Absolutely Continuous Spectrum for the Quasi-periodic Schrödinger Operator in Exponential Regime

Wencai Liu and Xiaoping Yuan*

School of Mathematical Sciences
Fudan University
Shanghai 200433, People’s Republic of China
12110180063@fudan.edu.cn
*Corresponding author: xpyuan@fudan.edu.cn

Abstract

Avila and Jitomirskaya prove that the quasi-periodic Schrödinger operator $H_{λ,v,α,θ}$ has purely absolutely continuous spectrum for $α$ in sub-exponential regime (i.e., $β(α) = 0$) with small $λ$, if $v$ is real analytic in a strip of real axis. In the present paper, we show that for all $α$ with $0 < β(α) < ∞$, $H_{λ,v,α,θ}$ has purely absolutely continuous spectrum with small $λ$, if $v$ is real analytic in strip $|ℑx| < Cβ$, where $C$ is a large absolute constant.

1 Introduction and the Main results

In the present paper, we study the quasi-periodic Schrödinger operator $H = H_{λ,v,α,θ}$ on $ℓ^2(ℤ)$:

$$(H_{λ,v,α,θ}u)_n = u_{n+1} + u_{n-1} + λv(θ + nα)u_n,$$  \hspace{1cm} (1.1)

where $v : T = ℜ/ℤ → ℜ$ is the potential, $λ$ is the coupling, $α$ is the frequency, and $θ$ is the phase. In particular, the almost Mathieu operator (AMO) is given by (1.1) with $v(θ) = 2 \cos(2πθ)$, denoted by $H_{1,α,θ}$.

For $λ = 0$, it is easy to verify that Schrödinger operator (1.1) has purely absolutely continuous spectrum ($[-2, 2]$) by Fourier transform. We expect the property (has purely absolutely continuous spectrum) preserves under sufficiently small perturbation, i.e., $λ$ is small. Usually there are two smallness about $λ$. One is perturbative, meaning that the smallness $λ$ depends not only on the potential $v$, but also on the frequency $α$; the other is non-perturbative, meaning that the smallness condition only depends on the potential $v$, not on $α$. 

1
It is well known that $H_{\lambda v, \alpha, \theta}$ has purely absolutely continuous spectrum for $\alpha \in \mathbb{Q}$ and all $\lambda$. Thus, unless stated otherwise, we always assume $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ in the present paper. We also assume $v$ is real analytic in a strip of real axis from now on.

The following notions are essential in the study of equation (1.1).

We say $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ satisfies a Diophantine condition $DC(\kappa, \tau)$ with $\kappa > 0$ and $\tau > 0$, if
\[
||k\alpha||_{\mathbb{R}/\mathbb{Z}} > \kappa |k|^{-\tau}
\]
for any $k \in \mathbb{Z} \setminus \{0\}$, where $||x||_{\mathbb{R}/\mathbb{Z}} = \min_{\ell \in \mathbb{Z}} |x - \ell|$. Let $DC = \cup_{\kappa > 0, \tau > 0} DC(\kappa, \tau)$. We say $\alpha$ satisfies Diophantine condition, if $\alpha \in DC$.

Let $\beta = \beta(\alpha) = \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}$, (1.2) where $\frac{p_n}{q_n}$ is the continued fraction approximants to $\alpha$. One usually calls set $\{\alpha \in \mathbb{R} \setminus \mathbb{Q} | \beta(\alpha) > 0\}$ exponential regime and set $\{\alpha \in \mathbb{R} \setminus \mathbb{Q} | \beta(\alpha) = 0\}$ sub-exponential regime. Notice that $\beta(\alpha) = 0$ for $\alpha \in DC$.

In [7], Eliasson treats (1.1) as a dynamical systems problem—reducibility of associated cocycles. He shows that such cocycles are reducible for a.e. spectrum, and gives good estimates for the non-reducible ones via a sophisticated KAM-type methods, which breaks the limitations of the earlier KAM methods, for instance, the work of Dinaburg and Sinai [6] (they need exclude some parts of the spectrum). As a result, Eliasson proves that $H = H_{\lambda v, \alpha, \theta}$ has purely absolutely continuous spectrum for all $\theta$, if $\alpha \in DC$ and $|\lambda| < \lambda_0(\alpha, v)$.

Clearly, Eliasson’s result is perturbative.

Bourgain and Jitomirskaya established the measure-theoretic version in non-perturbative regime, more precisely, they proves that for a.e. $\alpha$ and $\theta$, $H = H_{\lambda v, \alpha, \theta}$ has purely absolutely continuous spectrum if $|\lambda| < \lambda_0(v) (\lambda < 1)$, see [3], [4], [10] for some details. They approach this by classical Aubry duality and the sharp estimates of Green function in the regime of positive Lyapunov exponent. Bourgain list a example which suggests that the non-perturbative results in multifrequency is wrong [3].

In [2], Avila and Jitomirskaya firstly develop a quantitative version of Aubry duality (Lemma 3.3) for $\alpha \in DC$. As an application, they show that operator (1.1) has purely absolutely continuous spectrum in non-perturbative regime for $\alpha \in DC$ and all $\theta$, by reducing non-perturbative regime to Eliasson’s perturbative regime. In addition the sharp estimates of rotation number and transfer matrix ([11], [12]), Avila prove that $H_{\lambda v, \alpha, \theta}$ has purely absolutely continuous spectrum in non-perturbative regime if $\beta(\alpha) = 0$ [1].

---

1. $\lambda_0(\ast)$ means $\lambda_0$ depends on $\ast$.

2. The quasi-periodic Schrödinger operator in multifrequency $(k$ dimension, $k \geq 2)$ is given by $(H_{\lambda v, \alpha, \theta})_n = u_{n+1} + u_{n-1} + \lambda v(\theta + n\alpha)u_n$, where $\nu : \mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k \to \mathbb{R}$ is the potential and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ is such that $1, \alpha_1, \cdots, \alpha_k$ are independent over the rational numbers.
The present authors obtain the sharp estimate of rotation number in [15], and extend the quantitative version of Aubry duality to exponential regime [16]. Combining with Avila’s arguments in [1], we obtain the main theorem in the present paper.

**Theorem 1.1.** For irrational number $\alpha$ such that $0 < \beta(\alpha) < \infty$, if $v$ is analytic in strip $|\mathfrak{Im} x| < C\beta$, where $C$ is a large absolute constant, then there exists $\lambda_0 = \lambda_0(v, \beta)$ such that $H_{\lambda v, \alpha, \theta}$ has purely absolutely continuous spectrum if $|\lambda| < \lambda_0$.

### 2 Preliminaries

#### 2.1 Cocycles

Denote by $\text{SL}(2, \mathbb{C})$ the all complex $2 \times 2$-matrixes with determinant $1$. We say a function $f \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ if $f$ is well defined in $\mathbb{R}/\mathbb{Z}$, i.e., $f(x+1) = f(x)$, and $f$ is analytic in a strip of real axis. The definitions of $\text{SL}(2, \mathbb{R})$ and $C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ are similar to those of $\text{SL}(2, \mathbb{C})$ and $C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ respectively, except that the involved matrixes are real and the functions are real analytic.

A $C^\omega$-cocycle in $\text{SL}(2, \mathbb{C})$ is a pair $(\alpha, A) \in \mathbb{R} \times C^\omega(\mathbb{R}/\mathbb{Z}, \text{SL}(2, \mathbb{C}))$, where $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \text{SL}(2, \mathbb{C}))$ means $A(x) \in \text{SL}(2, \mathbb{C})$ and the elements of $A$ are in $C^\omega(\mathbb{R}/\mathbb{Z}, \mathbb{C})$. Sometimes, we say $A$ a $C^\omega$-cocycle for short, if there is no ambiguity. Note that all functions, cocycles in the present paper are analytic in a strip of real axis. Thus we often do not mention the analyticity, for instance, we say $A$ a cocycle instead of $C^\omega$-cocycle.

Given two cocycles $(\alpha, A)$ and $(\alpha, A')$, a conjugacy between them is a cocycle $B \in C^\omega(\mathbb{R}/\mathbb{Z}, \text{SL}(2, \mathbb{C}))$ such that

$$B(x + \alpha)^{-1}A(x)B(x) = A'.$$

The notion of real conjugacy (between real cocycles) is the same as before, except that we ask for $B \in C^\omega(\mathbb{R}/\mathbb{Z}, \text{PSL}(2, \mathbb{R}))$, i.e., $B(x+1) = \pm B(x)$ and $\det B = 1$. We say that cocycle $(\alpha, A)$ is reducible if it is conjugate to a constant cocycle.

The Lyapunov exponent for the cocycle $A$ is given by

$$L(\alpha, A) = \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n(x)\| dx,$$

where

$$A_n(x) = A(x + (n-1)\alpha)A(x + (n-2)\alpha) \cdots A(x).$$

We say cocycle $(\alpha, A)$ is bounded if $\sup_{n \geq 0, x \in \mathbb{R}} \|A_n(x)\| < \infty$.

We now consider the quasi-periodic Schrödinger operator $\{H_{\lambda v, \alpha, \theta}\}_{\theta \in \mathbb{R}}$. It is easy to verify that the spectrum of $H_{\lambda v, \alpha, \theta}$ does not depend on $\theta$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, thus we denote by $\Sigma_{\lambda v, \alpha}$.
Let
\[ S_{\lambda v, E} = \begin{pmatrix} E - \lambda v & -1 \\ 1 & 0 \end{pmatrix}. \]
We call \((\alpha, S_{\lambda v, E})\) Schrödinger cocycle.

Fix Schrödinger operator \(H_{\lambda v, \alpha, \theta}\), we define the Aubry model by
\[ \hat{H} = \hat{H}_{\lambda v, \alpha, \theta}, \]
\[ (\hat{H}\hat{u})_k = \sum_{k\in\mathbb{Z}} \lambda \hat{v}_k \hat{u}_{n-k} + 2 \cos(2\pi \theta + n\alpha) \hat{u}_n, \tag{2.4} \]
where \(\hat{v}_k\) is the Fourier coefficients of potential \(v\). If \(\alpha \in \mathbb{R}\setminus\mathbb{Q}\), the spectrum of \(\hat{H}_{\lambda v, \alpha, \theta}\) is also \(\Sigma_{\lambda v, \alpha}\) \([8]\).

### 2.2 The rotation number

Let \(A(\theta) = \begin{pmatrix} a(\theta) & b(\theta) \\ c(\theta) & d(\theta) \end{pmatrix}\), we define the map \(T_{\alpha, A} : (\theta, \varphi) \in \mathbb{T}\times\hat{\mathbb{Z}} \to (\theta + \alpha, \varphi_{\alpha, A}(\theta, \varphi)) \in \mathbb{T}\times\hat{\mathbb{Z}}\), with \(\varphi_{\alpha, A} = \frac{1}{2\pi} \arctan(\frac{c(\theta) + d(\theta) \tan 2\pi \theta}{a(\theta) + b(\theta) \tan 2\pi \theta})\), where \(\mathbb{T} = \mathbb{R}/\mathbb{Z}\). Assume now that \(A : \mathbb{R}/\mathbb{Z} \to \text{SL}(2, \mathbb{R})\) is homotopic to the identity, then \(T_{\alpha, A}\) admits a continuous lift \(\tilde{T}_{\alpha, A} : (\theta, \varphi) \in \mathbb{R}\times\mathbb{R} \rightarrow (\theta + \alpha, \tilde{\varphi}_{\alpha, A}(\theta, \varphi)) \in \mathbb{R}\times\mathbb{R}\) such that \(\tilde{\varphi}_{\alpha, A}(\theta, \varphi) \mod \frac{1}{2}\mathbb{Z} = \varphi_{\alpha, A}(\theta, \varphi) \mod \frac{1}{2}\mathbb{Z}\) is well defined on \(\mathbb{T}\times\frac{1}{2}\mathbb{T}\). The number \(\rho(\alpha, A) = \limsup_{n\to\infty} \frac{1}{n} (p_2 \circ \tilde{T}_{\alpha, A}^n(\theta, \varphi) - \varphi) \mod \frac{1}{2}\mathbb{Z}\), does not depend on the choices of \(\theta\) and \(\varphi\), where \(p_2(\theta, \varphi) = \varphi\), and is called the rotation number of \((\alpha, A)\) \([9], [14]\).

It’s easy to see that Schrödinger cocycle is homotopic to the identity, and let \(\rho_{\lambda v, \alpha}(E) \in [0, \frac{1}{2}]\) be the rotation number of Schrödinger cocycle \((\alpha, S_{\lambda v, E})\).

### 2.3 Spectral measure and the integrated density of states

Let \(H\) be a bounded self-adjoint operator on \(\ell^2(\mathbb{Z})\). Then \((H - z)^{-1}\) is analytic in \(\mathbb{C}\setminus\text{Spec}(H)\), where \(\text{Spec}(H)\) is the spectrum of \(H\), and we have for \(f \in \ell^2\)
\[ \mathfrak{S} \langle (H - z)^{-1} f, f \rangle = \mathfrak{S} z \cdot \| (H - z)^{-1} f \|^2, \]
where \(\langle \cdot, \cdot \rangle\) is the usual inner product in \(\ell^2(\mathbb{Z})\). Thus
\[ \phi_f(z) = \langle (H - z)^{-1} f, f \rangle \]
is an analytic function in the upper half plane with \(\mathfrak{S} \phi_f \geq 0\) (\(\phi_f\) is a so-called Herglotz function).

Therefore one has a representation
\[ \phi_f(z) = \langle (H - z)^{-1} f, f \rangle = \int_{\mathbb{R}} \frac{1}{x - z} d\mu_f(x), \tag{2.5} \]
where $\mu^f$ is the spectral measure associated to vector $f$.

Denote by $\mu^f_{\lambda,v,\theta}$ the spectral measure of operator $H_{\lambda,v,\theta}$ and vector $f$ as before. The integrated density of states $N_{\lambda,v,\alpha}$ is obtained by averaging the spectral measure $\mu^\alpha_{\lambda,v,\theta}$ with respect to $\theta$, where $e_0$ is the Dirac mass at $0 \in \mathbb{Z}$, i.e.,

$$N_{\lambda,v,\theta}(E) = \int_{\mathbb{R}/\mathbb{Z}} \mu^\alpha_{\lambda,v,\theta}(-\infty, E) d\theta.$$  

(2.6)

Between the integrated density of states $N_{\lambda,v,\theta}(E)$ and the rotation number $\rho_{\lambda,v,\theta}(E)$, there is the following relation [13]:

$$N_{\lambda,v,\theta}(E) = 1 - 2\rho_{\lambda,v,\theta}(E).$$  

(2.7)

### 3 Some known results

Let $\mu_{\lambda,v,\theta}$ be $\mu^f_{\lambda,v,\theta} + \mu^\alpha_{\lambda,v,\theta}$, where $e_i$ is the Dirac mass at $i \in \mathbb{Z}$. For simplicity, sometimes we drop the parameters dependence, for example, replacing $\mu_{\lambda,v,\theta}$ with $\mu$. Fix $A = S_{\lambda,v,E} = \begin{pmatrix} E - \lambda v & -1 \\ 1 & 0 \end{pmatrix}$. Below, $C$ is a large absolute constant and $c$ is a small absolute constant, which may change through the arguments, even when appear in the same formula. Denote by $C_*$ ($c_*$) a large (small) constant depending on $\lambda, v, \alpha$. Let $\| \cdot \|$ be the Euclidean norms, and denote $\| f \|_\eta = \sup_{|x| < \eta} \| f(x) \|$, $\| f \|_0 = \sup_{x \in \mathbb{R}} \| f(x) \|$.

**Lemma 3.1. (Lemma 2.4, [1])** Let $\mathcal{B}$ be the set of $E \in \mathbb{R}$ such that the cocycle $(\alpha, A)$ is bounded, then $\mu_{\mathcal{B}}$ is absolutely continuous.

**Lemma 3.2. (Lemma 2.5, [1])** We have $\mu(E - \epsilon, E + \epsilon) \leq C \epsilon \sup_{0 \leq s \leq \epsilon} \| A_s \|_0^2$.

Given $\epsilon_0 > 0$, we say $k$ is an $\epsilon_0$-resonance for $\theta$, if $\| 2\theta - k\alpha \|_{\mathbb{R}/\mathbb{Z}} \leq \epsilon_0$ and $\| 2\theta - k\alpha \|_{\mathbb{R}/\mathbb{Z}} = \min_{|\beta| \leq |k|} \| 2\theta - j\alpha \|_{\mathbb{R}/\mathbb{Z}}$.

Clearly, $0 \in \mathbb{Z}$ is an $\epsilon_0$-resonance. We order the $\epsilon_0$-resonances $0 = |n_0| < |n_1| \leq |n_2| \cdots$. We say $\theta$ is $\epsilon_0$-resonant if the set of $\epsilon_0$-resonances is infinite.

**Lemma 3.3. (Theorem 3.3, [2])** If $E \in \Sigma_{\lambda,v,\alpha}$, then there exists $\theta \in \mathbb{R}$ and a bounded solution of $\tilde{H}_{\lambda,v,\theta} \tilde{u} = E \tilde{u}$ with $\tilde{u}_0 = 1$ and $|\tilde{u}_k| \leq 1$.

Fix $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ such that $0 < \beta(\alpha) < \infty$. Let $\epsilon_0 = C_1\beta$, where $C_1$ is a large absolute constant, which is much larger than any absolute constant $C$, $c^{-1}$ emerging in the present paper. Set $h_1 = C_1\beta$, $h_2 = C_2\beta$. Fix $E \in \Sigma_{\lambda,v,\theta}$ below, and choose some $\theta = \theta(E)$ given by Lemma [3]. Denote $\{ n_j \}$ all the $\epsilon_0$-resonances for $\theta(E)$.

By the present authors’s arguments in [15],[16], if $v$ is analytic in strip $|\Im x| < C_2\beta$, where $C_2$ is a large absolute constant, then there exists $\lambda_0 = \lambda_0(v, \beta) > 0$ such that the following theorems hold for $|s| < \lambda_0$. 


Theorem 1.1. (Lemma 4.3, [16]) The Lyapunov exponent vanishes on \( \Sigma_{\lambda, v, 0} \), i.e., \( L(\alpha, S_{\lambda, v, E}) = 0 \) for all \( E \in \Sigma_{\lambda, v, 0} \).

Theorem 3.2. (Theorem 5.6, [16]) We have the following estimate,

\[
\|A_\lambda\|_0 \leq C_* e^{C_* n}, 0 \leq s \leq e^{C_* n}.
\] (3.1)

Theorem 3.3. (Corollary 6.2, [16]) The integrated density of states of \( H_{\lambda, v, 0} \) is \( 1/2 \)-Hölder continuous, that is \( N_{\lambda, v, \alpha}(J) \leq C_* |J|^{1/2} \) for any interval \( J \subset \mathbb{R} \).

Theorem 3.4. (Theorem 4.14, [16]) If \( \theta = \theta(E) \) has a \( \epsilon_0 \)-resonance \( n_j \), then there exists \( m_j \) with \( |m_j| \leq C |n_j| \) such that \( \|2 \rho_{\lambda, v, \alpha}(E) - m_j \alpha\|_{\mathbb{R}/\mathbb{Z}} \leq C_* e^{-\epsilon_0 |n_j|} \), or equivalently \( \|N_{\lambda, v, \alpha}(E) - m_j \alpha\|_{\mathbb{R}/\mathbb{Z}} \leq C_* e^{-\epsilon_0 |n_j|} \) by (2.7).

Theorem 3.5. (Theorem 5.8, [16]) If \( \theta(E) \) is not \( \epsilon_0 \)-resonant, then cocycle \( A \) is reducible.

Remark 3.1. In [15], the present authors only prove Theorem 3.4, 3.5 for AMO by quantitative version of Aubry duality in exponential regime. For the general quasi-periodic Schrödinger operator, the proof is similar if we use the quantitative version of Aubry duality for general potential \( v \) in [16].

4 Proof of Theorem 1.1

Lemma 4.1. For \( 0 < \epsilon < 1 \), \( N_{\lambda, v, \alpha}(E + \epsilon) - N_{\lambda, v, \alpha}(E - \epsilon) \geq C_* e^{2 \epsilon} \).

Proof: The lemma can be proved directly by Theorem 3.1 and 3.3. See the proof of Lemma 3.11 in [1] for details.

Proof of theorem 1.1 It is well known that it suffices to prove that \( \mu \) is absolutely continuous. Let \( \mathcal{B} \) be given by Lemma 3.1. Thus it suffices to show \( \mu(\Sigma_{\lambda, v, \alpha} \setminus \mathcal{B}) = 0 \). Let \( \mathcal{R} \) be the set of \( E \in E_{\lambda, v, \alpha} \) such that \( A \) is reducible. We have \( \mu(\mathcal{R} \setminus \mathcal{B}) = 0 \), since \( \mathcal{R} \setminus \mathcal{B} \) is a countable set and there is no eigenvalue in \( \mathcal{R} \) (see p.16 in [1] for details). Thus to prove the Theorem 1.1 it is sufficient to show that \( \mu(\Sigma_{\lambda, v, \alpha} \setminus \mathcal{R}) = 0 \).

Let \( K_m \subset \Sigma_{\lambda, v, \alpha} \), \( m \geq 1 \) be the set of \( E \) such that there exists \( \theta(E) \in \mathbb{R} \) given by Lemma 3.2 with a resonance \( 2^m \leq |n_j| \leq 2^{m+1} \). We will show that \( \sum \mu(K_m) < \infty \). By Theorem 3.5 \( \Sigma_{\lambda, v, \alpha} \setminus \mathcal{R} \subset \limsup K_m \), then \( \mu(\Sigma_{\lambda, v, \alpha} \setminus \mathcal{R}) = 0 \) by the fact \( \sum \mu(K_m) < \infty \) and the Borel-Cantelli Lemma.

For every \( E \in K_m \), let \( J_m(E) \) be an open \( \epsilon_m = C_* e^{-c_0 2^{m+1}} \) neighborhood of \( E \). By (3.1),

\[
\sup_{0 \leq s \leq C_* 2^{m+1}} \|A_s\|_0 \leq C_* e^{C_* 2^{m+1}}.
\] (4.1)
Take a finite subcover $\overline{K}_m \subset \bigcup_{j=0}^r J_m(E_j)$. Refining this subcover if necessary, we may assume that every $x \in \mathbb{R}$ is contained in at most 2 different $J_m(E_j)$.

By lemma 4.1, $N(J_m(E)) \geq c_s |J_m(E)|^2 \geq C_\ast e^{-\epsilon_0 2^m}$. By Theorem 3.4 if $E \in K_m$ then $|N(E) - k\alpha|_{\mathbb{R}/\mathbb{Z}} \leq C_\ast e^{-\epsilon_0 2^m}$ for some $|k| \leq C 2^m$, so there are at most $C_\ast 2^m$ intervals $J_m(E_j)$, i.e., $r \leq C_\ast 2^m$. Thus by (4.1) and Lemma 3.2,

$$\mu(\overline{K}_m) \leq \sum_{j=0}^r \mu(J_m(E_j)) \leq C_\ast 2^m e^{C \beta 2^m} e^{-c_\epsilon_0 2^m}, \quad (4.2)$$

which implies $\sum_m \mu(\overline{K}_m) < \infty$. □

Next, we will prove that the integrated density of states is absolutely continuous in perturbative regime for all $\alpha$ satisfying $0 < \beta(\alpha) < \infty$. We need a lemma first.

**Lemma 4.2.** (Corollary 1, [5]) If the Lyapunov exponent vanishes on $\Sigma_{\lambda,v,\alpha}$, then $H_{\lambda,v,\alpha,\theta}$ has purely absolutely continuous spectrum for almost $\theta$ if and only if the integrated density of states $N_{\lambda,v,\alpha}(E)$ is absolutely continuous.

**Theorem 4.1.** For irrational number $\alpha$ such that $0 < \beta(\alpha) < \infty$, if $v$ is analytic in strip $|\Im x| < C_2 \beta$, where $C_2$ is a large absolute constant, then there exists $\lambda_0 = \lambda_0(v,\beta)$ such that the integrated density of states $N_{\lambda,v,\alpha}(E)$ is absolutely continuous if $|\lambda| < \lambda_0$.

**Proof:** Using Theorem 3.1 and Lemma 4.2, $N_{\lambda,v,\alpha}(E)$ is absolutely continuous if and only if $H_{\lambda,v,\alpha,\theta}$ has purely absolutely continuous spectrum for almost every $\theta$. Together with Theorem 1.1 we finish the proof.

**References**

[1] A. Avila, The absolutely continuous spectrum of the almost Mathieu operator, arXiv preprint [arXiv:0810.2965], (2008).

[2] A. Avila, S. Jitomirskaya, Almost localization and almost reducibility, J. Eur. Math. Soc.12 (2010), 93-131.

[3] J. Bourgain, Green function estimates for lattice Schrödinger operators and applications, Ann. of Math. Studies 158, Univ. Press, Princeton, NJ, 2005.

[4] J. Bourgain, S. Jitomirskaya, Absolutely continuous spectrum for 1D quasiperiodic operators, Invent. math. 148 (2002), 453-463.
[5] D. Damanik, Lyapunov exponents and spectral analysis of ergodic Schrödinger operators: A survey of Kotani theory and its applications, Spectral Theory and Mathematical Physics: a Festschrift in honor of Barry Simon’s 60th Birthday, 539-563, Proc. Sympos. Pure Math. 76, Part 2, Amer. Math. Soc., Providence, RI, 2007.

[6] E. Dinaburg, Ya. Sinai, The one-dimensional Schrödinger equation with a quasi-periodic potential, Funct. Anal. Appl. 9 (1975), 279-289.

[7] L. H. Eliasson, Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation, Comm. Math. Phys. 146 (3) (1992), 447-482.

[8] A. Y. Gordon, S. Jitomirskaya, Y. Last, B. Simon, Duality and singular continuous spectrum in the almost Mathieu equation, Acta Math. 178 (1997), 169-183.

[9] M. Herman, Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d’un théorème d’Arnold et de Moser sur le tore de dimension 2, Comment. Math. Helv. 58 (1983), 453-502.

[10] S. Jitomirskaya, Metal-insulator transition for the almost Mathieu operator, Ann. of Math. 150 (2) (1999), 1159-1175.

[11] S. Jitomirskaya, Y. Last, Power-law subordinacy and singular spectra, i. half-line operators, Acta Math. 183 (2) (1999), 171-189.

[12] S. Jitomirskaya, Y. Last, Power law subordinacy and singular spectra, ii. line operators, Comm. Math. Phys. 211 (3) (2000), 643-658.

[13] R. Johnson, A review of recent work on almost periodic differential and difference operators, Acta Appl. Math. 1 (3) (1983), 241-261.

[14] R. Johnson, J. Moser, The rotation number for almost periodic potentials, Comm. Math. Phys. 84 (1982), 403-438.

[15] W. Liu, X. Yuan, Spectral Gaps of Almost Mathieu Operator in Exponential Regime, in preparation.
[16] W. Liu, X. Yuan, Hölder Continuity of the Spectral Measures for One-Dimensional Schrödinger Operator in Exponential Regime, in preparation.