Solvability of the divergence equation implies John via Poincaré inequality

Renjin Jiang, Aapo Kauranen and Pekka Koskela

Abstract. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded simply connected domain. We show that, for a fixed (every) \( p \in (1, \infty) \), the divergence equation \( \text{div} \ v = f \) is solvable in \( W_0^{1,p}(\Omega)^2 \) for every \( f \in L^p_0(\Omega) \), if and only if \( \Omega \) is a John domain, if and only if the weighted Poincaré inequality
\[
\int_{\Omega} |u(x) - u_\Omega|^q \, dx \leq C \int_{\Omega} |\nabla u(x)|^q \dist(x, \partial \Omega)^q \, dx
\]
holds for some (every) \( q \in [1, \infty) \). In higher dimensions similar results are proved under some additional assumptions on the domain in question.

1 Introduction

This paper is devoted to the study of geometric aspects of the solvability of the divergence equation. Our main tool is a weighted Poincaré inequality.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( p \in (1, \infty) \), and \( L^p_0(\Omega) \) be the space of all functions in \( L^p(\Omega) \) which have integral zero over \( \Omega \). The Sobolev space \( W^{1,p}(\Omega) \) is defined as
\[
W^{1,p}(\Omega) := \{ u \in L^p(\Omega) : \nabla u \in \mathcal{D}'(\Omega) \cap L^p(\Omega) \}
\]
with the norm
\[
\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.
\]
The Sobolev space \( W_0^{1,p}(\Omega) \) is then defined to be the closure of smooth functions with compact support in \( \Omega \) under the \( W^{1,p} \)-norm.

For \( f \in L^p_0(\Omega) \), a vector function \( v = (v_1, \ldots, v_n) \in L^p(\Omega)^n \) is a solution to the divergence equation \( \text{div} \ v = f \), if
\[
\int_{\Omega} v(x) \cdot \nabla g(x) \, dx = - \int_{\Omega} f(x) g(x) \, dx
\]
holds for each \( g \in W^{1,q}(\Omega) \), where \( q \) is the Hölder conjugate number of \( p \). We say that the divergence equation with Dirichlet boundary condition (\( \text{div}_{p,0} \), for short) is solvable,
if for each \( f \in L^p_0(\Omega) \), there exists \( v \in W^{1,p}_0(\Omega)^n \) such that \( \text{div} v = f \) holds in the above distributional sense, and there exists \( C > 0 \), independent of \( f \) such that

\[
\|v\|_{W^{1,p}(\Omega)^n} \leq C\|f\|_{L^p(\Omega)}.
\]

When \( \Omega \) has a Lipschitz boundary, it is well known that the divergence equation \( \text{div}_{p,0} \) is solvable for all \( p \in (1, \infty) \). There are several ways to prove this result, for instance, it can be proved via Functional Analysis, or via elementary constructions; see [3, 4, 5, 10, 19]. Recently, Acosta et al. [1] proved that \( \text{div}_{p,0} \) is solvable on John domains for all \( p \in (1, \infty) \) via a constructive approach.

On the other hand, if \( \Omega \) has an external cusp, it is known that the divergence equation \( \text{div}_{2,0} \) is not solvable in \( \Omega \); see [1].

Notice that \( p = 1 \) or \( p = \infty \), the divergence equation \( \text{div}_{p,0} \) does not necessarily admit a solution in \( W^{1,p}(\Omega)^n \) for \( p = 1 \) or for \( p = \infty \), even when \( \Omega \) is a cube; see [5].

It is natural to ask for necessary geometric conditions for the solvability of the divergence equation \( \text{div}_{p,0} \) for some (all) \( p \in (1, \infty) \). For domains satisfying a separation property introduced by Buckley and Koskela [6] (see Section 2 for the definition), it was shown by Acosta et al. [1] that the divergence equation \( \text{div}_{p,0} \) is solvable for \( p \in (1, n) \), if and only if \( \Omega \) is a John domain. Our result extends this to the case \( p > n \), and to the case \( p = n \) in some special cases.

Let us first recall the definition of a John domain. This terminology was introduced in [18], but these domains were studied already by F. John [14].

**Definition 1.1** (John domain). A bounded domain \( \Omega \subset \mathbb{R}^n \) with a distinguished point \( x_0 \in \Omega \) called a John domain if it satisfies the following “twisted cone” condition: there exists a constant \( C > 0 \) such that for all \( x \in \Omega \) there is a curve \( \gamma : [0, l] \rightarrow \Omega \) parametrised by arclength such that \( \gamma(0) = x \), \( \gamma(l) = x_0 \), and \( d(\gamma(t), \mathbb{R}^n \setminus \Omega) \geq Ct \) for all \( 0 \leq t \leq l \).

Observe that each Lipschitz domain is a John domain. Moreover, the boundary of a (planar) John domain may contain an interior cusp, while exterior cusps are ruled out.

For a mapping \( v = (v_1, v_2, \ldots, v_n) \in W^{1,1}_{\text{loc}}(\Omega)^n \), let \( Dv \) denote its weak differential. For \( x \in \Omega \), we denote by \( \rho(x) \) the distance from \( x \) to the boundary \( \partial \Omega \), i.e., \( \rho(x) := \text{dist}(x, \partial \Omega) \).

Our main result is the following theorem.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain that satisfies the separation property, \( n \geq 2 \). Then the following conditions are equivalent:

(i) \( \Omega \) is a John domain;

(ii) for some (every) \( p \in (1, n) \cup (n, \infty) \) and each \( f \in L^p_0(\Omega) \), there exists a solution \( v \in W^{1,p}_0(\Omega)^n \) to the equation \( \text{div} v = f \) with

\[
\|v\|_{W^{1,p}(\Omega)^n} \leq C\|f\|_{L^p(\Omega)};
\]

(iii) for some (every) \( p \in (1, \infty) \) and each \( f \in L^p_0(\Omega) \), there exists a solution \( v \in W^{1,p}_0(\Omega)^n \) to the equation \( \text{div} v = f \) with

\[
\|v\|_{L^p(\Omega)^n} + \|Dv\|_{L^p(\Omega)^{n\times n}} \leq C\|f\|_{L^p(\Omega)};
\]
Divergence equation and Poincaré inequality

(iv) for some (every) $p \in (1, \infty)$ and each $f \in L^p_0(\Omega)$, there exists a solution $v \in L^p(\Omega)^n$ to the equation $\text{div } v = f$ with

$$\|v\|_{L^p(\Omega)^n} \leq C\|f\|_{L^p(\Omega)}.$$ 

The meaning of “some (every)” in the statement above is that the given existence result for any fixed $p$ in the given parameter range actually implies the existence for every such $p$, under the assumptions of the theorem.

We have not been able to include the case $p = n$ in condition (ii). The case $p < n$ is proved in [1] by using Sobolev inequalities for $W^{1,p}_0$; our approach for $p > n$ is based on the fact that solutions in $W^{1,p}_0$ satisfy suitable Hardy inequalities. In Example 4.1 below, we construct a John domain where the divergence equation admits a solution in $W^{1,p}_0$, but the Hardy inequalities fail. However, we can include the case $p = n$ in Theorem 1.1 (ii) provided the complement of $\Omega$ is sufficiently thick on $\partial\Omega$; see Theorem 4.1 in Section 4.

Notice that each domain that is quasiconformally equivalent to a uniform domain $G$ satisfies the separation property. In particular, each simply connected plane domain satisfies the separation property; see [6].

**Corollary 1.1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Then for some (all) $p \in (1, \infty)$ and each $f \in L^p_0(\Omega)$, there exists a solution $v \in W^{1,p}_0(\Omega)^2$ to the equation $\text{div } v = f$ such that

$$\|v\|_{W^{1,p}(\Omega)^2} \leq C\|f\|_{L^p(\Omega)},$$

if and only if $\Omega$ is a John domain.

For $p = 2$, by duality, the solvability of the divergence equation with Dirichlet boundary condition is equivalent to the fact

$$\|f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{W^{1,2}_0(\Omega)^2}$$

for each $f \in L^2_0(\Omega)$; see [1] for instance. From Corollary 1.1 it follows that on a bounded simply connected domain $\Omega \subset \mathbb{R}^2$, for each $f \in L^2_0(\Omega)$, (1.1) holds if and only if $\Omega$ is a John domain.

Our main tool is the equivalence between the John condition and weighted Poincaré inequalities; see Theorem 2.1 below. To prove that solvability of the divergence equation $\text{div}_{p,0}$ implies John, Acosta et al. [1] used the characterization of Sobolev-Poincaré inequality from Buckley and Koskela [6]. As the Sobolev-Poincaré inequality only holds for $p \in [1, n)$, the authors were not able to deal with the case $p \geq n$ in the necessity of the John condition. To bypass this problem we generalize Buckley and Koskela’s characterization to the weighted setting. Precisely, the following special case of Theorem 2.1 below says that, for a domain $\Omega \subset \mathbb{R}^n$ satisfying the separation property, the weighted Poincaré inequality

$$\int_{\Omega} |u(x) - u_\Omega|^p \, dx \leq C \int_{\Omega} |\nabla u(x)|^p \rho(x)^p \, dx$$
holds for some (all) \( p \in [1, \infty) \), if and only if \( \Omega \) is a John domain. Using this together with the fact that solutions satisfy Hardy type inequalities, we obtain the desired result.

The paper is organized as follows. In Section 2, we show that our weighted Poincaré inequality implies the John condition. In Section 3, we study the divergence equation on John domains, and the main result Theorem 1.1 is proved in Section 4.

Throughout the paper, we denote by \( C \) positive constants which are independent of the main parameters, but which may vary from line to line. For \( p \in [1, n) \), its Sobolev conjugate \( \frac{np}{n-p} \) is denoted by \( p^* \); for each \( p \in (1, \infty) \), its Hölder conjugate \( \frac{p}{p-1} \) is denoted by \( p' \).

Corresponding to a to a function space \( X \), we denote its \( n \)-vector-valued analogs by \( X^n \).

## 2 The weighted Poincaré inequality

In this section, we give a generalization of Buckley and Koskela’s characterization from [6], which offers us the main tool for proving Theorem 1.1.

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, \( n \geq 2 \). We say that the Sobolev-Poincaré inequality \((SP_{p,p'})\), \( p \in [1, n) \), holds if there is a \( C > 0 \) such that for every \( u \in C^\infty(\Omega) \) we have that

\[
(SP_{p,p'}) \quad \left( \int_\Omega |u(x) - u_\Omega|^{p'} \, dx \right)^{1/p'} \leq C \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}.
\]

Above, \( u_\Omega \) denotes the integral average of \( u \) on \( \Omega \), i.e., \( u_\Omega = \frac{1}{\lambda_\Omega} \int_\Omega u \, dx \).

For \( \Omega \) satisfying the separation property (see Definition 2.1 below), Buckley-Koskela [6] have shown that it is a John domain if and only if \( \Omega \) supports a Sobolev-Poincaré inequality \((SP_{p,p'})\) for some (all) \( p \in [1, n) \).

Let us first recall the definition of separation property which was introduced in [6, 7]. Recall that for each \( x \in \Omega \), \( \rho(x) = d(x, \mathbb{R}^n \setminus \Omega) \).

**Definition 2.1** (separation property). We say that a domain \( \Omega \subset \mathbb{R}^n \) with a distinguished point \( x_0 \) has a separation property if there is a constant \( C_\gamma \) such that the following holds: For each \( x \in \Omega \) there is a curve \( \gamma : [0, 1] \to \Omega \) with \( \gamma(0) = x \), \( \gamma(1) = x_0 \), and such that for each \( t \) either \( \gamma([0, t]) \subset \Omega \) or \( \gamma([0, t]) \subset \mathbb{R}^n \setminus \Omega \), and such that for each \( y \in \gamma([0, t]) \) \( y \in \gamma([0, t]) \setminus B \) belongs to a different component of \( \Omega \setminus \partial B \) than \( x_0 \).

It follows from [6] that \( \Omega \) has a separation property if it is quasiconformally equivalent to a uniform domain \( G \). In particular, each simply connected planar domain satisfies a separation property.

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain satisfying the separation property, \( n \geq 2 \). Then the \((P_{p,q,b})\)-Poincaré inequality holds, i.e., for every \( u \in C^\infty(\Omega) \) we have that

\[
(P_{p,q,b}) \quad \left( \int_\Omega |u(x) - u_\Omega|^q \, dx \right)^{p/q} \leq C_0 \int_\Omega |\nabla u(x)|^p \rho(x)^b \, dx,
\]

for some (all) \((p, q, b)\) satisfying \( 1 \leq p \leq q < \infty \), \( \frac{n}{q} + 1 - \frac{n}{p} \geq 0 \) and \( b = p(\frac{n}{q} + 1 - \frac{n}{p}) \), if and only if \( \Omega \) is a John domain.
In what follows, we will call \((p, q, b)\) a Sobolev triple, if \((p, q, b)\) satisfies \(1 \leq p \leq q < \infty\), 
\[ \frac{b}{q} + 1 - \frac{n}{p} \geq 0 \] 
and \( b = p \left( \frac{a}{q} + 1 - \frac{n}{p} \right) \).

**Remark 2.1.** Notice that if \((p, q, b)\) is a Sobolev triple, then \( b \in [0, p] \). The two endpoint cases of \( b \) are of particular interest.

When \( b = 0 \), necessarily \( p \in [1, n) \) and \( q = p^* \); then \((P_{p,q,b})\)-Poincaré inequality is the Sobolev-Poincaré inequality \((SP_{p,p^*})\).

When \( b = p \), \( p \) equals \( q \) and takes values in \([1, \infty)\); we then denote \((P_{p,q,b})\)-Poincaré inequality by \((P_{p})\)-Poincaré inequality for convenience. The \((P_{p})\)-Poincaré inequality is the main tool for us to prove Theorem 1.1; see Section 4 below.

As each simply connected plane domain has a separation property, the following is an immediate corollary to Theorem 2.1.

**Corollary 2.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded simply connected domain. Then the \((P_{p,q,b})\)-Poincaré inequality holds for some (all) Sobolev triples \((p, q, b)\), if and only if \( \Omega \) is a John domain.

We will need the following characterization of a weighted Poincaré inequality from Hajlasz and Koskela [13, Theorem 1] (for non-weighted cases see Maz’ya [22]).

**Theorem 2.2** ([13]). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n \geq 2 \). Let \( 1 \leq p \leq q < \infty \) and \( b \geq 0 \). Then the following conditions are equivalent.

(i) There exists a constant \( C > 0 \) such that, for every \( u \in C^\infty(\Omega) \) it holds that

\[
\left( \int_\Omega |u(x) - u_\Omega|^q \, dx \right)^{1/q} \leq C \left( \int_\Omega |\nabla u(x)|^p \rho(x)^b \, dx \right)^{1/p}.
\]

(ii) For any fixed cube \( Q \subset \subset \Omega \), there exists a constant \( C > 0 \) such that

\[
\left( \int_\Omega |u(x)|^q \, dx \right)^{1/q} \leq C \left( \int_\Omega |\nabla u(x)|^p \rho(x)^b \, dx \right)^{1/p}
\]

whenever \( u \in C^\infty(\Omega) \) satisfies \( u|_Q = 0 \).

The following result generalizes [6, Theorem 2.1] to the setting of a weighted Sobolev-Poincaré inequality. Our proof follows the method of [6].

**Proposition 2.1.** Suppose that \((p, q, b)\) is a Sobolev triple, and that \( \Omega \) supports a \((P_{p,q,b})\)-Poincaré inequality. Fix a ball \( B_0 \subset \Omega \) and let \( w \in \Omega \). Then there exists a constant \( C = C(C_0, n, p, q, \Omega, B_0) \) such that

\[
\text{diam}(T) \leq Cd
\]

whenever \( T \) is a component of \( \Omega \setminus B(w, d) \) that does not intersect \( B_0 \).
Proof. From Theorem 2.2, the \((P_{p,q,b})\)-Poincaré inequality implies that for each Lipschitz function \(u\) that vanishes on \(B_0\), it holds that

\[
(\int_{\Omega} |u(x)|^q \, dx)^{p/q} \leq C_1 \int_{\Omega} |\nabla u(x)|^p \rho(x)^b \, dx,
\]

where \(C_1 = C(C_0, n, p, q, \Omega, B_0)\).

Let \(T\) be a component of \(\Omega \setminus B(w, d)\) that does not intersect \(B_0\). For each \(r \geq d\), set \(T(r) := T \setminus B(w, r)\); and for all \(r > s \geq d\), set \(A(s, r) := T(s) \setminus T(r)\).

If \(T(2d) = \emptyset\), then it is obvious \(\text{diam}(T) \leq 2d\). Otherwise, \(T(2d) \neq \emptyset\) and we continue with following steps.

**Claim 1.** \(|T(2d)| \leq C_2 d^n\), where \(C_2 = C(C_0, n, p, \Omega, B_0)\). Indeed, set

\[
u(x) := \begin{cases} 0, & \forall x \in \Omega \setminus T(d); \\ 1, & \forall x \in T(2d); \\ \frac{d(x, B(w, d))}{d}, & \forall x \in A(d, 2d). \end{cases}
\]

Then \(u\) is a Lipschitz function that vanishes on \(B_0\). The inequality (2.1) implies that

\[
|T(2d)|^{p/q} \leq \left( \int_{\Omega} |u(x)|^q \, dx \right)^{p/q} \leq C_1 \int_{\Omega} |\nabla u(x)|^p \rho(x)^b \, dx \leq C_1 \int_{A(d, 2d)} \rho(x)^b \, dx.
\]

As \(B(w, d)\) separates \(\Omega\), it follows that \(\rho(x) \leq 2d\) for each \(x \in B(w, d)\), and hence \(\rho(x) \leq 4d\) for each \(x \in B(w, 2d)\). Thus (2.2) implies that

\[
|T(2d)|^{p/q} \leq 4^b d^{b-p} C_3 A(d, 2d) dx \leq 4^b A_n d^{b-p},
\]

where \(A_n\) is the volume of the unit ball. As \(b = p \left( \frac{q}{q} + 1 - \frac{b}{p} \right)\), Claim 1 follows with \(C_2 := (4^{b+n}C_1 A_n)^{1/b} \).

Let \(r_0 := 2d\), for each \(j \geq 1\), choose \(r_j > r_{j-1}\) such that \(|T(r_j)| = 2^{-j}|T(2d)|\). Then \(|A(r_j, r_{j-1})| = 2^{-j}|T(2d)|\).

**Claim 2.** For each \(j \geq 1\), \(|r_j - r_{j-1}| \leq C_3 2^{-j/n} d\), where \(C_3 = C(C_0, n, p, q, \Omega, B_0)\). To prove this, let us consider two cases.

**Case 1.** If there exists \(x_j \in A(r_{j-1}, r_j)\) such that \(\rho(x_j) > C_4 2^{-j/n} d\), where \(C_4 := \left(6C_2/\omega_0\right)^{1/n}\), then \(|r_j - r_{j-1}| \leq 2C_4 2^{-j/n} d\).

Notice that \(T\) is a component of \(\Omega \setminus B(w, d)\), and \(B(x_j, C_4 2^{-j/n} d) \subset \Omega\) with center \(x_j \in A(r_{j-1}, r_j)\). Thus \(B(x_j, C_4 2^{-j/n} d) \setminus B(w, r_{j-1})\) is a subset of \(T\), which implies that the set \(B(x_j, C_4 2^{-j/n} d) \cap (B(w, r_{j-1}))\) is a subset of \(A(r_{j-1}, r_j)\).

Suppose towards a contradiction that \(|r_j - r_{j-1}| > 2C_4 2^{-j/n} d\). Then as \(x_j \in A(r_{j-1}, r_j)\), it follows that at least one third of \(B(x_j, C_4 2^{-j/n} d)\) is contained in \(A(r_{j-1}, r_j)\). We then have

\[
|A(r_{j-1}, r_j)| \geq \frac{1}{3} |B(x_j, C_4 2^{-j/n} d)| \geq \frac{1}{3} C_4^2 d^{2-1/n} \omega_n d^n \geq 2^{1-j} C_2 d^n > 2^{-j}|T(2d)| = |A(r_{j-1}, r_j)|,
\]

which is a contradiction.
which is a contradiction. This implies that $|r_j - r_{j-1}| \leq 2C_4 2^{-j/n} d$.

**Case 2.** If for each $x \in A(r_{j-1}, r_j)$, it holds that $\rho(x) \leq C_4 2^{-j/n} d$, then $|r_j - r_{j-1}| \leq (C_1 C_2^{1-\frac{\varepsilon}{\gamma}} C_3^{1/p})^{1/p} 2^{-j/n} d$.

In this case, similarly to Claim 1, we set

$$u(x) := \begin{cases} 0, & \forall x \in \Omega \setminus T(r_{j-1}); \\ 1, & \forall x \in T(r_j); \\ \frac{d(x, B(w, r_{j-1}))}{r_j - r_{j-1}}, & \forall x \in A(r_{j-1}, r_j), \end{cases}$$

and use the inequality (2.1) to obtain

$$|T(r_j)|^{p/q} \leq \left( \int_{\Omega} |u(x)|^q \, dx \right)^{p/q} \leq C_1 \int_{A(r_{j-1}, r_j)} \frac{\rho(x)^b}{|r_j - r_{j-1}|^p} \, dx,$$

which together with the facts $q \geq p$, $b = p(\frac{n}{q} + 1 - \frac{\varepsilon}{\gamma})$ and $|T(r_j)| = |A(r_{j-1}, r_j)| = 2^{-j}|T(2d)| \leq 2^{-j} d^p$ implies that

$$|r_j - r_{j-1}|^p \leq C_1 C_4^{b/n} d^b |A(r_{j-1}, r_j)|^{1-\frac{\varepsilon}{\gamma}} \leq C_1 C_4^{b/n} d^b |A(r_{j-1}, r_j)|^{1-\frac{\varepsilon}{\gamma}} C_2^{1-\frac{\varepsilon}{\gamma}} \leq C_1 C_2^{1-\frac{\varepsilon}{\gamma}} C_3^{b/p} d^p.$$ 

Hence, Claim 2 follows with $C_3 := \max\{2C_4, (C_1 C_2^{1-\frac{\varepsilon}{\gamma}} C_3^{1/p})^{1/p}\}$. Moreover, notice that $C_1$, $C_2$, $C_4$ and hence $C_3$ depend only on $C_0, n, p, q, \Omega, B_0$.

By using Claim 2, we finally obtain that

$$\text{diam} (T) \leq 2d + \sum_{j \geq 1} |r_j - r_{j-1}| \leq Cd,$$

where $C = C(C_0, n, p, q, \Omega, B_0)$, which completes the proof. \(\square\)

**Proof of Theorem 2.1.** If $\Omega$ is a John domain, then from [15, Theorem 2.1] it follows that the $(P_{p,q,b})$ holds for all Sobolev triples $(p, q, b)$ satisfying $1 \leq p \leq q < \infty$, $\frac{n}{q} + 1 - \frac{\varepsilon}{\gamma} \geq 0$ and $b = p(\frac{n}{q} + 1 - \frac{\varepsilon}{\gamma})$; also see [13].

For the converse we employ the argument from [6, Proof of Theorem 1.1] via Proposition 2.1. We sketch the proof for the sake of completeness.

Suppose that $(P_{p,q,b})$ holds for a Sobolev triple $(p, q, b)$. Fix $x \in \Omega$ and pick a curve $\gamma : [0, 1] \rightarrow \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$ as in Definition 2.1. According to [18, pp. 385-386] and [20, pp. 7-8], it is enough to show that $\text{diam} (\gamma([0, t])) \leq C \rho(\gamma(t))$.

Let $C_4$ be a constant as in Definition 2.1. If $\gamma([0, t]) \subset B(\gamma(t), C_4 \rho(\gamma(t)))$, the conclusion is obvious.
Otherwise the separation property implies that \( \partial B := \partial B(y(t), C_s \rho(y(t))) \) separates \( \gamma([0, t]) \setminus B \) from \( x_0 \). Let us consider two cases.

**Case 1.** If \( B \cap B_0 \neq \emptyset \), where \( B_0 := B(x_0, \rho(x_0)/2) \), then \( \gamma([0, t]) \subset B(y(t), C_5 \rho(y(t))) \) with \( C_5 = \frac{2 \text{diam}(\Omega)}{\rho(x_0)} C_s \). Indeed, as \( B \cap \partial \Omega \neq \emptyset \) and \( B \cap B(x_0, \rho(x_0)/2) \neq \emptyset \), it follows \( C_s \rho(y(t)) \geq \rho(x_0)/2 \). Hence \( \gamma([0, t]) \subset B(y(t), \text{diam}(\Omega)) \subset B(y(t), C_5 \rho(y(t))) \).

**Case 2.** If \( B \cap B_0 = \emptyset \), then \( \gamma([0, t]) \subset B(y(t), C_6 \rho(y(t))) \), where \( C_6 \) depends only on \( C_s, n, p, \Omega, B_0, C_s \). Let \( T \) be the component containing \( \gamma([0, t]) \setminus B \). As \( B \) separates \( \gamma([0, t]) \setminus B \) from \( x_0 \), \( T \) is a component of \( \Omega \setminus B \) that does not intersect \( B_0 \). By using Proposition 2.1, we see that \( \gamma([0, t]) \subset B(y(t), C_6 \rho(y(t))) \).

By letting \( C = \max\{C_s, C_5, C_6\} \), we obtain \( \text{diam}(\gamma([0, t])) \leq C \rho(y(t)) \), which completes the proof. \( \square \)

## 3 The divergence equation

In this section, we study the divergence equation on John domains.

**Theorem 3.1.** Let \( \Omega \) be a John domain in \( \mathbb{R}^n \), \( n \geq 2 \), and let \( q \in (1, \infty) \). Then for each \( f \in L^q_0(\Omega) \), there exists a solution \( u \in W^{1,q}_0(\Omega)^n \) to the equation \( \text{div} u = f \) in \( \Omega \). Moreover, there exists a constant \( C > 0 \), independent of \( f \), such that

\[
\|u\|_{L^q(\Omega)^n} + \|\nabla u\|_{L^q(\Omega)^n} \leq C \|f\|_{L^q(\Omega)}.
\]

**Remark 3.1.** Notice that on a bounded domain \( \Omega \), for \( q > n \), if \( u \in W^{1,q}_0(\Omega)^n \), then from the Hardy inequality \((H_0)\) it follows that

\[
\|u\|_{L^q(\Omega)^n} \leq C \|D u\|_{L^q(\Omega)};
\]

see Lemma 4.1 below. Thus the case \( q > n \) in Theorem 3.1 follows directly from Acosta et al. [1].

However, for \( q \leq n \), the Hardy inequality may fail even on a John domain. For instance, the domain \( B(0, 1) \setminus \{0\} \) does not admit the \( n \)-Hardy inequality, but it is a John domain; see [16]. Thus the main improvement in Theorem 3.1 is that for \( q \in (1, n] \), there are solutions \( u \) belong to \( W^{1,q}_0(\Omega)^n \) and satisfying (3.1).

For the proof, we need the following geometric decomposition from [8].

**Proposition 3.1** ([8]). Let \( \Omega \) be a John domain in \( \mathbb{R}^n \), \( n \geq 2 \). Let \( \{Q_j\}_j \) be a Whitney decomposition of \( \Omega \). Then there exists \( \sigma > 1 \) and a family of linear operators \( \{T_j\}_j \) such that for all \( p \in (1, \infty) \) and all \( f \in L^p_0(\Omega) \):

(i) \( \sum_j \chi_{Q_j} \leq C \chi_{\Omega} \);

(ii) \( \text{supp } T_j f \subset \sigma Q_j \) and \( T_j f \in L^p(\sigma Q_j) \);

(iii) \( f = \sum_{j \in I} T_j f \) in \( L^p(\Omega) \);

(iv) \( \sum_{j \in I} \int_{Q_j} |T_j f(x)|^p \, dx \leq C \int_{\Omega} |f(x)|^p \, dx \) for some \( C = C(p, n) > 0 \).
**Remark 3.2.** Notice that Duran et al. [9] also give an atomic decomposition via functional analysis; while the decomposition in Proposition 3.1 uses the geometric structure of $\Omega$, and does not depend on $p$.

**Proof of Theorem 3.1.** As discussed in Remark 3.1, we only need to consider the case $q \leq n$.

Suppose $f \in L^q_0(\Omega)$. We may choose a sequence $\{f_k\}_{k=1}^\infty \in L^q_0(\Omega) \cap L^\infty(\Omega)$ such that $f_k \to f$ in $L^q_0(\Omega)$. Let $\{Q_j\}_j$ be a Whitney covering of $\Omega$ as in Proposition 3.1. Applying Proposition 3.1 to each $f_k$, we see that $f_k = \sum_j T_j f_k$, where the decomposition holds in both $L^q_0(\Omega)\cap L^\infty(\Omega)$ and $L^q_0(\Omega)$. The same conclusion holds for $f = \sum_j T_j f$ in $L^q_0(\Omega)$.

By using [5, Theorem 2] (see also [8, Theorem 5.2]), on each cube $Q \subset \mathbb{R}^n$, there exists a linear operator $S$ that maps $L^q_0(Q)$ into $W^{1,p}_0(Q)$ for all $p \in (1, \infty)$, such that for each $g \in L^q_0(Q)$, $\text{div} (S g) = g$ and

$$\|D(S g)\|_{L^q(Q)} \leq C(n, q)\|g\|_{L^q(\partial Q)}.$$

Thus, by a translation and scaling argument, it follows that for each $j$, there exist a linear operator $S_j$ that maps $L^q_0(\sigma Q_j)$ into $W^{1,p}_0(\sigma Q_j)^n$ for each $p \in (1, \infty)$, and so that $\text{div} S_j T_j g = T_j g$ and

$$\|DS_j T_j g\|_{L^q(\sigma Q_j)} \leq C(n, p)\|T_j g\|_{L^q(\sigma Q_j)}$$

for every $g \in L^q_0(\Omega)$.

Write $u_j(x) := \sum_{j \in I} S_j T_j f(x)$ and $u_k(x) := \sum_{j \in I} S_j T_j f_k(x)$. As $\sum_{j \in I} \chi_{\sigma Q_j} \leq C \chi_{\Omega}$, we have $u, u_k \in W^{1,q}(\Omega)^n$, with

$$\int_{\Omega} |D(u_j(x))|^q \, dx \leq C \sum_{j \in I} \int_{\sigma Q_j} |D(S_j T_j f(x))|^q \, dx$$

$$\leq C \sum_{j \in I} \int_{\sigma Q_j} |T_j f(x)|^q \, dx \leq C \int_{\Omega} |f(x)|^q \, dx.$$

Moreover,

$$\int_{\Omega} \frac{|u(x)|}{\rho(x)^q} \, dx \leq C \sum_{j \in I} \ell(Q_j)^{-q} \int_{2 Q_j} |S_j T_j f(x)|^q \, dx$$

$$\leq C \sum_{j \in I} \int_{2 Q_j} |T_j f(x)|^q \, dx \leq C \int_{\Omega} |f(x)|^q \, dx.$$

The above two estimates prove (3.1).

It remains to show that $u(x) \in W^{1,q}_0(\Omega)^n$. Since $f_k \in L^\infty(\Omega)$, the Sobolev embedding theorem ensures that

$$\|S_j T_j f_k\|_{L^{\infty}(\sigma Q_j)} \leq C \ell(Q_j)^{1/2} \|T_j f_k\|_{L^{2n}(2 Q_j)} \leq C \ell(Q_j)^{1/2} \|f_k\|_{L^{2n}(\Omega)},$$

and hence, $|u_k(x)| \leq C \rho(x)^{1/2} \to 0$ as $x \to \partial \Omega$, which implies that $u_k \in W^{1,q}_0(\Omega)^n.$
4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We need the following Hardy inequality; see [2, 12, 16] for instance.

**Lemma 4.1** (Hardy inequality). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. If $p > n$, then there exists $C > 0$ such that the Hardy inequality $(H_p)$ holds for every $v \in W_0^{1,p}(\Omega)$.

$$(H_p) \quad \int_{\Omega} \frac{|v(x)|^p}{p(x)^p} \, dx \leq C \int_{\Omega} |\nabla v(x)|^p \, dx.$$ 

**Proof of Theorem 1.1.** Given a John domain $\Omega$, from [1] it follows that (ii) holds; from Theorem 3.1 it follows that (iii) holds and hence (iv) holds.

Conversely, suppose that separation property holds on $\Omega$. Let us first show that (ii) implies (i).

In the case $p \in (1, n)$, it follows from [1] that $\Omega$ is a John domain. Suppose now $p \in (n, \infty)$. Thus, for each $f \in L_0^p(\Omega)$, there is $u \in W_0^{1,p}(\Omega)$ satisfying $\text{div} u = f$ and

$$||Dv||_{L^p(\Omega)^{\infty}} \leq C ||f||_{L^p(\Omega)}.$$ 

Applying the Hardy inequality $(H_p)$ to $u \in W_0^{1,p}(\Omega)$, $p > n$, we see that

$$\int_{\Omega} \frac{|u(x)|^p}{p(x)^p} \, dx \leq C \int_{\Omega} |Du(x)|^p \, dx \leq C \int_{\Omega} |f(x)|^p \, dx.$$ 

Next, for each $u \in W^{1,p'}(\Omega)$ and each $f \in L^p(\Omega)$, where $1/p' + 1/p = 1$, it follows that

$$\left| \int_{\Omega} f(x)(u(x) - u_{\Omega}) \, dx \right| = \left| \int_{\Omega} f(x) (u(x) - f_{\Omega})(u(x) - u_{\Omega}) \, dx \right|$$

$$= \left| \int_{\Omega} u(x) \cdot \nabla (u(x) - u_{\Omega}) \, dx \right|$$

$$\leq \left( \int_{\Omega} \frac{|u(x)|^p}{p(x)^p} \, dx \right)^{1/p} \left( \int_{\Omega} |\nabla u(x)|^{p'} \rho(x)^{p'} \, dx \right)^{1/p'}$$

$$\leq C \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p} \left( \int_{\Omega} |\nabla u(x)|^{p'} \rho(x)^{p'} \, dx \right)^{1/p'}.$$
Taking the supremum over the set \( \{ f \in L^p(\Omega) : \|f\|_{L^p(\Omega)} \leq 1 \} \), we see that
\[
\int_\Omega |u(x) - u_\Omega|^p \, dx \leq C \int_\Omega |\nabla u(x)|' \rho(x)' \, dx,
\]
i.e., the \((P_{p'})\)-Poincaré inequality holds on \(\Omega\). By using Theorem 2.1 we see that \(\Omega\) is a John domain.

Let us show that (iv) implies (i), which further implies that (iii) implies (i). (iv) implies that for each \( f \in L^p_0(\Omega) \), there is \( u \in L^p(\Omega)^n \) satisfying \( \text{div} \, u = f \) and
\[
\|u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}.
\]

Then, using a duality argument as in (4.1), it follows the \((P_{p'})\)-Poincaré inequality holds on \(\Omega\). Using Theorem 2.1 again, we see that \(\Omega\) is a John domain, which completes the proof. \(\square\)

Notice Theorem 1.1 (ii) does not cover the borderline case \( p = n \). For \( 1 < p < n \), a calculation similar to the one in the proof of Theorem 1.1 was done in [1] relying on a Sobolev-Poincaré inequality. This does not work for \( p \geq n \) and we use Hardy inequality to bypass the problem in the case \( p > n \). In the case \( p = n \) we cannot rely on such an inequality without additional assumptions.

We can include the case \( p = n \) in Theorem 1.1 (ii) provided the complement of \(\Omega\) is sufficiently thick on \(\partial\Omega\). Precisely, it suffices to assume there exists \( \lambda > 0 \) such that \(\mathcal{H}_d^\lambda(\Omega^c \cap B(w, r)) \geq Cr^d\) for all \( w \in \partial\Omega \) and \( r > 0 \). Here \(\mathcal{H}_d^\lambda\) denotes \(\lambda\)-dimensional Hausdorff content; see [16]. For example, each simply connected plane domain satisfies this condition.

**Theorem 4.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain satisfying the separation property, \( n \geq 2 \). Suppose that there exists \( \lambda > 0 \) such that \(\mathcal{H}_d^\lambda(\Omega^c \cap B(w, r)) \geq Cr^d\) for all \( w \in \partial\Omega \) and \( r > 0 \).

Then \(\Omega\) is a John domain if and only if for some (all) \( p \in (1, \infty) \) and each \( f \in L^p_0(\Omega) \), there exists a solution \( v \in W^{1,p}_0(\Omega)^n \) to the equation \( \text{div} \, v = f \) with
\[
\|v\|_{W^{1,p}(\Omega)^n} \leq C\|f\|_{L^p(\Omega)^n}.
\]

**Proof.** By Theorem 1.1, we only need to show that if the divergence equation \( \text{div}_{n,0} \) is solvable, then \(\Omega\) is a John domain. In this case, from [17, 16], it follows that every \( v \in W^{1,n}_0(\Omega) \), there is a constant \( C \) such that
\[
(H_n) \quad \int_\Omega \frac{|v(x)|^n}{\rho(x)^n} \, dx \leq C \int_\Omega |\nabla v(x)|^n \, dx.
\]

Thus, for each \( f \in L^p_0(\Omega) \), there exists a solution \( v \in W^{1,n}_0(\Omega)^n \) to the equation \( \text{div} \, v = f \) such that
\[
\left( \int_\Omega \frac{|v(x)|^n}{\rho(x)^n} \, dx \right)^{1/n} \leq c\|v\|_{W^{1,n}(\Omega)^n} \leq C\|f\|_{L^p(\Omega)^n}.
\]
Arguing as in (4.1), we see that \((P_n)\)-Poincaré inequality holds on \(\Omega\), which implies \(\Omega\) is a John domain by Theorem 2.1.

On the other hand, we have the following example.

**Example 4.1.** For each \(1 < p \leq n\), there is John domain \(\Omega\) that satisfies the separation property, \(f \in L^p_0(\Omega)\) and \(v \in W^{1,p}_0(\Omega)\) so that \(\text{div } v = f\), and \(\|v\|_{L^p(\Omega)} = \infty\).

For simplicity, we only consider the case \(n = 2\); our reasoning easily extends to cover the higher dimensional case. Let \(p \in (1, 2]\) and set \(\Omega := B^2(0, 2) \setminus E\), where \(E \subset [0, 1]\) is a compact set so that \(\mathcal{H}^{2-p}(E) < \infty\), but \(\int_{B^2(0, 1)} d(x, E)^{-p} dx = \infty\). Fix \(\varphi \in C^\infty_0(B(0, 2))\) with \(\varphi(x) = x_1\) on \(B^2(0, 1)\). Then \(v = \nabla \varphi \in W^{1,p}_0(B^2(0, 2))\) is a solution to \(\text{div } v = \Delta \varphi\) on \(B^2(0, 2)\), and in particular, on \(\Omega = B^2(0, 2) \setminus E\). Moreover, \(\int_\Omega |v|^p dx = \infty\) and it is easy to check that \(v \in W^{1,p}_0(\Omega)\) via \(\mathcal{H}^{2-p}(E) < \infty\), and that \(\Omega\) satisfies the separation property.

**Proof of Corollary 1.1.** It was proved in [6] that each simply connected plane domain satisfies the separation property. Moreover, it is trivial that for each simply connected plane domain \(\Omega\), \(\mathcal{H}^\infty_0(\Omega \cap B(w, r)) \geq Cr\) for all \(w \in \partial \Omega\) and \(r > 0\). Hence, \(\Omega\) satisfies the requirements for Theorem 4.1, and Corollary 1.1 follows.

**Acknowledgment**

Jiang and Koskela were supported by the Academy of Finland Grants 131477 and 263850 and Kauranen was supported by The Finnish National Graduate School in Mathematics and its Applications.

**References**

[1] Acosta, G., Durán, R.G., Muschietti, M.A., Solutions of the divergence operator on John domains, Adv. Math. 206 (2006), 373-401.

[2] Ancona, A., On strong barriers and inequality of Hardy for domains in \(\mathbb{R}^n\), J. London Math. Soc. 34 (1986), 274C290.

[3] Arnold, D.N., Scott, L. R., Vogelius, M., Regular inversion of the divergence operator with Dirichlet boundary conditions on a polygon, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 15 (1988), 169-192.

[4] Auscher, P., Russ, E., Tchamitchian, P., Hardy Sobolev spaces on strongly Lipschitz domains of \(\mathbb{R}^n\), J. Funct. Anal. 218 (2005), 54-109.

[5] Bourgain, J., Brezis, H., On the equation \(\text{div} Y = f\) and application to control of phases, J. Amer. Math. Soc. 16 (2003), 393-426.

[6] Buckley, S., Koskela, P., Sobolev-Poincaré implies John, Math. Res. Lett. 2 (1995), 577-593.

[7] Buckley, S., Koskela, P., Criteria for imbeddings of Sobolev-Poincaré type, Internat. Math. Res. Notices (1996), 881-901.
Divergence equation and Poincaré inequality

[8] Diening L., Ružička M., Schumacher, K., A decomposition technique for John domains, Ann. Acad. Sci. Fenn. Math., 35 (2010), 87-114.

[9] Durán, R.G., Muschietti, M.A., Russ, E., Tchamitchian, P., Divergence operator and Poincaré inequalities on arbitrary bounded domains, Complex Var. Elliptic Equ. 55 (2010), 795-816.

[10] Duvaut G., Lions J.-L., Inequalities in Mechanics and Physics, Springer, 1976.

[11] Friedrichs K.O., On the boundary-value problems of the theory of elasticity and Korn’s inequality, Ann. of Math. 48 (2) (1947) 441-471.

[12] Hajłasz P., Pointwise Hardy inequalities, Proc. Amer. Math. Soc. 127 (1999), 417-423.

[13] Hajłasz P., Koskela P., Isoperimetric inequalities and imbedding theorems in irregular domains, J. London Math. Soc. (2) 58 (1998), 425-450.

[14] John F., Rotation and strain, Comm. Pure Appl. Math. 4 (1961) 391C414.

[15] Kilpeläinen T., Malý J., Sobolev inequalities on sets with irregular boundaries, Z. Anal. Anwendungen 19 (2000), 369-380.

[16] Koskela, P., Lehrbäck, J., Weighted pointwise Hardy inequalities, J. Lond. Math. Soc. (2) 79 (2009), 757-779.

[17] Lewis, J.L., Uniformly fat sets, Trans. Amer. Math. Soc. 308 (1988), 177-196.

[18] Martio, O., Sarvas, J., Injectivity theorems in plane and space, Ann. Acad. Sci. Fenn. Ser. A I Math. 4 (1979), 383-401.

[19] J. Nečas, Les méthodes directes en théorie des équations elliptiques. (French) Masson et Cie (Eds.), Paris; Academia, Editeurs, Prague, 1967.

[20] Näkki, R., Väisälä, J., John disks, Exposition. Math. 9 (1991), 3-43.

[21] Temam, R., Navier-Stokes equations. Theory and numerical analysis. Third edition. North-Holland Publishing Co., Amsterdam, 1984.

[22] Maz’ya V.G., Sobolev spaces (Springer, 1985).

Renjin Jiang\textsuperscript{1}, Aapo Kauranen\textsuperscript{2} & Pekka Koskela\textsuperscript{2}

1. School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Beijing 100875, People’s Republic of China

2. Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014 Finland

E-mail addresses: rejiang@bnu.edu.cn
aapo.p.kauranen@jyu.fi
pkoskela@maths.jyu.fi