A Recurrence Formula for Solutions of Burgers Equations

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Abstract

A Bäcklund transformation (BT) and a recurrence formula are derived by the homogeneous balance (HB) method. A initial problem of Burgers equations is reduced to a initial problem of heat equation by the BT, the initial problem of heat equation is resolved by the Fourier transformation method, substituting various solutions of the initial problem of the heat equation will yield solutions of the initial problem of the Burgers equations.

1 Introduction

The HB method is used to find solitary wave solutions and other kinds of exact solutions of nonlinear partial differential equations (PDEs)⁰¹, in this paper, the HB method is used first time with the help of the Fourier transformation method to solve some initial problems of Burgers equations which have been widely used in physical system.

This paper falls into 3 parts. In section 2, by the HB method we obtain a general BT³,⁴ which shows both the connection between solutions of Burgers equations itself and connection between solutions of Burgers and heat equations, its special case is Cole–Hopf transformation and especially

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through the BT a recurrence formula is generated for solutions of Burgers equations as

\[ u_{N+1} = \frac{u_{Nx} + v_{Nx}}{u_N + v_N} + u_N, \]
\[ v_{N+1} = \frac{u_{Ny} + v_{Ny}}{u_N + v_N} + v_N. \]

A reasonable initial problem of the Burgers equations is put out and reduced to an initial problem of the heat equation which is solved by Fourier transformation method in Sec.3. The solution of the initial problem of the heat equation with the aid of the Burgers yields the solution of the initial problem of the Burgers.

2 Burgers Equations

\[ u_t = u_{xx} + u_{yy} + 2uu_x + 2vu_y, \]
\[ v_t = v_{xx} + v_{yy} + 2uv_x + 2vv_y. \]  

(1)

This is the famous Burgers equations which have got much attention in recent years. In Ref.[7,8,9], high dimensional and high degree Burgers equations are studied, and a pair of exact solution of Burgers equations(1) is obtained in Ref.[4]. In this section we will present some BTs of Burgers equations(1) through the HB method.

According to the idea of HB method, we seek for its solution of the form:

\[ u = \frac{\partial f(\varphi)}{\partial x} + u_0 = f'(\varphi)\varphi_x + u_0, \]
\[ v = \frac{\partial g(\varphi)}{\partial y} + v_0 = g'(\varphi)\varphi_y + v_0, \]  

(2)

here \( f = f(\varphi) \) and \( g = g(\varphi) \) which are functions of single variable, and \( \varphi = \varphi(x, y, t) \) are all to be determined later. \( (u_0, v_0) \) are a pair of solutions of (1). By (2), it is easy to deduce that

\[ u_t = f''\varphi_x\varphi_t + f'\varphi_{xt} + u_{0t}, \]
\[ u_{xx} = f'''\varphi_x^2 + 3f''\varphi_x\varphi_{xx} + f'\varphi_{xxx} + u_{0xx}, \]
\[ u_{yy} = f'''\varphi_y^2 + 2f''\varphi_x\varphi_y + f'\varphi_{xyy} + u_{0yy}, \]
\[ 2uu_x = 2f'f''\varphi_x^3 + 2f^2\varphi_x\varphi_{xx} + 2f''\varphi_x^2u_0 + 2f'\varphi_{xx}u_0 + 2f'\varphi_xu_{0x} + 2u_0u_{0x}, \]
\[ 2vu_y = 2g'f''\varphi_y^3 + 2g'f'\varphi_y\varphi_{xy} + 2f''\varphi_y\varphi_yv_0 + 2f'\varphi_{xy}v_0 + 2g'\varphi_{0y}v_0 + 2vu_{0y}, \]  

(3)
\( v_t = g'' \varphi_y \varphi_t + g' \varphi_{yt} + v_0 t, \)

\( v_{xx} = g''' \varphi_y \varphi_x^2 + 2g'' \varphi_x \varphi_{xy} + g' \varphi_y \varphi_{xx} + g' \varphi_{yxx} + v_0 xx, \)

\( v_{yy} = g''' \varphi_y^3 + 3g'' \varphi_y \varphi_{yy} + g' \varphi_{yyy} + v_{0y}, \)

\[
2uv_y = 2f' g'' \varphi_x \varphi_y + 2f' g' \varphi_x \varphi_{xy} + 2g'' \varphi_x \varphi_y u_0 + 2g' \varphi_{xy} u_0 + 2f' \varphi_x v_0 x + 2u_0 v_0 x, \\
2vv_y = 2g' g'' \varphi_x \varphi_y^3 + 2g' \varphi_y \varphi_{yy} + 2g'' \varphi_y^2 v_0 + 2g' \varphi_y v_0 + 2g' \varphi_y v_0 y + 2v_0 v_0 y,
\]

(4)

Substituting (3) and (4) into (1), first setting the coefficients of \( \varphi_x^2 \) and \( \varphi_y^3 \) to zero, we obtain ODEs

\[
f''' + 2f' f'' = 0,
\]

\[
g''' + 2g' g'' = 0,
\]

which have solutions \( f = g = \ln \varphi \), thereby it holds that

\[
f''' + 2g' f'' = 0, \quad g''' + 2f' g'' = 0,
\]

\[
f'' + f'^2 = 0, \quad g'' + g'^2 = 0,
\]

by using (3)–(6), we obtain the expressions:

\[
u_t - u_{xx} - u_{yy} - 2uu_x - 2vu_y
\]

\[
= f''(\varphi_x \varphi_t - \varphi_x \varphi_{xx} - \varphi_x \varphi_{yy} - 2\varphi_x \varphi_y u_0 - 2\varphi_x^2 u_0)
\]

\[
+ f'(\varphi_x t - \varphi_{xxx} - \varphi_{xyy} - 2\varphi_{xx} u_0 - 2\varphi_x u_0 x - 2\varphi_{xy} v_0 - 2\varphi_y u_0 y),
\]

(7)

\[
v_t - v_{xx} - v_{yy} - 2uv_x - 2vv_y
\]

\[
= f''(\varphi_y \varphi_t - \varphi_y \varphi_{yy} - \varphi_y \varphi_{xx} - 2\varphi_y \varphi_x u_0 - 2\varphi_y^2 v_0)
\]

\[
+ f'(\varphi_y t - \varphi_{xxx} - \varphi_{yy} v_0 - 2\varphi_{xx} v_0 x - 2\varphi_{xy} v_0 y - 2\varphi_y v_0 y),
\]

(8)

It is easy to see that (2) are solutions of (1), provide that the right hand sides of (7) and (8) to be zero, hence set

\[
\varphi_t - \varphi_{xx} - \varphi_{yy} - 2\varphi_x u_0 - 2\varphi_y v_0 = 0,
\]

(9)

\[
\varphi_{xt} - \varphi_{xxx} - \varphi_{xyy} - 2\varphi_{xx} u_0 - 2\varphi_x u_0 x - 2\varphi_{xy} v_0 - 2\varphi_y u_0 y = 0,
\]

(10)

\[
\varphi_{yt} - \varphi_{yyy} - \varphi_{yy} v_0 - 2\varphi_{yy} v_0 y - 2\varphi_{xy} u_0 - 2\varphi_x v_0 x = 0,
\]

(11)

then the right hand sides of (7) and (8) respectively vanish. In fact (10) and (11) can be reduced into

\[
(\varphi_t - \varphi_{xx} - \varphi_{yy} - 2\varphi_x u_0 - 2\varphi_y v_0)_x + 2\varphi_y v_0 x - 2\varphi_y u_0 y = 0,
\]

(12)

\[
(\varphi_t - \varphi_{xx} - \varphi_{yy} - 2\varphi_x u_0 - 2\varphi_y v_0)_y + 2\varphi_x u_0 y - 2\varphi_x v_0 x = 0,
\]

(13)
thereby, (9)–(11) equal that
\[ \varphi_t - \varphi_{xx} - \varphi_{yy} - 2\varphi_x u_0 - 2\varphi_y v_0 = 0, \tag{14} \]

\[ u_0 y - v_0 x = 0, \tag{18} \]

here \( \varphi_x^2 + \varphi_y^2 \neq 0 \). Substituting \( f = g = \ln \varphi \) into (2), we obtain BT of Burgers equations (1)

\[ u = \varphi_x + u_0, \tag{15} \]
\[ v = \varphi_y + v_0, \tag{16} \]
\[ \varphi_t - \varphi_{xx} - \varphi_{yy} - 2\varphi_x u_0 - 2\varphi_y v_0, \tag{17} \]
\[ u_0 y - v_0 x = 0, \tag{18} \]

\[ \frac{\varphi}{\varphi}, \quad \frac{\varphi_x}{\varphi}, \quad \frac{\varphi_y}{\varphi}, \quad \varphi_t - \varphi_{xx} - \varphi_{yy} = 0, \tag{20} \]

which is usually called Cole–Hopf transformation.

3 Fourier Transformation

Provided the initial value problem of Burgers equations in what follows

\[ u_t = u_{xx} + u_{yy} + 2uu_x + 2uv_y, \tag{21} \]
\[ v_t = v_{xx} + v_{yy} + 2uv_x + 2vv_y, \tag{22} \]

\[ u|_{t=t_0} = s(x, y), \quad v|_{t=t_0} = k(x, y) \tag{23} \]
where \( s(x,y), k(x,y) \to 0 \) when \(|x|, |y| \to 0\).

By using the former result, we can take its solution to be as

\[
  u = \frac{\varphi_x}{\varphi} + u_0, \quad v = \frac{\varphi_y}{\varphi} + v_0, \tag{24}
\]

here \((u_0, v_0)\) are constant solutions of (21)–(22) and \( \varphi \) satisfies

\[
  \varphi_t - \varphi_{xx} - \varphi_{yy} - 2u_0\varphi_x - 2v_0\varphi_y = 0 \tag{25}
\]

then (21)–(23) can be rewritten into the initial value problem of the following form

\[
  \varphi_t - \varphi_{xx} - \varphi_{yy} - 2u_0\varphi_x - 2v_0\varphi_y = 0,
  \quad \varphi|_{t=t_0} = f(x, y), \tag{26}
\]

which

\[
  f(x, y) = e^{\int_0^x s(x, y)_{t0} dx + \int_0^y k(x, y)_{t0} dy} \tag{27}
\]

\((x_0, y_0)\) are arbitrary constants, \( s_1(x, y) = s(x, y) - u_0, k_1(x, y) = k(x, y) - v_0 \). Then if \( u_0, v_0 \) are arbitrary constant solutions of equations (1), by using of Fourier transformation, we obtain solutions of (26)–(27).

Now introduce the Fourier transformation simply. If \( f(x, y, t) \) is a function, \( \hat{f}(c_1, c_2, t) \) is used to denote its Fourier transformation as follow:

\[
  \hat{f}(c_1, c_2, t) = F[f(x, y, t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, t) e^{-ic_1x+ic_2y} dx dy \tag{28}
\]

and

\[
  f(x, y, t) = F^{-1}[\hat{f}(c_1, c_2, t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(c_1, c_2, t) e^{ic_1x+ic_2y} dc_1 dc_2 \tag{29}
\]

we’d like show its several properties which are used here:

I \quad F[a f + b g] = a F[f] + b F[g]

II \quad F\left(\frac{\partial^n \varphi}{\partial x^n \partial y^m}\right) = (ic_1)^n(ic_2)^m F[\varphi]

III \quad F[f(x-a, y-b, t)] = e^{-ic_1a-ic_2b} F[f(x,y,t)]

IV \quad F[f(ax, by, t)] = \frac{1}{|ab|} \hat{f}\left(\frac{a}{a}, \frac{b}{g}, t\right)

V \quad \text{if} \quad (f * g)(x, y, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-z_1, y-z_2, t) g(z_1, z_2, t) dz_1 dz_2, \text{ then}

\[
  F^{-1}[\hat{f} \hat{g}] = \frac{1}{2\pi} f * g
\]
here, \(a\) and \(b\) are arbitrary constants, \(m\) and \(n\) are arbitrary integers. In order to deal with the problem (26), the Fourier transformation is done for them, it is reduced to a order differential problem as following:

\[
\frac{d\hat{\varphi}}{dt} + c_1^2\hat{\varphi} + c_2^2\hat{\varphi} - 2iu_0c_1\hat{\varphi} - 2v_0c_2\hat{\varphi} = 0,
\]

\[
\hat{\varphi} \big|_{t=t_0} = \hat{f}(x, y),
\]

which admits solutions as

\[
\hat{\varphi}(c_1, c_2, t) = \hat{f}(c_1, c_2, t)e^{-\left(c_1^2 + c_2^2 - 2iu_0c_1 - 2iv_0c_2\right)(t-t_0)}
\]

set \(g = e^{-\left(c_1^2 + c_2^2 - 2iu_0c_1 - 2iv_0c_2\right)(t-t_0)}\) such that

\[
g(x, y, t) = F^{-1}[\hat{g}(c_1, c_2, t)] = \frac{1}{2(t-t_0)} e^{-\frac{(x-x_0)^2 + (y-y_0)^2}{4(t-t_0)}},
\]

so the solution of (26) is gained

\[
\varphi(x, y, t) = F^{-1}[\hat{g}] = \frac{1}{4\pi(t-t_0)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\int_{x_0}^{x} s_1(\tau_1, y_0) d\tau_1 + \int_{y_0}^{y} k_1(x-z_1, \tau_2) d\tau_2}.
\]

\[
e^{-\frac{(x_1-2iu_0)^2 + (x_2-2iv_0)^2}{4(t-t_0)}} dz_1 dz_2,
\]

hence \(u = \frac{\varphi_x}{\varphi} + u_0\), \(v = \frac{\varphi_y}{\varphi} + v_0\) which are solutions of (21)–(23) also can be got, detail discussion does not present here.

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