Entropons as collective excitations in active solids

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ABSTRACT
The vibrational dynamics of solids is described by phonons constituting basic collective excitations in equilibrium crystals. Here, we consider a non-equilibrium active solid, formed by self-propelled particles, which bring the system into a non-equilibrium steady-state. We identify novel vibrational collective excitations of non-equilibrium (active) origin, which coexist with phonons and dominate over them when the system is far from equilibrium. These vibrational excitations are interpreted in the framework of non-equilibrium physics, in particular, stochastic thermodynamics. We call them “entropons” because they are the modes of spectral entropy production (at a given frequency and wave vector). The existence of entropons could be verified in future experiments on dense self-propelled colloidal Janus particles and granular active matter, as well as in living systems, such as dense cell monolayers.

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I. INTRODUCTION
Active matter includes a broad range of systems composed of particles that locally convert energy from the environment into directed motion. The energy exchange with the environment, often induced by chemical reactions or self-imposed gradients, leads to self-propulsion of the particles and internally drives an active system out of equilibrium.

Dense systems of self-propelled particles are rather ubiquitous in nature and often form crystalline structures. Cell monolayers of the human body and dense colonies of bacteria are common examples. Moreover, active colloidal Janus particles may cluster and form dense crystallities, named “living crystals.” These active crystals show fascinating phenomena uncommon for equilibrium solids ranging from intrinsic spatial velocity correlations, spontaneous velocity alignment, and non-Gaussian velocity distributions to traveling crystals, collective rotations, and even flocking. Activity also shifts the equilibrium freezing transition, affects the nature of the two-dimensional melting scenario, and changes the relaxation properties of amorphous solids. Systematic studies of collective excitations in active solids are still in their infancy, and they remain poorly explored even in general non-equilibrium solids (beyond active matter). Unveiling how the nature of the vibrations of a solid changes out of equilibrium represents an open issue relevant in statistical as well as solid-state physics.

External forces or internal mechanisms that dissipate energy drive a system away from equilibrium and spontaneously produce entropy. While self-propulsion is generated by an uptake of energy from the environment, likewise active particles dissipate energy, leading to local entropy production. Quantifying the non-equilibrium character of active systems via this observable has represented a topic of central interest in recent years. This subject has been investigated numerically, simulating both active field theories, active particle dynamics in interacting systems, as well as colloids in the presence of an active bath. Particular attention has been devoted to phase-separated configurations where the main contribution to the spatial profile of the entropy production has been observed at the interface between dense and dilute phases. Conversely, analytical results for entropy production have been only derived for simple cases, such as the potential-free particle, and for near-equilibrium regimes through perturbative methods. Entropy production in active crystals, however, remains unexplored.
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II. MODEL

We study a two-dimensional crystal of N inertial active Brownian particles (ABP) in a square box of size $L$ with periodic boundary conditions. Each particle with mass $m$ evolves through underdamped dynamics\(^{50-81}\) for its position, $\mathbf{x}_i$, and velocity $\mathbf{v}_i = \dot{\mathbf{x}}_i$,

\[
\begin{align*}
 m\ddot{\mathbf{x}}_i &= -\gamma\dot{\mathbf{x}}_i + \mathbf{F}_i + \sqrt{2\gamma\xi_i + \mathbf{f}_i^0}, \\
 \dot{\theta}_i &= \sqrt{2D_I\eta_i},
\end{align*}
\]

where $\xi_i$ and $\eta_i$ are Gaussian white noises with zero average and unit variance. The term $\mathbf{f}_i^0 = \gamma\mathbf{x}_i$, models the active force, with $v_0$ being the swim velocity and $\mathbf{n}_i = (\cos \theta_i, \sin \theta_i)$ being the orientational unit vector, determined by an orientational angle $\theta_i$. The coefficients $\gamma$ and $T$ correspond to the friction coefficient and temperature of the solvent bath, respectively, and define the inertial time $\tau_I = m/\gamma$. $D_I$ is the rotational diffusion coefficient, which determines the persistence time, $\tau = 1/D_I$, i.e., the time that the particle needs to randomize its orientation. The single-particle dynamics is often described in terms of the so-called active temperature, $T_a = \frac{\tau}{\tau_I}$, that vanishes in the equilibrium limits, either $\tau \to 0$ or $v_0 \to 0$.

The interaction force $\mathbf{F}_i$ stems from a soft repulsive pair potential, $U_{\text{tot}} = \sum_j U(|\mathbf{x}_i - \mathbf{x}_j|)$, where

\[
U = 4\epsilon \left[ (d_0/r)^{12} - (d_0/r)^6 \right] + \epsilon\text{ if } r < d_02^{1/6} \text{ and zero otherwise (Weeks–Chandler–Andersen potential)}^{12},
\]

with $\epsilon$ and $d_0$ being the energy scale and the particle diameter, respectively. The packing fraction $\eta = \frac{\pi N}{6L^2}$ denotes the number density, with $N = 10^4$ and $L/\sigma = 88.6$ is chosen high enough to ensure a solid-like behavior characterized by a defect-free triangular lattice, as illustrated in Fig. 1. Further details are reported in Appendix C.

In the following, we chose an inertial regime such that $\tau_I \sim \tau$. Indeed, the inertial regime will play a key role in the spectral analysis of collective excitations.

III. RESULTS

A. Collective excitations and spectral entropy production

The non-equilibrium properties of the system are investigated by applying path-integral techniques to calculate the total entropy production rate, $\dot{s} = \lim_{t \to \infty} (\log (P_t/P_0))$ \(^{13,48,61}\), respectively (see Appendix A for details and Ref. 86 for a general review). The steady-state entropy production rate can be decomposed into its space-time Fourier spectrum as

\[
\dot{s} = \int_{\Omega} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sigma(\omega, q),
\]

where $q$ is the wave vector, $\omega$ is the frequency, and $\Omega$ represents the area of the first two-dimensional Brillouin zone.

The spectral entropy production $\sigma(\omega, q)$ is analytically predicted as (see Appendix A)

\[
\sigma(\omega, q) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{1}{2T} \left[ \langle \hat{\mathbf{F}}(\omega, -q) \cdot \hat{\mathbf{F}}(-\omega, -q) \rangle + \mathbf{C} \right]
\approx \frac{T_a}{T} \frac{K(\omega)}{\tau_I^2} \left| \frac{\omega^2}{\omega^2 - \omega^2(q)} \right|^2 + \omega^2,
\]

where the symbol $\mathbf{C}$ stands for complex conjugate. The vectors $\hat{\mathbf{F}}_d(-\omega, -q)$ and $\hat{\mathbf{F}}(-\omega, q)$ are the Fourier transforms of active force and velocity fields in the frequency and wave vector domains, respectively (see Appendix A for their definitions). The second line of Eq. (3) is obtained in the limit of a harmonic crystal and expresses...
The term $\tilde{\omega}(q)$ in Eq. (3) denotes the phonon dispersion relation of equilibrium solids, whose expression is reported in Appendix A for a triangular lattice. In general, $\tilde{\omega}(q) \propto \omega_0$, where $\omega_0 = \frac{1}{\hbar} \left( \hbar^2 (\nabla_x^2) + \frac{1}{\tau} \right)$ is the Einstein frequency of the solid determined by the derivative of the potential calculated at the average distance between neighboring particles, $x \sim 1/\sqrt{n}$.

As a main result of this paper, we characterize the collective excitations of the system by analytically calculating the dynamical correlation function of the particle displacement with respect to the unperturbed position of the lattice, $C_{\tilde{u}u}(\omega, q) = \lim_{t \to \infty} \langle \tilde{u}(\omega, q, t) \cdot \tilde{u}(\omega, q, 0) \rangle / t$. In a passive solid, $C_{\tilde{u}u}$ has a thermal origin ($\sim T$) and consists of a single term corresponding with phonons. In an active solid, $C_{\tilde{u}u}$ is formed by the sum of two contributions: with thermal and active origins from which we can identify two distinct collective excitations. We have

$$C_{\tilde{u}u}(\omega, q) = \frac{T}{\omega} \text{Im}[\tilde{R}_{\tilde{u}u}(\omega, q)] + \frac{T}{\gamma} \frac{K(\omega)}{\omega^2 - \tilde{\omega}^2(\omega, q)^2 + \omega^2},$$

(5)

where the second term $-\omega(\omega, q)/\omega^2$ coincides with the spectral entropy production given by Eq. (3) and Im[$\tilde{R}_{\tilde{u}u}(\omega, q)$] is the imaginary part of the response function due to a small perturbation, $h$, evaluated in the frequency and wave vector domains. The response is defined as $\tilde{R}_{\tilde{u}u}(\omega, q) = \delta(\tilde{u}(\omega, q))/\delta h(q)$, and one has

$$\text{Im}[\tilde{R}_{\tilde{u}u}(\omega, q)] = \frac{\omega\tau}{\gamma} \left( \omega^2 - \tilde{\omega}^2(\omega, q)^2 + \omega^2 \right).$$

(6)

Relation (5) indicates that collective excitations [through $C_{\tilde{u}u}(\omega, q) \sim (\tilde{u}(\omega, q) \cdot \tilde{u}(\omega, q))/t$] are determined by the sum of two contributions: (i) thermally excited crystal vibrations, independent of activity, that are identified with the phonons of equilibrium solids and (ii) additional vibrational contributions of the crystal that are purely induced by the activity. The latter are named entropons because each of them represent a mode of the spectral entropy production. We remark that the latter are dominant if $T\omega_0/\gamma > 1$ for a given $q$, far from equilibrium: There exists a typical $\tau$ (or $\tau_0$) at which the contribution of entropons becomes negligible compared to that of phonons. In the limit of vanishing active force ($T\omega_0 \to 0$), the response balances the lhs of Eq. (5) and the entropy production vanishes at equilibrium, as required. As a consequence, entropons disappear and thermal phonons remain the only collective excitations. Formula (5) can be interpreted as a Harada–Sasa relation applied to active solids at a given wave vector $q$ (see also Ref. 59 for a derivation of the Harada–Sasa relation in active field theories) and justifies the decomposition into conventional phonons with thermal origin and entropons, arising from entropy production.

The decomposition of the collective excitations in phonons and entropons has a deep physical meaning directly linked to emergent collective phenomena. Indeed, active systems at high density show velocity patterns and spatial velocity correlations, independently observed in numerical simulations in Refs. 20 and 23 and in experiments based on cell monolayers in Ref. 8. Since thermally excited phonons do not produce spatial patterns in real space, entropons are clear evidence of novel excitations in active solids.

B. Properties of entropons

A typical shape of $\sigma(\omega, q)/T$ is shown in Fig. 2(a) as a function of $\omega/\omega_0$ for a given $q$. A sharp peak occurs at a characteristic frequency $\omega^*(q)$. We identify this peak with an elementary excitation in the crystal and coin the term “entropon” to describe it, following the standard notation of elementary excitations in solids. Each entropon is identified with a peak of $\sigma(\omega, q)$.

Figures 2(b) and 2(c) show $\sigma(\omega, q)/T$ as a function of different values of $q$, revealing a good agreement between theory, Eq. (3), and numerical simulations. Close to the equilibrium, in the regime of small persistence time such that $\tau = 1/D_0 \ll 1/\omega_0$ [Fig. 2(b)], the peaks of $\sigma(\omega, q)/T$ occur at the phonon frequency $\omega^*(q) = \omega(q)$. In this regime, entropons have the same properties of phonons, but their amplitudes are small and proportional to $\tau$ (because of the prefactor $T\omega_0$). The active force behaves as an additional thermal source at effective temperature $T\omega_0$. In the opposite regime of large persistence time, $\tau = 1/D_0 \gg 1/\omega_0$ [Fig. 2(c)], the peaks of $\sigma(\omega, q)$ are shifted to $\omega^* < \omega(q)$. As a consequence, entropons are different from phonons since the crystal vibrations are now peaked at frequencies not coinciding with those typical of equilibrium solids. The frequency $\omega^*(q)$ that maximizes $\sigma(\omega, q)$ is reported in Fig. 2(d) as a function of $q$ for different rescaled persistence times, $\tau\omega_0$, while the difference $\omega(q) - \omega^*(q)$ is shown in Fig. 2(e) as a function of $\tau\omega_0$ for several values of $q$. Despite $\omega^*(q)$ linearly increasing with $q$ in the small persistence regime, a clear discrepancy from the linear law emerges in the large persistence regime for small $q$, where entropons follow a non-linear dispersion relation. As a consequence, the difference $\omega(q) - \omega^*(q)$ grows when $\tau\omega_0$ is increased much more as $q$ is decreased. Finally, the width of the peaks of $\sigma(\omega, q)$ increases with $\tau\omega_0$ [compare the curves with the same colors in Figs. 2(b) and 2(c)], implying shorter-lived excitations at high activities.

Figure 2(f) displays the integral over $\omega$ of the spectral entropy production, $s(q) = \int \omega \sigma(\omega, q) = \frac{\rho}{\tau_0} \left( \frac{\tau}{\tau_0 + \tau_0} \right) \frac{\omega_0}{\tau^2} \left( \frac{\omega}{\omega^*} \right)^{-1} \left[ 1 + \left( \frac{\omega}{\omega^*} \right)^2 \right]$, as a function of $\tau\omega_0$ for different $q$ to quantify the weight of each entropon. For $\tau\omega_0 \ll 1$, this observable is nearly $q$-independent and increases with $\tau\omega_0$: The larger the $\tau\omega_0$, the larger is the contribution of each entropon to $s(q)$. This linear behavior is due to the increase in the prefactor $T\omega_0/\tau$. For $\tau\omega_0 \geq 1$, the value of $s(q)/\tau\omega_0$ strongly depends on $q$, displaying higher values for smaller $q$. Interestingly, $s(q)/\tau\omega_0$ decreases as $\tau\omega_0$ is increased and, consequently, shows a non-monotonic behavior with a maximum that shifts for larger $\tau\omega_0$ as $q$ is decreased. To shed light on this surprising non-monotonicity, the height of the peak of the rescaled spectral entropy production, $\sigma(\omega^*, q)/T\omega_0$, is shown in Fig. 2(g) vs $\tau\omega_0$ for different values of $q$, revealing approximately the profile $1/(1 + b^2 \omega_0^2)$, where $b = b(q)$ is a fitting parameter. Considering formula (3), the height of the peak of $\sigma(\omega^*, q)$ is roughly determined by $\sigma(\omega^*, q)/T\omega_0 \sim K(\omega^*)$. By approximating $\omega \sim \omega^*$, we obtain

$$\sigma(\omega^*, q) \frac{T}{T\omega_0} \approx \frac{1}{1 + \omega^*(q)} \frac{1}{1 + \left( 1 + \frac{z}{\pi} \right)^2 \omega^2}.$$  

(7)
FIG. 2. Spectral entropy production. (a) Schematic representation of the spectral entropy production $\sigma(\omega, q) T/T_a$ to identify entropons as a peak in the spectrum. (b) and (c) $\sigma(\omega, q) T/T_a$ as a function of $\omega/\omega_E$ for different values of the rescaled wave vector $qd_0$ for $\tau\omega_E = 7 \times 10^{-2}, 7$, respectively. Points are obtained by simulations, while solid lines are obtained by Eq. (3). Dashed and solid vertical lines mark the phonon frequency $\bar{\omega}(q)/\omega_E$ and the maximum $\omega^*(q)/\omega_E$. (d) Frequency $\omega^*(q)/\omega_E$ peaked as a function of $qd_0$ for different values of the rescaled persistence time $\tau\omega_E$. The black dotted line is an eye guide to show a linear curve. (e) Difference between the frequency of the phonon, $\bar{\omega}(q) T/T_a$, and $\omega^*(q) T/T_a$ as a function of $\tau\omega_E = \omega_E/D_r$ for different values of $qd_0$. (f) Integrated entropy production, $s(q) T/(v_0^2\gamma)$, as a function of $\tau\omega_E$. (g) Maximal value of the entropy production, $\sigma(\omega^*, q) T/T_a$, as a function of $\tau\omega_E$ for different values of $qd_0$. Lines are obtained by fitting the function $1/(1 + b\tau^2\omega_E^2)$, where $b$ is a fitting parameter. Data with error bars are reported in Appendix C for (b) and (c), while the error for the other panels is smaller than the point size. The parameters are $N = 10^4$ and $\phi = 1.1$. 

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where $\xi$ is the correlation length of the spatial velocity correlation, \((\mathbf{v}(\mathbf{r}) \cdot \mathbf{v}(0))\), of an active solid and reads\(^{26}\)

$$\xi^2 = \frac{3}{2} \chi^2 \frac{\tau_0^2}{\tau + \tau_0^2} \omega_0^2. \quad (8)$$

Evaluating the denominator of Eq. (7) and requiring that the $q$-dependence is negligible, we determine the typical wave vectors at which $\sigma T/\xi$ starts decreasing.

$$q^2 = \frac{1}{\xi^2 (1 + \tau / \tau_0)}. \quad (9)$$

The wave-vectors with $q \leq q_0 \sim 1/\xi$ (length scales larger than $\xi$) provide the main contribution to the rescaled entropy, while those with $q \geq q_0 \sim 1/\xi$ (length scales smaller than $\xi$) have a much smaller weight. Since $\xi$ increases with $\tau \omega_0$, the modes with larger $q$ give a smaller contributions as $\tau \omega_0$ is increased.

Finally, we recall that the present analysis has been obtained in the underdamped regime, when the inertial time $\tau_i$ is of the same order or larger than the typical relaxation time of the solid, given by the inverse of the Einstein frequency $1/\omega_0$. Indeed, in the overdamped regime for $\tau_i \ll 1/\omega_0$, the spectrum shows only a single peak around $\omega \approx 0$, similarly to the case of equilibrium solids. This can be understood by looking at the denominator of Eq. (5) or Eq. (6) that suppresses the dependence from $\omega^2(q)$ for vanishing $\tau_i$.

### C. Total entropy production rate

By integrating over $\omega$ and $q$ and our prediction for $\sigma(\omega, q)$ [Eq. (2)], we can derive analytically the expression for the global density of entropy production rate of the solid, $\dot{s}$,

$$\dot{s} = \dot{s}_{\text{free}} \int \frac{dq}{\Omega} \frac{1}{1 + c \tau/t_0^2 \tilde{\omega}^2(q)}; \quad (10)$$

where $\dot{s}_{\text{free}} = \rho \tau_0^2 (\tau_0 + \tau)^{-1}$ is the density of entropy production rate of non-interacting active particles. $\dot{s}_{\text{free}}$ is proportional to the ratio between the active temperature ($T_a = v_0^2 \tau/2$) and the thermal temperature ($T$) and is a function of persistence time $\tau$ and inertial time $\tau_i$. The integral in Eq. (10) can be analytically expressed in terms of elliptic functions depending on the parameters of the model (see Appendix B). Figure 3(a) plots $\dot{s}$ as a function of $\tau \omega_0$, showing a non-monotonic behavior: in the small persistence regime ($\tau \omega_0 \ll 1$), $\dot{s} \sim 0$ because the system behaves as an inertial solid in equilibrium. Increasing $\tau \omega_0$, the system departs from equilibrium and $\dot{s}$ grows until reaching a maximum, roughly at $\tau \omega_0 \sim 1$. For further increase in $\tau \omega_0$, $\dot{s}$ decreases almost to zero. This decrease is consistent with the arrested states observed in dense active systems when $\tau \rightarrow \infty$\(^{53,88}\) for which $\dot{s} = 0$.

While the increase in $\dot{s}$ is expected when the system departs from equilibrium and is well explained by the increase (up to saturation) in the non-interacting entropy production rate $\dot{s}_{\text{free}} \sim \tau/\tau_0$, the physical picture behind the decrease in the large persistence regime can be understood by expressing $\dot{s}$ as a function of $\xi$. Here, we report the scaling behavior of $\dot{s}$ for $\xi \gg 1$,

$$\frac{\dot{s}}{\dot{s}_{\text{free}}} \sim \frac{\log(\xi)}{2\sqrt{3} \xi^2} \quad \text{for} \quad \xi \geq 1, \quad (11)$$

which is shown in Fig. 3(b). The decrease in $\dot{s}$ as $\xi$ increases suggests that the onset of spatial velocity correlations, characterizing active solids, reduces the entropy production rate. Coherent domains of strongly correlated velocities produce less entropy, while the incoherent behavior of the velocities of the particles determines a higher dissipation as if coherent motion somehow, minimizes the effective friction between different particles. This disorder-order picture has been employed to explain non-monotonic behaviors also in conservative dynamics displaying collective motion.\(^{39}\)

### IV. DISCUSSION

In conclusion, we have predicted new elementary vibrational excitations in active solids, termed entropons, because they are modes of the spectral entropy production. Our combined numerical and theoretical study revealed the properties of entropons, which coexist with phonons but dominate over phonons far from equilibrium when the active temperature is larger than the thermal one.
The concept of “entropons” as additional lattice vibrations has a broad generality that goes beyond monodisperse active crystals. It will certainly apply to binary crystals composed of active and passive particles. Moreover, we expect that in disordered dense systems, such as active glasses, dense active liquid crystals, and dense active colloids, entropons could play the dominant role of system excitations in determining entropy production. For instance, they could shed light on the activity-induced shift of the glass transition temperature.

Many experimental realizations of active crystals are available. Examples include confluent cell monolayers, dense assemblies of active colloids, as well as highly packed active granular systems, for which the solid structure has been achieved by connecting Hexbug particles by springs. Therefore, the existence of entropons can, in principle, be verified by analyzing particle trajectories in real space. To identify the contribution of entropons, it is crucial to have experimental access to the measurements of velocities and active forces, i.e., orientational angles. In this way, one can directly calculate the spectral entropy production in experiments using Eq. (3). Alternatively, entropons can be measured indirectly without knowing the active forces through Eq. (3) by evaluating the dynamical correlations of the particle displacement and subtracting the contribution of thermal phonons, i.e., the response function due to a small perturbation.

Despite derived in the active case, entropons will characterize more general non-equilibrium solids, as shown in Appendix E for a class of non-active solids driven out of equilibrium by temperature gradients. Therefore, the present paper reveals how concepts in the field of stochastic thermodynamics could have a determinant role in solid physics. For instance, we strongly believe that our field of stochastic thermodynamics could have a determinant role in solid physics. For instance, we strongly believe that our


draft (equal); Writing – review & editing (equal). U. Marini: Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal); A. Puglisi: Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal). Hartmut Löwen: Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal).

**DATA AVAILABILITY**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

**APPENDIX A: DERIVATION OF THE SPECTRAL ENTROPY PRODUCTION**

In this appendix, we derive the analytical expression for the spectral entropy production of the system, \(\sigma(\omega, q)\), in the frequency and wave vector domains, e.g., Eq. (3), and the expression for the dynamical correlation function of the displacement, e.g., Eq. (5).

1. **Dynamics in the Fourier space**

Before introducing the path-integral approach and defining the entropy production, it is useful to define the Fourier transform in frequency and wave vector domains of the dynamical variable of the system, namely, the displacement with respect to the unperturbed lattice position, \(u = x - x_0\), the velocity \(v\), and the active force \(f^a\),

\[
\hat{u}(\omega, q) = \int_{-T/2}^{T/2} dt \sum_{i=1}^{N} \hat{u}_i(t) e^{-i \omega q \cdot x_0 - i q \cdot x},
\]

\[
\hat{v}(\omega, q) = \int_{-T/2}^{T/2} dt \sum_{i=1}^{N} \hat{v}_i(t) e^{-i \omega q \cdot x_0 - i q \cdot x},
\]

\[
\hat{f}^a(\omega, q) = \int_{-T/2}^{T/2} dt \sum_{i=1}^{N} \hat{f}^a_i(t) e^{-i \omega q \cdot x_0 - i q \cdot x},
\]

where \(t\) is the time-window numerically used to define the Fourier transform, corresponding to the time window of the simulations.

By multiplying the dynamics (1) of the main text by \(e^{-i \omega q \cdot x_0 - i q \cdot x}\) and using definitions (A1a)–(A1c), the dynamics (1) can be expressed in the wave vector and frequency domains as

\[
- ma^2 \hat{u}(\omega, q) = \mathcal{P}(\omega, q) + \hat{\mathcal{I}}(\omega, q) + i \gamma w \hat{u}(\omega, q) + \sqrt{2T} \xi(\omega, q),
\]

(A2)

where \(\mathcal{P}(\omega, q) = -i \omega \hat{u}(\omega, q)\) and \(\hat{\mathcal{I}}(\omega, q)\) is a function of \(\hat{u}(\omega, q)\). The term \(\xi(\omega, q)\) is a white noise vector such that

\[
(\xi(\omega', q'), \xi(\omega, q)) = \delta(q + q') \delta(\omega + \omega').
\]

2. **Definition and implicit expression for the spectral entropy production**

The spectral entropy production \(\sigma(\omega, q)\) can be operatively calculated by using path-integral methods in frequency and wave vector domains starting from the path-integral definition of the entropy production rate \(s\).
\[ \dot{s} = \lim_{t \to \infty} \frac{1}{t} \left\{ \log \left[ \frac{P}{P_f} \right] \right\}, \quad (A3) \]

where \( P \) and \( P_f \) are the forward and backward trajectory expressed in terms of the forward and backward actions, \( A \) and \( A_f \), respectively, as

\[
P \sim e^{-A}, \quad (A4a)\]
\[
P_f \sim e^{-A_f}. \quad (A4b)\]

From the dynamics (A2) in wave vector and frequency domains, one can easily derive the actions \( A \) and \( A_f \), that read

\[
A = -\frac{1}{2} \int \frac{d\omega}{2\pi} \sum_q L(\omega, q)L(-\omega, -q), \quad (A5a)\]
\[
A_f = -\frac{1}{2} \int \frac{d\omega}{2\pi} \sum_q L_f(\omega, q)L_f(-\omega, -q), \quad (A5b)\]

where \( L(\omega, q) \) and \( L_f(\omega, q) \) are related to the forward and backward systems in \( q \) and \( \omega \), respectively. The former reads

\[
L(\omega, q) = \sqrt{\frac{m^2}{2\gamma}} \left[ -\omega^2 \hat{u}(\omega, q) - \frac{\hat{F}(\omega, q)}{m} \right]
- \frac{\hat{F}(\omega, q)}{m} + \frac{y}{m} \hat{\psi}(\omega, q), \quad (A6)\]

while the latter is given by

\[
L_f(\omega, q) = T L(\omega, q), \quad (A7)\]

with \( T \) being the operator that performs the time-reversal transformation (so that reverses the dynamics). To apply \( T \) to the Lagrangian, we need to account for the parity of the dynamical variables under \( \text{TRT} \). In particular, applying the time-reversal operator, \( T \) (that reverses the dynamics), we get

\[
T \hat{u}(\omega, q) = \hat{u}(\omega, q), \quad (A8a)\]
\[
T \hat{\psi}(\omega, q) = -\hat{\psi}(\omega, q), \quad (A8b)\]
\[
T \hat{F}(\omega, q) = \hat{F}(\omega, q), \quad (A8c)\]

since the displacement is even, the velocity is odd, and, finally, the active force is even under \( \text{TRT} \). While the choice for position and velocity is intuitive, the parity of \( \hat{F} \) could depend on the system considered. We observe that our even choice is fully justified, for instance, in those systems (such as colloids) where the active force arises from a local gradient of concentration, i.e., a position-dependent variable. In addition, we specify that the force \( \hat{F}(\omega, q) \) is even under \( \text{TRT} \), being a function of \( \hat{u}(\omega, q) \) only. In this way, \( L_f(\omega, q) \) is given by

\[
L_f(\omega, q) = \sqrt{\frac{m^2}{2\gamma}} \left[ -\omega^2 \hat{u}(\omega, q) - \frac{\hat{F}(\omega, q)}{m} \right]
- \frac{\hat{F}(\omega, q)}{m} + \frac{y}{m} \hat{\psi}(\omega, q). \quad (A9)\]

Combining the expressions for \( A \) and \( A_f \), taking the average, and dividing by \( t \), the total entropy production rate \( \dot{s} \) can be expressed as follows (after some algebraic manipulations):

\[
\dot{s} = \lim_{t \to \infty} \frac{1}{t} \int \frac{d\omega}{2\pi} \sum_q \frac{1}{2T} \left[ \langle \hat{\psi}(\omega, q) \hat{F}(\omega, q) \rangle \right.
+ \left. \langle \hat{\psi}(\omega, q) \hat{F}(\omega, q) \rangle \right], \quad (A10)\]

This leads to identifying the spectral entropy production as

\[
\sigma(\omega, q) = \lim_{t \to \infty} \frac{1}{t} \int \frac{d\omega}{2\pi} \sum_q \frac{1}{2T} \left[ \langle \hat{\psi}(\omega, q) \hat{F}(\omega, q) \rangle \right.
+ \left. \langle \hat{\psi}(\omega, q) \hat{F}(\omega, q) \rangle \right], \quad (A11)\]

which coincides with the right-hand side of Eq. (3) of the main text (first line). Note that Eq. (A11) is an implicit formula that can be calculated numerically.

We further remark that, as usual in the framework of stochastic thermodynamics, additional terms are obtained in the expression of \( \dot{s} \), for instance, a term proportional to \( \langle \hat{F} \cdot \hat{\psi} \rangle \) and another term proportional to \( \langle \omega \hat{V} \cdot \hat{\psi} \rangle \). These terms are boundary terms that do not contribute to the steady-state entropy production rate. Indeed,

\[
\int \frac{d\omega}{2\pi} \langle \hat{\psi}(\omega) \cdot \hat{F}(\omega) \rangle = \int dt \langle \hat{V}(t) \cdot \hat{F}(t) \rangle
= \int dt \frac{d}{dt} \left[ \frac{v(t)^2}{2} \right]. \quad (A12)\]

\[
\int \frac{d\omega}{2\pi} \langle \hat{\psi}(\omega) \cdot \hat{F}(\omega) \rangle = \int dt \langle \hat{V}(t) \cdot \hat{F}(t) \rangle
= -\int dt \langle \hat{V}(t) \cdot \nabla U(t) \rangle = -\int dt \frac{d}{dt} U(t). \quad (A13)\]

Dividing by \( t \) and taking the limit \( t \to \infty \), both terms vanish. However, the dependence on the force is implicitly contained in the expression for the entropy production rate, for instance, in the expression for the Einstein frequency [see Eq. (A15)].

**a. Discretization scheme for the path-integral approach**

To employ the path-integral approach, i.e., in writing the expressions for \( A \) and \( A_f \), we have adopted a continuous time formalism, assuming the Itô convention. Of course, since our system does not involve any multiplicative noise, the resulting entropy production rate is not affected by the choice of the integration scheme: Itô and Stratonovich conventions lead to the same results. The difference between Itô and Stratonovich conventions in the calculation of entropy production rate using path-integral approaches is described in detail in Refs. 49 and 85.

**3. Explicit expression for the spectral entropy production**

To obtain an analytical expression for \( \sigma(\omega, q) \), it is necessary to calculate the cross-correlation involved in Eq. (A11). In order to do that, we shall make two simplifying assumptions in the dynamics, Eq. (1) of the main text:
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The active Brownian particle (ABP) active force, $\mathbf{f}^a$, as an Ornstein–Uhlenbeck process for each particle evolving as

$$\tau \dot{q}^a_i = -\dot{q}^a_i + y_{0a} \sqrt{2 \tau} \xi_i,$$

(A16)

where $\xi_i$ is a vector of white noises such that $\langle \xi_i(t) \xi_j(0) \rangle = \delta(i)$. By using these approximations, the resulting dynamics reads

$$\dot{\mathbf{v}}_i = -\frac{\gamma}{m} \mathbf{v}_i - \frac{\omega_E^2}{m} \sum_j (\mathbf{u}_j - \mathbf{u}_i) + \frac{\mathbf{f}_m^a}{m} + \sqrt{2 \gamma T} \frac{m}{2} \xi_i,$$

(A17a)

The equations of motion in the Fourier Space become (switching from real space to wave vector)

$$\frac{d}{dt} \hat{\mathbf{v}}(\mathbf{q}) = \hat{\psi}(\mathbf{q}),$$

(A18a)

$$\frac{d}{dt} \hat{\psi}(\mathbf{q}) = -\frac{\gamma}{m} \hat{\psi}(\mathbf{q}) - \omega_E^2 \hat{\mathbf{u}}(\mathbf{q}) + \frac{\mathbf{f}_m^a}{m} + \sqrt{2 \gamma T} \frac{m}{2} \hat{\xi}(\mathbf{q}),$$

(A18b)

$$\tau \frac{d}{dt} \hat{\xi}(\mathbf{q}) = -\hat{\xi}(\mathbf{q}) + y_{0a} \sqrt{2 \tau} \hat{\mathbf{q}}(\mathbf{q}),$$

(A18c)

where the dispersion relation for a triangular lattice in two-dimensions, $\hat{\omega}(\mathbf{q})$, reads

$$\hat{\omega}^2(\mathbf{q}) = -2 \omega_E^2 \left[ \cos(q_x \hat{x}) + 2 \cos\left(\frac{\sqrt{3}}{2} q_x \hat{x} \right) \cos\left(\frac{\sqrt{3}}{2} q_x \hat{x} \right) \right] - 3.$$

(A19)

By applying the time-Fourier transform (switching from time to frequency) and recalling that $d/dt \rightarrow -i \omega$, it is possible to rewrite the dynamics in a compact form upon introducing the vector $\hat{\mathbf{a}}(\omega, \mathbf{q}) = (\hat{\mathbf{u}}(\omega, \mathbf{q}), \hat{\mathbf{v}}(\omega, \mathbf{q}), \hat{\mathbf{f}}^a(\omega, \mathbf{q}))$ and $\hat{\mathbf{w}}(\omega, \mathbf{q}) = m \hat{\mathbf{v}}(\omega, \mathbf{q})$, with $i$ being the imaginary unit and $T$ being the identity matrix. $\mathbf{M}$ is the dynamical matrix that rules the deterministic part of the dynamics that reads

$$\mathbf{M} = \begin{pmatrix} \hat{\omega}^2(\mathbf{q}) & \gamma T & -1 \frac{m}{T} \\ \gamma T & \frac{m}{T} & -\frac{1}{m} \frac{1}{T} \\ -1 \frac{m}{T} & -\frac{1}{m} \frac{1}{T} & 0 \end{pmatrix},$$

(A21)

and $\mathbf{w}$ is a vector of white noises such that

$$\langle \hat{\mathbf{w}}(\omega, \mathbf{q}) \hat{\mathbf{w}}(\omega', \mathbf{q}') \rangle = \mathbf{b} \cdot \mathbf{b}^T \delta(\omega + \omega') \delta(\mathbf{q} + \mathbf{q}'),$$

(A22)

where $T$ stems for transpose matrix and the noise matrix $\mathbf{b}$ reads

$$\mathbf{b} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2 \gamma T} y_{0a} \frac{m}{T} & 0 \\ 0 & 0 & \sqrt{2 \gamma T} y_{0a} \frac{m}{T} \end{pmatrix}. $$

(A23)

This equation can be rewritten as

$$\hat{\mathbf{M}}(\omega, \mathbf{q}) \cdot \hat{\mathbf{a}}(\omega, \mathbf{q}) = \mathbf{b} \cdot \hat{\mathbf{w}}(\omega, \mathbf{q}),$$

(A24)

where

$$\hat{\mathbf{M}}_q(\omega) = \hat{\mathbf{M}}(\omega, \mathbf{q}) - i\omega \mathbf{T}.$$ 

(A25)

From here, we can derive the dynamical correlation matrix as $\langle \hat{\mathbf{a}}(\omega, \mathbf{q}) \hat{\mathbf{a}}^T(\omega', \mathbf{q}') \rangle$ by multiplying the dynamics (A24) by $\mathbf{M}^{-1}$ on the left and by $\hat{\mathbf{a}}^T(\omega', \mathbf{q}')$ on the right to obtain

$$\langle \hat{\mathbf{a}}(\omega, \mathbf{q}) \hat{\mathbf{a}}^T(\omega', \mathbf{q}') \rangle = \hat{\mathbf{M}}(\omega, \mathbf{q})^{-1} \cdot \mathbf{b} \cdot \langle \hat{\mathbf{w}}(\omega, \mathbf{q}) \cdot \hat{\mathbf{a}}^T(\omega', \mathbf{q}') \rangle = \hat{\mathbf{M}}(\omega, \mathbf{q})^{-1} \cdot \mathbf{b} \cdot \mathbf{b}^T \cdot [\mathbf{M}^{-1}(\omega', \mathbf{q}')]^{-1} \delta(\omega + \omega') \delta(\mathbf{q} + \mathbf{q}').$$

(A26)

By applying this formula, we get the explicit expression for the dynamical correlations and, in particular, for the cross correlations occurring in the expression for $\sigma(\omega, \mathbf{q})$, i.e., Eq. (A11),

$$\frac{1}{2T} \int d\omega' \sum_q \left[ \langle \hat{\psi}(\omega, \mathbf{q}) \hat{\mathbf{f}}^a(\omega', \mathbf{q}') \rangle + \langle \hat{\psi}(\omega', \mathbf{q}') \hat{\mathbf{f}}^a(\omega, \mathbf{q}) \rangle \right]$$

$$= \frac{T_a \sqrt{\tau \omega^2}}{2T} \frac{T_a \sqrt{\tau \omega^2}}{\tau} \left[ \frac{\omega^2}{\tau^2} \delta(\omega + \omega') \delta(\mathbf{q} + \mathbf{q}') \right],$$

(A27)

where we have introduced the inertial time $\tau_i = m/y$, the active temperature $T_a = y_{0a} \tau$, and the shake function $K(\omega) = 1/(1 + \tau_i \omega^2)$. From this equation, we can straightforwardly identify the spectral entropy production as

$$\sigma(\omega, \mathbf{q}) = \frac{T_a \sqrt{\tau \omega^2}}{2T} \frac{T_a \sqrt{\tau \omega^2}}{\tau} \left[ \frac{\omega^2}{\tau^2} \delta(\omega + \omega') \delta(\mathbf{q} + \mathbf{q}') \right],$$

(A28)

which coincides with the main result of this paper, Eq. (3) (second line).
4. Derivation of the positional dynamical correlation

By using the compact expression for the dynamics (A26), one can derive all the dynamical correlations in frequency and wave vector domains and, in particular, the dynamical correlation of the displacement, \( C_{\hat{u},\hat{u}} \), defined as

\[
C_{\hat{u},\hat{u}}(\omega, \mathbf{q}) = \lim_{t \to \infty} \frac{1}{t} \langle \hat{u}(\omega, \mathbf{q}) \cdot \hat{u}(\omega, -\mathbf{q}) \rangle.
\]

(A29)

By performing simple algebraic calculations, one obtains

\[
C_{\hat{u},\hat{u}}(\omega, \mathbf{q}) = 2T \frac{\tau_I}{\tau_I^2(\omega^2 - \vec{\omega}^2(\mathbf{q}))^2 + \omega^2} + 2T \frac{\tau_I}{\tau_I(1 + \omega^2 \tau_I^2)} \frac{\tau_I^2}{\tau_I^2(\omega^2 - \vec{\omega}^2(\mathbf{q}))^2 + \omega^2} + \frac{T}{\tau_I},
\]

(A30)

The first term in the right-hand side of Eq. (A30) has a thermal origin \((\propto T)\) and is activity-independent, while the second term has a pure active origin being proportional to the active temperature.

We recognize that the last term in Eq. (A30) is proportional to the spectral entropy production \([i.e., -\dot{\sigma}(\omega, \mathbf{q})/\omega^2]\) after considering the shape function \(K(\omega)\) as in Eq. (4) of the main text. The first term in the right-hand side of Eq. (A30) is proportional to the imaginary part of the response function, \(R_{\hat{u},\hat{u}}\), obtained by perturbing the system by a small force \(\mathbf{h}\) and defined as

\[
R_{\hat{u},\hat{u}}(\omega, \mathbf{q}) = \frac{\delta\langle \hat{u}(\omega) \rangle}{\delta h(\omega)}|_{h=0}.
\]

(A31)

This observable explicitly reads

\[
\text{Im}[R_{\hat{u},\hat{u}}(\omega, \mathbf{q})] = \frac{\omega \tau_I}{\tau_I^2(\omega^2 - \vec{\omega}^2(\mathbf{q}))^2 + \omega^2}.
\]

(A32)

Therefore, the first term in the right-hand side of Eq. (A30) can be expressed as \(\propto \text{Im}[R_{\hat{u},\hat{u}}(\omega, \mathbf{q})]/\omega\). By using these identifications, we obtain Eq. (5) of the main text. Extending the brief comment in the main text on this relation, we outline that this formula is a relation of the Harada–Sasa form, i.e., a formula constraining dissipation (through entropy production), to response function and the correlation function of the displacement. At variance with the Harada–Sasa relation, developed in the case of particles in contact with a heat bath and driven out of equilibrium by ratcheting or constant forces, we have applied such a relation to active solids both in frequency and wave vector domains, and we have used such a relation to predict the existence of additional vibrational excitations, coexisting with phonons.

APPENDIX B: DERIVATION OF THE TOTAL ENTROPY PRODUCTION OF THE SYSTEM

The prediction for the spectral entropy production \(\sigma(\omega, \mathbf{q})\) allows us to calculate the entropy production of the system, \(\dot{s}\), explicitly as a function of the parameters of the model.

The integration over the frequency domain can be performed without approximations and yields the following result for \(\dot{s}\):

\[
\dot{s} = \frac{1}{N + 1} \frac{\Omega^2 \tau_f^2}{T} \sum_{\mathbf{q}} \frac{1}{1 + \frac{\Omega^2}{\tau_f^2} + \omega^2(\mathbf{q})^2}.
\]

(B1)

where, in the right-hand side, we have approximated the sum by an integral, introducing the symbol \(\Omega\) to denote the area of the Brillouin region associated with the triangular lattice.

The integral in the right-hand side of Eq. (B1) can be calculated through a simple change of variable, which allows us to rewrite the integral as follows:

\[
\dot{s} \approx \frac{\Omega^2 \tau_f^2}{T} \int_0^\infty \frac{dq}{\Omega} \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \int_{-\pi}^{\pi} \frac{dk_2}{2\pi} \frac{1}{1 - \pi r(k_1, k_2)},
\]

(B2)

where the function \(r(k)\) for a triangular lattice in two-dimensions reads

\[
r(k) = \frac{1}{3} \left[ \cos(k_1) + \cos(k_2) + \cos(k_1 + k_2) \right],
\]

(B3)

and the components of \(k\) are given in terms of the cartesian components of \(\mathbf{q}\) by the relations

\[
k_1 = \frac{1}{2} q_x + \frac{\sqrt{3}}{2} q_y,
\]

(B4)

\[
k_2 = \frac{1}{2} q_y - \frac{\sqrt{3}}{2} q_x.
\]

(B5)

The function \(z\) in Eq. (B2) is given by

\[
z = \frac{1}{1 + \frac{1+\psi/\tau_f}{\omega_0^2/\tau_f^2}},
\]

(B6)

The solution of the integral (B2) is known in terms of elliptic functions and allows us to express the entropy production rate \(\dot{s}\) as

\[
\dot{s} \approx \dot{s}_{\text{free}} \frac{1}{1 + \frac{6}{c} \pi \xi z K(a)},
\]

(B7)

where \(K(a)\) is the modified Bessel function of the first kind. The term \(\dot{s}_{\text{free}}\) coincides with the entropy production rate of a potential-free particle and is given by

\[
\dot{s}_{\text{free}} = \frac{\Omega^2 \tau_f^2}{T m} \frac{1}{1 + \tau_f^2},
\]

(B8)

and \(\xi\) is the correlation length of the spatial velocity correlation given by

\[
\xi^2 = \frac{3}{2} \frac{\omega_0^2}{1 + \tau_f^2} \frac{r^2}{1 + \tau_f^2},
\]

(B9)

Finally, \(z, c, \) and \(a\) are explicit functions of the parameters of the model through the expression for \(\xi\) and are given by

\[
z = \frac{1}{1 + \frac{r^2}{4\pi}}.
\]

(B9)
Fig. 4. Spectral entropy production, $\sigma(\omega, q) T/ T_a$, as a function of $\omega/ \omega_E$. Different panels show $\sigma(\omega, q) T/ T_a$ for different values of $\omega_E$ and different rescaled wave vector $q d$. (a)–(c) are obtained with $q d = 0.1, 0.2$, and $0.5$, respectively, for $\omega_E = 7$. These data correspond to those reported in Fig. 5(b) of the main text. (d)–(e) are obtained with $q d = 0.1, 0.2, 0.5$, respectively, for $\omega_E = 7 \times 10^{-2}$. These data correspond to those reported in Fig. 5(b) of the main text. Points with error bars are obtained from numerical simulations, while solid lines are theoretical predictions. Dashed, vertical colored lines mark the phonon frequency $\omega_d (q)$. The other parameters of the simulations are $v_0/(d_0 \omega_E) = 2 \times 10^{-1}$, $c/(m_d d_0^2 \omega_E^2) = 10^{-2}$, $\tau d_0 = 3.5$, and $T/ T_a = 10^{-4}$, where $T_a = v_0^2 \tau$, (as introduced and discussed in the text). In addition, $N = 10^4$ and $\phi = 1.1$.

\[ c = \frac{9}{2} - 3 + 2 \sqrt{3 + \frac{6}{2}}, \quad \text{(B10)} \]

\[ a = \frac{2 \left(3 + \frac{5}{2}\right)^{1/4}}{c^{1/2}}, \quad \text{(B11)} \]

Prediction (B7) has been used to calculate the theoretical curve in Fig. 4(a) of the main text. Expanding expression (B7) in powers of $1/\xi$, one gets Eq. (11) of the main text.

APPENDIX C: NUMERICAL DETAILS

1. Numerical details of the simulations

To integrate the dynamics (1a) and (1b) of the main text, we employ the Euler integration scheme, with a time step $\delta t = 10^{-3} \tau$, that explicitly reads

\[ \mathbf{x}_i(t + \delta t) = \mathbf{x}_i(t) + \delta t \mathbf{v}_i(t), \quad \text{(C1)} \]

\[ \mathbf{mv}_i(t + \delta t) = \mathbf{mv}_i(t) - \delta t \mathbf{v}_i(t) + \sqrt{\delta t} \sqrt{2 T_i} \mathbf{W}_i(t) + \delta t F(\{\mathbf{x}(t)\}) + \delta t \mathbf{F}_i(t), \quad \text{(C2)} \]

\[ \mathbf{F}_i(t) = v_0 \mathbf{y}_i, \quad \mathbf{y}_i = (\cos(\theta) \mathbf{n}_i, \sin(\theta) \mathbf{n}_i), \quad \text{(C3)} \]

\[ \mathbf{F}_i(t) = v_0 \mathbf{y}_i, \quad \mathbf{y}_i = (\cos(\theta) \mathbf{n}_i, \sin(\theta) \mathbf{n}_i), \quad \text{(C3)} \]

\[ \theta_i(t + \delta t) = \theta_i(t) + \sqrt{\delta t} \sqrt{2 T_i} Y_i(t), \quad \text{(C4)} \]

where $\mathbf{W}_i$ and $\mathbf{Y}_i$ are Gaussian random numbers with unit variance and zero average. Simulations were usually performed up to a final time of at least $t_{\text{max}} = 10^3 \tau$. The active solid contains $N = 10^4$ particles, at packing fraction $\phi = 1.1$, so that the size of the box reads $L = 85 d_0$. Positions are rescaled by the nominal particle diameter $d_0$, while time is rescaled by using Einstein frequency, $\omega_E$ (see definition in the main text). The dynamics is characterized by three typical times: the persistence time $\tau$ ($\tau = 1/D_0$), the inverse time $\tau \gamma = m/\gamma$, and the inverse of the Einstein frequency $1/\omega_E$ (typical of equilibrium solids). Their combinations give rise to two dimensionless parameters: $\tau \omega_E$ and $\tau_4 \omega_E$. In the main text, the latter is kept fixed to $\tau_4 \omega_E = 3.5$, and we have evaluated only the effect of the former parameter because it controls the non-equilibrium effects related to the spectral entropy production. In addition, the dynamics is characterized by the ratio between thermal and active temperature, $T/ T_a = 10^{-4}$, where $T_a = v_0^2 \tau$ (as introduced and discussed in the text). Finally, the rescaled swim velocity reads $v_0/(d_0 \omega_E) = 2 \times 10^{-1}$, and the rescaled energy constant is given by $\epsilon/(m d_0^2 \omega_E^2) = 10^{-4}$.

2. Measure of the spectral entropy production and error estimate

To measure the spectral entropy production $\sigma(\omega, q)$, we have considered Eq. (3) (first line) of the main text. To do so, we have calculated the Fourier transform of data both in frequency $\omega$ and...
FIG. 5. Displacement maps, $\Delta x_i(t) = x_i(t) - x_i(0)$, obtained for different values of the reduced persistent time: (a)–(c) with $\tau_{\omega_E} = 7$ and (d)–(f) with $\tau_{\omega_E} = 7 \times 10^{-2}$. Maps are reported for different duration times $\Delta t$: in particular, $\Delta t_{\omega_E} = 6 \times 10^{-1}$ in (a) and (d), $\Delta t_{\omega_E} = 6 \times 10^{-2}$ in (b) and (e), and $\Delta t_{\omega_E} = 1.2 \times 10^{-1}$ in (c) and (f). The other parameters of the simulations are $v_0/(d_0 \omega_E) = 2 \times 10^{-1}$, $\epsilon/(m d_0^2 \omega_E^2) = 10^{-2}$, $\tau_{I \omega_E} = 3.5$, and $T/T_a = 10^{-4}$, where $T_a = v_0^2 \gamma \tau$ (as introduced and discussed in the text). In addition, $N = 10^4$ and $\phi = 1.1$.

FIG. 6. Snapshots and mean-square displacements. (a)–(d) Snapshot configurations in the plane of motion for several values of the reduced persistent time $\tau_{\omega_E} = 7 \times 10^1, 7, 7 \times 10^{-1}, 7 \times 10^{-2}$ (from left to right). For presentation reasons, we are showing a small portion of the simulation box, i.e., a square with size $L/10$. Colors represent the direction of the self-propulsion, while hexagons are drawn as eye guides to emphasize the hexagonal order characterizing the triangular lattice. (e)–(h) Mean-square displacement, MSD(t), as a function of $t_{\omega_E}$ for $\tau_{\omega_E} = 7 \times 10^1, 7, 7 \times 10^{-1}, 7 \times 10^{-2}$ (from left to right). Error bars are obtained from the statistical error, while solid black lines are eye guides showing the scaling $\sim t$. The other parameters of the simulations are $v_0/(d_0 \omega_E) = 2 \times 10^{-1}$, $\epsilon/(m d_0^2 \omega_E^2) = 10^{-2}$, $\tau_{I \omega_E} = 3.5$, and $T/T_a = 10^{-4}$, where $T_a = v_0^2 \gamma \tau$ (as introduced and discussed in the text). In addition, $N = 10^4$ and $\phi = 1.1$. 
of the main text]. Finally, panels (a) and (d), (b) and (e), and (c) and (f) plot $q_0 d = 0.1, 0.2, \text{and } 0.5$, respectively.

**APPENDIX D: VISUALIZATION OF THE SYSTEM: SNAPSHOTS AND MEAN-SQUARE DISPLACEMENT**

In this section, we provide a visualization of the system in real space. In Figs. 3(a)–3(d), we report several snapshot configurations in the plane of motion of the system for different values of the reduced persistent time $t_{\text{re}} = 7 \times 10^1, 7 \times 10^2, 7 \times 10^3$. All of them display the typical hexagonal order characterizing two-dimensional crystals, as explicitly illustrated in Fig. 3, while the color gradient confirms the absence of spatial order in the direction of the active force, as expected.

To characterize the dynamics of the system, we also study the mean-square displacement, $\text{MSD}(t)$, defined as

$$\text{MSD}(t) = \langle (x(t) - x(0))^2 \rangle. \quad (D1)$$

In Figs. 3(e)–3(h), the $\text{MSD}(t)$ is plotted as a function of $t_{\text{re}}$ for the same values of $t_{\text{re}}$ shown in the snapshot configurations (same column). After transient regimes, for instance, showing the typical subdiffusive effects characterizing the dynamics of the solids, the system reaches a diffusive regime $\sim t$ outlined by the solid black line (which is a guide for the eyes). This analysis also confirms that the time window considered for the statistical analysis of the entropy production is enough to reach the steady state.

In Fig. 6, we also report the displacement maps $\Delta x(t) = x(t) - x(0)$ for different values of the reduced persistent time $t_{\text{re}}$ and for different duration times $t_{\text{de}}$, where $\Delta t = t - t_0$. For the smaller value of $t_{\text{de}}$, the displacement maps show weak structures that become more evident when $t_{\text{de}}$ increases. Spatial structures in the displacement maps are more evident and larger when $t_{\text{de}}$ increases.

**APPENDIX E: THE CONCEPT OF ENTROPONS BEYOND ACTIVE SOLIDS**

In this section, we show the generality of the concept of entropons that, in general, characterize a broad class of far from equilibrium solids, governed by a breaking of the fluctuation–dissipation theorem. This emphasizes the generality of entropons beyond active systems supporting our claim of generality expressed in the last sentence of the main paper.

Let us consider an elastic solid on a square lattice in two dimensions. The results could be easily generalized to arbitrary dimensions and arbitrary lattice structure, but we have chosen this case for simplicity. The particles are placed on a square lattice with periodic boundary conditions and connected by harmonic springs. The two cartesian components $x$ and $y$ of the displacement are driven by thermal baths at temperatures $T_x$ and $T_y$, respectively (with $T_x \neq T_y$), and mutually interact through a linear force,

$$F_i = -K \left[ \begin{array}{cc} -\lambda & \lambda \\ \lambda & -\lambda \end{array} \right] \left( \begin{array}{c} \Delta x_{ij} \\ \Delta y_{ij} \end{array} \right) \quad (E1)$$

where $\Delta x_{ij} = (x_i - x_j)$ and $\Delta y_{ij} = y_i - y_j$. While the force $F_i$ is symmetric for $x \rightarrow y$ and $y \rightarrow x$, the matrix of the coupling constants is not symmetric for $K \rightarrow \lambda$ and $\lambda \rightarrow K$. $K$ and $\lambda$ have a different physical meaning. Indeed, $K$ is the diagonal coupling, providing the amplitude of the usual vectorial harmonic force of components $(x_i - x_j, y_i - y_j)$. Physically, this term is a harmonic force that keeps fixed the solid structure. Instead, the constant $\lambda$ determines the amplitude of an additional force, usually absent in equilibrium solids, that has components $(y_i - y_j, x_i - x_j)$ and that couples the dynamics of the $x$ component to that of the $y$ component of the crystal. The force acting on the $x$ component is determined by the $y$ component of the displacement (and vice versa).

The dynamics of this non-equilibrium crystal read

$$\dot{v}_i^x = -\gamma v_i^x - K \sum_j (x_i - x_j) + \lambda \sum_j (y_i - y_j) + \sqrt{2T_x} \eta_i^x, \quad (E2a)$$

$$\dot{v}_i^y = -\gamma v_i^y - K \sum_j (y_i - y_j) + \lambda \sum_j (x_i - x_j) + \sqrt{2T_y} \eta_i^y, \quad (E2b)$$

where $\dot{x} = v_x, \dot{y} = v_y$, the sum runs over the first neighbors (four in the case of square lattice), $K$ represents the elastic constant of the solid, and $\lambda$ is a coupling constant mixing $x$ and $y$ components. The solid is pushed far from equilibrium by the temperature difference and the coupling between $x$ and $y$ coordinates, as known even in the case of two particles coupled by a harmonic spring.

Again, we can define the Fourier transform of the variables in the wave vector, $q$, and frequency domain, $\omega$. Using definitions (A1a) and (A1b), we define the vector of displacement, $\hat{q}(\omega, q) = (\hat{X}(\omega, q), \hat{Y}(\omega, q))$, around the lattice positions and velocity $\hat{v}(\omega, q) = (\hat{v}_x(\omega, q), \hat{v}_y(\omega, q))$. The transformed dynamics (E2) in Fourier space read

$$(-\omega^2 - i \omega y + \tilde{\omega}^2(q)) \hat{X}(\omega, q) + \frac{\lambda}{K} \tilde{\omega}^2(q) \hat{Y}(\omega, q) = \sqrt{2T_x} \tilde{\eta}^x(\omega, q), \quad (E3a)$$

$$(-\omega^2 - i \omega y + \tilde{\omega}^2(q)) \hat{Y}(\omega, q) + \frac{\lambda}{K} \tilde{\omega}^2(q) \hat{X}(\omega, q) = \sqrt{2T_y} \tilde{\eta}^y(\omega, q), \quad (E3b)$$

where $\omega(q)$ is given by

$$\omega(q)^2 = -2K \cos(q_x) \cos(q_y) - 2, \quad (E4)$$

and $\hat{v}(\omega, q)$ satisfies the relation

$$\hat{v}(\omega, q) = i \omega \hat{\omega}(\omega, q). \quad (E5)$$

Applying the method reported in the previous sections, i.e., Eq. (A3)–(A5), we can analytically calculate the spectral entropy production of the system, $\sigma(\omega, q)$. Indeed, the actions $\mathcal{A}$ and $\mathcal{A}_r$ read

$$\mathcal{A} = \int dq \int \frac{dq}{2\pi} \left[ \int \frac{dq}{4\pi} \left( (-\omega^2 + i \omega y + \tilde{\omega}^2(q)) \hat{X} + \frac{\lambda}{K} \tilde{\omega}^2(q) \hat{Y} \right)^2 \right]$$

$$+ \frac{1}{4T_x} \left[ (-\omega^2 + i \omega y + \tilde{\omega}^2(q)) \hat{Y} + \frac{\lambda}{K} \tilde{\omega}^2(q) \hat{X} \right]^2, \quad (E6a)$$

$$\mathcal{A}_r = \int dq \int \frac{dq}{2\pi} \left[ \int \frac{dq}{4\pi} \left( (-\omega^2 - i \omega y + \tilde{\omega}^2(q)) \hat{X} + \frac{\lambda}{K} \tilde{\omega}^2(q) \hat{Y} \right)^2 \right]$$

$$+ \frac{1}{4T_y} \left[ (-\omega^2 - i \omega y + \tilde{\omega}^2(q)) \hat{Y} + \frac{\lambda}{K} \tilde{\omega}^2(q) \hat{X} \right]^2, \quad (E6b)$$
having used the time-reversal transformation rule considered in Eq. (A8). With this method, we obtain
\[
\sigma(\omega, q) = -\lim_{t \to \infty} \frac{1}{T_x} \left\{ \frac{1}{T_x} \left\langle \dot{\psi}_x(\omega, q) \dot{\psi}(\omega, q) \right\rangle \right. \\
+ \left. \frac{1}{T_y} \left\langle \dot{\psi}_y(\omega, q) \dot{\psi}(\omega, q) \right\rangle \right\}.
\] (E7)

In this case, we can analytically evaluate the averages in Eq. (E7) by taking advantage of the linearity of the system. By adopting the same method employed in Eq. (A26), we calculate the analytical expression for the entropy production as a function of the parameters of the model,
\[
\sigma(\omega, q) = \gamma \left[ \frac{(T_x - T_y)^2}{T_x T_y} \right] \omega^2 \frac{1}{|D(\omega, q)|^2},
\] (E8)

where
\[
D(\omega, q) = \left( (-\omega^2 + \omega^2_i)^2 - \gamma^2 \omega^2 - \left( \frac{\lambda}{K} \omega^2(q) \right)^2 \right) \\
+ 2i\omega_r (-\omega^2_i + \omega^2(q)).
\] (E9)

This expression for the entropy production rate vanishes if \( \lambda = 0 \) but also if \( T_x = T_y \) (equilibrium limit).

In the same way, we can calculate the other elements of the dynamical correlation matrix that will shed light on the vibrational excitation of the solid. They are defined as
\[
C_{\dot{X}X}(\omega, q) = \lim_{t \to \infty} \frac{1}{T_x} \left\langle \dot{X}(\omega, q) \dot{X}(\omega, -q) \right\rangle,
\] (E10a)
\[
C_{\dot{Y}Y}(\omega, q) = \lim_{t \to \infty} \frac{1}{T_y} \left\langle \dot{Y}(\omega, q) \dot{Y}(\omega, -q) \right\rangle.
\] (E10b)

The dynamical correlations can be analytically calculated as a function of the parameters of the model, and we obtain the relation
\[
\frac{C_{\dot{X}X}(\omega, q)}{T_x} + \frac{C_{\dot{Y}Y}(\omega, q)}{T_y} = \frac{2}{\omega} \text{Im} \left( R_{\dot{X}X}(\omega, q) \right) + \frac{\sigma(\omega, q)}{\omega^2},
\] (E11)

where \( \sigma(\omega, q) \) is the spectral entropy production given by Eq. (E8), while \( \text{Im}(R_{\dot{X}X}(\omega, q)) \) and \( \text{Im}(R_{\dot{Y}Y}(\omega, q)) \) are the imaginary parts of the two diagonal elements of the response matrix that explicitly read
\[
\text{Im}(R_{\dot{X}X}(\omega, q)) = \text{Im}(R_{\dot{Y}Y}(\omega, q)) = \\
\frac{1}{|D(\omega, q)|} \text{Im} \left( -\omega^2 - i\omega r + \omega^2(q) \right) D^*(\omega, q),
\] (E12)

where the symbol * denotes the complex conjugate of a complex quantity.

The expression for the dynamical correlations, contained in Eq. (E11), provides the information on the vibrational excitations of the non-equilibrium solids described by Eq. (E2). This observable is formed by the sum of two terms: (i) an equilibrium term, proportional to the response matrix, that provides the contribution of phonons along \( x \) and \( y \) directions and (ii) a second term that is proportional to the entropy production of the system and can be identified as the contribution of entropons, being proportional to the spectral entropy production. Consistently with our picture, this term disappears in the equilibrium limit for \( \lambda = 0 \) and/or \( T_x = T_y \), i.e., when the entropy production rate vanishes.

The results of this section show that the picture of entropons goes beyond the case of active solids and characterizes more, in general, non-equilibrium solids reaching a steady state. In addition, in the case of solids driven out of equilibrium by the presence of different thermal baths, new vibrational excitations, encoding the spectral entropy production, coexist with thermal phonons.

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