Deriving dilaton potential in improved holographic QCD from chiral condensate

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We derive an explicit form of the dilaton potential in improved holographic QCD (IHQCD) from the QCD lattice data of the chiral condensate as a function of the quark mass. This establishes a data-driven holographic modeling of QCD — machine learning holographic QCD. The modeling consists of two steps for solving inverse problems. The first inverse problem is to find the emergent bulk geometry consistent with the lattice QCD simulation data at the boundary. We solve this problem with the refinement of neural ordinary differential equation, a machine learning technique. The second inverse problem is to derive a bulk gravity action with a dilaton potential such that its solution is the emergent bulk geometry. We solve this problem at non-zero temperature, and derive the explicit form of the dilaton potential. The dilaton potential determines the bulk action, the Einstein-dilaton system, thus we derive holographically the bulk system from the QCD chiral condensate data. The usefulness of the model is shown in the example of the prediction of the string breaking distance, whose value is found to be consistent with another lattice QCD data.

I. INTRODUCTION

Machine learning is one of the major methods in theoretical and experimental physics. The use is particularly effective in fields with vast amount of data, while it is still quite limited in formal aspects of theoretical physics.

For example in the research of AdS/CFT correspondence [1–3], in spite of numerous pairs of the bulk and the boundary QFT have been found, the strategy mainly uses symmetric properties of the physical systems, and is not based on the data science. However, let us note that if one formulates the AdS/CFT study as a problem of finding the bulk theory for a given boundary QFT data, it is a data science — the feature extraction of the vast amount of data of the QFT to interpret it as a higher-dimensional gravity theory. For this kind of problem, machine learning can help to find the bulk theory.

For QCD, we already have a rich variety of lattice and experimental data, with which we can try to solve this problem to find the holographic bulk gravity theory dual to QCD. The readers notice that this is an inverse problem. In the standard holographic modeling of QCD, one first come up with a gravity action, and solve it to find a metric function and a geometry. Then one put a favorite probe on it to compute quantities which is dual to the physical quantities in QCD, for example, putting a fundamental string gives a Wilson loop, or putting a probe scalar field gives a chiral condensate. On the other hand, what we really want in the holographic modeling of QCD is the gravity action, for a given QCD data. Therefore, this is an inverse problem.

It has been proven that some of the holographic modeling has been efficiently made by the use of machine learning [4–18]. In particular, the method proposed in [4, 5] regards the discretized bulk geometry as a neural network, and the network weights as the bulk spacetime metric. The data of the boundary QFT is the input data of the neural network. Thus the deep learning (DL), the machine learning using deep neural networks, has a similarity to the AdS/CFT correspondence, and works as a solver of the inverse problem. Once the training of the neural network is made, the bulk metric is determined automatically. This is called the AdS/DL correspondence.

Once the desired metric is obtained by the machine learning from the boundary QFT data, the remaining task is to find a gravity action whose solution is the metric function. The gravity system may have functional arbitrariness. For example in the improved holographic QCD [19, 20], a dilaton-gravity system is employed as the bulk action, and the dilaton potential allows the arbitrariness to host various QCD phenomena. In [16], we have explicitly derived a dilaton potential of a dilaton-gravity bulk system by requiring that its equations of motion are solved by the metric found in [12]. This metric is that of the AdS/QCD model found to be consistent with the $\rho$-meson spectral data in QCD experiments. Therefore, the double-layer inverse problem was solved in [12, 16] for this particular example of the hadronic spectra at zero temperature, and the data-driven modeling...
was complete for it.

In this paper, we study the non-zero temperature case to see whether this data-driven modeling of machine-learning holographic QCD works in more generic situations. Our goal is to derive a dilaton potential from the data in the non-zero temperature case. We find the dilaton potential of the bulk action, for the metric found in [6]. This metric is holographically consistent with the lattice QCD data of chiral condensate as a function of the quark mass. The solver is the neural ordinary differential equations (neural ODE) [21], a machine learning technology suitable for continuous systems rather than discretized systems. The gravity equations at non-zero temperature has less symmetries: the metric has two independent components to be determined (on the other hand, the zero-temperature metric has only one component to be determined). Thus the determination of the bulk action by its solution is even more nontrivial. We completely demonstrate that it is possible, and derive the dilaton potential in the bulk, only by using the explicit form of the metric solution.

With the obtained bulk action, we can compute the Wilson loop as a prediction of the holographic model. The model predicts that the string breaking distance at the temperature $T = 208$ [MeV] is $\delta_{W} \sim 0.5$ [fm]. This prediction turns out to be consistent with various lattice QCD simulation results, which shows a nontrivial consistency and the predictive power of the machine-learning holographic model and the AdS/DL correspondence.

The organization of this paper is as follows. In Sec. [I] we briefly review the results obtained in [6] for finding the metric from the chiral condensate data. We rerun the neural ODE with the metric ansatz dedicated to the improved holographic QCD. In Sec. [II] we use the metric to derive the dilaton potential. We analyze the asymptotic behavior of the dilaton potential. In Sec. [III] we calculate the holographic Wilson loop of the machine-learned model for our prediction and find that it agrees with various other lattice QCD results. Sec. [IV] is for our conclusion and discussions. Appendix [A] is dedicated for the neural ODE training details, and Appendix [B] is for the symmetric property of the fields used in this paper.

II. NEW RESULTS FOR EMERGENT METRIC IN ADS/DL

The bulk geometry of the holographic QCD describing the dual of the chiral condensate was obtained numerically in [6] by the use of the machine learning. The scheme was based on the AdS/DL correspondence [4]. This was refined in [13] in which, instead of using deep neural networks composed of piles of layers, the neural ODE [21] was employed which enabled the use of continuous layers, i.e. continuous bulk geometries in the AdS/CFT.

In this section, we report new results which improve the bulk geometries previously obtained in [13]. The major improvement is about the following three respects.

The first improvement is about the spatial range of the bulk geometry. Originally in [13] the reconstructed geometry by the neural ODE was in the range $0.1 \leq \eta \leq 1$ in the unit $L = 1$ where $L$ is the AdS radius, and here we extend the range to the region $0.01 \leq \eta \leq 1$. With this, the region near the horizon of the black hole (BH) in the bulk geometry is better approximated, as the BH horizon sits at $\eta = 0$.

The second improvement is about the prior knowledge about the metric function. The neural ODE uses a functional ansatz for the neural network weights which are bulk metric function $h(\eta)$ in our case. Here $h(\eta)$ is defined as, introducing a dilaton $\Phi$,

$$h(\eta) \equiv \partial_\eta \log \left( \sqrt{f g^3 e^{-\Phi}} \right), \quad (1)$$

in terms of a bulk metric in the string frame

$$ds^2 = -f(\eta) dt^2 + d\eta^2 + g(\eta) ds^2, \quad (2)$$

with the 3-D flat space metric $ds^2$. The metric [2] is assumed to interpolate AdS vacuum in the limit of $\eta = \infty$ with the AdS radius $L$,

$$\lim_{\eta \to \infty} \partial_\eta \log f(\eta) = \lim_{\eta \to \infty} \partial_\eta \log g(\eta) = \frac{2}{L} \quad (3)$$

and the BH horizon fixed at $\eta = 0$ as

$$\lim_{\eta \to 0} \eta \partial_\eta \log f(\eta) = 2, \quad \lim_{\eta \to 0} \eta \partial_\eta \log g(\eta) = 0. \quad (4)$$

As for the machine learning applied to the emergent metric, we employ the following ansatz for $h(\eta)$,

$$h(\eta) = h^{\text{ODE}}(\eta) \equiv \frac{1}{\eta} + b_1 \eta + b_3 \eta^3 + b_5 \eta^5. \quad (5)$$

The first term is the singularity due to the BH horizon, which is determined automatically for generic horizons with non-zero temperature. The remaining terms give our ansatz for the bulk geometry, where $b_1, b_3$ and $b_5$ are the coefficients which are trained in the neural ODE. Note that here we have only terms with odd powers in $\eta$, while in the work [13] all possible powers in the Taylor expansion were included. The reason of the current restriction of the powers is that in the Einstein dilaton system only the odd powers are allowed since the system has a certain $\mathbb{Z}_2$ parity. See Appendix [B] for the derivation of the restriction.

The third improvement is on the regularization. We select the form of the regularization suitable for the Einstein-dilaton system, see Appendix [A] for the details.

With these three improvements, we recapitulate the procedures provided in [13] to obtain the metric. Table [I] is the lattice QCD data which we use as the training data. It is the data of the chiral condensate $\langle \bar{q} q \rangle [\text{GeV}^3]$ as a function of the quark mass $m_q [\text{GeV}]$ at the temperature $T = 0.208$ GeV, obtained by the RBC–Bielefeld collaboration [22]. The network architecture and the holographic model are exactly the same as that of [13], except
with an unknown dilaton potential $V_D(\Phi)$ which is assumed to be smooth. The metric $(2)$ and the dilaton are supposed to be produced as a solution of this system. The aim of this paper is to determine the explicit dilaton potential $V_D(\Phi)$ with which the equations of motion of $(\ref{eq:5})$ is solved with the emergent metric given in the previous section.

The introduced dilaton field $\Phi$ satisfies the boundary conditions
\[
\lim_{\eta \to 0} \Phi = \Phi_0, \quad \lim_{\eta \to \infty} \Phi = \Phi_\infty \equiv 0, \tag{7}
\]
with a certain positive-definite value $\Phi_0$. In addition, $\Phi(\eta)$ is expected to be a monotonically decreasing function
\[
\partial_\eta \Phi(\eta) < 0 \quad \text{for} \ \eta > 0, \tag{8}
\]
and thus, there is a one-to-one mapping between the coordinate $\eta$ and dilaton $\Phi$: $\eta \in \mathbb{R}_{>0} \leftrightarrow \Phi = \Phi(\eta) \in (0, \Phi_0]$ as usually is in the AdS/CFT correspondence.

Let us derive the equations of motion of the system. Due to the presence of dilaton, we need to distinguish the metric in the string frame and that in the Einstein frame. We parameterize the metric in the Einstein frame as
\[
d s_E^2 = -f_E d t^2 + \rho(\eta)^2 d \eta^2 + g_E d x_m d x_m, \tag{9}
\]
\[
f_E = f e^{-\frac{1}{2} \Phi} = \exp \left( \frac{1}{2} (\psi(\eta) + 3(\chi(\eta))) \right), \tag{10}
\]
\[
g_E = g e^{-\frac{1}{2} \Phi} = \exp \left( \frac{1}{2} (\psi(\eta) - \chi(\eta)) \right). \tag{11}
\]

Two metrics $(2), (9)$ are related via a Weyl transformation, $ds_E^2 = e^{-\frac{4}{3} \Phi} ds_{st}^2$. Then we find that the equations of motion for $\psi, \chi, \Phi$ and $\rho$ result in, respectively,
\[
0 = 2\partial_\eta v + v^2 + w^2 + \frac{4}{9} \partial_\eta \Phi v + \frac{16}{9} (\partial_\eta \Phi)^2 - \frac{4}{3} V_D e^{-\frac{4}{3} \Phi}, \tag{12}
\]
\[
0 = \partial_\eta w + w v + \frac{2}{3} \partial_\eta \Phi w, \tag{13}
\]
\[
0 = \partial^2_\eta \Phi + v \partial_\eta \Phi + \frac{2}{3} (\partial_\eta \Phi)^2 + \frac{3}{8} \partial V_D e^{-\frac{4}{3} \Phi}, \tag{14}
\]
\[
0 = e^{\frac{4}{3} \Phi} \left( -\frac{3}{4} v^2 + \frac{3}{4} w^2 + \frac{4}{3} (\partial_\eta \Phi)^2 \right) + V_D, \tag{15}
\]
where we introduced $v \equiv \partial_\eta \psi, w \equiv \partial_\eta \chi$, and chose a gauge $\rho = e^{-\frac{4}{3} \Phi}$. The reason for the field redefinition $(10)$ and $(11)$ is that the resultant equations of motion are simpler and diagonalized.

\[^{1}\text{For the definition of the regularization and the details of the neural network architecture, see Appendix [4].}\]

\[^{2}\text{Note that the sign of the potential is reversed from the normal one here so that} \ V_D \ \text{will be positive almost everywhere. This follows the convention of previous studies.}\]

\[^{3}\text{We guess that any oscillating} \ \Phi(\eta) \ \text{background implies instability of the system against large fluctuations of} \ \Phi \ \text{and must be prohibited.}\]
We note here that the four equations of motion are not independent. In fact, one of them can be derived from the other three. Below, we briefly discuss the reason for that. Instead of introducing Eq. (15), let us write the metric in the Einstein frame as

\[ ds^2_{\text{Ed}} = -f_{\text{Ed}}dt^2 + \rho^2 d\xi^2 + g_{\text{Ed}}dx_m dx^m, \]

where an introduced coordinate \( \xi \) is related to \( \eta \) with a certain one-to-one mapping, satisfying

\[ d\xi = \rho^{-1} e^{-\frac{2}{3}\Phi} d\eta. \]  

By substituting metric (16) to anaction (6) and omitting surface terms, we obtain the following reduced action per unit four-dimensional volume

\[ S_{\text{rd}} = \int d\xi e^\psi \left\{ -\frac{1}{\rho} K + \rho V_D \right\}, \]

\[ K = -\frac{3}{4} (\partial_\xi \psi)^2 + \frac{3}{4} (\partial_\xi \chi)^2 + \frac{4}{3} (\partial_\xi \Phi)^2. \]

Here \( \rho \) is regarded as an einbein of this one-dimensional model and e.o.m of \( \rho \) gives

\[ 0 = \frac{\delta S_{\text{rd}}}{\delta \rho} = \frac{1}{\rho^2} K + V_D, \]

which is usually regarded as a constraint. By taking this constraint into account, an arbitrary solution of e.o.ms of this reduced model automatically satisfies those for the original five-dimensional model. General coordinate transformation invariance guarantees the following identity

\[ \rho \partial_\xi \left( \frac{\delta S_{\text{rd}}}{\delta \rho} \right) = \sum_{x=\psi,\chi,\Phi} \partial_x \chi \frac{\delta S_{\text{rd}}}{\delta \chi}. \]  

Therefore, for nontrivial configurations, one of three equations for \( \psi, \chi, \Phi \) is automatically satisfied if the others and the constraint are solved.

Let us obtain the consistent and independent set of equations of motion. By eliminating the potential \( V_D \) from Eq. (12) using Eq. (15), we obtain a simpler equation

\[ 0 = \partial_\eta v + w^2 + \frac{2}{3} \partial_\eta \Phi v + \frac{16}{9} (\partial_\eta \Phi)^2. \]  

We can choose Eqs. (15), (13) and (23) as independent equations, and if all of them are solved, Eq. (14) is automatically solved as mentioned above.

In order to solve the three equations systematically, we massage the set of equations further. Since the potential \( V_D \) appears only in Eq. (15) within those three equations, Eq. (15) can be regarded as an equation which implicitly determines the unknown \( V_D \) if the one-to-one mapping \[ holds. Since \( h(\eta) \) defined in Eq. (1) is calculated in terms of \( v(\eta) \) as

\[ h(\eta) = v(\eta) + \frac{5}{3} \partial_\eta \Phi(\eta), \]

using this \( h(\eta) \), \( \partial_\eta \Phi(\eta) \) in Eqs. (13) and (23) can be eliminated and we obtain

\[ 0 = \partial_\eta v + w^2 + \frac{6}{25} (v - h) \left( v - \frac{8}{3} h \right), \]

\[ 0 = \partial_\eta w + \frac{1}{5} (3v + 2h) w. \]

When a metric function \( h(\eta) \) is given, therefore, these two equations are used to determine the two functions \( v(\eta), w(\eta) \). The field \( v \) and \( w \) need to satisfy the boundary conditions

\[ \lim_{\eta \to 0} \eta v = 1, \quad \lim_{\eta \to 0} \eta w = 1, \]

\[ \lim_{\eta \to \infty} v = \frac{4}{L}, \quad \lim_{\eta \to \infty} w = 0. \]

Thus, in summary, our strategy to derive the dilaton potential is as follows. The input is the function \( h(\eta) \) which the neural ODE determined explicitly. First, we solve Eqs. (25) and (26) under the boundary conditions (27) and (28), then second, we obtain \( \Phi(\eta) \) by integrating Eq. (24), and finally we obtain the dilaton potential \( V_D \) by using Eq. (15).

As a side remark, note that if we set \( \Phi(\eta) = \Phi_0 = 0 \), the system reduces to one for the pure Einstein gravity and the exact solution for \( v, w \) is given by, with \( V_D = 12/L^2 \),

\[ v(\eta) = v_E(\eta) \equiv \frac{4}{L} \coth \frac{4\eta}{L}, \]

\[ w(\eta) = w_E(\eta) \equiv \frac{4}{L} \csch \frac{4\eta}{L}. \]

In this paper \( \Phi(\eta) \) takes a non-trivial configuration with \( \Phi_0 > 0 \).

**B. Asymptotics and extrapolation of \( h(\eta) \)**

In Sec. V \( h^{\text{ODE}}(\eta) \) was numerically determined but it covers the limited region \( 0.01 < \eta < 1 \) and it is technically hard to extend this region, whereas a solution of \( v, w \) may critically depend on the boundary conditions (27) and (28). Therefore, we have to seriously consider an asymptotic behavior of \( h(\eta) \) and need to extrapolate the trained \( h(\eta) \) with using an appropriate extrapolation function. In this subsection, we study the asymptotic regions of the spacetime: \( \eta \sim 0 \) and \( \eta \sim \infty \).
Focusing only on the mathematical aspects of this system of equations for \( v, w \), the function \( h \) can be arbitrarily imputed there, but as discussed bellow, some reasonable accompanying assumptions lead to several properties that the function must have.

At first, by assuming the smoothness of \( V_D(\Phi) \) around the point \( \Phi = \Phi_0 \) corresponding to the horizon \( \eta = 0 \) we can consider (Laurent series) expansions of \( v, w \) and \( h \) around \( \eta = 0 \) and define \( \mathbb{Z}_2 \) parities of those functions. All of the equations and the boundary conditions allow us to set \( v(\eta), w(\eta) \) and \( \partial_{\eta} \Phi(\eta) \) to all odd functions, and in fact, it can be shown that they must be so. See Appendix B. That is, they are expanded as

\[
v(\eta) = \frac{1}{\eta} + \sum_{n=1}^{\infty} a_n^{(v)} \eta^{2n-1}, \quad w(\eta) = \frac{1}{\eta} + \sum_{n=1}^{\infty} a_n^{(w)} \eta^{2n-1}, \quad \Phi(\eta) = \Phi_0 + \sum_{n=1}^{\infty} a_n^{(\Phi)} \eta^{2n},
\]

(31)

and especially, \( h(\eta) \) must be expanded as

\[
h(\eta) = \frac{1}{\eta} + \sum_{n=1}^{\infty} b_{2n-1} \eta^{2n-1},
\]

(32)

as we have already used in Eq. (5). Using the equations of motion, their first terms are related with each other as

\[
a_1^{(v)} = 2c_h + \tilde{c}_h, \quad a_1^{(w)} = -c_h + \tilde{c}_h, \quad a_1^{(\Phi)} = -\frac{9}{4} \tilde{c}_h, \quad b_1 = 2c_h - \frac{13}{2} \tilde{c}_h.
\]

(34)

where \( c_h \) and \( \tilde{c}_h \) are defined as

\[
c_h = \frac{2}{9} V_D(\Phi_0) e^{-\frac{4}{3} \Phi_0}, \quad \tilde{c}_h = \frac{1}{24} \frac{\partial V_D}{\partial \Phi}(\Phi_0) e^{-\frac{4}{3} \Phi_0}.
\]

(35)

This means that if the functional forms of \( V_D(\Phi) \) had been given, the coefficients would have been controlled only by the initial value \( \Phi_0 \) without introducing any other free parameters. In this paper, we are going backwards: we solve the equations of motion for the given \( h(\eta) \) and derive \( V_D \). In the following subsection, what we will do technically is to solve the Eqs. (25) and (26) by a numerical shooting with varying \( c_h \). Another parameter \( \tilde{c}_h \) will be determined automatically by the last equation in Eq. (34).

2. Asymptotic AdS behavior at \( \eta \sim \infty \)

Next, let us look at the asymptotic AdS region of the spacetime. The neural ODE provides \( h(\eta) \) only in the region \( \eta < L \), thus we need to find an appropriate extrapolation function of \( h(\eta) \) for the region \( \eta > L \). For this, we study the analytic behavior of the fields in the asymptotic AdS region.

The assumption of the AdS vacuum in the limit of \( \eta = \infty \) requires that the potential \( V_D \) behaves around the origin \( \Phi = 0 \) as

\[
V_D(0) = \frac{12}{L^2} \frac{\partial V_D}{\partial \Phi}(0) = 0.
\]

(36)

In this paper, we just assume\(^5\) that the dilaton mass \( m_\Phi \) around the vacuum satisfies

\[
m_\Phi^2 = -\frac{3}{8} \frac{\partial^2 V_D}{\partial \Phi^2}(0) > 0.
\]

(37)

Around this vacuum, \( \Phi \) and \( w \) behave as

\[
\Phi \sim c_\Phi^+ e^{-p_+ \eta}, \quad w \sim c_w e^{-\frac{4}{3} \eta},
\]

(38)

and then, \( v \) behaves as

\[
v \sim \frac{4}{L} - \frac{8c_\Phi^+}{3L} e^{-p_+ \eta} + \frac{c_w}{8L} e^{-\frac{4}{3} \eta},
\]

(39)

with constants \( c_\Phi^+, c_w \). Here \( p_+ \) is given by

\[
p_\pm = \frac{1}{L} \left( 2 \pm \sqrt{4 + m_\Phi^2 L^2} \right) \in \mathbb{R}.
\]

(40)

Therefore, \( h(\eta) \) obtained in Eq. (24) must behave as

\[
h(\eta) \sim h^{ex}(\eta) \equiv \frac{4}{L} \left( 1 - B_h e^{-p_\eta} \right)
\]

(41)

\(^5\) If we expect the potential to behave as \( V_D \sim \Lambda e^{2Q \Phi} \Phi^P \) for large \( \Phi \) with certain constants \( Q, P \) and \( \Lambda \), and the value \( \Phi_0 \) is also sufficiently large like \( O(10) \) even in the finite temperature, then \( b_1 \) is also expected to be

\[
b_1 \sim -\frac{13}{24} \left[ Q - \frac{32}{39} + \frac{P}{2 \Phi_0} \right] \Lambda e^{2(Q-\frac{4}{3})\Phi_0} \Phi^P.
\]

(33)

According to \([\]\), they have been predicted \( Q = 2/3, P = 1/2 \) and thus, this prediction indicates \( b_1 > 0 \), whereas its magnitude is smaller than \( O(10) \) with using \( \Lambda = O(10) \). On the other hand, if \( Q \sim 2\sqrt{3}/3, \) \( b_1 \) should take a negative value of much larger magnitude. Thus, the sign of \( b_1 \) has a lot to do with our expectations.

\(^6\) Note that the BF bound \( m_\Phi^2 \geq -4/L^2 \) for the stability of the AdS vacuum is weaker than this condition. Intuitively, if \( 0 \geq m_\Phi^2 \geq -4/L^2 \), then long-range interaction by dilaton distorts the asymptotic AdS spacetime to the extent that it becomes an obstacle for training \( h(\eta) \).
TABLE III. The values of $(p, B_h)$ determined to be smoothly connected to the neural ODE results.

| Trial | $p$       | $B_h$     |
|-------|-----------|-----------|
| #1    | 7.582     | 575.5     |
| #2    | 5.330     | 71.51     |

with certain constants $B_h, p$. Under the assumption of $m^2_\Phi \geq 0$, $p$ is given as $p = \min\{p_+, 8/L\}$ and takes a value in the narrow region

$$\frac{4}{L} < p \leq \frac{8}{L}. \quad (44)$$

After these analytical observations, we find that it is natural to extrapolate the trained $h(\eta) = h^{\text{ODE}}(\eta)$ in Eq. (41) to a region for large $\eta$ by using $h^{\text{ex}}(\eta)$ given in Eq. (41) as

$$h(\eta) = \begin{cases} 
    h^{\text{ODE}}(\eta) & \text{for } \eta < 1 \\
    h^{\text{ex}}(\eta) & \text{for } \eta \geq 1
\end{cases} \quad (45)$$

in the unit $L = 1$, where two parameters $p, B_h$ are set by requiring the smoothness of $h(\eta)$ at $\eta = 1$ as

$$h^{\text{ODE}}(1) = h^{\text{ex}}(1), \quad \partial_\eta h^{\text{ODE}}(1) = \partial_\eta h^{\text{ex}}(1). \quad (46)$$

Note that we have to check if $p$ satisfies $4 < p \leq 8$. In fact, many trained $h^{\text{ODE}}(\eta)$ with poor initial conditions have violated this with deriving too large $p$ and been rejected. Therefore we need to include regularizations for the neural ODE training, see Appendix A. Finally, with $h^{\text{ODE}}(\eta)$ given in Table I, we obtain reasonable data for $h^{\text{ex}}(\eta)$. We list it in Table III.

We show the profiles of $h(\eta)$ for trial #1 and trial #2 as thick solid lines in Fig. 1. One can see that there appears a deep valley in its profile. On the other hand, in the Einstein gravity case, $h_E$ is explicitly given as $h(\eta) = h_E(\eta) \equiv v_{\text{E}}(\eta)$ with $v_{\text{E}}(\eta)$ in Eq. (49), and is also plotted in Fig. 1 where there is no valley in this case. The resulting $h(\eta)$ with the neural ODE is, therefore, qualitatively different from that in the Einstein gravity case.

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7 The coefficient $B_h$ controls the whole functional form of $h(\eta)$. In fact, when $h(\eta)$ has a valley in its profile, $B_h$ must be positive. This is realized when $m^2_\Phi$ is not so large,

$$B_h = \frac{8 + 5p_+L}{12} c^2_\Phi > 0, \quad \text{for } 0 < m^2_\Phi < \frac{32}{L^2}. \quad (42)$$

because the assumption requires $c^2_\Phi > 0$. On the other hand, for $m^2_\Phi > \frac{32}{L^2}$, $B_h$ is negative as $B_h = -c^2_\Phi/32 < 0$ with $p = 8/L$.

8 If we consider the case with $0 > m^2_\Phi \geq -4/L^2$, $p$ is given by $p = p_-$ and thus satisfies

$$0 < p < \frac{2}{L}. \quad (43)$$

since $\Phi$ behaves as $\Phi \sim c_\Phi e^{-p_- \eta}$. In this case, there are technical difficulties to train the function $h(\eta)$ from the data in Table I,

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C. Deriving $V_D(\Phi)$ from $h(\eta)$

Here we are ready to calculate the unknown potential $V_D(\Phi)$ numerically from the trained $h(\eta)$ with the extrapolation [45]. This needs a numerical integration of the equations of motion by the shooting method regarding the unknown coefficient $c_h$, as is explained below.

Let us solve Eqs. (25) and (26) with this given $h(\eta)$ to obtain $v(\eta), w(\eta)$ for $\eta \in [\eta_{ir}, \eta_{uv}]$ in the unit $L = 1$. To be more precise, we take $\eta_{ir} = 10^{-5}$ as a near-horizon cut-off and $\eta_{uv} = 5$ as the asymptotic AdS cut-off, and apply the shooting method by integrating the equations from $\eta = \eta_{ir}$ to $\eta = \eta_{uv}$. At $\eta = \eta_{ir}$, according to Eq. (34), we take the initial values of $v, w$ as

$$v(\eta_{ir}) = \frac{1}{\eta_{ir}} \left[ \frac{1}{13} (30c_h - 2b_1) \eta_{ir} \right], \quad (47)$$

$$w(\eta_{ir}) = \frac{1}{\eta_{ir}} - \frac{1}{13} (9c_h + 2b_1) \eta_{ir}, \quad (48)$$

where $b_1$ is a given parameter appearing in the expansion of $h(\eta)$ and $c_h$ is regarded as a free parameter since its value is unknown. After integrating the differential equations, for large $\eta$ with $h \sim 4$, the value $v = h \sim 4$ turns out to be a saddle point since Eq. (25) reduces to be $\partial_\eta v \sim \frac{4}{\eta} (v - 4)$, whereas $w \sim 0$ is automatically satisfied since $w = 0$ is an attractor as long as $3v + 2h > 0$. Therefore we must adjust the value of $c_h$ precisely so that $v$ can satisfy

$$v(\eta_{uv}) = h(\eta_{uv}) \sim 4. \quad (49)$$

This is the numerical shooting. We start with some $O(1)$ value of $c_h$ and start varying the value until the equation $v(\eta_{uv}) = 4$ is attained. The procedure determines $v(\eta), w(\eta)$ numerically as plotted in Fig. 2. These two $v, w$ are positive definite. The numerically searched value

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9 From Eqs. (37) and (13) and the boundary conditions (28), in-
of $c_h$ is

$$c_h = \begin{cases} 
3.373 & \text{(Trial #1)} \\
3.135 & \text{(Trial #2)} 
\end{cases} \quad (51)$$

After this procedure, thus, we can calculate the value of dilaton by using Eq.(24),

$$\Phi(\eta) = \frac{3}{5} \int_{\eta}^{\eta_{ir}} d\eta' (v(\eta') - h(\eta')) , \quad (52)$$

and the value of the potential at $\eta$,

$$V_D(\Phi(\eta)) = \frac{3}{4} \frac{4}{\Phi} \left( v^2 - w^2 - \frac{16}{25} (v - h)^2 \right) , \quad (53)$$

by using Eq.(15). By combining them, finally we obtain a parametric representation of the potential $V_D(\Phi)$ for $\Phi \in [0, \Phi_0]$ as plotted in Fig.3. This concludes the complete determination of the bulk action by the QCD data of the chiral condensate, in the data-driven manner.

Let us study the obtained dilaton potential and discuss its physical implications. First, observe that, in Fig.3, each profile has a shallow valley. The resulting potential for trial #2 is approximated by the following polynomial except in the vicinity of the origin $\Phi$ corresponding to $\eta > 1$,

$$V_{fit}(\Phi) = 12 - 38.5534 \Phi^2 + 190.104 \Phi^3 - 366.759 \Phi^4 + 429.313 \Phi^5 - 280.434 \Phi^6 + 100.092 \Phi^7 - 14.7622 \Phi^8 \quad (54)$$

as shown by the dashed line in Fig.3. Here, we imposed condition (36), but did not force the polynomial to fit the resulting data near the origin, as the influence of the manual extrapolation should be dominant there.

We look at the large $\Phi$ behavior of the potential in more details. The exponent of the resulting potential for a domain $[\Phi(0.5), \Phi(\eta_{ir})] \sim [0.85, 1.5]$, that is, for relatively large $\Phi$, can be approximated by exponential functions as,

$$\frac{d}{d\Phi} \log V_D(\Phi) \sim 1.20 + \frac{3.60}{\Phi} - \frac{2.45}{\Phi^2} . \quad (55)$$

as seen in Fig.4. The large $\Phi$ behavior of the dilaton potential in the IHQCD is closely related to the hadron spectra. According to [19, 20], when $\Phi$ is sufficiently large, the Regge behavior of the glueball spectra leads to

$$\frac{d}{d\Phi} \log V_D(\Phi) \sim 2Q + \frac{P}{\Phi} . \quad (56)$$

with $Q = 2/3$ and $P = 1/2$. We may say that our value $Q$ is close to this, and the order of our value $P$ is consistent with this.
IV. PREDICTION: WILSON LOOP

Since we have obtained the hull bulk action, via the AdS/CFT dictionary we can compute various QCD quantities as a prediction of the holographic model. Here, as one of the basic predictions, we calculate the holographic Wilson loop, i.e. the quark-antiquark potential, and compare it with the lattice QCD results.

A. Temperature and metric

The calculation of the holographic Wilson loop needs the metric $f, g$ in the string frame, which are given by

$$f = e^{\frac{\psi + 3\chi}{2}} - \frac{\chi}{2} \Phi, \quad g = e^{\frac{\psi - w}{2}} + \frac{w}{2} \Phi.$$ (57)

Note that in the previous section we have determined only $v = \partial_\eta \psi$ and $w = \partial_\eta \chi$, so to get the metric components, we need the integration:

$$\psi(\eta) = -\int_{\eta}^{\infty} d\eta' \left( v(\eta') - \frac{4\eta'}{L} \right) + \frac{4\eta}{L} + 4\delta_T, \quad (58)$$

$$\chi(\eta) = -\int_{\eta}^{\infty} d\eta' w(\eta'). \quad (59)$$

Here, when we make the integration, we are careful about the integration constant. The second equation (59) does not have the integration constant since we have the boundary condition $\chi(\infty) = 0$ which follows from the fact that the metric goes asymptotically to the pure AdS metric: $f/g \to 1$ as $\eta \to \infty$. The first equation (58) has a nontrivial integration constant $\delta_T$, which actually includes the temperature dependence of the metric. This constant needs to be determined such that the Hawking temperature $T$ appearing in the standard formula

$$f \sim (2\pi T)^2 \eta^2 \quad \text{for} \quad \eta \sim 0 \quad (60)$$

is equal to the temperature of the QCD chiral condensate data which we used for the machine learning, $T = 208[\text{MeV}]$. Since the equation (60) deals with the coefficient of $\eta^2$ which is difficult to treat, let us introduce a function

$$F(\eta) \equiv \psi + 3\chi + \frac{2}{3} \Phi - \log \left( \tanh \frac{\eta}{L} \right) - \frac{\eta}{L},$$ (61)

which behaves at the two asymptotics as non-zero constants,

$$F(\eta) \sim \begin{cases} \delta_T & \text{for } \eta \gg L \\ \log(2\pi TL) & \text{for } \eta \sim 0. \end{cases} \quad (62)$$

Then using the integral expression

$$\log \left( \tanh \frac{\eta}{L} \right) = -\int_{\eta}^{\infty} d\eta' \frac{2}{L} \csch \frac{2\eta'}{L}, \quad (63)$$

we find

$$F(\eta) = \delta_T + \frac{2}{3} \Phi - \int_{\eta}^{\infty} d\eta' \left( \frac{v + 3w}{4} - \frac{2}{L} \csch \frac{2\eta'}{L} - \frac{1}{L} \right). \quad (64)$$

Therefore, using the expression for $F(0) = \log(2\pi TL)$, we need to determine the integration constant $\delta_T$ as

$$\delta_T = \delta_{\text{ref}} + \log(2\pi TL),$$ (65)

$$\delta_{\text{ref}} \equiv \int_{0}^{\infty} d\eta' \left( \frac{v + 3w}{4} - \frac{2}{L} \csch \frac{2\eta'}{L} - \frac{1}{L} \right) - \frac{2}{3} \Phi_0. \quad (66)$$

Using this formula, we can completely determine the metric components. Note that in reality we need to replace the integration region $(0, \infty)$ by $(\eta_{\text{ref}}, \eta_{\text{max}})$ for our numerical purpose. The resulting data are summarized in Table IV.

In Fig.5 we plot profiles of $f, g$ for trial #2 with taking $\delta_T = 0$ for simplicity, comparing them with those, $f = f_E, g = g_E$ calculated in the pure Einstein gravity case.

B. Holographic Wilson loop

Let us place a quark and an antiquark on the AdS boundary, keeping the quark-antiquark separation distance $d_{\text{W}}$, and consider a string hanging between them, of which the midpoint at $\eta = \eta_0$ is the deepest point of

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
Trial & $TL$ & $\Phi_0$ & $\delta_T$ & $d_{\text{W}}$ & $T_{\text{W}}$ & $T_{\text{W}}$ & $\Phi_0$ & $T_{\text{W}}$ \\
\hline
\#1 & 1.0761 & 1.4782 & -0.216 & 0.545 & 0.32 \\
\#2 & 0.751 & 1.4718 & -0.231 & 0.527 & 0.35 \\
\hline
\end{tabular}
\caption{Various resultant data.}
\end{table}
the string in the radial $\eta$ direction. According to [23, 24], the distance $d_W$ is given in terms of $\eta_0$ as

$$d_W = 2 \int_{\eta_0}^{\infty} \frac{d\eta}{\sqrt{f(\eta)g(\eta)}} \left( \frac{f(\eta_0)g(\eta_0)}{f(\eta_0)g(\eta_0) - f(\eta)g(\eta)} - 1 \right),$$

which is plotted in Fig.6 using the training results #1 and #2.

The quark-antiquark potential $V_W$ is calculated as

$$2\pi\alpha'V_W = 2\int_{\eta_0}^{\infty} d\eta \sqrt{f(\eta)} \left( \sqrt{\frac{f(\eta_0)g(\eta_0)}{f(\eta_0)g(\eta_0) - f(\eta)g(\eta)}} - 1 \right) \left( \frac{f(\eta_0)g(\eta_0)}{f(\eta_0)g(\eta_0) - f(\eta)g(\eta)} - 1 \right),$$

where the infinite energy of parallel two straight strings hanging from the AdS boundary to the endpoint at the horizon has been subtracted from the potential to make this expression finite. Since the string configuration is determined such that the free energy of the string is minimized, the connected string with $V_W > 0$ is not realized. If $V_W > 0$, the configuration of the two parallel straight strings is realized. The latter means the Debye screening of the quarks.

By combining the above two, we obtain a parametric representation of the function $V_W(d_W)$ as is plotted in Fig.7. The multivalued potential stems from the existence of two solutions for $d_W(\eta_0)$ in (67), see Fig.6. The upper branch is never realized, so one needs to look at only the lower branch in Fig.7. In addition, there is another configuration which is just a set of the straight parallel strings, giving $V_W = 0$. Thus, in Fig.7 once the lower branch goes above the $d_W$ axis, it is not realized and replaced by the line $V_W = 0$.

These $d_W$ and $V_W$ in the physical unit are obtained by the following formulas

$$d_W = d_W\bigg|_{L=1,\delta_T=0} \times e^{-\delta_{ref}},$$

$$\frac{2\pi\alpha'}{L^2} V_W = 2\pi\alpha' V_W\bigg|_{L=1,\delta_T=0} \times 2\pi T e^{\delta_{ref}},$$

and information of $e^{\delta_T} = 2\pi T L e^{\delta_{ref}}$ is easily restored.\(^{12}\)

Let us look at the short-distance and long-distance behavior of the obtained quark-antiquark potential. First, as for the short-distance behavior at small $d_W$ which corresponds to $\eta_0 \gg L$, we find

$$\frac{2\pi\alpha'}{L^2} V_W = - \frac{C_W^2}{d_W} + \text{const.},$$

where we have used

$$d_W = \frac{e^{-\delta_{ref}}}{2\pi T} C_W e^{-\frac{\eta_0}{\pi T}}, \quad C_W = \frac{(2\pi)^2}{\Gamma(\frac{1}{4})^2}. \tag{72}$$

This $1/d_W$ behavior is typical of conformal field theories, reflecting the fact that the string is in the pure AdS geometry. In fact, one can check that the quark potential is almost identical to that in the case of the pure Einstein gravity.

\(^{12}\) For instance, $\delta_{ref} = - \log(2\sqrt{2})$ in the Einstein gravity case with $\Phi_0 = 0$. 

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**FIG. 5.** Log plot for the metric components $f$ and $g$ for trial #2, and their comparison with the pure Einstein gravity case $f_E$ and $g_E$. We took $\delta_T = 0, L = 1$ for simplicity.

**FIG. 6.** Profiles of $d_W(\eta_0)$ with $L = 1$. Here $\delta_T = 0$ is formally taken for simplicity.

**FIG. 7.** Profiles of $V_W(d_W)$ in the Physical units.
We observe that the tension 

\[ T_W = 2\pi\alpha' L^2 \sim 0.35 \text{[GeV fm]} = 1.8 \text{[GeV fm}^{-1}] \]  

which take an almost fixed value for \( \eta_0 \) in the plateau. In fact, we find that this formula is consistent with Eq.(74) and Fig.8.

Finally, in Fig.9 we compare our results with those of the lattice calculation [26], adjusting the origin and the scale of the vertical axis. The lattice data shows that the string breaking distance at \( T \sim 200 \text{[MeV]} \) is estimated as \( \Delta \sim 0.5 \text{[fm]} \), which is in good agreement with our prediction \( d_W \) given in (75). This agreement ensures the validity of the present model of the emergent dilaton potential.

The long-distance behavior of the quark-antiquark potential differs significantly from the case of Einstein gravity. In fact, ours possesses the confining linear potential [13] as an intersection with \( V_W \equiv 0 \).

For trial\#2, the string breaking scale \( d_W \) determined\(^{13}\) is

\[ d_W = 0.587 \times e^{-\delta_{\text{ref}}} = 2.67 \text{[GeV}^{-1}] = 0.527 \text{[fm]}. \]  

This breaking distance is the physical prediction of the model.

Comparing Fig.5 and Fig.6, we find that this appearance of the linear potential is due to the existence of the plateau of the metric \( \sqrt{g} \) in Fig.5. Such a plateau does not appear in that for the Einstein gravity case. Note that Eq.(67) and Eq.(68) are defined for \( \eta > \eta_0 \) so that \( f(\eta)g(\eta) \) is a monotonically increasing function for all \( \eta > \eta_0 \) and the plateau immediately causes large value of \( d_W \) and \( V_W \). We can extract this large contribution from \( V_W \) using that in \( d_W \) by rewriting Eq.(68) as,

\[ 2\pi\alpha' V_W = \int f(\eta_0)\eta_0 d\eta \]

\[ -2 \int_{\eta_0}^{\infty} d\eta \left( \sqrt{f(\eta)} - \sqrt{f(\eta_0)} \right) \]

\[ -2 \int_0^{\eta_0} d\eta \sqrt{f(\eta)}. \]  

In this formula, only values of \( V_W \) and \( d_W \) drastically changes in the plateau and as a result we observe an almost linear potential there. We can read the tension \( T_W \) from this expression as

\[ T_W = \frac{2\pi\alpha'}{L^2} \sim \frac{\sqrt{f(\eta_0)\eta_0}}{L^2}, \]  

\[ \text{FIG. 9. Comparison between the lattice data [26] (black and grey symbols) and our result for trial \#2 (red dots and a red straight line) for the quark antiquark potential. The vertical axis is the potential, and the horizontal axis is the quark separation. The realized holographic potential is the lowest one among all the red dots and the red line at each value of the quark distance. To plot our result, we choose the value of \( \alpha' \) such that the long distance behavior of our model fits that of the lattice data, to find that our fitting value is \( L^2/2\pi\alpha' = 0.7 \). In addition, since our model is for \( T = 208 \text{[MeV]} \), we set the zero of our potential as 0.4 [GeV] in the lattice data plot, which is the red line.} \]
V. SUMMARY AND DISCUSSION

In this paper we have derived the dilaton potential (Fig. 3) in improved holographic QCD, from the lattice data of the chiral condensate as a function of the quark mass. First we have improved the neural ODE provided in [13] for obtaining the bulk metric from the chiral condensate data. Then requiring that the emergent machine-learned metric is a solution of the bulk equations of motion of an Einstein-dilaton system, we have derived the dilaton potential uniquely. This completes the bulk reconstruction program.

The result of the neural ODE fixes only a certain combination of the metric components. However, requiring that they solve the equations of motion, we can constrain the dilaton potential in the bulk action and in fact can derive it. This determines all the metric component and the dilaton profile uniquely, at the same time. We have used the determined metric to calculate a holographic Wilson loop, and have obtained a prediction of the QCD-string breaking distance $d \sim 0.5$ [fm] at our temperature value $T = 208$ [MeV] (see (75)). This value is almost identical to what is known in lattice simulations (see Fig. 9).

Let us make a brief comparison between the dilaton potential obtained by the chiral condensate in this work and the one obtained in [10] by the data of the hadron spectra. They are consistent with each other. In fact, our dilaton potential is defined in the range of $0 \leq \Phi \leq 1.5$ while the dilaton potential in [10] is in $0 \leq \Phi \leq O(20)$. The latter is insensitive to the detailed region near the origin. So naively we can combine these two dilaton potentials to form a consistent single dilaton potential. This unified Einstein-dilaton system can recover both of the chiral condensate at the non-zero temperature and the $\rho$-meson spectra at zero temperature.[14]

We should confess that it was unexpected that the string breaking distance took the right value in our model, because the confinement and Debye screening of the quarks is not directly related to the behavior of the chiral condensate in QCD. In fact, in the previous estimate of the string breaking distance by the machine-learned metric [6], the estimated value is quite different from our value. There are two reasons for that. First, in [6] a guess for a certain component of the metric was used, while in our present work we have derived all the components of the metric by requiring that they should be derived from a single Einstein-dilaton system. Another reason is that we have improved the machine learning to refine the metric. So we conclude that the guess of the metric component in [6] was too naive, and once the bulk system is specified the prediction of the string breaking distance goes quite well.

Finally let us discuss a subtle and remaining issue of the model. In this work we considered only the data at $T = 208$ [MeV], and in the future work it is desirable to include all lattice data at various values of the temperature, to find a unique and consistent bulk action from which all of the data is reproduced. Unfortunately this appears to be difficult, as follows. Assuming that the dilaton potential $V_D$ does not explicitly have a temperature dependence, we may calculate $h(\eta)$ from the Einstein-dilaton system with the potential $V_D(\Phi)$, and then we will find that $h(\eta)$ has no temperature dependence at all. This contradicts the temperature dependence of the chiral condensate. Therefore, to find the consistent model for any temperature, we may need to loosen our assumptions used in this paper (such as $m_D^2 > 0$), or we may need to consider more general bulk action with more terms. Other possibility would be to take into account the back reaction of the probe scalar field for the chiral condensate. These kinds of the generalized models deserve a detailed study.

Since the data-driven holographic modeling is now possible as we have demonstrated in this paper and in [10], the next task would be to find a unified holographic QCD model which reproduces all the physical observables of QCD. For this task, we need to compare various inversely-solved holographic models, as we have briefly tried above. Since QCD has infinite amount of data, the unification may need more machinery of deep learning.

ACKNOWLEDGMENTS

We would like to thank Hong-Ye Hu for discussions. The work of K.H. is supported in part by JSPS KAKENHI Grant Number JP22H05115, JP22H05111 and [16] Note that in this paper we have assumed $m_D^2 > 0$ which means $\Phi = 0$ is a top of a mountain in a sign-flipped potential. With this assumption, the dilaton profile is determined uniquely, so there will be no temperature dependence. Basically this is because the profile of the dilaton and the initial value $\Phi_0$ are uniquely fixed such that the radial "motion" of the dilaton, which starts at $\Phi_0$ at "time" $\eta = 0$, has to stop at the top of the sign-flipped dilaton potential hill at " time" $\eta = \infty$. Then the solutions of $\nu(\eta), \nu(\eta), \Phi(\eta)$ are uniquely determined by giving an initial value $\Phi_0$ (see Eq. (43)), and thus the chiral condensate does not depend on the temperature. Note that the explicit temperature dependence in the metric appears only as a common overall coefficient $e^{2L\eta} f, g$, as

$$
(f, g) = (f, g) \bigg|_{\eta = 0} e^{2L\eta} = (f, g) \bigg|_{T = T_{ref}} \times \left( \frac{T}{T_{ref}} \right)^2, \quad (78)
$$

with $T_{ref} \equiv e^{-h_{ref}/2\pi L}$. This overall factor is just the integration constant and does not appear in the expression of $h(\eta)$ which determines the chiral condensate. In this paper we have assumed $m_0^2 > 0$, otherwise the strategy we have taken here does not work well technically. On the other hand, if we have adopted the case $m_0^2 \leq 0$, $\Phi_0$ could be a continuous moduli of the solution and the temperature dependence can be encoded in $\Phi_0$.

15 This is a qualitative statement, and still needs a detailed confirmation. For example, our obtained AdS radius $L \sim 3.6$ [GeV$^{-1}$] $= 0.71$ [fm] at $T = 0.208$ [GeV] is slightly different from the value $L \sim 0.51$ [fm] at $T = 0$ obtained in [10].
Appendix A: Metrics trained by neural ODE

In this appendix, we describe the regularization methods and the training methods of our neural ODE in more details. The numerical data of the trained function $h$ is described in Table [II] with the definition [5].

In Fig. 10 we show our numerical results of the neural ODE. As the figure shows, the fit functions $h(\eta)$ reproduces the lattice QCD data well.

Let us describe our regularization method of the neural ODE. As mentioned in [13], the merit of the neural ODE is that there is no need of the smoothing regularization which was employed in the original AdS/DL research [11,15] to regard deep neural networks to be spacetimes. However, still we need some regularization to make sure that the asymptotic region of the function $h$ should approach the constant $h = 4$ so that the spacetime is asymptotically AdS$_5$. Furthermore, since we employ Einstein-dilaton system for the holographic QCD, the emergent metric needs to approach $h = 4$ in a specific manner. When the asymptotic behavior is approximated by $h(\eta) \approx 4 - 4B_0 \exp[-p\eta]$, the Einstein-dilaton system requires $4 < p \leq 8$ (see [14]).

To satisfy these two kinds of the requirement, we put the following regularization terms in the loss function $L$ of the neural ODE:

$$L_h = c_1(h(1) - 4)^2 + c_2\left(\frac{h'(1)}{h(1)} - 4\right)^2$$

where $c_1$ and $c_2$ are some positive numbers called regularization coefficients which are hyperparameters of the machine learning. The first term brings the trained metric function at $\eta = 1$ closer to the AdS value $h = 4$, and the second term brings the exponent $p$ closer to the value $p = 4$ which is the central value of the allowed region for $p$ in the Einstein-dilaton system.

During the training, we have to tune the initial conditions and the magnitude of the regularization for the training to be successful. The training architecture other than the regularizations is identical to that employed in [13]. Our successful training is obtained by the following procedure. First, we choose the initial conditions as follows:

$$L = 1.0, \quad \lambda = 0.3, \quad b_1 = -4.10, \quad b_3 = 7.00, \quad b_5 = 0.00.$$  \hspace{1cm} (A2)

These values for $b_1$, $b_3$ and $b_5$ are suggested from the functional form of $h$ of the training results given in [6,13]. We choose $c_1 = 0.01$ and $c_2 = 0.0000027$ (we tune the maximum value of $c_2$ for the training to be successful). Then after 10000 epochs of the training, we reach

$$L = 3.349, \quad \lambda = 0.008769, \quad b_1 = -3.819, \quad b_3 = 6.657, \quad b_5 = -0.4160. \hspace{1cm} (A3)$$

We use this data as another initial condition for the training with $c_1 = 0.01$ and $c_2 = 0.001$, and after 30000 epochs, we obtain the training result #1 given in Table [II]. Finally, using the result #1 as an initial condition, we make a training with $c_1 = 0.01$ and $c_2 = 0.003$ for 10000 epochs to find the training result #2 given in Table [II].

In summary, the difference between the results #1 and #2 is the strength of the second term in the regularization (A1). The training result #2 has a value of $B_0$ closer to $p = 4$, while both of #1 and #2 have the values of $p$ in the consistency range $4 < p \leq 8$.

As far as the training, we also note here that we checked if the resulting $\Phi(\eta)$ is monotonically decreasing function, as already declared in Eq.(5). Or equivalently,

$$v(\eta) > h(\eta) \quad \text{for} \quad \forall \eta, \hspace{1cm} (A4)$$

otherwise, $V_D(\Phi)$ turns out to be a multi-valued function unless something miraculous happens. With a given trained metric $h(\eta)$, Inequality (A4) is the second obstacle after Inequality (44) that the trained $h(\eta)$ has to overcome. In fact, before the modifications described in Sec.II many trained data for $h(\eta)$ with poor initial conditions, or a rough calculation accuracy caused multi-valued dilaton potential and were rejected.

Appendix B: $Z_2$ parity of functions

Even if the dilaton potential is unknown, an assumption that the potential is smooth leads to a certain $Z_2$ parity in the solution of $v, w$ and $\Phi$ around $\eta = 0$,

$$v(\eta) = v_{\text{odd}}(\eta) + \alpha_v \eta^n + \cdots,$$
$$w(\eta) = w_{\text{odd}}(\eta) + \alpha_w \eta^n + \cdots,$$
$$\Phi(\eta) = \Phi_{\text{even}}(\eta) + \alpha_{\Phi} \eta^{n+1} + \cdots. \hspace{1cm} (B1)$$

where $v_{\text{odd}}(\eta), w_{\text{odd}}(\eta)$ are odd functions with respect to $\eta$ and $\Phi_{\text{even}}(\eta)$ is even, and the rest of terms in the r.h.s are assumed to be not so, and $n, n', n'' \in 2\mathbb{Z} + 1$ are assumed to give the smallest powers within them, satisfying $n, n', n'' > -1$. By substituting these to the equations [12, 13] and [14] and taking an expansion around $\eta = 0$, we find that contributions coming from only $v_{\text{odd}}, w_{\text{odd}}$ and $\Phi_{\text{even}}$ give expansions of even powers. And thus focusing on the lowest order terms in the equations such that their order are not even, those contributions must give the linearized equations for the terms $\alpha_v \eta^n, \alpha_w \eta^{n'}$. JP22H01217. The work of T.S. is supported in part by JSPS KAKENHI Grant No. JP20J20628.
FIG. 10. The training results of the neural ODE. (a) and (b) are for our successful trial #1 and #2, respectively. In each figure, the left panel shows how the QCD lattice data (shown green) is successfully reproduced by the model (orange + green). Red lines are the original central points of the lattice QCD data. The right panel in each figure shows the trained metric function $h(\tilde{\eta})$ where $\tilde{\eta} = 1 - \eta$ (the region for the training is $0 \leq \eta \leq 1$ in the unit $L = 1$). At $\tilde{\eta} = 1$, the BH horizon exists, while $\tilde{\eta} = 0$ is supposed to be the AdS asymptotic region, thus approaches to $h = 4$.

and $\alpha_\Phi \eta^{n''+1}$ that are independent of the other contributions as,

$$0 = 2(n+1)\alpha_v \eta^{n-1} + 2\alpha_w \eta^{n''-1} + \frac{4}{3}(n''+1)\alpha_\Phi \eta^{n''-1} + \cdots$$  \hspace{1cm} (B2)

$$0 = (n'+1)\alpha_w \eta^{n'-1} + \alpha_v \eta^{n-1} + \frac{2}{3}(n''+1)\alpha_\Phi \eta^{n''-1} + \cdots$$  \hspace{1cm} (B3)

$$0 = (n''+1)^2\alpha_\Phi \eta^{n''-1} + 2a_1^{(\Phi)} \alpha_v \eta^{n+1} + \cdots$$  \hspace{1cm} (B4)

with using the leading terms of $v_{\text{odd}}, w_{\text{odd}}, \Phi_{\text{even}}$ as background fields,

$$v_{\text{odd}}(\eta) \sim \frac{1}{\eta}, \quad w_{\text{odd}}(\eta) \sim \frac{1}{\eta}, \quad \Phi_{\text{even}}(\eta) \sim \Phi_0 + a_1^{(\Phi)} \eta^2.$$  \hspace{1cm} (B5)

If $\{\alpha_v, \alpha_w, \alpha_\Phi\}$ are assumed to be non-trivial, the first two equations requires $n = n' = n''$ and $\alpha_v, \alpha_w \propto \alpha_\Phi$, whereas the last equation requires that $\alpha_\Phi$ vanishes and that is, $\alpha_v = \alpha_w = \alpha_\Phi = 0$. This inconsistency shows that $v(\eta), w(\eta)$ must be odd functions and $\Phi(\eta)$ are even. That is, $f, g$ and $\Phi$ have formally the following $\mathbb{Z}_2$ properties,

$$f(-\eta) = f(\eta), \quad g(-\eta) = g(\eta), \quad \Phi(-\eta) = \Phi(\eta).$$  \hspace{1cm} (B6)

although we do not consider the inside of the BH horizon for $\eta < 0$. Especially, $h(\eta)$ must be an odd function

$$h(-\eta) = -h(\eta).$$  \hspace{1cm} (B7)
[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231-252 (1998) doi:10.1023/A:1026654312961 [arXiv:hep-th/9711200 [hep-th]].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B 428, 105-114 (1998) doi:10.1016/S0370-2693(98)00377-3 [arXiv:hep-th/9802109 [hep-th]].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253-291 (1998) doi:10.4310/ATMP.1998.v2.n2.a2 [arXiv:hep-th/9802150 [hep-th]].

[4] K. Hashimoto, S. Sugishita, A. Tanaka and A. Tomiya, “Deep learning and the AdS/CFT correspondence,” Phys. Rev. D 98, no.4, 046019 (2018) doi:10.1103/PhysRevD.98.046019 [arXiv:1802.08313 [hep-th]].

[5] Y. Z. You, Z. Yang and X. L. Qi, “Machine Learning Spatial Geometry from Entanglement Features,” Phys. Rev. B 97, no.4, 045153 (2018) doi:10.1103/PhysRevB.97.045153 [arXiv:1709.01223 [cond-mat.dis-nn]].

[6] K. Hashimoto, S. Sugishita, A. Tanaka and A. Tomiya, “Deep Learning and Holographic QCD,” Phys. Rev. D 98, no.10, 106014 (2018) doi:10.1103/PhysRevD.98.106014 [arXiv:1809.10536 [hep-th]].

[7] H. Y. Hu, S. H. Li, L. Wang and Y. Z. You, “Machine Learning Holographic Mapping by Neural Network Renormalization Group,” Phys. Rev. Res. 2, no.2, 023369 (2020) doi:10.1103/PhysRevResearch.2.023369 [arXiv:1908.07366 [cs.LG]].

[8] K. Hashimoto, “AdS/CFT correspondence as a deep Boltzmann machine,” Phys. Rev. D 99, no.10, 106017 (2019) doi:10.1103/PhysRevD.99.106017 [arXiv:1903.04951 [hep-th]].

[9] X. Han and S. A. Hartnoll, “Deep Quantum Geometry of Matrices,” Phys. Rev. X 10, no.1, 011069 (2020) doi:10.1103/PhysRevX.10.011069 [arXiv:1906.08781 [hep-th]].

[10] J. Tan and C. B. Chen, “Deep learning the holographic black hole with charge,” Int. J. Mod. Phys. D 28, no.12, 1950153 (2019) doi:10.1142/S0218271819501530 [arXiv:1908.01470 [hep-th]].

[11] Y. K. Yan, S. F. Wu, X. H. Ge and Y. Tian, “Deep learning black hole metrics from shear viscosity,” Phys. Rev. D 102, no.10, 101902 (2020) doi:10.1103/PhysRevD.102.101902 [arXiv:2004.12112 [hep-th]].

[12] T. Akutagawa, K. Hashimoto and T. Sumimoto, “Deep Learning and AdS/QCD,” Phys. Rev. D 102, no.2, 026020 (2020) doi:10.1103/PhysRevD.102.026020 [arXiv:2005.02636 [hep-th]].

[13] K. Hashimoto, H. Y. Hu and Y. Z. You, “Neural ordinary differential equation and holographic quantum chromodynamics,” Mach. Learn. Sci. Tech. 2, no.3, 035011 (2021) doi:10.1088/2632-2153/abe527 [arXiv:2006.00712 [hep-th]].

[14] H. Y. Chen, Y. H. He, S. Lal and M. Z. Zaz, “Machine Learning Etudes in Conformal Field Theories,” [arXiv:2006.16114 [hep-th]].

[15] M. Song, M. S. H. Oh, Y. Ahn and K. Y. Kim, “AdS/Deep-Learning made easy: simple examples,” Chin. Phys. C 45, no.7, 073111 (2021) doi:10.1088/1674-1137/abfe36 [arXiv:2011.13726 [physics.class-ph]].

[16] K. Hashimoto, K. Ohashi and T. Sumimoto, “Deriving the dilaton potential in improved holographic QCD from the meson spectrum,” Phys. Rev. D 105, no.10, 106008 (2022) doi:10.1103/PhysRevD.105.106008 [arXiv:2108.08091 [hep-th]].

[17] J. Lam and Y. Z. You, “Machine learning statistical gravity from multi-region entanglement entropy,” Phys. Rev. Res. 3, no.4, 043199 (2021) doi:10.1103/PhysRevResearch.3.043199 [arXiv:2110.01115 [hep-th]].

[18] R. Katsube, W. H. Tam, M. Hotta and Y. Nambu, “Deep Learning Metric Detectors in General Relativity,” [arXiv:2206.03006 [gr-qc]].

[19] U. Gursoy and E. Kiritsis, “Exploring improved holographic theories for QCD: Part I,” JHEP 02, 032 (2008) doi:10.1088/1126-6708/2008/02/032 [arXiv:0707.1324 [hep-th]].

[20] U. Gursoy, E. Kiritsis and F. Nitti, “Exploring improved holographic theories for QCD: Part II,” JHEP 02, 019 (2008) doi:10.1088/1126-6708/2008/02/019 [arXiv:0707.1349 [hep-th]].

[21] T. Q. Chen, Y. Rubanova, J. Bettencourt, and D. K. Duben, “Neural Ordinary Differential Equations,” Advances in neural information processing systems (2018) 6571 [arXiv:1806.07366 [cs.LG]].

[22] W. Unger, “The Chiral Phase Transition of QCD with 2+1 Flavors : A lattice study on Goldstone modes and universal scaling,” PhD thesis, der Universitat Bielefeld (2010).

[23] J. M. Maldacena, “Wilson loops in large N field theories,” Phys. Rev. Lett. 80, 4859-4862 (1998) doi:10.1103/PhysRevLett.80.4859 [arXiv:hep-th/9803002 [hep-th]].

[24] S. J. Rey and J. T. Yee, “Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity,” Eur. Phys. J. C 22, 379-394 (2001) doi:10.1007/s100520010799 [arXiv:hep-th/9803001 [hep-th]].

[25] Y. Kinar, E. Schreiber and J. Schonnenschmid, “Q anti-Q potential from strings in curved space-time: Classical results,” Nucl. Phys. B 566, 103-125 (2000) doi:10.1016/S0550-3213(99)00652-5 [arXiv:hep-th/9811192 [hep-th]].

[26] P. Petreczky, “Quarkonium in Hot Medium,” J. Phys. G 37, 094009 (2010) doi:10.1088/0954-3899/37/9/094009 [arXiv:1001.5284 [hep-th]].