Degenerate Lah–Bell polynomials arising from degenerate Sheffer sequences

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Abstract

Umbral calculus is one of the important methods for obtaining the symmetric identities for the degenerate version of special numbers and polynomials. Recently, Kim–Kim (J. Math. Anal. Appl. 493(1):124521, 2021) introduced the $\lambda$-Sheffer sequence and the degenerate Sheffer sequence. They defined the $\lambda$-linear functionals and $\lambda$-differential operators, respectively, instead of the linear functionals and the differential operators of umbral calculus established by Rota. In this paper, the author gives various interesting identities related to the degenerate Lah–Bell polynomials and special polynomials and numbers by using degenerate Sheffer sequences, and at the same time derives the inversion formulas of these identities.

MSC: 11B73; 11B83; 05A19

Keywords: Degenerate Lah–Bell numbers and polynomials; The degenerate Sheffer sequence; The degenerate Bernoulli (Euler) polynomials; The degenerate Frobenius–Euler polynomials; The degenerate Daehee polynomials; The degenerate Bell polynomials

1 Introduction

It is important to note that many academics in the field of mathematics have been researching various degenerate versions of special polynomials and numbers not only in some arithmetic and combinatorial aspects but also in applications to differential equations, identities of symmetry and probability theory [9, 12, 14, 16–23], beginning with Carlitz's degenerate Bernoulli polynomials and the degenerate Euler polynomials [2].

Moreover, umbral calculus, established by Rota in the 1970s, was based on modern concepts such as linear functionals, linear operators, and adjoints [28]. Umbral calculus is one of the important methods for obtaining the symmetric identities for the degenerate version of special numbers and polynomials [5, 6, 24, 25, 28]. Recently, Kim–Kim [11] introduced the $\lambda$-Sheffer sequence and the degenerate Sheffer sequence. They defined the $\lambda$-linear functionals and $\lambda$-differential operators, respectively, instead of the linear functionals and the differential operators used by Rota [28]. Also, Kim et al. introduced the Lah–Bell polynomials and studied some identities of Lah–Bell polynomials [10, 25]. The two papers mentioned above inspired me. So, I focus on finding the noble identities of degenerate Lah–Bell polynomials in terms of quite a few well-known special polynomials.
and numbers arising from the degenerate Sheffer sequence. In addition, the author derives the inversion formulas of the identities obtained in this paper. They include the degenerate and other special polynomials and numbers such as Lah numbers, the degenerate falling factorial, the degenerate Bernoulli polynomials and numbers, degenerate Frobenius–Euler polynomials and numbers of order $r$, the degenerate Deahee polynomials, the degenerate Bell polynomials, and degenerate Stirling numbers of the first and second kinds.

Now, we give some definitions and properties needed in this paper.

The unsigned Lah number $L(n, k)$ counts the number of ways of all distributions of $n$ balls, labeled $1, 2, \ldots, n$, among $k$ unlabeled, contents-ordered boxes, with no box left empty and have an explicit formula

$$L(n, k) = \binom{n-1}{k-1} \frac{n!}{k!} \quad \text{(see [10, 25]).} \quad (1)$$

From (1), the generating function of $L(n, k)$ is given by

$$\frac{1}{k!} \left( \frac{t}{1-t} \right)^k = \sum_{n=k}^{\infty} L(n, k) \frac{t^n}{n!}, \quad (k \geq 0) \quad \text{(see [10, 25–27]).} \quad (2)$$

Recently, Lah–Bell polynomials were introduced by Kim–Kim to be

$$e^{(1-t^{-1})} = \sum_{n=0}^{\infty} B_n^x(t) \frac{t^n}{n!} \quad \text{(see [10]).} \quad (3)$$

When $x = 1$, $B_n^x = B_n^1(1)$ are called Lah–Bell numbers.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential function is defined by

$$e^{x \lambda \frac{t}{t-1}} = \sum_{n=0}^{\infty} \beta_n^{(x)}(\lambda) \frac{t^n}{n!} \quad \text{(see [2, 9, 12, 14, 16, 17, 19–23]).} \quad (4)$$

By Taylor expansion, we get

$$e^{x \lambda \frac{t}{t-1}} = \sum_{n=0}^{\infty} (x)_{n, \lambda} \frac{t^n}{n!} \quad \text{(see [9, 12, 14, 16, 17, 19–23]),} \quad (5)$$

where $(x)_{0, \lambda} = 1$, $(x)_{n, \lambda} = x(x-\lambda)(x-2\lambda) \cdots (x-(n-1)\lambda) \; (n \geq 1)$.

It is known that

$$(1-t)^{-m} = \sum_{l=0}^{\infty} \binom{-m}{l} (-1)^l t^l = \sum_{l=0}^{\infty} \langle m \rangle_l \frac{t^l}{l!} \quad \text{(see [1, 4]).} \quad (6)$$

where $\langle x \rangle_0 = 1$, $\langle x \rangle_n = x(x+1)(x+2) \cdots (x+n-1), \; (n \geq 1)$.

The degenerate Bernoulli polynomials and degenerate Euler polynomials of order $r$, respectively, are given by the generating functions

$$\left( \frac{t}{e^{\lambda \frac{t}{t-1}} - 1} \right)^r = \sum_{n=0}^{\infty} \beta_n^{(x)}(\lambda) \frac{t^n}{n!} \quad \text{(see [2, 7, 21])} \quad (7)$$
and

\[
\left( \frac{2}{e_{\lambda}(t) + 1} \right)^r e_{\lambda}^r(t) = \sum_{n=0}^{\infty} E_{n,\lambda}^{(r)}(x) \frac{t^n}{n!} \quad \text{(see [2, 7, 15, 18]).} \tag{8}
\]

We note that \( \beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0) \) and \( E_{n,\lambda}^{(r)} = E_{n,\lambda}^{(r)}(0) \) \((n \geq 0)\) are called degenerate Bernoulli and degenerate Euler numbers of order \( r \), respectively.

Kim et al. introduced the degenerate Frobenius–Euler polynomials of order \( r \) defined by

\[
\left( \frac{1 - u}{e_{\lambda}(t) - u} \right)^r e_{\lambda}^r(t) = \sum_{n=0}^{\infty} h_{n,\lambda}^{(r)}(x|u) \frac{t^n}{n!}, \quad (u \neq 1, u \in \mathbb{C}) \quad (k \geq 0) \quad \text{(see [15]).} \tag{9}
\]

When \( x = 0 \), \( h_{n,\lambda}^{(r)}(u) = h_{n,\lambda}^{(r)}(0|u) \) are called degenerate Frobenius–Euler numbers of order \( r \).

The degenerate Daehee polynomials are defined by

\[
\log_{e_{\lambda}}(1 + t) (1 + t)^r = \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{t^n}{n!} \quad \text{(see [9]).} \tag{10}
\]

Here \( \log_{e_{\lambda}}(1 + t) = \frac{1}{\lambda}((1 + t)^{\lambda} - 1) \) and \( \log_{e_{\lambda}}(e_{\lambda}(t)) = e_{\lambda}(\log_{e_{\lambda}}(t)) = t \).

When \( x = 0 \), \( D_{n,\lambda} = D_{n,\lambda}(0) \) are called degenerate Daehee numbers.

The Bell polynomials are defined by the generating function

\[
e^{(e^{x-1})} = \sum_{n=0}^{\infty} \frac{\text{Bel}_n(x)}{n!} t^n \quad \text{(see [8, 19, 22]).} \tag{11}
\]

Kim–Kim introduced the degenerate Bell polynomial given by

\[
e_{\lambda}^r(x, (t) - 1) = \sum_{l=0}^{\infty} \frac{\text{Bel}_{l,\lambda}(x)}{l!} t^l \quad \text{(see [19]).} \tag{12}
\]

For \( n \geq 0 \), it is well known that the Stirling numbers of the first and second kind, respectively, are given by

\[
(x)_n = \sum_{l=0}^{n} S_1(n, l)x^l \quad \text{and} \quad \frac{1}{k!}(\log(1 + t))^k = \sum_{n-k} S_1(n, k) \frac{t^n}{n!} \quad \text{(see [2, 3])} \tag{13}
\]

and

\[
x^n = \sum_{l=0}^{n} S_2(n, l)(x)_l \quad \text{and} \quad \frac{1}{k!}(e^t - 1)^k = \sum_{n-k} S_2(n, k) \frac{t^n}{n!} \quad \text{(see [2, 3]),} \tag{14}
\]

where \((x)_0 = 1, (x)_n = x(x - 1) \cdots (x - n + 1) \quad (n \geq 1)\).
Moreover, the degenerate Stirling numbers of the first and second kind, respectively, are given by

$$\binom{\lambda}{n} \sum_{l=0}^{n} S_{1,\lambda}(n, l) x^l$$ and

$$\frac{1}{k!} \left( \log_\lambda (1 + t) \right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!} \quad (k \geq 0) \text{ (see [12, 14])},$$

and

$$\binom{\lambda}{n} \sum_{l=0}^{n} S_{2,\lambda}(n, l) x^l$$ and

$$\frac{1}{k!} \left( e_\lambda (t) – 1 \right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!} \quad (k \geq 0) \text{ (see [12, 14]).}$$

For $k \in \mathbb{Z}$, Kim–Kim introduced the modified polyexponential function as

$$E_{\lambda}(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} \quad \text{(see [18, 21])}.$$  

By (17), we see that $E_{\lambda}(x) = e^x - 1$.

Kim–Jang considered the type 2 degenerate poly-Euler polynomials which are given by the generating function to be

$$E_{\lambda}(\log_\lambda(1 + 2t)) \left( e_\lambda(t) + 1 \right)^k = \sum_{n=0}^{\infty} E_{\lambda}(n, \lambda) \frac{t^n}{n!} \quad \text{(see [13]).}$$

When $x = 0$, $E_{\lambda}(0)$ are called type 2 degenerate poly-Euler numbers.

Since $E_{\lambda}(\log_\lambda(1 + 2t)) = 2t$, we see that $E_{\lambda}(x) = E_n(x)$ ($n \geq 0$) are the degenerate Euler polynomials.

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all power series in the variable $t$ over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \big| a_k \in \mathbb{C} \right\}.$$  

Let $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{P}^*$ be the vector space all linear functional on $\mathbb{P}$.

$$\mathbb{P}_n = \left\{ P(x) \in \mathbb{C}[x] \big| \deg P(x) \leq n \right\} \quad (n \geq 0).$$

Then $\mathbb{P}_n$ is an $(n + 1)$-dimensional vector space over $\mathbb{C}$.

Recently, Kim–Kim [11] considered the $\lambda$-linear functional and $\lambda$-differential operator as follows:

For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$ and a fixed nonzero real number $\lambda$, each $\lambda$ gives rise to the linear functional $\langle f(t)| \cdot \rangle_\lambda$ on $\mathbb{P}$, called $\lambda$-linear functional given by $f(t)$, which is defined
by
\[
\left\{ f(t) \right\}_{n, \lambda} = a_n, \quad \text{for all } n \geq 0 \text{ (see [11])}.
\]
and in particular \( \langle t^k \rangle_{n, \lambda} = n! \delta_{n,k} \), for all \( n, k \geq 0 \), where \( \delta_{n,k} \) is Kronecker's symbol.

For \( \lambda = 0 \), we observe that the linear functional \( \left\{ f(t) \right\}_{0} \) agrees with the one in \( \langle f(t)x^n \rangle = a_k, \quad (k \geq 0) \).

For each \( \lambda \in \mathbb{R} \), and each nonnegative integer \( k \), they also defined the differential operator on \( \mathbb{P} \) by
\[
\langle t^k \rangle_{n, \lambda} = \begin{cases} 
(n \lambda)(x)_{n-k, \lambda}, & \text{if } k \leq n, \\
0, & \text{if } k > n \text{ (see [11])}.
\end{cases}
\]
and for any power series \( f(t) = \sum_{k=0}^{\infty} a_k t^k \in \mathcal{F} \), \( \langle f(t) \rangle_{n, \lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k, \lambda} \) (\( n \geq 0 \)).

Let \( f(t) \) and \( g(t) \) be a delta series and an invertible series, respectively. Then there exist unique sequences \( s_{n, \lambda}(x) \) such that we have the orthogonality conditions
\[
\langle g(t)f(t)^k \rangle_{n, \lambda} = n! \delta_{n,k} \quad (n, k \geq 0) \text{ (see [11])}.
\]

The sequences \( s_{n, \lambda}(x) \) are called the \( \lambda \)-Sheffer sequences for \( (g(t), f(t)) \), which are denoted by \( s_{n, \lambda}(x) \sim (g(t), f(t))_\lambda \).

The sequence \( s_{n, \lambda}(x) \sim (g(t), f(t))_\lambda \) if and only if
\[
\frac{1}{g(f(t))} \frac{d^n}{dt^n} (f(t)) = \sum_{k=0}^{\infty} \frac{s_{k, \lambda}(x)}{k!} t^k, \quad (n, k \geq 0) \text{ (see [11])}.
\]

Assume that, for each \( \lambda \in \mathbb{R}^+ \) of the set of nonzero real numbers, \( s_{n, \lambda}(x) \) is \( \lambda \)-Sheffer for \( (g_{\lambda}(t), f_{\lambda}(t)) \). Assume also that \( \lim_{t \to 0} f_{\lambda}(t) = f(t) \) and \( \lim_{t \to 0} g_{\lambda}(t) = g(t) \), for some delta series \( f(t) \) and an invertible series \( g(t) \). Then \( \lim_{t \to 0} f_{\lambda}(t) = f(t) \), where is the compositional inverse of \( f(t) \) with \( \tilde{f}(t) = \tilde{f}(t) = t \). Let \( \lim_{t \to 0} s_{\lambda, \lambda}(x) = s_{\lambda}(x) \). In this case, Kim–Kim called this the family \( \{s_{n, \lambda}(x)\}_{\lambda \in \mathbb{R}^+, \lambda \geq 0} \) of \( \lambda \)-Sheffer sequences \( s_{n, \lambda} \) are the degenerate (Sheffer) sequences for the Sheffer polynomial \( s_n(x) \).

Let \( s_{n, \lambda}(x) \sim (g(t), f(t))_\lambda \) and \( r_{n, \lambda}(x) \sim (h(t), g(t))_\lambda \) (\( n \geq 0 \)). Then
\[
s_{n, \lambda}(x) = \sum_{k=0}^{n} \mu_{n,k} r_{k, \lambda}(x) \quad (n \geq 0),
\]
where \( \mu_{n,k} = \frac{1}{k!} \left( \frac{\partial^k}{\partial t^k} \frac{1}{g(f(t))} \langle h(t) \tilde{f}(t) \rangle \right)_{n, \lambda}(x) \quad (n, k \geq 0) \text{ (see [11]).} \)
2 Degenerate Lah–Bell polynomials arising from degenerate Sheffer sequences

In this section, we derive several identities between the degenerate Lah–Bell polynomials and some other polynomials arising from degenerate Sheffer sequences.

Kim–Kim introduced the degenerate Lah–Bell polynomials given by

$$e^t \left( \frac{1}{1-t} \right) = \sum_{n=0}^{\infty} B_{n,k}^{L} (x) \frac{t^n}{n!} \quad (n, k \geq 0) \text{ (see [10]).} \quad (26)$$

When $x = 1$, $B_{n,k}^{L} := B_{n,k}^{L} (1)$ are called the $n$th degenerate Lah–Bell numbers.

When $\lambda \to 0$, $\lim_{\lambda \to 0} B_{n,k}^{L} = B_{n}^{L}$ are the $n$th Lah–Bell numbers.

For $n \in \mathbb{N} \cup \{ 0 \}$ and $P(x) = \sum_{k=0}^{n} Z_{k} B_{k,n}^{L} (x) \in \mathbb{P}_{n}$, by using (23), we observe that

$$\left\langle \left( \frac{t}{1+t} \right)^{k} \left| P(x) \right\rangle \bigg|_{\lambda} = \sum_{l=0}^{n} Z_{l} \left\langle \left( \frac{t}{1+t} \right)^{k} \left| B_{k,l}^{L} (x) \right\rangle \bigg|_{\lambda} = \sum_{l=0}^{n} Z_{l} l! Z_{k,l} = k! Z_{k}. \quad (27)$$

From (27), we have

$$Z_{k} = \frac{1}{k!} \left( \frac{t}{1+t} \right)^{k} \left| P(x) \right\rangle \bigg|_{\lambda}. \quad (28)$$

Thus, we have

$$P(x) = \sum_{k=0}^{n} Z_{k} B_{k,n}^{L} (x) \quad \text{where} \quad Z_{k} = \frac{1}{k!} \left( \frac{t}{1+t} \right)^{k} \left| P(x) \right\rangle \bigg|_{\lambda}. \quad (29)$$

**Theorem 1** For $n \in \mathbb{N} \cup \{ 0 \}$, we have

$$B_{n,k}^{L} (x) = \sum_{k=0}^{n} L(n,k) B_{k,n}^{L} (x) = \sum_{k=0}^{n} \left( \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} (l)_{n} \right) (x)_{k,l}. \quad (30)$$

As the inversion formula of (30), we have

$$(x)_{n,k} = \sum_{k=0}^{n} (-1)^{n-k} L(n,k) B_{k,n}^{L} (x) = \sum_{k=0}^{n} \left( \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{l+n} (l)_{n} \right) B_{k,n}^{L} (x). \quad (31)$$

**Proof** From (5), (24) and (26), we consider the following two Sheffer sequences:

$$B_{n,k}^{L} (x) \sim \left( 1, -\frac{t}{1+t} \right) \lambda \quad \text{and} \quad (x)_{n,k} \sim (1, t)_{\lambda}. \quad (32)$$

From (6), (25) and (32), we have

$$B_{n,k}^{L} (x) = \sum_{k=0}^{n} \mu_{n,k} (x) B_{k,n}^{L} (x). \quad (33)$$
where

\[ \mu_{n,k} = \frac{1}{k!} \left( \frac{t}{1-t} \right)^k \langle x \rangle_{n,\lambda} = L(n,k), \]

or

\[ \mu_{n,k} = \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \left( \frac{1}{1-t} \right)^l \langle x \rangle_{n,\lambda} = \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} \langle l \rangle_n. \]  

(34)

Therefore, we have the identity (30).

To find the inversion formula of (30), let \( P(x) = (x)_{n,\lambda}. \) From (29),

\[ (x)_{n,\lambda} = \sum_{k=0}^n Z_k B^L_{k,\lambda}(x), \]  

where

\[ Z_k = \frac{1}{k!} \left( \frac{t}{1+t} \right)^k \langle x \rangle_{n,\lambda} = (-1)^{n-k} L(n,k), \]

or

\[ Z_k = \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^l \left( \frac{1}{1+t} \right)^l \langle x \rangle_{n,\lambda} = \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} (-1)^{l+n} \langle l \rangle_n. \]  

(36)

Therefore, from (35) and (36), we have the identity (31).  

\[ \Box \]

**Theorem 2** For \( n \in \mathbb{N} \cup \{0\} \) and \( r \in \mathbb{N}, \) we have

\[ B^L_{n,\lambda}(x) = \sum_{k=0}^n \sum_{l=0}^k \binom{n}{l} \binom{n-m-1}{m} L(l,k)L(n-m,l) \beta_{k,\lambda}(x). \]

(37)

As the inversion formula of (37), we have

\[ \beta_{n,\lambda}(x) = \sum_{k=0}^n \sum_{l=0}^k \binom{n}{l} \binom{n-m-1}{m} (-1)^l \binom{n-m}{m} \beta_{n-m,k} \beta_{k,\lambda}(x) \]

\[ = \sum_{k=0}^n \sum_{m=0}^n \binom{n}{m} (-1)^{n-m} L(n-m,k) \beta_{m,\lambda}(x) \]  

(38)

where \( \beta_{n,\lambda}(x) \) are the degenerate Bernoulli polynomials.

**Proof** From (7), (24) and (26), we consider the following two degenerate Sheffer sequences:

\[ B^L_{n,\lambda}(x) \sim \left( 1, \frac{t}{1+t} \right)_\lambda \quad \text{and} \quad \beta_{n,\lambda}(x) \sim \left( \frac{e\lambda(t) - 1}{t}, t \right)_\lambda. \]

(39)

From (2), (5) and (25), we have

\[ B^L_{n,\lambda}(x) = \sum_{k=0}^n \mu_{n,k} \beta_{k,\lambda}(x), \]  

(40)
where

\[
\mu_{n,k} = \frac{1}{k!} \left( \frac{e_{n,k} \left( \frac{t}{1-t} \right) - 1}{\left( \frac{t}{1-t} \right)^k} \right) \left( \frac{t}{1-t} \right) \left( x \right)_{n,\lambda}
\]

\[
= \left( \left( \frac{e_{n,k} \left( \frac{t}{1-t} \right) - 1}{\left( \frac{t}{1-t} \right)^k} \right) \left( \frac{t}{1-t} \right) \right) \left( x \right)_{n,\lambda}
\]

\[
= \sum_{l=k}^{n} \left( \begin{array}{l} n \\ l \end{array} \right) L(l, k) \left( \frac{e_{n,k} \left( \frac{t}{1-t} \right) - 1}{\left( \frac{t}{1-t} \right)^k} \right) \left( x \right)_{n-l,\lambda}
\]

To find the inversion formula of (37), let \( P(x) = \beta_{n,\lambda}(x) \). From (29), we have

\[
\beta_{n,\lambda}(x) = \sum_{k=0}^{n} Z_k B_{k,\lambda}^n(x) \quad (n \geq 0).
\]

where

\[
Z_k = \frac{1}{k!} \left( \frac{t}{1+t} \right)^k \left| \beta_{n,\lambda}(x) \right|_{\lambda}
\]

\[
= \frac{1}{k!} \sum_{l=0}^{k} \left( \begin{array}{l} k \\ l \end{array} \right) \left( \frac{t}{1+t} \right)^l \left| \beta_{n,\lambda}(x) \right|_{\lambda}
\]

\[
= \frac{1}{k!} \sum_{l=0}^{k} \left( \begin{array}{l} k \\ l \end{array} \right) \left( \frac{t}{1+t} \right)^l \sum_{\nu=0}^{n} \left( l+\nu-1 \atop \nu \right) \left| \sum_{m=0}^{n} \left( \begin{array}{l} n \\ m \end{array} \right) \beta_{m,\lambda}(x)_{n-m,\lambda} \right|_{\lambda}
\]

\[
= \frac{1}{k!} \sum_{l=0}^{k} \left( \begin{array}{l} k \\ l \end{array} \right) \left( \frac{t}{1+t} \right)^l \sum_{\nu=0}^{n} \left( l+\nu-1 \atop \nu \right) \sum_{m=0}^{n} \left( \begin{array}{l} n \\ m \end{array} \right) \beta_{m,\lambda}(x)_{n-m,\lambda}
\]

\[
= \frac{1}{k!} \sum_{\nu=0}^{n} \left( l+\nu-1 \atop \nu \right) \sum_{m=0}^{n} \left( \begin{array}{l} n \\ m \end{array} \right) \beta_{m,\lambda}(x)_{n-m,\lambda}
\]

Stated differently, we get

\[
Z_k = \frac{1}{k!} \left( \frac{t}{1+t} \right)^k \left| \beta_{n,\lambda}(x) \right|_{\lambda}
\]

\[
= \frac{1}{k!} \left( \frac{t}{1+t} \right)^k \sum_{m=0}^{n} \left( \begin{array}{l} n \\ m \end{array} \right) \beta_{m,\lambda}(x)_{n-m,\lambda}
\]
Therefore, from (42) and (43) we have the identity (38).

**Theorem 3** For \( n \in \mathbb{N} \cup \{0\} \), we have

\[
\mathcal{B}_{n,\lambda}^L(x) = \frac{1}{(1-u)^r} \sum_{k=0}^{n} \left( \sum_{l=0}^{k} \binom{n}{l} \binom{l+n-m}{n-m} (-1)^{l+n-m} \beta_{m,\lambda} (-1)^{n-m-k} L(n-m,k) \right) h_{n,\lambda}^{(r)}(x|u).
\] (44)

As the inversion formula of (44), we have

\[
h_{n,\lambda}^{(r)}(x|u) = \sum_{k=0}^{n} \left( \sum_{l=0}^{k} \binom{n}{l} \binom{l+n-m}{n-m} (-1)^{l+n-m} \beta_{m,\lambda} (-1)^{n-m-k} L(n-m,k) \right) \mathcal{B}_{n,\lambda}^L(x),
\] (45)

where \( h_{n,\lambda}^{(r)}(x|u) \) are the degenerate Frobenius–Euler polynomials of order \( r \).

**Proof** From (9), (24) and (26), we consider the following two degenerate Sheffer sequences:

\[
\mathcal{B}_{n,\lambda}^L(x) \sim \left( 1, \frac{t}{1+t} \right)_\lambda \quad \text{and} \quad h_{n,\lambda}^{(r)}(x|u) \sim \left( \left( \frac{e_{\lambda}(t)-u}{1-u} \right)^r, t \right)_\lambda.
\] (46)

By using (2), (5), (25) and (46), we have

\[
\mathcal{B}_{n,\lambda}^L(x) = \sum_{k=0}^{n} \mu_{n,k} h_{k,\lambda}^{(r)}(x|u),
\] (47)

Here

\[
\mu_{n,k} = \frac{1}{k!} \left( \frac{e_{\lambda}(\frac{t}{1+t})-u}{1-u} \right)^r \left( \frac{t}{1-t} \right)_\lambda^k \left( x \right)_n = \left( \left( \frac{e_{\lambda}(\frac{t}{1+t})-u}{1-u} \right)^r \left( \frac{t}{1-t} \right)_\lambda^k \left( x \right)_n \right)_\lambda
\] (48)
For $n$ Corollary 4

In the same way as (42) and (43), we have

where $E_{l,m,\lambda}(x)$

By the inversion formula of (51), we have

Therefore, from (47) and (48), we get the identity (44).

To find the inversion formula of (44), by (29), we have

In the same way as (42) and (43), we have

In another way, we can get

Therefore, from (49) and (50), we have the identity (45).

When $u = -1$ in Theorem 3, we have the following corollary.

**Corollary 4** For $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, we have

By the inversion formula of (51), we have

where $E_{l,m,\lambda}(x)$ are the degenerate Euler polynomials of order $r$. 
Theorem 5 For \( n \in \mathbb{N} \cup \{0\} \) and \( r \in \mathbb{N} \), we have

\[
B_{n,\lambda}^L(x) = \sum_{k=0}^{n} \left( \sum_{m=0}^{n} \sum_{j=0}^{n-m} \binom{n}{m} \frac{(1)_{j+1,\lambda}}{j+1} S_{2,\lambda}(l, k)L(m, l)L(n - m, l) \right) D_{k, \lambda}(x).
\] (52)

As the inversion formula of (52), we have

\[
D_{n, \lambda}(x) = \sum_{k=0}^{n} \left( \sum_{m=0}^{n} \sum_{j=0}^{n-m} \binom{n}{m} (-1)^{n-m-k} S_{2, \lambda}(n - m, l)L(n - m, k) D_{m, \lambda} \right) B_{k, \lambda}^L(x),
\] (53)

where \( D_{n, \lambda}(x) \) are the degenerate Daehee polynomials.

Proof From (10) and (24), we consider the following two degenerate Sheffer sequences:

\[
B_{n, \lambda}^L(x) \sim \left(1, \frac{t}{1+t}\right)_\lambda \quad \text{and} \quad D_{n, \lambda}(x) \sim \left(\frac{e_{\lambda}(t)}{t}, e_{\lambda}(t) - 1\right)_\lambda.
\] (54)

From (2), (16), (25) and (54), we have

\[
B_{n, \lambda}^L(x) = \sum_{k=0}^{n} \mu_{n,k} D_{k, \lambda}(x),
\] (55)

where

\[
\mu_{n,k} = \frac{1}{k!} \left( \frac{e_{\lambda}(t)}{t} - 1 \right) \left( \frac{t}{1-t} \right)^k \left( e_{\lambda}(t) - 1 \right)_\lambda.
\] (56)

Therefore, from (55) and (56), we get the identity (52).
To find the inversion formula of (52), from (29), we have

\[ D_{n,\lambda}(x) = \sum_{k=0}^{n} Z_k B_k^L(x). \]  

(57)

By using \((1 + t)^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) and the first equation of (15), we have

\[ Z_k = \left\langle \frac{1}{k!} \left( \frac{t}{1 + t} \right)^k \right| D_{n,\lambda}(x) \right\rangle_{\lambda} = \sum_{m=0}^{n} \binom{n}{m} D_{m,\lambda}(x) \sum_{j=0}^{n-m} \left( \frac{1}{k!} \left( \frac{t}{1 + t} \right)^k \right)_{\lambda} S_{1,\lambda}(n-m,j)(x)_{n-m,\lambda}. \]  

(58)

Therefore, from (57) and (58), we have the identity (53).

\[ \square \]

**Theorem 6** For \( n \in \mathbb{N} \cup \{0\} \), we have

\[ B^L_{n,\lambda}(x) = \sum_{k=0}^{n} \binom{n}{k} \left( \sum_{j=0}^{n} (-1)^{j-k} S_{2,\lambda}(n,j)L(l,k) \right) \text{Bel}_{k,\lambda}(x). \]  

(59)

As the inversion formula of (59), we have

\[ \text{Bel}_{n,\lambda}(x) = \sum_{k=0}^{n} \left( \sum_{j=0}^{n} (-1)^{j-k} S_{2,\lambda}(n,j)L(l,k) \right) B^L_{k,\lambda}(x), \]  

(60)

where \( \text{Bel}_{n,\lambda}(x) \) are the degenerate Bell polynomials.

**Proof** From (12), (24) and (26), we consider two degenerate Sheffer sequences as follows:

\[ B^L_{n,\lambda}(x) \sim \left( 1, \frac{t}{1 + t} \right)_{\lambda} \quad \text{and} \quad \text{Bel}_{n,\lambda}(x) \sim \left( 1, \log_{\lambda}(1 + t) \right)_{\lambda}. \]  

(61)

By using (2), (15), (25) and (61), we have

\[ B^L_{n,\lambda}(x) = \sum_{k=0}^{n} \mu_{n,k} \text{Bel}_{k,\lambda}(x), \]  

(62)
where
\[
\mu_{n,k} = \frac{1}{k!} \left( \log \left( 1 + \frac{t}{1 - t} \right) \right)^k \langle x \rangle^{(n)} \lambda
= \sum_{l=k}^{n} S_{1,\lambda}(l,k) \left( \frac{t}{1 - t} \right)^l \langle x \rangle^{(n)} \lambda \right) = \sum_{l=k}^{n} S_{1,\lambda}(l,k) L(n,l).
\]

Therefore from (62) and (63), we get the identity (59).

To find inversion formula of (59), from (29), we have
\[
\text{Bel}_{n,\lambda}(x) = \sum_{k=0}^{n} Z_k \text{B}_{L,\lambda}^k(x).
\]

From (5) and (16), we observe that
\[
\sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}(x) \frac{x^n}{n!} = e^x \left( e^{\lambda (t+1)} - 1 \right) = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} S_{2,\lambda}(n,l) \langle x \rangle \right) \frac{x^n}{n!}.
\]

Thus, by using Bel_{n,\lambda}(x) = \sum_{l=0}^{n} S_{2,\lambda}(n,l) \langle x \rangle and (6), we have
\[
Z_k = \frac{1}{k!} \left( \frac{t}{1 + t} \right)^k \langle \text{Bel}_{n,\lambda}(x) \rangle \lambda
= \left( \frac{1}{k!} \left( \frac{t}{1 + t} \right)^k \sum_{l=0}^{n} S_{2,\lambda}(n,l) \langle x \rangle \right) \lambda
\]
\[
= \sum_{l=0}^{n} S_{2,\lambda}(n,l) \left( \frac{1}{k!} \left( \frac{t}{1 + t} \right)^k \langle x \rangle \right) \lambda
= \sum_{l=0}^{n} S_{2,\lambda}(n,l)(-1)^{l-k} L(l,k).
\]

Therefore, from (64) and (65), we have the identity (60).

\[\square\]

**Theorem 7** For \( n \in \mathbb{N} \cup \{0\} \), we have
\[
\text{B}_{L,\lambda}^n(x) = \sum_{k=0}^{n} \left( \sum_{l=k}^{n} S_{2,\lambda}(l,k) L(n,l) \right)(x)_n \quad (n \geq 0).
\]

*Proof* Since \( e^x (\log(1+t)) = (1 + t)^x = \sum_{n=0}^{\infty} \langle x \rangle_n \frac{x^n}{n!} \), we have \( (x)_n \sim (1,e_x(t) - 1)_\lambda \).

Therefore, we consider the two degenerate Sheffer sequences as follows:
\[
\text{B}_{L,\lambda}^n(x) \sim \left( 1, \frac{t}{1 + t} \right)_\lambda \quad \text{and} \quad (x)_n \sim (1,e_x(t) - 1)_\lambda.
\]

Thus, from (2), (16) and (67), we have
\[
\text{B}_{L,\lambda}^n(x) = \sum_{k=0}^{n} \mu_{n,k}(x)_k \quad (n \geq 0),
\]

(68)
where

\[
\mu_{n,k} = \frac{1}{k!} \left( \left( e_t \left( \frac{t}{1-t} \right) - 1 \right) \left| (x)_{n,\lambda} \right| \right)_\lambda \\
= \sum_{l=k}^{n} S_{2,\lambda}(l,k) \left( \frac{t}{1-t} \right)^l \left| (x)_{n,\lambda} \right| = \sum_{l=k}^{n} S_{2,\lambda}(l,k)L(n, l). \tag{69}
\]

Therefore, from (68) and (69), we have the identity (66).

\[\square\]

**Theorem 8** For \( n \in \mathbb{N} \cup \{0\} \) and \( s \in \mathbb{N} \), we have

\[
E^{(i)}_{n,\lambda}(x) = \sum_{k=0}^{n} \mu_{n,k} B^L_{k,\lambda}(x), \tag{70}
\]

where \( E^{(i)}_{n,\lambda}(x) \) are type 2 degenerate poly-Euler polynomials.

**Proof** From (18), (24) and (26), we consider the following two degenerate Sheffer sequences as follows:

\[
B^L_{n,\lambda}(x) \sim \left( 1, \frac{t}{1+t} \right)_\lambda \quad \text{and} \quad E^{(i)}_{n,\lambda}(x) \sim \left( \frac{t(e_t(t) + 1)}{E_t(\log(1+2t))}, t \right)_\lambda. \tag{71}
\]

By using (2) and (25), we have

\[
E^{(i)}_{n,\lambda}(x) = \sum_{k=0}^{n} \mu_{n,k} B^L_{k,\lambda}(x), \tag{72}
\]

where

\[
\mu_{n,k} = \frac{1}{k!} \left( \frac{E_t(\log(1+2t))}{t(e_t(t) + 1)} \right) \left( \frac{t}{1+t} \right)^k \left| (x)_{n,\lambda} \right| \lambda \\
= \sum_{l=k}^{n} \left( \frac{n}{l} \right)(-1)^{l-k} L(l,k) \frac{E_t(\log(1+2t))}{t(e_t(t) + 1)} \left| (x)_{n-l,\lambda} \right| \lambda \tag{73}
\]

Therefore, from (72) and (73), we get the identity (70).

\[\square\]

**3 Conclusion**

The author represented the degenerate Lah–Bell polynomials in terms of quite a few well-known special polynomials and at the same time derived the inversion formulas of those identities by using the degenerate Sheffer sequences. We addressed the special polynomials and numbers: the degenerate falling factorial, the Lah numbers and the degenerate
Bernoulli polynomials; the Lah numbers and the degenerate Frobenius–Euler polynomials of order $r$; the Lah numbers and the degenerate Deahee polynomials; the Lah numbers and the degenerate Bell polynomials; the Lah numbers and the type 2 degenerate poly Euler polynomials. Therefore, the paper demonstrates that degenerate versions are not only applicable for number theory and combinatorics but also to symmetric identities, differential equations and probability theory. Building upon this, the author would like to further study into degenerate versions of certain special polynomials and numbers and their applications to physics, economics and engineering as well as mathematics.

Acknowledgements
The author would like to thank the referees for the detailed and valuable comments, which helped improve the original manuscript in its present form.

Funding
This work was supported by the Basic Science Research Program, the National Research Foundation of Korea, the Ministry of Education (NRF-2018R1D1A1B0704984).

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
The author declares that there is no ethical problem in the publication of this paper.

Competing interests
The author declares that they have no competing interests.

Consent for publication
The author want to publish this paper in this journal.

Authors’ contributions
The author was the only one contributing to the manuscript. The author read and approved the final manuscript.

Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 24 October 2020 Accepted: 29 November 2020 Published online: 07 December 2020

References
1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards Applied Mathematics Series, vol. 55. U.S. Government Printing Office, Washington (1964) xvi+1046 pp., For sale by the Superintendent of Documents
2. Carlitz, L.: Degenerate Stirling, Bernoulli and Eulerian numbers. Util. Math. 15, 51–88 (1979)
3. Carlitz, L.: Weighted Stirling numbers of the first and second kind. Fibonacci Q. 18, 147–162 (1980)
4. Comtet, L.: Advanced Combinatorics. The Art of Finite and Infinite Expansions. Reidel, Dordrecht (1974). Revised and enlarged edn, x+343 pp. ISBN: 90-277-0441-4 05-02
5. Dere, R., Simsek, Y.: Applications of umbral algebra to some special polynomials. Adv. Stud. Contemp. Math. (Kyungshang) 22(3), 433–438 (2012)
6. Ernst, T.: Examples of a $q$-umbral calculus. Adv. Stud. Contemp. Math. (Kyungshang) 16(1), 1–22 (2008)
7. Kim, D.S., Kim, T.: Some identities of Bernoulli and Euler polynomials arising from umbral calculus. Adv. Stud. Contemp. Math. (Kyungshang) 23(1), 159–171 (2013)
8. Kim, D.S., Kim, T.: Some identities of Bell polynomials. Sci. China Math. 58(10), 2095–2104 (2015)
9. Kim, D.S., Kim, T.: Some identities of degenerate Dahee numbers arising from certain differential equations. J. Nonlinear Sci. Appl. 10, 744–751 (2017)
10. Kim, D.S., Kim, T.: Lah–Bell numbers and polynomials. Proc. Jangjeon Math. Soc. 23(4), 577–586 (2020)
11. Kim, D.S., Kim, T.: Degenerate Sheffer sequence and $A$-Sheffer sequence. J. Math. Anal. Appl. 493(1), 124521 (2021)
12. Kim, D.S., Kim, T.: A note on a new type of degenerate Bernoulli numbers. Russ. J. Math. Phys. 27(2), 227–235 (2020)
13. Kim, H.K., Jang, L.-C.: Type 2 degenerate poly-Euler polynomials. Symmetry 12, 1011 (2020). https://doi.org/10.3390/sym12061011
14. Kim, T.: A note on degenerate Stirling polynomials of the second kind. Proc. Jangjeon Math. Soc. 20(3), 319–331 (2017)
15. Kim, T., Kim, D.S.: An identity of symmetry for the degenerate Frobenius–Euler polynomials. Math. Slovaca 68(1), 239–243 (2018)
16. Kim, T., Kim, D.S.: Some identities of extended degenerate $r$-central Bell polynomials arising from umbral calculus. Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat. 114, 1, 19 pp. (2020)
17. Kim, T., Kim, D.S.: Note on the degenerate gamma function. Russ. J. Math. Phys. 27(3), 352–358 (2020)
18. Kim, T., Kim, D.S.: Degenerate polyexponential functions and degenerate Bell polynomials. J. Math. Anal. Appl. 487(2), 124017, 15 pp. (2020)
19. Kim, T., Kim, D.S., Kim, H.Y., Kwon, J.: Some identities of degenerate Bell polynomials. Mathematics 8, 40, 8 pp. (2020)
20. Kim, T., Kim, D.S., Kim, H.Y., Lee, H., Jang, L.-C.: Degenerate Bell polynomials associated with umbral calculus. J. Inequal. Appl. 2020, 226, 15 pp. (2020)
21. Kim, T., Kim, D.S., Kwon, J., Lee, H.: Degenerate polyexponential functions and type 2 degenerate poly-Bernoulli numbers and polynomials. Adv. Differ. Equ. 2020, 168, 12 pp. (2020)
22. Kim, T., Kim, D.S., Lee, H., Kwon, J.: A note on some identities of new type degenerate Bell polynomials. Mathematics 7, 1086, 12 pp. (2019)
23. Kim, T., Kim, D.S., Lee, H., Park, J.W.: A note on degenerate r-Stirling numbers. J. Inequal. Appl. 2020, 225, 12 pp. (2020)
24. Kwaśniewski, A.K.: On ψ-umbral extensions of Stirling numbers and Dobinski-like formulas. Adv. Stud. Contemp. Math. (Kyungshang) 12(1), 73–100 (2006)
25. Ma, Y., Kim, D.S., Kim, T., Kim, H.Y., Lee, H.S.: Some identities of Lah–Bell polynomials. Adv. Differ. Equ. 2020, 510, 10 pp. (2020)
26. Nyul, G., Racz, G.: The r-Lah numbers. Discrete Math. 338, 1660–1666 (2015).
   https://doi.org/10.1016/j.disc.2014.03.029
27. Nyul, G., Racz, G.: Sums of r-Lah numbers and r-Lah polynomials. Ars Math. Contemp. 18(2), 211–222 (2020).
   https://doi.org/10.26493/1855-3974.1793.c4d
28. Roman, S.: The Umbral Calculus. Pure and Applied Mathematics, vol. 111, x+193 pp. Academic Press, New York (1984)