Perturbations and quasi-normal modes of black holes in Einstein-Aether theory

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We develop a new method for calculation of quasi-normal modes of black holes, when the effective potential, which governs black hole perturbations, is known only numerically in some region near the black hole. This method can be applied to perturbations of a wide class of numerical black hole solutions. We apply it to the black holes in the Einstein-Aether theory, a theory where general relativity is coupled to a unit time-like vector field, in order to observe local Lorentz symmetry violation. We found that in the non-reduced Einstein-Aether theory, real oscillation frequency and damping rate of quasi-normal modes are larger than those of Schwarzschild black holes in the Einstein theory.

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I. INTRODUCTION

General Relativity is based on local Lorentz invariance, yet recently there appeared a lot of attempts to go beyond local Lorentz symmetry \(^1\). Aether can be considered as locally preferred state of rest at each point of space-time due-to some unknown physics. Einstein-Aether theory is general relativity coupled to a dynamical time-like vector field \(u^a\), which is called “aether”. This theory is what comes instead of usual General Relativity when local Lorentz symmetry is broken. Namely, \(u^a\) breaks local boost invariance, while rotational symmetry in a preferred frame is preserved (see \([2]\) for a recent review). Different observable consequences of Einstein-Aether theory have been discussed recently \([3]\).

Quasi-normal modes have been studied recently extensively, because of their interpretation in Conformal Field Theory \([4]\) and possible, but yet unclear significance in Loop Quantum Gravity. At the same time, one of the most promising ways to check the theory of gravity as a fully non-linear theory is to observe the characteristic frequencies of black holes called quasi-normal modes \([5]\). QN frequencies are expected to be observed in the nearest future with the help of a new generation of gravitational antennas. This suggests a unique opportunity to test the spontaneous breaking of the local Lorentz invariance, through observing deviations of quasi-normal modes of black holes from their values in Einstein theory.

Therefore, we would like to know what comes instead of the well-known quasi-normal spectrum of black holes in Einstein-Aether theory. For this one must have a solution describing black holes in the Einstein-Aether theory. Fortunately such a solution has been recently obtained in \([6]\), yet, only numerically. That was one of the motivations for us for developing of the method for finding quasi-normal modes for potentials which are not known in analytical form, but are given only numerically in some region near a black hole (see Sec. III of this paper). In case of asymptotically anti-de Sitter space-times, the method for finding QN modes for metrics given by a set of differential equations was proposed in \([7]\).

We found here that quasi-normal modes for spherically symmetric black holes in the non-reduced Einstein-Aether theory have larger real oscillation frequency and damping rate than the corresponding QN modes in the locally Lorentz invariant Einstein gravity.

II. BASIC FORMULAS

The lagrangian of the full Einstein-Aether theory forms the most general diffeomorphism invariant action of the space-time metric \(g_{ab}\) and the aether field \(u^a\) involving no more than two derivatives given by

\[
L = - R - K_{mn}^{ab} \nabla_a u^m \nabla_b u^n - \lambda (g_{ab} u^a u^b - 1),
\]

here \(R\) is the Ricci scalar, \(\lambda\) is a Lagrange multiplier which provides the unit time-like constraint,

\[
K_{mn}^{ab} = c_1 g^{ab} g_{mn} + c_2 g^a_m g^b_n + c_3 g^{ab} g_{mn} + c_4 u^a u^b g_{mn},
\]

where \(c_i\) are dimensionless constants.

Spherical symmetry allows to fix \(c_4 = 0\). In this letter, following \([6]\), we shall consider the so-called non-reduced Einstein-Aether theory, for which \(c_3 = 0\), and we can use the field redefinition that fixes the coefficient \(c_2\) \([6]\):

\[
c_2 = - \frac{c_1^2}{2 - 4c_1 + 3c_1^2},
\]

so that \(c_1\) is the free parameter.

The metric for a spherically symmetric static black hole in Eddington-Finkelstein coordinates can be written in the form \([6]\):

\[
ds^2 = N(r)dt^2 - 2B(r)dtdr - r^2 d\Omega^2,
\]

where the functions \(N(r)\) and \(B(r)\) are given by numerical integration near the black hole event horizon \([6]\). One
can re-write this metric in a Schwarzschild like form:

\[ ds^2 = -N(r)dt^2 + \frac{B^2(r)}{N(r)}dr^2 + r^2d\Omega^2. \tag{3} \]

Now, let us consider test scalar and electromagnetic fields in the background of the black hole given by the metric [3]. Here we shall use general covariant generalizations of scalar and electromagnetic wave equations, i.e. we neglect interaction of these fields with aether. This can be well understood, since the background of a large astrophysical black hole has much more influence on propagation of fields during quasi-normal ringing than aether, at least not very far from the black hole. Yet, such phenomena as tales at asymptotically late times are probably affected by asymptotic regions of space-times as well as by a black hole itself.

Thus, the wave equations for test scalar \( \Phi \) and electromagnetic \( A_\mu \) fields are:

\[ (g^{\mu\nu}\sqrt{-g}\Phi)_{,\nu} = 0, \tag{4} \]

\[ ((\sigma_\alpha - A_{\alpha,\sigma})g^{\mu\nu}g^{\sigma\nu}\sqrt{-g})_{,\nu} = 0. \tag{5} \]

After making use of the metric coefficients [3], we found that the perturbation equations can be reduced to the wave-like form for the scalar and electromagnetic wave functions \( \Psi \) and \( \Psi_{el} \):

\[ \frac{d^2\Psi_i}{dr_*^2} + (\omega^2 - V_i(r))\Psi_i = 0, \quad dr_* = \frac{B(r)}{N(r)}dr. \tag{6} \]

The effective potentials take the form:

\[ V_s = N(r)\frac{\ell(\ell + 1)}{r^2} + \frac{1}{r} \frac{d}{dr_*} \frac{N(r)}{B(r)}. \tag{7} \]

\[ V_{el} = N(r)\frac{\ell(\ell + 1)}{r^2}. \tag{8} \]

Below we shall also use the particular case of the above metric when \( B(r) = 1 \) and \( N(r) = 1 - 2M/r \), what corresponds to an ordinary Schwarzschild black hole. Now we are in position to find quasi-normal spectrum for the above wave equations. Yet, as the function \( B(r) \), and \( N(r) \) are known only numerically, in the next section we shall have to develop a new technique for finding of quasi-normal modes in this case.

III. A NEW METHOD FOR CALCULATION OF QUASI-NORMAL MODES FOR POTENTIALS UNKNOWN ANALYTICALLY

For a wide class of problems of astrophysical interest, the dynamical wave equation has the form [6], where the effective potential has the form of some potential barrier that approaches constant values at the event horizon and spatial infinity. The general solution of the wave equation at infinity is

\[ \Psi = A_{in}\psi_{in} + A_{out}\psi_{out}, \quad \ell_* \rightarrow \infty. \tag{9} \]

The quasi-normal modes in this approach, by the definition, are the poles of the reflection coefficients \( A_{out}/A_{in} \). Out starting point is suggested by the WKB method [8] where the asymptotic solutions of the wave equation near the event horizon and near spatial infinity are matched with Taylor expansion near the peak of the potential, i.e. between two turning points \( V(r) - \omega^2 = 0 \). We state that the low laying quasi-normal modes are determined mainly by the behavior of the effective potential near its peak, while the behavior of the potential far from black hole is insignificant. Below we shall show that our statement is true. For this to happen, we shall consider the well-known potential for Schwarzschild black hole \( V(r) \) and also two other potentials which lay closely to the Schwarzschild potential near its maximum, but has very different behavior far from a black hole. These two potentials are chosen in the following way. We make a plot of the function \( V(r) \) of an analytic Schwarzschild potential, then we find some number of points for this potential \( V_s \) near its maximum which serve us a basis for our first potential \( V_{out} \) which is an interpolation of these points near the maximum by cubic splines. Second potential \( V_{fit} \) is a fit of the above plot near the maximum by a ratio of polynomial functions. The possibility of making use of the numerical interpolation of the potential near its maximum gives us possibility of finding QN spectrum of fields near the solutions which are not known in analytical form.

Let us explain the interpolation technique more precisely. Suppose we know some numerical solution for the metric. In other words, we are able to calculate the metric coefficients in a finite number of points with desired precision. Then, by using formulas [8], one can find

**Figure 1**: Potential for electromagnetic perturbations near Schwarzschild black hole (\( \ell = 2 \)) and the same potential interpolated numerically near its maximum. Even despite the behavior of the two potentials are very different in the full region of \( r \), except for a small region near black hole, low-laying quasi-normal modes for both potentials are very close.
the values of the potential at these points with the desired precision and any number of its derivatives. It is clear that the precision of the higher derivatives decreases very rapidly, so that, in order to find them with a good accuracy we have to find a lot of interpolation points very precisely. If the accuracy is not good enough, the calculations lead to some random values. Thus we feel that the potential has to be very smooth (and for physically relevant problems it is) to use the WKB approximation, because WKB method requires some number of the derivatives at the maximum of the effective potential. The 6-th order WKB formula reads

$$ V_{\ell} = \frac{iQ_0}{\sqrt{2Q_0}} - \sum_{i=2}^{6} \Lambda_i = n + \frac{1}{2}, \quad (10) $$

where the correction terms of the i-th WKB order \( \Lambda_i \) can be found in [7] and [10]. \( Q = V - \omega^2 \) and \( Q_0 \) means the i-th derivative of \( Q \) at its maximum. For the WKB formula of sixth order one needs twelve derivatives of the potentials.

Now let us see how the above programs of fit and interpolation work for the well-known case of the Schwarzschild black hole. In cases when we know the metric functions \( N(r) \) and \( B(r) \) numerically in some region near the black hole using these functions in the formulas for potentials [7] and [8] gives us numerical values for the potential near the black hole. The accurate plot of the effective potential [3] \( \ell = 2 \) is shown on Fig. [11] together with interpolated potential \( V_{\ell} \) and fit potential \( V_{fit} \) where the interpolation is made on the basis of 75 points from \( r = 2M \) until \( r = 5M \) and fit is made on the basis of 100 points found with 1% precision. Fit functions are chosen as fractions of two polynomials:

$$ N(r) = \frac{i=N_N}{\sum_{i=0}^{i=N_N} a_i^{(N)} r^i}, \quad B(r) = \frac{i=N_B}{1 + \sum_{i=1}^{i=N_B} b_i^{(B)} r^i}, $$

which are substituted into equations [7] and [8]. The numbers \( N_N \) and \( N_B \) determine the number of terms in the polynomials and are chosen in order to provide best convergence of the WKB series. Coefficients \( a_i^{(N)} \), \( b_i^{(N)} \), \( a_i^{(B)} \), \( b_i^{(B)} \) are determined by the fitting procedure. We see from the above plot, that at larger \( r \) the interpolation potential \( V_{int} \) grows, while the precise function and fit function are decreasing. In the region near black hole \( r = (2M, 5M) \) the difference between the potentials is negligible. The interpolation was performed by cubic spline with the help of Mathematica®. In order to diminish accumulating of an error, upon calculating of the first derivative of the interpolated potential, we had to interpolate also the obtained values for the first derivative and all twelve consequent derivatives up to the sixth order.

Let us look at the tables [11] The known 6-th order WKB quasi-normal frequencies are compared with those

| s=1 | Exact potential | 20 points interp. | 50 points interp. | Fit (10 points, 1%) | Fit (100 points, 1%) |
|-----|-----------------|------------------|------------------|-------------------|-------------------|
| \( \ell = 1 \), WKB3 | 0.491740-0.186212i | 0.491669-0.186217i | 0.491739-0.186216i | 0.491680-0.186493i | 0.491680-0.186181i |
| \( \ell = 1 \), WKB6 | 0.496380-0.185724i | 0.512474-0.174892i | 0.496625-0.184951i | 0.494266-0.184069i | 0.496285-0.185244i |
| \( \ell = 2 \), WKB3 | 0.914260-0.190136i | 0.914225-0.190132i | 0.914261-0.190132i | 0.910125-0.188874i | 0.914155-0.190069i |
| \( \ell = 2 \), WKB6 | 0.915187-0.189337i | 0.916333-0.189337i | 0.915134-0.190016i | 0.910338-0.188768i | 0.915098-0.189991i |
| \( \ell = 3 \), WKB3 | 2.095741-0.191968i | 2.095725-0.191978i | 2.095741-0.191968i | 2.089038-0.190693i | 2.095492-0.191861i |
| \( \ell = 3 \), WKB6 | 2.095826-0.191963i | 2.095818-0.191953i | 2.095826-0.191963i | 2.086121-0.190688i | 2.095575-0.191911i |

| s=10 | Exact potential | 20 points int. | 50 points int. | Fit (10 points, 1%) | Fit (100 points, 1%) |
|------|-----------------|----------------|----------------|-------------------|-------------------|
| \( \ell = 1 \), WKB3 | 0.582228-0.196903i | 0.582803-0.195801i | 0.582219-0.195995i | 0.579382-0.194684i | 0.582155-0.195097i |
| \( \ell = 1 \), WKB6 | 0.585919-0.195523i | 0.589883-0.192359i | 0.589540-0.195409i | 0.582932-0.194211i | 0.585754-0.195406i |
| \( \ell = 2 \), WKB3 | 0.966422-0.193601i | 0.966365-0.193592i | 0.966420-0.193616i | 0.961844-0.192317i | 0.966304-0.193575i |
| \( \ell = 2 \), WKB6 | 0.967284-0.193532i | 0.968078-0.192999i | 0.967284-0.193532i | 0.962685-0.192240i | 0.967166-0.193500i |
| \( \ell = 3 \), WKB3 | 1.350442-0.190924i | 1.350304-0.190928i | 1.350441-0.190925i | 1.344061-0.191738i | 1.350249-0.192922i |
| \( \ell = 3 \), WKB6 | 1.350732-0.190001i | 1.350850-0.192884i | 1.350739-0.192995i | 1.344377-0.191715i | 1.350569-0.192966i |
| \( \ell = 4 \), WKB3 | 1.734680-0.192790i | 1.734662-0.192802i | 1.734679-0.192793i | 1.726554-0.191569i | 1.734471-0.192766i |
| \( \ell = 4 \), WKB6 | 1.734831-0.192784i | 1.734848-0.192750i | 1.734834-0.192782i | 1.726704-0.191500i | 1.734622-0.192752i |
| \( \ell = 5 \), WKB3 | 2.119114-0.192678i | 2.119127-0.192688i | 2.119140-0.192678i | 2.109235-0.191935i | 2.118886-0.192646i |
| \( \ell = 5 \), WKB5 | 2.119224-0.192674i | 2.119219-0.192665i | 2.119224-0.192673i | 2.109316-0.191391i | 2.118969-0.192641i |
obtained by using formula (10) for interpolation function \( V_{int} \) and for fit function \( V_{fit} \). Within our approach we have two kinds of errors: error due-to WKB approximation, and error due-to approximation of the potential by fit or interpolation. The error because of the WKB approximation can be estimated simply by comparison of the QN values within different WKB orders. The initial deviations from the accurate potential due-to fit or interpolation induce larger errors for higher derivatives of the potential, and thereby, show themselves as a final error in the computed QNMs. Therefore, if one gets good convergence of the WKB series, it means that the deviations of the approximated potential from the accurate one are small enough, so that induced errors in higher derivatives are small. Thus, the criterium of goodness of a fit or interpolation is convergence of the WKB series. An interpolation, to give a good WKB convergence, can be improved in the following way: one should increase the density of interpolation points, and, at the same time, increase the accuracy with which each point is given. Unlike interpolation, the fit approximation is more economic and works better with increasing either number of fit points or accuracy with which each point is given.

Using the above mentioned procedure of improving of the fit and interpolation, we can, in principle, decrease the error due-to potential approximation until as small value as necessary. From tables I-II we see that larger number of points for interpolation or fit gives better accuracy as compared with data for the exact potential. From tables III-V we can see that real and imaginary parts of \( \omega \) is about 1% greater for \( c_1 = 0.1 \). When \( c_1 \) is growing up to \( c_1 = 0.77 \), both \( \text{Re}\omega \) and \( \text{Im}\omega \) are growing by about 10% and 20% respectively. For \( \ell \geq 1 \), the WKB method gives an error of about a fraction of a percent. Therefore our conclusion, that in the above region of \( c_1 \leq 0.77 \), the observed here effect of a few percents is much larger than the order of the WKB error. For the particular case of \( s = \ell = 0 \), the WKB error is about 5% (see Table V), therefore we cannot judge about the value of the aether effect in the case \( s = \ell = 0 \) for small \( c_1 \). Yet for \( c_1 \sim 0.7 \), the effect a few times greater than the error, and, qualitatively the behavior of the QN spectrum is the same: the larger \( c_1 \) leads to a larger real oscillation frequency and damping rate.

V. DISCUSSION

The quasinormal modes of black holes should be feasible for observation by new generation of gravitational antennas. This would make a great impact on gravitational physics, where gravitational waves have not been observed.

At the same time, the gravitational consequences of Local Lorentz symmetry violation must show itself in radiative processes around black holes. Gravitational radiation damping of binary pulsars orbits reproduces the weak field general relativity at the lowest post-Newtonian order [1]. The significant difference between Einstein and Einstein-Aether theories can show itself in the regime of strong field, i.e. in changing the characteristic quasi-normal spectrum of black holes. Thereby, aether, if exists, could be indirectly observed through observation of characteristic spectrum of black holes. This motivated us to perform the present study of quasi-normal modes for black holes in Einstein-Aether theory.

In this paper we developed a new method for finding quasi-normal modes for potentials determined only numerically in some region near a black hole. Then, the method has been applied here to numerical black hole solution in Einstein-Aether theory, which has been obtained recently in [3].
We showed that quasi-normal modes of Einstein-Aether black holes have larger damping rate and real oscillation frequency than QNMs of Schwarzschild black holes have. We considered here scalar and electromagnetic test fields in the background of a spherically symmetric black hole. The wave equation for gravitational perturbations could be found by consideration of perturbations of the complete system of Einstein-Aether equations. Then our approach for finding of the quasi-normal modes can be applied as well. Yet, it is well known that quasinormal modes in the eikonal approximation are the same for fields of different spin. Therefore we expect qualitatively the same results for gravitational perturbations as we observed for scalar and electromagnetic perturbations. If the breaking of Lorentz symmetry is not very small, i.e. $c_1$ is not very small, the deviation of the QNMs from their Schwarzschild values might be observed in the forthcoming experiments with gravitational antennas.

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Table III: Fundamental QN frequencies of the non-reduced theory for $s = 0$

| $\ell$ | $c_1$ | $\ell = 1$ | $\ell = 2$ | $\ell = 3$ | $\ell = 4$ | $\ell = 5$ |
|-------|-----|---------|---------|---------|---------|---------|
| 0.1   | 0.589693 | 0.197691 | 0.973933 | 0.195722 | 1.360108 | 0.194949 |
| 0.2   | 0.593814 | 0.193653 | 0.981227 | 0.197457 | 1.370409 | 0.196886 |
| 0.3   | 0.597713 | 0.203470 | 0.989398 | 0.201414 | 1.381943 | 0.200829 |
| 0.4   | 0.604163 | 0.207262 | 0.998783 | 0.205259 | 1.395215 | 0.204663 |
| 0.5   | 0.610153 | 0.212271 | 1.009919 | 0.210209 | 1.410990 | 0.209603 |
| 0.6   | 0.618482 | 0.218566 | 1.023707 | 0.216876 | 1.435653 | 0.216280 |
| 0.7   | 0.628421 | 0.229250 | 1.042296 | 0.226654 | 1.457064 | 0.226066 |
| 0.77  | 0.637216 | 0.240653 | 1.060028 | 0.236792 | 1.482451 | 0.236224 |

Table IV: Fundamental QN frequencies of the non-reduced theory for $s = 1$

| $\ell$ | $c_1$ | $\ell = 1$ | $\ell = 2$ | $\ell = 3$ | $\ell = 4$ | $\ell = 5$ |
|-------|-----|---------|---------|---------|---------|---------|
| 0.1   | 0.499500 | 0.187126 | 0.921357 | 0.192128 | 1.322831 | 0.193366 |
| 0.2   | 0.503184 | 0.188728 | 0.928321 | 0.193764 | 1.332882 | 0.195021 |
| 0.3   | 0.506385 | 0.192059 | 0.935562 | 0.197618 | 1.343774 | 0.198918 |
| 0.4   | 0.510456 | 0.195179 | 0.944082 | 0.201334 | 1.356438 | 0.202690 |
| 0.5   | 0.515081 | 0.199198 | 0.954092 | 0.206114 | 1.371426 | 0.207550 |
| 0.6   | 0.520683 | 0.204318 | 0.966339 | 0.212544 | 1.389924 | 0.214116 |
| 0.7   | 0.527472 | 0.211949 | 0.982634 | 0.221933 | 1.414826 | 0.223723 |
| 0.77  | 0.533464 | 0.219667 | 0.997944 | 0.231610 | 1.438525 | 0.233670 |

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Table V: Fundamental QN frequencies of the non-reduced theory for $s = 0$, $\ell = 0$

| $c_1$ | Poschl-Teller | WKB3 | WKB5 | WKB6 |
|-------|--------------|------|------|------|
| 0.1   | 0.230507 – 0.232576i | 0.209816 – 0.233510i | 0.210764 – 0.213968i | 0.221265 – 0.203869i |
| 0.2   | 0.231194 – 0.234727i | 0.209789 – 0.235548i | 0.212313 – 0.215742i | 0.222606 – 0.205766i |
| 0.3   | 0.232538 – 0.240341i | 0.210476 – 0.241535i | 0.212397 – 0.220352i | 0.224395 – 0.208570i |
| 0.4   | 0.233729 – 0.245740i | 0.210540 – 0.247090i | 0.215188 – 0.225894i | 0.222845 – 0.218132i |
| 0.5   | 0.235007 – 0.252888i | 0.209718 – 0.254241i | 0.220603 – 0.233407i | 0.220265 – 0.237365i |
| 0.6   | 0.236272 – 0.262898i | 0.207228 – 0.264133i | 0.229154 – 0.243557i | 0.217124 – 0.257051i |
| 0.7   | 0.237276 – 0.278203i | 0.201317 – 0.279282i | 0.238603 – 0.255254i | 0.224289 – 0.271544i |
| 0.77  | 0.237259 – 0.294753i | 0.193652 – 0.296415i | 0.240916 – 0.263198i | 0.241130 – 0.262964i |

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