COBORDISM OF FLAG BUNDLES

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Abstract. Let $G$ be a connected linear algebraic group over a field $k$ of characteristic zero and let $P$ be a parabolic subgroup of $G$ containing a fixed maximal torus $T$. For a scheme $X$ of finite type over $k$ and a principal $G$-bundle $E \to X$, we describe the rational algebraic cobordism of the flag bundle $E/P \to X$ in terms of the cobordism groups of $X$ and the classifying space $BT$. In particular, we obtain formulae for the algebraic cobordism groups of the various flag bundles associated to a vector bundle on a scheme. As a consequence, we describe the cobordism group of any principal bundle over a scheme. We also obtain similar formula for the higher Chow groups of flag bundles.

1. Introduction

Let $k$ be a field of characteristic zero. In this paper, we shall consider only those schemes which are quasi-projective over $k$. Based on the construction of the motivic algebraic cobordism spectrum $MGL$ by Voevodsky in the stable homotopy category of $k$, and the already known cobordism theory for complex manifolds [16], Levine and Morel [12] invented the algebraic cobordism theory $\Omega^*(-)$. The most important aspect of this theory is that $\Omega^*(-)$ is the universal oriented Borel-Moore homology theory in the category of $k$-schemes. In particular, it is the universal oriented cohomology theory in the category of smooth schemes over $k$.

As a consequence, many known theories, e.g., algebraic $K$-theory, Chow groups, can be directly obtained from the cobordism theory of Levine and Morel. This makes the question of describing the algebraic cobordism groups of various schemes interesting and important. Since this theory has been invented only some years ago, not many cases of computations of $\Omega^*(-)$ have been known. Levine and Morel showed that the coefficient ring $\Omega^*(k)$ is isomorphic to the known Lazard ring. They were also able to describe the algebraic cobordism of a projective bundle in terms of the cobordism group of the base scheme. The principal aim of this paper is to generalize this description to the case of arbitrary flag bundles. As a consequence, we also describe the cobordism groups of principal $G$-bundles over $k$-schemes.

In order to motivate our main results, we recall the following result, due to Borel and Leray, well known in algebraic topology and its analogue in algebraic geometry, due to Vistoli [19]. Assume $k = \mathbb{C}$ is the field of complex numbers and let $G$ be a connected and reductive complex algebraic group. We fix a maximal torus $T$ of $G$, a Borel subgroup $B$ of $G$ containing $T$ and let $W$ denote the Weyl group of $G$ with respect to $T$. Let $\hat{T}$ denote the character group of $T$ and let $\text{Sym}(\hat{T})$ denote the symmetric algebra of $\hat{T} \otimes \mathbb{Q}$ over $\mathbb{Q}$. Let $X$ be a complex manifold and let $E \to X$ be a principal $G$-bundle. The reader can think of it as a $G(\mathbb{C})$-fiber bundle over $X(\mathbb{C})$. Since the principal bundles are represented by the maps to the classifying

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spaces in topology, we immediately get the characteristic homomorphisms
\[
\text{Sym}(\hat{T})^W \cong H^*(BG, \mathbb{Q}) \xrightarrow{c_E} H^*(X, \mathbb{Q}) \text{ and } \text{Sym}(\hat{T}) \xrightarrow{\alpha_E} H^*(E/B, \mathbb{Q}).
\]
The homomorphism \(\alpha_E\) sends a character \(t\) of \(T\) to the first Chern class of the associated line bundle \(E/B \times \mathbb{A}^1 \rightarrow E/B\). This induces a homomorphism of \(\text{Sym}(\hat{T})\)-algebras
\[
(1.1) \quad H^*(X, \mathbb{Q}) \otimes_{\text{Sym}(\hat{T})^w} \text{Sym}(\hat{T}) \xrightarrow{\lambda_X} H^*(E/B, \mathbb{Q})
\]
and is an isomorphism.

Let \(X\) be now a scheme and let \(\text{CH}_*(X)\) denote the rational Chow group of algebraic cycles on \(X\) modulo the rational equivalence. Let \(A^*(X)\) be the Fulton-MacPherson bivariant cohomology ring of \(X\). Recall that this is a subring of the endomorphism ring of \(\text{CH}_*(X)\). In the same set up as above, Vistoli [19] showed that there are still the characteristic maps
\[
\text{Sym}(\hat{T})^W \xrightarrow{c_E} A^*(X) \text{ and } \text{Sym}(\hat{T}) \xrightarrow{\alpha_E} A^*(E/B)
\]
such that the induced map of \(\text{Sym}(\hat{T})\)-modules
\[
(1.2) \quad \text{CH}_*(X) \otimes_{\text{Sym}(\hat{T})^w} \text{Sym}(\hat{T}) \xrightarrow{\lambda_X} \text{CH}_*(E/B)
\]
is an isomorphism. This completely describes the Chow groups of the flag bundle in terms of the Chow group of the base.

In this paper, we study similar questions for the description of the higher Chow groups and more importantly, the algebraic cobordism groups of generalized flag bundles over any base scheme. The similar techniques can also be used to write down the description of the complex cobordism of flag bundles over complex manifolds. In case of the higher Chow groups, we obtain a more direct proof of the above formula using the localization sequence for these groups. In particular, this yields a different and simpler proof of Vistoli’s theorem for the Chow groups. The proof in the case of the cobordism becomes much more complicated, mainly due to the absence of the higher cobordism groups at present. In this case, we adapt some of the arguments of [19] to the case of cobordism. We now state our main results.

Let \(G\) be a connected linear algebraic group and let \(T, B\) and \(W\) be a fixed split maximal torus, a Borel subgroup containing the maximal torus and the associated Weyl group respectively. We shall often denote this datum by the quadruplet \((G, T, B, W)\). Let \(r\) denote the rank of \(T\). Let \(P\) be a parabolic subgroup of \(G\) containing \(B\) and let \(W_P\) denote the Weyl group of the Levi subgroup of \(P\) with respect to \(T\).

Let \(S(G)\) and \(C(G)\) denote the \(G\)-equivariant rational Chow ring and the algebraic cobordism ring of \(\text{Spec}(k)\) (see Section 3 below). Since we are interested in describing the higher Chow groups and cobordism groups with the rational coefficients, we make the convention that an abelian group \(A\) for us will actually mean \(A \otimes_{\mathbb{Z}} \mathbb{Q}\). Furthermore, we shall write the higher Chow groups and the cobordism groups cohomologically in this paper in the sense that \(\text{CH}^i(X, n)\) and \(\Omega^i(X)\) will mean the groups \(\text{CH}_{\dim(X)-i}(X, n)\) and \(\Omega_{\dim(X)-i}(X)\) respectively. For any scheme \(X\), we shall write the full Chow groups as
\[
\text{CH}^*(X) = \bigoplus_{i=0}^{\infty} \bigoplus_{n=0}^{\infty} \text{CH}^i(X, n).
\]

Let \(p : E \rightarrow X\) be a principal \(G\)-bundle and let \(\pi : E/P \rightarrow X\) be the flag bundle associated to the parabolic subgroup \(P\). We show in Section 3 below
that the algebraic cobordism groups $\Omega^*(X)$ and $\Omega^*(E/P)$ are modules over the rings $C(G)$ and $C(P)$ respectively. Moreover, it is known (cf. [10, Theorem 6.6]) that $C(G) = C(T)^W$ and $C(P) \cong C(T)^{WP}$. The similar methods also show that the higher Chow groups $\text{CH}^*(X, n)$ and $\text{CH}^*(E/P, n)$ are modules over the rings $S(G) = S^W$ and $S(P) = S^{WP}$ respectively. Now we have:

**Theorem 1.1.** The natural map of $C(P)$-modules

\[ \lambda_X : \Omega^*(X) \otimes_{C(G)} C(P) \to \Omega^*(E/P) \]

is an isomorphism. Moreover, it is an isomorphism of rings if $X$ is smooth.

**Theorem 1.2.** The natural map of $S(P)$-modules

\[ \alpha_X : \text{CH}^*(X) \otimes_{S(G)} S(P) \to \text{CH}^*(E/P) \]

is an isomorphism. This is an isomorphism of rings if $X$ is smooth.

As consequences of these results, we obtain the formulae (cf. Corollaries 6.6 and 8.2) for the cobordism and the higher Chow groups of principal bundles. We remark here that as we are working with the rational coefficients, the assumption about the maximal torus $T$ being split is not a necessary one. One can reduce to this case by the transfer arguments.

We conclude the introduction with a brief outline of the contents of this paper. We recall the definitions and some important properties of the ordinary and the equivariant algebraic cobordism in the next section. We use these fundamental properties to construct our map $\lambda_X$ in Section 3. We also deduce some functorial properties of this map with respect to morphisms between schemes. In section 4, we prove some algebraic results and use these together with some results of [10] to deduce our main result for algebraic cobordisms of trivial flag bundles over smooth schemes. In Section 5, we prove the surjectivity of $\lambda_X$ using the localization sequence for the cobordism, the corresponding result for the trivial flag bundles and an induction argument. We prove Theorem 3.4 in Section 6 by first proving it for the trivial flag bundles, which uses the proof of the similar result for the Chow groups of trivial bundles in [19], and then using a filtration argument for general schemes. The final proof of Theorem 1.1 is given in Section 7, where we deduce this from the case of flag bundles associated to the Borel subgroups. The last section is devoted to the proof of Theorem 1.2 where the main tool is the long exact localization sequence for the higher Chow groups.

2. Recollection of Ordinary and Equivariant Cobordism

In this section, we briefly recall the definitions and basic properties of the ordinary and equivariant algebraic cobordism.

2.1. Algebraic cobordism. Recall from [12] that for any scheme $X$ and any $i \in \mathbb{Z}$, the algebraic cobordism group $\Omega^i(X)$ is given by the quotient of the $\mathbb{Q}$-vector space $Z^i(X)$ on the classes of projective morphisms $[Y \xrightarrow{f} X]$, where $Y$ is a smooth scheme and $f$ has relative codimension $i = \dim(X) - \dim(Y)$. This quotient is obtained by the relations in $Z^i(X)$ defined by certain axioms like the dimension axiom, section axiom and the formal group law. It was later shown by Levine and Pandharipande [14] that $\Omega^i(X)$ can also be described as the quotient of $Z^i(X)$ by the subspace generated by those cobordism cycles which are given by the double point degeneration relation. In particular, there is a natural surjection $Z^*(X) \to \Omega^*(X)$. It also follows that $\Omega^*(X)$ is a graded $\mathbb{Q}$-vector space, where the grading is given by the codimension of a cobordism cycle. Moreover, $\Omega^i(X) = 0$ for $i > \dim(X)$ and $\Omega^i(X)$ could be non-zero for any $-\infty < i \leq \dim(X)$. In fact, the
exterior product on cobordism makes $\Omega^*(X)$ a graded ring for smooth $X$, which is a graded $\Omega^*(k)$-algebra. In general, $\Omega^*(X)$ is a graded $\Omega^*(k)$-module.

The following is the main result of Levine and Morel from which most of their other results on algebraic cobordism are deduced.

**Theorem 2.1.** The functor $X \mapsto \Omega_*(X)$ is the universal Borel-Moore homology on the category of $k$-schemes. In other words, it is universal among the homology theories on this category which have functorial push-forward for projective morphism, pull-back for smooth morphism (any morphism of smooth schemes), Chern classes for line bundles, and which satisfy Projective bundle formula, homotopy invariance, the above dimension, section and formal group law axioms. Moreover, for a $k$-scheme $X$ and closed subscheme $Z$ of $X$ of pure codimension $p$ with open complement $U$, there is a localization exact sequence

$$\Omega_*(Z) \to \Omega_*(X) \to \Omega_*(U) \to 0.$$ 

It was also shown in loc. cit. that the natural composite map

$$\Phi : L \to L \otimes Q \Omega^*(k) \to \Omega^*(k)$$

$$a \mapsto [a]$$

is an isomorphism of commutative graded rings. Here, $L$ is the Lazard ring which is a polynomial ring over $Q$ on infinite but countably many variables and is given by the quotient of the polynomial ring $Q[A_{ij} | (i, j) \in \mathbb{N}^2]$ by the relations, which uniquely define the universal formal group law $F_L$ of rank one on $L$. This formal group law is given by the power series

$$F_L(u, v) = u + v + \sum_{i,j \geq 1} a_{ij} u^i v^j,$$

where $a_{ij}$ is the equivalence class of $A_{ij}$ in the ring $L$. The Lazard ring is graded by putting the degree of $a_{ij}$ to be $1 - i - j$. In particular, one has $L_0 = Q, L_{-1} = Q a_{11}$ and $L_i = 0$ for $i \geq 1$, that is, $L$ is non-positively graded. We refer to loc. cit. for more properties of algebraic cobordism.

2.2. **Equivariant algebraic cobordism.** Let $(G, T, B, W)$ be the datum as above for a given connected linear algebraic group $G$ over $k$. For a scheme $X$ with a linear action of $G$, the equivariant algebraic cobordism of $X$ was defined by Deshpande [4] when $X$ is smooth and this was later defined and studied for all schemes in [10]. Since this is a new theory and since we shall have need for this here, albeit in a mild way, we briefly recall it.

For any integer $j \geq 0$, let $V_j$ be an $l$-dimensional representation of $G$ and let $U_j$ be a $G$-invariant open subset of $V_j$ such that the codimension of the complement $(V_j - U_j)$ in $V_j$ is at least $j$ and $G$ acts freely on $U_j$ such that the quotient $U_j/G$ is a quasi-projective scheme. Such a pair $(V_j, U_j)$ is called a **good** pair for the $G$-action corresponding to $j$. It is known that in our set up, good pairs always exist (cf. [4] Lemma 9]). Let $X_G$ denote the mixed quotient $X \times U_j$ of the product $X \times U_j$ by the diagonal action of $G$, which is free.

Let $X$ be a $k$-scheme of dimension $d$ with a $G$-action. Fix $j \geq 0$ and let $(V_j, U_j)$ be an $l$-dimensional good pair corresponding to $j$. Put

$$\Omega^G_l(X)_j = \frac{\Omega^G_{i+l-q} \left( X \times U_j \right)}{F_{d+l-g-j} \Omega^G_{i+l-q} \left( X \times U_j \right)}.$$ (2.1)
Here, $F_p \Omega_i(X)$ is the $p$th level of the Niveau filtration (cf. [10] Section 3) which is roughly given by the subspace of $\Omega_i(X)$ generated by the images of $\Omega_i(Z) \to \Omega_i(X)$ under the push-forward map, where $Z \to X$ is a closed subscheme of dimension at most $i$. It is known that $\Omega^G_i(X)$ is independent of the choice of the good pair $(V_j, U_j)$ and one defines

\begin{equation}
\Omega^G_i(X) := \lim_{\to j} \Omega^G_{i_j}(X).
\end{equation}

The reader should note from the above definition that unlike the ordinary cobordism, the equivariant algebraic cobordism $\Omega^G_i(X)$ can be non-zero for any $i \in \mathbb{Z}$. We let

\[ \Omega^G(X) = \bigoplus_{i \in \mathbb{Z}} \Omega^G_i(X) \]

and we let

\begin{equation}
\Omega^G_*(X) = \bigoplus_{i \in \mathbb{Z}} \Omega^G_i(X), \text{ where } \Omega^G_i(X) = \Omega^G_{\dim(X) - i}(X).
\end{equation}

The equivariant cobordism satisfies all those properties which are listed in Theorem 2.1 for the ordinary algebraic cobordism. Since we shall need some of these properties, we state them below for the sake of completeness.

**Theorem 2.2.** (cf. [10] Theorems 5.1, 5.4) The equivariant algebraic cobordism satisfies the following properties.

(i) Functoriality: The assignment $X \mapsto \Omega_*(X)$ is covariant for projective maps and contravariant for smooth maps of $G$-schemes. It is also contravariant for l.c.i. morphisms of $G$-schemes.

(ii) Homotopy: If $f : E \to X$ is a $G$-equivariant vector bundle, then $f^* : \Omega^G_*(X) \cong \Omega^G_*(E)$.

(iii) Chern classes: For any $G$-equivariant vector bundle $E \to X$ of rank $r$, there are equivariant Chern class operators $c^G_*(E) : \Omega^G_*(X) \to \Omega^G_{*+l}(X)$ for $1 \leq l \leq r$ which have same functoriality properties as in the non-equivariant case.

(iv) Free action: If $G$ acts freely on $X$ with quotient $Y$, then $\Omega^G_*(X) \cong \Omega_*(Y)$.

(v) Exterior Product: There is a natural product map

\[ \Omega^G_*(X) \otimes_{\mathbb{Z}} \Omega^G_*(X') \to \Omega^G_{*+r}(X \times X') \]

In particular, $\Omega^G_*(k)$ is a graded algebra and $\Omega^G_*(X)$ is a graded $\Omega^G_*(k)$-module for every $X \in \mathcal{V}_G$.

(vi) Projection formula: For a projective map $f : X' \to X$ in $\mathcal{V}_G$, one has for $x \in \Omega^G_*(X)$ and $x' \in \Omega^G_*(X')$, the formula $f_*(x' \cdot f^*(x)) = f_*(x') \cdot x$.

(vii) Localization sequence: For a $G$-invariant closed subscheme $Z \subset X$ with the complement $U$, there is an exact sequence

\[ \Omega^G_*(Z) \to \Omega^G_*(X) \to \Omega^G_*(U) \to 0. \]

For a $G$-equivariant vector bundle $E$ on $X$, we shall often denote the equivariant Chern class operators as $c^G_*(E) \cap -$. Note that these Chern classes behave like the Chern classes of the ordinary vector bundles on the ordinary cobordism of the mixed spaces defined before. In particular, if $\chi$ is a character of $G$ (which is just a $G$-equivariant line bundle on $\text{spec}(k)$), the above exterior product is explicitly described as

\begin{equation}
\Omega^G_*(X) \otimes_\mathbb{L} \Omega^*_*(k) \to \Omega^*_*(X)
\end{equation}
if \( p: X \to \text{Spec}(k) \) is the structure map. Here, \( L_\chi \) is the line bundle associated to \( \chi \) and \( c_1^G(L_\chi) \) is identified with the element \( c_1^G(L_\chi)(id) \) in the ring \( C(G) \).

We shall denote \( C(G) := \Omega^*(k) \) by \( \Omega^*(BG) \) and call it as the cobordism ring of the classifying space of \( G \). It is known from the universal property of the algebraic cobordism that for a complex linear algebraic group \( G \), there is a natural map of rings

\[
\rho_G: \Omega^*(BG) \to MU^*(BG),
\]

where \( MU^*(BG) \) is the rational complex cobordism ring of the topological classifying space of \( G(\mathbb{C}) \). Moreover, this realization map is in fact an isomorphism (cf. \cite{10}, Theorem 6.8). Thus, \( \Omega^*(BG) \) is truly the cobordism ring of the classifying space of \( G \).

If \( H \subset G \) is a closed subgroup, then for any \( G \)-scheme \( X \) and a good pair \((V_j, U_j)\), the \( G/H \)-fibration \( X_H \to X_G \) induces a natural restriction map

\[
r^G_H : \Omega^*_G(X) \to \Omega^*_H(X)
\]

which in particular gives a natural \( \mathbb{Q} \)-algebra homomorphism \( \Omega^*(BG) \to \Omega^*(BH) \). This restriction map in fact completely describes \( \Omega^*(BG) \) in terms of \( \Omega^*(BT) \) in the following way.

**Theorem 2.3.** (cf. \cite{10}, Theorem 6.6) The natural map \( \Omega^*_G(X) \to \Omega^*_T(X) \) induces an isomorphism of \( C(G) \)-modules

\[
\Omega^*_G(X) \xrightarrow{\cong} (\Omega^*_T(X))^W.
\]

In particular, one has \( C(G) \xrightarrow{\cong} C(T)^W \).

If \( T \) is a split torus of rank \( r \) and if \( \{\chi_1, \cdots, \chi_r\} \) is a \( \mathbb{Q} \)-basis of \( \hat{T} \) with associated line bundles \( \{L_{\chi_1}, \cdots, L_{\chi_r}\} \), then there is a natural ring isomorphism

\[
\mathbb{L}[[x_1, \cdots, x_r]] \xrightarrow{\cong} C(T)
\]

which maps \( x_i \) to the class of the first Chern class \( c_1(L_{\chi_i}) \) in \( \Omega^*(BT) \). Similarly, there is an isomorphism \( \mathbb{L}[[\gamma_1, \cdots, \gamma_n]] \xrightarrow{\cong} C(GL_n) \), where the image of \( \gamma_i \) is the \( i \)-th Chern class of the canonical rank \( n \) vector bundle on \( BGL_n \). Under the isomorphism \( C(GL_n) \cong C(T)^W \) of Theorem \( 2.3 \), the image of \( \gamma_i \) is the \( i \)-th elementary symmetric polynomial in the variables of \( C(T) \).

We also recall here that there is a similar relation between the Chow rings of \( BG \) and \( BT \) (cf. \cite{18}, \cite{5}), that is,

\[
S(G) = \text{CH}^*(BG) \xrightarrow{\cong} \text{CH}^*(BT)^W = S(T)^W
\]

and moreover

\[
S(GL_n) \cong \mathbb{Q}[\gamma_1, \cdots, \gamma_n] \leftrightarrow \mathbb{Q}[x_1, \cdots, x_n] = S(T).
\]

### 3. The homomorphism \( \lambda_X \)

In this section, we explain the homomorphism \( \lambda_X \) of Theorem \( 4.1 \) and then prove some functoriality properties of this map with respect to the maps between schemes. We consider the case of flag bundles associated to Borel subgroups of \( G \), from which the general case can easily be deduced (cf. Section \( 7 \)).

Let \( X \) be a scheme and let \( p: E \to X \) be a principal \( G \)-bundle and let \( \pi: E/B \to X \) be the flag bundle associated to the Borel subgroup \( B \). Since \( E \) is a \( G \)-scheme where \( G \) acts freely, it follows from Theorem \( 2.2 \) that \( \Omega^*(X) \cong \Omega^*_G(E) \).
is a $C(G)$-module. In the same way, $\Omega^*(E/T) \cong \Omega^*_T(E)$ is a $C(T)$-module (and hence a $C(G)$-module by restriction). On the other hand, $E/T \rightarrow E/B$ is a principal $B^u$-bundle, where $B^u$ is the unipotent radical of $B$. By [3, XXII, 5.9.5], $B^u$ has a finite filtration by normal subgroups whose successive quotients are the vector groups. A successive application of homotopy invariance now implies that $\Omega^*(E/B) \cong \Omega^*(E/T)$. Hence, $\Omega^*(E/B)$ is a $C(T)$-module. Thus, $\Omega^*(X)$ and $\Omega^*(E/B)$ are naturally $C(G)$ and $C(T)$-modules respectively, which defines the map $\lambda_X$ as $\lambda_X(w \otimes x) = x \cdot \pi^*(w)$.

Recall that $C(T)$ is the power series over $\mathbb{L}$ in the first Chern classes of the line bundles associated to the characters of $T$ and Theorem [2,3] implies that $C(G)$ is also generated by the first Chern classes of the $W$-invariant characters inside $C(T)$.

Using the description of these module structures in [2,4], we see that the map $\lambda_X$ is given by

$$\lambda_X : \Omega^*(X) \otimes_{C(G)} C(T) \rightarrow \Omega^*(E/B)$$

$$w \otimes c_1(\chi) \mapsto c_1(\chi) \cap \pi^*(w)$$

for a character $\chi$ of $T$. It is easy to see from this that this is an $\mathbb{L}$-algebra homomorphism if $X$ is smooth.

**Remark 3.1.** For readers who are little bit familiar with the language of quotient stacks and know that the $G$-equivariant line bundles on a $G$-scheme $X$ are same as ordinary line bundles on the quotient stack $[X/G]$, we can explain the above in this setup as follows. The principal $G$-bundle $E \rightarrow X$ uniquely gives rise to the following commutative diagram of morphisms.

$$\begin{array}{ccc}
E/B & \xrightarrow{\pi} & X \\
\downarrow q & & \downarrow \pi \\
BT & \rightarrow & BG,
\end{array}$$

where $BG$ is the quotient stack $[k/G]$. The map $\lambda_X$ is then given as $\lambda_X(w \otimes c_1(\chi)) = c_1(q^*(L_\chi)) \cap \pi^*(w)$.

In particular, if $E = G \times X \rightarrow X$ is a trivial principal bundle, then the map $X \rightarrow BG$ canonically factors through the structure map $X \rightarrow \text{Spec}(k) \rightarrow [k/G] = BG$. Hence, the map $\lambda_X$ in this case is given by

$$\Omega^*(X) \otimes_L (\mathbb{L} \otimes_{C(G)} C(T)) \rightarrow \Omega^*(E/B).$$

Here, the left term is identified as $\Omega^*(X) \otimes_L \Omega^*(G/B)$ by [10] Theorem 7.6] and $\Omega^*(X) \otimes_L \Omega^*(G/B) \xrightarrow{\lambda_X} \Omega^*(E/B)$ is simply the exterior product map.

To prove certain functoriality properties of $\lambda_X$, we need the following elementary result on the equivariant cobordism.

**Lemma 3.2.** Let $G$ be a linear algebraic group over $k$ and let $f : Y \rightarrow X$ and $g : Z \rightarrow X$ be projective and smooth morphisms of $G$-schemes respectively. Then the maps $f_* : \Omega^*_G(Y) \rightarrow \Omega^*_G(X)$ and $g^* : \Omega^*_G(X) \rightarrow \Omega^*_G(Z)$ are $C(G)$-linear.

**Proof.** We only give a sketch for the $C(G)$-linearity of $f_*$. The assertion about $g^*$ is similar and much simpler. Note that the $C(G)$-module structure on $\Omega^*_G(X)$ is given by the exterior product (cf. Theorem [2,2]). It suffices to show that

$$f_*(x \cdot w) = x \cdot f_*(w)$$

when $x$ and $w$ are generators of the corresponding cobordism groups $\Omega^*(U_j/G)$ and $\Omega^*((Y \times U_j)/G)$, where $(V_j, U_j)$ is any given good pair for $G$-action. So let
$W_1 \xrightarrow{s_1} U_j/G$ and $W_2 \xrightarrow{s_2} Y \xrightarrow{G} U_j$ be projective morphisms from smooth and connected schemes, representing the cobordism classes $x$ and $w$ respectively. Let $\widetilde{W}_1$ and $\widetilde{W}_2$ be the pull-backs of $W_1$ and $W_2$ to $U_j$ and $Y \times U_j$ respectively. By the definition of the push-forward and exterior product, we have

$$f_*(x \cdot w) = f_*\left(\left[\widetilde{W}_1 \times \widetilde{W}_2 \to Y \times (U_j \times U_j)\right]\right)$$

and the last term is same as the class of $x \cdot f_*(w)$ in $\Omega^*(X_G)$ which can be taken as $X \times (U_j \times U_j)$ because $(V_j \times V_j, U_j \times U_j)$ is also a good pair for the $G$-action. □

**Lemma 3.3.** Let $f : Y \to X$ and $g : Z \to X$ be respectively, the projective and the smooth morphisms. Let $p : E \to X$ be a principal $G$-bundle and let $E_Y$ and $E_Z$ denote its pull-backs to $Y$ and $Z$ respectively. Consider the following Cartesian diagrams of flag bundles.

$$\begin{array}{ccc}
E_Y/B & \xrightarrow{\pi} & E/B \\
\pi_Y \downarrow & & \downarrow \pi \\
Y & \xrightarrow{f} & X
\end{array}$$

Then the diagrams

$$\begin{array}{ccc}
\Omega^*(Y) \otimes_{C(G)} C(T) & \xrightarrow{\lambda_Y} & \Omega^*(E_Y/B) \\
\downarrow f_* \otimes \text{id} & & \downarrow \overline{f}_* \\
\Omega^*(X) \otimes_{C(G)} C(T) & \xrightarrow{\lambda_X} & \Omega^*(E/B)
\end{array}$$

$$\begin{array}{ccc}
\Omega^*(X) \otimes_{C(G)} C(T) & \xrightarrow{\lambda_X} & \Omega^*(E/B) \\
\downarrow g_* \otimes \text{id} & & \downarrow \overline{g}_* \\
\Omega^*(Z) \otimes_{C(G)} C(T) & \xrightarrow{\lambda_Z} & \Omega^*(E_Z/B)
\end{array}$$

are commutative.

**Proof.** To show the commutativity of the first square, we have

$$\overline{f}_* \circ \lambda_Y (w \otimes x) = \overline{f}_*(x \cdot \pi_Y^*(w))$$

By Lemma 3.2

$$= x \cdot \overline{f}_*(\pi_Y^*(w)) = x \cdot \pi^*(f_*(w)) = \lambda_X(f_*(w) \otimes x),$$

where the third equality follows from the fact that the first square in (3.4) is Cartesian with $\pi$ smooth and $f$ projective. The proof of the commutativity of the second square is similar. □
Let \((G, T, B, W)\) be as above where \(G\) is a connected linear algebraic group. Let \(G^u\) denote the unipotent radical of \(G\) and let \(L\) denote the corresponding quotient as a reductive group. Then any principal \(G\)-bundle \(E \to X\) canonically gives a principal \(L\)-bundle \(E_L = E/G^u \to X\). Moreover, as the Borel subgroup \(B\) contains \(G^u\), we see that \(E/B \cong E_L/B_L\), where \(B_L\) is the image of \(B\) which is a Borel subgroup of the reductive group \(L\). Since \(C(G) \cong C(L)\), as follows from the Levi decomposition and the homotopy invariance, we conclude that it is enough to consider the case when \(G\) is reductive in order to prove our main results. Hence for the rest of this paper, \(G\) will always denote a connected reductive group. We shall deduce Theorem 1.1 from the following result for the algebraic cobordism of the flag bundles associated to the Borel subgroup \(B\).

**Theorem 3.4.** Let \(p : E \to X\) be a principal \(G\)-bundle and let \(\pi : E/B \to X\) be the flag bundle associated to the Borel subgroup \(B\). The natural map of \(C(T)\)-modules

\[
\lambda_X : \Omega^*(X) \otimes_{C(G)} C(T) \to \Omega^*(E/B)
\]

is an isomorphism. Moreover, it is an isomorphism of rings if \(X\) is smooth.

4. SOME ALGEBRAIC REDUCTIONS

Let \((G, T, B, W)\) be as above and let \(T\) be a split torus of rank \(r\). This rank will be fixed throughout. We fix a basis \(\{\chi_1, \ldots, \chi_r\}\) of \(\hat{T}\) and let \(S = \text{Sym}(\hat{T}) = \mathbb{Q}[x_1, \ldots, x_r]\) be the polynomial algebra in the first Chern classes of the line bundles associated to the characters \(\{\chi_1, \ldots, \chi_r\}\). Let \(S^W \subset S\) be the subalgebra generated by the homogeneous polynomials which are invariant under the action of \(W\). This gives us a square of ring inclusions

\[
(\mathbb{L}[x_1, \ldots, x_r])^W \to \mathbb{L}[x_1, \ldots, x_r] \to C(G) \to C(T),
\]

which is Cartesian and where \(C(G)\) has been identified with \((C(T))^W\). We shall write \(\mathbb{L}[x_1, \ldots, x_r]\) simply as \(S_L\). Note that \(S_L\) and \(S^W_L\) are canonically isomorphic to \(\mathbb{L} \otimes_\mathbb{Q} S\) and \(\mathbb{L} \otimes_\mathbb{Q} S^W\) as \(\mathbb{L}\)-algebras. It is also known that \(S^W_L\) is a polynomial algebra over \(\mathbb{L}\) of rank \(r\). We shall denote the homogeneous generators of this subalgebra by \(\{\sigma_1, \ldots, \sigma_r\}\).

Let \(I\) be the ideal of \(S\) generated by the homogeneous elements of positive degree which are invariant under \(W\) and let \(\Lambda\) denote the ring \(S/I\). Then we see that \(\Lambda_L = \mathbb{L} \otimes_\mathbb{Q} \Lambda\) is canonically isomorphic to the \(\mathbb{L}\)-algebra \(\mathbb{L}[x_1, \ldots, x_r]/I\). We recall the following result from \cite{19} Lemma 1.2.

**Lemma 4.1.** The graded \(\mathbb{Q}\)-algebra \(\Lambda\) is finite. If \(N\) is the maximal integer for which \(\Lambda_N \neq 0\), then \(N = \dim(G/B)\). Moreover, the \(\mathbb{Q}\)-vector space \(\Lambda_N\) is one-dimensional, and if \(d\) is an integer, the homomorphism

\[
\Lambda_d \otimes \Lambda_{-d} \to \Lambda_N
\]

given by the multiplication in \(\Lambda\) is a perfect pairing of finite-dimensional \(\mathbb{Q}\)-vector spaces.

**Lemma 4.2.** Let \(A\) be a commutative ring and let \(I\) be an ideal of \(A\). Let \(J\) be a finitely generated ideal of \(A\). Then for any \(A\)-module \(M\), the natural maps of \(\hat{A}\)-modules

\[
\hat{J}M \to \hat{J}M, \quad \hat{M}/\hat{J}M \to \left(\hat{M}/\hat{J}M\right)
\]
are isomorphisms, where \( \hat{M} \) denotes the \( I \)-adic completion of \( M \).

**Proof.** Consider the exact sequence

\[
0 \to JM \to M \to \frac{M}{JM} \to 0.
\]

Since the topology on \( M \) is given by the descending chain \( M \supset IM \supset I^2M \supset \cdots \) of submodules, it follows from [15, Theorem 8.1] that

\[
0 \to \left( JM \right) \to \hat{M} \to \left( \frac{M}{JM} \right) \to 0
\]

is exact. Thus, we only need to show the first isomorphism to prove the lemma.

Suppose \( J = \sum_{i=1}^{n} a_i A \) and define

\[
\phi : M^n \to M
\]

\[
\phi(m_1, \ldots, m_n) = \sum_{i=1}^{n} a_i m_i.
\]

This makes the sequence

\[
M^r \to M \to \frac{M}{JM} \to 0
\]

exact. It again follows from [15, Theorem 8.1] that

\[
\hat{M}^r \xrightarrow{\hat{\phi}} \hat{M} \to \left( \frac{M}{JM} \right) \to 0
\]

is exact. On the other hand, \( \hat{\phi} \) is again given by \( \hat{\phi}(\hat{m}_1, \ldots, \hat{m}_n) = \sum_{i=1}^{n} a_i \hat{m}_i \). In other words, \( \text{Image}(\hat{\phi}) = J\hat{M} \). The first isomorphism now follows from this and (4.3). This proves the lemma.

**Corollary 4.3.** The natural homomorphisms of rings

\[
L \otimes \Lambda \xrightarrow{\varepsilon_L} \frac{L[x_1, \ldots, x_r]}{I} \to \frac{C(T)}{IC(T)} \to C(T) \otimes_{C(G)} L
\]

are isomorphisms.

**Proof.** We first observe that the cobordism ring \( C(T) \) is the inverse limit of the cobordism rings of the form \( (\Omega^*(BT))_{j \geq 0} \) on each of which the Weyl group acts. In particular, the action of \( W \) on \( C(T) \) is induced by its action on the polynomial ring \( L[x_1, \ldots, x_r] \) and \( C(T)^W \) is the inverse limit of the \( W \)-invariants in the inverse system \( (\Omega^*(BT))_{j \geq 0} \). Thus we see that a \( S^W \) can be written \( S^W = \frac{L[\sigma_1, \ldots, \sigma_r]}{L[x_1, \ldots, x_r]} \) and \( C(G) = C(T)^W \) is the subring of the power series ring \( L[[x_1, \ldots, x_r]] \) generated by the homogeneous polynomials \( \{\sigma_1, \ldots, \sigma_r\} \). Moreover, the ideal \( I \) in
$C(T)$ is the extension of the ideal $(\sigma_1, \cdots, \sigma_r)$ of $S^W = \mathbb{Q}[\sigma_1, \cdots, \sigma_r]$ which we also denote by $I$. Let $m$ denote the ideal $(x_1, \cdots, x_r)$ of $S_L$. Now we have

$$C(T) \otimes C(G) \mathbb{L} \cong C(T) \otimes C(G) \left( \frac{C(G)}{(\sigma_1, \cdots, \sigma_r)} \right) \otimes \frac{C(T)}{I_C(T)} \otimes \frac{S_L}{I(S_L)_m} \otimes \left( \frac{S_L}{I(S_L)} \right)_m \quad \text{(By Lemma 4.2)}.$$  

On the other hand,

$$(4.6) \quad \frac{S_L}{I(S_L)} \cong \mathbb{L} \otimes_{\mathbb{Q}} \left( \frac{\mathbb{Q}[x_1, \cdots, x_r]}{I} \right) \cong \prod_{j=1}^{s} \mathbb{L} \otimes_{\mathbb{Q}} A_j,$$

where each $A_j$ is an artinian local ring which is finite over $\mathbb{Q}$. In particular, the ideal $m$ is nilpotent in each of the factor $\mathbb{L} \otimes A_j$ and hence the last term in (4.6) is complete with respect to $m$. We conclude that $\frac{S_L}{I(S_L)}$ is complete in the $m$-adic topology. In particular, we obtain

$$C(T) \otimes C(G) \mathbb{L} \cong \frac{S_L}{I(S_L)} \otimes \frac{S_L}{I(S_L)}$$

and this completes the proof. $\square$

**Corollary 4.4.** Let $X \times G/B \to X$ be the trivial flag bundle. Then the map $\lambda_X$ is given by

$$\Omega^*(X) \otimes A_L \xrightarrow{\phi_X} \Omega^*(X) \otimes A_L \xrightarrow{\lambda_X} \Omega^*(E/B)$$

which is an $\mathbb{L}$-algebra isomorphism if $X$ is smooth.

**Proof.** The first assertion of the corollary follows directly from (3.3) and Corollary 4.3. If $X$ is smooth, this map is an $\mathbb{L}$-algebra homomorphism because so are the maps in (3.1) and Corollary 4.3. Moreover, it is an isomorphism by [10, Lemma 6.5] and [13, Theorem 3.1]. $\square$

### 5. Surjectivity of $\lambda_X$

We now let $p : E \to X$ be an arbitrary principal $G$-bundle and let $\pi : E/B \to X$ be the associated flag bundle. Using the above inclusions of the polynomial rings inside the power series rings, we get natural homomorphisms

$$(5.1) \quad \Omega^*(X) \otimes_{S_L} S_L \xrightarrow{\delta_X} \Omega^*(X) \otimes_{C(G)} C(T) \xrightarrow{\lambda_X} \Omega^*(E/B)$$

of $S_L$-modules, which are also $S_L$-algebra homomorphisms if $X$ is smooth.

As a first step towards proving Theorem 3.4, we show in this section that the map $\lambda_X$ is surjective. In fact, the proof that follows will show that the map $\phi_X$ is surjective. It will eventually turn out that both the maps $\phi_X$ and $\lambda_X$ are isomorphisms. We begin with the following elementary property of principal bundles and the local property of algebraic cobordism.
Lemma 5.1. Let $p : E \to X$ be a principal $G$-bundle and let $\pi : E/B \to X$ be the associated flag bundle. Then the $G$-action $G \times E \xrightarrow{\mu} E$ induces a commutative diagram

\begin{equation}
G/B \times E/B \xrightarrow{\mu'} E/B \\
\downarrow \pi' \downarrow \pi \\
E/B \xrightarrow{\pi} X
\end{equation}

which is Cartesian and where $\pi'$ is the projection to the second factor.

Proof. Since $E \xrightarrow{p} X$ is a principal $G$-bundle quotient of quasi-projective schemes, the action map $G \times E \xrightarrow{\mu} E$ induces a commutative diagram

\begin{equation}
G \times E \xrightarrow{\mu} E \\
\downarrow \nu' \downarrow \nu \\
E \xrightarrow{\nu} X
\end{equation}

which is Cartesian and where $\nu'$ is the projection to the second factor by the general properties of principal bundles (cf. [7, 0.10]).

Now, the map $\mu$ descends to a map $G \times E/B \xrightarrow{\nu'} E/B$. Moreover, this map is $B$-equivariant where $B$ acts trivially on $E/B$ and by left multiplication on $G$.

Taking the quotients, we get a canonical map $G/B \times E/B \xrightarrow{\mu'} E/B$ making the diagram (5.2) commute. It is now an easy exercise to check from (5.3) that this diagram is Cartesian too. \hfill \Box

Lemma 5.2. Let $f : X' \to X$ be a finite and étale morphism of smooth and connected schemes. Then there exists an open subscheme $U \subseteq X$ such that for the map $g = f|_U : U' = f^{-1}(U) \to U$, one has $g_*(1) = [k(X') : k(X)]$.

Proof. Let $\eta$ denote the generic point of $X$ and consider the Cartesian diagram

\begin{equation}
X' \xrightarrow{\iota_U} U' \xrightarrow{j'} X' \\
\downarrow h \downarrow g \downarrow f \\
\eta \xrightarrow{i} U \xrightarrow{j} X,
\end{equation}

where $U$ is any open subscheme of $X$. Since $X'$ is connected, we see that $X'_{\eta} = \text{Spec}(k(X'))$. Put $p = j \circ i$, $p' = j' \circ i'$ and $d = [k(X') : k(X)]$. It follows from [11, Lemma 4.7] that

$p^* \circ f_*(1) = h_* \circ p^*(1) = h_*(1) = d.$

Since the algebraic cobordism is generically constant by [11, Lemma 13.3, Corollary 13.4], there exists an open subscheme $U \subseteq X$ such that $j^* \circ f_*(1) = d$ in $\Omega^*(U)$. This in turn implies that

$g_*(1) = g_* \circ j''(1) = j^* \circ f_*(1) = d$

and this proves the lemma. \hfill \Box
Corollary 5.3. Let $f : X' \to X$ be a finite and étale morphism of smooth and connected schemes and consider the diagram (3.4). Then there exists an open subscheme $U \overset{j}{\to} X$ such that for $g = f|_{U'} : U'' = f^{-1}(U) \to U$, one has a commutative diagram

$$(5.5) \quad \Omega^*(U) \otimes_{C(G)} C(T) \xrightarrow{\varphi^*} \Omega^*(U') \otimes_{C(G)} C(T) \xrightarrow{\varphi^*} \Omega^*(U) \otimes_{C(G)} C(T)$$

such that the horizontal composite maps are multiplication by $[k(X') : k(X)]$.

**Proof.** We choose $U \overset{j}{\to} X$ as in Lemma 5.2. The commutativity of the diagram follows from Lemma 3.3. Moreover, as $f$ is finite and étale of degree $d$, it follows that $\bar{f}$ is also a morphism of the same type. We claim that $\bar{f}_*(1) = [k(X') : k(X)]$. To see this, we evaluate the required term as

$$\bar{f}_*(1) = \bar{f}_* \circ \pi^*_U(1) = \pi^*_U \circ g_*(1) = [k(X') : k(X)],$$

where the second equality follows from Lemma 5.2 and this proves the claim. The corollary now follows from the projection formula. \hfill \Box

**Proposition 5.4.** The map $\lambda_X$ is surjective for any scheme $X$.

**Proof.** We shall prove this by induction on the dimension of $X$. We can assume that $X$ is reduced. If $X$ is zero-dimensional, it is of the form $X = \text{Spec}(K)$, where $K$ is a finite product of finite field extensions of $k$. We prove the case when $X = \text{Spec}(k)$. The same proof applies for any finite extension of $k$. Now, there is a finite extension $k \hookrightarrow \ell$ such that $Y_1$ is of the form $G/B$. Hence the result holds for $X = \text{Spec}(\ell)$ by [10, Theorem 7.6]. The case of $\text{Spec}(k)$ now follows from Corollary 5.3.

If the map $\pi$ is of the form $X \times G/B \xrightarrow{\pi} X$ with $X$ smooth, the maps $\lambda_X$ and $\phi_X$ are in fact isomorphisms by (3.3), Corollary 4.4 [10, Lemma 6.5] and [13, Theorem 3.1]. In the general case, we can find an étale cover $X' \overset{j}{\to} X$ such that the base change $E/B \times_X X' \overset{\pi'}{\to} X'$ is the trivial flag bundle $X' \times G/B \to X'$.

We can now find a smooth and dense open subset $U \overset{j}{\to} X$ such that the map $U' = f^{-1}(U) \to U$ is finite and étale. Moreover, the flag bundle is still trivial on $U'$. Since $U$ is a disjoint union of smooth and connected schemes, the surjectivity of $\lambda_U$ follows from the case of the trivial bundle shown above and Corollary 5.3.

We now let $Z = X - U$ be the complement of $U$ in $X$ with the reduced closed subscheme structure and consider the diagram

$$(5.6) \quad \Omega^*(Z) \otimes_{C(G)} C(T) \xrightarrow{\lambda_Z} \Omega^*(X) \otimes_{C(G)} C(T) \xrightarrow{\lambda_X} \Omega^*(U) \otimes_{C(G)} C(T) \xrightarrow{\lambda_U} 0$$

which is commutative by Lemma 3.3 and whose rows are exact by Theorem 2.1. Since $U$ is open and dense, $Z$ is a closed subscheme of dimension which is strictly smaller than that of $X$. Hence the map $\lambda_Z$ is surjective by induction. We have shown above that $\lambda_U$ is surjective. Hence the map $\lambda_X$ is surjective too. \hfill \Box
Lemma 6.2. For any scheme $X$, there is a finite filtration by closed subschemes

$$0 = X^{n+1} \subset X^n \subset \cdots \subset X^1 \subset X^0 = X$$

such that $\psi_{U^i}$ is identity for each $0 \leq i \leq n$, where $U^i = (X^i - X^{i+1})$. 

Remark 5.5. Since the maps

$$\Omega^*(X) \otimes_{S^W} S \xrightarrow{\delta_X} \Omega^*(X) \otimes_{C(G)} C(T) \xrightarrow{\lambda_X} \Omega^*(E/B)$$

are isomorphisms for the trivial bundle $G/B \times X \xrightarrow{\pi} X$ for $X$ smooth by Corollary 4.4, exactly the same proof as for Proposition 5.4 shows that the maps $\delta_X$ and $\phi_X$ are also surjective for any scheme $X$.

6. Proof of Theorem 3.4

Recall from Section 4 that $\Lambda = S/I = \Lambda_0 \oplus \Lambda_1 \oplus \cdots \oplus \Lambda_N$ is the graded quotient of $S = \mathbb{Q}[x_1, \ldots, x_r]$ by the ideal $I$ which is generated by the homogeneous polynomials of positive degree which are invariant under $W$. It is clear that $\Lambda_0$ is one-dimensional over $\mathbb{Q}$ generated by the unit element of the ring. It also follows from Lemma 4.1 that $\Lambda_0$ is an one-dimensional $\mathbb{Q}$-vector space. We fix these two generators and denote them by $p_0 = 1$ and $p_N$ respectively. Let $p_0 = 1$ and $p_N$ be their homogeneous lifts in $S_0$ and $S_N$ respectively. For any scheme $X$, we consider the map

$$(6.1) \quad \psi : \Omega^*(X) \to \Omega^*(X)$$

$$\psi(x) = \pi_* \circ \phi_X(x \otimes p_N) = \pi_* \circ (c_1(p_N) \cap \pi^*(x))$$

where $\phi_X$ is the homomorphism in (6.1). We need the following property of this map.

Lemma 6.1. Let $Z \hookrightarrow X$ be a closed subscheme such that $\psi_Z$ is identity. Let $\Omega^*_X(Z) \subset \Omega^*(X)$ be the image of the map $\Omega^*_X(Z) \hookrightarrow \Omega^*(X)$. Then, $\psi_Z$ induces a map $\psi^Z_X : \Omega^*_X(Z) \to \Omega^*_Z(X)$ such that the diagram

$$(6.2) \quad \begin{array}{ccc}
\Omega^*_X(Z) & \to & \Omega^*(X) \\
\downarrow \psi^Z_X & & \downarrow \psi_X \\
\Omega^*_Z(X) & \to & \Omega^*(X)
\end{array}$$

is commutative and $\psi^Z_X$ is identity.

Proof. We consider the following diagram.

$$\begin{array}{ccc}
\Omega^*(Z) & \xrightarrow{p^Z_*} & \Omega^*(E_Z/B) \\
\downarrow i_* & & \downarrow i'_* \\
\Omega^*(X) & \xrightarrow{\pi'_*} & \Omega^*(E/B) \\
\downarrow i_* & & \downarrow i'_* \\
\Omega^*(Z) & \xrightarrow{p^*_N} & \Omega^*(E_Z/B) \\
\downarrow i_* & & \downarrow i_* \\
\Omega^*(X) & \xrightarrow{\pi_*} & \Omega^*(E/B) \\
\end{array}$$

The first square from the left clearly commutes as $\pi_Z$ is the pull-back of a smooth morphism. The second square commutes by Lemma 3.2. The third square is simply the commutativity of the push-forward maps. We conclude that the big outer square commutes. Since the top and the bottom composite horizontal arrows are $\psi_Z$ and $\psi_X$ respectively, we get the commutative square (6.2). Moreover, $\psi^Z_X$ is identity because $\psi_Z$ is so.

Lemma 6.2. For any scheme $X$, there is a finite filtration by closed subschemes

$$0 = X^{n+1} \subset X^n \subset \cdots \subset X^1 \subset X^0 = X$$

such that $\psi_{U^i}$ is identity for each $0 \leq i \leq n$, where $U^i = (X^i - X^{i+1})$. 

□
Proof. We prove this by induction on the dimension of $X$. If $X$ is zero-dimensional, we can use Corollary 5.3 and the argument in the proof of Proposition 5.4 to reduce to the case when $X = \text{Spec}(k)$ and $E/B = G/B$. We can apply Corollary 4.3 and [10] Theorem 7.6] to get a graded $L$-algebra isomorphism

\begin{equation}
\phi : L \otimes \Lambda \xrightarrow{\cong} \Omega^*(G/B).
\end{equation}

Let $S_\Lambda \twoheadrightarrow \Lambda$ denote the quotient map. Under the above isomorphism, we get for any homogeneous element $m \in \Lambda^d$,

\[
\psi_L(m) = \pi_* \phi \circ \eta(m \otimes p_N) = \pi_* \phi_k(m \otimes p_N) = \pi_* (\rho_N \cdot \pi^*(m)) = \pi_* (1 \otimes p_N) \cdot m,
\]

where the last equality holds by the projection formula. On the other hand, as $L^0$ is the one-dimensional vector space generated by the class of $[\text{Spec}(k)]$, we see that $\Omega^N(G/B) \cong L^0 \otimes \Lambda \xrightarrow{\cong} \mathbb{Q}[1 \otimes p_N]$ and $\pi_*(1 \otimes p_N)$ is simply the class of $[\text{Spec}(k)]$ in $L^0$, which is the identity element. This proves the zero-dimensional case. If $X$ is any smooth scheme and $Y = X \times G/B$, then it follows from Corollary 4.4 and the case of $X = \text{Spec}(k)$ that $\psi_X$ is identity.

In the general case, we can find an étale cover $f : X' \to X$ such that $E$ is trivial over $X$. Hence $E/B \times_X X' \stackrel{\cong}{\to} E/B \times X'$ and $\pi_{X'}$ is just the projection map by Lemma 5.1. Since $f$ is generically finite, we can find a dense open subset $U \subset X'$ such that $U$ is smooth and the map $g : U' = f^{-1}(U) \to U$ is finite and étale. The proof of Lemma 6.1 shows that the right square in the diagram

\begin{equation}
\begin{array}{ccc}
\Omega^*(U) & \xrightarrow{g_*} & \Omega^*(U') \\
\psi_U & & \psi_{U'} \downarrow \\
\Omega^*(U) & \xrightarrow{g_*} & \Omega^*(U')
\end{array}
\end{equation}

commutes. Since $g$ is étale, the similar argument shows that the left square commutes. Furthermore, we can apply Corollary 5.2 to choose the open subset $U$ so that $g^*$ is injective. We have shown above that $\psi_{U'}$ is identity. We conclude that $\psi_U$ must also be identity.

Put $X^1 = (X - U)$. Then $X^1$ is a closed subscheme of $X$ of dimension which is strictly less than that of $X$. The proof of the lemma now follows by induction. \qed

Proposition 6.3. For any scheme $X$, the homomorphism $\psi_X$ is an isomorphism.

Proof. We choose a finite filtration of $X$ as in Lemma 6.2. We have shown that $\psi_{(X - X^1)}$ is identity. Assume by induction that $\psi_{(X - X^m)}$ is an isomorphism and consider the diagram

\begin{equation}
\begin{array}{ccc}
0 & \to & \Omega^*_{(X - X^{m+1})} (X - X^{m+1}) \\
\psi_{(X - X^{m+1})} & & \psi_{(X - X^{m+1})} \downarrow \quad \downarrow \psi_{(X - X^{m+1})} \\
0 & \to & \Omega^*_{(X - X^{m+1})} (X - X^{m+1})
\end{array}
\end{equation}

which is commutative by Lemma 6.1. The top and the bottom rows are exact by Theorem 2.1. The left vertical map is identity by Lemmas 6.1 and 6.2. The right vertical map is isomorphism by induction. We conclude that the middle vertical map is an isomorphism too and the induction continues. \qed
Corollary 6.4. For any scheme $X$, the pull-back map $\Omega^*(X) \xrightarrow{\pi^*} \Omega^*(E/B)$ is injective.

Proof. This follows directly from Proposition 6.3.

Proposition 6.5. Let $X \times G/B \xrightarrow{\pi} X$ be the trivial flag bundle. Then the map $\lambda_X$ is an isomorphism.

Proof. The surjectivity of $\lambda_X$ and $\delta_X$ follows directly from Proposition 5.4 and Remark 5.5. So we only need to show the injectivity. By Remark 5.5 again, it suffices to show that $\phi_X$ is injective (and hence isomorphism). By Corollary 4.4, $\phi_X$ is same as the map of graded $\Lambda_{\mathbb{L}}$-modules

$$\Omega^*(X) \otimes \Lambda \xrightarrow{\phi_X} \Omega^*(E/B)$$

and this map factorizes the composite map $\theta_X$ as

$$\theta_X : \Omega^*(X) \otimes \mathbb{L} S \xrightarrow{\eta_X} \Omega^*(X) \otimes \Lambda \xrightarrow{\phi_X} \Omega^*(E/B).$$

It suffices to show that $\phi_X$ is injective on each graded piece of the left term in (6.6). So let

$$x \in (\Omega^*(X) \otimes \Lambda)^m = \bigoplus_{p=0}^N \Omega^{m-p} \otimes \Lambda_p.$$

be such that $\phi_X(x) = 0$.

Let $\{b_1^p, \ldots, b_s^p\}$ be a chosen $\mathbb{Q}$-basis of $\Lambda_p$ and let $\{u_1^{p_1}, \ldots, u_s^{p_1}\}$ their homogeneous lifts in $S$. We have $b_1^0 = \rho_0 = 1$ and $b_1^N = \rho_N$. Similarly, $u_1^0 = p_0 = 1$ and $u_1^N = p_N$. We can then write $x$ uniquely as

$$x = \sum_{p=0}^N x_p, \quad \text{where} \quad x_p = \sum_{i=1}^{s_p} x_i^p \otimes b_i^p.$$ 

We show inductively that $x_p = 0$ for each $p$. First of all,

$$\phi_X(x) = 0 \Rightarrow p_N \cdot \phi_X(x) = 0 \Rightarrow \theta_X(p_N \cdot x) = 0 \Rightarrow \phi_X(p_N \cdot x) = 0.$$ 

On the other hand, we have by Lemma 4.1

$$\rho_N \cdot x_p = \sum_{i=1}^{s_p} x_i^p \otimes (b_i^p \cdot \rho_N)$$

which is zero for $p > 0$ and $x_1^0 \otimes \rho_N$ for $p = 0$. Thus we conclude that

$$\phi_X(x) = 0 \Rightarrow \phi_X(x_1^0 \otimes \rho_N) = 0 \Rightarrow \pi_* \circ \phi_X(x_1^0 \otimes \rho_N) = 0.$$ 

But the last equality is equivalent to saying that $\psi_X(x_1^0) = 0$ and this implies that $x_1^0 = 0$ by Proposition 6.3. In particular, $x_0 = x_1^0 \otimes b_1^0 = 0$.

We assume by induction that $x_q = 0$ for $q < p$ with $p \geq 1$ and we show that $x_p$ is zero too. We prove this following the proof of [19] for the Chow groups. We fix an integer $1 \leq l \leq s_p$. By Lemma 4.1 there exists $c \in \Lambda_{N-p}$ such that

$$b_i^p \cdot c = \begin{cases} 0 & \text{if } i \neq l \\ \rho_N & \text{if } i = l. \end{cases}$$
In particular, we get
\[ c \cdot x = \sum_{j=p}^{N} \sum_{i=1}^{s_j} x_i^j \otimes (b_i^j \cdot c) \]
\[ = \sum_{i=1}^{s_p} x_i^p \otimes (b_i^p \cdot c) \]
\[ = x_i^p \otimes \rho_N, \]
where the second equality occurs because \( j + (N - p) > N \) for \( j > p \). We lift \( c \) to a homogeneous element \( e \) of \( S \). We then have as before,
\[ \phi_X(x) = 0 \Rightarrow e \phi_X(x) = 0 \Rightarrow \theta_X(e \cdot x) = 0 \Rightarrow \phi_X(c \cdot x) = 0. \]
We conclude from this that \( \phi_X(x) = 0 \Rightarrow \phi_X(x_i^p \otimes \rho_N) = 0 \), which in turn implies that
\[ \psi_X(x_i^p) = \pi_\ast \circ \phi_X(c_1(p_N) \cap \pi_\ast(x_i^p)) = \pi_\ast \circ \phi_X(x_i^p \otimes \rho_N) = 0. \]
It follows from Proposition 6.3 that \( x_i^p = 0 \) and the induction continues to show that \( x_p = 0 \). This completes the proof.

\[ \square \]

**Proof of Theorem 3.4.** The surjectivity of \( \lambda_X \) follows from Proposition 5.4. Furthermore, the map \( \delta_X \) is also surjective by Remark 5.5. Thus we only need to show that \( \phi_X \) is injective.

Put \( Y = E/B \). It follows from Lemma 5.1 that the pull-back \( Y \times_X Y \rightarrow Y \) is the trivial flag bundle \( G/B \times Y \rightarrow Y \). We now consider the diagram
\[ \Omega^\ast(X) \otimes_{S^W_L} S_L \xrightarrow{\phi_X} \Omega^\ast(E/B) \]
\[ \pi_\ast \circ \text{Id} \]
\[ \Omega^\ast(Y) \otimes_{S^W_L} S_L \xrightarrow{\phi_Y} \Omega^\ast(G/B \times Y) \]
which is commutative by Lemma 3.3. Since \( S_L \) is flat over \( S^W_L \), the left vertical arrow is injective by Corollary 6.4. The bottom horizontal arrow is injective by Proposition 6.5. We conclude that the top horizontal arrow is injective too. \( \square \)

**Corollary 6.6.** Let \( G \) be a connected linear algebraic group with a split maximal torus \( T \). If \( E \xrightarrow{p} X \) is a principal \( G \)-bundle over a scheme \( X \), then
\[ \Omega^\ast(E) \xrightarrow{\cong} \Omega^\ast(X) \otimes_{C(G)} L \cong \frac{\Omega^\ast(X)}{I \Omega^\ast(X)}, \]
where \( I \) is the ideal of \( C(G) \) generated by the Chern classes of \( G \)-homogeneous line bundles. This is an \( L \)-algebra isomorphism if \( X \) is smooth.

**Proof.** By [10], Theorem 7.4], the natural map
\[ \Omega^\ast_T(E) \otimes_{C(T)} L \rightarrow \Omega^\ast(E) \]
is an isomorphism, which is an \( L \)-algebra isomorphism if \( X \) is smooth. On the other hand, it follows from Theorem 2.2 that the term on the left is same as \( \Omega^\ast(E/B) \otimes_{C(T)} L \). The corollary now follows from Theorem 3.4. \( \square \)

**Corollary 6.7.** Let \( G \) be a connected algebraic group (not necessarily linear) over \( k \) and let \( G \xrightarrow{\alpha} A(G) \) be the albanese morphism of \( G \). Then there is an \( L \)-algebra isomorphism
\[ \Omega^\ast(G) \cong \frac{\Omega^\ast(A(G))}{I \Omega^\ast(A(G))}. \]
Proof. It follows immediately from Corollary 6.6 and the fundamental exact sequence
\[ 1 \to G_{\text{aff}} \to G \xrightarrow{\alpha_G} A(G) \to 1, \]
where \( G_{\text{aff}} \) is the largest connected linear algebraic subgroup of \( G \).

\[ \square \]

7. Cobordism of \( E/P \)

In this section, we complete the proof of Theorem 1.1 by deducing it from Theorem 3.4. This is done by using the following general technique, which the author learned from an unpublished note \([6]\) of Eddidin and Larsen. So let \( (G,T,B,W) \) be our given datum, where we have already reduced to the case where \( G \) is reductive. Since \( T \) is split, the group \( G \) is given by its root system \( \Phi(G,T) \). Let \( \Delta \) be a base of \( \Phi \) and \( B \) the corresponding Borel. By the well known theory of root system (cf. \([17]\)), for every subset \( I \) of \( \Delta \), there exists a corresponding parabolic subgroup \( P_I \supset B \) and every parabolic subgroup containing \( B \) is of this form. Let \( \Psi_I \) be the intersection of \( \Phi \) with the span of \( I \) and let \( P_I = M_I N_I \) be the corresponding Levi decomposition, where \( M_I \) contains \( T \) and \( \Phi(M_I,T) = \Psi_I \). The Weyl group of \( M_I \) with respect to \( T \) is the subgroup of \( W \) generated by the simple reflections corresponding to the elements of \( I \). The Borel subgroup \( B_I \) of \( M_I \) corresponding to the positive roots in \( \Psi_I \) is the intersection of \( B \) with \( M_I \) and hence the natural map \( P \to M_I \) gives an isomorphism \( P_I/B \xrightarrow{\cong} M_I/B_I \). Since every Borel subgroup is conjugate to \( B \), we can assume without loss of generality that \( P_I \) is same as \( P \). We then obtain tower of fibrations
\[ E \to E/N \to E/B \xrightarrow{\cong} E/P \xrightarrow{f} X. \]

We need the following Corollary of Theorem 3.4 to prove the case of parabolic flag bundles.

Corollary 7.1. Let \( (G,T,B,W) \) be the datum as above and let \( E \to X \) be a principal \( G \)-bundle. The natural map \( \Omega^*(X) \to \Omega^*(E/B)^W \) is an isomorphism.

Proof. This follows immediately from Theorem 3.4 using the fact that the trivial \( W \)-module \( \mathbb{Q} \) is a projective \( \mathbb{Q}[W] \)-module, and \( M^W = \text{Hom}_{\mathbb{Q}[W]}(\mathbb{Q},M) \) for any \( \mathbb{Q}[W] \)-module \( M \).

To prove Theorem 1.1 we notice that \( E/B \to E/P \) is a \( M/B \)-bundle, where \( B \) is a Borel in \( M \) containing \( T \). If \( W_P \) is Weyl group of \( M \) with respect to \( T \), we obtain
\[ \Omega^*(E/P) \otimes_{C(T)^{W_P}} C(T) \xrightarrow{\cong} \Omega^*(E/B) \]
by Theorem 3.4. Taking the \( W_P \)-invariants and using Corollary 7.1, we get
\[ \Omega^*(E/P) \xrightarrow{\cong} \Omega^*(E/B)^{W_P} \cong \Omega^*(X) \otimes_{C(T)^{W_P}} C(T)^{W_P}. \]
This completes the proof of Theorem 1.1.

\[ \square \]

8. Higher Chow groups of flag-bundles

Let \( X \) be a scheme and let \( p : E \to X \) be a principal \( G \)-bundle and let \( \pi : E/B \to X \) be the flag bundle associated to a Borel subgroup of \( G \). In this section, we describe the higher Chow groups of \( E \) and \( E/B \) in terms of the higher Chow groups of \( X \) and the characteristic classes of the maximal torus \( T \). From this, we obtain formula for the higher Chow groups of the flag bundles \( \pi : E/P \to X \).

Recall from \([1]\) that the higher Chow groups of \( X \) are given by the homology groups \( \text{CH}^i(X,n) = H_n(Z^i(X,\bullet)) \) of the cycle complex \( Z^i(X,\bullet) \) of codimension
i cycles in the simplicial spaces $X \times \Delta^n$. We refer to loc. cit. for more detail and for some standard functorial properties. For a scheme $X$ with $G$-action, the equivariant higher Chow groups $CH^*_G(X, \bullet)$ are defined in [5]. We refer the reader to [8] for more details about these groups. We denote by $CH^*(X)$, the full Chow groups $\bigoplus_{n=0}^\infty CH^*(X, n)$ of $X$. This is a $CH^*(k)$-algebra if $X$ is smooth.

Let $S = CH^*_T(k) \cong \mathbb{Q}[x_1, \ldots, x_r]$ and $SW = S(G) = CH^*_G(k) \cong \mathbb{Q}[\sigma_1, \ldots, \sigma_r]$ be as before. Then the same construction as in Section 3 (or as in [19]) gives a natural map of $S$-modules

\[(8.1) \quad \alpha_X : CH^*(X) \otimes_{SW} S \to CH^*(E/B).\]

Lemma 8.1. Let $E/B \times X \xrightarrow{\pi} X$ be the trivial flag bundle. Then $\alpha_X$ is an isomorphism.

Proof. We first assume $X = \text{Spec}(k)$. By [9, Theorem 1.6], the natural map $CH^*_G(G/B) \otimes_{SW} Q \to CH^*(G/B)$ is an isomorphism since $G/B$ is smooth and projective. On the other hand, we have $CH^*_G(G/B) \cong CH^*_T(k)$ by [8, Corollary 3.2]. In particular, we get

\[
CH^*(k) \otimes_{SW} S \cong (CH^*_T(k) \otimes_{S} Q) \otimes_{SW} S \\
\cong CH^*_T(k) \otimes_S (S \otimes_{SW} Q) \\
\cong CH^*_T(k) \otimes_{SW} Q \\
\cong CH^*(G/B),
\]

where the first isomorphism holds by [9, Theorem 1.6]. If $G/B \times X \xrightarrow{\pi} X$ is the trivial bundle, then we have

\[
CH^*(X) \otimes_{SW} S \cong (CH^*(X) \otimes_{CH^*(k)} CH^*(k)) \otimes_{SW} S \\
\cong CH^*(X) \otimes_{CH^*(k)} (CH^*(k) \otimes_{SW} S) \\
\cong CH^*(X) \otimes_{CH^*(k)} CH^*(G/B) \\
\cong CH^*(X \times G/B),
\]

where the third isomorphism follows from the case of $X = \text{Spec}(k)$ and the last isomorphism follows from [8, Lemma 3.6]. This completes the proof. \qed

Proof of Theorem 1.2. As in the case of cobordism, we can deduce the case of parabolic subgroups from the case of Borel subgroups. So we prove the result for the flag bundles of the type $E/B \to X$. We prove by induction on the dimension of $X$. If $X$ is zero-dimensional, this follows from the case of $X = \text{Spec}(k)$ shown above and the analogue of Corollary [5,3] for the higher Chow groups. In general, following the proof in the cobordism case, we can find a dense open subset $U \subset X$ and a finite étale cover $g : U' \to U$ such that the pull-back of $E/B$ is the trivial flag bundle on $U'$ and the result holds for $U'$ as shown above. We deduce the result for $U$ by the higher Chow groups analogue of Corollary [5,3]. Let $Z$ be the complement of $U$ in $X$ with the reduced closed subscheme structure. We get the following diagram of long exact localization sequences

\[
\begin{array}{ccc}
CH^*(U) \otimes S \otimes CH^*(Z) \otimes S & \xrightarrow{\alpha_U} & CH^*(X) \otimes S \otimes CH^*(U) \otimes S \\
\downarrow{\cong} & & \downarrow{\cong} \\
CH^*(E_U/B) \otimes CH^*(E_Z/B) & \xrightarrow{\alpha_Z} & CH^*(E/B) \otimes CH^*(E_U/B) \otimes CH^*(E_Z/B),
\end{array}
\]
where the tensor product in the top row is over the ring $S^W$. In particular, this row is exact by the flatness of $S$ over $S^W$. Since the dimension of $Z$ is strictly smaller than that of $X$, we see that the maps $\alpha_Z$ are isomorphisms by induction. We have shown above that the maps $\alpha_U$ are also isomorphisms. We conclude that $\alpha_X$ is an isomorphism.

**Corollary 8.2.** Let $G$ be a connected linear algebraic group with a split maximal torus $T$. Let $E \xrightarrow{p} X$ be a principal $G$-bundle over a scheme $X$. Then

$$\text{CH}^*(E) \xrightarrow{\sim} \text{CH}^*(X) \otimes_{S(G)} \mathbb{Q} \cong \frac{\text{CH}^*(X)}{I} \text{CH}^*(X),$$

where $I$ is the ideal of $S(G)$ generated by the Chern classes of $G$-homogeneous line bundles. This is a $\mathbb{Q}$-algebra isomorphism if $X$ is smooth.

**Proof.** The proof is exactly same as the proof of Corollary 6.6. □

The following corollary recovers a result of Brion [2, Proposition 2.8] for the ordinary Chow groups $\text{CH}^*(G,0)$ of connected algebraic groups as a special case.

**Corollary 8.3.** Let $G$ be a connected algebraic group (not necessarily linear) over $k$ and let $G \xrightarrow{\sim} A(G)$ be the albanese morphism of $G$. Then there is a $\mathbb{Q}$-algebra isomorphism

$$\text{CH}^*(G) \cong \frac{\text{CH}^*(A(G))}{I} \text{CH}^*(A(G)).$$

**Proof.** The proof is exactly same as the proof of Corollary 6.7. □

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