On transversality-type and regularity-type properties

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Abstract

We systematically investigate (sub)transversality, metric (sub)regularity and their relations, focusing on the primal characterizations. Our approach is different, since we work with sets instead of mappings and we do not make use of variational principles. This enables us to obtain new characterizations and most of the classical results in the field in a unified way as easy consequences. Moreover, we answer a question of A. Ioffe about finding a metric characterization of intrinsic transversality by showing its “almost” equivalence to tangential transversality.

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1 Introduction

Transversality is a classical concept of mathematical analysis and differential topology. Recently, it has proven to be useful in other research areas as well. As it is stated in [14], the transversality-oriented language is extremely natural and convenient in some parts of variational analysis, including subdifferential calculus and nonsmooth optimization, as well as in proving sufficient conditions for linear convergence of the alternating projections algorithm (cf. [10]).

There is no need to prove the importance of the metric regularity property in variational analysis – it is already a central concept, whose roots go back to classical results such as the implicit function theorem, Banach open mapping theorem and the theorems of Lyusternik and Graves.

In this paper, we systematically investigate transversality/subtransversality, metric regularity/subregularity and their relations, focusing on the primal characterizations of these properties. Moreover, we answer a question of A. Ioffe for finding a metric characterization of intrinsic transversality (Remark 6.1 in [14]) and some of the open questions from [4].

Transversality and metric regularity have been widely studied in the last decades – see e.g. the recent books [8] and [15]. The equivalence of transversality of sets and regularity of some associated maps is established 20 years ago in [12] and [13]. The other way around – reducing regularity properties to the corresponding transversality properties of two sets – is examined recently in [6] and [5]. Although it is known that the transversality-type and regularity-type properties are equivalent in some sense, we have not seen this clearly stated and properly exploited anywhere in the literature known to us.

In the considerations in this paper, the primal properties are transversality-type (not regularity-type as usual in the literature) and using them, we obtain regularity-type characterizations. We think that our approach is more straightforward and consistent as we work with sets, not mappings. The main technical result (Lemma 3.4) is formulated exactly to serve our needs and generality is not pursued. There are many similar assertions in the literature and their proofs all rely on variational principles (Ekeland variational principle or see e.g. [2], [3] or [16] for alternatives). Our result could be proved using them, but we prefer to prove it using transfinite induction. Its proof may seem longer, because it essentially contains the proof of the Ekeland variational principle, but in fact it is really direct to employ. In our understanding this kind of argument is natural and eliminates the need for
seeking for the “right” function in every particular case. A simple induction enables the transition to a global property from a local one in a straightforward manner.

We find new primal characterizations of transversality and subtransversality and use them to characterize metric regularity and subregularity. Our approach enables us to obtain most of the classical results in the field in a unified way as easy consequences.

Another type of criteria for metric (sub)regularity of set-valued mappings are those formulated in terms of slopes of suitable single-valued functions. This approach is initiated by Ioffe in [13] and fully developed in [1] (see also [20] for metric subregularity). In this paper we do not focus on such relations, but we characterize the transversality-type properties using the slopes of a suitable function. Primal characterizations of transversality-type properties and, respectively, their characterizations via the slope of the coupling function, help understand the exact relation between them.

In this way we obtain a metric characterization of intrinsic transversality and clarify the relationship

\[
\text{transversality} \implies \text{tangential transversality} \implies \text{intrinsic transversality} \implies \text{subtransversality}.
\]

This is done by showing the “almost” equivalence of intrinsic transversality and tangential transversality. Intrinsic transversality is introduced in [9] and [10] as a sufficient condition for local linear convergence of the alternating projections algorithm in finite dimensions, while tangential transversality is introduced in [4] as a sufficient condition for nonseparation of sets, tangential intersection properties and a Lagrange multiplier rule.

The paper is organized as follows: Some basic relations between subtransversality and subregularity and between transversality and regularity are obtained in Section 2. In Section 3 we state and prove our main technical result and use it to characterize subtransversality and subregularity in primal terms. Primal characterizations of transversality and regularity are obtained in Section 4. We provide characterizations of transversality-type properties in terms of the coupling function in Section 5. The “almost” equivalence of intrinsic transversality and tangential transversality is also shown in this section.

Throughout the paper if \((X, d)\) is a metric space, \(B_r(x_0)\) will denote the
open ball centered at \( x_0 \) with radius \( r \): \( B_r(x_0) := \{ x \in Y \mid d(x, x_0) < r \} \). The closed ball will be denoted by \( \overline{B}_r(x_0) \).

In what follows, for given metric spaces \( X \) and \( Y \), \( F \) is called a set-valued map between \( X \) and \( Y \), denoted by \( F : X \rightrightarrows Y \), if \( F : X \to 2^Y \). The graph of \( F \), denoted by \( \text{Gr} F \), is defined by

\[
\text{Gr} F := \{(x, y) \in X \times Y \mid y \in F(x)\}.
\]

The inverse map of \( F \), \( F^{-1} : Y \rightrightarrows X \) is defined by

\[
F^{-1}(y) := \{ x \in X \mid y \in F(x) \}, \text{ whenever } y \in Y.
\]

We endow the Cartesian product \( X \times Y \) of the metric spaces \( X \) and \( Y \), with the metric \( d((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2) \) for the sake of simplicity. The particular choice of the metric is relevant only to the constants involved. However, our goal in this paper is to derive qualitative results, so that we are not concerned with the constants.

## 2 Basic relations between (sub)transversality and (sub)regularity

In this section we obtain some rather straightforward but nevertheless important relations between subtransversality and subregularity and between transversality and regularity. We begin with the already classical definitions.

**Definition 2.1.** Let \( X \) and \( Y \) be metric spaces, \( F : X \rightrightarrows Y \) and \( (\bar{x}, \bar{y}) \in \text{Gr} F \). We say that \( F \) is metrically regular at \( (\bar{x}, \bar{y}) \) if there exist \( K > 0 \) and \( \delta > 0 \) such that for all \( x \in B_\delta(\bar{x}) \) and all \( y \in B_\delta(\bar{y}) \) the following inequality holds:

\[
d(x, F^{-1}(y)) \leq K d(y, F(x)).
\]

**Definition 2.2.** Let \( X \) and \( Y \) be metric spaces, \( F : X \rightrightarrows Y \) and \( (\bar{x}, \bar{y}) \in \text{Gr} F \). We say that \( F \) is metrically subregular at \( (\bar{x}, \bar{y}) \) if there exist \( K > 0 \) and \( \delta > 0 \) such that for all \( x \in B_\delta(\bar{x}) \) the following inequality holds:

\[
d(x, F^{-1}(\bar{y})) \leq K d(\bar{y}, F(x)).
\]

Assume that \( A \) and \( B \) are subsets of the normed space \( X \). Consider the function \( H_{A,B} : X \times X \to X \) defined as
\[ H_{A,B}(x_1, x_2) = \begin{cases} \{x_1 - x_2\}, & x_1 \in A, \; x_2 \in B \\ \emptyset, & \text{else} \end{cases} \]

**Definition 2.3.** Let $X$ be a Banach space, and $A$, $B$ be closed subsets of $X$. Let $\bar{x} \in A \cap B$. Then $A$ and $B$ are called transversal at $\bar{x}$ if $H_{A,B}$ is regular at $((\bar{x}, \bar{x}), 0)$.

**Definition 2.4.** Let $X$ be a Banach space, and $A$, $B$ be closed subsets of $X$. Let $\bar{x} \in A \cap B$. Then $A$ and $B$ are called subtransversal at $\bar{x}$ if $H_{A,B}$ is subregular at $((\bar{x}, \bar{x}), 0)$.

An equivalent characterization of transversality derived in [13] (cf. [19]) is

**Proposition 2.5.** Let $A$ and $B$ be closed subsets of the normed space $X$. $A$ and $B$ are transversal at $\bar{x} \in A \cap B$, if and only if there exists $K > 0$ and $\delta > 0$ such that

\[ d(x, (A - a) \cap (B - b)) \leq K(d(x, A - a) + d(x, B - b)) \]

for all $x \in \bar{B}_\delta(\bar{x})$ and $a, b \in \bar{B}_\delta(0)$.

Moreover, one observes that only one of the sets could be translated, i.e. we may take $a = 0$ and only vary $b$.

When $a$ and $b$ are fixed to be $0$ in the last definition, a similar characterization of subregularity is obtained (cf. [13]):

**Proposition 2.6.** Let $A$ and $B$ be closed subsets of the complete metric space $X$. $A$ and $B$ are subtransversal at $\bar{x} \in A \cap B$, if and only if there exists $K > 0$ and $\delta > 0$ such that

\[ d(x, A \cap B) \leq K(d(x, A) + d(x, B)) \]

for all $x \in B_\delta(\bar{x})$.

Thus we observe that $A$ and $B$ are transversal at $\bar{x} \in A \cap B$ if and only if the subtransversality inequality holds for $A - a$ and $B - b$ with constants $K$ and $\delta$ for all $x \in \bar{B}_\delta(\bar{x})$ and $a, b \in \bar{B}_\delta(0)$.

It is worth noting that while the definitions of transversality and subtransversality clearly make use of the linear structure, the characterization...
of subtransversality, given by Proposition 2.6, is purely metric. Thus, one may think of subtransversality as a metric concept. However, all characterizations of transversality use the linear structure.

The next theorem shows that regularity and subregularity could be characterized in terms of transversality and subtransversality (something very near is observed in [6] and [5], but it is not stated in this form).

**Theorem 2.7.** Let $F : X \Rightarrow Y$ be a set-valued mapping between the metric spaces $X$ and $Y$, and $(\bar{x}, \bar{y}) \in \text{Gr } F$. Define the sets $A := \text{Gr } F$ and $B := X \times \{\bar{y}\}$. Then $F$ is subregular at $(\bar{x}, \bar{y})$ if and only if $A$ and $B$ are subtransversal at $(\bar{x}, \bar{y})$.

**Proof.** Let the sets be subtransversal, that is there are $\delta > 0$ and $K_1 > 0$ such that

$$d((x, y), A \cap B) \leq K_1(d((x, y), A) + d((x, y), B))$$

for all $(x, y) \in \bar{B}_d((\bar{x}, \bar{y}))$. Observe that $A \cap B = \{(\hat{x}, \bar{y}) \mid \hat{x} \in F^{-1}(\bar{y})\}$. Let $x \in \bar{B}_d(\bar{x})$. Then

$$d((x, \bar{y}), A \cap B) = d(x, F^{-1}(\bar{y})).$$

On the other hand $d((x, \bar{y}), A) \leq d(\bar{y}, F(x))$ and $d((x, \bar{y}), B) = 0$, whence subtransversality implies

$$d(x, F^{-1}(\bar{y})) \leq K_1d(\bar{y}, F(x)),$$

hence $F$ is subregular at $(\bar{x}, \bar{y})$ with constants $K_1$ and $\delta$.

For the reverse direction, let $F$ be subregular at $(\bar{x}, \bar{y})$, that is there are $\delta > 0$ and $K_2 > 0$ such that

$$d(x, F^{-1}(\bar{y})) \leq K_2d(\bar{y}, F(x)).$$

Take $(x, y) \in \bar{B}_{\delta/3}((\bar{x}, \bar{y}))$ and $\varepsilon \in (0, \delta/3)$. Observe that $d((x, y), B) = d(y, \bar{y})$. Let $(x', y') \in A$ be such that $d(x, x') + d(y, y') \leq d((x, y), A) + \varepsilon$. Note that

$$d(x', x) \leq d((x', y'), (\bar{x}, \bar{y})) \leq d((x', y'), (x, y)) + d((x, y), (\bar{x}, \bar{y})) \leq d((x, y), A) + \varepsilon + d((x, y), (\bar{x}, \bar{y})) \leq \varepsilon + 2d((x, y), (\bar{x}, \bar{y})) \leq \delta.$$
Then
\[ d((x, y), A \cap B) = d(x, F^{-1}(\bar{y})) + d(y, \bar{y}) \leq d(x', F^{-1}(\bar{y})) + d(x, x') + d(y, \bar{y}) \]
\[ \leq K_2 d(\bar{y}, F(x')) + d(x, x') + d(y, \bar{y}) \leq K_2 d(y', \bar{y}) + d(x, x') + d(y, \bar{y}) \]
\[ \leq (K_2 + 1) d((x, y), B) + (K_2 + 1) d((x, y), A) + (K_2 + 1) \varepsilon \]

Letting \( \varepsilon \to 0 \) proves subtransversality with constants \( K_1 = K_2 + 1 \) and \( \delta/3 \).

**Corollary 2.8.** Let \( F : X \rightarrow Y, \ X \) and \( Y \) be metric spaces, and \((\bar{x}, \bar{y}) \in Gr F \) as above. Define the sets \( A := Gr F \) and \( B := x \times \{y\} \). Then \( F \) is regular at \((\bar{x}, \bar{y})\) if and only if there are constants \( \delta > 0 \) and \( K > 0 \) such that for any \((x, y) \in \bar{B}_\delta((\bar{x}, \bar{y}))\) and any \( \hat{y} \in \bar{B}_\delta(\bar{y}) \)

\[ d((x, y), A \cap B) \leq K(d((x, y), A) + d((x, y), B)) \] \( (2) \)

If in addition \( X \) and \( Y \) are normed spaces, then this is also equivalent to \( A \) and \( B := B_{\bar{y}} \) being transversal at \((\bar{x}, \bar{y})\).

**Proof.** Observe that in the first part of the proof above, we never made explicit use of the fact that \((\bar{x}, \bar{y}) \in A \cap B \). Pick \( \hat{y} \in \bar{B}_\delta(\bar{y}) \). The inequality \( (2) \) is satisfied with \( B_{\hat{y}} \) instead of \( B \), so that, according to Theorem 2.7, we arrive at \( d(x, F^{-1}(\hat{y})) \leq Kd(\hat{y}, F(x)) \) for all \( x \in \bar{B}_{\delta}(\bar{x}) \). Thus, we obtain regularity at \((\bar{x}, \bar{y})\).

For the other direction, again take \( \hat{y} \in \bar{B}_\delta(\bar{y}) \). Since \( d(x', F^{-1}(\hat{y})) \leq Kd(\hat{y}, F(x')) \), for \( x' \) near \( \bar{x} \), as in the proof of Theorem 2.7, we obtain

\[ d((x, y), A \cap B_{\hat{y}}) \leq (K + 1)(d((x, y), A) + d((x, y), B_{\hat{y}})) \]

for all \((x, y) \in \bar{B}_{\delta/3}((\bar{x}, \bar{y}))\).

If the spaces are normed, then \( B_{\hat{y}} = B + (x, \hat{y} - \bar{y}) \) for any \( x \). Thus the inequality \( (2) \) is the inequality defining transversality. \( \square \)

### 3 Primal characterizations of subtransversality and subregularity

In this section we obtain primal characterizations of subtransversality and subregularity. Our approach is to some extend motivated by the considera-
tions in the paper [4]. In it, the notion of tangential transversality is introduced as a sufficient condition for nonseparation of sets, tangential intersection properties and a Lagrange multiplier rule. The corresponding definition follows.

**Definition 3.1.** Let $A$ and $B$ be closed subsets of the metric space $X$. We say that $A$ and $B$ are tangentially transversal at $\bar{x} \in A \cap B$, if there exist $M > 0$, $\delta > 0$ and $\eta > 0$ such that for any two different points $x^A \in \overline{B}_\delta(\bar{x}) \cap A$ and $x^B \in \overline{B}_\delta(\bar{x}) \cap B$, there exist sequences $t_m \searrow 0$, $\{x^A_m\}_{m \geq 1} \subset A$ and $\{x^B_m\}_{m \geq 1} \subset B$ such that for all $m$

$$d(x^A_m, x^A) \leq t_m M, \quad d(x^B_m, x^B) \leq t_m M, \quad d(x^A_m, x^B_m) \leq d(x^A, x^B) - t_m \eta.$$ 

It can be reformulated equivalently.

**Proposition 3.2.** Let $A$ and $B$ be closed subsets of the metric space $X$. $A$ and $B$ are tangentially transversal at $\bar{x} \in A \cap B$, if and only if there exist $\delta > 0$ and $\zeta > 0$ such that for any two different points $x^A \in \overline{B}_\delta(\bar{x}) \cap A$ and $x^B \in \overline{B}_\delta(\bar{x}) \cap B$, there exist sequences $\{x^A_m\}_{m \geq 1} \subset A$ and $\{x^B_m\}_{m \geq 1} \subset B$ converging to $x^A$ and $x^B$ respectively, and such that for all $m$

$$d(x^A_m, x^B) \leq d(x^A, x^B) - \zeta \max\{d(x^A_m, x^A), d(x^B_m, x^B)\}$$

and $\max\{d(x^A_m, x^A), d(x^B_m, x^B)\} > 0$.

Now we introduce a weaker notion. Note that the main difference is that “there exists a sequence $\{t_n\}_{n=1}^\infty$ of positive reals tending to zero such that for every $t_n$ belonging to it . . .” is replaced by “there exists a positive real $\theta$ such that . . .”.

**Definition 3.3.** Let $A$ and $B$ be closed subsets of the metric space $X$ and $\bar{x} \in X$. We say that $A$ and $B$ have weak tangential transversality property (WTT) at $\bar{x}$ if there exist $\delta > 0$ and $M > 0$ such that $A \cap \overline{B}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \emptyset$, $B \cap \overline{B}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \emptyset$ and for any $x^A \in A \cap \overline{B}_{\delta}(\bar{x})$ and $x^B \in B \cap \overline{B}_{\delta}(\bar{x})$ with $x^A \neq x^B$, there exist $\theta > 0$, $\hat{x}^A \in A$ and $\hat{x}^B \in B$ such that

$$d(x^A, \hat{x}^A) \leq \theta M, \quad d(x^B, \hat{x}^B) \leq \theta M \quad \text{and} \quad d(\hat{x}^A, \hat{x}^B) \leq d(x^A, x^B) - \theta.$$ 

Equivalently, $A$ and $B$ have WTT at $\bar{x}$ if and only if there exist $\delta > 0$ and $M > 0$ such that $A \cap \overline{B}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \emptyset$, $B \cap \overline{B}_{\frac{\delta}{2(1+2M)}}(\bar{x}) \neq \emptyset$ and for any
Assume that there exist \( \delta > 0 \) and \( M > 0 \) such that for any \( x^A \in A \cap \bar{B}_\delta(\bar{x}) \) and \( x^B \in B \cap \bar{B}_\delta(\bar{x}) \) with \( f(x^A, x^B) \neq 0 \), there are \( \theta > 0 \), \( \hat{x}^A \in A \) and \( \hat{x}^B \in B \), such that \( d(x^A, \hat{x}^A) \leq M\theta \), \( d(x^B, \hat{x}^B) \leq M\theta \) and

\[
f(\hat{x}^A, \hat{x}^B) \leq f(x^A, x^B) - \theta.
\]

\( f \) is lower semicontinuous such that \( f(x, y) \leq d(x, y) \) for all \( x, y \in X \). Assume that there exist \( \delta > 0 \) and \( M > 0 \) such that for any \( x^A \in A \cap \bar{B}_\delta(\bar{x}) \) and \( x^B \in B \cap \bar{B}_\delta(\bar{x}) \) with \( f(x^A, x^B) \neq 0 \), there are \( \delta > 0 \), \( \hat{x}^A \in A \) and \( \hat{x}^B \in B \), such that \( d(x^A, \hat{x}^A) \leq M\theta \), \( d(x^B, \hat{x}^B) \leq M\theta \) and

\[
f(\hat{x}^A, \hat{x}^B) \leq f(x^A, x^B) - \theta.
\]

Fix \( x^A \in A \cap \bar{B}_{\frac{\delta}{1+2M}}(\bar{x}) \) and \( x^B \in B \cap \bar{B}_{\frac{\delta}{1+2M}}(\bar{x}) \). Then there exist \( \tilde{x}^A \in A \) and \( \tilde{x}^B \in B \), such that \( f(\tilde{x}^A, \tilde{x}^B) = 0 \), \( d(\tilde{x}^A, x^A) \leq Mf(x^A, x^B) \) and

\[
d(\tilde{x}^B, x^B) \leq Mf(x^A, x^B).
\]

\( f \) is lower semicontinuous such that \( f(x, y) \leq d(x, y) \) for all \( x, y \in X \). Assume that there exist \( \delta > 0 \) and \( M > 0 \) such that for any \( x^A \in A \cap \bar{B}_\delta(\bar{x}) \) and \( x^B \in B \cap \bar{B}_\delta(\bar{x}) \) with \( f(x^A, x^B) \neq 0 \), there are \( \delta > 0 \), \( \hat{x}^A \in A \) and \( \hat{x}^B \in B \), such that \( d(x^A, \hat{x}^A) \leq M\theta \), \( d(x^B, \hat{x}^B) \leq M\theta \) and

\[
f(\hat{x}^A, \hat{x}^B) \leq f(x^A, x^B) - \theta.
\]
(S3) \( d(x^A_\alpha, x^A_\gamma) \leq M(t_\alpha - t_\gamma) \) and \( d(x^B_\alpha, x^B_\gamma) \leq M(t_\alpha - t_\gamma) \) for each \( \gamma \leq \alpha \).

We implement our construction using induction on \( \alpha \). The process terminates when \( f(x^A_\alpha, x^B_\alpha) = 0 \) for some \( \alpha \), and this \( \alpha \) is named \( \alpha_0 \). We start with \( x^A_1 := x^A \in \overline{B}_{1+M}(\bar{x}) \cap A \), \( x^B_1 := x^B \in \overline{B}_{1+M}(\bar{x}) \cap B \) and \( t_1 = 0 \). It is straightforward to verify the inductive assumptions (S0)-(S3) for \( \beta \) and the process has not been terminated.

Assume that \( x^A_\beta \in \overline{B}_\delta(\bar{x}) \cap A \), \( x^B_\beta \in \overline{B}_\delta(\bar{x}) \cap B \) and \( t_\beta \) are constructed and (S1)-(S3) are true for all ordinals \( \beta \) less than \( \alpha \) and the process has not been terminated.

Let us first consider the case when \( \alpha \) is a successor ordinal, i.e. \( \alpha = \beta + 1 \).

As \( \beta < \alpha_0 \) (the process has not been terminated), we have \( f(x^A_\beta, x^B_\beta) \neq 0 \). Moreover (S0) holds, so we can apply the assumption in the statement of the theorem to obtain \( \theta > 0 \), \( \hat{x}^A_\beta \), \( \hat{x}^B_\beta \), and we define \( t_\alpha := t_\beta + \theta \), \( x^A_\alpha := \hat{x}^A_\beta \) and \( x^B_\alpha := \hat{x}^B_\beta \). Now we have \( x^A_\alpha \in A \), \( x^B_\alpha \in B \), \( d(x^A_\alpha, x^B_\alpha) \leq M\theta \), \( d(x^B_\alpha, x^B_\alpha) \leq M\theta \) and \( f(x^A_\alpha, x^B_\alpha) \leq f(x^A_\beta, x^B_\beta) - \theta \). Using the inductive assumption, we have

\[
 f(x^A_\alpha, x^B_\alpha) \leq f(x^A_\beta, x^B_\beta) - \theta \leq f(x^A, x^B) - t_\beta - \theta = f(x^A, x^B) - t_\alpha .
\]

Therefore, (S1) is verified for \( \alpha \).

Now the inequalities \( d(x^A_\alpha, x^A_\beta) \leq M\theta \), \( d(x^B_\alpha, x^B_\beta) \leq M\theta \) and the inductive assumption (S2) for \( \beta \) yield

\[
 d(x^A_\alpha, \bar{x}) \leq d(x^A_\alpha, x^A_\beta) + d(x^A_\alpha, x^A_\beta) \leq d(x^A_\alpha, \bar{x}) + t_\beta M + M\theta = d(x^A, \bar{x}) + t_\alpha M,
\]

\[
 d(x^B_\alpha, \bar{x}) \leq d(x^B_\alpha, x^B_\beta) + d(x^B_\alpha, x^B_\beta) \leq d(x^B_\alpha, \bar{x}) + t_\beta M + M\theta = d(x^B_\alpha, \bar{x}) + t_\alpha M.
\]

Thus (S2) is verified for \( \beta \). Using the estimate \( t_\beta \leq f(x^A, x^B) \leq d(x^A, x^B) \) from (S1) for \( \beta \), the assumption of the theorem and the above inequalities, we obtain

\[
 d(x^A_\alpha, \bar{x}) \leq d(x^A_\alpha, \bar{x}) + t_\alpha M \leq \frac{\delta + \frac{2\delta}{1 + 2M} = \delta}{1 + 2M} = \delta
\]

which means that \( x^A_\alpha \in \overline{B}_\delta(\bar{x}) \). Similarly \( x^B_\alpha \in \overline{B}_\delta(\bar{x}) \). Thus (S0) holds.

Now let \( \gamma < \alpha \). Then

\[
 d(x^A_\alpha, x^A_\gamma) \leq d(x^A_\beta, x^A_\gamma) + d(x^A_\alpha, x^A_\beta) \leq M(t_\beta - t_\gamma) + M\theta = M(t_\alpha - t_\gamma)
\]

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and in the same way
\[ d(x_\alpha^B, x_\gamma^B) \leq d(x_\beta^B, x_\gamma^B) + d(x_\alpha^B, x_\beta^B) \leq M(t_\beta - t_\gamma) + M\theta = M(t_\alpha - t_\gamma). \]

We have verified the inductive assumptions (S0)-(S3) for the case of a successor ordinal \( \alpha \).

We next consider the case when \( \alpha \) is a limit ordinal number. Let \( \beta < \alpha \) be arbitrary. Then \( \beta + 1 < \alpha \) too. Since the transfinite process has not stopped at \( \beta + 1 \), then \( f(x_\beta^A, x_\beta^B) > 0 \), and taking into account (S1) we obtain that \( t_\beta < f(x_\beta^A, x_\beta^B) \). Hence the increasing transfinite sequence \( \{t_\beta\}_{1 \leq \beta < \alpha} \) is bounded, and so it is convergent. We denote \( t_\alpha := \lim_{\beta \to \alpha} t_\beta \). Since \( d(x_\beta^A, x_\beta^B) \leq (t_\beta - t_\gamma)M \), the transfinite sequence \( \{x_\beta^A\}_{1 \leq \beta < \alpha} \) is fundamental. Hence there exists \( x_\alpha^A \) so that \( \{x_\beta^A\}_{1 \leq \beta < \alpha} \) tends to \( x_\alpha^A \) as \( \beta \) tends to \( \alpha \) with \( \beta < \alpha \). In the same way one can prove the existence of \( x_\alpha^B \) so that the transfinite sequence \( \{x_\beta^B\}_{1 \leq \beta < \alpha} \) tends to \( x_\alpha^B \) as \( \beta \) tends to \( \alpha \). To verify the inductive assumptions (S1)-(S3) for \( \alpha \), one can just take a limit for \( \beta \) tending to \( \alpha \) with \( \beta < \alpha \) in the same assumptions written for each \( \beta < \alpha \) (for (S1) one takes \( \lim \inf \) on the left and uses that this is greater than the value of \( f \) at the limit point, since the function is lower semicontinuous). For (S0) one uses that \( A \) and \( B \) are closed.

We have constructed inductively the transfinite sequences \( \{x_\beta^A\}_{\beta \leq \alpha} \subset A \), \( \{x_\beta^B\}_{\beta \leq \alpha} \subset B \) and \( \{t_\beta\}_{\beta \leq \alpha} \subset [0, +\infty) \). The process will terminate when \( f(x_\alpha^A, x_\alpha^B) = 0 \) for some \( \alpha \). Since \( f(x_\alpha^A, x_\alpha^B) \leq f(x_\alpha^A, x_\beta^B) - t_\alpha \) and the transfinite sequence \( t_\alpha \) is strictly increasing, the equality \( f(x_\alpha^A, x_\alpha^B) = 0 \) will be satisfied for some \( \alpha = \alpha_0 \) strictly preceding the first uncountable ordinal number. Indeed, the successor ordinals indexing the so constructed transfinite sequences form a countable set (because to every successor ordinal \( \alpha + 1 \) corresponds the open interval \( (t_\alpha, t_{\alpha+1}) \subset \mathbb{R} \), these intervals are disjoint and the rational numbers are countably many and dense in \( \mathbb{R} \)). Therefore, \( \alpha_0 \) is countably accessible. On the other hand-side, assuming the Axiom of countable choice, \( \omega_1 \) is not countably accessible. Hence our inductive process terminates before \( \omega_1 \).

Then we set \( \bar{x}^A := x_{\alpha_0}^A \in A \) and \( \bar{x}^B := x_{\alpha_0}^B \in B \) and because of (S1) we have that \( t_{\alpha_0} \leq f(x^A, x^B) \). Applying (S3) we obtain
\[ d(x_1^A, x_1^A) \leq M(t_{\alpha_0} - t_1) \leq Mf(x_1^A, x_1^B). \]

Similarly \( d(x_1^B, x_1^B) \leq Mf(x_1^A, x_1^B) \).

This completes the proof. \( \square \)

**Corollary 3.5.** Let \( A \) and \( B \) be closed subsets of the complete metric space \( X \) and \( \bar{x} \in X \). Let \( A \) and \( B \) have WTT property at \( \bar{x} \) with constants \( \delta \)
and $M$. Let us fix $x^A \in A$ with $d(x^A, \bar{x}) \leq \frac{\delta}{1 + 2M}$ and $x^B \in B$ with $d(x^B, \bar{x}) \leq \frac{\delta}{1 + 2M}$. Then, there exists $x^{AB} \in A \cap B$ with

$$d(x^{AB}, x^A) \leq Md(x^A, x^B) \quad \text{and} \quad d(x^{AB}, x^B) \leq Md(x^A, x^B).$$

**Proof.** Indeed, apply Lemma 3.4 for the function $f(x, y) := d(x, y)$.

Completeness is crucial in the above theorem. The next result shows that WTT is an equivalent characterization of subtransversality in the presence of completeness.

**Theorem 3.6.** Let $A$ and $B$ be closed subsets of the complete metric space $X$ and $\bar{x} \in X$. If $A$ and $B$ have WTT property at $\bar{x}$, then there exist $K > 0$ and $\delta > 0$ such that

$$d(x, A \cap B) \leq K(d(x, A) + d(x, B))$$

for all $x \in B_{\delta}(\bar{x})$.

If there exist $K > 0$ and $\delta > 0$ such that (3) holds for all $x \in B_{\delta}(\bar{x})$, $A \cap B_{\frac{\delta}{4K+10}}(\bar{x}) \neq \emptyset$ and $B \cap B_{\frac{\delta}{4K+10}}(\bar{x}) \neq \emptyset$, then $A$ and $B$ have WTT property at $\bar{x}$.

Moreover, if $\bar{x} \in A \cap B$, then $A$ and $B$ have WTT property at $\bar{x}$ if and only if $A$ and $B$ are subtransversal at $\bar{x}$.

**Proof.** Let $A$ and $B$ have WTT property with constants $M$, $\delta$. Let $\hat{\delta} := \frac{\delta}{8(1 + 2M)}$. Let $x \in B_{\delta}(\bar{x})$ and choose $\varepsilon \in (0, \hat{\delta})$. Then there exists $x^A \in A$, such that $d(x, x^A) \leq d(x, A) + \varepsilon$. We have that $d(x, A) \leq d(x, \bar{x}) + d(\bar{x}, A) \leq \hat{\delta} + \frac{\delta}{2(1 + 2M)} \leq 5\hat{\delta}$, so that $d(x, x^A) \leq 6\hat{\delta}$. Since $d(x, \bar{x}) \leq \hat{\delta}$, the triangle inequality implies

$$d(x^A, \bar{x}) \leq 7\hat{\delta} < \frac{\delta}{1 + 2M}.$$ 

Similarly, we find $x^B \in B$, such that $d(x, x^B) \leq d(x, B) + \varepsilon$ and

$$d(x^B, \bar{x}) < \frac{\delta}{1 + 2M}.$$ 

Then $x^A$ and $x^B$ satisfy the requirements in Corollary 3.5. Hence, there is $x^{AB} \in A \cap B$, such that

$$d(x^{AB}, x^A) \leq Md(x^A, x^B) \quad \text{and} \quad d(x^{AB}, x^B) \leq Md(x^A, x^B).$$
We estimate
\[
    d(x, A \cap B) \leq d(x, x^{AB}) \leq d(x, x^A) + d(x^{AB}) \leq d(x, A) + \varepsilon + M d(x^A, x^B)
\]
\[
    \leq d(x, A) + \varepsilon + M (d(x, A^C) + d(x^B))
\]
\[
    \leq d(x, A) + \varepsilon + M (d(x, A) + \varepsilon + d(x, B) + \varepsilon)
\]
\[
    \leq (M + 1) (d(x, A) + d(x, B)) + \varepsilon (1 + 2M)
\]

Letting \( \varepsilon \to 0 \) proves (3) with constants \( \delta \) and \( M + 1 \).

For the second part, let (3) hold with constants \( \delta \) and \( K \). Take \( x^A \in A \cap \overline{B}_\delta (\bar{x}) \) and \( x^B \in B \cap \overline{B}_\delta (\bar{x}) \) and for \( \varepsilon := d(x^A, x^B) > 0 \), find \( x^{AB} \in A \cap B \) such that
\[
d(x^A, x^{AB}) < d(x^A, A \cap B) + \varepsilon \leq K d(x^A, B) + \varepsilon
\]
\[
\leq K d(x^A, x^B) + \varepsilon = (K + 1) d(x^A, x^B).
\]

Then
\[
d(x^B, x^{AB}) \leq d(x^A, x^B) + d(x^A, x^{AB})
\]
\[
\leq d(x^A, x^B) + (K + 1) d(x^A, x^B) = (K + 2) d(x^A, x^B).
\]
Now WTT property follows with \( \hat{x}^A = \hat{x}^B = x^{AB} \), \( \theta = d(x^A, x^B) > 0 \) and \( M = K + 2 \), because proximity of \( A \) and \( B \) to \( \bar{x} \) is assumed.

\[\square\]

The next theorem is a primal characterization of subregularity (cf. Theorem 2.58 in [15] or Corollaries 5.8 and 5.9 in [20]).

**Theorem 3.7.** Let \( F : X \rightrightarrows Y \) be with closed graph and \((\bar{x}, \bar{y}) \in Gr F\), where \( X \) and \( Y \) are complete metric spaces. Then \( F \) is subregular at \((\bar{x}, \bar{y}) \in Gr F\) if and only if there exist constants \( \delta > 0 \) and \( \tau > 0 \) such that for all \((x, y) \in Gr F \cap \overline{B}_\delta ((\bar{x}, \bar{y}))\), there is \((\hat{x}, \hat{y}) \in Gr F \setminus \{(x, y)\}\), such that
\[
d(\hat{y}, \bar{y}) \leq d(y, \bar{y}) - \tau d((x, y), (\hat{x}, \hat{y}))
\]

**Proof.** According to Theorem 2.7 \( F \) is subregular at \((\bar{x}, \bar{y}) \) if and only if the sets \( A := Gr F \) and \( B := X \times \{\bar{y}\} \) are subtransversal at that point. Assume that they are subtransversal. Then, according to Theorem 3.6 WTT holds with some constants \( \delta \) and \( M \). Take \((x, y) \in A \cap \overline{B}_\delta ((\bar{x}, \bar{y}))\). Then \((x, y) \in B \cap \overline{B}_\delta ((\bar{x}, \bar{y}))\) and thus there exist \((\hat{x}, \hat{y}) \in A \) and \((\hat{x}_B, \hat{y}) \in B \) such that
\[
d((\hat{x}, \hat{y}), (\hat{x}_B, \hat{y})) \leq d((x, y), (x, \bar{y})) - \frac{1}{M} \max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\}
\]

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and max\{d((x, y), (x, \hat{y})), d(x, \hat{x}_B)\} > 0. If we assume that (\hat{x}, \hat{y}) = (x, y), then in particular \hat{y} = y. Thus
\[
d((x, y), (x, \hat{y})) - \frac{1}{M}\max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\} < d(y, \hat{y})
\]
and
\[
d(y, \hat{y}) \leq d((\hat{x}, \hat{y}), (\hat{x}_B, \hat{y})).
\]
This contradicts the earlier inequality, hence (\hat{x}, \hat{y}) \neq (x, y).
From here we obtain
\[
d(\hat{y}, \hat{y}) \leq d((\hat{x}, \hat{y}), (\hat{x}_B, \hat{y})) \leq d((x, y), (x, \hat{y})) - \frac{1}{M}\max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\}
\leq d(y, \hat{y}) - \tau d((x, y), (\hat{x}, \hat{y})),
\]
for \(\tau := \frac{1}{M}\).
Now assume that there exist constants \(\delta > 0\) and \(\tau > 0\) such that for all \((x, y) \in \text{Gr } F \cap \bar{B}_\delta((\hat{x}, \hat{y}))\), there is \((\hat{x}, \hat{y}) \in \text{Gr } F\), such that
\[
d(\hat{y}, \hat{y}) \leq d(y, \hat{y}) - \tau d((x, y), (\hat{x}, \hat{y})).
\]
Let the function \(f : (X \times Y) \times (X \times Y) \rightarrow [0, +\infty)\) be given by \(f((x_1, y_1), (x_2, y_2)) = d(y_1, y_2)\). Let \((x, y) \in A \cap \bar{B}_\delta((\hat{x}, \hat{y}))\) and \((x_B, \hat{y}) \in B\) with \(d(x_B, \hat{x}) \leq \delta\) be arbitrary. Then, there exists a point \((\hat{x}, \hat{y}) \in A\), satisfying the inequality.
For the points \((\hat{x}, \hat{y}) \in A\) and \((x_B, \hat{y}) \in B\) we estimate
\[
f((\hat{x}, \hat{y}), (x_B, \hat{y})) = d(\hat{y}, \hat{y}) \leq d(y, \hat{y}) - \tau d((x, y), (\hat{x}, \hat{y}))
= f((x, y), (x_B, \hat{y})) - \tau \max\{d((x, y), (\hat{x}, \hat{y})), d((x_B, \hat{y}), (x_B, \hat{y}))\},
\]
which means that we can apply Lemma 3.4 for \(A\) and \(B\) at \((\hat{x}, \hat{y})\) with function \(f\) and constants \(\delta\) and \(M := \frac{1}{\tau}\) if the starting points are sufficiently close to \((\hat{x}, \hat{y})\).
Let \(\hat{\delta} = \frac{\tau}{(\tau+2)(\tau+1)}\delta\) and take \(x \in \bar{B}_\delta(\hat{x})\). If \(d(\hat{y}, F(x)) \geq \tau \hat{\delta}\), then
\[
\frac{1}{\tau}d(\hat{y}, F(x)) \geq \hat{\delta} \geq d(x, \hat{x}) \geq d(x, F^{-1}(\hat{y})).
\]
Otherwise, take \(\varepsilon \in \left(0, \tau \hat{\delta} - d(\hat{y}, F(x))\right)\). Take \(y \in F(x)\) for which
\[
d(y, \hat{y}) \leq d(\hat{y}, F(x)) + \varepsilon \leq \tau \hat{\delta} < \frac{\tau}{\tau+2}\delta.
\]
For $M = 1/\tau$, we have $d(x, \bar{x}) \leq \frac{\delta}{1+2M}$ and $d(y, \bar{y}) \leq \frac{\delta}{1+2M}$.

Applying Lemma 3.1 to $(x,y) \in A$ and $(x,\bar{y}) \in B$, it follows that there exist points $(\bar{x}, \bar{y}) \in A$ and $(\bar{x}_B, \bar{y}) \in B$ such that $f((\bar{x}, \bar{y}), (\bar{x}_B, \bar{y})) = 0$, hence $\bar{y} = \bar{y}$, and $d((\bar{x}, \bar{y}), (x,y)) \leq M f((x,y), (\bar{x}, \bar{y})) = M d(y, \bar{y})$. Using that $\bar{x} \in F^{-1}(\bar{y})$ and the choice of $y$, we obtain that

$$d(x,F^{-1}(\bar{y})) \leq d(x, \bar{x}) \leq d((\bar{x}, \bar{y}), (x,y)) \leq M d(y, \bar{y})$$

$$\leq M d(\bar{y}, F(x)) + M \varepsilon$$

Letting $\varepsilon \to 0$, we obtain $d(x, F^{-1}(\bar{y})) \leq M d(\bar{y}, F(x))$ for all $x \in B_\delta(\bar{x})$. We have verified that $F$ is subregular at $(\bar{x}, \bar{y})$ by definition. \qed

4 Primal characterizations of transversality and regularity

We continue to obtain primal characterizations of transversality and regularity.

A direct consequence of the definition of transversality and Theorem 3.6 is a characterization of transversality in terms of “translated” subtransversality.

**Proposition 4.1.** Let $A$ and $B$ be closed subsets of the normed space $X$ and $\bar{x} \in A \cap B$. Then $A$ and $B$ are transversal at $\bar{x}$ if and only if there exist $\delta > 0$ and $M > 0$ such that for any $a \in B_\delta(0)$ and $b \in B_\delta(0)$, any $x^A \in A \cap B_\delta(\bar{x} + a)$ and $x^B \in B \cap B_\delta(\bar{x} + b)$ with $x^A \neq x^B$ there exist $\theta > 0$, $\hat{x}^A \in A$ and $\hat{x}^B \in B$ such that

$$\|x^A - \hat{x}^A\| \leq \theta M, \quad \|x^B - \hat{x}^B\| \leq \theta M$$

$$\|\hat{x}^A - \hat{x}^B - (a - b)\| \leq \|x^A - x^B - (a - b)\| - \theta.$$ 

**Proof.** Let $A$ and $B$ be transversal at $\bar{x}$ with constants $K$ and $\tilde{\delta}$. Denote $\delta = \delta/(4K+10)$. Then for all $a \in B_\delta(0)$ and $b \in B_\delta(0)$, the sets $A - a$ and $B - b$ have WTT property with constants $\delta$ and $M = K + 2$ according to Theorem 3.6.

Now let the sets satisfy the above property with constants $\delta$ and $M$. Thus for all $a \in B_{\frac{\delta}{2(K+2M)}}(0)$ and $b \in B_{\frac{\delta}{2(K+2M)}}(0)$, the sets $A - a$ and $B - b$ have WTT property with constants $\delta$ and $M$. Then, again Theorem 3.6 implies that

$$d(x, (A - a) \cap (B - b)) \leq (M + 1)(d(x, A - a) + d(x, B - b))$$
for all $x \in \overline{B}_\delta(\bar{x})$, which is precisely transversality.

Strengthening in one of the directions of this proposition gives a characterization of transversality in terms of “translated” tangential transversality.

**Proposition 4.2.** Let $A$ and $B$ be closed subsets of the Banach space $X$ and $\bar{x} \in A \cap B$. Then $A$ and $B$ are transversal at $\bar{x}$ if and only if there exist $\delta > 0$ and $M > 0$ such that for any $a \in \overline{B}_\delta(0)$ and $b \in \overline{B}_\delta(0)$, any $x^A \in B \cap \overline{B}_\delta(\bar{x} + a)$ and $x^B \in B \cap \overline{B}_\delta(\bar{x} + b)$ with $x^A \neq x^B$ there exist $\{u^A_n\}_{n \geq 1} \subset A$, $\{u^B_n\}_{n \geq 1} \subset B$ and $t_n \downarrow 0$ such that $x^A + t_n u^A_n \in A$, $x^B + t_n u^B_n \in B$

\[
\|u^A_n\| \leq M, \quad \|u^B_n\| \leq M \quad \text{and} \quad \|x^A - x^B + t_n(u^A_n - u^B_n) - (a - b)\| \leq \|x^A - x^B - (a - b)\| - t_n.
\]

**Proof.** The “if” direction is straightforward from Proposition 4.1.

For the converse, let $A$ and $B$ be transversal at $\bar{x}$. This means that the map $H := H_{AB}$ from (1) is regular at $((\bar{x}, \bar{x}), 0)$ with constants $\delta$ and $K$. Take $\delta < \delta/2$, $a \in \overline{B}_\delta(0)$ and $b \in \overline{B}_\delta(0)$ and $x^A$ and $x^B$.

Define $u = -\frac{x^A - x^B - (a - b)}{\|x^A - x^B - (a - b)\|}$ and choose a sequence $t_n \downarrow 0$ such that

\[
x^A - x^B + t_n u \in \overline{B}_\delta(0).
\]

For $n \geq 1$ consider $(x^A_n, x^B_n) \in H^{-1}(x^A - x^B + t_n v) \subset A \times B$ such that

\[
\|(x^A_n, x^B_n) - (x^A, x^B)\| \leq d((x^A, x^B), H^{-1}(x^A - x^B + t_n v)) + t_n.
\]

Denote

\[
(u^A_n, u^B_n) = \frac{1}{t_n}((x^A_n, x^B_n) - (x^A, x^B))
\]

so clearly $x^A + t_n u^A_n \in A$, $x^B + t_n u^B_n \in B$.

Moreover, metric regularity implies

\[
\|(u^A_n, u^B_n)\| \leq \frac{1}{t_n}d((x^A, x^B), H^{-1}(x^A - x^B + t_n v)) + 1 \leq \frac{1}{t_n}Kd(x^A - x^B + t_n u, H((x^A, x^B))) + 1 = K\frac{1}{t_n}\|x^A - x^B + t_n u - (x^A - x^B)\| + 1 = K + 1.
\]
Finally, we have that
\[
\|x^A + t_n u_n^A - (x^B + t_n u_n^B) - (a - b)\| = \|x^A_n - x^B_n - (a - b)\| = \|x^A - x^B - (a - b)\| - t_n.
\]

\[\square\]

**Remark 4.3.** In the above proposition we can obtain the (formally) stronger statement that there exists \(\lambda > 0\) such that the decreasing property holds for any \(t \in (0, \lambda]\) instead of the sequence \(\{t_n\}_{n=1}^\infty\) tending to zero from above.

**Remark 4.4.** Propositions 4.1 and 4.2 remain true if we consider translations in only one of the sets, i.e. we may take \(a = 0\) and only vary \(b\).

Next, we obtain necessary and sufficient primal conditions for metric regularity. The following theorem is a classical “rate of descent” characterization (cf. Theorem 2.50 in [15] or Theorem 7 in [17]).

**Theorem 4.5.** Let \(F : X \rightrightarrows Y\) be with closed graph and \((\bar{x}, \bar{y}) \in \text{Gr} F\), where \(X\) and \(Y\) are complete metric spaces. Then \(F\) is regular at \((\bar{x}, \bar{y})\) if and only if there exist \(\delta > 0\) and \(\tau > 0\) such that for all \((x, y) \in A \cap B \delta((\bar{x}, \bar{y}))\) and all \(v \in B \delta((\bar{y}))\), there is \((\hat{x}, \hat{y}) \in \text{Gr} F \setminus \{(x, y)\}\), such that
\[
d(\hat{y}, v) \leq d(y, v) - \tau d((x, y), (\hat{x}, \hat{y})).
\]

**Proof.** According to Corollary 2.8, \(F\) is regular at \((\bar{x}, \bar{y})\) if and only if there are constants \(\delta > 0\) and \(K > 0\) such that for any \((x, y) \in A \cap \text{Gr} F\) and any \(v \in \text{Gr} F\) it holds
\[
d((x, y), A \cap B_v) \leq K(d((x, y), A) + d((x, y), B_v)),
\]
where \(B_v := X \times \{v\}\).

Let \(F\) be regular at \((\bar{x}, \bar{y})\). Fix \(\hat{\delta} := \frac{\delta}{4K + 10}\), \((x, y) \in A \cap \text{Gr} \hat{\delta}((\bar{x}, \bar{y}))\) and \(v \in \text{Gr} \hat{\delta}(\bar{y})\). According to Theorem 3.6, \(A\) and \(B_v\) have WTT property at \((\bar{x}, \bar{y})\) with constants \(\hat{\delta}\) and \(M\). Hence, there exist \((\hat{x}, \hat{y}) \in A\) and \((\hat{x}^B, v) \in B_v\) such that
\[
d((\hat{x}, \hat{y}), (\hat{x}^B, v)) \leq d((x, y), (x, v)) - \frac{1}{M} \max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}^B)\}
\]
and \( \max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\} > 0 \). If we assume that \((\hat{x}, \hat{y}) = (x, y)\), then in particular \(\hat{y} = y\). Thus
\[
d((x, y), (x, v)) - \frac{1}{M} \max\{d((x, y), (\hat{x}, \hat{y})), d(x, \hat{x}_B)\} < d(y, v)
\]
and
\[
d(y, v) \leq d((\hat{x}, \hat{y}), (\hat{x}_B, v)).
\]
This contradicts the earlier inequality, hence \((\hat{x}, \hat{y}) \neq (x, y)\). From here we obtain
\[
d(\hat{y}, v) \leq d(y, v) - \tau d((x, y), (\hat{x}, \hat{y})),
\]
for \(\tau := \frac{1}{M}\).

Now, assume that there exist constants \(\delta > 0\) and \(\tau > 0\) such that for all \((x, y) \in \text{Gr } F \cap \overline{B}_\delta((\hat{x}, \hat{y}))\) and all \(v \in \overline{B}_\delta(\hat{y})\), there is \((\hat{x}, \hat{y}) \in \text{Gr } F \setminus \{(x, y)\}\), such that
\[
d(\hat{y}, v) \leq d(y, v) - \tau d((x, y), (\hat{x}, \hat{y})).
\]
Let the function \(f : (X \times Y) \times (X \times Y) \to [0, +\infty)\) be given by \(f((x_1, y_1), (x_2, y_2)) = d(y_1, y_2)\). Let us fix \((x, y) \in A \cap \overline{B}_{\delta/2}((\hat{x}, \hat{y}))\), \(v \in \overline{B}_{\delta/2}(\hat{y})\) and \((x_B^*, v) \in B_v\) with \(d(x_B^*, \bar{x}) \leq \delta/2\). Then, there exists a point \((\bar{x}, \bar{y}) \in A\), satisfying the above inequality.

For the points \((\hat{x}, \hat{y}) \in A\) and \((x_B, v) \in B_v\ we estimate
\[
f((\hat{x}, \hat{y}), (x_B, v)) = d(\hat{y}, v) \leq d(y, v) - \tau d((x, y), (\hat{x}, \hat{y}))
= f((x, y), (x_B, v)) - \tau \max\{d((x, y), (\hat{x}, \hat{y})), d((x_B, v), (x, v))\},
\]
which means that we can apply Lemma 3.4 for \(A\) and \(B_v\) at \((\bar{x}, \bar{y})\) with function \(f\) and constants \(\delta/2\) and \(M := \frac{1}{\tau}\) if the starting points are sufficiently close to \((\bar{x}, \bar{y})\).

Let \(\delta := \frac{\tau}{4(\tau + 2)(\tau + 1)}\delta\) and take \(v \in \overline{B}_\delta(\hat{y})\) and \(x \in \overline{B}_\delta(\bar{x})\). Applying Lemma 3.4 for \((\bar{x}, \bar{y}) \in A\) and \((\bar{x}, v) \in B_v\ we arrive at a point \(x_v \in F^{-1}(v)\) such that
\[
d(x_v, \bar{x}) \leq M d(\bar{y}, v) \leq \delta/\tau
\]
If \(d(v, F(x)) \geq \hat{\delta}(1 + \tau)\), then
\[
\frac{1}{\tau}d(v, F(x)) \geq \hat{\delta} + \frac{\hat{\delta}}{\tau} \geq d(x, \bar{x}) + d(x_v, \bar{x}) \geq d(x, x_v) \geq d(x, F^{-1}(v))
\]
Otherwise, take $\varepsilon \in \left(0, \hat{\delta}(\tau + 1) - d(\bar{y}, F(x))\right)$. Take $y \in F(x)$ for which
\[
d(y, v) \leq d(v, F(x)) + \varepsilon \leq \hat{\delta}(\tau + 1) \leq \frac{\tau}{4(\tau + 2)} \delta.
\]
Recall that $M = 1/\tau$, hence $d(x, \bar{x}) \leq \frac{\delta/2}{1+2M}$ and $d(y, \bar{y}) \leq d(y, v) + d(v, \bar{y}) \leq \tau/2(\tau + 2) \delta$.

Applying Lemma 3.4 to $(x, y) \in A$ and $(x, v) \in B$, it follows that there exist points $(\tilde{x}, \tilde{y}) \in A$ and $(\bar{x}, v) \in B$ such that $f((\tilde{x}, \tilde{y}), (\bar{x}, v)) = 0$, hence $\tilde{y} = v$, and $d((\tilde{x}, v), (x, y)) \leq M f((x, y), (x, v)) = M d(y, v)$. Using that $\tilde{x} \in F^{-1}(v)$ and the choice of $y$, we obtain that
\[
d(x, F^{-1}(v)) \leq d(x, \tilde{x}) \leq d((\tilde{x}, v), (x, y)) \leq M d(v, F(x)) + M \varepsilon
\]
Letting $\varepsilon \to 0$, we obtain $d(x, F^{-1}(v)) \leq M d(v, F(x))$ for all $x \in \bar{B}_{\delta}(\bar{x})$ and $v \in \bar{B}_{\delta}(\bar{y})$. We have verified that $F$ is regular at $(\tilde{x}, \tilde{y})$ by definition.

Using the above theorem, we establish a characterization of metric regularity of a map $F : X \rightrightarrows Y$, $X$ – complete metric space and $Y$ – Banach space, using its first order (contingent) variation $F^{(1)}(x, y)$. This is first done in [11] (see also Theorem 4.13 and Remark 4.14(c) in [1] for a proof in Banach spaces or [17] for an alternative proof). Given $(x, y) \in \text{Gr} F$, define $F^{(1)} : X \times Y \rightrightarrows Y$ by
\[
F^{(1)}(x, y) := \limsup_{t \to 0^+} \frac{F(\bar{B}_t(x)) - y}{t},
\]
where lim sup stands for the Kuratowski limit superior of sets. Equivalently, $v \in F^{(1)}(x, y)$ exactly when there exist sequences $t_n \to 0^+$, $v_n \to v$ and $(x_n, y_n) \in \text{Gr} F$ such that $d(x_n, x) \leq t_n$ and $y_n = y + t_n v_n$.

Our proof is done via a sequential characterization of metric regularity, which we have not seen stated anywhere in the literature.

**Corollary 4.6.** Let us consider $F : X \rightrightarrows Y$ with closed graph, where $X$ is a complete metric space and $Y$ is a Banach space. Then, the following are equivalent

(i) $F$ is regular at $(\bar{x}, \bar{y}) \in \text{Gr} F$
(ii) there exist $\delta > 0$ and $r > 0$ such that

$$B_r(0) \subset F^{(1)}(x, y) \text{ for all } (x, y) \in B_\delta(\bar{x}, \bar{y}) \cap \text{Gr } F$$

(iii) there exist $\delta > 0$ and $\tau > 0$ such that for all $(x, y) \in \text{Gr } F \cap \overline{B_\delta(\bar{x}, \bar{y})}$ and all $\hat{y} \in \overline{B_\delta(\bar{y})}$, there is a sequence $\{(x_n, y_n)\}_{n \geq 1} \subset \text{Gr } F \setminus \{(x, y)\}$ converging to $(x, y)$ such that for all $n$ it holds

$$\|y_n - \hat{y}\| \leq \|y - \hat{y}\| - \tau d((x_n, y_n), (x, y)).$$

Proof. We have that (iii) implies (i) by Theorem 3.5.

Next, we will show that (i) implies (ii). Let $F$ be regular at $(\bar{x}, \bar{y}) \in \text{Gr } F$. By definition there exist $K > 0$ and $\delta > 0$ such that for all $x \in B_\delta(\bar{x})$ and all $y \in B_\delta(\bar{y})$ the following inequality holds:

$$d(x, F^{-1}(y)) \leq Kd(y, F(x)).$$

Fix arbitrary $(x, y) \in B_\delta(\bar{x}, \bar{y}) \cap \text{Gr } F$, $v \in Y$ with $\|v\| < \frac{1}{K} =: r$ and a sequence $t_n \to 0_+$ such that $y_n := y + t_n v \in B_\delta(\bar{y})$. Then, there exist $\varepsilon > 0$ such that $\|v\| \leq \frac{1 - \varepsilon}{K}$. Moreover, for every $n \in \mathbb{N}$ there exists $x_n \in F^{-1}(y_n)$ such that $d(x, x_n) \leq d(x, F^{-1}(y_n)) + \varepsilon t_n$. Thus

$$d(x, x_n) \leq d(x, F^{-1}(y_n)) + \varepsilon t_n \leq Kd(y_n, F(x)) + \varepsilon t_n \leq K\|y_n - y\| + \varepsilon t_n \leq (1 - \varepsilon + \varepsilon)t_n = t_n.$$

Having that $(x_n, y_n) = (x_n, y + t_n v) \in \text{Gr } F$, $v \in F^{(1)}(x, y)$ by definition. We have shown that (ii) holds, if $F$ is regular at $(\bar{x}, \bar{y})$.

It remains to prove that (ii) implies (iii). Assume that (ii) holds. Let $(x, y) \in B_\delta(\bar{x}, \bar{y}) \cap \text{Gr } F$ and $\hat{y} \in B_\delta(\bar{y})$ be arbitrary. Let us denote $v := \rho \frac{\hat{y} - y}{\|\hat{y} - y\|}$ for some $\rho \in (0, r)$. Then, $v \in Y$ with $\|v\| = \rho$ and due to (ii) there exist sequences $t_n \to 0_+$, $v_n \to v$ and $(x_n, y_n) \in \text{Gr } F$ such that $d(x_n, x) \leq t_n$ and $y_n = y + t_n v_n$. Since $\rho t_n < 1$ for $n$ – large enough, we estimate

$$\|y_n - \hat{y}\| = \left\| y - \hat{y} + t_n \rho \frac{\hat{y} - y}{\|\hat{y} - y\|} \right\| = \|y - \hat{y}\| - t_n \rho.$$

Moreover, we have that $t_n \geq d(x_n, x) \geq \frac{d(x_n, x)}{\rho + 1}$ and $t_n = \frac{\|y_n - y\|}{\|v_n\|} \geq \frac{\|y_n - y\|}{\rho + 1}$ for $n$ – large enough. Therefore

$$\|y_n - \hat{y}\| \leq \|y - \hat{y}\| - \tau d((x_n, y_n), (x, y)).$$

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of Bouligand tangent cone, there exist sequences

Let us fix an arbitrary $\lambda$ such that for every $x, y \in \text{Gr} F$, define $DF(x|y) : X \Rightarrow Y$ as the map, whose graph is the (Bouligand) tangent cone $T_{\text{Gr} F}(x, y)$, i.e.

$$v \in DF(x|y)(u) \Leftrightarrow (u, v) \in T_{\text{Gr} F}(x, y).$$

**Corollary 4.7** (cf. Theorem 1.2 in [7] and Theorem 4.13 and Remark 4.14(b) in [1]). Let $F : X \Rightarrow Y$ and $(\bar{x}, \bar{y}) \in \text{Gr} F$, where $X$ and $Y$ are Banach spaces. Assume there exist $\delta > 0$ and $K > 0$ such that for any $(x, y) \in \text{Gr} F$ with $\|x - \bar{x}\| \leq \delta$, $0 < \|y - \bar{y}\| \leq \delta$ and any $v \in Y$, $\|v\| = 1$, holds

$$\inf \left\{ \|u\| : v \in DF(x|y)(u) \right\} \leq K.$$

Then $F$ is regular at $(\bar{x}, \bar{y}) \in \text{Gr} F$. The reverse direction is also true when $X$ is finite-dimensional.

**Proof.** For the first part, we have that for every $\varepsilon > 0$, every $(x, y) \in \text{Gr} F$ with $\|x - \bar{x}\| \leq \delta$, $0 < \|y - \bar{y}\| \leq \delta$ and for every $v \in Y$ with $\|v\| = 1$ there is $(u, v) \in T_{\text{Gr} F}(x, y)$ such that $\|u\| < K + \varepsilon$. From the definition of Bouligand tangent cone, there exist sequences $u_n \to u$, $v_n \to v$ and a sequence of positive $t_n$ tending to zero, such that $(x + t_n u_n, y + t_n v_n) \in \text{Gr} F$. Let us fix an arbitrary $\lambda \in (0, 1]$. We have that $\tau_n := \frac{t_n}{\lambda (K + \varepsilon)} \to 0$ and $(x + t_n u_n, y + t_n v_n) = (x + \tau_n \frac{\lambda u_n}{K + \varepsilon}, y + \tau_n \frac{\lambda v_n}{K + \varepsilon}) \in \text{Gr} F$. Without loss of generality we can assume that $\|u_n\| \leq K + \varepsilon$ for $n$ large enough. Taking into account that $d(x, x + \tau_n \frac{\lambda u_n}{K + \varepsilon}) \leq \lambda \tau_n \leq \tau_n$, we obtain that $\frac{\lambda u}{K + \varepsilon} \in F^{(1)}(x, y)$. Since the unit vector $v \in Y$, $\lambda \in (0, 1]$ and $\varepsilon > 0$ are arbitrary and $\|\frac{\lambda u}{K + \varepsilon}\| = \frac{\lambda}{K + \varepsilon}$, we obtain that $B_{\frac{\lambda}{K + \varepsilon}}(0) \subset F^{(1)}(x, y)$. Then, $F$ is regular at $(\bar{x}, \bar{y})$ due to Corollary 4.6.

For the reverse, let $X$ be finite-dimensional and $F$ be regular at $(\bar{x}, \bar{y})$ with constants $\delta$ and $K$. Let $(x, y) \in \text{Gr} F$ with $\|x - \bar{x}\| \leq \delta$ and $0 < \|y - \bar{y}\| \leq \delta$, $\varepsilon \in (0, \frac{1}{K})$ and $v \in Y$ with $\|v\| = 1$ be arbitrary. Then, we have that $w := (\frac{1}{K} - \varepsilon)v \in F^{(1)}(x, y)$ due to Corollary 4.6. That is, there exist sequences $t_n \to 0_+$, $w_n \to w$ and $(x_n, y_n) \in \text{Gr} F$ such that $\|x_n - x\| \leq t_n$ and $y_n = y + t_n w_n$. Moreover, since $X$ is finite-dimensional, we have $x_n = x + t_n p_0$, where $\tau := \frac{\lambda}{2(\rho + 1)}$.

The proof is complete. $\square$
where \( \|p_n\| \leq 1 \) which implies \( p_n \rightarrow_{n \to \infty} p \) (up to a subsequence, labeled in the same way) and \((p, w) \in T_{Gr} F\). Hence \((\frac{p}{\|p\|}, \frac{w}{\|w\|}) = (\frac{p}{K - \epsilon}, v) \in T_{Gr} F\) and \(\|\frac{p}{K - \epsilon}\| \leq \frac{1}{K - \epsilon}\). We have obtained that for any \( v \in Y, \|v\| = 1 \), it holds

\[
\inf \left\{ \|u\| \mid v \in DF(x|y)(u) \right\} \leq K.
\]

The proof is complete.

\[\square\]

5 Characterizations of transversality-type properties in terms of the coupling function

In this section we provide characterizations of subtransversality, transversality and tangential transversality in terms of the so-called coupling function. In this way we find a primal characterization of intrinsic transversality very close to the notion of tangential transversality. We begin by some necessary preliminaries.

For a set \( A \) in a metric space \( X \), we denote by \( \delta_A : X \rightarrow \mathbb{R} \cup \{+\infty\} \) its indicator function

\[
\delta_A(x) = \begin{cases} 
0, & \text{if } x \in A \\
+\infty, & \text{otherwise}.
\end{cases}
\]

**Definition 5.1.** Consider a metric space \( X \), a function \( f : X \rightarrow \mathbb{R} \cup \{-\infty, +\infty\} \) and a point \( \bar{x} \in X \) such that \( f(\bar{x}) \) is finite. The slope of \( f \) at \( \bar{x} \) is

\[
|\nabla f|^0(\bar{x}) := \limsup_{x \to \bar{x}} \frac{\max\{f(\bar{x}) - f(x), 0\}}{d(\bar{x}, x)}.
\]

The nonlocal slope is

\[
|\nabla f|^\circ(\bar{x}) := \sup_{x \neq \bar{x}} \frac{\max\{f(\bar{x}) - f(x), 0\}}{d(\bar{x}, x)}.
\]

For subsets \( A \) and \( B \) of the metric space \( X \), consider the so-called “coupling function” \( \phi : X \times X \rightarrow \mathbb{R} \cup \{+\infty\} \) introduced in [9] and defined as

\[
\phi(x, y) = \delta_A(x) + d(x, y) + \delta_B(y).
\]
Fix a point \( \bar{x} \in A \cap B \). First, we state two characterizations of sub-transversality and transversality in terms of the slope of the coupling function.

The following proposition is a reformulation of Theorem 3.6.

**Proposition 5.2.** Under completeness of the space \( X \), \( A \) and \( B \) are subtransversal at \( \bar{x} \) if and only if there exist \( \delta > 0 \) and \( \kappa > 0 \) such that for all \( x \in A \cap \overline{B_\delta(\bar{x})} \) and \( y \in B \cap \overline{B_\delta(\bar{x})} \) it holds

\[
|\nabla \phi|^\circ(x, y) = \sup_{(u, v) \neq (x, y)} \frac{\max\{\phi(x, y) - \phi(u, v), 0\}}{d((x, y), (u, v))} \geq \kappa.
\]

The following proposition is a reformulation of Proposition 4.1 and Proposition 4.2 is used for the equality of local and nonlocal slope.

**Proposition 5.3.** Under the additional assumption that \( X \) is a Banach space, \( A \) and \( B \) are transversal at \( \bar{x} \) if and only if there exist \( \delta > 0 \) and \( \kappa > 0 \) such that for all \( a \) and \( b \) with \( \|a\| \leq \delta \) and \( \|b\| \leq \delta \) and all \( x \in (A - a) \cap \overline{B_\delta(\bar{x})} \) and \( y \in (B - b) \cap \overline{B_\delta(\bar{x})} \) it holds

\[
|\nabla \phi_{a,b}|^\circ(x, y) = \sup_{(u, v) \neq (x, y)} \frac{\max\{\phi_{a,b}(x, y) - \phi_{a,b}(u, v), 0\}}{\|x - y\|} \geq \kappa
\]

where \( \phi_{a,b} \) denotes the coupling function of \( A - a \) and \( B - b \).

Moreover, we have that for all \( x \in (A - a) \cap \overline{B_\delta(\bar{x})} \) and \( y \in (B - b) \cap \overline{B_\delta(\bar{x})} \) the local and nonlocal slopes are equal: \( |\nabla \phi_{a,b}(x, y)| = |\nabla \phi_{a,b}|^\circ(x, y) \).

Next, we obtain a characterization of tangential transversality.

**Proposition 5.4.** The subsets \( A \) and \( B \) of the metric space \( X \) are tangentially transversal at \( \bar{x} \) if and only if there exist \( \delta > 0 \) and \( \kappa > 0 \) such that for any two different points \( x \in A \cap \overline{B_\delta(\bar{x})} \) and \( y \in B \cap \overline{B_\delta(\bar{x})} \) it holds

\[
|\nabla \phi|(x, y) = \limsup_{(u, v) \to (x, y)} \frac{\max\{\phi(x, y) - \phi(u, v), 0\}}{d((x, y), (u, v))} \geq \kappa.
\]

**Proof.** Let us assume that \( A \) and \( B \) are tangentially transversal at \( \bar{x} \in A \cap B \). Then, there exist \( M > 0 \), \( \delta > 0 \) and \( \kappa > 0 \) such that for any two different points \( x^A \in \overline{B_\delta(\bar{x})} \cap A \) and \( x^B \in \overline{B_\delta(\bar{x})} \cap B \), there exists a sequence \( \{t_m\} \),
\( t_m \downarrow 0 \), such that for every \( m \in \mathbb{N} \) there exist \( x_m^A \in A \) with \( d(x_m^A, x^A) \leq t_m M \), and \( x_m^B \in A \) with \( d(x_m^B, x^B) \leq t_m M \), and the following inequality holds true
\[
d(x_m^A, x_m^B) \leq d(x^A, x^B) - t_m 2M \kappa.
\]

This is equivalent to
\[
\frac{d(x^A, x^B) - d(x_m^A, x_m^B)}{t_m} \geq 2M \kappa.
\]

Using that \( d(x^A, x_m^A) \leq Mt_m \) and \( d(x^B, x_m^B) \leq Mt_m \), we have that
\[
\frac{d(x^A, x^B) - d(x_m^A, x_m^B)}{d(x^A, x_m^A) + d(x^B, x_m^B)} \geq \kappa.
\]

We have obtained that
\[
|\nabla \phi|(x^A, x^B) \geq \kappa
\]
for any two different points \( x^A \in \overline{\mathcal{B}}_\delta(\bar{x}) \cap A \) and \( x^B \in \overline{\mathcal{B}}_\delta(\bar{x}) \cap B \).

For the converse, we have that there exist \( \delta > 0 \) and \( \kappa > 0 \) such that
for any two different points \( x^A \in \overline{\mathcal{B}}_\delta(\bar{x}) \cap A \) and \( x^B \in \overline{\mathcal{B}}_\delta(\bar{x}) \cap B \) holds
\[
|\nabla \phi|(x^A, x^B) \geq \kappa.
\]
This means that there exist a sequence \( \{x_m^A\} \) tending to \( x^A \) and a sequence \( \{x_m^B\} \) tending to \( x^B \), such that
\[
\max\{\phi(x^A, x^B) - \phi(x_m^A, x_m^B), 0\} \geq \kappa / 2.
\]
Since \( \kappa \) is positive, we have that
\[
\frac{\phi(x^A, x^B) - \phi(x_m^A, x_m^B)}{d((x^A, x^B), (x_m^A, x_m^B))} \geq \frac{\kappa}{2}
\]
and by the definition of the coupling function it follows that that \( \{x_m^A\} \subset A \) and \( \{x_m^B\} \subset B \). We also have that
\[
\frac{d(x^A, x^B) - d(x_m^A, x_m^B)}{d(x^A, x_m^A) + d(x^B, x_m^B)} \geq \frac{\kappa}{2},
\]
since we consider the product metric space \( X \times X \) to be equipped with the sum metric.

By setting \( t_m := d(x^A, x_m^A) + d(x^B, x_m^B) \to 0 \), we obtain that
\[
d(x_m^A, x_m^B) \leq d(x^A, x^B) - t_m \frac{\kappa}{2}.
\]
Since \( d(x^A, x_m^A) \leq t_m \) and \( d(x^B, x_m^B) \leq t_m \), we have verified that \( A \) and \( B \) are tangentially transversal at \( \bar{x} \in A \cap B \).
We continue to show the “almost” equivalence of intrinsic transversality and tangential transversality. Intrinsic transversality is introduced in [9] and [10] as a sufficient condition for local linear convergence of the alternating projections algorithm in finite dimensions. Here is an equivalent definition in Banach spaces:

**Definition 5.5.** Let $X$ be a metric space. The closed sets $A, B \subset X$ are intrinsically transversal at the point $\bar{x} \in A \cap B$, if there exist $\delta > 0$ and $\kappa > 0$ such that for all $x^A \in B_\delta(\bar{x}) \cap A \setminus B$ and $x^B \in B_\delta(\bar{x}) \cap B \setminus A$ it holds true that

$$|\nabla \phi(x^A, x^B)| \geq \kappa.$$

This definition is equivalent to the original one, given in finite dimensional spaces (cf. Proposition 4.2 in [10]).

Due to Proposition 5.4 we have that the only difference between tangential transversality and intrinsic transversality is that in the original definition of tangential transversality the required condition should hold for all points of $A$ and $B$ (respectively) near the reference point, whereas in intrinsic transversality – only for points in $A \setminus B$ and $B \setminus A$ (respectively).

**Corollary 5.6.** Let $X$ be a metric space. The closed sets $A, B \subset X$ are intrinsically transversal at the point $\bar{x} \in A \cap B$, if and only if there exist $M > 0, \delta > 0$ and $\eta > 0$ such that for any two different points $x^A \in B_\delta(\bar{x}) \cap A \setminus B$ and $x^B \in B_\delta(\bar{x}) \cap B \setminus A$, there exist sequences $t_m \downarrow 0, \{x^A_m\}_{m \geq 1} \subset A$ and $\{x^B_m\}_{m \geq 1} \subset B$ such that for all $m$

$$d(x^A_m, x^A) \leq t_m M, \quad d(x^B_m, x^B) \leq t_m M, \quad d(x^A_m, x^B_m) \leq d(x^A, x^B) - t_m \eta.$$

In this way we answer a question of Prof. A. Ioffe about finding a metric characterization of intrinsic transversality.

The following example shows that although the difference is slight, the notion of tangential transversality is stronger than the one of intrinsic transversality.

**Example 5.7.** Consider the sets in $\mathbb{R}^2$,

$$A = \{(x, y) \mid y = 3x, \ x \geq 0\} \cup \left\{\left(\frac{1}{n}, \frac{2}{n}\right)\right\}_{n \geq 1}$$

and

$$B = \{(x, y) \mid y = x, \ x \geq 0\} \cup \left\{\left(\frac{1}{n}, \frac{2}{n}\right)\right\}_{n \geq 1}.$$
Apparently these two sets are intrinsically transversal at \((0,0)\), however they are not tangentially transversal, because there are isolated points of the intersection in every neighbourhood of the reference point.

We are also able to answer some of the questions posed in [4]:

1. **Tangential transversality is an intermediate property between transversality and subtransversality.** However, the exact relation between this new concept and the established notions of transversality, intrinsic transversality and subtransversality is not clarified yet.

   This question is now fully answered in the case of complete metric spaces. The characterizations of intrinsic transversality and tangential transversality show that the examples at the end of Section 6 in [10] may be used to prove that tangential transversality is strictly between transversality and subtransversality even in \(\mathbb{R}^d\).

2. **It would be useful to find some dual characterization of tangential transversality.**

   The original definition of intrinsic transversality is stated in dual terms (Definition 2.2 in [9] and Definition 3.1 in [10]) – the closed sets \(A, B \subset \mathbb{R}^d\) are intrinsically transversal at the point \(\bar{x} \in A \cap B\), if and only if there exist \(\delta > 0\) and \(\kappa > 0\) such that for all \(x^A \in \overline{B_\delta(\bar{x})} \cap A \setminus B\) and \(x^B \in \overline{B_\delta(\bar{x})} \cap B \setminus A\) it holds true that

   \[
   \max \left\{ d \left( \frac{x^A - x^B}{\| x^A - x^B \|}, N_B \left( x^B \right) \right), d \left( \frac{x^B - x^A}{\| x^B - x^A \|}, N_A \left( x^A \right) \right) \right\} \geq \kappa,
   \]

   where \(N_D(\bar{x})\) is the proximal or limiting normal cone to \(D\) at \(\bar{x}\).

   Replacing “\(x^A \in B_\delta(\bar{x}) \cap A \setminus B\) and \(x^B \in B_\delta(\bar{x}) \cap B \setminus A\)” with “\(x^A \in B_\delta(\bar{x}) \cap A, x^B \in B_\delta(\bar{x}) \cap B\) and \(x^A \neq x^B\)” we obtain a dual characterization of tangential transversality in finite dimensions.

   It is known that intrinsic transversality and subtransversality coincide for convex sets in finite-dimensional spaces (cf. Proposition 6.1 in [14] or Corollary 3.4 in [21] for an alternative proof). Both proofs exploit the dual characterizations of intrinsic transversality and subtransversality. Now we can easily obtain the stronger result

   **Corollary 5.8.** Let \(X\) be a Banach space. The closed convex sets \(A, B \subset X\) are tangentially transversal at the point \(\bar{x} \in A \cap B\), if and only if they are subtransversal at \(\bar{x}\).
Proof. It is enough to check, that if the sets are subtransversal, they are moreover tangentially transversal. According to the primal characterization obtained in Theorem 3.6, subtransversality implies WTT property with some constants $\delta$ and $M$. Let $x^A \in A \cap \overline{B}_\delta(\bar{x})$ and $x^B \in B \cap \overline{B}_\delta(\bar{x})$. Then there are $\hat{x}^A \in A$, $\hat{x}^B \in B$ and $\theta > 0$, such that

$$\|x^A - \hat{x}^A\| \leq \theta M, \quad \|x^B - \hat{x}^B\| \leq \theta M \quad \text{and} \quad \|\hat{x}^A - \hat{x}^B\| \leq \|x^A - x^B\| - \theta.$$  

Let $\{r_n\}_{n \geq 1} \subset (0, 1)$ be a sequence tending to 0. Since $A$ is convex, $x^A_n := (1 - r_n) x^A + r_n \hat{x}^A \in A$ for all $n \in \mathbb{N}$. Similarly for $x^B_n$. Then

$$\|x^A_n - x^B_n\| = \|(1 - r_n) (x^A - x^B) + r_n (\hat{x}^A - \hat{x}^B)\|$$

$$\leq (1 - r_n) \|x^A - x^B\| + r_n (\|x^A - x^B\| - \theta) = \|x^A - x^B\| - t_n$$

where $t_n = r_n \theta$. Moreover, for

$$w^A_n := \frac{x^A_n - x^A}{t_n}$$

we have

$$\|w^A_n\| = \frac{1}{t_n} \cdot r_n \|\hat{x}^A - x^A\| \leq \frac{1}{r_n \theta} r_n \theta M = M,$$

and similarly for $w^B_n$.

\[ \square \]

Thus intrinsic transversality also coincides with tangential transversality and subtransversality in the case of convex sets. This last equivalence is also straight-forward to obtain via function slopes characterizations – using that for convex functions the limiting slope and the nonlocal slope coincide (cf. e.g. Proposition 2.1(vii) in [20]), the result follows from Propositions 5.2 and 5.4.

We refer the reader to the papers [21] and [22] for a generalization of intrinsic transversality to Hilbert spaces, based on the normal structure. We have not explored the relation between our version of intrinsic transversality (Definition 5.5) and theirs (Definition 2(ii) in [21] and Definition 3 in [22]).

\section*{References}

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