The Intrinsic Bounds on the Risk Premium of Markovian Pricing Kernels

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Abstract
The risk premium is one of main concepts in mathematical finance. It is a measure of the trade-offs investors make between return and risk and is defined by the excess return relative to the risk-free interest rate that is earned from an asset per one unit of risk. The purpose of this article is to determine upper and lower bounds on the risk premium of an asset based on the market prices of options. One of the key assumptions to achieve this goal is that the market is Markovian. Under this assumption, we can transform the problem of finding the bounds into a second-order differential equation. We then obtain upper and lower bounds on the risk premium by analyzing the differential equation.

1 Introduction

The risk premium or market price of risk is one of main concepts in mathematical finance. The risk premium is a measure of the trade-offs investors make between return and risk and is defined by the excess return relative to the risk-free interest rate earned from an asset per one unit of risk. The risk premium determines the relation between an objective measure and a risk-neutral measure. An objective measure describes the actual stochastic dynamics of markets, and a risk-neutral measure determines the prices of options.

Recently, many authors have suggested that the risk premium (or, equivalently, objective measure) can be determined from a risk-neutral measure. Ross [33] demonstrated that the risk premium can be uniquely determined by a risk-neutral measure. His model assumes that there is a finite-state Markov process $X_t$ that drives the economy in discrete time $t \in \mathbb{N}$. Many authors have extended his model to a continuous-time setting using a Markov diffusion process $X_t$ with state space $\mathbb{R}$; see, e.g., [6], [9], [12], [23], [31] and [35]. Unfortunately, in the continuous-time model, the risk premium is not uniquely determined from a risk-neutral measure [18], [23].

To determine the risk premium uniquely, all of the aforementioned authors assumed that some information about the objective measure was known or restricted the process $X_t$ to some class. Borovicka, Hansen and Scheinkman [6] made the assumption that the process $X_t$ is stochastically stable under the objective measure. In [9], Carr and Yu assumed that the process $X_t$ is a bounded process. Dubynskiy and Goldstein [12] explored Markov diffusion models with reflecting boundary conditions.

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conditions. In [23], Hyungbin assumed that \( X_t \) is non-attracted to the left (or right) boundary under the objective measure. Qin and Linetsky [31] and Walden [35] assumed that the process \( X_t \) is recurrent under the objective measure. Without these assumptions, one cannot determine the risk premium uniquely.

The purpose of this article is to investigate the bounds of the risk premium. As mentioned above, without further assumptions, the risk premium is not uniquely determined, but one can determine upper and lower bounds on the risk premium. To determine these bounds, we need to consider how the risk premium of an asset is determined in a financial market.

In this paper, the underlying assumptions on the financial market are as follows. First, the financial market is defined as a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) having a standard one-dimensional Brownian motion \( B_t \). In addition, we assume that (the reciprocal of) the pricing kernel, denoted by \( L_t \), is separable in the sense that there exists a positive function \( \phi \in C^2(\mathbb{R}) \), a positive number \( \beta \) and a stochastic process \( X_t \) such that

\[
L_t = e^{\beta t} \phi(X_t) \phi^{-1}(X_0) .
\]

(1.1)

We will see that in this case the risk premium \( \theta_t \) is given by

\[
\theta_t = \frac{\sigma_t \phi'(X_t)}{\phi(X_t)} = \sigma_t \cdot (\phi' \phi^{-1})(X_t)
\]

(1.2)

where \( X_t \) follows the dynamics \( dX_t = \zeta_t dt + \sigma_t dB_t \). We refer to the process \( X_t \) as an indicator process. As an example, in the consumption-based capital asset pricing model [8,27], the risk premium of an asset is determined by the law of supply and demand and is expressed by

\[
\theta_t = -\frac{\sigma_t U''(c_t)}{U'(c_t)}
\]

(1.3)

where \( U(\cdot) \) is the utility of the representative agent in the market and \( c_t \) is the aggregate consumption, with \( dc_t = \zeta_t dt + \sigma_t dB_t \).

Second, the dynamics of the indicator process \( X_t \) under the risk-neutral measure is assumed to be known ex ante. Assume that there are ample options whose underlying processes are \( X_t \) in the market, thus we can obtain the dynamics of \( X_t \) under the risk-neutral measure. Let \( X_t \) follow the following dynamics:

\[
dX_t = k_t dt + \sigma_t dW_t
\]

where \( W_t \) is a Brownian motion under the risk-neutral measure. \( k_t \) and \( \sigma_t \) are assumed to be known.

Last, we assume that the market is Markovian and driven by the indicator process \( X_t \). More precisely, the indicator process \( X_t \) is a time-homogeneous Markovian diffusion process. Thus, \( k_t = k(X_t) \) and \( \sigma_t = \sigma(X_t) \) for some functions \( k(\cdot) \) and \( \sigma(\cdot) \). In addition, the interest rate \( r_t \) is determined by \( X_t \), i.e., there is a function \( r(\cdot) \) such that \( r_t = r(X_t) \).

The purpose of this article is to determine upper and lower bounds on \( \theta_t \) based on the prices of options in the market. Thus, \( k(\cdot), \sigma(\cdot) \) and \( r(\cdot) \) are assumed to be known. The positive function \( \phi(\cdot) \) and the positive number \( \beta \) are assumed to be unknown. The main idea is to determine the properties of all of the possible \( \phi(\cdot) \)'s and \( \beta \)'s and then to obtain upper and lower bounds on the possible \( (\phi \phi^{-1})(\cdot) \) values. From (1.2), we can determine the bounds of the risk premium \( \theta_t \).

The problem of determining the bounds of the risk premium can be transformed into a second-order differential equation. We will demonstrate that \( \phi(\cdot) \) satisfies the following differential equation:

\[
\mathcal{L} \phi(x) := \frac{1}{2} \sigma^2(x) \phi''(x) + k(x) \phi'(x) - r(x) \phi(x) = -\beta \phi(x)
\]
for some unknown positive number $\beta$. Thus, we can determine the bounds by investigating the bounds of $(\phi'\phi^{-1})(\cdot)$ for a positive solution $\phi(\cdot)$. It will be demonstrated that two special solutions of $Lh = 0$ play an important role for the bounds of the risk premium $\theta_t$.

The following provides an overview of this article. First, in Section 2, we review the consumption-based capital asset pricing model. In Section 3, we investigate the risk premium of an asset and see how the problem of determining the bounds of the risk premium is transformed into a second-order differential equation. We then state upper and lower bounds on the risk premium of an asset, which is the main result of this article, in Section 4. Finally, we explore examples in Section 5.

2 Model Setups

In this section, we review the consumption-based capital asset pricing model (CCAPM). For more details, refer to [8],[27].

2.1 Financial Markets

A financial market is defined as a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ having a Brownian motion $B_t$ with the filtration $\mathcal{F} = (\mathcal{F}_t)_{t=0}^{\infty}$ generated by $B_t$. All the processes in this article are assumed to be adapted to the filtration $\mathcal{F}$. $\mathbb{P}$ is the objective measure of this market.

**Assumption 1.** In the financial market, there are two assets: a risk-free asset $M_t$ and a risky asset $S_t$. It is assumed that

$$dM_t = r_t M_t \, dt, \quad M_0 = 1$$

with an interest rate process $r_t$. A risky asset $S_t$ is a positive continuous semi-martingale and follows

$$dS_t = \mu_t S_t \, dt + \nu_t S_t \, dB_t.$$

**Assumption 2.** The market $(\Omega, \mathcal{F}, \mathbb{P})$ with $(M_t, S_t)$ has no arbitrage and is complete.

This implies that for fixed $T > 0$, there exists a unique equivalent martingale measure $\mathbb{Q}$ with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that $\frac{S_t}{M_t}$ is a martingale under $\mathbb{Q}$ for $0 < t < T$. Put the Radon-Nikodym derivative

$$\Sigma_t = \frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t},$$

which is known to be a martingale process on $(\Omega, \mathcal{F}, \mathbb{P})$ for $0 < t < T$. We can write in the SDE form

$$d\Sigma_t = -\theta_t \Sigma_t \, dB_t$$

where

$$\theta_t := \frac{\mu_t - r_t}{\nu_t}$$

is the risk-premium or market price of risk. It is well-known that $W_t$ defined by

$$dW_t := \theta_t dt + dB_t$$

(2.1)

is a Brownian motion under $\mathbb{Q}$. We define the reciprocal of the pricing kernel by $L_t = \frac{\Sigma_t}{M_t}$. Using the Ito formula,

$$dL_t = (r_t + \theta_t^2) L_t \, dt + \theta_t L_t \, dB_t$$

$$= r_t L_t \, dt + \theta_t L_t \, dW_t$$

(2.3)

is obtained.
2.2 Indicator Process

In this section, we present the notion of an indicator process.

**Assumption 3.** There exist a positive function $\phi \in C^2(\mathbb{R})$, a positive number $\beta$ and a process $X_t$ in the market such that

$$L_t = e^{\beta t} \phi(X_t) \phi^{-1}(X_0).$$  \hspace{1cm} (2.4)

In this case, we say that $(\beta, \phi)$ is a principal pair of $X_t$. We refer to this process $X_t$ as an indicator process.

We assume that there are ample options in the market whose underlying processes are $X_t$ in the market. Thus we can obtain the dynamics of $X_t$ under the risk-neutral measure $Q$.

**Assumption 4.** The $Q$-dynamics of $X_t$ is assumed to be known.

The purpose of this article is to determine upper and lower bounds on the risk premium, or equivalently (the reciprocal of) on the pricing kernel $L_t$. For the remainder of this section, we clarify the meaning of the indicator process. As an example, we consider the consumption-based capital asset pricing model (CCAMP) [8],[27]. A standard argument of the CCAPM gives that (the reciprocal of) the pricing kernel $L_t$ is determined by the utility and discount rate for the representative agent in the market. More precisely,

$$L_t = e^{\beta t} \frac{U'(c_0)}{U'(c_t)}$$  \hspace{1cm} (2.5)

where $U(\cdot)$ is the utility of the representative agent, $\beta$ is the discount rate and $c_t$ is the aggregate consumption process of the market. Thus, the pricing kernel $L_t$ can be obtained by observing the aggregate consumption process $c_t$ when $U(\cdot)$ and $\beta$ are given.

However, this means of obtaining the pricing kernel has some flaws when put to practical use. The aggregate consumption data cannot be obtained immediately. These data are usually obtained annually or semi-annually from a statistical office or a statistics agency. Hence, it is impossible to find and calibrate the model of the aggregate consumption process immediately.

To overcome this obstacle, suppose there is a process $X_t$ in the financial market such that the dynamics of $X_t$ reflects the dynamics of the aggregate consumption process $c_t$ very well. Formally, there is a bijective map $i$ such that $c_t = i(X_t)$. Moreover, we assume that the data of $X_t$ can be obtained immediately. Then we can use this process $X_t$ to find $L_t$ instead of the aggregate consumption process.

There are several processes that can serve as indicator processes. One candidate is a stock market index process such as the Dow Jones Industrial Average and Standard & Poors (S&P) 500. In the long time interval, the stock market is procyclical with regard to the aggregate consumption process, which means that a stock market index process reflects the dynamics of aggregate consumption process well in the long-term. Data of a stock market index under the risk-neutral measure can be easily obtained from the financial market.

Another candidate is the short interest rate $r_t$. For a short time interval $t \in [0, T]$, it is well known that the short interest rate $r_t$ reflects the dynamics of the aggregate consumption process well. Moreover, $r_t$ has many advantages. We can obtain the data of $r_t$ immediately from the financial market. In addition, there are ample bonds and options whose underlying process is $r_t$ such that it is easily possible to obtain data pertaining to the dynamics of $r_t$ under the risk-neutral measure $Q$. We discuss the case in which the short rate serves as an indicator process in Section 5.2. In addition, a suitable (linear) combination of a short interest rate and stock market index processes can serve as an indicator process.
2.3 Markovian Structure

We assume that the market is efficient in the sense that the dynamics of $X_t$ is determined by information at the current time.

**Assumption 5.** Under the risk-neutral measure $\mathbb{Q}$, the indicator process $X_t$ is a time-homogeneous Markov diffusion process that satisfies the following stochastic differential equation:

$$dX_t = k(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = \xi.$$  

Here, $k(\cdot)$ and $\sigma(\cdot)$ satisfy some mild regularity conditions such that this stochastic differential equation has a strong solution. In addition, we assume that the range of $X_t$ is an open interval, that is, both endpoints are not attainable.

$k(\cdot)$ and $\sigma(\cdot)$ are assumed to be known by Assumption 4. We may assume that $\sigma(\cdot)$ is nonnegative. Otherwise, we replace $dW_t, dB_t, v_t$ and $\theta_t$ by $dW_t := \text{sign}(\sigma(X_t))\,dW_t, dB_t := \text{sign}(\sigma(X_t))\,dB_t, v_t = \text{sign}(\sigma(X_t))v_t$ and $\theta_t = \text{sign}(\sigma(X_t))\theta_t$.

**Assumption 6.** $\sigma(\cdot)$ is nonnegative.

**Assumption 7.** The short interest rate $r_t$ is determined by the indicator process $X_t$. More precisely, there is a positive function $r(\cdot)$ such that $r_t = r(X_t)$. It is assumed that $r(\cdot)$ is known.

Under the seven assumptions, the next section demonstrates how to transform the problem of determining the bounds of the risk premium into a second-order differential equation. We will also describe the properties of positive solutions of the differential equation.

3 Risk Premium

The purpose of this article is to determine upper and lower bounds on the risk premium $\theta_t$ of a risky asset $S_t$. First, we investigate how the risk premium $\theta_t$ is determined with the Markovian pricing kernel. Applying the Ito formula to (2.4), we have

$$dL_t = \left( \beta + \frac{1}{2}(\sigma^2 \phi'' \phi^{-1})(X_t) + (k \phi' \phi^{-1})(X_t) \right) L_t \,dt + (\sigma \phi' \phi^{-1})(X_t) L_t \,dW_t$$

and by (2.3), we know

$$dL_t = r(X_t) L_t \,dt + \theta_t L_t \,dW_t.$$  

By comparing these two equations, we obtain

$$\frac{1}{2}\sigma^2(x)\phi''(x) + k(x)\phi'(x) - r(x)\phi(x) = -\beta \phi(x)$$

and

$$\theta_t = (\sigma \phi' \phi^{-1})(X_t).$$  

*(3.1)*

Define a infinitesimal operator $\mathcal{L}$ by

$$\mathcal{L}\phi(x) = \frac{1}{2}\sigma^2(x)\phi''(x) + k(x)\phi'(x) - r(x)\phi(x).$$

**Theorem 3.1.** Under Assumption 1, let $(\beta, \phi)$ be a principal pair of $X_t$. In this case, $(\beta, \phi)$ satisfies $\mathcal{L}\phi = -\beta \phi$. 

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We also have the following theorem by (2.2) and (3.1).

**Theorem 3.2.** The risk premium is given by

\[ \theta_t = \theta(X_t) \]

where \( \theta(\cdot) := (\sigma \phi' \phi^{-1})(\cdot) \). Thus, \( dB_t = -\theta(X_t) \, dt + dW_t \).

This theorem explains the relation between the risk premium and the pricing kernel \( L_t \).

### 4 Intrinsic Bounds

The purpose of this section is to determine upper and lower bounds on the risk premium \( \theta_t \). If we know the function \( \phi(\cdot) \), then we can determine the risk premium precisely using Theorem 3.2. However, \( \phi(\cdot) \) cannot be observed in the market. The only information about \( \phi(\cdot) \) that we know is that it is a positive solution of \( L\phi = -\beta \phi \) for some positive number \( \beta \). Based on this information, we will determine lower and upper bounds on the risk premium instead of determining the risk premium precisely.

**Proposition 4.1.** Let \( \lambda < \beta \). Then, \( Lh = -\lambda h \) has two linearly independent positive solutions.

For a proof, see Appendix B. In particular, this proposition implies that for \( \lambda < \beta \), the equation \( Lh = -\lambda h \) has a positive solution. Thus,

\[ A_\lambda := \{ h'(\xi) \mid Lh = -\lambda h, h(\xi) = 1, h(\cdot) > 0 \} \]

is nonempty, where \( \xi := X_0 \).

**Notation.** For \( \lambda < \beta \), we define

\[ Z(\lambda) := \sup A_\lambda , \]
\[ z(\lambda) := \inf A_\lambda . \]

The next proposition implies that the supremum and infimum are achieved.

**Proposition 4.2.** For \( \lambda < \beta \), there are two functions \( H_\lambda \) and \( h_\lambda \) such that \( H_\lambda \) and \( h_\lambda \) are positive solutions of \( Lh = -\lambda h \), with \( H_\lambda(\xi) = h_\lambda(\xi) = 1 \) and \( H'_\lambda(\xi) = Z(\lambda), h'_\lambda(\xi) = z(\lambda) \).

See Appendix C for a proof. The two functions \( H_0 \) and \( h_0 \) will play a crucial role in determining the bounds of the risk premium \( \theta_t \). To see this, we need the following proposition.

**Proposition 4.3.** Let \( \lambda < \beta \). We have

\[ (H_\lambda h_\lambda^{-1})(x) \leq (\phi' \phi^{-1})(x) \leq (H'_\lambda H_\lambda^{-1})(x) \]

See Appendix D for a proof. This proposition gives upper and lower bounds on \( (\phi' \phi^{-1})(\cdot) \) when the value of \( \beta \) is given. However, the value of \( \beta \), the discount rate of the representative agent, cannot be observed from the market. The only information that we know about \( \beta \) is that it is positive. Thus, we can conclude that

\[ (h_0' h_0^{-1})(x) \leq (\phi' \phi^{-1})(x) \leq (H'_0 H_0^{-1})(x) . \]

By Assumption 6 and Theorem 3.2 we have the following theorem.

**Theorem 4.1.** (Intrinsic Bounds of Risk Premium)

Let \( \theta_t \) be the risk premium. Then,

\[ (\sigma h_0' h_0^{-1})(X_t) \leq \theta_t \leq (\sigma H'_0 H_0^{-1})(X_t) . \]

This theorem implies that we can determine the range of the risk premium when a risk-neutral measure is given. Upper and lower bounds can then be calculated using option prices.
5 Examples

5.1 Returns of Stock

In this section, we investigate the bounds of the risk premium when the indicator process \( X_t \) is (the log of) the stock price process \( S_t \). In practice, the S&P 500 index process, which can theoretically be regarded as a stock price process, is used as an indicator process \( \text{[2]} \). In this case, we can determine upper and lower bounds on the return of the stock process \( S_t \). Suppose that \( X_t = \ln S_t \) and the interest rate is constant \( r \). Under the risk-neutral measure, the dynamics of \( X_t \) is

\[
dX_t = \left( r - \frac{1}{2} \sigma^2(X_t) \right) dt + \sigma(X_t) dW_t.
\]

By Theorem \( \text{[4.1]} \) the risk premium satisfies

\[
(vh'_0h_0^{-1})(x) \leq \theta(x) \leq (vH'_0H_0^{-1})(x)
\]

where \( h_0 \) and \( H_0 \) are two positive solutions of

\[
\mathcal{L}h = \frac{1}{2} \sigma^2(x) h''(x) + \left( r - \frac{1}{2} \sigma^2(x) \right) h'(x) - rh(x) = 0
\]

from Proposition \( \text{[4.2]} \). We obtain upper and lower bounds on the return \( \mu_t \) by using (3.1).

\[
r + (\sigma^2h'_0h_0^{-1})(\ln S_t) \leq \mu_t \leq r + (\sigma^2H'_0H_0^{-1})(\ln S_t)
\]

Example 5.1. We explore the classical Black-Scholes model for stock price \( S_t \) where \( S_t = e^{X_t} \) and

\[
dX_t = \left( r - \frac{1}{2} v^2 \right) dt + v dW_t, \quad Y_0 = 0
\]

for \( v > 0 \). The infinitesimal operator is

\[
\mathcal{L}h(x) = \frac{1}{2} v^2 h''(x) + \left( r - \frac{1}{2} v^2 \right) h'(x) - rh(x).
\]

We want to find the positive solutions of \( \mathcal{L}h = 0 \) with \( h(0) = 1 \). The solutions are given by

\[
h(x) = ce^{-\frac{2v}{v}x} + (1 - c)e^x \quad \text{for } 0 \leq c \leq 1.
\]

Thus

\[
h_0(x) = e^{-\frac{2v}{v}x}, \quad H_0(x) = e^x.
\]

The risk premium \( \theta_t \) satisfies \(-\frac{2v}{v} \leq \theta_t \leq v\). The upper and lower bounds of the return \( \mu_t \) of \( S_t \) is given by

\[-r \leq \mu_t \leq r + v^2.
\]

5.2 Interest Rates

We explore the risk premium when the indicator process is an interest rate process. In this case, we assume that the risky asset \( S_t \) is a bond such that the Brownian motion of the risky asset price process coincides with that of the interest rate process. The reason that we make this assumption is that in the model setup presented in Section \( \text{[2]} \) the interest rate and risky asset are driven by a common Brownian motion. In a manner similar to that of the previous section, we can obtain upper and lower bounds on the return of the bond \( S_t \).

However, the bounds obtained in this manner are not useful in practice. In contrast with stock prices, the interest rate is usually recurrent under the objective measure. Among all possible values of the risk premium, there is only one risk premium such that the interest rate process is recurrent. Because one can determine the risk premium uniquely in this case, bounds on the risk premium will not be useful. For more details, see \( \text{[6]}, \text{[23]}, \text{[31]} \) and \( \text{[35]} \).
6 Conclusion

This article determined the possible range of the risk premium of the market using the market prices of options. One of the key assumptions to achieve this result is that the market is Markovian. Under this assumption, we can transform the problem of determining the bounds into a second-order differential equation. We then obtain the upper and lower bounds of the risk premium by analyzing the differential equation.

We illuminated how the problem of the risk premium is transformed into a problem described by a second-order differential equation. The risk premium is determined by

$$\theta_t = (\sigma \phi' \phi^{-1})(X_t)$$

with a positive function $\phi(\cdot)$. We demonstrated that $\phi(\cdot)$ satisfies

$$\mathcal{L} \phi = -\beta \phi$$

for some positive number $\beta$, where $\mathcal{L}$ is a second-order operator that is determined by option prices.

We demonstrated that two special solutions, $H_0$ and $h_0$ of $\mathcal{L} h = 0$ in Proposition 1.2 play a crucial role for determining upper and lower bounds on the risk premium. The risk premium $\theta_t$ satisfies

$$(\sigma h_0 h_0^{-1})(X_t) \leq \theta_t \leq (\sigma H_0 H_0^{-1})(X_t).$$

We also discussed how this result can be applied to determine the range of return of an asset.

The following extensions for future research are suggested. First, it would be interesting to extend the process $X_t$ to a multidimensional process. Second, it would be interesting to determine the bounds of the risk premium when the process $X_t$ is a non-Markov process. In this article, we discussed only a time-homogeneous Markov process. Third, it would be interesting to explore more general forms of the pricing kernel. We discussed only the case in which (the reciprocal of) the pricing kernel has the form $e^{\beta t} \phi(X_t)$.

A Bocher Theorem

We begin this section with the Bocher Theorem.

**Theorem A.1.** (Bocher) $u''(x) + p(x)u(x) = 0$

has a positive solution $u$ on a non-trivial interval (possibly unbounded) $I$ if and only if there is a $C^1$ solution $w(\cdot)$ on $I$ of

$$w'(x) + w^2(x) + p(x) \leq 0. $$

Refer to [5] and [14] for more details. The theorem above is useful to check the existence of the positive solution of $\mathcal{L} h = -\lambda h$. A solution $h$ of $\mathcal{L} h = -\lambda h$ can be expressed by $h = u q$, where $q(x) = e^{-\int_0^x \frac{k(y)}{\sigma^2(y)} dy}$ and $u$ is a solution of

$$u''(x) + \left(-\frac{d}{dx} \left( \frac{k(x)}{\sigma^2(x)} \right) - \frac{k^2(x)}{\sigma^4(x)} + \frac{2(-r(x) + \lambda)}{\sigma^2(x)} \right) u(x) = 0. $$

This can be shown by direct calculation. The problem of finding a positive solution $h$ of $\mathcal{L} h = -\lambda h$ can be transformed to the problem of finding a positive solution $u$ of $u''(x) + p(x)u(x) = 0$ for some $p(\cdot)$, to which we can apply the Bocher theorem.
Proof of Proposition 4.1

For convenience, we assume that $X_0 = 0$, the left boundary of the range of $X_t$ is $-\infty$ and the right boundary of the range of $X_t$ is $\infty$.

Proposition B.1. Let $h$ be a positive solution of $Lh = -\lambda h$ and write $h = vq$ where $q(x) := e^{-\int_0^x \frac{k(y)}{\sigma^2(y)} dy}$. A general solution of $Lh = -\lambda h$ is expressed by

$$h(x) \left( c_1 + c_2 \cdot \int_0^x v^{-2}(y) \, dy \right). \quad \text{(B.1)}$$

It can be proven by direct calculation.

Lemma B.1. Let $h$ be a positive solution of $Lh = -\lambda h$ with $h(0) = 1$. Suppose there exists a number $\beta > \lambda$ such that there is a positive solution $\phi$ of $L\phi = -\beta \phi$ with $\phi(0) = 1$. Then,

$$\int_{-\infty}^0 v^{-2}(y) \, dy \text{ is finite if } \phi'(0) \geq h'(0),$$

$$\int_{0}^{\infty} v^{-2}(y) \, dy \text{ is finite if } \phi'(0) \leq h'(0).$$

where $h = vq$ and $q(x) = e^{-\int_0^x \frac{k(y)}{\sigma^2(y)} dy}$.

Proof. Write $\phi = uq$. Assume that $\phi'(0) \geq h'(0)$, equivalently $u'(0) \geq v'(0)$. Define $\Gamma = \frac{u'}{u} - \frac{v'}{v}$. Then

$$\Gamma' = -\Gamma^2 - \frac{2v'}{v} \Gamma - \frac{2(\beta - \lambda)}{\sigma^2}.$$

Because $\Gamma(0) \geq 0$, we have that $\Gamma(x) > 0$ for $x < 0$, since if $\Gamma$ ever gets close to 0, then term $-\frac{2(\beta - \lambda)}{\sigma^2}$ dominates the right hand side of the equation. Choose $x_0$ with $x_0 < 0$. For $x < x_0$, we have

$$\frac{-2v'(x)}{v(x)} = \frac{\Gamma'(x)}{\Gamma(x)} + \frac{\Gamma(x)}{\Gamma(x)} + \frac{2(\beta - \lambda)}{\sigma^2(x)} \cdot \frac{1}{\Gamma(x)}.$$

Integrating from $x_0$ to $x$,

$$-2 \ln \frac{v(x)}{v(x_0)} = \ln \frac{\Gamma(x)}{\Gamma(x_0)} + \int_{x_0}^x \frac{\Gamma(y)}{\sigma^2(y)} \cdot \frac{1}{\Gamma(y)} \, dy$$

which leads to

$$\frac{v^2(x_0)}{v^2(x)} \leq \frac{\Gamma(x)}{\Gamma(x_0)} e^{\int_{x_0}^x \Gamma(y) \, dy}$$

for $x < x_0$. Thus,

$$\int_{-\infty}^{x_0} \frac{1}{v^2(y)} \, dy \leq \text{(constant)} \cdot \int_{-\infty}^{x_0} \Gamma(y) e^{\int_{x_0}^y \Gamma(w) \, dw} \, dy$$

$$= \text{(constant)} \cdot \left( 1 - e^{-\int_{-\infty}^{x_0} \Gamma(w) \, dw} \right)$$

$$\leq \text{(constant)} < \infty.$$

This implies that $\int_{-\infty}^0 v^{-2}(y) \, dy$ is finite. Similarly, we can show that $\int_0^{\infty} v^{-2}(y) \, dy$ is finite if $\phi'(0) \leq h'(0)$. \qed
We now prove Proposition 4.1.

Proof. Recall Appendix A. The existence of positive function $\phi$ is from Assumption 3, which is a positive solution of $L\phi = -\beta \phi$, explains that $Lh = -\lambda h$ has a positive solution $h$ for any $\lambda < \beta$. Thus, $Lh = -\lambda h$ has at least one positive solution.

We now show that there are two positive linearly independent solutions. By Proposition B.1, a general solutions of $Lh = -\lambda h$ is expressed by

$$h(x) = c_1 + c_2 \cdot \int_0^x v^{-2}(y) \, dy.$$  \hfill (B.2)

By Lemma B.1, at least one of $\int_{-\infty}^0 v^{-2}(y) \, dy$ and $\int_0^\infty v^{-2}(y) \, dy$ is finite. Thus we can suitably choose $c_1$ and $c_2 \neq 0$ such that the solution above is positive.

C Proof of Proposition 4.2

For convenience, we assume that $X_0 = 0$, the left boundary of the range of $X_t$ is $-\infty$ and the right boundary of the range of $X_t$ is $\infty$.

Proof. Fix a positive solution $h$ of $Lh = -\lambda h$ with $h(0) = 1$. By Proposition B.1, a general solution has the form

$$h(x) = e^{-\int_0^x \frac{k(y)}{\sigma^2(y)} \, dy} \cdot \int_0^x v^{-2}(y) \, dy,$$

where $q(x) := e^{-\int_0^x \frac{k(y)}{\sigma^2(y)} \, dy}$ and $h = qv$. We showed that at least one of $\int_{-\infty}^0 v^{-2}(y) \, dy$ and $\int_0^\infty v^{-2}(y) \, dy$ is finite in Lemma B.1. We may assume that $B := \int_{-\infty}^0 v^{-2}(y) \, dy$ is finite. A general (normalized to $h(0; c) = 1$) solution is expressed by

$$h(x; c) = h(x) \left(1 - c + c B \cdot \int_{-\infty}^x v^{-2}(y) \, dy\right),$$

which is a positive function for and only for $c$ with

$$0 \leq c \leq 1 \quad \text{if} \quad \int_0^\infty v^{-2}(y) \, dy = \infty,$$

$$-\frac{B}{D} \leq c \leq 1 \quad \text{if} \quad D := \int_0^\infty v^{-2}(y) \, dy < \infty.$$  

Using $h'(0; c) = h'(0) + \frac{c}{B}$, we have that

$$Z(\lambda) = h'(0) + \frac{1}{B} = h'(0; 1).$$

Similarly,

$$z(\lambda) = h'(0) \quad \text{if} \quad \int_0^\infty v^{-2}(y) \, dy = \infty,$$

$$z(\lambda) = h'(0) - \frac{1}{D} = h'(0; -\frac{B}{D}) \quad \text{if} \quad D := \int_0^\infty v^{-2}(y) \, dy < \infty.$$  

This completes the proof.
D \hspace{1em} \textbf{Proof of Proposition 4.3}

For convenience, we assume that $X_0 = 0$, the left boundary of the range of $X_t$ is $-\infty$ and the right boundary of the range of $X_t$ is $\infty$.

\textbf{Lemma D.1.} \textit{Recall the definitions of $H_\lambda$ and $h_\lambda$ in Proposition 4.2}. Let $q(x) := e^{-\int_0^x \frac{k(y)}{\sigma^2(y)} \, dy}$. Write $H_\lambda = V q$ and $h_\lambda = v q$. Then,

$$
\int_{-\infty}^0 V^{-2}(y) \, dy = \infty \quad \text{and} \quad \int_0^{\infty} v^{-2}(y) \, dy = \infty .
$$

\textit{Proof.} We only prove that $\int_{-\infty}^0 V^{-2}(y) \, dy = \infty$. By Proposition B.1 a general (normalized to $k(0; c) = 1$) solution of $L h = -\lambda h$ is expressed by

$$
h(x; c) := H_\lambda(x) \left( 1 + c \cdot \int_0^x V^{-2}(y) \, dy \right) ,
$$

thus we have

$$
h'(0; c) = H'_\lambda(0) + c .
$$

Since $H'_\lambda(0)$ is the maximum value by definition, we have that $c \leq 0$. Suppose that $\int_{-\infty}^0 V^{-2}(y) \, dy < \infty$ then we can choose a small positive number $c$ such that $h(x; c)$ is a positive function. This is a contradiction. \hfill \square

We now prove Proposition 4.3

\textit{Proof.} We only show the inequality of $x < 0$. First, we prove $(h'_\lambda h^{-1}_\lambda)(x) < (\phi'\phi^{-1})(x)$ for $x < 0$. Let $q(x) := e^{-\int_0^x \frac{k(y)}{\sigma^2(y)} \, dy}$. Write $\phi = u q$ and $h_\lambda = v q$. Then we have $\phi'(0) > h'_\lambda(0)$, equivalently $u'(0) > v'(0)$. It is because, if not, by Lemma B.1 we have

$$
\int_0^{\infty} v^{-2}(y) \, dy < \infty ,
$$

and this contradicts to Lemma D.1. Now we define $\gamma = \frac{u'}{u} - \frac{v'}{v}$. Then

$$
\gamma' = -\gamma^2 - \frac{2v'}{v} \gamma - \frac{2(\beta - \lambda)}{\sigma^2} .
$$

Because $\gamma(0) > 0$, we have that $\gamma(x) > 0$ for $x < 0$, since if $\gamma$ ever gets close to 0, then term $-\frac{2(\beta - \lambda)}{\sigma^2}$ dominates the right hand side of the equation. Thus for $x < 0$, it is obtained that $(v'\sigma^{-1})(x) < (u'\sigma^{-1})(x)$, which implies that $(h'_\lambda h^{-1}_\lambda)(x) < (\phi'\phi^{-1})(x)$.

We now prove $(\phi'\phi^{-1})(x) < (H'_\lambda H^{-1}_\lambda)(x)$. Write $H_\lambda = V q$. Then we have $\phi'(0) < H'_\lambda(0)$, equivalently $u'(0) < V'(0)$. Define $\Gamma := \frac{u'}{u} - \frac{V'}{V}$. Then

$$
\Gamma' = -\Gamma^2 - \frac{2V'}{V} \Gamma - \frac{2(\beta - \lambda)}{\sigma^2} .
$$

We claim that $\Gamma(x) < 0$ for $x < 0$. Suppose there exists $x_0 < 0$ such that $\Gamma(x_0) \geq 0$. Then for all $z < x_0$, it is obtained that $\Gamma(z) > 0$ since if $\Gamma$ ever gets close to 0, then term $-\frac{2(\beta - \lambda)}{\sigma^2}$ dominates the right hand side of the equation. For $z < x_0$, we have

$$
-\frac{2V'(z)}{V(z)} = \frac{\Gamma'(z)}{\Gamma(z)} + \Gamma(z) + \frac{2(\beta - \lambda)}{\sigma^2(z)} \cdot \frac{1}{\Gamma(z)} .
$$
Integrating from $x_0$ to $y$, 

$$-2 \ln \frac{V(y)}{V(x_0)} = \ln \frac{\Gamma(y)}{\Gamma(x_0)} + \int_{x_0}^{y} \frac{2(\beta - \lambda)}{\sigma^2(z)} \cdot \frac{1}{\Gamma(z)} \, dz$$

which leads to 

$$\frac{V^2(x_0)}{V^2(y)} \leq \frac{\Gamma(y)}{\Gamma(x_0)} e^{\int_{x_0}^{y} \Gamma(z) \, dz}$$

for $y < x_0$. Thus, 

$$\int_{-\infty}^{x_0} V^{-2}(y) \, dy \leq (\text{constant}) \cdot \int_{-\infty}^{x_0} \Gamma(y) \, e^{\int_{x_0}^{y} \Gamma(w) \, dw} \, dy$$

$$= (\text{constant}) \cdot \left(1 - e^{-\int_{-\infty}^{x_0} \Gamma(w) \, dw}\right)$$

$$\leq (\text{constant}) < \infty.$$ 

This implies that $\int_{-\infty}^{0} V^{-2}(y) \, dy$ is finite. This contradicts to Lemma D.1.

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