Fractals in Linear Ordinary Differential Equations

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Abstract

We prove the existence of fractal solutions to a class of linear ordinary differential equations. This reveals the possibility of chaos in the very short time limit of the evolution even of a linear one dimensional dynamical system.

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1 Introduction

Understanding the complexities in a nonlinear dynamical system is of great interest in contemporary sciences. Both chaos and fractals seem to offer two important ingredients towards this effort. Even a simple quadratic nonlinearity as represented by the logistic map is known to generate, for a sufficiently large control parameter, a fractal attractor, indicating the onset of the deterministic chaos in the model. A linear equation, on the other hand, is supposed to be immune to fractal attractors (solutions). In this paper we, however, report, on the contrary, a rather surprising fact: the simplest linear ODE of the form \( \frac{dy}{dt} = h(t)y \) admits fractal solutions, besides the standard exponential solution. This observation follows directly from our recent results indicating a deep relationship between the quantal geometric phase and the concept of time derived intrinsically from the (quantal) evolution\(^1\)(denoted I henceforth). We give a brief review of this relationship in Sec.2. In Sec.3, we present the existence proof of the fractal solution for a (time dependent) first order linear ODE.

2 Geometric phase and Time

In a time dependent quantal evolution the evolving state acquires a geometric phase apart from the usual dynamical phase. Let the evolution of the quantal state \( \psi(t) \) be described by the Schrodinger equation

\[
\frac{ih}{dt} \psi = H \psi
\]

where \( H \) is a time dependent Hamiltonian operator and \( t \) is the external Newtonian time. As time flows, the quantal state \( \psi (\langle \psi|\psi \rangle = 1) \) traces a trajectory in the Hilbert space \( \mathcal{H} \), considered as a U(1) principal bundle over the projective space \( \mathcal{P} \). Due to the irreducible quantal uncertainty the actual path must always be fluctuating in nature thus allowing the evolving state to follow a path intersecting 'near by' rays in \( \mathcal{P} \). If the state is assumed to move along a cyclic path in \( \mathcal{P} \) it returns eventually to the initial ray, but with a phase difference. In the case of an open path, the phase difference between the initial and final states equals to that of the cyclic path obtained by joining the initial and the final rays in \( \mathcal{P} \) by the shortest geodesic. The total phase \( \gamma \) so acquired by the state now consists of two components: \( \gamma = \gamma_d + \gamma_g \), the dominant 'dynamical' and the relatively small 'geometric' phases respectively. The dynamical phase \( \gamma_d = h^{-1} \int h(t)dt \) where \( h(t) = \langle \psi|H\psi \rangle \) denotes the mean energy in the state, is a consequence of the mean dynamical evolution in the state. This mean evolution could further be realized as a pure verticle displacement of the state along the the fibre of its ray in the associated Hilbert bundle. Further, the scale of the external time \( t \) is set by the mean energy : \( t \sim \frac{1}{h_0}, h_0 = h(t_0), t_0=\text{the initial time} \). As shown in I, the geometric phase \( \gamma_g = \int A \) where \( A = -i < \phi|d|\phi > \) and \( \phi = e^{i\gamma}\psi \); on the otherhand could be interpreted as an effect of the inherent fluctuations in the actual quantal evolution. Although, the nature of this phase is of course geometric, as encoded in the associated parallel transport law, it appears ,in any way, as a correction term in the mean energy, thus offering itself to an equivalent dynamical (Hamiltonian) treatment. The geometric phase thus provides naturally a new (microscopic) time scale in the evolution, having a conjugate relationship instead with the energy uncertainty, rather than with the mean energy. Following the Leibnizian view\(^2\), we call this time scale(variable) the intrinsic (geometric) time. Interestingly, this dynamical treatment of the geometric phase then allows one to write down a scaled Schrodinger equation which is suitable for an independent dynamical description of the
fluctuating state. The reproduction of a(n) (almost) self-similar replica of the Schrodinger equation in the scale of a fluctuation involves a sort of a 'renormalization' in the original equation which amounts to a subtraction of the mean (dynamical) motion in the state. It also turns out (see below) that the intrinsic time scale relates inversely (dually) to the extrinsic one in the limit of large fluctuations $O(1)$ (strong interaction) in the state. Thanks to the irreducible property of the quantal uncertainty, one can then iterate the above steps an unlimited number of times, thus arriving at an interesting observation: **quantal fluctuations are essentially fractals: having self-similar structures at all time scales.** As a corollary, the trajectory of the evolving nonstationary state and hence the complex state function itself must also be fractals(at least in the time variable). The later conclusion was not stated explicitly in I. We shall supply a proof in favour of this in the next section.

A remark is in order here. Quantum mechanics is a linear theory, with only a first order time derivative in the Schrodinger equation. Further, the (quantal) uncertainty principle is a direct outcome of this inherent linearity. Any linear time invariant model e.g., one in a signal processing system, also exhibits time-energy uncertainty. The above analysis should therefore be applicable not only in the domains stated above but also to any first order linear ODE with nontrivial time dependence. We recall that the (nonadiabatic) geometric phase was historically first discovered not in quantum mechanics but in optical polarisations having a linear Schrodinger equation-like guiding equation.

### 3 Main Results

Let us consider a first order linear ordinary differential equation of the form

$$i \frac{d}{dt} y = h(t) y$$

where $h(t)$ is a real function of the real parameter $t$. The factor $i$ in the l.h.s. of eq(2) is introduced to keep the analogy with the Schrodinger equation in view. This is not necessary, however, in general. The complex functions $y(t)$ are assumed to belong to a linear (Hilbert) space. The corresponding projective space thus consists of the rays $y' = e^{i\alpha} y, \alpha$ real. For the sake of clarity, we assume eq(2) to represent a linear time invariant dynamical system with mean 'energy' $h_0 = h(t_0)$ at the initial time $t = t_0$. Eq(2) is thus (form) invariant under the group of translations, which in turn guarantees the uniqueness of the standard exponential solution $y = \exp(-i \int_{t_0}^t h(t)dt)$. We however wish to present a new class of fractal solutions to eq(2) by extending the group of translations to the (modular) group $SL(2,\mathbb{R})$. We remark that the existence of a *nontrivial* geometric phase naturally provides the necessary window for introducing the duality transformation in a time dependent (linear) dynamical system. However, it turns out that the technique developed in I on the basis of a geometric phase is more general and could in fact be used in a situation with a vanishing geometric phase.

Note that in a quantum mechanical model $h(t)$ stands for a time dependent Hamiltonian operator. In an ordinary classical situation eq(2) may either be considered as a matrix equation e.g., the Fermi-Walker transport equation for the polarised light or be assumed to involve a background source of a motive force, to generate a nontrivial geometric phase. We however restrict the discussion to the simplest case where $y$ denotes a single complex function having continuous first derivative. We show that $SL(2,\mathbb{R})$ acts as an invariance group even in this apparently trivial equation.

Let us proceed mimicking the steps of I thus indicating briefly how an intrinsic time variable $\tau$ appears in eq(2). Because of the time translation invariance, it is sufficient to study the nature of the solution
close to the initial time which we now set to \( t = 0 \). Further, assume \( t \) to be a dimensionless variable measured in the unit of \( \frac{1}{\nu_0} \). Thus the Taylor expansion of \( h(t) \) close to 0 in eq(2) yields

\[
\frac{d}{dt} y = \left[ 1 + \nu(t) \right] y
\]  

(3)

The time dependent term in the rhs of this equation could be interpreted as the correction due to 'fluctuations' over the mean 'energy'. For a very short time interval (i.e., neglecting \( O(t^2) \) terms), fluctuations can be approximated as \( \nu(t) = \nu_1 t \). In the external time scale fluctuations thus scale as \( t \). However, \( \nu \) gives rise to a new time scale at the level of the first-order fluctuation, which follows from the equivalence of the description of the present model with that in I. Let us denote this intrinsic time scale by \( \tau \sim \frac{1}{\nu_1} \).

To make the analogy with I complete let us write the 'Born-Oppenheimer' ansatz \( y = y_0 y_1, y_0 = e^{-it} \), and introduce the intrinsic time \( \tau \) through

\[
\frac{d}{dt} - 1 \equiv t \frac{d}{dt} = -\frac{d}{d\tau}
\]  

(4)

We thus obtain

\[
-i \frac{d}{d\tau} y_1 = \nu_1 \left[ 1 + \nu_2(\tau) \right] y_1
\]  

(5)

Eq(5) has the following interpretation. The operator in the lhs of (4) subtracts out the mean (dynamical) evolution in (3) that manifests in the unit of the external time \( t(\sim 1) \). The residual (renormalized) evolution now consists only of the small scale fluctuations. However, in the scale of the intrinsic time \( \tau \sim \nu_1^{-1} \), an externally small fluctuation does appear substantially large. Eq(5) then tracks this 'internally large' fluctuation exactly in the same spirit as that of the original eq(3): i.e.; by identifying \( \nu_1 \) as the (scaled) mean 'energy' with a relatively small correction \( \nu_2(t) \) from the (2nd order) fluctuations. Further, the switching of the treatment from the external time scale down to the intrinsic scale amounts, in fact, to a transformation of the ordinary time \( t \) scale to a logarithmic scale:

\[
\tau = \ln|t|^{-1}
\]  

(6)

This is made explicit in the second equality in eq(4). Eq(5) thus offers a new approach in probing the very short time evolution of eq(2). Note that the standard exponential solution is obtained from the initial 'mean' solution \( y_0 \) near \( t = 0 \) through a combination of two operations: i) by improving upon the initial solution with the inclusion of the neglected terms in the Taylor (perturbation) series expansion (the Picard’s iteration), and then ii) by the successive applications of the translation group 'horizontally' along the ordinary \( t \)-axis. The first operation could indeed be achieved through the subtraction procedure (c.f., the lhs operator in eq(4)), but continuing to work instead in the original time (scale) \( t \). The new possibility that emerges in the present discussion is the following. Instead of following the evolution further 'horizontally', one could choose to dive 'vertically' down to the (intrinsic) logarithmic scale, thus climbing, as if, to a ripple of the background fluctuation. Consequently, the scale of the fluctuations gets sufficiently magnified (stretched), thus making a room for an (almost) independent treatment of the same, analogous to the mean motion.

To continue, we note that the (-) sign in eqs(4) and (5) is typically a consequence of the geometric origin of the intrinsic time introduced in I. It reveals a sort of 'relativity' between the extrinsic and the intrinsic treatments: the direction of traversal of a path as seen from the external time frame gets
reversed in the intrinsic scale. Note also that the subtraction operation of the mean energy in every unit of time (e.g., \( t \approx 1 \)) amounts to a folding (squeezing) on the solution space. Eqs (3)-(5) thus encode a set of successive stretching-twisting and folding operations; which can clearly be iterated ad infinitum; thus establishing, in turn, a hierarchy of self-similar structures in the very short time limit of the evolution. A number of observations can now be made.

1. Eq (6) represents a generalized duality relation between \( t \) and \( \tau \). In the limit of small \( \tau \), this reduces to \( t \approx \tau^{r-1}, \tau = 1 + \tau \). Thus in the limit of large fluctuations, the SL(2, R) group is realized as the exact invariance group of the ODE (2).

2. The duality eq (6) also tells us that an exact replica of the original eq (3) is reproduced after the second iteration when \( t \approx 0 \). Succintly, the complete iteration process can be expressed as

\[
\begin{align*}
\frac{d}{dt_n}y_n &= (-1)^n \nu_n [1 + \nu_{n+1}(t_n)] y_n \\
\frac{\nu_n t_n}{t_{n+1}} &= -\ln(\nu_n t_n), t_0 = t, \nu_0 = 1, n = 0, 1, 2, \ldots
\end{align*}
\]

In the limit \( t_n \to 0 \), the duality transformations reduce to the scaling relations \( t_{n+1} = \nu_n t_n \). For, \( t_n \to \nu_n^{-1} \) in the log scale \( \Rightarrow t_{n+1} = \nu_n(\nu_n^{-1} - t_n) \) and then replacing \( \nu_n^{-1} - t_n \) by \( t_n \) (because of the translation invariance) one obtains the result. Thus the set of nonlinear operations, as detailed above, defines a hyperbolic iterated function system (IFS) [3] with scaling (contractivity) factor \( \nu_1 = max(\nu_1, \nu_2) \), \( 0 \leq \nu_1 < 1 \), at each point \( t = t_0 \) of the real \( t \)-axis. The intended fractal solution of eq (2) \( t = t_0 \), our main result, is thus obtained as the unique attractor of this IFS.

3. Interestingly, the limiting value of the scaling factor \( \nu_n \), for any \( n \), at the fixed point \( t_n = 0 \) is given by the 'universal' value: \( \nu_\phi = \frac{\sqrt{5} - 1}{2} \), the golden mean. This follows from the necessary constraint that the system of equations (7) must coincide at the fixed point. One thus obtains the relations \( \nu_{n+1} = \nu_n^{-1}(1 + \nu_{n+1}) = \nu_n^{-1}[1 + \nu_{n+2}] = (\nu_n^{-1} \nu_{n+1}^{-1})(\nu_{n+1}^{-1} \nu_{n+2}^{-1}) \ldots = 1 \), by virtue of the duality \( t_n \sim \nu_n^{-1} \) but \( \nu_{n+1} \sim t_n \) \( \Rightarrow -\nu_n^{-1} \frac{dt_n}{d\nu_n} \sim 1 \).

4. As a consequence, the limiting form of the set of iterated equations (7), at the fixed point, is given by

\[
\begin{align*}
\frac{d}{dt_n}y_n &= (-1)^n \nu_g y_n \\
\frac{\nu_g}{t_n} &= t, t_{n+1} = \nu_g t_n, n > 0
\end{align*}
\]

These equations could be interpreted as one obtained by splitting the infinite degeneracies of the original equation \( i \frac{d}{dt} = y \), through the repeated applications of the nonlinear stretching and twisted-folding transformations at the fixed point. Indeed, this could be realized by introducing a partition of the unity: \( 1 = (1 - \nu_n) + \nu_n \), identifying the bracketed term as the 'mean' and the remainder as correction due to the fluctuations and then following the above steps. The apparent arbitrariness in the partition gets washed away at the fixed point, thus leading to the unique limiting system, eq (8). One thus obtains eq (8) as the attractor of the class of ODE considered here, under the nonlinear invariance group SL(2, R). An explicit form of the fixed point fractal solution could therefore be constructed in the form

\[
y = \exp(-i[t - \nu_g(t_1 - t_2 + t_3 - \ldots)])
\]

where the parameters \( t, t_n \) are treated as independent variables and each tends to \( 0 \) satisfying \( t \geq t_1 > t_2 > t_3 > \ldots \). By duality the point \( t_n = \nu_n^{-1} \) is mapped to \( t_{n+1} = 0 \), for each \( n \), ensuring a nontrivial sewing of the ordinary exponentials along the (internal) 'verticle' direction. However, the function (9) collapses to the simple form \( y \sim \exp(-i \nu_g t) \) in the very short time limit, provided one makes use of the scaling relations \( t_1 = t, t_{n+1} = \nu_g t_n \).
To interpret this result, we note that the original equation $\frac{dy}{d\tilde{t}} = y, \tilde{t} = -it$, in the log scale $|\tilde{t}| = \ln \tau^{-1}$, can be translated as the definition of the box-counting dimension of the real axis parametrized by $\tilde{t}$: $\ln y / \ln \tau = 1$, provided $y$ is identified as the total number of infinitesimal intervals needed to cover a finite segment of the $t$-axis. In the present case, writing $\nu_g = 1 - d, d = 3 - \sqrt{5}/2$, we get instead a nontrivial scaling law $y \approx \tau^{-(1-d)}$. Following I we interpret this result as one revealing a fractal structure in the real $t$ axis itself. The real $t$ axis thus behaves as a fat fractal with exterior dimension $d$. The golden mean is thus realized as the corresponding uncertainty exponent of this fat fractal.

4. The above discussions also clearly reveal the presence of the deterministic chaos in the very short time evolution of a linear dynamical system.

4 Final Remarks

We have presented a new approach in analyzing a (linear) differential equation. The results discussed here are expected to find interesting applications in a number of physical problems where a perturbative approach normally fails. Nevertheless, it’s striking how the golden mean emerges as a universal scaling constant in probing the short distance (time) structure of a linear dynamical system.

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