Lee classes on LCK manifolds with potential

Liviu Ornea¹, Misha Verbitsky²

Abstract
An LCK manifold is a complex manifold \((M,I)\) equipped with a Hermitian form \(\omega\) and a closed 1-form \(\theta\), called the Lee form, such that \(d\omega = \theta \wedge \omega\). An LCK manifold with potential is an LCK manifold with a positive Kähler potential on its cover, such that the deck group multiplies the Kähler potential by a constant. A Lee class of an LCK manifold is the cohomology class of the Lee form. We determine the set of Lee classes on LCK manifolds admitting an LCK structure with potential, showing that it is an open half-space in \(H^1(M,\mathbb{R})\). For Vaisman manifolds, this theorem was proven in 1994 by Tsukada; we give a new self-contained proof of his result.

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1 Introduction

A locally conformally Kähler (LCK) manifold is an Hermitian manifold $(M, I, g)$ equipped with an atlas $\{U_\alpha\}$ such that the restriction of $g$ to each $U_\alpha$ is conformally equivalent to some Kähler metric $g_\alpha$ defined only on $U_\alpha$, id est $g|_{U_\alpha} = e^{f_\alpha} g_\alpha$, where $f_\alpha \in C^\infty U_\alpha$. One can see that in this case the exterior derivatives of the conformal factors agree on intersections: $df_\alpha = df_\beta$ on $U_\alpha \cap U_\beta$, thus giving rise to a global closed 1-form $\theta$, called the Lee form.

Then the Hermitian form $\omega(x,y) := g(Ix,y)$ satisfies $d\omega = \theta \wedge \omega$ (see [DO] for an introduction to the subject).

Forgetting the complex structure, one arrives at the notion of locally conformally symplectic manifold (LCS, for short): a $2n$ dimensional real manifold endowed with a non-degenerate 2-form $\omega$ and a closed 1-form $\theta$ (also called Lee form) such that $d\omega = \theta \wedge \omega$.

Two subclasses of LCK manifolds are very important and rather well understood by now. The Vaisman manifolds, whose universal covers are Kähler cones over Sasakian manifolds (see Subsection 3), and LCK manifolds with potential, whose universal cover admits a Kähler metric with global, positive and automorphic potential (see Subsection 4 for the precise definition).

The cohomology class of the Lee form, called the Lee class, is the first cohomological invariant one encounters when dealing with the LCK manifolds. Let $(M, \theta, \omega)$ be a compact LCK manifold, and $[\theta] \in H^1(M, \mathbb{R})$ its Lee class. By Vaisman’s theorem (Theorem 2.10), $[\theta] = 0$ if and only if $M$ is of Kähler type.

For a compact Kähler manifold $X$, the subset of Kähler classes in $H^2(X, \mathbb{R})$ is the “Kähler cone”, and is one of the most important geometric features of a Kähler manifold. Similarly, we would like to have a description of the set of Lee classes on a given compact complex manifold which is known to admit LCK structures. It was already shown that in this case it cannot be a cone: indeed, by A. Otiman ([Ot, Theorem 3.11]), for an Inoue surface of class $S^0$, the set of Lee classes is a point.
For LCS structures, the set of the Lee classes is better understood, due to Eliashberg and Murphy, who proved that on any almost complex manifold with $H^1(M, \mathbb{Q}) \neq 0$, for any non-zero class $\alpha \in H^1(M, \mathbb{Q})$, there exists $C > 0$ such that $C\alpha$ is the Lee class of an LCS structure ([EM, Theorem 1.11]).

For complex surfaces with $b_1(M) = 1$, the set $\mathcal{L}$ of Lee classes of LCK structures was studied by Apostolov and Dloussky, who proved that $\mathcal{L}$ is either open or a point, [AD].

For higher dimensional LCK manifolds, the first important advance in this direction was due to K. Tsukada, who proved that the set of Lee classes on Vaisman manifolds is an open half-space ([Ts1, Theorem 5.1]), using the harmonic decomposition on Vaisman manifolds, due to T. Kashiwada, [Ka].

In this paper, we extend Tsukada’s theorem to compact LCK manifolds with potential of complex dimension greater than 3, using the following decomposition theorem for the first cohomology (Theorem 6.1):

$$H^1(M, \mathbb{C}) = H^{1,0}(M) \oplus \overline{H^{1,0}(M)} \oplus \langle \theta \rangle \quad (1.1)$$

where $H^{1,0}(M) \subset H^1(M, \mathbb{C})$ is the space of all closed holomorphic 1-forms, identified with a subspace in cohomology by Lemma 2.12. Tsukada proved this for Vaisman manifolds using the commutation formulae for Laplacians, and the harmonic decomposition for Vaisman manifolds.

I. Vaisman conjectured that $b_1(M)$ is odd-dimensional for any compact LCK manifold ([Va1, p. 535]); this famous conjecture was disproven by Oeljeklaus and Toma in [OT]. The decomposition (1.1) would imply that $b_1(M)$ is odd, hence the counterexample of Oeljeklaus-Toma does not satisfy (1.1). However, the natural map

$$H^{1,0}(M) \oplus \overline{H^{1,0}(M)} \oplus \langle \theta \rangle \longrightarrow H^1(M, \mathbb{C}) \quad (1.2)$$

is always injective (Lemma 2.12).

For LCK manifolds with potential, we deduce (1.1) from a deformation argument, by showing that an LCK manifold with potential $M_1$ obtained as a deformation of a Vaisman manifold $M_2$ satisfies $\dim H^{1,0}(M_1) \geq \dim H^{1,0}(M_2)$. Unless $\dim H^{1,0}(M_1) = \dim H^{1,0}(M_2)$, this would imply that $\dim H^{1,0}(M_1) > \frac{b_1(M_1)-1}{2}$, which is impossible because the map (1.2) is injective.

Notice that the equality $\dim H^{1,0}(M) = \frac{b_1(M)-1}{2}$ is valid for non-Kähler complex surfaces as well.

The decomposition (1.1) is the cornerstone for the description of the set of Lee classes on an LCK manifold with potential. Consider the linear map
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\( \mu : H^1(M, \mathbb{R}) \rightarrow \mathbb{R} \) vanishing on the codimension 1 subspace \( H^{1,0}(M) \oplus H^{0,1}(M) \subset H^1(M, \mathbb{R}) \) and positive on the Lee form. We prove that \( \xi \in H^1(M, \mathbb{R}) \) is a Lee class if and only if \( \mu(\xi) > 0 \) (Theorem 8.4).

Conventions: In the sequel, \((M, I)\) is a connected complex manifold of complex dimension \( n \geq 2 \). For an Hermitian metric \( g \), we shall denote with \( \omega(x, y) := g(Ix, y) \) the fundamental 2-form. We extend the action of the complex structure to \( k \)-forms by \( (I\eta)(x_1, \ldots, x_k) = (-1)^k \eta(Ix_1, \ldots, Ix_k) \), and we denote \( I\eta \) by \( \eta_c \). The complex differential \( d_c \) is defined as \( d_c = I^{-1}dI \).

We let \( d^* : \Lambda^k M \rightarrow \Lambda^{k-1} M \) be the metric adjoint of the exterior derivative.

2 LCK manifolds

We gather here the necessary background in LCK geometry. For details, please see [DO] and [OV4, OV2, OV1].

2.1 Definitions. Examples

Definition 2.1: \((M, I)\) is of locally conformally Kähler (LCK) type if it admits an Hermitian metric \( g \) whose fundamental form satisfies the equation

\[
    d\omega = \theta \wedge \omega
\]

for a closed 1-form \( \theta \) called the Lee form. Then \((M, I, \omega, \theta)\) is called an LCK manifold.

Remark 2.2:

1. The LCK condition is conformally invariant: if \( g \) is LCK with Lee form \( \theta \), then \( e^t g \) is LCK with Lee form \( \theta + dt \), hence to each conformal class of LCK metrics there corresponds a Lee class in \( H^1(M, \mathbb{R}) \).

2. In dimension \( n \neq 2 \), the equation (2.1) implies \( d\theta = 0 \).

3. Using (2.1), one can prove that the Lee form is determined in terms of \( I \) and \( \omega \) by \( \theta = -I \left( \frac{1}{n-1} d^* \omega \right) \).

4. If \( \theta \) is exact, the LCK manifold is called globally conformally Kähler (GCK). Usually, it is tacitly assumed that \( \theta \) is not exact.
In the sequel we will mostly use following definition, equivalent to Definition 2.1.

**Definition 2.3:** A complex manifold \((M, I)\) is LCK if and only if it admits a cover \((\tilde{M}, I)\) equipped with a Kähler metric \(\tilde{\omega}\) with respect to which the deck group of the cover acts by holomorphic homotheties.

**Definition 2.4:** The homothety character associated to a Kähler cover with deck group \(\Gamma\) is \(\chi : \Gamma \rightarrow \mathbb{R}^>0\), \(\chi(\gamma) = \gamma^* \tilde{\omega}/\tilde{\omega}\). The rank of \(\text{Im}(\chi)\) is the LCK rank of \((M, I, \omega)\).

**Remark 2.5:** Since \(\Gamma\) is a quotient group of \(\pi_1(M)\), we can consider \(\chi\) as a character on \(\pi_1(M)\). Let then \(L \rightarrow M\) be the local system associated to \(\chi\). It is a real line bundle and \(\theta\) can be viewed as a connection form in \(L\) which is thus flat. The line bundle \(L\) is also called the weight bundle of the LCK manifold.

**Definition 2.6:** The minimal Kähler cover of an LCK manifold corresponds to a group \(\Gamma\) on which \(\chi\) is injective (\(\Gamma\) does not contain \(\tilde{\omega}\)-isometries). This is the smallest cover admitting a Kähler metric which is conformal to the pullback of the LCK metric.

**Definition 2.7:** A differential form \(\alpha \in \Lambda^* \tilde{M}\) is called automorphic if \(\gamma^* \alpha = \chi(\gamma) \alpha\) for all \(\gamma \in \Gamma\).

**Remark 2.8:**

(i) Automorphic forms on \(\tilde{M}\) can be identified with \(L\)-valued forms on \(M\). In particular, since \(\pi^* \omega\) is \(\Gamma\)-invariant, \(\omega\) can be viewed as a section of \(\Lambda^{1,1}(M, L)\), and \(\omega^k\) as a section of \(\Lambda^{k,k}(M, L^\otimes k)\) etc.

(ii) Let \(d_\theta := d - \theta \wedge\). Then \(d_\theta \omega = 0\), hence \(\omega\) is a closed \(L\)-valued form.

(iii) The complex \((\Lambda^*, d_\theta)\) is elliptic, since \(d_\theta\) has the same symbol as \(d\), and its cohomology \(H^*_\theta(M)\) can be identified with the cohomology \(H^*(M, L)\) of the local system \(L\); it is called Morse-Novikov cohomology.

**Example 2.9:** The following manifolds admit an LCK structure: almost all known non-Kähler compact complex surfaces (see e.g. [VVO, Va1, Be, GO, Br]); Hopf manifolds: \((\mathbb{C}^n \setminus 0)/\langle A\rangle\), \(A \in \text{GL}(n, \mathbb{C})\) with eigenvalues
of absolute value $> 1$ (see e.g. [OV5]); some Oeljeklaus-Toma manifolds ([OT]); Kato manifolds ([IOP]) and some “toric Kato manifolds” ([IOPR]).

2.2 The dichotomy Kähler versus LCK

The next result, proven by Vaisman, shows that on compact complex manifolds, LCK and GCK metrics cannot coexist. For consistency, we provide a proof, slightly different from the original one.

**Theorem 2.10:** ([Va1]) Let $(M,\omega,\theta)$ be a compact LCK manifold, not globally conformally Kähler. Then $M$ does not admit a Kähler structure.

**Proof:** 

**Step 1:** That $M$ is not globally conformally Kähler means that $\theta$ is not cohomologous with zero, that is $\theta$ is not $d$-exact.

Let $d\omega = \omega \wedge \theta$, $\theta' = \theta + d\varphi$. Then

$$d(e^{\varphi}\omega) = e^{\varphi}\omega \wedge \theta + e^{\varphi}\omega \wedge d\varphi = e^{\varphi}\omega \wedge \theta'.$$

This means that we can replace the triple $(M,\omega,\theta)$ by $(M,e^{\varphi}\omega,\theta')$ for any 1-form $\theta'$ cohomologous to $\theta$.

**Step 2:** Assume that $M$ admits a Kähler structure. Then, by Hodge theory, $\theta$ is cohomologous to the sum of a holomorphic and an antiholomorphic form. After a conformal transformation (which changes $\theta$ in $\theta + d\varphi$) as in Step 1, we may assume that $\theta$ itself is the sum of a holomorphic and an antiholomorphic form.

**Step 3:** Then $dd^c\theta = \sqrt{-1}d\bar{\theta}d\theta = 0$, giving $dd^c(\omega^{n-1}) = \omega^{n-1} \wedge \theta \wedge I(\theta)$. Therefore $0 = \int_M dd^c(\omega^{n-1}) = \int Mass(\theta \wedge I(\theta))$, hence $\theta \wedge I(\theta) = 0$, thus $||\theta||^2 = 0$ and the initial metric is globally conformally Kähler. $\blacksquare$

Using similar techniques, we can prove:

**Lemma 2.11:** Let $(M,\theta,\omega)$ be a compact LCK manifold. Then the cohomology class $[\theta] \in H^1(M,\mathbb{R})$ cannot be represented by a form which is $d^c$-closed.

---

1 $(\mathbb{C}^n \setminus 0)/\langle A \rangle$ is called diagonal Hopf manifold when $A$ is diagonalizable, and non-diagonal Hopf manifold when $A$ is not diagonalizable.

2 Recall that the mass of a positive $(1,1)$-form $\eta$, denoted $Mass(\eta)$, is the volume form $\eta \wedge \omega^{n-1}$, [D].
Proof: Indeed, for each representative of $\theta$, this form can be realized as the Lee form for an LCK metric which is conformally equivalent to $\omega$. Therefore, it would suffice to show that $d^c \theta \neq 0$ for any compact LCK manifold $(M, \theta, \omega)$. If $d^c \theta = 0$, we would have $dd^c(\omega^{n-1}) = \omega^{n-1} \wedge \theta \wedge I(\theta)$, giving, as above, $0 = \int_M dd^c(\omega^{n-1}) = \int \text{Mass}(\theta \wedge I(\theta))$, hence $\theta \wedge I(\theta) = 0$, implying that $\theta = 0$.

This can be used to prove an important step in our decomposition theorem for $H^1(M)$, where $M$ is an LCK manifold with potential.

Lemma 2.12: Let $(M, \theta, \omega)$ be a compact LCK manifold, and $H^{1,0}(M)$ denote the space of closed holomorphic 1-forms on $M$. Then the natural map

$$H^{1,0}(M) \oplus \overline{H^{1,0}(M) + \langle \theta \rangle} \to H^1(M, \mathbb{C})$$

is injective, where $\langle \theta \rangle$ is the subspace generated by $\theta$.

Proof: A closed holomorphic form $\alpha$ belongs to $\ker d \cap \ker d^c$. Indeed, $\bar{\partial} \alpha = 0$ together with $d \alpha = 0$ implies $d^c \alpha = 0$. Therefore, if $\alpha \in H^{1,0}(M) + \overline{H^{1,0}(M)}$ is exact, one has $\alpha = df$ and $dd^c f = 0$, which is impossible by the maximum principle. However, if $\theta$ is cohomologous to a sum of holomorphic and antiholomorphic forms, this easily leads to a contradiction with Lemma 2.11. Indeed, suppose that $\theta = \alpha + df$, where $d \alpha = d^c \alpha = 0$. Making a conformal change, we obtain another LCK structure which has Lee form equal to $\alpha$. This is impossible, again by Lemma 2.11.

Corollary 2.13: Let $M$ be a compact LCK manifold, and $H^{1,0}(M)$ denote the space of closed holomorphic 1-forms on $M$. Then $\dim H^{1,0}(M) \leq \frac{b_1(M) - 1}{2}$. ■

3 Vaisman manifolds

The best understood subclass of LCK manifolds is the one with the Lee form which is parallel with respect to the Levi-Civita connection. They are called Vaisman manifolds.

If $(M, g, I, \theta)$ is Vaisman, the Lee field $\theta^\sharp$ is Killing and holomorphic; moreover, it commutes with $I \theta^\sharp$ ([Va2]). Denote by $\Sigma$ the holomorphic 1-dimensional foliation generated by $\theta^\sharp$ and $I \theta^\sharp$. It is called the canonical foliation (the motivation is given in the next theorem).
**Theorem 3.1:** Let $M$ be a compact Vaisman manifold, and $\Sigma \subset TM$ its canonical foliation. Then:

(i) $\Sigma$ is independent from the choice of the Vaisman metric ([Ts2]).

(ii) $d^c \theta = \omega - \theta \wedge I\theta$ ([Va2]) and the exact $(1,1)$-form $\omega_0 := d^c \theta$ is semi-positive ([Ve]). Therefore, $\Sigma = \ker \omega_0$, and $\omega_0$ is transversally Kähler with respect to $\Sigma$.

Since $\theta^\sharp$ is Killing and holomorphic, it generates a complex flow of $g$-isometries. These lift to holomorphic non-trivial homotheties of the Kähler metric on the universal cover $\tilde{M}$. This is in fact an equivalent definition of Vaisman-type manifolds, as the following criterion shows:

**Theorem 3.2: ([KO])** Let $(M, \omega, \theta)$ be a compact LCK manifold equipped with a holomorphic and conformal $\mathbb{C}$-action $\rho$ without fixed points, which lifts to non-isometric homotheties on the Kähler cover $\tilde{M}$. Then $(M, \omega, \theta)$ is conformally equivalent to a Vaisman manifold.

**Example 3.3:** A non-exhaustive list of examples of Vaisman manifolds comprises:

(i) Diagonal Hopf manifolds $(\mathbb{C}^n \setminus 0)/\langle A \rangle$ where $A$ is semi-simple and with eigenvalues $\alpha_i$ of absolute value $> 1$, [GO, OV5].

(ii) Elliptic complex surfaces (see [Be] for the complete classification of Vaisman compact surfaces; see also [VVO]).

(iii) All compact submanifolds of a Vaisman manifold ([Ve]).

**Remark 3.4:** The class of Vaisman manifolds is strict: neither the LCK Inoue surfaces, nor the non-diagonal Hopf manifolds can bear Vaisman metrics ([Be], [OV5]).

Recall that a form $\eta$ on a foliated manifold $(M, \Sigma)$ is called **basic** if it can be locally obtained as a pullback $\pi^* \eta_0$ from the leaf space of $\Sigma$, which is defined in a sufficiently small neighbourhood of every point $x \in M$.

The following claim is well known (and can be used as a definition of basic forms).
Claim 3.5: A form $\eta$ on $M$ is basic with respect to $\Sigma \subset TM$ if and only if for any vector field $X \in \Sigma$, one has $i_X(\eta) = \text{Lie}_X(\eta) = 0$, where $i_X$ denotes the contraction with $X$. □

Corollary 3.6: A closed form $\eta$ on $M$ is basic with respect to $\Sigma \subset TM$ if and only if $i_X(\eta) = 0$.

Proof: Follows from the Cartan formula $\text{Lie}_X(\eta) = i_X(d\eta) + d(i_X(\eta))$.

Further on, we need the following observation.

Proposition 3.7: Let $M$ be a compact Vaisman manifold, and $\eta$ a closed holomorphic 1-form on $M$. Then $\eta$ is basic with respect to the canonical foliation $\Sigma$ on $M$.

Proof: Let $n = \dim \mathbb{C} M$, and $\omega_0 \in \Lambda^{1,1}(M)$ the transversal Kähler form defined above. Since $\eta$ is closed and $\omega_0$ is exact, one has $\int_M \omega_0^{n-1} \wedge \eta \wedge \bar{\eta} = 0$. However, $-\sqrt{-1} \eta \wedge \bar{\eta}$ is a semi-positive form, and $\omega_0$ is strictly positive in the directions transversal to $\Sigma$. This implies that $-\sqrt{-1} \omega_0^{n-1} \wedge \eta \wedge \bar{\eta}$ is a positive volume form in every point $x \in M$ such that $\eta|_{T_x \tilde{M}}$ does not vanish on $\Sigma|_{T_x \tilde{M}}$. Since $\int_M \omega_0^{n-1} \wedge \eta \wedge \bar{\eta} = 0$, it follows that $\eta|_{\Sigma} = 0$ everywhere. By Corollary 3.6, $\eta$ is basic. □

4 LCK manifolds with potential

We now introduce the main object of study of this paper.

Definition 4.1: An LCK manifold has LCK potential if it admits a Kähler covering on which the Kähler metric has a global and positive potential function $\psi$ such that the deck group multiplies $\psi$ by a constant. In this case, $M$ is called LCK manifold with potential.

Example 4.2: All Vaisman manifolds are LCK manifolds with potential. Indeed, if $\pi : \tilde{M} \to M$ is the universal cover and $\theta$ is the Lee form on $M$, then $\pi^* \theta$ is an automorphic global Kähler potential for $\tilde{\omega}$. Also, the structure is inherited by all complex submanifolds of an LCK manifold with
potential. Among the non-Vaisman examples, we mention the non-diagonal Hopf manifolds, [OV4].

**Remark 4.3:** LCK Inoue surfaces, blow-ups of LCK manifolds and OT-manifolds cannot be LCK manifolds with potential, [Ot, Vu].

A wealth of examples is provided by the following fundamental result:

**Theorem 4.4:** ([OV2]) Let \((M, I, \omega, \theta)\) be a compact LCK manifold with potential. Then any small deformation \((M, I, t), \ t \in \mathbb{C}, \ |t| < \varepsilon,\) admit an LCK metric with potential. In particular, non-diagonal Hopf manifolds are LCK with potential.

**Remark 4.5:** By [OV3] (see also [I]), an LCK manifold with potential admits a conformal gauge such that

\[
d\theta^c = \omega - \theta \wedge \theta^c, \quad \text{where} \quad \theta^c(X) = -\theta(I X).
\]

We shall tacitly assume that the LCK metric is chosen in such a way that (4.1) holds.

We are especially interested in the LCK manifolds with potential of LCK rank 1, that is, LCK manifolds with potential admitting a Kähler \(\mathbb{Z}\)-covering. We showed in [OV2] that this is equivalent to the LCK potential being a proper function on the minimal Kähler cover. The minimal cover of a compact LCK manifold with proper potential is very nice from the complex and algebraic viewpoint:

**Theorem 4.6:** ([OV2, OV5]) Let \(M\) be a compact LCK manifold with proper potential, and \(\tilde{M}\) its Kähler \(\mathbb{Z}\)-covering. If \(\dim_{\mathbb{C}} M \geq 3\), then the metric completion \(\tilde{M}_{c}\) admits a structure of a complex variety, compatible with the complex structure on \(\tilde{M} \subset \tilde{M}_{c}\), and the complement \(\tilde{M}_{c} \setminus \tilde{M}\) is just one point. Moreover, \(\tilde{M}_{c}\) is an affine algebraic variety obtained as an affine cone over a projective orbifold.

**Remark 4.7:** Notice that \(\tilde{M}_{c}\) is indeed the Stein completion of \(\tilde{M}\) in the sense of [AS]. In the proof of our theorem, we used the filling theorem by Rossi and Andreotti-Siu ([Ros, AS]) which imposes the restriction \(\dim_{\mathbb{C}} > 2\).

By appearance, assuming that the potential is a proper function is a restrictive condition. However, this is not entirely true: as long as one is...
interested in the complex geometry (and not in the Riemannian one), one can always assume the LCK potential is proper.

**Theorem 4.8:** ([OV1, OV4]) Let \((M, \omega, \theta, \varphi)\) be a compact LCK manifold with improper LCK potential. Then \((\omega, \theta, \varphi)\) can be approximated in the \(C^\infty\)-topology by an LCK structure with proper LCK potential on the same complex manifold.

The following is one of the most important features of compact LCK manifolds with proper potential, of dimension greater than 3.

**Theorem 4.9:** Any compact LCK manifold \(M^n, n \geq 3\), admits a holomorphic embedding into a Hopf manifold \((\mathbb{C}^N \setminus 0)/\langle A \rangle\). A manifold \(M^n\) is Vaisman if and only if it admits an embedding to \((\mathbb{C}^N \setminus 0)/\langle A \rangle\), with the matrix \(A\) diagonalizable.

**Proof:** In [OV2, Theorem 3.4] it is shown that an LCK manifold with potential is embeddable into a Hopf manifold, and in [OV2, Theorem 3.6] it is shown that a Vaisman manifold is embeddable to a diagonal Hopf manifold. Conversely, in [OV5, Section 2.5] we show that a diagonal Hopf manifold is Vaisman, and in [Ve] it is shown that a positive-dimensional compact submanifold of a Vaisman manifold is Vaisman.

One of the most useful properties of compact LCK manifolds with potential in dimension greater than 3 is that their complex structure can be deformed to a complex structure that supports Vaisman metrics.

**Theorem 4.10:** ([OV3]) Let \((M, \omega, \theta)\), \(\dim_{\mathbb{C}} M \geq 3\), be a compact LCK manifold with proper potential. Then there exists a complex analytic deformation of \(M\) which admits a Vaisman metric.

**Remark 4.11:** A refinement of this result will be given in Theorem 7.3.
5 Algebraic cones and LCK manifolds with potential

5.1 Jordan-Chevalley decomposition

Further on, all algebraic groups are considered over \( \mathbb{C} \). For the definition and more reference on algebraic groups, please see [Hum].

**Definition 5.1:** An element of an algebraic group \( G \) is called **semisimple** if its image is semisimple for some exact algebraic representation of \( G \), and is called **unipotent** if its image is unipotent (that is, exponential of a nilpotent) for some exact algebraic representation of \( G \).

**Remark 5.2:** For any algebraic representation of an algebraic group \( G \), the image of any semisimple element is a semisimple operator, and the image of any unipotent element is a unipotent operator ([Hum, §15.3]).

**Theorem 5.3:** (Jordan-Chevalley decomposition), [Hum, §15.3]

Let \( G \) be an algebraic group, and \( A \in G \). Then there exists a unique decomposition \( A = SU \) of \( A \) in a product of commuting elements \( S \) and \( U \), where \( U \) is unipotent and \( S \) semisimple.

5.2 Algebraic cones

To better describe the universal cover of a compact LCK manifold with potential we need to introduce the closed and open algebraic cones.

**Definition 5.4:** A **closed algebraic cone** is an affine variety \( C \) admitting a \( \mathbb{C}^* \)-action \( \rho \) with a unique fixed point \( x_0 \), called the **origin**, and satisfying the following:

(i) \( C \) is smooth outside of \( x_0 \),

(ii) \( \rho \) acts on the Zariski tangent space \( T_{x_0}C \) with all eigenvalues \( |\alpha_i| < 1 \).

An **open algebraic cone** is a closed algebraic cone without the origin.

For the sake of completeness, we give a new and self-contained proof of the following basic result.
Theorem 5.5: ([OV5, Theorem 2.8]) Let $M = \hat{M}/\langle A \rangle$ be an LCK manifold with potential, with LCK rank 1, and $\hat{M}$ its Kähler $\mathbb{Z}$-covering. Then $\hat{M}$ is an open algebraic cone.

**Proof. Step 1:** Let $\hat{M}_c$ be the Stein completion of $\hat{M}$ equipped with an $A$-equivariant embedding to $\mathbb{C}^N$, where $A$ acts as a linear operator with all eigenvalues $|\alpha_i| < 1$.

Let $\mathcal{O}_{\mathbb{C}^N,0}$ denote the ring of germs of holomorphic functions in zero. Call a function $f \in \mathcal{O}_{\mathbb{C}^N,0}$ $A$-finite if the space $\langle f, A^* f, A^{2*} f, ... \rangle$ is finite-dimensional. A polynomial function is clearly $A$-finite. The converse is also true, because the Taylor decomposition of an $A$-finite function $f$ can only have finitely many components, otherwise the eigenspace decomposition of $f$ is infinite.

**Step 2:** We want to produce an explicit fundamental domain $U_0$ for the action of $\mathbb{Z} \cong \langle A \rangle = \{..., A^{-n}, A^{-n+1}, ..., A^{-1}, \text{Id}_{\mathbb{C}^n}, A, A^2, \ldots \}$ on $\mathbb{C}^N$, in such a way that $U_0 = V \setminus A(V)$, where $V$ is Stein. Let $B \subset \mathbb{C}^n$ be the unit ball. When the operator norm $\| A \|$ of $A$ is less than 1, one has $A(B) \subseteq B$, and $B \setminus A(B)$ is the fundamental domain which we can use. This would hold, for example, when $A$ is diagonalizable. On the other hand, the operator norm of a contraction can be bigger than 1. Consider for example the matrix $A = \begin{pmatrix} 1 & 1000 \\ 0 & \frac{1}{2} \end{pmatrix}$; its norm is at least 1000. Therefore, one should take more care when choosing the fundamental domain. Recall that any matrix over $\mathbb{C}$ admits a Jordan decomposition, and every Jordan cell

\[
\begin{pmatrix}
\alpha & 1 & 0 & \ldots & 0 \\
0 & \alpha & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & \alpha
\end{pmatrix}
\] (5.1)

is conjugate to

\[
\begin{pmatrix}
\alpha & \varepsilon & 0 & \ldots & 0 \\
0 & \alpha & \varepsilon & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \varepsilon \\
0 & 0 & 0 & \ldots & \alpha
\end{pmatrix}
\] (5.2)

(see e. g. Proposition 7.1 below). Writing $A$ in a Jordan basis and replacing each cell (5.1) with (5.2), for $\varepsilon$ sufficiently small, we obtain a contraction.
with an operator norm < 1 conjugate to $A$; then $A(B) \subseteq B$ is a fundamental
domain for the action of $\langle A \rangle$.

**Step 3:** Let $U_0$ be a fundamental domain for $A$ acting on $\mathbb{C}^N$. As
indicated in Step 2, every linear contraction is conjugate to an operator $A$
with operator norm < 1. The fundamental domain $U_0$ for $A$ with operator
norm < 1 can be obtained by taking an open ball $B \subset \mathbb{C}^n$ and removing
$A(B)$ from $B$. Denote by $U_n$ a copy of this domain obtained as $U_n := A^{-n}(U_0)$, and let $V_n := \{0\} \cup \bigcup_{i>n} U_i$. Since $V_n = A^{-n}(B)$, it is a Stein
domain in $\mathbb{C}^N$.

Let $A : H^0_b(\mathcal{O}_{U_0}) \rightarrow H^0_b(\mathcal{O}_{U_0})$ be the operator on the ring of bounded
holomorphic functions induced by the action of $A$ on $\mathbb{C}^N$. Clearly, this map
is compatible with the map $A : H^0(\mathcal{O}_{\tilde{M}_c}) \rightarrow H^0(\mathcal{O}_{\tilde{M}_c})$ constructed above; this is what allows us to denote them by the same letter.

**Step 4:** We are going to prove that the operator

$$A : H^0_b(\mathcal{O}_{U_0}) \rightarrow H^0_b(\mathcal{O}_{U_0})$$

is compact with respect to the topology defined by the sup-norm.\(^1\)

Let $\gamma : X \rightarrow X$ be a map taking a complex manifold $X$ to its precom-
pact subset. We prove that in this case the map $\gamma^* : H^0(\mathcal{O}_X) \rightarrow H^0(\mathcal{O}_X)$
is compact in the sup-topology.

For any $f \in H^0(\mathcal{O}_X)$ we have

$$|\gamma^* f|_{\sup} = \sup_{x \in \gamma(X)} |f(x)|.$$

This implies that $\gamma^*(f)$ is bounded. Therefore, for any sequence $\{f_i \in H^0(\mathcal{O}_X)\}$
converging in the $C^0$-topology, the sequence $\{\gamma^* f_i\}$ converges in the sup-topology. The set $B_C := \{v \in V \mid |v|_{\sup} \leq C\}$ is a normal family and hence, by Montel’s theorem, it is precompact in the $C^0$-topology ([D, Chapter I, Theorem 3.12]). Then $\gamma^* B_C$ is precompact in the sup-topology. This proves that the operator $\gamma^* : V \rightarrow V$ is compact.

**Step 5:** Let $\mathcal{I}(V_n)$ be the ideal of $\tilde{M}_c \cap V_n$ in $H^0_b(\mathcal{O}_{V_n})$. Recall that
$H^0_b(\mathcal{O}_X)$ is a Banach algebra, by Montel’s theorem ([D, Chapter IX, Propo-
sition 4.7]). By the Riesz-Schauder’s theorem ([F, Section 5.2]), a compact endomorphism of a Banach space admits a Jordan decomposition. Then

\(^1\)Recall that for a complex manifold $X$, the sup-topology on $H^0_b(\mathcal{O}_X)$ is the topology
given by the sup-norm, namely: $|f|_{\sup} := \sup_X |f|$. 

---

\(\text{L. Ornea, M. Verbitsky} \quad \text{Lee classes of LCK manifolds with potential} \)
A-finite vectors are finite linear combinations of the vectors from the Jordan cells. This implies that the set of $A$-finite functions in $\mathcal{I}(V_n)$ is dense in $\mathcal{I}(V_n)$, with respect to the sup-topology. On the other hand all $A$-finite functions can be holomorphically extended to $\mathbb{C}^N$ by automorphicity.

The base of $C^0$ (that is, compact-open) topology on $H^0(\mathcal{O}_{\mathbb{C}^N})$ is formed by translations of open sets consisting of all functions $f \in H^0(\mathcal{O}_{\mathbb{C}^N})$ which satisfy $|f| < C < \infty$ on a given compact, for some positive $C \in \mathbb{R}$. Therefore, it is the weakest topology such that its restriction to $H^0(\mathcal{O}_{\mathbb{C}^N})$ with sup-topology is continuous. This implies that any set of functions $S \subset H^0(\mathcal{O}_{\mathbb{C}^N})$ which is bounded on compacts and dense in $H^0(\mathcal{O}_{\mathbb{C}^N})$ for all $n$, is dense in $H^0(\mathcal{O}_{\mathbb{C}^N})$.

Since the space of $A$-finite functions is dense in $\mathcal{I}(V_n)$, the space $\mathcal{I}^A$ of $A$-finite functions in $\mathcal{I}$ is dense in $\mathcal{I}$ with respect to the $C^0$-topology. In particular, the set of common zeros of $\mathcal{I}^A$ coincides with $\tilde{M}_c \subset \mathbb{C}^N$.

**Step 6:** The $A$-finite functions are polynomials, as shown in Step 1. By Hilbert’s basis theorem, any ideal in the ring of polynomials is finitely generated. Therefore, the ideal $\mathcal{I}^A$ is finitely generated over polynomials. Let $f_1, ..., f_n$ be the set of its generators. By Step 2, the set of common zeros of $\mathcal{I}^A$ is $\tilde{M}_c \subset \mathbb{C}^N$; therefore, $\tilde{M}_c \subset \mathbb{C}^N$ is given by polynomial equations $f_1 = 0, f_2 = 0, ..., f_n = 0$.

**Step 7:** It remains to show that $\tilde{M}_c$ admits a holomorphic $\mathbb{C}^*$-action containing a contraction. Let $G$ be the Zariski closure of $\langle A \rangle$ in $\text{GL}(\mathbb{C}^N)$. This is a commutative algebraic group, acting on the variety $\tilde{M}_c \subset \mathbb{C}^N$. Let $A = SU$ be the Jordan-Chevalley decomposition for $A$, with $S, U \in G$. Since $G$ preserves $\tilde{M}_c$, the endomorphisms $S$ and $U \in \text{End}(\mathbb{C}^n)$ also act on $\tilde{M}_c$. Since the eigenvalues of $S$ are the same as the eigenvalues of $A$, it is a contraction. Let $G_S \subset G$, $G_S = e^{\mathbb{C}\log S}$ be a one-parametric subgroup containing $S$. We prove that $G_S$ can be approximated by subgroups of $G$ isomorphic to $\mathbb{C}^*$; then these subgroups also contain a contraction, and we are done.

Consider the map taking any $A_1 \in G$ to its unipotent component $U_1$. Since $G$ is commutative, this map is a group homomorphism. Therefore, its kernel $G_s$ (that is, the set of all semisimple elements in $G$) is an algebraic subgroup of $G$. A semisimple commutative algebraic subgroup of $\text{GL}(\mathbb{C}^N)$ is always isomorphic to $(\mathbb{C}^*)^k$ ([BT, Proposition 1.5]). The one-parametric subgroups $\mathbb{C}^* \subset (\mathbb{C}^*)^k$ are dense in $(\mathbb{C}^*)^k$ because one-parametric complex subgroups $\mathbb{C}^* \subset (\mathbb{C}^*)^k$ can be obtained as complexification of subgroups $S^1 \subset U(1)^k \subset (\mathbb{C}^*)^k$, and those are dense in $U(1)^k$. Therefore, the contrac-
section $S \in G_s = (\mathbb{C}^*)^k$ can be approximated by an element of $\mathbb{C}^*$ acting on $\tilde{M}_c$. ■

6 Hodge decomposition for $H^1(M)$ on LCK manifolds with potential

Any harmonic $r$-form, $r \leq n - 1$, on a compact $n$-dimensional Vaisman manifold $(M, \omega, \theta)$ can be uniquely written as a sum $\alpha + \theta \wedge \beta$ where $\alpha$ and $\beta$ are basic harmonic forms (see [Va2] or [OV6] for a different proof. In particular, the space of harmonic 1-forms on a compact Vaisman manifolds is identified with $\ker d \cap \ker d^c \oplus \langle \theta \rangle$.

For LCK manifolds with potential, such a decomposition is no longer available. Instead we can prove:

**Theorem 6.1:** Let $(M, \theta, \omega)$ be a compact LCK manifold with potential, and $H^1,0(M)$ denote the space of closed holomorphic 1-forms on $M$. Using Lemma 2.12, we consider $H^1,0(M) \oplus H^1,0(M) \oplus \langle \theta \rangle$ as a subspace in $H^1(M, \mathbb{C})$. Then $H^1(M, \mathbb{C}) = H^1,0(M) \oplus H^1,0(M) \oplus \langle \theta \rangle$.

**Proof:** To prove that the map

$$H^1,0(M) \oplus H^1,0(M) \oplus \langle \theta \rangle \longrightarrow H^1(M, \mathbb{C})$$

is surjective, it would suffice to show that $\dim_{\mathbb{C}} H^1,0(M) = \frac{b_1(M)-1}{2}$. We prove it by deforming $M$ to a Vaisman manifold $M_0$ and showing that $\dim H^1,0(M) = \dim H^1,0(M_0)$.

We first deform the LCK metric on $M$ to an LCK metric of LCK rank 1 (Theorem 4.8). This operation does not affect the complex structure on $M$, hence $\dim H^1,0(M)$ does not change, and it will suffice to prove that $\dim_{\mathbb{C}} H^1,0(M) = \frac{b_1(M)-1}{2}$ when $M$ is an LCK manifold with proper potential.

Let $\tilde{M}$ be the open algebraic cone associated with $M$ as in Theorem 5.5, and $A : \tilde{M} \rightarrow \tilde{M}$ the generator of the deck group. Applying the Jordan-Chevalley decomposition $A = SU$ as in Theorem 4.10, we can deform $\tilde{M}/(A)$ to the Vaisman manifold $M_0 := \tilde{M}/(S)$. To prove that $\dim_{\mathbb{C}} H^1,0(M) = \frac{b_1(M)-1}{2}$ it would suffice to show that all holomorphic, $S$-invariant 1-forms on $\tilde{M}$ are also $U$-invariant.

Consider $U$ as an automorphism of $M_0$. This automorphism is homotopy equivalent to the identity because $U = e^N$, where $N$ commutes with $S$. Since...
$U$ is a unipotent element of the group of automorphisms of the algebraic cone $\tilde{M}$, the action of $U_t := e^{tN}$ preserves $\tilde{M}$ and commutes with $S$, hence it is well defined on $M_0$. This gives a homotopy of $U = U_1$ to $\text{Id} = U_0$.

Since $U$ is homotopy equivalent to the identity, it acts trivially on $H^1(M_0)$, hence all $S$-invariant holomorphic forms on $\tilde{M}$ are also $SU$-invariant. This implies that $\dim_{\mathbb{C}} H^{1,0}(\tilde{M}) \geq \dim_{\mathbb{C}} H^{1,0}(M_0) = \frac{b_1(M_0) - 1}{2}$. The inequality in this expression is in fact an equality by Corollary 2.13. We thus proved Theorem 6.1. □

7 Approximating LCK with potential structures by Vaisman structures

We start with a linear algebra result which will be used in the proof of the main theorem of this section:

**Proposition 7.1:** Let $p \in \text{GL}(n, \mathbb{C})$ be a linear operator, and $p = su$ its Jordan decomposition, with $s$ semisimple, $u$ unipotent, and $su = us$. Then there exists a sequence $r_i \in \text{GL}(n, \mathbb{C})$ of operators commuting with $s$ and satisfying $\lim_{i \to \infty} r_i pr_i^{-1} = s$.

**Proof:** Since any operator is a sum of Jordan cells, it would suffice to prove Proposition 7.1 when $p$ is a single $k \times k$ Jordan cell,

$$p = \begin{pmatrix}
\alpha & 1 & 0 & \ldots & 0 \\
0 & \alpha & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & \alpha
\end{pmatrix}$$

In this case, $s = \text{const} \text{Id}$, hence it commutes with everything. Take

$$r_i = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & \varepsilon_i & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & \varepsilon_i^k
\end{pmatrix}$$
Then
\[
r_i p r_i^{-1} = \begin{pmatrix}
\alpha & \varepsilon_i & 0 & \ldots & 0 \\
0 & \alpha & \varepsilon_i & \ldots & 0 \\
0 & 0 & \varepsilon_i & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \varepsilon_i \\
0 & 0 & 0 & \ldots & \alpha
\end{pmatrix}
\]

Taking a sequence \( \varepsilon_i \) converging to 0, we obtain \( \lim_{i \to \infty} r_i p r_i^{-1} = s \).

The main result of this section, Theorem 7.3, gives a more precise description of the approximation in Theorem 4.10. In order to state it, we need to recall the notion of Teichmüller space. Recall first that the \( C^k \)-topology on the space of sections of a bundle \( B \to M \) is the topology of uniform convergence of \( b, \nabla b, \nabla^2 b, \ldots, \nabla^k b \) on compacts, for some connection \( \nabla \) on \( B \). The \( C^\infty \)-topology is the topology of uniform convergence of all derivatives. In other words, a set is open in the \( C^\infty \)-topology if it is open in all \( C^k \)-topologies. Now we can give:

**Definition 7.2:** Let \( \text{Comp} \) be the set of all integrable complex structures on \( M \), equipped with the \( C^\infty \)-topology, and \( \text{Diff}_0 \) the group of isotopies of \( M \), that is, the connected component of the group of diffeomorphisms of \( M \). The **Teichmüller space** of complex structures on \( M \) is the quotient \( \text{Teich} := \text{Comp} / \text{Diff}_0 \) equipped with the quotient topology.

**Theorem 7.3:** Let \((M, J)\) be an LCK manifold with potential, \( \dim_{\mathbb{C}} M \geq 3 \). Then there exists a Vaisman-type complex structure \((M, J_\infty)\) such that the point \([J_\infty]\) in the Teichmüller space \(\text{Teich}(M)\) of complex structures on \( M \) belongs to the closure of \([J] \in \text{Teich}(M)\). In other words, there exists a sequence of diffeomorphisms \( \nu_i \in \text{Diff}_0(M) \) such that \( \lim_{i \to \infty} \nu_i(J) = J_\infty \), where the limit is taken with respect to the \( C^\infty \)-topology on the space \( \text{Comp} \) of complex structures.

**Proof:** Without restricting the generality, we may assume that \((M, J)\) has LCK rank 1, and its \( \mathbb{Z} \)-cover is Kähler (Theorem 4.8). Fix an embedding of \((M, J)\) into the Hopf manifold \( H = \mathbb{C}^n \setminus \{0\} \) (Theorem 4.9).

Let \( A = u s \) be the Jordan-Chevalley decomposition for \( A \). By Proposition 7.1, there exists a sequence \( A_i = u_i s = r_i A r_i^{-1} \) of operators conjugated to \( A \) such that \( u_i \) converges to 0. Denote by \((H, I_i)\) the Hopf manifold \((H, I_i) := \mathbb{C}^n \setminus \{0\} / (A_i)\).

Since \((H, I_i)\) are all naturally isomorphic to \( H \), one obtains the embedding \( \varphi_i : (M, J) \to (H, I_i) \).
Since the operators $A_i = u_i s$ converge to $s$, the sequence $I_i \in \text{Comp}(H)$ converges to $I_\infty$, where $(H, I_\infty) := \frac{C^n \setminus \emptyset}{(s)}$.

Denote by $\gamma$ the generator of the monodromy acting on the Stein completion $\tilde{M}_c$ (Theorem 4.6), and $\varphi : \tilde{M}_c \rightarrow \mathbb{C}^n$ the embedding making the following diagram commutative

\[
\begin{array}{ccc}
\tilde{M}_c & \xrightarrow{\varphi} & \mathbb{C}^n \\
\downarrow {\gamma} & & \downarrow {A} \\
\tilde{M}_c & \xrightarrow{\varphi} & \mathbb{C}^n.
\end{array}
\]

Consider the map $\varphi_i := r_i \varphi r_i^{-1}$ as an embedding from $\tilde{M}_c$ to $\mathbb{C}^n$ making the following diagram commutative

\[
\begin{array}{ccc}
\tilde{M}_c & \xrightarrow{\varphi_i} & \mathbb{C}^n \\
\downarrow {\gamma} & & \downarrow {A_i} \\
\tilde{M}_c & \xrightarrow{\varphi_i} & \mathbb{C}^n.
\end{array}
\]

We use the same letter $\varphi_i$ to denote the embedding $(M, J) \rightarrow \frac{C^n \setminus \emptyset}{(A_i)}$ associated with $\varphi_i$. Since $\varphi(\tilde{M}_c)$ is $s$-invariant, and $A_i$ converge to $s$, the sequence $\varphi_i | \tilde{M}_c$ converges to $\varphi | \tilde{M}_c$, giving an embedding $\tilde{M}_c \rightarrow (H, I_\infty)$. The limit manifold $(M, J_\infty) = \tilde{M}_c$ is of Vaisman type, because it is embedded to a diagonal Hopf manifold (Theorem 4.9).

The maps $\varphi_i$ do not converge to $\varphi_\infty$ smoothly, because the sequence $\{r_i^{-1}\}$ is not bounded. However, the sequence $\{(\varphi_i(\tilde{M}_c), A_i)\}$ $C^\infty$-converges to $(\varphi_\infty(\tilde{M}_c), A)$ as a sequence of pairs

\[
(\text{algebraic subvariety } Z \subset \mathbb{C}^n, \text{ an automorphism } \psi \in \text{Aut}(Z));
\]

hence the corresponding points in Teich also converge. This is what we are going to show.

Let $S \subset \mathcal{O}_{\mathbb{C}^n}$, $\dim S = m$, be a finite-dimensional space generating the ideal of $\varphi_1(\tilde{M}_c)$ (Theorem 5.5). By Step 1 of the proof of Theorem 5.5, we may assume that all elements of $S$ are polynomials of degree less than $d$. Denote by $V \subset \mathcal{O}_{\mathbb{C}^n}$ the space of polynomials of degree $\leq d$, and let $X \subset \text{Gr}_m(V)$ be a subset of the Grassmannian of $m$-dimensional planes in $V$ consisting of all subspaces $W \subset V$ which generate an ideal $J_W \subset \mathbb{C}[\mathbb{C}^n]$
in the polynomial ring such that \( \mathbb{C}[\mathbb{C}^n]/J_W \) is isomorphic to the algebraic cone \( \tilde{M}_c \). \(^1\)

The sequence \( \{\sigma_i := \varphi_i(\tilde{M}_c)\} \) corresponds to points in \( X \) converging to \( \sigma_\infty := \varphi_\infty(\tilde{M}_c) \). This gives the convergence of the submanifolds \( \varphi_i(\tilde{M}) \) to \( \varphi_\infty(\tilde{M}) \) in \( \mathbb{C}^n \setminus 0 \). Indeed, consider the “universal fibration” over \( X \), with the fiber over \( W \in X \) being the algebraic cone associated with the ideal \( J_W \subset \mathcal{O}_{\mathbb{C}^n} \) generated by \( W \). The associated open cone fibration has smooth fibers. Indeed, any open algebraic cone is the total space of a \( \mathbb{C}^* \)-bundle over a projective manifold.

To finish the proof, we need to prove that the manifolds \( (M,J_i) = \varphi_i(\tilde{M}) \) smoothly converge to \( (M,J_\infty) = \varphi(\tilde{M}) \). This would follow if we prove that the corresponding cones in \( \mathbb{C}^n \) converge smoothly in each annulus \( B_R \setminus B_r \) around 0 (we need to restrict to the annulus, because the cone itself is singular around zero, hence it makes no sense to speak of \( C^\infty \)-convergence unless we remove a neighbourhood of the origin). Then \( (M,J_i) \) and \( (M,J_\infty) \) are quotients of the respective cones by \( A_i \) and \( A \)-actions respectively, and \( A_i \) converges to \( A \) in \( \text{GL}(n,\mathbb{C}) \).

However, the cones \( \varphi_i(\tilde{M}_c) \) are smooth in each annulus, and they converge to \( \tilde{M}_c \) in \( C^0 \)-topology (or in the Hausdorff metric) by construction. For smooth families of compact manifolds, the \( C^\infty \)-convergence of their fibers is automatic. To finish the proof, we replace the cone fibration over \( X \) by the corresponding fibration of compact complex orbifolds, which also converges to the central fiber. The fibers of a locally trivial fibration of compact orbifolds converge to the central fiber with all derivatives by Ehresmann’s theorem.

Let \( P \rightarrow X \) be the fibration with projective fibers over \( X \), obtained by taking \( \mathbb{C}^* \)-quotients of the tautological open cone fibration \( U \rightarrow X \). The fibration \( P \rightarrow X \) is locally trivial, because it is smooth and all its fibers are isomorphic projective orbifolds. The fibers of \( U \) are total spaces of the \( \mathbb{C}^* \)-bundles associated with \( \mathcal{O}(1) \) over fibers of \( P \). Then \( U \rightarrow X \) is smoothly locally trivial.

To obtain the convergence of the corresponding LCK manifolds, we notice that \( \lim_i A_i = s \), hence \( \lim_i \tilde{M}_i/\langle A_i \rangle = \tilde{M}/\langle s \rangle = (M,J_\infty) \).

**Corollary 7.4:** Let \( (M,J) \) be a compact complex manifold, \( \dim_{\mathbb{C}} M \geq 3 \), admitting an LCK structure with potential, and \( J_\infty \) the Vaisman-type complex structure on \( M \) obtained as in Theorem 7.3. Then any Vaisman-type

---

\(^1\)Interpreting \( X \) as a piece of the relevant Hilbert scheme, we obtain that it is an algebraic subvariety in \( \text{Gr}_m(V) \); we are not going to prove or use this observation.
Lee form on \((M, J_\infty)\) can be realized as the Lee form of an LCK structure with potential on \((M, J)\).

**Proof:** Let \((M, I, \omega, \theta)\) be a Vaisman structure on \(M\), with \(\omega = d^c \theta + \theta \wedge \theta^c\), and \(I_i\) a sequence of complex structures on \(M\) converging to \(I\), such that all \((M, I_i)\) are isotopic to \((M, J)\) as complex manifolds. Then the sequence \(\omega_i = I_i dI_i^{-1} \theta + \theta \wedge I_i \theta\) converges to

\[
d^c \theta + \theta \wedge \theta^c = I dI^{-1} \theta + \theta \wedge I \theta.
\]

Since positivity is an open condition, the \((1,1)\)-form \(\omega_i\) is positive for \(i\) sufficiently big. Then \((M, I_i, \omega_i, \theta)\) is LCK with potential, and \(\theta\) its Lee form. However, \(I_i\) is mapped to \(J\) by an isotopy which preserves the cohomology class of \(\theta\), hence \(\theta\) is a Lee class on \((M, J)\).

### 8 The set of Lee classes

#### 8.1 Opposite Lee forms on LCK manifolds with potential

As another preliminary result, we need the following non-existence claim. For Vaisman manifolds, it was obtained by K. Tsukada ([Ts1]).

**Proposition 8.1:** Let \((M, \theta, \omega)\) and \((M, \theta_1, \omega_1)\) be two LCK structures on the same compact complex manifold. Suppose that \((M, \theta, \omega)\) is an LCK structure with potential. Then \(\theta + \theta_1\) cannot be cohomologous to 0.

**Proof.** Step 1: If \([\theta]\) is the Lee class for an LCK structure with potential on \(M\), then \(a[\theta]\) is also a Lee class for one, for any \(a > 1\). To see this, consider the expression \(\omega = d^c \theta + \theta \wedge \theta^c\) (Remark 4.5) corresponding to the Kähler potential \(\phi\) on the Kähler cover \((\tilde{M}, \tilde{\omega}) \to (M, \theta, \omega)\), with \(\pi^* \theta = -d \log \phi\). Then \(\varphi^a\) is also a Kähler potential on \(M\),

\[
d d \varphi^a = \varphi^{a-2}(a \cdot \varphi d d^c \varphi + a(a - 1) d \varphi \wedge d^c \varphi).
\]

Indeed, the first summand \(a \varphi^{a-1} d d^c \varphi\) is Hermitian, because \(d d^c \varphi\) is Hermitian, and the second summand \(a(a - 1) d \varphi \wedge d^c \varphi\) is positive. The function \(\varphi^a\) is automorphic, hence it defines an LCK structure with potential on \(M\), and the corresponding Lee form is \(-d \log(\varphi^a) = a \theta\).

Step 2: Let \(\omega, \omega_1\) be LCK forms, and \(\theta, \theta_1\) the corresponding Lee forms. Suppose that \(k \theta + l \theta_1 = 0\). Then

\[
d(\omega^k \wedge \omega_1^l) = d \omega^k \wedge \omega_1^l + \omega^k \wedge d \omega_1^l = k \theta \wedge \omega^k \wedge \omega_1^l + l \theta_1 \wedge \omega^k \wedge \omega_1^l = 0.
\]
This computation can be interpreted as follows. Let $L$ be the weight bundle for $(M,\omega,\theta)$ and $L_1$ the weight bundle for $(M,\omega_1,\theta_1)$. Recall (Remark 2.8 (ii)) that $\omega$, $\omega_1$ are viewed as closed $L$- and $L_1$-valued forms. Then $\omega^k$ is a closed $L^\otimes k$-valued form, $\omega_1^l$ is a closed $L_1^\otimes l$-form, and $\omega^k \wedge \omega_1^l$ is a closed form with coefficients in the flat bundle $L^\otimes k \otimes L_1^\otimes l$, which is trivial.

Return now to the situation described in the assumptions of Proposition 8.1. Let $n = \dim_{\mathbb{C}} M$. Using Step 1, we replace the LCK structure $(\omega,\theta)$ by another LCK structure with potential in such a way that $\theta$ is replaced by $(n-1)\theta$. Then $(n-1)\theta_1 = -\theta$, and the volume form $\omega \wedge \omega_1^{n-1}$ is closed. However, $\omega$ is actually an exact $L$-valued form, because $\omega = d\theta(\theta^c)$, hence $\omega \wedge \omega_1^{n-1}$ is an exact $L \otimes L_1^\otimes(n-1)$-valued form. However, $L \otimes L_1^\otimes(n-1)$ is a trivial local system, which implies that $\omega \wedge \omega_1^{n-1}$ is exact.

We verify this with an explicit computation:

$$
\begin{align*}
    d(\theta^c \wedge \omega_1^{n-1}) &= d\theta^c \wedge \omega_1^{n-1} - \theta^c \wedge d\omega_1^{n-1} \\
    &= (\omega - \theta \wedge \theta^c) \wedge \omega_1^{n-1} - (n-1)\theta_1 \wedge \theta^c \wedge \omega_1^{n-1} \\
    &= \omega \wedge \omega_1^{n-1} - (\theta \wedge \theta^c + (n-1)\theta_1 \wedge \theta^c) \wedge \omega_1^{n-1} = \omega \wedge \omega_1^{n-1}.
\end{align*}
$$

We have shown that the positive volume form $\omega \wedge \omega_1^{n-1}$ on $M$ is exact, which is impossible.  

\section{The set of Lee classes on Vaisman manifolds}

To proceed, we need the following preliminary result, which might be of separate interest.

**Proposition 8.2:** Let $(M,\theta,\omega)$ be an LCK structure on a compact Vaisman manifold. Then $\theta$ is cohomologous to a Lee form of a Vaisman structure.

**Proof:** Let $X$ be the Lee field of a Vaisman structure $(M,\omega^V,\theta^V)$ on $M$, and $G$ the closure of the group generated by exponents of $X$ and $I(X)$. Since $X$ and $IX$ are Killing and commute, $G$ is a compact commutative Lie group, hence it is isomorphic to a compact torus. This group acts on $M$ by holomorphic isometries with respect to the Vaisman metric.

Averaging $\theta$ with the $G$-action, we obtain a $G$-invariant 1-form $\theta$, corresponding to another LCK structure in the same conformal class. Without restricting the generality, we may assume from the beginning that the form $\theta$ is $G$-invariant.

Now, the equation $d\omega = \omega \wedge \theta$ is invariant under the action of $G$, because $\theta$ is $G$-invariant; in other words, $d(g^*\omega) = g^*\omega \wedge \theta$, for all $g \in G$. This implies
that \( \omega \) averaged with \( G \) gives a form \( \omega^G \) which satisfies \( d(\omega^G) = \omega^G \wedge \theta \). We have constructed a \( G \)-invariant LCK structure \((M, \omega^G, \theta)\).

After lifting it to the Kähler cover \( \tilde{M} \) of \( (M, \omega^G, \theta) \), the group \( G \) becomes non-compact. Indeed, if it remained compact, it would act by isometries on the universal cover \( \tilde{M}_U \) of \( M \) as well, hence the action of \( G \) on \( M \) is lifted to the action of \( \tilde{G} \) on \( \tilde{M}_U \). This is impossible, however, because the lift of \( G \) to the Kähler cover associated with \((M, \omega^V, \theta^V)\) acts by non-trivial homotheties, hence \( G \) is lifted to an infinite cover \( \tilde{G} \to G \) effectively acting on \( \tilde{M}_U \).

We obtain that \( G \) acts by non-isometric homotheties on the Kähler cover associated with \((M, \omega^G, \theta)\). By Theorem 3.2, \((M, \omega^G, \theta)\) is actually Vaisman.

The following result was obtained by K. Tsukada ([Ts1]). We provide a new, simpler proof.

**Theorem 8.3:** Let \( M \) be a compact Vaisman manifold, and \( H^1(M) = H^{1,0}(M) \oplus H^{1,0}(M) \oplus (\theta) \) be the decomposition established in Theorem 6.1. Consider a 1-form \( \mu \in H^1(M)^* \) vanishing on \( H^{1,0}(M) \oplus H^{1,0}(M) \subset H^1(M) \) and satisfying \( \mu(\theta) > 0 \). Then a class \( \alpha \in H^1(M, \mathbb{R}) \) is a Lee class for some LCK structure if and only if \( \mu(x) > 0 \).

**Proof. Step 1:** We start by proving that any \( \alpha \in H^1(M, \mathbb{R}) \) satisfying \( \mu(x) > 0 \) can be realized as a Lee class.

From Remark 4.5, we have \( \omega = d^{c}\theta + \theta \wedge \imath \theta \). By Theorem 3.1 (ii), the form \( \omega_0 := d^{c}\theta \) is semi-positive: it vanishes on the canonical foliation \( \Sigma \) and is strictly positive in the transversal directions. Let \( u \in H^{1,0}(M) \oplus H^{1,0}(M) \). Then

\[
d^{c}(\theta + u) + (\theta + u) \wedge (\theta^c + u^c) = \omega_0 + (\theta + u) \wedge (\theta^c + u^c)
\]

is the sum of two semi-positive forms. Indeed, since \( u \) is \( d^{c}\)-closed, \( d^{c}(\theta+u) = \omega_0 \) is semi-positive; the form \( (\theta + u) \wedge (\theta^c + u^c) \) is semi-positive of rank 1 by definition.

By Proposition 3.7, \( u \) is basic. Since \( \theta + u \) is the sum of \( \theta \) and a basic form, the restriction of \( (\theta + u) \wedge (\theta^c + u^c) \) to \( \Sigma \) satisfies

\[
(\theta + u) \wedge (\theta^c + u^c) \big|_{\Sigma} = \theta \wedge \theta^c \big|_{\Sigma}.
\]

The sum \( \omega_0 + (\theta + u) \wedge (\theta^c + u^c) \) is strictly positive on all tangent vectors
$x \notin \Sigma$ because $\omega_0$ is positive on these vectors,\(^1\) and positive on $x \in \Sigma$ because $\theta \wedge \theta^c \big|_{\Sigma}$ is positive on such $x$.

**Step 2:** It remains to show that none of the classes $\alpha$ with $\mu(\alpha) \leq 0$ can be realized as a Lee class of an LCK structure. By Proposition 8.2, any Lee class on $M$ is the Lee class of a Vaisman metric. If $\mu(\alpha) = 0$, we can represent $\alpha$ by a $d, d^c$-closed form $\alpha_0$. This is impossible by Lemma 2.11.

**Step 3:** It remains to show that there are no LCK classes which satisfy $\mu(\alpha) < 0$. Suppose that such a class exists; by Proposition 8.2, it is the Lee class of a Vaisman manifold, hence it has an LCK potential. This is impossible, because two Lee classes for LCK structures with potential cannot sum to zero, by Proposition 8.1. ■

### 8.3 The set of Lee classes on LCK manifolds with potential

Now we can prove the main result of this paper.

**Theorem 8.4:** Let $(M, \theta, \omega)$ be a compact LCK manifold with potential, $\dim_C M \geq 3$, and $\mu : H^1(M, \mathbb{R}) \to \mathbb{R}$ a non-zero linear map vanishing on the space $H^{1,0}(M) \oplus H^{1,0}(M)$ which has codimension 1 by Theorem 6.1. Assume that $\mu(\theta) > 0$. Then $\xi \in H^1(M, \mathbb{R})$ is the Lee class of an LCK structure with potential on $M$ if and only if $\mu(\xi) > 0$.

**Proof:** Let $(M, I_\infty)$ be a Vaisman manifold, and $\{I_k\}$ the sequence of complex structures converging to $I_\infty$, such that all manifolds $(M, I_k)$ are isomorphic to $(M, I)$ (Theorem 7.3). Given an LCK metric with potential, choose the conformal gauge such that $\omega_\infty := d^c \theta_\infty + \theta_\infty \wedge \theta^c_\infty$ on $(M, I_\infty)$ (Remark 4.5). Then the form $I_k dI_k^{-1} \theta_\infty + \theta_\infty \wedge I_k (\theta_\infty)$ remains strictly positive for almost all manifolds $(M, I_k)$, because $\lim_k I_k = I_\infty$, and positivity is an open condition. This implies that $\theta_\infty$ is a Lee form on $(M, I_k)$, for $k$ sufficiently big. By Theorem 8.3 the set $\mathcal{L}$ of Lee classes on $(M, I_\infty)$ contains the half-space $\{u \in H^1(M, \mathbb{R}) \mid \mu_0(u) > 0\}$ for some linear map $\mu_0 : H^1(M, \mathbb{R}) \to \mathbb{R}$. By Proposition 8.1, $\mathcal{L}$ cannot be bigger than a closed half-space. However, $\mathcal{L}$ is open, because the condition “$d^c \theta + \theta \wedge \theta^c$ is Hermitian” is open in $\theta$, hence $\mathcal{L}$ is an open half-space. It remains only to show that $\mu_0$ is proportional to $\mu$. This would follow if we prove that

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\(^1\)When we say “a positive (1,1)-form $\alpha$ is positive on a vector $v$”, we mean that $\sqrt{-1} \alpha(v, I(v)) > 0$; a form is Hermitian if it is positive on all non-zero vectors.
ker $\mu = \ker \mu_0$. The space ker $\mu_0$ is the set of all classes $\alpha \in H^1(M, \mathbb{R})$ such that neither $\alpha$ nor $-\alpha$ are Lee classes, and ker $\mu$ are classes represented by $d, d^c$-closed forms. By Lemma 2.11, a Lee class of an LCK manifold cannot be represented by a $d^c$-closed form, which gives ker $\mu = \ker \mu_0$.

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