PRODUCTS OF COMPRESSIONS OF $k$th–ORDER SLANT TOEPLITZ OPERATORS TO MODEL SPACES

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Abstract. In this paper we investigate intertwining relations for compressions of $k$th–order slant Toeplitz operators to model spaces. We then ask when a product of two such compressions is a compression itself.

1. Introduction

Let $L^2 = L^2(\mathbb{T}, m)$ and $L^\infty = L^\infty(\mathbb{T}, m)$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle and $m$ is the normalized Lebesgue measure on $\mathbb{T}$. Fix a positive integer $k$. A $k$th–order slant Toeplitz operator with symbol $\varphi = \sum_{n=\infty}^{+\infty} a_n z^n \in L^\infty$ is the operator $U_\varphi : L^2 \to L^2$ represented with respect to the standard monomial basis by the doubly infinite matrix $[a_{ki-j}]_{i,j \in \mathbb{Z}}$ ($a_n$ being the $n$–th Fourier coefficient of $\varphi$). Clearly, for $k = 1$ such matrix is a doubly infinite Toeplitz matrix (has constant diagonals) and so in that case $U_\varphi = M_\varphi$ is the classical multiplication operator, $M_\varphi f = \varphi f$.

It is known that the $k$th–order slant Toeplitz operator $U_\varphi$ can be expressed as

$$U_\varphi f = W_k M_\varphi f, \quad f \in L^2,$$

where

$$W_k(z^n) = \begin{cases} z^m & \text{if } \frac{n}{k} = m \in \mathbb{Z}, \\ 0 & \text{if } \frac{n}{k} \notin \mathbb{Z}. \end{cases}$$

Moreover, $U_\varphi$ can be defined as above for any $\varphi \in L^2$, in which case it is densely defined (its domain contains $L^\infty$). However, $U_\varphi$ extends boundedly to $L^2$ if and only if $\varphi \in L^\infty$.

A systematic study of $k$th–order slant Toeplitz operators for $k = 2$ (called simply slant Toeplitz operators) was started by M. C. Ho [21] (see also [22][24]). He also considered compressions of slant Toeplitz operators to the classical Hardy space $H^2$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Such compressions were then investigated by T. Zegeye and S. C. Arora [38]. Slant Toeplitz operators have connections with wavelet theory and dynamical systems (see, e.g., [19][22][36]).

Generalized slant Toeplitz operators, that is, $k$th–order slant Toeplitz operators for $k \geq 2$, were introduced and studied by S. C. Arora and R. Batra [1][2]. Compressions to $H^2$ were also considered there.

Recall that the compression $V_\varphi : H^2 \to H^2$ of $k$th–order slant Toeplitz operator $U_\varphi$ is defined by

$$V_\varphi f = PU_\varphi f, \quad f \in H^2,$$
where $P$ is the Szegő projection. Again, $V_{\varphi}$ is densely defined for $\varphi \in L^2$ (its domain contains $H^\infty = H^2 \cap L^\infty$) and extends boundedly to $H^2$ if and only if $\varphi \in L^\infty$. For $k = 1$, $V_{\varphi} = T_{\varphi}$ is the classical Toeplitz operator, $T_{\varphi}f = P(\varphi f)$.

The more recent papers [12, 13] deal with slant Toeplitz operators in multivariable setting, while in [27, 28] the authors investigate commutativity of $k^{th}$-order slant Toeplitz operators.

In recent years, compressions of multiplication operators are intensely studied. In particular, compressions to model spaces, that is, subspaces of the form $K_\alpha = H^2 \ominus \alpha H^2$, where $\alpha$ is a nonconstant inner function: $\alpha \in H^\infty$ and $|\alpha| = 1$ a.e. on $\mathbb{T}$ (if $\alpha$ is constant, then $K_\alpha = \{0\}$). These subspaces are the non–trivial subspaces of $H^2$ which are invariant for the backward shift operator $S^* = T_{\bar{z}}$. They are also important in view of their connection with topics such as the B. Sz.-Nagy–C. Foias model theory [33, Chapter VI].

Since the model space $K_\alpha$ is a closed subspace of $H^2$, the functional $f \mapsto f^{(n)}(w)$ is bounded on $K_\alpha$ for each $n \in \mathbb{N}_0$ and $w \in \mathbb{D}$. Therefore, there exists a kernel function $k_{w,n}^\alpha \in K_\alpha$ such that $f^{(n)}(w) = \langle f, k_{w,n}^\alpha \rangle$ for all $f \in K_\alpha$. It is not difficult to see that $k_{w,n}^\alpha = P_\alpha k_{w,n}$, where $k_{w,n}(z) = \frac{\bar{\alpha} \alpha(z)}{1 - \bar{\alpha} \alpha}$ is in $H^2$ and $P_\alpha$ is the orthogonal projection from $L^2$ onto $K_\alpha$. Recall that $P_\alpha = P - M_\alpha P M_\alpha^*$ (see [15, Corollary 14.13]). In particular, $k_w(z) = k_{w,0}(z) = \frac{1}{1 - \bar{\alpha} \alpha}$ and $k_\alpha^w = P_\alpha k_{w,0}$ is given by

\begin{equation}
(1.1) \quad k_\alpha^w(z) = \frac{1 - \alpha \alpha(z)}{1 - \bar{\alpha} \alpha}, \quad z \in \mathbb{D}.
\end{equation}

Note that $k_\alpha^w$ belongs to $H^\infty$ for each $w \in \mathbb{D}$ and so $K_\alpha^\infty = K_\alpha \cap H^\infty$ is a dense subset of $K_\alpha$.

It is well known that $K_\alpha$ is preserved by the antilinear, isometric involution $C_\alpha : L^2 \to L^2$ (a conjugation) defined by

$$C_\alpha f(z) = \alpha(z) \overline{f(z)}, \quad |z| = 1,$$

(see [17] and [16, Chapter 8]). The function $\tilde{k}_{w,n}^\alpha = C_\alpha k_{w,n}^\alpha$ is called the conjugate kernel function. In particular, $\tilde{k}_w^\alpha = \tilde{k}_{w,0}^\alpha$ and a.e. on $\mathbb{T}$,

$$\tilde{k}_w^\alpha(z) = C_\alpha k_\alpha^w(z) = \frac{\alpha(z) - \alpha(w)}{z - w}$$

(the above formula is true also for all $z \in \mathbb{D} \setminus \{w\}$ with $\tilde{k}_w^\alpha(w) = \alpha'(w)$).

Finally, note that $K_\alpha$ is the set of all polynomials of degree at most $n - 1$. In that case $\dim K_\alpha = n$ and $\{1, z, \ldots, z^{n-1}\}$ is an orthonormal basis for $K_\alpha$. More details on properties and structure of model spaces can be found in [16] and [15].

For two inner functions $\alpha$, $\beta$ an asymmetric truncated Toeplitz operator $A_{\varphi}^{\alpha,\beta}$ with symbol $\varphi \in L^2$ is defined by

$$A_{\varphi}^{\alpha,\beta} f = P_\beta(\varphi f), \quad f \in K_\alpha^\infty.$$

Note that $A_{\varphi}^{\alpha,\beta}$ is densely defined since $K_\alpha^\infty$ is a dense subset of $K_\alpha$. Denote

$$T(\alpha, \beta) = \{A_{\varphi}^{\alpha,\beta} : \varphi \in L^2 \text{ and } A_{\varphi}^{\alpha,\beta} \text{ extends boundedly to } K_\alpha\}$$

and $T(\alpha) = T(\alpha, \alpha)$.

Truncated Toeplitz operators, i.e., operators $A_{\varphi}^{\alpha} = A_{\varphi}^{\alpha,\alpha}$ gained attention in 2007 with D. Sarason’s paper [31]. Under intensive study ever since, truncated Toeplitz operators proved to be a rich and interesting topic with many deep results and applications (see
Asymmetric truncated Toeplitz operators were introduced later in [39] and [5]. They were then studied for example in [7,20,25,26,29,30].

In this paper we continue the study of compressions of $k^{th}$-order slant Toeplitz operators to model spaces. Fix $k \in \mathbb{N}$. For two inner functions $\alpha$, $\beta$ and for $\varphi \in L^2$ we define

$$U_{\varphi}^{\alpha,\beta}f = P_{\beta}U_{\varphi}f = P_{\beta}W_{\alpha}(\varphi f), \quad f \in K_{\alpha}^{\infty},$$

and denote

$$S_k(\alpha, \beta) = \{U_{\varphi}^{\alpha,\beta} : \varphi \in L^2 \text{ and } U_{\varphi}^{\alpha,\beta} \text{ extends boundedly to } K_{\alpha}\}.$$  

Clearly, $S_1(\alpha, \beta) = \mathcal{T}(\alpha, \beta)$.

The class $S_k(\alpha, \beta)$ was first introduced in [31] where some basic algebraic properties of its elements were investigated. Here we focus on some commuting relations for operators from $S_k(\alpha, \beta)$ and on products of this kind of operators. Products of truncated Toeplitz and asymmetric truncated Toeplitz operators were considered in [10] and [37], respectively.

In Section 2 we introduce a shift invariance analogue which can be used to characterize operators from $S_k(\alpha, \beta)$.

In Section 3 we investigate some intertwining relations involving operators from $S_k(\alpha, \beta)$ and compressed shifts.

Sections 4–5 are devoted to products of operators from $S_k(\alpha, \beta)$ and related problems.

In what follows we will say that an inner function $\beta$ divides $\alpha$ if $\alpha/\beta = \alpha/\beta$ is also an inner function. In that case we will write $\beta \leq \alpha$. Moreover, for arbitrary inner functions $\alpha, \beta$ we will denote by $\gcd(\alpha, \beta)$ and $\text{lcm}(\alpha, \beta)$ their greatest common divisor and their least common multiple, respectively. See [3] for more details on the arithmetic of inner functions.

2. Shift invariance for operators from $S_k(\alpha, \beta)$

Recall that a bounded linear operator $A : K_{\alpha} \to K_{\beta}$ is shift invariant if

$$\langle Af, g \rangle = \langle Sf, Sg \rangle$$

for all $f \in K_{\alpha}$, $g \in K_{\beta}$ such that $Sf \in K_{\alpha}$ and $Sg \in K_{\beta}$. For $\alpha = \beta$ this notion was considered by D. Sarason in [33]. He proved that a bounded linear operator $A : K_{\alpha} \to K_{\alpha}$ is a truncated Toeplitz operator if and only if it is shift invariant. For the asymmetric case the notion of shift invariance was considered in [5] ($\beta \leq \alpha$), [29] ($\alpha, \beta$ - finite Blaschke products) and [20]. There it was proved that the operators from $\mathcal{T}(\alpha, \beta)$ are also characterized by shift invariance. Here we prove the following generalization of this result.

**Theorem 2.1.** Let $\alpha, \beta$ be two nonconstant inner functions and let $U : K_{\alpha} \to K_{\beta}$ be a bounded linear operator. Then $U \in S_k(\alpha, \beta)$, $k \in \mathbb{N}$, if and only if

$$\langle U S^k f, Sg \rangle = \langle U f, g \rangle$$

for all $f \in K_{\alpha}$, $g \in K_{\beta}$ such that $S^k f \in K_{\alpha}$ and $Sg \in K_{\beta}$.

As in the case of truncated Toeplitz operators, the proof is based on a characterization of operators from $S_k(\alpha, \beta)$ in terms of operators $S_{\alpha}$ and $S_{\beta}$. It was proved in [31] that a bounded linear operator $U : K_{\alpha} \to K_{\beta}$ belongs to $S_k(\alpha, \beta)$ if and only if there exist $\chi \in K_{\alpha}$ and $\psi_0, \ldots, \psi_{k-1} \in K_{\beta}$ such that

$$U - S_{\alpha}^k S_{\beta}^k = \tilde{k}_{\alpha}^0 \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes \tilde{k}_{0,j}^\alpha.$$
We also need the following lemma.

**Lemma 2.2.** Let $\alpha$ be a nonconstant inner function and let $m \in \mathbb{N}$. For each $f \in K_\alpha$ we have

(a) $S_\alpha^m f(z) = z^m f(z) - \sum_{j=0}^{m-1} \frac{1}{j!} \langle f, \tilde{k}_{0,j}^\alpha \rangle \alpha z^{m-1-j}, |z| = 1$;

(b) $(S_\alpha^*)^m f(z) = \overline{z}^m f(z) - \sum_{j=0}^{m-1} \frac{1}{j!} \langle f, \tilde{k}_{0,j}^\alpha \rangle \overline{z}^{m-j}, |z| = 1$.

**Proof.** Let $f \in K_\alpha$. Since $S_\alpha^m f = A_{zm}^\alpha$ and $P_\alpha = P - M_\alpha P M_\pi$, we get

$S_\alpha^m f = A_{zm}^\alpha = f - \alpha P(\overline{\alpha} z^m f)$.

Now

$$P(\overline{\alpha} z^m f) = P(z^{m-1} \overline{\alpha} zf) = P(z^{m-1} \overline{C_\alpha} f) = P\left( \sum_{j=0}^{\infty} (\overline{C_\alpha} f, \overline{z}^j) z^{m-1-j} \right)$$

$$= \sum_{j=0}^{m-1} \frac{1}{j!} (\overline{C_\alpha} f, \overline{z}^j) z^{m-1-j} = \sum_{j=0}^{m-1} \frac{1}{j!} (f, C_\alpha P_\alpha (\overline{z}^j)) z^{m-1-j}$$

$$= \sum_{j=0}^{m-1} \frac{1}{j!} \langle f, \tilde{k}_{0,j}^\alpha \rangle z^{m-1-j}$$

and (a) follows. Part (b) follows from Lemma 2(a) in [31] and the fact that for $f \in K_\alpha$ we have $(S_\alpha^*)^m f = (S_\alpha^*)^m f$.

Recall that for $f \in K_\alpha$ we have $Sf = zf \in K_\alpha$ if and only if $f \perp \tilde{k}_{0}^\alpha$. On the other hand, for $f \in K_\alpha$ we always have $S^* f \in K_\alpha$ and $\overline{zf} \in K_\alpha$ if and only if $f \perp \tilde{k}_{0}^\alpha$.

**Corollary 2.3.** Let $\alpha$ be a nonconstant inner function, let $m \in \mathbb{N}$ and let $f \in K_\alpha$. Then

(a) $z^m f(z) \in K_\alpha$ if and only if $f \perp \text{span}\{\tilde{k}_{0,j}^\alpha : j = 0, 1, \ldots, m-1\}$;

(b) $\overline{z}^m f(z) \in K_\alpha$ if and only if $f \perp \text{span}\{k_{0,j}^\alpha : j = 0, 1, \ldots, m-1\}$.

**Proof of Theorem 2.1.** Let $U \in S_k(\alpha, \beta)$ and take $f \in K_\alpha$, $g \in K_\beta$ such that $S^k f \in K_\alpha$, $Sg \in K_\beta$. By [31] Cor. 6] there are $\chi \in K_\alpha$ and $\psi_0, \ldots, \psi_{k-1} \in K_\beta$ such that (2.2) is satisfied. It follows from Corollary 2.3 that $f \perp \text{span}\{k_{0,j}^\alpha : j = 0, 1, \ldots, k-1\}$ and $g \perp \tilde{k}_{0}^\beta$, and therefore

$$\langle U f, g \rangle - \langle U S^k f, S g \rangle = \left\langle (U - S^* S^k) f, g \right\rangle = \left\langle \left( \tilde{k}_{0}^\beta \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes \tilde{k}_{0,j}^\alpha \right) f, g \right\rangle$$

$$= \langle f, \chi \rangle \cdot \langle \tilde{k}_{0}^\beta, g \rangle + \sum_{j=0}^{k-1} \langle f, \tilde{k}_{0,j}^\alpha \rangle \cdot \langle \psi_j, g \rangle = 0.$$ 

Assume now that $U$ satisfies (2.1). As above, for each $f \in K_\alpha$, $g \in K_\beta$ we have

$$\langle U f, g \rangle - \langle U S^k f, S g \rangle = \langle (U - S^* S^k) f, g \rangle.$$
Hence (2.1) means that the operator \( T = U - S^*_\beta US^k_\alpha \) maps \( f \perp M = \operatorname{span}\{\tilde{k}_{0,j}^\alpha : j = 0, 1, \ldots, k - 1\} \) to a function \( Tf \in N = \mathbb{C} \cdot \tilde{k}_0^\beta \). Thus

\[
(2.3) \quad P_{N^\perp}TP_{M^\perp} = 0,
\]

where \( P_{N^\perp} \) and \( P_{M^\perp} \) are orthogonal projections onto \( N^\perp = K_\beta \ominus N \) and \( M^\perp = K_\alpha \ominus M \), respectively. Clearly,

\[
P_{N^\perp} = I_{K_\beta} - c \cdot (\tilde{k}_0^\beta \otimes \tilde{k}_0^\beta)
\]

where \( c = \|\tilde{k}_0^\beta\|^{-1} \).

It can be verified that there exist complex numbers \( a_{i,j} \), \( 0 \leq i, j \leq k - 1 \), such that

\[
P_{M^\perp} = I_{K_\alpha} - \sum_{i,j=0}^{k-1} a_{i,j} \cdot (\tilde{k}_{0,i}^\alpha \otimes \tilde{k}_{0,j}^\alpha).
\]

Note that not only \( \{\tilde{k}_{0,j}^\alpha : j = 0, \ldots, k - 1\} \) are not orthogonal but they might not even be linearly independent, so it might happen that \( a_{i,j} = 0 \) for some \( i \) and \( j \). Still, (2.3) can be written as

\[
0 = (I_{K_\beta} - c \cdot (\tilde{k}_0^\beta \otimes \tilde{k}_0^\beta)) T \left( I_{K_\alpha} - \sum_{i,j=0}^{k-1} a_{i,j} \cdot (\tilde{k}_{0,i}^\alpha \otimes \tilde{k}_{0,j}^\alpha) \right)
\]

\[
= (I_{K_\beta} - c \cdot (\tilde{k}_0^\beta \otimes \tilde{k}_0^\beta)) \left( T - \sum_{i,j=0}^{k-1} a_{i,j} \cdot (T\tilde{k}_{0,i}^\alpha \otimes \tilde{k}_{0,j}^\alpha) \right)
\]

\[
= T - \sum_{i,j=0}^{k-1} a_{i,j} \cdot (T\tilde{k}_{0,i}^\alpha \otimes \tilde{k}_{0,j}^\alpha) - c \cdot (\tilde{k}_0^\beta \otimes T^*\tilde{k}_0^\beta) + c \sum_{i,j=0}^{k-1} a_{i,j} \cdot (T\tilde{k}_{0,i}^\alpha \tilde{k}_0^\beta) \cdot (\tilde{k}_0^\beta \otimes \tilde{k}_{0,j}^\alpha).
\]

Hence

\[
U - S^*_\beta US^k_\alpha = T = \tilde{k}_0^\beta \otimes (cT^*\tilde{k}_0^\beta) + \sum_{j=0}^{k-1} \left( \sum_{i=0}^{k-1} a_{i,j} \cdot (T\tilde{k}_{0,i}^\alpha - c\langle T\tilde{k}_{0,i}^\alpha, \tilde{k}_0^\beta \rangle \cdot \tilde{k}_0^\beta) \right) \otimes \tilde{k}_{0,j}^\alpha
\]

and \( U \) satisfies (2.2) with

\[
\chi = cT^*\tilde{k}_0^\beta \quad \text{and} \quad \psi_j = \sum_{i=0}^{k-1} a_{i,j} \cdot (T\tilde{k}_{0,i}^\alpha - c\langle T\tilde{k}_{0,i}^\alpha, \tilde{k}_0^\beta \rangle \cdot \tilde{k}_0^\beta).
\]

Thus by [31, Cor. 6], \( U \in \mathcal{S}_k(\alpha, \beta) \).

Note that if \( \dim K_\alpha \leq k \), then the set \( \{\tilde{k}_{0,j}^\alpha : j = 0, 1, \ldots, k - 1\} \) spans \( K_\alpha \) and so \( f \perp \operatorname{span}\{\tilde{k}_{0,j}^\alpha, j = 0, 1, \ldots, k - 1\} \) if and only if \( f = 0 \).

**Corollary 2.4.** Let \( \alpha, \beta \) be two nonconstant inner functions and assume that \( \dim K_\alpha = m < +\infty \). If \( k \geq m \), then every bounded linear operator from \( K_\alpha \) into \( K_\beta \) belongs to \( \mathcal{S}_k(\alpha, \beta) \).

### 3. Intertwining Properties for Operators from \( \mathcal{S}_k(\alpha, \beta) \)

Let \( \alpha \) and \( \beta \) be two nonconstant inner functions. By D. Sarasons commutant lifting theorem each bounded linear operator \( A : K_\alpha \to K_\alpha \) commuting with the compressed shift \( S_\alpha \) is of the form \( A = A^*_\varphi \) with \( \varphi \in H^\infty \). More generally, a bounded linear operator \( A : K_\alpha \to K_\beta \) satisfies

\[
(3.1) \quad S_\beta A = AS_\alpha
\]
if and only if \( A = A_{\varphi}^{\alpha,\beta} \) with \( \varphi \in H^\infty \) such that \( \beta \leq \alpha \varphi \) (see [3] Theorem III.1.16]). Recently, the authors in [6] proved using basic methods that \( A : K_\alpha \to K_\beta \) satisfies (3.1) if and only if \( A = A_{\varphi}^{\alpha,\beta} \) with \( \varphi \in \frac{\beta}{\gcd(\alpha,\beta)} K_{\gcd(\alpha,\beta)} \).

Here our goal is to describe (for any fixed positive integer \( k \)) all bounded linear operators \( U : K_\alpha \to K_\beta \) that satisfy

\[
S_\beta U = U S_\alpha^k.
\]

We use a reasoning similar to the one used in [6]. First recall that \( U : K_\alpha \to K_\beta \) belongs to \( S_k(\alpha, \beta) \) if and only if

\[
S_\beta U - U S_\alpha^k = k_0^\beta \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes \tilde{k}_0^\alpha
\]

for some \( \chi \in K_\alpha \) and \( \psi_0, \ldots, \psi_{k-1} \in K_\beta \) [31 Corollary 8(b)]. Thus each \( U \) satisfying (3.2) clearly belongs to \( S_k(\alpha, \beta) \). We therefore describe the operators from \( S_k(\alpha, \beta) \) satisfying (3.2).

**Proposition 3.1.** Let \( \alpha \) and \( \beta \) be two nonconstant inner functions and let \( \varphi \in L^2 \). Then

\[
(a) \quad S_\beta U_\varphi^{\alpha,\beta} - U_\varphi^{\alpha,\beta} S_\alpha^k = \sum_{j=0}^{k-1} \frac{1}{j!} P_\beta W_k(\alpha \varphi \varphi^k-1-j) \otimes \tilde{k}_0^\alpha - k_0^\beta \otimes P_\alpha(\varphi^k \varphi),
\]

\[
(b) \quad S_\beta U_\varphi^{\alpha,\beta} - U_\varphi^{\alpha,\beta} (S_\alpha^*)^k = \sum_{j=0}^{k-1} \frac{1}{j!} P_\beta W_k(\varphi \varphi^k-1-j) \otimes \tilde{k}_0^\alpha - k_0^\beta \otimes P_\alpha(\varphi \cdot W_\beta^k),
\]

where both equalities hold on \( K_\alpha^\infty \).

**Proof.** Let \( f \in K_\alpha^\infty \) and \( g \in K_\beta^\infty \). Then

\[
\langle S_\beta U_\varphi^{\alpha,\beta} f, g \rangle = \langle P_\beta W_k(\varphi f), S_\beta^* g \rangle = \langle W_k(\varphi f), S_\beta^* g \rangle = \langle \varphi f, W_\beta^* S_\alpha^* g \rangle
\]

\[
= \langle \varphi f, W_k(\varphi g - zg(0)) \rangle = \langle \varphi f, \tilde{z} W_\beta^* g \rangle - \langle \varphi f, \tilde{z} g(0) \rangle
\]

\[
= \langle \varphi \varphi^k f, W_\beta^* g \rangle - \langle f, \varphi \varphi^k \rangle g(0) = \langle \varphi \varphi^k f, W_\beta^* g \rangle - \langle f, P_\alpha(\varphi \varphi^k) \circ k_0^\beta, g \rangle
\]

\[
= \langle \varphi \varphi^k f, W_\beta^* g \rangle - \left( (k_0^\beta \otimes P_\alpha(\varphi \varphi^k)) f, g \right).
\]

Moreover, using Lemma 2.2(a), we get

\[
\langle U_\varphi^{\alpha,\beta} S_\alpha^k f, g \rangle = \langle \varphi S_\alpha^k f, W_\beta^* g \rangle = \langle \varphi \varphi^k f, W_\beta^* g \rangle - \sum_{j=0}^{k-1} \frac{1}{j!} \langle f, \tilde{k}_0^\alpha \rangle \cdot \langle \varphi \varphi^k-1-j, W_\beta^* g \rangle
\]

\[
= \langle \varphi \varphi^k f, W_\beta^* g \rangle - \sum_{j=0}^{k-1} \frac{1}{j!} \langle f, \tilde{k}_0^\alpha \rangle \cdot P_\beta W_k(\varphi \varphi^k-1-j, g)
\]

\[
= \langle \varphi \varphi^k f, W_\beta^* g \rangle - \left( \left( \sum_{j=0}^{k-1} \frac{1}{j!} P_\beta W_k(\varphi \varphi^k-1-j) \otimes \tilde{k}_0^\alpha \right) f, g \right).
\]

This completes the proof of (a).
To prove (b) note that, by Lemma \(2.2\)(a), \(S_\beta g = zg - \langle g, \tilde{k}_0^\beta \rangle \beta\). Hence, for \(f \in K_\alpha^\infty\) and \(g \in K_\beta^\infty\), we have

\[
\langle S_\beta^U \varphi_{\alpha, \beta} f, g \rangle = \langle W_k(\varphi f), S_\beta g \rangle = \langle W_k(\varphi f), zg \rangle - \langle W_k(\varphi f), \beta \rangle \cdot \langle g, \tilde{k}_0^\beta \rangle \\
= \langle \varphi f, W_k^* (zg) \rangle - \langle \varphi f, W_k^* \beta \rangle \cdot \langle g, \tilde{k}_0^\beta \rangle \\
= \langle \varphi \bar{z}^k f, W_k^* g \rangle - \left( \langle f, P_\alpha(\varphi \cdot W_k^* \beta) \rangle \tilde{k}_0^\beta, g \right) \\
= \langle \varphi \bar{z}^k f, W_k^* g \rangle - \left( \langle \tilde{k}_0^\beta \otimes P_\alpha(\varphi \cdot W_k^* \beta) \rangle, f \right, g \rangle.
\]

Moreover, by Lemma \(2.2\)(b),

\[
\langle U^\alpha_{\varphi} (S_{\alpha}^k) f, g \rangle = \langle \varphi (S_{\alpha}^k) f, W_k^* g \rangle = \langle \varphi \bar{z}^k f, W_k^* g \rangle - \sum_{j=0}^{k-1} \frac{1}{j!} \langle f, k_{0,j}^\alpha \rangle \cdot \langle \varphi \bar{z}^{k-j}, W_k^* g \rangle \\
= \langle \varphi \bar{z}^k f, W_k^* g \rangle - \sum_{j=0}^{k-1} \frac{1}{j!} \langle f, k_{0,j}^\alpha \rangle \cdot \langle P_\beta W_k (\varphi \bar{z}^{k-j}), g \rangle \\
= \langle \varphi \bar{z}^k f, W_k^* g \rangle - \left( \sum_{j=0}^{k-1} \frac{1}{j!} P_\beta W_k (\varphi \bar{z}^{k-j}) \otimes k_{0,j}^\alpha \right, f \rangle, g \rangle.
\]

and (b) follows. \(\square\)

**Corollary 3.2.** Let \(\alpha\) and \(\beta\) be two nonconstant inner functions and let \(\varphi \in H^2\).

(a) If \(W_k^* \beta \leq \alpha\) and \(U^\alpha_{\varphi} \in S_{k}(\alpha, \beta)\), then

\[
S_\beta U^\alpha_{\varphi} = U^\alpha_{\varphi} S_{\alpha}^k.
\]

(b) If \(\alpha \leq W_k^* \beta\) and \(U^\alpha_{\varphi} \in S_{k}(\alpha, \beta)\), then

\[
S_\beta U^\alpha_{\varphi} = U^\alpha_{\varphi} (S_{\alpha}^k)^k.
\]

**Proof.** Let \(\varphi \in H^2\).

(a) Since \(\overline{\varphi z^k} \in zH^2\), we have \(P_\alpha(\overline{\varphi z^k}) = 0\). If \(W_k^* \beta \leq \alpha\), then by \([31\text{, Lemma } 2.1(\text{g})]\) for each \(0 \leq j \leq k - 1\) we get

\[
P_\beta W_k (\alpha \varphi z^{k-1-j}) = W_k W_k^* (\alpha \varphi z^{k-1-j}) = 0,
\]

since here \(\alpha \varphi z^{k-1-j} \in (W_k^* \beta) \cdot H^2\). Thus (3.3) holds by Proposition 3.1(a).

(b) Here \(\overline{\varphi z^k} \in zH^2\) for each \(0 \leq j \leq k - 1\) and so

\[
P_\beta W_k (\overline{\varphi z^{k-j}}) = P_\beta P W_k (\overline{\varphi z^{k-j}}) = P_\beta W_k P(\overline{\varphi z^{k-j}}) = 0.
\]

Moreover, if \(\alpha \leq W_k^* \beta\), then \(W_k^* \beta \cdot \varphi \in \alpha H^2\) and so \(P_\alpha(W_k^* \beta \cdot \varphi) = 0\). Hence (3.4) follows from Proposition 3.1(b) (with \(\overline{\varphi}\) in place of \(\varphi\)). \(\square\)

Note that for \(k = 1\) the above corollary is Proposition 3.3 from [5].

In what follows we assume that \(\dim K_\alpha \geq k\) (if \(\dim K_\alpha \leq k\), then \(S_{k}(\alpha, \beta)\) contains all bounded linear operators from \(K_\alpha\) into \(K_\beta\)).

Let \(U = U^\alpha_{\varphi} \in S_{k}(\alpha, \beta)\) with \(\varphi \in L^2\). By Proposition 3.1, \(U\) satisfies (3.2) if and only if

\[
\sum_{j=0}^{k-1} \frac{1}{j!} P_\beta W_k (\alpha \varphi z^{k-1-j}) \otimes \tilde{k}_{0,j}^\alpha = k_0^\beta \otimes P_\alpha(\overline{\varphi z^k}).
\]
Since \( \dim K_\alpha \geq k \), \((3.5)\) holds if and only if there exist numbers \( c_0, c_1, \ldots, c_{k-1} \in \mathbb{C} \) such that
\[
\frac{1}{j!} P_\beta W_k(\alpha \varphi z^{k-1-j}) = c_j \cdot k_0^\beta \quad \text{for each } j \in \{0, 1, \ldots, k-1\}
\]
and
\[
P_\alpha(\varphi \psi) = \sum_{j=0}^{k-1} C_j k_0^\alpha.
\]
Note that \((3.7)\) happens if and only if
\[
P_\alpha(\alpha \varphi z^{k-1}) = P_\alpha C_\alpha(\varphi \psi) = C_\alpha P_\alpha(\varphi \psi) = C_\alpha \left( \sum_{j=0}^{k-1} C_j k_0^\alpha \right) = \sum_{j=0}^{k-1} C_j k_0^\alpha,
\]
that is, if and only if
\[
P_\alpha \left( \alpha \varphi z^{k-1} - \sum_{j=0}^{k-1} j! c_j z^j \right) = 0.
\]
This condition can be expressed as
\[
\alpha \varphi z^{k-1} - \sum_{j=0}^{k-1} j! c_j z^j \perp K_\alpha.
\]
Let us now consider \((3.6)\), which can be equivalently expressed as
\[
P_\beta \left( W_k(\alpha \varphi z^{k-1-j}) - j! c_j \right) = 0 \quad \text{for each } j \in \{0, 1, \ldots, k-1\},
\]
i.e.,
\[
W_k(\alpha \varphi z^{k-1-j}) - j! c_j \perp K_\beta \quad \text{for each } j \in \{0, 1, \ldots, k-1\}.
\]
For each \( 0 \leq j \leq k-1 \) denote
\[
\varphi_j = z^j W_k^* W_k(\alpha \varphi z^{k-1-j}).
\]
Observe that
\[
\varphi_j = M_{\psi_j} W_k^* \left( W_k(\alpha \varphi z^{k-1-j}) - j! c_j \right) + j! c_j z^j.
\]
Since \( M_{\psi_j} \) and \( W_k^* \) are isometries, \((3.9)\) is equivalent to
\[
\psi_j := \varphi_j - j! c_j z^j \perp z^j W_k^* K_\beta \quad \text{for each } j \in \{0, 1, \ldots, k-1\}.
\]
Recall that \( W_k^* W_k \) is the orthogonal projection from \( L^2 \) onto the closed linear span of \( \{ z^{km} : m \in \mathbb{Z} \} \). Hence the functions \( \psi_0, \psi_1, \ldots, \psi_{k-1} \) are pairwise orthogonal. The same is true for the subspaces \( W_k^* K_\beta, z W_k^* K_\beta, \ldots, z^{k-1} W_k^* K_\beta \). Moreover,
\[
W_k^* K_\beta \oplus z W_k^* K_\beta \oplus \ldots \oplus z^{k-1} W_k^* K_\beta = K_{W_k^*}
\]
(see \[31\] for details). It follows that \((3.10)\) is equivalent to
\[
\sum_{j=0}^{k-1} \psi_j = \sum_{j=0}^{k-1} \varphi_j - \sum_{j=0}^{k-1} j! c_j z^j \perp K_{W_k^*}.
\]
It is not difficult to verify that for each \( f \in L^2 \)
\[
f = \sum_{j=0}^{k-1} z^j W_k^* W_k(\varphi_j f).
\]
In particular,
\[ \sum_{j=0}^{k-1} \varphi_j = \sum_{j=0}^{k-1} z^j W_k^* W_k (z^j \alpha \varphi z^{k-1}) = \alpha \varphi z^{k-1}, \]
and (3.11) can be expressed as
\[ \alpha \varphi z^{k-1} - \sum_{j=0}^{k-1} j^1 c_j z^j \perp K_{W_k^*}. \]

Summing up, \( U = U_{\varphi}^{\alpha, \beta} \) satisfies (3.2) if and only if there exist \( c_0, c_1, \ldots, c_{k-1} \) such that (3.8) and (3.12) hold. Equivalently,
\[ \alpha \varphi z^{k-1} - \sum_{j=0}^{k-1} j^1 c_j z^j \in \text{span}\{K_\alpha, K_{W_k^*}\} = K_{\text{lcm}(\alpha, W_k^*)}, \]
that is,
\[ \alpha \varphi z^{k-1} - \sum_{j=0}^{k-1} j^1 c_j z^j \in \overline{zH^2 + \text{lcm}(\alpha, W_k^*)H^2}. \]

In other words, \( U_{\varphi}^{\alpha, \beta} \) satisfies (3.2) if and only if
\[
\varphi \in \alpha z^{k-1} K_{z^k} + \alpha z^k H^2 + \alpha z^{k-1} \text{lcm}(\alpha, W_k^*)H^2 = \alpha K_{z^k} + \alpha z^k H^2 + \frac{W_k^* \beta}{\gcd(\alpha, W_k^*)} H^2
= \alpha H^2 + z^{k-1} \frac{W_k^* \beta}{\gcd(\alpha, W_k^*)}, \text{lcm}(\alpha, W_k^*)H^2 + K_{\text{gcd}(\alpha, W_k^*)})
= \alpha H^2 + z^{k-1} (W_k^* \beta)H^2 + \frac{W_k^* \beta}{\gcd(\alpha, W_k^*)} K_{\text{gcd}(\alpha, W_k^*)}
\]
(\( K_{z^k} = \text{span}\{1, \ldots, z^{k-1}\} \)). We have thus proved

**Theorem 3.3.** Let \( \alpha, \beta \) be two nonconstant inner functions and let \( k \leq \dim K_\alpha \). Operator \( U = U_{\varphi}^{\alpha, \beta} \in \mathcal{S}_k(\alpha, \beta) \) satisfies
\[ S_\beta U_{\varphi}^{\alpha, \beta} = U_{\varphi}^{\alpha, \beta} S_\alpha^k \]
if and only if
\[ \varphi \in \alpha H^2 + z^{k-1} (W_k^* \beta)H^2 + \frac{W_k^* \beta}{\gcd(\alpha, W_k^*)} K_{\text{gcd}(\alpha, W_k^*)}. \]

As \( U_{\varphi}^{\alpha, \beta} = 0 \) for \( \varphi \in \alpha H^2 + z^{k-1} (W_k^* \beta)H^2 \) we get

**Theorem 3.4.** Let \( \alpha, \beta \) be two nonconstant inner functions, let \( k \leq \dim K_\alpha \) and let \( U : K_\alpha \to K_\beta \) be a bounded linear operator. Then
\[ S_\beta U = US_\alpha^k \]
if and only if \( U \in \mathcal{S}_k(\alpha, \beta) \) and \( U = U_{\varphi}^{\alpha, \beta} \) with
\[ \varphi \in \frac{z^{k-1} W_k^* \beta}{\gcd(\alpha, W_k^*)} K_{\text{gcd}(\alpha, W_k^*)}. \]

Note that for \( k = 1 \) we obtain Corollary 3.7 and Theorem 3.9 from [5]. Recall that in that case \( S_\beta A = AS_\alpha \) if and only if \( A \in \mathcal{T}(\alpha, \beta) \) and \( A = A_{\varphi}^{\alpha, \beta} \) with \( \varphi \in \frac{W_k^* \beta}{\gcd(\alpha, W_k^*)} K_{\text{gcd}(\alpha, \beta)}. \) In particular, each operator intertwining \( S_\beta \) and \( S_\alpha \) is an asymmetric truncated Toeplitz operator with an analytic symbol. Note that for \( k > 1 \) analicity of the symbol is not guaranteed.
Example 3.5. For any $k > 1$ let $a \in \mathbb{D} \setminus \{0\}$. Put $\alpha(z) = z^{2k}$ and $\beta(z) = z^2$. Then

$$W^*_k \beta = z^{2k} = \alpha = \gcd(\alpha, W^*_k \beta)$$

and

$$\alpha H^2 + \bar{z}^{-1} (W^*_k \beta) H^2 + \bar{z}^{-1} \frac{W^*_k \beta}{\gcd(\alpha, W^*_k \beta)} K_{\gcd(\alpha, W^*_k \beta)} = \bar{z}^{2k} H^2 + z^{k+1} H^2 + \bar{z}^{-1} K_{z^{2k}}.$$  

Hence $S_\beta U^{\alpha, \beta}_\varphi = U^{\alpha, \beta}_{\varphi} S^k_\alpha$ if for example $\varphi(z) = \bar{z}^{-k} - 1$.

Corollary 3.6. Let $\alpha, \beta$ be two nonconstant inner functions, $k \leq \dim K_\beta$ and let $U : K_\alpha \to K_\beta$ be a bounded linear operator. Then

$$(S^*_\beta)^k U = US^*_\alpha$$

if and only if $U^* \in S_k(\beta, \alpha)$ and $U = (U^{\beta, \alpha}_\varphi)^*$ with

$$\varphi \in \mathbb{C}^{k-1} \frac{W^*_\alpha}{\gcd(\beta, W^*_\alpha)} K_{\gcd(\beta, W^*_\alpha)}.$$  

Theorem 3.7. Let $\alpha, \beta$ be two nonconstant inner functions and let $k \leq \dim K_\alpha$. Operator $U = U^{\alpha, \beta}_\varphi \in S_k(\alpha, \beta)$ satisfies

$$S^*_\beta U^{\alpha, \beta}_\varphi = U^{\alpha, \beta}_\varphi (S^*_\alpha)^k$$

if and only if

$$\varphi \in \alpha H^2 + \bar{z}^{-1} (W^*_k \beta) H^2 + \left( \frac{\alpha}{\gcd(\alpha, W^*_k \beta)} K_{\gcd(\alpha, W^*_k \beta)} \right).$$

Proof. Let $U = U^{\alpha, \beta}_\varphi \in S_k(\alpha, \beta)$, where $\varphi \in L^2$. By Proposition 3.1, $U$ satisfies (3.13) if and only if

$$\sum_{j=0}^{k-1} \frac{1}{j} P_{k} W_k (\varphi \bar{z}^{k-j}) \otimes k^0_{0,j} = \tilde{K}^\beta_0 \otimes P_\alpha (\bar{\varphi} \cdot W^*_k \beta).$$

Since $\dim K_\alpha \geq k$, (3.14) holds if and only if there exist numbers $c_0, c_1, \ldots, c_{k-1} \in \mathbb{C}$ such that

$$\frac{1}{j} P_{k} W_k (\varphi \bar{z}^{k-j}) = c_j \cdot \tilde{K}^\beta_0$$

for each $j \in \{0, 1, \ldots, k-1\}$

and

$$P_\alpha (\bar{\varphi} \cdot W^*_k \beta) = \sum_{j=0}^{k-1} \tau_j k^\alpha_{0,j}.$$  

Equality (3.16) happens if and only if

$$\bar{\varphi} \cdot W^*_k \beta - \sum_{j=0}^{k-1} j! c_j z^j \perp K_\alpha.$$  

Moreover, (3.15) holds if and only if

$$W_k (\varphi \bar{z}^{k-j}) - j! c_j \beta \bar{z} \perp K_\beta$$

for each $j \in \{0, 1, \ldots, k-1\}$,

which is equivalent to

$$W_k W_k (\varphi \bar{z}^{k-j}) - j! c_j (W^*_k \beta) \bar{z}^k \perp W^*_k K_\beta$$

for each $j \in \{0, 1, \ldots, k-1\}$,

or

$$z^{k-j} W_k W_k (\varphi \bar{z}^{k-j}) - j! c_j (W^*_k \beta) \bar{z}^j \perp z^{k-j} W^*_k K_\beta$$

for each $j \in \{0, 1, \ldots, k-1\}$.
It follows that
\[ \varphi - (W_k^* \beta) \sum_{j=0}^{k-1} j! c_j z^j \perp zK_{W_k^*} \beta \]
and
\[ (W_k^* \beta) \varphi - \sum_{j=0}^{k-1} j! c_j z^j \perp (W_k^* \beta)zK_{W_k^*} = K_{W_k^*}, \]
which can also be expressed as
\[ (3.18) \]
\[ (W_k^* \beta) \varphi - \sum_{j=0}^{k-1} j! c_j z^j \perp K_{W_k^*}. \]
Summing up, \( U = U_{\varphi, \beta}^\alpha \) satisfies (3.13) if and only if there exist \( c_0, c_1, \ldots, c_{k-1} \) such that (3.17) and (3.18) hold, that is,
\[ (W_k^* \beta) \varphi - \sum_{j=0}^{k-1} j! c_j z^j \perp \text{span}\{K_\alpha, K_{W_k^*}\} = K_{\text{lcm}(\alpha, W_k^*)}. \]
Equivalently,
\[ (W_k^* \beta) \varphi \in K_{\cdot k} + \overline{zH^2 + \text{lcm}(\alpha, W_k^*)H^2}. \]
Thus, \( U_{\varphi, \beta}^\alpha \) satisfies (3.13) if and only if
\[ \varphi \in (W_k^* \beta)K_{\cdot k} + (W_k^* \beta)zH^2 + W_k^* \beta \cdot \text{lcm}(\alpha, W_k^*)H^2 \]
\[ = \overline{z^{k-1}(W_k^* \beta)K_{\cdot k} + z^{k-1}(W_k^* \beta)z^kH^2 + \frac{\alpha}{\text{lcm}(\alpha, W_k^*)H^2}} \]
\[ = \overline{z^{k-1}(W_k^* \beta)H^2 + \frac{\alpha}{\text{gcd}(\alpha, W_k^*)} \left( \text{gcd}(\alpha, W_k^*)H^2 + K_{\text{gcd}(\alpha, W_k^*)} \right)} \]
and so
\[ \varphi \in \overline{\alpha H^2 + z^{k-1}(W_k^* \beta)H^2 + \frac{\alpha}{\text{gcd}(\alpha, W_k^*)} \left( K_{\text{gcd}(\alpha, W_k^*)} \right)}, \]
which completes the proof.

Since \( U_{\varphi}^\alpha, \beta = 0 \) for \( \varphi \in \overline{\alpha H^2 + z^{k-1}(W_k^* \beta)H^2} \) and
\[ \left( \frac{\alpha}{\text{gcd}(\alpha, W_k^*)} \right) \left( K_{\text{gcd}(\alpha, W_k^*)} \right) = \overline{\alpha} \cdot \text{gcd}(\alpha, W_k^*) \left( K_{\text{gcd}(\alpha, W_k^*)} \right) = \overline{\alpha zK_{\text{gcd}(\alpha, W_k^*)}} \]
we get

**Theorem 3.8.** Let \( \alpha, \beta \) be two nonconstant inner functions, let \( k \leq \dim K_\alpha \) and let \( U : K_\alpha \to K_\beta \) be a bounded linear operator. Then
\[ S_{\beta}^k U = U(S_{\alpha}^*)^k \]
if and only if \( U \in S_k(\alpha, \beta) \) and \( U = U_{\varphi}^\alpha, \beta \) with
\[ \varphi \in \overline{\alpha zK_{\text{gcd}(\alpha, W_k^*)}}. \]

**Corollary 3.9.** Let \( \alpha, \beta \) be two nonconstant inner functions, let \( k \leq \dim K_\beta \) and let \( U : K_\alpha \to K_\beta \) be a bounded linear operator. Then
\[ S_{\beta}^k U = US_{\alpha} \]
if and only if \( U^* \in S_k(\beta, \alpha) \) and \( U = (U_{\varphi}^\beta, \alpha)^* \) with
\[ \varphi \in \overline{\beta zK_{\text{gcd}(\beta, W_k^*)}}. \]
4. Products of operators from $S_k(\alpha, \beta)$ with analytic or anti-analytic symbols

In this section we consider products of operators with analytic or anti-analytic symbols, which belong to $S_k(\alpha, \beta)$. We start with some auxiliary results.

Recall that for a nonconstant inner function $\alpha$ we have (see [34]):

\begin{equation}
I_{K_\alpha} - S_\alpha S_\alpha^* = k_0^\alpha \otimes k_0^\alpha
\end{equation}

and

\begin{equation}
I_{K_\alpha} - S_\alpha^* S_\alpha = \tilde{k}_0^\alpha \otimes \tilde{k}_0^\alpha.
\end{equation}

In what follows we will use the following lemma.

**Lemma 4.1.** Let $\alpha$ be a nonconstant inner function and let $m \in \mathbb{N}$. Then

\begin{equation}
I_{K_\alpha} - S_\alpha^m (S_\alpha^*)^m = \sum_{j=0}^{m-1} P_\alpha(z^j) \otimes P_\alpha(z^j) = \sum_{j=0}^{m-1} \left(\frac{1}{j!}\right)^2 (k_0^\alpha \otimes k_0^\alpha),
\end{equation}

\begin{equation}
I_{K_\alpha} - (S_\alpha^m) S_\alpha^m = \sum_{j=0}^{m-1} C_\alpha P_\alpha(z^j) \otimes C_\alpha P_\alpha(z^j) = \sum_{j=0}^{m-1} \left(\frac{1}{j!}\right)^2 (\tilde{k}_0^\alpha \otimes \tilde{k}_0^\alpha).
\end{equation}

**Proof.** To prove (4.3) observe that from (4.1) we have

\begin{equation*}
S_\alpha^j (S_\alpha^*)^j - S_\alpha^j (S_\alpha^*)^{j+1} = (S_\alpha^j k_0^\alpha) \otimes (S_\alpha^j k_0^\alpha)
\end{equation*}

for any nonnegative integer $j$. Adding the above equalities for $j = 0, 1, \ldots, m - 1$ we get

\begin{equation*}
I_{K_\alpha} - S_\alpha^m (S_\alpha^*)^m = \sum_{j=0}^{m-1} (S_\alpha^j k_0^\alpha) \otimes (S_\alpha^j k_0^\alpha),
\end{equation*}

where $S_\alpha^0 k_0^\alpha = k_0^\alpha$. Moreover, for any positive integer $j$,

\begin{equation}
S_\alpha^j k_0^\alpha = A_\alpha^j k_0^\alpha = P_\alpha(z^j - \overline{\alpha(0)}\alpha z^j) = P_\alpha(z^j) = \frac{1}{j!} P_\alpha(j! z^j) = \frac{1}{j!} k_0^\alpha
\end{equation}

and thus (4.3) holds. To prove (4.4) we apply $C_\alpha$ to (4.3) and use $C_\alpha$-symmetry of $S_\alpha$ ($C_\alpha S_\alpha C_\alpha = S_\alpha^*$, see [17]).

Let $k$ and $m$ be two arbitrary fixed positive integers.

**Proposition 4.2.** Let $\alpha, \beta$ and $\gamma$ be nonconstant inner functions and let $\varphi, \psi \in H^2$.

(a) Assume that $U^\beta_\varphi \gamma \in S_m(\beta, \gamma)$ and $U^\alpha_\psi \beta \in S_k(\alpha, \beta)$. If $W^*_{m, \gamma} \leq \beta$ and $W_k^* \beta \leq \alpha$, then $U = U^\beta_\varphi U^\alpha_\psi \beta \in S_{km}(\alpha, \gamma)$ and $U = U^\alpha_\psi \gamma \beta \in S_{km}(\alpha, \gamma)$ with

\[ \eta = \sum_{j=0}^{km-1} \frac{1}{j!} (W^*_{km, \gamma} k_0^\alpha) z^j. \]

(b) Assume that $U^\beta_\varphi \gamma \in S_m(\beta, \gamma)$ and $U^\alpha_\psi \beta \in S_k(\alpha, \beta)$. If $\alpha \leq W_k^* \beta$ and $\beta \leq W^*_{m, \gamma}$, then $U = U^\beta_\varphi U^\alpha_\psi \beta \in S_{km}(\alpha, \gamma)$ and $U = U^\alpha_\psi \gamma \beta \in S_{km}(\alpha, \gamma)$ with

\[ \zeta = \sum_{j=0}^{km-1} \frac{1}{j!} (W^*_{km, \gamma} \tilde{k}_0^\alpha) z^{j+1}. \]
Before giving a proof of Proposition 4.2, note that if a bounded linear operator $U : K_\alpha \to K_\beta$ satisfies (2.2) with some $\chi \in K_\alpha$ and $\psi_0, \ldots, \psi_{k-1} \in K_\beta$, then $U = U^{\alpha, \beta}_\varphi$ with

\[(4.6) \quad \varphi = (W^*_k \beta)z^d \chi + \alpha \sum_{j=0}^{k-1} (W^*_k \psi_j) j! z^{j+1}\]

(Theorem 1). Similarly, operators from $S_k(\alpha, \beta)$ can be characterized as follows (Corollary 7): $U \in S_k(\alpha, \beta)$ if and only if there exist functions $\chi \in K_\alpha$ and $\psi_0, \ldots, \psi_{k-1} \in K_\beta$ such that

\[(4.7) \quad U - S_\beta U(S_\alpha^*)^k = k_0^\beta \otimes \chi + \sum_{j=0}^{k-1} \psi_j \otimes k_0^\alpha, j.
\]

In that case, $U = U_{\alpha, \beta}^{\varphi}$ with

\[(4.8) \quad \varphi = \chi + \sum_{j=0}^{k-1} (W^*_k \psi_j) j! z^j\]

(Corollary 4).

**Proof of Proposition 4.2.** (a) Let $U = U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi$. By Corollary 3.2(a) and formula (4.3) (Lemma 4.1), we have

\[
U - S_\gamma U(S_\alpha^*)^{km} = U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi - S_\gamma U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi (S_\alpha^*)^{km} = U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi - U^{\beta, \gamma}_\varphi S_\beta^m U^{\alpha, \beta}_\psi (S_\alpha^*)^{km}
\]

\[
= U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi - U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi S_\alpha^{km} (S_\alpha^*)^{km} = U(I_{K_\alpha} - S_\alpha^{km} (S_\alpha^*)^{km})
\]

\[
= U \left( \sum_{j=0}^{km-1} \left( \frac{1}{j!} \right)^2 (k_{0,j}^\beta \otimes k_{0,j}^\alpha) \right) = \sum_{j=0}^{km-1} \left( \frac{1}{j!} \right)^2 (U k_{0,j}^\beta \otimes k_{0,j}^\alpha).
\]

Therefore $U = U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi = U^{\alpha, \gamma}_\eta \in S_{km}(\alpha, \gamma)$ (see (4.7)) where, by (4.8),

\[
\eta = \sum_{j=0}^{km-1} \frac{1}{j!} (W_{km} U k_{0,j}^\alpha) z^j.
\]

(b) Let $U = U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi$. By Corollary 3.2(b) and formula (4.4),

\[
U - S_\gamma^* U S_\alpha^{km} = U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi - S_\gamma^* U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi S_\alpha^{km} = U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi - U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi (S_\alpha^*)^{km} S_\alpha^{km}
\]

\[
= U(I - (S_\alpha^*)^{km} S_\alpha^{km}) = \sum_{j=0}^{km-1} \left( \frac{1}{j!} \right)^2 (U k_{0,j}^\alpha \otimes \tilde{k}_{0,j}^\alpha).
\]

Thus $U^{\beta, \gamma}_\varphi U^{\alpha, \beta}_\psi = U^{\alpha, \gamma}_\zeta \in S_{km}(\alpha, \gamma)$ (see (2.2)) where, by (4.6),

\[
\zeta = \bar{\pi} \sum_{j=0}^{km-1} \frac{1}{j!} (W^*_k U \tilde{k}_{0,j}^\alpha) z^{j+1}.
\]

\[\square\]

**Corollary 4.3.** Let $\alpha$, $\beta$ and $\gamma$ be nonconstant inner functions and let $\varphi, \psi \in H^2$.
(a) Assume that \( U_{\varphi}^{\beta, \gamma} \in \mathcal{S}_m(\beta, \gamma) \) and \( U_{\psi}^{\alpha, \beta} \in \mathcal{S}_k(\alpha, \beta) \). If \( W_m^* \gamma \leq \beta \) and \( W_k^* \beta \leq \alpha \), then \( U = U_{\varphi}^{\beta, \gamma} U_{\psi}^{\alpha, \beta} \in \mathcal{S}_{km}(\alpha, \gamma) \) and \( U = U_{\eta}^{\alpha, \gamma} \) with
\[
\eta = \sum_{j=0}^{km-1} W_{km}^* W_{km} P_{W_{km}^* \gamma}(W_k^* \varphi) P_{W_k^* \beta}(\psi z^j) \cdot \bar{\zeta}^j.
\]

(b) Assume that \( U_{\varphi}^{\beta, \gamma} \in \mathcal{S}_m(\beta, \gamma) \) and \( U_{\psi}^{\alpha, \beta} \in \mathcal{S}_k(\alpha, \beta) \). If \( \alpha \leq W_k^* \beta \) and \( \beta \leq W_m^* \gamma \), then \( U = U_{\varphi}^{\beta, \gamma} U_{\psi}^{\alpha, \beta} \in \mathcal{S}_{km}(\alpha, \gamma) \) and \( U = U_{\zeta}^{\alpha, \gamma} \) with
\[
\zeta = \alpha \sum_{j=0}^{km-1} W_{km}^* W_{km} P_{W_{km}^* \gamma}(\overline{W_k^* \varphi} \psi P_{\alpha}(\alpha \bar{\zeta}^{j+1})) \cdot \bar{z}^{j+1}.
\]

**Proof.** (a) Let \( U = U_{\varphi}^{\beta, \gamma} U_{\psi}^{\alpha, \beta} \), where \( \varphi, \psi \in H^2 \), \( U_{\varphi}^{\beta, \gamma} \in \mathcal{S}_m(\beta, \gamma) \) and \( U_{\psi}^{\alpha, \beta} \in \mathcal{S}_k(\alpha, \beta) \). Then for each \( f \in K_{\gamma}^* \),

\[
\langle \frac{1}{j!} U_{k_{0,j}}^\alpha, f \rangle = \langle U_{\varphi}^{\beta, \gamma} U_{\psi}^{\alpha, \beta} P_{\alpha}(z^j), f \rangle = \langle U_{\varphi}^{\beta, \gamma} P_{\alpha}(z^j), (U_{\varphi}^{\beta, \gamma})^* f \rangle
\]
\[
= \langle P_{W_k}(\psi P_{\alpha}(z^j)), P_{\beta}(\overline{W_m^* f}) \rangle = \langle \psi P_{\alpha}(z^j), W_k^* P_{\beta}(\overline{W_m^* f}) \rangle
\]
\[
= \langle P_{\alpha}(z^j), \overline{\psi P_{W_k^* \beta}(\overline{W_m^* f})} \rangle = \langle z^j, P_{\alpha}(\overline{\psi P_{W_k^* \beta}(\overline{W_m^* f})}) \rangle.
\]

Recall that \( W_k^* \beta \leq \alpha \), which implies that \( P_{W_k^* \beta}(\overline{W_m^* f}) \in K_{\alpha} \). Since \( \psi \in H^2 \), we see that \( P_{\alpha}(\overline{\psi P_{W_k^* \beta}(\overline{W_m^* f})}) \) is orthogonal to \( \alpha H^2 \) and so

\[
P_{\alpha}(\overline{\psi P_{W_k^* \beta}(\overline{W_m^* f})}) = P_{\alpha}(\overline{\psi P_{W_k^* \beta}(\overline{W_m^* f})} W_k^* W_m^* f) = P_{\alpha}(\overline{\psi P_{W_k^* \beta}(\overline{W_m^* f})} W_k^* W_m^* f).
\]

Hence, using the fact that \( W_k^* W_m^* = W_{km}^* \), we obtain

\[
\langle \frac{1}{j!} U_{k_{0,j}}^\alpha, f \rangle = \langle z^j, P_{\alpha}(\overline{\psi P_{W_k^* \beta}(\overline{W_m^* f})} W_{km}^* f) \rangle = \langle \psi z^j, W_{km}^* P_{\alpha}(\overline{\psi P_{W_k^* \beta}(\overline{W_m^* f})} W_{km}^* f) \rangle
\]
\[
= \langle W_k^* \varphi P_{\alpha}(z^j), W_{km}^* f \rangle = \langle P_{\gamma} W_{km}(W_k^* \varphi) P_{W_k^* \beta}(\psi z^j), f \rangle.
\]

Thus

\[
\frac{1}{j!} W_{km}^* U_{k_{0,j}}^\alpha = W_{km}^* P_{\gamma} W_{km}(W_k^* \varphi) P_{W_k^* \beta}(\psi z^j)
\]
\[
= W_{km}^* W_{km} P_{W_{km}^* \gamma}(W_m^* \varphi) P_{W_k^* \beta}(\psi z^j).
\]

By Proposition 1.2(a), \( U = U_{\eta}^{\alpha, \gamma} \in \mathcal{S}_{km}(\alpha, \gamma) \) with
\[
\eta = \sum_{j=0}^{km-1} W_{km}^* W_{km} P_{W_{km}^* \gamma}(W_m^* \varphi) P_{W_k^* \beta}(\psi z^j) \cdot \bar{\zeta}^j.
\]

(b) Let \( U = U_{\varphi}^{\beta, \gamma} U_{\psi}^{\alpha, \beta} \), where \( \varphi, \psi \in H^2 \), \( U_{\varphi}^{\beta, \gamma} \in \mathcal{S}_m(\beta, \gamma) \) and \( U_{\psi}^{\alpha, \beta} \in \mathcal{S}_k(\alpha, \beta) \). Then for each \( f \in K_{\gamma}^* \),

\[
\langle \frac{1}{j!} U_{k_{0,j}}^\alpha, f \rangle = \langle U_{\varphi}^{\beta, \gamma} U_{\psi}^{\alpha, \beta} C_{\alpha} P_{\alpha}(z^j), f \rangle = \langle U_{\varphi}^{\beta, \gamma} C_{\alpha} P_{\alpha}(z^j), (U_{\varphi}^{\beta, \gamma})^* f \rangle
\]
\[
= \langle P_{W_k}(\psi C_{\alpha} P_{\alpha}(z^j)), P_{\beta}(\varphi W_m^* f) \rangle = \langle \psi C_{\alpha} P_{\alpha}(z^j), W_k^* P_{\beta}(\varphi W_m^* f) \rangle
\]
\[
= \langle \psi C_{\alpha} P_{\alpha}(z^j), P_{W_k^* \beta}(\varphi W_m^* f) \rangle = \langle P_{W_k^* \beta}(\psi C_{\alpha} P_{\alpha}(z^j)), W_k^* W_m^* f \rangle.
\]
Here \( \alpha \leq W_k^* \beta \), which implies that
\[
P_{W_k^* \beta}(\overline{\psi C_{\alpha} P_{\alpha} z^j}) = P(\overline{\psi C_{\alpha} P_{\alpha} z^j}),
\]
and so
\[
\langle \frac{1}{j!} \overline{U_{\alpha}^k_{0,j}} f \rangle = \langle P(\overline{\psi C_{\alpha} P_{\alpha} z^j}), (W_k^* \varphi)W_k^*W_m^* f \rangle = \langle (W_k^* \varphi)\overline{\psi C_{\alpha} P_{\alpha} z^j}, W_m^* f \rangle = \langle P_{U_m} (\overline{W_k^* \varphi})\overline{P_{\alpha} C_{\alpha} z^j}, f \rangle.
\]
Thus
\[
\frac{1}{j!} W_k^* U_{\alpha}^k_{0,j} = W_k^* P_{U_m} (\overline{W_k^* \varphi})\overline{P_{\alpha} C_{\alpha} z^j}) = W_k^* W_m^* P_{U_m} W_{m^\gamma} ^* (\overline{W_k^* \varphi})\overline{P_{\alpha} (\alpha z^{j+1})}.
\]
By Proposition 4.3 b), \( U = U_{\varphi}^\alpha \gamma \in S_{km}(\alpha, \gamma) \) with
\[
\zeta = \frac{\alpha}{\alpha} \sum_{j=0}^{km-1} W_k^* W_m^* P_{U_m} W_{m^\gamma} ^* (\overline{W_k^* \varphi})\overline{P_{\alpha} (\alpha z^{j+1})} \cdot z^{j+1}.
\]

\[\square\]

Corollary 4.4. Let \( \alpha, \beta \) and \( \gamma \) be nonconstant inner functions and let \( \varphi, \psi \in H^2 \).
(a) Assume that \( A_{\varphi}^{\beta, \gamma} \in T(\beta, \gamma) \) and \( U_{\psi}^{\alpha, \beta} \in S_k(\alpha, \beta) \). If \( \gamma \leq \beta \) and \( W_k^* \beta \leq \alpha \), then
\[
U = A_{\varphi}^{\beta, \gamma} U_{\psi}^{\alpha, \beta} = U_{\varphi}^{\alpha, \gamma} \in S_k(\alpha, \gamma) \text{ with } \eta = \sum_{j=0}^{k-1} W_k^* W_m^* P_{U_m} W_{m^\gamma} ^* (\overline{W_k^* \varphi})\overline{P_{\alpha} (\alpha z^{j+1})} \cdot z^{j+1}.
\]
(b) Assume that \( U_{\varphi}^{\beta, \gamma} \in S_m(\beta, \gamma) \) and \( A_{\varphi}^{\alpha, \beta} \in T(\alpha, \beta) \). If \( W_m^* \gamma \leq \beta \leq \alpha \), then \( U = U_{\varphi}^{\beta, \gamma} A_{\varphi}^{\alpha, \beta} = U_{\varphi}^{\alpha, \gamma} \in S_m(\alpha, \gamma) \text{ with } \eta = \sum_{j=0}^{m-1} W_m^* W_m^* P_{U_m} W_{m^\gamma} ^* (\overline{W_m^* \varphi})\overline{P_{\alpha} (\alpha z^{j+1})} \cdot z^{j+1}.
\]
(c) Assume that \( A_{\varphi}^{\beta, \gamma} \in T(\beta, \gamma) \) and \( U_{\varphi}^{\alpha, \beta} \in S_k(\alpha, \beta) \). If \( \alpha \leq W_k^* \beta \) and \( \beta \leq \gamma \), then
\[
U = A_{\varphi}^{\beta, \gamma} U_{\varphi}^{\alpha, \beta} = U_{\varphi}^{\alpha, \gamma} \in S_k(\alpha, \gamma) \text{ with } \zeta = \alpha \sum_{j=0}^{k-1} W_k^* W_k^* P_{U_m} W_{m^\gamma} ^* (\overline{W_k^* \varphi})\overline{P_{\alpha} (\alpha z^{j+1})} \cdot z^{j+1}.
\]
(d) Assume that \( U_{\varphi}^{\beta, \gamma} \in S_m(\beta, \gamma) \) and \( A_{\varphi}^{\alpha, \beta} \in T(\alpha, \beta) \). If \( \alpha \leq \beta \leq W_m^* \gamma \), then \( U = U_{\varphi}^{\beta, \gamma} A_{\varphi}^{\alpha, \beta} = U_{\varphi}^{\alpha, \gamma} \in S_m(\alpha, \gamma) \text{ with } \zeta = \alpha \sum_{j=0}^{m-1} W_m^* W_m^* P_{U_m} W_{m^\gamma} ^* (\overline{W_m^* \varphi})\overline{P_{\alpha} (\alpha z^{j+1})} \cdot z^{j+1}.
\]
Note that under the assumptions from Proposition 4.2 a) for \( U = U_{\varphi}^{\gamma} U_{\psi}^{\alpha, \beta} \) we also have
\[
U - S_{\eta}^* U S_{\alpha}^{km} = U_{\varphi}^{\beta, \gamma} U_{\varphi}^{\alpha, \beta} - S_{\varphi}^* U_{\varphi}^{\beta, \gamma} U_{\psi}^{\alpha, \beta} S_{\alpha}^{km} = (I - S_{\varphi}^* S_{\psi}) U_{\varphi}^{\beta, \gamma} U_{\psi}^{\alpha, \beta} = (\overline{K_0} \otimes \overline{K_0}) U
\]
which implies that \( U = U_{\varphi}^{\beta, \gamma} U_{\psi}^{\alpha, \beta} = U_{\eta}^{\alpha, \gamma} \in S_{km}(\alpha, \gamma) \text{ with } \eta = (W_{km}^* \gamma) z^{km} U_{\varphi}^{\alpha, \gamma} \).

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Now for \( f \in K_0^\infty \) we have
\[
\langle U^*k_0^\gamma, f \rangle = \langle (U^\alpha)^*(U^\beta)^*k_0^\gamma, f \rangle = \langle (U^\alpha)^*k_0^\gamma, U^\alpha f \rangle
\]
\[
= \langle P_\beta(\varphi W_m^\star k_0^\gamma), P_\beta W_k(\psi f) \rangle = \langle W_k P_\beta(\varphi W_m^\star k_0^\gamma), \psi f \rangle
\]
\[
= \langle \psi P W_k^\star \beta W_k(\varphi W_m^\star k_0^\gamma), f \rangle = \langle P_\alpha(\psi P W_k^\star \beta((W_k^\star \varphi), W_{km}^\star k_0^\gamma), f \rangle
\]
\[
= \langle P_\alpha(\psi P W_k^\star \beta((W_k^\star \varphi) \cdot \bar{z}^{km} W_{km}^\star k_0^\gamma)), f \rangle = \langle P(\psi P W_k^\star \beta((W_k^\star \varphi) \cdot \bar{z}^{km} W_{km}^\star k_0^\gamma)), f \rangle.
\]

The last equality follows from the fact that \( \psi \in H^2 \) and \( W_k^\star \beta \leq \alpha \). Thus
\[
\eta = (W_k^\star \beta)\bar{z}^{km} P(\psi P W_k^\star \beta((W_k^\star \varphi) \cdot \bar{z}^{km} W_{km}^\star k_0^\gamma)).
\]

Analogously, under assumptions from part (b) of Proposition 4.2 for \( U = U^\beta^\varphi U^\alpha^\psi \) we have
\[
U - S_\gamma U(S_{\alpha}^* k_0^\beta = k_0^\gamma \otimes U^* k_0^\gamma
\]
and so \( U = U^\alpha^\gamma \in S_{km}(\alpha, \gamma) \) with
\[
\zeta = \overline{U^* k_0^\gamma}.
\]

Here for \( f \in K_0^\infty \) we have
\[
\langle U^* k_0^\gamma, f \rangle = \langle (U^\alpha)^*(U^\beta)^*k_0^\gamma, f \rangle = \langle (U^\alpha)^*k_0^\gamma, U^\alpha f \rangle
\]
\[
= \langle P_\beta(\varphi W_m^\star k_0^\gamma), P_\beta W_k(\psi f) \rangle = \langle \varphi W_m^\star k_0^\gamma, W_k P_\beta(\psi f) \rangle
\]
\[
= \langle W_k^\star(\varphi W_m^\star k_0^\gamma), P_\beta(\psi f) \rangle = \langle (W_k^\star \varphi) W_{km}^\star k_0^\gamma, P(\psi f) \rangle
\]
\[
= \langle \psi W_k^\star(\varphi) W_{km}^\star k_0^\gamma, f \rangle = \langle P_\alpha(\psi(\varphi) W_k^\star(\varphi)(1 - \gamma(0)W_{km}^\star k_0^\gamma), f \rangle
\]
\[
= \langle P_\alpha(\psi(\varphi)), f \rangle.
\]

This time we used the fact that \( \alpha \leq W_k^\star \beta \) (line 3) and \( \alpha \leq W_k^\star \beta \leq W_{km}^\star k_0^\gamma \) (the last equality). Thus
\[
\zeta = \overline{P_\alpha(\psi(\varphi))}.
\]

**Corollary 4.5.** Let \( \alpha, \beta \) and \( \gamma \) be nonconstant inner functions and let \( \varphi, \psi \in H^2 \).

(a) Assume that \( U^\alpha^\beta \in S_m(\beta, \gamma) \) and \( U^\alpha^\beta \in S_k(\alpha, \beta) \). If \( W_k^\star \gamma \leq \beta \) and \( W_k^\star \beta \leq \alpha \), then \( U = U^\beta^\gamma U^\alpha^\beta \in S_{km}(\alpha, \gamma) \) and \( U = U^\alpha^\gamma \) with
\[
\eta = (W_k^\star \beta)\bar{z}^{km} P(\psi P W_k^\star \beta((W_k^\star \varphi) \cdot \bar{z}^{km} W_{km}^\star k_0^\gamma)).
\]

(b) Assume that \( U^\beta^\gamma \in S_m(\beta, \gamma) \) and \( U^\alpha^\beta \in S_k(\alpha, \beta) \). If \( \alpha \leq W_k^\star \beta \) and \( \beta \leq W_m^\star \gamma \), then \( U = U^\beta^\gamma U^\alpha^\beta \in S_{km}(\alpha, \gamma) \) and \( U = U^\alpha^\gamma \) with
\[
\zeta = \overline{P_\alpha(\psi(\varphi))}.
\]

**Corollary 4.6.** Let \( \alpha, \beta \) and \( \gamma \) be nonconstant inner functions and let \( \varphi, \psi \in H^2 \).

(a) Assume that \( A^\beta^\gamma \in T(\beta, \gamma) \) and \( U^\alpha^\beta \in S_k(\alpha, \beta) \). If \( \gamma \leq \beta \) and \( W_k^\star \beta \leq \alpha \), then \( U = A^\beta^\gamma U^\alpha^\beta = U^\alpha^\gamma \in S_k(\alpha, \gamma) \) with \( \eta = (W_k^\star \gamma)\bar{z}^{km} P(\psi P W_k^\star \beta((W_k^\star \varphi) \cdot \bar{z}^{km} W_k^\star k_0^\gamma)).
\]

(b) Assume that \( A^\beta^\gamma \in S_m(\beta, \gamma) \) and \( U^\alpha^\beta \in T(\alpha, \beta) \). If \( W_m^\star \gamma \leq \beta \leq \alpha \), then \( U = A^\beta^\gamma U^\alpha^\beta = U^\gamma^\alpha \in S_m(\alpha, \gamma) \) with \( \eta = (W_m^\star \gamma)\bar{z}^{km} P(\psi P W_k^\star \beta((W_k^\star \varphi) \cdot \bar{z}^{km} W_{km}^\star k_0^\gamma)).
\]
(c) Assume that $A^{\beta,\gamma}_{\nu} \in \mathcal{T}(\beta, \gamma)$ and $U^{\alpha,\beta}_{\nu} \in S_{k}(\alpha, \beta)$. If $\alpha \leq W^*_k \beta$ and $\beta \leq \gamma$, then
\[ U = A^{\beta,\gamma}_{\nu} U^{\alpha,\beta}_{\nu} = U^{\alpha,\gamma}_{\zeta} \in S_{k}(\alpha, \gamma) \] with $\zeta = P_\alpha(\psi(W^*_k \varphi))$.

(d) Assume that $U^{\beta,\gamma}_{\nu} \in S_m(\beta, \gamma)$ and $A^{\alpha,\beta}_{\psi} \in \mathcal{T}(\alpha, \beta)$. If $\alpha \leq \beta \leq W^*_m \gamma$, then
\[ U = U^{\beta,\gamma}_{\nu} U^{\alpha,\beta}_{\psi} = U^{\alpha,\gamma}_{\zeta} \in S_m(\alpha, \gamma) \] with $\zeta = P_\alpha(\psi \varphi)$.

Finally, we can consider the case when $k = m = 1$ (see [37, Proposition 1]).

**Corollary 4.7.** Let $\alpha$, $\beta$ and $\gamma$ be nonconstant inner functions and let $\varphi, \psi \in H^2$.

(a) Assume that $A^{\beta,\gamma}_{\nu} \in \mathcal{T}(\beta, \gamma)$ and $A^{\alpha,\beta}_{\psi} \in \mathcal{T}(\alpha, \beta)$. If $\gamma \leq \beta \leq \alpha$, then $A = A^{\beta,\gamma}_{\nu} A^{\alpha,\beta}_{\psi} = A^{\alpha,\gamma}_{\eta} \in \mathcal{T}(\alpha, \gamma)$ with $\eta = P_\gamma(\varphi P_\beta(\psi))$.

(b) Assume that $A^{\beta,\gamma}_{\nu} \in \mathcal{T}(\beta, \gamma)$ and $A^{\alpha,\beta}_{\psi} \in \mathcal{T}(\alpha, \beta)$. If $\gamma \leq \beta \leq \alpha$, then $A = A^{\beta,\gamma}_{\nu} A^{\alpha,\beta}_{\psi} = A^{\alpha,\gamma}_{\eta} \in \mathcal{T}(\alpha, \gamma)$ with $\eta = \gamma z P_\beta(\psi P_\alpha(\varphi))$.

(c) Assume that $A^{\beta,\gamma}_{\nu} \in \mathcal{T}(\beta, \gamma)$ and $A^{\alpha,\beta}_{\psi} \in \mathcal{T}(\alpha, \beta)$. If $\alpha \leq \beta \leq \gamma$, then $A = A^{\beta,\gamma}_{\nu} A^{\alpha,\beta}_{\psi} = A^{\alpha,\gamma}_{\zeta} \in \mathcal{T}(\alpha, \gamma)$ with $\zeta = \alpha z P_\beta(\psi P_\alpha(\varphi))$.

(d) Assume that $A^{\beta,\gamma}_{\nu} \in \mathcal{T}(\beta, \gamma)$ and $A^{\alpha,\beta}_{\psi} \in \mathcal{T}(\alpha, \beta)$. If $\alpha \leq \beta \leq \gamma$, then $A = A^{\beta,\gamma}_{\nu} A^{\alpha,\beta}_{\psi} = A^{\alpha,\gamma}_{\zeta} \in \mathcal{T}(\alpha, \gamma)$ with $\zeta = P_\alpha(\psi \varphi)$.

5. Products of operators from $S_k(\alpha, \beta)$ with $L^2$ symbols

Let $\alpha$, $\beta$ be two nonconstant inner functions and fix $k \in \mathbb{N}$ such that $k \leq \dim K_\alpha$. Recall that in that case $U^{\alpha,\beta}_{\psi} = 0$ if and only if $\varphi \in \overline{\alpha H^2 + H^2 \mathcal{P}_{\beta}(\psi)}$. It follows that every operator from $S_k(\alpha, \beta)$ has a symbol of the form $\varphi_+ \varphi_-$, where $\varphi_+ \in K_\alpha$ and $\varphi_- = \sum_{j=1}^{k} z^j (W^*_k \varphi_j) \in z K W^*_k \beta$ for some $\varphi_j \in K_\beta$, $1 \leq j \leq k$. To see this note that
\[ L^2 = \overline{H^2} \oplus z H^2 = \overline{\alpha H^2} \oplus K_\alpha \oplus z (W^*_k \beta) H^2 \oplus z K W^*_k \beta. \]

Decomposing an arbitrary symbol $\varphi \in L^2$ accordingly as
\[ \varphi = \overline{\alpha h + \varphi_-} + (W^*_k \beta) zg + z \chi \quad (h, g \in H^2, \varphi_- \in K_\alpha, \chi \in K W^*_k \beta) \]
we get
\[ U^{\alpha,\beta}_{\psi} = U^{\alpha,\beta}_{\varphi_- + z \chi} \]

since
\[ z (W^*_k \beta) H^2 = \overline{\varphi_-} (W^*_k \beta) z H^2 \subset \overline{\varphi_-} (W^*_k \beta) H^2. \]

Now by the decomposition
\[ K W^*_k \beta = W^*_k (K_\beta) \oplus z W^*_k (K_\beta) \oplus \ldots \oplus z^{k-1} W^*_k (K_\beta) \]
(see formula (4.3) from [31]), there are functions $\varphi_1, \ldots, \varphi_k \in K_\beta$ such that $\chi = \sum_{j=1}^{k} z^{j-1} (W^*_k \varphi_j)$ and so

\[ (5.1) \quad \varphi_- + \varphi_+ = \overline{\varphi_-} + z \chi = \overline{\varphi_-} + \sum_{j=1}^{k} z^j (W^*_k \varphi_j). \]
Observe that this decomposition is orthogonal. Moreover, it can be shown that if \( U \in S_k(\alpha, \beta) \) has a symbol given by (5.1), then

\[
U - S_\beta U(S_\alpha^*)^k = k_0^\beta \otimes \varphi_- + \sum_{j=0}^{k-1} \frac{1}{j!} (S_\beta P_\beta W_k(z^{k-j} \varphi_+)) \otimes k_{0,j}^\alpha
\]

\[
= k_0^\beta \otimes \varphi_- + \sum_{j=0}^{k-1} \frac{1}{j!} (S_\beta \varphi_{k-j}) \otimes k_{0,j}^\alpha.
\]

See the proof of Theorem 2 in [31] for more detailed computations.

**Theorem 5.1.** Let \( \alpha, \beta \) and \( \gamma \) be nonconstant inner functions and let \( k, m \) be two fixed positive integers such that \( k \leq \dim K_\alpha \) and \( m \leq \dim K_\beta \). Assume that \( A = U^\beta_{\varphi_- + \varphi_+} \in S_m(\beta, \gamma) \) for some \( \varphi_- \in K_\beta, \varphi_+ \in zK_{W^m_\gamma} \) and \( B = U^\alpha_{\psi_- + \psi_+} \in S_k(\alpha, \beta) \) for some \( \psi_- \in K_\alpha, \psi_+ \in zK_{W^k_\beta} \). Then \( AB \in S_{km}(\alpha, \gamma) \) if and only if there exist \( \tau_p \in K_\gamma, p = 0, 1, \ldots, mk-1, \) and \( v \in K_\alpha \) such that

\[
\sum_{n=0}^{m-1} \frac{1}{n!} \langle Ak_{0,n}^\beta, S_\alpha^{kn} \varphi_- \rangle - \sum_{j=0}^{k-1} \left( \frac{1}{j!} (S_\gamma A\tilde{A}_{0,j}^\beta) \otimes (S_\alpha^k B^*\tilde{A}_{0,j}^\beta) \right) + \sum_{j=0}^{k-1} \frac{1}{j!} (S_\gamma A\tilde{A}_{0,j}^\beta) \otimes (S_\alpha^k B^*\tilde{A}_{0,j}^\beta)
\]

\[
= k_0^\gamma \otimes v + \sum_{p=0}^{km-1} \tau_p \otimes k_{0,p}^\alpha.
\]

In that case \( AB = U^\alpha_{\xi} \) with

\[
\xi = v + B^* \varphi_- - \sum_{n=0}^{m-1} \langle S_\beta^k k_0^\beta, S_\alpha^{kn} \varphi_- \rangle S_\alpha^{kn} \varphi_- - \sum_{n=0}^{m-1} \frac{1}{n!} \langle S_\beta^{n+1} P_\beta W_k(z^{k-l} \varphi_+), \varphi_- \rangle S_\alpha^{kn} k_{0,l}^\alpha
\]

\[
+ \sum_{p=0}^{km-1} (W_{km}^* \tau_p) \Re^p + \sum_{p=0}^{km-1} \sum_{n=0}^{m-1} (W_{km}^* \tau_p) \Re^p + \sum_{p=0}^{km-1} \sum_{n=0}^{m-1} (W_{km}^* \tau_p) \Re^p
\]

\[
- \sum_{p=0}^{km-1} \sum_{n=0}^{m-1} \left( \sum_{j=0}^{k-1} \frac{1}{j!} \langle S_\beta^{n+1} P_\beta W_k(z^{k-l} \varphi_+), k_{0,j}^\beta \rangle S_\gamma P_\gamma W_m(z^{m-j} \varphi_+) \right) \Re^p.
\]

**Proof.** Let \( A = U^\beta_{\varphi_- + \varphi_+} \in S_m(\beta, \gamma) \) with \( \varphi_- \in K_\beta, \varphi_+ \in zK_{W^m_\gamma} \) and \( B = U^\alpha_{\psi_- + \psi_+} \in S_k(\alpha, \beta) \) with \( \psi_- \in K_\alpha, \psi_+ \in zK_{W^k_\beta} \). As mentioned above, we then have

\[
A - S_\gamma A(S_\beta^*)^m = k_0^\gamma \otimes \varphi_- + \sum_{j=0}^{m-1} \frac{1}{j!} (S_\gamma P_\gamma W_m(z^{m-j} \varphi_+)) \otimes k_{0,j}^\alpha
\]

and

\[
B - S_\beta B(S_\alpha^*)^k = k_0^\beta \otimes \psi_- + \sum_{l=0}^{k-1} \frac{1}{l!} (S_\beta P_\beta W_k(z^{k-l} \varphi_+)) \otimes k_{0,l}^\alpha.
\]
By (5.3), for any nonnegative integer \( n \) we have

\[
S^n_\beta B(S^*_\alpha)^{kn} - S^{n+1}_\beta B(S^*_\alpha)^{(n+1)k} = (S^n_\beta k^{\beta}_0) \otimes (S^k_\alpha \psi_-) + \sum_{l=0}^{k-1} \frac{1}{l!} S^{n+1}_\beta P_\beta W_k(z^{k-l} \psi_+) \otimes S^k_\alpha k^{\alpha}_{0,l}.
\]

Adding the above for \( n = 0, 1, 2, \ldots, m-1 \), we obtain

(5.4)

\[
B - S^m_\beta B(S^*_\alpha)^{km} = \sum_{n=0}^{m-1} (S^n_\beta k^{\beta}_0) \otimes (S^k_\alpha \psi_-) + \sum_{n=0}^{m-1} \sum_{l=0}^{k-1} \frac{1}{l!} S^{n+1}_\beta P_\beta W_k(z^{k-l} \psi_+) \otimes S^k_\alpha k^{\alpha}_{0,l},
\]

Now the product \( AB \) belongs to \( S_{km}(\alpha, \gamma) \) if and only if there exist \( \Psi \in K_{\alpha} \) and \( \Phi_j \in K_{\gamma}, j = 0, 1, \ldots, km - 1 \), such that

(5.5)

\[
AB - S_\gamma AB(S^*_\alpha)^{km} = k^\gamma_0 \otimes \Psi + \sum_{j=0}^{km-1} \Phi_j \otimes k^{\alpha}_{0,j}.
\]

By Lemma 4.1, we have

\[
S_\gamma AB(S^*_\alpha)^{km} = S_\gamma A(S^*_\alpha)^m S^m_\beta B(S^*_\alpha)^{km} + S_\gamma A(I_{K_\beta} - (S^*_\alpha)^m S^m_\beta) B(S^*_\alpha)^{km} = S_\gamma A(S^*_\alpha)^m S^m_\beta B(S^*_\alpha)^{km} + S_\gamma A \left( \sum_{j=0}^{m-1} \left( \frac{1}{j!} \right)^2 (k^{\gamma}_{0,j} \otimes k^{\alpha}_{0,j}) \right) B(S^*_\alpha)^{km} = S_\gamma A(S^*_\alpha)^m S^m_\beta B(S^*_\alpha)^{km} + \sum_{j=0}^{m-1} \left( \frac{1}{j!} \right)^2 (S_\gamma A k^{\gamma}_{0,j} \otimes (S^k_\alpha B^* S^{kn}_{0,j})).
\]

Using (5.2) and (5.4) we obtain

\[
S_\gamma A(S^*_\alpha)^m S^m_\beta B(S^*_\alpha)^{km} = AB - \sum_{n=0}^{m-1} (AS^n_\beta k^{\beta}_0) \otimes (S^k_\alpha \psi_-) - \sum_{n=0}^{m-1} \sum_{l=0}^{k-1} \frac{1}{l!} AS^{n+1}_\beta P_\beta W_k(z^{k-l} \psi_+) \otimes S^k_\alpha k^{\alpha}_{0,l}
\]

\[
- k^\gamma_0 \otimes B^* \varphi_- - \sum_{j=0}^{m-1} \frac{1}{j!} (S_\gamma P_\gamma W_m(z^{m-j} \varphi_+)) \otimes B^* k^{\beta}_{0,j}
\]

\[
+ \sum_{n=0}^{m-1} (S^n_\beta k^{\beta}_0, \varphi_-) k^\gamma_0 \otimes (S^k_\alpha \psi_-)
\]

\[
+ \sum_{n=0}^{m-1} \sum_{l=0}^{k-1} \frac{1}{l!} (S^{n+1}_\beta P_\beta W_k(z^{k-l} \psi_+), \varphi_-) k^{\gamma}_0 \otimes S^k_\alpha k^{\alpha}_{0,l}
\]

\[
+ \sum_{j=0}^{m-1} \sum_{n=0}^{m-1} \frac{1}{j!} (S^n_\beta k^{\beta}_0 k^{\beta}_{0,j}) (S_\gamma P_\gamma W_m(z^{m-j} \varphi_+)) \otimes (S^k_\alpha \psi_-)
\]

\[
+ \sum_{n=0}^{m-1} \sum_{l=0}^{m-1} \sum_{j=0}^{m-1} (S^{n+1}_\beta P_\beta W_k(z^{k-l} \psi_+), k^{\beta}_{0,l}) (S_\gamma P_\gamma W_m(z^{m-j} \varphi_+)) \otimes S^k_\alpha k^{\alpha}_{0,l}.
\]

Note that

\[
S^k_\alpha k^{\alpha}_{0,l} = \frac{n}{(kn+l)!} k^\alpha_{0, kn+l}.
\]
It follows that

\[ AB - S_\gamma AB(S_\alpha^*)^{km} \]

\begin{align*}
&= k_0^\gamma \otimes \left( B^* \psi_+ - \sum_{n=0}^{m-1} \langle S_\beta^n k_0^\beta, \varphi_- \rangle S_\alpha^{kn} \psi_+ - \sum_{n=0}^{m-1} \sum_{l=0}^{k-1} \frac{1}{l!} \langle S_\beta^{n+1} P_\beta W_k(z^{k-l} \psi_+), \varphi_- \rangle S_\alpha^{kn} k_0^\alpha \right) \\
&\quad + \sum_{n=0}^{m-1} \sum_{l=0}^{k-1} \frac{1}{(kn+l)!} \sum_{j=0}^{n} \left( S_\beta^{n+1} P_\beta W_k(z^{k-j} \psi_+) \right) \otimes k_0^\alpha \\
&\quad - \sum_{n=0}^{m-1} \sum_{l=0}^{k-1} \frac{1}{(kn+l)!} \sum_{j=0}^{n} \left( S_\beta^{n+1} P_\beta W_k(z^{k-j} \psi_+) \right) \otimes k_0^\alpha \\
&= k_0^\gamma \otimes \sum_{p=0}^{k-1} \tau_p \otimes k_0^\alpha.
\end{align*}

Formula for the symbol of \( AB \) follows from (4.8). \( \square \)

**Corollary 5.2.** Let \( \alpha, \beta \) and \( \gamma \) be nonconstant inner functions and let \( k \) be a fixed positive integer such that \( k \leq \dim K_\alpha \). Assume that \( A = A_\alpha^\beta \varphi_+^\beta \in \mathcal{T}(\beta, \gamma) \) for some \( \varphi_- \in K_\beta, \varphi_+ \in zK_\gamma \) and \( B = U_{\varphi_-^\beta + \varphi_+}^\alpha \in S_k(\alpha, \beta) \) for some \( \psi_- \in K_\alpha, \psi_+ \in zK_{W_\beta^*}^* \). Then \( AB \in S_k(\alpha, \gamma) \) if and only if there exist \( \tau_p \in K_\gamma, p = 0, 1, \ldots, k-1 \), and \( v \in K_\alpha \) such that

\[ Ak_0^\beta \otimes \psi_- - S_\gamma A\tilde{k}_0^\beta \otimes S_\alpha^k B^* \tilde{k}_0^\beta + (S_\gamma P_\gamma(z \varphi_+)) \otimes (B^* k_0^\beta - \langle k_0^\beta, k_0^\beta \rangle \psi_-) \]

\[ = k_0^\gamma \otimes v + \sum_{p=0}^{k-1} \tau_p \otimes k_0^\alpha. \]
In that case \( AB = U^{\alpha,\beta} \) with
\[
\xi = \overline{\nu + B^*\psi_0} - \langle k_0^\alpha, \varphi_0 \rangle \psi_0 - \sum_{l=0}^{k-1} \frac{1}{l!} \langle S_{\beta} P_{\beta} W_k(z^{k-l}\psi_0), \varphi_0 \rangle k_{0,l}^\alpha \\
+ \sum_{p=0}^{k-1} (W_k^* \tau_p) p! \overline{\xi^p} + \sum_{p=0}^{k-1} (W_k^* A S_{\beta} P_{\beta} W_k(z^k \psi_0)) \overline{\xi^p} \\
- \sum_{p=0}^{k-1} \langle P_{\beta} W_k(z^{k-p}\psi_0), k_0^\beta \rangle S_{\gamma} P_{\gamma}(z\varphi_+ \overline{\xi^p}).
\]

**Corollary 5.3.** Let \( \alpha, \beta \) and \( \gamma \) be nonconstant inner functions and let \( m \) be a fixed positive integer such that \( m \leq \dim K_\beta \). Assume that \( A = U^{\beta,\gamma}_{\varphi_0+zK} \in S_m(\beta, \gamma) \) for some \( \varphi_0 \in K_\beta \), \( \varphi_+ \in zK W_{n+1} \) and \( B = A^{\alpha,\beta}_{\varphi_0+z\varphi_+} \in T(\alpha, \beta) \) for some \( \varphi_0 \in K_\alpha, \varphi_+ \in zK_\beta \). Then \( AB \in S_m(\alpha, \gamma) \) if and only if there exist \( \tau_p \in K_\gamma, p = 0, 1, \ldots, m - 1 \), and \( v \in K_\alpha \) such that
\[
\sum_{n=0}^{m-1} \frac{1}{n!} (A k_{0,n}^\beta) \otimes (S_{\alpha}^n \psi_-) - \sum_{n=0}^{m-1} \left( \frac{1}{n!} \right) \cdot (S_{\gamma} A k_{0,j}^\beta) \otimes (S_{\alpha}^n B^* k_{0,j}^\beta) \\
+ \sum_{j=0}^{m-1} \frac{1}{j!} (S_{\gamma} P_{\gamma} W_m(z^{m-j}\varphi_+)) \otimes \left( B^* k_{0,j}^\beta - \sum_{n=0}^{m-1} \langle S_{\beta} k_{0,n}^\beta, k_{0,j}^\beta \rangle S_{\alpha}^n \psi_- \right) \\
= k_0^\gamma \otimes v + \sum_{p=0}^{m-1} \tau_p \otimes k_{0,p}^\alpha.
\]

In that case \( AB = U^{\alpha,\beta,\gamma} \) with
\[
\xi = \overline{\nu + B^*\psi_0} - \langle S_{\alpha}^n k_{0,n}^\beta, \varphi_- \rangle S_{\alpha}^n \overline{\psi_-} - \sum_{n=0}^{m-1} \langle S_{\beta} P_{\beta} W_k(z^k \psi_0), \varphi_- \rangle S_{\alpha}^n k_{0,n}^\alpha \\
+ \sum_{p=0}^{m-1} (W_m^* \tau_p) p! \overline{\xi^p} + \sum_{p=0}^{m-1} (W_m^* A S_{\beta} P_{\beta} W_k(z^k \psi_0)) \overline{\xi^p} \\
- \sum_{p=0}^{m-1} \sum_{j=0}^{m-1} \left( \frac{1}{j!} \right) \langle S_{\beta}^n P_{\beta}(z^{m-j}\psi_0), k_{0,j}^\beta \rangle S_{\gamma} P_{\gamma} W_m(z^{m-j}\varphi_+) \overline{\xi^p}.
\]

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