Hardy and BMO spaces associated to divergence form elliptic operators

Steve Hofmann · Svitlana Mayboroda

Abstract Consider a second order divergence form elliptic operator $L$ with complex bounded measurable coefficients. In general, operators based on $L$, such as the Riesz transform or square function, may lie beyond the scope of the Calderón–Zygmund theory. They need not be bounded in the classical Hardy, BMO and even some $L^p$ spaces. In this work we develop a theory of Hardy and BMO spaces associated to $L$, which includes, in particular, a molecular decomposition, maximal and square function characterizations, duality of Hardy and BMO spaces, and a John–Nirenberg inequality.

Mathematics Subject Classification (2000) 42B30 · 42B35 · 42B25 · 35J15

S. Hofmann was supported by the National Science Foundation.

S. Hofmann
Department of Mathematics, University of Missouri at Columbia, Columbia, MO 65211, USA
e-mail: hofmann@math.missouri.edu

S. Mayboroda
Department of Mathematics, The Ohio State University, 231 W 18th Avenue, Columbus, OH 43210, USA
e-mail: svitlana@math.ohio-state.edu

Present Address:
S. Mayboroda
Department of Mathematics, Purdue University, W. Lafayette, IN 47907-2067, USA
e-mail: svitlana@math.purdue.edu
1 Introduction and statement of main results

Extensive study of classical real-variable Hardy spaces in $\mathbb{R}^n$ began in the early 1960s with the fundamental paper of Stein and Weiss [27]. Since then these classes of functions have played an important role in harmonic analysis, naturally continuing the scale of $L^p$ spaces to the range of $p \leq 1$. Although many real-variable methods have been developed (see especially the work of Fefferman and Stein [17]), the theory of Hardy spaces is intimately connected with properties of harmonic functions and of the Laplacian.

For instance, Hardy space $H^1(\mathbb{R}^n)$ can be viewed as the collection of functions $f \in L^1(\mathbb{R}^n)$ such that the Riesz transform $\nabla \Delta^{-1/2} f$ belongs to $L^1(\mathbb{R}^n)$. One also has alternative characterizations of $H^1(\mathbb{R}^n)$ by the square function and the non-tangential maximal function associated to the Poisson semigroup generated by Laplacian. To be precise, fix a family of non-tangential cones $\Gamma(x) := \{(y, t) \in \mathbb{R}^n \times (0, \infty) : |x - y| < t\}, x \in \mathbb{R}^n$, and define

$$S^\Delta f(x) = \left( \int \int_{\Gamma(x)} \left| t \nabla e^{-t\sqrt{\Delta}} f(y) \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad (1.1)$$

$$N^\Delta f(x) = \sup_{(y, t) \in \Gamma(x)} \left| e^{-t\sqrt{\Delta}} f(y) \right|. \quad (1.2)$$

Then $\|N^\Delta f\|_{L^1(\mathbb{R}^n)}$ and $\|S^\Delta f\|_{L^1(\mathbb{R}^n)}$ give equivalent norms in the space $H^1(\mathbb{R}^n)$, that is

$$\|N^\Delta f\|_{L^1(\mathbb{R}^n)} \approx \|S^\Delta f\|_{L^1(\mathbb{R}^n)} \approx \|f\|_{H^1(\mathbb{R}^n)}.$$

Consider now a general elliptic operator in divergence form with complex bounded coefficients. Let $A$ be an $n \times n$ matrix with entries

$$a_{jk} : L^\infty(\mathbb{R}^n) \rightarrow \mathbb{C}, \quad j = 1, \ldots, n, \quad k = 1, \ldots, n, \quad (1.4)$$

satisfying the ellipticity condition

$$\lambda |\xi|^2 \leq \Re e A\xi \cdot \bar{\xi} \quad \text{and} \quad |A\xi \cdot \bar{\zeta}| \leq \Lambda |\xi||\zeta|, \quad \forall \xi, \zeta \in \mathbb{C}^n, \quad (1.5)$$