Spectral Compressibility at the Metal-Insulator Transition of the Quantum Hall Effect

Rochus Klesse and Marcus Metzler
Institut für Theoretische Physik der Universität zu Köln
D-50937 Köln, Germany

Since the development of the classical Random Matrix Theory (RMT) by Wigner, Dyson, Mehta and others the statistics of energy levels of complex quantum systems have become the subject of research in many areas of physics, from the study of atomic nuclei to the investigation of disordered metals and quantum chaos [1]. The remarkable property the level statistics of these systems have in common is their universality, regardless of the microscopic details of the particular system.

In the following we will focus on the spectral properties of systems undergoing a metal-insulator transition. In this case at least two entirely different types of level statistics are involved. In the insulating phase the probability density of an eigenstate is almost completely localized within a comparatively small volume. As long as the spatial extension of any two states is small compared to their distance, they are independent in the same way as two states of two separate systems. Consequently, in the localized regime the energy levels are uncorrelated and therefore governed by Poisson statistics. For example, the probability to find two consecutive levels separated by an energy \( \epsilon = s \Delta \), \( \Delta \) being the average level spacing, is given by the Poisson-distribution \( P(s) = \exp(-s/s) \). The number variance \( \Sigma_2(N) \equiv \langle (n - \langle n \rangle)^2 \rangle \) of an energy interval which on average contains \( N = \langle n \rangle \) levels is \( \Sigma_2(N) = N \), according to the central limit theorem.

In the metallic or delocalized phase, on the other hand, the eigenstates are extended over the entire system. In this case the disorder potential causes the levels to repel each other. This level repulsion leads to a considerable rigidity of the spectrum with respect to fluctuations in the level density: the number variance increases only logarithmically with the number of levels, \( \Sigma_2(N) \propto \ln(N) \) and the spectral compressibility defined by

\[
\chi = \lim_{N \to \infty} \lim_{L \to \infty} \frac{d\Sigma_2(N)}{dN},
\]

vanishes; in the delocalized regime the spectra are incompressible. The level spacing distribution is described by the so called Wigner surmise, \( P(s) \propto s^\beta \exp(-c_\beta s^2) \), where \( \beta \) is a number of order one which depends on the symmetry of the system’s Hamiltonian [2]. The factor \( s^\beta \), missing in the Poisson statistics, reflects the strong level-repulsion.

At the mobility edge, where the two phases with their different kinds of spectra meet, things become more complicated and have given rise to extended investigations and controversial discussions concerning the critical level statistics [3–8]. In the vicinity of the transition energy the probability density of a state is neither localized on a small confined area nor smeared out almost homogeneously over the whole system, but forms a self-similar measure which fluctuates very strongly on all length scales. It is best described in terms of multifractality [3].

In recent publications Chalker, Lerner and Smith presented a treatment on level distributions in disordered systems, which in contrast to earlier works takes care of a possible non-trivial structure of the eigenstates [9]. A central result of their work is a relation between the spectral form factor \( K(t) \) (i.e. the Fourier transformed of the two-level-correlation function \( R(s) \)) and the ensemble averaged quantum return probability \( p(t) \) of a state initially confined to a small volume. Using this result and from scaling theory that \( p(t) \propto t^{-D_2/d} \), \( D_2 < d \) being the fractal (correlation) dimension [10], Chalker et al. [11] derived the critical spectral compressibility

\[
\chi = \frac{\eta}{2d} \quad (\eta < 1),
\]

where the anomalous diffusion exponent \( \eta \) is related to \( D_2 \) by \( \eta = d - D_2 \) [12].

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*Work performed within the research program of the Sonderforschungsbereich 341, Köln-Aachen-Jülich
In spite of the Poisson-like behavior of the number variance $\Sigma_2(N)$, there still is a strong level repulsion for small level separations. This is obvious in the spacing distribution function $P(s)$, which deviates only in the tail from the Wigner surmise $\cosh^2$ (Fig. 3). Hence, the non-vanishing compressibility must be due to level density fluctuations on larger level separations.

Up to now, Eq. (2) could not be confirmed directly by numerical calculations. The numerical results achieved for the Anderson transition in $d = 3$ dimensions [13] are compatible with a linear increase of the number variance with $N$ but their rather large numerical uncertainties do not allow to strictly exclude other dependencies. In particular, the slope of $\Sigma_2(N)$ for large $N$ could not be determined precisely. Besides, the value of the multifractal exponent $D_2$ for the 3d-MIT is not known very accurately [14]. In case of the MIT in two-dimensional quantum Hall systems the only numerical simulation [15] presented so far shows no significant linear contribution to $\Sigma_2(N)$.

The difficulties in determining the compressibility are due to the necessity to investigate level separations that are large compared to the average level spacing. Since only the critical region (which is usually a small portion of the entire spectrum) can be used for the statistics, one has to go to very large system sizes to produce a sufficient number of critical levels. Unfortunately, the widths of the critical region achievable with present computer capacities extends over no more than a few hundred levels, so that the results are strongly affected by incalculable finite size effects. This can of course not be compensated by a large number of disorder realizations. Consequently, to study critical level statistics numerically another way has to be found to produce larger critical spectra.

In this communication we show that by using the Chalker-Coddington network model [10] for the Quantum Hall Effect the problems we just mentioned can be avoided. As pointed out by Fertig in a detailed semiclassical analysis leading to a description very similar to that in [10], an energy dependent unitary operator $U(E)$ associated to the network model offers an alternative method for the numerical calculations of energy spectra and eigenstates [17]. However, here we do not use $U(E)$ for determining the real energy spectrum $E_n$, but calculate the eigenvalues $e^{i\omega}$ of $U(E)$ itself, whereby the energy $E$ is fixed to the critical value $E_c$. The obtained quasi-spectrum $\omega$ is governed by the same statistics as the energy spectrum at $E_c$ and is therefore suitable for our purposes. By using this new method it is possible to improve the statistics considerably and to confirm that $\chi = \eta/2d$ to a high degree of accuracy.

Our numerical approach is closely related to that of Edrei et al. [18], where the concept of a network model has been used for calculating wave propagation through random media. Actually, the definition of network states and operator used here are in principle identical to those in [18]. However, the main difference are the boundary conditions. Whereas in [18] open systems are treated, since they focused on transmission amplitudes, here we use closed systems in order to get information about the energy level distribution.

The network-model for the Quantum Hall Effect [14] is based on ideas developed in the early eighties [19] for the description of the Anderson transition in terms of scattering theory. It provides a semiclassical description of a 2d electron in a quantizing magnetic field and a smooth disorder potential with correlation length $\lambda$ large compared to the magnetic length $l_c$. The electron executes a fast cyclotron motion on a circle of radius $l_c$ around a guiding center, which drifts slowly along a contour $\gamma$ of constant energy, $V(\gamma) = E \equiv E^* - \hbar \omega_c/2$. At saddle-points of the potential with energies close to $E$ the electron tunnels with an appreciable probability between different contours.

The motion of electrons along the contours is depicted by one-dimensional, unidirectional channels, called links. The electron tunneling between them is represented by $2 \times 2$ scattering matrices $S = \{ t_{ml} \}$, which connect the complex current amplitudes $\psi_k, \psi_l$ of incoming links to those in outgoing links $\psi_m, \psi_n$ (Fig. 3). Due to the random length of the links between neighboring saddlepoints an electron acquires random phases $\phi_j$, which we absorb into the scattering coefficients $t_{ml}$.

The coefficients $t_{ml}$ depend on the electron energy $E$ and can, in principle, be determined by semiclassical methods for a given disorder potential [17]. For a saddle-point at zero energy and tunneling energy $E_t$ the tunneling amplitudes are $T = |t_{mk}|^2 = |t_{nl}|^2 = (1 + \exp(-E/E_t))^{-1}$ and $R = |t_{ml}|^2 = |t_{nk}|^2 = 1 - T$. Moreover, the random phases $\phi_{ml} = \arg(t_{ml})$ depend in a rather complicated manner on the energy $E [17]$. A state $\Psi$ of the network is given by its complex amplitudes $\psi_j$ on the links, $\Psi = \{ \psi_j \}$. It is stationary at energy $E$ if the scattering condition

$$\psi_m = t_{mk}(E)\psi_k + t_{ml}(E)\psi_l$$

is satisfied at each saddle point.

Defining an unitary operator $U(E)$ by

$$U(E)\psi_i = t_{mi}(E)\psi_m + t_{ni}(E)\psi_n,$$

where $\psi_i = \{ \delta_{ji} \} j$, this condition can be written as

$$U(E)\Psi = \Psi.$$

The energy enters parametrically via the coefficients $t_{ml}(E)$. This equation has non-trivial solutions only for discrete energies $E_n$. According to Fertig [17], these energies $E_n$ are eigenenergies of the system and their eigenvectors $\Psi_n$ determine the amplitudes of the corresponding eigenstates on the different equipotential contours (links). Hence, under certain circumstances the
scattering condition offers an alternative method of determining numerically eigenenergies and states of an electron in an disordered system, as it was first — to the best of our knowledge — pointed out by Fertig. This method was utilized in [26] for the numerical calculation of critical eigenstates in quantum Hall systems. (Like in [26], Eq. (3) can also be taken as a definition of time evolution for states at energy $E$, $\Psi(t+\tau) \equiv U(E)\Psi(t)$, whereby the energy dispersion is neglected.)

Before we proceed it is necessary to consider the length and energy scales determining the critical region. The averaged level spacing $\Delta$ is related to the system size $L$ and magnetic length $l_c$ via $\Delta \sim \Gamma(l_c/L)^2$, $\Gamma$ being the width of the disorder broadened Landau band. Close to the critical energy $E_c = 0$ the localization length $\xi$ is $\xi \sim |\lambda/E|^{-\nu}$. Therefore, the width of the critical energy region is $\Delta_c \sim (\lambda/l)^2 (L/\lambda)^{2-1/\nu}$.

Note that even for a fixed ratio $L/\lambda$ the number $N_c$ can be enhanced by increasing the ratio $\lambda/l_c$. We emphasize that in the Chalker-Coddington model this ratio is arbitrarily large, so that the number of critical levels is not restricted, which makes the model in principle very convenient for investigations of critical level statistics. However, the calculation of the energies $E_n$ by solving the non-linear Eq. (3) is a difficult numerical task and not suitable for practical purposes.

Therefore, in order to determine spectral statistics let us discuss instead of Eq. (3) the eigenvalue problem

$$U(E)\Psi_l(E) = e^{i\omega(E)}\Psi_l(E).$$

For a given energy $E$ the unimodular eigenvalues $e^{i\omega(E)}$ define quasi-energies $\omega_l(E), l = 1, \ldots, M = \text{dim} U(E).$ They are smooth functions of the energy and do not cross each other, following a theorem by von Neumann and Wigner [23]. According to Eq. (3), the intersection points of the curves $\omega_l(E)$ with the lines $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$ determine the energy levels $E_n$. The flow of the levels $\omega_l(E)$ obeys two symmetries: First, the intersection points $E_n'$ with shifted lines $\omega' = \Omega, \Omega \pm 2\pi, \ldots$ must exhibit the same statistics as the original spectrum $E_n$, since the corresponding transformed operator $U'(E) = e^{-i\Omega}U(E)$ belongs to the same universality class as $U(E)$ ($U'$ deviates from $U$ only by a global phase shift, which has no influence on the statistical properties). Second, as long as the critical regime is not left, $|E| < \Delta_c$, the statistical properties of the flow $\omega_l(E)$ can not change significantly with energy, because such a change would be accompanied by a new energy scale inbetween $\Delta$ and $\Delta_c (\ll E_i \ll \Gamma)$, which makes no physical sense.

Due to this homogeneity in both directions and due to the strong repulsion of the $\omega_l(E)$ they must behave as depicted in Fig. 3. The average slope of the curves varies neither strongly with the level number $l$ nor with the energy $E$. Further, this homogeneity implies that the intersection points $\omega_l$ with a cut $c$ crossing the band of curves show essentially the same statistics, independent of the precise position of $c$. For this reason, instead of the real energy spectrum $E_n$ one can also use a quasi-spectrum $\omega_l(E)$ with $E$ within the critical regime for an analysis of the critical level statistics [24]. A big advantage of this method is the simple fact that the $\omega_l(E)$ are far better numerically accessible than the real energies $E_n$. They can be calculated by solving the linear eigenvalue problem (3) with standard numerical methods.

For our calculations we used closed networks of $50 \times 50$ saddle-points with periodic boundaries in one and reflecting in the other direction. The transmission amplitudes were set to the critical value $T = 1/2$ for models describing the transition point and to $T_\iota = (1+\exp(\pm E/E_\iota))^{-1}$ with $E/E_\iota = 10$ for non-critical systems with strongly localized states. The disorder is represented by random scattering phases $\phi_m = \arg(t_{m\iota})$. Note that the calculations are done at constant energies $E = E_c$ and $E = 10E_\iota$, respectively, hence we did not have to know the energy dependence of the phases $\phi_m(E)$. From this settings we obtained random network operators at the critical point, $U(E = E_c)$, and deep in the localized regime, $U(E = 10E_\iota)$, of dimension $M = 2 \times 50 \times 50$. Diagonalizing them by standard numerical methods yields critical ($E = E_c$) and non-critical ($E = 10E_\iota$) quasi-spectra of $M = 5000$ levels each.

As explained in the considerations given above, the critical level statistics can be determined by analyzing the critical quasi-spectra. Although the statistics do not change within the quasi-spectra one has to take into account that the total number of quasi-levels per spectrum is fixed to $M$. So, when calculating the number variance $\Sigma_2(N)$ one has to confine oneself to interval sizes $\Delta \omega$ with averaged level number $N = \langle n \rangle_{\Delta \omega}$ small compared to $M$. We checked by numerical simulations with Poisson distributed levels that at a total number of $M = 5000$ levels deviations from the expected number variance $\Sigma_2(N) = N$ are negligible for $N < 300$.

For the determination of the critical number variance $\Sigma_2(N)$ we divided the quasi-spectra of 40 different disordered critical network operators $U(E_i)$ into non-overlapping intervals of length $\Delta \omega = (2\pi/5000)N$, $N$ ranging from 1 to 300. For each $N$ this results in an ensemble of $M_N = 40 \times M/N$ intervals with level numbers $n_i$ and $\langle n_i \rangle = N$, from which we calculate the level number variance $\Sigma_2(N) = M^{-1} \sum_{i=1}^{M} (n_i - N)^2$.

The results plotted in Fig. 3 show a clearly linear behavior of the number variance $\Sigma(N)$ for a wide range of $N$. Eq. (5) predicts a slope of $\eta/2d = (2 - D_2)/2d = 0.125 \pm 0.01$, where we have taken $D_2 = 1.5 \pm 0.05$ from independent numerical calculations of critical states [23,24,21]. A least square fit of our data yields a slope $0.124 \pm 0.006$, which agrees with the prediction very well.
The dashed lines mark the range of the expected fluctuations of \( \Sigma_2(n) \) due to the finite number \( m_N \) of intervals, calculated via the \( \chi^2 \)-distribution for \( \alpha = 0.8, 0.2 \). This clearly indicates that the deviations of the data from the straight line are not systematic but due to statistical fluctuations.

The spacing distribution \( P(s) \) plotted in the inset of Fig. 3 shows for small spacings \( s \) a WD-type behavior for a GUE ensemble (dashed line), \( P(s) \propto s^2 \), indicating strong level repulsion for small level spacing.

We use the same procedure as for the critical quasi-spectra for five non-critical quasi-spectra at \( E = 10E_c \) in the strongly localized regime. Here \( \Sigma_2(N) \) follows a straight line of slope 1, as it should be, since in this region the levels are Poisson-distributed.

To summarize, the recently derived relation between the multifractal exponent \( \eta = d - D_2 \) of eigenstates and the spectral compressibility at the mobility edge, \( \chi = \eta/2d \), has been confirmed numerically for the integer quantum Hall delocalization transitions. This has been done by introducing a new method to investigate spectral properties of disordered systems.

We would like to thank János Hajdu, Bodo Huckestein and Martin Janssen for valuable discussions and the research program Sonderforschungsbereich 341, Köln-Aachen-Jülich for their support.

[1] see e.g., M. L. Mehta, Random Matrices, 2nd ed. (Academic Press, New York, 1991), and references therein.
[2] B. I. Shklovskii, B. Shapiro, B. R Sears, P. Lambrianides, H. B. Shore, Phys. Rev. B 47, 11487 (1993).
[3] B. L. Altshuler, I. Zherekeshov, S. Kotochigova, B. Shklovskii, JETP 67, 625 (1988).
[4] V. E. Kravtsov, I. V. Lerner, B. L. Altshuler, A. G. Aronov, Phys. Rev. Lett. 72, 888 (1994).
[5] A. G. Aronov, A. D. Mirlin, Phys. Rev. B 51, 6131 (1995).
[6] V. E. Kravtsov, I. V. Lerner, Phys. Rev. Lett. 74, 2563 (1995).
[7] J. T. Chalker, V. E. Kravtsov, I. V. Lerner, JETP Lett. 64, 386 (1996).
[8] D. Braun, G. Montambaux, Phys. Rev. B 52, 13903 (1995).
[9] M. Janssen, Int. J. Mod. Phys. 8, 943 (1994).
[10] J. T. Chalker, I. V. Lerner, R. S. Smith, Phys. Rev. Lett 77, 554 (1996); J. Math. Phys. 37, 5061 (1996).
[11] J. T. Chalker, G. J. Daniell, Phys. Rev. Lett. 61, 593 (1988); B. Huckestein, L. Schweitzer, Phys. Rev. Lett. 72, 713 (1994); L. T. Brandes, B. Huckestein, L. Schweitzer Ann. Physik 5, 633 (1996).
[12] M. Janssen, Ph.D. Thesis, Universität zu Köln, 1990; J. Hajdu, M. Janssen, in: G. Györgyi, I. Kondor, L. Sasvári, T. Tel (eds.), From Phase Transitions to Chaos, World Scientific, Singapore, 1992.
[13] S. N. Evangelou, Phys. Rev. B 49, 16805 (1994); Y. Ono, T. Ohtsuki, B. Kramer, J. Phys. Soc. Jpn 65, 6 (1996).
[14] The values for \( D_2 \) ranges from 1.3 up to 1.9, C. M. Soukoulis, E. N. Economou, Phys. Rev. Lett. 52, 565 (1984).
[15] M. Feingold, Y. Avishai, R. Berkovits, Phys. Rev. B 52, 8400 (1995).
[16] J. T. Chalker, P. D. Coddington, J. Phys. C 21, 2665 (1988).
[17] H. A. Fertig, Phys. Rev. B 38, 996 (1988).
[18] I. Edrei, M. Kaveh, B. Shapiro, Phys. Rev. Lett. 62, 2120 (1989).
[19] P. W. Anderson, Phys. Rev. B 23, 4828 (1981); B. Shapiro, Phys. Rev. Lett. 48, 823 (1982).
[20] B. Huckestein, R. Klesse, Phys. Rev. B 55, R7303 (1997).
[21] R. Klesse, Ph.D. Thesis, Universität zu Köln, 1996.
[22] M. Metzler, Ph.D. Thesis, Universität zu Köln, 1996.
[23] J. v. Neumann, E. Wigner, Physikalische Zeitschrift 30, 467 (1929).
[24] A similar approach taking advantage of a relation between scattering phases and energy-levels was used by Jalabert and Pichard in another context, R. A. Jalabert, J.-L. Pichard, J. Phys. I France 5, 287 (1995).

In order to get a better understanding of the quasi-energies \( \omega_l \) and their eigenfunctions \( \Psi_l \) we refer to recent numerical works [22, 20]. In dynamical simulations [2], where \( U(E_c) \) is interpreted as evolution operator for microscopic time steps — this means that the \( \omega_l(E_c) \) are thought of as actual eigenenergies \( E_l \) with eigenstates \( \Psi_l(E_c) \) —, the well known critical anomalous diffusion behavior shows up in good agreement with the predictions of scaling theory [1]. Moreover, the correlations in the local amplitudes of eigenfunctions \( \Psi_l(E_c) \) were found to obey the same scaling behavior in the spatial and quasi-energy difference \( \Delta \omega = |\omega_l - \omega_l| \) as those of real critical eigenfunctions in real energy differences [22, 20]. This suggests to interpret the quasi-spectrum \( \omega_l(E) \) of \( U(E) \) as a spectrum, which is statistically equivalent to the excitation spectrum of the real system in the vicinity of the energy \( E \).

[26] R. Klesse, M. Metzler, Euro Phys. Lett., 32, 229 (1995).
[27] W. Pook, M. Janssen, Z. Phys. 82, 295 (1991).

FIG. 1. The Chalker-Coddington network. At each saddle point a scattering matrix \( S \) describes the transition from incoming to outgoing states. The operator \( U \) maps each incoming link amplitude to the two outgoing links according to the transmission coefficients \( t_{nl1}, t_{nl2}, \ldots \).

FIG. 2. The eigenvalues \( \exp(i\omega_l(E)) \) of \( U(E) \) as functions of the energy \( E \). The intersections with the lines \( \omega = 2\pi z, z \in Z \) determine the energy-levels \( E_n \), those with the \( E = E_n \)-line the quasi-energies \( \omega_l(E_c) \).
FIG. 3. The level number variance $\Sigma_2$ plotted against the average number of levels $N$ at the critical point (◇) and in the localized regime (□). The solid line has a slope of 0.124, the dashed lines mark the range of the expected statistical fluctuations ($\chi^2_\alpha$, $\alpha = 0.8, 0.2$). The dotted line has a slope of 1. The inset shows the level spacing distribution $P(s)$ at the critical point (◇) and the WD-distribution for GUE (dashed line).
