MAXIMIZING 2-INDEPENDENT SETS IN 3-UNIFORM HYPERGRAPHS

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Abstract. There has been interest recently in maximizing the number of independent sets in graphs. For example, the Kahn-Zhao theorem gives an upper bound on the number of independent sets in a $d$-regular graph. Similarly, it is a corollary of the Kruskal-Katona theorem that the lex graph has the maximum number of independent sets in a graph of fixed size and order. In this paper we solve two equivalent problems.

The first is: what 3-uniform hypergraph on a ground set of size $n$, having at least $t$ edges, has the most 2-independent sets? Here a 2-independent set is a subset of vertices containing fewer than 2 vertices from each edge. This is equivalent to the problem of determining which graph on $n$ vertices having at least $t$ triangles has the most independent sets. The (hypergraph) answer is that, ignoring some transient and some persistent exceptions, a $(2, 3, 1)$-lex style 3-graph is optimal.

We also discuss the problem of maximizing the number of $s$-independent sets in $r$-uniform hypergraphs of fixed size and order, proving some simple results, and conjecture an asymptotically correct general solution to the problem.

1. Introduction

For many years, there has been interest in finding the maximum size of a variety of substructures (such as independent sets or matchings) in a graph satisfying certain conditions. In recent years, there has been increased interest in extremal questions about the number of these sub-structures. That is, rather than asking for the size of the largest independent set, one could ask which graph has the most independent sets, given some set of conditions. In fact, many extremal problems for the the number of independent sets have been studied. A classic example is the Kahn-Zhao theorem, proved initially by Kahn [4] in the bipartite case, and then extended to the general case by Zhao [11].

Theorem 1 (Kahn-Zhao). If $G$ is a $d$-regular graph then $\text{ind}(G)$, the number of independent sets in $G$, satisfies

$$\text{ind}(G) \leq (2d+1) \frac{2^n}{2^n} = (\text{ind}(K_{d,d})) \frac{2^n}{2^n}$$

where $K_{d,d}$ is the complete balanced bipartite graph on $2d$ vertices.

In particular, if $2d$ divides $n$, the $d$-regular graph with the most independent sets is a disjoint union of complete balanced bipartite graphs. In a different vein, one could consider the independent set maximization problem for graphs having $n$ vertices and $e$ edges. It has been shown (see, e.g., [1]) that the Kruskal-Katona Theorem [7, 5] implies that the lex graph, $\mathcal{L}(n, e)$, has the greatest number of independent sets among graphs having $n$ vertices and $e$ edges. The lex graph, $\mathcal{L}(n, e)$ is the graph that has vertex set $[n]$ and edge set the first $e$ sets in the lex (or dictionary) order, $<_L$, on $\binom{n}{2}$.

It is natural to try to extend these extremal results for the number of independent sets to hypergraphs. A hypergraph $\mathcal{H}$ is an ordered pair $(\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ where $\mathcal{V}(\mathcal{H})$ is a vertex set and $\mathcal{E}(\mathcal{H})$ is a set of edges where each edge is a subset of $\mathcal{V}(\mathcal{H})$. Typically we abuse notation...
and refer to a hypergraph as its edge set, writing, for example, \( E \in \mathcal{H} \) to mean \( E \in \mathcal{E} ( \mathcal{H} ) \) and \( \mathcal{H} + E \) to mean \( ( \mathcal{V} ( \mathcal{H} ) , \mathcal{E} ( \mathcal{H} ) ) \cup \{ E \} \). A hypergraph is \( r \)-\textit{uniform} if all edges have size \( r \). For convenience we’ll often call an \( r \)-uniform hypergraph an \( r \)-\textit{graph}.

In a graph, an independent set is a subset of vertices containing at most one vertex from each edge. In an \( r \)-graph for \( r > 2 \), it makes sense to consider allowing more than one vertex from the independent set to be in each edge.

**Definition.** For an \( r \)-graph \( \mathcal{H} = ( \mathcal{V} , \mathcal{E} ) \) and an integer \( s \) with \( 1 \leq s \leq r \), a set \( I \subset \mathcal{V} \) is \( s \)-\textit{independent} if \( |I \cap E| < s \) for all \( E \in \mathcal{E} \). We let \( I_s ( \mathcal{H} ) \) denote the set of \( s \)-independent sets of a hypergraph \( \mathcal{H} \) and set \( i_s ( \mathcal{H} ) = |I_s ( \mathcal{H} )| \).

There has been some research on independent sets in hypergraphs, mostly focused on determining algorithms for finding independent sets in hypergraphs (see, e.g., [10]) or on finding the independent set of largest size (see, e.g., [6]). However, some extremal questions about the number of independent sets in hypergraphs have been addressed. In [2] Cutler and Radcliffe give an asymptotically best possible upper bound on the number of \( s \)-independent sets in an \( r \)-uniform hypergraph of fixed size and order. Since they use a version of the hypergraph regularity lemma, their results only apply to graphs with a large number of vertices.

It is also the case that maximizing 1-independent sets and \( r \)-independent sets in \( r \)-uniform hypergraphs with \( n \) vertices and \( e \) edges is straightforward. Defining the \textit{lex} \( r \)-\textit{graph} \( \mathcal{L}_r ( n , e ) \) to be the \( r \)-graph with vertex set \([n]\) and edge set the first \( e \) sets in the lex ordering\(^1\) on \( \binom{[n]}{r} \), the Kruskal-Katona Theorem implies the following:

**Theorem 2.** Let \( i_r ( \mathcal{H} ) \) be the number of \( r \)-independent sets in \( \mathcal{H} \). If \( \mathcal{H} \) is an \( r \)-uniform hypergraph with \( n \) vertices and \( e \) edges then

\[
i_r ( \mathcal{H} ) \leq i_r ( \mathcal{L}_r ( n , e ) )\]

The \textit{colex} \( r \)-\textit{graph} \( \mathcal{C}_r ( n , e ) \) is the \( r \)-graph with vertex set \([n]\) and edge set the first \( e \) sets in the colex order\(^2\) on \( \binom{[n]}{r} \).

**Theorem 3.** If \( \mathcal{H} \) is an \( r \)-graph on \( n \) vertices with \( e \) edges then

\[
i_1 ( \mathcal{H} ) \leq i_1 ( \mathcal{C}_r ( n , e ) )\]

This theorem follows immediately from the simple lemma below.

**Lemma 4.** For a hypergraph \( \mathcal{H} \) let \( S ( \mathcal{H} ) \) be the set of isolated vertices in \( \mathcal{H} \), and let \( s ( \mathcal{H} ) = |S ( \mathcal{H} )| \).

\begin{enumerate}
  \item \( i_1 ( \mathcal{H} ) = 2^{s ( \mathcal{H} )} \).
  \item If \( \mathcal{H} \in \mathcal{H}_r ( n , e ) \) then \( s ( \mathcal{H} ) \leq s ( \mathcal{C}_r ( n , e ) ) \).
\end{enumerate}

**Proof.** For the first, note that a set \( D \) is 1-independent in a hypergraph \( \mathcal{H} \) if and only if \( |A \cap E| < 1 \) for all \( E \in \mathcal{E} ( \mathcal{H} ) \), i.e. \( A \subseteq S ( \mathcal{H} ) \). Thus \( i_1 ( \mathcal{H} ) = 2^{s ( \mathcal{H} )} \). For the second, note that trivially \( s ( \mathcal{H} ) \geq m \) requires \( e \leq \binom{n-m}{r} \), so

\[
s ( \mathcal{H} ) \leq \max \{ m : e \leq \binom{n-m}{r} \}.
\]

On the other hand \( \mathcal{C}_r ( n , e ) \) achieves the bound on the right. \( \Box \)

\(^1\)The lex ordering, \( <_L \), on \( \binom{[n]}{r} \) is defined by \( A <_L B \) if and only if \( \min \{ A \Delta B \} \subseteq A \).

\(^2\)The colex ordering, \( <_L \), on \( \binom{[n]}{r} \) is defined by \( A <_L B \) if and only if \( \max \{ A \Delta B \} \subseteq B \).
Remark. If \( e \) is not of the form \( \binom{k}{r} \) for any \( k \) then there are many graphs having the same number of isolated vertices as the colex graph. In fact, if \( \binom{k-1}{r} < e < \binom{k}{r} \) then any \( e \)-subset of \( \binom{k}{r} \) for \( K \) a \( k \)-set has the maximum number of isolated vertices.

1.1. Our problem. The problem we consider in this paper can be phrased in two ways. If we write \( H_r(n,e) \) for the family of \( r \)-uniform hypergraphs with \( n \) vertices and \( e \) edges, then from one perspective we are determining

\[
\max \{ i_2(H) : H \in H_3(n,e) \}
\]

for all values of \( n \) and \( m \). The other perspective is a graph-theoretic one. If \( H \) is a 3-uniform hypergraph on vertex set \( V \) we can consider the graph \( G = \partial_2 H \) with edge set

\[
E(G) = \left\{ xy \in \binom{V}{2} : \exists F \in E(H) \text{ s.t. } xy \subseteq F \right\}.
\]

A set \( I \subseteq V \) is 2-independent in \( H \) if and only if does not overlap with any edge of \( H \) in at least 2 vertices. But this is precisely the same as requiring that \( I \) is an independent set of \( G \). Each edge of \( H \) gives a triangle in \( G \) (though not necessarily vice versa). From this perspective we are trying to determine

\[
\max \{ i(G) : n(G) = n, \ k_3(G) \geq e \},
\]

where we write \( k_3(G) \) for the number of triangles in \( G \). For completeness we carefully prove the equivalence of these two problems.

**Lemma 5.** For all \( n, m \in \mathbb{N} \) we have

\[
\max\{i_2(H) : H \text{ is a 3-uniform hypergraph on vertex set } [n] \text{ with } e(H) = e\} = \max\{i(G) : G \text{ is a graph on vertex set } [n] \text{ with } k_3(G) \geq e\}.
\]

**Proof.** To prove that the left hand side is at most the right we just take \( H \) to attain the maximum on the left and let \( G = \partial_2 H \). We have \( k_3(G) \geq e(H) = e \) and \( i(G) = i_2(H) \). In the other direction, take a graph \( G \) maximizing the right hand side. Let \( K_3(G) \) be the 3-uniform hypergraph on \( [n] \) whose edges are the vertex sets of triangles in \( G \). By hypothesis \( e(K_3(G)) \geq e \), so we can take \( H \) to be an arbitrary spanning sub-hypergraph of \( K_3(G) \) having exactly \( e \) edges. We get

\[
i_2(H) = i(\partial_2 H) \geq i(G),
\]

since \( \partial_2 H \) is a spanning subgraph of \( G \). \( \square \)

Phrased in this way some of the difficulties of the problem are laid bare. To find the 2-independent sets of \( H \) of size \( t \) we need to first take the lower shadow of \( H \) to find \( G \), and then take the upper shadow of \( E(G) \) on level \( t \); the 2-independent sets are those not in this upper shadow. The twin demands on \( H \) of having not too large a lower shadow \( G \), which in turn has not too large an upper shadow \( \partial G \), are in conflict. For \( H \) to have small lower shadow, it should look as much like a colex initial segment as possible. For \( G \) to have small upper shadow it should look as much like the lex graph as possible.

We state here our main theorem, using some undefined terms that will be clarified later and giving less detail than we do in later sections.
Main Theorem. With a finite number of persistent exceptions (that appear for all values of \( n \)), and a finite number of transient exceptions (that only appear for \( n \leq 31 \)) the maximum number of independent sets in a graph \( G \), subject to having at least \( m \) triangles, is achieved either by the lex graph with the fewest edges subject to having at least \( m \) triangles, or the lexish graph with the fewest edges subject to having at least \( m \) triangles.

Equivalently, and subject to the same exceptions, the maximum number of 2-independent sets in a 3-uniform hypergraph with \( e \) edges on \( n \) vertices is achieved either by the \((2, 3, 1)\)-lex hypergraph or the \((2, 3, 1)\)-lexish hypergraph having \( e \) edges.

We have chosen in this paper to present the hypergraph as our fundamental object for the purposes of proving the main theorem. Later we will meet the downset associated with a shifted hypergraph \( H \). This is (essentially) the edge set of \( G = \partial_2 H \).

We introduce \( \pi \)-lex uniform hypergraphs (for any permutation \( \pi \)) in Section 2. In Section 4 we state our main theorem more explicitly (Theorem 10).

We begin the proof of Theorem 10 in Section 3 by providing background on shifted hypergraphs and proving that an \( r \)-graph attaining the maximum number of \( s \)-independent sets can be found among the shifted hypergraphs. In Section 5 we introduce a way to draw a shifted 3-graph as a “nice” subset of a 3-dimensional cube and discuss a way to count the number of 2-independent sets lost when an edge is added to a shifted 3-graph. Using this we restate the problem yet again, in language useful for our proof. In Sections 6 and 7 we introduce a set of local moves that do not decrease the number of 2-independent sets. In Sections 8 and 9 we use these lemmas to determine which cases are left to prove by computation. Finally, we prove Theorem 10 in Section 10.

1.2. Conventions. We describe here some conventions that apply throughout our paper.

- It will be convenient for us to use a slightly non-standard ground set for our hypergraphs: we let \([n] = \{0, 1, \ldots, n - 1\}\), and we will consider all our hypergraphs to have vertex set \([n]\) for some \( n \).
- We will often need to describe finite sets of integers by listing their elements. Whenever we do so we do so in increasing order. Thus when we write \( A = \{a_1, a_2, \ldots, a_k\} \) we will always assume that \( a_1 < a_2 < \cdots < a_k \).

2. Orderings on \( k \)-sets and \( \pi \)-lex Graphs

In order to state our results we need to describe a number of orderings on \( r \)-sets of integers and some associated \( r \)-graphs. These graphs are an extension of the idea of lex and colex graphs to \( r \)-graphs for \( r > 2 \). Recall that the lex order, \(<_L\), on finite subsets of \( \mathbb{N} \) is defined by \( A <_L B \) if \( \min(AB) \in A \). The colex order, \(<_C\), is defined by \( A <_C B \) if \( \max(AB) \in B \). We create the lex \( r \)-graph, \( \mathcal{L}_r(n, e) \), is the \( r \)-graph with vertex set \([n]\) and edge set the initial segment in the lex order on \( \binom{[n]}{r} \) of length \( e \). Similarly, the colex \( r \)-graph, \( \mathcal{C}_r(n, e) \), is the \( r \)-graph with vertex set \([n]\) and edge set the initial segment in the colex order on \( \binom{[n]}{r} \) of length \( e \).

Example. The first few edges in the lex ordering on \( \binom{[n]}{2} \) are
\[
\{0, 1\}, \{0, 2\}, \ldots, \{0, n - 1\}, \{1, 2\}, \{1, 3\}, \ldots, \{1, n - 1\}, \{2, 3\}, \ldots
\]
and the first few edges in the colex ordering on are
\[
\{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 3\}, \{1, 3\}, \{2, 3\}, \{0, 4\}, \{1, 4\}, \ldots
\]
Note that initial segments of colex do not depend on the size of the ground set, unlike those of the lex ordering. Sets that are early in the lex ordering have small least elements, and sets that are early in the colex ordering have small greatest elements. This idea will help in understanding \( \pi \)-lex graphs.

In \( r \)-graphs for \( r > 2 \) we can define other natural orders on \( \binom{[n]}{r} \) leading to other \( r \)-graphs. In fact, we can define \( r! \) orderings. While these orderings seem very natural we have not seen them introduced elsewhere.

**Definition.** Consider a permutation \( \pi = (\pi_1, \ldots, \pi_k) \) and let \( A = \{a_1, a_2, \ldots, a_k\} \) and \( B = \{b_1, b_2, \ldots, b_k\} \) be sets in \( \binom{[n]}{k} \). We define the \( \pi \)-lex order on \( \binom{[n]}{k} \) by \( A <_\pi B \) if for the least \( i \) for which \( a_{\pi_i} \neq b_{\pi_i} \) we have \( a_{\pi_i} < b_{\pi_i} \).

Given a permutation \( \pi \), define the \( \pi \)-lex \( r \)-graph with \( n \) vertices and \( e \) edges to be the \( r \)-graph on vertex set \( [n] \) with edge set forming an initial segment of the \( \pi \)-lex order on \( \binom{[n]}{r} \) of length \( e \).

**Example.** The lex ordering on \( \binom{[n]}{3} \) is \( \pi \)-lex for \( \pi = (1, 2, 3) \) and the colex ordering on \( \binom{[n]}{3} \) is \( \pi \)-lex for \( \pi = (3, 2, 1) \). The \( \pi \)-lex ordering that will be particularly important to us is the \( (2,3,1) \)-lex ordering. The first few sets in the \( (2,3,1) \)-lex ordering on \( \binom{[n]}{3} \) are:

\[
\{0, 1, 2\}, \{0, 1, 3\}, \ldots, \{0, 1, n - 1\}, \{0, 2, 3\}, \{1, 2, 3\}, \{0, 2, 4\}, \{1, 2, 4\}, \ldots \]

\[
\{0, 2, n - 1\}, \{1, 2, n - 1\}, \{0, 3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{0, 3, 5\}, \quad \{1, 3, 5\}, \{2, 3, 5\}, \ldots, \{0, 3, n - 1\}, \{1, 3, n - 1\}, \{2, 3, n - 1\}, \{0, 4, 5\}, \{1, 4, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \ldots, \{0, 4, n - 1\}, \{1, 4, n - 1\}, \{2, 4, n - 1\}, \{3, 4, n - 1\}, \ldots. \]

Notice that sets that are small in the \( (2,3,1) \)-lex ordering have their second greatest element being small.

There is a natural partial ordering on \( \binom{[n]}{k} \) that will also be relevant, that we call the compression ordering. Given \( A = \{a_1, a_2, \ldots, a_k\} \) and \( B = \{b_1, b_2, \ldots, b_k\} \) we let \( A \approx B \) if \( a_i \leq b_i \) for all \( i \). Equivalently, \( A \approx B \) if and only if for all \( x \in \mathbb{R} \) we have \( |A \cap (-\infty, x]| \geq |B \cap (-\infty, x]| \). The following simple lemma will be useful later.

**Lemma 6.** If \( A_1, B_1 \) are \( s \)-sets with \( A_1 \approx B_1 \), \( A_2, B_2 \) are \((r-s)\)-sets with \( A_2 \approx B_2 \) and \( A_1 \cap A_2 = B_1 \cap B_2 = \emptyset \) then \( A \approx B \) where \( A = A_1 \cup A_2 \) and \( B = B_1 \cup B_2 \).

**Proof.** For all \( x \in \mathbb{R} \) we have

\[
|A \cap (-\infty, x]| = |A_1 \cap (-\infty, x]| + |A_2 \cap (-\infty, x]| 
\geq |B_1 \cap (-\infty, x]| + |B_2 \cap (-\infty, x]| = |B \cap (-\infty, x]|.
\]

\( \square \)

### 3. Shifted Hypergraphs

Since threshold graphs appear as an answer to many extremal questions in graphs, the concept of a “threshold hypergraph” should be useful when answering similar questions in hypergraphs. While there are many equivalent definitions of threshold graphs (see \( \S \)), in \( \S \) Reiterman, Rödl, Šiňajová, and Tůma show that the extensions of three of the equivalent definitions of threshold graphs are not equivalent for \( r \)-graphs with \( r > 2 \). The version that
will be useful to us is the notion of *shifted hypergraphs*, introduced in [3]. We will show that \( s \)-independent sets in \( r \)-graphs are maximized by shifted hypergraphs and use this fact restate the problem.

**Definition.** Given a set \( A \subset [n] \) and \( i, j \in [n] \) such that \( A \cap \{i, j\} = \{i\} \) define \( A_{i \to j} = (A \setminus \{i\}) \cup \{j\} \).

**Definition.** Consider a hypergraph \( \mathcal{H} \) with vertex set \( [n] \) and edge set \( \mathcal{E} \). For \( 0 \leq j < i \leq n-1 \) define the \((i, j)\)-shift \( S_{i \to j} \) as follows:

- for each \( E \in \mathcal{E} \),
  \[
  S_{i \to j}(E) = \begin{cases} 
  E_{i \to j} & \text{if } E \cap \{i, j\} = \{i\}, \\
  E & \text{otherwise}
  \end{cases}
  \]
- let \( S_{i \to j}(\mathcal{E}) = \{S_{i \to j}(E) : E \in \mathcal{E}\} \cup \{E : E, S_{i \to j}(E) \in \mathcal{E}\} \).

For a hypergraph \( \mathcal{H} \) on vertex set \( [n] \), we will write \( \mathcal{H}_{i \to j} \) to mean the hypergraph on vertex set \( [n] \) with edge set \( S_{i \to j}(\mathcal{E}(\mathcal{H})) \).

Thus, \( \mathcal{H}_{i \to j} \) is a hypergraph with the same number of edges as \( \mathcal{H} \) with the same sizes, but where we have replaced \( i \) with \( j \) whenever possible.

**Definition.** A hypergraph \( \mathcal{H} = ([n], \mathcal{E}) \) is *shifted* if and only if \( \mathcal{H}_{i \to j} = \mathcal{H} \) for all \( 0 \leq j < i \leq n-1 \).

We will extend the definition of \( \mathcal{H}_{i \to j} \) slightly and set \( \mathcal{H}_{i \to i} = \mathcal{H} \) for all \( i \in [n] \). In the next definition we extend again to apply a number of shifts at once.

**Definition.** Given an \( r \)-graph \( \mathcal{H} \) and \( k \)-sets \( A \preceq B \) with \( A = \{a_1, a_2, \ldots, a_k\} \), \( B = \{b_1, b_2, \ldots, b_k\} \) we define

\[
\mathcal{H}_{B \to A} = (\cdots((\mathcal{H}_{b_1 \to a_1})_{b_2 \to a_2})\cdots)_{b_k \to a_k}.
\]

We will use this definition in Section 5. In particular, we will use the fact that if we apply a shift from all the vertices in one edge to another \( r \)-set of vertices, \( A \), then \( A \) will be in the edge set of the shifted graph. We prove this in the next lemma.

**Lemma 7.** If \( \mathcal{H} \) is an \( r \)-graph on \([n]\) and \( A \preceq B \) are \( r \)-sets with \( B \in \mathcal{H} \) then \( A \in \mathcal{H}_{B \to A} \).

**Proof.** We’ll prove it by (reverse) induction on the parameter

\[
\ell = \max\{j : a_i = b_i \text{ for all } i \leq j\}.
\]

If \( \ell = r \) then \( A = B \) and there is nothing to prove. If \( \ell = r - 1 \) then \( A = B_{b_r \to a_r} \) and \( \mathcal{H}_{B \to A} = \mathcal{H}_{b_r \to a_r} \). It is clear from the definition of shifting that \( A \in \mathcal{H}_{B \to A} \). Suppose then that \( \ell < r - 1 \). Note that \( a_{\ell+1} \neq b_{\ell+1} \). Consider \( B' = B \triangle \{a_{\ell+1}, b_{\ell+1}\} = B_{b_{\ell+1} \to a_{\ell+1}} \).

We have \( A \preceq B' \preceq B \). Since all earlier compressions have no effect we have \( \mathcal{H}_{B \to A} = \mathcal{H}_{b_{\ell+1} \to a_{\ell+1}} \). By the definition of shifting we know that \( B' \in \mathcal{H}_{b_{\ell+1} \to a_{\ell+1}} \) since \( B \in \mathcal{H} \). This implies by induction that \( A \in (\mathcal{H}_{b_{\ell+1} \to a_{\ell+1}})_{B' \to A} = \mathcal{H}_{B \to A} \), as required. \( \square \)

3.1. **Shifted Hypergraphs Maximize \( s \)-independent Sets.** In this section we will show that for any \( r, s, n, \) and \( e \) we can find a \( r \)-graph maximizing the number of \( s \)-independent sets in \( \mathcal{H}_s(n, e) \) among the shifted hypergraphs. In the next proof we will construct an injection from the set of \( s \)-independent sets in some hypergraph \( \mathcal{H} \) to the set of \( s \)-independent sets in the shift \( \mathcal{H}_{i \to j} \). Note that in the next lemma we need not assume that the hypergraph is uniform.
Lemma 8. Let $\mathcal{H}$ be a hypergraph with vertex set $[n]$ and let $0 \leq j < i < n$. Then for all $s$,

\[ i_s(\mathcal{H}_{i \rightarrow j}) \geq i_s(\mathcal{H}). \]

Proof. We will define an injection from $\mathcal{I}_s(\mathcal{H}) \setminus \mathcal{I}_s(\mathcal{H}_{i \rightarrow j})$ to $\mathcal{I}_s(\mathcal{H}_{i \rightarrow j}) \setminus \mathcal{I}_s(\mathcal{H})$. Let $I$ be an independent set in $\mathcal{I}_s(\mathcal{H}) \setminus \mathcal{I}_s(\mathcal{H}_{i \rightarrow j})$. If $j \notin I$ we have $|I \cap S_{i \rightarrow j}(E)| \leq |I \cap E|$ for all $E \in \mathcal{E}$ and so $j \in I$. Similarly, $i \notin I$, because if $I$ is $s$-independent in $\mathcal{H}$ and $i, j \in I$ then $I$ is $s$-independent in $\mathcal{H}_{i \rightarrow j}$. Define $f : \mathcal{I}_s(\mathcal{H}) \setminus \mathcal{I}_s(\mathcal{H}_{i \rightarrow j}) \to \mathcal{I}_s(\mathcal{H}_{i \rightarrow j}) \setminus \mathcal{I}_s(\mathcal{H})$ by $f(I) = I_{j \rightarrow i}$. This is clearly an injection so we need only show that $I_{j \rightarrow i} \in \mathcal{I}_s(\mathcal{H}_{i \rightarrow j}) \setminus \mathcal{I}_s(\mathcal{H})$. Let $F \in \mathcal{E}(\mathcal{H}_{i \rightarrow j})$ and consider $|I_{j \rightarrow i} \cap F|$.

Recall $\mathcal{E}(\mathcal{H}_{i \rightarrow j}) = \{S_{i \rightarrow j}(E) : E \in \mathcal{E}(\mathcal{H})\} \cup \{E : E, S_{i \rightarrow j}(E) \in \mathcal{E}(\mathcal{H})\}$. Suppose $F \in \{S_{i \rightarrow j}(E) : E \in \mathcal{E}(\mathcal{H})\}$. Then either

- $F = E$ for some $E \in \mathcal{E}(\mathcal{H})$ because $E \cap \{i, j\} \neq \{i\}$ and so $S_{i \rightarrow j}(E) = E$ or
- $F = E_{i \rightarrow j}$ for some $E \in \mathcal{E}(\mathcal{H})$

Suppose $F \in \{E : E, S_{i \rightarrow j}(E) \in \mathcal{E}(\mathcal{H})\}$. It’s possible that $E$ and $S_{i \rightarrow j}(E)$ are in $\mathcal{E}(\mathcal{H})$ for two reasons:

- $S_{i \rightarrow j}(E) = E$ because $E \cap \{i, j\} \neq \{i\}$ (which is the same as the first case above) or
- $S_{i \rightarrow j}(E) = E_{i \rightarrow j}$ but $E_{i \rightarrow j} \in \mathcal{E}(\mathcal{H})$

So the proof will be in three cases.

(1) Suppose that $F = E$ for some $E \in \mathcal{E}(\mathcal{H})$ such that $E \cap \{i, j\} \neq \{i\}$. If $E \cap \{i, j\} = \emptyset$ then

\[ |I_{j \rightarrow i} \cap F| = |I_{j \rightarrow i} \cap E| = |I \cap E| < s. \]

If $E \cap \{i, j\} = \{j\}$ then

\[ |I_{j \rightarrow i} \cap F| = |I_{j \rightarrow i} \cap E| < |I \cap E| < s. \]

If $E \cap \{i, j\} = \{i, j\}$ then

\[ |I_{j \rightarrow i} \cap F| = |I_{j \rightarrow i} \cap E| = |I \cap E| < s. \]

(2) Suppose that $F = E_{i \rightarrow j}$ for some $E \in \mathcal{E}(\mathcal{H})$. Then

\[ |F \cap I_{j \rightarrow i}| = |E_{i \rightarrow j} \cap I_{j \rightarrow i}| = |E \cap I| < s. \]

(3) Suppose that $F = E$ for some $E \in \mathcal{E}(\mathcal{H})$ such that $E \cap \{i, j\} = \{i\}$ and $E_{i \rightarrow j} \in \mathcal{E}(\mathcal{H})$. Then

\[ |F \cap I_{j \rightarrow i}| = |E \cap I_{j \rightarrow i}| = |E_{i \rightarrow j} \cap I| < s. \]

Therefore $I_{j \rightarrow i} \in \mathcal{I}_s(\mathcal{H}_{i \rightarrow j})$. It remains to show that $I_{j \rightarrow i} \notin \mathcal{I}_s(\mathcal{H})$. Since $I \notin \mathcal{I}_s(\mathcal{H}_{i \rightarrow j})$ there exists $E \in \mathcal{E}(\mathcal{H}_{i \rightarrow j})$ such that $|I \cap E| \geq s$. It must be the case that $E = F_{i \rightarrow j}$ for some $F \in \mathcal{H}$ and $E \neq F$. Then

\[ s \leq |I \cap E| = |I_{j \rightarrow i} \cap E_{j \rightarrow i}| = |I_{j \rightarrow i} \cap F|. \]

Thus, $I_{j \rightarrow i} \notin \mathcal{I}_s(\mathcal{H})$. So, $|\mathcal{I}_s(\mathcal{H}) \setminus \mathcal{I}_s(\mathcal{H}_{i \rightarrow j})| \leq |\mathcal{I}_s(\mathcal{H}_{i \rightarrow j}) \setminus \mathcal{I}_s(\mathcal{H})|$. Therefore,

\[ |\mathcal{I}_s(\mathcal{H})| \leq |\mathcal{I}_s(\mathcal{H}_{i \rightarrow j})|. \]

\[ \square \]

Corollary 9. A hypergraph maximizing the number of $s$-independent sets among all hypergraphs with $n$ vertices and $e$ edges can be found among the shifted hypergraphs.
Figure 1. A visualization for a downset for a hypergraph with 7 vertices.
The hypergraph could have edge set \{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 2, 3\}\} or
\{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 2, 3\}, \{1, 2, 3\}\}.

Proof. Let \(t(H) = \sum_{E \in E(H)} \sum_{i \in E} i\). Pick \(H\) with the maximal number of \(s\)-independent sets and
\(t(H)\) minimal. Let \(0 \leq j < i \leq n\). Note \(H_{i \rightarrow j}\) has the same number of vertices and edges as
\(H\) and \(i_s(H_{i \rightarrow j}) \geq i_s(H)\) by Lemma 8. Thus, we must have \(H_{i \rightarrow j} = H\), else \(t(H_{i \rightarrow j}) < t(H)\)
contradicting the definition of \(H\). So \(H\) is a shifted hypergraph maximizing the number of
\(s\)-independent sets. \(\square\)

For the remainder of the paper we will focus on shifted hypergraphs.

4. Formal Statement of Main Result

Theorems 2 and 3 answer the question of which 3-graphs have the most 3-independent
sets and 1-independent sets, respectively. Our main result answers the question of which
3-graphs have the most 2-independent sets. We need some preliminary definitions before we
state the theorem.

As shown in Section 3, we need only consider shifted hypergraphs. It will turn out that
the feature of a shifted 3-graph \(H\) that determines \(i_2(H)\) is the collection of its edges that
contain 0. We make the following definition so that we can state our main result, but we
discuss the topic more extensively in Section 5.

Definition. Given a shifted 3-graph \(H\) the downset of \(H\) is the set
\[D(H) = \{(i, j) : \{0, i, j\} \in H\} \]
This is indeed a downset in the poset
\[B_n = \{(i, j) : 1 \leq i < j \leq n - 1\} \subseteq \{1, 2, \ldots, n - 1\}^2\]
with the product order.

Associating hypergraphs to downsets is a many to one relationship. A hypergraph \(H\) has
exactly one downset, but given a downset \(D\), there are often many (shifted) hypergraphs
that have downset \(D\). An example of how we visualize the downset is shown in Figure 1. A
cell \((i, j)\) is shaded provided that \(\{0, i, j\} \in H\). The downset of a hypergraph differs from
the lower shadow \(\partial_2(H)\) introduced in Section 1.1 in that the edges in \(\partial_2(H)\) that contain 0
are not shown in the downset—they are implied.
In Section 2 we introduced \((2, 3, 1)\)-lex 3-graphs. The maximizers of 2-independent sets in \(H_3(n, e)\) are generally \((2, 3, 1)\)-lex graphs. We describe the 3-graphs that are maximizers by their downsets in the following definition.

**Definition.** We say that a shifted 3-graph \(H\) is \((2, 3, 1)\)-lex style if its downset \(D = D(H)\) satisfies

- \(D\) is an initial segment in lex order, or
- \(D\) is a downset in \(B_n\) that is an initial segment in lex order missing one edge.

The possible downsets of \((2, 3, 1)\)-lex style 3-graphs are shown in Figure 2.

**Remark.** All \((2, 3, 1)\)-lex graphs are \((2, 3, 1)\)-lex style as a consequence of having a downset that is an initial segment in lex order have the property that we can arrange the edges not in the base layer so that they form an initial segment in \((2, 3, 1)\)-lex order. Notice that if \(D\) is a downset in \(B_n\) that is an initial segment in lex order missing one edge then that edge must correspond to the top cell in the second to last column. This is shown in the right downset in Figure 2.

Theorem 10 says, roughly, that hypergraphs that have downsets that are \((2, 3, 1)\)-lex style maximize 2-independent sets. In the following theorem we describe the non-\((2, 3, 1)\)-lex style hypergraphs that maximize 2-independent sets by their lower shadow graph.

**Theorem 10.** Let \(H\) be a 3-graph on \(n\) vertices with \(e\) edges where \(n \geq 32\). Then there exists a 3-graph \(G\) with \(n\) vertices and \(e\) edges such that

\[i_2(H) \leq i_2(G),\]

where \(G\) is either \((2, 3, 1)\)-lex style or \(G\) has \(\partial_2(G)\) coming from one of the following set of 5 persistent exceptions:

\[\mathcal{P}_n = \{(K_3 \vee E_1) \cup E_{n-5}, (K_2 \vee E_{n-5}) \cup E_2, (K_2 \vee E_{n-4}) \cup E_1, K_3 \vee E_{n-4}, K_4 \vee E_{n-5}\} .\]

When \(n < 32\) there are 16 possible downsets of hypergraphs that maximize 2-independent sets that are not \((2, 3, 1)\)-lex style or in \(\mathcal{P}_n\). These downsets are shown in Table 1.

To complete the picture, we state the equivalent theorem for the graph problem. We need a definition first.
Remark. For define the minimal edge of set in $L$. It is also true that $J$ is the unique $\preceq$-minimal ($\preceq$-set in $\mathcal{L}(n, e)$) such that $|E \cap I| \geq s$. By Lemma 6 we have $E_0(I) = I_s \cup J \preceq E_s \cup F$. Now by Lemma 7 since $E \in \mathcal{H}$, we have $E_0(I) \in \mathcal{H}_{E \to E_0(I)} = \mathcal{H}$, the last equality holding since $\mathcal{H}$ is shifted.

### Table 1. All exceptions to the maximizer being $(2, 3, 1)$-lex style when $n < 32$. 

| $n$ | $\partial_2(\mathcal{H})$ |
|-----|-----------------|
| 7   | $K_5$           |
| 8   | $K_5$, $K_6$, $K_7$ |
| 9   | $K_5$, $K_6$, $K_7$, $K_8$, $K_9 - K_{1,6}$ |
| 10  | $K_9$           |
| 11  | $K_{10}$, $K_{11} - K_{1,9}$ |
| 12  | $K_{11}$        |
| 14  | $K_{13}$, $K_{13} - e$ |
| 16  | $K_{15}$        |

**Definition.** A graph with $n$ vertices and $e$ edges is **lexish** if it is either the lex graph $\mathcal{L}(n, e)$ or else $\mathcal{L}(n, e) - f$ where $f$ is the edge $(i - 1)n$ where $i$ is such that $\{1, 2, \ldots, i + 1\}$ is the unique largest clique in $\mathcal{L}(n, e)$.

**Theorem 11.** Let $H$ be a graph on $n$ vertices with $t$ triangles where $n \geq 32$. Then there exists a graph $G$ on $n$ vertices such that $k_3(G) \geq t$ and $i(G) \geq i(H)$ and moreover $G$ is either a lex graph, a lexish graph, or $(K_2 \lor E_t) \cup E_{n-t-2}$.

### 5. Counting 2-independent Sets in Shifted 3-graphs

In this section we will develop a way to count 2-independent sets in shifted 3-graphs. This will result in a translation of the problem to an optimization problem that is easier to visualize.

**Definition.** Given $r \geq s \geq 2$, suppose $I \subseteq [n]$ is a set of size at least $s$. Let $I_s$ be the $s$-set consisting of the $s$ smallest elements of $I$, and let $J$ be the $r - s$ smallest elements of $[n] \setminus I_s$. Define the **minimal edge** of $I$ to be $E_0(I) = I_s \cup J$. Note that $E_0(I)$ is the unique $\preceq$-minimal set in $\binom{[n]}{r}$ that has $|E \cap I| \geq s$.

**Remark.** For $r = 3, s = 2$ and $I \subseteq [n]$ of size at least 2, the minimal edge of $I$ is $E_0(I) = \{a_1, a_2, b\}$ where $a_1$ and $a_2$ are the two smallest elements of $I$ and $b = \min\{i \in [n] : i \neq a_1, a_2\}$.

The purpose of defining the minimal edge of a set $I$ is that $I$ is $s$-independent in a shifted $r$-graph $\mathcal{H}$ exactly when $E_0(I)$ is not in $\mathcal{H}$.

**Lemma 12.** Let $\mathcal{H}$ be a shifted $r$-graph and consider a set $I \subseteq [n]$ with $|I| \geq s$. The set $I$ is $s$-independent in $\mathcal{H}$ if and only if $E_0(I) \notin \mathcal{E}(\mathcal{H})$.

**Proof.** Suppose that $I$ is an $s$-independent set. Then $E_0(I) \notin \mathcal{E}(\mathcal{H})$ since $|I \cap E_0(I)| \geq s$. Suppose now that $I$ is not an $s$-independent set. There exists an edge $E \in \mathcal{H}$ such that $|E \cap I| \geq s$. Let $E_s$ be the set of the $s$ smallest elements of $E \cap I$, and $F$ be $E \setminus E_s$. Note that, with the notation of the previous definition, $I_s \preceq E_s$, since $I_s$ is the unique $\preceq$-minimal $s$ set in $I$. It is also true that $J \preceq F$. To see this note first that $F \subseteq [n] \setminus I_s$; any $x \in F \cap I_s$ would have to be one of the $s$ smallest elements of $E \cap I$, hence in $E_s$, a contradiction. Now $J \preceq F$ since $J$ is the unique $\preceq$-minimal $(r - s)$-set in $[n] \setminus I_s$. By Lemma 9 we have $E_0(I) = I_s \cup J \preceq E_s \cup F$. Now by Lemma 7 since $E \in \mathcal{H}$, we have $E_0(I) \in \mathcal{H}_{E \to E_0(I)} = \mathcal{H}$, the last equality holding since $\mathcal{H}$ is shifted. $\square$
Corollary 13. Let $I \subset [n]$ with $|I| \geq s$. Suppose $\mathcal{H}' = \mathcal{H} + E$ and that $\mathcal{H}'$ and $\mathcal{H}$ are shifted $r$-graphs. Then $I \in \mathcal{I}_s(\mathcal{H}) \setminus \mathcal{I}_s(\mathcal{H}')$ if and only if $E_0(I) = E$.

Proof. By Lemma 12, $I \in \mathcal{I}_s(\mathcal{H})$ if and only if $E_0(I) \notin \mathcal{H}$ and $I \notin \mathcal{I}_s(\mathcal{H}')$ if and only if $E_0(I) \in \mathcal{H}'$. Thus, $I \in \mathcal{I}_s(\mathcal{H}) \setminus \mathcal{I}_s(\mathcal{H}')$ if and only if $E_0(I) = E = \mathcal{H}' \setminus \mathcal{H}$. \hfill $\square$

Now we are able to calculate the number of sets that are lost when an edge is added to a shifted hypergraph.

Lemma 14. Let $\mathcal{H}$ be a shifted 3-graph on vertex set $[n]$, let $E = \{i, j, k\}$ and suppose that $\mathcal{H}' = \mathcal{H} + E$ is also shifted. Then

$$i_2(\mathcal{H}') = i_2(\mathcal{H}) - c_{ijk}$$

where

$$c_{ijk} = \begin{cases} 2^{n-1} & \text{if } \{i, j, k\} = \{0, 1, 2\} \\ 2^{n-k} & \text{if } i = 0, j = 1 \text{ and } k \neq 2 \\ 2^{n-k-1} & \text{if } i = 0 \text{ and } j > 1 \\ 0 & \text{if } i \neq 0 \end{cases}$$

Remark. We will refer to $c_{ijk}$ as the cost of the edge $\{i, j, k\}$.

Proof. By Corollary 13, $I \in i_2(\mathcal{H}) \setminus i_2(\mathcal{H}')$ if and only if $E_0(I) = E$. Thus, to determine the cost of an edge $E$ we must count the number of sets $I$ such that $E_0(I) = E$.

If $E = \{0, 1, 2\}$ we are counting sets such that $E_0(I) = \{0, 1, 2\}$. These are exactly those sets having two smallest elements 0 and 1, 0 and 2, or 1 and 2. The number of sets with this property is $2^{n-2} + 2^{n-3} + 2^{n-3} = 2^{n-1}$. Thus, $c_{012} = 2^{n-1}$.

Suppose that $\{0, 1, k\}$ is added to a hypergraph where $k \neq 2$. Here we count sets $I$ such that $E_0(I) = \{0, 1, k\}$. These are the sets with smallest elements 0 and $k$ or 1 and $k$. The number of sets with this property is $2^{n-k-1} + 2^{n-k-1} = 2^{n-k}$. Thus $c_{01k} = 2^{n-k}$ for $k \neq 2$.

Suppose now $E = \{0, j, k\}$ with $j > 1$. Here, $E_0(I) = E$ if and only if the two smallest elements of $I$ are $j$ and $k$. There are $2^{n-k-1}$ of these meaning $c_{0jk} = 2^{n-k-1}$ when $j > 1$.

Finally, if $0 \notin E$ then it is not one of the edges of the form $E = \{a_1, a_2, b\}$ where $b = \min\{i \in [n] : i \neq a_1, a_2\}$. Thus, the cost of $\{i, j, k\}$ where $i \neq 0$ is 0. \hfill $\square$

Note that $\sum_{i<j<k} c_{ijk} = 2^n - (n + 1)$ meaning that $i_2(K_n^3) = n + 1$ where $K_n^3$ is the complete 3-graph on $n$ vertices. The 2-independent sets in $K_n^3$ are the empty set and all the singletons.

Let $\mathcal{H}$ be a 3-graph with vertex set $[n]$. We will visualize $\mathcal{H}$ by letting its edges be $1 \times 1 \times 1$ cubes labeled by the vertices in the edge in increasing order. Then we can think of these $1 \times 1 \times 1$ cubes inside an $(n - 2) \times (n - 2) \times (n - 2)$ cube labeled as in Figure 3. Figure 4 shows the edges of the complete hypergraph on 7 vertices inside a $5 \times 5 \times 5$ cube with the visible cubes labeled.

![Figure 3. The labeling of the cube. The shaded tetrahedron](image-url)
Lemma 14 says that, assuming the hypergraph is shifted, any edge that does not contain 0 is “free”, i.e., adding such an edge does not cost us any independent sets. More rigorously, if \( E = \{i, j, k\} \) with \( i \neq 0 \) we have \( i_2(\mathcal{H}) = i_2(\mathcal{H} + E) \). In the cube picture this means that any edge that is not in the bottom layer is free. For this reason, we focus on the downset of \( \mathcal{H} \). The downset of \( \mathcal{H} \) corresponds to edges in the base layer. Figure 5 shows the cube where we have suppressed the first dimension and show only the edges with non-zero costs.

Figure 5. Edges in base layer, \( B_7 \).

We will call each of the squares in \( B_n \) a cell and label it \((a, b)\) if the edge associated to that square is \( \{0, a, b\} \).

Recall that we are restricting ourselves to shifted hypergraphs as we can find a maximizer among the shifted hypergraphs. By definition a shifted hypergraph \( \mathcal{H} \) on vertex set \([n]\) satisfies the following condition: if \( \{a, b, c\} \in \mathcal{E}(\mathcal{H}) \) then \( \{i, j, k\} \in \mathcal{E}(\mathcal{H}) \) whenever \( i \leq a, j \leq b, \) and \( k \leq c \). In \( B_n \) this says that if \( \{0, b, c\} \in \mathcal{E}(\mathcal{H}) \) then \( \{0, j, k\} \in \mathcal{E}(\mathcal{H}) \) for all \( j \leq b \) and \( k \leq c \). That is, if we include a cell \((b, c)\) in our hypergraph, we must also include all cells that are to the left or below.

Each cell has an associated cost as given in Lemma 14 and an associated amount of space: the number of edges we could get for that cost, given that taking those edges results in a shifted hypergraph. The cost and space for cells in \( B_7 \) are given in Figure 6.
Figure 6. At left the cost of each cell in $B_7$, at right the space in each cell.

For $D$, a collection of cells, let $C(D)$ be the cost of those cells and $S(D)$ be the amount of room in those cells.

Remark. The space of a cell $(i, j)$ is $i$. We chose $[n] = \{0, 1, \ldots, n-1\}$ for this reason.

Our goal, finding a 3-graph on $n$ vertices having $e$ edges with the maximum number of 2-independent sets, can be rephrased as follows: find a downset $D$ in $B_n$ such that $C(D)$ is minimized subject to the condition that $S(D) \geq e$.

For the rest of the paper we will only be concerned with the shape of the downset in the bottom layer. Given a downset in $B_n$ that has enough space to accommodate the number of edges we need we can arrange the edges in higher layers to get a shifted 3-graph (often in several ways). When we discuss the number of 2-independent sets in $D \subseteq B_n$ we mean the number of 2-independent sets in any $H$ that has downset $D$.

Finally we introduce an order on downsets in $B_n$. For downsets $D$ and $D'$ we say that $D$ is lex-less than $D'$, or $D <_L D'$, if

$$\min_{\text{Lex}} D \Delta D' \in D.$$  

Here $\min_{\text{Lex}} D \Delta D'$ means the minimum cell in $D \Delta D'$ under the lex ordering on cells in $B_n$.

Definition. A downset $D$ in $B_n$ is an optimal downset if, for some $e$, $D$ minimizes $C(D)$ among all downsets with space at least $e$ and it is the earliest downset in lex order to do so.

6. Local Moves

In this section we show certain downsets in $B_n$ do not have as many 2-independent sets as the downset associated to a $(2, 3, 1)$-lex style 3-graph. Our strategy is to show that, given a downset $D$ that is not $(2, 3, 1)$-lex style, there exists a downset $D'$ such that $S(D') \geq S(D)$ and $C(D') \leq C(D)$ and $D' <_L D$. That is, we will show that some downsets that are not $(2, 3, 1)$-lex style are not optimal downsets. We’ll call the switch from $D$ to $D'$ a local move.

To talk about the local moves we first need the definition of corner.

Definition. For a downset $D$ the cell $(a, b)$ is a corner of $D$ if it is a maximal element of $D$.

The rest of this section is organized into three subsections, one for each of the three types of local moves we will perform. In Section 6.1 we will perform “one cell moves”, that is, local moves in which we remove only one cell from $D$. In Section 6.2 we will perform “column
moves” which are local moves in which we remove a column-like subset of the downset $D$. Finally in Section 6.3 we consider a local move that removes a large subset of cells.

6.1. One Cell Moves. First we will consider some local moves where we exchange one cell of a downset $D$ for two cells in $B_n \setminus D$. To do this, we first define the horizontal distance vector of a downset.

**Definition.** For a downset $D$, let $(o_1, o_2, \ldots, o_k)$ be the sequence of the first coordinates of the corners written in increasing order and let the horizontal distance vector be $H(D) = (o_2 - o_1, o_3 - o_2, \ldots, o_k - o_{k-1})$.

**Lemma 15.** Let $D$ be a downset with horizontal distance vector $(d_1, d_2, \ldots, d_k)$ where $d_i = o_{i+1} - o_i$, the difference between the first coordinates of consecutive corners. If $3 \leq d_i \leq \frac{a_i+3}{2}$ then $D$ is not optimal.

**Proof.** Let $(a, b)$ and $(c, d)$ be consecutive corners and suppose $3 \leq c - a \leq \frac{c+3}{2}$. Since the previous corner is $(a, b)$ we can remove cell $(c, d)$ and replace it with cells $(a+1, d+1)$ and $(a+2, d+1)$ and still have a downset. Let $D' = D - (c, d) + (a+1, d+1) + (a+2, d+1)$. The move from $D$ to $D'$ is illustrated in Figure 7.

![Figure 7. Move occurring in the proof of Lemma 15 for consecutive corners](image)

Note the room of cell $(c, d)$ is $c$ and the room in the replacement cells is collectively $2a + 3$. Since $c - a \leq \frac{c+3}{2}$ we have $c \leq 2a + 3$ and so there is at least much space in $D'$. Moreover, the cost of each of the replacement cells is half the cost of $(c, d)$ and so $C(D) = C(D')$. Finally $D' <_L D$. Therefore such a $D$ is not optimal. □

Lemma 15 says that in an optimal downset the horizontal distance between two corners is either small (less than 3) or is large (about half the larger amount of space). Let’s consider first when the horizontal distance between corners is small. When the horizontal distance between two corners is 1 we will say there is a *short stair* and when the horizontal distance between two consecutive corners is 2 we will say there is a *long stair*.

**Lemma 16.** Consider a downset $D$ with horizontal distance vector $H(D)$. If $H(D)$ has three consecutive 1’s, two consecutive 2’s, or an adjacent 1 and 2 then $D$ is not an optimal downset.
Proof. In Figure 8 we show the downsets resulting from the horizontal distance vectors having three consecutive 1’s, two consecutive 2’s, a 1 followed by 2, and a 2 followed by 1. In each case we can show that there is a downset with at least as much space and less cost that is earlier in lex order.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{From left to right, 3 short stairs, 2 long stairs, 1 long stair followed by a short stair, and 1 short stair followed by a long stair. The vertical drops may be of any height at least 1. We create downsets that are earlier in lex order by removing cells marked \times and replacing them with cells marked \checkmark.}
\end{figure}

Suppose that the horizontal distance vector has three consecutive 1’s. Name the corresponding corners \((i, a), (i+1, b), (i+2, c),\) and \((i+3, d)\) and note \(a > b > c > d\). Consider the downset \(D' = D - (i+3, d) + (i+1, b+1) + (i+2, c+1)\). Since \((i+1) + (i+2) = 2i + 3 > i + 3\) we have \(S(D') > S(D)\). Moreover, since \(a > b > c\), the cost of \((i+2, c)\) is at most half the cost of the cell \((i+3, d)\) and the cost of the cell \((i+1, b+1)\) is at most a fourth of the cost of the cell \((i+3, d)\). Therefore \(C(D') < C(D)\).

The proof for each of the other cases is similar. \(\square\)

From Lemma 16 we know that in an optimal downset the only possible “staircases” are 1 long stair, 1 short stair, or 2 short stairs. Note that these are exactly the types of staircases that appear at the end of a downset of a \((2, 3, 1)\)-lex style hypergraph. Our next lemma describes the types of vertical drops that can appear in these transitions.

**Lemma 17.** Suppose \(D\) is a downset with corners \((a, b), (a+1, c)\) and \((a+2, d)\). If \(b - c > 1\) then \(D\) is not an optimal downset. Similarly, if \(D\) is a downset with corners \((a, b)\) and \((a+2, c)\) where \(b - c > 1\) then \(D\) is not an optimal downset.

**Proof.** First consider a downset \(D\) with corners \((a, b), (a+1, c)\), and \((a+2, d)\). If \(b - c > 1\) then \(D' = D - (a+2, d) + (a+1, c+1) + (a+1, c+2)\) is a downset with \(C(D') < C(D)\), \(S(D') > S(D)\), and \(D' \prec L D\). For a downset \(D\) with corners \((a, b)\) and \((a+2, c)\) if \(b - c > 1\) then \(D' = D - (a+2, c) + (a+1, c+1) + (a+1, c+2)\) is a downset with \(C(D') < C(D)\), \(S(D') > S(D)\), and \(D' \prec L D\). \(\square\)

Lemmas 15, 16, and 17 allow us to say that optimal downsets have small groups of corners that are “far” apart. The small groups (or “transitions”) look like those in Figure 9 where the unlabeled drops are arbitrary.

We will say that a downset ends with stairs if the last entry of the horizontal distance vector is a 1 or a 2. Lemmas 15 and 16 say that if a downset ends with stairs, then it ends with 2 short stairs, 1 short stair, or 1 long stair. In the next lemma we address downsets that end with 2 short stairs or 1 long stair and are not \((2, 3, 1)\)-lex style.

**Lemma 18.** Suppose that \(D\) is not \((2, 3, 1)\)-lex style. If \(D\) ends with 2 short stairs or 1 long stair then \(D\) is not an optimal downset.
Proof. Suppose $D$ ends with 2 short stairs or 1 long stair, and there exists an earlier corner, as shown in the first two downsets in Figure 10. In each of these cases we can replace the last corner (marked with $\times$) with two earlier cells (marked with $\checkmark$) which cost strictly less and have at least as much space.

Suppose that $D$ ends with 2 short stairs or 1 long stair and there does not exist an earlier corner. If the top stair $(i, j)$ has $j = n - 1$ then $D$ is $(2, 3, 1)$-lex style. Otherwise we can replace the last corner (marked with $\times$) with two earlier cells (marked with $\checkmark$) which have at least as much space and cost at most as much. This results in a downset that is earlier in $(2, 3, 1)$-lex order.

6.2. Column Moves. In this section we apply moves in which a subset of the cells in the last column of the downset are traded for a row. These moves will be used on downsets that have that their last corner $(i, j)$ satisfies $j - i \geq \lfloor \log_2(i) \rfloor$. Since having a corner $(i, j)$ means the number of cells in column $j$ is $j - i$ this is ensuring that the last column of the downset has at least $\lfloor \log_2(i) \rfloor$ cells.

Lemma 19. Suppose the last corner of a downset $D$ is $(i, j)$ where $j - i \geq \lfloor \log_2(i) \rfloor$, and $i \geq 5$. If $(i, j)$ is the only corner and $j < n - 1$ then $D$ is not an optimal downset.

Proof. Let $t = \lfloor \log_2(i) \rfloor$ and define $L = \{(i, h) : j - t + 1 \leq h \leq j\}$ and $R = \{(h, j + 1) : 1 \leq h \leq i - 2\}$. Consider $D' = D - L + R$. Note that we add all possible cells in the row except for one (see Figure 11).
Computing the cost of $L$ and $R$ we have
\[
C(L) = 2^{n-j-1} + 2^{n-j} + \cdots + 2^{n-j+1+t-1} \\
= 2^{n-j-1}(2^t - 1) \\
= 2^{n-j-2}(2^{\lceil \log_2(i) \rceil + 1} - 2)
\]
and
\[
C(R) = 2^{n-j-2}(i - 3) + 2^{n-j-1} = 2^{n-j-2}(i - 1).
\]
Since $2^{\lceil \log_2(i) \rceil + 1} - 2 \geq i - 1$, we have $C(D) \geq C(D')$. Moreover,
\[
S(L) = i \cdot \lceil \log_2(i) \rceil
\]
and
\[
S(R) = \frac{(i - 2)(i - 1)}{2}.
\]
So $S(D') \geq S(D)$ when $i \geq 9$ or $i = 7$. Since $D' <_L D$ we are done if $i \geq 9$ or $i = 7$.

In the cases where $i = 5, 6$ or 8 we add all possible cells in the row. That is, we let $R = \{(h, j + 1) : 1 \leq h \leq i - 1\}$ and leave $L$ the same. The downsets $D' = D - L + R$ each have at most the cost of $D$, at least the space of $D$, and $D' <_L D$. \qed

**Lemma 20.** Suppose a downset $D$ does not end in stairs and has last corner $(i, j)$ where $j - i \geq \lceil \log_2(i) \rceil$ and $i \geq 6$. If there is an earlier corner then $D$ is not an optimal downset.

**Proof.** Since $D$ does not end in stairs, all previous corners $(k, m)$ have $k < \frac{i - 3}{2}$. Choose $(k, m)$ to be the second to last corner. Let $t = \lceil \log_2(i) \rceil$ and consider
\[
D' = D - \{(i, h) : j - t + 1 \leq h \leq j\} + \{(h, j + 1) : k + 1 \leq h \leq i - 1\}.
\]
That is, we consider the downset $D'$ in which we remove $t$ cells from the last column and replace them with the available cells at height $j + 1$. This move is shown in Figure 12.
Figure 12. Column move in the proof of Lemma 20

The cost of the column is
\[ 2^{n-j-1} + \ldots + 2^{n-j-1+i-1} = 2^{n-j-1}(2^i - 1) = 2^{n-j-2}(2^\lceil \log_2(i) \rceil + 1 - 2) > 2^{n-j-2}(i - 2). \]

Note there are at most \( i - 2 \) cells in the row (since there is a previous corner) and the cost of each cell is \( 2^{n-j-2} \). Thus, the cost of the row is strictly less than the cost of the column.

The space in the column is exactly \( i \lceil \log_2(i) \rceil \) and the space in the row is
\[
S(\{(h, j+1) : k+1 \leq h \leq i-1\}) = (k+1) + (k+2) + \cdots + (i-1)
\]
\[
= \frac{(i-1)i}{2} - \frac{k(k+1)}{2}
\]
\[
\geq \frac{(i-1)i}{2} - \frac{\frac{i-1}{2} \cdot \frac{i-2}{2}}{2}
\]
\[
= \frac{3}{8}i^2 + \frac{i}{4} - 1
\]
since \( k < \frac{i-3}{2} \). So \( S(D') \geq S(D) \) when \( i \geq 6 \). Therefore \( D \) is not an optimal downset. \( \square \)

**Corollary 21.** Suppose that a downset \( D \) does not end in stairs, is not \((2, 3, 1)\)-lex style, and has last corner \((i, j)\) where \( j - i \geq \lceil \log_2(i) \rceil \) and \( i \geq 5 \). Then \( D \) is not an optimal downset.

**Proof.** Consider such a downset \( D \). If there is no previous corner then \( j < n - 1 \) since \( D \) is not \((2, 3, 1)\)-lex style. Thus, \( D \) is not optimal by Lemma 19. If \( D \) has a previous corner \((i', j')\) then \( i' < \frac{i-3}{2} \) by Lemma 15. Then \( i \geq 6 \), else such a previous corner can not exist. By Lemma 20 \( D \) is not optimal. \( \square \)

Corollary 21 deals with downsets that do not end in stairs and have that the last column is tall. In the next lemmas, we will deal with downsets that end with stairs and the column of the top stair is tall. By Lemmas 16 and 18 we only need to consider downsets that end in one short stair.

**Lemma 22.** Suppose that the last corner of a downset \( D \) is \((i', j')\) and the first corner is \((i, j)\) with \( i = i' - 1 \). If \( j - i \geq \lceil \log_2(i) \rceil + 1 \) and \( i \geq 6 \) then \( D \) is not an optimal downset.
Proof. Let \( t = \lfloor \log_2(i) \rfloor \). For \( h \in \{j - t + 1, \ldots, j\} \) let \( \ell(h) \) be the greatest integer such that \( (\ell(h), h) \in D \). Note \( \ell(h) \in \{i, i'\} \). Let \( L = \{(\ell(h), h) : j - t + 1 \leq h \leq j\} \). These are cells in \( B_n \) since \( j - i \geq \lfloor \log_2(i) \rfloor + 1 \) and \( i' \leq i + 1 \). Let \( R = \{(m, j + 1) : 1 \leq m \leq i - 2\} \). Consider \( D' = D - L + R \) as shown in Figure 13.

Since the cost of a cell (with the exception of those in the first column) only depends on the height of the cell, the cost argument is exactly the same as that of Lemma 19. Moreover,

\[
S(L) \leq i + (i + 1)(\lfloor \log_2(i) \rfloor - 1) = i \cdot \lfloor \log_2(i) \rfloor + \lfloor \log_2(i) \rfloor - 1
\]

and

\[
S(R) = \frac{(i - 2)(i - 1)}{2}.
\]

Thus, \( S(D') \geq S(D) \) when \( i \geq 10 \) or \( i = 7 \).

In the cases where \( i = 6, 8 \) or \( 9 \) we add all possible cells in the row. That is, we let \( R = \{(h, j + 1) : 1 \leq h \leq i - 1\} \) and leave \( L \) the same. The downsets \( D' = D - L + R \) each have at most the cost and at least the space of \( D \) and are earlier in lex order. Therefore, for \( i \geq 6 \), such an \( D \) is not an optimal downset. \[Q.E.D.\]

In the next lemma consider the case where a downset ends with one short stair, there is an earlier corner, and the column of the top stair is tall.

**Lemma 23.** Suppose that the last corner of a downset is \((i', j')\), the second to last corner is \((i, j)\) where \( i = i' - 1 \) and there is an earlier corner with space less than \( \frac{i - 3}{2} \). If \( i \geq 6 \) and \( j - i \geq \lfloor \log_2(i) \rfloor + 1 \) then \( D \) is not an optimal downset.

Proof. As in the proof of Lemma 22 let \( t = \lfloor \log_2(i) \rfloor \), for each \( h \in \{j - t + 1, \ldots, j\} \) let \( \ell(h) \) be the greatest integer such that \( (\ell(h), h) \in D \), and let \( L = \{(\ell(h), h) : j - t + 1 \leq h \leq j\} \). Let \( R = \{(h, j + 1) : 1 \leq h \leq i - 1\} \cap \left(B_n \setminus D\right) \). Consider \( D' = D - L + R \), shown in Figure 14.
Since the cost of a cell (with the exception of those in the first column) only depends on the height of the cell, the cost argument is exactly the same as that of Lemma 20. Moreover, 
\[ S(L) \leq (i + 1)(\lfloor \log_2(i) \rfloor - 1) + i = (i + 1)(\lfloor \log_2(i) \rfloor) - 1. \]

As in the proof of Lemma 20, 
\[ S(R) \geq \frac{3i^2}{8} + \frac{i}{4} - 1. \]

So \( S(R) > S(L) \) for \( i \geq 6 \) when \( i \neq 8 \). If \( i = 8 \) one can take \( \lfloor \log_2(i) \rfloor - 1 \) cells for \( L \) to show \( D \) is not optimal. Since \( C(D') \leq C(D) \) and \( S(D') \geq S(D) \) the downset \( D \) is not optimal. \( \square \)

**Corollary 24.** Suppose that a downset \( D \) is not \((2, 3, 1)\)-lex style, ends in one short stair, and the top stair \((i, j)\) has \( j - i \geq \lfloor \log_2(i) \rfloor + 1 \) with \( i \geq 6 \). Then \( D \) is not an optimal downset.

**Proof.** If \( D \) has no other corners then \( D \) is not optimal by Lemma 22. Now suppose \( D \) ends in one short stair and has a previous corner, call it \((k, m)\). Note \( k \leq i - 3 \), else \( D \) would end in two short stairs. If \( k \geq \frac{i - 3}{2} \) then \( 3 \leq i - k \leq \frac{i + 3}{2} \) and so \( D \) is not optimal by Lemma 15. Therefore, \( k < \frac{i - 3}{2} \). By Lemma 23 \( D \) is not an optimal downset. \( \square \)

### 6.3. Larger Moves

In this section we will consider moves that are very similar to those in the previous section. We will trade a number of cells from the right side of a downset for the cells in the next row up. The difference is that we allow the removed cells to come from multiple columns. The removed cells will be those that are largest in the lex order on cells. For two cells \((i, j)\) and \((m, k)\) we say \((i, j) \leq (m, k)\) in lex order if and only if \( i < m \) or \( i = m \) and \( j \leq k \).

**Lemma 25.** Suppose that \( D \) is a downset that does not end in stairs with last corner \((i, j)\) such that \( j - i < \lfloor \log_2(i) \rfloor \), \( i \geq 16 \) and \( j \leq n - 3 \). Then \( D \) is not an optimal downset.

**Proof.** We will prove that there exists a downset \( D' \) such that \( C(D') \leq C(D) \), \( S(D') \geq S(D) \) and that \( D' <_L D \). First we will consider the case where \( 2 \leq j - i \). Let \( T \) be the \( \lfloor \frac{i}{7} \rfloor \) greatest
cells of $D$ under lex order. Let $\ell$ be such that $i - \ell + 1$ is the least amount of space in any cell of $T$. That is, $T$ occupies $\ell$ columns.

Let $R$ be the cells in $B_n \setminus D$ at height $j + 1$ and $j + 2$ and with space at most $c = i - \ell$. Since $j \leq n - 3$, there are available cells at both height $j + 1$ and $j + 2$. So

$$R = \{(h, k) : 1 \leq h \leq c, j + 1 \leq k \leq j + 2\} \cap (B_n \setminus D).$$

The sets of cells $R$ and $T$ are shown in Figure 15. Let $D' = D - T + R$.

First we will compare $C(T)$ and $C(R)$. When $i \geq 18$ the size of $T$ is at least 9. So the average cost of a cell in $T$ is at least $2^{n-j}$ and

$$C(T) \geq 2^{n-j} \cdot \left\lfloor \frac{i}{2} \right\rfloor \geq 2^{n-j} \left( \frac{i}{2} - \frac{1}{2} \right) = (i - 1) \cdot 2^{n-j-1}.$$ 

The cost of $R$ is greatest if there are no previous corners and $c = i - 2$. This gives the following upper bound on $C(R)$:

$$C(R) \leq 3[(i - 3)2^{n-j-3} + 2^{n-j-2}] = 3(i - 1)2^{n-j-3}.$$ 

Since $C(T) \geq 4(i - 1)2^{n-j-3} \geq 3(i - 1)2^{n-j-3} \geq C(R)$ we have $C(D') \leq C(D)$ when $i \geq 18$. When $i = 16$ and $i = 17$ we verify by computer that a downset satisfying the constraints is not optimal.

Now we compare $S(T)$ and $S(R)$. Since $T$ must occupy at least 3 columns and $|T| = \left\lfloor \frac{i}{2} \right\rfloor$,

$$S(T) \leq i \left\lfloor \frac{i}{2} \right\rfloor - \left( \left\lfloor \frac{i}{2} \right\rfloor - (\lfloor \log i \rfloor - 1) \right) - \left( \left\lfloor \frac{i}{2} \right\rfloor - (\lfloor \log i \rfloor - 1 + \lfloor \log i \rfloor) \right)$$

$$= (i - 2) \left\lfloor \frac{i}{2} \right\rfloor + 3 \lfloor \log i \rfloor - 2.$$
Note $\ell$ is greatest when $j - i$ is least. When $j - i = 2$, if $\ell = \lceil \sqrt{i} \rceil$ then $T$ could have up to $(\lceil \sqrt{i} \rceil + 1)^2/2$ cells and
\[
\left( \lceil \sqrt{i} \rceil + 1 \right)^2 / 2 \geq \left( \sqrt{i} \right)^2 / 2 = \frac{i}{2} \geq \left\lfloor \frac{i}{2} \right\rfloor.
\]
So, $\ell \leq \lceil \sqrt{i} \rceil$ and $c = i - \ell \geq i - \lceil \sqrt{i} \rceil$. Let $a$ be the space in the previous corner at height $j + 1$ (letting $a = 0$ if there is no previous corner at height $j + 1$) and let $b$ be the space in the previous corner at height $j + 2$ (letting $b = 0$ if there is no previous corner at height $j + 2$). Allowing both previous corners to have space $\lceil \frac{i - 4}{2} \rceil$ gives a lower bound on $S(R)$:
\[
S(R) \geq [(a + 1) + (a + 2) + \cdots + c] + [(b + 1) + (b + 2) + \cdots + c]
\]
\[
= \frac{c(c + 1)}{2} - a(a + 1) + \frac{c(c + 1)}{2} - b(b + 1)
\]
\[
\geq \left( i - \lceil \sqrt{i} \rceil \right) \left( i - \lfloor \sqrt{i} \rfloor + 1 \right) - \left\lfloor \frac{i - 4}{2} \right\rfloor \left\lfloor \frac{i - 2}{2} \right\rfloor.
\]
So $S(R) \geq i \left\lfloor \frac{i}{2} \right\rfloor \geq S(T)$ when $i > 16$. When $i = 16$, $T$ uses exactly 3 columns and so our upper bound for $R$ can be improved and still $S(R) \geq S(T)$. Moreover, $D' \lessdot D$. Therefore, $D$ is not an optimal downset in the case where $j - i \geq 2$.

When $j - i = 1$ we let $T$ be the $\lfloor \frac{i}{2} \rfloor - 1$ greatest cells in lex ordering and keep $R$ the same. Via similar computations we get $S(R) \geq S(T)$, $C(T) \leq C(R)$, and $D' \lessdot D$ for $i \geq 16$. \qed

Lemma 25 dealt with downsets that did not end in stairs, but the last column was short. We now do a similar move when there is a short stair and the top stair’s column is short.

**Lemma 26.** Suppose that a downset $D$ ends in one short stair and is not $(2,3,1)$-lex style. If the last two corners $(i', j')$ and $(i, j)$ with $i = i' - 1$ satisfy $j - i < \lceil \log_2(i) \rceil + 1$, $j \leq n - 3$, and $i \geq 16$ then $D$ is not an optimal downset.

**Proof.** Again we will prove that there exists a downset $D'$ such that $C(D') \leq C(D)$, $S(D') \geq S(D)$ and that $D' \lessdot D$. Let $T$ be the $\lfloor \frac{i}{2} \rfloor$ greatest cells of $D$ under the lex order. Let $\ell$ be such that $i - \ell + 1$ is the least amount of space in any cell of $T$. So $T$ occupies $\ell + 1$ columns. The sets of cells $R$ and $T$ are shown in Figure 16. Let $D' = D - T + R$.

By an identical argument to Lemma 25 we get $C(D') \leq C(D)$. We also use a nearly identical argument to compare $S(T)$ and $S(R)$. This time we use that the maximum space in any cell of $T$ is $i + 1$ and $c \geq i - (\lceil \sqrt{i} \rceil - 1)$ and conclude that $S(R) \geq S(T)$. Since we also have $D' \lessdot D$, $D$ is not an optimal downset. \qed

In the previous lemmas, we moved $\lfloor \frac{i}{2} \rfloor$ cells to two rows in the case that two rows were available. In the next lemmas, we address if there is only one available row by moving $\lfloor \frac{i}{4} \rfloor$ cells to 1 row.

**Lemma 27.** Suppose that $D$ is a downset that does not end stairs with last corner $(i, j)$ such that $j - i < \lceil \log_2(i) \rceil$, $i \geq 23$ and $j = n - 2$. Then $D$ is not an optimal downset.

**Proof.** We will prove that there exists a downset $D'$ such that $C(D') \leq C(D)$, $S(D') \geq S(D)$ and that $D' \lessdot D$. Let $T$ be the $\lfloor \frac{i}{4} \rfloor$ greatest cells of $D$ under the lex ordering. Let $\ell$ be such that $i - \ell + 1$ is the least amount of space in any cell of $T$. That is, $T$ occupies $\ell$ columns.
Let $R$ be the cells in $B_n \setminus D$ at height $n - 1$ and with space at most $c = \min\{i - \ell, i - 4\}$. Letting $a$ be the space in the previous corner (and $a = 0$ if there is no previous corner), $R = \{(h, n - 1) : a + 1 \leq h \leq c\}$. The sets of cells $R$ and $T$ are shown in Figure 17. Let $D' = D - T + R$.

First we will compare $C(T)$ and $C(R)$. When the size of $T$ is at least 9, the average cost of a cell in $T$ is at least 4 and

$$C(T) \geq 4 \cdot \left\lfloor \frac{i}{4} \right\rfloor \geq 4 \left(\frac{i}{4} - \frac{3}{4}\right) = (i - 3).$$

Moreover, since $c \leq i - 4$,

$$C(R) \leq 1 \cdot (i - 5) + 2 = (i - 3).$$

When $5 \leq |T| \leq |R|$ we verify by computer that a downset satisfying the constraints is not optimal. Therefore, when $i \geq 20$, $C(T) \geq C(R)$ and so $C(D') \leq C(D)$. 
Now we compare \( S(T) \) and \( S(R) \). Since the space in any cell of \( T \) is at most \( i \) and \( |T| = \left\lfloor \frac{i}{4} \right\rfloor \),

\[
S(T) \leq \left\lfloor \frac{i}{4} \right\rfloor \cdot i \leq \frac{i^2}{4}.
\]

Note \( \ell \) is greatest when \( j - i \) is least. When \( j - i = 1 \), if \( \ell = \lceil \sqrt[4]{i^2} \rceil \) then \( T \) has at least \( \frac{\sqrt{i^2}}{2} \) cells. So, \( c = \min\{i - 4, i - \ell\} \geq i - \sqrt{\frac{i^2}{2}} - 1 \).

Using a similar argument to that in Lemma 25,

\[
S(R) \geq \frac{3i^2}{8} - \frac{i^{3/2}}{\sqrt{2}} + \frac{i}{2} + \frac{1}{2} \sqrt{\frac{i}{2}} - 1.
\]

So \( S(R) \geq \frac{3i^2}{8} - \frac{i^{3/2}}{\sqrt{2}} + \frac{i}{2} + \frac{1}{2} \sqrt{\frac{i}{2}} - 1 \geq \frac{i^2}{4} \geq S(T) \) when \( i \geq 23 \), we know that \( S(D') \geq S(D) \). Since \( D' <_L D \), such a \( D \) is not an optimal downset. \( \square \)

In the final lemma for this section we consider downsets similar to those of Lemma 27, but end in one short stair.

**Lemma 28.** Suppose that a downset \( D \) ends in one short stair. If the last two corners \((i', j')\) and \((i, j)\) with \( i = i' - 1 \) satisfy \( j - i < \lceil \log_2(i) \rceil + 1 \), \( i \geq 23 \), and \( j = n - 2 \) then \( D \) is not an optimal downset.

**Proof.** Using the same setup as in the proof of Lemma 27 let \( R \) be the cells in \( B_n \setminus D \) at height \( n - 1 \) and with space at most \( c = \min\{i - \ell, i - 4\} \). Allowing a previous corner to have space \( a \) (and setting \( a = 0 \) if there is no previous corner), \( R = \{(h, n - 1) : a + 1 \leq h \leq c\} \). By Lemma 18 we know \( a \leq \frac{i - 4}{2} \). The sets of cells \( R \) and \( T \) are shown in Figure 18. Let \( D' = D - T + R \).

![Figure 18](image-url) An example of a downset \( D \) with \( R \) and \( T \) as described in the proof of Lemma 28.
When $23 \leq i < 32$, we get that $c = i - 4$ and by a counting argument similar to the one in Lemma 27 we get

$$S(R) \geq \frac{(i-4)(i-3)}{2} - \frac{(i-2)^2}{2} \geq \left\lfloor \frac{i}{2} \right\rfloor (i+1) \geq S(T).$$

When $i \geq 32$ by a similar argument again we get $c \geq i - \sqrt{\frac{3}{2}}$ and find $S(R) \geq S(T)$ again.

If $i \geq 23$ then the average cost of a cell is at least 4 and by an identical argument to that of Lemma 27, $C(D') \leq C(D)$.

Finally, $D' \not\leq D$ and so such a $D$ is not an optimal downset.

\[\square\]

7. Narrow Downsets and Persistent Exceptions

Many of our lemmas thus far required that the last corner $(i, j)$ has $i \geq c$ for some small $c$. In this section we will deal with the “narrow” cases, that is, where $i < c$. The first lemma deals with the case where $D$ does not end in stairs and the second lemma when $D$ ends in stairs.

There are some optimal downsets that are not $(2, 3, 1)$-lex style which appear as optimal downsets for all $n$. We define $C_n = \{[2, 1], [n-5, n-6], [n-4, n-5], [n-3, n-4, n-5], [n-3, n-4, n-5, n-6]\}$. Let $H(C_n)$ be the hypergraphs generated by the partitions in $C_n$.

**Lemma 29.** Suppose that $D$ is a downset in $B_n$ for $n \geq 10$. Suppose $D$ does not end in stairs, $D$ is not $(2, 3, 1)$-lex style, $D \not\in C_n$, and the last corner of $D$ is $(i, j)$. If $i < 5$ then $D$ is not optimal.

**Proof.** Throughout this proof we use the fact that $j > i$ and that if $i < 5$ then there can be no previous corners by Lemma 15. If $i = 1$ then $D$ is $(2, 3, 1)$-lex style. If $i = 2$ and $j \geq n - 3$ then $D$ is $(2, 3, 1)$-lex style or $D \in C_n$. If $i = 2$ and $4 \leq j \leq n - 4$ then $D - \{(2, j), (2, j - 1)\} + \{(1, j + 1), (1, j + 2), (1, j + 3), (1, j + 4)\}$ shows $D$ is not optimal. If $i = 3$ then $D \in C_n$.

Suppose $i = 3$. If $j \geq n - 2$ then $D \in C_n$ or $D$ is $(2, 3, 1)$-lex style. If $5 \leq j \leq n - 3$ then $D - \{(3, j), (3, j - 1)\} + \{(1, j + 1), (1, j + 2), (2, j + 1), (2, j + 2)\}$ shows that $D$ is not optimal. If $j = 4$ then, recalling $n \geq 10$, we see that $D - \{(2, 4), (3, 4)\} + \{(1, k) : 5 \leq k \leq 9\}$ shows $D$ is not optimal.

Finally, suppose $i = 4$. If $j \geq n - 2$ then $D$ is $(2, 3, 1)$-style or $D \in C_n$. If $5 \leq j \leq n - 3$ then $D - \{(4, j), (4, j - 1)\} + \{(2, j + 1), (2, j + 2), (3, j + 1), (3, j + 2)\}$ shows $D$ is not optimal. If $j = 5$ then $D - \{(5, 4), (3, 5), (3, 4)\} + \{(m, n) : 1 \leq m \leq 2, 6 \leq n \leq 9\}$ shows $D$ is not optimal.

\[\square\]

**Lemma 30.** Suppose that $D$ is a downset that is not $(2, 3, 1)$-lex style, that $D$ ends in one short stair, and the second to last corner $(i, j)$ has $i \leq 6$. Then $D$ is not optimal.

**Proof.** Let $(i, j)$ be the second to last corner and $(i', j')$ be the last corner. By Lemma 15 any previous corner must have space less than $\frac{i-3}{2}$. Since $i \leq 5$ in all cases there are no previous corners. Since $D$ is not $(2, 3, 1)$-lex style we know $j < n - 1$. If $i = 1$ and $j = n - 2$ then $D$ is $(2, 3, 1)$-lex style. If $i = 1$ and $j < n - 2$ then there are at least two empty rows. Then $D - (i', j') + (i, j + 1) + (i, j + 2)$ has the same cost and space and is earlier in lex order. If $i = 2$ then $D - (i', j') + (1, j + 1) + (2, j + 1)$ has the same cost of space and costs less. If $3 \leq i \leq 5$ then $D - (i', j') + (1, j + 1) + (2, j + 1) + (3, j + 1)$ has at least as much space and at most the cost and is earlier in lex order.
8. Downset Extensions

In this section we will consider a downset $D$ in $B_n$ inside $B_{\ell}$ for $\ell > n$. We will show that if $D$ is not optimal in $B_n$ then $D$ is not optimal in $B_{\ell}$ either, and a similar lemma for when $D$ is optimal.

**Definition.** Given a downset $D$ in $B_n$, let the extension of $D$, denoted $\overline{D}$, be the downset in $B_{n+1}$ where $(i, j) \in \overline{D}$ if and only if $(i, j) \in D$.

**Lemma 31.** Suppose $D$ is not an optimal downset in $B_n$. Then the extension of $D$ is not optimal in $B_{n+1}$.

**Proof.** Suppose that $D$ is not an optimal downset in $B_n$ and let $\overline{D}$ be the extension of $D$. Then there exists a downset $D'$ in $B_n$ such that $S(D') \geq S(D)$, $C(D') \leq C(D)$, and $D'$ is earlier in lex order. We claim that $\overline{D}$ is not an optimal downset in $B_{n+1}$. Consider $\overline{D}$. Then

$$S(\overline{D}) = \sum_{(i,j) \in \overline{D}} i = \sum_{(i,j) \in D'} i = S(D') \geq S(D) = \sum_{(i,j) \in A} i \geq \sum_{(i,j) \in \overline{D}} i = S(\overline{D}).$$

Recall the cost of a cell $(i, j) \in D$ with $i \neq 1$ is $2^{n-j-1}$. The cost of the same cell $(i, j)$ in $\overline{D}$ is $2^{(n+1)-j-1} = 2^{2n-j-1}$. This works similarly when $i = 1$ and thus the cost of $D$ is half the cost of its extension. So,

$$C(\overline{D}) = 2C(D') \geq 2C(D) = C(\overline{D}).$$

Moreover, $\overline{D'} <_L \overline{D}$ since $D' <_L D$ and our definition for the lex ordering on downsets is independent of $n$. Therefore, if $D$ is not an optimal downset in $B_n$ then its extension is not an optimal downset in $B_{n+1}$. \qed

**Lemma 32.** Let $D$ be a downset in $B_n$ with first corner $(a,b)$ where $n-1-b \geq 4$ and last corner $(i,j)$ where $i \geq 6$. Then $D$ is not optimal.

**Proof.** Let $D$ be such a downset. We will construct a downset $D'$ that has at least as much space and costs at most as much. Let $T$ be the $\lfloor i/2 \rfloor$ greatest cells of $D$ under lex ordering. If $(c,d)$ is the top cell in column $c$, let $S(c) = \{(c,d+1), (c,d+2), (c,d+3), (c,d+4)\}$ and let

$$S = \bigcup_{c=1}^{\lfloor i/2 \rfloor} S(c).$$

Let $D' = D - T + S$. First we claim that $C(D') \leq C(D)$. If $(c,d)$ is the top cell in column $c$ there is a corresponding cell $(e,f)$ in $T$ such that $f < d$ and so, if $c \neq 1$, then $C(S(c)) = 2^{n-d-1} - 2^{n-d-5} \leq 2^{n-f-1} = C((e,f))$. This argument holds for each column of $S$ with a distinct cell of $T$. The cost of the first column is double, but there is at least one cell with height at most $b-1$ in $T$ that accounts for this.

Next we claim $S(D') \geq S(D)$. In the adding of 4 rows we get that the new space is

$$4 \cdot \frac{(\lfloor i/2 \rfloor)(\lfloor i/2 \rfloor + 1)}{2}.$$ 

The space in the removed cells is at most $i \cdot (\lfloor i/2 \rfloor)$ so $S(D') \geq S(D)$. Finally, $D' <_L D$. \qed

**Corollary 33.** For $n \geq 10$, if $D$ is optimal in $B_n$ and $D \neq [2,1]$ then $D$ is not optimal in $B_{n+4}$.  

\[26\]
Lemma 34. Suppose that \( D \) does not end in stairs and has last corner \( (i, j) \) with \( j - i < \lfloor \lg i \rfloor \), \( j = n - 2 \). Then \( D \) is not optimal for any \( n \geq 32 \).

Proof. By Lemma 27 we know such a \( D \) is not optimal when \( i \geq 23 \). Since \( j - i < \lfloor \lg i \rfloor \) and \( i < 23 \) then \( j - i \leq 4 \). Thus, \( i \leq 22 \) and \( j \leq 26 \). Since \( j = n - 2 \) we know \( n \leq 28 \). If \( D \) is not optimal in \( B_{28} \) then \( D \) is not optimal in \( B_n \) for any \( n \geq 28 \). If \( D \) is optimal in \( B_{28} \) then \( D \) is not optimal in \( B_n \) for \( n \geq 32 \). 

Lemma 35. Suppose that \( D \) does not end in stairs and has last corner \( (i, j) \) with \( j \leq n - 3 \) and \( j - i < \lfloor \lg i \rfloor \). Then \( D \) is not optimal for any \( n \geq 30 \) or \( D = [2, 1] \).

Proof. We know such a \( D \) is not optimal for \( i \geq 16 \). Thus, for any optimal \( i \leq 15 \) so \( j - i \leq 3 \) and \( j \leq 18 \). If there is no previous corner then \( D \) fits inside \( B_{19} \) so isn’t optimal for \( n \geq 23 \). If there is a previous corner then we can replace \( (i, j) \) with a cell at every level not in the first column so a previous corner \( (i', j') \) has to have \( j' < j + i/2 \leq 18 + (15/2) = 25.5 \). So \( D \) fits inside \( B_{26} \) and is not optimal for \( n \geq 30 \). 

Lemma 36. Suppose \( D \) is a downset that ends with one short stair with the top stair being \( (i, j) \). Additionally assume \( j - i < \lfloor \lg i \rfloor + 1 \), \( j = n - 2 \). If \( n \geq 32 \) then \( D \) is not optimal.

Proof. By Lemma 28 we know such a \( D \) is not optimal if \( i \geq 23 \). Suppose \( i \leq 22 \). Since \( j - i < \lfloor \lg i \rfloor + 1 \), \( j - i \leq 4 \). So \( j \leq 26 \) and since \( j = n - 2 \), \( n \leq 28 \). If \( D \) is not optimal in \( B_{28} \) then \( D \) is not optimal in \( B_n \) for any \( n \geq 28 \). If \( D \) is optimal in \( B_{28} \) then \( D \) is not optimal in \( B_n \) for \( n \geq 32 \).

Lemma 37. Suppose \( D \) is a downset that ends with one short stair with the top stair being \( (i, j) \). Additionally, assume \( j - i < \lfloor \lg i \rfloor + 1 \) and \( j \leq n - 3 \). If \( n \geq 30 \) then \( D \) is not optimal.

Proof. By Lemma 26 we know such a downset is not optimal if \( i \geq 16 \). Since \( j < \lfloor \lg i \rfloor + i + 1 \) and \( i \leq 15 \) we know \( j \leq 18 \). If there is no previous corner then \( D \) fits inside \( B_{19} \). Suppose there is a previous corner. Note that we can replace \( (i, j) \) with one cell at each height and still save on cost. Since \( i < 16 \) if we have a previous corner \( (i', j') \) with \( j' \geq j + 8 \) then there is a downset \( D \) with lower cost, at least as much space, and is earlier in lex order. Thus, \( j' < 18 + 8 = 26 \) and \( D \) fits inside \( B_{26} \). By Lemmas 31 and 33 we know \( D \) is not optimal if \( n \geq 30 \). 

10. Proof of Theorem

Proposition 38. Suppose \( \mathcal{H} \in \mathcal{H}(n, e) \) is not \((2, 3, 1)\)-lex style. Let \( D = D(\mathcal{H}) \) and suppose \( D \) ends in stairs. If \( \mathcal{H} \) is optimal then \( n < 32 \).

Proof. The cases for this proof are outlined in Figure 8. By Lemma 16, \( D \) ends in one short stair, two short stairs, or one long stair. By Lemma 18 no optimal \( D \) ends in two short stairs or one long stair.
Suppose that $D$ ends in one short stair and let $(i, j)$ be the second to last corner. If $i < 6$ then $D$ is not optimal by Lemma 30. Suppose that $i \geq 6$. If $j - i \geq \lceil \lg i \rceil + 1$ then $D$ is not optimal by Corollary 24.

So, suppose $j - i < \lceil \lg i \rceil + 1$. If $j = n - 1$ then $D$ is $(2, 3, 1)$-lex style. If $j = n - 2$ then $n < 32$ by Lemma 36. Finally, if $j \leq n - 3$ then $n < 30$ by Lemma 37. Therefore, if such an $\mathcal{H}$ is optimal then $n < 32$. □

**Proposition 39.** Suppose $\mathcal{H} \in \mathcal{H}(n, e)$ is not $(2, 3, 1)$-lex style. Let $D = D(\mathcal{H})$ and suppose $D$ does not end in stairs. If $\mathcal{H}$ is optimal then $D \in \mathcal{P}_n$ or $n < 32$.

**Proof.** Let $(i, j)$ be the last corner of $D$. Suppose that $j - i \geq \lceil \lg i \rceil$. When $i \geq 5$, Corollary 21 tells us that $D$ is not optimal. If $i < 5$ then, by Lemma 29, $D \in \mathcal{P}_n$, $D$ is not optimal, or $n < 10$.

Now suppose that $j - i < \lceil \lg i \rceil$. If $j = n - 1$ then $D$ must be $(2, 3, 1)$-lex style. If $j = n - 2$ then by Lemma 34, $n < 32$. Finally, if $j \leq n - 3$ then $n \leq 30$ or $D \in \mathcal{P}_n$ by Lemma 35. □

**Proof of Theorem 10.** When $n \geq 32$ we find the only optimal 3-graphs are $(2, 3, 1)$-lex style or are in $\mathcal{P}_n$ by Propositions 38 and 39. When $n < 32$ we find all hypergraphs that maximize 2-independent sets using a computer search which leads us to $(2, 3, 1)$-lex style graphs or those with shadow graphs shown in Table 1. □

11. Conclusion

We have found the maximum number of $s$-independent sets in $n$ vertex 3-uniform hypergraphs with $e$ edges for all possible $n, e$ and $s$. While the answer is straightforward for $s = 1$ and $s = 3$, the answer for $s = 2$ requires a generalization of lex and colex graphs to $\pi$-lex graphs. Sadly the result is not as straightforward as saying that the optimal hypergraphs are $(2, 3, 1)$-lex initial segments. Even the generalization to $(2, 3, 1)$-lex style doesn’t cover
all the cases. There are both transient and persistent exceptions that are not $(2, 3, 1)$-lex style.

It still seems to us possible that asymptotically we can give a good characterization of the $r$-graph on $n$ vertices having $e$-edges having the fewest $s$-independent sets. The following conjecture is a strengthened version of the main theorem (Theorem 5) of [2].

**Conjecture 40.** Fix $1 \leq s \leq r$ and $\eta > 0$. Let $H$ be a hypergraph on $n$ vertices with $e$ edges (where $e < (1 - \eta)\binom{n}{r}$) having the maximum number of $s$-independent sets. Let $P(e)$ be the initial segment of $\left[\binom{n}{r}\right]$ in the $(r-s+1, r-s+1, \ldots, r, 1, 2, \ldots, s)$-lex order. Then

$$i_s(H) \leq (1 + o(1))i_s(P(e)).$$

The case $r = 3, s = 2$ is a consequence of our main theorem. For all $r$ the cases $s = r$ and $s = 1$ are also proved. The case $s = r$ is a special case of Theorem 2. The case $s = 1$ is true because the argument of Lemma 4 applies equally well to the initial segments in $(r, 1, 2, \ldots, r-1)$-lex.

One open problem to consider, which is probably very hard, is the level sets problem. For instance, one could try to determine which 3-uniform hypergraph with $n$ vertices and $m$ edges maximizes the number of 2-independent sets of size $t$. Our result doesn’t answer this question. As an example, we know that for 12 vertices and 10 edges, the $(2, 3, 1)$-lex hypergraph maximizes the number of 2-independent sets in total, however, this graph does not maximize the 2-independent sets of size 2 (at the very least the colex graph does better).

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