Thermodynamics of Quantum Isolated Horizons with model Hamiltonians

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Following a recent proposal, we consider the most general structure possible for the Hamiltonian operator associated with the Quantum Isolated Horizon (QIH) with explanations of the underlying physical motivations. After a brief overview of the microcanonical ensemble results, thermodynamic analysis with this model Hamiltonian is presented considering usual canonical energy ensemble for fixed number of punctures. It is shown that for a quantum spacetime admitting a thermodynamically stable QIH as its internal boundary, there must exist a specific bound on the Barbero-Immirzi parameter if the entropy of the QIH obeys the Bekenstein-Hawking Area Law. Arguing that the known classical results must follow in the correspondence limit, the model is fixed, yielding the energy spectrum of the QIH.

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I. INTRODUCTION

The classical Isolated Horizon (IH) is a null inner boundary of spacetime, foliated by 2-spheres, with specific local properties consistent with General Theory of Relativity and is a generalization of the teleological concept of event horizons to more realistic and dynamical situations that are expected to occur in Nature [1–5]. The quantization of IH phase space [6, 7] in the Loop Quantum Gravity (LQG) framework provides the topology of Quantum Isolated Horizon (QIH), at a particular time slice, to be that of 2-sphere punctured by the spin network describing the bulk quantum geometry. The QIH degrees of freedom belong to the Hilbert space of an SU(2) Chern-Simons (CS) theory coupled to the punctures, acting as sources. Recently it has been shown [8, 9] that the microcanonical entropy of a QIH can be completely written in terms of the two macroscopic parameters, \(k\) and \(N\), which fully characterizes the macrostates of the QIH, applying standard statistical mechanical methods and using the knowledge of the physical degrees of freedom of the QIH belonging to the Hilbert space of the CS theory. To mention, \(k\) is the level of the CS theory and \(N\) is the total number of punctures on the QIH. It has been also shown that the Barbero-Immirzi (BI) parameter(\(\gamma\)) [10–13] must be bounded within a certain range of values for the Bekenstein-Hawking area law (BHAL) [14] to be valid. However, by studying a system in the microcanonical ensemble we only get a statistical viewpoint. The true thermodynamic analysis of a system begins only in the canonical or grand canonical ensemble where thermal fluctuations are allowed. But, to proceed, we need to know the Hamiltonian and the corresponding energy spectrum associated with the QIH and there is none in the present literature. Although there is a well established notion of classical energy associated with the IH which obeys first law [3], but in the quantum theory there is no known Hamiltonian or any energy spectrum associated with a QIH resulting from a true quantization. Hopefully one can quantize the classical energy associated with IH [3] someday. In this work, we do not attempt to do this quantization. Instead, based upon well motivated physical arguments we propose the most general structure of the Hamiltonian that a QIH can have and perform an extensive thermodynamic analysis, especially concerning stability of QIHs. Considering the model Hamiltonian to yield the known results of classical thermodynamics, the unknown coefficients are suitably chosen and hence we obtain the Hamiltonian operator and the corresponding energy spectrum of a thermodynamically stable QIH.

Here is an outline of the subject matter of this paper. In section (II), we discuss the model Hamiltonian, hence the corresponding energy spectrum, for QIH proposed in a companion paper [15], based on the knowledge of the theory of QIH and area operator in LQG framework. The structure of the Hamiltonian ensures that it is gauge invariant, self-adjoint and commutes with the area operator implying that the mean area of the QIH is a constant of motion, which signifies that the constant classical area property of IH is emergent. To be specific, the energy contribution from a single puncture is written as a power series in the area contribution from a single puncture. In section (III), we review the results of [8, 9] in a nutshell and

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recast the microcanonical entropy in terms of mean energy and number of punctures of the QIH. In section (IV), we construct the canonical partition function for a spacetime admitting a QIH as its inner boundary by the thermal holographic approach introduced in [16] and generalized for the grand canonical ensemble in [17]. Having the explicit structure of the Hamiltonian and the corresponding energy spectrum at our disposal we perform the thermodynamic analysis, especially the entropy, the expression for the equilibrium temperature and the specific heat are calculated. It is shown that the BI parameter must have specific bound for a thermodynamically stable QIH obeying the BHAL, depending on the energy spectrum of the QIH. The expressions for the temperature and specific heat being plagued with the unknown coefficients of the model, we make suitable choices of these coefficients so as to match the expression of equilibrium temperature with some known results of the classical theory. Firstly, in section (IV A), a simplified model of the energy spectrum is studied where the single puncture energy contribution is related to the single puncture area contribution by a power law only, which helps us fixing the coefficients. In section (IV B), we study a special case of this power law spectrum considering its importance in current literature. In section (IV C), using the determined coefficients back into the general energy spectrum the analysis is completed. As a result we obtain the proper bound on the BI parameter for thermodynamically stable QIH obeying BHAL, the expression for the equilibrium temperature and finally the general Hamiltonian operator and corresponding energy spectrum of the QIH. Finally, we conclude with a discussion in section (V) explaining the relevance of this work as far as the literature of black hole thermodynamics is concerned.

II. MODEL HAMILTONIAN AND ENERGY SPECTRUM OF QIH

Even though the first law associated with an IH [3] gives us the notion of a well defined classical energy associated with the horizon, there has never been an attempt to deal with the energy spectrum of a QIH until recently, in [18], it has been proposed that the area spectrum of the QIH, with some scale factor is itself the energy spectrum of the QIH. But, the proposal was based on some ad hoc arguments and semiclassical approximations and above all, QIHs with such energy spectrum has been shown to be thermodynamically unstable [19]. This particular case will be discussed in details in one of the forthcoming sections with separate attention in view of its importance in current literature.

In this work, we shall not deal with the quantization of the classical energy, but we shall take the quantum theory as the starting point and consider a model Hamiltonian operator for the QIH, hence obtaining a well defined energy spectrum for the same, proposed in a companion paper [15]. The proposed model will be fixed so as to match the known classical results. In this section, we shall review and discuss the relevant details and physical motivations behind considering the model Hamiltonian proposed in [15].

As we begin with the quantum theory, the available knowledge at our disposal and relevant in this context are the quantum geometric area operators and area spectrum in LQG [20, 21] and the theory of QIH [6, 7]. The area operator is a gauge invariant, self-adjoint observable in LQG defined for any arbitrary two dimensional surface \((S)\) embedded in the three dimensional spatial manifold \((\Sigma)\) obtained from a specific foliation of the four dimensional spacetime manifold \((M \equiv R \otimes \Sigma)\) by some preferred time evolution vector field \([20, 21]\). The area operator and the corresponding spectrum for a QIH is completely known to us. For a given number of punctures \((N)\), the Hilbert space structure of the QIH is given by \(\mathcal{H}_{q_1} \otimes \cdots \otimes \mathcal{H}_{q_N}\), where \(q_i\) is the spin at the \(l\)-th puncture and \(1/2 \leq q_i \leq k/2, \forall l \in [1, N], k\) being the level of the associated CS theory [6, 7, 22] and ‘Inv’ stands for gauge invariance.

For a given spin sequence, the area spectrum is given by

\[
\hat{A}|j_1, \cdots, j_N\rangle = 8\pi c^2 g_p^2 \sum_{l=1}^{N} \sqrt{j_l(j_l+1)}|j_1, \cdots, j_N\rangle
\]

(1)

The structure of the Hilbert space of the QIH prompts us to write down the area operator as

\[
\hat{A} \equiv \hat{A}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{I}_{j_N} + \hat{I}_{j_1} \otimes \hat{A}_{j_2} \otimes \cdots \otimes \hat{I}_{j_N} + \cdots + \hat{I}_{j_1} \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{A}_{j_N}
\]

(2)

Looking at the description of a QIH in the LQG framework, it is quite explicit that the punctures i.e. the intersection points of the bulk spin network with the IH, are the elementary quantum building blocks of the QIH [6, 7]. It is supported by the fact that all the essential properties of the QIH are manifested at the punctures. The connections are flat everywhere on the horizon except at the punctures. The holonomies of the connection are nontrivial only along the disjoint loops enclosing the punctures, otherwise trivially equal to unity. Last but not the least, all the essential and relevant properties of the area operator (e.g. self adjointness, gauge invariance, etc.) are indeed manifested by these individual punctures or intersection points [20, 21]. This particular way of looking at issues related to QIH is due to [15] and one needs to look into
it to understand the motivations behind this viewpoint of attacking the problem initially from a quantum perspective.

Now, the scenario at once compels us to think of any operator belonging to the QIH Hilbert space as being contributed from the individual punctures and should have a structure like that of the area operator in (2). Hence, it follows that the Hamiltonian operator for the QIH should be written as

$$\hat{H}_S \equiv \hat{H}_j \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{I}_{j_N} + \hat{I}_j \otimes \hat{H}_{j_2} \otimes \cdots \otimes \hat{I}_{j_N} + \cdots + \hat{I}_j \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{H}_{j_N}$$

(3)

The area operator is a gauge invariant observable follows from the fact that when the bulk spin network pierces the surface $S$, the puncture is assigned with an $SU(2)$ spin and the area contribution from that puncture is nothing but the Casimir of the $SU(2)$ gauge group of the underlying theory [20, 21]. Hence, we can argue that the ‘powers’ of the Casimir being also gauge invariant can as well be the contribution from a single puncture to some other gauge invariant observable smeared over the QIH, say the Hamiltonian. So, we propose that the most general form of the contribution of the Hamiltonian from a single puncture to be of the form

$$\hat{H}_j \equiv \ell_p \sum_{n=0}^{\Lambda} b_n \hat{A}_j^n$$

(4)

where $\Lambda$ is a necessary cut-off and $b_n$ are coefficients with proper dimension (to be discussed shortly). Hence, the Hamiltonian operator for the QIH can now be written as

$$\hat{H}_S \equiv \ell_p \sum_{n=0}^{\Lambda} b_n \left( \hat{A}_j^n \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{I}_{j_N} + \hat{I}_j \otimes \hat{A}_j^n \otimes \cdots \otimes \hat{I}_{j_N} + \cdots + \hat{I}_j \otimes \hat{I}_{j_2} \otimes \cdots \otimes \hat{A}_j^n \right)$$

(5)

Besides the gauge invariance of the above operator, there is yet another property which follows from its construction. It is straightforward to see that it commutes with the area operator

$$\left[ \hat{H}_S, \hat{A} \right] \equiv 0$$

(6)

The crucial implication that it has, due to this commutativity, is that, if there is some evolution parameter $\tau$ with respect to which the QIH evolves, then it is evident that

$$\frac{d}{d\tau} \langle \Psi | \hat{A} | \Psi \rangle = \frac{1}{i\hbar} \langle \Psi | \left[ \hat{H}_S, \hat{A} \right] | \Psi \rangle = 0$$

(7)

i.e. the expectation value of the area operator of a QIH is a constant of motion. This is perfectly consistent with the fact that in the correspondence limit, the most crucial classical property of the corresponding classical Isolated Horizon i.e. fixed classical area, emerges as a consequence of the construction of the model Hamiltonian. Further, as a result of this commutativity in eq.(6), we can also say that the states of the quantum CS theory describing the local quantum degrees of freedom of the QIH, are simultaneous eigenstates of the area operator and the model Hamiltonian for the QIH. Thus, we have successfully justified the proposal of the model Hamiltonian for the QIH based on strong physical motivations which can be briefly restated as follows :-

- The punctures are the most fundamental and elementary constituents of the QIH which collectively provide an effective description of the IH in the correspondence limit.
- The model Hamiltonian shares all the necessary and relevant properties of the area operator e.g. gauge-invariance, self-adjointness, etc.
- The model Hamiltonian and the area operator associated with the QIH have simultaneous eigenstates which are those of the CS theory coupled to punctures.
- The structure of the model Hamiltonian ensures that the constant area property of IH emerges in the correspondence limit.

Now, from (4), the single puncture contribution to the energy spectrum can be easily written as

$$E_j = \ell_p \sum_{n=0}^{\Lambda} b_n A_j^n = \ell_p \sum_{n=0}^{\Lambda} a_n(\gamma) C_j^n$$

(8)
where \( \text{Dim}[b_n] = \ell_p^{-2n} \) and \( a_n(\gamma) = b_n(8\pi\gamma\ell_p^2)^n \). For simplicity, in our model, we consider \( n \) to take only integral values and \( n \geq 0 \) to ensure incremental monotonicity of the energy spectrum with the area contribution of a single puncture. Since this kind of model Hamiltonian or energy spectrum of the QIH has not been studied previously in literature and any property of such spectrum is hitherto unknown, to avoid any problem with the convergence of such spectrum we have used the cut-off parameter(\( \Lambda \)) \textit{a priori}. We shall see that, at least from the thermodynamic viewpoint, we can assert that such a cut-off is indeed required and should emerge automatically from a true quantization of the horizon energy, if can be done anyhow. This is because, as we proceed, the parameter \( \Lambda \) will come out to be directly related to the equilibrium temperature of the QIH and one does not expect it to diverge for thermodynamically stable systems.

Of course there remain several questions to be answered about this model Hamiltonian as far as its candidature of being the true Hamiltonian of the QIH, which will result from a true quantization of the theory, is concerned, resolving which is itself a mammoth task to complete with various technical subtleties to overcome. As far as this work is concerned, our aim is to analyze the thermodynamic properties of the QIH considering the most general structure of the Hamiltonian one can construct from a pure quantum viewpoint devoid of any classical notions.

Although the physical motivations behind the proposal of the model Hamiltonian has been discussed in details but the motivations behind the approach from quantum to classical rather than trying to quantize the classical theory is beyond the scope of this paper and are worth having separate explanations. To have a taste of the motivations behind adopting this particular viewpoint, which is meant for tackling other problems besides the issue of the Hamiltonian, as far as the theories of IH and QIH are concerned, one can see [15].

### III. MICROCANONICAL ENSEMBLE

Recently it has been argued in [9] that taking into account all possible spins on the QIH, the CS level \( k \) and the total number of punctures \( N \) are the only macroscopic parameters which characterize the macrostates of the QIH. Thus, defining the microcanonical ensemble for given \( k \) and \( N \), the microcanonical entropy of QIH in terms of these two macroscopic parameters is given by\([8, 9]\)

\[
S_{MC} = \frac{\lambda k}{2} + N\sigma
\]

(9)

The distribution for the dominant spin configuration is given by

\[
s^*_j = N(2j+1)\exp[-\lambda C_j - \sigma]
\]

(10)

For given \( k \) and \( N \), \( \lambda \) and \( \sigma \) are solutions of the following two equations:

\[
\exp[\sigma] = \sum (2j+1)\exp[-\lambda C_j]
\]

(11a)

\[
k/2 = N\sum C_j(2j+1)\exp[-\lambda C_j - \sigma]
\]

(11b)

where \( C_j = \sqrt{j(j+1)} \). In the appropriate limits \([8, 9]\) the above two equations reduce to

\[
\sigma = \log\left(\frac{2}{\lambda^2} - 1\right)
\]

(12a)

\[
k/N = \frac{8}{\lambda(2 - \lambda^2)}
\]

(12b)

We have considered here only the first order approximation. Second order calculation gives rise to logarithmic corrections[9] which is irrelevant in this analysis. The consistency of the eq.(12) demands \( \lambda^2 < 2 \) (since \( k, N > 0 \)) and arguing \( \sigma \) to be a negative definite quantity [8] the allowed range of \( \lambda \) is given by

\[
1 < \lambda < \sqrt{2}
\]

(13)

As far as the thermodynamical aspect is concerned it is necessary to write the microcanonical entropy in terms of the average area and the number of punctures, first shown in [18]. Using the relation \( k = A_{cl}/4\pi\gamma\ell_p^2 \approx A^*/4\pi\gamma\ell_p^2 \) [23] the eq.(9) can be cast in the following form

\[
S_{MC} = \frac{\lambda A^*}{8\pi\gamma\ell_p^2} + N\sigma
\]

(14)
For each given value of \( k/N \) we can find a positive value of \( \lambda \) from eq.\((12)\) such that it remains within the range given by \((13)\). Hence, there exists a unique value of \( \gamma \) given by \( \lambda/2\pi \) for each given value of \( k/N \) such that we obtain area term to obey the BHAL. Hence, the allowed range of \( \gamma \) is given by

\[
0.159 < \gamma < 0.225
\]

One can look into \cite{8} for a detailed explanation on this issue.

Now, it follows from eq.\((8)\) that the energy eigenvalue of the QIH in a state designated by the spin configuration \( \{s_j\} \) will be given by

\[
\hat{H}_S|\{s_j\}\rangle = \ell_p \sum_j \sum_{n=0}^\Lambda a_n(\gamma) s_j C_j^n |\{s_j\}\rangle
\]

One can look into \cite{9} for an explanation of how \(|\{s_j\}\rangle\) forms the eigenstates of the QIH and thus a generic quantum state of the QIH can be written as \(|\Psi_S\rangle = \sum_{\{s_j\}} c_{\{s_j\}} |\{s_j\}\rangle\). Hence, the expectation value of the Hamiltonian operator or the mean energy for the QIH is given by

\[
\langle \hat{H}_S \rangle = \langle \Psi_S | \hat{H}_S | \Psi_S \rangle
\]

\[
= \ell_p \sum_{\{s_j\}} \omega(\{s_j\}) \sum_j \sum_{n=0}^\Lambda a_n(\gamma) s_j C_j^n
\]

\[
= \ell_p N \sum_{s_j} \sum_{n=0}^\infty a_n(\gamma) (2j+1) C_j^n \exp(-\lambda C_j - \sigma)
\]

\[
= \ell_p N \frac{\lambda^2}{(2 - \lambda^2)} \sum_{n=0}^\infty a_n(\gamma) \left[ \frac{2\Gamma(n+2)}{\lambda^{2+n}} - \delta_n^0 \right]
\]

\[
= \frac{\lambda^2 F(\Lambda, \lambda, \gamma)}{(2 - \lambda^2)} \ell_p N
\]

where \( F(\Lambda, \lambda, \gamma) = \sum_{n=0}^\Lambda a_n(\gamma) \left[ \frac{2\Gamma(n+2)}{\lambda^{2+n}} - \delta_n^0 \right] \), which is obviously a positive definite function of \( \lambda \). Also one should note that \( \lambda^2 < 2 \) is the most crucial and indispensable inequality of this theory which imposes an upper bound on the parameter \( \lambda \). It follows from the validity of the eq.\((18)\) and is also the reality condition for \( \sigma \).

Now, one can show by a straightforward calculation similar to the one for deriving eq.\((18)\) that

\[
E^* = \ell_p \sum_{s_j} \sum_{n=0}^\infty a_n(\gamma) s_j^* C_j^n
\]

\[
= \ell_p N \sum_{s_j} \sum_{n=0}^\infty a_n(\gamma) (2j+1) C_j^n \exp(-\lambda C_j - \sigma)
\]

\[
= \ell_p N \frac{\lambda^2}{(2 - \lambda^2)} \sum_{n=0}^\infty a_n(\gamma) \left[ \frac{2\Gamma(n+2)}{\lambda^{2+n}} - \delta_n^0 \right]
\]

\[
= \frac{\lambda^2 F(\Lambda, \lambda, \gamma)}{(2 - \lambda^2)} \ell_p N
\]

where \( F(\Lambda, \lambda, \gamma) = \sum_{n=0}^\Lambda a_n(\gamma) \left[ \frac{2\Gamma(n+2)}{\lambda^{2+n}} - \delta_n^0 \right] \), which is obviously a positive definite function of \( \lambda \). Also one should note that \( \lambda^2 < 2 \) is the most crucial and indispensable inequality of this theory which imposes an upper bound on the parameter \( \lambda \). It follows from the validity of the eq.\((18)\) and is also the reality condition for \( \sigma \).

Now, one can show by a straightforward calculation similar to the one for deriving eq.\((18)\) that

\[
A^* = \frac{32\pi\gamma}{\lambda(2 - \lambda^2)} \ell_p N
\]

Hence, from eq.\((18)\) and eq.\((19)\) we have a relation between the mean values of energy and area of the QIH given by

\[
E^* = \frac{\xi(\Lambda, \lambda, \gamma)}{\ell_p} A^*
\]

where the quantity \( \xi(\Lambda, \lambda, \gamma) = \frac{\lambda^3 F(\Lambda, \lambda, \gamma)}{32\pi\gamma} \) can be explicitly written as

\[
\xi(\Lambda, \lambda, \gamma) = \frac{\lambda^3}{32\pi\gamma} \sum_{n=0}^\Lambda a_n(\gamma) \left[ \frac{2\Gamma(n+2)}{\lambda^{2+n}} - \delta_n^0 \right]
\]
Now, let us recollect that the $\gamma$-fit is required only to manifest the BHAL. Otherwise $S_{MC}$ will be given by eq.(14), the Lagrange multipliers being functions of $k$ and $N$. So, we shall always make the $\gamma$-fit after performing all the calculations. The microcanonical entropy without the $\gamma$-fit, given by eq.(14) can be expressed in terms of the equilibrium energy of the QIH using eq.(20) as

$$S_{MC} = \frac{\lambda}{8\pi\gamma\xi\ell_p}E^* + N\sigma$$

(22)

Therefore, inverse temperature is defined in the microcanonical ensemble as

$$\beta = \frac{\partial S_{MC}}{\partial E^*}|_N = \frac{\lambda}{8\pi\gamma\xi\ell_p}$$

(23)

which enables us to write the microcanonical entropy in the familiar thermodynamic form as

$$S_{MC} = \beta E^* + N\sigma$$

(24)

Hence, from eq.(23) and using eq.(21), the expression for the temperature can be written as

$$T = \frac{8\pi\gamma\xi\ell_p}{\lambda} = \frac{\ell_p}{4} \lambda^2 \sum_{n=0}^{\Lambda} a_n(\gamma) \left[ \frac{2\Gamma(n+2)}{\lambda^{2+n}} - \delta_n^{(0)} \right]$$

(25)

Hence, we have derived the expression for the temperature from the microcanonical ensemble thermodynamics. When we consider the system in thermal equilibrium with a heat bath at some particular temperature in the canonical ensemble, it is ensured that the common temperature is defined as above and is treated as a free parameter rather than being a derived quantity.

IV. CANONICAL ENSEMBLE

Now, to get more insight about the thermodynamical properties of QIH we shall switch over to the canonical ensemble and follow the arguments made in [16, 17]. To begin with, we must write down the all important canonical partition function for spacetimes admitting QIH as its internal boundary, which is defined as

$$Z_C = Tr(\exp{-\beta \hat{H}})$$

(26)

where we have considered an ensemble of spacetimes, admitting QIH as an inner boundary, in contact with a heat bath with inverse temperature $\beta$. $H$ is the Hamiltonian for the system. Now, unlike the classical theory, the degrees of freedom of the quantum theory are independent as far as the bulk and the boundary are concerned [6, 7]. The Hilbert space for the quantum spacetime, with QIH as the inner boundary($S$) of the bulk quantum geometry($B$), can be written as $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$. As a result we can write the Hamiltonian for the thermodynamic system as

$$\hat{H} \equiv \hat{H}_S \otimes \hat{I}_B + \hat{I}_S \otimes \hat{H}_B$$

(27)

where $\hat{I}$ are the identity operators for the corresponding Hilbert spaces designated by the suffixes. Similarly, the wave function for a quantum spacetime, admitting QIH as an internal boundary, can be written as

$$|\Psi\rangle = \sum_{S,B} C_{SB}|\Psi_S\rangle \otimes |\Psi_B\rangle$$

(28)

It should be noted that the Hilbert space, the quantum states, etc. are all results of the kinematic quantization of classical degrees of freedom in a suitable choice of time slicing. Now, from the knowledge of the Hamiltonian formulation of General Relativity(GR) we know that, GR being a covariant theory, the bulk Hamiltonian is zero. Hence, we can write the quantum version of this constraint as

$$\hat{H}_B|\Psi_B\rangle \approx 0$$

(29)
Rewriting the partition function in eq. (26) in terms of the bulk and the boundary sectors using eq. (27), eq. (28) and applying the quantum Hamiltonian constraint given by eq. (29) we obtain
\[ Z_C = \sum_{S,B} |C_{SB}|^2 \langle \Psi_B | \otimes \langle \Psi_S | \exp - \beta (\hat{H}_S \otimes \hat{I}_B + \hat{I}_S \otimes \hat{H}_B) | \Psi_S \rangle \otimes | \Psi_B \rangle \]
\[ = \sum_{S} |\tilde{C}_S|^2 \langle \Psi_S | \exp - \beta \hat{H}_S | \Psi_S \rangle \]
where $|\tilde{C}_S|^2 = \sum_B |C_{SB}|^2 \langle \Psi_B | \Psi_B \rangle$. Now, one should not consider this QIH Hamiltonian ($\hat{H}_S$) to be that of the quantum CS theory. Being a topological field theory, the Hamiltonian of the CS theory vanishes. This would have resulted in the partition function to be zero! On the other hand there is another Hamiltonian defined on the IH which provides a well defined notion of classical energy and satisfies the first law under the Hamiltonian evolution of space in time [3]. But, till now this notion of energy has no quantum version. One can see [15] for a discussion on this issue and the prediction of a possible way to handle this problematic situation i.e. the quantum to classical viewpoint. The proposal of the model Hamiltonian, which we shall use here as $\hat{H}_S$ is an outcome of that line of thought. The eigenvalues of this Hamiltonian will give us the quantum energy spectrum of the QIH and the eigenvalue equation can be written as
\[ \hat{H}_S | \Psi_S \rangle = E_S | \Psi_S \rangle \]
Hence, the partition function can be simply written as
\[ Z_C = \sum_{\{s\}} \Omega[\{s\}] \exp - \beta E_{\{s\}} \]
In eq. (31), $|\tilde{C}_S|^2$ denotes the probability of finding the QIH at a particular surface state irrespective of the states of the bulk quantum geometry. Now, we do not have a quantum version of the Hamiltonian of the horizon introduced in [3]. But in the previous section we have argued that similar to the area spectrum of the QIH in the LQG framework, we can construct an energy spectrum of the QIH by virtue of the additive contributions of the noninteracting punctures. Thus, using the energy spectrum given by eq. (16) and rewriting the partition function as sum over spin configuration we get
\[ Z_C = \sum_{\{s\}} \Omega[\{s\}] \exp - \beta E_{\{s\}} \]
The state having energy $E_{\{s\}}$ exhibits $\Omega[\{s\}]$-fold degeneracy which in the continuum limit can be viewed to be the density of states. Since $\omega[\{s^*\}] \simeq 1$ which alternatively implies that $\Omega[\{s^*\}]$ is very very large compared to the number of microstates for other configurations, we can write the exact partition function as the sum of the dominant part and the negligible fluctuations as follows
\[ Z_C = \Omega[\{s^*\}] \exp - \beta E_{\{s^*\}} + \text{sub-dominant terms} \]
\[ = \Omega[\{s^*\}] \exp - \beta E^* + \delta \]
\[ = Z^*_C + \delta \]
where $E_{\{s^*\}} = \sum_j s^*_j E_j = E^*$ and $\delta$ denotes the contributions from the sub-dominant configurations contributing to the thermal fluctuations. Having calculated the partition function it is straightforward to show that the ensemble average of the energy of the QIH comes out to be
\[ \bar{E} = - \frac{\partial}{\partial \beta} \log Z_C \simeq E^* \]
Now, taking logarithm of both sides of eq. (32) and using the definition of canonical entropy $S_C = \log Z_C + \beta \bar{E}$ and using $\bar{E} \simeq E^*$ from eq. (17) it is straightforward to show that
\[ S_C = S_{MC} + \log(1 + \delta/Z^*_C) \]
where $S_{MC}$ is given by eq. (24). This is a familiar result previously shown in literature in the context of black hole thermodynamics in [16, 24–30], but in a different mathematical approach where the fluctuations appear as terms of Taylor expansion about the equilibrium (see [31]). It can be also shown by this method [24–30] that the argument of the logarithm is equivalent to the specific heat of the thermodynamic system.
under consideration as a consequence of which the canonical entropy becomes ill defined for systems having negative definite specific heat. Here, having constructed the energy spectrum of the QIH we shall directly calculate the specific heat of the QIH and examine its stability for the allowed values of $\gamma$.

By definition, the specific heat is given by

$$C = \frac{\partial E^*}{\partial T} \bigg|_N = \frac{d\lambda}{dT} \frac{\partial E^*}{\partial \lambda} \bigg|_N$$

One can easily check that

$$\frac{dT}{d\lambda} = -\frac{\lambda p}{2} \sum_{n=0}^{\Lambda} a_n(\gamma) \left[ \frac{n\Gamma(n+2)}{\lambda^{n+2}} + \delta^0_n \right]$$

$$\frac{\partial E^*}{\partial \lambda} = \frac{\lambda}{(2-\lambda^2)^2} \sum_{n=0}^{\Lambda} a_n(\gamma) \left[ 2\Gamma(n+3) \left( \lambda^2 - \frac{2n}{n+2} \right) + 4\delta^0_n \right] N$$

Hence, it follows that the specific heat for the QIH is given by

$$C = -\frac{2N}{(2-\lambda^2)^2} \frac{\sum_{n=0}^{\Lambda} a_n(\gamma) \left[ 2\Gamma(n+3) \left( \lambda^2 - \frac{2n}{n+2} \right) + 4\delta^0_n \right]}{\sum_{n=0}^{\Lambda} a_n(\gamma) \left[ \frac{n\Gamma(n+2)}{\lambda^{n+2}} + \delta^0_n \right]}$$

From the above expression for the specific heat we can not get any idea about the stability of the QIH since the $a_n$-s are completely unknown to us as we have only assumed such a general energy spectrum for a consistent thermodynamic analysis of the stable QIHs. But, since we are involved in a calculation of a purely quantum origin, we can use some known classical or semiclassical results as inputs so as to get an idea of the unknown quantities in this quantum theory.

A. Single Puncture Energy Spectrum as a power law

As far as this work is concerned we have only tried to construct a model energy spectrum for the QIH which is not the result of a true quantization of the horizon energy. So, we shall try to look at some of the consequences of this model energy spectrum by making some specific choices. Let us investigate the special case where the single puncture energy spectrum follows a power law of the single puncture area contribution i.e. we choose $b_n = \eta_m \delta^0_n \ell^p 2^n$ with $m \geq 1$. The single puncture energy spectrum given by eq.(8) written as a polynomial gets reduced to a power law given by

$$E_j = \ell^p m A_j^m = \ell^p m \eta_m (8\pi\gamma)^m C_j^m$$

where $\eta_m$ is an unknown positive constant. As a result of this choice, one can also show from eq.(35) that the expression for specific heat is now reduced to

$$C = \frac{4N}{(2-\lambda^2)^2} \left( 1 + \frac{2m}{m+2} \right) \left( \frac{2m}{m+2} - \lambda^2 \right)$$

It is interesting to see that

$$\lim_{m \to \infty} C = \frac{4N}{(2-\lambda^2)}$$

This implies that any finite value of $m$, which is obvious for a physical theory ($m \to \infty$ is unphysical), the value of $\lambda^2$ will be certainly bounded from above by the constant $\lambda_m^2 < 2$ for the QIH to have local thermodynamic stability i.e. $C > 0$, where $\lambda_m^2 = \frac{2m}{m+2}$. Hence, there is a range $\lambda_m^2 < \lambda^2 < 2$ for which the theory does not admit a thermodynamically stable QIH. The range of the allowed values of $\lambda$ for a given power law $E_j \propto A_j^m$ of the single puncture energy spectrum is given by

$$C < 0, \quad \lambda_m < \lambda < \sqrt{2}$$

$$C > 0, \quad 1 < \lambda < \lambda_m$$
where \( \lambda_m = \sqrt{\frac{\Lambda m}{m+2}} \) and the lower bound of \( \lambda \) for \( C > 0 \) follows from the inequality (13). As a consequence of the power law, the expression of the temperature given by eq.(25) is now reduced to

\[
T = \frac{1}{2} \eta m \ell_p \Gamma(m+2) \left[ \frac{8 \pi \gamma}{\lambda} \right]^m
\]  

(40)

For the BHAL to be valid, we must have \( \lambda = 2 \pi \gamma \) which reduces eq.(40) to

\[
T = \frac{1}{2} \eta m 4^m \Gamma(m+2) \ell_p
\]  

(41)

Hence, the temperature clearly carries the exponent of the power law which governs the single puncture energy spectrum and that is quite disturbing as far as our knowledge of usual thermodynamics is concerned. But, since we have proposed a model energy spectrum, we shall demand that this model must consistently yield the known results of the classical IH thermodynamics by choosing the coefficients \( \eta_m \) appropriately.

**Choosing the coefficients :** Just for the moment if we go back to the general single puncture energy spectrum given by eq.(8), one can argue that the coefficients must decrease with increasing \( m \) so as to make the energy spectrum convergent. But, most importantly, we must get rid of the parameter governing the single puncture energy spectrum from appearing in the expression of the temperature in eq.(41). Hence, from this power law analysis we have a good reason to choose \( \eta_m = \eta 4^{-m} \Gamma(m+2)^{-1} \). \( \eta \) is a scaling constant of the energy spectrum. As a result the temperature in eq.(40) is now given by

\[
T = \frac{1}{2} \eta \ell_p \left[ \frac{2 \pi \gamma}{\lambda} \right]^m
\]  

(42)

which, for the BHAL to be valid i.e. for \( \lambda = 2 \pi \gamma \), reduces to

\[
T = \frac{1}{2} \eta \ell_p
\]  

(43)

That the choice of the coefficients is appropriate can be understood from the fact that the temperature is now independent of the single puncture energy spectrum of the QIH. Also, the fact that ambiguous parameter \( \eta \) enters the expression for temperature is perfectly consistent with the scaling ambiguity manifested by the surface gravity(which is related to temperature) in the first law of the classical IH resulting from the time evolution ambiguity of the classical phase space [3].

The general conclusion one can draw from the above analysis is that, for a QIH, if the BHAL is considered to be valid, the allowed range of \( \gamma \) is given by

\[
C < 0, \quad \frac{\lambda m}{2 \pi} < \gamma < \frac{1}{\sqrt{2 \pi}}
\]  

(44a)

\[
C > 0, \quad 0.159 < \gamma < \frac{\lambda m}{2 \pi}
\]  

(44b)

where the lower bound on \( \gamma \) for \( C > 0 \) is obtained from the inequality (15).

Hence, from the above findings, we can make the following statement :

*A theory of thermodynamically stable QIH having the single puncture energy spectrum given by \( E_j \propto A_j^{-m} \), whose entropy obeys the BHAL, can only allow values of the Barbero-Immirzi parameter bounded by the range given by \( 0.159 < \gamma < \frac{1}{2 \pi} \sqrt{\frac{2m}{m+2}} \).*

As far as the full range of allowed values of \( \gamma \), given by eq.(15), is concerned, the above statement has an immediate consequence :

\[
\frac{1}{2 \pi} \sqrt{\frac{2m}{m+2}} > 0.159 \quad \Rightarrow \quad m > 1 \quad \text{for} \quad C > 0
\]  

(45)

i.e. a QIH exhibiting the single puncture energy spectrum as \( E_j \propto A_j^{-m} \) can be thermodynamically stable if and only if \( m > 1 \). In other words, if one models the horizon energy spectrum considering the single puncture energy exhibiting the power law \( E_j \propto A_j^{-m} \) with \( m = 1 \), there will not be any possible value of \( \gamma \) to have a consistent quantum theory of spacetime admitting a thermodynamically stable QIH in the LQG framework.
B. A Specific Case: $E_j \propto A_j$

Even though from the analysis in the previous section, it is quite evident that for $m = 1$, there does not exist any allowed value of $\gamma$ so as to admit a thermodynamically stable QIH manifesting BHAL, in this section, we study this special case of our general single puncture energy spectrum in eq.(8) as a particular example, which is of immense importance as far as contemporary literature is concerned [18, 32, 33].

We have shown in section(III) that if the single puncture energy contribution is written as a polynomial of the single puncture area contribution, then the average energy of the QIH is proportional to its average area. Similar calculations and result follow for power law spectrum where only the proportionality constant is changed. It can be trivially derived as the calculation of temperature shown in subsection(IV A). The average or mean values are taken to be approximately equal to the classical values(correspondence limit) and hence, we have here obtained $E_{cl} \propto A_{cl}$ with a very general underlying energy spectrum of the QIH.

On the other hand, beginning from the classical theory, if one shows that $E_{cl} = \eta_1 A_{cl}, \eta_1$ being some proportionality constant, there is no other choice than to write $E \equiv \eta_1 A$ in the quantum theory which has actually been done in [18, 32, 33]. This is tantamount to writing the corresponding energy spectrum contribution from a single puncture as $E_j = \eta_1 A_j$, which is only a very specific choice ($m = 1$) as far as our power law analysis in subsection(IV A) is concerned.

But, since some important results have been obtained and crucial predictions have been made from this particular choice $E_j \propto A_j$ in [18, 32, 33], it is worth studying the results for this specific case separately. We shall use $m = 1$ in the the results following from the detailed thermodynamic analysis of the power law energy spectrum of the QIH presented in our work in subsection(IV A) and analyze the consequences with proper explanations.

The specific heat given by eq.(37), is dependent on the exponent of the single puncture energy spectrum power law and for $m = 1$ it yields the result corresponding to $E_j = \eta_1 A_j$ which is given by

$$C = \frac{12N}{(2 - \lambda^2)^2} \left(\frac{2}{3} - \lambda^2\right) = N\lambda^2 \frac{d^2\sigma}{d\lambda^2}$$

where $\sigma$ is given by eq.(12a). Since, we have $\lambda = 2\pi \gamma$, the above expression for specific heat reduces to $C = N\gamma^2 d^2\sigma/d\gamma^2$, which is the expression for specific heat obtained in [18](correct expression to be found in the erratum).

Hence, for the QIH, having the single puncture energy spectrum $E_j \propto A_j$, we must have $\lambda < \sqrt{2/3} \Rightarrow \gamma < \frac{1}{\sqrt{\pi}} \approx 0.130$ for the QIH to have local thermodynamic stability i.e. $C > 0$. One can check that upper bound on $\lambda$ and $\gamma$ are less than their corresponding lowest possible allowed values given by eq.(13) and eq.(15).

Thus, we can conclude that there is no possible value of $\gamma$ that allows a consistent quantum theory admitting a thermodynamically stable QIH, with single puncture energy spectrum $E_j \propto A_j$, as the inner boundary of spacetime. The thermodynamic instability of a QIH with this particular energy spectrum has been confirmed earlier in a somewhat less rigorous manner in [19].

C. Revisiting the Complete Energy Spectrum

Having done all these analyses in the previous sections, now we at least have got a hint about the coefficients $a_n$ so as to get an explicit structure of the full energy spectrum of the QIH. Since we only proposed the single puncture energy contribution given by eq.(8) which resulted in the QIH energy spectrum in eq.(16), we did not have any idea about the coefficients $a_n$. But, considering the choice of the coefficients $a_n$ from the power law analysis shown above, we can propose the single puncture energy spectrum to be

$$E_j = \eta \ell_p \sum_{n=0}^{\Lambda} \frac{A_j^n}{4^n \Gamma(n+2)}$$

(46)

from which it follows that the energy spectrum for a QIH will be given by

$$\hat{H}_S \mid \{s_j\} = \eta \ell_p \sum_{j} \sum_{n=0}^{\Lambda} \frac{1}{4^n \Gamma(n+2)} s_j A_j^n \mid \{s_j\}\rangle$$

$$= \eta \ell_p \sum_{j} \sum_{n=0}^{\Lambda} \frac{(2\pi \gamma)^n s_j C_j^n}{\Gamma(n+2)} \mid \{s_j\}\rangle$$

(47)
where we have used $A_j = 8\pi\gamma C_j$. The form of the coefficients can be explicitly written as

$$a_n(\gamma) = \eta \frac{(2\pi\gamma)^n}{\Gamma(n+2)}$$

where $\eta$ is some unknown positive constant. Considering the energy spectrum of the QIH given by eq.(47) and considering the validity of the BHAL ($\lambda = 2\pi\gamma$), the specific heat comes out to be

$$C = \frac{2N(\Lambda^2 + 5\Lambda + 8)}{(1 - 2\pi^2 \gamma^2)(\Lambda^2 + 5\Lambda + 8)} \left[ \frac{\Lambda^2 + \Lambda}{\Lambda^2 + 5\Lambda + 8} - 2\pi^2 \gamma^2 \right]$$

and the expression of the equilibrium temperature is given by

$$T = \frac{1}{2\eta} \Lambda \ell_p \left[ 1 + \frac{(1 - 2\pi^2 \gamma^2)}{\Lambda} \right]$$

Hence, we can conclude that the QIH having the energy spectrum given by eq.(47) is thermodynamically stable if and only if

$$\gamma < \frac{1}{\sqrt{2\pi}} \sqrt[1/2]{\frac{\Lambda^2 + \Lambda}{\Lambda^2 + 5\Lambda + 8}}$$

The point to be noted in this above calculation is that, although the limit $\Lambda \to \infty$ leads to the result $\gamma < \frac{1}{\sqrt{2\pi}}$, which is absolutely consistent with the full range of $\gamma$, the cause of concern is the expression for the temperature, given by eq.(50), which diverges in this limit. Hence, at this moment we can not model the single puncture energy spectrum with some convergent series with infinite terms and the cut off ($\Lambda$) seems to be absolutely necessary.

Now, as we are investigating the canonical ensemble thermodynamics with a model Hamiltonian, we shall expect to yield the results already known by choosing the parameters suitably. We shall refer to two cases [3] and [18]. In [3], the surface gravity having a rescaling ambiguity is fixed by choosing a vector field which will reduce to the Killing vector fields in the stationary situations. On the other hand, in [18, 40] it has been somehow argued that the local observers who remains stationary with respect to the horizon [40] will observe a universal surface gravity. Hence, to explain these two scenarios we fix our model by choosing $\eta = \left[ 1 + \frac{(1 - 2\pi^2 \gamma^2)}{\Lambda} \right]^{-1}$ and thus eq.(50) reduces to

$$T = \frac{1}{2} \Lambda \ell_p$$

from which the surface gravity can be obtained by using $\kappa = 2\pi T$. Now it is up to the reader to decide which of the scenarios to prefer, [3] or [18, 40]. Choosing $\Lambda$ as per requirement would give the underlying general energy spectrum of the QIH given by

$$\hat{H}_S|\{s_j\} = \ell_p \sum_j \sum_{n=0}^{\Lambda} \sum_{n=0}^{\Lambda} \frac{(2\pi\gamma)^n}{\Gamma(n+2)} C_j^n \{s_j\}$$

which can yield the notion of classical energy associated with the Isolated Horizon given in [3] or [18, 40].

V. DISCUSSION

In this concluding section we shall make some remarks on the upper bound on $\gamma$, followed by some comments on the model Hamiltonian and the corresponding energy spectrum of the QIH used here following the proposal in [15] and finally we shall point out the relevance of this work as far as the literature of black hole thermodynamics is concerned.

The thermodynamics associated with the kind of energy spectrum of a QIH considered here, has never been studied earlier in literature. This is due to the complete different viewpoint of going from quantum to classical regime presented in [15] motivated by some deep underlying issues regarding the theories of IH and QIH. Generally, one considers the horizon energy as a function of the horizon area (e.g. power law). This intuition works in our mind due to our instinctive affinity to look at a quantum theory through the
classical spectacles. To be more explicit, in numerous cases of the study of black holes the mass formula for known black hole solutions are expressed in terms of the area and addressed as the mass spectrum of the black hole [24-30]. In fact, many a times, in such formulae, the area spectrum of LQG is used directly in the classical formula and the mass of the black hole is considered to be quantized \[25, 27, 34\] which is of course not a true quantization of the horizon energy and also devoid of any physical justification, apart from being an ad hoc assumption. A genuine energy spectrum for a QIH should be derived by quantization of the classical notion of horizon energy similar to the quantization of area, volume and length resulting in the corresponding operators in quantum gravity [35–39]. The other alternative is to propose one based on solid physical arguments, which we have done. Interestingly, the result provided by our model energy spectrum in eq.(20) from a purely quantum statistical and thermodynamical perspective, is capable of explaining the results of [32, 33, 40, 41] or [3]. It remains to be seen whether it is possible to construct the corresponding Hamiltonian operator for the QIH yielding the kind of energy spectrum discussed here from the classical theory, where of course A will be known a priori.

Now, as far as the thermodynamic aspect of this paper is concerned, it is worth mentioning that unlike \([23, 42, 43]\) we deal with usual canonical energy ensemble as we have an explicit structure of the Hamiltonian. We need not use a ‘Boltzmann-like’ factor \(e^{-\alpha A}\) in the canonical partition function[23], accompanied by a fictitious conjugate parameter \(\alpha\), alongside the Boltzmann factor \(e^{-\beta E}\). As far as \([42, 43]\) are concerned, it is a pure area ensemble involving only \(e^{-\alpha A}\) in the canonical partition function and devoid of the Boltzmann factor \(e^{-\beta E}\). All of these approaches are significant and interesting by their own virtue. But none of them actually attacks the problem of black hole horizon thermodynamics following usual canonical energy ensemble due to the lack of knowledge of the Hamiltonian and the energy spectrum associated with the horizon. This is where the use of the proposed model Hamiltonian repay the benefits and allow us to follow usual canonical energy ensemble approach to thermodynamic analysis of QIHs.

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