A CLASS OF LORENTZIAN MANIFOLDS WITH INDECOMPOSABLE HOLONOMY GROUPS

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Abstract. In this paper a class of $S^1$-bundles whose total space admits a nowhere vanishing recurrent lightlike vector field with respect to a Lorentzian metric is considered. It is shown that the Lorentzian metric can be modified such that its associated restricted holonomy group becomes indecomposable and reducible. Then, we apply basic Hodge theory to construct examples with Hermitian screen holonomy. Finally, examples with various properties are provided (e.g., complete pp-waves and indecomposable Lorentzian manifolds not homeomorphic to a product with a 1-dimensional space).

1. Introduction

For a Lorentzian manifold $(X, g)$ of dimension $n + 2$ let $Hol^0(X, g)$ be the connected component of its holonomy group. By Wu’s theorem [Bes87] $(X, g)$ is locally a product of semi-Riemannian manifolds if its holonomy representation decomposes. Therefore, we may focus on Lorentzian manifolds with indecomposable restricted holonomy groups. In contrast to the positive definite case $Hol^0(X, g) \subset SO_0(1, n + 1)$ does not need to be irreducible if it is indecomposable. In fact $SO_0(1, n + 1)$ is the only connected irreducible subgroup of $SO_0(1, n + 1)$ (see [DSO01]).

The action of a reducible indecomposable subgroup of $SO_0(1, n + 1)$ on $\mathbb{R}^{1,n+1}$ leaves a degenerate subspace $W$ invariant and we get an invariant lightlike line $W \cap W^\perp$. Under the action of $Hol^0(X, g)$ this corresponds locally to a lightlike subbundle $\Xi \subset TX$ of rank one which is spanned by a non-vanishing recurrent lightlike vector field. If $v$ is a lightlike vector in $\mathbb{R}^{1,n+1}$ spanning the invariant line, then $Hol^0(X, g) \subset Stab(\mathbb{R} \cdot v) \subset SO_0(1, n + 1)$. It can be shown that $Stab(\mathbb{R} \cdot v) \cong (\mathbb{R}^* \times SO(n)) \ltimes \mathbb{R}^n$. If we choose a basis $(v, e_1, \ldots, e_n, z)$ of $\mathbb{R}^{n+2}$ satisfying $g(e_i, e_j) = \delta_{ij}$, $g(v, z) = 1$ and $g(v, v) = g(z, z) = 0$ then the Lie algebra $(\mathbb{R} \otimes so(n)) \ltimes \mathbb{R}^n$ of $(\mathbb{R}^* \times SO(n)) \ltimes \mathbb{R}^n$ is given by

$$\left\{ \begin{pmatrix} a & w^T & 0 \\ 0 & A & -w \\ 0 & 0 & -a \end{pmatrix} : a \in \mathbb{R}, \ A \in so(n), \ w \in \mathbb{R}^n \right\}.$$

Being a subalgebra of a compact Lie algebra the projection

$$g := pr_{so(n)}(h) \subset so(n)$$

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1Here we say that a vector field $V$ is recurrent if $\nabla V = \alpha \otimes V$ for some 1-form $\alpha$ where $\nabla$ is the Levi-Civita connection of $(X, g)$.
is compact and therefore reductive. Hence we have $g = z(g) + [g, g]$ where $z(g)$ is the center of $g$. Bérard-Bergery and Ikemakhen have shown in [BBI93] that a reducible indecomposable subalgebra $h$ of $(\mathbb{R} \oplus \mathfrak{so}(n)) \rtimes \mathbb{R}^n$ belongs to one of four types. The full classification of Lorentzian holonomy algebras has been found by Leistner in [Lei07]. He proved that $g$ acts as the holonomy representation of a Riemannian manifold.

**Theorem 1.1** (Bérard-Bergery & Ikemakhen [BBI93], Leistner [Lei07]). Let $h$ be an indecomposable Lorentzian holonomy algebra.

1. Then $h$ belongs to one of the following types:
   - Type 1: $h = (\mathbb{R} \oplus g) \rtimes \mathbb{R}^n$
   - Type 2: $h = g \rtimes \mathbb{R}^n$
   - Type 3:
     $$h = \begin{cases} 
     \begin{pmatrix} \varphi(A) & w^T & 0 \\ 0 & A & w \\ 0 & 0 & -\varphi(A) \end{pmatrix} : A \in g, w \in \mathbb{R}^n \end{cases}$$
     where $\varphi : g \to \mathbb{R}$ is an epimorphism satisfying $\varphi|_{[g, g]} = 0$.
   - Type 4: There is $0 < \ell < n$ such that $\mathbb{R}^n = \mathbb{R}^\ell \oplus \mathbb{R}^{n-\ell}$, $g \subset \mathfrak{so}(\ell)$ and
     $$h = \begin{cases} 
     \begin{pmatrix} 0 & \psi(A)^T & 0 & w^T & 0 \\ 0 & 0 & -\psi(A) \\ 0 & 0 & A & -w \\ 0 & 0 & 0 & 0 \end{pmatrix} : A \in g, w \in \mathbb{R}^\ell \end{cases}$$
     for some epimorphism $\psi : g \to \mathbb{R}^{n-\ell}$ satisfying $\psi|_{[g, g]} = 0$.

2. The projection $g = pr_{\mathfrak{so}(\ell)}(h)$ is the holonomy algebra of a Riemannian manifold.

All possible holonomy groups can be constructed by Lorentzian metrics on $\mathbb{R}^{n+2}$ (see [Gal06]). The problem to construct topologically nontrivial examples with certain properties is largely open. A first approach has been made in [BM08] where non-trivial globally hyperbolic examples of the form $X = \mathbb{R}^2 \times M^n$ have been derived. Recently, globally hyperbolic examples for most Lorentzian holonomies have been found in [Baz09].

In this paper we study a class of Lorentzian metrics on the total space of $S^1$-bundles whose holonomy group is reducible but indecomposable. All manifolds are assumed to be connected without boundary.

2. **Lorentzian manifolds with indecomposable holonomy group**

2.1. **Global properties.**

In the following let $(X, g)$ be a Lorentzian manifold whose (full) holonomy group is indecomposable and reducible. The screen bundle $S$ of $(X, g)$ is defined as $S = \text{Coker}(\Xi : I \to \Xi^\perp)$. From $\nabla^{(X, g)}$ we derive the induced connection $\nabla^S$ on $S$. Since $\text{Hol}(X, g) \subset (\mathbb{R}^* \times O(n)) \rtimes \mathbb{R}^n$ we may define $G := \text{pr}_{O(n)}(\text{Hol}(X, g))$. It can be shown that $\text{Hol}(S, \nabla^S) = G$ and $\text{hol}(S, \nabla^S) = g$ [AB08].
In order to study the geometry of \((S, \nabla^S)\) it is convenient to study a non-canonical realization of \(S\) as a distribution in \(TX\) given by a non-canonical splitting \(s\) of the exact sequence

\[
0 \rightarrow \Xi \rightarrow \Xi^\perp \rightarrow S \rightarrow 0.
\]

Hence, we may define \(S := s(S)\). Since \(\Xi \subset S^\perp\) there is a uniquely defined isotropic distribution \(\Theta \subset S^\perp\) of rank one with the following property: If \(V \in \Gamma(U \subset X, \Xi)\) then there exists \(Z \in \Gamma(U \subset X, \Theta)\) such that \(g(V, Z) = 1\).

The Levi-Civita connection on \((X, g)\) induces connections \(\nabla^\Xi\) on the sub-bundles \(\Xi\) and \(S\) given by

\[
\nabla^\Xi := pr_\Xi \circ \nabla|_\Xi \quad \text{and} \quad \nabla^S := pr_S \circ \nabla|_S.
\]

Moreover, the canonical bundle morphism \(S \rightarrow S\) is easily shown to be an isomorphism such that \(\nabla^S = F^* \nabla^S\), i.e., \(Hol(S, \nabla^S) = G\).

In order to understand the global structure of a Lorentzian manifold with indecomposable and reducible holonomy we can apply the following

**Theorem 2.1** (Tischler [Tis70]). Let \(X\) be compact manifold admitting a nowhere vanishing closed 1-form. Then there is a fiber bundle \(X \rightarrow S^1\).

If \(Hol(X, g)\) is indecomposable and reducible and \(Hol^0(X, g)\) is of type 2 or 4 we derive a globally defined parallel lightlike vector field \(V\) on its time-orientation cover \((X^T, g^T)\). In this case the nowhere vanishing 1-form \(\theta := g^T(V, \cdot)\) on \(X^T\) is closed and Tischler’s theorem applies if \(X\) is compact.

In particular, all integral manifolds of \(\Xi^\perp\) are diffeomorphic and \(X\) is covered by \(\mathbb{R} \times Y\).

**Definition 2.2.** A manifold \(X^n\) is totally twisted if there is no homeomorphism \(X \rightarrow \mathbb{R} \times Y\) or \(X \rightarrow S^1 \times Y\) where \(Y\) is of dimension \(n - 1\).

In the next section we will construct totally twisted Lorentzian manifolds whose (full) holonomy representation is indecomposable and reducible such that \(Hol^0(X, g)\) is of type 1 or 2. In the non-compact case the first Betti number for these manifolds will be zero.

### 2.2. Local properties.

If \(Hol^0(X, g)\) is indecomposable and reducible we have a locally defined recurrent lightlike vector field \(V\) around \(p \in X\). It is shown in [Wal50] that we can find local coordinates \((x, y^1, \ldots, y^n, z)\) in \(U \ni p\) such that

\[
g = 2dxdz + u_idy^i dz + f dz^2 + g_{\alpha\beta} dy^\alpha dy^\beta
\]

and \(\frac{\partial}{\partial z} \in \Xi\) on \(U\) where \(u_i, f \in C^\infty(U)\) and \(\frac{\partial u_i}{\partial z} = \frac{g_{i\alpha}}{\partial z} = 0\). Local coordinates of this form will be called Walker coordinates.

**Proposition 2.3.** Let \((X, g)\) be a Lorentzian manifold such that \(\text{hol}_{loc}^0(X, g)\) indecomposable and reducible for all \(p \in X\). Then

1. \(Hol(X, g)\) is indecomposable and reducible.
2. \(\Xi\) admits a global nowhere vanishing section if and only if \((X, g)\) is time-orientable.

\(^2\)The relation of the curvatures of \(\nabla^\Xi\) and \(\nabla^S\) to the holonomy of \((X, g)\) has been studied in [Bcz05].
(3) $\text{Hol}^0(X,g)$ is of type 2 or 4 if and only if there is a $\text{hol}^0_p(X,g)$-invariant non-zero vector for all $p \in X$.

**Proof.** Since $\text{hol}^0_p(X,g) = \text{hol}_p(U_\alpha, g|_{U_\alpha})$ for some neighborhood $U_\alpha \ni p$ we have $\exists U_\alpha|_{U_\alpha \cap U_\beta} = \exists U_\beta|_{U_\alpha \cap U_\beta}$, i.e., there is a $\text{Hol}(X,g)$-invariant isotropic distribution on $X$.

If $(X,g)$ is time-orientable we may locally choose future pointing sections $V_\alpha \in \Gamma(U_\alpha, \Xi)$ and use a partition of unity. Conversely, if $\Xi$ admits a global nowhere vanishing section $V$, so does $\Theta$. If $Z \in \Gamma(X,\Theta)$ denotes this section then $\nabla \cdot (V - Z)$ is a timelike unit vector field.

For the last statement assume $\nabla \cdot V|_{U_\alpha} = 0$ for some local sections $V|_{U_\alpha} \in \Gamma(U_\alpha, \Xi)$. If $V \in \Gamma(X,\Xi)$ is nowhere vanishing then $V|_{U_\alpha} = \lambda U_\alpha V|_{U_\alpha}$ and

$$\nabla \cdot V|_{U_\alpha} = d(\log(\lambda U_\alpha))(\cdot) \lambda U_\alpha V|_{U_\alpha} = d(\log(\lambda U_\alpha))(\cdot) V|_{U_\alpha}.$$  

By construction $d(\log(\lambda U_\alpha)) = d(\log(\lambda U_\beta))$ on $U_\alpha \cap U_\beta$, i.e., $\nabla \cdot V = \alpha(\cdot) V$ for some closed 1-form $\alpha$.

However, the existence of a covering of $X$ by indecomposable Walker coordinates does not imply reducibility of $\text{Hol}^0(X,g)$\(^3\). For any given Walker coordinates an integrable realization of the screen bundle is given by $S := \text{span}\{\frac{\partial}{\partial y^\alpha}\}$. In this case $\Xi = \text{span}\{\frac{\partial}{\partial x^\alpha}\}$ and $\Theta := \text{span}\{Z\}$ with $Z := \frac{1}{2}(g^{\alpha\beta} u^\alpha u^\beta - f)\frac{\partial}{\partial x^\alpha} - g^{\alpha\beta} u^\alpha \frac{\partial}{\partial y^\beta} + \frac{\partial}{\partial z}$. In particular, the parallel transport equations immediately imply $\text{Hol}(M_{zz}, \nabla M_{zz}) \subset \text{Hol}(S, \nabla S)$ where $M_{xx}$ is the Riemannian submanifold given by the $\frac{\partial}{\partial y^\alpha}$-coordinates\(^4\). Moreover, any realization of the screen bundle is of the form $\text{span}\{\partial_\alpha - v_\alpha \partial_0\}$ and integrable if and only if $\partial_\alpha v_\beta = \partial_\beta v_\alpha + v_\alpha \partial_\delta v_\beta - v_\beta \partial_\delta v_\alpha = 0$.

The holonomy of any given Walker coordinates is indecomposable and not of type 4 if for some $p \in U$ and any $\alpha$ there exists $\beta$ such that

$$g(R(\partial_\alpha - u_\alpha \partial_0, \partial_\beta - \frac{1}{2}f \partial_0)) = 0.$$  

A long computation shows

$$g(R(Y_\alpha, \partial_\beta - \frac{1}{2}f \partial_0)Y_\beta, \partial_\beta - \frac{1}{2}f \partial_0) = \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta} - \frac{1}{2}(\Gamma^\gamma_{\alpha\beta} \frac{\partial f}{\partial y^\gamma} + \partial_\alpha \frac{\partial f}{\partial x^\beta} \partial_\gamma + \partial_\beta \frac{\partial f}{\partial x^\alpha} \partial_\gamma - \partial_\alpha \partial_\beta \frac{\partial^2 f}{\partial x^\gamma} - F),$$

where $F$ is a polynomial in $u_\alpha, g_{\alpha\beta}, g^{\alpha\beta}$ and its first and second derivatives\(^5\). E.g., $f := \frac{1}{2}(F + 1)\sum (y^\alpha)^2$ provides Walker coordinates of type 2 and $f := \frac{1}{2}x(F + 1)\sum (y^\alpha)^2$ provides Walker coordinates of type 1 or 3. Later we will refer to such a choice of $f$ as sufficiently generic.

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\(^3\)A counterexample can be constructed as follows: Let $f_1, f_2 \in C^\infty(\mathbb{R})$ such that $f_1|_{-\infty,-1} = f_2|_{1,\infty} = 1$ and $f_1|_{-\infty,1} = f_2|_{-\infty,1} = 0$. On $\mathbb{R}^3$ we define $g = 2dz dx + y^2 f_1(z)dz^2 + y^2 f_2(z)dx^2 + dy^2$. Then $\text{Hol}^0(\mathbb{R}^3, g) = SO_0(1,2)$.

\(^4\)Note however, that all indecomposable Lorentzian holonomies have been realized in $\text{Galb}[\mathbb{R}^3]$ by Walker coordinates for which $\text{Hol}(M_{zz}, \nabla M_{zz}) = 0$.

\(^5\)In fact, $F$ is given by

$$F = \partial_\gamma (\Gamma^\gamma_{\alpha\beta}) - \partial_\gamma (\Gamma^\gamma_{\alpha\beta}) u_\alpha + \partial_\alpha (u_\gamma) u_\beta + \frac{1}{2} \partial_\alpha (u_\gamma) \omega^\beta_\gamma + \frac{1}{2} \omega^\alpha_\beta \Gamma^\gamma_{\alpha\beta} u_\beta - \frac{1}{2} \omega^\alpha_\beta \Gamma^\gamma_{\alpha\beta} u_\gamma,$$

where $\omega^\alpha_\beta := g^{\alpha\gamma}(\partial_\gamma u_\alpha - \partial_\alpha u_\beta + \frac{\partial}{\partial z})$. 
3. The total space of an $S^1$-bundle as a Lorentzian manifold

First, we will construct a Lorentzian metric on the total space of an $S^1$-bundle over a base manifold admitting a nowhere vanishing closed 1-form. Under this metric the vertical vector field on the total space becomes recurrent. Then we will give conditions under which the restricted holonomy representation becomes indecomposable. The idea is based on the following well-known observation: The exact sequence of sheaves:

$$0 \rightarrow \mathbb{Z} \rightarrow C^\infty_M \xrightarrow{\exp} C^\infty_M \rightarrow 0$$

provides the isomorphism:

$$c_1 : \{\text{iso. classes of complex line bundles on } M\} \rightarrow H^2(M, \mathbb{Z}).$$

As usual we write $[\frac{\psi}{2\pi}] \in H^2(M, \mathbb{Z})$ if $\psi$ is a closed 2-form and $[\frac{\varphi}{2\pi}] \in \text{Im}(H^2(M, \mathbb{Z}) \rightarrow H^2_{dR}(M, \mathbb{R}))$. We state the main construction method:

**Proposition 3.1.** Let $(M, g)$ be a Riemannian manifold and $\eta$ a nowhere vanishing closed 1-form on $M$. Let $\psi$ be a 2-form on $M$ with $[\frac{\psi}{2\pi}] \in H^2(M, \mathbb{Z})$. Then there exists an $S^1$-bundle $\pi : X \rightarrow M$ satisfying $c_1(X \rightarrow M) = [\frac{\psi}{2\pi}]$ and

1. There is a global non-vanishing 1-form $\theta$ on $X$ such that:

$$\tilde{g}_f := 2\theta \pi^* \eta + f \cdot \pi^* \eta^2 + \pi^* g$$

defines a Lorentzian metric on $X$ for any $f \in C^\infty(X)$.

2. Given $p \in X$ and a local 1-form $\phi$ with $\psi = d\phi$ there are local coordinates $(x, y^1, \ldots, y^n, z)$ around $p$ such that:

$$\tilde{g}_f = 2 dx dz + (u_i + 2g_{i(n+1)}) dy^i dz + (f + \frac{u_{n+1}}{2} + g_{(n+1),(n+1)}) dz^2 + g_{ij} dy^i dy^j$$

where $2\phi = u_i dy^i + u_{n+1} dz$.

3. The $U(1)$-action of $X \rightarrow M$ acts by isometries on $(X, \tilde{g}_f)$ if $f$ is constant on the fibers.

4. The vertical vector field is a global lightlike vector field which is parallel if and only if $f$ is constant on the fibers.

**Proof.** Consider the smooth complex line bundle $L \rightarrow M$ given by $c_1^{-1}(-[\frac{\psi}{2\pi}])$ and some Hermitian metric $h$ on $L$. The curvature endomorphism of the Chern connection $\nabla^C_L$ of $(L, h)$ is given by the closed imaginary $(1,1)$-form $iF_h$. Moreover, $[iF_h] = -2\pi i c_1(L)$. Hence, $F_h - \psi = df$ is exact and $\nabla_L := \nabla^C_L - if$ is another Hermitian connection on $L$ with respect to $h$ and its curvature endomorphism is given by $i\psi$.

The metric $h$ provides a $U(1)$-reduction of the $\text{GL}(1, \mathbb{C})$-bundle $(L, h)$. Since $\nabla_L$ is Hermitian it reduces as well. In this way, we derive an $S^1$-bundle $X := \{v \in L : h(v, v) = 1\} \rightarrow M$ together with the $U(1)$-connection $\nabla_L$.

Consider the 1-form $\eta$ on $M$. By Frobenius’ theorem we can find for all $x \in M$ local coordinates $(y_1, \ldots, y_n, z)$ on some neighborhood $U \ni x$ such that $\eta = dz$. Moreover, we may assume that $X \rightarrow M$ is trivial over $U$ and
ψ = dφU. Consider a unit length section sU : U → L such that\(^6\)
\[ \nabla_L sU = iφ_U \otimes sU. \]
Using the section sU we may define local coordinates \((x^0, \ldots, x^{n+1}) := (x, y^1, \ldots, y^n, z)\) given by \(e^{ix^0}sU(y^1, \ldots, y^n, z)\).

We have to construct the 1-form \(θ\) on \(V\). Consider a unit length section \(s\) such that \(ψ = φ_U - φ_0\). Given the local coordinates defined by \(sU\) and \(sV\) we observe
\[ e^{ix^0}sU = e^{i(x^0+c)}sV = e^{i(x^0+gUV+c)}sU, \]
i.e., \(dx_U - dx_V = dgUV = φ_U - φ_0\). From this equation we conclude that \(dx_U + φ_U\) glues to a global non-vanishing 1-form \(θ\) on \(X\). Moreover, \(dθ = π^∗ψ\), i.e., the pullback of \(ψ\) is an exact form on \(X\).

We have to show that \(g_f := 2θ π^∗η + f \cdot π^∗η^2 + π^∗g\) is a Lorentzian metric. This can be checked in the given local coordinate expression
\[
\tilde{g}_f = 
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & g_{i1} & \cdots & g_{in} & g_{i(n+1)} + u_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & g_{n1} & \cdots & g_{nn} & g_{n(n+1)} + u_n \\
1 & g_{(n+1)1} + u_1 & \cdots & g_{(n+1)n} + u_n & f + u_{n+1} + g_{(n+1)(n+1)}
\end{pmatrix}
\]
and we conclude \(\det(\tilde{g}_{ij}) < 0\) since \((g_{ij})_{1 \leq i,j \leq n}\) is the Riemannian metric \(g\) restricted to the submanifold \(\{(y_1, \ldots, y_n, \text{const.})\}\).

If \(f \in C^∞(X)\) is constant on the fibers the \(U(1)\)-action of the bundle leaves \(\tilde{g}_f\) invariant since \(θ\) is the connection 1-form of \(∇_L\) and all other terms in \(\tilde{g}_f\) are pullbacks.

By definition of \(\tilde{g}_f\) the vertical vector field is lightlike. Using the local coordinate expression for \(\tilde{g}_f\) we compute
\[
Γ^k_{i0} = \frac{1}{2} δ_{i(n+1)} δ_{k0} \frac{∂f}{∂x^0} \quad i, k ∈ \{0, \ldots, n+1\}.
\]
Therefore, the vertical vector field is parallel if and only if \(f\) is constant on the fibers.

As we have already observed any time-orientable Lorentzian manifold whose holonomy representation is of type 2 or 4 admits a nowhere vanishing closed 1-form. Hence, Prop. \(\bullet\) provides a method to construct new examples from old.

Up to diffeomorphism \(X\) depends only on the choice of the class \([\frac{g_f}{2π}]\) ∈ \(H^2(M,\mathbb{Z})\). However, the Lorentzian metric \(g_f\) depends on the particular representative \(ψ ∈ [ψ] ∈ H^2(M,\mathbb{R})\). This will be important in the following sections.

\(^{6}\)We can find such a section without any restriction: If \(t : U → L\) is any unit length section we have \(∇_L t = iα \otimes t\) for some 1-form \(α\). Hence, \(α - φ_U = df\) and \(s_U := e^{-iφ_U}\) has the connection form \(iφ_U\).
Example 3.4. Let $M := \mathbb{R}^3 \setminus \{(0,0,-1),(0,0,1)\}$ and let $0 \neq [\frac{x}{2}] \in H^2(M,\mathbb{Z})$. Define $\gamma := \frac{\partial}{\partial x}$ on $M$ and construct $(X, \tilde{g}_f)$ as in Prop. 3.1 with $f \in C^\infty(X)$ sufficiently generic. Then $X$ is totally twisted with $b_1(X) = 0$ and $\text{hol}(X, \tilde{g}_f)$ is of type 2 if $\frac{\partial f}{\partial x} \neq 0$ and otherwise of type 1.

Proof. From the long exact sequence of homotopy groups for the fibration $X \to M$ we conclude that $\pi_1(X)$ is a finite torsion group. Moreover, Gysin’s sequence implies $H^3(X,\mathbb{R}) = \mathbb{R}^2$. Using the K"unneth formula we derive a contradiction unless $X$ is totally twisted. \[\square\]

A compact totally twisted example will be constructed in Ex. 5.2.

By Tischler’s theorem $M$ cannot be compact and simply connected if we want to apply Prop. 3.1. However, if $M = N \times L$ with $\dim L = 1$ and $N$ not necessarily compact we may define $\gamma = \frac{\partial}{\partial x}$ and consider $[\frac{y}{2}] \in H^2(N,\mathbb{Z})$. In this case $X = \tilde{X} \times L$ where $\tilde{X}$ is the total space of the
bundle corresponding to \([\frac{\psi}{12}]\). Let \(g\) be a Riemannian metric on \(N\). Then we consider the Lorentzian metric

\[
\tilde{g}_f := 2\theta dz + f \cdot dz^2 + \pi^* g
\]

for some function \(f \in C^\infty(X)\), where \(z\) is the coordinate on \(L\). In this situation we say \((X, \tilde{g}_f)\) is of toric type\(^\text{\footnotemark}\).

\footnotetext{\(\text{If } M = N \times S^1 \text{ then } X \text{ is a torus bundle over } N \text{ where one direction in the fibers is trivial.} \)}

**Proposition 3.5.** Let \((X, \tilde{g}_f)\) be of toric type and \(\pi : \tilde{X} \rightarrow M\) the corresponding \(S^1\)-bundle. Then:

- The horizontal distribution in \(TX\) is isomorphic to the screen bundle,
- \(\mathfrak{hol}(M, g) \subset G\),
- \(\mathfrak{hol}_{\nu}(M, g) \subset \mathfrak{hol}(S, \nabla S)\).

**Proof.**

1. We have \(X = \tilde{X} \times L\) with \(\dim L = 1\). If \(\frac{\partial}{\partial z}\) is the global coordinate field on \(L\) and \(V\) the vertical vector field of \(\tilde{X}\) the screen bundle may be identified with \(S = \text{span}\{V, Z := -\frac{1}{2}V + \frac{\partial}{\partial z}\}\). Since \(\tilde{g}_f = 2dzdz + u_i dy^i dz + g_{ij} dy^i dy^j + f dz^2\) we observe

\[
\tilde{g}_f(V, \frac{\partial}{\partial y^i} - u_i \frac{\partial}{\partial x}) = \tilde{g}_f(Z, \frac{\partial}{\partial y^i} - u_i \frac{\partial}{\partial x}) = 0,
\]

i.e., \(Y_t := \frac{\partial}{\partial y^i} - u_i \frac{\partial}{\partial x} \in S\). However, a simple computation shows that the horizontal space of \(\tilde{X}\) is spanned by \(\{Y_t\}\).

2. Fix \((p, q) \in X\) and let \(x := \pi(p)\). To each \(a \in \mathfrak{hol}(M, g)\) we construct a loop \(\tilde{\gamma} : I \rightarrow X\) on which parallel displacement induces \(a \in G\). More precisely, let \(\gamma : [0, 1] \rightarrow M\) be a loop with \(\gamma(0) = x\) and let \(\tilde{\delta} : [0, 1] \rightarrow \tilde{X}\) with \(\tilde{\delta}(0) = p\) be its horizontal lift with \(\tilde{u} := \tilde{\delta}(1)\). If \(u \neq p\) let \(\tilde{\beta}\) be the integral curve of the vertical field in the fiber \(\pi^{-1}(x)\) connecting \(u\) and \(p\). We define

\[
\tilde{\gamma} := \begin{cases} 
(\tilde{\delta} \ast \tilde{\beta}, q) & \text{if } u \neq p, \\
(\tilde{\delta}, q) & \text{otherwise}.
\end{cases}
\]

Let \(v \in T_xM\) and let \(v_t\) be its parallel displacement along \(\gamma\). Write \(\tilde{v}_t\) for the horizontal lift of \(v_t\). First, we consider the parallel displacement \(w_t\) of \(\tilde{v}_0\) along \(\delta = (\tilde{\delta}, q) : I \rightarrow X\). We show \(\tilde{v}_t = pr_S(w_t)\). Clearly, the set \(J \subset I\) on which this equation holds is non-empty and closed. In order to show that \(J \subset I\) is open we may use local coordinates. For \(1 \leq \alpha, \beta, k \leq n\) we have \(w_t^{(n+1)} = \delta_t^{(n+1)} = \Gamma^{i}_{\alpha \beta} = 0\) and

\[
0 = w_t^k + \Gamma^k_{ij} \delta_t^i w_t^j = \tilde{w}_t^k + \Gamma^k_{\alpha \beta} \tilde{\gamma}_t^\alpha \tilde{w}_t^\beta = \tilde{w}_t^k + \tilde{\Gamma}^k_{\alpha \beta} \tilde{\gamma}_t^\alpha \tilde{w}_t^\beta,
\]

where \(\tilde{\Gamma}^k_{\alpha \beta}\) are the Christoffel symbols of \((M, g)\). This shows \(w_t^k = v_t^k\), i.e., \(pr_S(w_t) = \tilde{v}_t\).

Assume \(u \neq p\) and consider the parallel displacement of a vector \(\tilde{v} \in \Xi^{\frac{1}{2}}_\nu\) along \(\beta = (\tilde{\beta}, q)\). Again we can work in a local coordinate chart and conclude \(\tilde{v}_t^k = \text{const}\).
4. **Examples and Hermitian Screen Bundles**

Given the results from the last section we can construct explicit examples of non-trivial Lorentzian manifolds with indecomposable, reducible holonomy representations. From Proposition 3.5 we immediately derive

**Example 4.1.** Let \((M,g)\) be a Riemannian manifold such that \(\mathfrak{hol}(M,g) = \mathfrak{so}(n)\). If \((X,\tilde{g}_f)\) is of toric type over \((M,g)\) and if \(f \in C^\infty(X)\) is sufficiently generic

\[
\mathfrak{hol}(X,\tilde{g}_f) = \begin{cases} 
\mathfrak{so}(n) \ltimes \mathbb{R}^n & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\
(\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n & \text{otherwise.}
\end{cases}
\]

In order to construct other examples we compute the curvature of \((S,\nabla^S)\) in case that \((X,\tilde{g}_f)\) is of toric type.

\[
\nabla_{\partial_k} Y_j = \left( \Gamma^k_{ij} - u_j \Gamma^k_{0i} \right) \frac{\partial}{\partial x^k} - \frac{\partial u_j}{\partial x^k} \frac{\partial}{\partial x^k}
\]

implies \(\nabla_{\partial_k} Y_j = -\frac{\partial u_j}{\partial x^k} \frac{\partial}{\partial x^k} = 0\) and for \(\alpha \in \{1, \ldots, n\}\) we have \(\nabla^S_{\partial_i} Y_j = \text{pr}_S(\nabla_{\partial_i} Y_j) = \Gamma^k_{ij} Y_k\). Hence,

\[
R^{S}(Y_i, Y_j) Y_k = \text{pr}_S(R(\partial_i, \partial_j) \partial_k) = \text{pr}_S(R(M,g)(\partial_i, \partial_j) \partial_k).
\]

The same way we conclude \(R^{S}(\partial_0, Y_j) Y_k = R^{S}(\partial_0, Z) Y_k = 0\).

Any almost complex structure \(J\) on \(M\) induces an almost complex structure \(\tilde{J}\) on the screen bundle \(S\) since \(J\) can be lifted to the horizontal bundle.

The same way we can lift other tensors to the screen bundle and conclude

**Lemma 4.2.** Let \((M,g)\) be a Riemannian manifold and \([\omega] \in H^2(M,\mathbb{Z})\). Let \((X = \tilde{X} \times L, g_f)\) be of toric type where \(\tilde{X} \to M\) is the \(S^1\)-bundle corresponding to \(\frac{\omega}{2\pi}\) and \(f \in C^\infty(X)\) is sufficiently generic. The following holds:

1. \(H^0(S,\nabla^S) = 0 \iff (M,g)\) is flat and \(\nabla^{(M,g)} \psi = 0\).
2. If \((M,J,g)\) is Kähler then \(\nabla^S \tilde{J} = 0 \iff \psi \in \Lambda^{1,1}(M,J)\).
3. If \((M,J,g)\) is Kähler with a parallel holomorphic volume form \(\Omega\) then \(\nabla^S \tilde{\Omega} = 0 \iff \psi \in \Lambda^{1,1}(M,J)\) is a primitive form\(^8\).
4. If \((M,J_1, J_2, J_3, g)\) is hyperkähler then \(\nabla^S \tilde{J}_1 = \nabla^S \tilde{J}_2 = \nabla^S \tilde{J}_3 = 0 \iff \psi \in \Lambda^{1,1}(M,J_1) \cap \Lambda^{1,1}(M,J_2)\).
5. If \((M,H,g)\) is quaternion-Kähler then \(\tilde{H}\) is a \(\nabla^S\)-parallel rank 3 subbundle of \(S\) if \(\psi \in \Gamma(H^{1,1})\)\(^9\).

---

\(^8\)Remember, a 2-form \(\psi\) on \((M,J,g)\) is primitive if \(\Lambda \psi = \sum_{i=1}^{\dim(M,J)} \psi(e_i, J e_i) = 0\) where \(\Lambda\) is the dual Lefschetz operator.

\(^9\)Here we say \((M,H,g)\) is quaternion-Kähler if \(H\) is locally spanned by almost complex structures \(J_1, J_2, J_3 = J_1 J_2\) such that \(g\) is Hermitian with respect to \(J_1, J_2, J_3\) and \(\nabla J_i \in H\). The bundle \(H^{1,1} \subset \Lambda^2 TM\) is locally defined as the intersection \(\Lambda^{1,1}(U_a, J_1) \cap \Lambda^{1,1}(U_a, J_2)\).
Proof.

(1) By Proposition 3.5 \( H^0(S, \nabla^S) = 0 \) implies that \((M, g)\) is flat. Moreover, using local coordinates on \((M, g)\) such that \(\Gamma^\gamma_{\alpha\beta} = 0\) we have
\[
R^{\nabla^S}(Y_i, Z)Y_k = \sum_\alpha \nabla^S_{Y_i}(\psi(\partial_k, \partial_\alpha)Y_\alpha) = \sum_\alpha (Y_i(\psi(\partial_k, \partial_\alpha)))Y_\alpha
\]
\[
= \sum_\alpha ((\nabla_{\partial_\beta}\psi)(\partial_k, \partial_\alpha))Y_\alpha,
\]
i.e., \(R^{\nabla^S} = 0 \iff R^{(M, g)} = 0\) and \(\nabla\psi = 0\).

(2) If \((M, J, g)\) is Kähler let \((y^1, \ldots, y^{2m})\) are local coordinates on \(M\) such that \(\partial_{2k} = J(\partial_{2k-1})\) and \(Y_j\) the horizontal lift of \(\partial_j\). Since \(\nabla^{(M, g)}J = 0\) the only non-vanishing \(\nabla^S J\) can be \(\nabla^S Y_j\). However,
\[
\langle \nabla^S (\tilde J Y_j), Y_\ell \rangle = g^{\ell\alpha} \psi(J(\partial_j), \partial_\alpha)\langle Y_k, Y_\ell \rangle = \psi(J(\partial_j), \partial_\ell)
\]
and
\[
\langle \tilde J (\nabla^S Y_j), Y_\ell \rangle = -\langle \nabla^S Y_j, \tilde J Y_\ell \rangle = -g^{\ell\alpha} \psi(\partial_j, \partial_\alpha)\langle Y_k, \tilde J Y_\ell \rangle = -\psi(\partial_j, J\partial_\ell),
\]
i.e., \(\nabla^S Y_j = 0 \iff \psi(J\partial_j, \partial_\ell) + \psi(\partial_j, J\partial_\ell) = 0\).

(3) We have to compute \(\nabla Z \tilde \Omega\). For \(1 \leq k \leq m\) define \(Z_k := \frac{i}{2}(Y_k - i\tilde J Y_k)\). A short computation shows
\[
(\nabla Z \tilde \Omega)(Z_1, \ldots, Z_m) = \sqrt{-1}(\Lambda\psi)\tilde \Omega(Z_1, \ldots, Z_m),
\]
i.e., \(\nabla^S \tilde \Omega = 0 \iff \Lambda\psi = 0\).

(4) This follows from the second statement.

(5) Clearly, \(\tilde H\) is \(\nabla^S\)-parallel if \(\psi \in H^{1,1}\) by the second statement.

(6) If \((M, \phi, g)\) is a G2-manifold with a parallel positive 3-form \(\phi\) then
\[
(\nabla^S \tilde \phi)(Y_\alpha, Y_\beta, Y_\gamma) = C_{24}(\psi \otimes \phi)(\partial_\alpha, \partial_\beta, \partial_\gamma)
+ C_{24}(\psi \otimes \phi)(\partial_\beta, \partial_\gamma, \partial_\alpha)
+ C_{24}(\psi \otimes \phi)(\partial_\gamma, \partial_\alpha, \partial_\beta)
= BI(C_{24}(\psi \otimes \phi))(\partial_\alpha, \partial_\beta, \partial_\gamma).
\]

(7) If \((M, \Omega, g)\) is a Spin(7)-manifold with a parallel admissible 4-form \(\Omega\) then
\[
(\nabla^S \tilde \Omega)(Y_\alpha, Y_\beta, Y_\gamma, Y_\delta) = C_{24}(\psi \otimes \Omega)(\partial_\alpha, \partial_\beta, \partial_\gamma, \partial_\delta)
- C_{24}(\psi \otimes \Omega)(\partial_\beta, \partial_\gamma, \partial_\delta, \partial_\alpha)
+ C_{24}(\psi \otimes \Omega)(\partial_\gamma, \partial_\delta, \partial_\alpha, \partial_\beta)
- C_{24}(\psi \otimes \Omega)(\partial_\delta, \partial_\alpha, \partial_\beta, \partial_\gamma)
= AB(C_{24}(\psi \otimes \Omega))(\partial_\alpha, \partial_\beta, \partial_\gamma, \partial_\delta).
\]
The lemma above provides sufficient conditions for a toric type Lorentzian manifold to have specified screen holonomy. Using basic Hodge theory we construct explicit examples with Hermitian screen holonomy.

For any complex manifold \( X \) we write
\[
H^{1,1}(X, \mathbb{Z}) := \text{Im}(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})) \cap H^{1,1}(X),
\]
\[
H^{1,1}(X, \mathbb{Q}) := \text{Im}(H^2(X, \mathbb{Q}) \to H^2(X, \mathbb{C})) \cap H^{1,1}(X).
\]
If \( X \) is a compact Kähler manifold the Lefschetz theorem on \((1,1)\)-classes implies \( H^{1,1}(X, \mathbb{Z}) = \text{NS}(X) \) where \( \text{NS}(X) \) is the Neron-Severi group of \( X \) defined as the image of
\[
\text{Pic}(X) = H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \xrightarrow{\cdot c} H^2(X, \mathbb{C}).
\]
By Lemma 4.2 we have \( \text{Hol}(S, \nabla^S) \subset U(n) \) if \( \psi \in \Lambda^{1,1}(M, J) \). By definition \( [\psi] \in H^2(M, \mathbb{Z}) \). Therefore \( \text{Hol}(S, \nabla^S) \subset U(n) \) if \( [\psi] \in \text{NS}(M, J) \). It is not difficult to construct examples over \( \mathbb{C}P^n \). In the non-symmetric case we may apply the following

**Corollary 4.3.** Let \((M^{2n}, J)\) be a compact simply-connected irreducible Kähler manifold with \( c_1(M, J) < 0 \) and \( J \) its Einstein-Kähler metric. Let \( \alpha \in H^2(M, \mathbb{Z}) \) be a Hodge class, e.g., \(-c_1(M, J)\). If \((X = \tilde{X} \times L, \tilde{g}_f)\) is of toric type over \((M, J, g)\) where \( \tilde{X} \to M \) is constructed using a representative of \( \alpha \) and if \( f \in C^\infty \) is sufficiently generic then
\[
\text{hol}(X, \tilde{g}_f) = \begin{cases} (\mathbb{R} \times u(n)) \times \mathbb{R}^{2n} & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ (\mathbb{R} \oplus u(n)) \times \mathbb{R}^{2n} & \text{otherwise.} \end{cases}
\]

**Proof.** By Aubin-Yau theorem we have an Einstein-Kähler metric which is unique up to homothety and \(-c_1(M, J)\) is a Hodge class. Since \((M, J, g)\) is compact and simply connected with negative Einstein constant it is not symmetric and w.l.o.g. we have \( \text{hol}(M, J, g) = u(n) \). Hence, the statement follows from Proposition 3.5. \( \blacksquare \)

Next we construct Lorentzian manifolds such that \( \text{Hol}(S, \nabla^S) = SU(n) \). In the following we say \((M, J, g)\) is a Calabi-Yau manifold if \( M \) is a compact Kähler manifold with \( \text{Hol}(M, J, g) = SU(n) \). Since \((M, J, g)\) is compact Kähler the Laplace operator commutes with the dual Lefschetz operator \( \Lambda \) and we can define the primitive cohomology group
\[
H^{1,1}_{\text{prim}}(M, J) := \text{Ker}(\Lambda : H^{1,1}(M, J) \to \mathbb{C}).
\]
Moreover, the Lefschetz decomposition implies
\[
H^{1,1}(M, J, \mathbb{R}) = \mathbb{R}[\omega] \oplus H^{1,1}_{\text{prim}}((M, J), \mathbb{R}).
\]
Let \( \check{\Omega} = h \cdot \check{\Omega} \) for some nowhere-vanishing function \( h \in C^\infty(X) \). We say \((\check{J}, \check{\Omega})\) defines an \( SU(n)\)-structure on \((S, \nabla^S)\) if \( \nabla^S \check{J} = \nabla^S \check{\Omega} = 0 \). Clearly, this implies \( \text{Hol}(S, \nabla^S) \subset SU(n) \). Moreover, we conclude

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10 Explicit examples can be found in Yau77.
Corollary 4.4. Let \((M, J, g)\) be a Calabi-Yau manifold. Let \((X = \tilde{X} \times L, \tilde{g}_f)\) be of toric type over \((M, J, g)\) where \(\tilde{X} \to M\) is constructed using a representative \(\alpha\) of some \([\alpha] \in NS(M, J)\) and \(f \in C^\infty(X)\) is sufficiently generic. Suppose \(\Lambda[\alpha] \in \mathbb{Z}\) or \(L = \mathbb{R}\). Then \((\tilde{J}, e^{-\sqrt{-1} \langle \Lambda \alpha \rangle} \tilde{\Omega})\) defines an \(SU(n)\)-structure on \((\tilde{S}, \nabla^\tilde{S})\) if and only if \(\alpha\) is the harmonic representative of \([\alpha]\). In this case we have

\[
\text{hol}(X, \tilde{g}_f) = \begin{cases} 
\mathfrak{su}(n) \times \mathbb{R}^{2n} & \text{if } \partial_f / \partial z = 0, \\
(\mathbb{R} \oplus \mathfrak{su}(n)) \times \mathbb{R}^{2n} & \text{otherwise}.
\end{cases}
\]

Proof. If \([\alpha] \in NS(M, J)\) and \(\alpha \in [\alpha]\) then \(\Lambda[\alpha] = \text{const.}\) by the Lefschetz decomposition and \(\nabla^2 e^{-\sqrt{-1} \langle \Lambda \alpha \rangle} \tilde{\Omega} = 0\) by Lemma 4.2 where \(z\) is the coordinate on \(L\). Moreover, for the harmonic representative of \([\alpha]\) we have \(\nabla^2 e^{-\sqrt{-1} \langle \Lambda \alpha \rangle} \tilde{\Omega} = 0\). The converse is implied by the \(\partial \bar{\partial}\)-lemma and the Kähler identities. 

Remark 4.5.

(1) Let \((M, J, g)\) be a Calabi-Yau \(n\)-fold such that \(n \geq 3\). In this case \([\omega] \in H^2(M, \mathbb{Q})\) is its Kähler class. Then \(\dim_{\mathbb{Q}} H^{1,1}_{\text{prim}}((M, J), \mathbb{Q}) = b_2 - 1\) (see [We158]), i.e., \(\text{rk}(H^{1,1}_{\text{prim}}((M, J), \mathbb{Z})) = b_2 - 1\). Moreover, if \((X, \tilde{g}_f)\) is as above and \(\alpha \in [\alpha] \in H^{1,1}_{\text{prim}}((M, J), \mathbb{Z})\) is harmonic then \((\tilde{J}, \tilde{\Omega})\) defines an \(SU(n)\)-structure on \((\tilde{S}, \nabla^\tilde{S})\).

(2) If \((M, J, g)\) is an exceptional \(K3\)-surface \(\Box\) we can choose a basis \((b_1, b_2, [\omega], e_1, \ldots, e_{19})\) of \(H^2(M, \mathbb{Q})\) such that

\[
\text{span}_{\mathbb{C}}\{b_1, b_2\} = H^{2,0}(M, J) \oplus H^{0,2}(M, J).
\]

Using the intersection form and the Gram-Schmidt algorithm we derive a basis \(\tilde{c}_1, \ldots, \tilde{c}_{19}\) of \(H^{1,1}_{\text{prim}}((M, J), \mathbb{Z})\), i.e., \(\text{rk}(H^{1,1}_{\text{prim}}((M, J), \mathbb{Z})) = 19\). Hence, the same remark applies for any harmonic \(\alpha \in [\alpha] \in H^{1,1}_{\text{prim}}((M, J), \mathbb{Z})\).

(3) Since \(\mathfrak{su}(2) = \mathfrak{sp}(1)\) Corollary 4.4 gives examples with symplectic screen holonomy. 

Finally, we construct Lorentzian manifolds such that \(\text{Hol}(S, \nabla^S) = Sp(n)\). In the following a simply connected, compact Kähler manifold \(X\) with \(H^{2,0}(X) = \mathbb{C}[\sigma]\) where \(\sigma\) is everywhere non-degenerate is called holomorphic symplectic. We write \(\rho(X)\) for the Picard number of a Kähler manifold \(X\).

Theorem 4.6 (K. Oguiso \[Ogu03\]. Let \(X\) be a holomorphic symplectic manifold with \(b_2 = N + 2\). Then, for each integer \(0 \leq k \leq N\) there exists a holomorphic symplectic manifold \(X'\) such that \(X\) and \(X'\) are deformation equivalent and \(\rho(X') = k\). 

Using Oguiso’s theorem we can find a holomorphic symplectic structure with maximal Picard number on the differentiable manifold underlying any holomorphic symplectic manifold. Hence, we derive plenty of examples by the following

\[\Box\] We call a \(K3\)-surface exceptional if its Picard number is maximal.
Corollary 4.7. Let \((M^{4n}, J)\) be a holomorphic symplectic manifold with \(b_2 \geq 4\) and \(\rho(M, J) = b_2 - 2\). Then, there exists an irreducible hyperkähler structure \((M, J, J_2, J_3, g)\) with Kähler class \([\omega] \in H^2(M, \mathbb{Q})\) and \(0 \neq [\psi] \in H^{1,1}(M, J) \cap H^{1,1}(M, J_2) \cap H^2(M, \mathbb{Z})\) on \(M\). Moreover, if \((X = X \times L, \tilde{g}_f)\) is of toric type over \((M, J, g)\) where \(X \to M\) is constructed using the harmonic representative of \([\psi]\) and \(f \in C^\infty(X)\) is sufficiently generic then

\[
\mathfrak{hol}(X, \tilde{g}_f) = \begin{cases} \mathfrak{sp}(n) \times \mathbb{R}^{4n} & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ (\mathbb{R} \oplus \mathfrak{sp}(n)) \times \mathbb{R}^{4n} & \text{otherwise.} \end{cases}
\]

Proof. We can find a Kähler class \([\omega] \in H^2(M, \mathbb{Q})\) since \(\rho(M, J)\) is maximal and Beauville’s theorem [Bea83] [Prop. 4.2] implies the existence of a hyperkähler structure \((M, J, J_2, J_3, g)\) on \(M\) where \([\omega]\) is the Kähler class of \((M, J, g)\). We define operators

\[\text{ad} J_i : \Lambda^0 J_i^q M \to \Lambda^{0+q} M \quad \text{with} \quad \eta \mapsto (p - q) \sqrt{\mathbb{I} - \eta}.\]

It is shown in [Ver95] [Prop. 2.1] that the Lie algebra \(\mathfrak{g}_M\) generated by \(\text{ad} J_1, \text{ad} J_2, \text{ad} J_3\) is isomorphic to \(\mathfrak{su}(2)\). Moreover, its action commutes with the Laplace operator and therefore induces an \(\mathfrak{su}(2)\)-action on the cohomology of \(M\). In particular, [Ver95] [Prop. 5.2] implies \(H_{inv} = H_{prim}^{1,1}(M)\) where \(H_{inv}\) is the space of all \(\mathfrak{g}_M\) invariant elements in \(H^2(M)\). Finally, if \(\alpha\) is the harmonic representant of some \([\alpha] \in H^2(M)\) then by definition \(\alpha \in \Lambda^1 J_1 M \cap \Lambda^1 J_2 M\) if and only if \([\alpha] \in H_{inv}\). Using the Hodge-Riemann bilinear form we derive a basis \(c_1, \ldots, c_{b_2-3} \in H^2(M, \mathbb{Q})\) of \(H_{prim}^{1,1}(M, J)\).

5. PP-WAVES AND COMPLETENESS

Now we are in the position to give examples of non-trivial spaces realizing the holonomy algebras \(\mathbb{R} \times \mathbb{R}^n\) and \(\mathbb{R}^n\). We say a Lorentzian manifold \((X, \tilde{g})\) with indecomposable and reducible (full) holonomy representation is a pp-wave if \(\mathfrak{hol}(X, \tilde{g}) = \mathbb{R} \times \mathbb{R}^n\) and a pp-wave if \(\mathfrak{hol}(X, \tilde{g}) = \mathbb{R}^n\). In particular, a pp-wave or a pp-wave is parallelizable if and only if it is time-orientable and \(\text{Hol}(S, \nabla^S)\) is connected. Lemma 4.2 implies

Example 5.1. Let \(T^n = S^1 \times \ldots \times S^1\) be the \(n\)-dimensional torus and \(g\) the standard flat Riemannian metric on \(T^n\). The standard coordinates induce a global trivialization \((\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y^n})\) of \(TT^n\) and \(dy^1 \wedge dy^2\) is the volume form of \(T^2 \to T^n\). On \(T^n\) we choose the \(S^1\)-bundle \(\tilde{X} \to T^n\) defined by \(cdy^1 \wedge dy^2\). If \((X = X \times L, \tilde{g}_f)\) is of toric type and \(f \in C^\infty(X)\) is sufficiently generic then

\[
\mathfrak{hol}(X, \tilde{g}_f) = \begin{cases} \mathbb{R}^n & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ \mathbb{R} \times \mathbb{R}^n & \text{otherwise.} \end{cases}
\]

Example 5.1 does not provide a 3-dimensional example. However, the only possible reducible indecomposable holonomy algebras in dimension 3 are \(\mathbb{R}\) and \(\mathbb{R} \times \mathbb{R}\). We conclude
Example 5.2. Let $T^2$ be the flat torus with standard coordinates $(y^1, y^2)$. Define $X \to T^2$ using the volume form and Proposition 3.1 with $\eta = dy^2$. If $f \in C^\infty(X)$ is sufficiently generic then

$$\text{hol}(X, \tilde{g}_f) = \begin{cases} \mathbb{R} & \text{if } \frac{\partial f}{\partial x} \equiv 0, \\ \mathbb{R} \times \mathbb{R} & \text{otherwise.} \end{cases}$$

In particular, $X$ is totally twisted.

Proof. If $X = S^1 \times Y$ then Gysin’s sequence implies $b_2(X) = b_2(Y) + b_1(Y) = 2$ and $b_1(X) = 1 + b_1(Y) = 2$, i.e., $\chi(Y) = 1$ and $b_2(Y) = 1$. Finally, the classification of closed surfaces implies a contradiction. □

Finally, we will construct complete pp-waves using the following

Corollary 5.3. Let $T^{n+1}$ be the flat torus with $\psi := dy^1 \wedge dz$ and $\eta := dz$ where $(y^1, \ldots, y^n, z)$ are the standard coordinates. If $(X, \tilde{g}_f)$ is constructed as in Proposition 3.1 with $f \in C^\infty(T^{n+1})$ sufficiently generic then $(X, \tilde{g}_f)$ is a complete compact pp-wave.

Proof. We have to show that the geodesics are defined for all $t \in \mathbb{R}$. Our approach is motivated by [CFS03]. Let $F_{n+1} : \mathbb{R}^{n+1} \to T^{n+1}$ be the universal covering map and consider the diagram

We write $g := (F_{n+1}^* \circ F_1 \times \text{id})^* \tilde{g}_f$. Then

$$g = 2dxdz + (y^1 + f + 1)dz^2 + \sum_{i=1}^n (dy^i)^2.$$

Let $\gamma(t) = (x(t), y^i(t), z(t))$ be a curve on $\mathbb{R}^{n+2}$ of constant energy $E_{\gamma} := g(\dot{\gamma}, \dot{\gamma})$. We compute

$$0 = \dot{z} + \Gamma^{n+1}_{ij} \dot{x}^i \dot{x}^j = \ddot{z},$$

$$0 = \dot{y}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k + \dot{y}^i + \frac{z^2}{2} \frac{\partial f}{\partial y^i} + \delta^i_1,$$

$$0 = \dot{x} + \Gamma^0_{ij} \dot{x}^i \dot{x}^j = \ddot{x} - \frac{1}{2} \frac{\partial f}{\partial z} \dot{z}^2 + \sum_{i=1}^n \frac{\partial f}{\partial y^i} \dot{z} \dot{y}^i.$$

Hence $\dot{z} = A$ is constant. Let $\gamma_2$ be the projection of $\gamma$ to $\mathbb{R}^n \subset \mathbb{R}^{n+2}$ given by the $(y^i)$ coordinates. Then

$$(5.1) \quad \frac{\nabla}{dt} \dot{\gamma}_2 = A^2 \left( \text{grad}_{\gamma} f + \frac{\partial}{\partial y^1} \right) \quad \text{on } (\mathbb{R}^n, \langle \cdot, \cdot \rangle).$$
Assume \( \gamma_2 \) is defined for all \( t \in \mathbb{R} \). Since \( E_\gamma = 2\dot{x} \dot{z} + (y^1 + f + 1) \dot{z}^2 + \sum_{i=1}^{n} (\dot{y}^i)^2 \), we conclude \( x(t) = x(0) + t \) if \( \dot{x} = 0 \) and

\[
x(t) = x_0 + \frac{1}{2A} \int_0^t E_\gamma - g(\gamma_2, \gamma_2) - A^2(f(\gamma_2(s)) + 1 + y^1(s)) ds
\]

otherwise. In order to show the existence of \( \gamma_2 \) for all \( t \in \mathbb{R} \) we define \( \alpha(t) := (\gamma_2, \dot{\gamma}_2) \) and

\[
F(x_1, \ldots, x_{2n}) := (x_n, \ldots, x_{2n}, \frac{A^2}{2}(\partial_1 f + 1), \frac{A^2}{2} \partial_2 f, \ldots, \frac{A^2}{2} \partial_n f).
\]

Then (5.1) is equivalent to \( \dot{\alpha} = F(\alpha) \). Let \( C := \sup_{T_{n+1}} |f| + \sup_{T_{n+1}} |\nabla f| \). If \( \alpha \) is not defined for all \( t \in \mathbb{R} \) then it must leave any compact set. However, \( \alpha(t) = \alpha(t_0) + \int_{t_0}^t F(\alpha(s)) ds \) and Gronwall’s lemma imply that \( \alpha \) is bounded on any \([t_0, t_1] \) since

\[
\|F(x)\|^2 \leq \sum_{j=1}^{n} (x_{j+n})^2 + \frac{A^4}{4}(C^2 + 2C + 1) \leq \|x\|^2 + \frac{A^4}{4}(C^2 + 2C + 1).
\]

Hence, \( \gamma_2 \) is defined for all \( t \in \mathbb{R} \).

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