Superfluid Hydrodynamics of an Electron Gas in a Superstrong Magnetic Field

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We derive the equations of hydrodynamics of a fully polarized electron gas placed in a strong magnetic field. These equations reveal the existence of solitons – immobile or propagating domain wall-like defects whose plane is perpendicular to the field direction. The solitons are used to construct weakly excited stationary states, and novel nonuniform persistent current states of the system.

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Understanding the properties of matter subject to an extremely strong magnetic field [1] has fundamental importance. At fields of order $10^9$ T the interaction of atomic electrons with the external magnetic field begins to dominate over their Coulomb interaction which results in a novel kind of atomic physics [4]. In semiconductors and semimetals due to the smallness of the effective mass of the charge carriers and dielectric screening of the Coulomb interaction [5], the corresponding field is substantially smaller, and accessible in the laboratory [6].

Since a uniform magnetic field localizes the motion of a charged particle in the plane perpendicular to the field direction, and does not affect the motion along the field [7], strong fields can be used to study essentially one-dimensional physics using three-dimensional samples. The one-dimensionality is most pronounced if the field is strong enough to pull all the carriers into the lowest Landau level. In this limit in an interacting system a variety of phases of one-dimensional nature such as charge-density waves [8], Wigner crystals [9], and excitonic insulators [8] have been predicted, and a uniform-density phase (adiabatically evolving from the noninteracting limit) occupies only a portion of the phase diagram [10]. Recent work [11] has demonstrated that this phase has Luttinger-liquid-like properties.

In this Letter, by studying elementary excitations, we show that the uniform-density phase behaves as a unidirectional superfluid. The governing equations of hydrodynamics have a rich mathematical structure which owes its existence to the one-dimensional physics of the problem. A detailed treatment will be presented elsewhere [12]. Below we restrict ourselves only to the salient physics effects.

We start with reminding the reader of the method of constructing the ground state of a noninteracting electron gas of density $n$ placed in a fully polarizing magnetic field $H$ pointing along the $x$ direction of a coordinate system. The motion of electrons is confined within infinitely long Landau tubes parallel to the field direction: the motion perpendicular to the tube axis is localized within the magnetic length $l = (\hbar c/\epsilon e H)^{1/2}$ while the motion along the tube is free. The degeneracy of the lowest Landau level, i.e. the number of Landau tubes in the system of cross-sectional area $S$ perpendicular to the field direction is given by $eHS/2\pi\hbar c = S/2\pi l^2$ which implies that the cross-sectional area of the individual tube is $2\pi l^2$. Therefore the magnetic flux confined within the tube is the flux quantum $\hbar c/\epsilon e$, and thus each Landau tube can be thought of as a magnetic field line. Hence the quantum state of every electron can be characterized by specifying the Landau tube it belongs to, and the wavevector $k_x$ describing the free motion along the field. Because of the Pauli principle, for a given Landau tube all the states with $|k_x| \leq k_F$ are occupied while all the states having $|k_x| > k_F$ are empty. The Fermi wave vector $k_F$ can be found as $k_F = \pi n\rho$ where $\rho$ is the linear electron density in the tube, which can be computed by dividing the total number of electrons $N$ by the number of Landau tubes $S/2\pi l^2$, and by the tube length $L$. Thus

$$\rho = 2\pi l^2 n, \quad k_F = 2\pi^2 l^2 n, \quad \text{and the Fermi energy can be found as} \quad \epsilon_F = \hbar^2 k_F^2 / 2m = (2\pi^2 l^2 n)^2 / 2m \quad \text{where} \ m \quad \text{is the electron mass.}$$

The expression for the ground-state energy

$$E_0 = \frac{V}{(2\pi l)^2} \int_{-k_F}^{k_F} \frac{\hbar^2 k_x^2}{2m} dk_x = \frac{V(2\pi^2 l^2 n)^3}{6m} \quad (1)$$

(where $V = SL$ is the system volume) can be rewritten as $(L\pi^2 h^2 / 6m)(2\pi^2 l^2 n)^3$ times $S/2\pi l^2$: the ground-state energy of a one-dimensional spinless Fermi gas of linear density $\rho = 2\pi l^2 n$, multiplied by the total number of Landau tubes. The condition of being at the lowest Landau level is satisfied if the Fermi energy lies below the bottom of the second Landau level $\hbar eH/\epsilon mc$ [3]. This leads to the condition $n l^3 \leq 1/\pi^2 \sqrt{7}$ which gives the range of applicability of Eq. (1). In what follows we will adopt the stronger condition $n l^3 \ll 1$ which also allows us to neglect transitions to the higher Landau levels.

The presence of the Coulomb interaction introduces a new length scale into the problem, the “exciton” Bohr radius $a = \hbar^2 / \epsilon mc^2$, where $\epsilon$ is the dielectric constant of the medium. In astrophysical applications when $\epsilon = 1$, and $m = m_e$, the electron mass in vacuum, $a$ reduces to Bohr’s radius. In the limit $n l^3 \ll 1$ the average distance between the electrons belonging to the same Landau tube, $1/2\pi l^2 n$, is much bigger than the average distance between the nearest tubes $l$. Thus the electrons belonging to the same Landau tube interact much more weakly among themselves than the electrons belonging...
to different tubes. Therefore the physical picture of independent Landau tubes survives if the Fermi energy is much bigger than the Coulomb interaction of electrons confined to different tubes. The latter (in an electrically neutral system) is estimated \[1\] as the plasmon energy \(\hbar\omega_p\) where \(\omega_p^2 = 4\pi ne^2/\epsilon_0m_e\) is the plasma frequency. Combining the requirement \(\epsilon_F \gg \hbar\omega_p\) with the condition that all electrons are at the bottom of the lowest Landau level, we arrive at the concentration interval
\[\frac{(a/l)^{8/3}}{\epsilon} \ll n a^3 \ll (a/l)^3\] (2)
which is the range of applicability of our theory. The interval \(\epsilon\) exists if the exciton Bohr radius \(a\) is much bigger than the magnetic length \(l\); the magnetic fields satisfying this condition are referred to as superstrong \(\epsilon\). Both length scales coincide at the field \(H_0 = m^2e^3c/\epsilon^2\hbar^2\).

In astrophysical applications \((m = m_e, \epsilon = 1)\) \(H_0 = 2.35 \times 10^{10}G\), while in condensed matter applications the corresponding field is \((\epsilon m_e/m)^2 \sim 10^4 - 10^6\) times smaller, and for \(a/l \sim 5 - 10\) the density range \(\epsilon\) exists in doped semiconductors and semimetals.

We postulate that the long-wavelength physics of a polarized electron gas placed in a slowly varying external potential \(V(r)\) is described by the energy functional of the complex order parameter \(\Psi(r,t)\):

\[
F = \int dV \left[ \frac{\hbar^2}{2m} \left| \frac{d\Psi}{dx} \right|^2 + V(r)|\Psi|^2 + \frac{(2\pi^2\hbar^2)^2}{6m} |\Psi|^4 \right]
\] (3)

The physical meaning of \(\Psi\) is that \(|\Psi|^2 = n(r,t)\) gives the electron density which is a function of position \(r\) and time \(t\). If \(\Psi\) is restricted to be real, then \(\Psi\) can be recognized as a starting point of a generalized Thomas-Fermi density functional theory \(\Psi\); for the translationally-invariant case the density is uniform, and then \(\Psi\) reduces to the ground-state energy \(\epsilon\). The second term of \(\Psi\) is the energy in the external field. The derivative term, known as “Weizsacker inhomogeneity correction” \(\Psi\), which is a lowest-order term of a gradient expansion, takes into account the fact that deviations from uniformity in the field direction are costly. Density functionals similar to \(\Psi\) are used to describe static properties of multielectron atoms placed in very strong magnetic fields \(\Psi\).

We further postulate that the order parameter \(\Psi\) evolves according to the equation of motion
\[
i\hbar\partial_t \Psi = \frac{\delta F}{\delta \Psi} = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(r) + \frac{(2\pi^2\hbar^2)^2}{2m} |\Psi|^4 \right] \Psi
\] (4)

In the representation \(\Psi = n^{1/2} \exp i\theta\) and \(v = (\hbar/m)\partial_x\theta\), Eq.\(\Psi\) reduces to the system of two coupled equations:
\[\partial_t n + \partial_x (nv) = 0\] (5)
\[m(\partial_t v + v\partial_x v) = -\partial_x |V + \frac{(2\pi^2\hbar^2)^2}{2m} n^2 - \hbar^2 \partial_x^2 / 2m\sqrt{n}|\] (6)

In the long-wavelength limit when the last term in the right-hand side of \(\Psi\) can be dropped, the system of equations \(\Psi\) and \(\Psi\) can be identified with those of hydrodynamics of an ideal fluid \(\Psi\) whose pressure and density are related by the equation of state \(p = (2\pi^2\hbar^2)^2 n^3 / 3m\).

The variable \(v = (\hbar/m)\partial_x\theta\) has a meaning of the superfluid velocity which only has a component along the field direction. We note that the equations of hydrodynamics of an electron gas in a supermagnetic field could be written down from the outset and the density dependence of pressure can be computed from the ground-state energy \(\Psi\) as \(p = -\delta E/dV\). This provides an important test of the underlying equations \(\Psi\) and \(\Psi\); specifically, selection of the gradient term of \(\Psi\) in the Weizsacker form plays a central role in getting the correct hydrodynamical limit. We stress that hydrodynamical description requires two fields, density and velocity; that is why the order parameter \(\Psi\) must be complex.

**Spectrum of small density oscillations:** Let us look for solutions to \(\Psi\) (with \(V = 0\)) of the form \(\Psi(x,t) = \psi(x,t) \exp(-i\mu t/\hbar)\). The function \(\psi(x,t)\) then obeys the equation
\[
i\hbar\partial_t \psi = \frac{\hbar^2}{2m} \left[ -\partial_x^2 + (2\pi^2\hbar^2)^2 (|\psi|^4 - \psi_0^4) \right] \psi
\] (7)
where the chemical potential \(\mu = \pi^2\hbar^2 (2\pi^2\hbar^2)^2 \psi_0^4 / 2m\) is selected so that the ground state corresponds to the uniform electron density \(n_0 = \psi_0^2\). Consider a small oscillation of \(\psi\) around its constant value \(\psi_0\): \(\psi = \psi_0 + A \exp[i(qx - \omega t)] + B^* \exp[-i(qx - \omega t)]\), where \(A\) and \(B^*\) are small complex amplitudes, and \(q\) is the wavevector along the field direction. Substituting the expression for \(\psi\) into \(\Psi\), linearizing, and solving the resulting system of two equations, we find for the dispersion law
\[
(\hbar^2)^2 = (\hbar^2 q^2 / 2m)^2 + (\hbar cq)^2
\] (8)

The spectrum takes the Bogoliubov form \(\Psi\): for large momenta the dispersion law coincides with the free-particle energy \(\hbar^2 q^2 / 2m\), while at low momenta we get instead the phonon dispersion \(\omega = cq\), where \(c = 2\pi^2\hbar n_0 / m\) is the sound velocity. We note that this velocity coincides with the hydrodynamical expression \(c = (dp/\rho d\theta)^{1/2}\) for a gas with equation of state \(p = (2\pi^2\hbar^2)^2 n^3 / 3m\). The Landau criterion applied to \(\Psi\) then implies \(\Psi\) that the critical velocity of superfluid motion along the direction of magnetic field equals the sound velocity.

**Solitons:** - Without the external potential \((V = 0)\) Eq.\(\Psi\) has soliton solutions \(\Psi\). These describe moving (gray soliton) or standing (dark soliton) density depressions accompanied by a corresponding profile of phase \(\Psi\). Because of the system uniformity in the plane perpendicular to the magnetic field, the solitons are domain
wall-like defects whose plane is perpendicular to the direction of magnetic field; the solitons can only propagate along (or opposite) to the field direction (Figure 1).

\[
\tau = (2 \theta \text{ phase (y, z) field direction is labeled (y, z) The phase profile (not shown) is a smeared step function (10); the step position coincides with that of the density dip.}
\]

FIG. 1. Schematic illustration of the density profile due to the soliton: electron density depends only on the coordinate \(x\) in the direction of the magnetic field \(H\); transverse to the field direction is labeled \((y, z)\). The phase profile (not shown) is a smeared step function (10); the step position coincides with that of the density dip.

In dimensionless variables \(\nu = n/n_0, y = 2\pi l^2 n_0 x, \tau = (2\pi^2 l^2 n_0)^2 H t/l, \zeta = y - \beta \tau, \) the density \((\nu)\) and phase \((\theta)\) profiles for the soliton are given by (10)

\[
\nu = 1 - \frac{3(1 - \beta^2)}{2 + (1 + 3\beta^2)^{1/2}} \cosh[2(1 - \beta^2)^{1/2}\zeta] \tag{9}
\]

\[
2\theta = \cos^{-1} \left[ \frac{(3\beta^2/\nu) - 1}{(1 + 3\beta^2)^{1/2}} \right] \tag{10}
\]

where the dimensionless velocity \(\beta\) is measured in units of the sound velocity. These formulas imply that as \(\beta\) varies between zero and unity, the soliton becomes more shallow, and the phase shift across the soliton \(\Delta \theta = \cos^{-1}\left[(3\beta^2 - 1)/(1 + 3\beta^2)^{1/2}\right]\) changes from \(-\pi\) (\(\beta = 0\)) to zero (\(\beta = 1\)); the soliton solution ceases to exist for velocities exceeding the sound velocity. Microscopically the domain wall is composed of aligned point-like solitons belonging to different Landau tubes, and there is a positive energy per unit area associated with the planar defect. It can be calculated as the energy of the soliton belonging to the Landau tube (computed in (14), divided by the cross-sectional area of the Landau tube \(2\pi l^2\):

\[
\sigma(\beta) = \frac{\sqrt{3}(\pi l n_0)^2}{m} (1 - \beta^2) \ln \left[ \frac{2 + [3(1 - \beta^2)^{1/2}]}{(1 + 3\beta^2)^{1/2}} \right] \tag{11}
\]

The positivity of the surface energy (11) guarantees the stability of the domain wall (10), (11) against transverse fluctuations.

Stationary states: Solitons form a natural language to describe weakly excited states of the system. For example, let us look at electron gas placed inside a layer of macroscopic thickness \(L\); the plane of the layer is perpendicular to the direction of the magnetic field. If the electrons cannot leave the layer, then in the ground state the density profile will look as shown schematically in Figure 2 (left): the density stays constant almost everywhere except for the narrow vicinity of the walls where it sharply plunges to zero.

![FIG. 2. Sketch of the first (left) and the second (right) excited states of electron gas in a box.](image)

It is convenient to think of these two dips as having one dark soliton present in the system. It is also convenient to take the convention that what is shown in Figure 2 (left) is a first excited state with respect to the strictly uniform state which we will call the ground state. Thus the first excited state has just one soliton present in the system. Then the second excited state will have two solitons [Figure 2 (right)]. The argument can be extended to any number of solitons. For example, if there are \(Q\) dark solitons present, then as long as soliton tails do not overlap, \(Q \ll l^2 n_0 L\), the system energy is given by \(E(Q) = E_0 + S \sigma(0) Q\) where \(E_0\) is the ground-state energy (9). A similar conclusion is applicable to the case of a ring-shaped sample with the only constraint that now \(Q\) must be even because the phase shift due to the dark soliton is \(\pm \pi\). Having an even number of dark solitons will guarantee that the overall change of phase of the order parameter upon going around the ring is multiple of \(2\pi\), and thus the order parameter is unique.

Because translational symmetry is not broken for the ring geometry, dark soliton states do not exhaust all possible excited states of the system: gray solitons can also be involved. However it will be easier to anticipate their role after discussing the persistent current states.

Persistent currents: Persistent currents are only possible in the ring geometry. Experimentally the magnetic field directed along the circumference of a ring-shaped sample can be generated by an electric current flowing along the symmetry axis perpendicular to the plane of the ring. In the ring geometry the phase of the order parameter can only change by a multiple of \(2\pi\). The simplest way to accommodate a nonzero phase change is to have a constant phase gradient along the ring (17). This will lead to a spatially uniform persistent current state in which all the electrons are moving around the ring with the same velocity. This state can be also viewed as a stationary state.

There is also a novel nonuniform way of accommodating the phase difference. The corresponding persistent current states can be constructed from gray solitons, and they have a series of remarkable properties. For example, imagine we want to insert into the ring \(Q\) solitons to accommodate an overall phase shift of \(2\pi\). Then the phase shift due to the individual soliton must be \(2\pi/Q\).
This will in turn determine the soliton velocity $\beta$ which will then fix the surface energy $\beta$. An example of four gray solitons moving along the ring is sketched in Figure 3.

The energy of the train consisting of $Q$ solitons running around the ring of circumference $L$ is given by $E(Q) = E_0 + S\sigma L/\beta(Q)|Q|$ as long as soliton tails do not overlap, $Q \ll (1 - \beta^2)^{1/2}n_0L$. In the reference frame moving with the soliton train the solitons will be standing but the background electron gas will be moving with a constant velocity. This gives an example of a stationary state involving gray solitons.

For the uniform moving state the total angular momentum of the system is quantized in units of $N\hbar$ where $N$ is the total particle number $[17]$. We speculate that for the soliton train moving state the total angular momentum might be a fraction of $N\hbar$. Our contention is based on the observation that while the soliton is moving, the electron motion is localized only in the vicinity of the soliton core - merely a tiny fraction of particles are involved in the motion.

In applying these results to the real system one has to keep in mind that since a soliton is a density depletion, for solitons which do not travel fast enough, the density at the soliton core can drop below the left bound of $\beta$. Therefore our results are strictly applicable only to solitons whose velocity is larger than some critical velocity. Future work is necessary to understand how interactions between the electrons belonging to different Landau tubes affect the slow solitons. One possibility is that they convert the depleted soliton core into a two-dimensional Wigner crystal. Another possibility is a modulational instability of the soliton in which Coulomb repulsion of the electrons decreases at the expense of increase of the surface energy $\beta$.

The unique mathematical structure of the underlying equations $[3]$ which so far did not enter the discussion deserves a special commentary. First we note that in the hydrodynamical limit when the last term of $[3]$ can be neglected (and $V = 0$) the problem of arbitrary motion of a one-dimensional gas with the equation of state $p \sim n^3$ can be solved in a closed form $[18]$. Second, (even for $V \sim x^2$) equations $[4]$-$[9]$ have self-similar solutions $[19]$ which only exist for the equation of state $p \sim n^3$. Physically some of them describe free expansion of the electron gas initially confined along the field direction $[10]$.

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