ABEL SUMMATION OF RAMANUJAN-FOURIER SERIES

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ABSTRACT. This paper is a replacement for the earlier paper by this author [9] which attempted to prove (1) under very general conditions. This failed because the continuity necessary to employ the Bochner-Fejér kernel was not established. This replacement paper uses Abel summation to prove (1) is correct, but under less general conditions.

The main result of this paper is:

Given two arithmetic functions, \(f(n)\) and \(g(n)\), of bounded variation having point-wise convergent Ramanujan-Fourier (R-F) expansions of:

\[
f(n) = \sum_{q=1}^{\infty} a_q c_q(n)
\]

\[
g(n) = \sum_{q=1}^{\infty} b_q c_q(n)
\]

and

\[
\sum_{q=1}^{\infty} |a_q| \phi(q) z^q < \infty
\]

\[
\sum_{q=1}^{\infty} |b_q| \phi(q) z^q < \infty
\]

for all \(z\) in the open interval, \(0 < z < 1\),

Then:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) g(n \pm m) = \sum_{q=1}^{\infty} a_q b_q c_q(m)
\]

When applied to the auto-correlation function of an arithmetic function, \(f(n)\), the result is the Wiener-Khichtine theorem as applied to Ramanujan-Fourier series. Namely:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) f(n \pm m) = \sum_{q=1}^{\infty} \|a_q\|^2 c_q(m)
\]

Equation (1) is the central relationship to be proven within the works of H. G. Gadiyar and R. Padma, [4], [5], [6], and [7], as related to the following conjectures in number theory:

1. The Hardy-Littlewood prime \(k\)-tuple conjecture for \(k=2\).
2. The twinned prime conjecture.
3. The strong Goldbach conjecture.
4. The Sophie Germaine primes conjecture.

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1. Introduction

One of the remarkable achievements of Ramanujan[1], Hardy[2] and Carmichael[3] was the development of Ramanujan-Fourier series which converge (point-wise) to an arithmetic function. The Ramanujan-Fourier series, for an arithmetic function, \( f(n) \), can be given by

\[
f(n) = \sum_{q=1}^{\infty} a_q c_q(n)
\]

where the Ramanujan sum, \( c_q(n) \), is defined as

\[
c_q(n) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{2\pi i n k / q}
\]

and \((k,q)\) is the greatest common divisor of \( k \) and \( q \). The Ramanujan-Fourier coefficient, \( a_q \), is given by:

\[
a_q = \frac{1}{\phi(q)} \lim_{N \to \infty} \sum_{n=1}^{N} f(n) c_q(n)
\]

where \( \phi(q) \) is the Euler totient function.

Some of the properties of \( c_q(n) \) and the mean value of \( c_q(n) \) are:

a) \( c_1(n) = 1 \) for all \( n \).

b) \( c_q(0) = \phi(q) \) for all \( q \).

c) \( c_q(n) = \begin{cases} \phi(q) & q \mid n \\ -1 & q / n \end{cases} \)

d) \( c_q(n) = c_q(-n) \) for all \( q \) and all \( n \).

e) \( \lim_{N \to \infty} \sum_{n=1}^{N} c_q(n) = \begin{cases} 1 & q = 1 \\ 0 & \text{otherwise} \end{cases} \)

f) \( \lim_{N \to \infty} \sum_{n=1}^{N} c_{q_1}(n)c_{q_2}(n \pm m) = \begin{cases} c_{q_1}(m) & q_1 = q_2 \\ 0 & q_1 \neq q_2 \end{cases} \)

Given two arithmetic functions, \( f(n) \) and \( g(n) \), having point-wise convergent Ramanujan-Fourier (R-F) expansions of:

\[
f(n) = \sum_{q=1}^{\infty} a_q c_q(n) \quad \text{and} \quad g(n) = \sum_{q=1}^{\infty} b_q c_q(n)
\]

the author had hoped to prove in [9] that:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n)g(n \pm m) = \sum_{q=1}^{\infty} a_q b_q c_q(m)
\]
without restrictions on $f(n)$, $g(n)$, $a_q$, or $b_q$.

When applied to the auto-correlation function equation, (2) yields (1); the Wiener-Khinchine theorem as applied to Ramanujan-Fourier series.

The Weiner-Khinchin formula for Ramanujan-Fourier series is used to prove the results found in [4] and [5]. Theorem 2 proves (in many cases) this central relationship:

The value of the auto-correlation of an arithmetic function is the sum of the squares of the R-F coefficients for that arithmetic function.
2. Proof of Theorem 2

Definition 1. A function, \( f(z) : z \in S \subset \mathbb{R} \rightarrow \mathbb{C} \), is bounded if there exists a radius, \( \leq R_B \), about the origin such that for all \( z \in S \) \( |f(z)| \leq R_B \).

Theorem 2. If two arithmetic functions, \( f(n) \) and \( g(n) \), have point-wise convergent Ramanujan-Fourier (R-F) expansions of:

\[
f(n) = \sum_{q=1}^{\infty} a_q c_q(n) \quad \text{and} \quad g(n) = \sum_{q=1}^{\infty} b_q c_q(n)
\]

where \( f(n) \) and \( g(n) \) are bounded in the sense of Definition 1 and

\[
\sum_{q=1}^{\infty} |a_q \phi(q) z^q| < \infty \quad \text{and} \quad \sum_{q=1}^{\infty} |b_q \phi(q) z^q| < \infty
\]

for all \( z \) in the open interval, \( 0 < z < 1 \),

Then:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) g(n \pm m) = \sum_{q=1}^{\infty} a_q b_q c_q(m)
\]

Proof. The outline of the proof is to use the boundedness of the arithmetic functions to demonstrate the convolution has a finite limit and then to use the Abel summation result from Lemma 2 to assign a value to that limit.

Since, \( f(n) \) and \( g(n) \) are of bounded variation, the product is also of bounded variation. Thus there exist a constant, \( 0 \leq R_B \) such that

\[
|f(n) g(m)| \leq R_B
\]

for all \( n, m \in S \).

\[
\left| \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) g(n \pm m) \right| \leq \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |f(n) g(n \pm m)|
\]

\[
\leq \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} R_B
\]

\[
\leq R_B \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} 1
\]

\[
\leq R_B
\]

Thus,

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) g(n \pm m)
\]

converges to some finite limit, \( L \), within the radius, \( |L| \leq R_B \).

By Lemma 2

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) g(n \pm m) \quad \text{is Abel summable to} \quad \sum_{q=1}^{\infty} a_q b_q c_q(m)
\]
By Abel’s theorem the finite limit of \( \text{\text{(5)}} \) is equal to the number assigned by Abel summation.

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) g(n \pm m) = \sum_{q=1}^{\infty} a_q b_q c_q(m)
\]

□
3. Proof of Lemma A

Definition 3. Given an arithmetic function, \( f(z, n) \), with a point-wise, convergent Ramanujan-Fourier (R-F) expansion of
\[
f(z, n) = \sum_{q=1}^{\infty} a_q z^q c_q(n)
\]
then the partial summation, \( f_Q(z, n) \), is defined as:
\[
f_Q(z, n) = \sum_{q=1}^{Q} a_q z^q c_q(n)
\]

Lemma A. Given an arithmetic function, \( f(z, n) \), with an R-F expansion of
\[
f(z, n) = \sum_{q=1}^{\infty} a_q z^q c_q(n)
\]
where, for any fixed \( z \) in the open interval, \( 0 < z < 1 \), the summation
\[
\sum_{q=1}^{\infty} |a_q| z^q \phi(q)
\]
converges to some finite value,

Then, the sequence of functions, \( f_Q(z, n) \) converges uniformly in \( n \) to \( f(z, n) \) as \( Q \to \infty \).

Proof.

(6) \[
f(z, n) = \sum_{q=1}^{\infty} a_q z^q c_q(n)
\]

(7) \[
f(z, n) = \sum_{q=1}^{Q} a_q z^q c_q(n) + \sum_{q=Q+1}^{\infty} a_q z^q c_q(n)
\]

(8) \[
f(z, n) - \sum_{q=1}^{Q} a_q z^q c_q(n) = \sum_{q=Q+1}^{\infty} a_q z^q c_q(n)
\]

(9) \[
\left| f(z, n) - \sum_{q=1}^{Q} a_q z^q c_q(n) \right| = \left| \sum_{q=Q+1}^{\infty} a_q z^q c_q(n) \right|
\]

(10) \[
\left| f(z, n) - \sum_{q=1}^{Q} a_q z^q c_q(n) \right| \leq \sum_{q=Q+1}^{\infty} |a_q z^q c_q(n)|
\]

Since \(|c_q(n)| \leq \phi(q)\) for all \( q \) and all \( n \), equation (10) becomes:

(11) \[
\left| f(z, n) - \sum_{q=1}^{Q} a_q z^q c_q(n) \right| \leq \sum_{q=Q+1}^{\infty} |a_q| z^q \phi(q)
\]
By hypothesis $\sum_{q=1}^{\infty} |a_q| z^q \phi(q)$ converges. Thus, for every $\epsilon$ such that $0 < \epsilon < 1$, there exists a $Q_\epsilon$ such that $\sum_{q=Q+1}^{\infty} |a_q| z^q \phi(q) < \epsilon$ for all $Q$ where $Q_\epsilon \leq Q$

Applying this to (11) yields:

\[
(12) \quad \left| f(z, n) - \sum_{q=1}^{Q_\epsilon} a_q z^q c_q(n) \right| \leq \epsilon
\]

By the definition of $f_{Q_\epsilon}(z, n)$ equation (12) becomes

\[
(13) \quad \lim_{Q_\epsilon \to \infty} |f(z, n) - f_{Q_\epsilon}(z, n)| \leq \epsilon
\]

Since $\epsilon$ can be arbitrarily small and the selection of $Q_\epsilon$ is without regard to $n$, the sequence of functions, $f_{Q_\epsilon}(z, n)$, converges to $f(z, n)$ uniformly in $n$ as $Q_\epsilon \to \infty$
4. Proof of Lemma B

Lemma B. Given two arithmetic functions, \( f(n) \) and \( g(n) \), with point-wise convergent Ramanujan-Fourier expansions of:

\[
\begin{align*}
  f(n) &= \sum_{q=1}^{\infty} a_q c_q(n) \\
  g(n) &= \sum_{q=1}^{\infty} b_q c_q(n)
\end{align*}
\]

and

\[
\left| \sum_{q=1}^{\infty} |a_q| \phi(q) z^q \right| < \infty \quad \text{and} \quad \left| \sum_{q=1}^{\infty} |b_q| \phi(q) z^q \right| < \infty
\]

for any \( z \) in the open interval, \( 0 < z < 1 \).

Then

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n)g(n) \]  

is Abel summable to:

\[
\sum_{q=1}^{\infty} a_q b_q c_q(m)
\]

Proof. For any fixed \( z \) in the open interval \( 0 < z < 1 \) it is possible to create two related functions, \( f(z, n) \) and \( g(z, n) \) with point-wise convergent R-F expansions of:

\[
\begin{align*}
  f(z, n) &= \sum_{q=1}^{\infty} a_q z^q c_q(n) \\
  g(z, n) &= \sum_{q=1}^{\infty} b_q z^q c_q(n)
\end{align*}
\]

From the definitions in (15) it follows that as the value of \( z \) pushes up to 1 that:

\[
\lim_{z \to 1^{-}} f(z, n) = f(n) \quad \text{and} \quad \lim_{z \to 1^{-}} g(z, n) = g(n)
\]

By hypothesis the sums, \( \sum_{q=1}^{\infty} |a_q| z^q \phi(q) \) and \( \sum_{q=1}^{\infty} |b_q| z^q \phi(q) \), converge.

Thus, by Lemma A there exist two constants, \( Q_1 \) and \( Q_2 \), such that:

\[
|f(z, n) - f_Q_1(z, n)| < \epsilon \quad \text{and} \quad |g(z, n \pm m) - g_Q_2(z, n \pm m)| < \epsilon
\]

Selecting \( Q_\epsilon \) as max \((Q_1, Q_2)\) yields:

\[
|f(z, n) - f_{Q_\epsilon}(z, n)| < \epsilon \quad \text{and} \quad |g(z, n \pm m) - g_{Q_\epsilon}(z, n \pm m)| < \epsilon
\]

equation (18) can also be stated as:

\[
f_{Q_\epsilon}(z, n) - \epsilon \leq f(z, n) \leq f_{Q_\epsilon}(z, n) + \epsilon
\]

Beginning with the R-F expansions of \( f(z, n) \) and \( g(z, n \pm m) \), the convolution of \( f(z, n) \) and \( g(z, n \pm m) \) is given by:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(z, n)g(z, n \pm m) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left[ \sum_{q_1=1}^{\infty} a_{q_1} z^{q_1} c_{q_1}(n) \sum_{q_2=1}^{\infty} b_{q_2} z^{q_2} c_{q_2}(n \pm m) \right]
\]

Applying the left hand inequality from eqrefeq:CoreInequality:AsBoundedInterval to (20) yields:
(21) \[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(z, n) \overline{g(z, n \pm m)} \leq \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left[ \epsilon + \sum_{q_1=1}^{Q} a_{q_1} z^{q_1} c_{q_1}(n) \right] + \sum_{q_2=1}^{Q} b_{q_2} z^{q_2} c_{q_2}(n \pm m) \]

(22) \[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(z, n) \overline{g(z, n \pm m)} \leq \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left[ \sum_{q_1=1}^{Q} a_{q_1} z^{q_1} c_{q_1}(n) \sum_{q_2=1}^{Q} b_{q_2} z^{q_2} c_{q_2}(n \pm m) \right] + \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left[ \epsilon + \sum_{q_1=1}^{Q} a_{q_1} z^{q_1} c_{q_1}(n) \right] + \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left[ \epsilon + \sum_{q_2=1}^{Q} b_{q_2} z^{q_2} c_{q_2}(n \pm m) \right] + \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left[ \epsilon^2 \right]
\]

(23) \[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(z, n) \overline{g(z, n \pm m)} \leq \sum_{q_1=1}^{Q} \sum_{q_2=1}^{Q} a_{q_1} b_{q_2} z^{q_1} z^{q_2} \left[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_{q_1}(n) c_{q_2}(n \pm m) \right] + \epsilon \sum_{q_1=1}^{Q} a_{q_1} z^{q_1} \left[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_{q_1}(n) \right] + \epsilon \sum_{q_2=1}^{Q} b_{q_2} z^{q_2} \left[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_{q_2}(n \pm m) \right] + \epsilon^2 \left[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} 1 \right]
\]

Applying the limit process, \( N \to \infty \) and using the properties of \( c_q(n) \) listed in the introduction equation (23) becomes:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(z, n) \overline{g(z, n \pm m)} \leq \sum_{q=1}^{Q} a_q b_q z^{2q} c_q(m) + \epsilon \left[ a_1 z^1 + b_1 z^1 \right] + \epsilon^2
\]

(24)

Thus the upper bound is given by:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(z, n) \overline{g(z, n \pm m)} \leq \left( \sum_{q=1}^{Q} a_q b_q z^{2q} c_q(m) \right) + \epsilon \left( a_1 + b_1 \right) z + \epsilon^2
\]

(25)
Starting with the right hand inequality of eqref:CoreInequality:AsBoundedInterval and using similar reasoning it can be shown that the lower bound is given by:

(26) \[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(z, n) g(z, n \pm m) \geq \left( \sum_{q=1}^{Q} a_q b_q z^{2q} c_q(m) \right) - \epsilon (a_1 + b_1) z + \epsilon^2 \]

Combining the upper and lower bounds of (25) and (26) yields:

(27) \[ \left( \sum_{q=1}^{Q} a_q b_q z^{2q} c_q(m) \right) - \epsilon (a_1 + b_1) z + \epsilon^2 \leq \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(z, n) g(z, n \pm m) \leq \left( \sum_{q=1}^{Q} a_q b_q z^{2q} c_q(m) \right) + \epsilon (a_1 + b_1) z + \epsilon^2 \]

Since \( \epsilon \) can be chosen to be arbitrarily small and both \( a_1 \) and \( b_1 \) are finite due to the pointwise convergence of the R-F expansions for \( f(n) \) and \( g(n) \), applying the limit process: \( \epsilon \to 0 \), which implies \( Q \epsilon \to \infty \), to (27) yields:

(28) \[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(z, n) g(z, n \pm m) = \sum_{q=1}^{\infty} a_q b_q z^{2q} c_q(m) \]

Letting \( z \) push up to 1 in equation (28) yields:

(29) \[ \lim_{z \to 1} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(z, n) g(z, n \pm m) = \lim_{z \to 1} \sum_{q=1}^{\infty} a_q b_q z^{2q} c_q(m) \]

Thus:

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) g(n \pm m) \]

is Abel summable to:

\[ \sum_{q=1}^{\infty} a_q b_q c_q(m) \]
5. Applicability of Theorem 2 to [4] and [5]

The arithmetic function used in both [4] and [5] is:

\[ \frac{\phi(n)}{n} \Lambda(n) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n) \]  

From this \( a_q = \frac{\mu(q)}{\phi(q)} \) and

\[
\sum_{q=1}^{\infty} \left| \frac{\mu(q)}{\phi(q)} \right| z^q \phi(q) = \sum_{q=1}^{\infty} \left| \frac{\mu(q)}{\phi(q)} \right| z^q \phi(q) = \sum_{q=1}^{\infty} |\mu(q)| z^q \\
\leq \sum_{q=1}^{\infty} z^q \leq \frac{z}{1-z} \]

For all \( z \) in the open interval, \( 0 < z < 1 \)

\[ \sum_{q=1}^{\infty} \left| \frac{\mu(q)}{\phi(q)} \right| z^q \phi(q) \leq \frac{z}{1-z} < \infty \]

Thus, Theorem 2 applies to the autocorrelation of the arithmetic function:

\[ \frac{\phi(n)}{n} \Lambda(n) \]

For paper [4] on twinned primes,

\[ \lim_{N \to \infty} \sum_{n=1}^{N} \left( \frac{\phi(n)\Lambda(n)}{n} \right) \left( \frac{\phi(n+h)\Lambda(n+h)}{n+h} \right) \]

is Abel summable to \( C(h) \) which is defined in equation 7 of [4] as:

\[ \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\phi^2(q)} c_q(h) = C(h) \]

Similarly, for paper [5] on Sophie German primes,

\[ \lim_{N \to \infty} \frac{\Phi_{(a,b,l)}(N)}{N} \]

is Abel-summable to the expression \( S \) defined in equation 16 of [5].
6. Conclusions

The problem with proving \( (2) \), the strong form of the Wiener-Khinchin theorem, is two fold.

The first problem is the question:

Does the limit

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) f(n \pm m)
\]

exist or not?

The second problem is the question:

If the limit in (36) exists, then what is the value of that limit?

Theorem 2 answers the second question, but not the first. If the limit in (36) exists, then the value of that limit is

\[
\sum_{q=1}^{\infty} ||a_q||^2 c_q(m)
\]

The author hopes to apply Abel summation to almost periodic functions to form a Poisson-like kernel which is analogous to the Bochner-Fejér kernel found in [8] for Cesàro summation of a.p. functions.

It is hoped that with such a Poisson-like kernel in hand, the kernel can be used to prove the convergence question for equation (2) and (a fortiori) equation (2). Once the conditions for convergence for (2) are understood and proved, then Theorem 2 can be used to determine the value to which the function(s) converge.
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