GLOBAL EXISTENCE AND ASYMPTOTIC DECAY OF SOLUTIONS TO THE NON-ISENTROPIC EULER-MAXWELL SYSTEM

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Abstract. The non-isentropic compressible Euler-Maxwell system is investigated in $\mathbb{R}^3$ in the present paper, and the $L^q$ time decay rate for the global smooth solution is established. It is shown that the density and temperature of electron converge to the equilibrium states at the same rate $(1+t)^{-\frac{1}{4}}$ in $L^q$ norm.

Keywords: Non-isentropic Euler-Maxwell equations, Globally smooth solution, Asymptotic behavior

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1. Introduction and main results

We investigate the asymptotic behavior of globally smooth solutions for the (rescaled) non-isentropic Euler-Maxwell systems, which takes the following form (see Appendix A and [1, 3, 6, 7, 19]):

\begin{equation}
\begin{aligned}
\begin{cases}
\partial_t n + \nabla \cdot (nu) &= 0, \\
\partial_t u + (u \cdot \nabla)u + \nabla \theta + \theta \nabla \ln n + u &= -(E + u \times B), \\
\partial_t \theta + u \cdot \nabla \theta + \frac{2}{3} \theta \nabla \cdot u &= \frac{1}{3} |u|^2 - (\theta - 1), \\
\partial_t E - \nabla \times B &= nu, \\
\partial_t B + \nabla \times E &= 0, \\
\nabla \cdot E &= 1 - n, \quad \nabla \cdot B = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3.
\end{cases}
\end{aligned}
\end{equation}

Here, $n, u, \theta$ denote the scaled macroscopic density, mean velocity vector and temperature of the electrons and $E, B$ the scaled electric field and magnetic field. They are functions of a three-dimensional position vector $x \in \mathbb{R}^3$ and of the time $t > 0$. The fields $E$ and $B$ are coupled to the particles through the Maxwell equations and act on the particles via the Lorentz force $E + u \times B$.

In this paper, we are interested in the asymptotics and global existence of smooth solutions of system (1.1) with the initial conditions :

\begin{equation}
(n, u, \theta, E, B)|_{t=0} = (n_0, u_0, \theta_0, E_0, B_0), \quad x \in \mathbb{R}^3,
\end{equation}
which satisfies the compatible condition

\begin{equation}
\nabla \cdot E_0 = 1 - n_0, \quad \nabla \cdot B_0 = 0, \quad x \in \mathbb{R}^3.
\end{equation}

The Euler-Maxwell system \((1.1)\) is a symmetrizable hyperbolic system for \(n, \theta > 0\). It is known that the Cauchy problem \((1.1)-(1.2)\) has a local smooth solution when the initial data are smooth. In a simplified one dimensional isentropic Euler-Maxwell system, the global existence of entropy solutions has been given in \([2]\) by the compensated compactness method. For the three dimensional isentropic Euler-Maxwell system, the existence of global smooth solutions with small amplitude to the Cauchy problem in the whole space and to the periodic problem in the torus is established by Peng et al in \([15]\) and Ueda et al in \([17]\) respectively, and the decay rate of the smooth solution when \(t\) goes to infinity is obtained by Duan in \([4]\) and Ueda et al in \([18]\). For asymptotic limits with small parameters, see \([13, 14]\) and references therein. Recently, Yang et al in \([19]\) consider the diffusive relaxation limit of the three dimensional non-isentropic Euler-Maxwell system.

In this paper, we consider large time asymptotics and global existence of the smooth solutions to the non-isentropic Euler-Maxwell system. Our main goal here is to establish the global existence of smooth solutions around a constant state \((1, 0, 1, 0, 0)\), which is a equilibrium solution of system \((1.1)\), and the decay rate of the global smooth solutions in time for the system \((1.1)\). Our main results read as follows:

**Theorem 1.1.** Let \(s \geq 4\) be an integer, and \(1.3\) hold. There exist \(\delta_0 > 0\) and a constant \(C_0\) such that if

\[\|\|n_0 - 1, u_0, \theta_0 - 1, E_0, B_0\|\|_s \leq \delta_0,\]

then the initial problem \((1.1)-(1.2)\) has a unique global solution \([n(t, x), u(t, x), \theta(t, x), E(t, x), B(t, x)]\) with

\[\{n - 1, u, \theta - 1, E, B\} \in C^1([0, T); H^{s-1}(\mathbb{R}^3)) \cap C([0, T); H^s(\mathbb{R}^3))\]

and

\[\sup_{t \geq 0} \|\|n(t) - 1, u(t), \theta(t) - 1, E(t), B(t)\|\|_s \leq C_0 \|\|n_0 - 1, u_0, \theta_0 - 1, E_0, B_0\|\|_s.\]

Furthermore, there exist \(\delta_1 > 0\) and a constant \(C_1\) such that if

\[\|\|n_0 - 1, u_0, \theta_0 - 1, E_0, B_0\|\|_{L^1} + \|\|u_0, E_0, B_0\|\|_{L^1} \leq \delta_1,\]

then \([n(t, x), u(t, x), \theta(t, x), E(t, x), B(t, x)]\) satisfies

\[\|\|n(t) - 1, \theta(t) - 1\|\|_{L^q} \leq C_1 (1 + t)^{-\frac{q}{2} + \frac{3}{4}},\]

\[\|E(t)\|_{L^q} \leq C_1 (1 + t)^{-2 + \frac{3}{q}},\]

\[\|u(t), B(t)\|_{L^q} \leq C_1 (1 + t)^{-\frac{q}{2} + \frac{3}{4}},\]

with \(t \geq 0\) and \(2 \leq q \leq \infty\).

The proof of Theorem \([1.1]\) is based on the careful energy methods and the Fourier multiplier technique. This is divided into three key steps: The first key step is to establish the global a priori high order Sobolev’s energy estimates in time by using the careful energy methods and the skew-symmetric dissipative structure (see \([17]\) of Euler-Maxwell system, which concludes the global existence results in Theorem \([1.1]\). Due to the complexity of non-isentropic case caused by the coupled energy equations, the technique of symmetrizer is used here to remove this difficulty when we deal with the Euler part of the Euler-Maxwell system so as to obtain the energy estimates for the the density, the velocity and the temperature while the special symmetric
structure of the Maxwell system is used when we obtain the energy estimates for the electromagnetic fields for the Maxwell part of the Euler-Maxwell system, which is different from the method used by [3,17] to deal with the isentropic case of compressible Euler-Maxwell system. The second key step is to obtain the $L^p - L^q$ time decay rate of the linearized operator for the non-isentropic Euler-Maxwell system by using the Fourier technique, used first by Kawashima in [9] and extended then to the other problems, see [5,4,18] and the references therein. Here we first apply energy method in the Fourier space to obtain the basic $L^\infty$ estimates for the Fourier transform of the solution. Then we solve the dissipative linear wave system of three order by the Fourier technique. We find that the ‘error’ functions $\rho, \Theta, \nabla \cdot u$ satisfy the same dissipative linear wave equation which is of order three and is different from that of the isentropic Euler-Maxwell system. We also find an interesting phenomenon that the estimate on $B$ which is depending on the temperature $\Theta$ is different from that of the isentropic Euler-Maxwell equation. Since the linear system involved here is of order three, it is complex to obtain the time decay rate of the linear system by the Fourier analysis. Fortunately, we can obtain the elaborate spectrum structure of the eigenvalue equations of this linear system of order three, which yield to the desired time decay rate for the linearized system. The third step of the proof is to obtain the time decay rate in Theorem 1.1 by combining the previous two steps and apply the energy estimate technique to the nonlinear problem satisfied by the error functions, whose solutions can be represented by the solution-semigroup operator for the linearized problem by using the Duhamel Principle. This concludes the time asymptotic stability results in Theorem 1.1.

For the later use in this paper, we give some notations. For any integer $s \geq 0$, $H^s, \hat{H}^s$ denote the Sobolev space $H^s(\mathbb{R}^3)$ and the $s$-order homogeneous Sobolev space, respectively. Set $L^2 = H^0$. The norm of $H^s$ is denoted by $\|\cdot\|_s$ with $\|\cdot\|_0 = \|\cdot\|$. $f \sim g$ denotes $\gamma f \leq g \leq \frac{1}{\gamma} f$ for a constant $0 < \gamma < 1$. $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{R}^3)$, i.e.

$$\langle f, g \rangle = \int_{\mathbb{R}^3} f(x)g(x)dx, \quad f = f(x), \quad g = g(x) \in L^2(\mathbb{R}^3).$$

We set $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3} = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ for a multi-index $\alpha = [\alpha_1, \alpha_2, \alpha_3]$. For an integrable function $f : \mathbb{R}^3 \to \mathbb{R}$, its Fourier transform is defined by

$$\hat{f}(k) = \int_{\mathbb{R}^3} e^{-ix \cdot k} f(x)dx, \quad x \cdot k := \sum_{j=1}^{3} x_jk_j, \quad k \in \mathbb{R}^3,$$

where $i = \sqrt{-1} \in \mathbb{C}$ is the imaginary unit. For two complex numbers or vectors $a$ and $b$, $(a|b) := a \cdot \overline{b}$ denotes the dot product of $a$ with the complex conjugate of $b$. We also use $\Re(a)$ to denote the real part of $a$.

This paper is organized as follows. In Section 2, the transformation of system (1.1) is presented. In Section 3, we obtain the existence and uniqueness of global solutions. In Section 4, we study the linearized homogeneous equations to get the $L^p - L^q$ decay property and the explicit representation of solutions. Lastly, in Section 5, we investigate the decay rates of solutions of the nonlinear system (2.2) and complete the proof of Theorem 1.1.

2. Transformation of system (1.1)

Suppose $[n(t, x), u(t, x), \theta(t, x), E(t, x), B(t, x)]$ to be a smooth solution to the initial problem of the non-isentropic Euler-Maxwell equations (1.1) with initial data (1.2) which satisfies (1.3). We introduce the transformation

$$n(t, x) = 1 + \rho(t, x), \quad \theta(t, x) = 1 + \Theta(t, x).$$

(2.1)
Then, the system (1.1) becomes
\[
\begin{aligned}
\partial_t \rho + \nabla \cdot ((1 + \rho)u) &= 0, \\
\partial_t u + (u \cdot \nabla)u + \nabla \Theta + \frac{1 + \Theta}{1 + \rho} \nabla \rho &= -(E + u \times B) - u, \\
\partial_t \Theta + u \cdot \nabla \Theta + \frac{2}{3}(1 + \Theta) \nabla \cdot u &= \frac{1}{3} |u|^2 - \Theta, \\
\partial_t E - \nabla \times B &= (1 + \rho)u, \\
\partial_t B + \nabla \times E &= 0, \\
\nabla \cdot E &= -\rho, \quad \nabla \cdot B = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3,
\end{aligned}
\]
with initial data
\[
U|_{t=0} = [\rho, u, \Theta, E, B]|_{t=0} = U_0 := [\rho_0, u_0, \Theta_0, E_0, B_0], \quad x \in \mathbb{R}^3,
\]
which satisfies the compatible condition
\[
\nabla \cdot E_0 = -\rho_0, \quad \nabla \cdot B_0 = 0, \quad x \in \mathbb{R}^3,
\]
with \(\rho_0 = n_0 - 1\). In the following, we set \(s \geq 4\). Besides, for \(U = [\rho, u, \Theta, E, B]\), we use \(E_s(U(t)), \ E^h_s(U(t))\) to denote the energy functional, the high-order energy functional, the dissipation rate and the high-order dissipation rate, respectively. Here,
\[
(2.5) \quad E_s(U(t)) \sim \|\rho, u, \Theta, E, B\|_s^2,
\]
\[
(2.6) \quad E^h_s(U(t)) \sim \|\nabla \rho, u, \Theta, E, B\|_{s-1}^2,
\]
\[
(2.7) \quad D_s(U(t)) \sim \|\rho, u, \Theta\|_s^2 + \|\nabla [E, B]\|_{s-2}^2 + \|E\|_2^2,
\]
and
\[
(2.8) \quad D^h_s(U(t)) \sim \|\nabla \rho, u, \Theta\|_{s-1}^2 + \|\nabla [E, B]\|_{s-2}^2.
\]
Then, for the initial problem (2.2)-(2.3), we obtain

**Proposition 2.1.** Assume that \(U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0]\) satisfies (2.4). Then, there exist \(E_s(\cdot)\) and \(D_s(\cdot)\) such that the following holds true. If \(E_s(U_0) > 0\) is small enough, then, for any \(t \geq 0\), the initial problem (2.2)-(2.3) has a unique nonzero global solution \(U = [\rho, u, \Theta, E, B]\) which satisfies
\[
(2.9) \quad U \in C^1([0, T); H^{s-1}(\mathbb{R}^3)) \cap C([0, T); H^s(\mathbb{R}^3)),
\]
and
\[
(2.10) \quad E_s(U(t)) + \gamma \int_0^t D_s(U(y))dy \leq E_s(U_0).
\]
Furthermore, we investigate the time decay rates of solutions in Proposition 2.1 under some extra conditions on the given initial data \(U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0]\). For this purpose, we define \(\epsilon_s(U_0)\) as
\[
(2.11) \quad \epsilon_s(U_0) = \|U_0\|_s + \|[u_0, E_0, B_0]\|_{L^1}
\]
for \(s \geq 4\). Then, we have
Proposition 2.2. Assume that (2.4) holds for given initial data $U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0]$. If $\epsilon_{s+2}(U_0)$ is small enough, then $U = [\rho, u, \Theta, E, B]$ satisfies
\begin{equation}
\|U(t)\|_s \leq C\epsilon_{s+2}(U_0)(1 + t)^{-\frac{s}{2}}
\end{equation}
for $t \geq 0$. Moreover, if $\epsilon_{s+6}(U_0)$ is small enough, then $U = [\rho, u, \Theta, E, B]$ also satisfies
\begin{equation}
\|\nabla U(t)\|_{s-1} \leq C\epsilon_{s+6}(U_0)(1 + t)^{-\frac{s}{2}}
\end{equation}
for $t \geq 0$.

Proposition 2.3. Assume that (2.4) holds for given initial data $U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0]$. If $\epsilon_{13}(U_0)$ is small enough, then $U = [\rho, u, \Theta, E, B]$ satisfies
\begin{equation}
\|\rho(t), \Theta(t)\|_{L^q} \leq C(1 + t)^{-\frac{1}{4}}
\end{equation}
(2.14)
\begin{equation}
\|u(t), E(t)\|_{L^q} \leq C(1 + t)^{-\frac{2}{3} + \frac{3}{2q}}
\end{equation}
(2.15)
and
\begin{equation}
\|B(t)\|_{L^q} \leq C(1 + t)^{-\frac{3}{4} + \frac{3}{4q}}
\end{equation}
(2.16)
for $2 \leq q \leq \infty$ and $t \geq 0$.

Lastly, one can obtain Theorem 1.1 from Proposition 2.1 and Proposition 2.3. Therefore, we prove the three previous Propositions in the rest of this paper.

3. Global solutions for system (2.2)

We review Moser-type calculs inequalities in Sobolev spaces and the local existence of smooth solutions for symmetrizable hyperbolic equations, which will be used in the proof of our main theorem.

Lemma 3.1. (Moser-type calculus inequalities, see [10, 11]) Let $s \geq 1$ be an integer. Suppose $v \in H^s(\mathbb{R}^3)$, $\nabla v \in L^\infty(\mathbb{R}^3)$ and $v \in H^{s-1}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Then for all multi-index $\alpha$ with $|\alpha| \leq s$, we have $\partial^\alpha uv \in L^2(\mathbb{R}^3)$ and
\begin{equation}
\|\partial^\alpha (uv) - u\partial^\alpha v\| \leq C_s(\|\nabla u\|_{L^\infty} \|D^{s-1}v\| + \|D^s u\|_{L^\infty}v),
\end{equation}
where
\begin{equation}
\|D^s u\| = \sum_{|\alpha| = s} \|\partial^\alpha u\|.
\end{equation}
Moreover, if $s > \frac{5}{2}$, then the embedding $H^{s-1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ is continuous and we have
\begin{equation}
\|\partial^\alpha (uv) - u\partial^\alpha v\| \leq C_s \|u\|_{L^\infty} \|s-1\|v, \|uv\|_{s-1} \leq C_s \|u\|_{s-1} \|v\|_{s-1}.
\end{equation}

Lemma 3.2. (Local existence of smooth solutions, see [3, 11]) Let $s > \frac{5}{2}$ and $(\rho_0, u_0, \Theta_0, E_0, B_0) \in H^s(\mathbb{R}^3)$. Then there exist $T > 0$ and a unique smooth solution $(u, \theta, \Theta, E, B)$ to the Cauchy problem (2.3) satisfying $(u, \theta, \Theta, E, B) \in C^1([0, T); H^{s-1}(\mathbb{R}^3)) \cap C([0, T); H^s(\mathbb{R}^3))$.

3.1. Preliminary results. In this subsection, we will prove the following a priori estimates

Theorem 3.1. Assume that $U = [\rho, u, \Theta, E, B] \in C^1([0, T); H^{s-1}(\mathbb{R}^3)) \cap C([0, T); H^s(\mathbb{R}^3))$ is smooth for $T > 0$ with
\begin{equation}
\sup_{0 \leq t \leq T} \|U(t)\|_s \leq \delta
\end{equation}
(3.1)

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for \( \delta \leq \delta_0 \) with \( \delta_0 \) sufficiently small and suppose \( U \) to be the solution of the system (2.2) for \( t \in (0, T) \). Then, there exist \( \mathcal{E}_s(\cdot) \) and \( \mathcal{D}_s(\cdot) \) such that

\[
\frac{d}{dt} \mathcal{E}_s(U(t)) + \gamma \mathcal{D}_s(U(t)) \leq C[\mathcal{E}_s(U(t))^{\frac{1}{2}} + \mathcal{E}_s(U(t))]\mathcal{D}_s(U(t))
\]

for \( 0 \leq t \leq T \).

**Proof.** We will use five steps to finish the proof. The step 1 is to estimate the Euler part and the Maxwell part of the Euler-Maxwell system, respectively. Steps 2-4 is to obtain the dissipative estimates for \( \rho, E \) and \( B \) by using the skew-symmetric structure of the Euler-Maxwell system.

**Step 1.** It holds that

\[
\frac{d}{dt} \sum_{|\alpha| \leq s} [\langle A^I_0(W_I) \partial^\alpha W_I, \partial^\alpha W_I \rangle + \|\partial^\alpha W_{II}\|^2] + \|u\|^2 + \frac{1}{3} \|\Theta\|^2 \leq C \|W\|_s \|W_I\|^2_s,
\]

where

\[
W_I = \begin{pmatrix} \rho \\ u \\ \Theta \end{pmatrix}, \quad W_{II} = \begin{pmatrix} E \\ B \end{pmatrix}, \quad W = \begin{pmatrix} W_I \\ W_{II} \end{pmatrix}, \quad A^I_0(W_I) = \begin{pmatrix} \frac{1+\Theta}{1+\rho} & 0 & 0 \\ 0 & (1+\rho)I_3 & 0 \\ 0 & 0 & \frac{3+\rho}{2(1+\Theta)} \end{pmatrix}.
\]

In fact, Set

\[
A^I_j(W_I) = \begin{pmatrix} u_{ij} & (1+\rho)e_j^T & 0 \\ \frac{1+\Theta}{1+\rho}e_j & u_{ij}I_3 & e_j \\ 0 & \frac{3}{2}(1+\Theta)e_j^T & u_j \end{pmatrix}, \quad j = 1, 2, 3,
\]

\[
K_I(W) = \begin{pmatrix} 0 \\ -(E + u \times B) \\ 0 \end{pmatrix}, \quad K_2(W) = \begin{pmatrix} 0 \\ -u \\ \frac{1}{2}|u|^2 - \Theta \end{pmatrix}.
\]

Then the first three equations of (2.2) for \( W_I \) can be rewritten under the form

\[
\partial_t W_I + \sum_{j=1}^3 A^I_j(W_I) \partial_{x_j} W_I = K_I(W) + K_2(W).
\]

It is clear that system (3.4) for \( W_I \) is symmetrizable hyperbolic when \( 1 + \rho, 1 + \Theta > 0 \). More precisely, since we consider small solutions defined in a time interval \([0, T]\) with \( T > 0 \), (3.1) implies that \( \| [\rho, \Theta] \|_{L^\infty(\{(0, T) \times \mathbb{R}^3\})} \leq C_s \| [\rho, \Theta] \|_s \leq C_s \delta \leq \frac{1}{2} \). Then \( \frac{1}{2} \leq 1 + \rho, 1 + \Theta \leq \frac{3}{2} \). It follows that \( A^I_j(W_I) \) is symmetric positive definite and \( \tilde{A}^I_j(W_I) = A^I_0(W_I)A^I_j(W_I) \) are symmetric for all \( 1 \leq j \leq 3 \). This choice of \( A^I_j(W_I) \) will simplify energy estimates.

For \( |\alpha| \leq s \), differentiating equations (3.4) with respect to \( x \) and multiplying the resulting equations by the symmetrizer matrix \( A^I_0(W_I) \), one has

\[
A^I_0(W_I) \partial_t \partial^\alpha W_I + \sum_{j=1}^3 A^I_0(W_I)A^I_j(W_I) \partial_{x_j} \partial^\alpha W_I = A^I_0(W_I) \partial^\alpha \left( K_I(W) + K_2(W) \right) + J_\alpha,
\]

where \( J_\alpha \) is defined by

\[
J_\alpha = -\sum_{j=1}^3 A^I_0(W_I) [\partial^\alpha (A^I_j(W_I) \partial_{x_j} W_I) - A^I_j(W_I) \partial^\alpha (\partial_{x_j} W_I)].
\]
Applying Lemma 3.1 to $J_\alpha$, we get
\begin{equation}
\|J_\alpha\| \leq C\left(\|\nabla A_j^I(W_I)\|_{L^\infty} + \|D^s A_j^I(W_I)\| + \|\partial_{x_j} W_I\|_{L^\infty}\right)
\leq C\left(\|W_I\|_s + \|W_I\|_s\right) \|W_I\|_s \leq C\|W_I\|^2_s.
\end{equation}
Taking the inner product of equations (3.5) with $\partial^3 W_I$ and using the fact that the matrix $\tilde{A}_j^I(W_I)$ is symmetric, we have
\begin{equation}
\frac{d}{dt} \langle A_0^I(W_I)\partial^\alpha W_I, \partial^\alpha W_I \rangle = 2\langle J_\alpha, \partial^\alpha W_I \rangle + \langle \text{div} A_1^I(W_I)\partial^\alpha W_I, \partial^\alpha W_I \rangle
+ 2\langle A_0^I(W_I)\partial^\alpha W_I, \partial^\alpha K_I(W) + \partial^\alpha K_2(W) \rangle,
\end{equation}
where
\[
\text{div} A_1^I(W_I) = \partial_i A_0^I(W_I) + \sum_{j=1}^3 \partial_{x_j} A_j^I(W_I) = \partial_i A_0^I(W_I) + \partial_\Theta A_0^I(W_I)\partial_i \Theta + \sum_{j=1}^3 (A_j^I)'(W_I)\partial_{x_j} W_I.
\]
Using the first equation in (2.2) and Lemma 3.1 we have
\begin{equation}
\|\partial_\rho\rho\|_{L^\infty} \leq C\|\partial_\rho\rho\|_s \leq C\|\nabla \cdot ((1 + \rho)u)\|_{s-1} \leq C(1 + \|\rho\|_s)\|u\|_s.
\end{equation}
Then
\begin{equation}
\|\text{div} A_1^I(W_I)\|_{L^\infty} \leq C(1 + \|W_I\|_s)\|W_I\|_s.
\end{equation}
Now let us estimate each term on the right hand side of (3.7). For the first two terms, by Cauchy-Schwarz inequality and using estimates (3.6) and (3.9), we have
\begin{equation}
\langle J_\alpha, \partial^\alpha W_I \rangle + \langle \text{div} A_1^I(W_I)\partial^\alpha W_I, \partial^\alpha W_I \rangle \leq C(1 + \|W_I\|_s)\|W_I\|_s \leq C\|W\|_s\|W_I\|_s.
\end{equation}
For the third term of the right hand side of (3.7), by using the definition of $A_0^I(W_I)$, $K_I(W)$ and $K_2(W)$, we obtain
\begin{equation}
2\langle A_0^I(W_I)\partial^\alpha W_I, \partial^\alpha K_I(W) + \partial^\alpha K_2(W) \rangle
= -2\langle (1 + \rho)\partial^\alpha u, \partial^\alpha u \rangle - 2\|\partial^\alpha u, \partial^\alpha E \| - 2\langle \partial^\alpha u, \partial^\alpha (u \times B) \rangle - 2\langle \rho \partial^\alpha u, \partial^\alpha E \rangle
\end{equation}
\begin{equation}
- 2\langle \rho \partial^\alpha u, \partial^\alpha (u \times B) \rangle + \langle \frac{1 + \rho}{2(1 + \Theta)} \partial^\alpha \Theta, \partial^\alpha (|u|^2) \rangle - \langle \frac{3(1 + \rho)}{2(1 + \Theta)} \partial^\alpha \Theta, \partial^\alpha \Theta \rangle
\leq -\|\partial^\alpha u\|^2 - \frac{1}{4}\|\partial^\alpha \Theta\|^2 - 2\langle \partial^\alpha u, \partial^\alpha E \rangle + C\|W\|_s\|W_I\|_s^2.
\end{equation}
Next we write the system for $W_{II}$ as:
\[
\partial_i W_{II} + \sum_{j=1}^3 A_j^{II}\partial_{x_j} W_{II} = ((1 + \rho)u, 0)^T,
\]
where
\[
A_j^{II} = \begin{pmatrix} 0 & L_j^T \\ L_j & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
For $|\alpha| \leq s$, differentiating the fourth and fifth equations of (2.2) with respect to $x$, we get
\begin{equation}
\begin{cases}
\partial_t \partial^\alpha E - \nabla \times \partial^\alpha B = \partial^\alpha [(1 + \rho)u], \\
\partial_t \partial^\alpha B + \nabla \times \partial^\alpha E = 0.
\end{cases}
\end{equation}
By the vector analysis formula
\[ \nabla \cdot (f \times g) = (\nabla \times f) \cdot g - (\nabla \times g) \cdot f, \]
one term appearing in Sobolev energy estimates vanishes, i.e.
\[ \int_{\mathbb{R}^3} \left( - \nabla \times \partial^\alpha B \cdot \partial^\alpha E + \nabla \times \partial^\alpha E \cdot \partial^\alpha B \right) dx = \int_{\mathbb{R}^3} \nabla \cdot (\partial^\alpha E \times \partial^\alpha B) dx = 0. \]

Hence, the standard energy estimate for $\text{(3.12)}$ together with Lemma 3.1 yields
\[ \frac{d}{dt} \| \partial^\alpha W_{11} \|^2 = 2\langle \partial^\alpha E, \partial^\alpha u \rangle + 2\langle \partial^\alpha E, \partial^\alpha (\rho u) \rangle \leq 2\langle \partial^\alpha E, \partial^\alpha u \rangle + C\| W \|_s \| W_l \|_s^2. \]

Adding $\text{(3.7)}, \text{(3.10)}, \text{(3.11)}$ and $\text{(3.13)}$, summing up for all $|\alpha| \leq s$, we get $\text{(3.3)}$.

**Step 2.** It holds that
\[ \frac{d}{dt} \sum_{\alpha \leq s-1} \left( \frac{1}{2(1+\rho)} \partial^\alpha \rho - \partial^\alpha \nabla \cdot u, \partial^\alpha \rho \right) + \gamma \| \rho \|_s^2 \leq C\left( \| W \|_s \| W_l \|_s^2 + \|[u, \Theta]\|_s^2 \right). \]

In fact, for $|\alpha| \leq s-1$. Differentiating the second equation of $\text{(2.2)}$ with respect to $t$ and taking the inner product of the resulting equation with $\partial^\alpha \nabla \rho$, we have
\[ \langle \frac{1+\Theta}{1+\rho} \partial^\alpha \nabla \rho, \partial^\alpha \nabla \rho \rangle + \langle \partial^\alpha E, \partial^\alpha \nabla \rho \rangle = -\langle \partial^\alpha \left( \frac{1+\Theta}{1+\rho} \nabla \rho \right) - \frac{1+\Theta}{1+\rho} \partial^\alpha \nabla \rho, \partial^\alpha \nabla \rho \rangle - \langle \partial^\alpha \partial_t u, \partial^\alpha \nabla \rho \rangle \\
- \langle \partial^\alpha (u \cdot \nabla u + \nabla \Theta + u \times B), \partial^\alpha \nabla \rho \rangle - \langle \partial^\alpha u, \partial^\alpha \nabla \rho \rangle. \]

Let us estimate each term in $\text{(3.15)}$. First, noting that $\frac{1}{2} \leq 1+\rho, 1+\Theta \leq \frac{3}{2}$, we have
\[ \langle \frac{1+\Theta}{1+\rho} \partial^\alpha \nabla \rho, \partial^\alpha \nabla \rho \rangle + \langle \partial^\alpha E, \partial^\alpha \nabla \rho \rangle = \langle \frac{1+\Theta}{1+\rho} \partial^\alpha \nabla \rho, \partial^\alpha \nabla \rho \rangle + \langle \partial^\alpha \rho, \partial^\alpha \rho \rangle \geq C^{-1}(\| \partial^\alpha \nabla \rho \|^2 + \| \partial^\alpha \rho \|^2). \]

By Lemma 3.1 we obtain
\[ \| \partial^\alpha \left( \frac{1+\Theta}{1+\rho} \nabla \rho \right) - \frac{1+\Theta}{1+\rho} \partial^\alpha \nabla \rho \| \leq C\| W_l \|_s^2. \]

Then,
\[ \langle \partial^\alpha \left( \frac{1+\Theta}{1+\rho} \nabla \rho \right) - \frac{1+\Theta}{1+\rho} \partial^\alpha \nabla \rho, \partial^\alpha \nabla \rho \rangle \leq C\| W \|_s \| W_l \|_s^2. \]

Obviously,
\[ -\langle \partial^\alpha \partial_t u, \partial^\alpha \nabla \rho \rangle = \frac{d}{dt} \langle \partial^\alpha \nabla \cdot u, \partial^\alpha \rho \rangle - \langle \partial^\alpha \nabla \cdot u, \partial^\alpha \partial_t \rho \rangle. \]

Then, from $\text{(3.3)}$ we have
\[ \langle \partial^\alpha \nabla \cdot u, \partial^\alpha \partial_t \rho \rangle \leq \langle \partial^\alpha \nabla \cdot u \| \| \partial^\alpha \partial_t \rho \| \leq \| u \|_s^2 + \| W \|_s \| W_l \|_s^2. \]

Hence,
\[ \langle \partial^\alpha \partial_t u, \partial^\alpha \nabla \rho \rangle \leq \frac{d}{dt} \langle \partial^\alpha \nabla \cdot u, \partial^\alpha \rho \rangle + \| u \|_s^2 + \| W \|_s \| W_l \|_s^2. \]
From (3.1), one has
\[
\left| \langle \partial^\alpha (u \cdot \nabla u + \nabla \Theta + u \times B), \partial^\alpha \nabla \rho \rangle \right| \\
\leq (\| \partial^\alpha (u \cdot \nabla u) \| + \| \partial^\alpha (u \times B) \| + \| \partial^\alpha \nabla \Theta \|) \| \partial^\alpha \nabla \rho \| \\
\leq C(\| u \|_{s-1} \| u \|_s + \| u \|_{s-1} B \|_{s-1} + \| u \|_{s-1} + \| \Theta \|_s) \| \rho \|_s \\
\leq C\| W \|_s \| W_I \|_s^2 + \varepsilon \| \rho \|_s^2 + C_\varepsilon (\| u \|_s^2 + \| \Theta \|_s^2).
\]

(3.19)

Now we establish the uniform estimates for the last term on the right hand side of (3.15). We get
\[
-\langle \partial^\alpha u, \partial^\alpha \nabla \rho \rangle = -\langle \partial^\alpha \left( \frac{\partial_t \rho}{1 + \rho} \right) - \frac{\partial^\alpha \partial_t \rho}{1 + \rho}, \partial^\alpha \rho \rangle - \langle \frac{u \cdot \nabla \rho}{1 + \rho}, \partial^\alpha \rho \rangle \\
- \langle \partial_t \partial^\alpha \rho, \partial^\alpha \rho \rangle - \langle \partial_t \left( \frac{u \cdot \nabla \rho}{1 + \rho} \right), \partial^\alpha \rho \rangle,
\]
with
\[
-\langle \partial_t \partial^\alpha \rho, \partial^\alpha \rho \rangle = -\frac{1}{2} \frac{d}{dt} \left( \frac{1}{1 + \rho} \partial^\alpha \rho, \partial^\alpha \rho \right) + \frac{1}{2} \langle \partial_t \left( \frac{1}{1 + \rho} \right) \partial^\alpha \rho, \partial^\alpha \rho \rangle \\
= -\frac{1}{2} \frac{d}{dt} \left( \frac{1}{1 + \rho} \partial^\alpha \rho, \partial^\alpha \rho \right) + \frac{1}{2} \left( \frac{1}{1 + \rho} \right)^2 \partial_t \rho \partial^\alpha \rho, \partial^\alpha \rho \rangle.
\]

(3.20)

Obviously,
\[
\left| \left( \frac{1}{(1 + \rho)^2} \partial_t \rho \partial^\alpha \rho, \partial^\alpha \rho \right) \right| \leq C \| \rho \|_{s-1} \| u \|_s \| \rho \|_s \leq C \| W \|_s \| W_I \|_s^2.
\]

(3.22)

Using (3.8) and (3.1), we have
\[
\left| \langle \frac{1}{(1 + \rho)^2} \partial_t \rho \partial^\alpha \rho, \partial^\alpha \rho \rangle \right| \leq C \| \partial_t \rho \|_{L^\infty} \| \partial^\alpha \rho \|^2 \leq C \| W \|_s \| W_I \|_s^2.
\]

(3.23)

By Lemma 3.1 and the continuous embedding \( H^{s-1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3) \), one has
\[
\left\| \partial^\alpha \left( \frac{\partial_t \rho}{1 + \rho} \right) - \frac{\partial^\alpha \partial_t \rho}{1 + \rho} \right\| \leq C(\| \nabla \left( \frac{1}{1 + \rho} \right) \|_{L^\infty} \| \partial_t \rho \|_{s-2} + \| D^{s-1} \left( \frac{1}{1 + \rho} \right) \| \| \partial_t \rho \|_{L^\infty} \\
\leq C(\| \nabla \rho \|_{s-1} \| \partial_t \rho \|_{s-2} + \| \rho \|_{s-1} \| \partial_t \rho \|_{s-1}) \\
\leq C \| \rho \|_s \| \partial_t \rho \|_{s-1} \\
\leq C(1 + \| \rho \|_s) \| u \|_s \| \rho \|_s.
\]

Then, since (3.1), we have
\[
\left| \langle \partial^\alpha \left( \frac{\partial_t \rho}{1 + \rho} \right) - \frac{\partial^\alpha \partial_t \rho}{1 + \rho}, \partial^\alpha \rho \rangle \right| \leq C \| W \|_s \| W_I \|_s^2.
\]

(3.24)

Similarly,
\[
\left\| \partial^\alpha \left( \frac{u \cdot \nabla \rho}{1 + \rho} \right) - \frac{u \cdot \partial^\alpha \nabla \rho}{1 + \rho} \right\| \leq C(\| \nabla \left( \frac{u}{1 + \rho} \right) \|_{L^\infty} \| \nabla \rho \|_{s-2} + \| D^{s-1} \left( \frac{u}{1 + \rho} \right) \| \| \nabla \rho \|_{L^\infty}) \leq C \| W_I \|_s \| \rho \|_s.
\]

Therefore,
\[
\left| \langle \partial^\alpha \left( \frac{u \cdot \nabla \rho}{1 + \rho} \right) - \frac{u \cdot \partial^\alpha \nabla \rho}{1 + \rho}, \partial^\alpha \rho \rangle \right| \leq C \| W \|_s \| W_I \|_s^2.
\]

(3.25)

Thus, combining (3.15)-(3.25), we get
\[
\frac{d}{dt} \left( \frac{1}{1 + \rho} \partial^\alpha \rho - \partial^\alpha \nabla \cdot u, \partial^\alpha \rho \right) + C^{-1}(\| \partial^\alpha \nabla \rho \|^2 + \| \partial^\alpha \rho \|^2) \leq C \| W \|_s \| W_I \|_s^2 + \varepsilon \| \rho \|_s^2 + C_\varepsilon (\| u \|_s^2 + \| \Theta \|_s^2).
\]
Summing up this inequality for all $|\alpha| \leq s - 1$ and taking $\varepsilon > 0$ small enough, we obtain (3.11).

Step 3. It holds that

$$\frac{d}{dt} \sum_{|\alpha| \leq s - 1} \langle \partial^\alpha u, \partial^\alpha E \rangle + \gamma \|E\|_s^2 \leq C\|E, B\|_s \|u\|_s^2 + \|E\|_{s-2}^2$$

(3.26)

$$+ C\|\rho, u, \Theta\|_s^2 + C\|u\|_s\|\nabla B\|_{s-2}.$$

In fact, for $|\alpha| \leq s - 1$, applying $\partial^\alpha$ to the second equation of (2.2), multiplying it by $\partial^\alpha E$, taking integration in $x$ and then using the fourth equation of (2.2) implies

$$\frac{d}{dt} \langle \partial^\alpha u, \partial^\alpha E \rangle + \|\partial^\alpha E\|^2$$

$$= - \langle \partial^\alpha (u \cdot \nabla u) \partial^\alpha E \rangle - \langle \partial^\alpha (1 + \frac{\Theta}{1 + \rho}) \nabla \rho \rangle, \partial^\alpha E \rangle - \langle \partial^\alpha u, \partial^\alpha E \rangle$$

$$- \langle \partial^\alpha (u \times B) \partial^\alpha E \rangle + \|\partial^\alpha u, \nabla \partial^\alpha B\| + \|\partial^\alpha u\|^2 + \|\partial^\alpha (\rho u)\|, \partial^\alpha u \rangle. $$

Furthermore, using Cauchy-Schwarz inequality, one has

$$\frac{d}{dt} \langle \partial^\alpha u, \partial^\alpha E \rangle + \gamma \|\partial^\alpha E\|^2$$

$$= C \left( \|E\|_s \|u\|_s^2 + \|u\|_s\|\nabla B\|_{s-2} \right) + C\|B\|_s \left( \|u\|_s^2 + \|E\|_{s-1}^2 \right). $$

Therefore, taking summation of the previous estimate over $|\alpha| \leq s - 1$, one has (3.26).

Step 4. It holds that

$$\frac{d}{dt} \sum_{|\alpha| \leq s - 2} \langle - \nabla \times \partial^\alpha E, \partial^\alpha B \rangle + \gamma \|\nabla B\|_{s-2} \leq C \|[u, E]\|_s \|u\|_s^2 + C\|\rho\|_s \|\nabla u\|_{s-1}^2.$$

(3.27)

In fact, for $|\alpha| \leq s - 2$, applying $\partial^\alpha$ to the fourth equation of (2.2), multiplying it by $-\partial^\alpha \nabla \times B$, taking integration in $x$ and then using the fifth equation of (2.2) gives

$$\frac{d}{dt} \langle - \nabla \times \partial^\alpha E, \partial^\alpha B \rangle + \|\nabla \times \partial^\alpha B\|^2 = \|\nabla \times \partial^\alpha E\|^2 - \langle \partial^\alpha u, \nabla \times \partial^\alpha B \rangle - \langle \partial^\alpha (\rho u), \nabla \times \partial^\alpha B \rangle.$$ 

Furthermore, using Cauchy-Schwarz inequality and taking summation over $|\alpha| \leq s - 2$, one has (3.27). Where we also used

$$\|\partial^\alpha \partial_i B\| = \|\partial_i \Delta^{-1} \nabla \times (\nabla \times \partial^\alpha B)\| \leq C \|\nabla \times \partial^\alpha B\|$$

for each $1 \leq i \leq 3$, due to the fact that $\nabla \cdot B = 0$ and $\partial_i \Delta^{-1} \nabla$ is bounded from $L^p$ to $L^p$ for $1 < p < \infty$, see [16].

Step 5. Next, based four previous steps, we will prove (3.2). We define the energy functional as

$$E_s(U(t)) = \sum_{|\alpha| \leq s} \left( \langle A^\alpha_0(W_I) \partial^\alpha W_I, \partial^\alpha W_I \rangle + \|\partial^\alpha W_{II}\|^2 \right)$$

$$+ K_1 \sum_{|\alpha| \leq s - 1} \langle \frac{1}{2(1 + \rho)} \partial^\alpha \rho - \partial^\alpha \nabla \cdot u, \partial^\alpha \rho \rangle + K_2 \sum_{|\alpha| \leq s - 1} \langle \partial^\alpha u, \partial^\alpha E \rangle$$

$$+ K_3 \sum_{|\alpha| \leq s - 2} \langle - \nabla \times \partial^\alpha E, \partial^\alpha B \rangle,$$

where, constants $0 < K_3 < K_2 < K_1 < 1$ is to be chosen later. Notice the fact that $A^\alpha_0(W_I)$ is positively and that as soon as $0 < K_j \ll 1_{j=1,2,3}$ is sufficiently small, then $E_s(U(t)) \sim \|U\|_s^2$
Global existence and asymptotic decay of solutions to the Non-isentropic Euler-Maxwell system

In this section, so as to obtain the time-decay rates of solutions to the nonlinear system \( (2.2) \) in the next section, we consider the following initial problem on the linearized homogeneous equations corresponding to system \( (2.2) \):

\[
\begin{cases}
\partial_t \rho + \nabla \cdot u = 0, \\
\partial_t u + \nabla \rho + \nabla \Theta + E + u = 0, \\
\partial_t \Theta + \frac{\Theta}{3} \nabla \cdot u + \Theta = 0, \\
\partial_t E - \nabla \times B - u = 0, \\
\partial_t B + \nabla \times E = 0, \\
\nabla \cdot E = -\rho, \quad \nabla \cdot B = 0, \quad t > 0, x \in \mathbb{R}^3,
\end{cases}
\]

with initial data

\[
U_{|t=0} = U_0 := [\rho_0, u_0, \Theta_0, E_0, B_0], \quad x \in \mathbb{R}^3,
\]

which satisfies the compatible condition

\[
\nabla \cdot E_0 = -\rho_0, \quad \nabla \cdot B_0 = 0, \quad x \in \mathbb{R}^3.
\]

In this section, we usually use \( U = [\rho, u, \Theta, E, B] \) to denote the solution of the linearized homogeneous system \( (4.1) \).

4.1. **Pointwise time-frequency estimate.** In this subsection, we utilize the energy method to the initial problem \( (4.1)-(4.3) \) in the Fourier space to present that there is a time-frequency Lyapunov functional which is equivalent to \( |\hat{U}(t,k)|^2 \) and furthermore its dissipation rate could be represented by itself. The main result of this subsection is presented in the following.
Firstly, one can acquire from the first five equations of the system (4.6) that

\( \partial_t \hat{\rho} + ik \cdot \hat{u} = 0, \)
\( \partial_t \hat{u} + ik \hat{\rho} + ik \hat{\Theta} + \hat{E} + \hat{u} = 0, \)
\( \partial_t \hat{\Theta} + \frac{2}{3} ik \cdot \hat{u} + \hat{\Theta} = 0, \)
\( \partial_t \hat{E} - ik \times \hat{B} - \hat{u} = 0, \)
\( \partial_t \hat{B} + ik \times \hat{E} = 0, \)
\( ik \cdot \hat{E} = -\hat{\rho}, \quad ik \cdot \hat{B} = 0, \quad (t, k) \in (0, \infty) \times \mathbb{R}^3. \)

Multiplying the second equation of the system (4.6) by \( ik \hat{\rho} \), utilizing integration by parts in \( t \) and replacing \( \partial_t \hat{\rho} \) by the first equation of the system (4.6), we have

\( \partial_t (\hat{u} | ik \hat{\rho}| + (1 + |k|^2)|\hat{\rho}|^2) = |k \cdot \hat{u}|^2 - |k|^2 \hat{\Theta} \cdot \hat{\rho} + ik \cdot \hat{u} \hat{\rho}. \)

Multiplying the second equation of the system (4.6) by \( ik \hat{\Theta} \), utilizing integration by parts in \( t \) and replacing \( \partial_t \hat{\Theta} \) by the third equation of the system (4.6), we also have

\( \partial_t (\hat{u} | ik \hat{\Theta}| + |k|^2|\hat{\Theta}|^2) = \frac{2}{3} |k \cdot \hat{u}|^2 + 2ik \cdot \hat{u} \hat{\Theta} - |k|^2 \hat{\rho} \hat{\Theta} - \hat{\rho} \hat{\Theta}, \)

putting it together with (4.8) and taking the real part after utilizing the Cauchy-Schwarz inequality, one has

\( \partial_t \mathcal{R} \left( \hat{u} | ik \left( \hat{\rho} + \hat{\Theta} \right) \right) + \gamma |\hat{\rho}|^2 \leq C |k \cdot \hat{u}|^2 + C |\hat{\Theta}|^2. \)

Multiplying it by \( \frac{1}{1 + |k|^2} \), one can obtain

\( \frac{\partial_t \mathcal{R} \left( \hat{u} | ik \left( \hat{\rho} + \hat{\Theta} \right) \right)}{1 + |k|^2} + \frac{\gamma |\hat{\rho}|^2}{1 + |k|^2} \leq C (|\hat{u}|^2 + |\hat{\Theta}|^2). \)

Similarly, multiplying the second equation of the system (4.6) by \( \hat{E} \), utilizing integration by parts in \( t \) and replacing \( \partial_t \hat{E} \) by the fourth equation of the system (4.6), we have

\( \partial_t \left( \hat{u} | \hat{E} | + \left( |\hat{E}|^2 + |k \cdot \hat{E}|^2 \right) \right) = |\hat{u}|^2 - \hat{\Theta} \hat{\rho} - ik \times \hat{B} \cdot \hat{u} - \hat{u} \cdot \hat{E}. \)
Lastly, we define the time-frequency Lyapunov functional as
\[ E \frac{\partial_t |k|^2 \mathcal{R}(\hat{u} | \hat{E})}{(1 + |k|^2)^2} + \frac{\gamma |k|^2 (|\hat{E}|^2 + |\hat{k} \cdot \hat{E}|^2)}{(1 + |k|^2)^2} \leq C \left( \frac{1}{1 + |k|^2} (\hat{\rho} | k + \hat{\Theta}^2) + |k|^2 \mathcal{R}(\frac{-ik \times \hat{B} \cdot \hat{u}}{1 + |k|^2}) \right) \]
\[ \frac{|k|^2 \mathcal{R}(\hat{u} | \hat{E})}{(1 + |k|^2)^2} \]

Similarly, from the fourth and fifth equations of the system (4.6), one has
\[ \partial_t \left( -ik \times \hat{B} | \hat{E} \right) + |k \times \hat{B}|^2 = |k \times \hat{E}|^2 - (ik \times \hat{B} | \hat{u} \right), \]
which after utilizing Cauchy-Schwarz inequality and multiplying it by \( \frac{1}{(1 + |k|^2)^2} \), gives
\[ \frac{\partial_t \mathcal{R}(\hat{u} | \hat{E})}{(1 + |k|^2)^2} + \gamma |k \times \hat{B}|^2 \leq \frac{|k|^2 |\hat{E}|^2}{(1 + |k|^2)^2} + C |\hat{u}|^2. \]

Lastly, we define the time-frequency Lyapunov functional as
\[ \mathcal{E}(\hat{U}(t, k)) = \left[ \hat{\rho}, \hat{u}, \frac{\sqrt{6}}{2} \hat{\Theta}, \hat{E}, \hat{B} \right]^2 + K_1 \frac{\mathcal{R}(\hat{u} | ik (\hat{\rho} + \hat{\Theta}))}{1 + |k|^2} \]
\[ + K_2 \frac{|k|^2 \mathcal{R}(\hat{u} | \hat{E})}{(1 + |k|^2)^2} + K_3 \frac{\mathcal{R}(\hat{u} | \hat{E})}{(1 + |k|^2)^2}, \]
where, constants \( 0 < K_3 \approx K_2 \approx K_1 \approx 1 \) are to be determined later. Notice that as soon as \( 0 < K_j \approx 1 \) \( j=1,2,3 \) be sufficiently small, then (4.4) holds true. Moreover, by setting \( 0 < K_3 \ll K_2 \ll K_1 \ll 1 \) be sufficiently small with \( K_3^3 \ll K_3 \), taking the summation of (4.7), (4.9) \( \times K_1 \), (4.11) \( \times K_2 \) and (4.12) \( \times K_3 \), one has
\[ \partial_t \mathcal{E}(\hat{U}(t, k)) \leq \frac{\gamma |\hat{\rho}|^2}{1 + |k|^2} + \frac{\gamma |\hat{q}|^2}{(1 + |k|^2)^2} |\hat{E}, \hat{B}|^2 \leq 0, \]
where we have used the Cauchy-Schwarz inequality as follows
\[ \frac{K_2}{|k|^2 \mathcal{R}(\hat{u} | \hat{E})}{(1 + |k|^2)^2} \leq \frac{K_2^3 |\hat{u}|^2}{2(1 + |k|^2)^2} + \frac{K_2^3 |\hat{B}|^2}{2(1 + |k|^2)^2}. \]

Then, noticing \( \mathcal{E}(\hat{U}) \sim |\hat{U}|^2 \) and
\[ \frac{\gamma |\hat{k}|^2}{(1 + |k|^2)^2} \leq \frac{\gamma |\hat{\rho}|^2}{1 + |k|^2} + \frac{\gamma |\hat{q}|^2}{1 + |k|^2} |\hat{E}, \hat{B}|^2, \]
one can obtain (4.5) from (4.13). Now, we have finished the proof of Theorem 4.1.

It is straightforward from Theorem 4.1 to obtain the pointwise time-frequency estimate on the norm of \( |\hat{U}(t, k)| \) in terms of the given initial data norm of \( |\hat{U}_0(k)| \).
Corollary 4.1. Suppose $U(t, x), (t, x) \in (0, \infty) \times \mathbb{R}^3$ to be a solution to the initial problem (4.1)-(4.3). Then, there exist constants $\gamma > 0, C > 0$ such that

$$|\hat{U}(t, k)| \leq Ce^{-\frac{|\gamma k^2 t|}{(1 + |k|^2)^{\gamma}}} |\hat{U}_0(k)|$$

holds for $t \geq 0$ and $k \in \mathbb{R}^3$.

4.2 $L^p - L^q$ time-decay property.

Formally, the solution to the initial problem (4.1)-(4.2) is presented as

$$U(t) = e^{tL}U_0,$$

here, $e^{tL}$ is named as the linear solution operator for $t \geq 0$. The main result of this subsection, whose proof will be omitted here, is stated as follows; see [1], [5], [9].

Theorem 4.2. Let $1 \leq p, r \leq 2 \leq q \leq \infty, l \geq 0$ and an integer $m \geq 0$. Define

$$[l + 3(\frac{1}{r} - \frac{1}{q})]^+ = \begin{cases} l & \text{if } r = q = 2 \text{ and } l \text{ is an integer,} \\ [l + 3(\frac{1}{r} - \frac{1}{q})]^- + 1 & \text{otherwise,} \end{cases}$$

where, we use $[\cdot]_-$ to denote the integer part of the argument. Assume $U_0$ satisfies (4.3). Then, $e^{tL}$ satisfies the following time-decay property:

$$\|\nabla^m e^{tL}U_0\|_{L^q} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{m}{2}} \|U_0\|_{L^p} + C(1 + t)^{-\frac{1}{2}} \|\nabla^m + [l + 3(\frac{1}{r} - \frac{1}{q})]^+] \cdot U_0\|_{L^r}$$

for $t \geq 0$, here $C = C(p, q, r, l, m)$.

4.3 Representation of solutions. We investigate the explicit solution $U = [\rho, u, \Theta, E, B] = e^{tL}U_0$ to the initial problem (4.1)-(4.2) with the compatible condition (4.3) or equivalently the equations (4.6) in Fourier space in this subsection. We find that $\rho, \Theta, \nabla \cdot u$ satisfy the same equation which is of order three and is different from that of the isentropic Euler-Maxwell system. The main purpose is to prove Theorem 4.3 presented at the bottom of this subsection.

From the first three equations of (4.1) and $\nabla \cdot E = -\rho$, one has

$$\partial_{ttt}\rho + 2\partial_{tt}\rho - \frac{5}{3}\partial_t \Delta \rho + 2\partial_t \rho + \rho - \Delta \rho = 0,$$

with initial data

$$\begin{cases} \rho|_{t=0} = \rho_0 = -\nabla \cdot E_0, \\ \partial_t \rho|_{t=0} = -\nabla \cdot u_0, \\ \partial_{ttt}\rho|_{t=0} = \nabla \cdot u_0 + \Delta \rho_0 - \rho_0 + \Delta \Theta_0. \end{cases}$$

After taking Fourier transform in $x$ for (4.18) and (4.19), one has

$$\partial_{ttt}\hat{\rho} + 2\partial_{tt}\hat{\rho} + \left(2 + \frac{5}{3}|k|^2\right) \partial_t \hat{\rho} + (1 + |k|^2) \hat{\rho} = 0,$$

with initial data

$$\begin{cases} \hat{\rho}|_{t=0} = \hat{\rho}_0 = -ik \cdot \hat{E}_0, \\ \partial_t \hat{\rho}|_{t=0} = -ik \cdot \hat{u}_0, \\ \partial_{ttt}\hat{\rho}|_{t=0} = -(1 + |k|^2) \hat{\rho}_0 + ik \cdot \hat{u}_0 - |k|^2 \hat{\Theta}_0. \end{cases}$$
The characteristic equation of (4.20) is

\[ F(\lambda) := \lambda^3 + 2\lambda^2 + \left(2 + \frac{5}{3}|k|^2\right)\lambda + 1 + |k|^2 = 0. \]

For the roots of the previous characteristic equation and their properties, we obtain

**Lemma 4.1.** Suppose \(|k| \neq 0\). Then, \(F(\lambda) = 0, \lambda \in \mathbb{C}\) has a real root \(\sigma = \sigma(|k|) \in (-1, -\frac{3}{5})\) and two conjugate complex roots \(\lambda_{\pm} = \beta \pm i\omega\) with \(\beta = \beta(|k|) \in (-\frac{7}{10}, -\frac{1}{2})\) and \(\omega = \omega(|k|) \in (\frac{\sqrt{3}}{3}, +\infty)\) which satisfy

\[
\beta = -1 - \frac{\sigma}{2}, \quad \omega = \frac{1}{2}\sqrt{3\sigma^2 + 4\sigma + 4 + \frac{20}{3}|k|^2}.
\]

\(\sigma, \beta, \omega\) are smooth in \(|k| > 0\), and \(\sigma(|k|)\) is strictly increasing over \(|k| > 0\), with

\[
\lim_{|k| \to 0} \sigma(|k|) = -1, \quad \lim_{|k| \to \infty} \sigma(|k|) = -\frac{3}{5}.
\]

Furthermore, the asymptotic behavior in the following hold true:

\[
\sigma(|k|) = -1 + O(1)|k|^2, \quad \beta(|k|) = \frac{1}{2} - O(1)|k|^2, \quad \omega(|k|) = \frac{\sqrt{3}}{2} + O(1)|k|
\]

whenever \(|k| \leq 1\) is sufficiently small, and

\[
\sigma(|k|) = -\frac{3}{5} - O(1)|k|^{-2}, \quad \beta(|k|) = -\frac{7}{10} + O(1)|k|^{-2}, \quad \omega(|k|) = O(1)|k|
\]

whenever \(|k| \geq 1\) is sufficiently large. Here and in the sequel, we use \(O(1)\) to denote a generic strictly positive constant.

**Proof.** Let \(|k| \neq 0\). Firstly, we search the possibly existing real root for equation \(F(\lambda) = 0\) in \(\lambda \in \mathbb{R}\). From that

\[ F'(\lambda) = 3\lambda^2 + 4\lambda + 2 + \frac{5}{3}|k|^2 = 3(\lambda + \frac{2}{3})^2 + \frac{2}{3} + \frac{5}{3}|k|^2 > 0, \]

and \(F(-1) = -\frac{3}{5}|k|^2 < 0, \quad F(-\frac{2}{3}) = \frac{38}{125} > 0\), one can obtain that equation \(F(\lambda) = 0\) indeed has one and only one real root denoted by \(\sigma = \sigma(|k|)\) which satisfies \(-1 < \sigma < -\frac{3}{5}\). After taking derivative of \(F(\sigma(|k|)) = 0\) in \(|k|\), one has

\[
\sigma'(|k|) = \frac{-|k||2 + \frac{10}{3}\sigma|}{3\sigma^2 + 4\sigma + 2 + \frac{5}{3}|k|^2} > 0,
\]

so that \(\sigma(\cdot)\) is strictly increasing over \(|k| > 0\). Since \(F(\sigma) = 0\) can be represented as

\[
\sigma \left[ \frac{\sigma(\sigma + 2)}{2 + \frac{5}{3}|k|^2} + 1 \right] = -\frac{1 + |k|^2}{2 + \frac{4}{3}|k|^2},
\]

then \(\sigma\) has limits \(-1\) and \(-\frac{3}{5}\) as \(|k| \to 0\) and \(|k| \to \infty\), respectively.

\(F(\sigma(|k|)) = 0\) is also equivalent with

\[
\sigma + 1 = \frac{\frac{2}{3}|k|^2 + (\sigma + 1)^2}{(\sigma + 1)^2 + 1 + \frac{5}{3}|k|^2}
\]

or

\[
\sigma + \frac{3}{5} = \frac{-\frac{1}{5} + \frac{3}{5}\sigma(\sigma + 2)}{\sigma(\sigma + 2) + 2 + \frac{5}{3}|k|^2}.
\]
Therefore, it follows that $\sigma(|k|) = -1 + O(1)|k|^2$ whenever $|k| < 1$ is sufficiently small and $\sigma(|k|) = -\frac{2}{3} - O(1)|k|^{-2}$ whenever $|k| \geq 1$ is sufficiently large. Next, let us search roots of $F(\mathcal{X}) = 0$ in $\mathcal{X} \in \mathbb{C}$. Since $F(\sigma) = 0$ with $\sigma \in \mathbb{R}$, $F(\mathcal{X}) = 0$ can be decomposed as

$$F(\mathcal{X}) = (\mathcal{X} - \sigma) \left[ \left( \mathcal{X} + 1 + \frac{\sigma}{2} \right)^2 + \frac{3}{4} \sigma^2 + \sigma + \frac{5}{3}|k|^2 + 1 \right] = 0.$$ 

Then, there exist two conjugate complex roots $\mathcal{X}_\pm = \beta \pm i\omega$ which satisfy

$$\left( \mathcal{X} + 1 + \frac{\sigma}{2} \right)^2 + \frac{3}{4} \sigma^2 + \sigma + \frac{5}{3}|k|^2 + 1 = 0.$$ 

By solving the previous equation, one can obtain that $\beta = \beta(|k|), \omega = \omega(|k|)$ take the form of (4.22). It is straightforward from the asymptotic behavior of $\sigma(|k|)$ at $|k| = 0$ and $\infty$ to obtain that of $\beta(|k|), \omega(|k|)$. Then, we have finished the proof of Lemma 4.1. \qed

Based on Lemma 4.1 we define the solution of (4.20) as

$$(4.23) \quad \hat{\rho}(t, k) = c_1(k)e^{\alpha t} + e^{\beta t}(c_2(k)\cos\omega t + c_3(k)\sin\omega t),$$

where $c_j(k), 1 \leq j \leq 3,$ is to be chosen by (4.21) later. Again using $\nabla \cdot E = -\rho,$ (4.23) implies

$$(4.24) \quad \tilde{k} \cdot \hat{E}(t, k) = i|k|^{-1} \left( c_1(k)e^{\alpha t} + e^{\beta t}(c_2(k)\cos\omega t + c_3(k)\sin\omega t) \right).$$

Here and in the sequel $\tilde{k} = \frac{k}{|k|}.$ In fact, (4.23) implies

$$(4.25) \quad \begin{bmatrix} \rho_{|t=0} \\ \partial_t \rho_{|t=0} \\ \partial_{tt} \rho_{|t=0} \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 0 \\ \sigma & \beta & \omega \\ \sigma^2 & \beta^2 - \omega^2 & 2\beta \omega \end{bmatrix}.$$ 

It is directly to obtain that

$$\det A = \omega \left[ \omega^2 + (\sigma - \beta)^2 \right] = \omega \left( 3\sigma^2 + 4\sigma + 2 + \frac{5}{3}|k|^2 \right) > 0$$

and

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} (\beta^2 + \omega^2) \omega & -2\beta \omega & \omega \\ \sigma(\sigma - 2\beta) \omega & 2\beta \omega & -\omega \\ \sigma(\beta^2 - \omega^2 - \sigma) \omega & \omega^2 + \sigma^2 - \beta^2 & \beta - \sigma \end{bmatrix}.$$ 

From (4.25) and (4.21), one has

$$[c_1, c_2, c_3]^T = \frac{1}{3\sigma^2 + 4\sigma + 2 + \frac{5}{3}|k|^2} \begin{bmatrix} \beta^2 + \omega^2 - \left( 1 + |k|^2 \right) \\ \sigma^2 - 2\sigma \beta + \left( 1 + |k|^2 \right) \\ \sigma(\beta^2 - \omega^2 - \sigma) - \left( \beta - \sigma \right) (1 + |k|^2) \end{bmatrix} \frac{|k|}{\omega} \left( \beta^2 - \sigma^2 - \omega^2 + \beta - \sigma \right) \frac{\sigma - \beta |k|^2}{|k|^2} \begin{bmatrix} \hat{\rho}_0 \\ \tilde{k} \cdot \hat{u}_0 \\ \hat{\Theta}_0 \end{bmatrix}.$$ 

Where, we use $[,]^T$ to denote the transpose of a vector. Making further simplifications with the form of $\beta$ and $\omega$, we have

$$[c_1, c_2, c_3]^T = \frac{1}{3\sigma^2 + 4\sigma + 2 + \frac{5}{3}|k|^2} \begin{bmatrix} (\sigma + 1)^2 + \frac{2}{3}|k|^2 \\ 2\sigma^2 + \sigma + 1 + |k|^2 \\ \frac{\sigma^2 + \sigma + (1 + |k|^2) - \frac{5}{3}|k|^2}{\omega} \frac{|k|}{\omega} \left( \frac{2}{3}\sigma^2 + \frac{5}{3}\sigma + 1 + \frac{5}{3}|k|^2 \right) \end{bmatrix} \frac{|k|}{\omega} \left( 1 + \frac{5}{3}\sigma \right) \begin{bmatrix} \hat{\rho}_0 \\ \tilde{k} \cdot \hat{u}_0 \\ \hat{\Theta}_0 \end{bmatrix}.$$
Similarly, from the first three equations of (4.1) and \( \nabla \cdot E = -\rho \), one has

\[
\partial_{ttt}\Theta + 2\partial_{tt}\Theta + \left(2 + \frac{5}{3}|k|^2\right) \partial_t\Theta + (1 + |k|^2) \Theta = 0,
\]

with initial data

\[
\begin{aligned}
(4.27) \qquad & \begin{cases}
\Theta|_{t=0} = \hat{\Theta}_0, \\
\partial_t\Theta|_{t=0} = \frac{2}{3}ik \cdot \hat{u}_0, \\
\partial_{tt}\Theta|_{t=0} = -\frac{2}{3} (1 + |k|^2) \hat{\rho}_0 + \frac{4}{3}ik \cdot \hat{u}_0 + \left(1 - \frac{2}{3}|k|^2\right) \Theta_0.
\end{cases}
\end{aligned}
\]

From Lemma 4.1 one can define the solution of (4.27) as

\[
(4.29) \qquad \Theta(t, k) = c_4(k)e^{\sigma t} + e^{\beta t} (c_5(k) \cos \omega t + c_6(k) \sin \omega t),
\]

where \( c_j(k), \ 4 \leq j \leq 6, \) is to be chosen by (4.28) later. In fact, (4.29) implies

\[
[c_4, c_5, c_6]^T \begin{bmatrix}
\sigma^2 + 2\sigma + \frac{2}{3} |k|^2 \\
2\sigma^2 + 3\sigma + \frac{2}{3} |k|^2 \\
-\frac{1}{\omega}(\sigma^2 - \frac{2}{3}|k|^2) + \frac{2}{\omega}|k|^2
\end{bmatrix}
\begin{bmatrix}
\hat{\rho}_0 \\
\hat{k} \cdot \hat{u}_0 \\
\hat{\Theta}_0
\end{bmatrix} = 0.
\]

Similarly, again from the first three equations of (4.1) and \( \nabla \cdot E = -\rho \), one also has

\[
\partial_{ttt}(\tilde{k} \cdot \hat{u}) + 2\partial_{tt}(\tilde{k} \cdot \hat{u}) + \left(2 + \frac{5}{3}|k|^2\right) \partial_t(\tilde{k} \cdot \hat{u}) + (1 + |k|^2) (\tilde{k} \cdot \hat{u}) = 0.
\]

Initial data is given by

\[
(4.31) \qquad \begin{aligned}
& (\tilde{k} \cdot \hat{u})|_{t=0} = (\tilde{k} \cdot \hat{u})_0, \\
& \partial_t(\tilde{k} \cdot \hat{u})|_{t=0} = -i\left(1 + |k|^2\right) \hat{\rho}_0 - \tilde{k} \cdot \hat{u}_0 - i|k|\Theta_0, \\
& \partial_{tt}(\tilde{k} \cdot \hat{u})|_{t=0} = i\left(1 + |k|^2\right) \hat{\rho}_0 - \frac{5}{3}|k|^2 \tilde{k} \cdot \hat{u}_0 + 2i|k|\Theta_0.
\end{aligned}
\]

After tenuous computation, one has

\[
(4.33) \qquad \tilde{k} \cdot \hat{u}(t, k) = c_7(k)e^{\sigma t} + e^{\beta t} (c_8(k) \cos \omega t + c_9(k) \sin \omega t),
\]

with

\[
[c_7, c_8, c_9]^T \begin{bmatrix}
\sigma^2 + 2\sigma + \frac{2}{3} |k|^2 \\
\sigma^2 - \frac{2}{3}|k|^2 \\
-\left(1 + \sigma\right)|k|^{-1}\left(1 + |k|^2\right) i
\end{bmatrix}
\begin{bmatrix}
\hat{\rho}_0 \\
\tilde{k} \cdot \hat{u}_0 \\
\Theta_0
\end{bmatrix} = 0.
\]
Next, for \((t, k) \in (0, \infty) \times \mathbb{R}^3\), let us solve
\[
\begin{aligned}
M_1(t, k) &= -\tilde{k} \times (\tilde{k} \times \hat{u}(t, k)), \\
M_2(t, k) &= -\tilde{k} \times (\tilde{k} \times \hat{E}(t, k)), \\
M_3(t, k) &= -\tilde{k} \times (\tilde{k} \times \hat{B}(t, k)).
\end{aligned}
\]
Taking the curl for the second, fourth and fifth equations of the system \(\text{(4.1)}\), one has
\[
\begin{aligned}
\partial_t(\nabla \times u) + \nabla \times E + \nabla \times u &= 0, \\
\partial_t(\nabla \times E) - \nabla \times (\nabla \times B) - \nabla \times u &= 0, \\
\partial_t(\nabla \times B) + \nabla \times (\nabla \times E) &= 0.
\end{aligned}
\]
Taking Fourier transform in \(x\) for the previous system, it follows that
\[
\begin{aligned}
\partial_t M_1 &= -M_1 - M_2, \\
\partial_t M_2 &= M_1 + ik \times M_3, \\
\partial_t M_3 &= -ik \times M_2.
\end{aligned}
\]
Initial data is given as
\[
[M_1, M_2, M_3]|_{t=0} = [M_{1,0}, M_{2,0}, M_{3,0}].
\]
Here,
\[
M_{1,0} = -\tilde{k} \times (\tilde{k} \times \hat{u}_0), \quad M_{2,0} = -\tilde{k} \times (\tilde{k} \times \hat{E}_0), \quad M_{3,0} = -\tilde{k} \times (\tilde{k} \times \hat{B}_0).
\]
It straightforward to get
\[
\partial_{tt} M_2 + \partial_t M_2 + \left(1 + |k|^2\right) \partial_t M_2 + |k|^2 M_2 = 0,
\]
with initial data
\[
\begin{aligned}
M_2|_{t=0} &= M_{2,0}, \\
\partial_t M_2|_{t=0} &= M_{1,0} + ik \times M_{3,0}, \\
\partial_{tt} M_2|_{t=0} &= -M_{1,0} - (1 + |k|^2) M_{2,0}.
\end{aligned}
\]
The characteristic equation of \(\text{(4.35)}\) is
\[
F_*(\mathcal{X}) := \mathcal{X}^3 + \mathcal{X}^2 + \left(1 + |k|^2\right) \mathcal{X} + |k|^2 = 0.
\]
For the roots of the previous characteristic equation and their properties, we have

**Lemma 4.2.** Suppose \(|k| \neq 0\). Then, \(F_*(\mathcal{X}) = 0\), \(\mathcal{X} \in \mathbb{C}\) has a real root \(\sigma_* = \sigma_*(|k|) \in (-1, 0)\) and two conjugate complex roots \(\mathcal{X}_\pm = \beta_* \pm i \omega_*\) with \(\beta_* = \beta_*(|k|) \in (-\frac{1}{2}, 0)\) and \(\omega_* = \omega_*(|k|) \in (\sqrt{\frac{3}{2}}, +\infty)\) which satisfy
\[
\beta_* = -\frac{1}{2} - \frac{\sigma_*}{2}, \quad \omega_* = \frac{1}{2} \sqrt{3\sigma_*^2 + 2\sigma_* + 3 + 4|k|^2}.
\]
\(\sigma_*, \beta_*, \omega_*\) are smooth in \(|k| > 0\), and \(\sigma_*(|k|)\) is strictly decreasing over \(|k| > 0\), with
\[
\lim_{|k| \to 0} \sigma_*(|k|) = 0, \quad \lim_{|k| \to \infty} \sigma_*(|k|) = -1.
\]
Furthermore, the asymptotic behavior as follows hold true:
\[
\sigma_*(|k|) = -O(1)|k|^2, \quad \beta_*(|k|) = -\frac{1}{2} + O(1)|k|^2, \quad \omega_*(|k|) = \frac{\sqrt{3}}{2} + O(1)|k|
\]
whenever \(|k| \leq 1\) is sufficiently small, and
\[
\sigma_*(|k|) = -1 + O(1)|k|^{-2}, \quad \beta_*(|k|) = -O(1)|k|^{-2}, \quad \omega_*(|k|) = O(1)|k|
\]
whenever $|k| \geq 1$ is sufficiently large.

Similarly as before, we obtain

$$M_1(t,k) = -\frac{c_{10}(k)}{1 + \sigma} \rho e^{\sigma t} - \frac{c_{11}(k)}{(1 + \beta)^2 + \omega^2} e^{\beta t} [(1 + \beta) \cos \omega t + \omega \sin \omega t]$$ (4.38)

$$- \frac{c_{12}(k)}{(1 + \beta)^2 + \omega^2} e^{\beta t} [(1 + \beta) \sin \omega t - \omega \cos \omega t],$$ (4.39)

and

$$M_2(t,k) = c_{10}(k) e^{\sigma t} + e^{\beta t} [c_{11}(k) \cos \omega t + c_{12}(k) \sin \omega t],$$ (4.40)

with

$$[c_{10}, c_{11}, c_{12}]^T = \frac{1}{3\sigma^2 + 2\sigma + 1 + |k|^2} \left[ \begin{array}{c} \sigma I_3 \\ -\sigma I_3 \\ \frac{\sigma^2 + 3\sigma + 1 + |k|^2}{\omega} I_3 \\ \frac{\sigma^2 + 3\sigma + 1 + |k|^2}{2\omega} I_3 \\ \frac{\sigma^2 + 3\sigma + 1 + |k|^2}{\omega} I_3 \\ \frac{\sigma^2 + 3\sigma + 1 + |k|^2}{2\omega} I_3 \\ (\sigma + 1)i k x \\ -(\sigma + 1)i k x \\ \frac{3\sigma + 3\sigma + 1 + |k|^2}{\omega} I_3 \\ \frac{3\sigma + 3\sigma + 1 + |k|^2}{2\omega} I_3 \\ \frac{3\sigma + 3\sigma + 1 + |k|^2}{\omega} I_3 \\ \frac{3\sigma + 3\sigma + 1 + |k|^2}{2\omega} I_3 \end{array} \right] \left[ \begin{array}{c} M_{1,0} \\ M_{1,0} \\ M_{1,0} \\ M_{1,0} \\ M_{1,0} \\ M_{1,0} \\ M_{1,0} \\ M_{1,0} \\ M_{1,0} \\ M_{1,0} \\ M_{1,0} \\ M_{1,0} \end{array} \right].$$ (4.41)

Now, one can obtain the explicit representation of $\dot{U} = [\rho, \dot{u}, \dot{\Theta}, \dot{E}, \dot{B}]$ in the following from the previous computations.

**Theorem 4.3.** Suppose $U = [\rho, u, \Theta, E, B]$ to be the solution of the initial problem (1.1)- (1.2) on the linearized homogeneous equations with initial data $U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0]$ which satisfies (1.3). For $(t,k) \in (0,\infty) \times \mathbb{R}^3$ with $|k| \neq 0$, we have the decomposition

$$\left[ \begin{array}{c} \dot{\rho}(t,k) \\ \dot{u}(t,k) \\ \dot{\Theta}(t,k) \\ \dot{E}(t,k) \\ \dot{B}(t,k) \end{array} \right] = \left[ \begin{array}{c} \rho(t,k) \\ u(t,k) \\ \Theta(t,k) \\ E(t,k) \\ B(t,k) \end{array} \right] + \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right],$$ (4.42)

here $\dot{u}_\parallel, \dot{u}_\perp$ are defined as

$$\dot{u}_\parallel = \vec{k} \cdot \dot{u}, \quad \dot{u}_\perp = -\vec{k} \times (\vec{k} \times \dot{u}) = (I_3 - \vec{k} \otimes \vec{k}) \dot{u},$$

and likewise for $\dot{E}_\parallel, \dot{E}_\perp$ and $\dot{B}_\perp$. Denote

$$\left[ \begin{array}{c} M_1(t,k) \\ M_2(t,k) \\ M_3(t,k) \end{array} \right] := \left[ \begin{array}{c} \dot{u}_\parallel(t,k) \\ \dot{E}_\parallel(t,k) \\ \dot{B}_\parallel(t,k) \end{array} \right], \quad \left[ \begin{array}{c} M_{1,0}(k) \\ M_{2,0}(k) \\ M_{3,0}(k) \end{array} \right] := \left[ \begin{array}{c} \dot{u}_{0,\parallel}(t,k) \\ \dot{E}_{0,\parallel}(t,k) \\ \dot{B}_{0,\parallel}(t,k) \end{array} \right].$$ (4.43)

Then, there exit matrices $G_{8 \times 8}(t,k)$ and $G_{9 \times 9}(t,k)$ such that

$$\left[ \begin{array}{c} \rho(t,k) \\ u(t,k) \\ \Theta(t,k) \\ E(t,k) \end{array} \right] = G_{8 \times 8}(t,k) \left[ \begin{array}{c} \rho_0(k) \\ u_{0,\parallel}(k) \\ \Theta_0(k) \\ E_{0,\parallel}(k) \end{array} \right].$$ (4.44)
and

\[
\begin{bmatrix}
M_1(t, k) \\
M_2(t, k) \\
M_3(t, k)
\end{bmatrix} = G_{9 \times 9}^{II}(t, k) \begin{bmatrix}
M_{1,0}(t) \\
M_{2,0}(t) \\
M_{3,0}(t)
\end{bmatrix},
\]

where \(G_{8 \times 8}^{I}\) is explicitly determined by representations (4.23), (4.33), (4.29), (4.24) for \(\hat{\rho}(t, k), v_i(t, k), \hat{\Theta}(t, k), E_i(t, k)\) with \(c_i(k)\), \(1 \leq i \leq 9\) defined by (4.26), (4.30), (4.34) in terms of \(\hat{\rho}_0(k), \hat{u}_{i,0}(k), \hat{\Theta}_0(k), \hat{E}_{i,0}(k)\) \((\hat{k} | k|^{-1} \hat{\rho}_0)\); and \(G_{9 \times 9}^{II}\) is determined by the representations (4.38), (4.39), (4.40) for \(M_1(t, k), M_2(t, k), M_3(t, k)\) with \(c_{10}(k), c_{11}(k)\) and \(c_{12}(k)\) defined by (4.41) in terms of \(M_{1,0}(k), M_{2,0}(k), M_{3,0}(k)\).

4.4 Refined \(L^p - L^q\) time-decay property. We utilize Theorem 4.3 to acquire some refined \(L^p - L^q\) time-decay property for every component of the solution \(U = [\rho, u, \Theta, E, B]\) in this subsection. For this purpose, we first search the subtle time-frequency pointwise estimates on \(\hat{U} = [\hat{\rho}, \hat{u}, \hat{\Theta}, \hat{E}, \hat{B}]\) as follows

**Lemma 4.3.** Suppose \(U = [\rho, u, \Theta, E, B]\) to be the solution of the linearized homogeneous equations (4.11) with initial data \(U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0]\) which satisfy (4.3). Then, there exist constants \(\gamma > 0, C > 0\) such that for \((t, k) \in (0, \infty) \times \mathbb{R}^3\),

\[
|\hat{\rho}(t, k)| \leq C e^{-\frac{\gamma}{2} t} |\hat{\rho}_0, \hat{u}_0, \hat{\Theta}_0|,
\]

\[
|\hat{u}(t, k)| \leq C e^{-\frac{\gamma}{2} t} \left| \hat{\rho}_0(t, k), \hat{u}_0(t, k), \hat{\Theta}_0(t, k), \hat{E}_0(t, k) \right| + C \left| \hat{u}_0(t, k), \hat{E}_0(t, k), \hat{B}_0(t, k) \right| \cdot \begin{cases}
 e^{-\gamma t} + |k| e^{-\gamma |k|^2 t} & \text{if } |k| \leq 1, \\
 e^{-\gamma t} + \frac{1}{|k|} e^{-\gamma |k|^2 t} & \text{if } |k| > 1,
\end{cases}
\]

\[
|\hat{\Theta}(t, k)| \leq C e^{-\frac{\gamma}{2} t} \left| \hat{\rho}_0, \hat{u}_0, \hat{\Theta}_0 \right|,
\]

\[
|\hat{E}(t, k)| \leq C e^{-\frac{\gamma}{2} t} \left| \hat{u}_0(t, k), \hat{\Theta}_0(t, k), \hat{E}_0(t, k) \right| + C \left| \hat{u}_0(t, k), \hat{E}_0(t, k), \hat{B}_0(t, k) \right| \cdot \begin{cases}
 e^{-\gamma t} + |k| e^{-\gamma |k|^2 t} & \text{if } |k| \leq 1, \\
 |k|^{-2} e^{-\gamma t} + \frac{1}{|k|} e^{-\gamma |k|^2 t} & \text{if } |k| > 1,
\end{cases}
\]

and

\[
|\hat{B}(t, k)| \leq C \left| \hat{u}_0(t, k), \hat{E}_0(t, k), \hat{B}_0(t, k) \right| \cdot \begin{cases}
 |k| e^{-\gamma t} + |k| e^{-\gamma |k|^2 t} & \text{if } |k| \leq 1, \\
 |k|^{-1} e^{-\gamma t} + \frac{1}{|k|} e^{-\gamma |k|^2 t} & \text{if } |k| > 1,
\end{cases}
\]

**Proof.** Firstly, we search the upper bound of \(\hat{\rho}\) defined as (4.23). In fact, from Lemma 4.1 it is directly to check (4.26) to obtain

\[
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} = \begin{bmatrix}
O(1) |k|^2 & -O(1) |k|^3 i & -O(1) |k|^2 \\
O(1) & O(1) |k|^3 i & O(1) |k|^2 \\
O(1) & -O(1) |k| & -O(1) |k|^2
\end{bmatrix} \begin{bmatrix}
\hat{\rho}_0 \\
\hat{k} \cdot \hat{u}_0 \\
\hat{\Theta}_0
\end{bmatrix},
\]

as \(|k| \to 0\), and

\[
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} = \begin{bmatrix}
O(1) & -O(1) |k|^{-1} i & -O(1) \\
O(1) |k|^{-1} i & O(1) \\
O(1) |k|^{-1} & -O(1) i & O(1) |k|^{-1}
\end{bmatrix} \begin{bmatrix}
\hat{\rho}_0 \\
\hat{k} \cdot \hat{u}_0 \\
\hat{\Theta}_0
\end{bmatrix}.
\]
as $|k| \to \infty$. Then, after putting the previous computations into (4.26), one has
\[
\dot{\hat{\rho}}(t, k) = \left( O(1)|k|^2 \hat{\rho}_0 - O(1)|k|^3 i \tilde{k} \cdot \hat{u}_0 - O(1)|k|^2 \hat{\Theta}_0 \right) e^{\sigma t} \\
+ \left( O(1)\hat{\rho}_0 + O(1)|k|^3 i \tilde{k} \cdot \hat{u}_0 + O(1)|k|^2 \hat{\Theta}_0 \right) e^{\beta t} \cos \omega t \\
+ \left( O(1)\hat{\rho}_0 - O(1)|k| i \tilde{k} \cdot \hat{u}_0 - O(1)|k|^2 \hat{\Theta}_0 \right) e^{\beta t} \sin \omega t,
\]
as $|k| \to 0$, and
\[
\dot{\hat{\rho}}(t, k) = \left( O(1)\hat{\rho}_0 - O(1)|k|^{-1} i \tilde{k} \cdot \hat{u}_0 - O(1)\hat{\Theta}_0 \right) e^{\sigma t} \\
+ \left( O(1)\hat{\rho}_0 + O(1)|k|^{-1} i \tilde{k} \cdot \hat{u}_0 + O(1)\hat{\Theta}_0 \right) e^{\beta t} \cos \omega t \\
+ \left( O(1)|k|^{-1} \hat{\rho}_0 - O(1)|k|^{-1} \tilde{k} \cdot \hat{u}_0 + O(1)|k|^{-1} \hat{\Theta}_0 \right) e^{\beta t} \sin \omega t,
\]
as $|k| \to \infty$. Therefore, one can obtain (4.46). Similarly, one can get (4.48) and the first term on the right hand side of (4.49).

Next, we search the upper bound of $\tilde{u}_|(t, k)$ defined as (4.33). In fact, from Lemma 4.1, it is directly to check (4.34) to get

\[
\begin{pmatrix}
c_7 \\
c_8 \\
c_9
\end{pmatrix} = \begin{pmatrix}
O(1) & -O(1) & O(1)|k|i & -O(1)|k|^2 \\
-O(1) & O(1) & -O(1)|k|i & O(1)|k|^2 \\
O(1) & -O(1) & -O(1)i & O(1)\left(1 - |k|^2\right)
\end{pmatrix} \begin{pmatrix}
\hat{\rho}_0 \\
\tilde{k} \cdot \hat{u}_0 \\
\hat{\Theta}_0 \\
\tilde{k} \cdot \tilde{E}_0
\end{pmatrix}
\]
as $|k| \to 0$, and

\[
\begin{pmatrix}
c_7 \\
c_8 \\
c_9
\end{pmatrix} = \begin{pmatrix}
O(1)\left(1 - |k|^{-1}i\right) & -O(1) & O(1)|k|^{-1}i \\
O(1)\left(|k|^{-2} - |k|^{-1}i\right) & O(1) & -O(1)|k|^{-1}i \\
O(1)\left(|k|^{-1} - i\right) & -O(1)|k|^{-1} & -O(1)|k|^{-1}i
\end{pmatrix} \begin{pmatrix}
\hat{\rho}_0 \\
\tilde{k} \cdot \hat{u}_0 \\
\hat{\Theta}_0
\end{pmatrix}
\]
as $|k| \to \infty$. Therefore, after putting the previous computations into (4.33), one has
\[
\tilde{k} \cdot \hat{u}(t, k) = \left( O(1)\hat{\rho}_0 - O(1)\tilde{k} \cdot \hat{u}_0 + O(1)|k|i\hat{\Theta}_0 - O(1)|k|^2 i\tilde{k} \cdot \tilde{E}_0 \right) e^{\sigma t} \\
+ \left( -O(1)\hat{\rho}_0 + O(1)\tilde{k} \cdot \hat{u}_0 - O(1)|k|i\hat{\Theta}_0 + O(1)|k|^2 i\tilde{k} \cdot \tilde{E}_0 \right) e^{\beta t} \cos \omega t \\
+ \left( O(1)\hat{\rho}_0 - O(1)\tilde{k} \cdot \hat{u}_0 - O(1)i\hat{\Theta}_0 + O(1)\left(1 - |k|^2\right) \tilde{k} \cdot \tilde{E}_0 \right) e^{\beta t} \sin \omega t,
\]
as $|k| \to 0$, and
\[
\tilde{k} \cdot \hat{u}(t, k) = \left( O(1)\left(1 - |k|^{-1}i\right) \hat{\rho}_0 - O(1)\tilde{k} \cdot \hat{u}_0 - O(1)|k|^{-1}i\hat{\Theta}_0 \right) e^{\sigma t} \\
+ \left( O(1)\left(|k|^{-2} - |k|^{-1}i\right) \hat{\rho}_0 + O(1)\tilde{k} \cdot \hat{u}_0 - O(1)|k|^{-1}i\hat{\Theta}_0 \right) e^{\beta t} \cos \omega t \\
+ \left( O(1)\left(|k|^{-1} - i\right) \hat{\rho}_0 - O(1)|k|^{-1} \tilde{k} \cdot \hat{u}_0 - O(1)|k|^{-1}i\hat{\Theta}_0 \right) e^{\beta t} \sin \omega t,
\]
as $|k| \to \infty$. Then, from above computations, one obtain the first term on the right hand side of (4.47). Similarly, we get (4.50) and the second term on the right hand side of both (4.47) and (4.49). Now, we have finished the proof of Lemma 4.3.

With the help of Lemma 4.3, one can refine the time-decay property for the solution $U = [\rho, u, \Theta, E, B]$ obtained in Theorem 4.2 in the following.
Theorem 4.4. Let \( 1 \leq p, r \leq 2 \leq q \leq \infty, l \geq 0 \) and an integer \( m \geq 0 \). Assume \( U(t) = e^{tL}U_0 \) is the solution of the initial problem (4.1)-(4.2) with initial data \([\rho_0, u_0, \Theta_0, E_0, B_0]\) which satisfies (4.3). Then, \( U = [\rho, u, \Theta, E, B] \) satisfies

\[
\| \nabla^m \rho(t) \|_{L^q} \leq Ce^{-\frac{t}{2}} \left( \| [\rho_0, u_0, \Theta_0] \|_{L^p} + \| \nabla^m \left[ \frac{3}{4} \left( \frac{1}{2} - \frac{1}{q} \right) \right] \left[ [\rho_0, u_0, \Theta_0] \right] \|_{L^r} \right),
\]

(4.51)

\[
\| \nabla^m u(t) \|_{L^q} \leq Ce^{-\frac{t}{2}} \left( \| [\rho_0, \Theta_0] \|_{L^p} + \| \nabla^m \left[ \frac{3}{4} \left( \frac{1}{2} - \frac{1}{q} \right) \right] \left[ [\rho_0, \Theta_0] \right] \|_{L^r} \right) + C(1 + t)^{-\frac{3}{4} \left( \frac{1}{2} - \frac{1}{q} \right)} \left\| \nabla^m \left[ \frac{1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) \right] \left[ [u_0, E_0, B_0] \right] \right\|_{L^r},
\]

(4.52)

\[
\| \nabla^m \Theta(t) \|_{L^q} \leq Ce^{-\frac{t}{2}} \left( \| [\rho_0, u_0, \Theta_0] \|_{L^p} + \| \nabla^m \left[ \frac{3}{4} \left( \frac{1}{2} - \frac{1}{q} \right) \right] \left[ [\rho_0, u_0, \Theta_0] \right] \|_{L^r} \right),
\]

(4.53)

\[
\| \nabla^m E(t) \|_{L^q} \leq C(1 + t)^{-\frac{3}{4} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{m+1}{2}} \left\| [u_0, \Theta_0, E_0, B_0] \right\|_{L^p} + C(1 + t)^{-\frac{k}{2}} \left\| \nabla^m \left[ \frac{1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) \right] + [u_0, \Theta_0, E_0, B_0] \right\|_{L^r},
\]

(4.54)

\[
\| \nabla^m B(t) \|_{L^q} \leq C(1 + t)^{-\frac{3}{4} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{m}{2}} \left\| [u_0, E_0, B_0] \right\|_{L^p} + C(1 + t)^{-\frac{k}{2}} \left\| \nabla^m \left[ \frac{1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) \right] + [u_0, E_0, B_0] \right\|_{L^r},
\]

(4.55)

for \( t \geq 0 \), here \( C = C(p, q, r, l, m) \) and \( [1 + 3(\frac{1}{r} - \frac{1}{q})]_+ \) is defined as (4.16).

Proof. We only give the estimate for \( \Theta \). Take \( 1 \leq p, r \leq 2 \leq q \leq \infty \) and an integer \( m \geq 0 \), it follows from Hausdorff-Young inequality with \( \frac{1}{q} + \frac{1}{q'} = 1 \) and (4.18) that

\[
\| \nabla^m \Theta(t) \|_{L^q_x} \leq C \| k^m \Theta(t) \|_{L^q_x} \leq Ce^{-\frac{t}{2}} \left( \| k^m [\hat{\rho}_0, \hat{u}_0, \hat{\Theta}_0] \|_{L^q([k] \leq 1)} + \| k^m [\hat{\rho}_0, \hat{u}_0, \hat{\Theta}_0] \|_{L^q([k] \geq 1)} \right).
\]

(4.56)

Now let us estimate each term on the right hand side of (4.56). For the first term, fixing \( \varepsilon > 0 \) sufficiently small and using Holder inequality \( \frac{1}{q'} = \frac{1}{p'} + \frac{\varepsilon}{q'} \) with \( \frac{1}{p'} + \frac{\varepsilon}{q'} = 1 \), one has

\[
\left\| k^m [\hat{\rho}_0, \hat{u}_0, \hat{\Theta}_0] \right\|_{L^q([k] \leq 1)} = \left\| k^m \left[ \frac{\varepsilon}{p'q'} \left( 3 - \varepsilon \right) \right] [\hat{\rho}_0, \hat{u}_0, \hat{\Theta}_0] \right\|_{L^q([k] \leq 1)} \leq \left\| k^m \left[ \frac{\varepsilon}{p'q'} \left( 3 - \varepsilon \right) \right] [\hat{\rho}_0, \hat{u}_0, \hat{\Theta}_0] \right\|_{L^q([k] \leq 1)} \leq C \left\| \left[ \frac{\varepsilon}{p'q'} \left( 3 - \varepsilon \right) \right] [\hat{\rho}_0, \hat{u}_0, \hat{\Theta}_0] \right\|_{L^q([k] \leq 1)} \leq C \left\| [\hat{\rho}_0, \hat{u}_0, \hat{\Theta}_0] \right\|_{L^p([k] \leq 1)} \leq C \| [\rho_0, u_0, \Theta_0] \|_{L^p}.
\]
For the second term, using H"older inequality $\frac{1}{q} = \frac{1}{p} + \frac{r'}{rq'}$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\varepsilon > 0$ small enough, we have
\[
\left\| |k|^m \left[ \hat{\rho}_0, \hat{u}_0, \hat{\Theta}_0 \right] \right\|_{L^q(|k| \geq 1)} \leq C \left\| |k|^{-\frac{r'}{rq'}}(3+\varepsilon) \left| |k|^m + \frac{r'}{rq'}(3+\varepsilon) \left[ \hat{\rho}_0, \hat{u}_0, \hat{\Theta}_0 \right] \right\|_{L^q(|k| \geq 1)}
\]
which together the above estimate implies \((4.53)\). Similarly as before, we can get \((4.51)\), \((4.52)\), \((4.54)\) and \((4.55)\). Then, we finished the proof of Theorem 4.4 \(\square\)

Based on Theorem 4.4, we list some particular cases as follows for later use.

**Corollary 4.2.** Assume $U(t) = e^{tL}U_0$ is the solution of the initial problem \((4.1)-(4.2)\) with initial data $U = [\rho_0, u_0, \Theta_0, E_0, B_0]$ which satisfies \((4.3)\). Then, $U = [\rho, u, \Theta, E, B]$ satisfies
\[
\begin{align*}
\|\rho(t)\| &\leq C e^{-\frac{t}{2}} \|[\rho_0, u_0, \Theta_0]\|, \\
\|u(t)\| &\leq C e^{-\frac{t}{2}} \|[\rho_0, \Theta_0]\| + C(1 + t)^{-\frac{3}{2}} \|[u_0, E_0, B_0]\|_{L^1 \cap H^2}, \\
\|\Theta(t)\| &\leq C e^{-\frac{t}{2}} \|[\rho_0, u_0, \Theta_0]\|, \\
\|E(t)\| &\leq C(1 + t)^{-\frac{3}{2}} \|[u_0, \Theta_0, E_0, B_0]\|_{L^1 \cap H^3}, \\
\|B(t)\| &\leq C(1 + t)^{-\frac{3}{2}} \|[u_0, E_0, B_0]\|_{L^1 \cap H^2},
\end{align*}
\]
with
\[
\begin{align*}
\|\rho(t)\|_{L^\infty} &\leq C e^{-\frac{t}{2}} \|[\rho_0, u_0, \Theta_0]\|_{L^2 \cap H^2}, \\
\|u(t)\|_{L^\infty} &\leq C e^{-\frac{t}{2}} \|[\rho_0, \Theta_0]\|_{L^1 \cap H^2} + C(1 + t)^{-2} \|[u_0, E_0, B_0]\|_{L^1 \cap H^3}, \\
\|\Theta(t)\|_{L^\infty} &\leq C e^{-\frac{t}{2}} \|[\rho_0, u_0, \Theta_0]\|_{L^2 \cap H^2}, \\
\|E(t)\|_{L^\infty} &\leq C(1 + t)^{-2} \|[u_0, \Theta_0, E_0, B_0]\|_{L^1 \cap H^6}, \\
\|B(t)\|_{L^\infty} &\leq C(1 + t)^{-\frac{3}{2}} \|[u_0, E_0, B_0]\|_{L^1 \cap H^5},
\end{align*}
\]
and
\[
\begin{align*}
\|\nabla B(t)\| &\leq C(1 + t)^{-\frac{5}{2}} \|[u_0, E_0, B_0]\|_{L^1 \cap H^4}, \\
\|\nabla^s [E(t), B(t)]\| &\leq C(1 + t)^{-\frac{3}{2}} \|[u_0, \Theta_0, E_0, B_0]\|_{L^2 \cap H^{s+3}},
\end{align*}
\]
holds for any $t \geq 0$.

5. **Time-decay rates for system \((2.2)\)**

In this section, let us give the proof of Proposition 2.2 and Proposition 2.3. For the solution $U = [\rho, u, \Theta, E, B]$ of the nonlinear initial problem \((2.2)-(2.3)\), we search the time-decay rates.
of the energy \(\|U(t)\|_{L^2}^2\) and the high-order energy \(\|\nabla U(t)\|_{H^{-1}}^2\) in the first two subsections. In the last subsection, the time-decay rates in \(L^q\) with \(2 \leq q \leq \infty\) for every component \(\rho, u, \Theta, E\) and \(B\) of the solution \(U\) are presented.

In the following, since we shall utilize the linear \(L^p - L^q\) time-decay property of the homogeneous equations (4.1) investigated in the section above to the nonlinear equations (2.2), we rewrite (2.2) in the following form:

\[
\begin{align*}
\partial_t \rho + \nabla \cdot u &= g_1, \\
\partial_t u + \nabla \rho + \nabla \Theta + E + u &= g_2, \\
\partial_t \Theta + \frac{2}{3} \nabla \cdot u + \Theta &= g_3, \\
\partial_t E - \nabla \times B - u &= g_4, \\
\partial_t B + \nabla \times E &= 0, \\
\nabla \cdot E &= -\rho, \quad \nabla \cdot B = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3,
\end{align*}
\]

with

\[
\begin{align*}
g_1 &= -\nabla \cdot (\rho u), \\
g_2 &= -(u \cdot \nabla) u - \left(\frac{1 + \Theta}{1 + \rho} - 1\right) \nabla \rho - u \times B, \\
g_3 &= -u \cdot \nabla \Theta - \frac{2}{3} \Theta \nabla \cdot u + \frac{1}{3} |u|^2, \\
g_4 &= \rho u.
\end{align*}
\]

Then, by the Duhamel principle, the solution \(U\) can be formally written as

\[
U(t) = e^{tL}U_0 + \int_0^t e^{(t-y)L}[g_1(y), g_2(y), g_3(y), g_4(y), 0]dy,
\]

here, \(e^{tL}\) is defined as (4.15).

**Remark 5.1.** In the time integral term of (5.3), since \([g_1(y), g_2(y), g_3(y), g_4(y), 0]\) satisfies compatible condition (4.3), it makes sense that \(e^{(t-y)L}\) acts on \([g_1(y), g_2(y), g_3(y), g_4(y), 0]\) for \(0 \leq y \leq t\).

### 5.1. Decay rate for the energy functional

In this subsection, let us search the time-decay estimate (2.12) in Proposition 2.2 for the energy \(\|U(t)\|_{L^2}^2\). We begin with the following Lemma which can be seen straightforward from the proof of Theorem 3.1

**Lemma 5.1.** Suppose \(U = [\rho, u, \Theta, E, B]\) to be the solution of the initial problem (2.2) - (2.3) with \(U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0]\) which satisfies (2.3) obtained by Proposition 2.1. If \(E_s(U_0)\) is small enough, then for any \(t \geq 0\), it holds that

\[
\frac{d}{dt} E_s(U(t)) + \gamma D_s(U(t)) \leq 0.
\]

Based on Lemma 5.1, one can check that

\[
(1 + t)^{1/2} E_s(U(t)) + \gamma \int_0^t (1 + y)^{1/2} D_s(U(y))dy
\]

\[
\leq E_s(U_0) + l \int_0^t (1 + y)^{1/2} E_s(U(y))dy
\]

\[
\leq E_s(U_0) + Cl \int_0^t (1 + y)^{1/2} \left(\|B(y)\|^2 + D_{s+1}(U(y))\right)dy,
\]

where
where we have used $E_s(U(t)) \leq \|B(t)\|^2 + D_{s+1}(U(t))$. Using (5.4) again, one has
\[
E_{s+2}(U(t)) + \gamma \int_0^t D_{s+2}(U(s))dy \leq E_{s+2}(U_0)
\]
and
\[
(1 + t)^{l-1}E_{s+1}(U(t)) + \gamma \int_0^t (1 + y)^{l-1}D_{s+1}(U(y))dy
\]
\[
\leq E_{s+1}(U_0) + C(1 - l) \int_0^t (1 + y)^{l-2} \left( \|B(y)\|^2 + D_{s+2}(U(y)) \right)dy.
\]
Therefore, by iterating the previous estimates, we have
\[
(1 + t)^l E_s(U(t)) + \gamma \int_0^t (1 + y)^l D_s(U(y))dy
\]
\[
\leq C E_{s+2}(U_0) + C \int_0^t (1 + y)^{l-1} \|B(y)\|^2 dy
\]
for $1 < l < 2$.

Now, let us estimate the integral term on the right hand side of (5.5). Applying the last linear estimate on $B$ in (5.5) to (5.3), one has
\[
\|B(t)\| \leq C(1 + t)^{-\frac{3}{2}} \|[u_0, E_0, B_0]\|_{L^1 \cap H^2} + C \int_0^t (1 + t - y)^{-\frac{3}{2}} \|[g_2(y), g_4(y)]\|_{L^1 \cap H^2} dy.
\]
It is directly to check that for any $0 \leq y \leq t$,
\[
\|[g_2(y), g_4(y)]\|_{L^1 \cap H^2} \leq C \|U(y)\|^2 \leq C E_s(U(y)) \leq C(1 + t)^{-\frac{3}{2}} E_{s,\infty}(U(t)),
\]
where $E_{s,\infty}(U(t)) := \sup_{0 \leq y \leq t} (1 + y)^{\frac{3}{2}} E_s(U(y))$. Plugging this into (5.6) implies
\[
\|B(t)\| \leq C(1 + t)^{-\frac{3}{2}} \left( \|[u_0, E_0, B_0]\|_{L^1 \cap H^2} + E_{s,\infty}(U(t)) \right).
\]

Next, we prove the uniform-in-time bound of $E_{s,\infty}(U(t))$ which implies the decay rates of the energy functional $E_s(U(t))$ and thus $\|U(t)\|^2$. In fact, by choosing $l = \frac{3}{2} + \varepsilon$ in (5.5) with $\varepsilon > 0$ sufficiently small and using (5.7), it follows that
\[
(1 + t)^{\frac{3}{2} + \varepsilon} E_s(U(t)) + \gamma \int_0^t (1 + y)^{\frac{3}{2} + \varepsilon} D_s(U(y)) dy
\]
\[
\leq C E_{s+2}(U_0) + C(1 + t)^{\varepsilon} \left( \|[u_0, E_0, B_0]\|_{L^1 \cap H^2}^2 + [E_{s,\infty}(U(t))]^2 \right),
\]
which yields
\[
(1 + t)^{\frac{3}{2}} E_s(U(t)) \leq C \left( E_{s+2}(U_0) + \|[u_0, E_0, B_0]\|_{L^1}^2 + [E_{s,\infty}(U(t))]^2 \right),
\]
and thus
\[
E_{s,\infty}(U(t)) \leq C \left( \epsilon_{s+2}(U_0)^2 + [E_{s,\infty}(U(t))]^2 \right),
\]
since $\epsilon_{s+2}(U_0) > 0$ is small enough, it holds that $E_{s,\infty}(U(t)) \leq C\epsilon_{s+2}(U_0)^2$ for any $t \geq 0$, which gives $\|U(t)\|_s \leq C E_s(U(t))^{\frac{1}{2}} \leq C\epsilon_{s+2}(U_0)(1 + t)^{-\frac{3}{4}}$, that is (2.12). □

5.2. Decay rate for the high-order energy functional. In this subsection, we seek the decay estimate of the high-order energy functional $\|\nabla U(t)\|^2_{s-1}$, that is (2.13) in Proposition 2.2. For that, we are reduced to establish the time-decay estimates on $\|\nabla B\|$ and $\|\nabla^s[E, B]\|$ with the help of the following Lemma.
Lemma 5.2. Suppose $U = [\rho, u, \Theta, E, B]$ to be the solution of the initial problem (2.2)–(2.8) with $U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0]$ which satisfies (2.4) obtained in Proposition 2.1. If $E_s(U_0)$ is small enough, then there exist the high-order energy functional $E_s^h(\cdot)$ and the high-order dissipation rate $D_s^h(\cdot)$ such that

$$
\frac{d}{dt} E_s^h(U(t)) + \gamma D_s^h(U(t)) \leq C \|\nabla B\|^2,
$$

holds for any $t \geq 0$.

Proof. The proof is very similar to the proof of Theorem 3.1. In fact, after letting $|\alpha| \geq 1$, then corresponding to (3.3), (3.14), (3.26) and (3.27), it can also be tested that

$$
\frac{d}{dt} \sum_{1 \leq |\alpha| \leq s} \langle A_0^I(W_t) \partial^\alpha W_t, \partial^\alpha W_t \rangle + \|\partial^\alpha W_{tt}\|^2 + \|\nabla u\|^2_{s-1} + \frac{1}{3} \|
abla \Theta\|^2_{s-1} \leq C \|W\|_{s} \|W_t\|_{s},
$$

$$
\frac{d}{dt} \sum_{1 \leq |\alpha| \leq s-1} \langle \frac{1}{2(1+\rho)} \partial^\alpha \rho - \partial^\alpha \nabla \cdot u, \partial^\alpha \rho \rangle + \gamma \|\nabla \rho\|^2_{s-1} \leq C \|W\|_{s} \|W_t\|_{s}^2 + \|
abla^2 u\|^2_{s-2} + \|
abla \Theta\|^2_{s-1},
$$

$$
\frac{d}{dt} \sum_{1 \leq |\alpha| \leq s-1} \langle \partial^\alpha u, \partial^\alpha E \rangle + \gamma \|\nabla^2 E\|^2_{s-2} \leq C \|\nabla u\|^2_{s-1} + \|\nabla^2 \Theta\|^2_{s-2} + C \|U\|_{s} \|U_t\|_{s} + C \|
abla^2 u\|_{s-2} \|\nabla B\|_{s-2}
$$

and

$$
\frac{d}{dt} \sum_{1 \leq |\alpha| \leq s-2} \langle -\nabla \times \partial^\alpha E, \partial^\alpha B \rangle + \gamma \|\nabla^2 B\|^2_{s-3} \leq C \||\nabla u, \nabla^2 E\|_{s-3}^2 + C \|U\|_{s} \|\nabla [\rho, u]\|_{s-1}.
$$

Now, in the similar way as in Step 5 of Theorem 3.1. We define the high-order energy functional as

$$
E_s^h(U(t)) = \sum_{1 \leq |\alpha| \leq s} \langle A_0^I(W_t) \partial^\alpha W_t, \partial^\alpha W_t \rangle + \|\partial^\alpha W_{tt}\|^2
$$

$$
+ K_1 \sum_{1 \leq |\alpha| \leq s-1} \langle \frac{1}{2(1+\rho)} \partial^\alpha \rho - \partial^\alpha \nabla \cdot u, \partial^\alpha \rho \rangle
$$

$$
+ K_2 \sum_{1 \leq |\alpha| \leq s-1} \langle \partial^\alpha u, \partial^\alpha E \rangle + K_3 \sum_{1 \leq |\alpha| \leq s-2} \langle -\nabla \times \partial^\alpha E, \partial^\alpha B \rangle.
$$

Similarly, one can take $0 < K_3 \ll K_2 \ll K_1 \ll 1$ be sufficiently small with $K_2^3 \ll K_3$, that is, $E_s^h(\cdot)$ is really a high-order energy functional which satisfies (2.8), and moreover, the summation of the four above estimates with coefficients corresponding to (5.9) implies (5.8) with $D_s^h(\cdot)$ defined as (2.8). We have finished the proof of Lemma 5.2. \hfill \Box

Based on Lemma 5.2, one can check that

$$
\frac{d}{dt} E_s^h(U(t)) + \gamma E_s^h(U(t)) \leq C \|\nabla B\|^2 + \|\nabla^s [E, B]\|^2,
$$

which implies

$$
E_s^h(U(t)) \leq E_s^h(U_0) e^{-\gamma t} + C \int_0^t e^{-\gamma(t-y)} \left(\|\nabla B(y)\|^2 + \|\nabla^s [E(y), B(y)]\|^2\right) dy.
$$
Now, let us estimate the time integral term on the right hand side of the previous inequality. Firstly, we have

**Lemma 5.3.** Suppose \( U = [\rho, u, \Theta, E, B] \) to be the solution of the initial problem (2.2)-(2.3) with \( U_0 = [\rho_0, u_0, \Theta_0, E_0, B_0] \) which satisfies (2.4) obtained in Proposition 2.1. If \( \epsilon_{N+6}(U_0) \) is small enough, then

\[
\|\nabla B(t)\|^2 + \|\nabla^s [E(t), B(t)]\|^2 \leq C\epsilon_{s+6}(U_0)^2 (1 + t)^{-\frac{5}{2}},
\]

for any \( t \geq 0 \).

**Proof.** The proof is similar to that of the isentropic case in [4]. Apply the first linear estimate on \( \nabla B \) in (4.59) to (5.3) so that

\[
\|\nabla B(t)\| \leq C(1 + t)^{-\frac{5}{2}} \|[u_0, E_0, B_0]\|_{L^1 \cap H^4} + C \int_0^t (1 + t - y)^{-\frac{5}{2}} \|[g_2(y), g_4(y)]\|_{L^1 \cap H^4} dy
\]

\[
\leq C(1 + t)^{-\frac{5}{2}} \|[u_0, E_0, B_0]\|_{L^1 \cap H^4} + C \int_0^t (1 + t - y)^{-\frac{5}{2}} \|U(y)\|_{L^4}^2 dy
\]

\[
\leq C(1 + t)^{-\frac{5}{2}} \|[u_0, E_0, B_0]\|_{L^1 \cap H^4} + C \int_0^t (1 + t - y)^{-\frac{5}{2}} \epsilon_{s+6}(U_0)^2 (1 + y)^{-\frac{5}{2}} dy
\]

\[
\leq C\epsilon_{s+6}(U_0)(1 + t)^{-\frac{5}{2}}.
\]

Similarly, applying the second linear estimate on \( \nabla^s [E(t), B(t)] \) in (4.59) to (5.3), one has

\[
\|\nabla^s [E(t), B(t)]\| \leq C(1 + t)^{-\frac{5}{2}} \|[u_0, \Theta_0, E_0, B_0]\|_{L^2 \cap H^{s+3}} + C \int_0^t (1 + t - y)^{-\frac{5}{2}} \|[g_2(y), g_3(y), g_4(y)]\|_{L^2 \cap H^{s+3}} dy
\]

\[
\leq C(1 + t)^{-\frac{5}{2}} \|[u_0, \Theta_0, E_0, B_0]\|_{L^2 \cap H^{s+3}} + C \int_0^t (1 + t - y)^{-\frac{5}{2}} \|U(y)\|_{L^4}^2 dy
\]

\[
\leq C(1 + t)^{-\frac{5}{2}} \|[u_0, \Theta_0, E_0, B_0]\|_{L^2 \cap H^{s+3}} + C \int_0^t (1 + t - y)^{-\frac{5}{2}} \epsilon_{s+6}(U_0)^2 (1 + y)^{-\frac{5}{2}} dy
\]

\[
\leq C\epsilon_{s+6}(U_0)(1 + t)^{-\frac{5}{2}}.
\]

Where we have used (2.12) and the smallness of \( \epsilon_{s+6}(U_0) \). Now, we have finished the proof of Lemma 5.3. \( \square \)

Then, by putting (5.11) into (5.10), we have

\[
| \mathcal{E}_N^b(U(t)) | \leq | \mathcal{E}_N^b(U_0) | e^{-\lambda t} + C\epsilon_{N+6}(U_0)^2 (1 + t)^{-\frac{5}{2}}.
\]

Since \( \mathcal{E}_N^b(U(t)) \sim \|\nabla U(t)\|_{H^s-1}^2 \) holds true for any \( t \geq 0 \), (2.16) follows. Therefore, we have finished the proof of Proposition 2.2.

### 5.3 Decay rate in \( L^q \)

In this subsection, we will search the decay rates of solutions \( U = [\rho, u, \Theta, E, B] \) in \( L^q \) with \( 2 \leq q \leq +\infty \) to the initial problem (2.2)-(2.3) by proving Proposition 2.3. Throughout this subsection, we always assume that \( \epsilon_{13}(U_0) > 0 \) is small enough. First, for \( s \geq 4 \), Proposition 2.2 shows that if \( \epsilon_{s+2}(U_0) \) is small enough,

\[
\|U(t)\|_{s} \leq C\epsilon_{s+2}(U_0)(1 + t)^{-\frac{5}{4}},
\]

and if \( \epsilon_{s+6}(U_0) \) is small enough,

\[
\|\nabla U(t)\|_{s-1} \leq C\epsilon_{s+6}(U_0)(1 + t)^{-\frac{5}{4}}.
\]
Now, let us establish the estimates on $B$, $[u, E]$ and $[\rho, \Theta]$ in the following.

**Estimate on $\|B\|_{L^q}$**. For $L^2$ rate, it is straightforward from (5.12) to obtain

$$
\|B(t)\| \leq C \epsilon_6(U_0)(1 + t)^{-\frac{q}{2}}.
$$

For $L^\infty$ rate, by utilizing $L^\infty$ estimate on $B$ in (4.58) to (5.3), we have

$$
\|B(t)\|_{L^\infty} \leq C(1 + t)^{-\frac{q}{2}} \|[u_0, E_0, B_0]\|_{L^1 \cap \dot{H}^3} + C \int_0^t (1 + t - y)^{-\frac{q}{2}} \|[g_2(y), g_4(y)]\|_{L^1 \cap \dot{H}^5} dy.
$$

Because of (5.12),

$$
\|g_2(t), g_4(t)\|_{L^1 \cap \dot{H}^5} \leq C \|U(t)\|_6^2 \leq C \epsilon_6(U_0)^2 (1 + t)^{-\frac{3}{2}},
$$

one has

$$
\|B(t)\|_{L^\infty} \leq C \epsilon_6(U_0)(1 + t)^{-\frac{3}{2}}.
$$

Therefore, by $L^2 - L^\infty$ interpolation

$$(5.14) \quad \|B(t)\|_{L^q} \leq C \epsilon_6(U_0)(1 + t)^{-\frac{q}{2} + \frac{6}{q}},
$$

for $2 \leq q \leq \infty$.

**Estimate on $\|[u, E]\|_{L^q}$**. For $L^2$ rate, utilizing the $L^2$ estimate on $u$ and $E$ in (4.57) to (5.3), we have

$$
\|u(t)\| \leq C(1 + t)^{-\frac{q}{2}} \| [\rho_0, \Theta_0] \| + \|[u_0, E_0, B_0]\|_{L^1 \cap \dot{H}^2} + C \int_0^t (1 + t - y)^{-\frac{q}{2}} \|[g_1(y), g_3(y)]\| + \|[g_2(y), g_4(y)]\|_{L^1 \cap \dot{H}^2} dy
$$

and

$$
\|E(t)\| \leq C(1 + t)^{-\frac{q}{2}} \|[u_0, \Theta_0, E_0, B_0]\|_{L^1 \cap \dot{H}^3} + C \int_0^t (1 + t - y)^{-\frac{q}{2}} \|[g_2(y), g_3(y), g_4(y)]\|_{L^1 \cap \dot{H}^3} dy,
$$

Because of (5.12),

$$
\|[g_1(t), g_3(t)]\| + \|[g_2(t), g_3(t), g_4(t)]\|_{L^1 \cap \dot{H}^3} \leq C \|U(t)\|_4^2 \leq C \epsilon_6(U_0)^2 (1 + t)^{-\frac{3}{2}},
$$

it holds that

$$
\|[u(t), E(t)]\| \leq C \epsilon_6(U_0)(1 + t)^{-\frac{3}{2}}.
$$

For $L^\infty$ rate, utilizing the $L^\infty$ estimates on $u$ and $E$ in (4.58) to (5.3), we have

$$
\|u(t)\|_{L^\infty} \leq C(1 + t)^{-2} \| [\rho_0, \Theta_0] \|_{L^1 \cap \dot{H}^2} + \|[u_0, E_0, B_0]\|_{L^1 \cap \dot{H}^3} + C \int_0^t (1 + t - y)^{-2} \|[g_1(y), g_3(y)]\|_{L^1 \cap \dot{H}^2} + \|[g_2(y), g_4(y)]\|_{L^1 \cap \dot{H}^3} dy
$$

and

$$
\|E(t)\|_{L^\infty} \leq C(1 + t)^{-2} \|[u_0, \Theta_0, E_0, B_0]\|_{L^1 \cap \dot{H}^3} + C \int_0^t (1 + t - y)^{-2} \|[g_2(y), g_3(y), g_4(y)]\|_{L^1 \cap \dot{H}^3} dy.
$$

Since

$$
\|[g_1(t), g_3(t)]\|_{\dot{H}^2} + \|[g_2(t), g_3(t), g_4(t)]\|_{\dot{H}^3 \cap \dot{H}^6} \leq C \|\nabla U(t)\|_6^2 \leq \epsilon_{13}(U_0)^2 (1 + t)^{-\frac{5}{2}},
$$

and

$$
\|[g_1(t), g_2(t), g_3(t), g_4(t)]\|_{L^1} \leq C \|U(t)\| ([u(t)] + \|\nabla U\|) \leq \epsilon_{10}(U_0)^2 (1 + t)^{-2},
$$

then, it holds that

$$
\|[u(t), E(t)]\|_{L^\infty} \leq C \epsilon_{13}(U_0)^2 (1 + t)^{-2}.
$$
Then, by $L^2 - L^\infty$ interpolation
\begin{equation}
\|u(t), E(t)\|_{L^q} \leq C\epsilon_3(U_0)(1 + t)^{-2 + \frac{3}{2q}}.
\end{equation}
for $2 \leq q \leq \infty$.

Estimate on $\|\rho, \Theta\|_{L^q}$. For $L^2$ rate, utilizing the $L^2$ estimates on $\rho$ and $\Theta$ in (4.57) to (5.3), we have
\begin{equation}
\|\rho(t) - u(t), \Theta(t)\| \leq C\epsilon_3(U_0)(1 + t)^{-2 + \frac{3}{2q}},
\end{equation}
then (5.6) implies the slower decay estimate
\begin{equation}
\|\rho(t), \Theta(t)\| \leq C\epsilon_3(U_0)(1 + t)^{-2 + \frac{3}{2q}}.
\end{equation}
Furthermore, after estimating $\|g_1(t), g_2(t), g_3(t)\|$ and utilizing the previous slower decay estimate, one has
\begin{align*}
\|g_1(t), g_2(t), g_3(t)\| & \leq C\|\nabla U(t)\|_1^2 + \|u(t)\| \|B(t)\|_{L^\infty} + \|\nabla U(t)\|) \leq C\epsilon_10(U_0)^2(1 + t)^{-\frac{3}{2}},
\end{align*}
then (5.16) implies the slower decay estimate
\begin{equation}
\|\rho(t), \Theta(t)\| \leq C\epsilon_10(U_0)(1 + t)^{-\frac{3}{2}}.
\end{equation}
For $L^\infty$ rate, by utilizing the $L^\infty$ estimates on $\rho$ and $\Theta$ in (4.58) to (5.3),
\begin{equation}
\|\rho(t), \Theta(t)\|_{L^\infty} \leq C\epsilon_3(U_0)(1 + t)^{-\frac{3}{2}},
\end{equation}
It is straightforward to check
\begin{align*}
\|g_1(t), g_2(t), g_3(t)\|_{L^3 \cap H^2} & \leq C\|\nabla U(t)\|_1 (\|\rho(t), \Theta(t)\|) + \|u(t), B(t)\|_{L^\infty} + \|\nabla [\rho(t), u(t), \Theta(t)]\|_{L^\infty}) \leq C\epsilon_3(U_0)(1 + t)^{-\frac{3}{2}},
\end{align*}
which yields from (5.18) that
\begin{equation}
\|\rho(t), \Theta(t)\|_{L^\infty} \leq C\epsilon_3(U_0)(1 + t)^{-\frac{3}{2}}.
\end{equation}
Then, by $L^2 - L^\infty$ interpolation
\begin{equation}
\|\rho(t), \Theta(t)\|_{L^q} \leq C\epsilon_3(U_0)(1 + t)^{-\frac{3}{2}},
\end{equation}
for $2 \leq q \leq \infty$. Therefore, (5.20), (5.15) and (5.14) give (2.14), (2.15) and (2.16), respectively.

Now, we have finished the proof of Proposition 2.3. \hfill \Box

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Appendix A
In a medium, the Euler-Maxwell system including natural collision terms is written as the following (nonconservative) form (see [1 3 6 7]):

\[(A.1) \quad \partial_t n + \nabla \cdot (nu) = 0,\]
\[(A.2) \quad m [\partial_t (nu) + \nabla \cdot (nu \otimes u)] + k \nabla (n\theta) = -qn(E + u \times B) - \frac{mn\nu}{\tau_p},\]
\[(A.3) \quad \partial_t \theta + u \cdot \nabla \theta + \frac{2}{3} \theta \nabla \cdot u = \frac{k_0}{n} \nabla \cdot (n\nabla \theta) - \frac{2m|u|^2}{3k} \left( \frac{1}{2\tau_\omega} - \frac{1}{\tau_p} \right) - \frac{1}{\tau_\omega} (\theta - \theta_\ast),\]
\[(A.4) \quad \epsilon \partial_t E - \mu^{-1} \nabla \times B = \frac{q}{m} nu, \quad \epsilon \nabla \cdot E = b(x) - \frac{q}{m} n,\]
\[(A.5) \quad \partial_t B + \nabla \times E = 0, \quad \nabla \cdot B = 0,\]

for \((t, x) \in (0, +\infty) \times \mathbb{R}^3\). Here, \(\epsilon > 0, \mu > 0\) and \(m > 0\) are the permittivity of the medium, the permeability of the medium and the particle mass, respectively. In vacuum, \(\epsilon = \epsilon_0, \mu = \mu_0\) with \(c = (\epsilon_0\mu_0)^{-\frac{1}{2}}\) being the speed of light. \(q\) is the electronic charge, \(m\) is the effective electron mass, \(k\) is Boltzmann’s constant, \(\tau_p\) is the momentum relaxation time, \(\tau_\omega\) is the energy relaxation time, \(k_0\) is a constant multiplier (with the variable density) of heat conduction. The function \(\theta_\ast(x)\) is the ambient device temperature, and the function \(b(x)\) stands for the prescribed density of positive charged background ions (doping profile). In this paper, we assume \(q = m = k = \epsilon = \mu = \tau_p = \tau_\omega = \theta_\ast(x) = b(x) = 1\) and \(k_0 = 0\) for the sake of simplicity. It’s well known that (A.2) is equivalent to

\[(A.6) \quad n\partial_t u + n(u \cdot \nabla)u + n\nabla \theta + \theta \nabla n + nu = -n(E + u \times B),\]

then, we obtain the desired Euler-Maxwell system of the form (1.1).

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