Matrix Product Eigenstates for One-Dimensional Stochastic Models and Quantum Spin Chains

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Abstract

We show that all zero energy eigenstates of an arbitrary $m$–state quantum spin chain Hamiltonian with nearest neighbor interaction in the bulk and single site boundary terms, which can also describe the dynamics of stochastic models, can be written as matrix product states. This means that the weights in these states can be expressed as expectation values in a Fock representation of an algebra generated by $2m$ operators fulfilling $m^2$ quadratic relations which are defined by the Hamiltonian.

Key words: Reaction-diffusion systems, quantum spin chains, matrix product states

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\section{Introduction}

A number of problems in many particle systems have been studied with the help of so called matrix product states. The idea of this technique is to express physical quantities such as ground state wave functions or correlation functions as products of operators acting on an auxiliary space and fulfilling algebraic relations defined by the Hamiltonian of the system. Introduced in the context of lattice animals \cite{1} the technique has been used to find ground states of various quantum spin chains \cite{2}-\cite{4}.

As it has been shown by Derrida, Evans, Hakim, and Pasquier \cite{5} the steady state of the one species asymmetric exclusion process with open boundaries can be written as a matrix product state. Later works study this process in more detail \cite{6}-\cite{8}. Examples for asymmetric exclusion processes with two species were investigated in \cite{9,10} with the help of the same technique. The algebra used in these studies is generated by as many operators as states a single site can take. It has the important property that it leads to recurrence relations for the steady state of systems of different lattice lengths. For other problems this algebra had to be generalized by enclosing additional operators. This was first done in the study of the dynamics of the asymmetric exclusion process \cite{11,12}. Another example is the algebra for the reaction–diffusion model studied in \cite{13} which is generated by twice as many operators as the one from \cite{5}. In this Fock–algebra the recursion property is lost. The same is true for the algebra used in \cite{14}-\cite{17} for stochastic models with parallel updating.

One of the questions coming up naturally in this context is the following: To which kind of problems can the matrix technique be applied? In other words: Does a Hamiltonian, which describes either a quantum spin system or a stochastic process, need to have any particular property in order to have matrix product eigenstates? In this paper we will prove the following proposition: \textit{Any zero energy eigenstate of a Hamiltonian with nearest neighbor interaction in the bulk and single site boundary terms can be written as a matrix product state with respect to the Fock–algebra (3.2)-(3.3), which is that of Ref. \cite{13}.} The technique of matrix product states was called matrix product ansatz in the literature. What we will see is that this technique, in the form of (3.2)-(3.3), is not an ansatz but rather an identical reformulation of the eigenvector equation for the zero energy eigenstate.

The bulk of the paper is organized as follows: In Section 2 we define the class of Hamiltonians and in Section 3 we give the proof of the proposition. We conclude with some remarks on matrix product techniques in Section 4.

\section{Definition of a class of models}

The Hamiltonian we are going to consider in this paper is of the form

\begin{equation}
H = h^{(L)} + \sum_{j=1}^{N-1} h_{j,j+1} + h^{(R)}
\end{equation}

with

\begin{equation}

h^{(L)} = h^{(l)} \otimes I^{\otimes (N-1)} , \quad h^{(R)} = I^{\otimes (N-1)} \otimes h^{(r)}
\end{equation}
and \( h_{j,j+1} = I^\otimes(j-1) \otimes h \otimes I^\otimes(N-j-1) \) \( (2.3) \)

where \( I \) is the \( m \times m \)-identity matrix, \( h \) is an \( m^2 \times m^2 \)-matrix describing a two site interactions in the bulk, and \( h^{(l)}, h^{(r)} \) are \( m \times m \)-matrices defining single-site boundary terms. This kind of Hamiltonian appears as well in the study of one–dimensional stochastic systems as in the study of quantum spin chains. We will now describe both applications in more detail.

**Stochastic model:** Let us consider a one-dimensional lattice with \( N \) sites, each of which can be in either of \( m \) states. A configuration on the lattice is completely defined by the set of occupation numbers \( \{ s_i \} = s_1, s_2, \ldots, s_N \) with \( s_i = 1, \ldots, m \ \forall \ i = 1, \ldots, N \). The system evolves stochastically. During an infinitesimal time step \( dt \) its configuration \( \{ s_i \} \) can change to a configuration \( \{ s'_i \} \) with a probability \( r(\{ s_i \}, \{ s'_i \}) dt \) where the \( r(\{ s_i \}, \{ s'_i \}) \) are referred to as rates. This process can be described in terms of a rate equation which reads

\[
\partial_t P(\{ s_i \}, t) = \sum_{\{ s'_i \} \neq \{ s_i \}} \left[ r(\{ s'_i \}, \{ s_i \}) P(\{ s'_i \}, t) - r(\{ s_i \}, \{ s'_i \}) P(\{ s_i \}, t) \right] \ (2.4)
\]

where \( P(\{ s_i \}, t) = P(s_1, s_2, \ldots, s_N, t) \) is the probability of finding the configuration \( s_1, s_2, \ldots, s_N \) at time \( t \). Throughout this paper we restrict ourselves to dynamics where the configuration can change only at two adjacent sites at a time and the rate for such a change depends only on these two sites. The rates are assumed to be independent of the position in the bulk of system. At the boundaries, i.e. at sites 1 and \( N \), we assume additional processes to take place.

It is convenient to introduce a vector notation by writing

\[
| P(t) \rangle = \sum_{\{ s_i \}} P(s_1, s_2, \ldots, s_N, t) \left| s_1 \right\rangle \otimes \left| s_2 \right\rangle \otimes \cdots \otimes \left| s_N \right\rangle \quad (2.5)
\]

with

\[
| k \rangle = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1 \\
0 \\
\vdots \\
0 \\
\vdots \\
0
\end{pmatrix} \left\downarrow \begin{align*}
&1 \\
&k \\
&m
\end{align*} \right.
\ (2.6)

In terms of these vectors the rate equation reads

\[
\partial_t | P(t) \rangle = -H | P(t) \rangle \quad (2.7)
\]

where \( H \) is an \( m^N \times m^N \)-matrix which is defined in terms of the rates. We call \( H \) the Hamiltonian here. For the processes we are studying it has the structure of \( \text{(2.1)-(2.3)} \). For stochastic models the matrices \( H \) have a particular property: They have to have vanishing column sums because the total probability has to be conserved. This implies that there is a
zero energy eigenstate for every lattice length $N$ which is just the steady state of the system. It is denoted by $|P_N\rangle$ and obeys the relation
\[ H |P_N\rangle = 0. \] (2.8)
In Section 3 we will discuss a representation of these states.

Quantum spin chain: Consider a chain of length $N$ with a spin $s$-particle sitting on each site. Suppose there is an interaction between adjacent particles and some surface fields acting on sites 1 and $N$. Then the system dynamics is described by a Schrödinger equation with a Hamiltonian of type (2.1)-(2.3) if we chose a spin $s$-representation and $m = 2s + 1$. The Hamiltonian generally does not have vanishing columns sums like in the stochastic case, though it has to be hermitian. This condition does not effect the construction of matrix product states below. Although the physical meaning of the Schrödinger equation is different from the meaning of the rate equation (2.7) it has the same mathematical structure. And just as for the stochastic model we are often interested in the ground state of the Hamiltonian $H$.

3 Matrix product states

Let us now turn to the matrix product states [5]-[13]. We introduce an auxiliary vector space $V_a$ and we define $2m$ operators $D_s$ and $X_s$ with $s = 1, 2, ..., m$ acting on $V_a$ as well as two vectors $|W\rangle$ and $<V|$ in $V_a$. (Vectors in the auxiliary space are denoted by $|\ldots\rangle$ in contrast to the vectors in the configuration space which are denoted by $|\ldots\rangle$.) Next we define a vector $|\tilde{P}_N\rangle$ as
\[ |\tilde{P}_N\rangle = < W | \left( \begin{array}{c} D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_m \end{array} \right)^{\otimes N} | V \rangle. \] (3.1)
where $\otimes$ stands for the direct product in the configuration space, so that the above equation reads in terms of its components: $\tilde{P}_N(s_1, s_2, ..., s_N) = < W | D_{s_1} D_{s_2} ... D_{s_{N-1}} D_{s_N} | V \rangle$. We call states of type $|\tilde{P}_N\rangle$ matrix product states.

Let $H$ be a Hamiltonian of type (2.1)-(2.3) which has a zero energy eigenstate for all lattice lengths $N$ (a Hamiltonian describing a stochastic model always has this property). Then we can state the following proposition:

(i) If $D_s$, $X_s$ and $|W\rangle$, $<V|$ fulfill the following relations
\[ h \left[ \begin{array}{c} \begin{array}{c} D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_m \end{array} \otimes \begin{array}{c} D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_m \end{array} \end{array} \right] = \left[ \begin{array}{c} X_1 \\ X_2 \\ \cdot \\ \cdot \\ X_m \end{array} \otimes \begin{array}{c} D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_m \end{array} \right] - \left[ \begin{array}{c} D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_m \end{array} \otimes \begin{array}{c} X_1 \\ X_2 \\ \cdot \\ \cdot \\ X_m \end{array} \right], \] (3.2)
Let us define for any number $|s|$ the vector $|s\rangle$ at position 1 + $D$ orthogonal basis vectors operators $D_s \langle s\rangle$ transposed $\langle s\rangle W$.

The expectation values of products of different length in the operator algebra defined in [5]-[7], the relations (3.2)-(3.3) do not lead to recurrence relations for $|s\rangle$.

The mechanism is basically the same as the one worked out in Ref. [5]. In Appendix A we explain it in detail.

Before we come to the proof of the statement we stress again that, in contrast to the algebra defined in [5]-[7], the relations (3.2)-(3.3) do not lead to recurrence relations for expectation values of products of different length in the operators $D_s$ and $X_s$.

For any vector $|s\rangle$ solving $H |s\rangle = 0$ one can find operators $D_s$, $X_s$ ($s = 1, 2, ..., m$) and vectors $|\psi\rangle$, $|\phi\rangle$ in some space $V_a$, such that $|s\rangle$ can be represented as a matrix product state $|s\rangle$ defined by Eq. (3.2) and relations (3.3) are fulfilled in $V_a$.

The proof of (ii) is based on a site by site cancellation of terms when $H$ is applied on $|s\rangle$. The mechanism is basically the same as the one worked out in Ref. [5]. In Appendix A we explain it in detail.

In order to prove (ii) we construct a representation of operators and vectors fulfilling the Fock-algebra (3.2)-(3.3) using the eigenvectors $|s\rangle$ of systems with lengths $N = 1, 2, ..., M$. Let us define for any number $M = 1, 2, ..., M$-dimensional space $V_M$ with a set of orthogonal basis vectors $|s_1, s_2, ..., s_M\rangle$ as well as the one-dimensional space $V_0$ with the basis vector |. We define the space $V_a$ as the direct sum of all $V_M$ with $M = 0, 1, 2, ..., M$. The above basis vectors may be written as infinite column vectors with a 1 at position $1 + s_1 + s_2 + s_3 + ... + s_M$ and a 0 at all other positions, and their transposed $|s_1, s_2, ..., s_M\rangle$ can be written as the corresponding row vectors. Next we define operators $D_s$ and $X_s$ as well as vectors $|\psi\rangle$ and $|\phi\rangle$ by means of their action on all the $|s_1, s_2, ..., s_M\rangle$ which provides a matrix representation if we write the $|s_1, s_2, ..., s_M\rangle$ as columns. The $D_s$ we define by

$$D_s |s_1, s_2, ..., s_M\rangle = |s_1, s_2, s_3, ..., s_M\rangle$$

for $N = 1, 2, ..., M$ and $s_i = 1, 2, ..., m$.

The $X_s$ act on the basis vectors of $V_{N-1}$, i.e. on the $|s_1, s_2, ..., s_{N-1}\rangle$, as

$$\left( \begin{array}{c} N-1 \text{ numbers} \\ 1, 1, ..., 1 > \\ 1, 1, ..., 2 > \end{array} \right) \otimes \left( \begin{array}{c} X_1 \\ X_2 \\ . \\ . \\ X_m \end{array} \right) = \left( \begin{array}{c} m, m, ..., m-1 > \\ m, m, ..., m > \end{array} \right)$$

for $N = 1, 2, ..., m$. 

\[
< W | h^{(l)} \left( \begin{array}{c} D_1 \\ D_2 \\ . \\ D_m \end{array} \right) = - < W | h^{(r)} \left( \begin{array}{c} X_1 \\ X_2 \\ . \\ X_m \end{array} \right) and \quad h^{(r)} \left( \begin{array}{c} D_1 \\ D_2 \\ . \\ D_m \end{array} \right) |V> = \left( \begin{array}{c} X_1 \\ X_2 \\ . \\ X_m \end{array} \right) |V>, \quad (3.3)
\]

then the vector $|\tilde{P}_N\rangle$ defined by Eq. (3.4) solves the equation for the zero energy eigenstate, i.e.,

$$H |\tilde{P}_N\rangle = 0.$$
\[
\left( \begin{array}{c}
D_1 \\
D_2 \\
\vdots \\
D_m
\end{array} \right)^\otimes N 
\left( h_{1,2} + h_{2,3} + \ldots + h_{N-1,N} + h^{(R)} \right) | >
\]

for \( N = 1, 2, \ldots \)

where \( h_{i,i+1} \) and \( h^{(R)} \) act on the N-fold tensor product in Eq. (3.6) according to their definitions (2.2) and (2.3). Note that each choice of \( N \) gives a different set of basis vectors in the column on the l.h.s., and taking successively \( N = 1, 2, \ldots \) all basis vectors will occur in this column exactly once. Hence, the above equation defines the action of each \( X_s \) on each basis vector in a consistent way. The vectors \( |V > \) and \( < W | \) we define by

\[
|V > = | >
\]

\[
< W|s_1, s_2, \ldots, s_N > = P_N(s_1, s_2, \ldots, s_N)
\]

for \( N = 1, 2, \ldots \) and \( s_i = 1, 2, \ldots, m \)

where \( P_N(s_1, s_2, \ldots, s_N) \) are the components of \( |P_N > \). The above definitions fix the \( D_s \), \( X_s \) and the \( < W | \), \( |V > \) completely up to a constant \( < W | > \). This constant can be fixed arbitrarily, since it does not enter into any result.

We have to prove now that these operators and vectors lead back to \( |P_N > \) by means of Eq. (3.4), i.e., that \( \tilde{P}_N \) is equal to \( |P_N > \), and that they obey the Fock–algebra (3.2)-(3.3). Let us begin the first proof by rewriting (3.5) as

\[
D_{s_1}D_{s_2}\ldots D_{s_N}| > = |s_1, s_2, \ldots, s_N >
\]

for \( N = 1, 2, \ldots \) and \( s_i = 1, 2, \ldots, m \).

(The above relation gives a more intuitive meaning of the representation we have chosen: A basis vector \( |s_1, s_2, \ldots, s_M > \) is created by applying the sequence \( D_{s_1}D_{s_2}\ldots D_{s_M} \) on \( | > \).)

Furthermore, using (3.8) we get:

\[
< W|D_{s_1}D_{s_2}\ldots D_{s_N}| > = P_N(s_1, s_2, \ldots, s_N)
\]

for \( N = 1, 2, \ldots \) and \( s_i = 1, 2, \ldots, m \).

Rewriting (3.10) as a direct product of vectors in the configuration space and using \( | > = |V > \) yields

\[
< W| \left( \begin{array}{c}
D_1 \\
D_2 \\
\vdots \\
D_m
\end{array} \right)^\otimes N |V > = |P_N >
\]

for \( N = 1, 2, \ldots \).

which is what we were to prove.
In order to show that above representation fulfills the algebra \((3.2)-(3.3)\) we rewrite the definition \((3.6)\) using Eq. \((3.9)\) and multiply the column vector \begin{pmatrix} D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_m \end{pmatrix}\) from the left:

\[
\begin{pmatrix} D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_m \end{pmatrix} ^\otimes n \otimes \begin{pmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ X_m \end{pmatrix} = [h_{n+1,n+2} + h_{n+2,n+3} + \ldots + h_{N-1,N} + h^{(R)}] \begin{pmatrix} D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_m \end{pmatrix} ^\otimes N
\]

for \(n = 0, 1, \ldots, N - 1\) and \(N = 1, 2, \ldots\). The right equation of \((3.3)\) is nothing but the case \(n = 0\), \(N = 1\) of above relation. Consequently it holds. Next we use Eq. \((3.12)\) to compute

\[
\begin{pmatrix} D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_m \end{pmatrix} ^\otimes n \otimes \begin{pmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ X_m \end{pmatrix} = h_{1,2} \begin{pmatrix} D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_m \end{pmatrix} ^\otimes N
\]

i.e.,

\[
\begin{pmatrix} D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_m \end{pmatrix} ^\otimes (N-2) = 0 \quad \text{for} \quad N = 2, 3, \ldots.
\]
Since the products of the $D_a$ span the whole space $V_a$, Eq. (3.2) must hold in $V_a$. What remains to be proven is the left equation of (3.3). We multiply Eq. (3.12) for $n = 0$ by $\langle W|$ from the left and use Eq. (3.11):

$$< W| \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix} \otimes \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_m \end{pmatrix}^{\otimes(N-1)} | > = [h_{1,2} + h_{2,3} + \ldots + h_{N-1,N} + h^{(R)}] | P_N \rangle. \quad (3.15)$$

Writing now the Hamiltonian of an $N$-site system as $H = h^{(L)} + h_{1,2} + h_{2,3} + \ldots + h_{N-1,N} + h^{(R)}$ and using $H|P_N\rangle = 0$, the above relation can be written as:

$$< W| \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix} \otimes \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_m \end{pmatrix}^{\otimes(N-1)} | > = -h^{(L)} | P_N \rangle. \quad (3.16)$$

Using again Eq. (3.11) yields

$$< W| \left\{ h^{(L)} \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_m \end{pmatrix} + \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix} \right\} \otimes \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_m \end{pmatrix}^{\otimes(N-1)} | > = 0 \quad (3.17)$$

for $N = 1, 2, \ldots$.

Consequently the left equation of (3.3) holds in $V_a$. This completes the proof that the matrices defined by Eq. (3.5)-(3.8) represent the algebra (3.2)-(3.3) and hence the proof of proposition (ii).

Let us add some remarks on the proof. We proved that the Fock algebra (3.2)-(3.3) has a nontrivial representation if an eigenstate with zero energy exists for all lattice lengths. The construction of the matrices obeying (3.2) can be done without any demand for special properties of $H$. (An associative algebra generated by $2m$ generators with $m^2$ quadratic relations is always a well defined mathematical object.) The special property of $H$, i.e. the existence of zero energy eigenstates $| P_N(s_1, s_2, \ldots, s_N) \rangle$, was only needed for the construction of a non trivial scalar product involving the vectors $< W|$ and $| V >$. More precisely, the vectors $| P_N(s_1, s_2, \ldots, s_N) \rangle$ only enter the definition of the vector $< W|$ and the equation which determines $< W|$ is just the equation for the zero energy eigenstate.

Due to the lack of recurrence relations the scalar product in (3.2)-(3.3) is not completely fixed and can be chosen independently in each vector space $V_M$. For that reason it would be
sufficient to require that at least for one lattice length $N$ the Hamiltonian has a zero energy eigenstate. In that case $< W |$ is nontrivial only on the subspace $V_N$.

One may define a finite dimensional $V_a$ as the direct sum over all $V_M$ with $M \leq N_{\text{max}}$ for some number $N_{\text{max}}$. A matrix representation of the algebra (3.2)-(3.3) can then be found by assuming Eq. (3.3)-(3.8) for $N = 1, 2, ..., N_{\text{max}}$ as well as $D_s |s_1, s_2, ..., s_{N_{\text{max}}}, > = 0$ and $X_s |s_1, s_2, ..., s_{N_{\text{max}}}, > = 0$. Then Eq. (3.1) provides a matrix product representation for all eigenvectors $|P_N \rangle$ with $N \leq N_{\text{max}}$.

It is of course also possible to express eigenstate of non zero eigenvalues as matrix product state by adding an appropriate shift to the Hamiltonian. This is done by replacing $h, h^{(l)}$, and $h^{(r)}$ in Eqs. (3.2)-(3.3) by $h - \epsilon$, $h^{(l)} - \epsilon^{(l)}$, and $h^{(r)} - \epsilon^{(r)}$, respectively. In this case the Hamiltonian $H$ in Eq. (3.4) has to be replaced by $H - E(N)$ with $E(N) = \epsilon^{(l)} + \epsilon^{(r)} + (N - 1)\epsilon$ if $N$ is the lattice length. The vector $|P_N \rangle$ is then an eigenstate with energy $E(N)$. However, if $\epsilon^{(l)}, \epsilon^{(r)}$, and $\epsilon$ are independent of $N$, Eq. (3.4) will have a non trivial solution only for a finite number of lattice lengths $N$ and $< W |$ will therefore be non trivial only on the corresponding subspaces $V_N$ of $V_a$.

4 Concluding Remarks

We proved that any zero energy eigenstate of a Hamiltonian of the form (2.1) can be written as a matrix product state with respect to the Fock representation of the algebra (3.2)-(3.3). The proof has been done by showing that the operators as well as the scalar product can be defined in a non trivial way. The equations defining the scalar product are exactly the equations for the zero energy eigenstate. This shows that by application of the algebra on an abstract level, i.e. by using only the relations (3.2)-(3.3), one can not gain any insight into the form of the eigenstates. One ends up with nothing but a reformulation of the equation for the zero energy eigenstate. Therefore the so called matrix product ansatz, i.e. the application of the Fock–algebra (3.2)-(3.3) to an eigenvector problem, is not really an ansatz, it is an identity. The eigenvector problem is not transformed into another problem, it remains unchanged. However, the situation becomes different if one considers a more special form of the algebra as done in [14] where the operators $X_s$ are replaced by numbers.

We checked that our proposition is also true for the Fock–algebra describing stochastic models with parallel updating [14].

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A Appendix: Proof of eigenvalue equation for matrix product state

Following the line of Ref. [5] we prove proposition (i), i.e., we show that the states (3.1) solve the eigenvalue equation (3.4) if (3.2)-(3.3) are fulfilled. Using the definition (2.1) we write

\[ H = h^{(l)} \otimes I^{\otimes(N-1)} + \sum_{j=1}^{N-1} I^{\otimes(j-1)} \otimes h \otimes I^{\otimes(N-j-1)} \]

\[ + I^{\otimes(N-1)} \otimes h^{(r)} \]  

(A.1)

Next we define a state \( |P_N(k) \rangle \) as

\[ |P_N(k) \rangle = \langle W | \begin{pmatrix} D_1 \otimes (k-1) \\ . \\ . \\ D_m \end{pmatrix} \otimes \begin{pmatrix} X_1 \\ . \\ . \\ X_m \end{pmatrix} \otimes \begin{pmatrix} D_1 \otimes (N-k) \\ . \\ . \\ D_m \end{pmatrix} |V \rangle . \]  

(A.2)

Applying now the terms occurring in Eq. (A.1) on the state

\[ |\tilde{P}_N \rangle = \langle W | \begin{pmatrix} D_1 \otimes N \\ . \\ . \\ D_m \end{pmatrix} |V \rangle \]  

(A.3)

yields

\[ h^{(l)} \otimes I^{\otimes(N-1)} |\tilde{P}_N \rangle = -|P_N(1) \rangle \]  

(A.4)

\[ I^{\otimes(j-1)} \otimes h \otimes I^{\otimes(N-j-1)} |\tilde{P}_N \rangle = |P_N(j) \rangle - |P_N(j+1) \rangle \]  

(A.5)

\[ I^{\otimes(N-1)} \otimes h^{(r)} |\tilde{P}_N \rangle = |P_N(N) \rangle \]  

(A.6)

where the algebra (3.2)-(3.3) was used. Applying now the full operator \( H \) (see Eq. (A.1)) on the state \( |\tilde{P}_N \rangle \) we get

\[ H |\tilde{P}_N \rangle = -|P_N(1) \rangle + \sum_{j=1}^{N} \left( |P_N(j) \rangle - |P_N(j+1) \rangle \right) + |P_N(N) \rangle \]

\[ = 0 \]  

(A.7)

which is what we were to show.
References

[1] V. Hakim and J. P. Nadal, J. Phys. A **16** (1983) L213
[2] M. Fannes, B. Nachtergale and R. F. Werner, Rev. Brasil. Fisica **19** (1989) 469
[3] M. Fannes, B. Nachtergale and R. F. Werner, Comm. Math. Phys. **144** (1992), 443
[4] A. Klümper, A. Schadschneider and J. Zittarz, J. Phys. A **24** (1991) L955
[5] B. Derrida, M. R. Evans, V. Hakim and V. Pasquier, J. Phys. A **26** (1993) 1493
[6] S. Sandow, Phys. Rev. E **50** (1994) 2660
[7] F. H. L. Essler and V. Rittenberg, J. Phys. A **29** (1996), 3375
[8] The matrix ansatz technique as well as some results are reviewed in: B. Derrida, M. R. Evans, *The Asymmetric Exclusion Model: Exact Results Through a Matrix Approach*, to be published in *Nonequilibrium Statistical Mechanics in One Dimension* ed. V. Privman, Cambridge University Press (1996)
[9] B. Derrida, S. A. Janowsky, J. L. Lebowitz and E. R. Speer, Europhys. Lett. **22** (1993) 651
[10] M. R. Evans, D. P. Foster, C. Godrèche and D. Mukamel, J. Stat. Phys. **80** (1995) 69
[11] R. B. Stinchcombe and G. M. Schütz, Phys. Rev. Lett. **75** (1995) 140
[12] G. M. Schütz, *Stochastic Reaction-Diffusion Processes, Operator Algebras and Integrable Quantum Spin Chains*, to appear in the Proceedings of the Satellite Meeting to Statphys 19 on Statistical Models, Yang-Baxter Equation and Related Topics, at Nankai University, Tianjin (August 1995), eds. F.Y. Wu and M.L. Ge, (World Scientific, Singapore)
[13] H. Hinrichsen, S. Sandow and I. Peschel, J. Phys. A **29** (1996), 2643
[14] H. Hinrichsen, J. Phys. A **29** (1996), 3659
[15] N. Rajewski, A. Schadschneider and M. Schreckenberg, J.Phys. A **26** (1996), L305
[16] A. Honecker and I. Peschel, Matrix-Product States for a One-Dimensional Lattice Gas with Parallel Dynamics, preprint cond-mat/9606053
[17] H. Hinrichsen and S. Sandow, Deterministic Exclusion Process with a Stochastic Defect: Matrix Product Ground States, preprint cond-mat 9611134, to appear in J.Phys. A (1997)