A PANSIOT-TYPE SUBWORD COMPLEXITY THEOREM FOR AUTOMORPHISMS OF FREE GROUPS

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Abstract. Inspired by Pansiot’s work on substitutions, we prove a similar theorem for automorphisms of a free group \( F \) of finite rank: if a right-infinite word represents an attracting fixed point of an automorphism of \( F \), the subword complexity of \( X \) is equivalent to \( n \), \( n \log \log n \), \( n \log n \), or \( n^2 \). The proof uses combinatorial arguments analogue to Pansiot’s as well as train tracks. We also define the recurrence complexity of \( X \), and we apply it to laminations. In particular, we show that attracting laminations have complexity equivalent to \( n \), \( n \log \log n \), \( n \log n \), or \( n^2 \) (to \( n \) if the automorphism is fully irreducible).

1. Introduction

Let \( A = \{a_1, \ldots, a_d\} \) be a finite set, called an alphabet. A finite word (on \( A \)) of length \( n \geq 1 \) is a finite sequence \( x_1 \ldots x_n \in A^n \). The free monoid on \( A \) is the set \( A^* = \bigcup_{n \geq 0} A^n \) consisting of all finite words together with the empty word \( \varepsilon \in A^0 \) (of length 0), equipped with the concatenation law. Elements of \( A^\mathbb{N} \) are (right)-infinite words. Elements of \( A^\mathbb{Z} \) are bi-infinite words.

Given an infinite word \( X = x_1 \ldots x_k \ldots \) or a bi-infinite word \( X = \ldots x_{-1} x_0 x_1 \ldots x_k \ldots \) on a finite alphabet \( A \), its complexity (or subword complexity, factor complexity) is the function counting, for each \( n \geq 1 \), the number of factors (or subwords) of length \( n \) in \( X \):

\[ p_X(n) = \# \{w \in A^n \mid \exists i, w = x_i x_{i+1} \ldots x_{i+n-1}\}. \]

Similarly, given a set \( E \) of words (if these words are all finite, we require that \( E \) be infinite), the complexity of \( E \) is:

\[ p_E(n) = \# \{w \in A^n \mid \exists x \in E, \exists i, w = x_i x_{i+1} \ldots x_{i+n-1}\}. \]

The complexity is submultiplicative, in the sense that \( p_X(n + n') \leq p_X(n)p_X(n') \), so that the limit

\[ h(X) = \lim_{n \to +\infty} \frac{\log p_X(n)}{n \log d} \]

exists: it is the well-known topological entropy of \( X \). In this sense, the complexity of a word \( X \) can be seen as a finer dynamical invariant than entropy, which is particularly relevant in the case of 0 entropy (i.e. when the complexity is subexponential).

Indeed, the interest in complexity has historically focused on low complexity words; the intuition is that a word with low complexity should be “simple.” For instance, the words with bounded complexity are exactly the eventually periodic words. More is true, as shown by Hedlund and Morse \cite{28}: \( X \in A^\mathbb{N} \) is eventually periodic if and only if there exists \( n \in \mathbb{N} \) such that \( p_X(n) \leq n \).

Just one step further, we find the Sturmian words. These have been largely studied (see for instance \cite{25} chapter 2, \cite{30}) as they appear as the natural coding of lines in the plane \( \mathbb{R}^2 \) with irrational slope: such a line cuts the entire grid (joining the points of \( \mathbb{Z}^2 \) by horizontal and vertical lines), giving rise to a bi-infinite word in the letters \( h \) (horizontal) and \( v \) (vertical). In terms of complexity \cite{28}, a word \( X \in A^\mathbb{Z} \) is Sturmian if and only if \( p_X(n) = n + 1 \) for all \( n \in \mathbb{N} \).

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This result has motivated an abundant literature on words of low complexity – see for instance [11, 9, 14] and references therein – with two kind of approaches: on one side determining the kind of functions that can appear as the complexity of a word – see for instance [8, 14, 27] – and on the other side developing methods for, given a word (whether arising from arithmetics, dynamical systems, combinatorics, and so on), computing (or estimating) its complexity – see for instance [7]. Examples thus play a central role in these investigations.

In the present paper, we are interested in a result of Pansiot [29] about infinite words obtained by iterating a non-erasing substitution on a letter.

A substitution σ on a finite alphabet A may be viewed as an endomorphism of the free monoid A*. Concretely, σ is defined by the data σ(a) ∈ A* for all a ∈ A. It is non-erasing if σ(a) ≠ ε for all a ∈ A.

If σ is non-erasing, then, for all a ∈ A such that σ(a) has length at least 2 and the first letter of σ(a) is a, the word σ^k(a) is a prefix of σ^{k+1}(a) and the length of σ^k(a) grows to infinity. Hence the sequence σ^k(a) converges to an infinite word X ∈ A* (of which every σ^k(a) is a prefix).

Pansiot’s theorem asserts that the complexity of such an X is bounded, linear (equivalent to n), quadratic (equivalent to n^2), equivalent to n log n, or equivalent to n log log n (in this setting, “f(n) equivalent to g(n)” means that there exist two positive constants c_1 and c_2 such that c_1g(n) ≤ f(n) ≤ c_2g(n) for all n). This result strengthens a previous result of Ehrenfeucht, Lee, and Rozenberg [12] who showed, in terms of D0L systems, that the complexity of X is at most quadratic (the total complexity introduced in Definition 3.1 is similar to the complexity of D0L languages). We refer to [9] for a detailed proof of Pansiot’s theorem and a lot of complements.

The main purpose of this paper is to extend Pansiot’s theorem to arbitrary automorphisms of a free group F of finite rank. Its main results were announced in [22].

There are several key differences between a free monoid and a free group. In particular, a free group F is much more flexible: a free monoid has finitely many automorphisms (one may only permute the generators), while the automorphism group Aut(F) is a big group which is currently under active investigation (see [32] for instance).

In fact, one can view a free group F as an abstract object rather than a set of words. This is analogous to linear algebra, where one studies n-dimensional vector spaces over K rather than just K^n. With this point of view, our Pansiot-type theorem appears as a theorem about an element X belonging to the boundary ∂F of F which is fixed under the action of an automorphism.

One may view ∂F as the space of ends of F, or its Gromov boundary, see for instance [6, 10, 17, 24]. Topologically, ∂F is homeomorphic to a Cantor set, and F ∪ ∂F should be viewed as a compactification of F (equipped with a word metric). There is a natural action of Aut(F) on F ∪ ∂F by homeomorphisms (see [10]).

Algebraically, we choose a free basis A of F. Elements of F are reduced words on the alphabet E = A ∪ A^-1 (reduced means that no letter is followed by its inverse). Elements X of ∂F are represented by reduced right-infinite words X_A. A sequence in F ∪ ∂F converges if, for each integer s, the s-prefixes converge.

Given X ∈ ∂F, the complexity of the word X_A representing it in the basis A depends on A, but in a very controlled way. If B is another basis, the complexity functions p_A, p_B of X_A, X_B are equivalent (denoted p_A ∼ p_B) in the following sense: there exists a positive integer C such that p_A(n) ≤ Cp_B(Cn) and p_B(n) ≤ Cp_A(Cn) for all n (Proposition 2.11). This basic fact may also be expressed as saying that two right-infinite words X_A, Y_A which differ by an automorphism α of F (i.e. Y_A = α(X_A)) have equivalent complexities.

The complexity p_x of an element X ∈ ∂F is therefore well-defined up to equivalence. In particular, it makes sense to say that the complexity of X is linear (equivalent to n), quadratic (equivalent to n^2), equivalent to n log n, equivalent to n log log n.

If α ∈ Aut(F) and X ∈ ∂F is the limit of a sequence α^p(g), with g ∈ F, as p → +∞, as is the case in Pansiot’s situation, then X is fixed by α.
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Conversely, if $X \in \partial F$ is fixed by $\alpha$, there are several possibilities (see [15]). First, $X$ may belong to the boundary of the fixed subgroup $\text{Fix} \alpha = \{ g \in F \mid \alpha(g) = g \}$. This subgroup may have rank 2 or more, and there is no hope of saying anything general about $p_X$ in this case. If $X \notin \partial \text{Fix} \alpha$, however, it is either attracting or repelling for the action of $\alpha$ on $F \cup \partial F$ (see [15]).

We can now state our main result. The various possibilities are illustrated in Section 2.1.

**Theorem 1.1** (Theorem 4.2). If $X \in \partial F$ is fixed by $\alpha$ and $X \notin \partial \text{Fix} \alpha$, then the complexity $p_X$ of $X$ is linear, quadratic, equivalent to $n \log n$, or equivalent to $n \log \log n$ (bounded cannot occur).

**Remark 1.2.** In this statement, equivalence may be understood as defined above, or as Lipschitz-equivalence: two functions $f$ and $g$ are Lipschitz-equivalent if there exists $C > 0$ such that $\frac{1}{C} \leq \frac{f(x)}{g(x)} \leq C$ for all $x$ (see Remark 2.8).

**Corollary 1.3.** Let $\alpha$ be an automorphism of a finitely generated free group $F$, and $g \in F$. If $\alpha^p(g)$ converges to an element $X \in \partial F$ as $p \to +\infty$, then the complexity $p_X$ of $X$ is bounded, linear, quadratic, equivalent to $n \log n$, or equivalent to $n \log \log n$.

One may be more precise, under additional assumptions on $\alpha$ (see Section 4 for definitions).

**Corollary 1.4** (Corollary 4.4). Under the assumptions of Theorem 1.1 or Corollary 1.3:

- If $\alpha$ is fully irreducible, the complexity of $X$ is linear.
- If $\alpha$ is atoroidal, the complexity of $X$ cannot be quadratic (it is equivalent to $n$, $n \log n$, or $n \log \log n$).
- If $\alpha$ is polynomially growing, the complexity of $X$ is bounded or quadratic.

We do not know whether our results may be extended to injective endomorphisms.

Let us now give the main ideas of the proof of Theorem 1.1. Unfortunately, one cannot prove them by simply viewing $\alpha$ as a substitution on $A \cup A^{-1}$ and applying Pansiot’s theorem, because of cancellation.

Indeed, cancellation is another key difference between free monoids and free groups. When applying $\alpha$ to a (finite or infinite) word, one replaces each letter by its image, but one also has to make the word reduced, by successively cancelling all subwords of the form $\beta \beta^{-1}$ or $\beta^{-1} \beta$ with $\beta \in A$ (the order in which the removals are performed is irrelevant).

As an example, consider the so-called Tribonacci substitution on the alphabet $\{a, b, c]\$, given by $\sigma(a) = ab$, $\sigma(b) = ac$, $\sigma(c) = a$. It defines a fully irreducible automorphism $\alpha$ of the free group $F(a, b, c)$, whose cube fixes the infinite word

$$X = \lim_{p \to +\infty} \omega^{-3p}(a^{-1}) = a^{-1}ba^{-1}bc^{-1}ba^{-1}ba^{-1}bc^{-1}bc^{-2}bc^{-1}b \ldots$$

This word is a repelling fixed point of $\alpha^3$, and a lot of cancellation occurs when $\alpha$ is applied to it. By Corollary 1.3, the complexity of $X$ is linear.

From a technical point of view, here are two typical exemples (known as INP’s and exceptional paths) which do not appear when considering substitutions and force us to use different techniques.

Consider the automorphism $\alpha_1$ of the free group $F(a, b)$ on two generators $a, b$ sending $a$ to $ab$ and $b$ to $bab$. It sends the commutator $[a, b] = aba^{-1}b^{-1}$ to $ababb^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$, which reduces to $aba^{-1}b^{-1}$: the element $[a, b]$ is fixed. Now consider the automorphism $\alpha_2$ of $F(a, b, c)$ sending $a$ to $a$, $b$ to $ba$, and $c$ to $ca^2$. Applying powers of $\alpha_2$ to $bc^{-1}$ sends it to $ba^{-1}c^{-1}$, then to $ba^{-2}c^{-1}$, $ba^{-3}c^{-1}$, ... Note that $ba^{-k}c^{-1}$ is an infinite sequence of words which cannot be written as a concatenation $uv$ such that no cancellation occurs between $\alpha_2(u)$ and $\alpha_2(v)$.

**Exceptional paths** such as $ba^{-k}c^{-1}$, with $k$ unbounded, are the key obstruction to applying Pansiot’s theorem directly. In Subsection 3.3, exceptional paths will be ruled out and we will be able to use substitutions to shorten a proof. This technique was first used by
R. Gupta [18] to estimate the frequency of a path $\gamma$. Note, however, that the substitution $\zeta_\gamma$ that she uses depends on $\gamma$.

The standard way to control cancellation is to use train tracks. Train tracks for automorphisms of free groups were introduced by Bestvina-Handel [4] and improved by Bestvina-Feighn-Handel [3]. We use the completely split train tracks constructed by Feighn-Handel [18]. See Section 2.6 for definitions, and [3] for examples of train tracks.

Any automorphism of a free group may be represented by a map $f$ from a graph to itself. For instance, $\alpha_1$ above is represented on a rose with two petals labelled $a$ and $b$ (see Figure 1 below). The map $f$ wraps the first half of the edge labelled $a$ over the whole edge and the second half over the edge labelled $b$. It sends the $b$-edge to a path consisting of 3 edges ($a$, then $b$, then $a$).

The idea of train tracks is to represent a given automorphism $\alpha$ by a continuous map $f$ on a graph $G$ with possibly more than one vertex ($f$ represents $\alpha$ in the sense that we can identify the fundamental group of $G$ with $F$ so that $\alpha$ is the automorphism induced by $f$). The map sends each vertex to a vertex, each edge to a path consisting of one or more edges. The point is to construct $f$ in such a way that as little cancellation as possible occurs when powers of $f$ are applied to a given edge.

Using [18], we represent $\alpha$ by a completely split train track map $f : G \to G$. Any $X \in \partial F$ is represented by a ray $\rho$ in $G$: a locally injective map from $[0, \infty)$ to $G$ sending each interval $[p, p+1]$ onto an edge. Up to equivalence, the complexity of $X$ may be computed by counting the complexity of $\rho$, i.e. the number of subpaths of length $n$ (see Lemma 2.13).

Proposition 3.3 states that the complexity of $\rho$ is bounded or satisfies the conclusion of Theorem 1.1. Its proof combines combinatorial arguments used by Pansiot and geometric arguments using the properties of completely split train tracks. Theorem 1.1 follows fairly directly from Proposition 3.3, see Section 3.

In Section 3 we also associate to any $X \in \partial F$ a recurrence complexity $p^{\text{rec}}_X$, by only counting words of length $n$ that appear infinitely often in $X$, and we show:

**Theorem 1.5** (Theorem 5.6). If $X$ is as in Theorem 1.1 or Corollary 1.3, its recurrence complexity $p^{\text{rec}}_X$ is bounded, linear, quadratic, equivalent to $n \log n$, or equivalent to $n \log \log n$. If $p^{\text{rec}}_X$ is not quadratic, it is equivalent to $p_X$.

On the other hand, if $\alpha$ is the automorphism of $\mathbb{F}_3$ defined by $a \mapsto a, b \mapsto ba, c \mapsto cb$, the fixed point $X = \lim_{n \to \infty} \alpha^n(c) = cbaba^2 \cdots$ has quadratic complexity but linear recurrence complexity (see Example 5.2).

One may associate to any $X \in \partial F$ a lamination $L_X$ (see Section 5.4), and $p_X^{\text{rec}}$ is the complexity of $L_X$. Theorem 1.1 and Corollary 1.3 thus control the complexity of $L_X$, for $X$ fixed by $\alpha$. This applies in particular to the attracting laminations constructed by Bestvina-Feighn-Handel in [3].

**Theorem 1.6** (Corollary 5.9). Let $\Phi \in \text{Out}(\mathbb{F}_n)$. The complexity of any attracting lamination of $\Phi$ is linear, quadratic, equivalent to $n \log n$, or equivalent to $n \log \log n$. It is linear if $\Phi$ is fully irreducible, at most $n \log n$ if $\Phi$ is atoroidal.

### 2. Preliminaries

#### 2.1. Examples

We first give examples illustrating the various possibilities in Theorem 1.1 and features of its proof. They may be viewed as positive automorphisms of $F$, or as invertible substitutions. We selected them so that they illustrate train track maps and their strata. We refer to Sections 2.6 and 3 for definitions.

We use as a building block the following automorphism $\omega$ of the free group $F(a,b)$ on two generators $a, b$: it sends $a$ to $ab$ and $b$ to $bab$. It is realized geometrically by a pseudo-Anosov homeomorphism of a punctured torus. The infinite word $X = ab^2ab^2abab \cdots = \lim_{n \to \infty} \omega^n(a)$ is an attracting fixed point whose complexity is linear.

The obvious map representing $\omega$ on a rose with two petals is a completely split train track map. There is one exponential (EG) stratum, consisting of the whole graph. The
commutator \([a, b] = aba^{-1}b^{-1}\) is fixed under \(\omega\), it is represented by an INP. Similarly, in
the examples given below, the obvious map on a rose is a completely split train track map.

We now consider a free group \(F(a, b, c, d)\) of rank 4, which we view as \(F(a, b) * F(c, d)\).

The automorphism \(\alpha : \begin{cases} a \mapsto ab \\ b \mapsto bab \\ c \mapsto cd \\ d \mapsto dcd \end{cases}\)
acting on each of the two free factors as \(\omega\) has two unrelated exponential strata (consisting of the \((a, b)\)-edges and the \((c, d)\)-edges respectively). There is no divergence (in the sense of Definition 3.2): under iteration of \(\omega\), all
four letters \(a, b, c, d\) grow like \(\lambda^n\), with \(\lambda\) the largest eigenvalue of the matrix \(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\). By
this we mean that, for \(x \in \{a, b, c, d\}\), the length of \(\omega^n(x)\) is equivalent to \(\lambda^n\) as \(n \to +\infty\).
Just as above, the fixed points \(X = \lim_{n \to \infty} \alpha^n(a)\) and \(Y = \lim_{n \to \infty} \alpha^n(c)\) have linear complexity.

We now create divergence by making the image of the \((a, b)\)-stratum go through the \((c, d)\)-stratum. Define \(\alpha_{1,1} : \begin{cases} a \mapsto a c b \\ b \mapsto bab \\ c \mapsto cd \\ d \mapsto dcd \end{cases}\). Because of the letter \(c\) in the image of \(a\),
the two strata interact. The letters \(a\) and \(b\) grow like \(n\lambda^n\), whereas \(c\) and \(d\) still grow like \(\lambda^n\). There is polynomial divergence, and the attracting fixed point \(\lim_{n \to \infty} \alpha_{1,1}^n(a)\) has complexity \(n \log \log n\) (up to equivalence).

To create exponential divergence, we use \(\omega^2\) in the \((a, b)\)-stratum. Define
\[
\alpha_{2,1} : \begin{cases} a \mapsto ab \ c \ bab \\ b \mapsto bab \ ab \ bab \\ c \mapsto cd \\ d \mapsto dcd \end{cases}.
\]
Now \(a\) and \(b\) grow like \((\lambda^2)^n\), and the complexity of \(\lim_{n \to \infty} \alpha_{2,1}^n(a)\) is \(n \log n\).

All these examples are positive automorphisms. In general, though, automorphisms of \(F\) send the generators to words containing both the generators and their inverses. Here is a simple example, with the fixed point \(X = \lim_{n \to \infty} \alpha_0^n(e) = ecd^{-1} f cbab^{-1} a^{-1} d^{-1} f \cdots\)
containing exceptional paths (such as \(cd^{-1}\)). The complexity of \(X\) is quadratic. On the
other hand, the fixed point \(Y = \lim_{n \to \infty} \alpha_0^n(c) = c[a, b]^{-1} \in \text{Fix}_0\) belongs to \(\partial\text{Fix}_0\) (because \(c[a, b]^{-1} \in \text{Fix}_0\) and has bounded complexity.

\[
\alpha_0 : \begin{cases} a \mapsto ab \\ b \mapsto bab \\ c \mapsto c[a, b] \\ d \mapsto d[a, b]^{-1} \\ e \mapsto e cd^{-1} f \\ f \mapsto fe cd^{-1} f \end{cases}.
\]

2.2. Graphs. Let \(G\) be a finite connected graph. Let \(E\) be the set of oriented edges, equipped with the fixed-point free involution \(e \mapsto \bar{e}\) reversing the orientation of edges. We write \(o(e)\) for the origin of \(e\), and \(t(e)\) for its terminal point, so that \(t(e) = o(\bar{e})\). The
graph is not assumed to be simplicial: there may be an edge with \(o(e) = t(e)\) (a loop), and
distinct edges \(e, e'\) with \(o(e') = o(e)\) and \(t(e') = t(e)\). When drawing pictures, we choose a
representative for each pair \((e, \bar{e})\) and represent it with an arrow and a label (see Figure 1).

We assume that the first Betti number \(N\) of \(G\) (equal to \(a_1 - a_0 + 1\) with \(a_0\) the number of vertices and \(a_1\) the number of non-oriented edges) is at least 2, so that the fundamental group \(F = \pi_1(G)\), a free group of rank \(N\), is non-abelian.
We view $G$ as a topological space, with universal covering a tree $\tilde{G}$. One recovers $G$ as the quotient $G/F$, with $F$ acting on $\tilde{G}$ freely by covering transformations. Two distinct vertices of $G$ are joined by a unique segment, whose length is the number of edges that it contains.

2.3. Edge paths and rays.

**Definition 2.1** (Edge path). An edge path in $G$ is a word $\gamma = e_1 \ldots e_n$ on the alphabet $E$ which is reduced ($e_{i+1} \neq \bar{e}_i$) and satisfies $o(e_{i+1}) = t(e_i)$ for $i = 1, \ldots, n - 1$. The inverse path is $\bar{\gamma} = \bar{e}_n \ldots \bar{e}_1$ (we sometimes write $\gamma^{-1}$ instead of $\bar{\gamma}$).

The length $|\gamma|$ of $\gamma$ is the integer $n$. The path is trivial if $n = 0$.

A subpath of $\gamma$ is a path of the form $e_re_{r+1}\ldots e_s$, with $1 \leq r \leq s \leq n$.

For $k \leq n$, the $k$-prefix of $\gamma$ is $e_1 \ldots e_k$, the $k$-suffix is $e_{n-k+1} \ldots e_n$. A subpath is interior if it is neither a prefix nor a suffix.

**Example 2.2.** Suppose that $G$ is a rose with $N$ petals: it has one vertex and $N$ pairs of oriented edges $(e_i, \bar{e}_i)$. An edge path is a reduced word on the alphabet $\{e_1, \ldots, e_N, \bar{e}_N\}$. If $G$ is a theta graph (see Figure 1), an edge path is a word on the alphabet $\{a, \bar{a}, b, \bar{b}, c, \bar{c}\}$ in which, for instance, a letter $a$ may only be followed/preceded by $\bar{b}$ or $\bar{c}$ (from a symbolic point of view, the set of infinite edge paths in this theta graph can be identified with a subshift of finite type of the full shift on the alphabet $\{a, \bar{a}, b, \bar{b}, c, \bar{c}\}$, namely the subshift consisting of words not containing any subword $x\bar{x}, \bar{xx}, xy, \bar{xy}$ for $x, y \in \{a, b, c\}$).

**Definition 2.3** (Ray). A ray $\rho$ in $G$ is a right-infinite word $e_1 e_2 \ldots$ on $E$ satisfying the conditions $e_{i+1} \neq \bar{e}_i$ and $o(e_{i+1}) = t(e_i)$ for each $i$. The origin of $\rho$ is the origin of its initial edge $e_1$.

For $k \geq 0$, the $k$-truncation $\rho_k$ of $\rho$ is the ray $e_{k+1}e_{k+2}\ldots$.

Two rays $\rho, \rho'$ are equivalent if they have a common truncation $\rho_k = \rho'_{k'}$, with $k, k' \geq 0$.

We view a non-trivial edge path $\gamma$ geometrically as a map from $[0, n]$ to $G$ sending $[i-1, i]$ onto $e_i$. This map is usually called an immersion because, $\gamma$ being a reduced word, it is locally injective. A ray is an immersion from $[0, \infty)$ to $G$.

An edge path $\gamma$ may be lifted to an oriented segment of length $n$ in $\tilde{G}$. Two oriented segments in $\tilde{G}$ project to the same edge path if and only if they differ by the action of an element of $F$.

When viewed in $\tilde{G}$, a ray $\rho$ joins a vertex (its origin) to an end of $\tilde{G}$. Any locally compact space $X$, such as $\tilde{G}$, has a space of ends $E(X)$, defined using complements of compact subsets (see for instance [10, 16]) The space $E(\tilde{G})$ is homeomorphic to a Cantor set. More concretely, one may view $E(\tilde{G})$ as the space of all rays with a fixed origin $\bar{v}$. The end of $\rho$ is the only ray $\rho_\infty$ with origin $\bar{v}$ such that $\rho_\infty \cap \rho_0$ is infinite. The set of equivalence classes of rays in $G$, as defined above, may be identified with the quotient of $E(\tilde{G})$ by the natural action of $F$.

2.4. Complexity of rays.

**Definition 2.4** (Complexity). Let $\rho$ be a ray as in Definition 2.3. It is a right-infinite word on the alphabet $E$, and we let its (subword, factor) complexity $p_\rho$ be the function associating to an integer $n \geq 1$ the number of distinct subwords of length $n$. 
The function $p_\rho$ is non-decreasing. It is convenient to define it for $x$ a positive real number (not just an integer) by setting $p_\rho(x) = p_\rho([x])$. It is a standard fact \cite{28} that $p_\rho(n) \geq n + 1$ if $\rho$ is not ultimately periodic.

**Definition 2.5 (Equivalent functions).** Given two non-decreasing functions $p$ and $q$ defined on $(0, \infty)$, we write $p \preceq q$ if there exists $C > 0$ such that $p(x) \leq Cq(Cx)$ for all $x$. We say that $p$ and $q$ are equivalent, denoted $p \sim q$, if $p \preceq q$ and $q \preceq p$.

**Definition 2.6 (Linear, quadratic).** We say that $p$ is linear if it is equivalent to $n$, quadratic if it is equivalent to $n^2$.

**Example 2.7.** If $\rho$ is a ray, the complexity of a truncation $\rho_k$ is equivalent to the complexity of $\rho$, so equivalent rays have equivalent complexities.

**Remark 2.8 (Lipschitz-equivalence).** When discussing growth, we will consider Lipschitz-equivalence: $p$ and $q$ are Lipschitz-equivalent if there exists $C > 0$ such that $\frac{1}{C}p \leq q \leq C\rho$.

Clearly Lipschitz-equivalence implies equivalence in the sense of Definition 2.5, though $\lambda^n$ and $\mu^n$ are equivalent but not Lipschitz-equivalent for $\lambda \neq \mu$. On the other hand, the converse (equivalence implies Lipschitz-equivalence) is true if $p$ is one of the functions $1, n, n^2, n \log n, n \log \log n$, so in most of our statements about complexity equivalence may be understood in either sense.

If $e$ is an edge of $G$ with distinct endpoints, collapsing $e$ to a point yields a graph $G'$ having the same first Betti number, with a projection map $\pi : G \to G'$. A ray $\rho$ in $G$ yields a ray $\pi(\rho)$ in $G'$, obtained by erasing the letters $e, \bar{e}$; if we view $\rho$ as a map from $[0, \infty)$ to $G$, then $\pi(\rho)$ is just a reparametrization of $\pi \circ \rho$.

**Lemma 2.9.** Let $\pi : G \to G'$ be the map obtained by collapsing an edge $e$ of $G$ with distinct endpoints. If $\rho$ is any ray in $G$, the complexities of $\pi(\rho)$ and $\rho$ are equivalent.

**Example 2.10.** Suppose that $G$ is a theta graph, as in Example 2.2 (see Figure 1). Then $\rho$ is a right-infinite word on $\{a, \bar{a}, b, \bar{b}, e, \bar{e}\}$. The lemma claims that erasing all letters $c, \bar{c}$ does not change the complexity, up to equivalence.

**Proof.** The key observation is that, in a ray, no letter $e$ or $\bar{e}$ may be followed (or preceded) by $e$ or $\bar{e}$. Any subpath of $\pi(\rho)$ of length $\leq n$ is therefore the image of a subpath of $\rho$ of length $\leq 2n - 1$, and recalling that the functions are non-decreasing we may write

$$\frac{n}{2}p_{\pi(\rho)}(\frac{n}{2}) \leq \sum_{m=1}^{n} p_{\pi(\rho)}(m) \leq \sum_{m=1}^{2n-1} p_{\rho}(m) \leq 2np_{\rho}(2n).$$

In the other direction, the image of a subpath of $\rho$ of length $\leq n$ is a subpath of $\pi(\rho)$ of length $\leq n$, and at most 4 subpaths may have a given projection: if $\pi(\gamma_1) = \pi(\gamma_2)$, one passes from $\gamma_1$ to $\gamma_2$ by inserting or deleting $e$ or $\bar{e}$ (but not both) at the beginning and/or end of $\gamma_1$. Thus

$$\frac{n}{2}p_{\rho}(\frac{n}{2}) \leq \sum_{m=1}^{n} p_{\rho}(m) \leq 4 \sum_{m=1}^{n} p_{\pi(\rho)}(m) \leq 4np_{\pi(\rho)}(n).$$

The lemma follows.

2.5. **Complexity in free groups.** A non-abelian finitely generated free group $F$ has a boundary $\partial F$ (see for instance \cite{2, 11, 17, 24}). It may be viewed as the space of ends of the group $F$, or as the Gromov boundary of the hyperbolic group $F$ (it is homeomorphic to a Cantor set). One should view $F \cup \partial F$ as a compactification of $F$, and an element $X$ of $\partial F$ as a limit of elements of $F$. There are natural actions of $F$ (by left-translations) and of the automorphism group $\text{Aut}(F)$ on $\partial F$.

Concretely, if we fix a free basis $A = \{a_1, \ldots, a_N\}$ of $F$, then an element $X \in \partial F$ is just a reduced right-infinite word $X_A$ on the alphabet $A^{\pm 1}$ consisting of the $a_i$’s and their inverses, which we denote by $\bar{a}_i$. The action of $\alpha \in \text{Aut}(F)$ on $X$ consists in replacing each $a_i$, $\bar{a}_i$ by its image by $\alpha$ (with $\alpha(\bar{a}_i) = \alpha(a_i)^{-1}$), and reducing, i.e. deleting subwords
of the form $a_i a_i \bar{a}_i$ or $a_i \bar{a}_i$ (the order in which the deletions are performed is irrelevant). The conjugation by $g$ acts on $X$ as left multiplication by $g$.

Geometrically, choose an isomorphism $\varphi$ between $F$ and the fundamental group $\pi_1(G,v)$ of a graph $G$ as in Subsection 2.2 with $v$ a vertex (such an isomorphism is called a marking of $G$; changing the marking of $G$ is equivalent to applying an automorphism of $F$).

The boundary of $F$ is then identified to the set of immersed rays in $G$ starting at $v$, and also to the set of rays in $\tilde{G}$ starting at a given lift $\tilde{v}$ of $v$ (the space of ends $\mathcal{E}(\tilde{G})$. One recovers the previous point of view (right-infinite words on the alphabet $G$ of a graph $\tilde{G}$ whose edges represent the elements of a free basis $A$ (see Example 2.2). In this case, $\tilde{G}$ is the Cayley graph of $F$ with respect to $A$.

We wish to associate a complexity function to any $X \in \partial F$. The obvious idea is to choose a basis $A$ and define the complexity of $X$ as that of $X_A$, but we have to control how it changes if we use a different basis $B$. Equivalently (considering the automorphism taking $A$ to $B$), we must compare the complexities of $X_A$ and $\alpha(X)_A$ for $\alpha \in \text{Aut}(F)$.

**Proposition 2.11.** Let $A$ and $B$ be two free bases of $F$.

The complexities of the right-infinite reduced words $X_A$ and $X_B$ representing a given $X \in \partial F$ in $A$ and $B$ respectively are equivalent.

If $\alpha \in \text{Aut}(F)$, the words $X_A$ and $\alpha(X)_A$ representing $X$ and $\alpha(X)$ respectively have equivalent complexities.

Recalling the previous discussion, we get:

**Corollary 2.12.** Any element $X \in \partial F$ has a complexity function $p_X$ which is well-defined up to equivalence.

The proposition is a special case of the following general fact. As explained above, any $X \in \partial F$ may be represented by a ray with origin $v$ in any graph $G$ with a marking $\varphi : F \to \pi_1(G,v)$. The complexity of $X_A$ is the complexity of the ray representing $X$ in the rose associated to $A$.

**Lemma 2.13.** Given $X \in \partial F$, the complexity of the ray representing it in $G$ is, up to equivalence, independent of the choices of $G$ and $\varphi$. In particular, it is equivalent to $p_X$.

**Proof.** This is a direct consequence of Lemma 2.9 First, using collapse moves as in the lemma, one may reduce to the case when the graphs are roses. One must then check that the complexity is invariant under a change of the marking (the isomorphism $\varphi$).

One way of seeing this is by observing that a change of marking is just an element of $\text{Aut}(F)$. There is a standard set of generators for $\text{Aut}(F)$, known as elementary Nielsen transformations (see for instance [5, 20]). Up to renumbering the generators $a_1, \ldots, a_N$ of $F$, there are two Nielsen transformations: the first one maps $a_i$ to $a_i^{-1}$ and leaves the other $a_i$’s fixed (so does not change the complexity), the second one maps $a_1$ to $a_1 a_2$ and leaves the other $a_i$’s fixed. A transformation of the second kind may be realized geometrically by applying a collapse move preceded by the inverse of a collapse move (blowing up the vertex of a rose into an edge); for instance, collapsing either the $a$-edge or the $b$-edge in the theta graph of Figure 1 produces two roses whose markings differ by a Nielsen automorphism. One concludes using Lemma 2.9

**Remark 2.14.** Let $f : G' \to G$ be a $k$-sheeted covering map between finite graphs, for some $k \geq 2$. Since any edge path in $G$ has exactly $k$ lifts to $G'$, the complexity of a ray in $G'$ is equivalent to the complexity of its image in $G$.

If $J$ is a finitely generated subgroup of $F$, its boundary $\partial J$ naturally embeds into $\partial F$. Moreover, by a well-known theorem by M. Hall (see for instance [26, 31]), $J$ is a free factor of a finite index subgroup $F' \subset F$. Let $F = \pi_1(G)$, and let $f : G' \to G$ be the covering map associated to the inclusion $F' \subset F$. The remark made above implies that given $X \in \partial J$, its complexity in $F$ is equivalent to its complexity in $F'$, hence to its complexity in $J$.

2.6. **Train track maps.** Let $\alpha$ be an automorphism of a finitely generated free group $F$. It represents an element $\Phi$ of the group of outer automorphisms $\text{Out}(F) = \text{Aut}(F)/\text{Inn}(F)$,
with $\text{Inn}(F)$ the group of inner automorphisms (conjugations). It is customary to represent $\alpha$ (or rather $\Phi$) by a continuous map $f : G \to G$ (a homotopy equivalence), with $G$ a finite connected graph whose fundamental group is isomorphic to $F$.

Saying that $f$ represents $\alpha$ must be understood as follows. As we do not wish to choose basepoints, a homotopy equivalence of $G$ does not induce an automorphism of $F = \pi_1(G)$, but only an outer automorphism. We require that the element of $\text{Out}(F)$ induced by $f$ be the image $\Phi$ of $\alpha$ in $\text{Out}(F)$.

We will use \textit{completely split train tracks}, introduced in [13] elaborating on [4] and [3]. A completely split train track map is a map $f : G \to G$ satisfying many properties. We describe the properties that we shall need.

Unless mentioned otherwise, all definitions and properties below may be found in [13], where it is proved that any automorphism $\alpha \in \text{Aut}(F)$ has a power $\alpha^k$ which may be represented by a completely split train track map $f$ on some graph $G$ with $\pi_1(G) \cong F$ (note that Theorem 1.1 is true for $\alpha$ if they are true for $\alpha^k$).

If $Z$ is a finite set and $\varphi : Z \to Z$ is any map, there is a power $\psi$ of $\varphi$ such that $\psi(z)$ is fixed by $\psi$ for any $z \in Z$. This observation allows us to assume that many attributes of paths are fixed by $f$.

The train track map $f$ maps a vertex of $G$ to a vertex, an edge to a non-trivial edge path.

The restriction of $f$ to an edge path $\gamma$ may fail to be locally injective (because $f(e_{i+1})$ and $f(e_i)$ may start with the same edge), and we define $f^k_\gamma(\gamma)$ as the edge path obtained by tightening the image: $f^k_\gamma(\gamma)$ is homotopic to $f(\gamma)$ relative endpoints and locally injective (there is no backtracking); replacing $f(\gamma)$ by $f^k_\gamma(\gamma)$ is similar to reducing a word. The path $\gamma$ is \textit{pre-trivial} if some $f^k_\gamma(\gamma)$ is a trivial path.

\textbf{Example 2.15.} When $G$ is a rose with $N$ petals, we orient the edges and label them by letters which we view as generators of $F_N$. The map $f$ then corresponds to an endomorphism of $F_N$. For instance, the automorphism of $F_3$ sending $a$ to $a$, $b$ to $ba$, $c$ to $cb$ is represented by a map $f$ fixing the edge labelled $a$, mapping the $b$ edge to the edge path $ba$, and the $c$ edge to the edge path $cb$.

All paths in $G$ will be edge paths, so we often drop the word “edge”.

A \textit{splitting} of an edge path $\gamma$ is a decomposition of $\gamma$ as a concatenation of subpaths $\gamma = \gamma_1 \cdots \gamma_p$ such that $f^k_{\gamma_1}(\gamma_1) \cdots f^k_{\gamma_p}(\gamma_p)$ for all $k \geq 1$: no cancellation occurs between the tightened images of $\gamma_i$ and $\gamma_{i+1}$ when applying a power of $f$. We define a splitting $\gamma_1 \cdots \gamma_p \cdots$ of a ray $\rho$ similarly.

A \textit{Nielsen path} (NP) is a non-trivial edge path $\gamma$ such that $f^k_\gamma(\gamma) = \gamma$ (unlike other authors, we do not need to consider paths whose endpoints are not vertices). An edge $e$ is a Nielsen path if and only if it is fixed: $f(e) = e$. A path is \textit{pre-Nielsen} if some $f^k_\gamma(\gamma)$, with $k \geq 0$, is Nielsen.

There is a filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_K = G$ representing $G$ as an increasing sequence of (not necessarily connected) subgraphs. The union of the edges contained in $G_r$ but not in $G_{r-1}$ is a subgraph $H_r$ called the \textit{r-th stratum}. We say that edges in $H_r$ have \textit{height} $r$. The height $h(\gamma)$ of an edge path is the maximal height of its edges.

The filtration is compatible with $f$, in the sense that $f(G_r) \subset G_r$; equivalently, the image of an edge of height $r$ is an edge path of height $\leq r$.

There are three types of strata.

\begin{itemize}
  \item $H_r$ is a \textit{0-stratum} if $f(H_r) \subset G_{r-1}$. All components of a 0-stratum are contractible (they are subtrees of $G$).
  \item $H_r$ is an \textit{EG stratum} if, for any edge $e$ of $H_r$, the image $f(e)$ has a splitting $e_1ue_2$ with $e_1,e_2$ edges in $H_r$ (the path $u$ may be trivial); this is not the standard definition, but it is equivalent to it after raising $f$ to a power if needed. No cancellation between edges in $H_r$ occurs when applying $f$ to $f^k_\gamma(e)$, for $k \geq 1$ and $e$ an edge of $H_r$ (edges of $H_r$ are $r$-legal). In particular, the length of $f^k_\gamma(e)$ grows exponentially (EG means exponentially growing).
\end{itemize}
The last possibility is an NEG stratum: $H_r$ consists of a single edge $e$, and $f(e)$ is an edge path with a splitting of the form $eu$ for some loop $u$ contained in $G_{r-1}$. The stratum, and the edge $e$, is fixed if $f(e) = e$, linear if $u$ is pre-Nielsen (in particular, not pre-trivial).

An edge is called a 0-edge, an EG edge, an NEG edge depending on the nature of the stratum containing it.

A Nielsen path is indivisible, and called an INP, if it is not a concatenation of two Nielsen paths. Following tradition, we do not consider a fixed edge $e$ as an INP (it becomes divisible if paths whose endpoints are not vertices are allowed). Any Nielsen path is a finite concatenation of INP’s and fixed edges.

If $\gamma$ is an INP of height $r$, the stratum $H_r$ cannot be a 0-stratum. If it is EG, then the initial and terminal edges of $\gamma$ are distinct edges belonging to $H_r$, and (up to changing the orientation) $\gamma$ is the only INP of height $r$. If $H_r$ is NEG, then $H_r$ is a linear edge $e$ and (up to orientation) $\gamma = eu^p\bar{e}$ with $p \neq 0$, where $u$ is a Nielsen path and $f(e) = eu^d$ with $d > 0$. Implicit in the notation $u^d$ is the fact that one can concatenate copies of $u$: if $u = e_1 \ldots e_n$, then the terminal point of $e_n$ is the origin of $e_1$, and $e_n \neq e_1$; when $d < 0$, the notation $u^d$ means $\bar{u}^{-d}$.

An exceptional path is a path of the form $\gamma = eu^p\bar{e}'$ where $u$ is a Nielsen path, $p \in \mathbb{Z}$, and $e, e'$ are linear edges such that $f(e) = eu^d$ and $f(e') = e' u^d$ with $d, d' > 0$; unlike [13], we require $d \neq d'$: if $d = d'$ we view $eu^p\bar{e}'$ as an INP, not as an exceptional path. Note that $f_\sharp(\gamma)$ is an exceptional path if $\gamma$ is one.

A non-trivial path $\gamma$ is a connecting path if it is contained in a 0-stratum and its endpoints belong to an EG stratum $H_r$. A connecting path cannot be pre-trivial. Since 0-strata have contractible components, there is a bound for the length of connecting paths.

A complete splitting of a path $\gamma$ is a splitting $\gamma = \gamma_1 \ldots \gamma_p$ such that each $\gamma_i$, called a term (or splitting unit) of the splitting, is one of the following:

- a single edge in an EG or NEG stratum;
- an exceptional path;
- an indivisible Nielsen path (INP);
- a connecting path contained in some $f^k_\sharp(e)$, with $k \geq 1$ and $e$ an edge in an EG or NEG stratum.

A term is never pre-trivial. There is a uniform bound for the length of terms which are not exceptional or INP’s of the form $eu^p\bar{e}$ with $e$ linear.

A path is completely split if it is non-trivial and has a complete splitting. This complete splitting is unique. We say that its terms are the terms of $\gamma$. A subpath $\gamma_r \ldots \gamma_s$, which is a union of terms of the complete splitting of $\gamma$, will be called a full subpath of $\gamma$. It is interior if $r > 1$ and $s < p$.

If a subpath of a completely split $\gamma$ is an INP or an exceptional path, it is contained in a term of the complete splitting of $\gamma$ (which is exceptional or an INP), see the proof of Lemma 4.11 in [13].

The image of any EG or NEG edge is completely split, and its complete splitting is a refinement of the splitting $e_1ue_2$, $eu$ or $\bar{u}e$ mentioned above. The tightened image of a completely split path $\gamma_1 \ldots \gamma_p$ is completely split, and its complete splitting refines the splitting $f_\sharp(\gamma_1) \ldots f_\sharp(\gamma_p)$.

If $\gamma$ is an arbitrary edge path, there is an integer $k$ such that $f^k_\sharp(\gamma)$ is completely split.

Generally speaking, many results are proved by induction on height, using the following observations. If $\gamma$ is a connecting path contained in $H_i$, then $f_\sharp(\gamma) \subset G_{i-1}$. If $e$ is an edge in an NEG stratum $H_i$, then $f(e)$ equals $eu$ or $ue$ with $u \subset G_{i-1}$.

2.7. Growth. Let $f$ be a completely split train track map as above. The growth of an edge path $\gamma$ is the Lipschitz-equivalence class of the function $n \mapsto |f^n_\sharp(\gamma)|$ (see Remark 2.8). A path is growing if $|f^n_\sharp(\gamma)|$ is unbounded (this is equivalent to $\gamma$ not being pre-Nielsen or pre-trivial). Since some $f^k_\sharp(\gamma)$ is completely split, and the growth of a completely split path is the maximal growth of its terms, it suffices to understand the growth of terms.
An INP, a fixed edge do not grow. An exceptional path, and a linear edge as defined above, grow linearly. If $\gamma$ is a connecting path, $f(\gamma)$ has lower height and one can understand the growth of $\gamma$ by induction. Induction also applies if $e$ is an NEG edge since $f^n_{\rho}(e) = eu_{\rho}f_{\rho}(u)\ldots f_{\rho}^{-1}(u)$ if $f(e) = eu$; in particular, $e$ has polynomial growth of degree $k+1$ if $u$ has polynomial growth of degree $k$. As previously mentioned, an EG edge grows exponentially. More precisely:

**Proposition 2.16.** The growth function of any edge path $\gamma$ is Lipschitz-equivalent to $n^d\lambda^n$ for some $\lambda \geq 1$ and some integer $d \geq 0$. If $\gamma$ is growing and its growth is polynomial ($\lambda = 1$), some $f_{\rho}^{\kappa}(\gamma)$ is completely split and there is a term which is an exceptional path or a linear edge.

The first assertion is proved in the appendix of [21]. The second one follows from remarks made above, using induction on height. We say that $n^d\lambda^n$ is the growth type of $\gamma$.

Similarly, if $\alpha \in \text{Aut}(F)$ and $g \in F$, the length of $\alpha^n(g)$ is Lipschitz-equivalent to some $n^d\lambda^n$ as $n \to \infty$ (see the appendix of [21]).

### 3. Computing the complexity

Let $f : G \to G$ be a completely split train track map. Let $e$ be an edge such that $f(e) = e \cdot \gamma$ for some completely split path $\gamma$. The union of the paths $f_{\rho}(\gamma)$ is a completely split ray $\rho = e\gamma f_{\rho}(\gamma)\ldots f_{\rho}^{k}(\gamma)\ldots$ whose complexity we shall compute. For technical reasons, we will also consider the total complexity of $\gamma$, defined as follows.

**Definition 3.1** (Total complexity). We define the total complexity $p_{\rho}(n)$ of a path $\gamma$ as the number of distinct paths of length $n$ appearing as a subpath of some $f_{\rho}^{k}(\gamma)$ (this is similar to the complexity of DOL languages studied in [12]).

Clearly $p_{\gamma} \leq p_{\rho}$ for $\gamma$ and $\rho$ as above. See Lemma [5.3] for a converse.

Consider all growing terms in the complete splittings of the paths $f_{\rho}^{k}(\gamma)$, and their growth types $n^d\lambda^n$.

**Definition 3.2** (Divergence). Following [29], we say that $\gamma$ (or $\rho$) is non-divergent if all growing terms have the same $\lambda$ and $d$, exponentially divergent if two different $\lambda$’s occur, polynomially divergent if all $\lambda$’s are the same but two different $d$’s occur (this is equivalent to saying that either all $\lambda$’s are equal to 1 and $n^2$ occurs, or all $\lambda$’s are equal, different from 1, and $n\lambda^n$ occurs).

The key result is the following.

**Proposition 3.3.** Let $e$ with $f(e) = e \cdot \gamma$ and $\rho = e\gamma f_{\rho}(\gamma)\ldots f_{\rho}^{k}(\gamma)\ldots$ be as above.

1. If $\gamma$ does not grow (it is pre-Nielsen), the complexity $p_{\rho}(n)$ is bounded (this is obvious).
2. If all growing terms of the paths $f_{\rho}^{k}(\gamma)$ grow exponentially, $p_{\rho}(n)$ is equivalent to $n$, $n\log\log n$, or $n\log n$, depending on whether $\gamma$ is non-divergent, polynomially divergent, or exponentially divergent.
3. Otherwise, $p_{\rho}(n)$ is quadratic (equivalent to $n^2$).

Equivalence may be understood as Lipschitz-equivalence (see Remark [2.8]).

**Remark 3.4.** The same results hold for the total complexity $p_{\gamma}(n)$, except that in order to get quadratic complexity we sometimes need $\gamma$ to be “long enough” (see Lemma [5.8]).

For instance, consider the automorphism of $F_2$ sending $a$ to $a$ and $b$ to $ba$, and the corresponding map on a rose. The $b$ edge is linear but the total complexity of the path $bbaba^2$, which is of the form $\delta(f(\delta) f^2(\delta))$, does have quadratic total complexity.
The remainder of this section is devoted to the proof of Proposition 3.3. Many combinatorial arguments are similar to Pansiot’s, but we need some preliminary technical results.

It is easy to check that Proposition 3.3 (as well as most of the statements that we shall make) is true for $f$ provided that it is true for some power (note that a power of a completely split train track map is also one). We may therefore replace $f$ by a power whenever convenient.

We distinguish between geometric subpaths of $\rho$ and abstract paths, i.e. words on the alphabet consisting of the edges of $G$: a given abstract path may appear several times in $\rho$.

3.1. Preliminary results. In this subsection we let $f : G \to G$ be a completely split train track map, and $\gamma$ an arbitrary edge path.

**Lemma 3.5** (Stabilization of prefixes). There exists a power $g$ of $f$ such that, given any edge path $\gamma$, there is $C > 0$ such that, for any $n$, the $n$-prefix of $g_k^n(\gamma)$, viewed as an abstract path, is independent of $f$ for $k > Cn$.

**Proof.** After replacing $\gamma$ by some $f_i(\gamma)$, we may assume that $\gamma$ is completely split. There is a uniform linear lower bound for the length of $f_k(\tau)$, for $\tau$ a growing term, so we may assume that $\gamma$ is a single growing term. Choose $g$ such that $g(e)$ and $g_2^1(e)$ have the same initial edge $a_e$ if $e$ is any EG edge.

The result is clear if $\gamma$ is an exceptional path, and also if it is an exceptional path, and also if it is an EG edge $e$ such that $f(e) = eu$ because $g_k^1(\gamma)$ is a prefix of $g_k^{k+1}(\gamma)$. If $\gamma$ is an EG edge $e$, then $g_k^1(\gamma)$ starts with $g_k^{k-1}(a_e)$; in this case, stabilization occurs for $k = O(\log n)$. If $\gamma$ is an exceptional path $\gamma$ with $f(e) = we$, the result is clear if $w$ is pre-Nielsen, and follows by induction on height if $w$ is growing. We also use induction if $\gamma$ is a connecting path. \qed

**Lemma 3.6** (Nielsen paths are short). Let $\gamma$ be a completely split path such that no $f_k(\gamma)$ has a term which is a linear edge or an exceptional path. There is a number $C$ such that, if $\delta$ is any non-growing full subpath of some $f_k(\gamma)$, then $\delta$ has length $|\delta| \leq C$.

The assumption on $\gamma$ implies that it is pre-Nielsen or grows exponentially. More precisely, any term appearing in the complete splitting of some $f_k(\gamma)$ is either non-growing (it is an INP, a fixed edge, or a pre-Nielsen connecting path) or exponentially growing (a single edge or a connecting path). There is a uniform bound for the length of growing terms.

**Proof.** Replacing $f$ by a power, we may assume that the image of any pre-Nielsen connecting path is a Nielsen path. Given any full subpath $\delta \subset f_k(\gamma)$, let $n_\delta$ be the number of terms of $\delta$ which are pre-Nielsen connecting paths. If $e$ is an edge of $f_k(\gamma)$ adjacent to $\delta$ (there may be 0, 1 or 2 such edges), let $h(e)$ be the height of $e$ (i.e. $h(e) = i$ if $e \subset H_i$), and define $c_\delta$ as the sum of the $h(e)$’s (with $c(\delta) = 0$ if there is no adjacent edge). Note that $c_{f_1(\delta)} \leq c_\delta$ because no term is pre-trivial.

We choose $A$ bigger than $|f_2(\tau)|$ for any pre-Nielsen connecting path $\tau$, and $B$ bigger than $|f_2(\tau)| + An_{f_2(\tau)}$ for any growing term $\tau$. Let $\Delta(\delta) = |\delta| + An_{f_2(\tau)} + Bc_\delta$. We show by induction on $k$ that, for any maximal non-growing full subpath $\delta \subset f_k(\gamma)$, one has $\Delta(\delta) \leq C$ for some constant $C$ (depending only on $\gamma$) subject to two requirements.

The first requirement is that the result should hold for $k = 0$ (there are only finitely many $\delta$’s to consider). The second one will be specified later.

Let $\delta$ be a maximal non-growing full subpath of $f_k(\gamma)$, with $k \geq 1$. We consider the minimal full subpath of $f_k^{k-1}(\gamma)$ whose image contains $\delta$. It is a union of terms, and every interior term is non-growing. We therefore write it as $b_1 \hat{\delta} b_2$, with $b_1, b_2$ growing terms and $\hat{\delta}$ a union of non-growing terms ($b_1, b_2, \hat{\delta}$ may be empty). Note that $\hat{\delta}$ is maximal as a non-growing full subpath of $f_k^{k-1}(\gamma)$, even if $b_1, b_2$ are empty, so $\Delta(\hat{\delta}) \leq C$ by the induction hypothesis.
There is a uniform bound for $|b_1|$ and $|b_2|$, so paths $\delta$ such that $\hat{\delta}$ is empty have bounded length, and the second requirement on $C$ is that $\Delta(\delta) \leq C$ should hold for paths $\delta$ such that $\hat{\delta}$ is empty. From now on, we consider $\delta$ such that $\hat{\delta}$ is non-empty.

First note that $|f_\gamma(\hat{\delta})| = An_{f_\gamma(\hat{\delta})} - |\hat{\delta}| + An_{\delta}$. Indeed, $\hat{\delta}$ consists of fixed terms and pre-Nielsen connecting paths. These connecting paths map onto Nielsen paths, so $n_{f_\gamma(\hat{\delta})} = 0$ and the inequality follows from the choice of $A$.

If $b_1$ and $b_2$ are empty, we have $\delta = f_\gamma(\hat{\delta})$ and $c_{f_\gamma(\hat{\delta})} \leq c_\delta$, so $\Delta(\delta) \leq \Delta(\hat{\delta}) \leq C$. Assume that, say, $b_1$ is non-empty and $b_2$ is empty (the argument when both are non-empty is similar). We write $f_\gamma(b_1) = b\delta_1$, where $b\gamma$ is a growing term of the complete splitting of $f_\gamma(b_1)$ and $\delta = \delta_1 f_\gamma(\hat{\delta})$ (both $b\gamma$ and $\delta_1$ are non-empty). We then have

$$|\delta| + An_{\delta} = |\delta_1| + An_{\hat{\delta}} + |f_\gamma(\hat{\delta})| + An_{f_\gamma(\hat{\delta})} \leq B + |\hat{\delta}| + An_{\hat{\delta}}$$

since $|\delta_1| + An_{\hat{\delta}} \leq |f_\gamma(b_1)| + An_{f_\gamma(b_1)} \leq B$ by our choice of $B$.

It therefore suffices to prove that the terminal edge $e'$ of $b\gamma$ (an edge adjacent to $\delta$) has strictly lower height than the terminal edge $e$ of $b_1$ (an edge adjacent to $\hat{\delta}$): this implies $c_\delta \leq c_{\delta} - 1$ and $\Delta(\delta) \leq \Delta(\hat{\delta}) \leq C$. Now $b_1$ is a growing term and $f_\gamma(b_1)$ ends with a non-growing term, so there are only two possibilities: either $b_1$ is a connecting path, or $b_1 = e$ is a NEG stratum $H_e$ and $f(e) = eu$ with $u \in G_{r-1}$. In both cases $e'$ has smaller height (in the second case, note that $u$ is growing because there is no linear edge, hence contains $e'$).

As in [29], we shall use the following fact:

**Lemma 3.7.** Given an integer $n$, let $P_n$ be the number of integers $p$ such that

$$C_1 p^{d_1} \lambda^p_1 \leq n \leq C_2 p^{d_2} \lambda^p_2,$$

where $d_1, d_2$ are positive integers, $C_1, C_2 > 0$, and $1 < \lambda_1 \leq \lambda_2$.

There exist positive numbers $M_1, M_2$ such that

$$M_1 \leq \frac{P_n}{\phi(n)} \leq M_2$$

for $n \geq 2$, with:

- $\phi(n) = 1$ if $\lambda_1 = \lambda_2$, $d_1 = d_2$, and $C_1 < C_2$ (no divergence);
- $\phi(n) = \log n$ if $\lambda_1 = \lambda_2$ and $d_1 \leq d_2$ (polynomial divergence);
- $\phi(n) = \log n$ if $\lambda_1 < \lambda_2$ (exponential divergence).

\[\square\]

3.2. Upper bound. We first prove upper bounds for the complexity $p_\gamma(n)$, with $\rho$ as in Proposition 3.3. Replacing $f$ by a power, we may assume that Lemma 3.5 applies.

We fix a number $K$ which is large compared to $|\gamma|$, the constant $C$ provided by Lemma 3.6 if it applies to $\gamma$, and the length of terms which are not exceptional paths or INP’s.

By abuse, we view $f_\gamma$ as a map from $\rho$ to $\rho$ sending $e$ to $e\gamma$ and $f_\gamma^k(\gamma)$ to $f_\gamma^{k+1}(\gamma)$. Consider $n > K$ and a subpath $w_0$ of length $n$ in $\rho$. Let $w_1$ be the smallest full subpath of $\rho$ such that $f_\gamma(w_1)$ contains $w_0$. Then define $w_2$ similarly and keep going until the first $p$ for which $|w_{p+1}| \leq K < |w_p|$ (see Figure 2). Such a $p$ exists because $n > K > |\gamma|$.

Denote by $n^\rho \lambda^p$ the growth type of $\gamma$. The path $w_p$ is the smallest full subpath $\nu$ of $\rho$ such that $f_\gamma(\nu)$ contains $w_0$. Since $|w_p|$ is bounded and $|u_0| = n$, this gives an inequality of the form $n \leq C_1 p^{d_1} \lambda^p$, hence a lower bound for $p$ in terms of $n$ and the growth type $n^\rho \lambda^p$ of $\gamma$ (with $C_1$ depending only on $f, \gamma$ and $K$).

We now consider the complete splitting of $w_p$. There are two cases. Both may occur for subpaths $w_0$ representing the same abstract path, but in order to bound $p_\gamma$ up to equivalence we may choose one for every abstract $w_0$.

- First suppose that $w_p$ has an interior growing term $\tau$, and call $x$ its origin. We then have $f_\gamma^p(\tau) \subset u_0$, hence an inequality $C_2 p^{d_1} \lambda^p \leq n$ yielding an upper bound for $p$ in terms
of $n$ and the growth type $n^d \lambda^m x$ of $\tau$. The combination of the lower and upper bounds restrict $p$ to an interval whose size $S_n$ depends on the divergence of $\gamma$.

Recall that we want an upper bound for $p_\rho(n)$, the number of distinct abstract paths $w_0$ of length $n$ contained in $\rho$. Such a path is determined by the abstract path $w_p$, the number $p$, and the position of $f^p(x)$ within $w_0$. There are boundedly many possibilities for $w_p$ and $x$, and $n + 1$ possibilities for the position of $f^p(x)$, so that $nS_n$ is an upper bound for $p_\rho$ (up to equivalence).

We shall use Lemma 3.7 to estimate $S_n$, which is the number of integers $p$ satisfying inequalities of the form $C_2 2^p \lambda^p \leq n \leq C_1 p^d \lambda^p$. First, since $\gamma$ grows at least linearly, $S_n$ is at most linear and thus $p_\rho$ is at most quadratic. Now suppose, as in the second assertion of Proposition 3.3, that all growing terms of the paths $f^k_x(\gamma)$ grow exponentially.

If $\gamma$ is non-divergent, the bounds are of the form $C_2 2^p \lambda^p \leq n \leq C_1 p^d \lambda^p$, with $\lambda > 1$ and the same $d$, $\lambda$ on both sides. By Lemma 3.7 this implies that $S_n$ is uniformly bounded, and $p_\rho$ is at most linear. If the divergence is polynomial, we have $C_2 p^d \lambda^p \leq n \leq C_1 p^d \lambda^p$, with the same $d$, $\lambda$ on both sides, and $S_n = O((\log \log n)$. If the divergence is exponential, we have $C_2 p^d \lambda^p \leq n \leq C_1 p^d \lambda^p$ and $S_n = O(\log n)$.

• The second case is when the complete splitting of $w_0$ has no interior growing term. By Lemma 3.6 and our choice of $K$, this cannot happen if all growing terms of the paths $f^k_x(\gamma)$ grow exponentially. It thus suffices to show that $p_\gamma$ is at most quadratic.

If $w_0$ is a single term, it must be an exceptional path (a trivial case), so we can assume that $w_p$ may be split at a vertex $x$ as a concatenation $w_p = vv'$ (one of $v, v'$ may be non-growing). As in the first case we have $f^p(x) \in w_0$. The path $w_0$ consists of a suffix of $f^k_x(v)$ and a prefix of $f^k_x(v')$, and the quadratic bound for $p_\gamma$ follows from stabilization of prefixes (Lemma 3.8).

### 3.3. Quadratic lower bound.
In this subsection we shall deduce the third assertion of Proposition 3.3, a quadratic lower bound for $p_\rho$, from the following lemma. Recall (Definition 3.1) that the total complexity $p_\delta(n)$ of a path $\delta$ is the number of distinct paths of length $n$ appearing as a subpath of some $f^k_x(\delta)$.

**Lemma 3.8.** The total complexity $p_\delta(n)$ is at least quadratic if $\delta$ is an edge path with a complete splitting refining a splitting of one of the following forms:

- $\nu \theta w$, where $\theta$ is an exceptional path and $v, w$ are growing;
- $\nu \theta w$, where $\theta$ is a linear edge with $f(\theta) = \theta u$, the paths $v, w$ are growing, and $w$ has at least two growing terms in its complete splitting.

**Proof.** In the first case we write $\theta = eu^v e'$, with $f(e) = eu^d$ and $f(e') = e' u^d$. We then have $f^k_x(\delta) = v_k e u^a_k e' w_k$ where $a_k$ grows linearly and $|v_k|, |w_k|$ grow at least linearly. Given $n$, consider for all $k$ such that $|a_k| u_k \geq n/2$ the subpaths $\eta$ of length $n$ of $f^k_x(\delta)$.
containing the subpath $e^a w e'$. The number of subpaths thus constructed is equivalent to $n^k$. They are all distinct as abstract paths: $\eta$ determines $k$ because the subpath $u^a k$ may be characterized as the only maximal subpath of $\eta$ whose height is less than $h(e)$ and $h(e')$, and whose length is at least $n/2$.

In the second case, we write $f^k_\sharp(\delta) = v_k \theta u_k^k v_k$. We now consider $k$ such that $k|u| \geq n/2$ and subpaths $\eta$ of length $n$ containing $\theta u_k^k$. To show that they are distinct, we consider the maximal subpath of $\eta$ whose height is less than $h(\theta)$ and whose length is at least $n/2$. There is an added difficulty because $u$ may be a prefix of $w_k$.

Write $w = xyz$ with $x$ a (possibly empty) Nielsen path, $y$ a growing term, and $z$ growing. Using induction on height, we may assume that $y$ is not a connecting path. If it is an EG edge, or an exceptional path, or an NEG edge with $f(y) = y\tau$, every prefix of $w_k$ of the form $u^k x \tau^k$. The number of these paths is equivalent to $n^k$ because $z$ is growing.

Suppose that $\gamma$ is as in Proposition 3.3 and neither 1 nor 2 applies. By Proposition 2.16, there is $k_0$ such that, for $k \geq k_0$, the path $f^k_\sharp(\gamma)$ contains a subpath $\theta_k$ which is an exceptional path or a linear edge. We apply the lemma to $\theta = \theta_{k_0+2}$ (with the obvious modifications if $f(\theta)$ equals $u\theta$ rather than $\theta u$).

3.4. A lower bound from divergence. There remains to prove the lower bounds in the second assertion of Proposition 3.3. Our assumptions imply that all terms appearing in the complete splitting of $f^k_\sharp(\gamma)$ are INP’s, edges which are not linear, or connecting paths. There are no exceptional paths, and the length of INP’s is bounded. This allows us to shorten the proof by introducing a substitution as in [15] and applying Pansiot’s theorem.

Let $R$ be the alphabet consisting of $e$ and all terms $\delta$ appearing in the complete splittings of the paths $f^k_\sharp(\gamma)$. As explained above, $R$ is finite.

The map $f$ induces a substitution $\tau$ on the alphabet $R$: a term $b$ is sent to the word in $R^*$ given by the complete splitting of $f_\sharp(b)$. Applying powers of $\tau$ to $e$ produces a $\tau$-fixed right-infinite word $Y$ in the alphabet $R$ corresponding to the complete splitting of $\rho$. The complexity of $Y$ is controlled by Pansiot’s theorem, we just need to relate it to that of $\rho$.

Let $C$ be a bound for the length (as a path) of all terms in $R$. We can associate to any subword of length $n$ in $Y$ the corresponding completely split subpath of $\rho$. It has length between $n$ and $C n$. Uniqueness of the complete splitting ensures that this map is injective, so that

$$p_Y(n) \leq \sum_{m=n}^{C n} p_\rho(m).$$

Conversely, consider a subpath $\gamma$ of length $n$ in $\rho$. Let $\hat{\gamma}$ be the smallest full subpath containing it. This yields a word of length at most $n$ in $R^*$ (different subpaths which are equal as abstract paths may produce different words, but we do not care). At most $C$ different paths can produce a given $\hat{\gamma}$, and we get

$$p_\rho(n) \leq \frac{1}{C} \sum_{m=1}^{n} p_\rho(m).$$

As in the proof of Lemma 2.10, we conclude that $p_\rho$ is equivalent to $p_Y$ and we can apply Pansiot’s theorem to compute $p_\rho$. Moreover, the growth of a term (as a path) is equivalent to that of the letter of $R$ representing it under $\tau$, so the divergence used in the proposition is the same as the divergence that appears in Pansiot’s theorem.

This concludes the proof of Proposition 3.3.
4. Complexity of fixed points on the boundary

Let \( \alpha \) be an automorphism of a finitely generated free group \( F \). As explained in Subsection 2.7, it acts on the boundary \( \partial F \), and we let \( X \in \partial F \) be a fixed point. We are interested in the complexity \( p_X \) of \( X \), which is well-defined up to equivalence by Corollary 2.12.

There are three possibilities for \( X \) (see [13]). First, \( \alpha \) may have a non-trivial fixed subgroup \( \text{Fix}_\alpha = \{g \in F \mid \alpha(g) = g\} \), and \( X \) may be a point of the boundary of \( \text{Fix}_\alpha \) (the most trivial example being when \( \alpha \) is the identity); in this case \( X \) may be fairly arbitrary, and there is no hope of describing its complexity. By Proposition 1.1 of [13], fixed points \( X \) which do not belong to \( \partial \text{Fix}_\alpha \) are either attracting or repelling (attracting for \( \alpha^{-1} \)) for the action of \( \alpha \) on \( F \cup \partial F \).

Remark 4.1. If \( X \in \partial F \) is rational, i.e. of the form \( g^\infty = \lim_{n \to +\infty} g^n \) for some \( g \in F \), then \( \alpha(X) = \alpha(g)^\infty \), so \( \alpha(X) = X \) implies \( \alpha(g) = g \) and \( X \in \partial \text{Fix}_\alpha \). Thus an attracting or repelling fixed point \( X \) cannot be rational: viewed as a right-infinite word, it is not ultimately periodic. In particular, its complexity is at least linear by [28].

Theorem 4.2. Let \( \alpha \) be an automorphism of a finitely generated free group \( F \), and let \( X \in \partial F \) be fixed by \( \alpha \). If \( X \notin \partial \text{Fix}_\alpha \) (i.e. \( X \) is attracting or repelling), its complexity \( p_X \) is equivalent to \( n, n \log \log n, n \log n \), or \( n^2 \).

For a given \( \alpha \), there are only finitely many attracting/repelling fixed points up to the action of \( \text{Fix}_\alpha \) by [10]. On the other hand, if we fix \( \Phi \in \text{Out}(F) \) and vary the representative \( \alpha \), the theorem applies to infinitely many \( X \) (see Example 5.10).

Theorem 4.2 is a rewording of Theorem 1.1. It implies Corollary 1.3 because any \( X = \lim_{p \to \infty} \alpha^p(g) \) must be rational or attracting ([19], Théorème 2.15). It is an immediate consequence of Proposition 4.3 and the following lemma.

Lemma 4.3. Let \( X \) be as in the theorem. There exist a completely split train track map \( f : G \to G \) and a ray \( \rho \) in \( G \) such that:

- the complexity of \( \rho \) is equivalent to the complexity of \( X \);
- there is an edge \( e \) such that \( f(e) = e \cdot \sigma \) with \( \sigma \) a growing completely split path, and \( \rho = e \cdot \sigma \cdot f_2(\sigma) \cdot f_3(\sigma) \cdot \ldots \).

We do not claim that \( f \) represents (a power of) \( \alpha \), or even that \( \pi_1(G) \simeq F \).

Proof. This relies on Lemma 4.36 (2) of [13]. To apply it, we need \( X \) to be attracting and \( \alpha \) to be principal (see Definition 3.1 of [13]), so we start by modifying \( \alpha \). To make \( X \) attracting, we replace \( \alpha \) by \( \alpha^{-1} \) if needed. To make \( \alpha \) principal, we replace it by the automorphism of \( F * \mathbb{F}_2 \) equal to \( \alpha \) on \( F \) and the identity on \( \mathbb{F}_2 \). We then raise it to a power, so that it is represented by a completely split train track map \( f : G \to G \) (Theorem 4.28 of [13]). We view \( X \) as an attracting fixed point of the modified \( \alpha \).

We now apply Lemma 4.36 (2) of [13]. It provides a vertex \( x \) of \( G \) with \( f(x) = x \) and a non-linear edge \( e \) with origin \( x \) such that \( e \subset f_2(e) \subset f_3(e) \subset \ldots \) is an increasing sequence of paths whose union is a ray \( \rho \) representing \( X \). We let \( \sigma \) be the path such that \( f(e) = e \cdot \sigma \).

Recall (see for instance [20, 21]) that \( \alpha \) is fully irreducible (also known as iwip) if no free factor of \( F \) other than \( \{1\} \) or \( F \) is mapped to a conjugate by a power of \( \alpha \), atoroidal if there is no non-trivial \( \alpha \)-periodic conjugacy class in \( F \), polynomially growing if every \( g \in F \) grows polynomially under iteration of \( \alpha \) (equivalently, every edge path grows polynomially under iteration of \( f \)).

Corollary 4.4. Let \( X \) be an attracting or repelling fixed point of \( \alpha \) in \( \partial F \setminus \partial \text{Fix}_\alpha \).

- If \( \alpha \) is fully irreducible, the complexity of \( X \) is linear.
- If \( \alpha \) is atoroidal, the complexity of \( X \) is \( O(n \log n) \).
- If \( \alpha \) is polynomially growing, the complexity of \( X \) is quadratic.
The functions $\rho$

Proof. This follows from Proposition 5.3. If $\alpha$ is fully irreducible, there is no divergence ($G$ consists of a single EG stratum). If $\alpha$ is atoroidal, there is no linear edge or exceptional path $\gamma$ (as the path $u$ appearing in the definition of $\gamma$ given in Subsection 2.6 would define a fixed conjugacy class), so $p_X$ cannot be quadratic. If $\alpha$ is polynomially growing, we must be in the third case of the proposition.

Remark 4.5. Using Remark 2.11, one can show that the complexity is quadratic also if $\alpha$ is arbitrary but $X$ belongs to the boundary of a polynomial subgroup (i.e. a subgroup $J$ which is invariant under an automorphism $\beta$ representing the same outer automorphism as $\alpha$, with $\beta|_J$ polynomially growing, see Proposition 1.4 of [21]).

5. Recurrence complexity

Let $f : G \to G$ be a completely split train track map.

5.1. Preliminaries. The complexity function $p_\rho$ of a ray is defined by counting the abstract paths appearing at least once in $\rho$. We also defined the (total) complexity $p_\gamma$ of a path $\gamma$ by counting the paths appearing in some $f^k_\gamma(\gamma)$. We now define the recurrence complexity function $p^{\text{rec}}_\rho$ by counting the paths appearing infinitely often.

Definition 5.1 (Recurrence complexity). Given a ray $\rho$ in $G$, we define the recurrence complexity function $p^{\text{rec}}_\rho(n)$ as the number of abstract subpaths of length $n$ appearing infinitely often in $\rho$.

Given an edge path $\gamma$, we define $p^{\text{rec}}_\gamma(n)$ as the number of abstract subpaths of length $n$ appearing in infinitely many $f^k_\gamma(\gamma)$.

Lemma 2.4 is true for recurrence complexity, so that any element $X \in \partial F$ has a recurrence complexity $p^{\text{rec}}_X$, well-defined up to equivalence.

Example 5.2. The recurrence complexity may be smaller than the complexity: if $\alpha$ is the automorphism of $F_3$ (or invertible substitution on 3 letters) defined by $a \mapsto a, b \mapsto ba, c \mapsto cb$, the fixed point $X = \lim_{p \to \infty} \alpha^p(c) = cbba^2 \cdots$ and the path $\gamma = bbab^2$ have quadratic complexity and linear recurrence complexity (subwords appearing infinitely often contain $b$ at most once).

Lemma 5.3. Let $\rho = e.\sigma.f_1(\sigma).f_2^2(\sigma).f_3^4(\sigma) \ldots$ be as in Lemma 4.3, and let $\gamma = \sigma.f_2(\sigma).f_3^2(\sigma)$. The functions $p_X$ and $p_\gamma$ are equivalent, and $p_\rho \sim p_X^{\text{rec}} \sim p_\gamma^{\text{rec}} \geq n$.

Proof. Clearly $p_\gamma \leq p_X$. To prove $p_X \lesssim p_\gamma$, we have to control the subpaths $\tau$ of $\rho$ which are not contained in any $f^k_\gamma(\gamma)$. Note that any such $\tau$ contains some $f^k_\gamma(\sigma)$, hence has to be long if it is far from the origin of $\rho$. More precisely, there is a function $g$, depending on the growth type of $\sigma$, such that $\tau$ is contained in the $g(|\tau|)$-prefix of $\rho$, and we have $p_X(n) \leq p_\gamma(n) + O(g(n))$.

To compute $g$, we have to estimate the distance from the subpath $f^k_\gamma(\sigma)$ to the origin of $\rho$ in terms of its length, in other words to bound $\sum_{k=1}^\ell |f^k_\gamma(\sigma)|$ in terms of $|f^\ell_\gamma(\sigma)|$.

Let $k^d \lambda^k$ be the growth type of $\sigma$. If $\lambda > 1$, then $\sum_{k=1}^\ell k^d \lambda^k = O(\ell^d \lambda^\ell)$ and $g$ is linear, so $p_X(n) \leq p_\gamma(n) + O(n)$, hence $p_X \lesssim p_\gamma$. If $\lambda = 1$, then $\sum_{k=1}^\ell k^d = O(\ell^{d+1})$ and $g(n) \sim n^{d/\lambda}$. We deduce $p_X(n) \leq p_\gamma(n) + O(n^{d/\lambda})$, and $p_X \lesssim p_\gamma$ because $d/\lambda \leq 2$ and $p_\gamma$ is quadratic by Lemma 4.8.

Clearly $p_X \geq p^{\text{rec}}_X \geq p^{\text{rec}}_\gamma$, and $p^{\text{rec}}_X \lesssim p^{\text{rec}}_\gamma$ because any path contributing to $p^{\text{rec}}_X$ appears arbitrarily far from the origin of $\rho$. The recurrence complexity of $X$ is at least linear because the function $p^{\text{rec}}_\rho$ is increasing: if $p^{\text{rec}}_\rho(n+1) = p^{\text{rec}}_\rho(n)$ for some $n$, some truncation $p_k$ of $\rho$ satisfies $p_{p_k}(n+1) = p_{p_k}(n)$ hence is ultimately periodic by [28], a contradiction.

5.2. Recurrence complexity of paths.

Proposition 5.4. The recurrence complexity $p^{\text{rec}}_\gamma$ of any path $\gamma$ in $G$ is equivalent to $1, n, n \log \log n, n \log n,$ or $n^2$.

We start by the following simple remark.
Lemma 5.5. Assume that γ splits as γ1,γ2. If pγ1rec ≥ n or pγ2rec ≥ n, then pγrec is the maximum of pγ1rec, pγ2rec.

Proof. This follows from stabilisation of prefixes (Lemma 5.3). Words of length n appearing in f^k_2(γ) appear in f^k_1(γ1) or in f^k_2(γ2) or contain the image of the splitting point. By Lemma 5.3, neglecting the third type causes a linear error in pγrec.

Proof of Proposition 5.4. We first consider the recurrence complexity of terms other than connecting paths. Fixed edges, linear edges, INP’s and exceptional paths have bounded recurrence complexity. Now let e be a non-linear and non-fixed edge in an EG or NEG stratum. Since e appears as a term in infinitely many images f^k_2(e), its recurrence complexity pγrec is equal to its total complexity pγ, which is unbounded and satisfies one of the conclusions of the proposition by Proposition 5.3.

Now suppose that γ is completely split, and no term is a connecting path. If every term of γ has bounded recurrence complexity, one easily checks that pγrec is bounded or linear. Otherwise, using Lemma 5.5 one shows by induction on the number of terms that the proposition holds for γ.

This proves the proposition when no connecting path occurs, since any γ has a completely split image f^k_2(γ). To handle connecting paths, one shows by induction on height that connecting paths satisfy the proposition, and moreover any connecting path γ with bounded recurrence complexity has an image f^k_2(γ) whose complete splitting only contains fixed edges, linear edges, INP’s and exceptional paths. One then argues as before.

5.3. Recurrence complexity of fixed points.

Theorem 5.6. Let α be an automorphism of a finitely generated free group F, and let X ∈ ∂F be fixed by α. Assume X ∉ ∂Fixα.

- The recurrence complexity pXrec is equivalent to n, n log log n, n log n, or n^2.
- If the usual complexity pX is not quadratic, in particular if α is fully irreducible or atoroidal, then pXrec ∼ pX.

Proof. Represent X by ρ = σ.f^1(σ).f^2(σ).f^3(σ).... as in Lemma 4.3. The first assertion follows directly from Proposition 5.4 and Lemma 5.3. We prove the second one.

Each f^k_2(σ) has a complete splitting. Since the complexity is not quadratic, there is no linear edge or exceptional path. All growing terms grow exponentially and there is a bound K for the length of growing terms and the length of non-growing full subpaths (see Lemma 3.6). By merging terms, we can construct a splitting of each f^k_2(σ), hence of ρ, such that all pieces are growing and have length at most 3K.

Consider all pieces b, b’ such that b b’ appears in this splitting of ρ. We truncate ρ, by removing some initial path σ.f_2(σ)...f^j_2(σ) so that, if b b’ appears, it appears infinitely often. We write the splitting of f^j_2+1(σ)f^j_2+2(σ) by pieces as b_1...b_r.

As j and k vary, the paths f^j_2(b_j b_{j+1}) cover (the truncated) ρ with overlaps so, as in the proof of Lemma 5.3, exponential growth implies that we can compute the usual complexity of X, up to a linear error, by considering only subpaths of ρ contained in some f^j_2(b_j b_{j+1}). Since every b_j b_{j+1} appears infinitely often in the splitting of ρ constructed above, we conclude pXrec ∼ pX.

5.4. Complexity of laminations. Laminations are an important tool in the theory of automorphisms of free groups. We refer to [11] for a thorough discussion of various viewpoints: algebraic laminations, symbolic laminations, laminary languages.

In this subsection we explain that any lamination has a complexity function, well-defined up to equivalence. Moreover, any X ∈ ∂F generates a lamination LX. If X is fixed by some automorphism α ∈ Aut(F), we use Theorem 5.6 to control the complexity of LX.

This applies in particular to the attracting laminations defined in [3], see Corollary 5.9 below.

To define a lamination simply, we fix a free basis A of F. A (symbolic) lamination is a non-empty set L of reduced bi-infinite words z = (z_i)_{i ∈ Z} in the alphabet E = A ∪ A^−1.
(called the leaves of the lamination) which is closed (in the product topology on $E^\mathbb{Z}$),
invariant under the shift $\sigma$ (defined by $(\sigma(z))_i = z_{i+1}$), and invariant under the flip $\tau$
derived by $(\tau(z))_i = (z_{-i})^{-1}$.

As in Subsection 2.5, there is an action of $\text{Out}(F)$ on the set of laminations: replace
each letter in $E$ by its image, and reduce (bounded backtracking guarantees that this
is well-defined, see [11]); inner automorphisms act trivially, so this is really an action of
$\text{Out}(F)$. Similarly, one may express a lamination $L$ in a different basis $B$: replace each
letter in $E$ by its expression as a word on $B^{\pm 1}$ and reduce.

The complexity $p_L$ of a lamination $L$ is defined by counting all the subwords appearing
in some leaf of $L$. Arguing as in Subsection 2.5, one sees that $p_L$ is well-defined up to

Now let $X$ be an element of $\partial F$, viewed as a right-infinite word $X_A$ on the alphabet
$E$. One defines the language $\mathcal{L}_X$ associated to $X$ as the set of all finite words $u$ such
that $u$ or $u^{-1}$ appears infinitely often as a subword of $X_A$ (note that $\mathcal{L}_X$ does not change
if we truncate $X$). It is a laminary language in the sense of [11]. One then defines the
lamination $L_X$ associated to $X$ as the set of words $z = (z_i)_{i \in \mathbb{Z}}$ as above such that any
finite subword of $z$ belongs to $\mathcal{L}_X$. The assignment $X \mapsto L_X$ is equivariant with respect
to the action of $\text{Out}(F)$.

It is clear from this construction that, given $X \in \partial F$, the complexity of $L_X$ is equivalent
to the recurrence complexity of $X$. We may therefore use Theorem 5.3.

**Theorem 5.7.** Let $\alpha$ be an automorphism of a finitely generated free group $F$, and let
$X \in \partial F$ be fixed by $\alpha$. Assume $X \notin \partial \text{Fix}(\alpha)$. Let $L_X$ be the lamination
corresponding to $X$.

- The complexity of $L_X$ is equivalent to $n, n \log \log n, n \log n$, or $n^2$.
- If the usual complexity $p_X$ of $X$ is not quadratic, in particular if $\alpha$ is fully irreducible or atoroidal, then the complexity of $L_X$ is equivalent to $p_X$. \hfill $\square$

We conclude by giving examples illustrating this theorem. Recall (Example 5.2) that the
complexity of $L_X$ may differ from that of $p_X$.

**Example 5.8 (Attracting laminations).** Let $f : G \to G$ be a completely split train track
map representing $\Phi \in \text{Out}(F)$ as in Subsection 2.5. Let $H_r$ be an exponential stratum.
Replacing $f$ by a power if needed, we can find an edge $e$ in $H_r$ such that $f(e)$ splits as
$euf_2$ with $e \in H_r$ an edge in $H_r$. The union of the increasing sequence $e \subset f(e) \subset f_2^2(e) \cdot \cdot \cdot \subset f_n^2(e)$ is a ray $\rho$ in $G$ which defines an element $X \in \partial F$ (well-defined up to the action
of $F$ on $\partial F$ by left-multiplication). This element $X$ is an attracting fixed point of some
representative $\alpha$ of $\Phi$ in $\text{Aut}(F)$, and the lamination $L_X$ associated to $X$ is the attracting
lamination associated to the stratum $H_r$ (see [2, 34]). Similarly, if $e$ is an edge in an NEG
stratum with $f(e) = eu$, and $u$ is growing, then $X = euf_2(u)f_2^2(u)\cdot \cdot \cdot$ is an attracting
fixed point of a representative $\alpha$.

**Corollary 5.9.** Let $\Phi \in \text{Out}(F_n)$. The complexity of any attracting lamination of $\Phi$ is
linear, quadratic, equivalent to $n \log n$, or equivalent to $n \log \log n$. It is linear if $\Phi$ is fully irreducible, at most $n \log n$ if $\Phi$ is atoroidal. \hfill $\square$

The representatives $\alpha$ constructed above are rather special. “Most” representatives $\alpha$
of $\Phi$ have exactly one attracting fixed point (see [23]), but it is not given by the previous
construction.

**Example 5.10.** Let $F$ be the free group on two generators $a, b$. Let $\Phi$ be the outer
automorphism of $F$ represented by $\alpha$ sending $a$ to $ab$ and $b$ to $bab$ (it is realized geometrically
by a pseudo-Anosov homeomorphism of a punctured torus and has one EG stratum). The
infinite word $X = ab^2a^2bab\cdot \cdot \cdot = \lim_{n \to \infty} a^n(a)$ is an attracting fixed point and the
associated lamination is the attracting lamination of $\Phi$.

Let $\gamma = [a, b]$; it is fixed by $\alpha$. For $k \geq 1$, define $\alpha_k$ by $\alpha_k(g) = \alpha^k \alpha(g) \gamma^{-k} a^{-1}$.
It is another representative of $\Phi$, with attracting fixed point $X_k = a \gamma^k \alpha(a) \gamma^k \alpha^{-2}(a) \cdot \cdot \cdot$.
The laminary language $\mathcal{L}_{X_k}$ consists of all words $w$ such that $w^{\pm 1}$ is contained in some
$\alpha^n(a) \gamma^k \alpha^{n+1}(a)$.
This example shows that, for a given $\Phi$, Theorem 5.7 may apply to infinitely many different laminations.

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