Meson-Meson Scattering in Relativistic Constraint Dynamics

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Abstract. Dirac’s relativistic constraint dynamics have been successfully applied to obtain a covariant nonperturbative description of QED and QCD bound states. We use this formalism to describe a microscopic theory of meson-meson scattering as a relativistic generalization of the nonrelativistic quark-interchange model developed by Barnes and Swanson.

In order to rule out false signals for the onset of the formation of a quark-gluon plasma one needs a reliable relativistic meson-meson scattering formalism. For example the dissociative process $\pi + J/\psi \to D + D^*$ could, by rapidly taking the $J/\psi$ out of the picture, mimic the suppression of $J/\psi$ production thought to occur in a quark-gluon plasma at high temperatures [1]-[4]. A nonrelativistic formalism for such a process in the microscopic quark-interchange picture of meson-meson scattering was developed by Barnes and Swanson [5] and later supplemented by a detailed quark model by Wong, Barnes, and Swanson [6]. Here we show how to extend the quark-interchange model to the relativistic domain using constraint dynamics, which has been successfully applied to two-body bound state problems in QED [7], QCD [8], and to two-body nucleon-nucleon scattering [9].

The two-body relativistic wave equations of constraint dynamics can be derived from the Bethe Salpeter Equation. They have their origins however in classical relativistic mechanics where one starts with two mass shell constraints and introduces interactions $\Phi_i$ (here world scalar interactions)

$$\mathcal{H}_i^0 = p_i^2 + m_i^2 \to p_i^2 + M_i^2 \equiv \mathcal{H}_i \equiv p_i^2 + m_i^2 + \Phi_i(x_1 - x_2, p_1, p_2); \quad i = 1, 2,$$

in such a way that the constraints are compatible $\{H_1, H_2\} \approx 0$. These constraints in turn imply that the interaction potentials satisfy a relativistic third-law condition

$$\Phi_1 = \Phi_2 = \Phi(x_{12\perp}, p_1, p_2) \equiv \Phi_w,$$

and that they depend, as Fig. 1a indicates, on its “perped” component $x_{12\perp} = (x_1 - x_2)_{\perp}$ perpendicular to the total momentum $P = p_1 + p_2$. The relative time is covariantly eliminated since in the CM system $r_{12} = \sqrt{x_{12\perp}^2 - r_{12\perp}^2}; \quad t_1 - t_2 = 0.$
For two particles with spins, one has two Dirac equations \cite{10} (here given for minimal scalar and vector interactions) instead of two generalized mass shell constraints,

\[ S_i \psi \equiv \gamma_5 i (\gamma_i \cdot (p_i - \tilde{A}_i)) + m_i + \tilde{S}_i \psi = 0, \quad i = 1, 2. \]  

(3)

Their compatibility \([S_1, S_2] \psi = 0\) is guaranteed if supersymmetry is added to the conditions that applied in the two body spinless case. The vector and scalar interactions each depend on underlying invariant functions \(A(r)\) and \(S(r)\). The compatibility condition leads to an automatic incorporation of correct spin-dependent recoil terms,

\[ \tilde{A}_i^\mu = \tilde{A}_i^\mu (A(r), x_{\perp}, p_1, p_2, \gamma_1, \gamma_2); \quad \tilde{S} = \tilde{S}_1 (S(r), A(r), x_{\perp}, p_1, p_2, \gamma_1, \gamma_2). \]  

(4)

The two-body Dirac equations can be put into a simple and local 4-component Schrödinger-like form. In the case of lowest order QED, they have an exact solution \cite{11} for singlet positronium that agrees with standard perturbative results. Thus, they are less likely to produce spurious results when applied to QCD.

Using such a formalism, we obtain very good results for the entire meson spectrum from the light pion to the heavy upsilon states \cite{8}. The nonperturbative structures in our equations provide for chiral symmetry in the sense that the pion (although not its excited states or the \(\rho\)) behave like a Goldstone boson.

The compatibility conditions for four spinless particles, \(\{H_i, H_j\} \approx 0\), unlike their two body counterpart, are not tractable as the set of two-body momenta are not separately conserved and three- and four-body interactions are needed for full compatibility,

\[ H_{i0} = p_i^2 + m_i^2 \rightarrow H_i \equiv p_i^2 + m_i^2 + \sum_{j \neq i} \Phi_{ij} (x_{ij\perp}) + \sum_{j \neq k \neq i} \Phi_{ijk} + \Phi_{1234}, i = 1, 2, 3, 4 \]  

(5)

Previously, we used Dirac’s constraint dynamics to obtain a Hamiltonian formulation of the relativistic \(N\)-body problem in a separable two-body basis in which the particles interact pair-wise through scalar and vector interactions by neglecting the many-body interactions \cite{12}. The resultant \(N\)-body Hamiltonian is relativistically covariant and can be separated in terms of the center-of-mass and the relative motion of any two-body subsystem. The two-body wave functions can be used as basis states to evaluate reaction matrix elements in the general \(N\)-body problem. In such a formalism, there is however the difficulty of determining the commutation relations involving the creation or annihilation operators of particles that belong to different composites.
Sazdjian [13] has found alternatively that compatibility can be obtained if one demands that the two-body interactions depend on the component of the relative coordinates transverse to the total momentum of the four-body system instead of the two-body system. In this formalism, one introduces the transverse \((T)\) component

\[ x_{ijT} = x_{ij} + (x_{ij} \cdot P) \frac{P}{\sqrt{-P^2}}, \]

where \(P = p_1 + p_2 + p_3 + p_4\), \(x_{ij} = x_i - x_j\), and one assumes \(\Phi_{ij} = \Phi_{ij}(x_{ijT})\). This formalism is suited for bound systems. Below we shows its adaptation to the scattering problem.

We review the nonrelativistic approach, starting with the orthogonality and completeness conditions

\[
\langle p'_1, p'_2|p''_1, p''_2 \rangle = \delta^3(p'_1 - p''_1)\delta^3(p'_2 - p''_2),
\]

\[
1_{12p} = \int d^3p'_1 d^3p'_2|p'_1, p'_2\rangle\langle p'_1, p'_2|,
\]

so that with the wave function defined by \(\langle p'_1, p'_2|M(P)\rangle = \delta^3(p'_{12} - P_{12})\tilde{\psi}_P(p'_{12})\), the scalar products of meson wave functions is simply given by

\[
\langle M(Q)|M(P)\rangle = \delta^3(P' - Q) \int d^3p_{12}\tilde{\psi}_Q(p'_{12})\tilde{\psi}_P(p'_{12}).
\]

Using the above orthogonality, completeness conditions, and wave function, we can construct the meson scattering amplitude for the reaction \(P_{12} + P_{34} \rightarrow Q_{14} + Q_{32}\) shown in Fig. 1(b) in terms of the momentum matrix elements of the interaction potential,

\[
\langle M(Q_{14})M(Q_{23}); Q|V(x_{13})|M(P_{12})M(P_{34}); P\rangle
\]

\[
= \int d^3q'_1 d^3q'_2 d^3p'_1 d^3p'_3 \langle q'_1, q'_2, q'_3, q'_4|V(x_{13})|p'_1, p'_2, p'_3, p'_4\rangle
\]

\[
\times \tilde{\psi}_{Q_{14}}(q'_{14})\tilde{\psi}_{Q_{32}}(q'_{32})\tilde{\psi}_{P_{12}}(p'_{12})\tilde{\psi}_{P_{34}}(p'_{34}).
\]

The coordinate completeness condition then leads finally to the results of Barnes, Swanson and Wong in [5] and [6],

\[
\langle M(Q_{14})M(Q_{23}); Q|V(x_{13})|M(P_{12})M(P_{34}); P\rangle
\]

\[
= \delta^3(Q_{14} + Q_{32} - P_{12} - P_{34}) \int d^3p'_1 d^3q'_1 \tilde{V}(p'_1 - q'_1)\tilde{\psi}_{Q_{14}}(q'_1 - \frac{m_1}{M_{14}}Q_{14})
\]

\[
\times \tilde{\psi}_{Q_{32}}(\frac{m_2}{M_{32}}Q_{32} - P_{12} + P'_1)\tilde{\psi}_{P_{12}}(p'_1 - \frac{m_1}{M_{12}}P_{12})\tilde{\psi}_{P_{34}}(q'_1 - Q_{14} + \frac{m_4}{M_{34}}P_{34}).
\]

We now display an analogous formalism in the relativistic case in which, as in the nonrelativistic case, the key ingredients are the scalar product, orthogonality, and completeness conditions.

Scalar products [14] in the relativistic case are complicated by the fact that the relativistic effective potentials are energy dependent (as occurs for example in the one-body Klein-Gordon equation). In a general case this may lead to important contributions but in the Born approximation that we follow here it can be ignored.
We introduce here the completeness and orthogonality conditions,

\[ 1_{12p} = \int d^4p_1 d^4p_2 |p_1, p_2, \hat{P}| \delta (p_1 \cdot \hat{P} + w) \delta (p_2 \cdot \hat{P}) |p_1\rangle, \]

\[ \langle p_1', p_2' | p_1, p_2 \rangle = \delta^4 (p_1' - p_1, \tau) \delta^4 (p_2' - p_1, \tau), \]

\[ \delta^4 (p_i' - p_i, \tau) \equiv \int dr_i \delta^4 (p_i' - p_i + r \hat{P}_1) \exp (-ir_i \tau), \]

where \( p' = (\varepsilon_2 p_1' - \varepsilon_1 p_2')/w \), and \( P_{12} = p_1' + p_2' \). Unlike the nonrelativistic case there is here a \( \hat{P} \), showing the dependence on the particular two-body state.

Using the wave function (\( \hat{n} \) an arbitrary time-like unit vector)

\[ \langle x_1', x_2' | M(\hat{P}_{12}) \rangle \equiv \sqrt{\frac{(\hat{P}_{12} \cdot \hat{n})}{(2\pi)^3}} \exp i (P_{12} \cdot X_{12}) \psi_{12} (x_{12\perp}) \]

and completeness conditions, we obtain the scalar product

\[ \langle M(Q_{12}) | M(P_{12}) \rangle = \delta^4 (P_{12} - Q_{12}, \tau) (-\hat{P}_1 \cdot \hat{n}) \int d^4 x_1' \delta (x_1' \cdot \hat{P}_1) \psi_{12}^* (x_{12\perp}) \psi_{12} (x_{12\perp}). \]

The derivation of the meson-meson scattering amplitude parallels its nonrelativistic counterpart until one gets to the the momentum space matrix element of the potential \( \langle q_1', q_2', q_3, q_4 | \Phi (x_{13}) | p_1', p_2', p_3', p_4' \rangle \) that is analogue of the nonrelativistic matrix element \( \langle q_1', q_2', q_3, q_4 | V (x_{13}) | p_1', p_2', p_3', p_4' \rangle \). The problem is that the bra and ket momentum states in the relativistic expression belong to different mesons.

The initial state orthogonality condition below (final state condition is similar) shows the explicit dependence on meson momenta \( P_{12} \) and \( P_{34} \),

\[ \langle p''_1, p''_2, p''_3, p''_4, \hat{P}_{12}, \hat{P}_{34} | p_1', p_2', p_3', p_4', \hat{P}_{12}, \hat{P}_{34} \rangle \]

\[ \equiv \int dr_1 dr_2 dr_3 dr_4 \delta^4 (p''_1 - p' + r_1 \hat{P}_{12}) \delta^4 (p''_2 - p_2 + r_2 \hat{P}_{12}) \]

\[ \times \delta^4 (p''_3 - p_3 + r_3 \hat{P}_{34}) \delta^4 (p''_4 - p_4 + r_4 \hat{P}_{34}) \exp (-i (r_1 + r_2 + r_3 + r_4) \tau). \]

Our postulate for different sets of mesons in the bra and ket states is one that uses the total four momentum unit vector \( \hat{n} \) of the four quark system in place of the constituent four momenta,

\[ \langle q_1', q_2', q_3, q_4 | \Phi (x_{13}) | p_1', p_2', p_3', p_4' \rangle \equiv \langle q_1', q_2', q_3, q_4; \hat{Q}_1; \hat{Q}_2; \hat{Q}_3; \hat{Q}_4 | p_1', p_2', p_3', p_4', \hat{P}_{12}, \hat{P}_{34} \rangle \]

\[ \equiv \int dr_1 dr_2 dr_3 dr_4 \delta^4 (q_1' - p_1 + r_1 \hat{n}) \delta^4 (q_3' - p_3 + r_3 \hat{n}) \]

\[ \times \delta^4 (q_2' - p_2 + r_2 \hat{n}) \delta^4 (q_4' - p_4 + r_4 \hat{n}) \exp (-i (r_1 + r_2 + r_3 + r_4) \tau). \]

The physical assumption this reflects is that in the collision process the individual mesons lose their identity and momentarily we have a four body system as described by the Sazdjian formalism. In a like manner the momentum matrix element of the potential

\[ \langle q_1', q_2', q_3, q_4 | \Phi (x_{13}) | p_1', p_2', p_3', p_4' \rangle \]

\[ = \langle q_1', q_2', q_3, q_4 | \Phi (x_{13} \cdot (1 + \hat{n} \hat{n})) | p_1', p_2', p_3', p_4' \rangle \]

is one that reflects the Sazdjian hypothesis of coordinate dependence only through its component perpendicular to the total momentum. We thus have a hybrid model in which
the meson wave functions have the usual two-body perped variable ($\perp$) dependence but the potential has the four-body transversality ($T$) dependence. The diagram in Fig. (1b) details the hybrid nature of the two combined constraint formalisms.

We obtain finally an expression that is a relatively simple three-dimensional but covariant generalization of the nonrelativistic expression given in Eq. (11)

$$
\langle M(Q_{14}) M(Q_{23}) ; Q | \Phi(x_{13T}) | M(P_{12}) M(P_{34}) ; P \rangle \\
= \sqrt{\hat{n} \cdot \hat{P}_{12} \hat{n} \cdot \hat{Q}_{14}} \delta^4(P - Q, \tau) \int \delta(\hat{P}_{12} \cdot p_{12}) \delta(q_{14} \cdot \hat{Q}_{14}) \\
\times \tilde{\psi}_{Q_{14}}(q_{14}^\prime) \tilde{\psi}_{Q_{32}}(q_{32}^\prime) \tilde{\psi}_{P_{12}}(p_{12}^\prime) \tilde{\psi}_{P_{34}}(p_{34}^\prime) \Phi[(p_{13}^\prime - q_{13}^\prime) T] d^4 q_{14} d^4 p_{12} d^4 p_{34} 
$$

(17)

Our aim now is to apply the relativistic quark model wave functions we have developed to compute meson-meson cross sections.

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