Abstract: We study the quark number susceptibility in holographic QCD with a finite chemical potential or under an external magnetic field at finite temperature. We first consider the quark number susceptibility with the chemical potential. We observe that approaching $T_c$ from high temperature regime, $\chi_q/T^2$ develops a peak as we increase the chemical potential, which confirms recent lattice QCD results. We discuss this behavior in connection with the existence of the critical end point in the QCD phase diagram. We also consider the quark number susceptibility under the external magnetic field. We predict that the quark number susceptibility exhibits a blow-up behavior at low temperature as we raise the value of the magnetic field. We finally spell out some limitations of our study.

Keywords: Gauge-gravity correspondence, QCD
1. Introduction

Understanding the QCD phase diagram is one of the important problems in modern theoretical physics; see [1] for some recent reviews. An important feature of the QCD phase diagram is the existence of the critical end point (CEP) where the first order phase transition, from the hadronic phase to the quark gluon plasma (QGP), ends. Based on the fact that at the critical point thermodynamic observables such as susceptibilities diverge and the order parameter fluctuates on long wavelengths, basic ideas to observe CEP are suggested in [2].

One of the important signals of QGP formation is the fluctuation of conserved charges such as baryon number or electric charge [3, 4]. The quark (or baryon) number susceptibility, which measures the response of QCD to a change of the quark chemical potential is one of them [3, 5]. There have been many studies to calculate the quark number susceptibility in various model studies [6, 7, 8] and lattice simulations [9, 10, 11, 12]. In [13], the quark number susceptibility at finite temperature is studied with AdS/QCD models. The existence of a peak in the quark number susceptibility near $T_c$ is confirmed by recent lattice QCD calculations based on the Taylor expansion with respect to the quark (or baryon) chemical potential [14, 15]. This implies the existence of CEP, at which the first order phase transition terminates in the $(\mu_q, T)$ plane of the QCD phase diagram, see [16] for a
review. Lattice QCD estimates that the location of the CEP would be $T^E = 165 - 175$ MeV and $\mu_B^E = 250 - 400$ MeV [16]. The existence of the CEP has been also investigated in various QCD-based model studies [17].

The AdS/CFT correspondence [18, 19, 20] is a powerful tool to investigate strongly coupled gauge theories including critical phenomena. Using this correspondence, we can obtain physical quantities in gauge theories from calculations in gravity side. Although the correspondence between QCD and gravity theory is not known, we can obtain much insights by using this correspondence. In fact, many properties are shared by various gauge theories, especially in hydrodynamic limit. Hydrodynamic properties can be read off by introducing a small perturbations into the bulk fields [21, 22]. This gives small fluctuations to the fluid in the gauge theory side. Physical quantities, like transport coefficients, can be calculated by considering the responses to these small perturbations; see [23, 24] and references therein.

In order to study dynamics of quarks (or baryons) in the gauge theory side, we have to introduce the $U(1)$ baryon symmetry in the gauge theory side. According to the Gubser-Klebanov-Polyakov/Witten relation [19, 20], a global symmetry in the gauge theory side corresponds to a gauge symmetry in the gravity side. To consider the $U(1)$ baryon symmetry, we have to introduce a $U(1)$ gauge field in the bulk. This kind of models can be constructed from the string theory setup. In D3/D7 setup, for example, D7-branes are considered as the flavor brane and gauge fields on D7-branes play the role of the bulk gauge field. This model has $\mathcal{N} = 2$ supersymmetry, and hence, the corresponding gauge theory is $\mathcal{N} = 2$ supersymmetric QCD (SQCD). Even though the real QCD does not have supersymmetry, this discrepancy is expected to be solved by universality mentioned above.

In this paper, we study one of the promising QGP probes by using the AdS/CFT correspondence. The primary goal is to calculate the quark number susceptibility at finite temperature with a finite quark chemical potential. Our approach has some ups and downs. Contrary to the lattice QCD considerations, we can calculate the susceptibility with arbitrary values of the chemical potential. However, we are not able to study the quark number susceptibility in confined phase, which will be discussed at the end of the section 3.1. Moreover, our study based on AdS/CFT is inherently suffering from $1/N_c$ corrections. Unfortunately, a systematic way of collecting all those corrections has not been established. With this caution in mind, we first revisit the quark number susceptibility at finite temperature, and then we generalize it with the chemical potential. For this we work in the AdS black hole and Reissner-Nordström-AdS backgrounds.

In addition to this, we evaluate the quark number susceptibility at finite temperature with a constant magnetic field. The basic motivation is due to the observation that the constant magnetic field enhances the dynamical chiral symmetry breaking [25]. On top of it, recently it is argued that sufficiently large magnetic fields are created in heavy ion collisions [26], and so our study may be tested in a terrestrial experiment. Therefore it is interesting to study the phase diagram of QCD as a function of the magnetic field, and so the quark number susceptibility.
2. Retarded Green functions and quark number susceptibility

In this section, we briefly summarize the prescription for the Minkowskian correlator in the AdS/CFT correspondence, and then define the quark number susceptibility through the correlator. We here follow the prescription proposed in [21]. We work on the following 5D background,

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{uu}(du)^2, \]  

(2.1)

where \( x^\mu \) and \( u \) are the 4D and radial coordinate, respectively. We refer the boundary at \( u = 0 \) and the horizon at \( u = 1 \). Let us consider a solution of an equation of motion in this 5D background. Suppose the solution is given by

\[ \phi(u, x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx}f_k(u)\phi^0(k), \]

(2.2)

where \( f_k(u) \) is normalized such that \( f_k(0) = 1 \) at the boundary. After putting the equation of motion back into the action, the on-shell action might be reduced to surface terms

\[ S[\phi^0] = \int \frac{d^4k}{(2\pi)^4} \phi^0(-k)\mathcal{G}(k, u)\phi^0(k) \bigg|_{u=0}^{u=1}. \]

(2.3)

Here, the function \( \mathcal{G}(k, u) \) can be written in terms of \( f_{\pm k}(u) \) and \( \partial_u f_{\pm k}(u) \). Accommodating Gubser-Klebanov-Polyakov/Witten relation [19, 20] to Minkowski spacetime, Son and Starinets proposed the formula to get the retarded Green functions,

\[ G_R(k) = 2\mathcal{G}(k, u) \bigg|_{u=0}, \]

(2.4)

where the incoming boundary condition at the horizon is imposed. In this paper, we consider correlators of \( U(1) \) currents \( J_\mu(x) \), where \( J_\mu(x) \) is the vector current of quark field or quark number current. Now we define the precise form of the retarded Green functions which we discuss later:

\[ G_{\mu \nu}(k) = -i \int d^4x e^{-ikx} \theta(t) \langle [J_\mu(x), J_\nu(0)] \rangle. \]

(2.5)

The quark number susceptibility was proposed as a probe of the QCD chiral phase transition at zero chemical potential [3, 5],

\[ \chi_q = \frac{\partial n_q}{\partial \mu_q}. \]

(2.6)

Later it has been shown that the quark number susceptibility can be rewritten in terms of the retarded Green function through the fluctuation-dissipation theorem [3, 5],

\[ \chi_q(T, \mu) = -\lim_{k \to 0} \text{Re} \left( G_t(\omega = 0, k) \right), \]

(2.7)

where \( G_{\mu \nu}(\omega, k) \) is the retarded Green function, which is defined in (2.5).
3. Reissner-Nordström-AdS background

In this section, we briefly review the Reissner-Nordström-AdS (RN-AdS) background in 5D spacetime first. Afterwards, we explain the interpretation of RN-AdS spacetime in terms of the D3/D7-brane setting.

We consider the Einstein-Maxwell action with negative cosmological constant:

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left( R - 2\Lambda \right) - \frac{1}{4e^2} \int d^5x \sqrt{-g} F_{mn} F^{mn},$$

(3.1)

where we denote the gravitation constant and the cosmological constant as $\kappa^2 = \frac{8\pi G}{5}$ and $\Lambda$, respectively. The $U(1)$ gauge field strength is given by

$$F_{mn}(x) = \partial_m A_n(x) - \partial_n A_m(x).$$

The RN-AdS background is a solution of the equations of motion derived from this action, and it is given by

$$ds^2 = r^2 \left( \frac{1}{l^2} - f(r)(dt)^2 + (d\vec{x})^2 \right) + \frac{l^2}{r^2 f(r)} (dr)^2,$$

(3.2a)

$$A_t = -\frac{Q}{r^2} + \mu,$$

(3.2b)

with

$$f(r) = 1 - \frac{m l^2}{r^4} + \frac{q^2 l^2}{r^6}, \quad \Lambda = -\frac{6}{l^2}, \quad e^2 = \frac{2Q^2}{3q^2} \kappa^2.$$

The parameters $l$, $m$, and $q$ are the AdS radius, mass and charge, while $Q$ and $\mu$ are $U(1)$ charge and constant which is interpreted as the chemical potential. Since the gauge potential $A_t(x)$ must vanish at the horizon, the charge $Q$ and the chemical potential $\mu$ are related as

$$\mu = \frac{Q}{r_+^2}.$$

(3.3)

The horizons of the RN-AdS black hole are located at the zero for $f(r)^{1}$,

$$f(r) = 1 - \frac{m l^2}{r^4} + \frac{q^2 l^2}{r^6} = \frac{1}{r_0} \left( r^2 - r_+^2 \right) \left( r^2 - r_-^2 \right) \left( r^2 - r_0^2 \right),$$

(3.4)

where the explicit forms of the horizon radii are given by

$$r_+^2 = \left( \frac{m}{3q^2} \left( \cos \left( \frac{\theta}{3} + \frac{4}{3}\pi \right) \right) \right)^{-1},$$

(3.5a)

$$r_-^2 = \left( \frac{m}{3q^2} \left( \cos \left( \frac{\theta}{3} \right) \right) \right)^{-1},$$

(3.5b)

$$r_0^2 = \left( \frac{m}{3q^2} \left( \cos \left( \frac{\theta}{3} + \frac{2}{3}\pi \right) \right) \right)^{-1},$$

(3.5c)

with $r_+^2 + r_-^2 = -r_0^2$. Here

$$\theta = \arctan \left( \frac{3\sqrt{3}q^2 \sqrt{4m^3 l^2 - 27q^4}}{2m^3 l^2 - 27q^4} \right).$$

(3.4)

In order to define the horizon, the charge $q$ must satisfy a relation $q^4 \leq 4m^3 l^2 / 27$. 

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The positions expressed by $r_+$ and $r_-$ correspond to the outer and inner horizon, respectively. It is useful to notice that the charge $q$ can be expressed in terms of $\theta$ and $m$ by

$$q^4 = \frac{4m^3l^2}{27} \sin^2 \left(\frac{\theta}{2}\right).$$

The outer horizon takes a value in

$$\sqrt{\frac{m}{3}} l \leq r_+^2 \leq \sqrt{ml},$$

where the upper bound and the lower bound correspond to the case for $q = 0$ and the extremal case, respectively.

The temperature is defined from the conical singularity free condition around the horizon $r_+$,

$$T = \frac{r_+^2 f'(r_+)}{4\pi l^2} = \frac{r_+}{\pi l^2} \left(1 - \frac{1}{2} \frac{q^2 l^2}{r_+^6}\right) = \frac{1}{2\pi b} \left(1 - \frac{a}{2}\right), \quad (> 0),$$

where

$$a \equiv \frac{q^2 l^2}{r_+^6}, \quad b \equiv \frac{l^2}{2r_+^3}.$$ (3.7)

In the limit $q \to 0$, these parameters go to

$$a \to 0, \quad b \to \frac{l^{3/2}}{2m^{1/4}},$$

and the temperature becomes

$$T \to T_0 = \frac{m^{1/4}}{\pi^{3/2}}.$$ (3.8)

It might be useful to rewrite the parameters $a$ and $b$ in terms of the temperature and the chemical potential,

$$a = 2 - \frac{4}{1 + \sqrt{1 + 4(\bar{\mu}/T)^2}}, \quad b = \left(\frac{1}{\pi T}\right) \frac{1}{1 + \sqrt{1 + 4(\bar{\mu}/T)^2}},$$

where we denote $\bar{\mu} \equiv \mu \sqrt{8\pi G_5/(3(\pi e)^2)}$.

For later convenience, we shall introduce new dimensionless coordinate $u \equiv r_+^2/r^2$ which is normalized by the outer horizon. In this coordinate system, the horizon and the boundary are located at $u = 1$ and $u = 0$, respectively. The background metric (3.2a) can be rewritten as

$$ds^2 = \frac{l^2}{4b^2u^2} \left(-f(u)(dt)^2 + (d\vec{x})^2\right) + \frac{l^2}{4u^2 f(u)} (du)^2,$$ (3.9)

with

$$f(u) = (1 - u)(1 + u - au^2).$$

Now let us explain the interpretation of this RN-AdS spacetime in terms of the brane setting. We consider $N_c$ D3-branes and treat them as a gravitational background i.e.
AdS$_5 \times S^5$. In order to consider the baryon charge at the boundary theory, we introduce $N_f$ D7-branes. The D7-branes are wrapping on $S^3$ of $S^5$, and we neglect this $S^3$ dependence. Here we use the probe approximation for the D7-branes, so that a backreaction from “the D7-brane tension” is neglected. Then the effective action becomes that for 5D gauge theory. The baryon current corresponds to $U(1)$ gauge field on this flavor D7-branes. The 5D effective action of the system might be given by the action (3.1) with

$$\frac{l^3}{\kappa^2} = \frac{N_c^2}{4\pi^2}, \quad \frac{l}{e^2} = \frac{N_c N_f}{4\pi^2}. \quad (3.10)$$

The baryon charge which can be identified by $Q$ and the charge of RN-AdS $q$ are then related by [27]

$$q^2 = \frac{2}{3} \frac{N_f}{N_c} l^2 Q^2. \quad (3.11)$$

By using the relation (3.11), one might understand that AdS leads RN-AdS through a backreaction from “the $U(1)$ baryon charge” on D7-branes. However this does not necessarily imply that $N_c$ and $N_f$ are of the same order of magnitude. The baryon charge is provided by open strings between D3- and D7-branes. We can introduce large numbers of these strings even for small $N_f$. Then, the geometry receives the backreaction from the charge even though the D7-branes itself is treated as a probe. We here consider the case in which $N_f$ is finite while the charge $Q$ is large. This can be understood in the large $N_c$ context through the relation (3.11). The finite contribution of the charge $q$ of RN-AdS could be only considerable if the large baryon charge $Q(\propto \sqrt{N_c})$ is taken in the large $N_c$.

We will evaluate the quark number susceptibility using the hard wall model [28, 29] and soft wall model [30] with the RN-AdS background. Note that the Hawking-Page type transition in both models is studied in [31]. In the hard wall model to ensure the confinement a sharp infrared (IR) cutoff is introduced, while in the soft wall model a dilaton-like field is coupled to the 5D bulk action. At high temperature, due to the Hawking-Page type transition discussed in [31], AdS black hole background is stable. In this case the black hole horizon completely covers up the IR cutoff of the hard wall model, while in the soft wall model an IR scale, which is associated with the dilaton-like field, is still visible. So the only energy scale in the hard wall model at high temperature is the temperature itself. In the soft wall model, however, we have two energy scales, temperature and the IR scale. A consequence of this is that the quark number susceptibility in the hard wall model turns out to be temperature-independent, while that in the soft wall model shows non-trivial temperature dependence at high temperature. The latter is close to the observations made in lattice QCD and also in QCD models.

### 3.1 Quark number susceptibility in hard wall model

In this subsection, we shall discuss the quark number susceptibility in the hard wall model on the RN-AdS background. The 5D action of the gauge field, which is dual to the 4D quark number current $j_\mu(x) = \bar{q}(t, \vec{x}) \gamma_\mu q(t, \vec{x})$, is

$$S = -\frac{1}{4g_5^2} \int d^5x \sqrt{-g} F_{mn} F^{mn}, \quad (3.12)$$
where $g_5$ is the 5D gauge coupling constant. In this work we consider two different values of the gauge coupling constant: $g_5^2 = 12\pi^2/N_c$ from the hard wall model \cite{28, 29} and $g_5^2 = 4\pi^2 l/(N_c N_f)$ from D3/D7.

To obtain the quark number susceptibility, we use the action (3.12) with the metric and gauge perturbations around the classical background (3.2a) and (3.2b). Since the Green function which provides the quark number susceptibility (2.7) is the current-current correlator $G_t t(k)$, we here need to consider the scalar type in the metric perturbation. We follow the procedure in \cite{24} and refer to the result

$$G_t t(\omega, k) = -\frac{lk^2}{4g_5^2(1 + a)b^2} \left\{ \frac{9a}{k^2 - 3\omega^2} + \frac{2(2 - a)^2 b}{D_p(\omega, k)} \right\}, \quad (3.13)$$

where

$$D_p(\omega, k) = 2(2 + a)bk^2 - 4i(1 + a)\omega - (2 - a)^2 bD_+ \omega^2,$$

with

$$D_+ = \frac{2}{(2 - a)^4(1 + 4a)^{3/2}} \left\{ -27(2 - a)a^2 \sqrt{1 + 4a} \right.$$  

$$\left. + 4(1 + 4a)^{3/2}(1 + a)^3 \log(2 - a) \right.$$  

$$\left. + 4(2 - 2a + 41a^2)(1 + a)^2 K_1(1) \right\},$$

$$K_1(1) = \frac{1}{2} \log(2 - a) - \log \left(1 - \frac{2a}{1 + \sqrt{1 + 4a}}\right).$$

By using the formula (2.7), we obtain

$$\chi_q(T, \mu) = \frac{l}{2g_5^2 b^2} \left( \frac{2 + 5a}{2 + a} \right), \quad (3.14)$$

where $a$ and $b$ are given through the definition (3.8),

$$b^{-1} = \pi T \left( 1 + \sqrt{1 + \frac{16l}{3(N_c g_5)^2}} \left( \frac{\mu}{T} \right)^2 \right), \quad a = 2 - 4\pi Tb. \quad (3.15)$$

In the charge free case $\mu = 0$, our result agrees with that in \cite{13}.

Before going on further with the quark number susceptibility, we briefly discuss the Hawking-Page type transition. As in \cite{31}, there exists Hawking-Page type transition in the hard wall model and soft wall model. At low temperature in confined phase, thermal AdS dominates the partition function, while at high temperature in deconfined phase, AdS black hole geometry dominates. Therefore, the quark number susceptibility is described by the AdS black hole background at high temperature and by the thermal AdS at low temperature. The critical temperature for deconfinement is $\sim 120$ MeV in the hard wall model and $T_c \sim 190$ MeV in the soft wall model \cite{31}. A similar critical temperature was estimated as $T_c \sim 210$ MeV by using the soft metric model \cite{32}. Note that the value of the critical temperature depends on the number of quark flavors, quark mass and quark number
density [33, 34]. To obtain the critical temperature in the present case, we have to do the Hawking-Page type analysis with charged thermal AdS and RN-AdS backgrounds. Since the charged thermal AdS background has not been discovered, we could not precisely fix the critical temperature for the deconfinement transition. For simplicity, we assume that the critical temperature is around the value obtained in [31]: \( T_c = 100 \text{ MeV} \) in the hard wall model and \( T_c = 200 \text{ MeV} \) in the soft wall model.

The result in (3.14) is shown in Figure 1, where the gauge coupling of D3/D7 has been used. We confirmed that the result with the gauge coupling from the hard wall model shows a similar behavior. Below \( T_c \) we adopt the quark number susceptibility calculated in the thermal AdS background [13]. As well known, the thermal AdS would not exhibit the temperature dependence. Therefore, the quark number susceptibility would be a temperature independent constant, and it turns out zero [13]. For high temperature regime, we use different values of the chemical potential, \( \mu = 0.0 \sim 0.8 \text{ GeV} \). Approaching \( T_c \) from high temperature regime, we observe in Figure 1 that \( \chi_q/T^2 \) shows a blow-up behavior as we increase \( \mu \), which may indicate the existence of the CEP in the QCD phase diagram. Compared to lattice QCD, our study has an advantage that we do not need to do a Taylor expansion with respect to the chemical potential. However, our study has a limitation that we cannot study the temperature dependence of the quark number susceptibility below the critical temperature, which is a generic problem in a model study based on the AdS/CFT due to large \( N_c \) nature. Also, the RN-AdS is describing a QCD-like or QGP-like system. For instance the relation between the chemical potential and its conjugate charge (or number) density is given by \( Q \sim \mu T^2 \), see (3.3). In a realistic system like QGP, the relation in general takes the following form \( Q \sim a\mu^3 + b\mu T^2 \), where \( a \) and \( b \) are constants.

![Figure 1: \( \chi_q/(N_cN_fT^2) \) in the hard wall model for varying \( \mu \text{(GeV)} \) with \( N_c = 3 \) and \( N_f = 2 \).](image-url)
3.2 Quark number susceptibility in soft wall model

We shall work in the soft wall model \[30\]. We consider the following action

\[ S = -\frac{1}{4g^2} \int d^5x \sqrt{-g} \ e^{-\Phi} F_{mn} F^{mn}, \]

(3.16)

with the dilaton-like field \( \Phi(x) \) together with \( U(1) \) gauge field \( A_m(x) \). We make use of the RN-AdS geometry (3.2a) and (3.2b) as the background.

In \[30\], the dilaton like field is given by \( \Phi(x) = \frac{l^4}{\tilde{c}} \frac{\tilde{c}}{r^2} \). Rewriting it in terms of \( u \), we have

\[ \Phi(x) = cu, \]

(3.17)

where \( c \equiv \frac{l^4}{\tilde{c}} \frac{\tilde{c}}{r^2} \). We shall work in \( A_u(x) = 0 \) gauge and use the Fourier decomposition

\[ A_\mu(t, z, u) = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t + ikz} A_\mu(k, u), \]

(3.18)

where we choose the momenta which are along the \( z \)-direction. Equations of motion with respect to \( A_t(u) \) and \( A_u(u) \) are given as,

\[ 0 = A'''_t + \left( e^{-\Phi} f \right)' A'_t - \frac{b^2}{uf} \left( k^2 A_t + k\omega A_z \right), \]

(3.19a)

\[ 0 = \omega A'_t + k f A'_z, \]

(3.19b)

where the prime stands for the derivative with respect to \( u \). An equation of motion for \( A_z(u) \) can be derived from (3.19a) and (3.19b). For \( A_x(u) \) and \( A_y(u) \), one can obtain decoupled second order ordinary differential equations. Since we are interested in the time-time component of the retarded Green function to calculate the quark number susceptibility, we will not consider \( A_x(u) \) and \( A_y(u) \) hereafter.

3.2.1 Solution for \( A_t \)

From the equations (3.19a) and (3.19b), we obtain an equation of motion

\[ 0 = A'''_t + \left( e^{-\Phi} f \right)' A''_t + \left( \frac{b^2}{uf} - \frac{(\Phi' uf)'}{uf} \right) A'_t. \]

(3.20)

Since the equation (3.20) is an ordinary second order differential equation for \( A'_t(u) \) with a regular singularity at the horizon \( u = 1 \), we first write the solution as \( A'_t(u) = (1 - u)^\nu F(u) \) where \( F(u) \) might be a regular function at the horizon. The parameter \( \nu \) could be fixed as

\[ \nu = -i \frac{\omega}{4\pi T}, \]

(3.21)

by imposing the incoming wave condition.

Now we solve the equation of motion in the hydrodynamic regime i.e. small \( \omega \) and \( k \) compared with the temperature \( T \). In order to do the perturbative analysis it might be useful to factorize the dilaton factor from \( F(u) \),

\[ F(u) = e^{\Phi(u)} \bar{F}(u), \]

(3.22)
so that the equation (3.20) can be expressed as a simpler form

$$0 = \left( e^\Phi u f \tilde{F}' \right)' + i \omega \frac{2b}{2-a} e^\Phi u (1 + u - au^2) \tilde{F}' + i \omega \frac{b}{2-a} \left( e^\Phi u (1 + u - au^2) \right)' \tilde{F}$$

$$+ \frac{\omega^2 b^2}{(2-a)^2 (1 + u - au^2)} \left((2-a)^2 + (1-a)(3-a)u \right)$$

$$+ (1 - 4a + a^2)u^2 - a(2 - a)u^3 + a^2 u^4 \right) \tilde{F}$$

$$- k^2 b^2 e^\Phi \tilde{F}.$$ 

(3.23)

The function $\tilde{F}(u)$ is now expanded as

$$\tilde{F}(u) = F_0(u) + \omega F_\omega(u) + k^2 F_{k^2}(u) + \mathcal{O}(\omega^2, \omega k^2),$$

(3.24)

where the functions $F_0(u), F_\omega(u)$ and $F_{k^2}(u)$ are determined by imposing suitable boundary conditions. The solution can be obtained recursively. The result is as follows:

$$F_0(u) = C, \quad \text{(const.)} \quad \text{(3.25a)}$$

$$F_\omega(u) = i C b e^c \left\{ E_i(-cu) + K_1(u) - K_1(0) ight\}$$

$$- \frac{e^{-c}}{2-a} \left( E_i(c(1-u)) - E_i(c) - \log(1-u) \right), \quad \text{(3.25b)}$$

$$F_{k^2}(u) = - \frac{C b^2 e^c}{c} \left\{ E_i(-cu) + K_1(u) - K_1(0) ight\}$$

$$- \frac{e^{-c}}{2-a} \left( E_i(c(1-u)) - E_i(c) - \log(1-u) \right)$$

$$- \frac{(1+a)e^{-c}}{2(2-a)\sqrt{1+4a}} \left( \log \left( \frac{1 + 1 - 2au}{1 \sqrt{1+4a}} \right) - \log \left( \frac{1 - 1 - 2au}{1 \sqrt{1+4a}} \right) \right)$$

$$- e^{-c} \log u + \frac{(1-a)e^{-c}}{2(2-a)} \log(1 + u - au^2) \right\}, \quad \text{(3.25c)}$$

with

$$K_1(u) = \frac{1}{2(2-a)\sqrt{1+4a}} \left\{ e^{-\frac{c}{2a}(1+\sqrt{1+4a})} \left( (1+a) - (1-a)\sqrt{1+4a} \right) \right\}$$

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2 The derivation of the solutions is given in Appendix A.

3 The function $Ei(x)$ is an exponential integral

$$Ei(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt.$$ 

This can be expanded as follows:

$$E_i(x) = \gamma + \log x + x + \mathcal{O}(x^2),$$

where $\gamma$ is the Euler constant.
\[ \times E_i \left( \frac{c}{2a} (1 + \sqrt{1 + 4a} - 2au) \right) \\
- e^{-\frac{b}{c} (1 - \sqrt{1 + 4a})} \left( (1 + a) + (1 - a)\sqrt{1 + 4a} \right) \\
\times E_i \left( \frac{c}{2a} (1 - \sqrt{1 + 4a} - 2au) \right) \right]. \]

Let us consider the integration constant \( C \). This could be estimated in terms of the boundary values of the fields

\[
\lim_{u \to 0} A_t(u) = A_t^0, \quad \lim_{u \to 0} A_z(u) = A_z^0.
\]

Using the equation of motion (3.19a), a relation

\[
\lim_{u \to 0} \left( uf \left( A''_t - \Phi' A'_t \right) \right) = b^2 \left( k^2 A_t^0 + \omega k A_z^0 \right)
\]

should hold. Therefore we may fix the constant \( C \) as

\[
C = \frac{b \left( k^2 A_t^0 + \omega k A_z^0 \right)}{e^c \left( i\omega - \frac{b}{c} (1 - e^{-c}) k^2 \right)}.
\]

One can see the existence of the hydrodynamic pole in the complex \( \omega \)-plane.

Near the boundary the obtained solution leads a relation between the radial derivative of the fields and its boundary values

\[
A'_t(u) = b^2 \left( k^2 A_t^0 + \omega k A_z^0 \right) \log u \\
+ \frac{1}{e^c \left( i\omega - \frac{b}{c} (1 - e^{-c}) k^2 \right)} \left\{ b \left( k^2 A_t^0 + \omega k A_z^0 \right) + O(\omega k^2, k^4) \right\} + O(u).
\]

One can also obtain a similar relation for \( A'_z(u) \) through the equation (3.19b).

### 3.2.2 Retarded Green functions

We now proceed to evaluate the Minkowski correlators. An on-shell action can be obtained from (3.16),

\[
S_0[A] = \frac{l}{4g_s^2 b^2} \int \frac{d^4k}{(2\pi)^4} e^{-\Phi(u)} \left( A_t(-k, u) A'_t(k, u) - f(u) A_z(-k, u) A'_z(k, u) \right) \bigg|_{u=1}^{u=0}.
\]

By using the relation (2.4) and the definition (2.3), we can read off the correlators in the hydrodynamic approximation,

\[
G_{tt}(\omega, k) = \frac{l}{2g_s^2 b} \frac{k^2}{e^c \left( i\omega - Dk^2 \right)},
\]

\[
G_{tt}(\omega, k) = -\frac{l}{2g_s^2 b} \frac{\omega k}{e^c \left( i\omega - Dk^2 \right)},
\]

\[
G_{zz}(\omega, k) = \frac{l}{2g_s^2 b} \frac{\omega^2}{e^c \left( i\omega - Dk^2 \right)}.
\]
where we have introduced the following local counter term in the boundary to remove the logarithmic singularity:

\[ S_{ct} = \frac{l}{8g_5^2} \log \varepsilon \int d^4x \sqrt{-g^{(4)}} F_{\mu\nu} F^{\mu\nu}. \]

The constant \( D \) is the diffusion constant

\[
D = \frac{b}{c}(1 - e^{-c}). \tag{3.31}
\]

Using the correlator \( 3.30a \), we can obtain the quark number susceptibility \( \chi_q \) in terms of the temperature and the chemical potential through the definition \( 2.7 \),

\[
\chi_q(T, \mu) = \frac{l}{2g_5^2b^2} \left( \frac{c}{e^c - 1} \right). \tag{3.32}
\]

In the charge free case it reduces to

\[
\chi_q(T) = \frac{2\pi^2 T^2}{g_5^2} \left( \frac{c}{e^c - 1} \right), \tag{3.33}
\]

which is different from \[13\]. We confirmed that \( 3.33 \) is correct starting from the 5D AdS-Schwarzschild background. In terms of \( \tilde{c}(=cr^2_+/l^4) \), the susceptibility \( 3.32 \) is given by

\[
\chi_q(T, \mu) = \frac{2l\tilde{c}}{g_5^2(e^{4b^2\tilde{c}} - 1)}. \tag{3.34}
\]

In the soft wall model \[30\], \( \tilde{c} \) is fixed by the rho meson mass. In the present case we cannot use hadronic observables such as masses or couplings to fix the constant since we are working in the black hole phase, where light mesons are to be melted away. Here we take another route to fix it. We compare our \( c_2 \) defined below, equation \( 3.35 \), with that from lattice QCD \[14\] at \( T = T_c \) and choose \( \tilde{c} \) to reproduce the lattice result.

\[
\chi_q/T^2 = \sum_{n} 2n(2n - 1)c_{2n}(\mu/T)^{2(n-1)}. \tag{3.35}
\]

The quark number susceptibility with the chemical potential is shown in Figure 2 where the gauge coupling from D3/D7 has been used. For high temperature regime, we use different values of the chemical potential, \( \mu = 0.0 \sim 1.6 \) GeV. Again we find that the quark number susceptibility shows a blow-up behavior as we lower the temperature to the critical temperature, thereby indicating the existence of the CEP in the \( (\mu, T) \) plane QCD phase diagram.

### 3.3 Implication in QCD phase diagram

Here we discuss our results in the light of QCD phase diagram. As well known, the nature of the QCD transition does depend on the number of quark flavors and the value of the quark mass. For pure \( SU(3) \) gauge theory with no quarks, it is the first order. In the case of two massless and one massive quarks, the transition is the second order at zero or
small quark chemical potentials, and it becomes the first order as we increase the chemical potential. The point where the second order transition becomes the first order is called tricritical point. With physical quark masses of up, down, and strange, the second order at zero or low chemical potential becomes the crossover, and the tricritical point turns into the critical end point, see [16] for reviews.

Now what can we say about the QCD phase structure based on our study? Our approach is based on AdS/CFT, and so it inherently probes QCD (or QCD-like theory) at large $N_c$. The transition suggested by our study is manifestly first order as shown in the figures, 1 and 2, where the quark number susceptibility shows a discontinuous jump at the transition point. This discontinuity is there since we use thermal AdS at low temperature and RN-AdS at high temperature. This first order nature of the transition would be the end of story as long as we don’t consider $1/N_c$ corrections. Though we may not be able to assemble all the leading $1/N_c$ corrections in a consistent way, we discuss some speculation on what could come out with those corrections. For zero chemical potential, as shown in Figs. 1 and 2, we have two lines at low and high temperature regimes. There is a hope that with $1/N_c$ corrections those two straight lines are connected with no discontinuity at zero or small chemical potentials since the discontinuous gap is not much big. If this happens, the transition could be the second order or the crossover. When we increase the chemical potential, this possibility does not seem plausible since the quark number susceptibility in high temperature regime blows up near the transition point and also the transition point will move to the left, i.e. the transition temperature decreases with the chemical potential [33, 34]. This means that the quark number susceptibility at low temperature, which is zero with no $1/N_c$ corrections, should increase very rapidly to realize smooth change at the transition temperature. With $1/N_c$ corrections, this will not be possible. Therefore at high chemical potential, the first order nature of the transition will persist regardless of the presence of $1/N_c$ corrections, while for small chemical potentials it may
change to the second order or the crossover due to the $1/N_c$ corrections. However, we emphasize here again that this is just a speculation.

In short, the transition from our study is the first order, and it may, however, become the second order or the crossover with $1/N_c$ corrections.

4. Quark number susceptibility under magnetic field

In this section, we study the quark number susceptibility with an external magnetic field turned on. The basic motivation is due to the observation that the constant magnetic field enhances the dynamical chiral symmetry breaking $\langle \bar{q}q \rangle \sim |eB|$ and generates the dynamical quark mass $m_{\text{dyn}}^q = f(|eB|)$ \[25\]. Using an effective low energy QCD model, linear sigma model with quarks, the authors of \[35\] showed that with increasing magnetic field the QCD transition changes from the crossover to the first order. This implies that there exists the CEP in $(B, T)$ plane as we raise the magnetic field. Therefore, we would expect that the external magnetic field affects the behavior of the quark number susceptibility at finite temperature. For instance, the presence of the CEP means a peak in the value of quark number susceptibility as we increases the magnetic field. So the main motivation of our study with the magnetic field is to observe if the peak appears with the magnetic field. In addition, a recent study \[26\] shows that sufficiently large magnetic fields are likely created in relativistic heavy ion collisions, and so our study may be tested in experiments.

To calculate the quark number susceptibility at finite temperature, we consider here the non-extremal AdS$_5 \times S^5$,

$$ds^2 = \frac{l^2(\pi T)^2}{u^2} \left( -f(u)(dt)^2 + (d\vec{x})^2 \right) + \frac{l^2}{4u^2f(u)}(du)^2 + l^2d\Omega_5^2,$$  \hspace{1cm} (4.1)

where $f(u) = 1 - u^2$ and $T$ is the temperature. The gauge field comes from the probe D7-brane whose action reads

$$S_{D7} = -N_f T_7 \int d^8x \ e^{-\phi} \sqrt{-\det(G_{MN} + 2\pi\alpha' F_{MN})},$$  \hspace{1cm} (4.2)

where $T_7 = 1/(2\pi l_s^8)$ and $e^\phi = g_s$. $G_{MN}(x)$ is the induced metric which we consider as the trivial one. Here the external magnetic field $B$ enters the action as \[36\]

$$F_{MN} = F_{MN}^{(0)} + F_{MN}, \quad F_{xy}^{(0)} = B.$$  \hspace{1cm} (4.3)

Then, wrapping the D7-brane on $S^3$ and taking \[4\]

$$E_{MN} = G_{MN} + F_{MN}^{(0)},$$  \hspace{1cm} (4.4)

we get the 5D action \[5\]

$$S_{5D} = -\frac{1}{4g_5^2} \int d^4x \ du \sqrt{-\det(E_5)} \ (E_5^{-1})^{ml}(E_5^{-1})^{kn}F_{mn}F_{lk},$$  \hspace{1cm} (4.5)

\[4\]One may be tempted to use the hard wall or soft wall model for simplicity. In this case, however, the magnetic field does not affect the equations of motion for gauge fields.

\[5\]Since $(E_5^{-1})^{mn}$ is not symmetric, there exist additional terms in the action, which are the powers of $(E_5^{-1})^{mn}F_{nm}$. However, our choice of 4-momentum, $k^\mu = (\omega, 0, 0, k)$, makes those terms vanish.
where \( g_5 = \sqrt{4\pi^2/(N_c N_f)} \) and we have defined an inverse of 5D part of \( E_{MN} \) i.e. \((E_5^{-1})^m l = E_{5m}(E_5^{-1})^l m = \delta^m_l \). As we did in the previous section, we take the gauge \( A_u(x) = 0 \) and the same Fourier decomposition as \((3.18)\). The resulting equations of motion with respect to \( A_t(u) \) and \( A_u(u) \) lead

\[
\begin{align*}
0 &= X_{t z}^{-1}(X_{tu} A'_l)' - (k^2 A_t + k \omega A_z), \quad (4.6a) \\
0 &= \omega A'_t + k f A'_z, \quad (4.6b)
\end{align*}
\]

where we have defined

\[
X_{t z} = \sqrt{-\det(E_5)(E_5^{-1})^{t t}(E_5^{-1})^{z z}}, \quad X_{tu} = \sqrt{-\det(E_5)(E_5^{-1})^{t t}(E_5^{-1})^{u u}}. \quad (4.7)
\]

Differentiating the equation \((4.6a)\) with respect to \( u \) and using the equation \((4.6b)\), we obtain

\[
0 = \left(X_{t z}^{-1}(X_{tu} A'_l)'ight)' - (k^2 - \omega^2 f^{-1}) A'_l, \quad (4.8)
\]

Now using the hydrodynamic expansion \( A'_t(u) = (1 - u)\nu \left(F_0(u) + \omega F_\omega(u) + k^2 F_{k^2}(u) + \mathcal{O}(\omega^2, \omega k^2)\right)\), where \( \nu = -i \omega/(4\pi T) \) as the incoming wave condition, we get the equations of motion for \( F_0(u) \) and \( F_{k^2}(u) \), respectively,

\[
\begin{align*}
0 &= (X_{t z}^{-1}(X_{tu} F_0)')', \quad (4.9a) \\
0 &= (X_{t z}^{-1}(X_{tu} F_{k^2})')' - F_0. \quad (4.9b)
\end{align*}
\]

In \((4.9a)\), \( X_{tu} F_0(u) \) should be a constant (\( \equiv C_0 \)) to avoid the singularity at \( u = 1 \) due to \( X_{t z}^{-1}(u) \rightarrow 0 \) as \( u \rightarrow 1 \). As a result, we obtain

\[
F_0 = C_0 X_{tu}^{-1}. \quad (4.10)
\]

Using this solution, the equation \((4.9b)\) is recasted as

\[
X_{t z}^{-1}(X_{tu} F_{k^2})' = C_0 S(u) \quad \text{with} \quad S(u) = \int_1^u du' X_{tu}^{-1}(u'). \quad (4.11)
\]

Then, we insert the above solutions into \((4.6a)\) to obtain

\[
\begin{align*}
k^2 A_t + k \omega A_z &= X_{t z}^{-1}(X_{tu} A'_l)' \\
&= X_{t z}^{-1}(X_{tu} k^2 F_{k^2})' + \mathcal{O}(\omega) \\
&= k^2 C_0 S(u) + \mathcal{O}(\omega). \quad (4.12)
\end{align*}
\]

Since we will take \( \omega = 0 \), this equation determines \( A_t(u) \). Note that \( A_t(u) \) obtained from this procedure is the zeroth order term of the series solution, since we are substituting \( A'_l(u) \) evaluated to \( \mathcal{O}(\omega, k^2) \). In fact, we should integrate the equation \((4.11)\) once more to get the solution of \( \mathcal{O}(k^2) \), which cannot be done analytically. However, to get the susceptibility, only the zeroth order solution is needed since \( k \rightarrow 0 \) limit of Green function.
with $\omega = 0$ implies the contribution of the zeroth order only. Thus, from (1.12), the zeroth order solution is now

$$A_t(u) = C_0 S(u) = A_0^\mu S(u)/S(0), \quad (4.13)$$

where $A_0^\mu \equiv A_\mu(u)|_{u=0}$.

As a result, we get the retarded Green function and the quark number susceptibility

$$\chi_q = -\lim_{k \to 0} \text{Re} \left( G_{t \ell}(\omega = 0, k) \right) = \frac{1}{g_5^2} [S(0)]^{-1}. \quad (4.14)$$

Then, using the explicit form of $X_{t\mu}(u)$ from (4.7), we obtain

$$\chi_q(T, B) = \frac{1}{g_5^2} \left[ \int_1^0 \frac{du}{X_{t\mu}} \right]^{-1}
= \frac{1}{g_5^2} \left[ \int_1^0 \frac{-ldu}{2\sqrt{(2\pi\alpha'B)^2 u^2 + (\pi T)^4}} \right]^{-1}
= \frac{4\pi\alpha' B}{g_5^2 l} \left[ \text{arcsinh} \left( \frac{2\pi\alpha'B}{(\pi T)^2} \right) \right]^{-1}. \quad (4.15)$$

When $B = 0$, we observe that $\chi_q/T^2 \sim T^0$ (constant), which agrees with the result of the hard wall model at $\mu = 0$. This should be so since the hard wall model action used is nothing but the leading term of Dirac-Born-Infeld action for the D7 probe brane after perturbative expansions of the action in terms of $\alpha'$.

The plot of $\chi_q/T^2$ for varying $B(\equiv 2\pi\alpha' B)$ is given in Figure 3, where we take $B = (0, 1, 2, 3, 4)$ from bottom to top. We find that the quark number susceptibility increases rapidly with increasing $B$ as we lower $T$ from high temperature regime. Note that since D3/D7 model does not exhibit the confined phase, we plot our quark number susceptibility in entire finite temperature regime. This observation itself is very new, though the blow-up behavior at low temperature could be expected by the study of (35). By studying the magnetic field dependence of a modified potential, the authors of (35) showed that with increasing magnetic field the QCD transition changes from the crossover to the first order, which implies the existence of the CEP, and so the diverging behavior of the quark number susceptibility. In this sense, our study support indirectly the result obtained in (35). But, our study made in this section would not address the QCD transition itself due to the absence of the confined phase in D3/D7 model. To improve this defect and to see if the blow-up behavior is universal regardless of the gravity background, we may consider D4/D6 or D4/D8 model, which is relegated to a future study including the effect of a finite quark mass. Note that the finite quark mass seems soften the peak in the quark number susceptibility (37).

Before closing this section, we compare $\chi_q/T^2$ in D3/D7 and D4/D8 (38) at finite temperature with no magnetic field. We prepare a basic setup to calculate the quark number susceptibility with an external magnetic field in D4/D8 in Appendix B. In (39), from the point of view of the chiral symmetry breaking, an external magnetic field was
Figure 3: $\chi_q/(N_c N_f T^2)$ for varying $\tilde{B} = 2\pi \alpha' B$ with $N_c = 3$ and $N_f = 2$. Here $\tilde{B} = (0, 1, 2, 3, 4)$ from bottom to top.

considered in D4/D8. Although we do not provide the result in Appendix B, we can see the behavior resulting from D4/D8 system by look through the $B = 0$ case. In that case, Chern-Simons terms do not contribute and we can easily get the result

$$\chi_q(T) \sim \left[ \int_1^0 \frac{-du}{\sqrt{T^6 u - 1}} \right]^{-1} \sim T^3. \tag{4.16}$$

This result is, however, different from what we obtained in D3/D7 system. The different $T$ dependence of $\chi_q$ come from the different $T$ dependence of the horizons and from the different exponents of $u$ in the integrands. The dimensionful parameter $l$ compensates the different powers of $T$. On the baryonic density and susceptibilities in D4/D8 model, we refer to \[10\].

5. Summary

We studied the quark number susceptibility with the finite quark chemical potential or under the external magnetic field at high temperature. We first considered the hard wall model in the RN-AdS background and observed that as we lower the temperature starting from a high temperature, the quark number susceptibility shows a peak with increasing $\mu$. When $\mu = 0$, however, the quark number susceptibility is independent of the temperature both at low and high temperatures. To improve this at least in high temperature regime, we move to the soft wall model and found that at high temperature the quark number susceptibility increases with the temperature for $\mu = 0$, and observed a similar blow-up behavior as we lowered the temperature with increasing chemical potential. This peak may imply the existence of the CEP in QCD phase diagram on $(\mu, T)$ plane. We discussed implication of our results to the QCD phase diagram in 3.3, where we concluded that the transition from our study is the first order, and it may, however, become the second order or the crossover with $1/N_c$ corrections.
We then calculated the quark number susceptibility under the external magnetic field. As we raise the magnitude of the magnetic field, we observed a similar rising-up behavior as we lowered the temperature with increasing magnetic field. It will be interesting if this behavior in \((B, T)\) plane is to be confirmed or disconfirmed by lattice QCD or any other studies.

Finally, we compared our D3/D7 model study with D4/D8 at high temperature to find

\[ \chi_q \sim T^2, \quad \text{D3/D7}, \]
\[ \chi_q \sim T^3, \quad \text{D4/D8}. \]

We observed that the \(T\) dependence of \(\chi_q\) is sensitive to the background geometry. In QGP, \(\chi_q/T^2\) saturates to a constant value, ideal gas limit, at sufficiently high temperature. It is interesting to see that D3/D7 model shows this feature. This might imply that the blow-up behavior in \((\mu, T)\) or \((B, T)\) plane may vary with the gravity background. Therefore it will be interesting to see if the blow-up behavior is universal. On top of this, finite quark mass effect on the quark number susceptibility is also interesting to investigate. These issues will be addressed in a future study.

Before closing the paper, we remark some limitations of our study. In our study, we can calculate the susceptibility with arbitrary values of the chemical potential. However, we are not able to study the temperature dependence of the quark number susceptibility in confined phase due to the Hawking-Page type transition. Therefore, we evaluate the quark number susceptibility only in high temperature regime, deconfined phase. In addition, our study based on AdS/CFT is inherently suffering from \(1/N_c\) corrections. Unfortunately, a systematic way of collecting all those corrections has not been established.

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A. Perturbative solution of \(A_t\) in the soft wall model

Plugging the expansion (3.24) into the equation (3.23), one can read off the one for \(F_0(u)\),

\[ 0 = \left( e^\Phi u F_0' \right)' . \]  

(A.1)

A general solution is given by

\[ F_0(u) = C_0 + D_0 \left\{ E_i(-cu) - \frac{e^{-c}}{2-a} E_i(c(1-u)) + K_1(u) \right\} . \]  

(A.2)

where \(C_0\) and \(D_0\) are integration constants. Imposing the regular condition at the horizon, the solution should be

\[ F_0(u) = C_0 \equiv C. \]  

(const.)  

(A.3)
The constant \( C \) could be estimated later.

By using this solution, one can get an equation for \( F_\omega(u) \),

\[
0 = \left( e^{\Phi} u F_\omega' \right)' + i\omega \frac{b}{2 - a} \left( e^{\Phi} u (1 + u - au^2) \right)'.
\] (A.4)

A general solution is

\[
F_\omega(u) = C_1 + D_1 \left\{ E_i(-cu) + K_1(u) \right\} - \frac{e^{-c}}{2 - a} \left\{ D_1 E_i(c(1 - u)) - iCbe^c \log(1 - u) \right\}.
\] (A.5)

Removing the singularity at the horizon, the integration constant \( D_1 \) should be

\[ D_1 = iCbe^c. \]

In order to fix the another constant \( C_1 \), it might be convenient to impose a condition at the boundary,

\[
\left[ F_\omega(u) - E_i(-cu) \lim_{u \to 0} \left( \frac{F_\omega(u)}{E_i(-cu)} \right) \right]_{u=0} = 0,
\]

so that finite terms at the boundary could be removed. The final form of the solution is

\[
F_\omega(u) = iCbe^c \left\{ E_i(-cu) + K_1(u) - K_1(0) - \frac{e^{-c}}{2 - a} \left( E_i(c(1 - u)) - E_i(c) - \log(1 - u) \right) \right\}.
\] (A.6)

A differential equation for \( F_{k^2}(u) \) is

\[
0 = \left( e^{\Phi} u F_{k^2}' \right)' - Cb^2 e^\Phi.
\] (A.7)

A general solution can be obtained as

\[
F_{k^2}(u) = C_2 + D_2 \left\{ E_i(-cu) + K_1(u) \right\} - \frac{e^{-c}}{2 - a} \left\{ D_2 E_i(c(1 - u)) + \frac{Cb^2 e^c}{c} \log(1 - u) \right\}
\]

\[
+ \frac{C b^2}{2(2 - a)c} \left\{ \frac{1 + a}{\sqrt{1 + 4a}} \log \left( \frac{1 + \frac{1 - 2au}{\sqrt{1 + 4a}}}{1 - \frac{1 - 2au}{\sqrt{1 + 4a}}} \right) + 2(2 - a) \log u - (1 - a) \log(1 + u - au^2) \right\},
\] (A.8)

and the constant \( D_2 \) can be fixed as

\[ D_2 = -\frac{Cb^2 e^c}{c}. \]
The remaining constant $C_2$ might be determined to eliminate finite contributions at the boundary. The solution then becomes

$$F_{k2}(u) = -\frac{C_2 e^c}{c} \left\{ E_i(-cu) + K_1(u) - K_1(0) \right. \\
- \frac{e^{-c}}{2 - a} \left( E_i(c(1 - u)) - E_i(c) - \log(1 - u) \right) \\
- \frac{(1 + a)e^{-c}}{2(2 - a)\sqrt{1 + 4a}} \left( \log \left( \frac{1 + \frac{1 - 2au}{\sqrt{1 + 4a}}}{1 + \frac{1}{\sqrt{1 + 4a}}} \right) - \log \left( \frac{1 - \frac{1 - 2au}{\sqrt{1 + 4a}}}{1 - \frac{1}{\sqrt{1 + 4a}}} \right) \right) \\
- e^{-c} \log u + \frac{(1 - a)e^{-c}}{2(2 - a)} \log(1 + u - au^2) \right\}. \quad (A.9)$$

B. D4/D8 brane system with the external magnetic field

In this section we shall discuss an effect of a constant magnetic field to the quark number susceptibility. We here consider the D4/D8 brane system with the constant magnetic field.

We first introduce the bulk background geometry of $N_c$ D4-branes in type IIA superstring theory with the compactification on a circle. There exist two different phases i.e. confined and deconfined phases. We here take the deconfined phase. The background is then given by

$$ds^2 = \left( \frac{U}{R} \right)^{3/2} \left( -f(U)(dt)^2 + (d\vec{x})^2 + (dx_4)^2 \right) + \left( \frac{R}{U} \right)^{3/2} \left( \frac{(dU)^2}{f(U)} + U^2 d\Omega_4^2 \right), \quad (B.1a)$$
$$e^\phi = g_s \left( \frac{U}{R} \right)^{3/4}, \quad (B.1b)$$

where the $\phi(x)$ is the dilaton field. $d\Omega_4^2$ is the metric of the four sphere and $R$ is the curvature radius of the background which is expressed in terms of the string coupling $g_s$ and the string length $l_s = \sqrt{\alpha'}$, 

$$R^3 = \pi g_s N_c l_s^3.$$ 

The function $f(U)$ is given by

$$f(U) = 1 - \left( \frac{U_T}{U} \right)^3, \quad (B.2)$$

and the temperature can be read off as

$$T = \frac{3}{4\pi} \frac{U_T^{1/2}}{R^{3/2}}. \quad (B.3)$$

Following Sakai and Sugimoto [38], we introduce the probe D8-brane which sits in the transverse direction to the compactified one $x_4$. In the deconfined phase where the $x_4-U$
subspace forms a cylinder, the D8-brane might be a straight line which simply follows the geodesic from $U = U_T$ to infinity,

$$
\text{d}s^2_{\text{D8}} = \left( \frac{U}{R} \right)^{3/2} \left( - f(U) dt^2 + (d\vec{x})^2 + \left( R \right)^{3/2} \left( \frac{dU}{f(U)} U^2 d\Omega_3^2 \right) \right). \tag{B.4}
$$

The action for the D8-brane consists of the sum of the DBI and the Chern-Simons actions. The DBI action is given by

$$
S_{\text{DBI}} = -T_8 \int d^9 x \ e^{-\phi} \sqrt{-\det(G_{MN} + 2\pi\alpha' F_{MN})}, \tag{B.5}
$$

where $T_8 = 1/(2\pi)^8 l_9^9$ is the D8-brane tension and $G_{MN}(x)$ is the induced metric (B.4). We put the constant magnetic field $B$ as the background of the $U(1)$ gauge field in the D8-brane and consider small fluctuations,

$$
F_{MN} = F_{MN}^{(0)} + F_{MN} \quad \text{with} \quad F_{xy}^{(0)} = B. \tag{B.6}
$$

It might be useful to collect the background fields as

$$
E_{MN} = G_{MN} + 2\pi\alpha' F_{MN}^{(0)}. \tag{B.7}
$$

Integrating over the four-sphere, we can then obtain the following action for the fluctuations from the DBI action (B.5),

$$
S_{5D} = -\frac{N_c R}{96\pi^3 \alpha'} \int d^4 x \ dU \sqrt{-\det(E_5)} \ \left( \frac{U}{R} \right)^{1/4} (E_5^{-1})^{ml}(E_5^{-1})^{kn} F_{mn} F_{lk}, \tag{B.8}
$$

where the indices $m$ and $n$ run through $t, 1, 2, 3, U$ and an inverse of 5D part of $E_{MN}$ has been defined i.e. $(E_5^{-1})^{ml} E_{5ln} = E_{5ml}(E_5^{-1})^{ln} = \delta_n^m$. We set the four sphere components of the gauge fields to be zero. The 5D Chern-Simons action arises after an integration of the RR four form over the four sphere on the D8-brane,

$$
S_{\text{CS}} = -i \frac{N_c}{48\pi^2} \int A \wedge F \wedge F. \tag{B.9}
$$

As we did in the main part of the paper, we work on the gauge $A_U(x) = 0$ and use the same Fourier decomposition as (3.18). Equations of motion for $A_t(U)$ and $A_z(U)$ are then given by

$$
0 = \alpha \left\{ \left( \frac{U}{R} \right) g(U) A'_t - \left( \frac{U}{R} \right)^{-2} g(U) \left( k^2 A_t + \omega k A_z \right) \right\} - i \beta B A'_z, \tag{B.10a}
$$

$$
0 = \alpha \left\{ \left( \frac{U}{R} \right) g(U) f(U) A'_z + \left( \frac{U}{R} \right)^{-2} g(U) \left( \omega^2 A_z + \omega k A_t \right) \right\} - i \beta B A'_t, \tag{B.10b}
$$

$$
0 = \alpha \left( \frac{U}{R} \right) g(U) \left( \omega A'_t + k f(U) A'_z \right) - i \beta B \left( \omega A_z + k A_t \right), \tag{B.10c}
$$

where

$$
g(U) = \sqrt{\left( \frac{U}{R} \right)^3 + (2\pi\alpha' B)^2}. \tag{B.11}
$$
The constants $\alpha$ and $\beta$ are defined by
\[
\alpha = \frac{N_c R}{24 \pi^3 \alpha'}, \quad \beta = \frac{N_c}{8 \pi^2}.
\]
The equations (B.10a) and (B.10c) imply (B.10b).

In order to solve the set of equations, it is standard to introduce the master variable,
\[
Z(U) = \omega A_z(U) + k A_t(U). \tag{B.12}
\]
The master equation becomes the following form:
\[
0 = Z'' + \left( \frac{1}{U} + \frac{g'(U)}{g(U)} + \frac{\omega^2 f'(U)}{f(U)(\omega^2 - k^2 f(U))} \right) Z' + \left( \frac{U}{R} \right)^{-3} \frac{\omega^2 - k^2 f(U)}{f^2(U)} Z
\]
\[-i \frac{\beta B}{\alpha} \left( \frac{U}{R} \right)^{-1} \frac{\omega k f'(U)}{f(U)g(U)(\omega^2 - k^2 f(U))} Z
\]
\[+ \frac{\beta^2 B^2}{\alpha^2} \left( \frac{U}{R} \right)^{-2} \frac{Z}{f(U)g^2(U)}. \tag{B.13}
\]
It might be convenient to introduce a dimensionless radial coordinate $u \equiv U_T / U$. The locations of the horizon and the boundary correspond to $u = 1$ and $u = 0$, respectively. By using this coordinate, the master equation (B.13) becomes
\[
0 = Z'' + \left( \frac{1}{u} + \frac{g'(u)}{g(u)} + \frac{\omega^2 f'(u)}{f(u)(\omega^2 - k^2 f(u))} \right) Z' + \left( \frac{3}{4\pi T} \right)^2 \frac{\omega^2 - k^2 f(u)}{u f^2(u)} Z
\]
\[+ \frac{3\pi \alpha' B}{u f(u)g(u)(\omega^2 - k^2 f(u))} Z
\]
\[+ (3\pi \alpha' B)^2 \frac{Z}{u^2 f(u)g^2(u)}. \tag{B.14}
\]
with
\[
f(u) = 1 - u^3, \quad g(u) = \sqrt{\left( \frac{4\pi RT}{3} \right)^6 \frac{1}{u^3} + (2\pi \alpha' B)^2},
\]
where the prime now implies the derivative with respect to $u$. We can impose the incoming wave condition at the horizon,
\[
Z(u) = (1 - u)^{-i \frac{\omega}{4\pi T}} F(u), \tag{B.15}
\]
where the function $F(u)$ should be regular at the horizon. The master equation then becomes that for the function $F(u)$,
\[
0 = F'' + \left( \frac{1}{u} + \frac{g'(u)}{g(u)} + \frac{\omega^2 f'(u)}{f(u)(\omega^2 - k^2 f(u))} + i \frac{\omega}{4\pi T} \frac{1}{1 - u} \right) F'
\]
\[+ \left\{ i \frac{\omega}{4\pi T} \frac{1}{1 - u} \left( \frac{1}{1 - u} + \frac{1}{u} + \frac{g'(u)}{g(u)} + \frac{\omega^2 f'(u)}{f(u)(\omega^2 - k^2 f(u))} \right)
\]
\[+ \left( \frac{1}{4\pi T} \right)^2 \left( - \frac{\omega^2}{(1 - u)^2} + \frac{9(\omega^2 - k^2 f(u))}{u f^2(u)} \right) \right\} F
\]
\[+ i 3\pi \alpha' B \frac{\omega k f'(u)}{u f(u)g(u)(\omega^2 - k^2 f(u))} F + (3\pi \alpha' B)^2 \frac{F}{u^2 f(u)g^2(u)}. \tag{B.16}
\]
Multiplying the factor \((\omega^2 - k^2 f(u))\) to the equation above, we could apply the hydrodynamics approximation. The function \(F(u)\) can be expanded as

\[
F(u) = F_0(u) + \omega F_\omega(u) + k F_k(u) + \mathcal{O}(\omega^2, k^2, \omega k).
\] (B.17)

The order \(\mathcal{O}(\omega k)\) in the expansion of the master equation, we can fix the function \(F_0(u)\) as

\[
F_0(u) = 0,
\] (B.18)

Equations for \(F_k(u)\) and \(F_\omega(u)\) can be read off from the \(\mathcal{O}(\omega^2 k)\) and \(\mathcal{O}(\omega k^2)\), respectively,

\[
0 = \left( u f(u) g(u) F_k'(u) \right) + i 3 \pi \alpha' B f'(u) F_\omega(u) + (3 \pi \alpha' B)^2 \frac{F_k(u)}{u g(u)},
\] (B.19)

\[
0 = \left( u g(u) F_\omega'(u) \right) + i 3 \pi \alpha' B \left( \frac{1}{f(u)} \right)' F_k(u) + (3 \pi \alpha' B)^2 \frac{F_\omega(u)}{u f(u) g(u)}.
\] (B.20)

References

[1] R. Stock, [arXiv:0909.0601[nucl-ex]]; Z. Fodor and S.D. Katz, [arXiv:0908.3341[hep-ph]].

[2] M.A. Stephanov, K. Rajagopal and E.V. Shuryak, Phys. Rev. Lett. 81 (1998) 4816, [arXiv:hep-ph/9806219].

[3] L. McLerran, Phys. Rev. D36 (1987) 3291.

[4] S. Jeon and V. Koch, Phys. Rev. Lett. 85 (2000) 2076, [arXiv:hep-ph/0003168]; M. Asakawa, U.W. Heinz and B. Muller, Phys. Rev. Lett. 85 (2000) 2072, [arXiv:hep-ph/0003169]; M. Prakash and I. Zahed, Phys. Rev. Lett. 69 (1992) 3282.

[5] S. Gottlieb, W. Liu, D. Toussaint, R.L. Renken and R.L. Sugar, Phys. Rev. Lett. 59 (1987) 2247.

[6] T. Kunihiro, Phys. Lett. B271 (1991) 395.

[7] P. Chakraborty, M.G. Mustafa and M.H. Thoma, Eur. Phys. J. C23 (2002) 591, [arXiv:hep-ph/0111022].

[8] M. Harada, Y. Kim, M. Rho and C. Sasaki, Nucl. Phys. A727 (2003) 437, [arXiv:hep-ph/0207012].

[9] S. Gottlieb, W. Liu, D. Toussaint, R.L. Renken and R.L. Sugar, Phys. Rev. D38 (1988) 2888.

[10] R.V. Gavai, J. Potvin and S. Sanielevici, Phys. Rev. D40 (1989) 2743.

[11] S.A. Gottlieb, et al, Phys. Rev. D55 (1997) 6852, [arXiv:hep-lat/9612020].

[12] S. Ejiri, F. Karsch and K. Redlich, Phys. Lett. B633 (2006) 275, [arXiv:hep-ph/0509051].

[13] K. Jo, Y. Kim, H.K. Lee and S.-J. Sin, JHEP 0811 (2008) 040, [arXiv:0810.0063[hep-ph]].

[14] C.R. Allton, M. Doring, S. Ejiri, S.J. Hands, O. Kaczmarek, F. Karsch, E. Laermann and K. Redlich, Phys. Rev. D71 (2005) 054508, [arXiv:hep-lat/0501030].
[15] S. Ejiri, C.R. Allton, M. Doring, S.J. Hands, O. Kaczmarek, F. Karsch, E. Laermann and K. Redlich, Nucl. Phys. A774 (2006) 837, [arXiv:hep-ph/0509361]; R.V. Gavai and S. Gupta, Phys. Rev. D71 (2005) 114014, [arXiv:hep-lat/0412035].

[16] S. Gupta, [arXiv:0909.4630[nucl-ex]]; R. Gavai and S. Gupta, PoS LAT2005 (2006) 160, [arXiv:hep-lat/0509151]; M.A. Stephanov, Prog. Theor. Phys. Suppl. 153 (2004) 139, Int. J. Mod. Phys. A20 (2005) 4387, [arXiv:hep-ph/0402115].

[17] Y. Hatta and T. Ikeda, Phys. Rev. D67 (2003) 014028, [arXiv:hep-ph/0210284]; P. Costa, C.A. de Sousa, M.C. Ruivo and Y.L. Kalinovsky, Phys. Lett. B647 (2007) 431, [arXiv:hep-ph/0701135]; C. Sasaki, B. Friman and K. Redlich, Phys. Rev. D75 (2007) 074013, [arXiv:hep-ph/0611147]; B.-J. Schaefer and J. Wambach, Phys. Rev. D75 (2007) 085015, [arXiv:hep-ph/0603256]; P. Costa, M.C. Ruivo and C.A. de Sousa, Phys. Rev. D77 (2008) 096001, [arXiv:0801.3417[hep-ph]]; K. Redlich, B. Friman and C. Sasaki, J. Phys. G35 (2008) 044013, [arXiv:0712.2926[hep-ph]]; K. Fukushima, Phys. Rev. D77 (2008) 114028, [arXiv:0803.3318[hep-ph]]; M. He, J.-F. Li, W.-M. Sun and H.-S. Zong, Phys. Rev. D79 (2009) 036001, [arXiv:0811.1835[hep-ph]].

[18] J.M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, [Int. J. Theor. Phys. 38 (1999) 1113], [arXiv:hep-th/9711200].

[19] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys. Lett. B428 (1998) 105, [arXiv:hep-th/9802109].

[20] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253, [arXiv:hep-th/9802150].

[21] D.T. Son and A.O. Starinets, JHEP 0209 (2002) 042, [arXiv:hep-th/0205051].

[22] G. Policastro, D.T. Son and A.O. Starinets, JHEP 0209 (2002) 043, [arXiv:hep-th/0205052].

[23] X.-H. Ge, Y. Matsuo, F.-W. Shu, S.-J. Sin and T. Tsukioka, Prog. Theor. Phys. 120 (2008) 833, [arXiv:0806.4460[hep-th]]; Y. Matsuo, S.-J. Sin, S. Takeuchi and T. Tsukioka, [arXiv:0910.3722[hep-th]].

[24] Y. Matsuo, S.-J. Sin, S. Takeuchi, T. Tsukioka and C.-M. Yoo, Nucl. Phys. B820 (2009) 593, [arXiv:0901.0610[hep-th]].

[25] V.P. Gusynin, V.A. Miransky and I.A. Shovkovy, Phys. Rev. Lett. 73 (1994) 3499, Erratum-ibid. 76 (1996) 1005, [arXiv:hep-ph/9405262]; K.G. Klimenko, Theor. Math. Phys. 89 (1992) 1161, [Teor. Mat. Fiz. 89 (1991) 211]; D.K. Hong, Y. Kim and S.-J. Sin, Phys. Rev. D54 (1996) 7879, [arXiv:hep-th/9603157].

[26] D.E. Kharzeev, L.D. McLerran and H.J. Warringa, Nucl. Phys. A803 (2008) 227, [arXiv:0711.0850[hep-ph]].

[27] S.-J. Sin, JHEP 0710 (2007) 078, [arXiv:0707.2719[hep-th]].

[28] J. Erlich, E. Katz, D.T. Son and M.A. Stephanov, Phys. Rev. Lett. 95 (2005) 261602, [arXiv:hep-ph/0501128].
[29] L. Da Rold and A. Pomarol, Nucl. Phys. B721 (2005) 79, [arXiv:hep-ph/0501218].

[30] A. Karch, E. Katz, D.T. Son and M.A. Stephanov, Phys. Rev. D74 (2006) 015005, [arXiv:hep-ph/0602229].

[31] C.P. Herzog, Phys. Rev. Lett. 98 (2007) 091601, [arXiv:hep-th/0608151].

[32] O. Andreev and V.I. Zakharov, Phys. Lett. B645 (2007) 437, [arXiv:hep-ph/0607026].

[33] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, Phys. Rev. D60 (1999) 104026, [arXiv:hep-th/9904197].

[34] C.A. Ballon Bayona, H. Boschi-Filho, N.R.F. Braga and L.A. Pando Zayas, Phys. Rev. D77 (2008) 046002, [arXiv:0705.1529[hep-th]].
Y. Kim, B.H. Lee, S. Nam, C. Park and S.-J. Sin, Phys. Rev. D76 (2007) 086003, [arXiv:0706.2525[hep-ph]]; R.-G. Cai and N. Ohta, Phys. Rev. D76 (2007) 106001, [arXiv:0707.2013[hep-th]]; W.-Y. Wen, [arXiv:0707.4116[hep-th]]; O. Bergman, G. Lifschytz and M. Lippert, JHEP 0711 (2007) 056, [arXiv:0708.0326[hep-th]]; M. Rozali, H.-H. Shieh, M. Van Raamsdonk and J. Wu, JHEP 0801 (2008) 053, [arXiv:0708.1322[hep-th]]; Y. Kim, T. Misumi and I.J. Shin, [arXiv:0911.3205[hep-ph]].

[35] E.S. Fraga and A.J. Mizher, Phys. Rev. D78 (2008) 025016, [arXiv:0804.1452[hep-ph]].

[36] V. G. Filev, C.V. Johnson, R.C. Rashkov and K.S. Viswanathan, JHEP 0710 (2007) 019, [arXiv:hep-th/0701001]; V. G. Filev, JHEP 0804 (2008) 088, [arXiv:0706.3811[hep-th]]; O. Bergman, G. Lifschytz and M. Lippert, JHEP 0805 (2008) 007, [arXiv:0802.3720[hep-th]], Phys. Rev. D79 (2009) 105024, [arXiv:0806.0366[hep-th]].

[37] M. Cheng, et al, Phys. Rev. D79 (2009) 074505, [arXiv:0811.1006[hep-lat]].

[38] T. Sakai and S. Sugimoto, Prog. Theor. Phys. 113 (2005) 843, [arXiv:hep-th/0412141], Prog. Theor. Phys. 114 (2005) 1083, [arXiv:hep-th/0507073].

[39] C.V. Johnson and A. Kundu, JHEP 0812 (2008) 053, [arXiv:0803.0038[hep-th]], JHEP 0907 (2009) 103, [arXiv:0904.4320[hep-th]].

[40] K.-Y. Kim and J. Liao, Nucl. Phys. B822 (2009) 201, [arXiv:0906.2978[hep-th]].