Enumeration of $n$-connected ominoes inscribed in an abacus using ECO

Eman F Mohommed
Al mustansiriyah University\ College of Education- Iraq
Emial emanfatel@uomustansiriyah.edu.iq

Abstract. ECO is enumeration of combinatorial objects based on local expansion called succession rule which can be translate in to generating function. This research aims to use the design structure of nested chain abacus $\mathfrak{R}$ proposed by Mohommed in 2019, 2020 to construct a succession rule. Furthermore, based on this rule, generating function will be development.

Keywords: James abacus diagram, Enumeration, Succession Rule, generating function.

1. Introduction
A finite number of unite squares are said to be connected if they are adjacent edge to edge. This connected shape has been made used as a part of mainstream riddles since the year 1907 but the first researcher to study it systematically is Solomon Golomb in 1954. He proposed a connection of $n$ squares adjacent edge to edge with a connected internal as polyominoes as well as a unit square as ominoes. In advancing in this field, Klarners Konstant (1966) defines a connectivity of finite number of unit square devoid of cat point as $n$-ominoes\(^4\). The number $n$-connected shape that can be achieved from $n$ unit squares is included in the list of unsolved problem which is known as enumeration problem\(^5\). This $n$-connected ominoes is one of the most popular areas in the field of combinatorial mathematics having a very long history right from 19\(^{th}\) century which has also drawn the attention of researchers from other fields of chemistry and physics. Particularly, the former establishes a relationship with $n$-connected ominoes by given a definition to equivalent objects namely animals which is acquired by taking the cantor of the cells of $n$-connected ominoes\(^6\). These models are permitted to simplify the description of phenomena similar to phase transitions\(^7\) or percolation\(^8\) and Ising model\(^9\). The problem of characterizing and enumerating this model has been examined by these phenomena. This enumeration problem is associated with an equivalent of $n$-connected ominoes object known as cell growth which begins from one omino and continue to grow gradually by the addition of omino at every step to its periphery\(^5\). Usually, closed-form expressions for the size of most classes of $n$-connected ominoes are unknown. In dealing with this type of problem, the easiest thing to do is to solve a similar or simpler problem which will give us the understanding or idea that can be used to solve the original problem.

In the study of $n$-connected ominoes, the enumeration of simpler subsets of $n$-connected ominoes has been the most effective line of research. The set of $n$-
connected ominoes can be reduced until it is solvable by enforcing additional restrictions, such as directedness or convexity as mentioned by Vowlomb\textsuperscript{10,11,12}. However, all these researchers do not present the graphical representation called abacus in representing the \(n\)-connected ominoes and its family. Subsequently, the problem of characterizing and enumerating sets described in terms of some constraints is not considered by the abacus properties. The abacus called nested chain abacus \(\mathcal{R}\) used to represented the \(n\)-connected ominoes\textsuperscript{3}.

2. Generating Function with Respect to chains

This section employs the design structure of nested chain abacus to construct a succession rule. Based on Theorem 2, Theorem 3, Theorem 4 and Corollary 1 in\textsuperscript{1}, two sequences \(P^\text{Rec-P}_p\) and \(P^\text{Rec}_p\) for rectangle path nested chain abacus and rectangular nested chain abacus respectively can be obtained from the number of positions in each chain. In Theorem 2.1 we created the first sequence with rectangle-path nested chain abacus.

**Theorem 1.1** Let \(\mathcal{R}\) be the rectangle-path nested chain abacus with \(e\) column, \(r\) rows and \(c\) chains.

1. If \(e < r\) then,

\[
P^\text{Rec-P}_p = \begin{cases} 
  r - e + 1 & \text{if } \rho = 1, \\
  2P^\text{Rec-P}_1 + 6 & \text{if } \rho = 2, \\
  p^\text{Rec-P}_p + 8 & \text{if } \rho \geqslant 3.
\end{cases}
\]

2. If \(r < e\) then,

\[
P^\text{Rec-P}_p = \begin{cases} 
  e - r + 1 & \text{if } \rho = 1, \\
  2P^\text{Rec-P}_1 + 6 & \text{if } \rho = 2, \\
  p^\text{Rec-P}_p + 8 & \text{if } \rho \geqslant 3.
\end{cases}
\]

where \(p^\text{Rec-P}_p\) be the number of positions in chain \(i\) and \(\rho = c - i + 1\) for \(1 \leqslant i \leqslant c\).

**Proof.**

1. Based on\textsuperscript{1}, the number of position in chain \(c\) is

\[
re - c(2r + 2e) + \sum_{i=1}^{c} 4(2i - 1).
\]

Since \(c = \frac{e+1}{2}\), then,

\[
k = re - \frac{e - 1}{2}(2r + 2e) + 8 \sum_{i=1}^{c-1} i - \sum_{i=1}^{c-1} 4 + \sum_{i=1}^{c-1} i + 4 \frac{e - 1}{2}
\]

2.
Since $\sum_{i=1}^{c-1} i = \frac{(c-1)c}{2}$, then, $k = e - r + 1$.

Thus,

$$P_{1}^{Rec-P} = r - e + 1.$$ 

Since chain $c - 1$ is a rectangular chain the number of positions in chain $c - 1$ is

$$2r - 2e + 8 = 2(r - e + 1) + 6 = 2P_{1}^{Rec-P} + 6.$$ 

The difference between two rectangular chains in $\mathcal{R}$ is 8 then

$$P_{c-2}^{Rec-P} - P_{c-3}^{Rec-P} = P_{c-2}^{Rec-P} - P_{c-3}^{Rec-P} = \ldots = 8.$$ 

$$P_{3}^{Rec-P} - P_{2}^{Rec-P} = P_{4}^{Rec-P} - P_{3}^{Rec-P} = \ldots = 8.$$ 

Hence

$$4P_{\rho}^{Rec-P} = P_{\rho-1}^{Rec-P} + 8A$$ 

2. Follow directly by proof 1 of Theorem 2.1.

Next we developed the second sequence with rectangular nested chain.

**Theorem 5.25** Let $\mathcal{R}$ be the rectangular nested chain abacus with $e$ column, $r$ rows and $c$ chains.

$$P_{\rho}^{Rec} = \begin{cases} 
2r - 2e + 1 & \text{if } \rho = 1, \\
P_{\rho-1}^{Rec} + 8 & \text{if } \rho > 2.
\end{cases}$$

$$P_{\rho}^{Rec} = \begin{cases} 
2e - 2r + 1 & \text{if } \rho = 1, \\
P_{\rho-1}^{Rec} + 8 & \text{if } \rho > 2.
\end{cases}$$

where $P_{\rho}^{Rec}$ be the number of positions in chain $i$ and $\rho = c - i + 1$ for $1 \leq i \leq c$.

**Proof.**

1. Since $\mathcal{R}$ is a rectangular nested chain abacus then, chain $c$ derived by two consecutive columns. The number of position in chain $i$ is

$$2e + 2r - 4(2i - 1),$$

and $c = \frac{e}{2}$, respectively. Thus,

$$2e + 2r - 4(2 \frac{e}{2} - 1)$$

and

$$P_{1}^{Rec} = 2r - 2e + 1.$$ 

Then, there is arithmetical sequence for the number of positions in the chains with common difference of succession equal to (-8). So,

$$P_{\rho} = P_{\rho-1}^{Rec} + 8$$

where $\rho > 2$.

2. Follow directly by proof 1.

3. Succession Rule
A succession rule, $\Omega$, is a system $((a); \wp)$, consisting of an axiom $a$ and a set, $\wp$, of productions or rewriting of rules defined on a set of labels $M \subseteq \mathbb{N}^+$. 

$$\Omega = \{ a \rightarrow (c_1(k))(c_2(k))(c_3(k)) \cdots (c_t(k)); \cdots \cdots \cdots (1)$$

where $a \in M$ is a constant and the $c_i$ are functions $M \rightarrow M$.

One of the main properties of a succession rule is the consistency principle, i.e. each label $(\wp)$ must produce exactly $\wp$ elements. A succession rule induces, and is suitably represented by, a generating tree whose root is labelled by the axiom $(a)$, and a node labelled $(k)$ produces at the next level $k$ sons labelled by $(c_1(k), \ldots, c_k(k))$ respectively (which in turn will produce $(c_1(k), \ldots, c_k(k))$ sons, etc.). The succession rule produces a sequence, $\{f_n\}_n$, of positive integers, where $f_n$ is the number of nodes at level $n$ of the generating tree and its generating function is denoted by $f_\Omega = \sum_{n\geq0} f_n x^n$.

We will construct our succession rule starting from a single chain, which will grow step by step by adding $C_i$ beads, where $C_i$ is the number of positions in chain $i$ as shown in Figure 1 which illustrates the number of nested chain abacus in levels 1 and 2.

![Figure 1: First levels of the generating tree of $\Omega$ if the first chain consists of one position.](image)

Based on the Figure 1, we constructed the generating tree where $L_n$ corresponds to the number of nested chain abacus at level $n$ for $n \geq 1$.

**Lemma 3.1** The number of nested chain abacus in the generating tree is done by adding one chain at level $N$ where $1 \leq N \leq n$ is

$$ll_0 = 1$$
$$ll_1 = p_2$$
$$ll_2 = p_2(p_2 + 8)$$
$$ll_3 = p_2(p_2 + 8)(p_2 + 16)$$
$$ll_4 = p_2(p_2 + 8)(p_2 + 16)(p_2 + 24)$$
$$\vdots$$
$$ll_n = \prod_{k=0}^{n-1} (p_2 + 8k)$$
where \( L_n \) is the number of nested chain abacus in level \( n \) and \( n \geq 0 \).

**Proof.** Since we are starting from single chain, then, there is a nested chain abacus in \( L_0 \). Next, we will add full chain with \( P_2 \) beads where the number of the beads in the second chain depends on the structure of the nested chain abacus (see Theorems 5.24 and 5.25). Thus, there are \( P_2 \) of the nested chain abacus in \( L_1 \). Continue to add full chain with \( P_3 \) beads where \( P_3 = P_2 + 8 \). Thus, there are \( P_2(P_2 + 8) \) nested chain abacus. Hence, there are \( \prod_{k=0}^{n-1} (P_2 + 8k) \) nested chain abacus at level \( n \). Figure 2 illustrates to Lemma 3.1.

Figure 2 illustrates to Lemma 3.1.

\[ \Omega = \{ P_2 \} = P_2 + k8 \quad [P_2 + 8(k + 1)]^{P_2 + 8k}, \quad \ldots \quad (3) \]

**4. Generating Function**

A succession rule, \( \Omega \), defines a sequence of positive integers \( f_n, n \geq 0 \), \( f_n \) being the number of the nodes at level \( n \) in the generating tree. In succession equation (4), all elements are changed following to \( (P_2 + 8k) \). Thus,

\[
f_\Omega(x) = \sum_{n \geq 0} f_n x^n
\]

\[
f_\Omega(x, y) = \sum_{n \geq 0, k \geq 1} f_{n,k} x^n y^k \quad \ldots \quad (4)
\]

Using this succession rule (where power notation denotes the repeating number of levels), since \( f_{n,k} = 0, k \neq P_2 + 8n \) \( (n \geq 0) \). Since the starting number of the tree is
$P_2$, the first term is $x^0 y^{P_2}$ we can transfer (5) as shown below

\[
 f_\Omega(x, y)l = \sum_{n \geq 0, k \geq 0} f_{n, P_2 + 8k} \\
l = x^0 y^{P_2} + \sum_{n \geq 1, k \geq 0} f_{n, P_2 + 8k} x^n y^{P_2 + 8k} \\
l = y^{P_2} + x \sum_{n \geq 0, k \geq 0} f_{n, P_2 + 8k} x^n (P_2 + 8k) (y^{P_2 + 8(k+1)}) \\
l = y^{P_2} + x \sum_{n \geq 0, k \geq 0} f_{n, P_2 + 8k} x^n (P_2 + 8k). y^{P_2 + 8(k+1)} \\
l = y^{P_2} + xy^9 \sum_{n \geq 0, k \geq 0} f_{n, P_2 + 8k} x^n (P_2 + 8k). y^{P_2 + 8k - 1} \\
l = y^{P_2} + xy^9 \frac{\partial}{\partial y} \left( \sum_{n \geq 0, k \geq 0} f_\Omega(x, y) \right) \\
l = y^{P_2} + xy^9 \frac{\partial n(x, y)}{\partial y} \quad \text{................... (5)}
\]

Thus,

\[
 \frac{\partial f_\Omega(x, y)}{\partial y} - \frac{1}{xy^9} f_\Omega(x, y)l = -\frac{y^{P_2 - 9}}{x} \quad \text{.................. (6)}
\]

The solution of $\frac{dy}{dx} + q(x)y = Q(x)$ is $y = e^{-\int q(x)dx} (c + \int Q(x)e^{\int q(x)dx} dx)$.

Let $q(y) = -\frac{1}{xy^9}$, $Q(y) = -\frac{y^{P_2 - 9}}{x}$, then,

\[
 f_\Omega(x, y)l = e^{\frac{1}{xy^9}} \left(c + \int -\frac{y^{P_2 - 9}}{x} e^{\frac{1}{xy^9}} dy \right) \\
l = e^{\frac{1}{8x^9}} \left(c - \lim_{x \to +0} e^{\frac{1}{8x^9}} \right) \quad \text{................... (7)}
\]

\[
 l = \lim_{x \to +0} e^{\frac{1}{8x^9}} \left(c - \lim_{x \to +0} \frac{e^{\frac{1}{8x^9}}}{x} \right) \\
= e^0 \times (c - 0). \quad \text{............................. (8)}
\]

Let $z = \frac{1}{x}, x \to +0, z \to +\infty$,

\[
 \lim_{x \to +0} e^{\frac{1}{8x^9}} l = \lim_{z \to +\infty} ze^{-z} \\
l = \lim_{z \to +\infty} \frac{1}{z^y} \\
l = \lim_{z \to +\infty} (e^z)^y \\
l = \lim_{z \to +\infty} e^{1/z} \\
= 0. \quad \text{............................. (9)}
\]
Thus,
\[
\lim_{x \to +0} f_\Omega(x, 1) l = \lim_{x \to +0} e^{\frac{1}{bx}} \left( c - \lim_{x \to +0} e^{-\frac{1}{bx}} \frac{1}{x} \right)
\]
\[
l = \lim_{x \to +0} e^{\frac{1}{bx}}(c - 0)
\]
\[
l = e^{-\infty} \times (c - 0) = 0. \quad \text{……………………………… (10)}
\]

So from this, \(c=0\) and
\[
f_\Omega(x, y) l = e^{\frac{1}{bxy^3}} \left( c - \int y^{P_2 - 0} e^{-\frac{1}{bxy^3}} dy \right)
\]
\[
l = -e^{\frac{1}{bxy^3}} \left( \int y^{P_2 - 0} e^{-\frac{1}{bxy^3}} dy \right).
\]

From this equation, we can find the generating functions such as \(f_\Omega(x) = f_\Omega(x, 1)\).

**Example 3.1** Suppose that \(P_1 = 6, P_2 = 14\) and \(P_3 = P_2 + 8 = 22, ...\) By employing programming code the generating function is
\[
f(x) = 1 + 14x + 308x^2 + 6776x^3 + O(x).
\]

5. **Conclusion**

The crux of this work is to generating function with respect to chain is proposed.

**References**

[1] Mohommed EF. Constructing a Nested Chain in James Abacus Diagram. InJournal of Physics: Conference Series 2019 Sep (Vol. 1294, No. 3, p. 032019). IOP Publishing.

[2] Mohommed, Eman F. "Topological Structure of Nested Chain Abacus." *Iraqi Journal of Science* (2020): 153-160.

[3] Mohommed, Eman F., Haslinda Ibrahim, and Nazihah Ahmad. "Enumeration of n-connected oniommes inscribed in an abacus." JP Journal of Algebra of Number Theory and applications 39, no. 6 (2017): 843-874.

[4] Klärner, A. Enumeration involving sums over compositions (Unpublished doctoral dissertation). University of Alberta, Edmonton, 1966.

[5] Harary F. Unsolved problems in the enumeration of graphs. this issue. 1960:63.

[6] Bousquet-Mélou, Mireille, and Andrew Rechnitzer. Lattice animals and heaps of dimers. Discrete Mathematics 258.1-3 (2002): 235-274.

[7] H. N. V. Temperley. Combinatorial problems suggested by the statistical mechanics of domains and of rubber-like molecules, Phys. review, vol. 103, no. 2, (1956), 1 2 16.

[8] J.M. Hammersley, Percolation processes II: the connective constant, Proc. Cambridge. Philos. Soc., vol. 53, (1957), 642–645.

[9] Tomei, Carlos, and Tania Vieira. "The kernel of the adjacency matrix of a rectangular mesh. Discrete and Computational Geometry 28, no. 3 (2002): 411-425.

[10] Castiglione, G., Frosini, A., Restivo, A., & Rinaldi, S. Enumeration of l-convex polyominoes by rows and columns. Theoretical Computer Science, 2005, 347(1-2), 336–352.

[11] Chow, S., & Ruskey, F. Gray codes for column-convex polyominoes and a new class of distributive lattices. Discrete Mathematics, 2009, 309(17), 5284–5297.

[12] Duchêne, E., Rinaldi, S., & Schaeffer, G. The number of z-convex polyominoes. Advances in Applied Mathematics, 2008, 40(1), 54–72.

[13] Bacchelli, S., Ferrari, L., Pinzani, R., & Sprugnoli, R. Mixed succession rules: The commutative case. Journal of Combinatorial Theory, Series A, 2010, 117(5), 568–582.