Elliptic flow from quark coalescence: mass ordering or quark number scaling?

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We show that either mass ordering or quark number scaling of anisotropic flow can result from quark coalescence, depending on the nature of phase space correlations at hadronization. Quark number scaling signals nonequilibrium dynamics because it can only appear when hydrodynamic correlations break down. However, the scaling does not hold for all nonthermal distributions, and is compatible with covariant transport theory only if remarkable cancellations occur at RHIC.

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Introduction and conclusions. The goal of recent nuclear collision experiments at the Relativistic Heavy Ion Collider (RHIC) is to create extremely hot and dense nuclear matter and determine its properties. An important experimental probe in this quest is elliptic flow, \( v_2 \equiv \langle \cos(2\phi) \rangle \), the second Fourier moment of the azimuthal momentum distribution. Its dependence on impact parameter \( b \), transverse momentum, \( p_T \) and particle species provides unique constraints on the equation of state (EOS) and collision dynamics.

An amazing wealth of experimental \( v_2(p_T) \) data is available from \( Au + Au \) at \( \sqrt{s_{NN}} = 130 \) and 200 GeV at RHIC for charged hadrons \( \pi^\pm, K^\pm \), and \( p + \bar{p} \), \( K^0 \), \( \Lambda + \bar{\Lambda} \), and \( \Xi^- + \bar{\Xi}^+ \) and \( \Omega^- + \bar{\Omega}^+ \). Below \( p_T \lesssim 1 - 1.5 \) GeV, and for \( b \lesssim 8 \) fm, these data can be reproduced fairly well using ideal (Euler) hydrodynamics. In that theory, flavor dependence enters mainly through the mass of the particles, and the \( v_2(p_T) \) mass ordering pattern depends on the EOS. Remarkably, the data indicate a first-order deconfinement phase transition to the so called quark-gluon plasma (QGP) phase.

In contrast, a spectacular quark number scaling pattern \( v_2(p_T) \) is predicted from hadronization via quark coalescence. In the coalescence approach mesons (baryons) form from two (three) co-moving quarks/antiquarks, leading to a unique meson-baryon differentiation in the flow pattern.

Though mesons tend to be lighter than baryons, quark number scaling is in general different from mass ordering. Comparison between precise elliptic flow \( v_2(p_T) \) data for heavy mesons, such as the \( \phi \) (1020) or \( K^{*+} \) (892), and light baryons (e.g., protons or lambdas) could distinguish between the two behaviors.

In view of the above difference, the realization that for thermal constituent distributions coalescence reduces to statistical hadronization generated significant controversy. If, in that case, hadronization is the same both from coalescence and hydrodynamics, how could the resulting hadronic momentum anisotropies be different?

In this paper we show that there is no contradiction at all. Either of the two scaling behaviors can result from quark coalescence, depending on the type of phasespace correlations present at hadronization. Thermal constituent distributions imply unique coordinate-momentum correlations, for which mass ordering follows from coalescence. For quark number scaling to appear, departure from hydrodynamic behavior is essential. The growing experimental evidence \( K, \Lambda, \Xi \) for quark number scaling in the intermediate transverse momentum region \( p_T \sim 2 - 5 \) GeV at RHIC is a signal of the breakdown of hydrodynamics above \( p_T > 2 \) GeV, which complements and corroborates other indications, such as the saturation of elliptic flow.

Quark number scaling emerges only under certain conditions. Either only a subset of all possible local momentum anisotropies can be present, or important cancellations between contributions by the different anisotropies need to occur. Only the latter possibility is compatible with covariant transport theory, in which high-\( p_T \) particles are emitted in a strongly preferred direction locally.

Here we consider momentum anisotropies right after hadronization but before resonance decays. Secondary production affects significantly the final anisotropies in hydrodynamics, but seems to influence little the quark number scaling pattern. At large \( p_T \gtrsim 5 \) GeV, contributions from jet fragmentation (ignored here) also need to be considered.

In addition, this study considers coalescence on a 3D spacetime hypersurface. In coalescence from diffuse 4D freezeout distributions in spacetime, constituent spacetime and space-momentum correlations influence the final hadron distributions in a more complex way.

Anisotropic flow. Momentum anisotropies can be characterized via the Fourier expansion of the freezeout source distribution \( S(x, p_T, y) \equiv dN/d^4x d^2p_T dy \) in terms of the momentum azimuthal angle \( \phi \)

\[
S(x, p_T, y) \equiv S_0(x, p_T, y) \left[ 1 + 2 \sum_{n=1}^{\infty} \Re \left( v_n(x, p_T, y) e^{-in\phi} \right) \right] (1)
\]

Here \( p_T \equiv p_{T}(\cos \phi, \sin \phi) \), \( \phi = 0 \) is the impact parameter direction, \( y \equiv 0.5 \ln(E + p_z)/(E - p_z) \) is the rapidity, and \( v_n(x, p_T, y) \) is the \( n \)-th order momentum anisotropy coefficient for particles emitted from an in-
finitesimal spacetime volume around \( x \). The real and imaginary parts of \( v_n \) correspond to \( \cos(n\phi) \) and \( \sin(n\phi) \) terms. The final observable anisotropy \( \bar{v}_n \) is the weighted average

\[
\bar{v}_n(p_T, y) = \frac{\int d^4x S_0(x, p_T, y) v_n(x, p_T, y)}{\int d^4x S_0(x, p_T, y)},
\]

which is always real due to reflection symmetry across the collision plane \( \phi \rightarrow -\phi \) (for spherically symmetric nuclei). In addition, for a symmetric collision system, all odd \( \bar{v}_n \) vanish at midrapidity \( y \approx 0 \) due to \( \phi \rightarrow (\pi - \phi) \) symmetry.

In the following rapidity arguments will be often suppressed for brevity.

**Momentum anisotropies from ideal hydrodynamics.**

Two main assumptions in approaches based on ideal hydrodynamics are local kinetic equilibrium and sudden freezeout on a 3D spacetime hypersurface. In such a case

\[
S(x, \vec{p}) = p^n \sigma(x) \delta(x^\mu \sigma(x) - x_0^\alpha \sigma_n(x_0)) f(x, \vec{p})
\]

where \( \sigma(x) \) is the normalized normal vector of the hypersurface, while \( x_0 \) is an arbitrary point on the hypersurface\(^\text{31}\), and

\[
f_{th}(x, \vec{p}) = \frac{g}{(2\pi)^3} \left\{ e^{[p^\mu u_\mu(x)-\mu(x)]/T(x)} + a_s \right\}^{-1},
\]

where \( g \) is the degeneracy factor, and \( a_s = -1, 1, 0 \) respectively for bosons, fermions, or Boltzmann statistics. \( T, \mu, \) and \( u_\mu \) are the local temperature, chemical potential, and flow velocity.

Momentum anisotropies have two sources, the hypersurface \((p \cdot \sigma)\) and the flow profile \((p \cdot u)\). For hypersurfaces typically assumed in analytic studies, such as constant time or constant \( \tau = \sqrt{T - \vec{v}^2} \) hypersurfaces, \( p \cdot \sigma \) is independent of \( \phi \). Therefore, for simplicity we focus here on anisotropies due to the flow field only. Note, in general, \( \phi \) dependence can appear even for azimuthally symmetric hypersurfaces\(^\text{25}\).

The anisotropy coefficients in general depend on the particle species, mainly because of the difference in mass. Statistics and the chemical potentials only play a role at very low \( m_T \equiv \sqrt{m^2 + p_T^2} \sim \mu, T \). (At freezeout at RHIC, \( T \sim 100 - 130 \text{ MeV} \), while \( \mu \) increases from \( \sim 80 \text{ MeV} \) for pions to \( \sim 350 \text{ MeV} \) for heavy particles\(^\text{14}\).)

To demonstrate the mass dependence, consider not very low momenta, so that Boltzmann statistics is applicable. The flow field can be characterized by a local transverse and longitudinal rapidity component, \( \bar{v}_T(x) \equiv v_T(\cos \Phi_T, \sin \Phi_T) \) and \( \bar{y}(x) \), as \( u^\mu(x) \equiv (\text{ch} y, \bar{v}_T, \text{sh} \bar{y})/\sqrt{T - \bar{v}_T^2} \). It is simple to show that

\[
v_n(x, p_T, y) = e^{in\Phi_n(x)} \frac{I_n(s_{\bar{y}T}(x) p_T/T(x))}{I_0(s_{\bar{y}T}(x) p_T/T(x))},
\]

where \( s_{\bar{y}T} \equiv v_T/\sqrt{1 - v_T^2} \) and \( \{I_n\} \) are the modified Bessel functions. Remarkably, the local anisotropy depends only on \( p_T \), and therefore is the same for all particles and rapidities. Note, at low \( p_T \),

\[
|v_n| \approx (s_{\bar{y}T} p_T/2T)^n/n! \propto p_T^n,
\]

as dictated by general analyticity arguments\(^\text{27}\), while at high \( p_T \), \( |v_n| \approx 1 - \text{const} \times T/p_T \rightarrow 1 \).

The coordinate-averaged flow coefficients\(^\text{2}\), on the other hand, in general decrease in magnitude with particle mass. This follows from the properties of the weight \( S_0 \), which via\(^\text{3}\) corresponds to the distribution

\[
f_0 = \exp \left[ -\frac{m_T \text{ch} (y - \bar{y}) \text{ch}_{\bar{y}T}(x)}{T(x)} \right] I_0\left( p_T s_{\bar{y}T}(x) \right)
\]

While \( |v_n| \) monotonically increases with the radial flow \( y_T \), the only mass dependent part in \( f_0 \), the exponential, decreases with \( y_T \). The decrease is sharper for larger mass. Therefore for heavier particles, smaller \( v_n \) values get preferred. Though this argument ignores the \( x \)-dependence of the cosine term from the exponential in\(^\text{3} \), which in principle could reverse the rise of \( \text{Re} v_n \) with \( y_T \) and therefore affect the mass dependence pattern, the general trend prevails in practice.

For more details of the derivations in the case of \( \text{Re} v_2 \), see\(^\text{27}\), for example.

**Anisotropic flow from quark coalescence.**

In the quark coalescence approach, constituent quarks/antiquarks that are close in phasespace can combine to form hadrons. Assuming coalescence occurs on a 3D spacetime hypersurface, the hadron binding energies are small, and coalescence is a relatively rare process\(^\text{16, 17}\), the invariant hadron momentum distribution can be expressed\(^\text{14, 15, 24}\) in terms of the constituent phasespace distributions and the hadron Wigner functions.

Ignoring variations of phasespace distributions on length and momentum scales corresponding to a typical hadron \((\sim 1 \text{ fm and } \sim 200 \text{ MeV})\), the phasespace distributions of mesons and baryons created via \( \alpha \beta \rightarrow M \) and \( \alpha \beta \gamma \rightarrow B \) can be written as

\[
f_B(x, \vec{p}) = \frac{(2\pi)^3 g_a}{g_{\alpha \beta}} \frac{f_\alpha(x, \vec{p}_\alpha) f_\beta(x, \vec{p}_\beta) f_\gamma(x, \vec{p}_\gamma)}{g_{\alpha \beta} g_{\gamma}}
\]

\[
f_M(x, \vec{p}) = \frac{(2\pi)^3 g_m}{g_a g_{\beta}} \frac{f_\alpha(x, \vec{p}_\alpha) f_\beta(x, \vec{p}_\beta)}{g_{\alpha \beta}} .
\]

Here \( g \) is the degeneracy of the particle (spin and color), while \( \vec{p}_i \parallel \vec{p}_\alpha \) and the hadron momentum is shared roughly in proportion to constituent mass\(^\text{17}\). For hadrons composed of \( u, d, \) and \( s \) quarks, the sharing is approximately equal because of the relatively small difference between \( m_{u,d} \approx 0.3 \text{ GeV} \) and \( m_s \approx 0.5 \text{ GeV} \). On the other hand, in hadrons that also contain a heavy quark, e.g., \( D \) mesons or the \( \Lambda_c \), the heavy quark carries most of the momentum.
A direct implication of \( (7) \) is

\[
\begin{align*}
    v_{n,B}(x, p_T) &= \sum_{i=0,\beta} v_{n,i}(x, p_{T,i}) + \Delta v_{n,B}(x, p_T) \\
    v_{n,M}(x, p_T) &= \sum_{i=0,\beta} v_{n,i}(x, p_{T,i}) + \Delta v_{n,M}(x, p_T), \quad (8)
\end{align*}
\]

where to leading order in \( v_{n,i} \), the corrections \( \Delta v_{n,M} \) and \( \Delta v_{n,B} \) are

\[
\Delta v_n = \sum_{i \neq j, k=1}^{\infty} v_{n+k,i} v_{k,j} + \sum_{i<j, k=1}^{n-1} v_{n-k,i} v_{k,j} + O\left(\{v_{k,i}\}^3\right) \quad (9)
\]

with arguments identical to those in \( (8) \) dropped for brevity. Thus, if the corrections are small, \( v_n \) is additive, and in the absence of quark flavor dependence scales with quark number \( \frac{1}{n!} \).

Assuming, as for example in \( \ref{th:QCD} \), that the spatial and momentum dependence of phasespace distributions factorize, the local \( v_n \) coefficients are the same everywhere and therefore the scaling carries over to the observable spacetime averages \( v_\bar{n} \). In this case, the local \( v_n \) must of course be real, and at RHIC the odd ones must vanish at midrapidity. The neglect of the correction terms \( \Delta V_2 \) for elliptic flow is then justified because the data show \( v_3 \lesssim v_2 \lesssim 0 \).

On the other hand, the scaling does not hold for arbitrary distributions. For example, for thermal (Boltzmann) constituent phasespace distributions, coalescence \( \ref{coalescence} \) gives thermal hadron distributions because, for the weak bound states assumed, \( m_M \simeq m_\alpha + m_\beta, \quad m_B \simeq m_\alpha + m_\beta + m_\gamma \). Therefore, anisotropies are the same as from hydrodynamics, i.e., depend on the hadron mass. This also follows from a direct calculation. For mesons, the kernel is \( f_0 \rightarrow f_0, f_{0,\beta}[1 + \sum_{m=1} v_{n,\alpha} v_{n,\beta} + c.c.] \), which is identical to the hydrodynamic one with \( m = m_M \). The exponential part is the same because \( p_T = p_{T,\alpha} + p_{T,\beta} \), momenta are shared such that \( m_T \simeq m_{T,\alpha} + m_{T,\beta} \), and therefore also \( y \simeq y_\alpha \simeq y_\beta \). The rest of the kernel and all meson flow anisotropies also agree, as can be shown using the exact flow addition formula

\[
v_{n,M} = \sum_{i=0,\beta} v_{n,i} + \sum_{k=1}^{\infty} (v_{n+k,\alpha} v_{k,\beta} + \alpha \leftrightarrow \beta) + \sum_{k=1}^{n-1} v_{n-k,\alpha} v_{k,\beta} \\
= 1 + \sum_{k=1}^{\infty} (v_{k,\alpha} v_{k,\beta} + c.c.) \quad (10)
\]

and the Bessel function addition theorem. The direct proof for baryons is analogous but more involved.

Thus, for thermal constituent distributions, momentum anisotropies from quark coalescence depend on mass, and not quark number. There is an intuitive way to see why this must be so. In the frame where the flow velocity is zero, the momentum distribution is isotropic for all constituents. Therefore, in that frame all \( v_{n,i} \) vanish, and hence \( \ref{coalescence} \) gives no hadron anisotropies. The only source of anisotropy then is the Lorentz boost back to the laboratory frame, which depends only on the particle mass, momentum, and the boost velocity. This is also true even if hydrodynamics breaks down, as long as distributions are of the form \( f(x, p) = g(p \cdot u(x, x)) \). Quark number scaling can emerge only if, in any frame, at least one of the constituent distributions is anisotropic.

In the thermal case, quark number scaling is violated because the nonlinear terms \( \ref{th:QCD} \) are important. Consider for example \( v_2 \), at low \( p_T \) so that \( \ref{coalescence} \) is justified. To leading \( O(p_T^2) \) order, both the linear and the \( v_1 v_1 \) terms contribute. In fact the latter give as much as half the meson \( v_2 \), and two-thirds for baryons. For higher-order flow coefficients, all \( v_{n-k} v_k \) terms contribute at leading order in \( p_T \).

Therefore, one class of freezeout distributions (besides uniform local anisotropy \( v_n \)) for which the observable \( v_\bar{n} \) scales with quark number is when

\[
|v_n| \gg |v_{n-k}| |v_k| \quad (11)
\]

in the spacetime region where most particles are emitted from. In this case, there is (approximate) scaling locally and, because the formula is linear, scaling is preserved upon averaging. Note, a small amount of other anisotropies, for example \( v_1 \sim v_2^{3/2} \) besides a pure \( v_2 \), can even help compensate the denominator in \( \ref{th:QCD} \), which otherwise tends to reduce the flow relative to the scaling expectation.

The scaling may also hold even if the nonlinear terms are important. However, that requires a high degree of fortuitous cancellations at RHIC, as illustrated in Fig. \ref{fig:flow_results}. Results from the covariant parton transport model MPC \( \ref{MPC} \) are shown for \( Au + Au \) at \( \sqrt{s_{NN}} = 200 \text{ GeV} \) with \( b = 8 \text{ fm} \), from a calculation identical to the one in \( \ref{MPC} \) with a gluon-gluon cross section \( \sigma_{gg} = 10 \text{ mb} \). In the left panel local constituent quark anisotropies up to fourth order are plotted as a function of transverse momentum, averaged over the first quadrant of the transverse plane \( (0 < \phi_x < \pi/2, \text{ where } x_T = x_T(\cos \phi_x, \sin \phi_x)) \). Coefficients that vanish upon a full spacetime average due to symmetry, in particular \( v_1 \) and \( |v_2| \), are surprisingly large. Therefore, \( \ref{th:QCD} \) does not hold and, for example, the denominator in \( \ref{th:QCD} \) exceeds 2 above \( p_T,i \gtrsim 1 \text{ GeV} \).

The origin of these large anisotropies is shown in the right panel of Fig. \ref{fig:flow_results} where we plot the \( \phi \) distributions at midrapidity for \( 2 < p_T < 3 \text{ GeV} \), averaged over four wedges in the transverse plane \( \phi_x \in [k\pi/8, (k+1)\pi/8] \), with \( k = 0,1,2,3 \). Instead of small harmonic modulations over a uniform background, the distributions are strongly peaked because high-\( p_T \) particles can only escape from a surface layer of the reaction region. In this
case, anisotropies from coalescence change their character completely. Consider, for simplicity, Gaussian constituent distributions around an azimuthal direction $\phi_0$, 

$$S_i(x, p_T, y) = C_i(x, p_T, y) \exp\left[-(\phi - \phi_0)^2 / (2\sigma_i^2)\right].$$

For not too wide $\sigma \lesssim 2/\sqrt{n + 1}$, the constituent and hadron anisotropies

$$v_{n,i} \simeq e^{in\phi_0} e^{-n^2\sigma_i^2/2}, \quad v_{n,h} \simeq e^{in\phi_0} e^{-n^2(\sum_i \sigma_i^{-2})},$$

scale as $(const)^{-n^2}$. This is very different from both $(13)$ and $(11)$. For example, if all constituents have the same width $\sigma$, we have

$$|v_{n,B}(3p_T)| \simeq |v_{n,q}(p_T)|^{1/3}, \quad |v_{n,M}(2p_T)| \simeq |v_{n,q}(p_T)|^{1/2},$$

which gives for $|v_{n,q}| \gtrsim 0.1$ smaller baryon/meson flow ratios, and for $|v_{n,q}| > 0.25$ also smaller hadron flows, than $v_{n,B}(3p_T) = 3v_{n,q}(p_T)$, $v_{n,M}(2p_T) = 2v_{n,q}(p_T)$ from linear scaling $(13)$.

The observable averages $\{\tilde{v}_n\}$ are, of course, determined by the interplay between variations in the local emission angle, the width, and constituent density. It is quite remarkable that the end result in a dynamical coalescence approach $(12)$ is only a modest $\sim 20\%$ and $\sim 30\%$ reduction of pion and proton elliptic flow at RHIC relative to quark number scaling.

The above results underscore the importance of local momentum anisotropies in hadronization via quark coalescence. Flow addition formulas apply locally, even though only spacetime averages of the final anisotropies can be observed. Odd-order $\{\tilde{v}_n\}$ and the sine terms play a major role, even at midrapidity. These terms have been ignored in analytic studies so far $(10, 30)$ but are naturally incorporated in a dynamical approach $(21)$.

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