Descent equations for superamplitudes

Mathew Bullimore\textsuperscript{\textcopyright} and David Skinner\textsuperscript{\textcopyright}

\textsuperscript{\textcopyright}Rudolf Peierls Centre for Theoretical Physics, 1 Keble Road, Oxford, OX1 3NP, UK
\textsuperscript{\textcopyright}Perimeter Institute for Theoretical Physics, 31 Caroline St., Waterloo, ON, N2L 2Y5, Canada

Abstract

At loop level in planar $\mathcal{N} = 4$ SYM, the dual superconformal symmetry of tree amplitudes is lost. This is true even if one uses a supersymmetry preserving regulator, and even for finite quantities that remain dual conformally invariant. We examine this breaking from the dual point of view of the super Wilson Loop, tracing it to the difference between supersymmetries of the self-dual and of the full theories. We show that the anomaly is controlled by a descent equation that determines the derivative of an $\ell$-loop amplitude in terms of a single non-trivial integral of an $(\ell - 1)$-loop amplitude. We propose that this equation can be used recursively to construct multi-loop amplitudes in a way that makes their transcendentality manifest.
1 Introduction

Tree level scattering amplitudes in $\mathcal{N} = 4$ SYM are invariant under the action of the infinite dimensional Yangian algebra $\mathcal{Y} [\mathfrak{psu}(2, 2|4)]$ \[1\]. This statement is highly constraining: the tree amplitudes are completely determined by this Yangian, together with knowledge of their behaviour in collinear limits \[3, 4\].

At loop level, much of this symmetry is broken. Broken symmetries can still place powerful constraints on the amplitudes, provided the structure of the breaking is understood. For example, the infra-red divergences of scattering amplitudes that violate conformal invariance take a universal exponential form \[5, 6\], and demanding consistency with this structure played a key role in the construction of the Bern-Dixon-Smirnov ansatz \[7\] for all-loop planar MHV amplitudes. This ansatz also provides a particular solution to the anomalous Ward identity for dual special conformal transformations \[8\] that follows from the duality between planar scattering amplitudes and null polygonal Wilson Loops. The Ward identity then states that, once the BDS ansatz is factored out, any remainder must be a dual conformal invariant.

In this paper, we will be concerned with the dual supersymmetry of scattering amplitudes, discovered at tree level in \[1\]. All known $\ell$ loop N$^k$MHV amplitudes in planar $\mathcal{N} = 4$ SYM may be written as

$$ \mathcal{M} = \sum (\text{leading singularity}) \times (\text{bosonic integral}) $$

where the leading singularities are Yangian invariants \[9, 12\] and the bosonic integrals generally require regularization. If the external data for the scattering process is given in terms of momentum supertwistors \[13, 14\], then the superconformal supercharges $Q$ and $\bar{S}$ may be represented by the first order differential operators

$$ Q_{\text{ext}} = \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial \chi_i} \quad \text{and} \quad \bar{S}_{\text{ext}} = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial \chi_i}, $$

acting on this unconstrained data. It is immediately clear that these generators annihilate any $\mathcal{M}$ of the form (1), since they annihilate both the leading singularities and the purely bosonic integrals. However, the conjugate supercharges

$$ \bar{Q}_{\text{ext}} = \sum_{i=1}^{n} \chi_i \frac{\partial}{\partial \mu_i} \quad \text{and} \quad S_{\text{ext}} = \sum_{i=1}^{n} \chi_i \frac{\partial}{\partial \lambda_i} $$

fail to annihilate the loop integrals since they differentiate with respect to the bosonic variables. Thus, as was observed in \[3\], these $\bar{Q}$ and $S$ supersymmetries are broken at the quantum level.

There are a number of reasons why this $\bar{Q}$ anomaly seems particularly puzzling. Firstly, the breaking of $\bar{Q}$ apparently has nothing to do with the fact that loop amplitudes require regularization. Even finite quantities, such as the remainder function or the ratio
\( \mathcal{M}/\mathcal{M}_{\text{MHV}} \) of the superamplitude to the MHV amplitude, fail to be annihilated by (3), despite being fully dual conformally invariant. Furthermore, from the point of view of the duality to Wilson Loops, these dual superconformal charges are just the ordinary superconformal charges of the dual theory. As stressed in [3], we usually expect to be able to find a regularization scheme that preserves Poincaré supersymmetry \( \hat{Q} \) and \( \bar{S} \), rather than the chiral half of the superalgebra consisting of \( Q \) and \( S \) that is preserved here.

Secondly, instead of stripping off the MHV tree amplitude and passing to momentum twistor space, one could equally choose to remove a factor of the MHV tree amplitude and work in dual momentum twistor space (i.e. the dual projective space). Then the rôles would be reversed: for the same reasons as above, one would find that \( Q \) and \( S \) (represented by \( \chi \partial/\partial \bar{Z} \)) fail to annihilate the resulting expression, while \( \bar{Q} \) and \( \bar{S} \) (\( \sim \bar{Z} \partial/\partial \chi \)) would be preserved.

These points strongly suggest that the failure of (3) to annihilate loop amplitudes is strongly tied to the representation of scattering amplitudes in a (dual) chiral superspace. To try to circumvent this, in [15], Caron-Huot constructed a non-chiral extension of the super Wilson Loop that is (dual) supersymmetric\(^1\) (see also [16]). However, while this non-chiral super Wilson Loop is undoubtedly a fascinating object in its own right, it is no longer dual purely to the amplitudes: its \( \bar{\theta}_i \) expansion involves a large number of additional terms that are responsible for restoring \( \bar{Q} \) symmetry, but whose independent meaning is not clear.

In this paper, we show that the \( \bar{Q} \) anomaly can be understood purely within the context of the chiral superloop / superamplitude duality. More precisely, working in the Wilson Loop context and treating \( \bar{Q} \) as a regular supersymmetry, in section 2 we show that the generators (3) correspond to field transformations that are symmetries of the self-dual sector of \( \mathcal{N} = 4 \) SYM only; they do not even preserve the classical action of full \( \mathcal{N} = 4 \) SYM. In section 3 we propose a Ward identity that states that the full (and completely standard) \( \bar{Q} \) transformations are indeed symmetries of the all-orders chiral superloop; the failure of (3) to annihilate the superloop is compensated by the action of the difference

\[
\bar{Q}^{(1)} \equiv \bar{Q}_{\text{full}} - \bar{Q}^{(0)}
\]

between the supercharges in the full and self-dual theories. In section 4 we perform a simple test of this Ward identity, using it to reconstruct (the symbol of) the 1-loop MHV amplitude.

We believe this interpretation of the \( \bar{Q} \) anomaly is very natural from the point of view of the super Wilson Loop. From the point of view of the scattering amplitude however, we will see that the Ward identity mixes different orders in perturbation theory. Thus, although the complete planar S-matrix is fully dual supersymmetric, this is not true of individual \( \ell \)-loop \( N^k \)MHV amplitudes. In more detail, the \( \bar{Q} \) non-invariance of an \( \ell \)-loop \( N^k \)MHV amplitude will be seen to be corrected by a term coming from the \((\ell - 1)\)-loop \( N^{k+1} \)MHV amplitude. (Note that understanding the \( \bar{Q} \) anomaly thus requires the superloop / superamplitude

\(^1\)At least for Poincaré supersymmetry. The superconformal algebra constrains \{\( Q, S \)\} = \( K \), so any \( Q \)-invariant process with a \( K \) anomaly must also be anomalous under \( S \).
duality; it cannot be seen purely within the duality between bosonic Wilson Loops and MHV amplitudes).

In section 5 we investigate the structure of the Ward identity in more detail. We show that we can view $\bar{Q}_{\text{ext}}$ and $\bar{Q}^{(1)}$ as generating a descent procedure that governs the structure of the $\bar{Q}_{\text{ext}}$ anomaly. In this procedure, $k$ and $\ell$ play the roles of ghost number and form degree. The $\bar{Q}^{(1)}$ action can indeed be understood as an operation carried out on a superloop with one extra vertex, taken in a particular collinear limit. The resulting descent equation powerfully constrains the form of multiloop $\mathcal{N}=4$MHV amplitudes: the $\bar{Q}_{\text{ext}}$ variation – and hence the first order derivative – of higher loop amplitudes is determined in terms of a single integral of lower loop ones.

It is quite remarkable that only a single (non-trivial) integral is involved. To increase the loop order by one we usually expect to have to perform a four-dimensional integral over another loop momentum, or Wilson Loop vertex. However, there is much redundancy in this description and in fact all known $\ell$-loop amplitudes in planar $\mathcal{N}=4$ SYM obey an extension of the Kotikov - Lipatov principle [17] for the cusp anomalous dimension, which states that they have transcendentality $2\ell$; i.e., they can be expressed as only $2\ell$-fold iterated integrals of rational functions. The descent equations presented here make this transcendentality manifest.

The fact that the complete S-matrix is invariant under dual supersymmetry while individual amplitudes are not is strongly reminiscent of work of Korchemsky & Sokatchev [3], of Sever & Vieira [18] and of Beisert et al. [4, 19]. These authors give a careful study of the action of the original superconformal generators on the scattering amplitude, and show that certain generators should be ‘corrected’ to ensure the tree amplitude remains invariant even when external states become collinear. Essentially the same corrections account for the violation of superconformal symmetry at loop level, once collinear singularities between external and internal states are considered. Since the dual Poincaré supercharge $\bar{Q}$ considered here coincides [1] with the original superconformal supercharge $\bar{s} \sim \eta \partial / \partial \lambda$ that receives corrections in the collinear limit, one suspects there must be a close connection between the story here and that of [3, 4, 18, 19]. We finish by elucidating this relation in section 6.

\textit{Note added:} While this paper was in preparation, we became aware of the work [20] by Simon Caron-Huot and Song He, which has some overlap with the work presented here.

## 2 Supersymmetries of self-dual $\mathcal{N} = 4$ SYM

In [21, 23] the duality between scattering amplitudes (divided by the MHV tree) and null polygonal Wilson Loops in planar $\mathcal{N} = 4$ SYM was extended beyond the MHV sector to the full superamplitude at the level of the four-dimensional integrand. This was achieved
by considering the correlation function\(^2\)

\[
W[C] \equiv \frac{1}{N} \left\langle \text{Tr } \mathcal{P} \exp \left( i \oint_C A \right) \right\rangle \tag{5}
\]

of the trace of the holonomy in the fundamental representation of a certain connection\(^3\)

\[
A_\alpha(x, \theta) = A_{\alpha\dot{\alpha}}(x, \theta) dx^{\alpha\dot{\alpha}} + \Gamma_{\alpha\dot{\alpha}}(x, \theta) d\theta^{\alpha\dot{\alpha}} \tag{6}
\]

in chiral superspace, whose detailed form will be given later. The super Wilson Loop in \((5)\) is computed around a curve \(C\) that is the lift

\[
x_i(t) = x_i + t(x_{i+1} - x_i) \quad \theta_i(t) = \theta_i + t(\theta_{i+1} - \theta_i) \tag{7}
\]

of the null polygonal contour to chiral superspace, where

\[
x_{i+1} - x_i = \lambda_i \bar{\lambda}_i \quad \text{and} \quad \theta_{i+1} - \theta_i = \lambda_i \eta_i . \tag{8}
\]

Expanding \((5)\) in powers of the fermionic coordinates \(\eta_i\) allows for arbitrary external helicities in the scattering process. The calculation is in fact most easily carried out using a twistor formulation of both the operator and the \(N=4\) action \([24]\), because the twistor contour solves the constraints \((8)\) automatically. In particular, it has been shown that BCFW recursion relations for the scattering amplitude \([12,25]\) follow from the loop equations for this super Wilson Loop \([23]\).

One of the most striking aspects of this duality is that the complete tree-level super-amplitude was found to arise from taking the Wilson Loop correlator \(\langle W \rangle_{sd}\) in self-dual \(N=4\) SYM only. In space-time, this theory is given by the action \([26,27]\)

\[
S_{sd} = \frac{1}{g^2} \int d^4x \text{ Tr} \left( G^{\alpha\beta} F_{\alpha\beta} + 2i \bar{\psi}_{\alpha a} D^{\alpha a} \psi_{\alpha} + \frac{1}{2} D_{\mu} \phi^{ab} D^{\mu} \phi_{ab} + \bar{\psi}_{\alpha} \left[ \phi^{ab}, \bar{\psi}_{\alpha} \right] \right) \tag{9}
\]

where \(G_{\alpha\beta} = G_{\beta\alpha}\) represents an anti self-dual two-form \(G\) in the adjoint representation of the gauge group. \(G\) is an auxiliary field whose equation of motion \(F_{\alpha\beta} = 0\) constrains the anti self-dual part of the Yang-Mills fieldstrength \(F = dA - i[A,A]\) to vanish. Self-dual Yang-Mills is an integrable theory in four dimensions \([28]\) and so has no scattering amplitudes. It is therefore quite remarkable that it nonetheless knows about the complete classical S-matrix of full Yang-Mills via the correlation function \((5)\).

The self-dual action \((9)\) possesses \(N = 4\) superconformal symmetry. In particular, focussing on the Poincaré supersymmetries \(\delta_{sd} = eQ + \bar{e}\bar{Q}\), it is straightforward to check

\(^2\)We use conventions in which the (bosonic) covariant derivative \(D = d - iA\).

\(^3\)Here, \(a = 1, \ldots, 4\) indexes the R-symmetry, while \(\alpha = 0, 1\) and \(\dot{\alpha} = \dot{0}, \dot{1}\) are left and right Weyl spinor indices.
that (9) is invariant under the field transformations [26,29,30]
\[ \delta_{sd} A = -i |e^a| \bar{\psi}_a \]
\[ \delta_{sd} |\psi_a| = D \phi_{ab} |e^b| + i | \bar{e}_a | F^+ \]
\[ \delta_{sd} \phi_{ab} = -i \epsilon_{abcd} (e^c \psi^d) + i (\bar{e}_a \psi_a - [\bar{e}_b \psi_a]) \] (10)
\[ \delta_{sd} |\psi^a| = i G |e^a| + \frac{i}{2} [\phi^{ab}, \phi_{bc}] |e^c| + [\bar{e}_b | D \phi_{ab} \]
\[ \delta_{sd} G_{\alpha\beta} = \epsilon^a_{(\alpha} \left[ \psi^b_{\beta)}, \phi_{ab} \right] + [\bar{e}_a | D_{(\alpha} \psi_{\beta)}^b] \].

These transformations leave the action invariant (up to total derivatives), but as usual represent the supersymmetry algebra only up to the field equations of (9) and field dependent gauge transformations.

From the purposes of this paper, the most important feature of the transformations (10) is that they do not coincide with the field transformations that generate supersymmetries of full (non self-dual) \( \mathcal{N} = 4 \) SYM.

More precisely, if the action of full \( \mathcal{N} = SYM \) is written in Chalmers & Siegel form
\[ S_{full} = S_{sd} + S_{MHV} \] (11)
with \( S_{sd} \) as in (9) and
\[ S_{MHV} = \frac{1}{g^2} \int d^4x \text{ Tr} \left( -\frac{1}{2} G^{\alpha\beta} G_{\alpha\beta} + \psi^{aa} \phi_{ab} \phi_{ba} + \frac{1}{8} [\phi^{ab}, \phi^{cd}] [\phi_{ab}, \phi_{cd}] \right) \], (12)
then the \( Q \) supersymmetries of the full theory are exactly the same as in (10) (terms proportional to \( \epsilon \)). However, for the \( \bar{Q} \) supersymmetries we have
\[ \bar{Q}_{full} = \bar{Q}^{(0)} + \bar{Q}^{(1)} \] (13)
where \( \bar{Q}^{(0)} \) are the transformations of the self-dual theory given in (10) (terms proportional to \( \bar{\epsilon} \)). The difference \( \bar{Q}^{(1)} \) acts trivially on \( \phi, \psi \) and the auxiliary field \( G \), but non-trivially on the gluon and the positive helicity states of the gluino:
\[ \delta^{(1)} A = i |\psi^a| |\bar{e}_a| \]
\[ \delta^{(1)} |\bar{\psi}_a| = -i \left[ \bar{e}_c | [\phi^{cb}, \phi^{ab}] \right]. \] (14)

Since \( S_{MHV} = S_{MHV}[\phi, \psi, G] \), we immediately see that \( \delta^{(1)} S_{MHV} = 0 \). Invariance of the self-dual action under the self dual supersymmetries (\( \delta_{sd} S_{sd} = 0 \)) and of the full action under the full supersymmetries (\( \delta_{full} S_{full} = 0 \)) then implies that
\[ \delta_{sd} S_{MHV} + \delta^{(1)} S_{sd} = 0 \] (15)

This field remains auxiliary in the full theory (11), and is fixed to be the anti self-dual part of the fieldstrength \( G_{\alpha\beta} = F_{\alpha\beta} \). The \( Q \) transformation \( \delta G = \epsilon |\psi, \phi| \) agrees with the standard transformation of \( F_{\alpha\beta} \) upon using the \( \psi \) equation of motion, but note that the \( Q \) transformations in (10) remain symmetries of \( S_{sd} + S_{MHV} \) without the use of field equations.
One may verify that only this combination - rather than the individual terms - vanishes. In other words, self-dual $\mathcal{N} = \text{Yang-Mills}$ is not invariant under the supersymmetries of the full theory, nor is the full theory invariant under the supersymmetries of the self-dual theory.

### 3 Ward Identities for super Wilson Loops

In the rest of the paper, we will show that the difference between the self-dual and full $\bar{Q}$ supersymmetries is responsible for the anomaly in this symmetry for loop amplitudes. The reason the difference between self-dual and full supersymmetries is related to the difference between (dual) supersymmetries of tree and loop amplitudes is that, while the self-dual correlator $W_{\text{sd}}$ corresponds to the tree-level S-matrix, to obtain quantum corrections to the scattering amplitude, we must instead compute the super Wilson Loop correlator in the full theory. In particular, every time one calls upon $S_{\text{MHV}}$ to provide a vertex for diagrams contributing to this full correlator, the loop order of the corresponding amplitude calculation is increased – these are chiral Lagrangian insertions in the language of $[31]$. Calling upon $S_{\text{MHV}}$ a total of $\ell$ times yields a contribution to the superloop corresponding to a piece of the $\ell$-loop scattering amplitude. In twistor space, $S_{\text{MHV}}$ becomes an infinite sum of MHV vertices $[24]$. The axial gauge Feynman diagrams of the twistor Wilson Loop are the planar duals of MHV diagrams for the scattering amplitude $[21]$ while including the effect of these vertices in the loop equations $[23]$ generates the correction to the tree-level BCFW recursion relations, promoting them to the all-loop integrand recursion relation of $[12]$.

Now, when one acts on the external data of a Wilson Loop correlator with the operator

$$\bar{Q}_{\text{ext}} = \sum_i \theta_i \frac{\partial}{\partial x_i} + \eta_i \frac{\partial}{\partial \bar{\lambda}_i} = \sum_i \chi_i \frac{\partial}{\partial \mu_i}$$

(either on chiral superspace-time or in twistor space), it is important to understand which $\bar{Q}$ this corresponds to. The choice is easy: since $\bar{Q}_{\text{ext}}$ annihilates tree amplitudes and since these are computed by the expectation value of the super Wilson Loop in the self-dual theory only, $\bar{Q}_{\text{ext}}$ must act on the fields as the self-dual transformations. For only then do we have the Ward identity

$$\sum_i \chi_i \frac{\partial}{\partial \mu_i} W[C_n] = \frac{1}{N} \left. \left[ \right. \bar{Q}^{(0)} \right| \text{Tr} \left. P \exp \left( i \oint_{C_n} A \right) \right]_{\text{sd}} = 0$$

since it is $\bar{Q}^{(0)}$, and not the full $\bar{Q}$, that generates a symmetry of the self-dual theory.

To check that $\bar{Q}_{\text{ext}}$ really does generate only the self-dual supersymmetries, note that the geometric action $Z^I \partial / \partial Z^I$ of the superconformal group on supertwistor space generates an action on the twistor superfield

$$\mathcal{A}(Z, \chi) = a(Z) + \chi^a \Phi_a(Z) + \frac{1}{2!} \chi^a \chi^b \Phi_{ab}(Z) + \frac{\epsilon_{abcd}}{3!} \chi^a \chi^b \chi^c \Psi^d(Z) + \frac{\epsilon_{abcd}}{4!} \chi^a \chi^b \chi^c \chi^d g(Z)$$

(18)
in the usual way
\[ \delta A = \epsilon^I J Z^I \frac{\partial A}{\partial Z^J} . \] (19)
by Lie derivation along \( V = \epsilon^I J Z^I \partial / \partial Z^J \). These are manifest symmetries of the holomorphic Chern-Simons action \[ S = \frac{1}{g^2} \int_{\mathbb{C}P^3} \text{D}^{3/4} Z \wedge \text{Tr} \left( A \wedge \bar{\partial} A + \frac{2i}{3} A \wedge A \wedge A \wedge A \right) \] (20)
that corresponds to the self-dual action \( (9) \) on twistor space. In particular, the Poincaré \( \bar{Q} \) transformations act on the twistor fields as
\[ \delta a(Z) = 0 \quad \delta \bar{\Psi}_a(Z) = \bar{\epsilon}_a^\ast \frac{\partial a(Z)}{\partial \mu^a} \]
\[ \delta \Phi_{ab}(Z) = \bar{\epsilon}_a^\ast \frac{\partial \Phi_{ab}(Z)}{\partial \mu^a} \quad \delta \Psi^a(Z) = \bar{\epsilon}_b^\ast \frac{\partial \Phi^{ab}(Z)}{\partial \mu^a} \]
\[ \delta g(Z) = \bar{\epsilon}_a^\ast \frac{\partial g(Z)}{\partial \mu^a} \] (21)
so that the lowest component \( a(Z) \) is left invariant. Under the (Abelian)\(^5\) Penrose transform, the self-dual part of the space-time fieldstrength is
\[ F_{\dot{\alpha} \dot{\beta}}(x) = \oint (\lambda d\lambda) \frac{\partial^2 a}{\partial \mu^a \partial \mu^b} \bigg|_{\mu = x\lambda} \] (22)
and so
\[ \delta F_{\dot{\alpha} \dot{\beta}}(x) = \oint (\lambda d\lambda) \frac{\partial^2 \delta a}{\partial \mu^a \partial \mu^b} \bigg|_{\mu = x\lambda} = 0 . \] (23)
This is in agreement with \( (10) \), but is incompatible with \( (14) \). Therefore, the geometric transformation \( (16) \) of the external twistor data indeed generates the field transformations that are supersymmetries of only the self-dual theory.

Now, if we act with the same operator \( (16) \) on the external data of a Wilson Loop correlator in full \( \mathcal{N} = 4 \) SYM – \( i.e., \) including loop corrections to the amplitude – then the Ward identity \( (17) \) receives a correction, becoming
\[ \sum_i \chi^i \frac{\partial}{\partial \mu_i} W[C_n] = \frac{1}{N} \left[ \left[ \bar{Q}^{(0)}, \text{Tr} P \exp \left( i \oint C_n \bar{A} \right) \right] \right] \text{full} \]
\[ = - \frac{1}{N} \left[ \left[ \bar{Q}^{(1)}, \text{Tr} P \exp \left( i \oint C_n \bar{A} \right) \right] \right] \text{full} \] (24)
reflecting the fact that it is \( \bar{Q}_{\text{full}} = \bar{Q}^{(0)} + \bar{Q}^{(1)} \) that generates a symmetry of the full theory. The non-zero right hand side of \( (24) \) measures the failure of the full \( \mathcal{N} = 4 \) action to be invariant under chiral supersymmetry transformations.

\(^5\)The transformations \( (21) \) are the twistor space transformations of the component fields even in the non-Abelian case. The non-linearities in the space-time transformations \( (10) \) arise from non-linearities in the non-Abelian Penrose transform.
Equation (24) is one of the main claims of this paper. It is simply the assertion that the correlator of the super Wilson Loop in the full quantum theory is invariant under $\mathcal{N} = 4$ Poincaré supersymmetry. Equivalently, all-loop scattering amplitudes are exactly invariant under dual Poincaré supersymmetry. However, beyond tree level these dual supersymmetries are not generated by the straightforward action of (16).

In the following sections, we will examine the structure of (24) more closely and test it in a few simple examples. We will see that it provides a straightforward route to compute the symbol of loop level scattering amplitudes directly, without recourse to the integrand. Let us first conclude this section with a few clarifying remarks. Firstly, the fact that loop corrections to the scattering amplitude come only from vertices drawn from $S_{\text{MHV}}$ suggests we rescale the fields so that the action becomes

$$S_{\text{full}} = S_{\text{sd}} + g^2 S_{\text{MHV}},$$

with $S_{\text{sd}}$ and $S_{\text{MHV}}$ now being independent of the coupling constant. The required rescalings are uniquely determined to be

$$A \rightarrow A \quad |\bar{\psi}_a\rangle \rightarrow g^2 |\bar{\psi}_a\rangle \quad \phi_{ab} \rightarrow g \phi_{ab} \quad \langle \psi^a | \rightarrow g^2 \langle \psi^a | \quad G \rightarrow g^2 G.$$

We also rescale $|\theta^a\rangle \rightarrow g^{-\frac{1}{2}} |\theta^a\rangle$ to ensure that the superconnection $A$ itself remains independent of the coupling. With this normalization, which was used in $^{21-23}$, the perturbative expansion of the super Wilson Loop correlator matches that of the amplitude order by order in $g^2$. Having rescaled $\theta$, we also rescale $\bar{\theta} \rightarrow g^2 \bar{\theta}$ so that $x + \theta \bar{\theta}$ is unchanged. If we finally perform a compensating rescaling

$$\langle \epsilon^a | \rightarrow g^{-\frac{1}{2}} \langle \epsilon^a | \quad \text{and} \quad |\bar{\epsilon}_a\rangle \rightarrow g^\frac{1}{2} |\bar{\epsilon}_a\rangle$$

in the parameters of the supersymmetry transformations, we find that the self-dual transformations of equation (10) remain independent of the coupling constant, while the transformations of (14) become proportional to $g^2$:

$$\delta^{(1)}A = ig^2 |\psi^a\rangle \langle \bar{\epsilon}_a|$$

$$\delta^{(1)}|\bar{\psi}_a\rangle = -i g^2 |\bar{\epsilon}_a\rangle \left[ \phi^{cb} , \phi_{ba} \right].$$

With this normalization, the Ward identity (24) becomes

$$\sum_i \chi^i \frac{\partial}{\partial \mu_i} W[C_n] \bigg|_{g^2} = -\frac{1}{N} \left\langle \left[ \bar{Q}^{(1)} , \text{Tr} \exp \left( i \oint C_n A \right) \right] \right\rangle_{g^2}$$

so that dual supersymmetry transformations $\bar{Q}_{\text{full}}$ mix different orders of perturbation theory from the point of view of the amplitude. This is the reason the $\bar{Q}$ anomaly will be useful: the derivative, and hence the symbol$^6$ of higher loop amplitudes may be read off if we understand the $\bar{Q}^{(1)}$ action on lower loop ones.

$^6$See e.g. $^{33,34}$ for an introduction to symbols of transcendental functions.
Secondly, although we have focussed on the anomaly in dual Poincaré supersymmetry, a similar story is true for the dual superconformal symmetry $S_{a\alpha}$. There is again a difference $\delta S_{a\alpha} \equiv S_{\text{full}}^a - S_{\text{sd}}^a$ between the self-dual and full supercharges, and again the self-dual action is invariant only under the self-dual transformations, while the full action is invariant only under the full transformations. An important difference between the Poincaré and conformal supersymmetries is that the Ward identity (24) does not hold for the loop amplitudes, because of collinear / infra-red singularities. Indeed, the superconformal algebra enforces

$$\{ \bar{\epsilon}_a^i, S_{b\beta} \} = \delta^i_a K^{\beta\delta} \quad \text{and} \quad [K^{\beta\delta}, \bar{Q}_{a\alpha}] = \delta^\beta_\alpha S^{\beta a},$$

so any quantity with a $K$ anomaly – such as the scattering amplitude – cannot be invariant even under the action of the full superconformal generator. However, since $\bar{Q}_{\text{full}}$ is a symmetry, the second equation in (30) shows that the superconformal anomaly for $S_{\text{full}}^a$ must be governed by a simple supersymmetry transformation of the anomalous Ward identity for dual conformal transformations [8]. Conversely, quantities that are functions purely of dual conformal cross-ratios (such as the ratio function $R \equiv M/\mathcal{M}_{\text{MHV}} = W_n/(W_n|\chi=0)$ or the ratios

$$\frac{\langle W_n \rangle \langle W_m \rangle}{\langle W_{\text{top}} \rangle \langle W_{\text{bot}} \rangle}$$

of (super) Wilson Loops considered in [35–37] should be fully (dual) superconformal under the action of both $\bar{Q}_{\text{full}}$ and $S_{\text{full}}$.

4 A simple check in the Abelian case

In this section we perform an explicit 1-loop check of the Ward identity (24) for the Abelian theory. We will see that computing the rhs of this Ward identity quickly allows one to deduce the 1-loop MHV amplitude.

In the Abelian case, the only non-trivial $\bar{Q}^{(1)}$ transformation is

$$\delta^{(1)} A = i |\psi^a\rangle |\bar{\epsilon}_a\rangle$$

for the photon. Furthermore, the only appearance of $A$ in the Abelian superconnection is in the same place as for a bosonic connection:

$$A(x, \theta)|_{\text{Abelian}} = A_{\alpha\delta} dx^{\alpha\delta} + \text{terms independent of } A.$$  

Using the facts that the adjoint representation is trivial and that holonomy based at some point $x$ is the same as the Wilson Loop operator, the Abelian superloop varies under $\bar{Q}^{(1)}$ as

$$[\bar{\epsilon} \cdot \bar{Q}^{(1)}, W[C_{\alpha}]] = -\frac{1}{N} \left\langle \int [\bar{e}_a dx] |\psi^a\rangle \exp \left( i \oint A \right) \right\rangle;$$

an insertion of $\Psi$ at some point $x$ on the loop.

Like the MHV amplitude itself, this correlation function diverges and requires regularization. A convenient way to regularize is by framing the loop; that is, we choose a
non-null vector field $v$ normal to $C$ and point split divergent contributions by translation along this vector field (see figure [1]). An important property of this regularization is that it preserves the $Q$ and $S$ supersymmetries of the chiral superloop [23]. It is closely related to the finite ratios of null polygonal Wilson Loops considered by [35–37], since to lowest order in $g^2$, it amounts to computing the cross-correlator

$$\frac{\langle W[C] W[C'] \rangle}{\langle W[C] \rangle \langle W[C'] \rangle}$$

where $C$ is the original null polygon and $C'$ is the polygon obtained by translating $C$ infinitesimally along $v$. Because $v$ is non-null, no vertex of $C$ is null-separated from any vertex of $C'$. We make the convention that $x_i$ label the vertices of $C$, while $x_j$ label the vertices of $C'$.

Since we are only interested in the variation of the 1-loop MHV amplitude, it suffices to compute the correlator (34) only to first non-trivial order in $\theta$, and only using the self-dual theory. To this order the superconnection is simply

$$A(x, \theta) = A + i\theta^a \bar{\psi}_a^\alpha + \mathcal{O}(\theta^2)$$

and so the only possible contribution is from a single fermion exchanged between the two copies $C$ and $C'$ of the loop, as in figure [2]. Inserting the fermion propagator

$$\langle \psi_\alpha^a(x) \bar{\psi}_\beta^b(y) \rangle = i\delta_\beta^a (x-y)_{\alpha\dot{\beta}} (x-y)^4$$

and performing the integrals around both copies of the loop, the Ward identity gives to

$$7$$
Figure 2: In the Abelian theory at order $g^2$, the $\bar{Q}(1)$ variation receives contributions only from a single fermion propagator stretched between the two copies of the framed loop. In [15], Caron-Huot showed that exactly this diagram resides at order $\chi\bar{\chi}$ in the non-chiral extension of the supersymmetric Wilson Loop. Here we have discovered the same object purely within the chiral superloop dual to the superamplitude.

lowest order

$$
\sum_i \bar{\epsilon} \cdot \chi_i \frac{\partial}{\partial \mu_i} W_{\text{framed}}[C_n] = \sum_{i,j} \left\langle \int_{x_i}^{x_{i+1}} [\bar{\epsilon}_a(dx)\psi^a(x)] \int_{x_j}^{x_{j+1}} [\bar{\psi}_b(dy)\theta^b] \right\rangle + (i \leftrightarrow j)
$$

$$
= \sum_{i,j} \int_0^1 ds \int_0^1 dt \frac{[i \bar{\epsilon}_a] \chi_a^b (i|x_{ij}|j)}{(x_{ij} + sx_{i+1} - ty_{j+1})^4} + (i \leftrightarrow j) \quad (38)
$$

$$
= \sum_{i,j} \frac{(i-1, i, i+1, i, j)}{(i-1, i, i+1, i, j)} \chi_a^b \log \frac{x_{ij+1}^2 x_{i+1 j+1}^2}{x_{ij} x_{i+1 j+1}^2} + (i \leftrightarrow j)
$$

where $(i, j, k, l)$ denotes the $\text{SL}(4;\mathbb{C})$-invariant contraction $\varepsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D$. In this equation, $i$ and $j$ run around the two copies of the framed loop and the sum over $(i \leftrightarrow j)$ accounts for the fact that we could perform the $\bar{Q}(1)$ variation on either copy. By construction $x_{ij}^2 \neq 0$ for all pairs of vertices $x_i$ and $x_j$ the summand in (38) is always well-defined even in four dimensions.

The momentum twistor fermions $\chi_i$ may be varied independently, so we immediately deduce that at order $\theta^0$ the 1-loop symbol of this framed Wilson Loop is

$$
\mathbf{S}(W_{\text{framed}}[C_n]|_{g^2}) = \sum_{i,j} X_i \cdot X_{i+1} X_{i+1} \cdot X_j \otimes (i-1, i, i+1, j) + (i \leftrightarrow j)
$$

$$
= \sum_{i,j} X_i \cdot X_j \otimes \frac{(i-1, i, i+1, j-1)(i-2, i-1, i, j)}{(i-1, i, i+1, j)(i-2, i-1, i, j-1)} + (i \leftrightarrow j). \quad (39)
$$

This is the same as the symbol of the 1-loop cross-correlator of two (bosonic) Wilson Loops computed in [36] from a gluon exchanged between the two loops. Since the 1-loop MHV

---

8Here, the twistor line $X_i$ is the line $(i, i-1)$ joining twistors $Z_i$ and $Z_{i-1}$ whereas in [36] $X_i$ denotes the line $(i+1, i)$. 

---
amplitude may be computed by stretching a single gluon across the Wilson Loop, we have verified that the chiral superloop (or scattering amplitude) obeys the (dual) super Ward identity \[^{[24]}\] , at least to 1-loop order in the MHV sector\[^{[9]}\].

We emphasize that we do not expect this statement to strongly depend on the choice of regularization scheme. For example, in dimensional regularization, the four particle 1-loop MHV amplitude is given by \[^{[38]}\]

\[
M_{n=4}^{\text{MHV}} = -\frac{2}{\epsilon^2} \left[ \left( \frac{\mu^2}{-s} \right)^\epsilon + \left( \frac{\mu^2}{-t} \right)^\epsilon \right] - \log^2 \frac{s}{t} + \pi^2 + O(\epsilon), \tag{40}
\]

where \( d = 4 - 2\epsilon > 4. \) Acting with \( \bar{Q}_{\text{ext}} = \sum \chi_i \partial / \partial \mu_i \) and ignoring order \( \epsilon \) corrections gives

\[
[\bar{\epsilon} \cdot \bar{Q}_{\text{ext}}, M_{n=4}^{\text{MHV}}] = -\frac{1}{\epsilon} \left[ \frac{(\bar{\epsilon}_a 234) \chi_i^a}{(234)} + \text{cyclic} \left[ \left( \frac{\mu^2}{-s} \right)^\epsilon + \left( \frac{\mu^2}{-t} \right)^\epsilon \right] \right]. \tag{41}
\]

One readily finds the same result by using the dimensionally regularized fermion propagator

\[
\langle \Psi(x)^a_{\alpha} \bar{\Psi}(y)_{\dot{\alpha}b} \rangle = \delta^a_b \frac{(x - y)_{\alpha\dot{\alpha}}}{((x - y)^2 + i0)^{2-\epsilon}} \tag{42}
\]

in (38).

Finally, as an alternative method, we could have equally computed \([\bar{\epsilon} \cdot \bar{Q}_{\text{ext}}, M_{\text{MHV}}]\) using only the self-dual supersymmetry transformations \( [10] \). The only part of the action that is not invariant under \( \bar{Q}(0) \) is \( S_{\text{MHV}} \), given in (12). In the Abelian case, this MHV action reduces to \( \frac{g^2}{2} \int d^4y \ G_{\beta\gamma} G^{\beta\gamma} \). Therefore, upon integrating by parts in the path integral we have

\[
\sum_i \chi_i \frac{\partial}{\partial \mu_i} W[C_n] = -i \langle [\bar{Q}(0), S_{\text{MHV}}] \exp \left( i \oint_{C_n} A \right) \rangle \tag{43}
\]

\[
= -i g^2 \int d^4y \langle \partial_{\alpha\beta} \psi^a_{\gamma}(y) G^{\beta\gamma}(y) \exp \left( i \oint_{C_n} A \right) \rangle.
\]

A further integration by parts\[^{[10]}\] transfers the \( y \)-derivative to \( G \), producing a term proportional to the Abelian equation of motion for \( G \). The correlation function is then non-vanishing only when this equation of motion is localized to the Wilson Loop contour, and again leaves us with an insertion of \( \psi \) on \( C_n \).

## 5 Descent equations

In the Abelian case, it was straightforward to compute the right hand side \( \langle [\bar{Q}^{(1)}, W] \rangle \) of the Ward identity directly. However, calculating this correlation function was really no
simpler than the original computations of \cite{32,40} for the 1-loop MHV amplitude itself. To understand the anomaly in \( \bar{Q}_{\text{ext}} \) beyond one loop, or beyond the MHV sector, we must consider the non-Abelian theory. A direct computation then becomes even less appealing, both because the variation of the non-Abelian superconnection is more complicated, and because many more diagrams contribute.

To do better, in this section we will reformulate the right hand side of the Ward identity as an operation that may be carried out purely at the level of the external twistor data. The resulting equation may be interpreted as a descent equation that controls the structure of the dual supersymmetry \( \bar{Q} \) anomaly.

In the non-Abelian case, the superconnection \( A \) is determined by the constraints
\[
\lambda^\alpha \lambda^\beta \left[ D_{\alpha \dot{\alpha}} , D_{\beta \dot{\beta}} \right] = 0 \\
\lambda^\alpha \lambda^\beta \left\{ D_{\alpha a} , D_{\beta b} \right\} = 0 
\]
that state that \( A \) is integrable along super null rays \( \approx \mathbb{R}^{14} \). Up to order \( \theta^4 \), these constraints are solved by \cite{21,30}
\[
A(x, \theta) = A(x) + i \langle \theta^a \rangle [\bar{\psi}_a + \frac{1}{2} \langle \theta^a \rangle D\phi_{ab} - \frac{1}{3!} \epsilon_{abcd} \langle \theta^a \rangle \langle \theta^b \rangle D\langle \theta^c \psi^d \rangle] \\
+ \frac{1}{4!} \epsilon_{abcd} \langle \theta^a \rangle \langle \theta^b \rangle D\langle \theta^c \rangle G\langle \theta^d \rangle + \cdots
\]
\[
|\Gamma_a(x, \theta)\rangle = \frac{i}{2} \phi_{ab} |\theta^b\rangle - \frac{1}{3} \epsilon_{abcd} \langle \theta^b \rangle \langle \theta^c \psi^d \rangle + \frac{i}{8} \epsilon_{abcd} \langle \theta^b \rangle \langle \theta^c \rangle G\langle \theta^d \rangle + \cdots .
\]
Note that the fermionic component of the superconnection \( |\Gamma_a\rangle \) depends only on the component fields \( \{ \phi, \psi, G \} \) and so is unaffected by the non-Abelian \( \bar{Q}^{(1)} \) transformation
\[
\delta^{(1)} A = i \langle \psi^a \rangle [\bar{\epsilon}_a] \\
\delta^{(1)} |\bar{\psi}_a\rangle = -\frac{i}{2} [\bar{\epsilon}_c] [\phi^{cb}, \phi_{ba}] .
\]
Therefore, in the non-Abelian case we find that
\[
\delta^{(1)} \left[ \text{Tr P exp} \left( i \oint_{C_n} A \right) \right] = i \text{Tr} \left( \oint \delta^{(1)} A(x, \theta) \text{Hol}(x, \theta)[\bar{A}; C_n] \right)
\]
where
\[
\delta^{(1)} A = \left( i \langle \psi^a \rangle [\bar{\epsilon}_a] + \frac{1}{2} \langle \theta^a \rangle [\bar{\epsilon}_c] \left( \phi^{cb}, \phi_{ba} \right) + \frac{1}{2} \langle \theta^a \rangle [\bar{\epsilon}_c] \left( [\langle \theta^b \psi^c \rangle, \phi_{ab}] + \cdots \right) \right) dx \\
+ \epsilon_{abcd} \left( -\frac{1}{3!} \langle \theta^a \rangle [\bar{\epsilon}_c] \left( [\langle \theta^b \psi^c \rangle, \langle \theta^c \psi^d \rangle] + \frac{1}{4!} \langle \theta^a \rangle [\bar{\epsilon}_c] \left( [\langle \theta^b \psi^c \rangle, \langle \theta^c \rangle G\langle \theta^d \rangle] + \cdots \right) \right) dx .
\]
As at the end of the previous section, an alternative way to arrive at the same result is again to note that
\[
\sum_i \chi_i \frac{\partial}{\partial \mu_i} W = \left\langle [\bar{\epsilon} \cdot \bar{Q}^{(0)}, W[C_n]] \right\rangle = -i \left\langle [\bar{\epsilon} \cdot \bar{Q}^{(0)}, S_{\text{MHV}}] W[C_n] \right\rangle .
\]
In the non-Abelian case, the $\bar{Q}^{(0)}$ variation of $S_{\text{MHV}}$ is, schematically,
\[
\delta^{(0)} S_{\text{MHV}} \sim \int d^4 y \, \text{Tr} \left( \psi^a \bar{\epsilon}_a \times [\text{eom for } G] + \bar{\epsilon}_c \left[ \phi^{cb}, \phi_{ba} \right] \times [\text{eom for } \psi^a] \right). \tag{50}
\]
This insertion of the $G$ and $\psi^a$ equations of motion would vanish in the absence of any further operator insertions, but fails to vanish because of the Wilson Loop, where it becomes localized. Since $G$ is conjugate to $A$ and $\psi^a$ is conjugate to $\bar{\psi}^a$, the net effect is to insert a copy of the superconnection $A$ on the Wilson Loop contour, where every occurrence of $A$ and $\bar{\psi}^a$ is replaced by $\psi^a$ and $[\phi, \phi]$, respectively. This is equivalent to an insertion of the $\bar{Q}^{(1)}$ transformation (48).

Inserting the expansion (48) into (47) and directly computing the resulting correlator is clearly not the way to proceed. However, such Wilson loop correlators with operator insertions integrated along edges often arise from deformations of the contour, for example, collinear limits play an important role in the Wilson loop OPE. We wish to show that, instead of involving an unknown correlator, the super Ward identity (24) can be reformulated as
\[
\sum_i \bar{\epsilon}_i \cdot \chi_i \frac{\partial}{\partial \mu_i} W[C_n] = g^2 \sum_i \int_{[0, \infty] \times S^1} V \dot{D}^{(3)} \mathcal{Z}_{n+1}(\ldots, i, \mathcal{Z}, i+1, \ldots), \tag{51}
\]
in terms of an $(n+1)$-point superloop. Here, $V = \bar{\epsilon}_a \chi^a \partial/\partial \mu^a$ is the vector field on twistor space that generates the usual $\bar{Q}^{(0)}$ transformations, while $\dot{D}^{(3)} \mathcal{Z}$ is the standard holomorphic measure on the Calabi-Yau superspace [32] so that
\[
V \dot{D}^{(3)} \mathcal{Z} = (\bar{\epsilon}_a, \mathcal{Z}, d\mathcal{Z}, d\mathcal{Z}) d^4 \chi \chi^a. \tag{52}
\]
The Grassmann integral should be performed with the new $\chi$ treated as independent of the other $\chi_i$s. Bosonically, the additional twistor is constrained to lie in the plane $(i-1, i, i+1)$ and so may be parametrized as
\[
Z = Z_i + p(Z_{i-1} + qZ_{i+1}) \tag{53}
\]
whereupon the bosonic part of the measure becomes $(\bar{\epsilon}_a, i-1, i, i+1) dq dp d\mu$. With this parametrization, the contour extracts the residue of the integrand at $p = 0$, and integrates $q$ from 0 to $\infty$ [11]. This contour ensures that the line $(i \mathcal{Z})$ corresponds to the insertion point of $\delta^{(1)} A$ in space-time; see figure 3.

11 The coordinate $q$ is related to the space-time parametrization $x(t) = x_i + t(x_{i+1} - x_i)$ by
\[
t = \frac{q(i+1 \bar{i})}{(i-1 \bar{i}) - q(i+1 \bar{i})}.
\]
In the Lorentzian case, $Z$ should lie in the intersection of $(i-1, i, i+1)$ with $\mathbb{P}^3 := \{ Z \in \mathbb{CP}^3 | Z \cdot \bar{Z} = 0 \}$, where the dot implies the SU(2, 2) metric appropriate for the Lorentzian conformal group. In this case the $q$ contour should be $0 \leq |q| \leq \infty$ in the direction $\text{arg}(q) = \frac{1}{2} \log (\frac{Z_{i-1} \bar{Z}_{i+1} / Z_{i+1} \bar{Z}_{i-1} \cdot Z_{i-1}})$. 

15
Figure 3: The \((n+1)\)-point superloop is integrated over a contour that fixes \(Z \rightarrow Z_i\) and also causes the line \(X\) to move in the plane \((i-1, i, i+1)\) between the lines \((i-1, i)\) and \((i, i+1)\). This corresponds to a point \(x\) that is integrated along the edge of the space-time Wilson Loop between \(x_i\) and \(x_{i+1}\).

The importance of equation [51] is that it provides us with a representation of the action of \(\bar{Q}^{(1)}\) on the external data. We have written [51] in the coupling constant normalization [26] adapted to agree with perturbation theory of the amplitude, making it clear that the formula is recursive; \(W_{n+1}\) only needs to be known to order \(g^{2\ell-2}\) in order to know the left hand side to order \(g^{2\ell}\). More precisely, the differential of the Wilson Loop on the left hand side lowers its transcendentality by one, while on the right the integral \(dq\) over the contour with boundary increases the transcendentality by one (both the Cauchy pole and the Grassmann integration preserve transcendentality, as they may be performed on the rational Yangian invariants - leading singularities - in front of the loop integrals). In this way the fact that \(\ell\)-loop amplitudes have transcendentality only \(2\ell\), in accordance with the Kotikov-Lipatov principle [17], is made manifest.

On the other hand, the Grassmann degree of the left hand side is increased by one, while that of the right is decreased by three. Consequently, contributions to the \(\bar{Q}^{(0)}\) variation of an \(N^k\)MHV amplitude are compensated by the \(\bar{Q}^{(1)}\) transformation of an \(N^{k+1}\)MHV amplitude.

Putting these observations together and recalling that

\[ (\bar{Q}^{(0)})^2 = (\bar{Q}^{(1)})^2 = 0 \quad \text{and} \quad \bar{Q}^{(0)} \bar{Q}^{(1)} + \bar{Q}^{(1)} \bar{Q}^{(0)} = 0 \, , \]

we see that we can view the Ward identity [24] or [51] as a descent equation that governs the anomaly in \(\sum_i \lambda_i \partial / \partial \mu_i\). Following the usual argument, since \(\bar{Q}^{(0)} \mathcal{M}^{\text{tree}} = 0\) we have

\[ \bar{Q}^{(0)} \bar{Q}^{(1)} \mathcal{M}^{\text{tree}} = -\bar{Q}^{(1)} \bar{Q}^{(0)} \mathcal{M}^{\text{tree}} = 0 \, , \]

so that

\[ \bar{Q}^{(1)} \mathcal{M}^{\text{tree}} = \bar{Q}^{(0)} \tilde{M} \text{ for some } \tilde{M}, \text{ which in [51] is identified as the 1-loop amplitude } \tilde{M} = \mathcal{M}^{1\text{-loop}}. \]

Continuing the descent procedure generates (derivatives of) higher loop amplitudes.

\[ \footnote{\text{We assume that } \bar{Q}^{(0)} \text{ has trivial cohomology, at least at MHV level } k + \frac{1}{4} \text{ with } k \in \mathbb{Z}.} \]
Before demonstrating that (51) is equivalent to (47), let us first gain some familiarity with it by recovering the result of section 4. Up to order $g^2$ and $\theta$, the descent equation gives

$$
\sum_i \bar{e}_i \chi_i \frac{\partial}{\partial \mu_i} W_{\text{MHV}}^{1-\text{loop}}(1, \ldots, n) = \sum_i \int V \cdot D^{3|4} Z W_{\text{NMHV}}^{\text{tree}}(i, Z, i+1, \ldots) .
$$

(56)

Using the momentum twistor MHV expression

$$W_{\text{NMHV}}^{\text{tree}} = \sum_{k<j} [*, k, k+1, j, j+1]
$$

of the NMHV tree amplitude [41] (corresponding to a single twistor superpropagator stretched across the twistor superloop [21]), we note that the only $R$-invariants in this sum that have a pole as $Z \rightarrow Z_i$ are $[*, i, Z, j, j+1]$ for some $j$. We thus find

$$
\sum_i \int V \cdot D^{3|4} Z W_{\text{NMHV}}^{\text{tree}}(\ldots, i, Z, i+1, \ldots) = \sum_{i,j} \int V \cdot D^{3|4} Z [*, i, Z, j, j+1]
$$

$$=
\sum_{i,j} \int (\bar{e}_a, Z, dZ, dZ)(j, j+1, *, i) \chi_a^b(Z, j, j+1, *) + \chi_a^b(j+1, *, i, Z) + \chi_a^{i+1}(*, i, Z, j)
$$

$$= \sum_{i,j} \int dq \ (\bar{e}_a, i-1, i, i+1)
$$

$$\frac{\chi^a_b(i, j, j+1, *)(j, j+1, *, i)}{(Z(s), j, j+1, *)(*, i, Z(s), j)} + \frac{\chi^a_{i+1}(i, j, j+1, *)(j, j+1, *, i)}{(Z(s), j, j+1, *)(j+1, *, i, Z(s))}
$$

(58)

where in going to the second line we have performed the Grassmann integral and in going to the third we used the explicit parametrization and performed the $p$ contour integral. (In doing this, note that the $\chi_i$ term has a double pole at $p = 0$, but no residue.) Collecting terms proportional to $\chi_j$ gives

$$
\sum_{i,j} \int dq \ (\bar{e}_a, i-1, i, i+1)
$$

$$\frac{(i, j-1, j, j+1)}{(i, Z(s), j, j+1) (i, Z(s), j-1, j)} \chi_j^a = \sum_{i,j} \frac{(\bar{e}_a, i-1, i, i+1)}{(i, j-1, i, i+1)} \log \frac{X_i Y_{j+1} X_{i+1} Y_j}{X_i Y_j X_{i+1} Y_{j+1}}
$$

(59)

in agreement with (38).\(^{13}\)

We now turn to relating our descent equation to the space-time correlator (47). To do so, we must recall a few facts about the twistor space formulation of the superloop. In [21,23] it was shown that the superloop could be defined in twistor space as a product of holomorphic frames around the nodal curve $(Z_1 Z_2) \cup (Z_2 Z_3) \cup \cdots \cup (Z_n Z_1)$ that corresponds to the null polygon $C_n$ in space-time. (We will also call this twistor curve $C_n$.)

\(^{13}\)We assume for simplicity that the reference supertwistor $Z_* = (Z_*, 0)$; any non-vanishing $\chi_i$ may be verified to cancel around the sum.

\(^{14}\)Here we have been a little cavalier with the regularization. One may check that the exact expression (38), including the $i \leftrightarrow j$ term, is reproduced from the descent equations for the framed superloop (55).
frame $h(x, \theta; \lambda)$ is a smooth gauge transformation that defines a holomorphic trivialization of the twistor gauge bundle over the line $X$; i.e., $h^{-1} \circ (\bar{\partial} - i\mathcal{A})|_X \circ h = \bar{\partial}|_X$, so that $h$ obeys

$$\langle \bar{\partial} - i\mathcal{A} \rangle|_X h = 0 \quad (60)$$

where $\bar{\partial}|_X$ is the $\bar{\partial}$-operator with respect to $\lambda$.

For a given $\mathcal{A}$, equation (60) uniquely defines the frame only up to gauge transformations $h(x, \theta; \lambda) \to h(x, \theta; \lambda)g(x, \theta)$, where $g$ must be globally holomorphic in, and hence independent of, $\lambda$. In particular, we can use this freedom to pick a frame

$$U_X(Z, Z_i) := h(x, \theta; \lambda)h^{-1}(x, \theta; \lambda_i) \quad (61)$$

that is normalised to be the identity at a some point $Z_i$ on the line $X$. The twistor space Wilson Loop used in [21,23] is then the product

$$W[C_n] = \frac{1}{N} \langle \text{Tr} \left( \cdots U_{X,i+1}(Z_{i+1}, Z_i)U_{X,i+1}(Z_i, Z_{i-1}) \cdots \right) \rangle \quad (62)$$

of these holomorphic frames around $C_n$. This product computes a complex analogue of the trace of the holonomy of the partial, or (0,1), connection $\bar{\partial} - i\mathcal{A}$ around the holomorphic curve in twistor space.

The space-time superconnection may be recovered from these holomorphic frames in the standard way [21,42]. In particular, to recover $\langle \Gamma^a(x, \theta) \rangle$ one first shows that, although $h$ itself depends smoothly on $\lambda$, the combination $\langle \lambda \partial^a h^{-1} \rangle$ is in fact globally holomorphic. Since it clearly has homogeneity +1, Liouville’s theorem implies that it must be linear, so that

$$\lambda^a \frac{\partial h^{-1}}{\partial \theta^{\alpha a}} h = i\lambda^a \Gamma^a(x, \theta) \quad (63)$$

for some field $\Gamma^a$ that depends only on space-time. If we use this field to define a fermionic covariant derivative $\langle \lambda D_a \rangle = \langle \lambda | \partial_a - i\Gamma_a \rangle$ projected along $|\lambda\rangle$, then (63) immediately implies that $D_{\alpha a}$ satisfies the integrability constraint in (44), from which the full $\Gamma^a$ may be reconstructed. In addition, multiplying both (63) and its bosonic counterpart on the right by $h^{-1}$, we see that $h^{-1}$ also obeys the defining equation for the super null Wilson Line in space-time. They must thus agree, up to a gauge transformation.

In fact, by pairing the holomorphic frames differently around the curve as

$$U_{X,i+1}(Z_{i+1}, Z_i) U_X(Z_i, Z_{i-1}) \cdots h(x_{i+1}; \lambda_{i+1}) h^{-1}(x_{i+1}, \lambda_{i+1}) h(x_i, \lambda_i) h^{-1}(x_i, \lambda_{i-1}) h(x_{i-1}, \lambda_{i-1}) \cdots, \quad (64)$$

both the twistor and space-time superloops may be exhibited simultaneously. If we use
this observation in [47] we find that

\[
\delta^{(1)} W = \frac{i}{N} \oint_{C_n} dx^a \delta^{(1)} A \left[ \cdots \exp \left( i \int_{x}^{x_{i+1}} A \right) \delta^{(1)} A \left( x \right) \exp \left( i \int_{x_{i}}^{x} A \right) \cdots \right] \\
= \frac{i}{N} \oint_{C_n} dx^a \delta^{(1)} A \left[ \cdots h^{-1}(x_{i+1}, \lambda_i) h(x, \lambda_i) \delta^{(1)} A \left( x \right) h^{-1}(x, \lambda_i) h(x, \lambda_i) \cdots \right] \\
= \frac{i}{N} \oint_{C_n} dx^a \delta^{(1)} A \left[ \cdots U_{X+1}(Z_{i+1}, Z_i) h(x, \lambda_i) \delta^{(1)} A \left( x \right) h^{-1}(x, \lambda_i) U_X(Z_i, Z_{i-1}) \cdots \right]
\]

so that all the holomorphic frames except those immediately adjacent to the insertion may be paired into Us.

Let us now relate this to the recursive formula (51). Writing the \((n+1)\)-point superloop in terms of the expectation value of the product

\[
\cdots U(Z_{i+1}, Z) U(Z, Z_i) U(Z_i, Z_{i-1}) \cdots
\]

of holomorphic frames, we must carry out the integral over the additional supertwistor \(Z\). The only piece that depends on \(Z\) is \(U(i+1, Z) U(Z, i)\) and a little thought shows that the only contributions to the contour integral come from the Grassmann integrals acting on \(U(Z, i)\), so that field insertions get trapped between \(Z_i\) and \(Z\) as we take the residue where \(Z \to Z_i\).

Focusing on the Poincaré supersymmetry \(\bar{e}^a \tilde{Q}^a\), the bosonic measure in (52) becomes

\[
(\bar{e}_a, Z, dZ, dZ) = [\bar{e}_a d\mu \langle \lambda d\lambda \rangle] = [\bar{e}_a d\lambda \langle \lambda d\lambda \rangle].
\]

To compute the effect of the Grassmann integration on \(U(Z, i)\) we must replace the Grassmann integrals \(d\chi \equiv \partial/\partial \chi\) by an operation on the \(\theta\)s, because \(U_X(Z, i)\) depends on these fermions only through its dependence on the line \((x, \theta)\). If we pull back a function \(f(Z)\) on super twistor space to the spin bundle by setting \([\mu] = x|\lambda\) and \(\chi = \theta|\lambda\), then

\[
\left( \frac{\partial f}{\partial \chi^a} \right)_{\chi = \theta|\lambda} = \frac{1}{\langle \rho|\lambda \rangle} \rho^a \frac{\partial f}{\partial \theta^a}
\]

for an arbitrary reference spinor \(|\rho\rangle\) that defines a choice of lift of \(\partial/\partial \chi\) to the spin bundle. By definition \(\theta = (\chi|\lambda - \chi|\lambda)/\langle \lambda i\rangle\), so to ensure we do not pick up contributions from \(\chi_i = \theta|i\), we must choose the lift \(|\rho\rangle = |i\rangle\), since \(\langle i \partial/\partial \theta \rangle \chi_i = \langle i \partial/\partial \theta \rangle \theta|i\rangle = 0\).

From the defining equation (60) we see that

\[
0 = \frac{\partial}{\partial \theta^a} \left[ (\bar{\partial} - iA)|_X U_X(Z, Z_i) \right] \\
= (\bar{\partial} - iA)|_X U_X(Z, Z_i) - i|\lambda \rangle \frac{\partial A}{\partial \chi^a} U_X(Z, Z_i),
\]

where we understand that the \((\bar{\partial} - iA)\)-operator is always pulled back to \(X\), so that in particular \(A\) depends on \(\theta\) only through \(\chi = \theta|\lambda\). Given that \(U\) solves (60), this is solved
in terms of the integral

$$|\partial_a U_X(Z, Z_i)\rangle = i \int_X \langle \lambda' d\lambda \rangle \langle \lambda \rangle \frac{\partial A}{\partial \chi^a} U_X(Z, Z') |\lambda' \rangle U_X(Z', Z_i)$$

(70)

over another point $Z' \in X$.

The fermionic measure $d^4 \chi^a$ means that we never get contributions from the term of order $\chi^4$ in the holomorphic frame $U_X(Z, Z_i)$. However, it is convenient to temporarily include such contributions and then later project them out. Thus, combining equations (67), (68) and (70) we consider the following expression involving three fermionic derivatives

$$\int (\bar{\epsilon}_a, Z, dZ, dZ) \frac{\varepsilon^{abcd}}{3!} \frac{\partial^3}{\partial \chi^b \partial \chi^c \partial \chi^d} W(\ldots, i, Z, i+1, \ldots)$$

$$= -\frac{i}{3! N} \int [\bar{\epsilon}_a d\chi / \lambda'] \langle \lambda d\lambda \rangle \text{Tr} \left[ \cdots \varepsilon^{abcd} (\langle i \partial_0 / \lambda \rangle \langle i \partial_0 / \lambda \rangle) / (\langle i \lambda \rangle)^2 \int_X \langle \lambda' d\lambda' \rangle \langle \lambda \lambda' \rangle \frac{\partial A}{\partial \chi^a} U_X(Z, Z') \frac{\partial A}{\partial \chi^a} U_X(Z', Z_i) \cdots \right]$$

$$= -\frac{i}{3! N} \int_{x_i}^{x_{i+1}} dt [\bar{\epsilon}_a] \langle \lambda d\lambda \rangle \text{Tr} \left[ \cdots \varepsilon^{abcd} (\langle i \partial_0 / \lambda \rangle \langle i \partial_0 / \lambda \rangle) / (\langle i \lambda \rangle)^2 \int_X \langle \lambda' d\lambda' \rangle \langle \lambda \lambda' \rangle \frac{\partial A}{\partial \chi^a} U_X(Z, Z') \frac{\partial A}{\partial \chi^a} U_X(Z', Z_i) \cdots \right]$$

(71)

where in going to the last line we used the fact that $dx = |i| |i| \text{dt}$ on the $i$th edge. It is now straightforward to perform the contour integral setting $\langle i \lambda \rangle = 0$, which leaves us with

$$-\frac{i}{N} \int_{x_i}^{x_{i+1}} dt [\bar{\epsilon}_a] \langle \lambda d\lambda \rangle \text{Tr} \left[ \cdots \varepsilon^{abcd} (\langle i \partial_0 / \lambda \rangle \langle i \partial_0 / \lambda \rangle) / (\langle i \lambda \rangle)^2 \int_X \langle \lambda' d\lambda' \rangle \langle \lambda \lambda' \rangle \frac{\partial A}{\partial \chi^a} U_X(Z, Z') \frac{\partial A}{\partial \chi^a} U_X(Z', Z_i) \cdots \right]$$

(72)

as a residue.

We can make sense of this expression if we note that, since $U(Z, Z_i)$ is normalized to be the identity when $Z = Z_i$, if we evaluate the equation

$$\langle \lambda \Gamma_a(x, \theta) \rangle = \langle \lambda \partial_a U_X^{-1}(Z, Z_i) \rangle U_X(Z, Z_i)$$

(73)

at $Z = Z_i$, the space-time connection $|\Gamma_a\rangle$ defined in the gauge specified by these normalized holomorphic frames must obey $\langle i \Gamma_a(x, \theta) \rangle = 0$. Hence in this gauge we have

$$|\Gamma_a(x, \theta) \rangle = |i\rangle \gamma_a(x, \theta; \lambda_i)$$

(74)

where $\gamma_a$ is a fermionic Lorentz scalar that depends smoothly on $\lambda_i$ (the data of the gauge choice) as well as on $(x, \theta)$. Using (70) in (73) shows that

$$\gamma_a(x, \theta; \lambda_i) = \int_X \langle \lambda d\lambda \rangle \langle i \lambda \rangle \frac{\partial A}{\partial \chi^a} U_X(Z_i, Z') \frac{\partial A}{\partial \chi^a} U_X(Z', Z_i)$$

(75)

exactly as appears in (72).

The existence of a gauge in which $|\Gamma_a\rangle = |\lambda\rangle \gamma_a$ has a remarkable consequence that was also exploited in [43]. From the integrability conditions (44), the only non-vanishing part of the fermionic supercurvature is

$$\mathcal{W}_{ab} = i c^{ab} \{ D_{aa}, D_{bb} \} = \partial^a_{[a} \Gamma_{b]} - i \{ \Gamma^a_{a}, \Gamma_{ab} \}$$

(76)
but if $|\Gamma_a| \propto |i|$ then the final anticommutator vanishes, even in the non-Abelian case. Furthermore, in this gauge we have

$$W_{cd} = \partial^a [\Gamma_d^a] = \langle i \partial^a | \gamma_{cd} \rangle$$

$$\lambda^a_i D_{ab} W_{cd} = \langle i \partial_b | (i \partial_c) \gamma_{cd} \rangle .$$

(77)

Using these together with (75) in (72), we see that

$$\frac{\varepsilon_{abcd}}{3!} \int (\bar{\epsilon}_a, Z, dZ, dZ) \frac{\partial^3}{\partial X^b \partial X^c \partial X^d} W(\cdots, i, Z, i+1, \cdots)$$

$$= -\frac{i}{3! N} \int_{x_i} dx^a \alpha \langle \text{Tr} \left[ \cdots U_{X_{i+1}}(Z_{i+1}, Z_i) \bar{\epsilon}_{ab} D_{ab} W_{cd} U_{X_i}(Z_i, Z_{i-1}) \cdots \right] \rangle$$

(78)

in terms of the covariant derivative of the fermionic supercurvature.

At present, the fields in (78) are expressed using a gauge that is natural on twistor space, but somewhat obscure on space-time. To obtain an expression purely in terms of space-time fields, we should transform to the gauge (\bar{\chi}) \varepsilon which coincides with the \bar{\chi} in the expression in (71) is chosen so as to put the fields in this gauge. Then since $U_X(Z, Z_i) = h(x, \theta; \lambda) h^{-1}(x, \theta; \lambda)$, the definition (73) of the space-time superconnection shows that if we replace $U_X(Z, Z_i) \to U_X(Z, Z_i) h^{-1}(x, \theta; \lambda) = h(x, \theta; \lambda)$, then the adjoint-valued derivative of the supercurvature in (77) transforms as

$$D_{ab} W_{cd} \to h(x, \theta, \lambda) D_{ab} W_{cd} h^{-1}(x, \theta; \lambda) .$$

(79)

Comparing equations (78) & (65), we have shown that, once transformed to space-time, the expression in (71) is

$$\frac{i}{N} \left\langle \text{Tr} \left( \oint \frac{\varepsilon_{abcd}}{3!} [\bar{\epsilon}_a \partial x |D_b W_{cd}] \text{Hol}_{\langle x, \theta | [A_n]} \right) \right\rangle .$$

(80)

Before identifying this insertion with the variation $\delta^{(1)} \Lambda(x, \theta)$ given in (48), there is one final subtlety to address. In replacing the Grassmann integral $\int (d^4 \chi |\bar{\epsilon}_a | x^a \cdots)$ by the derivative $[\bar{\epsilon}_a (\partial^a / \partial \chi)^a (\cdots)$, we should really set $\chi = 0$ after taking the three derivatives. This merely expresses the fact that, due to the explicit $\chi$ in the measure, the superloop itself can only be expanded to order $(\chi)^3$. From the superconnection (45) we find that

$$W_{ab}(x, \theta) = i \phi_{ab} - \varepsilon_{abcd} \langle \theta^c \psi^d \rangle + \frac{1}{2} \varepsilon_{abcd} \langle \theta^c | G | \theta^d \rangle + \frac{i}{4} \left[ \phi_{ac}, \phi_{bd} \right] \langle \theta^c \theta^d \rangle + \cdots$$

and

$$\frac{\varepsilon_{abcd}}{3!} |D_b W_{cd} \rangle = i \langle \theta^a \rangle + G | \theta^a \rangle + \frac{1}{2} | \theta^a \rangle \left[ \phi^{ab}, \phi_{ba} \right] + \cdots ,$$

(81)

which coincides with the $\bar{Q}^{(1)}$ variation $\delta^{(1)} \Lambda$ in (48) once we set to zero those components of the covariant derivative that originated from the $(\chi)^4$ term in the holomorphic frame.
in the expansion above, this removes the term proportional to $G$. This may be achieved by inserting the projector

$$P^e_a = \delta^e_a - \frac{1}{4}(\theta^e \partial_a).$$

(82)

We have thus demonstrated that

$$\sum_i \bar{\epsilon} \cdot \chi_i \frac{\partial}{\partial \mu_i} W[C_n] = i \text{Tr} \left( \oint \delta^{(1)} A(x, \theta) \text{Hol}(x, \theta) [A; C_n] \right)$$

$$= i \text{Tr} \left( \oint \frac{\varepsilon^{abcd}}{3!} [\bar{\epsilon}_e | dx | P^e_a D_b] \text{Hol}(x, \theta) [A; C_n] \right)$$

$$= g^2 \sum_i \int_{[0,\infty] \times S^1} V_\alpha D^{3|4} Z W(\ldots, i, Z, i + 1, \ldots),$$

(83)

as claimed.

### 6 Interpretation for amplitudes

In this paper we have examined the failure of loop amplitudes to be annihilated by the dual superconformal generator $\bar{Q} = \sum_i \chi_i \partial / \partial \mu_i$. From the point of view of the dual superloop, this is an ordinary supercharge, and we have used this perspective throughout the paper. However, it is also natural to wonder how our results, in particular the descent equation

$$\sum_i \chi_i \frac{\partial}{\partial \mu_i} W[C_n] = g^2 \sum_i \int V_\alpha D^{3|4} Z W(\ldots, i, Z, i + 1, \ldots),$$

(84)

arise from the point of view of scattering amplitudes, and how they are to be interpreted there.

They key to understanding this is to note that the particular dual Poincaré supercharge $\bar{Q}$ we have studied actually coincides with the original superconformal $\bar{s}$ supercharge \cite{1}. This supercharge may be represented on the $n$-particle on-shell momentum space as

$$\bar{s} = \sum_i \eta_i \frac{\partial}{\partial \lambda_i},$$

(85)

where $\eta_i^a = (\chi_i^a (i \ i+1) + \text{cyclic}) / \langle i-1 | i \ i+1 \rangle$ in terms of the (momentum) twistor fermions $\chi_i$ we have used in the rest of the paper. As shown in \cite{3,4,18,19}, even the $n$-particle MHV tree amplitude is not strictly invariant under the transformations (85) because of potential contributions to $\bar{s}$ at poles in $\lambda_i$ when external states become collinear. In our paper, we have assumed the initial superloop contour $C_n$ to be generic, so this tree-level failure is invisible.

However, \cite{3,18,19} further showed that the same phenomenon is responsible for the violation of $\bar{s}$ by loop amplitudes, where the collinearity is now between an external momentum and a loop momentum. The authors of \cite{18,19} further showed that the naive
Figure 4: From the point of view of the amplitudes, corrections to the original superconformal generator $\bar{s} = \bar{Q}$ arise when a loop momentum becomes collinear with some external momentum $p_i$. At one loop this contribution may be represented by a dispersion integral of the cut diagram on the left. From the point of view of the superloop, the same contribution arises as a particular BCFW decomposition of the $(n+1)$-point tree amplitude inside the descent equations.

action of (85) on the loop amplitude could be ‘corrected’ by deforming the way this generator acts on the fields. The required deformation includes a contribution that at 1-loop can be expressed as a dispersion integral of a rational function obtained from acting on the unitarity cut of the 1-loop amplitude in the limit that one of the cut loop propagators becomes collinear with one of the external momenta adjacent to it. See figure 4 for an illustration. In this figure, the displayed propagators are understood to be cut, so that

$$ (x - x_i)^2 = (x - x_{i+1})^2 = (x - x_{j+1})^2 = 0 .$$

The dispersion integral is over the fraction of momenta shared by the collinear states that is carried away by the external edge, while the operator $\bar{s}_3$ is the classical $\bar{s}$ operator acting on the three particle MHV amplitude indicated in the figure (which is not zero in this collinear regime).

To relate this to the descent equation (83) for the superloop, we make use of a particular BCFW decomposition of the $(n+1)$-point superloop. Consider the BCFW deformation

$$ Z \rightarrow Z(r) \equiv Z + rZ_{i+1}$$

where $r$ is the deformation parameter. Applying this deformation to $W(\ldots, i, Z, i+1, \ldots)$ we see that as $r$ varies, the only twistor line to be affected is $(iZ)$. For the superloop in

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15Beisert et al. also identified two other sources of contribution to the $\bar{s}$ anomaly in loops. The first arises because the amplitudes require regularization which inevitably breaks superconformal symmetry. However, it is less clear that one cannot find a regulator that preserves the Poincaré supersymmetry of the Wilson Loop (= dual supersymmetry of the amplitude). More practically, this contribution will vanish in any finite quantity such as the ratio function. The final potential contribution only arises if two or more external legs become collinear. We have ignored this possibility by our genericity assumption on $C_n$. It would clearly be interesting to revisit this issue.

16In [23] BCFW recursion for scattering amplitudes was identified with a particular version of the Migdal-Makeenko equations for Wilson Loops.
the self-dual theory (tree-level scattering amplitude) we have the BCFW decomposition
\[ W(\ldots, i, Z, i+1, \ldots) = W(\ldots, i, i+1, \ldots) \]
\[ + \sum_{j=i+2}^{i-2} [i, Z, i+1, j, j+1] W(j+1, \ldots, i, Z_j^x) W(Z_j^{x}, Z_j^2, i+1, \ldots, j), \quad (88) \]

where \( Z_j^x \) is the value of \( Z(r) \) when the deformed line intersects \((j, j+1)\), and where \( Z_j^x \) is the intersection point. Using the fact that as \( Z \) moves, it always lies in the plane \((i-1, i, i+1)\), this intersection point is simply the intersection\(^{17}\)
\[ Z_j^x = (i-1, i+1) \cap (j, j+1) \]
while
\[ Z_j^2 = (Z i+1) \cap (j, j+1) \]
is the shifted point.

It is now easy to connect this to the amplitude picture discussed above. The first, ‘homogeneous’ term in the recursion is independent of \( Z \) and cannot contribute to the integral on the right hand side of \((83)\). For the other terms, both the Grassmann integrals and the contour integral setting \( Z \rightarrow Z_i \) may be performed using the explicit R-invariant in \((88)\). Noting that the shifted point \( Z_j^2 \) also reduces to \( Z_i \) in this limit, we see that the integrand of the descent equation reduces to a product of two smaller Wilson Loops \( W(j+1, \ldots, i, Z_j^x) W(Z_j^x, i, i+1, \ldots, j) \) that share the line \( X = (iZ_j^x) \), times a coefficient that is left over from the R-invariant.

This situation is illustrated on the right of figure 4. The line \( X \) intersects the lines \((i-1, i), (i, i+1) \) and \((j, j+1) \) and so obeys the cut conditions \((86)\). This line corresponds to the location of the operator insertion \( \delta^{(1)} \Lambda(x, \theta) \) from the point of the superloop, and is the dual region momentum corresponding to the loop momentum in the amplitude. That \( X \) should be associated with a loop momentum in the scattering amplitude is particularly natural if we recall that quantum corrections to the amplitude correspond to insertions of \( S_{\text{MHV}} \) in the superloop, and that \( \delta^{(1)} \Lambda \) can be obtained from the \( \bar{Q}^{(0)} \) transformation of \( S_{\text{MHV}} \) as discussed around \((50)\). The kinematics of the three-particle MHV amplitude – all three particles sharing a common \( \lambda \) spinor – is also reflected in the figure as the triple intersection at \( Z_i \). The momentum in the cut propagator between \( p_j \) and \( p_{j+1} \) is determined in terms of \( Z_j^x \). Finally, the dispersion integral for the amplitude becomes the integral over the line \( X \) lying in the plane \((i-1, i, i+1) \), corresponding to integrating the insertion of \( \Lambda \) along the edge of the Wilson Loop.

We believe a similar story will be true for multi-loop amplitudes (or more precisely their finite ratio & remainder functions) provided one uses the all-loop extension of the BCFW recursion relations discovered in \[12\].

\(^{17}\)The condition that these lines and planes do intersect is non-trivial in the superspace. It is ensured by the R-invariant prefactor \([i, Z, i+1, j, j+1]\), which may be interpreted as a fermionic \( \delta \)-function with support only when its five arguments lie on a common \( \mathbb{CP}^3 \subset \mathbb{CP}^3/4 \). See \[41\] for further discussion.
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