EXACT DIMENSIONALITY AND LEDRAPPIER-YOUNG FORMULA FOR THE FURSTENBERG MEASURE

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Abstract. Assuming strong irreducibility and proximality, we prove that the Furstenberg measure, corresponding to a finitely supported measure on the general linear group of a finite dimensional real vector space, is exact dimensional. We also establish a Ledrappier-Young type formula for its dimension. The general strategy of the proof is based on the argument given by Feng for the exact dimensionality of self-affine measures.

1. Introduction

1.1. Background and the main result. Let \( V \) be a real vector space with \( 2 \leq \dim V < \infty \). Fix an inner product \( \langle \cdot, \cdot \rangle \) on \( V \), and denote the induced norm by \( |\cdot| \). For a linear subspace \( W \) of \( V \) denote by \( P(W) \) its projective space. For \( x, y \in P(V) \) set,

\[
d(x, y) = \left(1 - \langle x, y \rangle^2\right)^{1/2},
\]

where \( x \in \mathfrak{F} \) and \( y \in \mathfrak{F} \) are unit vectors. It is easy to verify that this defines a metric on \( P(V) \).

The general linear group of \( V \) acts on \( P(V) \) in a natural way by setting,

\[
A x = Ax \quad \text{for} \quad A \in GL(V) \quad \text{and} \quad x \in P(V).
\]

Let \( \mu \in \mathcal{M}(GL(V)) \), where for a standard Borel space \( X \) the collection of Borel probability measures on \( X \) is denoted by \( \mathcal{M}(X) \). We say that \( \nu \in \mathcal{M}(P(V)) \) is \( \mu \)-stationary if,

\[
\nu(F) = \int A \nu(F) \, d\mu(A) \quad \text{for every Borel set} \quad F \subset P(V),
\]

where \( A \nu \) is the push-forward of \( \nu \) via the map \( \mathfrak{F} \to A \mathfrak{F} \). Since \( P(V) \) is compact there always exists at least one \( \mu \)-stationary measure.

Write \( S_\mu \) for the smallest closed subsemigroup of \( GL(V) \) such that \( \mu(S_\mu) = 1 \). Suppose from now on that \( S_\mu \) is strongly irreducible and proximal. The first assumption means that there does not exist a finite family of proper nonzero linear subspaces \( W_1, \ldots, W_k \) of \( V \) such that

\[
A(\bigcup_{i=1}^k W_i) = \bigcup_{i=1}^k W_i \quad \text{for all} \quad A \in S_\mu.
\]

The second assumption means that there exist \( A_1, A_2, \ldots \in S_\mu \) and \( \alpha_1, \alpha_2, \ldots \in \mathbb{R} \) such that \( \{\alpha_n A_n\}_{n \geq 1} \) converges to a rank 1 endomorphism of \( V \) in the norm topology. From these assumptions it follows that there exists a unique \( \nu \in \mathcal{M}(P(V)) \)

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which is $\mu$-stationary. It is called the Furstenberg measure corresponding to the distribution $\mu$. For a proof see [BL Theorem III.3.1] or [BQ Proposition 4.7].

The main purpose of this paper is to establish the exact dimensionality of the Furstenberg measure, under the additional assumption of $\mu$ being finitely supported. A Borel probability measure $\theta$ on a metric space $X$ is said to be exact dimensional if there exists a number $\alpha \geq 0$ such that,

$$\lim_{r \downarrow 0} \frac{\log \theta(B(x, r))}{\log r} = \alpha$$

for $\theta$-a.e. $x \in X$, where $B(x, r)$ is the closed ball in $X$ with centre $x$ and radius $r$. If $\theta$ is exact dimensional then the number $\alpha$ is denoted $\dim \theta$, is called the dimension of $\theta$ and is equal to the value given to $\theta$ by other commonly used notions of dimension (see [Fa, Chapter 10]). In particular $\dim \theta$ is equal to the Hausdorff dimension of $\theta$, which is denoted $\dim_H \theta$ and defined by

$$\dim_H \theta = \inf \left\{ \dim_H F : F \subset X \text{ is Borel with } \theta(F) > 0 \right\},$$

where $\dim_H F$ is the Hausdorff dimension of $F$.

When $\dim V = 2$ the exact dimensionality of the Furstenberg measure $\nu$ was already established in previous works, without assuming that $\mu$ is finitely supported. It was shown by Ledrappier (see [Led]) that in this case the function,

$$x \mapsto \frac{\log \nu(B(x, r))}{\log r},$$

converges in $\nu$-probability to the value $h_F(\nu)/(\lambda_0 - \lambda_1)$ as $r \to 0$. Here $\lambda_0 > \lambda_1$ are the Lyapunov exponents corresponding to $\mu$ (see the next section), and $h_F(\nu)$ is the Furstenberg entropy of $\nu$ which is defined by

$$h_F(\nu) = \int \int \log \frac{dA\nu}{d\nu}(x, \mu) dA\nu(x) d\mu(A).$$

More recently, Hochman and Solomyak [HS] (see the discussion below) have shown that $\nu$ is exact dimensional with,

$$\dim \nu = h_F(\nu)/(\lambda_0 - \lambda_1),$$

whenever $\dim V = 2$. In a recent paper Lessa [Les] has extended these results to disintegrations along certain 1-dimensional foliations, of stationary measures on the space of complete flags.

In this paper we establish the exact dimensionality of $\nu$ also in higher dimensions. The following theorem is our main result.

**Theorem 1.1.** Let $\mu \in \mathcal{M}(\text{GL}(V))$ be finitely supported, and suppose that $S_\mu$ is strongly irreducible and proximal. Let $\nu \in \mathcal{M}(\text{P}(V))$ be the Furstenberg measure corresponding to $\mu$. Then $\nu$ is exact dimensional, and $\dim \nu$ satisfies a Ledrappier-Young type dimension formula.

The precise formula satisfied by $\dim \nu$ will be given in the next section. Its name comes from the work of Ledrappier and Young [LY], in which they have obtained a formula, in terms of conditional entropies and Lyapunov exponents, for the local dimensions along stable and unstable manifolds of invariant measures of $C^2$ smooth diffeomorphisms.

Let us provide some more background and mention other related results. A measure $\theta$ on $\mathbb{R}^d$ is said to be self-affine if it is stationary with respect to a finitely
supported measure $\rho$ on the semigroup of affine invertible contractions of $\mathbb{R}^d$. If $\rho$ is supported on the semigroup of contracting similarities, then $\theta$ is said to be self-similar. The support of the measure $\rho$ is sometimes referred to as an iterated function system (IFS).

Self-affine measures and Furstenberg measures share various features. For instance, both can be realised as the image of a Bernoulli measure on the symbolic space under an appropriate equivariant map. In the case of self-affine measures this map is called the coding map. For the Furstenberg measure it is called the Furstenberg boundary map (see the next section). Additionally, both types of measures can be represented as a weighted average of distorted copies of themselves, which are sometimes referred to as cylinder measures. The Furstenberg measure on the 1-dimensional projective space resembles a self-similar measure on the real line. In higher dimensions the Furstenberg measure resembles a self-affine measure, for which the linear parts of the maps in the IFS satisfy irreducibility and proximality assumptions similar to ours.

Exact dimensionality plays an important role in the study of stationary fractal measures, and the question of whether every self-affine measure satisfies this property has received a lot of attention. In [F1H], by introducing a notion of projection entropy, Feng and Hu have proved that every self-similar measure on $\mathbb{R}^d$ is exact dimensional, with dimension given by the projection entropy divided by the Lyapunov exponent. In fact they have shown this, more generally, for the push-forward of any ergodic measure under the coding map.

In [BK] Bárány and Käenmäki proved that every planar self-affine measure is exact dimensional. Moreover, they proved this for every self-affine measure on $\mathbb{R}^d$ with $d$ distinct Lyapunov exponents, and showed that its dimension is given by a Ledrappier-Young type formula. Additionally, under further assumptions, they established this for projections under the coding map of quasi-Bernoulli measures.

Lastly, in a recent paper Feng [F5] has managed to provide a complete solution for this problem, and proved that all self-affine measures are exact dimensional and satisfy a Ledrappier-Young type formula. In fact he was able to show this for projections of general ergodic measures and to systems which are only average contracting. The general strategy for our proof of Theorem 1.1 is based on Feng’s argument.

Besides their intrinsic interest, exact dimensionality and Ledrappier-Young type formulas have played an important role in some recent and significant developments in the dimension theory of fractal measures. Hochman [Ho1, Ho2] has shown that, under a mild exponential separation assumption on the maps in the IFS and an additional irreducibility assumption in higher dimensions, the dimension of a self-similar measure is equal to its natural upper bound. If $\rho$ is the corresponding finitely supported measure on the contracting similarities, this upper bound is equal to the minimum between the dimension of the ambient space and the quotient obtained by dividing the Shannon entropy of $\rho$ by its Lyapunov exponent. To be more precise regarding the exponential separation assumption, it requires the existence of an $\epsilon > 0$ such that for all $n \geq 1$ the distance between two distinct compositions of length $n$ of map from the IFS is at least $\epsilon^n$. These works rely on the exact dimensionality of self-similar measures.

Bárány, Hochman and Rapaport [BHR] have proved for planar self-affine measures that if one assumes strong irreducibility and proximality for the linear parts
of the maps in the IFS, and that the supports of the cylinder measures are disjoint, then the dimension is equal to its natural upper bound. In this case the natural upper bound is the minimum between 2 and a quantity known as the Lyapunov dimension, which generalises the entropy divided by exponent formula. The condition regarding the disjointness of the supports is usually referred to as the strong separation condition (SSC). Hochman and Rapaport [HR] have later established this statement under a much milder exponential separation assumption instead of the SSC. Both of these results rely on the exact dimensionality and Ledrappier-Young formula for planar self-affine measures.

Lastly, in [HS] Hochman and Solomyak have proved their main result while relying on the exact dimensionality of the 1-dimensional Furstenberg measure, which they establish in the same paper. Stated in the notation of Theorem 1.1, this result says that if \( \mu \) is finitely supported, \( S_\mu \) is strongly irreducible and proximal, and the matrices in the support of \( \mu \) satisfy an exponential separation condition, then \( \dim \nu \) is equal to its natural upper bound, where \( \nu \) is the Furstenberg measure on \( P(\mathbb{R}^2) \). The natural upper bound in this case is the minimum between 1 and the Shannon entropy of \( \mu \) divided by the difference of the two Lyapunov exponents.

1.2. Dimension formulas. In this section we provide a precise statement for the dimension formula satisfied by the Furstenberg measure \( \nu \). We also give similar formulas for typical projections and slices of \( \nu \). First we need some more definitions and notations.

Let \( \mu \in M(GL(V)) \) be finitely supported. Then there exist a finite index set \( \Lambda \), distinct elements \( \{A_l\}_{l \in \Lambda} \) of \( GL(V) \), and a probability vector \( p = (p_l)_{l \in \Lambda} \) with strictly positive coordinates, such that

\[
\mu = \sum_{l \in \Lambda} p_l \delta_{A_l}.
\]

Here \( \delta_{A_l} \in M(GL(V)) \) is the Dirac mass at \( A_l \). As before, denote by \( S_\mu \) the smallest closed subsemigroup of \( GL(V) \) such that \( \mu(S_\mu) = 1 \), and suppose that \( S_\mu \) is strongly irreducible and proximal. Let \( \nu \in M(P(V)) \) be the Furstenberg measure corresponding to \( \mu \).

Write \( \Omega = \Lambda^\mathbb{Z} \) and equip \( \Omega \) with its Borel \( \sigma \)-algebra, generated by the cylinder sets. Let \( \beta \) be the Bernoulli measure on \( \Omega \) corresponding to the probability vector \( p \), that is \( \beta = p^\mathbb{Z} \). Let \( \sigma : \Omega \to \Omega \) be the left shift map, i.e.

\[
(\sigma \omega)_n = \omega_{n+1} \text{ for } \omega \in \Omega \text{ and } n \in \mathbb{Z}.
\]

From our assumptions on \( S_\mu \) it follows (see [BQ] Lemma 2.17 and Proposition 4.7) that there exists a Borel map \( \pi : \Omega \to P(V) \) such that,

1. \( \pi \) depends only on the nonnegative coordinates of \( \Omega \);
2. \( \pi \omega = A_{\omega_0} \pi \sigma \omega \) for \( \beta \)-a.e. \( \omega \).

4
(3) the distribution of \( \pi \) with respect to \( \beta \) is equal to \( \nu \), that is \( \pi \beta = \nu \);
(4) for \( \beta \)-a.e. \( \omega \),
\[
\lim_{n \to \infty} A_{\omega_0} \cdots A_{\omega_n} \nu = \delta_{\pi(\omega)},
\]
where \( \delta_{\pi(\omega)} \) is the Dirac mass at \( \pi(\omega) \) and the convergence is in the weak-* topology.

The map \( \pi \) is often called the Furstenberg boundary map.

By the Oseledets’ multiplicative ergodic theorem \([O]\), applied to the ergodic system \((\Omega, \beta, \sigma^{-1})\) and the matrix cocycle \( \omega \to A_{\omega^{-1}} \), there exist positive integers \( s, d_0, \ldots, d_s \), real numbers \( \lambda_0 > \ldots > \lambda_s \) and linear subspaces,
\[
V = V_{\omega}^{-1} \supset V_{\omega}^{0} \supset \cdots \supset V_{\omega}^{s} = \{0\}
\]
for \( \omega \in \Omega \), such that,
1. \( \dim V_{\omega}^i = \sum_{k=i+1}^{s} d_k \) for \( \omega \in \Omega \) and \( -1 \leq i \leq s \);
2. the map \( \omega \to (V_{\omega}^i)_{i=1}^{s} \) is Borel measurable and depends only on the negative coordinates of \( \Omega \);
3. \( V_{\omega}^{-1} = A_{\omega^{-1}} V_{\omega}^{0} \) for \( \beta \)-a.e. \( \omega \) and each \( -1 \leq i \leq s \);
4. for \( \beta \)-a.e. \( \omega \) and each \( 0 \leq i \leq s \),
\[
\lim_{n \to \infty} \frac{1}{n} \log |A_{\omega^{-n}} \cdots A_{\omega^{-1}} x| = \lambda_i
\]
for \( x \in V_{\omega}^{i-1} \setminus V_{\omega}^{i} \).

The numbers \( \lambda_0, \ldots, \lambda_s \) are called the Lyapunov exponents corresponding to \( \nu \). For \( 0 \leq i \leq s \) the integer \( d_i \) is called the multiplicity of \( \lambda_i \). Note that from our assumptions on \( S_\nu \) it follows that \( d_0 = 1 \) (see \([BL\) Theorem III.6.1]). We set,
\[
\hat{\lambda}_i = \lambda_i - \lambda_0 \quad \text{for} \quad 1 \leq i \leq s
\].

Remark 1.2. Let \( \theta \) be the distribution of the random flag \( (V_{\omega}^i)_{i=1}^{s} \). That is, for every Borel subset \( B \) of the flag manifold,
\[
\theta(B) = \beta\{ \omega : (V_{\omega}^i)_{i=1}^{s} \in B \}
\].
Write \( \mu^- \) for the distribution,
\[
\sum_{\lambda \in \Lambda} \mu_{\lambda} \delta_{A_{\lambda}^{-1}} \in \mathcal{M}(\text{GL}(V))
\].
Then from the identities \( V_{\omega}^{-1} = A_{\omega^{-1}} V_{\omega}^{0} \) it follows that \( \theta \) is \( \mu^- \)-stationary. Note that in general, our assumptions do not guarantee the uniqueness of a \( \mu^- \)-stationary measure on the flag manifold.

For a proper linear subspace \( W \) of \( V \) write \( P_{W^\perp} \) for the orthogonal projection onto \( W^\perp \). Note that \( P_{W^\perp} \) defines a map from \( P(V) \setminus P(W) \) to \( P(W^\perp) \) by setting,
\[
P_{W^\perp}(\overline{\sigma}) = P_{W^\perp} \overline{\sigma}
\]
for \( \overline{\sigma} \in P(V) \setminus P(W) \).

Let \( \zeta_W \) be the partition of \( P(V) \setminus P(W) \) such that for \( \overline{\sigma} \in P(V) \setminus P(W) \),
\[
\zeta_W(\overline{\sigma}) = \{ \overline{\tau} \in P(V) \setminus P(W) : P_{W^\perp} \overline{\tau} = P_{W^\perp} \overline{\sigma} \}
\].

Here \( \zeta_W(\overline{\sigma}) \) denotes the unique element of \( \zeta_W \) which contains \( \overline{\sigma} \). Since \( W \neq V \), and because \( S_\nu \) is strongly irreducible, is follows that \( \nu(P(W)) = 0 \) (see \([BL\) Proposition III.2.3]). Hence \( P_{W^\perp} \) defines a Borel map on \( P(V) \) outside a set of zero \( \nu \)-measure, and the disintegration
\[
\{ \nu^{\overline{\sigma}} \}_{\overline{\sigma} \in P(V)} \subset \mathcal{M}(P(V)),
\]
of \( \nu \) with respect to the measurable partition \( \zeta_W \), is \( \nu \)-a.e. well defined (see Section 2.3). Note that if \( W \) is of codimension 1 then \( \nu^W = \nu \) for \( \nu \)-a.e. \( \mathcal{F} \).

Denote by \( \mathcal{P} \) the partition of \( \Omega \) according to the 0-coordinate, that is
\[
\mathcal{P} = \{ \{ \omega \in \Omega : \omega_0 = l \} : l \in \Lambda \}.
\]
Write \( \mathcal{B} \) for the Borel \( \sigma \)-algebra of \( \mathcal{P}(V) \). For a proper linear subspace \( W \) of \( V \), write \( H_\beta(\mathcal{P} | \pi^{-1}P_{W}^{-1}\mathcal{B}) \) for the conditional entropy of \( \mathcal{P} \) given \( \pi^{-1}P_{W}^{-1}\mathcal{B} \) with respect to \( \beta \) (see Section 2.2). It is well defined since the identity \( \pi \beta = \nu \) implies that the composition \( P_{W,\beta} \circ \pi \) defines a Borel map on \( \Omega \) outside of a set of zero \( \beta \)-measure. Thus for \( 0 \leq i \leq s \) we can set,
\[
(1.2) \quad H_i = \int H_\beta(\mathcal{P} | \pi^{-1}P_{(V^i)^\perp}^{-1}\mathcal{B}) \, d\beta(\omega) .
\]

Note that since the subspaces \( V^0_\omega \) are of codimension 1, the \( \sigma \)-algebras \( \pi^{-1}P_{(V^0_\omega)^\perp}^{-1}\mathcal{B} \) are trivial with respect to \( \beta \). This implies that \( H_0 = H(p) \), where \( H(p) \) is the entropy of the probability vector \( p \). Also observe that, since \( V^{i+1}_\omega \subset V^i_\omega \) for \( 0 \leq i < s \) and \( \omega \in \Omega \), the \( \sigma \)-algebras which appear in the definition of \( H_{i+1} \) are finer than the ones which appear in the definition of \( H_i \). This implies that \( H_{i+1} \leq H_i \) for \( 0 \leq i < s \).

We are now ready to state our dimension formulas for \( \nu \), its projections and its slices.

**Theorem 1.3.** Suppose that \( \mu \) is finitely supported, and that \( S_\mu \) is strongly irreducible and proximal. Let \( \nu \) be the Furstenberg measure corresponding to \( \mu \). Then, in the notations above, for \( \beta \)-a.e. \( \omega \in \Omega \), \( \nu \)-a.e. \( \mathcal{F} \in \mathcal{P}(V) \) and every \( 0 \leq i < k \leq s \), the following statements are satisfied.

1. \( \nu \) is exact dimensional with,
\[
\dim \nu = \sum_{j=0}^{s-1} \frac{H_{j+1} - H_j}{\lambda_{j+1}} ;
\]
2. \( P_{(V^i)^\perp} \nu \) is exact dimensional with,
\[
\dim P_{(V^i)^\perp} \nu = \sum_{j=0}^{k-1} \frac{H_{j+1} - H_j}{\lambda_{j+1}} ;
\]
3. \( \nu^V_{\mathcal{F}} \) is exact dimensional with,
\[
\dim \nu^V_{\mathcal{F}} = \sum_{j=i}^{s-1} \frac{H_{j+1} - H_j}{\lambda_{j+1}} ;
\]
4. \( P_{(V^i)^\perp} \nu^V_{\mathcal{F}} \) is exact dimensional with,
\[
\dim P_{(V^i)^\perp} \nu^V_{\mathcal{F}} = \sum_{j=i}^{k-1} \frac{H_{j+1} - H_j}{\lambda_{j+1}} .
\]

**Remark 1.4.** For every \( \omega \in \Omega \) the subspace \( V^s_\omega \) is trivial and \( V^0_\omega \) is of codimension 1. Hence \( P_{(V^s)^\perp} \) is the identity and \( \nu^V_{\mathcal{F}} = \nu \) for \( \nu \)-a.e. \( \mathcal{F} \). Thus in order to prove Theorems 1.3 and 1.1 it is enough to establish part 4 of Theorem 1.3.
The above theorem yields a dimension conservation result for the Furstenberg measure. Let $W$ be a proper linear subspace of $V$, and let $\theta \in \mathcal{M}(P(V))$ be with $\theta(P(W)) = 0$. Following Furstenberg [Ful], we say that $\theta$ is dimension conserving with respect to $P_{W^\perp}$ if,
\[
\dim_H P_{W^\perp} \theta + \dim_H \theta_{W^\perp} = \dim_H \theta \quad \text{for } \theta\text{-a.e. } \mathbf{x},
\]
where $\dim_H$ is as defined in (1.1). The following corollary follows directly from parts (1)-(3) of Theorem 1.3.

**Corollary 1.5.** Assume the conditions of Theorem 1.3 are satisfied. Then $\nu$ is dimension conserving with respect to $P_{(V_2)^\perp}$ for $\beta$-a.e. $\omega \in \Omega$ and every $0 \leq i \leq s$.

For self-affine measures results analogous to Corollary 1.5 were obtained in [BK] and [Fe]. It is worth pointing out that in [FJ] Falconer and Jin proved that self-similar measures on $\mathbb{R}^d$ with finite rotation groups are dimension conserving with respect to any orthogonal projection. For self-similar sets with finite rotation groups this result was first obtained by Furstenberg [Ful], who introduced this notion.

### 1.3. The Lyapunov dimension

In this section we introduce the upper bound for $\dim \nu$, which was mentioned in the discussion at the end of Section 1.1. We continue to use the notations from the previous section, and assume the conditions of Theorem 1.3 are satisfied. As before write $H(p)$ for the entropy of the probability vector $p$. We also set,
\[
L_i = -\sum_{j=1}^i \hat{\lambda}_j d_j \quad \text{for } 0 \leq i \leq s.
\]

**Definition 1.6.** Let $m = m(\mu)$ be such that,
\[
m = \max\{0 \leq i \leq s : H(p) \geq L_i\},
\]
and write,
\[
\dim_{LY} \mu = \begin{cases} 
\sum_{j=1}^m d_j + \frac{H(p) - L_m}{-\hat{\lambda}_{m+1}} & \text{if } m < s \\
\dim V - 1 & \text{if } m = s
\end{cases}.
\]

We call the number $\dim_{LY} \mu$ the Lyapunov dimension corresponding to $\mu$.

This definition is analogous to the one given by Jordan, Pollicott and Simon in [JPS] for the Lyapunov dimension $\dim_{LY} \theta$ of a self-affine measure $\theta$ on $\mathbb{R}^d$. It was shown there that $\dim_{LY} \theta$ is always an upper bound for $\dim_H \theta$, and that if the linear parts of the maps in the IFS are fixed and all have norm strictly less than $1/2$, then
\[
\dim_H \theta = \min\{\dim_{LY} \theta, d\}
\]
for Lebesgue a.e. selection of the translations.

Now let $\Delta$ be the set of numbers of the form $-\sum_{i=1}^s \hat{\lambda}_i^{-1} x_i$, where $x_1, \ldots, x_s$ are nonnegative real numbers which satisfy,
\[
\sum_{i=1}^s x_i \leq H(p) \quad \text{and } x_i \leq -\hat{\lambda}_i d_i \quad \text{for } 1 \leq i \leq s.
\]

From,
\[
0 > \hat{\lambda}_1 > \ldots > \hat{\lambda}_s \quad \text{and } \sum_{i=1}^s d_i = \dim V - 1,
\]
it follows easily that,

\[ \dim_{LY} \mu = \max \Delta. \]  

On the other hand, as a simple consequence of part (4) of Theorem 1.3 (see Lemma 6.5),

\[ 0 \leq H_i - H_{i+1} \leq -\tilde{\lambda}_{i+1} d_{i+1} \text{ for } 0 \leq i < s. \]

Also note that,

\[ H(p) = H_0 \geq \sum_{i=0}^{s-1} (H_i - H_{i+1}). \]

Combining the last inequality with (1.4) and part (1) of Theorem 1.3, we obtain that \( \dim \nu \) is a member of \( \Delta \). This together with (1.3) yields the following corollary.

**Corollary 1.7.** Assume the conditions of Theorem 1.3 are satisfied, then \( \dim \nu \leq \dim_{LY} \mu \).

As mentioned in Section 1.1, it is reasonable to expect for the equality \( \dim \nu = \dim_{LY} \mu \) to hold under an additional exponential separation assumption.

### 1.4. About the proof of Theorem 1.3

The starting point of the argument is the observation that, under our standing assumptions, the matrices

\[ A_{\omega^{-n} \ldots \omega^{-1}} := A_{\omega^{-n}} \cdots A_{\omega^{-1}} \]

contract the projective space, outside of the set \( P(V^0_\omega) \), for \( \beta \)-a.e. \( \omega \) and for \( n \geq 1 \) large. Here \( V^0_\omega \) is the linear hyperplane obtained by the Oseledets’ theorem. This contraction property follows from the fact that the multiplicity of the top Lyapunov exponent is equal to 1, and it makes it possible to employ techniques used in the study of self-affine measures on \( \mathbb{R}^d \).

As mentioned before, the general idea of the proof is based on Feng’s argument for the exact dimensionality of self-affine measures. Nevertheless our proof contains nontrivial differences and new features. Some of these come from the fact that the matrices \( A_{\omega^{-n} \ldots \omega^{-1}} \) only contract most of the projective space. In the self-affine case, or even in the more general average contracting case considered in [Fe], these matrices uniformly contract all of the Euclidean space for \( \beta \)-a.e. \( \omega \).

In order to deal with this issue, for \( \beta \)-a.e. \( \omega \) we use the Oseledets splitting of \( V \) at \( \omega \) (see Section 2.6) in order to construct a coordinate chart for \( P(V) \), whose domain contains \( \pi \omega \) and on which the matrices \( A_{\omega^{-n} \ldots \omega^{-1}} \) are uniformly contracting. Recall that \( \pi \) is the Furstenberg boundary map. For this approach to succeed we have to make sure that, for \( n \geq 1 \) large, the lines \( \pi \sigma^n \omega \) do not become too close to the boundary of the coordinate domains. This is achieved by applying a result of Guivarc’h [Gu] (see Section 2.5). It yields a regularity property for the Furstenberg measure \( \nu \), which controls the \( \nu \)-measure of neighbourhoods of projective spaces of linear hyperplanes of \( V \).

As in [Fe], a key part of the argument involves the estimation of the so-called transverse dimensions, which are the local dimensions of projections of certain conditional measures of \( \beta \). These conditional measures correspond to measurable partitions \( \xi_0, \ldots, \xi_s \) of \( \Omega \), which are similar to the ones constructed in [Fe]. When they are projected via \( \pi \), one obtains the conditional measures of \( \nu \) which appear in the statement of Theorem 1.3. In order to deal with the estimation of the transverse dimensions we employ an idea used in [Fe], which involves an induced dynamics and...
makes it possible to focus on trajectories where the angles between the Oseledets subspaces are not too small.

The result of Guivarc’h, which provides the regularity property for \( \nu \), requires the strong irreducibility and proximality assumptions. These assumptions also insure that the multiplicity of the top Lyapunov exponent is \( 1 \), a fact which is crucial for our development. Strong irreducibility also implies the necessary fact that \( \nu(\mathcal{P}(W)) = 0 \) for every proper linear subspace \( W \) of \( V \), though this may be regarded as a very mild form of the regularity property. The assumption of \( \mu \) being finitely supported is needed in order to carry out entropy computations and to guarantee the integrability of a certain dominating function (see Lemma 5.9 and Remark 5.10 following it). It seems reasonable to expect for the exact dimensionality of \( \mu \)-stationary measures to hold under weaker assumptions than ours, but this will probably require a different method of proof.

**Structure of the paper.** In Section 2 we develop necessary notations and background. In Section 3 we construct the coordinate charts mentioned above, and prove some related auxiliary results. In Section 4 we construct the measurable partitions \( \xi_0, \ldots, \xi_s \), and derive some necessary properties of them. In Section 5 we estimate the transverse dimensions. In Section 6 we complete the proof of our main result Theorem 1.3.

### 2. Preliminaries

#### 2.1. General notations.

For a metric space \( X \), \( x \in X \) and \( r > 0 \), we denote by \( B(x, r) \) the closed ball in \( X \) with centre \( x \) and radius \( r \). Given a set \( Y \), a map \( \phi : Y \to X \), \( y \in Y \) and \( r > 0 \), we often write \( B_\phi(y, r) \) in place of \( \phi^{-1}(B(\phi(y), r)) \).

It will sometimes be convenient to use the little-o notation. For parameters \( \alpha_1, \ldots, \alpha_k \) we write \( o(\alpha_1, \ldots, \alpha_k)(n) \) in order to denote an unspecified function \( f : \mathbb{N} \to \mathbb{R} \), which depends on \( \alpha_1, \ldots, \alpha_n \) and satisfies \( \frac{1}{n}f(n) \to 0 \) as \( n \to \infty \).

#### 2.2. Conditional information and entropy.

We give here the definitions and basic properties of the entropy and information functions. For more details see [Pa, Section 2] for instance.

Let \( (X, \mathcal{B}, \rho) \) be a probability space. For a sub-\( \sigma \)-algebra \( \mathcal{F} \) of \( \mathcal{B} \) and \( f \in L^1(\rho) \) we denote the conditional expectation of \( f \) given \( \mathcal{F} \) by \( \mathbb{E}_\rho(f \mid \mathcal{F}) \). Given a finite measurable partition \( \mathcal{E} \) of \( X \) we write \( I_\rho(\mathcal{E} \mid \mathcal{F}) \) for the conditional information of \( \mathcal{E} \) given \( \mathcal{F} \). That is,

\[
I_\rho(\mathcal{E} \mid \mathcal{F}) = -\sum_{E \in \mathcal{E}} 1_E \log \mathbb{E}_\rho(1_E \mid \mathcal{F}),
\]

where \( 1_E \) is the indicator function of \( E \). The conditional entropy of \( \mathcal{E} \) given \( \mathcal{F} \) is denoted \( H_\rho(\mathcal{E} \mid \mathcal{F}) \) and defined by,

\[
H_\rho(\mathcal{E} \mid \mathcal{F}) = \int I_\rho(\mathcal{E} \mid \mathcal{F}) \, d\rho.
\]

When \( \mathcal{F} \) is the trivial \( \sigma \)-algebra we write \( H_\rho(\mathcal{E}) \) in place of \( H_\rho(\mathcal{E} \mid \mathcal{F}) \).

If \( \mathcal{G} \) is a sub-\( \sigma \)-algebra of \( \mathcal{F} \),

\[
H_\rho(\mathcal{E} \mid \mathcal{F}) \leq H_\rho(\mathcal{E} \mid \mathcal{G}).
\]
If $L^1(\rho)$ is separable as a metric space, and $C$ is another finite measurable partition of $X$,

\begin{equation}
I_\rho(E \vee C \mid F) = I_\rho(E \mid F) + I_\rho(C \mid F \vee \hat{E}).
\end{equation}

Here $E \vee C$ is the common refinement of $E$ and $C$, and $\hat{E}$ is the $\sigma$-algebra generated
by $E$. Integrating the last equality we obtain,

\begin{equation}
H_\rho(E \vee C \mid F) = H_\rho(E \mid F) + H_\rho(C \mid F \vee \hat{E}).
\end{equation}

If $T : X \to X$ is measure preserving,

\begin{equation}
I_\rho(E \mid F) \circ T = I_\rho(T^{-1}E \mid T^{-1}F).
\end{equation}

2.3. Disintegration of measures. We give the necessary facts regarding disintegration of measures. For more details see [EW, Section 5].

We call a measurable space $(X, B)$ a Borel space, if $X$ is a Borel subset of a compact metric space $X$ and $B$ is the restriction of the Borel $\sigma$-algebra of $X$ to $X$. We denote the collection of probability measures on $(X, B)$ by $\mathcal{M}(X)$. If $\rho \in \mathcal{M}(X)$ we say that $(X, B, \rho)$ is a Borel probability space.

Suppose $(X, B)$ is a Borel space. Given a partition $\xi$ of $X$ into measurable sets and $x \in X$, we write $\xi(x)$ for the unique element of $\xi$ which contains $x$. A subset $F$ of $X$ is said to be $\xi$-saturated if it contains $\xi(x)$ for every $x \in F$. The sub-$\sigma$-algebra of $B$ determined by $\xi$ is denoted $\hat{\xi}$ and defined by,

$$
\hat{\xi} = \{ F \in B : F \text{ is } \xi\text{-saturated} \}.
$$

We say that $\xi$ is a measurable partition if it is generated by a countable collection of measurable sets. That is if there exist $F_1, F_2, \ldots \in B$ such that,

$$
\xi(x) = \bigcap_{x \in F_n} F_n \cap \bigcap_{x \notin F_n} (X \setminus F_n) \text{ for all } x \in X.
$$

If $(Y, F)$ is another Borel space, $\zeta$ is a measurable partition of $Y$ and $\varphi : X \to Y$ is measurable, then

$$
\varphi^{-1}\zeta := \{ \varphi^{-1}\zeta(y) : y \in Y \}
$$

is easily seen to be a measurable partition of $X$. In particular this is the case for the partition $\{\varphi^{-1}\{y\}\}_{y \in Y}$ into level sets of $\varphi$.

**Theorem 2.1.** Let $(X, B, \rho)$ be a Borel probability space and let $\xi$ be a measurable partition of $X$. Then there exists a collection $\{\rho_x^\xi\}_{x \in X} \subset \mathcal{M}(X)$ such that,

1. for every $f \in L^1(\rho),$

\begin{equation}
\int f \, d\rho_x^\xi = E_\rho(f \mid \hat{\xi})(x) \text{ for } \rho\text{-a.e. } x;
\end{equation}

2. $\rho_x^\xi(\xi(x)) = 1$ for $x \in X$;

3. $\rho_x^\xi = \rho_y^\xi$ for $x, y \in X$ with $\xi(x) = \xi(y)$.

Moreover, these properties uniquely determine $\{\rho_x^\xi\}_{x \in X}$ up to a set of zero $\rho$-measure. We call the collection $\{\rho_x^\xi\}_{x \in X}$ the disintegration of $\rho$ with respect to the partition $\xi$.

**Lemma 2.2.** Let $(X, B, \rho)$ be a Borel probability space and let $\xi$ and $\zeta$ be a measurable partitions of $X$. Suppose that $\xi$ is finer that $\zeta$, that is $\xi(x) \subset \zeta(x)$ for all $x \in X$. Then for $\rho$-a.e. $x$,

$$
(\rho_y^\zeta)_y = (\rho_x^\xi)_y = \rho_y^\xi \text{ for } \rho_x^\xi\text{-a.e. } y.
$$
Lemma 2.3. Let \((X, \mathcal{B}, \rho)\) and \((Y, \mathcal{F}, \tau)\) be Borel probability spaces, let \(T : X \to Y\) be measure preserving and let \(\xi\) be a measurable partition of \(Y\). Then,
\[
T \rho_2^{\{x\}} = \tau_2^{x} \quad \text{for } \rho\text{-a.e. } x.
\]

We say that a complete separable metric space \(Y\) is a Besicovitch space if the Besicovitch covering lemma (see e.g. [Ma]) holds in \(Y\). Besicovitch spaces include, for instance, Euclidean spaces and compact finite-dimensional Riemannian manifolds. The following lemma is stated in [Fe, Lemma 2.5]. Its proof for the case \(Y = \mathbb{R}^d\) is given in [FH, Lemma 3.3]. Recall the notation \(B^\phi(x, r)\) from Section 2.1.

Lemma 2.4. Let \(\phi : X \to Y\) be a measurable mapping from a Borel probability space \((X, \mathcal{B}_X, \rho)\) to a Besicovitch space \(Y\). Denote by \(\mathcal{B}_Y\) the Borel \(\sigma\)-algebra of \(Y\). Let \(\xi\) be a measurable partition of \(X\) and let \(A \in \mathcal{B}_X\). Then for \(\rho\)-a.e. \(x \in X\),
\[
\lim_{r \downarrow 0} \frac{\rho_2^\xi(B^\phi(x, r) \cap A)}{\rho_2^\xi(B^\phi(x, r))} = E_\rho(1_A | \xi \vee \phi^{-1}(\mathcal{B}_Y))(x).
\]

2.4. A metric on the projective space. Recall that \(V\) is a real vector space with \(2 \leq \dim V < \infty\). Fix an inner product \(\langle \cdot, \cdot \rangle\) on \(V\), and denote its induced norm by \(\| \cdot \|\). Given a linear subspace \(W\) of \(V\), write \(P(W)\) for its projective space and \(P_W\) for the orthogonal projection onto \(W\) (by definition \(P(\{0\}) = 0\)). For \(0 \neq x \in V\) denote by \(\mathbf{x}\) the unique element of \(P(V)\) which contains \(x\). For \(0 \leq k \leq d\) write \(\text{Gr}(k, V)\) for Grassmannian manifold of \(k\)-dimensional linear subspaces of \(V\).

Let \(V^*\) be the dual of \(V\). Denote by \(A^2(V)\) the vector space of alternating 2-forms on \(V^*\). Let \(\langle \cdot, \cdot \rangle\) be the inner product on \(A^2(V)\) which satisfies,
\[
\langle x_1 \wedge x_2, y_1 \wedge y_2 \rangle = \det \begin{pmatrix} \langle x_1, y_1 \rangle & \langle x_1, y_2 \rangle \\ \langle x_2, y_1 \rangle & \langle x_2, y_2 \rangle \end{pmatrix} \quad \text{for } x_1, x_2, y_1, y_2 \in V.
\]

We denote the norm induced by this inner product by \(\| \cdot \|\). Given an endomorphism \(T\) of \(V\), write \(A^2T\) for the endomorphism of \(A^2(V)\) which satisfies,
\[
A^2T(x \wedge y) = (Tx) \wedge (Ty) \quad \text{for } x, y \in V.
\]

It is easy to verify that if \(P : V \to V\) is an orthogonal projection, then \(A^2P\) is also an orthogonal projection (defined on \(A^2(V)\)).

For \(\mathbf{x}, \mathbf{y} \in P(V)\) write,
\[
d(\mathbf{x}, \mathbf{y}) = \left(1 - \langle x, y \rangle^2\right)^{1/2},
\]
where \(x \in \mathbf{x}\) and \(y \in \mathbf{y}\) are unit vectors. It is easy to verify that this defines a metric on \(P(V)\). Note that,
\[
d(\mathbf{x}, \mathbf{y}) = |x|^{-1}|y|^{-1}\|x \wedge y\| \quad \text{for any } 0 \neq x \in \mathbf{x} \text{ and } 0 \neq y \in \mathbf{y}.
\]

For a subset \(Y \subset P(V)\) we set,
\[
d(\mathbf{x}, Y) = \inf\{d(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in Y\}.
\]

The following simple lemma will be used in Section 3.

Lemma 2.5. Let \(W \neq \{0\}\) be a proper linear subspace of \(V\) and let \(\mathbf{x}, \mathbf{y} \in P(V)\). Suppose that \(\mathbf{x}, \mathbf{y} \notin P(W^\perp)\), then
\[
d(P_W \mathbf{x}, P_W \mathbf{y}) \leq d(\mathbf{x}, P(W^\perp))^{-1}d(\mathbf{y}, P(W^\perp))^{-1}d(\mathbf{x}, \mathbf{y}).
\]
Proof. Let \( x \in \mathcal{F} \) and \( y \in \overline{\mathcal{F}} \) be with \( |x| = |y| = 1 \). If \( \mathcal{F} \in P(W) \),
\[
|P_W x| = |x| = 1 = d(\mathcal{F}, P(W^\perp)) .
\]
If \( \mathcal{F} \notin P(W) \),
\[
d(\mathcal{F}, P(W^\perp)) \leq d(\mathcal{F}, P(W^\perp)) = |P_W x|^{-1} \|x \land P_W x\|
= |P_W x|^{-1} \|P_W x \land P_W x\| = |P_W x| \cdot d(\mathcal{F}, P(W^\perp)) = |P_W x| .
\]
Similarly we always have \( |P_W y| \geq d(\mathcal{F}, P(W^\perp)) \). Additionally, since \( \Lambda^2 P_W \) is an orthogonal projection,
\[
\|P_W x \land P_W y\| = \|\Lambda^2 P_W (x \land y)\| \leq \|x \land y\| .
\]
Hence,
\[
d(\mathcal{F}, P(W^\perp)) = |P_W x|^{-1} |P_W y|^{-1} \|P_W x \land P_W y\|
\leq d(\mathcal{F}, P(W^\perp))^{-1} d(\mathcal{F}, P(W^\perp))^{-1} \|x \land y\|
= d(\mathcal{F}, P(W^\perp))^{-1} d(\mathcal{F}, P(W^\perp))^{-1} d(\mathcal{F}, P(W^\perp)) ,
\]
which completes the proof of the lemma. \( \square \)

2.5. The Furstenberg measure and the boundary map. Recall from Section 1.2 that \( \Lambda \) is a finite index set, \( \{A_i\}_{i \in \Lambda} \) are distinct elements of \( GL(V) \), \( p = (p_i)_{i \in \Lambda} \) is a probability vector with strictly positive coordinates and,
\[
\mu = \sum_{i \in \Lambda} p_i \delta_{A_i} \in \mathcal{M}(GL(V)) .
\]
As before, let \( S_\mu \) be the smallest closed subsemigroup of \( GL(V) \) such that \( \mu(S_\mu) = 1 \). We shall always assume from now on that \( S_\mu \) is strongly irreducible and proximal. Let \( \nu \) be the Furstenberg measure corresponding to \( \mu \), which means that \( \nu \) is unique \( \mu \)-stationary member of \( \mathcal{M}(P(V)) \).

Write \( \Omega = \Lambda^2 \) and let \( \sigma : \Omega \to \Omega \) be the left shift map. That is,
\[
(\sigma \omega)_n = \omega_{n+1} \text{ for } \omega \in \Omega \text{ and } n \in \mathbb{Z} .
\]
Denote by \( \mathcal{P} \) the partition of \( \Omega \) according to the 0-coordinate, i.e.
\[
\mathcal{P} = \{ \{ \omega \in \Omega : \omega_0 = l \} : l \in \Lambda \} .
\]
For integers \( m \leq n \) set,
\[
\mathcal{P}_m^n = \bigvee_{j=m}^n \sigma^{-j} \mathcal{P} .
\]
The atoms of these partitions are called the cylinder sets of \( \Omega \). We equip \( \Omega \) with the \( \sigma \)-algebra generated by its cylinder sets, which makes it into a Borel space. Let \( \beta \) be the Bernoulli measure on \( \Omega \) corresponding to the probability vector \( p \), that is \( \beta = \overline{p}^\mathcal{P} \). The triple \( (\Omega, \beta, \sigma) \) is an invertible ergodic measure preserving system.

As mentioned in Section 1.2 from our assumptions on \( S_\mu \) we obtain the following statement. Given a finite word \( l_1 \cdots l_n \) over the alphabet \( \Lambda \), we write \( A_{l_1} \cdots A_{l_n} \) in place of \( A_{l_1} \cdots A_{l_n} \).

**Theorem 2.6.** There exist a Borel set \( \Omega_0 \subseteq \Omega \), with \( \sigma(\Omega_0) = \Omega_0 \) and \( \beta(\Omega_0) = 1 \), and a Borel map \( \pi : \Omega_0 \to P(V) \), called the Furstenberg boundary map, such that:

1. \( \pi \) depends only on the nonnegative coordinates of \( \Omega \);
2. \( \pi \omega = A_{\omega_0} \pi \sigma \omega \) for \( \omega \in \Omega_0 \);
The following theorem, due to Guivarc’h [Gu Theorem 7], is used in Lemma 2.8 to bound the mass given by $\nu = \pi \beta$ to neighbourhoods of projective spaces of hyperplanes. A proof of this theorem can also be found in [BQ, Theorem 14.1].

**Theorem 2.7.** Assume, as we do, that $\mu$ is finitely supported and that $S_\mu$ is strongly irreducible and proximal. Then there exist $0 < \alpha \leq 1$ and $1 < C_0 < \infty$ such that for all $y \in V$ with $|y| = 1$,

$$\int \left( \frac{|x|}{|\langle x, y \rangle|} \right)^\alpha d\pi \beta(x) \leq C_0.$$

**Remark.** Theorem 2.7 remains true if instead of assuming that $\mu$ is finitely supported it is assumed that it has a finite exponential moment.

**Lemma 2.8.** There exist $0 < \alpha \leq 1$ and $1 < C < \infty$ such that for every $W \in \text{Gr}(\dim V - 1, V)$ and $r > 0$,

$$\pi \beta \{ \pi : d(\pi, \text{P}(W)) \leq r \} \leq Cr^\alpha.$$

**Proof.** Let $\alpha$ and $C_0$ be as in Theorem 2.7. Fix $W \in \text{Gr}(d - 1, V)$ and $0 < r < 1$. Let $y \in W^\perp$ be with $|y| = 1$, and $x \in V$ be with $|x| = 1$ and $d(\pi, \text{P}(W)) \leq r$. There exists $w \in W$ which satisfies $|w| = 1$ and $d(\pi, \text{P}(W)) \leq r$. We have,

$$r^2 \geq d(\pi, \text{P}(W))^2 = 1 - \langle x, w \rangle^2 = (1 - \langle x, w \rangle)(1 + \langle x, w \rangle).$$

Thus, by replacing $w$ with $-w$ if necessary, we may assume that $r^2 \geq 1 - \langle x, w \rangle$. Hence,

$$|x - w|^2 = 2 - 2 \langle x, w \rangle \leq 2r^2,$$

and so,

$$|x|^{-1} \cdot |\langle x, y \rangle| = |\langle x - w, y \rangle| \leq |x - w| \leq 2^{1/2}r.$$

From this we get,

$$\beta \{ \omega : d(\omega, \text{P}(W)) \leq r \} \leq \pi \beta \left\{ \pi : |x|^{-1} \cdot |\langle x, y \rangle| \leq 2^{1/2}r \right\} \leq 2^{\alpha/2}r^{\alpha} \cdot \int (|x| \cdot |\langle x, y \rangle|^{-1})^\alpha d\pi \beta(x) \leq 2^{\alpha/2}C_0r^\alpha,$$

which completes the proof of the lemma with $C = C_0 2^{\alpha/2}$. \hfill \square

### 2.6. Oseledets’ multiplicative ergodic theorem.

The following statement follows directly from Oseledets theorem (e.g., see [Ku, Section 3]), applied to the system $(\Omega, \beta, \sigma^{-1})$ and the matrix cocycle $\omega \to A_{\omega^{-1}}$, and by removing a set of zero $\beta$-measure from $\Omega_0$ without changing the notation (while still maintaining $\sigma(\Omega_0) = \Omega_0$).

**Theorem 2.9.** There exist positive integers $s, d_0, \ldots, d_s$, with $\dim V = d_0 + \ldots + d_s$, and real numbers $\lambda_0 > \ldots > \lambda_s$, so that for every $\omega \in \Omega_0$ there exist linear subspaces $E^0_\omega, \ldots, E^s_\omega \subset V$ such that,
(1) \( V = \bigoplus_{i=0}^{s} E^i \) and \( \dim E^i = d_i \) for \( 0 \leq i \leq s \);
(2) \( E^i_{\sigma^{-1} \omega} = A_{\omega^{-1}} E^i_\omega \) for \( 0 \leq i \leq s \);
(3) for \( 0 \leq i \leq s \) and \( 0 \neq x \in E^i_\omega \),
\[
\lim_{n \to \infty} \frac{1}{n} \log |A_{\omega^{-n} \ldots \omega^{-1}} x| = \lambda_i,
\]
with uniform convergence on any compact subset of \( E^i_\omega \setminus \{0\} \);
(4) for \( 0 \leq i \leq s \),
\[
\lim_{n \to \infty} \frac{1}{n} \max_{x \in E^i_{\omega^{-n}} \setminus \{0\}} \log |A_{\omega^{-n} \ldots \omega^{-1}} x| = \lambda_i,
\]
\[
\lim_{n \to \infty} \frac{1}{n} \min_{x \in E^i_{\omega^{-n}} \setminus \{0\}} \log |A_{\omega^{-n} \ldots \omega^{-1}} x| = \lambda_i;
\]
(5) \( \lim_{n \to -\infty} \frac{1}{n} \log \kappa(\sigma^n \omega) = 0 \), where for \( \eta \in \Omega_0 \)
\[
\kappa(\eta) = \min \{d(x, \gamma) : 0 \neq x \in \bigoplus_{i \in I} E^i_\eta, \ 0 \neq \gamma \in \bigoplus_{j \in J} E^j_\eta \text{ and } I \cap J = \emptyset \};
\]
(6) the map \( \omega \to E^i_\omega \) is Borel measurable for each \( 0 \leq i \leq s \).

**Remark 2.10.** By the strong irreducibility and proximality of \( S_\mu \) it follows that \( d_0 = 1 \) (see [BL, Theorem III.6.1]). This fact will play an important role in our development.

**Remark 2.11.** The numbers \( \lambda_0, \ldots, \lambda_s \) are called the Lyapunov exponents corresponding to \( \mu \). For \( 0 \leq i \leq s \) the integer \( d_i \) is called the multiplicity of \( \lambda_i \). The decomposition \( V = \bigoplus_{i=0}^{s} E^i_\omega \) is called the Oseledets splitting of \( V \) at \( \omega \). The subspaces \( E^0_\omega, \ldots, E^s_\omega \) are called the Oseledets subspaces corresponding to \( \omega \).

**Remark 2.12.** Property (4) of Theorem 2.9 is not stated in [Ru] in its present form. On the other hand, it follows from the development carried out there that for \( \omega \in \Omega_0 \), \( 0 \leq i \leq s \) and \( 0 \neq x \in E^i_\omega \),
\[
\lim_{n \to \infty} \frac{1}{n} \log A^{i}_{\omega^{-n}} = -\lambda_i,
\]
with uniform convergence on any compact subset of \( E^i_\omega \setminus \{0\} \). This together with property (2) easily imply property (4).

For \( 1 \leq i \leq s \) write \( \lambda_i' = \lambda_i - \lambda_0 \). For \( 0 \leq i \leq s \) and \( \omega \in \Omega_0 \) set \( V^i_\omega = \bigoplus_{k=i+1}^{s} E^k_\omega \). Note that \( V^0_\omega \) is of codimension 1. Also note that \( V^s_\omega = \{0\} \) and that for \( 0 \leq i < s \),
\[
V^i_\omega = \{ x \in V : \lim_{n \to \infty} \frac{1}{n} \log |A_{\omega^{-n} \ldots \omega^{-1}} x| \leq \lambda_i + 1 \}.
\]
This shows that the Borel maps \( \omega \to V^i_\omega \) depend only on the negative coordinates of \( \omega \). It is worth pointing out that this is not true for the Oseledets subspaces \( E^0_\omega, \ldots, E^s_\omega \).

### 3. Construction of local coordinates

In this section we use the Oseledets splittings \( V = \bigoplus_{i=0}^{s} E^i_\omega \) in order to construct coordinate charts \( g_\omega \) for \( P(V) \), whose domains are \( P(V) \setminus P(V^0) \). We then derive some useful properties for these charts. First we show that the domains just mentioned are neighbourhoods of the points \( \pi \omega \), and that \( \pi \sigma^n \omega \) does not escape exponentially fast to the boundary of the domains as \( n \to \infty \).
3.1. Coordinate neighbourhoods for \( \pi \omega \).

**Lemma 3.1.** We have,
\[
\beta \{ \omega : \pi \omega \in P(V^0_\omega) \} = 0.
\]
Thus, by removing a subset of zero \( \beta \)-measure from \( \Omega_0 \) without changing the notation, we may assume that \( \pi \omega \notin P(V^0_\omega) \) for all \( \omega \in \Omega_0 \).

**Proof.** Since \( S_n \) is strongly irreducible we have \( \pi \beta(P(W)) = 0 \) for \( W \in \text{Gr(dim } V - 1, V) \) (see [BL] Proposition III.2.3). Recall that \( \pi \) depends only on the nonnegative coordinates and that \( \omega \to V^0_\omega \) depends only on the negative coordinates. Thus, since \( \beta \) is a Bernoulli measure,
\[
\beta \{ \omega : \pi \omega \in P(V^0_\omega) \} = \int \pi \beta(P(V^0_\omega)) \, d\beta(\omega) = 0,
\]
which is what we wanted. \( \square \)

**Lemma 3.2.** For \( \beta \)-a.e. \( \omega \in \Omega_0 \),
\[
(3.1) \quad \lim_{n \to \infty} \frac{1}{n} \log d(\pi \sigma^n \omega, P(V^0_{\sigma^n \omega})) = 0.
\]
Thus, by removing a subset of zero \( \beta \)-measure from \( \Omega_0 \) without changing the notation, we may assume that (3.1) holds for all \( \omega \in \Omega_0 \).

**Proof.** Let \( \epsilon > 0 \) and for \( n \geq 1 \) set,
\[
F_n = \{ \omega \in \Omega_0 : d(\pi \omega, P(V^0_\omega)) < e^{-n \epsilon} \}.
\]
As in the proof of the previous lemma, since \( \beta \) is a Bernoulli measure,
\[
\beta(F_n) = \int \pi \beta(\pi : d(\pi \omega, P(V^0_\omega)) \leq e^{-n \epsilon}) \, d\beta(\omega).
\]
Let \( \alpha \) and \( C \) be as in Lemma 2.8. Then since \( \sigma \) preserves \( \beta \),
\[
\beta(\sigma^{-n} F_n) = \beta(F_n) \leq Ce^{-n \epsilon}.
\]
Thus, by the Borel-Cantelli lemma, for \( \beta \)-a.e. \( \omega \) there exists \( N_{\epsilon, \omega} \geq 1 \) such that,
\[
d(\pi \sigma^n \omega, P(V^0_{\sigma^n \omega})) \geq e^{-n \epsilon} \quad \text{for all } n \geq N_{\epsilon, \omega},
\]
which completes the proof of the lemma. \( \square \)

3.2. The coordinate maps \( g_\omega \). For \( \omega \in \Omega_0 \) and \( 0 \leq i \leq s \) let \( L^i_\omega \) be the linear projection of \( V \) onto \( E^i_\omega \) with respect to the splitting \( \bigoplus_{k=0}^{s} E^i_\omega \). That is for \( x \in V \),
\[
x = \sum_{i=0}^{s} L^i_\omega(x) \text{ with } L^i_\omega(x) \in E^i_\omega \text{ for } 0 \leq i \leq s.
\]
Write,
\[
L^0_\omega(x) = (L^0_\omega(x), \ldots, L^s_\omega(x)).
\]
For every \( \omega \in \Omega_0 \) fix a unit vector \( u^0_\omega \) in \( E^0_\omega \). Recall that \( \dim E^0_\omega = 1 \), and so \( E^0_\omega = \text{span}\{u^0_\omega\} \). Let \( f^0_\omega : V \to \mathbb{R} \) be the linear functional with,
\[
L^0_\omega(x) = f^0_\omega(x)u^0_\omega \text{ for } x \in V.
\]
Note that \( f^0_\omega(x) = 0 \) if and only if \( x \in V^0_\omega \). For \( \pi \in \text{P}(V) \setminus \text{P}(V^0_\omega) \) set,
\[
g^i_\omega(\pi) = L^i_\omega(x)/f^0_\omega(x) \text{ for } 0 \leq i \leq s,
\]
and,
\[
g_\omega(\pi) = (g^0_\omega(\pi), \ldots, g^s_\omega(\pi)).
\]
Many times we shall use the fact that \( L_\omega(x) = g_\omega(\varphi) \) for \( x \in \varphi \) with \( f_\omega^0(x) = 1 \).
Given a vector \( v = (v^0, \ldots, v^s) \), with \( v^i \in E^i_\omega \) for \( 0 \leq i \leq s \), write
\[
\|v\| = \max_{0 \leq i \leq s} |v^i|.
\]
Note that \( \|g_\omega(\varphi)\| \geq 1 \) for all \( \varphi \in P(V) \setminus P(V^0_\omega) \).

3.3. **Useful properties.** Recall from (2.6) that the subspaces \( V^i_\omega \) can be characterised in terms of the growth rate of \( |A_{\omega^-i-n}x| \). The following lemma provides a similar characterisation for certain foliations of \( P(V) \), which are defined in terms of the maps \( g_\omega \). Recall from Section 2.6 that for \( 1 \leq i \leq s \) we write \( \lambda_i = \lambda_i - \lambda_0 \).

**Lemma 3.3.** Let \( \omega \in \Omega_0 \) and \( \varphi, \varphi \in P(V) \setminus P(V^0_\omega) \) be with \( \varphi \neq \varphi \). Let \( 0 \leq i < s \) be such that,
\[ g_{\omega}^{i+1}(\varphi) \neq g_{\omega}^{i+1}(\varphi) \text{ and } g_{\omega}^{k}(\varphi) = g_{\omega}^{k}(\varphi) \text{ for } 0 \leq k \leq i. \]
Then,
\[ \lim_{n \to \infty} \frac{1}{n} \log(d(A_{\omega^-i-n} \varphi, A_{\omega^-i-n} \varphi)) = \tilde{\lambda}_{i+1}. \]

**Proof.** Since \( \varphi, \varphi \notin P(V^0_\omega) \) there exist \( x \in \varphi \) and \( y \in \varphi \) with \( f_\omega^0(x) = f_\omega^0(y) = 1 \).

Write,
\[
v_x = \sum_{k=0}^{i} L^k_\omega x, \quad w_x = \sum_{k=i+1}^{s} L^k_\omega x, \quad v_y = \sum_{k=0}^{i} L^k_\omega y, \quad w_y = \sum_{k=i+1}^{s} L^k_\omega y.
\]
By the definition of \( i \) we have \( v_x = v_y \), hence
\[
(x \wedge y) = v_x \wedge v_y + v_x \wedge w_y + w_x \wedge v_y + w_x \wedge w_y = v_x \wedge (w_y - w_x) + w_x \wedge w_y.
\]

For \( n \geq 1 \) set \( A_{\omega,n} := A_{\omega^-n} \). Note that \( L^0_\omega v_x \neq 0 \) and \( L^{i+1}_\omega (w_y - w_x) \neq 0 \) by the definition of \( i \). Additionally,
\[
L^k_\omega w_x = L^k_\omega w_y = 0 \text{ for } 0 \leq k \leq i.
\]
Combining these facts together with part (3) of Theorem 2.9 gives,
\[ |A_{\omega,n} v_x| = e^{n\lambda_0 + o_\omega \varphi(n)}, \]
\[ |A_{\omega,n} (w_y - w_x)| = e^{n\lambda_{i+1} + o_\omega \varphi(n)}, \]
and
\[ \|A_{\omega,n} v_x \wedge A_{\omega,n} w_y\| \leq e^{2n\lambda_{i+1} + o_\omega \varphi(n)}. \]

Note that,
\[ v_x \in \bigoplus_{k=0}^{i} E^k_\omega \quad \text{and} \quad w_y - w_x \in \bigoplus_{k=i+1}^{s} E^k_\omega. \]
so by part (2) of Theorem 2.9
\[ A_{\omega,n} v_x \in \bigoplus_{k=0}^{i} E^k_{\omega^-n} \quad \text{and} \quad A_{\omega,n} (w_y - w_x) \in \bigoplus_{k=i+1}^{s} E^k_{\omega^-n}. \]

Hence by part (5) of Theorem 2.9
\[ 1 \geq \left| \frac{A_{\omega,n} v_x \wedge A_{\omega,n} (w_y - w_x)}{|A_{\omega,n} v_x| \cdot |A_{\omega,n} (w_y - w_x)|} \right| \geq \kappa(\omega^{-n}) = e^{o_\omega(n)}. \]
Thus by (3.3) and (3.4),
\[ \|A_{\omega,n} v_x \wedge A_{\omega,n} (w_y - w_x)\| = e^{(\lambda_0 + \lambda_{i+1}) + o_\omega \varphi(n)}. \]
From this, (3.2) and (3.5),

\[ ||A_{\omega,n}x \wedge A_{\omega,n}y|| = e^{n(\lambda_0 + \lambda_i + 1) + o_{\omega}(\pi(n))}. \]  

From \( L_0^0 x \neq 0, L_0^0 y \neq 0 \) and part (3) of Theorem 2.9,

\[ |A_{\omega,n}x| = e^{n\lambda_0 + o_{\omega}(\pi(n))} \quad \text{and} \quad |A_{\omega,n}y| = e^{n\lambda_0 + o_{\omega}(\pi(n))}. \]

Hence by (3.4),

\[ d(A_{\omega,n}, A_{\omega,n}) = |A_{\omega,n}x|^{-1} |A_{\omega,n}y|^{-1} ||A_{\omega,n}x \wedge A_{\omega,n}y|| = e^{n(\lambda_i + 1 - \lambda_0) + o_{\omega}(\pi(n))}. \]

Since \( \bar{\lambda}_i + 1 = \lambda_i + 1 - \lambda_0 \), this completes the proof of the lemma. \( \square \)

In the remaining part of this section we show that \( d(\overline{\pi}, P(V_0^0)) \) is comparable with \( ||g_\omega(\overline{\pi})||_{-2}^{-1} \) in a manner depending on \( \kappa(\omega) \). For this we need the following lemma.

**Lemma 3.4.** Let \( \omega \in \Omega_0 \) and \( x \in V \) be given. Then,

\[ |x| \geq 2^{-s/2} \kappa(\omega)^s ||L_\omega(x)||_\infty. \]

**Proof.** For \( 0 \leq k \leq s \) set \( x_k = \sum_{i=0}^k L_\omega^i(x) \). We show by induction that for every \( 0 \leq k \leq s \),

\[ |x_k| \geq 2^{-k/2} \kappa(\omega)^k ||L_\omega(x_k)||_\infty. \]

Since \( x_s = x \) this will prove the lemma. Note that \( ||L_\omega(x_0)||_\infty = |x_0| \), hence (3.7) holds for \( k = 0 \).

Let \( 0 \leq k < s \) be such that (3.7) is satisfied for \( k \). If \( L_\omega^{k+1}(x) = 0 \) then \( x_{k+1} = x_k \), and so (3.7) holds also for \( k + 1 \). If \( x_k = 0 \) then \( |x_{k+1}| = ||L_\omega(x_{k+1})||_\infty \), and so (3.7) clearly holds for \( k + 1 \). It follows that we may assume that \( L_\omega^{k+1}(x) \neq 0 \) and \( x_k \neq 0 \).

Write \( u = L_\omega^{k+1}(x)/|L_\omega^{k+1}(x)| \) and \( v = x_k/|x_k| \). From \( u \in E_\omega^{k+1}, v \in \oplus_{i=0}^k E_\omega^i \) and the definition of \( \kappa(\omega) \),

\[ 1 - |\langle u, v \rangle| = \frac{d(\overline{\pi}, \overline{\pi})^2}{1 + |\langle u, v \rangle|} \geq \kappa(\omega)^2/2. \]

Thus,

\[ |x_{k+1}|^2 = \langle x_k + L_\omega^{k+1}(x), x_k + L_\omega^{k+1}(x) \rangle \]
\[ \geq |x_k|^2 + |L_\omega^{k+1}(x)|^2 - 2|x_k| \cdot |L_\omega^{k+1}(x)| \cdot |\langle u, v \rangle| \]
\[ = (|x_k|^2 + |L_\omega^{k+1}(x)|^2)(1 - |\langle u, u \rangle|) + |\langle v, u \rangle| (|x_k| - |L_\omega^{k+1}(x)|)^2 \]
\[ \geq 2^{-1}\kappa(\omega)^2(|x_k|^2 + |L_\omega^{k+1}(x)|^2). \]

From this and since (3.7) holds for \( k \),

\[ |x_{k+1}|^2 \geq 2^{-1}\kappa(\omega)^2(2^{-k}\kappa(\omega)^{2k}||L_\omega(x_k)||_\infty^2 + |L_\omega^{k+1}(x)|^2) \]
\[ \geq 2^{-k-1}\kappa(\omega)^{2(k+1)}||L_\omega(x_{k+1})||_\infty^2. \]

This shows that (3.7) holds for \( k + 1 \), which completes the induction and the proof of the lemma. \( \square \)

**Lemma 3.5.** Let \( \omega \in \Omega_0 \) and \( \overline{\pi} \in P(V) \setminus P(V_0^0) \) be given. Then,

\[ d(\overline{\pi}, P(V_0^0)) \leq s2^{s}\kappa(\omega)^{-s}||g_\omega(\overline{\pi})||_{-2}^{-1}. \]
\textbf{Proof.} Write $M$ for $\|g_\omega(\varpi)\|_\infty$. If $M = 1$ the statement is trivial, so we may assume that $M > 1$. Let $x \in \varpi$ be with $f_0^0(x) = 1$ and set $y = \sum_{i=1}^s L_i^0(x)$. From $f_0^0(x) = 1$ and $M > 1$ it follows that $g_\omega(\varpi) = L_\omega(x)$ and $\|L_\omega(y)\|_\infty = M$. Thus by Lemma 3.4

$$|x|, |y| \geq 2^{-s/2} \kappa(\omega)^s M.$$  

Additionally,

$$x \wedge y = (u_\omega^0 + y) \wedge y = u_\omega^0 \wedge y = \sum_{i=1}^s (u_\omega^0 \wedge L_i^0(x)).$$

Hence,

$$\|x \wedge y\| \leq \sum_{i=1}^s \|u_\omega^0 \wedge L_i^0(x)\| \leq sM.$$  

From $y \in V_\omega^0$ and these estimates we obtain,

$$d(\varpi, P(V_\omega^0)) \leq d(\varpi, \varpi) = |x|^{-1}|y|^{-1}\|x \wedge y\| \leq s2^s \kappa(\omega)^{-2s} M^{-1},$$

which completes the proof of the lemma. \hfill \Box

\textbf{Lemma 3.6.} Let $\omega \in \Omega_0$ and $\varpi \in P(V) \setminus P(V_\omega^0)$ be given. Then,

$$d(\varpi, P(V_\omega^0)) \geq (2s)^{-1} \kappa(\omega)\|g_\omega(\varpi)\|_\infty^{-1}.$$  

\textbf{Proof.} Let $x \in \varpi$ be with $f_0^0(x) = 1$, $y \in V_\omega^0$ be with $|y| = 1$ and $z \in (V_\omega^0)^\perp$ be with $|z| = 1$. Recall that $V_\omega^0$ is of codimension 1, so $z$ spans $(V_\omega^0)^\perp$. For every $w \in V_\omega^0$ with $0 < |w| \leq 1,$

$$1 - |\langle u_\omega^0, w/\|w\|\rangle| = \frac{d(w_\omega^0, \varpi)}{1 + |\langle u_\omega^0, w/\|w\|\rangle|} \geq \kappa(\omega)^2/2.$$  

Hence,

$$|\langle u_\omega^0, w\rangle| \leq |\langle u_\omega^0, w/\|w\|\rangle| \leq 1 - \kappa(\omega)^2/2.$$  

From this and,

$$u_\omega^0 = \langle u_\omega^0, z\rangle z + PV_\omega^0 u_\omega^0,$$

it follows,

$$1 = |u_\omega^0|^2 = \langle u_\omega^0, z\rangle^2 + \langle u_\omega^0, PV_\omega^0 u_\omega^0\rangle \leq \langle u_\omega^0, z\rangle^2 + 1 - \kappa(\omega)^2/2.$$  

This together with $f_0^0(x) = 1$ implies,

$$|\langle x, z\rangle| = |\langle u_\omega^0, z\rangle| \geq \kappa(\omega)/2.$$  

Now since $\|z \wedge y\| = 1$, $\langle y, z\rangle = 0$ and $|y| = 1$,

$$\|x \wedge y\| \geq |\langle x \wedge y, z \wedge y\rangle| = \left| \det \begin{pmatrix} \langle x, z\rangle & \langle x, y\rangle \\ \langle y, z\rangle & \langle y, y\rangle \end{pmatrix} \right| = |\langle x, z\rangle| \geq \kappa(\omega)/2.$$  

From this and since,

$$|x| \leq s\|L_\omega(x)\|_\infty = s\|g_\omega(\varpi)\|_\infty,$$

we get,

$$d(\varpi, \varpi) = |x|^{-1}\|x \wedge y\| \geq (2s)^{-1} \kappa(\omega)\|g_\omega(\varpi)\|_\infty^{-1}.$$  

Since $y$ is an arbitrary unit vector in $V_\omega^0$ this completes the proof of the lemma. \hfill \Box
4. Construction and properties of measurable partitions

In this section we construct measurable partitions for $\Omega_0$, similar to the ones defined in Section 4. We then establish some useful properties for these partitions and their corresponding conditional measures.

4.1. Construction. Let $\xi_0$ be the partition of $\Omega_0$ according to the negative coordinates. That is for $\omega \in \Omega_0$,

$$\xi_0(\omega) = \{ \eta \in \Omega_0 : \eta_j = \omega_j \text{ for } j \leq -1 \}.$$ 

Recall that for $0 \leq i \leq s$ the Borel map $\omega \rightarrow V^i_\omega$ depends only on the negative coordinates (see Section 2.6). Thus $V^i_\omega = V^i_\eta$ whenever $\xi_0(\omega) = \xi_0(\eta)$. Also recall that $\pi \omega \notin \mathcal{P}(V^0_\omega)$ for all $\omega \in \Omega_0$ (see Lemma 3.1). Hence for each $0 \leq i \leq s$ the map which takes $\omega \in \Omega_0$ to $P(V^i_\omega) \pi \omega \in \mathcal{P}(V^i_\omega)$ is well defined. It is clear that this map is Borel measurable. For $1 \leq i \leq s$ let $\xi_i$ be the partition of $\Omega_0$ such that for every $\omega \in \Omega_0$,

$$\xi_i(\omega) = \{ \eta \in \xi_0(\omega) : P(V^{i-1}_\omega) \pi \eta = P(V^{i-1}_\omega) \pi \omega \}. $$

It is easy to see that for every $0 \leq i \leq s$ there exists a Borel space $Y_i$ and a Borel map $\varphi_i : \Omega_0 \rightarrow Y_i$, such that the partition $\{ \varphi_i^{-1}(y) \}_{y \in Y_i}$ into the level sets of $\varphi_i$ is equal to $\xi_i$. Thus $\xi_0, ..., \xi_s$ are measurable partitions (see Section 2.3).

4.2. Properties of the partitions. The following lemma shows that it is possible to describe the partitions $\xi_i$ in terms of the coordinate maps $g_\omega$.

Lemma 4.1. Let $0 \leq i \leq s$ and $\omega, \eta \in \Omega_0$ be given, and suppose that $\xi_0(\omega) = \xi_0(\eta)$. Then $\xi_i(\omega) = \xi_i(\eta)$ if and only if $g^k_\omega(\pi \eta) = g^k_\omega(\pi \omega)$ for $0 \leq k \leq i$.

Proof. Since $g^0_\omega(\pi \eta)$ and $g^0_\omega(\pi \omega)$ are both equal to $u^0_\omega$, the lemma holds trivially when $i = 0$. Since $V^i_\omega = \emptyset$, we have $\xi_i(\omega) = \xi_i(\eta)$ if and only if $\pi \omega = \pi \eta$, which clearly holds if and only if $g_\omega(\pi \eta) = g_\omega(\pi \omega)$. Thus the lemma is also clear when $i = s$, and so we may assume that $1 \leq i < s$.

Let $x_\omega \in \pi \omega$ and $x_\eta \in \pi \eta$ be with $f^0_\omega(x_\omega) = f^0_\omega(x_\eta) = 1$. Since $L^k_\omega x_\omega, L^k_\omega x_\eta \in V^i_\omega$ for each $1 \leq k \leq s$,

$$P(V^i_\omega) x_\omega = \sum_{k=0}^{i} P(V^i_\omega) L^k_\omega x_\omega \quad \text{and} \quad P(V^i_\omega) x_\eta = \sum_{k=0}^{i} P(V^i_\omega) L^k_\omega x_\eta.$$

Moreover, it is not hard to see that,

$$P(V^i_\omega) \text{ is injective on } E^k_\omega \text{ for each } 0 \leq k \leq i,$$

and that,

$$V = V^i_\omega \oplus (\oplus_{k=0}^{i} P(V^i_\omega) E^k_\omega) .$$

We have $\xi_i(\omega) = \xi_i(\eta)$ if and only if $P(V^i_\omega) \pi \eta = P(V^i_\omega) \pi \omega$, which holds if and only if $P(V^i_\omega) x_\eta = c P(V^i_\omega) x_\omega$ for some $0 \neq c \in \mathbb{R}$. By (4.1) this holds if and only if,

$$\sum_{k=0}^{i} P(V^i_\omega) L^k_\omega x_\eta = \sum_{k=0}^{i} c P(V^i_\omega) L^k_\omega x_\omega \text{ for some } 0 \neq c \in \mathbb{R}.$$

But by (4.2) and (4.3) $f^0_\omega(x_\omega) = f^0_\omega(x_\eta) = 1$ this holds if and only if there exists $c \neq 0$ with $g^k_\omega(\pi \eta) = c g^k_\omega(\pi \omega)$ for $0 \leq k \leq i$. Since $g^0_\omega(\pi \eta) = g^0_\omega(\pi \omega) = u^0_\omega$ this completes the proof of the lemma.

\footnote{\textit{□}}
The following lemma describes the partitions $\sigma^{-n}\xi_i$. Its proof is similar to that of \cite[Lemma 4.4(1)]{Fe}. Recall the finite partitions $P^m_n$ from Section \ref{sec:finite-partitions}.

**Lemma 4.2.** Let $\omega \in \Omega_0$, $0 \leq i \leq s$ and $n \geq 1$ be given. Then,

$$\xi_i(\omega) \cap P^m_{n-1}(\omega) = \sigma^{-n}(\xi_i(\sigma^n\omega)).$$

As a consequence,

$$\xi_i \cap P^m_{n-1} = \sigma^{-n}\xi_i.$$  

**Proof.** Let $\eta \in \xi_i(\omega) \cap P^m_{n-1}(\omega)$, then $\eta_j = \omega_j$ for $j < n$ and by Lemma \ref{lem:finite-partitions},

$$g^k_\sigma(\pi\eta) = g^k_\sigma(\pi\omega) \text{ for } 0 \leq k \leq i.$$

Thus, since $A_{\omega_0, \ldots, \omega_{n-2}} = A_{\eta_0, \ldots, \eta_{n-2}}$, by part (2) of Theorem \ref{thm:finite-partitions} and and Lemma \ref{lem:finite-partitions},

$$\lim_{m \to \infty} \frac{1}{n + m} \log d(A_{\omega_0, \ldots, \omega_{n-2}\pi\sigma^n\eta, A_{\omega_0, \ldots, \omega_{n-2}\pi\sigma^n\omega})$$

$$\leq \lim_{m \to \infty} \frac{1}{m} \log d(A_{\omega_0, \ldots, \omega_{n-2}\pi\eta, A_{\omega_0, \ldots, \omega_{n-2}\pi\omega}) \leq \lambda_{i+1},$$

(wher $\lambda_{i+1}$ is interpreted as $-\infty$ in the case $i = s$). Now another application of Lemma \ref{lem:finite-partitions} gives,

$$g^k_{\sigma^n\omega}(\pi\sigma^n\eta) = g^k_{\sigma^n\omega}(\pi\sigma^n\omega) \text{ for } 0 \leq k \leq i.$$

This together with Lemma \ref{lem:finite-partitions} implies $\sigma^n\eta \in \xi_i(\sigma^n\omega)$, which shows

$$\xi_i(\omega) \cap P^m_{n-1}(\omega) \subset \sigma^{-n}(\xi_i(\sigma^n\omega)).$$

The reverse containment is proven similarly, which completes the proof of the lemma. \hfill $\square$

For $n \geq 1$ and $\epsilon > 0$ set,

$$Q_{n, \epsilon} = \{ \omega \in \Omega_0 : d(\pi\sigma^j\omega, P(\sigma^j)) \geq e^{-j\epsilon} \text{ for } j \geq n \}.$$

For $1 \leq i \leq s$, $\omega \in \Omega_0$ and $r > 0$ write,

$$\Gamma_i(\omega, r) = \{ \eta \in \xi_0(\omega) : d(P(V_j)\pi\omega, P(V_j)\pi\eta) \leq r \}.$$  

The following proposition, whose statement resembles that of \cite[Lemma 4.4(2)]{Fe}, will be used in Section \ref{sec:main-result} when we prove our main result.

**Proposition 4.3.** Let $1 \leq j \leq s$ and $0 \leq i < j$ be given. Then for every $\epsilon > 0$ there exists a Borel map $N_\epsilon : \Omega_0 \to \mathbb{N}$ such that for $\omega \in \Omega_0$ and $n \geq N_\epsilon(\omega)$,

$$Q_{n, \epsilon} \cap \xi_i(\omega) \cap P^m_{n-1}(\omega) \subset \Gamma_j(\omega, e^{n(\lambda_{i+1}+5\epsilon)}).$$

**Proof.** Let $0 < \epsilon < -\lambda_1/3$ and $\omega \in \Omega_0$ be given. Let $n \geq 1$ be large with respect to $\epsilon$ and $\omega$ in a manner described during the proof. Since the maps $\pi$ and $\omega \to E^k_\omega$ are all Borel measurable, it will be clear that the conditions imposed on how large $n$ should be, are all Borel measurable as well.

Let $\eta \in Q_{n, \epsilon} \cap \xi_i(\omega) \cap P^m_{n-1}(\omega)$. From Lemma \ref{lem:finite-partitions} it follows that $\sigma^n\eta \in \xi_i(\sigma^n\omega)$. Thus $V^0_{\sigma^n\eta} = V^0_{\sigma^n\omega}$ and by Lemma \ref{lem:finite-partitions},

$$g^k_{\sigma^n\omega}(\pi\sigma^n\eta) = g^k_{\sigma^n\omega}(\pi\sigma^n\eta) \text{ for } 0 \leq k \leq i.$$

Let $x_{\sigma^n\omega} \in \pi\sigma^n\omega$ and $x_{\sigma^n\eta} \in \pi\sigma^n\eta$ be with $f^0_{\sigma^n\omega}(x_{\sigma^n\omega}) = f^0_{\sigma^n\omega}(x_{\sigma^n\eta}) = 1.$
By part 3 of Theorem 2.9 and by assuming that \( \delta \) is sufficiently large with respect to \( \epsilon \) and \( \omega \), we get \( \kappa(\sigma^n \omega) \geq -\epsilon \delta/(2s) \). From \( \eta \in Q_{n,\epsilon}, \; V_{\sigma^n \eta}^0 = V_{\sigma^n \omega}^0 \) and Lemma 3.5
\[
e^{-\epsilon \delta} \leq d(\pi \sigma^n \eta, P(V_{\sigma^n \omega}^0)) \leq 2^{s} \kappa(\sigma^n \omega)^{-2s} \| g_{\sigma^n \omega}(\pi \sigma^n \eta) \|_\infty^{-1}.
\]
Thus,
\[
(4.7) \quad \| L_{\sigma^n \omega, x_{\sigma^n \eta}} \|_\infty = \| g_{\sigma^n \omega}(\pi \sigma^n \eta) \|_\infty \leq 2^{s} \kappa^{2s} \| \epsilon \delta \|.
\]
From Lemma 3.2 and by assuming that \( \delta \) is large enough with respect to \( \omega \) and \( \epsilon \), we get \( d(\pi \sigma^n \omega, P(V_{\sigma^n \omega}^0)) \geq -\epsilon \delta \). Hence the same argument as above gives,
\[
(4.8) \quad \| L_{\sigma^n \omega, x_{\sigma^n \omega}} \|_\infty = \| g_{\sigma^n \omega}(\pi \sigma^n \omega) \|_\infty \leq 2^{s} \kappa^{2s} \| \epsilon \delta \|.
\]
Write \( A_{\omega,n} \) for \( A_{\omega_0 \ldots \omega_{n-1}} \) and note that from \( \eta \in P_{\omega}^{n-1}(\omega) \) it follows that \( A_{\omega,n} = A_{\omega_0 \ldots \omega_{n-1}}. \) If \( u_0^0 \notin P((V^1_{\omega})^\perp) \) then,
\[
1 - |P_{V^1_{\omega}} u_0^0|^2 = 1 - \left| \left< u_0^0, P_{V^1_{\omega}} u_0^0 \right> \right| = \frac{d(u_0^0, P_{V^1_{\omega}} u_0^0)^2}{1 + \left| \left< u_0^0, P_{V^1_{\omega}} u_0^0 \right> \right|} \geq \kappa(\omega)^2/2.
\]
Thus,
\[
1 = |u_0^0|^2 = |P_{V^1_{\omega}} u_0^0|^2 + |P_{(V^1_{\omega})^\perp} u_0^0|^2 \leq 1 - \kappa(\omega)^2/2 + |P_{(V^1_{\omega})^\perp} u_0^0|^2,
\]
which gives \( |P_{(V^1_{\omega})^\perp} u_0^0|^2 \geq \kappa(\omega)/2 \). Note that this inequality holds trivially if \( u_0^0 \in P((V^1_{\omega})^\perp) \) have \( |P_{(V^1_{\omega})^\perp} u_0^0| \geq \kappa(\omega)/2 \). By part 2 of Theorem 2.9 it follows that \( A_{\omega,n} u_0^0 \in \overline{u_0^0} \), hence
\[
|P_{(V^1_{\omega})^\perp} A_{\omega,n} u_0^0| \geq \frac{1}{2} \kappa(\omega) |A_{\omega,n} u_0^0|.
\]
Now from this, from part 4 of Theorem 2.9 and from (4.7),
\[
|P_{(V^1_{\omega})^\perp} A_{\omega,n} x_{\sigma^n \eta}| = |P_{(V^1_{\omega})^\perp} A_{\omega,n} u_0^0| + \sum_{k=1}^{s} |P_{(V^1_{\omega})^\perp} A_{\omega,n} L_{\sigma^n \omega, x_{\sigma^n \eta}}|
\geq \frac{1}{2} \kappa(\omega) |A_{\omega,n} u_0^0| - \sum_{k=1}^{s} |A_{\omega,n} L_{\sigma^n \omega, x_{\sigma^n \eta}}|
= e^{n \lambda_0 + o_\omega(n)} - \sum_{k=1}^{s} e^{n \lambda_k + o_\omega(n)} |L_{\sigma^n \omega, x_{\sigma^n \eta}}|
= e^{n \lambda_0 + o_\omega(n)} - \sum_{k=1}^{s} e^{n \lambda_k + o_\omega(n)} 2^{s} \kappa^{2s} \| \epsilon \delta \|
= e^{n \lambda_0 + o_\omega(n)},
\]
where the last equality follows from \( \epsilon < -\hat{\lambda}_1/3 \). Similarly by using (4.8) we obtain,
\[
|P_{(V^1_{\omega})^\perp} A_{\omega,n} x_{\sigma^n \omega}| \geq e^{n \lambda_0 + o_\omega(n)}.
\]
Next we estimate the norm of,
\[
A^2 P_{(V^1_{\omega})^\perp} (A_{\omega,n} x_{\sigma^n \omega} \land A_{\omega,n} x_{\sigma^n \eta}),
\]
where \( A^2 P_{(V^1_{\omega})^\perp} \) is defined in (2.3). Write,
\[
v_{\sigma^n \omega} = \sum_{k=0}^{i} L_{\sigma^n \omega}^k (x_{\sigma^n \omega}) \quad \text{and} \quad w_{\sigma^n \omega} = \sum_{k=i+1}^{s} L_{\sigma^n \omega}^k (x_{\sigma^n \omega}),
\]
where
Lemma 4.4. Lemma is similar to that of [F, Lemma 4.6].

given at the introduction in (1.2) (see Lemma 6.4). The proof of the following
expression is the conditional entropy of
\[ P \]
\[ \sum_{k=0}^{s} L_{\sigma}^{k}(x_{\sigma}^{n}) \]
and similarly,
\[ w_{\sigma}^{n} = \sum_{k=1}^{s} L_{\sigma}^{k}(x_{\sigma}^{n}) \]
From (4.6) we get \( v_{\sigma}^{n} = v_{\sigma}^{n}. \) Hence,
\[ x_{\sigma}^{n} \land x_{\sigma}^{n} = (v_{\sigma}^{n} + w_{\sigma}^{n}) \land (v_{\sigma}^{n} + w_{\sigma}^{n}) \]
(4.9) \[ = v_{\sigma}^{n} \land w_{\sigma}^{n} \land v_{\sigma}^{n} \land w_{\sigma}^{n} \land w_{\sigma}^{n} \land w_{\sigma}^{n} \]
By applying part (1) of Theorem 2.9 and then (4.7), it follows that for each \( 0 \leq k \leq s, \)
\[ |A_{\omega,n}L_{\sigma}^{k}(x_{\sigma}^{n})| = e^{n\lambda_{k} + o_{\omega}(n)} |L_{\sigma}^{k}(x_{\sigma}^{n})| \leq e^{n(\lambda_{k} + 2\epsilon) + o_{\omega}(n)}, \]
and similarly by (4.8),
\[ |A_{\omega,n}L_{\sigma}^{k}(x_{\sigma}^{n})| \leq e^{n(\lambda_{k} + 2\epsilon) + o_{\omega}(n)}. \]
Thus from (4.9) we get,
\[ \| A_{\omega,n}x_{\sigma}^{n} \land A_{\omega,n}x_{\sigma}^{n} \| \leq e^{n(\lambda_{0} + \lambda_{i+1} + 4\epsilon) + o_{\omega}(n)}. \]
Since \( P_{V_{\omega}^{2}} \) is an orthogonal projection the same holds for \( A^{2}P_{V_{\omega}^{2}} \). Hence,
\[ \| A^{2}P_{V_{\omega}^{2}}(A_{\omega,n}x_{\sigma}^{n} \land A_{\omega,n}x_{\sigma}^{n}) \| \leq e^{n(\lambda_{0} + \lambda_{i+1} + 4\epsilon) + o_{\omega}(n)}. \]
Now set \( T = P_{V_{\omega}^{2}}A_{\omega,n}. \) Then from the last inequality, from part (2) of Theorem
2.9 and by the lower bounds on \( |T_{x_{\sigma}^{n}}| \) and \( |T_{x_{\sigma}^{n}}| \) obtained above,
\[ d(P_{V_{\omega}^{2}}, \pi \omega, P_{V_{\omega}^{2}}, \pi \eta) = d(T \pi \sigma^{n}, \pi \sigma^{n}) \]
\[ = |T_{x_{\sigma}^{n}}|^{-1}|T_{x_{\sigma}^{n}}|^{-1} \| T_{x_{\sigma}^{n}} \land T_{x_{\sigma}^{n}} \| \]
\[ \leq e^{-2n\lambda_{0} + o_{\omega}(n)} e^{n(\lambda_{0} + \lambda_{i+1} + 4\epsilon) + o_{\omega}(n)} \]
\[ = e^{n(\lambda_{i+1} + 4\epsilon) + o_{\omega}(n)}. \]
Thus, by assuming that \( n \) is sufficiently large with respect to \( \omega \) and \( \epsilon \) we get,
\[ \eta \in \Gamma_{i}(\omega, e^{n(\lambda_{i+1} + 5\epsilon)}), \]
which completes the proof of the proposition. \( \square \)

For \( 0 \leq i \leq s \) we write \( H_{i} \) in place of \( H_{i}(P | \xi_{i}), \) where recall that the last
expression is the conditional entropy of \( P \) given the \( \sigma \)-algebra \( \xi_{i} \) (see Sections 2.2
and 2.3). It is easy to verify that this definition of \( H_{i} \) is consistent with the one
given at the introduction in (1.2) (see Lemma 6.4). The proof of the following
Lemma is similar to that of [F, Lemma 4.6].

**Lemma 4.4.** Let \( 0 \leq i \leq s, \) then for \( \beta \)-a.e. \( \omega \in \Omega_{0} \) and each \( n \geq 1, \)
\[ \log \beta_{\omega}^{n} \left( P_{0}^{n-1}(\omega) \right) = I_{\beta}(P_{0}^{n-1} | \xi_{i})(\omega) = \sum_{j=0}^{n-1} I_{\beta}(P | \xi_{i})(\sigma^{j} \omega), \]
and,
\[ -\frac{1}{n} \log \beta_{\omega}^{n} \left( P_{0}^{n-1}(\omega) \right) = H_{i} \] for \( \beta \)-a.e. \( \omega \).
Proof. By Lemma 4.2 we have \( \xi_i \vee \mathcal{P} = \sigma^{-1} \xi_i \). It is easy to verify that this implies,

\[
\hat{\xi}_i \vee \hat{\mathcal{P}} = \sigma^{-1} \hat{\xi}_i,
\]

which means that for every \( B \in \hat{\xi}_i \vee \hat{\mathcal{P}} \) there exists \( B' \in \sigma^{-1} \hat{\xi}_i \) with \( \beta(B \Delta B') = 0 \) and vice versa. From this, together with (2.2) and (2.1) in Section 2.2 it follows that for \( n \geq 1 \),

\[
I_\beta(P_0^{n-1} \mid \hat{\xi}_i) = I_\beta(P \mid \hat{\xi}_i) + I_\beta(P_1^{n-1} \mid \hat{\xi}_i \vee \hat{\mathcal{P}})
\]

\[
= I_\beta(P \mid \hat{\xi}_i) + I_\beta(\sigma^{-1}P_0^{n-2} \mid \sigma^{-1} \hat{\xi}_i)
\]

\[
= I_\beta(P \mid \hat{\xi}_i) + I_\beta(P_0^{n-2} \mid \hat{\xi}_i) \circ \sigma.
\]

Iterating this we get,

\[
I_\beta(P_0^{n-1} \mid \hat{\xi}_i) = \sum_{j=0}^{n-1} I_\beta(P \mid \hat{\xi}_i) \circ \sigma^j.
\]

Additionally, by the definitions of the conditional information and measures (see Theorem 2.1),

\[
- \log \beta_\xi(P_0^{n-1}(\omega)) = I_\beta(P_0^{n-1} \mid \hat{\xi}_i)(\omega) \text{ for } \beta\text{-a.e. } \omega,
\]

which gives (4.10). Birkhoff’s ergodic theorem combined with (4.10) implies (4.11), which completes the proof of the lemma.

The following two lemmas will be used in Section 6 when we prove our main result. Recall the sets \( Q_{n,\epsilon} \) from (3.2).

Lemma 4.5. For \( \epsilon > 0 \) and \( 0 \leq i \leq s \),

\[
\lim_{n \to \infty} \frac{\beta_\xi(Q_{m,\epsilon} \cap P_0^{n-1}(\omega))}{\beta_\xi(P_0^{n-1}(\omega))} = 1 \text{ for } \beta\text{-a.e. } \omega \in \Omega_0.
\]

Proof. From Lemma 2.2 and Theorem 2.1 it follows that for \( n, m \geq 1 \),

\[
\frac{\beta_\xi(Q_{m,\epsilon} \cap P_0^{n-1}(\omega))}{\beta_\xi(P_0^{n-1}(\omega))} = E_\beta(1_{Q_{m,\epsilon}} \mid \hat{\xi}_i \vee P_0^{n-1}(\omega)) \text{ for } \beta\text{-a.e. } \omega.
\]

Note that the sequence of \( \sigma \)-algebras,

\[
\{ \hat{\xi}_i \vee P_0^{n-1} \}_{n \geq 1},
\]

increases to the Borel \( \sigma \)-algebra of \( \Omega \). Thus, from (4.12) and the increasing martingale theorem (see [Pa, Section 2.1]),

\[
\lim_{n \to \infty} \frac{\beta_\xi(Q_{m,\epsilon} \cap P_0^{n-1}(\omega))}{\beta_\xi(P_0^{n-1}(\omega))} = 1_{Q_{m,\epsilon}}(\omega) \text{ for } \beta\text{-a.e. } \omega.
\]

Additionally we have \( Q_{m,\epsilon} \subset Q_{n,\epsilon} \) whenever \( n \geq m \), hence

\[
\liminf_{n \to \infty} \frac{\beta_\xi(Q_{m,\epsilon} \cap P_0^{n-1}(\omega))}{\beta_\xi(P_0^{n-1}(\omega))} \geq 1_{Q_{m,\epsilon}}(\omega) \text{ for } \beta\text{-a.e. } \omega.
\]

Now since by Lemma 3.2

\[
\Omega_0 = \cup_{m \geq 1} Q_{m,\epsilon},
\]

this completes the proof of the lemma. \( \square \)

Recall the sets \( \Gamma_k(\omega, r) \) from (4.5).
Lemma 4.6. Let $1 \leq k \leq s$ and $0 \leq i \leq s$ be given, and let $F \subset \Omega_0$ be a Borel set. Then for $\beta$-a.e. $\omega \in F$,
\[
\lim_{r \downarrow 0} \frac{\beta^k_\omega (\Gamma_k(\omega, r) \cap F)}{\beta^k_\omega (\Gamma_k(\omega, r))} > 0 .
\]

Proof. Define a map $\phi_k : \Omega_0 \to P(V)$ by $\phi_k(\omega) = P(V^k_\omega \cap \pi \omega)$. Recall that $\pi \omega \notin P(V^k_\omega)$ for each $\omega \in \Omega_0$, so $\phi_k$ is well defined. Since $\pi$ and $\omega \to V^k_\omega$ are Borel measurable the same holds for $\phi_k$. It follows directly from the definitions that for $r > 0$ and $\omega \in \Omega_0$,
\[
\Gamma_k(\omega, r) = \xi_0(\omega) \cap \mathcal{B}^\phi_k(\omega, r),
\]
where the notation $\mathcal{B}^\phi_k(\omega, r)$ was defined in Section 2.4. Write $\mathcal{B}$ for the Borel $\sigma$-algebra of $P(V)$. Then by Lemma 2.3 it follows that for $\beta$-a.e. $\omega$,
\[
\lim_{r \downarrow 0} \frac{\beta^k_\omega (\Gamma_k(\omega, r) \cap F)}{\beta^k_\omega (\Gamma_k(\omega, r))} = E_\beta (1_F | \xi_i \vee \phi_k^{-1} \mathcal{B}) (\omega) .
\]

Now, by the definition of the conditional expectation, it follows easily (see [FH, Lemma 3.10]) that for $\beta$-a.e. $\omega \in F$,
\[
E_\beta (1_F | \xi \vee \phi_k^{-1} \mathcal{B}) (\omega) > 0 ,
\]
which completes the proof of the lemma. \hfill \Box

The following lemma, which is similar to [FH, Lemma 4.5(3)], will be used in the next section.

Lemma 4.7. Let $0 \leq i \leq s$ and $n \geq 1$ be given. Then for $\beta$-a.e. $\omega$ we have for any Borel set $F \subset \Omega_0$,
\[
\beta^n_\omega(\sigma^{-n}F \cap \mathcal{P}^{n-1}_0(\omega)) = \beta^n_{\sigma^n \omega}(F) \beta^n_\omega(\mathcal{P}^{n-1}_0(\omega)) .
\]

Proof. By Lemma 2.3
\[
(4.13) \quad \sigma^{-n}\beta^n_{\sigma^n \omega} = \beta^n_\omega \beta^{-n}_i \text{ for } \beta\text{-a.e. } \omega .
\]

By Lemmas 4.2 and 4.2 it follows that for $\beta$-a.e. $\omega$,
\[
\beta^{-n}_\omega \beta^n_i = \beta^n_{\xi_i \vee \mathcal{P}^{n-1}_0} = (\beta^n_{\xi_i \vee \mathcal{P}^{n-1}_0}) \text{ for } \beta\text{-a.e. } \eta .
\]
Thus for $\beta$-a.e. $\omega$ and any $F \subset \Omega_0$ Borel,
\[
\beta^{-n}_\omega \beta^n_i (F) = \frac{\beta^n_{\omega} (F \cap \mathcal{P}^{n-1}_0(\omega))}{\beta^n_{\omega} (\mathcal{P}^{n-1}_0(\omega))} .
\]

From this and (4.13) it follows that for $\beta$-a.e. $\omega$ and any $F \subset \Omega_0$ Borel,
\[
\frac{\beta^n_{\omega} (\sigma^{-n}F \cap \mathcal{P}^{n-1}_0(\omega))}{\beta^n_{\omega} (\mathcal{P}^{n-1}_0(\omega))} = \beta^{-n}_\omega \beta^n_i (\sigma^{-n}F) = \sigma^{-n} \beta^n_{\sigma^n \omega}(\sigma^{-n}F) = \beta^n_{\sigma^n \omega}(F) ,
\]
which completes the proof of the lemma. \hfill \Box
5. Transverse dimensions

In this section we prove an inequality for the transverse dimensions. Recall that,

\[ H_i = H_\beta(P \mid \hat{\xi}_i) \text{ for } 0 \leq i \leq s, \]

and that for \( 1 \leq i \leq s, \omega \in \Omega_0 \) and \( r > 0, \)

\[ \Gamma_i(\omega, r) = \{ \eta \in \xi_0(\omega) : d(P_{(V^\perp_\omega)}, \pi\omega, P_{(V^\perp_\omega)} \pi\eta) \leq r \}. \]

We also set,

\[ \vartheta_{i-1}(\omega) = \liminf_{r \downarrow 0} \frac{\log \beta^\vartheta \vartheta_{i-1}(\Gamma_i(\omega, r))}{\log r}. \]

Following [Fe] we call \( \vartheta_0, \ldots, \vartheta_{s-1} \) the transverse dimensions of \( \beta \). The purpose of this section is to prove the following proposition. Its proof is a modification of that of [Fe, Proposition 5.1].

**Proposition 5.1.** For \( 1 \leq i \leq s \) and \( \beta \)-a.e. \( \omega, \)

\[ \vartheta_{i-1}(\omega) \geq \frac{H_i - H_{i-1}}{\lambda_i}. \]

### 5.1. Preparations for the proof of Proposition 5.1

For \( 1 \leq i \leq s, \omega \in \Omega_0 \) and \( r > 0 \) set,

\[ T_i(\omega, r) = \{ \eta \in \xi_{i-1}(\omega) : |g^\vartheta_\omega(\pi\omega) - g^\vartheta_{\pi\eta}| \leq r \}. \]

In this subsection we mainly study the relation between the sets \( \Gamma_i(\omega, r) \) and \( T_i(\omega, r) \). Later we establish other facts which will be needed for the proof of Proposition 5.1. We start with the following containment.

**Lemma 5.2.** Let \( 1 \leq i \leq s, \omega \in \Omega_0 \) and \( r > 0 \) be given. Then,

\[ T_i(\omega, r) \subset \Gamma_i(\omega, s^32^{2+s}\|g^\vartheta_{\pi\omega}\|_\infty, k(\omega)^{-2s-2}r). \]

**Proof.** Let \( \eta \in T_i(\omega, r) \), and let \( x_\omega \in \pi\omega \) and \( x_\eta \in \pi\eta \) be with \( f^0_\omega(x_\omega) = f^0_\omega(x_\eta) = 1 \).

First we show that,

\[ |P_{(V^\perp_\omega)} x_\omega| |P_{(V^\perp_\omega)} x_\eta| \geq s^{-1}2^{-1-(s/2)}k(\omega)^{s+1}. \]

We prove this inequality only for \( x_\eta \); the proof for \( x_\omega \) is similar. By Lemma 5.3,

\[ |x_\eta| \geq 2^{-s/2}k(\omega)^s\|g^\vartheta_{\pi\eta}\|_\infty. \]

If \( \pi\eta \in P((V^\perp_\omega)^\perp) \) then since \( \|g^\vartheta_{\pi\eta}\|_\infty \geq 1, \)

\[ |P_{(V^\perp_\omega)} x_\eta| = |x_\eta| \geq 2^{-s/2}k(\omega)^s, \]

and so we may assume that \( \pi\eta \notin P((V^\perp_\omega)^\perp) \). Now since \( P_{(V^\perp_\omega)} \pi\eta \in P(V^\perp_\omega), \)

\[ d(\pi\eta, P(V^\perp_\omega)) \leq d(\pi\eta, P_{(V^\perp_\omega)} \pi\eta) \]

\[ = |x_\eta|^{-1} |P_{(V^\perp_\omega)} x_\eta|^{-1} ||x_\eta \wedge P_{(V^\perp_\omega)} x_\eta|| \]

\[ \leq 2^{s/2}k(\omega)^{-s} \|g^\vartheta_{\pi\eta}\|^{-1}_\infty |P_{(V^\perp_\omega)} x_\eta x_\eta \wedge P_{(V^\perp_\omega)} x_\eta|| \]

\[ = 2^{s/2}k(\omega)^{-s} \|g^\vartheta_{\pi\eta}\|^{-1}_\infty |P_{(V^\perp_\omega)} x_\eta|.| \]

This together with Lemma 5.6 gives,

\[ (2s)^{-1}k(\omega)^{-s} \|g^\vartheta_{\pi\eta}\|^{-1}_\infty \leq d(\pi\eta, P(V^\perp_\omega)) \leq 2^{s/2}k(\omega)^{-s} \|g^\vartheta_{\pi\eta}\|^{-1}_\infty |P_{(V^\perp_\omega)} x_\eta|, \]

which implies 5.1.
Now set $M = \|g_\omega \pi \omega\|_\infty$ and let us show that,

\begin{equation}
    \|P_(V^i_\omega)^\perp x_\omega \wedge P_(V^i_\omega)^\perp x_\eta\| \leq 2sMr .
\end{equation}

Write $y = \sum_{k=0}^{i-1} g^k_\omega \pi \omega$. From $\eta \in T_i(\omega, r) \subset \xi_{i-1}(\omega)$ and Lemma 4.1, it follows,

$$y = \sum_{k=0}^{i-1} g^k_\omega \pi \eta \quad \text{and} \quad |g^k_\omega \pi \omega - g^k_\omega \pi \eta| \leq r .$$

Since $g^k_\omega \pi \eta \in V^i_\omega$ for all $i < k \leq s$,

$$P_(V^i_\omega)^\perp x_\eta = P_(V^i_\omega)^\perp \sum_{k=0}^{s} g^k_\omega \pi \eta = P_(V^i_\omega)^\perp (y + g^k_\omega \pi \eta) .$$

Similarly,

$$P_(V^i_\omega)^\perp x_\omega = P_(V^i_\omega)^\perp (y + g^k_\omega \pi \omega) .$$

Since $P_(V^i_\omega)^\perp$ is an orthogonal projection the same holds for $A^2P_(V^i_\omega)^\perp$ (which is defined in (2.3)). Hence,

$$\|P_(V^i_\omega)^\perp x_\omega \wedge P_(V^i_\omega)^\perp x_\eta\| = \|A^2P_(V^i_\omega)^\perp ((y + g^k_\omega \pi \omega) \wedge (y + g^k_\omega \pi \eta))\|$$

$$\leq \|(y + g^k_\omega \pi \omega) \wedge (y + g^k_\omega \pi \eta)\|$$

$$\leq \|y \wedge (g^k_\omega \pi \omega - g^k_\omega \pi \eta\| + \|g^k_\omega \pi \omega \wedge (g^k_\omega \pi \eta - g^k_\omega \pi \omega)\|$$

$$\leq |y| \cdot g^k_\omega \pi \omega - g^k_\omega \pi \omega \| + |g^k_\omega \pi \omega \| \cdot g^k_\omega \pi \eta - g^k_\omega \pi \omega \|$$

$$\leq 2sMr .$$

Combining (5.1) with (5.2) we obtain,

$$d(P_(V^i_\omega)^\perp \pi \omega, P_(V^i_\omega)^\perp \pi \eta) = \|P_(V^i_\omega)^\perp x_\omega\|^{-1} \|P_(V^i_\omega)^\perp x_\eta\|^{-1} \|P_(V^i_\omega)^\perp x_\omega \wedge P_(V^i_\omega)^\perp x_\eta\|$$

$$\leq s^{3+2s} M \cdot \kappa(\omega)^{-2s-2} ,$$

which completes the proof of the lemma. \hfill \Box

The containment in the other direction, which is proven in Lemma 5.5, requires a bit more work. For $\omega \in \Omega_0$ and $0 \leq i \leq s$ we write $W_\omega^i$ for $\otimes_{k=0}^i E_\omega^k$.

**Lemma 5.3.** Let $1 \leq i \leq s$ be given, then for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that the following holds. Let $\omega \in \Omega_0$ be with $\kappa(\omega) \geq \epsilon$. Then for every $x, y \in W_\omega^i$,

$$\|A^2P_(V^i_\omega)^\perp (x \wedge y)\| \geq \delta \|x \wedge y\| .$$

**Proof.** Since $V^s_\omega = \{0\}$ for every $\omega \in \Omega_0$, the lemma holds trivially when $i = s$. Assume by contradiction that the lemma is false for some $1 \leq i < s$ and $\epsilon > 0$. Then for every $n \geq 1$ there exist $\omega_n \in \Omega_0$ and $x_n, y_n \in W_\omega^i$ such that $\kappa(\omega_n) \geq \epsilon$, $\|x_n \wedge y_n\| = 1$ and $\|A^2P_(V^i_\omega)^\perp (x_n \wedge y_n)\| \leq \frac{1}{n}$. Note that from $\|x_n \wedge y_n\| = 1$ it follows that,

$$x_n \wedge y_n = |x_n|^{-1}|P_(W_\omega^i)^\perp y_n|^{-1}(x_n \wedge P_(W_\omega^i)^\perp y_n) .$$

From this, from $P_(W_\omega^i)^\perp y_n \in W_\omega^i$, and by replacing the vectors $x_n$ and $y_n$ with the vectors $|x_n|^{-1}x_n$ and $|P_(W_\omega^i)^\perp y_n|^{-1}P_(W_\omega^i)^\perp y_n$ if necessary, it follows that they may assume to begin with that $|x_n| = |y_n| = 1$.

Recall that by part (1) of Theorem 2.3 we have $\dim E_\omega^k = d_k$ for $\omega \in \Omega_0$ and $0 \leq k \leq s$. Set $q_1 = \sum_{k=0}^i d_k$ and $q_2 = \sum_{k=i+1}^s d_k$, then $W_\omega^i \in \text{Gr}(q_1, V)$ and $V_\omega^i \in \text{Gr}(q_2, V)$ for $n \geq 1$. By moving to a subsequence without changing the
notation, we may assume that there exist \( W \in \text{Gr}(q_1, V) \), \( U \in \text{Gr}(q_2, V) \) and \( x, y \in W \) such that \( W^i_{\omega_1} \nrightarrow W, V^i_{\omega_2} \nrightarrow U, x_n \nrightarrow x \) and \( y_n \nrightarrow y \). Since \( \kappa(\omega_n) \geq \epsilon \) for \( n \geq 1 \), it follows from the definition of \( \kappa \) that,

\[
d(w, x) \geq \epsilon \quad \text{for all } w \in W \text{ and } x \in U.
\]

In particular we have \( V = W \oplus U \). From,

\[
\|A^2P_{(V^i_{\omega_n})^+}(x_n \wedge y_n)\| \leq \frac{1}{n} \quad \text{and} \quad \|x_n \wedge y_n\| = 1 \quad \text{for } n \geq 1,
\]

it follows that \( P_{U^i} x \wedge P_{U^i} y = 0 \) and \( x \wedge y \neq 0 \).

Since \( P_{U^i} x \wedge P_{U^i} y = 0 \) there exists \( c_x, c_y \in \mathbb{R} \), not both 0, such that \( c_x P_{U^i} x + c_y P_{U^i} y = 0 \). From \( x \wedge y \neq 0 \) if it follows that \( c_x x + c_y y \neq 0 \), which shows that the restriction of \( P_{U^i} \) to \( W \) is not injective. But this clearly contradicts \( V = W \oplus U \), which completes the proof of the lemma.

\[\Box\]

Lemma 5.4. Let \( 0 \leq i < s \) be given, then for every \( \epsilon > 0 \) there exists \( \delta = \delta(\epsilon) > 0 \) such that the following holds. Let \( \omega \in \Omega_0, w \in W^i_\omega \) and \( v_1, v_2 \in V^i_\omega \) be with \( \kappa(\omega) \geq \epsilon \), \( \epsilon \leq |w| \leq \epsilon^{-1} \) and \( |v_1| \leq \epsilon^{-1} \). Then,

\[
\|(w + v_1) \wedge v_2\| \geq \delta |w + v_1| \cdot |v_2|.
\]

Proof. Assume by contradiction that the lemma is false for some \( 0 \leq i < s \) and \( \epsilon > 0 \). Then for every \( n \geq 1 \) there exist \( \omega_n \in \Omega_0, w_n \in W^i_{\omega_n} \) and \( v_1,n, v_2,n \in V^i_{\omega_n} \) such that \( \kappa(\omega_n) \geq \epsilon, \epsilon \leq |w_n| \leq \epsilon^{-1}, |v_1,n| \leq \epsilon^{-1}, |v_2,n| = 1 \) and,

\[
\|(w_n + v_1,n) \wedge v_2,n\| \leq \frac{1}{n} |w_n + v_1,n|.
\]

Set \( q_1 = \sum_{k=0}^{i} d_k \) and \( q_2 = \sum_{k=i+1}^{s} d_k \). By moving to a subsequence without changing the notation we may assume that there exist \( W \in \text{Gr}(q_1, V), U \in \text{Gr}(q_2, V) \), \( w \in W \) and \( u_1, u_2 \in U \) such that \( W^i_{\omega_n} \nrightarrow W, V^i_{\omega_n} \nrightarrow U, w_n \nrightarrow w, v_1,n \nrightarrow u_1 \) and \( v_2,n \nrightarrow u_2 \).

As in the proof of Lemma 5.3 since \( \kappa(\omega_n) \geq \epsilon \) for \( n \geq 1 \) we have \( V = W \oplus U \). From \( |w_n| \geq \epsilon \) and \( |v_2,n| = 1 \) for \( n \geq 1 \) it follows that \( w \neq 0 \) and \( u_2 \neq 0 \). Since,

\[
\frac{1}{n} |w_n + v_1,n| \leq 2/(\epsilon n) \quad \text{for } n \geq 1,
\]

it is also clear that \( (w + u_1) \wedge u_2 = 0 \). Thus there exists \( c \in \mathbb{R} \) such that \( w = cu_2 - u_1 \). But this contradicts \( V = W \oplus U \) and \( w \neq 0 \), which completes the proof of the lemma.

\[\Box\]

Lemma 5.5. For every \( \epsilon > 0 \) there exists \( M = M(\epsilon) > 1 \) such that the following holds. Let \( 1 \leq i \leq s, \omega \in \Omega_0 \) and \( r > 0 \) be given. Suppose that \( \kappa(\omega) \geq \epsilon \), \( \|g_{\omega} \pi \omega\|_{\infty} \leq \epsilon^{-1} \) and \( r < M^{-1} \), then

\[
\Gamma_i(\omega, r) \cap \xi_{i-1}(\omega) \subset T_i(\omega, Mr).
\]

Proof. Let \( \epsilon > 0 \), let \( \delta > 0 \) be small with respect to \( \epsilon \) and \( s \), and let \( M > 1 \) be large with respect to \( \delta \). Let \( 1 \leq i \leq s, \omega \in \Omega_0 \) and \( r > 0 \), and suppose that \( \kappa(\omega) \geq \epsilon \), \( \|g_{\omega} \pi \omega\|_{\infty} \leq \epsilon^{-1} \) and \( r < M^{-1} \). Let \( \eta \in \Gamma_i(\omega, r) \cap \xi_{i-1}(\omega) \) and fix \( x_\omega \in \pi \omega \) and \( x_\eta \in \pi \eta \) with \( f^0_\omega(x_\omega) = f^0_\eta(x_\eta) = 1 \).

Write \( y = \sum_{k=0}^{i-1} g_{\omega,k}^{k} \pi \omega \). From \( \eta \in \xi_{i-1}(\omega) \) and Lemma 5.1 it follows that \( y = \sum_{k=0}^{i-1} g_{\omega,k}^{k} \pi \omega \). Since \( g_{\omega,k}^{k} \pi \omega, g_{\eta,k}^{k} \pi \eta \in V^i_\omega \) for all \( i < k \leq s \),

\[
P_{(V^i_\omega)^+} x_\omega = P_{(V^i_\omega)^+} (y + g_{\omega}^{i} \pi \omega) \quad \text{and} \quad P_{(V^i_\omega)^+} x_\eta = P_{(V^i_\omega)^+} (y + g_{\eta}^{i} \pi \eta).
\]
Thus, by Lemma \[5.3\] and by assuming that $\delta$ is small enough with respect to $\epsilon$,
\[
\|P_{(V^i)} x_\omega \cap P_{(V^i)} x_\eta\| = \|A^2 P_{(V^i)} ((y + g^i_\omega \pi \omega) \cap (y + g^i_\omega \pi \eta))\| \\
\geq \delta \| (y + g^i_\omega \pi \omega) \cap (y + g^i_\omega \pi \eta)\| \\
= \delta \| (y + g^i_\omega \pi \omega) \cap (g^i_\omega \pi \eta - g^i_\omega \pi \omega)\| .
\]
(5.3)
By Lemma \[3.4\] and $|L^0_\omega y| = 1$,
\[
|y| \geq 2^{-s/2} \kappa(\omega)^s \|L_\omega(y)\|_{\infty} \geq 2^{-s/2} \epsilon^s .
\]
From $\|g^i_\omega \pi \omega\|_{\infty} \leq \epsilon^{-1}$ it follows that $|y| \leq \kappa \epsilon^{-1}$ and $|g^i_\omega \pi \omega| \leq \epsilon^{-1}$. Thus by (5.3), Lemma \[5.4\] and by assuming that $\delta$ is small enough with respect to $\epsilon$ and $s$,
\[
\|P_{(V^i)} x_\omega \cap P_{(V^i)} x_\eta\| \geq \delta^2 |y + g^i_\omega \pi \omega| \cdot |g^i_\omega \pi \eta - g^i_\omega \pi \omega| .
\]
From Lemma \[3.4\] we get $|y + g^i_\omega \pi \omega| \geq 2^{-s/2} \epsilon^s$. Hence we may assume that,
\[
\|P_{(V^i)} x_\omega \cap P_{(V^i)} x_\eta\| \geq \delta^3 |g^i_\omega \pi \eta - g^i_\omega \pi \omega| .
\]
Additionally,
\[
|P_{(V^i)} x_\omega| \leq |x_\omega| \leq (s + 1)\|g^i_\omega \pi \omega\|_{\infty} \leq (s + 1)\epsilon^{-1} \leq \delta^{-1} .
\]
Thus from $\eta \in \Gamma_i(\omega, r)$,
\[
r \geq d(P_{(V^i)} x_\omega, P_{(V^i)} x_\eta) \\
\geq |P_{(V^i)} x_\omega|^{-1} |P_{(V^i)} x_\eta|^{-1} \|P_{(V^i)} x_\omega \cap P_{(V^i)} x_\eta\| \\
\geq \delta^4 \cdot |P_{(V^i)} x_\eta|^{-1} \cdot |g^i_\omega \pi \eta - g^i_\omega \pi \omega| .
\]
(5.4)
Now assume by contradiction that $|g^i_\omega \pi \eta| > 2\|g^i_\omega \pi \omega\|_{\infty}$, then
\[
|g^i_\omega \pi \eta - g^i_\omega \pi \omega| \geq |g^i_\omega \pi \eta| - \|g^i_\omega \pi \omega|_{\infty} \geq \frac{1}{2} |g^i_\omega \pi \eta| .
\]
Also, by assuming that $\delta$ is small enough with respect to $\epsilon$ and $s$,
\[
|P_{(V^i)} x_\eta| = |P_{(V^i)} (y + g^i_\omega \pi \eta)| \leq |y| + |g^i_\omega \pi \eta| \leq \delta^{-1} |g^i_\omega \pi \eta| .
\]
Hence by (5.4),
\[
r \geq \delta^5 \cdot |g^i_\omega \pi \eta|^{-1} \cdot \frac{1}{2} |g^i_\omega \pi \eta| = \delta^5 / 2 .
\]
But by assuming that $M > 2\delta^{-5}$ this contradicts $r < M^{-1}$, and so we must have
\[
|g^i_\omega \pi \eta| \leq 2\|g^i_\omega \pi \omega|_{\infty} \leq 2\epsilon^{-1} .
\]
This gives,
\[
|P_{(V^i)} x_\eta| \leq |y| + |g^i_\omega \pi \eta| \leq (s + 2)\epsilon^{-1} \leq \delta^{-1} ,
\]
and so by (5.4),
\[
r \geq \delta^5 |g^i_\omega \pi \eta - g^i_\omega \pi \omega| .
\]
Assuming $M \geq \delta^{-5}$ this gives $\eta \in T_i(\omega, Mr)$, which completes the proof of the lemma.

The advantage of working with the sets $T_i(\omega, r)$ is that they behave relatively well with respect to the shift $\sigma$. This is displayed by the following lemma. For $1 \leq i \leq s$, $\omega \in \Omega_0$ and $n \geq 0$ set,
\[
a^i_{\omega, n} := \frac{1}{|A_{\omega_0 \ldots \omega_{n-1}} \nu^0_{\omega_\sigma}|} \cdot \min_{x \in E_{\omega_0 \ldots \omega_{n-1}}(x)} \|A_{\omega_0 \ldots \omega_{n-1}} x\| .
\]
Lemma 5.6. Let $1 \leq i \leq s$, $\omega \in \Omega_0$, $n \geq 0$ and $r > 0$ be given. Then,

$$T_i(\omega, a_{\omega,n}^i r) \cap \mathcal{P}_0^{n-1}(\omega) \subset \sigma^{-n}T_i(\sigma^n \omega, r).$$

Proof. Write $A_{\omega,n}$ for $A_{\omega_0,\omega_{n-1}}$. By part (2) of Theorem 2.9,

$$E^k_\eta = A_{\omega,n} E^{k,n}_\eta \text{ for } 0 \leq k \leq s.$$ 

Since dim $E^0_\eta = 1$ for all $\eta \in \Omega_0$, there exists $b_{\omega,n} = \pm 1$ such that,

$$\frac{A_{\omega,n} u_{\sigma^n \omega}^0}{|A_{\omega,n} u_{\sigma^n \omega}^0|} = b_{\omega,n} u_\omega^0.$$

Let $\eta \in T_i(\omega, a_{\omega,n}^i r) \cap \mathcal{P}_0^{n-1}(\omega)$ and fix $x_{\sigma^n \eta} \in \pi\sigma^n \eta$ with $f^0_{\sigma^n \omega}(x_{\sigma^n \eta}) = 1$. We have,

$$A_{\omega,n} x_{\sigma^n \eta} = A_{\omega,n} u_{\sigma^n \omega}^0 + \sum_{k=1}^s A_{\omega,n} L^k_{\omega,n} x_{\sigma^n \eta} = |A_{\omega,n} u_{\sigma^n \omega}^0| b_{\omega,n} u_\omega^0 + \sum_{k=1}^s A_{\omega,n} L^k_{\omega,n} x_{\sigma^n \eta}.$$ 

From this and (5.5),

$$L^i_{\omega,n} x_{\sigma^n \eta} = A_{\omega,n} L^i_{\sigma^n \omega} x_{\sigma^n \eta} \text{ and } f^0_i(A_{\omega,n} x_{\sigma^n \eta}) = |A_{\omega,n} u_{\sigma^n \omega}^0| b_{\omega,n}.$$ 

Now note that $\pi \eta = A_{\omega,n} \pi \sigma^n \eta$, and so $0 \neq A_{\omega,n} x_{\sigma^n \eta} \in \pi \eta$. This implies,

$$g^i_{\omega,n} \pi \eta = (L^i_{\omega,n} A_{\omega,n} x_{\sigma^n \eta}) / f^0_i(A_{\omega,n} x_{\sigma^n \eta}) = |A_{\omega,n} u_{\sigma^n \omega}^0|^{-1} b_{\omega,n}^{-1} A_{\omega,n} L^i_{\omega,n} x_{\sigma^n \eta} = |A_{\omega,n} u_{\sigma^n \omega}^0|^{-1} b_{\omega,n}^{-1} A_{\omega,n} g^i_{\sigma^n \omega}(\pi \sigma^n \eta).$$

A similar argument gives,

$$g^i_{\omega,n} \pi \omega = |A_{\omega,n} u_{\sigma^n \omega}^0|^{-1} b_{\omega,n}^{-1} A_{\omega,n} g^i_{\sigma^n \omega}(\pi \sigma^n \omega).$$

From these formulas, the definition of $a_{\omega,n}^i$ and $\eta \in T_i(\omega, a_{\omega,n}^i r)$, we get

$$a_{\omega,n}^i r \geq |g^i_{\omega,n} \pi \omega - g^i_{\omega,n} \pi \eta| = |A_{\omega,n} u_{\sigma^n \omega}^0|^{-1} |A_{\omega,n} (g^i_{\sigma^n \omega}(\pi \sigma^n \omega) - g^i_{\sigma^n \omega}(\pi \sigma^n \eta))| \geq a_{\omega,n}^i.$$ 

This, together with Lemma 4.2, implies that $\sigma^n \eta \in T_i(\sigma^n \omega, r)$, which completes the proof of the lemma. \hfill \square

The following theorem, which will be used in the next subsection, is due to Maker [Mak].

Theorem 5.7. Let $(X, \mathcal{B}, \rho, T)$ be an ergodic measure preserving system and let $h, h_1, h_2, \ldots \in L^1(\rho)$. Suppose that $h_n(x) \overset{\rho}{\rightharpoonup} h(x)$ for $\rho$-a.e. $x$ and that $\sup_{n \geq 1} |h_n|$ is integrable. Then,

$$\frac{1}{n} \sum_{j=0}^{n-1} h_{n-j} \circ T^j(x) \overset{\rho}{\rightharpoonup} \int h \, d\rho \quad \text{for } \rho\text{-a.e. } x.$$
5.2. Proof of Proposition 5.1. We are now ready to begin the proof of Proposition 5.1. Let $0 < \epsilon < 1$ be small and let $M > 1$ be large with respect to $\epsilon$ and $s$. Set,

$$F_0(\epsilon) = \{ \omega \in \Omega_0 : \kappa(\omega) \geq \epsilon \text{ and } \|g_{\omega, \pi \omega}\|_\infty \leq \epsilon^{-1} \},$$

then $\beta(F_0(\epsilon)) > 0$ by assuming that $\epsilon$ is sufficiently small. By part 4 of Theorem 2.9 there exist an integer $N = N(\epsilon, M) \geq 1$ and a Borel set $F = F(\epsilon, M) \subset F_0(\epsilon)$, such that $\beta(F) \geq (1 - \epsilon)\beta(F_0(\epsilon)) > 0$ and for every $1 \leq i \leq s$,

$$(5.6) \quad M^{-1} \geq a_{i, n}^1 \geq M \exp(n(\lambda_i - \epsilon)) \text{ for } \omega \in F \text{ and } n \geq N.$$  

It is clear that $\beta(F) \to 1$ as $\epsilon \to 0$. Thus in order to prove Proposition 5.1 it suffices to show that for each $1 \leq i \leq s$,

$$(5.7) \quad \vartheta_{i-1}(\omega) \geq \frac{H_i - H_{i-1}}{\lambda_i - \epsilon} \text{ for } \beta\text{-a.e. } \omega \in F.$$  

By the Poincaré recurrence theorem, and by removing a Borel set of zero $\beta$-measure from $F$, we may assume that

$$(5.8) \quad \# \{ n \geq 1 : \sigma^n N \omega \in F \} = \infty \text{ for all } \omega \in F.$$  

Let $\sigma_F : F \to F$ be the transformation induced by $\sigma^N$ on the set $F$. That is $\sigma_F(\omega) = \sigma^{Nr_F(\omega)}(\omega)$ for $\omega \in F$, where

$$r_F(\omega) = \inf \{ n \geq 1 : \sigma^n N \omega \in F \}.$$  

Let $\beta_F$ be the Borel probability measure on $F$ which satisfies,

$$\beta_F(D) = \beta(F \cap D)/\beta(F) \text{ for any Borel set } D \subset F.$$  

Since $(\Omega, \sigma^N, \beta)$ is an ergodic measure preserving system the same is true for the system $(F, \sigma_F, \beta_F)$ (e.g. see [EW, Lemma 2.43]).

For $\omega \in F$ and $1 \leq i \leq s$ set,

$$\ell(\omega) = N r_F(\omega) \quad \text{and} \quad \rho(i, \omega) = \exp(\ell(\omega)(\lambda_i - \epsilon)).$$

In the proof of the following lemma we are going to use the auxiliary results obtained in Section 5.1.

Lemma 5.8. Let $\omega \in F$, $1 \leq i \leq s$ and $0 < r \leq 1$ be given. Then,

$$\xi_{i-1}(\omega) \cap \Gamma_1(\omega, \rho(i, \omega)r) \cap \mathcal{P}_0^{(\omega)}(\omega) \subset \sigma^{-\ell(\omega)}\Gamma_1(\sigma_F \omega, r).$$

Proof. Write,

$$L = \xi_{i-1}(\omega) \cap \Gamma_1(\omega, \rho(i, \omega)r) \cap \mathcal{P}_0^{(\omega)}(\omega).$$

By (5.6) and since $\ell(\omega) \geq N$,

$$M^{-1} \geq a_{i, \ell(\omega)}^1 \geq M \rho(i, \omega).$$

Since $F \subset F_0(\epsilon)$, we have $\kappa(\omega) \geq \epsilon$ and $\|g_{\omega, \pi \omega}\|_\infty \leq \epsilon^{-1}$. From this, $\rho(i, \omega)r < M^{-1}$, Lemma 5.5 and by assuming that $M$ is sufficiently large with respect to $\epsilon$,

$$L \subset T_1(\omega, M^{1/2} \rho(i, \omega)r) \cap \mathcal{P}_0^{(\omega)}(\omega).$$

Thus from $M \rho(i, \omega) \leq a_{i, \ell(\omega)}^1$ and Lemma 5.6

$$(5.9) \quad L \subset T_1(\omega, M^{-1/2} a_{i, \ell(\omega)}^1 r) \cap \mathcal{P}_0^{(\omega)}(\omega) \subset \sigma^{-\ell(\omega)}T_1(\sigma^{\ell(\omega)} \omega, M^{-1/2} r).$$

Write,

$$R = s^3 2^s \|g_{\sigma^{\ell(\omega)} \omega, \pi \sigma^{\ell(\omega)} \omega}\|_\infty \kappa(\sigma^{\ell(\omega)} \omega)^{-2s-2}.$$
Then by Lemma 5.2
\[ T_i(\sigma^\ell(\omega), M^{-1/2}r) \subset \Gamma_i(\sigma^\ell(\omega), RM^{-1/2}r). \]

Since \( \sigma^\ell(\omega) \in F \) we have,
\[ k(\sigma^\ell(\omega)) \geq \epsilon \quad \text{and} \quad \|g_{\sigma^\ell(\omega)}\pi\sigma^\ell(\omega)\|_\infty \leq \epsilon^{-1}. \]

Hence, by taking \( M \) to be sufficiently large with respect to \( \epsilon \) and \( s \) we may assume that \( RM^{-1/2} \leq 1 \). From this, \( (5.9) \) and \( (5.10) \) we now get,
\[ L \subset \sigma^{-\ell(\omega)}\Gamma_i(\sigma^\ell(\omega), r). \]

Since \( \sigma^\ell(\omega) = \sigma_F \omega \) this completes the proof of the lemma.

The rest of the proof of Proposition 5.1 is similar to the proof of [Fe, Proposition 5.1]. For completeness and clarity we essentially provide full details. We shall need to establish some more lemmas before we can continue with the proof of \( (5.7) \).

For \( n \geq 1 \) write, \[ F_n = \{ \omega \in F : r_F(\omega) = n \}. \]

The following lemma will enable us to apply Maker’s ergodic theorem, which was stated above.

**Lemma 5.9.** Let \( 1 \leq i \leq s \) be given. Then for \( \beta \)-a.e. \( \omega \in F \),
\[ \lim_{r \downarrow 0} \log \frac{\beta_{\omega}^{t(-1)}(\Gamma_i(\omega, r) \cap P_{0}^{(\omega^{-1})}(\omega))}{\beta_{\omega}^{t(-1)}(\Gamma_i(\omega, r))} = - \sum_{0 \leq j < \infty} \hat{I}_j(\Gamma_i(\omega, r)). \]

Furthermore, set
\[ q(\omega) = - \inf_{r > 0} \log \frac{\beta_{\omega}^{t(-1)}(\Gamma_i(\omega, r) \cap P_{0}^{(\omega^{-1})}(\omega))}{\beta_{\omega}^{t(-1)}(\Gamma_i(\omega, r))}, \]
then \( q \geq 0 \) and \( q \in L^1(\beta_F) \).

**Proof.** For \( \omega \in F \) and \( r > 0 \) with \( \beta_{\omega}^{t(-1)}(\Gamma_i(\omega, r)) > 0 \) write,
\[ \alpha(\omega, r) = \log \frac{\beta_{\omega}^{t(-1)}(\Gamma_i(\omega, r) \cap P_{0}^{(\omega^{-1})}(\omega))}{\beta_{\omega}^{t(-1)}(\Gamma_i(\omega, r))}. \]

As in the proof of Lemma 4.16 define a Borel map \( \phi_i : \Omega_0 \rightarrow P(V) \) by \( \phi_i(\omega) = P_{(V_\omega)}^{\pi_\omega}. \) Note that,
\[ \Gamma_i(\omega, r) = \xi_0(\omega) \cap B^{\phi_i}(\omega, r) \quad \text{for} \quad \omega \in \Omega_0 \quad \text{and} \quad r > 0. \]

From this and \( F = \cup_{k \geq 1} F_k \), we get that for \( \beta \)-a.e. \( \omega \in F \),
\[ \alpha(\omega, r) = \log \frac{\beta_{\omega}^{t(-1)}(B^{\phi_i}(\omega, r) \cap P_{0}^{(\omega^{-1})}(\omega))}{\beta_{\omega}^{t(-1)}(B^{\phi_i}(\omega, r))} = \sum_{k \geq 1} \sum_{A \in \mathcal{P}_{0}^{\pi_{\omega}}} 1_{F_k \cap A}(\omega) \log \frac{\beta_{\omega}^{t(-1)}(B^{\phi_i}(\omega, r) \cap A)}{\beta_{\omega}^{t(-1)}(B^{\phi_i}(\omega, r))}. \]

Denote the Borel \( \sigma \)-algebra of \( P(V) \) by \( \mathcal{B} \). It is easy to verify that,
\[ \widehat{\xi_i^{-1}} \vee \phi_i^{-1}(\mathcal{B}) = \xi_i. \]
From this and Lemma 2.4 it follows that for $\beta$-a.e. $\omega \in F$,

$$
\lim_{r \downarrow 0} \alpha(\omega, r) = \sum_{k \geq 1} \sum_{A \in \mathcal{P}_k^{N-1}} 1_{F_k \cap A}(\omega) \log E_\beta(1_A \mid \hat{\xi}_i \vee \phi_i^{-1}(B))(\omega)
$$

$$
= \sum_{k \geq 1} 1_{F_k}(\omega) \sum_{A \in \mathcal{P}_k^{N-1}} 1_A(\omega) \log E_\beta(1_A \mid \hat{\xi}_i)(\omega)
$$

$$
= -\sum_{k \geq 1} 1_{F_k}(\omega) I_\beta(\mathcal{P} \mid \hat{\xi}_i)(\sigma^j \omega).
$$

This together with (4.10) shows that for $\beta$-a.e. $\omega \in F$,

$$
\lim_{r \downarrow 0} \alpha(\omega, r) = -\sum_{k \geq 1} \sum_{j=0}^{kN-1} I_\beta(\mathcal{P} \mid \hat{\xi}_i)(\sigma^j \omega),
$$

which is the first part of the lemma.

The proof of $q \in L^1(\beta_F)$ is exactly the same as the proof of the analogous fact in [Fe, Proposition 5.5], and is therefore omitted. □

Remark 5.10. It is worth pointing out that if in our main result, $\mu$ is only assumed to be discrete and with finite Shannon entropy (instead of being finitely supported), then the argument in [Fe, Proposition 5.5] which gives the integrability of $q$ does not seem to work.

We continue towards proving (5.7). Since $(\Omega, \sigma^N, \beta)$ is ergodic, a classical result due to Kac [Ka] gives,

$$
(5.11) \quad \int_F r_F \, d\beta_F = \beta(F)^{-1}.
$$

The following Lemma follows directly from [Fe Lemma 2.11], the ergodicity of $(F, \sigma_F, \beta_F)$ and (5.11).

**Lemma 5.11.** Let $h \in L^1(\beta)$ and set,

$$
\tilde{h}(\omega) = \sum_{0 \leq j < N_{r_F}(\omega)} h(\sigma^j \omega) \quad \text{for } \omega \in F.
$$

Then $\tilde{h} \in L^1(\beta_F)$ and,

$$
\int \tilde{h} \, d\beta_F = \frac{N}{\beta(F)} \int h \, d\beta.
$$

Recall that for $\omega \in F$ and $1 \leq i \leq s$ we write,

$$
\ell(\omega) = N_{r_F}(\omega) \text{ and } \rho(i, \omega) = \exp(\ell(\omega) (\hat{\lambda}_i - \epsilon)).
$$

Set $\rho_0(i, \omega) = 1$ and for $n \geq 1$ set,

$$
\rho_n(i, \omega) = \prod_{k=0}^{n-1} \rho(i, \sigma_F^k \omega),
$$

where $\sigma_F^k := (\sigma_F)^k$. 32
Lemma 5.12. Let \(1 \leq i \leq s\) be given. Then for \(\beta\)-a.e. \(\omega \in F\),

\[
\lim_{n \to \infty} \frac{\log \rho_n(i, \omega)}{\log \rho_{n-1}(i, \omega)} = 1.
\]

Proof. Let \(M \geq 1\) be an integer. Since \(\sigma_F \beta_F = \beta_F\) and by (5.11),

\[
\sum_{n \geq 1} \beta_F\{\omega : r_F(\sigma^F_n \omega) \geq nM^{-1}\} = \sum_{n \geq 1} \beta_F\{r_F \geq nM^{-1}\}
\]

= \(\int M r_F \, d\beta_F = M/\beta(F) < \infty\).

From this and the Borel-Cantelli lemma it follows that for \(\beta\)-a.e. \(\omega \in F\) there exists \(N_\omega \geq 1\) such that,

\[
\frac{1}{n} r_F(\sigma^n_F \omega) < M^{-1} \quad \text{for all} \quad n \geq N_\omega,
\]

which shows,

\[
\lim_{n \to \infty} \frac{1}{n} r_F(\sigma^n_F \omega) = 0 \quad \text{for \(\beta\)-a.e. \(\omega \in F\).}
\]

The lemma now follows easily from this and since \(\lim \log \rho_n(i, \omega) \geq n(\epsilon - \lambda_i)\) for each \(n \geq 1\).

We resume with the proof of Proposition [5.1]. Fix \(1 \leq i \leq s\), and recall that our aim is to show (5.7).

For \(n \geq 1\) and \(\beta\)-a.e. \(\omega \in F\) we can set,

\[
K_n(\omega) = \log \frac{\beta^{\xi_{i-1}}_\omega(\Gamma_i(\omega, \rho_n(i, \omega)))}{\beta^{\xi_{i-1}}_{\sigma_F \omega}(\Gamma_i(\sigma_F \omega, \rho_{n-1}(i, \sigma_F \omega)))},
\]

\[
G_n(\omega) = \log \frac{\beta^{\xi_{i-1}}_\omega(\Gamma_i(\omega, \rho_n(i, \omega)) \cap \sigma^{\ell}(\omega) \cup P_0^{(\omega)-1}(\omega))}{\beta^{\xi_{i-1}}_{\sigma_F \omega}(\Gamma_i(\sigma_F \omega, \rho_{n-1}(i, \sigma_F \omega)))},
\]

and,

\[
R_l(\omega) = \sum_{0 \leq j < \lambda_F} I_\beta(\mathcal{P} | \tilde{\xi}_i)(\sigma^l \omega) \quad \text{for} \quad l = i, i - 1.
\]

Then for \(\beta\)-a.e. \(\omega \in F\),

\[
K_n(\omega) + G_n(\omega) = \log \frac{\beta^{\xi_{i-1}}_\omega(\Gamma_i(\omega, \rho_n(i, \omega)) \cap \sigma^{\ell}(\omega) \cup P_0^{(\omega)-1}(\omega))}{\beta^{\xi_{i-1}}_{\sigma_F \omega}(\Gamma_i(\sigma_F \omega, \rho_{n-1}(i, \sigma_F \omega)))}.
\]

Hence by Lemma [5.8],

\[
K_n(\omega) + G_n(\omega) \leq \log \frac{\beta^{\xi_{i-1}}_\omega(\Gamma_i(\sigma_F \omega, \rho_{n-1}(i, \sigma_F \omega)) \cap \sigma^{\ell}(\omega) \cup P_0^{(\omega)-1}(\omega))}{\beta^{\xi_{i-1}}_{\sigma_F \omega}(\Gamma_i(\sigma_F \omega, \rho_{n-1}(i, \sigma_F \omega)))}.
\]

From this and Lemma [4.7] we get that for \(\beta\)-a.e. \(\omega \in F\),

\[
K_n(\omega) + G_n(\omega) \leq \log \beta^{\xi_{i-1}}_\omega(P_0^{(\omega)-1}(\omega))
\]

= \(\sum_{k=1}^\infty 1_{F_k}(\omega) \log \beta^{\xi_{i-1}}_\omega(P_0^{kN-1}(\omega))\).

Thus by (H.10),

\[
K_n(\omega) + G_n(\omega) \leq -\sum_{k=1}^\infty 1_{F_k}(\omega) \sum_{j=0}^{kN-1} I_{\beta}(\mathcal{P} | \tilde{\xi}_{i-1})(\sigma^j \omega) = -R_{i-1}(\omega).
\]
It follows that for $\beta$-a.e. $\omega \in F$ and any $n \geq 1$,

$$- \log \beta^{\xi_{i-1}}(\Gamma_i(\omega, \rho_n(i, \omega))) = \left( - \sum_{j=0}^{n-1} K_{n-j}(\sigma^j F) - \log \beta^{\xi_{i-1}}(\Gamma_i(\sigma^j F, 1)) \right) \geq - \sum_{j=0}^{n-1} K_{n-j}(\sigma^j F) \geq \sum_{j=0}^{n-1} (G_{n-j}(\sigma^j F) + R_{i-1}(\sigma^j F)) \cdot$$

(5.12)

By Lemma 5.11 it follows that for $l = i, i - 1$ we have $R_l \in L^1(\beta F)$ with,

$$\int R_l \, d\beta_F = \frac{N}{\beta(F)} \int 1_{\beta}(P | \hat{\xi}_l) \, d\beta = \frac{N}{\beta(F)} H_{i}(P | \hat{\xi}_l) = \frac{N}{\beta(F)} H_{i-1}.$$  

From this and Birkhoff’s ergodic theorem it follows that for $\beta$-a.e. $\omega \in F$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} R_{i-1}(\sigma^j F) = \frac{N}{\beta(F)} H_{i-1}. \quad (5.13)$$

By Lemma 5.9 it follows that for $\beta$-a.e. $\omega \in F$,

$$\lim_{n \to \infty} G_{n}(\omega) = -R_{i}(\omega) \text{ and } \sup_{n \geq 1} |G_n| \in L^1(\beta F).$$

Thus from Theorem 5.7 we get that for $\beta$-a.e. $\omega \in F$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} G_{n-j}(\sigma^j F) = \int -R_i \, d\beta_F = - \frac{N}{\beta(F)} H_i \cdot \quad (5.14)$$

For $\omega \in F$ and $n \geq 1$,

$$\log \rho_n(i, \omega) = \sum_{j=0}^{n-1} \log \rho(i, \sigma^j F) = (\hat{\lambda}_i - \epsilon) N \sum_{j=0}^{n-1} r_F(\sigma^j F) \cdot$$

From this, Birkhoff’s theorem and (5.11), it follows that for $\beta$-a.e. $\omega \in F$,

$$\lim_{n \to \infty} - \frac{1}{n} \log \rho_n(i, \omega) = (\epsilon - \hat{\lambda}_i) N \int r_F \, d\beta_F = \frac{(\epsilon - \hat{\lambda}_i) N}{\beta(F)} \cdot$$

Now from (5.12), (5.13), (5.14) and the last equality, we get that for $\beta$-a.e. $\omega \in F$,

$$\liminf_{n \to \infty} \frac{\log \beta^{\xi_{i-1}}(\Gamma_i(\omega, \rho_n(i, \omega)))}{\log \rho_n(i, \omega)} \geq \lim_{n \to \infty} \frac{1}{n} \left( \sum_{j=0}^{n-1} (G_{n-j}(\sigma^j F) + R_{i-1}(\sigma^j F)) \right) - \frac{1}{n} \log \rho_n(i, \omega) = \frac{H_i - H_{i-1}}{\hat{\lambda}_i - \epsilon}. \quad (5.15)$$

This together with Lemma 5.12 shows that for $\beta$-a.e. $\omega \in F$,

$$\liminf_{r \downarrow 0} \frac{\log \beta^{\xi_{i-1}}(\Gamma_i(\omega, r))}{\log r} \geq \frac{H_i - H_{i-1}}{\hat{\lambda}_i - \epsilon},$$

which gives (5.7) and completes the proof of Proposition 5.1.
6. Proof of the main result

In this Section we complete the proof of our main result Theorem 1.3. For \(1 \leq k \leq s\), \(0 \leq i \leq k\) and \(\omega \in \Omega_0\) set,

\[
\pi_{i,k}(\omega) = \limsup_{r \downarrow 0} \frac{\log \beta_{i,k}^\omega(\Gamma_k(\omega, r))}{\log r}
\]

and

\[
\gamma_{i,k}(\omega) = \liminf_{r \downarrow 0} \frac{\log \beta_{i,k}^\omega(\Gamma_k(\omega, r))}{\log r}
\]

Recall that for \(0 \leq i < s\) we write,

\[
\vartheta_i(\omega) = \liminf_{r \downarrow 0} \frac{\log \beta_{i+1,k}^\omega(\Gamma_{i+1}(\omega, r))}{\log r}
\]

The proof of the following proposition is a modification of the argument used in [Fe, Section 6, Proof of (C3)]. That argument in turn follows the lines of the proof of [FH, Theorem 2.11], which was adapted from the original proof of [LY] Lemma 11.3.1.

**Proposition 6.1.** For \(1 \leq k \leq s\), \(0 \leq i < k\) and \(\beta\)-a.e. \(\omega\),

\[
\gamma_{i+1,k}(\omega) + \vartheta_i(\omega) \leq \gamma_{i,k}(\omega).
\]

**Proof.** Assume by contradiction that the proposition is false. Recall that for \(\omega \in \Omega_0\) we have \(\pi \omega \notin P(V_0^\omega)\). Since \((V_0^\omega)^+ \subset (V_k^\omega)^+\) this implies that also \(P(V_k^\omega)^+ \pi \omega \notin P(V_0^\omega)\). Thus, since we assume that the proposition is false, there exist \(1 \leq k \leq s\), \(0 \leq i < k\), \(0 \leq \epsilon < 1\) and \(F \subset \Omega_0\), such that \(\beta(F) > 0\) and for \(\omega \in F\),

\[
d(P(V_k^\omega)^+ \pi \omega, P(V_0^\omega)) \geq \epsilon \text{ and } \gamma_{i+1,k}(\omega) + \vartheta_i(\omega) > \gamma_{i,k}(\omega).
\]

There exist \(\alpha > 0\) and real numbers \(\gamma_{i,k}, \gamma_{i+1,k}, \vartheta_i\) such that,

\[
\gamma_{i+1,k} + \vartheta_i > \gamma_{i,k} + \alpha,
\]

and for any \(\rho > 0\) there exists \(F_\rho \subset F\) with \(\beta(F_\rho) > 0\), so that for \(\omega \in F_\rho\),

\[
(6.1) \quad |\gamma_{i,k}(\omega) - \gamma_{i,k}| < \rho/2 \quad |\gamma_{i+1,k}(\omega) - \gamma_{i+1,k}| < \rho/2 \quad |\vartheta_i(\omega) - \vartheta_i| < \rho/2.
\]

Fix \(0 < \rho < \alpha/2\), then there exist \(N_1 \geq 1\) and \(F'_\rho \subset F_\rho\) with \(\beta(F'_\rho) > 0\) and,

\[
(6.2) \quad \beta_{i+1,k}^\omega(\Gamma_k(\omega, 2\epsilon^{-n})) \leq \epsilon^{-n(\gamma_{i+1,k} - \rho)} \text{ for } \omega \in F'_\rho \text{ and } n \geq N_1.
\]

By Lemma 4.3 there exist \(c > 0\), \(N_2 \geq N_1\) and \(F''_\rho \subset F_\rho\) such that \(\beta(F''_\rho) > 0\) and,

\[
(6.3) \quad \frac{\beta_{i+1,k}^\omega(\Gamma_k(\omega, e^{-n}) \cap F'_\rho)}{\beta_{i+1,k}^\omega(\Gamma_k(\omega, e^{-n}))} > c \text{ for } \omega \in F''_\rho \text{ and } n \geq N_2.
\]

Let \(\omega \in F''_\rho\) and \(n \geq N_2\), and write

\[
B_{\omega,n} = \{ \eta \in \xi_i(\omega) : \xi_{i+1}(\eta) \cap F'_\rho \cap \Gamma_k(\omega, e^{-n}) \neq \emptyset \}.
\]

Let us show that \(B_{\omega,n} \subset \Gamma_{i+1}(\omega, e^{-2}-e^{-n})\). Given \(\eta \in B_{\omega,n}\) there exists

\[
\zeta \in \xi_{i+1}(\eta) \cap F''_\rho \cap \Gamma_k(\omega, e^{-n}).
\]

Since \(\xi_0(\zeta) = \xi_0(\eta) = \xi_0(\omega),\)

\[
V_{\eta}^{\omega} = V_{\eta}^{\omega} = V_{\zeta}^{\omega} \text{ for all } 0 \leq j \leq s.
\]

If \(i + 1 < s\) then from \(\omega, \zeta \in F''_\rho \subset F\),

\[
d(P(V_{\omega}^j)^+ \pi \omega, P(V_{\omega}^{j+1})) \geq d(P(V_{\omega}^j)^+ \pi \omega, P(V_{\omega}^0)) \geq \epsilon,
\]
and,
\[ d(P_{V^{i+1}_+} \pi \zeta, P(V^{i+1}_\omega)) \geq d(P_{V^{j+1}_+} \pi \zeta, P(V^0_\omega)) = d(P_{V^{j+1}_+} \pi \zeta, P(V^0_\omega)) \geq \varepsilon. \]

Since \( i < k \) we have \( (V^{i+1}_\omega)^\perp \subset (V^k_\omega)^\perp \). From this and Lemma 2.5
\[ d(P_{V^{i+1}_+} \pi \zeta, P(V^{i+1}_+ \pi \omega)) = d(P_{V^{j+1}_+} \pi \zeta, P(V^{j+1}_+ \pi \omega)) \leq e^{-2} d(P_{V^{j+1}_+} \pi \zeta, P(V^0_\omega). \]

Note that since \( V^0_\omega = \{0\} \) this inequality is trivial when \( i+1 = s \). Since \( \zeta \in \xi_{i+1}(\eta) \) we have \( P(\xi_{i+1}) = P(\xi_{i+1} \pi \zeta). \) Now combining these facts with \( \zeta \in \Gamma_k(\omega, e^{-n}) \) we get,
\[ d(P_{V^{i+1}_+} \pi \eta, P(V^{i+1}_+ \pi \omega)) = d(P_{V^{j+1}_+} \pi \zeta, P(V^{j+1}_+ \pi \omega)) \leq e^{-2} d(P_{V^{j+1}_+} \pi \zeta, P(V^0_\omega)) \leq e^{-2} e^{-n}, \]
which shows \( B_{\omega,n} \subset \Gamma_{i+1}(\omega, e^{-2} e^{-n}). \)

Next let us show that,
\[ \beta_{\eta}^{\xi+1}(\Gamma_k(\omega, e^{-n}) \cap F'_{\rho}) \leq e^{-n}(\omega_{k+1}) \quad \text{for } \eta \in B_{\omega,n}. \]

Let \( \eta \) and \( \zeta \) be as in the last paragraph. Since \( d(P_{V^{j+1}_+} \pi \zeta, P(V^{j+1}_+ \pi \omega) \leq e^{-n}, \gamma_k(\omega, e^{-n}) \cap F_{\rho}^0 \subset \Gamma_k(\xi_{i+1}, 2 e^{-n}). \)

From this, \( \beta_{\eta}^{\xi+1} = \beta_{\xi}^{\xi+1} \) and (6.2),
\[ \beta_{\eta}^{\xi+1}(\Gamma_k(\omega, e^{-n}) \cap F'_{\rho}) = \beta_{\xi}^{\xi+1}(\Gamma_k(\omega, e^{-n}) \cap F'_{\rho}) \leq \beta_{\xi}^{\xi+1}(\Gamma_k(\xi_{i+1}, 2 e^{-n})) \leq e^{-n}(\omega_{k+1}^{i+1}). \]

which gives (6.4).

Now from (6.3), \( B_{\omega,n} \subset \Gamma_i(\omega, e^{-2} e^{-n}) \) and (6.4), it follows that for \( \beta \)-a.e. \( \omega \in F''_{\rho} \) and every \( n \geq N_2, \)
\[ \beta_{\omega}^{\xi}(\Gamma_k(\omega, e^{-n})) \leq c^{-1} \beta_{\omega}^{\xi}(\Gamma_k(\omega, e^{-n}) \cap F'_{\rho}) \leq e^{-1} \int_{B_{\omega,n}} \beta_{\omega}^{\xi+1}(\Gamma_k(\omega, e^{-n}) \cap F'_{\rho}) d\beta_{\omega}^{\xi}(\eta) \leq c^{-1} \beta_{\omega}^{\xi}(\Gamma_k(\omega, e^{-2} e^{-n})) e^{-n}(\omega_{k+1}). \]

Thus, by taking logarithm on both sides, dividing by \( -n \) and letting \( n \) tend to infinity,
\[ \gamma_{\omega, \xi}(\omega) \geq \vartheta_i(\omega) + \gamma_{\omega, \xi+1, k} - \rho \quad \text{for } \beta \text{-a.e. } \omega \in F''_{\rho}. \]

By (6.1) we now get,
\[ \gamma_{\omega, \xi} + 2 \rho \geq \vartheta_i + \gamma_{\omega, \xi+1, k}. \]

But this contradicts \( \rho < \alpha/2 \) and \( \gamma_{\omega, \xi+1, k} + \vartheta_i > \gamma_{\omega, \xi} + \alpha \), which completes the proof of the proposition. \( \square \)

The proof of the following proposition follows the lines of the argument used in [Fe, Section 6, Proof of (C2)]. That argument in turn is modified from [LY] §10.2 and the proof of [EH] Theorem 2.11.
Proposition 6.2. For \(1 \leq k \leq s\), \(0 \leq i < k\) and \(\beta\)-a.e. \(\omega\),
\[
\frac{H_{i+1} - H_i}{\lambda_{i+1}} \geq \tau_{i,k}(\omega) - \tau_{i+1,k}(\omega).
\]

Proof. Assume by contradiction that the proposition is false. Then there exist
\(1 \leq k \leq s\), \(0 \leq i < k\) and \(F \subset \Omega_0\) with \(\beta(F) > 0\) and,
\[
\frac{H_{i+1} - H_i}{\lambda_{i+1}} < \tau_{i,k}(\omega) - \tau_{i+1,k}(\omega) \text{ for } \omega \in F.
\]
Thus there exist \(\alpha > 0\) and real numbers \(\tau_{i,k}\) and \(\tau_{i+1,k}\) such that,
\[
\frac{H_{i+1} - H_i}{\lambda_{i+1}} < \tau_{i,k} - \tau_{i+1,k} - \alpha,
\]
and for any \(\epsilon > 0\) there exists \(B_\epsilon \subset F\) with \(\beta(B_\epsilon) > 0\), so that for \(\omega \in B_\epsilon\),
\[
|\tau_{i,k}(\omega) - \tau_{i,k}| < \epsilon/2 \quad \text{and} \quad |\tau_{i+1,k}(\omega) - \tau_{i+1,k}| < \epsilon/2.
\]
Fix \(0 < \epsilon < -\lambda_{i+1}/6\), and for \(\omega \in \Omega_0\) and \(n \geq 1\), write,
\[
D_{\omega,n} = \Gamma_k(\omega, e^{n(\tilde{\lambda}_{i+1} + 5\epsilon)}).
\]
Recall the sets \(Q_{n,\epsilon}\) defined in (4.3). By removing a subset of zero \(\beta\)-measure
from \(B_\epsilon\) without changing the notation, it follows that there exists a Borel function \(n_0 : B_\epsilon \to \mathbb{N}\) such that for \(\omega \in B_\epsilon\) and \(n \geq n_0(\omega)\),
(1) \(\log \frac{\beta_{i+1}^{*}(D_{\omega,n})}{n(\lambda_i + 5\epsilon)} < \gamma_{i+1,k} + \epsilon;\)
(2) \(-\frac{1}{n} \log \beta_{i+1}^{*}(P_{0}^{n-1}(\omega)) > H_{i+1} - \epsilon\) (by Lemma 4.4);
(3) \(Q_{n,\epsilon} \cap \xi_i(\omega) \cap P_{0}^{n-1}(\omega) \subset D_{\omega,n}\) (by Proposition 4.3);
(4) \(-\frac{1}{n} \log \beta_{i+1}^{*}(\tilde{Q}_{n,\epsilon} \cap \tilde{P}_{0}^{n-1}(\omega)) < H_{i} + \epsilon\) (by Lemmas 4.3 and 4.5);
(5) \(\log \beta_{i+1}^{*}(D_{\omega,n} \cap \Delta) > c;\)
(6) \(\log \beta_{i+1}^{*}(\tilde{\xi}_i(\omega, 2e^{n(\lambda_{i+1} + 5\epsilon)})) > \gamma_{i,k} - \epsilon;\)
(7) \(-\frac{\log n}{\epsilon} < c;\)
Fix \(\omega \in \Delta\) such that all of the conditions (1)–(7) are satisfied with \(n = n(\omega) \geq n_0\) such that,
(6.6) \(\beta_{i+1}^{*}(D_{\omega,n} \cap \Delta) > c\beta_{i+1}^{*}(D_{\omega,n}) > c\exp(n(\tilde{\lambda}_{i+1} + 5\epsilon)(\gamma_{i+1,k} + \epsilon)).\)
Write,
\[
\mathcal{E} = \{P \in P_{0}^{n-1} : P \cap \xi_i(\omega) \cap D_{\omega,n} \cap \Delta \neq \emptyset\},
\]
and,
\[
\mathcal{E}' = \{P \in P_{0}^{n-1} : P \cap \xi_{i+1}(\omega) \cap D_{\omega,n} \cap \Delta \neq \emptyset\}.
\]
From \(\xi_{i+1}(\omega) \subset \xi_i(\omega)\) it follows that \(\mathcal{E}' \subset \mathcal{E}\).
Given \(P \in \mathcal{E}'\) there exists \(\eta \in \xi_{i+1}(\omega) \cap D_{\omega,n} \cap \Delta\) with \(P = P_{0}^{n-1}(\eta)\). Hence from (2),
\[
\beta_{i+1}^{*}(P) = \beta_{i+1}^{*}(P_{0}^{n-1}(\eta)) < \exp(-n(H_{i+1} - \epsilon)).
\]
Thus,
\[ \beta_{\xi+1}^g(D_{\omega,n} \cap \Delta) \leq \sum_{P \in \mathcal{E}'} \beta_{\omega}^{\xi+1}(P) < |\mathcal{E}'| \exp(-n(H_{i+1} - \epsilon)). \]

This together with (6.6) implies,
\[ |\mathcal{E}'| > c \exp(n(\hat{\lambda}_{i+1} + 5\epsilon)(\bar{\tau}_{i+1,k} + \epsilon)) \exp(n(H_{i+1} - \epsilon)). \]

Let us show that,
\[ Q_{n,\epsilon} \cap \xi_i(\omega) \cap P \subset \Gamma_k(\omega, 2e^{n(\hat{\lambda}_{i+1} + 5\epsilon)}) \text{ for } P \in \mathcal{E}, \]
and,
\[ \beta_{\omega}^\xi(Q_{n,\epsilon} \cap P) > e^{-n(H_i + \epsilon)} \text{ for } P \in \mathcal{E}. \]

Given $P \in \mathcal{E}$ there exists $\eta \in P \cap \xi_i(\omega) \cap D_{\omega,n} \cap \Delta$. Since $\eta \in D_{\omega,n}$ we have,
\[ d(P(V_{\gamma}^\perp \pi \eta, P(V_{\gamma}^\perp \pi \eta)) \leq e^{n(\hat{\lambda}_{i+1} + 5\epsilon)}. \]

By (3) it follows,
\[ d(P(V_{\gamma}^\perp \pi \eta, P(V_{\gamma}^\perp \pi \tau)) \leq e^{n(\hat{\lambda}_{i+1} + 5\epsilon)} \text{ for } \zeta \in Q_{n,\epsilon} \cap \xi_i(\omega) \cap P_n^{n-1}(\eta). \]

Thus,
\[ Q_{n,\epsilon} \cap \xi_i(\omega) \cap P = Q_{n,\epsilon} \cap \xi_i(\eta) \cap P_n^{n-1}(\eta) \subset \Gamma_k(\omega, 2e^{n(\hat{\lambda}_{i+1} + 5\epsilon)}), \]
\[ \text{which gives } (6.8). \] Since $\eta \in \Delta$ it follows by (11),
\[ \beta_{\omega}^\xi(Q_{n,\epsilon} \cap P) = \beta_{\eta}^\xi(Q_{n,\epsilon} \cap P_n^{n-1}(\eta)) > e^{-n(H_i + \epsilon)}, \]
\[ \text{which gives } (6.9). \]

From (6.8), (6.9), $\mathcal{E}' \subset \mathcal{E}$ and (6.7) we now get,
\[ \beta_{\omega}^\xi(\Gamma_k(\omega, 2e^{n(\hat{\lambda}_{i+1} + 5\epsilon)})) \geq \sum_{P \in \mathcal{E}} \beta_{\omega}^\xi(Q_{n,\epsilon} \cap P) \geq |\mathcal{E}'| e^{-n(H_i + \epsilon)} \geq c \exp(n(\hat{\lambda}_{i+1} + 5\epsilon)(\bar{\tau}_{i+1,k} + \epsilon)) \exp(n(H_{i+1} - \epsilon) - n(H_i + \epsilon)). \]

This together with (6) gives,
\[ c \exp(n(\hat{\lambda}_{i+1} + 5\epsilon)(\bar{\tau}_{i+1,k} + \epsilon) + nH_{i+1} - nH_i - 2n\epsilon) < \exp(n(\bar{\tau}_{i,k} - \epsilon)(\hat{\lambda}_{i+1} + 5\epsilon)). \]

Now by taking logarithm on both sides and by dividing by $n$ it follows from $\mathcal{E}$ that,
\[ (\hat{\lambda}_{i+1} + 5\epsilon)(\bar{\tau}_{i+1,k} + \epsilon) + H_{i+1} - H_i - 3\epsilon < (\bar{\tau}_{i,k} - \epsilon)(\hat{\lambda}_{i+1} + 5\epsilon). \]

Since this holds for arbitrarily small $\epsilon > 0$ we obtain,
\[ \frac{H_{i+1} - H_i}{\hat{\lambda}_{i+1}} \geq \bar{\tau}_{i+1,k} - \bar{\tau}_{i,k}, \]
which contradicts (6.5) and completes the proof of the proposition. \[ \square \]

Combining Propositions 5.1, 6.1 and 6.2 together, we obtain the following.

**Claim 6.3.** For $1 \leq k \leq s$ and $0 \leq i \leq k$ we have,
\[ \bar{\tau}_{i,k}(\omega) = \gamma_{i,k}(\omega) = \sum_{j=i}^{k-1} \frac{H_{j+1} - H_j}{\hat{\lambda}_{j+1}} \text{ for } \beta\text{-a.e. } \omega. \]
Proof. Fix $1 \leq k \leq s$. We prove the claim by backward induction on $i$. By the definition of $\xi_k$ it follows that for $\beta$-a.e. $\omega$,

$$P(V_k^\perp) \cdot \pi \eta = P(V_k^\perp) \cdot \pi \omega \quad \text{for $\beta$-a.e. } \eta.$$  

From this it follows directly that $\nabla_{k,k}^\perp(\omega) = \gamma_{k,k}(\omega) = 0$ for $\beta$-a.e. $\omega$, which gives (6.10) in the case $i = k$.

Now let $0 \leq i < k$ and suppose that (6.10) has been proven for $i + 1$. By Proposition 6.2,

$$\frac{H_{i+1} - H_i}{\lambda_{i+1}} \geq \nabla_{i,k} \omega \geq \nabla_{i+1,k} \omega \quad \text{for } \beta \text{-a.e. } \omega,$$

by Proposition 6.1,

$$\vartheta_i(\omega) \geq \frac{H_{i+1} - H_i}{\lambda_{i+1}} \quad \text{for } \beta \text{-a.e. } \omega,$$

and by Proposition 6.1,

$$\sum_{i+1,k} \omega + \vartheta_i(\omega) \leq \gamma_{i,k}(\omega) \quad \text{for } \beta \text{-a.e. } \omega.$$

Combining these facts with the induction hypothesis, we obtain that for $\beta$-a.e. $\omega$,

$$\nabla_{i,k}(\omega) - \sum_{j=i+1}^{k-1} \frac{H_{j+1} - H_j}{\lambda_{j+1}} = \nabla_{i,k}(\omega) - \nabla_{i+1,k}(\omega) \leq \frac{H_{i+1} - H_i}{\lambda_{i+1}} \leq \vartheta_i(\omega) \leq \gamma_{i,k}(\omega) - \sum_{j=i+1}^{k-1} \frac{H_{j+1} - H_j}{\lambda_{j+1}}.$$

This proves (6.10) also for $i$, which completes the proof of the claim. $\square$

We are finally ready to complete the proof of Theorem 1.3 which follows easily from the last claim. Recall from Section 2.5 that $\nu = \pi \beta$ is the Furstenberg measure corresponding to $\mu = \sum_{l \in A} p_l \delta_{A_l}$. Also, recall from Section 1.2 that for a proper linear subspace $W$ of $V$ the partition $\zeta_W$ of $P(V) \setminus P(W)$ is define by,

$$\zeta_W(\pi) = \{ \pi \in P(V) \setminus P(W) : P_W + \pi = P_W \cap \pi \}. $$

By remark 1.4 in Section 1.2 in order to prove Theorem 1.3 we only need to establish part (4) of that theorem, whose statement we now recall.

**Theorem.** For $\beta$-a.e. $\omega$, $\nu$-a.e. $\pi$ and every $0 \leq i < k \leq s$, the measure $P(V_k^\perp) \cdot \nu_{\zeta_V^k}^\perp$ is exact dimensional with,

$$\dim P(V_k^\perp) \cdot \nu_{\zeta_V^k}^\perp = \sum_{j=i}^{k-1} \frac{H_{j+1} - H_j}{\lambda_{j+1}}.$$  

**Proof.** Let $Z_{\geq 0}$ and $Z_{<0}$ denote the sets of nonnegative and negative integers respectively. Write $\Omega^+$ for the space of sequences $(\omega_n)_{n \geq 0} \in \Lambda^{Z_{\geq 0}}$, and $\Omega^-$ for the space of sequences $(\omega_n)_{n < 0} \in \Lambda^{Z_{<0}}$. We equip each of these spaces with its Borel
\[ \sigma \text{-algebra generated by cylinder sets. Let } q^+ : \Omega \to \Omega^+ \text{ and } q^- : \Omega \to \Omega^- \text{ be the projections onto the nonnegative and negative coordinates respectively. Note that since } \beta \text{ is a Bernoulli measure the maps } q^+ \text{ and } q^- \text{ are independent as random elements on } (\Omega, \beta). \]

Write \( \beta^+ \) and \( \beta^- \) for the Bernoulli measures corresponding to \( p \) on \( \Omega^+ \) and \( \Omega^- \) respectively, that is \( \beta^+ = p^{\mathbb{Z}^+} \) and \( \beta^- = p^{\mathbb{Z}^-} \).

Recall that \( \pi \) only depends on the nonnegative coordinates. Thus there exists a Borel map \( \pi_+ : \Omega^+ \to P(V) \) such that \( \pi \omega = \pi_+ q^+ \omega \) for \( \omega \in \Omega_0 \). Since \( \nu = \pi \beta \) it follows that \( \nu = \pi_+ \beta^+ \). Also, recall that for each \( 0 \leq i \leq s \) the map which takes \( \omega \in \Omega_0 \) to the \( V_i \) depends only on the negative coordinates. Thus we may write \( V_i \) in place of \( V_i^0 \) for \( \omega \in \Omega_0 \).

For a proper linear subspace \( W \) of \( V \) write \( \xi_W \) for the partition of \( \Omega^+ \setminus \pi_+^{-1} P(W) \), such that for \( \omega \) in this set,

\[ \xi_W(\omega) = \{ \eta \in \Omega^+ \setminus \pi_+^{-1} P(W) : P_{W^+} \pi_+ \eta = P_{W^+} \pi_+ \omega \} . \]

Note that,

\[ \xi_W(\omega) = \pi_+^{-1} \xi_W(\pi_+ \omega) . \]

Since \( W \neq V \) we have,

\[ \beta^+ (\pi_+^{-1} P(W)) = \nu(P(W)) = 0, \]

so the conditional measures \( \{ (\beta^+)^{\xi_W} \}_{\omega \in \Omega^+} \subset \mathcal{M}(\Omega^+) \) are \( \beta^+ \text{-a.e. defined.} \)

From \( \nu = \pi_+ \beta^+, \) (6.11) and Lemma 2.3,

\[ \pi_+ (\beta^+)^{\xi_W} = \nu^{\xi_W} \text{ for } \beta^+ \text {-a.e. } \omega . \]

Since \( \beta = \beta^- \times \beta^+ \), it is easy to verify (by using [RW] Proposition 5.19 for instance) that for each \( 0 \leq i \leq s \), it follows that for \( 0 \leq i \leq s \) and \( \beta \text{-a.e. } \omega , \)

\[ \beta^i_\omega = \delta_{q^- \omega} \times (\beta^+)^{\xi_{q^+ i} - \omega} \text{ for } \beta \text{-a.e. } \omega . \]

From these facts together with \( \pi = \pi_+ q^+ \), it follows that for \( 0 \leq i \leq s \) and \( \beta \text{-a.e. } \omega , \)

\[ \pi^i_\omega \beta^i_\omega = \pi_+ (\beta^+)_{\pi_+^{-1} P(W)}^i - \omega = (\nu)^{\xi_{q^+ i} - \omega} . \]

Now let \( 0 \leq i < k \leq s \) be given and write,

\[ \alpha = \sum_{j=i}^{k-1} \frac{H_{j+1} - H_j}{\lambda_{j+1}} . \]

From Claim 6.3 and the definitions of \( \pi_{i,k} \) and \( \gamma_{i,k} \), it follows that for \( \beta \text{-a.e. } \omega , \)

\[ \lim_{r \downarrow 0} \frac{\log \beta^i_\omega (\Gamma_k(\omega, r))}{\log r} = \alpha . \]

By the definition of \( \Gamma_k(\omega, r) \) this implies that for \( \beta \text{-a.e. } \omega , \)

\[ \lim_{r \downarrow 0} \frac{\log P_{\pi_+ q^+ \omega}(B(P_{\pi_+ q^+ \omega}(\pi_+ \omega, r)))}{\log r} = \alpha . \]
From this and (6.13) it follows that for \( \beta \)-a.e. \( \omega \),
\[
\lim_{r \downarrow 0} \frac{\log P_{(V^k_q, \omega)}(\nu_\pi \gamma^\omega(B(P(V^k_q)\perp \pi^+ q^\omega, r)))}{\log r} = \alpha.
\]
From this, since \( \pi^+ q^+ \beta = \nu \) and since \( q^+ \) and \( q^- \) are \( \beta \)-independent elements, it follows that for \( \beta^-\)-a.e. \( \omega \), \( \nu \)-a.e. \( x \),
\[
\lim_{r \downarrow 0} \frac{\log P_{(V^k_q, \omega)}(\nu_\pi \gamma^\omega(B(P(V^k_q)\perp \pi^+ q^+ \omega, r)))}{\log r} = \alpha.
\]
Note that for \( \beta^-\)-a.e. \( \omega \), \( \nu \)-a.e. \( x \) and \( \nu \)-a.e. \( y \) the last equality holds with \( y \) in place of \( x \). Since for \( \beta^-\)-a.e. \( \omega \), \( \nu \)-a.e. \( x \), \( \nu_\pi \gamma^\omega \) is \( \nu_\pi \gamma^\omega \)-a.e. \( y \), this completes the proof of the theorem. \( \square \)

In the next lemma we show that the different definitions for \( H^i \), given in Section 1.2 and Section 4, yield the same value.

**Lemma 6.4.** Let \( B \) be the Borel \( \sigma \)-algebra of \( P(V) \). Then for \( 0 \leq i \leq s \) we have,
\[
H^i_{\beta}(P | \hat{\xi}_i) = \int H^i_{\beta}(P | \pi^{-1} P^{-1}(V^i_q) B) d\beta(\omega).
\]

**Proof.** We use here the notations introduced in the last proof. Let \( P^+ \) be the partition of \( \Omega^+ \) according to the 0-coordinate. Given \( \theta \in \mathcal{M}(\Omega^+) \) it will be convenient to write \( H(P^+; \theta) \) in place of \( H^i_{\beta}(P^+) \). By the definitions of the conditional measures and entropy, and by (6.12), we get
\[
H^i_{\beta}(P | \hat{\xi}_i) = \int H_{\beta \xi_i}(P) d\beta(\omega)
\]
\[
= \int H \left( P^+; \left( \beta^+ \right)_{\omega_i}^{\xi^i_{\nu_\pi \gamma^\omega}} \right) d\beta^+(\omega_1) d\beta^-(-\omega_2)
\]
\[
= \int H_{\beta^+}(P^+ | \xi^i_{\nu_\pi \gamma^\omega}) d\beta^-(\omega_2)
\]
\[
= \int H_{\beta^+}(P | \pi^{-1} P^{-1}(V^i_q) B) d\beta(\omega),
\]
which completes the proof of the lemma. \( \square \)

As a Corollary of Theorem 1.3 we can now prove the following lemma, which was used in Section 1.3 when the Lyapunov dimension was discussed. Recall the numbers \( d_0, \ldots, d_s \) from Theorem 2.9

**Lemma 6.5.** Let \( 0 \leq i < s \) be given, then
\[
0 \leq H^i - H_{i+1} \leq -\lambda_{i+1} d_{i+1}.
\]

**Proof.** For \( \omega \in \Omega_0 \) and \( \pi \in P(V) \setminus P(V^i_q) \),
\[
\xi^i_{\nu_\pi \gamma^\omega} = P(\pi \oplus V^i_q) \setminus P(V^i_q).
\]
Hence,
\[
(6.14) \quad P_{(V^i_q \perp \nu_\pi \gamma^\omega)}(\xi^i_{\nu_\pi \gamma^\omega}) \subset P_{(V^i_q \perp \nu_\pi \gamma^\omega)}(\pi \oplus P_{(V^i_q \perp \nu_\pi \gamma^\omega)}(V^i_q)).
\]
Note that,
$$\dim P(V_{i+1}^j \perp V^j_\omega) = \dim V^j_\omega - \dim V^j_{i+1} = d_{i+1},$$
and so,
$$P(R(V_{i+1}^j \perp P(V_{i+1}^j \perp V^j_\omega)),$$
is a smooth $d_{i+1}$-dimensional manifold. This together with (6.14) gives,
$$\dim_H \left( P(V_{i+1}^j (\zeta V^j_\omega(\mathcal{F})) \right) \leq d_{i+1},$$
where $\dim_H$ stands from Hausdorff dimension.

Additionally, for $\beta$-a.e. $\omega$ and $\nu$-a.e. $x$,
$$P(V_{i+1}^j \perp \zeta V^j_\omega(\mathcal{F})) = 1.$$
Hence if $P(V_{i+1}^j \perp \zeta V^j_\omega(\mathcal{F}))$ is also exact dimensional, then from (6.15) and (1.1) we obtain that its dimension is at most $d_{i+1}$. It now follows from part (1) of Theorem 1.3 that for $\beta$-a.e. $\omega$ and $\nu$-a.e. $x$,
$$0 \leq H_i - H_{i+1} = -\lambda_{i+1} \dim P(V_{i+1}^j \perp \zeta V^j_\omega(\mathcal{F})) \leq -\lambda_{i+1} d_{i+1},$$
which completes the proof of the lemma. □

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43