BELL INEQUALITIES AND PSEUDO-FUNCTIONAL DENSITIES

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Abstract

A local hidden variables model with pseudo-functional probability density function restricted to a binary-event probability space is able to reproduce the quantum correlation in an Einstein-Podolsky-Rosen-Bohm-Aharonov experiment.

1. Introduction

In physics, Bell inequalities (BI’s) are of fundamental importance because they solved Einstein’s worries about completeness of quantum mechanics. In the mid-fifties, Bohm reformulated Einstein’s incompleteness arguments into a correlation between spin-states of spatially separated particles, originally in the singlet state. BI’s refer to this situation.

Research, with or without BI’s, pointed at the inconsistency of adding local hidden variables (LHV’s) to quantum mechanics. This raises questions about BI’s themselves. If they appear unnecessary they might also not eliminate all LHV’s allowed in standard probability theory. Moreover, if this is demonstrated by construction of a valid model, LHV’s cannot be inconsistent with quantum correlation.

\[ \int P(\vec{b}, \vec{d}) = d\lambda \rho(\lambda) A(\vec{b}, \lambda) B(\vec{d}, \lambda) \]

The correlation between measurements of two spins, using local hidden variable(s), \( \lambda \), is

Here, \( \rho(\lambda) \) is the probability density function (PDF). Functions \( A(\vec{a}, \lambda) \) and \( B(\vec{b}, \lambda) \), \( \vec{a} \rightarrow \vec{b} \rightarrow \lambda = 1 \), represent the result of the measurement (ideally \( \pm 1 \)) at two distant spin measurement devices.

Because \( A \) is independent of parameter vector \( \vec{b} \) of the device \( D_B \), and vice versa, locality is maintained.

\[ | P(\vec{b}, \vec{d}) - P(\vec{b}, \vec{d}) | + | P(\vec{b}, \vec{d}) + P(\vec{b}, \vec{d}) | \leq 2 \]

For regular probability densities, the following inequality holds.\(^{10}\)

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The quantum correlation, \( P(\vec{a}, \vec{b}) = (\vec{a} \cdot \vec{b}) \), violates the inequality. Hence, a distinction between hidden variable predictions and quantum mechanical predictions is possible.

In this letter we present a classical probabilistic singular LHV’s model which meets the following:

\[
\rho = \rho(\lambda) \geq 0, \forall \lambda \in \Lambda,
\]

\[
\langle \rho \rangle = 1,
\]

\[
A = A(\vec{p}, \lambda) = +I, B = B(\vec{p}, \lambda) = \pm I,
\]

\[
|\langle \rho A \rangle| = 0, |\langle \rho B \rangle| = 0,
\]

\[
|\langle \rho A^2 \rangle| = |\langle \rho B^2 \rangle| = 1,
\]

\[
\langle \rho AB \rangle = -(\vec{p} \cdot \vec{b}),
\]

following:

with \( \langle f \rangle = \int d\lambda f(\lambda) \). An extra condition is that the density is associated with a genuine probability space.

\[
P_f(x) = \begin{cases} 
1/x, & x > 0 \\
0, & x \leq 0
\end{cases}
\]

In the probability density, the pseudo-function

is used which represents the positive branch of the principal value function.\(^{10}\) Here, \( \theta(x) = 1 \), when \( x > 0 \), \( \theta(x) = 1/2 \), when \( x = 0 \), while, \( \theta(x) = 0 \), when \( x < 0 \).

Moreover, two integrals are crucial. Let us inspect \( \lambda \in [-\beta, +\beta] \), with \( \beta = \exp(3^{3/4}) = 1.5507 \).

\[
\int_{-\beta}^{+\beta} d\lambda P_f \frac{\theta(\lambda)}{\lambda} = -\rho \int_{0}^{\beta} d\lambda \text{overlambda} = C = 3^{3/4},
\]

We find that
with _p Hadamard's finite part. The integral of the product of the pseudo-function and the sign function, defined here as sign(x)=+1, x≥0, while sign(x)=-1, x<0, is when f=exp((C-1)/2)≈0.7553, C=3-3/4≈0.4387. Note that f<β. The choice of parameters will become clear later on.

Subsequently, the model is presented and Eq. (3) is verified. First, the integrations are understood as

\[ \prod_{nq=1}^{3} \sum_{\lambda_{\tau}=1}^{4} \int_{-\beta}^{+\beta} d\lambda_{\tau} \int_{-\infty}^{+\infty} d\chi \prod_{\eta_{s}=1}^{-1} d\eta_{s} <\bullet \cdots \bullet >. \]

Hence, the variables nq, λτ and ηs, in the specified ranges, are postulated.

Here, Γ(n1,n2,λ1,...,λ4,χ,η1,η2) is true when the following conditions apply. In the first place, nq must only run through the complete set \{1,2,3\}, excluding all other sets. Secondly, λτ must only run through the complete interval [-β,β], excluding other intervals. Thirdly, the χ must run through the complete real interval (-∞,∞) excluding other intervals, subsets of (-∞,∞). Fourthly, ηs must only run through the complete set [-1,1], excluding all other intervals. If one of the associated sets that defines the range of a particular hidden variable is unequal to the one indicated, i.e., to \{1,2,3\}, [-β,β], (-∞,∞), or, [-1,1], and/or the variable lies outside the indicated set, then Γ(n1,...,η2) is false. Note that it is not unusual that intervals determine a PDF.

Second, the singular probability density function is specified by

\[ \rho = \left\{ \begin{array}{ll} \frac{1}{4} \prod_{\tau=1}^{4} P_{\tau} \frac{\theta(\lambda_{\tau})}{\lambda_{\tau}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\chi^{2}}{2}\right), & \Gamma = \text{true}, \\ 0, & \Gamma = \text{false} \end{array} \right. \]

In addition, let us specify a delta function for sets X and Y as \( \delta(X,Y)=1 \), when X=Y, while \( \delta(X,Y)=0 \) when X≠Y. Moreover, the universal event is given by \( \Xi=\{1,2,3\}^{3} \times [-\beta,\beta]^{4} \times (-\infty,\infty) \times [-1,1]^{5} \), with \( \{1,2,3\}^{3} \times \{1,2,3\} \times \{1,2,3\} \times \{1,2,3\}, \) etc, whereby \( \times \) is the Cartesian product, and {} is the empty set.

From the previous specifications, we see a binary-event probability space associated to ρ. Accordingly, this space is written as \( (Q,\_P_{\rho}[\bullet]) \), with \( \_={\Xi,\{\}} \) and \( P_{\rho}[Q]=<\delta(Q,\Xi)\rho>, Q\in\_. \)
Hence additivity, \( P(Q) + P(Q^c) = 1 \), with \( Q^c = \mathbb{Z} \setminus Q \), (\( Q \cap Q^c = \{ \} \)), is warranted in \( (Q, \_\, , P_\rho) \), while the other basic axioms (Ref. 10, page 22) also hold for \( (Q, \_\, , P_\rho[\bullet]) \). Hence, an elementary probability space can be associated to the density \( \rho \), from which valid probability measures can be obtained. Observe that \( \text{BI}'s \) do not forbid binary probability spaces, such as \( (Q, \_\, , P_\rho[\bullet]) \), to be associated to densities to be used in Eq. (1). Moreover, the discreteness of the spin space coincides with the probability space.

\[
A = \text{sign}(\chi) \text{sign} \{ \delta_{\alpha_1 \alpha_2} a_{\alpha_1} \text{sign}(f - \lambda_1) \text{sign}(f - \lambda_2) \cdot \eta_1 \},
\]

Fourth, in the model the functions \( A \) and \( B \) are given by

\[
B = -\text{sign}(\chi) \text{sign} \{ \delta_{\alpha_1 \alpha_3} b_{\alpha_3} \text{sign}(f - \lambda_3) \text{sign}(f - \lambda_4) \cdot \eta_3 \},
\]

and with \( \delta_{x,y} = 1 \) when \( x = y \), while \( \delta_{x,y} = 0 \) when \( x \neq y \), and \( A^2 = B^2 = 1 \).

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\chi \exp(-\chi^2/2) \text{sign}(\chi) = 0,
\]

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\chi \exp(-\chi^2/2)(\text{sign}(\chi))^2 = 1.
\]

We first may observe that \( \rho \geq 0 \). Second, it follows that \( \langle \rho \rangle = 1 \). Third, because it follows that \( \langle \rho A \rangle = \langle \rho B \rangle = 0 \), while \( \langle \rho A^2 \rangle = \langle \rho B^2 \rangle = 1 \). Fourth, because \( a^\alpha_1 \cdot b^\alpha_2 = 1 \), the density

\[
\langle \rho AB \rangle = -\sum_{k=1}^4 \sum_{m=1}^4 \sum_{n=1}^4 \delta_{k,m} \delta_{k,n} a_m b_n
\]

\[
-\int_{-\beta}^{+\beta} d\lambda P_f \frac{\theta(\lambda)}{\lambda} \text{sign}(f - \lambda) f',
\]

entails that we also have

from which it follows that \( \langle \rho AB \rangle = -(a_1 b_1 + a_2 b_2 + a_3 b_3) = -(a \cdot b) \).

For completeness, the conflict with \( \text{BI}'s \) emerges from operations with absolute signs leading to \( \text{BI}'s \). In our case we have
which is contradictory. Hence, no straightforward path may lead here to BI's.

This concludes the proof that a valid local hidden variables model is possible that remains within the bounds of classical probability theory. Because of this, the model cannot beforehand be dismissed as unphysical.

References

1. J. S. Bell, Physics 1, 195 (1964).
2. J. F. Clauser and A. Shimony, Rep. Prog. Phys. 41, 1881 (1978).
3. A. Aspect, P. Grangier and G. Roger, Phys. Rev. Lett. 49, 91 (1982).
4. A. Einstein, N. Rosen and B. Podolsky, Phys. Rev. 47, 777 (1935).
5. D. Bohm and Y. Aharonov, Phys. Rev. 108, 1070 (1957).
6. E.S. Fry, T. Walther and S. Li, Phys. Rev. A52, 4381 (1995).
7. D. M. Greenberger, M. A. Horne and A. Zeilinger, *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, Edited by M. Kafos (Kluwer, Dordrecht 1989), p. 74.
8. A. M. Mood, F. A. Graybill and D. C. Boes, *Introduction to the theory of Statistics* (MacGraw-Hill Singapore, 1974).
9. A. Einstein in: *Albert Einstein, Philosopher, Scientist*, edited by P. A. Schilp (Library of Living Philosophers, Evanston, Illinois, 1949), p. 85.
10. M. J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions*, Camb. Univ. Press. Cambridge (1958).