Exact cosmological solutions in modified Brans–Dicke theory

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Abstract

In this paper, we obtain exact cosmological vacuum solutions for an extended FLRW homogeneous and isotropic Brans–Dicke (BD) universe in five dimensions for all values of the curvature index. Then, by employing the equations associated with a modified Brans–Dicke theory (MBDT) \cite{1}, we construct the physics on a four-dimensional hypersurface. We show that the induced matter obeys the equation of state of a fluid of barotropic type. We discuss the properties of such induced matter for some values of the equation of state parameter and analyze in detail their corresponding solutions. To illustrate the cosmological behaviors of the solutions, we contrast our solutions with those of the standard BD theory. We find that, in the MBDT scenario, it is impossible to find a physically acceptable solution associated with the negative curvature for both the dust-dominated and radiation-dominated universes. However, for spatially flat and closed universes, we argue that our obtained solutions are more general than those associated with the standard BD theory and, moreover, they contain a few classes of solutions which have no analog in the BD cosmology. For these particular cases, we further compare the results with those extracted in the context of the induced-matter theory (IMT) and general relativity (GR). Furthermore, we discuss in detail the time behaviors of the cosmological quantities and compare them with recent observational data. We find a favorable range for the deceleration parameter associated with a matter-dominated spatially flat universe (for the late times), which is compatible with recent corresponding observational results.

Keywords: FLRW cosmology, modified Brans–Dicke theory, induced-matter theory, extra dimension, scalar–tensor theories

(Some figures may appear in colour only in the online journal)
1. Introduction

The scalar–tensor theories (for a complete review for applications in cosmology, see, e.g., [2]) have been proposed based on the main idea which asserts that the gravitational coupling is time dependent, an idea related to the large number hypothesis by Dirac [3, 4]. He suggested that the gravitational constant $G$ decreases with the age of the universe, an assumption that can be in agreement with a few geological facts [5, 6]. Then, Dirac’s idea was developed by other physicists by constructing new versions of the scalar–tensor theories in which the gravitational constant has been replaced by a scalar field [7, 8], and consequently it must satisfy a generalized conservation law proposed in the theory. In 1961, the simplest version of the scalar–tensor theories was proposed by Brans and Dicke, primarily motivated by cosmology and Mach’s principle [9].

Using different methods, exact cosmological solutions have been obtained for the isotropic FLRW models, the anisotropic Bianchi types I–IX and the related Kantowski–Sachs models by assuming various kinds of ordinary matter in the context of the BD theory (see, e.g., [2, 10–22] and references therein).

Recently, it has been shown that, by applying a specific reductional procedure for the conventional BD theory in $(D + 1)$-dimensional space-time, an MBDT in $D$ dimensions is obtained [1]. The MBDT has four sets of field equations, of which two sets, regardless of the geometrical origin of the matter and scalar potential, are mathematically similar to those derived from the BD standard action with a scalar potential. One set of the mentioned MBDT field equations is the extended version of the conservation equation introduced in induced-matter theory (IMT); see, e.g., [23–26]. Finally, the fourth equation retrieved in MBDT has no analog in the conventional BD theory or in scalar–tensor theories.

To gain an insight into the physical features of the MBDT and compare the properties of its geometrically induced matter and scalar potential with the ordinary ones assumed in the context of the BD theory as well as with observational data, several investigations have been presented by considering both the spatially flat FLRW universe (in four [27, 28] and arbitrary [1] dimensions) and the Bianchi type I cosmology [29]. We should note that, to our knowledge, the solutions of the non-flat FLRW space-time have not been obtained in either the higher-dimensional BD theory or the MBDT scenario. Moreover, the solutions herein associated with flat space are wider than the ones obtained in [1]. It is important to note that the induced scalar potential and matter in the MBDT lead us to an accelerating universe in late times (as will be seen in this work) and we do not need to add a scalar potential by hand as in some phenomenological scenarios in the context of the DB theory [30, 31].

The aim of this paper will be to employ the MBDT to reduce the five-dimensional FLRW solutions (for all values of the curvature) on a suitably projected four-dimensional space-time. We will show that the induced matter obeys the barotropic equation of state and consequently this geometrically originated matter can be examined for particular choices of well known matter types in the universe. Subsequently, the properties of obtained extended solutions will be analyzed and then they will be compared with the ones obtained from the four-dimensional BD theory (with or without a scalar potential) as well as with observational data. We will further show that all the solutions can be considered as generalized versions of the well known solutions of the BD theory (e.g., the Dehnen–Obregón, Nariai, O’Hanlon–Tupper solutions) in four-dimensional space-time.

In this work, the expressions for physical quantities will not be obtained either by using the conformal time or by defining other variables to explain the behavior of the scale factor and scalar field, but instead we will discuss the behaviors from their corresponding direct physical expressions.
Our work is organized as follows. In section 2, we review the MBDT set-up. In section 3, by assuming a five-dimensional FLRW universe (with all values of the curvature), we solve the field equations associated with the standard BD theory in vacuum. In section 4, by means of the MBDT setting, we construct the physics on a four-dimensional hypersurface. We show that the induced matter obeys the barotropic equation of state and consequently it can play the role of either the ordinary matter or dark matter–dark energy in the universe. By assuming a few known values for the equation of state parameter, we discuss the properties of the solutions and compare them with those obtained in the context of the BD theory as well as with observational data. Finally, in section 5, we review the main results and add some new discussion.

2. Modified Brans–Dicke theory in four dimensions

Let us in this section briefly review the MBDT set-up [1, 28].

In analogy to the four-dimensional standard BD theory, the action associated with a five-dimensional BD theory, in the Jordan frame, can be written as

\[ S^{(5)} = \int d^5x \sqrt{|G^{(5)}|} \left[ \phi R^{(5)} - \frac{\omega}{\phi} G^{ab}(\nabla_a \phi)(\nabla_b \phi) + 16\pi L^{(5)}_{\text{matter}} \right], \]  (2.1)

where the Latin indices run from zero to four, \( G^{(5)} \) is the determinant of the metric \( G_{ab} \) associated with a five-dimensional space-time, \( R^{(5)} \) is the curvature scalar and \( \nabla_a \) stands for the covariant derivative in the five-dimensional space-time. Moreover, \( \phi \) is the BD scalar field, \( \omega \) is the adjustable dimensionless BD coupling parameter and we have chosen \( c = 1 \). In addition, \( L^{(5)}_{\text{matter}} \) is the Lagrangian density associated with the ordinary matter, which is minimally coupled to the BD scalar field.

The field equations, derived from the action (2.1), are given by

\[ G^{(5)}_{ab} = \frac{8\pi}{\phi} T^{(5)}_{ab} + \frac{\omega}{\phi^2} \left[ (\nabla_a \phi)(\nabla_b \phi) - \frac{1}{2} G_{ab}(\nabla^c \phi)(\nabla_c \phi) \right] + \frac{1}{\phi} (\nabla_a \nabla_b \phi - G_{ab} \nabla^2 \phi) \]  (2.2)

and

\[ \nabla^2 \phi = \frac{8\pi T^{(5)}}{3\omega + 4}, \]  (2.3)

where \( \nabla^2 = \nabla_a \nabla^a \), \( T_{ab} \) is the energy–momentum tensor (EMT) of the ordinary matter fields in a five-dimensional space-time and \( T^{(5)} = G^{ab} T^{(5)}_{ab} \).

By means of the reduction procedure for the BD theory, the field equations associated with the four-dimensional MBDT can be obtained on a hypersurface. In these field equations, geometrically induced terms will play the role of the ordinary matter sources and scalar potential in an extended version of the standard BD theory\(^1\). Let us be more precise. By applying the well known line element\(^2\) [24]

\[ ds^2 = G_{ab}(x^\alpha) dx^a dx^b = g_{\mu l}(x^\alpha, l) dx^\mu dx^\nu + \epsilon \psi^2 (x^\alpha, l) dl^2, \]  (2.4)

the BD field equations (2.2) and (2.3) are induced on the hypersurface \( \Sigma_0 (l = l_0 = \text{constant}) \), which is orthogonal to the unit vector \( n^a = \delta^a_4 / \psi \) where \( n_a n^a = \epsilon \). In this respect, the MBDT,

\(^1\) In the Lagrangian density associated with the original BD theory, there is no scalar potential.

\(^2\) The Greek indices run from 0 to 3 and \( l \) stands for the non-compact coordinate associated with the fifth dimension. The indicator \( \epsilon = \pm 1 \) is chosen such that the extra dimension can be either time-like or space-like, and \( \psi = \psi(x^\alpha, l) \) is another scalar field.
which can be regarded as a generalized version of the IMT, BD theory or GR, conveys four field equations. One pair of these equations reproduces conventional four-dimensional BD field equations but with the specificity that an induced scalar potential is present, in which the induced EMT and the mentioned scalar potential have a geometrical origin. The other pair has no analog in the standard four-dimensional BD theory (for a more detailed presentation of the MBDT, see [1]). We will now explain how our equations for a four-dimensional universe are obtained from a five-dimensional BD setting, on a four-dimensional hypersurface, directly and explicitly computing all terms, by means of dimensional reduction and projection, i.e. all elements having a direct geometric origin, computed consistently and none taken as an ad hoc assumption. Namely, in the next section, we will use

\[
G^{(4)}_{\mu\nu} = \frac{8\pi}{\phi} (S_{\mu\nu} + T^{(BD)}_{\mu\nu}) + \frac{\omega}{\phi^2} \left[ (D_\mu\phi)(D_\nu\phi) - \frac{1}{2} g_{\mu\nu} (D_\alpha\phi)(D^\alpha\phi) \right] \\
+ \frac{1}{\phi} [D_\mu D_\nu \phi - g_{\mu\nu} D^2 \phi] - g_{\mu\nu} \frac{V(\phi)}{2\phi},
\]

\[= 8\pi \frac{\omega}{\phi^2} \left[ (D_\mu\phi)(D_\nu\phi) - \frac{1}{2} g_{\mu\nu} (D_\alpha\phi)(D^\alpha\phi) \right] \\
+ \frac{1}{\phi} [D_\mu D_\nu \phi - g_{\mu\nu} D^2 \phi] - g_{\mu\nu} \frac{V(\phi)}{2\phi},\]

(2.5)

where \(D_\alpha\) is the covariant derivative on the hypersurface,

\[
D^2 \phi = \frac{8\pi}{2\omega + 3} (S + T^{(BD)}) + \frac{1}{2\omega + 3} \left[ \phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \right],
\]

(2.6)

where \(D^2 \equiv D_\alpha D^\alpha\),

\[
S_{\mu\nu} \equiv T^{(5)}_{\mu\nu} - g_{\mu\nu} \left[ \frac{(\omega + 1) T^{(5)}}{3\omega + 4} - \frac{\epsilon T^{(5)}}{\psi^2} \right]
\]

(2.7)

is the effective matter obtained from the five-dimensional EMT and

\[
T^{[BD]}_{\mu\nu} = T^{[IMT]}_{\mu\nu} + T^{(\phi)}_{\mu\nu} + \frac{1}{16\pi} g_{\mu\nu} V(\phi)
\]

(2.8)

is an induced EMT on a four-dimensional space-time in which

\[
\frac{8\pi}{\phi} T^{[IMT]}_{\mu\nu} \equiv D_\mu D_\nu \psi - \frac{\epsilon}{2\psi^2} \left( \psi' g'_{\mu\nu} \psi - g_{\mu\nu} + g^{\alpha\beta} g_{\mu\alpha} g_{\nu\beta} - \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta} \right) \\
- \frac{\epsilon g_{\mu\nu}}{8\psi^2} [g^{\alpha\beta} g_{\alpha\beta}'] + (g^{\alpha\beta} g_{\alpha\beta}')^2],
\]

(2.9)

\[
\frac{8\pi}{\phi} T^{(\phi)}_{\mu\nu} \equiv \frac{\epsilon \phi'}{2\psi^2} \left( g_{\mu\nu}' + g_{\mu\nu} \left( \frac{\omega \phi'}{\phi} - \frac{g^{\alpha\beta} g_{\alpha\beta}'}{\phi} \right) \right).
\]

(2.10)

where a prime denotes a derivative with respect to the fifth coordinate, \(l\). The quantity \(V(\phi)\) is an induced scalar potential on the hypersurface, which is obtained from
The first term of the induced EMT, i.e. $T_{\mu\nu}^{[\text{IMT}]}$, which bears a resemblance to the one introduced in IMT [23, 24], is the fifth part of the metric (2.4) which is geometrically induced on the hypersurface. Whereas the $\phi$-part, i.e. $T_{\phi\phi}^{[5]}$, is composed of the BD scalar field and its derivatives with respect to the fifth coordinate, it has no analog in IMT. Moreover, the induced scalar potential is provided completely from the geometry of the fifth dimension rather than adding an extra term by hand, such as the ones usually employed in phenomenological applications of the scalar–tensor theories in astrophysics/cosmology; see, e.g., [31] and references therein.

In section 4, we will use these geometrical intrinsic as well as elegant advantages of the MBDT to retrieve BD cosmological features on the hypersurface for a homogeneous and isotropic Friedmann universe and compare them with the results of conventional BD theory, other scalar–tensor theories and also observational data.

3. Exact solutions of Brans–Dicke cosmology in a five-dimensional space-time

We start with a five-dimensional extended version of the Friedmann–Lemaître–Robertson–Walker (FLRW) universe in which there is no ordinary matter, i.e., $T_{00}^{(5)} = 0$. We will find the exact solutions in the five-dimensional space-time and then, in section 4, by applying the set-up reviewed in the previous section, we concentrate on the cosmological solutions further MBDT projected on a four-dimensional hypersurface.

Let us work with a metric whose components depend only on comoving time. Thus, we consider the line element

$$
dS^2 = -dr^2 + a^2(t) \left[ \frac{dr^2}{1 - \lambda r^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right] + \epsilon \psi^2(t) dl^2, \tag{3.1}\n$$

where, without loss of generality, we can set the spatial curvature constant as $\lambda = -1, 0, 1$, which correspond to open, flat and closed universes, respectively; $(t, r, \theta, \phi)$ are the coordinates associated with a four-dimensional space-time whose spatial sections are spherically symmetric. $a(t)$ and $\psi(t)$ are cosmological scale factors.

Employing equations (2.2) and (2.3) with the line element (3.1) gives us the dynamical field equations in a five-dimensional space-time, namely

$$\ddot{\phi} + \dot{\phi} \left[ \frac{3\dot{a}}{a} + \frac{\dot{\psi}}{\psi} \right] = 0, \tag{3.2}\n$$

$$\frac{\dot{a}}{a} \left( \frac{a}{\dot{a}} + \frac{\dot{\phi}}{\phi} \right) + \frac{\lambda}{a^2} = -\dot{\phi} \left( \frac{\dot{a}}{a} + \frac{\dot{\psi}}{\psi} \right) + \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2, \tag{3.3}\n$$

$$\frac{2\ddot{a}}{a} + \frac{\ddot{a}}{a} + \frac{2\dot{\psi}}{\psi} = \frac{\lambda}{a^2} + \frac{\ddot{\phi}}{\phi} = -\dot{\phi} \left( \frac{2\ddot{a}}{a} + \frac{\omega}{2} \frac{\dot{\phi}}{\phi} + \frac{\dot{\psi}}{\psi} \right) - \frac{\dot{\phi}}{\phi}, \tag{3.4}\n$$

The first term of the induced EMT, i.e. $T_{\mu\nu}^{[\text{IMT}]}$, which bears a resemblance to the one introduced in IMT [23, 24], is the fifth part of the metric (2.4) which is geometrically induced on the hypersurface. Whereas the $\phi$-part, i.e. $T_{\phi\phi}^{[5]}$, is composed of the BD scalar field and its derivatives with respect to the fifth coordinate, it has no analog in IMT. Moreover, the induced scalar potential is provided completely from the geometry of the fifth dimension rather than adding an extra term by hand, such as the ones usually employed in phenomenological applications of the scalar–tensor theories in astrophysics/cosmology; see, e.g., [31] and references therein.

In section 4, we will use these geometrical intrinsic as well as elegant advantages of the MBDT to retrieve BD cosmological features on the hypersurface for a homogeneous and isotropic Friedmann universe and compare them with the results of conventional BD theory, other scalar–tensor theories and also observational data.
An overdot denotes the derivative with respect to the cosmic time and we have assumed the BD scalar field to depend only on the cosmic time. Equation (3.2) has been derived from (2.3), and equations (3.3)–(3.5) have been, respectively, obtained by setting \( a = 0 = b \), \( a = b = 1, 2, 3 \) and \( a = 4 = b \) in equation (2.2).

We should note that the coupled nonlinear equations (3.2)–(3.5) are not independent. To solve these equations, let us consider a well known assumption: we employ Dirac’s hypothesis that states that the gravitational constant and the scale factor of the universe should be related to each other by a power-law relation \( [11–13] \). As in the BD theory the scalar field is proportional to the inverse of the gravitational constant (see equations (4.14) and (4.15)), we can write

\[
\phi(t) = C\left[a(t)\right]^n. \tag{3.6}
\]

In order to obtain consistent solutions, the above assumption leads us to consider also a power law between \( a(t) \) and \( \psi(t) \), as

\[
\psi(t) = \alpha \left[a(t)\right]^{-\beta}. \tag{3.7}
\]

In relations (3.6) and (3.7), \( C > 0 \) (corresponding to attractive gravity) and \( \alpha > 0 \) are constants which are not independent and they can be determined in an arbitrary fixed time; \( n \) and \( \beta \) are parameters which must satisfy the field equations\(^3\) (3.2)–(3.5).

From (3.2), after a simple integration, we get

\[
\dot{\phi}a^3\psi = A, \tag{3.8}
\]

where \( A \neq 0 \) is a constant; plugging \( \phi \) and \( \psi \) from (3.6) and (3.7) into (3.8), we obtain

\[
a(t) = \left[\frac{A}{C\alpha}\right]\left(3 + n - \beta\right)t^{\frac{1}{3}n + \beta}, \tag{3.9}
\]

where \( 3 + n - \beta \neq 0 \) and we have assumed that at the initial time \( t_i = 0 \) the scale factor vanishes. By employing the power-law solutions (3.6) and (3.7) in (3.3), we obtain

\[
\left(\frac{\dot{a}}{a}\right)^2 = \beta\left[1, + \frac{n}{3} + n\left(1 - \frac{\omega}{6}\right) + 1\right] + \frac{\lambda}{a^2} = 0. \tag{3.10}
\]

In what follows, we will investigate all the possible solutions associated with the flat space (\( \lambda = 0 \)) and non-flat spaces (\( \lambda^2 = 1 \)).

3.1. Flat space solutions

For the flat space, by substituting \( \lambda = 0 \) into (3.10), we obtain the following algebraic equation:

\[
-\frac{\omega}{6}n^2 + \left(1 - \frac{\beta}{3}\right)n + (1 - \beta) = 0. \tag{3.11}
\]

\(^3\) From current particle physics data, it can be assumed that, for an extra coordinate having lengthlike nature, it must be very small at the present time [32, 33], i.e., it must contract with time. However, in this paper, to obtain a comprehensive list of solutions, we will also investigate the cases in which \( \beta < 0 \).
Equation (3.11) gives $n$ in terms of $\beta$ as

\[ n = \frac{1}{\omega} [(3 - \beta) \pm \sqrt{\beta^2 - 6\beta(1 + \omega) + 3(2\omega + 3)}]. \]  

(3.12)

Thus, the general solutions for the flat space are given by relations (3.6), (3.7) and (3.9), in which $n$ and $\beta$ are not independent and are related according to (3.12). These solutions are classified into two sets: (i) the power-law solution when $n = \beta - 3$, in which $n$ and $\beta$ are not independent and they are related to each other by (3.12); (ii) the exponential solution that is obtained when $n = \beta - 3$. To retrieve the second class of solutions, we have to start from the field equations (3.2)–(3.5). It is straightforward to show that

\[ a(t) = a_i e^{\frac{\beta}{\omega}t}, \quad \phi(t) = C a_i e^{\frac{\beta}{\omega}t}, \quad \psi(t) = \alpha a_i^{-3} e^{\frac{3\omega}{\omega^2 - 4\omega}}. \]  

(3.13)

where $a_i$ is an integration constant. For this particular case, from (3.11), $\beta$ in terms of the BD coupling parameter is given by

\[ \beta = \frac{3(1 + \omega) \pm \sqrt{-3(5 + 4\omega)}}{2 + \omega}, \]  

(3.14)

where, when $\omega \to -2$, for the upper case (plus sign) $\beta$ goes to unity, while for the lower sign it goes to minus infinity. Namely, for the former we get a contracting fifth dimension with time, while for the latter the fifth dimension increases with time.

### 3.2. Non-flat space solutions

For $\lambda^2 = 1$, by employing (3.9) in (3.10) we obtain an algebraic relation which is satisfied for all values of $t$ provided that

\[ n = \beta - 2. \]  

(3.15)

Thus, from (3.9) the general solutions associated with the non-flat space can be briefly summarized as

\[ a(t) = \frac{1}{\beta - 2} \left( \frac{A}{C a_i} \right)^t, \quad \phi(t) = C [a(t)]^{\beta - 2} = C \left[ \frac{1}{\beta - 2} \left( \frac{A}{C a_i} \right)^t \right]^{\beta - 2}, \]  

\[ \psi(t) = \alpha [a(t)]^{-\beta} = \alpha \left[ \frac{1}{\beta - 2} \left( \frac{A}{C a_i} \right)^t \right]^{-\beta}, \]  

(3.16)

where

\[ \left( \frac{A}{C a_i} \right)^2 = \frac{6\lambda(\beta - 2)^2}{\omega(\beta - 2)^2 + 2\beta(\beta - 2) + 6}. \]  

(3.17)

Besides, by employing relations (3.5), (3.15)–(3.17), it is easy to show that

\[ \omega = -\frac{2(\beta^2 - 2\beta + 3)}{(\beta - 2)^2}, \]  

(3.18)

where in the particular case $\beta = 2$, $|\omega| \to \infty$ and consequently the BD scalar field takes a constant value. In this case, we cannot find a finite value for the scale factor $a(t)$. Namely, in this case, when $|\omega| \to \infty$ there is no vacuum solution. As when the BD coupling parameter goes to infinity, the BD theory may reduce to GR, thus the above result is in agreement with the fact that there are no vacuum solutions associated with the FLRW universe in GR (without a cosmological constant term) for $\lambda^2 = 1$. 

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In the special case where the scalar factor of the fifth dimension is a constant, i.e., \( \psi = \text{constant} \), by setting \( \beta = 0 \), equation (3.17) gives

\[
\left( \frac{A}{c_0} \right)^2 = \frac{4\lambda}{2\omega + 1}. \tag{3.19}
\]

Therefore, for the cases with positive curvature (\( \lambda = 1 \)) and negative curvature (\( \lambda = -1 \)), we must choose \( \omega > -3/2 \) and \( \omega < -3/2 \), respectively. From (3.19), for both of these cases, we have

\[
a(t) = \pm \frac{t}{\sqrt{\left| 1 + \frac{2\omega}{\lambda} \right|}}, \quad \text{and} \quad \phi(t) = \frac{C|1 + \frac{2\omega}{\lambda}|}{t^2}. \tag{3.20}
\]

4. Effective Brans–Dicke cosmology on a four-dimensional hypersurface

In this section, we will employ the MBDT set-up reviewed in section 2 for the obtained solutions of the previous section to construct the physics on a four-dimensional space-time. Then, we evaluate the properties of the corresponding solutions and compare them with those of the standard BD theory.

The non-vanishing components of the induced matter on the hypersurface can be obtained by substituting the components of the metric (3.1) and the isotropic BD scalar field in relation (2.8), as

\[
\frac{8\pi}{\phi} T_{i0}^{\text{BD}} = -\frac{\dot{\phi}}{\phi} + \frac{V(\phi)}{2\phi}, \tag{4.1}
\]

\[
\frac{8\pi}{\phi} T_{ij}^{\text{BD}} = \frac{\ddot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi} \frac{\dot{V}(\phi)}{V(\phi)}, \tag{4.2}
\]

where \( i = 1, 2, 3 \) (with no sum) and the induced potential \( V(\phi) \) will be derived using the differential equation (2.11). From (4.2), it is clear that the different components of \( T_{ij}^{\text{BD}} \) are equal, thus the induced matter can be considered as a perfect fluid.

The induced scalar potential is evaluated on the four-dimensional hypersurface by substituting the solutions (3.6), (3.7), (3.9) and (3.16) into (2.11). Thus, we obtain\(^4\)

\[
\frac{dV}{d\phi} \bigg|_{\Sigma_0} = \begin{cases} 
-2\frac{\dot{A}}{A} (1 + \omega) \frac{\beta}{\lambda} \left( \frac{A}{c_0} \right)^2 \phi^{\frac{2(3+n-\beta)}{\lambda}} & \text{for } \lambda = 0, \\
-2C \frac{\dot{A}}{A} (1 + \omega) \left( \frac{\beta}{\lambda} \right) \left( \frac{A}{c_0} \right)^2 \phi^{\frac{2(3+n-\beta)}{\lambda}} & \text{for } \lambda = -1, +1.
\end{cases} \tag{4.3}
\]

For a non-flat space, integrating the lower differential equation of (4.3), by assuming that the constants of integration are zero, we obtain

\(^4\) We should note that some quantities appearing in different solutions have different values. For instance, the quantity \( \frac{\dot{A}}{A} \), which is seen in the upper and lower relations of the induced scalar potential (4.3), has different values in terms of the corresponding parameters, such that this quantity for the non-flat space is related to the other quantities by (3.17), whereas we have not obtained any general relation for this quantity in the case of the flat space.
Moreover, in this case, scalar potential leads to inconsistency for applying the energy conditions, we opt to abandon the solutions associated with this case of the induced scalar potential.

From the upper equation of (4.3), the power-law scalar potential associated with the flat space is given by

\[
V(\phi) = \begin{cases} 
-48\beta C(1 + \omega) & \text{for } \beta = 4 \\
-2C\ln(1 + \omega) & \text{for } \beta \neq 2, 4,
\end{cases}
\]

where \(A\), according to relation (3.17), is a function of the parameters \(\beta, \omega\) and the curvature constant \(\lambda\).

The power-law scalar potential (i.e. the lower relation in (4.4) for non-flat space and (4.5) for flat space, whose solutions, in what follows, will be discussed in this paper), vanishes in the particular cases where either \(\omega = -1\) or \(\beta = 0\).

For the power-law scalar potential, by employing (3.9) and (3.16) for flat and non-flat spaces, we obtain the induced scalar potential versus cosmic time as

\[
V(\phi) = \begin{cases} 
2\beta C(1 + \omega) + \left(\frac{A}{C\alpha}\right)^{\frac{1}{\beta}} & \text{for } \lambda = 0, \\
-2\left(1 + \omega\right)\left(\frac{A}{C\alpha}\right)^{\frac{1}{\beta}} & \text{for } \lambda = -1, +1; \beta \neq 2, 4,
\end{cases}
\]

In order to write the components of the induced matter (associated with the power-law cases) on the hypersurface, we employ (3.6), (3.7), (3.9), (3.17), (3.16) and (4.6), and relations (4.1) and (4.2). Thus, the energy density \(\rho_{BD} \equiv -T_{00}^{[BD]}\) and the isotropic pressures \(p_{BD} = p_{I} \equiv T_{I}^{[BD]}\) for the non-flat and flat spaces are given, respectively, by

\[
\rho_{BD} = \rho_{0}t^{\beta-4}, \quad p_{BD} = p_{0}t^{\beta-4}, \quad \text{for } \lambda = -1, +1,
\]

where

\[
\rho_{0} \equiv \frac{C}{8\pi} \left(\frac{A}{C\alpha}\right)^{\frac{1}{\beta}} \left[\frac{(\beta + 1)(\beta - 4) + (1 + \omega)(\beta - 2)^2}{(\beta - 4)(\beta - 2)^{\beta-2}}\right],
\]

and (for the flat space)

\[
\rho_{BD} = -\frac{C\beta}{8\pi} \left[\frac{A(n - \beta + 3)}{Cn\alpha}\right]^{\frac{1}{\beta}} \left[\frac{6(\beta - 3) + n(\omega + 2\beta - 9)}{(n - 2\beta + 6)(n - \beta + 3)^2}\right]t^{\frac{\beta}{\beta - 2} - 2},
\]

\[
p_{BD} = \frac{C\beta}{8\pi} \left[\frac{A(n - \beta + 3)}{Cn\alpha}\right]^{\frac{1}{\beta}} \left[\frac{(1 + \omega)n^2 + n - 2\beta + 6}{(n - 2\beta + 6)(n - \beta + 3)^2}\right]t^{\frac{\beta}{\beta - 2} - 2}.
\]

As the logarithmic effective potential leads to inconsistency for applying the energy conditions, we opt to abandon the solutions associated with this case of the induced scalar potential.

In the special case where \(\omega = -1\), the BD theory corresponds to the low energy limit of the bosonic string theory. Moreover, in this case, scalar–tensor theories contain certain similarities with supergravity and string theory [2, 34].
The above geometrical effective matter relations give a barotropic equation of state associated with an induced perfect fluid for all the cases. Namely,

\[ p_{\text{BD}} = W_{\text{BD}} \rho_{\text{BD}}, \]  

where

\[ W_{\text{BD}} = \begin{cases} 
\frac{2(\beta - 3) - n(\omega_n + n + 1)}{5(\beta - 3) + n(\omega_n + 2\beta - 9)} & \text{for } \lambda = 0, \\
\frac{(\beta - 4) - (1 + \omega_n)(\beta - 2)^2}{(\beta + 1)(\beta - 4) + (1 + \omega_n)(\beta - 2)^2} & \text{for } \lambda = -1, +1. 
\end{cases} \]  

By using the relations (3.9), (3.18), (4.12) and (4.13), it is straightforward to show that the induced matter (on the four-dimensional hypersurface) is conserved, namely \( \dot{\rho}_{\text{BD}} + 3(\dot{a}/a)(\rho_{\text{BD}} + P_{\text{BD}}) = 0 \). This result is of relevance in the context of scalar–tensor theories, because it allows us to state that the BD scalar field does couple minimally with the induced matter, and thus the model herein does respect the principle of equivalence.

As we would like to study, in particular, the behavior of the gravitational coupling, we review herewith general features associated with it. As the field equation (2.5) corresponds to the conventional BD field equations, Newton’s gravitational constant (in a four-dimensional space-time) can be read

\[ G_N(\phi) = \frac{1}{\phi}. \]  

Moreover, we should note that the above definition has been used in cosmological application, rather than in the analysis in the context of the post-Newtonian formalism [35], where the relation

\[ G_{\text{eff}} = \frac{2(\omega + 2)}{2\omega + 5} \frac{1}{\phi} \]  

has been defined for spherically symmetric solutions [2, 36-38] and employed in the Solar System experiments. However, we should note that the rates of variation of \( G_N \) and \( G_{\text{eff}} \) have the same features.

In what follows, we will elaborate on the MBDT retrieved above for concrete values of the equation of state parameter. After probing the properties of the geometrical induced matter, we compare the results with those obtained from the conventional BD theory and observational data.

### 4.1. Vacuum cosmologies

One of the main questions in the context of scalar–tensor theories is whether non-trivial vacuum (absence of ordinary matter) solutions exist or not. Such solutions are very interesting because, when \( t \) asymptotically tends to zero, the fluid-filled FLRW solutions approach the vacuum solutions for wide ranges of the varying BD coupling parameter [39]. After the seminal papers by Brans and Dicke, to our knowledge, this question was investigated by O’Hanlon and Tupper [10]. They found the cosmological non-static vacuum solutions for the BD theory for spatially flat FLRW metric and addressed questions such as the validity of the Mach principle and Birkhoff theorem in the BD theory. However, for the non-flat FLRW space, they only found solutions for spatial values of the BD coupling parameter, \( \omega = -3/2, -4/3, 0. \) Subsequently, the whole class of solutions associated with the latter (non-flat) case were obtained by Dehnen and Obregón by assuming a power-law relation between the scale factor and the effective gravitational constant [13].
In what follows, we investigate the simple case in which there are no effects of the geometrical induced matter and scalar potential on the MBDT four-dimensional universe. In this case, we expect to obtain solutions which would be similar to those obtained in the context of the conventional BD theory for a four-dimensional FLRW universe.

4.1.1. Flat space

For the flat space \((\lambda = 0)\) and non-flat spaces \((\lambda^2 = 1)\) the only manner in which to obtain the vacuum cosmology on the hypersurface is setting \(\psi = \alpha = \text{constant}\). Therefore, for the first class of solutions (power-law solutions), by substituting \(\beta = 0\) into the solutions associated with the flat space (i.e. relations \((3.6)\), \((3.7)\) and \((3.9)\)) in which \(n\) and \(\beta\) are related to each other by \((3.12)\), after calculations, we obtain

\[
 a(t) = \left[\left(-\frac{A}{2C\alpha}\right)(1 \pm 3\sqrt{1 + 2\omega/3})\right]^n, \quad \text{where } r_{\pm} = \frac{(1 + \omega) \pm \sqrt{1 + 2\omega/3}}{4 + 3\omega}, 
\]

\[(4.16)\]

\[
 \phi(t) = C [a(t)]^{\frac{1}{2}(1+\sqrt{1+2\omega/3})}, \quad \text{(4.17)}
\]

where we have assumed \(1 + 2\omega/3 \geq 0\) and \(\omega \geq -4/3\) (or \(n \geq -3\)). For this case, from \((4.3)\), without loss of generality, we can set the induced scalar potential equal to zero, and thus, by using relations \((4.1)\) and \((4.2)\), we find that the induced matter also vanishes on the hypersurface. The relations \((4.16)\) and \((4.17)\) correspond to the well known O’Hanlon–Tupper solution \([10, 40]\) which has been obtained for a four-dimensional spatially flat FRW universe in the standard BD theory where the ordinary matter is absent. Let us present the above solutions in terms of Hubble parameter \(H = \dot{a}/a\), and the age of the universe, \(t_0\). From \((4.16)\), we obtain

\[
 t_0 = \frac{r_{\pm}}{H}. \quad \text{(4.18)}
\]

---

7 For convenience, from now on, we will investigate the solutions associated with the flat and non-flat spaces in separate parts.
Therefore, relations (4.16) and (4.17) can be written as

\[ a(t) = a_0 \left( \frac{t}{t_0} \right)^{s_b} \quad \text{with} \quad a_0 = \left[ \left( -\frac{A}{2CH^0} \right)(1 \mp \sqrt{1 + 2\omega/3}) \right]^m, \quad (4.19) \]

\[ \phi(t) = \phi_0 \left( \frac{t}{t_0} \right)^{m_+} \quad \text{with} \quad \phi_0 = \left[ \left( -\frac{A}{2CH^0} \right)(1 \mp \sqrt{1 + 2\omega/3}) \right]^{m_+}, \]

\[ m_\pm = \frac{1 \pm \sqrt{3(3 + 2\omega)}}{4 + 3\omega}, \quad (4.20) \]

where, from the present values of the scale factor, Hubble constant and scalar field (i.e. \( a_0, H^0 \) and \( \phi_0 \)), we can determine the constant \( A/Co \).

Let us now discuss the behavior of the gravitational coupling, according to relation (4.20), when \( \omega > -3/2 \) and \( \phi_0 > 0 \). In\(^8\) figure 1 (left-hand panel), we have plotted the behavior of \( s_\pm \equiv -m_\pm \), the power of the cosmic time associated with the gravitational coupling. As is seen, for either the upper case (plus sign) (only when the BD coupling parameter is restricted to \(-3/2 < \omega < -4/3\) or lower case (minus sign), the behavior of the gravitational coupling is contrary to Dirac’s hypothesis, whereas for the upper case when \( \omega > -4/3 \) the gravitational coupling decreases with cosmic time, which is in accordance with Dirac’s hypothesis. Regarding the behavior of the scale factor, we see that when \( \omega \) is restricted as \(-3/2 < \omega < -4/3\), we obtain \( r_s > 1 \) (see figure 1, right-hand panel), which corresponds to an accelerating behavior of the scale factor for the upper case. The scale factor of the universe decelerates when either \( \omega > -3/2 \) (for the lower case) or \( \omega > 0 \) (for the upper case). From (4.18) for the upper case, we see that when the BD coupling parameter is larger than \(-3/2 \) and approaches \(-4/3 \) the value of the age of the universe takes large values. For the second class (exponential solution), by setting \( \beta = 0 \) in relations (3.13) and (3.14), we find the solutions associated with \( \omega = -4/3 \) for the upper case (plus sign) as

\[ a(t) = a_1 e^{-\frac{w}{3}}, \quad \phi(t) = Ca_1^{-3} e^{\frac{w}{3}} \psi(t) = \alpha = \text{constant}, \quad (4.21) \]

whereas the lower case does not give an acceptable solution.

In other words, we have

\[ a(t) = a_1 e^{Ht}, \quad \phi(t) = Ca_1^{-3} e^{-3Ht}, \quad \text{with} \quad H = -\frac{A}{3Co} = \text{constant}, \quad (4.22) \]

which indicates that the O’Hanlon–Tupper solution in the limiting case \( \omega = -4/3 \) approaches the de Sitter space, likewise the conventional BD theory; see, e.g., [2]. Similar to the first class of the flat space with \( \beta = 0 \), for this case also the components of the induced matter and the scalar potential vanish. We should note that, similar to the standard BD theory, the solution (4.22) is the unique de Sitter solution associated with the flat and vacuum space; it is different from the one obtained in GR with a minimally coupled scalar field, in which that scalar field takes a constant value. By assuming that \( \phi > 0 \), from (4.22), we see that the BD scalar field decreases with cosmic time and thus the effective gravitational coupling (4.14) increases with time; such a behavior is contrary to Dirac’s hypothesis.

4.1.2. Non-flat space. It should be noted that there are no vacuum solutions associated with FLRW space with \( \lambda = \pm 1 \) in GR in the absence of the cosmological constant [41]. However,

\(^8\) It will be of interest to discuss, for reason of convenience, the behavior of these and other quantities by the least possible number of figures. However, unfortunately, as we will see, for some cases, because of different ranges and scales used, it is not a feasible task.
such solutions have been found in the BD theory [15, 41, 42]. In what follows, we would like to investigate the corresponding solutions in MBDT.

In the case of the non-flat space where $\beta = 0$, we obtain $\psi = \alpha = \text{constant}$ and the scale factor and scalar field are given by (3.20). For this case, from relations (2.11), (3.7), (4.1) and (4.2), we find that the induced matter and scalar potential vanish on the hypersurface. The cosmology associated with this case is the same as that obtained by Dehnen and Obregón in [13]. With $t_0 = 1/H$, the age of the universe at the present time$^9$, the solutions (3.20) can be given by

$$a(t) = a_0 \left( \frac{t}{t_0} \right) \text{ and } \phi(t) = \phi_0 \left( \frac{t_0}{t} \right)^2,$$  

(4.23)

in which the present values associated with the scale factor and the BD scalar field are

$$a_0 = C \frac{1/H}{\sqrt{1 + \frac{2\omega}{3}}}, \text{ and } \phi_0 = C \frac{1 + 2\omega}{1/H^2}. \quad (4.24)$$

We should note that, by determining $t_0$, $a_0$ and $\phi_0$, the solutions associated with the closed and open universes cannot be distinguished if the BD coupling parameter is restricted to $\omega > -3/2$ and $\omega < -3/2$, respectively.

### 4.2. Dust cosmologies

Dust solutions associated with the FLRW universe in the context of the BD theory (with vanishing scalar potential) have been investigated in [9, 11]. In [44, 45], some properties of the mentioned solutions have been discussed for large values of $\omega$. Furthermore, there are a few publications in which the solutions have been obtained in terms of the conformal time and other variables different from the scale factor and the BD scalar field [2, 46–48].

#### 4.2.1. Flat space

For this case, by solving equations (4.13) (upper case) with $W_{BD} = 0$ together with the general equation (3.11) associated with the flat space, we obtain two classes of solutions as

$$n^\pm = \pm \sqrt{-3(1 + \omega)} \frac{1 + \omega}{1 + \omega}, \quad \beta^\pm = \frac{1}{2} \left( 3 \pm \sqrt{-3(1 + \omega)} \frac{1 + \omega}{1 + \omega} \right). \quad (4.25)$$

Thus, by substituting the above values for $\beta$ and $n$ into (3.6), (3.9), the first relation of (4.6) and (4.9), the solutions associated with the flat space are given by

$$a^\pm = a_0^\pm t^{r^\pm}, \quad a_0^\pm = \left[ \frac{A}{2C\alpha} (1 - \sqrt{-3(1 + \omega)}) \right]^{r^\pm}, \quad (4.26)$$

$$\phi^\pm = \phi_0^\pm t^{m^\pm}, \quad \phi_0^\pm = C \left[ \frac{A}{2C\alpha} (1 - \sqrt{-3(1 + \omega)}) \right]^{m^\pm}, \quad (4.27)$$

$$V^\pm = V_0^\pm \left[ \frac{1}{(1 + \omega)} \right] \left[ \frac{1}{(1 + \omega)} \right]^{m^\pm}, \quad (4.28)$$

$^9$ The age of the universe estimated by best fit to the Planck 2013 data [43] has been reported to be $13.813 \pm 0.058$ billion years.
where
\[ r^\pm = \frac{2[(1 + \omega) \pm \sqrt{-\frac{1 + \omega}{3}}]}{4 + 3 \omega}, \]
\[ m^\pm = n^\pm r^\pm = \frac{2[1 \pm \sqrt{3(1 + \omega)}]}{4 + 3 \omega}, \]
\[ V_0^\pm = \frac{C}{3}[3(1 + \omega) \pm \sqrt{-3(1 + \omega)}]\left(\frac{A}{C\alpha}\right)^{\frac{2[(1 + \omega) \pm \sqrt{-3(1 + \omega)}]}{4 + 3 \omega}}, \]
\[ \rho_0^\pm = \frac{-C}{4\pi(4 + 3 \omega)}\left[\frac{5 + 4 \omega \pm \sqrt{-\frac{1 + \omega}{3}}}{4 + 3 \omega}\right]\left[\frac{A}{2C\alpha}(1 \pm \sqrt{-3(1 + \omega)})\right]^{\frac{2[(1 + \omega) \pm \sqrt{-3(1 + \omega)}]}{4 + 3 \omega}}, \]
\[ \omega = -4/3, \omega < -1 \text{ and the parameters } r^\pm \text{ and } m^\pm \text{ are related to each other as } 3r^\pm + m^\pm = 2. \]

For this case, the scale factor of the fifth dimension is given by
\[ \psi^\pm = \alpha\left[\frac{-A}{2C\alpha}(1 \pm \sqrt{-3(1 + \omega)})\right]^{-1}. \]

As \( a, \rho_{BD}, \phi \) and \( \psi \) must take positive values for an arbitrary fixed time, we can find the allowed ranges for the BD coupling parameter for these solutions. We should note that, to have reasonable solutions, \( C \) and \( \alpha \) must take positive values, but \( A \) can take either positive or negative values, each of which gives two classes of solutions, one for the upper case and another for the lower case. Let us categorize the resulting solutions in terms of the sign of \( A \).

**Case I: \( A < 0 \).** For the lower case (minus sign), as the induced energy density and the scale factor do not take real values in an arbitrary fixed time, we abstain from considering them.

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**Figure 2.** The left-hand panel shows the behavior of \( a_0^+ \) in the parameter space \((\omega, a_0^+)\). In the right-hand panel, blue, red and green curves, respectively, correspond to \( \phi, \psi \) and \( \rho_{BD} \) versus \( \omega \) for an arbitrary fixed time when \( A < 0 \). The mentioned quantities are associated with the spatially flat FLRW universe, which is filled with a dust fluid. We have set \( A = -1, C = \alpha = 1 \). Note that when \( \omega \) tends to \(-4/3\), \( a_0^+ \) takes very small values while \( \rho_0^+ \) takes very large values.
as acceptable solutions. However, for the upper case (plus sign), when the BD coupling parameter was restricted to $\omega < -4/3$, we found that $a_0^+$, $\rho_0^+$, $\phi_0^+$ and $\psi_0^+$ take positive real values (see figure 2) and consequently this case can be a physical solution. In this setting, we see that for the allowed values of $\omega$, $r^+ > 1$, i.e., we obtain an accelerating universe. In addition, $m^+ < 0$, which means that the BD coupling parameter decreases with the cosmic time. Namely, the gravitational coupling increases with time, which is not in agreement with Dirac’s hypothesis. Moreover, the induced energy density and the fifth dimension decrease with time.

**Case II: $A > 0$.** For the lower case (minus sign) the quantities $a$, $\phi$, $\psi$ and $\rho_{BD}$ (in an arbitrary fixed time) take positive real values when the BD coupling parameter is restricted to $\omega < -1$, whereas for the upper case (plus sign) we find a narrow range for the BD coupling parameter which gives acceptable solutions. Although in this case $a$, $\phi$ and $\psi$ (for an arbitrary time) take positive real values when $-4/3 < \omega < -19/16$ the allowed range is $-19/16 < \omega < -1$, in which the solution is acceptable. In figure 3, for the corresponding allowed ranges, we have shown the behavior of the mentioned quantities versus $\omega$ for an arbitrary fixed time.

Let us summarize briefly the behavior of the quantities associated with this case for the corresponding allowed ranges. (i) We found that for the lower case the scale factor of the universe decelerates with cosmic time, while we have a contracting scale factor for the upper case. (ii) Both $m^+$ and $m^-$ take positive values, namely, the BD scalar field always increases with cosmic time, and consequently the gravitational coupling decreases with time, which is in agreement with Dirac’s hypothesis. (iii) For the lower case, both the
induced energy density and the scale factor of the fifth dimension decrease with time. However, for the upper case, the induced energy density increases with time while the fifth dimension contracts with time.

4.2.2. Non-flat space. For this case, by considering the lower equation of (4.13) and solving $W_{BD} = 0$, we obtain two values for $\beta$ which both depend only on the BD coupling parameter as

$$
\beta_i = \frac{(5 + 4\omega) + \chi_i}{2(1 + \omega)}, \quad i = 1, 2, \quad \chi_1 \equiv \sqrt{-(7 + 8\omega)}, \quad \chi_2 \equiv -\sqrt{-(7 + 8\omega)},
$$

where $\omega \leq -7/8$ and $\omega \neq -1$. We should note that, when $\omega$ tends to $-1$, $\beta_1$ and $\beta_2$ go to $\infty$ and 4, respectively.

By substituting $\beta_i$ from (4.35) into (3.17), we obtain

$$
X_i \equiv \left[\left(\frac{\Lambda}{C\alpha}\right)^2\right] = \frac{3\lambda[(3 - 8\omega) + 7\chi_i]}{(4 + \omega)(11 + 8\omega)}.
$$

Replacing $\beta_i$ and $X_i$ from (4.35) and (4.36) into (3.16), the lower relation of (4.6) and (4.7), we obtain four classes of mathematical solutions as

$$
a_i^\pm = a_0^\pm t, \quad \phi_i^\pm = \phi_0^\pm \frac{1 + \chi_i}{2(1 + \omega)}, \quad \psi_i^\pm = \psi_0^\pm \frac{(5 + 4\alpha + 4\chi_i)}{(1 + \omega)},
$$

$$
\rho_{\text{BD}}^\pm = \rho_0^\pm \frac{(5 + 4\chi_i + 4\alpha)}{(1 + \omega)}, \quad V_i^\pm = V_0^\pm \frac{(5 + 4\chi_i + 4\alpha)}{(1 + \omega)}.
$$

where

$$
a_0^\pm = \pm \frac{1}{\sqrt{3}} \sqrt{X_i}(1 - \chi_i), \quad \phi_0^\pm = C[a_0^\pm \frac{(5 + 4\chi_i + 4\alpha)}{(1 + \omega)}], \quad \psi_0^\pm = \alpha[a_0^\pm \frac{(5 + 4\chi_i + 4\alpha)}{(1 + \omega)}],
$$

$$
\rho_{\text{BD}}^\pm = -\frac{C}{16\pi} \left[\pm \sqrt{X_i} \right]^{\frac{1 + \chi_i}{(1 + \omega)}} \left[\frac{1 + \chi_i}{2(1 + \omega)}\right]^{\frac{(5 + 4\alpha + 4\chi_i)}{(1 + \omega)}} \times \left\{\frac{4\omega^2[16 + 8\omega + \chi_i] + 6\omega[23 + \chi_i] + (53 + \chi_i)}{(1 + \omega)^2}\right\},
$$

$$
V_{\text{BD}}^\pm = -C \left[\pm \sqrt{X_i} \right]^{\frac{1 + \chi_i}{(1 + \omega)}} \left[\frac{1 + \chi_i}{2(1 + \omega)}\right]^{\frac{(5 + 4\alpha + 4\chi_i)}{(1 + \omega)}} \left[-\frac{(11 + 20\omega + 8\omega^2 + \chi_i)}{(1 + \omega)^3}\right].
$$

In what follows, we will investigate each case separately, discussing the acceptable solutions associated with each case. However, we should point out a few general features which are common to all the solutions. The first point is that as the quantity $\chi_i$ appears in all the solutions, so, for every acceptable solution, the BD coupling parameter must be restricted to $\omega < -7/8$. The second point is that, as $\sqrt{X_i}$ also appear in relations of $a_0$, to obtain acceptable solutions the right-hand side of (4.36) must take positive real values. As $X_i$ depends on (further $\omega$) the curvature index, we have to discuss its behavior by considering also the sign of $\lambda$.

First solution ($i = 1$). In this case, for the closed universe we find that $X_1$ takes positive values when $\omega > -11/8$, while for the open universe it takes positive values when $\omega < -11/8$ (see figure 4, left-hand panel). Therefore, by considering the allowed range from

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10 Indeed, we have two kinds of solution specified with $i = 1, 2$ (the first solution with $i = 1$ and the second solution with $i = 2$), which, in turn, have two cases specified as an upper case (plus sign) and a lower case (minus sign).
we find that the acceptable range for the closed universe will be $-\frac{11}{8} < \omega < -\frac{7}{8}$ while that for the open universe will be $\omega < -\frac{11}{8}$.

Second solution $(i = 2)$. For this case, $X_2$ takes positive values when $\omega < -\frac{7}{8}$. Moreover, $\chi_2$ also takes real values for this range, thus this range can be acceptable for the closed universe. However, for the open universe $X_2$ takes negative values (see the right-hand panel of figure 4), thus there is no acceptable solution for this case.

Within the above discussion, we identified allowable ranges for $\omega$. Now, we can apply these to find solutions associated with the open and closed universes, separately.

Open universe.

First solution. Up to now, in order to have allowable values for $\chi_1$ and $X_1$, we found that the BD coupling parameter must be restricted to $\omega < -\frac{11}{8}$. However, in this range $\alpha_{10}^+ \quad \text{and} \quad \rho_{10}^+$ must take also positive values. The numerical results show that, in this range, $\alpha_{10}^- < 0$, while $\alpha_{10}^+ > 0$; see figure 5, left-hand panel. Namely, the upper solution (plus sign) is not acceptable, while the lower solution (minus sign) can be an acceptable solution provided that $\rho_{10}^- > 0$. However, just by looking at the relation of $\rho_{10}^-$ in (4.38), we find that $\rho_{10}^-$ for this range never takes only real values and it has also an imaginary
part. In figure 5 (right-hand panel), we have plotted this imaginary part. Thus, for this case, there is no physically acceptable solution.

Second solution. As mentioned before, \( X_2 < 0 \); there is no acceptable solution for this case. Therefore, the reduced cosmology arising from the MBDT field equations, for the FLRW open universe which is filled with dust fluid, just gives a few mathematical solutions, which cannot be physically acceptable solutions.

– Closed universe.

First solution. As discussed above, for this case, the allowed range is \(-11/8 < \omega < -7/8\). Now, we should examine this range to see whether it can generate positive real values for \( \rho_{10}^+ \) and \( a_{10}^+ \) or not. For the upper case (plus sign), we see that the range \(-1 < \omega < -7/8\) gives \( a_{10}^+ > 0 \). We can see that, for this range, \( \rho_{10}^+ > 0 \). In figure 6 (left-hand panel), we have plotted the behavior of \( a_{10}^+ \) and \( \rho_{10}^+ \) versus \( \omega \). Our numerical results show that, in this allowable range, \( a_{10}^+ \) always takes very small real values which can be physically acceptable. From (4.38), as \( C > 0 \) and \( \alpha > 0 \), thus, \( \psi_{10}^+ > 0 \) and \( \phi_{10}^+ > 0 \). Finally, we conclude that the first solution (upper case) in the range \(-1 < \omega < -7/8\) is an acceptable solution for the closed universe. However, for the lower case, although in the range \(-11/8 < \omega < -7/8\), we obtain \( a_{10}^+ > 0 \); but in this range, \( \rho_{10}^+ \) does not take real values, thus we cannot accept this case as a physical solution. Let us investigate the behavior of the quantities for the upper case in terms of cosmic time. From relations (3.16), the lower relation of (4.6) and (4.7), we see that \( a(t) \) increases versus time, linearly; the behavior of \( \beta_1 - 4, \beta_1 - 2 \) and \( -\beta_1 \) versus the allowable \( \omega \) determines the behavior of the quantities associated with this case. For \(-1 < \omega < -7/8\), we find that \( \beta_1 - 2 \) and \( \beta_1 - 4 \) take positive real values, while \(-\beta_1 \) always takes negative values. Namely, \( \phi_{10}^+ (t) \) and \( \rho_{10}^+ \) increase versus time, while \( \psi_{10}^+ \) decreases versus time. As the fifth dimension contracts with the cosmic time, this is
favorable. Moreover, as the BD coupling parameter increases versus time, thus the gravitational coupling decreases with time, which is in agreement with Dirac’s hypothesis.

Second solution. For this case, by considering the allowable values for $\chi_2$ and $\sqrt{X_2}$, we obtained the allowable range $\omega < -7/8$. Let us probe this range and investigate if positive real values for the scale factor and the induced energy density in an arbitrary fixed time can be obtained. From (4.38), for $\omega < -7/8$, we find that $a_{20}^+ > 0$ while $a_{20}^- < 0$; see the right-hand panel of figure 6. Namely, the lower case does not give an acceptable solution. Now, we should determine the behavior of the induced energy density for the upper case. By means of numerical methods, it is straightforward to show that $\rho_{20}^+$ takes positive real values in the allowable range; see the right-hand panel of figure 6. As we have shown that the second solution (upper case) is an acceptable solution, determining the behavior of the quantities versus time will be a worthwhile task. Similarly to the previous case, $a(t)$ increases linearly with time, and the behavior of $\beta_2 - 2$, $\beta_2 - 4$ and $-\beta_2$ versus $\omega$ determines the behavior of the other quantities versus time. We can easily show that for $\omega < -1$ $\beta_2 - 4$ takes negative values, while for $-1 < \omega < -7/8$ it takes positive values. Namely, for the former range the induced energy density decreases with time, whereas for the latter range it increases with time. However, $\beta_2 - 4$ for all the values which are taken from $\omega < -7/8$ ($\omega \approx -1$) produces positive real values, which leads to an increasing BD scalar field with cosmic time. Thus, the gravitational coupling decreases with time, which is again in agreement with Dirac’s hypothesis. Finally, as $\beta_2$ always takes positive values, the fifth dimension contracts with cosmic time.

4.3. Radiation cosmologies

The radiation-dominated universe in the context of the standard BD theory has been investigated in [17, 18, 49–51]. In what follows, we will find the specific solutions associated with the radiation-dominated universe in MBDT and discuss the properties of corresponding quantities and their differences from those obtained with standard BD theory.

4.3.1. Flat space. By setting $W_{BD} = 1/3$ in the upper relation of (4.13) and solving this equation together with the one obtained for the flat space in five dimensions, i.e. equation (3.11), we obtain $\beta = \frac{3 + 2\omega}{2(1 + \omega)}$, $n = \frac{1}{1 + \omega}$, (4.39)

By substituting the above values of $n$ and $\beta$ into relations (3.6), (3.7) and (3.9), as well as the upper relation of (4.6) and (4.10), we obtain the following solutions:

$$a(t) = a_0 t^{\frac{4 (3 + \omega)}{8 + \omega}}, \phi(t) = \phi_0 t^{\frac{3 + \omega}{3 + \omega}}, V = V_0 t^{\frac{3 + \omega}{3 + \omega}}, \rho_{BD} = \rho_0 t^{-\frac{6(1+\omega)}{8+\omega}}.$$ (4.40)

11 We should note that when $\omega = -1$, $\beta_2$ goes to infinity, thus we exclude the value $\omega = -1$ from this solution.

12 The other solution is $n = 0$, $\beta = 1$ in which the BD scalar field takes a constant value. To find the exact solution for this case, we must start from equations (3.2)–(3.5). Solving them by assuming a power law for the scale factor gives $a(t) = a_0 t^{1/2}$ and $\psi(t) = \psi_0 t^{1}$ (where $a_0$ and $\psi_0$ are constants), which is the unique solution of the Einstein field equations in vacuum for the spatially flat FLRW metric in a five-dimensional space-time.
where the constants $a_0$, $f_0$, $V_0$ and $\rho_0$ are given by

\[
a_0 = \left[ A(5 + 4\omega) \frac{2^{\frac{2\omega}{2\omega + 1}}}{2C\alpha} \right],
\]

\[
f_0 = C\left[ A(5 + 4\omega) \frac{2^{\frac{2\omega}{2\omega + 1}}}{2C\alpha} \right],
\]

\[
V_0 = \frac{C(3 + 2\omega)}{4} \left( \frac{A}{2C\alpha} \right)^{\frac{2\omega}{2\omega + 1}} (5 + 4\omega)^{\frac{2\omega}{2\omega + 1}},
\]

\[
\rho_0 = \frac{3A(3 + 2\omega)}{32\pi\alpha} \left[ \frac{A(5 + 4\omega)}{2C\alpha} \right]^{\frac{2\omega}{2\omega + 1}},
\]

and the constants $A$, $C$ and $\alpha$ can be determined by knowing the age of the universe and its energy density at the present time. The scale factor of the fifth dimension is given by

\[
\psi'(t) = \alpha \left[ \frac{A(5 + 4\omega)t}{2C\alpha} \right]^{\frac{2\omega}{2\omega + 1}}.
\]

Let us first find the allowable ranges of $\omega$ for this case. As $a_0$, $f_0$, $V_0$ and $\rho_0$ should take positive real values, from relations (4.41) and (4.44), we find the allowed ranges of the BD.
coupling parameter, which depends on the sign of $A$. Therefore, in what follows, we would investigate two separate solutions.

**Case I:** $A < 0$. In this case, when the BD coupling parameter is restricted to $\omega < -5/4$ the quantities $a_0$, $\phi_0$ and $\psi_0$ take positive real values, whereas $\rho_0$ takes positive real values when $\omega < -3/2$. Therefore, the allowed range for this case is $\omega < -3/2$; see the upper panels of figure 7. For this allowed range, it is straightforward to show that $0 < 2(1 + \omega)/(5 + 4\omega) < 1$. More precisely, the scale factor of the universe decelerates with cosmic time. Furthermore, the induced energy density, BD scalar field and fifth dimension decrease with time.

**Case II:** $A > 0$. In this case, to obtain positive real values for the corresponding quantities (in an arbitrary fixed time), $\omega$ must be restricted to $\omega > -5/4$ (see the lower panels of figure 7), which gives reasonable results for the radiation-dominated universe as follows. It is easy to show the following. (i) For $-5/4 < \omega < -1$ we obtain $2(1 + \omega)/(5 + 4\omega) < 0$, while for $\omega > -1$ we obtain $0 < 2(1 + \omega)/(5 + 4\omega) < 1$. The former range (of the power of the cosmic time associated with the scale factor) leads us to a spatially flat universe whose scale factor contracts with the cosmic time, whereas the latter leads to a decelerating universe. (ii) The energy density increases with the cosmic time for $-5/4 < \omega < -1$, whereas it decreases with time when $\omega > -1$. (iii) The BD scalar field increases with the cosmic time; in this case, the gravitational coupling decreases with time, which is in agreement with Dirac’s hypothesis. (iv) $\psi(t)$ also decreases with cosmic time for all allowed values of $\omega$. The above properties imply that the MBDT scenario, by assuming the mentioned constraint on the BD coupling parameter, can be a relevant model for obtaining a radiation-dominated universe when $\omega > -1$, in which the induced matter can play properly the role of ordinary matter in the universe.

### 4.3.2. Non-flat space

For this case, by setting $W_{\text{BD}} = 1/3$ in the lower relation of (4.13), it is easy to show that

$$\beta = \frac{4(3 + 2\omega)}{5 + 4\omega}, \quad (4.46)$$

where $\omega = -5/4$.

By substituting $\beta$ from (4.46) into (3.17), we get

$$\left(\frac{A}{C_0}\right)^2 = \frac{4\lambda}{(3 + 2\omega)(11 + 8\omega)}, \quad (4.47)$$

where $\omega = -3/2$, $-11/8$. By employing relations (4.46) and (4.47) for the general solutions associated with the non-flat space, namely (3.16), we obtain

$$a^\pm = a_0^\pm t, \quad \phi^\pm = \phi_0^\pm e^{\pm \omega t}, \quad \psi^\pm = \psi_0^\pm e^{\pm \omega t}, \quad \rho^\pm = \rho_0^\pm e^{\pm \omega t}, \quad V^\pm = V_0^\pm t^{\frac{3(3 + 2\omega)}{3 + 2\omega}}, \quad (4.48)$$

where

$$a_0^\pm = \pm (5 + 4\omega) \sqrt[3]{\frac{\lambda}{(3 + 2\omega)(11 + 8\omega)}}. \quad (4.49)$$

---

13 The other solution for this case is $\beta = 2$, which, according to (3.16), yields a constant value for the BD scalar field and infinite value for the scale factor. As discussed in section 3, in this case the BD theory may reduce to GR and thus such a result is reasonable.
In what follows, we discuss the behavior of the quantities for closed and open universes separately.

– Open universe

For this case, by substituting $\lambda = -1$ into (4.49), we find that, in order to obtain real values for the square root, the BD coupling parameter must be restricted to $-3/2 < \omega < -11/8$. On the other hand, $a_0$ must take positive values, thus for the upper case (plus sign) we get $\omega > -5/4$, which does not overlap with the previous allowable range of $\omega$. Therefore, there is no consistent physical solution for the upper case. However, for the lower case (minus sign), $\omega$ must be restricted to $\omega < -5/4$, which has an overlap with the previously allowed range. Namely, in this case, the range $-3/2 < \omega < -11/8$ not only gives real values for $\Lambda/C\alpha$ but also produces positive values for $a_0$. Nonetheless, this is not enough to have an acceptable physical solution. More concretely, other quantities such as $\rho_0^\pm$, $\phi_0^\pm$, and $\psi_0^\pm$ take real positive values in the mentioned range of $\omega$. However, just by investigating $\rho_0^\pm$, we find that it takes imaginary values in the mentioned range. Finally, we can conclude that the set of solutions (4.48) for an open universe are only valid as mathematical solutions, namely, for the FRLW universe with $\lambda = -1$ (when $\phi$ and $\psi$ are related to the scale factor with a power-law equation), the MBDT scenario does not yield any physical solution.

– Closed universe

By substituting $\lambda = 1$ into relations (4.48)–(4.52) we obtain two classes of solutions for this case. Let us describe the corresponding quantities for this case by finding the allowable range for the BD coupling parameter. From (4.49), we see that to have real values for the square root $\omega$ must be restricted to either $\omega < -3/2$ or $\omega > -11/8$. By
considering these allowable ranges of $\omega$, in what follows, we investigate the solutions associated with the upper case (plus sign) and lower case (minus sign), separately.

**Upper sign.** In this case, from (4.49), we find that $a_0^+$ takes positive values if the BD coupling parameter is restricted to $\omega > -5/4$. For this allowed range of $\omega$, as $C > 0$ and $\alpha > , \rho_{BD}, \phi$ and $\psi$ take positive real values for an arbitrary fixed time; see figure 8. Consequently, the acceptable physical solutions associated with a closed universe with the upper sign (which is filled with radiation) are given by (4.48), in which $\omega$ must be larger than $-5/4$. It is also noteworthy to describe the time behavior of quantities associated with this case. From (4.48), it is straightforward to show that for $\omega > -5/4$, (i) as $\phi_0^+ > 0$ and $2/(5 + 4\omega) > 0$, the BD scalar field always increases with the cosmic time, hence the gravitational coupling decreases with cosmic time, which is in agreement with Dirac’s hypothesis; (ii) as $\rho_0^+ > 0$, the induced energy density increases with time when $-5/4 < \omega < -1$, while it decreases with time when $\omega > -1$; (iii) $\psi^+(t)$ always decreases with the cosmic time, which is a relevant outcome.

**Lower sign.** For this case, $a_0^-$ takes positive values when the BD coupling parameter is restricted to either $-11/8 < \omega < -5/4$ or $\omega < -3/2$. However, similarly to the lower case for the open universe, we cannot obtain real values for the energy density in arbitrary fixed time when $-11/8 < \omega < -5/4$. For the other range, i.e. $\omega < -3/2$, from (4.50) we see that (by assuming $C > 0$ and $\alpha > 0$) $\psi^-$ and $\phi$ always take positive real values, whereas from (4.52) we find that $\rho_0^-$ does not take real values. Thus, this case is not a physically acceptable solution.

5. Summary and discussion

The MBDT [1] has four sets of field equations. One set corresponds to the generalized version of the conservation law introduced in IMT [23, 24]. Another set is a nonlinear differential equation associated with the scale factor of the extra dimension, which has no analog in the standard BD theory. Finally, the other two sets can be related to the conventional BD action, but with specific scalar potential, in four dimensions in which the matter and the scalar potential have an intrinsic geometrical origin. This induced matter is composed of three parts; see equation (2.8). The first part is a function of the first and second derivatives of the metric components with respect to the extra dimension. The second and third parts are functions of the BD scalar field, metric components and their derivatives with respect to the extra dimension.

Achieving unification of matter and geometry has been claimed as the main motivation for introducing a large extra dimension in IMT [52]. The MBDT can be considered as an extended version of the IMT, hence the motivation that has been followed to also consider a large extra dimension to construct the MBDT scenario. However, in MBDT, further the geometrically induced matter, there is also an induced scalar potential which has also been of interest to discuss in MBDT. This scalar potential has been employed either to yield an accelerating universe [1, 27, 28] (by assuming a spatially flat FLRW universe) or to obtain more general solutions for a Bianchi type I model [29]. The main objective of our work herein was to employ the MBDT scenario to obtain new extended solutions associated with both the spatially flat FLRW universe (by assuming more generalized power-law solutions than ones assumed in the previous works) and the non-flat space with respect the corresponding solutions in the context of the conventional BD theory.
In this paper, we started from the geometry of the five-dimensional bulk and then constructed the physics on the projected four-dimensional hypersurface. More precisely, by considering a five-dimensional FLRW universe (without ordinary matter) with all values of the curvature index, we have derived the equations of the standard BD theory whose scalar field depends only on the cosmic time. Then, by assuming Dirac’s hypothesis, which claims that the gravitational constant should be connected to the scale factor of the universe in a power-law relation, we have solved the equations of motion. Our results show that there is a general solution for the non-flat space (see equations (3.1)) and two kinds of solution for the flat space, the general power-law solution and exponential solution. We have also discussed some particular cases of these solutions when either $\omega$ or $\beta$ takes special values.

Subsequently, we have employed the MBDT set-up (reviewed in section 2) to construct the physics on a four-dimensional space-time. First, we found the relations associated with the induced matter and induced scalar potential in general cases. Our results have shown that the induced scalar potential is in either logarithmic or power-law form. As the former leads to an inconsistency to apply the energy conditions, we abstained from proceeding to study it. However, the latter yielded the equations of state for the barotropic matter for all values of the curvature index. Such induced matter, in which all the terms emerge from the geometry of the fifth dimension, has interesting properties. Namely, it obeys the conservation law similarly to the ordinary matter in the standard BD theory. Thus, it respects the principle of equivalence. Besides, it is worthwhile to emphasize that the induced scalar potential, which contributes to construct the induced matter and consequently the behavior of the BD scalar field, has been induced from the geometry (as a fundamental concept) rather than adding it by hand to the action.

As there is a non-zero scalar potential, the relations associated with the scale factor, the BD scalar field and the components of the induced matter, which depend effectively on $\beta$ and/or $\omega$, are more generalized than those of the conventional BD theory.

We proceeded to study the properties of the induced matter on the hypersurface as consequence of the effects of the geometry of the fifth dimension and contrast them with the ones reported for ordinary matter in the BD theory. For this, we have concentrated on known types of the which, in what follows, we summarize and compare with those obtained from the conventional BD theory, IMT and GR.

1. Vacuum solutions. For both the flat and non-flat spaces, the only manner in which to obtain vacuum solutions on a four-dimensional hypersurface is setting $\beta = 0$, i.e. assuming $\psi = \text{constant}$. In this case, we found that the induced scalar potential and the components of the induced matter vanish. We have shown that the solutions associated with the model herein are similar to those obtained in the standard BD theory. Namely, we obtained exactly the same solutions as obtained by O’Hanlon and Tupper [10] and Dehnen and Obregón [13] in the context of the standard BD theory in vacuum. By means of numerical analysis, we further investigated the time behavior of the scale factor, BD scalar field and gravitational constant for each solution. We should note that the calculations associated with the vacuum case can be an appropriate procedure to test the correctness of the results produced by the MBDT scenario. For all the solutions associated with the vacuum case, we see that the power-law assumption between the scale factor of the fifth dimension and the BD scalar field, as well as the scale factor of the universe, is in contradiction with Dirac’s hypothesis. The other general consequences of these solutions have shown that, similarly to the conventional BD theory, the Birkhoff theorem and the Mach principle, which implies that the matter in the universe determines the value of the gravitational coupling, are not valid in the context of the MBDT.
2. Dust solutions. For a flat space, we found two classes of mathematical solutions (the upper and lower cases, see equations (4.26)-(4.29)), which depend on the sign of integration constant $A$. We have discussed the effects of the sign of $A$ on the solutions associated with the upper and lower cases. By assuming $A < 0$, we found that only one of the solutions (the upper case) is physically acceptable when $\omega < -4/3$. For this case, we have shown that the scale factor of the universe accelerates, while the BD scalar field and induced energy density decrease, with cosmic time. Let us compare the result with the observational data: from relations (4.30), it is straightforward to show that, when $-2 < \omega < -3/2$, then $r^+$ can be restricted as $1.56 \lesssim r^+ \lesssim 2.94$ or, equivalently, the deceleration parameter (at present time) $q_0^+ = (1 - r^+)/r^+$ is restricted to $-0.66 \lesssim q_0^+ \lesssim -0.36$, which is in agreement with the recent observational measurements [53].

For $A > 0$, we found that both of the solutions can be physically acceptable. We have also discussed the properties of the quantities for this case. We found that, for both the upper and lower solutions, the gravitational constant decreases with cosmic time, which is in agreement with Dirac’s hypothesis. It is noteworthy that, for all of the mentioned dust solutions associated with the flat space, the fifth dimension decreases with cosmic time, which can be an interesting result in the context of higher-dimensional models.

Let us summarize and further discuss solutions (4.25)-(4.34) for large values of $\omega$. For these cases (upper case with $A < 0$ and lower case with $A > 0$), without loss of generality, we can set $A = Cn\alpha$. As $C$ and $\alpha$ always take positive values and $n$ takes negative and positive values for the former and latter cases, respectively, $A$, takes negative and positive values for the former and latter, respectively. From this, it is easy to show that when $\omega \rightarrow -\infty$, we obtain $a(t) = (3t/2)^{4/3}, \phi = C, \psi = (2\alpha/3)r^{-1}$ and $\rho = (C/(3\pi))r^{-2}$. Namely, the results are in agreement with those obtained from IMT and thus with GR.

For non-flat space, $\lambda^2 = 1$, we obtained four different classes of mathematical solutions; see the set of relations (4.35)-(4.38). By further probing of the properties of the solutions associated with the open universe, we discovered that there is no physically acceptable solution for it. This result is in accordance with the one obtained for the standard BD theory [11].

However, for the closed universe, among four classes of mathematical solutions, we have shown that only two of them can be physically acceptable. In one class of these solutions (first solution, upper case) the BD coupling parameter must be restricted to $-1 < \omega < -7/8$, whereas for the other case we found that the allowed range is $\omega < -7/8$ (second solution, upper case). In comparison with the corresponding solution in the standard BD theory (with vanishing scalar potential) [11], we find that there is no analog for the former case. However, in the latter case, the allowed range for $\omega$ in our model is replaced with $\omega < -2$ [11], in which probing the solutions associated with the large values of $\omega$ needs careful calculations [44, 45]. We can further prove the results of our second solution (upper case) when $\omega \rightarrow -\infty$. In this case, we obtain $\phi = C = constant$, $a(t) = (\sqrt{3}/2)t, \rho(t) = C/(\pi t^2)$ and $\psi(t) = 2\alpha/(3t^2)$, which may coincide with the results obtained for the corresponding solutions in IMT [24]. For both the physically acceptable solutions, we have found that, for the corresponding allowable ranges of $\omega$, the fifth dimension and the gravitational coupling decrease with the cosmic time and thus these solutions can be of interest.

3. Radiation solutions.

For the spatially flat FLRW universe, depending on the sign of $A$, we obtained two
different classes of acceptable solutions. The first class of the solutions corresponds to
$A < 0$, in which the allowed range is $\omega < -3/2$. For this case, the scale factor of
the universe decelerates with cosmic time, and the BD scalar field, the induced energy
density and the fifth dimension decrease with time. The second class, which corresponds
to $A > 0$, implies a different range of the BD coupling parameter, i.e. $\omega > -5/4$. In this
case, we found that the mentioned range gives physically acceptable solutions, but in the
range $\omega > -1$ we obtained a decelerating universe; the induced energy density and the
fifth dimension decrease with cosmic time; and the BD coupling parameter increases with
cosmic time, which is in agreement with Dirac’s hypothesis. When $[\omega]$ takes large values,
without loss of generality, by assuming $A = \pm C n_\alpha$ (where $C > 0$ and $\alpha > 0$) in
relations (4.39)–(4.45), when $\omega \rightarrow \pm \infty$, it is straightforward to show that for both of the
solutions we obtain $\phi = C = \text{constant}$, $\psi(t) = (\sqrt{2} \alpha/2)t^{-1/2}$, $a(t) = (2t)^{1/2}$ and
$\rho = (3C/32\pi)t^{-3}$, which are in agreement with the solutions associated with the FLRW
universe (which is filled with radiation) in IMT and thus in GR.

For the non-flat space, we obtained two classes of mathematical solutions for each the
open and closed universes. For an open universe, both classes give neither positive nor
real values for the induced energy density, thus they cannot be physically acceptable
solutions. For a closed universe, only one of the solutions (the upper case) is acceptable
when $\omega > -1$. For this class of the solutions, we have presented the properties of the
corresponding quantities. For instance, we have shown that the gravitational constant,
the induced energy density and $\psi(t)$ decrease with the cosmic time, whereas the BD scalar
field increases with time, which is in agreement with Dirac’s hypothesis. All of the
properties for this class of solutions show that it is a physically allowed cosmological
model for a radiation-dominated universe. When $\omega \rightarrow \infty$, from relations (4.46)–(4.52),
we obtain $\phi = C = \text{constant}$, $a(t) = t$, $\psi(t) = \alpha t^{-2}$ and $\rho(t) = (3C/4\pi)t^{-3}$, which are in
agreement with the corresponding ones obtained in IMT.

Finally, it is worthwhile to mention a few points regarding the MBDT setting [1] and the
obtained solutions of our work herein.

– Perhaps it would be a good idea to presume the MBDT as a powerful and a fundamental
setting to obtain the exact solutions which correspond to those of the standard BD theory
as well as a few particular types of the scalar–tensor theories. It is clear that such a task
can be achieved only by considering the two sets of the MBDT field equations which
correspond to those derived from the standard BD action including a scalar potential. As
mentioned, there are two other sets which have no analog in the standard BD theory.
How can we interpret these field equations? Under what conditions do these
interpretations relate to those obtained in IMT?

– We should emphasize that the solutions of this manuscript are more extended than the
corresponding ones obtained in the standard BD theory as well as those obtained in [1].
Moreover, a few classes of our solutions have no analog to those obtained by means of the
corresponding conventional setting. We should note that the solutions herein can still
be generalized by assuming ordinary matter in the bulk and/or supposing that the BD
scalar field and the components of the metric are also functions of (further the cosmic
time) the extra coordinate $l$.

– In order to obtain the solutions herein, we have not introduced conformal time or other
variables different from the scale factor and the BD scalar field; see, e.g., [2, 46–48].

– As our MBDT setting [1] has been formulated in arbitrary dimensions, it can be
employed not only for obtaining the reduced solutions (for a specified model) in $(3 + 1)$
dimensions but also it can be examined for deriving the solutions in $(2 + 1)$ dimensions.
This manner of studying lower-dimensional gravity theories (see, e.g., [54] and references therein) can be of interest in probing a relationship between such theories and the standard four-dimensional standard BD theory, as well as introducing a procedure of producing exact solutions in (2 + 1) dimensions that are, as might be expected, related to the vacuum (3 + 1)-dimensional solutions. Subsequently, we can also investigate what happen for the behaviors of the physical quantities in the particular cases, especially when the BD coupling parameter goes to infinity.

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