Biharmonic hypersurfaces in hemispheres

Matheus Vieira

Abstract
In this paper, we consider the Balmuş-Montaldo-Oniciuc’s conjecture in the case of hemispheres. We prove that a compact non-minimal biharmonic hypersurface in a hemisphere of $S^{n+1}$ must be the small hypersphere $S^n\left(\frac{1}{\sqrt{2}}\right)$, provided that $n^2 - H^2$ does not change sign.

1 Introduction
It is well known that minimal hypersurfaces can be seen as hypersurfaces whose canonical inclusion is a harmonic map. Thus, it is natural to study hypersurfaces whose canonical inclusion is a biharmonic map, the so called biharmonic hypersurfaces (for more information, see Section 2). From the point of view of finding new examples and classification results, the theory of biharmonic hypersurfaces seems to be more interesting when the ambient space has positive curvature. There are many papers studying biharmonic hypersurfaces in the sphere such as [1], [2], [4], [5], [8], [12], [13], [14] and [15].

In [1], Balmuş-Montaldo-Oniciuc conjectured that non-minimal biharmonic hypersurfaces in the unit Euclidean sphere $S^{n+1}$ must be open parts of the small hypersphere $S^n\left(\frac{1}{\sqrt{2}}\right)$ or of the generalized Clifford tori $S^{n_1}\left(\frac{1}{\sqrt{2}}\right) \times S^{n_2}\left(\frac{1}{\sqrt{2}}\right)$ with $n_1 + n_2 = n$ and $n_1 \neq n_2$. These are the canonical examples of non-minimal biharmonic hypersurfaces in the sphere. Note that the generalized Clifford tori cannot lie in a hemisphere. Thus, it is natural to ask whether a compact non-minimal biharmonic hypersurface in a hemisphere of $S^{n+1}$ must be the small hypersphere $S^n\left(\frac{1}{\sqrt{2}}\right)$. We give an affirmative answer to this question when $n^2 - H^2$ does not change sign.

Theorem 1. Given a compact biharmonic hypersurface $M^n$ in a closed hemisphere of $S^{n+1}$, if $n^2 - H^2$ does not change sign, then either $M^n$ is the equator of the hemisphere or it is the small hypersphere $S^n\left(\frac{1}{\sqrt{2}}\right)$.

Note that the condition $n^2 - H^2 \geq 0$ is satisfied by biharmonic hypersurfaces in $S^{n+1}$ with constant mean curvature (this follows from Theorem 2 and the Cauchy-Schwarz inequality).

The case $n^2 - H^2 \leq 0$ was proved in a direct way by Balmuş-Oniciuc (Corollary 3.3 in [2]). Our proof is completely different. It is based on a formula
for the bilaplacian of the restriction of a function defined on the ambient space (Theorem 4).

The author would like to thank Detang Zhou, Cezar Oniciuc and Dorel Fetcu for their support.

2 Preliminaries

Let us recall some concepts and describe conventions of the paper.

Consider a Riemannian manifold $M$ with Levi-Civita connection $\nabla$. The Riemann curvature is defined by

$$ Riem \left( u, v \right) w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w. $$

The Ricci curvature is defined by

$$ Ric \left( u, v \right) = tr_M Riem \left( \cdot, u \right) v. $$

Fix a function $f$ on $M$. The Hessian of $f$ is defined by

$$ \nabla \nabla f \left( u, v \right) = \langle \nabla_u \nabla f, v \rangle. $$

The Laplacian of $f$ is defined by

$$ \Delta f = tr_M \nabla \nabla f. $$

Now consider a hypersurface $M$ in a Riemannian manifold $\bar{M}$ with normal vector $N$. The second fundamental form is defined by

$$ A \left( u, v \right) = \langle \bar{\nabla}_u v, N \rangle. $$

The mean curvature is defined by

$$ H = tr_M A. $$

Note that we do not normalize the mean curvature. We use a bar above the quantities related to $\bar{M}$ such as the Levi-Civita connection $\bar{\nabla}$, the Riemann curvature $\bar{Riem}$, the Ricci curvature $\bar{Ric}$ and the Hessian $\bar{\nabla} \bar{\nabla} f$ of a function $\bar{f}$ on $\bar{M}$. Given a vector field $\bar{v}$ on $\bar{M}$, we denote its projection into the tangent bundle $TM$ and normal bundle $NM$ by $\bar{v}^T$ and $\bar{v}^N$, respectively. Note that $\bar{v}^N = \langle \bar{v}, N \rangle N$.

A map $\phi : M \rightarrow \bar{M}$ between Riemannian manifolds is called harmonic if it is a critical point of the functional

$$ E \left( \phi \right) = \int_M |d\phi|^2. $$

Critical points of this functional satisfy

$$ \tau \left( \phi \right) = 0, $$
where
\[ \tau(\phi) = tr_M \nabla d\phi. \]

It is well known that a hypersurface \( M \) in \( \bar{M} \) is minimal if and only if the canonical inclusion is a harmonic map. For more information about harmonic maps and submanifolds, see [7].

A map \( \phi : M \to \bar{M} \) between Riemannian manifolds is called biharmonic if it is a critical point of the functional
\[ E_2(\phi) = \int_M |\tau(\phi)|^2. \]

Critical points of this functional satisfy
\[ \tau_2(\phi) = 0, \]
where
\[ \tau_2(\phi) = \Delta \tau(\phi) + tr_M \bar{Riem} (\tau(\phi), d\phi) d\phi. \]

A hypersurface \( M \) in \( \bar{M} \) is called biharmonic if the canonical inclusion is a biharmonic map. For more information about biharmonic maps and submanifolds, see [9] and [10].

The next result was applied to submanifolds of the sphere for the first time by Oniciuc (Theorem 3.1 in [14]). Later, in the case of hypersurfaces, it was extended to general ambient spaces by Ou (Theorem 2.1 in [16]). See also Remark 4.10 in [11]. This is an important result in the theory of biharmonic submanifolds.

**Theorem 2.** ([14], [16]) A hypersurface \( M \) in a Riemannian manifold \( \bar{M} \) is biharmonic if and only if \( B_N \) and \( B_T \) vanish, where
\[ B_N = \Delta H - H |A|^2 + H \bar{Ric}(N,N), \]
and
\[ B_T = 2A(\nabla H) + H\nabla H - 2H (\bar{Ric}(N))^T. \]

Note that we identify \((1,1)\) tensors and \((0,2)\) tensors. In a local orthonormal frame \( \{e_i\} \) on \( M \), we have
\[ A(\nabla H) = \sum_i A(\nabla H, e_i) e_i, \]
and
\[ (\bar{Ric}(N))^T = \sum_i \bar{Ric}(N,e_i) e_i. \]

We prove the next known result for the sake of completeness.

**Lemma 3.** For a hypersurface \( M \) in a Riemannian manifold \( \bar{M} \) and a function \( f \) on \( M \), we have
\[ \nabla \nabla f(u,v) = \nabla \nabla \bar{f}(u,v) + \langle \nabla \bar{f}, N \rangle A(u,v), \]
where \( f = \bar{f}|M \). In particular,
\[ \Delta f = tr_M \nabla \nabla f + \langle \nabla \bar{f}, N \rangle H. \]
Proof. Take a local orthonormal frame \{e_i\} on M such that \(\nabla_{e_i} e_j = 0\) at a fixed point of \(M\). At this point, we have
\[
\nabla \nabla f (e_i, e_j) = e_i e_j f
\]
and
\[
\nabla \nabla \bar{f} (e_i, e_j) = e_i e_j \bar{f} - \left\langle \nabla \bar{f}, (\nabla_{e_i} e_j)^N \right\rangle
\]
\[
= e_i e_j \bar{f} - \left\langle \nabla \bar{f}, N \right\rangle A (e_i, e_j).
\]
Combining the above equations, we get the result.

The proof of Theorem 4 is based on the next result.

**Theorem 4.** For a hypersurface \(M\) in Riemannian manifold \(\bar{M}\) and a function \(\bar{f}\) on \(\bar{M}\), we have
\[
\Delta \Delta \bar{f} = \Delta \left(\text{tr}_M \nabla \nabla \bar{f} \right) + H \Delta \left\langle \nabla \bar{f}, N \right\rangle + 2 \left\langle \nabla \left\langle \nabla \bar{f}, N \right\rangle, \nabla H \right\rangle + \left\langle \nabla \bar{f}, N \right\rangle \Delta H.
\]
where \(\bar{f} = f | M\).

Proof. By Lemma 3, we have
\[
\Delta \bar{f} = \text{tr}_M \nabla \nabla \bar{f} + \left\langle \nabla \bar{f}, N \right\rangle H.
\]
Taking the Laplacian, we have
\[
\Delta \Delta \bar{f} = \Delta \left(\text{tr}_M \nabla \nabla \bar{f} \right) + H \Delta \left\langle \nabla \bar{f}, N \right\rangle + 2 \left\langle \nabla \left\langle \nabla \bar{f}, N \right\rangle, \nabla H \right\rangle + \left\langle \nabla \bar{f}, N \right\rangle \Delta H.
\]
Take a local orthonormal frame \{e_i\} on \(M\). We have
\[
e_i \left\langle \nabla \bar{f}, N \right\rangle = \left\langle \nabla_{e_i} \nabla \bar{f}, N \right\rangle + \left\langle \nabla \bar{f}, \nabla_{e_i} N \right\rangle
\]
\[
= \nabla \nabla \bar{f} (e_i, N) - A (e_i, \nabla \bar{f}).
\]
We find that
\[
\Delta \Delta \bar{f} = \Delta \left(\text{tr}_M \nabla \nabla \bar{f} \right) + H \Delta \left\langle \nabla \bar{f}, N \right\rangle + 2 \nabla \nabla \bar{f} (\nabla H, N)
\]
\[
- 2 A (\nabla H, \nabla \bar{f}) + \left\langle \nabla \bar{f}, N \right\rangle \Delta H.
\]
We can assume that \(\nabla_{e_i} e_j = 0\) at a fixed point of \(M\). At this point, we have
\[
\Delta \left\langle \nabla \bar{f}, N \right\rangle = \sum_i e_i e_i \left\langle \nabla \bar{f}, N \right\rangle
\]
\[
= \sum_i e_i \left( \nabla \nabla \bar{f} (e_i, N) \right) - \sum_i e_i A (e_i, \nabla \bar{f})
\]
\[
= \sum_i \left( \nabla_{e_i} \nabla \nabla \bar{f} \right) (e_i, N) + \sum_i \nabla \nabla \bar{f} \left( \left( \nabla_{e_i} e_i \right)^N, N \right)
\]
\[
+ \sum_i \nabla \nabla \bar{f} (e_i, \nabla_{e_i} N) - \sum_i \left( \nabla_{e_i} A \right) (e_i, \nabla \bar{f}) - \sum_i A (e_i, \nabla_{e_i} \nabla \bar{f}).
\]
Using the expressions of $B$ we have
\[
\left( \nabla_{e_i} \bar{\nabla} f \right) (e_i, N) = \left( \nabla_{e_i} \bar{\nabla} f \right) (N, e_i) = \left( \nabla_N \bar{\nabla} f \right) (e_i, e_i) - \langle \bar{\text{Riem}} (e_i, N) e_i, \nabla \bar{f} \rangle,
\]
and the Codazzi equation
\[
\left( \nabla_{e_i} A \right) (e_i, \nabla f) = \left( \nabla_{e_i} A \right) (\nabla f, e_i) = \left( \nabla_{\nabla f} A \right) (e_i, e_i) + \langle \bar{\text{Riem}} (e_i, \nabla f) e_i, N \rangle,
\]
we have
\[
\Delta \langle \nabla \bar{f}, N \rangle = tr_M \left( \bar{\nabla}_N \bar{\nabla} f \right) + \nabla \left( \bar{\nabla} f, N \right) + H \bar{\nabla} \bar{f} (N, N)
- \langle \bar{\nabla} \bar{\nabla} f, A \rangle - \langle \nabla f, \nabla H \rangle + \bar{\text{Ric}} (\nabla f, N) - \langle \nabla \nabla f, A \rangle.
\]
Using the fact that
\[
\bar{\text{Ric}} (\bar{\nabla} f, N) = \langle \bar{\nabla} f, N \rangle \bar{\text{Ric}} (N, N) + \bar{\text{Ric}} (\nabla f, N),
\]
and Lemma 3 we have
\[
\Delta \langle \nabla \bar{f}, N \rangle = tr_M \left( \nabla_N \nabla \bar{f} \right) + \nabla \langle \nabla \bar{f}, N \rangle \bar{\text{Ric}} (N, N) + H \nabla \bar{f} (N, N)
- 2 \langle \nabla \nabla \bar{f}, A \rangle - \langle \nabla f, \nabla H \rangle + 2 \bar{\text{Ric}} (\nabla f, N) - \langle \nabla \bar{f}, N \rangle |A|^2.
\]
We find that
\[
\Delta \Delta f = \Delta \left( tr_M \bar{\nabla} \bar{\nabla} f \right) + H tr_M \left( \bar{\nabla}_N \bar{\nabla} f \right) + \nabla \langle \nabla \bar{f}, N \rangle \bar{\text{Ric}} (N, N)
+ H^2 \bar{\nabla} \bar{\nabla} f (N, N) - 2 H \langle \bar{\nabla} \bar{\nabla} f, A \rangle - H \langle \nabla f, \nabla H \rangle + 2 H \bar{\text{Ric}} (\nabla f, N)
- H \langle \nabla \bar{f}, N \rangle |A|^2 + 2 \nabla \nabla \bar{f} (\nabla H, N) - 2 A \langle \nabla H, \nabla f \rangle + \langle \nabla \bar{f}, N \rangle \Delta H
= \Delta \left( tr_M \bar{\nabla} \bar{\nabla} f \right) + H tr_M \left( \bar{\nabla}_N \bar{\nabla} f \right) + H^2 \bar{\nabla} \bar{\nabla} f (N, N) - 2 H \langle \bar{\nabla} \nabla \bar{f}, A \rangle
+ 2 \nabla \nabla \bar{f} (\nabla H, N) + \langle \nabla \bar{f}, N \rangle \left( 2 \bar{\text{Ric}} (N, N) - H |A|^2 + \Delta H \right)
+ \langle -H \nabla H + 2 H \bar{\text{Ric}} (N) - 2 A \langle \nabla H \rangle, \nabla f \rangle.
\]
Using the expressions of $B_T$ and $B_N$, we get the result. \hfill \square

\section{Proof of the main result}

Proof of Theorem \[1\]

\textit{Proof.} Consider the position vector $X$ of $R^{n+2}$. Since $M$ lies in a closed hemisphere, we can take a fixed vector $V$ in $R^{n+2}$ such that
\[
\langle X, V \rangle |M| \geq 0.
\]
Consider \( f = \langle X, V \rangle | \mathcal{M} \) and \( \bar{f} = \langle X, V \rangle | S^{n+1} \). We know that
\[
\bar{\nabla} \bar{\nabla} \bar{f} = -\bar{f} g_{S^{n+1}},
\]
and
\[
\bar{\nabla} \bar{\nabla} \bar{\nabla} \bar{f} = -d\bar{f} \otimes \bar{g}_{S^{n+1}}.
\]
By Theorem 2 and Theorem 4, we have
\[
\Delta \Delta f = \Delta (\text{tr} \bar{\nabla} \bar{\nabla} \bar{f}) + H \text{tr} (\bar{\nabla} N \bar{\nabla} \bar{f}) + H^2 \bar{\nabla} \bar{\nabla} \bar{f} (N, N)
\]
\[
- 2H \langle \bar{\nabla} \bar{f}, A \rangle + 2 \bar{\nabla} \bar{f} (\bar{\nabla} H, N)
\]
\[
= \Delta (\text{tr} \bar{\nabla} \bar{f}) - nH \langle \bar{\nabla} \bar{f}, N \rangle - H^2 f + 2H^2 f
\]
\[
= \Delta (\text{tr} \bar{\nabla} \bar{f}) - nH \langle \bar{\nabla} \bar{f}, N \rangle + H^2 f.
\]
Integrating and using the divergence theorem, we have
\[
0 = 0 - \int_{\mathcal{M}} nH \langle \bar{\nabla} \bar{f}, N \rangle + \int_{\mathcal{M}} H^2 f.
\]
By Lemma 3, we have
\[
\Delta f = \text{tr} \bar{\nabla} \bar{\nabla} \bar{f} + \langle \bar{\nabla} \bar{f}, N \rangle H
\]
\[
= -nf + \langle \bar{\nabla} \bar{f}, N \rangle H.
\]
Multiplying by \( n \), integrating and using the divergence theorem, we have
\[
0 = - \int_{\mathcal{M}} n^2 f + \int_{\mathcal{M}} nH \langle \bar{\nabla} \bar{f}, N \rangle.
\]
Combining the above equations, we have
\[
\int_{\mathcal{M}} (n^2 - H^2) f = 0.
\]
Since \( n^2 - H^2 \) does not change sign, we have
\[
(n^2 - H^2) f = 0.
\]
If \( H^2 = n^2 \) everywhere, by Theorem 2.10 in [1], we conclude that \( \mathcal{M} \) is the small hypersphere \( S^n \left( \frac{1}{\sqrt{2}} \right) \). Otherwise, we have \( f = 0 \) on an open subset of \( \mathcal{M} \), that is, an open subset of \( \mathcal{M} \) lies in the equator of the hemisphere. In this case, by the fact that this subset is minimal and Theorem 1.3 in [3], we find that the whole \( \mathcal{M} \) is minimal and we conclude that \( \mathcal{M} \) is the equator of the hemisphere.
4 Further applications

Consider the position vector $X$ and the canonical basis $\{E_i\}_{i=1}^{n+2}$ of $R^{n+2}$.

In [17], Takahashi proved that a hypersurface $M^n$ in $S^{n+1}$ is minimal if and only if the functions $x_i = \langle X, E_i \rangle |M^n$ satisfy $\Delta x_i = -nx_i$. As an application of Theorem 4, we have a similar result for biharmonic hypersurfaces.

**Theorem 5.** A hypersurface $M^n$ in $S^{n+1}$ is biharmonic if and only if the restriction of the cartesian coordinates $x_i = \langle X, E_i \rangle |M^n$ of $R^{n+2}$ to $M^n$ satisfy

$$\Delta \Delta x_i = (n^2 + H^2) x_i - 2nH \langle N, E_i \rangle \quad (1 \leq i \leq n + 2).$$

*Proof.* Consider $f = \langle X, E_i \rangle |M$ and $\tilde{f} = \langle X, E_i \rangle |S^{n+1}$. We know that

$$\nabla \nabla \tilde{f} = -\tilde{f} g_{S^{n+1}},$$

and

$$\nabla \nabla \nabla f = -df \otimes \tilde{g}_{S^{n+1}}.$$

By Theorem 4 we have

$$\Delta \Delta f = \Delta (tr_M \nabla \nabla \tilde{f}) + H tr_M (\nabla_N \nabla \nabla \tilde{f}) + H^2 \nabla \nabla \tilde{f} \langle N, N \rangle$$

$$- 2H \langle \nabla \nabla \tilde{f}, A \rangle + 2 \nabla \nabla \tilde{f} \langle \nabla H, N \rangle + \langle B_N N - B_T, \nabla \tilde{f} \rangle$$

$$= -n \Delta f - nH \langle \nabla \tilde{f}, N \rangle - H^2 f + 2H^2 \tilde{f} + \langle B_N N - B_T, \nabla \tilde{f} \rangle$$

$$= -n \Delta f - nH \langle \nabla \tilde{f}, N \rangle + H^2 f + \langle B_N N - B_T, \nabla \tilde{f} \rangle.$$

By Lemma 2 we have

$$\Delta f = tr_M \nabla \nabla \tilde{f} + \langle \nabla \tilde{f}, N \rangle H$$

$$= -nf + \langle \nabla \tilde{f}, N \rangle H.$$

We find that

$$\Delta \Delta f = (n^2 + H^2) f - 2nH \langle \nabla \tilde{f}, N \rangle + \langle B_N N - B_T, \nabla \tilde{f} \rangle.$$

Since $\nabla \tilde{f} = \text{proj}_{TS^{n+1}} E_i$, we have

$$\Delta \Delta x_i = (n^2 + H^2) x_i - 2nH \langle N, E_i \rangle + \langle B_N N - B_T, E_i \rangle \quad (1 \leq i \leq n + 2).$$

The result follows from Theorem 4. \qed

This result was obtained in a completely different way by Caddeo-Montaldo-Oniciuc (Proposition 4.1 in [3]). The statement is different, but equivalent.

As an application of Theorem 5, we find a sufficient condition for a biharmonic hypersurface in $S^{n+1}$ to be minimal.

**Corollary 6.** A biharmonic hypersurface $M^n$ in $S^{n+1}$ is minimal, provided that there is a function $\phi$ on $M^n$ such that the restriction of the cartesian coordinates $x_i = \langle X, E_i \rangle |M^n$ of $R^{n+2}$ to $M^n$ satisfy

$$\Delta \Delta x_i = \phi x_i \quad (1 \leq i \leq n + 2).$$
Proof. Since $\Delta \Delta x_i = \phi x_i$, by Theorem 5, we have

$$2nH \langle N, E_i \rangle = (n^2 + H^2 - \phi) \langle X, E_i \rangle \quad (1 \leq i \leq n + 2).$$

We find that

$$2nHN = (n^2 + H^2 - \phi) X.$$

Since $N$ is tangent to $S^{n+1}$ and $X$ is normal to $S^{n+1}$, we have $H = 0$. \qed

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Departamento de Matemática, Universidade Federal do Espírito Santo, Vitória, Brazil. E-mail: matheus.vieira@ufes.br