Theoretical model of static semi-ellipsoidal droplet on a horizontal surface

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Abstract: The objective of this work is to establish a theoretical static force equilibrium equation of the semi-ellipsoidal droplet on a horizontal wall by analyzing the forces acting on the differential element and deducing the expressions of surface tension as well as the internal pressure acting on the droplet. The relationship between the static spreading radius, the density, the volume, the surface tension coefficient and the contact angle of the droplet is given. The results from the equation are compared with the experimental data and show a good agreement. The data from the numerical simulation using the volume of fluid method also agree well with those from the theoretical equation.

The collision processes of droplet on wall are widespread in industrial and agricultural production as well as in nature [1-4], such as the collision of raindrops on the surface of airplanes and trains, the collision of droplets in the fire-fighting [5] and in the spray cooling, the fuel spray impingement in the cylinder of internal combustion engine [6] and the hitting of droplet on the blades in the turbo-machinery [7] etc. The process of droplet colliding with wall is an important research content of the two-phase flow. It is also a basic content of heat and mass transfer studies.

Predecessors have done a lot of researches in the process of droplet impacting on wall. In the aspect of theoretical studies, Chandra [8] theoretically analyzed the maximum diameter of droplet hitting on wall according to the energy conservation, and got an expression of the maximum spreading diameter of droplet. Pasandideh et al. [9] improved the existing model by taking the viscous boundary layer into account and ignoring the gas-liquid surface energy of droplet. Vafaei et al. [10] thought that the axial symmetric droplet did not meet the initial Young equation. [11] They gave an equation of contact angle depending on the mass of droplet and properties of the three phases by analyzing the surface tension and the gravity. Borisov et al. [12] analyzed the adhesive attraction between droplet and wall. They got a theory that the viscous force does not have to be considered for droplets in minor diameter. They also concluded that the $We$ number and the wetting angle have majority influences on the motion state of droplet after the collision on wall. In the experiments, Sikalo et al. [13-14] got the droplet deformation at different collision speed and the viscosity after collision on the paraffin and

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They also got the effect of collision speed and viscosity on the droplet spreading. Bi et al. \cite{15} did an experiment about the spreading characteristics of droplet impacting on solid surface. They got a result that the viscosity has an important influence on spreading of droplet. They also found that the surface tension plays a crucial role for the retraction process. In the meanwhile, the collision speed influences the largest spreading factor. In the numerical simulation, Li Chunxi et al. \cite{16} established the film thickness and concentration evolution model by using lubrication theory. They got the spreading characteristics of droplet and the influence law of the substrate structural parameters. Li Yanpeng et al. \cite{17} simulated the flow processes of droplet impacting on a curved surface with low impact energy. To sum up, the researches on droplet impingement tends to use experimental method and simulation method. Because of the complexity of droplet flow, there are huge difficulties in analyzing the stresses acting on droplet theoretically. Thus, few relative studies had been done on this area.

This article analyzes the forces on the differential element of the static semi-ellipsoidal droplet on flat wall, the mathematical expression deduction of surface tension and internal pressure on the element, the force equilibrium equation of static droplet, the static spreading radius and the height of the semi-ellipsoidal droplet on horizontal wall by solving the equation numerically, and the comparison of the radius and height from the equation with experimental data and numerical simulation results.

1. Theoretical model of static spreading radius

In certain conditions, the droplet oscillates once it collides with wall under the action of the internal pressure, surface tension and inertial force. The droplet spread out along the wall at the beginning of collision. When the droplet reaches its maximum spreading radius, the spreading speed reduces to zero and the droplet begins to retract. Similarly, once it retracts to the minimum spreading radius, the retraction speed becomes zero the spreading restarts. Under the action of viscous damping dissipation, the amplitude of spreading and retraction decreases gradually. In the last, the droplet reaches the stationary state. At this time, the spreading radius of the droplet is the static spreading radius.

1.1 Droplet geometric model and stress analysis

There are three types of contact angle of the droplet with the wall, as shown in Fig.1. Fig.1a, Fig.1b and Fig.1c show the contact angles less than, equal to and larger than 90° respectively. The direction of the contact force is same as the direction of the contact angle.

![Figure 1. Contact angle of actual droplet.](image)

Under certain conditions, the shape of a real droplet is close to semi-ellipsoidal. \cite{18} Therefore, this
article assumes the droplet having the shape of semi-ellipsoid. This article also assumes that
the magnitude and the direction of contact force acting on semi-ellipsoidal droplet are same as that acting
on real droplet.

In static state, the semi-ellipsoidal geometry model of a droplet is shown in Fig. 2a. The radius of
droplet in the horizontal direction is \( r \). The height is \( z \). A differential fluid element is taken in the
semi-ellipsoid. The differential angle of the element is \( d\alpha \). The angle between the tangent line at point
\((r', z')\) and the axis \( r \) is \( \beta \). The angle between the tangent line at point \((r'+dr', z'+dz')\) and the axis \( r \) is
\( \beta + d\beta \).

The shape equation of semi-ellipsoidal droplet is:

\[
\frac{r'^2}{r^2} + \frac{z'^2}{z^2} = 1 \quad \text{or} \quad \left\{ \begin{array}{l} r' = r \cos(\phi) \\ z' = z \sin(\phi) \end{array} \right. \quad \text{if} \quad r > 0, z > 0, 0 < \phi < \frac{\pi}{2}
\]  

(1)

The relationship between the slope and the angle $\beta$ is:

\[
\frac{dz'}{dr'} = -\tan \beta
\]

(2)

The stresses acting on the droplet element are shown in Fig. 2 b. $F_{\sigma}$ is the surface tension acting
on the curved liquid surface of element and $F_{p}$ is the internal pressure acting on the side face of
element. Their expressions are as follows:

The magnitude of surface tension:

- The left edge of surface: $F_{\sigma 1} = \sigma r'd\alpha$  \hspace{1cm} (3)
- The right edge of surface: $F_{\sigma 2} = \sigma (r'+dr')d\alpha$  \hspace{1cm} (4)
- The front edge of surface: $F_{\sigma 3} = \sigma dl$  \hspace{1cm} (5)
The rear edge of surface: \( F_{\alpha z} = \sigma dl \) (6)

Where, \( l \) is the 1/2 circumference of a semi-ellipsoidal droplet, \( l = \frac{\pi z}{2} + r - z \)

The magnitude of internal pressure:

Internal average pressure applied on the side face of element is:

\[
p = 2\sigma k + \frac{1}{2} \left[ \rho g z + \rho g (z - z') \right]
\] (7)

Where:

\( 2\sigma k \) — the Laplace pressure, i.e., the pressure difference between the inside and the outside of the curved liquid surface of element

\( k \) — the curvature at point \((0, z)\), \( k = \frac{z}{r^2} \)

\[ \frac{1}{2} \left[ \rho g z + \rho g (z - z') \right] \] — The average static pressure, i.e., the average gravitational pressure between point \((r', z')\) and point \((r', 0)\).

The left side of element: \( F_{p1} = p r' z' d\alpha \) (8)

The right side of element: \( F_{p1}' = (p + dp) (r' + dr')(z' + dz') d\alpha \) (9)

The front side of element: \( F_{p2} = pz' dr' \) (10)

The rear side of element: \( F_{p2}' = pz' dr' \) (11)

Adding up the projections of above forces (equation (3) to (6), and (8) to (11)) in the \( r' \)-axis direction and omitting the higher order infinitesimal, we can obtain the static force equilibrium equation of element:

\[
(-\sigma r' \sin \beta d\alpha d\beta + \sigma \cos \beta d\alpha dr') - 2\sigma dl \sin \frac{d\alpha}{2} +
\]

\[
(-pr' d\alpha dz' - pz' d\alpha dr' - r' z' d\alpha dp) + 2pz' dr' \sin \frac{d\alpha}{2} = 0
\]

The contact force between droplet and wall:

\[
F_{\sigma} = \int_0^{2\pi} \sigma r \cos \theta d\alpha = 2\pi \sigma r \cos \theta \] (13)

Integrating Eq. (12) in the entire droplet and adding Eq. (13), we can get:

\[
\int \left[ (-\sigma r' \sin \beta d\alpha d\beta + \sigma \cos \beta d\alpha dr') - 2\sigma dl \sin \frac{d\alpha}{2} \right] +
\int \left[ (-pr' d\alpha dz' - pz' d\alpha dr' - r' z' d\alpha dp) + 2pz' dr' \sin \frac{d\alpha}{2} \right] + 2\pi \sigma r \cos \theta = 0
\] (14)

Solving Eq. (1) and (2) simultaneously, we can get the relationship between the geometric parameters of droplet:
Substituting Eq. (15) to (17) into Eq. (14) and simplifying the result, we can obtain the integral static force equilibrium equation of droplet:

\[ -2\pi r(1 - \cos \theta) + \frac{9\rho \gamma \sigma}{4r^4} + \left( \frac{3}{4} - \frac{1}{\pi} \right) \frac{3\rho \gamma^2 \rho}{2r^3} = 0 \]  

Eq. (18) gives the theoretical relationship between the static spreading radius, the density, the volume, the surface tension coefficient and the contact angle of droplet. In Eq. (18), the first item is the resultant surface tension, and its direction is towards the center of droplet, which plays a decisive role in droplet retraction; and other items are the resultant internal pressure, and its direction is away from the center, which plays a leading role in droplet spreading. When droplet is in static, the resultant surface tension is equal to the resultant internal pressure, but their direction is opposite. In this paper, Eq. (18) is solved by using the method of dichotomy to obtain the static spreading radius.

2. Numerical simulation of static spreading radius

The oscillation process of droplet on wall is simulated by VOF method of commercial software Fluent\textsuperscript{[19]}. When the oscillation stops under the action of viscosity damping, the droplet reaches its stationary state, and the static spreading radius can be obtained. The droplet may have different initial velocity, i.e., the oscillation process is different, but its final static spreading radius is the same.

Fig. 3 shows the axial symmetric region for numerical calculation. The radius and the height of the region are 3mm and 2 mm respectively. The uniform quadrilateral grid has 0.015 mm side length which meets the requirements of the precision of numerical calculation. The pressure boundary is considered as an atmospheric pressure. The wall condition is considered as no relative slipping between the droplet and the wall. The coupling algorithm of droplet velocity and pressure is the PISO method. The iteration converges when the residual is less than $10^{-6}$ with the time step is $4 \times 10^{-6}$s.
The droplet is water and its initial shape is hemispherical. The ambient medium is air. The gas and the liquid are both assumed incompressible in the calculation. The physical parameters and initial conditions are shown in Table 1. Where $\mu$ is the coefficient of viscosity, or the dynamic viscosity, $\sigma$ is the coefficient of surface tension, and $v_0$ is the initial spreading speed.

| Table 1. Physical parameters and initial conditions of the droplet |
|-------------------------|-------|--------|-----------|-------|
| Droplet                 | $\rho$ | $\mu$ | $\sigma$ | $\theta$ | $V$  | $v_0$ |
| kg m$^{-3}$             | Pa s   | N m$^{-1}$ | °       | m$^3$  | m s$^{-1}$ |
| Water                   | 1000   | 0.001  | 0.073    | 64     | $2.62 \times 10^{-10}$ | 0.5 |

3. Model Validation

3.1. Experimental verification of the theoretical model

In order to verify the reliability of the theoretical model of static spreading radius, the comparison between the shape of the theoretical droplet and that of the experiment is shown in Fig. 4. The experimental result comes from the literature [20]. The droplet volume is $8.18 \times 10^{-9}$ m$^3$, the measured equilibrium contact angle is $95^\circ$, the static spreading radius is 1.56 mm, and the rest of the parameters are in reference [20]. The static spreading radius from Eq. (18) is 1.588 mm, the relative error between theory and experiment is 1.79%. The static height of the theoretical models and the experimental droplets are 1.550 mm and 1.59 mm respectively, and the relative error is -2.52%. The theoretical calculations and experimental results are fairly close, though the static spreading radius of theoretical model is slightly larger than that of the experimental data. It can be seen easily from Fig. 4 that the theoretical shape of droplet is in good agreement with that of the experiment.

3.2. Comparisons between numerical simulation and theoretical model
The droplet static shapes of theoretical model and of numerical simulation are shown in Fig. 5. The droplet volume is $2.62 \times 10^{-10} \text{ m}^3$. The contact angle of Fig. 5(a), Fig. 5 (b) and Fig. 5 (c) are 85°, 90° and 95° respectively. It can be seen easily that when the contact angle is 85°, the static spreading radius of theoretical model is slightly less than that of the numerical simulation; when the contact angle is 90°, their results are in good agreement; when the contact angle is 95°, the static spreading radius of theoretical model is slightly greater than that of the numerical simulation.

**Figure 5.** Comparison of the static spreading form.
The static spreading radii of the theoretical model and of the numerical simulation are shown in Table 2. It can be seen from Table 2 that when $64^\circ < \theta < 90^\circ$, the radius will become smaller as the contact angle becomes larger, and the absolute value of the error between the theoretical model and the numerical simulation becomes smaller, but the error is negative; when the droplet contact angle is $90^\circ$, the error is -0.4%, the absolute value of the error is smallest; when $90^\circ < \theta < 105^\circ$, the radius will become smaller as the contact angle becomes larger, the error becomes positive and larger. The maximum absolute value of the error of the static spreading radius is within 13%.

### Table 2. Static spreading radii of simulation and of theory.

| Program | Contact angle (°) | Static spreading radii (mm) | Error (%) |
|---------|------------------|-----------------------------|-----------|
|         |                  | Simulation solutions | Theoretical solutions |       |
| 1       | 64               | 0.617                      | 0.538      | -12.80%  |
| 2       | 70               | 0.591                      | 0.528      | -10.66%  |
| 3       | 85               | 0.525                      | 0.507      | -3.43%   |
| 4       | 90               | 0.503                      | 0.501      | -0.40%   |
| 5       | 95               | 0.482                      | 0.496      | 2.90%    |
| 6       | 105              | 0.440                      | 0.486      | 10.45%   |

The static height of theoretical model can be calculated from Eq. (19). The results of the static height of theoretical model and of numerical simulation are listed in Table 3. It can be seen from the results that when $64^\circ < \theta < 90^\circ$, the height will become larger as the contact angle becomes larger, and the error between theoretical model and numerical simulation becomes smaller and maintains positive; when the droplet contact angle is $90^\circ$, the error is 0.2% and minimum; when $90^\circ < \theta < 105^\circ$, the height will become larger as the contact angle becomes larger, the error is negative and the absolute value of the error becomes larger. The maximum absolute error of the static height is also within 13%.

### Table 3. Static height of simulation and of theory.

| Program | Contact angle (°) | Static height (mm) | Error (%) |
|---------|------------------|--------------------|-----------|
|         |                  | Simulation solutions | Theoretical solutions |       |
| 1       | 64               | 0.388              | 0.432      | 11.34%   |
| 2       | 70               | 0.412              | 0.448      | 8.74%    |
| 3       | 85               | 0.476              | 0.486      | 2.10%    |
| 4       | 90               | 0.497              | 0.498      | 0.20%    |
It can be seen from Table 2 and Table 3 that the relative errors of the radius and of the static height have different sign, but their absolute values are within 13%. This fact shows that the theoretical solutions and the numerical results are in good agreement. There are differences in shape between the semi-ellipsoidal droplet and the real droplet. The edge of semi-ellipsoid droplet is vertical with the wall, and a contact force is applied along the direction of contact angle of real droplet to simulate the forces acting on the droplet by the wall. When the contact angle of the real droplet is 90 °, the semi-elliptical model is closest to the actual droplet, so the errors between theoretical formula and numerical simulation are minimal. When the contact angle of real droplet is close to 90 °, the semi-elliptical model is also close to the real droplet. And the absolute values of error between theoretical model and numerical simulation are smaller. So that the theoretical model and the numerical simulation have a higher fit.

Comparing the results of theoretical model with that of experiment and of numerical simulation, we can find that there is a good agreement between the three. This shows that the model of force acting on the static semi-ellipsoidal droplet is correct and can be used farther to study the motion state of the droplet.

Solving Eq. (18), we can find that the static spreading radius has nothing to do with the initial velocity of droplet, and that the static spreading radius increases with the increasing of the density and the volume of droplet and decreases with the increasing of the surface tension coefficient and the contact angle.

4. Conclusion

Based on the purely theoretical stress analysis of the stationary state of semi-ellipsoidal droplet, we get the relationship between the radius and the density, the volume, the surface tension coefficient and the contact angle. Comparing the theoretical model results with the experimental and the numerical results, we get the following conclusions:

1. The theoretical relationship of static spreading radius can be used to analyze the spreading radius and the static height of droplet on the horizontal wall, and to calculate accurately the size of area covered by droplet;

2. The error of static spreading radius between theoretical model and experiment result is 1.79%, and the error of static height is 2.52%. In the range of the contact angle which has been simulated, the error absolute values of the radius and of the height between theoretical model and numerical simulation have the same changing rule and are less than 13%. When the contact angle is 90 °, the errors of the radius and of the height are minimum values, and the values are 0.4% and 0.4% respectively;

3. The analysis on the theoretical model of semi-ellipsoidal static droplet shows that the static spreading radius has nothing to do with the initial speed of droplet impacting on the wall; the static spreading radius increases with the increasing of droplet density and volume and decreases with the increasing of surface tension coefficient and contact angle.
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