Impurity spins randomly distributed at the surfaces and interfaces of superconducting wires are known to cause flux noise in Superconducting Quantum Interference Devices (SQUIDs), providing a dominant mechanism for decoherence in flux-tunable superconducting qubits. While flux noise is well characterized experimentally, the microscopic model underlying spin dynamics remains a great puzzle. Available first-principles theories are too computationally expensive to capture spin diffusion over large length scales, hindering comparisons between microscopic models and experimental data. In contrast, third principles approaches lump spin dynamics into a single phenomenological spin-diffusion operator $D\nabla^2$, that is not able to describe the quantum noise regime and connect to microscopic models and different disorder scenarios such as spin clusters. Here we propose an intermediate "second principles" method to describe general spin dissipation and flux noise in the quantum regime. It leads to the interpretation that flux noise arises from the density of paramagnon excitations at the edge of the superconducting wire, with paramagnon-paramagnon interactions leading to spin diffusion, and interactions between paramagnons and other degrees of freedom such as phonons, electron gas, and two-level systems leading to spin energy relaxation. At high frequency $\omega$ we obtain an upper bound for flux noise, showing that the (super)Ohmic noise observed in experiments is not originating from interacting spin impurities. We apply the method to Heisenberg models in two dimensional square lattices with a random distribution of vacancies, with nearest-neighbor spins coupled by constant exchange. Explicit numerical calculations of flux noise shows that it follows the observed power law $A/\omega^\alpha$, with amplitude $A$ and exponent $\alpha$ depending on temperature and inhomogeneities such as spatial confinement and disorder. These results are compared to experiments in niobium and aluminium devices. The method establishes a connection between flux noise experiments and microscopic Hamiltonians with the goal of identifying relevant microscopic mechanisms and guiding strategies for reducing flux noise.

I. INTRODUCTION

While progress in experimental realization of quantum computers based on superconducting wires and Josephson junctions has been remarkable [1], the noise level in current devices greatly reduces their capacity to solve problems, and washes out their “quantum advantage”. One key issue is the trade off between scalability and flux noise. Qubit frequency tunability is essential to circumvent the frequency crowding problem faced by superconducting circuits with more than 100 qubits. This requires the addition of Superconducting Quantum Interference Devices (SQUIDs) to the circuit, increasing their sensitivity to flux noise [2]. A similar issue plagues SQUID qubits, in that additional qubit interconnection increases the impact of flux noise [3].

The origin of flux noise in superconducting devices remains unknown, although there is consensus that it arises from the dynamics of spin centers (magnetic impurities) near the superconducting wires [4–10] (see Fig. 1). This conclusion is supported by experiments showing that the amplitude of the noise follows the Curie susceptibility law $\chi(\omega = 0) \propto 1/(T - T_{CW})$ which rules out the spin glass scenario. Instead, [10] measured $T_{CW} \approx -10$ mK $< 0$ indicating proximity to an antiferromagnetic phase, while [12] measured $T_{CW} \approx +5$ mK $> 0$ indicating proximity to a ferromagnetic phase.

Measurements of flux noise [10, 11, 13, 14] are quite puzzling. They reveal approximate $\tilde{S}_\Phi(\omega) \propto 1/\omega^\alpha$ frequency dependence over several decades of frequency, and show a sizable asymmetry $\tilde{S}_\Phi(\omega) = \tilde{S}_\Phi(\omega) - \tilde{S}_\Phi(-\omega)$ due to quantum noise (spontaneous emission) [10]. To our knowledge all theories of flux noise available in the literature are semiclassical so they are not able to describe this asymmetry [7, 11, 15]. Moreover, flux noise was shown to become either Ohmic $\propto \omega$ [16] or super-Ohmic $\propto \omega^3$ above 4 GHz [10, 17], and a key open question is whether or not this high frequency contribution is also due to impurity spins.

The main mechanism for spin energy relaxation in ferromagnetic metals, the so called Gilbert damping, relies on magnetic excitations (magnons) decaying into electron-hole pair excitations in the metal. In a superconductor these are exponentially suppressed at temperatures much lower than the superconducting energy gap, making the Gilbert damping constant exponentially small [18]. In addition, the weak magnetic fields in superconducting devices implies the spin-orbit and hyperfine spin-flip rates are close to zero [4]. The only remaining
Describing spin diffusion from “first principles”, i.e. by integrating the Heisenberg equations of motion for a model of interacting spins, is a well known challenge of theoretical physics [22]. The standard method is what we call “third-principles approach”: It assumes the spins can be described by a continuous magnetization density \( \mathbf{M}(\mathbf{r}, t) \) that satisfies the phenomenological equation \( \frac{\partial \mathbf{M}}{\partial t} = D \nabla^2 \mathbf{M} \), with \( D \) the spin diffusion constant. With all physical properties lumped into a phenomenological constant \( D \), the third-principles theory can not establish a connection to microscopic model spin Hamiltonians, spin spatial distributions, and the impact of inhomogeneity and clusters.

A serious shortcoming of the third principles approach is that it requires the assumption of a hard boundary condition such as \( \mathbf{M}(\mathbf{r}, t) = 0 \) at wire edges and the surface of spin clusters in order to ensure total spin conservation across the boundaries [7, 12]. A hard boundary condition like this is unjustifiable, and in fact is known to be violated in magnetic systems due to the appearance of confined surface or edge magnons [25]. Developing a theory of spin dynamics that properly accounts for the boundary effects is of crucial importance because flux noise is known to be dominated by spins at the edge of the wire where the supercurrent is maximum [5, 24].

The purpose of this article is to propose a “second principles” theoretical framework for spin dissipation (diffusion plus relaxation) that includes quantum noise and is more “microscopic” than the usual third-principles approach. The goal is to establish a connection between flux noise measured in experiments and microscopic spin Hamiltonians, without the prohibitive computational cost associated to the first-principles approach.

Below we describe general theoretical results, and then present explicit numerical calculations of flux noise for the particular case of the Heisenberg model with nearest neighbour interactions in the paramagnetic phase \( T > T_{\text{c}}(\text{max}) \). Our calculations are done in a finite spin lattice with a random distribution of vacancies, showing explicit predictions for spatial confinement (wire edges) and disorder due to random distribution of vacancies across the wire’s surface as well as wide distributions of individual spin-flip rates \( \Gamma_i \).

II. MODEL FOR FLUX NOISE AND LINEAR RESPONSE THEORY

We start by describing the impact of wire currents on impurity spins and how it leads to a general expression for flux noise. The magnetic moment of an impurity spin is given by \( -g \mu_B \mathbf{s}_i \), where \( g \approx 2 \) is the \( g \)-factor, \( \mu_B \) is the Bohr magneton, and \( \mathbf{s}_i \) is a dimensionless spin operator for an impurity located at position \( \mathbf{R}_i \). It couples to the superconducting wire current density by producing a flux

\[
\hat{\Phi} = -\sum_i x_i \mathbf{F}(\mathbf{R}_i) \cdot \hat{s}_i, \tag{1}
\]

This is the desire to know what is the impact of nonhomogeneous spin distributions, such as impurity spin clustering [14, 15].
where the sum goes over all sites $\mathbf{R}_i$ of a virtual square lattice containing $N$ sites. The variable $x_i = 1$ when there is a spin at the virtual site, and $x_i = 0$ otherwise, and the spin density is $\sigma = \sum x_i/N = N_s/N$, where $N_s$ is the number of spins. The flux vector $\mathbf{F}(t)$ is directly proportional to $B_1(\mathbf{r})$, the magnetic field produced by the wire’s current density: $\mathbf{F}(t) = g\mu_B B_1(t)/I$, where $I$ is the total current flowing through the wire.

The wire’s current in turn affects spins by imprinting an external local field

$$\mathbf{h}_i = -g\mu_B B_1(\mathbf{R}_i) = -IF(\mathbf{R}_i), \quad (2)$$

that couples to the spin according to $\mathcal{H}_c = -\sum_j x_i \mathbf{h}_i \cdot \mathbf{s}_i$, so $\mathbf{h}_i$ has dimensions of energy. When the local field $\mathbf{h}_i$ is time dependent, the spins respond according to

$$\langle \hat{s}_i^a(t) \rangle_{h\neq0} = \langle \hat{s}_i^a(t) \rangle_{h=0} + \sum_{j,b} \int_{-\infty}^{\infty} dt' \chi_{ij}^{ab}(t-t')\hat{s}_{j}^{b}(t'), \quad (3)$$

where $a, b = x, y, z$ and the dynamical susceptibility is given by the linear response formula,

$$\chi_{ij}^{ab}(t-t') = x_i x_j \int_{-\infty}^{\infty} dt e^{i\omega t} \langle [\hat{s}_i^a(t), \hat{s}_j^b(t')] \rangle. \quad (4)$$

Defining spin noise as

$$\tilde{S}_{ij}^{ab}(\omega) = x_i x_j \int_{-\infty}^{\infty} dt e^{i\omega t} \langle [\hat{s}_i^a(t), \hat{s}_j^b(0)] \rangle, \quad (5)$$

and using Eq. (4) we obtain the general relationship between susceptibility and spin noise,

$$\tilde{\chi}_{ij}^{ab}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \frac{1}{\omega' - \omega - i\eta} \tilde{S}_{ij}^{ab}(\omega'), \quad (6)$$

where $\eta \to 0^+$. Taking the imaginary part and using the fact that $\tilde{S}_{ij}^{ab}(\omega) = \tilde{S}_{ji}^{ab}(\omega^*)$, we get the fluctuation-dissipation theorem for spins,

$$\tilde{S}_{ij}^{ab}(\omega) = \frac{2\hbar}{1 - e^{-\hbar\omega/k_BT}} \frac{1}{2i} [\tilde{\chi}_{ij}^{ab}(\omega) - \tilde{\chi}_{ji}^{ba}(\omega^*)]. \quad (7)$$

The flux noise is then given by

$$\tilde{S}_{\Phi}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \delta \Phi(t) \delta \Phi(0) \rangle = \sum_{i,j,a,b} F^a(\mathbf{R}_i) \tilde{S}_{ij}^{ab}(\omega) F^b(\mathbf{R}_j), \quad (8)$$

where $\delta \Phi(t) = \Phi(t) - \langle \Phi \rangle$ denotes flux fluctuation.

### III. THEORY OF SPIN DYNAMICS

Our goal is to compute flux noise for the general Heisenberg quantum spin Hamiltonian,

$$\mathcal{H} = -\frac{1}{2} \sum_{i,j} x_i x_j J_{ij} \mathbf{s}_i \cdot \mathbf{s}_j - \sum_i x_i \mathbf{h}_i \cdot \mathbf{s}_i, \quad (9)$$

plus spin energy relaxation due to other degrees of freedom such as phonons, electron-hole excitations, and two-level system defects [19]. Here $J_{ij}$ is the exchange interaction between spins $i$ and $j$, that can be ferromagnetic ($J_{ij} > 0$) or antiferromagnetic ($J_{ij} < 0$), and $\mathbf{h}_i$ is the local field defined in Eq. (2).

#### A. Static mean-field theory

In mean-field theory we neglect higher order fluctuations by approximating $\langle \hat{s}_i^a \hat{s}_j^b \rangle \approx \langle \hat{s}_i^a \rangle \langle \hat{s}_j^b \rangle$. We simplify the notation by writing $\langle \hat{s}_i \rangle = s_i$, i.e. the spin vector without a hat denotes the average of the spin operator (a real vector). The mean-field approximation is exactly the same as the “classical spin model” used by many authors, e.g. [15]. An additional approximation in mean-field theory is to assume the system’s entropy can be written as a sum of single spin entropies [26]:

$$\langle S \rangle = k_B \sum_i x_i \left[ \ln 2 - 2|s_i|^2 - \frac{4}{3}(|s_i|^2)^2 + O((|s_i|^2)^3) \right]. \quad (10)$$

This expression is specific to spin-1/2 impurities. Note that cutting the expansion to fourth order affects the result only when $T \ll T_{\text{mag}}$, where $T_{\text{mag}}$ is a critical temperature for a phase where $s_i > 0$. The free energy is thus given by:

$$\mathcal{F} = \langle \mathcal{H} \rangle - T\langle S \rangle = -\frac{1}{2} \sum_{i,j} x_i x_j J_{ij} s_i s_j - \sum_i x_i \mathbf{h}_i \cdot \mathbf{s}_i - (k_BT) \sum_i x_i \left[ \ln 2 - 2|s_i|^2 - \frac{4}{3}(|s_i|^2)^2 \right]. \quad (11)$$

Usually thermal equilibrium is realized by the set of $s_i$ that leads to the global minimum of the free energy. For example, take $x_i = 1$ for all $i$, $J_{ij} = J > 0$ for nearest neighbors and zero otherwise, and a lattice with periodic boundary conditions (b.c.). In this case the global minimum of $\mathcal{F}$ is realized by $s_i = s^\text{eq}$ for all $i$ (the ferromagnetic homogeneous state), with free energy given by

$$\frac{\mathcal{F}}{N} = \left( 2k_BT - \frac{zJ}{2} \right) (s^\text{eq})^2 + \frac{4}{3} k_BT (s^\text{eq})^4 - k_BT \ln 2, \quad (12)$$

where $z$ is the number of nearest neighbors for each site of the lattice. From this expression we see that a global minimum with $s^\text{eq} > 0$ appears only when the first term changes sign, leading to critical temperature $k_BT_{\text{mag}} = k_BT_{\text{CW}} = zJ/4$. The same calculation can be done for $J < 0$ when the lattice can be partitioned into two sublattices with one being n.n. to the other. For this case the global minimum of $\mathcal{F}$ is realized by $s_i = +s^\text{eq}$ for one sublattice and $s_i = -s^\text{eq}$ for the other (antiferromagnetic homogeneous state). This leads to $k_BT_c = z|J|/4$. However, $k_BT_{\text{CW}} = -z|J|/4$ because the magnetic susceptibility does not have a singularity at $T_c$ (it's the staggered susceptibility that is singular at $T_c$). These are the
well-known mean-field results for phase transitions in the spin-1/2 Heisenberg model [26].

### B. Dynamical mean-field theory

Based on fundamental theories of spin dynamics [26] we propose the following generalized equation of motion for the spins,

$$\frac{\partial s_i}{\partial t} = \frac{1}{\hbar} s_i \times H_i - \sum_j D_{ij} H_j - \Gamma_i (s_i - s_i^{\text{inst eq}}), \tag{13}$$

expected to be valid for frequencies smaller than a cut-off $\Omega_c$ to be discussed later. In addition to the usual spin precession, this includes a discrete version of the intra-spin dissipation operator $D_{ij}$, together with isotropic spin energy relaxation $\Gamma_i$. The $\Gamma_i$ drives $s_i$ towards its “instantaneous equilibrium” value

$$s_i^{\text{inst eq}} = s_i^{\text{eq}} + \sum_j \chi_{ij}(0) \delta h_j(t), \tag{14}$$

which is time-dependent due to local field dynamics, $h_i(t) = h_i^{\text{eq}} + \delta h_i(t)$, where $h_i^{\text{eq}}$ is the static part. The quantities $s_i^{\text{eq}}$ do not depend on time, they are thermal equilibrium averages calculated assuming $\delta h_i(t) = 0$, i.e. they only depend on $h_i^{\text{eq}}$ and other static free energy parameters; $\chi_{ij}^{eq}(0) = \partial (s_i^{eq})^a / \partial h_j^b$ is the $\omega = 0$ susceptibility, assumed isotropic ($\propto \delta_{ab}$) to be consistent with our Hamiltonian (9). We call Eq. (14) the “instantaneous approximation”, because it assumes that the other degrees of freedom causing spin energy relaxation relax much faster than the spins themselves, so that the spin system remains in thermal equilibrium with the other non-spin degrees of freedom at all times. Note how Eq. (14) introduces the $\omega = 0$ susceptibility self-consistently into the equation of motion (13).

The internal spin field is defined as

$$H_i = -\frac{\partial F}{\partial s_i}, \tag{15}$$

$$= x_j \left\{ \sum_j x_j \hat{J}_{ij} s_j + h_i - 4k_B T \left[ 1 + \frac{4}{3} s_i^2 \right] s_i \right\},$$

and the thermal equilibrium spin averages $s_i^{eq}$ are determined by imposing time independence, $\delta h_i(t) = 0$ and $\partial s_i / \partial t = 0$ for all $i$. This implies $s_i^{eq}$ must be found by solving the system of equations

$$\frac{1}{\hbar} s_i^{eq} \times H_i^{eq} - \sum_j D_{ij} H_j^{eq} = 0, \tag{16}$$

where $H_i^{eq}$ is Eq. (15) with $s_i = s_i^{eq}$ and $h_i = h_i^{eq}$. Note how Eq. (16) is always satisfied for $H_i^{eq} = 0$, a smooth local minimum of the free energy. However, other solutions with $H_i^{eq} \neq 0$ may arise in the presence of site-dependent local fields $h_i^{eq}$.

The three terms in the right hand side of Eq. (13) correspond to reactive dynamics, intra-spin-system dissipation (e.g. diffusion), and spin energy relaxation due to other degrees of freedom, respectively. The reactive term is non-dissipative, it does not change sign under time reversal $t \rightarrow -t$ so it has the same symmetry as the left hand side. The second and third terms on the right hand side do change sign under time reversal, leading to an irreversible approach to thermal equilibrium (the arrow of time). These terms must be added ad hoc to the linearized equations of motion so that the zeroth (attainment of thermal equilibrium) and second (entropy always increases) laws of thermodynamics are obeyed. That is, the system is able to reach thermal equilibrium, and the free energy always decreases as a function of time when the system is in contact with a thermal reservoir.

A few notes about the microscopic origin of spin dissipation $D_{ij}$ are warranted. The normal modes of Eq. (13) are called magnons and paramagnons, to be defined below. The reactive terms of Eq. (13) describe the dynamics of noninteracting (para)magnons obtained by the mean-field approximation. Exactly the same results are obtained from different methods, e.g. using a Holstein-Primakoff transformation to convert spin operators into Bosonic creation/destruction operators; transforming Hamiltonian (9) and keeping contributions that are quadratic in these Bosonic operators leads to the same magnon modes as Eq. (13) with $D_{ij} = \Gamma_i = 0$ [25]. However, the higher order terms that are neglected in this quadratic approximation can be interpreted as describing (para)magnon-(para)magnon interactions. The introduction of $D_{ij} \neq 0$ serves to account for these interactions phenomenologically.

### C. Calculation of dynamical susceptibility

For small deviations from equilibrium we write $h_i = h_i^{eq} + \delta h_i(t)$, and $s_i = s_i^{eq} + \delta s_i(t)$, where both $\delta h_i(t)$ and $\delta s_i(t)$ are small time-dependent perturbations.

We plug these into the equation of motion (13), drop non-linear terms such as $(\delta s_i)^2$, $(\delta h_i(t))^2$, and use Eq. (16) to simplify:
\[
\frac{\partial}{\partial t}(\delta s_i) = \frac{1}{\hbar} s_i^{\text{eq}} \times \left\{ (\delta h_i) + \sum_j x_j J_{ij}(\delta s_j) - 4k_B T \left[ 1 + \frac{4}{3}(s_{i}^{\text{eq}})^2 \right] (\delta s_i) - \frac{32}{3} k_B T [s_i^{\text{eq}} \cdot (\delta s_i)] \right\} \\
- \sum_j D_{ij} x_j \left\{ (\delta h_j) + \sum_k x_k J_{jk}(\delta s_k) - 4k_B T \left[ 1 + \frac{4}{3}(s_{j}^{\text{eq}})^2 \right] (\delta s_j) - \frac{32}{3} k_B T [s_j^{\text{eq}} \cdot (\delta s_j)] \right\} \\
+ \frac{1}{\hbar} (\delta s_i) \times H_i^{\text{eq}} - \Gamma_i(\delta s_i) + \Gamma_i \sum_j \chi_{ij}(0)(\delta h_j).
\]

(17)

We assume \( s_i^{\text{eq}} = s_i^{\text{eq}} \hat{z} \) and \( H_i^{\text{eq}} = H_i^{\text{eq}} \hat{z} \) and break this down into two equations, one for \( \delta s_i^z \) obtained by dot product with \( \hat{z} \) on both sides of Eq. (17), and the other for \( \delta s_i^x = \delta s_i^y = i \delta s_i^x \) obtained by dot product with \( \hat{x} + i \hat{y} \). Taking the time Fourier transform we get two decoupled equations:

\[
(\omega - \mathbf{P}_d) \cdot \delta \tilde{z} = i(\mathbf{\Gamma} \cdot \tilde{\chi}_0 - \mathbf{D}) \cdot \delta \tilde{h}, \tag{18a}
\]

\[
(\omega - \mathbf{M}) \cdot \delta \tilde{s}^+ = i(\mathbf{\Gamma} \cdot \tilde{\chi}_0 - \mathbf{D}) - \frac{1}{\hbar} s^{\text{eq}} \cdot \delta \tilde{h}^+, \tag{18b}
\]

where \( \delta \tilde{z}, \delta \tilde{s}^+ \) and \( \delta \tilde{h}, \delta \tilde{h}^+ \) are \( N \)-component column vectors, and \( \mathbf{P}, \mathbf{\Gamma}, \tilde{\chi}_0, \mathbf{D}, s^{\text{eq}} \) are \( \mathbf{N} \times \mathbf{N} \) matrices. They are defined by \( [\mathbf{P}]_{ij} = x_i \delta_{ij}, [\mathbf{\Gamma}]_{ij} = x_i \Gamma_{ij}, [\tilde{\chi}_0]_{ij} = \tilde{\chi}_{ij}(0), [\mathbf{D}]_{ij} = x_i x_j \Gamma_{ij}, \) and \( [s^{\text{eq}}]_{ij} = x_i s^{\text{eq}}_{ij} \).

The matrices \( \mathbf{P} \) and \( \mathbf{M} \) are the paramagron and magnon matrices, respectively. They are given by

\[
\mathbf{P} = -i \left\{ \mathbf{I} + \frac{4}{3} (s^{\text{eq}})^2 \right\} \tag{19a}
\]

\[
\mathbf{M} = -i \left\{ \mathbf{I} + \frac{4}{3} (s^{\text{eq}})^2 \right\} \tag{19b}
\]

where \( [\mathbf{I}]_{ij} = x_i x_j \delta_{ij}, \) and \( [H^{\text{eq}}]_{ij} = H^{\text{eq}} \delta_{ij} \).

The eigenvalues of \( \mathbf{P} \) and \( \mathbf{M} \) are paramagron and magnon frequencies, respectively. While paramagnons cause spin fluctuations along \( s^{\text{eq}} \), the magnons cause fluctuations perpendicular to \( s^{\text{eq}} \). In the paramagnetic phase \( (T > T^{\text{mag}}) \) with zero external fields, \( s^{\text{eq}} = H^{\text{eq}} = 0 \) and the matrices \( \mathbf{P} \) and \( \mathbf{M} \) become identical, signaling the presence of isotropic spin fluctuations (i.e. paramagnons are three-fold degenerate). From now on we shall focus our discussion on this paramagnetic regime, so we only need to consider the spectrum of \( \mathbf{P} \).

Diagonalize \( \mathbf{P} \) with a transformation \( \mathbf{U} \) such that

\[
\mathbf{U}^{-1} \cdot \mathbf{P} \cdot \mathbf{U} = \mathbf{P}_d = -i \sum_m \gamma_m \tilde{\epsilon}_m \otimes \tilde{\epsilon}_m^T, \tag{20}
\]

where \( m \) labels the paramagron mode with frequency \(-i \gamma_m \), with \( \tilde{\epsilon}_m \) unit column vectors, \( \tilde{\epsilon}_m^T = (0, \ldots, 0, 1, 0, \ldots, 0) \), etc. All elements of \( \mathbf{P} \) are pure complex, therefore the \( \mathbf{U} \) and \( \mathbf{U}^{-1} \) can be chosen to have real elements. Take the complex conjugate of Eq. (20) and use \( \mathbf{P}^* = -\mathbf{P} \) to see that \( \gamma_0 = \gamma_m \).

Apply \( \mathbf{U}^{-1} \) on both sides of Eq. (18a),

\[
(\omega - \mathbf{P}_d) \cdot \mathbf{U}^{-1} \cdot \delta \tilde{z} = i \mathbf{U}^{-1} \cdot (\mathbf{\Gamma} \cdot \tilde{\chi}_0 - \mathbf{D}) \cdot \delta \tilde{h}, \tag{21}
\]

and invert the diagonal matrix to get an exact expression for the susceptibility:

\[
(\delta \tilde{z}) = i \sum_m \tilde{\epsilon}_m \otimes \tilde{\epsilon}_m^T \cdot \mathbf{U}^{-1} \cdot (\mathbf{\Gamma} \cdot \tilde{\chi}_0 - \mathbf{D}) \cdot \delta \tilde{h}. \tag{22}
\]

Transformation (20) implies that column vector \( \mathbf{v}_m = \mathbf{U} \cdot \tilde{\epsilon}_m \) is the right eigenvector of \( \mathbf{P} \) associated to the eigenvalue \(-i \gamma_m \). Similarly, the line vector \( \mathbf{v}_m^T = \tilde{\epsilon}_m^T \cdot \mathbf{U}^{-1} \) is the left eigenvector of \( \mathbf{P} \) associated to the same eigenvalue. We have \( \mathbf{v}_m \cdot \mathbf{v}_m' = \delta_{mm'} \), but the set \( \{ \mathbf{v}_m \} \) is not mutually orthogonal because \( \{ \mathbf{P}, \mathbf{P}^\dagger \} \neq 0 \) (\( \mathbf{P} \) is not normal).

Using these results, Eqs. (18a) and (22) imply the following exact expression for the isotropic dynamical susceptibility:

\[
\chi^{ab}(\omega) = i (\omega - \mathbf{P}^{-1})^{-1} \cdot (\mathbf{\Gamma} \cdot \tilde{\chi}_0 - \mathbf{D}) \delta_{ab} \tag{23a}
\]

\[
= i \sum_m \mathbf{v}_m \otimes \mathbf{v}_m^T \cdot [\mathbf{v}_m^{-1} \cdot (\mathbf{\Gamma} \cdot \tilde{\chi}_0 - \mathbf{D})] \delta_{ab}, \tag{23b}
\]

valid for \( T > T^{\text{mag}} \) with \( a, b = x, y, z \). The dynamical susceptibility has poles at the paramagron frequencies \( \omega = -i \gamma_m \), and these contribute to dissipation and noise. The paramagnons are said to be purely dissipative because \( \gamma_m \) is real, leading to \( e^{-i \omega t} = e^{-i \gamma_m t} \) for the decay of the spin excitations. For \( T > T^{\text{mag}} \) we have \( \gamma_m \geq 0 \) because the paramagnetic phase is stable.

At high frequency \( \omega \gg \text{Max}_m \{ \gamma_m \} \), Eq. (23b) leads to

\[
\chi^{ab}(\omega) \approx \frac{i \mathbf{\Gamma} \cdot \tilde{\chi}_0}{\omega} \mathbf{D} \delta_{ab}, \tag{24}
\]

because \( \sum_m \mathbf{v}_m \otimes \mathbf{v}_m^{-1} = \mathbf{I} \). As Eq. (13) has an upper frequency cut-off, Eq. (24) should be taken as an upper bound on the modulus of the susceptibility.

Now consider the opposite limit, \( \omega \rightarrow 0 \). Since \( \chi^{ab}(0) = \tilde{\chi}_0 \delta_{ab} \), setting \( \omega = 0 \) in Eq. (23a) leads to \( (\mathbf{P} + \mathbf{I}) \cdot \tilde{\chi}_0 = \mathbf{D} \), and using Eq. (19a) we get

\[
\chi^{ab}(\omega = 0) = \tilde{\chi}_0 \delta_{ab} = (4k_B T I - J)^{-1} \delta_{ab}. \tag{25}
\]

This is the generalized Curie-Weiss susceptibility for a nonhomogeneous spin system (valid for \( T > T^{\text{mag}} \)).
Exact analytical results can be obtained for the special case of a translation-invariant spin system. If the system is close to being translation-invariant, e.g. only a few vacancies are present so that \( \sigma = \langle x_i \rangle_i \leq 1 \), and \( \Gamma_i \) does not depend appreciably on \( i \), a homogeneous approximation (HA) can be proposed. The HA replaces \( x_i, \Gamma_i, J_{ij} \), and \( D_{ij} \) by their spatial averages \( \bar{\sigma}, \bar{\Gamma}, \bar{J}_{ij} \) and \( \bar{D}_{ij} \) (the latter two depending only on \( \bar{R}_i - \bar{R}_j \)), making the problem analytically solvable. In this case Appendix A shows that the exact susceptibility can be obtained by spatial Fourier transformation,

\[
\chi^{ab}(q, \omega) = \frac{i}{\omega + i} \left\{ -\bar{D}(q) \left[ 4k_B T - \bar{J}(q) \right] + \bar{\Gamma} \right\}, \tag{26}
\]

where \( \bar{D}(q) = \sum_v D_{i,v} e^{-iq \cdot v} \) and \( \bar{J}(q) = \sum_m \bar{J}_{i,m} e^{-iq \cdot v} \). The paramagnon modes are labelled by \( q \in 1^{st} \) Brillouin zone, each with frequency eigenvalue

\[
\gamma_m = \gamma_q = -\bar{D}(q) \left[ 4k_B T - \bar{J}(q) \right] + \bar{\Gamma}, \tag{27}
\]

and associated right and left eigenvectors \( v_m = e_q, \quad v_n^{-1} = e_q^* / N_s \), respectively, where \( e_q = (e^{-iq \cdot R_0}, \ldots, e^{-iq \cdot R_{N_s-1}}) \), and \( N_s \) is the number of occupied sites forming a translation-invariant lattice.

When \( \omega \to 0 \), Eq. (26) leads to

\[
\tilde{\chi}^{ab}(q, 0) = \frac{1}{4k_B T - \bar{D}(q)} \delta_{ab}, \tag{28}
\]

where \( T_{CW}(q) = \bar{J}(q)/(4k_B T) \) is the Curie-Weiss temperature in Fourier space.

Without translation invariance, e.g. in the presence of spin clusters, Eq. (25) shows that the \( \omega = 0 \) susceptibility may have several different temperature poles \( T_{CW}(0) \), each associated with different clusters having different \( J_{ij} \) or number of neighbours. However, \( T_{CW} \) only depends on spin-spin interaction, it does not depend on relaxation parameters \( \Gamma_i \).

\section{V. Paramagnon Flux Density in the Presence of Spin Diffusion from an Infinite Plane of Spins}

At this point we consider the first model for \( D_{ij} \) and \( \rho_o(\gamma) \). Assume the spin system is an infinite square lattice with no vacancies (homogeneous). Moreover, assume the superconducting wire is also infinite with flux vector given by the “edge model” shown in Fig. 1:

\[
\mathbf{F}(r) = F_0 (\delta_{x,-W/2} - \delta_{x,W/2}) \mathbf{z}, \tag{34}
\]

where \( W \) is the wire width and \( \delta_{x,-W/2} \) are Kronecker delta functions. Equation (34) gives a good description of thin-film wires where it is shown that \( \mathbf{F} \) is sharply peaked at the wire edges [24].

The simplest model for \( D_{ij} \) is to emulate the third-principles theory. To do this, assume the phenomenological paramagnon relaxation rates are given by

\[
\gamma_q = \frac{D(T)q^2}{W^2} + \Gamma, \quad \text{with } D(T) \text{ the spin diffusion constant.}\]

In Eq. (27) shows that this \( \gamma_q \) is obtained by a choice of \( D_{ij} \) that has the following Fourier representation at low \( q \):

\[
\bar{D}(q) = -\frac{D(T)q^2}{4(k_B T - J)}. \tag{35}
\]

Plug these into Eq. (32) along with \( \tilde{\chi}(q, 0) \approx 1/[4(k_B T - J)] \) and \( |\mathbf{F}(q)|^2 = 4F_0^2 N_s^2 \sin^2 (q_s W/2) \delta_{q_s,0} \) to get
\[ \rho_\Phi(\gamma) = \frac{4F_0^2N_s^2}{N_s\gamma} \int_{-\pi/a_0}^{\pi/a_0} dq \sin^2 \left( \frac{q_z W}{2} \right) \frac{\Gamma + Dq^2}{4(k_BT - J)} \left( \gamma - Dq^2 - \Gamma \right) \]

where \( \theta(x) \) is the Heaviside step function, with \( \gamma_{\text{min}} = \Gamma \) and \( \gamma_{\text{max}} = D(\pi/a_0)^2 + \Gamma \) defining the region where \( \rho_\Phi(\gamma) \) is nonzero. The sine squared represents interference between the two edges of the wire; in most cases this averages out to 1/2 either because of small fluctuations in wire shape or frequency resolution during integration over \( \gamma \). Apart from this, \( \rho_\Phi(\gamma) \) follows a power law frequency

\[ \rho_\Phi(\gamma) = \frac{C(T)}{1 - e^{-\frac{\gamma}{\omega}}} C(T) \frac{b_\Phi(\omega)}{\omega^\alpha}, \]

where

\[ b_\Phi(\omega) = \int_{\gamma_{\text{min}}/\omega}^{\gamma_{\text{max}}/\omega} dx \frac{x^{1-\alpha}}{1 + x^2}. \]

Thus, when \( \gamma_{\text{min}} \ll \omega \ll \gamma_{\text{max}}, b_\Phi(\omega) \approx \pi/[2 \sin (\pi\alpha/2)] \) and the flux noise scales as a power law in frequency with the same exponent \( \alpha \), \( \bar{S}_\Phi(\omega) \propto 1/\omega^\alpha \).

We emphasize that the \( \alpha = 1/2 \) obtained in this section is a direct consequence of assuming long-wavelength diffusion in an infinite and homogeneous (spin density \( \sigma = 1 \)) lattice of spins. The next section sets up a method to investigate the impact of spatial confinement and disorder.

VI. SPECIFICATION OF DISSIPATION MATRIX \( D_{ij} \) IN THE PRESENCE OF CONFINEMENT AND DISORDER

To go beyond the third-principles assumption of long wavelength spin diffusion we need to come up with a specification for \( D_{ij} \) that respects several physical constraints. To do this, we take inspiration from random walk models in a lattice. The key idea is that exchange interaction \( J_{ij} \) is the main driver for each random walk step, a spin “flip-flop”. A sequence of many flip-flops will lead to diffusion. The constraint of total spin conservation motivates our postulation of the following spin dissipation matrix:

\[ D_{ij} = \frac{d_0(T)}{\hbar J_c} \left( x_i x_j |J_{ij}| - \delta_{ij} \sum_k x_i x_k |J_{ik}| \right). \]

Here \( d_0(T) \) is a function of temperature to be determined by fitting the theory to experiments (note \( d_0(T) \) is dimensionless). This is introduced to account for critical behaviour of the spin diffusion constant near \( T_c \) [27]. The quantity

\[ \bar{J}_c = \frac{1}{N_c} \sum_{j,k \in \text{cluster } c} x_j x_k |J_{jk}| \]

is the average exchange times coordination number for \( c^{th} \) cluster, the cluster that contains spin \( i \). Such a cluster is defined as the set of all spins \( j \) such that either \( J_{ij} \neq 0 \) or there exists a set of sites \( k_1, k_2, \ldots, k_n \) such that \( J_{ik_1}, J_{ik_2}, J_{ik_2}, \ldots, J_{ik_n} \neq 0 \). \( N_c \) is the number of spins in the isolated cluster \( c \). The following motivates this choice:

1. When the local external field \( h_i \) and spin relaxation rate \( \Gamma_i \) are both zero, Eq. (40) preserves total spin. For each isolated spin cluster, \( \sum_{i,j \in \text{cluster } c} D_{ij} H_j = 0 \), therefore summing Eq. (13) over all spins in a cluster leads to \( \frac{\partial}{\partial t} (\sum_{i \in \text{cluster } c} s_i) = 0 \), so the total spin in each cluster is a constant of the motion. Therefore, we do not need to assume hard boundary conditions [7, 12] to describe confined systems such as spin clusters and wire edges.

2. This choice for \( D_{ij} \) gives rise to diffusion in the long wavelength regime. E.g. for the homogeneous nearest-neighbor model in the square lattice with \( J_{ij} = J, J_{i,i+v} = J \) for \( v = \pm a\hat{x}, \pm a\hat{y} \), we get

\[ -\sum_{i,j} \delta(r - R_i) D_{ij} H_j = \frac{d_0(T) a^2 k_BT}{\hbar} \nabla^2 M(r), \]

where we assumed high temperature \( (k_BT \gg J \) and \((s_i)^2 \ll 1 \) and took the continuum limit by defining the magnetization density \( M(r) = -\sum_i s_i \delta(r - R_i) \). Equation (42) may be compared to experiments that show spin diffusion constant increasing with temperature [11].

3. Consider the time derivative of the free energy when \( \delta h_i(t) = 0 \) in each isolated spin cluster:
\[
\frac{\partial F}{\partial t} = \sum_i \frac{\partial F}{\partial s_i} \cdot \frac{\partial s_i}{\partial t} = \frac{d_0(T)}{\hbar J_c} \sum_i H_i \cdot \sum_j x_i x_j |J_{ij}|(H_j - H_i) + \sum_i \Gamma_i H_i \cdot (s_i - s_i^{eq}) \\
= -\frac{d_0(T)}{\hbar J_c} \sum_{i<j} x_i x_j |J_{ij}|(H_j - H_i)^2 - \sum_i \Gamma_i (F(s_i) - F(s_i^{eq}) + \mathcal{O}[(\delta s_i)^2]).
\]

The first term on the RHS is always negative, showing that our choice for \(D_{ij}\) always tends to decrease the free energy as time increases (i.e. it obeys the 2nd law of thermodynamics). This justifies our use of modulus of \(J_{ij}\) in Eq. (40).

The second term on the RHS of Eq. (43) is negative provided that the deviation out of equilibrium is small and \(F(s_i^{eq})\) is a local minimum of the free energy. Therefore, the coupling to other nonspin degrees of freedom \(\Gamma_i\) pushes the system towards a local minimum of the free energy, without subjecting the system to the constraint of spin conservation.

The fact that \(D_{ij}\) conserves spin may be expressed more generally as \(\mathbf{1}_T \cdot \mathbf{D} = 0_T\), where \(\mathbf{1}\) is the column vector with all \(N\) components equal to 1. When in addition \(\mathbf{F} = 0\), the spin conservation law holds and this has additional implications.

When \(\mathbf{F} = 0\) all contributions to the paramagnon matrix (19a) have \(D_{ij} = 0\) on the left, so follows that the \(m = 0\) mode defined by \(\tilde{\delta}_0^{-1} = \mathbf{1}_T \cdot \delta s_i^{eq} = 0\) is a left eigenvector of \(\mathbf{P}\) associated to \(\gamma_0 = 0\). This is true for general \(J_{ij}\).

Because \(v_0^{-1} \cdot \mathbf{D} = 0_T\), \(m = 0\) does not contribute to the sum in the dynamical susceptibility Eq. (23b). This occurs as a direct consequence of the conservation law \(\frac{\partial}{\partial t} (\mathbf{1}_T \cdot \delta s_i^{eq}) = 0\), so we say \(m = 0\) is the nondissipative Goldstone paramagnon.

### VII. NUMERICAL EVALUATION OF THE PARAMAGNON FLUX DENSITY: HEISENBERG MODEL IN THE 2D SQUARE LATTICE WITH A RANDOM DISTRIBUTION OF VACANCIES

For numerical calculations with our proposed \(D_{ij}\) in Eq. (40), we consider a \(20 \times 20\) (\(N = 400\)) “virtual” square lattice. We randomly populate \(N_s \leq N\) sites with spins, yielding spin density \(\sigma = N_s/N\). The remaining unoccupied virtual sites are called vacancies, see Fig. 1. All calculations below are done with open boundary condition (b.c.) along \(x\), and periodic b.c. along \(y\), describing spins confined within the region of the SC wire.

All calculations assume the nearest-neighbour (n.n.) Heisenberg model in a square lattice with \(J_{ij} = J \sum_{\nu} \delta_{i,j+\nu} + \nu\) for \(\nu = \pm a_0 \hat{x}, \pm a_0 \hat{y}\). Divide matrix (19a) by \(d_0(T)/J\). Then the eigenvalues \(\gamma_m\) are expressed in units of \(k_B T/J\). That way the problem is now specified by \(k_B T/J\), \(\hbar \Gamma,/(d_0(T)/J)\), and \(\hbar \omega/(d_0(T)/J)\). As a result, when \(\Gamma_i = 0\), the calculated exponent \(\alpha\) is independent of the choice of \(d_0(T)\) (although \(\gamma_{\min}, \gamma_{\max}\) do depend on \(d_0(T)\)).

Figure 2 shows explicit calculations of \(\rho_{\Phi}(\gamma)\) using Eq. (31) with numerical calculations of the eigenvalues \(\gamma_m\) of \(\mathbf{P}\), with the \(\delta(x)\) function approximated by a Gaussian with standard deviation \(0.1 \gamma_{\max}\). The flux vector was given by the “edge model”, see Eq. (34) and Fig. 1. Results for \(k_B T/J = 12\) and \(\Gamma_i = 0\) are shown for both \(J > 0\) (FM) and \(J < 0\) (AFM), using spin densities \(\sigma = 1, 0.75, 0.5\). The latter two are averaged over 512 random instances, each containing a different distribution of vacancies in the virtual lattice with the same \(N_s = \sigma N\). These results demonstrate that \(\rho_{\Phi}(\gamma)\) remains a power law in \(\gamma\), even in the presence of confinement (open b.c.) and spatial disorder (\(\sigma < 1\)). However, the corresponding exponent \(\alpha > 1/2\), contrasting to the case without confinement/disorder (Section V).

Figure 3 shows explicit calculations of the frequency exponent \(\alpha\) appearing in both \(\rho_{\Phi}(\gamma)\) and \(\tilde{S}_{\Phi}(\omega)\) as a function of \(T\) and \(\sigma\), with all other parameters like in Fig. 2. It is seen that \(\alpha\) decreases with \(T\) for the FM model, and has the opposite behavior for the AFM model. At high \(T\), both FM/AFM models lead to \(\alpha \approx 0.7\) for \(\sigma = 1\); this demonstrates the important of confinement. As \(\sigma\) decreases from 1, \(\alpha\) further deviates from its infinite/homogeneous value of 1/2. This demonstrates the impact of disorder. Note how the dependence of \(\alpha\) on \(\sigma\) is nonmonotonic.

Interestingly, several experiments with niobium devices measure \(\alpha = 0.7\) [3, 28].

### VIII. DISORDER DUE TO WIDE DISTRIBUTION OF RELAXATION RATES \(\Gamma_i\) AND SPIN-SPIN INTERACTIONS

Another model for disorder is to assume an arbitrary distribution of relaxation rates \(\Gamma_i\). When \(J_{ij} = D_{ij} = 0\), \(\rho_{\Phi}(\gamma)\) and \(S_{\Phi}(\omega)\) can be computed exactly. The eigenvalues of \(\mathbf{P}\) are simply \(\gamma_i = \Gamma_i\), and the eigenvectors are the unit column vectors \(e_i\). Using Eq. (31) the paramagnon
FIG. 2. Explicit calculations of paramagnon flux density $\rho_\Phi(\gamma)$ for a confined system of spins ($20 \times 20$ virtual lattice with open b.c. along $x$), using the edge model for the flux vector (Eq. (34) and Fig. 1). Results are shown for $k_BT/|J| = 12$ and $\Gamma_i = 0$ for (a) $J > 0$ (FM) and (b) $J < 0$ (AFM), for spin densities $\sigma = 1, 0.75, 0.5$. The cases with $\sigma < 1$ were averaged over 512 random instances, each containing a different distribution of vacancies in the virtual lattice with the same $N_s = \sigma N$. These plots demonstrate that $\rho_\Phi(\gamma)$ remains a power law in $\gamma$, even in the presence of confinement (open b.c.) and disorder ($\sigma < 1$). However, the exponent $\alpha > 1/2$, in contrast to $\alpha = 1/2$ in the absence of confinement and disorder.

FIG. 3. Explicit calculations of flux noise frequency exponent $\alpha$ as a function of temperature for $\Gamma_i = 0$ in the n.n. Heisenberg model with confinement and disorder, with all other parameters as in Fig. 2. (a) FM case with $J > 0$, (b) AFM case with $J < 0$. Note how $\alpha$ decreases (increases) with $T$ for the FM (AFM) cases. In all cases $\alpha > 1/2$ demonstrating the relevance of confinement and disorder. The dependence of $\alpha$ on $\sigma$ is nonmonotonic.

The flux density becomes

$$
\rho_\Phi(\gamma) = \frac{1}{\gamma} \sum_i |F_i|^2 \Gamma_i \chi_0 \delta(\gamma - \Gamma_i)
$$

$$
= \frac{1}{4k_BT} \sum_i |F_i|^2 \int d\Gamma \rho(\Gamma) \frac{\Gamma \delta(\gamma - \Gamma)}{\gamma}
$$

$$
= \frac{1}{4k_BT} \sum_i |F_i|^2 \rho(\gamma),
$$

where in the second line we plugged Eq. (25) and took an average using $\rho(\Gamma)$, the probability density for rates $\Gamma_i$. The justification for wide distributions of spin-flip rates $\Gamma_i$ for spin impurities is given in [4, 19]. Each spin interacts with one or more amorphous two-level systems (TLSs), leading to the cross-relaxation rate (joint spin
for numerical calculations using the second-principles theory, the spin $\frac{1}{f}$ Eq. (30) to get the flux noise action $J$ in semiconducting devices. The spin $\frac{1}{f}$ is the spin equivalent of the well-known $\frac{1}{f}$ we will refer to this model as the spin $\frac{1}{f}$ model by choosing $\Gamma_i = \Gamma_{\text{max}} \exp (-\lambda_{\text{max}} r_i)$ with $\lambda_{\text{max}} = 20$ and $r_i \in [0, 1]$ a random number generated for each of the 400 sites.

The calculations should be compared to the case of $J_{ij} = 0$, that has $\alpha = 1$ (Eq. (47)). As shown in Fig. 4, the addition of a wide distribution of $\Gamma_i$ to the interacting spin system increases $\alpha$ in the low to intermediate $T$ range. At high $T$, the impact of $\Gamma_i$ is washed out.

IX. SIMPLE EXPRESSIONS FOR FLUX NOISE FOR COMPARISON TO EXPERIMENTS

The previous sections showed that the paramagnon flux density scales as a power law in $\gamma$, $\rho_\Phi(\gamma) = C/\gamma^\alpha$, for $\gamma_{\text{min}} \ll \gamma \ll \gamma_{\text{max}}$, and $\rho_\Phi(\gamma) \approx 0$ outside this range. The exponent $\alpha$ and amplitude $C$ depend on temperature, confinement, and disorder.

As discussed above Eq. (26), when $\sigma \leq 1$ and $\Gamma_i$ is nearly uniform we can use the homogeneous approximation (HA) described in Appendix A to obtain simpler analytical expressions. The HA replaces $x_i$, $\Gamma_i$, $J_{ij}$, and $D_{ij}$ by their spatial averages, $\langle \sigma \rangle$, $\langle \Gamma_i \rangle$, $\langle J_{ij} \rangle$, and $\langle D_{ij} \rangle$, making the problem analytically solvable. In particular, $\rho_\Phi(\gamma)$ is approximated by the translation-invariant Eq. (32). Con-
sider its integral over all $\gamma$:

$$\int d\gamma p_\Phi(\gamma) = \frac{1}{N_s} \sum_q |\vec{F}(q)|^2 \frac{1}{\gamma_q} \left[ \Gamma \chi(q,0) - \tilde{D}(q) \right]$$

$$= \frac{1}{N_s} \sum_q \frac{|\vec{F}(q)|^2}{4k_BT - J(q)}$$

$$= \frac{\sigma \sum_i |F_i|^2}{4(k_BT - \sigma J)}, \quad \text{(48)}$$

where in the second line we used Eqs. (27) and (28), and in the third line we assumed the typical scale for variations in $\vec{F}(q)$ is much larger than the lattice spacing $a_0$, so that $J(q)$ can be approximated by $J(0) = 4\sigma J$. Now equate this to $\int d\gamma C/\gamma^\alpha \approx C_{\gamma_{\text{max}}}^{-\alpha}/(1 - \alpha)$ (valid for $\alpha < 1$) in order to obtain an expression for amplitude $C$; plug this into Eq. (38) to get

$$\tilde{S}_\Phi(\omega, T) \approx \frac{\hbar \omega}{1 - e^{-\omega/T}} \frac{\alpha(1 - \alpha) \sum_i |F_i|^2 h_\alpha(\omega)}{2 \gamma_{\text{max}}(k_BT - \sigma J)} \left| \frac{\omega}{\sigma J} \right|^\alpha. \quad \text{(49)}$$

When $\alpha \lesssim 1$, the temperature dependence of $\gamma_{\text{max}} \propto d_0(T)T$ is washed out, leading to a simple relation valid for $\hbar \omega \ll k_BT$:

$$\frac{\tilde{S}_\Phi(\omega, T)}{\tilde{S}_\Phi(\omega, T) \approx \sigma J/k_BT \approx \sigma J/k_BT - \sigma J} \approx \frac{k_BT}{k_BT - \sigma J}. \quad \text{(50)}$$

Note how within HA and for $\alpha \lesssim 1$ this noise amplitude ratio is independent of $d_0(T)$ and other details such as values of $\Gamma_i$.

Figure 5 shows this dependence for the FM ($\tilde{J} > 0$) and AFM ($\tilde{J} < 0$) cases, and compares Eq. (50) to numerical evaluation with $T = 10J/k_BT$ and $\hbar \omega/|\tilde{J}| = 0.1$ chosen as the high temperature and frequency, respectively, and $d_0(T) = 1$, $\Gamma_i = 0$. It shows that the amplitude of $\tilde{S}_\Phi(\omega, T)$ for a given $\omega$ decreases with increasing $T$ for the FM model. For the AFM model it instead increases with $T$. Extrapolating to lower temperatures (including $T < 0$ for AFM) allows determination of the value of $\sigma J$ that models the spins.

The singularity in $\tilde{S}_\Phi(\omega)$ for the FM model as $T$ gets close to $T_{\text{mag}}^{\text{max}} = \sigma J/k_BT$ is due to the formation of clusters of spins with nonzero magnetization (short range order) [14, 24]. As seen in Fig. 5, numerical results deviate from Eq. (50) demonstrating the effect of confinement, disorder and the value of $\alpha$ ($\alpha < 1$ for most $T$, see Fig. 3) on the $T$ dependence of spin clusters.

Similar considerations apply to the antisymmetric flux noise

$$\tilde{S}_\Phi^-(\omega) = \tilde{S}_\Phi(\omega) - \tilde{S}_\Phi(-\omega)$$

$$= \frac{\sigma(1 - \alpha) \sum_i |F_i|^2 \hbar^\alpha h_\alpha(\omega)}{2 \gamma_{\text{max}}(k_BT - \sigma J)} \left| \frac{\omega}{\sigma J} \right|^{1-\alpha}. \quad \text{(51)}$$

This is seen to have weak frequency dependence when $\alpha \lesssim 1$. Plotting $1/\tilde{S}_\Phi^-(\omega)$ as a function of $T$ yields a straight line that extrapolates to zero at $T = \sigma J < 0$.

In Quintana et al. [10] (inset of Fig. 3) this procedure reveals $\sigma J = -10$ mK.

---

**FIG. 5.** Temperature dependence of flux noise amplitude. Due to the formation of magnetized spin clusters, the noise amplitude for the FM model has a singularity at $T = T_{\text{mag}}^{\text{max}} = \sigma J/k_BT$. As a result, the FM $\tilde{S}_\Phi(\omega)$ decreases with increasing $T$. For the AFM model ($\tilde{J} < 0$), $\tilde{S}_\Phi(\omega)$ is instead slightly increasing with $T$. The solid curves are based on Eq. (50), with dashed ones representing simple extrapolation to determine $\sigma J$ for the AFM model. Data points are numerical evaluation with the second principles theory, using $T = 10J/k_BT$ and $\hbar \omega/|\tilde{J}| = 0.1$ as the high $T$ and low $\omega$, respectively, and $d_0(T) = 1$, $\Gamma_i = 0$.

---

**X. DISCUSSION AND CONCLUSIONS**

Several experiments in SQUID-based devices [3, 10, 11, 14, 17, 28] have concluded that flux noise follows the empirical law

$$\tilde{S}_\Phi(\omega) = \frac{A}{\omega^\alpha}, \quad \text{(52)}$$

where amplitude $A$ and exponent $\alpha < 1$ are both temperature-dependent. Here we developed a theory of flux noise due to interacting spins that is able to calculate $A$ and $\alpha$ for realistic model impurity spin systems with disorder due to vacancies and wide distributions of spin relaxation rates, and for spins confined in bounded regions such as SC wires.

To achieve this we needed to develop a method that does not rely on the “third-principles” diffusion operator $D \nabla^2$. Our “second-principles” method instead assumes lattice sites are coupled by a dissipation matrix $D_{ij}$, that accounts for the spin conservation law exactly. We described results in terms of a general $D_{ij}$, and also proposed a particular form for $D_{ij}$ based on fundamental principles such as the second law of thermodynamics. This allowed us to write $D_{ij}$ in terms of the exchange interaction between spins, connecting our theory to model spin Hamiltonians.

A central concept is the interpretation of flux noise in terms of (para)magnon excitations. Flux noise is shown to be directly related to the density of edge
(para)magnons, the ones that lead to fluctuations at the superconducting wire edges, where spin flips cause the largest flux changes to the device. While parameter \( D_{ij} \) accounts for the interactions between (para)magnons, the rates \( \Gamma_i \) describe interactions between spins and other degrees of freedom such as phonons, electron gas excitations, and two-level system defects. Both \( D_{ij} \) and \( \Gamma_i \) seem to be required to explain experiments.

We can separate experiments in two groups, the ones using niobium and the ones using aluminum devices. Measurements in niobium devices show both amplitude \( A \) and exponent \( \alpha \) decreasing with increasing \( T \) (Fig. 3 of [14]). According to our Eq. (50) and Fig. 5 this \( A(T) \) requires a ferromagnetic model \( (J > 0) \). Our calculated \( \alpha(T) \) for \( \Gamma_i = 0 \) ranges from 0.9 at low \( T \) to 0.7 at high \( T \) for a confined system (spins only on top of the wire) with a small number of vacancies \( (\sigma \lesssim 1) \), but the exponent can approach 0.5 when the spin system extends beyond the wire (Eq. (36)). In comparison, experimental measurement shows \( \alpha(T) \) going from 0.8 to 0.4 with increasing \( T \) [14]. Therefore, a n.n. Heisenberg model with \( J > 0 \) and \( \sigma \lesssim 1 \) and low \( \Gamma_i \ll J/h \) provides a reasonable model.

Much less data exists for aluminum devices, but one experiment definitely shows that their \( A(T) \) and \( \alpha(T) \) are qualitatively different from niobium. In [10], \( A(T) \) is shown to increase with \( T \) and based on our Fig. 5 an antiferromagnetic model \( (J < 0) \) is required. The authors reached the same conclusion by measuring asymmetric noise \( S_6(\omega) \) and extrapolating to \( T < 0 \) to get \( T_{CW} = -10 \) mK. According to our interpretation this implies \( \sigma J/k_B = -10 \) mK. The measured exponent \( \alpha = 0.96 \pm 0.05 \) can not be explained by our theory with \( \Gamma_i = 0 \) (Fig. 3b). However, Fig. 4b shows that introducing a wide distribution of \( \Gamma_i \)'s (the spin 1/f model) makes \( \alpha \approx 1 \) at low temperatures. This indicates the necessity of a model with coexisting parameters \( \Gamma_i \sim J/h \) in aluminum devices. Measurements of \( \alpha(T) \) over a wide temperature range are not yet available to confirm this scenario.

In addition to low frequency flux noise, experiments also measure Ohmic \( (\propto \omega) \) [16] or super-Ohmic \( (\propto \omega^3) \) [10, 17] flux noise in the GHz range. We now argue that this behaviour can not arise from interacting spins alone.

Our theory is fundamentally based on the assumption of “hydrodynamics”, i.e. that spin degrees of freedom can be described by the classical equation of motion (13) [29]. As a result, it overestimates the noise for \( \omega > \Omega_c \), where \( \Omega_c \) is a high-frequency cut-off. The cut-off \( \Omega_c \) can be estimated from exact calculations of the moments of the noise spectrum at \( T \rightarrow \infty \). Calculations for the 3d Heisenberg model [21] suggests \( h\Omega_c \sim 10J/h \) for our 2d case. When \( \omega > \Omega_c \), \( S_\omega(\omega) \) drops off faster than \( 1/(\omega) \), so that our Eq. (29) becomes an upper bound for flux noise. In Eq. (33) this upper bound was shown to be \( \propto 1/\omega \) when quantum noise is included. As the \( 1/\omega \) upper bound holds for all interacting spin models, this allows us to conclude that the high frequency Ohmic [16] or super-Ohmic [10, 17] flux noise observed in SQUIDs can not originate from a model of interacting impurity spins. A likely source is the normal resistance due to excited quasiparticles, either by thermal or nonequilibrium sources such as cosmic rays [30].

In conclusion, we developed a “second principles” theory of flux noise due to interacting spins that is able to account for the confinement and disorder present in realistic impurity spins systems on superconducting devices. The theoretical framework allows explicit prediction of the amplitude and exponent of flux noise due to different wire geometries and spin disorder scenarios, such as random vacancies and wide distributions of spin-flip rates due to interactions with amorphous TLSs. Comparing numerical results to experiments allowed us to specify different spin Hamiltonians for niobium and aluminum devices. Generalizations of the theory that include time dependent external currents and fields can be used to design of optimal control strategies that reduce the impact of flux noise on quantum devices.

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**Appendix A: Exact solution of the homogeneous case and homogeneous approximation**

When the spin system is translation-invariant, the “second principles” method yields exact analytic expressions for the spin noise for general \( D_{ij} \) using spatial Fourier transforms. This is the case when the system has periodic b.c., and the vacancies are organized in a regular sublattice of the full virtual lattice. The homogeneous case (no vacancies, \( \sigma = 1 \) with periodic b.c. is the most relevant example.

When the system is not translation invariant, we may take spatial averages over parameters \( x_i, D_{ij}, J_{ij}, \) and \( \Gamma_i \) in order to force its equation of motion to become exactly solvable. In this case the resulting analytic solution is called homogeneous approximation (HA).

In the paramagnetic phase \( (T > T_{\rm{CW}}) \), the EOM Eq. (17) becomes

\[
\frac{\partial}{\partial t}(\delta s^a) = -\sum_j D_{ij} x_j \left[ (\delta h_j^a) + \sum_k x_k J_{jk}(\delta s_k^a) \right] \\
-4k_B T(\delta s_j^a) - \Gamma_i(\delta s_i^a) + \Gamma_i \sum_j \chi_{ij}(0)(\delta h_j^a),
\]

for \( a = x, y, z \). If this is not translation invariant, replace
\[ x_i, D_{ij}, \Gamma_i \text{ by their average values:} \]
\[ \bar{x} = \frac{1}{N} \sum_i x_i = \sigma, \quad (A2a) \]
\[ \tilde{D}_v = \frac{1}{N\sigma^2} \sum_i x_i x_{i+v} D_{i,i+v}, \quad (A2b) \]
\[ \tilde{J}_v = \frac{1}{N\sigma^2} \sum_i x_i x_{i+v} J_{i,i+v}, \quad (A2c) \]
\[ \bar{\Gamma} = \frac{1}{N \sigma^2} \sum_i \bar{\Gamma}_i, \quad (A2d) \]
\[ \bar{\chi}_0(0) = \frac{1}{N\sigma^2} \sum_i \bar{\chi}_{i,i+v}(0). \quad (A2e) \]

Equation (17) becomes
\[ \frac{\partial}{\partial t} (\delta \chi_{\sigma}^a) = -\sum_v \sigma \tilde{D}_v \left[ (\delta h_{i,v}^a) + \sum_v \sigma \tilde{J}_v (\delta \chi_{i+v,v}^a) \right], \quad (A3) \]
\[ -4k_B T (\delta \chi_{i,v}^a) = \frac{1}{\gamma} \sum_v \bar{\chi}_v(0)(\delta \chi_{i+v,v}^a). \quad (A4) \]

Take the Fourier transform in both sides
\[ \frac{\partial}{\partial t} (\delta \chi_{\sigma}^a(\omega)) = \int dt \left[ e^{-i\omega t} \frac{\partial}{\partial t} (\delta \chi_{\sigma}^a) \right], \quad (A5) \]

where
\[ \tilde{D}(q) = \sigma \sum_v \tilde{D}_v e^{-iq \cdot v}, \quad (A6a) \]
\[ \tilde{J}(q) = \sigma \sum_v \tilde{J}_v e^{-iq \cdot v}. \quad (A6b) \]

This result implies the zero-frequency susceptibility,
\[ \tilde{\chi}_{ab}(q, \omega = 0) = \frac{1}{4k_B T - T_{\text{CW}}(q)}, \quad (A7) \]

with Curie-Weiss temperature \( T_{\text{CW}}(q) = \frac{1}{4k_B} \tilde{J}(q) \).

Using the fluctuation-dissipation theorem the spin noise in Fourier space becomes exactly equal to
\[ \tilde{S}_{ab}(q, \omega) = \frac{2\hbar \omega}{1 - e^{-\hbar \omega/k_B T}} \]
\[ \times \left[ \frac{\tilde{\Gamma}_0(0) - \tilde{D}(q)}{\omega^2 + \tilde{J}(q) [4k_B T - \tilde{J}(q)] + \bar{\Gamma}} \right], \quad (A8) \]

In the notation of Eq. (23b), the paramagnon modes are labelled by \( m = q \in 1^{st} \) Brillouin zone, each with frequency eigenvalue \( \gamma_m = \gamma_q = -\tilde{D}(q) \left[ 4k_B T - \tilde{J}(q) \right] + \bar{\Gamma}. \quad (A9) \)

Note that we assumed the presence of \( N_s \) \( R_i \)'s forming a translation-invariant lattice, so there are \( N_s \) \( q \)'s in the 1st Brillouin zone. As a consequence Eq. (A8) does not have \( \sigma \) appearing explicitly in the numerator.

Similar to what was done in Section IV, we can define a mode density to interpret Eq. (A8),
\[ \tilde{S}_{ab}(q, \omega) = \frac{2\pi \hbar \omega}{1 - e^{-\frac{\hbar \omega}{k_B T}}} \int d\gamma \frac{\gamma/\pi}{\omega^2 + \gamma^2} \rho^{ab}(\gamma, q). \quad (A10) \]

The \( \rho^{ab}(\gamma, q) \) is called paramagnon wavevector density (in contrast to the paramagnon flux density defined by Eq. (31)). It’s given by
\[ \rho^{ab}(\gamma, q) = \frac{1}{\gamma} \left[ \tilde{\Gamma}_0(0) - \tilde{D}(q) \right] \delta(\gamma - \gamma_q) \delta_{ab}. \quad (A11) \]

For the n.n. Heisenberg model in the 2d square lattice we get
\[ \tilde{D}(q) = -\frac{d_0(T)}{h} \left[ \sin^2 \left( \frac{q_x a_0}{2} \right) + \sin^2 \left( \frac{q_y a_0}{2} \right) \right], \quad (A12a) \]
\[ \tilde{J}(q) = 2\sigma J \left[ \cos (q_x a_0) + \cos (q_y a_0) \right]. \quad (A12b) \]

We remark that \( \tilde{D}(q) \) does not depend on \( \sigma \) because from Eq. (41) \( J_c = 4\sigma |J| \), so that \( \sigma \) cancels out in the definition of \( \tilde{D}(q) \). These results lead to the effective diffusion constant in the HA,
\[ D_{\text{hom}}(q) \equiv -\tilde{D}(q) \left[ 4k_B T - \tilde{J}(q) \right] \]
\[ = \frac{d_0(T)a_0^2}{h} \left[ k_B T - \sigma J \left( \frac{1 - q^2 a_0^2}{4} \right) \right] + O(q^4). \quad (A13) \]

Finally, in the HA approximation the flux noise with edge flux vector Eq. (34) is given by
\[ \tilde{S}_b(\omega) = 4F^2 N_{sx} N_{sy} \sum_q \sin^2 \left( \frac{q W}{2} \right) \tilde{S}_{zz}(q \hat{x}, \omega), \quad (A14) \]

where \( q = \frac{2\pi}{N_{sx} a_0} (n - \frac{N_{sx}}{2}) \) with \( n = 0, 1, \ldots, N_{sx} - 1 \), where \( N_s = N_{sx} N_{sy} \) is the number of occupied sites.

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