KITAEV’S LATTICE MODEL AND TURAEV-VIRO TQFTS

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ABSTRACT. In this paper, we examine Kitaev’s lattice model for an arbitrary complex, semisimple Hopf algebra. We prove that this model gives the same topological invariants as Turaev-Viro theory. Using the description of Turaev-Viro theory as an extended TQFT, we prove that the excited states of the Kitaev model correspond to Turaev-Viro theory on a surface with boundary.

INTRODUCTION

In [Kit2003], Kitaev introduced a series of quantum codes on a surface which are suitable for fault-tolerant quantum computation. The best-known of these is the famous toric code, which is based on $\mathbb{Z}_2$. The toric code is simple to describe, but is not general enough to allow for universal quantum computation. One may consider a similar model based on any finite group $G$ or more generally, any finite-dimensional semisimple Hopf algebra (see [BMCA2010]). Given a group $G$, Kitaev constructs a Hilbert space on a triangulated surface and a Hamiltonian, whose ground state is a topological invariant of the surface. The excited states of this Hamiltonian form a mathematical model of anyons, particles which live in two-dimensions and which have been proposed as a model for fault-tolerant quantum computing. The Kitaev model was studied extensively in [BMCA2010], where the authors present the model in the general Hopf algebra setting.

The string-net model was introduced by Levin and Wen in the context of condensed matter physics [LW2005]; in a different language (and in greater generality), this model was also described by Walker in [KW2006]. An excellent exposition of this model and its applications to quantum computing is [KKR2010], and a careful mathematical description can be found in [Kir2011], where the author proves that the string-net model is isomorphic to Turaev-Viro theory.

The relationship between the work of Levin-Wen and Kitaev was discussed in [BA2009]. Finally, a thorough analysis of Kitaev’s model for a finite group is presented in [BD2008], which includes a very detailed description of so-called ribbon operators. It should be noted that these papers are geared toward physicists and are somewhat difficult to read for mathematicians.

The main goal of this paper is to describe Kitaev’s model in a fashion that is easily understandable to mathematicians and to relate it to Turaev-Viro invariants. We begin with the construction of Kitaev’s model for a general (semisimple) Hopf algebra $R$. We carefully review the construction of a Hilbert space and Hamiltonian

\[1\] This is related to the fact that $\mathbb{Z}_2$ is abelian.

\[2\] It is possible to consider the situation in even more generality. The category of representations of a semisimple Hopf algebra forms a spherical category, but not every spherical category arises this way. One may consider lattice models which start with an arbitrary spherical category, in this paper, we will consider categories of representations.
We then examine excited states of the Hamiltonian. These correspond to higher eigenstates of the Hamiltonian, and decompose into a protected space which is a topological invariant of Σ and certain local excitation spaces, which correspond to irreducible representations of \( D(R) \), the Drinfeld double of \( R \). We show that the excited states of the Kitaev model correspond to the Turaev-Viro and string-net models for surfaces with boundary (see [BalK2010], [Kir2011]). This part of the paper is new.

1. HOPF ALGEBRAS

1.1. Basic definitions. Throughout the paper, we denote by \( R \) a finite-dimensional Hopf algebra over \( \mathbb{C} \) with

- multiplication \( \mu_R : R \otimes R \to R \),
- unit \( \eta_R : \mathbb{C} \to R \),
- comultiplication \( \Delta_R : R \to R \otimes R \)
- counit \( \epsilon_R : R \to \mathbb{C} \)
- antipode \( S_R : R \to R \)

We will drop the subscript \( R \) when there is no ambiguity.

We will use the Sweedler notation, writing \( \Delta(x) = x' \otimes x'' \), \( \Delta^2(x) = x' \otimes x'' \otimes x''' \), etc.; summation will be implicit in these formulas. If the number of factors is large, we will also use the alternative notation writing \( \Delta^{(n-1)}(x) = x^{(1)} \otimes x^{(2)} \otimes \cdots \otimes x^{(n)} \).

We will denote by \( R^* \) the dual Hopf algebra. We will use Greek letters \( \alpha, \beta, \ldots \) for elements of \( R^* \). We will also use the Sweedler notation for comultiplication in \( R^* \), writing \( \Delta_R^*(\alpha) = \alpha' \otimes \alpha'' \); thus,

\[
\langle \alpha' \otimes \alpha'', x_1 \otimes x_2 \rangle = \langle \alpha, x_1 x_2 \rangle
\]

where \( \langle \rangle \) stands for the canonical pairing \( R^* \otimes R \to \mathbb{C} \).

From now on, we will also assume that \( R \) is semisimple. The following theorem shows that in fact this condition can be replaced by one of several equivalent conditions.

**Theorem 1.1.** [LR1987] Let \( R \) be a finite-dimensional Hopf algebra over a field of characteristic zero. Then the following are equivalent:

1. \( R \) is semisimple.
2. \( R^* \) is semisimple.
3. \( S_R^2 = \text{id} \).

1.2. Haar integral. Let \( R \) be as described above. Then we have a distinguished element in \( R \) called the Haar integral which is defined by the following conditions:

1. \( hx = xh = \epsilon_R(x)h \) for all \( x \in R \).
2. \( h^2 = h \).

The following theorem lists important properties of the Haar integral.

**Theorem 1.2.** Let \( R \) be a semisimple, finite-dimensional Hopf algebra. Then

1. \( h \) exists and is unique.
2. \( S(h) = h \).
For any \( n \), the element
\[
h_n = \Delta^{(n-1)}(h) \in R^{\otimes n}
\]
is cyclically invariant. In particular, for \( n = 2 \), we have \( \Delta^{op}(h) = \Delta(h) \).

**Lemma 1.3.** The Haar integral acts by 1 in the trivial representation of \( R \) and by 0 in any irreducible non-trivial representation.

**Proof.** This is an immediate consequence of the definition of Haar integral. \( \square \)

**Corollary 1.4.** Let \( W_1, \ldots, W_n \) be finite-dimensional representations of \( R \). Then \( h_n = \Delta^{n-1}(h) \) acts in \( W_1 \otimes \cdots \otimes W_n \) by projection onto the invariant subspace
\[
\{ w \in W_1 \otimes \cdots \otimes W_n \mid \Delta^{n-1}(x)w = \varepsilon(x)w \} \simeq \text{Hom}_R(1, W_1 \otimes \cdots \otimes W_n)
\]

1.3. **Representations.** Since \( R \) is semisimple, any representation of \( R \) is completely reducible. We will denote by \( V_i, i \in I \), a set of representatives of isomorphism classes of irreducible representations of \( R \); we will also use the notation \( d_i = \dim V_i \). In particular, the trivial one-dimensional representation of \( R \) will be denoted \( 1 = V_0 \).

We will frequently use the notation
\[
\langle W_1, \ldots, W_n \rangle = \text{Hom}_R(1, W_1 \otimes \cdots \otimes W_n) \simeq \{ w \in W_1 \otimes \cdots \otimes W_n \mid \Delta^{n-1}(x)w = \varepsilon(x)w \}
\]
(compare with Corollary 1.4).

Given a representation \( V \) of \( R \), we can define the dual representation \( V^* \). As a vector space \( V^* \) is the ordinary dual space to \( V \). The action of \( R \) is defined by
\[
\langle x\alpha, v \rangle = \langle \alpha, S(x)v \rangle
\]
where \( \alpha \in V^*, v \in V, \) and \( x \in R \). Note that since \( S^2 = \text{id} \), the usual vector space isomorphism \( V^{**} \simeq V \) is an isomorphism of representations; thus, \( V^{**} \) is canonically isomorphic to \( V \) as a representation of \( R \).

Since a dual of an irreducible representation is irreducible, we have an involution \( \vee : I \rightarrow I \) such that \( V_i^* \simeq V_i^\vee \). This isomorphism is not canonical; however, one does have a canonical isomorphism
\[
V_i \otimes V_i^* \simeq V_i^\vee \otimes V_i^\vee.
\]

For any two representations \( V, W \) of \( R \), we denote by \( \text{Hom}_R(V, W) \) the space of \( R \)-morphisms from \( V \) to \( W \). We have a non-degenerate pairing
\[
\langle \varphi, \psi \rangle = \langle 1 \rightarrow (V \otimes V^*) \xrightarrow{\varphi \otimes \psi} W \otimes W^* \rightarrow 1 \rangle
\]
given by
\[
\text{Hom}_R(V, W) \otimes \text{Hom}_R(V^*, W^*) \rightarrow \mathbb{C}
\]

By semisimplicity, we have a canonical isomorphism
\[
R \cong \bigoplus_{i \in I} \text{End}_\mathbb{C}(V_i) = \bigoplus_i V_i \otimes V_i^*.
\]

It is convenient to write the Hopf algebra structure of \( R \) in terms of this isomorphism.

**Lemma 1.5.** Under isomorphism \( \rho \), multiplication, comultiplication, unit, counit, and antipode of \( R \) are given by
• **Multiplication:** \( \mu_R : V_i \otimes V_i^* \otimes V_j \otimes V_j^* \xrightarrow{1 \otimes \text{ev} \otimes 1} \delta_{i,j} V_i \otimes V_i^* \), where ev : \( V_i^* \otimes V_i \rightarrow \mathbb{C} \) is the evaluation map.

• **Comultiplication:** Let \( \varphi_\alpha \) be a basis for \( \text{Hom}_R(V_i, V_j \otimes V_k) \) and \( \varphi_\alpha^* \in \text{Hom}_R(V_i^*, V_j^* \otimes V_k^*) \) the dual basis with respect to the pairing given by (1.4). Then

\[
\Delta_R : \bigoplus_i V_i \otimes V_i^* \sum d_i \varphi_\alpha \otimes \varphi_\alpha^* \rightarrow \bigoplus_{j,k} V_j \otimes V_k \otimes V_j^* \otimes V_k^*
\]

\[
\xrightarrow{T_{23} \circ T_{34}} \bigoplus_{j,k} V_j \otimes V_j^* \otimes V_k \otimes V_k^*
\]

• **Unit:** \( \eta_R = \bigoplus_i \dim V_i \sum_{j=1}^{\dim V_i} v_j \otimes v_j^* = \sum_i \text{id}_{V_i}, \) where \( \{v_j\} \) is a basis for \( V_i \) and \( \{v_j^*\} \) is the dual basis.

• **Counit:** For \( a \otimes b \in V_i \otimes V_i^* \), \( \epsilon_R(a \otimes b) = \delta_{i,0} b(a) \)

• **Antipode:** If \( a \otimes b \in V_i \otimes V_i^* \), then

\[
S(a \otimes b) = b \otimes a \in V_i^* \otimes V_i \simeq V_i^\vee \otimes V_i^\vee
\]

(see (1.3)).

In this language, the Haar integral is given by the canonical element \( 1 \in V_0 \otimes V_0^* \).

1.4. **Graphical calculus.** We will frequently use graphical presentation of morphisms between representations of \( R \). We will use the same conventions as in [Kir2011], representing a morphism \( \varphi : W_1 \otimes \ldots W_k \rightarrow W'_1 \otimes \cdots \otimes W'_l \) by a tangle with \( k \) strands labeled \( W_1, \ldots, W_k \) at the top and \( l \) strands labeled \( W'_1, \ldots, W'_l \) at the bottom. We will also use the usual cap and cup tangles to represent evaluation and coevaluation morphisms.

1.5. **Dual Hopf algebra.** Given a semisimple Hopf algebra \( R \), we will define the following version of the dual Hopf algebra

\[
\overline{R} = (R^{\text{op}})^*
\]

where \( R^{\text{op}} \) denotes the algebra \( R \) with opposite multiplication.

Since comultiplication in \( \overline{R} \) is opposite to comultiplication in \( R^* \), notation \( \alpha', \alpha'' \) is ambiguous. We adopt the following convention: notation \( \alpha', \alpha'' \) (or, equivalently, \( \alpha^{(1)}, \alpha^{(2)}, \ldots \)) always refers to comultiplication in \( R^* \). Thus, comultiplication in \( \overline{R} \) is given by

\[
\Delta_{\overline{R}}(\alpha) = \alpha'' \otimes \alpha'.
\]

Note that \( \overline{R} \) is canonically isomorphic to \( R \) as a vector space but as a Hopf algebra, has opposite multiplication and comultiplication. Thus, the map

\[
S : R \rightarrow \overline{R}
\]

is an isomorphism of Hopf algebras.

Note that by Theorem 1.1, \( \overline{R} \) is also semisimple and thus has a unique Haar integral. We will denote by

\[
\bar{h} \in \overline{R}
\]

the Haar integral of \( \overline{R} \).
Lemma 1.6. Let \( R \) be a semisimple Hopf algebra. Then the Haar integral of \( \overline{R} \) is given by

\[
\langle \overline{h}, x \rangle = \frac{1}{\dim \overline{R}} \text{tr}_R(x)
\]

where \( \text{tr}_R(x) = \sum d_i \text{tr}_{V_i}(x) \) is the trace of action of \( x \) in the (left or right) regular representation.

We can also rewrite the Haar integral of \( R \) in terms of the isomorphism \( R \cong \bigoplus V_i \otimes V_i^* \) (see (1.5)).

Lemma 1.7. Let \( x_k = v_k \otimes w_k \in V_k \otimes V_k^* \); using isomorphism (1.5), each \( x_k \) can be considered as element of \( R \). Then

\[
\langle \overline{h}, x_1 \ldots x_n \rangle = \begin{cases} 
\frac{d_i}{\dim \overline{R}} (v_1, w_n) (v_1, v_2) \ldots (w_{n-1}, v_n), & i_1 = i_2 = \cdots = i_n = i, \\
0, & \text{otherwise}
\end{cases}
\]

(see Figure 1).

\[
\langle \overline{h}, (v_1 \otimes w_1) \ldots (v_n \otimes w_n) \rangle = \frac{d_i}{\dim \overline{R}} v_1 w_1 v_2 w_2 \cdots v_n w_n
\]

**Figure 1.** The Haar integral of \( \overline{R} \)

1.6. Regular action. With have two obvious actions of \( R \) on itself, called left and right regular actions:

- **\( L_x \):** \( y \mapsto xy \)
- **\( R_x \):** \( y \mapsto yS(x) \)

Note that these actions commute.

In a similar way, the dual Hopf algebra \( \overline{R} \) can also be endowed with two commuting actions of \( \overline{R} \).

Using (1.2), we can also define two actions of \( R \) on \( \overline{R} \):

- **\( L^*_x \):** \( \lambda \mapsto x.\lambda \), where \( \langle x.\lambda, y \rangle = \langle \lambda, S(x)y \rangle \)
- **\( R^*_x \):** \( \lambda \mapsto \lambda.S(x) \), where \( \langle \lambda.x, y \rangle = \langle \lambda, yS(x) \rangle \)

and two actions of \( \overline{R} \) on \( R \):

- **\( L^*_\alpha \):** \( x \mapsto \alpha.x := (\alpha, S(x'))x'' \), so that \( \langle \lambda, \alpha.x \rangle = \langle S(\alpha)\lambda, x \rangle \)
- **\( R^*_\alpha \):** \( x \mapsto x.\alpha := (\alpha', x')x'' \), so that \( \langle \lambda, x.\alpha \rangle = \langle \lambda S(\alpha), x \rangle \)

Note that notation \( x.\alpha \) is ambiguous, as it can mean \( L^*_\alpha(x) \) or \( R^*_\alpha(x) \). In most cases the meaning will be clear from the context.

Note that we can also define left and right action of the Hopf algebra \( \overline{R} \) on \( \overline{R} \). It is easy to see that these actions are given by

- **\( L_t \):** \( \lambda \mapsto \lambda t \), where \( t \in \overline{R} \) is considered as an element of \( R \) via trivial vector space isomorphism.
- **\( R_t \):** \( \lambda \mapsto S(t)\lambda \), where \( t \in \overline{R} \) is considered as an element of \( R \) via trivial vector space isomorphism.
We will use these actions (together with left and right regular actions of $R$ on itself) repeatedly throughout the rest of the paper. All the operators we will discuss can be defined in terms of them.

**Lemma 1.8.** Let $\overline{h}$ be the Haar integral of $\overline{R}$. Consider the left regular action of $\overline{h}$ on $R^\otimes n$.

\begin{equation}
L^*_h: R^\otimes n \to R^\otimes n
\end{equation}

$x_n \otimes \cdots \otimes x_1 \mapsto \overline{h}^{(n)}x_n \otimes \cdots \otimes \overline{h}^{(1)}x_1 = \langle \overline{h}, S(x_n' \ldots x_1') \rangle x_n'' \otimes \cdots \otimes x_1''$

(recall that comultiplication in $\overline{R}$ is given by $\Delta^{(n-1)} = \alpha^{(n)} \otimes \cdots \otimes \alpha^{(1)}$).

Then, after identifying each copy of $R$ with $\bigoplus V_i \otimes V_i^*$ (see (1.5)), $L^*_h$ is given by the following picture:

\[
\sum_{i_1, \ldots, i_n, j_1, \ldots, j_n, k} \frac{d_{i_1} \ldots d_{i_n} d_k}{\dim R} \sum_{\alpha, \beta, \ldots} \sum_{i_n^*, i_n} k \quad \sum_{j_1^*, j_1} k
\]

where $\varphi^\alpha, \varphi^\alpha$ are as in Lemma 1.5.

Note that the crossings in the picture are just permutation of factors (there is no braiding in the category $\text{Rep} R$) and the whole map is not a morphism of representations.

**Proof.** Follows by combining the formula for multiplication and comultiplication in $R$ (Lemma 1.5) and Lemma 1.7.

\[\square\]

1.7. **Drinfeld double.** Given a finite-dimensional semisimple Hopf algebra $R$, there is a well-known way to construct from it a quasitriangular Hopf algebra $D(R)$. This new Hopf algebra, called the Drinfeld Double of $R$, has numerous applications in representation theory and physics. We will review some basic details of the construction. For a much more detailed description and proofs see, e.g. [Kas], [ES2002].

Let $R$ be a semisimple, finite-dimensional Hopf algebra; as before, let $\overline{R} = (R^{op})^*$. 

**Theorem 1.9.** The following operations define on the vector space $R \otimes \overline{R}$ a structure of a Hopf algebra. This Hopf algebra will be denoted $D(R)$ and called Drinfeld double of $R$. 

(1) Multiplication:
\[(x \otimes \alpha) \cdot (y \otimes \beta) = xy'' \otimes \alpha'' \beta,\]
where \(\alpha'' \in \overline{R}\) is defined by
\[
\langle \alpha'', v \rangle = \langle \alpha, y'' z S^{-1} (y') \rangle
\]
(1.9)

(2) Unit: \(1_{D(R)} = 1_R \otimes 1_{\overline{R}}\)

(3) Comultiplication: \(x \otimes \alpha \mapsto (x' \otimes \alpha'') \otimes (x'' \otimes \alpha')\)

(4) Counit: \(x \otimes \alpha \mapsto \epsilon_R (x) \epsilon_{\overline{R}} (\alpha)\)

(5) Antipode: \(S(x \otimes \alpha) = S(\alpha) S(x) = S(x'') S(\alpha) S(x)\)

Remark 1.10. This definition follows [ES2002] and differs slightly from the one given in [Kas], where \(D(R)\) is defined as \(R \otimes R\). However, it is not difficult to show that these definitions are equivalent.

Recall (see [Kas]) that \(D(R)\) has a canonical quasitriangular structure, with \(R\)-matrix \(R = \sum x_\alpha \otimes x_\alpha\), where \(x_\alpha, x_\alpha\) are dual bases in \(R, \overline{R}\) respectively. Thus, the category of finite-dimensional representations of \(D(R)\) has a structure of a braided ribbon category. Moreover, it is known that the category of representations of \(D(R)\) is in fact equivalent to the so-called Drinfeld Center of the category of representations of \(R\) (see [Mug2003a, Mug2003b]). In particular, for any representation \(Y\) of \(D(R)\) and a representation \(V\) of \(R\), we have a functorial isomorphism
\[
V \otimes Y \mapsto Y \otimes V
\]
(1.10)
\[
v \otimes y \mapsto \sum_\alpha x^\alpha y \otimes x_\alpha \nu
\]
This map (or sometimes its inverse) will be called the half-braiding.

Lemma 1.11. Let \(R\) be a semisimple, finite-dimensional Hopf algebra. Then \(D(R)\) is semisimple with Haar integral given by
\[
h_{D(R)} = h_R \otimes h_{\overline{R}}
\]
(1.11)
Moreover, both \(h, \overline{h}\) are central in \(D(R)\).

1.8. Action of \(D(R)\) on \(R \otimes R\). In this section, we define an action of \(D(R)\) on \(R \otimes R\). This will be used in the future.

Lemma 1.12. For \(\alpha \in R, \alpha \in \overline{R}\), define the operators \(p_\alpha: R \otimes R \to R \otimes R\) and \(q_\alpha: R \otimes R \to R \otimes R\) by
\[
p_\alpha (u \otimes v) = a' u \otimes v S(a'')
\]
\[
q_\alpha (u \otimes v) = \langle \alpha, S(u' v') \rangle a'' \otimes v'' = \alpha'' u \otimes a' v.
\]
Then these operators satisfy the commutation relations of \(D(R)\): the map
\[
D(R) \to \text{End}(R \otimes R)
\]
\[
\alpha \otimes \alpha \mapsto p_\alpha q_\alpha
\]
is a morphism of algebras.
Proof. This follows by explicit computation \((\text{BMCA2010})\), using the following formulas:

\[
\alpha(xy) = x''(\alpha_x y) = (y\alpha x)y'', \quad \text{where}
\]

\[
\langle \alpha_x, z \rangle = \langle \alpha, zS(x') \rangle
\]

\[
\langle y\alpha, z \rangle = \langle \alpha, S(y')z \rangle
\]

\[\square\]

2. Kitaev’s Model

In this section, we look at Kitaev’s lattice model. This model is a generalization of the well-known toric code; we get a theory for any finite-dimensional semisimple Hopf algebra \(R\) over \(\mathbb{C}\). We begin with a compact, oriented surface \(\Sigma\) with a fixed cell decomposition \(\Delta\).\(^3\) We will assign to \((\Sigma, \Delta)\) a finite-dimensional Hilbert space \(\mathcal{H}_K(\Sigma)\) and introduce a Hamiltonian consisting of local operators. The ground state of this Hamiltonian is useful for quantum computation. We will later show see that this ground state can be identified with the Turaev-Viro vector space \(Z_{TV}(\Sigma)\). This obviously implies that the ground state is a topological invariant of \(\Sigma\); in particular, it does not depend on the cell decomposition \(\Delta\).

From now on, we fix a choice of finite-dimensional, semisimple Hopf algebra \(R\).

2.1. Crude Hilbert space. Given a compact oriented surface \(\Sigma\) with a cell decomposition \(\Delta\), we denote by \(E\) the set of (unoriented) edges of \(\Delta\). Then for any choice \(o\) of orientation of each edge of \(\Sigma\), we define the space

\[
\mathcal{H}_K(\Sigma, \Delta, o) = \bigotimes_E R
\]

We will graphically represent a vector \(\bigotimes x_e \in \mathcal{H}_K\) by writing the vector \(x_e\) next to each edge \(e\).

So defined vector space depends on the choice of orientation. However, in fact vector spaces coming from different orientations are canonically isomorphic. Namely, if \(o\) and \(o'\) are two orientations that differ by reversing orientation of a single edge \(e\), we identify

\[
\mathcal{H}_K(o) \to \mathcal{H}_K(o')
\]

\[
x_e \mapsto S(x_e)
\]

(see Figure 2). Note that since \(S^2 = \text{id}\), this isomorphism is well defined. This shows that all spaces \(\mathcal{H}_K(o)\) for different choices of orientation are canonically isomorphic to each other; thus, we will drop the choice of orientation from our formulas writing just \(\mathcal{H}_K(\Sigma, \Delta)\).

The Hilbert space \(\mathcal{H}_K\) is clearly not a topological invariant; in particular, its dimension depends on number of edges in \(\Delta\).

\(^3\)Some papers begin instead with a surface \(\Sigma\) with an embedded graph \(\Gamma\). This is clearly equivalent data; the graph \(\Gamma\) corresponds to the 1-skeleton of the cell decomposition.
2.2. **Vertex and plaquette operators.** We now define a collection of operators on $\mathcal{H}_K$; in the next section, we will use them to construct the Hamiltonian on $\mathcal{H}_K$. As before, we fix a closed oriented surface $\Sigma$ and a choice of cell decomposition $\Delta$.

**Definition 2.1.** Let $(\Sigma, \Delta)$ be a surface with a cell decomposition. A *site* $s$ is a pair $(v, p)$ where $v$ is a vertex of $\Delta$ and $p$ is an adjacent plaquette (face).

A typical site is shown in Figure 3. We will depict a site as a green line connecting a vertex to the center of an adjacent plaquette. Equivalently, if we superimpose the dual lattice, a site connects a vertex in the lattice to an adjacent vertex in the dual lattice.

![Figure 3](image)

**Figure 3.** The site $s = (v, p)$ is drawn as a green line connecting $v$ and the center of $p$.

At each vertex $v$, we have a natural counterclockwise cyclic ordering of edges incident to $v$. Similarly, given a plaquette (2-cell) $p$, we have the *clockwise* cyclic ordering of edges on $\partial p$.

**Definition 2.2.** Given a site $s = (v, p)$ of the cell decomposition $\Delta$ and an element $a \in R$, the vertex operator $A_{v, p}^a : \mathcal{H}_K(\Sigma, \Delta) \to \mathcal{H}_K(\Sigma, \Delta)$ is defined by

$$A_{v, p}^a : x \mapsto S(a) x,$$

where the edges incident to $v$ are indexed *counterclockwise* starting from $p$.

In the definition above, the edges incident to $v$ are all pointing away from the vertex. It is easy to see, using (2.2), that if any edge is oriented towards the vertex, then left action would be replaced by the right action: instead of $a^{(i)} x_i$, we would have $x_i S(a^{(i)})$.

In a similar way, one defines the plaquette operators.
Definition 2.3. Given a site \( s = (v, p) \) of the cell decomposition \( \Delta \) and an element \( \alpha \in R \), the plaquette operator \( B^\alpha_{p,v} : H_K(\Sigma, \Delta) \to H_K(\Sigma, \Delta) \) is defined by

\[
B^\alpha_{p,v} : \quad p \xrightarrow{\alpha.(x)} \alpha.(x)\alpha.(x)_{x_1} \xrightarrow{\alpha.(x)_{x_2}} \alpha.(x)_{x_3} = \langle \alpha, S(x_1' \ldots x_1') \rangle
\]

where \( \alpha.x \) stands for left action of \( R \) on \( R \) as defined in Section 1.6.

In the definition above, the edges surrounding \( p \) are all given a clockwise orientation (even though the indices go counterclockwise). It is easy to see, using (2.2), that if any edge is oriented counterclockwise, then left action would be replaced by the right action: instead of \( \alpha.(x)_{x_i} \), we would have \( x_i . S(\alpha.(x)) \).

Theorem 2.4.

1. If \( v, w \) are distinct vertices, then the operators \( A^a_v \), \( A^b_w \) commute for any \( a, b \in R \).
   Similarly, if \( p, q \) are distinct plaquettes, then the operators \( B^\alpha_p \), \( B^\beta_q \) commute for any \( \alpha, \beta \in \overline{R} \).
2. If \( v, p \) are not incident to one another, then operators \( A^a_v \), \( B^\alpha_p \) commute.
3. For a given site \( s = (v, p) \), the operators \( A^a_{v,p} \), \( B^\alpha_{p,v} \) satisfy the commutation relations of Drinfeld double of \( R \): the map

\[
(2.3) \quad \rho_s : D(R) \to \text{End}(H_K(\Sigma, \Delta))
\]

\[
(2.4) \quad a \otimes \alpha \mapsto A^a_v B^\alpha_p
\]

is an algebra morphism.

Proof. (1) The operators \( A_v \), \( A_w \) obviously commute if the edges incident to \( v \) and those incident to \( w \) are disjoint. We therefore assume that \( v \) and \( w \) are adjacent, i.e. at least one edge connects them. Clearly, we need only to check that the actions of \( A_v \), \( A_w \) commute on their common support. Suppose such an edge \( e \) is oriented so that it points from \( v \) to \( w \). Then \( A_v \) acts on the corresponding copy of \( R \) via the left regular representation, and \( A_w \) acts on \( e \) via the right regular representation. These are obviously commuting actions. The proof for plaquette operators is similar.

(2) Obvious.

(3) Follows from the following generalization of Lemma 1.12, proof of which we leave to the reader.

Lemma 2.5. Let \( X \) be a representation of \( R \), and \( Y \) – a representation of \( \overline{R} \). For \( a \in R \), \( \alpha \in \overline{R} \), define the operators \( p_a, q_\alpha \in \text{End}(R \otimes X \otimes Y \otimes R) \)
by
\[ p_\alpha (u \otimes x \otimes y \otimes v) = a' u \otimes a'' x \otimes y \otimes v S(a''') \]
\[ q_\alpha (u \otimes x \otimes y \otimes v) = a'''. u \otimes x \otimes a''. y \otimes a'. v \]

Then these operators satisfy the commutation relations of \( D(R) \): the map
\[ D(R) \to \text{End}(R \otimes X \otimes Y \otimes R) \]
\[ a \otimes \alpha \mapsto p_\alpha q_\alpha \]
is a morphism of algebras.

\[ \square \]

2.3. Duality. The \( A \) and \( B \) projectors are dual to one another in the following sense. Consider a dual theory, in which we begin with the dual cell decomposition \( \Delta^* \) with edge orientation inherited from \( \Delta \) as shown in Figure 4 and the dual Hopf algebra \( \overline{R} \).

![Figure 4](image.png)

**Figure 4.** The convention for orienting edges of the dual graph is shown above. Here the solid black edge is from the original cell decomposition \( \Delta \) and the dashed blue arrow belongs to the dual one \( \Delta^* \).

We get the Hilbert space \( \overline{H}_K \) which may be identified with \( H^*_K \) using the evaluation pairing \( \text{ev}: R \otimes R \to \mathbb{C} \). Note that the vertices in \( \Delta^* \) correspond to plaquettes in \( \Delta \) and vice versa. The following lemma shows that the vertex operators from one theory correspond naturally to the plaquette operators from the other.

**Lemma 2.6.** Under the natural pairing \( \langle ., . \rangle \) of \( \overline{H}_K \) and \( H_K \), we have
\[ \langle y, B^\alpha_s x \rangle = \langle A^{S(\alpha)} y, x \rangle \]
\[ \langle y, A^\alpha_s x \rangle = \langle B^\alpha y, x \rangle \]
where \( x \in H_K, y \in \overline{H}_K, a \in R, \alpha \in \overline{R} \) and \( s \) is a site (both in \( \Delta \) and \( \Delta^* \)).

**Proof.** Let \( s \) be a site; label edges of \( \Delta, \Delta^* \) around \( s \) as shown in Figure 5. Then
\[ \langle y, B^\alpha_s x \rangle = \langle y_1, \alpha^{(1)} x_1 \rangle \ldots \langle y_n, \alpha^{(n)} x_n \rangle \]
\[ = \langle S(\alpha^{(1)}) y_1, x_1 \rangle \ldots \langle S(\alpha^{(n)}) y_n, x_n \rangle \]
\[ = \langle S(\alpha)^{(1)} y_1, x_1 \rangle \ldots \langle S(\alpha)^{(1)} y_n, x_n \rangle \]
\[ = \langle A^{S(\alpha)} y, x \rangle \]
(recall that comultiplication in \( \overline{R} \) is given by \( \Delta^{(n-1)} R = \beta^{(n)} \otimes \ldots \otimes \beta^{(1)} \).)
The second identity is proved similarly. \[ \square \]
2.4. **The groundspace.** Let $\mathcal{H}_K(\Sigma, \Delta)$ be as in Section 2.1. Consider the following special case of the vertex and plaquette operators:

\begin{align*}
A_v &= A_h^v \\
B_p &= B_{\bar{h}}^p
\end{align*}

where $h \in R, \bar{h} \in \overline{R}$ are the Haar integrals of $R, \overline{R}$.

Note that since $\Delta^n(h)$ is cyclically invariant (see Theorem 1.2), the operator $A_v$ only depends on the vertex $v$ and not on the choice of the adjacent plaquette $p$ (which was used before to construct the linear ordering of the edges adjacent to $v$); similarly, $B_p$ only depends on the choice of $p$.

Using these operators, we define the Hamiltonian $H : \mathcal{H}_K \to \mathcal{H}_K$ by

\begin{equation}
H = \sum_v (1 - A_v) + \sum_p (1 - B_p)
\end{equation}

The most important property of this Hamiltonian is that it consists of commuting operators.

**Theorem 2.7.**

1. All operators $A_v, B_p$ commute with each other.
2. Each of these operators is idempotent: $A_v^2 = A_v, B_p^2 = B_p$.

**Proof.** Immediately follows from Theorem 2.4 and $h^2 = h$, $h$ is central (Theorem 1.2). \hfill $\square$

The Hamiltonian (2.7) is a sum of these local projectors and since they all commute, $H$ is diagonalizable.

**Definition 2.8.** The ground state $K_R(\Sigma, \Delta)$ of Kitaev’s model is the zero eigenspace of $H$:

\begin{equation}
K_R(\Sigma, \Delta) = \{ x \in \mathcal{H}_K(\Sigma, \Delta)| Hx = 0 \}.
\end{equation}

It is easy to see that $x \in K_R(\Sigma)$ iff $A_v x = B_p x = x$ for every vertex $v$ and plaquette $p$.

We will show below that up to a canonical isomorphism, the groundspace does not depend on the choice of the cell decomposition.
3. Turaev–Viro and Levin-Wen models

In this section, we give an overview of two other theories: Turaev–Viro and Levin-Wen (stringnet) models. All results of this section are known and given here just for the readers convenience.

We will mostly follow the approach and notation of our earlier papers [BalK2010, Kir2011], to which the reader is referred for more detail and references.

Throughout the section, we let \( \mathcal{A} \) be a spherical fusion category, i.e. a fusion category together with a functorial isomorphism \( V \cong V^{**} \) satisfying appropriate properties. We will denote by \( \{V_i, i \in I\} \) the set of representatives of isomorphism classes of simple objects in \( \mathcal{A} \), and by \( d_i \dim V_i \) the categorical dimension of \( V_i \). We will also use the notation \( D = \sqrt{\sum d_i^2} \).

Note that for every semisimple finite-dimensional Hopf algebra \( R \), the category \( \mathcal{A} = \text{Rep}(R) \) of finite-dimensional representations of \( R \) is a spherical fusion category, and the notation \( V_i, d_i \) agree with the notation of Section 1. In this case, \( D^2 = \dim R \).

3.1. Turaev–Viro model. Let \( \mathcal{A} \) be a spherical fusion category as above. Then one can define a 3-dimensional TQFT \( Z^A_{TV} \), called the Turaev–Viro model; it was originally defined in [TV1992] and generalized to arbitrary spherical categories by Barrett and Westbury [BW1996]. In particular, for any closed oriented surface \( \Sigma \) this theory gives a vector space \( Z^A_{TV}(\Sigma) \), defined as follows.

First, we choose a cell decomposition \( \Delta \) of \( \Sigma \). A coloring of edges of \( \Delta \) is a choice, for every oriented edge \( e \) of \( \Delta \) of a simple object \( l(e) \) so that \( l(e) = l(e)^\ast \).

We define the state space

\[
H^A_{TV}(\Sigma, \Delta) = \bigoplus_l \bigotimes_C H(C, l)
\]

where \( l \) is a coloring of edges of \( \Delta \), \( C \) is a 2-cell of \( \Delta \), and

\[
H(C, l) = \langle l(e_1), l(e_2), \ldots, l(e_n) \rangle, \quad \partial C = e_1 \cup e_2 \cdots \cup e_n
\]

where the edges \( e_1, \ldots, e_n \) are taken in the counterclockwise order on \( \partial C \) as shown in Figure 6.

![Figure 6. State space for a cell](image)

Next, given a cobordism \( M \) between two surfaces \( \Sigma, \Sigma' \) with cell decompositions, one can define an operator \( Z(M): H^A_{TV}(\Sigma, \Delta) \to H^A_{TV}(\Sigma', \Delta') \); it is defined using a cell decomposition of \( M \) but can be shown to be independent of the choice of the decomposition (see [BalK2010 Theorem 4.4]). In particular, taking \( M = \Sigma \times I \), we get an operator \( Z(\Sigma \times I): H^A_{TV}(\Sigma, \Delta) \to H^A_{TV}(\Sigma, \Delta) \) which can be shown to be a projector. We now define the Turaev–Viro space associated to \( \Sigma \) as

\[
Z^A_{TV}(\Sigma, \Delta) = \text{Im}(Z(M \times I)).
\]
It can be shown that for any two cell decompositions $\Delta, \Delta'$ of the same surface $\Sigma$, we have a canonical isomorphism $Z^A_{TV}(\Sigma, \Delta) \simeq Z^A_{TV}(\Sigma, \Delta')$ (see [BalK2010]); thus, this space is determined just by the surface $\Sigma$. Therefore, we will omit $\Delta$ in the notation, writing just $Z^A_{TV}(\Sigma)$.

3.2. Stringnet model. There is also another way of constructing a vector space associated to an oriented closed surface $\Sigma$; this construction was introduced in the papers of Levin and Wen [LW2005]. We will refer to it as stringnet model (or sometimes as Levin-Wen model). In this section we give an overview of this model, following the conventions of [Kir2011].

In this model, we again begin with a spherical fusion category $A$ and consider colored graphs $\Gamma$ on $\Sigma$. Edges of the graph should be oriented and colored by objects of $A$ (not necessarily simple); vertices are colored by morphisms $\psi_v \in \text{Hom}_A(1, V_1 \otimes \cdots \otimes V_n)$, where $V_i$ are colors of edges incident to $v$ taken in counterclockwise order and with outward orientation (if some edges come with inward orientation, the corresponding $V_i$ should be replaced by $V^*_i$).

**Figure 7. Labeling of colored graphs**

We will follow the conventions of [Kir2011]: in particular, if a graph contains a pair of vertices, one with outgoing edges labeled $V_1, \ldots, V_n$ and the other with edges labeled $V^*_n, \ldots, V^*_1$, and the vertices are labeled by the same letter $\alpha$ (or $\beta$, or $\ldots$) it will stand for summation over the dual bases:

\[
\sum_{i \in \text{Irr}(A)} \alpha_i \varphi_{\alpha} 
\]

where $\varphi_{\alpha} \in \langle V_1, \ldots, V_n \rangle$, $\varphi^\alpha \in \langle V^*_n, \ldots, V^*_1 \rangle$ are dual bases with respect to pairing (1.4).

We then define the stringnet space

\[
H^{str}(\Sigma) = \text{Formal linear combinations of colored graphs on } \Sigma / \text{Local relations}
\]

Local relations come from embedded disks in $\Sigma$; the precise definition can be found in [Kir2011]. Here we only give one local relation which will be useful in the future:

\[
\sum_{i \in \text{Irr}(A)} d_i \alpha_i = V_1 \cdots V_n
\]
The following result has been stated in a number of papers; a rigorous proof can be found in [Kir2011].

**Theorem 3.1.** Let $A$ be a spherical fusion category and $\Sigma$ a closed oriented surface. Then one has a canonical isomorphism $Z^A_{TV}(\Sigma) \simeq H_{\text{str}}(\Sigma)$.

In fact, we will need a more detailed version of the theorem above. Namely, let $\Delta$ be a cell decomposition of $\Sigma$. Let $\Sigma - \Delta^0$ be the surface with punctures obtained by removing from $\Sigma$ all vertices of $\Delta$ and let $H_{\Delta}^\text{str} = H_{\text{str}}(\Sigma - \Delta^0)$ be the corresponding stringnet space. Then one has the following results.

**Theorem 3.2.**

1. The natural map $H_{\Delta}^\text{str} \rightarrow H_{\text{str}}$ induces an isomorphism $H_{\text{str}} \simeq \text{Im}(B^s) = \{ x \in H_{\Delta}^\text{str} \mid B^s_p x = x \ \forall p \} \subset H_{\Delta}^\text{str}$ where $B^s_p = \prod_{p} B^s_p$, $p$ runs over the set of vertices of $\Delta$ and $B^s_p: H_{\Delta}^\text{str} \rightarrow H_{\Delta}^\text{str}$ is the operator which adds to a colored graph a small loop around puncture $p$ as shown below. (The superscript $\text{str}$ is introduced to avoid confusion with the plaquette operators $B_p$ in Kitaev’s model; relation between the two operators is clarified below.)

\[
\sum_i \frac{d_i}{D^2} \begin{array}{ccc}
  & * & \\
  \text{p} & & i
\end{array}
\]

Figure 8. Operator $B^s_p$

2. One has a natural isomorphism $H_{TV}(\Sigma, \Delta) \simeq H_{\Delta}^\text{str}$

3. Under the isomorphism of the previous part, the operator associated to the cylinder $Z_{TV}(\Sigma \times I): H_{TV}(\Sigma, \Delta) \rightarrow H_{TV}(\Sigma, \Delta)$ is identified with the projector $B^s = \prod_{p} B^s_p: H_{\Delta}^\text{str} \rightarrow H_{\Delta}^\text{str}$.

The proof of this theorem can be found in [Kir2011]; obviously, it implies Theorem 3.1. Note that the isomorphism constructed in the proof requires a non-trivial choice of normalizations; see [Kir2011] for details.

4. The main theorem: closed surface

In this section, we prove the first main result in the paper, identifying the ground space $K_R(\Sigma, \Delta)$ of Kitaev model with the vector space $Z^A_{TV}$ of the Turaev–Viro TQFT with the category $A = \text{Rep}(R)$.

**Theorem 4.1.** Let $R$ be a finite-dimensional semisimple Hopf algebra. Then for any closed, oriented surface $\Sigma$ with a cell decomposition $\Delta$ one has a canonical isomorphism $K_R(\Sigma, \Delta) \cong Z^A_{TV}(\Sigma)$, where $Z^A_{TV}$ is the Turaev–Viro TQFT based on the category $A = \text{Rep}(R)$.

For example, on the sphere $S^2$, the ground state is one-dimensional, or non-degenerate in physics terminology.

The proof of this theorem occupies the rest of this section. For brevity, we will denote the Hilbert space of Kitaev model just by $\mathcal{H}_K$, dropping $\Sigma, \Delta$ from the notation.
Recall that \( K_R \subset \mathcal{H}_K \) was defined as the ground space of the Hamiltonian. We begin by introducing an intermediate vector space \( \mathcal{H}_A \) such that \( K_R \subset \mathcal{H}_A \subset \mathcal{H}_K \). Namely, we let
\[
\mathcal{H}_A = \ker(\sum (1 - A_v)) = \{ x \in \mathcal{H}_K \mid A_v x = x \ \forall \ v \} \subset \mathcal{H}_K
\]
where \( A_v \) are vertex operators \( (2.6) \) and the sum is over all vertices \( v \) of \( \Delta \).

Since \( A_v, B_p \) commute, the \( B_p \) operators preserve subspace \( \mathcal{H}_A \subset \mathcal{H}_K \). The following equality is obvious from the definitions:
\[
K_R = \{ x \in \mathcal{H}_A \mid B_p x = x \ \forall \ p \}
\]
where \( p \) ranges over all 2-cells of \( \Delta \).

We can now formulate the first lemma relating Kitaev’s model with the Turaev–Viro TQFT.

**Lemma 4.2.** One has a natural isomorphism
\[
\mathcal{H}_A(\Sigma, \Delta) \simeq \mathcal{H}_{TV}(\Sigma, \Delta^*)
\]
where \( \Delta^* \) is the dual cell decomposition.

**Proof.** Recall that we have an isomorphism \( R \simeq \bigoplus V_i \otimes V_i^* \) (see (1.5)). Using this isomorphism, we can give an equivalent description of the vector space \( \mathcal{H}_K \). Namely, let us denote by \( E^o r \) the set of oriented edges of \( \Delta \), i.e. pairs \( e = (e, \text{orientation of } e) \); for such an oriented edge \( e \), we denote by \( \bar{e} \) the edge with opposite orientation.

Then we can rewrite the definition of \( \mathcal{H}_K \) as follows:
\[
\mathcal{H}_K = \bigoplus \bigotimes_l \mathcal{H}_v
\]
where the sum is over all colorings of edges of \( \Delta \) and tensor product is over all oriented edges \( e \) of \( \Delta \); thus, every unoriented edge \( e \) appears in this tensor product twice, with opposite orientations. We will illustrate a vector \( v = \bigotimes v_e \) by drawing two oriented half-edges in place of every (unoriented) edge \( e \) and writing the corresponding vector \( v_e \) next to each half-edge, as shown in Figure 9.

![Figure 9](image)

Re-arranging the factors of (4.2), we can write
\[
\mathcal{H}_K = \bigoplus \bigotimes_l \mathcal{H}_v
\]
where the product is over all vertices \( v \) of the cell decomposition \( \Delta \) and
\[
\mathcal{H}_v = l(e_1) \otimes \cdots \otimes l(e_n)
\]
where the \( l(e_1), \ldots, l(e_n) \) are the colors of edges incident to \( v \) taken in counterclockwise order with outgoing orientation.
In this language, the vertex operator $A_v$ acts on $\mathcal{H}_v$ by
\[
A_v(a_1 \otimes \cdots \otimes a_n) = h^{(1)}a_1 \otimes \cdots \otimes h^{(n)}a_n
\]
By Corollary 1.4 we see that therefore the image of $\prod A_v$ is the space
\[
\mathcal{H}_A = \bigoplus_l \bigotimes_v (l(e_1), \ldots, l(e_n))
\]
where, as before, $l(e_1), \ldots, l(e_n)$ are the colors of edges incident to $v$ taken in counterclockwise order with outgoing orientation.

Since vertices of $\Delta$ correspond to 2-cells of $\Delta^*$, this gives an isomorphism
\[
\theta: \mathcal{H}_A \simeq \bigoplus_l \bigotimes_v H_{TV}(C_v, l) = H_{TV}(\Sigma, \Delta^*)
\]
where $C_v$ is the 2-cell of $\Delta^*$ corresponding to vertex $v$ of $\Delta$.

However, for reasons that will become clear in the future, we will rescale this isomorphism and define
\[
\tilde{\theta} = \sqrt{d_l} \theta
\]
where, for any choice of simple coloring $l$ of edges, $\sqrt{d_l}$ acts on $\bigotimes C_v H(C_v, l)$ by multiplication by the factor
\[
\prod_e \sqrt{d_{l(e)}}
\]
where the product is over all unoriented edges $e$ of $\Delta^*$.

Figure 10 shows the composition map $\mathcal{H}_A \overset{\tilde{\theta}}{\rightarrow} H_{TV}(\Sigma, \Delta^*) \simeq H_{str}^{\Delta^*}$ (cf. Theorem 3.2).

**Figure 10.** Isomorphism $\mathcal{H}_A \simeq H_{str}^{\Delta^*}$. Asterisk * shows the puncture obtained by removing a vertex of $\Delta^*$.

**Lemma 4.3.** Under the isomorphism of Lemma 4.2, the operator $B = \prod_p B_p: \mathcal{H}_A \rightarrow \mathcal{H}_A$ is identified with the operator $Z_{TV}(\Sigma \times I): H_{TV}(\Sigma, \Delta^*) \rightarrow H_{TV}(\Sigma, \Delta^*)$.

**Proof.** We will prove it in the language of stringnets: combining isomorphism $\mathcal{H}_A \simeq H_{TV}(\Sigma, \Delta^*)$ (see Lemma 4.2) and $H_{TV}(\Sigma, \Delta^*) \simeq H_{str}^{\Delta^*}$ (see Theorem 3.2), we get an isomorphism $\mathcal{H}_A \simeq H_{str}^{\Delta^*}$, and it suffices to prove that under this isomorphism, the plaquette projector $B_p$ of Kitaev model is identified with the projector $B_p^{str}$ of...
the stringnet model. To avoid complicated notation, we write explicitly the proof in the case shown in Figure 10.

Using Lemma 1.8, we see that the projector \( B_p \) of Kitaev model can be described as follows: if \( x \in H_A \) is as shown in Figure 10 then

\[
\bar{\theta}(B_p x) = \sum_{k,j_1,\ldots,j_5} \frac{d_k}{\text{dim } R^{j_1 \ldots j_5}} \alpha^{i_1} \alpha^{i_2} \alpha^{i_3} \alpha^{i_4} \alpha^{i_5}
\]

Now we can use local relation (3.4) in stringnet space to transform it as follows:

\[
\bar{\theta}(B_p x) = \sum_k \frac{d_k}{\text{dim } R} \bar{\theta}(x)
\]

(recall that \( A = \text{Rep } R, D^2 = \text{dim } R \)).

Combining Lemma 4.2, Lemma 4.3, we get the statement of the theorem.

**Corollary 4.4.** The space \( K_R(\Sigma, \Delta) \) is independent of the choice of cell decomposition \( \Delta \).

5. **Excited states and Turaev–Viro theory with boundary**

In the previous section, we constructed a Hamiltonian on the Hilbert space \( H_K(\Sigma, \Delta) \). The Hamiltonian had a special form; it was expressed as a sum of local commuting projectors. We saw that the ground state was naturally isomorphic to that in Turaev-Viro theory. In this section we study higher eigenstates of the Hamiltonian, which are typically called excited states. Physically, excited states are interpreted as “quasiparticles” (anyons) of various types sitting on the surface \( \Sigma \). Excited states can also be described in Turaev-Viro theory, viewed as an extended 3-2-1 TQFT; a particle in this language corresponds to a puncture in the surface with certain boundary conditions.

5.1. **Excited states in Kitaev model.** As before, let \( \Sigma \) be a closed surface with a cell decomposition \( \Delta \).

Recall (see Definition 2.1) that a site of \( \Delta \) is a pair \( s = (v, p) \) of a vertex and incident edge.
Lemma 5.2. Each site \( s = (v, p) \) and \( s' = (v', p') \) are said to be disjoint if \( v \) is not incident to \( p' \) and \( v' \) is not incident to \( p \) (which in particular implies that \( v \neq v' \) and \( p \neq p' \)). More generally, we call a collection of \( n \) sites disjoint if any two among them are disjoint.

The following result immediately follows from Theorem 2.4.

Lemma 5.2. Each site \( s \) defines an action \( \rho_s \) of \( D(R) \) on \( \mathcal{H}_K; \) if \( s, s' \) are disjoint sites, then these actions commute.

From this perspective, the ground state \( K_R(\Sigma) \) has the trivial representation of \( D(R) \) attached to every site.

In physics language, a representation \( V \) of \( D(R) \) at a site \( s \) models a particle of type \( V \) at \( s \), with the trivial representation corresponding to the absence of a particle. Thus, the ground state has no particles at all at any site; it is called the vacuum state.

Now suppose we fix a collection of \( n \) disjoint sites \( S = \{s_1, \ldots, s_n\}, \) \( s_i = (v_i, p_i) \). For a vertex \( v \), we will write \( v \in S \) if \( v \) is one of the vertices \( v_i \), and similarly for a plaquette \( p \).

Define the operator \( H_S: \mathcal{H}_K(\Sigma, \Delta) \to \mathcal{H}_K(\Sigma, \Delta) \) by

\[
(5.1) \quad H_S = \sum_{v \in S} (1 - A_v) + \sum_{p \notin S} (1 - B_p)
\]

Let

\[
(5.2) \quad \mathcal{L}(\Sigma, \Delta, S) = \ker(H_S) = \{ x \in \mathcal{H}_K \mid A_v x = x \forall v \notin S, \quad B_p x = x \forall p \notin S \}
\]

We think of \( \mathcal{L}(\Sigma, \Delta, S) \) as the space of \( n \) particles fixed at sites \( s_1, \ldots, s_n \) on the surface; for brevity, we will frequently drop \( \Sigma \) and \( \Delta \) from the notation, writing just \( \mathcal{L}(S) \). Our next goal is to describe this space.

By Lemma 5.2, we have an action of the algebra \( D(R)^{\otimes S} \) on \( \mathcal{L}(S) \). Since the algebra \( D(R)^{\otimes S} \) is semisimple, we can write

\[
(5.3) \quad \mathcal{L}(s_1, \ldots, s_n) = \bigoplus_{Y_1, \ldots, Y_n} (Y_1^* \otimes Y_2^* \otimes \cdots \otimes Y_n^*) \otimes \mathcal{M}(\Sigma, Y_1, \ldots, Y_n)
\]

where \( Y_1, \ldots, Y_n \in \text{Irr}(D(R)) \) are irreducible representations of \( D(R) \) and \( \mathcal{M}(\Sigma, Y_1, \ldots, Y_n) \) is some vector space. (Note that \( \mathcal{M} \) also depends on the cell decomposition \( \Delta \) and the set of sites \( S \); we will usually suppress it in the notation.) The algebra \( D(R)^{\otimes S} \) acts in an obvious way on the tensor product \( Y_1^* \otimes \cdots \otimes Y_n^* \) and acts trivially on the space \( \mathcal{M} \).

The space \( \mathcal{M}(\Sigma, Y_1, \ldots, Y_n) \) is called the protected subspace in [Kit2003]. It is unaffected by local operators, as suggested above, but we can act on it (in a suitable sense), by nonlocal operators, such as creating, interchanging or annihilating particles. For example, there is a natural action of the braid group on \( \mathcal{M} \), which, with suitable starting data, is capable of performing universal quantum computation.

Our next goal will be relating the protected space \( \mathcal{M}(\Sigma, Y_1, \ldots, Y_n) \) with the Turaev–Viro and stringnet model for surfaces with boundary.

5.2. Rewriting the protected space. Throughout this section, \( \Sigma, S, Y \) are as in the previous section.

First, recall that the space \( \mathcal{L}(s_1, \ldots, s_n) \) has a natural structure of a \( D(R)^{\otimes n} \) module, where the \( i \)-th copy of \( D(R) \) acts on site \( s_i \). Since for any collection
\(Y_i \in \text{Irr}(D(R))\), the vector space \(Y_1 \boxtimes \cdots \boxtimes Y_n\) also has a natural structure of \(D(R)^{\otimes n}\)-module, we can define the action of a \(D(R)^{\otimes n}\) on the space

\[(Y_1 \boxtimes \cdots \boxtimes Y_n) \otimes \mathcal{L}(s_1, \ldots, s_n)\]

using Hopf algebra structure of \(D(R)^{\otimes n}\).

Using the decomposition of \(\mathcal{L}(s_1, \ldots, s_n)\) from (5.3), we can extract the protected space \(\mathcal{M}\):

\[(5.4) \quad \mathcal{M}(\Sigma, \mathbf{Y}) \cong [(Y_1 \boxtimes \cdots \boxtimes Y_n) \otimes \mathcal{L}(s_1, \ldots, s_n)]^{D(R)^{\otimes n}}\]

Equivalently, consider the vector space

\[\mathcal{H}_K(\Sigma, \Delta, \mathbf{Y}) = (Y_1 \boxtimes \cdots \boxtimes Y_n) \otimes \mathcal{H}_K(\Sigma, \Delta)\]

where \(\mathcal{H}_K(\Sigma, \Delta)\) is the crude Hilbert space defined in Section 2.1. We will graphically represent vectors in this space by writing a vector \(x_e \in R\) next to each oriented edge \(e\), and also drawing, for every site \(s_i\), a green segment connecting \(v\) and center of the plaquette \(p\) (as in Figure 9) labelled by \(y_i\), as shown in Figure 11.

**Figure 11.** Graphical presentation of a vector in \(\mathcal{H}_K(\Sigma, \Delta, \mathbf{Y})\)

For every vertex \(v\) and \(a \in R\), define the operator \(\hat{A}_v^a : \mathcal{H}_K(\Sigma, \Delta, \mathbf{Y}) \to \mathcal{H}_K(\Sigma, \Delta, \mathbf{Y})\) by \(\hat{A}_v^a = \text{id}_Y \otimes \hat{A}_v^a\) if \(v \notin \{s_1, \ldots, s_n\}\) and by the figure below if \(v = v_k \in S\):

Similarly, for any plaquette \(p\) and \(\alpha \in \mathcal{P}\), define the operator \(\hat{B}_p^\alpha : \mathcal{H}_K(\Sigma, \Delta, \mathbf{Y}) \to \mathcal{H}_K(\Sigma, \Delta, \mathbf{Y})\) by \(\hat{B}_p^\alpha = \text{id}_Y \otimes \hat{B}_p^\alpha\) if \(p \notin \{s_1, \ldots, s_n\}\) and by the figure below if \(p = p_i \in S\) (recall that comultiplication in \(\mathcal{P}\) is given by \(\Delta(\alpha) = \alpha'' \otimes \alpha'\):
It is easy to see that then the operators $\tilde{A}_v$, $\tilde{B}_p$ satisfy the relations of Theorem 2.4; in particular, for any site $s = (v, p)$ (including the sites $s_1, \ldots, s_n$), the operators $\tilde{A}_v$, $\tilde{B}_p$ satisfy the relations of Drinfeld double. It follows from the definition of $L$ and (5.4) that

$$M(\Sigma, \Delta, Y) = \{ x \in H_K(\Sigma, \Delta, Y) | \tilde{A}_v^h x = \tilde{B}_p^h x = x \forall v, p \}$$

5.3. Turaev–Viro theory surfaces with boundary. We recall the definition of Turaev–Viro model for surfaces with boundary, following [BalK2010]. As before, let $A$ be a spherical fusion category. Let $C$ be the Drinfeld center of $A$; as is well known, in the example $A = \text{Rep}(R)$, we have $C = \text{Rep}(D(R))$. We have an obvious forgetful functor $F: C \to A$ which has an adjoint $I: A \to C$ (see details in [BalK2010]).

We will use colored graphs on surfaces where some of the lines are colored by elements of the Drinfeld double. When drawing such graphs, we will show objects of $Z(A)$ by double green lines and the half-braiding isomorphism $\varphi_Y: Y \otimes V \to V \otimes Y$ by crossing as in Figure 12.

![Graphical presentation of the half-braiding $\varphi_Y: Y \otimes V \to V \otimes Y$](image)

**Figure 12.** Graphical presentation of the half-braiding $\varphi_Y: Y \otimes V \to V \otimes Y$, $Y \in \text{Obj } Z(A)$, $V \in \text{Obj } A$

Now let $\Sigma_0$ be a surface with $n$ boundary components, together with a choice of marked point $p_a$ on each boundary component. Consider the new surface $\Sigma$ obtained by gluing to $\Sigma_0$ $n$ copies of the standard 2-disk. This is a closed surface; moreover, each cell decomposition $\Delta$ of $\Sigma_0$ gives rise to a cell decomposition of $\Sigma$ obtained by adding to $\Delta$ each of the glued disks as a 2-cell. These cells will be called embedded disks.

We can now define the state space for such a surface. Namely, let $l$ be a coloring of edges of $\Delta$ by simple objects and let $Y = \{Y_1, \ldots, Y_n\}$ be a collection of objects
of \( C \), one object for each boundary component of \( \Sigma_0 \). Then we define the state space
\[
H_{TV}(\Sigma_0, \Delta, Y, l) = \bigotimes_C H_{TV}(C, l)
\]
where the product is over all 2-cells of \( \Delta \) (including the embedded disks) and
\[
H_{TV}(C, l) = \begin{cases} \langle Y_a, l(e_1), l(e_2), \ldots, l(e_n) \rangle & C = D_a \text{ – an embedded disk} \\ \langle l(e_1), l(e_2), \ldots, l(e_n) \rangle & C \text{ – an ordinary 2-cell of } N \end{cases}
\]
where \( e_1, e_2, \ldots \) are edges of \( C \) traveled counterclockwise; for the embedded disks, we also require that we start with the marked point \( p_a \); for ordinary 2-cells of \( \Delta \) the choice of starting point is not important.

As before, we now define
\[
(5.6) \quad H_{TV}(\Sigma_0, \Delta, Y) = \bigoplus_l \bigotimes_C H_{TV}(C, l)
\]
where \( C \) runs over the set of all 2-cells (including the embedded disks) and the sum is taken over all equivalence classes of colorings \( l \) of edges of \( \Delta \).

It has been shown in [BalK2010] that for a suitably defined notion of a cobordism between such surfaces with embedded disks, every cobordism \( M: \Sigma_1 \to \Sigma_2 \) (together with a cell decomposition extending the cell decompositions of \( \Sigma_1, \Sigma_2 \)) gives rise to a linear operator
\[
Z_{TV}(M): H(\Sigma_1, \Delta, Y) \to H(\Sigma_2, \Delta, Y)
\]
which does not depend on the choice of the cell decomposition of \( M \), so that composition of cobordisms corresponds to composition of linear operators. Thus, we can repeat the same steps as before and define TV theory for surfaces with boundary by
\[
Z_{TV}(\Sigma_0, Y) = \text{Im}(A)
\]
where \( A = Z_{TV}(\Sigma_0 \times I): H_{TV}(\Sigma_0, \Delta, Y) \to H_{TV}(\Sigma, \Delta, Y) \).

It has been shown in [BalK2010] that this defines a 3-2-1 TQFT; in particular, so defined vector space does not depend on the choice of cell decomposition \( \Delta \).

Moreover, it is possible to compute this vector space explicitly. For example, if \( \Sigma_0 = S^2 \) is sphere with \( n \) boundary components, then
\[
Z_{TV}(\Sigma_0, Y_1, \ldots, Y_n) \cong \text{Hom}_C(1, Y_1 \otimes \cdots \otimes Y_n)
\]

5.4. Stringnet for surfaces with boundary. We can now describe the stringnet model as an extended theory, in which we allow surfaces with boundary. We give an overview of the theory, referring the reader to [Kir2011] for a detailed description.

Recall that given a spherical category \( \mathcal{A} \), we defined the notion of a colored graph \( \Gamma \) on an oriented surface \( \Sigma_0 \). For a surface with boundary, we consider colored graphs which may terminate on the boundary, and the legs terminating on the boundary should be colored by objects of \( \mathcal{A} \). Thus, every colored graph \( \Gamma \) defines a collection of points \( B = \{b_1, \ldots, b_n\} \subset \partial \Sigma_0 \) (the endpoints of the legs of \( \Gamma \)) and a collection of objects \( V_b \in \text{Obj } \mathcal{A} \) for every \( b \in B \): the colors of the legs of \( \Gamma \) taken with outgoing orientation. We will denote the pair \( (B, \{V_b\}) \) by \( V = \Gamma \cap \partial \Sigma \)
and call it *boundary value*. Similar to the closed case, we can define, for a fixed boundary value $\mathbf{V}$, the stringnet space

$$H_{str}^{st}(\Sigma_0, \mathbf{V}) = \left( \frac{\text{formal combinations of colored graphs with boundary value } \mathbf{V}}{\text{local relations}} \right)$$

It was shown in [Kir2011] that boundary conditions actually form a category $\tilde{C}(\partial \Sigma_0)$ so that $H_{str}^{st}(\Sigma_0, \mathbf{V})$ is functorial in $\mathbf{V}$. Moreover, if we denote by $\tilde{C}(\partial \Sigma_0)$ the pseudo-abelian completion of this category, then one has an equivalence

$$J: \tilde{C}(S^1) \simeq C(\partial \Sigma_0) \rightarrow \text{Vec}$$

where $\tilde{C} = Z(A)$ is the Drinfeld center of $A$ and $I: A \rightarrow \tilde{C}$ is the adjoint of the forgetful functor $Z(A) \rightarrow A$. Thus, if $\partial \Sigma_0$ is a union of $n$ circles, then a choice of parametrization $\psi: \partial \Sigma_0 \simeq S^1 \sqcup \cdots \sqcup S^1$ gives rise to an equivalence of categories $\tilde{C}(\partial \Sigma_0) \simeq C^{\mathbb{G}_n}$.

Since any functor $\tilde{C} \rightarrow \text{Vec}$ naturally extends to a functor of the pseudo-abelian completion $\tilde{C} \rightarrow \text{Vec}$, we can define the stringnet space $H_{str}^{st}(\Sigma_0, \mathbf{Y})$ for any $\mathbf{Y} \in \tilde{C}(\partial \Sigma_0)$. Equivalently, given a surface $\Sigma_0$ together with a parametrization $\psi$ of the boundary components, we can define the vector space $H_{str}^{st}(\Sigma_0, \psi, \mathbf{Y})$, where $\mathbf{Y} = \{Y_1, \ldots, Y_n\}$, $Y_a \in Z(A)$.

The space $H_{str}^{st}(\Sigma_0, \psi, \mathbf{Y})$ admits an alternative definition. Namely, let $\Sigma$ be the closed surface obtained by gluing to $\Sigma_0$ a copy of the standard 2-disk $D$ along each boundary circle $(\partial \Sigma_0)_a$ of $\Sigma_0$, using parametrizations $\psi_a$. So defined, the surface comes with a collection of marked points $p_a = \psi_a^{-1}(p)$, where $p = (1,0)$ is the marked point on $S^1$. Moreover, for every point $p_a$ we also have a distinguished “tangent direction” $v_a$ at $p_a$ (in PL setting, we understand it as a germ of an arc starting at $p_a$), namely the direction of the radius connecting $p$ with the center of the disk $D$. We will refer to the collection $(\Sigma, \{p_a\}, \{v_a\})$ as an extended surface. It is easy to see that given $(\Sigma, \{p_a\}, \{v_a\})$, the original surface $\Sigma_0$ and parametrizations $\psi_a$ are defined uniquely up to a contractible set of choices.

For such an extended surface and a choice of collection of objects $\mathbf{Y} = \{Y_1, \ldots, Y_n\}$, $Y_a \in Z(A)$, define

$$(5.7) \quad \hat{H}_{str}(\Sigma, \mathbf{Y}) = \text{VGraph}^\prime(\Sigma, \mathbf{Y})/(\text{Local relations})$$

where $\text{VGraph}^\prime(\Sigma, \mathbf{Y})$ is the vector space of formal linear combinations of colored graphs on $\Sigma$ such that each colored graph has an uncolored one-valent vertex at each point $p_a$, with the corresponding edge coming from direction $v_a$ (i.e., in some neighborhood of $p_a$, the edge coincides with the corresponding arc) and colored by the object $F(Y_a)$ as shown in Figure 13 and local relations are defined in the same way as before: each embedded disk $D \subset \Sigma$ not containing the special points $p_a$ gives rise to local relations.

![Figure 13. Colored graphs in a neighborhood of marked point](image)

The following lemma is a reformulation of results of [Kir2011].
Lemma 5.3. Let $\Sigma_0$ be a compact surface with $n$ boundary components, $\psi: \partial \Sigma_0 \simeq S^1 \sqcup \cdots \sqcup S^1$ — a parametrization of the boundary, and $Y = \{Y_1, \ldots, Y_n\}$, $Y_a \in Z(A)$ — a choice of boundary conditions. Then one has a canonical isomorphism

$$H^{\text{str}}(\Sigma_0, \psi, Y) = \{x \in \hat{H}^{\text{str}}(\Sigma, Y) \mid B^\text{str}_{p_a} x = x \ \forall a\}$$

where for each marked point $p_a$, the operator $B^\text{str}_{p_a}$ is defined by

\[
p_a \mapsto \sum_i d_i \frac{1}{D^2} \quad \text{Figure 14. Operator } B_p \text{ for a marked point}
\]

The following is the main result of [Kir2011].

Theorem 5.4. Let $\Sigma_0$ be a compact surface with $n$ boundary components, $\psi: \partial \Sigma_0 \simeq S^1 \sqcup \cdots \sqcup S^1$ — a parametrization of the boundary. Then for any $Y = \{Y_1, \ldots, Y_n\} \in Z(A)^{\otimes n}$, one has a canonical functorial isomorphism

$$Z_{TV}(\Sigma_0, \psi, Y) \cong H^{\text{str}}(\Sigma_0, \psi, Y)$$

where, as before, $\Sigma$ is obtained from $\Sigma_0$ by gluing disks along the boundary.

As before, we will need a more detailed construction of the isomorphism of this theorem, parallel to the description for closed surfaces given in Theorem 3.2. Namely, let $\Delta_0$ be a cell decomposition of $\Sigma_0$ such that for every boundary component $(\partial \Sigma_0)_a$, the corresponding marked point $p_a = \psi^{-1}(0,1)$ is a vertex of $\Delta_0$. By adding to $\Delta_0$ a disk $D_a$ for each boundary component, we get a cell decomposition $\Delta$ of closed surface $\Sigma$.

Let $\Sigma - \Delta^0$ be the surface with punctures obtained by removing from $\Sigma$ all vertices of $\Delta$ (this includes the marked points $p_a$). Let $\hat{H}^{\text{str}}(\Sigma - \Delta^0, Y)$ be the stringnet space defined by boundary condition of Figure 13 near puncture $p_a$ (and trivial boundary condition near all other punctures). Then one has the following results.

Theorem 5.5.

1. One has an isomorphism

$$H^{\text{str}}(\Sigma, Y) \cong \{x \in \hat{H}^{\text{str}}(\Sigma - \Delta^0, Y) \mid B^\text{str}_p x = x \ \forall p \in \Delta^0\} \subset \hat{H}^{\text{str}}(\Sigma - \Delta^0, Y)$$

where $B^\text{str}_p: H^{\text{str}}_\Delta \to H^{\text{str}}_\Delta$ is the operator which adds to a colored graph a small loop around puncture $p$ as shown in Figure 8, Figure 13.

2. One has a natural isomorphism $H_{TV}(\Sigma_0, \Delta_0, Y) \cong H^{\text{str}}(\Sigma - \Delta^0, Y)$

3. Under the isomorphism of the previous part, the operator associated to the cylinder $Z_{TV}(\Sigma \times I)$; $H_{TV} \to H_{TV}$ is identified with the projector $B^\times = \prod_p B_p: \hat{H}^{\text{str}}(\Sigma - \Delta^0, Y) \to \hat{H}^{\text{str}}(\Sigma - \Delta^0, Y)$.

The proof of this theorem can be found in [Kir2011]; obviously, it implies Theorem 5.4.
6. Comparison of Kitaev, Turaev–Viro and Levin-Wen models for surfaces with boundary

In this section, we establish the relation between the protected space for Kitaev’s model and the Turaev–Viro (and thus the Levin-Wen) space for surfaces with boundary, extending Theorem 4.1 to surfaces with boundary.

6.1. Statement of the main theorem. As before, we fix a semisimple Hopf algebra $R$ over $\mathbb{C}$ and denote $\mathcal{A} = \text{Rep}(D(R)) \cong Z(\mathcal{A})$.

Throughout the section, we fix a choice of a compact oriented surface $\Sigma$ (without boundary) and a cell decomposition $\Delta$ of $\Sigma$. We also fix a finite collection of disjoint sites $S = \{s_1, \ldots, s_n\}$ and a finite collection $Y = \{Y_1, \ldots, Y_n\}$ of irreducible representations of $D(R)$.

We denote by $\Delta^*$ the dual cell decomposition of $\Sigma$. Then each site $s_i = (v_i, p_i)$ defines a cell $D_i$ (containing $v_i$) of $\Delta^*$ and a marked point $p_i$ on the boundary of $D_i$.

Denote by $\Sigma_0$ the surface with boundary and marked points, obtained by removing from $\Sigma$ the interiors of $D_1, \ldots, D_n$. Clearly, in this situation $\Sigma$ can be obtained from $\Sigma_0$ by gluing the disks $D_1, \ldots, D_n$.

**Theorem 6.1.** Let $\Sigma, \Sigma_0, Y$ be as above. Then one has a canonical functorial isomorphism

$$M(\Sigma, \Delta, Y) \cong Z_{TV}(\Sigma_0, Y) \cong H^{str}(\Sigma_0, \{p_i\}, Y),$$

where $M(\Sigma, \Delta, Y)$ is the protected space defined by (5.3).

The following example is instructive.

**Example 6.2.** Let $\Sigma$ be the sphere with $n$ sites labeled by $Y_1, \ldots, Y_n$. Then $Z_{TV}(\Sigma_0, Y_1, \ldots, Y_n) \cong \text{Hom}_{D(R)}(1, Y_1 \otimes \cdots \otimes Y_n)$ [BalK2010]. It follows that for $n = 1$, $M(Y)$ is one-dimensional if $Y$ is trivial one-dimensional representation of $D(R)$, and $M(Y) = 0$ if $Y$ is non-trivial irreducible representation of $D(R)$; thus, $\mathcal{L}(s_1) = M(s_1, 1)$ is one dimensional, i.e. there are no single particle excitations on the sphere. For $n = 2$, $Z_{TV}(\Sigma_0, Y, Z) = \text{Hom}_{D(R)}(1, Y \otimes Z) = 0$, unless $Z \cong Y^*$. It follows that two-particle excitations on the sphere consist of a particle of type $Y$ at one site and a particle of type $Y^*$ at another site.

The proof of the theorem occupies the rest of this section. We begin with some preliminary results.

6.2. Lemma on Haar integral. We will need the following technical lemma.

**Lemma 6.3.** Let $Y$ be a representation of $D(R)$, and let $\bar{h} \in R$ be the Haar integral of $R$.

Consider the map

$$Y \otimes R \rightarrow Y$$

$$y \otimes r \mapsto \bar{h}' \langle \bar{h}', r \rangle$$

where $\lambda, y$ stands for the action of $R$ on $Y$. 
Then under the isomorphism $R \simeq \bigoplus V_i \otimes V_i^*$, this map is identified with the map pictured below

\begin{equation}
\sum_i \frac{d_i}{\dim R} \sum \alpha \langle \bar{h}, x_\alpha \rangle \langle \bar{h}', \bar{h}'' \rangle \langle x_\alpha, v \rangle \langle f \rangle \rightarrow V_i
\end{equation}

where the upper crossing is just the permutation of factors (note that it is not a morphism of modules).

Proof. By definition of the $R$-matrix (1.10), the map (6.1) is given by

\[
y \otimes v \otimes f \mapsto \frac{d_i}{\dim R} \sum \alpha \langle \bar{h}, x_\alpha \rangle \langle \bar{h}', \bar{h}'' \rangle \langle x_\alpha, v \rangle \langle f \rangle = \sum \alpha \langle \bar{h}, x_\alpha \rangle \langle \bar{h}', \bar{h}'' \rangle \langle x_\alpha, v \rangle \langle f \rangle
\]

where $v \in V_i$, $f \in V^*_i$, and $x_\alpha, x_n$ are dual bases in $R$, $R^*$.

Since for any $\lambda \in R$, we have $\langle \lambda, xr \rangle = \langle \lambda', x \rangle \langle \lambda'', r \rangle$, this can be rewritten as

\[
\sum \alpha \langle \bar{h}, x_\alpha \rangle \langle \bar{h}', \bar{h}'' \rangle \langle x_\alpha, v \rangle \langle f \rangle = \langle \bar{h}', y \rangle \langle \bar{h}'' \rangle \langle v \rangle \langle f \rangle
\]

Since $\Delta(\bar{h})$ is symmetric ($\bar{h}' \otimes \bar{h}'' = \bar{h}'' \otimes \bar{h}'$), we get the statement of the lemma. □

Combining this with the formula for multiplication and comultiplication in $R$ under the isomorphims $R \simeq \bigoplus V_i \otimes V_i^*$, we get the following corollary, generalizing Lemma 1.8

**Corollary 6.4.** Consider the map

\[
Y \otimes R^{\otimes n} \rightarrow Y \otimes R^{\otimes n}
y \otimes x_n \otimes \cdots \otimes x_1 \mapsto \bar{h}^{(n+1)} y \otimes \bar{h}^{(n)} x_n \otimes \cdots \otimes \bar{h}^{(1)} x_1
= \bar{h}'' y \otimes \langle \bar{h}', S(x_n \ldots x_1) \rangle x_n'' \otimes \cdots \otimes x_1''
\]
Then under the isomorphism $R \simeq \bigoplus V_i \otimes V_i^*$, this map is identified with the map pictured below.

$$y \otimes x_n \otimes \cdots \otimes x_1 \mapsto \sum_{i_1, \ldots, i_n, j_1, \ldots, j_n, k} d_{i_1} \cdots d_{i_n} d_k \dim R \sum_{\alpha, \beta, \ldots}$$

6.3. Proof of the main theorem. We can now complete the proof of Theorem 6.1 by combining results of the two previous subsections.

By (5.5), the space $M$ can be obtained from the space $H_K(\Sigma, \Delta, Y)$ by applying projectors $A_v, B_p$. Let us consider the intermediate space obtained by $A_v$ projectors only:

$$H_A(\Sigma, \Delta, Y) = \{ x \in H_K(\Sigma, \Delta, Y) | \tilde{A}^h_v x = x \forall v \}$$

Lemma 6.5. We have an isomorphism

$$H_A(\Sigma, \Delta, Y) \cong H_{TV}(\Sigma, \Delta^*, Y^*) \cong \hat{H}^{str}(\Sigma - \Delta^*, Y)$$

where the space on the right is the string net space on $\Sigma$, with the centers of plaquettes removed, and boundary condition $Y_i$ at site $s_i$ (cf. (5.7)).

Proof. The proof repeats with necessary changes the proof of Lemma 4.2. Namely, the same arguments as in the proof of Lemma 4.2 show that

$$H_A(\Sigma, \Delta, Y) = \bigoplus_l \bigotimes_v H_A(v, l)$$

where the product is over all vertices of $\Delta$ and

$$H_v(v, l) = \begin{cases} (l(e_1), \ldots, l(e_n)), & v \notin S \\ (Y_i, l(e_1), \ldots, l(e_n)), & v = v_i \in S \end{cases}$$

where $e_1, \ldots, e_n$ are edges starting at $v$, in counterclockwise order. Thus, we see that we have a natural isomorphism $H_A(v, l) = H_{TV}(C_v, l)$, where $C_v$ is the 2-cell of the dual cell decomposition $\Delta^*$ corresponding to $v$. Note that it also holds in the case when $v \in S$, in which case $C_v$ is the embedded disk $D_i$.

Thus, we have a natural isomorphism

$$\theta: H_A(\Sigma, \Delta, Y) \simeq H_{TV}(\Sigma, \Delta^*, Y)$$

As in the proof of Lemma 4.2, we choose to rescale it to define

$$\tilde{\theta} = \sqrt{d_i} \theta$$
Figure 15 shows the composition map $\mathcal{H}_A(\Sigma, \Delta, Y) \cong H_{TV}(\Sigma, \Delta^*, Y) \simeq \hat{H}_{str}(\Sigma - \Delta^0, Y)$.

\begin{align*}
&v_1 \in V_{i1}, w_1 \in V_{i1}^*, \ldots, y \in Y \\
&\varphi_1 = y \otimes w_5 \otimes u_1 \otimes v_1, \ldots
\end{align*}

**Figure 15.** Isomorphism $\mathcal{H}_A \simeq H_{str}^\times$. Asterisk * shows the puncture obtained by removing a vertex of $\Delta^*$.

Lemma 6.6. Under the isomorphism of the previous lemma, the operators $B_p$ of Kitaev’s model (for all $p$, including $p \in S$) are identified with the operators $B^*_p$ of stringnet model.

**Proof.** For $p \notin S$, the proof is the same as in Lemma 4.3. For $p \in S$, it follows from Corollary 6.3.

Taken together, these two lemmas immediately imply Theorem 6.1.

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