Privacy Amplification by Mixing
and Diffusion Mechanisms

Borja Balle, Gilles Barthe\(^{1,2}\), Marco Gaboardi\(^{3}\), and Joseph Geumlek\(^{4}\)

\(^{1}\)MPI-SP
\(^{2}\)IMDEA Software Institute
\(^{3}\)University at Buffalo, SUNY
\(^{4}\)U.C. San Diego

October 29, 2019

Abstract

A fundamental result in differential privacy states that the privacy guarantees of a mechanism are preserved by any post-processing of its output. In this paper we investigate under what conditions stochastic post-processing can amplify the privacy of a mechanism. By interpreting post-processing as the application of a Markov operator, we first give a series of amplification results in terms of uniform mixing properties of the Markov process defined by said operator. Next we provide amplification bounds in terms of coupling arguments which can be applied in cases where uniform mixing is not available. Finally, we introduce a new family of mechanisms based on diffusion processes which are closed under post-processing, and analyze their privacy via a novel heat flow argument. On the applied side, we generalize the analysis of “privacy amplification by iteration” in Noisy SGD and show it admits an exponential improvement in the strongly convex case, and study a mechanism based on the Ornstein–Uhlenbeck diffusion process which contains the Gaussian mechanism with optimal post-processing on bounded inputs as a special case.

1 Introduction

Differential privacy (DP) \cite{Dwork2006} has arisen in the last decade into a strong de-facto standard for privacy-preserving computation in the context of statistical analysis. The success of DP is based, at least in part, on the availability of robust building blocks (e.g., the Laplace, exponential and Gaussian mechanisms) together with relatively simple rules for analyzing complex mechanisms built out of these blocks (e.g., composition and robustness to post-processing). The inherent tension between privacy and utility in practical applications has sparked a renewed interest into the development of further rules leading to tighter privacy bounds. A trend in this direction is to find ways to measure the privacy introduced by sources of randomness that are not accounted for by standard composition rules. Generally speaking, these are referred to as privacy amplification rules, with prominent examples being amplification by subsampling \cite{Chaudhuri2006,Chaudhuri2011,Li2012,Beimel2013,Beimel2014,Bun2015,Balle2019}.

1
Motivated by these considerations, in this paper we initiate a systematic study of privacy amplification by stochastic post-processing. Specifically, given a DP mechanism \( M \) producing (probabilistic) outputs in \( \mathcal{X} \) and a Markov operator \( K \) defining a stochastic transition between \( \mathcal{X} \) and \( \mathcal{Y} \), we are interested in measuring the privacy of the post-processed mechanism \( K \circ M \) producing outputs in \( \mathcal{Y} \). The standard post-processing property of DP states that \( K \circ M \) is at least as private as \( M \). Our goal is to understand under what conditions the post-processed mechanism \( K \circ M \) is strictly more private than \( M \). Roughly speaking, this amplification should be non-trivial when the operator \( K \) “forgets” information about the distribution of its input \( M(D) \). Our main insight is that, at least when \( \mathcal{Y} = \mathcal{X} \), the forgetfulness of \( K \) from the point of view of DP can be measured using similar tools to the ones developed to analyze the speed of convergence, i.e. mixing, of the Markov process associated with \( K \).

In this setting, we provide three types of results, each associated with a standard method used in the study of convergence for Markov processes. In the first place, Section 3 provides DP amplification results for the case where the operator \( K \) satisfies a uniform mixing condition. These include standard conditions used in the analysis of Markov chains on discrete spaces, including the well-known Dobrushin coefficient and Doeblin’s minorization condition [Levin and Peres, 2017]. Although in principle uniform mixing conditions can also be defined in more general non-discrete spaces [Del Moral et al., 2003], most Markov operators of interest in \( \mathbb{R}^d \) do not exhibit uniform mixing since the speed of convergence depends on how far apart the initial inputs are. Convergence analyses in this case rely on more sophisticated tools, including Lyapunov functions [Meyn and Tweedie, 2012], coupling methods [Lindvall, 2002] and functional inequalities [Bakry et al., 2013].

Following these ideas, Section 4 investigates the use of coupling methods to quantify privacy amplification by post-processing under Rényi DP [Mironov, 2017]. These methods apply to operators given by, e.g., Gaussian and Laplace distributions, for which uniform mixing does not hold. Results in this section are intimately related to the privacy amplification by iteration phenomenon studied in [Feldman et al., 2018] and can be interpreted as extensions of their main results to more general settings. In particular, our analysis unpacks the shifted Rényi divergence used in the proofs from [Feldman et al., 2018] and allows us to easily track the effect of iterating arbitrary noisy Lipschitz maps. As a consequence, we show an exponential improvement on the privacy amplification by iteration of Noisy SGD in the strongly convex case which follows from applying this generalized analysis to strict contractions.

Our last set of results concerns the case where \( K \) is replaced by a family of operators \( (P_t)_{t \geq 0} \) forming a Markov semigroup [Bakry et al., 2013]. This is the natural setting for continuous-time Markov processes, and includes diffusion processes defined in terms of stochastic differential equations [Øksendal, 2003]. In Section 5 we associate (a collection of) diffusion mechanisms \( (M_t)_{t \geq 0} \) to a diffusion semigroup. Interestingly, these mechanisms are, by construction, closed under post-processing in the sense that \( P_s \circ M_t = M_{s+t} \). We show the Gaussian mechanism falls into this family – since Gaussian noise is closed under addition – and also present a new mechanism based on the Ornstein-Uhlenbeck process which has better mean squared error than the standard Gaussian mechanism (and matches the error of the optimally post-processed Gaussian mechanism with bounded inputs). Our main result on diffusion mechanisms provides a generic Rényi DP guarantee based on an intrinsic notion of sensitivity derived from the geometry induced by the semigroup. The proof relies on a heat flow argument reminiscent of the analysis of mixing in diffusion processes based on...
2 Background

We start by introducing notation and concepts that will be used throughout the paper. We write
\[ n = \{1, \ldots, n\}, \quad a \wedge b = \min\{a, b\} \quad \text{and} \quad [a]_+ = \max\{a, 0\}. \]

**Probability.** Let \( \mathbb{X} = (\mathbb{X}, \Sigma, \lambda) \) be a measurable space with sigma-algebra \( \Sigma \) and base measure \( \lambda \). We write \( \mathcal{P}(\mathbb{X}) \) to denote the set of probability distributions on \( \mathbb{X} \). Given a probability distribution \( \mu \in \mathcal{P}(\mathbb{X}) \) and a measurable event \( E \subseteq \mathbb{X} \) we write \( \mu(E) = \mathbb{P}[X \in E] \) for a random variable \( X \sim \mu \), denote its expectation under \( f : \mathbb{X} \to \mathbb{R}^d \) by \( \mathbb{E} [f(X)] \), and can get back its distribution as \( \mu = \text{Law}(X) \). Given two distributions \( \mu, \nu \) (or, in general, arbitrary measures) we write \( \mu \ll \nu \) to denote that \( \mu \) is absolutely continuous with respect to \( \nu \), in which case there exists a Radon-Nikodym derivative \( \frac{d\mu}{d\nu} \). We shall reserve the notation \( p_\mu = \frac{d\mu}{d\lambda} \) to denote the density of \( \mu \) with respect to the base measure. We also write \( \mathcal{C}(\mu, \nu) \) to denote the set of couplings between \( \mu \) and \( \nu \); i.e. \( \pi \in \mathcal{C}(\mu, \nu) \) is a distribution on \( \mathcal{P}(\mathbb{X} \times \mathbb{X}) \) with marginals \( \mu \) and \( \nu \). The support of a distribution is \( \text{supp}(\mu) \).

**Markov Operators.** We will use \( \mathcal{K}(\mathbb{X}, \mathbb{Y}) \) to denote the set of Markov operators \( K : \mathbb{X} \to \mathcal{P}(\mathbb{Y}) \) defining a stochastic transition map between \( \mathbb{X} \) and \( \mathbb{Y} \) and satisfying that \( x \mapsto K(x)(E) \) is measurable for every measurable \( E \subseteq \mathbb{Y} \). Markov operators act on distributions \( \mu \in \mathcal{P}(\mathbb{X}) \) on the left through \( (\mu K)(E) = \int K(x)(E) \mu(dx) \), and on functions \( f : \mathbb{Y} \to \mathbb{R} \) on the right through \( (Kf)(x) = \int f(y)K(x, dy) \), which can also be written as \( (Kf)(x) = \mathbb{E} [f(X)] \) with \( X \sim K(x) \). The kernel of a Markov operator \( K \) (with respect to \( \lambda \)) is the function \( k(x, \cdot) = \frac{dK(x)}{d\lambda} \) associating with \( x \) the density of \( K(x) \) with respect to a fixed measure.

**Divergences.** A popular way to measure dissimilarity between distributions is to use Csiszár divergences \( D_\phi(\mu \| \nu) = \int \phi(\frac{d\mu}{d\nu}) d\nu \), where \( \phi : \mathbb{R}_+ \to \mathbb{R} \) is convex with \( \phi(1) = 0 \). Taking \( \phi(u) = \frac{1}{2} |u - 1| \) yields the total variation distance \( TV(\mu, \nu) \), and the choice \( \phi(u) = [u - e^\varepsilon]^+ \) with \( \varepsilon \geq 0 \) gives the hockey-stick divergence \( D_{\varepsilon^+}(\mu \| \nu) \), which satisfies
\[
D_{\varepsilon^+}(\mu \| \nu) = \int \left[ \frac{d\mu}{d\nu} - e^\varepsilon \right]_+ d\nu = \int [p_\mu - e^\varepsilon p_\nu]_+ d\lambda = \sup_{E \subseteq \mathbb{X}} \left( \mu(E) - e^\varepsilon \nu(E) \right) .
\]
It is easy to check that \( \varepsilon \mapsto D_{\varepsilon^+}(\mu \| \nu) \) is monotonically decreasing and \( D_1 = TV \). All Csiszár divergences satisfy joint convexity \( D((1 - \gamma)\mu_1 + \gamma \mu_2 \| (1 - \gamma)\nu_1 + \gamma \nu_2) \leq (1 - \gamma)D(\mu_1 \| \nu_1) + \gamma D(\mu_2 \| \nu_2) \) and the data processing inequality \( D(\mu K \| \nu K) \leq D(\mu \| \nu) \) for any Markov operator \( K \). Rényi divergences\(^1\) are another way to compare distributions. For \( \alpha > 1 \) the Rényi divergence of order \( \alpha \) is defined as \( R_\alpha(\mu \| \nu) = \frac{1}{\alpha - 1} \log \int \left( \frac{d\mu}{d\nu} \right)^\alpha d\nu \), and also satisfies the data processing inequality. Finally, to measure similarity between \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \) we sometimes use the \( \infty \)-Wasserstein distance:
\[
W_\infty(\mu, \nu) = \inf_{\pi \in \mathcal{C}(\mu, \nu)} \inf \{ w \geq 0 : \|X - Y\| \leq w \text{ holds almost surely for } (X, Y) \sim \pi \} .
\]

**Differential Privacy.** A mechanism \( M : \mathbb{D}^n \to \mathcal{P}(\mathbb{X}) \) is a randomized function that takes a dataset \( D \in \mathbb{D}^n \) over some universe of records \( \mathbb{D} \) and returns a (sample from) distribution \( M(D) \). We write \( D \simeq D' \) to denote two databases differing in a single record. We say that \( M \) satisfies\(^2\)

---

1. Rényi divergences do not belong to the family of Csiszár divergences.
2. This divergence characterization of DP is due to Bakry et al., 2013. Barthe and Olmedo, 2013.
3 Amplification From Uniform Mixing

We start our analysis of privacy amplification by stochastic post-processing by considering settings where the Markov operator $K$ satisfies one of the following uniform mixing conditions.

**Definition 1.** Let $K \in \mathcal{K}(\mathbb{X}, \mathbb{Y})$ be a Markov operator, $\gamma \in [0,1]$ and $\varepsilon \geq 0$. We say that $K$ is:

1. $\gamma$-Dobrushin if $\sup_{x,x'} TV(K(x), K(x')) \leq \gamma$,
2. $(\gamma, \varepsilon)$-Dobrushin if $\sup_{x,x'} D_{\varepsilon}(K(x)||K(x')) \leq \gamma$,
3. $\gamma$-Doeblin if there exists a distribution $\omega \in \mathcal{P}(\mathbb{Y})$ such that $K(x) \geq (1 - \gamma)\omega$ for all $x \in \mathbb{X}$,
4. $\gamma$-ultra-mixing if for all $x, x' \in \mathbb{X}$ we have $K(x) \ll K(x')$ and $\frac{dK(x)}{dK(x')} \geq 1 - \gamma$.

Most of these conditions arise in the context of mixing analyses in Markov chains. In particular, the Dobrushin condition can be tracked back to [Dobrushin, 1956], while Doeblin’s condition was introduced earlier [Doeblin, 1937] (see also [Nummelin, 2004]). Ultra-mixing is a strengthening of the Dobrushin condition can be tracked back to [Dobrushin, 1956], while Doeblin’s condition was introduced earlier [Doeblin, 1937] (see also [Nummelin, 2004]). Ultra-mixing is a strengthening of Doeblin’s condition used in [Del Moral et al., 2003]. The $(\gamma, \varepsilon)$-Dobrushin is, on the other hand, new and is designed to be a generalization of Dobrushin tailored for amplification under the hockey-stick divergence.

It is not hard to see that Dobrushin’s is the weakest among these conditions, and in fact we have the implications summarized in Figure 1 (see Lemma 9). This explains why the amplification bounds in the following result are increasingly stronger, and in particular why the first two only provide amplification in $\delta$, while the last two also amplify the $\varepsilon$ parameter.

**Theorem 1.** Let $M$ be an $(\varepsilon, \delta)$-DP mechanism. For a given Markov operator $K$, the post-processed mechanism $K \circ M$ satisfies:

1. $(\varepsilon, \delta')$-DP with $\delta' = \gamma \delta$ if $K$ is $\gamma$-Dobrushin,
2. $(\varepsilon, \delta')$-DP with $\delta' = \gamma \delta$ if $K$ is $(\gamma, \varepsilon)$-Dobrushin with $\bar{\varepsilon} = \log(1 + \frac{\varepsilon - 1}{\delta})$,
3. $(\varepsilon', \delta')$-DP with $\varepsilon' = \log(1 + \gamma(\varepsilon - 1))$ and $\delta' = \gamma(1 - e^{\varepsilon' - \varepsilon}(1 - \delta))$ if $K$ is $\gamma$-Doeblin,
4. $(\varepsilon', \delta')$-DP with $\varepsilon' = \log(1 + \gamma(\varepsilon - 1))$ and $\delta' = \gamma \delta e^{\varepsilon' - \varepsilon}$ if $K$ is $\gamma$-ultra-mixing.

A few remarks about this result are in order. First we note that (2) is stronger than (1) since the monotonicity of hockey-stick divergences implies $TV = D_1 \geq D_\varepsilon$. Also note how in the results above we always have $\varepsilon' \leq \varepsilon$, and in fact the form of $\varepsilon'$ is the same as obtained under amplification by subsampling when, e.g., a $\gamma$-fraction of the original dataset is kept. This is not a coincidence since the proofs of (3) and (4) leverage the overlapping mixtures technique used to analyze amplification by subsampling in [Balle et al., 2018]. However, we note that for (3) we can have $\delta' > 0$ even with $\delta = 0$. In fact the Doeblin condition only leads to an amplification in $\delta$ if $\gamma \leq \frac{\delta e^{\varepsilon'} - 1}{(1 - \delta)(\varepsilon' - 1)}$.

We conclude this section by noting that the conditions in Definition 1, despite being quite natural, might be too stringent for proving amplification for DP mechanisms on, say, $\mathbb{R}^d$. One way to see this is to interpret the operator $K : \mathbb{X} \rightarrow \mathcal{P}(\mathbb{Y})$ as a mechanism and to note that the uniform

\[ D_\infty(\mu||\nu) = \mu(\text{supp}(\mu) \setminus \text{supp}(\nu)) \]
mixing conditions on $K$ can be rephrased in terms of local DP (LDP) properties (see Table 1 for property translations) where the supremum is taken over any pair of inputs (instead of neighboring ones). This motivates the results on next section, where we look for finer conditions to prove amplification by stochastic post-processing.

4 Amplification From Couplings

In this section we turn to coupling-based proofs of amplification by post-processing under the Rényi DP framework. Our first result is a measure-theoretic generalization of the shift-reduction lemma in [Feldman et al., 2018] which does not require the underlying space to be a normed vector space. The main differences in our proof are to use explicit couplings instead of the shifted Rényi divergence which implicitly relies on the existence of a norm (through the use of $W_{\infty}$), and replace the identity $U + W - W = U$ between random variables which depends on the vector-space structure with a transport operators $H_{\pi}$ and $H_{\pi'}$ which satisfy $\mu H_{\pi'} H_{\pi} = \mu$ in a general measure-theoretic setting.

Given a coupling $\pi \in C(\mu, \nu)$ with $\mu, \nu \in \mathcal{P}(X)$ we construct a transport Markov operator $H_{\pi} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ with kernel $h_{\pi}(x, y) = \frac{p_{\pi}(x, y)}{p_{\mu}(x)}$, where $p_{\pi} = \frac{d\pi}{d\lambda \otimes \lambda}$ and $p_{\mu} = \frac{d\mu}{d\lambda}$. It is immediate to verify from the definition that $H_{\pi}$ is a Markov operator satisfying the transport property $\mu H_{\pi} = \nu$ (see Lemma 16).

**Theorem 2**. Let $\alpha \geq 1$, $\mu, \nu \in \mathcal{P}(\mathcal{X})$ and $K \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$. For any distribution $\omega \in \mathcal{P}(X)$ and coupling $\pi \in C(\omega, \mu)$ we have

$$R_\alpha(\mu K \parallel \nu K) \leq R_\alpha(\omega \parallel \nu) + \sup_{x \in \text{supp}(\nu)} R_\alpha((H_{\pi} K)(x) \parallel K(x)).$$

Note that this result captures the data-processing inequality for Rényi divergences since taking $\omega = \mu$ and the identity coupling yields $R_\alpha(\mu K \parallel \nu K) \leq R_\alpha(\mu \parallel \nu)$. The next examples illustrate the use of this theorem to obtain amplification by operators corresponding to the addition of Gaussian and Laplace noise.

**Example 1 (Iterated Gaussian)**. We can show that (1) is tight and equivalent to the shift-reduction lemma [Feldman et al., 2018] on $\mathbb{R}^d$ by considering the simple scenario of adding Gaussian noise to the output of a Gaussian mechanism. In particular, suppose $M(D) = \mathcal{N}(f(D), \sigma_2^2 I)$ for some function $f$ with global $L_2$-sensitivity $\Delta$ and the Markov operator $K$ is given by $K(x) = \mathcal{N}(x, \sigma_2^2 I)$.
The post-processed mechanism is given by \((K \circ M)(D) = \mathcal{N}(f(D), (\sigma_1^2 + \sigma_2^2) I)\), which satisfies \((\alpha, \frac{\alpha \Delta^2}{2(\sigma_1^2 + \sigma_2^2)})\)-RDP. We now show how this result also follows from Theorem \[2\] Given two datasets \(D \approx D'\) we write \(\mu = M(D) = \mathcal{N}(u, \sigma_1^2 I)\) and \(\nu = M(D') = \mathcal{N}(v, \sigma_1^2 I)\) with \(\|u - v\| \leq \Delta\). We take \(\omega = \mathcal{N}(w, \sigma_1^2 I)\) for some \(w\) to be determined later, and couple \(\omega\) and \(\mu\) through a translation \(\tau = u - w\), yielding a coupling \(\pi\) with \(p_\pi(x, y) \propto \exp(-\frac{\|x - w\|^2}{2\sigma_1^2})\|y = x + \tau\) and a transport operator \(H_\pi\) with kernel \(h_\pi(x, y) = \|y = x + \tau\). Plugging these into (1) we get

\[
R_\alpha(\mu K \| \nu K) \leq \frac{\alpha \|w - v\|^2}{2\sigma_1^2} + \sup_{x \in \mathbb{R}^d} R_\alpha(K(x + \tau) \| K(x)) = \frac{\alpha}{2} \left( \frac{\|w - v\|^2}{\sigma_1^2} + \frac{\|u - w\|^2}{\sigma_2^2} \right).
\]

Finally, taking \(w = \theta u + (1 - \theta)v\) with \(\theta = (1 + \frac{\sigma_2^2}{\sigma_1^2})^{-1}\) yields \(R_\alpha(\mu K \| \nu K) \leq \frac{\alpha \Delta^2}{2(\sigma_1^2 + \sigma_2^2)}\).

**Example 2** (Iterated Laplace). To illustrate the flexibility of this technique, we also apply it to get an amplification result for iterated Laplace noise, in which Laplace noise is added to the output of a Laplace mechanism. We begin by noting a negative result that there is no amplification in the \((\varepsilon, 0)\)-DP regime.

**Lemma 3** Let \(M(D) = \text{Lap}(f(D), \lambda_1)\) for some function \(f : \mathbb{D} \to \mathbb{R}\) with global \(L_1\)-sensitivity \(\Delta\) and let the Markov operator \(K\) be given by \(K(x) = \text{Lap}(x, \lambda_2)\). The post-processed mechanism \((K \circ M)\) does not achieve \((\varepsilon, 0)\)-DP for any \(\varepsilon < \frac{\Delta}{\max(\lambda_1, \lambda_2)}\). Note that \(M\) achieves \((\frac{\Delta}{\lambda_1}, 0)\)-DP and \(K(f(D))\) achieves \((\frac{\Delta}{\lambda_2}, 0)\)-DP.

However, the iterated Laplace mechanism \(K \circ M\) above still offers additional privacy in the relaxed RDP setting. An application of (1) allows us to identify some of this improvement. Recall from [Mironov 2017, Corollary 2] that \(M\) satisfies \((\alpha, \frac{\alpha}{\lambda_1} \log g_\alpha(\frac{\alpha}{\lambda_1}))\)-RDP with \(g_\alpha(z) = \frac{\alpha}{\lambda_1} \exp(z(\alpha - 1)) + \frac{\alpha}{2\alpha - 1} \exp(-za)\). As in Example 1 we take \(\omega = \text{Lap}(w, \lambda_1)\) for some \(w\) to be determined later, and couple \(\omega\) and \(\mu\) through a translation \(\tau = u - w\). Through (1) we obtain

\[
R_\alpha(\mu K \| \nu K) \leq \frac{1}{\alpha - 1} \log \left( g_\alpha \left( \frac{\|w - v\|}{\lambda_1} \right) \right) + \sup_{x \in \mathbb{R}} R_\alpha(K(x + \tau) \| K(x)) \leq \frac{1}{\alpha - 1} \log \left( g_\alpha \left( \frac{\|w - v\|}{\lambda_1} \right) g_\alpha \left( \frac{\|u - w\|}{\lambda_2} \right) \right).
\]

In the simple case where \(\lambda_1 = \lambda_2\), an amplification result is observed from the log-convexity of \(g_\alpha\), since \(g_\alpha(a)g_\alpha(b) \leq g_\alpha(a + b)\). When \(\lambda_1 \neq \lambda_2\), certain values of \(w\) still result in amplification, but they depend nontrivially on \(\alpha\). However, we also observe that this improvement vanishes as \(\alpha \to \infty\), since the necessary convexity also vanishes. In the limit, the lowest upper bound offered by (1) for \(R_\infty\) (which reduces to \((\varepsilon, 0)\)-DP) matches the \(\frac{\Delta}{\max(\lambda_1, \lambda_2)}\) result of Lemma 3.

**Example 3** (Lipschitz Kernel). As a warm-up for the results in Section 111 we now re-work Example 1 with a slightly more complex Markov operator. Suppose \(\psi\) is an \(L\)-Lipschitz map and let \(K(x) = \mathcal{N}(\psi(x), \sigma_2^2 I)\). Taking \(M\) to be the Gaussian mechanism from Example 1 we will show that the post-processed mechanism \(K \circ M\) satisfies \((\alpha, \frac{\alpha \Delta^2}{2\sigma_2^2})\)-RDP with \(\sigma_2^2 = \sigma_1^2 + \frac{\Delta^2}{\sigma_1^2}\). To prove this bound, we instantiate the notation from Example 1 and use the same coupling strategy to obtain

\[
R_\alpha(\mu K \| \nu K) \leq \frac{\alpha}{2} \left( \frac{\|w - v\|^2}{\sigma_1^2} + \sup_{x \in \mathbb{R}^d} \frac{\|\psi(x + \tau) - \psi(x)\|^2}{\sigma_2^2} \right) \leq \frac{\alpha}{2} \left( \frac{\|w - v\|^2}{\sigma_1^2} + L^2 \frac{\|u - w\|^2}{\sigma_2^2} \right),
\]

\(\text{That is, } \|\psi(x) - \psi(y)\| \leq L\|x - y\|\) for any pair \(x, y\).
where the second inequality uses the Lipschitz property. As before, the result follows from taking \( w = \theta u + (1 - \theta)v \) with \( \theta = (1 + \frac{\sigma^2}{L^2\sigma_1^2})^{-1} \). This example shows that we get amplification (i.e. \( \sigma^2 > \sigma_2^2 \)) for any \( L < \infty \) and \( \sigma_2 > 0 \), although the amount of amplification decreases as \( L \) grows. On the other hand, for \( L < 1 \) the amplification is stronger than just adding Gaussian noise (Example \([1]\).

### 4.1 Amplification by Iteration in Noisy Projected SGD with Strongly Convex Losses

Now we use Theorem \([2]\) and the computations above to show that the proof of privacy amplification by iteration \([Feldman et al., 2018, Theorem 22]\) can be extended to explicitly track the Lipschitz coefficients in a “noisy iteration” algorithm. In particular, this allows us to show an exponential improvement on the rate of privacy amplification by iteration in noisy SGD when the loss is strongly convex. To obtain this result we first provide an iterated version of Theorem \([2]\) in \( \mathbb{R}^d \) with Lipschitz Gaussian kernels. This version of the analysis introduces an explicit dependence on the \( W_\infty \) distances along an “interpolating” path between the initial distributions \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \) which could later be optimized for different applications. In our view, this helps to clarify the intuition behind the previous analysis by amplification by iteration.

**Theorem 4.** Let \( \alpha \geq 1, \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \) and let \( \mathbb{K} \subseteq \mathbb{R}^d \) be a convex set. Suppose \( K_1, \ldots, K_r \in \mathcal{K}(\mathbb{R}^d, \mathbb{R}^d) \) are Markov operators where \( Y_i \sim K_i(x) \) is obtained as \( Y_i = \Pi_\mathbb{K}(\psi_i(x) + Z_i) \) with \( Z_i \sim \mathcal{N}(0, \sigma^2I) \), where the maps \( \psi_i : \mathbb{K} \rightarrow \mathbb{R}^d \) are \( L \)-Lipschitz for all \( i \in [r] \). For any \( \mu_0, \mu_1, \ldots, \mu_r \in \mathcal{P}(\mathbb{R}^d) \) with \( \mu_0 = \mu \) and \( \mu_r = \nu \) we have

\[
R_\alpha(\mu K_1 \cdots K_r \| \nu K_1 \cdots K_r) \leq \frac{\alpha L^2}{2\sigma^2} \sum_{i=1}^r L^{2(r-i)} W_\infty(\mu_i, \mu_{i-1})^2.
\]

Furthermore, if \( L \leq 1 \) and \( W_\infty(\mu, \nu) = \Delta \), then

\[
R_\alpha(\mu K_1 \cdots K_r \| \nu K_1 \cdots K_r) \leq \frac{\alpha \Delta^2 L^{r+1}}{2r\sigma^2}.
\]

Note how taking \( L = 1 \) in the bound above we obtain \( \frac{\alpha \Delta^2}{2\sigma^2} = O(1/r) \), which matches \([Feldman et al., 2018, Theorem 1]\). On the other hand, for \( L \) strictly smaller than 1, the analysis above shows that the amplification rate is \( O(L^{r+1}/r) \) as a consequence of the maps \( \psi_i \) being strict contractions, i.e. \( \|\psi_i(x) - \psi_i(y)\| < \|x - y\| \). For \( L > 1 \) this result is not useful since the sum will diverge; however, the proof could easily be adapted to handle the case where each \( \psi_i \) is \( L_i \)-Lipschitz with some \( L_i > 1 \) and some \( L_i < 1 \). We now apply this result to improve the per-person privacy guarantees of noisy projected SGD (Algorithm \([1]\)) in the case where the loss function is smooth and strongly convex.

A function \( f : \mathbb{K} \subseteq \mathbb{R}^d \rightarrow \mathbb{R} \) defined on a convex set is \( \beta \)-smooth if it is continuously differentiable and \( \nabla f \) is \( \beta \)-Lipschitz, i.e., \( \|\nabla f(x) - \nabla f(y)\| \leq \beta \|x - y\| \), and is \( \rho \)-strongly convex if the function \( g(x) = f(x) - \frac{\rho}{2} \|x\|^2 \) is convex. When we say that a loss function \( \ell : \mathbb{K} \times \mathbb{D} \rightarrow \mathbb{R} \) satisfies a property (e.g. smoothness) we mean the property is satisfied by \( \ell(\cdot, z) \) for all \( z \in \mathbb{D} \). Furthermore, we recall from \([Feldman et al., 2018]\) that a mechanism \( M : \mathbb{D}^n \rightarrow \mathbb{K} \) satisfies \( (\alpha, \epsilon) \)-RDP at index \( i \) if \( R_\alpha(M(D)||M(D')) \leq \epsilon \) holds for any pair of databases \( D \) and \( D' \) differing on the \( i \)th coordinate.

---

\[7\] Here \( \Pi_\mathbb{K}(x) = \arg\min_{y \in \mathbb{K}} \|x - y\| \) denotes the projection operator onto the convex set \( \mathbb{K} \subseteq \mathbb{R}^d \).
Let \( \Theta(\log(x_i)) \) the same optimization error as non-private SGD while providing privacy for all individuals. [Fieldman et al., 2018, Corollary 31] one can show that, in the strongly convex setting, running [Øksendal, 2003] for details. This class arising from the Ornstein-Uhlenbeck diffusion process includes the well-known Gaussian mechanism, and also identify another interesting mechanism in [Bakry et al., 2013] and take a more abstract approach to Markov diffusion semigroups. We will show this class of operators in [Øksendal, 2003] for details.

Since [Feldman et al., 2018, Theorem 23] shows that for smooth Lipschitz loss functions the guarantee at index \( i \) of NoisyProjSGD is given by \( \epsilon_i = O\left(\frac{C^2}{(n-i)\sigma^2}\right) \), our result provides an exponential improvement in the strongly convex case. This implies, for example, that using the technique in [Feldman et al., 2018, Corollary 31] one can show that, in the strongly convex setting, running \( \Theta(\log(d)) \) additional iterations of NoisyProjSGD on public data is enough to attain (up to constant factors) the same optimization error as non-private SGD while providing privacy for all individuals.

**5 Diffusion Mechanisms**

Now we go beyond the analysis from previous sections and simultaneously consider a family of Markov operators \( \mathbf{P} = (P_t)_{t \geq 0} \) indexed by a continuous parameter \( t \) and satisfying the semigroup property \( P_t P_s = P_{t+s} \). Such \( \mathbf{P} \) is called a **Markov semigroup** and can be used to define a family of output perturbation mechanisms \( M_t^f(D) = P_t(f(D)) \) which are closed under post-processing by \( \mathbf{P} \) in the sense that \( P_s \circ M_t^f = M_{t+s}^f \). The semigroup property greatly simplifies the analysis of privacy amplification by post-processing, since, for example, if we show that \( M_t^f \) satisfies \((\alpha, \epsilon(t))\)-RDP, then this immediately provides RDP guarantees for any post-processing of \( M_t \) by any number of operators in \( \mathbf{P} \). The main result of this section provides such privacy analysis for mechanisms arising from symmetric diffusion Markov semigroups in Euclidean space. We will show this class includes the well-known Gaussian mechanism, and also identify another interesting mechanism in this class arising from the Ornstein-Uhlenbeck diffusion process.

Roughly speaking, a **diffusion** Markov semigroup \( \mathbf{P} = (P_t)_{t \geq 0} \) on \( \mathbb{R}^d \) corresponds to the case where \( X_t \sim P_t(x) \) defines a Markov process \( (X_t)_{t \geq 0} \) arising from a (time-homogeneous Itô) **stochastic differential equation** (SDE) of the form \( X_0 = x \) and \( dX_t = u(X_t)dt + v(X_t)dW_t \), where \( W_t \) is a standard \( d \)-dimensional Wiener process, and the **drift** \( u : \mathbb{R}^d \to \mathbb{R}^d \) and **diffusion** \( v : \mathbb{R}^d \to \mathbb{R}^{d \times d} \) coefficients satisfy appropriate regularity assumptions.\(^8\) In this paper, however, we shall follow [Bakry et al., 2013] and take a more abstract approach to Markov diffusion semigroups. We synthesize this approach by making a number of hypotheses on \( \mathbf{P} \) that we discuss after introducing two core concepts from the theory of Markov semigroups.

\(^8\)The details are not relevant here since we work directly with semigroups satisfying Assumption \( \square \). We refer to [Øksendal, 2003] for details.
In the context of a Markov semigroup $\mathbf{P}$, the action of the Markov operators $P_t$ on functions can be used to define the generator $L$ of the semigroup as the operator given by $Lf = \frac{d}{dt}(P_tf)|_{t=0}$. In particular, for a diffusion semigroup arising from the SDE $dX_t = u(X_t)dt + v(X_t)dW_t$ it is well-known that one can write the generator as $Lf = \langle u, \nabla f \rangle + \frac{1}{2} \text{tr}(vv^\top, H(f))$, where $H(f)$ is the Hessian of $f$ and the second term is a Frobenius inner product. Using the generator one also defines the so-called carré du champ operator $\Gamma(f,g) = \frac{1}{2}(L(fg) - fLg - gLf)$. This operator is bilinear and non-negative in the sense that $\Gamma(f,f) \geq 0$. The carré du champ operator can be interpreted as a device to measure how far $L$ is from being a first-order differential operator, since, e.g., if $L = \sum_i a_i \frac{\partial}{\partial x_i}$ then $L(fg) = fLg + gLf$ and therefore $\Gamma(f,g) = 0$. The operator $\Gamma$ can also be related to notions of curvature/contractivity of the underlying semigroup [Bakry et al., 2013].

Below we illustrate these concepts with the example of Brownian motion; but first we formally state our assumptions on the semigroup.

**Assumption 1**. Suppose the Markov semigroup $\mathbf{P} = (P_t)_{t \geq 0} \subset \mathcal{K}(\mathbb{R}^d, \mathbb{R}^d)$ satisfies the following:

1. There exists a unique non-negative invariant measure $\lambda$; that is, $\lambda P_t = \lambda$ for all $t \geq 0$. When the invariant measure is finite we normalize it to be a probability measure.
2. The operators $P_t$ admit a symmetric kernel $p_t(x,y) = p_t(y,x)$ with respect to the invariant measure. Equivalently, the invariant measure $\lambda$ is reversible for the Markov process $X_t$.
3. The generator $L$ satisfies the diffusion property $L\phi(f) = \phi'(f)Lf + \phi''(f)\Gamma(f)$ for any differentiable $\phi : \mathbb{R} \to \mathbb{R}$. This is a chain rule property saying that $L$ is a second-order differential operator without constant terms.

**Example 4** (Brownian Motion). The simplest diffusion process is the Brownian motion given by the simple SDE $dX_t = \sqrt{2}dW_t$, which corresponds to the semigroup $\mathbf{P}$ given by $P_t(x) = \mathcal{N}(x,2t)$. In this case, the mechanism $M^f_t(D) = P_t(f(D))$ is a Gaussian mechanism with variance $\sigma^2 = 2t$ and therefore satisfies $(\alpha, \frac{\alpha^2}{4t})$-RDP, where $\Delta$ is the global $L_2$-sensitivity of $f$. A direct substitution with $u = 0$ and $v = \sqrt{2}I$ shows that the semigroup’s generator is the standard Laplacian in $\mathbb{R}^d$, $L = \nabla^2 = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$, and a simple calculation yields the expression $\Gamma(f,g) = \langle \nabla f, \nabla g \rangle$ for the carré du champ operator. Now we check that $\mathbf{P}$ satisfies the conditions in Assumption 1. First, we recall that Brownian motion has the Lebesgue measure $\lambda$ on $\mathbb{R}^d$ as its unique invariant measure; this happens to be a non-finite measure. With respect to $\lambda$, the semigroup has kernel $p_t(x,y) \propto \exp\left(-\frac{||x-y||^2}{4t}\right)$ which is clearly symmetric. Finally, we use the chain rule for the gradient to verify that

$$Lf = \nabla^2 \phi(f) = \nabla \phi'(f) \nabla f = \phi''(f) \langle \nabla f, \nabla f \rangle + \phi'(f) \nabla^2 f = \phi''(f) \Gamma(f) + \phi'(f)Lf.$$  

Now we turn to the main result of this section, which provides a privacy analysis for the diffusion mechanism $M^f_t$ associated with an arbitrary symmetric diffusion Markov semigroup. The key insight behind this result is that the carré du champ operator of the semigroup provides a measure $\Lambda(t)$ of intrinsic sensitivity for the mechanism $M^f_t$ defined as:

$$\Lambda(t) = \sup_{D \simeq D'} t \int_0^\infty \kappa_{f(D),f(D')}(s) ds, \quad \text{where} \quad \kappa_{x,x'}(t) = \sup_{y \in \mathbb{R}^d} \Gamma \left( \log \frac{p_t(x,y)}{p_t(x',y)} \right).$$

**Theorem 1**. Let $f : \mathbb{D}^n \to \mathbb{R}^d$ and let $\mathbf{P} = (P_t)_{t \geq 0}$ by a Markov semigroup on $\mathbb{R}^d$ satisfying Assumption 1. If the mechanism $M^f_t(D) = P_t(f(D))$ has intrinsic sensitivity $\Lambda(t)$, then it satisfies $(\alpha, \alpha \Lambda(t))$-RDP for any $\alpha > 1$ and $t > 0$.  

Example 5 (Brownian Motion, Continued). To illustrate the use of Theorem 6, we show how it can be used to recover the privacy guarantees of the Gaussian mechanism through its connection with Brownian motion. We let \( P \) be the semigroup from Example 4 and start by using \( \Gamma(f) = \|\nabla f\|^2 \) to compute \( \kappa_{x,x'}(t) \) as follows:

\[
\Gamma \left( \log \frac{p_t(x,y)}{\tilde{p}_t(x',y')} \right) = \left\| \nabla_y \left( \frac{\|x' - y\|^2 - \|x - y\|^2}{4t} \right) \right\|^2 = \frac{\|x - x'\|^2}{4t^2} .
\]

Now we use \( \int_{t}^{\infty} \frac{1}{s} ds = \frac{1}{2} \) and \( \Delta^2 = \sup_{D \approx D'} \| f(D) - f(D') \|^2 \) to see that the mechanism associated with \( P \) has intrinsic sensitivity \( \Lambda(t) = \Delta^2 t \), yielding the privacy guarantee from Example 4.

5.1 The Ornstein-Uhlenbeck Mechanism

Beyond Brownian motion, another well-known diffusion process is the Ornstein-Uhlenbeck process with parameters \( \theta, \rho > 0 \) given by the SDE \( dX_t = -\theta X_t dt + \sqrt{2\rho} dW_t \). This diffusion process is associate with the semigroup \( P = (P_t)_{t \geq 0} \) given by \( P_t(x) = N(e^{-\theta t}x, \frac{\rho^2}{\theta^2} (1 - e^{-2\theta t}) I) \). One interpretation of this diffusion process is to think of \( X_t \) as a Brownian motion with variance \( \rho^2 \) applied to a mean reverting flow that pulls a particle towards the origin at a rate \( \theta \). In particular, the mechanism \( M_t f(D) \) is given by releasing \( e^{-\theta t} f(D) + N(0, \frac{\rho^2}{\theta^2} (1 - e^{-2\theta t})) \).

Taking the limit \( t \to \infty \) one sees that the (unique) invariant measure of \( P \) is the Gaussian distribution \( \lambda = N(0, \frac{\rho^2}{\theta^2} I) \). From the SDE characterization of this process it is easy to check that its generator is \( Lf = \rho^2 \nabla^2 f - \theta \langle x, \nabla f \rangle \) and the associated carré du champ operator is \( \Gamma(f,g) = \rho^2 \langle \nabla f, \nabla g \rangle \). Thus, \( P \) satisfies conditions (1) and (3) in Assumption 1.

To check the symmetry condition we apply a change of measure to the Gaussian density \( \tilde{p}_t(x,y) \) of \( P_t \) with respect to the Lebesgue measure to get its density w.r.t. \( \lambda \):

\[
\hat{p}_t(x,y) = \tilde{p}_t(x,y) \propto \exp \left( \frac{\theta \|y - e^{-\theta t} x\|^2}{2\rho^2 (1 - e^{-2\theta t})} \right) = \exp \left( \frac{-\theta \|x\|^2 - 2e^{\theta t}(x,y) + \|y\|^2}{2\rho^2 (e^{2\theta t} - 1)} \right) ,
\]

where \( \hat{p}_\lambda \) is the density of \( \lambda \) w.r.t. the Lebesgue measure. Thus, Theorem 6 yields the following.

Corollary 7. Let \( f : \mathbb{R}^d \to \mathbb{R}^d \) have global \( L_2 \)-sensitivity \( \Delta \) and \( P = (P_t)_{t \geq 0} \) be the Ornstein-Uhlenbeck semigroup with parameters \( \theta, \rho \). For any \( \alpha > 1 \) and \( t > 0 \) the mechanism \( M_t f(D) = P_t(f(D)) \) satisfies \( (\alpha, \alpha \Lambda(t)) \)-RDP with \( \Lambda(t) = \frac{\theta \Delta^2 t}{2\rho^2 (e^{2\theta t} - 1)} \).

The Ornstein-Uhlenbeck mechanism is not an unbiased mechanism since \( \mathbb{E}[M_t f(D)] = e^{-\theta t} f(D) \). This bias is the reason why the privacy guarantee in Corollary 7 exhibits a rate \( O(e^{-2\theta t}) \), while, for example, the Brownian motion mechanism only exhibits a rate \( O(t^{-1}) \). In particular, the Ornstein-Uhlenbeck mechanism achieves its privacy not only by introducing noise, but also by shrinking \( f(D) \) towards a data-independent point (the origin in this case); this effectively corresponds to reducing the sensitivity of \( f \) from \( \Delta \) to \( e^{-\theta t} \Delta \). This provides a way to trade-off variance and bias in the mean-squared error (MSE) incurred by privately releasing \( f(D) \) in a similar way that can be achieved by post-processing the Gaussian mechanism when \( f(D) \) is known to be bounded.

To formalize this result we define the mean squared error \( \mathcal{E}_{OU}(\theta, \rho, t) \) of the Ornstein-Uhlenbeck mechanism with parameters \( \theta, \rho \) at time \( t \), which is given by:

\[
\mathcal{E}_{OU}(\theta, \rho, t) \triangleq \mathbb{E}[\| f(D) - M_t f(D) \|^2] = (1 - e^{-\theta t})^2 \| f(D) \|^2 + \frac{\rho^2}{\theta} (1 - e^{-2\theta t}) .
\]

(4)
Similarly, we define $E_{GM}(\theta, \rho, t)$ as the mean squared error of a Gaussian mechanism with the same privacy guarantees as $M^f_t$ with parameters $\theta, \rho$. In particular, we have $E_{GM}(\theta, \rho, t) = d\tilde{\sigma}^2$, where $\tilde{\sigma}^2 \triangleq \frac{\rho^2 (e^{2\theta t} - 1)}{\theta}$ (cf. Corollary 7). We also note the post-processed Gaussian mechanism (PGM) $D \mapsto \beta(f(D) + \mathcal{N}(0, \tilde{\sigma}^2 I))$ which multiplies the output by a scalar $\beta$ optimized to minimize the MSE under the condition $\|f(D)\| \leq R$ yields error $E_{PGM}(\theta, \rho, t) \leq E_{GM}(\theta, \rho, t)(1 + \frac{d\tilde{\sigma}^2}{R^2})^{-1}$.

**Theorem** Suppose $f : \mathbb{D}^d \rightarrow \mathbb{R}^d$ has global $L_2$-sensitivity $\Delta$ and satisfies $\sup_D \|f(D)\| \leq R$. If $\theta R^2 \leq 4d\rho^2$ then we have $\frac{\mathcal{E}_{OU}(\theta, \rho, t)}{\mathcal{E}_{GM}(\theta, \rho, t)} \leq 1$ for all $t \geq 0$ and $\lim_{t \to \infty} \frac{\mathcal{E}_{OU}(\theta, \rho, t)}{\mathcal{E}_{GM}(\theta, \rho, t)} = 0$. In particular, taking $\theta = \log \left(1 + \frac{d\Delta^2}{2eR^2}\right)$ and $\rho^2 = \frac{\theta d\Delta^2}{2e(e^{2\theta t} - 1)}$ with $\epsilon > 0$, the mechanism $M^f_t$ satisfies $(\alpha, \alpha\epsilon)$-RDP at time $t = 1$ and we have $\frac{\mathcal{E}_{OU}(\theta, \rho, 1)}{\mathcal{E}_{GM}(\theta, \rho, 1)} \leq \left(1 + \frac{d\Delta^2}{2eR^2}\right)^{-1}$.

This result not only shows that the Ornstein-Uhlenbeck mechanism is uniformly better than the Gaussian mechanism for any level of privacy, but also shows that in this mechanism the error always stays bounded and can attain the same level of error as the Gaussian mechanism with optimal post-processing. To see this note that with the choices of parameters made in the second statement give $E_{GM}(\theta, \rho, 1) = \frac{d\Delta^2}{2e}$ and therefore $\mathcal{E}_{OU}(\theta, \rho, 1) \leq \frac{d\Delta^2 R^2}{2eR^2 + d\Delta^2}$, which behaves like $O(R^2)$ with $\Delta$ constant and either $\epsilon \to 0$ or $d \to \infty$.

**6 Conclusion**

We have undertaken a systematic study of amplification by post-processing. Our results yield improvements over recent work on amplification by iteration, and introduce a new Ornstein-Uhlenbeck mechanism which is more accurate than the Gaussian mechanism. In the future it would be interesting to study applications of amplification by post-processing. One promising application is *Hierarchical Differential Privacy*, where information is released under increasingly strong privacy constraints (e.g. to a restricted group within a company, globally within a company, and finally to outside parties).

**Acknowledgements** MG was partially supported by NSF grant CCF-1718220.

**Acknowledgments**

Marco Gaboardi’s work was partially supported by NSF grant #1718220.

**References**

D. Bakry, I. Gentil, and M. Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348. Springer Science & Business Media, 2013.

B. Balle, G. Barthe, and M. Gaboardi. Privacy amplification by subsampling: Tight analyses via couplings and divergences. In *Advances in Neural Information Processing Systems 31: Annual Conference on Neural Information Processing Systems 2018, NeurIPS 2018, 3-8 December 2018, Montréal, Canada.*, pages 6280–6290, 2018.
B. Balle, J. Bell, A. Gascón, and K. Nissim. The privacy blanket of the shuffle model. *CoRR*, abs/1903.02837, 2019. URL [http://arxiv.org/abs/1903.02837](http://arxiv.org/abs/1903.02837).

G. Barthe and F. Olmedo. Beyond differential privacy: Composition theorems and relational logic for f-divergences between probabilistic programs. In *International Colloquium on Automata, Languages, and Programming*, pages 49–60. Springer, 2013.

A. Beimel, K. Nissim, and U. Stemmer. Characterizing the sample complexity of private learners. In *Proceedings of the 4th conference on Innovations in Theoretical Computer Science*, pages 97–110. ACM, 2013.

A. Beimel, H. Brenner, S. P. Kasiviswanathan, and K. Nissim. Bounds on the sample complexity for private learning and private data release. *Machine learning*, 94(3):401–437, 2014.

S. Bubeck. Convex optimization: Algorithms and complexity. *Foundations and Trends® in Machine Learning*, 8(3-4):231–357, 2015.

M. Bun, K. Nissim, U. Stemmer, and S. Vadhan. Differentially private release and learning of threshold functions. In *Foundations of Computer Science (FOCS), 2015 IEEE 56th Annual Symposium on*, pages 634–649. IEEE, 2015.

K. Chaudhuri and N. Mishra. When random sampling preserves privacy. In *Annual International Cryptology Conference*, pages 198–213. Springer, 2006.

A. Cheu, A. D. Smith, J. Ullman, D. Zeber, and M. Zhilyaev. Distributed differential privacy via shuffling. In *Advances in Cryptology - EUROCRYPT 2019 - 38th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Darmstadt, Germany, May 19-23, 2019, Proceedings, Part I*, pages 375–403, 2019.

J. E. Cohen, Y. Iwasa, G. Rautu, M. B. Ruskai, E. Seneta, and G. Zbaganu. Relative entropy under mappings by stochastic matrices. *Linear algebra and its applications*, 179:211–235, 1993.

P. Del Moral, M. Ledoux, and L. Miclo. On contraction properties of Markov kernels. *Probability theory and related fields*, 126(3):395–420, 2003.

R. L. Dobrushin. Central limit theorem for nonstationary Markov chains. I. *Theory of Probability & Its Applications*, 1(1):65–80, 1956.

W. Doeblin. Sur les propriétés asymptotiques de mouvements rÉgis par certains types de chaÎnes simples (suite et fin). *Bulletin mathématique de la Société Roumaine des Sciences*, 39(2):3–61, 1937. ISSN 12203858. URL [http://www.jstor.org/stable/43769812](http://www.jstor.org/stable/43769812).

C. Dwork, F. McSherry, K. Nissim, and A. Smith. Calibrating noise to sensitivity in private data analysis. In *Theory of cryptography*, pages 265–284. Springer, 2006.

Ú. Erlingsson, V. Feldman, I. Mironov, A. Raghunathan, K. Talwar, and A. Thakurta. Amplification by shuffling: From local to central differential privacy via anonymity. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2468–2479. SIAM, 2019.
V. Feldman, I. Mironov, K. Talwar, and A. Thakurta. Privacy amplification by iteration. In 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS), pages 521–532. IEEE, 2018.

S. P. Kasiviswanathan, H. K. Lee, K. Nissim, S. Raskhodnikova, and A. Smith. What can we learn privately? SIAM Journal on Computing, 40(3):793–826, 2011.

D. A. Levin and Y. Peres. Markov chains and mixing times, volume 107. American Mathematical Soc., 2017.

N. Li, W. Qardaji, and D. Su. On sampling, anonymization, and differential privacy or, k-anonymization meets differential privacy. In Proceedings of the 7th ACM Symposium on Information, Computer and Communications Security, pages 32–33. ACM, 2012.

T. Lindvall. Lectures on the coupling method. Courier Corporation, 2002.

S. P. Meyn and R. L. Tweedie. Markov chains and stochastic stability. Springer Science & Business Media, 2012.

I. Mironov. Rényi differential privacy. In 30th IEEE Computer Security Foundations Symposium, CSF 2017, Santa Barbara, CA, USA, August 21-25, 2017, pages 263–275, 2017.

E. Nummelin. General irreducible Markov chains and non-negative operators, volume 83. Cambridge University Press, 2004.

B. Øksendal. Stochastic differential equations. In Stochastic differential equations, pages 65–84. Springer, 2003.

M. Raginsky. Strong data processing inequalities and Φ-Sobolev inequalities for discrete channels. IEEE Transactions on Information Theory, 62(6):3355–3389, 2016.

Y.-X. Wang, B. Balle, and S. Kasiviswanathan. Subsampled rényi differential privacy and analytical moments accountant. In Proceedings of the 22nd International Conference on Artificial Intelligence and Statistics (AISTATS), 2019.
A Proofs for Section 3 (Amplification From Uniform Mixing)

Lemma 9. The implications in Figure 1 hold.

Proof. That $(\gamma, \varepsilon)$-Dobrushin implies $\gamma$-Dobrushin follows directly from $D_{e^\varepsilon}(K(x)\|K(x')) \leq TV(K(x), K(x'))$.

To see that $\gamma$-Dobrushin implies $\gamma$-Dobrushin we observe that the kernel of a $\gamma$-Dobrushin operator must satisfy $\inf_x k(x, y) \geq (1 - \gamma)p_\omega(y)$ for any $y$. Thus, we can use the characterization of $TV$ in terms of a minimum to get

\[
TV(K(x), K(x')) = 1 - \int (k(x, y) \wedge k(x', y))\lambda(dy) \leq 1 - (1 - \gamma) \int p_\omega(y)\lambda(dy) = \gamma.
\]

Finally, to get the $\gamma$-Dobrushin condition for an operator $K$ satisfying $\gamma$-ultra-mixing we recall from [Del Moral et al., 2003, Lemma 4.1] that for such an operator we have that $K(x) \geq (1 - \gamma)\tilde{\omega}K$ is satisfied for any probability distribution $\tilde{\omega}$ and $x \in \text{supp}(\tilde{\omega})$. Thus, taking $\tilde{\omega}$ to have full support we obtain Dobruein's condition with $\omega = \tilde{\omega}K$.

For convenience, we split the proof of Theorem 1 into four separate statements, each corresponding to one of the claims in the theorem.

Recall that a Markov operator $K \in \mathcal{K}(X, \mathcal{Y})$ is $\gamma$-Dobrushin if $\sup_{x,x'} TV(K(x), K(x')) \leq \gamma$.

Theorem 10. Let $M$ be an $(\varepsilon, \delta)$-DP mechanism. If $K$ is a $\gamma$-Dobrushin Markov operator, then the composition $K \circ M$ is $(\varepsilon, \gamma\delta)$-DP.

Proof. This follows directly from the strong Markov contraction lemma established by Cohen et al. [1993] in the discrete case and by Del Moral et al. [2003] in the general case (see also Raginsky, 2016). In particular, this lemma states that for any divergence $D$ in the sense of Csiszar we have $D(\mu K\|\nu K) \leq \gamma D(\mu\|\nu)$. Letting $\mu = M(D)$ and $\nu = M(D')$ for some $D \simeq D'$ and applying this inequality to $D_{e^\varepsilon}(\mu K\|\nu K)$ yields the result.

Next we prove amplification when $K$ is a $(\gamma, \varepsilon)$-Dobrushin operator. Recall that a Markov operator $K \in \mathcal{K}(X, \mathcal{Y})$ is $(\gamma, \varepsilon)$-Dobrushin if $\sup_{x,x'} D_{e^\varepsilon}(K(x)\|K(x')) \leq \gamma$. We will require the following technical lemma in the proof of Theorem 13.

Lemma 11. Let $\mu \perp \nu$ denote the fact $\text{supp}(\mu) \cap \text{supp}(\nu) = \emptyset$. If $K$ is $(\gamma, \varepsilon)$-Dobrushin, then we have

\[
\sup_{\mu \perp \nu} D_{e^\varepsilon}(\mu K\|\nu K) \leq \gamma.
\]

Proof. Note that the condition on $\gamma$ can be written as $\sup_{x,x'} D_{e^\varepsilon}(\delta_x K\|\delta_{x'} K) \leq \gamma$. This shows that by hypothesis the condition already holds for the distributions $\delta_x \perp \delta_{x'}$ with $x \neq x'$. Thus, all we need to do is prove that these distributions are extremal for $D_{e^\varepsilon}(\mu K\|\nu K)$ among all distributions with $\mu \perp \nu$. Let $\mu \perp \nu$ and define $U = \text{supp}(\mu)$ and $V = \text{supp}(\nu)$. Working in the discrete setting for simplicity, we can write $\mu = \sum_{x \in U} \mu(x)\delta_x$, with an equivalent expression for $\nu$. Now we use the joint convexity of $D_{e^\varepsilon}$ to write

\[
D_{e^\varepsilon}(\mu K\|\nu K) \leq \sum_{x \in U} \mu(x)D_{e^\varepsilon}(\delta_x K\|\nu K) \leq \sum_{x \in U} \sum_{x' \in V} \mu(x)\nu(x')D_{e^\varepsilon}(\delta_x K\|\delta_{x'} K) \leq \sup_{x \neq x'} D(\delta_x K\|\delta_{x'} K) \leq \gamma.
\]
Lemma 12. Let \( a \land b \triangleq \min\{a, b\} \). Then we have
\[
\mathcal{D}_{e^\varepsilon}(\mu \| \nu) = 1 - \int (p_\mu(x) \land e^\varepsilon p_\nu(x)) \lambda(dx) .
\]

Proof. Define \( A = \{ x : p_\mu(x) \leq e^\varepsilon p_\nu(x) \} \) to be set of points where \( \mu \) is dominated by \( e^\varepsilon \nu \), and let \( A^c \) denote its complementary. Then we have the identities
\[
\int (p_\mu \land e^\varepsilon p_\nu) d\lambda = \int_A d\mu + e^\varepsilon \int_{A^c} d\nu ,
\]
\[
\int [p_\mu - e^\varepsilon p_\nu]_+ d\lambda = \int_{A^c} d\mu - e^\varepsilon \int_A d\nu .
\]
Thus we obtain the desired result since
\[
\mathcal{D}_{e^\varepsilon}(\mu \| \nu) + \int (p_\mu \land e^\varepsilon p_\nu) d\lambda = \int [p_\mu - e^\varepsilon p_\nu]_+ d\lambda + \int (p_\mu \land e^\varepsilon p_\nu) d\lambda = \int_{A^c} d\mu + \int_A d\mu = 1 .
\]

Theorem 13. Let \( M \) be an \((\varepsilon, \delta)\)-DP mechanism and let \( \varepsilon' = \log (1 + \frac{e^\varepsilon - 1}{\delta}) \). If \( K \) is a \((\gamma, \varepsilon')\)-Dobrushin Markov operator, then the composition \( K \circ M \) is \((\varepsilon, \gamma\delta)\)-DP.

Proof. Fix \( \mu = M(D) \) and \( \nu = M(D') \) for some \( D \simeq D' \) and let \( \theta = \mathcal{D}_{e^\varepsilon}(\mu \| \nu) \leq \delta \). We start by constructing overlapping mixture decompositions for \( \mu \) and \( \nu \) as follows. First, define the function \( f = p_\mu \land e^\varepsilon p_\nu \) and let \( \omega \) be the probability distribution with density \( p_\omega = \frac{f}{\int f d\lambda} = \frac{1}{1 - \theta} \), where we used Lemma 12. Now note that by construction we have the inequalities
\[
p_\mu - (1 - \theta) p_\omega = p_\mu - p_\mu \land e^\varepsilon p_\nu \geq 0 ,
p_\nu - \frac{1 - \theta}{e^\varepsilon} p_\omega = p_\nu - p_\nu \land e^{-\varepsilon} p_\mu \geq 0 .
\]
Assuming without loss of generality that \( \mu \neq \nu \), these inequalities imply that we can construct probability distributions \( \mu' \) and \( \nu' \) such that
\[
\mu = (1 - \theta) \omega + \theta \mu' ,
\]
\[
\nu = \frac{1 - \theta}{e^\varepsilon} \omega + \left(1 - \frac{1 - \theta}{e^\varepsilon}\right) \nu' .
\]
Now we observe that the distributions \( \mu' \) and \( \nu' \) defined in this way have disjoint support. To see this we first use the identity \( p_\mu = (1 - \theta)p_\omega + \theta p_{\mu'} \) to see that
\[
p_{\mu'}(x) > 0 \equiv p_\mu(x) - (1 - \theta)p_\omega(x) > 0 \equiv p_\mu(x) - p_\mu(x) \land e^\varepsilon p_\nu(x) > 0 \equiv p_\mu(x) > e^\varepsilon p_\nu(x) .
\]
Thus we have \( \text{supp(} \mu' \text{)} = \{ x : p_\mu(x) > e^\varepsilon p_\nu(x) \} \). A similar argument applied to \( p_\nu \) shows that on the other hand \( \text{supp(} \nu' \text{)} = \{ x : p_\nu(x) < e^\varepsilon p_\mu(x) \} \), and thus \( \mu' \perp \nu' \).

Finally, we proceed to use the mixture decomposition of \( \mu \) and \( \nu \) and the condition \( \mu' \perp \nu' \) to bound \( \mathcal{D}_{e^\varepsilon}(\mu K \| \nu K) \) as follows. By using the mixture decompositions we get
\[
\mu - e^\varepsilon \nu = \theta \mu' - e^\varepsilon \left(1 - \frac{1 - \theta}{e^\varepsilon}\right) \nu' = \theta (\mu' - e^\varepsilon \nu') ,
\]
15
Theorem 14. Let $K$ be an $(\varepsilon, \delta)$-DP mechanism. If $K$ is a $\gamma$-Doobian Markov operator, then the composition $K \circ M$ is $(\varepsilon', \delta')$-DP with $\varepsilon' = \log(1 + \gamma(e^\varepsilon - 1) + \delta' = \gamma(1 - e^{\varepsilon' - \varepsilon}(1 - \delta))$.

Proof. Fix $\mu = M(D)$ and $\nu = M(D')$ for some $D \simeq D'$. Let $\omega$ be a witness that $K$ is $\gamma$-Doobian and let $K_\omega$ be the constant Markov operator given by $K_\omega(x) = \omega$ for all $x$. Doobian's condition $K(x) \geq (1 - \gamma)\omega = (1 - \gamma)K_\omega(x)$ implies that the following is again a Markov operator:

$$\tilde{K} = \frac{K - (1 - \gamma)K_\omega}{\gamma}.$$ 

Thus, we can write $K$ as the mixture $K = (1 - \gamma)K_\omega + \gamma\tilde{K}$ and then use the advanced joint convexity property of $D_{\varepsilon'}$ [Balle et al., 2018, Theorem 2] with $\varepsilon' = \log(1 + \gamma(e^\varepsilon - 1))$ to obtain the following:

$$D_{\varepsilon'}(\mu K\|\nu K) = D_{\varepsilon'}((1 - \gamma)\omega + \gamma\mu\tilde{K})(1 - \gamma)\omega + \gamma\nu\tilde{K})$$

$$= \gamma D_{\varepsilon'}(\mu\tilde{K}||\omega + \beta\nu\tilde{K})$$

$$\leq \gamma \left((1 - \beta)D_{\varepsilon'}(\mu\tilde{K}||\omega) + \beta D_{\varepsilon'}(\mu\tilde{K}||\nu\tilde{K})\right),$$

where $\beta = e^{\varepsilon' - \varepsilon}$. Finally, using the immediate bounds $D_{\varepsilon'}(\mu\tilde{K}||\nu\tilde{K}) \leq D_{\varepsilon'}(\mu||\nu)$ and $D_{\varepsilon'}(\mu\tilde{K}||\omega) \leq 1$, we get

$$D_{\varepsilon'}(\mu K\|\nu K) \leq \gamma(1 - e^{\varepsilon' - \varepsilon} + e^{\varepsilon' - \varepsilon} \delta).$$

Our last amplification result applies to operators satisfying the ultra-mixing condition of Del Moral et al. [2003]. We say that a Markov operator $K \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$ is $\gamma$-ultra-mixing if for all $x, x' \in \mathcal{X}$ we have $K(x) \leq K(x')$ and $\frac{\partial K(x)}{\partial K(x')} \geq 1 - \gamma$. The proof strategy is based on the ideas from the previous proof, although in this case the argument is slightly more technical as it involves a strengthening of the Dobinian condition implied by ultra-mixing that only holds under a specific support.

Theorem 15. Let $M$ be an $(\varepsilon, \delta)$-DP mechanism. If $K$ is a $\gamma$-ultra-mixing Markov operator, then the composition $K \circ M$ is $(\varepsilon', \delta')$-DP with $\varepsilon' = \log(1 + \gamma(e^\varepsilon - 1))$ and $\delta' = \gamma\delta e^{\varepsilon' - \varepsilon}$.

Proof. Fix $\mu = M(D)$ and $\nu = M(D')$ for some $D \simeq D'$. The proof follows a similar strategy as the one used in Theorem 14 but coupled with the following consequence of the ultra-mixing property: for any probability distribution $\omega$ and $x \in \text{supp}(\omega)$ we have $K(x) \geq (1 - \gamma)\omega K$ [Del Moral et al., 2003].
We use this property to construct a collection of mixture decompositions for $K$ as follows. Let $\alpha \in (0,1)$ and take $\tilde{\omega} = (1-\alpha)\mu + \alpha \nu$ and $\omega = \tilde{\omega}K$. By the ultra-mixing condition and the argument used in the proof of Theorem 14, we can show that

$$K - \frac{1}{\gamma}(1-\gamma)K$$

is a Markov operator from $\text{supp}(\mu) \cup \text{supp}(\nu)$ into $\mathbb{X}$. Here $K_\omega$ is the constant Markov operator $K_\omega(x) = \omega$. Furthermore, the expression for $K$ and the definition of $\omega$ imply that

$$\tilde{\omega}K = \frac{\tilde{\omega}K - (1-\gamma)\tilde{\omega}K_\omega}{\gamma} = \omega.$$  (5)

Now note that the mixture decompositions $\mu K = (1-\gamma)\omega + \gamma \tilde{\omega}K$ and $\nu K = (1-\gamma)\omega + \gamma \tilde{\omega}K$ and the advanced joint convexity property of $D_{e^{\varepsilon'}}$ [Balle et al., 2018, Theorem 2] with $\varepsilon' = \log(1+\gamma(e^\varepsilon - 1))$ yield

$$D_{e^{\varepsilon'}}(\mu K\|\nu K) \leq \gamma \left( (1-\beta)D_{e^{\varepsilon'}}(\mu \tilde{\omega}K\|\omega) + \beta D_{e^{\varepsilon'}}(\mu \tilde{\omega}K\|\nu \tilde{\omega}K) \right)$$

$$\leq \gamma \left( (1-\beta)D_{e^{\varepsilon'}}(\mu \tilde{\omega}K\|\omega) + \beta D_{e^{\varepsilon'}}(\mu \|\nu) \right)$$

$$\leq \gamma \left( (1-\beta)D_{e^{\varepsilon'}}(\mu \tilde{\omega}K\|\omega) + \beta \delta \right),$$

where $\beta = e^{\varepsilon'} - \varepsilon$. Using (5) we can expand the remaining divergence above as follows:

$$D_{e^{\varepsilon'}}(\mu \tilde{\omega}K\|\omega) = D_{e^{\varepsilon'}}(\mu \tilde{\omega}K\|\tilde{\omega}K) \leq D_{e^{\varepsilon'}}(\mu\|\tilde{\omega}) \leq \alpha D_{e^{\varepsilon'}}(\mu\|\nu) \leq \alpha \delta,$$

where we used the definition of $\tilde{\omega}$ and joint convexity. Since $\alpha$ was arbitrary, we can now take the limit $\alpha \to 0$ to obtain the bound $D_{e^{\varepsilon'}}(\mu K\|\nu K) \leq \gamma \delta e^{\varepsilon' - \varepsilon}$. \hfill \Box

**Proof of Theorem 1** It follows from Theorems 10, 13, 14 and 15. \hfill \Box

**B Proofs for Section 4 (Amplification From Couplings)**

**Lemma 16.** The transport operator $H_\pi$ with $\pi \in \mathcal{C}(\mu,\nu)$ satisfies $\mu H_\pi = \nu$.

**Proof.** Take an arbitrary event $E$ and note that:

$$(\mu H_\pi)(E) = \int_\mathbb{X} H_\pi(x)(E)\mu(dx) = \int_\mathbb{X} \int_E h_\pi(x,y)\mu(dx)\lambda(dy) = \int_\mathbb{X} \int_E \frac{p_\pi(x,y)}{p_\mu(x)}\mu(dx)\lambda(dy)$$

$$= \int_\mathbb{X} \int_E p_\pi(x,y)\lambda(dx)\lambda(dy) = \int_E p_\nu(y)\lambda(dy) = \nu(E),$$

where we used the coupling property $\int_\mathbb{X} p_\pi(x,y)\lambda(dx) = p_\nu(y)$. \hfill \Box

**Theorem 2** Let $\alpha \geq 1$, $\mu, \nu \in \mathcal{P}(\mathbb{X})$ and $K \in \mathcal{K}(\mathbb{X},\mathbb{Y})$. For any distribution $\omega \in \mathcal{P}(\mathbb{X})$ and coupling $\pi \in \mathcal{C}(\omega,\mu)$ we have

$$R_\alpha(\mu K\|\nu K) \leq R_\alpha(\omega\|\nu) + \sup_{x \in \text{supp}(\nu)} R_\alpha((H_\pi K)(x)||K(x)).$$  (1)
Proof. Let $\omega \in \mathcal{P}(\mathbb{X})$ and $\pi \in \mathcal{C}(\omega, \mu)$ be as in the statement, and let $\pi' = C(\mu, \omega)$. Note that taking $H_\pi$ and $H_{\pi'}$ to be the corresponding transport operators we have $\mu = \mu H_{\pi'} H_\pi = \omega H_\pi$. Now, given a $\lambda \in \mathcal{P}(\mathbb{X} \times \mathbb{X})$ let $\Pi_2(\lambda) = \int \lambda(dx, \cdot)$ denote the marginal of $\lambda$ on the second coordinate. In particular, if $\mu \otimes K$ denotes the joint distribution of $\mu$ and $\mu K$, then we have $\Pi_2(\mu \otimes K) = \mu K$. Thus, by the data processing inequality we have

$$R_\alpha(\mu K || \nu K) = R_\alpha(\omega H_\pi K || \nu K) = R_\alpha(\Pi_2(\omega \otimes H_\pi K)||\Pi_2(\nu \otimes K)) \leq R_\alpha(\omega \otimes H_\pi K || \nu \otimes K).$$

The final step is to expand the RHS of the derivation above as follows:

$$e^{(\alpha - 1)R_\alpha(\omega \otimes H_\pi K || \nu \otimes K)} = \iint \left( \frac{d(\omega \otimes H_\pi K)}{d(\nu \otimes K)} \right)^\alpha \nu(dx) K(x, dy)$$

$$= \iint \left( \frac{\rho_{\omega}(x) \int \rho_{\pi}(x, dz) k(z, y)}{\rho_{\nu}(x) k(x, y)} \right)^\alpha \nu(dx) K(x, dy)$$

$$= \iint \left( \frac{\rho_{\omega}(x)}{\rho_{\nu}(x)} \left( \int \rho_{\pi}(x, dz) k(z, y) \right) \right)^\alpha \nu(dx) K(x, dy)$$

$$\leq \left( \int \left( \frac{\rho_{\omega}(x)}{\rho_{\nu}(x)} \right)^\alpha \nu(dx) \right) \left( \sup_x \int \left( \int \rho_{\pi}(x, dz) k(z, y) \right)^\alpha k(x, y) \right) K(x, dy)$$

$$= e^{(\alpha - 1)R_\alpha(\omega || \nu)} \cdot e^{(\alpha - 1)\sup_x R_\alpha((H_\pi K(x))||K(x))},$$

where the supremums are taken with respect to $x \in \text{supp}(\nu)$. \hfill \Box

Lemma 3 Let $M(D) = \text{Lap}(f(D), \lambda_1)$ for some function $f : \mathbb{D} \to \mathbb{R}$ with global $L_1$-sensitivity $\Delta$ and let the Markov operator $K$ be given by $K(x) = \text{Lap}(x, \lambda_2)$. The post-processed mechanism $(K \circ M)$ does not achieve $(\varepsilon, 0)$-DP for any $\varepsilon < \frac{\Delta}{\max\{\lambda_1, \lambda_2\}}$. Note that $M$ achieves $(\frac{\Delta}{\lambda_1}, 0)$-DP and $K(f(D))$ achieves $(\frac{\Delta}{\lambda_2}, 0)$-DP.

Proof. This can be shown by directly analyzing the distribution arising from the sum of two independent laplace variables. Let $\text{Lap}(2, \lambda_1, \lambda_2)$ denote this distribution. In the following equations, we assume $x > 0$. Due to symmetry around the origin, densities at negative values can be found by looking instead at the corresponding positive location.

$$\text{Lap}(x; \lambda_1, \lambda_2) = \int_{-x}^{x} \frac{1}{2\lambda_1} \exp\left( -\frac{|-t|}{\lambda_1} \right) \frac{1}{2\lambda_2} \exp\left( -\frac{|t|}{\lambda_2} \right) dt$$

$$= \frac{1}{4\lambda_1 \lambda_2} \int_{-\infty}^{\infty} \exp\left( -\frac{\lambda_2|\lambda_2| - \lambda_1|t|}{\lambda_1 \lambda_2} \right) dt$$

$$= \frac{1}{4\lambda_1 \lambda_2} \left( \int_{-\infty}^{0} e^{-\frac{\lambda_2|x-t|}{\lambda_1 \lambda_2}} dt + \int_{0}^{x} e^{-\frac{\lambda_2(t-x)+\lambda_1 t}{\lambda_1 \lambda_2}} dt + \int_{x}^{\infty} e^{-\frac{\lambda_2(t-x)+\lambda_1 t}{\lambda_1 \lambda_2}} dt \right)$$

$$= \frac{1}{4\lambda_1 \lambda_2} \left( \int_{-\infty}^{0} e^{-\frac{\lambda_2|x-t|}{\lambda_1 \lambda_2}} dt + \int_{0}^{x} e^{-\frac{\lambda_2(t-x)+\lambda_1 t}{\lambda_1 \lambda_2}} dt + \int_{x}^{\infty} e^{-\frac{\lambda_2(t-x)+\lambda_1 t}{\lambda_1 \lambda_2}} dt \right)$$

$$= \frac{1}{4\lambda_1 \lambda_2} \left( \frac{e^{-\lambda_2|x-t|\lambda_1 t}}{(\lambda_1 + \lambda_2) / \lambda_1 \lambda_2} \bigg|_{t=0}^{t=x} + \int_{0}^{x} e^{-\frac{\lambda_2(t-x)+\lambda_1 t}{\lambda_1 \lambda_2}} dt + \int_{x}^{\infty} e^{-\frac{\lambda_2(t-x)+\lambda_1 t}{\lambda_1 \lambda_2}} dt \right) \bigg|_{t=-\infty}^{t=\infty}$$
The integration on the middle term varies between the cases $\lambda_1 = \lambda_2$ and $\lambda_1 \neq \lambda_2$. Finishing this derivation and replacing $x$ with $|x|$ to account for both positive and negative values, we get a complete expression for our $Lap2(\lambda_1, \lambda_2)$ density.

$$Lap2(x; \lambda_1, \lambda_2) = \begin{cases} \frac{1}{4} \left( \frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1-\lambda_2} \right) e^{-\frac{|x|}{\lambda_1}} + \left( \frac{1}{\lambda_1+\lambda_2} - \frac{1}{\lambda_1-\lambda_2} \right) e^{-\frac{|x|}{\lambda_2}} & \text{if } \lambda_1 \neq \lambda_2 , \\ \frac{1}{\lambda_1} e^{-\frac{|x|}{\lambda_1}} (\lambda_1 + |x|) & \text{if } \lambda_1 = \lambda_2 . \end{cases}$$  \hspace{1cm} (6)

To finish this lemma, we need to derive the best $(\epsilon, 0)$-DP guarantee offered by adding noise from $Lap2(\lambda_1, \lambda_2)$. From the post-processing property of DP and the commutivity of additive mechanisms, we know this guarantee is upper-bounded by $\Delta / \max \{ \lambda_1, \lambda_2 \}$. A direct computation of $\lim_{x \to \infty} \log(Lap2(x; \lambda_1, \lambda_2)/Lap2(x + \Delta; \lambda_1, \lambda_2))$ results in $\Delta / \max \{ \lambda_1, \lambda_2 \}$ in both cases of equation (6). This arises from the limit depending entirely on the dominating term with the largest exponent. Therefore, this lower-bounds the privacy guarantee by the same value. Thus we can conclude this is the exact level of $(\epsilon, 0)$-DP guarantee offered by this mechanism.

\[\square\]

**Theorem 4**: Let $\alpha \geq 1$, $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and let $K \subseteq \mathbb{R}^d$ be a convex set. Suppose $K_1, \ldots, K_r \in \mathcal{K}(\mathbb{R}^d, \mathbb{R}^d)$ are Markov operators where $Y_i \sim K_i(x)$ is obtained as $Y_i = \Pi_K(\psi_i(x) + Z_i)$ with $Z_i \sim \mathcal{N}(0, \sigma^2 I)$, where the maps $\psi_i : K \to \mathbb{R}^d$ are $L$-Lipschitz for all $i \in [r]$. For any $\mu_0, \mu_1, \ldots, \mu_r \in \mathcal{P}(\mathbb{R}^d)$ with $\mu_0 = \mu$ and $\mu_r = \nu$ we have

$$R_\alpha(\mu K_1 \cdots K_r \| \nu K_1 \cdots K_r) \leq \frac{\alpha L^2}{2\sigma^2} \sum_{i=1}^{r} L^{2(r-i)} W_\infty(\mu_i, \mu_{i-1})^2 .$$  \hspace{1cm} (2)

Furthermore, if $L \leq 1$ and $W_\infty(\mu, \nu) = \Delta$, then

$$R_\alpha(\mu K_1 \cdots K_r \| \nu K_1 \cdots K_r) \leq \frac{\alpha \Delta^2 L^{r+1}}{2r\sigma^2} .$$  \hspace{1cm} (3)

The proof of Theorem 4 relies on the following technical lemma about the effect of a projected Lipschitz Gaussian operator on the $\infty$-Wasserstein distance between two distributions.

**Lemma 17**: Let $K \subseteq \mathbb{R}^d$ be a convex set and $\psi : K \to \mathbb{R}^d$ be $L$-Lipschitz. Suppose $K \in \mathcal{K}(\mathbb{R}^d, \mathbb{R}^d)$ is a Markov operator where $Y \sim K(x)$ is obtained as $Y = \Pi_K(\psi(x) + Z)$ with $Z \sim \mathcal{N}(0, \sigma^2 I)$. Then, for any $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ we have $W_\infty(\mu K, \nu K) \leq LW_\infty(\mu, \nu)$.

**Proof**: Let $\pi \in \mathcal{C}(\mu, \nu)$ be a witness of $W_\infty(\mu, \nu) = \Delta$. We construct a witness of $W_\infty(\mu K, \nu K) \leq L \Delta$ as follows: sample $(X, X') \sim \pi$ and $Z \sim \mathcal{N}(0, \sigma^2 I)$ and then let $Y = \Pi_K(\psi(X) + Z)$ and $Y' = \Pi_K(\psi(X') + Z)$. It is clear from the construction that $\text{Law}((Y, Y')) \in \mathcal{C}(\mu K, \nu K)$. Furthermore, by the Lipschitz assumption on $\psi$ and that fact that the map $\Pi_K$ is contractive, the following holds almost surely:

$$\|Y - Y'\| \leq \|\psi(X) - \psi(X')\| \leq L \|X - X'\| \leq L \Delta .$$  \hspace{1cm} \(\square\)
Proof of Theorem \[7\] We prove \[2\] by induction on \(r\). For the base case \(r = 1\) we apply Theorem \[2\] with \(\omega = \nu\) and a coupling \(\pi \in \mathcal{C}(\nu, \mu)\) witnessing that \(W_\infty(\mu, \nu) = \Delta\). This choice of coupling guarantees that for any \(x \in \text{supp}(\nu)\) we have \(\text{supp}(H_{\pi}(x)) \subseteq B_\Delta(x)\), where \(B_\Delta(x)\) is the ball of radius \(\Delta\) around \(x\). Note also that \((H_{\pi}K_1)(x) = H_{\pi}(x)K_1\). Thus, from \[1\] we obtain, using Hölder’s inequality and the monotonicity of the logarithm, that:

\[
R_\alpha(\mu K_1 \| \nu K_1) \leq \sup_{x \in \text{supp}(\nu)} R_\alpha((H_{\pi}K_1)(x) \| K_1(x)) \leq \sup_{x \in \text{supp}(\nu)} \sup_{y \in \text{supp}(H_{\pi}(x))} R_\alpha(K_1(y) \| K_1(x)) \leq \sup_{\|x-y\| \leq \Delta} R_\alpha(K_1(y) \| K_1(x)).
\]

Now note that the Markov operator \(K_1\) can be obtained by post-processing \(\tilde{K}_1(x) = \mathcal{N}(\psi_1(x), \sigma^2 \mathbb{I})\) with the projection \(\Pi_\Delta\). Thus, by the data processing inequality we obtain

\[
\sup_{\|x-y\| \leq \Delta} R_\alpha(K_1(y) \| K_1(x)) \leq \sup_{\|x-y\| \leq \Delta} R_\alpha(\tilde{K}_1(y) \| \tilde{K}_1(x)) = \sup_{\|x-y\| \leq \Delta} \frac{\alpha \|\psi_1(x) - \psi_1(y)\|^2}{2\sigma^2} \leq \frac{\alpha \Delta^2 L^2}{2\sigma^2}.
\]

For the inductive case we suppose that \[2\] holds for some \(r \geq 1\) and consider the case \(r + 1\), in which we need to bound \(R_\alpha(\mu K_1 \cdots K_{r+1} \| \nu K_1 \cdots K_{r+1})\). Let \(\mu_0, \mu_1, \ldots, \mu_{r+1}\) be a sequence of distributions with \(\mu_0 = \mu\) and \(\mu_{r+1} = \nu\). Applying \[1\] with \(\omega = \mu K_1 \cdots K_r\) and some coupling \(\pi \in \mathcal{C}(\mu K_1 \cdots K_r, \mu K_1 \cdots K_r)\) we have

\[
R_\alpha(\mu K_1 \cdots K_{r+1} \| \nu K_1 \cdots K_{r+1}) \leq R_\alpha(\mu K_1 \cdots K_r \| \nu K_1 \cdots K_r) + \sup_{x \in \text{supp}(\nu K_1 \cdots K_r)} R_\alpha((H_{\pi}K_{r+1})(x) \| K_{r+1}(x)).
\]

By the inductive hypothesis, the first term in the RHS above can be bounded as follows:

\[
R_\alpha(\mu K_1 \cdots K_r \| \nu K_1 \cdots K_r) \leq \frac{\alpha L^2}{2\sigma^2} \sum_{i=1}^{r} L^{2(r-i)} W_\infty(\mu_{i+1}, \mu_i)^2 \leq \frac{\alpha L^2}{2\sigma^2} \sum_{i=2}^{r+1} L^{2(r+1-i)} W_\infty(\mu_i, \mu_{i-1})^2.
\]

To bound the second term we assume the coupling \(\pi\) is a witness of \(W_\infty(\mu_1 K_1 \cdots K_r, \mu K_1 \cdots K_r) = \Delta'\), in which case a similar argument to the one we used in the base case yields:

\[
\sup_x R_\alpha((H_{\pi}K_{r+1})(x) \| K_{r+1}(x)) \leq \sup_{x \in \text{supp}(H_{\pi}(x))} R_\alpha(K_{r+1}(y) \| K_{r+1}(x)) \leq \sup_{\|x-y\| \leq \Delta'} R_\alpha(K_{r+1}(y) \| K_{r+1}(x)) \leq \frac{\alpha \Delta'^2 L^2}{2\sigma^2} \leq \frac{\alpha L^{2r+2} W_\infty(\mu_1, \mu)^2}{2\sigma^2},
\]

where the last inequality follows from Lemma \[17\] Plugging the last three inequalities together we
Finally obtain
\[
R_\alpha(\mu K_1 \cdots K_{r+1} || \nu K_1 \cdots K_{r+1}) \leq \frac{\alpha L^2r+2 L_{\infty}(\mu_1, \mu_0)^2}{2\sigma^2} + \frac{\alpha L^2}{2\sigma^2} \sum_{i=2}^{r+1} L^{2(r+1-i)} W_\infty(\mu_i, \mu_{i-1})^2 \]
\[
= \frac{\alpha L^2}{2\sigma^2} \sum_{i=1}^{r+1} L^{2(r+1-i)} W_\infty(\mu_i, \mu_{i-1})^2 .
\]

When \( L \leq 1 \), we can obtain (3) from (2) as follows. First, construct a sequence of distributions \( \mu_0, \ldots, \mu_r \) such that \( \Delta_i \triangleq W_\infty(\mu_i, \mu_{i-1}) = \Delta_0 L^i \) for \( i \in [r] \), where \( \Delta_0 = \frac{1}{\Gamma - L} \) is a normalization constant chosen such that \( \sum_{i \in [r]} \Delta_i = 1 \). With this choice plugged into (2) we obtain
\[
R_\alpha(\mu K_1 \cdots K_r || \nu K_1 \cdots K_r) \leq \frac{\alpha L^2}{2\sigma^2} \sum_{j=1}^{r} \Delta_j^2 L^{2j} = \frac{\alpha \Delta^2}{2\sigma^2} \sum_{j=1}^{r} \left( \frac{L^{-1} - L^2}{L^{-1} - L^j} \right)^2 \leq \frac{\alpha \Delta^2}{\sigma^2} L^{r+1} \phi(L)^2 .
\]

Now we note the function \( \phi(L) \) defined above is increasing in \([0,1]\) and furthermore \( \lim_{L \to 1} \phi(L) = \frac{1}{r} \), which can be checked by applying L'Hôpital's rule twice. Thus, we can plug the inequality \( \phi(L) \leq \frac{1}{r} \) above to obtain (3).

But we still need to show that a sequence \( \mu_0, \ldots, \mu_r \) with \( \Delta_i \) as above exists. To construct such a sequence we let \( \pi \in \mathcal{C}(\mu, \nu) \) be a witness of \( W_\infty(\mu, \nu) = \Delta \), take random variables \( (X, X') \sim \pi \), and define \( \mu_i = \text{Law}((1 - \theta_i)X + \theta_i X') \) with \( \theta_i = \frac{\Delta_i}{\Delta} \sum_{j=1}^{i} L^j = \frac{1}{1 - L^i} \). Clearly we get \( \mu_0 = \text{Law}(X) = \mu \) and \( \mu_r = \text{Law}(X') = \nu \).

To see that \( W_\infty(\mu_i, \mu_{i-1}) = \Delta_0 L^i \) we construct a coupling between \( \mu_i \) and \( \mu_{i-1} \) as follows: sample \( (X, X') \sim \pi \) and let \( Y = (1 - \theta_i)X + \theta_i X' \) and \( Y' = (1 - \theta_{i-1})X + \theta_{i-1} X' \). Clearly we have \( \text{Law}((Y, Y')) \in \mathcal{C}(\mu_i, \mu_{i-1}) \). Furthermore, with probability one the following holds:
\[
\|Y - Y'\| = \|(\theta_{i-1} - \theta_i)X - (\theta_{i-1} - \theta_i)X'\| = \frac{\Delta_0}{\Delta} L^i \|X - X'\| \leq \Delta_0 L^i ,
\]
where the last inequality uses that \( \pi \) is a witness of \( W_\infty(\mu, \nu) = \Delta \). This concludes the proof. \( \square \)

**Theorem 5.** Let \( \ell : \mathbb{K} \times \mathbb{D} \to \mathbb{R} \) be a \( C \)-Lipschitz, \( \beta \)-smooth, \( \rho \)-strongly convex loss function. If \( \eta \leq \frac{2}{\beta + \rho} \), then NoisyProjSGD(\( D, \ell, \eta, \sigma, \xi_0 \)) satisfies \( (\alpha, \alpha \epsilon_i) \)-RDP at index \( i \), where \( \epsilon_i = \frac{2C^2}{\sigma^2} (n - i) \alpha \) and \( \epsilon_i = \left( \frac{\alpha}{\sigma^2} \right)^2 \left( \frac{n - i}{\alpha} \right)^{n - 1} \) for \( 1 \leq i \leq n - 1 \).

To prove Theorem 5 we will use the following well-known fact about convex optimization: gradient iterations on a strongly convex function are strict contractions. The lemma below provides an expression for the contraction coefficient.

**Lemma 18.** Let \( \mathbb{K} \subseteq \mathbb{R}^d \) be a convex set and suppose the function \( f : \mathbb{K} \to \mathbb{R} \) is \( \beta \)-smooth and \( \rho \)-strongly convex. If \( \eta \leq \frac{2}{\beta + \rho} \), then the map \( \psi(x) = x - \eta \nabla f(x) \) is \( L \)-Lipschitz on \( \mathbb{K} \) with \( L = \sqrt{1 - \frac{2\beta \rho}{\beta + \rho}} < 1 \).

**Proof.** This follows from a standard calculation in convex optimization; see e.g. [Bubeck, 2015, Theorem 3.12]. We reproduce the proof here for completeness. Recall from [Bubeck, 2015, Lemma 3.11] that if a function \( f \) is \( \beta \)-smooth and \( \rho \)-strongly convex, then for any \( x, y \in \mathbb{K} \) we have
\[
\frac{\beta \rho}{\beta + \rho} \|x - y\|^2 + \frac{1}{\beta + \rho} \|\nabla f(x) - \nabla f(y)\|^2 \leq \langle \nabla f(x) - \nabla f(y), x - y \rangle .
\]
Using this inequality, one can show the following:

\[
\|\psi(x) - \psi(y)\|^2 = \|(x - \eta \nabla f(x)) - (y - \eta \nabla f(y))\|^2 \\
= \|x - y\|^2 + \eta^2 \|\nabla f(x) - \nabla f(y)\|^2 - 2\eta \langle \nabla f(x) - \nabla f(y), x - y \rangle \\
\leq \left(1 - \frac{2\eta^2 \rho}{\beta + \rho}\right) \|x - y\|^2 + \eta \left(\frac{2}{\beta + \rho}\right) \|\nabla f(x) - \nabla f(y)\|^2 \\
\leq \left(1 - \frac{2\eta^2 \rho}{\beta + \rho}\right) \|x - y\|^2 ,
\]

where the last inequality uses our assumption on \(\eta\). 

**Proof of Theorem 5.** Fix \(1 \leq i \leq n - 1\) and let \(D \simeq D'\) be two datasets differing on the \(i\)th coordinate. Let \(\xi \triangleq \xi_{i-1} \in \mathcal{P}(\mathbb{R}^d)\) represent the distribution of \(x_{i-1}\) in the execution of Algorithm 1 with input \(D\). Since \(D\) and \(D'\) differ only on the \(i\)th coordinate, the distribution of \(x_{i-1}\) on input \(D'\) is also \(\xi\). Now let \(\psi_0(x) = x - \eta \nabla_x \ell(x, z_i),\ \psi_0'(x) = x - \eta \nabla_x \ell(x, z_i')\), and \(\psi_j(x) = x - \eta \nabla_x \ell(x, z_{i+j})\) for \(j \in [r]\) with \(r = n - i\). Defining the Markov operators \(K_j, j \in \{0, \ldots, r\}\), where \(Y_j \sim K_j(x)\) is given by \(K_j(x) = \Pi_K(\psi_j(x) + Z)\) with \(Z \sim N(0, \sigma^2 I)\), we immediately obtain that the distribution of the output \(x_n\) of NoisyProjSGD(\(D, \ell, \eta, \sigma\)) can be written as \(\xi_K K_1 \cdots K_r\). Similarly, the distribution of the output of NoisyProjSGD(\(D', \ell, \eta, \sigma\)) can be written as \(\xi_{K_0} K_1 \cdots K_r\), where \(K_0(x) = N(\psi_0(x), \eta^2\sigma^2 I)\). Therefore, to obtain the Rényi differential privacy of NoisyProjSGD(\(D, \ell, \eta, \sigma\)) at index \(i\) we need to bound \(R_\alpha(\xi_K K_1 \cdots K_r \parallel \xi_{K_0} K_1 \cdots K_r)\).

With the goal to apply Theorem 5, we first define \(\mu = \xi_K 0\) and \(\nu = \xi_{K_0}'\) and use the Lipschitz assumption on \(\ell\) to conclude that \(W_\infty(\mu, \nu) \leq 2\eta C\). Indeed, consider the coupling \(\pi \in \mathcal{C}(\mu, \nu)\) obtained by sampling \((Y, Y') \sim \pi\) as follows: sample \(X \sim \xi\) and \(Z \sim N(0, \eta^2 \sigma^2 I)\), and then let \(Y = \Pi_K(\psi_0(X) + Z)\) and \(Y' = \Pi_K(\psi_0'(X) + Z)\). Now, since \(\ell(\cdot, z_i)\) and \(\ell(\cdot, z_i')\) are both \(C\)-Lipschitz and \(\Pi_K\) is contractive, we see that the following holds almost surely under \(\pi\):

\[
\|Y - Y'\| \leq \|\psi_0(X) - \psi_0'(X)\| = \eta \|\nabla_x \ell(X, z_i) - \nabla_x \ell(X, z_i')\| \\
\leq \eta \left(\|\nabla_x \ell(X, z_i)\| + \|\nabla_x \ell(X, z_i')\|\right) \leq 2\eta C .
\]

Thus, \(W_\infty(\mu, \nu) \leq 2\eta C\) as claimed.

Next we note that the assumption \(\eta \leq \frac{2}{\beta + \rho}\) together with Lemma 18 imply that \(\psi_j, j \in [r]\), are all \(L\)-Lipschitz with \(L = \sqrt{1 - \frac{2\eta^2 \rho}{\beta + \rho}} < 1\). Thus we can apply Theorem 5 with \(\Delta = 2\eta C\) to obtain

\[
R_\alpha(\xi_K K_1 \cdots K_r \parallel \xi_{K_0} K_1 \cdots K_r) \leq \frac{2\eta^2 C^2 L^n - i + 1}{(n - i)\eta^2 \sigma^2} = \frac{2\eta C^2}{(n - i)\sigma^2} \left(1 - \frac{2\eta \beta \rho}{\beta + \rho}\right)^{\frac{n - i + 1}{2}} .
\]

This concludes the analysis of the case \(i < n\).

For the case \(i = n\) we need to bound \(R_\alpha(\xi_K 0 \parallel \xi_{K_0}')\), where now \(\xi\) is the distribution of \(x_{n-1}\), and the operators \(K_0\) and \(K_0'\) are defined as above. By Hölder’s inequality, monotonicity of the logarithm, the contractiveness of \(\Pi_K\) and the Lipschitz assumption on \(\ell\) we have

\[
R_\alpha(\xi_K 0 \parallel \xi_{K_0}') \leq \sup_{x \in \text{supp}(\xi)} R_\alpha(K_0(x) \parallel K_0'(x)) \leq \sup_{x \in \mathbb{R}^d} R_\alpha(K_0(x) \parallel K_0'(x)) \\
\leq \sup_{x \in \mathbb{R}^d} \frac{\alpha \eta^2 \|\nabla_x \ell(x, z_n) - \nabla_x \ell(x, z_n')\|^2}{2\eta^2 \sigma^2} \leq \frac{2\alpha C^2}{\sigma^2} .
\]

\(\square\)
C Proofs for Section 5 (Diffusion Mechanisms)

**Theorem 6.** Let \( f : \mathbb{D}^n \to \mathbb{R}^d \) and let \( P = (P_t)_{t \geq 0} \) be a Markov semigroup on \( \mathbb{R}^d \) satisfying Assumption 1. If the mechanism \( M^f_t(D) = P_t(f(D)) \) has intrinsic sensitivity \( \Lambda(t) \), then it satisfies \((\alpha, \alpha \Lambda(t))\)-RDP for any \( \alpha > 1 \) and \( t > 0 \).

The proof of Theorem 6 relies, first of all, on the following lemma.

**Lemma 19.** Let \( \varphi : [t, \infty) \to \mathbb{R} \) be a function satisfying \( \varphi(s) > 0 \) and \( \lim_{s \to \infty} \varphi(s) = 1 \). Suppose there exists a function \( \kappa(s) \) and a constant \( c > 0 \) such that for all \( s \geq t \) we have \( \frac{d}{ds}\varphi(s) \geq -c\kappa(s)\varphi(s) \). Then \( \varphi(t) \leq \exp\left( c \int_t^{\infty} \kappa(s) ds \right) \).

**Proof.** The bound follows from a direct application of the fundamental theorem of calculus. Indeed, noting \( \lim_{s \to \infty} \log \varphi(s) = 0 \), we have

\[
-\log \varphi(t) = \lim_{s \to \infty} \log \varphi(s) - \log \varphi(t) = \int_t^{\infty} \left( \frac{d}{ds} \log \varphi(s) \right) ds
= \int_t^{\infty} \left( \frac{d}{ds} \frac{\varphi(s)}{\varphi(s)} \right) ds \geq -c \int_t^{\infty} \kappa(s) ds.
\]

\(\square\)

In order to apply this lemma to bound the Rényi DP of the diffusion mechanism \( M^f_t \) we will need to compute the derivative with respect to \( t \) of the Rényi divergence between \( P_t(x) \) and \( P_t(x') \). To be able to evaluate this derivative we will use some well-known relations between the kernel \( p_t(x,y) \) of a semigroup with invariant measure \( \lambda \) and its generator \( L \), as well as further calculus rules for the carré du champ operator \( \Gamma \). We now introduce the required properties without proof and recall they are standard facts in the theory of symmetric diffusion processes (see, e.g., [Bakry et al., 2013]), and in particular they hold for any Markov semigroup satisfying Assumption 1.

1. (Reversible Fokker-Planck Equation) For any \( x, y, t \) we have \( \frac{d}{dt} p_t(x,y) = L_y p_t(x,y) \), where \( L_y \) denotes the generator operating on \( y \mapsto p_t(x,y) \).

2. (Integration by Parts) We have \( \int \Gamma(f,g) d\lambda = -\int (Lf)g d\lambda \) for any \( f, g \) where the integrals are defined.

3. (Chain Rule for \( \Gamma \)) For any differentiable function \( \phi \) we have \( \Gamma(\phi(f), g) = \phi'(f)\Gamma(f,g) \) for any functions \( f, g \) where the terms are defined.

4. (Product Rule for \( \Gamma \)) We have \( \Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h) \) for any functions \( f, g, h \) where the terms are defined.

**Proof of Theorem 6.** Let us define the function \( \phi(u) = u^\alpha \) for \( \alpha > 1 \) and note that the derivatives of \( \phi \) satisfy the following identities:

\[
\phi'(u) = \alpha \frac{\phi(u)}{u}, \quad (7)
\]
\[
\phi''(u) = \alpha(\alpha - 1) \frac{\phi(u)}{u^2}, \quad (8)
\]
\[
-u\phi''(u) = \frac{d}{du} \left( \phi(u) - u\phi'(u) \right). \quad (9)
\]
Now fix datasets $D \simeq D'$ and let $x = f(D)$ and $x' = f(D')$. With this notation we have $M_t^f(D) = P_t(x)$, $M_t^f(D') = P_t(x')$ and $R_\alpha(P_t(x\|P_t(x'))) = \frac{1}{\alpha - 1} \log \varphi(t)$, where we defined

$$\varphi(t) \triangleq \int \phi \left( \frac{p_t(x, y)}{p_t(x', y)} \right) p_t(x', y) \lambda(dy) .$$

Since $P$ has a unique invariant measure $\lambda$, then we must have $\lim_{t \to \infty} \frac{p_t(x, y)}{p_t(x', y)} = 1$ for any $x, y$, and therefore $\lim_{t \to \infty} \varphi(t) = 1$. Thus, by Lemma \[9\] to obtain the desired bound it suffices to show that the inequality $\frac{d}{dt} \varphi(t) \geq -\alpha(\alpha - 1)\kappa_{x,x'}(t)\varphi(t)$ holds for $t > 0$.

We will now show that this inequality is indeed satisfied. For simplicity, let use define the notation $p_t(y) \triangleq p_t(x, y)$, $q_t(y) \triangleq p_t(x', y)$, $r_t(y) \triangleq \frac{p_t(y)}{q_t(y)}$ and $\partial_t \triangleq \frac{d}{dt}$. With these, we now can apply the properties of $P$ and $\phi$ to compute the derivative of $\varphi(t)$ as follows\[10\]

$$\partial_t \varphi(t) = \int \partial_t (\phi(r_t)q_t)$$

$$= \int \phi'(r_t)(\partial_t r_t)q_t + \phi(r_t)(\partial_t q_t)$$

$$= \int \phi'(r_t)(Lp_t)q_t - (Lq_t)p_t + \phi(r_t)(Lq_t)$$

$$= \int \phi'(r_t)(Lp_t) + (\phi(r_t) - r_t\phi'(r_t))(Lq_t)$$

$$= -\int \Gamma(\phi'(r_t), p_t) + \Gamma(\phi(r_t) - r_t\phi'(r_t), q_t)$$

$$= -\int \phi''(r_t)\Gamma(\Gamma(r_t, p_t) + \Gamma(\phi(r_t) - r_t\phi'(r_t), q_t)$$

$$= -\int \phi''(r_t)\Gamma(r_t, p_t) - r_t\phi''(r_t)\Gamma(r_t, q_t)$$

$$= -\alpha(\alpha - 1) \int \frac{\phi(r_t)}{r_t^2} (\Gamma(r_t, p_t) - r_t\Gamma(r_t, q_t))$$

$$= -\alpha(\alpha - 1) \int \phi(r_t)q_t \left( \frac{q_t\Gamma(r_t, p_t) - p_t\Gamma(r_t, q_t)}{p_t^2} \right)$$

The last step in the proof is to verify the following identify, which follows from the rules of calculus\[10\]

10 All integrals in this derivation are with respect to the invariant measure $d\lambda$, which is omitted for convenience.
under $\Gamma$:

$$
\Gamma(\log r_t, \log r_t) = \frac{1}{r_t^2} \Gamma(r_t, r_t)
$$

by Chain Rule for $\Gamma$, 

$$
= \frac{1}{r_t^2} \Gamma(\frac{r_t}{qt}, \log r_t)
$$

by definition of $r_t$, 

$$
= \frac{1}{r_t^2} \left( \frac{1}{qt} \Gamma(r_t, p_t) + p_t \Gamma(\frac{r_t}{qt}) \right)
$$

by Product Rule for $\Gamma$, 

$$
= \frac{1}{r_t^2} \left( \frac{1}{qt} \Gamma(r_t, p_t) - \frac{p_t}{qt} \Gamma(r_t, qt) \right)
$$

by Chain Rule for $\Gamma$, 

$$
= \frac{q_t \Gamma(r_t, p_t) - p_t \Gamma(r_t, qt)}{p_t^2}
$$

by definition of $r_t$.

Now we finally put the last two derivations together to conclude that

$$
\frac{d}{dt} \varphi(t) = -\alpha(\alpha - 1) \int \phi \left( \frac{p_t(x, y)}{p_t(x', y)} \right) p_t(x', y) \Gamma \left( \log \frac{p_t(x, y)}{p_t(x', y)} \right) \lambda(dy)
$$

$$
\geq -\alpha(\alpha - 1) \kappa_{x,x'}(t) \int \phi \left( \frac{p_t(x, y)}{p_t(x', y)} \right) p_t(x', y) \lambda(dy)
$$

$$
= -\alpha(\alpha - 1) \kappa_{x,x'}(t) \varphi(t).
$$

\square

**Corollary 7.** Let $f : \mathbb{D}^n \to \mathbb{R}^d$ have global $L_2$-sensitivity $\Delta$ and $P = (P_t)_{t \geq 0}$ be the Ornstein-Uhlenbeck semigroup with parameters $\theta, \rho$. For any $\alpha > 1$ and $t > 0$ the mechanism $M^f_t(D) = P_t(f(D))$ satisfies $(\alpha, \alpha \Lambda(t))$-RDP with $\Lambda(t) = \frac{\theta \Delta^2}{2 \rho^2 e^{2\theta t} - 1}$.

**Proof.** Using the expression of the kernel of $P_t$ with respect to the invariant measure $\lambda$ we first compute

$$
\log \left( \frac{p_t(x, y)}{p_t(x', y)} \right) = \frac{\theta e^{\theta t} (x - x', y)}{\rho^2 (e^{2\theta t} - 1)}.
$$

Next we use the expression $\Gamma(f) = \rho^2 \|\nabla f\|^2$ for the carré du champ operator to obtain

$$
\kappa_{x,x'}(t) = \frac{\theta^2 e^{2\theta t} \|x - x'\|^2}{\rho^2 (e^{2\theta t} - 1)^2}.
$$

Applying the easily verifiable integral formula

$$
\int_t^\infty \frac{e^{2\theta s}}{(e^{2\theta s} - 1)^2} ds = \frac{1}{2\theta (e^{2\theta t} - 1)}
$$

in the definition of $\Lambda(t)$ yields the desired result. \square
Theorem 8. Suppose \( f : \mathbb{D}^n \rightarrow \mathbb{R}^d \) has global \( L_2 \)-sensitivity \( \Delta \) and satisfies \( \sup_D \| f(D) \| \leq R \). If \( \theta R^2 \leq 4 \rho^2 \) then we have \( \frac{\mathcal{E}_{\text{OU}}(\theta, \rho, t)}{\mathcal{E}_{\text{GM}}(\theta, \rho, t)} \leq 1 \) for all \( t \geq 0 \) and \( \lim_{t \to \infty} \frac{\mathcal{E}_{\text{OU}}(\theta, \rho, t)}{\mathcal{E}_{\text{GM}}(\theta, \rho, t)} = 0 \). In particular, taking \( \theta = \log \left( 1 + \frac{d \Delta^2}{2e R^2} \right) \) and \( \rho^2 = \frac{\theta \Delta^2}{2e(e^{2\theta} - 1)} \) with \( \epsilon > 0 \), the mechanism \( M_t \) satisfies \((\alpha, \alpha \epsilon)\)-RDP at time \( t = 1 \) and we have \( \frac{\mathcal{E}_{\text{OU}}(\theta, \rho, 1)}{\mathcal{E}_{\text{GM}}(\theta, \rho, 1)} \leq \left( 1 + \frac{d \Delta^2}{2e R^2} \right)^{-1} \).

Proof. First note that at time \( t = 0 \) we have \( \mathcal{E}_{\text{OU}}(\theta, \rho, 0) = \mathcal{E}_{\text{GM}}(\theta, \rho, 0) = 0 \). Thus, to see that \( \mathcal{E}_{\text{OU}}(\theta, \rho, t) \leq \mathcal{E}_{\text{GM}}(\theta, \rho, t) \) for \( t > 0 \) it is enough to check that \( \frac{d}{dt} \mathcal{E}_{\text{OU}}(\theta, \rho, t) \leq \frac{d}{dt} \mathcal{E}_{\text{GM}}(\theta, \rho, t) \) for \( t \geq 0 \). Indeed, differentiating (4), this follows from the boundedness of \( f \) and \( \theta R^2 \leq 4 \rho^2 \) by noting:

\[
\frac{d}{dt} \mathcal{E}_{\text{OU}}(\theta, \rho, t) \leq 2 \rho^2 e^{-2\theta t} + 2 \theta R^2 e^{-\theta t} (1 - e^{-\theta t}) \leq 2 \rho^2 e^{-2\theta t} + 8 \rho^2 e^{-\theta t} (1 - e^{-\theta t}) = \frac{d}{dt} \mathcal{E}_{\text{GM}}(\theta, \rho, t),
\]

where the last two steps use the inequality \( 4e^s - 3 \leq e^{4s}, s \geq 0 \), and the definition of \( \tilde{\sigma}^2 \). To see that the ratio converges to 0 we just observe that the limit of \( \mathcal{E}_{\text{OU}}(\theta, \rho, t) \) is finite while \( \mathcal{E}_{\text{GM}}(\theta, \rho, t) \) grows to infinity as \( t \to \infty \).

The privacy bound in the case with a fixed level of privacy at \( t = 1 \) follows from directly from Corollary 7. The error bound follows substituting the chosen parameters in the expression for the mean squared error. In the first place, we use the definitions of \( \tilde{\sigma}^2 \) and \( \rho^2 \) to get

\[
\mathcal{E}_{\text{GM}}(\theta, \rho, 1) = d \tilde{\sigma}^2 = \frac{d \rho^2 (e^{2\theta} - 1)}{\theta} = \frac{d \Delta^2}{2e}.
\]

On the other hand, substituting the choice for \( \rho \) on the error of the Ornstein-Uhlenbeck mechanism and using the boundedness of \( f \) we get

\[
\mathcal{E}_{\text{OU}}(\theta, \rho, 1) \leq (1 - e^{-\theta})^2 R^2 + \frac{d \rho^2}{\theta} (1 - e^{-2\theta}) = (1 - e^{-\theta})^2 R^2 + \frac{d \Delta^2}{2e} e^{-2\theta}.
\]

Finally, plugging the choice of \( \theta \) in this last expression yields:

\[
(1 - e^{-\theta})^2 R^2 + \frac{d \Delta^2}{2e} e^{-2\theta} = \frac{R^2 \left( \frac{d \Delta^2}{2e R^2} \right)^2 + R^4 \frac{d \Delta^2}{2e R^2}}{\left( R^2 + \frac{d \Delta^2}{2e R^2} \right)^2} = \frac{d \Delta^2}{2e} \frac{1}{1 + \frac{d \Delta^2}{2e R^2}}.
\]

\( \square \)