CERTAIN TOPICS ON THE LIE ALGEBRA \( gl(\lambda) \) REPRESENTATION THEORY

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Introduction.

0.1. The aim of this work is to give a review of recent results in the representation theory of the Lie algebra \( gl(\lambda) \) and their applications to the new combinatorial identities (see 0.11).

The \( gl(\lambda) \) is an infinite-dimensional Lie algebra, dependent on the parameter \( \lambda \in \mathbb{C} \), which is a continuous version of the Lie algebra \( gl_\infty \). It was introduced by B.L. Feigin in [1] for calculating the homology ring of the Lie algebra of differential operators on a line and has several equivalent definitions. Here we shall consider three of them.

0.2. First, it is the Lie algebra constructed from the associative algebra of twisted differential operators (see [7]) on \( \mathbb{C}P^1 \). The differential operator of order \( \leq k \) on the line bundle \( \mathcal{L} \) is a global object which locally acts on the holomorphic sections of this bundle and \( \ldots [[[D, f_1], f_2] \ldots f_{k+1}] = 0 \) for any \( k+1 \) holomorphic functions (these functions are considered as zero-order operators \( f_i : \Gamma(\mathcal{L}) \to \Gamma(\mathcal{L}) \)). Any line bundle on \( \mathbb{C}P^1 \) is \( \mathcal{O}(n) \) for a certain \( n \in \mathbb{Z} \), and we obtain for each \( n \in \mathbb{Z} \) an associative algebra of twisted differential operators (for \( n = 0 \) they are ordinary differential operators). In fact, such an algebra of twisted differential operators exists for any \( n = \lambda \in \mathbb{C} \) (irrespective of the fact that the corresponding line bundle does not exist for a nonintegral \( \lambda \)).

Recall the construction of this algebra (see [6, 7] for details). We denote by \( \mathcal{D}_\mathcal{O}(1) \) a sheaf of (ordinary) differential operators on the total space of the bundle \( \mathcal{O}(1) \) on \( \mathbb{C}P^1 \), and let \( E \) be a Euler vector field on fibers of this bundle. We denote by \( \mathcal{D}_E \) the subsheaf in \( \mathcal{D}_\mathcal{O}(1) \) consisting of operators commuting with \( E \), and then \( \{\mathcal{D}E - \lambda \mathcal{D} \mid \mathcal{D} \in \mathcal{D}_E\} \) is a two-sided ideal in \( \mathcal{D}_E \), and we denote by \( \text{Dif}_\lambda \) the (associative) quotient algebra of \( \mathcal{D}_E \) over this ideal. Then for \( \lambda \in \mathbb{Z} \) \( \text{Dif}_\lambda \) coincides with the algebra of differential operators on the bundle \( \mathcal{O}(\lambda) \). We can give the following heuristic explanation for this fact. \( \text{Dif}_\lambda \) are differential operators which must act on the sections of the “degree of homogeneity \( \lambda \)” (\( \lambda \in \mathbb{C} \)). Using two holomorphic sections \( \gamma_1 \) and \( \gamma_2 \) of the bundle \( \mathcal{O}(1) \), we can construct two functions \( \tilde{\gamma}_1 \) and \( \tilde{\gamma}_2 \), linear along the fibers, on which \( E \) acts with the eigenvalue 1. From the sheaf \( \tilde{\gamma}_1 \mathcal{O}(1) + \tilde{\gamma}_2 \mathcal{O}(1) \) we must take all sections of the degree of homogeneity \( \lambda \), and \( \text{Dif}_\lambda \) is what acts on these sections. The subalgebra \( \mathcal{D} \) consists of differential operators which preserve the proper decomposition of the operator \( E \), and the factorization with respect to the ideal \( \{\mathcal{D}E - \lambda \mathcal{D} \mid \mathcal{D} \in \mathcal{D}_E\} \) is the isolation of the component corresponding to the eigenvalue \( \lambda \).
From this point of view \( gl(\lambda) = \text{Lie}(\text{Diff}) \).

0.3. We have the representation \( p_{\lambda} : U(\mathfrak{sl}_2) \to \text{Diff}_\lambda(\mathbb{C}P^1) \) for all \( \lambda \in \mathbb{C} \) (see [6, 7]). In order to construct it, we represent \( \mathcal{O}(1) \) as \( \text{SL}_2(\mathbb{C}) \times \tilde{\mathbb{C}} \), where \( \tilde{\mathbb{C}} \) is the corresponding 1-dimensional representation of the Borel subgroup \( B \subset \text{SL}_2 \). Then \( \text{SL}_2(\mathbb{C}) \) acts on the total space of the bundle \( \mathcal{O}(1) \), and the Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \) is mapped into the vector fields on \( \mathcal{O}(1) \). This is a Lie algebra homomorphism, and therefore, the map \( p_{\lambda} : U(\mathfrak{sl}_2) \to \text{Diff}_\lambda \) is well-defined. The Bellinson–Bernstein theorem [6] states in this simplest case that \( p_{\lambda} \) is surjective and that the kernel of \( p_{\lambda} \) is a two-sided ideal in \( U(\mathfrak{sl}_2) \) generated by \( \Delta - \frac{\lambda(\lambda+2)}{2} \), where \( \Delta = e \cdot f + e + \frac{\lambda+1}{2} \in U(\mathfrak{sl}_2) \) is the Casimir operator [4]. On the Verma \( \mathfrak{sl}_2 \)-module with the highest weight \( \lambda \) the operator \( \Delta \) acts by the multiplication by \( \frac{\lambda(\lambda+2)}{2} \). Thus

\[
\text{gl}(\lambda) = \text{Lie}\left(U(\mathfrak{sl}_2)/\left(\frac{\lambda(\lambda+2)}{2}\right)\right).
\]

0.4. The name of the Lie algebra \( \text{gl}(\lambda) \) itself implies a connection with the Lie algebra \( \text{gl}_n \). We shall now give its last definition from which it is clear, in particular, in what sense \( \text{gl}(\lambda) \) is a Lie algebra of matrices of complex dimension.

Let us consider an \( n \)-dimensional irreducible \( \mathfrak{sl}_2 \)-module \( V \). There are a surjection \( U(\mathfrak{sl}_2) \to \text{gl}(V) \) and a principal \( \mathfrak{sl}_2 \)-subalgebra \( \mathfrak{sl}_2 \subset U(\mathfrak{sl}_2) \to \text{gl}(V) \) in \( \text{gl}(V) \). Being a \( \mathfrak{sl}_2 \)-module, \( \text{gl}(V) \cong \text{gl}_n \cong \bigoplus_{i=0}^{n-1} \mathfrak{a}_i \), where \( \mathfrak{a}_i \) is an irreducible \((2i+1)\)-dimensional \( \mathfrak{sl}_2 \)-module. Being a Lie algebra, \( \text{gl}_n \) is generated by \( \mathfrak{a}_0, \mathfrak{a}_1, \) and \( \mathfrak{a}_2 \), and for a sufficiently large \( n \) relations of a fixed degree, depend on \( n \) analytically (these relations were described in [1]). This makes it possible to consider \( n \) to be a complex number \( \lambda \) by substituting \( \lambda \) for \( n \) in all relations, and in this way define the structure of the Lie algebra on \( \bigoplus_{i=0}^{n} \mathfrak{a}_i \) dependent on \( \lambda \in \mathbb{C} \). It is easy to show (see [1] and 1.1 in Ch. 2) that this definition is equivalent to the preceding ones.

0.5. The Lie algebra \( \text{gl}(\lambda) \) is \( \mathbb{Z} \)-graded, i.e., \( \text{gl}(\lambda)^i = \{ \xi \in \text{gl}(\lambda) \mid [h, \xi] = 2i\xi \} \), where \( h \in \mathfrak{sl}_2 \subset \text{gl}(\lambda) \), and there exists the Cartan decomposition (see [14]) \( \text{gl}(\lambda) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \), where \( \mathfrak{n}_- = \bigoplus_{i<0} \text{gl}(\lambda)^i, \mathfrak{h} = \text{gl}(\lambda)^0, \mathfrak{n}_+ = \bigoplus_{i>0} \text{gl}(\lambda)^i. \) This Cartan decomposition is consistent with the decomposition of \( \text{gl}_n \) (see 0.4). However, an essential difference from the classical Lie algebra is that \( \mathfrak{n}_\pm \) cannot be decomposed into a direct sum of one-dimensional \( h \)-invariant subspaces. For instance, \( \mathfrak{n}_+^{(1)} \) is entirely \( h \)-invariant (it cannot be decomposed even into a direct sum of two proper subspaces) and is a space of simple positive root vectors. From this point of view \( \text{gl}(\lambda) \) is a Lie algebra with a continual system of roots.

This can be illustrated by a simple example. A parabolic subalgebra in \( \text{gl}(\lambda) \) corresponding to the roots \( \alpha_1, \ldots, \alpha_k \in \mathbb{C} \) is the subalgebra \( \mathfrak{p}_{\alpha_1, \ldots, \alpha_k} \) generated by \( \mathfrak{n}_+, \mathfrak{h}, \) and \( \mathfrak{n}_-^{(-1)} = \{ P(h) f \mid P(h) = (h-a_1) \cdots (h-a_k) P_1(h), P_1 \in \mathbb{C}[h] \} \). (Here we use the second definition of \( \text{gl}(\lambda) \) as Lie \( \left(U(\mathfrak{sl}_2)\right)\left(\frac{-\lambda(\lambda+2)}{2}\right), \) see 0.3.) For \( \lambda \in \mathbb{Z} \), there exists a surjection \( \text{gl}(\lambda) \to \text{gl}_{\lambda+1} \) (see 0.4) and under this surjection almost all these parabolic subalgebras pass into one (independent of \( \alpha_1, \ldots, \alpha_k \) ) parabolic subalgebra in \( \text{gl}_{n+1} \). In the classical case, parabolic subalgebras correspond to subsets of simple positive roots.

0.6. The subject of our study is representations with the highest weight (see [8, 14, 4]) of the Lie algebra \( \text{gl}(\lambda) \) all of whose levels (relative to the \( \mathbb{Z} \)-grading, see 0.5) are finite-dimensional. Thus, we study the representations \( V \) of the Lie algebra \( \text{gl}(\lambda) \) which satisfy the following conditions:
(1) there exists a vector \( v \in V \) such that
\[ n_+ v = 0, \quad \text{and} \quad h v = \chi(h) v \quad \text{for all} \quad h \in \mathfrak{h} \]

(2) \( V = U(gl(\lambda)) \cdot v \)

(3) all levels of \( V \) are finite-dimensional.

Clearly, this representation exists not for all \( \chi \in \mathfrak{h}^* \). For the general \( \chi \), the corresponding Verma module is irreducible (and has infinite-dimensional levels). In fact, any representation of this kind with the \( k \)-dimensional first level is a quotient of the representation induced with a certain character from \( p_{\alpha_1, \ldots, \alpha_2} \) (see 1.3 in Ch. 2). The space of characters is \( k \)-dimensional and we obtain a \( (2k) \)-parametric family of representations. For the general parameters, these representations are irreducible, and our first objective is to find the parameters for which these representations are reducible.

In this review, we only consider a 2-parameter family of representations with 1-dimensional first level.

0.7. We follow the method of Kac and Radul [2] who studied the representations of a Lie algebra close to \( gl(\lambda) \), namely that of the differential operators on a circle.

The inclusion \( \varphi_s : gl(\lambda) \hookrightarrow gl_{\infty,s} \), where \( gl_{\infty,s} \) is a Lie algebra analytically depending on \( s \in \mathbb{C} \) and isomorphic for the general \( s \) to the Lie algebra \( gl_{\infty} \) of generalized Jacobian matrices, is defined for any \( s \in \mathbb{C} \) (see 2.1 of Ch. 2). It turns out that \( \varphi_s \) can be extended to the mapping \( \varphi^O_s : gl^O(\lambda) \rightarrow gl_{\infty,s} \), which is surjective. Here the Lie algebra \( gl^O(\lambda) \) is a certain completion of \( gl(\lambda) \), and the \( gl(\lambda) \)-invariant subspace of any \( gl_{\infty,s} \)-module is also \( gl^O(\lambda) \)-invariant (and hence \( gl_{\infty,s} \)-invariant).

This reduces the problem of the irreducibility of \( gl(\lambda) \)-modules to the corresponding problem of the irreducibility of \( gl_{\infty,s} \)-modules, and the latter problem can easily be reduced to the problem for \( \mathfrak{g}l_{\infty} \). (We easily pass to central extension since \( H^2(gl(\lambda), \mathbb{C}) = 0 \)).

0.8. In Ch. 1, we consider a “model” situation, namely, the representation \( Ind_\mu \) of the Lie algebra \( \mathfrak{g}l_{\infty} \) induced from the largest parabolic subalgebra. This representation has a zero highest weight and a central charge \( \mu \in \mathbb{C} \). For the modules \( Ind_\mu \), we find all singular vectors and describe the Jantzen filtration (see [13]) in terms of irreducible \( gl^{(1)}_{\mathbb{Z}} \oplus gl^{(2)}_{\mathbb{Z}} \)-modules. (Here \( gl^{(1)}_{\mathbb{Z}} \) and \( gl^{(2)}_{\mathbb{Z}} \) are the “upper” and “lower” subalgebras in \( \mathfrak{g}l_{\infty} \), see Fig. 1 in Ch. 1.) We also find the determinant of Shapovalov’s form (see [15]) on all levels as a function of \( \mu \). As an obvious consequence, we obtain formulas for the character of the irreducible first consecutive quotient module of the Jantzen filtration. For example, in the case of \( \mu = 1 \) we get the classical Euler identity (see [10])

\[
\frac{1}{\prod_{i \geq 1} (1 - q^i)} = 1 + \sum_{k \geq 1} \frac{q^k}{(1 - q)^2} \cdot \cdots \cdot (1 - q^k)^2; \]

in the case of \( \mu \in \mathbb{Z}_{>1} \), we get the “higher” Euler identities and, in the case of \( \mu \in \mathbb{Z}_{\leq -1} \), we get expressions for the characters of the corresponding irreducible
representations. Thus, for $\mu = -1$, we find that the character of the irreducible $\hat{gl}_\infty$-module with zero highest weight and the central charge $-1$ is

$$
\chi_{-1} = 1 + \sum_{k \geq 1} \frac{q^k}{(1-q^2)(1-q^3)\cdots(1-q^k)^2}.
$$

We prove that the consecutive quotient modules of the Jantzen filtration of the representation $Ind_\mu$ are simple and that the terms of the Jantzen filtration exhaust all submodules of $Ind_\mu$; in particular, we obtain explicit expression for the characters of the highest irreducible consecutive quotient modules of the Jantzen filtration.

Considering the Lie algebra $gl_{2n}$ instead of $\hat{gl}_\infty$, we get the “finite forms” of all the identities and formulas.

Another expression for the determinant of Shapovalov’s form of the representation $Ind_\mu$ was obtained by Jantzen (see [13]).

0.9. There exist $\hat{gl}_{\infty,s}$-modules $Ind_{\mu,s}$ similar to the $\hat{gl}_\infty$-modules $Ind_\mu$. Considering the inverse images $\theta^*_s(Ind_{\mu,s})$ of the $\hat{gl}_{\infty,s}$-modules $Ind_{\mu,s}$ under the embedding $\theta_s : gl(\lambda) \hookrightarrow \hat{gl}_{\infty,s}$, we obtain a 2-parameter family of representations of the Lie algebra $gl(\lambda)$ with a 1-dimensional first level. As was noted in 0.7, we can solve the problem of irreducibility for these representations. It turns out, however, that in this way we can get representations induces from all parabolic subalgebras of the corresponding dimension, except for two (for the general $\lambda$). The determinant of Shapovalov’s form of the representation $\theta^*_s(Ind_{\mu,s})$ is a product, and when a parabolic subalgebra approaches the exceptional one, some factors have a simple zero or pole. It follows from the irreducibility theorem (for the general $\lambda$) of the representation of $gl(\lambda)$ induced from two exceptional parabolic subalgebras as that the order of the zero is equal to the order of the pole. Equating the corresponding numbers on all levels, we get a local identity (see 0.11).

Next, in order to prove the irreducibility theorem of the representations of $gl(\lambda)$ induced from exceptional subalgebras (for the general $\lambda$), we consider the embedding $\theta_s : gl(\lambda) \hookrightarrow \hat{gl}_{\infty,s}(C[t]/t^2)$; the highest weight of the representation induced from an exceptional parabolic subalgebra is equal to the highest weight of the inverse image, under $\theta_s$, of a certain induced representation $I$ of $\hat{gl}_{\infty,s}(C[t]/t^2)$, and since the completions $\theta^*_s : gl(\lambda) \hookrightarrow \hat{gl}_{\infty,s}(C[t]/t^2)$ are surjective, it follows, by analogy with Subsec 0.7, that this reduces the problem to the calculation of the character of the irreducible quotient of the module $I$. This is the subject of Ch. 2.

0.10. In Ch. 3, we consider the Lie algebras $gl(\lambda)$, $\lambda \in \mathbb{C}$, as those corresponding to the finite points of the Riemannian sphere $S^2$. In this case, in the neighborhood of the point $\{\infty\} \in S^2$, the Lie algebra $gl(\lambda)$ is deformed into the Lie algebra of regular functions on a nondegenerate symplectic leaf of a standard foliation in $sl_2^*$ with an induced Poisson bracket (Lie algebras of functions for all nondegenerate leaves are isomorphic). Thus, we assume that the point $\{\infty\} \in S^2$ is associated with a Lie algebra and the whole family of Lie algebras on $S^2$ is decomposed into an infinite direct sum of line bundles on $S^2$.

By choosing, at every point, an induced representation of the corresponding Lie algebra, which depends holomorphically on the point, we can obtain a situation in which an arbitrary induced representation of the Lie algebra of functions on a
hyperboloid sits at the point \( \{\infty\} \). Then the determinant of Shapovalov’s form of some level is a holomorphic section of a certain line bundle on \( S^2 \), its Chern class can be easily found. On the other hand, this Chern class is equal to the sum of zeros with multiplicities of the determinant of Shapovalov’s form over all points of the sphere (on this level). We prove the irreducibility theorem of induced representations of the Lie algebra of functions on a hyperboloid for the generic values of the parameters, and then the sum is extended only to the finite points of the sphere at which we can calculate these multiplicities by the methods given in Chs. 1 and 2. Combining these calculations for all levels, we obtain the \textit{global identity} (see Subsec 0.11).

In fact, we prove the irreducibility of induced representations (for the general parameters) of the Lie algebra of functions on a cone, that is, on a degenerate leaf of a foliation in \( sl^*_2 \), and this implies an assertion for nondegenerate leaves (the Lie algebras of functions on all nondegenerate leaves are isomorphic). The Lie algebra of functions on a cone is nilpotent up to with subalgebra \( sl_2 \) algebras of functions on all nondegenerate leaves are isomorphic. The Lie algebra of functions on a cone is nilpotent up to with subalgebra \( sl_2 \), i.e., the functions that have the point \( 0 \in \mathbb{C}^3 \) a zero of a certain degree form an ideal at it, and we apply Kirillov’s theory [5] according to which representations of nilpotent Lie algebras induced from the largest subalgebras are irreducible.

0.11. The local identity:

\[
\left. \frac{d}{da} \left[ \prod_{i \geq 1} \frac{1}{(1 - a \cdot q^i)} \right] \right|_{a=1} = \sum_{\text{over all } \text{Young diagrams } D_{l_1, \ldots, l_k}} \#D_{l_1, \ldots, l_k} \cdot q^{\sum l_i \cdot i^2} \cdot (\chi(D_{l_1, \ldots, l_k}))^2.
\]

The global identity:

\[
\left. \frac{d}{da} \left[ \frac{1}{(1 - q)} \cdot \prod_{i=1}^{\infty} \frac{1}{(1 - a^2 q^i)^2} \cdot \prod_{i=k+1}^{\infty} \frac{1}{(1 - a^4 q^i)^3} \right] \right|_{a=1} + 2 \sum_{k+1 \geq 1} \left. \frac{d}{da} \left[ \prod_{i=1}^{k+1} \frac{1}{(1 - q_i)} \cdot \prod_{i=k+1}^{\infty} \frac{1}{(1 - q_i)^{k+1} (1 - a q^i)^{k+1-i}} \right] \right|_{a=1} - \sum_{\text{over all } \text{Young diagrams } D_{l_1, \ldots, l_k}} \left\{ \text{the length of the “central” diagonal } D_{l_1, \ldots, l_k} \right\} \cdot q^{\sum l_i \cdot i^2} \cdot (\chi(D_{l_1, \ldots, l_k}))^2.
\]

Here \( D_{l_1, \ldots, l_k} \) is the Young diagram consisting of blocks \( 1 \times l_1, 2 \times l_2, \ldots, k \times l_k \); the “central” diagonal is the diagonal beginning at its upper left vertex (see Fig. 1), \( \#D_{l_1, \ldots, l_k} \) is the number of squares in \( D_{l_1, \ldots, l_k} \).

\( \chi(D_{l_1, \ldots, l_k}) \) is the corresponding “semiinfinite” character defined as follows: consider the Lie algebra \( gl_{\infty} \) of the matrices \( (a_{ij}) \), \( i, j = 1, \ldots, \infty \) and let \( \alpha_1^\vee, \alpha_2^\vee, \ldots \) be its simple coroots. Then \( \chi(D_{l_1, \ldots, l_k}) \) is the character of the irreducible \( gl_{\infty} \)-module with the highest weight \( \chi \) such that \( \chi(\alpha_1^\vee) = l_1, \ldots, \chi(\alpha_k^\vee) = l_k, \chi(\alpha_{k+1}^\vee) = \ldots = 0 \). For example, if \( D \) is a single block \( 1 \times k \) or \( k \times 1 \), then \( \chi(D) = \frac{1}{(1-q^2) \ldots (1-q^k)}. \) In general, \( \chi(D_{l_1, \ldots, l_k}) \) can be easily found from Weyl’s formula [4].
Fig. 1. The Young diagram $D_{l_1, \ldots, l_k}$ and its “central” diagonal

The left-hand sides of these identities appear in combinatorial identities connected with plane partitionings (see [10]).

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Chapter 1.

Representations of the Lie Algebras $gl_{2n}$ and $\hat{gl}_\infty$

Induced from the Largest Parabolic Subalgebra

1. Reducibility and Singular Vectors.

1.0. There are two commuting subalgebras $gl_n$ in the Lie algebra $gl_{2n}$, namely, the “upper” subalgebra and the “lower” one (see Fig. 1). We denote them by $gl_{2n}^{(1)}$ and $gl_{2n}^{(2)}$. There are also two Abelian subalgebras $\mathfrak{A}_+$ and $\mathfrak{A}_-$.

Fig. 1.

Let $\alpha_{-n+1}^{\vee}, \ldots, \alpha_{-1}^{\vee}; \alpha_0^{\vee}; \alpha_1^{\vee}, \ldots, \alpha_{n-1}^{\vee}$ be simple coroots. We shall consider the character $\chi_\mu : \mathfrak{h} \to \mathbb{C}$ ($gl_{2n} = n_- \oplus \mathfrak{h} \oplus n_+$) such that $\chi_\mu(\alpha_0^{\vee}) = \mu$, $\chi_\mu(\alpha_i^{\vee}) = 0$ for
\( i \neq 0 \). It is clear that this is a necessary and sufficient condition for determining the representation \( \text{Ind}_\mu \) induced from \( p = \mathfrak{gl}_n \oplus \mathfrak{A}_+ \oplus \mathfrak{gl}_n \) with the highest weight \( \chi_\mu \). Thus, we have defined the family of representations \( \text{Ind}_\mu (\mu \in \mathbb{C}) \) of the Lie algebra \( \mathfrak{gl}_n \). They are the main object of our study in Ch 1. We introduce certain notations: \( x_{ij} \) and \( y_{ij} \) are elements of \( \mathfrak{A}_+ \) and \( \mathfrak{A}_- \), respectively \((i, j = 1 \ldots n)\), in the case of \( \mathfrak{A}_+ \), the numbering goes upward and to the right and, in the case of \( \mathfrak{A}_- \), it goes downward and to the left. For example, \( y_{11} \) is the only element of \( \mathfrak{A}_- \) graded \(-1\).

Next, let \( A_k \) be a \( k \times k \) matrix in the upper right corner in \( A_k = (y_{ij}) \), \( i, j = 1 \ldots k \), and \( \text{Det}_k \in \mathcal{U}(\mathfrak{A}_-) \) be the determinant of the matrix \( A_k \). For example, \( \text{Det}_1 = y_{11} \), \( \text{Det}_2 = y_{12} \cdot y_{21} - y_{11} \cdot y_{22} \) and so on. Let \( v \) be the highest weight vector of \( \text{Ind}_\mu \).

**Theorem 1.** The main result of Sec. 1 is the following theorem

The representations \( \text{Ind}_\mu \) of the Lie algebra \( \mathfrak{gl}_n \) are reducible for \( \mu \in \mathbb{Z}_{\geq -n+1} \) and irreducible for other \( \mu \in \mathbb{C} \). For \( \mu \in \mathbb{Z}_{\geq 0} \), the singular vectors in \( \text{Ind}_\mu \) are

\[
\text{Det}_1^{\mu+1} \cdot v, \text{Det}_2^{\mu+2} \cdot v, \ldots, \text{Det}_n^{\mu+n} \cdot v.
\]

For \(-n+1 \leq \mu \leq 0 \) and \( \mu \in \mathbb{Z} \), the singular vectors in \( \text{Ind}_\mu \) are

\[
\text{Det}_{-\mu+1} \cdot v, \text{Det}_{-\mu+2} \cdot v, \ldots, \text{Det}_n^{\mu+n} \cdot v.
\]

The remaining part of Sec. 1 is devoted to the proof of this theorem.

**Remark.** \( \text{Det}_k \) are weight elements of \( \mathcal{U}(\mathfrak{A}_-) \) of the weight, \(- (k\alpha_0 + (k - 1)(\alpha_{-1} + \alpha_1) + \ldots (\alpha_{-k+1} + \alpha_{-k} - 1)) \). For the \( \mathfrak{I} \)-specialization, this weight is equal to \(-k^2\).

**1.2.** Let \( \mathfrak{gl}_n^{(1)} = \mathfrak{n}_+^{(1)} \oplus \mathfrak{h}^{(1)} \oplus \mathfrak{n}_-^{(1)} \), \( \mathfrak{gl}_n^{(2)} = \mathfrak{n}_+^{(2)} \oplus \mathfrak{h}^{(2)} \oplus \mathfrak{n}_-^{(2)} \), and \( v \) be the highest weight vector in \( \text{Ind}_\mu \). Any element of \( \text{Ind}_\mu \) has the form \( \xi \cdot v \), where \( \xi \in \mathcal{U}(\mathfrak{A}_-) = S^*(\mathfrak{A}_-) \). If \( \xi \cdot v \) is a singular vector, then

\[
[e, \xi] \cdot v \in \mathcal{U}(\mathfrak{A}_-) \cdot (\mathfrak{n}_+^{(1)} \oplus \mathfrak{n}_-^{(2)})v,
\]

where \( e \in \mathfrak{n}_+ \subset \mathfrak{gl}_2 \) is arbitrary.

The main observation is that \( \mathcal{U}(\mathfrak{A}) \) is invariant with respect to the \( \text{ad}\)-action of \( \mathfrak{n}_+^{(1)} \) and \( \mathfrak{n}_-^{(2)} \) i.e., we search for vectors in \( \text{Ind}_\mu \) which are singular for \( \mathfrak{gl}_n^{(1)} \oplus \mathfrak{gl}_n^{(2)} \) and then find the values of \( \mu \) when these singular vectors pass to 0 upon the application of \( e_0 = x_{11} \). To be more precise, if \( e_{-1}, \ldots, e_{-n+1}, (e_1, \ldots, e_{n-1}) \) are root vectors from \( \mathfrak{gl}_n^{(1)} \) (\( \mathfrak{gl}_n^{(2)} \)) corresponding to the simple positive roots \( \alpha_{-1}, \ldots, \alpha_{-n+1}(\alpha_1, \ldots, \alpha_{n-1}) \), then the operators \( \text{ad}(e_i) \) for \( i < 0 \) shift the \((|i|+1)\)th column of \( \mathfrak{A}_- \) (reckoning from the center) to the right by 1 and for \( i > 0 \) they shift the \((i+1)\)th row upward by 1. In other words, on the space \( \mathcal{U}(\mathfrak{A}_-) = C[y_{ij}; i, j = 1, \ldots, n] \) \( \text{ad}(e_i) \) for \( i < 0 \) acts by the vector field \( \sum_{s=1}^n y_{s,i} \frac{\partial}{\partial y_{s,|i|+1}} \) and for \( i > 0 \) by the vector field \( \sum_{s=1}^n y_{i,s} \frac{\partial}{\partial y_{i+1,s}} \).

**1.3.** The following lemma refers to the action of the operators \( \text{ad}(\mathfrak{n}_+^{(1)}) \) and \( \text{ad}(\mathfrak{n}_-^{(2)}) \) on \( \mathcal{U}(\mathfrak{A}_-) \).
Lemma 1. The monomials $\text{Det}^{k_1} \cdot \ldots \cdot \text{Det}^{k_n} \cdot v$ and only these vectors are $gl_n^{(1)} \oplus gl_n^{(2)}$-singular vectors in $\text{Ind}_\mu$ (for any $\mu$).

Proof. The fact that these monomials are $gl_n^{(1)} \oplus gl_n^{(2)}$-singular vectors follows from the results of Subsec. 1.2. Conversely, let $\xi \cdot v$ be a $gl_n^{(1)} \oplus gl_n^{(2)}$-singular vectors in $\text{Ind}_\mu$ ($\xi \in S^*(A_-)$). We shall consider the minimal rectangular domain in $A_-$ with the vertex in the central (upper right) corner which includes all elements which are contained in the notation of $\xi$. Suppose, for example, that it is horizontal side is not smaller than the vertical one and is $l$ long. Then there are no more than $l$ squares $y_{l1}, \ldots, y_{lt}$, in the $l$th column of this rectangle, we shall apply the elements from $n_+^{(1)}$ that shift the $l$th column to the right by $1, \ldots, (l-1)$ squares (if the vertical side is greater than or equal to the horizontal one, then the lower row must be shifted upward by the elements from $n_+^{(2)}$). We regard as variables only $y_{l1}, \ldots, y_{lt}$, i.e., we assume that the other variables have general values. We search for a nontrivial linear expression from $y_{l1}, \ldots, y_{lt}$ which vanish under the corresponding $(l-1)$ vector fields. Let $\xi = \sum_{i=1}^{t} a_i y_{i,l}$ be this expression, and then

$$
\begin{align*}
\sum_{i=1}^{l} a_i \bar{y}_{i,l-1} &= 0 \\
\sum_{i=1}^{l} a_i \bar{y}_{i,1} &= 0
\end{align*}
$$

where $a_i$ are unknowns and $\bar{y}_{i,j}$ are $y_{i,j}$ regarded as constants of the general position. It is clear that then the solution $\{a_i\}$ is unique, and therefore,

$$
\xi = \sum_{i=1}^{t} a_i y_{i,l} = \text{Det} \begin{pmatrix} y_{1,l} & \bar{y}_{1,l-1} & \ldots & \bar{y}_{1,1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{l,l} & \bar{y}_{l,l-1} & \ldots & \bar{y}_{l,1} \end{pmatrix} = A.
$$

Indeed, let us consider an $l$-dimensional vector space $V = \{y_{1,l}, \ldots, y_{l,l}\}$. For every point of $V$ there is a hyperplane going through it and vanishing under these $l-1$ vector fields. We draw in $V$ a straight line intersecting these hyperplanes transversally. Clearly, the function which is vanishing under the fields is uniquely defined by its restriction to this straight line. However, (2) defines a linear function of the parameter of this straight line and any polynomial of the parameter of the straight line lies in $A$ (see (3)). Then we seek the $gl_n^{(1)} \oplus gl_n^{(2)}$-singular weight vectors, and any vector of this kind has in accordance with that has been proved already, the form $C \cdot \text{Det}_l^k \cdot v$, where $C$ is expression from the least square. We can apply the preceding arguments to the expression $C$ and obtain Lemma 1.
1.4. The next step is to find when \(e_0\) acts by zero on the expression \(\operatorname{Det}_1^{(k)} \cdot \ldots \cdot \operatorname{Det}_n^{(k)} \cdot v\) for finding the \(gl_2n\)-singular vectors. We shall first calculate \([e_0, \operatorname{Det}_k] (e_0 = x_{11})\).

Let, as in Subsec 1.0, \(A_k\) be the matrix \((y_{ij})\) \(i, j = 1 \ldots k\). We denote by \(\tilde{A}_k\) the matrix \((y_{ij})\) \(i, j = 2 \ldots k\). This is a \((k - 1) \times (k - 1)\) matrix. Suppose, furthermore, that \(z_i^+\) is an element from \(n^{(1)}\) lying in the \(i\)th column above \(y_{1i}\) and \(z_j^-\) is an element from \(n^{(2)}\) lying in the \(j\)th row on the right of \(y_{j1}\). We write \(\operatorname{Det}_k = \operatorname{Det} \tilde{A}_k\) first with respect to the upper row and then with respect to the right column. We obtain

\[
\operatorname{Det} A_k = (-1)^{1+k} y_{11} \cdot \operatorname{Det} \tilde{A}_k + \sum_{i,j=2\ldots k} (-1)^{i+j} y_{1i} y_{j1} \cdot \operatorname{Det} \tilde{A}_{ij},
\]

where \(\tilde{A}_{ij}\) is the matrix \(\tilde{A}_k\) without the \(i\)th row. The matter is that only \(y_{1i}\) and \(y_{j1}\) do not commute with \(e_0\).

We have

\[
[e_0, \operatorname{Det}_k] = (-1)^{1+k} \operatorname{Det} \tilde{A}_k \cdot \alpha_0^\vee + \sum_{i,j=2\ldots k} (-1)^{i+j} \operatorname{Det} \tilde{A}_{ij} (z_i^+ \cdot y_{j1} - y_{1i} z_j^-).
\]

Since \([z_i^+, y_{j1}] = -y_{j1}\), we have

\[
[e_0, \operatorname{Det}_k] = (-1)^{1+k} \operatorname{Det} \tilde{A}_k \cdot \alpha_0^\vee + \sum_{i,j=2\ldots k} (-1)^{i+j-1} \cdot y_{j1} \cdot \operatorname{Det} \tilde{A}_{ij} + \sum_{i,j=2\ldots k} (-1)^{i+j} \cdot \operatorname{Det} \tilde{A}_{ij} (y_{j1} z_i^+ - y_{1i} z_j^-).
\]

However,

\[
\sum_{i,j=2\ldots k} (-1)^{i+j-1} y_{j1} \cdot \tilde{A}_{ij} = (-1)^{k+1} \cdot (k - 1) \cdot \operatorname{Det} \tilde{A}_k.
\]

Therefore from (4) and (5) we have

\[
[e_0, \operatorname{Det}_k] = (-1)^{1+k} \cdot \operatorname{Det} \tilde{A}_k \cdot (\alpha_0^\vee + k - 1) + \sum_{i,j=2\ldots k} (-1)^{i+j} \cdot \operatorname{Det} \tilde{A}_{ij} (y_{j1} \cdot z_i^+ - y_{1i} z_j^-).
\]

It is clear that the second term in (6) belongs to \(\mathcal{U}(\mathfrak{sl}_n) \cdot n^{(1)} \oplus \mathcal{U}(\mathfrak{sl}_n) \cdot n^{(2)}\) and therefore acts by zero on the highest weight vector \(v\) in \(\text{Ind}_\mu\). We have proved the following lemma.

**Lemma 2.** \(\operatorname{Det}_k \cdot v\) is a singular vector in \(\text{Ind}_\mu\) for \(\mu = -k + 1\).

Obviously, \(\operatorname{Det}_1^{k+1} \cdot v\) is a singular vector in \(\text{Ind}_\mu\) for \(\mu = k \geq 0\). We have thus proved that for \(\mu \in \mathbb{Z}_{\geq -n+1}\) the representations \(\text{Ind}_\mu\) are reducible.

Note the following important consequence: let \(\mu = -k + 1 + t\), where \(t\) is small. It follows from (6) that \([e_0, \operatorname{Det}_k \cdot v\) is a first-order infinitesimal. This is also true in the case of a singular vector of the form \(\operatorname{Det}_k^{k+1} \cdot v\) for \(\mu = k \geq 0\).

1.5. Here we calculate \([e_0, \operatorname{Det}_k]\).
Lemma 3.
\[
\begin{align*}
[z_i^+, \text{Det}_k] &= 0, \\
[z_i^-, \text{Det}_k] &= 0, (i = 2 \ldots k).
\end{align*}
\]

Proof. \(ad(z_i^+)\) acts by a vector field on \(S^*(\mathfrak{g}_-)\) which shifts the first column to the \(i^{th}\) one and \(ad(z_i^-)\) acts by a vector field which shifts the first row to the \(i^{th}\) one. Hence, for \(i \leq k\), we obtain the determinant of a matrix with two similar columns (rows).

Lemma 4.
\[
[e_0, \text{Det}_k] = (-1)^{1+k} \cdot l \cdot (\text{Det}_{\tilde{A}_k} \cdot \text{Det}_k^{l-1} + \text{Det}_{k} \cdot \text{Det}_{\tilde{A}_k} \cdot (\alpha_0^\vee + k - l - 1) + x),
\]
where \(x \in \mathcal{U}(\mathfrak{g}_-) \cdot (n_-(1) \oplus \mathcal{U}(\mathfrak{g}_-) \cdot n_-(2))\).

Proof. The weight of \(\text{Det}_k\) is equal to \(-\left( k\alpha_0 + (k-1)(\alpha_{-1} + \alpha_1) + \ldots + (\alpha_{-k+1} + \alpha_{k-1}) \right) = \Theta_k; \Theta_k(\alpha_0^\vee) = -2\). According to Lemma 3 and relation (6), we have
\[
[e_0, \text{Det}_k] = (-1)^{1+k} \cdot \left[ \text{Det}_{\tilde{A}_k} \cdot (\alpha_0^\vee + k - 1) \cdot \text{Det}_k^{l-1} + \\
\text{Det}_k \cdot \text{Det}_{\tilde{A}_k} \cdot (\alpha_0^\vee + k - 1) \cdot \text{Det}_k^{n-2} + \ldots \right] + x,
\]
where \(x \in \mathcal{U}(\mathfrak{g}_-) \cdot (n_-(1) \oplus n_-(2))\).

\[
(\alpha_0^\vee + k - 1) \cdot \text{Det}_k = \text{Det}_k \left( \alpha_0^\vee + k - 1 + \Theta_k(\alpha_0^\vee) \right) = \text{Det}_k(\alpha_0^\vee + k - 3).
\]

Therefore,
\[
[e_0, \text{Det}_k^l] = (-1)^{1+k} \left( \text{Det}_{\tilde{A}_k} \cdot \text{Det}_k^{l-1} \right) \cdot \left( (\alpha_0^\vee + k - 1) + (\alpha_0^\vee + k - 3) + \ldots + (\alpha_0^\vee + k - 1 - 2(l - 1)) \right) + x.
\]

The part dependent on \(\alpha_0^\vee\) is equal to \((\alpha_0^\vee + k - l)\).

Corollary 1. (1) \(\text{Det}_k^l \cdot v\) is a singular vector in \(\text{Ind}_\mu\) only for \(\mu = l - k\).

(2) If \(\mu = l - k + t\), where \(t \in \mathbb{C}\) is small, then \([e_0, \text{Det}_k^l] \cdot v\) is the first-order infinitesimal with respect to \(t\).
1.6. Let \( \xi \cdot v = \text{Det}^{l_1}_1 \cdots \text{Det}^{l_k}_k \cdot v \) be a singular vector in \( \text{Ind} \mu \) with \( l_k \neq 0 \). We denote \( \text{Det}_k = (-1)^{k+1} \cdot l_k \cdot \text{Det} A_k \), and then

\[
\left[ e_0, \text{Det}^{l_1}_1 \cdots \text{Det}^{l_k}_k \right] = \\
\text{Det}_1 \cdot \text{Det}^{l_1-1}_1 \text{Det}^{l_2}_2 \cdots \text{Det}^{l_k}_k \left( \alpha_0^\vee + 1 - l_1 - 2(l_2 + l_3 + \ldots + l_k) \right) + \\
\text{Det}^{l_1}_1 \text{Det}^{l_2-1}_2 \text{Det}^{l_3}_3 \cdots \text{Det}^{l_k}_k \left( \alpha_0^\vee + 2 - l_2 - 2(l_3 + \ldots + l_k) \right) + \\
\cdots + \text{Det}^{l_1}_1 \cdots \text{Det}^{l_{k-1}}_{k-1} \cdot \text{Det}^{l_k-1}_k \cdot (\alpha_0^\vee + k - l_k) + x,
\]

(7)

where \( x \in U(\mathfrak{g}\mathfrak{l}_n) (\mathfrak{n}_-^{(1)} \oplus \mathfrak{n}_-^{(2)}) \).

We shall consider all determinants as elements of the ring \( \mathbb{C}[y_{ij} \colon i, j = 1 \ldots n] \). The last term in this ring cannot be divided by \( \text{Det}^{l_k}_k \) but can only be divided by \( \text{Det}^{l_{k-1}}_{k-1} \), and therefore, if \( \xi v \) is a singular vector in \( \text{Ind} \mu \), then either the last term in (7) (with the exception for \( x \) acts on \( v \) by zero or all the others. It follows from (7) that the parts in the parentheses, dependent on \( \alpha_0^\vee \), have the form \( (\alpha_0^\vee - a_s) \), where \( a_s \) increases. Since \( l_k \neq 0 \) by hypothesis, the last term acts on \( v \) by zero. Hence follows the part of Theorem 1 stating that \( \mu \in \mathbb{Z}_{\geq n+1} \). Let us consider the other terms. The last term cannot be divided by \( \text{Det}^{l_{k-1}}_{k-1} \) but can only be divided by \( \text{Det}^{l_{k-1}}_{k-1} \) and the other terms can be divided by \( \text{Det}^{l_{k-1}}_{k-1} \). Therefore this term acts on \( v \) by zero, and this is impossible since the sequence \( a_s \) increases. This proves that the vectors \( \text{Det}^{l_k}_k \cdot v \) exhaust all singular vectors in \( \text{Ind} \mu \) and also proves Theorem 1.

2. Shapovalov’s Form on \( \text{Ind} \mu \).

2.0. Let us consider any semisimple Lie algebra \( g \) and assume that \( M \) is a certain Verma module over \( g \). Then the contragradient module \( M' \) has the same highest weight as \( M \), and therefore, there is a unique \( g \)-homomorphism \( \alpha : M \rightarrow M' \). On every level \( \nu_i \) of the module \( M \), the map \( \alpha \) defines the bilinear form \( \alpha_k : V_k \otimes V_k \rightarrow \mathbb{C} \). The form \( \alpha_k \) is known as Shapovalov’s form of the Verma module \( M \). We have \( g = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \) and, for \( f_i \in \mathfrak{n}_- \), denote \( e_i = \omega(f_i) \), where \( \omega \) is the Chevalley involution. It is clear that \( (f_{i_1} f_{i_2} \cdots f_{i_k} v, f_{j_1} f_{j_2} \cdots f_{j_l} v) = e_{i_1} \cdots e_{i_k} f_{j_1} \cdots f_{j_l} v \). Next, the form \( \alpha_k \) is symmetric. The vector \( v \in V_k \) lies in a certain proper submodule in \( M \) if and only if \( \langle v, v \rangle = 0 \) for all \( w \in V_k \). If Shapovalov’s form is nondegenerate on \( V_k \), then the maximal submodule in \( M \) does not intersect \( \bigoplus_{i \leq k} V_i \). Let us consider a quotient module \( L \) of the module \( M \),

\[ L = M / M_1. \]

Then \( (V_k \cap M_1) \perp V_k \), and therefore, there exists Shapovalov’s form \( \alpha_k : L \otimes L \rightarrow \mathbb{C} \) with the properties given above. In particular, \( \alpha_k \) is nondegenerate if and only if \( L \) is irreducible.

Let us consider Shapovalov’s form on \( \text{Ind} \mu \). For \( \mu \notin \mathbb{Z}_{\geq n+1} \) it is nondegenerate. The determinant of Shapovalov’s form is not uniquely defined. However, if we regarded this determinant as a function of \( \mu \), then the multiplicity of zero is uniquely defined.

Our aim is to determine the multiplicity of zero of the determinant of Shapovalov’s form on the levels of \( \text{Ind} \mu \).
2.1. Lemma 1 states that \( w = \det_{1}^{k_{1}} \cdots \det_{n}^{k_{n}} \cdot v \) is a \( gl_{n}^{(1)} \oplus gl_{n}^{(2)} \)-singular vector in \( Ind_{\mu} \) for any \( k_{1}, \ldots, k_{n} \geq 0 \) and these monomials exhaust \( gl_{n}^{(1)} \oplus gl_{n}^{(2)} \)-singular vectors.

**Lemma 5.** For different \( (k_{1}, \ldots, k_{n}) \) the \( gl_{n}^{(1)} \oplus gl_{n}^{(2)} \)-weights of the corresponding monomials \( w \) are different.

**Proof.** The proof is obvious.

This very simple result has a fundamental significance.

**Lemma 6.** \( Ind_{\mu} = \oplus \) (of irreducible finite-dimensional \( gl_{n}^{(1)} \oplus gl_{n}^{(2)} \)-modules with different highest weights).

**Proof.** We have only to prove that \( n_{\perp}^{(1)} \oplus n_{\perp}^{(2)} \) acts on \( Ind_{\mu} \) locally finitely. The operators \( adn_{\perp}^{(1)}(adn_{\perp}^{(2)}) \) shift any element \( y_{ij} \) to the left (downward) and the elements of the \( n \)th column (of the \( n \)th row) commute with \( n_{\perp}^{(1)}(n_{\perp}^{(2)}) \).

**Example.** The \( gl_{n}^{(1)} \oplus gl_{n}^{(2)} \)-modules corresponding to \( \det_{1} \cdot v, \det_{2} \cdot v, \ldots, \det_{n} \cdot v \) are, respectively,

\[
\Lambda^{n-1}V \otimes \Lambda^{n-1}V, \Lambda^{n-2}V \otimes \Lambda^{n-2}V, \ldots, \Lambda^{1}V \otimes \Lambda^{1}V, \mathbb{C},
\]

where \( V \) is the tautological \( n \)-dimensional representation of \( gl_{n} \).

2.2. Since the weight of all arising \( gl_{n}^{(1)} \oplus gl_{n}^{(2)} \)-modules are different, under the mapping \( \alpha : Ind_{\mu} \to Ind_{\mu}^{p} \) that defines Shapovalov’s form every irreducible \( gl_{n}^{(1)} \oplus gl_{n}^{(2)} \)-module \( L \) passes into \( L' \). It follows that all irreducible \( gl_{n}^{(1)} \oplus gl_{n}^{(2)} \)-modules arising in \( Ind_{\mu} \) are pairwise orthogonal in the sense of Shapovalov’s form. Thus, if we choose a base on a certain level of \( Ind_{\mu} \) that agrees with the decomposition of \( Ind_{\mu} \) into the direct sum of \( gl_{n}^{(1)} \oplus gl_{n}^{(2)} \)-modules, then the matrix of Shapovalov’s form is block-diagonal in this base. Therefore, the determinant of Shapovalov’s form is equal to the product of the determinants of these blocks.

Next, let \( w = \det_{1}^{k_{1}} \cdots \det_{n}^{k_{n}} \cdot v \) be a singular vector \( gl_{n}^{(1)} \oplus gl_{n}^{(2)} \)-module \( L \).

A typical vector of the intersection of \( L \) with a certain level of \( Ind_{\mu} \) has the form \( f_{i_{1}} \cdots f_{i_{n}} w \), where \( f_{i_{1}}, \ldots, f_{i_{n}} \in n_{\perp}^{(1)} \oplus n_{\perp}^{(2)} \). The corresponding block in Shapovalov’s matrix consists of elements \( w' e_{j_{1}} \cdots e_{j_{k}} f_{i_{1}} \cdots f_{i_{n}} w \), where \( e \) and \( w' \) are images of \( f \) and \( w \) under the Chevalley involution. Since \( n_{\perp}^{(1)} n_{\perp}^{(2)} \) acts trivially on \( w \), we find that the block of Shapovalov’s matrix on the \( j \)th level of \( Ind_{\mu} \), corresponding to the \( gl_{n}^{(1)} \oplus gl_{n}^{(2)} \)-module \( L \), is equal to \( \langle w, w \rangle p_{j}(L) \cdot A \), where \( p_{j}(L) \) is the dimension of the intersection of the \( j \)th level of \( Ind_{\mu} \) with \( L \) and \( A \) is the matrix of Shapovalov’s form of the \( gl_{n}^{(1)} \oplus gl_{n}^{(2)} \)-module \( L \) on this level. Note that here only \( \langle w, w \rangle \) depends on \( \mu \) and \( \det A \neq 0 \) since \( L \) is irreducible. Thus the problem of determining the multiplicity of the zero of the determinant of Shapovalov’s form (on any level) reduces to the search for \( p_{w}(\mu) = \langle w, w \rangle \) for any monomial \( w = \det_{1}^{k_{1}} \cdots \det_{n}^{k_{n}} \cdot v \).

2.3. The direct calculation of \( \langle w, w \rangle \), even for \( w = \det_{k} \cdot v \) is impossible, and therefore, we do the following: first, we prove that \( \deg p_{w}(\mu) = \sum_{i=1}^{n} i \cdot k_{i} \) (which
was to be expected), and second, we find possible roots of \( p_w(\mu) \) and their maximal possible multiplicities. It turns out that the sum of these multiplicities is exactly equal to \( \sum_{i=1}^{n} i \cdot k_i \).

**Lemma 7.** \( p_w(a) = 0 \Leftrightarrow w \) belongs to the submodule in \( \text{Ind}_a \).

**Proof.** If \( w \) belongs to the submodule, then it is clear that \( \langle w, w \rangle = 0 \). On the other hand, in the block base (see Subsec. 2.2) of a certain level of \( \text{Ind}_\mu \) \( w \) is a \( 1 \times 1 \) block since \( w \) is the only element of some \( gl_n^{(1)} \oplus gl_n^{(2)} \)-module \( L \) on its level. Therefore, \( \langle w, w \rangle = 0 \Rightarrow \langle w_1, w \rangle = 0 \) for any \( w_1 \) from the corresponding level. Consequently, \( w \) belongs to the proper submodule in \( \text{Ind}_\mu \).

It follows from Lemma 7 and Theorem 1 that \( p_w(\mu) \) can have only integer \( \geq -n + 1 \) roots. Let us first consider \( w = \text{Det}_k \cdot v \).

**Lemma 8.** For \( w = \text{Det}_k \cdot v \) \( p_w(\mu) \) can only have the roots

\[-k + 1, \ldots, -1, 0\]

(a total of \( k \) roots, and \( p_w \) is a of degree \( k \) polynomial; however, there can be multiplicities).

**Proof.** It follows from Lemma 7 that if \( p_w(a) = 0 \), then we can obtain a certain singular vector in \( \text{Ind}_a \) by acting on \( w \) by operators from \( \mathfrak{n}_+ \subset gl_{2n} \).

1. \( a \geq -k + 1 \).

The action of \( \text{ad} \mathfrak{A}_+ \) cannot send the element from the \( k \times k \) square of \( A_k \) into \( \mathfrak{A}_- \), and the first singular vector in \( \text{Ind}_{-k} \) is \( \text{Det}_{k+1} \cdot v \).

2. \( a \leq 0 \).

It is true that for every \( w = \text{Det}_1^1 \ldots \text{Det}_n^k \cdot v \), the action \( \mathfrak{A}_+ \) cannot increase the degree of \( w \) with respect to any \( y_{ij} \in \mathfrak{A}_- \). Let us prove this for \( w = \text{Det}_k \cdot v \) and \( y_{11} = f_0 \).

**Fig. 2**

\( \text{Det}_k \) is the sum of monomials each of which contains exactly one element from the first row and exactly one element from the first column (they can coincide). Let \( z_1^+, z_2^+, z_3^+, \ldots \) be elements lying in the first column of \( \mathfrak{A}_- \) outside of \( \mathfrak{A}_- \) and \( z_1^-, z_2^-, z_3^-, \ldots \) be elements lying in the first row of \( \mathfrak{A}_- \) outside of \( \mathfrak{A}_- \) (see Fig. 2). The reader can easily verify that the element \( f_0 \) can be obtained in only two ways:

1. we apply \( \text{ad}(x_{ia}) \) to \( y_{11} \), obtain \( z_1^{i-1} \), and apply \( \text{ad}(z_1^{i-1}) \) to \( y_{11} \);
2. we apply \( \text{ad}(x_{ai}) \) to \( y_{11} \), obtain \( z_1^{i-1} \), and apply \( \text{ad}(z_1^{i-1}) \) to \( y_{11} \).
Thus, one element from the first row and one element from the first column must be used in the formation of $f_0$. In general, an element from the $i$th row and an element from the $j$th column must be used in the formation of the element $y_{ij}$. Therefore, in the process of application of elements from $\mathfrak{A}_+$ to $\text{Det}_k$, every monomial will contain not more than one element from every row and every column.

For $\mu = 1$, the singular vectors $f^1_0, \ldots$ in $\text{Ind}_1$ contain $f_0$ of the $\geq 2$ degree, and therefore, is impossible to obtain them from $\text{Det}_k$ by the application of elements from $\mathfrak{A}_+$.

The proof of Lemma 8 implies the following corollary.

**Corollary 2.**

1. For any monomial $w = \text{Det}_1^{k_1} \cdot \ldots \cdot \text{Det}_n^{k_n} \cdot v$ the application of elements from $\mathfrak{A}_+$ cannot increase the degree of $w$ with respect to any $y_{ij} \in \mathfrak{A}_-$,

2. the roots of $p_w(\mu)$ lie in the interval $[-l+1, (\sum k_i) - 1]$, where $l$ is the largest $i$ for which $k_i \neq 0$.

2.4. Here we find an upper estimate for the multiplicities of the roots of $p_w(\mu)$ given by Corollary 2 (2).

**Lemma 9.** If $\mu = a$ is a root of multiplicity $d$ of the polynomial $p_w(\mu)$, then, for $\mu = a + t$ ($t$ is small), $(w, \cdot)_a + t$ is either equal to 0 or is an infinitesimal of the order of exactly $d$ with respect to $t$. In other words, for any $e_{j_1}, \ldots, e_{j_s} \in \mathfrak{n}_+ \subset \mathfrak{gl}_{2n}$, $(e_{j_1} \cdot \ldots \cdot e_{j_s} \cdot w) \cdot v$ is either 0 or an infinitesimal of order $d$.

**Proof.** This is similar to the proof of Lemma 7.

It follows from Lemma 9 that the required multiplicity is equal to the multiplicity of zero for $\mu = a + t$ in the expression $(e_{j_1} \cdot \ldots \cdot e_{j_s} \cdot w) \cdot v$ for all $e_{j_1}, \ldots, e_{j_s} \in \mathfrak{n}_+ \subset \mathfrak{gl}_{2n}$.

**Lemma 10.** The maximal possible multiplicity of the root $\mu = a$ in $p_w(\mu)$ is not larger than the number of singular vectors in $\text{Ind}_a$ lying at the levels not exceeding the level of $w$.

**Proof.** According to Corollary 1 (2), every singular vector increases the order with respect to $t$ by unity. On the other hand, if $\tilde{w} \in \text{Ind}_a$ is not a singular vector, then $e_{js} \tilde{w}$ in $\text{Ind}_{a+t}$ is not an infinitesimal and not zero for some $e_{j_s} \in \mathfrak{n}_+ \subset \mathfrak{gl}_{2n}$. Iterating this process we derive the statement of the lemma from Lemma 9.

In conjunction with Corollary 2 (1) we get

**Lemma 11.** The maximal possible multiplicity of the root $\mu = a$ in $p_w(\mu)$ is equal to the number of singular vectors in $\text{Ind}_a$ lying not lower than $w$ and having (in the notation $\text{Det}_p^r \cdot v$) a degree with respect to any $y_{ij} \in \mathfrak{A}_-$ not higher than $w$.

**Examples.** In Subsec. 2.5 we shall prove that $\text{deg } p_w(\mu) = \sum_{i=1}^n i \cdot k_i$.

(i) $w = \text{Det}_k \cdot v$.

For $-k+1 \leq a \leq 0$, the conditions of Lemma 11 are satisfied only by the singular vector $\text{Det}_{-a+1} \cdot v$, and therefore, every one of the possible $k$ roots has a multiplicity not higher than 1, and since $\text{deg } p_w(\mu) = k$, we have $p_w(\mu) = \mu \cdot (\mu + 1) \cdot \ldots \cdot (\mu + k - 1)$ with an accuracy up to a constant.

(ii) $w = \text{Det}_2 \cdot \text{Det}_3 \cdot v$.

The possible roots of $p_w(\mu)$ are

$+1, 0, -1, -2$
for $\mu = -2$, singular vectors in $\text{Ind}_\mu$ are $\text{Det}_3 \cdot v, \text{Det}_1^2 \cdot v, \ldots$; the conditions of Lemma 11 are satisfied only by $\text{Det}_3 \cdot v$, the maximal multiplicity is 1;
for $\mu = -1$ singular vectors are $\text{Det}_2 \cdot v, \text{Det}_1^3 \cdot v, \ldots$; the conditions of Lemma 11 are satisfied only by $\text{Det}_2 \cdot v$, the maximal multiplicity is 1;
for $\mu = 0$, singular vectors are $\text{Det}_1 \cdot v, \text{Det}_2^2 \cdot v, \text{Det}_3^3 \cdot v, \ldots$; the conditions of Lemma 11 are satisfied only by $\text{Det}_1 \cdot v, \text{Det}_2^2 \cdot v$; the maximal multiplicity is 2;
for $\mu = +1$, singular vectors are $\text{Det}_2^1 \cdot v, \text{Det}_3^2 \cdot v, \ldots$; the conditions of Lemma 11 are satisfied only by $\text{Det}_2^1 \cdot v$; the maximal multiplicity is 1.

On the other hand, $\deg p_w = 5$, and therefore

$$p_w(\mu) = (\mu - 1) \cdot \mu^2 \cdot (\mu + 1)(\mu + 2).$$

with the accuracy up to a constant.

**Lemma 12.** For $w = \text{Det}_1^{k_1} \cdot \ldots \cdot \text{Det}_n^{k_n} \cdot v$ we have

$$\deg p_w(\mu) = \sum_{i=1}^{n} i \cdot k_i.$$

The proof will be given in Subsec 2.5.

The situation encountered in the examples given above is the same for any $w$. Indeed, we can use another technique to calculate the sum of the maximal possible multiplicities. We can obtain $\text{Det}_n^{k_n}, \text{Det}_n^{k_n-1}, \ldots, \text{Det}_n, \text{Det}_{n-1}^{k_{n-1}}, \ldots, \text{Det}_1^1, \ldots, \text{Det}_1$, from $\text{Det}_n^{k_n}$, certainly, for different $\mu$. We have the total of $n \cdot k_n$ singular vectors. Repeating this reasoning for $\text{Det}_1^{k_i}$ for any $i$, we find that the sum of the maximal possible multiplicities is $\sum_{i=1}^{n} i \cdot k_i$.

We have proved

**Lemma 13.** The sum of multiplicities calculated with the use of Lemma 11 is $\sum_{i=1}^{n} i \cdot k_i$.

It is clear that we can find $p_w(\mu)$ (with an accuracy to within the multiplication by a constant) for any $w = \text{Det}_1^{k_1} \cdot \ldots \cdot \text{Det}_n^{k_n} \cdot v$.

There are two ways of formulating the answer in a concise form (the second form is the main one), namely,

**Technique 1.** For every $\text{Det}_k$, we must write the corresponding polynomial $p_k(\mu) = \mu(\mu + 1) \ldots (\mu + k - 1)$ and shift this $\text{Det}_k$ to the right, to the end of the monomial; when shifting by one, the determinant (of the first degree) $p_k(\mu)$ is replaced by $p_k(\mu - 1)$. We must shift all the polynomials to the end of the monomial and then multiply them. (We must do this in the order

$$w = \text{Det}_1^{k_1} \cdot \ldots \cdot \text{Det}_n^{k_n} \cdot v).$$

**Technique 2.** If we consider $w$ to be the highest weight of the corresponding $gl_n^{(2)}$-module, then a certain Young diagram $\mathcal{D}(w)$ corresponds to it (here $n \gg \deg w$).
We must cut $\mathcal{D}(w)$ by oblique straight lines under the angle of 45° as is shown in Fig. 3.

The number of squares on a certain diagonal is the multiplicity of the corresponding root, from the smaller to the larger roots if we cut upward. We denote by
\[ w = \text{Det}_2 \cdot \text{Det}_3 \cdot v \]

\[ p_w(\mu) = (\mu - 1)\mu^2(\mu + 1)(\mu + 2) \]

Fig. 3

\[ l(D) \] the number of diagonals and by \( n_1, \ldots, n_{l(D)} \) the number of squares on the diagonals if we cut upward. Then

\[ p_w(\mu) = \prod_{i=1}^{l(D)} (\mu + k(D) - i)^{n_i}, \tag{8} \]

where \( k(D) \) is the maximal height of \( D(w) \) ( \( p_w(\mu) \) with an accuracy to within the multiplication by a constant). Thus, the length of the “central” diagonal that passes through the upper left vertex of \( D(w) \) is equal to the multiplicity of the root \( \mu = 0 \).

2.5. Here we prove Lemma 12. As we have mentioned, the direct calculation of \( p_w(\mu) \) is impossible, even for \( w = \text{Det}_k \cdot v \). However, it is very easy to calculate explicitly the highest coefficient and prove that it is not equal to zero.

Let \( w = \text{Det}_1^k \cdot \ldots \cdot \text{Det}_n^k \cdot v \). We shall calculate \( \langle w, w \rangle \). We denote by \( p_i^k \) the monomials, the terms of \( \text{Det}_l \), taken with the signs \((i = 1, \ldots, l)\) and by \( q_i^k \) the images of \( p_i^k \) under the Chevalley involution. The sign that appears due to the Chevalley involution does not depend on the choice of a definite monomial but only depends on \( l \). Only the terms

\[ \left( (q_{i_n}^{j_n} \ldots q_{i_1}^{j_1}) (q_{n-1}^{j_{n-1}} \ldots q_{n-1}^{j_1}) \ldots (p_{n-1}^{j_{n-1}} \ldots p_{n-1}^{j_1}) (p_n^{j_n} \ldots p_n^{j_1}) \right) \cdot v, \tag{9} \]

contribute to the highest monomial namely, the terms which are “symmetric” with respect to \( p \) and \( q \). Due to this symmetry, the signs of all monomials are cancelled out. Now we set \( \tilde{q}_i^\sigma = \prod_{i=1}^l x_{\sigma(i)i} \) for \( p_i^\sigma = \prod_{i=1}^l y_{\sigma(i)i} \). Then, when we replace \( q \) by \( \tilde{q} \), relation (9) remains valid with an accuracy to within the common sign dependent on \( w \). Now we shift \( \tilde{q} \) to the right and, as a result, no signs appear. We have proved the lemma.

3. The Jantzen Filtration: Formulas for the Characters of Irreducible Representations and Combinatorial Identities.
3.0. The considerations of Sec. 2 result in numerous formulas and identities. Indeed, let \( a \in \mathbb{Z}_{\geq -n+1} \) be a critical value of the parameter, and let \( \text{Ind}_a^{(1)} \) be the kernel of Shapovalov’s form on \( \text{Ind}_a \). Then \( \text{Ind}_a^{(1)} \) is a submodule and \( \text{Ind}_a / \text{Ind}_a^{(1)} \) is irreducible. Thus, from the results of Sec. 2 we can deduce formulas for the characters of irreducible representations of \( gl_{2n} \) and \( gl_\infty \). For example, at \( \mu = 1 \) for \( \hat{gl}_\infty \) the character of the irreducible representation of \( \hat{gl}_\infty \) with zero highest weight and the central charge 1 is equal, by the Weyl formula, to \( \prod_{i \geq 1} \frac{1}{(1 - q^i)} \), and from the results of Sec. 2 the same character is equal (see Subsec. 3.2) to

\[
1 + \sum_{k \geq 1} \frac{q^{k^2}}{(1 - q) \cdots (1 - q^k)^2}.
\]

Thus we obtain the Euler identity

\[
\prod_{i \geq 1} \frac{1}{(1 - q^i)} = 1 + \sum_{k \geq 1} \frac{q^{k^2}}{(1 - q) \cdots (1 - q^k)^2}.
\]

If we set \( \mu = 2, 3, \ldots \), then we get the “higher analogs” of identity (10). On the other hand, for \( \mu = -1, -2, \ldots \), we get relations for the character of the corresponding irreducible representation \( \text{Ind}_\mu / \text{Ind}_\mu^{(1)} \) (see (17), (18) below). Note that by considering \( gl_{2n} \) rather than \( gl_\infty \), we obtain the “finite form” of identity (10) and of the highest identities (see Subsec. 3.2).

3.1. Definition. For \( k \in \mathbb{Z}_{\geq 1} \), we set \( \text{Ind}_a^{(k)} = \{ w \in \text{Ind}_a \mid \langle w, w_1 \rangle_{a+t} \text{ is divisible by } t^k \text{ for any } w_1 \in \text{Ind}_a \} \). \( \text{Ind}_a^{(k)} \) is the \( k \)th term of the Jantzen filtration. \( \text{Ind}_a^{(k)} \) is a submodule in \( \text{Ind}_a \) for any integer \( k \geq 1 \), and \( \text{Ind}_1 \supset \text{Ind}_a^{(1)} \supset \text{Ind}_a^{(2)} \supset \ldots \) is the Jantzen filtration of the module \( \text{Ind}_a \).

Lemma 14 (A description of the Jantzen filtration of \( \text{Ind}_a \)). Let \( a \) be a critical value of the parameter. Then \( \text{Ind}_a^{(k)} \) is the union of all irreducible \( gl_n^{(1)} \oplus gl_n^{(2)} \)-modules, the scalar square \( p_w(\mu) \) of whose highest weight vector \( w \) has \( \mu = a \) a root of multiplicity \( \geq k \).

Proof. According to Lemma 9, the condition imposed on \( w \) formulated in the lemma is equivalent to the fact that \( w \in \text{Ind}_a^{(k)} \). However, since \( \text{Ind}_a^{(k)} \) is a \( gl_{2n} \)-submodule, \( \text{Ind}_a^{(k)} \) contains the whole irreducible \( gl_n^{(1)} \oplus gl_n^{(2)} \)-module with the weight vector \( w \).

We have proved the following relationship (that always exists) of the Jantzen filtration with zero multiplicity of the determinant of Shapovalov’s form.

Corollary 3. Let \( V_j \) be \( j \)th level of \( \text{Ind}_a \). Then the multiplicity of zero (with respect to \( \mu \) at the point \( \mu = a \)) of the determinant of Shapovalov’s form on the \( j \)th level is

\[
\dim(V_j \cap \text{Ind}_a^{(1)}) + \dim(V_j \cap \text{Ind}_a^{(2)}) + \ldots
\]

(the sum is finite).

Theorem 2 (Jantzen’s conjecture). The consecutive quotient modules of the Jantzen filtration are semisimple.

Proof. Due to the existence of the \( gl_n^{(1)} \oplus gl_n^{(2)} \)-action on \( \text{Ind}_a \), as distinct from the case of the Verma modules, in our case, Jantzen’s conjecture can be proved by means of elementary reasoning.
The restriction of Shapovalov’s form to $\text{Ind}_a^{(k)}$ is divisible by $t^k$. We divide it by $t^k$ and then set $t = 0$. Then this form induces an invariant symmetric form on $\text{Ind}_a^{(k)}/\text{Ind}_a^{(k+1)}$ with values in $\mathbb{C}$. We denote it by $\langle \cdot, \cdot \rangle_k$. The form $\langle \cdot, \cdot \rangle_k$ on $\text{Ind}_a^{(k)}/\text{Ind}_a^{(k+1)}$ is nondegenerate. Indeed, assume that we consider the $l$th level $V_l$ of the module $\text{Ind}_a$, and let $\xi \in V_l$ belong to the kernel of $\langle \cdot, \cdot \rangle_k$. Then $\xi = \xi_1 + \ldots + \xi_s$, where $\xi_1, \ldots, \xi_s$ are components of $\xi$ relative to the decomposition of $V_l$ into the direct sum of the intersections of $V_l$ with irreducible $gl_{n(1)} \oplus gl_{n(2)}$-module $V_l^{(1)}, \ldots, V_l^{(s)}$.

Assume, for example, that $\xi_1 \neq 0$. It follows from 2.2 that $V_l^{(1)} \perp V_l^{(2)}, \ldots, V_l^{(1)} \perp V_l^{(s)}$. Let $w$ be the highest weight vector of the $gl_{n(1)} \oplus gl_{n(2)}$-module corresponding to $V_l^{(1)}$. Then, in accordance with Lemma 14, $\langle w, w \rangle$ is divisible by $t^k$ and not divisible by $t^{k+1}$, and therefore, the form $\langle \cdot, \cdot \rangle_k$ on this $gl_{n(1)} \oplus gl_{n(2)}$-module is nondegenerate.

Therefore, $\langle \xi, \xi \rangle_k \neq 0$ for a certain $\xi' \in V_l^{(1)}$, and consequently $\langle \xi, \xi' \rangle_k \neq 0$.

Suppose now that $N \subset \text{Ind}_a^{(k)}/\text{Ind}_a^{(k+1)}$ is a certain $gl_{2n}$-submodule. To prove the theorem, it is sufficient to show that $N \cap N^\perp = 0$ ($N^\perp$ in the sense of $\langle \cdot, \cdot \rangle_k$). Let $\xi \in N \cap N^\perp$ and $\xi \neq 0$. Assume, as above, that $\xi_1 \in V_l^{(1)}$ is nonzero. Being a $gl_{n(1)} \oplus gl_{n(2)}$-module, $N$ contains all $V_l^{(1)}$, and the reasoning that proves the nondegeneracy of $\langle \cdot, \cdot \rangle_k$ allows us to find $\xi'_1 \in V_l^{(1)} \subset N$ such that $\langle \xi_1, \xi'_1 \rangle = \langle \xi, \xi'_1 \rangle$ is nonzero, and therefore, $\xi$ is not orthogonal to $N$.

**Theorem 3.** The submodules generated by singular vector in $\text{Ind}_a$ constitute the complete list of submodules in $\text{Ind}_a$. They are embedded into each other and the $k$th submodule coincides with $\text{Ind}_a^{(k)}$. The consecutive quotient modules $\text{Ind}_a^{(k)}/\text{Ind}_a^{(k+1)}$ are simple.

**Proof.**

(1) There is exactly one singular vector in $\text{Ind}_a^{(k)}/\text{Ind}_a^{(k+1)}$. In fact only some highest weight vector of the irreducible $gl_{n(1)} \oplus gl_{n(2)}$-module can be such a singular vector. Now the assertion follows from Corollary 2 (1) of Sec. 2.

(2) It follows from (1) and Theorem 2 that the consecutive quotient modules of the Jantzen filtration are simple. Indeed, assume that $\text{Det}_p \cdot v$ is a singular vector in $\text{Ind}_a$ and that it belongs to $\text{Ind}_a^{(k)}$. Then $\text{Det}_p^{q+1} \cdot v$ is the next singular vector and it belongs to $\text{Ind}_a^{(k+1)}$. $\text{Det}_p^{q+1} \cdot v$ belongs to $\text{Ind}_a^{(k)}$ and, since the representation $\text{Ind}_a^{(k)}/\text{Ind}_a^{(k+1)}$ is simple,

$$\text{Det}_p^{q+1} \cdot v = U(\mathfrak{h}_+) \cdot \text{Det}_p^q \cdot v + \alpha,$$

where $\alpha \in \text{Ind}_a^{(k+1)}$. It follows from Lemma 14 and Lemma 11 that $\alpha = 0$ from the grading considerations. Therefore, $\text{Det}_p^{q+1} \cdot v$ belongs to the submodule in $\text{Ind}_a$ generated by the singular vector $\text{Det}_p^q \cdot v$, and this implies that the singular vector $\text{Det}_p^{q+1} \cdot v$ also belongs to this module. This proves Theorem 3.

3.2. Here we consider formulas following from the fact that $\text{Ind}_a/\text{Ind}_a^{(1)}$ is an irreducible module for $a > 0$.

First of all, let us consider a representation of $\widehat{gl}_\infty$ (or of $gl_{\infty, \text{fin}}$) with a zero highest weight and the central charge $\mu = 1$ (or with one zero label respectively). Then $\text{Det}_1^1 \cdot v$ is the first singular vector, and it follows from Lemma 11
that outside of $\text{Ind}_1^{(1)}$ there are $gl^{(1)}_\mathbb{F} \oplus gl^{(2)}_\mathbb{F}$-modules with the highest weight vectors $v, \text{Det}_1 \cdot v, \text{Det}_2 \cdot v, \ldots$ and only these vectors. It is clear that the character of the irreducible $gl^{(2)}_\mathbb{F}$-module with the highest vector $\text{Det}_k \cdot v$ is equal to

$$\frac{1}{(1 - q)(1 - q^2) \ldots (1 - q^k)},$$

and the weight of $\text{Det}_k$ is equal to $k^2$. Hence,

$$\prod_{i \geq 1} (1 - q^i) = 1 + \sum_{k \geq 1} \frac{q^k}{(1 - q)^2 \cdots (1 - q^k)^2}.$$  

(relation (10), the Euler identity).

Let us write out the “finite form” of the last identity, replacing $\hat{gl}_\infty$ by $gl_{2n}$: only $v, \text{Det}_1 \cdot v, \ldots, \text{Det}_n \cdot v$ remain. The character of the irreducible $gl^{(2)}_n$-module with the highest weight $\text{Det}_k$ is equal to

$$\frac{(1 - q^{n-k+1}) \cdots (1 - q^n)}{(1 - q) \cdots (1 - q^k)},$$

and the character of the corresponding irreducible $gl_{2n}$-module is equal to

$$\frac{(1 - q^{n+1}) \cdots (1 - q^{2n})}{(1 - q) \cdots (1 - q^n)}.$$  

We have

$$\frac{(1 - q^{n+1}) \cdots (1 - q^{2n})}{(1 - q) \cdots (1 - q^n)} = 1 + \sum_{k=1}^{n} \frac{q^k \cdot (1 - q^{n-k+1})^2 \cdots (1 - q^n)^2}{(1 - q^2) \cdots (1 - q^k)^2} (n \in \mathbb{Z}_{>0}).$$  

In what follows, we shall not write out the finite forms of the formulas and identities.

For $\mu = l \geq 1$, the first singular vector in the $\hat{gl}_\infty$-module $\text{Ind}_\mu$ is $\text{Det}_1^{l+1} \cdot v$, and therefore, the highest weight vectors of the $gl^{(1)}_\mathbb{F} \oplus gl^{(2)}_\mathbb{F}$-modules which lie outside of $\text{Ind}_\mu^{(1)}$ are the monomials $w = \text{Det}_{k_1}^{s_1} \cdots \text{Det}_{s_l}^{k_l} \cdot v$ with $k_1 + \ldots + k_s \leq l$. Let $D(w)$ be the corresponding Young diagram, and let $\chi(w)$ be the character of the corresponding irreducible $gl^{(2)}_\mathbb{F}$-module. Then

$$\frac{1}{(1 - q)(1 - q^2) \ldots (1 - q^l)(1 - q^{l+1}) \cdots} = 1 + \sum_{\{w|k_1 + \ldots + k_s \leq l\}} q^{k_1 + 4k_2 + \ldots + s^2k_s} \cdot (\chi(w))^2.$$  

We write this explicitly for $l = 2$. Here either $w = \text{Det}_i \cdot v$ or $w = \text{Det}_i \cdot \text{Det}_j \cdot v$ for $i \leq j$. In the second case, the diagram $D(w)$ is shown in Fig. 4:
\[ \chi(Det_i \cdot Det_j) = \prod_{k=1}^{i} \frac{(1 - q^{i-k+1}) \cdots (1 - q^k)}{(1 - q^k) \cdots (1 - q^{i-k+1})} \]

We have

\[ \prod_{s \geq 1} (1 - q^s)^2 = 1 + \sum_{k \geq 1} \frac{q^{k^2}}{(1 - q^2)^2 \cdots (1 - q^{k^2})^2} + \]

\[ \sum_{j \geq 1} \frac{q^{j^2}}{(1 - q^2)^2 \cdots (1 - q^{j^2})^2} \]

Finally,

\[ \prod_{i \geq 1} (1 - q^i)^{\alpha_i} = 1 + \sum_{w} q^{k_1 + 4k_2 + \cdots + s^2 k_s} (\chi(w))^2, \]

where \( w = Det_{k_1} \cdots Det_{k_s} \cdot v \) and \( \chi(w) \) is the character of the irreducible \( gl(2) \)-module with the corresponding Young diagram. Note that in (16) the summation is carried out over all Young diagrams.

3.3. Let us consider the case \( \mu = -1, -2, -3, \ldots \). For example, if \( \mu = -1 \), then \( w \) which lie in \( Ind_{-1} \setminus Ind_{-1}^{(1)} \) are \( Det_1 \cdot v, Det_2 \cdot v, Det_3 \cdot v, \ldots \). We denote by \( \chi_{-1} \) the character of the irreducible \( gl_\infty \)-module with the zero highest weight and the central charge \(-1\). We have

\[ \chi_{-1} = 1 + \sum_{k \geq 1} q^{k} \cdot \frac{1}{(1 - q^2)^2 \cdots (1 - q^{k})^2}. \]

For \( \mu = -l \), the first singular vector in \( Ind_{-l} \) is \( Det_{l+1} \cdot v \), and therefore, the general case is as follows. We denote \( W_l = \{ w = Det_{k_1} \cdots Det_{k_s} : k_i \geq 0 \} \). Then

\[ \chi_{-l} = \sum_{w \in W_l} q^{k_1 + 4k_2 + \cdots + l^2 k_l} (\chi(w))^2. \]

3.4. Here we use the whole Jantzen filtration, not only its first term. According to Theorem 3, the module \( Ind_{-1}^{(k)} / Ind_{-1}^{(k+1)} \) is irreducible.

3.4.1. Let \(-k \in \mathbb{Z}_{\leq 0}\), and then the singular vectors in \( Ind_{-k} \) are \( Det_{k+1} \cdot v, Det_{k+2} \cdot v, \ldots \). It is clear that \( Ind_{-k}^{(l)} / Ind_{-k}^{(l+1)} \) is an irreducible module with the highest weight vector \( Det_{l+1} \cdot v \) (\( l \geq 1 \)). We denote the corresponding weight by \( \Lambda_l^- \). We have

\[
\begin{align*}
\Lambda_l^- (\alpha_0^+) &= -k - 2l \\
\Lambda_l^- (\alpha_{l+1}^+) &= l, \quad l \geq 1 \\
\end{align*}
\]

the other \( \Lambda_l^- (\alpha_j^+) = 0 \).
The highest weight vectors of the $gl(1)^{\infty} \oplus gl(2)^{\infty}$-modules $w$ lying in $Ind_{-k}^{l} \setminus Ind_{-k}^{l+1}$ are

$$W_{l}^{-} = \left\{ \text{Det}_{1}^{k_{1}} \cdot \text{Det}_{2}^{k_{2}} \cdot \ldots \cdot v \mid \sum_{s \geq k+l} k_{s} \geq l, \quad \text{and} \quad \sum_{s \geq k+l+1} k_{s} \leq l \right\}.$$  

Then, as usual,

$$(20) \quad \chi_{\infty}(\Lambda_{l}^{-}) = \sum_{w \in W_{l}^{-}} q^{-l(k+l)^{2}+k_{1}+4k_{2}+\ldots} \cdot (\chi(w))^{2}. \quad \tag{20}$$

3.4.2. Let $k \in \mathbb{Z}_{\geq 0}$. The singular vectors in $Ind_{k}$ are $\text{Det}_{1}^{k+1} \cdot v, \text{Det}_{2}^{k+2} \cdot v, \ldots$. The highest weight vector of $Ind_{k}^{l} / Ind_{k}^{l+1}$ is $\text{Det}_{1}^{k+l} \cdot v$. The corresponding highest weight

$$\Lambda_{l}^{+}(\alpha_{i}^{\vee}) = \begin{cases} -k - 2l & \text{if } \alpha_{i}^{\vee} = \alpha_{0}^{\vee} \\ k + l, & \text{if } l \geq 1 \\ 0, & \text{the other } \alpha_{j}^{\vee} = 0 \end{cases} \quad \tag{21}$$

and $gl(1)^{\infty} \oplus gl(2)^{\infty}$, the highest weight vectors $w$ lying in $Ind_{k}^{l} \setminus Ind_{k}^{l+1}$ are

$$W_{l}^{+} = \left\{ \text{Det}_{1}^{k_{1}} \cdot \text{Det}_{2}^{k_{2}} \cdot \ldots \cdot v \mid \sum_{s \geq k+l} k_{s} \geq k+l, \quad \text{and} \quad \sum_{s \geq k+l+1} k_{s} \leq k+l \right\}.$$  

Then

$$(22) \quad \chi_{\infty}(\Lambda_{l}^{+}) = \sum_{w \in W_{l}^{+}} q^{-l(k+l)^{2}+k_{1}+4k_{2}+\ldots} \cdot (\chi(w))^{2}. \quad \tag{22}$$

Chapter 2.

The Lie Algebra $gl(\lambda)$: Irreducible Representations and the Local Identity

1. Introduction: the Lie Algebra $gl(\lambda)$ and Induced Representations.

1.0. In Chapters 2 and 3 we deal with the theory of representations of the Lie algebras $gl(\lambda) (\lambda \in \mathbb{C})$ and of the Lie algebra of functions on a hyperboloid. We prove the equivalence of the irreducibility of certain representations of these Lie algebras to the local identity (in Ch. 2) and to the global identity (in Ch. 3) with power series (see Subsec. 0.11). The local identity connected with the representations of the Lie algebra $gl(\lambda)$ have the form

$$\frac{d}{da} \left( \prod_{i \geq 1} \frac{1}{(1-aq^{i})^{i}} \right) \bigg|_{a=1} =$$

$$\sum_{\text{over all } w=\text{Det}_{1}^{k_{1}} \cdot \ldots \cdot \text{Det}_{k}^{k_{k}}} \# D(w) \cdot q^{\sum_{i \geq 1} i^{2}} \cdot (\chi(w))^{2}. \quad \tag{1}$$
Appendix B).

Then Theorem 2 stating that for the general valov’s form of the representations $\theta$ by continuity Lie algebra $\mathfrak{gl}(\lambda)$ from two exceptional parabolic subalgebras for the general representations. Thus, we get the local identity (1).

Let us define the associative algebra $U\lambda$ as a quotient algebra $U(sl_2)/ \left( \Delta - \frac{\lambda(\lambda + 2)}{2} \right)$, where $U(sl_2)$ is the universal enveloping algebra of the Lie algebra $sl_2(\mathbb{C})$, $\lambda \in \mathbb{C}$, and $\Delta = ef + fe + \frac{h^2}{2} \in U(sl_2)$ is the Casimir operator. It is easy to show that the
Lie algebra (with the bracket \([a, b] = a \cdot b - b \cdot a\) constructed using the associative algebra \(U_\lambda\) coincides with the Lie algebra \(gl(\lambda)\).

Indeed, \(\Delta\) is the central element in \(U(sl_2)\) which acts on the representation of \(sl_2\) with the highest weight \(\lambda\) by the multiplication by \(\frac{\lambda(\lambda + 2)}{2}\). Thus, for \(\lambda \in \mathbb{Z}_{\geq 0}\), we have the homomorphism of Lie algebras

\[
\gamma(\lambda) \to \gamma(V_{\lambda + 1}),
\]

where \(V_{\lambda + 1}\) is an irreducible \((\lambda + 1)\)-dimensional \(sl_2\)-module with the highest weight \(\lambda\). This mapping is surjective. We denote its kernel by \(J_\lambda\). Then \(\gamma(\lambda)/J_\lambda \cong \gamma_{\lambda + 1}\) and \(J_\lambda = \bigoplus_{i=\lambda + 1} \pi_i\) with respect to the adjoined action of \(sl_2 \subset U(sl_2)\). However, for this definition of \(\gamma(\lambda)\), all relations a priori depend analytically on \(\lambda\).

The Lie algebra \(sl_2\) is \(\{e, h, f\}\) is injected into \(gl(\lambda)\).

We set

\[
\gamma(\lambda)^l = \{v \in \gamma(\lambda) \mid [h, v] = 2lv, \ l \in \mathbb{Z}\}
\]

and

\[
\gamma_+ = \bigoplus_{i \geq 0} \gamma(\lambda)^i; \ \gamma = \gamma(\lambda)^0; \ \gamma_- = \bigoplus_{i < 0} \gamma(\lambda)^i.
\]

We have \(\gamma(\lambda) = \gamma_+ \oplus \gamma \oplus \gamma_-\). Note that for \(\lambda \in \mathbb{Z}_{\geq 0}\), the corresponding decomposition of \(\gamma_{\lambda + 1}\) is \(\gamma(\lambda)/J_\lambda = \gamma_+ \oplus \gamma \oplus \gamma_-\) coincides with the standard Cartan decomposition of \(\gamma_{\lambda + 1}\). In this text, we consider representations of \(\gamma(\lambda)\) with the highest weight vector, i.e., a vector \(v\) such that

\[
\gamma_+ v = 0, \quad hv = \chi(h) \cdot v \quad \text{for } h \in \gamma.
\]

It is clear that the decomposition \(\gamma(\lambda) = \bigoplus_{i \in \mathbb{Z}} \gamma(\lambda)^i\) defines the grading on \(\gamma(\lambda)\).

Therefore, every representation with the highest weight vector is \(\mathbb{Z}_{\geq 0}\)-graded.

The simplest representations with the weight vector, namely, the Verma modules, have infinite-dimensional levels since the subalgebra \(\gamma_-\) is generated by the infinite-dimensional subset \(\gamma(\lambda)^{-1}\) and the Cartan algebra \(\gamma\) acts on the levels not semi-simply. (The latter fact is true since \(\gamma(\lambda)^1 \cdot (\gamma(\lambda)^{-1})\) cannot be decomposed into the direct sum of \(\gamma\)-invariant one-dimensional subspaces but is only entirely \(\gamma\)-invariant.)

In order to get a describable theory of representations, we must consider the highest weights \(\chi\) for which the corresponding Verma module has a sufficient number of singular vectors and can be factorized to a representation with finite-dimensional levels. We shall say that a representation with finite-dimensional levels is \emph{quasifinite}. These are representations induced from parabolic subalgebras and any quasifinite representation of \(\gamma(\lambda)\) is a quotient of a representation induced from some parabolic subalgebra (see Subsec. 1.3).

In addition to the \(\mathbb{Z}\)-grading \(\gamma(\lambda) = \bigoplus_{k \in \mathbb{Z}} \gamma(\lambda)^k\), there exists a filtration \(\gamma(\lambda)_k = \bigoplus_{i=0}^{k+1} \pi_i\). The vector space \(\gamma(\lambda)^1 \cdot (\gamma(\lambda)^{-1})\) is an analog of the space of positive \(i=0\) (negative) simple root vectors. It is \(\gamma\)-invariant and cannot be decomposed into the direct sum of proper \(\gamma\)-invariant subspaces. Note that \(\gamma(\lambda)^{\pm 1}\) generates \(\gamma_\pm\).
1.2. Parabolic subalgebras in $gl(\lambda)$ are constructed as follows. Let us consider, in the “space of negative simple root vectors” $gl(\lambda)^{-1} = \{ P(h) \cdot f, P(h) \in \mathbb{C} [h] \}$, a subspace of codimension $k$ consisting of elements of the form $P(h) \cdot f$, where

$$P = (h - \alpha_1) \ldots (h - \alpha_k)P_1, \quad P_1 \in \mathbb{C} [h].$$

We denote by $\tilde{n}_-$ the Lie subalgebra in $n_-$ generated by this subspace.

**Lemma 1.** $\tilde{n}_- \oplus h \oplus n_+ is a Lie subalgebra in gl(\lambda)$.

**Definition.** The subalgebra $p_{\alpha_1, \ldots, \alpha_k} = \tilde{n}_- \oplus h \oplus n_+ \subset gl(\lambda)$ is parabolic corresponding by the roots $\alpha_1, \ldots, \alpha_k$.

Assume now that $\theta : p_{\alpha_1, \ldots, \alpha_k} \rightarrow \mathbb{C}$ is a one-dimensional representation. Note that this is equivalent to the determination of the character $\theta : h \rightarrow \mathbb{C}$ such that

$$\theta|_{h \cap [\tilde{n}_-, n_+]} = 0.\tag{2}$$

The space of these characters $\theta$ is $(k + 1)$-dimensional.

Indeed, $h \cap [\tilde{n}_-, n_+] = h \cap [\tilde{n}^{-1}_-, n_-^+]$, and therefore $(1) \Leftrightarrow \theta|_{\tilde{n}_- \cap (h - \alpha_1) \ldots (h - \alpha_k) P_1 f} = 0$ for all $P_1 \in \mathbb{C} [h]$. It follows immediately that $\theta$ can be uniquely determined from $\theta(1), \theta(h), \ldots, \theta(h^k)$ (see Remark 1).

Now we set

$$L_{\alpha_1, \ldots, \alpha_k} \theta = U(gl(\lambda)) \bigotimes_{U(p_{\alpha_1, \ldots, \alpha_k})} \mathbb{C}.\tag{3}$$

These representations of $gl(\lambda)$ are known as generalized Verma modules.

**Proposition 1.** (1) $\tilde{n}_- = \bigoplus_{l \geq 1} \tilde{n}_-^l, \quad (\tilde{n}_-^l = \tilde{n}_- \cap gl(\lambda)^{-l})$, where $\tilde{n}_-^{l+1} = \left\{ \prod_{i=1}^{k} (h - \alpha_i)(h - \alpha_i + 2) \ldots (h - \alpha_i + 2l) \right\} \cdot P_1 \cdot f^{l+1}, \quad P_1 \in \mathbb{C} [h]$.\n
(2) The $q$-character of the representation $L_{\alpha_1, \ldots, \alpha_k} \theta$ is equal to $((1 - q)(1 - q^2)^2(1 - q^3)^3 \ldots)^{-k}$.

**Proof.** First, (2) follows from (1). The subspace $a$ which is a complement of $\tilde{n}_-$ in $n_-$ has $k \cdot l$ elements of grading $-l$, and therefore, (2) follows from the Poincaré–Birkhoff–Witt theorem. To obtain (1), we note that in the associative algebra $U(sl_2)/\left( \Delta - \frac{\lambda(\lambda + 2)}{2} \right)$, and, consequently, in $gl(\lambda)$, the relation $f \cdot P(h) = P(h + 2) \cdot f$ holds true for any $P(h) \in \mathbb{C} [h]$.

1.3. Thus, we have constructed a $(2k + 1)$-parameter family of representations of $gl(\lambda)$ with a $k$-dimensional first level. We can immediately prove the following proposition.

**Proposition 2.** Any quasifinite representation of $gl(\lambda)$ with a $k$-dimensional first level, generated by one highest weight vector, is a quotient representation of $L_{\alpha_1, \ldots, \alpha_k} \theta$ for certain $\alpha_1, \ldots, \alpha_k$ and $\theta : h \rightarrow \mathbb{C}$.

**Proof.** Let us consider the quasifinite representation $N$ of $gl(\lambda)$ with a $k$-dimensional first level, $v$ being its highest weight vector.
We shall consider
\[ J = \{ \xi \in gl(\lambda) \mid \xi v = \mu(\xi) v, \ \mu(\xi) \in \mathbb{C} \}. \]
It is clear that \( J \) is a subalgebra in \( gl(\lambda) \), \( J \supset \mathfrak{h} \bigoplus \mathfrak{n}_+ \). Let \( p(h) : f \in J \cap \mathfrak{n}_- \). Then \([p(h) \cdot f, p_1(h)] = p(h)(p_1(h + 2) - p_1(h))f \in J \cap \mathfrak{n}_-\), and therefore, \( J \cap \mathfrak{n}_- = \{p(h)f, \ p \in I\} \), where \( I \) is an ideal in \( \mathbb{C}[h] \).

It is clear that \( J \cap \mathfrak{n}_- \neq 0 \) since the first level of \( N \) is finite-dimensional and \((J \cap \mathfrak{n}_-)v = 0\) from considerations of grading.

Therefore, \( N \) is a quotient representation of the representation induced from the parabolic subalgebra \( p_{\alpha_1, \ldots, \alpha_k} \), where \( I = \{(h - \alpha_1) \ldots (h - \alpha_k)P_1, \ P_1 \in \mathbb{C}[h]\} \).

1.4. Remarks.

1) \( sl(\lambda) = \bigoplus_{i \geq 0} \pi_i \) is a subalgebra in \( gl(\lambda) \), and we do not distinguish between the highest weights \( \chi \) and \( \chi' \) such that \( \chi|_{sl(\lambda) \cap \mathfrak{h}} = \chi'|_{sl(\lambda) \cap \mathfrak{h}} \). In this sense, the representation does not depend on \( \chi(1) \), and we define a representation with a \( k \)-dimensional first level by \( 2 \cdot k \) (not by \( 2k + 1 \)) parameters.

2) We can immediately show that for the general choice of \( 2 \cdot k \) parameters, the corresponding representation of \( gl(\lambda) \) (for a fixed \( \lambda \)) with the character \((1 - q)(1 - q^2)^2(1 - q^3)^3 \ldots)^{-k} \) is irreducible. We do not prove this statement here since it follows from the theorems given in Sec. 2.

2. Embeddings into \( gl_{\infty} \) and Critical Highest Weights.

2.0. In this section, we define the embeddings \( \theta_s : gl(\lambda) \hookrightarrow gl_{\infty,s} \) (see Subsec. 2.1) and investigate the problem of reducibility of the representations \( \theta_s^*(\text{Ind}_{\mu,s}) \) \( (\mu, s \in \mathbb{C}) \). It follows from Proposition 2 that for the given \( s, \mu \in \mathbb{C} \), there exist \( \alpha \), \( \chi(h) \in \mathbb{C} \), and a representation
\[ L_{\alpha, \lambda} \rightarrow \theta_s^*(\text{Ind}_{\mu,s}), \]
and both representations have the same character equal to \( \prod_{i \geq 1} \frac{1}{(1 - q^i)^i} \). Therefore, the irreducibility of \( L_{\alpha, \lambda} \) is equivalent to the irreducibility of \( \theta_s^*(\text{Ind}_{\mu,s}) \), this way we obtaining all parameters \( \alpha \in \mathbb{C} \), except for two or one (according as \( \lambda \)).

2.1. The Lie algebra \( gl(\lambda) \) can be constructed in a standard way with the use of the associative algebra \( U_\lambda = U(sl_3)/\left( \Delta - \frac{\lambda(\lambda + 2)}{2} \right) \). This associative algebra can be described as follows: we fix the algebra \( A = \mathbb{C}[h] \), and being a vector space,
\begin{align*}
U_\lambda &= \mathbb{C}[h] \bigoplus e\mathbb{C}[h] \bigoplus e^2\mathbb{C}[h] \bigoplus \ldots \\
&\bigoplus f\mathbb{C}[h] \bigoplus f^2\mathbb{C}[h] \bigoplus \ldots.
\end{align*}

The following relations hold true:
\begin{align*}
h e &= e(h + 2) \\
h f &= f(h - 2) \\
e f &= T_1(h) = \frac{1}{2} \left( h - \frac{h^2}{2} + \frac{\lambda(\lambda + 2)}{2} \right) \\
e f &= T_2(h) = \frac{1}{2} \left( -h - \frac{h^2}{2} + \frac{\lambda(\lambda + 2)}{2} \right).
\end{align*}
Let
\[ \sigma_1 : p(h) \mapsto p(h + 2) \text{ and } \sigma_2 : p(h) \mapsto p(h - 2) \]
be two automorphisms of the algebra \( A = \mathbb{C} [h] \), and then
\[
\begin{align*}
\sigma_1 T_1 &= T_2 \\
\sigma_2 T_2 &= T_1
\end{align*}
\]
where \( p, T_1, T_2 \in \mathbb{C} [h] \).

Let us now consider the associative algebra \( \text{Mat}_\infty \) of generalized Jacobian matrices. We set
\[
A = \prod_{i=-\infty}^{\infty} \mathbb{C} (i), \text{ where } \mathbb{C} (i) \text{ is a copy of the algebra } \mathbb{C}.
\]
Being a vector space,
\[
\text{Mat}_\infty = A \bigoplus eA \bigoplus e^2 A \bigoplus e^3 A \bigoplus \ldots \\
\bigoplus fA \bigoplus f^2 A \bigoplus f^3 A \bigoplus \ldots
\]

\[
\begin{align*}
ef &= fe = 1 \in A \\
H \cdot e &= e \cdot D(H), \text{ where } \\
D : A &\rightarrow A - \text{ a shift to the right by unity.}
\end{align*}
\]

(Here \( A \) are diagonal matrices, \( e \) is a diagonal of unities which is above the principal, and \( f \) is a diagonal of unities which is below the principal diagonal.)

The main difference between \( \text{Mat}_\infty \) and \( U_\lambda \) is that \( T_1 = T_2 = 1 \) in (7), and then, as in (5), \( T_i \neq 1 \).

It follows that we can construct the mapping of the associative algebras \( \varphi_s : U_\lambda \rightarrow \text{Mat}_{\infty, s} \), where, being a vector space,
\[
\text{Mat}_{\infty, s} = A \bigoplus eA \bigoplus e^2 A \bigoplus e^3 A \bigoplus \ldots \\
\bigoplus fA \bigoplus f^2 A \bigoplus f^3 A \bigoplus \ldots
\]

where \( A = \prod_{i=-\infty}^{\infty} \mathbb{C} (i) \) and
\[
\begin{align*}
H \cdot e &= e \cdot D(H) \\
H \cdot f &= f \cdot D^{-1}(H) \\
ef &= (T_1(s - 2i))_{i \in \mathbb{Z}} \in A \\
fe &= (T_2(s - 2i))_{i \in \mathbb{Z}} \in A,
\end{align*}
\]
\( s \in \mathbb{C} \).
Mat\(_{\infty,s}\) is an associative algebra by virtue of the relation \(\sigma_1 T_1 = T_2\). In order to construct this mapping, we must construct the mapping \(\varphi_s : \mathbb{C}[h] \to \prod_{i=-\infty}^{\infty} \mathbb{C}(i)\) such that the diagram

\[
\begin{array}{ccc}
\mathbb{C}[h] & \xrightarrow{\varphi_s} & \prod_{i=-\infty}^{\infty} \mathbb{C}(i) \\
\sigma_1 \downarrow & & \downarrow D \\
\mathbb{C}[h] & \xrightarrow{\varphi_s} & \prod_{i=-\infty}^{\infty} \mathbb{C}(i) \\
\end{array}
\]

is commutative.

It follows that if \(\varphi_s(h)_0 = s \in \mathbb{C}(0)\), then \(\varphi_s(h)_i = s - 2i \in \mathbb{C}(i)\). Thus

\[
(10) \quad \varphi_s(p(h))_i = p(s - 2i) \in \mathbb{C}(i),
\]

where \(p \in \mathbb{C}[h]\).

**Lemma 2.** \(\varphi_s : U_\lambda \to Mat_{\infty,s}\) is an embedding.

We have defined the embedding of the corresponding Lie algebras \(\varphi_s : gl(\lambda) \hookrightarrow gl_{\infty,s}\). We will define the associative algebra \(Mat_{\infty,i}(i \in \mathbb{Z})\) by relations (8) and (9), where, instead of the relations for \(ef\) and \(fe\) in (9), we set for \(j \neq i\) and \((ef)_i = 0; ef \in A, (ef)_j = 1\) for \(fe \in A, (fe)_j = 1\) for \(j \neq i + 1\) and \((fe)_{i+1} = 0\). Clearly, the algebra \(Mat_{\infty,i}\) is associative, and we denote by \(gl_{\infty,i}\) the corresponding Lie algebra.

In what follows, we distinguish between two cases, namely,

\[
\begin{align*}
(i) & \quad T_1(s - 2i) = 0 \text{ for a certain } i \in \mathbb{Z}, \\
(ii) & \quad T_1(s - 2i) \neq 0 \text{ for all } i \in \mathbb{Z}.
\end{align*}
\]

**Lemma 3.** For the general \(\lambda\),

\(Mat_{\infty,s} \cong Mat_{\infty,i}\) in case (i),

\(Mat_{\infty,s} \cong Mat_{\infty}\) in case (ii).

**Proof.** The map \(\Psi_s : Mat_{\infty,s} \to Mat_{\infty}\) is defined:

\[
\begin{align*}
& H \mapsto H \\
& f \mapsto f \\
& e \mapsto (T_1(s - 2i), i \in \mathbb{Z}) \cdot e,
\end{align*}
\]

which is an isomorphism in case (ii). This proves (ii), and case (i) can be proved by analogy.

The assumption concerning the general \(\lambda\) is that \(T_1\) does not have two roots \(t_1\) and \(t_2\) such that \(t_1 - t_2 \in 2\mathbb{Z}\). When these two roots exist, we have \(Mat_{\infty,s} \cong Mat_{\infty,i,j}\) for certain \(s \in \mathbb{C}\), where the last associative algebra can be determined in an obvious way.

**Definition.** The associative algebra \(U^\lambda_O\) can be constructed from relations (4) and (5) where \(A = \mathbb{C}[h]\) is replaced by \(A = O\), the algebra of holomorphic functions \(\mathbb{C} \to \mathbb{C}\). The Lie algebra \(gl^O(\lambda)\) is the Lie algebra corresponding to the associative algebra \(U^\lambda_O\).
Lemma 4. The embedding $\varphi_s : gl(\lambda) \hookrightarrow gl_{\infty,s}$ can be continued to the map $\varphi_s^\infty : gl^\infty(\lambda) \rightarrow gl_{\infty,s}$ which is surjective.

Proof. The first statement is obvious and the second statement follows from the fact that for the discrete sequence $(s - 2i)_{i \in \mathbb{Z}}$ there exists a holomorphic function from $\mathbb{C}$ into $\mathbb{C}$ which assumes the given values at the points of this sequence.

2.2. There is a 2-cocycle $\alpha$ of the Lie algebra $gl_{\infty,s}$ defined as

\begin{equation}
\alpha(E_{ij}, E_{ji}) = \begin{cases} P_j - i(s - 2i) & \text{for } i \leq 0, j \geq 1 \\ 0 & \text{otherwise} \end{cases}
\end{equation}

\begin{equation}
\alpha(E_{ij}, E_{kl}) = 0 \text{ for } i \neq l \text{ and } j \neq k.
\end{equation}

Here $P_l(h) = e^l f^l$, $E_{ij} = 1_i e^{j-i}$ for $i < j$, where $1_i \in \prod_{i=-\infty}^{\infty} \mathbb{C}$ is a sequence with 1 is the $i$th position and 0 in the other positions. Note that in case (ii) (see (11)) the cocycle $\alpha$ is the inverse image for $\Psi_s$ (see (12)) of a standard cocycle on $gl_{\infty}$ and in case (i) the cocycle $\alpha$ is cohomologous to zero.

The $\widehat{gl}_{\infty,s}$-module $Ind_{\mu,s}$ can be defined in a standard way as a quotient module of the Verma module $M_{\mu}$ with the zero highest weight and the central charge $\mu \in \mathbb{C}$, by analogy with the module of $Ind_{\mu}$ in the $\widehat{gl}_{\infty}$-case (see Ch. 1).

Lemma 5.

1. If $T_i(s - 2i) = 0$ for a certain $i \in \mathbb{Z}$ (case (i)), then $Ind_{\mu,s}$ is reducible for all $\mu \in \mathbb{C}$.
2. If $T_i(s - 2i) \neq 0$ for all $i \in \mathbb{Z}$ (case (ii)) and $\mu \in \mathbb{Z}$, then $Ind_{\mu,s}$ is reducible.
3. If $T_i(s - 2i) \neq 0$ for all $i \in \mathbb{Z}$ (case (ii)) and $\mu \notin \mathbb{Z}$, then $Ind_{\mu,s}$ is irreducible.

Proof. According to Lemma 3, statement (1) reduces to the corresponding statement concerning $gl_{\infty,(i)}$, and (2) and (3) reduce to the corresponding statements concerning $\widehat{gl}_{\infty}$. Cases (2) and (3) follow directly from Theorem 1, Ch. 1; in general, all the statements follow from the results of Sec. 3 of this Chapter, where we calculate the determinant of Shapovalov’s from of the representation $\theta^*_s(Ind_{\mu,s})$. However, we give here the direct proof of (1) for convenience.

We set $e_l = 1_l \cdot e$; $f_l = 1_{l+1} \cdot f$. These are the positive and the negative root vectors in $gl_{\infty,(i)}$. For $gl_{\infty,(i)}$ we have

\begin{equation}
[e_i, f_i] = 0.
\end{equation}

We set $x = [\ldots[f_0, f_1, f_2], \ldots, f_i]$ for $i \geq 0$ and $x = [\ldots[f_i, f_{i+1}, f_{i+2}], \ldots, f_0]$ for $i \leq 0$
In Fig. 1 $x$ is a hatched element. Let $v$ be the highest weight vector in $\text{Ind}_{\mu,s}$. Then $xv$ is a singular vector in $\text{Ind}_{\mu,s}$. Indeed, it is sufficient to verify that

\begin{equation}
  e_0 x v = 0 \text{ in } \text{Ind}_{\mu,s}
\end{equation}

and

\begin{equation}
  e_i x v = 0 \text{ in } \text{Ind}_{\mu,s},
\end{equation}

since if $j \neq 0$, $i$ then $[e_j, x] = 0$ in $\mathfrak{gl}_{\infty,(i)}$. In order to prove (15), we note that $[e_0, x]$ lies outside of the angle hatched in Fig. 2, and therefore, $[e_0, x]v = 0$ in $\text{Ind}_{\mu,s}$; (16) follows from the permutability of $e_i$ with all $f_j$, $j \in \mathbb{Z}$ in $U(\mathfrak{gl}_{\infty,(i)})$.

2.3. We can show that $H^2(\mathfrak{gl}(\lambda), \mathbb{C}) = 0$, i.e., $\mathfrak{gl}(\lambda)$ does not have any nontrivial central extensions. Therefore, the embedding $\hat{\phi}_s : \hat{\mathfrak{gl}}(\lambda) \hookrightarrow \hat{\mathfrak{gl}}_{\infty,s}$ with an induced cocycle on $\mathfrak{gl}(\lambda)$ defines the embedding $\theta_s : \mathfrak{gl}(\lambda) \hookrightarrow \hat{\mathfrak{gl}}_{\infty,s}$. From (13) it is easy to find

\begin{equation}
  \begin{cases}
    \theta_s(h) = \varphi_s(h) + T_1(s) \cdot c \\
    \theta_s(3h^2 - 2c\lambda) = \varphi_s(3h^2 - 2c\lambda) + (2s - 2)T_1(s) \cdot c,
  \end{cases}
\end{equation}

where $c_\lambda = \frac{\lambda(\lambda + 2)}{2}$, $T_1(s) = \frac{1}{2} \left( h - \frac{h^2}{2} + c_\lambda \right)$, and $c$ is the central in $\hat{\mathfrak{gl}}_{\infty,s}$. On the other hand, $\theta(1) = 1 + \varepsilon \cdot c$, and in a nondegenerate case, the embeddings $\theta_s$ are parametrized by the choice of $\varepsilon \in \mathbb{C}$. There is the commutative diagram

\[
\begin{array}{ccc}
\hat{gl}^O(\lambda) & \xrightarrow{\varphi_s^O} & \hat{gl}_{\infty,s} \\
\uparrow t & & \\
gl(\lambda) & & \\
\end{array}
\]

where $t$ is a natural embedding which preserves the grading and $\varphi_s^O$ is surjective. We shall denote the grading by a superscript.

**Lemma 6.** Let $V$ be a quasifinite $\mathfrak{gl}(\lambda)$-module. Then the action of $\mathfrak{gl}(\lambda)^k$ on $V$ for $k \neq 0$ can be naturally continued to the action of $\hat{\mathfrak{gl}}^O(\lambda)^k$ on $V$.

**Corollary.** Any $\mathfrak{gl}(\lambda)$-invariant subspace $W$ of the module $\text{Ind}_{\mu,s}$ is invariant with respect to $\hat{\mathfrak{gl}}_{\infty,s}$.

**Proof of the Corollary.** Since $\varphi_s^O : \hat{\mathfrak{gl}}^O(\lambda)^k \rightarrow \hat{\mathfrak{gl}}_{\infty,s}^k$ is surjective, it follows, in accordance with Lemma 6 that $W$ is also $\hat{\mathfrak{gl}}_{\infty,s}^k$-invariant for $k \neq 0$, whence follows the $\hat{\mathfrak{gl}}_{\infty,s}$-invariance.

**Proof of the Lemma [2].** Let $V$ be a graded quasifinite representation: $V = \bigoplus_{p \geq 0} V_p$, $\dim V_p < \infty$. We shall consider $\text{Hom}(V, V) := \bigoplus_{p,q \geq 0} \text{Hom}(V_p, V_q)$ with the topology of the direct sum of finite-dimensional spaces. We can assume that $V_p$
are normed spaces, and then the norm $\|\cdot\|_{p,q}$ is induced on $\text{Hom} (V_p, V_q)$. We shall show that for $k \neq 0$, the mapping $\text{gl}(\lambda)^k \rightarrow \text{Hom} (V, V)$ is continuous; for which purpose we shall show that $\|e^k h^n\|$ is bounded in $\text{Hom} (V_p, V_{p+k})$ for fixed $k$, $p$, and an arbitrary $n$. We note that $e^k \in \text{Hom} (V_p, V_{p+k})$, and $(ad h^2 - k^2) \in \text{End} (\text{Hom} (V_p, V_{p+k}))$.

We have $\|e^k h^n\|_{p,p+k} \leq A \cdot \alpha^n$, where $A = \|e^k\|$ and $\alpha = \frac{\|ad h^2 - k^2\|}{2k}$, and therefore $\|e^k f(h)\|_{p,p+k} = \|\sum_{n \geq 0} f_n e^k h^n\|_{p,p+k} \leq \sum_{n \geq 0} |f_n| \cdot \|e^k h^n\|_{p,p+k} \leq A \cdot \sum_{n \geq 0} |f_n| \cdot \alpha^n = A \cdot \varphi(f)(\alpha)$.

The statement of the Lemma follows from the fact that $\varphi : \sum f_i z^i \mapsto \sum |f_i| z^i$ is continuous and that $O$ is a completion of $\mathbb{C}[z]$ in the topology of the uniform convergence on compact sets.

2.4. We have proved that the $\text{gl}(\lambda)$-module $\theta^s_\mu(\text{Ind}_\mu, \ast)$ is reducible for $\mu \in \mathbb{Z}$ or for $T_1(s - 2i) = 0$ for a certain $i \in \mathbb{Z}$ and irreducible otherwise. Its highest weight $\chi$ is

\[
\begin{cases}
\chi(1) = \varepsilon \cdot \mu \\
\chi(h) = T_1(s) \cdot \mu \\
\chi(3h^2 - 2c_\lambda) = (2s - 2)T_1(s) \cdot \mu.
\end{cases}
\]

(18)

Let us compare this with the expression for the representation induced from $p_\alpha$ with the given $\chi(1)$. $\chi(h)$:

\[
\begin{cases}
\chi(1) = \chi(1) \\
\chi(h) = \chi(h) \\
\chi(3h^2 - 2c_\lambda) = 2(1 + \alpha) \cdot \chi(h).
\end{cases}
\]

(19)

We compare (12) with (13) and express $(\alpha, \chi(h))$ parameters in terms of the $(s, \mu)$ parameters. If $T_1(s) \neq 0$; then

\[
\begin{aligned}
s &= \alpha + 2 \\
\mu &= \chi(h)/T_1(\alpha + 2).
\end{aligned}
\]

(20)

It is now clear that if $T_1(\alpha + 2) = 0$, then, for all $\chi(h)$, the corresponding representation of $\text{gl}(\lambda)$ does not have the $(s, \mu)$-parametrization.

We have proved the following theorem

**Theorem 1.** For $T_1(\alpha + 2) \neq 0$ the representation of the Lie algebra $\text{gl}(\lambda)$, induced from the parabolic subalgebra $p_\alpha$, with the highest weight $\chi$, is reducible if

1. $T_1(\alpha + 2i) = 0$ for the integer $i \neq 1$

or

2. for $\frac{\chi(h)}{T_1(\alpha + 2)} \in \mathbb{Z}$.

In other cases, this representation is irreducible.
Theorem 2. For all $\chi(h) \neq 0$, the representation of the Lie algebra $gl(\lambda)$ induced from $p_\alpha$ is reducible for $T_1(\alpha + 2) = 0$ if $T_1(\alpha + 2i) = 0$ for a certain integer $i \neq 1$ and irreducible otherwise.

As we shall see in Sec. 3, the local identity (1) is equivalent to this theorem. The proof of the Theorem 2 will be given in Sec. 4.

2.5. Remark. Using the technique developed, we can easily show that the $gl(\lambda)$-module $\bigotimes_{i=1}^k L_{\alpha_i, \chi_i}$ does not coincide with the generalized Verma module with the same highest weight if $\alpha_i - \alpha_j \in 2\cdot Z$ for certain $i \neq j$ and coincides with it otherwise. This module is reducible if $L_{\alpha_i, \chi_i}$ is reducible for a certain $i$ or if $\alpha_i - \alpha_j \in 2\cdot Z$ for certain $i \neq j$ and irreducible otherwise.

3. The Local Identity.

3.0. We fix a general (see Theorem 2 in Subsec. 2.4) $\lambda \in \mathbb{C}$, and let $\alpha$ be such that $T_1(\alpha + 2) = 0$. It is clear that the determinant of Shapovalov’s form depends analytically on $\alpha$ and $\chi(h)$ (in the sense refined below this determinant is not uniquely defined). Therefore, Theorem 2 from Subsec. 2.4 is equivalent to the statement that for $T_1(s) \sim t$ and $\mu \sim \frac{1}{t}$ as $t \to 0$ (see (20)), the corresponding limit of the determinant of Shapovalov’s form does not depend on $t$, i.e., tends to a certain finite nonzero number.

3.1. Chevalley involution. Let $\omega$ be the Chevalley involution of the Lie algebra $sl_2$: $\omega(e) = -f$, $\omega(f) = -e$, $\omega(h) = -h$. There exist two continuations of $\omega$ to $U(sl_2)$ as a Lie algebra, namely,

(i) $\omega_1(1) = 1$, $\omega_1|_{sl_2} = \omega$, $\omega_1(a \cdot b) = \omega_1(a) \cdot \omega_1(b)$;

(ii) $\omega_2(1) = -1$, $\omega_2|_{sl_2} = \omega$, $\omega_2(a \cdot b) = -\omega_2(b) \cdot \omega_2(a)$.

For our purposes (i.e., for the existence of a map from the Verma module into a countergradient module) the condition $\omega|_b = -Id$ is necessary, which is satisfied for $\omega_2$. It is also clear that $\omega_2$ defines the involution of the Lie algebra $gl(\lambda)$ which is known as the Chevalley involution, and is denoted by $\omega$.

3.2. Knowing the expression for the determinant of Shapovalov’s form of the $\hat{gl}_\infty$-module of $Ind_\mu$ as a function of $\mu \in \mathbb{C}$, we want to find the expression for the determinant of Shapovalov’s form of the $gl(\lambda)$-module $\theta_s(1)(Ind_{\mu,s})$ as a function of $\mu, s \in \mathbb{C}$.

We denote $\Gamma_s = \Psi_s \varphi_s$ (see (10), (12)), $\Gamma_s : U_\lambda \hookrightarrow Mat_\infty (s \in \mathbb{C})$. Recall that $\Gamma_s$ is defined as follows:

$$
\begin{align*}
ed &\mapsto \left( T_1(s - 2i), \ i \in \mathbb{Z} \right) \cdot e \\
p(h) &\mapsto \left( p(s - 2i), \ i \in \mathbb{Z} \right) \\
f &\mapsto f.
\end{align*}
$$
Assuming that $s \in \mathbb{C}$, we define the holomorphic functions $\delta_{i,s}$ on $\mathbb{C}$ by the condition

$$\begin{cases}
\delta_{i,s}(s-2i) = 1, \\
\delta_{i,s}(s-2j) = 0 \text{ for } j \neq i.
\end{cases}$$

(Clearly, these functions are not uniquely defined.)

For our purposes $\{f^k \delta_{i,s}\} \subset \mathfrak{n}_-^{(-k)} \subset gl^O(\lambda)$ is more convenient than the standard base $\{f^k h^l\}$ since under the surjection $\Gamma^O_s : gl^O(\lambda) \to gl_{\infty}$ the elements of the first set pass into the standard base $\{E_{ij}, i > j\} \subset \mathfrak{n}_- \subset gl_{\infty}$ (and this is very convenient for calculating of Shapovalov’s form). In this case, the multiplicities of the roots of the determinant of Shapovalov’s form do not change when $gl(\lambda)$ is replaced by $gl^O(\lambda)$.

3.3. It is clear that the base on the $k$th level of the induced representation of $gl(\lambda)$ with a one-dimensional first level is formed by the vectors

$$w = (f^{i_1} \delta_{j_1,s}) \cdot \ldots \cdot (f^{i_p} \delta_{j_p,s}) \cdot v,$$

where $v$ is the highest weight vector, $i_1 + \ldots + i_p = k$, and

$$j_q \in [-i_q + 1, 0] \text{ for all } q.$$ We want to calculate Shapovalov’s form in this base. It is clear that

$$\Phi_{gl^O(\lambda)}(w_1, w_2) = \Phi_{gl_{\infty}}(\Gamma^O_s w_1, \Gamma^O_s w_2),$$

where the $gl(\lambda)$-module on the left-hand side is the inverse image of the $\hat{gl}_{\infty}$-module for $\Gamma_s$ on the right-hand side. In explicit form,

$$\Phi_{gl^O(\lambda)}(w_1, w_2) =$$

$$\pm (\delta_{j'_1,s} e^{i'_1}) \cdot \ldots \cdot (\delta_{j'_p,s} e^{i'_p}) \cdot (f^{i_1} \delta_{j_1,s}) \cdot \ldots \cdot (f^{i_p} \delta_{j_p,s}) v$$

(for the representation of $gl(\lambda)$) =

$$\pm \Gamma^O_s (\delta_{j'_1,s} e^{i'_1}) \cdot \ldots \cdot \Gamma^O_s (\delta_{j'_p,s} e^{i'_p}) \cdot \Gamma^O_s (f^{i_1} \delta_{j_1,s}) \cdot \ldots \cdot \Gamma^O_s (f^{i_p} \delta_{j_p,s}) v$$

(for the representation of $gl^O(\lambda)$).

We have

$$\Gamma^O_s (f^i \delta_{j,s}) = \Gamma^O_s (\delta_{j,s}) \cdot \Gamma^O_s (e) \cdot \ldots \cdot \Gamma^O_s (e) =$$

$$\mathbb{I}_j \cdot (T) \cdot e \cdot (T) \cdot e \cdot \ldots \cdot (T) \cdot e,$$

where $\mathbb{I}_j \in \prod_{i \in \mathbb{Z}} \mathbb{C}(i)$ is a sequence with 1 in the $j$th place and with 0 in the other places and $(T) = \left( T_i(s-2i), \ i \in \mathbb{Z} \right) \in \prod_{i \in \mathbb{Z}} \mathbb{C}(i)$. On the other hand,

$$\Gamma^O_s (f^i \delta_{j,s}) = f^i \cdot \mathbb{I}_j.$$
It is obvious from (26) that

\[(28) \quad \Gamma_s^C(\alpha_j, e^i) = \left( \prod_{l=0}^{i-1} T_1(s - 2j - 2l) \right) \cdot \Pi_j \cdot e^i,\]

and the determinant of Shapovalov’s form of the \(gl(\lambda)\)-module \(\theta_\lambda(\text{Ind}_{\mu_s}, s)\) is

\[(29) \quad \det \Phi_{gl(\lambda)}(s, \mu) = \prod_{w \text{ over all } k \text{th level (22)}} \prod_{m=1}^{p} \prod_{i=0}^{i_p - 1} T_1(s - 2j_m - 2l) \cdot \det \Phi_{gl(\lambda)}(\mu).\]

Recall (see (23)) that \(j_q \in [-i_q + 1, 0]\).

3.4. Assume now that \(t \to 0\), and let \(T_1(s) \sim t^l\) and \(\mu \sim 1/t\) (see Subsec. 3.0).

Then \(\det \Phi_{gl_{\infty}}(\mu) \sim 1/t^{\deg_k}\), where \(\deg_k\) is the degree of \(\det \Phi_{gl_{\infty}}(\mu)\) for \(\mu\) on the \(k\)th level. On the other hand, according to condition (23), the multiplicity of zero of the product in relation (29) for a general \(\lambda\) is equal to \(p\) for every \(w = (f^1, \delta_{j_1, s}) \cdot \ldots \cdot (f^r, \delta_{j_r, s})\) since \(T_1(s - 2l)\) has 0 only for \(i = 0\).

Thus, according to Lemma 12 from Subsec. 2.2 of Ch. 1, Theorem 2 from Subsec. 2.4 is equivalent to the local identity (1).

4. Proof of the Local Identity: Embeddings of \(gl(\lambda)\) into \(\hat{gl}_{\infty}(\mathbb{C}[t]/t^2)\).

4.0. In Sec. 3, we have derived the local identity (1) from Theorem 2. This section is devoted to the proof of the theorem itself. First, Theorem 2 follows from the following weaker result.

**Theorem 3.** For a general \(\lambda\), \(T_1(\alpha + 2) = 0\), and \(\chi(h) \neq 0\), the representation of the Lie algebra \(gl(\lambda)\) induced from \(p_\alpha\) is irreducible.

Indeed, the local identity follows from Theorem 3 by analogy with Sec. 3. However, the conditions of Theorem 2 are precisely the conditions which are necessary for the arguments from Sec. 3, in order to have no additional zeros.

In this section, we prove Theorem 3. As we have seen in Sec. 2, for \(T_1(\alpha + 2) = 0\), the representation of the Lie algebra \(gl(\lambda)\) induced from \(p_\alpha\) does not have the \((s, \mu)\)-parametrization.

However, we can obtain its highest weight by considering the embeddings \(\theta_\lambda : gl(\lambda) \hookrightarrow \hat{gl}_{\infty}(\mathbb{C}[t]/t^2)\) (see below) and the inverse images of the \(\hat{gl}_{\infty}(\mathbb{C}[t]/t^2)\)-modules \(\text{Ind}_{\mu_1, \mu_2, s}\) for \(\theta_\lambda(\text{Ind}_{\mu_1, \mu_2, s})\). (Here \(\mu_1, \mu_2\) are two central charges.) For certain \(s, \mu_1, \mu_2\), this representation has the same highest weight as that induced from the exceptional parabolic subalgebra \(p_\alpha\), and we prove that the irreducible quotient module for these \(\mu_1, \mu_2, s\) of the module \(\theta_\lambda(\text{Ind}_{\mu_1, \mu_2, s})\) has, for \(\mu_2 \neq 0\), a character not smaller than \(\prod_{i=1}^{\infty} (1 - q^i)^{-1}\). On the other hand, this character is not greater than \(\prod_{i=1}^{\infty} (1 - q^i)^{-1}\) since there exists a representation induced from \(p_\alpha\) with the corresponding highest weight. Hence Theorem 3 follows.
4.1. We shall consider the “finite” algebra \( gl_{2n}(C[t]/t^2) \) instead of \( \hat{gl}_\infty(C[t]/t^2) \) and then pass to the limit. We denote by \( \overline{gl}_{2n} := gl_{2n}t \), \( \overline{gl}_{2n} \) the Abelian Lie algebra, being a \( gl_{2n} \)-module, \( \overline{gl}_{2n} \) is isomorphic to the adjoint action on \( gl_{2n} \). There are two subalgebras, \( gl_n^{(1)} \) and \( gl_n^{(2)} \), in \( gl_{2n} \), and they commute with each other, and there are also two Abelian subalgebras \( \mathfrak{A}_+ \) and \( \mathfrak{A}_- \) (see Fig. 1 in Ch. 1). We assume a similar notation for \( \overline{gl}_{2n} \), \( \overline{gl}_{2n} = n_- \oplus h \oplus n_+ \), \( gl_n^{(1)} = n^{(1)} \oplus h^{(1)} \oplus n^{(1)}_+ \) and so on.

4.1.1. The associative algebra \( \text{Mat}_\infty(C[t]/t^2) \) is described as follows.
We denote by \( A = \prod_{i \in \mathbb{Z}} (C[t]/t^2) \), the direct product of algebras. Being a vector space,
\[
\text{Mat}_\infty(C[t]/t^2) = A \oplus eA \oplus e^2A \oplus e^3A \oplus \ldots \oplus fA \oplus f^2A \oplus f^3A \oplus \ldots ,
\]
and the relations
\[
\begin{cases}
H \cdot e = e \cdot D(H) \\
H \cdot f = f \cdot D^{-1}(H) \\
e \cdot f = (\ldots, 1, 1, 1, \ldots) \in A \\
f \cdot e = (\ldots, 1, 1, 1, \ldots) \in A,
\end{cases}
\]
are satisfied, where \( D : A \to A \) is a shift to the right by 1.

We denote by \( a_i \in A \) a sequence with \( t \) in the \( i \)-th place and 1 in the other places. Then, replacing the last two conditions in (30) by
\[
\begin{cases}
e \cdot f = a_i \in A \\
f \cdot e = a_{i+1} \in A,
\end{cases}
\]
we get the associative algebra \( \text{Mat}_{\infty,i}(C[t]/t^2) \). We denote by \( gl_{\infty,i}(C[t]/t^2) \) the corresponding Lie algebra. It is clear that there exists a “finite” analog of \( gl_{2n,i}(C[t]/t^2) \).

4.1.2. Let \( \alpha_0^\vee, \alpha_0 \) be the “central” coroots in \( gl_{2n}(C[t]/t^2) \). We shall considers the \( gl_{2n}(C[t]/t^2) \)-module of \( Ind_{\mu_1, \mu_t,s} \) induced from \( p \oplus \overline{p} \) with the highest weight \( \chi \) such that \( \chi(\alpha_0^\vee) = \mu_1, \chi(\alpha_0) = \mu_t \). Here \( p = gl_n^{(1)} \oplus \mathfrak{A}_+ \oplus gl_n^{(2)} \) and \( \overline{p} = gl_n^{(1)} \oplus \mathfrak{A}_+ \oplus gl_n^{(2)} \). We denote by \( Ind_{\mu_1, \mu_t,s} \) a similar module over \( gl_{2n,i}(C[t]/t^2) \) \( \hat{gl}_{\infty,i}(C[t]/t^2) \), respectively.

**Lemma 7.** The \( q \)-character of the irreducible quotient module of the \( \hat{gl}_{\infty,0}(C[t]/t^2) \)-module \( Ind_{\mu_1, \mu_t,s} \) is greater or equal to \( \prod_{i \geq 1} (1 - q^i)^{-1} \) for \( \mu_t \neq 0 \).

**Remark.** We shall see in Subsec. 4.2.3 that for \( \mu_t \neq 0 \) this character is equal to \( \prod_{i \geq 1} (1 - q^i)^{-1} \).
4.1.3. Proof. Let us consider the action \( gl(1) \oplus gl(2) \subset gl_{\infty,0}(\mathbb{C}[t]/t^2) \) on the module \( Ind_{\mu_1,\mu_2,s} \). Being a vector space, \( Ind_{\mu_1,\mu_2,s} \cong S^*({\mathfrak{A}_-}) \otimes S^*({\mathfrak{A}_-}) \). According to what was stated in Ch. 1 (Lemma 6 in Sec. 2), the \( gl(1) \oplus gl(2) \)-module \( S^*({\mathfrak{A}_-}) \) is isomorphic to the direct sum of all irreducible modules \( L_w \otimes L_w \) generated by the highest weight vectors \( w = Det_{k_1} \cdots Det_{k_i} v \) (here \( v \) is the highest weight vector in \( Ind_{\mu_1,\mu_2,s} \) and \( Det_k \in S^*({\mathfrak{A}_-}) \) is the determinant of the matrix \( A_k = (y_{ij}, i, j = 1 \ldots k) \)). Therefore, \( Ind_{\mu_1,\mu_2,s} \cong S^*({\mathfrak{A}_-}) \otimes S^*({\mathfrak{A}_-}) \) is isomorphic to the direct sum of the tensor products of irreducible \( gl(1) \oplus gl(2) \)-modules \( L_w \otimes L_w \otimes (L_w \otimes L_w) \), each of which, in turn, can be decomposed into the direct sum of irreducible \( gl(1) \oplus gl(2) \)-modules. We denote by \( w \cdot v \) the highest weight vector of \( \hat{g}l_{\infty,0}(\mathbb{C}[t]/t^2) \) and therefore, the vectors \( \hat{g}l_{\infty,0}(\mathbb{C}[t]/t^2) \) are obviously singular vectors of the \( gl(1) \oplus gl(2) \)-module \( \hat{g}l_{\infty,0}(\mathbb{C}[t]/t^2) \). On the other hand, it follows that the intersection of the greatest proper submodule in \( Ind_{\mu_1,\mu_2,s} \) with \( S^*({\mathfrak{A}_-}) \otimes 1 \subset S^*({\mathfrak{A}_-}) \) is zero. This proves Lemma 7 since \( S^*({\mathfrak{A}_-}) = \bigoplus_w L_w \otimes L_w \) as a \( gl(1) \oplus gl(2) \)-module, see Ch. 1, Sec. 1.

4.1.4. Remarks.

1. The vectors \( \hat{g}l_{\infty,0}(\mathbb{C}[t]/t^2) \) and therefore, the vectors \( \hat{g}l_{\infty,0}(\mathbb{C}[t]/t^2) \) are not \( \hat{g}l_{\infty,0}(\mathbb{C}[t]/t^2) \)-singular in \( Ind_{\mu_1,\mu_2,s} \).

2. Clearly, the \( \hat{g}l_{\infty,0}(\mathbb{C}[t]/t^2) \) does not depend on \( \mu_1 \) in the obvious sense.

4.2. Recall that \( gl(\lambda) = \text{Lie } (U_\lambda) \), where \( U_\lambda = U(sl_2) \), and \( \Delta \) is the Casimir operator \( e \cdot f + f \cdot e + h \cdot h/2 \in U(sl_2) \), \( \lambda \in \mathbb{C} \). Being a vector space,

\[
U_\lambda = \mathbb{C}[h] \oplus \mathbb{C}[h] \oplus e \cdot \mathbb{C}[h] \oplus e^2 \cdot \mathbb{C}[h] \oplus \ldots \oplus f \cdot \mathbb{C}[h] \oplus f^2 \cdot \mathbb{C}[h] \oplus \ldots
\]

\( \{e, f, h\} \) is a standard basis in \( sl_2 \). The relations

\[
\begin{align*}
  h \cdot e &= e \cdot (h + 2) \\
  h \cdot f &= f \cdot (h - 2) \\
  e \cdot f &= T_1(h) = \frac{1}{2} \left( h - \frac{h^2}{2} + \frac{\lambda(\lambda + 2)}{2} \right) \\
  f \cdot e &= T_2(h) = \frac{1}{2} \left( -h - \frac{h^2}{2} + \frac{\lambda(\lambda + 2)}{2} \right).
\end{align*}
\]

4.2.1. We define the Mat\( ^*_\infty(\mathbb{C}[t]/t^m) \) algebra by relations (30) (setting \( A = \)…
\[ \prod_{i \in \mathbb{Z}} (\mathbb{C} \left[ t \right]/t^m)_i \] and replacing the last two conditions in (30) by

\[
\begin{cases}
  e \cdot f = \left( T_1(s-2i+t), \ i \in \mathbb{Z} \right) \in A \\
  f \cdot e = \left( T_2(s-2i+t), \ i \in \mathbb{Z} \right) \in A,
\end{cases}
\]

\[ m \in \mathbb{Z}_{\geq 1}, \ s \in \mathbb{C}. \]

In order to construct the mapping \( \mathcal{U}_\lambda \) in \( \text{Mat}^*_\infty(\mathbb{C}[t]/t^m) \), it suffices to determine the mapping \( \varphi_s : \mathbb{C}[h] \to A \) such that the diagram

\[
\begin{array}{ccc}
\mathbb{C}[h] & \xrightarrow{\varphi_s} & A \\
\sigma \downarrow & & \Downarrow D \\
\mathbb{C}[h] & \xrightarrow{\varphi_s} & A
\end{array}
\]

is commutative \((\sigma(p(h)) = p(h + 2))\). Then \( \varphi_s \) is extended to \( \mathcal{U}_\lambda \), being the homomorphism of an associative algebra, with the use of the relations \( \varphi_s(e) = e \), \( \varphi_s(f) = f \).

It immediately follows from (34) that \( \varphi_s(p(h)) = (p(s-2i+t), \ i \in \mathbb{Z}) \in A \). Obviously, \( \varphi_s : \mathcal{U}_\lambda \to \text{Mat}^*_\infty(\mathbb{C}[t]/t^m) \) is an embedding.

4.2.2. Let \( m = 2 \).

**Lemma 8.** For a general \( \lambda \) we have (i) \( \text{Mat}^*_\infty(\mathbb{C}[t]/t^2) \cong \text{Mat}^*_\infty(\mathbb{C}[t]/t^2) \) if \( T_1(s-2i) \neq 0 \) for all \( i \in \mathbb{Z} \).

(ii) \( \text{Mat}^*_\infty \cong \text{Mat}^*_\infty, i \) if \( T_1(s-2i) = 0 \)

**Proof.** This is obvious; see also Lemma 3.

We denote by \( gl^*_\infty(\mathbb{C}[t]/t^m) \) the Lie algebra constructed with the use of the associative algebra \( \text{Mat}^*_\infty(\mathbb{C}[t]/t^m) \).

The cocycle \( \alpha \) on the Lie algebra \( gl^*_\infty(\mathbb{C}[t]/t^2) \) with the values in \( \mathbb{C}[t]/t^2 \) is constructed in the standard way,

\[
\alpha(E_{ij}, E_{jk}) = \begin{cases} \ P_j \cdot (s-2i) & \text{for } i \leq 0, \ j \geq 1 \\ 0, & \text{otherwise} \end{cases}
\]

\[ \alpha(E_{ij}, E_{kl}) = 0 \text{ for } i \neq l \text{ and } j \neq k \]

and is continued by \( \mathbb{C}[t]/t^2 \)-linearity (here \( P_t(h) = e^j \cdot f^i, E_{ij} = 1 i \cdot e^j \) for \( i < j \)).

In case (i) of Lemma 8, this cocycle is the inverse image of the standard cocycle on \( gl^*_\infty(\mathbb{C}[t]/t^2) \) and, in case (ii) of Lemma 8, the corresponding central extension is the direct sum of the trivial and the one-dimensional extension.

4.2.3. We set \( h_k = [e, f h^{k-1}] \) in \( gl(\lambda) \) and then, in the base \( \{ h_k \} \), the highest weight of the representation induced from the parabolic subalgebra \( p_\alpha \) is equal to

\[
\begin{cases}
\chi(h_1) = \chi(h) \\
\chi(h_2) = \alpha \chi(h) \\
\chi(h_3) = \alpha^2 \chi(h) \\
\end{cases}
\]
Next, the highest weight of the representation \( \theta_s^* (Ind_{\mu_1, \mu_1, s}) \) (\( \theta_s: gl(\lambda) \rightarrow \hat{gl}_\infty^s (\mathbb{C} [t]/t^2) \)) is equal, by relation (35), to

\[
\begin{align*}
\chi(h_1) &= T_1(s) \cdot \mu_1 + T_1'(s) \mu_t \\
\chi(h_2) &= T_1(s) \cdot (s - 2) \mu_1 + (T_1'(s) \cdot (s - 2) + T_1(s)) \mu_t \\
\chi(h_3) &= T_1(s) \cdot (s - 2)^2 \mu_1 + (T_1'(s) \cdot (s - 2)^2 + 2(s - 2) \cdot T_1(s)) \mu_t \\
\chi(h_4) &= T_1(s) \cdot (s - 2)^3 \mu_1 + (T_1'(s) \cdot (s - 2)^3 + 3(s - 2)^2 \cdot T_1(s)) \mu_t
\end{align*}
\] (37)

Under the embedding \( \theta_s: gl(\lambda) \rightarrow \hat{gl}_\infty^s (\mathbb{C} ) \) the highest weight of the module \( \theta_s^* (Ind_{\mu_1}) \) is equal to

\[
\begin{align*}
\chi(h_1) &= T_1(s) \cdot \mu \\
\chi(h_2) &= (s - 2) \cdot T_1(s) \cdot \mu
\end{align*}
\] (38)

Comparing (36) with (38), we see that if \( T_1(\alpha + 2) = 0 \) the corresponding highest weight exists in (36) and does not exist in (38) for all \( \chi(h) \) (since \( s = \alpha + 2 \)).

We set \( T_1(s) = 0 \) in (37); then we can easily pass from (37) to (36) by setting \( \alpha = s - 2, T_1(\alpha + 2) = 0, \mu_1 \) being arbitrary, \( \chi(h) = T_1'(s) \cdot \mu_t \). The main fact here is that (for the general \( \lambda \)) \( T_1(h) \) has no multiple roots, i.e., \( T_1(s) = 0 \Rightarrow T_1'(s) \neq 0 \).

4.2.4. Thus, \( T_1(s) = 0 \), and we are in the situation of (ii) of Lemma 8, \( \hat{gl}_\infty^s (\mathbb{C} [t]/t^2) \cong \hat{gl}_{\infty,0}^s (\mathbb{C} [t]/t^2) \). Lemma 7 states that the character of the irreducible quotient module of the \( \hat{gl}_{\infty,0}^s (\mathbb{C} [t]/t^2) \)-module \( Ind_{\mu_1, \mu_1, s} \) for \( \mu_t \neq 0 \) is greater or equal to \( \prod_{i \geq 1} (1 - q^i)^{-1} \) and we need the following lemma.

**Lemma 9** [2]. Any \( gl(\lambda) \)-invariant subspace of the module \( Ind_{\mu_1, \mu_1, s} \) is invariant with respect to \( \hat{gl}_\infty^s (\mathbb{C} [t]/t^2) \) (for all \( s \in \mathbb{C} \)).

**Proof.** See the corollary of Lemma 6.

Lemma 7, Lemma 8, and Lemma 9 imply Theorem 3 (see Subsec. 4.0).

4.2.5. **Remarks.**

1. It follows from relation (37) that all representations of \( gl(\lambda) \) induced from the parabolic subalgebras \( p_{\alpha, \alpha}, \alpha \in \mathbb{C} \) (generated by \( n_+, h, \) and \( \{ f(h - \alpha)^2 p(h), p(h) \in \mathbb{C} [h] \} \)) have the form \( \theta_s^* (Ind_{\mu_1, \mu_1, s}) \).

Indeed, we have the recurrent condition

\[
\chi(h_k) = 2\alpha \chi(h_{k-1}) - \alpha^2 \chi(h_{k-2});
\]

imposed on the highest weight \( \chi \) of the representation induced from \( p_{\alpha, \alpha}, \alpha^k, \) and \( k \alpha^{k-1} \) are the corresponding basic solutions.

2. Let \( T_1(s) = 0 \) and \( T_1(h) = a(h - s) + b(h - s)^2 \) \( (a \neq 0) \).

We set

\[
\mu_1 = \frac{1}{a} \cdot \frac{1}{(h - s)^2}; \quad \mu_t = \frac{1}{b} \cdot \frac{1}{(h - s)}.
\]
Then, as $h \to s$, relations (37) give the limiting values, and we find that for $T_1(\alpha + 2) = 0$, the representations induced from $p_{\alpha, \alpha}$ do not have notation (37). We could have obtained them, by analogy with Subsec. 4.2.3, as the inverse images under the inclusion of $gl(\lambda)$ into $\hat{gl}_{\infty}(\mathbb{C}[t]/t^3)$.

Next, for obtaining identity (1) we used in Sec. 3 not the whole expression for the determinant of Shapovalov’s form on the levels of the $\hat{gl}_{\infty}$-module of $Ind_\mu$ but only its degrees with respect to $\mu$ on the levels. However, the “highest” local identities obtained in this way are simply equal to the degrees of identity (1) (see also Appendix B).

Chapter 3.

The Lie Algebra of Functions on a Hyperboloid: the Global Identity

1. The Flat Deformation in $gl(\lambda)$ and the Principal Vector Bundle on $S^2$.

1.1. If we assume that the Lie algebra $gl(\lambda)$ corresponds to the point $\lambda \in \mathbb{C} = S^2 \setminus \{\infty\}$, then the Lie algebra of functions on a hyperboloid corresponds to the point $\{\infty\} \subset S^2$. To be more precise, we shall consider the standard symplectic foliation over $sl_2(\mathbb{C})^*$. We call the general symplectic leaf of this foliation, which is a nondegenerate quadratic surface in $\mathbb{C}^3$, a hyperboloid. The induced Poisson bracket defines the structure of the Lie algebra on regular functions on a symplectic leaf. These Lie algebras are isomorphic for all nondegenerate symplectic leaves.

There exists a flat deformation of the Lie algebras of functions on a hyperboloid into the Lie algebra $gl(\lambda_t)$ in the neighborhood of $t = 0$, where $\lambda_t \cdot (\lambda_t + 2) = \frac{t^2}{\lambda(\lambda + 2)}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ is fixed. We fix $t \neq 0$. We shall consider the Lie algebra $sl_2(\mathbb{C})$ as an algebra with the generators $e, h, f$, and relations $[e, f] = 2te, [h, e] = 2tf, [h, f] = -2tf$. We denote by $U_\lambda(t)$ the quotient algebra of the corresponding universal enveloping algebra with respect to the ideal generated by the central element $2ef - th + \frac{h^2}{2} - \frac{\lambda(\lambda + 2)}{2}, \quad \text{for } t \neq 0,$

$$U_\lambda(t) \cong U_{\lambda_t} = U(sl_2) \big/ \left( \Delta - \frac{\lambda(\lambda + 2)}{t^2} \right),$$

where $\Delta$ is the standard Casimir operator in $U(sl_2)$. The corresponding Lie algebra with the bracket $[a, b] = a \cdot b - b \cdot a$ is isomorphic to the Lie algebra $gl(\lambda_1)$.

We denote by $\text{Lie}^t U_\lambda(t)$ the corresponding Lie algebra with the bracket $[a, b] = \frac{a \cdot b - b \cdot a}{t}.$ The Lie algebra $\mathfrak{s} = \lim_{t \to 0^+} \text{Lie}^t U_\lambda(t)$ is isomorphic to the Lie algebra of regular functions on the symplectic leaf $\left\{2ef + \frac{h^2}{2} = \frac{\lambda(\lambda + 2)}{2} \right\}$ of the foliation on $sl_2^*$ with an induced Poisson bracket. (If we simplify this construction without factorizing with respect to the central element, then we get a Poisson bracket on regular functions on $sl_2^*$, i.e., on $\mathbb{C}[e, f, h]$.)

1.2. We denote $U = S^2 \setminus \{\infty\}$ and $V = S^2 \setminus \{0\}$ and assume that the parameter $t$ runs over all points from $S^2$. Thus we described in Subsec. 1.1 what we have in
U. We denote by $e_U, f_U, h_U$ the elements $e, f, h$ from Subsec. 1.1 respectively. Suppose now that the Lie algebra $gl(\lambda_t)$ corresponds to the point $t \in V = S^2 \setminus \{0\}$ ($gl(0)$ corresponds to the point $\{\infty\}$). We denote the elements $e, f, h$ appearing in the definition of $gl(\lambda_t)$ by $e_V, f_V, h_V$ respectively. We want to calculate the corresponding transition functions. First, we note that the Lie subalgebra $sl_2$ is globally defined, i.e. the transition functions are identical. In general, the isomorphism $\alpha : U_\lambda(t) \to U_\lambda$ is defined as $\alpha : e^k h^l \mapsto t^{k+l} \cdot e^k h^l$ etc, and $\beta : \xi \mapsto \frac{\xi}{t}$ under the isomorphism $\beta : \text{Lie}^t U_\lambda(t) \to \text{Lie} U_\lambda(t)$ for any $\xi$. Therefore on $U \cap V$ we have

$$
\frac{e_U^k h_U^l}{e_V^k h_V^l} = t^{k+l-1}.
$$

Thus the family of our Lie algebras on $S^2$ is decomposed into the (infinite) direct sum of line bundles on $S^2$.

1.3. At every point $t \in S^2$ we choose a parabolic subalgebra of the corresponding Lie algebra which contains $n_+$ holomorphically dependent on $t$. We shall only consider parabolic subalgebras of the “maximal size”, i.e. subalgebras such that the first level of the induced representation is one-dimensional. Recall that for a fixed $t$ these subalgebras depend on one complex parameter $\alpha$ and the subalgebra $p_\alpha$ is generated by $n_+, h$, and

$$
n^\alpha = \left\{ p(h) \cdot (h - \alpha) \cdot f, \ p(h) \in \mathbb{C}[h] \right\}.
$$

Furthermore, for every $t \in S^2$ we consider the representation induced from the corresponding parabolic subalgebra for a certain value of the highest weight $\chi_t(h)$ for all $t \in S^2$. Then we can regard the spaces of representations as the direct sum of line bundles on $S^2$ and

$$
\sum_{k \geq 0} \left( \sum_{l=1}^{n_k} \alpha^{C_{l,k}} \right) q^k = \frac{1}{(1 - q)} \frac{1}{(1 - a q^2)(1 - a^2 q^2)} \frac{1}{(1 - a^2 q^3)(1 - a^3 q^3)(1 - a^4 q^3)} \ldots.
$$

Here $n_k$ is the dimension of the $k$th level of the induced representation and $C_{1,k}; \ldots; C_{n_k,k}$ are Chern classes of the line bundles into whose direct sum the $k$th level of the representation is decomposed. (Every monomial $f^p h^q$ has a Chern class equal to $p + q - 1$ in addition to its grading equal to $p$. In this sense, the last relation is the refinement of our relation for the character $\prod_{i=1} \frac{1}{(1 - q^i)}$ with due account of the Chern class.)

The quadratic form, namely, Shapovalov’s form, is invariantly defined in every fiber of a definite level. Clearly, its determinant is not uniquely defined. However, we choose a holomorphic frame of the bundle of the $k$th level subject to the decomposition of this bundle into a direct sum. Suppose that $U, V$ are our coordinate maps on $S^2$ and $\xi_{1,U} ; \ldots ; \xi_{n_k,U}$ and $\xi_{1,V} ; \ldots ; \xi_{n_k,V}$ are the corresponding sections in these coordinates. Then $\frac{\xi_{i,U}}{\xi_{i,V}} = t^{c_{i,k}}$ and

$$
\frac{\det ||\langle \xi_{i,U}, \xi_{j,U}\rangle||}{\det ||\langle \xi_{i,U}, \xi_{j,V}\rangle||} = t^{2 \sum_{i=1}^{n_k} c_{i,k}}.
$$

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Therefore, the determinant of Shapovalov’s form is a (not uniquely defined) holomorphic section of the line bundle over \( S^2 \) with the Chern class \( 2 \sum_{i=1}^{n_k} C_{i,k} \) (on the \( k \)th level). We denote this Chern class by \( C_k \). Then

\[
\sum_{k \geq 1} C_k \cdot q^k = \frac{d}{da} \left[ \frac{1}{1 - q} \cdot \frac{1}{(1 - a^2 q^2)(1 - a^4 q^2)} \cdot \frac{1}{(1 - a^4 q^3)(1 - a^6 q^3)(1 - a^8 q^3)} \cdots \right] \bigg|_{a=1}.
\]

This is a simple consequence of relation (1).

1.4. We choose the subalgebras \( p_{\alpha,t} \) and characters \( \chi_t(h) \) for all \( t \in S^2 \) such that for \( t = 0 \) (corresponding to the Lie algebra of functions on a hyperboloid) \( p \) and \( \chi(h) \) are in the general position. In this case we can calculate the Chern class of the bundle of the determinant of Shapovalov’s form in two ways, namely, as in Subsec. 1.3 and as the sum of zeros of the determinant of Shapovalov’s form with multiplicities (the multiplicities of zeros are uniquely defined). In the next section, we shall calculate the Chern class with the second technique, using the results of Chs. 1 and 2 on the assumption of the irreducibility for \( t = 0 \). Equating the corresponding generating function to expression (2), we obtain an identity with power series which is equivalent to the irreducibility for the general parameters of the induced representations of the Lie algebra of functions on a hyperboloid. We shall prove a more exact result in Sec. 3 (see Theorem 1).

2. The Main Calculation.

2.1. **Lemma 1.** Let \( \alpha, \chi \in \mathbb{C} \) be arbitrary. We set \( \alpha_t = \frac{\alpha}{t} \), \( \chi_t(h_V) = \chi \) in the coordinates on \( V = S^2 \setminus \{0\} \). Then the corresponding limiting parameters on \( U = S^2 \setminus \{\infty\} \) as \( t \to 0 \), are equal to \( \alpha_0 = \alpha \) and \( \chi_0(h_U) = \chi \) (in the coordinates on \( U \)).

**Proof.** The direct verification.

Thus, for any parameters of the induced representation for \( t = 0 \), we have a holomorphic continuation to the whole sphere.

*Everywhere in what follows \( \alpha \) and \( \chi \) are considered to be of the general position.*

We shall need relations that connect the \((s, \mu)\)-parametrization of the induced representations of \( gl(\lambda) \) with the one-dimensional first level with the \((\alpha, \chi(h))\)-parametrization (see Subsec. 2.4 of Ch. 2). We have

\[
\begin{cases}
T_1(s)\mu = \chi(h) \\
(2s - 2)T_1(s)\mu = 2(1 + \alpha)\chi(h).
\end{cases}
\]

The meaning of this is that in the \((s, \mu)\)-parametrization the reducibility condition can be naturally formulated:

\[
\begin{cases}
(i) \quad \mu \in \mathbb{Z} \\
(ii) \quad T_1(s + 2i) = 0 \text{ for a certain } i \in \mathbb{Z}.
\end{cases}
\]
(recall that $T_1(s) = \frac{1}{2} \left(s - \frac{s^2}{2} + \frac{\lambda(\lambda + 2)}{2}\right)$. For $T_1(s) \neq 0$ it follows from (3) that
\[ s = \alpha + 2, \text{ and } \mu = \chi(h)/T_1(\alpha + 2). \]

Let $\chi(h)$ be arbitrary and $T_1(\alpha + 2) = 0$. In this case, the corresponding representation does not have the $(s, \mu)$-parametrization. Thus, under the conditions of Lemma 1, we have on $V = S^2 \setminus \{0\}$

\[
\begin{align*}
(i) & \quad \chi(h) \bigg| T_{1,t} \left( \frac{\alpha}{t} + 2 \right) \in \mathbb{Z}; \\
(ii) & \quad T_{1,t} \left( \frac{\alpha}{t} + 2i \right) = 0 \text{ for a certain } i \in \mathbb{Z};
\end{align*}
\]

\[ T_{1,t}(s) = \frac{1}{2} \left(s - \frac{s^2}{2} + \frac{\lambda(\lambda + 2)}{2t^2}\right). \]

2.2. Thus, all points $t \in S^2 \setminus \{0\}$ at which the determinant of Shapovalov’s form has a zero are given by formulas (5), (i) and (ii). In addition, it follows from Theorem 1, Sec. 3 (see below) that for $t = 0$ the determinant of Shapovalov’s form does not vanish (for the general $\alpha, \chi(h)$).

Our main purpose is to determine the multiplicities of zeros according as $t \in S^2$. For example, in case (5), (i) we must connect the multiplicity of the zero of the determinant of Shapovalov’s form for $\hat{gl}_\infty$ for the integral central charge $\mu$, as a function of $\mu$, found in Ch. 1 with the multiplicity of the zero as a function of $t \in S^2$. In fact, the results of Ch. 1 and 2 are sufficient for all cases ((5), (i) and (ii)).

Recall (see (29) Ch. 2) that the determinant of Shapovalov’s form of the $gl(\lambda)$-module $\theta^*_\mu(\text{Ind}_{\mu,s})$ as a function of $s$ and $\mu$, is given by the relation

\[
\det \Phi_{gl(\lambda)}(s, \mu) = \left( \prod_{\text{over all } w \text{ from the } k\text{th level}} \prod_{m=1}^{p} \prod_{l=0}^{i_p-1} T_1(s - 2jm - 2l) \right) \times \det \Phi_{\hat{gl}_\infty}(\mu).
\]

Here we set
\[ w = (f_{i_1}\delta_{j_1}, s) \cdot \ldots \cdot (f_{i_p}\delta_{j_p}, s) \cdot v, \]
where $i_1 + \ldots + i_p = k$ and

\[ j_q \in [-i_q + 1, 0]. \]

2.3. Here we find the multiplicities of zeros $t \in S^2 \setminus \{0\}$ in cases (5), (i) and (ii).

2.3.1. In case (5), (i) on the assumption that $\alpha$ and $\lambda$ are general $T_{1,t_0} \left( \frac{\alpha}{t_0} + 2i \right) \neq 0$ is nonzero for any $i \in \mathbb{Z}$, and therefore the product in (6) does not vanish. Thus, if $\mu_0 \in \mathbb{Z}$, then the multiplicity of the root $t_0$ according as $t \in S^2$ in the determinant of Shapovalov’s form of the $k$th level for $gl(\lambda t_0)$ is equal to the multiplicity of the root $\mu_0$ according as $\mu \in \mathbb{C}$ in the determinant of Shapovalov’s form of the $k$th level of $\hat{gl}_\infty$. 
2.3.2. Case (5), (ii) for $i = 1$ was considered in Ch. 2, namely, in this case the corresponding representation $(\alpha, \chi(h))$ does not have the $(s, \mu)$-parametrization, and $t = t_0 \mu \sim \frac{1}{t - t_0}$ in the deleted neighborhoods. Therefore, here, in relation (6),

$$\det \Phi_{\mathfrak{gl}_\infty}(\mu) \sim \frac{1}{(t - t_0)^{\deg_s}},$$

where $\deg_s$ is the degree of $\det \Phi_{\mathfrak{gl}_\infty}(\mu)$ with respect to $\mu$ on the $k$th level. On the other hand, according to condition (7), the multiplicity of zero in (6) for the general $\alpha, \lambda$ is equal to $p$ for every $w = (f^1 \delta^1_j , \ldots , (f^p \delta^p_j , s) \cdot v$,

since $T_1(s - 2i)$ is zero only for $i = 0$.

Thus, in accordance with Lemma 12 from Subsec. 2.2 of Ch. 1, we see that the irreducibility of the representation of the Lie algebra $\mathfrak{gl}(\lambda)$, induced from $p_\alpha$, for $T_1(\alpha + 2) = 0$ and the general $\lambda$, is equivalent to the following local identity from Ch. 2:

$$\left. \frac{d}{da} \left( \prod_{i \geq 1} \frac{1}{(1 - a q^i)^i} \right) \right|_{a = 1} = \sum_{w = \text{Det}_1^{i_1} \cdots \text{Det}_k^{i_k}} \# D(w) \cdot q^{\sum_{i} i^2} \cdot \left( \chi(w) \right)^2,$$

where $\# D(w)$ is the number of squares in the Young diagram corresponding to $w$ (see Introduction, Fig. 1) and $\chi(w)$ is the corresponding semiinfinite character (see Subsec. 0.11 or Ch. 1).

2.3.3. Let us consider case (5), (ii) for $i \neq 1$. In this case, the corresponding representation (for the general $\alpha, \lambda$) belongs to the $(s, \mu)$-parametrization and the part of relation (6) dependent on $\mu$ does not have any zeros of poles. Furthermore, let $|i - 1| = k_+$. Then the generating function

$$\sum_{k \geq 0} \left\{ \begin{array}{l}
\text{the multiplicity of zero in the neighborhood of } t = t_0 \\
\text{of the determinant of Shapovalov's form on the}
\text{kth level}
\end{array} \right\} \cdot q^k$$

is equal, by virtue of relations (6), (7), to

$$\left. \frac{d}{da} \left[ \prod_{i = 1}^{k_+} \frac{1}{(1 - q^i)^i} \cdot \prod_{i = k_+ + 1}^{\infty} \frac{1}{(1 - q^i)^{k_+} \cdot (1 - a q^i)^{i-k_+}} \right] \right|_{a = 1}. $$

2.4. Here we shall obtain a global identity equivalent to the theorem on the irreducibility, for the general parameters, of the induced representation of the Lie algebra of functions on a hyperboloid (with the one-dimensional first level).

None of the values of $t \in S^2 \setminus \{0\}$ meets condition (5), (i) for $\mu = 0$ and a one pair of points on $S^2 \setminus \{0\}$ meets it for every $\mu \in \mathbb{Z} \setminus \{0\}$. 
According to Subsec. 2.3.1, the corresponding generating function for the sum of multiplicities of zeros of the determinant of Shapovalov’s form of the kth level for all \( t \in S^2 \setminus \{0\} \) that satisfy condition (5), (i) with \( \mu \neq 0 \) is equal to

\[
(10) \quad 2 \sum_{a \in \mathbb{Z} \setminus \{0\}} \left\{ \begin{array}{l}
\text{the multiplicity of the root } \mu = a \\
\text{in the determinant of Shapovalov’s form for } g\lambda_\infty \text{ on the } k\text{th level}
\end{array} \right\} \cdot q^k.
\]

For every \( k_+ \neq 0 \) (see Subsec. 2.3.3) condition (5), (ii) is satisfied by four points on \( S^2 \setminus \{0\} \) and for \( k_+ = 0 \) it is satisfied by two points.

It follows from Theorem 1, Sec. 3 (see below) that for the general \( \alpha, \lambda \) the corresponding induced representation of the Lie algebra of functions on a hyperboloid is irreducible. Then the Chern class found in (2) is equal to the sum of multiplicities of zeros of the determinant of Shapovalov’s form for \( t \in S^2 \setminus \{0\} \). We have

\[
\frac{d}{da} \left[ \prod_{i=1}^{\infty} \frac{1}{(1-aq^i)} \right] \bigg|_{a=1} = 1 + 2 \sum_{k \geq 1} \sum_{a \in \mathbb{Z} \setminus \{0\}} \left\{ \begin{array}{l}
\text{the multiplicity of the root } \mu = a \\
\text{in the determinant of Shapovalov’s form of } g\lambda_\infty \text{ on the } k\text{th level}
\end{array} \right\} \cdot q^k +
\]

\[
(11) \quad 2 \sum_{k_+ \geq 1} \frac{d}{da} \left[ \prod_{i=1}^{k_+} \frac{1}{(1-aq^i)} \cdot \prod_{i=k_++1}^{\infty} \frac{1}{(1-aq^i)^{1-k_+}} \right] \bigg|_{a=1}.
\]

In accordance with the local identity from Subsec. 2.3.1, the sum of the two last but one terms is equal to 0. However, in form (11) we easily arrive at the following form of the global identity.

**Global identity:**

\[
\frac{d}{da} \left[ \prod_{i=1}^{\infty} \frac{1}{(1-aq^i)^{1-k_+}} \right] \bigg|_{a=1} +
\]

\[
2 \sum_{k_+ \geq 1} \frac{d}{da} \left[ \prod_{i=1}^{k_+} \frac{1}{(1-q^i)} \cdot \prod_{i=k_++1}^{\infty} \frac{1}{(1-q^i)^{1-k_+}} \right] \bigg|_{a=1} -
\]

\[
(12) \quad \sum_{\text{over all } \text{Young diagrams } \mu \in \text{Det}_1^{\infty} \ldots \text{Det}_b^{\infty}} \left\{ \begin{array}{l}
\text{in } p_w(\mu) = \langle w, w' \rangle \\
\text{the multiplicity of the root } \mu = 0
\end{array} \right\} \cdot q^\sum i \cdot i^2 \cdot \left( \chi(w) \right)^2.
\]

According to relation (8) from Ch. 1, the multiplicity of the root \( \mu = 0 \) in \( p_w(\mu) = \langle w, w' \rangle \) is equal to the length of the “central” diagonal in the corresponding Young diagram that starts from the upper left vertex (see Fig. 1, Introduction).
Recall the definition of the semi infinite character $\chi(w)$. Let us consider the Lie algebra of finite matrices $(a_{ij})$, $i,j = 1 \ldots \infty$. Suppose that $\alpha_1^\vee, \alpha_2^\vee, \ldots$ are the corresponding simple coroots. Then $\chi(w)$ is the character of the irreducible representation of this Lie algebra with the highest weight $\chi : \chi(\alpha_1^\vee) = l_1, \ldots, \chi(\alpha_k^\vee) = l_k$, $\chi(\alpha_{k+1}^\vee) = \ldots = 0$. Thus $\chi(w) = \frac{1}{(1-q) \cdots (1-q^k)}$ for $w = \text{Det}_k$ as well as for $w = \text{Det}_1$.

3. Proof of the Global Identity: Reduction to the Nilpotent Case.

3.0. According to the results of Sec. 2, the global identity (12) is equivalent to the irreducibility of representations of the Lie algebra $\mathfrak{s}$ of functions on the hyperboloid $\{2ef + \frac{h^2}{2} = \lambda(\lambda + 2)\}$, $\lambda(\lambda + 2) \neq 0$ induced from the parabolic subalgebras $\mathfrak{p}_\alpha$ with the highest weight $\chi$ for the general values of $\alpha$ and $\chi(h)$. We denote by $\mathfrak{s}_0$ the Lie algebra of functions on the cone $\{2ef + \frac{h^2}{2} = 0\}$ which is a degenerate symplectic leaf of the foliation in $\mathfrak{sl}_2(\mathbb{C})^*$. It is clear that it suffices to prove a similar statement for $\mathfrak{s}_0$ (of the Lie algebra of functions on all nondegenerate symplectic leaves are isomorphic). But the Lie algebra $\mathfrak{s}_0$ is nilpotent with an accuracy to within the subalgebra $\mathfrak{sl}_2$ contained in it, and we can apply Kirillov’s theory to it (see Appendix A).

3.1. The following theorem is the main result of this section.

**Theorem 1.** The representation of the Lie algebra $\mathfrak{s}$ induced from the parabolic subalgebra $\mathfrak{p}_\alpha$ with the highest weight $\chi \in \mathfrak{h}^*$ is reducible for

1. $\chi(h) = 0$

or

2. $\alpha^2 = \lambda(\lambda + 2)$ and irreducible otherwise.

**Remarks.**

1. The highest weight $\chi$ of the representation induced from $\mathfrak{p}_\alpha$ can be uniquely determined with an accuracy to within $\chi(1)$ from $\chi(h)$.

2. For $\alpha^2 = \lambda(\lambda + 2)$ the corresponding subalgebra $\mathfrak{p}_\alpha \subset \mathfrak{s}$ is not maximal: it is contained in the subalgebra $\mathfrak{p}'_\alpha = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \{f^k(h-\alpha)p(h), p(h) \in \mathbb{C}[h], k \geq 1\}$.

The remaining part of Sec. 3 is devoted to the proof of the theorem.

3.2. Theorem 1 is a simple corollary of the following weaker result.

**Theorem 2.** For a general $\alpha$, $\chi(h)$ the representation of the Lie algebra $\mathfrak{s}$ induced from $\mathfrak{p}_\alpha$ with the highest weight $\chi \in \mathfrak{h}^*$ is irreducible.

We saw in Sec. 2 that the irreducibility for the general $\alpha$, $\chi(h)$ is equivalent to the global identity which can be violated only if conditions (5), (i)–(ii) are satisfied by a not maximal of points (see Subsec. 2.4), i.e., if $T_t, t(\frac{\alpha}{t} + 2) = \frac{1}{2}(\frac{\alpha}{t} - \frac{\alpha^2}{2t^2} + \frac{\lambda(\lambda + 2)}{2t^2})$ does not contain terms with $\frac{1}{t^2}$, i.e., $\alpha^2 = \lambda(\lambda + 2)$, or for $\chi(h) = 0$.

It is clear that the nonsatisfaction of this identity for certain $\alpha$, $\chi(h)$ means that the determinant of Shapovalov’s form of the corresponding representation vanishes. Therefore Theorem 1 follows from Theorem 2.

Below we shall prove Theorem 2.
3.3. Since the determinant of Shapovalov’s form is an analytic function of \( \lambda \in \mathbb{C} \), Theorem 2 is a corollary of the following theorem on the representations of the Lie algebra \( \mathfrak{so}_0 \).

**Theorem 3.** The representations of the Lie algebra \( \mathfrak{so}_0 \), induced from \( \mathfrak{p}_\alpha \), with the highest weight \( \chi \in \mathfrak{h}^* \) are irreducible for \( \alpha \), \( \chi(h) \neq 0 \).

The proof of Theorem 3 is given in Subsecs. 3.4–3.5.

3.4. The functions on the cone \( \{2ef + h^2/2 = 0\} \), which have the point \( 0 \in \mathbb{C}^3 \) as a zero of order \( \geq k \), form an ideal \( I_k \subset \mathfrak{so}_0 \). The Lie algebra \( I_2 \) is nilpotent, and \( I_3 = \{I_2, I_2\} \), \( I_4 = \{I_2, I_3\} \) etc.

For nilpotent finite-dimensional Lie algebra Kirillov’s theorem [5] states that the representations induced from the maximal parabolic subalgebras are irreducible. To be more precise, let \( \mathfrak{g} \) be such a Lie algebra, \( \chi \in \mathfrak{g}^* \). Let \( \mathfrak{b} \) be a maximal subalgebra in \( \mathfrak{g} \) on which the bilinear form \( \langle x, y \rangle = \chi(\{x, y\}) \), \( x, y \in \mathfrak{g} \) is zero. Then \( \chi \) defines the one-dimensional \( \mathfrak{b} \)-module \( \mathbb{C}_\chi \) and the induced representation \( U(\mathfrak{g}) \otimes \mathbb{C} \chi \) is irreducible. We shall prove this theorem in Appendix A.

Suppose now that \( \chi \in \mathfrak{h}^* \) is the highest weight of the representation of the algebra \( \mathfrak{so}_0 \) induced from \( \mathfrak{p}_\alpha \). We continue it to the functional on \( \mathfrak{so}_0 \) by setting \( \chi|_{\mathfrak{n}_-} = \chi|_{\mathfrak{n}_-} = 0 \).

**Lemma 2.** Let \( \alpha \), \( \chi(h) \neq 0 \). Then the intersection \( I_2 \cap \mathfrak{p}_\alpha \) is the maximal subalgebra in \( I_2 \) on which the form \( \langle x, y \rangle = \chi|_{I_2}(\{x, y\}) \); \( x, y \in I_2 \) is zero.

**Proof.** It is obvious that \( I_2 \cap \mathfrak{p}_\alpha = (I_2 \cap \mathfrak{n}_+) \oplus (I_2 \cap \mathfrak{n}_-) \oplus \{f^k(h - \alpha)p(h), \, k \geq 2, \, p(h) \in \mathbb{C}[h]\} \oplus \{fh(h - \alpha)p(h), \, p(h) \in \mathbb{C}[h]\} \). Let \( \tilde{\mathfrak{p}}_\alpha \) be the maximal subalgebra in \( I_2 \) which contains \( \mathfrak{p}_\alpha \cap I_2 \) and on which the form \( \langle x, y \rangle \) is zero. We must prove that \( \tilde{\mathfrak{p}}_\alpha = \mathfrak{p}_\alpha \).

(i) Let \( f^k(h - \alpha)p(h) \in \tilde{\mathfrak{p}}_\alpha \), where \( l < k \) and \( p(h) \) is relatively prime to \( (h - \alpha) \). Commuting this element with \( h^2, h^3, \ldots \in I_2 \cap \mathfrak{p}_\alpha \), we see that \( f^k(h - \alpha)p(h)p_1(h) \in \tilde{\mathfrak{p}}_\alpha \) for any \( p_1(h) \in \mathbb{C}[h] \). In particular, \( f^k(h - \alpha)^{k-l} \in \tilde{\mathfrak{p}}_\alpha \).

(ii) If \( fh \in \tilde{\mathfrak{p}}_\alpha \), then \( \chi(\{fh, e\}) = 0 \chi(\{f(h - \alpha), e\}) \) is also zero. Hence contrary to the hypothesis of the lemma, either \( \alpha \) or \( \chi(h) \) is equal to zero.

(iii) Assume that \( f^k(h - \alpha)^{k-l} \in \tilde{\mathfrak{p}}_\alpha \) (see (i)), \( \{f^k(h - \alpha)^{k-1}, eh\} = -k f^{k-1} h^2(h - \alpha)^{k-1} + (k - 1) f^{k-1} h^2(h - \alpha)^{k-2} \cdot h \cdot e + 2k f^{k} e(h - \alpha)^{k-1} \) lies in \( \tilde{\mathfrak{p}}_\alpha \) since \( eh \in \mathfrak{p}_\alpha \). The first and third terms lie in \( \mathfrak{p}_\alpha \) and, hence, \( f^{k-1} h^3(h - \alpha)^{k-2} \) lies in \( \tilde{\mathfrak{p}}_\alpha \). According to (i), it follows that \( f^{k-1} (h - \alpha)^{k-2} \in \tilde{\mathfrak{p}}_\alpha \). Thus, lowering the degree step by step, we arrive at a contradiction with (ii).

3.5. **Proof of Theorem 2:**

**Lemma 3.** Let \( \alpha \), \( \chi(h) \neq 0 \). Then the representation \( U(I_2) \otimes \mathbb{C} \chi|_{I_2} \) of the Lie algebra \( I_2 \) is irreducible.

**Proof.** In order to use Kirillov’s theorem (see Subsec. 3.4), it suffices to reduce the consideration to a finite-dimensional case. The quotient algebra \( I_2/I_k \) is nilpotent and finite-dimensional and the corresponding induced representation of this Lie algebra is irreducible according to Kirillov’s theorem. However, for \( k \gg 0 \) the factorization does not affect the levels which are considerably smaller than \( k \). In particular, there are no \( I_2 \)-singular vectors on them.
Remark. Let $b_k$ be the corresponding maximal subalgebra in $I_2/I_k$. It is not true that $b_k = I_2 \cap p_\alpha/I_k \cap p_\alpha$; in particular, $I_{k-1}/I_k \subset b_k$. However, it is true for the intersections with $n_l$, where $l \ll k$.

The space of the representation of the Lie algebra $s_0$ induced from $p_\alpha$ can be written as

$$V_1 = \{\text{monomials of } f^i h^j, \ j < i\},$$

or as

$$V_2 = \{\text{monomials of } f^i h^j, \ i \geq 2, \ j < i, \ and \ f h\}.$$  

There exists an isomorphism $V_1 \cong V_2$ which commutes with the action of $s_0$. However, the restriction of $V_2$ to $I_2$ coincides with the representation of $s_0$, induced from $I_2 \cap p_\alpha$, which is irreducible according to Lemma 3 if $\alpha, \chi(h) \neq 0$.

**Appendix A.** The following result due to A. A. Kirillov [5]:

**Theorem.** Let $g$ be a finite dimensional nilpotent Lie algebra $\chi \in g^*$, and let $b$ be a maximal subalgebra in $g$ such that the restriction of the bilinear form $(x \mid y) = \chi([x, y])$ on $g$ to $b$ is zero. Then the induced representation $M_\chi = U(g) \otimes_{U(b)} C_\chi$ is irreducible.

To make the reasoning of Chapter 3 self-contained, we shall prove it in this Appendix.

We begin with with obvious lemmas.

**Lemma 1.** Denote $b_0 = b$ and $b = \text{norm} b_{i-1}$. Then there exists an integer $n$ such that $b_n = g$ and inclusions $b = b_0 \subset b_1 \subset \ldots \subset b_n = g$ are strict.

**Lemma 2.** If $x \in b_i$, $y \in b_j$ $(i, j \geq 1)$ then $[x, y] \in b_{i+j-1}$.

Using filtration on $g$ from Lemma 1, we can define a filtration

$$C = U_0 \subset U_1 \subset U_2 \subset \ldots$$

on the universal enveloping algebra $U(g)$. It follows easily from Lemma 2 that the adjoint algebra corresponding to this filtration is commutative. Hence, any $b \in b$ defines the mapping

$$\text{ad}(b) : U_k/U_{k-1} \to U_{k-1}/U_{k-2}.$$  

**Lemma 3.** For any $u \in U_k/U_{k-1}$, $k \geq 2$, there exists $b \in b$ such that $\text{ad}(b)(u) \neq 0$ in $U_{k-1}/U_{k-2}$.

**Proof of the Theorem.**

Let $v$ be the highest vector in $M_\chi$. Suppose that for some $x \in b_1$ $xv$ lies in a proper submodule of $M_\chi$. Then there exists $b \in b$ for which $\chi([x, b]) \neq 0$. Indeed, if $\chi([x, b]) = 0$ for all $b \in b$ we can join $x$ to $b$ and, hence, $b$ is not maximal.

Then $bv = \alpha v$ for some $\alpha \in C$, and $[b, x]v = bxv - \alpha xv$ lies in a proper submodule and, hence, this submodule coincides with $M_\chi$, contrary to the fact that it is proper.

Now, assume that there exists a proper submodule $M$ in $M_\chi$, and let $u \cdot v$ be some element of $M$, where $u \in U_k \setminus U_{k-1}$. According to Lemma 3, there exists $b \in b$ such that $\text{ad}(b)(u) \neq 0$ as an element of $U_{k-1}/U_{k-2}$, and therefore $\text{ad}(b)(u)v$ is a nonzero element of $M$ with lower filtration. Iterating this process, we shall arrive to the case in the beginning of the proof.
Appendix B. In fact, using methods of Ch. 1 one can easily prove the following generalization of identity (16) of Ch. 1 and of the local identity (see 0.11)

\[
\prod_{i \geq 1} \frac{1}{(1 - a \cdot q^i)^i} = \sum_{\text{over all Young diagrams } D_{l_1, \ldots, l_k}} a^{\#D_{l_1, \ldots, l_k}} \cdot q^{\sum l_i \cdot i^2} \cdot (\chi(D_{l_1, \ldots, l_k}))^2
\]

(the notation is as in 0.11). Therefore, in the reasoning of Sec. 4 of Chapter 2 imbedding into \(gl_\infty(\mathbb{C}[t]/t^2)\) can be replaced by more elementary considerations, using additional grading. This proves both the local identity and its generalization (1) for higher derivators. For the time being, the author does not know any generalizations of the global identity (see 0.11) for higher derivators.

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