On a Certain Stratification of the Gauge Orbit Space

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Abstract. For a principal SU($n$)-bundle over a compact manifold of dimension 2, 3, 4, we determine the orbit types of the action of the gauge group on the space of connections modulo pointed local gauge transformations. We find that they are given by Howe subgroups of SU($n$) for which a certain characteristic equation is solvable. Depending on the base manifold, this equation leads to a linear, bilinear, or quadratic Diophantine equation.

Keywords: classification, gauge orbit space, nongeneric strata, orbit types, pointed gauge group, stratification

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1. Introduction

Despite the many successes of gauge theory there is still a variety of open problems to be solved. Some of them are connected with the structure of the gauge orbit space. That one cannot circumvent to study this space was first brought to the attention of physicists by the discovery of what was later called the Gribov ambiguity [9]. It originates from the nontrivial bundle structure of the factorization by local gauge transformations [23]. Another peculiarity is that the gauge orbit space is a stratified space rather than a smooth manifold. It consists of a generic stratum and several nongeneric strata that form singularities [1, 13]. Whereas the generic stratum was studied intensively in the early 1980s, leading to a geometric understanding of the Faddev-Popov procedure [4] and of anomalies [3], the role of nongeneric strata is not clarified yet. There are several partial results and conjectures [2, 7], however, a systematic study has still to be undertaken. In the present letter we make a step in this direction. We derive an explicit description of the particular stratification which is induced by the orbit types of the action of the gauge group (assumed to be SU($n$)) on the space of connections modulo pointed local gauge transformations. The intention of this is to provide a framework in which problems can be given a concrete formulation and some of the hypotheses about the role of nongeneric strata can be tested.
The letter is organized as follows. In Section 2, we recall the construction of the group action to be considered. In Section 3, we give a characterization of stabilizers in terms of Howe subgroups. The latter are described in Section 4 and the determination of the set of orbit types is accomplished in Section 5. In Section 6, we discuss the result for base manifolds $S^4$, $S^2 \times S^2$, and $\mathbb{CP}^2$.

2. Preliminaries

Let $G$ be a compact connected Lie group, $M$ a compact connected orientable Riemannian manifold, and $P$ a principal $G$-bundle over $M$. Let $\mathcal{A}^k$ and $\mathcal{G}^k$ denote the spaces of connection forms and gauge transformations of $P$, respectively, of Sobolev class $k$. Gauge transformations will be viewed as $G$-space morphisms $P \to G$. For $2k > \dim M$, $\mathcal{A}^k$ is an affine Hilbert space and $\mathcal{G}^{k+1}$ is a Hilbert Lie group and the action of $\mathcal{G}^{k+1}$ on $\mathcal{A}^k$, given by

$$A(g) = g^{-1}Ag + g^{-1}dg, \quad A \in \mathcal{A}^k, g \in \mathcal{G}^{k+1},$$

is smooth [15, 23]. The quotient topological space $\mathcal{M}^k = \mathcal{A}^k / \mathcal{G}^{k+1}$ is known as the gauge orbit space of $P$. It represents the space of physical degrees of freedom for any gauge theory without matter defined on $P$. Note that $\mathcal{M}^k$ does not depend essentially on the technical parameter $k$, because using a smoothing argument one can show that for $l \leq k$, $\mathcal{M}^k$ is open and dense in $\mathcal{M}^l$.

As usual for Lie group actions on manifolds, $\mathcal{M}^k$ need not be a smooth manifold. One has, however, the following general construction for a group $H$ acting on a manifold $X$. First, recall that the stabilizers of points on the same orbit in $X$ are conjugate in $H$. Thus, there exists a map which assigns to each orbit in $X/H$ the conjugacy class of stabilizers of its representatives. It is denoted by Type. The disjoint decomposition

$$X/H = \bigcup_{\tau \in \text{Type}(X/H)} \text{Type}^{-1}(\tau),$$

is called orbit type decomposition of $X/H$. The set $\text{Type}(X/H)$ carries a natural partial ordering: $\tau \leq \tau'$ iff there exist respective representatives $S, S'$ such that $S \subseteq S'$. It was shown in [13] that $\text{Type}(\mathcal{A}^k / \mathcal{G}^{k+1})$ is countable and that the subsets $\text{Type}^{-1}(\tau)$ are manifolds. Moreover, for any $\tau \in \text{Type}(\mathcal{A}^k / \mathcal{G}^{k+1})$,

$$\text{Type}^{-1}(\tau) \text{ is open and dense in } \bigcup_{\tau' \geq \tau} \text{Type}^{-1}(\tau').$$
These properties are condensed in the notion of 'stratification', see [13]. An explicit description of \( \text{Type}(A^k/G^{k+1}) \) for \( G = SU(n) \) and \( \dim M \leq 4 \) was derived in [19, 20]. In this letter an analogous result will be provided for a coarser stratification of \( M^k \). It is obtained by viewing \( M^k \) as the orbit space of another smooth Lie group action, constructed as follows. Let \( p_* \in P \) be fixed. Since, for our choice of \( k \), the elements of \( G^{k+1} \) are continuous, the Lie group homomorphism

\[
\phi^k : G^{k+1} \to G, \quad g \mapsto g(p_*)
\]

exists. Its kernel, denoted by \( G^{k+1}_* \), is known as the group of pointed local gauge transformations. It is a normal Lie subgroup and the quotient \( G^{k+1}/G^{k+1}_* \) is a Lie group isomorphic, via \( \phi^k \), to \( G \) [5]. Through this isomorphism, the residual action of \( G^{k+1}/G^{k+1}_* \) on \( B^k := A^k/G^{k+1}_* \) defines an action of \( G \) on \( B^k \). Explicitly, for \( a \in G \) and \( \omega \in B^k \),

\[
\omega(a) = [A(g)]_* , \quad (3)
\]

where \( A \in A^k, g \in G^{k+1} \) such that \( [A]_* = \omega, \phi^k(g) = a \). Note that the orbit space \( B^k/G \) is homeomorphic to \( M^k \) [6]. It is known that \( B^k \) is a smooth manifold and that \( A^k \) is a smooth locally trivial principal \( G^{k+1}_* \)-bundle over \( B^k \) [15]. Using local triviality, the action of \( G \) on \( B^k \) can be seen to be smooth. It was shown in [14] that the orbit type decomposition (1) of \( M^k \) induced from \( G \)-action on \( B^k \) is again a stratification and that it is encoded, in the sense of (2), in \( \text{Type}(B^k/G) \).

For \( \omega \in B^k \) and \( A \in A^k \), let \( G^\omega \) and \( G^{k+1}_A \) denote the stabilizers under \( G \)-action and \( G^{k+1}_* \)-action, respectively. If \( \omega = [A]_* \), (3) implies

\[
\phi^k(G^{k+1}_A) = G^\omega . \quad (4)
\]

Thus, the stratification of \( M^k \) induced from \( \text{Type}(B^k/G) \) is coarser than that induced from \( \text{Type}(A^k/G^{k+1}) \). In particular, \( \phi^k \) projects to a surjective map

\[
\text{Type}(A^k/G^{k+1}) \to \text{Type}(B^k/G) . \quad (5)
\]

In the sequel, we are going to determine \( \text{Type}(B^k/G) \) for \( G = SU(n) \) and \( \dim M = 2, 3, 4 \).

3. Characterization of Stabilizers

Let \( \omega \in B^k \) and \( A \in A^k \) such that \( \omega = [A]_* \). Let \( P_A \) and \( H_A \) denote, respectively, the holonomy bundle and holonomy group of \( A \), based at \( p_* \). Let \( C_G(\cdot) \) denote the centralizer in \( G \). According to (4) and the well-known relation \( \phi^k(G^{k+1}_A) = C_G(H_A) \) [8, 16],

\[
G^\omega = C_G(H_A) . \quad (6)
\]
DEFINITION 3.1. A subgroup $H \subseteq G$ is called Howe iff $H = C_G(K)$ for some subset $K \subseteq G$. A (smooth) reduction of $P$ to a subgroup $H \subseteq G$ is called holonomy-induced iff it possesses a (smooth) connected reduction to some subgroup $H_0$ which obeys $C^2_G(H_0) = H$.

Note that Howe subgroups can be equivalently characterized by the property $H = C^2_G(H)$. Moreover, by definition, the structure group of a holonomy-induced bundle reduction is always Howe.

PROPOSITION 3.2. Assume $\dim M \geq 2$. For a subgroup $S \subseteq G$, the following assertions are equivalent:

(a) $S$ is a stabilizer of $G$-action on $B^k$,

(b) $S$ is Howe and a holonomy-induced reduction of $P$ to $C_G(S)$ exists.

Proof. (a) $\Rightarrow$ (b): Due to (6), $S = C_G(H_A)$ for some $A \in A^k$. Hence, $S$ is Howe. For $k$ large enough, $A$ is of class $C^1$, so that $P_A$ is of class $C^2$. Due to standard smoothing theory, $P_A$ is $C^2$-isomorphic to some smooth reduction $Q$ (obviously connected) of $P$ to $H_A$. The extension $Q \cdot C^2_G(H_A)$ is a holonomy-induced reduction of $P$ to the subgroup $C^2_G(H_A) = C_G(S)$.

(b) $\Rightarrow$ (a): By assumption, there exists a smooth connected reduction $Q$ of $P$ to some subgroup $H$ obeying $C^2_G(H) = C_G(S)$. Since $S$ is Howe, this implies $C_G(H) = S$. Being connected, $Q$ is the holonomy bundle, and $H$ the holonomy group, of some smooth connection $A$ on $P$ [12] (this requires $\dim M \geq 2$). Then (6) yields $G_{[A]^*} = S$. $\square$

Remark. (i) The subgroups $S$ and $C_G(S)$ form a so-called reductive Howe dual pair in $G$. Such pairs play an important role in the representation theory of Lie groups [10].

(ii) Proposition 3.2 implies, in particular, that $\text{Type}(B^k/G)$ does not depend on $k$.

(iii) While stabilizers of $G$-action on $B^k$ are characterized by the mere existence of certain bundle reductions, those of $G^{k+1}$-action on $A^k$ are characterized by the reductions themselves, see [19].

According to Proposition 3.2, $\text{Type}(B^k/G)$ is a subset of $\text{Howe}(G)$, the set of conjugacy classes of Howe subgroups of $G$, and its partial ordering is induced from the latter. Thus, the determination of $\text{Type}(B^k/G)$ proceeds through that of $\text{Howe}(G)$.

4. The Howe Subgroups of $SU(n)$

General references for the determination of $\text{Howe}(G)$ are [17, 18, 21]. The case of $SU(n)$, however, is simpler than the general case. Let
Sub\(_s(M_n(\mathbb{C}))\) denote the set of unital \(*\)-subalgebras of \(M_n(\mathbb{C})\), the algebra of complex \((n \times n)\)-matrices, modulo conjugacy under \(U(n)\).

**Lemma 4.1.** Intersection with \(SU(n)\) yields a 1-1 relation between \(Sub_s(M_n(\mathbb{C}))\) and Howe\((SU(n))\).

*Proof.* Notice that any \(L \in Sub_s(M_n(\mathbb{C}))\) is spanned by its unitary elements which are also unitary in \(M_n(\mathbb{C})\). Hence \(C_{M_n(\mathbb{C})}(L) = C_{M_n(\mathbb{C})}(L \cap SU(n))\). Using this and the double commutant theorem one can establish the 1-1 relation on the level of subalgebras and subgroups. It obviously survives the passage to conjugacy classes. \(\square\)

The set \(Sub_s(M_n(\mathbb{C}))\) can be described as follows. Let \(K(n)\) denote the collection of pairs \(J = (k, m)\) of sequences \(k = (k_1, \ldots, k_r), m = (m_1, \ldots, m_r), r = 1, 2, 3, \ldots, n\), consisting of positive integers such that \(k \cdot m = \sum_{i=1}^r k_i m_i = n\). Any \(J \in K(n)\) defines a decomposition

\[
\mathbb{C}^n = \left( \mathbb{C}^{k_1} \otimes \mathbb{C}^{m_1} \right) \oplus \cdots \oplus \left( \mathbb{C}^{k_r} \otimes \mathbb{C}^{m_r} \right)
\]

and an associated injective homomorphism

\[
\prod_{i=1}^r M_{k_i}(\mathbb{C}) \to M_n(\mathbb{C}), \quad (D_1, \ldots, D_r) \mapsto \bigoplus_{i=1}^r D_i \otimes 1_{m_i}.
\]

The image will be denoted by \(M_J(\mathbb{C})\) and its intersection with \(SU(n)\) by \(SU(J)\). We introduce an equivalence relation on \(K(n)\): \(J \sim J'\) iff they differ by a simultaneous permutation of \(k\) and \(m\). Let \(\tilde{K}(n)\) denote the set of equivalence classes. As a basic fact, the map \(J \mapsto M_J(\mathbb{C})\) induces a bijection from \(\tilde{K}(n)\) onto \(Sub_s(M_n(\mathbb{C}))\). Thus, Lemma 4.1 yields

**Proposition 4.2.** The map \(J \mapsto SU(J)\) induces a bijection from \(\tilde{K}(n)\) onto Howe\((SU(n))\). \(\square\)

*Example.* For \(J = ((1), (n))\) and \(((n), (1))\), \(SU(J)\) is the center and the whole group, respectively. For \(J = ((1, \ldots, 1), (1, \ldots, 1))\), \(SU(J)\) is the maximal torus of \(SU(n)\). For \(J = ((2, 3), (1, 1)) \in K(5)\), \(SU(J) = S(U2 \times U3)\), the symmetry group of the standard model. In the grand unified \(SU(5)\)-model this is the subgroup to which \(SU(5)\) is broken by the Higgs mechanism. For details on the structure of \(SU(J)\), see [19].

Next, consider the partial ordering of Howe\((SU(n))\), which obviously coincides with that of \(Sub_s(M_n(\mathbb{C}))\). The latter can be described in terms of inclusion matrices or Bratteli diagrams, see [20, Lemma 3.1]. Here we only give operations to create direct successors. Since Howe\((SU(n))\) is finite this will allow us to reconstruct the partial ordering and to draw Hasse diagrams. Let \(J = (k, m) \in K(n)\). Consider the following two operations.
Splitting: Choose $i$ such that $m_i \neq 1$ and choose positive integers $m_{i1}, m_{i2}$ such that $m_i = m_{i1} + m_{i2}$. Define $J' = (k', m')$, where

$$k' = (k_1, \ldots, k_{i-1}, k_i, k_{i+1}, \ldots, k_r),$$
$$m' = (m_1, \ldots, m_{i-1}, m_{i1}, m_{i2}, m_{i+1}, \ldots, m_r).$$

By construction, $J' \in K(n)$ and $M_J(C) \subseteq M_{J'}(C)$, where $M_{k_i} \subseteq M_{k_i}(C) \times M_{k_i}(C)$ diagonally and all the other factors coincide.

Merging: Choose $i < j$ such that $m_i = m_j$. Define $J' = (k', m')$ by

$$k' = (k_1, \ldots, k_{i-1}, k_i + k_j, k_{i+1}, \ldots, \widehat{k_j}, \ldots, k_r),$$
$$m' = (m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, \widehat{m_j}, \ldots, m_r).$$

Here $\widehat{\cdot}$ means that the entry is omitted. Again, $J' \in K(n)$ and $M_J(C) \subseteq M_{J'}(C)$, where, up to conjugacy, $M_{k_i}(C) \times M_{k_j}(C) \subseteq M_{k_i+k_j}(C)$ and all other factors coincide.

Although the following seems to be well known, the only reference the authors are aware of is [22].

**PROPOSITION 4.3.** Let $J \in K(n)$. The labels $J'$ of the direct successors of $SU(J)$ are obtained by applying all possible splitting and merging operations to $J$. \hfill $\Box$

**Remark.** (i) When applying splitting and merging operations to $J$ one can restrict oneself to those that yield inequivalent $J'$.

(ii) Obviously, taking the centralizer inverts the partial ordering relation. Thus, Proposition 4.3 also yields an algorithm to create direct predecessors.

**Example.** Consider $SU(4)$. The center has label $J = ((1), (4))$. Two splitting operations can be applied, yielding $J'_1 = ((1, 1), (1, 3))$ and $J'_2 = ((1, 1), (2, 2))$. At the next stage, a splitting operation can be applied to $J'_1$, yielding $J''_1 = ((1, 1, 1), (1, 1, 2))$. Two splitting operations can be applied to $J'_2$. Their results are equivalent to $J''_1$. This means that $SU(J'_1)$ and $SU(J'_2)$ have common direct successor $SU(J''_1)$. Furthermore, a merging operation can be applied to $J''_2$, yielding $((2, 2))$. Continuing the procedure one can easily construct the Hasse diagram of Howe($SU(n)$) for $n = 4$ and, similarly, for any other value of $n$. The results for $n = 2, \ldots, 5$ are shown in Figure 6.3 at the end. Note that the diagrams are symmetric w.r.t. reflection at the vertical central axis and simultaneous interchange of $k$ and $m$. Of course, this is due to Remark (ii) above.
The following lemma was proved in [19, Thm. 6.2].

**LEMMA 5.1.** Any reduction of a principal $\text{SU}(n)$-bundle to a Howe subgroup is holonomy-induced. \(\Box\)

**Remark.** Lemma 5.1 does not hold, for example, for $\text{SO}(n)$.

**LEMMA 5.2.** For $J = (k, m)$, $C_{\text{SU}(n)}(\text{SU}(J))$ is conjugate to $\text{SU}(J^c)$, where $J^c = (m, k)$.

For the reductions of $P$ to $\text{SU}(J^c)$, the following classification was derived in [19]. Let the symbol $\langle \rangle$ denote the greatest common divisor of the integers enclosed. Define integers $\tilde{k}_i$ by $\tilde{k}_i = k_i$, $i = 1, \ldots, r$. Let $H^{\text{even}}(M, \mathbb{Z})$ denote the even degree part of the integral cohomology ring $H^*(M, \mathbb{Z})$. We introduce the notation

$$H^{(\mathbf{m})}(M, \mathbb{Z}) = \{ \alpha \in \prod_{i=1}^{r} H^{\text{even}}(M, \mathbb{Z}) \mid \alpha_i^{(0)} = 1, \alpha_i^{(2j)} = 0 \text{ for } j > m_i \}.$$  

Here $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\alpha_i^{(2j)}$ denotes the component of $\alpha_i$ of degree $2j$. Note that each of the $\alpha_i$ can be viewed as the (total) Chern class of a $\text{U}(m_i)$-bundle over $M$. Finally, let $c(P)$ denote the (total) Chern class of $P$ and $\beta_{(k)} : H^1(M, \mathbb{Z}_{(k)}) \to H^2(M, \mathbb{Z})$ the Bockstein homomorphism associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_{(k)} \to 0$ of coefficient homomorphisms. Consider the system of equations

$$\sum_{i=1}^{r} \tilde{k}_i \alpha_i^{(2)} = \beta_{(k)}(\xi), \quad (7)$$
$$\alpha_1^{k_1} \cdots \alpha_r^{k_r} = c(P) \quad (8)$$

in the indeterminates $\alpha \in H^{(\mathbf{m})}(M, \mathbb{Z})$ and $\xi \in H^1(M, \mathbb{Z}_{(k)})$. The following was stated in [19] as Theorem 5.16.

**LEMMA 5.3.** Assume $\dim M \leq 4$. The reductions of $P$ to the subgroup $\text{SU}(J^c)$ are classified, up to isomorphy, by the solutions of (7), (8). \(\Box\)

**Remark.** Eq. (7) is a relation between the characteristic classes which classify principal $\text{SU}(J^c)$-bundles. It emerges from their construction. Eq. (8) represents the condition that the $\text{SU}(J^c)$-bundle labelled by $\alpha$, $\xi$ is a reduction of $P$.

Eq. (8) actually contains two equations, sorted by degree,

$$\sum_{i=1}^{r} k_i \alpha_i^{(2)} = 0, \quad (9)$$
where $c_2(P)$ denotes the second Chern class of $P$. Note that (9) already follows from (7), because $\langle k \rangle \beta_{\langle k \rangle} = 0$. For our purposes, it suffices to know whether the system (7), (8) has a solution or not. Let $H_F^{(m)}(M, \mathbb{Z})$ denote the torsion-free part of $H^{(m)}(M, \mathbb{Z})$.

**Lemma 5.4.** The system of Eqs. (7), (8) possesses a solution if and only if Eq. (8) possesses a solution $\alpha \in H_F^{(m)}(M, \mathbb{Z})$.

**Proof.** Let $\alpha \in H_F^{(m)}(M, \mathbb{Z})$ be a solution of (8). Then, due to (9), choosing $\xi = 0$ yields a solution to (7). Conversely, let $\alpha, \xi$ be a solution of (7) and (8). Decompose $\alpha = \alpha_T + \alpha_F$ into torsion and torsion-free part. Eq. (9) is satisfied by $\alpha_T$ and $\alpha_F$ independently. By orientability of $M$, $\alpha_T$ does not contribute to (10). It follows that $\alpha_F$ solves (8). \(\square\)

Let $K(P)$ denote the subset of $K(n)$ of elements $J = (k, m)$ for which (8) possesses a solution in $H_F^{(m)}(M, \mathbb{Z})$. Since simultaneous permutations of $k$ and $m$ do not affect this property, we can pass to the set of equivalence classes, which will be denoted by $\hat{K}(P)$.

**Theorem 5.5.** Assume $\dim M = 2, 3, 4$. Then the map $J \mapsto SU(J)$ induces a bijection from $\hat{K}(P)$ onto Type($B^k/G$).

**Proof.** Let $J \in K(n)$. It suffices to check that $J \in K(P)$ iff $SU(J)$ is a stabilizer. Due to Proposition 3.2 and Lemma 5.2, $SU(J)$ is a stabilizer iff $P$ admits a holonomy-induced reduction to $SU(J^c)$. According to Lemma 5.1, one can omit holonomy-induced here. Then the assertion follows from Lemmas 5.3 and 5.4. \(\square\)

As a result, the determination of Type($B^k/G$) is reduced to a discussion of the solvability of the system of equations (9), (10). Let us remark that, contrary to that, the elements of Type($A^k/G^{k+1}$) are characterized by the solutions $\alpha, \xi$ themselves (cf. Lemma 5.3 and Remark (iii) after Proposition 3.2). On the level of the data $J, \alpha, \xi$, the map (5) reads $(J, \alpha, \xi) \mapsto J$.

6. Examples

In dimensions 2, 3, any principal SU($n$)-bundle is trivial, hence can be reduced to any of the subgroups SU($J$). Therefore, Type($B^k/G$) =
Howe(SU(n)). In dimension 4, Eq. (10) may give rise to a linear, bilinear, or quadratic Diophantine equation. Examples for these 3 types are provided by $M = S^4$, $S^2 \times S^2$, and $\mathbb{CP}^2$, respectively. For all of them, $H^{(m)}(M, \mathbb{Z}) = H_F^{(m)}(M, \mathbb{Z})$.

6.1. Base Manifold $M = S^4$

Since $H^2(S^4, \mathbb{Z}) = 0$, Eq. (9) is trivially satisfied. We parametrize $c_2(P) = c_P \gamma$ and $\alpha^{(4)}_i = b_i \gamma$, $i = 1, \ldots, r$, where $\gamma$ is a generator of $H^4(S^4, \mathbb{Z})$. Eq. (10) yields the linear Diophantine equation

$$\sum_{i=1}^r k_i b_i = c_P.$$  \hfill (11)

Recall that $b_i \in \mathbb{Z}$ if $m_i \neq 1$ and $b_i = 0$ otherwise. Thus, (11) has a solution iff $c_P$ is a multiple of $\langle k^o \rangle$, where $k^o$ is obtained from $k$ by deleting all members $k_i$ for which $m_i = 1$. The case $k^o = \emptyset$ can be consistently incorporated by putting $\langle \emptyset \rangle = 0$. Denoting $\langle k^o \rangle$ by $d_{S^4}(J)$, $K(P) = \{ J \in K(n) \mid d_{S^4}(J) \text{ divides } c_P \}$. \hfill (12)

6.2. Base Manifold $M = S^2 \times S^2$

Let $\gamma$ be a generator of $H^2(S^2, \mathbb{Z})$. Then $H^2(M, \mathbb{Z})$ and $H^4(M, \mathbb{Z})$ are generated by $\gamma \times 1, 1 \times \gamma$ and $\gamma \times \gamma$, respectively. We expand $\alpha^{(2)}_i = a_{1i} \gamma \times 1 + a_{2i} 1 \times \gamma$ and $c_2(P) = c_P \gamma \times \gamma$. Eqs. (9), (10) read

$$\sum_{i=1}^r k_i a_{li} = 0, \quad l = 1, 2,$$  \hfill (13)

$$\langle k^o \rangle b + \sum_{i,j=1}^r k_i (k_j - \delta_{ij}) a_{1i} a_{2j} = c_P.$$  \hfill (14)

Here $b \in \mathbb{Z}$ is an indeterminate (thus we made use of our previous result) and $\delta_{ij}$ denotes the Kronecker symbol. Since the case $r = 1$ is trivial, we may assume $r \geq 2$. Then the set of solutions of Eq. (13) can be parametrized by integers $t_{l,pq}$, $1 \leq p < q \leq r$, as follows [24]:

$$a_{li} = - \sum_{m=1}^{i-1} \tilde{k}_m t_{l,mi} + \sum_{m=i+1}^r \tilde{k}_m t_{l,im}, \quad i = 1, \ldots, r, l = 1, 2.$$  

Unless $r = 2$, the parametrization is not 1-1, but it generates all solutions, which suffices for our purposes. Insertion into (14) yields the
bilinear Diophantine equation

\[
(k^o)b + \sum_{\substack{1 \leq m < i \leq r \atop 1 \leq n < j \leq r}} L_{mi,nj} t_{1,mi} t_{2,nj} = c_P ,
\]

where

\[
L_{mi,nj} = \langle k \rangle \tilde{k}_m \tilde{k}_i \tilde{k}_j (\delta_{mj} - \delta_{ij}) + \tilde{k}_j (\delta_{ni} - \delta_{mn}) .
\]

It is well known that a bilinear form over \( \mathbb{Z} \) can take as value any multiple of the greatest common divisor of its coefficients [24]. Denote the latter by \( \langle L \rangle \). Then (15) has a solution iff \( c_P \) is a multiple of \( d_{S^2 \times S^2}(J) = \langle \langle k^o \rangle, \langle L \rangle \rangle \). The case \( r = 1 \) can be consistently incorporated by setting \( L = \emptyset \). Thus,

\[
K(P) = \{ J \in K(n) \mid d_{S^2 \times S^2}(J) \text{ divides } c_P \} .
\]

As \( d_{S^2 \times S^2}(J) \) divides \( d_{S^4}(J) \), there are 'more' orbit types over \( S^2 \times S^2 \) than over \( S^4 \). Note that (16) holds for \( M = T^4 \) (the 4-torus), too.

To compute \( \langle L \rangle \), observe that, besides 0 and up to a sign, there are two types of coefficients, namely,

\[
L_{mi,nj} = -\langle k \rangle \tilde{k}_m \tilde{k}_i \tilde{k}_j , \quad 1 \leq m < i < j \leq r \\
L_{mi,mi} = -\langle k \rangle \tilde{k}_m \tilde{k}_i (\tilde{k}_m + \tilde{k}_i) , \quad 1 \leq m < i \leq r .
\]

Thus, \( \langle L \rangle \) is just the greatest common divisor of all these numbers. As an example, consider \( J = ((4,4,6),(1,1,2)) \in K(20) \). Here \( \langle k \rangle = 2 \) and \( \langle k^o \rangle = 6 \). The relevant coefficients are \( 2k_1 \tilde{k}_2 \tilde{k}_3 = 2^3 \cdot 3 \), \( 2k_1 \tilde{k}_2 (\tilde{k}_1 + \tilde{k}_2) = 2^3 \), and \( 2k_1 \tilde{k}_3 (\tilde{k}_1 + \tilde{k}_3) = 2^2 \cdot 3 \cdot 5 \). They yield \( \langle L \rangle = 4 \). Hence, \( d_{S^2 \times S^2}(J) = 2 \).

The numbers \( d_{S^4}(J) \) and \( d_{S^2 \times S^2}(J) \) for \( J \in K(n), n = 2, \ldots, 5 \), are given in Figure (6.3). Using (12) and (16), the respective Hasse diagrams of Type(\( B^k \)/\( G \)) can be read off directly from this figure.

### 6.3. Base Manifold \( M = \mathbb{C}P^2 \)

Let \( \gamma \) be a generator of \( H^2(\mathbb{C}P^2, \mathbb{Z}) \). Then \( H^4(\mathbb{C}P^2, \mathbb{Z}) \) is generated by \( \gamma^2 \). With \( \alpha_i^{(2)} = a_i \gamma \) and \( c_2(P) = c_P \gamma^2 \), Eqs. (9) and (10) read

\[
\sum_{i=1}^{r} k_i a_i = 0 ,
\]

\[
(k^o)b + \frac{1}{2} \sum_{i,j=1}^{r} k_i (k_j - \delta_{ij}) a_i a_j = c_P ,
\]
where \( b \in \mathbb{Z} \) is an indeterminate. Notice that all the coefficients are integral. Thus, here we are facing a quadratic Diophantine equation or, phrased differently, the representation problem for a quadratic form over \( \mathbb{Z} \). Since a profound discussion, apart from some simple examples, requires methods from number theory (which is beyond the scope of this letter), it will be given elsewhere. Let us only consider the cases \( n = 2 \) and \( n = 3 \). For \( n = 2 \), only \( J = ((1, 1), (1, 1)) \) needs to be considered. By eliminating \( a_2 \), (17) becomes \(-a_1^2 = c_P\). Thus, the orbit type labelled by \( J \) is present iff \(-c_P\) is a square. Next, consider \( n = 3 \). Type \( J = ((1, 1), (1, 2)) \) is always present, because here \( \langle k^\circ \rangle = 1 \). For type \( J = ((2, 1), (1, 1)) \), (17) becomes \(-3a_1^2 = c_P\). Thus, this orbit type is present iff \(-c_P\) is 3 times a square. Finally, for \( J = ((1, 1), (1, 1, 1)) \), after elimination of \( a_3 \), (17) reads

\[-(a_1^2 + a_1a_2 + a_2^2) = c_P. \tag{18}\]

Here it is no longer obvious for which \( c_P \) this equation has a solution. Of course, since the l.h.s. is negative definite, for any given \( c_P \), only finitely many values of \( a_1, a_2 \) have to be tested, so one could use the help of a computer. In fact, for some \( J \) this might be the only way to solve the problem. For (18), however, more elaborate arguments [11] show that it is solvable, and hence type \( J \) is present in the gauge orbit space, iff \( c_P \leq 0 \) and

(i) \( c_P \neq -3^m(3n + 2), \forall m, n \in \mathbb{Z}, \ m \geq 0. \)

(ii) in the decomposition of \(-c_P\) into prime factors, any prime with \( p = 5 \) or \( 11 \mod 12 \) appears to an even power.

Thus, a solution exists for \(-c_P = 1, 3, 4, 7, 9, 12 \) etc., but not for \( 2, 5, 6, 8, 10, 11 \) etc. While (i) determines an arithmetic progression of bundles \( P \) for which type \( J \) is not present, (ii) picks out additional such \( P \) in a sporadic manner. Note that, while (ii) is a condition whose form is peculiar to binary quadratic forms, (i) is an analogue of the condition appearing in the famous result of Gauß that a positive integer is a sum of 3 squares iff it is not of the form \( 4^m(8n + 7) \).

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Figure 1. Hasse diagrams of Howe(SU(n)) for n = 2, . . . , 5. All edges are directed from left to right. Vertices are labelled by J = (k, m), written as (k_1k_2 . . . |m_1m_2 . . .) omitting commas, and numbers d_{S^4}(J); d_{S^2 × S^2}(J).

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