Extended Palatini action for general relativity and the natural emergence of the cosmological constant

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Abstract

In the Palatini action of general relativity the connection and the metric are treated as independent dynamical variables. Instead of assuming a relation between these quantities, the desired relation between them is derived through the Euler-Lagrange equations of the Palatini action. In this manuscript we construct an extended Palatini action, where we do not assume any a priori relationship between the connection, the covariant metric tensor, and the contravariant metric tensor. Instead we treat these three quantities as independent dynamical variables. We show that this action reproduces the standard Einstein field equations depending on a single metric tensor. We further show that in this formulation the cosmological constant has an additional theoretical significance. Normally the cosmological constant is added to the Einstein field equations for the purpose of having general relativity be consistent with cosmological observations. In the formulation presented here, the nonvanishing cosmological constant also ensures the self-consistency of the theory.

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I. INTRODUCTION

In metric theories of gravity, complete information about the spacetime geometry is encoded in the components of the metric tensor $g_{\mu\nu}$. Yet a complete field theory on a curved spacetime also requires a spacetime connection $\Gamma^\alpha_{\beta\gamma}$ needed for the definition of the covariant derivative, and a contravariant metric tensor $g^{\mu\nu}$ needed for the construction of norms of covariant vector fields. In the Einstein-Hilbert action of general relativity (GR) it is assumed that the spacetime connection is the Levi-Civita connection, determined by the metric alone, and furthermore it is assumed that the contravariant tensor $g^{\mu\nu}$ that appears in the combination $g^{\mu\nu}R_{\mu\nu}$ in the Einstein-Hilbert action is equal to the inverse matrix of the covariant metric tensor $g_{\mu\nu}$ also appearing in this action. This paper will discuss removing these implicit assumptions and having the relation between the connection $\Gamma^\alpha_{\beta\gamma}$ and the metric $g_{\mu\nu}$, and the relation between the contravariant metric tensor $g^{\mu\nu}$ and the covariant metric tensor $g_{\mu\nu}$ become predictions of the theory. The first of these assumptions is removed by the Palatini action [1]. In this action the connection and the metric are considered to be independent dynamical variables, thereby producing two sets of Euler-Lagrange equations. One set of equations are the Einstein’s field equations (depending on the connection and the metric), and the other set of equations ensures that the connection is metric compatible, and therefore equals to the Levi-Civita connection. The Palatini formalism has been useful in extensions and modification of GR and also in quantization of gravity, see for example Refs. [2, 3, 4, 5, 6]. In this paper, we extend Palatini’s formalism and construct an extended Palatini action in which the connection $\Gamma^\alpha_{\beta\gamma}$, the contravariant metric tensor $g^{\mu\nu}$, and the covariant metric tensor $g_{\mu\nu}$ that appear in the action are considered to be independent dynamical variables. This action may find similar application in modification or quantization of GR.

Our extended Palatini action reproduces GR provided that the cosmological constant is nonvanishing. This is a rather unusual property of this formalism. Usually, the cosmological constant is added to the Einstein equations in order that the predictions of GR be consistent with the observed accelerated expansion of the Universe, but it does not serve an additional theoretical purpose. In the formulation presented here, the cosmological constant emerges naturally as an ingredient that ensures that the theory is self-consistent, and that the Einstein field equations follow from the Euler-Lagrange equations. It is interesting to
note that the cosmological constant has a similar theoretical significance in a very different purely affine gravitational action due to schrödinger [7].

II. ACTION AND FIELD EQUATIONS

To see how one arrives to the gravitational action presented in this paper, it is instructive to consider first the Einstein-Hilbert action. This action is a functional of the metric alone and it reads

\[ S_{EH}[g_{\mu\nu}] = \frac{1}{16\pi} \int g^{\mu\nu} R_{\mu\nu}(g) \sqrt{- \det(g_{\alpha\beta})} d^4x , \]  

(2.1)

where we set \( G = c = 1 \) throughout. Here there are two implicit assumptions. First, it is assumed that the spacetime connection is the Levi-Civita connection, so that the Ricci tensor \( R_{\mu\nu}(g) \) which is normally determined by the connection, is now determined by the metric. Second, it is assumed that the contravariant tensor \( g^{\mu\nu} \) equals to the inverse matrix of the metric tensor \( g_{\mu\nu} \). The Palatini action provides an alternative action, which reproduces GR without making the first implicit assumption. This action reads

\[ S_{Palatini}[g_{\mu\nu}, \Gamma^\alpha_{\beta\gamma}] = \frac{1}{16\pi} \int g^{\mu\nu} R_{\mu\nu}(\Gamma) \sqrt{- \det(g_{\alpha\beta})} d^4x . \]  

(2.2)

Here the action is a functional of both the metric and the connection¹, and the Ricci tensor \( R_{\mu\nu}(\Gamma) \) by its definition depends only on the connection. Assuming that the connection is symmetric \( \Gamma^\alpha_{[\beta\gamma]} = 0 \) the Euler-Lagrange equations give rise to the Einstein’s field equations together with the following equations

\[ \nabla_\epsilon g^{\mu\nu} + g^{\mu\nu} \left[ \partial_\epsilon \log \sqrt{- \det(g_{\alpha\beta}) - \Gamma_\alpha^{\alpha\epsilon}} \right] = 0 , \]  

(2.3)

where \( \nabla_\epsilon \) denotes the standard covariant derivative depending on the connection. Relation (2.3) imply that connection is metric compatible \( \nabla_\epsilon g^{\mu\nu} = 0 \), and so it ensures that the connection equals to the Levi-Civita connection.

We now extend the Palatini action and remove the assumption that the tensor \( g^{\mu\nu} \) equals to the inverse matrix of the metric \( g_{\mu\nu} \). For this purpose we replace the inverse metric tensor

¹ For completeness we also need to supplement \( S_{EH} \) and \( S_{Palatini} \) with a matter action \( S_M[\psi, g_{\mu\nu}] \), where \( \psi \) collectively denotes the matter fields. This introduces some ambiguity in the case of the Palatini action, where it is possible to have a matter action which is also a functional of the connection (e.g. in a tetrad formulation for fermions). Here for simplicity we assume that the matter action depends only on the matter fields and the metric.
$g^{\mu \nu}$ in the Palatini action (2.2) with a new independent dynamical variable denoted $\tilde{g}^{\mu \nu}$. To recover the Einstein field equations we also multiply the Lagrangian density by $16(g_{\mu \nu}\tilde{g}^{\mu \nu})^{-2}$ and introduce a cosmological constant. The resultant gravitational action reads

$$S_G[g_{\alpha \beta}, \tilde{g}^{\mu \nu}, \Gamma^\rho_{\sigma \eta}] = \frac{1}{\pi} \int \left[ \tilde{g}^{\alpha \beta} R_{\alpha \beta}(\Gamma) - 2\Lambda \right] (g_{\mu \nu}\tilde{g}^{\mu \nu})^{-2} \sqrt{-\det(g_{\rho \sigma})} d^4x. \quad (2.4)$$

Here $g_{\alpha \beta}$, $\tilde{g}^{\mu \nu}$ and $\Gamma^\rho_{\beta \gamma}$ are independent dynamical variables and we assume that the connection is symmetric in its lower indices. Notice that from its definition this action has two independent metric tensors: a contravariant tensor $\tilde{g}^{\mu \nu}$ and a covariant tensor $g_{\mu \nu}$. For clarity we shall not raise or lower indices with any of these metric tensors. In addition we introduce the following matter action

$$S_M = S_M[\tilde{g}^{\mu \nu}, \psi]. \quad (2.5)$$

Here $\psi$ collectively denote the matter fields. Notice that this action does not depend on the metric $g_{\mu \nu}$. The complete action is given by

$$S = S_G + S_M. \quad (2.6)$$

The action $S$ gives rise to the following Euler-Lagrange equations

$$(\tilde{g}^{\alpha \beta} R_{\alpha \beta} - 2\Lambda)(g^{\mu \nu} - \frac{4}{d}\tilde{g}^{\mu \nu}) = 0, \quad (2.7)$$

$$(g^{\alpha \beta} g_{\alpha \beta} - 2\Lambda) = \frac{\pi d^2(\tilde{g}/g)^{1/2}}{2} T_{\alpha \beta} \quad (2.8)$$

$$(\nabla_\epsilon \tilde{g}^{\alpha \beta} + \tilde{g}^{\alpha \beta} [\partial_\epsilon \log(d^{-2}\sqrt{-g}) - \Gamma^\gamma_{\gamma \epsilon}] = 0, \quad (2.9)$$

$$\frac{\delta S_M}{\delta \psi} = 0. \quad (2.10)$$

Here we have introduced the tensors $g^{\alpha \beta}$ and $\tilde{g}_{\mu \nu}$ which are defined by $g^{\gamma \alpha}g_{\alpha \beta} = \delta^\gamma_\beta$ and $\tilde{g}^{\gamma \alpha}\tilde{g}_{\alpha \beta} = \delta^\gamma_\beta$, respectively. For brevity we have also introduced the notation $d \equiv g_{\rho \sigma}\tilde{g}^{\rho \sigma}$, $\tilde{g} \equiv \det(\tilde{g}_{\alpha \beta})$, and $g \equiv \det(g_{\alpha \beta})$. In addition, we have defined the energy-momentum tensor $T_{\alpha \beta}$ through

$$\delta S_M = -\frac{1}{2} \int T_{\alpha \beta} \delta \tilde{g}^{\alpha \beta} \sqrt{-\tilde{g}} d^4x.$$

Contracting Eq. (2.8) with $\tilde{g}^{\alpha \beta}$ gives

$$R_{\alpha \beta} \tilde{g}^{\alpha \beta} = -\frac{\pi}{2} d^2(\tilde{g}/g)^{1/2} T_{\mu \nu} \tilde{g}^{\mu \nu} + 4\Lambda.$$
Substituting this equation into Eq. (2.7) yields
\[ 2\Lambda - \frac{\pi}{2} d^2 (\tilde{g}/g)^{1/2} T_{\alpha\beta} \tilde{g}^{\alpha\beta} (g^{\mu\nu} - \frac{4}{d} \tilde{g}^{\mu\nu}) = 0. \] (2.11)

Notice that a contraction of Eq. (2.11) with the metric $g_{\mu\nu}$ yields an identity, thus providing a consistency check for this equation. To understand the consequences of Eq. (2.11) let us consider first an empty spacetime region with a nonvanishing cosmological constant, so that $T_{\alpha\beta} = 0$. In this region Eq. (2.11) is reduced to
\[ \Lambda (g^{\mu\nu} - \frac{4}{d} \tilde{g}^{\mu\nu}) = 0. \] (2.12)

Eq. (2.12) provides a relation between the metric $g_{\alpha\beta}$ and the metric $\tilde{g}^{\alpha\beta}$ provided that the cosmological constant does not vanish. Notice that if the cosmological constant does vanish the remaining Euler-Lagrange equations (2.8, 2.9, 2.10) are insufficient to determine any of the metric tensors. We now demand that theory be sufficiently general and admit solutions with empty regions of spacetime. By virtue of the above analysis this requirement forces us to impose a nonvanishing cosmological constant $\Lambda \neq 0$. Next, let us focus our attention to a region with matter, where $T_{\alpha\beta} \neq 0$. Here there are three possibilities: First, the expression in the square brackets in Eq. (2.11) may be different from zero. In this case Eq. (2.12) remains valid. Second, the expression in the square brackets may be different from zero everywhere except for a submanifold with dimensionality smaller than four (e.g. this expression may vanish on a three dimensional hypersurface). In this case, by continuity, Eq. (2.12) remains valid. Third, the expression in the square brackets may vanish in a four dimensional submanifold. In this case Eq. (2.12) does not follow from Eq. (2.11). However, this case is nongeneric and demands fine tuning, and so it does not correspond to a realistic physical configuration. Furthermore, given such a fine tuned configuration, we can normally introduce an infinitesimal (and so experimentally unobserved) modification to $T_{\alpha\beta}$ such that the expression in the square brackets would no longer vanish. For these reasons we discard these pathological field configurations. We now proceed to derive the Einstein field equations.

It is useful to rewrite (2.11) as
\[ \tilde{g}_{\mu\nu} = \frac{4}{d} g_{\mu\nu}. \] (2.13)

Substituting Eq. (2.13) into Eq. (2.9) gives
\[ \nabla_\epsilon \tilde{g}^{\alpha\beta} + \tilde{g}^{\alpha\beta} \left[ \partial_\epsilon \log \sqrt{-\tilde{g}} - \Gamma^\gamma_{\gamma\epsilon} \right] = 0. \] (2.14)
Eq. (2.14) has the same form as Eq. (2.3) produced by the standard Palatini action (2.2) and so it implies that the connection equals to the Levi-Civita connection evaluated with the metric $\tilde{g}^{\mu\nu}$. Substituting Eq. (2.13) into Eq. (2.8) yields the standard Einstein field equations

$$R_{\alpha\beta}(\tilde{g}) - \frac{1}{2} \tilde{g}_{\alpha\beta} \tilde{g}^{\mu\nu} R_{\mu\nu}(\tilde{g}) + \tilde{g}_{\alpha\beta} \Lambda = 8\pi T_{\alpha\beta}.$$  

(2.15)

Here $R_{\alpha\beta}(\tilde{g})$ denotes the Ricci tensor evaluated from the metric $\tilde{g}_{\alpha\beta}$. We see that Eq. (2.15) depends on the metric $\tilde{g}_{\mu\nu}$ and is independent of the metric $g_{\mu\nu}$. In other words we have found that the contravariant metric tensor $\tilde{g}^{\mu\nu}$ that appears in the action $S$ equals to the inverse matrix of the metric $\tilde{g}_{\mu\nu}$ that satisfies the Einstein field equations (2.15). Notice that the Euler-Lagrange equations (2.7, 2.8, 2.9, 2.10) have been reduced to three differential equations (2.10, 2.14, 2.15) and one algebraic relation (2.13). The fact that these differential equations depend only on the variables $\tilde{g}^{\mu\nu}$ and $\psi$ but are independent of the variable $g_{\mu\nu}$, signals that all of the independent physical degrees of freedom in this theory are encoded in the fields $\tilde{g}^{\mu\nu}$ and $\psi$. Furthermore, the absence of the metric $g_{\mu\nu}$ from the field equation that depend on the matter fields, Eqs. (2.10, 2.15), means that the metric $g_{\mu\nu}$ can not have any observational consequences. To further clarify this point notice that using Eq. (2.13) we can express the metric $g_{\mu\nu}$ as $g_{\mu\nu} = \Omega^2(x) \tilde{g}_{\mu\nu}$, where $\Omega^2(x)$ is a conformal factor. However, the conformal factor $\Omega^2(x)$ can not be determined by the Euler-Lagrange equations (2.7, 2.10). To see this, notice that the action $S$ is invariant under the conformal transformation $g_{\mu\nu} \rightarrow \Omega^2(x) g_{\mu\nu}$, where $\psi$ and $\tilde{g}^{\mu\nu}$ are kept fixed; implying that this conformal transformation is a symmetry of the theory. This ambiguity in the determination of the metric $g_{\mu\nu}$, as opposed to the well posed initial value formulation for the metric $\tilde{g}_{\mu\nu}$ reinforces the interpretation that $\tilde{g}_{\mu\nu}$ is the only physical metric in this theory.

### III. CONCLUSIONS

By extending the Palatini action, we have shown that the Einstein field equations can be derived from a gravitational action that does not make any prior assumptions about the relation between the contravariant metric and the covariant metric. In this theory the Einstein field equations (2.15) are valid only if the cosmological constant does not vanish.
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