Abstract: In this talk we review the harmonic space formulation of the twistor transform for the supersymmetric self-dual Yang-Mills equations. The recently established harmonic-twistor correspondence for the N-extended supersymmetric gauge theories is described. It affords an explicit construction of solutions to these equations which displays a remarkable matreoshka-like structure determined by the $N=0$ core.

1 Introduction

The Yang-Mills self-duality (SDYM) equations are well known Lorentz invariant four dimensional exactly solvable nonlinear systems. Remarkably, these equations afford generalisation to the super self-duality equations for extended super Yang-Mills theories without spoiling their integrability properties. The extended super self-duality equations are therefore further examples of exactly solvable Lorentz invariant four dimensional systems; and the Penrose-Ward twistor transform [1], so sucessful for the self-duality equations in complexified four-dimensional space, may be generalised to extended superspaces. The original twistor transform and its supersymmetric generalisations have been found to have a clear and tractable formulation in the language of “harmonic spaces”. We therefore call them “harmonic-twistor correspondences”.

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For the N-extended supersymmetric self-duality equations, moreover, this harmonic space formulation of the twistor transform reveals a remarkable “matreoshka”-like structure: Much of the structure of an N-extended self-dual theory is determined by its lower-N sub-theory; and ultimately, by the non-supersymmetric N=0 core. In particular, given any solution of the N=0 self-duality equations, its most general supersymmetric extension may be recursively constructed. The problem of finding the general local solution of the $N > 0$ super self-duality equations therefore reduces to finding the general solution of the N=0 self-duality equations. The latter completely determines the general N=1 solution, which in turn determines the N=2 solution, and so on. A further consequence of the matreoshka phenomenon is the vanishing of many conserved currents for super self-dual systems, for instance the vanishing of the Yang-Mills stress tensor for N=0 self-dual fields is reflected in the vanishing of the extended supergauge theory supercurrents which contain the stress tensor and its superpartners.

Harmonic (super)spaces contain additional coordinates: harmonics or twistors, which we denote by commuting spinors $u^{\pm}_\alpha$. The origin of this enlargement is the fact that harmonic spaces are cosets of the (super) Poincare group by some subgroup of the rotation group, whereas customary (super)space coordinates parametrise the coset of the (super) Poincare group by the entire rotation group. For global considerations harmonics need to be considered as coordinates on the four-dimensional (super)conformal group factored by its maximal parabolic subgroup. In this talk, however, we limit ourselves to local aspects of the self-duality equations.

Originally, harmonic superspaces were introduced as appropriate tools for the construction of unconstrained off-shell N = 2 and 3 super Yang-Mills theories; and involved the ‘harmonisation’ of the internal unitary groups of supersymmetry, with each particular case ($N = 2, 3$) requiring individual consideration. For the (super) self-duality restrictions, however, one harmonises the rotation group instead. This being N-independent, the harmonisation is universal; and in contrast to the previous aim of constructing off-shell theories, the main aim of the study of the self-dual restrictions is the investigation of the on-shell theory, viz. to solve the (super) self-duality equations of motion.

The self-duality conditions have recently attracted a great deal of renewed interest in view of their reductions to lower-dimensional completely integrable systems and the prospect of unifying lower-dimensional solution methods under the banner of the SDYM twistor transform. The programme has by now advanced rather far, with most known integrable systems having been rederived by the abovementioned reduction. Moreover, there have also appeared papers dealing with reductions of super self-duality equations. Our considerations suggest the interesting possibility that completely integrable supersymmetric systems are merely further layers of the self-dual matreoshka.

The main purpose of these lecture notes is to review the harmonic-space formulation of the twistor transform. In section 2 we discuss this formulation for the N=0 case. In section 3 we discuss the super self-duality conditions and in section 4 we review the generalisation of the harmonic-twistor correspondence to N-extended super
self-duality equations for all $N > 0$. The latter yields, in particular, a representation of all possible symmetries of these equations, including an important subgroup of diffeomorphisms of the analytic subspace of harmonic superspace. In section 5 we discuss the solution matreoshka: Given an N=0 solution, we show that a purely algorithmic procedure yields solutions of higher N theories.

2 Self-duality as harmonic space analyticity

The usual self-duality condition for the Yang-Mills field strength

$$F_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma},$$

basically says that the (0,1) part of the gauge field vanishes. This is better expressed in terms of 2-spinor notation in the form: $f_{\dot{a} \dot{\beta}} = 0$ which is equivalent to the statement that the field strengths curvature only contains the (1,0) Lorentz representation, i.e.

$$[\mathcal{D}_{\dot{a} \dot{\alpha}}, \mathcal{D}_{\dot{\beta} \dot{\beta}}] = \epsilon_{\dot{a} \dot{\beta}} f_{\dot{a} \dot{\beta}}.$$

Now multiplying (2) by two commuting spinors $u_{\dot{\alpha}}$, $u_{\dot{\beta}}$ mentioned in the Introduction, one can compactly represent it as the vanishing of a curvature

$$[\nabla_{\dot{\alpha}}^+, \nabla_{\dot{\beta}}^+] = 0,$$

where $\nabla_{\dot{\alpha}}^+ \equiv u_{\dot{\alpha}}^+ \nabla_{\dot{a} \dot{\alpha}}$, with linear system

$$\nabla_{\dot{\alpha}}^+ \phi = 0.$$

This is precisely the Belavin-Zakharov-Ward linear system for SDYM. Now the $u_{\dot{\alpha}}$ are actually harmonics [4] on $S^2$ and it is better to consider these equations in an auxiliary space (‘harmonic space’) with coordinates $\{x^{\pm \alpha} \equiv x^{\dot{\alpha} \dot{\alpha}} u_{\dot{\alpha}}^+, u_{\dot{\alpha}}^+: u_{\dot{\alpha}}^+ u_{\dot{\alpha}}^- = 1\}$, where the harmonics are defined up to a $U(1)$ phase (see [4, 2, 3]), and gauge covariant derivatives

$$\nabla_{\dot{\alpha}}^+ = \partial_{\dot{\alpha}}^+ + A_{\dot{\alpha}}^+ = \partial_{\dot{x}^{\dot{-\alpha}}} + A_{\dot{\alpha}}^+. $$

In this space (3) is actually not equivalent to the self-duality conditions. We also need

$$[D^{++}, \nabla_{\dot{\alpha}}^+] = 0,$$

where $D^{++}$ is a harmonic space derivative which acts on negatively-charged harmonic space coordinates to yield their positively-charged counterparts, i.e. $D^{++} u_{\dot{\alpha}}^- = u_{\dot{\alpha}}^+$, $D^{++} x^{-\alpha} = x^{+\alpha}$, whereas $D^{++} u_{\dot{\alpha}}^+ = D^{++} x^{\alpha} = 0$. In ordinary $x$-space, when the harmonics are treated as parameters, the condition (6) is actually incorporated in the definition of $\nabla_{\dot{\alpha}}^+$ as a linear combination of the covariant derivatives. The system (3,6) is now equivalent to SDYM and has been considered by many authors, e.g. [8, 9, 10, 11, 12]; the equivalence holding in spaces of signature (4,0) or (2,2), or in complexified space. In this regard, we should note that for real spaces, our understanding is completely clear for the Euclidean signature. For the (2,2) signature, the situation is richer and more
intricate due to the noncompact nature of the rotation group. On the one hand, there appear infinite dimensional representations, and on the other hand, novel subgroups (in particular, the parabolic ones) as well as new cosets (some of them rather intriguing). Our present considerations concern only those signature (2,2) configurations which may be obtained by Wick rotation of (4,0) configurations.

Now, in (6) the covariant derivative (5) has pure-gauge form
\[ \nabla^+ = \partial^+ + \varphi \partial^+ \varphi^{-1}, \tag{7} \]
and \( D^{++} \) is ‘short’ i.e. has no connection. This choice of frame is actually inherited from the four-dimensional x-space and is not the most natural one for harmonic space. We may however change coordinates to a basis in which \( \nabla^+ \alpha \) is ‘short’ and \( D^{++} \) is ‘long’ (i.e. acquires a Lie-algebra-valued connection) instead. Namely,
\[ \nabla^+_\alpha = \partial^+_\alpha, \]
\[ D^{++} = D^{++} + V^{++}, \tag{8} \]
a change of frame tantamount to a gauge transformation by the ‘bridge’ \( \varphi \) in (4). In this basis the SDYM system (3,6) remarkably takes the form of a Cauchy-Riemann (CR) condition
\[ \frac{\partial}{\partial x^{-\alpha}} V^{++} = 0 \tag{9} \]
expressing independence of half the x-coordinates. In virtue of passing to this basis the nonlinear SDYM equations (1) are in a sense trivialised: Any ‘analytic’ (i.e. satisfying (9)) function \( V^{++} = V^{++}(x^{+\alpha}, u^{\pm}) \) corresponds to some self-dual gauge potential. From any such \( V^{++} \), by solving the linear equation
\[ D^{++} \varphi = \varphi V^{++} \tag{10} \]
for the bridge \( \varphi \), a self-dual vector potential may be recovered from the harmonic expansion:
\[ \varphi \partial^+_\alpha \varphi^{-1} = u^{+\dot{\alpha}} A_{\alpha \dot{\alpha}}; \tag{11} \]
the linearity in the harmonics \( u^{+\dot{\alpha}} \) being guaranteed by (6). An important comment: It follows from (10) that
\[ D^{++} \det \varphi = \det \varphi \ tr V^{++}. \]
Therefore, for semisimple gauge groups (\( tr V^{++} = 0 \)) we have
\[ D^{++} \det \varphi = 0. \tag{12} \]
We may therefore either solve (11) for a unimodular bridge, or without worrying about the determinant we may substract traces in (11) when calculating the connection.

Solving (11) for an arbitrary analytic gauge algebra valued function \( V^{++} \) yields the general self-dual solution. This correspondence between self-dual gauge potentials and holomorphic prepotentials \( V^{++} \) is just a transparent formulation of the Penrose-Ward twistor correspondence for SDYM and is a convenient tool for the explicit construction
of local solutions of the self-duality equations. For instance the 1-instanton BPST solution

\[ A^j_{\alpha\dot{a}i} = \frac{1}{\rho^2 + x^2} \left( \frac{1}{2} x_{\alpha\beta} \delta^j_i + \epsilon_{ia} x^j_{\dot{a}} \right), \]  

(13)
corresponds to the analytic function \([13, 14]\)

\[ (u^{++j})_i = \frac{x^{ij} x^j_i}{\rho^2} \]  

(14)
via the bridge

\[ (\varphi_0)_i^j = \left( 1 + \frac{x^2}{\rho^2} \right)^{-\frac{i}{2}} \left( \delta^j_i + \frac{x^{ij} x^j_i}{\rho^2} \right). \]  

(15)

Furthermore, in the analytic subspace of harmonic space (with coordinates \(\{x^{+\alpha}, u^+_\alpha\}\)), there exists an especially simple presentation of the infinite-dimensional symmetry group acting on solutions of the self-duality equations. It is the (apparently trivial) transformation \(V^{++} \to V^{++'} = g^{++}\), where \(g^{++}\) depends in an arbitrary way on \(V^{++}\) and its derivatives as well as on the analytic coordinates themselves, modulo gauge transformations \(V^{++} \to e^{-\lambda}(V^{++} + D^{++})e^\lambda\), where \(\lambda\) is also an arbitrary analytic function. The situation is the same for any extended supersymmetric gauge theory, as we discuss in section 4.

### 3 Super self-duality

Since extended super Yang-Mills theories are massless theories, the components are classified by helicity and we have the following representation content in theories up to \(N=3\):

\[
\begin{array}{cccccc}
\text{helicity} & 1 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & -1 \\
N = 0 & f_{\alpha\beta} & & & & & \\
N = 1 & f_{\alpha\beta} & \lambda_\alpha & & \lambda_{\dot{a}} & f_{\dot{a}\dot{b}} & \\
N = 2 & f_{\alpha\beta} & \lambda^i_\alpha & W_i & \lambda_{\dot{a}i} & f_{\dot{a}\dot{b}} & \\
N = 3 & f_{\alpha\beta} & \lambda^i_\alpha & W_i & \chi_{\dot{a}} & \chi_\alpha & W^i & \lambda_{\dot{a}i} & f_{\dot{a}\dot{b}} & \end{array}
\]  

(16)

In real Minkowski space fields in the left and right triangles are related by CPT conjugation but in complexified space or in a space with signature (4,0) or (2,2), we may set fields in one of the triangles to zero without affecting fields in the other triangle. If we set all the fields in the right (left) triangle to zero, the equations of motion reduce to the super (anti-) self-duality equations. For instance, the equations of motion for the \(N=3\) theory take the form

\[
\begin{align*}
\epsilon^{\dot{a}\gamma} & \mathcal{D}_{\alpha\gamma} f_{\dot{a}\dot{b}} + \epsilon^{\beta\gamma} \mathcal{D}_{\gamma\dot{b}} f_{\alpha\beta} = \{\lambda_{\alpha i}, \lambda^i_{\dot{b}}\} + \{\chi_{\alpha i}, \chi_{\dot{b}}\} + [W_i, \mathcal{D}_{\alpha\beta} W^j] + [W^i, \mathcal{D}_{\alpha\beta} W_i] \\
\epsilon^{\dot{a}\gamma} & \mathcal{D}_{\alpha\gamma} \lambda_{\dot{a}i} = -\epsilon_{ijk} [\lambda^j_{\alpha i}, W^k] + [\lambda_{\alpha i}, W_i] \\
\epsilon^{\gamma\dot{a}} & \mathcal{D}_{\gamma\dot{a}} \lambda^i_{\beta j} = -\epsilon_{ijk} [\lambda^j_{\beta i}, W^k] + [\chi_{\beta j}, W^i] \\
\epsilon^{\gamma\dot{a}} & \mathcal{D}_{\gamma\dot{a}} \chi_{\dot{a}} = -[\lambda^i_{\dot{a}}, W^k] \\
\epsilon^{\gamma\dot{a}} & \mathcal{D}_{\gamma\dot{a}} \chi_\alpha = -[\lambda^i_{\alpha}, W^k] \\
\mathcal{D}_{\alpha\beta} & \mathcal{D}^{\alpha\beta} W_i = -2 [[W^j, W_i], W^j] + [[W^j, W^j], W_i] + \frac{1}{2} \epsilon_{ijk} [\lambda^j_{\alpha i}, \lambda^k_{\dot{a}j}] + \{\lambda_{\alpha i}, \chi_{\dot{a}j}\} \\
\mathcal{D}_{\alpha\beta} & \mathcal{D}^{\alpha\beta} W^i = -2 [[W^j, W^i], W^j] + [[W^j, W^j], W^i] + \frac{1}{2} \epsilon_{ijk} [\lambda^j_{\dot{a}i}, \lambda_{\dot{a}k}] + \{\lambda^i_{\dot{a}i}, \chi_\alpha\}
\end{align*}
\]  

(17)
On setting the fields in the right-hand triangle to zero, we obtain
\begin{align}
\epsilon_{\beta\gamma} \mathcal{D}_{\alpha\beta} f_{\alpha\beta} &= 0 \\
\epsilon_{\gamma\beta} \mathcal{D}_{\gamma\beta} \lambda_{\beta} &= 0 \\
\epsilon_{\lambda\delta} \mathcal{D}_{\alpha\gamma} \lambda_{\delta} &= -[\lambda_{\alpha}, W_{k}] \\
\mathcal{D}_{\alpha\beta} \mathcal{D}^{\alpha\beta} W_{i} &= \frac{1}{2} \epsilon_{ijk} \{\lambda_{\alpha}, \lambda_{\alpha}^k\}. 
\end{align} 
\tag{18}

We see that the spin 1 source current actually factorises into parts from the two triangles, so it manifestly vanishes for super self-dual solutions. The first equation in (18) is just the Bianchi identity for self-dual field-strengths. So apart from the self-duality condition (1), we have one equation for zero-modes of the covariant Dirac operator in the background of a self-dual vector potential (having (1) as integrability condition) and two further non-linear equations. However, as we shall describe in the following sections, any given self-dual vector potential actually determines the general (local) solution of the rest of the equations. This is the most striking consequence of the matreoshka phenomenon: the N=0 core determining the properties of the higher-N theories. Another consequence is that many conserved currents identically vanish in the super self-dual sector. For instance, since self-duality always implies the source-free second order Yang-Mills equations, the spin 1 source current vanishes for the entire matreoshka. Moreover, the usual Yang-Mills stress tensor clearly vanishes for self-dual fields:

\[ T_{\alpha\beta} \equiv f_{\alpha\beta} f_{\alpha\beta} = 0; \]

and as a consequence of this, once one has put on further layers of the matreoshka, the supercurrents generating supersymmetry transformations, which contain the stress tensor as well as its superpartners also identically vanish for super self-dual fields. In fact, just as the stress tensor factorises into parts from the two triangles in (16), all the supercurrents also factorise in this way. This is best seen in superspace language. The full (non-self-dual ) super Yang-Mills theories are conventionally described using super field-strengths defined by the following curvature constraints

\[ N = 1 : \quad [\mathcal{D}_{\alpha}, \mathcal{D}_{\alpha\beta}] = \epsilon_{\alpha\beta} w_{\alpha}, \quad [\mathcal{D}_{\beta}, \mathcal{D}_{\alpha\beta}] = \epsilon_{\beta\alpha} \bar{w}_{\beta}; \]

\[ N = 2 : \quad \{\mathcal{D}_{\alpha i}, \mathcal{D}_{\beta j}\} = \epsilon_{ij} \epsilon_{\alpha\beta} W, \quad \{\mathcal{D}_{\alpha i}, \mathcal{D}_{\beta j}\} = \epsilon_{ij} \epsilon_{\alpha\beta} \bar{W}; \]

\[ N = 3 : \quad \{\mathcal{D}_{\alpha i}, \mathcal{D}_{\beta j}\} = \epsilon_{ijk} \epsilon_{\alpha\beta} W^k, \quad \{\mathcal{D}_{\alpha i}, \mathcal{D}_{\beta j}\} = \epsilon_{ijk} \epsilon_{\alpha\beta} \bar{W}^k. \]
\tag{19}

In terms of these superfields the supercurrents take the form

\[ N = 1 : \quad V_{\alpha} = w_{\alpha} \bar{w}_{\alpha}; \]

\[ N = 2 : \quad V = W \bar{W}; \]

\[ N = 3 : \quad V^i_j = W^i \bar{W}^j - \frac{1}{3} \delta^i_j W^k \bar{W}^k; \]
\tag{20}

and the super self-duality equations (eqs. (18) and their lower-N truncations) take the compact forms

\[ \bar{w}_{\alpha} = 0, \quad W = 0, \quad W^k = 0; \]
\tag{21}

which manifestly demonstrate the vanishing of the supercurrents (21).
4 Super self-duality as harmonic space analyticity

In N-independent form, (21) can be conveniently written as the following restrictions of the conventional representation-defining constraints for super Yang-Mills [15]:

\[
\begin{align*}
\{\mathcal{D}_i^\alpha, \mathcal{D}_j^\beta\} &+ \{\mathcal{D}_i^\alpha, \mathcal{D}_\beta^i\} = 0 \\
\{\mathcal{D}_{ai}, \mathcal{D}_{bj}\} &\equiv 0 = [\mathcal{D}_{ai}, \nabla_{a\beta}] \\
\{\mathcal{D}_{aj}, \mathcal{D}_i^\beta\} &\equiv 2\delta_j^i \nabla_{a\beta} \\
\end{align*}
\]

(22)

In harmonic superspaces with coordinates

\[
\{x^{\pm\alpha} \equiv u^{\pm\alpha} \hat{x}^{\hat{\alpha}}, \bar{\vartheta}_{\pm i} \equiv u^{\pm\alpha} \bar{\vartheta}_{\hat{a}^i}, \vartheta^{\alpha i}, u^{\pm\alpha}\};
\]

these take the form

\[
\begin{align*}
\{\mathcal{D}_{ai}, \mathcal{D}_{bj}\} &\equiv 0 = \{\bar{\mathcal{D}}^{+i}, \bar{\mathcal{D}}^{+j}\} \\
[\nabla_{+\alpha}, \nabla_{+\beta}] &\equiv 0 = [\bar{\mathcal{D}}^{+i}, \nabla_{+\alpha}] \\
\{\mathcal{D}_{aj}, \mathcal{D}_i^+\} &\equiv 2\delta_j^i \nabla_{+\alpha} \\
[\mathcal{D}_{ai}, \nabla_{+\alpha}] &\equiv 0,
\end{align*}
\]

(23)

where the gauge covariant derivatives are given by

\[
\begin{align*}
\mathcal{D}_{ai} &\equiv \mathcal{D}_{ai} + A_{ai} \\
\mathcal{D}^{+i} &\equiv \mathcal{D}^{+i} + A^{+i} \\
\nabla_{+\alpha} &\equiv \partial_{+\alpha} + A_{+\alpha},
\end{align*}
\]

(24)

and satisfy the equations

\[
[D^{++}, \mathcal{D}_{ai}] = [D^{++}, \bar{\mathcal{D}}^{+i}] = [D^{++}, \nabla_{+\alpha}] = 0.
\]

(25)

The equations (23,24) are equivalent to (22) and (23) are consistency conditions for the following system of linear equations

\[
\begin{align*}
\mathcal{D}_{ai}\varphi &\equiv 0 \\
\mathcal{D}^{+i}\varphi &\equiv 0 \\
\nabla_{+\alpha}\varphi &\equiv 0,
\end{align*}
\]

(26)

This system is extremely redundant, \(\varphi\) allowing the following transformation under the gauge group

\[
\varphi \rightarrow e^{-\tau(x^{a \hat{a}}, \bar{\vartheta}_{\hat{a}^i}^{\hat{\alpha}})\varphi}e^{\lambda(x^{a \hat{a}}, \bar{\vartheta}_{\hat{a}^i}^{\hat{\alpha}})},
\]

(27)

where \(\tau\) and \(\lambda\) are arbitrary functions of the variables shown, without affecting the constraints (23). These constraints therefore allow an economic choice of chiral-analytic basis in which the bridge \(\phi\) and the prepotential \(V^{++}\) depend only on positively \(U(1)\)-charged, barred Grassmann variables, viz. \(\bar{\vartheta}_{\hat{a}^i}^{\hat{\alpha}}\), being independent of \(\vartheta^{\alpha i}\) and \(\bar{\vartheta}_{\hat{a}^i}^{-}\). In this basis, \(\varphi\) too is independent of \(\vartheta^{\alpha i}\) and \(\bar{\vartheta}_{\hat{a}^i}^{-}\); its non-analyticity manifesting itself in its dependence on \(x^{-\alpha}\). Moreover, consistently with the commutation relations (23), the covariant spinor derivatives take the form \(\mathcal{D}_{ai} = \frac{\partial}{\partial x^{a \hat{a}}}, \bar{\mathcal{D}}^{i} = 2\bar{\vartheta}^{ai}\nabla_{+\alpha}^{+}\). The super self-duality conditions (23,24) are therefore equivalent to the same system of equations as the \(N=0\) SDYM equations, viz. (18), except that now \(\varphi\) and \(A_{+\alpha}\) are superfields.
depending on \( \{x^\pm\alpha, \bar{\vartheta}_i^\pm, u_\alpha^\pm\} \). As for the N=0 case, we may express this system in the form of analyticity conditions for the harmonic space connection superfield \( V^{++} \):

\[
\frac{\partial}{\partial x^{-\alpha}} V^{++}(x^\alpha, \bar{\vartheta}^+ i, u_\alpha^\pm) = 0.
\] (28)

The super SDYM systems are thus equivalent to the CR-like conditions (28); and fields solving for instance (18) may be obtained by inserting solutions \( \varphi \) of the equation

\[
D^{++} \varphi(x^\pm\alpha, \bar{\vartheta}_i^\pm, u_\alpha^\pm) = \varphi(x^\pm\alpha, \bar{\vartheta}_i^+ u_\alpha^\pm) V^{++}(x^\alpha, \bar{\vartheta}_i^+, u_\alpha^\pm)
\] (29)

into the expression

\[
\varphi \partial^+_\alpha \varphi^{-1} = u^{+\alpha} A_{\alpha\bar{\alpha}}(x^{\alpha\bar{\alpha}}, \bar{\vartheta}_i^\pm),
\] (30)

(the left side being guaranteed to be linear in \( u^+ \)), and expanding the superfield vector potential on the right thus:

\[
A_{\alpha\bar{\beta}}(x, \bar{\vartheta}) = A_{\alpha\bar{\beta}}(x) + \bar{\vartheta}^\lambda \lambda_{\alpha}(x) + \epsilon^{ijk} \bar{\vartheta}_{\alpha\beta} \bar{\vartheta}_{\alpha\gamma} \nabla_{\alpha\beta} W_k(x) + \epsilon^{ijk} \bar{\vartheta}_{\alpha\beta} \bar{\vartheta}_{\alpha\gamma} \nabla_{\alpha\beta} \chi_{\gamma},
\] (31)

to obtain the component multiplet satisfying (18). In fact as we have already mentioned, any N=0 solution completely and recursively determines its higher-N extensions. We shall describe this solution matreoshka in the next section.

The most general infinite-dimensional group of transformations of super-self-dual solutions acquires a transparent form in the analytic harmonic superspace with coordinates \( \{x^\pm\alpha, \bar{\vartheta}^+ i, u_\alpha^\pm\} \). As for the \( N = 0 \) case (see the comment at the end of sec.2) it is given by the transformation

\[
V^{++} \rightarrow V^{++'} = g^{++}(V^{++}, x^\alpha, \bar{\vartheta}^+ i, u_\alpha^\pm),
\] (32)

where \( g^{++} \) is an arbitrary doubly \( U(1) \)-charged analytic algebra-valued functional, modulo gauge transformations \( V^{++} \rightarrow e^{-\lambda}(V^{++} + D^{++})e^\lambda \), where \( \lambda \) is also an arbitrary analytic function. This group has an interesting subgroup of transformations

\[
V^{++} \rightarrow V^{++'} = V^{++}(x^+ i', \bar{\vartheta}^+ i', u'),
\] (33)

induced by diffeomorphisms of the analytic harmonic superspace

\[
x^\alpha' = x^\alpha'(x^+ i, \bar{\vartheta}^+, u), \bar{\vartheta}^+ i' = \bar{\vartheta}^+ i'(x^+ i, \bar{\vartheta}^+, u), u' = u'(x^+ i, \bar{\vartheta}^+, u).
\] (34)

It would be of value to know how this group is realised in ordinary superspace and how it contains the Bäcklund transformations of (14), which correspond to a class of transformations (32) with \( g^{++} = g^{++}(V^{++}) \), a functional of \( V^{++} \) only.

As we have seen, the equation (28) encodes all the super SDYM systems, independently of the extension \( N \). The action for SDYM suggested by (10) is therefore immediately generalisable to arbitrary \( N \) thus:

\[
S = \int d^4x \ d\bar{\vartheta}_1^+ \ldots d\bar{\vartheta}_N^+ \ d^2u \ \text{tr} \ (\partial^+ \alpha \zeta_\alpha^{-3-N} \varphi^{-1} D^{++} \varphi),
\] (35)

which on varying the auxilliary field \( \zeta_\alpha^{-3-N} \) yields the CR condition (28). Although this Lagrange multiplier appears to be dynamical, it does not represent any additional
physical degrees of freedom because of the following argument due to [16]. On varying \( \varphi \), we obtain
\[
\varphi^{-1} D^{++} [\varphi \partial^{+\alpha} \zeta^{3-N}_\alpha \varphi^{-1}] = 0
\]
which is actually tantamount to
\[
\partial^{+\alpha} \zeta^{3-N}_\alpha = 0.
\]
All local solutions of this equations have the form \( \zeta^{3-N}_\alpha = \partial^{+}_\alpha y^{-4-N} \) with arbitrary \( y^{-4-N} \). However \( \zeta^{3-N}_\alpha \) occurs in the action via \( \partial^{+\alpha} \zeta^{3-N}_\alpha \), so it is defined only modulo the addition of \( \partial^{+\beta} y^{3-N}_{[\alpha\beta]} \). This arbitrariness in \( \zeta \) precisely balances its degree of freedom, so the action \( (35) \) describes no unwanted propagating modes. The action for the N=1 theory presented in [2] is just \( (35) \) in a different coordinate frame.

5 The solution matreoshka

We now discuss the solution of \( (29) \). Our main result is that given a solution of the N=0 equation \( (10) \), which we rewrite as
\[
D^{++} \varphi_b(x^{\pm\alpha}, u^{\pm}_\alpha) = \varphi_b(x^{\pm\alpha}, u^{\pm}_\alpha) v^{++}(x^{+\alpha}, u^{\pm}_\alpha),
\]
the solution of the supersymmetric system can be completely determined. Let us consider an N=1 bridge \( \varphi \) in the form
\[
\varphi = e^{\bar{\vartheta}^+ \varphi^-(x^{\pm\alpha}, u^{\pm}_\alpha)} \varphi_b(x^{\pm\alpha}, u^{\pm}_\alpha),
\]
where \( \varphi_b \) is some (presumably to be known) solution of \( (30) \), and
\[
V^{++}(x^{+\alpha}, \bar{\vartheta}^+, u^{\pm}_\alpha) = v^{++}(x^{+\alpha}, u^{\pm}_\alpha) + \bar{\vartheta}^+ v^+(x^{+\alpha}, u^{\pm}_\alpha),
\]
some arbitrary analytic superfield. In virtue of \( (29) \) the unknown function \( \psi^- \) satisfies
\[
D^{++} \psi^- = \varphi_b v^+ \varphi_b^{-1},
\]
a first-order equation in which the right-hand side is some known function. It is therefore manifestly integrable, determining the N=1 bridge \( \varphi \) from which the superfield vector potential may be obtained:
\[
A^{+\alpha}_\alpha = - \partial^{+\alpha} \varphi_b \varphi_b^{-1} - \bar{\vartheta}^+ \nabla^{+\alpha} \psi^-.
\]
The coefficient of \( \bar{\vartheta}^+ \) in the above superfield vector potential is precisely the spinor field \( \lambda_\alpha \) satisfying the Dirac equation in the background of component vector potential \( A^{+\alpha}_\alpha = - \partial^{+\alpha} \varphi_b \varphi_b^{-1} \). This N=1 bridge may now be dressed up to an N=2 bridge:
\[
\varphi = e^{\bar{\vartheta}^+ \varphi^-_2 (\bar{\vartheta}^+ x^{\pm\alpha})} e^{\bar{\vartheta}^+ \varphi^-_1 (x^{\pm\alpha})} \varphi_b = (1 + \bar{\vartheta}^2 \varphi^-_2 (x^{\pm\alpha}) + \bar{\vartheta}^2 \bar{\vartheta}^1 + \varphi^- \varphi^-_{21} (x^{\pm\alpha})) (1 + \bar{\vartheta}^1 \varphi^- / (x^{\pm\alpha})) \varphi_b(x^{\pm\alpha})
\]
and the N=2 analytic prepotential may be expanded thus
\[
V^{++}(x^{+\alpha}, \bar{\vartheta}^i) = \left( v^{++} + \bar{\vartheta}^{+1} v^+_1 (x^{+\alpha}) \right) \left( v^{++} + \bar{\vartheta}^{+1} v^+_2 (x^{+\alpha}) \right).
\]
Once again, in virtue of (29) the unknown functions in this ansatz for \( \varphi \) satisfy first-order equations which afford explicit integration; and the N=2 super self-dual multiplet may be explicitly constructed. Now given an N=2 solution we can promote it to an N=3 solution using the matreoshukan ansatz

\[
\varphi = e^{\beta^3 + \psi^-_3 (\beta^{2+}, \beta^{1+}, x^{\pm \alpha})} e^{\beta^2 + \psi^-_2 (\beta^{1+}, x^{\pm \alpha})} e^{\beta^1 + \psi^-_1 (x^{\pm \alpha})} \varphi_b, \tag{43}
\]

where \( \psi^-_3 \) and \( \psi^-_2 \) are N=2 and N=1 superfields respectively; this form clearly breaking the internal SU(3) invariance, just as the N=2 ansatz (41) breaks the internal SU(2) invariance. Expanding the superfield \( \psi^-_2 \) as in (41) and \( \psi^-_3 \) as follows:

\[
\psi^-_3 (\bar{\theta}^+_2, \bar{\theta}^+_1, x^{\pm \alpha}) = \psi^-_3 (x^{\pm \alpha}) + \bar{\theta}^{2+} \psi^-_{32} (x^{\pm \alpha}) + \bar{\theta}^{1+} \psi^-_{31} (x^{\pm \alpha}) + \bar{\theta}^{2+} \bar{\theta}^{1+} \psi^-_{321} (x^{\pm \alpha}),
\]

and the N=3 analytic superfield thus:

\[
V^{++}(x^{+\alpha}, \bar{\theta}^{++}) = \left( V^{++}(x^{+\alpha}) + \bar{\theta}^{1+} v^+_1 (x^{+\alpha}) + \bar{\theta}^{2+} (v^+_2 (x^{+\alpha}) + \bar{\theta}^{1+} v^++_2 (x^{+\alpha})) \right) + \bar{\theta}^{3+} (v^+_3 (x^{+\alpha}) + \bar{\theta}^{1+} v^++_3 (x^{+\alpha}) + \bar{\theta}^{2+} (v^++_3 (x^{+\alpha}) + \bar{\theta}^{1+} v^++^+_3 (x^{+\alpha}))), \tag{44}
\]

again yields a system of first-order equations for the unknown functions in (43), thus allowing the explicit construction of the N=3 self-dual multiplet. This matreoshka structure in which successively higher N-superfields are parametrised as N=1 superfields with (N-1)-superfield ‘components’ is very reminiscent of the Cayley-Dixon procedure of describing division algebras: a complex number as a complex combination of two reals, a quaternion as a complex combination of two complex numbers; and an octonion as a complex combination of two quaternions.

As an example let us take the \( \varphi_b \) for the BPST instanton (13) and the simplest \( v^+ \) linear in \( x^+ \) and having a constant spinorial parameter \( \zeta_i \) of dimension [cm]\(^{-\frac{3}{2}}\):

\[
(v^+)^i_j = x^+^j \zeta_i + x^+_i \zeta^j. \tag{45}
\]

This yields

\[
\psi^-_{ij} = \left( 1 + \frac{x^2}{\rho^2} \right)^{-1} \left( x^{-j} \zeta_i + \left( 1 + \frac{x^2}{\rho^2} \right) x^{-i} \zeta^j - \frac{1}{\rho^2} x^{-i} x^+ x^+_i \zeta^j \right), \tag{46}
\]

from which the vector potential may now be found to be

\[
A^i_{aai} = \frac{1}{\rho^2 + x^2} \left( \frac{1}{2} x_{aa} \delta^i_j + \epsilon_{iaa} x^i_j \right) + \bar{\psi}^i \rho^4 \frac{\epsilon_{iaa} \zeta^j + \delta^i_j \zeta_i}{(\rho^2 + x^2)^2}. \tag{47}
\]

In fact this is solution is related to the N=0 one we started with by a supertranslation with parameter \( \rho^2 \zeta^\alpha \):

\[
x^+^\alpha \rightarrow x^+^\alpha + \bar{\theta}^{++} \rho^2 \zeta^\alpha.
\]

Similarly, using another \( v^+ \) linear in \( x^+ \), but of the form

\[
(v^+)^i_j = x^+^p c_{pik} \epsilon^{kj}, \tag{48}
\]
where $c_{pk}$ is a totally symmetric tensor parameter having, like the parameter $\zeta$ of the previous example, dimension $[cm]^{-\frac{3}{2}}$. This yields

$$A_{\alpha\dot{\alpha}i}^{j} = \frac{1}{\rho^2 + x^2} \left( \frac{1}{2} x_{\alpha\dot{\alpha}} \delta^j_i + \epsilon_{i\alpha} x^j_{\dot{\alpha}} \right) + \bar{\eta}_\dot{\alpha} c_{\alpha in} \epsilon^{nj} \left( 1 + \frac{x^2}{\rho^2} \right),$$

(49)

a potential not related to the N=0 one by any symmetry transformation. This simple solution, however, does not vanish asymptotically.

Now choosing a $v^+$ quadratic in $x^+$ with constant spinorial parameter $\bar{\eta}^{\dot{\alpha}}$ of dimension $[cm]^{-\frac{5}{2}}$:

$$(v^+)_{ij} = x^+ j x^+ i u_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}},$$

yields the self-dual vector potential

$$A_{\alpha\dot{\alpha}i}^{j} = \frac{1}{\rho^2 + x^2} \left( \frac{1}{2} x_{\alpha\dot{\alpha}} \delta^j_i + \epsilon_{i\alpha} x^j_{\dot{\alpha}} \right) + \bar{\eta}_\dot{\alpha} \rho^4 (\rho^2 + x^2)^2 (\epsilon_{i\alpha} x^j_{\dot{\beta}} - \delta^j_{\dot{\alpha}} x^i_{\dot{\beta}}) \bar{\eta}^{\dot{\beta}},$$

(50)

related to the N=0 one by a superconformal transformation with parameter $\rho^2 \bar{\eta}_\dot{\alpha}$

$$x^{+\alpha} \rightarrow x^{+\alpha} \left( 1 - \rho^2 \bar{\eta}_\dot{\alpha} u^{-\dot{\alpha}} \bar{\eta}^+ \right)$$

(51)

and is precisely the solution discussed by [17].

These are just some particular examples of our solution generating technique [3]; our method, however, describes all local solutions of the super self-duality equations.

6 Conclusion

To conclude we mention some prospects of this approach to self-duality. The vanishing supergauge supercurrents are just the non-gravitational sources for the spin 2 field in supergravity theories. This indicates that the situation in self-dual supergravity is very similar and that our matreshka is part of a much larger, albeit more intricate, supergravity matreshka. This gives rise to the prospect of obtaining hyper-kähler manifolds with additional spinorial structure.

Going in the other direction, recent interest in self-duality has concentrated around the Ward conjecture [3] that all lower dimensional solvable systems are reductions of SDYM; and our solution matreshka promises to yield new (supersymmetric) solvable systems, together with their solutions, by truncation of the analytic data. This would yield a unification of the various existing methods of solving two dimensional systems as different manifestations of the harmonic-twistor correspondence for SDYM.

As we have seen, the spin 1 source currents of all super self-dual theories vanish because they factorise into parts from the two triangles in [13]. It turns out that we can solve the full (non-self-dual) super Yang-Mills equations, in other words restore these source currents, by intermingling self-dual and anti-self-dual holomorphic data [13]; and this works exactly for the N=3 case. Work on the explicit construction of non-self-dual N=3 solutions is in progress.

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