Quantum group symmetries and completeness for $A^{(2)}_{2n}$ open spin chains

Ibrahim Ahmed, Rafael I Nepomechie and Chunguang Wang

Physics Department, PO Box 248046, University of Miami, Coral Gables, FL 33124, United States of America
E-mail: ibrahimahmed@miami.edu, nepomechie@miami.edu and c.wang22@umiami.edu

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Abstract
We argue that the Hamiltonians for $A^{(2)}_{2n}$ open quantum spin chains corresponding to two choices of integrable boundary conditions have the symmetries $U_q(B_n)$ and $U_q(C_n)$, respectively. We find a formula for the Dynkin labels of the Bethe states (which determine the degeneracies of the corresponding eigenvalues) in terms of the numbers of Bethe roots of each type. With the help of this formula, we verify numerically (for a generic value of the anisotropy parameter) that the degeneracies and multiplicities of the spectra implied by the quantum group symmetries are completely described by the Bethe ansatz.

Keywords: quantum integrability, quantum group, Bethe ansatz

(Some figures may appear in colour only in the online journal)

1. Introduction

Interesting new connections of integrable quantum spin chains to integrable quantum field theory, conformal field theory (CFT) and string theory, as well as to condensed matter physics, continue to be found. A case in point concerns the $A^{(2)}_n$ family of models [1–4], which has recently been revisited by Vernier et al [5–7]. For example, it was argued in [5] that the $A^{(2)}_2$ model [1] has a regime where the continuum limit is a certain non-compact CFT, the so-called black hole sigma model [8, 9].

Another interesting feature of these models is that they can have quantum group symmetries (see e.g. [10, 11]), provided that the boundary conditions are suitable. For the closed chains with periodic boundary conditions studied in [5–7], such symmetries can be realized
only indirectly; however, quantum group symmetries can be realized directly in open chains [12].

Motivated in part by these recent developments, we have set out to revisit the quantum group symmetries of the \(A_n^{(2)}\) family of models. We therefore focus instead on open chains; and, for concreteness, we restrict here to the even series \(A_{2n}^{(2)}\), leaving the odd series \(A_{2n-1}^{(2)}\) for a future publication. It has long been known that, for one simple set of integrable boundary conditions, the former models have \(U_q(B_n)\) symmetry [13, 14].

We argue here that—surprisingly—the \(A_{2n}^{(2)}\) models have \(U_q(C_n)\) symmetry for another set of integrable boundary conditions. (The symmetry for the case \(n = 1\) was already noticed in [15], but the symmetry for the general case \(n > 1\) had remained unexplored until now.) The symmetries (both \(U_q(B_n)\) and \(U_q(C_n)\)) determine the degeneracies and multiplicities of the spectra, which are completely described by the Bethe ansatz solutions.

The outline of this paper is as follows. In section 2 we briefly review the construction of the integrable \(A_{2n}^{(2)}\) open quantum spin chains that are the focus of this paper. In section 3 we show that the Hamiltonians for the two cases of interest can be expressed as sums of two-body terms. We use this fact in section 4 to demonstrate that the Hamiltonians have quantum group symmetries, which in turn determine the degeneracies and multiplicities of the spectra. In section 5 we briefly review the Bethe ansatz solutions of the models, and we obtain a formula for the Dynkin labels of the Bethe states, part of whose proof is sketched in an appendix. In section 6 we use this formula to help verify numerically that the Bethe ansatz solutions completely account for the degeneracies and multiplicities implied by the quantum group symmetries. In section 7 we briefly summarize our conclusions, and list some interesting open problems.

2. The models

We briefly review here the construction of the integrable \(A_{2n}^{(2)}\) open quantum spin chains that will turn out to have quantum group symmetries. The basic ingredients are the \(R\)-matrix and \(K\)-matrices, which are used to construct a commuting transfer matrix that contains the integrable Hamiltonian.

2.1. \(R\)-matrix

The \(R\)-matrix is a matrix-valued function \(R(u)\) of the so-called spectral parameter \(u\) that maps \(V \otimes V\) to itself, where here \(V\) is a \((2n + 1)\)-dimensional vector space, which is a solution of the Yang–Baxter equation (YBE) on \(V \otimes V \otimes V\)

\[ R_{12}(u - v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u - v). \]  

(2.1)

We use the standard notations \(R_{12} = R \otimes 1, R_{23} = 1 \otimes R, R_{13} = P_{23} R_{12} P_{23}\), where \(P\) is the permutation matrix on \(V \otimes V\)

\[ P = \sum_{\alpha, \beta = 1}^{2n+1} e_{\alpha \beta} \otimes e_{\beta \alpha}. \]  

(2.2)

and \(e_{\alpha \beta}\) are the \((2n + 1) \times (2n + 1)\) elementary matrices with elements \((e_{\alpha \beta})_{ij} = \delta_{\alpha i} \delta_{\beta j}\).

We focus here on the \(R\)-matrix (A.2) that is associated with the fundamental representation of \(A_{2n}^{(2)}\) [2–4] with anisotropy parameter \(\eta\), which is a generalization of the Izergin–Korepin
that is associated with $A^{(2)}_2$. Besides satisfying the YBE, this $R$-matrix enjoys several additional important properties, among them $PT$ symmetry

$$R_{21}(u) = P_{12} R_{12}(u) \mathcal{P} = R_{12}^{(2)}(u),$$

(2.3)

unitarity

$$R_{12}(u) R_{21}(-u) = \xi(u) \xi(-u) \mathbb{1} \otimes \mathbb{1},$$

(2.4)

where $\xi(u)$ is given by

$$\xi(u) = 2 \sinh(u^2 - 2\eta) \cosh(u^2 - 2(n + 1)\eta),$$

(2.5)

regularity

$$R(0) = \xi(0) \mathcal{P},$$

(2.6)

and crossing symmetry

$$R_{12}(u) = V_1 R_{21}^{(2)}(-u - \rho) V_1 = V_2^* R_{12}^{(2)}(-u - \rho) V_2^*,$$

(2.7)

where $\rho = -i\pi - 2(2n + 1)\eta$; and the matrix $V$, which is given by (A.6), satisfies $V^2 = \mathbb{1}$.

2.2. $K$-matrices

The matrix $K^-(u)$, which maps $\mathcal{V}$ to itself, is a solution of the boundary Yang–Baxter equation (BYBE) on $\mathcal{V} \otimes \mathcal{V}$ [16–19]

$$R_{12}(u - v) K_1^-(u) R_{21}(u + v) K_2^-(v) = K_2^-(v) R_{12}(u + v) K_1^-(u) R_{21}(u - v).$$

(2.8)

The matrix $K^-(u)$ is assumed to have the regularity property

$$K^-(0) = \kappa \mathbb{1}.$$

(2.9)

Similarly, $K^+(u)$ satisfies [17, 18]

$$R_{12}(-u + v) K_1^{+t_1}(u) M_1^{-1} R_{21}(-u - v - 2\rho) M_1 K_2^{+t_2}(v)$$

$$= K_2^{+t_2}(v) M_1 R_{12}(-u - v - 2\rho) M_1^{-1} K_1^{+t_1}(u) R_{21}(-u + v),$$

(2.10)

where the matrix $M$ is defined by

$$M = V^* V,$$

(2.11)

and is given by (A.7). If $K^-(u)$ is a solution of the BYBE (2.8), then [17, 18]

$$K^+(u) = K^{-t}(-u - \rho) M$$

(2.12)

is a solution of (2.10).

We consider here two different sets of $K$-matrices:

$$(I) : \quad K^-(u) = \mathbb{1}, \quad K^+(u) = M,$$

(2.13)

$$(II) : \quad K^-(u) = K(u), \quad K^+(u) = K(-u - \rho) M.$$

(2.14)

The fact that $K^-(u) = \mathbb{1}$ is a solution of the BYBE was noted in [13]. The matrix $K(u)$ in (2.14) is the diagonal matrix given by

$$K(u) = \text{diag}(k_1(u), \ldots, k_{2n+1}(u)),$$

(2.15)
where
\[
k_j(u) = \begin{cases} 
e^{-u [i \cosh \eta + \sinh(u - 2n\eta)]} & j = 1, \ldots, n \\
i \cosh(u + \eta) - \sinh(2n\eta) & j = n + 1 \\
e^{2[ i \cosh \eta + \sinh(u - 2n\eta)]} & j = n + 2, \ldots, 2n + 1 \end{cases}.
\] (2.16)

where \( \epsilon \) can have the values \( \pm 1 \), but for concreteness we henceforth set \( \epsilon = +1 \). This \( K \)-matrix has the regularity property (2.9) with
\[
\kappa = i \cosh \eta - \sinh(2n\eta).
\] (2.17)

The solution (2.15) and (2.16) of the BYBE (2.8) for the case \( n = 1 \) was found in [13], and the generalization for \( n > 1 \) was found in [20, 21].

### 2.3. Transfer matrix and Hamiltonian

The transfer matrix \( t(u) \) for an integrable open quantum spin chain with \( N \) sites, which acts on the quantum space \( \mathcal{V}^\otimes N \), is given by [17]
\[
t(u) = \text{tr}_a K^+_a(u) T_a(u) K^-_a(u) \hat{T}_a(u),
\] (2.18)
where the monodromy matrices are defined by
\[
T_a(u) = R_{aN}(u) R_{aN-1}(u) \cdots R_{a1}(u), \quad \hat{T}_a(u) = R_{1a}(u) \cdots R_{N-1a}(u) R_{N0}(u),
\] (2.19)
and the trace in (2.18) is over the auxiliary space, which we denote by \( a \). The various properties satisfied by the \( R \) and \( K \) matrices can be used to show that the transfer matrix satisfies the fundamental commutativity property [17]
\[
[t(u), t(v)] = 0 \text{ for all } u, v.
\] (2.20)

The corresponding integrable open chain Hamiltonian \( \mathcal{H} \) is given (up to multiplicative and additive constants) by \( t'(0) \), which evidently satisfies
\[
[\mathcal{H}, t(u)] = 0.
\] (2.21)

More explicitly, one finds [17]
\[
\mathcal{H} = \sum_{k=1}^{N-1} h_{k,k+1} + \frac{1}{2\kappa} K^-_0(0) + \frac{1}{\text{tr}K^+_0(0)} \text{tr}_0 K^+_0(0) h_{N0},
\] (2.22)
where the two-site Hamiltonian \( h_{k,k+1} \) is given by
\[
h_{k,k+1} = \frac{1}{\xi(0)} \mathcal{T}_{k,k+1} R'_{k,k+1}(0).
\] (2.23)

### 3. Simplification of the Hamiltonian

We show here that the boundary terms in the Hamiltonian (2.22) can be simplified for the two sets of \( K \)-matrices (2.13) and (2.14) in such a way that the Hamiltonians are expressed as sums of two-body terms, which will allow us to demonstrate their quantum group invariance in the following section. The key step in this simplification is a \( K \)-matrix identity (3.1), which is reminiscent of Sklyanin’s ‘less obvious’ isomorphism given by equations (17) and (18) in [17], and the Ghoshal–Zamolodchikov boundary crossing-unitarity relation, see equations (3.33) and (3.35) in [19].
3.1. An identity for the $K$-matrix

A useful identity is

$$\text{tr}_1 K_1^{-1} (-u - \rho) M_1 R_{12}(2u) \mathcal{P}_{12} = f(u) V_2 K_2^{-1}(u) V_2,$$

(3.1)

where $f(u)$ is a scalar function. The remainder of this section is devoted to proving this identity. Readers who are more interested to see how this identity can be used to simplify the boundary terms in the Hamiltonian may skip directly to section 3.2.

It is helpful to recall (see e.g. [22]) that the crossing symmetry (2.7) can be used to show that the $R$-matrix degenerates at $u = -\rho$ to a projector onto a one-dimensional subspace,

$$\tilde{P}_{12}^- \equiv \frac{1}{(2n + 1) \xi(0)} R_{12}(-\rho) = \frac{1}{(2n + 1)} V_1 \mathcal{P}_{12}^0 V_1,$$

(3.2)

which obeys

$$(\tilde{P}_{12}^-)^2 = \tilde{P}_{12}^-$$

(3.3)

and

$$\tilde{P}_{12}^- A_{12} \tilde{P}_{12}^- = \text{tr}_{12} \left( \tilde{P}_{12}^0 A_{12} \right) \tilde{P}_{12}^-,$$

(3.4)

where $A$ is an arbitrary matrix acting on $\mathcal{V} \otimes \mathcal{V}$. This projector is not symmetric,

$$\tilde{P}_{21}^- \equiv \mathcal{P}_{12} \tilde{P}_{12}^- \mathcal{P}_{12} = (\tilde{P}_{12}^-)^{\dagger} \neq \tilde{P}_{12}^-.$$

(3.5)

We also recall that

$$V_1 R_{12}(u) V_1 = V_2 R_{21}(u) V_2.$$

(3.6)

The starting point of the proof is the BYBE (2.8), where we set $v = -u - \rho$ and use the definition (3.2) to obtain

$$R_{12}(2u + \rho) K_1^{-1}(u) \tilde{P}_{21}^- K_2^{-1}(u) = K_2^{-1}(u) \tilde{P}_{12}^- K_1^{-1}(u) R_{21}(2u + \rho).$$

(3.7)

With the help of the relations

$$\tilde{P}_{21}^- = V_1^0 V_2^0 \tilde{P}_{12}^- V_1^0 V_2^0$$

(3.8)

and

$$R_{21}(2u + \rho) = V_1^0 V_2^0 R_{12}(2u + \rho) V_1^0 V_2^0$$

(3.9)

that follow from (3.6), we arrive at

$$R_{12}(2u + \rho) K_1^{-1}(u) V_1^0 V_2^0 \tilde{P}_{12}^- V_1^0 V_2^0 = K_2^{-1}(u) V_1^0 V_2^0 R_{12}(2u + \rho) V_1^0 V_2^0.$$

(3.10)

Multiplying both sides on the right by $\tilde{P}_{12}^-$ and using the projector property (3.4), we obtain

$$R_{12}(2u + \rho) K_1^{-1}(u) V_1^0 V_2^0 \tilde{P}_{12}^- K_2^{-1}(u) V_1^0 V_2^0 = g(u) K_2^{-1}(u) (-u - \rho) \tilde{P}_{12}^-,$$

(3.11)

where $g(u)$ is some scalar function. Multiplying both sides, on both the right and the left, by the permutation matrix $\mathcal{P}_{12}$, and then using the crossing equation (2.7) and the expression (3.2) for $\tilde{P}_{12}^-$, we obtain

$$V_1^0 R_{12}^0 (-2u - 2\rho) K_2^{-1}(u) V_1^0 V_2^0 \mathcal{P}_{12}^0 V_1^0 = g(u) K_2^{-1}(u) (-u - \rho) V_1^0 \mathcal{P}_{12}^0 V_1^0.$$

(3.12)
Taking the trace of both sides over the first space, we arrive at
\[ \text{tr}_1 R_{12}^2 (-2u - 2\rho) K_2^-(u) V_2^1 V_2^2 \mathcal{P}_{12}^\dagger = g(u) \text{tr}_1 K_1^-(u - \rho) V_1^1 \mathcal{P}_{12}^\dagger V_1^2, \]  
which can be simplified to
\[ \text{tr}_1 K_1^-(u) M_1 R_{12}^2 (-2u - 2\rho) \mathcal{P}_{12} = g(u) V_2 K_2^-(u) V_2. \]  
(3.13)
Replacing \( u \mapsto -u - \rho \) and setting \( f(u) = g(-u - \rho) \), we finally obtain (3.1).

### 3.2. Simplified Hamiltonians

We now proceed to simplify the boundary terms in the Hamiltonian (2.22) using the identity (3.1), which can be rewritten as
\[ \text{tr}_1 K_1^+(u) P_{12} R_{21}^2 (2u) = f(u) V_2 K_2^+(u) V_2 \]  
(3.15)
for diagonal \( K^\pm \)-matrices that are related by (2.12).

#### 3.2.1. Set I.
For the first set of \( K \)-matrices (2.13), the identity (3.15) immediately implies that
\[ \text{tr}_1 M_1 P_{12} R_{21}^2 (2u) = f(u) I_2. \]  
(3.16)
Differentiating this relation with respect to \( u \) and then setting \( u = 0 \), we obtain the result
\[ \text{tr}_1 M_1 P_{12} R_{21}^2 (0) \propto I_2 \]  
(3.17)
(see also [13, 23]) and therefore
\[ \text{tr}_0 K_0^+(0) h_{N0} = \text{tr}_0 M_0 h_{N0} \propto \text{tr}_0 M_0 P_{N0} R_{N0}^\dagger (0) \propto I_N, \]  
(3.18)
i.e. the corresponding boundary term is proportional to the identity matrix. Moreover, since \( K^-(0) = I \), the boundary term with \( K^-(0) \) evidently vanishes.

In short, the two boundary terms in the expression (2.22) for the Hamiltonian can be dropped. The Hamiltonian for the set I therefore reduces to a sum of two-site Hamiltonians [13]
\[ \mathcal{H}^{(I)} = \sum_{k=1}^{N-1} h_{kk+1}. \]  
(3.19)
Its relation to the transfer matrix (2.18) is given by
\[ \mathcal{H}^{(I)} = \frac{1}{c_1} t'(0) + c_2 \mathcal{T}^N, \]  
(3.20)
with
\[ c_1 = 4^{N+1} \sinh((2n + 1)\eta) \cosh((2n - 1)\eta) \sinh^{2N-1}(2\eta) \cosh^{2N}((2n + 1)\eta), \]
\[ c_2 = \frac{\cosh((6n + 1)\eta)}{2 \sinh((4n + 2)\eta) \cosh((2n - 1)\eta)}. \]  
(3.21)
The Hamiltonian (3.19) is Hermitian for real, but not for imaginary, values of \( \eta \).
3.2.2. Set II. We turn now to the second set of $K$-matrices (2.14). Setting $u = 0$ in the identity (3.15), and using the regularity properties (2.6) and (2.9), we obtain

$$f(0) = \frac{1}{\kappa} \xi(0) tr K^+(0).$$

(3.22)

Moreover, differentiating the identity (3.15) with respect to $u$ and then setting $u = 0$, we obtain

$$2 tr_1 K_1^+(0) P_{12} R_{21}'(0) + \ldots = f(0) V_2 K_2^{-'}(0)V_2 + \ldots,$$

(3.23)

where the ellipses represent terms that are proportional to the identity, which we drop. Using the explicit form of the $K$-matrix (2.15) and (2.16), we observe that

$$VK^{-'}(0)V = -K_2^{-'}(0) + \mu U + \nu I,$$

(3.24)

where

$$\mu = 2(i \sinh \eta - \cosh(2n\eta)), \quad \nu = 2 \cosh(2n\eta), \quad U = e_{n+1,n+1}.$$  

(3.25)

Substituting (3.22) and (3.24) into (3.23), we arrive at the identity

$$\frac{1}{\xi(0) tr K^+(0)} tr_1 K_1^+(0) P_{12} R_{21}'(0) = -\frac{1}{2\kappa} K_2^{-'}(0) + \frac{\mu}{2\kappa} U_2 + \ldots,$$

(3.26)

The Hamiltonian (2.22) for the set II therefore reduces to the form

$$H^{(\text{II})} = \sum_{k=1}^{N-1} h_{k,k+1} + \frac{1}{2\kappa} \left[ K_1^{-'}(0) - K_N^{-'}(0) \right] + \frac{\mu}{2\kappa} U_N.$$  

(3.27)

Let us define a new two-site Hamiltonian $\tilde{h}_{k,k+1}$ as follows

$$\tilde{h}_{k,k+1} \equiv h_{k,k+1} + \frac{1}{2\kappa} \left[ K_1'(0) - K_N'(0) \right].$$

(3.28)

We conclude that, up to a term proportional to $U_N$, the Hamiltonian again reduces to a sum of two-site Hamiltonians,

$$H^{(\text{II})} = \sum_{k=1}^{N-1} \tilde{h}_{k,k+1} + \frac{\mu}{2\kappa} U_N.$$  

(3.29)

Its relation to the transfer matrix (2.18) is given by

$$H^{(\text{II})} = \frac{1}{c_1} t'(0) + c_2 \otimes U_N,$$

(3.30)

with

$$c_1 = 2^{2N+1} (\cosh \eta + i \sinh(2n\eta))^2 \sinh((4n + 2)\eta) \cosh((2n + 3)\eta) [\sinh(2\eta) \cosh((2n + 1)\eta)]^{2N-1},$$

$$c_2 = \frac{\cosh((6n + 5)\eta)}{2 \sinh((4n + 2)\eta) \cosh((2n + 3)\eta)} + \frac{i \cosh(2n\eta)}{\cosh \eta + i \sinh(2n\eta)}. $$

(3.31)

The Hamiltonian (3.29) is not Hermitian for either real or imaginary values of $\eta$. 

7
4. Quantum group symmetries

We first review the $U_q(B_n)$ symmetry of the Hamiltonian corresponding to the first set of $K$-matrices (2.13). We then argue that the Hamiltonian corresponding to the second set of $K$-matrices (2.14) has the quantum group symmetry $U_q(C_n)$.

4.1. Set I: $U_q(B_n)$ symmetry

It was already argued in [13] that the Hamiltonian $H(I)$ (3.19) corresponding to the first set of $K$-matrices (2.13) has $U_q(B_n)$ symmetry. It was subsequently shown in [14] (generalizing the arguments in [24] for the XXZ chain) that this symmetry extends to the full transfer matrix $t(u)$ (2.18). Here we explicitly construct the coproduct of the generators, and show that they commute with the Hamiltonian.

For the vector representation of $B_n = O(2n+1)$, in the so-called orthogonal basis, the Cartan generators $\{H_1, \ldots, H_n\}$ are given by the diagonal matrices

$$H_\alpha = e_{\alpha, \alpha} - e_{2n+2-\alpha, 2n+2-\alpha}, \quad \alpha = 1, 2, \ldots, n,$$

(4.1)

and the generators $\{E_1^\pm, \ldots, E_n^\pm\}$ corresponding to the simple roots are given by

$$E_\alpha^+ = e_{\alpha, \alpha+1} + e_{2n+1-\alpha, 2n+2-\alpha}, \quad E_\alpha^- = E_\alpha^+ \tau, \quad \alpha = 1, 2, \ldots, n.$$  (4.2)

Indeed, these generators satisfy

$$[H_i, E_j^\pm] = \pm \alpha_{i,j} E_j^\pm, \quad i, j = 1, 2, \ldots, n,$$

(4.3)

where $\{\alpha^{(1)}, \ldots, \alpha^{(n)}\}$ are the simple roots of $B_n$ in the orthogonal basis (see e.g. [26])

$$\alpha^{(1)} = (1, -1, 0, \ldots, 0),$$
$$\alpha^{(2)} = (0, 1, -1, 0, \ldots, 0),$$
$$\vdots$$
$$\alpha^{(n-1)} = (0, \ldots, 0, 1, -1),$$
$$\alpha^{(n)} = (0, \ldots, 0, 1).$$

Let us define the following coproduct for these generators

$$\Delta(H_j) = H_j \otimes \mathbb{1} + \mathbb{1} \otimes H_j,$$
$$\Delta(E_j^\pm) = E_j^\pm \otimes e^{i\pi H_j} e^{\eta(H_j - H_{j+1})} \otimes E_j^\pm,$$

(4.5)

where $j = 1, \ldots, n$ with $H_{n+1} \equiv 0$. We observe that

$$\Omega_{ij} \Delta(E_j^+) \Delta(E_j^-) - \Delta(E_j^-) \Delta(E_j^+) \Omega_{ij} = \delta_{ij} \frac{q^{\Delta(H_j) - \Delta(H_{j+1})} - q^{-\Delta(H_j) + \Delta(H_{j+1})}}{q - q^{-1}},$$

(4.6)

where $q = e^{2\eta}$ and

$$\Omega_{ij} = \begin{cases} e^{i\pi H_{\max}(j)} \otimes \mathbb{1} & |i - j| = 1 \vspace{0.1cm} \\ \mathbb{1} \otimes \mathbb{1} & |i - j| \neq 1 \end{cases}.$$  (4.7)

The two-site Hamiltonian (2.23) commutes with the coproducts (4.5)

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1 Explicit matrix representations for the generators can be obtained from e.g. [25] or Maple.
\[ [\Delta(H_j), h_{1,2}] = [\Delta(E_j^\pm), h_{1,2}] = 0, \quad j = 1, \ldots, n. \] 

(4.8)

Since the \(N\)-site Hamiltonian is given (3.19) by the sum of two-site Hamiltonians, it follows that the \(N\)-site Hamiltonian commutes with the \(N\)-fold coproducts

\[ [\Delta(N)(H_j), \mathcal{H}^{(j)}] = [\Delta(N)(E_j^\pm), \mathcal{H}^{(j)}] = 0, \quad j = 1, \ldots, n. \] 

(4.9)

This provides an explicit demonstration of the \(U_q(B_n)\) invariance of the Hamiltonian \(\mathcal{H}^{(j)}\).

### 4.1.1. Degeneracies and multiplicities for \(U_q(B_n)\)

One of the important consequences of the \(U_q(B_n)\) symmetry of the Hamiltonian is that the energy eigenstates form irreducible representations of this algebra. For generic values of \(\eta\) (i.e. \(\eta \neq i\pi/p\), where \(p\) is a rational number), the representations are the same as for the classical algebra \(B_n\). The generalization of the familiar Clebsch-Gordan theorem from \(A_1 = SU(2)\) to \(B_n\) implies that the \(N\)-site Hilbert space has a decomposition of the form

\[ \mathcal{V}^{(2n+1)\otimes N} = \bigoplus_j d^{(j,N,n)} \mathcal{V}^{(j)}, \] 

(4.10)

where \(\mathcal{V}^{(j)}\) denotes an irreducible representation of \(B_n\) with dimension \(j\) (=degeneracy of the corresponding energy eigenvalue) and \(d^{(j,N,n)}\) is its multiplicity. Here we specify the irreducible representations by their dimensions, and we allow for the possibility that there can be more than one inequivalent irreducible representation with a given dimension. For example, \(B_2\) has a \(35\) and a \(35'\).

The first few cases are as follows (see e.g. [26]):

**\(B_1\):** \(N = 2\):

\[ 3 \otimes 3 = 1 \oplus 3 \oplus 5 \]

\[ = [0] \oplus [2] \oplus [4] \]

\[ N = 3 : \quad 3 \otimes 3 \otimes 3 = 1 \oplus 3 \cdot 3 \oplus 2 \cdot 5 \oplus 7 \]

\[ = [0] \oplus [3][2] \oplus [2][4] \oplus [6] \]  

(4.11)

**\(B_2\):** \(N = 2\):

\[ 5 \otimes 5 = 1 \oplus 10 \oplus 14 \]

\[ = [0,0] \oplus [0,2] \oplus [2,0] \]

\[ N = 3 : \quad 5 \otimes 5 \otimes 5 = 3 \cdot 5 \oplus 10 \oplus 30 \oplus 2 \cdot 35 \]

\[ = 3[1,0] \oplus [0,2] \oplus [3,0] \oplus 2[1,2] \]  

(4.12)

**\(B_3\):** \(N = 2\):

\[ 7 \otimes 7 = 1 \oplus 21 \oplus 27 \]

\[ = [0,0,0] \oplus [0,1,0] \oplus [2,0,0] \]

\[ N = 3 : \quad 7 \otimes 7 \otimes 7 = 3 \cdot 7 \oplus 35 \oplus 77 \oplus 2 \cdot 105 \]

\[ = 3[1,0,0] \oplus [0,0,2] \oplus [3,0,0] \oplus 2[1,1,0] \]  

(4.13)

\(^2\)For later reference, we also present the tensor-product decompositions in terms of the Dynkin labels \([a_1, \ldots, a_n]\) of the representations.
We have verified numerically that the Hamiltonian as well as the transfer matrix for set I (2.13) have exactly these degeneracies and multiplicities for generic values of $\eta$, which provides further evidence of their $U_q(B_n)$ invariance.

4.2. Set II: $U_q(C_n)$ symmetry

For the vector representation of $C_n = Sp(2n)$ in the orthogonal basis, the Cartan generators are given by

$$\tilde{H}_\alpha = \tilde{e}_{\alpha,\alpha} - \tilde{e}_{2n+1-\alpha,2n+1-\alpha}, \quad \alpha = 1, 2, \ldots, n,$$

and the generators corresponding to the simple roots are given by

$$\tilde{E}_\alpha^+ = \tilde{e}_{\alpha,\alpha+1} + \tilde{e}_{2n-\alpha,2n+1-\alpha}, \quad \alpha = 1, 2, \ldots, n-1,$$

$$\tilde{E}_n^+ = \tilde{e}_{n,n+1},$$

and $\tilde{E}_\alpha^- = \tilde{E}_\alpha^+$, where $\tilde{e}_{\alpha,\beta}$ are the elementary $(2n) \times (2n)$ matrices. These generators satisfy

$$[\tilde{H}_i, \tilde{E}_j^\pm] = \pm \alpha_i^{(j)} \tilde{E}_j^\pm, \quad i, j = 1, 2, \ldots, n,$$

where $\{\alpha^{(1)}, \ldots, \alpha^{(n)}\}$ are the simple roots of $C_n$ in the orthogonal basis

$$\alpha^{(1)} = (1, -1, 0, \ldots, 0),$$

$$\alpha^{(2)} = (0, 1, -1, 0, \ldots, 0),$$

$$\vdots$$

$$\alpha^{(n-1)} = (0, \ldots, 0, 1, -1),$$

$$\alpha^{(n)} = (0, \ldots, 0, 2).$$

c.f. (4.4).

Let us now consider the Hamiltonian $\mathcal{H}^{(II)}$ (3.29) corresponding to the second set of $K$-matrices (2.14). The appearance of $U_q(C_n)$ symmetry in this spin chain can be understood as a sort of ‘breaking’ of $B_n$ down to $C_n$. That is, we consider an embedding of $C_n$ in $B_n$, such that the vector space $V^{(2n+1)}$ at each site, which forms a $(2n+1)$-dimensional irreducible representation of $B_n$, decomposes into the direct sum of the $2n$-dimensional and 1-dimensional irreducible representations of $C_n$.

$$V^{(2n+1)} \cong W^{(2n)} \oplus W^{(1)}.$$

(4.18)

We construct the corresponding generators of $C_n$ on $V^{(2n+1)}$ by starting from the vector representation of the $C_n$ generators in terms of $(2n) \times (2n)$ matrices (4.14) and (4.15), and then inserting a column of 0’s between columns $n$ and $n+1$, and a row of 0’s between rows $n$ and $n+1$, thereby arriving at a set of $(2n+1) \times (2n+1)$ matrices. That is,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ C & 0 & D \end{pmatrix},$$

(4.19)

where $A, B, C, D$ represent $n \times n$ matrices.
In short, we henceforth represent the generators of $C_n$ by $(2n + 1) \times (2n + 1)$ matrices, such that the Cartan generators are given by the diagonal matrices
\[ H_\alpha = e_{\alpha, \alpha} - e_{2n+2-\alpha, 2n+2-\alpha}, \quad \alpha = 1, 2, \ldots, n, \] (4.20)
and the generators corresponding to the simple roots are given by
\[ E^+_\alpha = e_{\alpha, \alpha+1} + e_{2n+1-\alpha, 2n+1-\alpha}, \quad \alpha = 1, 2, \ldots, n-1, \]
\[ E^-_n = e_{n,n+2}, \] (4.21)
and $E^+_n = E^+_1$. Comparing with the corresponding expressions for the generators of $B_n$ (4.1) and (4.2), we see that they are exactly the same, except for $E^\pm_n$.

Below we shall also need another pair of generators of $C_n$, which we denote by $E^\pm_0$:
\[ E^+_0 = e_{1,2n+1}, \quad E^-_0 = e_{2n+1,1}, \] (4.22)
which are related to $E^\pm_n$ as follows
\[ E^\pm_n = \begin{cases} E^\pm_0 & n = 1 \\ -\frac{1}{2}[[[E^\pm_0, E^\pm_1], E^\pm_1], E^\pm_1] & n = 2 \\ \frac{1}{4}[[[[E^\pm_0, E^\pm_1], E^\pm_1], E^\pm_1], E^\pm_1] & n = 3, \\ \vdots & \vdots \\ (-\frac{1}{2})^{n-1}[[[[[E^\pm_0, E^\pm_1], E^\pm_1], \ldots, E^\pm_{n-1}], E^\pm_{n-1}] & n \end{cases} \] (4.23)
where the final line has a $2(n-1)$-fold multiple commutator.

The Cartan generators have the usual coproduct
\[ \Delta(H_j) = H_j \otimes I + I \otimes H_j, \quad j = 1, \ldots, n, \] (4.24)
and we propose the following coproducts for the first $n-1$ raising/lowering operators
\[ \Delta(E^\pm_j) = E^\pm_j \otimes e^{i\pi H_{j+1}} + e^{i\pi H_j} e^{-2\pi(H_j-H_{j+1})} \otimes E^\pm_j, \quad j = 1, \ldots, n-1. \] (4.25)

We have not succeeded to find such a simple expression for the coproduct for $E^\pm_n$. However, we observe that the generators $E^\pm_0$ (4.22) do have a simple coproduct
\[ \Delta(E^\pm_0) = E^\pm_0 \otimes I + e^{2\pi H_1} \otimes E^\pm_0. \] (4.26)
Hence, using (4.23), we obtain the result
\[ \Delta(E^\pm_n) = (-\frac{1}{2})^{n-1}[[[[\Delta(E^\pm_0), \Delta(E^\pm_1)], \Delta(E^\pm_1)], \Delta(E^\pm_1)], \Delta(E^\pm_1)]. \] (4.27)
These expressions for the coproducts satisfy the coassociativity property \cite{11}\(^\ast\)
\[ (\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta. \] (4.28)

\(^\ast\) An earlier version of this paper had a different expression for $\Delta(E^\pm_n)$, which did not satisfy the coassociativity property.
We observe the following relations for $1 \leq i, j \leq n$:

\[
\Delta(E^+_i) \Delta(E^-_j) - e^{4\eta} \Delta(E^-_i) \Delta(E^+_j) = \frac{e^{-4\eta(\Delta(H^i) - \Delta(H^+_i))} - \mathbb{I} \otimes \mathbb{I}}{e^{-4\eta} - 1},
\]

\[
e^{2\eta} \Omega_{ij} \Delta(E^+_i) \Delta(E^-_j) = \Delta(E^-_i) \Delta(E^+_j) \Omega_{ij}, \quad |i - j| = 1,
\]

\[
\Delta(E^+_i) \Delta(E^-_j) = \Delta(E^-_i) \Delta(E^+_j) \quad |i - j| \geq 2,
\]

(4.29)

where $\Omega_{ij}$ is given by (4.7).

By construction, the coproducts (4.24)–(4.27) commute with the ‘new’ two-site Hamiltonian (3.28)

\[
[\Delta(H^i), \tilde{h}_{1,2}] = [\Delta(E^\pm_i), \tilde{h}_{1,2}] = 0, \quad j = 1, \ldots, n.
\]

(4.30)

Moreover, all the generators (whose row $(n + 1)$ and column $(n + 1)$ are null, as in (4.19)) evidently commute with $U = e^{q_{n+1} L_{n+1}}$. Since the $N$-site Hamiltonian is given (3.29) by the sum of two-site Hamiltonians and a term proportional to $U_N$, it follows that the $N$-site Hamiltonian commutes with the $N$-fold coproducts

\[
[\Delta(H^N), \mathcal{H}^{(H)}] = [\Delta(E^\pm_N), \mathcal{H}^{(H)}] = 0, \quad j = 1, \ldots, n,
\]

(4.31)

which implies the $U_q(C_n)$ invariance of the Hamiltonian $\mathcal{H}^{(H)}$. We conjecture that this symmetry also extends to the full transfer matrix. The symmetry for the case $n = 1$ (note that $C_1 = A_1$) was first noted in [15].

4.2.1. Degeneracies and multiplicities for $U_q(C_n)$. The $U_q(C_n)$ invariance of the Hamiltonian implies that, for generic values of $\eta$, the $N$-site Hilbert space has a decomposition of the form (see equation (4.10))

\[
\mathcal{W}^{(2n)} \oplus \mathcal{W}^{(1)} \cong \bigoplus_j d^{(jN,n)} \mathcal{W}^{(j)},
\]

(4.32)

where $\mathcal{W}^{(j)}$ denotes an irreducible representation of $C_n$ with dimension $j$ (=degeneracy of the corresponding energy eigenvalue) and $d^{(jN,n)}$ is its multiplicity.

The first few cases are as follows (see again e.g. [26]):

\[
\begin{align*}
C_1 = A_1 : \quad & N = 2 : \quad (2 \oplus 1)^{\otimes 2} = 2 \cdot 1 \oplus 2 \cdot 2 \oplus 3 \\
& \quad = 2[0] \oplus 2[1] \oplus 2[2]
\end{align*}
\]

\[
\begin{align*}
& N = 3 : \quad (2 \oplus 1)^{\otimes 3} = 4 \cdot 1 \oplus 5 \cdot 2 \oplus 3 \cdot 3 \oplus 4 \\
& \quad = 4[0] \oplus 5[1] \oplus 3[2] \oplus 3[3]
\end{align*}
\]

(4.33)

\[
\begin{align*}
C_2 : \quad & N = 2 : \quad (4 \oplus 1)^{\otimes 2} = 2 \cdot 1 \oplus 2 \cdot 4 \oplus 5 \oplus 10 \\
& \quad = 2[0,0] \oplus 2[1,0] \oplus 2[0,1] \oplus 2[2,0]
\end{align*}
\]

\[
\begin{align*}
& N = 3 : \quad (4 \oplus 1)^{\otimes 3} = 4 \cdot 1 \oplus 6 \cdot 4 \oplus 3 \cdot 5 \oplus 3 \cdot 10 \oplus 2 \cdot 16 \oplus 20 \\
& \quad = 4[0,0] \oplus 6[1,0] \oplus 3[0,1] \oplus 3[2,0] \oplus 2[1,1] \oplus 3[3,0]
\end{align*}
\]

(4.34)
We have verified numerically that the Hamiltonian as well as the transfer matrix for set II (2.14) have exactly these degeneracies and multiplicities for generic values of \( \eta \), which provides further evidence of their \( U_q(C_n) \) invariance.

### 5. Bethe ansatz

Our discussion so far has not made use of the integrability of the models. However, this integrability has been exploited to obtain Bethe ansatz solutions of the models corresponding to sets I (2.13) and II (2.14) in [27] and [28], respectively.

Here we study how the quantum group symmetry of these models is reflected in their Bethe ansatz solutions. Our main result is a formula for the Dynkin label uniquely characterizes an irreducible representation, and in particular determines its dimension, which is the degeneracy of the corresponding eigenvalue. The number of distinct solutions of the Bethe equations with \( n \) Bethe roots determines the multiplicity. We readily verify numerically in section 6 that, in this way, the patterns of degeneracies and multiplicities predicted by the quantum group symmetry (4.11)–(4.13) and (4.33)–(4.35) are completely accounted for by the Bethe ansatz solutions.

#### 5.1. Review of the Bethe ansatz solutions

Before presenting our formula for the Dynkin labels, we briefly summarize here the Bethe ansatz solutions of the models. The Bethe states, which we denote by

\[
|A^{(m_1, \ldots, m_n)}\rangle = |\{u_1^{(1)}, \ldots, u_{m_1}^{(1)}\}, \ldots, \{u_1^{(n)}, \ldots, u_{m_n}^{(n)}\}\rangle
\]

depend on \( n \) sets of Bethe roots \( \{u_1^{(1)}, \ldots, u_{m_1}^{(1)}\}, \ldots, \{u_1^{(n)}, \ldots, u_{m_n}^{(n)}\} \), which are solutions of the following \( n \) sets of Bethe equations [27, 28]

---

4 The solution of the \( A_n^{(2)} \) family of integrable quantum spin chains has a long history. The initial work was for closed chains with periodic boundary conditions. The case \( n = 1 \) (corresponding to the Izergin–Korepin model) was first solved using the analytical Bethe ansatz approach [29, 30], which gave the eigenvalues (but not the eigenvectors) of the transfer matrix. This approach was subsequently extended to \( n > 1 \) in [31]. The algebraic Bethe ansatz for the case \( n = 1 \), which gave also the eigenvectors of the transfer matrix, was formulated in the important work [32]. The seminal work of Sklyanin [17] made it possible to generalize these results to open \( A_n^{(1)} \) chains. The case \( n = 1 \) with the first set of \( K \)-matrices (2.13) was solved using the analytical Bethe ansatz approach in [33], and this approach was subsequently extended to \( n > 1 \) in [27]. The algebraic Bethe ansatz for the case \( n = 1 \) was developed in [34, 35]. Finally, the algebraic Bethe ansatz for \( n > 1 \) with general diagonal \( K \)-matrices [20, 21] was formulated in [28]. An analytical Bethe ansatz approach for the case \( n = 1 \) with general non-diagonal \( K \)-matrices has recently been formulated in [36]. Other related work includes [37–44].
\[ e^{2N}(u_k^{(1)}) = \prod_{j=1, j \neq k}^{m_u} e_2(u_k^{(1)} - u_j^{(1)}) e_2(u_k^{(1)} + u_j^{(1)}) \prod_{j=1}^{m_u} e_{-1}(u_k^{(1)} - u_j^{(2)}) e_{-1}(u_k^{(1)} + u_j^{(2)}) \]

\[ k = 1, \ldots, m_1, \]

\[ 1 = \prod_{j=1}^{m_u-1} e_{-1}(u_k^{(l)} - u_j^{(l-1)}) e_{-1}(u_k^{(l)} + u_j^{(l-1)}) \prod_{j=1, j \neq k}^{m_u} e_2(u_k^{(l)} - u_j^{(l)}) e_2(u_k^{(l)} + u_j^{(l)}) \]

\[ \times \prod_{j=1}^{m_u+1} e_{-1}(u_k^{(l)} - u_j^{(l+1)}) e_{-1}(u_k^{(l)} + u_j^{(l+1)}), \quad k = 1, \ldots, m_1, \quad l = 2, \ldots, n - 1, \]

\[ \chi(u_k^{(n)}) = \prod_{j=1}^{m_u} e_{-1}(u_k^{(n)} - u_j^{(n-1)}) e_{-1}(u_k^{(n)} + u_j^{(n-1)}) \]

\[ \times \prod_{j=1, j \neq k}^{m_u} e_2(u_k^{(n)} - u_j^{(n)}) e_{-1}(u_k^{(n)} - u_j^{(n)} + i\pi) e_2(u_k^{(n)} + u_j^{(n)}) e_{-1}(u_k^{(n)} + u_j^{(n)} + i\pi), \quad k = 1, \ldots, m_u. \]

(5.2)

where here we use the compact notation

\[ e_k(u) = \frac{\sinh(\frac{u}{2} + \eta k)}{\sinh(\frac{u}{2} - \eta k)}. \]

(5.3)

and

\[ \chi(u) = \left\{ \begin{array}{ll} 1 & \text{for } U_q(B_n) \\ \left( \frac{\sinh(\frac{u+\eta + \pi}{2})}{\sinh(\frac{u-\eta + \pi}{2})} \right)^2 & \text{for } U_q(C_n) \end{array} \right. \]

(5.4)

The above equations are for \( n > 1 \). For \( n = 1 \), the Bethe equations are given by

\[ e_1^{2N}(u_k^{(1)}) \chi(u_k^{(1)}) = \prod_{j=1, j \neq k}^{m_u} e_2(u_k^{(1)} - u_j^{(1)}) e_{-1}(u_k^{(1)} - u_j^{(1)} + i\pi) \]

\[ \times e_2(u_k^{(1)} + u_j^{(1)}) e_{-1}(u_k^{(1)} + u_j^{(1)} + i\pi), \quad k = 1, \ldots, m_1. \]

(5.5)

The Bethe states are certain simultaneous eigenstates of the transfer matrix \( t(u) \) (2.18) and the Cartan generators \( \Delta_{(N)}(H_i) \) (4.1), (4.5), (4.20) and (4.24),

\[ t(u) |\Lambda^{(m_1, \ldots, m_n)}(u)\rangle = \Lambda^{(m_1, \ldots, m_n)}(u) |\Lambda^{(m_1, \ldots, m_n)}\rangle, \]

\[ \Delta_{(N)}(H_i) |\Lambda^{(m_1, \ldots, m_n)}\rangle = h_i |\Lambda^{(m_1, \ldots, m_n)}\rangle, \quad i = 1, \ldots, n. \]

(5.6)

The eigenvalues of the transfer matrix are given by [27, 28]
\[ \begin{align*}
\Lambda^{(m_1, \ldots, m_n)}(u) &= A^{(m_0)}(u) \psi_1(u) \frac{\sinh(u - 2(2n + 1)\eta) \cosh(u - (2n - 1)\eta)}{\sinh(u - 2\eta) \cosh(u - (2n + 1)\eta)} \left[ 2 \sinh(\frac{u}{2} - 2\eta) \cosh(\frac{u}{2} - (2n + 1)\eta) \right]^{2N} \\
&+ C^{(m_0)}(u) \tilde{\psi}_1(u) \frac{\sinh u}{\sinh(u - 4m\eta)} \cosh(u - (2n + 1)\eta) \left[ 2 \sinh(\frac{u}{2} - (2n - 1)\eta) \right]^{2N} \\
&+ \left\{ w(u) \psi_2(u) B^{(m_0)}_w(u) + \sum_{n=1}^{N} z_{\eta}(u) \psi_1(u) B^{(m_0, m_{n+1})}_l(u) + \tilde{z}_{\eta}(u) \tilde{\psi}_1(u) \tilde{B}^{(m_0, m_{n+1})}_l(u) \right\} \\
&+ \left[ 2 \sinh(\frac{u}{2} - (2n + 1)\eta) \right]^{2N}. \quad (5.7)
\end{align*} \]

where

\[ A^{(m_0)}(u) = \prod_{j=1}^{m_0} \frac{\sinh(\frac{u}{2} - u^{(1)}_j) + \eta}{\sinh(\frac{u}{2} - u^{(1)}_j) - \eta} \frac{\sinh(\frac{u}{2} + u^{(1)}_j + \eta)}{\sinh(\frac{u}{2} + u^{(1)}_j) + \eta}, \quad (5.8) \]

\[ C^{(m_0)}(u) = A^{(m_0)}(-u - \rho) = \prod_{j=1}^{m_0} \frac{\cosh(\frac{u}{2} - u^{(1)}_j) - 2(n + 1)\eta}{\cosh(\frac{u}{2} - u^{(1)}_j) - \eta} \frac{\cosh(\frac{u}{2} + u^{(1)}_j) - 2(2n + 1)\eta}{\cosh(\frac{u}{2} + u^{(1)}_j) + 2\eta}, \quad (5.9) \]

\[ B^{(m_0, m_{n+1})}_l(u) = \prod_{j=1}^{m_0} \frac{\sinh(\frac{u}{2} - u^{(1)}_j) - (l + 2)\eta}{\sinh(\frac{u}{2} - u^{(1)}_j) - l\eta} \frac{\sinh(\frac{u}{2} + u^{(1)}_j) - (l + 2)\eta}{\sinh(\frac{u}{2} + u^{(1)}_j) - l\eta} \times \prod_{j=1}^{m_{n+1}} \frac{\sinh(\frac{u}{2} - u^{(1)}_{j+1}) - (l + 1)\eta}{\sinh(\frac{u}{2} - u^{(1)}_{j+1}) - (l + 1)\eta} \frac{\sinh(\frac{u}{2} + u^{(1)}_{j+1}) - (l + 1)\eta}{\sinh(\frac{u}{2} + u^{(1)}_{j+1}) - (l + 1)\eta}, \quad (5.10) \]

\[ \tilde{B}^{(m_0, m_{n+1})}_l(u) = B^{(m_0, m_{n+1})}_l(-u - \rho), \quad l = 1, \ldots, n - 1, \]

\[ B^{(m_0)}_n(u) = \prod_{j=1}^{m_0} \frac{\sinh(\frac{u}{2} - u^{(n)}_j) - (n + 2)\eta}{\sinh(\frac{u}{2} - u^{(n)}_j) - n\eta} \frac{\sinh(\frac{u}{2} + u^{(n)}_j) - (n + 2)\eta}{\sinh(\frac{u}{2} + u^{(n)}_j) - n\eta} \times \frac{\cosh(\frac{u}{2} - u^{(n)}_j) - (n - 1)\eta}{\cosh(\frac{u}{2} - u^{(n)}_j) - (n - 1)\eta} \frac{\cosh(\frac{u}{2} + u^{(n)}_j) - (n + 1)\eta}{\cosh(\frac{u}{2} + u^{(n)}_j) - (n + 1)\eta}, \quad (5.11) \]

and

\[ z_l(u) = \frac{\sinh(u - 2(2n + 1)\eta) \cosh(2n - 1\eta)}{\sinh(u - 2\eta) \cosh(u + 2n - 1\eta)}, \quad l = 1, \ldots, n - 1, \]

\[ \tilde{z}_l(u) = z_l(-u - \rho), \quad l = 1, \ldots, n - 1, \]

\[ w(u) = \frac{\sinh(u - 2(2n + 1)\eta)}{\sinh(u - 2\eta) \cosh(u - 2n - 1\eta)}. \quad (5.12) \]
The Bethe states have been constructed in [28] using the nested algebraic Bethe ansatz approach. The ‘double-row’ monodromy matrix
\[
T_{\rho}(u) = T_{\rho}(u) K_{\rho}^{-1}(u) \hat{T}_{\rho}(u)
\]
can be written as a \((2n + 1) \times (2n + 1)\) matrix in the auxiliary space whose matrix elements are operators on the quantum space \(V^{\otimes N}\)
\[
T_{\rho}(u) = \begin{pmatrix}
A_1(u) & B_2(u) & B_3(u) & \ldots & B_{2n}(u) & F(u) \\
* & * & * & \ldots & * & * \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
* & * & * & \ldots & * & * \\
G(u) & C_2(u) & C_3(u) & \ldots & C_{2n}(u) & A_{2n+1}(u)
\end{pmatrix}
\]
\((2n + 1) \times (2n + 1)\)

The basic idea is to construct the Bethe states using the \(B_i(u)\) operators (as well as others) as creation operators acting on the reference state
\[
|0\rangle = \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}^{\otimes N}
\]

We conjecture that the (on-shell) Bethe states are highest-weight states of the quantum group
\[
\Delta_{(N)}(E^+_j) \Lambda^{(m_1, \ldots, m_k)} = 0, \quad i = 1, \ldots, n,
\]
as is the case for other integrable open quantum spin chains with quantum group symmetry (see e.g. [12, 14, 27, 33, 45–49]). However, a proof of this conjecture is beyond the scope of this paper. As a consequence of (5.18), degenerate eigenvectors (i.e. linearly independent eigenvectors of the transfer matrix \(t(u)\) whose corresponding eigenvalues coincide with the eigenvalue \(\Lambda^{(m_1, \ldots, m_k)}(u)\) of the Bethe state \(|\Lambda^{(m_1, \ldots, m_k)}\rangle\) which are obtained by acting on the Bethe state with the lowering operators \(\Delta_{(N)}(E^-_j)\) form an irreducible representation of the algebra that is uniquely characterized by the (highest) weights of the Bethe state, known as the Dynkin label.
5.2. Dynkin labels of the Bethe states

We propose that the Dynkin label \([a_1, \ldots, a_n]\) corresponding to a Bethe state \(|\Lambda^{(m_1, \ldots, m_n)}\rangle\) whose Bethe roots have cardinalities \((m_1, \ldots, m_n)\) is given for \(n > 1\) by

\[
\begin{align*}
a_1 &= N - 2m_1 + m_2, \\
a_i &= m_{i-1} - 2m_i + m_{i+1}, & i = 2, \ldots, n - 1, \\
a_n &= \begin{cases} 2(m_{n-1} - m_n) & \text{for } U_q(B_n) \\ m_{n-1} - m_n & \text{for } U_q(C_n) \end{cases}.
\end{align*}
\tag{5.19}
\]

For \(n = 1\),

\[
a_1 = \begin{cases} 2(N - m_1) & \text{for } U_q(B_1) \\ N - m_1 & \text{for } U_q(C_1) \end{cases}.
\tag{5.20}
\]

It is convenient to divide the proof of this result into two parts. The first part of the proof is the relation of the eigenvalues \((h_1, \ldots, h_n)\) of the Cartan generators to the cardinalities \((m_1, \ldots, m_n)\) of the Bethe roots

\[
\begin{align*}
h_1 &= N - m_1, \\
h_i &= m_{i-1} - m_i, & i = 2, 3, \ldots, n.
\end{align*}
\tag{5.21}
\]

This relation, which was proposed in [27], is the same as for the closed \(A_{2n}^{(2)}\) chain [31]. Its proof is sketched in appendix B.

The second part of the proof is the relation of the Dynkin label \([a_1, \ldots, a_n]\) to the eigenvalues \((h_1, \ldots, h_n)\) of the Cartan generators

\[
\begin{align*}
a_i &= h_i - h_{i+1}, & i = 1, 2, \ldots, n - 1, \\
a_n &= \begin{cases} 2h_n & \text{for } U_q(B_n) \\ h_n & \text{for } U_q(C_n) \end{cases}.
\end{align*}
\tag{5.22}
\]

This relation originates from the definition of Dynkin label (see e.g. [26])

\[
(h_1, \ldots, h_n) = \sum_{j=1}^{n} a_j \omega_j,
\tag{5.23}
\]

where \(\omega_j\) are the fundamental weights. In the orthogonal basis in which we work (recall equations (4.4) and (4.17)), the fundamental weights are given by

\[
\begin{align*}
\omega_1 &= (1, 0, 0, 0, \ldots, 0), \\
\omega_2 &= (1, 1, 0, 0, \ldots, 0), \\
\omega_3 &= (1, 1, 1, 0, \ldots, 0), \\
& \quad \vdots \\
\omega_{n-1} &= (1, 1, 1, \ldots, 1, 0), \\
\omega_n &= \begin{cases} (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) & \text{for } U_q(B_n) \\ (1, 1, \ldots, 1) & \text{for } U_q(C_n) \end{cases}.
\end{align*}
\tag{5.24}
\]

Substituting these expressions for the fundamental weights into (5.23), we see that
can be used to restrict the Bethe equation (5.2) ∈ ⩾ for the set I (2.13), and in tables B7 of the representation corresponding to the Dynkin label coincides with the degeneracy 6. Bethe equations with the given cardinality of Bethe roots by direct diagonalization, and the multiplicity (mult) i.e. the number of solutions of the Bethe equation (5.2) (or, equivalently, of the transfer matrix (2.18) at some generic value of −t obtained directly and from the reported solutions of the Bethe equations using the values of (m1, . . . , mn) for which solutions of the Bethe equation (5.2) with a given value of N can be expected.

6. Numerical check of completeness

We present solutions (m(1), . . . , m(n)) of the A2n Bethe equation (5.2) for small values of n and N and a generic value of η (namely, η = −0.1i) in tables B1–B6 for set I (2.13), and in tables B7–B12 for set II (2.14). Each table also displays the cardinalities (m1, . . . , mn) of the Bethe roots, the corresponding Dynkin label [a1, . . . , ar] obtained using the formula (5.19), the degeneracy (‘deg’) of the corresponding eigenvalue of the Hamiltonian H(I) and H(U) (or, equivalently, of the transfer matrix t(u) at some generic value of u) obtained by direct diagonalization, and the multiplicity (‘mult’) i.e. the number of solutions of the Bethe equations with the given cardinality of Bethe roots.

We observe that, for each solution of the Bethe equations in these tables, the dimension of the representation corresponding to the Dynkin label coincides with the degeneracy. Moreover, the degeneracies and multiplicities predicted by the quantum group symmetry (4.11)–(4.13) and (4.33)–(4.35) are completely accounted for by the Bethe ansatz solutions.

The eigenvalues of the Hamiltonians H(I) (3.19) and H(U) (3.29), as well as the eigenvalues of the transfer matrix t(u) (2.18) for the two sets (2.13)–(2.14) at some generic value of u, are not displayed in the tables in order to minimize their size. Nevertheless, we have computed these eigenvalues both directly and from the reported solutions of the Bethe equations using

Inverting the relations (5.25), we arrive at the desired result (5.22).

The main result (5.19) and (5.20) follows immediately from the two relations (5.21) and (5.22).

Since the Dynkin labels are nonnegative ai ≥ 0, the result (5.19) can be inverted to deduce the values of (m1, . . . , mn) for which solutions of the Bethe equation (5.2) with a given value of N can be expected.

Figure 1. Dynkin diagrams for (a) A2(n) (b) Bn (c) Cn.
(5.14) and (5.7)–(5.13), respectively; and we find perfect agreement between the results from these two approaches.

7. Conclusions

We have argued that the $A_{2n}^{(2)}$ integrable open quantum spin chains with the boundary conditions specified by (2.13) and (2.14) have the quantum group symmetries $U_q(B_n)$ and $U_q(C_n)$, respectively, see equations (4.9) and (4.31). A key point of this argument is that the Hamiltonians can be expressed as sums of two-body terms, see (3.19) and (3.29). In hindsight, the appearance of $B_n$ and $C_n$ can be inferred from the extended Dynkin diagram for $A_{2n}^{(2)}$ (see figure 1): removing the rightmost or leftmost nodes yields the Dynkin diagrams for the subalgebras $B_n$ or $C_n$, respectively.

We have also found a formula (5.19) for the Dynkin label of a Bethe state; the Dynkin label uniquely characterizes an irreducible representation, and in particular determines its dimension, which is the degeneracy of the corresponding eigenvalue. With the help of this formula, we have verified numerically (for a generic value of $\eta$) that the degeneracies and multiplicities implied by the quantum group symmetry (4.11)–(4.13) and (4.33)–(4.35) are completely accounted for by the Bethe ansatz solutions, see tables B1–B6 and B7–B12, respectively. Similar results have recently been noted for the simpler case of the $U_q(A_1)$-invariant spin-1/2 chain [12] at generic values of $q$ in [50].

Several interesting problems remain to be addressed, including the following: proving that the transfer matrix $t(u)$ for the set II (2.14) has $U_q(C_n)$ symmetry; showing that the Bethe states have the highest weight property (5.18); and investigating the case that $q$ is a root of unity (non-generic values of $\eta$). We also note that the sets (2.13) and (2.14) do not exhaust the possible integrable diagonal boundary conditions [20, 21]. We expect that models with these other boundary conditions will have ‘less’ quantum group symmetry, which nevertheless may be worth exploring. It may also be interesting to find explicit formulas for the multiplicities in the tensor product decompositions of $B_n$ (4.10) and $C_n$ (4.32) in terms of the Dynkin labels $[a_1, \ldots, a_n]^\mathbb{R}$. These multiplicities should—remarkably—coincide with the number of solutions of the Bethe equation (5.2) at generic values of $\eta$ for the corresponding (5.19) values of $m_1, \ldots, m_n$.

Acknowledgments

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\footnote{For the case of $A_1$, such a formula is well known, see e.g. equation (2.8) in [50]. Some recent progress on this problem was reported in [51, 52].}
Appendix A. The $A^{(2)}_{2n}$ R-matrix

The R-matrix associated with the fundamental representation of $A^{(2)}_{2n}$ was found by Bazhanov [2, 3] and Jimbo [4]. We follow the latter reference; however, as in [27], we use the variables $u$ and $\eta$ instead of $x$ and $k$, respectively, which are related as follows:

$$x = e^u, \quad k = e^{2\eta}. \quad (A.1)$$

The R-matrix is given by$^9$

$$R(u) = c(u) \sum_{\alpha \neq \alpha'} e_{\alpha\alpha} \otimes e_{\alpha'\alpha} + b(u) \sum_{\alpha \neq \beta, \beta'} e_{\alpha\alpha} \otimes e_{\beta\beta'}$$

$$+ (e(u) \sum_{\alpha < \beta, \alpha \neq \beta'} + \bar{e}(u) \sum_{\alpha > \beta, \alpha \neq \beta'}) e_{\alpha\beta} \otimes e_{\beta\alpha} + \sum_{\alpha, \beta} a_{\alpha\beta}(u) e_{\alpha\beta} \otimes e_{\alpha'\beta'}, \quad (A.2)$$

with

$$c(u) = 2 \sinh(u/2 - 2\eta) \cosh(u/2 - (2n + 1)\eta),$$

$$b(u) = 2 \sinh(u/2) \cosh(u/2 - (2n + 1)\eta),$$

$$e(u) = -2e^{-u} \sinh(2\eta) \cosh(u/2 - (2n + 1)\eta),$$

$$\bar{e}(u) = e^u e(u), \quad (A.3)$$

$$a_{\alpha\beta}(u) = \begin{cases} 
\sinh(u - (2n - 1)\eta) + \sinh((2n - 1)\eta) & \alpha = \beta, \alpha \neq \alpha', \\
\sinh(u - (2n + 1)\eta) + \sinh((2n - 1)\eta) + \sinh((2n + 3)\eta) - \sinh((2n + 1)\eta) & \alpha = \beta, \alpha = \alpha', \\
-2e^{((2n + 1) + 2(\alpha - \beta))\eta} e^{-u} \sinh(u/2) \sinh(2\eta) & \alpha < \beta, \alpha \neq \beta', \\
2e^{((2n + 1) - 2(\beta - \alpha))\eta} e^{-u} \sinh((2n + 3)\eta) \sinh(2\eta) - 2e^{((2n + 1) - 2\beta)\eta} \cosh((2n + 2) - 2\beta)\eta) \sinh(2\eta) & \alpha < \beta, \alpha = \beta', \\
2e^{((2n + 1) + 2(\alpha - \beta))\eta} e^{-u} \sinh(u/2) \sinh(2\eta) - 2e^{((2n + 1) - 2\beta)\eta} \cosh(2\beta)\eta) \sinh(2\eta) & \alpha > \beta, \alpha \neq \beta', \\
2e^{u - 2\beta\eta} \sinh((2n + 1) - 2\beta)\eta) \sinh(2\eta) - 2e^{((2n + 1) - 2\beta)\eta} \cosh(2\beta)\eta) \sinh(2\eta) & \alpha > \beta, \alpha = \beta', \\
\end{cases}$$

where

$$\tilde{\alpha} = \begin{cases} 
\alpha + \frac{1}{2} & 1 \leq \alpha < n + 1 \\
\alpha & \alpha = n + 1 \\
\alpha - \frac{1}{2} & n + 1 < \alpha \leq 2n + 1 
\end{cases}, \quad (A.4)$$

$$\alpha' = 2n + 2 - \alpha, \quad (A.5)$$

$^9$This expression for the R-matrix differs from the one given in [4] by the overall factor $2e^{u - (2n + 3)\eta}$. 

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This $R$-matrix has crossing symmetry (2.7), where $V$ is given by\(^\text{10}\)
\[
V = \sum_\alpha e^{\alpha_\alpha} \delta_{\alpha, \alpha'} + \sum_{\alpha < \alpha'} e^{-(2n+1-2\alpha)} \eta e^{\alpha}_{\alpha\alpha'} + \sum_{\alpha > \alpha'} e^{(2n+1-2\alpha') \eta} e^{\alpha}_{\alpha\alpha'}.
\] (A.6)

The matrix $M = V^2$ is therefore given by the diagonal matrix
\[
M = \text{diag}(e^{\alpha_\alpha}), \quad \alpha = 1, 2, \ldots, 2n+1.
\] (A.7)

**Appendix B. Eigenvalues of the Cartan generators**

We sketch here a proof of the relation (5.21)
\[
h_1 = N - m_1, \quad h_i = m_{i-1} - m_i, \quad i = 2, 3, \ldots, n,
\] (B.1)
based on the nested algebraic Bethe ansatz solution [28]. Since the argument is somewhat intricate, it is helpful to first consider some special cases. Hence, as a first warm-up, we consider the case $A_2^{(2)}$ in appendix B.1; and then, as a second warm-up, we consider the case $A_4^{(2)}$ in appendix B.2. Finally, we consider the general case $A_{2n}^{(2)}$ in appendix B.3\(^\text{11}\).

**B.1. $A_2^{(2)}$**

For the case $n = 1$, the Bethe states are given by
\[
|\Lambda^{(m_1)}\rangle = B_2(u^{(1)}) \cdots B_2(u^{(1)}_{m_1}) |0\rangle + \ldots,
\] (B.2)

where $B_2(u)$ is the operator appearing in the double-row monodromy matrix (5.16), and $|0\rangle$ is the reference state (5.17). The ellipsis denotes contributions from terms that also depend on the operator $F(u)$, which here and below we assume can be safely ignored. Using the facts\(^\text{12}\)
\[
[H_1, B_2(u)] = -B_2(u), \quad H_1 |0\rangle = N |0\rangle,
\] (B.3)
we immediately see that
\[
H_1 |\Lambda^{(m_1)}\rangle = (N - m_1) |\Lambda^{(m_1)}\rangle.
\] (B.4)
Therefore $h_1 = N - m_1$, in agreement with (B.1).

**B.2. $A_4^{(2)}$**

We now consider the case $n = 2$, where nesting first appears. The (first-level) Bethe states are given by
\[
|\Lambda^{(m_1, m_2)}\rangle = f_{i_1 \cdots i_{m_1}} B_i(u^{(1)}_{i_1}) \cdots B_{i_{m_1}}(u^{(1)}_{m_1}) |0\rangle + \ldots,
\] (B.5)

where $i_1, \ldots, i_{m_1} \in \{2, 3, 4\}$, $f_{i_1 \cdots i_{m_1}}$ are coefficients that are still to be determined, and summation over repeated indices is understood.

\(^\text{10}\) We take this opportunity to correct several typos in the corresponding equation (59) in [27].

\(^\text{11}\) The proof of (B.1) presented here supersedes the discussion given in appendix B of [27].

\(^\text{12}\) In order to lighten the notation, here and below we drop the notation $\Delta_{ij}$ for the Cartan generators on $k$ sites.
Let $n_i$ denote the number of $B_i(u)$ operators appearing in $|\Lambda^{(m_1, m_2)}\rangle$ (B.5). Evidently,
\[ m_1 = n_2 + n_3 + n_4. \]  
(B.6)

Using the facts
\[ [H_1, B_j(u)] = -B_j(u), \quad j = 2, 3, 4, \quad H_1|0\rangle = N|0\rangle, \]  
(B.7)
we obtain
\[ H_1|\Lambda^{(m_1, m_2)}\rangle = (N - n_2 - n_3 - n_4)|\Lambda^{(m_1, m_2)}\rangle, \]  
(B.8)
which, in view of (B.6), again implies $h_1 = N - m_1$.

Moreover, using the facts
\[ [H_2, B_j(u)] = \begin{cases} 
B_j(u) & \text{for } j = 2 \\
-B_j(u) & \text{for } j = 4 \\
0 & \text{otherwise}
\end{cases}, \quad H_2|0\rangle = 0, \]  
(B.9)
we obtain
\[ H_2|\Lambda^{(m_1, m_2)}\rangle = (n_2 - n_4)|\Lambda^{(m_1, m_2)}\rangle, \]  
(B.10)
which implies
\[ h_2 = n_2 - n_4. \]  
(B.11)

The coefficients in (B.5) are given by the scalar product\(^\text{13}\)
\[ f_{u_1 \ldots u_{m_1}} = \left( e_{r_1} \otimes \cdots \otimes e_{r_{m_1}} \right) |\tilde{\psi}\rangle, \]  
(B.12)
where $|\tilde{\psi}\rangle$ is the second-level state
\[ |\tilde{\psi}\rangle = B_2(u_1^{(2)}) \cdots B_2(u_{m_2}^{(2)})|\tilde{0}\rangle + \ldots \]  
(B.13)

\(^{13}\)Since the transfer matrix is symmetric (see appendix B in [33]), its left and right eigenvectors are each other’s transpose.
where $\tilde{B}_2(u)$ are the $A_2^{(2)}$ creation operators constructed as in (5.15) with $n = 1$ except with inhomogeneous monodromy matrices (the inhomogeneities are given by $\{u_{1}^{(1)}, \ldots, u_{m_1}^{(1)}\}$). Moreover,

$$|\tilde{0}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes^{m_1} ,$$

(B.14)

and

$$|e_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad |e_3\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad |e_4\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

(B.15)

Let $\tilde{H}_1$ denote the Cartan generator for the case $A_2^{(2)}$, and let us now evaluate its matrix element

$$\left( \langle e_{1i} | \otimes \cdots \otimes \langle e_{m_1} | \right) \tilde{H}_1 |\tilde{\psi}\rangle$$

(B.16)

in two different ways. To compute the action of $\tilde{H}_1$ to the right, we use $\tilde{H}_1 |\tilde{\psi}\rangle = (m_1 - m_2) |\tilde{\psi}\rangle$, similarly to (B.4). To compute the action of $\tilde{H}_1$ to the left, we use the fact

$$\tilde{H}_1 |e_j\rangle = \begin{cases} |e_j\rangle & \text{for } j = 2 \\ -|e_j\rangle & \text{for } j = 4 \\ 0 & \text{otherwise} \end{cases}$$

(B.17)
and therefore
\[
\left( |e_i| \otimes \cdots \otimes |e_{i_n}| \right) \tilde{H}_1 = \left( |e_i| \otimes \cdots \otimes |e_{i_n}| \right) (n_2 - n_4).
\]  
(B.18)

We conclude that
\[
(n_2 - n_4)f_{i_1 \cdots i_n} = (m_1 - m_2)f_{i_1 \cdots i_n},
\]  
(B.19)

which implies that \(f_{i_1 \cdots i_n}\) is zero unless
\[
n_2 - n_4 = m_1 - m_2.
\]  
(B.20)

Recalling (B.11), we conclude that \(h_2 = m_1 - m_2\), in agreement with (B.1).
\[ A^{(2)}_{2n} \]

In order to treat the general case, it is necessary to adopt a more systematic (but unfortunately significantly heavier) notation. We therefore write the Bethe states as

\[ |\Lambda^{(m_1,\ldots,m_n)}\rangle = f^{(1)}_{i_1^{(1)} \ldots i_{m_1}^{(1)}} |\psi^{(1)}_{i_1^{(1)} \ldots i_{m_1}^{(1)}}\rangle. \tag{B.21} \]

**B.3. First level.** The first-level states are given by

\[ |\psi^{(1)}_{i_1^{(1)} \ldots i_{m_1}^{(1)}}\rangle = B^{(1)}_{i_1^{(1)}} (u^{(1)}_{1^{(1)}}) \cdots B^{(1)}_{i_{m_1}^{(1)}} (u^{(1)}_{m_1^{(1)}}) |0^{(1)}\rangle + \ldots, \tag{B.22} \]

where \( i_1^{(1)}, \ldots, i_{m_1^{(1)}} \in \{2, \ldots, 2n\} \); and \( B^{(1)}_i (u) \equiv B_i (u) \) and \( |0^{(1)}\rangle \equiv |0\rangle \) are given by (5.16) and (5.17), respectively.

Letting \( n^{(1)}_i \) denote the number of \( B^{(1)}_i (u) \) operators appearing in (B.22), we have

\[ m_1 = n^{(1)}_2 + \ldots + n^{(1)}_{2n}. \tag{B.23} \]

For \( H^{(1)}_i \equiv H_i \), we have for \( i = 1 \):

\[ [H^{(1)}_1, B^{(1)}_j (u)] = -B^{(1)}_j (u), \quad j = 2, \ldots, 2n, \quad H^{(1)}_1 |0^{(1)}\rangle = N |0^{(1)}\rangle; \tag{B.24} \]

and for \( i > 1 \):

\[ [H^{(1)}_i, B^{(1)}_j (u)] = \begin{cases} B^{(1)}_j (u) & \text{for } j = i \\ -B^{(1)}_j (u) & \text{for } j = 2n + 2 - i \\ 0 & \text{otherwise} \end{cases}, \quad H^{(1)}_i |0^{(1)}\rangle = 0. \tag{B.25} \]

Therefore

\[ H^{(1)}_i |\Lambda^{(m_1,\ldots,m_n)}\rangle = (N - n^{(1)}_2 - \ldots - n^{(1)}_{2n}) |\Lambda^{(m_1,\ldots,m_n)}\rangle, \]

\[ H^{(1)}_i |\Lambda^{(m_1,\ldots,m_n)}\rangle = (n^{(1)}_i - n^{(1)}_{2n + 2 - i}) |\Lambda^{(m_1,\ldots,m_n)}\rangle, \quad i = 2, \ldots, n, \tag{B.26} \]

which implies

\[ h_1 = N - m_1, \]

\[ h_i = n^{(1)}_i - n^{(1)}_{2n + 2 - i}, \quad i = 2, \ldots, n. \tag{B.27} \]
Table B10. $U_q(C_2), N = 3$

| $m_1$ | $m_2$ | $a_1$ | $a_2$ | deg | mult | $\{u_1^{(1)}\}$ | $\{u_2^{(2)}\}$ |
|-------|-------|-------|-------|-----|------|-----------------|-----------------|
| 0     | 0     | 3     | 0     | 20  | 1    | —               | —               |
| 1     | 0     | 1     | 1     | 16  | 2    | 0.115986        | —               |
|       |       |       |       |     |      | 0.351133        | —               |
| 1     | 1     | 2     | 0     | 10  | 3    | 1.584176        | 1.70996         |
|       |       |       |       |     |      | 0.321003        | 0.760756        |
|       |       |       |       |     |      | 0.112316        | 0.701168        |
| 2     | 1     | 0     | 1     | 5   | 3    | 0.382283, 0.963791 | 1.34411         |
|       |       |       |       |     |      | 0.118089, 1.05603 | 1.3902          |
|       |       |       |       |     |      | 0.113785, 0.333555 | 0.2923          |
| 2     | 2     | 1     | 0     | 4   | 6    | 0.397606, 0.688759 | 0.9169 ± 0.307663i |
|       |       |       |       |     |      | 1.23957 ± 0.466025i | 1.39288 ± 0.481069i |
|       |       |       |       |     |      | 0.119249, 1.66217i | 0.711593i, 2.20269i |
|       |       |       |       |     |      | 0.116831, 0.865494 | 0.934114 ± 0.250442i |
|       |       |       |       |     |      | 0.385256, 1.71269i | 0.61563i, 2.28518i |
|       |       |       |       |     |      | 0.117124, 0.36247i | 0.3474i, 2.68405i |
| 3     | 3     | 0     | 0     | 1   | 4    | 0.988238, 0.860023 ± 0.700064i | 1.18442, 1.06721 ± 0.745058i |
|       |       |       |       |     |      | 0.425069, 0.848958, 1.5453i | 1.13679, 0.680288i, 2.20976i |
|       |       |       |       |     |      | 0.113851, 0.338372, 1.44183i | 0.279454 ± 0.211351i, 2.39497i |
|       |       |       |       |     |      | 0.120953, 0.94247, 1.51754i | 1.1893, 0.757077i, 2.14786i |
Table B11. \( U_q(C_3), N = 2 \)

| \( m_1 \) | \( m_2 \) | \( m_3 \) | \( a_1 \) | \( a_2 \) | \( a_3 \) | deg mult | \( u_k^{(1)} \) | \( u_k^{(2)} \) | \( u_k^{(3)} \) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 2 | 0 | 0 | 21 | 1 | — | — |
| 1 | 0 | 0 | 0 | 1 | 0 | 14 | 1 | 0.201347 | — |
| 1 | 1 | 1 | 1 | 0 | 0 | 6 | 2 | 0.190268 | 0.796966 | 1.04547 |
| 2 | 2 | 2 | 0 | 0 | 0 | 1 | 2 | 0.9507 ± 0.448287i | 1.21099 ± 0.473011i | 1.36739 ± 0.485421i |

B.3.2. Second level. The coefficients in (B.21) are given by the scalar product

\[
\langle e_i^{(1)} | \otimes \cdots \otimes e_i^{(1)} \rangle \langle \psi^{(2)} | \otimes \cdots \otimes (u_i^{(2)}) f_j^{(2)} \rangle = (B.28)
\]

where the second-level states are given by

\[
|\psi^{(2)}\rangle_{\ell_1^{(2)} \cdots \ell_{n_2}^{(2)}} = B_{\ell_1^{(2)} \cdots \ell_{n_2}^{(2)}}^{(2)} (u_i^{(2)}) |0^{(2)}\rangle + \ldots ,
\]

where \( \ell_1^{(2)}, \ldots, \ell_{n_2}^{(2)} \in \{2, \ldots, 2n - 2\} \); \( B_{\ell_1^{(2)}}^{(2)}(u) \) are the (inhomogeneous) creation operators for \( A_{2n-2}^{(2)} \); and

\[
|0^{(2)}\rangle = \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}_{2n-1}
\]

Moreover,

\[
|e_2^{(1)}\rangle = \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}_{2n-1}, \quad \ldots, \quad |e_{2n}^{(1)}\rangle = \begin{pmatrix}
0 \\
0 \\
\vdots \\
1
\end{pmatrix}_{2n-1}
\]

We have that

\[
m_2 = n_2^{(2)} + \ldots + n_{2n-2}^{(2)}.
\]

and hence

\[
H_1^{(2)} |\psi^{(2)}\rangle_{\ell_1^{(2)} \cdots \ell_{n_2}^{(2)}} = (m_1 - m_2) |\psi^{(2)}\rangle_{\ell_1^{(2)} \cdots \ell_{n_2}^{(2)}},
\]

\[
H_i^{(2)} |\psi^{(2)}\rangle_{\ell_1^{(2)} \cdots \ell_{n_2}^{(2)}} = (n_i^{(2)} - n_2^{(2)}_{2n-i}) |\psi^{(2)}\rangle_{\ell_1^{(2)} \cdots \ell_{n_2}^{(2)}}, \quad i = 2, \ldots, n - 1.
\]

Furthermore,

\[
H_i^{(2)} |e_j^{(1)}\rangle = \begin{cases}
|e_j^{(1)}\rangle & \text{for } j = i + 1 \\
-|e_j^{(1)}\rangle & \text{for } j = 2n + 1 - i \\
0 & \text{otherwise}
\end{cases}
\]

Evaluating the matrix element
| $m_1$ | $m_2$ | $m_3$ | $a_1$ | $a_2$ | $a_3$ | deg | mult  | $\{u_k^{(1)}\}$ | $\{u_k^{(2)}\}$ | $\{u_k^{(3)}\}$ |
|-------|-------|-------|-------|-------|-------|------|-------|---------------|---------------|---------------|
| 0     | 0     | 0     | 0     | 3     | 0     | 0    | 56    | 1             | —             | —             |
| 1     | 0     | 0     | 1     | 1     | 1     | 0    | 64    | 2             | 0.115986      | —             |
|       |       |       |       |       |       |      |       |               | 0.351133      | —             |
| 1     | 1     | 1     | 2     | 0     | 0     | 21   | 3     | 0.113011      | 0.78232       | 1.03469       |
|       |       |       |       |       |       |      |       |               | 0.326182      | 1.07712       |
|       |       |       |       |       |       |      |       |               | 1.86199       | 2.10901       |
| 2     | 1     | 0     | 0     | 0     | 1     | 14   | 3     | 0.115986, 0.351133 | 0.331791      | —             |
| 2     | 1     | 1     | 0     | 1     | 0     | 14   | 3     | 0.117608, 1.20956 | 1.58891       | 1.71382       |
|       |       |       |       |       |       |      |       |               | 0.11417, 0.336229 | 0.298556       | 0.751895      |
|       |       |       |       |       |       |      |       |               | 0.372422, 1.13284 | 1.54735        | 1.67608       |
| 2     | 2     | 2     | 1     | 0     | 0     | 6    | 6     | 0.377177, 0.835326 | 1.03506 ± 0.320648i | 1.23102 ± 0.37667i |
|       |       |       |       |       |       |      |       |               | 1.41135 ± 0.538002i | 1.59277 ± 0.555019i | 1.70587 ± 0.566549i |
|       |       |       |       |       |       |      |       |               | 0.409958, 1.35937i | 0.414206i, 1.74096i | 0.791369i, 2.1359i |
|       |       |       |       |       |       |      |       |               | 0.116641, 0.980323 | 1.05068 ± 0.260276i | 1.24798 ± 0.342251i |
|       |       |       |       |       |       |      |       |               | 0.118386, 0.376545 | 0.367, 1.49213i | 0.65911i, 2.2416i |
|       |       |       |       |       |       |      |       |               | 0.121155, 1.30785i | 0.572787i, 1.67608i | 0.885614i, 2.04744i |
| 3     | 3     | 3     | 0     | 0     | 0     | 1    | 4     | 0.443222, 0.937877, 1.23794i | 1.24501i, 0.44955i, 1.66845i | 1.41773i, 0.832685i, 2.09128i |
|       |       |       |       |       |       |      |       |               | 1.11346, 0.953596 ± 0.779036i | 1.3403, 1.18988 ± 0.830487i | 1.47811i, 1.32949 ± 0.861223i |
|       |       |       |       |       |       |      |       |               | ?                         | ?                 | ?             |
|       |       |       |       |       |       |      |       |               | ?                         | ?                 | ?             |
\[
\left( (e^{(1)}_{i_{\ell_1}}) \otimes \cdots \otimes (e^{(1)}_{i_{\ell_q}}) \right) H^{(2)}_i |\psi^{(2)}\rangle_{i_{\ell_2} \cdots i_{\ell_q}} f^{(2)}_{i_{\ell_2} \cdots i_{\ell_q}} \]  

in two different ways by acting with \( H^{(2)}_i \) to both the left and the right, we obtain for \( i = 1 \)

\[
n^{(1)}_2 - n^{(1)}_{2n} = m_1 - m_2,
\]

and for \( i > 1 \)

\[
n^{(1)}_{i+1} - n^{(1)}_{2n+1-i} = n^{(2)}_i - n^{(2)}_{2n-i}, \quad i = 2, \ldots, n - 1.
\]

\textbf{B.3.3. Level } k. \text{ At level } k = 2, 3, \ldots, n - 1, \text{ we have}

\[
f^{(k-1)}_{i_{\ell_1} \cdots i_{\ell_{k-1}}} = \left( (e^{(k-1)}_{i_{\ell_1}}) \otimes \cdots \otimes (e^{(k-1)}_{i_{\ell_{k-1}}}) \right) |\psi^{(k)}\rangle_{i_{\ell_2} \cdots i_{\ell_{k-1}}} f^{(k)}_{i_{\ell_2} \cdots i_{\ell_{k-1}}},
\]

where the level-\( k \) states are given by

\[
|\psi^{(k)}\rangle_{i_{\ell_1} \cdots i_{\ell_k}} = B_{i_{\ell_1}}^{(k)} (u_{i_{\ell_1}}) \cdots B_{i_{\ell_k}}^{(k)} (u_{i_{\ell_k}}) |0^{(k)}\rangle + \ldots,
\]

where \( i_1, \ldots, i_m \in \{2, \ldots, 2n - 2k + 2\}; B_{i_{\ell}}^{(k)} (u) \) are the (inhomogeneous) creation operators for \( \mathcal{A}^{(2)}_{2n-2k+2}; \) and

\[
|0^{(k)}\rangle = \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}_{2n-2k+3} \otimes m_{k-1},
\]

Moreover,

\[
|e^{(k-1)}_{2} = \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}_{2n-2k+3} \quad \text{and} \quad |e^{(k-1)}_{2n-2k+4} = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1
\end{pmatrix}_{2n-2k+3}.
\]

We have that

\[
m_k = n^{(k)}_2 + \ldots + n^{(k)}_{2n-2k+2}.
\]

Also,

\[
[H^{(k)}_i, B^{(k)}_j (u)] = -B^{(k)}_j (u), \quad j = 2, \ldots, 2n - 2k + 2, \quad H^{(k)}_i |0^{(k)}\rangle = m_{k-1} |0^{(k)}\rangle;
\]

and for \( i > 1 \):

\[
[H^{(k)}_i, B^{(k)}_j (u)] = \begin{cases}
B^{(k)}_j (u) & \text{for } j = i \\
-B^{(k)}_j (u) & \text{for } j = 2n - 2k + 4 - i \\
0 & \text{otherwise}
\end{cases}, \quad H^{(k)}_i |0^{(k)}\rangle = 0.
\]
Hence

\[ H_i^{(k)} | \psi^{(k)} \rangle_{i, \ldots, i_n} = (m_{k-1} - m_k) | \psi^{(k)} \rangle_{i, \ldots, i_n}, \]

\[ H_i^{(k)} | \psi^{(k)} \rangle_{i, \ldots, i_n} = (n_i - n_{2n-2k+4-i}) | \psi^{(k)} \rangle_{i, \ldots, i_n}, \quad i = 2, \ldots, n - k + 1. \]  \tag{B.45}

Furthermore,

\[ H_i^{(k)} | e^{(k-1)} \rangle = \begin{cases} | e^{(k-1)} \rangle & \text{for } j = i + 1 \\ -| e^{(k-1)} \rangle & \text{for } j = 2n - 2k + 5 - i \\ 0 & \text{otherwise} \end{cases} \tag{B.46} \]

Evaluating the matrix element

\[ \left( \left( e^{(k-1)} \right)_{i_{i-1}} \otimes \cdots \otimes \left( e^{(k-1)} \right)_{i_{n_{i-1}}} \right) H_i^{(k)} | \psi^{(k)} \rangle_{i, \ldots, i_n} f_i^{(k)}_{i_{i-1}} \]

in two different ways by acting with \( H_i^{(k)} \) to both the left and the right, we obtain

\[ n_2^{(k-1)} - n_2^{(k-1)} = m_{k-1} - m_k, \]

\[ n_1^{(k-1)} - n_1^{(k-1)} = n_i^{(k)} - n_{2n-2k+4-i}, \quad i = 2, \ldots, n - k + 1. \]  \tag{B.48}

**B.3.4. Level n.** At the final level \( k = n \), we have

\[ f_i^{(n-1)}_{i_{i-1}, \ldots, i_{n_{i-1}}} = \left( e^{(n-1)} \right)_{i_{i-1}} \otimes \cdots \otimes \left( e^{(n-1)} \right)_{i_{n_{i-1}}} | \psi^{(n)} \rangle, \]  \tag{B.49}

where the level-\( n \) states are given by

\[ | \psi^{(n)} \rangle = B_2^{(n)} (u_1^{(n)}) \cdots B_2^{(n)} (u_{m_1}^{(n)}) | \psi^{(0)} \rangle + \ldots, \]  \tag{B.50}

where \( B_i^{(n)} (u) \) are the (inhomogeneous) creation operators for \( A_2^{(2)} \). Moreover,

\[ | \psi^{(0)} \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \mathbf{m}_{n-1}, \]  \tag{B.51}

and

\[ | \psi^{(n-1)} \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad | \psi_3^{(n)} \rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad | \psi_4^{(n)} \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]  \tag{B.52}

We have that

\[ H_i^{(n)} | \psi^{(n)} \rangle = (m_{n-1} - m_n) | \psi^{(n)} \rangle \]  \tag{B.53}

and

\[ H_i^{(n)} | e^{(n-1)} \rangle = \begin{cases} | e^{(n-1)} \rangle & \text{for } j = 2 \\ -| e^{(n-1)} \rangle & \text{for } j = 4 \\ 0 & \text{otherwise} \end{cases} \tag{B.54} \]
Evaluating the matrix element
\[
\left( |e^{(n-1)}_{i} \rangle \otimes \cdots \otimes |e^{(n-1)}_{i_{n-1}} \rangle \right) H^{(n)}_{1} |\psi^{(n)} \rangle
\]
(B.55)
in two different ways, we obtain
\[
n_{2}^{(n-1)} - n_{4}^{(n-1)} = m_{n-1} - m_{n}.
\]
(B.56)

Combining all the results (B.27), (B.36), (B.37), (B.48) and (B.56), we obtain the desired relations (B.1). Indeed, one can see that
\[
h_{i} = n_{i+2-k}^{(k-1)} - n_{2k+4-k-1}^{(k-1)}, \quad k = 2, \ldots, n,
\]
(B.57)

which gives for \( i = k \)
\[
h_{k} = n_{2(k-1)}^{(k-1)} - n_{2k-2k+4}^{(k-1)} = m_{k-1} - m_{k},
\]
(B.58)

where the second equality follows from (B.48).

ORCID

Rafael I Nepomechie https://orcid.org/0000-0003-1000-3400

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