Students’ Ways of Thinking on Geometry

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Abstract. Ways of Thinking (WoT) is a fundamental ability that is part of the mental act that students must learn to solve geometry problems. The purpose of this research was to describe how WoT is used to solve geometry problems. This research used the descriptive qualitative methodology with a holistic case study design and data collection techniques such as tests, interviews, observation, documentation, and triangulation. The subjects in this research were 35 fourth-grade students from Sambas Regency Public Elementary Schools, Indonesia. The research focused on three students with varying levels of cognitive abilities: low, medium, and high. The results showed that students with high, medium, and low cognitive capabilities performed mental acts with the corresponding WoT. Students with high, medium and low cognitive capabilities were more likely to work on questions that required interpretation. Besides, it also revealed that errors committed students were learning errors, reading errors, careless errors, conceptual errors, and procedure errors. The implementation of WoT on the topic of geometry can be utilized as an alternate reference for mathematics teachers when developing teaching materials for mathematics learning to increase students’ mathematical knowledge.

Keywords: ways of thinking, types of studentss errors, geometry

Introduction

Geometry is a branch of mathematics that is taught with the aim that students can develop ways of thinking and understanding in solving problems of everyday life (Irfan & Andika, 2020). The ability to think geometrically is required for learning geometry research (Wardhani, 2015). Geometry can assist students with visualization, conceptualization, critical thinking, problem-solving, deductive reasoning, argumentation, and logical evidence (Jupri, 2017; Seah, 2015). However, some students find geometric materials difficult to comprehend when learning mathematics (Fauzi et al., 2019; MdYunus et al., 2019). Dealing with the empirical data, it is clear that students have difficulty constructing original and correct structures, are less comprehensive, take a long time, and have difficulty maintaining the solution (Noto et al., 2019).

The results of the author’s preliminary study of fourth-grade students at State Elementary School 2 in the Sambas Regency, Indonesia. Revealed some findings that: 1) geometry material is difficult to understand because there are many formulas, symbols, and formal operations; 2) some students have difficulty in understanding word problem; 3) some students struggle to translate everyday sentences into mathematical sentences, and 4) some students have a learning disability because the teacher does not understand how they think. Every student generally has had mathematical abilities in ways of thinking and understanding since childhood, both at high
and low levels. In this case, the students' ways of thinking (WoT) is closely related to mental acts (Ikhwanudin et al., 2019). Students' reasoning involves mental acts such as interpreting, conjecturing, inferring, proving, explaining, compiling, generalizing, applying, predicting, classifying, finding, and solving problems (Harel, 2008). Furthermore, mathematics consists of two complementary parts. The first part is a collection or structure of axioms, definitions, theorems, proofs, problems, and solutions. This first section contains ways of thinking. The second part was ways of understanding which are the characteristics of mental acts that produce products of ways of thinking (Harel, 2020).

The statement mentioned above implies the following relationships: Mental act is a characteristic of thinking according to the problem at hand. The flow of thought that is formed by a mental act is a way of thinking (WoT). Ways of understanding (WoU) are formed as the flow of thought continues and comes into contact with certain contexts of forming an understanding (Ikhwanudin & Suryadi, 2018). Students, in this case, are required to comprehend why and how algorithmic calculations are used, not just memorize (Kaminski, 2002; Mohamed & Johnny, 2010). Students who understand concepts well enough to generalize and transfer their knowledge using higher-order thinking processes are considered to have good conceptual understanding skills, as opposed to students who simply memorize (Steinberg et al., 2004).

Harel's perspective on ways of thinking (WoT) and ways of understanding (WoU) is one of many that have been expressed through various thought processes. Harel emphasized the importance of allowing for mental acts in the learning environment, such as the formation of a continuous flow of thinking (ways of thinking) on mathematical objects that lead to comprehension. He also studied mathematics instructors with the Interview Communication Map (ICM), an exploratory framework for uncovering the triadic cycle in clinical-based interviews, such as mental act, WoT, and WoU (Koichu & Harel, 2007; Nurhasanah et al., 2021). In designing and evaluating interviews, Interview Communication Map (ICM) helps analysts to think about the nature of interview data (Hunting, 1997; Koichu & Harel, 2007). Harel's ideas, particularly mental acts, WoT, and WoU, are used to examine students' anticipation and predictions in solving algebraic problems (Lim, 2006). More research has shown how students' ways of thinking can influence their academic performance (Çimer & Ursavaş, 2012; Harel, 2008; Ikhwanudin & Suryadi, 2018; Lee et al., 2020; Lockwood & Weber, 2015; Neuenschwander, 2013; Nurhasanah et al., 2021; Owens, 2014; Radović et al., 2019; Sari et al., 2019).

Furthermore, using Harel's theory comprehensively can analyze students' thinking processes in detail. This theory is essential to serve as a line of thought that can be utilized by students and teachers in the process of learning mathematics (Harel, 2008). Other theories related
to the thought process of constructing mathematical objects in mental acts include Piaget's theory explaining that understanding mathematical objects, begins with mental acts related to mathematical objects in the environment (Harel, 2008; Loward S. Friedman & Miriam W. Schuctack, 2006). These mental acts are internalized into processes, then encapsulated in objects. A process is a cognitive unit in the success of mathematical thinking (Ndraha, 2015).

Some research on expert-presented modes of thinking and hypotheses on thinking processes include identifying eighth-grade studentss' ways of thinking (WoT) and Ways of understanding (WoU) concerning generalization (Oflaz & Demircioğlu, 2018); The Duality, Necessity, and Repeated Reasoning Principle (DNR) perspective on mathematics curriculum and instruction explain that mathematics is the union of two categories of knowledge, namely ways of understanding (WoU) and ways of thinking (WoT) as well as generalizations of definitions, proofs, and proof schemes (Harel, 2008). In a study conducted by Lim (2006), it was found that understanding how students think when solving math problems might assist teachers in the learning process of teaching consistently and appropriately, specifically by understanding how students think. Lim (2006) analyzes studentss' anticipation and predictions in solving algebra problems using Harrel's theories of a mental act, WoT, and WoU.

However, research related to studentss' thinking in solving geometry problems in aspects of traditional Malay houses at Elementary Schools in Sambas Regency has not been studied by authors. Thus, the authors are interested to examine the context of the Sambas Malay traditional house due to mathematics and culture being related to each other (Rosa & Orey, 2011). According to Fitroh (2020), the 2013 curriculum includes the improvement in studentss' mathematical ability to culture. Furthermore, studentss' ways of thinking in Sambas Regency are needed to formulate conjectures of students' knowledge to be able to be applied and interpreted in solving mathematical problems in the real world and used to make logical decisions and provide appropriate reasoning. This is related to the process of constructing mathematical objects where the learning situation must allow mental act, which results in the formation of a continuous flow of thinking so that a flow of thinking is obtained that leads to an understanding of mathematical objects (Arnon et al., 2014; Brousseau, 2002; Jamilah et al., 2020). Thus, the research questions are formulated as follows: How are the studentss' thinking tendencies which include interpreting, explaining, and problems-solving on geometry?; What are the types of errors made by students in completing the ways of thinking test on geometry?

Method

The type of this study is qualitative research. This method was used because the authors intended to find out what the students thought about the geometry of the Sambas Malay traditional
house. The method utilized in this study places a greater emphasis on interpretive studies for data analysis. A holistic case study design was used in the research to describe numerous field findings relevant to research issues (Oaks et al., 2013).

The research procedures consisted of preparation and implementation. School selection and instrument development were validated by mathematicians as part of the planning stage. Designing research instruments, validating the instrument, collecting data, analyzing data, and generating reports were all steps of implementation. The research focused on three students with varying levels of cognitive abilities: low, medium, and high. State Elementary School 2 in the Sambas district was chosen because of its location which is close to the traditional Malay house of Sambas. Besides, the students also have learned geometry. The findings of students' responses to the thinking tool were assessed by three students based on their abilities, namely one high-ability student, one medium-ability student, and one low-ability student. The grouping criteria based on the mean and standard deviation of students' daily math test scores are presented in Table 1.

| No | Students’ ability | Criteria                  |
|----|-------------------|---------------------------|
| 1  | high ability      | \( X \geq \bar{X} + SD \) |
| 2  | medium ability    | \( \bar{X} + SD < X < \bar{X} - SD \) |
| 3  | low ability       | \( X \leq \bar{X} - SD \) |

The criteria for grouping high, medium, and low ability students are based on Table 1. Where \( \bar{X} \) is the average value of the initial mathematical ability, the author describes. The SD of the initial mathematical ability value is the standard deviation.

The research was conducted by the authors as the main instrument concerning the data in the form of tests and interviews as supporting instruments. The examination can be described as a method of deriving information about nonverbal numerical types (Best & Kahn, 2006; Cohen et al., 2007). The purpose of the test is to evaluate a person's knowledge and abilities in a specific field or subject (Freankel et al., 2012). The test was created in the form of a description of one question, as shown in Table 2, and given to 35 students in grade IV to describe how they thought about solving geometry problems.

Interviews were also conducted to identify the in-deep meaning of everything that has been done on the instrument that students use to organize their experiences and make sense of their surroundings (Bassey, 1999; Hatch, 2002). The type of interview used in this study was semi-structured. The interview aimed to learn more about the students’ capacity to comprehend mathematical thinking process events and occurrences (Sugiyono, 2016). Tests and interviews were conducted on different days to collect accurate subject data. To analyze the test and interview data, the triangulation method was used, which included: 1) data reduction (data selection process,
removing unnecessary information, and organizing interview results); 2) data presentation, specifically interview data; and 3) conclusions drawn from all of the data in the test and interview results (Miles et al., 2014). Table 2 presents the results of the geometry test or interpreting, explaining, and problem-solving indicators held in the Sambas Malay traditional house.

Table 2. Indicators for questioning, interpreting, explaining, and problem-solving

| Test instruments | Indicator | Ways of Thinking (WoT) |
|------------------|-----------|------------------------|
| 1. Masnah's house is one of the traditional Malay houses that is still preserved, has a roof that looks like a flat trapezoid from the front and back. The front side of the roof of Masnah's house has a parallel side length of 32 meters and 20 meters, respectively. The total area of the two sides of the roof is 884m². | Interpreting | WoT: Ways of thinking is a cognitive characteristic of mental actions that emerges from an test of observations, in which students understand the word problem, verbal forms, and mathematical images and symbols in a variety of ways. |
| a. Create a sketch of Masnah's roof's shape using the sizes mentioned in the problem! | Interpreting | WoT: Ways of thinking is a cognitive characteristic of mental actions that emerges from an test of observations, in which students understand the word problem, verbal forms, and mathematical images and symbols in a variety of ways. |
| b. How the sides of the house can form a flat roof! Give your explanation! Explain! | Explaining | WoT: Students who use ways of thinking to describe mathematical concepts can use both verbal and writing communication to express their mathematical thoughts. |
| c. Which part of the roof of Masnah's house is not known and determine the size of the roof! | Problem-solving | WoT: Students model mathematics using a problem-solving approach technique and their mathematical knowledge based on the illustration of the shape of the roof to solve difficulties in the form of story problems. |

Results and Discussion

The data and discussions in this research focus on students’ ways of thinking and the types of student errors, with levels of cognitive ability namely high (A-1), medium (R-2), and low (R-3) in completing word problem about geometry in the context of a traditional Malay house of Sambas.

Interpreting

The results of students’ answers were analyzed using interpretation indicators divided into high, medium, and low ability levels. The student’s answer of the high ability level, student A-1 are described in Figure 2. Students A-1 were observed executing mental act interpretation by interpreting word problem relating to the design of the roof of Masnah’s traditional Malay house with a flat shape as presented in Figure 2 on the answer sheet. Student A-1 interpreted the roof of
the typical Malay sambas houses into an image, specifically a trapezoid image, which is a cognitive property of mental act interpreting (ways of thinking).

Figure 2. The answers of students A-1 for question 1a

Furthermore, the authors conducted interviews to analyze in-depth the results of the answers from tests carried out by high ability students on interpreting indicators as follows:

Authors : Look at this question, from the word problem given, do you understand about the meaning of this question?
Student : Yes, I do.
Authors : Can you sketch the roof of Masnah’s traditional Malay house?
Student : Yes, I can.
Authors : Can you sketch the shape of the roof of Masnah’s traditional Malay house?
Student :

Authors : Can you explain which side is 20 meters and which side is 32 meters?
Student : The length of 20 meters is the top side, while the bottom side of the trapezoid has the length of 32 meters

Students were able to translate the meaning of word problem into acceptable visual representations, according to the results of interviews with A-1. Student A-1 could incorporate information from word problem into geometry concepts with logical explanations, according to their comprehension. This study's findings are consistent with the research by Ikhwanudin & Suryadi (2018) who revealed that gifted students solve fraction issues by creating visuals and picturing the difficulties at hand. This is in line with the findings of Amril et al (2020), who claimed that students used a range of strategies of thinking when interpreting the mental act, including visual representation. Students S-2 with the medium level of ability answers the question 1a as shown in Figure 3.

Figure 3. The answers of students S-2 for question 1a

Figure 3 presents the results of the analysis of students' performance when solving mathematical problems related to interpreting indicators which include interpreting word problem related to the Sambas' traditional house results from the context of geometry. According to the findings of students' work, S-2 with medium ability accurately translates the roof of the traditional
Malay Sambas home into the form of a picture, namely a trapezoid image. The authors also conducted interviews to analyze in-depth the results of student's work of medium ability on interpreting indicators as follows:

Authors : Look at this number, from the word problem given, do you understand the meaning of this question?
Student : Yes, I got it
Authors : Can you sketch the shape of the roof of Masnah’s house?
Student : Yes, I can
Authors : Can you sketch the shape of the roof of Masnah’s house?
Student :

Authors : Can you explain which side is 20 meters long and which side is 32 meters long?
Student : The length of the shorter side of the trapezoid is 20 meters, while the 32 meters is the length of the other side of the trapezoid.

Based on the results of interviews with S-2, it was identified that S-2 was able to explain well by connecting the initial information on the form of word problem into the mathematical context related to the concept of geometry. It is proved that the students' way of thinking to produce statements in achieving a meaning from verbal into images is correct. The findings in this study are in line with the results of research by Becker & Rivera (2005) who found that students' visual ways of thinking (images) in obtaining solutions to solving geometry problems must consider the characteristics of structural forms. Figure 4 shows the results of the student R-3 answer with low ability levels for question 1a.

Based on the answer of students R-3 with low-level cognitive abilities, it was identified that students interpreted the word problem on the roof of the traditional Malay Sambas house into the form of a flat shape image, namely a trapezoid. Furthermore, the authors conducted interviews to analyze in-depth of students' answers as follows.

Authors : Please, sketch the shape of the roof of the house!
Student :

Authors : Can you tell which side is 20 meters long and which side is 32 meters long?
Student : 32 meters on the top, and 20 meters on the right.
Authors : Are you sure the answer is correct?
Student : Yes, I'm sure.
Based on the interviews, it was found that student wrong in determining the length of the side of the roof of the traditional Malay Sambas house, namely the trapezoid. The error made by R-3 was a misread direction error. This occurred because students were wrong in reading the direction of the information, giving rise to the wrong meaning of the information in the questions. It was identified that students continued to write down the information contained in the questions in the answers given to students. Student R-3 also experienced an error in writing the length of the parallel side of the trapezoid.

Several research findings on interpretation indicators were found based on data analysis results, student A-1) with high-level cognitive abilities able to discern the meaning of word problem into visual. Then, in the context of geometry, students S-2 with a medium level of cognitive were recognized as being able to comprehend word problem connected to the outcomes of the sambas Malay traditional home. Meanwhile, students R-3 with low cognitive abilities were able to interpret the word problem about the shape of the roof of a Malay sambas house into a flat shape image. The findings of this study, it was discovered that students' errors were misread direction errors. In line with research conducted by Nolting (2002), it was stated that the information contained in the questions was not understood by the students. According to Clements and Sarama (2010), students made errors while reading the questions so they could not understand the sentences in the questions as a whole.

**Explaining**

Explaining is one form of reasoning. The results of the answers A-1 as high-level cognitive ability students on the explaining indicator are presented in Figure 5.

![Figure 5. The answers of students A-1 for question 1b](image)

Based on Figure 5, It was identified that student A-1 understood the questions and answered them through explanation. The author conducted interviews with A-1 to analyze in-depth the results of the A-1 answers as follows:

**Authors**: After you sketch the roof of the house, try to observe each side of the flat shape (trapezoid), What do you think about the sides? Please, explain!

**Student**: The top and bottom sides do not have the same length.

**Authors**: Alright, are you sure your answer is correct? (While pointing to the answer sheet)

**Student**: I am sure.
Authors : The top and bottom sides, are they parallel? How about the left and right sides?
Student : The top and bottom sides are parallel. For the right side and the left side, they are opposite, teacher, they are not parallel.
Authors : Do you have difficulty in answering this question?
Student : I don’t.

Based on the interviews, it was identified that A-1 can explain the connection of the roof of the house into a trapezoidal flat shape. The student also able to explain logically the sides of the trapezoid oral and written. According to interviews, the student explained the trapezoid’s sides fluently and correctly. The student can respond directly and provide appropriate explanations. This occurs because the incoming stimulus is consistent with the scheme. The findings of this study are consistent with the findings of Ikhwanudin et al (2019), who found that explaining is closely related to the ability to express mathematical ideas through oral, demonstration, and writing in solving mathematical problems.

The answer of the student S-2 from medium-level cognitive ability students is presented in Figure 6.

![Figure 6. The answers of students S-2 for question 1b](image)

It is shown in Figure 6 that S-2 couldn’t understand the questions where mathematical ideas described in his answer contained incorrect reasons. Interviews with the S-2 were also conducted to analyze in-depth his answer as follows:

Authors : After you sketch the roof of the house, try to observe each side of the flat shape. What do you think about the sides? Please explain!
Student : The top and bottom sides of the roof’s house are flat. The flat shape is a trapezoid.
Authors : Alright, are you sure your answer is correct? (While pointing to the Answer sheet)
Student : (Silent, did not answer)
Authors : Did you have any difficulty in doing this question?
Student : Yes, I had some difficulties.

Based on interview, it was identified that students could not explain correctly the sides of the trapezoid. During the interview process, the S-2 was confused to rephrase the problem of his own words or sentences. The S-2 was identified had an error in explaining the sides of the trapezoid. The findings of the research are in line with the research conducted by Nazariah et al (2017) which stated that students with a medium level of cognitive mathematical abilities, the method used to solve mathematical problems, tend to give long answers but are less accurate.

The answers of the R-3 from low cognitive ability student are presented in Figure 7.
Based on Figure 7, students R-3 with low-level cognitive abilities could not explain the answers to the problems given correctly. It was identified that students unable to reconstruct the knowledge that student has, with new knowledge, which can be used to solve problems related to the sides of the trapezoid. Furthermore, the author conducted interviews with a student R-3 as follow.

Authors : After you sketch the roof of the house, try to observe each side of the flat shape (trapezoid). What do you think about the sides? Please, explain!
Student : They have the same length up and downside
Authors : Alright, are you sure your answer is correct? (While pointing to the answer sheet)
Student : I’m confused, ma’am
Authors : Did you have any difficulty in doing this question?
Student : It was difficult to explain the sides of the trapezoid. There was not enough time ma’am while the questions must be answered and submitted.

Based on interviews it was found that student R-3 could not give the correct answers. This is because a student confused in giving explanations and giving wrong answers. The student stated that he did not have enough time to learn geometry related to the characteristics and properties of a trapezoid.

Overall, based on the results of data analysis, A-1 with high-level cognitive abilities was able to explain logically related to the sides of the trapezoid through oral and written and S-2 with medium-level cognitive abilities was also not able to explain mathematical ideas through writing containing explanations. Meanwhile, R-3 with low-level cognitive abilities was unable to reconstruct the knowledge he has to solve problems. Thus, the students only simply gave illogical explanations. In conclusion, the findings are in line with the research conducted by Wiryoatmojo et al (2013) which stated that when students experience an error in the thinking process in solving mathematical problems, it will cause the answer to be wrong.

Problem-Solving

The third mental act found from the analysis of students’ answers is an indicator of problem-solving. The answer of student A-1 from high-level cognitive abilities student is presented in Figure 8. Based on Figure 8, it was identified that A-1 was able to sort out information relevant to the problem and presented it in variable form. The students’ way of thinking was right in choosing the formula for the area of a trapezoid \( L = \frac{1}{2} \times \text{number of parallel sides} \times \text{time} \). Furthermore, the student can also construct relationships between variables. However, it was
found that the final answer written by A-1 was not correct, the actual height of the trapezoid should be 17 meters, not 34 meters. In addition, the conclusion of the answer given by A-1 is not correct. In this case, A-1 has made an error in the test procedure where the student is completing the question at the end.

The author also conducted interviews to analyze in-depth the answers from subjects A-1 on problem-solving indicators with as follow:

Authors : Pay attention to this question, how Would you solve it?
Student : I will solve this problem by using the formula for the area of a trapezoid
Authors : Can you mention what is information provided and asked in the question?
Student : The length of side 1 is 32 meters, the length of side 2 is 20 meters, the area of both sides is 884 meters
Authors : Can you make a mathematical model of this story problem?
Student : Yes, ma'am. Let the length of side 1 be the length of side 2 be b, let the area be L.
Authors : OK, let's continue. Can you rewrite the steps for the solution?
Student : The area of the trapezoid is 884 meters, add up the top side with the bottom side 20 +32, so the result obtained will be 52. Apply it into the area formula L = (a + b ) x t
Find the height t =884/26
Authors : So, how many meters high is Masnah's roof?
Student : 34 meters.
Authors : Are you sure the answer is correct?
Student : Yes, I'm sure.
Authors : Do you have another way to solve this problem, to get the roof height of Masnah's traditional Malay house?
Student : No, ma’am.
Based on interview, A-1 knew what was known and what was asked from the given geometry problem. Student A-1 was also able to rephrase the problem by making a mathematical model. However, from interviews conducted by the author, it was found that students could not solve the problem with the correct solution because there were early stages that were wrongly done by the student.

The steps of the problem-solving process in the given geometric problem should be for the total area of the two sides of the roof, which is 884, the area of side 1 must be calculated first, from the roof of the house by 884 divided by 2 to produce a trapezoidal area of 442 meters. In this case, the students did not check his work. The findings in this study in line with Kurniati & Annizar (2017) who found that students with different cognitive abilities will produce different problems, especially in terms of understanding. Furthermore, the types of research errors that have been carried out by these authors are in line with the research conducted by Jupri et al (2014) which found errors made by students in solving mathematical problems given related to conceptual errors. The errors occurred because the student does not truly understand the rules in mixed operations, namely the addition and multiplication of numbers.

The answer of S-2 with medium-level cognitive abilities is presented in Figure 9.

![Figure 9. The answers of students S-2 for question 1c](image)

A problem-solving approach is a cognitive characteristic of mental act problem-solving. Based on Figure 9 which presents about S-2 answers with a medium level of cognitive ability, it was identified that students had understood the questions and had answered using a problem-solving approach, in the form of multiplication and division operations strategies. However, from the students' answers, it was found that the student had made errors in the test procedure in solving geometry. Furthermore, the author conducted interviews with the S-2 to analyze in-depth the students' answers to the problem-solving indicators as follows:

**Authors**: Observe the results of your work again. Can you explain what is known and asked from this question?

**Student**: It is known that the length is 32 meters and 20 meters (the student looks confused). The area is also mentioned in the question (student does not mention about the wide). Then the question asked is the high.

**Authors**: Can you make a mathematical model?

**Student**: (Silent and did not give answer)
Authors: Alright, let's continue. Please do re-check whether the area used is 884 meters?

Student: (Silent)

Authors: Ok, let's continue. Can you rewrite the steps for the solution?

Student: Enter into the formula area \( L = (\text{length of the top} + \text{length of the bottom}) \times \text{height} \)

\[
884 = x (32 + 20) \times \text{height} \\
884 = x 52 \times \text{height} \\
884 = 26 \times \text{height} \\
\text{Height} = \frac{884}{26}
\]

Authors: So, how many meters high is Masnah's roof?

Student: 34 meters

Authors: Are you sure the answer is correct?

Student: (silent... did not give any answer)

Authors: Do you have another way to solve this problem, to find the height of the roof of Masnah's house?

Student: No, ma'am.

Authors: So what is the conclusion?

Student: The roof of the house is in the form of a trapezoid with a pointed side at the top and a height of 34 meters

Authors: Are you sure the answer is correct?

Student: No, ma'am, I did not re-check the questions that I worked on.

Authors: Did you have any difficulty in doing this question?

Student: No, but I'm a bit confused about the area of the two sides of the trapezoid which is 884 meters, what should I do then?

Based on interviews it is identified that the student was able to determine what is known and what is asked from the geometry problem. However, it was found in this research that students confused in expressing the problem (wrong in making mathematical models). The student could not provide solutions correctly according to the calculation algorithm because he has made an error in determining the area of the trapezoid. During the interview, it was revealed that the student was not sure about the solution he made. This is because the student did not check the process of the stages involved in working on geometry problems. The findings of this study are in line with the research conducted by Hasanah et al. (2009) which stated that the factors that influence the differences in students' abilities in terms of understanding are students' mastery of related mathematical concepts. Nihayah (2019), found that students paid less attention to simple things in solving problems.

The analysis R-3 answers with low-level cognitive abilities on the problem-solving indicators are presented in Figure 10.

Figure 10. The answers of students R-3 for question 1c
Figure 10 shows the answer of R-3 with low-level cognitive abilities. It was identified that the students did not carry out the mathematization process. Furthermore, R-3 could not determine the relationship of the trapezoidal formula and left the answer blank by giving the wrong solution. This is because R-3 with low-level cognitive abilities did not understand the concepts or principles needed to solve geometry problems. An interview with R-3 who has low-level cognitive abilities as follows:

**Authors**: Look again at the results of the answers you did, is your answer correct?
**Student**: No, ma’am.
**Authors**: Do you have another way to solve this problem, to get the roof height of Masnah’s house?
**Student**: I don’t know ma’am.
**Authors**: Did you have any difficulty in doing this question?
**Student**: Yes, I did.

Based on interviews it was found that students could not determine what was known and what was asked. It was also identified that student unable to express the problems and they even could not make mathematical models. This happened because students could not construct knowledge with the geometry concepts they have learned to use in solving geometry problems. It was also identified that R-3 was stuck on a problem and spent a lot of time. Furthermore, R-3 answered without giving a reason and left the answer sheet blank.

Based on the data analysis of students’ answers, several research findings were obtained including A-1 with high-level cognitive abilities which tend to be able to sort out information relevant to the problem and then present it into variable form, choose the right formula, and construct relationships between variables. S-2 with a medium level of cognitive ability able to understand how to use a problem-solving approach, in the form of multiplication and division operations strategies. Meanwhile, R-3 who has low-level cognitive abilities does not carry out the mathematization process and cannot construct relationships between word problem about the trapezoid-shaped roof of the house into the trapezoidal area formula. From the students’ answers, the errors found included errors due to lack of accuracy, and procedures errors where the students were stuck on a problem and spent a lot of time. Students answered without giving a reason and left the answer sheet blank, errors in making mathematical models, not understanding the concepts or principles needed to solve geometry problems, and not doing re-check.

This type of error is in line with Edo et al (2013) who found that students cannot formulate problems mathematically, as can be seen when students cannot solve problems completely. In this case, students use their method called instinct. This finding aligns with Ikhwandwin et al (2019) who found that students’ WoT in understanding fraction material. It was found that students were wrong with applying the equation procedure to addition and multiplication operations of fractions because of the lack of understanding of the concept of fractions. Nolting (2002) found that errors
during the test procedure take place when students cannot solve the problem until the final step, turn the correct test answer into one, get stuck on a problem, spend a lot of time, make careless errors, answer without giving a reason, and left the answer sheet blank. The mathematical concepts built by students are not intact or are called construction holes. Nurhasanah et al. (2021) revealed that understanding the wrong or incomplete concept resulted in illogical or incorrect WoT. Furthermore, in Harel's theory, a person's way of thinking determines how to understand a concept or problem. The implication of this finding is students who have a positive correlation between WoT and WoU, have an impact on improving learning outcome achievement (Nurhasanah et al., 2021).

Conclusion

The following conclusions were obtained from the findings of that type of data analysis. WoT has revealed several interpretations of mathematical symbols, ways of explaining, and problem-solving the veracity of the results provided in solving geometry challenges. Students are more likely to work on questions about interpreting indicators in general. High-ability students think in a variety of ways and can interpret, explain, and problem-solving information in a variety of ways. Medium-ability students, on the other hand, have less capacity for describing things than high-ability students. Low-ability students, struggled with a variety of WoT, especially signs of explaining and understanding. This finding implies that the more diverse a student's thinking is the better at solving geometry problems he or she will be.

There are several types of errors made by students, including errors that occur when students read the direction of information about the problem incorrectly, causing the information contained in the problem to have the wrong meaning (misread direction error), errors that occur when students are careless in working on the questions (careless errors), and errors that occur when students are unable to read the directions of information about the problem (misread direction error), and errors during the test procedure take place when students cannot solve the problem until the final step, turn the correct test answer into one, get stuck on a problem, spend a lot of time, answer without giving a reason, and leave the answer sheet blank (procedure error). Ways of thinking (WoT) is very important learn to fully understand students’ thinking in learning geometry.

The results of this research have implications for developing strategies to explore and expand geometry. These findings also have implications for improving students' abilities to solve mathematical problems by increasing their capacity and diversity of thinking. Furthermore, the difficulties faced by students have often been related to medium and low ability levels so then this research has implications for incompleteness in solving geometry problems. In this current
research, the authors recommend developing a learning design that emphasizes comprehension, scaffolding, and remediation for students who often make errors in solving geometry problems.

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