ON PSEUDOCOMPACT INVERSE PRIMITIVE (SEMI)TOPOLOGICAL SEMIGROUPS

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ABSTRACT. We study the structure of inverse primitive pseudocompact semitopological and topological semigroups. We find conditions when the maximal subgroup of an inverse primitive pseudocompact semitopological semigroup $S$ is a closed subset of $S$ and describe the topological structure of such semiregular semitopological semigroups. Later we describe the structure of pseudocompact topological Brandt $\lambda^0$-extensions of topological semigroups and semiregular (quasi-regular) primitive inverse topological semigroups. In particular we show that inversion in a quasi-regular primitive inverse pseudocompact topological semigroup is continuous. Also an analogue of Comfort–Ross Theorem proved for such semigroups: a Tychonoff product of an arbitrary family of primitive inverse semiregular pseudocompact semitopological semigroups with closed maximal subgroups is pseudocompact. We describe the structure of the Stone-Čech compactification of a Hausdorff primitive inverse countably compact semitopological semigroup $S$ such that every maximal subgroup of $S$ is a topological group.

1. Introduction and preliminaries

Further we shall follow the terminology of [8, 9, 13, 24, 30]. By $\mathbb{N}$ we shall denote the set of all positive integers.

A semigroup is a non-empty set with a binary associative operation. A semigroup $S$ is called inverse if for any $x \in S$ there exists a unique $y \in S$ such that $x \cdot y \cdot x = x$ and $y \cdot x \cdot y = y$. Such the element $y$ in $S$ is called inverse to $x$ and is denoted by $x^{-1}$. The map assigning to each element $x$ of an inverse semigroup $S$ its inverse $x^{-1}$ is called the inversion.

For a semigroup $S$ by $E(S)$ we denote the subset of idempotents of $S$, and by $S^1$ (resp., $S^0$) we denote the semigroup $S$ the adjoined unit (resp., zero) (see [9] Section 1.1). Also if a semigroup $S$ has zero $0_S$, then for any $A \subseteq S$ we denote $A^* = A \setminus \{0_S\}$.

For a semilattice $E$ the semilattice operation on $E$ determines the partial order $\leq$ on $E$:

$$e \leq f \iff ef = fe = e.$$  

This order is called natural. An element $e$ of a partially ordered set $X$ is called minimal if $f \leq e$ implies $f = e$ for $f \in X$. An idempotent $e$ of a semigroup $S$ without zero (with zero) is called primitive if $e$ is a minimal element in $E(S)$ (in $(E(S))^*$).

Let $S$ be a semigroup with zero and $\lambda \geq 1$ be a cardinal. On the set $B_\lambda(S) = (\lambda \times S \times \lambda) \cup \{0\}$ we define a semigroup operation as follows

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in \lambda$ and $a, b \in S$. If $S$ is a monoid, then the semigroup $B_\lambda(S)$ is called the Brandt $\lambda$-extension of the semigroup $S$ [13]. Obviously, $J = \{0\} \cup \{(\alpha, \Theta, \beta): \Theta$ is the zero of $S\}$ is an ideal of $B_\lambda(S)$. We put $B_\lambda^0(S) = B_\lambda(S)/J$ and we shall call $B_\lambda^0(S)$ the Brandt $\lambda^0$-extension of the semigroup $S$ with zero [16]. Further, if $A \subseteq S$ then we shall
denote \( A_{\alpha,\beta} = \{ (\alpha, s, \beta) : s \in A \} \) if \( A \) does not contain zero, and \( A_{\alpha,\beta} = \{ (\alpha, s, \beta) : s \in A \setminus \{0\} \} \cup \{0\} \) if \( 0 \in A \), for \( \alpha, \beta \in \lambda \). If \( \mathcal{I} \) is a trivial semigroup (i.e., \( \mathcal{I} \) contains only one element), then by \( \mathcal{I}^0 \) we denote the semigroup \( \mathcal{I} \) with the adjoined zero. Obviously, for any \( \lambda \geq 2 \) the Brandt \( \lambda^0 \)-extension of the semigroup \( \mathcal{T}^0 \) is isomorphic to the semigroup of \( \lambda \times \lambda \)-matrix units and any Brandt \( \lambda^0 \)-extension of a semigroup with zero contains the semigroup of \( \lambda \times \lambda \)-matrix units. Further by \( B_{\lambda} \) we shall denote the semigroup of \( \lambda \times \lambda \)-matrix units and by \( B_{\lambda}(1) \) the subsemigroup of \( \lambda \times \lambda \)-matrix units of the Brandt \( \lambda^0 \)-extension of a monoid \( S \) with zero.

A semigroup \( S \) with zero is called 0-simple if \( \{0\} \) and \( S \) are its only ideals and \( S^2 \neq \{0\} \), and completely 0-simple if it is 0-simple and has a primitive idempotent [9]. A completely 0-simple inverse semigroup is called a Brandt semigroup [24]. By Theorem II.3.5 [24], a semigroup \( S \) is a Brandt semigroup if and only if \( S \) is isomorphic to a Brandt \( \lambda \)-extension \( B_{\lambda}(G) \) of a group \( G \).

Let \( \{ S_\iota : \iota \in \mathcal{I} \} \) be a disjoint family of semigroups with zero such that \( 0_\iota \) is zero in \( S_\iota \) for any \( \iota \in \mathcal{I} \). We put \( S = \{0\} \cup \bigcup \{ S_\iota^* : \iota \in \mathcal{I} \} \), where \( 0 \notin \bigcup \{ S_\iota^* : \iota \in \mathcal{I} \} \), and define a semigroup operation “\( \cdot \)” on \( S \) in the following way

\[
s \cdot t = \begin{cases} st, & \text{if } st \in S_\iota^* \text{ for some } \iota \in \mathcal{I}; \\ 0, & \text{otherwise}. \end{cases}
\]

The semigroup \( S \) with the operation “\( \cdot \)” is called an orthogonal sum of the semigroups \( \{ S_\iota : \iota \in \mathcal{I} \} \) and in this case we shall write \( S = \sum_{\iota \in \mathcal{I}} S_\iota \).

A non-trivial inverse semigroup is called a primitive inverse semigroup if all its non-zero idempotents are primitive [24]. A semigroup \( S \) is a primitive inverse semigroup if and only if \( S \) is an orthogonal sum of Brandt semigroups [24, Theorem II.4.3]. We shall call a Brandt subsemigroup \( T \) of a primitive inverse semigroup \( S \) maximal if every Brandt subsemigroup of \( S \) which contains \( T \), coincides with \( T \).

In this paper all topological spaces are Hausdorff. If \( Y \) is a subspace of a topological space \( X \) and \( A \subseteq Y \), then by \( \text{cl}_Y(A) \) and \( \text{int}_Y(A) \) we denote the topological closure and interior of \( A \) in \( Y \), respectively.

A subset \( A \) of a topological space \( X \) is called regular open if \( \text{int}_X(\text{cl}_X(A)) = A \).

We recall that a topological space \( X \) is said to be

- semiregular if \( X \) has a base consisting of regular open subsets;
- quasiregular if for any non-empty open set \( U \subset X \) there exists a non-empty open set \( V \subset U \) such that \( \text{cl}_X(V) \subseteq U \);
- compact if each open cover of \( X \) has a finite subcover;
- sequentially compact if each sequence \( \{x_\iota\}_{\iota \in \mathbb{N}} \) of \( X \) has a convergent subsequence in \( X \);
- countably compact if each open countable cover of \( X \) has a finite subcover;
- countably compact at a subset \( A \subseteq X \) if every infinite subset \( B \subseteq A \) has an accumulation point \( x \) in \( X \);
- countably pracompact if there exists a dense subset \( A \) in \( X \) such that \( X \) is countably compact at \( A \);
- pseudocompact if each locally finite open cover of \( X \) is finite;
- \( k \)-space if a subset \( F \subset X \) is closed in \( X \) if and only if \( F \cap K \) is closed in \( K \) for every compact subspace \( K \subseteq X \).

According to Theorem 3.10.22 of [13], a Tychonoff topological space \( X \) is pseudocompact if and only if each continuous real-valued function on \( X \) is bounded. Also, a Hausdorff topological space \( X \) is pseudocompact if and only if every locally finite family of non-empty open subsets of \( X \) is finite. Every compact space and every sequentially compact space are countably compact, every countably compact space is countably pracompact, and every countably pracompact space is pseudocompact (see [24]).

We recall that the Stone-Čech compactification of a Tychonoff space \( X \) is a compact Hausdorff space \( \beta X \) containing \( X \) as a dense subspace so that each continuous map \( f : X \to Y \) to a compact Hausdorff space \( Y \) extends to a continuous map \( \overline{f} : \beta X \to Y \) [13].

A (semi)topological semigroup is a Hausdorff topological space with a (separately) continuous semigroup operation. A topological semigroup which is an inverse semigroup is called an inverse topological
**semigroup.** A **topological inverse semigroup** is an inverse topological semigroup with continuous inversion. We observe that the inversion on a topological inverse semigroup is a homeomorphism (see [12 Proposition II.1]). A Hausdorff topology $\tau$ on a (inverse) semigroup $S$ is called **(inverse) semigroup** if $(S, \tau)$ is a topological (inverse) semigroup. A **paratopological (semitopological) group** is a Hausdorff topological space with a jointly (separately) continuous group operation. A paratopological group with continuous inversion is a **topological group**.

Let $\mathcal{STSG}_0$ be a class of semitopological semigroups.

**Definition 1.1** ([15]). Let $\lambda \geq 1$ be a cardinal and $(S, \tau) \in \mathcal{STSG}_0$ be a semitopological monoid with zero. Let $\tau_B$ be a topology on $B_\lambda(S)$ such that

a) $(B_\lambda(S), \tau_B) \in \mathcal{STSG}_0$; and

b) for some $\alpha \in \lambda$ the topological subspace $(S_{\alpha,\alpha}, \tau_B|_{S_{\alpha,\alpha}})$ is naturally homeomorphic to $(S, \tau)$.

Then $(B_\lambda(S), \tau_B)$ is called a **topological Brandt $\lambda$-extension of $(S, \tau)$ in $\mathcal{STSG}_0$**.

**Definition 1.2** ([16]). Let $\lambda \geq 1$ be a cardinal and $(S, \tau) \in \mathcal{STSG}_0$. Let $\tau_B$ be a topology on $B_\lambda^0(S)$ such that

a) $(B_\lambda^0(S), \tau_B) \in \mathcal{STSG}_0$;

b) the topological subspace $(S_{\alpha,\alpha}, \tau_B|_{S_{\alpha,\alpha}})$ is naturally homeomorphic to $(S, \tau)$ for some $\alpha \in \lambda$.

Then $(B_\lambda^0(S), \tau_B)$ is called a **topological Brandt $\lambda^0$-extension of $(S, \tau)$ in $\mathcal{STSG}_0$**.

Later, if $\mathcal{STSG}_0$ coincides with the class of all semitopological semigroups we shall say that $(B_\lambda^0(S), \tau_B)$ (resp., $(B_\lambda(S), \tau_B)$) is called a **topological Brandt $\lambda^0$-extension** (resp., a **topological Brandt $\lambda$-extension**) of $(S, \tau)$.

Algebraic properties of Brandt $\lambda^0$-extensions of monoids with zero, non-trivial homomorphisms between them, and a category whose objects are ingredients of the construction of such extensions were described in [21]. Also, in [18] and [21] a category whose objects are ingredients in the constructions of finite (resp., compact, countably compact) topological Brandt $\lambda^0$-extensions of topological monoids with zeros were described.

Gutik and Repovš proved that any 0-simple countably compact topological inverse semigroup is topologically isomorphic to a topological Brandt $\lambda$-extension $B_\lambda(H)$ of a countably compact topological group $H$ in the class of all topological inverse semigroups for some finite cardinal $\lambda \geq 1$ [20]. Also, every 0-simple pseudocompact topological inverse semigroup is topologically isomorphic to a topological Brandt $\lambda$-extension $B_\lambda(H)$ of a pseudocompact topological group $H$ in the class of all topological inverse semigroups for some finite cardinal $\lambda \geq 1$ [19]. Next Gutik and Repovš showed in [20] that the Stone-Čech compactification $\beta(T)$ of a 0-simple countably compact topological inverse semigroup $T$ has a natural structure of a 0-simple compact topological inverse semigroup. It was proved in [19] that the same is true for 0-simple pseudocompact topological inverse semigroups.

In the paper [7] the structure of compact and countably compact primitive topological inverse semigroups was described and was showed that any countably compact primitive topological inverse semigroup embeds into a compact primitive topological inverse semigroup.

Comfort and Ross in [10] proved that a Tychonoff product of an arbitrary non-empty family of pseudocompact topological groups is a pseudocompact topological group. Also, they proved there that the Stone-Čech compactification of a pseudocompact topological group has a natural structure of a compact topological group. Ravsky in [28] generalized Comfort–Ross Theorem and proved that a Tychonoff product of an arbitrary non-empty family of pseudocompact paratopological groups is pseudocompact.

In the paper [17] it is described the structure of pseudocompact primitive topological inverse semigroups and it is shown that the Tychonoff product of an arbitrary non-empty family of pseudocompact primitive topological inverse semigroups is pseudocompact. Also, there is proved that the Stone-Čech compactification of a pseudocompact primitive topological inverse semigroup has a natural structure of a compact primitive topological inverse semigroup.
In this paper we study the structure of inverse primitive pseudocompact semitopological and topological semigroups. We find conditions when a maximal subgroup of an inverse primitive pseudocompact semitopological semigroup $S$ is a closed subset of $S$ and describe the topological structure of such semiregular semigroup. Later we describe structure of pseudocompact topological Brandt $\lambda^0$-extensions of topological semigroups and semiregular (quasi-regular) primitive inverse topological semigroups. In particular we show that the inversion in a quasi-regular primitive inverse pseudocompact topological semigroup is continuous. Also an analogue of Comfort–Ross Theorem is proved for such semigroups: the Tychonoff product of an arbitrary non-empty family of primitive inverse semiregular pseudocompact semitopological semigroups with closed maximal subgroups is a pseudocompact space. We describe the structure of the Stone–Čech compactification of a Hausdorff primitive inverse countably compact semitopological semigroup $S$ such that every maximal subgroup of $S$ is a topological group.

2. PSEUDOCOMPACT TOPOLOGICAL BRANDT $\lambda^0$-EXTENSIONS OF TOPOLOGICAL SEMIGROUPS AND PRIMITIVE INVERSE SEMITOPOLOGICAL SEMIGROUPS

Given a topological space $(X, \tau)$ Stone \cite{31} and Katětov \cite{22} consider the topology $\tau_r$ on $X$ generated by the base consisting of all regular open sets of the space $(X, \tau)$. This topology is called the regularization of the topology $\tau$. If $(X, \tau)$ is a paratopological group then $(X, \tau_r)$ is a $T_3$ paratopological group \cite[Ex. 1.9]{25}, \cite[p. 31]{26}, and \cite[p. 28]{26}.

Lemma 2.1 (\cite[Theorem 1.7]{3}). Each paratopological group that is a dense $G_\delta$-subset of a regular pseudocompact space is a topological group.

We recall that a group $G$ endowed with a topology is left (resp. right) ($\omega$-)precompact, if for each neighborhood $U$ of the unit of $G$ there exists a (countable) finite subset $F$ of $G$ such that $FU = G$ (resp. $UF = G$). It is easy to check (see, for instance, \cite[Proposition 3.1]{25} or \cite[Proposition 2.1]{25}) that a paratopological group $G$ is left precompact if and only if $G$ is right precompact, so we shall call left precompact paratopological groups as precompact. Moreover, it is well-known \cite{1} that a Hausdorff topological group $G$ is precompact if and only if $G$ is a subgroup of a compact topological group. Theorem 1 from \cite{5} implies the following

Lemma 2.2. A Hausdorff topological group $G$ is precompact if and only if for any neighborhood $W$ of the unit of the group $G$ there exists a finite set $F \subset G$ such that $G = FWF$.

Lemma 2.3. Let $S$ be a Hausdorff left topological semigroup, $0$ be a right zero of the semigroup $S$ and $G = S \setminus \{0\}$ be a subgroup of the semigroup $S$. Then $0$ is an isolated point of the semigroup $S$ provided one of the following conditions holds:

(1) the group $G$ is left precompact;
(2) the group $G$ is a pseudocompact paratopological group;
(3) the group $G$ is left $\omega$-precompact pseudocompact;
(4) $S$ is a pseudocompact topological semigroup;
(5) $S$ is a topological semigroup and for each neighborhood $U \subset G$ of the unit of the group $G$ there exists a finite subset $F$ of the group $G$ such that $G = FU^{-1}U$.

Proof. Assume the converse. Put $\mathcal{F} = \{U \cap G: U \subset S$ is a neighbourhood of the point $0\}$. Since $0$ is a non-isolated point of the semigroup $S$, the family $\mathcal{F}$ is a filter. Let $x \in G$ be an arbitrary element and $U$ be an arbitrary member of the filter $\mathcal{F}$. Since $x0 = 0$ and left shifts on the semigroup $S$ are continuous, there exists a member $V$ of the filter $\mathcal{F}$ such that $xV \subset U$. Then $V \subset x^{-1}U$, so $x^{-1}U \in \mathcal{F}$. Since $S$ is Hausdorff, there exists a neighborhood $W \subset G$ of the unit such that $G \setminus W \in \mathcal{F}$.

Now we separately consider the cases.

(1) Since the group $G$ is left precompact, there exists a finite subset $F$ of the group $G$ such that $FW = G$. But then

$$\emptyset = G \setminus \bigcup_{x \in F} xW = \bigcap_{x \in F} x(G \setminus W) \in \mathcal{F},$$

where $\mathcal{F}$ is a filter generated by all open sets of $G$ containing $0$.
a contradiction.

(2) Since the regularization \(G_r\) of the group \(G\) is a pseudocompact \(T_3\) (and, hence, a regular) paratopological group, \(G_r\) is a topological group by Lemma \([2.2]\). Therefore \(G_r\) is precompact. Thus there exists a finite subset \(F\) of the group \(G\) such that \(F \cdot \text{cl}_G(W) = G\). But then
\[
\emptyset = G \setminus \bigcup_{x \in F} x \cdot \text{cl}_G(W) = \bigcap_{x \in F} x(G \setminus \text{cl}_G(W)) \subseteq F,
\]
a contradiction.

(3) Since the group \(G\) is left \(\omega\)-precompact, there exists a countable subset \(C = \{c_n : n \in \mathbb{N}\}\) of the group \(G\) such that \(CW = G\). For each positive integer \(n\) put \(C_n = \{c_i : 1 \leq i \leq n\}\) and \(V_n = G \setminus C_n W\). Since the family \(F\) is a filter we have that \(V_n \in F\). Since 0 is a non-isolated point of the semigroup \(S\), its neighbourhood \(\text{int}_G(V_n)\) is a non-empty open subset of the space \(G\). Since the space \(G\) is pseudocompact, there exists a point \(x \in \bigcap_{n \in \mathbb{N}} \text{cl}_G(\text{int}_G(V_n))\). Since \(G = CW\) we conclude that there exists a positive integer \(n\) such that \(x \in C_n W\). But
\[
c_n W \cap \text{cl}_G(\text{int}_G(V_n)) \subseteq c_n W \cap \text{cl}_G(V_n) = c_n W \cap \text{cl}_G(G \setminus C_n W) = c_n W \cap (G \setminus C_n W) = \emptyset,
\]
a contradiction.

(4) At first we suppose that the space of the semigroup \(S\) is regular. Lemma \([2.1]\) implies that \(G\) is a topological group. If the group \(G\) is left precompact then 0 is an isolated point of the semigroup \(S\) by Case (1). So we assume the group \(G\) is not left precompact. By Lemma \([2.2]\) there exists a neighbourhood \(W_0 \subset G\) of the unit such that \(G \neq F_0 W_0 F_0\) for each finite subset \(F_0\) of the group \(G\). The multiplication on the semigroup \(S\) is continuous. Hence there exists a member \(V_1\) of the filter \(F\) such that \(V_2 \subset G \setminus W\). Moreover, there exist a symmetric open neighbourhood \(W_1\) of the unit and a member \(V_2\) of the filter \(F\) such that \(W_1^5 V_2 \subset V_1\) and \(W_1^4 \subset W_0\). Let \(C\) be a maximal subset of the set \(G \setminus V_2\) such that \(W_1^2 c \cap W_1^2 c' = \emptyset\) for distinct elements \(c, c'\) of the set \(C\). If \(z\) is an arbitrary element of the set \(G \setminus V_2\) then \(W_1^2 c \cap W_1^2 z = \emptyset\) for an element \(c\) of the set \(C\). Hence \(G \setminus V_2 \subset W_1^2 C\). Put \(F = \{c \in C : W_1 c \cap V_2 = \emptyset\}\). Then we have that \(C \setminus F \subset W_1 V_2\) and hence \(G \setminus V_2 \subset W_1^2 C \subset W_1^4 F\). Then we get that \(G \setminus V_1 \subset G \setminus V_2 \subset W_1^4 F \cup W_1 V_2\) and hence \(G \setminus V_1 \subset W_1^4 F\), because \(W_1 V_2 \subset V_1\). Since \(e \notin G \setminus W \supset V_1^2 \supset (G \setminus W_1^4 F)^2\), we see that \(x(G \setminus W_1^4 F) \neq e\) for each element \(x \in G \setminus W_1^4 F\). Then we have that \((G \setminus W_1^4 F)^{-1} \subset W_1^4 F\) and hence \(G \subset W_1^4 F \cup F^{-1} W_1^4\).

Since \(W_1^4 \subset W_0\) we conclude that the set \(F\) is infinite. Let \(C'\) be an arbitrary countable infinite subset of the set \(F\). Since the space \(S\) is pseudocompact we have that there exists a point \(x_0 \in S\) such that each neighbourhood \(V'\) of the point \(x_0\) intersects infinitely many members of the family \(\{W_1 c : c \in C'\}\) of the open subsets of the space \(S\). Clearly, \(x_0 \neq 0\). Then \(x_0 \in G\). Put \(V' = W_1 x_0\). Then there exist distinct elements \(c, c'\) of the set \(C'\) such that \(W_1 c \cap W_1 x_0 = \emptyset\) and \(W_1 c \cap W_1 x_0 = \emptyset\). This implies \(x_0 \in W_1^2 c \cap W_1^2 c' \neq \emptyset\), a contradiction.

Now we consider the case when the space of the semigroup \(S\) is not necessarily regular. We claim that the regularization \(S_r\) of the semigroup \(S\) is a regular topological semigroup.

Indeed, let \(U = \text{int}_S(\text{cl}_S(U))\) be an arbitrary regular open subset of the space \(S\) and \(x \in U\) be an arbitrary point. If \(x \neq 0\) then there exists an open neighbourhood \(W \subset G\) of the unit such that \(x W^2 \subset U\). Then \(x \in x W^2 \subset x W \cap \text{cl}_S(W) \subset \text{cl}_S(U)\). Since translations by elements of the group \(G\) are homeomorphisms of the space, the set \(x W \cap \text{cl}_S(W)\) is open, and hence
\[
x \in x W \subset \text{cl}_S(x W) \subset x W \cap \text{cl}_S(W) \subset \text{int}_S(\text{cl}_S(U))\).
\]
If \(x = 0\) then there exist an open neighbourhood \(W \subset G\) of the unit and an open neighbourhood \(V \subset G\) of \(x\) such that \(W V \subset U\). Then \(x \in V \subset W V \subset W \cap \text{cl}_S(V) \subset \text{cl}_S(U)\). We have that \(x \in V \subset \text{int}_S(\text{cl}_S(U))\). Let \(x \neq y \in \text{cl}_S(V)\) be an arbitrary point. Then \(W y \subset \text{cl}_S(V)\) is an open neighbourhood of \(y\). Hence \(y \in \text{int}_S(\text{cl}_S(U))\). Therefore the space of the \(S_r\) is regular.

Now we show that the multiplication on the semigroup \(S_r\) is continuous. Indeed, let \(x, y \in S\) be arbitrary points and \(O_{xy} = \text{int}_S(\text{cl}_S(O_{xy})) \ni xy\) be an arbitrary regular open subset of the space \(S\). There exist open subsets \(O_x \subset x, O_y \ni y\) of the semigroup \(S\) such that \(O_x O_y \subset O_{xy}\). Since the multiplication on the semigroup \(S\) is continuous, \(\text{cl}_S(O_x) \cdot \text{cl}_S(O_y) \subset \text{cl}_S(O_{xy})\). Let \(x' \in \text{cl}_S(O_x)\),
Remark 2.4. Authors do not know, if a counterpart of Lemma 2.3 holds when the group $G$ is a countably compact semitopological group.

Proposition 2.5. Let $S$ be a Hausdorff semitopological semigroup such that $S$ is an orthogonal sum of the family $\{B^0(\lambda_i) (S_i) : i \in \mathcal{F}\}$ of topological Brandt $\lambda^0$-extensions of semitopological monoids with zeros. Then for every non-zero element $(\alpha_i, g_i, \beta_i) \in (S_i)_{\alpha_i, \beta_i} \subseteq B^0(\lambda_i) (S_i) \subseteq S$ there exists an open neighbourhood $U_{(\alpha, g, \beta)}$ of $(\alpha, g, \beta)$ in $S$ such that $U_{(\alpha, g, \beta)} \subseteq (S_i)_{\alpha, \beta}$ and hence every subset $(S_i)_{\alpha, \beta}$ is an open subset of $S$.

Proof. Suppose the contrary that $U_{(\alpha, g, \beta)} \not\subset (S_i)_{\alpha, \beta}$ for every open neighbourhood $U_{(\alpha, g, \beta)}$ of $(\alpha, g, \beta)$ in $S$. Hausdorffness of $S$ implies that there exists an open neighbourhood $V_{(\alpha, g, \beta)}$ of $(\alpha, g, \beta)$ in $S$ such that $0 \notin V_{(\alpha, g, \beta)}$. By separate continuity of the semigroup operation in $S$ we have that there exists an open neighbourhood $W_{(\alpha, g, \beta)}$ of $(\alpha, g, \beta)$ in $S$ such that

$$W_{(\alpha, g, \beta)} \cdot (\beta, e, \alpha) \subseteq V_{(\alpha, g, \beta)} \quad \text{and} \quad (\alpha, e, \alpha) \cdot W_{(\alpha, g, \beta)} \subseteq V_{(\alpha, g, \beta)}.$$

Then condition $W_{(\alpha, g, \beta)} \not\subset (S_i)_{\alpha, \beta}$ implies that either $W_{(\alpha, g, \beta)} \cdot (\beta, e, \alpha) \not\subset 0$ or $(\alpha, e, \alpha) \cdot W_{(\alpha, g, \beta)} \not\subset 0$, a contradiction. The obtained contradiction implies the statement of the proposition.

Corollary 2.6. Let $S$ be a Hausdorff primitive inverse semitopological semigroup and $S$ be an orthogonal sum of the family $\{B^0(\lambda_i) (G_i) : i \in \mathcal{F}\}$ of semitopological Brandt semigroups with zeros. Then the following statements hold:

(i) for every non-zero element $(\alpha_i, g_i, \beta_i) \in (G_i)_{\alpha_i, \beta_i} \subseteq B^0(\lambda_i) (G_i) \subseteq S$ there exists an open neighbourhood $U_{(\alpha, g, \beta)}$ of $(\alpha, g, \beta)$ in $S$ such that $U_{(\alpha, g, \beta)} \subseteq (G_i)_{\alpha, \beta}$ and hence every subset $(G_i)_{\alpha, \beta}$ is an open subset of $S$;

(ii) every non-zero idempotent of $S$ is an isolated point of $E(S)$.

Proof. Assertion (i) follows from Proposition 2.5 and (ii) follows from (i).

Proposition 2.7. Let $S$ be a Hausdorff countably compact semitopological semigroup such that $S$ is an orthogonal sum of the family $\{B^0(\lambda_i) (S_i) : i \in \mathcal{F}\}$ of topological Brandt $\lambda^0$-extensions of semitopological monoids with zeros. Then for every open neighbourhood $U(0)$ of zero 0 in $S$ the set of pairs of indices $(\alpha_i, \beta_i)$ such that $(S_i)_{\alpha_i, \beta_i} \not\subset U(0)$ is finite. Moreover, every maximal topological Brandt $\lambda^0$-extension $B^0(\lambda_i) (S_i)$, $i \in \mathcal{F}$, is countably compact.

Proof. Suppose to the contrary that there exists an open neighbourhood $U(0)$ of the zero 0 in $S$ such that $(S_i)_{\alpha_i, \beta_i} \not\subset U(0)$ for infinitely many pairs of indices $(\alpha_i, \beta_i)$. Then for every such $(S_i)_{\alpha_i, \beta_i}$ we choose a point $x_{\alpha_i, \beta_i} \in (S_i)_{\alpha_i, \beta_i} \setminus U(0)$ and put $A = \bigcup\{x_{\alpha_i, \beta_i}\}$. Then $A$ is infinite and Proposition 2.3 implies that the set $A$ has no accumulation point in $S$. This contradicts Theorem 3.10.3 of [13]. The obtained contradiction implies the first statement of the proposition.
The second statement follows from Proposition 2.9 because by Theorem 3.10.4 of [13] every closed subspace of a countably compact space is countably compact. □

Proposition 2.9 implies the following corollary:

Corollary 2.8. Let $S$ be a Hausdorff primitive inverse countably compact semitopological semigroup and $S$ be an orthogonal sum of the family $\{B_{\lambda}(G_i) : i \in \mathcal{I}\}$ of semitopological Brandt semigroups with zeros. Then for every open neighbourhood $U(0)$ of zero 0 in $S$ the set of pairs of indices $(\alpha_i, \beta_i)$ such that $(S_i)_{\alpha_i, \beta_i} \not\subseteq U(0)$ is finite. Moreover, every maximal Brandt subsemigroup $B_{\lambda}(G_i)$, $i \in \mathcal{I}$, is countably compact.

Proposition 2.9. Let $S$ be a Hausdorff pseudocompact semitopological semigroup such that $S$ is an orthogonal sum of the family $\{B^0_{\lambda}(S_i) : i \in \mathcal{I}\}$ of topological Brandt $\lambda^0$-extensions of semitopological monoids with zeros. Then

(i) every maximal topological Brandt $\lambda^0$-extension $B^0_{\lambda}(S_i)$, $i \in \mathcal{I}$, is pseudocompact;

(ii) the subspace $(S_i)_{\alpha_i, \beta_i}$ is pseudocompact for all $\alpha_i, \beta_i \in \lambda_i$.

Proof. (i) Let $\mathcal{F} = \{U_\alpha : \alpha \in \mathcal{I}\}$ be an infinite family of open non-empty subsets of $B^0_{\lambda}(S_i)$. If 0 is contained in infinitely many members of the family $\mathcal{F}$ then it is not locally finite. In the opposite case the family $\mathcal{F}$ contains an infinite subfamily $\mathcal{F}'$ no member of which contains 0. Since the space $S$ is pseudocompact, there exists a point $x \in S$ such that each neighbourhood of $x$ intersects infinitely many members the family $\mathcal{F}'$. Suppose that $x \in U = B^0_{\lambda}(S_j) \setminus \{0\}$ for some index $j \neq i$. By Proposition 2.5 $U$ is an open subset of $S$. But $U \cap U_\alpha = \emptyset$ for each member $U_\alpha$ of the family $\mathcal{F}'$. Hence $x \in B^0_{\lambda}(S_i)$, a contradiction. Thus the family $\mathcal{F}'$ is not locally finite in $B^0_{\lambda}(S_i)$.

(ii) Since the semigroup operation in $S$ is relatively continuous we have that the map $f_{\alpha_i, \beta_i} : S \to S : x \mapsto (\alpha_i, 1_{S_i}, \alpha_i) \cdot x \cdot (\beta_i, 1_{S_i}, \beta_i)$ is continuous too, and hence $(S_i)_{\alpha_i, \beta_i}$ is a pseudocompact subspace of $S$ as a continuous image of a pseudocompact space. □

Proposition 2.9 implies the following corollary:

Corollary 2.10. Let $S$ be a Hausdorff primitive inverse pseudocompact semitopological semigroup and $S$ be an orthogonal sum of the family $\{B_{\lambda}(G_i) : i \in \mathcal{I}\}$ of semitopological Brandt semigroups with zeros. Then

(i) every maximal Brandt semigroup $B_{\lambda}(G_i)$, $i \in \mathcal{I}$, is pseudocompact;

(ii) $(G_i)^0_{\alpha_i, \beta_i}$ is pseudocompact for all $\alpha_i, \beta_i \in \lambda_i$.

Proposition 2.11. Let $S$ be a semiregular pseudocompact semitopological semigroup such that $S$ is an orthogonal sum of the family $\{B^0_{\lambda}(S_i) : i \in \mathcal{I}\}$ of topological Brandt $\lambda^0$-extensions of semitopological monoids with zeros. Then for every open neighbourhood $U(0)$ of zero 0 in $S$ the set of pairs of indices $(\alpha_i, \beta_i)$ such that $(S_i)_{\alpha_i, \beta_i} \not\subseteq U(0)$ is finite.

Proof. Since the semigroup $S$ is semiregular, there exists a regular open neighbourhood $V(0)$ of the zero 0 in $S$ such that $V(0) \subseteq U(0)$. Let $\mathcal{A} = \{(\alpha_i, \beta_i) : (S_i)_{\alpha_i, \beta_i} \not\subseteq V(0)\}$. Let $(\alpha_i, \beta_i) \in \mathcal{A}$ be an arbitrary pair. The set $(S_i)_{\alpha_i, \beta_i} = (S_i)^*_{\alpha_i, \beta_i} \setminus \text{cl}_S V(0)$ is a non-empty open subset of the topological space $S$. Indeed, in the opposite case $(S_i)_{\alpha_i, \beta_i} \subseteq \text{cl}_S V(0)$ and since by Proposition 2.5 the set $(S_i)^*_{\alpha_i, \beta_i}$ is open and the set $V(0)$ is regular open, we have $(S_i)_{\alpha_i, \beta_i} \subseteq \text{int}_S(\text{cl}_S V(0)) = V(0)$, a contradiction. We can easily check that the family $\mathcal{P} = \{(S_i)^*_{\alpha_i, \beta_i} : (\alpha_i, \beta_i) \in \mathcal{A}\}$ is a locally finite family of open subsets of the topological space $S$. Since the topological space $S$ is pseudocompact, the family $\mathcal{P}$ is finite, so the family $\mathcal{A}$ is finite too. □

Proposition 2.11 implies the following

Corollary 2.12. Let $S$ be a semiregular primitive inverse pseudocompact semitopological semigroup and $S$ be an orthogonal sum of the family $\{B_{\lambda}(G_i) : i \in \mathcal{I}\}$ of semitopological Brandt semigroups with zeros. Then for every open neighbourhood $U(0)$ of zero 0 in $S$ the set of pairs of indices $(\alpha_i, \beta_i)$ such that $(G_i)^0_{\alpha_i, \beta_i} \not\subseteq U(0)$ is finite.
The structure of primitive Hausdorff pseudocompact topological inverse semigroup is described in [17]. There is proved that every primitive Hausdorff pseudocompact topological inverse semigroup $S$ is topologically isomorphic to the orthogonal sum $\sum_{i \in I} B_{\lambda_i}(G_i)$ of topological Brandt $\lambda_i$-extensions $B_{\lambda_i}(G_i)$ of pseudocompact topological groups $G_i$ in the class of topological inverse semigroups for some finite cardinals $\lambda_i \geq 1$. Also there is described a base of the topology of a primitive Hausdorff pseudocompact topological inverse semigroup. Similar results for primitive Hausdorff countably compact topological inverse semigroups and Hausdorff compact topological inverse semigroups were obtained in [7].

The following example shows that counterparts of these results do not hold for primitive Hausdorff compact (and hence countably compact and pseudocompact) semitopological inverse semigroups with continuous inversion.

**Example 2.13.** Let $\mathbb{Z}(+)$ be the discrete additive group of integers and $\mathcal{O} \notin \mathbb{Z}(+)$. We put $\mathbb{Z}^0$ be $\mathbb{Z}(+)$ with adjoined zero $\mathcal{O}$ and define on $\mathbb{Z}^0$ the topology of the one-point Alexandroff compactification with the remainder $\mathcal{O}$. Simple verifications show that $\mathbb{Z}^0$ is a Hausdorff compact semitopological inverse semigroup with continuous inversion.

We fix an arbitrary cardinal $\lambda \geq 1$. Define a topology $\tau_B$ on $B_\lambda^0(\mathbb{Z}^0)$ as follows:

(i) all non-zero elements of $B_\lambda^0(\mathbb{Z}^0)$ are isolated points;
(ii) the family $\mathcal{P}(0) = \{U(\alpha, \beta, n) : \alpha, \beta \in \lambda, n \in \mathbb{N}\}$, where

$$U(\alpha, \beta, n) = B_\lambda^0(\mathbb{Z}^0) \setminus \{-n, -n + 1, \ldots, n - 1, n\}_{\alpha, \beta},$$

forms a pseudobase of the topology $\tau_B$.

Simple verifications shows that $(B_\lambda^0(\mathbb{Z}^0), \tau_B)$ is a Hausdorff compact semitopological inverse semigroup with continuous inversion, and moreover the space $(B_\lambda^0(\mathbb{Z}^0), \tau_B)$ is homeomorphic to the one-point Alexandroff compactification of the discrete space of cardinality $\max\{\lambda, \omega\}$ with the remainder zero of the semigroup $B_\lambda^0(\mathbb{Z}^0)$.

**Theorem 2.14.** Let $S$ be a Hausdorff primitive inverse countably compact semitopological semigroup and $S$ be an orthogonal sum of the family $\{B_{\lambda_i}(G_i) : i \in I\}$ of semitopological Brandt semigroups with zeros. Suppose that for every $i \in I$ there exists a maximal non-zero subgroup $(G_i)_{\alpha_i, \alpha_i}$, $\alpha_i \in \lambda_i$, such that at least one of the following conditions holds:

1. the group $(G_i)_{\alpha_i, \alpha_i}$ is left precompact;
2. $(G_i)_{\alpha_i, \alpha_i}$ is a pseudocompact paratopological group;
3. the group $(G_i)_{\alpha_i, \alpha_i}$ is left $\omega$-precompact pseudocompact;
4. the semigroup $S_{\alpha_i, \alpha_i} = (G_i)_{\alpha_i, \alpha_i} \cup \{0\}$ is a topological semigroup.

Then the following assertions hold:

(i) every maximal subgroup of $S$ is a closed subset of $S$ and hence is countably compact;
(ii) for every $i \in I$ the maximal Brandt semigroup $B_{\lambda_i}(G_i)$ is a countably compact topological Brandt $\lambda$-extension of a countably compact semitopological group $G_i$;
(iii) if $\mathcal{B}_{(\alpha_i, e_i, \alpha_i)}$ is a base of the topology at the unit $(\alpha_i, e_i, \alpha_i)$ of a maximal non-zero subgroup $(G_i)_{\alpha_i, \alpha_i}$ of $S$, $i \in I$, such that $U \subseteq (G_i)_{\alpha_i, \alpha_i}$ for any $U \in \mathcal{B}_{(\alpha_i, e_i, \alpha_i)}$, then the family

$$\mathcal{B}_{(\beta_i, x, \gamma_i)} = \{(\beta_i, x, \alpha_i) \cdot U \cdot (\alpha_i, e_i, \gamma_i) : U \in \mathcal{B}_{(\alpha_i, e_i, \alpha_i)}\}$$

is a base of the topology of $S$ at the point $(\beta_i, x, \gamma_i) \in (G_i)_{\beta_i, \gamma_i} \subseteq B_{\lambda_i}(G_i)$, for all $\beta_i, \gamma_i \in \lambda_i$;
(iv) the family

$$\mathcal{B}_0 = \{S \setminus ((G_i)_{\alpha_i, \beta_i_1} \cup \cdots \cup (G_k)_{\alpha_k, \beta_k}) : i_1, \ldots, i_k \in I, \alpha_{i_k}, \beta_{i_k} \in \lambda_{i_k}, k \in \mathbb{N}, \{(\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_k}, \beta_{i_k})\} \text{ is finite}\}$$

is a base of the topology at zero of $S$. 
Proof. (i) Fix an arbitrary maximal subgroup $G$ of $S$. Without loss of generality we can assume that $G$ is a non-zero subgroup of $S$. Then there exists a maximal Brandt subsemigroup $B_{\lambda_i}(G_i)$, $i \in \mathcal{I}$, which contains $G$. The separate continuity of the semigroup operation in $S$ implies that for all $\alpha_i, \beta_i, \gamma_i, \delta_i \in \lambda_i$ the map $\psi_{\alpha_i, \beta_i} : S \to S$ defined by the formula $\psi_{\alpha_i, \beta_i}(x) = (\gamma_i, e_i, \alpha_i) \cdot x \cdot (\beta_i, e_i, \delta_i)$, where $e_i$ is unit of the group $G_i$, is continuous. Since for all $\alpha_i, \beta_i, \gamma_i, \delta_i \in \lambda_i$ the restrictions $\psi_{\alpha_i,\beta_i}^{\gamma_i,\delta_i} : (G_i)_{\gamma_i,\delta_i} \to (G_i)_{\alpha_i,\beta_i}$ are bijective continuous maps we conclude that $(G_i)_{\alpha_i,\beta_i}$ and $(G_i)_{\gamma_i,\delta_i}$ are homeomorphic subspace of $S$, and moreover the semitopological subgroups $(G_i)_{\alpha_i,\beta_i}$ and $(G_i)_{\gamma_i,\delta_i}$ are topologically isomorphic for all indices $\alpha_i, \gamma_i \in \lambda_i$. Therefore we have that $G$ is topologically isomorphic to the semitopological subgroup $(G_i)_{\alpha_i,\beta_i}$ for any $\alpha_i \in \lambda_i$. For any $\alpha_i, \beta_i \in \lambda_i$ we put $S_{\alpha_i,\beta_i} = (G_i)_{\alpha_i,\beta_i} \cup \{0\}$. Then we have that for all $\alpha_i, \beta_i, \gamma_i, \delta_i \in \lambda_i$ the restrictions $\psi_{\alpha_i,\beta_i}^{\gamma_i,\delta_i} : S_{\gamma_i,\delta_i} \to S_{\alpha_i,\beta_i}$ are homeomorphic subspace of $S$, and moreover the semitopological subgroups $S_{\alpha_i,\beta_i}$ and $S_{\gamma_i,\delta_i}$ are topologically isomorphic for all indices $\alpha_i, \gamma_i \in \lambda_i$. Now Lemma 2.3 implies that 0 is an isolated point in $S_{\alpha_i,\beta_i}$. Indeed, if one of Conditions (1)-(3) of the theorem is satisfied then we can directly apply Lemma 2.3 and if Condition (4) of the theorem is satisfied then we observe that for each $\lambda_i$ and $\alpha_i \in \lambda_i$ the isomorphism $S_{\alpha_i,\beta_i}$ of $S$ is countably compact as a retract of $S$ and hence $S_{\alpha_i,\beta_i}$ is pseudocompact and therefore Lemma 2.3. By Corollary 2.6 we get that $(G_i)_{\alpha_i,\beta_i}$ is a closed subspace of $S$ and by Theorem 3.10.4 from [13] is countably compact, and hence so is $G$, too.

(ii) The arguments presented in the proof of the assertion (i) imply that for every $i \in \mathcal{I}$ the maximal Brandt semigroup $B_{\lambda_i}(G_i)$ is a topological Brandt $\lambda$-extension of a countably compact semitopological group $G_i$. By Corollary 2.6 we have that for every $i \in \mathcal{I}$ the maximal Brandt semigroup $B_{\lambda_i}(G_i)$ is a closed subset of $S$ and by Theorem 3.10.4 from [13] is countably compact.

Assertion (iii) follows from (ii).

(iv) follows from Corollary 2.8 and assertions (i) and (ii). \qed

Then for every open neighbourhood $U(0)$ of zero 0 in $S$ the set of pairs of indices $(\alpha_i, \beta_i)$ such that $(S_i)_{\alpha_i,\beta_i} \not\subseteq U(0)$ is finite.

The proof of the following theorem is similar to the proof of Theorem 2.14 using Corollary 2.10 and Proposition 2.11.

**Theorem 2.15.** Let $S$ be a Hausdorff primitive inverse pseudocompact semitopological semigroup and $S$ be an orthogonal sum of the family $\{B_{\lambda_i}(G_i) : i \in \mathcal{I}\}$ of semitopological Brandt semigroups with zeros. Let for every $i \in \mathcal{I}$ there exists a maximal non-zero subgroup $(G_i)_{\alpha_i,\beta_i}$, $\alpha_i \in \lambda_i$, such that at least one of the following conditions holds:

1. the group $(G_i)_{\alpha_i,\beta_i}$ is left $\omega$-precompact;
2. $(G_i)_{\alpha_i,\beta_i}$ is a pseudocompact paratopological group;
3. the group $(G_i)_{\alpha_i,\beta_i}$ is left $\omega$-precompact pseudocompact;
4. the semigroup $S_{\alpha_i,\beta_i} = (G_i)_{\alpha_i,\beta_i} \cup \{0\}$ is a topological semigroup.

Then the following assertions hold:

(i) every maximal subgroup of $S$ is an open-and-closed subset of $S$ and hence is pseudocompact;
(ii) for every $i \in \mathcal{I}$ the maximal Brandt semigroup $B_{\lambda_i}(G_i)$ is a pseudocompact topological Brandt $\lambda$-extension of a pseudocompact semitopological group $G_i$;
(iii) if $\mathcal{B}_{(\alpha_i,\beta_i,\gamma_i)}$ is a base of the topology at the unit $(\alpha_i, e_i, \alpha_i)$ of a maximal non-zero subgroup $(G_i)_{\alpha_i,\beta_i}$ of $S$, $i \in \mathcal{I}$, such that $U \subseteq (G_i)_{\alpha_i,\beta_i}$ for any $U \in \mathcal{B}_{(\alpha_i,\beta_i,\gamma_i)}$, then the family $\mathcal{B}_{(\beta_i, x, \gamma_i)} = \{(\beta_i, x, \alpha_i) \cdot U \cdot (\alpha_i, e_i, \gamma_i) : U \in \mathcal{B}_{(\alpha_i,\beta_i,\gamma_i)}\}$

is a base of the topology at the point $(\beta_i, x, \gamma_i) \in (G_i)_{\beta_i,\gamma_i} \subseteq B_{\lambda_i}(G_i)$, for all $\beta_i, \gamma_i \in \lambda_i$;

if in addition the topological space $S$ is semiregular then
Proposition 2.17. The space $B$ for all $\lambda \in \mathbb{N}$ and similarly

$$B_0 = \{ S \setminus \left( (G_{i_1})_{\alpha_{i_1}, \beta_{i_1}} \cup \cdots \cup (G_{i_k})_{\alpha_{i_k}, \beta_{i_k}} \right) : i_1, \ldots, i_k \in \mathcal{I}, \alpha_{i_k}, \beta_{i_k} \in \lambda_i, k \in \mathbb{N}, \{(\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_k}, \beta_{i_k}) \} \text{ is finite} \}$$

is a base of the topology at zero of $S$.

The following example shows that in the case of primitive Hausdorff pseudocompact semitopological inverse semigroups with compact maximal subgroups and continuous inversion the statement (iii) of Theorem 2.15 doesn’t hold.

Example 2.16. Let $\lambda$ be an infinite cardinal and $T$ be the unit circle with the usual multiplication of complex numbers and the usual topology $\tau_T$. It is obvious that $(T, \tau_T)$ is a topological group. The base of the topology $\tau_B$ on the Brandt semigroup $B_\lambda(T)$ we define as follows:

1) for every non-zero element $(\alpha, x, \beta)$ of the semigroup $B_\lambda(T)$ the family

$$B_{(\alpha, x, \beta)} = \{ (\alpha, U(x), \beta) : U(x) \in B_T(x) \},$$

where $B_T(x)$ is a base of the topology $\tau_T$ at the point $x \in T$, is the base of the topology $\tau_T$ at $(\alpha, x, \beta) \in B_\lambda(T)$;

2) the family

$$B_0 = \{ U(\alpha_1, \beta_1; \ldots ; \alpha_n, \beta_n; x_1, \ldots, x_k) : \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in \lambda, x_1, \ldots, x_k \in T, n, k \in \mathbb{N} \},$$

where

$$U(\alpha_1, \beta_1; \ldots ; \alpha_n, \beta_n; x_1, \ldots, x_k) = B_\lambda(T) \setminus (T_{\alpha_1, \beta_1} \cup \cdots \cup T_{\alpha_n, \beta_n} \cup \{ (\alpha, x_1, \beta) : \alpha, \beta \in \lambda, i = 1, \ldots, k \}),$$

is the base of the topology $\tau_T$ at zero $0 \in B_\lambda(T)$.

Simple verifications show that $(B_\lambda(T), \tau_B)$ is a non-semiregular Hausdorff pseudocompact topological space for every infinite cardinal $\lambda$. Next we shall show that the semigroup operation on $(B_\lambda(T), \tau_B)$ is separately continuous. The proof of the separate continuity of the semigroup operation in the cases $0 \cdot 0$ and $(\alpha, x, \beta) \cdot (\gamma, y, \delta)$, where $\alpha, \beta, \gamma, \delta \in \lambda$ and $x, y \in T$, is trivial, and hence we only consider the following cases:

$$(\alpha, x, \beta) \cdot 0 \quad \text{and} \quad 0 \cdot (\alpha, x, \beta).$$

Then we have that

$$(\alpha, x, \beta) \cdot U(\beta, \beta_1; \ldots ; \beta_n; \alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; x_1, \ldots, x_k) \subseteq \{ 0 \} \cup \bigcup \{ T_{\alpha, \gamma} \setminus \{ (\alpha, x_1, \gamma), \ldots, (\alpha, x_k, \gamma) \} : \gamma \in \lambda \setminus \{ \beta_1, \ldots, \beta_n \} \} \subseteq U(\alpha, \beta_1; \ldots ; \alpha_n, \beta_n; \alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; x_1, \ldots, x_k) \subseteq U(\alpha_1, \beta_1; \ldots ; \alpha_n, \beta_n; x_1, \ldots, x_k)$$

and similarly

$$U(\alpha_1, \alpha; \ldots ; \alpha_n, \alpha; \alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; x_1, \ldots, x_k) \cdot (\alpha, x, \beta) \subseteq \{ 0 \} \cup \bigcup \{ T_{\gamma, \beta} \setminus \{ (\gamma, x_1, \beta), \ldots, (\gamma, x_k, \beta) \} : \gamma \in \lambda \setminus \{ \alpha_1, \ldots, \alpha_n \} \} \subseteq U(\alpha_1, \beta; \ldots ; \alpha_n, \beta; \alpha_1, \beta_1; \ldots; \alpha_n, \beta_n; x_1, \ldots, x_k) \subseteq U(\alpha_1, \beta_1; \ldots ; \alpha_n, \beta_n; x_1, \ldots, x_k),$$

for all $U(\alpha_1, \beta_1; \ldots ; \alpha_n, \beta_n; x_1, \ldots, x_k), U(\alpha_1, \beta_1; \ldots ; \alpha_n, \beta_n; x_1, \ldots, x_k) \in B_0$. This completes the proof of separate continuity of the semigroup operation in $(B_\lambda(T), \tau_B)$.

Proposition 2.17. The space $(B_\lambda(T), \tau_B)$ is countably pracompact if and only if $\lambda \leq c$. 
Proof. ($\Leftarrow$) Suppose that $\lambda \leq c$. Then there exists a countable dense subgroup $H$ in $T$. Let $\mathcal{H}$ be the family of all distinct conjugate classes of subgroup $H$ in $T$. Since the subgroup $H$ is countable we conclude that the cardinality of $\mathcal{H}$ is $c$. This implies that there exist one-to-one (not necessary bijective) map $f: \lambda \times \lambda \rightarrow \mathcal{H}$: $(\alpha, \beta) \mapsto g_{\alpha,\beta}.H$. Then by the definition of the topology $\tau_B$ we have that $A = \bigcup_{\alpha,\beta \in \lambda} (g_{\alpha,\beta}H)_{\alpha,\beta}$ is a dense subset of the topological space $(B_\lambda(T), \tau_B)$. Fix an arbitrary infinite countable subset $Q$ of $A$. If the set $Q \cap \mathbb{T}_{\alpha,\beta}$ is infinite for some $\alpha, \beta \in \lambda$ then compactness of $T$ implies that $Q$ has an accumulation point in $\mathbb{T}_{\alpha,\beta}$, and hence in $(B_\lambda(T), \tau_B)$. In the other case the definition of the topology $\tau_B$ implies that zero $0$ is an accumulation point of $Q$. Therefore we have that the space $(B_\lambda(T), \tau_B)$ is countably compact at $A$, and hence it is countably pracom pact.

($\Rightarrow$) Suppose that there exists a cardinal $\lambda > c$ such that the space $(B_\lambda(T), \tau_B)$ is countably pracom pact. Then there exists a dense subset $A$ in $(B_\lambda(T), \tau_B)$ such that the space $(B_\lambda(T), \tau_B)$ is countably compact at $A$. The definition of the topology $\tau_B$ implies that $A \cap \mathbb{T}_{\alpha,\beta}$ is dense subset in $\mathbb{T}_{\alpha,\beta}$ for all $\alpha, \beta \in \lambda$. Since $\lambda > c$ and $|\mathbb{T}| = c$ we conclude that there exists a point $x \in \mathbb{T}$ such that $(\alpha, x, \beta) \in A$ for infinitely many distinct pairs $(\alpha, \beta)$ of indices in $\lambda$. Put $K = \{(\alpha, \beta) \in \lambda \times \lambda: (\alpha, x, \beta) \in A\}$. The definition of the topology $\tau_B$ implies that for every infinite countable subset $K_0 \subseteq K$ the set $\{(\alpha, x, \beta): (\alpha, \beta) \in K_0\}$ has no accumulation point in $(B_\lambda(T), \tau_B)$, a contradiction.

The proof of the following proposition is similar to Proposition 2.8 from [17].

Proposition 2.18. Let $S$ be a semiregular pseudocompact (Hausdorff countably compact) semitopological semigroup such that $S$ is an orthogonal sum of the family $\{B_0^0(S_i): i \in \mathcal{I}\}$ of topological Brandt $\lambda^0_{\mathcal{I}}$-extensions of semitopological monoids with zeros, i.e. $S = \sum_{i \in \mathcal{I}} B_0^0(S_i)$. Then the following assertions hold:

(i) the topological space $S$ is regular if and only if the space $S_i$ is regular for each $i \in \mathcal{I}$;

(ii) the topological space $S$ is Tychonoff if and only if the space $S_i$ is Tychonoff for each $i \in \mathcal{I}$;

(iii) the topological space $S$ is normal if and only if the space $S_i$ is normal for each $i \in \mathcal{I}$.

The following theorem characterizes pseudocompact topological Brandt $\lambda^0$-extensions of topological monoids with zero in the class of Hausdorff topological semigroups.

Theorem 2.19. A topological Brandt $\lambda^0$-extension $(B_0^0(S), \tau_B)$ of a topological monoid $(S, \tau_S)$ with zero in the class of Hausdorff topological semigroups is pseudocompact if and only if cardinal $\lambda$ is finite and the space $(S, \tau_S)$ is pseudocompact.

Proof. ($\Leftarrow$) The continuity of the semigroup operation in $(B_0^0(S), \tau_B)$ implies that for all $\alpha, \beta, \gamma, \delta \in \lambda$ the map $\psi_{\alpha,\beta,\gamma,\delta}: B_0^0(S) \rightarrow B_0^0(S)$ defined by the formula $\psi_{\alpha,\beta,\gamma,\delta}(x) = (\gamma, 1_S, \alpha) \cdot x \cdot (\beta, 1_S, \delta)$, where $1_S$ is unit of the semigroup $S$, is continuous. Since for all $\alpha, \beta, \gamma, \delta \in \lambda$ the restrictions $\psi_{\alpha,\beta,\gamma,\delta}|_{S_{\alpha,\beta}}: S_{\alpha,\beta} \rightarrow S_{\gamma,\delta}$ and $\psi_{\alpha,\beta,\gamma,\delta}|_{S_{\gamma,\delta}}: S_{\gamma,\delta} \rightarrow S_{\alpha,\beta}$ are bijective continuous maps we conclude that $S_{\alpha,\beta}$ and $S_{\gamma,\delta}$ are homeomorphic subspaces of $(B_0^0(S), \tau_B)$. Therefore we have that the space $(B_0^0(S), \tau_B)$ is a union of finitely many copies of a pseudocompact topological space $(S, \tau_S)$, and hence it is pseudocompact.

($\Rightarrow$) Suppose that a topological Brandt $\lambda^0$-extension $(B_0^0(S), \tau_B)$ of a topological monoid $(S, \tau_S)$ with zero in the class of topological semigroups is pseudocompact. Then by Proposition 2.18(ii) the space $(S, \tau_S)$ is pseudocompact.

Suppose the contrary that there exists a pseudocompact topological Brandt $\lambda^0$-extension $(B_0^0(S), \tau_B)$ of a topological monoid $(S, \tau_S)$ with zero in the class of Hausdorff topological semigroups such that the cardinal $\lambda$ is infinite. Then the Hausdorffness of $(B_0^0(S), \tau_B)$ implies that for every $\alpha \in \lambda$ there exist open disjoint neighbourhoods $U_0$ and $U_{(\alpha, 1_S, \alpha)}$ of zero and $(\alpha, 1_S, \alpha)$ in $(B_0^0(S), \tau_B)$, respectively. Without loss of generality we can assume that $U_{(\alpha, 1_S, \alpha)} = (U(1_S))_{\alpha,\alpha}$ for some open neighbourhood $U(1_S)$ of unit $1_S$ in $(S, \tau_S)$ (see Proposition 2.5). By continuity of the semigroup operation in $(B_0^0(S), \tau_B)$ we have that there exists an open neighbourhood $V_0$ of zero in $(B_0^0(S), \tau_B)$ such that $V_0 \cdot V_0 \subseteq U_0$. Also the continuity of the semigroup operation in $(S, \tau_S)$ implies that there exists an open neighbourhood $V(1_S)$ of unit $1_S$ in $(S, \tau_S)$ such that $V(1_S) \cdot V(1_S) \subseteq U(1_S)$ in $S$. 

ON PSEUDOCOMPACT INVERSE PRIMITIVE (SEMI)TOPOLOGICAL SEMIGROUPS

11
Then the pseudocompactness of \((B^0_\lambda(S), \tau_B)\) implies that zero 0 is an accumulation point of each infinite subfamily of \(\{(V(1_S))_{\alpha, \beta}: \alpha, \beta \in \lambda\}\) and hence \(V_0 \cap (V(1_S))_{\alpha, \beta} = \emptyset\) only for finitely many pairs if indices \((\alpha, \beta)\). Hence by the definition of the semigroup operation on \(B^0_\lambda(S)\) we have that \((V_0, V_0) \cap U_{(\alpha,1,\beta,0)} \neq \emptyset\). This contradicts the assumption \(U_0 \cap U_{(\alpha,1,\beta,0)} = \emptyset\). The obtained contradiction implies that cardinal \(\lambda\) is finite. \(\square\)

Theorem 2.19 implies the following corollary:

**Corollary 2.20.** A pseudocompact topological Brandt \(\lambda^0\)-extension of a topological inverse monoid with zero in the class of Hausdorff topological semigroups is a topological inverse semigroup.

The following example shows that there exists a compact topological semigroup with a non-pseudocompact topological Brandt \(2^\alpha\)-extension in the class of topological semigroups and hence the counterpart of Theorem 2.19 does not necessarily hold for semigroups without non-zero idempotent.

**Example 2.21.** Let \(X\) be any infinite Hausdorff compact topological space. Fix an arbitrary \(z \in X\) and define the semigroup operation on \(X\) in the following way: \(x \cdot y = z\) for all \(x, y \in X\). It is obvious that this operation is continuous on \(X\) and \(z\) is zero of \(X\). The set \(X\) endowed with such an operation is called a semigroup with zero-multiplication. We define the topology \(\tau_B\) on the Brandt \(2^\alpha\)-extension \(B^0_\lambda(X)\) of the semigroup \(X\) as follows:

(i) the family \(B(0) = \{U_{1,1} \cup U_{2,2}: U \in B(z)\}\), where \(B(z)\) is a base of the topology of \(X\) at \(z\), is the base of topology \(\tau_B\) at zero of \(B^0_\lambda(X)\);

(ii) for \(i = 1, 2\) and any \(x \in X \setminus \{z\}\) the family \(B_i(x, z) = \{U_{i, i}: U \in B(x)\}\), where \(B(x)\) is a base of the topology of \(X\) at the point \(x\), is the base of topology \(\tau_B\) at the point \((i, x, i) \in B^0_\lambda(X)\);

(iii) all points of the subsets \(X_{1,2}^*\) and \(X_{2,1}^*\) are isolated points in \((B^0_\lambda(X), \tau_B)\).

It is obvious that \(B^0_\lambda(X)\) is a semigroup with zero-multiplication. Simple verifications show that \(\tau_B\) is a Hausdorff topology on \(B^0_\lambda(X)\). Hence \((B^0_\lambda(X), \tau_B)\) is a topological semigroup and \((B^0_\lambda(X), \tau_B)\) is a topological Brandt \(2^\alpha\)-extension of \(X\) in the class of topological semigroups. Since \(X_{1,2}^*\) and \(X_{2,1}^*\) are discrete open-and-closed subspaces of \((B^0_\lambda(X), \tau_B)\) we have that the topological space \((B^0_\lambda(X), \tau_B)\) is not pseudocompact.

Also, the following example shows that there exists a compact topological semigroup \(S\) such that for every infinite cardinal \(\lambda\) there exists a compact (and hence pseudocompact) topological Brandt \(\lambda^0\)-extension \(B^0_\lambda(S)\) of the semigroup \(S\) in the class of topological semigroups.

**Example 2.22.** Let \(X\) be a compact topological semigroup which defined in Example 2.21 and \(\lambda\) be an arbitrary infinite cardinal. We define the topology \(\tau_B\) on the Brandt \(\lambda^0\)-extension \(B^0_\lambda(X)\) of the semigroup \(X\) as follows:

(i) the family \(B_B(0) = \{U_A(0) = \bigcup_{(\alpha, \beta) \in (\lambda \times \lambda) \setminus A} X_{\alpha, \beta} \cup \bigcup_{(\gamma, \delta) \in \gamma \times \delta} (U(z))_{\gamma, \delta}: A\) is a finite subset of \(\lambda \times \lambda\) and \(U(z) \in B_X(z)\}\), where \(B_X(z)\) is a base of the topology \(x \in X\), is the base of topology \(\tau_B\) at zero of \(B^0_\lambda(X)\);

(ii) for all \(\alpha, \beta \in \lambda\) and any \(x \in X \setminus \{z\}\) the family \(B_{(\alpha, x, \beta)} = \{U_{\alpha, \beta}: U \in B_X(x)\}\), where \(B_X(x)\) is a base of the topology of \(X\) at the point \(x\), is the base of topology \(\tau_B\) at the point \((\alpha, x, \beta) \in B^0_\lambda(X)\).

It is obviously that \(B^0_\lambda(X)\) is a semigroup with zero-multiplication. Simple verifications show that \(\tau_B\) is a Hausdorff compact topology on \(B^0_\lambda(X)\). Hence \((B^0_\lambda(X), \tau_B)\) is a topological semigroup and \((B^0_\lambda(X), \tau_B)\) is a compact topological Brandt \(\lambda^0\)-extension of \(X\) in the class of topological semigroups.

The following proposition extends Theorem 2.19.

**Proposition 2.23.** Let \(S\) be a Hausdorff pseudocompact topological semigroup such that \(S\) is an orthogonal sum of the family \(\{B^0_\lambda(S_i): i \in \mathcal{I}\}\) of topological Brandt \(\lambda^0\)-extensions of topological semigroups with zeros, i.e. \(S = \sum_{i \in \mathcal{I}} B^0_\lambda(S_i)\). If for some \(i \in \mathcal{I}\) the semigroup \(S_i\) has a non-zero idempotent then cardinal \(\lambda_i\) is finite.
Proof. Suppose the contrary that there exists $i \in \mathcal{I}$ such that the cardinal $\lambda_i$ is infinite. Let $e$ be a non-zero idempotent of $S_i$. Then the Hausdorffness of $S$ implies that for every $\alpha_i \in \lambda_i$ there exist open disjoint neighbourhoods $U_0$ and $U_{(\alpha_i, e, \alpha_i)}$ of zero and $(\alpha_i, e, \alpha_i)$ in $S_i$ respectively. By continuity of the semigroup operation in $S$ we have that there exists an open neighbourhood $V_{(\alpha_i, e, \alpha_i)}$ of $(\alpha_i, e, \alpha_i)$ in $S$ such that $(\alpha_i, e, \alpha_i) \cdot V_{(\alpha_i, e, \alpha_i)} \cdot (\alpha_i, e, \alpha_i) \subseteq U_{(\alpha_i, e, \alpha_i)}$. This implies that $V_{(\alpha_i, e, \alpha_i)} \subseteq (S^*_i)_{\alpha_i, \alpha_i}$. Therefore without loss of generality we can assume that $U_{(\alpha_i, e, \alpha_i)} = (U(e))_{\alpha_i, \alpha_i}$ for some open neighbourhood $U(e)$ of the idempotent $e$ in $S_i$. By continuity of the semigroup operation in $S$ we have that there exists an open neighbourhood $V_0$ of zero in $S$ such that $V_0 \cdot V_0 \subseteq U_0$. Also the continuity of the semigroup operation in $S_i$ implies that there exists an open neighbourhood $V(e)$ of the idempotent $e$ in $S_i$ such that $V(e) \cdot V(e) \subseteq U(e)$ in $S_i$.

Then the pseudocompactness of $S$ implies that zero 0 is an accumulation point of each infinite subfamily of $\{(V(1_S))_{\alpha_i, \beta_i} : \alpha_i, \beta_i \in \lambda_i, i \in \mathcal{I}\}$ and hence $V_0 \cap (V(1_S))_{\alpha_i, \beta_i} = \varnothing$ only for finitely many pairs if indices $(\alpha_i, \beta_i)$ from $\lambda_i$, $i \in \mathcal{I}$. Hence by the definition of the semigroup operation on $S$ we have that $(V_0 \cdot V_0) \cap U_{(\alpha_i, e, \alpha_i)} \neq \varnothing$. This contradicts the assumption $U_0 \cap U_{(\alpha_i, e, \alpha_i)} = \varnothing$. The obtained contradiction implies that cardinal $\lambda_i$ is finite. \hfill \Box

Theorem 2.15 and Proposition 2.23 imply the following:

**Theorem 2.24.** Let $S$ be a Hausdorff primitive inverse pseudocompact topological semigroup and $S$ be an orthogonal sum of the family $\{B_{\lambda_i}(G_i) : i \in \mathcal{I}\}$ of topological Brandt semigroups with zeros. Then the following assertions hold:

(i) every cardinal $\lambda_i$ is finite;

(ii) every maximal subgroup of $S$ is open-and-closed subset of $S$ and hence is pseudocompact;

(iii) for every $i \in \mathcal{I}$ the maximal Brandt semigroup $B_{\lambda_i}(G_i)$ is a pseudocompact topological Brandt $\lambda$-extension of a pseudocompact paratopological group $G_i$;

(iv) if $\mathcal{B}_{(\alpha_i, e, \alpha_i)}$ is a base of the topology at the unity $(\alpha_i, e, \alpha_i)$ of a maximal non-zero subgroup $(G_i)_{\alpha_i, \alpha_i}$ of $S$, $i \in \mathcal{I}$, such that $U \subseteq (G_i)_{\alpha_i, \alpha_i}$ for any $U \in \mathcal{B}_{(\alpha_i, e, \alpha_i)}$, then the family

$\mathcal{B}_{(\beta_i, x, \gamma_i)} = \{(\beta_i, x, \gamma_i) : \beta_i, x, \gamma_i \in \lambda_i\}
\mathcal{B}_{(\beta_i, x, \gamma_i)}
$ is a base of the topology at the point $(\beta_i, x, \gamma_i) \in (G_i)_{\beta_i, \gamma_i} \subseteq B_{\lambda_i}(G_i)$, for all $\beta_i, \gamma_i \in \lambda_i$;

if in addition the topological space $S$ is semiregular then

(v) the family

$\mathcal{B}_0 = \{S \setminus ((G_{i_1})_{\alpha_{i_1}, \beta_{i_1}} \cup \cdots \cup (G_{i_k})_{\alpha_{i_k}, \beta_{i_k}}) : i_1, \ldots, i_k \in \mathcal{I}, \alpha_{i_k}, \beta_{i_k} \in \lambda_{i_k},
\text{ } k \in \mathbb{N}, \{(\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_k}, \beta_{i_k})\} \text{ is finite}\}
\mathcal{B}_0
$ is a base of the topology at zero of $S$.

The following example shows that statement (v) of Theorem 2.24 does not necessarily hold when the topological space of the semigroup $S$ is functionally Hausdorff and countably pracompact but it is not semiregular.

**Example 2.25.** One of the authors in the paper [28, Example 3] constructed a functionally Hausdorff $\omega$-pracompact first countable paratopological group $(G, \tau_R)$ such that each power of $(G, \tau_R)$ is countably pracompact but $(G, \tau_R)$ is not a topological group. Moreover, the group $(G, \tau_R)$ contains an open dense subsemigroup $S$. Let $\mathcal{I}$ be an infinite set of indices. For any $i \in \mathcal{I}$ put $\lambda_i$ be any finite cardinal $\geq 1$. Let $B_{\lambda_i}(G)$ be the algebraic Brandt $\lambda_i$-extension of the algebraic group $G$ for each $i \in \mathcal{I}$. Put $R(G, \{\lambda_i\}_{i \in \mathcal{I}}) = \sum_{i \in \mathcal{I}} B_{\lambda_i}(G)$. Also for any subset $C$ of the group $G$ and all $i, i_1, \ldots, i_k \in \mathcal{I}, k \in \mathbb{N}$, put

$B_{\lambda_i}(C) = \{0\} \cup \{(\alpha_{i_i}, x, \beta_i) \in B_{\lambda_i}(G) : x \in C, \alpha_i, \beta_i \in \lambda_i\},
R(C, \{\lambda_i\}_{i \in \mathcal{I}}) = \sum_{i \in \mathcal{I}} B_{\lambda_i}(C)
$ and $U(i_1, \ldots, i_k) = R(S, \{\lambda_i\}_{i \in \mathcal{I}}) \setminus ((B_{\lambda_1}(S))^* \cup \cdots \cup (B_{\lambda_k}(S))^*)$.

We define the topology $\tau_{RB}$ on $R(G, \{\lambda_i\}_{i \in \mathcal{I}})$ in the following way:
Theorem 2.26. inverse pseudocompact topological semigroups and this follows from the next two propositions.

It is obvious that $(R(G, \{\lambda_i\} \subset \mathcal{F}), \tau_{RB})$ is a Hausdorff topological space, but since $S$ is a dense open subsemigroup of $(G, \tau_R)$ we conclude that $(R(G, \{\lambda_i\} \subset \mathcal{F}), \tau_{RB})$ is not semiregular. Also, since the space $(G, \tau_R)$ is functionally Hausdorff and $G_{\beta,\gamma}$ is an open-and-closed subspace of $(R(G, \{\lambda_i\} \subset \mathcal{F}), \tau_{RB})$, for all $\beta_i, \gamma_i \in \lambda_i$, the space $(R(G, \{\lambda_i\} \subset \mathcal{F}), \tau_{RB})$ is functionally Hausdorff too.

Now, the definition of the semigroup $R(G, \{\lambda_i\} \subset \mathcal{F})$ implies that

$$U(i_1, \ldots, i_k) \cdot B_{\lambda_m}(G) = B_{\lambda_m}(G) \cdot U(i_1, \ldots, i_k) = \{0\},$$

for each $i_m \in \{i_1, \ldots, i_k\}$ and $U(i_1, \ldots, i_k) \cdot U(i_1, \ldots, i_k) \subseteq U(i_1, \ldots, i_k)$ for all $i_1, \ldots, i_k \in \mathcal{F}$, $k \in \mathbb{N}$, because $S$ is a subsemigroup of the group $G$. This and the continuity of the group operation in $(G, \tau_R)$ imply that the semigroup operation in $(R(G, \{\lambda_i\} \subset \mathcal{F}))$ is continuous.

We claim that the topological space $(R(G, \{\lambda_i\} \subset \mathcal{F}), \tau_{RB})$ is countably paracompact. Indeed, there exists a set $A \subset S \subset G$ such that $A$ is dense in the space $(G, \tau_R)$ and this space is countably compact at $A$ [23, Example 3]. Then the set $R(A, \{\lambda_i\} \subset \mathcal{F})$ is dense in $R(G, \{\lambda_i\} \subset \mathcal{F})$. We claim that the space $R(G, \{\lambda_i\} \subset \mathcal{F})$ is countably compact at $R(A, \lambda, \mathcal{F})$. Indeed, let $A'$ be an arbitrary countable infinite subset of the set $R(A, \{\lambda_i\} \subset \mathcal{F})$. If $0$ is not an accumulation point of the set $A'$ then there exist indices $i_1, \ldots, i_k \in \mathcal{F}$ such that the set $U(i_1, \ldots, i_k) \cap A'$ is finite. Since $A \subset S$ then $A' \subset R(A, \{\lambda_i\} \subset \mathcal{F}) \subset R(S, \{\lambda_i\} \subset \mathcal{F})$ and the set $A' \cap \{ (B_{\lambda_i}(S))^{\ast} \cup \cdots \cup (B_{\lambda_i}(S))^{\ast} \} \subset A' \cap U(i_1, \ldots, i_k)$ is infinite. Since for each $1 \leq j \leq k$ the cardinal $\lambda_j$ is finite then there exists an index $1 \leq j \leq k$ and elements $\alpha, \beta \in \lambda_j$ such that an intersection $A' \cap S_{\alpha,\beta} \subset B_{\lambda_j}(A) \subset B_{\lambda_j}(G)$ is infinite. Since the space $G$ is countably compact at $A$, $B_{\lambda_j}(G)$ is countable compact at $B_{\lambda_j}(A)$. Therefore the set $A' \cap S_{\alpha,\beta}$ has an accumulation point in $B_{\lambda_j}(G)$.

Unlike to functional Hausdorffness the quasiregularity guaranties most stronger properties of primitive inverse pseudocompact topological semigroups and this follows from the next two propositions.

Theorem 2.26. Let $S$ be a quasiregular primitive inverse pseudocompact topological semigroup and $S$ be an orthogonal sum of the family $\{B_{\lambda_i}(G_i): i \in \mathcal{F}\}$ of topological Brandt semigroups with zeros. Then the family

$$\mathcal{B}_0 = \{ S \setminus \{(G_{i_1})_{\alpha_{i_1},\beta_{i_1}} \cup \cdots \cup (G_{i_k})_{\alpha_{i_k},\beta_{i_k}}: i_1, \ldots, i_k \in \mathcal{F}, \alpha_{i_k},\beta_{i_k} \in \lambda_{i_k}, k \in \mathbb{N}, \{ (\alpha_{i_1},\beta_{i_1}), \ldots, (\alpha_{i_k},\beta_{i_k}) \} \text{ is finite} \}$$

is a base of the topology at zero of $S$.

Proof. Suppose to the contrary that there exists an open subset $W \ni 0$ of $S$ such that $U \not\subseteq W$ for any $U \in \mathcal{B}_0$. There exists an open neighbourhood $V \subseteq W$ of zero in $S$ such that $V \cdot V \cdot V \subseteq W$. Since every non-zero maximal subgroup of $S$ is an open-and-closed subset of $S$ and the space $S$ is pseudocompact, there exist finitely many indices $i_1, \ldots, i_k \in \mathcal{F}$ such that $V \cap (S \setminus \{(B_{\lambda_i}(S))^{\ast} \cup \cdots \cup (B_{\lambda_k}(S))^{\ast}\})$ is a dense open subset of the space $S \setminus \{(B_{\lambda_i}(S))^{\ast} \cup \cdots \cup (B_{\lambda_k}(S))^{\ast}\}$. Then every non-zero maximal subgroup of $S$ is a quasi-regular space and hence by Proposition 3 of [23] (see also [29]) every maximal subgroup of $S$ is a topological group. Now, Proposition 2.5 of [17] implies that $V \cdot V \cdot V \supseteq S \setminus \{(B_{\lambda_i}(S))^{\ast} \cup \cdots \cup (B_{\lambda_k}(S))^{\ast}\} \not\subseteq W$. The obtained contradiction implies the theorem.

Since inversion on a quasiregular pseudocompact paratopological group is continuous, Theorems 2.24 and 2.26 imply the following corollary:

Corollary 2.27. Inversion on a quasiregular primitive inverse pseudocompact topological semigroup is continuous.
Remark 2.28. Example 1 of [6] shows that in general case inversion on a quasi-regular inverse countably compact topological semigroup which maximal subgroups are topological groups is not continuous. Also Corollary 2.27 and Proposition 2.8 from [17] imply that the space of a quasi-regular primitive inverse pseudocompact topological semigroup is Tychonoff.

Also, Corollary 2.27 implies

**Corollary 2.29.** Every quasi-regular pseudocompact Brandt topological semigroup is a topological inverse semigroup.

Theorem 2.14 implies the following:

**Theorem 2.30.** Let $S$ be a Hausdorff primitive inverse countably compact topological semigroup and $S$ be an orthogonal sum of the family $\{B_{\lambda_i}(G_i) : i \in \mathcal{I}\}$ of topological Brandt semigroups with zeros. Then the family

$$\mathcal{B}_0 = \{S\setminus (G_i)^{\alpha_i,\beta_i_1} \cup \cdots \cup (G_i)^{\alpha_i,\beta_i_k} : i_1, \ldots, i_k \in \mathcal{I}, \alpha_i, \beta_i \in \lambda_i, k \in \mathbb{N}, \{(\alpha_i, \beta_i_1), \ldots, (\alpha_i, \beta_i_k)\} \text{ is finite}\}$$

is a base of the topology at zero of $S$.

Definition 1.1, Theorem 2.30 and arguments presented in the proof of Theorem 2.14 imply the following corollary:

**Corollary 2.31.** Inversion on a Hausdorff primitive inverse countably compact topological semigroup $S$ is continuous if and only if every maximal subgroup of $S$ is a topological group.

Remark 2.32. One of the authors, using a result of P. Koszmider, A. Tomita and S. Watson [23], constructed a MA-example of a Hausdorff countably compact paratopological group failing to be a topological group [27, 28].

3. Products of pseudocompact inverse primitive semitopological semigroups and the their Stone-$\check{C}$ech compactifications

**Proposition 3.1.** Let $X$ be a pseudocompact topological space and $Y$ be a sequentially compact topological space. Then $X \times Y$ is pseudocompact.

**Proof.** Let $\{U_n : n \in \mathbb{N}\}$ be a family of non-empty open subsets of the space $X \times Y$. Let $n$ be a natural number. There exist non-empty open subsets $V_n \subset X$ and $W_n \subset Y$ such that $V_n \times W_n \subset U_n$. Choose a point $y_n \in W_n$. Since the space $Y$ is sequentially compact, the sequence $\{y_n : n \in \mathbb{N}\}$ has a subsequence $\{y_{n_k} : k \in \mathbb{N}\}$ convergent to a point $y \in Y$. Since the space $X$ is pseudocompact, there exists a point $x \in X$ such that $x \in \text{cl}(V_{n_k})$ for infinitely many numbers $k$. Then $(x, y) \in \text{cl}(U_n)$ for infinitely many numbers $n$. \qed

**Proposition 3.2.** Let $X$ be a Hausdorff pseudocompact topological space. Then $X \times Y$ is pseudocompact for any pseudocompact $k$-space $Y$.

**Proof.** It suffice to observe that every non-empty open subset of the Cartesian product $X \times Y$ contains an open subset $U \times V$, where $U$ and $V$ are non-empty open subset of $X$ and $Y$, respectively, and then Lemma 3.10.12 of [13] implies the statement of the proposition. \qed

Proposition 3.2 implies the following two corollaries.

**Corollary 3.3.** The Cartesian product $X \times Y$ of a pseudocompact space $X$ and compactum $Y$ is pseudocompact.

**Corollary 3.4.** The Cartesian product $X \times Y$ of a pseudocompact space $X$ and a pseudocompact sequential space $Y$ is pseudocompact.

**Proposition 3.5.** Let $S$ be a primitive semitopological inverse semigroup such that every maximal subgroup is a pseudocompact paratopological (topological) group. Then $S$ is a continuous image of

\footnote{not necessarily a homomorphic image}
the product $\tilde{E}_S \times G_S$, where $\tilde{E}_S$ is a compact semilattice and $G_S$ is a pseudocompact paratopological (topological) group provided one of the following conditions holds:

1. $S$ is semiregular pseudocompact;
2. $S$ is Hausdorff countably compact.

**Proof.** We only consider the case when $S$ is a semiregular pseudocompact space and every maximal subgroup of $S$ is a paratopological group because in case (2) the proof is similar.

By Theorem 2.15 the topological semigroup $S$ is topologically isomorphic to the orthogonal sum $\sum_{i \in I} B_{\lambda_i}(G_i)$ of topological Brandt $\lambda_i$-extensions $B_{\lambda_i}(G_i)$ of pseudocompact paratopological groups $G_i$ in the class of Hausdorff semitopological semigroups for some cardinals $\lambda_i \geq 1$ and the family defined by formula (1) determines the base of a topology at zero of $S$.

Fix an arbitrary $i \in I$. Then by Corollary 2.12 the space $E(B_{\lambda_i}(G_i))$ is compact. First we consider the case when the cardinal $\lambda_i$ is finite. Suppose that $|E(B_{\lambda_i}(G_i))| = n_i + 1$ for some integer $n_i$. Then we have that $\lambda_i = n_i \geq 1$. On the set $E_i = (\lambda_i \times \lambda_i) \cup \{0\}$, where $0 \notin \lambda_i \times \lambda_i$ we define the binary operation in the following way

$$(\alpha, \beta) \cdot (\gamma, \delta) = \begin{cases} (\alpha, \beta), & \text{if } (\alpha, \beta) = (\gamma, \delta); \\ 0, & \text{otherwise}, \end{cases}$$

and $0 \cdot (\alpha, \beta) = (\alpha, \beta) \cdot 0 = 0 \cdot 0 = 0$ for all $\alpha, \beta, \gamma, \delta \in \lambda_i$. Simple verifications show that $E_i$ with such defined operation is a semilattice and every non-zero idempotent of $E_i$ is primitive. If the cardinal $\lambda_i$ is infinite then on the set $E_i = (\lambda_i \times \lambda_i) \cup \{0\}$ we define the semilattice operation in the similar way.

By $\tilde{E}_S$ we denote the orthogonal sum $\sum_{i \in I} E_i$. It is obvious that $\tilde{E}_S$ is a semilattice and every non-zero idempotent of $\tilde{E}_S$ is primitive. We determine on $\tilde{E}_S$ the topology of the Alexandroff one-point compactification $\tau_A$: all non-zero idempotents of $\tilde{E}_S$ are isolated points in $\tilde{E}_S$ and the family

$$\mathcal{B}(0) = \{U: U \ni 0 \text{ and } \tilde{E}_S \setminus U \text{ is finite} \}$$

is the base of the topology $\tau_A$ at zero $0 \in \tilde{E}_S$. Simple verifications show that $\tilde{E}_S$ with the topology $\tau_A$ is a Hausdorff compact topological semilattice. Later we denote $(\tilde{E}_S, \tau_A)$ by $\tilde{E}_S$.

Let $G_S = \prod_{i \in I} G_i$ be the direct product of pseudocompact paratopological groups $G_i$, $i \in I$, with the Tychonoff topology. Then Proposition 24 from [28] implies that $G_S$ is a pseudocompact paratopological group. Also by Corollary 3.3 we have that the product $\tilde{E}_S \times G_S$ is a Hausdorff compact topological semilattice. Later we denote $(\tilde{E}_S \times G_S, \tau)$ by $\tilde{E}_S \times G_S$.

For every $i \in I$ by $\pi_i: G_S = \prod_{i \in I} G_i \to G_i$ we denote the projection on the $i$-th factor.

Now, for every $i \in I$ we define the map $f_i: E_i \times G_S \to B_{\lambda_i}(G_i)$ by the formula $f_i((\alpha, \beta), g) = (\alpha, \pi_i(\beta), g)$ and $f_i(0, g) = 0_i$ is zero of the semigroup $B_{\lambda_i}(G_i)$, and put $f = \bigcup_{i \in I} f_i$. It is obvious that the map $f: \tilde{E}_S \times G_S \to S$ is well defined. The definition of the topology $\tau_A$ on $\tilde{E}_S$ implies that for every $((\alpha, \beta), g) \in E_i \times G_i \subseteq \tilde{E}_S \times G_i$ the set $\{(\alpha, \beta)\} \times G_i$ is open in $\tilde{E}_S \times G_S$ and hence the map $f$ is continuous at the point $((\alpha, \beta), g)$. Also for every $U(0) = S \setminus (B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup \cdots \cup B_{\lambda_{i_n}}(G_{i_n}))$ the set $f^{-1}(U(0)) = (\tilde{E}_S \setminus (\lambda_{i_1} \times \lambda_{i_1}) \cup \ldots \cup (\lambda_{i_n} \times \lambda_{i_n})) \times G_S$ is open in $\tilde{E}_S \times G_S$, and hence the map $f$ is continuous.

We observe that in the case when all maximal subgroups of $S$ are topological groups then $G_S = \prod_{i \in I} G_i$ is a pseudocompact topological group by Comfort–Ross theorem (see Theorem 1.4 in [10]).

Also, in the case of a Hausdorff semitopological semigroup $S$ the proof is similar.

The following theorem is a counterpart of Comfort–Ross Theorem for primitive pseudocompact semitopological inverse semigroups.

**Theorem 3.6.** Let $\{S_j: j \in I\}$ be a non-empty family of primitive semitopological inverse semigroups such that for each $j \in I$ the semigroup $S_j$ is either semiregular pseudocompact or Hausdorff countably compact, and moreover each maximal subgroup of $S_j$ a pseudocompact paratopological group. Then the direct product $\prod_{j \in I} S_j$ with the Tychonoff topology is a pseudocompact semitopological inverse semigroup.
Theorem 3.8. Let \( S \) be a primitive inverse topological semigroup. Then \( S \) is a continuous (not necessarily homomorphic) image of the product \( E_S \times G_S \), where \( E_S \) is a compact semilattice and \( G_S \) is a pseudocompact paratopological group provided one of the following conditions holds:

1. \( S \) is semiregular pseudocompact;
2. \( S \) is Hausdorff countably compact.

Proof. Since the direct product of the non-empty family of semitopological inverse semigroups is a semitopological inverse semigroup, it is sufficient to show that the space \( \prod_{j \in J} S_j \) is pseudocompact. For each \( j \in J \) let \( \tilde{E}_{S_j}, G_{S_j} \), and \( f_j : \tilde{E}_{S_j} \times G_{S_j} \to S_j \) be the semilattice, the group and the map, respectively, defined in the proof of Proposition 3.5. Since the space \( \prod_{j \in J} \tilde{E}_{S_j} \times G_{S_j} \) is homeomorphic to the product \( \prod_{j \in J} \tilde{E}_{S_j} \times \prod_{j \in J} G_{S_j} \) we conclude that by Theorem 3.2.4 from [13], Corollary 8.3 and Proposition 24 from [28] the space \( \prod_{j \in J} \tilde{E}_{S_j} \times G_{S_j} \) is pseudocompact. Now, since the map \( \prod_{j \in J} f_j : \prod_{j \in J} \left( \tilde{E}_{S_j} \times G_{S_j} \right) \to \prod_{j \in J} S_j \) is continuous we have that \( \prod_{j \in J} S_j \) is a pseudocompact topological space. \( \square \)

The proofs of the following two propositions are similar to Proposition 3.5 and Theorem 3.6 and they generalize Proposition 2.11 and Theorem 2.12 from [17].

Proposition 3.7. Let \( S \) be a primitive inverse topological semigroup. Then \( S \) is a continuous (not necessarily homomorphic) image of the product \( E_S \times G_S \), where \( E_S \) is a compact semilattice and \( G_S \) is a pseudocompact paratopological group provided one of the following conditions holds:

1. \( S \) is semiregular pseudocompact;
2. \( S \) is Hausdorff countably compact.

Theorem 3.8. Let \( \{S_i : i \in J \} \) be a non-empty family of primitive inverse semiregular pseudocompact (Hausdorff countably) topological semigroups. Then the direct product \( \prod_{j \in J} S_j \) with the Tychonoff topology is a pseudocompact inverse topological semigroup.

Let a Tychonoff topological space \( X \) be a topological sum of subspaces \( A \) and \( B \), i.e., \( X = A \bigoplus B \). It is obvious that every continuous map \( f : A \to K \) from \( A \) into a compact space \( K \) (resp., \( f : B \to K \) from \( B \) into a compact space \( K \)) extends to a continuous map \( \tilde{f} : X \to K \). This implies the following proposition:

Proposition 3.9. If a Tychonoff topological space \( X \) is a topological sum of subspaces \( A \) and \( B \), then \( \beta X \) is equivalent to the topological sum \( \beta A \bigoplus \beta B \).

The following theorem follows from Corollary 227 and Theorem 3.2 of [17], and it describes the structure of the Stone-Čech compactification of a primitive inverse pseudocompact quasi-regular topological semigroup.

Theorem 3.10. Let \( S \) be a primitive inverse pseudocompact quasi-regular topological semigroup. Then the Stone-Čech compactification of \( S \) admits a structure of primitive topological inverse semigroup with respect to which the inclusion mapping of \( S \) into \( \beta S \) is a topological isomorphism. Moreover, \( \beta S \) is topologically isomorphic to the orthogonal sum \( \sum_{i \in J} B_{\lambda_i}(\beta G_i) \) of topological Brandt \( \lambda_i \)-extensions \( B_{\lambda_i}(\beta G_i) \) of compact topological groups \( \beta G_i \) in the class of topological inverse semigroups for some finite cardinals \( \lambda_i \geq 1 \).

Theorem 3.11. Let \( S \) be a regular primitive inverse countably compact semitopological semigroup and \( S \) be an orthogonal sum of the family \( \{B_{\lambda_i}(G_i) : i \in J \} \) of semitopological Brandt semigroups with zeros. Suppose that for every \( i \in J \) there exists a maximal non-zero subgroup \( (G_i)_{\alpha_i} \), \( \alpha_i \in \lambda_i \), such that at least one of the following conditions holds:

1. the group \( (G_i)_{\alpha_i} \) is left precompact;
2. the group \( (G_i)_{\alpha_i} \) is left \( \omega \)-precompact pseudocompact;
3. the semigroup \( S_{\alpha_i} = (G_i)_{\alpha_i} \cup \{0\} \) is a topological semigroup.

Then the Stone-Čech compactification of \( S \) admits a structure of primitive inverse semitopological semigroup with continuous inversion with respect to which the inclusion mapping of \( S \) into \( \beta S \) is a topological isomorphism. Moreover, \( \beta S \) is topologically isomorphic to the orthogonal sum \( \sum_{i \in J} B_{\lambda_i}(\beta G_i) \)
of compact topological Brandt \(\lambda_i\)-extensions \(B_{\lambda_i}(\beta G_i)\) of compact topological groups \(\beta G_i\) in the class of semitopological semigroups for some cardinals \(\lambda_i \geq 1\).

**Proof.** By Theorem 2.14 the semigroup \(S\) is topologically isomorphic to an orthogonal sum \(\sum_{i \in \mathcal{F}} B_{\lambda_i}(G_i)\) of topological Brandt \(\lambda_i\)-extensions \(B_{\lambda_i}(G_i)\) of countably compact paratopological groups \(G_i\) in the class of semitopological semigroups for some cardinals \(\lambda_i \geq 1\), such that any non-zero \(\mathcal{H}\)-class of \(S\) is an open-and-closed subset of \(S\), and the family \(\mathcal{B}(0)\) defined by formula (11) determines a base of the topology at zero 0 of \(S\). Since the space \(S\) regular any non-zero \(\mathcal{H}\)-class of \(S\) is an open-and-closed subset of \(S\) we have that every maximal subgroup of \(S\) is a topological group [28, Proposition 3]. Hence we get that \(S\) is topologically isomorphic to the orthogonal sum \(\sum_{i \in \mathcal{F}} B_{\lambda_i}(G_i)\) of topological Brandt \(\lambda_i\)-extensions \(B_{\lambda_i}(G_i)\) of countably compact topological groups \(G_i\) in the class of semitopological semigroups for some cardinals \(\lambda_i \geq 1\). Then by Proposition 2.18 the topological space of the semigroup \(S\) is Tychonoff, and hence the Stone-Čech compactification of \(S\) exists.

By Theorem 3.9 \(S \times S\) is a pseudocompact topological space. Now by Theorem 1 of [14], we have that \(\beta(S \times S)\) is equivalent to \(\beta S \times \beta S\), and hence by Theorem 1.1 of [28], \(S\) is a subsemigroup of the compact semitopological semigroup \(\beta S\).

By Proposition 3.9 for every non-zero \(\mathcal{H}\)-class \((G_i)_{k,l}, k, l \in \lambda_i\), we have that \(\text{cl}_{\beta S}((G_i)_{k,l})\) is equivalent to \(\beta(G_i)_{k,l}\), and hence it is equivalent to \(\beta G_i\). Therefore we may naturally consider the space \(\sum_{i \in \mathcal{F}} B_{\lambda_i}(\beta G_i)\) as a subspace of the space \(\beta S\). Suppose that \(\sum_{i \in \mathcal{F}} B_{\lambda_i}(\beta G_i) \neq \beta S\). We fix an arbitrary \(x \in \beta S \setminus \sum_{i \in \mathcal{F}} B_{\lambda_i}(\beta G_i)\). Then Hausdorffness of \(\beta S\) implies that there exist open neighbourhoods \(V(x)\) and \(V(0)\) of the points \(x\) and the zero 0 in \(\beta S\), respectively, and there exist finitely many indices \(i_1, \ldots, i_k \in \mathcal{F}\) and finitely many pairs of indices \((\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_k}, \beta_{i_k})\) such that \(V(0) \cap \beta S \supseteq \beta S \setminus \bigcup_{i \in \mathcal{F}} B_{\lambda_i}(G_i)_{\alpha_i, \beta_i}\). Then we have that

\[
V(x) \cap \beta S \subseteq \bigcup_{i \in \mathcal{F}} B_{\lambda_i}(G_i)_{\alpha_i, \beta_i} \subseteq \bigcup_{i \in \mathcal{F}} B_{\lambda_i}(\beta G_i)_{\alpha_i, \beta_i}.
\]

But this contradicts that \(x\) is an accumulation point of \(\sum_{i \in \mathcal{F}} B_{\lambda_i}(\beta G_i)\) in \(\beta S\), which does not belong to \(\sum_{i \in \mathcal{F}} B_{\lambda_i}(\beta G_i)\), because \((\beta G_i)_{\alpha_i, \beta_i} \cup \cdots \cup (\beta G_i)_{\alpha_i, \beta_i}\) is a compact subset of \(\beta S\).

Recall [11] that a Bohr compactification of a semitopological semigroup \(S\) is a pair \((b, B(S))\) such that \(B(S)\) is a compact semitopological semigroup, \(b: S \rightarrow B(S)\) is a continuous homomorphism, and if \(g: S \rightarrow T\) is a continuous homomorphism of \(S\) into a compact semitopological semigroup \(T\), then there exists a unique continuous homomorphism \(f: B(S) \rightarrow T\) such that the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{b} & B(S) \\
\downarrow{g} & & \downarrow{f} \\
T & \xrightarrow{} & T
\end{array}
\]

commutes. In the sequel, similar as in the general topology by the Bohr compactification of a semitopological semigroup \(S\) we shall mean not only pair \((b, B(S))\) but also the compact semitopological semigroup \(B(S)\).

The definitions of the Stone-Čech compactification and the Bohr compactification, and Theorem 3.11 imply the following corollary:

**Corollary 3.12.** Let \(S\) be a Hausdorff primitive inverse countably compact semitopological semigroup such that every maximal subgroup of \(S\) is a pseudocompact topological group and \(S\) be an orthogonal sum of the family \(\{B_{\lambda_i}(G_i): i \in \mathcal{F}\}\) of semitopological Brandt semigroups with zeros. Then the Bohr compactification of \(S\) admits a structure of primitive inverse semitopological semigroup with continuous inversion with respect to which the inclusion mapping of \(S\) into \((b, B(S))\) is a topological isomorphism. Moreover, \((b, B(S))\) is topologically isomorphic to the orthogonal sum \(\sum_{i \in \mathcal{F}} B_{\lambda_i}(\beta G_i)\) of topological Brandt \(\lambda_i\)-extensions \(B_{\lambda_i}(\beta G_i)\) of compact topological groups \(\beta G_i\) in the class of semitopological semigroups for some cardinals \(\lambda_i \geq 1\).
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