Equivalence of Weighted Anchored and ANOVA Spaces of Functions with Mixed Smoothness of Order one in $L_p$

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Abstract

We consider $\gamma$-weighted anchored and ANOVA spaces of functions with mixed first order partial derivatives bounded in a weighted $L_p$ norm with $1 \leq p \leq \infty$. The domain of the functions is $D^d$, where $D \subseteq \mathbb{R}$ is a bounded or unbounded interval. We provide conditions on the weights $\gamma$ that guarantee that anchored and ANOVA spaces are equal (as sets of functions) and have equivalent norms with equivalence constants uniformly or polynomially bounded in $d$. Moreover, we discuss applications of these results to integration and approximation of functions on $D^d$.

1 Introduction

This paper studies the equivalence of anchored and ANOVA spaces, a research initiated in [5] in an abstract setting for reproducing kernel Hilbert spaces. It provides extensions of the results obtained in [6, 8]. The two major differences between those papers and the current one are the following. First of all, the domain of functions in the former two papers is $[0,1]^d$, whereas we allow now for $D^d$ with any (bounded or unbounded) interval $D \subseteq \mathbb{R}$. To simplify the presentation we assume that $0 = \min D$ throughout this paper. Secondly, the standard $L_p$ norms were used in the former papers whereas we consider now mixed $L_p$-$\ell_q$ norms with $1 \leq p, q \leq \infty$ that are based on a probability density function $\psi$ that is positive a.e. on $D$.

We now describe briefly the $\gamma$-weighted anchored and ANOVA norms and spaces, which are studied in detail in Sections 2 and 3. Consider first $d = 2$ and $p = q = 1$, and, in order to simplify the presentation in the introduction, a function $f : D^2 \to \mathbb{R}$ with continuous mixed partial derivatives of order one. For a family $\gamma = (\gamma_u)_{u \subseteq \{1,2\}}$ of positive
reals the $\gamma$-weighted anchored norm of $f$ is given by

$$
\|f\|_{W_{\gamma,1,1,\psi,\gamma}} = \gamma_0^{-1} |f(0,0)| + \gamma_1^{-1} \int_D \left| \frac{\partial}{\partial x_1} f(x_1,0) \right| \psi(x_1) \, dx_1
$$

$$
+ \gamma_2^{-1} \int_D \left| \frac{\partial}{\partial x_2} f(0,x_2) \right| \psi(x_2) \, dx_2
$$

$$
+ \gamma_3^{-1} \int_D \int_D \left| \frac{\partial^2}{\partial x_1 \partial x_2} f(x_1,x_2) \right| \psi(x_1) \psi(x_2) \, dx_1 \, dx_2,
$$

and the $\gamma$-weighted ANOVA norm is given by

$$
\|f\|_{W_{A,1,1,\psi,\gamma}} = \gamma_0^{-1} \left| \int_D \int_D f(x_1,x_2) \psi(x_2) \psi(x_1) \, dx_2 \, dx_1 \right|
$$

$$
+ \gamma_1^{-1} \int_D \left| \int_D \frac{\partial}{\partial x_1} f(x_1,x_2) \psi(x_2) \, dx_2 \right| \psi(x_1) \, dx_1
$$

$$
+ \gamma_2^{-1} \int_D \left| \int_D \frac{\partial}{\partial x_2} f(x_1,x_2) \psi(x_1) \, dx_1 \right| \psi(x_2) \, dx_2
$$

$$
+ \gamma_3^{-1} \int_D \int_D \left| \frac{\partial^2}{\partial x_1 \partial x_2} f(x_1,x_2) \right| \psi(x_1) \psi(x_2) \, dx_1 \, dx_2,
$$

provided that the respective integrals are finite. For $1 < p \leq \infty$ or $1 < q \leq \infty$, the definition of the $\gamma$-weighted anchored and ANOVA norms are appropriately changed.

For arbitrary $d \in \mathbb{N}$ and a family $\gamma = (\gamma_u)_{u \subseteq \{1, \ldots, d\}}$ of non-negative weights the norms of functions of $d$ variables have $2^d$ terms involving various mixed partial derivatives $\partial^{[u]} / \prod_{j \in u} \partial x_j$, each weighted by $\gamma_u^{-1}$ for $u \subseteq \{1, \ldots, d\}$. In particular, if $\gamma_u = 0$ then the corresponding term involving $\partial^{[u]} / \prod_{j \in u} \partial x_j$ is assumed to be zero.

Roughly speaking, the $\gamma$-weighted anchored space $W_{\gamma,1,1,\psi,\gamma}$ is the Banach space of functions with finite anchored norm, and the ANOVA space $W_{A,1,1,\psi,\gamma}$ is the Banach space of functions with finite ANOVA norm. Actually there a two different ways how to rigorously define these spaces, even for $D = [0,1]$ and $\psi = 1$, which is most frequently studied in the literature. The first approach is to consider weak derivatives, but then the problem arises how to define the section (for the anchored norm) or the integral (for the ANOVA norm) of a weak derivative on a set of measure zero; we refer to [6] for a solution in the case $D = [0,1]$, $p = q$, and $\psi = 1$. The second approach, which is presented here and which is well suited for the application of interpolation theory, is based on smoothing and superposition of functions with suitable integrability properties. This approach, however, does not immediately yield an intrinsic characterization of the whole space via differentiability properties.

We only state here that $p$ and $\psi$ have to satisfy certain integrability conditions, see (1) and (2), for the spaces to be well defined, and that the spaces are continuously embedded into the space of continuous functions on $D^d$. The integrability conditions reveal, in particular, that $m_\psi < \infty$ and $\kappa_\psi < \infty$ for

$$
m_\psi = \int_D y \psi(y) \, dy
$$
and
\[ \kappa_\psi = \operatorname{ess \sup}_{t \in D} \int_{D \cap (t, \infty)} \frac{\psi(y) \, dy}{\psi(t)} \]
are necessary for the anchored and the ANOVA space to be well defined for any \( p \).

The main aim of this paper is to compare the spaces \( W_{\eta,p,q,\psi,\gamma} \) and \( W_{A,p,q,\psi,\gamma} \) and their norms, see Section 4. It turns out that \( W_{\eta,p,q,\psi,\gamma} = W_{A,p,q,\psi,\gamma} \) if and only if \( \gamma \) satisfies the condition
\[ \gamma_w > 0 \quad \text{implies that} \quad \gamma_u > 0 \quad \text{for all} \quad u \subseteq w, \]
which is assumed to hold for the rest of the introduction. Let
\[ \iota_{p,q,\psi,\gamma} : W_{A,p,q,\psi,\gamma} \hookrightarrow W_{\eta,p,q,\psi,\gamma} \]
denote the embedding operator from the ANOVA space into the anchored space, which, together with its inverse, is continuous due to the closed graph theorem. We show that both, \( W_{\eta,p,q,\psi,\gamma} \) and \( W_{A,p,q,\psi,\gamma} \) are isometrically isomorphic to a \( \gamma \)-weighted \( \ell_q \) sum of spaces \( L_{p,\psi}(D^u) \). The latter consist of equivalence classes of real-valued functions on \( D^u \), with norms given by
\[ \| g \|_{L_{p,q}(D^u)} = \left( \int_{D^u} |g(x)|^p \prod_{j=1}^d \psi(x_j) \, dx \right)^{1/p} \]
for \( 1 \leq p < \infty \), with the usual modification for \( p = \infty \). We employ these isometric isomorphisms to show that
\[ \| \iota_{p,q,\psi,\gamma} \| = \| \iota_{p,q,\psi,\gamma}^{-1} \|. \]
Moreover, we provide explicit expressions for the norm \( \| \iota_{p,q,\psi,\gamma} \| \) of the embedding in terms of \( \gamma \), \( m_\psi \), and \( \kappa_\psi \) in the four extremal cases corresponding to \( p,q \in \{1, \infty\} \). In all other cases we use complex interpolation theory to get upper bounds for \( \| \iota_{p,q,\psi,\gamma} \| \).

Observe that \( \| \iota_{p,q,\psi,\gamma} \| \) depends on the number \( d \) of variables only via the family \( \gamma \) of weights. In Section 5 we study, for a number of different classes of weights, when the norms \( \| \iota_{p,q,\psi,\gamma} \| \) of the embeddings are uniformly bounded in \( d \). If this holds, then we say that the \( \gamma \)-weighted anchored and ANOVA spaces are equivalent independently of the dimension.

The paper is organized as follows. Section 2 provides basic facts used in the paper. The \( \gamma \)-weighted anchored and ANOVA spaces and norms are discussed in Section 3. The main results are stated in Section 4. Applications of the results to special classes of weights can be found in Section 5. Applications of the results from Sections 4 and 5 to integration and approximation of functions are discussed in Section 6.

### 2 Notation and Basic Facts

Let \( D \subseteq [0, \infty) \) be an interval with \( 0 \in D \), i.e.,
\[ D = [0, T) \text{ for some } T \in (0, \infty] \quad \text{or} \quad D = [0, T] \text{ for some } T \in (0, \infty), \]
and let 
\[ \psi : D \to [0, \infty) \]
be a probability density function that is positive almost everywhere.

Let \( d \) be a positive integer. In what follows we will use \( u, v \), and \( w \) to denote subsets of \([1 : d]\), where
\[ [1 : d] = \{1, \ldots, d\}, \]
and we will denote the complement of \( u \) in \([1 : d]\) by \( u^c \). We will also use \( t, x, y, \) and \( z \) to denote points from \( D^d \), and we will often use the following notation
\[ [x_u; t_{uc}] = (y_1, \ldots, y_d) \quad \text{with} \quad y_j = \begin{cases} x_j & \text{if } j \in u, \\ t_j & \text{if } j \in u^c. \end{cases} \]

For \( u \neq \emptyset \) we will write \( D^u \) to denote the set of points \( x_u = (x_j)_{j \in u} \) with \( x_j \in D \). To simplify the notation we will often write \( x_u; t_{uc} \) instead of \([x_u; t_{uc}]\).

In the sequel, let \( u \neq \emptyset \). We will often consider real-valued functions on \( D^d \) that only depend on the variables with indices from \( u \). To simplify the notation we will identify any such function with a function on \( D^u \) in the canonical way; vice versa, functions of the latter kind are identified with functions on \( D^d \).

In the sequel, the integrability index \( p \) satisfies \( 1 \leq p \leq \infty \), and \( p' \) denotes its conjugate index given by \( \frac{1}{p} + \frac{1}{p'} = 1 \). Furthermore, let
\[ \psi_u(t_u) = \prod_{j \in u} \psi(t_j). \]

Let \( L_{p, \psi}(D^u) \) be the Banach space of functions on \( D^u \) with the corresponding weighted \( L_p \) norm. For \( 1 \leq p < \infty \) this norm is given by
\[ \|g\|_{L_{p, \psi}(D^u)} = \left( \int_{D^u} |g(t_u)|^p \psi_u(t_u) \, dt_u \right)^{1/p}. \]

For \( p = \infty \) we have the usual modification
\[ \|g\|_{L_{\infty, \psi}(D^u)} = \|g\|_{L_{\infty}(D^u)} = \sup_{t_u \in D^u} |g(t_u)|, \]
which does not depend on \( \psi \). The space of locally \( p \)-integrable functions on \( D^u \) is given by
\[ L_{p, \text{loc}}(D^u) = \{ f : D^u \to \mathbb{R} : f|_{[0,\infty)} \in L_p([0,\infty)) \text{ for all } x \in D^u \}. \]

Here \([0, \infty)\) denotes the half-open interval in \( D^u \) with lower left corner \( 0 \) and upper right corner \( \infty \).

By \( 1_{[0,\infty)} \) we denote the indicator function of the interval \([0, \infty) \subseteq \mathbb{R} \). Consider
\[ K_0(x, t) = 1_{[0,\infty)}(t) \quad \text{and} \quad K_A(x, t) = 1_{[0,\infty)}(t) - \psi(t), \]
where \( x, t \in D \) and
\[ \psi(t) = \int_t^\infty \psi(y) \, dy = \int_D K_0(y, t) \psi(y) \, dy. \]
Define
\[ K_{\hat{\mathfrak{n}},u}(x_u, t_u) = \prod_{j \in u} K_{\hat{\mathfrak{n}}}(x_j, t_j) \quad \text{and} \quad K_{A,u}(x_u, t_u) = \prod_{j \in u} K_A(x_j, t_j). \]

Both, \( K_{\hat{\mathfrak{n}},u} \) and \( K_{A,u} \), will be used as integral kernels in the construction of the anchored and the ANOVA spaces. At first we study the basic integrability properties.

**Lemma 1**

(i) We have
\[ \int_{D^u} |g_u(t_u)| K_{\hat{\mathfrak{n}},u}(x_u, t_u) \, dt_u < \infty \]
for every \( g_u \in L_{p,\psi}(D^u) \) and every \( x_u \in D^u \) if and only if
\[ \begin{aligned} p &= \infty \quad \text{or} \\ p < \infty \quad \text{and} \quad \psi^{-1/p} \in L_{p'}(D). \end{aligned} \]

(ii) We have
\[ \int_{D^u} |g_u(t_u)||K_{A,u}(x_u, t_u)| \, dt_u < \infty \]
for every \( g_u \in L_{p,\psi}(D^u) \) and every \( x_u \in D^u \) if and only if
\[ \begin{aligned} p &= \infty \quad \text{and} \quad \bar{\psi} \in L_1(D) \quad \text{or} \\ p < \infty \quad \text{and} \quad \psi^{-1/p} \in L_{p'}(D) \quad \text{and} \quad \bar{\psi} \cdot \psi^{-1/p} \in L_{p'}(D). \end{aligned} \]

**Proof.** For \( \star \in \{ \hat{\mathfrak{n}}, A \} \), \( g_u \in L_{p,\psi}(D^u) \), and \( x_u \in D^u \) we have
\[ \int_{D^u} |g_u(t_u)||K_{\star,u}(x_u, t_u)| \, dt_u = \int_{D^u} |g_u(t_u)| \frac{|K_{\star,u}(x_u, t_u)|}{\psi_u(t_u)} \psi_u(t_u) \, dt_u. \]

Lemma 21 from the Appendix shows that
\[ \int_{D^u} |g_u(t_u)||K_{\star,u}(x_u, t_u)| \, dt_u < \infty \quad \text{for all} \quad g_u \in L_{p,\psi}(D^u) \quad \text{and} \quad x_u \in D^u \]
if and only if
\[ \frac{K_{\star,u}(x_u, \cdot)}{\psi_u} \in L_{p',\psi}(D^u) \quad \text{for all} \quad x_u \in D^u, \]
which is equivalent to
\[ \frac{K_{\star}(x, \cdot)}{\psi} \in L_{p',\psi}(D) \quad \text{for all} \quad x \in D. \]  \( \star \)

Furthermore, we use \( K_A(0, \cdot) = -\bar{\psi} \) to conclude that (3) with \( \star = A \) is equivalent to (3) with \( \star = \hat{\mathfrak{n}} \) and
\[ \bar{\psi}/\psi \in L_{p',\psi}(D). \]
Consider $\star = \phi$. Then we have (3) if and only if

$$\psi^{-1} \in L_{p'}^{\text{loc}}(D),$$

which yields the claim in (i).

Consider $\star = A$. To establish the claim in (ii) one easily verifies that (4) is equivalent to $\overline{\psi} \in L_1(D)$ if $p = \infty$ and to $\overline{\psi} \cdot \psi^{-1/p} \in L_{p'}(D)$ if $p < \infty$. \hfill $\square$

We comment on the conditions (1) and (2).

**Remark 2** Obviously, (2) implies (1), and Lemma 1 yields the following monotonicity property. If one of these conditions is satisfied for $p = p_1$ and $\psi$, then it also holds for the same density $\psi$ and every $p > p_1$. Given (2), we obtain $\overline{\psi} \in L_1(D)$, and therefore

$$m_\psi = \int_D \overline{\psi}(t) \, dt = \int_D y \overline{\psi}(y) \, dy \in (0, \infty). \quad (5)$$

**Remark 3** Consider the assumption (2) in the case $p < \infty$. If $D$ is compact, then $L_{p'}^{\text{loc}}(D) = L_{p'}(D)$ and $\psi^{-1/p} \in L_{p'}(D)$ implies $\overline{\psi} \cdot \psi^{-1/p} \in L_{p'}(D)$. Therefore we have equivalence of (1) and (2) for compact sets $D$. If $D$ is not compact, then $\overline{\psi} \cdot \psi^{-1/p} \in L_{p'}(D)$ implies $\psi^{-1/p} \in L_{p'}^{\text{loc}}(D)$.

**Example 4** Consider the case of a bounded interval $D$ with $T = 1$ for simplicity. Let

$$\psi(t) = (\alpha + 1) \cdot (1 - t)^\alpha$$

for $\alpha > -1$, so that

$$\overline{\psi}(t) = (1 - t)^{\alpha + 1}.$$  

The following facts are easily verified with the help of Remark 3. If $D = [0, 1)$, then (2) holds true for every $p$. If $D = [0, 1]$, then (2) holds true if and only if $p > \alpha + 1$ or $p = 1$ and $\alpha = 0$.

**Example 5** Consider the unbounded interval $D = [0, \infty)$. Let

$$\psi(t) = (\alpha - 1) \cdot (1 + t)^{-\alpha}$$

for $\alpha > 1$, so that

$$\overline{\psi}(t) = (1 + t)^{1-\alpha}.$$  

Clearly, we have (1) for every $p$. With the help of Remark 3 we easily verify that (2) holds true if and only if $\alpha > 2$ and $p > 1 + 1/(\alpha - 2)$.

**Example 6** Consider again $D = [0, \infty)$. Let

$$\psi(t) = c \cdot \exp(-b \cdot t^a),$$

where $a, b > 0$ and $c = 1/ \int_0^\infty \exp(-b \cdot t^a) \, dt$. Clearly, we have (1) for every $p$. We claim that (2) holds true if and only if $a \geq 1$ or $p > 1$. Note that

$$\lim_{t \to \infty} \frac{\overline{\psi}(t)}{t^{1-a} \exp(-b \cdot t^a)} = c/(ab) \in (0, \infty),$$

which follows from L'Hôpital's rule. Once more, it remains to apply Remark 3.
For the rest of this section let $\star \in \{\emptyset, A\}$ and assume that (1) is satisfied if $\star = \emptyset$ and that (2) is satisfied if $\star = A$. For $g_u \in L_{p,\psi}(D^u)$ we put
\[
T_{\star,u}(g_u) = \int_{D^u} g_u(t_u) K_{\star,u}(\cdot, t_u) \, dt_u,
\]
which is well defined due to Lemma 1.

**Remark 7** By $f^{(u)}$ we mean $f^{(u)} = \prod_{j \in u} \frac{\partial}{\partial x_j} f$, where $\frac{\partial}{\partial x_j} f$ denotes the distributional derivative of $f$ with respect to $x_j$. Lemma 1 and Remark 2 imply that $L_{p,\psi}(D^u) \subseteq L^1_{\text{loc}}(D^u)$. It follows that $f = T_{\star,u}(g_u)$ with any $g_u \in L_{p,\psi}(D^u)$ has a weak derivative $f^{(u)}$ and $f^{(u)} = g_u$.

On the space $C(D^d)$ of continuous real-valued functions on $D^d$ we consider the topology of uniform convergence on compact subsets.

**Lemma 8** We have $T_{\star,u}(g_u) \in C(D^d)$ for every $g_u \in L_{p,\psi}(D^u)$, and the mapping
\[
T_{\star,u} : L_{p,\psi}(D^u) \to C(D^d)
\]
is linear, continuous, and one-to-one.

**Proof.** We obtain $T_{\star,u}(g_u) \in C(D^d)$ from
\[
| (T_{\star,u}(g_u))(x_u) - (T_{\star,u}(g_u))(y_u) | \leq \int_{D^u} |g_u(t_u)| |K_{\emptyset,u}(x_u, t_u) - K_{\emptyset,u}(y_u, t_u)| \, dt_u
\]
and $g_u \in L^1_{\text{loc}}(D^u)$, see Remark 7. Linearity of $T_{\star,u}$ obviously holds. To prove continuity of $T_{\star,u}$ it suffices to show that
\[
\sup_{x \in [0,y]} \left\| K_{\star}(x, \cdot) \right\|_{L_{p,\psi}(D)} < \infty
\]
for every $y \in D$, cf. the proof of Lemma 1. The latter property holds true due to (1) if $\star = \emptyset$ or (2) if $\star = A$. It remains to show that $T_{\star,u}$ is one-to-one. For this purpose consider $g_u$ such that $T_{\star,u}(g_u) = 0$. Remark 7 yields $g_u = 0$, which completes the proof. $\square$

### 3 Anchored and ANOVA Spaces

As previously, let $\star \in \{\emptyset, A\}$, and assume that (1) is satisfied if $\star = \emptyset$ and that (2) is satisfied if $\star = A$.

For $u \neq \emptyset$ we define the spaces
\[
F_{\star,p,\psi,u} = T_{\star,u}(L_{p,\psi}(D^u)).
\]
For $u = \emptyset$ and $c \in \mathbb{R}$ we put $L_p(D^u) = L_{p,\psi}(D^u) = \mathbb{R}$ with $\|c\|_{L_{p,\psi}(D^u)} = |c|$, and $T_{\star,u}(c)$ denotes the constant function with value $c$. Hence $F_{\star,p,\psi,\emptyset} = T_{\star,\emptyset}(L_{p,\psi}(D^\emptyset))$ is the space of constant functions on $D^d$. 

7
Let $u \subseteq [1 : d]$ and $f \in F_{*,p,\psi,u}$. Note that $f(x)$ depends on $x$ only through $x_u$. Furthermore,

$$f(x) = 0 \text{ if } x_j = 0 \text{ for any } j \in u,$$

if $\star = \emptyset$, and

$$\int_D f(x) \psi(x_j) \, dx_j = 0 \text{ if } j \in u,$$

if $\star = \emptyset$, for any $f \in F_{*,p,\psi,u}$. This leads directly to the following lemma; see [13, Thm. 2.1] for a general result on decomposition of functions.

**Lemma 9** Let $f_{*,u}, \tilde{f}_{*,u} \in F_{*,p,\psi,u}$ for $u \subseteq [1 : d]$. Then we have

$$\sum_{u \subseteq [1 : d]} f_{*,u} = \sum_{u \subseteq [1 : d]} \tilde{f}_{*,u} \quad \text{if and only if} \quad f_{*,u} = \tilde{f}_{*,u} \quad \text{for all } u \subseteq [1 : d].$$

We identify the elements $(f_{*,u})_{u \subseteq [1 : d]}$ of the direct sum $\bigoplus_{u \subseteq [1 : d]} F_{*,p,\psi,u}$ with the continuous functions $\sum_{u \subseteq [1 : d]} f_{*,u}$ on $D^d$, which is possible due to Lemma 9. The representation

$$f = \sum_{u \subseteq [1 : d]} f_{*,u} = \sum_{u \subseteq [1 : d]} T_{*,u}(g_{*,u})$$

with $g_{*,u} \in L_{p,\psi}(D^u)$ and $f_{*,u} = T_{*,u}(g_{*,u})$ is called the **anchored decomposition** of $f$ in the case $\star = \emptyset$ and the **ANOVA decomposition** of $f$ in the case $\star = \emptyset$. We provide an explicit relation between the corresponding functions $g_{*,u}$ in these two decompositions. Put

$$\psi_{\text{mp}}(t_w) = \prod_{j \in w} \overline{\psi}(t_j).$$

For convenience of notation we set $\int_{D^w} f(t_w) \, dt_w = f$ for $w = \emptyset$.

**Lemma 10** Assume that (2) is satisfied. If $w \subseteq u^c$ and $g_{*,u^w,w} \in L_{p,\psi}(D^{u^w,w})$, then

$$g_{*,u^w,w}(x_u; t_w) \cdot \overline{\psi}_{\text{mp}}(t_w) \in L_1(D^w) \text{ for a.e. } x_u \text{ if } u \neq \emptyset \quad \text{and} \quad \int_{D^w} g_{*,u^w,w}(x_u; t_w) \cdot \overline{\psi}_{\text{mp}}(t_w) \, dt_w \in L_{p,\psi}(D^u).$$

Moreover, let $g_{\emptyset,u}, g_{A,u} \in L_{p,\psi}(D^u)$ for every $u$. Then

$$\sum_{u \subseteq [1 : d]} T_{\emptyset,u}(g_{\emptyset,u}) = \sum_{u \subseteq [1 : d]} T_{A,u}(g_{A,u})$$

if and only if

$$g_{\emptyset,u} = \sum_{w \subseteq u^c} (-1)^{|w|} \int_{D^w} g_{A,u^w,w}(x_u; t_w) \overline{\psi}_{\text{mp}}(t_w) \, dt_w$$

and

$$g_{A,u} = \sum_{w \subseteq u^c} \int_{D^w} g_{\emptyset,u^w,w}(x_u; t_w) \overline{\psi}_{\text{mp}}(t_w) \, dt_w.$$
Proof. In the proof of (7) we consider the non-trivial case \( w \neq \emptyset \). Let \( h \in L_{p', \psi}(D^u) \), and put
\[
\tilde{h}(x_u; t_w) = g_{*, u \cup w}(x_u; t_w) h(x_u).
\]
It follows that \( \tilde{h}(\cdot; t_w) \psi_u(\cdot) \in L_1(D^u) \) for a.e. \( t_w \) and
\[
\int_{D^u} |\tilde{h}(x_u; t_w)| \psi_u(x_u) \, dx_u \in L_{p, \psi}(D^w).
\]
Consider the case \( p < \infty \), and put \( c = \| \psi \cdot \psi^{-1/p} \|_{L_{p'}(D)} < \infty \), see (2). We obtain
\[
\int_{D^w} \int_{D^u} |g_{*, u \cup w}(x_u; t_w)| \overline{\psi}_w(t_w) \, dt_w \cdot |h(x_u)| \psi_u(x_u) \, dx_u
\]
\[
= \int_{D^w} \int_{D^u} |\tilde{h}(x_u; t_w)| \psi_u(x_u) \, dx_u \cdot \psi_{1/p}(t_w) / \psi_{1/p}(t_w) \, dt_w
\]
\[
\leq \left\| \int_{D^u} |\tilde{h}(x_u; \cdot)| \psi_u(x_u) \, dx_u \cdot \psi_{1/p}(\cdot) \right\|_{L_p(D^u)} \cdot c^{\|w\|}
\]
\[
< \infty.
\]
Lemma 21 from the Appendix shows that
\[
\int_{D^w} |g_{*, u \cup w}(x_u; t_w)| \overline{\psi}_w(t_w) \, dt_w \in L_{p, \psi}(D^u),
\]
and hereby we get (7). For \( p = \infty \) the proof of (7) is straightforward and thus omitted.

Now we prove the equivalence of (8) and (9). Take any \( f_{A,u} = T_{A,u}(g_{A,u}) \in F_{A,p,\psi,u} \). Then
\[
f_{A,u}(x) = \int_{D^u} g_{A,u}(t_u) K_{A,u}(x_u, t_u) \, dt_u
\]
\[
= \int_{D^u} g_{A,u}(t_u) \sum_{v \subseteq u} K_{\cap, v}(x_v, t_v) (-1)^{|u| - |v|} \overline{\psi}_u(t_u \setminus v) \, dt_u
\]
\[
= \sum_{v \subseteq u} (-1)^{|u| - |v|} \int_{D^u} K_{\cap, v}(x_v, t_v) \left( \int_{D^u \setminus v} g_{A,u}(t_v; t_u \setminus v) \overline{\psi}_u(t_u \setminus v) \, dt_u \setminus v \right) \, dt_v
\]
\[
= \sum_{v \subseteq u} T_{\cap, v}(h_{u,v}),
\]
where
\[
h_{u,v}(t_v) = (-1)^{|u| - |v|} \int_{D^u \setminus v} g_{A,u}(t_v; t_u \setminus v) \overline{\psi}_u(t_u \setminus v) \, dt_u \setminus v.
\]
Note that the appearing integrals are well defined by (7). Furthermore, we get
\[
\sum_{u \subseteq [1:d]} T_{A,u}(g_{A,u}) = \sum_{u \subseteq [1:d]} \sum_{v \subseteq u} T_{\cap, v}(h_{u,v}) = \sum_{v \subseteq [1:d]} \sum_{m \subseteq v^c} T_{\cap, v}(h_{v \cup m, v})
\]
\[
= \sum_{v \subseteq [1:d]} T_{\cap, v} \left( \sum_{m \subseteq v^c} h_{v \cup m, v} \right).
\]
Given (8), Lemma 8 and Lemma 9 imply (9). Conversely, if \( g_{\cap, u} \) is given by (9), then we obtain (8). The equivalence of (8) and (10) can be established in the same way. \( \square \)
Example 11 Assume that (2) is satisfied, and let $\eta_u \in \mathbb{R}$ as well as $g_u(\mathbf{x}) = \prod_{j \in u} g(x_j)$, where $g \in L_{p, \psi}(D)$. Moreover, let

$$f_* = \sum_{u \subseteq \{1: d\}} \eta_u T_{*, u}(g_u). \quad (11)$$

The right-hand side in (11) is the anchored or ANOVA decomposition, respectively, of $f_*$, and its components $\eta_u T_{*, u}(g_u)$ are of tensor product form. We use Lemma 10 to compute the ANOVA decomposition of $f_\delta$ and the anchored decomposition of $f_A$. Since $\overline{\psi}(t) = K_\delta(x, t) - K_A(x, t)$, we get $g \cdot \overline{\psi} \in L_1(D)$ from Lemma 1. Put

$$c = \int_D g(t) \cdot \overline{\psi}(t) \, dt.$$ 

We obtain

$$g_{A, u} = \sum_{w \subseteq u} \eta_{u, w} c^{\lvert w \rvert} \cdot g_u,$$

and therefore

$$f_\delta = \sum_{u \subseteq \{1: d\}} \sum_{w \subseteq u} \eta_{u, w} \cdot c^{\lvert w \rvert} \cdot T_{A, u}(g_u)$$

is the ANOVA decomposition of $f_\delta$. In the same way we obtain

$$f_A = \sum_{u \subseteq \{1: d\}} \sum_{w \subseteq u} \eta_{u, w} \cdot (-c)^{\lvert w \rvert} \cdot T_{A, u}(g_u)$$

as the anchored decomposition of $f_A$. In both cases, the components of the new decomposition are again of tensor product form.

For $u \neq \emptyset$ we endow the spaces $F_{*, p, \psi, u}$ with the norms

$$\|T_{*, u}(g_u)\|_{F_{*, p, \psi, u}} = \|g_u\|_{L_{p, \psi}(D^u)},$$

which are well defined due to Lemma 8. Moreover, the space $F_{*, p, \psi, \emptyset}$ of constant functions is equipped with its natural norm.

Consider a family $\gamma = (\gamma_u)_{u \subseteq \{1: d\}}$ of non-negative numbers, called weights, and put

$$\mathcal{U}_\gamma = \{u \subseteq \{1: d\} : \gamma_u > 0\}.$$

Henceforth we assume that $\mathcal{U}_\gamma \neq \emptyset$. Let $1 \leq q \leq \infty$. For any family $(a_u)_u = (a_u)_{u \in \mathcal{U}_\gamma}$ of real numbers we put

$$\|(a_u)_u\|_q = \left(\sum_{u \in \mathcal{U}_\gamma} |a_u|^q\right)^{1/q}.$$
if $q < \infty$ and
\[
|(a_u)_u|_\infty = \max_{u \in \Gamma} |a_u|
\]
if $q = \infty$.

We endow the function spaces
\[
W_{*,p,q,\psi,\gamma} = \bigoplus_{u \in \Gamma} F_{*,p,\psi,u}
\]
with the following $\gamma$-weighted anchored and ANOVA norms. For $f \in W_{*,p,q,\psi,\gamma}$ given by \((6)\), i.e., $f_{*,u} = 0$ for $u \neq \Upsilon_{\gamma}$, we put
\[
\|f\|_{W_{*,p,q,\psi,\gamma}} = \left( \|f_{*,u}\|_{F_{*,p,\psi,u}/\gamma_u} \right)_u = \left( \|g_{*,u}\|_{L_p(\psi(D^u))/\gamma_u} \right)_u.
\]
Note that $\| \cdot \|_{W_{*,\infty,q,\psi,\gamma}}$ does not depend on $\psi$. Clearly, all spaces $W_{*,p,q,\psi,\gamma}$, equipped with the corresponding norm $\| \cdot \|_{W_{*,p,q,\psi,\gamma}}$, are Banach spaces, which are continuously embedded into $C(D^d)$, see Lemma 8. In particular, every point evaluation $f \mapsto f(x)$ on $W_{*,p,q,\psi,\gamma}$ is continuous. We refer to $W_{\gamma,p,q,\psi,\gamma}$ and $W_{A,p,q,\psi,\gamma}$ as $\gamma$-weighted anchored and ANOVA spaces, respectively.

Remark 12 In the particular case $D = [0,1]$, $\psi = 1$, and $p = q$, the spaces $W_{*,p,p,1,\gamma}$ have been studied in \([6]\). Actually, these spaces were introduced via differentiability properties of their elements $f \in L_p(D^d)$, namely, the existence of all weak derivatives $f^{(u)}$ in $L_p(D^d)$ together with
\[
\gamma_u = 0 \quad \Rightarrow \quad f^{(u)}(\cdot_u;0_w) = 0
\]
in the case $* = \gamma$ and
\[
\gamma_u = 0 \quad \Rightarrow \quad \int_{D^u} f^{(u)}(\cdot_u;\cdot_w) \, dt_w = 0
\]
in the case $* = A$. See \([6]\, \text{Lem. 3, Rem. 5}\) for the definition of $f^{(u)}(\cdot_u;\cdot_w)$ and \([6]\, \text{Prop. 11}\) for the equivalence of the approaches from \([6]\) and from the present paper in the particular case mentioned above. Moreover, the components of the anchored decomposition of $f$ are given by
\[
f_{\gamma,u} = \int_{D^u} f^{(u)}(\cdot_u;0_w) K_{\gamma,u}(\cdot_u;\cdot_u) \, dt_u
\]
and the components of the ANOVA decomposition of $f$ are given by
\[
f_A,u = \int_{D^u} \int_{D^w} f^{(u)}(\cdot_u;\cdot_w) \, dt_w \, K_{A,u}(\cdot_u;\cdot_u) \, dt_u,
\]
see \([6]\, \text{Prop. 7}\). These results should extend to the case of a general domain $D$, a general weight function $\psi$, and arbitrary $p$ and $q$. 

11
4 Equivalence of Norms

Let \( p, q \in [1, \infty] \) and let \( \gamma = (\gamma_u)_{u \subseteq [1:d]} \) be a family of weights such that \( \Upsilon_{\gamma} \neq \emptyset \). Furthermore, we assume (2).

Similar to [6, Prop. 13], we have the following proposition, whose proof is provided for completeness.

**Proposition 13** The \( \gamma \)-weighted anchored and ANOVA spaces are equal, i.e., \( W_{\cap, p,q,\psi,\gamma} = W_{A, p,q,\psi,\gamma} \) (as vector spaces), if and only if the following holds:

\[
\gamma_w > 0 \quad \text{implies that} \quad \gamma_u > 0 \quad \text{for all} \quad u \subseteq w. \tag{12}
\]

Moreover, if (12) does not hold then \( W_{\cap, p,q,\psi,\gamma} \not\subseteq W_{A,p,q,\psi,\gamma} \) and \( W_{A,p,q,\psi,\gamma} \not\subseteq W_{\cap, p,q,\psi,\gamma} \).

**Proof.** As follows from Lemma 10, every term \( f_{\cap,w} \) in the anchored decomposition of \( f \in W_{\cap, p,q,\psi,\gamma} \) is a linear combination of functions \( f_{A,u} \in F_{A,p,\psi,u} \) with \( u \subseteq w \). Hence (12) implies \( W_{\cap, p,q,\psi,\gamma} \subseteq W_{A,p,q,\psi,\gamma} \). Using the same argument, we derive \( W_{A,p,q,\psi,\gamma} \subseteq W_{\cap, p,q,\psi,\gamma} \) from (12).

Suppose now that (12) does not hold for some \( w \). Consider \( f(x) = \prod_{j \in w} x_j \), which corresponds to \( f = f_{\cap} \) with \( g = 1 \) as well as \( \eta_w = 1 \) and \( \eta_u = 0 \) for \( u \neq w \) in Example 11. Therefore \( f \in W_{\cap, p,q,\psi,\gamma} \) with ANOVA decomposition

\[
f = \sum_{u \subseteq w} c_{[w]-[u]} \cdot T_{A,u}(g_u).
\]

Since \( c = m_\psi \neq 0 \), see (5), and \( T_{A,u}(g_u) \neq 0 \) for every \( u \), but \( \gamma_u = 0 \) for some \( u \subseteq w \), we obtain \( f \notin W_{A,p,q,\psi,\gamma} \).

The fact that \( W_{A,p,q,\psi,\gamma} \) is not a subset of \( W_{\cap, p,q,\psi,\gamma} \) can be shown in a similar way by considering \( f(x) = \prod_{j \in w} (x_j - m_\psi) \).

From now on we assume that (12) holds. Let

\[
i_{p,q,\gamma}: W_{A,p,q,\psi,\gamma} \hookrightarrow W_{\cap, p,q,\psi,\gamma} \quad \text{and} \quad i_{p,q,\gamma}^{-1}: W_{\cap, p,q,\psi,\gamma} \hookrightarrow W_{A,p,q,\psi,\gamma}
\]

denote the embedding operators, which are continuous due to the closed graph theorem. To simplify the notation we will often write

\[
i_{p,q} \quad \text{and} \quad i_{p,q}^{-1}.
\]

We are interested in the norms of these operators.

Let \( \star \in \{ \cap, A \} \). Recall that, by definition, the operator

\[
T_{\star,u}: L_{p,\psi}(D^u) \rightarrow F_{\star,p,\psi,u}
\]

is an isometric isomorphism. Define the weighted space

\[
\ell_{q,\gamma}([L_{p,\psi}(D^u)]_u) = \bigoplus_{u \in \Upsilon_{\gamma}} L_{p,\psi}(D^u)
\]
endowed with the norm
\[ \| (g_u)_u \| = \left( \| g_u \|_{L_{p,\psi}(D^u)} / \gamma_u \right)_u^q. \]

Again, by definition, the mapping
\[ T_* : \ell_{q,\gamma}((L_{p,\psi}(D^u))_u) \rightarrow W_{*,p,q,\psi,\gamma}, \]
given by
\[ T_* ((g_u)_u) = \sum_{u \in U_\gamma} T_{*,u}(g_u), \]
is an isometric isomorphism. Define the continuous operator
\[ \delta_{p,q} = \delta_{p,q,\psi,\gamma} : \ell_{q,\gamma}((L_{p,\psi}(D^u))_u) \rightarrow \ell_{q,\gamma}((L_{p,\psi}(D^u))_u) \]
by \( \delta_{p,q} = T_{\delta}^{-1} \circ \iota_{p,q} \circ T_{A}. \) In other words, \( \delta_{p,q} \) is defined such that the diagram
\[ W_{\delta,p,q,\psi,\gamma} \xleftarrow{T_{\delta}} \ell_{q,\gamma}((L_{p,\psi}(D^u))_u) \xrightarrow{T_{\delta}^{-1}} \ell_{q,\gamma}((L_{p,\psi}(D^u))_u) \]
is commutative. Since \( T_{\delta} \) and \( T_{A} \) are isometric isomorphisms, we have
\[ \| \delta_{p,q} \| = \| \iota_{p,q} \| \quad \text{and} \quad \| \delta_{p,q}^{-1} \| = \| \iota_{p,q}^{-1} \|. \] (13)
Furthermore, (9) and (10) yield the explicit representation
\[ \delta_{p,q} ((g_u)_u) = \left( (-1)^{|u|} \sum_{w \subseteq u^c} \int_{D^w} (-1)^{|u|+|w|} g_{u \cup w}(\cdot_u; t_w) \overline{\psi}_w(t_w) \, dt_w \right)_u \] (14)
and
\[ \delta_{p,q}^{-1} ((g_u)_u) = \left( \sum_{w \subseteq u^c} \int_{D^w} g_{u \cup w}(\cdot_u; t_w) \overline{\psi}_w(t_w) \, dt_w \right)_u. \] (15)

The following result was first obtained in [10] for \( D = [0,1] \) and \( \psi = 1. \)

**Proposition 14** We have
\[ \| \iota_{p,q,\psi,\gamma} \| = \| \iota_{p,q,\psi,\gamma}^{-1} \| = \| \delta_{p,q,\psi,\gamma} \| = \| \delta_{p,q,\psi,\gamma}^{-1} \|. \] (16)

**Proof.** Define the operator
\[ S : \ell_{q,\gamma}((L_{p,\psi}(D^u))_u) \rightarrow \ell_{q,\gamma}((L_{p,\psi}(D^u))_u) \]
by
\[ S ((g_u)_u) = \left( (-1)^{|u|} g_u \right)_u. \]
By definition of $S$ and by (14) as well as (15) the diagram
\[
\begin{array}{ccc}
\ell_{q,\gamma}((L_p,\psi)(D^u))_u & \xrightarrow{S} & \ell_{q,\gamma}((L_p,\psi)(D^u))_u \\
\uparrow_{S^{-1}} & & \downarrow_{S} \\
\ell_{q,\gamma}((L_p,\psi)(D^u))_u & \xrightarrow{S} & \ell_{q,\gamma}((L_p,\psi)(D^u))_u
\end{array}
\]
is commutative. Again $S$ is an isometric isomorphism. Hence we get the claim. \qed

Next we compute the norms of the embeddings in the extremal case $p, q \in \{1, \infty\}$. Afterwards we use interpolation to find upper bounds for the general case. In what follows we use the convention that $\frac{-1}{2} = 0$.

4.1 The Case $p, q \in \{1, \infty\}$

Recall that for $p = 1$ we have
\[
\kappa_\psi = \frac{\|\psi/\psi\|_{L_\infty(D)}}{\in (0, \infty)},
\]
see (2). Furthermore, recall the definition of $m_\psi$ in (5). In the particular case from Remark 12, where we have $m_\psi = 1/2$ and $\kappa_\psi = 1$, the following result was obtained in [6, Thm. 14].

**Theorem 15** For $p, q \in \{1, \infty\}$ we have
\[
\|i_{p,q,\psi,\gamma}\| = \|i_{p,q,\psi,\gamma}^{-1}\| = C_{p,q,\psi,\gamma},
\]
where
\[
C_{p,q,\psi,\gamma} = \begin{cases}
\max_{u \subseteq [1:d]} \sum_{\nu \subseteq u} m_{\psi}^{\nu} \frac{\gamma_{\nu \cup u}}{\gamma_u} & \text{for } p = \infty \text{ and } q = \infty, \\
\max_{\nu \subseteq [1:d]} \sum_{\mu \subseteq \nu} m_{\psi}^{\nu - \mu} \frac{\gamma_{\mu}}{\gamma_u} & \text{for } p = \infty \text{ and } q = 1, \\
\max_{\nu \subseteq [1:d]} \sum_{\mu \subseteq \nu} \kappa_{\psi}^{\nu - \mu} \frac{\gamma_{\mu}}{\gamma_u} & \text{for } p = 1 \text{ and } q = 1, \\
\max_{\nu \subseteq [1:d]} \sum_{\mu \subseteq \nu} \kappa_{\psi}^{\nu \cup u} \frac{\gamma_{\nu \cup u}}{\gamma_u} & \text{for } p = 1 \text{ and } q = \infty.
\end{cases}
\]

**Proof.** According to Proposition 14 is suffices to consider either $i_{p,q,\psi,\gamma}$ or $i_{p,q,\psi,\gamma}^{-1}$. In the sequel,
\[
f = \sum_{u \in \Upsilon_\gamma} T_{A,u}(g_{A,u}) = \sum_{u \in \Upsilon_\gamma} T_{\phi,u}(g_{\phi,u})
\]
with $g_{A,u}, g_{\phi,u} \in L_{p,\psi}(D^u)$.
Case $p = q = \infty$

Applying (10), we get

$$
\|f\|_{W_{\infty, \infty, \psi, \gamma}} = \max_{u \in U\gamma} \frac{\|g_{A,u}\|_{L_\infty(D^u)}}{\gamma_u}
= \max_{u \in U\gamma} \text{ess sup}_{x_u \in D^u} \left| \frac{1}{\gamma_u} \sum_{v \subseteq U^c} \int_{D^v} g_{\delta, u,w}(x_u; t_v) \overline{\psi}_v(t_v) \, dt_v \right|
\leq \max_{u \in U\gamma} \left( \frac{\|g_{\delta, u,w}\|_{L_\infty(D^u)}}{\gamma_u} \right) \cdot \sum_{v \subseteq U^c} \frac{\gamma_u}{\gamma_v} m_{\psi}^{[v]}
= \||f\|_{W_{\infty, \infty, \psi, \gamma}} C_{\infty, \infty, \psi, \gamma}.
$$

This proves that $\|t_{\infty, \infty}^{-1}\| \leq C_{\infty, \infty, \psi, \gamma}$.

We next prove that there is equality. In fact, for $f_{\delta, \psi}$ according to Example 11 with $g = 1$ and $\eta_v = \gamma_v$ we obtain $\|f_{\delta, \psi}\|_{W_{\infty, \infty, \psi, \gamma}} = 1$ and $\|f_{\delta, \psi}\|_{W_{\infty, \infty, \psi, \gamma}} = C_{\infty, \infty, \psi, \gamma}$.

Case $p = \infty$ and $q = 1$

Applying (10) again, we have

$$
\|f\|_{W_{\infty, 1, \psi, \gamma}} = \sum_{u \in U\gamma} \frac{1}{\gamma_u} \|g_{A,u}\|_{L_\infty(D^u)}
= \sum_{u \in U\gamma} \text{ess sup}_{x_u \in D^u} \left| \sum_{v \subseteq U^c} \int_{D^v} g_{\delta, u,w}(x_u; t_v) \overline{\psi}_v(t_v) \, dt_v \right|
\leq \sum_{u \in U\gamma} \frac{1}{\gamma_u} \sum_{v \subseteq U^c} \|g_{\delta, u,w}\|_{L_\infty(D^u)} \frac{m_{\psi}^{[v]}}{\gamma_v}
= \sum_{u \in U\gamma} \sum_{v \subseteq U^c} \frac{\gamma_u}{\gamma_v} \frac{\|g_{\delta, u,w}\|_{L_\infty(D^u)}}{\gamma_u} m_{\psi}^{[v]}
\leq \||f\|_{W_{\infty, 1, \psi, \gamma}} C_{\infty, 1, \psi, \gamma},
$$

which proves $\|t_{\infty, 1}^{-1}\| \leq C_{\infty, 1, \psi, \gamma}$.

Let $\omega$ be such that

$$
\sum_{u \subseteq \omega} m_{\psi}^{[u]} \frac{\gamma_v}{\gamma_u} = C_{\infty, 1, \psi, \gamma},
$$

and consider $f_A$ according to Example 11 with $g = 1$ as well as $\eta_v = \gamma_v$ and $\eta_u = 0$ for $u \neq \omega$. Clearly, the ANOVA norm of $f_A$ is equal to one, and $\|f_A\|_{W_{\infty, 1, \psi, \gamma}} = C_{\infty, 1, \psi, \gamma}$. Therefore $\|t_{\infty, 1}\| \geq C_{\infty, 1, \psi, \gamma}$.
Case $p = q = 1$

Applying (10), we have

$$
\|f\|_{W^{A,1,1,\psi,\gamma}} = \sum_{u \subseteq U} \frac{1}{\gamma_u} \|g_{A,u}\|_{L_1(\psi(D^a))}
$$

$$
= \sum_{u \subseteq U} \frac{1}{\gamma_u} \int_D \left| \int_{D^a} g_{\psi,A,u}(x_u, t_{1b}) \bar{\psi}_{1b}(t_{1b}) \, dt_{1b} \right| \psi_u(x_u) \, dx_u
$$

$$
\leq \sum_{u \subseteq U} \sum_{w \subseteq u} \int_D \int_{D^a} \frac{|g_{\psi,A,u}(x_u, t_{1b})|}{\gamma_u} \bar{\psi}_{1b}(x_u, t_{1b}) \psi_{1b}(t_{1b}) \, dt_{1b} \, dx_u.
$$

Estimating $\bar{\psi}/\psi$ by $\kappa_{\psi}$ and replacing $u \cup w$ by $v$, we get that $w = v \setminus u$ and

$$
\|f\|_{W^{A,1,1,\psi,\gamma}} \leq \sum_{v \subseteq U} \frac{1}{\gamma_v} \|g_{\psi,v}\|_{L_1(\psi(D^a))} \cdot \left( \sum_{u \subseteq v} \frac{\kappa_{\psi} |v| - |u|}{\gamma_u} \right)
$$

$$
\leq \|f\|_{W^{A,1,1,\psi,\gamma}} C_{1,1,\psi,\gamma}.
$$

This proves that $\|l_{1,1}^{-1}\| \leq C_{1,1,\psi,\gamma}$.

We now show that $\|l_{1,1}^{-1}\| = C_{1,1,\psi,\gamma}$. Consider a sequence of non-negative functions $G_n \in L_1(D)$ such that

$$
\int_D G_n(x) \, dx = 1 \quad \text{and} \quad \lim_{n \to \infty} \int_D G_n(x) \frac{\bar{\psi}(x)}{\psi(x)} \, dx = \kappa_{\psi}.
$$

For instance,

$$
G_n(x) = \frac{1}{\lambda(K_n)} 1_{K_n}(x), \quad \text{where} \quad K_n \subseteq \left\{ x \in D : \frac{\bar{\psi}(x)}{\psi(x)} \geq \kappa_{\psi} - \frac{1}{n} \right\},
$$

(17)

has positive and finite Lebesgue measure $\lambda(K_n)$.

Define

$$
g_n(t) = \frac{G_n(t)}{\psi(t)}. \tag{18}
$$

Of course, $\|g_n\|_{L_1(\psi(D))} = 1$. Moreover, $m_n$ defined by

$$
m_n = \int_D g_n(t) \bar{\psi}(t) \, dt
$$

satisfies $\lim_{n \to \infty} m_n = \kappa_{\psi}$.

Let $w \subseteq [1 : d]$ be such that

$$
C_{1,1,\psi,\gamma} = \sum_{u \subseteq w} \frac{\kappa_{\psi} |w| - |u|}{\gamma_u} \frac{\gamma_w}{\gamma_u}.
$$
Let $f_n$ be given according to Example 11 with $g = g_n, \eta_w = \gamma_w$, and $\eta_u = 0$ for $u \neq w$. It follows that $f_n \in W_{n,1,\psi,\gamma}$ with $\|f_n\|_{W_{n,1,\psi,\gamma}} = 1$ and $c = m_n$. Moreover, the ANOVA decomposition of $f_n$ is given by

$$f_n = \gamma_w \sum_{u \subseteq w} m_n^{\|u\| - |u|} T_{A,u}(g_u).$$

Consequently,

$$\|f_n\|_{W_{A,1,\psi,\gamma}} = \sum_{u \subseteq w} m_n^{\|u\| - |u|} \frac{\gamma_w}{\gamma_u},$$

which converges to $C_{1,1,\psi,\gamma}$ as $n \to \infty$. This completes the proof that $\|s_{1,1}\| = C_{1,1,\psi,\gamma}$.

**Case $p = 1$ and $q = \infty$**

Here we have

$$\|f\|_{W_{A,1,\infty,\psi,\gamma}} = \max_{u \subseteq A} \frac{1}{\gamma_u} \int_{D^u} \left| \sum_{v \subseteq w} \int_{D^v} g_{n,w}(x_u, t_u) \psi_v(t_v) \frac{\overline{\psi_v}(t_v)}{\psi_v(t_v)} dt_v \right| \psi_u(x_u) dx_u$$

$$\leq \max_{u \subseteq A} \sum_{v \subseteq w} \frac{\|g_{n,w}\|_{L_1(D^u)}}{\gamma_{u,v}} \frac{\gamma_{w,v}}{\gamma_u} K_{\psi} |v|^{1/p}$$

$$\leq C_{1,\infty,\psi,\gamma} \|f\|_{W_{n,1,\infty,\psi,\gamma}}$$

and $\|s_{1,1}\| \leq C_{1,\infty,\psi,\gamma}$, as needed.

To prove that $\|s_{1,\infty}\|$ is equal to $C_{1,\infty,\psi,\gamma}$ it is enough to consider $f_A$ as in Example 11 with $g = g_n$ as in (18) and $\eta_u = (-1)^{|u|} \gamma_u$. Then $\|f_A\|_{W_{A,1,\infty,\psi,\gamma}} = 1$. By using the anchored decomposition of $f_A$, we see that

$$\|f_A\|_{W_{n,1,\infty,\psi,\gamma}} = \max_{u \subseteq [1,d]} \sum_{v \subseteq u} m_n^{\|u\| - |u|} \frac{\gamma_{u,v}}{\gamma_u},$$

which converges to $C_{1,\infty,\psi,\gamma}$ as $n \to \infty$. □

### 4.2 General Case $p, q \in [1, \infty]$

In this section we follow the approach from [8], as we use complex interpolation to obtain upper bounds for $\|s_{p,q,\psi,\gamma}\| = \|s_{p,q,\psi,\gamma}^{-1}\|$ for $p, q \in [1, \infty]$. We assume here that (2) holds for $p = 1$. Then it also holds for $p > 1$, so all function spaces considered below are well defined.

We work with complex valued functions since the direct application of the complex interpolation method needs complex scalars. By considering real and imaginary parts separately, the definition of the spaces $W_{p,q,\psi,\gamma}$ can be extended to complex valued functions on $D^d$. Derivatives and integrals are applied to both parts. The results of the previous subsection remain valid. Indeed, the lower bounds for the norms obviously also hold for complex valued functions. Moreover, the proofs of the upper bounds remain valid...
also in the complex case. This is due to the fact that the inequalities used are triangle inequalities, which are also valid for complex scalars.

By Proposition 14 it is enough to consider \( \| p_q, \psi, \gamma \| \). The next theorem provides the general interpolation result for these norms, cf. [8, Sec. 4].

**Theorem 16** Let \( p, q, p_0, q_0, p_1, q_1 \in [1, \infty] \) and \( \theta \in [0, 1] \) be such that

\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.
\]

Then

\[
\| p_q, \psi, \gamma \| \leq \| p_0, q_0, \psi, \gamma \|^{1 - \theta} \| p_1, q_1, \psi, \gamma \|^{\theta}.
\]

**Proof.** By Proposition 14 it is enough to prove the corresponding result for \( L_{p,q,\psi,\gamma} \) instead of \( p_q, \psi, \gamma \). The relevant results for spaces of type \( \ell_{q, \gamma} ((A_u)_u) \) with \( A_u = L_{p,q}(D^u) \) are Theorem 1.18.1 (formula (4)) in [17], i.e.,

\[
[\ell_{q_0}(A_j), \ell_{q_1}(B_j)]_\theta = \ell_q ([A_j, B_j]_\theta),
\]

for Banach spaces \( A_j, B_j \) and \( 1/q = (1 - \theta)/q_0 + \theta/q_1 \), complemented by the following Remark 2, and Theorem 1.18.6.2 (formula (15)) in [17], i.e.,

\[
[L_{p_0}(A), L_{p_1}(A)]_\theta = L_p(A),
\]

for a Banach space \( A \) and \( 1/p = (1 - \theta)/p_0 + \theta/p_1 \) with \( 0 < \theta < 1 \). All these interpolation identities are to be understood with equality of the norms. Together they prove the claim in the theorem. \( \square \)

**Remark 17** For future applications it might be useful to use different weight sequences for different \( p,q \). As explained in [8], this is possible without much difficulties by also interpolating the weights. Additionally, it is also possible to interpolate the weight functions \( \psi \).

In the particular case where \( D = [0, 1], \psi = 1 \) and \( p = q \) the upper bound for \( \| p_p, \psi, \gamma \| \) from the next result was obtained in [8, Thm. 2].

**Theorem 18** For \( 1 \leq p \leq q \leq \infty \) we have

\[
\| p_q, \psi, \gamma \| = \| p_{p, \psi, \gamma} \| \leq C_{1/p}^{1/p - 1/q} \cdot C_{1,1}^{1/q} \cdot C_{1,1,\infty}^{1-1/p}.
\]

For \( 1 \leq q \leq p \leq \infty \) we have

\[
\| p_q, \psi, \gamma \| = \| p_{p, \psi, \gamma} \| \leq C_{1/q}^{1/q - 1/p} \cdot C_{1,p}^{1/p} \cdot C_{1,1,\infty}^{1-1/q}.
\]

**Proof.** For simplicity, we abbreviate \( C_{p,q} = C_{p,q,\psi,\gamma} \). In a first step, applying Theorem 16 in the case \( p_0 = q_0 = \infty, p_1 = q_1 = 1, \theta = \frac{1}{p} \), we get

\[
\| p_{p,p,\gamma} \| \leq C_{1,1}^{1/p} \cdot C_{1,1,\infty}^{1-1/p}.
\]
In the case \( p < q \), determine \( r \in [1, \infty] \) and \( \theta \in [0, 1] \) via the equations
\[
\frac{1}{p} = \frac{1 - \theta}{1} + \frac{\theta}{r} \quad \text{and} \quad \frac{1}{q} = \frac{1 - \theta}{\infty} + \frac{\theta}{r}
\]
and obtain, again applying Theorem 16,
\[
\| \tilde{j}_{p,q,\psi,\gamma} \| \leq C_{1,\infty}^{1-\theta} \cdot C_{r,r}^\theta \leq C_{1,\infty}^{1-\theta} \cdot C_{1,1}^{\theta/(1-1/r)} = C_{1,\infty}^{1/p-1/q} \cdot C_{1,1}^{1/q} \cdot C_{1,\infty}^{1-1/p}.
\]
In the case \( q < p \), we similarly get
\[
\| \tilde{j}_{p,q,\psi,\gamma} \| \leq C_{1,\infty}^{1/q-1/p} \cdot C_{1,1}^{1/p} \cdot C_{1,\infty}^{1-1/q}.
\]

Adopting the proof technique of a part of Theorem 1 in [10], we can show the following lower bound. For notational convenience we put \( \psi^{1/\infty} = 1 \).

**Theorem 19** Let
\[
B_p = \| \overline{\psi}/\psi^{1/p} \|_{L_p'(D)}.
\]

For all \( p, q \in [1, \infty] \)
\[
\| \tilde{j}_{p,q,\psi,\gamma} \| \geq \sup \left( \sum_{w \subseteq u} c_{u,w} \gamma_{u,w} \frac{B_p^{|w|}}{|\gamma_u|_{u,q}} \right)_{u,q},
\]
where the supremum is taken over all families \( (c_u)_u = (c_u)_{u \in U_\gamma} \) of non-negative real numbers.

**Proof.** Consider a sequence of functions \( G_n \in L_p(D) \) such that \( \| G_n \|_{L_p(D)} = 1 \) and
\[
\lim_{n \to \infty} \int_D G_n(t) \overline{\psi(t)}^{1/p(t)} \, dt = B_p.
\]

In the case \( p = 1 \) we may choose this sequence as in (17). In the case \( p > 1 \) it suffices to consider a single function \( G = G_n \), since \( L_p(D) \) is the dual space of \( L_{p'}(D) \). Define
\[
g_n(t) = \frac{G_n(t)}{\overline{\psi(t)}^{1/p(t)}}
\]
and
\[
m_n = \int_D g_n(t) \overline{\psi(t)} \, dt
\]
to obtain \( g_n \in L_p,\psi(D) \) with \( \| g_n \|_{L_p,\psi(D)} = 1 \) and \( \lim_{n \to \infty} m_n = B_p \). Let \( f_\cap \) be given by Example 11 with \( g = g_n \) and \( \bar{\eta}_u = c_u \gamma_u \), where \( c_u \in \mathbb{R} \). Clearly \( c = m_n \) and
\[
\| f_\cap \|_{W_{A,p,q,\psi,\gamma}} = |(c_u)_{u,q}|.
\]
Furthermore,
\[
\| f_\cap \|_{W_{A,p,q,\psi,\gamma}} = \left( \sum_{w \subseteq u} c_{u,w} \gamma_{u,w} m_n^{|w|}/|\gamma_u|_{u,q} \right)_{u,q}.
\]
Let \( n \) tend to \( \infty \) to complete the proof.

\[\square\]
5 Uniform and Polynomial Equivalence of Norms

So far the number $d$ of variables was fixed. In this section we study the equivalence of the norm for varying $d$. More precisely, for $d \in \mathbb{N}$ we have a family

$$\gamma[d] = (\gamma_{d,u})_{u \subseteq [1:d]}$$

of weights that satisfy (12) for every $d$. Furthermore, we assume that (2) is satisfied so that

$$W_{n,p,q,\psi,\gamma[d]} = W_{A,p,q,\psi,\gamma[d]}$$

for all $d \in \mathbb{N}$ as vector spaces. Of course, the norm

$$\|i_{p,q,\psi,\gamma[d]}\|\quad = \quad \|i_{p,q,\psi,\gamma[d]}^{-1}\|$$

of the embeddings depends on $d$ in all non-trivial cases.

**Definition 20**

(i) The weighted anchored and ANOVA norms are uniformly equivalent if

$$\sup_{d \in \mathbb{N}} \|i_{p,q,\psi,\gamma[d]}\| \quad < \quad \infty.$$  

(ii) The weighted anchored and ANOVA norms are polynomially equivalent if there exist $\tau > 0$ such that

$$\|i_{p,q,\psi,\gamma[d]}\| \quad = \quad O\left(d^\tau\right).$$

The smallest (or infimum of) such $\tau$ is called the exponent of polynomial equivalence of the norms.

We now consider special classes of weights to see when there is uniform or polynomial equivalence. The explicit formulas for $C_{p,q,\psi,\gamma[d]} = \|i_{p,q,\psi,\gamma[d]}\|$ for $p, q \in \{1, \infty\}$ according to Theorem 15 are very similar to those obtained in [6] for $D = [0, 1]$ and $\psi = 1$. Since applying them with Theorem 18 is rather straightforward, we will omit the proofs of upper bounds of $\|i_{p,q,\psi,\gamma[d]}\|$. Corresponding lower bounds can be obtained by applying techniques from [10] to Theorem 19. This is why proofs of some lower bounds are also omitted. We begin with the product weights that are the most commonly used in the literature.

*Product weights*, introduced in [10], have the following form

$$\gamma_u = \prod_{j \in u} \gamma_j$$

for positive numbers $\gamma_j$.

We have

$$C_{\infty,1,\psi,\gamma[d]} = C_{\infty,\infty,\psi,\gamma[d]} = \prod_{j=1}^d (1 + m_\psi \gamma_j)$$
and
\[ C_{1,1,\psi,\gamma} = C_{1,\infty,\psi,\gamma} = \prod_{j=1}^{d} (1 + \kappa_{\psi} \gamma_j). \]

Hence the conditions
\[ \sum_{j=1}^{\infty} \gamma_j < \infty \quad \text{and} \quad \sup_{d \in \mathbb{N}} \sum_{j=1}^{d} \gamma_j \ln(d+1) < \infty \]  

are sufficient for the corresponding uniform and polynomial equivalences for all \( p, q \in [1, \infty] \). From Theorem 19, one can conclude that (22) are also necessary for corresponding uniform and polynomial equivalences for any \( p, q \in [1, \infty] \), cf. [10, Prop. 3].

**Product order-dependent weights**, introduced in [12], have the form
\[ \gamma_{d,u} = \left( |u|! \right)^{\beta_1} \cdot \prod_{j \in u} \frac{c}{j^{\beta_2}} \quad \text{with} \quad 0 < \beta_1 < \beta_2 \quad \text{and} \quad c > 0. \]

The following lower bound holds for every \( p, q \) and \( \tau > 0 \)
\[ \|i_{p,q,\psi,\gamma}|d\| = \Omega(d^\tau), \]

asymptotically for \( d \to \infty \), cf. [6] Prop. 18] for the extremal cases \( p = q = 1 \) and \( p = q = \infty \) and cf. [10] Prop. 8] for the general case. Hence there is no polynomial equivalence for any \( p, q \in [1, \infty] \).

**Finite order weights**, introduced in [3], are such that
\[ \gamma_{d,u} = 0 \quad \text{if} \quad |u| > r \]

for a given fixed number \( r \geq 1 \). We consider the following special weights
\[ \gamma_u = \omega^{|u|} \quad \text{if} \quad |u| \leq r, \]

where \( \omega \) is a positive number. For \( q = 1 \), we have
\[ C_{1,1,\psi,\gamma} = (1 + \kappa_{\psi} \omega)^r \quad \text{and} \quad C_{\infty,1,\psi,\gamma} = (1 + m_{\psi} \omega)^r, \]

whereas for \( q = \infty \), we have
\[ C_{p,\infty,\psi,\gamma} = \Theta(d^p) \quad \text{for} \ p \in \{1, \infty\}. \]

Using Theorem 18 we conclude that for arbitrary \( p \) and \( q \)
\[ \|i_{p,q,\psi,\gamma}|d\| = O\left(d^{r(1-1/p)}\right). \]

Hence we have at least polynomial equivalence with the exponent bounded by \( r(1-1/p) \). For \( q = 1 \), we have uniform equivalence. Actually, the exponent of polynomial equivalence
is precisely \( r(1 - 1/q) \). Indeed, use the lower bound in Theorem 19 for \( c_u = 1 \) if \(|u| = r\) and \( c_u = 0 \) otherwise. If \(|u| \leq r\) we obtain

\[
\sum_{w \subseteq u^c} c_{u,w} \frac{B_p^{|w|}}{\gamma_u} \cdot \sum_{|w| = |u|} = \left( \frac{d - |u|}{r - |u|} \right) \cdot (\sigma_p)^{1/q}.
\]

For \( 1 \leq q < \infty \) this yields

\[
\|\hat{\mathbf{i}}_{p,q,\gamma[d]}\| \geq \left( \sum_{s=0}^{r} (\omega B_p)^{q(r-s)} \cdot \frac{(d-s)^q}{(r-s)^q} \cdot \frac{d^q}{s^q} \right)^{1/q}.
\]

Considering only the term with \( s = 0 \) this gives

\[
\|\hat{\mathbf{i}}_{p,q,\gamma[d]}\| \geq (\omega B_p)^{r} \cdot \left( \frac{d}{r} \right)^{1-1/q} = \Omega \left( \frac{d^{1-1/q}}{r^{1-1/q}} \right).
\]

This estimate is valid for \( q = \infty \), too.

Consider finite diameter weights, introduced by Creutzig (see [2] and [15]), of the form

\[
\gamma_u = \begin{cases} 
\omega^{|u|} & \text{if diam}(u) \leq r, \\
0 & \text{if diam}(u) > r,
\end{cases}
\]

where \( \text{diam}(u) = \max_{i,j \in u} |i - j| \), where \( \text{diam}(\emptyset) = 0 \), by convention. As in [10],

\[
C_{1,1,\gamma[d]} = (1 + \omega \kappa_\psi)^{r+1} \quad \text{and} \quad C_{\infty,1,\gamma[d]} = (1 + \omega m_\psi)^{r+1},
\]

whereas

\[
C_{p,\infty,\gamma[d]} = \Theta(d) \quad \text{for } p \in \{1, \infty\}.
\]

By applying interpolation we get \( \|\hat{\mathbf{i}}_{p,q,\gamma[d]}\| = O \left( d^{1-1/q} \right) \) for all \( p, q \). Similar to the proof of Proposition 7 in [10] one can show that the above bound is sharp, i.e.,

\[
\|\hat{\mathbf{i}}_{p,q,\gamma[d]}\| = \Theta \left( d^{1-1/q} \right),
\]

which means polynomial equivalence with the exponent \( 1 - 1/q \).

Finally, consider special dimension-dependent weights

\[
\gamma_{d,u} = d^{-|u|}.
\]

introduced in [7]. Then for \( q \in \{1, \infty\}, \)

\[
C_{1,q,\gamma[d]} = (1 + \kappa_\psi/d)^d \leq \exp(\kappa_\psi) \quad \text{and} \quad C_{\infty,q,\gamma[d]} = (1 + m_\psi/d)^d \leq \exp(m_\psi).
\]

The interpolation yields uniform equivalence for all \( p \) and \( q \).
6 Applications to Integration and Approximation

A thorough study of applications of embedding results to high- and infinite-dimensional integration in the setting of reproducing kernel Hilbert spaces with product weights is carried out in [4]. This abstract approach covers the particular case \( p = q = 2 \) with product weights \( \gamma_u \) in the setting of the present paper. A number of new error estimates and new tractability results could be obtained in [4] by transferring known results for the anchored setting to the ANOVA setting or vice versa. Roughly speaking, the anchored setting is known to be very well suited for the analysis of deterministic algorithms, while the ANOVA setting is much preferable for the analysis of randomized algorithms.

The results of the present paper allow to transfer results between the anchored and the ANOVA setting beyond Hilbert spaces and product weights. Unfortunately, we are only aware of few results for the non-Hilbert space setting or the corresponding weighted discrepancies, see [1, 9, 10, 11, 14, 16, 18]. In the sequel we discuss the transfer of results from [9, 10, 11, 18] from the anchored setting to the ANOVA setting.

At first, we illustrate how to transfer the tractability results from [9]. The results there are formulated in terms of the weighted star discrepancy. Via Koksma-Hlawka duality, this corresponds to results for uniform integration on \( D^d = [0,1]^d \) in \( W_{\psi,1,1,\gamma} \) in the case \( \psi = 1 \).

For product weights satisfying the condition \( \sum_{j=1}^{\infty} \gamma_j < \infty \), Theorem 3 in [9] shows that this problem is strongly tractable, i.e., the number \( N(\varepsilon,d) \) of sample points needed to achieve an error \( \varepsilon > 0 \) can be bounded by \( C\varepsilon^{-\beta} \) with absolute constants \( C, \beta > 0 \) independent of the dimension \( d \). Moreover, it is also shown there that the exponent of strong tractability, that is the infimum over all possible \( \beta \), is 1. The results in Section 5 show that anchored and ANOVA norms are uniformly equivalent. Hence we immediately obtain that we also have strong tractability of uniform integration on \( D^d = [0,1]^d \) in \( W_{A,1,1,\psi,\gamma} \) in the case \( \psi = 1 \). The algorithms achieving this are QMC-algorithms using superpositions of digital nets over \( \mathbb{Z}_2 \). For details of the construction we refer to [9] and the references therein.

For general weights, Corollary 1 in [9] shows that integration is polynomially tractable with \( \varepsilon \)-exponent 2 and \( d \)-exponent 0 under the condition

\[
C_\gamma = \sup_{d \in \mathbb{N}} \max_{u \subseteq [1:d]} \gamma_{d,u} \sqrt{|u|} < \infty.
\]

This condition is satisfied for bounded finite order weights, for finite diameter weights and for the dimension dependent weights in (24). More exactly, we have the estimate

\[
N(\varepsilon,d) \leq CC_\gamma (1 + \log d) \varepsilon^{-2}
\]

with some constant \( C > 0 \). From the results in Section 5 we infer that the same holds for uniform integration on \( D^d = [0,1]^d \) in \( W_{A,1,1,\psi,\gamma} \) in the case \( \psi = 1 \) for the special finite order weights defined in (23), for finite diameter weights, and for the special dimension dependent weights (24).

The anchored spaces studied in this paper were also considered in [18] in the context of function approximation. Actually, functions of infinitely-many variables are studied in [18], however, if we define \( \gamma_u = 0 \) for every finite set \( u \subseteq \mathbb{N} \) with \( u \setminus [1 : d] \neq \emptyset \) in the
setting of the latter paper, we obtain functions on \( D^d \). For a given probability density \( \omega : D \to \mathbb{R}_+ \) and a real \( 1 \leq s \leq \infty \) one is interested in approximating \( f \in W_{\alpha,p,q,\psi,\gamma} \) with the error measured in a norm \( \| \cdot \|_G \) satisfying the following condition. If \( s < \infty \), then

\[
\| f \|_G \leq \left( \sum_{u \in U_\gamma} \| f_{\gamma,u} \|_{G_u}^s \right)^{1/s}
\]

for the anchored decomposition

\[
f = \sum_{u \in U_\gamma} f_{\gamma,u},
\]

where

\[
\| f_u \|_{G_u} = \left( \int_{D^{|u|}} |f_u(x_u)|^s \prod_{j \in u} \omega(x_j) \, dx_u \right)^{1/s}.
\]

For \( s = \infty \) the condition is modified in the usual way. Almost optimal algorithms and sharp complexity bounds for such approximation problems were derived in [18]; however, only for the anchored spaces. The uniform equivalence studied in the current paper allows to transfer the result of [18] to the case of ANOVA spaces resulting in almost optimal algorithms and sharp complexity bounds. The multivariate decomposition method, which is particularly tuned to the anchored setting at a first glance, turns out to be almost optimal also in the ANOVA setting.

Now we turn to the results from [10, 11]. For problems with large number \( d \) of variables, one may try to replace the original functions \( f \) by functions \( f_k \) with only \( k \ll d \) variables, namely,

\[
f_k(x) = f(x_1, \ldots, x_k, 0, 0, \ldots, 0).
\]

As shown in [10, 11], for weighted integration and weighted \( L_s \) approximation, as above, and for modest error demands \( \varepsilon \), one can truncate the dimension with \( k = k(\varepsilon) \) being very small. This holds for problems defined on anchored spaces \( W_{\alpha,p,q,\psi,\gamma} \). Moreover, in general, this desirable property does not hold for ANOVA spaces \( W_{A,p,q,\psi,\gamma} \). However, if the anchored and ANOVA spaces are uniformly equivalent then also in the setting of ANOVA spaces, one can use functions \( f_k(\varepsilon) \) with \( k(\varepsilon) \) only slightly larger than the corresponding truncation dimension for the anchored spaces.

7 Appendix

As previously, \( 1 \leq p \leq \infty \), and \( p' \) denotes its conjugate, \( 1/p + 1/p' = 1 \). Furthermore, let \((\Omega, \mathcal{A}, \mu)\) denote a probability space.

**Lemma 21** Let \( g : \Omega \to \mathbb{R} \) be a measurable function. Then we have \( g \in L_{p'}(\mu) \) if and only if \( fg \in L_1(\mu) \) for all \( f \in L_p(\mu) \).

**Proof.** Suppose that \( fg \in L_1(\mu) \) for all \( f \in L_p(\mu) \). For \( p = \infty \) the function \( f \), given by \( f(x) = g(x)/|g(x)| \) if \( g(x) \neq 0 \) and \( f(x) = 0 \) otherwise, is in \( L_\infty(\mu) \). Hence \( |g| = fg \in L_1(\mu) \), implying \( g \in L_1(\mu) \).
Now let $1 \leq p < \infty$. For $n \in \mathbb{N}$ let $g_n : \Omega \to \mathbb{R}$ be the function equal to $g(x)$ if $|g(x)| \leq n$ and $g_n(x) = 0$ otherwise. Then $g_n$ is bounded and $g_n \to g$ almost everywhere. In particular $g_n \in L_p' (\mu)$, so the functionals $I_n$ defined by $I_n f = \int_{\Omega} f g_n \, d\mu$ are linear and bounded on $L_p(\mu)$. By assumption and the dominated convergence theorem, the limit

$$\lim_{n \to \infty} I_n f = \lim_{n \to \infty} \int_{\Omega} f g_n \, d\mu = \int_{\Omega} f g \, d\mu$$

exists and is finite for every $f \in L_p(\mu)$. So the bounded linear functionals $I_n$ converge pointwise to the linear functional $I$ given by $I f = \int_{\Omega} f g \, d\mu$. The Banach-Steinhaus Theorem implies that $I$ is a bounded functional on $L_p(\mu)$. Since $L_p(\mu)$ is the dual space of $L_p'(\mu)$, we obtain $g \in L_p'(\mu)$.

Hölder’s inequality immediately yields the reverse implication. □

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