Application of Current Algebra in Three Pseudoscalar Meson Decays of $\tau$ Lepton

L. Beldjoudi and Tran N. Truong

Centre de Physique Théorique *
Ecole Polytechnique, 91128 Palaiseau, France

ABSTRACT

The decays of $\tau \to 3\pi \nu$ and $\tau \to \pi K^* \nu, K \rho \nu$ are calculated using the hard pion and kaon current algebra and assuming the Axial-Vector meson dominance of the hadronic axial currents. Using the experimental data on their masses and widths, the $\tau$ decay branching ratios into these channels are calculated and found to be in a reasonable agreement with the experimental data. In particular, using the available Aleph data on the $3\pi$ spectrum, we determine the $A_1$ parameters, $m_A = 1.24 \pm 0.02 \text{GeV}$, $\Gamma_A = 0.43 \pm 0.02 \text{ GeV}$; the hard current algebra calculation yields a $3\pi$ branching ratio of $19 \pm 3\%$. 

* Laboratoire Propre du CNRS UPR A.0014
Application of hard current algebra in $\tau$ decay was initiated soon after its discovery [1, 2, 3]. It was also pointed out that in a related calculation, using the hard current algebra technique, the cross sections for $e^+e^- \rightarrow 4\pi^\pm$ can be calculated and agree with the data in the 1 GeV region within a factor of 4 instead of a factor of 10 using the usual soft current algebra. This technique was extended to the $Ke_4$ decays with an implementation of the unitarity in the 2 pion channels [4], and was later used with success in the resolution of the $\eta \rightarrow 3\pi$ problem [5].

The technique of the hard pion current algebra consists in using the PCAC and the Lehmann Symanzik Zimmerman reduction formula [6] by contracting out the pseudoscalar fields. One would get the expressions for single, double or triple equal time commutator relations (ETCR) whose Fourier transforms are the physical matrix elements involving soft pion emissions. The remaining terms, which cannot be calculated by current algebra technique, involve higher order in the pseudoscalar meson momenta and therefore do not contribute in the hypothetical world where the pseudoscalars have zero four momenta. In the physical world where their four momenta do not vanish, corrections must be taken into account in an approximate manner by using substracted dispersion relations; the substruction constants are given by the current algebra low energy theorems and the substruction points are made at the scales where these current algebra low energy theorems are valid [7].

To illustrate our point let us consider the $e^+e^- \rightarrow 4\pi$. By contracting out the 2 S-wave pions we obtain the single and the double ETCR. The single ETCR term is related to the physical matrix element $\tau \rightarrow 3\pi\nu$ where the axial vector meson $A_1$ dominates; the double ETCR is related to the physical pion form factor which is dominated by the vector meson $\rho$. This technique enables us to incorporate the effect of the heavy fields $\rho$ and $A_1$ which are the main features of the low energy QCD and in a way which is consistent with chiral symmetry and experimental data. This method is however not unique.

Instead of this approach, it is now a fashion to use the effective Chiral Lagrangian which treats elegantly the low energy theorems involving the Nambu-Goldstone bosons; quantum corrections are treated perturbatively; this method is effectively a power series expansion in momenta and hence cannot take into account of the resonance effect. One is forced to accept the fact that Chiral Perturbation Theory, as is usually practiced, described only the low energy tail of the resonance which is certainly not the gross feature of the physics involved. The remedy for this approach is to build in the theory the heavy fields $\rho$ and $A_1$ and possibly also a heavy $\sigma$ fields as was done by a number of authors [8, 9]. It is suggested here
that the loop corrections should be treated unsystematically in the bubble chain approximation in order to make these heavy fields unstable in a way which is consistent with the unitarity requirement. This procedure is the same as the usual way of handling the W and Z propagators in the standard model calculation. One then can incorporate many nice features of the old Vector Meson Dominance models but taking into account also of the low energy chiral properties of the pseudoscalars. This approach will be explored in the future.

The purpose of this letter is to study the simpler processes $\tau \to 3\pi \nu$ and $\tau \to \pi K^* \nu, K \rho \nu$ and leave the more complicated process $e^+ e^- \to 4\pi^\pm$ or $\tau \to 4\pi \nu$ for a future publication. We do not expect to achieve here the precision of the order of 10% or better which is usually obtained for soft pion emission processes like $Kl_2, Kl_3$ and $Kl_4$ or the S wave $\pi N$ scattering lengths etc. This is so because the matrix elements depend on the scalar product of pion momenta to that of the current which is large in the physical region and considerable correction has to be made in order to reach the chiral limit.

Using the hadronic properties (widths and masses) of the Axial-mesons $A_1, Q_1$ and $Q_2$ and treat them as unstable particles in the usual way, their branching ratios and spectra in $\tau$ decays are calculated and found to be in good agreement with the experimental data.

We begin first by recalling the well-known formula [10] for the ratio

$$R_H = \frac{\Gamma(\tau \to H^- \nu)}{\Gamma(\tau \to e\nu\bar{\nu})} = \frac{6\pi}{m_\tau^2} \left( \frac{\cos^2 \theta_c}{\sin^2 \theta_c} \right) \int_{m_H^2}^{m_\tau^2} dQ^2 \left( 1 - \frac{Q^2}{m_\tau^2} \right) \left( a_0(Q^2) + (1 + 2\frac{Q^2}{m_\tau^2}) a_1(Q^2) \right)$$

(1)

where $a_0(Q^2)$ and $a_1(Q^2)$ are respectively the spin 0 and spin 1 part of the hadronic spectral functions and $\theta_c$ is the Cabbibo angle. The expression multiplying with $\cos^2 \theta_c$ is for the $\Delta S = 0$ hadronic tau decay and that multiplying with $\sin^2 \theta_c$ is for $\Delta S = 1$ hadronic tau decay.

We want to study the matrix element for the axial vector hadronic current involving three pseudoscalar mesons. For clarity, we first make the simplified approximation that the system of three pseudoscalar mesons can be represented by a pseudoscalar and a vector meson. This assumption is reasonable because in the usual angular momentum decomposition one pair of the pseudoscalars have to be in the relative P state and will be shown to be dominated by the vector meson. We
assume furthermore that the axial hadronic currents are dominated by the Axial
vector mesons $A_1$, $Q_1$ and $Q_2$, just the same as the vector hadronic currents are
dominated by the vector mesons $\rho$ and $K^*$. A more precise study of $\tau \rightarrow 3\pi \nu$ is
given, treating the vector meson $\rho$ as a resonant $2\pi$ state.

I) $\tau \rightarrow 3\pi \nu$ Decay

Let us first consider the $\Delta S = 0$ decay. As mentioned above we approximate
the $3\pi$ state by a $\pi\rho$ state. The most general matrix element can be written as:

$$
\langle \pi^-(k)\rho^0(p)|A_{\mu}^{1-i2}(0)|0\rangle = f_1(Q^2)\epsilon_\mu + \epsilon.k ((k + p)_\mu f_2(Q^2) + (k - p)_\mu f_3(Q^2))
$$

where $Q^2 = (k + p)^2$ and $\epsilon$ is the polarization vector of $\rho$. $f_1$, $f_2$, and $f_3$ are complex
form factors and are only functions of $Q^2$. Current algebra soft pion theorem, which
is obtained by taking the limit $k_\mu \rightarrow 0$, gives only information on $f_1$ but not on
the other 2 form factors. In an explicit model, it was shown that they contribute
little to the $\tau \rightarrow \pi\rho\nu$. Interested readers are referred to the original article [5]. (We
assume here that the decay constant of $\pi'$ is sufficiently small and hence can be
neglected). Using the standard low energy current algebra theorem and taking the
limit $k_\mu \rightarrow 0$ we have:

$$
\lim_{k_\mu \rightarrow 0} \langle \pi^-(k)\rho^0(p)|A_{\mu}^{1-i2}(0)|0\rangle = -\sqrt{2}\frac{f_\rho}{f_\pi}\epsilon_\mu(p)
$$

(3)

where $f_\pi = 93 MeV$, and $f_\rho$ is defined by the rate of $\rho \rightarrow e^+e^-$. Using the experimental data [11] we obtain, $f_\rho = 0.118 GeV^2$. This value of $f_\rho$ is equivalent to
writing approximatively the pion form factor as

$$
F_\pi(s) = m_\rho^2(1+\delta s/m_\rho^2)/\left(m_\rho^2 - s - im_\rho\Gamma_\rho(s)\right).
$$

A good fit to the experimental data is obtained with $\delta = 0.2$. In fact the more general form of Eq(3) reads

$$
\lim_{k_\mu \rightarrow 0} \langle \pi^-(k)\pi^+(q_1)\pi^-(q_2)|A_{\mu}^{1-i2}(0)|0\rangle = -\sqrt{2}\frac{f_\rho}{f_\pi}F_\pi(s)(q_1 - q_2)_\mu
$$

(4)

For convenience we shall first use Eq(3). The $3\pi$ matrix element below the $\rho\pi$ threshold can be straightforwardly obtained from Eq(4). Using Eq(2) in (3) we have:

$$
f_1(m_\rho^2) = -\sqrt{2}\frac{f_\rho}{f_\pi}
$$

(5)
Let us start with the narrow width approximation for the $A_1$ propagator. Using $A_1$ dominance for the form factor we have:

$$f_1(Q^2) = -\sqrt{2} \frac{f_\rho}{f_\pi} \frac{(m_A^2 - m_\rho^2)}{m_A^2 - Q^2}$$  \hspace{1cm} (6)

The generalisation of Eq[6] to take into account of the unstable nature of $A_1$ can be straightforwardly made. Using the $A_1$ dominance hypothesis for the axial current, the general expression for $f_1(Q^2)$ is:

$$f_1(Q^2) = -\sqrt{2} \frac{f_\rho}{f_\pi} \frac{m_A^2 - m_\rho^2 - \pi(m_\rho^2)}{m_A^2 - Q^2 - \pi(Q^2)}$$  \hspace{1cm} (7)

where we use the standard prescription for describing an unstable particle, with $\pi(Q^2)$ being the self energy operator of the $A_1$ resonance and is obtained by the bubble summation of the $\pi\rho$ intermediate states, similarly to the treatment of the $W$ and $Z$ propagators in the standard model. In order to have the usual Breit Wigner description of a resonance, we must make a twice subtracted dispersion relation with $\text{Re}[\pi(m_A^2)] = \text{Re}[\pi'(m_A^2)] = 0$ [12] where $m_A$ is the $A_1$ mass:

$$\text{Re}[\pi(Q^2)] = \frac{(Q^2 - m_A^2)^2}{\pi} \int_{(m_\rho + m_\pi)^2}^{\infty} dz \frac{\text{Im}[\pi(z)] - \text{Im}[\pi(m_A^2)] - (z - m_A^2) \text{Im}[\pi'(m_A^2)]}{(z - m_A^2)^2(z - Q^2)}$$  \hspace{1cm} (8-a)

$$\text{Im}[\pi(Q^2)] = \frac{g_{A\rho\pi}^2 \sqrt{\lambda(Q^2, m_\rho^2, m_\pi^2)}}{8\pi Q^2} \left(1 + \frac{\lambda(Q^2, m_\rho^2, m_\pi^2)}{12m_\rho^2 Q^2}\right)$$  \hspace{1cm} (8-b)

where we define the $\pi^0 \rho^+ A_1^-$ vertex as $g_{A\rho\pi} e(A).e(\rho)$, $\lambda(Q^2, m_\rho^2, m_\pi^2) = (Q^2 - (m_\rho + m_\pi)^2)(Q^2 - (m_\rho - m_\pi)^2)$, and P stands for the principal part integration. The dispersion integral (8-a) can be written as:

$$\text{Re}[\pi(Q^2)] = h(Q^2) - h(m_A^2) - (Q^2 - m_A^2)h'(m_A^2)$$  \hspace{1cm} (9)

where

$$h(Q^2) = \frac{Q^4}{\pi} \int_{(m_\rho + m_\pi)^2}^{\infty} dz \frac{\text{Im}[\pi(z)]}{z^2(z - Q^2)}$$

$$h(Q^2) = -\frac{g_{A\rho\pi}^2}{8\pi} \left(\frac{I_1(Q^2)}{12m_\rho^2} + \frac{5m_\rho^2 - m_\pi^2}{6m_\rho^2} I_2(Q^2) + \frac{(m_\rho^2 - m_\pi^2)^2}{12m_\rho^2} I_3(Q^2)\right)$$

4
$I_1(Q^2)$, $I_2(Q^2)$ and $I_3(Q^2)$ are defined in the appendix. Using Eq[7] the spectral functions $a_0(Q^2)$ and $a_1(Q^2)$ can be straightforwardly calculated:

\begin{align*}
    a_1(Q^2) &= \frac{|f_1(Q^2)|^2}{8\pi Q^2} \sqrt{\lambda(Q^2, m_{\rho}^2, m_{\pi}^2)} \left( 1 + \frac{\lambda(Q^2, m_{\rho}^2, m_{\pi}^2)}{12m_{\rho}^2Q^2} \right)^3 \\
    a_0(Q^2) &= \frac{|f_1(Q^2)|^2}{4\pi m_{\rho}^2} (Q^2/m_{A}^2 - 1)^2 \left( \frac{\sqrt{\lambda(Q^2, m_{\rho}^2, m_{\pi}^2)}}{2Q^2} \right)
\end{align*}

We want now to generalize Eq(8-b) and Eq(10) to take into account of the unstable nature of the $\rho$ meson, i.e we want to treat it as a resonant $2\pi$ P state. Instead of Eq(10) we have now:

\begin{equation}
    Im[\bar{\pi}(Q^2)] = \frac{g_{A\rho\pi}^2}{8\pi} \frac{(\sqrt{Q^2} - m_{\pi})^2}{\pi} \int_{4m_{\pi}^2} ds \frac{m_{\rho}\Gamma_\rho(s)}{(s - m_{\rho}^2)^2 + m_{\rho}^2\Gamma_\rho(s)} \frac{\sqrt{\lambda(Q^2, s, m_{\pi}^2)}}{Q^2} \left( 1 + \frac{\lambda(Q^2, s, m_{\pi}^2)}{12sQ^2} \right)
\end{equation}

Instead of Eq(10) we have:

\begin{align*}
    a_1(Q^2) &= \frac{2f_\rho^2}{f_\pi^2} \frac{1}{8\pi Q^2} \frac{(\sqrt{Q^2} - m_{\pi})^2}{\pi} \int_{4m_{\pi}^2} ds \frac{m_{\rho}\Gamma_\rho(s)}{(s - m_{\rho}^2)^2 + m_{\rho}^2\Gamma_\rho(s)} \frac{\sqrt{\lambda(Q^2, s, m_{\pi}^2)}}{Q^2} \\
    &\quad \left( 1 + \frac{\lambda(Q^2, s, m_{\pi}^2)}{12sQ^2} \right)^3 \left| \frac{m_{A}^2 - s - \bar{\pi}(s)}{m_{A}^2 - Q^2 - \bar{\pi}(Q^2)} \right|^2 \\
    a_0(Q^2) &= \frac{2f_\rho^2}{f_\pi} \frac{1}{4\pi} (Q^2/m_{A}^2 - 1)^2 \frac{1}{\pi} \int_{4m_{\pi}^2} ds \frac{m_{\rho}\Gamma_\rho(s)}{s(s - m_{\rho}^2)^2 + m_{\rho}^2\Gamma_\rho(s)} \left( \frac{\sqrt{\lambda(Q^2, s, m_{\pi}^2)}}{2Q^2} \right)^3 \\
    &\quad \left| \frac{m_{A}^2 - s - \bar{\pi}(s)}{m_{A}^2 - Q^2 - \bar{\pi}(Q^2)} \right|^2
\end{align*}

(11)

(12)

where $\Gamma_\rho(s)$ is the $\rho$ width; in terms of the coupling constant $g_{\rho\pi\pi}$ we have $m_{\rho}\Gamma_\rho(s) = g_{\rho\pi\pi}^2 (\sqrt{1 - 4m_{\pi}^2/s} \, /s)^3$. From experimental data on the $\rho$ width, we obtain $g_{\rho\pi\pi} = 6$.

$Re[\bar{\pi}(Q^2)]$ is calculated as in eq[8-a] with $Im[\pi(Q^2)]$ replaced by $Im[\bar{\pi}(Q^2)]$ given in Eq[11]. $Re[\bar{\pi}(Q^2)]$ is calculated numerically. The total branching ratio for $3\pi$ is obtained by multiplying Eqs(10) and (12) by a factor of 2.
Using the narrow width approximation for the $\rho$ resonance, the $\pi\rho$ spectrum for $m_A = 1.2 GeV$ and $\Gamma_A = 0.35, 0.4, 0.45$ GeV is shown in Fig (1). The corresponding $R$ ratios are respectively $2, 1.8, 1.6$ which are much larger than the experimental data. Because of the $\rho$ narrow width approximation, we cannot calculate the $3\pi$ spectrum below the $\pi\rho$ threshold $s = (m_\rho + m_\pi)^2$. Furthermore the $\pi\rho$ spectrum is found to be inaccurate near to the $\pi\rho$ threshold.

We show below these disagreements with the experimental data are due to the zero width approximation for the $\rho$ resonance.

In Fig(2), $Re[\pi(Q^2)]$ in the $\pi\rho$ approximation, and the more correct form $Re[\tilde{\pi}(Q^2)]$ are shown. It is seen that $Re[\tilde{\pi}(Q^2)]$ has a much smoother behavior than the approximate one $Re[\pi(Q^2)]$ near the $\pi\rho$ threshold. Note that at the current algebra low energy theorem limit $s = m_\rho^2$, $Re[\tilde{\pi}(Q^2)]$ is smaller than $Re[\pi(Q^2)]$.

In Fig 3, 4 and 5, the $3\pi$ spectra are plotted as a function of the $3\pi$ invariant mass where, for clarity, we normalise the theoretical prediction to the number of events. The data are taken from the ALEPH group [15] for illustration purpose. Because the experimental correction for acceptance has not been made, the experimental results should be considered as preliminary. In table 1 the values of $R_{3\pi} = \frac{\Gamma(\tau \rightarrow 3\pi\nu)}{\Gamma(\tau \rightarrow e\nu\bar{\nu})}$ are given as a function of $\Gamma_A$ and $m_A$.

It is seen that $R_{3\pi}$ is an increasing function of $m_A$ and decreasing function of $\Gamma_A$.

If we assume that the acceptance correction to the ALEPH data was negligible, our best fit to the $3\pi$ spectrum gives the following ranges of $m_A$ and $\Gamma_A$: $m_A = 1.24 \pm 0.02 GeV$, $\Gamma_A = 0.43 \pm 0.02$ GeV.

Using these results, the calculated branching ratio for $\tau \rightarrow 3\pi\nu$ is $19 \pm 3\%$. The central value corresponds to our best fit which is shown in fig(6). This value agrees with the ALEPH branching ratio $19.14 \pm 0.48 \pm 0.44\%$.

The main difference between our results and the recent studies [16,17] of the $\tau \rightarrow 3\pi\nu$ is that in our approach we can make more predictions due to the application of the low energy current algebra theorem. That is, using the experimental mass and width of the axial vector meson $A_1$, we can predict the $\tau \rightarrow A_1\nu$ branching ratio. A more ambitious approach to reduce the $A_1$ parameters is to predict the $A_1$ width as a function of its mass, similarly to the KSRF relation to the $\rho$ meson[18]. This calculation will be dealt in a forthcoming paper[19].
II) $\tau \to K\rho \nu$ and $K^*\pi \nu$ Decays

The study of the $\Delta S = 1$ decays is more complicated due to the presence of the two axial vector mesons $Q_1(1270)$ and $Q_2(1400)$. It is an experimental fact that $Q_1$ is coupled strongly to the $\rho K$ and weakly to $\pi K^*$ while $Q_2$ couples strongly to $\pi K^*$ and weakly to $K\rho$. Using the experimental data $B.R(Q_1 \to K\rho) = 42\%$, $BR(Q_1 \to \pi K^*) = 16\%$, $B.R(Q_2 \to K\rho) = 3\%$ and $BR(Q_2 \to \pi K^*) = 94\%$ together with the total width $\Gamma_{Q_1} = 90 MeV$, $\Gamma_{Q_2} = 170 MeV$ we obtain (in unit of GeV): $|g_{Q_1\rho K}| = 2.6\pm0.20$, $|g_{Q_2\rho K}| = 0.62\pm0.30$, $|g_{Q_1\pi K^*}| = 1.18\pm0.20$, $|g_{Q_2\pi K^*}| = 3.66\pm0.10$

Since $Q_1$ and $Q_2$ are a linear combination of the states from two different octets, one cannot make use of the SU(3) symmetry to fix the signs of the coupling constants $g$ [13].

Similarly to Eq(3), we can derive the low energy theorems for the matrix element $\langle \rho^0(p)K^-|A_{\mu}^{\pi^0-\pi^0}(0)|0\rangle$ by taking the kaon soft, and the low energy theorem for the matrix element $\langle \pi^-(k)K^*o(p)|A_{\mu}^{\pi^0-\pi^0}(0)|0\rangle$ by taking the pion soft.

In the narrow width approximation for $Q_1$ and $Q_2$ resonances, we would expect to have the following low energy theorems:

$$f_{Q_1} \frac{g_{Q_1\rho K}}{m_1^2 - m_\rho^2} + f_{Q_2} \frac{g_{Q_2\rho K}}{m_2^2 - m_\rho^2} = - \frac{f_\rho}{f_{K^+}}$$

$$f_{Q_1} \frac{g_{Q_1\pi K^*}}{m_1^2 - m_{K^*}^2} + f_{Q_2} \frac{g_{Q_2\pi K^*}}{m_2^2 - m_{K^*}^2} = \frac{f_{K^*}}{f_{\pi^+}}$$

(13)

where $f_{Q_1}$, $f_{Q_2}$ are similarly defined as $f_\rho$.

Eq(13) are strictly not correct because of the unitarity constraints due to the complication of the coupled channel problem. A correct treatment of this problem will be the subject of a separate publication. Because the experimental data on $\tau \to K^-\rho^0\nu$ and $\tau \to \pi^-K^*o\nu$ are very crude, we make the simple approximation that $Q_1$ couples only to $\rho K$ and $Q_2$ to $K^*\pi$ and therefore neglect the mixing of these 2 channels. Solving Eq[9] for $f_{Q_1}$ and $f_{Q_2}$ we have, $f_{Q_1} = 0.27 GeV^2$ and $f_{Q_2} = 0.39 GeV^2$.

Similarly to $f_1(s)$ defined in Eq(2), instead of Eq(7), we now have:

$$f_{K^0}(Q^2) = - \frac{f_\rho}{f_{K^+}} \frac{m_1^2 - m_\rho^2 - \pi_1(m_\rho^2)}{m_1^2 - Q^2 - \pi_1(Q^2)}$$

$$f_{K^*\pi}(Q^2) = \frac{f_{K^*}}{f_{\pi^+}} \frac{m_2^2 - m_{K^*}^2 - \pi_2(m_{K^*}^2)}{m_2^2 - Q^2 - \pi_2(Q^2)}$$

(14)
where $\pi_1(Q^2)$ and $\pi_2(Q^2)$ satisfy a twice substracted dispersion relation with $Re[\pi_1(m_1^2)] = Re[\pi_1'(m_1^2)] = 0$ and $Re[\pi_2(m_2^2)] = Re[\pi_2'(m_2^2)] = 0$. It should be noted that because Kaons are made of up, down and strange quark, the chiral (soft) Kaon limit must be taken simultaneously with the soft pion i.e we must use the $SU(3)_L \times SU(3)_R$ limit. Because the phase space for $K\rho$ is tiny as compared with that of $K^*\pi$ in $Q_1$ decays, we must make a correction in the expression for $\pi_1(s)$ to take into account of this special situation. More explicitly we have:

$$Im[\pi_1(Q^2)] = \frac{3}{2} g_{Q_2\rho K}^2 \frac{(\sqrt{Q^2} - m_\rho)^2}{8\pi} \int ds \frac{m_\rho \Gamma_\rho(s)}{(s - m_\rho^2)^2 + m_\rho^2 \Gamma_\rho(s)^2} \sqrt{\lambda(Q^2, s, m_K^2)} $$

$$\left(1 + \frac{\lambda(Q^2, s, m_K^2)}{12s Q^2}\right) + \frac{3}{4} g_{Q_2\pi K^*}^2 \frac{1}{\pi} \int ds \frac{m_K \Gamma_{K^*}(s)}{(s - m_{K^*}^2)^2 + m_K^2 \Gamma_{K^*}(s)^2} $$

$$\left[\frac{\sqrt{\lambda(Q^2, s, m_{K^*}^2)}}{Q^2} \left(1 + \frac{\lambda(Q^2, s, m_{K^*}^2)}{12s Q^2}\right) + \frac{(m_{\pi}^2 - m_{K^*}^2)\sqrt{\lambda(Q^2, s, m_{\pi}^2)}}{6s \lambda(s, m_{\pi}^2, m_{K^*}^2)}Q^4\right]$$

(15)

It is a good approximation to make a $\delta$ function for the $K^*$ propagator on the right hand side of the equation (11). Similarly we can make the $\delta$ function approximation for the $\rho$ and $K^*$ propagators.

$$Im[\pi_2(Q^2)] = \frac{3}{2} g_{Q_2\rho K}^2 \frac{\sqrt{\lambda(Q^2, m_\rho^2, m_{K^*}^2)}}{8\pi} \frac{\sqrt{\lambda(Q^2, m_\rho^2, m_{K^*}^2)}}{Q^2} \left(1 + \frac{\lambda(Q^2, m_\rho^2, m_{K^*}^2)}{12m_\rho^2 Q^2}\right) + $$

$$\frac{3}{4} g_{Q_2\pi K^*}^2 \frac{\sqrt{\lambda(Q^2, m_{K^*}^2, m_{\pi}^2)}}{8\pi} \frac{\sqrt{\lambda(Q^2, m_{K^*}^2, m_{\pi}^2)}}{Q^2} \left(1 + \frac{\lambda(Q^2, m_{K^*}^2, m_{\pi}^2)}{12m_{K^*}^2 Q^2}\right)$$

where $\Gamma_{K^*}(s)$ is the $K^*$ width. A straightforward calculation gives:

$m_{K^*} \Gamma_{K^*}(s) = g_{K^*\pi K}^2 \frac{\sqrt{\lambda(s, m_{K^*}^2, m_{\pi}^2)}}{32\pi} / s^2$. From the experimental data on the $K^*$ width, we obtain $g_{K^*\pi K} = 4.48$.

The self energy operators $\pi_1$ and $\pi_2$ can be straightforwardly calculated using dispersion relations. The corresponding spectral functions are easily calculated.
\[ a_1(Q^2) = \frac{\rho^2}{K^2} \frac{1}{8\pi Q^2} \int ds \left( \frac{m_\rho \Gamma_\rho(s)}{(s - m_\rho^2)^2 + m_\rho^2 \Gamma_\rho(s)^2} \right) \left( \frac{\sqrt{\lambda(Q^2, s, m_\pi^2)}}{Q^2} \right) \left( 1 + \frac{\lambda(Q^2, s, m_\pi^2)}{12sQ^2} \right) \left| \frac{m_1^2 - s - \pi_1(s)}{m_1^2 - Q^2 - \pi_1(Q^2)} \right|^2 \]

\[ + \frac{\rho^{*2}}{K^{*2}} \frac{1}{8\pi Q^2} \int ds \left( \frac{m_{K^*} \Gamma_{K^*}(s)}{(s - m_{K^*}^2)^2 + m_{K^*}^2 \Gamma_{K^*}(s)^2} \right) \left| \frac{m_2^2 - s - \pi_2(s)}{m_2^2 - Q^2 - \pi_2(Q^2)} \right|^2 \]

\[ \left[ \frac{\sqrt{\lambda(Q^2, s, m_{K^*}^2)}}{Q^2} \right] \left( 1 + \frac{\lambda(Q^2, s, m_{K^*}^2)}{12sQ^2} \right) + \frac{(m_\pi^2 - m_{K^*}^2)^2 \sqrt{\lambda(Q^2, s, m_{K^*}^2)^3}}{6s\lambda(s, m_{\pi}^2, m_{K^*}^2)Q^4} \right] \]

(16)

and a similar expression for \( a_0 \). Numerical calculation gives:

\[ BR(\tau \to \rho^0 K^- \nu) = 0.1\% \]

\[ BR(\tau \to \pi^- K^{0*} \nu) = 0.4\% \]

These results are in good agreement with the values of TPC/Two-Gamma collaboration [14]: \( B(\tau \to K^{0*} \pi^- \nu, \text{neutrals}) = 0.51\pm0.2\pm0.13 \) and \( B(\tau \to K^- \pi^+ \pi^- \nu) = 0.7 \pm 0.2 \). The \( K^- \rho^0 \nu \) mode is therefore consistent with zero.

ACKNOWLEDGMENT

We are grateful to André Rougé and Henri Videau for providing us with the Aleph Collaboration experimental data on \( \tau \to 3\pi \nu \) decay.

APPENDIX A

\[ I_1(s) = -\frac{s^2}{\pi} P \int_{(m_{\pi} + m_{\rho})^2}^{\infty} \frac{\sqrt{\lambda(z, m_{\pi}^2, m_{\rho}^2)}}{z^2(z - s)} dz \]
\[ I_2(s) = -\frac{s^2}{\pi} P \int_{(m_\pi + m_\rho)^2}^\infty \frac{\sqrt{\lambda(z, m_\pi^2, m_\rho^2)}}{z^3(z - s)} dz \]

\[ I_3(s) = -\frac{s^2}{\pi} P \int_{(m_\pi + m_\rho)^2}^\infty \frac{\sqrt{\lambda(z, m_\pi^2, m_\rho^2)}}{z^4(z - s)} dz \]

These integrals are conveniently expressed in terms of a generating function

\[ \psi(s) = -\frac{\lambda(s, m_\pi^2, m_\rho^2)}{2} P \int_{(m_\pi + m_\rho)^2}^\infty \frac{dz}{\sqrt{\lambda(z, m_\pi^2, m_\rho^2)}(z - s)} \]

For convenience we give the analytic continuation of this function to other regions.

\[ \psi(s) = \begin{cases} 
\sqrt{\lambda(s, m_\pi^2, m_\rho^2)} \log \left( \frac{\sqrt{s - (m_\pi + m_\rho)^2} + \sqrt{s - (m_\pi - m_\rho)^2}}{2 \sqrt{m_\pi m_\rho}} \right) & \text{if } s \geq (m_\pi + m_\rho)^2 \\
-\sqrt{\lambda(s, m_\pi^2, m_\rho^2)} \log \left( \frac{\sqrt{s - (m_\pi + m_\rho)^2} + \sqrt{s - (m_\pi - m_\rho)^2}}{2 \sqrt{m_\pi m_\rho}} \right) & \text{if } s \leq (m_\pi - m_\rho)^2 \\
|\lambda(s, m_\pi^2, m_\rho^2)| \arctan \left( \sqrt{\frac{s - (m_\pi - m_\rho)^2}{s - (m_\pi + m_\rho)^2}} \right) & \text{if not} 
\end{cases} \]

\[ I_1(s) = \frac{2}{\pi} (\psi(s) - \psi(0) - s\psi'(0)) \]

\[ I_2(s) = \frac{2}{\pi s} (\psi(s) - \psi(0) - s\psi'(0) - \frac{s^2}{2} \psi''(0)) \]

\[ I_3(s) = \frac{2}{\pi s^2} (\psi(s) - \psi(0) - s\psi'(0) - \frac{s^2}{2} \psi''(0) - \frac{s^3}{6} \psi'''(0)) \]
\[ \Gamma_A = 0.35 \]

| mass (in GeV) | \( R_{3\pi} \) |
|---------------|----------------|
| 1.15          | 1.02           |
| 1.175         | 1.12           |
| 1.2           | 1.20           |
| 1.25          | 1.40           |

\[ \Gamma_A = 0.4 \]

| mass (in GeV) | \( R_{3\pi} \) |
|---------------|----------------|
| 1.15          | 0.88           |
| 1.175         | 0.96           |
| 1.2           | 1.04           |
| 1.26          | 1.20           |

\[ \Gamma_A = 0.45 \]

| mass (in GeV) | \( R_{3\pi} \) |
|---------------|----------------|
| 1.15          | 0.76           |
| 1.175         | 0.84           |
| 1.2           | 0.9            |
| 1.25          | 1.05           |

Table 1: Results for \( R_{3\pi} \) defined in (1) for different \( A_1 \) parameters.

REFERENCES

[1] T.N Pham, C.Roiesnel and T.N Truong, *Phys. Rev. Lett.* **41** (1978) 371

[2] T.N Pham, C.Roiesnel and T.N Truong, *Phys. Lett.* **78B** (1978) 623

[3] T.N Pham, C.Roiesnel and T.N Truong, *Phys. Lett.* **80B** (1978) 119

[4] T.N Truong, *Phys. Lett.* **99B** (1981) 154

[5] C.Roiesnel and T.N Truong, *Nucl. Phys.* **B187** (1981) 293; Ecole Polytechnique preprint A515-0982, unpublished; C.Roiesnel, Thèse d’Etat, Université de Paris Sud (Orsay) (1982).

[6] H. Lehmann, K. Symanzik and W. Zimmermann *Nuovo Cimento* **1** (1955) 205.
[7] R.E. Marshak, Riazuddin and C.P. Ryan, *Theory of Weak Interactions in Particle Physics*, Wiley-Interscience (1969).

[8] M.Lacombe, B.Loiseau, R. Vinh Mau, W.N.Cottingham *Phy. Rev.* **D38**(1988)1491.

[9] M. Bando, T. Kugo, K. Yamawaki, *Phy. Report* (1988)219 and *references therein*

[10] Y.S. Tsai, *Phy. Rev.* **D4**(1971)2821.

[11] Particle Data Group, *Phy. Rev.* **D45**(1992).

[12] G. Gounaris and J.J Sakurai, *Phy. Rev. Lett.* **21**(1968)244.

[13] M. Suzuki, *Phy. Rev.* **D47**(1993)1252.

[14] TPC/Two-Gamma Collaboration, D.A. Bauer et al., Report No. LBL-33037, (1993) (unpublished).

[15] Aleph Collaboration *Z.Phy* **C59**(1990) 369.

[16] J.H Kuhn, A.Santamaria *Z.Phy* **C48**(1990) 445.

[17] N. Isgur, C. Morningstar, C.Reader *Phy. Rev.* **D39**(1989)1357. M.G. Bowler *Phy.Lett.* **182B**(1981)400

[18] K.Kawarabayashi and M. Suzuki, *Phy. Rev. Lett.* **16**(1966)255. Riazuddin and Fayyazudddin, *Phy. Rev.* **147**(1966)1071.

[19] L. Beldjoudi, H.Ngoc Long and T.N Truong Polytechnique preprint
FIGURE CAPTIONS

**Fig.1** Prediction for $\tau \rightarrow 3\pi \nu$ spectrum as a function of the $3\pi$ invariant mass in the narrow width approximation for the $\rho$ resonance ($2\pi$ P state = $\rho$). Dashed/solid/dot-dashed curves correspond respectively to $\Gamma_A = 0.35$, 0.4, 0.45 GeV, the $A_1$ mass is fixed to 1.2 GeV. The experimental data are those of Aleph group [15].

**Fig.2** Comparison between $Re[\pi(Q^2)]$ and $Re[\bar{\pi}(Q^2)]$ (solid/dashed curves) as defined in (8-a) and (11) (in units of $GeV^2$).

**Fig.3** Calculation of the $3\pi$ invariant mass spectrum for $\tau \rightarrow 3\pi\nu$ decay using $\Gamma_A = 0.35 GeV$. Long-dashed/dot-dashed/short-dashed/solid curves correspond respectively to $m_A = 1.15, 1.175, 1.2, 1.26$ GeV.

**Fig.4** Idem using $\Gamma_A = 0.4$ GeV.

**Fig.5** Idem using $\Gamma_A = 0.45$ GeV.

**Fig.6** Our best fit for the $\tau \rightarrow 3\pi\nu$ spectrum corresponding to $m_A = 1.24$ GeV, $\Gamma_A = 0.43$ GeV.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9411424v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9411424v1
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9411424v1
This figure "fig1-4.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9411424v1
This figure "fig1-5.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9411424v1
This figure "fig1-6.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9411424v1