tilde{v}^\lambda + \tilde{z}, \tilde{v}^\lambda + \tilde{z})\|_{\tilde{H}^0} \|\tilde{v}^\lambda\|_{\tilde{H}^2}$$

$$\leq C(1 + |\lambda|) (\|\tilde{v}^\lambda\|_{\tilde{H}^{1/2}} \|\tilde{v}^\lambda\|_{\tilde{H}^{3/2}} + \|\tilde{v}^\lambda\|_{\tilde{H}^{1/2}} \|\tilde{v}^\lambda\|_{\tilde{H}^{3/2}} \|\tilde{z}\|_{\tilde{H}^{3/2}} + \|\tilde{z}\|_{\tilde{H}^{3/2}}$$
\[ + \|\tilde{z}\|_{\dot{H}^\frac{3}{2}}^2 \|\tilde{v}^\lambda\|_{\dot{H}^\frac{3}{2}} \]
\[ \leq \frac{\nu}{2} \|\tilde{v}^\lambda\|_{\dot{H}^1}^2 + C_\nu (1 + \lambda^2) \left( \|\tilde{v}^\lambda\|_{\dot{H}^1}^2 + \|\tilde{z}\|_{\dot{H}^\frac{3}{2}}^2 \right) \]

Therefore, with \( g := \|\tilde{v}^\lambda\|_{\dot{H}^1}^2 + \|\tilde{z}\|_{\dot{H}^\frac{3}{2}}^2 \in L^1(0, T) \) by the previous Proposition, we have
\[ \frac{d}{dt} \|\tilde{v}^\lambda(t)\|_{\dot{H}^1}^2 + \nu \|\tilde{v}^\lambda(t)\|_{\dot{H}^1}^2 \leq C_\nu (1 + \lambda^2) g(t) \|\tilde{v}^\lambda(t)\|_{\dot{H}^1}^2 + C_\nu (1 + \lambda^2) \|\tilde{z}(t)\|_{\dot{H}^\frac{3}{2}}^4 \] (25)
and as before we get
\[ \sup_{0 \leq t \leq T} \|\tilde{v}^\lambda(t)\|_{\dot{H}^1}^2 \leq \|\tilde{x}\|_{\dot{H}^1}^2 e^{C_\nu (1 + \lambda^2) \int_0^T g(t)dt} + C_\nu (1 + \lambda^2) \int_0^T e^{C_\nu (1 + \lambda^2) \int_0^T g(t)dt} \|\tilde{z}(t)\|_{\dot{H}^\frac{3}{2}}^4 dt \]
and integrating in time 25 we get
\[ \nu \int_0^T \|\tilde{v}^\lambda(t)\|_{\dot{H}^1}^2 dt \leq \|\tilde{x}\|_{\dot{H}^1}^2 + C_\nu (1 + \lambda^2) \left[ \sup_{0 \leq t \leq T} \|\tilde{v}^\lambda(t)\|_{\dot{H}^1}^2 \int_0^T g(t)dt + \int_0^T \|\tilde{z}(t)\|_{\dot{H}^\frac{3}{2}}^4 dt \right]. \]

With these bounds on the \( L^\infty(0, T; \dot{H}^1) \) and \( L^2(0, T; \dot{H}^\frac{3}{2}) \)-norms, we can get estimates for the time derivative, i.e. \( \frac{d}{dt} \tilde{u}^\lambda \in L^2(0, T; \dot{H}^0) \), in order to conclude the proof as before.

Since \( \tilde{v}^\lambda \in C([0, T]; \dot{H}^1) \cap L^2(0, T; \dot{H}^\frac{3}{2}) \subset L^4(0, T; \dot{H}^\frac{3}{2}) \) and by Proposition 1 \( \tilde{z} \in C([0, T]; \dot{H}^\frac{3}{2}) \), we get that \( \tilde{u}^\lambda \in C([0, T]; \dot{H}^1) \cap L^4(0, T; \dot{H}^\frac{3}{2}) \).

The continuous dependence on the initial data is obtained as in the previous proposition. Let as consider two solutions with different initial data and the equation 24 for the difference. Then, with usual procedure and using Lemma 2.1
\[ \frac{1}{2} \frac{d}{dt} \|U^\lambda(t)\|_{\dot{H}^1}^2 + \nu \|U^\lambda(t)\|_{\dot{H}^1}^2 \]
\[ = -\langle \tilde{B}^\lambda(U^\lambda(t), \tilde{u}^\lambda(1)(t)), \tilde{U}^\lambda(t) \rangle \]
\[ \leq \|\tilde{B}^\lambda(U^\lambda(t), \tilde{u}^\lambda(1)(t))\|_{\dot{H}^1} \|\tilde{U}^\lambda(t)\|_{\dot{H}^1} \]
\[ \leq C(1 + |\lambda|) \left( \|U^\lambda(t)\|_{\dot{H}^\frac{3}{2}}^2 + \|\tilde{u}^\lambda(2)(t)\|_{\dot{H}^\frac{3}{2}} \right) \|U^\lambda(t)\|_{\dot{H}^2} \]
\[ \leq C(1 + |\lambda|) \left( \|\tilde{u}^\lambda(1)(t)\|_{\dot{H}^\frac{3}{2}} + \|\tilde{u}^\lambda(2)(t)\|_{\dot{H}^\frac{3}{2}} \right) \|U^\lambda(t)\|_{\dot{H}^2} \]
\[ \leq C(1 + |\lambda|) \left( \|\tilde{u}^\lambda(1)(t)\|_{\dot{H}^\frac{3}{2}} + \|\tilde{u}^\lambda(2)(t)\|_{\dot{H}^\frac{3}{2}} \right) \|U^\lambda(t)\|_{\dot{H}^2} \]
\[ \leq \frac{\nu}{2} \|U^\lambda(t)\|_{\dot{H}^2}^2 + C(1 + \lambda^2) \left( \|\tilde{u}^\lambda(1)(t)\|_{\dot{H}^\frac{3}{2}} + \|\tilde{u}^\lambda(2)(t)\|_{\dot{H}^\frac{3}{2}} \right) \|U^\lambda(t)\|_{\dot{H}^1}^2. \]

Using Gronwall’s lemma, with usual procedure we get
\[ \|\tilde{u}^\lambda(1) - \tilde{u}^\lambda(2)\|_{C([0, T]; \dot{H}^1)} \leq C \|\tilde{x}^{(1)} - \tilde{x}^{(2)}\|_{\dot{H}^1}, \]
for a suitable constant depending on \( T, \lambda, \nu \) and \( \|\tilde{u}^\lambda(i)\|_{L^4(0, T; \dot{H}^\frac{3}{2})}, i = 1, 2 \). This gives continuous dependence on the initial data and thus Feller property in \( \dot{H}^1 \). \( \square \)

The previous results are classical and we could have skipped the details of the proof, quoting previous results for stochastic 2D Navier-Stokes equations. We have given all the details in order to show that the technique used in previous papers for the stochastic 2D Navier-Stokes equations is successful also for our system 19.
With this remark in mind, we can now state another regularity result, based on the technique of [13], thanks to the properties 17-18 of the bilinear operator $\tilde{B}^\lambda$.

**Proposition 4.** Let $\alpha \in \mathbb{N}, \alpha \geq 2$.

Assume $\mathcal{R}(G) \subseteq \dot{H}^{\alpha+\varepsilon}$ for some $\varepsilon > 0$. Then for arbitrary $\tilde{x} \in \dot{H}^\alpha$, there exists a unique process $\tilde{u}^\lambda$ solution to system 19 over $[0, T]$ with the initial condition $\tilde{u}^\lambda(0) = \tilde{x}$, such that

$$\tilde{u}^\lambda \in C([0, T]; \dot{H}^\alpha)$$

$\mathbb{P}$-a.s.; it is a Markov and Feller process in $\dot{H}^\alpha$.

We recall that $\dot{H}^\alpha \subset |C(D)|^2$ for $\alpha > 1$. Therefore, by means of the latter Proposition we can define (for each fixed time and space) the quantity $\tilde{u}^\lambda(t, l\xi) - \tilde{u}^\lambda(t, 0)$ appearing in the structure functions (for $\lambda \neq 0$ and for $\lambda = 0$ as well).

### 3.2. Existence of invariant measures.

The existence of an invariant measure is obtained by means of Krylov-Bogoliubov method (see e.g. [10]). This requires Feller property in $\dot{H}^0$ and a tightness result. In particular, following Chow (see Theorem 2.2 in [6]) it is enough to prove the tightness for the time averages in the following form

$$\lim_{T \to \infty} \sup_{R > 0} \frac{1}{T} \int_0^T \mathbb{P}\{\|\tilde{u}^\lambda(t)\|_{\dot{H}^1} > R\} dt = 0.$$  

(26)

for some $T_0 > 0$. This provides that there exists an invariant measure with support in $\dot{H}^1$.

To prove it, we first show that the solution $\tilde{u}^\lambda$ on the time interval $[0, T]$ fulfills

$$\mathbb{E}\|\tilde{u}^\lambda(t)\|_{\dot{H}^\alpha}^2 + 2\nu \mathbb{E} \int_0^t \|\tilde{u}^\lambda(s)\|_{\dot{H}^\alpha}^2 \, ds = \|\tilde{x}\|_{\dot{H}^\alpha}^2 + 2\|G\|_{HS}^2 t$$  

(27)

for any $t \in (0, T]$. This can be done as in [26] (Theorem 1.2, Ch X), by means of Itô formula for $d\|\tilde{u}^\lambda(t)\|_{\dot{H}^\alpha}^2$; indeed, assuming $H0$ and using $\langle B^\lambda(\tilde{u}^\lambda, \tilde{u}^\lambda), \tilde{u}^\lambda \rangle = 0$ for $\tilde{u}^\lambda \in \dot{H}^1$, we get

$$\|\tilde{u}^\lambda(t)\|_{\dot{H}^\alpha}^2 + 2\nu \int_0^t \|\tilde{u}^\lambda(s)\|_{\dot{H}^\alpha}^2 \, ds = \|\tilde{x}\|_{\dot{H}^\alpha}^2 + 2\int_0^t \langle \tilde{u}^\lambda(s), \tilde{G}d\tilde{\beta}(s) \rangle + 2\|G\|_{HS}^2 t$$

and we can proceed as in [26] to obtain 27.

Now, taking the solution of 19 on the time interval $[0, T]$ with $\tilde{x} = 0$, relationship 27 becomes

$$\mathbb{E}\|\tilde{u}^\lambda(T)\|_{\dot{H}^\alpha}^2 + 2\nu \mathbb{E} \int_0^T \|\tilde{u}^\lambda(s)\|_{\dot{H}^\alpha}^2 \, ds = 2\|G\|_{HS}^2 T$$

giving

$$\frac{1}{T} \int_0^T \mathbb{E}\|\tilde{u}^\lambda(t)\|_{\dot{H}^\alpha}^2 \, dt \leq \frac{1}{\nu} \|G\|_{HS}^2.$$

Then, by Chebyshev inequality we have

$$\frac{1}{T} \int_0^T \mathbb{P}\{\|\tilde{u}^\lambda(t)\|_{\dot{H}^1} > R\} dt \leq \frac{1}{T} \int_0^T \mathbb{E}\|\tilde{u}^\lambda(t)\|_{\dot{H}^\alpha}^2 \, dt \leq \frac{1}{\nu} \frac{\|G\|_{HS}^2}{T}$$

proving 26.

Therefore we have proven the following result.

**Theorem 3.1.** Assume H0.

Then, there exists at least one invariant measure $\mu^\lambda$ for system 19. Moreover $\mu^\lambda(\dot{H}^1) = 1$. 

3.3. Uniqueness of invariant measures. In this section we prove that there exists a unique invariant measure $\mu^\lambda$ and its support is in $[C(D)]^2$, which is important to define the structure functions as we explained after Proposition 4. We actually prove that $\mu^\lambda(\hat{H}^2) = 1$ under suitable assumption on the covariance of the noise.

Uniqueness of the invariant measure can be proven by different methods (see [22]). Here we follow [10]: we fix $\alpha \geq 2$ (as in Proposition 4) and show that the Markov semigroup $\{P_t^\lambda\}_{t \geq 0}$ is irreducible and strongly Feller in $\hat{H}^\alpha$. By means of Khasminski and Doob theorems we get uniqueness of the invariant measure, which is strongly mixing and equivalent to all the transition functions.

Let us recall the definitions. Irreducibility in $\hat{H}^\alpha$ means that $P^\lambda(t, \tilde{x}, \Gamma) > 0$ for any $t > 0$, $\tilde{x} \in \hat{H}^\alpha$ and open non-empty subset $\Gamma$ of $\hat{H}^\alpha$. The Markov semigroup is strongly Feller in $\hat{H}^\alpha$ if $P_t^\lambda : B_0(\hat{H}^\alpha) \to C_b(\hat{H}^\alpha)$ for any $t > 0$.

We shall prove in the next two subsections the following result, for each fixed $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{N}, \alpha \geq 2$.

**Theorem 3.2.** Assume that the operator $G$ is injective and there exists $\varepsilon > 0$ such that

$$H^{\alpha+1} \subseteq \mathcal{R}(G) \subseteq H^{\alpha+\varepsilon}.$$  

Then the Markov semigroup $P_t^\lambda$ is irreducible and strongly Feller in $\hat{H}^\alpha$. Therefore there exists a unique invariant measure $\mu^\lambda$ of the equation 19; it is supported on $\hat{H}^\alpha$, it is equivalent to each transition probability $P^\lambda(t, \tilde{x}, \cdot)$ and

$$\lim_{t \to +\infty} P^\lambda(t, \tilde{x}, \Gamma) = \mu^\lambda(\Gamma) \quad \forall \Gamma \in B(\hat{H}^\alpha)$$  

for arbitrary $\tilde{x} \in \hat{H}^\alpha$.

We will obtain this result by merging Propositions 5 and 7. Notice that if $H^{\alpha+1} \subseteq \mathcal{R}(G) \subseteq H^{\alpha+\varepsilon}$ then $\mathcal{R}(G)$ is dense in $\hat{H}^\alpha$, since $H^{\alpha+1}$ is densely embedded into $H^\alpha$.

For example, our assumption is fulfilled if we choose $G = A^{-p}$ with $p > 1$; in this case $\mathcal{R}(G) = H^{2p}$ and therefore there exists an integer $\alpha \geq 2$ for which the assumptions of Theorem 3.2 are fulfilled. This gives a full noise with suitable space regularity. Otherwise, one could prove uniqueness of the invariant measure with a degenerate noise as in [20, 21].

We divide the proof into two parts; first, we consider irreducibility and then the strong Feller property.

3.3.1. Irreducibility. Irreducibility in $\hat{H}^\alpha$ means that, starting from any $\tilde{x} \in \hat{H}^\alpha$, there is a strictly positive probability to be at any time $t > 0$ in any non-empty open subset of $\hat{H}^\alpha$. It is enough to check it for any open ball $B(\tilde{y}, \rho) = \{\tilde{x} \in \hat{H}^\alpha : \|\tilde{x} - \tilde{y}\|_{\hat{H}^\alpha} < \rho\}$. Therefore we have to check that

$$P^\lambda(t, \tilde{x}, B(\tilde{y}, \rho)) \equiv \mathbb{P}\{\|u^\lambda(t; \tilde{x}) - \tilde{y}\|_{\hat{H}^\alpha} < \rho\} > 0$$

for any $\tilde{x}, \tilde{y} \in \hat{H}^\alpha$ and $t, \rho > 0$. We proceed as in [17, 12].

We define $\tilde{u}_s : [0, t] \to \hat{H}^\alpha$ linking $\tilde{x}$ to $\tilde{y}$ as

$$\tilde{u}_s(s) = \begin{cases} e^{-s\hat{A}\tilde{x}} & s \in [0, \frac{1}{4}t] \\ e^{-(t-s)\hat{A}\tilde{y}} & s \in \left[\frac{3}{4}t, t\right] \\ \tilde{u}_s\left(\frac{t}{4}\right) + \frac{s - \frac{t}{2}}{\frac{3}{4}t - \frac{t}{4}} \left(\tilde{u}_s\left(\frac{3}{4}t\right) - \tilde{u}_s\left(\frac{1}{4}t\right)\right) & s \in \left[\frac{1}{4}t, \frac{3}{4}t\right] \end{cases}$$
This belongs to $C([0, t]; \tilde{H}^\alpha)$. Then we consider the (unique) solution $\tilde{v}_\alpha^\lambda$ of
\[
\frac{d\tilde{v}_\alpha^\lambda}{dt}(t) + \nu A\tilde{v}_\alpha^\lambda(t) = -\tilde{B}^\lambda(\tilde{u}_*(t), \tilde{u}_*(t)); \quad \tilde{v}_\alpha^\lambda(0) = \tilde{x}
\]
By Lemma 2.1 the r.h.s. belongs to $C([0, t]; \tilde{H}^{\alpha-1})$. Then $\tilde{v}_\alpha^\lambda \in C([0, t]; \tilde{H}^\alpha) \cap L^2(0, t; \tilde{H}^{\alpha+1})$. Finally we set $\tilde{z}_\alpha^\lambda = \tilde{u}_* - \tilde{v}_\alpha^\lambda \in C([0, t]; \tilde{H}^\alpha)$ and thus the equation fulfilled by $\tilde{v}_\alpha^\lambda$ can be written also as
\[
\frac{d\tilde{v}_\alpha^\lambda}{dt} + \nu A\tilde{v}_\alpha^\lambda + \tilde{B}^\lambda(\tilde{v}_\alpha^\lambda + \tilde{z}_\alpha^\lambda, \tilde{v}_\alpha^\lambda + \tilde{z}_\alpha^\lambda) = 0; \quad \tilde{v}_\alpha^\lambda(0) = \tilde{x}
\]
Now we prove a continuous dependence of $\tilde{v}_\alpha^\lambda$ on $\tilde{z}$ in equation 22. This is a deterministic result and is proven for any integer $\alpha \geq 2$.

**Lemma 3.3.** We are given $\tilde{x} \in \tilde{H}^\alpha$ and $\tilde{z}_1, \tilde{z}_2 \in C([0, T]; \tilde{H}^\alpha)$. Let $\tilde{v}_\alpha^\lambda \in C([0, T]; \tilde{H}^\alpha)$ be the solution of
\[
\frac{d\tilde{v}_\alpha^\lambda}{dt}(t) + \nu A\tilde{v}_\alpha^\lambda(t) + \tilde{B}^\lambda(\tilde{v}_\alpha^\lambda(t) + \tilde{z}_\alpha^\lambda(t), \tilde{v}_\alpha^\lambda(t) + \tilde{z}_\alpha^\lambda(t)) = 0, \quad \tilde{v}_\alpha^\lambda(0) = \tilde{x}
\]
for $i = 1, 2$. Then, there exists a constant $C$ (depending on $T, \nu, \lambda, \|\tilde{v}_1^\alpha + \tilde{z}_1^\alpha\|^2_{\tilde{H}^\alpha}$ and $\|\tilde{v}_2^\alpha + \tilde{z}_2^\alpha\|^2_{\tilde{H}^\alpha}$) such that
\[
\|\tilde{v}_\alpha^\lambda - \tilde{v}_\alpha^\lambda\|^2_{C([0, T]; \tilde{H}^\alpha)} \leq C\|\tilde{z}_1 - \tilde{z}_2\|^2_{C([0, T]; \tilde{H}^\alpha)}.
\]

**Proof.** Set $\tilde{V}^\lambda = \tilde{v}_\alpha^\lambda - \tilde{v}_\alpha^\lambda$ and $\tilde{Z} = \tilde{z}_1 - \tilde{z}_2$. Then $\tilde{V}^\lambda$ satisfies
\[
\frac{d\tilde{V}^\lambda}{dt}(t) + \nu A\tilde{V}^\lambda(t) + \tilde{B}^\lambda(\tilde{V}^\lambda(t) + \tilde{Z}(t), \tilde{V}^\lambda(t) + \tilde{Z}(t)) \\
+ \tilde{B}^\lambda(\tilde{v}_\alpha^\lambda(t) + \tilde{z}_\alpha^\lambda(t), \tilde{v}_\alpha^\lambda(t) + \tilde{z}_\alpha^\lambda(t)) = 0 \quad (29)
\]
with $\tilde{V}^\lambda(0) = 0$. We multiply this equation by $\tilde{A}^\alpha \tilde{V}^\lambda(t)$ and integrate on the domain:
\[
\frac{1}{2} \frac{d}{dt}\|\tilde{V}^\lambda(t)\|^2_{\tilde{H}^\alpha} \quad + \nu\|\tilde{V}^\lambda(t)\|^2_{\tilde{H}^\alpha+1} \\
= -\langle \tilde{A}^{\frac{\alpha-1}{2}} \tilde{B}(\tilde{V}^\lambda(t) + \tilde{Z}(t), \tilde{v}_\alpha^\lambda(t) + \tilde{z}_\alpha^\lambda(t)), \tilde{A}^{\frac{\alpha+1}{2}} \tilde{V}^\lambda(t) \rangle \\
- \langle \tilde{A}^{\frac{\alpha-1}{2}} \tilde{B}(\tilde{v}_\alpha^\lambda(t) + \tilde{z}_\alpha^\lambda(t), \tilde{V}^\lambda(t) + \tilde{Z}(t)), \tilde{A}^{\frac{\alpha+1}{2}} \tilde{V}^\lambda(t) \rangle.
\]
We estimate the bilinear terms by means of Lemma 2.1 iii)
\[
\|\tilde{B}^\lambda(\tilde{V}^\lambda + \tilde{Z}, \tilde{v}_\alpha^\lambda + \tilde{z}_\alpha^\lambda) + \tilde{B}^\lambda(\tilde{v}_\alpha^\lambda + \tilde{z}_1, \tilde{V}^\lambda + \tilde{Z})\|_{\tilde{H}^{\alpha-1}} \\
\leq C \left[\|\tilde{V}^\lambda + \tilde{Z}\|_{\tilde{H}^\alpha} + \|\tilde{v}_\alpha^\lambda + \tilde{z}_\alpha^\lambda\|_{\tilde{H}^\alpha}\right] \|\tilde{V}^\lambda + \tilde{Z}\|_{\tilde{H}^\alpha}
\]
so that the r.h.s. is bounded by
\[
C \left[\|\tilde{V}^\lambda + \tilde{Z}\|_{\tilde{H}^\alpha} + \|\tilde{v}_\alpha^\lambda + \tilde{z}_\alpha^\lambda\|_{\tilde{H}^\alpha}\right] \|\tilde{V}^\lambda + \tilde{Z}\|_{\tilde{H}^\alpha}.
\]
By Young inequality this is bounded by
\[
\frac{\nu}{2}\|\tilde{V}^\lambda\|^2_{\tilde{H}^\alpha} + C\nu \left[\|\tilde{V}^\lambda + \tilde{Z}\|_{\tilde{H}^\alpha} + \|\tilde{v}_\alpha^\lambda + \tilde{z}_\alpha^\lambda\|_{\tilde{H}^\alpha}\right] \|\tilde{V}^\lambda + \tilde{Z}\|_{\tilde{H}^\alpha}.
\]
Therefore, setting $\phi = \|\tilde{v}_\alpha^\lambda + \tilde{z}_1\|^2_{\tilde{H}^\alpha} + \|\tilde{v}_\alpha^\lambda + \tilde{z}_2\|^2_{\tilde{H}^\alpha} \in C([0, T])$, we obtain
\[
\frac{d}{dt}\|\tilde{V}^\lambda(t)\|^2_{\tilde{H}^\alpha} \leq C\phi(t)\|\tilde{V}^\lambda(t)\|^2_{\tilde{H}^\alpha} + C\phi(t)\|\tilde{Z}(t)\|^2_{\tilde{H}^\alpha}.
\]
Since $\tilde{V}^\lambda(0) = 0$, Gronwall lemma gives the required result. \(\square\)
Since $\tilde{u}^\lambda - \tilde{u}_* = \tilde{u}^\lambda - \tilde{u}_*^\lambda + \tilde{z} - \tilde{z}_*^\lambda$, from the triangle inequality and the latter lemma it follows that there exists a constant $\tilde{C}$ such that
\[
\|\tilde{u}^\lambda - \tilde{u}_*\|_{C([0,t];\tilde{H}^\alpha)} \leq \tilde{C}\|\tilde{z} - \tilde{z}_*^\lambda\|_{C([0,t];\tilde{H}^\alpha)}.
\]

Now we come back to estimate $\mathbb{P}\{\|\tilde{u}^\lambda(t; \tilde{u}_*^\lambda(0) = \tilde{x}) - \tilde{y}\|_{\tilde{H}^\alpha} < \rho\}$. Since the initial data is always $\tilde{x}$, in the sequel we drop it for simplicity. We have
\[
\{\|\tilde{u}^\lambda(t) - \tilde{y}\|_{\tilde{H}^\alpha} < \rho\} = \{\|\tilde{u}^\lambda(t) - \tilde{u}_*(t)\|_{\tilde{H}^\alpha} < \rho\}
\supseteq \{\|\tilde{u}^\lambda - \tilde{u}_*\|_{C([0,t];\tilde{H}^\alpha)} < \rho\}
\supseteq \{\|\tilde{z} - \tilde{z}_*^\lambda\|_{C([0,t];\tilde{H}^\alpha)} < \frac{\rho}{\tilde{C}}\}
\]
(30)

where $\tilde{u}^\lambda, \tilde{z}$ are processes and $\tilde{u}_*, \tilde{z}_*^\lambda$ are deterministic functions. We want to show that the latter term is strictly positive. Properties of the Ornstein-Uhlenbeck process $\tilde{z}$ have been given in Proposition 1. If we assume in addition that $\mathcal{R}(G)$ is dense in $\tilde{H}^\alpha$, then by Proposition 2.7. in [23] we have that the closure of the support law of is $C_0([0,t];\tilde{H}^\alpha)$ (when the subscript 0 denotes that the initial value vanishes). Therefore the law of the process $\tilde{z}$ is a full measure in $C_0([0,t];\tilde{H}^\alpha)$, i.e.
\[
\mathbb{P}\{\|\tilde{z} - \tilde{\zeta}\|_{C([0,t];\tilde{H}^\alpha)} < \rho\} > 0
\]
for any $\tilde{\zeta} \in C_0([0,t];\tilde{H}^\alpha)$ and $\rho > 0$. Keeping in mind 30 we obtain
\[
\mathbb{P}\{\|\tilde{u}^\lambda(t) - \tilde{y}\|_{\tilde{H}^\alpha} < \rho\} > 0.
\]

Summing up, for any $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{N}, \alpha \geq 2$ we have proved

**Proposition 5.** Assume $\mathcal{R}(G) \subseteq H^{\alpha+\varepsilon}$ for some $\varepsilon > 0$ and $\mathcal{R}(G)$ dense in $H^\alpha$. Given any $\tilde{x}, \tilde{y} \in \tilde{H}^\alpha$, $t > 0$ and $\rho > 0$ we have
\[
P^\lambda(t, \tilde{x}, B(\tilde{y}, \rho)) > 0
\]
i.e. the Markov semigroup $P^\lambda_t$ is irreducible in $\tilde{H}^\alpha$.

3.3.2. **Strong feller.** The second property of the Markov semigroup we have to check is the strongly Feller property in $\tilde{H}^\alpha$, i.e. $P^\lambda_t : B_b(\tilde{H}^\alpha) \rightarrow C_b(\tilde{H}^\alpha)$ for any $t > 0$.

We already proved that $\{P^\lambda_t\}_{t \geq 0}$ is Feller in $\tilde{H}^\alpha$. By the mean value theorem, we would get that it is Lipschitz Feller if we were able to estimate the derivative of $P^\lambda_t \psi$. This is not true, but as in [13] we can prove it for a modified version of 19
\[
\begin{aligned}
d\tilde{u}^{\lambda,(R)}(t) + \tilde{A}\tilde{u}^{\lambda,(R)}(t)dt + &\quad \Theta_R(\|\tilde{u}^{\lambda,(R)}(t)\|_{\tilde{H}^\alpha})\tilde{B}^{\lambda}(\tilde{u}^{\lambda,(R)}(t), \tilde{u}^{\lambda,(R)}(t)) dt = \tilde{G}d\tilde{\beta}(t)
\end{aligned}
\]
(31)

where cut-off function $\Theta_R$ is a $C^\infty$ function equal to 1 in $[-R,R]$ and 0 outside $[-R-1, R+1]$.

By means of Bismut-Elworthy-Li’s formula, we prove that the Markov semigroup associated to 31 is Lipschitz Feller in $\tilde{H}^\alpha$.

**Proposition 6.** Assume that for some $\alpha \in \mathbb{N}$ with $\alpha \geq 2$ the operator $G$ is injective with $H^{\alpha+\varepsilon} \subseteq \mathcal{R}(G) \subseteq H^{\alpha+2\varepsilon}$ for some $\varepsilon > 0$.

Then, for every $\lambda \in \mathbb{R}$ and $t, R > 0$ there exists a constant $L = L(\lambda, R, t)$ such that
\[
\left| P^\lambda_t(\psi(\tilde{x}) - P^\lambda_t(\psi(\tilde{y})) \right| \leq L \|\tilde{x} - \tilde{y}\|_{\tilde{H}^\alpha}
\]
for all $\tilde{x}, \tilde{y} \in \tilde{H}^\alpha, \psi \in C_b(\tilde{H}^\alpha)$ with $\|\psi\|_b \leq 1$.

Moreover, $P^\lambda_t(\psi)$ is Lipschitz continuous for arbitrary $\psi \in B_b(\tilde{H}^\alpha)$. 

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The proof is the same as in [17, 13], based on the properties of the bilinear operator $B^*$ given in Lemma 2.1.

Moreover, from the estimates of Section 3.1 and Proposition 3.3, we have that $u^\lambda, u^{\lambda,(R)} \in C([0, T]; \dot{H}^\alpha)$ $P$-a.s., and
\[
\sup_{\|\tilde{x}\|_{\dot{H}^\alpha} \leq M} \sup_{0 \leq t \leq T} \|u^{\lambda,(R)}(t, \tilde{x})\|_{\dot{H}^\alpha} < \infty,
\]
Thus the processes $u^\lambda(\cdot, \tilde{x})$ and $u^{\lambda,(R)}(\cdot, \tilde{x})$ coincide in the ball $B_R := \{v : \|v\|^2_{\dot{H}^\alpha} \leq R\}$. Therefore one proves that
\[
\lim_{R \to \infty} \|P^{\lambda,(R)}(t, \tilde{x}, \cdot) - P^\lambda(t, \tilde{x}, \cdot)\|_{\text{var}} = 0 \tag{32}
\]
uniformly with respect to $\tilde{x}$ in bounded sets of $\dot{H}^\alpha$, where $\|\cdot\|_{\text{var}}$ denotes the total variation norm of a measure.

Now, passing to the limit as $R \to \infty$, we obtain the strong Feller property for the principal equation 19. Indeed,
\[
\|P^\lambda(t, \tilde{x}, \cdot) - P^\lambda(t, \tilde{y}, \cdot)\|_{\text{var}} \leq \|P^\lambda(t, \tilde{x}, \cdot) - P^{\lambda,(R)}(t, \tilde{x}, \cdot)\|_{\text{var}}
+ \|P^{\lambda,(R)}(t, \tilde{x}, \cdot) - P^{\lambda,(R)}(t, \tilde{y}, \cdot)\|_{\text{var}} + \|P^{\lambda,(R)}(t, \tilde{y}, \cdot) - P^\lambda(t, \tilde{y}, \cdot)\|_{\text{var}}
\]
We fix any $\varepsilon > 0$. From 32, there exists $R_\varepsilon > 0$ such that
\[
\|P^{\lambda,(R_\varepsilon)}(t, \tilde{x}, \cdot) - P^{\lambda,(R_\varepsilon)}(t, \tilde{y}, \cdot)\|_{\text{var}} < \varepsilon,
\]
\[
\|P^{\lambda,(R_\varepsilon)}(t, \tilde{x}, \cdot) - P^{\lambda,(R_\varepsilon)}(t, \tilde{y}, \cdot)\|_{\text{var}} < \varepsilon.
\]
On the other hand, from Proposition 6 there exists $\delta_\varepsilon > 0$ such that
\[
\|P^{\lambda,(R_\varepsilon)}(t, \tilde{x}, \cdot) - P^{\lambda,(R_\varepsilon)}(t, \tilde{y}, \cdot)\|_{\text{var}} < \delta_\varepsilon
\]
for all $\tilde{x}, \tilde{y}$ with $\|\tilde{x} - \tilde{y}\|_{\dot{H}^\alpha} < \delta_\varepsilon$.

Thus, given any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that
\[
\|P^\lambda(t, \tilde{x}, \cdot) - P^\lambda(t, \tilde{y}, \cdot)\|_{\text{var}} < 3\varepsilon
\]
for all $\tilde{x}, \tilde{y}$ with $\|\tilde{x} - \tilde{y}\|_{\dot{H}^\alpha} < \delta_\varepsilon$. Therefore we have proven the strong Feller property.

**Proposition 7.** Assume that for some $\alpha \in \mathbb{N}$ with $\alpha \geq 2$ the operator $G$ is injective and $H^{\alpha+1} \subseteq \mathcal{R}(G) \subseteq H^{\alpha+\varepsilon}$ for some $\varepsilon > 0$.

Then, $P^\lambda_t$ is strong Feller in $\dot{H}^\alpha$.

### 4. Continuous dependence of the invariant measure wrt to the parameter $\lambda$.

In this section we show the continuous dependence of the invariant measure $\mu^\lambda$ on the parameter $\lambda$. In particular we are interested in the case of $\lambda \to 0$.

We have a first result. Let us fix the family of the unique invariant measures $\mu^\lambda$ (given in Section 3), and consider the limit when $\lambda \to 0$.

**Proposition 8.** The family of invariant measures $\{\mu^\lambda\}_{\lambda \neq 0}$ is tight in $\dot{H}^{1-\varepsilon}$ for any $\varepsilon > 0$. Therefore there exists a measure $m$ on $H^{1-\varepsilon}$ and a sequence $\{\mu^{\lambda_n}\}_{n \in \mathbb{N}}$ (with $\lambda_n \to 0$ as $n \to \infty$) weakly converging to $m$ in $\dot{H}^{1-\varepsilon}$, i.e.
\[
\lim_{n \to \infty} \int f d\mu^{\lambda_n} = \int f dm \quad \forall f \in C_b(\dot{H}^{1-\varepsilon}).
\]
Finally, the supports of $m$ and $\mu^{\lambda_n}$ are contained in $\dot{H}^1$. 
Proposition 9. The family \( \{ \tilde{u}^{\lambda}_{st} \}_{\lambda \in \mathbb{R}} \) of stationary processes solving 19 is tight in \( L^2_{loc}(0, \infty; \tilde{H}^0) \cap C([0, \infty); \tilde{H}^{-1}) \).

Thus there exists a new probability basis \((\tilde{\Omega}, \tilde{F}, \tilde{P})\), a sequence \( \{ \tilde{u}^{\lambda_n}_{st} \} \) of stationary processes and a limit process \( \tilde{u}^{0}_{st} \) defined on it with values in \( L^2_{loc}(0, \infty; \tilde{H}^0) \cap C([0, \infty); \tilde{H}^{-1}) \) and solving 19 with parameter \( \lambda_n \) and 0 respectively, such that \( \tilde{u}^{\lambda_n}_{st} \) and \( \tilde{u}^{\lambda_n}_{st} \) have the same law and

\[
\lim_{n \to \infty} \tilde{u}^{\lambda_n}_{st} = \tilde{u}^{0}_{st} \quad \text{in} \quad \tilde{H}^0 \cap C([0, \infty); \tilde{H}^{-1}) \quad \tilde{P} - a.s.
\]

Finally, the process \( \tilde{u}^{0}_{st} \) is a stationary process in \( \tilde{H}^0 \).

In particular, for any \( t \geq 0 \) we have

\[
\lim_{n \to \infty} \tilde{u}^{\lambda_n}_{st}(t) = \tilde{u}^{0}_{st}(t) \quad \text{in} \quad \tilde{H}^{-1} \quad \tilde{P} - a.s.
\]

Since the law of \( \tilde{u}^{\lambda_n}_{st}(t) \) is \( \mu^{\lambda_n} \) and the law of \( \tilde{u}^{0}_{st}(t) \) is \( \mu^0 \), we have that

\[
\lim_{n \to \infty} \mu^{\lambda_n} = \mu^0 \quad \text{weakly in} \quad \tilde{H}^{-1}.
\]

Since there exists a unique invariant measure for the system 6 with \( \lambda = 0 \), we get that any sequence extracted from \( \{ \mu^{\lambda} \}_{\lambda \in \mathbb{R}} \) weakly converges to \( \mu^0 \) in \( \tilde{H}^{-1} \) as \( \lambda \to 0 \).

Bearing in mind Proposition 8, this identifies \( m \) with \( \mu^0 \) as measures on Borelian sets of \( \tilde{H}^{-1} \). Since we know that both \( m \) and \( \mu^0 \) are supported on \( \tilde{H}^1 \) indeed, we get that \( m = \mu^0 \) as measures on Borelian subsets of \( \tilde{H}^1 \).

Theorem 4.1. Let \( \varepsilon > 0 \) be given. For any sequence \( \{ \mu^{\lambda_n} \}_{n \in \mathbb{N}} \) (with \( \lambda_n \to 0 \) as \( n \to \infty \)) we have

\[
\lim_{n \to \infty} \mu^{\lambda_n} = \mu^0 \quad \text{weakly in} \quad \tilde{H}^{1-\varepsilon}.
\]
4.1. Convergence of stationary solutions. We now prove Proposition 9. For simplicity we drop the subindex and denote by \( u^\lambda \) the stationary solution of 19 whose marginal at any fixed time is the unique invariant measure \( \mu^\lambda \).

The proof is based on two steps: first we show that the sequence of laws of \( \tilde{u}^\lambda \), \( \lambda > 0 \), is tight; then we pass to the limit in a suitable way and get that the limit process is a weak solution of system 6 (in the probabilistic sense).

Let us recall some of the estimates performed in Section 3 by means of the Itô formula: for \( t \geq 0 \)
\[
\|\tilde{u}^\lambda(t)\|_{H^0}^2 + 2\nu \int_0^t \|\tilde{u}^\lambda(s)\|_{H^1}^2 ds = \|\tilde{u}^\lambda(0)\|_{H^0}^2 + 2 \int_0^t (\tilde{u}^\lambda(s), \tilde{G}d\tilde{\beta}(s)) + 2\|G\|^2_{HS} t.
\]
(35) Now, using the Burkholder-Davis-Gundy inequality and taking the expected values yields a uniform estimate with respect to \( \lambda \), that is
\[
\sup_{\lambda \in \mathbb{R}} \left[ \mathbb{E} \sup_{0 \leq t \leq T} \|\tilde{u}^\lambda(t)\|_{H^0}^2 + \nu \mathbb{E} \int_0^T \|\tilde{u}^\lambda(s)\|_{H^1}^2 ds \right] \leq C.
\]

Now, we write equation 19 in the integral form
\[
\dot{\tilde{u}}^\lambda(t) = \tilde{u}^\lambda(0) - \int_0^t [\nu \tilde{A}\tilde{u}^\lambda(s) + \tilde{B}^\lambda (\tilde{u}^\lambda(s), \tilde{u}^\lambda(s))] ds + \tilde{G}\tilde{\beta}(t), \quad t > 0.
\]
We have from Lemma 2.1 and using an interpolation estimate
\[
\int_0^T \|\tilde{B}^\lambda(\tilde{u}^\lambda(s), \tilde{u}^\lambda(s))\|_{\dot{H}^{-1}}^2 ds \leq C^2(1 + |\lambda|)^2 \int_0^T \|\tilde{u}^\lambda(s)\|^4_{H^2} ds
\]
\[
\leq C^2(1 + |\lambda|)^2 \int_0^T \|\tilde{u}^\lambda(s)\|^2_{H^0} \|\tilde{u}^\lambda(s)\|^2_{H^1} ds
\]
\[
\leq C^2(1 + |\lambda|)^2 \|\tilde{u}^\lambda\|^2_{L^\infty(0,T;\dot{H}^0)} \|\tilde{u}^\lambda\|^2_{L^2(0,T;\dot{H}^1)}
\]
Therefore, by usual estimations (see, e.g., [16]) we get that there exist constants \( C \) such that
\[
\sup_{\lambda \in \mathbb{R}} \mathbb{E} \int_0^T \tilde{A}\tilde{u}^\lambda(s) ds \|_{W^{1,2}(0,T;\dot{H}^{-1})} \leq C
\]
\[
\sup_{|\lambda| \leq 1} \mathbb{E} \int_0^T \tilde{B}^\lambda (\tilde{u}^\lambda(s), \tilde{u}^\lambda(s)) ds \|_{W^{1,2}(0,T;\dot{H}^{-1})} \leq C
\]
\[
\mathbb{E}\|\tilde{G}\tilde{\beta}(t)\|^2_{W^{1,2}(0,T;\dot{H}^0)} \leq C(\alpha)
\]
for all \( \alpha \in (0, \frac{1}{2}) \). Therefore, for any finite \( T \)
\[
\sup_{|\lambda| \leq 1} \mathbb{E}\|\tilde{u}^\lambda\|^2_{W^{1,2}(0,T;\dot{H}^{-1})} < \infty.
\]
On the other hand, we already know from the previous estimate above that
\[
\sup_\lambda \mathbb{E}\|\tilde{u}^\lambda\|^2_{L^2(0,T;\dot{H}^1)} < \infty.
\]
Using that the space \( L^2(0,T;\dot{H}^1) \cap W^{1,2}(0,T;\dot{H}^{-1}) \) is compactly embedded in \( L^2(0,T;\dot{H}^0) \) and in \( C([0,T];\dot{H}^{-1}) \) (see [26] Ch IV, Theorem 4.1), it follows that the sequence of laws of processes \( \{u^\lambda\}_\lambda \) is tight in \( L^2(0,T;\dot{H}^0) \cap C([0,T];\dot{H}^{-1}) \).

With the usual procedure (see, e.g., [16], we get that the sequence of laws of processes \( \{u^\lambda\}_\lambda \) is tight in \( L^2_{loc}(0,\infty;\dot{H}^0) \cap C([0,\infty);\dot{H}^{-1}) \).
From Prokhorov and Skorohod theorems follows that there exists a basis $(\bar{\Omega}, \bar{F}, \bar{P})$ (with expectation $\bar{E}$), and on this basis, $L^2_{loc}(0, \infty; \bar{H}^0) \cap C([0, \infty); \bar{H}^{-1})$-valued random variables $\bar{u}^0, \bar{u}^\lambda$, such that $\mathcal{L}(\bar{u}^\lambda) = \mathcal{L}(\bar{u}^0)$ and for each finite $T$

$$\lim_{\lambda_n \to 0} \bar{u}^\lambda = \bar{u}^0 \quad \text{in} \quad L^2(0, T; \bar{H}^0) \cap C([0, T]; \bar{H}^{-1}) \quad \bar{P} - \text{a.s.} \quad (36)$$

Moreover, each process $\bar{u}^\lambda$ satisfies the same estimates as $\bar{u}^0$ since they have the same law; hence

$$\sup_{\lambda \in \mathbb{R}} \left[ \bar{E} \sup_{0 \leq t \leq T} \| \bar{u}^\lambda(t) \|^2_{\bar{H}^0} + \nu \bar{E} \int_0^T \| \bar{u}^\lambda(s) \|^2_{\bar{H}^1} ds \right] \leq C.$$  

The fact that the limit process $\bar{u}^0$ solves system 6 follows by passing to the limit on the system 19, see [16] and [4].

In addition, $\bar{u}^\lambda$ and $\bar{u}^0$ have the same law; then $\bar{u}^\lambda$ is a stationary process. By the convergence $\bar{P}$-a.s. in $C([0, \infty); \bar{H}^{-1})$ we get that also $\bar{u}^0$ is a stationary process in $\bar{H}^{-1}$.

Finally, from the estimate above, we have that

$$\bar{u}^0 \in L^\infty(0, T; \bar{H}^0) \quad \bar{P} - \text{a.s.}$$

Then, for $T < \infty$ almost each path $\bar{u}^0 \in C([0, T]; \bar{H}^{-1}) \cap L^\infty(0, T; \bar{H}^0)$; thus it is weakly continuous in $\bar{H}^0$, i.e. we have for any $\phi \in \bar{H}^0$

$$\lim_{t \to t_0} \int_D \bar{u}^0(t) \phi \, dx = \int_D \bar{u}^0(t_0) \phi \, dx \quad \bar{P} - \text{a.s.}$$

and for any $t \in [0, T]$

$$\| \bar{u}^0(t) \|_{\bar{H}^0} \leq \| \bar{u}^0 \|_{L^\infty(0, T; \bar{H}^0)} \quad \bar{P} - \text{a.s.}$$

(see [24] p 263).

Hence, for every $t \geq 0$, the mapping $\bar{\omega} \mapsto \bar{u}^0(t, \bar{\omega})$ is well defined from $\bar{\Omega}$ to $\bar{H}^0$ and it is weakly measurable. Since $\bar{H}^0$ is a separable Banach space, it is strongly measurable (see [27] p 131). Therefore, it is meaningful to speak about the law of $\bar{u}^0(t)$ in $\bar{H}^0$. The stationarity of $\bar{u}^0$ in $\bar{H}^0$ has to be understood in this sense.  

5. Conclusions. In this paper, we investigated the statistics of a nonlinear model, the stochastic Navier-Stokes system 1 versus its linear counterpart given by the stochastic passive scalar equation 2. We coupled them by introducing a parameter $\lambda \in \mathbb{R}$ and obtained a joint system 9. After rescaling the joint system, we can get a symmetric system 10. Moreover the system being symmetric implies that the averages computed on each component of the system are the same. These averages are computed with respect to the invariant measure of the system.

The main goal of the paper was to study the existence, uniqueness of invariant measures for system 9 and their properties with respect to the parameter $\lambda$ in particular its continuous dependence when $\lambda \to 0$.

We proved that the joint system 9 has a unique, ergodic invariant measure $\mu^\lambda$ for any $\lambda \in \mathbb{R}$. Then, when $\lambda \to 0$, we proved that $\mu^\lambda \to \mu^0$ where $\mu^0$ is the unique invariant measure of joint system 9 for $\lambda = 0$ which is the joint system 6. As a consequence, the statistical properties obtained for the symmetric system 10 translate to the joint system 6. More precisely the statistical properties of 1 are similar to 2 but are simpler to compute. All our results are given for a non degenerate noise but can be extended for a degenerate noise.
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