MIXED MODEL OF INDUCED QCD

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Abstract

The problems with the $Z_N$ symmetry breaking in the induced QCD are analyzed. We compute the Wilson loops in the strong coupling phase, but we do not find the $Z_N$ symmetry breaking, for arbitrary potential. We suggest to bypass this problem by adding to the model a heavy fermion field in a fundamental representation of $SU(N)$. Remarkably, the model still can be solved exactly by the Riemann-Hilbert method, for arbitrary number $N_f$ of flavors. At $N_f \ll N \to \infty$ there is a new regime, with two vacuum densities. The $Z_N$ symmetry breaking density satisfies the linear integral equation, with the kernel, depending upon the old density. The symmetry breaking requires certain eigenvalue condition, which takes some extra parameter adjustment of the scalar potential.

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1 Introduction

1.1 Critical phenomena in induced QCD

The first few months of investigation of induced QCD \[1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13\] revealed some nice features of this new lattice gauge model, as well as some problems. The functional integral of this model reads

\[
Z = \prod_x \int D\Phi_x \exp \left(-N \text{ tr } [U(\Phi_x)] \prod_{<xy>} I[\Phi_x, \Phi_y]\right),
\]

(1.1)

with the Itzykson-Zuber integral

\[
I[\Phi_x, \Phi_y] \equiv \int D\Omega_{xy} \exp \left[N \text{ tr } [\Phi_x \Omega_{xy} \Phi_y \Omega_{xy}^\dagger]\right],
\]

(1.2)

where the scalar field \( \Phi_x \) is the \( N \otimes N \) hermitean matrix and the gauge field \( \Omega_{xy} \) is the \( N \otimes N \) unitary matrix. For the purposes of induction of QCD at large scales there is nothing wrong with the \( U(1) \) subgroup of \( U(N) \), therefore we leave the trace of \( \Phi \) finite, and \( \det \Omega \neq 1 \) to simplify the equations.

The first nice feature is the unique possibility to solve the model analytically at \( N = \infty \) by reducing it to the nonlinear integral equation (master field equation) for the vacuum density \( \rho(\lambda) \) of eigenvalues of the scalar field \( \Phi \), and the linear one for the fluctuation \( \delta \rho_x(\lambda) \).

The dimension \( D \) of the lattice enters as a parameter in these equations. The critical phenomena are quite rich: there are infinite number of fixed points \( \rho_\alpha(\lambda) = |\lambda|^\alpha \), with different critical indices \( \alpha = 2n + 1 \pm \frac{1}{2} \arccos \frac{D}{3D-2} \). Explicit massive solutions, interpolating between pairs of the fixed points \( \rho_\alpha(\lambda) \) and \( \rho_{2k-\alpha}(\lambda) \) were found in the last paper \[11\].

In particular, the masses of scalar excitations scale as

\[
m^2_{\text{phys}} \propto B^\delta; \quad \delta = \frac{1}{2} + \frac{k - 2m - 1}{2(\alpha - k)},
\]

(1.3)

where \( B \) is the coefficient at the perturbation

\[
\rho(\lambda) = |\lambda|^\alpha + B|\lambda|^{2k-\alpha}; \quad \alpha > k.
\]

At the critical point this coefficient vanishes, which is needed to suppress the operator of the lower scaling dimension. One would expect this \( B \) to vanish as some linear superposition of deviations of the parameters of the scalar field potential \( U(\Phi) = \frac{1}{2}m^2\Phi^2 + \frac{1}{4!}\lambda_0\Phi^4 + \ldots \) from the critical values. As it was discussed in \[11\], the multicritical point of the type \( \alpha, k, m \) is realized at \( \alpha > 2m + 1 \) when the terms \( \mathcal{O}(\Phi^{2m}) \) in the scalar potential become relevant in the wave equation.

The model, therefore, has a nontrivial thermodynamics with calculable indexes in arbitrary dimension. The forbidden interval is \( \frac{1}{2} < D < 1 \), and the solutions at \( D > 1 \)
show no pathologies, contrary to the string models. The string models can be regarded as linear realizations of the gauge symmetry in the same lattice gauge model, with the unitary measure replaced by the Gaussian one, like in the Weingarten model. This leads to the singularity at $D = 1$ with the forbidden interval $D > 1$.

The difference between the unitary and the Gaussian measure is known to produce an extra structure at a random surface in the strong coupling expansion. In the weak coupling expansion, the usual perturbative QCD is hopefully recovered in the vicinity of the unit element, which arises as the classical solution for the unitary measure, but has no special role in case of the Gaussian one.

In a way, this field theory of the field $\rho_x(\lambda)$, with extra continuum coordinate $\lambda$ represents the theory of extended objects, with infinite internal motion in continuum limit, when the support of eigenvalues extends to the whole real axis of $\lambda$. This infinite internal motion is the only hope to recover the perturbative QCD with the space independent master field. As it was mentioned in [1], the spectral integrals over $\lambda$ in the strong coupling expansion reproduce the lattice Feynman graphs of the large $N$ theory, much in the same way, as it took place in reduced Eguchi-Kawai models [14]. The correspondence between the master field of our model and quenched and twisted reduced models was discussed in the recent paper [13].

1.2 $Z_N$ problem and loop averages

These were nice features. The problems arise when one tries to introduce quarks. Within the large $N$ expansion the quarks act as infinitesimal perturbation, which do not change the vacuum of the theory. All the physics of the conventional $1/N$ expansion is described by the Wilson loops $W(C)$ in fundamental representation, averaged over QCD vacuum. The quark confinement corresponds to the global $Z_N$ symmetry of this vacuum. The dynamical realization of this symmetry in QCD is the (minimal) area law $W(C \to \infty) \sim \exp(-\sigma A(C))$ at large distances, combined with asymptotic freedom $W(C \to 0) \sim 1$ at small ones.

It is not clear, how to get this behavior of the Wilson loop in our model. As it was noted already in the first paper [1], and discussed at length in subsequent papers [8, 9, 10], the effective action for the gauge field after elimination of the heavy scalars involves only absolute values of the Wilson loops. This leads to the spurious local $Z_N$ symmetry in this model, as well as in any other model, built from the traces in the adjoint representation of the gauge group. As a consequence, we get the local confinement: the Wilson loop vanishes, unless the loop folds on itself, so that the minimal area equals zero:

$$W(C) = \delta_{0,A(C)}$$ (1.5)

3Actually, we get the local $U(1)$ symmetry, but we could always lower it to $Z_N$ by fixing the determinant of $\Omega$. 

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Formally, this corresponds to infinite string tension $\sigma = \infty$.

Recently, the loop equations and the Eguchi-Kawai reduction were studied for our model in [13]. For the quadratic potential $U(\Phi)$, the solution of the loop equations was found, in which the adjoint Wilson loop, as well as the fundamental one, obey the trivial "zero area law" ([13]), up to $O\left(\frac{1}{N^2}\right)$ corrections. The path amplitude of scalar field,

$$\langle \Phi | G(L) | \Phi \rangle \equiv \left\langle \Phi_{x_0} \prod_{k=0}^{L-1} \Omega_{x_k,x_{k+1}} \otimes \Omega^\dagger_{x_k,x_{k+1}} \Phi_{x_L} \right\rangle = \Lambda^L \left\langle \text{tr} \Phi^2 \right\rangle,$$  \hspace{1cm} (1.6)

where the matrix $\Phi$ is treated as a vector in adjoint representation. It is implied, that the path $\{x_0, x_1, \ldots, x_L\}$ has no backtracking steps, and should it have, those steps would not count, as the matrices would cancel. So, this amplitude depends upon the algebraic length $|\Gamma|$, $L(\Gamma)$ of the path $\Gamma$, (as opposed to the usual lattice length $|\Gamma| = \#$ of steps). The explicit formula for $\langle \text{tr} \Phi^2 \rangle$, (in our normalization of $m_0$),

$$\langle \text{tr} \Phi^2 \rangle = \frac{2(2D-1)}{m_0^2(D-1) + D \sqrt{m_0^4 + 4(1-2D)}},$$  \hspace{1cm} (1.7)

coincides with the semi-circle solution, found previously [9] from the master field equation.

The solution for $\Lambda$ reads

$$\Lambda = \sqrt{\frac{1}{4} m_0^4 + (1-2D) + \frac{1}{2} m_0^2},$$  \hspace{1cm} (1.8)

These solutions, however, fail to produce any critical phenomena, which are necessary to match perturbative QCD. Apparently, it is impossible to induce QCD without self-interaction of the scalar field (if it is possible at all!).

The usual propagator $\langle \Phi | G(x,y) | \Phi \rangle$ could be obtained from this one by summing over all paths

$$\langle \Phi | G(x,y) | \Phi \rangle = \sum_{\Gamma_{xy}} m_0^{-2|\Gamma_{xy}|} \langle \Phi | G(L(\Gamma_{xy})) | \Phi \rangle$$  \hspace{1cm} (1.9)

This sum is calculable, but the results were not presented. Presumably, they would agree with the solution of the corresponding wave equation [3, 11], inverting the Gaussian part of effective Lagrangean for $\delta \rho_x(\lambda)$.

The same quantities could be found for arbitrary potential from the master field equation, by noting, that the one-link integrals factorize over the path, so that the general invariant propagator

$$\langle \lambda | G(L) | \mu \rangle \equiv \left\langle \frac{1}{\lambda - \Phi_{x_0}} \prod_{k=0}^{L-1} \Omega_{x_k,x_{k+1}} \otimes \Omega^\dagger_{x_k,x_{k+1}} \frac{1}{\mu - \Phi_{x_L}} \right\rangle \hspace{1cm} (1.10)$$

$$= \left\langle \frac{1}{\lambda - \Phi_{x_0}} \prod_{k=0}^{L-1} (\Omega_{x_k,x_{k+1}} \otimes \Omega^\dagger_{x_k,x_{k+1}}) \frac{1}{\mu - \Phi_{x_L}} \right\rangle \Phi.$$
where $\langle \rangle_\Omega$ denotes the one-link averages with the measure (1.2), at fixed scalar field,  
and $\langle \rangle_\Phi$ denotes the scalar field average. As before, $\Phi$ is treated as a vector in adjoint 
representation. We could write the following identity,

$$
\Omega_{xy} \otimes \Omega_{xy}^\dagger \Omega = I \frac{1}{\lambda - \Phi_y} I \frac{1}{\lambda - \nabla \Phi_x} I = G_{\lambda}(\Phi_x)
$$

where $\nabla \Phi \equiv \frac{1}{N} \frac{\partial}{\partial \Phi}$,

$$
G_{\lambda}(\Phi) = (\lambda - \nabla \Phi_x - F(\Phi_x))^{-1},
$$

and

$$
F(\Phi) = \left[ \frac{1}{N} \frac{\partial \ln I [\Phi_x, \Phi_y]}{\partial \Phi_x} \right]_{\Phi_y = \Phi_x} = \frac{U'(\Phi)}{2D} - \frac{1}{D} \varphi \int d\nu \frac{\rho(\nu)}{\nu - \Phi}.
$$

The details could be found in previous papers, or below, where we repeat the same 
computations for the mixed model. The function $G_{\lambda}(\Phi)$ was computed at $N = \infty$ in [2]

$$
G_{\lambda}(\mu) = \frac{1}{\lambda - R(\mu)} \Re \exp \left( \int \frac{d\nu}{\pi(\nu - \mu - i0)} \arctan \frac{\pi \rho(\nu)}{\lambda - R(\nu)} \right),
$$

with

$$
R(\nu) = \frac{U'(\nu)}{2D} + \frac{D - 1}{D} \varphi \int d\mu \frac{\rho(\mu)}{\nu - \mu}.
$$

Let us represent this function in terms of dispersion integral

$$
G_{\lambda}(\Phi) = \varphi \int d\mu \frac{\langle \mu | \Lambda | \lambda \rangle}{\mu - \Phi}
$$

with the kernel $\langle \mu | \Lambda | \lambda \rangle$ given by inverse dispersion relation

$$
\langle \nu | \Lambda | \lambda \rangle = \varphi \int \frac{d\mu}{\pi^2} \frac{G_{\lambda}(\mu)}{\nu - \mu},
$$

then we see, that the one-link average acts as the operator $\Lambda$

$$
\langle \Omega_{xy} \otimes \Omega_{xy}^\dagger \rangle_\Omega = \int d\mu \langle \mu | \Lambda | \lambda \rangle \frac{1}{\mu - \Phi_x}
$$

which yields the following result for the propagator

$$
\frac{\langle \nu | G(L) | \mu \rangle}{N} = \int d\lambda \int d\phi \frac{\rho(\phi)}{(\nu - \phi)(\lambda - \phi)} \langle \lambda | \Lambda^L | \mu \rangle
$$

In general, there is always the trivial eigenvalue $\Lambda_0 = 1$, corresponding to the unit matrix 
as an eigenfunction. The other eigenvalues must be less than 1, otherwise there would be a phase transition.

In particular, according to [3]

$$
\langle \Omega_{xy} \otimes \Omega_{xy}^\dagger \rangle_\Omega \Phi_y = F(\Phi_x)
$$
For the quadratic potential, according to [5], $F(\Phi) = \Lambda \Phi$, so that there is a trivial linear eigenfunction $\Phi$ with the eigenvalue $\Lambda$. This is in complete agreement with the loop equations.

The same arguments lead to the following result for the adjoint loop average

$$W^a(C) = \delta_{0,L(C)} + \frac{1}{N^2} \left[ -1 + \left( 1 - \delta_{0,L(C)} \right) \text{tr} \Lambda^{L(C)} \right]$$

(1.21)

The leading term at $N = \infty$, as before, arises for the backtracking loop, and it corresponds to the infinite string tension, regardless the solution for the scalar field density. The $\frac{1}{N^2}$ correction describes the perimeter law for the pointlike "mesons", propagating along the adjoint (double) path. The wave equation, effectively summing over all these paths, was derived in the previous papers [3, 11].

As for the fundamental Wilson loop, in the large $N$ limit, we could take the $SO(2N+1)$ gauge group instead of the $SU(N)$. The large $N$ saddle point equations would be the same, up to $O\left(\frac{1}{N}\right)$ corrections. It is also known, that the loop equations [14] of the complete gauge theory, with the Yang-Mills or Wilson terms, coincide with the same accuracy for the $SU(N)$ and $SO(2N+1)$ gauge groups.

On the other hand, there is no center in the $SO(2N+1)$ group, hence the Wilson loop would not vanish. In virtue of the factorization property [14] $\langle \text{tr} A \text{tr} B \rangle = \langle \text{tr} A \rangle \langle \text{tr} B \rangle + O(1)$ the fundamental Wilson loop would be equal to the square root of the adjoint Wilson loop, in agreement with suggestion of [9]. At $N = \infty$, therefore, both satisfy the zero area law, unless we invoke the subtle mechanism of the spontaneous generation of the fundamental Wilson loops in effective gauge action [7, 9, 13], which is not clear how to do.

### 1.3 Mixed model

At this point it is worth recalling, that our model is somewhat artificial. There was no physical reason in the choice of the adjoint representation of the matter field [1] to induce QCD. We did so, simply because at that time this seemed to be the only model with correct counting of powers of $N$, which could be solved by orthogonal polynomial technique. Later, when the model was actually solved, the more powerful technique of the Riemann-Hilbert equations was found. Now, when we are no longer limited to the Itzykson-Zuber integral, the time came to extend the model.

We add to the model the heavy fermion constituent field, $\Psi_x$, with $N_f \ll N \rightarrow \infty$ flavor components, and solve this mixed model exactly. The physical quarks $q_x$ could be introduced later, at larger spatial scales. The $Z_N$ symmetry, as we show, can be spontaneously broken in this solution, therefore the physical, light quarks would be able to propagate in this vacuum.

By adjusting the parameters, we could interpolate between the local confinement and
free quark regimes, which gives us more chances to induce QCD. At least, the most obvious objection is now eliminated.

The functional integral of the mixed model reads

$$\prod_x \int D\Phi_x \exp -N \text{tr} \left[ U(\Phi_x) \right] \int D\Psi_x \exp N \text{tr} \left[ M\Psi_x \bar{\Psi}_x \right] \int D\Omega \exp N \text{tr} \left[ \Phi_x \Omega_{xy} \Phi_y \Omega_{xy}^\dagger + \Omega_{xy} \Psi_H \bar{\Psi}_x + \Omega_{xy}^\dagger \Psi_H \bar{\Psi}_y \right],$$

where $\Psi_x(\bar{\Psi}_x)$ is $N \otimes N'(N' \otimes N)$ matrix. The second dimension $N' = N_s N_f$ where $N_f$ is the number of flavors, and $N_s = 2^{\lfloor D/2 \rfloor}$ is the number of spin components. The matrices $H_{x,x+\mu} = \frac{1}{2} (1 + \gamma_\mu)$ act on the spin components.

The physical quark confinement in this case would correspond to the situation, where the masses of the $\bar{\Psi}q, \bar{\Psi}\Psi$ mesons would stay in the lattice cutoff range in the local limit, when the masses of physical $\bar{q}q$ mesons go to zero in lattice units. In terms of the Wilson loops, there would always be the perimeter law, with large decrement, due to $\bar{\Psi}$ following the quark along the loop, but hopefully there would also be the term with area law, due to the induced gluon planar graphs filling the loop of the light quark.

## 2 Collective fields and classical dynamics at large $N$

### 2.1 Two densities

The solution of the mixed model starts with the observation, that one-link integrals

$$I = \int d\Omega \exp N \text{tr} \left[ \Phi_1 \Omega \Phi_2 \Omega^\dagger + \Omega \Psi_2 H_{12} \bar{\Psi}_1 + \Omega^\dagger \Psi_1 H_{21} \bar{\Psi}_2 \right],$$

depend, in virtue of the gauge invariance, only upon the densities $\rho_1, \rho_2, \sigma_1, \sigma_2$, where

$$\rho_x(\lambda) = \frac{1}{N} \text{tr} \delta(\lambda - \Phi_x), \quad \langle i\alpha|\sigma_x(\lambda)|j\beta \rangle = \frac{1}{N} \text{tr} \delta(\lambda - \Phi_x) \Psi_x^{i\alpha} \bar{\Psi}_x^{j\beta}.$$  

with the spin indexes $\alpha, \beta$ and flavor indexes $i, j$ fixed. We shall treat $\sigma$ as $N' \otimes N'$ matrix.

This allows one to completely eliminate $\Psi$ from the problem, by the local change of variables\(^4\). Let us introduce the $N' \otimes N'$ matrix Lagrange multiplier $\epsilon(\lambda)$ for the constraint (2.3), then we have to compute the integral

$$\int D\Psi \int D\sigma \int D\epsilon \exp \left[ \text{tr}' \left( \Psi \epsilon(\lambda) \bar{\Psi} \right) - N \int d\lambda \text{tr}\epsilon(\lambda)\sigma(\lambda) \right],$$

\(^4\)This simple observation, applied to the old induced model, with quarks in place of $\Psi$, immediately rules out the hypothesis, that it could induce QCD without the spontaneous breaking of $Z_N$ symmetry.
where $\text{tr}'$ corresponds to the spin and flavor trace, and the $\epsilon$ integration goes along imaginary axis. The notation $\epsilon(\Phi)$ is used for the matrix-valued function, in practice this is used in the basis where $\Phi$ is diagonal, where it means the diagonal matrix of $\epsilon(\lambda_i)$, where $\lambda_i$ are the eigenvalues of $\Phi$.

Integrating over $\Psi$ first, we find

$$
\int D\Psi \exp \left[ \text{tr}' \left( \Psi \epsilon(\Phi) \bar{\Psi} \right) \right] \propto \prod_{i=1}^{N} \det \epsilon(\lambda_i) = \exp \left[ N \int d\lambda \rho(\lambda) \text{tr}' \ln \epsilon(\lambda) \right]. \tag{2.5}
$$

This yield the extra local term in effective action for the $\epsilon, \sigma$ variables,

$$
\delta S_{\text{eff}}(\sigma, \epsilon, \rho) = N \int d\lambda \text{tr}' (\epsilon(\lambda) \sigma(\lambda) - \rho(\lambda) \ln \epsilon(\lambda)). \tag{2.6}
$$

The local field $\epsilon(\lambda)$ can be eliminated from equations of motion

$$
\epsilon(\lambda) = \rho(\lambda) (\sigma(\lambda))^{-1}, \tag{2.7}
$$

which yields

$$
\delta S_{\text{eff}}(\sigma, \rho) = \text{const} + N \int d\lambda \rho(\lambda) (-N' \ln \rho(\lambda) + \text{tr}' \ln \sigma(\lambda)). \tag{2.8}
$$

The extra term in effective action for the $\rho$ field was computed in [3]. It starts from $O(N)$ terms in the large $N$ limit, which can be neglected in the leading order under consideration, provided $1 \ll N_f$. Therefore, the above term in effective action serves both densities.

### 2.2 Classical equations

We assume, that $1 \ll N_f \ll N$, so that the flavor corrections are important, but still the classical equations for the $N' \otimes N'$ matrix $\sigma$ could be applied. Our objective now is to derive the set of equations for the one-link integral, and solve them together with these classical equations. For the space independent master fields $\rho, \sigma$ the classical equations read

$$
\frac{2D}{N^2} \frac{d}{d\lambda} \frac{\delta \ln I}{\delta \rho(\lambda)} + 2\Re V'(\lambda) = U'(\lambda) + \frac{1}{N} \left( -N' \frac{\rho'(\lambda)}{\rho(\lambda)} + \text{tr}' \frac{\sigma'(\lambda)}{\sigma(\lambda)} \right), \tag{2.9}
$$

$$
\frac{2D}{N^2} \frac{d}{d\lambda} \frac{\delta \ln I}{\delta \sigma(\lambda)} = -M + \frac{1}{N} \rho(\lambda) (\sigma(\lambda))^{-1}. \tag{2.10}
$$

The first classical equation follows from variation of the total action with respect to $\psi(\lambda)$, where $\delta \rho = \psi'(\lambda)$, the extra derivative being introduced to preserve the normalization condition $\int d\lambda \rho(\lambda) = 1$. The boundary conditions are $\psi(\pm \infty) = 0$. This is the same equation as the old one, except for the last term, coming from effective action. The potential

$$
V'(z) = \int d\nu \frac{\rho(\nu)}{z - \nu}. \tag{2.11}
$$
and the values at the real axis are understood as limits from the upper half plane

\[
V'(\lambda + i0) = -\pi \rho(\lambda) + \varphi \int d\nu \frac{\rho(\nu)}{\lambda - \nu}
\] (2.12)

The second classical equation is new. It follows from the variation of the total action in \(\sigma(\lambda)\), with spinor and flavor indexes implied everywhere. The mass term and the scalar density are proportional to the unit matrix, and the second term in general case involves the inverse of \(\sigma\) matrix. We assume, that the vacuum densities \(\rho_x(\lambda)\) and \(\sigma_x(\lambda)\) are spatially homogeneous and the second one is proportional to the unit matrix, in virtue of the symmetry of the model.

### 2.3 Schwinger-Dyson identities

Apart from these classical equations, which are valid only at \(N \to \infty\), there are some identities, which are satisfied by the one-link integral. The simplest is the one, used in [2] to solve the scalar model,

\[
\frac{1}{I} \text{tr} (\lambda - \nabla \Phi_1)^{-1} I = \text{tr} (\lambda - \Phi_2)^{-1},
\] (2.13)

where

\[
(\nabla \Phi)_{ij} = \frac{1}{N} \frac{\partial}{\partial \Phi_{ji}}.
\] (2.14)

One may rewrite this identity as follows

\[
\text{tr} (\lambda - \nabla \Phi_1 - F(\Phi_1))^{-1} = \text{tr} (\lambda - \Phi_2)^{-1},
\] (2.15)

where

\[
F(\Phi_1) = \nabla \Phi_1 \ln I,
\] (2.16)

is the matrix-valued function. In the same way, as in [2, 3], we find

\[
2D F(\lambda) = \frac{2D}{N^2} \frac{d}{d\lambda} \frac{\delta \ln I}{\delta \rho_1(\lambda)} = U'(\lambda) - 2\Re V'(\lambda) + \frac{1}{N} \left( -N' \frac{\rho(\lambda)}{\rho(\lambda)} + \text{tr} \frac{\sigma'(\lambda)}{\sigma(\lambda)} \right)
\] (2.17)

In order to compute another variation of \(I\), we note, that in virtue of unitarity of \(\Omega\),

\[
\text{tr} (\lambda - \nabla \Phi_1)^{-1} \frac{\partial I}{\partial \Psi_1} \Psi_1 = \text{tr} (\lambda - \Phi_2)^{-1} \Psi_2 \frac{\partial I}{\partial \Psi_2},
\] (2.18)

with sum over spins and flavors implied. On the other hand, from definition of \(\sigma(\lambda)\) (for \(\Psi = \Psi_1, \Psi_2\))

\[
\frac{\partial I}{\partial \Psi} = \frac{1}{N} \frac{\delta I}{\delta \sigma(\Phi)}, \quad \frac{\partial I}{\partial \Psi} = \frac{1}{N} \frac{\delta I}{\delta \sigma(\Phi)} \Psi,
\] (2.19)
which yields, after moving $I$ to the left, shifting $\nabla \rightarrow \nabla + F$, replacing the trace on the right by the spectral integral,

$$\frac{1}{N} \text{tr} (\lambda - \nabla \phi - F(\Phi))^{-1} \eta(\Phi) = \int d\mu \rho(\mu) \frac{\eta(\mu)}{\lambda - \mu}$$  \hspace{1cm} (2.20)

with

$$\eta(\mu) = \frac{1}{N^2} \text{tr} \sigma(\mu) \frac{\delta \ln I}{\delta \sigma(\mu)}$$  \hspace{1cm} (2.21)

From now on we are going to deal with classical solution $\sigma(\lambda)$ which has no indices in spin space. Using the classical equation (2.10)

$$\eta(\mu) = \frac{N'}{2D} \left(-M\sigma(\mu) + \frac{1}{N} \rho(\mu)\right).$$  \hspace{1cm} (2.22)

We cannot directly represent the left of (2.20) in terms of the spectral integral, as there are derivatives $\nabla \phi$, which prevent us from diagonalizing $\Phi$. The problem is of the same kind, as in the pure induced model, which we solved before. We have an identical problem in the present model, in the scalar sector, described by (2.13), so let us solve it first.

### 2.4 First master field equation

Let us introduce the matrix-valued function $G^0_\lambda(\Phi)$ as solution of the differential equation

$$\nabla \phi G^0_\lambda(\Phi) = -1 + (\lambda - F(\Phi)) G^0_\lambda(\Phi),$$  \hspace{1cm} (2.23)

then we have the following normalization condition from (2.13)

$$\text{tr} \left(G^0_\lambda(\Phi) + \frac{1}{\Phi - \lambda}\right) = 0$$  \hspace{1cm} (2.24)

which holds identically for any $\lambda$. We choose $\lambda$ to belong to the support of the eigenvalues, and take the real solution for $G^0$, corresponding to the principle value prescription.

In the large $N$ limit, the derivative $\nabla \phi$ acts as the following integral [2]

$$\nabla \phi G^0_\lambda(\Phi) = \int d\mu \rho(\mu) \frac{G^0_\lambda(\mu) - G^0_\lambda(\Phi)}{\mu - \Phi}$$  \hspace{1cm} (2.25)

which reduces the problem to the Riemann-Hilbert integral equation

$$1 + \varphi \int d\mu \rho(\mu) \frac{G^0_\lambda(\mu)}{\mu - \nu} = (\lambda - R(\nu))G^0_\lambda(\nu),$$  \hspace{1cm} (2.26)

$$R(\nu) = F(\nu) + \Re V''(\nu) = \frac{U'(\nu)}{2D} + \frac{D - 1}{D} \Im V'(\nu) + \frac{N_f N_s}{2N D} \left(\frac{\rho'(\nu)}{\rho(\nu)} - \frac{\sigma'(\nu)}{\sigma(\nu)}\right).$$  \hspace{1cm} (2.27)
For the analytic function
\[ T^0_\lambda(z) = 1 + \int d\mu \rho(\mu) \frac{G^0_\lambda(\mu)}{\mu - z}, \] (2.28)
which is defined in the upper half plane and extended to the whole plane by the symmetry relation
\[ T_\lambda(z) = \overline{T_\lambda(z)}, \] (2.29)
this is the boundary problem
\[ \frac{T^0_\lambda(\nu + i0)}{T^0_\lambda(\nu - i0)} = \frac{\lambda - R(\nu) + i\pi \rho(\nu)}{\lambda - R(\nu) - i\pi \rho(\nu)} \] (2.30)
with the well known solution
\[ T^0_\lambda(z) = \exp \left( \int \frac{d\nu}{\pi(\nu - z)} \arctan \frac{\pi \rho(\nu)}{\lambda - R(\nu)} \right). \] (2.31)
The solution for \( G^0_\lambda \) can be obtained from the real part of the complex conjugate boundary values
\[ T^0_\lambda(\nu \pm i0) = (\lambda - R(\nu) \pm i\pi \rho(\nu)) G^0_\lambda(\nu); \quad G^0_\lambda(\nu) = \frac{\Re T^0_\lambda(\nu + i0)}{\lambda - R(\nu)}. \] (2.32)

As long as we are interested only in the equation for density, we do not need this expression, but rather can use the asymptotic formula
\[ \lim_{z \to \infty} z \left( 1 - T^0_\lambda(z) \right) = \int d\mu \rho(\mu) G^0_\lambda(\mu) = \Re V'(\lambda), \] (2.33)
where the last relation followed from (2.24). On the other hand, we can easily find the same quantity from the explicit solution for \( T^0_\lambda(z) \) which yields the first master field equation
\[ \varphi \int d\mu \left( \frac{\pi \rho(\mu)}{\mu - \lambda} + \arctan \frac{\pi \rho(\mu)}{\lambda - R(\mu)} \right) = 0, \] (2.34)
which differs from the old one by the last term in definition (2.27) of \( R(\lambda) \).

2.5 The second master field equation

Let us now derive the second master field equation. Repeating the same steps, we find for the new function
\[ G_\lambda(\Phi) = (\lambda - \nabla \Phi - F(\Phi))^{-1} \eta(\Phi) \] (2.35)
the following differential equation
\[ \nabla \Phi G_\lambda(\Phi) = -\eta(\Phi) + (\lambda - F(\Phi)) G_\lambda(\Phi), \] (2.36)
and the normalization condition
\[ \int d\mu \rho(\mu) G_\lambda(\mu) = \varphi \int d\mu \frac{\rho(\mu) \eta(\mu)}{\lambda - \mu} \] (2.37)
Furthermore, at \( N = \infty \) we arrive at the integral equation

\[
\eta(\nu) + \varphi \int d\mu \rho(\mu) \frac{G_\lambda(\mu)}{\mu - \nu} = (\lambda - R(\nu))G_\lambda(\nu),
\]  

(2.38)

with the same \( R(\nu) \) as before. The first term adds a new element to the problem. Now, the analytic function

\[
T_\lambda(z) = i \int \frac{d\mu}{\pi} \frac{\eta(\mu)}{z - \mu} + \int d\mu \rho(\mu) G_\lambda(\mu) \mu - \nu = (\lambda - R(\nu))G_\lambda(\nu),
\]  

(2.39)

in the upper half plane of \( z \), and it is continued to the lower half plane by the symmetry, as before. The boundary problem

\[
T_\lambda(\nu \pm i0) = (\lambda - R(\nu) \pm i\pi \rho(\nu)) G_\lambda(\nu) \pm i \varphi \int \frac{d\mu}{\pi} \frac{\eta(\mu)}{\nu - \mu},
\]  

(2.40)

is now an inhomogeneous one. It can be solved by substitution

\[ T_\lambda(z) = T_\lambda^0(z)J_\lambda(z), \]

(2.41)

which provides the following boundary problem for \( J_\lambda \),

\[
(\lambda - R(\nu) \pm i\pi \rho(\nu)) \left( J_\lambda(\nu \pm i0)G_\lambda^0(\nu) - G_\lambda(\nu) \right) = \pm i \varphi \int \frac{d\mu}{\pi} \frac{\eta(\mu)}{\nu - \mu}. \]

(2.42)

This equation defines the imaginary part

\[
\Im J_\lambda(\nu + i0) = \frac{1}{G_\lambda^0(\nu)} \frac{(\lambda - R(\nu))}{(\lambda - R(\nu))^2 + \pi^2 \rho^2(\nu)} \varphi \int \frac{d\mu}{\pi} \frac{\eta(\mu)}{\nu - \mu}. \]

(2.43)

so that we could restore \( J_\lambda \) from dispersion relation

\[
J_\lambda(z) = \int \frac{d\nu}{\pi} \frac{\Im J_\lambda(\nu + i0)}{\nu - z}. \]

(2.44)

There are no subtraction terms here, as this function should decrease as \( z^{-1} \) to satisfy the normalization condition. The coefficient in front of \( z^{-1} \) can be found from above equations; this yields the following relation

\[
\int \frac{d\nu}{\pi} \Im J_\lambda(\nu + i0) = -\varphi \int \frac{d\nu}{\pi} \eta(\nu) + \varphi \int \frac{d\nu}{\lambda - \nu} \eta(\nu) \]

(2.45)

The real part of this relation provides us with the second master field equation

\[
\varphi \int d\nu \left( \frac{\rho(\nu)\eta(\nu)}{\lambda - \nu} - \frac{1}{G_\lambda^0(\nu)} \frac{(\lambda - R(\nu))}{(\lambda - R(\nu))^2 + \pi^2 \rho^2(\nu)} \varphi \int \frac{d\mu}{\pi^2} \frac{\eta(\mu)}{\nu - \mu} \right) = 0 \]

(2.46)

The imaginary part yields

\[
0 = \int d\nu \eta(\nu) \]

(2.47)
or, substituting the explicit formulas,

\[ \int d\nu \sigma(\nu) = \frac{1}{N^M} \]  

(2.48)

which is quite a surprise. The ”heavy quark condensate” is trivially related to the bare mass. This is, actually, the lattice artifact, nothing to do with critical phenomena. It is convenient to renormalize \( \sigma(\lambda) \rightarrow \frac{1}{N^M} \sigma(\lambda) \), \( \eta(\lambda) \rightarrow \frac{N^N}{2D M} \eta(\lambda) \), so that

\[ \eta(\lambda) = \rho(\lambda) - \sigma(\lambda); \quad \int d\nu \rho(\nu) = \int d\nu \sigma(\nu) = 1. \]  

(2.49)

The expression for \( R(\lambda) \) would not change, as it involves only the logarithmic derivative of \( \sigma(\lambda) \), and the above equation for \( \eta(\lambda) \) would not change, being linear.

Now, it is easy to see, that there is always the \( Z_N \) symmetric solution

\[ \sigma(\lambda) = \rho(\lambda) \]  

(2.50)

of the master field equations, with the same scalar density \( \rho(\lambda) \) as before. The extra term in \( R(\lambda) \) exactly vanishes in this case.

This is the old vacuum. One may readily check, that the average \( \langle \Omega \rangle_\Omega = 0 \) in this case. In general case, this average is linearly related to \( \eta(\lambda) \). As for the adjoint averages, those would be the same, as before, as it follows from above equations. The \( \frac{N^N}{N} \) corrections drop from \( R(\lambda) \), and \( \rho(\lambda) \) is the same, so the heavy fermions decouple at infinite \( N \) and small \( \frac{N_f}{N} \).

However, with proper adjustment of parameters of the scalar potential, the linear master field equation for \( \eta(\lambda) \) (with \( \sigma = \rho \) in \( R \)) could have a nonzero solution. This is the spontaneous breaking of the \( Z_N \) symmetry in our model. After this bifurcation point, the new vacuum, with two different densities would be stable. Presumably, this one does induce QCD, but that remains to be seen.

Let us stress once again, that this solution does not apply to the Veneziano limit \( N_f \sim N \), as in this case the classical equations for the matrix field \( \sigma \) are no longer valid. However, the first correction in \( \frac{N_f}{N} \) which we found, already breaks the \( Z_N \) symmetry of the vacuum.

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4  Note added

When this paper was finished, a new paper [15] appeared, where the trivial $D = 1$ and Gaussian solutions were discussed at great length. Also, the completeness of the Schwinger-Dyson equations, leading to the master field equation, was questioned. Unfortunately, this objection is based on a misunderstanding. The authors of [15] forgot about the gauge invariance, in virtue of which the Itzykson-Zuber integral $I[\Phi_x, \Phi_y]$ depend on $\text{tr} \Phi^n_x$, $\text{tr} \Phi^m_y$, but cannot depend upon $\text{tr} \Phi^n_x \Phi^m_y$, as they suggest. The subgroup $P_N$ of the gauge group independently permutes the eigenvalues $\Phi^{(i)}_x$ and $\Phi^{(j)}_y$, which eliminates the terms like $\sum_j \Phi^{(j)}_x \Phi^{(j)}_y$.

So, the gauge invariance plays the role of the "mixed" Schwinger-Dyson equations, they were worried about. With the gauge invariant Ansatz, the Schwinger-Dyson equations unambiguously determine the Taylor expansion of the logarithmic derivative $F$ of the Itzykson-Zuber integral, as it was discussed in [2]. Actually, the uniqueness of the reconstruction of $F$ is irrelevant, since we do not solve the Schwinger-Dyson identities for $F$, but rather substitute the classical equation for $F$ into these equations, to obtain the master field equation for $\rho$. All we need here, is the correct Schwinger-Dyson equation, the completeness is not used. As for the other "lessons" from the Gaussian and one-dimensional models, I doubt their relevance to the problem of induction of QCD.

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