Cover It Up! Bipartite Graphs Uncover Identifiability in Sparse Factor Analysis

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Abstract

Despite the popularity of factor models with sparse loading matrices, little attention has been given to formally address identifiability of these models beyond standard rotation-based identification such as the positive lower triangular constraint. To fill this gap, we present a counting rule on the number of nonzero factor loadings that is sufficient for achieving generic uniqueness of the variance decomposition in the factor representation. This is formalized in the framework of sparse matrix spaces and some classical elements from graph and network theory. Furthermore, we provide a computationally efficient tool for verifying the counting rule. Our methodology is illustrated for real data in the context of post-processing posterior draws in Bayesian sparse factor analysis.

Keywords: Computational complexity; factor analysis; shrinkage prior; sparsity; variance identification

JEL classification: C11, C38, C63

1 Introduction

Ever since the pioneering work of Thurstone [1931, 1935], factor analysis has been a popular method to model the covariance matrix $V(y_t)$ of correlated, multivariate observations $y_t$ of dimension $m$,

\begin{align}
V(y_t \mid \beta, \Psi, \Sigma_f) &= \beta \Sigma_f \beta^\top + \Psi, \\
\mathbb{E}(y_t \mid \beta, f_t, \Psi) &= \beta f_t, \quad \mathbb{E}(f_t) = 0, \\
V(y_t \mid \beta, f_t, \Psi) &= \Psi, \quad V(f_t) = \Sigma_f = I_r,
\end{align}

where $f_t$ is $r$-dimensional, $\beta$ is $m \times r$-dimensional, $\Psi = \text{diag}(\sigma_1^2, \ldots, \sigma_m^2)$, and $\Sigma_f$ is the identity matrix in the paper at hand. As discussed by the comprehensive textbooks of Gorsuch [1983] and Anderson

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factor analysis is an umbrella term for a large number of heuristic or statistical methods, which handle various aspects of this underdetermined task. The researcher has to make many decisions during the analysis, and, unfortunately, this extensive and complex toolbox is often applied unprofessionally by practitioners, according to recurrent reviews [Ford et al., 1986, Conway and Huffcutt, 2003, Howard, 2016].

On the other hand, there are common ways to structure a factor analysis into subtasks. Following the title of the textbook by Thompson [2004], it is customary to first separate any factor analysis into two stages. During exploratory factor analysis (EFA), decisions are made, among others, about the number of factors \( r \), whether correlated or orthogonal factors are more appropriate, the identifying rotation of the factor loadings \( \beta \), and potential insignificant entries in \( \beta \), i.e. fixed zeros \( \beta_{ij} = 0 \). Then, conditionally on EFA, during confirmatory factor analysis, factor loadings \( \beta \) and factor scores \( f_t \) are estimated and hypothesis tests are conducted. In this setting, we advocate the standardization of a further step within EFA: a check on the unique identifiability of the idiosyncratic variances \( \Psi = \text{diag}(\sigma_1^2, \ldots, \sigma_m^2) \).

Classical results about generic global and local identification by Shapiro [1985] and Bekker and ten Berge [1997] solve the question of variance identification in what we call dense factor analysis (i.e., no structural zeros in \( \beta \)) by referring to the Ledermann bound [Ledermann, 1937]. However, these results do not apply in a zero-inflated model space because the exact zeros may fix \( \beta \) in a locally or even globally non-variance-identified space. A simple example is \( \beta = (0, \alpha, 0)^\top \) with \( r = 1 \) and \( m = 3 \). This factor can be eliminated and built into the idiosyncratic variance \( \sigma_2^2 \) for all \( \alpha \neq 0 \).

Identification becomes vital when the number of factors \( r \) is estimated. Reiersøl [1950] shows that if there is a solution to a factor analysis, i.e. a successfully estimated model with an assumed number of factors \( k = r \), then there are infinitely many solutions with more factors \( k > r \). Consequently, it is easy to overestimate the number of factors if parameters are not uniquely identified. We demonstrate this in Section 4.

1.1 Insignificant Factor Loadings

In this paper, we are concerned with the sparse latent factor model introduced by West [2003]. He describes the model as a hierarchical prior distribution, which we also use in Section 4, and applies it as a variable selection tool to tame a model with potentially many latent factors (referred to as “large \( p \”)).

Our motivation also falls under variable selection. However, we consider the model useful even outside the realm of large \( p \); rather, we see sparsity as a natural part of the usual practice of latent factor analysis.

Henceforward, assume that at the end of EFA \( r \) is specified and \( \Sigma_f = I_r \) is picked and, most importantly for us, a zero-nonzero pattern is fixed for \( \beta \). We formalize the insignificance of some entries in \( \beta \) by the sparse space \( \Theta_{sp} = (\delta, \mathbb{R}^{\sum_{i,j} \delta_{ij}}) \), which is a pair of a binary matrix \( \delta \in \{0, 1\}^{m \times r} \) and a real vector space of dimension \( \sum_{i,j} \delta_{ij} \). \( \Theta_{sp} \) is equipped with a probability measure that is absolutely continuous with respect to the \( \sum_{i,j} \delta_{ij} \)-dimensional Lebesgue-measure. Then, a matrix \( \beta \) is said to be generated by \( \Theta_{sp} \) if for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq r \) we have that \( \delta_{ij} = 0 \) implies \( \beta_{ij} = 0 \). On the other hand, the set of \( \beta \) matrices that are generated by \( \Theta_{sp} \) and have other zeros than \( \delta \) is a zero probability event. Finally, \( \Theta_{sp} \) is said to be represented by \( \delta \).
Typically, a factor analysis starts in an unrestricted space with loading matrix \( \beta_1 \in \mathbb{R}^{m \times r} \). Then, the researcher moves into their preferred space \( \beta_2 = \beta_1 U(\beta_1) \) via the orthonormal rotation matrix \( U(\beta_1) \), which, as indicated, often depends on \( \beta_1 \) (e.g., the varimax rotation [Kaiser, 1958] or the generalized lower triangular rotation [Frühwirth-Schnatter and Lopes, 2018]). The preferred space is where signal and noise are separated, resulting in the sparse space \( \Theta_{sp} \). In the Bayesian paradigm, spike-and-slab priors make this procedure explicit and exact via Bernoulli conditional distributions [West, 2003, Lucas et al., 2006, Ghahramani et al., 2007, Frühwirth-Schnatter and Lopes, 2010, 2018, Kaufmann and Schuhmacher, 2019], while continuous spike-and-slab priors [Rocková and George, 2017] approximate it. More generally applicable methods emulating a sparse space include heuristic thresholding [Bhattacharya and Dunson, 2011, Legramanti et al., 2020], thresholding in relation with sample size [Gorsuch, 1983, “salient loading”], highlighting [Fabrigar et al., 1999], and color coding [Bolfarine et al., 2021, Zhao et al., 2016]. There are examples in the connected literature of principal component analysis as well [McCrae and Costa, 1997, Leckman et al., 1997].

In this paper, identifiability of \( \Psi \) is examined based on the classical rule by Anderson and Rubin [1956]. Others have developed variants on the Anderson-Rubin rule for specific cases [Dunn, 1973, Bollen, 1989], for dedicated but correlated factors [Conti et al., 2014] and for correlated idiosyncratic errors [Williams, 2020]. Our main contribution is two-fold: we prove that a counting rule for zero-nonzero pattern of \( \beta \) is generically equivalent to the Anderson-Rubin rule, and we provide a proof and an algorithm for this counting rule to be efficiently verified. Further, we introduce bipartite graphs to factor analysis and provide empirical evidence for the usefulness of variance identification in reducing the estimated number of factors.

The paper is structured as follows. The simple counting rule of Anderson and Rubin [1956, Thm 5.5] is generalized in Section 2. The efficient verification of the counting rule is non-trivial, and it is handled in Section 3. After a numerical illustration in Section 4, the paper concludes in Section 5.

1.2 Note on Proof Techniques

In this section, we take a detour and introduce classical notions and results from graph theory for completeness. In the following sections there are two theorems, and in both of them the zero-nonzero pattern is represented by a customized undirected finite bipartite graph. Then, there are three parts to the proofs: first, the appropriateness of the representation needs to be shown and, at the same time, the task is translated into the language of graph theory; second, a solution to the translated task is found as a minimal vertex cover or a maximal matching (see definitions below); third, when applicable, efficient computation of the solution is outlined.

**Definition 1** (Graph and Weighted Graph). A graph \( G = (V, E) \) is a pair of a set of vertices (also called nodes) \( V \) and a set of undirected edges \( E \subseteq \{ \{u, v\} : u \in V, v \in V \setminus \{u\} \} \). A vertex-weighted graph \( G = (V, E, w) \) is a graph extended with a weight mapping \( w : V \mapsto \mathbb{R} \).

A graph is a special case of a vertex-weighted graph where \( w(v) = 1 \) for each \( v \in V \).

**Definition 2** (Bipartite Graph). A bipartite graph \( B = (V_1, V_2, E) \) is a triplet of two disjoint sets of vertices \( V_1 \) and \( V_2 \) and a set of undirected edges \( E \subseteq \{ \{u, v\} : u \in V_1, v \in V_2 \} \).
Note that a bipartite graph $B = (V_1, V_2, E_B)$ is isomorphic to the graph $G = (V, E_G)$ for $V = V_1 \cup V_2$ and $E_G = E_B$.

**Definition 3** (Bipartite Adjacency Matrix and Generated Bipartite Graph). Given a binary matrix $\delta$, an equivalent graph representation is the bipartite graph $B = (V_{\text{col}}, V_{\text{row}}, E_B)$, where vertices of $V_{\text{col}}$ and $V_{\text{row}}$ correspond to the columns and, respectively, rows of $\delta$, and an edge is drawn between $u \in V_{\text{col}}$ and $v \in V_{\text{row}}$ if the corresponding matrix element $\delta_{u,v}$ is nonzero. Then, $\delta$ describes which pairs of vertices of $B$ are adjacent (i.e., connected by an edge) and therefore $\delta$ is called the adjacency matrix of $B$. Turned around, $B$ is called the generated bipartite graph of $\delta$.

Figure 1 shows a binary matrix $\delta$ and its generated bipartite graph $B = (V_{\text{col}}, V_{\text{row}}, E_B)$ with $V_{\text{col}} = \{u_1, u_2, u_3, u_4\}$ and $V_{\text{row}} = \{v_1, v_2, v_3, v_4\}$. The edge set is $E_B = \{\{u_1, v_1\}, \{u_1, v_2\}, \{u_1, v_4\}, \{u_2, v_2\}, \{u_2, v_3\}, \{u_3, v_3\}, \{u_4, v_4\}\}$.

**Definition 4** (Matching). A matching in graph $G = (V, E)$ is a set of pairwise disconnected edges $M \subseteq E$; i.e., no two edges of $M$ have the same endpoint. A maximum matching in $G$ is a matching with the maximal number of edges among all matchings in $G$. For a subset of vertices $S \subseteq V$, an $S$-saturating matching is a matching in $G$ that covers all vertices of $S$.

Note that there may exist matchings in $G$ that cannot be extended to larger matchings but are not maximum matchings. In Figure 1, there are many matchings: e.g., $\emptyset$, $\{u_1, v_1\}$, and $\{u_1, v_4\}, \{u_2, v_3\}$ is one that cannot be extended further; moreover, $\{\{u_1, v_2\}, \{u_2, v_2\}\}$ is not a matching because $v_2$ appears twice as end point. $\{\{u_i, v_i\}\}_{i=1}^4$ is the only maximum matching, and it is also $V_1$-saturating. Furthermore, if we remove edge $\{u_2, v_2\}$ from $E_B$, then there is no $V_1$-saturating matching but two maximum matchings of size three.

**Definition 5** (Vertex Cover). A vertex cover in graph $G = (V, E)$ is a set of vertices $C \subseteq V$ such that every edge in $E$ has an endpoint in $C$. A minimum vertex cover in $G$ is a vertex cover with the minimal set size among all vertex covers in $G$. A minimum vertex cover in weighted graph $G = (V, E, w)$ is a vertex cover with the minimal total weight among all vertex covers in $G$.

Note that there may exist vertex covers in $G$ that cannot be reduced to smaller vertex covers but are not minimum vertex covers. In Figure 1, there are many vertex covers: e.g., $V_{\text{row}} \cup V_{\text{col}}$, and $\{u_2, u_3, v_1, v_2, v_4\}$ is one that cannot be reduced further; $\{u_1, u_2\}$ is not a vertex cover because both end points of edge $\{u_3, v_3\}$ are missing from the set. There are several minimum vertex covers, e.g., $V_{\text{col}}$, all of them of size four.
**Theorem** (König [1931] and Egerváry [1931]). *The size of a maximum matching is equal to the size of a minimum vertex cover in bipartite graphs.*

In Figure 1, both the maximum matching and the minimum vertex cover are of size four. If we remove edge \( \{u_2, v_2\} \) from \( E_B \), then both reduce to size three.

**Theorem** (Marriage Theorem by Hall [1935]). *In any bipartite graph \( B = (V_1, V_2, E) \), there exists a \( V_1 \)-saturating matching if and only if for every subset \( W \subseteq V_1 \), the neighborhood of \( W \) in \( V_2 \) is a larger set than \( W \).*

An exemplary counterfactual statement: if we remove edge \( \{u_2, v_2\} \) from Figure 1, then there is no \( V_{col} \)-saturating matching, and the neighborhood of \( W = \{u_2, u_3\} \) becomes the smaller set \( \{v_3\} \) at the same time.

For a more detailed introduction to graph theoretic notions, see chapters 3.1 and 3.2 of West [2001].

## 2 Sufficient Condition for Generic Global Variance Identification

Anderson and Rubin [1956] provide a sufficient condition for variance identification based on the following row deletion property.

**Definition 6** (Row Deletion Property, \( RD(r,s) \)). Let \( \beta \in \mathbb{R}^{m \times r} \). If any \( s > 0 \) rows of \( \beta \) are removed, then the remaining rows can be grouped into two matrices of rank \( r \).

Note that \( m \geq 2r + s \) is implied by \( RD(r,s) \).

**Theorem** (Anderson and Rubin [1956, Theorem 5.1]). \( RD(r,1) \) is a sufficient condition for identification of \( \Psi \).

The theorem holds for every \( \beta \in \mathbb{R}^{m \times r} \) and thus also for a matrix with zero entries. This makes it relevant for sparse matrix spaces.

The \( s > 1 \) case appears in overfitting factor models, when specific factors are allowed, as introduced by Tumura and Sato [1980].

**Theorem** (Tumura and Sato [1980, Theorem 1]). Assume that \( RD(r,s) \) holds and the factor decomposition of equation (1) exists with rank \( r \). Denote the solution by \( \beta_r \). If there is another factor decomposition with \( \beta_{r+s} \) of rank \( r + s \), then there exists a rotation \( G \) such that \( \beta_{r+s} G = [\beta_r \ S] \), where \( S \) is a collection of specific factors; i.e. \( SS^T \) is diagonal.

Later, Sato [1992] introduces a counting rule that is necessary for the row deletion property.

**Definition 7** (3-5-7-9 Counting Rule, \( CR(r,s) \)). Let \( \delta \) be a binary matrix of size \( m \times r \). For every \( 1 \leq q \leq r \), every submatrix containing \( q \) columns of \( \delta \) has at least \( 2q + s \) nonzero rows.
Theorem (Sato [1992, Theorem 3.4]). A necessary condition for RD\((r,s)\) is that CR\((r,s)\) holds for the zero-nonzero pattern \(\delta\) of \(\beta G\), where \(G\) is any \(r \times r\) unitary matrix.

There are three issues with applying this Theorem to verify identification of \(\Psi\) for a given sparse \(\beta\): the exponential number of submatrices within CR\((r,s)\), the infinite number of rotations \(G\), and that this is a necessary condition for a sufficient condition. Frühwirth-Schnatter and Lopes [2018] overcome the latter two in sparse matrix spaces \(\Theta_{sp}\) with probability one, although they need to assume that \(\delta\) is a generalized lower triangular matrix. In the next section, we show that RD\((r,s)\) and CR\((r,s)\) are equivalent almost everywhere without further assumptions. We apply this result in Section 4 to restrict model search to a space of variance identified models in a Bayesian setting.

Henceforth, let us consider a sparse matrix space \(\Theta_{sp} = (\delta, \mathbb{R}^{\sum_{i,j} \delta_{ij}})\) and a matrix \(\beta\) generated by \(\Theta_{sp}\). Following Shapiro [1985] and Bekker and ten Berge [1997], we consider generic variance identification in this section; however, we adapt it to our sparse setting.

**Definition 8** (Generic Global Variance Identification). Model (1) is globally variance identified at \((\beta, \Psi)\) if the only diagonal covariance matrix \(\Omega\), for which \(V(y) - \Omega\) is also a covariance matrix and \(\text{rank}(V(y) - \Omega) \leq \text{rank}(V(y) - \Psi)\), is \(\Omega = \Psi\). The model is generically variance identified within \(\Theta_{sp}\) if it is globally variance identified except for a set of Lebesgue-measure zero in \(\mathbb{R}^{\sum_{i,j} \delta_{ij}} \times (\mathbb{R}^{+})^{m}\), which is the product of the domains of \(\beta\) and \(\Psi\).

Note that the theorem of Anderson and Rubin ascertains point-wise (not generic) global variance identification.

We start with a simple lemma.

**Lemma 1.** Assume that \(\delta\) satisfies CR\((r,s)\). If any \(s > 0\) rows are deleted from \(\delta\), then the remaining matrix satisfies CR\((r,0)\).

Note that the inverse of this statement is false. If zero rows are added to a CR\((r,0)\)-matrix, the extended matrix still only satisfies CR\((r,0)\).

In the following, we assume that \(m \geq 2r + s\) and \(\delta\) satisfies CR\((r,s)\). By the end of this section, we show that if \(s\) rows are deleted from \(\delta\) and accordingly from \(\beta\), then, with probability one in \(\Theta_{sp}\), the remaining rows of \(\beta\) can be grouped into two matrices \(\beta_A^r\) and \(\beta_B^r\) such that both have rank \(r\). With Definition 9 and Lemma 2, we consider first a square matrix that will later be one of the two matrices of rank \(r\) in Definition 6.

**Definition 9** (RCM Matrix). A row-column-matching (RCM) matrix space of dimension \(r\) is a sparse matrix space \(\Theta_{sp}^r = (\delta^r, \mathbb{R}^{\sum_{i,j} \delta_{ij}^r})\) with the following property: \(\delta^r\) is \(r \times r\)-dimensional, and in the bipartite graph \(B = (V_{col}, V_{row}, E_B)\) generated from \(\delta^r\) there is a \(V_{col}\)-saturating matching. An RCM matrix is a square matrix generated by an RCM matrix space.

Since the number of rows and columns is the same, a \(V_{col}\)-saturating matching is also a \(V_{row}\)-saturating matching in \(B\). Importantly, this matching directly corresponds to a reordered diagonal of ones in \(\delta^r\), as depicted in Figure 2.
Figure 2: Left: example RCM space of dimension 4, represented via its zero-nonzero pattern $\delta$. Right: generated bipartite graph $B = (V_{col}, V_{row}, E_B)$ of $\delta$. One $V_{col}$-saturating matching and the corresponding reordered diagonal are shown in boldface.

**Lemma 2.** Let the $r \times r$-dimensional matrix $\beta^{\circ}$ be generated by a sparse matrix space $\Theta^{\circ}_{sp}$, and consider the following two conditions.

(a) $\beta^{\circ}$ is non-singular.

(b) $\Theta^{\circ}_{sp}$ is an RCM space.

Then, (a) implies (b) everywhere and (b) implies (a) almost everywhere in $\Theta^{\circ}_{sp}$.

**Proof.** The proof is based on the Leibniz formula for determinants. Note that the matching specified in Definition 9 is a set of $r$ nonzero variables in $\beta^{\circ}$ that are located in pairwise different rows and columns. Consequently, for an $r \times r$ square matrix, the product of the elements of the matching constitutes one summand in the Leibniz formula (potentially after a sign switch). If $\beta^{\circ}$ is non-singular, then its determinant is nonzero, therefore there has to be a nonzero summand and $\Theta^{\circ}_{sp}$ is an RCM space. If $\Theta^{\circ}_{sp}$ is an RCM space, then $\beta^{\circ}$ has a nonzero summand with probability one. There may be further summands, but only finitely many and they all include further nonzero variables from $\beta^{\circ}$, and therefore their sum is nonzero with probability one. Hence, $\det(\beta^{\circ}) \neq 0$ and $\beta^{\circ}$ is non-singular with probability one.

The following statement is a straightforward implication of Lemma 2, which connects the row deletion property to RCM spaces. Let us fix $s$ arbitrary rows that are removed from $\beta$. We will denote the remaining rows by $\beta^{\text{rem}}$ and the induced remainder objects by $\delta^{\text{rem}}$ and $\Theta^{\text{rem}}_{sp}$.

**Corollary 3.** Consider the following two conditions.

(a) $\beta^{\text{rem}}$ satisfies $RD(r,0)$.

(b) Two distinct groups of $r$ rows of $\delta^{\text{rem}}$ can be selected to form square matrices $\delta^{\text{A}}$ and $\delta^{\text{B}}$ such that the sparse matrix spaces represented by $\delta^{\text{A}}$ and $\delta^{\text{B}}$ are both RCM matrix spaces.

Then, (a) implies (b) everywhere and (b) implies (a) almost everywhere in $\Theta^{\text{rem}}_{sp}$.

We demonstrate in Figure 3 the ingredients to prove that $CR(r,s)$ is sufficient for Corollary 3(b). On the left hand side of the figure, $\delta$ satisfies $CR(3,2)$. We have to examine every $\delta^{\text{rem}}$ that results from removing two rows of $\delta$. Consider now removing the first and sixth rows. Then, we need to
Figure 3: Demonstration of the proof of Theorem 4 for \( r = 3 \) and \( s = 2 \). The \( \delta \) on the left hand side satisfies CR\((r,s)\). Consequently, after we remove any two rows, e.g., the first and the sixth in this instance, there is a sufficiently large matching in the graph representation of \( \delta_{\text{rem}} \) used within the proof.

For our purposes it is enough to prove that a matching of size \( 2r \) exists in \( B_{\text{surf}} \) if \( \delta_{\text{rem}} \) satisfies CR\((r,0)\): such a matching necessarily connects every column and their duplicates to unique rows, therefore it automatically specifies two disjoint RCM submatrices \( \delta_A \) and \( \delta_B \) of \( \beta_{\text{rem}} \) needed for Corollary 3(b). According to König’s theorem, it is sufficient to prove that the smallest vertex cover is of size \( 2r \). Note that a trivial vertex cover consists of all the columns and their duplicates, and there are exactly \( 2r \) vertices in this vertex cover. If there was a smaller vertex cover, then it would include \( k \) rows and exclude \( l \) columns (including potential duplicates), where \( k < l \). Since this would be a vertex cover, this would mean that all edges that go from those \( l \) columns are covered by the \( k \) rows. However, since \( \delta_{\text{rem}} \) satisfies CR\((r,0)\), the \( l \) columns and duplicates should neighbor at least \( l \) rows: if the net number of columns is \( l/2 \), they already neighbor at least \( 2 \cdot l/2 = l \) rows; if the net number of columns is larger, then even more rows. Therefore \( k \geq l \): a contradiction. This concludes the proof. The statement is summarized in Theorem 4.

**Theorem 4.** If \( \delta \) satisfies the counting rule CR\((r,s)\), then the extended row deletion property RD\((r,s)\) generically holds within \( \Theta_{sp} \).

Note that due to Sato’s theorem [Sato, 1992], the counting rule is still necessary for the row deletion property.
Now we are ready to connect the 3-5-7-9 counting rule with the Anderson-Rubin theorem. This is the foundation of our framework that is put to use in Section 4.

**Corollary 5.** If $\delta$ satisfies the counting rule $CR(r, 1)$, then Model (1) is generically globally variance identified within $\Theta_{sp}$.

**Remark.** Given a binary matrix $\delta$, the verification of $CR(r, 1)$ is a finite task. However, a naïve approach may need to visit $2^r-1$ matrices in order to make a decision. The combinatorial explosion quickly becomes an issue in practice as $r$ increases. In Section 3, we establish the applicability of Theorem 4 for large models.

**Remark.** The counting rule is not necessary. The following sparse space is an example where the counting rule does not hold but the model is generically globally variance identified.

$$
\beta = \begin{pmatrix}
\beta_{11} & 0 & 0 \\
\beta_{21} & \beta_{22} & 0 \\
0 & \beta_{32} & \beta_{33} \\
0 & \beta_{41} & 0 \\
0 & \beta_{52} & 0 \\
0 & 0 & \beta_{63}
\end{pmatrix},
\quad
V(y) = \begin{pmatrix}
v_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & v_2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & v_4 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & v_5 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & v_6
\end{pmatrix}.
$$

To see this, observe that all factor loadings can generically be computed (up to sign switches in each column) from the lower triangular elements of $V(y)$, e.g., $\beta_{11} = \sqrt{c_{21}c_{41}/c_{42}}$, and then all elements of $\Psi$ can generically be computed given the factor loadings and the diagonal of $V(y)$.

## 3 Efficient Verification of the Sufficient Condition

We extend the previous section and describe an algorithm that verifies $CR(r, 1)$ without visiting all the non-empty submatrices of $\delta$. In our construction, we do not just consider the nonzero rows of $\delta$; rather, we focus on unions of nonzero rows and nonzero columns such that the submatrix of $\delta$ that is left out of the union is a zero matrix. We first show that $CR(r, 1)$ is equivalent to all such unions being “not too small,” or, in other words, that the smallest such union is not too small. In the second half of the proof, we show how this optimization task can be solved efficiently.

As a starting point, consider two special cases: zero columns and zero rows can be removed from the sparsity matrix $\delta$. First, when zero columns are removed, then the unnecessary factors have to be discarded as well, and $r$ has to be adjusted. Second, the addition or removal zero rows does not influence rank conditions in tall matrices; therefore, $RD(r, 1)$ holds for $\beta$ if and only if it holds for $\beta$ without its zero rows. In the remaining part of the derivation, we assume that every row and column of $\delta$ has at least one nonzero element.

The main part of the proof consists of two steps:

1. Reformulate $CR(r, 1)$ as a minimum weighted vertex cover (MWVC) task, and
2. Show that the MWVC at hand can be solved efficiently via a polynomial algorithm.
Step 1. Let $B_{\text{adj}} = (V_{\text{col}}, V_{\text{row}}, E_{\text{adj}})$ be the bipartite graph that is generated from $\delta$. Furthermore, we equip the vertices of $B_{\text{adj}}$ with weights: elements of $V_{\text{col}}$ and $V_{\text{row}}$ are weighted with $2r + 1$ and, respectively, with $r$. The top row of Figure 4 shows an example. Then, the following lemma provides the basis for the polynomial algorithm. Terminology introduced in Definitions 1 and 5 is used.

**Lemma 6.** $\delta$ satisfies $CR(r, 1)$ if and only if the total weight $M^*$ of the MWVC in $B_{\text{adj}}$ is at least $r(2r + 1)$.

The intuition behind the vertex cover is that the submatrix formed by the rows and columns that are left out is a zero matrix in $\delta$.

*Proof of Lemma 6.* To prove this lemma, first note that $M^* \leq r(2r + 1)$ always holds. Indeed, the set $V_{\text{col}}$ has total weight $(2r + 1)r + r \cdot 0$ and it is a vertex cover. Another interesting case is the opposite: $V_{\text{row}}$ is also always a vertex cover with weight $(2r + 1) \cdot 0 + rm \geq r(2r + 1)$. Now we turn to the statement.

For the first direction of the proof, assume that $CR(r, 1)$ does not hold; i.e., there exists a submatrix $\delta_q$ made of $1 \leq q \leq r$ columns of $\delta$ with at most $2q$ nonzero rows. Then $M^* < r(2r + 1)$. Indeed, the $2q$ rows of $\delta_q$ and the $r - q$ columns outside of $\delta_q$ constitute a vertex cover. Therefore, $M^* \leq (2r + 1)(r - q) + r \cdot 2q = r(2r + 1) - q < r(2r + 1)$.

For the opposite direction, we assume that $CR(r, 1)$ holds, and we would like to show that the objective function takes always at least the aforementioned value $r(2r + 1)$. Let us take any vertex cover and denote by $k$ and $l$ the number of columns and, respectively, rows that are included in the vertex cover. For $k = r$, the total weight is at least $r(2r + 1)$. Now, consider $0 \leq k \leq r - 1$. There are $r - k$ columns in $\delta$ excluded from the vertex cover; the submatrix constructed from these columns contains at least $2(r - k) + 1$ nonzero rows. Hence, in order to cover these nonzero rows, we must have $l \geq 2(r - k) + 1$. This means that the total weight for this setting evaluates to $(2r + 1)k + rl \geq (2r + 1)k + r(2(r - k) + 1) = r(2r + 1) + k \geq r(2r + 1)$. Since our argument holds for all vertex covers, we have shown that $M^* \geq r(2r + 1)$.

Remark. In its current form, the proof cannot be extended to $CR(r, s)$ by choosing different weights for $V_{\text{col}}$ or $V_{\text{row}}$. In particular, if $x$ denotes the ratio of vertex weights in $V_{\text{col}}$ and $V_{\text{row}}$ (i.e., $x = (2r + 1)/r$ above), then it can be shown that $x > 2 + s - 1$ and $x \leq 2 + s/r$ are both necessary for the proof. This interval is non-empty only if $s \leq 1$.

Step 2. For the purposes of this section, we refer to notions from network theory.

**Definition 10 (Network).** A network is a set of nodes combined with a set of directed edges between the nodes. There are two distinguished nodes: the source node $s$ and the sink node $t$. Each edge has a weight, also called capacity, which is a non-negative number.

**Definition 11 (Cut in a Network).** A cut is a partition of the nodes into two sets $A$ and $B$, where $s \in A$ and $t \in B$. Correspondingly, cut-set is the set of edges that go from $A$ to $B$. Equivalently, a cut-set is a set of edges whose removal disconnects the source node of the network from the sink node of the network. For simplicity, we refer to cut-sets as cuts as well. The value of a cut is the sum of weights of the edges in the cut. A minimal cut is a cut whose value is minimal among all cuts in the network.
To introduce the algorithm, we reformulate the MWVC task as a minimal network cut problem. We first construct the network $N$. Its nodes are the source node $s$, one node for every element of $F$ and one for every element of $Y$, and the sink node $t$. There are three groups of edges in $N$: for every node $u \in V_{\text{col}}$, an edge goes from $s$ to $u$ with weight $2r + 1$; for every pair $u,v$ with $\delta_{v,u} = 1$, an edge goes from $u$ to $v$ in $N$ with infinite weight; and for every node $v \in V_{\text{row}}$, an edge goes from $v$ to $t$ with weight $r$. Figure 4 shows an example of the construction. Importantly, the size of $N$ is polynomial in the size parameters $m$ and $r$ of the model as $N$ has $m + r + 2$ nodes and less than $m + r + mr$ edges. In the remaining part of this section, we will show that the minimal cut of $N$ is equal to $M^\star$. Then, we are finished, since the minimal cut can be computed in $P(r,m) = O((m + r + 2)^2(m + r + mr))$ steps using Dinic’s algorithm [Tarjan, 1987, chapter 8], where $O$ is the Big-O notation.

In the above MWVC task, let us denote by $S \subseteq V_{\text{col}} \cup V_{\text{row}}$ a set of vertices. If $S$ is a vertex cover, then it corresponds to a finite-value cut $C$: for every included column $u \in S \cap V_{\text{col}}$ (respectively, included row $v \in S \cap V_{\text{row}}$), the corresponding edge going from $s$ to $u$ (respectively, from $v$ to $t$) is included in $C$, and nothing else. $C$ has finite value because it only contains finite-weighted edges. More concretely, the value of $C$ coincides with the weight $M^\star$ of the MWVC. $C$ is also a cut, since for each edge $u \rightarrow v$ going from $V_{\text{col}}$ to $V_{\text{row}}$, either $s \rightarrow u$ or $v \rightarrow t$ is included in $C$; that is the implication of $S$ being a vertex cover. Conversely, each finite valued cut corresponds to a vertex cover by the same construction.

Since there exists at least one vertex cover (e.g., all vertices), there exists a minimum, and, as we have seen, its total weight is equal to the value of a minimal cut in $N$. What we have proven is summarized in Theorem 7.

**Theorem 7.** Property $CR(r,1)$ can be verified algorithmically in $P(r,m)$ steps, where $P(r,m)$ is a polynomial in $r$ and $m$. □

This theorem and its proof give a means for efficient verification of generic variance identification in any sparse factor analysis based on the Anderson-Rubin row deletion theorem.

**Remark.** As noted above, $P(r,m) = O((m + r + 2)^2(m + r + mr))$, where $O$ is the Big-O notation. For fixed $r$, this is $O(m^3)$, and for fixed $m$, this is $O(r^3)$. Furthermore, if we look back to Theorem 4 and consider the removal of $s$ rows as a sequential procedure, first $s - 1$ rows and then another, then we get a polynomial time (in $r$ and $m$ for fixed $s$) algorithm for $CR(r,s)$ the following way. It is easy to see that $\delta$ is $CR(r,s)$ if and only if after removing any $s - 1$ rows the remaining binary matrix is $CR(r,1)$. This trivially gives rise to an algorithm with complexity $O(m^{s-1})P(r,m)$, which may be practical for $s = 2$ or $s = 3$ for small $m$.

## 4 Numerical Illustration

We demonstrate that missing variance identification may unnecessarily inflate the estimated number of factors during EFA. Bayesian estimation is applied, which allows us to emulate matrix sparsity using a hierarchical prior distribution on $\beta$. With that, we compare two estimation strategies. Under the first

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1This network construction was inspired by Goemans [2008].
Figure 4: An example $\delta$ (top left) with $r = 3$ and $m = 8$, and a corresponding bipartite graph $B_{adj}$ and network $N$ (bottom). Binary variables $u_i$ and $v_j$ correspond to the columns and rows of $\delta$, respectively, $i = 1, 2, 3$, $j = 1, \ldots, 8$. In $B_{adj}$, vertices of $V_{col}$ and $V_{row}$ have weight 7 and 3, respectively. In $N$, edges between $s$ and $V_{col}$, between $V_{col}$ and $V_{row}$, and between $V_{row}$ and $t$ have weight 7, $\infty$, and 3, respectively.
one, variance identification as a step is ignored. Alternatively, the prior on $\beta$ is restricted to the sufficient condition of Section 2 for variance identification. Two measures are considered for the comparison: out-of-sample predictive likelihood and the number of factors used during prediction.

### 4.1 Setup

Weekly returns of 17 currencies against the EUR are investigated between January, 2003, and December, 2005. The series include the currencies of big trading partners of the Eurozone (Australian Dollar, Canadian Dollar, British Pound, Hong Kong Dollar, Japanese Yen, South Korean Won, New Zealand Dollar, Russian Ruble, Turkish Lira, and US Dollar), and important local partners (Swiss Franc, Czech Koruna, Danish Krone, Norwegian Krone, Polish Zloty, Romanian Leu, and Swedish Krona). The chosen time period mostly avoids large international crises and heavy-tailed return distributions, as depicted in Figure 5, which renders the static latent factor model of Equation (1) appropriate for its analysis.

For this demonstration, however, we make the simplifying parametric assumption that observations and factors follow a multivariate Gaussian distribution:

$$y_t \sim N_m(\beta f_t, \Psi), \quad f_t \sim N_r(0, \Sigma_f).$$  \tag{2}

Furthermore, to facilitate variance identification through $\text{CR}(r, 1)$, we follow the tradition of West [2003] and introduce indicator variables $\delta_{ij} \in \{0, 1\}$ for every factor loading as parameters to estimate. This also gives us the opportunity to easily embed Equation (2) into a potentially overfitting factor model, where the indicator matrix $\delta = \{\delta_{ij}\}, i = 1, \ldots, 17, \text{ and } j = 1, \ldots, r_{\text{max}}$, attains the maximal width $r_{\text{max}} = 8$ for this data set.

Following Conti et al. [2014], and Kaufmann and Schuhmacher [2019], Bayesian posterior sampling is applied with a conjugate prior on $\beta$ and $\Psi$, combined with column-wise shrinkage on the indicators:

$$\beta_{ij} \mid \delta_{ij} = 0 \equiv 0,$$

$$\beta_{ij} \mid \delta_{ij} = 1 \sim N(0, \sigma_i^2),$$

$$\sigma_i^2 \sim IG(c_0, C_0),$$

$$\delta_{ij} \sim Ber(\tau_j),$$

$$\tau_j \sim B(a_0, b_0),$$  \tag{3}

where $IG(c_0, C_0)$ denotes the inverted Gamma distribution with kernel density $x^{-c_0-1} \exp(-C_0/x)$, $Ber(\tau_j)$ is the Bernoulli distribution with success probability $\tau_j$, and $B(a_0, b_0)$ is the Beta distribution with kernel density $x^{a_0-1}(1-x)^{b_0-1}$. The choice of $\sigma_i^2$ as the variance lets $\beta_{ij}$ capture potential scaling differences between the observation series. The fairly vague prior $(c_0, C_0) = (1, 0.3)$ is adopted from Legramanti et al. [2020], where $i = 1, \ldots, m$. Following Frühwirth-Schnatter and Lopes [2018], $(a_0, b_0) = (0.2, 0.6)$ is chosen first, which, Ghahramani et al. [2007] argue, is an example of a sticky prior and achieves shrinkage within each column. Later, the uniform prior $(a_0, b_0) = (1, 1)$ is also considered for comparison.

\footnote{$2r_{\text{max}} + 1 \leq m = 17$, where $m$ is the number of observation series, is essential for variance identification via $\text{RD}(r, 1)$, and therefore also via $\text{CR}(r, 1)$.}
Figure 5: Data set of 17 exchange rates against EUR.
4.2 Prediction exercise

Denote by \( \theta = (\beta, \delta, \{\sigma_i^2\}, \{\tau_j\}) \) the collection of parameters to be estimated. After one run of Markov chain Monte Carlo (MCMC) sampling, we compare two strategies in this section. Under the first one, variance identification is ignored and every draw \( \tilde{\theta} \) is kept. Alternatively, and this is implemented as a post-processing step after MCMC, we only keep realizations \( \tilde{\theta} \) such that \( \tilde{\delta} \) satisfies CR(\( \tilde{r}, 1 \)), where \( \tilde{r} \leq r_{\text{max}} \) is the number of nonzero columns in \( \tilde{\delta} \), and compute summaries from the filtered sample. This way, due to the hierarchical setup of equation (3), we restrict \( \delta | y \) such that \( \tilde{\Psi} | (y, \tilde{\delta}) \) is (generically) identified. Given that \( \tilde{\delta} \) is accepted for further use, realizations of \( (\tilde{\beta}, \tilde{\Psi}) | (y, \tilde{\delta}) \) is used for prediction. Note that the predictive density of model (2) is invariant to rotations of \( \tilde{\beta} \), therefore we do not take steps toward the identification of \( \tilde{\beta} \).

Estimation is done on 52 weekly returns, and the log-predictive-likelihood is estimated for the next weekly return as the mean of the realized log-predictive-likelihoods. Then, the time window is shifted by one week, and estimation and prediction are repeated. The procedure is done 100 times, which covers approximately two years of weekly predictions under a moving window regime.

4.3 Results

The top chart of Figure 6 shows the difference in log-predictive-likelihoods between an unidentified and an identified sparse matrix space via Corollary 3 when applying column shrinkage to \( \delta \). Restricting the prior to identified patterns does not significantly affect predictive performance of the factor model. The bottom chart displays the estimated number of factors \( \tilde{r} \) for both priors and the two strategies throughout the shifting periods. Variance identification consistently reduces \( \tilde{r} \) by 0.6–1.1 for the shrinkage prior. The difference is also significant: not just the posterior mean but the entire posterior distribution shifts downwards in all time periods as variance identification is incorporated into EFA.

As a simple prior sensitivity check, we consider a uniform prior \( (a_0, b_0) = (1, 1) \) on the column-specific shrinkage parameter \( \tau_j, j = 1, \ldots, r_{\text{max}} \), and compare it to the aforementioned shrinkage prior. As shown in Figure 6, a significant increase in \( \tilde{r} \) results from the reduced shrinkage in both columns. At the same time, the bottom row provides evidence for variance identification continuing to reject too large models while maintaining predictive performance (results not reported here for the uniform prior for brevity).

In our experience, the share of variance identified matrices increases in the posterior sample with more shrinkage, and this is reflected in Figure 7, which shows the posterior proportion of variance identified \( \delta \) matrices under the two prior specifications. The shrinkage prior prefers either close to empty or close to full columns a priori, separately for each column. In contrast, the uniform prior produces close to half full columns a priori. This strongly spills over to the posterior distribution for this data set as can be seen from the proportions. The counting rule is more likely satisfied with more crowded columns, which results in less than every 10th draw to be accepted in most time periods under the uniform prior, making the estimation procedure for \( \tilde{r} \) rather inefficient. At the same time, between 20 and 40% are accepted in most time periods under the shrinkage prior.
Figure 6: Top: Log-predictive-likelihood (LPL) without variance identification minus LPL with the prior restricted to variance identified models. Dots represent the difference in posterior means and the gray intervals are the ±2 posterior standard deviation credible region under the unrestricted prior. The uniform prior results in a very similar image. Bottom: Posterior distribution of the number of factors through time for the shrinkage prior (top; \(a_0 = 0.2\) and \(b_0 = 0.6\)) and the uniform prior (bottom; \(a_0 = 1\) and \(b_0 = 1\)), and with (left) and without (right) variance identification. The size of the gray bubbles represents the posterior probability, and only every fourth period is shown for improved legibility. The black line and dots highlight the posterior mean for every time period.
5 Conclusion

In this paper, we have provided a computationally efficient tool for generic variance identification in sparse factor models. To that end, we have introduced the framework of sparse matrix spaces and some classical elements from graph and network theory, which are rarely used in the econometrics literature. As demonstrated through a numerical illustration with real data, this new approach can be used to reduce the estimated number of factors while retaining predictive performance of the sparse factor model.

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