Hawking–Page phase transitions in four-dimensional Einstein–Gauss–Bonnet gravity

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The Hawking–Page (HP) phase transitions of the anti-de Sitter black holes in the extended phase space are studied in a novel four-dimensional Einstein–Gauss–Bonnet (4EGB) gravity, which is proposed by rescaling the Gauss–Bonnet (GB) coupling constant $\alpha \rightarrow \alpha/(d-4)$ in $d$ dimensions and redefining the four-dimensional gravity in the limit $d \rightarrow 4$. The GB term shows nontrivial contributions to both black hole mass and entropy simultaneously, and decreases the HP phase transition temperature $T_{HP}$. Moreover, the HP phase transitions can happen only within a range of pressure in the 4EGB gravity. For the charged black holes, $T_{HP}$ also decreases with the electric potential in the grand canonical ensemble. A general discussion of the HP phase transitions in the Einstein, GB, and 4EGB gravities is also presented.

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I. INTRODUCTION

Black hole physics is one of the central topics in modern physics, as it exhibits that a black hole should be considered as a complicated system with temperature and entropy [1]. The establishment of the four laws of black hole thermodynamics clearly indicates the profound relationship of thermodynamics, classical gravity, and quantum mechanics, and will help to our final understanding of quantum gravity [2].

In the last decade, black hole thermodynamics in the extended phase space has attracted increasing attentions [3]. The basic motivation is to construct the effective black hole volume $V$ and pressure $p$ and to restore the $p-V$ term that is absent in black hole thermodynamics. This purpose can be realized in the anti-de Sitter (AdS) space with a negative cosmological constant $\Lambda$. Since the gravitational potential in the AdS space increases at large distances, it offers a positive pressure $p$. If $\Lambda$ is further allowed to run, $p$ will become a varying thermodynamic quantity, and the effective black hole volume $V$ can be defined as the conjugate variable of $p$. By this means, the missing $p-V$ term reappears in the first law of black hole thermodynamics, but in the form $V \, dp$, not the usual work term $-p \, dV$. Therefore, the black hole mass should not be identified as internal energy, but rather as enthalpy. In the framework of the extended phase space, an AdS black hole shows significant similarities to a non-ideal fluid, especially in their phase transitions and critical phenomena. These remarkable observations have induced a large number of relevant works. See Ref. [4] for the recent progresses and references.

The famous Hawking–Page (HP) phase transition possesses an important position among the black hole phase transitions [5]. It was first studied between the Schwarzschild–AdS black hole and the thermal AdS background, and then extended to the charged AdS [i.e., Reissner–Nordström–AdS (RN–AdS)] black hole [6]. Moreover, it was also interpreted as a confinement–deconfinement phase transition of gauge fields in the AdS/CFT duality [7]. In general, the partition function of the black hole–thermal AdS system is dominated by the thermal AdS phase or the black hole phase at low or high temperatures respectively. Therefore, above the phase transition temperature $T_{HP}$, the thermal AdS gas will collapse into a black hole via a first-order phase transition. The HP phase transition has been widely investigated in the literature, with emphasis in the extended phase space [8–17]. Currently, the research topics are mainly focused in various modified gravity theories.

One of the important modified gravity theories is the Gauss–Bonnet (GB) gravity, which is a special case of the most general Lovelock gravity, with the GB term $G$ being the leading-order correction to the Einstein–Hilbert action and including the Einstein gravity as the low energy and small curvature limit. Albeit $G$ is quadratic in curvature tensors, the equations of gravitational field are still of second-order and thus avoid ghosts automatically. However, the studies of the GB gravity are usually restricted to high-dimensional physics, as in a four-dimensional manifold, the GB term reduces to a topological invariant and its integral just corresponds to the Euler characteristic of the manifold. From this point of view, the GB term cannot influence space-time structure, global charges, and their conjugate potentials, so it has no dynamical effect and is thus always disregarded in four dimensions, unless coupled to other matter fields. The explorations of the GB gravity in the extended phase space can be found in Refs. [18–32].

Recently, a novel four-dimensional Einstein–Gauss–Bonnet (4EGB) gravity theory was proposed by Glavan and Lin and was applied to maximally symmetric space-time, spherically symmetric black hole, and cosmology [33]. Their basic idea was to rescale the GB coupling constant $\alpha \rightarrow \alpha/(d-4)$ in $d$-dimensional space-time and to redefine four-dimensional gravity in the limit $d \rightarrow 4$. With this rescaling, the GB term was shown to bypass the Lovelock theorem, to be free from the Ostrograd-
sky instability, and to contribute nontrivial effects to the gravitational dynamics even in four dimensions. This work rapidly aroused intensive research interests, such as black hole solutions [34–41], relativistic stars [42, 43], gravitational lensing [44, 45], stability [46–49], radiation and accretion [50, 51], cosmic censorship conjecture [52, 53], quasi-normal modes [54, 55], black hole shadows [56–58], thermodynamics and phase transitions [59–63], cosmological applications [64–67], and observational constraints [68–70]. Meanwhile, it also received severe criticisms [71–75] (e.g., the vacuum is not well-defined), and several variants of Ref. [33] have already been constructed to deal with these troubles [76–78]. Nevertheless, at present it is well worth investigating the various applications of this 4EGB gravity and exploring its relevant effects to the most extent.

The basic purpose of this paper is to study the HP phase transitions in the extended phase space in the 4EGB gravity. This paper is a successive research of our previous work [16], in which we investigated the HP phase transitions of four-dimensional AdS black holes in the Einstein and GB gravities, and found that the HP temperature decreases at large electric potentials and angular velocities and also decreases with the GB coupling constant. The main improvements in our present work are threefold. First, in the GB gravity, only the black hole entropy receives a shift from the GB term, but the black hole mass is unaltered. However, in the 4EGB gravity, both the black hole entropy and mass are affected by the GB term, as it influences black hole thermodynamics and dynamics simultaneously. Second, the correction to the black hole entropy is more complicated in the 4EGB gravity than the GB gravity. Third, in the Einstein and GB gravities, the minimal black hole temperature is always positive, but in the 4EGB gravity, it is allowed to reach zero. All these ingredients naturally lead to the evident similarities and also distinct dissimilarities among the Einstein, GB, and 4EGB gravities in the HP phase transitions, and we will systematically consider all these issues in this paper.

This paper is organized as follows. In Sect. II, we explain the 4EGB gravity and discuss the extended phase space and the HP phase transition in more detail. In Sects. III and IV, the HP phase transitions of the Schwarzschild–AdS and RN–AdS black holes are studied in the 4EGB gravity in order. We conclude in Sect. V. In this paper, we work in the natural system of units and set $c = G_N = \hbar = k_B = 1$, but the $d$-dimensional gravitational constant $G_d$ is kept without loss of generality.

### II. BLACK HOLE THERMODYNAMICS IN THE 4EGB GRAVITY

In this section, in the framework of the 4EGB gravity, we outline the thermodynamic properties of the RN–AdS black holes in the extended phase space and discuss the HP phase transition in more detail.

#### A. Novel 4EGB gravity

We start from the action of the gravitational field of $d$-dimensional GB gravity,

$$\int \frac{d^d x}{16 \pi G_d} (R + \alpha G),$$

where $\alpha$ is the GB coupling constant, and $G$ is the GB term,

$$G := R_{\mu \nu \lambda \rho} R^{\mu \nu \lambda \rho} - 4 R_{\mu \nu} R^{\mu \nu} + R^2,$$

where $R_{\mu \nu \lambda \rho}$ is the Riemann tensor, $R_{\mu \nu}$ is the Ricci tensor, and $R$ is the Ricci scalar. The equation of motion can be directly achieved from the variation with respect to the metric tensor,

$$\frac{R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R}{g_{\mu \nu}} + \frac{\alpha}{2} H_{\mu \nu} = 8 \pi G_d T_{\mu \nu},$$

where $T_{\mu \nu}$ is the energy–momentum tensor of matter field, and $H_{\mu \nu} = 2 R R_{\mu \nu} - 4 R_{\mu \lambda \nu \rho} R^{\lambda \rho} + 2 R_{\mu \lambda \nu \rho} R^{\lambda \nu \rho} - 4 R_{\mu \lambda} R^{\lambda \nu} - g_{\mu \nu} G / 2$. The trace of Eq. (2) is

$$\left(1 - \frac{d}{2}\right) R + \left(1 - \frac{d}{4}\right) \alpha G = 8 \pi G_d T.$$

Hence, we clearly see that the GB term has no effect on gravitational dynamics in four dimensions. However, if $\alpha$ is rescaled as

$$\alpha \rightarrow \frac{\alpha}{d - 4},$$

the vanishing factor in front of the GB term in Eq. (3) disappears, and a novel 4EGB gravity can be redefined in the limit $d \rightarrow 4 [33]$. This operation quite resembles the dimensional regularization in renormalization theories. In this way, the GB term will have nontrivial effect on gravitational dynamics even in four dimensions.

In the same way, the action of the gravitational field of the RN–AdS black hole in the 4EGB gravity reads [34]

$$\int d^d x \frac{\sqrt{-g}}{16 \pi G_d} \left[ R - F_{\mu \nu} F^{\mu \nu} + \frac{(d - 2)(d - 1)}{l^2} + \frac{\alpha}{d - 4} G \right],$$

where $F_{\mu \nu}$ is the electromagnetic tensor and $l$ is the AdS curvature radius. In four dimensions, the static and spherically symmetric metric solution is

$$ds^2 = - f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

where $f(r) = 1 - \frac{2M}{r} + \frac{\alpha}{d - 4} \frac{r^2}{l^2}$, and $M$ is the mass of the black hole.
with the solution as [34]

\[ f(r) = 1 + \frac{r^2}{2a} \left[ 1 \pm \sqrt{1 + 4\alpha \left( \frac{2M}{r^2} - \frac{Q^2}{r^4} - \frac{1}{2} \right) } \right], \tag{4} \]

where \( M \) and \( Q \) are the black hole mass and electric charge. Below, we only choose the minus sign in front of the square root. Thus, when \( \alpha \to 0 \), \( f(r) \) reduces to the RN–AdS black hole solution,

\[ f(r) \to 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{L^2}. \tag{5} \]

B. Black hole thermodynamics in the extended phase space

From Eq. (4), the event horizon radius \( r_+ \) of the RN–AdS black hole can be determined as the largest root of \( f(r) = 0 \). Then, the black hole mass can be extracted as

\[ M = \frac{3r_+^2 + 3Q^2 + 3\alpha + 8\pi p r_+^4}{6r_+} = M(r_+, p, Q, \alpha), \tag{6} \]

where \( p \) is the effective thermodynamic pressure in the extended phase space,

\[ p = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi l^2}. \]

In the limit of vanishing charge and pressure, from Eq. (6), we have \( r_+ = M + \sqrt{M^2 - \alpha} \), and this sets an upper bound of \( \alpha \) as \( \alpha < M^2 \). Moreover, as in extra-dimensional physics, \( \alpha \) is proportional to the inverse string tension with positive coefficient [79], we set the lower bound of \( \alpha \) to be 0 in this work for simplicity. For the discussion with a negative \( \alpha \), see Ref. [56].

Next, the Hawking temperature of the RN–AdS black hole is

\[ T = \frac{f'(r_+)}{4\pi} = \frac{r_+^2 - Q^2 - \alpha + 8\pi p r_+^4}{4\pi r_+(r_+^2 + 2\alpha)} = T(r_+, p, Q, \alpha). \tag{7} \]

By rewriting Eq. (7), we can establish the equation of state of the RN–AdS black hole in the extended phase space, like that of a non-ideal fluid,

\[ p = \frac{4\pi r_+(r_+^2 + 2\alpha)T - r_+^2 + Q^2 + \alpha}{8\pi r_+^4}. \tag{8} \]

Furthermore, the entropy of the RN–AdS black hole can be calculated as [80]

\[ S = \int_{r_+}^{r_{+0}} \frac{1}{T} \left( \frac{\partial M}{\partial r_+} \right)_{p, Q, \alpha} dr_+ = \pi r_+^2 + 4\pi \alpha \ln \frac{r_+}{L_0} = S(r_+, \alpha), \tag{9} \]

where \( L_0 \) is an integral constant to be determined later. Hence, the black hole entropy receives a logarithmic correction in the 4EGB gravity. If we solve \( r_+ = r_+(S, \alpha) \) from Eq. (9) and substitute it into Eq. (6), we can reexpress the RN–AdS black hole mass in terms of the thermodynamic variables as \( M = M(r_+(S, \alpha), p, Q, \alpha) \).

A straightforward differentiation of \( M \) yields the first law of black hole thermodynamics in the extended phase space,

\[ dM = TV dP + \Phi dQ + X \alpha, \tag{10} \]

where \( T, V, \Phi, \) and \( X \) are the Hawking temperature, thermodynamic volume, electric potential at event horizon, and conjugate variable of \( \alpha \) respectively,

\[ T = \left( \frac{\partial M}{\partial S} \right)_{p, Q, \alpha} = \left( \frac{\partial M}{\partial r_+} \right)_{p, Q, \alpha}, \tag{11} \]

\[ V = \left( \frac{\partial M}{\partial P} \right)_{S, p, Q, \alpha} = \left( \frac{\partial M}{\partial r_+} \right)_{r_+, p, Q, \alpha} = \frac{4\pi r_+^3}{3}, \tag{12} \]

\[ \Phi = \left( \frac{\partial M}{\partial Q} \right)_{S, p, \alpha} = \left( \frac{\partial M}{\partial r_+} \right)_{r_+, p, \alpha} = \frac{Q}{r_+}. \tag{13} \]

\[ X = \left( \frac{\partial M}{\partial \alpha} \right)_{S, p, Q} = \left( \frac{\partial M}{\partial r_+} \right)_{r_+, p, Q} = \left( \frac{\partial M}{\partial \alpha} \right)_{r_+, p, Q} = \frac{r_+^2 - Q^2 - \alpha + 8\pi p r_+^4}{8\pi r_+^4} \left[ \left( \frac{\partial \ln L_0}{\partial \ln r_+} \right) S - \ln \frac{r_+}{L_0} \right] + \frac{1}{2r_+}. \tag{14} \]

Substituting Eqs. (11)–(14) into the Smarr relation (i.e., the Gibbs–Duhem relation in traditional thermodynamics),

\[ M = 2TS - 2pV + \Phi Q + 2X \alpha, \]

in order to guarantee this integral form of the first law of black hole thermodynamics, we need

\[ \left( \frac{\partial \ln L_0}{\partial \ln r_+} \right) S = \frac{1}{2}. \]

This condition fixes the integral constant \( L_0 \) in Eq. (9) as \( L_0 = \sqrt{\alpha} \), so we finally arrive at the RN–AdS black hole entropy in the 4EGB gravity [61, 81],

\[ S = \pi r_+^2 + 4\pi \alpha \ln \frac{r_+}{L_0}. \tag{15} \]

This result can also be obtained via the Iyer–Wald formula [82].

From Eqs. (6) and (15), we find that, in the 4EGB gravity, the GB term influences both the black hole mass and entropy. This is more complicated than the case in the GB gravity, in which the GB term merely modifies the black hole entropy by a shift \( 4\alpha \). Consequently, the HP phase transitions in the 4EGB gravity will exhibit richer thermodynamic behaviors than those in the GB gravity.
C. HP phase transition

Because the $p-V$ term in Eq. (10) is $V dp$, the distinct character of black hole thermodynamics in the extended phase space is that the black hole mass should be identified as enthalpy. Therefore, the corresponding thermodynamic potential of interest should be the Gibbs free energy. Furthermore, since the thermal AdS background is neutral, it is impossible for a black hole with fixed charge $Q$ to undergo the HP phase transition due to the conservation of charge, so all the relevant discussions must be performed in a grand canonical ensemble with fixed electric potential $\Phi$ ($Q$ is allowed to vary). In the grand canonical ensemble, the Gibbs free energy $G$ of the RN–AdS black hole should be constructed as

$$G = M - TS - \Phi Q.$$

Substituting Eqs. (6), (7), (13), and (15) into the above equation, we have

$$G = \frac{3(1 - \Phi^2)r_+^2 + 3\alpha + 8\pi p r_+^4}{6r_+} - \frac{(1 - \Phi^2)r_+^2 - \alpha + 8\pi p r_+^4}{4r_+(r_+^2 + 2\alpha)} \left( r_+^2 + 4\alpha \ln \frac{r_+}{\alpha} \right)$$

$$= G(r_+, p, \Phi, \alpha).$$

Then, we can solve $r_+ = r_+(T, p, \Phi, \alpha)$ from Eq. (7), substitute the result into Eq. (16), and eventually reexpress the Gibbs free energy in its usual form,

$$G = G(T, p, \Phi, \alpha).$$

The analytical solution of $G(T, p, \Phi, \alpha)$ does exist in principle, but we omit it here for its unnecessary lengthy expression.

Due to the Hawking radiation, a stable large black hole (with positive heat capacity and large event horizon radius) can exchange energy and establish the equilibrium with the thermal AdS background. In the black hole–thermal AdS system, first, the Gibbs free energy of the thermal AdS background is zero, since the total number of thermal gas particles is not conserved; second, the Gibbs free energy of the RN–AdS black hole will be shown to decrease with temperature. Hence, below or above the HP temperature, the thermal AdS phase or the black hole phase with the global minimum of Gibbs free energy is thermodynamically preferred respectively, so the criterion of the HP phase transition is

$$G = 0,$$

and the HP temperature $T_{HP}$ can be fixed accordingly.

III. HP PHASE TRANSITIONS OF THE SCHWARZSCHILD–ADS BLACK HOLES

In this section, we discuss the HP phase transitions of the Schwarzschild–AdS black holes in the extended phase space. Our discussions consist of two steps: first, to determine the HP temperature $T_{HP}$ as a function of pressure $p$; second, to determine the global phase structure of the black hole–thermal AdS system (i.e., to figure out the dependence of Gibbs free energy $G$ on temperature $T$). For comparison, we study the relevant issues both in the Einstein and 4EGB gravities. In the Einstein gravity, the detailed results can be found in our previous work [16], and we only refer to them when needed. Below, we focus on the 4EGB gravity and explore its corrections to the Einstein gravity.

A. Einstein gravity

We start our discussions in the Einstein gravity with $\alpha = 0$. The intermediate variable in calculations in Ref. [16] is the black hole entropy $S$, but now is the event horizon radius $r_+$. These two calculational methods are equivalent when $\alpha = 0$. However, when $\alpha \neq 0$, the procedure in Ref. [16] will intrinsically not work, as we see from Eq. (15) that the dependence of $S$ on $r_+$ is not polynomial, while all the expressions in Eqs. (6), (7), and (16) are written in terms of $r_+$. Therefore, we still need to briefly repeat the calculations here.

In the Einstein gravity, from Eqs. (7) and (16), the Schwarzschild–AdS black hole temperature and Gibbs free energy reduce to

$$T = \frac{1 + 8\pi p r_+^2}{4\pi r_+},$$

$$G = \frac{3r_+ - 8\pi p r_+^3}{12}.$$

Hence, at the HP phase transition point, from the criterion in Eq. (17), we obtain $r_+ = \sqrt{3/(8\pi p)}$. Substituting it into Eq. (18), the HP temperature is

$$T_{HP} = \sqrt{\frac{8p}{3\pi}}.$$

Actually, the $T_{HP}$–$p$ curve is just the coexistence line in the phase diagram. More importantly, as $p$ has no terminal point in Eq. (20), the HP phase transition can happen at all pressures, without a critical point, so it is more like a solid–liquid phase transition, rather than a liquid–gas one.

Then, from Eq. (18), we can solve $r_+$ in terms of $T$ and $p$,

$$r_+(T, p) = \frac{1}{4\pi p} \left( \pi T \pm \sqrt{\pi^2 T^2 - 2\pi p} \right),$$

where $\pm$ correspond to large and small black holes with different event horizon radii respectively. Moreover, from Eq. (21), the Schwarzschild–AdS black hole must have a positive minimal temperature as

$$T_0 = \sqrt{\frac{2p}{\pi}}.$$
In fact, \((T_0, r_+(T_0)) = (\sqrt{2p/\pi}, \sqrt{1/(8p)})\) is exactly the meeting point of the \(r_+ - T\) curves of large and small black holes, as shown in Fig. 1. Because the black hole heat capacity satisfies \(C_p = T(\partial S/\partial T) = T(\partial S/\partial r_+)(\partial r_+/\partial T)\), we observe from Fig. 1 that the large Schwarzschild–AdS black hole with positive \(C_p\) is thermodynamically stable and can thus establish the equilibrium with the thermal AdS background, but a small one is on the contrary.

Next, substituting Eq. (21) into Eq. (19), we obtain the Gibbs free energies of large and small Schwarzschild–AdS black holes,

\[
G(T, p) = \frac{\sqrt{\pi T^2 - p} \pm T\sqrt{\pi^2 T^2 - 2\pi p}}{24\sqrt{2\pi p^2}} \times \left(4p - \pi T^2 + T\sqrt{\pi^2 T^2 - 2\pi p}\right).
\]

The \(G - T\) curves are shown in Fig. 2, and both curves decrease with temperature, meeting at \(T_0\) with a cusp. For the unstable small black hole, its \(G - T\) curve is concave and is always above the \(T\)-axis, without the HP phase transition. However, for the stable large black hole, its \(G - T\) curve is convex and crosses the \(T\)-axis at the HP temperature \(T_{HP}\). Below or above \(T_{HP}\), the thermal AdS phase with vanishing \(G\) or the black hole phase with negative \(G\) is globally preferred respectively. At \(T_{HP}\), there is a discontinuity in the derivatives of the \(G - T\) curves, indicating that the HP phase transition is of first-order.

Since the unstable small black holes cannot be in equilibrium with the thermal AdS background, we will not mention their HP phase transitions anymore.

FIG. 1: The event horizon radii of the Schwarzschild–AdS black holes as a function of temperature (LBH and SBH stand for large and small black holes respectively). At a given pressure \(p\), the black hole temperature has a positive minimum \(T_0 = \sqrt{2p/\pi}\). The large black holes are thermodynamically stable, as they have positive \(r_+ - T\) slopes and positive heat capacities.

![FIG. 1](image1)

FIG. 2: The Gibbs free energies of large and small Schwarzschild–AdS black holes as a function of temperature, with a fixed pressure \(p = 0.1\). There is no HP phase transition for the small black hole, as its \(G - T\) curve is always above the \(T\)-axis. At low or high temperatures, the thermal AdS phase or the large black hole phase is thermodynamically preferred respectively, with the global minimum of \(G\) (thick black line), and their \(G - T\) curves intersect at the HP temperature \(T_{HP}\), corresponding to a first-order phase transition.

![FIG. 2](image2)

B. 4EGB gravity

Now, we take into account the GB term and discuss its effects on the HP phase transitions in the 4EGB gravity. From Eqs. (7) and (16), the black hole temperature and Gibbs free energy are modified to

\[
T = \frac{r_+^2 - \alpha + 8\pi p r_+^4}{4\pi r_+^2},
\]

\[
G = \frac{3r_+^2 + 3\alpha + 8\pi p r_+^4}{6r_+} - \frac{r_+^2 - \alpha + 8\pi p r_+^4}{4r_+(r_+^2 + 2\alpha)} \left(r_+^2 + 4\alpha \ln \frac{r_+}{\sqrt{\alpha}}\right).
\]

Before the relevant discussions, we should first point out an important difference between the Einstein and 4EGB gravities. In the former, from Eq. (20), there is no lower or upper bound of \(p\), and the HP phase transition can happen at all pressures. However, in the latter, \(p\) has both lower and upper bounds, making the analyses more complicated, so we must determine the range of \(p\) in advance. This range comes from three aspects. First, we expect the HP phase transition above the critical pressure of the Schwarzschild–AdS black hole, otherwise a swallowtail behavior will appear in the \(G - T\) curve, and the black hole will undergo a van der Waals-like phase transition, which is beyond the scope of our present work. This requirement will set a lower bound of \(p\). Second, we will see immediately that, when \(\alpha \neq 0\), the minimal black
hole temperature can reach zero, so the minimal black hole entropy should be evaluated as \( S(0, p, \alpha) \). The positivity of \( S(0, p, \alpha) \) will set an upper bound of \( p \). Third, since the Gibbs free energy decreases with temperature, if the HP phase transition happens, \( G(0, p, \alpha) \) should also be positive, but we will find that this condition is satisfied automatically. Altogether, \( p \) has both lower and upper bounds simultaneously.

First, the critical pressure \( p_c \) can be fixed by the equation of state in Eq. (8),

\[
\left( \frac{\partial p}{\partial r^+} \right)_{T, \alpha} = \left( \frac{\partial^2 p}{\partial r^+ \partial r^+} \right)_{T, \alpha} = 0,
\]

and these conditions set the lower bound of \( p \),

\[
p > p_c = \frac{15 - 8\sqrt{3}}{288\pi\alpha} \approx 0.00126. \tag{27}
\]

Second, when \( T = 0 \), from Eq. (23), we obtain

\[
r^+ = \sqrt{\frac{1 + 32\pi\alpha p - 1}{16\pi p}}. \tag{25}
\]

Substituting Eq. (25) into Eq. (15), we need

\[
S(0, p, \alpha) = \frac{\sqrt{1 + 32\pi\alpha p - 1}}{16p} + 2\pi\alpha \ln \frac{\sqrt{1 + 32\pi\alpha p - 1}}{16\pi\alpha p} > 0. \tag{26}
\]

This inequality can only be solved numerically,

\[
p < \frac{0.0238}{\alpha}. \tag{26}
\]

Third, substituting Eq. (25) into Eq. (24), we easily find

\[
G(0, p, \alpha) = \frac{\sqrt{1 + 32\pi\alpha p - 1 + 32\pi\alpha p}}{12\sqrt{\pi p(\sqrt{1 + 32\pi\alpha p - 1})}} > 0, \tag{26}
\]

so this inequality does not provide any further constraint. In all, the range of \( p \) can be summarized as

\[
0.00126 < \alpha p < 0.0238. \tag{27}
\]

In the following, we restrict our discussions within this range, in which the black holes can have and only have the HP phase transitions.

Now, we return to the HP phase transitions in the 4EGB gravity. By the same procedure in Sect. III A, the \( T_{\text{HP}}-p \) curves can be numerically plotted in Fig. 3. We find that, for any non-vanishing \( \alpha \), the HP phase transitions can only happen in the range in Eq. (27), and at a given pressure, the HP temperature \( T_{\text{HP}} \) decreases with \( \alpha \).

Next, from Eq. (23), the \( r^+-T \) curves are shown in Fig. 4. We see that, totally different from Fig. 1, the minimal temperatures of the Schwarzschild–AdS black holes can approach zero now. This is the distinct difference between the Einstein and 4EGB gravities, and the reason is that there is a negative term \(-\alpha \) in the numerator in Eq. (23), but not in Eq. (18).

The difference of the minimal black hole temperature causes the \( G-T \) curves in the 4EGB gravity obviously different from those in the Einstein gravity in Fig. 2. Here, all the \( G-T \) curves set out from the \( G \)-axis with \( T = 0 \), instead of \( T_0 \). From Eq. (26), the intercept on the \( G \)-axis \( G(0, p, \alpha) \) increases with \( \alpha \). Meanwhile, the
intercept on the $T$-axis (i.e., the HP temperature $T_{HP}$) decreases with $\alpha$, consistent with Fig. 3.

![Graph showing Gibbs free energies of large Schwarzschild-AdS black holes as a function of temperature in the 4EGB gravity, with a fixed pressure $p = 0.1$ and different values of $\alpha$. All the $G-T$ curves start from the $G$-axis, with the vanishing black hole temperature. Moreover, from the intersection points of the $G-T$ curves across the $T$-axis, the HP temperature $T_{HP}$ decreases with $\alpha$. The $G-T$ curves in the Einstein gravity are also shown for comparison.](image)

FIG. 5: The Gibbs free energies of the large Schwarzschild-AdS black holes as a function of temperature in the 4EGB gravity, with a fixed pressure $p = 0.1$ and different values of $\alpha$. All the $G-T$ curves start from the $G$-axis, with the vanishing black hole temperature. Moreover, from the intersection points of the $G-T$ curves across the $T$-axis, the HP temperature $T_{HP}$ decreases with $\alpha$. The $G-T$ curves in the Einstein gravity are also shown for comparison.

IV. HP PHASE TRANSITIONS OF THE RN–ADS BLACK HOLES

In this section, we continue our discussions on the HP phase transitions of the RN–AdS black holes in the Einstein and 4EGB gravities. Before detailed calculations, one important issue should be stressed. On account of the conservation of charge and the neutrality of thermal AdS background, the HP phase transitions of the charged AdS black holes must be considered in the grand canonical ensemble with fixed electric potential.

A. Einstein gravity

The discussions in the Einstein gravity are in parallel with Sect. III A. The results were also given in Ref. [16], and we only list them briefly. The metric of the RN–AdS black hole is shown in Eq. (5), from which we can determine the range of black hole charge as $Q < M$ and the range of electric potential as $\Phi < 1$.

From Eqs. (7) and (16), the RN–AdS black hole temperature and Gibbs free energy read

$$T = \frac{1 - \Phi^2 + 8\pi p r_+^2}{4\pi r_+},$$

$$G = M - TS - \Phi Q = \frac{3(1 - \Phi^2)r_+ - 8\pi p r_+^3}{12}.$$  

From Eq. (17), at the HP phase transition point, $r_+ = \sqrt{3(1 - \Phi^2)/(8\pi p)}$, and the HP temperature is

$$T_{HP} = \frac{8p}{3\pi}(1 - \Phi^2).$$

Again, there is no bound of $p$, so the HP phase transition can happen at all pressures, and at a given pressure, $T_{HP}$ decreases with $\Phi$.

Next, from Eq. (28), we have

$$r_+(T, p) = \frac{1}{4\pi p} \left[ \pi T \pm \sqrt{\pi^2 T^2 - 2\pi p (1 - \Phi^2)} \right],$$

and the minimal black hole temperature is

$$T_0 = \frac{2p}{\pi}(1 - \Phi^2).$$

The $r_+ - T$ curves are plotted in Fig. 6, which are similar to those in Fig. 1.

![Graph showing the event horizon radii of the RN–AdS black holes as a function of temperature in the 4EGB gravity, with a fixed pressure $p = 0.1$ and different values of $\Phi$.](image)

FIG. 6: The event horizon radii of the RN–AdS black holes as a function of temperature in the 4EGB gravity, with a fixed pressure $p = 0.1$ and different values of $\Phi$. Similar to the Schwarzschild–AdS case, the RN–AdS black hole also has a minimal temperature, but modified by the electric potential as $T_0 = \sqrt{2p(1 - \Phi^2)}/\pi$.

Substituting Eq. (31) into Eq. (29), the Gibbs free energies of the large and small RN–AdS black holes are

$$G(T, p, \Phi) = \sqrt{\pi T^2 - p(1 - \Phi^2) \pm T \sqrt{\pi^2 T^2 - 2\pi p(1 - \Phi^2)}}$$

$$\times \left[ 4p(1 - \Phi^2) - \pi T^2 \mp T \sqrt{\pi^2 T^2 - 2\pi p(1 - \Phi^2)} \right].$$
The $G$–$T$ curves are shown in Fig. 7. We see that the RN–AdS black hole still has a minimal temperature $T_0$, and the HP temperature $T_{\text{HP}}$ decreases with $\Phi$, consistent with Eq. (30).

FIG. 7: The Gibbs free energies of large and small RN–AdS black holes as a function of temperature, with a fixed pressure $p = 0.1$ and different values of $\Phi$. The HP temperature $T_{\text{HP}}$ decreases with the electric potential $\Phi$, as shown in Eq. (30).

B. 4EGB gravity

Finally, we move on to the most general case, the HP phase transitions of the RN–AdS black holes in the 4EGB gravity. From Eqs. (7) and (16), we have

\begin{align}
T &= \frac{(1 - \Phi^2)r_+^2 - \alpha + 8\pi pr_+^4}{4\pi r_+(r_+^2 + 2\alpha)}, \\
G &= \frac{3(1 - \Phi^2)r_+^2 + 3\alpha + 8\pi pr_+^4}{6r_+} \\
&\quad - \frac{(1 - \Phi^2)r_+^2 - \alpha + 8\pi pr_+^4}{4r_+(r_+^2 + 2\alpha)} \left( r_+^2 + 4\alpha \ln \frac{r_+}{\sqrt{\alpha}} \right). 
\end{align}  

(32)

(33)

Again, we need to determine the range of pressure $p$ in advance. Following the discussions in Sect. III B, first, $p$ should be greater than its lower bound (i.e., critical pressure $p_c$), such that there is only HP phase transition, but no van der Waals-like one. Second, from $S(0, p, \Phi, \alpha) > 0$, we can fix the upper bound of $p$. Third, the condition $G(0, p, \Phi, \alpha) > 0$ still does not provide further constraint. Now, the range of $p$ depends not only on the GB coupling constant $\alpha$, but also on the electric potential $\Phi$, as numerically shown in Fig. 8. We find that, with $\Phi$ increasing, the range of $p$ becomes larger, so the HP phase transitions can happen in a larger range of pressure.

With the above preparations, the investigations of the HP phase transitions of the RN–AdS black holes in the 4EGB gravity are straightforward. In the following, we choose the electric potential $\Phi = 0.5$ and plot the $T_{\text{HP}}$–$p$, $r_+–T$, and $G$–$T$ curves in Figs. (9)–(11) respectively.

FIG. 8: The range of pressure of the RN–AdS black holes in the HP phase transitions in the 4EGB gravity. To be convenient and consistent with Eq. (27), we show the relation of $\alpha p$ and $\Phi$, and the HP phase transitions can happen only in the shadowed area. When $\Phi = 0$, the intercepts 0.00126 and 0.0238 recover the range in the Schwarzschild–AdS case in Eq. (27). With $\Phi$ increasing, the range of $p$ allowed for the HP phase transition becomes larger.

FIG. 9: The HP temperature of the RN–AdS black holes as a function of pressure. The solid line is the coexistence line in the Einstein gravity. In the 4EGB gravity, pressure $p$ has both lower and upper bounds, with the detailed values in the figure. At a fixed pressure $p$, $T_{\text{HP}}$ decreases with both $\alpha$ and $\Phi$.

From Figs. (9)–(11), we observe that the HP temperature $T_{\text{HP}}$ decreases with both $\alpha$ and $\Phi$, and the minimal black hole temperature can reach zero. These general
potential. Otherwise, if we chose the canonical ensemble with fixed charge, the situations would become different, but such choice is not quite physically meaningful, see the Appendix in Ref. [16].

Last, we briefly comment the HP phase transitions of the rotating AdS black holes in the 4EGB gravity. Unfortunately, the rotating black hole metric solution in the 4EGB gravity is still unavailable. Although there have been some attempts via the Newman–Janis algorithm [37, 57], the effective rotating solution is not obtained by solving the field equations. Meanwhile, from the results in Sects. III and IV and also from our experience in Ref. [16], except the mathematical complexities caused by rotation, we do not expect more new physical results.

V. CONCLUSION

Black hole thermodynamics in the extended phase space has received intensive research interests, as there are much richer thermodynamic phenomena due to the introduction of the effective black hole volume and pressure. For example, the HP phase transitions in the extended phase space are widely studied within and beyond the Einstein gravity.

The gravity theory with the GB term is the minimal extension of the Einstein gravity. In four-dimensional GB gravity, the GB term is a topological invariant and is irrelevant to gravitational dynamics. However, it changes black hole thermodynamics via correcting the black hole entropy and thus influences the HP phase transition. These were the results in our previous work in Ref. [16].

Recently, a novel 4EGB gravity theory was realized by rescaling the GB coupling constant $\alpha \rightarrow \alpha/(d-4)$ in $d$ dimensions and redefining the four-dimensional gravity in the limit $d \rightarrow 4$ [33]. In such a way, the GB term has nontrivial dynamical effects even in four dimensions, and both the black hole mass and entropy are altered accordingly. As a result, the HP phase transition will exhibit new characters in the 4EGB gravity, and this is the research topic of our present work.

In this paper, we study the HP phase transitions of the Schwarzschild–AdS and RN–AdS black holes in the 4EGB gravity. We should emphasize that the similarities to the Schwarzschild–AdS case are based upon the fact that we utilize the grand canonical ensemble with fixed electric features are similar to those in the Schwarzschild–AdS case in the 4EGB gravity, only with the detailed values of $T_{HP}$ and the range of $p$ modified by $\Phi$.

Till now, we complete the discussions of the HP phase transitions of the RN–AdS black holes in the 4EGB gravity. We should emphasize that the similarities to the Schwarzschild–AdS case are based upon the fact that we utilize the grand canonical ensemble with fixed electric

FIG. 10: The event horizon radii of the RN–AdS black holes as a function of temperature in the 4EGB gravity, with a fixed pressure $p = 0.1$, fixed electric potential $\Phi = 0.5$, and different values of $\alpha$. The minimal black hole temperature is zero.

FIG. 11: The Gibbs free energies of the large RN–AdS black holes as a function of temperature in the 4EGB gravity, with a fixed pressure $p = 0.1$, fixed electric potential $\Phi = 0.5$, and different values of $\alpha$. The $G$–$T$ curves in the Einstein gravity are also shown for comparison.
the GB term only changes the black hole entropy, but not mass. Moreover, $p$ has only an upper bound, and the minimal black hole temperature is still $T_0$, but not zero. Therefore, we can conclude that the HP phase transition behaviors in the GB gravity lie exactly between the Einstein and 4EGB gravities.

In summary, the Einstein, GB, and 4EGB gravities form a series, in which the influences from the GB term become greater and greater. These influences result in the tighter and tighter bounds of pressure, within which the HP phase transitions can happen. Altogether, we wish to provide a general picture of the HP phase transitions with GB term to the most extent.

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