Research Article

Xinfu Li*

Standing waves to upper critical Choquard equation with a local perturbation: Multiplicity, qualitative properties and stability

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Abstract: In this article, we consider the upper critical Choquard equation with a local perturbation

\[ -\Delta u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u + \mu |u|^{q-2}u, \quad u \in \mathbb{R}^N, \]

where \( N \geq 3 \), \( \mu > 0, \alpha > 0, \lambda \in \mathbb{R}, \alpha \in (0, N) \), \( p = p = \frac{N+\alpha}{N-2} \), \( q \in \left( 2, 2 + \frac{4}{N} \right) \), and \( I_\alpha = \frac{C}{|x|^N} \) with \( C > 0 \). When \( \mu a^{1-\gamma} \leq (2K)^{q-2p} \) with \( \gamma = \frac{N}{2} - \frac{N-\alpha}{q} \) and \( K \) being some positive constant, we prove

1. Existence and orbital stability of the ground states.
2. Existence, positivity, radial symmetry, exponential decay, and orbital instability of the “second class” solutions.

This article generalized and improved parts of the results obtained for the Schrödinger equation.

Keywords: normalized solutions, multiplicity, symmetry, stability, upper critical exponent

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1 Introduction and main results

In this article, we study standing waves of prescribed mass to the Choquard equation with a local perturbation

\[ i \partial_t \psi + \Delta \psi + (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi + \mu |\psi|^{q-2}\psi = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \]  

(1.1)

where \( N \geq 3, \psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, \mu > 0, \alpha \in (0, N), I_\alpha \) is the Riesz potential defined for every \( x \in \mathbb{R}^N \setminus \{0\} \) by

\[ I_\alpha(x) = \frac{A_\alpha(N)}{|x|^{N-\alpha}}, \quad A_\alpha(N) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{q-2p}{2}\right)^{N/2} a^2}, \]  

(1.2)

with \( \Gamma \) denoting the Gamma function (see [1], p. 19), \( p \) and \( q \) will be defined later.

The equation (1.1) has several physical origins. When \( N = 3, p = 2, \alpha = 2, \) and \( \mu = 0 \), (1.1) was investigated by Pekar in [2] to study the quantum theory of a polaron at rest. In [3], Choquard applied it as an approximation to Hartree-Fock theory of one component plasma. It also arises in multiple particle systems [4] and quantum mechanics [5]. When \( p = 2 \), equation (1.1) reduces to the well-known Hartree equation.

* Corresponding author: Xinfu Li, School of Science, Tianjin University of Commerce, Tianjin 300134, People’s Republic of China, e-mail: lxylxf@tjcu.edu.cn

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The Choquard equation (1.1) with or without a local perturbation has attracted much attention nowadays, see [6–11] for the local existence, global existence, blow up, and more in general dynamical properties.

Standing waves to (1.1) are solutions of the form \( \psi(t, x) = e^{-it}u(x) \), where \( \lambda \in \mathbb{R} \) and \( u : \mathbb{R}^N \to \mathbb{C} \). Then \( u \) satisfies the equation

\[
- \Delta u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u + \mu |u|^{q-2}u, \quad x \in \mathbb{R}^N.
\] (1.3)

When looking for solutions to (1.3) one choice is to fix \( \lambda < 0 \) and to search for solutions to (1.3) as critical points of the action functional

\[
J(u) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla u|^2 - \frac{\lambda}{2} |u|^2 - \frac{1}{2p} (I_\alpha * |u|^p)|u|^p - \frac{\mu}{q} |u|^q \right) dx,
\]

see for example [12–16] and references therein. Another choice is to fix the \( L^2 \)-norm of the unknown \( u \), that is, to consider the problem

\[
\begin{cases}
- \Delta u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u + \mu |u|^{q-2}u, \quad x \in \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |u|^2 = a
\end{cases}
\] (1.4)

with fixed \( a > 0 \) and unknown \( \lambda \in \mathbb{R} \). In this direction, define on \( H^1(\mathbb{R}^N) \) the energy functional

\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q.
\]

It is standard to check that \( E \in C^1 \) under some assumptions on \( p \) and \( q \), and a critical point of \( E \) constrained to

\[
S_a = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 = a \right\}
\]

gives rise to a solution to (1.4). Such solution is usually called a normalized solution of (1.3) on \( S_a \), which is the aim of this article.

For future reference, we recall.

**Definition 1.1.** We say that \( u \) is a normalized ground state to (1.3) on \( S_a \) if

\[
E(u) = c_a^e = \inf_{v \in S_a} \left\{ E(v) : v \in S_a, \quad \left( E|_{S_a} \right)'(v) = 0 \right\}.
\]

The set of the normalized ground states will be denoted by \( G_a \).

**Definition 1.2.** \( G_a \) is orbitally stable if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that, for any \( \psi_0 \in H^1(\mathbb{R}^N) \) with \( \inf_{v \in G_a} \| \psi_0 - v \|_{H^1} < \delta \), we have

\[
\inf_{v \in G_a} \| \psi(t, \cdot) - v \|_{H^1} < \epsilon \quad \text{for any } t > 0,
\]

where \( \psi(t, x) \) denotes the solution to (1.1) with initial value \( \psi_0 \).

A standing wave \( e^{-it}u \) is strongly unstable if for every \( \epsilon > 0 \) there exists \( \psi_0 \in H^1(\mathbb{R}^N) \) such that \( \| \psi_0 - u \|_{H^1} < \epsilon \) and \( \psi(t, x) \) blows up in finite time.

When studying normalized solutions to the Choquard equation

\[
- \Delta u = \lambda u + (I_\alpha * |u|^p)|u|^{p-2}u, \quad x \in \mathbb{R}^N,
\] (1.5)
the $L^2$-critical exponent $p^* = 1 + \frac{2 + a}{N - 2}$, the Hardy-Littlewood-Sobolev upper critical exponent $\bar{p} = \frac{N + a}{N - 2}$, and lower critical exponent $p = \frac{N + a}{N}$ play an important role. For $p < p^*$, the existence of normalized ground state to (1.5) was studied by Cazenave and Lions [17] and Ye [18] by considering the minimizer of $E$ constrained on $S_a$. Cazenave and Lions [17] also studied the orbital stability of the normalized ground states set by using the concentration compactness principle. For $p^* < p < \bar{p}$, the functional $E$ is no longer bounded from below on $S_a$. By considering the minimizer of $E$ constrained on the Pohožaev set, Luo [19] obtained the existence and instability of normalized ground state to (1.5). For $p = \bar{p}$, by scaling invariance, the result is delicate, see [17] and [18] for details. See [20–22] for studies of Choquard equation with general nonlinearity. For (1.5) with $p = \bar{p}$, Moroz and Van Schaftingen [23] showed that (1.5) has no solutions in $H^1(\mathbb{R}^N)$ for fixed $\lambda < 0$. While Gao and Yang [24] obtained the solution to the equation

$$-\Delta u = \left( I_a * |u|^\bar{p} \right) |u|^\bar{p} - 2 u, \quad x \in \mathbb{R}^N \tag{1.6}$$

in $D^{1,2}(\mathbb{R}^N)$. So it is interesting to study the normalized solutions to (1.5) with $p = \bar{p}$ under a local perturbation $\mu |u|^q u$, namely equation (1.4). In a recent article, Li [25] considered the existence and symmetry of solutions to (1.4) with $p = \bar{p}$ and $2 + \frac{a}{N} \leq q < 2 + \frac{4}{N}$. Note that $2 + \frac{a}{N}$ is the $L^2$-critical exponent in studying normalized solutions to the Schrödinger equation

$$-\Delta u = \lambda u + |u|^q u, \quad x \in \mathbb{R}^N. \tag{1.7}$$

We should point out that Liu and Shi [10] studied the existence and orbital stability of ground states to (1.4) with $p = p^*$ and $2 < q < 2 + \frac{4}{N}$. Before stating the main results of this article, we make some notations. In the following, we assume $p = \bar{p} = \frac{N + a}{N - 2}$ in (1.4). Set

$$S_a := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \left( \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{\int_{\mathbb{R}^N} \left( I_a * |u|^\bar{p} \right) |u|^\bar{p} \right)^{1/\bar{p}}, \tag{1.8}$$

$$K = \frac{2 \bar{p} - q \gamma_q}{2 \bar{p} \gamma_q} \left( \frac{\bar{p} (2 - q \gamma_q) C_{N, q}^a \bar{p}^2}{q (\bar{p} - 1)} \right)^{\frac{3 - 2 \bar{p}}{N - q \gamma_q}} \tag{1.9}$$

with $\gamma_q = \frac{N}{2} - \frac{N}{q}$ and $C_{N, q}$ defined in Lemma 2.1,

$$\rho_0 = \left( \frac{\bar{p} (2 - q \gamma_q) S_a}{2 \bar{p} - q \gamma_q} \right)^{\frac{2}{N - q \gamma_q}}, \tag{1.10}$$

$$B_{\rho_0} = \{ u \in H^1(\mathbb{R}^N) : ||\nabla u||^2 < \rho_0 \}, \quad V_a = S_a \cap B_{\rho_0}, \quad m_a = \inf_{u \in V_a} E(u).$$

Now we state the first two main results of this article.

**Theorem 1.3.** Let $N \geq 3$, $\alpha \in (0, N)$, $2 < q < 2 + \frac{4}{N}$, $p = \bar{p}$, $\mu > 0$, $\alpha > 0$, $\mu a^{\frac{\alpha}{\alpha + 1}} \leq (2K)^{\frac{2}{\alpha + 1}}$. Then

1. $E_{S_a}$ has a critical point $\bar{u}$ at negative level $m_a < 0$, which is an interior local minimizer of $E$ on the set $V_a$.

2. $m_a = c^a_\alpha$ (that is, $\bar{u}$ is a ground state to (1.4)), and any other ground state to (1.4) is a local minimizer of $E$ on $V_a$.

3. $G_a$ is compact, up to translation.

4. $c^a_\alpha$ is reached by a positive and radially symmetric non-increasing function.

5. For any $u \in G_a$, there exists $\lambda < 0$ such that $u$ satisfies (1.4).
Theorem 1.4. Let the assumptions in Theorem 1.3 hold, \( \alpha \geq N - 4 \) (i.e., \( \bar{p} \geq 2 \)) and \( \alpha < N - 2 \). Then the set \( \mathcal{G}_a \) is orbitally stable.

To prove Theorems 1.3 and 1.4, we follow the strategy of [26]. In the proofs, a special role will be played by the Pohožaev set

\[ \mathcal{P}_a = \{ u \in \mathcal{S}_a : P(u) = 0 \}, \]

where

\[ P(u) = \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} (I_\alpha \ast |u|^\beta)|u|^\beta - \mu \gamma \int_{\mathbb{R}^N} |u|^\gamma. \]

The set \( \mathcal{P}_a \) is quite related to the fiber map

\[ \Psi_a(x) = E(u) = \frac{1}{2} \tau^2 |\nabla u|^2_2 - \frac{1}{2p} \tau^\alpha \int_{\mathbb{R}^N} (I_\alpha \ast |u|^\beta)|u|^\beta - \frac{\mu}{q} \tau^\gamma \|u\|^\gamma_\gamma, \quad (1.11) \]

where

\[ u_\epsilon(x) = \tau^\epsilon u(\tau x), \quad x \in \mathbb{R}^N, \quad \epsilon > 0. \]

The fiber map \( \Psi_a(x) \) is introduced by Jeanjean in [30] for the Schrödinger equation and is well studied by Soave in [31]. According to \( \Psi_a(x) \), \( \mathcal{P}_a = \mathcal{P}_{a,+} \cup \mathcal{P}_{a,-} \), where

\[ \mathcal{P}_{a,+} = \{ u \in \mathcal{P}_a : E(u) < 0 \}, \quad \mathcal{P}_{a,-} = \{ u \in \mathcal{P}_a : E(u) > 0 \} \]

if \( \mu a^{-\gamma} \leq (2K)^{\frac{N+\alpha}{N-a}} \), see Lemmas 3.3 and 4.4.

The ground state \( u_* \) obtained in Theorem 1.3 lies on \( \mathcal{P}_{a,+} \) and can be characterized by

\[ E(u_*) = \inf_{u \in \mathcal{P}_{a,+}} E(u) = \inf_{u \in \mathcal{P}_a} E(u) = m_a. \]

The critical point \( u_* \) which will be obtained in the following theorem lies on \( \mathcal{P}_{a,-} \) and can be characterized by

\[ E(u_*) = \inf_{u \in \mathcal{P}_{a,-}} E(u) = \inf_{u \in \mathcal{P}_a} E(u). \]

Precisely,

Theorem 1.5. Let \( N \geq 3, \alpha \in (0, N) \), \( 2 < q < 2 + \frac{4}{N} \), \( p = \bar{p}, \mu > 0, \alpha > 0, \mu a^{-\gamma} \leq (2K)^{\frac{N+\alpha}{N-a}} \). Then there exists a second solution \( u_* \) to (1.4) which satisfies

\[ 0 < E(u_*) = \inf_{u \in \mathcal{P}_{a,-}} E(u) < m_a + \frac{2 + \alpha}{2(N + \alpha)} \frac{N+\alpha}{N-a}. \]

In particular, \( u_* \) is not a ground state.

Remark 1.6. Note that the result is new in the case \( \mu a^{-\gamma} = (2K)^{\frac{N+\alpha}{N-a}} \). There is not corresponding result even to the Schrödinger equation (1.7). During the proof, the lower bound of \( \inf_{u \in \mathcal{P}_a} E(u) \) obtained in Lemma 4.4 plays an important role. The proof of Lemma 4.4 is interesting. Maybe it gives us some insights to consider the case \( \mu a^{-\gamma} > (2K)^{\frac{N+\alpha}{N-a}} \).

We combine the methods used in [27] and [29] to prove Theorem 1.5. Precisely, we first use the mountain pass lemma to obtain a Palais-Smith sequence \( \{ u_n \} \) of \( E \) on \( \mathcal{S}_a \cap H^1(\mathbb{R}^N) \) with \( P(u_n) \to 0 \) and \( E(u_n) \to M(a) \) as \( n \to \infty \), see Lemma 4.1. Second, by using the Pohožaev constraint method and the Schwartz rearrangement, we can show that

\[ M(a) = M(a) = \inf_{u \in \mathcal{P}_{a,-}} E(u) = \inf_{u \in \mathcal{P}_a} E(u), \]

see Lemma 4.2. Third, by using the radial symmetry of \( u_n \) and the bounds of \( \inf_{u \in \mathcal{P}_a} E(u) \), we can show that \( \{ u_n \} \) converges to a solution to (1.4). In the proof, to obtain the upper bound of \( \inf_{u \in \mathcal{P}_a} E(u) \) is a difficult task.
When $N \geq 5$ and $\bar{p} < 2$, the methods used in [29] cannot threat the nonlocal term $(I_\alpha * |u|^{\bar{p}})u|^{\bar{p}-2}u$ directly, see Lemma 4.5. Inspired by [27], by using the radially non-increasing of $u$, and by calculating the nonlocal term carefully (see (4.34)), we can choose $|y_i|$ satisfying (4.30) and (4.31). Based on which, we can give the upper bound of $\inf_{v \in \mathcal{P}_N} E(u)$ when $N \geq 5$ and $\bar{p} < 2$, see Lemma 4.6.

The following result is about the positivity, radial symmetry, and exponential decay of the “second class” solution.

**Theorem 1.7.** Assume the conditions in Theorem 1.5 hold. Let $u$ be a solution to (1.4) with $E(u) = \inf_{v \in \mathcal{P}_N} E(v)$, then

1. $|u| > 0$;
2. There exist $x_0 \in \mathbb{R}^N$ and a non-increasing positive function $v : (0, \infty) \to \mathbb{R}$ such that $|u(x)| = v(|x - x_0|)$ for almost every $x \in \mathbb{R}^N$;
3. If $\alpha \geq N - 4$ (i.e., $\bar{p} \geq 2$), then $|u|$ has exponential decay at infinity:

$$|u(x)| \leq Ce^{-\delta|x|}, \quad |x| \geq r_0,$$

for some $C > 0$, $\delta > 0$, and $r_0 > 0$.

The positivity is obtained by using the properties of $\Psi(\tau)$ and $\inf_{v \in \mathcal{P}_N} E(v)$. The symmetry is obtained by using the theories of polarization and the fact that $\inf_{v \in \mathcal{P}_N} E(v)$ is a mountain pass level value. This method is motivated by [32]. The exponential decay follows the radial symmetry, the estimate of $(I_\alpha * |u|^{\bar{p}})$, and the exponential decay studied in [33] to the Schrödinger equation. Theorem 1.7 plays an important role in proving the following result.

Theorem 1.8 is about the instability of the “second class” solution, which is very new in the existing research. As we know, most existing results are about the instability of a solution but not all solutions.

**Theorem 1.8.** Assume the conditions in Theorem 1.4 hold. Let $u$ be a solution to (1.4) with $E(u) = \inf_{v \in \mathcal{P}_N} E(v)$, then $\lambda < 0$ and the associated standing wave $e^{-i\lambda t}u$ is strongly unstable.

**Remark 1.9.** The conditions $\alpha \geq N - 4$ (i.e., $\bar{p} \geq 2$) and $\alpha < N - 2$ in Theorems 1.4 and 1.8 are added for obtaining the local existence of solution to (1.1), see Lemma 6.6. The condition $\alpha \geq N - 4$ in Theorem 1.8 is also needed to prove the exponential decay of $u$ (see (3) in Theorem 1.7), which is used to show that $|x|u \in L^2(\mathbb{R}^N)$.

The condition $\bar{p} \geq 2$ is added since the nonlinearity $(I_\alpha * |u|^{\bar{p}})|u|^{\bar{p}-2}u$ is singular when $\bar{p} < 2$. We do not know whether it can be removed. While, the condition $\alpha < N - 2$ is added for technical reason, and we guess it can be removed.

This article is organized as follows. In Section 2, we cite some preliminaries. Sections 3–5 are devoted to the proofs of Theorems 1.3, 1.5, and 1.7, respectively. In Section 6, we first give a local existence result, and then prove Theorems 1.4 and 1.8.

**Notation:** In this article, it is understood that all functions, unless otherwise stated, are complex valued, but for simplicity we write $L^r(\mathbb{R}^N)$, $W^{1,r}(\mathbb{R}^N)$, $H^1(\mathbb{R}^N)$ $D^{1,2}(\mathbb{R}^N)$, etc. For $1 \leq r < \infty$, $L^r(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm $\|u\|_r^r = \int_{\mathbb{R}^N} |u|^r$. $W^{1,r}(\mathbb{R}^N)$ is the usual Sobolev space endowed with the norm $\|u\|_r^r = \|\nabla u\|_r^r + \|u\|_r$, $H^1(\mathbb{R}^N) = W^{1,2}(\mathbb{R}^N)$ and $\|u\|_2^2 = \|\nabla u\|_2^2$, $D^{1,2}(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$.

$H^1_0(\mathbb{R}^N)$ denotes the subspace of functions in $H^1(\mathbb{R}^N)$ which are radially symmetric with respect to zero. $S_{\alpha} = S_2 \cap H^1_0(\mathbb{R}^N), C_1, C_2, \ldots$ denote positive constants, whose values can change from line to line. The notation $A \leq B$ means that $A \leq CB$ for some constant $C > 0$. If $A \leq B \leq A$, we write $A \approx B$. 

2 Preliminaries

The following Gagliardo-Nirenberg inequality can be found in [34].

**Lemma 2.1.** Let $N \geq 1$ and $2 < p < 2^*$, then the following sharp Gagliardo-Nirenberg inequality holds for any $u \in H^1(\mathbb{R}^N)$, where the sharp constant $C_{N,p}$ is

$$
C_{N,p}^p = \frac{2p}{2N + (2 - N)p} \left( \frac{2N + (2 - N)p}{N(p - 2)} \right)^{\frac{N - 2}{2}} \frac{1}{\|Q_p\|_2^p}
$$

and $Q_p$ is the unique positive radial solution of equation

$$
-\Delta Q + Q = |Q|^{p-2}Q.
$$

The following well-known Hardy-Littlewood-Sobolev inequality can be found in [35].

**Lemma 2.2.** Let $N \geq 1$, $p$, $r > 1$, and $0 < \beta < N$ with $1/p + (N - \beta)/N + 1/r = 2$. Let $u \in L^p(\mathbb{R}^N)$ and $v \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(N, \beta, p)$, independent of $u$ and $v$, such that

$$
\left\| \int \frac{u(x)v(y)}{|x - y|^{N - \beta}} \, dx \right\| \leq C(N, \beta, p) \|u\|_p \|v\|_r.
$$

If $p = r = \frac{2N}{N + \beta}$, then

$$
C(N, \beta, p) = C_p(N) = \frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{N + \beta}{2}\right)} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N}{2} - \frac{\beta}{2}\right)} \frac{1}{\|Q_p\|^N}.
$$

**Remark 2.3.** (1). By the Hardy-Littlewood-Sobolev inequality above, for any $v \in L^s(\mathbb{R}^N)$ with $s \in (1, N/\alpha)$, $I_\alpha * v \in L^{Ns/\alpha}(\mathbb{R}^N)$ and

$$
\|I_\alpha * v\|_{L^{Ns/\alpha}} \leq C \|v\|_{L^s},
$$

where $C > 0$ is a constant depending only on $N$, $\alpha$, and $s$.

(2). By the Hardy-Littlewood-Sobolev inequality above and the Sobolev embedding theorem, we obtain

$$
\int (I_\beta * |u|^p)|u|^p \leq C \left( \int \frac{2p}{2N + (2 - N)p} \right)^{1 + \beta/N} \leq C \|u\|_{H^\beta(\mathbb{R}^N)}^{2p}
$$

for any $p \in [1 + \beta/N, (N + \beta)(N - 2)]$ if $N \geq 3$ and $p \in [1 + \beta/N, +\infty)$ if $N = 1, 2$, where $C > 0$ is a constant depending only on $N, \beta$, and $p$.

The following fact is used in this article (see [36]).

**Lemma 2.4.** Let $N \geq 3$, $\alpha \in (0, N)$, and $p \in \left[ \frac{N + \alpha}{N}, \frac{N + \alpha}{N - 2} \right]$. Assume that $\{w_n\}_{n=1}^{\infty} \subset H^1(\mathbb{R}^N)$ satisfying $w_n \rightharpoonup w$ weakly in $H^1(\mathbb{R}^N)$ as $n \to \infty$, then

$$(I_\alpha * |w_n|^p)|w_n|^{p-2}w_n \rightharpoonup (I_\alpha * |w|^p)|w|^{p-2}w$$

weakly in $H^1(\mathbb{R}^N)$ as $n \to \infty$.

The following lemma is used in this article, see [33] for its proof.
Lemma 2.5. Let $N \geq 3$ and $1 \leq t < +\infty$. If $u \in L^t(\mathbb{R}^N)$ is a radial non-increasing function (i.e., $0 \leq u(x) \leq u(y)$ if $|x| \geq |y|$), then one has
\[ |u(x)| \leq |x|^{-N/|t|} \left( \frac{N}{|S^{N-1}|} \right)^{1/|t|} \|u\|_t, \quad x \neq 0, \]
where $|S^{N-1}|$ is the area of the unit sphere in $\mathbb{R}^N$.

The following Pohožaev identity is cited from [13], where the proof is given for $\lambda > 0$ but it clearly extends to $\lambda \in \mathbb{R}$.

Lemma 2.6. Let $N \geq 3$, $\alpha \in (0, N)$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $p \in \left[ \frac{N + \alpha}{N}, \frac{N + 2 + \alpha}{N - 2} \right]$ and $q \in [2, 2']$. If $u \in H^1(\mathbb{R}^N)$ is a solution to (1.3), then $u$ satisfies the Pohožaev identity
\[ \frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 = \frac{N}{2} \int_{\mathbb{R}^N} |u|^2 + \frac{N + \alpha}{2p} \int_{\mathbb{R}^N} (I_\lambda * |u|^p)|u|^p + \frac{\mu N}{q} \int_{\mathbb{R}^N} |u|^q, \]
which combined with the Pohožaev identity from Lemma 2.6 gives that $P(u) = 0$.

Lemma 2.7. Assume the conditions in Lemma 2.6 hold. If $u \in H^1(\mathbb{R}^N)$ is a solution to (1.3), then $P(u) = 0$.

Proof. Multiplying (1.3) by $u$ and integrating over $\mathbb{R}^N$, we derive
\[ \int_{\mathbb{R}^N} |\nabla u|^2 = \lambda \int_{\mathbb{R}^N} |u|^2 + \int_{\mathbb{R}^N} (I_\lambda * |u|^p)|u|^p + \mu \int_{\mathbb{R}^N} |u|^q, \]
which combined with the Pohožaev identity from Lemma 2.6 gives that $P(u) = 0$. \qed

3 Existence of normalized ground state standing waves

In this section, we prove Theorem 1.3. We first study the lower bound of $E(u)$. By (1.8) and Lemma 2.1, we obtain, for any $u \in S_\alpha$,
\[ E(u) \geq \frac{1}{2} \|\nabla u\|^2 - \frac{1}{2p} \int_{\mathbb{R}^N} |\nabla u(x)|^2 \left( \sum_{j \neq \alpha} \theta_j \right) - \frac{\mu}{q} \left( \frac{q}{\alpha} \right)^{\frac{q}{2}} \int_{\mathbb{R}^N} |\nabla u|^q = \|\nabla u\|^2 f_{\mu, \alpha}(\|\nabla u\|^2) \]  
with
\[ f_{\mu, \alpha}(\rho) := \frac{1}{2} \left( 1 - \frac{1}{2p} \right) S^N_\alpha \rho^{p - 2} - \frac{\mu}{q} \left( \frac{q}{\alpha} \right)^{\frac{q}{2}} \rho - \frac{\mu}{q} \left( \frac{q}{\alpha} \right)^{\frac{q}{2}} \rho^{-\frac{q}{2}}, \quad \rho \in (0, \infty). \]

Next we study the properties of $f_{\mu, \alpha}(\rho)$.

Lemma 3.1. Let $N \geq 3$, $a \in (0, N)$, $\mu > 0$, $p > a$, $q \in \left( 2, 2 + \frac{a}{N} \right)$, and $K$ be defined in (1.9). Then
\[ \max_{\rho > 0} f_{\mu, \alpha}(\rho) \begin{cases} > 0, & \text{if } \mu a^{-\frac{a}{N}} < (2K)^{\frac{q}{2} - \frac{a}{N}}; \\ = 0, & \text{if } \mu a^{-\frac{a}{N}} = (2K)^{\frac{q}{2} - \frac{a}{N}}; \\ < 0, & \text{if } \mu a^{-\frac{a}{N}} > (2K)^{\frac{q}{2} - \frac{a}{N}}. \end{cases} \]

Proof. By the definition of $f_{\mu, \alpha}(\rho)$, we have that
\[ f'_{\mu, \alpha}(\rho) = -\frac{p - 1}{2p} S^N_\alpha \rho^{p - 2} - \frac{\mu}{q} \left( \frac{q}{\alpha} \right)^{\frac{q}{2}} \rho^{-\frac{q}{2}}. \]
Hence, the equation \( f_{\mu,a}(\rho) = 0 \) has a unique solution given by
\[
\rho_{\mu,a} = \left( \frac{p \mu(2 - q \rho_a)}{q(p - 1) a^\mu q_{\infty}^{\frac{q(1 - \mu)}}} \right)^{\frac{2}{2-q}}.
\] (3.4)

Taking into account that \( f_{\mu,a}(\rho) \to -\infty \) as \( \rho \to 0^+ \) and \( f_{\mu,a}(\rho) \to -\infty \) as \( \rho \to +\infty \), we obtain that \( \rho_{\mu,a} \) is the unique global maximum point of \( f_{\mu,a}(\rho) \) and the maximum value is
\[
\max_{\rho \geq 0} f_{\mu,a}(\rho_{\mu,a}) = f_{\mu,a}(\rho_{\mu,a}) = \frac{1}{2} - K_{\mu a} + \int_0^{\rho_{\mu,a}} \frac{q(1 - \mu)}{2} \rho^{1-\mu} d\rho,
\]
which implies that (3.3) holds.

**Lemma 3.2.** Let \( N \geq 3, a \in (0, N), \mu > 0, p = \bar{p}, \) and \( q \in (2, 2 + \frac{4}{N}) \). If \( a_1 > 0 \) and \( \rho_1 > 0 \) are such that \( f_{\mu,a}(\rho_1) \geq 0 \), then for any \( a_2 \in (0, a_1) \), we have
\[
f_{\mu,a}(\rho_2) > 0 \text{ for } \rho_2 \in \left( a_2^\mu \rho_1, \rho_1 \right).
\] (3.5)

**Proof.** It is obvious that \( f_{\mu,a}(\rho_1) > f_{\mu,a}(\rho_2) \geq 0 \), and by direct calculation,
\[
f_{\mu,a}(\rho_2) = \frac{1}{2} - \frac{1}{2p} S_a \bar{p} \rho_2^{1-\mu} - \frac{\mu \bar{p} q_{\infty}^{\frac{q(1-\mu)}}}{q_{\infty}^2} \rho_2^{1-\mu} = f_{\mu,a}(\rho_2) \geq 0.
\]
It follows from the properties of \( f_{\mu,a}(\rho) \) studied in Lemma 3.1 that (3.5) holds.

By Lemma 3.1, the domain \( \{(\mu, a) \in \mathbb{R}^2 : \mu > 0, a > 0\} \) is divided into three parts \( \Omega_1, \Omega_2, \) and \( \Omega_3 \) by the curve \( \mu a^\mu = (2K)^{\frac{q_{\infty}^{1-q}}{2q}} \) with
\[
\Omega_1 = \left\{(\mu, a) \in \mathbb{R}^2 : \mu > 0, a > 0, \mu a^\mu < (2K)^{\frac{q_{\infty}^{1-q}}{2q}}\right\},
\]
\[
\Omega_2 = \left\{(\mu, a) \in \mathbb{R}^2 : \mu > 0, a > 0, \mu a^\mu = (2K)^{\frac{q_{\infty}^{1-q}}{2q}}\right\},
\]
and
\[
\Omega_3 = \left\{(\mu, a) \in \mathbb{R}^2 : \mu > 0, a > 0, \mu a^\mu > (2K)^{\frac{q_{\infty}^{1-q}}{2q}}\right\}.
\]
In this article, we will consider the domain \( \Omega_1 \cup \Omega_2 \). For fixed \( \mu > 0 \), define \( a_0 \) such that
\[
\mu a_0^\mu = (2K)^{\frac{q_{\infty}^{1-q}}{2q}}.
\] (3.6)
Then \( \Omega_1 \cup \Omega_2 = \{(\mu, a) \in \mathbb{R}^2, \mu > 0, 0 < a \leq a_0\} \). Note that \( \rho_0 \) defined in (1.10) is \( \rho_{\mu,a,a} \), and by Lemmas 3.1 and 3.2, \( f_{\mu,a}(\rho_0) = 0 \) and \( f_{\mu,a}(\rho_0) > 0 \) for \( a \in (0, a_0) \). Hence, \( \inf_{u \in S_a} E(u) \geq 0 \). Moreover, \( V_a \) is a potential well, see Lemma 3.4.

For future use, we study the properties of \( \Psi_\lambda(\tau) \) defined in (1.11).

**Lemma 3.3.** Let \( N \geq 3, a \in (0, N), \mu > 0, p = \bar{p}, q \in (2, 2 + \frac{4}{N}) \), and \( a \in (0, a_0) \). Then for every \( u \in S_a \), the function \( \Psi_\lambda(\tau) \) has exactly two critical points \( \tau_u^\ast \) and \( \tau_u \) with \( 0 < \tau_u^\ast < \tau_u \). Moreover,
(1) \( \tau_u^\ast \) is a local minimum point for \( \Psi_\lambda(\tau) \), \( E(u_{\tau_u^\ast}) < 0 \), and \( u_{\tau_u^\ast} \in V_a \).
(2) \( \tau_u \) is a global maximum point for \( \Psi_\lambda(\tau) \), \( \Psi_\lambda(\tau) < 0 \) for \( \tau > \tau_u \), and
\[ E(u_{n}) \geq \inf_{u \in \mathcal{V}_{a}} E(u) \geq 0. \]

In particular, if \( a \in (0, a_0) \), then \( \inf_{u \in \mathcal{V}_{a}} E(u) > 0 \).

(3) \( \Psi_{a}^{\prime}(r_{a}) < 0 \) and the maps \( u \in S_{a} \mapsto r_{a} \in \mathbb{R} \) is of class \( C^{1} \).

**Proof.** The proof can be obtained by modifying the proof of ([27], Lemma 2.4) in a trivial way. So we omit it. \( \square \)

**Lemma 3.4.** Let \( N \geq 3 \), \( a \in (0, N) \), \( p = \bar{p} \), \( q \in \left( 2, 2 + \frac{4}{N} \right) \), \( \mu > 0 \), and \( a \in (0, a_0) \). Then

(1) \( m_{a} = \inf_{u \in \mathcal{V}_{a}} E(u) < 0 \leq \inf_{u \in \mathcal{V}_{a}} E(u) \).

(2) If \( m_{a} \) is reached, then any ground state to (1.4) is contained in \( V_{a} \).

**Proof.**

(1) In view of Lemma 3.3, we just need to prove \( \inf_{u \in \mathcal{V}_{a}} E(u) < 0 \). For any fixed \( u \in S_{a} \), let \( u_{x}(x) \) and \( \Psi(\tau) \) be defined in (1.12) and (1.11), respectively. It is obvious that \( \|V u_{x}\|_{2}^{2} \to 0 \) and \( E(u_{x}) = \Psi(\tau) \to 0^{-} \) as \( \tau \to 0^{-} \). Hence, we can choose \( \tau_{0} > 0 \) sufficiently small such that \( u_{\tau_{0}} \in V_{a} \) and \( E(u_{\tau_{0}}) < 0 \).

(2) Let \( u \in V_{a} \) be such that \( E(u) = m_{a} \). By (1), \( u \) is a solution to (1.4). Let \( v \) be any ground state to (1.4). Then \( E(v) \leq E(u) = m_{a} < 0 \), and by Lemma 2.7, \( P(v) = 0 \). Consequently, by Lemma 3.3, \( \tau_{v}^{*} = 1 \) and \( v = v_{e} \in V_{a} \). \( \square \)

**Lemma 3.5.** Let \( N \geq 3 \), \( a \in (0, N) \), \( p = \bar{p} \), \( q \in \left( 2, 2 + \frac{4}{N} \right) \), and \( \mu > 0 \). Then

(1) \( a \in (0, a_0) \mapsto m_{a} \) is a continuous mapping.

(2) Let \( a \in (0, a_0] \). We have for every \( a_{1} \in (0, a) : m_{a} \leq m_{a_{1}} + m_{a_{1} - a} \), and if \( m_{a_{1}} \) or \( m_{a_{1} - a} \) is reached, then the inequality is strict.

**Proof.** The proof can be done by modifying the proof of ([26], Lemma 2.6) in a trivial way. So we omit it. \( \square \)

The following result will both imply the existence of a ground state to (1.4) and will be a crucial step to derive the orbital stability of the set \( G_{a} \).

**Proposition 3.6.** Let \( N \geq 3 \), \( a \in (0, N) \), \( p = \bar{p} \), \( q \in \left( 2, 2 + \frac{4}{N} \right) \), \( \mu > 0 \), and \( a \in (0, a_0) \). If \( \{u_{n}\} \subset B_{p_{0}} \) is such that \( \|u_{n}\|_{2}^{2} \to a \) and \( E(u_{n}) \to m_{a} \), then, up to translation, \( u_{n} \) converges to \( u \in V_{a} \) strongly in \( H^{1}(\mathbb{R}^{N}) \).

**Proof.** Since \( \{u_{n}\} \subset B_{p_{0}} \) and \( \|u_{n}\|_{2}^{2} \to a \), we obtain that \( \{u_{n}\} \) is bounded in \( H^{1}(\mathbb{R}^{N}) \). We claim that

\[
\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^{N}} \int_{B_{r}(y)} |u_{n}(x)|^{2} \, dx > 0. \tag{3.7}
\]

If it is false, \( \|u_{n}\|_{2} \to 0 \) as \( n \to \infty \) by Lions' vanishing lemma, see ([37], Lemma 1.21). By using (1.8) and \( \{u_{n}\} \subset B_{p_{0}} \), we obtain that

\[
E(u_{n}) = \frac{1}{2} \|\nabla u_{n}\|_{2}^{2} - \frac{1}{2p} \int_{\mathbb{R}^{N}} \left( I_{a} \ast |u_{n}|^{p} \right) |u_{n}|^{p} + o_{n}(1)
\]

\[
\geq \frac{1}{2} \|\nabla u_{n}\|_{2}^{2} - \frac{1}{2p} S_{a}^{p} \|\nabla u_{n}\|_{2}^{2p} + o_{n}(1)
\]

\[
= \|\nabla u_{n}\|_{2}^{2} \left( \frac{1}{2} - \frac{1}{2p} S_{a}^{p} \|\nabla u_{n}\|_{2}^{2p-2} \right) + o_{n}(1)
\]

\[
\geq \|\nabla u_{n}\|_{2}^{2} \left( \frac{1}{2} - \frac{1}{2p} S_{a}^{p} \|\nabla u_{n}\|_{2}^{2p-1} \right) + o_{n}(1).
\]
Since \( f_{\mu, a_0}(\rho_0) = 0 \), we have that
\[
\frac{1}{2} - \frac{1}{2p} S_{\rho_0} \rho_0^{-p} = \frac{H C_{\delta, \rho_0}^{q(1-\gamma)}}{q \rho_0^{2-2}} > 0.
\]
Consequently, \( E(u_n) \geq c_n(1) \), which contradicts \( E(u_n) \to m_a < 0 \).

So (3.7) holds. Going if necessary to a subsequence, there exists a sequence \( y_n \) such that,
\[
\begin{align*}
&u_n \rightharpoonup \tilde{u} \quad \text{weakly in } H^1(\mathbb{R}^N), \\
&u_n \to \tilde{u} \quad \text{in } L^q(\mathbb{R}^N), \\
&\nabla u_n \to \nabla \tilde{u} \quad \text{in } L^p(\mathbb{R}^N).
\end{align*}
\]

and
\[
\int_{\mathbb{R}^N} (|I_{a_n}*|u_n|^p|u_n|^p = \int_{\mathbb{R}^N} (|I_{a_n}*|\tilde{u}_n|^p|\tilde{u}_n|^p = \int_{\mathbb{R}^N} (|I_{a_n}*|u|^p|u|^p + \int_{\mathbb{R}^N} (|I_{a_n}*|v_n|^p|v_n|^p + o_n(1).\]

Consequently,
\[
E(u_n) = E(\tilde{u}_n) = E(u) + E(v_n) + o_n(1).
\]

Next, by repeating word by word the proof of Theorem 2.5 in [26], we can show that \( |v_n| \to 0 \) and \( |\nabla v_n| \to 0 \) as \( n \to \infty \). Thus, \( \tilde{u}_n \to u \in V \) strongly in \( H^1(\mathbb{R}^N) \).

**Proof of Theorem 1.3.** (1), (2), and (3) follow from Proposition 3.6 and Lemma 3.4. To prove (4), we let \(|\bar{u}|^*\) denote the Schwartz rearrangement of \(|\bar{u}|\). Then
\[
\|\bar{u}\|^2_2 = \|\bar{u}\|^2_2 = a, \quad \|\nabla \bar{u}\|^2_2 = \|\nabla \bar{u}\|^2_2 = \|\nabla \bar{u}\|^2_2 < \rho_0, \quad \|\bar{u}\|^2_2 = \|\bar{u}\|^2_2,
\]

\[
\int_{\mathbb{R}^N} (|I_{a_n}*|\bar{u}|^p|\bar{u}|^p \geq \int_{\mathbb{R}^N} (|I_{a_n}*|\bar{u}|^p|\bar{u}|^p.
\]

These imply that \( |\bar{u}|^* \in V \) and \( E(|\bar{u}|^*) \leq E(\bar{u}) = m_a \). By the definition of \( m_a \), we know \( m_a \) is attained by the positive and radially symmetric non-increasing function \( |\bar{u}|^* \). Finally, we prove (5). By using the equation (1.4), \( P(u) = 0, 0 < \gamma < 1, \) and \( \mu > 0 \), we obtain
\[
\lambda a = \|\nabla u\|^2_2 - \int_{\mathbb{R}^N} (|I_{a_n}*|u|^p|u|^p - \mu |u|^q_2 = \mu(\gamma - 1)\|u\|^2_2 < 0,
\]

which implies \( \lambda < 0 \). The proof is complete.

**4 Existence of mountain pass-type normalized standing waves**

In this section, we prove Theorem 1.5. First, we use the mountain pass lemma to obtain a special Palais-Smale sequence. Now we set
\[
M_\lambda(a) = \inf_{g \in \Gamma(a) \in (0, \infty)} \max_{t \in \mathbb{R}} E(g(t)),
\]

where
\[
\Gamma(a) = \{ g \in C([0, \infty), S_{a(t)} : g(0) \in P_{a, t}, \exists t \text{ s.t. } g(t) \in E_m \text{ for } t \geq t \}
\]

with \( E_c = \{ u \in H^1(\mathbb{R}^N) : E(u) < c \} \). Then we have
Lemma 4.1. Let \( N \geq 3, \alpha \in (0, N), p = \bar{p}, q \in \left(2, 2 + \frac{4}{N}\right), \mu > 0, \alpha > 0, \mu a^{\frac{q-1}{q}} \leq (2K)^{\frac{q-2}{4q-4}}. Then there exists a Palais-Smale sequence \( \{u_n\} \subset S_{a, r} \) for \( E_{\bar{u}} \) at level \( M_\alpha(a) \), with \( P(u_n) \to 0 \) as \( n \to \infty. \)

**Proof.** We follow the strategy introduced in [30] and consider the functional \( \tilde{E} : \mathbb{R}_+ \times H^1(\mathbb{R}^N) \to \mathbb{R} \) defined by

\[
\tilde{E}(\tau, u) = E(u) = \Psi_\tau(t).
\]

Define

\[
\tilde{M}_\alpha(a) = \inf_{\tilde{g} \in \tilde{E}(a) \cap \Omega} \max_{t \geq t_\tilde{g}} \tilde{E}(\tilde{g}(t)),
\]

where

\[
\tilde{E}(a) = \left\{ \tilde{g} \in \mathcal{C}(0, \infty), \mathbb{R}_+ \times S_{a, r} : \tilde{g}(0) \in V_a \cap E_0, \exists \tilde{g} \text{ s.t. } \tilde{g}(t) \in \left(1, E_{2m_a}\right), t \geq t_\tilde{g} \right\}.
\]

Similar to Lemma 3.3 in [27], we can show that \( \tilde{M}_\alpha(a) = M_\alpha(a) \) and

\[
\tilde{M}_\alpha(a) = \inf_{\tilde{g} \in \tilde{E}(a) \cap \Omega} \max_{t \geq t_\tilde{g}} \tilde{E}(\tilde{g}(t)) \geq \max\left\{ \tilde{E}(\tilde{g}(0)), \tilde{E}(\tilde{g}(t_\tilde{g})) \right\}.
\]

Then repeating word by word the proof of Proposition 1.10 in [27], we can obtain a Palais-Smale sequence \( \{u_n\} \subset S_{a, r} \) for \( E_{\bar{u}} \) at level \( M_\alpha(a) \), with \( P(u_n) \to 0 \) as \( n \to \infty. \) \( \square \)

Next we study the value of \( M_\alpha(a) \). For this aim, we set

\[
M(a) = \inf_{g \in \Gamma(a) \cap \Omega} \max_{t \geq t_g} E(g(t)),
\]

where

\[
\Gamma(a) = \left\{ g \in \mathcal{C}[0, \infty), S_a : g(0) \in V_a \cap E_0, \exists g \text{s.t. } g(t) \in E_{2m_a}, t \geq t_g \right\}. \tag{4.1}
\]

**Lemma 4.2.** Let \( N \geq 3, \alpha \in (0, N), p = \bar{p}, q \in \left(2, 2 + \frac{4}{N}\right), \mu > 0, \alpha > 0, \mu a^{\frac{q-1}{q}} \leq (2K)^{\frac{q-2}{4q-4}}. Then

\[
M_\alpha(a) = M(a) = \inf_{\tau_{u_{-}}} \inf_{g \in \Gamma}(a) \max_{t \geq t_g} E(g(t)) \geq \inf_{g \in \Gamma}(a) E(g(t_0)) \geq \inf_{u \in \mathcal{P}_{a, r}} E(u).
\]

**Proof.** Obviously, \( M_\alpha(a) \geq M(a) \).

For any \( g(t) \in \Gamma(a) \), since \( g(0) \in V_a, E(g(0)) < 0, \) and \( E(g(t_0)) < 2m_a < m_a \), by Lemma 3.3, we have \( \tau_{g(0)} > 1 \) and \( \tau_{g(t_0)} < 1 \). So by the continuity of \( g(t) \) and of \( u \mapsto \tau_u \), we know that there exists \( t_0 \) such that \( \tau_{g(t_0)} = 1 \), i.e., \( g(t_0) \in \mathcal{P}_{a, r} \). Thus,

\[
M(a) = \inf_{g \in \Gamma(a) \cap \Omega} \max_{t \geq t_g} E(g(t)) \geq \inf_{g \in \Gamma(a)} E(g(t_0)) \geq \inf_{u \in \mathcal{P}_{a, r}} E(u).
\]

For any \( u \in \mathcal{P}_{a, r} \), let \( |u|^* \) be the Schwartz rearrangement of \( |u| \). Since \( \|u^*\|_t = \|u\|_t \) with \( t \in [1, \infty) \), \( \|\nabla(|u|^*)\|_2 \leq \|\nabla u\|_2 \) and

\[
\int_{\mathbb{R}^N} \left(I_u + (|u|^*)^\beta\right)(|u|^*)^\beta \geq \int_{\mathbb{R}^N} \left(I_u + |u|^\beta\right)|u|^\beta,
\]

we obtain that \( \Psi_{|u|^*}(r) \leq \Psi_u(r) \) for any \( r \in [0, \infty) \). Let \( \tau_u \) be defined by Lemma 3.3 such that \( P(u_{\tau_u}) = 0 \). Then

\[
E(u) = \Psi_u(1) = \Psi_{|u|^*}(\tau_u) \geq \Psi_{|u|^*}(\tau_{|u|^*}) \geq \Psi_{|u|^*}(\tau_{|u|^*}).
\]

Since \( u_{\tau_u}^* \in \mathcal{P}_{a, r} \cap H^1_0(\mathbb{R}^N) \), we have that

\[
E(u) \geq \inf_{\mathcal{P}_{a, r}} E(u).
\]
By the arbitrariness of \( u \), we obtain that
\[
\inf_{\mathcal{F}_{\mu}} E(u) \geq \inf_{\mathcal{F}_{\mu} \cap \mathcal{H}_0^1(\mathbb{R}^k)} E(u). \tag{4.2}
\]

For any \( u \in \mathcal{F}_{\mu} \cap \mathcal{H}_0^1(\mathbb{R}^N) \), define
\[
g_a(t) = u_t + t' u_s,
\]
where \( t' \) is defined by Lemma 3.3. Then \( g_a(t) \in \Gamma(a) \) and
\[
E(u) = \max_{t \in (0, \infty)} E(g_a(t)) \geq M_a(a),
\]
which implies that \( \inf_{\mathcal{F}_{\mu} \cap \mathcal{H}_0^1(\mathbb{R}^N)} E(u) \geq M_a(a) \). The proof is complete. \( \square \)

When \( \mu \frac{q-1}{q} \leq (2K)^{\frac{q-2}{p-q}}, \) the lower bound of \( \inf_{u \in \mathcal{F}_{\mu}} E(u) \) is already studied in Lemma 3.3. For clarity, we restate it in the following lemma.

**Lemma 4.3.** Let \( N \geq 3, \ a \in (0, N), \ p = \bar{p}, \ 2 < q < 2 + \frac{a}{N}, \ \mu > 0, \ a > 0, \) and \( \mu \frac{q-1}{q} \leq (2K)^{\frac{q-2}{p-q}}. \) Then \( \inf_{u \in \mathcal{F}_{\mu}} E(u) > 0. \)

When \( \mu \frac{q-1}{q} > (2K)^{\frac{q-2}{p-q}}, \) by Lemma 3.3, \( \inf_{u \in \mathcal{F}_{\mu}} E(u) \geq 0. \) Now we prove that the strict inequality holds.

**Lemma 4.4.** Let \( N \geq 3, \ a \in (0, N), \ p = \bar{p}, \ 2 < q < 2 + \frac{a}{N}, \ \mu > 0, \ a > 0, \) and \( \mu \frac{q-1}{q} > (2K)^{\frac{q-2}{p-q}}. \) Then \( \inf_{u \in \mathcal{F}_{\mu}} E(u) > 0. \)

**Proof.** Suppose by contradiction that \( \inf_{u \in \mathcal{F}_{\mu}} E(u) = 0. \) Since by Lemma 4.2, \( \inf_{\mathcal{F}_{\mu}} E(u) = \inf_{\mathcal{F}_{\mu} \cap \mathcal{H}_0^1(\mathbb{R}^N)} E(u), \) there exists \( \{u_n\} \subset \mathcal{F}_{\mu} \cap \mathcal{H}_0^1(\mathbb{R}^N) \) such that \( P(u_n) = 0 \) and \( E(u_n) = A_n, \) where \( A_n \to 0 \) as \( n \to \infty. \) By using \( E(u_n) = A_n, \) \( P(u_n) = 0, \) \( \|u_n\|^2 = a, \) (1.8), and Lemma 2.1, we obtain that
\[
\begin{aligned}
\|\nabla u_n\|^2 &= \frac{2\bar{p} - q\bar{q}}{\bar{p} - 1} |u_n|^q + C_1 A_n \\
\|\nabla u_n\|^2 &= \frac{2\bar{p} - q\bar{q}}{\bar{p} - 1} \int_{\mathbb{R}^N} (I_a + |u_n|^\bar{p}) |u_n|^\bar{p} - C_2 A_n \leq \frac{2\bar{p} - q\bar{q}}{\bar{p} - 1} S_\bar{p}^\bar{p} \|\nabla u_n\|^2 + C_2 A_n,
\end{aligned}
\tag{4.3}
\]
where \( C_1 \) and \( C_2 \) are some positive constants. Consequently, \( \liminf_{n \to \infty} \|\nabla u_n\|^2 > 0 \) and
\[
\begin{aligned}
\|\nabla u_n\|^2 &\leq \left( \frac{2\bar{p} - q\bar{q}}{\bar{p} - 1} \right) \left( \frac{q^2 - 2}{2\bar{p} - q\bar{q}} \right) S_\bar{p}^\bar{p} + o_1(1), \\
\|\nabla u_n\|^2 &\leq \frac{\bar{p}(2 - q\bar{q})}{2\bar{p} - q\bar{q}} S_\bar{p}^\bar{p} + o_1(1),
\end{aligned}
\]
which implies that
\[
\begin{aligned}
\|\nabla u_n\|^2 &\leq \rho_0 + o_1(1), \\
\|\nabla u_n\|^2 &\geq \rho_0 + o_1(1).
\end{aligned}
\]
Hence, \( \|\nabla u_n\|^2 \to \rho_0 \) as \( n \to \infty, \) which combined with (4.3) gives that
\[
\begin{aligned}
\|u_n\|^q &= C_0 \|\nabla u_n\|^{q-1} \|\nabla u_n\|^q, \\
\int_{\mathbb{R}^N} (I_a + |u_n|^\bar{p}) |u_n|^\bar{p} &\to S_\bar{p}^\bar{p} \|\nabla u_n\|^2.
\end{aligned}
\tag{4.4}
as \( n \to \infty \). That is, \( \{u_n\} \subset H^1_0(\mathbb{R}^N) \) is a minimizing sequence of
\[
\frac{1}{C_{N,q}^u} \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\|\nabla u\|_{L^2}^2 + \|u\|_{L^p}^{2(1-\frac{2}{p})}}{\|u\|_{L^q}^q}
\] (4.5)
and
\[
S_u := \inf_{u \in D^{1,\infty}(\mathbb{R}^N) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\int_{\mathbb{R}^N} \left( (|\nabla u|^q + |u|^q)^{\frac{1}{q'}} \right)^{1/q'}}.
\] (4.6)

Since \( \{u_n\} \subset H^1_0(\mathbb{R}^N) \) is bounded, there exists \( u_0 \in H^1_0(\mathbb{R}^N) \) such that \( u_n \rightharpoonup u_0 \) weakly in \( H^1(\mathbb{R}^N) \), \( u_n \to u_0 \) strongly in \( L^p(\mathbb{R}^N) \) with \( t \in (2, 2^\ast) \), and \( u_n \to u_0 \) a.e. in \( \mathbb{R}^N \). By the weak convergence, we have \( \|u_0\|_{L^q}^q \leq \|u_n\|_{L^q}^q \) and \( \|\nabla u_0\|_2^2 \leq \|\nabla u_n\|_2^2 \). Consequently, \( u_0 \) is a minimizer of (4.5) and \( u_n \to u_0 \) strongly in \( H^1(\mathbb{R}^N) \). By Theorem B in [34], \( u_0 \) is the ground state of the equation
\[
\frac{(q-2)N}{4} \Delta u - \left( 1 + \frac{(q-2)(2-N)}{4} \right) u + |u|^{q-2}u = 0.
\] (4.7)

By using (4.6) and \( u_n \to u_0 \) strongly in \( H^1(\mathbb{R}^N) \), we obtain that \( u_0 \) is a minimizer of \( S_u \). So \( u_0 \) is of the form
\[
u_0 = C \left( \frac{b}{b^2 + |x|^2} \right)^{\frac{N-2}{2}},
\] (4.8)
where \( C > 0 \) is a fixed constant and \( b \in (0, \infty) \) is a parameter, see [24]. (4.8) contradicts to (4.7). Thus, \( \inf_{u \in \mathcal{F}_u} E(u) > 0 \).

The next two lemmas are about the upper bound of \( \inf_{u \in \mathcal{F}_u} E(u) \).

**Lemma 4.5.** Let \( N \geq 3 \), \( a \in (0, N) \), \( p = \bar{p}, \ q \in \left( 2, 2 + \frac{4}{N} \right) \), \( \mu > 0 \), \( a > 0 \), and \( \mu a^{\frac{4}{N-2}} \leq (2K)^{\frac{N-p}{p(N-2)}}. \) If \( N \geq 5 \), we further assume that \( \bar{p} \geq 2 \) (i.e., \( a \geq N - 4 \)). Then
\[
\inf_{u \in \mathcal{F}_u} E(u) < m_a + \frac{2 + a}{2(N+a)} S_{a/a}^{Na/a}.
\]

**Proof.** For any \( \varepsilon > 0 \), we define
\[
u_\varepsilon(x) = \varphi(x) U_\varepsilon(x),
\] (4.9)
where \( \varphi(x) \in C_c^\infty(\mathbb{R}^N) \) is a cut off function satisfying: (a) \( 0 \leq \varphi(x) \leq 1 \) for any \( x \in \mathbb{R}^N \); (b) \( \varphi(x) \equiv 1 \) in \( B_1 \); (c) \( \varphi(x) \equiv 0 \) in \( \mathbb{R}^N \setminus B_2 \). Here, \( B_s \) denotes the ball in \( \mathbb{R}^N \) of center at origin and radius \( s \).
\[
U_\varepsilon(x) = \left( \frac{N(N-2)\varepsilon^2}{(\varepsilon^2 + |x|^2)^\frac{N-2}{2}} \right)^{\frac{N-2}{2}}.
\]
where \( U_\varepsilon(x) \) is the extremal function of the minimizing problem (1.8). In [24], they proved that \( S_a = \frac{S}{(A_a(N) C_0(N))^{1/p}} \), where \( A_a(N) \) is defined in (1.2), \( C_0(N) \) is in Lemma 2.2, and
\[
S := \inf_{u \in D^{1,\infty}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |
abla u|^2}{\left( \int_{\mathbb{R}^N} |u|^{2N/(N-2)} \right)^{N/(N-2)}}.
\]
By [38] (see also [37]), we have the following estimates.
\[
\int_{\mathbb{R}^N} |
abla u_\varepsilon|^2 = S^2 + O(\varepsilon^{N-2}), \quad N \geq 3,
\] (4.10)
and

\[
\int_{\mathbb{R}^n} |u_\epsilon|^2 = \begin{cases} 
K_0 \epsilon^2 + O(\epsilon^{n-2}), & N \geq 5, \\
K_0 \epsilon^2 \ln \epsilon + O(\epsilon^2), & N = 4, \\
K_0 \epsilon + O(\epsilon^2), & N = 3,
\end{cases}
\tag{4.11}
\]

where \( K_0 > 0 \). By direct calculation, for \( t \in (2, \infty) \), there exists \( K_1 > 0 \) such that

\[
\int_{\mathbb{R}^n} |u_\epsilon|^t \geq (N(N - 2))^{\frac{-2}{N - 2}} K_1 e^{-K_1|\ln \epsilon|} \left( \frac{1}{1 + |x|^2} \right)^{\frac{2(N - 2)}{N - 2}} dx
\tag{4.12}
\]

Moreover, similar to that [39] and [24], by direct computation, we have

\[
\int_{\mathbb{R}^n} (I_a \ast |u_\epsilon|^p)|u_\epsilon|^p \geq (A_n(N)C_n(N))^\frac{N+a}{2} S_n^\frac{N+a}{2} + O(\epsilon^{\frac{N}{2}}).
\tag{4.13}
\]

Let \( u_\epsilon \) be a positive and radially symmetric non-increasing ground state to (1.4). For \( t \geq 0 \), we define

\[
\tilde{u}_{\epsilon,t} = u_\epsilon + tu_\epsilon \quad \text{and} \quad u_{\epsilon,t} = \left( a^{-\frac{1}{2}}\|\tilde{u}_{\epsilon,t}\|_2 \right)^{\frac{N}{2}} \tilde{u}_{\epsilon,t} \left( a^{-\frac{1}{2}}\|\tilde{u}_{\epsilon,t}\|_2 x \right).
\tag{4.14}
\]

Then

\[
\int_{\mathbb{R}^n} |\tilde{u}_{\epsilon,t}|^2 = a, \quad \int_{\mathbb{R}^n} |\nabla \tilde{u}_{\epsilon,t}|^2 = \int_{\mathbb{R}^n} |\nabla \tilde{u}_{\epsilon,t}|^2,
\]

\[
\int_{\mathbb{R}^n} (I_a \ast |\tilde{u}_{\epsilon,t}|^p)|\tilde{u}_{\epsilon,t}|^p = \int_{\mathbb{R}^n} (I_a \ast |\tilde{u}_{\epsilon,t}|^p)|\tilde{u}_{\epsilon,t}|^p,
\]

\[
\int_{\mathbb{R}^n} |\tilde{u}_{\epsilon,t}|^q = \left( a^{-\frac{1}{2}}\|\tilde{u}_{\epsilon,t}\|_2 \right)^{\frac{q}{2}} \int_{\mathbb{R}^n} |\tilde{u}_{\epsilon,t}|^q.
\]

Since \( \tilde{u}_{\epsilon,t} \in S_a \), by Lemma 3.3, there exists a unique \( \tau_{\epsilon,t} > 0 \) such that \( \tilde{u}_{\epsilon,t} \in P_{a,t} \), which implies that

\[
(\tau_{\epsilon,t})^{-2+q} \|\tilde{u}_{\epsilon,t}\|_2^2 = (\tau_{\epsilon,t})^{-2+q} \int_{\mathbb{R}^n} (I_a \ast |\tilde{u}_{\epsilon,t}|^p)|\tilde{u}_{\epsilon,t}|^p + \mu \|\tilde{u}_{\epsilon,t}\|_q^q.
\tag{4.15}
\]

Since \( \tilde{u}_{\epsilon,0} = u_\epsilon \in P_{a,t} \), by Lemma 3.3, \( \tau_{\epsilon,0} > 1 \). By (4.10), (4.13), and (4.15), \( \tau_{\epsilon,t} \to 0 \) as \( t \to +\infty \) uniformly for \( \epsilon > 0 \) sufficiently small. Since \( \tau_{\epsilon,t} \) is unique by Lemma 3.3, it is standard to show that \( \tau_{\epsilon,t} \) is continuous for \( t \geq 0 \), which implies that there exists \( t_0 > 0 \) such that \( \tau_{\epsilon,t} = 1 \). Consequently, \( \inf_{u \in P_{a,t}} E(u) \leq \sup_{t \geq 0} E(\tilde{u}_{\epsilon,t}) \) for any \( \epsilon \) small enough. By (4.10)–(4.13), and the expression

\[
E(\tilde{u}_{\epsilon,t}) = \frac{1}{2} \|\nabla \tilde{u}_{\epsilon,t}\|_2^2 - \frac{1}{2p} \int_{\mathbb{R}^n} (I_a \ast |\tilde{u}_{\epsilon,t}|^p)|\tilde{u}_{\epsilon,t}|^p - \frac{\mu}{q} \left( a^{-\frac{1}{2}}\|\tilde{u}_{\epsilon,t}\|_2 \right)^q \|\tilde{u}_{\epsilon,t}\|_q^q,
\tag{4.16}
\]

we have \( E(\tilde{u}_{\epsilon,t}) \to m_a \) as \( t \to 0 \), and

\[
E(\tilde{u}_{\epsilon,t}) \leq t \|\nabla u_\epsilon\|_2^2 - \frac{1}{2p} \int_{\mathbb{R}^n} (I_a \ast |u_\epsilon|^p)|u_\epsilon|^p \to -\infty
\]
as \( t \to +\infty \) uniformly for \( \epsilon > 0 \) sufficiently small. Hence, there exists \( t_0 > 0 \) large enough and \( \epsilon_0 > 0 \) small enough such that
\[ E(\tilde{u}_{e,t}) < m_a + \frac{2 + a}{2(N + a)} \frac{N + a}{a} \]

for \( t < \frac{1}{t_0} \) and \( t > t_0 \) uniformly for \( 0 < \varepsilon < \varepsilon_0 \).

Next we estimate \( E(\tilde{u}_{e,t}) \) for \( \frac{1}{t_0} < t < t_0 \). By using the inequalities

\[ (a + b)^r \geq a^r + ra^{-1}b + b^r, \quad a > 0, b > 0, r \geq 2, \]

and

\[ (a + b)^r \geq a^r + ra^{-1}b + rabs^{-1} + b^r, \quad a > 0, b > 0, r \geq 3, \]

we obtain that

\[
\|\nabla \tilde{u}_{e,t}\|^2 = \|\nabla u_x\|^2 + 2t \int_{\mathbb{R}^N} \nabla u_x \nabla \tilde{u}_{e,t} + \|\nabla (tu_x)\|^2, \tag{4.17}
\]

\[
\|\tilde{u}_{e,t}\|^2 \geq \|u_x\|^2 + \|tu_x\|^2 + qt \int_{\mathbb{R}^N} |u_x|^q u_x, \tag{4.18}
\]

\[
\|\tilde{u}_{e,t}\|^2 = \|u_x\|^2 + 2t \int_{\mathbb{R}^N} u_x u_x + \|tu_x\|^2,
\]

\[
\left( a^{\frac{1}{2}} \|\tilde{u}_{e,t}\|^2 \right)^2 = 1 + \frac{2t}{a} \int_{\mathbb{R}^N} u_x u_x + \frac{t^2}{a} \|u_x\|^2, \tag{4.19}
\]

\[
\int_{\mathbb{R}^N} \left( I_a \ast |u_x\|^p \right) |\tilde{u}_{e,t}|^p \\
\geq \int_{\mathbb{R}^N} \left( I_a \ast |u_x\|^p \right) |u_x|^p + 2pt \int_{\mathbb{R}^N} \left( I_a \ast |u_x\|^p \right) |u_x|^q u_x + 2\bar{p} \int_{\mathbb{R}^N} \left( I_a \ast |tu_x\|^p \right) |tu_x|^q u_x + \int_{\mathbb{R}^N} \left( I_a \ast |tu_x\|^p \right) |tu_x|^p \\
+ \int_{\mathbb{R}^N} \left( I_a \ast |tu_x\|^p \right) |tu_x|^p \tag{4.20}
\]

for \( N = 3 \), and

\[
\int_{\mathbb{R}^N} \left( I_a \ast |\tilde{u}_{e,t}\|^p \right) |\tilde{u}_{e,t}|^p \geq \int_{\mathbb{R}^N} \left( I_a \ast |u_x\|^p \right) |u_x|^p + \int_{\mathbb{R}^N} \left( I_a \ast |tu_x\|^p \right) |tu_x|^p + 2pt \int_{\mathbb{R}^N} \left( I_a \ast |u_x\|^p \right) |u_x|^q u_x \tag{4.21}
\]

for \( N \geq 4 \), and \( N \geq 5 \), \( \bar{p} \geq 2 \).

By the positivity of \( u_x \), we have

\[
\int_{\mathbb{R}^N} u_x u_x = \int_{B_1} \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{x^2}{2\sigma^2}} \int_0^1 \frac{1}{\sqrt{2\pi \tau}} e^{-\frac{x^2}{2\tau}} r^{N-1} dr = e^{rac{N-2}{4}} \left( \frac{1}{\varepsilon} \right)^2 \varepsilon^{rac{N-2}{2}}. \tag{4.22}
\]

By (4.11), (4.19), (4.22), and the inequality \( (1 + t)^a \geq 1 + at \) for \( t \geq 0 \) and \( a < 0 \), we obtain that

\[
\left( a^{\frac{1}{2}} \|\tilde{u}_{e,t}\|^2 \right)^{\frac{q_{\gamma} - q}{2}} = \left( 1 + \frac{2t}{a} \int_{\mathbb{R}^N} u_x u_x + \frac{t^2}{a} \|u_x\|^2 \right)^{\frac{q_{\gamma} - q}{2}} \geq 1 + \frac{q_{\gamma} - q}{2} \left( \frac{2t}{a} \int_{\mathbb{R}^N} u_x u_x + \frac{t^2}{a} \|u_x\|^2 \right). \tag{4.23}
\]

Case \( N = 3 \). Noting that \( u_x \) satisfies the equation

\[-\Delta u_x = \lambda u_x + \left( I_a \ast |u_x|^p \right) |u_x|^p u_x + \mu |u_x|^q u_x, \]
with $\lambda < 0$ and $\lambda a = \mu(\gamma_q - 1)||u||^p$ (see (3.8)), and by using (4.16), (4.17), (4.18), (4.20), and (4.23), we obtain that

$$E(\tilde{u}_{e,t}) \leq \frac{1}{2}||\nabla u_t||^2 + t \int_{\mathbb{R}^N} \nabla u_t \nabla u_e + \frac{1}{2}||\nabla (tu_e)||^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_a * |u_t|^p)|u_t|^p$$

$$- t \int_{\mathbb{R}^N} (I_a * |u_t|^p)|u_t|^\beta u_e - \int_{\mathbb{R}^N} (I_a * |tu_e|^p)|tu_e|^\beta u_e$$

$$- \frac{1}{2p} \int_{\mathbb{R}^N} (I_a * |tu_e|^p)|tu_e|^{\beta} - \frac{\mu}{q}||u_t||_q^q - \frac{\mu}{q}||tu_e||_q^q$$

$$- \mu t \int_{\mathbb{R}^N} |u_t|^{\beta - 1} u_e - \frac{\mu(\gamma_q - 1)}{a} \int_{\mathbb{R}^N} u_t u_e + \frac{\mu(1 - \gamma_q)t^2}{2a} ||u_t||_q^q ||\tilde{u}_{e,t}||_q^q$$

$$= E(u_e) + E(tu_e) + t\lambda \int_{\mathbb{R}^N} u_t u_e - \frac{\mu(1 - \gamma_q)}{a} ||\tilde{u}_{e,t}||_q^q \int_{\mathbb{R}^N} u_t u_e - \frac{\mu(1 - \gamma_q)t^2}{2a} ||u_t||_q^q ||\tilde{u}_{e,t}||_q^q$$

$$- \int_{\mathbb{R}^N} (I_a * |tu_e|^p)|tu_e|^{\beta - 1} u_e$$

$$= m_a + E(tu_e) + \frac{\mu(1 - \gamma_q)}{a} (||\tilde{u}_{e,t}||_q^q - ||u_t||_q^q) \int_{\mathbb{R}^N} u_t u_e + \frac{\mu(1 - \gamma_q)t^2}{2a} ||u_t||_q^q ||\tilde{u}_{e,t}||_q^q$$

$$- \int_{\mathbb{R}^N} (I_a * |tu_e|^p)|tu_e|^{\beta - 1} u_e.$$

By direct calculation, we have

$$\int_{\mathbb{R}^N} (I_a * |u_t|^p)|u_t|^\beta u_e \geq \int_{\mathbb{R}^N} (I_a * |u_t|^p)|u_t|^{\beta - 1}$$

$$\geq \int_{\mathbb{R}^N} |U(x)|^p |U(y)|^{\beta - 1} |x - y|^{N-\alpha} dx dy$$

$$= \varepsilon^{\frac{N-\alpha}{2}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^\frac{N-\alpha}{2} + |y|^{N-\alpha}(1 + |y|^\frac{N-\alpha}{2}))))} dx dy$$

$$\geq \varepsilon^{\frac{N-\alpha}{2}},$$

$$||\tilde{u}_{e,t}||_q^q - ||u_t||_q^q = ||u_t + tu_e||_q^q - ||u_t||_q^q \leq \int_{\mathbb{R}^N} |u_t|^{\beta - 1} tu_e + ||tu_e||_q^q,$$

and similar to (4.22),

$$\int_{\mathbb{R}^N} u_t^{\beta - 1} u_e \leq \int_{\mathbb{R}^N} U_j(x) \leq \varepsilon^{\frac{N-\alpha}{2}}.$$

By (4.10), (4.11), (4.13), (4.22), (4.24), (4.25), (4.26), and (4.27), we obtain

$$E(\tilde{u}_{e,t}) \leq m_a + \frac{t^2}{2} \left( S_1^N + O(\varepsilon^{N-2}) \right) - \frac{t^2}{2p} \left( A_0(N)C_0(N) \frac{N+a}{N} + O(\varepsilon^{N-a}) \right)$$

$$- \frac{\mu}{q} t^q ||u_t||_q^q + O(\varepsilon^{N-2}) + O(\varepsilon^{\frac{N-a}{2}})||u_t||_q^q + O(||u_t||_q^q) - Ce^{\frac{N-a}{2}}$$

$$< m_a + \frac{t^2}{2} S_1^N - \frac{t^2}{2p} \left( A_0(N)C_0(N) \frac{N+a}{N} \right)$$

$$\leq m_a + \frac{2 + \alpha}{2(N + \alpha)} S_1^N$$

for $\frac{1}{\varepsilon_0} < t < t_0$ uniformly for $\varepsilon \in (0, \varepsilon_0)$ small enough.
Case $N \geq 4$. Similar to case $N = 3$, by using (4.16), (4.17), (4.18), (4.21), and (4.23), we have

$$E(\tilde{u}_e, t) \leq m_a + E(tu_e) + \frac{\mu(1 - \eta_0)}{a} \left( \| \tilde{u}_e, t \|_q^q - \| u_e \|_q^q \right) \int_{\mathbb{R}^N} u_e u_e + \frac{\mu(1 - \eta_0) t^2}{2a} \| u_e \|_2^2 \| \tilde{u}_e, t \|_q^q, \tag{4.28}$$

Thus, by using (4.10), (4.11), (4.12), (4.13), (4.22), (4.26), (4.27), and (4.28), we obtain

$$E(\tilde{u}_e, t) \leq m_a + \frac{t^2}{2} \left( S^2 + O(e^{N - 2}) \right) - \frac{t^{2p}}{2p} \left( (A_d(N)C_d(N))^{\frac{N + r}{2}} S^2 \right) + \frac{\mu}{q} \| u_e \|_q^q + O(e^{N - 2}) \| u_e \|_q^q + O(\| u_e \|_2^2) \tag{4.29}$$

$$< m_a + \frac{t^2}{2} S^2 - \frac{t^{2p}}{2p} (A_d(N)C_d(N))^{\frac{N + r}{2}} S^2 + \frac{\mu}{q} \| u_e \|_q^q$$

$$\leq m_a + \frac{2 + \alpha}{2(N + \alpha)} S_{N + r}^N \tag{4.30}$$

for $\frac{1}{t_0} < t < t_0$ uniformly for $\varepsilon \in (0, \varepsilon_0)$ small enough. The proof is complete. \qed

**Lemma 4.6.** Let $N \geq 4$, $\alpha \in (0, N)$, $p = \rho$, $q \in \left( 2, 2 + \frac{\lambda}{N} \right)$, $\mu > 0$, $a > 0$, and $\mu a^{\frac{\rho - \mu}{\rho - 2}} \leq (2K)^{\frac{\rho - \mu}{\rho - 2}}$. Then

$$\inf_{u \in I_{\varepsilon}} E(u) < m_a + \frac{2 + \alpha}{2(N + \alpha)} S_{N + r}^N. \tag{4.31}$$

**Proof.** Step 1. Let $u_\varepsilon$ and $u_\varepsilon$ be defined in Lemma 4.5. We claim that for any $\varepsilon > 0$, there exists $\varepsilon_0 \in \mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} u_\varepsilon(x - \varepsilon y) u_\varepsilon(x) dx \leq \| u_\varepsilon \|_2^2 \tag{4.30}$$

and

$$\int_{\mathbb{R}^N} \nabla u_\varepsilon(x - \varepsilon y) \cdot \nabla u_\varepsilon(x) dx \leq \| u_\varepsilon \|_2^2. \tag{4.31}$$

Indeed, since $u_\varepsilon$ is radial and non-increasing, by Lemma 2.5, we obtain that

$$\int_{\mathbb{R}^N} u_\varepsilon(x - y) u_\varepsilon(x) dx \leq \left( \frac{N}{|\lambda|} \right)^{1/2} \sqrt{\lambda} \int_{\mathbb{R}^N} |x - y|^{-N/2} u_\varepsilon(x) dx. \tag{4.32}$$

Noting that $\text{supp}(u_\varepsilon) \subset B_2$, and by using the Hölder inequality, we have, for $|y| > 10$,

$$\int_{\mathbb{R}^N} u_\varepsilon(x - y) u_\varepsilon(x) dx \leq \int_{B_2} \left( \frac{|y|}{2} \right)^{-N/2} u_\varepsilon(x) dx \leq \left( \frac{|y|}{2} \right)^{-N/2} \| u_\varepsilon \|_{L^1}^2 \tag{4.33}$$

which combined with (4.11) implies that (4.30) holds for $\varepsilon_0$ large enough.

Noting that for any $y \in \mathbb{R}^N$, $u_\varepsilon(x - y)$ is a solution to the equation

$$-\Delta u = \lambda u + \left( I_\varepsilon + |u|^{\beta - 2} \right) u \tag{4.34}$$

with some $\lambda < 0$, we obtain that

$$\int_{\mathbb{R}^N} \nabla u_\varepsilon(x - y) \cdot \nabla u_\varepsilon(x) dx \leq \int_{\mathbb{R}^N} \left( I_\varepsilon + |u_\varepsilon(x - y)|^{\beta - 2} \right) u_\varepsilon(x - y) u_\varepsilon(x) dx + \mu \int_{\mathbb{R}^N} |u_\varepsilon(x - y)|^{\beta - 2} u_\varepsilon(x - y) u_\varepsilon(x) dx. \tag{4.35}$$
Similar to the proof of (4.30), we have
\[
\mu \int \frac{1}{|x-y|^{N-2}} u(x-y) u(x) dx \leq \frac{1}{2} \|u_c\|_2^2
\]  
(4.33)
for \(y\) large enough. Now, for \(x \in B_2\) and \(|y| > 100\), we calculate
\[
\int \frac{1}{|x-z|^{N-a}} u_c(z-y)^p dz = \left( \int_{|x-z|^{N-a}} + \int_{B_{10}(y) \setminus B_{10}(y)/2} \right) \frac{1}{|x-z|^{N-a}} u_c(z-y)^p dz \leq I_1 + I_2 + I_3, 
\]  
(4.34)
It follows from \(x \in B_2\), \(|y| > 100\), and \(z \in \mathbb{R}^N \setminus B_{10}(y)\) that
\[
|x-z| \geq |z| - |x| \geq \frac{1}{2} |z| \geq \frac{1}{2} |y|
\]

and
\[
|x-z| \geq \frac{1}{2} |z| \geq \frac{1}{4} (|z| + |y|) \geq \frac{1}{4} |z| - |y|. 
\]
By using Lemma 2.5 with \(t = 2\), for any \(\delta \in (0, \min\{N - a, Np/2 - a\})\), we have
\[
I_1 \leq \int_{|x-z|^{N-a}} \frac{1}{|y|^\delta (|z| - |y|)^{N-a-\delta}} |z-y|^{-\frac{Np}{2}} \|u_c(z-y)\|_2^\delta dz \leq \frac{1}{|y|^\delta}. 
\]  
(4.35)
For \(x \in B_2\), \(|y| > 100\), and \(z \in B_{10}(y) \setminus B_{10}(y)/2\), we obtain that \(|z-x| \leq 8|y|\) and \(|z-y| \geq \frac{|y|}{2}\). By using Lemma 2.5 with \(t = 2\), we have
\[
I_2 \leq \int_{B_{10}(y) \setminus B_{10}(y)/2} \frac{1}{|x-z|^{N-a} (|y|/2)^{\delta}} \|u_c(z-y)\|_2^\delta dz \leq |y|^{-\frac{Np}{2}}. 
\]  
(4.36)
For \(x \in B_2\), \(|y| > 100\), and \(z \in B_{10}(y) \setminus B_{10}(y)/2\), we obtain that \(|z-x| \geq \frac{|y|}{3}\) and \(|z-y| \leq \frac{|y|}{2}\). Then by using Lemma 2.5 with \(t = 2^*\) and \(\|u_c\|_2^* \leq C\), we obtain that
\[
I_3 \leq \int_{B_{10}(y) \setminus B_{10}(y)/2} \frac{1}{|x-z|^{N-a} (|y|/2)^{\delta}} dz \leq \int_{B_{10}(0) \setminus B_{10}(0)/2} \frac{1}{|y|^{N-a} (|y|/3)^{\delta}} dz \leq \frac{1}{|y|^{N-a}} \int_0^{\frac{|y|}{2}} r^{-\frac{Np}{2}} r^{N-1} dr \leq |y|^{\frac{Np}{2}}. 
\]  
(4.37)
By using (4.34), (4.35), (4.36), and (4.37), similar to the proof of (4.30), we obtain
\[
\int_{\mathbb{R}^n} \left( I_a + |u, (x - y)|^2 \right) |u, (x - y)|^2 - u(x, y) u(x) dx \leq \frac{1}{2} \|u\|_2^2
\]  
(4.38)
for \(y\) large enough. In view of (4.32), (4.33), and (4.38), we complete the proof of (4.31).

**Step 2.** Let \(\gamma\) be given in Step 1 such that (4.30) and (4.31) hold. As in (4.14), we define
\[
\hat{u}_{e, t} = u(x - \gamma) + tu(x) \quad \text{and} \quad \hat{u}_{e, t} = \left( a - \frac{1}{2} \|\hat{u}_{e, t}\|_2^2 \right)^{\frac{N-2}{2}} \hat{u}_{e, t} \left( a - \frac{1}{2} \|\hat{u}_{e, t}\|_2^2 \right)^{\frac{N-2}{2}}.
\]
Similar to the proof of Lemma 4.5, there exists \(t_0 > 0\) large enough and \(\varepsilon_0 > 0\) small enough such that
\[
E(\hat{u}_{e, t}) < m_a + \frac{2 + a}{2(N + a)} S_m^{N+a}
\]
(4.39)
for \(t < \frac{1}{t_0}\) and \(t > t_0\) uniformly for \(0 < \varepsilon < \varepsilon_0\).

By using the inequality
\[
(a + b)^r \geq a^r + b^r, \quad a > 0, b > 0, \ r \geq 1,
\]
(4.10), (4.11), (4.12), (4.13), (4.30), and (4.31), similar to (4.29), we obtain
\[
E(\hat{u}_{e, t}) \leq \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2N} \int_{\mathbb{R}^n} \nabla u, (x - y) \nabla u(x) + \frac{1}{2} \|\nabla (tu, e, t)\|_2^2 - \frac{1}{2p} \int_{\mathbb{R}^n} \left( I_a + |u, |^p \right) |u, |^p
\]
\[
- \frac{1}{2p} \int_{\mathbb{R}^n} \left( I_a + |tu, |^p \right) |tu, |^p - \frac{p}{q} \|u, \|_q^q - \frac{\mu}{q} \|tu, \|_q^q
\]
\[
- \frac{\mu(\gamma - 1)}{2} \left( \frac{2t}{a} \int_{\mathbb{R}^n} u(x, y) u_t(x) + \frac{t^2}{a} \|u, e, t, \|_q^q \right) \|u, e, t\|_q^q
\]
\[
\leq E(u) + E(tu) + O(\|u, e, t, \|_2^2) + O(\|u, e, t, \|_2^2) \|u, e, t, \|_q^q
\]
\[
< m_a + \frac{2 + a}{2(N + a)} S_m^{N+a}
\]
(4.40)
for \(\frac{1}{t_0} < t < t_0\) uniformly for \(\varepsilon \in (0, \varepsilon_0)\) small enough. In view of (4.39) and (4.40), we complete the proof. \(\square\)

The next lemma is about the convergence of the Palais-Smale sequence.

**Lemma 4.7.** Assume \(N \geq 3, a \in (0, N), p \equiv p, q \in \left( \frac{2 + \frac{4}{N}}{2}, \frac{2}{N} \right), a > 0, \mu > 0, \) and \(\mu a^{\frac{q-1}{p-q}} \leq (2K)^{\frac{q-1}{p-q}}. \) Let \(\{u_n\} \subset S_{n,t} \) be a Palais-Smale sequence for \(E|_{S_n} \) at level \(c, \) with \(P(u_n) \to 0 \) as \(n \to \infty. \) If
\[
0 < c < m_a + \frac{2 + a}{2(N + a)} S_m^{N+a},
\]
then up a subsequence, \(u_n \to u \) strongly in \(H^1(\mathbb{R}^N), \) and \(u \) is a radial solution to (1.4) with \(E(u) = c \) and some \(\lambda < 0. \)

**Proof.** The proof is divided into four steps.

**Step 1.** We show \(\{u_n\} \) is bounded in \(H^1(\mathbb{R}^N). \) It follows from \(P(u_n) = o_n(1) \) and \(E(u_n) = c + o_n(1) \) that
\[
E(u_n) = \left( \frac{1}{2} - \frac{1}{2p} \right) \|\nabla u_n\|_2^2 + \left( \frac{\gamma - 1}{2} \right) \|u_n\|_q^q + o_n(1).
\]
Since \( q \ell < 2 < 2\bar{p} \), by using the Gagliardo-Nirenberg inequality, we obtain that
\[
\left( \frac{1}{2} - \frac{1}{2\bar{p}} \right) \| \nabla u_n \|_2^2 \leq c + \left( \frac{1}{q} - \frac{Y_{\bar{p}}}{2\bar{p}} \right) \| u_n \|_q^q + o_n(1)
\]
\[
\leq c + \left( \frac{1}{q} - \frac{Y_{\bar{p}}}{2\bar{p}} \right) \mu C_{\bar{p}, q}^2 \| \nabla u_n \|_{2^*}^{2^*} + o_n(1),
\]
which implies that \( \| \nabla u_n \|_2^2 \) is bounded. Since \( \{ u_n \} \subset S_\alpha \), we obtain that \( \{ u_n \} \) is bounded in \( H^1(\mathbb{R}^N) \).

There exists \( u \in H^1_0(\mathbb{R}^N) \) such that, up to a subsequence, \( u_n \rightharpoonup u \) weakly in \( H^1(\mathbb{R}^N) \), \( u_n \rightarrow u \) strongly in \( L^t(\mathbb{R}^N) \) with \( t \in (2, 2^*) \) and \( u_n \rightarrow u \) a.e. in \( \mathbb{R}^N \).

**Step 2. We claim that \( u \neq 0 \).** Suppose by contradiction that \( u \equiv 0 \). By using \( E(u_n) = c + o_n(1), P(u_n) = o_n(1), \| u_n \|_q^q = o_n(1) \), and (1.8), we obtain that
\[
E(u_n) = \left( \frac{1}{2} - \frac{1}{2\bar{p}} \right) \| \nabla u_n \|_2^2 + o_n(1)
\]

and
\[
\| \nabla u_n \|_2^2 = \int_{\mathbb{R}^N} \left( I_a * |u_n|^{\bar{p}} \right) |u_n|^{\bar{p}} + o_n(1) \leq (S_{\alpha}^{-2} \| \nabla u_n \|_2^2)^{\bar{p}} + o_n(1).
\]

(4.41)

Since \( c > 0 \), we obtain \( \liminf_{n \to \infty} \| \nabla u_n \|_2^2 > 0 \) and hence
\[
\limsup_{n \to \infty} \| \nabla u_n \|_2^2 \geq S_{\alpha}^{1+\alpha}.
\]

Consequently,
\[
c = \lim_{n \to \infty} \left( \frac{1}{2} - \frac{1}{2\bar{p}} \right) \| \nabla u_n \|_2^2 + o_n(1) \geq \frac{2 + \alpha}{2(N + \alpha)} S_{\alpha}^{1+\alpha},
\]

which contradicts to
\[
c < m_\alpha + \frac{2 + \alpha}{2(N + \alpha)} S_{\alpha}^{1+\alpha}
\]

and \( m_\alpha < 0 \). So \( u \neq 0 \).

**Step 3. We show \( u \) is a solution to (1.3) with some \( \lambda < 0 \).** Since \( \{ u_n \} \) is a Palais-Smale sequence of \( E|_{S_\alpha} \), by the Lagrange multipliers rule, there exists \( \lambda_n \) such that
\[
\int_{\mathbb{R}^N} (\nabla u_n \cdot \nabla \varphi - \lambda_n u_n \varphi) - (I_a * |u_n|^{\bar{p}}) |u_n|^{\bar{p}} - \mu |u_n|^{q-2} u_n \varphi = o_n(1) \| \varphi \|_{\bar{p}}
\]

(4.42)

for every \( \varphi \in H^1(\mathbb{R}^N) \). The choice \( \varphi = u_n \) provides
\[
\lambda_n a = \| \nabla u_n \|_2^2 - \int_{\mathbb{R}^N} \left( I_a * |u_n|^{\bar{p}} \right) |u_n|^{\bar{p}} - \mu \| u_n \|_q^q + o_n(1)
\]

(4.43)

and the boundedness of \( \{ u_n \} \) in \( H^1(\mathbb{R}^N) \) implies that \( \lambda_n \) is bounded as well; thus, up to a subsequence \( \lambda_n \rightharpoonup \lambda \in \mathbb{R} \). Furthermore, by using \( P(u_n) = o_n(1), (4.43), \mu > 0, \gamma_0 \in (0, 1), \) and \( u_n \rightharpoonup u \) weakly in \( H^1(\mathbb{R}^N) \), we obtain that
\[
-\lambda_n a = \mu (1 - \gamma_0) \| u_n \|_q^q + o_n(1)
\]

and then
\[
-\lambda a \geq \mu (1 - \gamma_0) \| u \|_q^q > 0,
\]
which implies that $\lambda < 0$. By using (4.42) and Lemma 2.4, we obtain that
\[
\int_{\mathbb{R}^N} (\nabla u \cdot \nabla \varphi - \lambda u \varphi - (I_\alpha \ast |u|^\beta)|u|^{\beta-2}u \varphi - \mu|u|^{q-2}u \varphi)
= \lim_{n \to \infty} \int_{\mathbb{R}^N} (\nabla u_n \cdot \nabla \varphi - \lambda_n u_n \varphi - (I_\alpha \ast |u_n|^\beta)|u_n|^{\beta-2}u_n \varphi - \mu|u_n|^{q-2}u_n \varphi) \tag{4.44}
= \lim_{n \to \infty} o(1)\|\varphi\|_{H^1} = 0,
\]
which implies that $u$ satisfies the equation
\[
-\Delta u = \lambda u + (I_\alpha \ast |u|^\beta)|u|^{\beta-2}u + \mu|u|^{q-2}u. \tag{4.45}
\]
Thus, $P(u) = 0$ by Lemma 2.7.

**Step 4. We show** $u_n \to u$ **strongly in** $H^1(\mathbb{R}^N)$. Set $v_n = u_n - u$. Then we have
\[
\|u_n\|_2^2 = \|u\|_2^2 + \|v_n\|_2^2 + o_n(1), \quad \|\nabla u_n\|_2^2 = \|\nabla u\|_2^2 + \|\nabla v_n\|_2^2 + o_n(1), \tag{4.46}
\]
\[
\|u_n\|_q^q = \|u\|_q^q + \|v_n\|_q^q + o_n(1) = \|u\|_q^q + o_n(1) \tag{4.47}
\]
and
\[
\int_{\mathbb{R}^N} (I_\alpha \ast |u_n|^\beta)|u_n|^\beta = \int_{\mathbb{R}^N} (I_\alpha \ast |u|^\beta)|u|^\beta + \int_{\mathbb{R}^N} (I_\alpha \ast |v_n|^\beta)|v_n|^\beta + o_n(1),
\]
which combined with $P(u_n) = o_n(1)$ and $P(u) = 0$ gives that
\[
\|\nabla v_n\|_2^2 = \int_{\mathbb{R}^N} (I_\alpha \ast |v_n|^\beta)|v_n|^\beta + o_n(1). \tag{4.48}
\]
Similarly to (4.41), we infer that
\[
\limsup_{n \to \infty} \|\nabla v_n\|_2^2 \geq \frac{N+2}{2N+2} \quad \text{or} \quad \liminf_{n \to \infty} \|\nabla v_n\|_2^2 = 0.
\]
If $\limsup_{n \to \infty} \|\nabla v_n\|_2^2 \geq \frac{N+2}{2N+2}$, then by using the fact that $u$ satisfies (4.45), $\|u\|_2^2 \leq a$, (4.46), (4.47), and Lemma 3.5, we obtain that
\[
E(u_n) = \left(1 - \frac{1}{2N} \right)\|u_n\|_2^2 + \left(\frac{1}{2} - \frac{1}{q} \frac{1}{2} \right)\|u_n\|_q^q + o_n(1)
= \left(1 - \frac{1}{2N} \right)\|u\|_2^2 + \left(\frac{1}{2} - \frac{1}{q} \frac{1}{2} \right)\|u\|_q^q + \left(1 - \frac{1}{2N} \right)\|\nabla v_n\|_2^2 + o_n(1)
\geq m_\alpha + \frac{2 + a}{2(N + a)} \frac{N+2}{2N+2} + o_n(1)
\geq m_\alpha + \frac{2 + a}{2(N + a)} \frac{N+2}{2N+2} + o_n(1),
\]
which contradicts $E(u_n) = c + o_n(1)$ and $c < m_\alpha + \frac{2 + a}{2(N + a)} \frac{N+2}{2N+2}$. Thus,
\[
\liminf_{n \to \infty} \|\nabla v_n\|_2^2 = 0
\]
holds. So up to a subsequence, $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\mathbb{R}^N)$. Choosing $\varphi = u_n - u$ in (4.42) and (4.44), and subtracting, we obtain that
\[
\int_{\mathbb{R}^N} (|\nabla (u_n - u)|^2 - \lambda|u_n - u|^2) \to 0.
\]
Since $\lambda < 0$, we get that $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$. The proof is complete. $\square$

**Proof of Theorem 1.5.** It is a direct result of Lemmas 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, and 4.7. $\square$
5 Positivity, symmetry, and exponential decay of solution \( u \) to (1.4) with \( E(u) = \inf_{\mathcal{P}_{a-}} E(\nu) \)

In this section, we prove Theorem 1.7. For future use, we first give the following result.

**Lemma 5.1.** Let \( N \geq 3, \ a \in (0, N), \ p = \bar{p}, \ q \in \left(2, 2 + \frac{4}{N}\right), \ \mu > 0, \ \alpha > 0, \) and \( \mu a^{\frac{\mu-1}{\gamma-1}} \leq (2K)^{\frac{q-2}{p-2}}. \) If \( u \in \mathcal{P}_{a-} \) such that \( E(u) = \inf_{\mathcal{P}_{a-}} E(\nu), \) then \( u \) satisfies the equation (1.4) with some \( \lambda < 0. \)

**Proof.** By the Lagrange multiplier rule, there exist \( \lambda \) and \( \eta \) such that \( u \) satisfies

\[
-\Delta u - \left(I_a + |u|^{\bar{p}}\right)|u|^\beta u - \mu|u|^{q^*-2}u = \lambda u + \eta \left[-2\Delta u - 2\bar{p}\left(I_a + |u|^{\bar{p}}\right)|u|^\beta u - \mu q_q u|u|^{q^*-2}u\right],
\]

or equivalently,

\[
-(1-2\eta)\Delta u = \lambda u + (1-\eta\bar{p})\left(I_a + |u|^{\bar{p}}\right)|u|^\beta u + \mu(1-\eta q_q)|u|^{q^*-2}u.
\]

Next we show \( \eta = 0. \) Similar to the definition of \( \mathcal{P}_0 \) (see Lemma 2.7), we obtain

\[
(1-2\eta)\|\nabla u\|_2^2 - (1-\eta\bar{p})\int_{\mathbb{R}^N} \left(I_a + |u|^{\bar{p}}\right)|u|^\beta u - \mu(1-\eta q_q)|u|^{q^*-2}u = 0,
\]

which combined with \( P(u) = 0 \) gives that

\[
\eta \left[2\|\nabla u\|_2^2 - 2\bar{p} \int_{\mathbb{R}^N} \left(I_a + |u|^{\bar{p}}\right)|u|^\beta u - \mu q_q^2\|u\|_q^2\right] = 0.
\]

If \( \eta \neq 0, \) then

\[
2\|\nabla u\|_2^2 - 2\bar{p} \int_{\mathbb{R}^N} \left(I_a + |u|^{\bar{p}}\right)|u|^\beta u - \mu q_q^2\|u\|_q^2 = 0,
\]

which combined with \( P(u) = 0 \) gives that

\[
\begin{cases}
\mu q_q^2(2\bar{p} - q_q)\|u\|_2^2 = (2\bar{p} - 2)\|\nabla u\|_2^2, \\
(q_q - 2\bar{p}) \int_{\mathbb{R}^N} \left(I_a + |u|^{\bar{p}}\right)|u|^\beta u = (q_q - 2)\|\nabla u\|_2^2.
\end{cases}
\]

Hence,

\[
E(u) = \frac{(\bar{p} - 1)(q_q - 2)}{2\bar{p} q_q} \|\nabla u\|_2^2 < 0,
\]

which contradicts to \( \inf_{\mathcal{P}_{a-}} E(\nu) \geq 0, \) see Lemma 3.3. So \( \eta = 0. \)

It follows from (5.1) with \( \eta = 0, \) \( P(u) = 0, \) \( 0 < q < 1, \) and \( \mu > 0 \) that

\[
\lambda a = \|\nabla u\|_2^2 - \int_{\mathbb{R}^N} \left(I_a + |u|^{\bar{p}}\right)|u|^\beta u - \mu\|u\|_q^2 = \mu(q_q - 1)\|u\|_q^2 < 0,
\]

which implies \( \lambda < 0. \) The proof is complete. \( \square \)

Next, we study the positivity of the solution \( u \) to (1.4) with \( E(u) = \inf_{\mathcal{P}_{a-}} E(\nu). \) By Lemma 5.1, it is enough to prove the following result.

**Proposition 5.2.** Let \( N \geq 3, \ a \in (0, N), \ p = \bar{p}, \ q \in \left(2, 2 + \frac{4}{N}\right), \ \mu > 0, \ \alpha > 0, \) and \( \mu a^{\frac{\mu-1}{\gamma-1}} \leq (2K)^{\frac{q-2}{p-2}}. \) If \( u \in \mathcal{P}_{a-} \) such that \( E(u) = \inf_{\mathcal{P}_{a-}} E(\nu), \) then \( |u|_{r_\nu} \in \mathcal{P}_{a-} \) and \( E(|u|_{r_\nu}) = \inf_{\mathcal{P}_{a-}} E(\nu). \) Moreover, \( |u|_{r_\nu} > 0 \) in \( \mathbb{R}^N. \)
Proof. It follows from $|\nabla u|^2 \leq |\nabla u|^2$ that $\Psi(u(\tau)) \leq \Psi(u(\tau))$ for any $\tau > 0$. By Lemma 3.3, we have

$$E(u(\tau)) = \Psi(u(\tau)) \leq \Psi(u(\tau)) = E(u).$$

Since $|u|_{\alpha,N} \in \mathcal{P}_{\alpha,N}$, we obtain that $E(|u|_{\alpha,N}) = \inf_{\mathcal{P}_{\alpha,N}} E(v)$. By Lemma 5.1, there exists $\lambda < 0$ such that $|u|_{\alpha,N}$ satisfies the equation

$$-\Delta u = \lambda u + (I_n * |u|^p)|u|^{p-2}u + \mu|u|^{q-2}u.$$

Since $|u|_{\alpha,N}$ is continuous by Theorem 2.1 in [13], the strong maximum principle implies that $|u|_{\alpha,N} > 0$ in $\mathbb{R}^N$.

Next, we study the radial symmetry of the solution $u$ to (1.4) with $E(u) = \inf_{\mathcal{P}_N} E(v)$. We follow the arguments of [32], which relies on polarization. So we first recall some theories of polarization [40,23,41].

Assume that $H \subset \mathbb{R}^N$ is a closed half-space and that $\sigma_H$ is the reflection with respect to $\partial H$. The polarization $u^H : \mathbb{R}^N \to \mathbb{R}$ of $u : \mathbb{R}^N \to \mathbb{R}$ is defined for $x \in \mathbb{R}^N$ by

$$u^H(x) = \begin{cases} \max(u(x), u(\sigma_H(x))), & \text{if } x \in H, \\ \min(u(x), u(\sigma_H(x))), & \text{if } x \notin H. \end{cases}$$

Lemma 5.3. (Polarization and Dirichlet integrals, Lemma 5.3 in [40]). Let $H \subset \mathbb{R}^N$ be a closed half-space. If $u \in H^1(\mathbb{R}^N)$, then $u^H \in H^1(\mathbb{R}^N)$, and

$$\int_{\mathbb{R}^N} |\nabla u^H|^2 = \int_{\mathbb{R}^N} |\nabla u|^2.$$

Lemma 5.4. (Polarization and nonlocal integrals, Lemma 5.3 in [23]). Let $\alpha \in (0, N)$, $u \in L^{2N/\alpha}(\mathbb{R}^N)$, and $H \subset \mathbb{R}^N$ be a closed half-space. If $u \geq 0$, then

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x)u(y)}{|x-y|^{N-\alpha}} \, dx \, dy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^H(x)u^H(y)}{|x-y|^{N-\alpha}} \, dx \, dy,$$

with equality if and only if either $u^H = u$ or $u^H = u \circ \sigma_H$.

Lemma 5.5. (Symmetry and polarization, Proposition 3.15 in [41], Lemma 5.4 in [23]). Assume that $u \in L^2(\mathbb{R}^N)$ is non-negative. There exist $x_0 \in \mathbb{R}^N$ and a non-increasing function $v : (0, \infty) \to \mathbb{R}$ such that for almost every $x \in \mathbb{R}^N$, $u(x) = v(|x-x_0|)$ if and only if for every closed half-space $H \subset \mathbb{R}^N$, $u^H = u = u \circ \sigma_H$.

Now we are ready to prove the radial symmetry result.

Proposition 5.6. Let $N \geq 3$, $\alpha \in (0, N)$, $p = \tilde{p}$, $q \in \left(2, 2 + \frac{4}{N}\right)$, $\mu > 0$, $\alpha > 0$, and $\mu a^{\frac{p^2}{q} - q} \leq (2K)^{\frac{N-4}{2}}$. If $u$ is a positive solution to (1.4) with $E(\tilde{u}) = \inf_{\mathcal{P}_N} E(v)$, then there exist $x_0 \in \mathbb{R}^N$ and a non-increasing positive function $v : (0, \infty) \to \mathbb{R}$ such that $u(x) = v(|x-x_0|)$ for almost every $x \in \mathbb{R}^N$.

Proof. By Lemmas 2.7 and 3.3, $u \in \mathcal{P}_{\alpha,N}$. Let $\Gamma(\alpha)$ be defined in (4.1), $\tau_1 \geq 0$ be small enough such that $u_{\tau_1} \in V_\alpha$ and $E(u_{\tau_1}) < 0$. Then $g_\alpha(t) = u_{\tau_1 + \alpha} \in \Gamma(\alpha)$, $g_\alpha(t) = u_{\tau_1 + \alpha} \geq 0$ for every $t \geq 0$, $E(g_\alpha(t)) = E(u) = \inf_{\mathcal{P}_N} E(v)$ for any $t \in (0, \infty) \setminus \{\tau_1 - \tau_1\}$.

For every closed half-space $H$ define the path $g^H_u : [0, \infty) \to S_h$ by $g^H_u(t) = (g_u(t))^H$. By Lemma 5.3 and $\|u\|_\infty = \|u\|_\infty$, with $r \in [1, \infty)$, we have $g^H_u \in C([0, \infty), S_h)$. By Lemmas 5.3 and 5.4, we obtain that $g^H_u(0) \in V_\alpha$ and $E(g^H_u(t)) \leq E(g_u(t))$ for every $t \in [0, \infty)$ and thus $g^H_u \in \Gamma(\alpha)$. Hence,

$$\max_{t \in [0, \infty)} E(g^H_u(t)) \geq \inf_{v \in \mathcal{P}_N} E(v).$$
Since for every $t \in ([0, \infty) \setminus \{\tau - \tau\})$, 
$$E(g_u^H(t)) \leq E(g_u^H(t)) < E(u) = \inf_{v \in \mathcal{P}_u} E(v),$$
we deduce that 
$$E(g_u^H(\tau - \tau)) = E(u^H) = \inf_{v \in \mathcal{P}_u} E(v).$$
Hence $E(u^H) = E(u)$, which implies that 
$$\int_{\mathbb{R}^N} \left( I_a \ast |u|^\beta \right) |u|^\beta = \int_{\mathbb{R}^N} \left( I_a \ast |u|^\beta \right) |u|^\beta.$$ 
By Lemma 5.4, we have $u^H = u$ or $u^H = u \circ \sigma_H$. By Lemma 5.5, we complete the proof. \hfill $\Box$

**Proposition 5.7.** Let $N \geq 3, \alpha \in (0, N), p = \tilde{p}, q \in \left(2, 2 + \frac{4}{N}\right)$, $\mu > 0$, $a > 0$, $\mu a^{\alpha - \frac{4}{N}} \leq (2K)^{\frac{N-2p}{N-4}}$, and $\alpha \geq N - 4$ (i.e., $\tilde{p} \geq 2$). If $u$ is a positive solution to (1.4) with $E(u) = \inf_{\mathcal{P}_u} E(v)$, then $u$ has exponential decay at infinity: 
$$u(x) \leq Ce^{-\delta|x|}, \quad |x| \geq r_0,$$ 
for some $C > 0$, $\delta > 0$, and $r_0 > 0$.

**Proof.** By Lemmas 2.7 and 5.1, there exists $\lambda < 0$ such that $u$ satisfies the equation 
$$-\Delta u = \lambda u + \left( I_a \ast |u|^\beta \right) |u|^\beta - u + \mu |u|^\beta - u.$$ (5.2)
By Proposition 5.6, there exist $x_0 \in \mathbb{R}^N$ and a non-increasing positive function $\nu : (0, \infty) \to \mathbb{R}$ such that $u(x) = \nu(|x - x_0|)$ for almost every $x \in \mathbb{R}^N$. Hence, $w = u(x + x_0)$ is a positive and radially non-increasing solution to (5.2). Similar to the estimate of (4.34), there exists $r_0 > 0$ such that 
$$\left( I_a \ast |w|^\beta \right)(x) = C \int_{\mathbb{R}^N} \frac{|w(x - z)|^\beta}{|z|^N} dz \leq -\frac{\lambda}{2}$$ 
for $|x| > r_0$. Hence, if $\tilde{p} > 2$, there exists $C > 0$ such that $w$ satisfies 
$$-\Delta w \leq \lambda w + Cw^{\tilde{p} - 2}w + \mu w^{\tilde{p} - 2}w, \quad |x| \geq r_0,$$ 
and if $\tilde{p} = 2$, $w$ satisfies 
$$-\Delta w \leq \frac{\lambda}{2}w + \mu w^{\tilde{p} - 2}w, \quad |x| \geq r_0.$$ 
Now, repeating word by word the proof of Lemma 2 in [33], we can show that $w$ decays exponentially at infinity. The proof is complete. \hfill $\Box$

**Proof of Theorem 1.7.** By Proposition 5.2, $w = |u|^\beta \in \mathcal{P}_u$ is a positive solution to (1.4) with $E(w) = \inf_{v \in \mathcal{P}_u} E(v)$. Hence, $w$ has exponential decay at infinity by Proposition 5.7, and by Proposition 5.6, there exist $x_0 \in \mathbb{R}^N$ and a non-increasing positive function $\nu : (0, \infty) \to \mathbb{R}$ such that $w = \nu(|x - x_0|)$ for almost every $x \in \mathbb{R}^N$. The proof is complete by using the fact $|u(x)| = (\tau_{|u|}^{N/2})(\frac{x}{\tau_{|u|}}).$ \hfill $\Box$

### 6 Dynamical studies to the equation (1.1)

In this section, we first study the local existence, global existence, and the finite time blow up to the Cauchy problem (1.1), and then study the stability and instability of the standing waves obtained in Sections 3 and 4.
6.1 Local existence

In this subsection, we consider the local existence to the Cauchy problem

\[
\begin{cases}
    i\partial_t \varphi + \Delta \varphi + \left( I_\alpha + |\varphi|^\beta \right) |\varphi|^{\beta-2} \varphi + \mu |\varphi|^{q-2} \varphi, & (t, x) \in \mathbb{R} \times \mathbb{R}^N, \\
    \varphi(0, x) = \varphi_0(x) \in H^1(\mathbb{R}^N), & x \in \mathbb{R}^N.
\end{cases}
\]  

(6.1)

**Definition 6.1.** Let \( N \geq 3 \). The pair \((p, r)\) is said to be Schrödinger admissible, for short \((p, r) \in S\), if

\[
\frac{2}{p} + \frac{N}{r} = \frac{N}{2}, \quad p, r \in [2, \infty].
\]

Define

\[
(p_1, r_1) = \left( \frac{2(N + \alpha)}{N - 2} = 2\bar{p}, \frac{2N(N + \alpha)}{N\alpha + 4 + N^2 - 2N} \right)
\]

and

\[
(p_2, r_2) = \left( \frac{4q}{(q - 2)(N - 2)}, \frac{Nq}{q + N - 2} \right).
\]

Then \((p_1, r_1), (p_2, r_2) \in S\) by direct calculation. For such defined admissible pairs, we define the spaces

\[
\|\varphi\|_{Y_T} = \|\varphi\|_{Y^p_{r_2}, T} + \|\varphi\|_{Y^p_{r_1}, T}, \quad \text{and} \quad \|\varphi\|_{X_T} = \|\varphi\|_{X^p_{r_2}, T} + \|\varphi\|_{X^p_{r_1}, T},
\]

(6.4)

where, for any \( p, r \in (1, \infty) \),

\[
\|\varphi(t, x)\|_{Y^p_{r_1}, T} = \left( \int_0^T \|\varphi(t, \cdot)\|_p^p \, dt \right)^{1/p}
\]

and

\[
\|\varphi(t, x)\|_{X^p_{r_1}, T} = \left( \int_0^T \|\varphi(t, \cdot)\|_q^q \, dt \right)^{1/p}.
\]

**Definition 6.2.** Let \( T > 0 \). We say that \( \varphi(t, x) \) is an integral solution of the Cauchy problem (6.1) on the time interval \([0, T]\) if \( \varphi \in C([0, T], H^1(\mathbb{R}^N)) \cap X_T \), and \( \varphi(t, x) = e^{it\Delta} \varphi_0(x) - i \int_0^t e^{i(t-s)\Delta} g(\phi(s, x)) \, ds \) for all \( t \in (0, T) \),

where \( g(\varphi) = g_1(\varphi) + g_2(\varphi) \), \( g_1(\varphi) = (L_\alpha + |\varphi|^\beta)|\varphi|^{\beta-2} \varphi \), and \( g_2(\varphi) = \mu |\varphi|^{q-2} \varphi \).

Let us recall Strichartz’s estimates that will be useful in the sequel (see, e.g., ([7], Theorem 2.3.3 and Remark 2.3.8) and [42] for the endpoint estimates).

**Lemma 6.3.** Let \( N \geq 3 \), \((p, r)\), and \((\bar{p}, \bar{r}) \in S\). Then there exists a constant \( C > 0 \) such that for any \( T > 0 \), the following properties hold:

(i) For any \( u \in L^2(\mathbb{R}^N) \), the function \( t \mapsto e^{it\Delta} u \) belongs to \( Y^p_{\bar{r}, T} \cap C([0, T], L^2(\mathbb{R}^N)) \) and \( \|e^{it\Delta} u\|_{Y^p_{\bar{r}, T}} \leq C\|u\|_2 \).

(ii) Let \( F(t, x) \in Y^p_{\bar{r}, T}, \) where we use a prime to denote conjugate indices. Then the function

\[
t \mapsto \Phi_F(t, x) = \int_0^t e^{i(t-s)\Delta} F(s, x) \, ds
\]

belongs to \( Y^p_{p, T} \cap C([0, T], L^2(\mathbb{R}^N)) \) and \( \|\Phi_F\|_{Y^p_{p, T}} \leq C\|F\|_{Y^p_{\bar{r}, T}} \).
(iii) For every \( u \in H^1(\mathbb{R}^N) \), the function \( t \mapsto e^{itA}u \) belongs to \( X_{p,p,T} \cap C([0,T],H^1(\mathbb{R}^N)) \) and 
\[
|e^{itA}u|_{X_{p,p,T}} \leq C|u|_{H^1}.
\]

**Lemma 6.4.** Let \( N \geq 3, p = \bar{p}, p \geq N - 4 \) (i.e., \( \bar{p} \geq 2 \)), \( p < N - 2 \), \( (p_1, \tau) \) be defined in (6.2) and \( g(\varphi) \) be defined in Definition 6.2. Then for every \( (\bar{p}, \bar{\tau}) \in S \) there exists a constant \( C > 0 \) such that for every \( T > 0 \),
\[
\left\| \int_0^t e^{i(t-s)\Delta} \nabla g_i(\varphi(s)) \right\|_{Y_{p,i,T}} \leq C\|\nabla \varphi\|^{2p-1}_{H^{p_1,\tau_1,T}}
\]
and
\[
\left\| \int_0^t e^{i(t-s)\Delta}[g_i(\varphi(s)) - g_i(\psi(s))] \right\|_{Y_{p,i,T}} \leq C\left(\|\nabla \varphi\|^{2p-2}_{H^{p_1,\tau_1,T}} + \|\nabla \psi\|^{2p-2}_{H^{p_1,\tau_1,T}}\right)\|\varphi - \psi\|_{V_{p_1,\tau_1,T}}.
\]

**Proof.** By using
\[
|\nabla (|\varphi|^p)| \leq |\varphi|^{p-1}|\nabla \varphi| \quad \text{and} \quad |\nabla (|\varphi|^{p-2}\varphi)| \leq |\varphi|^{p-3}|\nabla \varphi|,
\]
we obtain that
\[
|\nabla g_i(\varphi)| \leq \left| (I_{p} + |\varphi|^p)\varphi|^{p-2}\nabla \varphi \right| + \left| (I_{p} + |\varphi|^{p-2}\nabla \varphi)\varphi|^{p-2}\varphi \right| = I_1 + I_2.
\]
By using
\[
|\varphi|^p - |\psi|^p \leq (|\varphi| + |\psi|)^{p-1}|\varphi - \psi|
\]
and
\[
|\varphi|^{p-2}\varphi - |\psi|^{p-2}\psi \leq (|\varphi|^{p-2} + |\psi|^{p-2})|\varphi - \psi|
\]
we obtain that
\[
|g_i(\varphi) - g_i(\psi)| \leq \left| (I_{p} + |\varphi|^p)(|\varphi|^{p-2}\varphi - |\psi|^{p-2}\varphi) \right| + \left| (I_{p} + |\varphi|^{p-2}\nabla \varphi)(\varphi|^{p-2}\varphi - \psi) \right| \leq \left| (I_{p} + |\varphi|^p)(|\varphi|^{p-2} + |\psi|^{p-2})(\varphi - \psi) \right| + \left| (I_{p} + (|\varphi| + |\psi|)^{p-1})(\varphi - \psi) \right|\psi|^{p-2}\varphi
\]
\[
= I_3 + I_4.
\]

Case \( p > 2 \). Set
\[
a_i = \frac{2N}{N - 2 - \alpha} \quad \text{and} \quad q_i = \frac{2N(N + \alpha)}{(\alpha + 4 - N)(N - 2 + \alpha)},
\]
then
\[
\frac{Na_i}{N + ad_i} \leq \left( 1, \frac{N}{a} \right), \quad \frac{Na_i}{N + ad_i} \bar{p} = (\bar{p} - 2)q_i = N_i^* = \frac{N_i}{N - r_i}, \quad \frac{1}{r_i} = \frac{1}{a_i} + \frac{1}{q_i} + \frac{1}{r_i}.
\]
By using the Hölder inequality, the Hardy-Littlewood-Sobolev inequality, and the Sobolev embedding \( W^{1,\alpha}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \), we have
\[
\|I_1\|_{r_i} \leq \|I_{p} + |\varphi|^p\|_{N_i} \||\varphi|^{p-2}\nabla \varphi\|_{r_i} \leq \|\varphi\|_{N_i} \||\varphi|^{p-2}\nabla \varphi\|_{r_i} = \|\varphi\|^{p-2}_{N_i} \|\varphi\|_{r_i} \leq \|\nabla \varphi\|^{2p-2}_{r_i}. \tag{6.8}
\]
Set
\[ a_2 = \frac{2N}{N - \alpha} \quad \text{and} \quad q_2 = \frac{2N(N + \alpha)}{(\alpha + 2)(N - 2 + \alpha)}, \]
then
\[ \frac{1}{r_1^i} = \frac{1}{a_2} + \frac{1}{1 + q_2}, \quad \frac{Na_2}{N + a a_2} \in \left( 1, \frac{N}{\alpha} \right), \quad \frac{1}{r_1} = \frac{1}{1 + q_2}, \quad \text{and} \quad (\tilde{p} - 1)q_2 = r_1'. \]

By using the Hölder inequality, the Hardy-Littlewood-Sobolev inequality, and the Sobolev embedding \( W^{1,N}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N) \), we have
\[
\left\| I_2 \right\|_{r_1^i} \leq \left\| I_4 \right\| + \left( \left\| \varphi \right\|^{p - 1} \right) \left( \left\| \varphi \right\|^q \right) \left\| I_3 \right\|_{q_2} \quad \leq \left\| \varphi \right\|^{p - 1} \left\| \nabla \varphi \right\|^{\frac{N}{p - 1}q_2} \quad \leq \left\| \varphi \right\|^{p - 1} \left\| \nabla \varphi \right\| \left\| \varphi \right\|^{\frac{N}{p - 1}q_2} \quad \leq \left\| \varphi \right\|^{\frac{N}{p - 1}q_2}. \tag{6.9} \]

By using (6.7), (6.8), and (6.9), we have
\[ \left\| \nabla g(t, \varphi) \right\|_{L_{p_1, r_1}^1} = \left( \int_0^T \left\| g(t) \right\|_{L_{p_1, r_1}^1} \, dt \right)^{\frac{1}{r_1}} \leq \left( \int_0^T \left\| \nabla \varphi(t) \right\|_{L_{p_1, r_1}^1} \, dt \right)^{\frac{1}{r_1}} = \left\| \nabla \varphi \right\|_{L_{p_1, r_1}^1}. \]

Hence, by Lemma 6.3(ii), we obtain that
\[ \left\| \int_0^t e^{(t - s) \Delta} \nabla g(t) \, ds \right\|_{L_{p_1, r_1}^1} \leq \left\| \nabla g(t) \right\|_{L_{p_1, r_1}^1} \leq \left\| \nabla \varphi \right\|_{L_{p_1, r_1}^1}; \]
that is, (6.5) holds.

Similar to (6.8) and (6.9), we obtain that
\[ \left\| I_3 \right\|_{r_1^i} \leq \left( \left\| \varphi \right\|_{r_1^i} + \left\| \psi \right\|_{r_1^i} \right)^{2p - 2} \left\| \varphi - \psi \right\|_{r_1^i} \tag{6.10} \]
and
\[ \left\| I_4 \right\|_{r_1^i} \leq \left( \left\| \varphi \right\|_{r_1^i} + \left\| \psi \right\|_{r_1^i} \right)^{2p - 2} \left\| \varphi - \psi \right\|_{r_1^i}. \tag{6.11} \]

By using the Hölder inequality, we have
\[ \left\| I_3 \right\|_{L_{p_1, r_1}^1} \leq \left( \int_0^T \left( \left\| \varphi(t) \right\|_{r_1^i} + \left\| \psi(t) \right\|_{r_1^i} \right)^{2p - 2} \left\| \varphi(t) - \psi(t) \right\|_{r_1^i} \, dt \right)^{\frac{1}{r_1}} \]
\[ \leq \left( \int_0^T \left( \left\| \varphi(t) \right\|_{r_1^i} + \left\| \psi(t) \right\|_{r_1^i} \right)^{p_1} \, dt \right)^{\frac{1}{r_1}} \left( \int_0^T \left\| \varphi(t) - \psi(t) \right\|_{r_1^i} \, dt \right)^{\frac{1}{r_1}} \]
\[ \leq \left\| \nabla \varphi \right\|_{L_{p_1, r_1}^1} + \left\| \nabla \psi \right\|_{L_{p_1, r_1}^1} \left\| \varphi(t) - \psi(t) \right\|_{L_{p_1, r_1}^1}. \]

Hence, by Lemma 6.3 (ii), we obtain (6.6) holds.

Case \( \tilde{p} = 2 \). Similar to case \( \tilde{p} > 2 \), just in the estimate of \( \left\| I_3 \right\|_{r_1^i} \) and \( \left\| I_4 \right\|_{r_1^i} \) by choosing \( q_1 = \infty \) and \( a_1 = \frac{2N}{N - 2 + \alpha} \), we have (6.8), (6.9), (6.10), and (6.11) hold and then (6.5) and (6.6) hold. The proof is complete. \( \square \)
The following lemma is cited from [26].

**Lemma 6.5.** Let \( N \geq 3, q \in (2, 2^*), (p_2, r_2) \) be defined in (6.3) and \( g_2(\varphi) \) be defined in Definition 6.2. Then for every \((\tilde{p}, \tilde{r}) \in S\) there exists a constant \(C > 0\) such that for every \(T > 0\),

\[
\left\| \int_0^T e^{itNq} \nabla g_2(\varphi(s)) ds \right\|_{\tilde{p}, \tilde{r}, T} \leq CT^{\left(\frac{N(2^*-2)}{4} - 1\right)} \left( \|\nabla \varphi\|_{\tilde{p}, \tilde{r}, T}^{q-1} + \|\nabla \psi\|_{\tilde{p}, \tilde{r}, T}^{q-2} \right) \|\varphi - \psi\|_{\tilde{p}, \tilde{r}, T},
\]

and

\[
\left\| \int_0^T e^{itNq} [g_2(\varphi(s)) - g_2(\psi(s))] ds \right\|_{\tilde{p}, \tilde{r}, T} \leq CT^{\left(\frac{N(2^*-2)}{4} - 1\right)} \left( \|\nabla \varphi\|_{\tilde{p}, \tilde{r}, T}^{q-2} + \|\nabla \psi\|_{\tilde{p}, \tilde{r}, T}^{q-2} \right) \|\varphi - \psi\|_{\tilde{p}, \tilde{r}, T}.
\]

Similar to the proof of Lemma 3.7 in [26], we have the following result.

**Lemma 6.6.** For all \(R, T > 0\) the metric space \((B_{R,T}, d)\) is complete, where

\[B_{R,T} := \{ u \in X_T : \|u\|_{X_T} \leq R \} \quad \text{and} \quad d(u, v) = \|u - v\|_{X_T}.
\]

Now, we are ready to prove the following local existence result.

**Proposition 6.7.** There exists \(Y_0 > 0\) such that if \(\varphi_0 \in H^1(\mathbb{R}^N)\) and \(T \in (0, 1]\) satisfy

\[
\|e^{it\Delta} \varphi_0\|_{X_T} \leq Y_0,
\]

then there exists a unique integral solution \(\varphi(t, x)\) to (6.1) on the time interval \([0, T]\). Moreover, \(\varphi(t, x) \in X_{p,r,T}\) for every \((p, r) \in S\) and satisfies the following conservation laws:

\[
E(\varphi(t)) = E(\varphi_0), \quad \|\varphi(t)\|_2 = \|\varphi_0\|_2, \quad \text{for all} \quad t \in [0, T].
\]

**Proof.** By modifying the proof of Proposition 3.3 in [26], we can show that there exists a unique integral solution \(\varphi(t, x)\) to (6.1) on the time interval \([0, T]\) and \(\varphi(t, x) \in X_{p,r,T}\) for every \((p, r) \in S\). The proofs of the conservation laws (6.13) follow the proofs of Propositions 1 and 2 in [43], which can be repeated mutatis mutandis in the context of (6.1). \(\square\)

### 6.2 Orbital stability

Now we prove Theorem 1.4.

**Proof of Theorem 1.4.** Since in the context of (6.1), we have the local existence result (Proposition 6.7), the proof of Theorem 1.4 can be done by repeating word by word Section 4 in [26] and we omit it. \(\square\)

### 6.3 Orbital instability

In this subsection, we prove Theorem 1.8. For this aim, we first give the following result.
Lemma 6.8. Assume \( N \geq 3, \alpha \in (0, N), p = \tilde{p}, q \in \left(2, 2 + \frac{4}{N}\right), \alpha \geq N - 4 \) (i.e., \( \tilde{p} \geq 2 \)), and \( \alpha < N - 2 \). Let \( u \in S_\alpha \) be such that \( E(\psi(t)) < \inf_{\mathbb{R}_+} E(v) \) and let \( \tau_u \) be the unique global maximum point of \( \Psi(t) \) determined in Lemma 3.3. If \( \tau_u < 1 \), and \( |x|u \in L^2(\mathbb{R}^N) \), then the solution \( \psi(t, x) \) of (1.1) with initial value \( u \) blows up in finite time.

Proof. We claim that
\[
\text{if } u \in S_\alpha \text{ and } \tau_u \in (0, 1), \text{ then } P(u) \leq E(u) - \inf_{\mathbb{R}_+} E(v). \tag{6.14}
\]
Indeed, by using the equality
\[
\Psi(\tau_u) = \Psi(1) + \Psi''(1)(\tau_u - 1) + \Psi''(\xi)(\tau_u - 1)^2, \quad \text{for some } \xi \in (\tau_u, 1),
\]
and noting that \( P(u) < P(\tau_u) = 0, \Psi''(\xi) < 0 \) for \( \xi > \tau_u, \Psi''(1) = P(u), \) and \( \Psi(1) = E(u), \) we obtain that
\[
\inf_{\mathbb{R}_+} E(v) \leq \Psi(\tau_u) \leq E(u) - P(u),
\]
which implies that (6.14) holds.

Now, let us consider the solution \( \psi(t, x) \) with initial value \( u \). By Proposition 6.7, \( \psi(t, x) \in C([0, T_{\max}), H^1(\mathbb{R}^N)) \), where \( T_{\max} \in (0, +\infty) \) is the maximal lifespan of \( \psi(t, x) \). Since by assumption \( \tau_u < 1 \), and the map \( u \mapsto \tau_u \) is continuous, we deduce that \( \tau_{\psi(t)} < 1 \) as well for \( t \) small, say \( t \in [0, t_0] \). By (6.14), the assumption \( E(u) < \inf_{\mathbb{R}_+} E(v) \), and the conservation laws of mass and energy, we obtain that for \( t \in [0, t_0] \),
\[
P(\psi(t)) \leq E(\psi(t)) - \inf_{\mathbb{R}_+} E(v) = E(u) - \inf_{\mathbb{R}_+} E(v) < -\delta.
\]
Hence, \( P(\psi(t)) \leq -\delta \) and then \( \tau_{\psi(t)} < 1 \). Hence, by continuity, the above argument yields
\[
P(\psi(t)) \leq -\delta, \quad \text{for any } t \in [0, T_{\max}).
\]
To obtain a contradiction we recall that, since \( |x|u \in L^2(\mathbb{R}^N) \) by assumption, by the virial identity (see Proposition 6.5.1 in [7]), the function
\[
\Phi(t) = \int_{\mathbb{R}^N} |x|^2|\psi(t, x)|^2 \, dx
\]
is of class \( C^2 \), with \( \Phi''(t) = 8P(\psi(t)) \leq -8\delta \) for every \( t \in [0, T_{\max}) \). Therefore,
\[
0 \leq \Phi(t) \leq \Phi(0) + \Phi''(0)t - 4\delta t^2 \quad \text{for every } t \in [0, T_{\max}).
\]
Since the right hand side becomes negative for \( t \) large, this yields an upper bound on \( T_{\max} \), which in turn implies finite time blow up.

\[\square\]

Proof of Theorem 1.8. By Lemmas 2.7 and 5.1, \( u \) satisfies (1.4) with some \( \lambda < 0 \). Next, we prove the strong instability of \( e^{-\lambda t}u(x) \). For \( s > 1 \), let \( u_s = s^N/2u(sx) \) and \( \psi_s(t, x) \) be the solution to (1.1) with initial value \( u_s \). We have \( u_s \to u \) strongly in \( H^1(\mathbb{R}^N) \) as \( s \to 1^+ \), and hence it is sufficient to prove that \( \psi_s \) blows up in finite time. Let \( \tau_{u_s} \) be defined by Lemma 3.3. Clearly \( \tau_{u_s} = s^{-1} < 1 \), and by the definition of \( \tau_{u_s} \),
\[
E(u_s) < E(u_{s, \tau_{u_s}}) = E(u) = \inf_{\mathbb{R}_+} E(v).
\]
By Theorem 1.7 (3), \( |x|u_s \in L^2(\mathbb{R}^N) \). Hence, by Lemma 6.8, \( \psi_s \) blows up in finite time. The proof is complete.

\[\square\]

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