\begin{abstract}
The Jones polynomial $V_L(t)$ for an oriented link $L$ is a one-variable Laurent polynomial link invariant discovered by Jones. For any integer $n \geq 3$, we show that: (1) the difference of Jones polynomials for two oriented links which are $C_n$-equivalent is divisible by $(t-1)^n (t^2 + t + 1) (t^2 + 1)$, and (2) there exists a pair of two oriented knots which are $C_n$-equivalent such that the difference of the Jones polynomials for them equals $(t-1)^n (t^2 + t + 1) (t^2 + 1)$.
\end{abstract}

\section{Introduction}

The Jones polynomial $V_L(t) \in \mathbb{Z}[t^{\pm 1/2}]$ is an integral Laurent polynomial link invariant for an oriented link $L$ defined by the following formulae:

\begin{align*}
V_O(t) &= 1, \\
t^{-1}V_{L_+}(t) - tV_{L_-}(t) &= \left(t^{1/2} - t^{-1/2}\right)V_{L_0}(t),
\end{align*}

where $O$ denotes the trivial knot and $L_+$, $L_-$ and $L_0$ are oriented links which are identical except inside the depicted regions as illustrated in Fig. 1.1. The triple of oriented links $(L_+, L_-, L_0)$ is called a skein triple. Jones also showed the following property of the Jones polynomials for oriented knots.

\begin{theorem}(Jones \cite[Proposition 12.5]{Jones}) For any two oriented knots $J$ and $K$, $V_J(t) - V_K(t)$ is divisible by $(t-1)^2 (t^2 + t + 1)$.
\end{theorem}

On the basis of Theorem 1.1, for an oriented knot $K$, Jones called the polynomial $W_K(t) = \{1 - V_K(t)\} / (t-1)^2 (t^2 + t + 1)$ a simplified polynomial and made a table of the simplified polynomials for knots up to 10 crossings \cite{Jones}. In particular, if $K$ is the right-handed trefoil knot then $W_K(t) = 1$. So the polynomial $(t-1)^2 (t^2 + t + 1)$ is maximal as a divisor of the difference of Jones polynomials of any pair of two oriented knots.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{skein_triple}
\caption{Skein triple $(L_+, L_-, L_0)$}
\end{figure}
Our purpose in this paper is to examine the difference of Jones polynomials for two oriented links which are $C_n$-equivalent, where a $C_n$-equivalence is an equivalence relation on oriented links introduced by Habiro [8] and Gusarov [5] independently as follows. For a positive integer $n$, a $C_n$-move is a local move on oriented links as illustrated in Fig. 1.2 if $n \geq 2$, and a $C_1$-move is a crossing change. Two oriented links are said to be $C_n$-equivalent if they are transformed into each other by $C_n$-moves and ambient isotopies. By the definition of a $C_n$-move, it is easy to see that a $C_n$-equivalence implies a $C_{n-1}$-equivalence. Note that a $C_2$-move equals a delta move introduced by Matveev [17] and Murakami-Nakanishi [20] independently as illustrated in Fig. 1.3 (1), and a $C_3$-move equals a clasp-pass move introduced by Habiro [7] as illustrated in Fig. 1.3 (2). A $C_n$-move is closely related to the Vassiliev invariants of oriented links [26], [2], [1], [25]. It is known that if two oriented links are $C_n$-equivalent then they have the same Vassiliev invariants of order $\leq n - 1$, and specially for oriented knots, the converse is also true [9], [5].

Now let us generalize Theorem 1.1 to oriented links which are $C_n$-equivalent.

**Theorem 1.2.**

1. If two oriented links $L$ and $M$ are $C_2$-equivalent, then $V_L(t) - V_M(t)$ is divisible by $(t - 1)^2 (t^2 + t + 1)$.

2. For any integer $n \geq 3$, if two oriented links $L$ and $M$ are $C_n$-equivalent, then $V_L(t) - V_M(t)$ is divisible by $(t - 1)^n (t^2 + t + 1) (t^2 + 1)$.

**Figure 1.2.** $C_n$-move ($n \geq 2$)

**Figure 1.3.** (1) Delta move, (2) Clasp-pass move, (3) Pass move
We remark that Theorem 1.2 (1) was also observed in [4, Theorem 2] for oriented knots by using the Kauffman bracket. Since any two oriented knots are $C_2$-equivalent [20], Theorem 1.1 is deduced from Theorem 1.2 (1).

In the case of $n \geq 3$, we show the maximality of $(t - 1)^n (t^2 + t + 1) (t^2 + 1)$ as a divisor of the difference of Jones polynomials for oriented links which are $C_n$-equivalent as follows. Let $J_n$ and $K_n$ be two oriented knots as illustrated in Fig. 1.4. Note that $J_n$ and $K_n$ are transformed into each other by a single $C_n$-move, see Fig. 1.5. Then we have the following.

**Theorem 1.3.**

$$V_{J_n}(t) - V_{K_n}(t) = (-1)^{n+1} (t - 1)^n (t^2 + t + 1) (t^2 + 1).$$

![Figure 1.4. Oriented knots $J_n$ and $K_n$ ($n \geq 3$)](image)

![Figure 1.5. $J_n$ and $K_n$ are transformed into each other by a single $C_n$-move](image)

In section 2, we prove Theorem 1.2 and give its applications to the study of the difference of Vassiliev invariants of order $\leq n$ for two oriented links which are $C_n$-equivalent. In section 3, we prove Theorem 1.3 without knowing $V_{J_n}(t)$ and $V_{K_n}(t)$ individually by applying Kanenobu’s formula for the difference of Jones polynomials for two oriented knots which are transformed into each other by a single $C_n$-move (Lemma 3.1) and a $C_n$-move which does not change the knot type (Lemma 3.2).
2. Proof of Theorem 1.2

We recall the following results about the special values of the Jones polynomial. Here an r-component oriented link \( L \) is said to be proper if \( \text{lk}(K, L \setminus K) \equiv 0 \pmod{2} \) for each component \( K \) of \( L \), where \( \text{lk} \) denotes the linking number, and the Arf invariant is a link invariant introduced in [24] defined for only proper links.

**Lemma 2.1.** Let \( L \) be an \( r \)-component oriented link. Then the following holds.

1. \( (6) \ (12.1) ) \ V_L(1) = ( -2 )^{r-1} \).
2. \( (6) \ (12.4) ) \ V_L \left( e^{2 \pi \sqrt{-1}/3} \right) = (-1)^{r-1} \).
3. (Murakami [19]) \( V_L(\sqrt{-1}) = \left( \sqrt{-2} \right)^{r-1} \cdot (-1)^{\text{Arf}(L)} \) if \( L \) is proper, and 0 if \( L \) is nonproper, where Arf denotes the Arf invariant.

For an oriented link \( L \), we denote the \( l \)-th derivative at 1 of the Jones polynomial \( V_L(t) \) by \( V_L^{(l)}(1) \). It is known that \( V_L^{(1)}(1) \) is a Vassiliev invariant of order \( \leq l \) [12]. Then we have the following.

**Lemma 2.2.** Let \( L \) and \( M \) be two oriented \( r \)-component links and \( n \) an integer with \( n \geq 2 \). Then \( V_L^{(n)}(1) = V_M^{(n)}(1) \) for \( l = 1, 2, \ldots, n-1 \) if and only if \( V_L(t) - V_M(t) \) is divisible by \( (t-1)^n \). In case \( n \geq 3 \), \( V_L^{(n)}(1) = V_M^{(n)}(1) \) if and only if \( V_L(t) - V_M(t) \) is divisible by \( (t-1)^{n-1} \). In case \( n \geq 4 \), \( V_L^{(n)}(1) = V_M^{(n)}(1) \) if and only if \( V_L(t) - V_M(t) \) is divisible by \( (t-1)^n \).

**Proof.** The ‘if’ part is clear because \((t-1)^n \) divides \( V_L(t) - V_M(t) \). We show the ‘only if’ part by the induction on \( n \). Assume that \( n = 2 \). By Lemma 2.1 (1) and (2), there exists a polynomial \( g(t) \in \mathbb{Z} [ t^{\pm 1/2} ] \) such that

\[
V_L(t) - V_M(t) = (t^1 - 1) g(t).
\]

Then by differentiating both sides in (2.1), we have

\[
(2.2) \quad V_L^{(1)}(t) - V_M^{(1)}(t) = 3t^2 g(t) + (t^3 - 1) g^{(1)}(t).
\]

Thus by the assumption and (2.2), we have \( g(1) = 0 \). This implies that \( V_L(t) - V_M(t) \) is divisible by \( (t-1)^2 \). Next assume that \( n \geq 3 \) and \( V_L^{(r)}(1) = V_M^{(r)}(1) \) for \( l = 0, 1, \ldots, n-1 \). By the induction hypothesis, it follows that there exists a polynomial \( h(t) \in \mathbb{Z} [ t^{\pm 1/2} ] \) such that

\[
(2.3) \quad V_L(t) - V_M(t) = (t-1)^{n-1} (t^2 + 1) h(t).
\]

Let us denote \( (t^2 + 1) h(t) \) by \( \tilde{h}(t) \). Then by (2.3) and the assumption, we have

\[
(2.4) \quad 0 = V_L^{(n-1)}(1) - V_M^{(n-1)}(1) = (n-1)! \tilde{h}(1).
\]

Thus we have \( 0 = \tilde{h}(1) = 3h(1) \), namely \( h(1) = 0 \). This implies that \( t-1 \) divides \( h(t) \), therefore we have the desired conclusion. \( \square \)

**Remark 2.3.** For an \( r \)-component oriented link \( L \), it is known that

\[
V_L^{(r)}(1) = -3(-2)^{r-2} \text{Lk}(L)
\]

if \( r \geq 2 \) and 0 if \( r = 1 \), where \( \text{Lk} \) denotes the total linking number, that is the summation of all pairwise linking numbers of \( L \) [6] (12.2]). Thus Lemma 2.2 implies Theorem 1.1 in case \( r = 1 \), and \( \text{Lk}(L) = \text{Lk}(M) \) if and only if \( V_L(t) - V_M(t) \) is divisible by \( (t-1)^2 (t^2 + 1) \) in case \( r \geq 2 \).

**Lemma 2.4.** For an integer \( n \geq 3 \), if two oriented links \( L \) and \( M \) are \( C_n \)-equivalent, then \( V_L(t) - V_M(t) \) is divisible by \( t^2 + 1 \).
Thus we have the result.

Proof. Let \( L \) and \( M \) be two \( r \)-component oriented links which are \( C_n \)-equivalent. Then \( L \) and \( M \) are \( C_3 \)-equivalent. Note that a \( C_3 \)-move = a clasp-pass move can be realized by a single pass move \([13]\) as illustrated in Fig. 13 (3), and a pass move does not change the Arf invariant of a proper link \([20]\) Appendix. If \( L \) is proper, then \( M \) is also proper because \( L \) and \( M \) are also \( C_2 \)-equivalent and a \( C_2 \)-move does not change the pairwise linking numbers. Then by Lemma 2.1 (3), we have \( V_L(\sqrt{-1}) = V_M(\sqrt{-1}) \). If \( L \) is nonproper, then \( M \) is also nonproper and by Lemma 2.1 (3), we have \( V_L(\sqrt{-1}) = 0 = V_M(\sqrt{-1}) \).

Proof of Theorem 1.2. As we mentioned before, if two oriented \( r \)-component links \( L \) and \( M \) are \( C_n \)-equivalent then \( V_L^{(l)}(1) = V_M^{(l)}(1) \) for \( l = 1, 2, \ldots, n - 1 \). By combining this fact with Lemma 2.2, we have (1) in case \( n = 2 \), and by combining this fact with Lemma 2.2 and Lemma 2.4, we have (2).

As an application, we give alternative short proofs for two theorems shown by H. A. Miyazawa. Note that these theorems were proved by fairly combinatorial argument, that is, by making up a list of oriented \( C_n \)-moves carefully and checking the congruence for each of the cases. First we show the following as a direct consequence of Theorem 1.2 (2).

**Theorem 2.5.** (H. A. Miyazawa [18] Theorem 1.5) For an integer \( n \geq 3 \), if two oriented links \( L \) and \( M \) are \( C_n \)-equivalent, then it follows that

\[
V_L^{(n)}(1) \equiv V_M^{(n)}(1) \pmod{6 \cdot n!}.
\]

**Proof.** Assume that two oriented links \( L \) and \( M \) are \( C_n \)-equivalent. Then by Theorem 1.2 (2), there exists a polynomial \( f(t) \in \mathbb{Z}[t^{\pm 1/2}] \) such that

\[
V_L(t) - V_M(t) = (t - 1)^n \left( t^2 + t + 1 \right) \left( t^2 + 1 \right) f(t).
\]

Let us denote \( (t^2 + t + 1) \left( t^2 + 1 \right) f(t) \) by \( \tilde{f}(t) \). Then by (2.5), we have

\[
V_L^{(n)}(1) - V_M^{(n)}(1) = n! \cdot \tilde{f}(1) = n! \cdot 6 f(1).
\]

Thus we have the result.

Miyazawa also showed the best possibility of Theorem 2.5 by exhibiting two pairs of two oriented knots which are \( C_n \)-equivalent whose differences of the Jones polynomials do not equal \( 6 \cdot n! \) but the greatest common divisor of them is \( 6 \cdot n! \). The best possibility of Theorem 2.5 may also be given by two oriented knots \( J_n \) and \( K_n \) in Theorem 1.5 whose difference of the Jones polynomials exactly equals \( 6 \cdot n! \). Such an example was also observed by Horiuchi [10].

On the other hand, the **Conway polynomial** \( \nabla_L(z) \in \mathbb{Z}[z] \) is an integral polynomial link invariant for an oriented link \( L \) defined by the following formulae:

\[
\nabla_O(z) = 1, \quad \nabla_{L_+}(z) - \nabla_{L_-}(z) = z \nabla_{L_0}(z),
\]

where \( (L_+, L_-, L_0) \) is a skein triple in Fig. 11 [8]. Note that

\[
V_L(-1) = \nabla_L(-2\sqrt{-1})
\]

and the absolute value of \( V_L(-1) \) is known as the determinant of \( L \). We denote the coefficient of \( z^l \) in \( \nabla_L(z) \) by \( a_l(L) \). Then it is known that the Conway polynomial
of an $r$-component oriented link $L$ is of the following form

\begin{equation}
\nabla_L(z) = \sum_{i \geq 0} a_{r+2i-1}(L)z^{r+2i-1}.
\end{equation}

It is known that $a_l(L)$ is a Vassiliev invariant of order $\leq l$ \cite{11}. Thus if two oriented links $L$ and $M$ are $C_n$-equivalent, then $a_l(L) = a_l(M)$ for $l \leq n - 1$. In the case of $l = n$, Miyazawa showed the following. Note that in the case of oriented knots, this had been obtained by Ohyama-Ogushi \cite{21}.

**Theorem 2.6.** (H. A. Miyazawa \cite{18} Theorem 1.3) For an integer $n \geq 3$, if two oriented links $L$ and $M$ are $C_n$-equivalent, then it follows that

\[ a_n(L) \equiv a_n(M) \pmod{2}. \]

*Proof.* Let $L$ and $M$ be two $r$-component oriented links which are $C_n$-equivalent. If $n \equiv r \pmod{2}$, then by (2.7) we have $a_n(L) = a_n(M) = 0$. Assume that $n \not\equiv r \pmod{2}$. Then by Theorem 1.2 (2) and (2.6), there exists a polynomial $W(t) \in \mathbb{Z}[t^{\pm 1/2}]$ such that

\[ (-1)^n \cdot 2^{n+1} \cdot W(-1) = V_L(-1) - V_M(-1) = \nabla_L(-2\sqrt{-1}) - \nabla_M(-2\sqrt{-1}) = \sum_{i \geq 1} \left\{ a_{n+2i-2}(L) - a_{n+2i-2}(M) \right\} \cdot (-2\sqrt{-1})^{n+2i-2}. \]

This implies

\[ 0 \equiv \{ a_n(L) - a_n(M) \} \cdot 2^n \pmod{2^{n+1}} \]

and therefore $a_n(L) - a_n(M)$ must be even. \hfill \Box

Miyazawa showed that Theorem 2.6 is also best possible. Furthermore, Ohyama-Yamada proved that for an integer $n \geq 2$, if two oriented knots $J$ and $K$ are transformed into each other by a single $C_{2n}$-move then $a_{2n}(J) - a_{2n}(K) = 0, \pm 2$ \cite{23} Theorem 1.3].

**3. Proof of Theorem 1.3**

We show three lemmas needed to prove the Theorem 1.3. The first lemma is Kanenobu’s formula for the difference of Jones polynomials for two oriented links which are transformed into each other by a single $C_n$-move. Let $L$ and $M$ be two oriented links which are transformed into each other by a single $C_n$-move as illustrated in Fig. 12. Let $c_{j1}, c_{j2}$ ($j = 2, 3, \ldots, n$) and $c_1$ be crossings of $L$ as illustrated in Fig. 5.1. We denote the sign of $c_1$ by $\varepsilon_1$ and the sign of $c_{j1}$ by $\varepsilon_j$ ($j = 2, 3, \ldots, n$). Let $L[\delta_2, \delta_3, \ldots, \delta_n]$ be the link obtained from $L$ by smoothing the crossing $c_1$, smoothing the crossing $c_{j1}$ if $\delta_j = 1$, and changing the crossing $c_{j1}$ and smoothing the crossing $c_{j2}$ if $\delta_j = -1$ ($j = 2, 3, \ldots, n$). Then the following formula holds.

**Lemma 3.1.** (Kanenobu \cite{11} (4.10))

\[ V_L(t) - V_M(t) = \left( \prod_{i=1}^n \varepsilon_i \right) t^{\sum_{i=1}^n \varepsilon_i - \frac{n}{2}} (t - 1)^n \sum_{\delta_2, \delta_3, \ldots, \delta_n = \pm 1} \left( \prod_{j=2}^n \delta_j \right) V_{L[\delta_2, \delta_3, \ldots, \delta_n]}(t). \]
Next we show the second lemma. For an integer \( n \geq 3 \), let \( L'_n \) and \( M'_n \) be two links as illustrated in Fig. 3.2, where \( T \) is an arbitrary 2-string tangle which are same for both links. Note that \( L'_n \) and \( M'_n \) are transformed into each other by a single \( C_n \)-move. Then we have the following.

**Lemma 3.2.** \( L'_n \) and \( M'_n \) are ambient isotopic.

**Proof.** See Fig. 3.3.

![Figure 3.2. Two links \( L'_n \) and \( M'_n \) \((n \geq 3)\)](image)

Lemma 3.2 gives a new example of a \( C_n \)-move which does not change the knot type. Such an example was first discovered by Ohyama-Tsukamoto [22].

The third lemma is a calculation of the Jones polynomial for a \((2, -m)\)-torus knot or link \( N_m \) for a non-negative integer \( m \) as illustrated in Fig. 3.6 (1). Note that such a calculation has been already known, see [14, pp. 37], [16, Lemma 2.1] for example. However we state a formula and give a proof for reader’s convenience.

**Lemma 3.3.**

\[
V_{N_m}(t) = \left(-t^{-\frac{1}{2}}\right)^m \left(-t^\frac{1}{2} - t^{-\frac{1}{2}}\right) + \frac{\left(-t^{\frac{1}{2} - \frac{3m}{2}}\right)\{1 - (-t)^m\}}{1 + t}.
\]

**Proof.** Note that \( N_0 \) is the trivial 2-component link and \( N_1 \) is the trivial knot. Then we can check the formula directly for \( m = 0, 1 \). Assume that \( m \geq 2 \). Then
Figure 3.3. $L'_n$ and $M'_n$ are ambient isotopic

we obtain the skein triple $(N_{m-2}, N_m, N_{m-1})$ easily and therefore we have

\[(3.1) \quad t^{-1}V_{N_{m-2}}(t) - tV_{N_m}(t) = \left(t^{1/2} - t^{-1/2}\right)V_{N_{m-1}}(t).\]
By (3.1), we have

\begin{equation}
(3.2) \quad V_{N_{m}}(t) + t^{-\frac{1}{2}}V_{N_{m-1}}(t) = t^{-\frac{1}{2}}\left\{ V_{N_{m-1}}(t) + t^{-\frac{1}{2}}V_{N_{m-2}}(t) \right\} = \left( t^{-\frac{1}{2}} \right)^{m-1} \{ V_{N_{1}}(t) + t^{-\frac{1}{2}}V_{N_{0}}(t) \} = -t^{1-\frac{3m}{2}}.
\end{equation}

Then by (3.2), we have

\begin{align*}
V_{N_{m}}(t) &= -t^{-\frac{1}{2}}V_{N_{m-1}}(t) - t^{1-\frac{3m}{2}} \\
&= \left( -t^{-\frac{1}{2}} \right)^{m} V_{N_{0}}(t) + \sum_{i=1}^{m} \left( -t^{1-\frac{3m}{2}} \right)^{i-1} \\
&= \left( -t^{-\frac{1}{2}} \right)^{m} \left( -t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right) + \frac{\left( -t^{1-\frac{3m}{2}} \right)^{m} \{ 1 - (t)^{m} \}}{1 + t}.
\end{align*}

\[ \square \]

**Proof of Theorem 3.3** First we show in the case of \( n = 3 \) and 4. If \( n = 3 \), by a calculation (with the help of [15]) we have

\begin{align*}
V_{J_{3}}(t) &= t^{-1} - 2 + 4t - 4t^{2} + 5t^{3} - 5t^{4} + 3t^{5} - 2t^{6} + 7t^{7}, \\
V_{K_{3}}(t) &= t^{-1} - 1 + 2t - 2t^{2} + 2t^{3} - 2t^{4} + 5t.
\end{align*}

Then we have

\[ V_{J_{3}}(t) - V_{K_{3}}(t) = (t - 1)^{3} \left( t^{2} + t + 1 \right) \left( t^{2} + 1 \right). \]

If \( n = 4 \), by a calculation we have

\begin{align*}
V_{J_{4}}(t) &= -t^{-2} + 4t^{-1} - 8 + 13t - 15t^{2} + 17t^{3} - 16t^{4} + 12t^{5} - 8t^{6} + 4t^{7} - t^{8}, \\
V_{K_{4}}(t) &= -t^{-2} + 4t^{-1} - 7 + 10t - 11t^{2} + 12t^{3} - 10t^{4} + 7t^{5} - 4t^{6} + t^{7}.
\end{align*}

Then we have

\[ V_{J_{4}}(t) - V_{K_{4}}(t) = -(t - 1)^{4} \left( t^{2} + t + 1 \right) \left( t^{2} + 1 \right). \]

From now on, we assume that \( n \geq 5 \). Since \( \epsilon_{i} = 1 \) for any \( i \), we have

\begin{equation}
(3.3) \quad V_{J_{n}}(t) - V_{K_{n}}(t) = t^{\frac{2}{2}} (t - 1)^{n} \sum_{\delta_{2}, \ldots, \delta_{n} = \pm 1} \left( \prod_{j=2}^{n} \delta_{j} \right) V_{J_{n}}[\delta_{2}, \ldots, \delta_{n}](t).
\end{equation}

If \( \delta_{2} = -1 \), we can see that \( J_{n} [-1, \delta_{3}, \ldots, \delta_{n-1}, 1] \) and \( J_{n} [-1, \delta_{3}, \ldots, \delta_{n-1}, -1] \) are ambient isotopic, see Fig. 3.4. Thus by (3.3), we have

\begin{equation}
(3.4) \quad V_{J_{n}}(t) - V_{K_{n}}(t) = t^{\frac{2}{2}} (t - 1)^{n} \sum_{\delta_{3}, \ldots, \delta_{n} = \pm 1} \left( \prod_{j=3}^{n} \delta_{j} \right) V_{J_{n}}[1, \delta_{3}, \ldots, \delta_{n}](t).
\end{equation}

Let \( k \) be an integer satisfying \( 3 \leq k \leq n - 2 \). Note that \( k \) is also satisfied with \( 3 \leq n - k + 1 \leq n - 2 \). Then we can see that \( J_{n} [1, \ldots, 1, -1, \delta_{k+1}, \ldots, \delta_{n}] \) is ambient isotopic to \( L_{n-k+1}^{1} [\delta_{k+1}, \ldots, \delta_{n}] \) for some 2-string tangle \( T_{k} \), see Fig. 3.5, where
\[ \delta_{n+1} \cdots \delta_2 \delta_1 = \delta_3 \delta_{n-1} \delta_n \]

Figure 3.4. \( J_n [-1, \delta_3, \ldots, \delta_{n-1}, 1] \) and \( J_n [-1, \delta_3, \ldots, \delta_{n-1}, -1] \) are ambient isotopic

\[ L'_{n-k+1} \text{ and } M'_{n-k+1} \] are corresponding knots as illustrated in Fig. 3.2. Then by Lemma 3.2, we have \( L'_{n-k+1} \text{ and } M'_{n-k+1} \) are ambient isotopic and therefore

\[ \sum_{\delta_{k+1}, \ldots, \delta_n = \pm 1} \left( \prod_{j=k+1}^{n} \delta_j \right) V_{J_n[1, \ldots, 1, -1, \delta_{k+1}, \ldots, \delta_n]}(t) \]

\[ = \left\{ V_{L'_{n-k+1}}(t) - V_{M'_{n-k+1}}(t) \right\} \bigg/ \left( - \prod_{i=k+1}^{n} \varepsilon_i \right) t^{-1 + \sum_{i=k+1}^{n} \varepsilon_i - \frac{1}{2} (n-k)} (t - 1)^{n-k} \]

\[ = 0. \]

Thus by (3.4) and (3.5), we have

\[ (3.6) \quad V_J(t) - V_K(t) = t^{\frac{3}{2}} (t - 1)^n \sum_{\delta_{n-1}, \delta_n = \pm 1} \delta_{n-1} \delta_n V_{J_n[1, \ldots, 1, -1, \delta_{n-1}, \delta_n]}(t). \]

\[ \delta_{n+1} \cdots \delta_2 \delta_1 = \delta_3 \delta_{n-1} \delta_n \]

\[ T_k \]

Figure 3.5. \( J_n[1, \ldots, 1, -1, \delta_{k+1}, \ldots, \delta_n] \) is ambient isotopic to \( L'_{n-k+1}[\delta_{k+1}, \ldots, \delta_n] \) for some 2-string tangle \( T_k \)

We can see easily that \( J_n[1, \ldots, 1, -1, -1] \) is ambient isotopic to \( N_{n-3} \), \( J_n[1, \ldots, 1, -1, 1] \) is ambient isotopic to the split union of \( N_{n-4} \) and the trivial knot, and \( J_n[1, \ldots, 1, -1] \) is ambient isotopic to the connected sum of \( N_{n-3} \), the Hopf link with linking number
1 and the Hopf link with linking number $-1$. Thus we have

\begin{align*}
(3.7) \quad V_{J_n[1, \ldots, 1, -1]}(t) &= V_{N_{n-3}}(t), \\
(3.8) \quad V_{J_n[1, \ldots, 1, -1, 1]}(t) &= (-t^{\frac{1}{2}} - t^{-\frac{1}{2}}) V_{N_{n-4}}(t), \\
(3.9) \quad V_{J_n[1, \ldots, 1, -1, 1, 1]}(t) &= (-t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right) V_{N_{n-3}}(t).
\end{align*}

Further, $J_n[1, \ldots, 1]$ is ambient isotopic to the oriented link as illustrated in Fig. 3.6 (2), where $m = n - 4$. We obtain the skein triple $(N_{n-3}, J_n[1, \ldots, 1], N_{n-4})$ by changing and smoothing the marked crossing in Fig. 3.6. Thus we have

\begin{equation}
(3.10) \quad V_{J_n[1, \ldots, 1]}(t) = t^{-2} V_{N_{n-3}}(t) - t^{-1} \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right) V_{N_{n-4}}(t).
\end{equation}

By combining with (3.6), (3.7), (3.8), (3.9) and (3.10), we have

\begin{equation}
(3.11) \quad V_{J_n}(t) - V_{K_n}(t) = \left(t^{\frac{1}{2}} (t - 1)^n \left(-1 - t^2 \right) V_{N_{n-3}}(t) + \left(t^{\frac{1}{2}} + t^{-\frac{1}{2}} \right) V_{N_{n-4}}(t) \right) + \left(t^{\frac{1}{2}} (t - 1)^n (1 + t^2) \right) \left(-V_{N_{n-3}}(t) + t^{-\frac{1}{2}} V_{N_{n-4}}(t) \right).
\end{equation}

Here, by Lemma 3.3 we also have

\begin{equation}
(3.12) \quad -V_{N_{n-3}}(t) + t^{-\frac{1}{2}} V_{N_{n-4}}(t) = (-1)^{n-3} t^{-\frac{1}{2}} \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right) - \frac{(-1)^{10} t^{\frac{3n}{2}}}{1 + t} \left(1 - (-t)^{n-3} \right) + \frac{(-1)^{10} t^{\frac{3n}{2}}}{1 + t} \left(1 - (-t)^{n-4} \right) + \frac{(-1)^{n-4} t^{\frac{2n}{2}} + (-1)^{n-6} t^{\frac{2n}{2}}}{1 + t} + (-1)^{n+1} t^{\frac{n}{2}} (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) t^{\frac{1}{2}} + t^{-\frac{1}{2}}.
\end{equation}

By (3.11) and (3.12), we have the desired conclusion. □

\begin{figure}[h]
\centering
\begin{tabular}{cc}
\includegraphics[width=0.4\textwidth]{figure1.png} & \includegraphics[width=0.4\textwidth]{figure2.png} \\
\textbf{(1)} & \textbf{(2)}
\end{tabular}
\caption{Figure 3.6. (1) $(2, -m)$-torus knot or link $N_m$. (2) A link ambient isotopic to $J_n[1, \ldots, 1, 1]$ if $m = n - 4$}
\end{figure}
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