QUANTUM BLACK HOLES AS ATOMS

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In some respects the black hole plays the same role in gravitation that the atom played in the nascent quantum mechanics. This analogy suggests that black hole mass $M$ might have a discrete spectrum. I review the physical arguments for the expectation that black hole horizon area eigenvalues are uniformly spaced, or equivalently, that the spacing between stationary black hole mass levels behaves like $1/M$. This sort of spectrum has also emerged in a variety of formal approaches to black hole quantization by a number of workers (with some notable exceptions). If true, this result indicates a distortion of the semiclassical Hawking spectrum which could be observable even for macroscopic black holes. Black hole entropy suggests that the mentioned mass levels should be degenerate to the tune of an exponential in $M^2$, as first noted by Mukhanov. This has implications for the statistics of the radiation. I also discuss open questions: whether radiative decay will spread the levels beyond recognition, whether extremal black holes can be described by this scheme, etc. I then describe an elementary algebra for the relevant black hole observables, an outcome of work by Mukhanov and myself, which reproduces the uniformly spaced area spectrum.

1 Introduction

We have all been taught that at the Planck scale $M_P = \hbar^{1/2} \approx 1.2 \times 10^{20} \text{MeV}$ or $\ell_P = \hbar^{1/2} \approx 1.6 \times 10^{-33} \text{cm}$ (I use units with Newton’s $G = c = 1$ and denote the charge of the electron by $-e$), quantum gravity effects must become important. But this scale is so extreme by laboratory standards that it would seem one shall never be able to put quantum gravity to the test in the laboratory. One may, however, ask: is it possible that by some recondite effect quantum gravity may make itself felt well below the Planck energy (well above the Planck length)?

Research in black hole physics has uncovered several mysteries: Why is the statistical black hole entropy proportional to the horizon area? What happens to the information in black hole evaporation? How to resolve those mysteries in a simple way?

Here I would like to discuss a scheme for eliciting some answers to these questions. This is a partial and, as yet, informal scheme, not a quantum theory of gravity. It derives from an early attempt of mine to quantize black holes, and a reformulation of it by Mukhanov. In classical general relativity the mass spectrum of black holes is a continuum. The scheme suggests that in quantum theory the black hole mass spectrum must be discrete and highly degenerate in the sense that the black hole horizon area is restricted to equispaced levels whose degeneracy corresponds, by the usual Boltzmann–Einstein formula, to the black hole entropy associated with each area eigenvalue. The scheme makes use of rather common ideas from quantum mechanics and field theory, while keeping the classical limit of things in sight. I shall begin with a physical discussion, and become a bit formal in the sequel.

In setting out to give a quantum description of black holes, a primary question (first asked by Wheeler in the late 1960’s) is what is the complete set of quantum
numbers required to describe a black hole in a stationary quantum state. Quantum numbers are first and foremost attributes of elementary particles. Now an elementary object with mass below $M_P$ has its gravitational radius tucked below its Compton wavelength; it is thus properly termed "elementary particle". By contrast an elementary object with mass above $M_P$ has its Compton wavelength submerged under the gravitational radius; it is best called a black hole. The seeming discontinuity between the two occasioned by the emergence of the horizon is not really there because at the Planck scale the spacetime geometry should be quite fuzzy. So there is no in–between regime here, and by continuity the smallest black holes should be quite like elementary particles, and should merit description by a few quantum numbers like mass, charge, spin, etc.

How do things change as the black hole gets larger? For macroscopic black holes within general relativity, Wheeler enunciated long ago his highly influential "no hair" conjecture: a black hole is parametrized only by quantities, such as mass, spin angular momentum and charge, which are subject to Gauss–type laws. One may add magnetic charge to Wheeler's list because it is subject to a Gauss law, and there are generalization of the Kerr–Newman solution with magnetic charge alongside electric charge. I have already argued that the many "hairy" black hole solutions discovered in the last decade do not provide us with additional quantum numbers for black holes. Hence I assume here that the only quantum numbers of a stationary black hole state are mass, electric charge, magnetic monopole and spin.

2 Mass Spectrum of a Black Hole

I shall thus focus on black hole eigenstates of mass $\hat{M}$, electric charge $\hat{Q}$, magnetic monopole $\hat{G}$, spin angular momentum $\hat{J}^2$ and $\hat{J}_z$ and, of course, linear momentum $\hat{P}$. This last can be set to zero if one agrees to work in the black hole’s center of mass. The eigenvalue spectra of $\hat{Q}, \hat{G}, \hat{J}^2, \hat{J}_z$ are well known. By making the standard assumption that this last set of operators are mutually commuting, one may immediately establish the mass spectrum for the extremal black holes.

The classical extremal Kerr–Newman black hole is defined by the constraint

$$M^2 = Q^2 + G^2 + J^2/M^2$$

Solving for $M$, discarding the negative root solution (it gives imaginary $M$), and replacing in this expression $Q \to qe, G \to gh/2e$ and $J^2 \to j(j+1)h^2$ with $q, g$ integers and $j$ a nonnegative integer or half–integer, one enforces the quantization of charge, magnetic monopole and angular momentum, and obtains the mass eigenvalues first found by Mazur

$$M_{qgj} = M_P \left[ q^2 e^2/2 + g^2 h^2/8e^2 + \sqrt{(q^2 e^2/2 + g^2 h^2/8e^2)^2 + j(j+1)} \right]^{1/2}. \quad (2)$$

For nonextremal black holes I shall avail myself of the classical relation between mass and area of the Kerr–Newman black hole

$$M^2 = \frac{A}{16\pi} \left( 1 + \frac{4\pi(Q^2 + G^2)}{A} \right)^2 + \frac{4\pi J^2}{A} \quad (3)$$
One should note that only the parameter domain
\[ A \geq \sqrt{(Q^2 + G^2)^2 + (8\pi)^2J^2} \]  
(4)
is physical because only there does the usual expression for \( A \) as a function of \( M, Q, G \) and \( J \) hold.

In converting Eq. 3 to a quantum relation between the operators \( \hat{M}, \hat{Q}, \hat{G}, \hat{J} \) one faces the problem of factor ordering. Now the area of a black hole should be invariant under rotations of its spin; since \( \hat{J} \) is the generator of such rotations, one sees that \( [\hat{A}, \hat{J}] = 0 \). Similarly, area should remain invariant under gauge transformation whose generator is, as usual, the charge \( \hat{Q} \). Hence \( [\hat{A}, \hat{Q}] = 0 \). Duality invariance of the Einstein–Maxwell equations would then suggest that \( [\hat{A}, \hat{G}] = 0 \). Hence one may merely replace the parameters in Eq. 3 by the corresponding operators:

\[ \hat{M}^2 = \left[ \frac{\hat{A}}{16\pi} \left( 1 + \frac{4\pi(\hat{Q}^2 + \hat{G}^2)}{\hat{A}} \right)^2 + \frac{4\pi \hat{J}^2}{\hat{A}} \right] \Theta \left( \hat{A} - (\hat{Q}^2 + \hat{G}^2)^2 + (8\pi)^2\hat{J}^2 \right) \]  
(5)

This formula allows one to read off eigenvalues of \( \hat{M} \) from those of \( \hat{A} \), the charges and the angular momentum; it was first used in this sense long ago. The Heavyside \( \Theta \) (step) function enforces the physical restriction Eq. 4; when this last is violated, a zero mass eigenvalue is predicted, which means there is no such black hole.

3 Horizon Area as an Adiabatic Invariant

Later in this presentation I derive the eigenvalues of \( \hat{A} \) from a quantum operator algebra. However, much insight can be gleaned by using a simple physical approach inspired by the similarity of horizon area to an adiabatic invariant in mechanics.

What is an adiabatic invariant?

A physical system governed by a hamiltonian \( H(q, p, \lambda) \) which depends on a time dependent parameter \( \lambda(t) \) is said to undergo an adiabatic change if \( \lambda \) varies on a timescale long compared to the longest timescale \( T \) of the internal (oscillatory) motions : \( \lambda^{-1}d\lambda/dt \ll T^{-1} \). Any dynamical quantity \( A(q, p) \), a function or functional of the \( p \)'s and \( q \)'s, which changes little during the time \( T \) while \( H \) accumulates a significant total change, is said to be an adiabatic invariant. Ehrenfest showed that for a quasiperiodic system, all Jacobi action integrals of the form \( A = \oint p dq \) are adiabatic invariants. In particular, for an harmonic oscillator with slowly time–varying frequency \( \omega(t) \) (say a pendulum that oscillates with small amplitude while the string is lengthened slowly), the Jacobi integral equals \( 2\pi E/\omega \). Thus when the spring constant varies on a timescale \( \gg \omega^{-1} \), \( E/\omega \) remains constant even while \( E \) changes sizeably.

The subject is interesting here because one can understand the adiabatic invariance of \( E/\omega \) in quantum terms. For an harmonic oscillator in a stationary state labeled by quantum number \( n \), \( E/\omega = (n + \frac{1}{2})\hbar \) One expects \( n \) to remain constant during an adiabatic change because the perturbations imposed on the system have frequencies \( \ll \omega \), so that transitions between states of different \( n \) are strongly suppressed. Therefore, the ratio \( E/\omega \) is preserved. In the Bohr–Sommerfeld theory (old
quantum mechanics), all Jacobi actions are quantized in integers: $$\oint p\,dq = 2\pi n\hbar.$$ The above logic then clarifies why the classical Jacobi actions are adiabatic invariants.

Ehrenfest generalized this insight: any classical adiabatic invariant (action integral or not) corresponds to a quantum entity with discrete spectrum. The rationale is that an adiabatic change, by virtue of its slowness, is expected to lead only to continuous changes in the system, not to jumps that change a discrete quantum number. The preservation of the value of the quantum entity would then explain the classical invariant property.

Ehrenfest’s hypothesis can be used profitably in many problems. As an illustration consider a relativistic particle of rest mass $$m$$ and charge $$e$$ spiralling in a magnetic field $$B$$. One knows that the Larmor spiralling frequency is

$$\Omega = \frac{e|B|}{\gamma_L m} = \frac{e|B|}{E}, \hspace{1cm} (6)$$

where $$\gamma_L$$ is Lorentz’s gamma factor, and $$E$$ the total energy. When $$B$$ varies in space or in time slowly over one Larmor radius $$r$$ or over one Larmor period $$2\pi/\Omega$$, there exists an adiabatic invariant of the form

$$\Phi = \pi|B|r^2, \hspace{1cm} (7)$$

namely the magnetic flux through one loop of orbit is invariant. Now rewrite the energy

$$E = m\left(1 - \dot{r}^2 - \dot{z}^2 - r^2\Omega^2\right)^{-1/2} \hspace{1cm} (8)$$

by replacing $$\dot{z} \rightarrow p_z/m\gamma_L$$, taking into account that $$\dot{r}$$ is nearly vanishing, and replacing $$\Omega$$ and $$r$$ by means of Eq. (6) and Eq. (7) to get

$$E^2 = m^2 + p_z^2 + e^2r^2B^2 = m^2 + p_z^2 + e^2\Phi B/\pi \hspace{1cm} (9)$$

By Ehrenfest’s hypothesis, in the quantum problem $$\Phi$$ should have a discrete spectrum. Thus for fixed $$p_z$$, $$E^2$$ should be quantized, possibly with uniformly spaced eigenvalues. And indeed, the exact solution of the relativistic Landau problem with the Klein–Gordon equation leads to the spectrum

$$E^2 = m^2 + p_z^2 + e\hbar B(2n + 1); \hspace{1cm} n = 0, 1, \cdots \hspace{1cm} (10)$$

which justifies the prediction from the Ehrenfest hypothesis.

I shall now argue that the horizon area of a nonextremal Kerr–Newman black hole shows signs of being the analog of a mechanical adiabatic invariant. Application of Ehrenfest’s hypothesis in combination with a dynamic calculation will then lead to the spectrum for the horizon area operator.

Consider a Reissner–Nordström black hole of mass $$M$$ and charge $$Q$$. One shoots in radially a classical point particle of charge $$\varepsilon$$ with (conserved) total energy adjusted to the value $$E = \varepsilon Q/r_H$$, where $$r_H$$ is the radius of the black hole in Boyer–Lindquist coordinates. In Newtonian terms this particle should marginally reach the horizon where its potential energy just exhausts the total energy.
of the exact equation of motion supports this conclusion: the particle’s motion has
a turning point at the horizon. Because of this, the assimilation of particle by the
black hole takes place especially slowly: it is an adiabatic process.

Now the area of the horizon is originally

\[ A = 4\pi r_H^2 = 4\pi \left( M + \sqrt{M^2 - Q^2} \right)^2 \]  

and the (small) change inflicted on it by the absorption of the particle is

\[ \Delta A = \theta_{RN}^{-1} \cdot (\Delta M - Q \Delta Q/r_H) \]  

with

\[ \theta_{RN} \equiv \frac{1}{2} A^{-1} \sqrt{M^2 - Q^2} \]  

Thus if the black hole is not extremal so that \( \theta_{RN} \neq 0 \), \( \Delta A = 0 \) because \( \Delta M = E \) while \( \Delta Q = \varepsilon \) and \( E = \varepsilon Q/r_H \). Therefore, the horizon area is invariant in the course
of an adiabatic change of the black hole. But this conclusion does not extend to the
extremal black hole: when \( Q = M, \sqrt{M^2 - Q^2} \) in Eq. [11] is unchanged to \( O(\varepsilon^2) \)
during the absorption, so that \( \Delta A = 8\pi ME \neq 0 \).

As a second example consider a Kerr black hole of mass \( M \) and angular mo-
momentum \( J \). Send onto it a scalar wave of the form \( Y_{\ell m}(\theta, \phi)e^{-i\omega t} \). It is known
that for \( \omega \approx \Omega m \), the absorption coefficient has the form

\[ \Gamma = K_{\omega \ell m}(M, J) \cdot (\omega - \Omega m) \]  

where

\[ \Omega \equiv \frac{J/M}{r_H^2 + (J/M)^2} \]  

is the rotational angular frequency of the hole, while \( K_{\omega \ell m}(M, J) \) is a positive
coefficient. If one chooses \( \omega = m \Omega \), the wave is perfectly reflected. By choosing
\( \omega - \Omega m \) slightly positive, one arranges for a small fraction of the wave to get
absorbed. If the reflected wave is repeatedly reflected back towards the black hole
by a suitable large spheroidal mirror surrounding it, one can arrange for a sizable
fraction of the wave’s energy and angular momentum to get absorbed eventually.
But since this takes place over many cycles of reflection, the change in the hole is
an adiabatic one.

Now he horizon area of the Kerr black hole is

\[ A = 4\pi \left[ \left( M + \sqrt{M^2 - (J/M)^2} \right)^2 + (J/M)^2 \right] \]  

and small changes of it are given by

\[ \Delta A = \theta_K^{-1} \cdot (\Delta M - \Omega \Delta J) \]  

where

\[ \theta_K \equiv \frac{1}{2} A^{-1} \sqrt{M^2 - (J/M)^2} \]
The overall changes $\Delta M$ and $\Delta J$ must stand in the ratio $\omega/m$. This can be worked out from the energy–momentum tensor, but is simplest appreciated by thinking of the wave as made of quanta, each with energy $\hbar \omega$ and angular momentum $\hbar m$. Since I chose $\omega \approx \Omega m$, it follows from Eq. (17) that if the black hole is not extremal, $\Delta A \approx 0$ to the accuracy of the former equality. Evidently, here too horizon area is invariant during adiabatic changes. This conclusion is inapplicable to the extremal black hole for reasons similar to those in our first example.

The two examples, and the one to follow in the next section, support the conjecture that for a nonextremal Kerr-Newman black hole, the horizon area $A$ is, classically, an adiabatic invariant. By taking Ehrenfest’s hypothesis seriously, I conclude that the horizon area of a nonextremal Kerr-Newman quantum black hole, $\hat{A}$, just like that of the extremal black hole, must have a discrete eigenvalue spectrum.

4 Spacing and Multiplicity of the Area Levels

In view of the last conclusion I write the horizon area eigenvalues as

$$a_n = f(n); \quad n = 1, 2, 3, \ldots$$  \hspace{1cm} (19)

The function $f$ must clearly be positive and monotonically increasing (this last just reflects the ordering of eigenvalues by magnitude). However, one cannot infer from all this that $f$ is linear in its argument. For not every adiabatic invariant has a simply quantized spectrum: if $K$ is an adiabatic invariant, and its quantum counterpart $\hat{K}$ has a uniformly spaced spectrum, $K^2$ is still an adiabatic invariant, but $\hat{K}^2$’s spectrum is not uniformly spaced. At any rate, in light of Eq. (3) and the quantization of charge, magnetic monopole, and angular momentum, this result implies that the nonextremal Kerr-Newman black hole mass has a discrete spectrum. Its form will be elucidated in Sec. 5.

To elucidate the spacing of the area levels, I elaborate here on Christodoulou’s prescient question: can the assimilation of a point particle by a Kerr black hole be made reversibly in the sense that all changes of the black hole are undone by absorption of a suitable second particle? This is a good question because black hole horizon area cannot decrease, so that any process which causes it to increase is irrevocably irreversible. Christodoulou’s answer, later generalized to the Kerr–Newman black hole, was that the process is reversible if the (point) particle, which may be electrically charged and carry angular momentum, is injected at the horizon from a turning point in its orbit. In this case the horizon area (or equivalently the irreducible mass) is left unchanged, so that the effects on the black hole can be undone by a second reversible process which adds charges and angular momentum opposite in sign to those added by the first. Our Reissner–Nordström example in Sec. 3 is a special case of a Christodoulou reversible process. For Kerr–Newman black holes, Christodoulou’s reversible process furnishes one with a further example of the adiabatic invariance of horizon area (after all, the capture from a turning point is slow - adiabatic). One can check that Christodoulou and Ruffini’s calculation proves reversibility only for nonextremal black holes.

All the preceding is purely classical theory: in the Christodoulou–Ruffini process the particle follows a classical path, and must be a point particle in order for
its absorption to leave the area unchanged. How would quantum theory modify things? I do not intend anything so complicated as solution of a Schrödinger–like equation. But as a concession to quantum theory let me ascribe to the particle a finite radius \( b \) while continuing to assume, in the spirit of Ehrenfest’s theorem in quantum mechanics, that the center of mass of the particle follows a classical trajectory. Recalculating \( a \ la \) Christodoulou–Ruffini for a particle of mass \( \mu \) one finds that absorption of the particle now necessarily involves an increase in area. This is minimized if the particle is captured when its center of mass is at a turning point of its motion a proper distance \( b \) away from the horizon:

\[
(\Delta A)_{\text{min}} = 8\pi \mu b. \tag{20}
\]

This last conclusion fails for extremal black holes because the analog of the quantity \( \theta_K \) in Eq. 18 diverges in that case. The minimal increase in area is not Eq. 20, but a quantity dependent on \( M, Q, G \) and \( J \).

The limit \( b \to 0 \) of Eq. 20 recovers Christodoulou’s reversible process for the nonextremal black holes. However, a quantum point particle is subject to quantum uncertainty. If it is known to be at the horizon with high accuracy, its radial momentum is highly uncertain; this prevents the turning point condition from being fulfilled. And, of course, a relativistic quantum point particle cannot even be localized to better than a Compton length \( \hbar/\mu \). Thus in quantum theory the limit \( b \to 0 \) is not a legal one. One can get an idea of the smallest possible increase in horizon area in the quantum theory by putting \( b \to \xi \hbar/\mu \) in Eq. 20, where \( \xi \) is a number of order unity. Thus

\[
(\Delta A)_{\text{min}} = 8\pi \xi \hbar = \alpha \ell P^2. \tag{21}
\]

Surprisingly, for nonextremal black holes, \( (\Delta A)_{\text{min}} \) turns out to be independent of the black hole characteristics \( M, Q, G \) and \( J \).

The fact that for nonextremal black holes there is a minimum area increase as soon as one allows quantum nuances to the problem, suggests that this \( (\Delta A)_{\text{min}} \) corresponds to the spacing between eigenvalues of \( \hat{A} \) in the quantum theory. And the fact that \( (\Delta A)_{\text{min}} \) is a universal constant suggests that the spacing between eigenvalues is a uniform spacing. For it would be strange indeed if that spacing were to vary, say, as mass of the black hole, and yet the increment in area resulting from the best approximation to a reversible process would contrive to come out universal, as in Eq. 21, by involving a number of quantum steps inversely proportional to the eigenvalue spacing. I thus conclude that for nonextremal black holes the spectrum of \( \hat{A} \) is

\[
a_n = \alpha \ell P^2 (n + \eta); \quad \eta > -1; \quad n = 1, 2, \ldots \tag{22}
\]

where the condition on \( \eta \) excludes nonpositive area eigenvalues.

Now Eqs. 21, 22 fail for an extremal Kerr–Newman black hole, so one cannot deduce as above that its area eigenvalues are evenly spaced. This is entirely consistent with Eq. 3 according to which the area spectrum is then very complicated.
5 Demystifying Black Hole’s Entropy Proportionality to Area

The reader will have noticed that the previous arguments have said nothing about entropy; the discussion has been at the level of mechanics, not statistical physics. I shall now make use of Eq. 22 to understand, in a pleasant and intuitive way, the mysterious proportionality between black hole entropy and horizon area.

The quantization of horizon area in equal steps brings to mind an horizon formed by patches of equal area $\alpha\ell_P^2$, which get added one at a time. It is unnecessary - even detrimental - to think of a specific shape or localization of these patches. It is their standard size which is important, and which makes them all equivalent. This patchwork horizon can be regarded as having many degrees of freedom, one for each patch. After all, the concept “degree of freedom” emerges for systems whose parts can act independently, and here the patches can be added to the patchwork one at a time. In quantum theory degrees of freedom independently manifest distinct states. Since the patches are all equivalent, each will have the same number of quantum states, say, $k$. Therefore, the total number of quantum states of the horizon is

$$N = k^{A/\alpha\ell_P^2}$$

where $k$ is a positive integer and the effects of the $\eta$ zero point in Eq. 22 are glossed over in this, intuitive, argument.

Nobody assures one that the $N$ states are all equally probable. But if I assume that the $k$ states of each patch–degree of freedom are all equally likely, then all $N$ states are equally probable. In that case the statistical (Boltzmann) entropy associated with the horizon is $\ln N$ or

$$S_{BH} = \frac{\ln k}{\alpha\ell_P^2}$$

Thus is the proportionality between black hole entropy and horizon area justified in simple terms. One may interject that even if not all $k$ states are equally probable, one can still use Eq. 24 provided $k$ is regarded as an effective number of equally probable states. Comparison of Eq. 24 with Hawking’s coefficient in the black hole entropy calibrates the constant $\alpha$:

$$\alpha = 4\ln k$$

The above argument is meant to demystify the direct proportionality of black hole entropy and horizon area. It depends crucially on the uniformly spaced area spectrum. The logic leading to the number of states Eq. 23 has occasionally been used without regard to any particular area spectrum but these early arguments lack conviction because their partition of the horizon into equal area cells would be without basis if the desired result (entropy $\propto$ area) were not known.

Mukhanov’s alternate route to Eqs. 24 and 25 starts from the accepted formula relating black hole area and entropy. In the spirit of the Boltzmann–Einstein formula, he views $\exp(S_{BH})$ as the degeneracy of the particular area eigenvalue because $\exp(S_{BH})$ quantifies the number of microstates of the black hole that correspond to a particular external macrostate. Since black hole entropy is determined
by thermodynamic arguments only up to an additive constant, one writes, in this approach, \( S_{BH} = A/4\ell_p^2 + \text{const.} \). Substitution of the area eigenvalues from Eq. 22 gives the degeneracy corresponding to the \( n \)-th area eigenvalue:

\[
g_n = \exp \left( \frac{a_n}{4\ell_p^2} + \text{const.} \right) = g_1 e^{\alpha(n-1)/4}
\]

(26)

As stressed by Mukhanov, since \( g_n \) has to be integer for every \( n \), this is only possible when

\[
g_1 = 1, 2, \cdots \quad \text{and} \quad \alpha = 4 \times \{ \ln 2, \ln 3, \cdots \}
\]

(27)

The simplest option for \( g_1 \) would seem to be \( g_1 = 1 \) (nondegenerate black hole ground state). Here the additive constant in Eq. 26 must be retained: were it zero, the area \( a_1 \) would also vanish which seems an odd thing for a black hole. Just this case was studied in Ref. 16; it is a bit ugly in that the eigenvalue law Eq. 22 and the black hole entropy include related but undetermined additive constants.

The next case, \( g_1 = 2 \) (doubly degenerate black hole ground state), no longer requires the ugly additive constant in the black hole entropy to keep \( a_1 \) from vanishing. With this constant set to zero and the choice \( \alpha = 4 \ln 2 \) corresponding to \( k = 2 \), Eqs. 22 and 26 require that \( \eta = 0 \) so that one is rid of the second ugly constant as well. The area spectrum is

\[
a_n = 4\ell_p^2 \ln 2 \cdot n; \quad n = 1, 2, \cdots
\]

(28)

This spectrum, which I shall adopt henceforth, is good for nonextremal Kerr–Newman black holes. The corresponding degeneracy of area eigenvalues

\[
g_n = 2^n
\]

(29)

corresponds to a doubling of the degeneracy as one passes from one area eigenvalue to the next largest. Mukhanov thought of this multiplicity as the number of ways in which a black hole in the \( n \)-th area level can be made by first making a black hole in the ground state, and then proceeding to “excite it” up the ladder of area levels in all possible ways. Danielsson and Schiffer considered this multiplicity as representing rather the number of ways the black hole with area \( a_n \) can “decay” down the staircase of levels to the ground state.

To what extent do these intuitively physical predictions correspond to formal results from existing quantum gravity schemes? The uniformly spaced area spectrum was first proposed in 1975. Starting with Kogan’s 1986 string theoretic argument, a number of formal calculations, most in the last few years, have recovered this form of the spectrum. Mention may be made of the quantum membrane approaches of Maggiore and of Lousto which establish the uniformly spaced levels as the base for excitations of the black hole.

There are also several canonical quantum gravity treatments of a shell or ball of dust collapsing on its way to black hole formation. Those by Schiffer and Peleg get the uniformly spaced area spectrum. But Berezin, as well as Dolgov and Khriplovich, obtain mass spectra for the ensuing black hole which correspond to
discrete area spectra with nonuniform spacing (in Berezin’s approach the levels are infinitely degenerate). Other canonical quantum gravity approaches by Louko and Mäkelä, Barvinskiĭ and Kunstatter, Mäkelä and Kastrup treat a spherically symmetric vacuum spacetime endowed with dynamics by some subtlety, and also come up with a uniformly spaced area spectrum. There is, however, no general agreement on the spacing of the levels. The analogous treatment of the charged black hole by Mäkelä and Repo gets a nonuniform area spectrum.

Mention must also be made of the loop quantum gravity determination of the black hole area spectrum by Barreira, Carfora and Rovelli and by Krasnov. It leads to a discrete spectrum of complex form and highly nonuniform spacing. An exception to this qualification is the spectrum for the extremal neutral Kerr black hole where the aforementioned determination, assuming it can be applied to a nonspherical black hole, would concur with the Mazur spectrum Eq. (I. Khriplovich pointed this out to me during the meeting).

This set of contradictory conclusions seems to certify the view held by many that none of the existing formal schemes of quantum gravity is as yet a quantum theory of gravity.

6 The Black Hole Line Emission Spectrum

The particular area spectrum Eq. (28) implies, by virtue of the Christodoulou–Ruffini relation Eq. (3), a definite discrete mass spectrum. For zero charge and angular momentum the mass spectrum is of the form

\[ M \propto \sqrt{n}; \quad n = 1, 2 \quad (30) \]

implying the level spacing

\[ \omega_0 \equiv \Delta M/\hbar = (8\pi M)^{-1} \ln 2 \quad (31) \]

This simple result is in agreement with Bohr’s correspondence principle: “transition frequencies at large quantum numbers should equal classical oscillation frequencies”, because a classical Schwarzschild black hole displays ‘ringing frequencies’ which scale as \( M^{-1} \), just as Eq. (31) would predict. This agreement would be destroyed if the area eigenvalues were unevenly spaced. Indeed, the loop gravity spectrum mentioned in Sec. 5 fails this correspondence principle test.

By analogy with atomic transitions, a black hole at some particular mass level would be expected to make a transition to some lower level with emission of a quantum (or quanta) of any of the fields in nature. The corresponding line spectrum - very different from the Hawking semiclassical continuum - was first discussed in Ref. 1 and further analyzed much later. It comprises lines with all frequencies which are integral multiples of \( \omega_0 \). An elementary estimate gives the strength of the successive lines as falling off roughly as \( \exp(-8\pi M\omega/\hbar) \), so that only a few lines - those within the Hawking peak of the semiclassical emission - will be easily visible. The statistics of quanta in the radiation are reasonable: the number of quanta of a given kind in a given line emitted over a fixed time interval is Poisson distributed.

Most important, as Mukhanov was first to remark, this simple spectrum provides a way to make quantum gravity effects detectable even for black holes well
above the Planck mass: the uniform frequency spacing of the black hole lines occurs at all mass scales, and the unit of spacing is inversely proportional to the black hole mass over all scales. Of course, for very massive black holes, one would expect all the lines to become dim and unobservable (just as in the semiclassical description the Hawking radiance intensity goes down as $1/M^2$), but there should be a mass regime (primordial mini–black holes?) well above Planck’s for which the first few uniformly spaced lines should be detectable under optimum circumstances. It is thus clearly important to understand clearly the nature of the line spectrum.

The first natural question is whether natural broadening of the lines will not smear the spectrum into a continuum. First explored by Mukhanov, this issue has been revisited recently by both of us. By the usual argument the broadening of a line, $\delta \omega$, should be of order $\tau$, the typical time (as measured at infinity) between transitions of the black hole from level to level. One may thus estimate the rate of loss of black hole mass as

$$\frac{dM}{dt} \approx -\frac{\hbar \omega_0}{\tau} = -\frac{\hbar \ln 2}{8\pi M \tau}$$

(32)

Alternatively, one can estimate $dM/dt$ by assuming, in accordance with Hawking’s semiclassical result, that the radiation is black body radiation, at least in its intensity. Taking the radiating area as $4\pi(2M)^2$ and the temperature as $\bar{\hbar}/8\pi M$ one gets

$$\frac{dM}{dt} = -\frac{\gamma \hbar}{15360\pi M^2}$$

(33)

where $\gamma$ is a fudge factor that summarizes the grossness of our approximation. By comparing Eq. (33) with Eq. (32) one infers $\tau$ which then gives

$$\frac{\Delta \omega}{\omega_0} \sim 0.019 \gamma$$

(34)

Mukhanov and I regard $\gamma$ to be of order unity, which would make the natural broadening weak and the line spectrum sharp. More recently Mäkelä has estimated a much larger value, and claimed that the line spectrum effectively washes out into a continuum. He views this as a welcome development because it brings the ideas about black hole quantization, as here described, into consonance with Hawking’s smooth semiclassical spectrum.

Mäkelä uses Page’s estimate of black hole luminosity which takes into account the emission of several species of quanta, whereas our value $\gamma \sim 1$ is based on one species. It is, of course, true that a black hole will radiate all possible species, not just one. This is expected to enhance $\gamma$ by an order or two over the naive value. But it is also true that because the emission is, in the first instance, in lines, part of the frequency spectrum is thus blocked, which should lead to a reduced value for $\gamma$ in Eq. (33). Mukhanov and I consider the two tendencies to partly compensate, and expect $\gamma$ to exceed its putative value of unity by no more than an order of magnitude. According to Eq. (34) this should leave the emission lines unblended.

The above is not to say that the emission spectrum should be purely a line spectrum. Multiple quanta emission in one transition will also contribute a continuum. To go back to atomic analogies, the transition from the 2s to the 1s states of
atomic hydrogen, being absolutely forbidden by one–photon emission, occurs with
the long lifetime of 8 s by two–photon emission; the photon spectrum is thus a con-
utinuum over the relevant frequency range. Likewise, some multiple quanta emission
should accompany transitions of the black hole between its mass levels thus forming
a continuum that would compete with the line spectrum (at the conference G. Lavrelashvili reminded me of this). However, no reason is known why one–quantum
transition would be forbidden in the black hole case. Thus my expectation, again
based on the atomic analogy, is that most of the energy will get radiated in one–
quantum transitions which give lines. Thus the spectrum, in first approximation,
should be made up of lines sticking quite clearly out of a lowly continuum.

In atomic physics emission spectra display a hierarchy of splittings which can
be viewed as reflecting the hierarchical breaking of the various symmetries. Thus in
atomic hydrogen the \( O(4) \) symmetry of the Coulomb problem, which is reflected in
the Rydberg–Bohr spectrum, is broken by relativistic effects (spin–orbit interaction
and Thomas precession) thus giving rise to fine structure splitting of lines. But
even an exact relativistic treatment in the framework of Dirac’s equation leaves
the 2s and the 2p levels perfectly degenerate. They are split by a minute energy
by vacuum polarization effects and the Lamb shift. In addition, the rather weak
interaction of the electron with the nuclear proton’s magnetic moment leads to a
small hyperfine splitting of members of some of the other fine structure multiplets.
The very simple spectrum Eq. 30 is analogous to the hydrogenic Rydberg–Bohr
spectrum. Are there any splittings of the lines here discussed?

There is certainly room for splitting because of the \( 2^n \)–fold degeneracy of the
levels. The question is whether there is some breaking of symmetry, analogous to
the ones taking place in the atomic case, which would actually split the black hole
lines. To answer the question one obviously needs more detailed information about
the way the black hole mass spectrum arises in the context of the various symmetries
than we have heretofore elicited from our simple arguments. The desire to find out
more about this question is the main impetus behind the algebraic approach to be
described in Sec. 6.

Before leaving the subject I mention some related puzzles which cannot be
fully resolved purely by analogy with atomic physics. For instance, according to
Eq. 31, a Schwarzschild black hole cannot emit quanta at frequencies below \( \omega_0 \);
by the usual argument of microscopic reversibility, it should not absorb below this
frequency. However, classically a Schwarzschild black hole absorbs all frequencies,
albeit with decreasing crosssection as the frequencies become small compared to
\( M^{-1} \). Ref. 1 offered a solution to this paradox based on the observation that at the
low frequencies envisaged, the \textit{classical} absorptivity of the hole is so small that the
expected amount of energy absorbed is always below one quantum’s worth, unless
the energy of the incident wave much exceeds \( \hbar \omega_0 \). Thus the classical description,
which conflicts with the quantum description, must fail unless enough energy is
incident to elicit a quantum jump of the black hole by \textit{many quanta} absorption
(analogous to many–photon processes in nonlinear optical media). This anomalous
absorption would be interpreted in classical theory as the expected absorption of
sub–threshold frequencies.

The above resolution, by invoking a many–quantum process, brings back the
specter of many quantum emission and an emission continuum to compete with the line spectrum, and makes the more urgent the task of proceeding beyond mere analogies in the description of the emission process.

7 Algebraic Approach to the Quantum Black Hole

In quantum theory one usually obtains spectra of operators from the algebra they obey. For instance, Pauli obtained the complete spectrum of hydrogen in non-relativistic theory from the $O(4)$ algebra of the relevant operators. This approach sidesteps the question of constructing the wavefunctions for the states. I will now describe an axiomatic algebra, whose genesis goes back to joint work with Mukhanov, which describes the quantum black hole and gives an area spectrum identical to the one found above. It thus supports the results obtained previously, and illuminates the question of level splitting.

Sec. 2 introduced some of the relevant operators for a black hole: mass $\hat{M}$, horizon area $\hat{A}$, charge $\hat{Q}$, monopole $\hat{G}$ and angular momentum $\hat{J}$. The spectrum of $\hat{Q}$ is $\{ q \epsilon; q = \text{integer} \}$, that of $\hat{J}^2$ $\{ j(j + 1)\hbar^2 \}$, while that of $\hat{J}_z$ is $\{-j\hbar, -(j - 1)\hbar, \cdots, (j - 1)\hbar, j\hbar \}$ with $j$ a nonnegative integer or half-integer. In all that follows I shall ignore $\hat{G}$ for brevity. The first axiom is:

**Axiom 1**: Horizon area is represented by a positive semi-definite operator $\hat{A}$ with a discrete spectrum $\{ a_n; n = 0, 1, 2 \cdots \}$. The degeneracy of the eigenvalue $a_n$, denoted $g(n)$, is independent of the $j, m$ and $q$.

Discreteness of the area spectrum, as suggested by the adiabatic invariant character of horizon area, is formalized in this axiom; it is not here proved. One imagines the eigenvalues to be arranged so that $a_0 = 0$ corresponds to the vacuum $|\text{vac}\rangle$ (state devoid of any black holes) while the rest of the $a_n$ are arranged in order of increasing value. Since I do not refer to $\hat{G}$ in what follows, no confusion will arise with the use of $g$ for degeneracy. I take $g(0) = 1$.

Because $\hat{A}, \hat{Q}, \hat{J}^2$ and $\hat{J}_z$ mutually commute, one can infer the spectrum of $\hat{M}$ from that of $\hat{A}$ directly from the Christodoulou–Ruffini formula Eq. 3 which, as mentioned in Sec. 2, does not suffer from factor ordering ambiguities. But the triviality of the algebra precludes one learning anything about the spectrum of $\hat{A}$ itself.

This motivates the introduction of creation operators for black holes in their various states. In view of the similarities between black hole and elementary particles, it seems not farfetched to imagine black holes as particles of some field, and then creation operators appear naturally. Our second axiom is thus

**Axiom 2**: There exist operators $\hat{R}_{njmqs}$ with the property that $\hat{R}_{njmqs}|\text{vac}\rangle$ is a one black hole state with horizon area $a_n$, squared spin $j(j + 1)\hbar^2$, $z$–component of spin $m\hbar$, charge $q\epsilon$ and internal quantum number $s$. All one–black hole states are spanned by the basis $\{ \hat{R}_{njmqs}|\text{vac}\rangle \}$.

I do not purport to construct $\hat{R}_{njmqs}$. The internal quantum numbers are necessary because from the black hole entropy one knows that each state seen by an external observer corresponds to many internal states; these need to be distinguished
by additional quantum numbers (below called variously \( s, t \) or \( r \)). In the interest of clarity in the equations, I shall, when no misunderstanding can arise, write \( \hat{R}_\kappa \) or just \( \hat{R} \) for \( \hat{R}_{\kappa s} \).

Commutation of the operators now available creates more operators. If this process continues indefinitely, no information can be obtained from the algebra unless additional assumptions are made. Faith that it is possible to elucidate the physics from the algebra leads me to require closure of the algebra (which I suppose to be linear in analogy with many physically successful algebras) at an early stage. This is formalized in

**Axiom 3:** The operators \( \hat{A}, \hat{J}, \hat{Q}, \hat{R}_\kappa \) and \( [\hat{A}, \hat{R}_\kappa] \) form a closed, linear, infinite-dimensional nonabelian algebra.

Now the physical interpretation of the operators \( \hat{R}_\kappa \) leaves little choice regarding their commutators with the operators \( \hat{A}, \hat{Q} \) and \( \hat{J} \). The algebra of \( \hat{A} \) will be the subject of the Sec. 8. Here I start with \( \hat{J} \). Since \( \hat{R}_{\kappa s}|\text{vac}\rangle \) is defined as a state with spin quantum numbers \( j \) and \( m \), the collection of such states with fixed \( j \) and all allowed \( m \) must transform among themselves under rotations of the black hole like the spherical harmonics \( Y_{jm} \) (or the corresponding spinorial harmonic when \( j \) is half-integer). Since \( |\text{vac}\rangle \) must obviously be invariant under rotation, one learns that the \( \hat{R}_{\kappa s} \) may be taken to behave like an irreducible spherical tensor operator of rank \( j \) with the usual \( 2j + 1 \) components labeled by \( m \).

This means that

\[
[\hat{J}_z, \hat{R}_\kappa] = m_\kappa \hbar \hat{R}_\kappa
\]

(35)

and

\[
[\hat{J}_\pm, \hat{R}_\kappa] = \sqrt{j_\kappa(j_\kappa + 1) - m_\kappa(m_\kappa \pm 1)} \hbar \hat{R}_\kappa m_\kappa \pm 1
\]

(36)

where \( \hat{J}_\pm \) are the well known raising and lowering operators for the \( z \)-component of spin. To check these commutators I first operate with Eq. 35 on \( |\text{vac}\rangle \) and take into account that \( \hat{J}_z|\text{vac}\rangle = 0 \) (the vacuum has zero spin) to get

\[
\hat{J}_z \hat{R}_\kappa |\text{vac}\rangle = m_\kappa \hbar \hat{R}_\kappa |\text{vac}\rangle
\]

(37)

Also from the relation \( \hat{J}^2 = (\hat{J}_z \hat{J}_+ + \hat{J}_- \hat{J}_+)/2 + \hat{J}_0^2 \), one can work out \( [\hat{J}^2, \hat{R}_\kappa] \) and operate with it on \( |\text{vac}\rangle \); after double use of Eqs. 35 and 36 one gets

\[
\hat{J}^2 \hat{R}_{\kappa s} |\text{vac}\rangle = j_\kappa(j_\kappa + 1)\hbar^2 \hat{R}_{\kappa s} |\text{vac}\rangle
\]

(38)

Of course both of these results were required by the definition of \( \hat{R}_{\kappa s}|\text{vac}\rangle \).

Moving on one recalls that \( \hat{Q} \) is the generator of (global) gauge transformations of the black hole, which means that for an arbitrary real number \( \chi \), \( \exp(i\chi \hat{Q}) \) elicits a phase change \( \chi \) of the black hole state:

\[
\exp(i\chi \hat{Q}) \hat{R}_{\kappa s} |\text{vac}\rangle = \exp(i\chi q_\kappa e) \hat{R}_{\kappa s} |\text{vac}\rangle
\]

(39)

This equations parallels

\[
\exp(i\phi \hat{J}_z/\hbar) \hat{R}_{\kappa s} |\text{vac}\rangle = \exp(i\phi m_\kappa) \hat{R}_{\kappa s} |\text{vac}\rangle
\]

(40)
which expresses the fact that $\hat{J}_z$ is the generator of rotations of the spin about the $z$ axis. Thus by analogy with Eq. 35, one may settle on the commutation relation

$$[\hat{Q}, \hat{R}_{\kappa s}] = q_\kappa e R_{\kappa s}. \quad (41)$$

Operating with this on the vacuum (recall that $\hat{Q}|\text{vac}\rangle = 0$) gives

$$\hat{Q} \hat{R}_{\kappa s}|\text{vac}\rangle = q_\kappa e R_{\kappa s}|\text{vac}\rangle \quad (42)$$

so that $R_{\kappa s}|\text{vac}\rangle$ is indeed a one black hole state with definite charge $q_\kappa e$, as required.

In addition to Eqs. 35-36 and 41 one would like to determine $[\hat{A}, \hat{R}_{\kappa s}]$, but since it is unclear what kind of symmetry transformation $\hat{A}$ generates, a roundabout route is indicated.

### 8 Algebra of the Area Observable

Consider the Jacobi identity

$$[\hat{A}, [\hat{V}, \hat{R}_\kappa]] + [\hat{V}, [\hat{R}_\kappa, \hat{A}]] + [\hat{R}_\kappa, [\hat{A}, \hat{V}]] = 0 \quad (43)$$

valid for three arbitrary operators $\hat{A}, \hat{V}$ and $\hat{R}_\kappa$. Suppose one replaces $\hat{V}$ in turn by $\hat{J}_z$, $\hat{J}_\pm$ and $\hat{Q}$, and makes use of Eqs. 35-36 and 41 as well as the commutativity of $\hat{J}_z$, $\hat{J}_\pm$, $\hat{Q}$, and $\hat{A}$ to obtain the three commutators

$$[\hat{J}_z, [\hat{A}, \hat{R}_{\kappa m}]] = m_\kappa \hbar [\hat{A}, \hat{R}_{\kappa m}],$$

$$[\hat{J}_\pm, [\hat{A}, \hat{R}_{\kappa m}]] = \sqrt{j_\kappa (j_\kappa + 1) - m_\kappa (m_\kappa \pm 1)} \hbar [\hat{A}, \hat{R}_{\kappa m} \pm 1],$$

$$[\hat{Q}, [\hat{A}, \hat{R}_{\kappa m}]] = q_\kappa e [\hat{A}, \hat{R}_{\kappa m}]. \quad (44)$$

Thus, for fixed $j$, $m$, and $q$, a particular $[\hat{A}, \hat{R}_{njmqs}]$ has commutators of the same form as all the $\hat{R}_{njmqs}$ with various $n$ and $s$. Hence one can write generically

$$[\hat{A}, \hat{R}_{\kappa s}] = \sum_{n\lambda t} h_{\kappa s}^{\lambda t} \hat{R}_{\lambda s} + \hat{T}_{\kappa s} \quad (45)$$

where $n_\lambda$ belongs to the set $\lambda$, the $h_{\kappa s}^{\lambda t}$ are constants and $\hat{T}_{\kappa s}$ is an operator independent of all the $\hat{R}_{\kappa s}$ (for otherwise it could be lumped with them in the r.h.s.). Eq. 15 is really a definition of $\hat{T}_{\kappa s}$, which operator obviously mimics the behavior of $\hat{R}_{\kappa s}$ under rotations and gauge transformations.

Operating with Eq. 45 on the vacuum (and remembering that $\hat{A}|\text{vac}\rangle = 0$ because of the postulated $a_0 = 0$) one gets

$$a_\kappa \hat{R}_{\kappa s}|\text{vac}\rangle = \sum_{n\lambda t} h_{\kappa s}^{\lambda t} \hat{R}_{\lambda s}|\text{vac}\rangle + \hat{T}_{\kappa s}|\text{vac}\rangle \quad (46)$$

Now because the $\hat{R}_{\lambda s}|\text{vac}\rangle$ with various $n_\lambda$ and $t$ are independent, one must set

$$h_{\kappa s}^{\lambda t} = a_\kappa \delta_{n_\kappa} n_\lambda \delta_s t \quad \text{and} \quad \hat{T}_{\kappa s}|\text{vac}\rangle = 0 \quad (47)$$
so that finally

$$[\hat{A}, \hat{R}_{\kappa s}] = a_\kappa \hat{R}_{\kappa s} + \hat{T}_{\kappa s}$$

(48)

with the $\hat{T}_{\kappa s}$ operators annihilating the vacuum. The appearance of these new operators requires one to understand something about their commutation relations.

Since under rotations and gauge transformations $\hat{T}_{\kappa s}$ transforms just like $\hat{R}_{\kappa s}$, one can take the commutators of $\hat{T}_{\kappa s}$ with $\hat{J}_z, \hat{J}_\pm$, and $\hat{Q}$ to parallel Eqs. 35, 36, and 42. Then by the same argument that led to Eqs. 41, one finds that $[\hat{A}, \hat{T}_{\kappa s}]$ transforms just like $\hat{R}_{\kappa s}$. Now since by Eq. 48 $[\hat{A}, \hat{R}_{\kappa s}]$ can be replaced by $\hat{T}_{\kappa s}$ and $\hat{R}_{\kappa s}$, then by Axiom 3 $[\hat{A}, \hat{T}_{\kappa s}]$ must be expressible as a linear combination of the operators $\hat{A}$, $\hat{J}$, $\hat{Q}$, $\hat{R}_{\kappa s}$ and $\hat{T}_{\kappa s}$ which transforms like $[\hat{A}, \hat{T}_{\kappa s}]$ under rotations and gauge transformations. The generic one is

$$[\hat{A}, \hat{T}_{\kappa s}] = \sum_{n, \lambda, t} \left( B_{\kappa s}^{\lambda t} \hat{T}_{\lambda t} + C_{\kappa s}^{\lambda t} \hat{R}_{\lambda t} \right) + a_\kappa \delta_q^0 \left[ \delta_j^0 (D\hat{Q} + E\hat{A}) + \delta_j^1 F\hat{J}_m \right]$$

(49)

Here $\kappa$ and $\lambda$ share common $j, m$ and $q$, the coefficients $B_{\kappa s}^{\lambda t}, C_{\kappa s}^{\lambda t}, D, E$ and $F$ are structure constants ($D$ and $E$ may depend on $n_\kappa$, $s$ and $F$ on $m_\kappa$ as well), and $\hat{J}_m$ are the spherical tensor components of the vector operator $\hat{J}$, namely $\hat{J}_\pm/\sqrt{2}$ and $\hat{J}_z$. The prefactor $a_\kappa$ is added for later convenience.

Upon operation with Eq. 49 on $|\text{vac}\rangle$, the only surviving terms are those on the r.h.s. involving the $\hat{R}_{\kappa s}$ because of Eq. 47 and the fact that the vacuum bears no charge or angular momentum and has zero area eigenvalue. Thus necessarily $C_{\kappa s}^{\lambda t} \equiv 0$ because $\hat{R}_{\lambda t}|\text{vac}\rangle$ cannot vanish.

One can also give an informal argument that either all $\hat{T}_{\kappa s} \equiv 0$ or all $B_{\kappa s}^{\lambda t} \equiv 0$. Consider a pair of orthogonal one–black hole states, $|x\rangle$ and $|y\rangle$, of the sort $\hat{R}_{\kappa s}|\text{vac}\rangle$. The matrix element of Eq. 49 between these two states is

$$\sum_{n, \lambda, t} B_{\kappa s}^{\lambda t} \langle y | \hat{T}_{\lambda t} | x \rangle = (a_y - a_x) \langle y | \hat{T}_{\kappa s} | x \rangle$$

(50)

because $\langle y | \hat{Q} | x \rangle$, etc. drop out by the orthogonality of $|x\rangle$ and $|y\rangle$. It is necessary to adjust $q_y = q_x + q_\lambda$ so that $\langle y | \hat{T}_{\lambda t} | x \rangle$ shall not vanish trivially. This follows from the fact that in analogy with Eq. 41, $[\hat{Q}, \hat{T}_{\lambda t}] = q_\lambda \hat{e} \hat{T}_{\lambda t}$. The matrix element of this last equation is $(q_y - q_x - q_\lambda) \langle y | \hat{T}_{\kappa s} | x \rangle = 0$, which upholds the claim. In like way one must take $m_y = m_x + m_\lambda$.

According to Eq. 50, the “vector” whose components are $\langle y | \hat{T}_{\lambda t} | x \rangle$ for all $\lambda, t$ is an eigenvector of the matrix $B_{\kappa s}^{\lambda t}$. This is true for every $|x\rangle$ if one adjusts $|y\rangle$ in accordance with the mentioned constraints. But in a framework where we truncate the infinite dimensional problem to a finite dimensional one, the number of such eigenvectors exceeds the dimension of $B_{\kappa s}^{\lambda t}$ since for every pair $\kappa, s$ one can choose a $|x\rangle$ with the same quantum numbers, but then one is still free to choose $a_y$ and $s_y$ in specifying $|y\rangle$. The surplus may mean that the eigenvectors constructed as above often vanish. For instance, $\langle x | \hat{T}_{\kappa s} | y \rangle$ might vanish unless $a_y = a_x$ and $s_y = s_x$. Then we would have exactly the right number of (nontrivial) eigenvectors. But according to Eq. 50, all the corresponding eigenvalues would then vanish. A matrix all whose eigenvalues vanish must vanish, and so in this eventuality $B_{\kappa s}^{\lambda t} \equiv 0$. 

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One can escape the above conclusion if the \( \langle y \mid \hat{T}_{\kappa s} \mid x \rangle \) with \( |y\rangle \) properly adjusted as above is the same vector up to normalization for several different \( |x\rangle \)'s. This does not look likely. The other escape clause is for all the \( \langle y \mid \hat{T}_{\kappa s} \mid x \rangle \) to vanish, which by completeness of the states \( |x\rangle \) and \( |y\rangle \) means that \( \hat{T}_{\kappa s} = 0 \). Then Eq. 50 is satisfied trivially. In both of the eventualities, the term involving \( B_{\kappa s \lambda t} \hat{T}_{\lambda t} \) in Eq. 49 drops out.

Accepting this one defines a new creation operator

\[
\hat{R}^\text{new}_{\kappa s} \equiv \hat{R}_{\kappa s} + (a_{\kappa})^{-1} \left\{ \hat{T}_{\kappa s} + \delta_{\kappa 0} [\delta_{\kappa 0} (D\hat{Q} + E\hat{A}) + \delta_{\lambda 1} F_{\lambda t}] \right\}
\]

(51)

Since \( \hat{T}_{\kappa s}, \hat{A}, \hat{J}_m \) and \( \hat{Q} \) all anhilate \( |\text{vac}\rangle \), it is seen that \( \hat{R}^\text{new}_{\kappa s} \) creates the same one–black hole state as \( \hat{R}_{\kappa s} \). But the \( \hat{R}^\text{new}_{\kappa s} \) turn out to satisfy simpler commutation relations. Substituting in \( [\hat{A}, \hat{R}^\text{new}_{\kappa s}] \) from Eqs. 48 and 49 one gets the commutator

\[
[\hat{A}, \hat{R}^\text{new}_{\kappa s}] = a_{\kappa} \hat{R}^\text{new}_{\kappa s}
\]

(52)

which is reminiscent of Eqs. 35 and 41. Henceforth I use only \( \hat{R}^\text{new}_{\kappa s} \) but drop the "new".

9 Algebraic Derivation of the Area Spectrum

Operating with \( R_{\kappa s} \hat{R}_{\lambda t} \) on \( |\text{vac}\rangle \) and simplifying the result with Eq. 52 gives

\[
\hat{A}\hat{R}_{\kappa s}\hat{R}_{\lambda t}|\text{vac}\rangle = \hat{R}_{\kappa s}(\hat{A} + a_{\kappa})\hat{R}_{\lambda t}|\text{vac}\rangle = (a_{\kappa} + a_{\lambda})\hat{R}_{\kappa s}\hat{R}_{\lambda t}|\text{vac}\rangle
\]

(53)

so that the state \( \hat{R}_{\kappa s}\hat{R}_{\lambda t}|\text{vac}\rangle \) has horizon area equal to the sum of the areas of the states \( \hat{R}_{\kappa s}|\text{vac}\rangle \) and \( \hat{R}_{\lambda t}|\text{vac}\rangle \). Analogy with field theory might lead one to believe that \( \hat{R}_{\kappa s}\hat{R}_{\lambda t}|\text{vac}\rangle \) is just a two–black hole state, in which case the result just obtained would be trivial. But in fact, the axiomatic approach allows other possibilities.

Recall Eqs. 33, 41 and 52, namely

\[
[X, \hat{R}_\kappa] = x_\kappa \hat{R}_\kappa \quad \text{for} \quad X = \{\hat{A}, \hat{Q}, \hat{J}_z\}
\]

(54)

The Jacobi identity, Eq. 113, can then be used to infer that

\[
[X, [\hat{R}_\kappa, \hat{R}_\lambda]] = (x_\kappa + x_\lambda)[\hat{R}_\kappa, \hat{R}_\lambda]
\]

(55)

which makes it clear that \( [\hat{R}_\kappa, \hat{R}_\lambda] \) has the same transformations under rotations and gauge transformations as a single \( \hat{R}_\mu \) with

\[
x_\mu \equiv x_\kappa + x_\lambda
\]

(56)

Axiom 3 then allows one to conclude that (\( \varepsilon_{\kappa \lambda} \) and \( \epsilon_{\kappa \lambda} \) are structure constants)

\[
[\hat{R}_\kappa, \hat{R}_\lambda] = \sum_{\mu} \left( \varepsilon_{\kappa \lambda \mu} \hat{R}_\mu + \epsilon_{\kappa \lambda \mu} \hat{T}_\mu \right) + a_\mu \delta_0 \left[ \delta_0 (D\hat{Q} + E\hat{A}) + \delta_1 F_{\lambda t} \right]
\]

(57)
where \( j, m, q \in \mu \). Although closure was postulated with respect to the old \( \hat{R}'s \), we use the new \( \hat{R}'s \) here. This causes no difficulty because the two differ only by a superposition of \( \hat{T}'s \), and such terms have been added anyway.

When one operates with Eq. 57 on \(|\text{vac}\rangle\) one gets
\[
[\hat{R}_\kappa, \hat{R}_\lambda]|\text{vac}\rangle = |\bullet\rangle (58)
\]

where \(|\bullet\rangle\) stands for a one–black hole state, a superposition of states with various \( \mu t \).

Were \( \hat{R}_\kappa \hat{R}_\lambda |\text{vac}\rangle \) purely a two–black hole state as suggested by the field–theoretic analogy, one could not get Eq. 58. Inevitably
\[
\hat{R}_\kappa \hat{R}_\lambda |\text{vac}\rangle = |\bullet \bullet\rangle + |\bullet\rangle (59)
\]

with \(|\bullet \bullet\rangle\) a two–black hole state, symmetric under exchange of the \( \kappa s \) and \( \lambda t \) pairs. The superposition of one and two–black hole states means that the rule of additivity of eigenvalues, Eq. 56, applies to one black hole as well as two:
\[
\text{the sum of two eigenvalues of } \hat{Q}, \hat{J}_z \text{ or } \hat{A} \text{ of a single black hole is also a possible eigenvalue of a single black hole.}
\]

For charge or \( z \)–spin component this rule is consistent with experience with quantum systems whose charges are always integer multiples of the fundamental charge (which might be a third of the electron’s), and whose \( z \)–spins are integer or half integer multiples of \( \hbar \). This agreement serves as a partial check of our line of reasoning.

In accordance with Axiom 1, let \( a_1 \) be the smallest nonvanishing eigenvalue of \( \hat{A} \). Then Eq. 59 says that any positive integral multiple \( na_1 \) (which can be obtained by repeatedly adding \( a_1 \) to itself) is also an eigenvalue. This spectrum of \( \hat{A} \) agrees with that found in Sec. 5 by heuristic arguments. But the question is, are there any other area eigenvalues in between the integral ones (this has a bearing on the question of whether splitting of the levels found in Sec. 5 is at all possible) ?

To answer this query, I write down the hermitian conjugate of Eq. 52:
\[
[\hat{A}, \hat{R}_\kappa^\dagger] = -a_\kappa \hat{R}_\kappa^\dagger (60)
\]

Then
\[
\hat{A} \hat{R}_\kappa^\dagger \hat{R}_\lambda |\text{vac}\rangle = \left( \hat{R}_\kappa^\dagger \hat{A} - a_\kappa \hat{R}_\kappa^\dagger \right) \hat{R}_\lambda |\text{vac}\rangle = (a_\lambda - a_\kappa) \hat{R}_\kappa^\dagger \hat{R}_\lambda |\text{vac}\rangle (61)
\]

Thus differences of area eigenvalues appear as eigenvalues in their own right. Since \( \hat{A} \) has no negative eigenvalues, if \( n_\lambda \leq n_\kappa \), the operator \( \hat{R}_\kappa^\dagger \) must annihilate the one–black hole state \( \hat{R}_\lambda |\text{vac}\rangle \) and there is no black hole state \( \hat{R}_\kappa^\dagger \hat{R}_\lambda |\text{vac}\rangle \). By contrast, if \( n_\kappa < n_\lambda \), \( \hat{R}_\kappa^\dagger \) obviously lowers the area eigenvalue of \( \hat{R}_\lambda \). There is thus no doubt that \( \hat{R}_\kappa^\dagger \hat{R}_\lambda |\text{vac}\rangle \) is a purely one–black hole state (a “lowering” operator cannot create an extra black hole: Eq. 61 shows that \( \hat{R}_\kappa^\dagger \) annihilates the vacuum). In conclusion, positive differences of one–black hole area eigenvalues are also allowed area eigenvalues for one black hole.

If there were fractional eigenvalues of \( \hat{A} \), one could, by substracting a suitable integral eigenvalue, get a positive eigenvalue below \( a_1 \), in contradiction with \( a_1 \)’s definition as lowest positive area eigenvalue. Thus the set \( \{na_1; \ n = 1, 2, \cdots\} \) comprises the totality of \( \hat{A} \) eigenvalues for one black hole, in complete agreement

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with the heuristic arguments of Sec. 5 (but the algebra by itself cannot set the area scale \(a_1\)).

What about the degeneracy of area eigenvalues? According to Axiom 1, \(g(n)\), the degeneracy of the area eigenvalue \(na_1\), is independent of \(j, m\) and \(q\). Thus for fixed \(\{n_\kappa, j_\kappa, m_\kappa, q_\kappa\}\) where not all of \(j_\kappa, m_\kappa\) and \(q_\kappa\) vanish, there are \(g(n_\kappa)\) independent one–black hole states \(|\hat{R}_s\rangle\) distinguished by the values of \(s\). Analogously, the set \(\{n_\lambda = 1, j_\lambda = 0, m_\lambda = 0, q_\lambda = 0\}\) specifies \(g(1)\) independent states \(|\hat{R}_s\rangle\), all different from the previous ones because not all quantum numbers agree. One can thus form \(g(1) \cdot g(n_\kappa)\) one–black hole states, \(|\hat{R}_s\rangle{|\text{vac}\rangle}\), with area eigenvalues \((n_\kappa + 1)a_1\) and charge and angular momentum just like the states \(\hat{R}_s{|\text{vac}\rangle}\). If these new states are independent, their number cannot exceed the total number of states with area \((n_\kappa + 1)a_1\), namely \(g(n_\kappa + 1) \geq g(1) \cdot g(n_\kappa)\). Iterating this inequality starting from \(n_\kappa = 1\) one gets

\[
g(n) \geq g(1)^n \tag{62}
\]

The value \(g(1) = 1\) is excluded because one knows that there is some degeneracy. Thus the result here is consistent with the law \(23\) which we obtained heuristically. In particular, it supports the idea that the degeneracy grows exponentially with area. The specific value \(g(1) = 2\) used in Sec. 5 requires further input.

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