Pairbreaking effect of correlated impurities in a superfluid Fermi liquid

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Abstract

The conventional theory of superconducting alloys does not take into account the discrete character of impurities. Experimental data for superfluid $^3$He in aerogel and for some high-$T_c$ superconductors reveal a significant discrepancy between the observed temperatures, $T_c$, of their transitions in the superfluid or superconducting states and those predicted theoretically. Here a theoretical scheme is presented for finding corrections to the $T_c$ originating from spatial correlations between impurities. Analysis is limited to the Ginzburg and Landau temperature region. The shift of $T_c$ with respect to the pure material is represented as a series in concentration of the impurities, $x$. In the first order on $x$, the conventional mean-field result for lowering $T_c$ is recovered. The contribution of correlations enters the second-order term. It is expressed via the structure factor of the ensemble of impurities. For superfluid $^3$He in a silica aerogel, the sign of the correction corresponds to an enhancement of the $T_c$, so that the resulting pair-breaking effect of impurities is weakened. When the correlation radius of the impurities, $R$, exceeds the coherence length of the superfluid, $\xi_0$, the contribution of correlations to the shift of $T_c$ acquires a factor, $\sim (R/\xi_0)^2$, and the weakening of the pair-breaking effect becomes appreciable. The presented scheme is applied to the superfluid $^3$He in aerogel.

Keywords: correlations, impurities, superfluidity, superconductivity, aerogel

1. Introduction

Impurities can significantly change the properties of a superconductor. For example, it is possible to change a type of superconductor from first to second by increasing the concentration of impurities. In the case of an unconventional Cooper pairing, impurities lower the transition temperature, $T_c$. The word ‘unconventional’ means that at this type of Cooper pairing, the gauge symmetry and some other symmetries of a normal phase are broken. Transition temperature is one of the most important thermodynamic characteristics of a superconductor. It is also one of the first quantities to be measured in experiments. The Abrikosov–Gorkov (AG) [1] theory of superconducting alloys renders a quantitative description of the lowering of the transition temperature and of other properties of superconducting alloys in terms of one parameter: a transport mean-free path, $l_n$, of electrons in a given material, which in turn is inversely proportional to the number of impurities per unit volume $n$, or their concentration, $x$. In the AG theory it is assumed that the distribution of impurities is completely random (i.e., their positions are not correlated). The effect of correlations on the thermodynamic properties of superconductors is of the second order of concentration, and in most cases it can be neglected in comparison with the principal mean-field effect, which is linear in $x$. However, there may be special reasons for an enhancement of the contribution of correlations.

In the last 20 to 30 years, many unconventional superconductors were discovered, including high-$T_c$ materials. Doping with impurities is a significant part of the processes of their preparation. The pair-breaking effect of impurities in high-$T_c$ superconductors is observed experimentally [2, 3], but the lowering of $T_c$ in many cases is smaller than that, as predicted by the AG theory (cf [4] and references therein). This tendency is interpreted in [4] as a manifestation of the inhomogeneity of the condensate, which can be appreciable in superconductors with a very short coherence length.

Doping metallic superconductors changes not only the concentration and distribution of scattering centers, but also
other parameters like the concentration of charge carriers, the lattice constant, etc., which in turn change the $T_c$. A special effort has to be made to separate the effects of different factors. These difficulties are not present in another physical object where the effect of correlation of impurities is observed: it is superfluid $^3$He in aerogel. Liquid $^3$He becomes superfluid at temperatures on the order of 1 mK. $^3$He is an unconventional superfluid; Cooper pairs are formed in a state with angular momentum $l = 1$ and spin $s = 1$, so that the rotational symmetries both in spin and in the orbital space are broken. The properties of pure (or bulk) superfluid $^3$He are thoroughly investigated experimentally, and most of them are quantitatively interpreted [5] so that changes introduced by impurities can be accurately separated. All that makes liquid $^3$He a favorable object in the study of the effects of impurities, except that introducing impurities into superfluid $^3$He is not a trivial task. Potential floating impurities at these low temperatures stick to the walls of the container. To remain in the bulk of the liquid, impurities have to form a self-supporting structure, so this in experiments, high-porosity aerogels are used [6–8]. The majority of the data are obtained with silica aerogel, which consists of very thin (diameter $d \approx 3$ nm) strands of SiO$_2$. Scattering of quasiparticles by the silica strands renders a finite mean-free path, $l_{nr}$, and lowers the transition temperature in the superfluid state. Instead of concentration, aerogels are traditionally characterized by porosity, $P$ (i.e., by the fraction of space which is not occupied by the strands). In most of the experiments with $^3$He, silica aerogels with $P \approx 98\%$ are used. In theoretical models the strands forming aerogel are often treated as chains of silica balls with radius $\rho \approx d/2$; these balls play the part of scattering centers. The centers can be treated as impurities in the conventional theory as long as $\rho \ll \xi_0$, where $\xi_0$ is the coherence length of the superfluid.

An essentially new property of the aerogel is that to form a rigid structure, the positions of the centers have to be correlated in space. The correlation function is an intrinsic property of the aerogel; it does not depend on a substance filling its pores, and it can be directly measured by x-ray scattering. The presence and importance of correlations were realized at the very beginning of the investigation of $^3$He in aerogel [9]. In particular, correlations were suggested to be responsible for deviations of the observed dependence of $\frac{\Delta E}{T}$ on $\frac{1}{T}$ from that predicted by the AG theory. To solve the disagreement, more complicated theoretical models of $^3$He in aerogel were suggested [10]. The isotropic inhomogeneous scattering model (IISM) combines the description in terms of impurities with the description in terms of pores. This model improves the agreement with the data for $^3$He, but it has certain drawbacks. The IISM exploits uncontrollable approximations, and the exact physical meaning of the fitting parameters is not clear. Another successful model, phenomenological IISM, is an heuristic interpolation between the limiting situations of impurities and pores. It is not clear also how the IISM can be generalized for application to correlated impurities in superconductors.

In the present paper, correlations are treated within the perturbative approach developed by Larkin and Ovchinnikov for superconductors with a scalar order parameter [11]. The effect of quenched disorder is introduced in the Ginzburg and Landau equations via position-dependent coefficients. Random deviations of the coefficients from their average values are treated as perturbations. This approach is more standard and more universal than any previously used. A preliminary account of the application of this approach to the superfluid $^3$He in silica aerogel was published before [12]. In comparison with [12], in this paper a more general and more realistic formulation is used. In particular the anisotropy of the coherence length of $^3$He and the possible anisotropy of scattering cross sections of impurities are taken into account. Better estimation of a region of applicability of the present approach is given, and the process of summation of the perturbation series is fully reconsidered.

2. Phenomenology

Pair-breaking impurities make the condensate of Cooper pairs spatially nonuniform. In the vicinity of the transition temperature, $T_c$, free energy of this superconductor can be written as a Ginzburg and Landau functional with the coefficients, which are random functions of the coordinate. For a condensate with the scalar order parameter $\Psi(r)$:

$$
F_0[\Psi(r)] = F_0 + \int \left\{ a(r) |\Psi(r)|^4 + \frac{1}{2} b(r) |\Psi(r)|^2 + c(r) \left| \nabla \Psi(r) \right|^2 \right\} \, dr.
$$

(1)

In view of further applications, the magnetic field is not taken into account in this functional. For a situation when fluctuations of coefficients $a(r)$, $b(r)$, $c(r)$ at their average values are small, Larkin and Ovchinnikov [11] have analyzed the effect of inhomogeneities on the thermodynamic and electromagnetic properties of a superconductor with the scalar order parameter treating the fluctuations as a perturbation. According to their analysis, the strongest effect on the average value of the order parameter and on the shift of the transition temperature have fluctuations of the coefficient $a(r)$, which can be expressed in terms of fluctuations of the local transition temperature, $T_c(r)$: $a(r) = a(T - T_c(r))$. The contribution of these fluctuations is singular at $T \rightarrow T_c$. Contributions of fluctuations of $b(r)$ and $c(r)$ are regular in this limit; they can be neglected so that the average over the ensemble of impurities, values $(b)$ and $(c)$, can be used in the equations. If $T_{c0}$ is the temperature of transition in the absence of impurities, then $a(r) = a(T - T_{c0})$, where $T = (T_{c0} - T_c)/T$ and $\tau(r) = (T_{c0} - T_c(r))/T$ is a random local shift of $T_c$. After introducing dimensionless variables $\Psi = \Psi_0 \psi$ with $\Psi_0^2 = aT(b)$, and $\xi_0^2 = (c)/aT$ or $\xi_0^2 = \frac{\xi_0^2}{20}$, the equation for the extremum of the functional (1) takes the form

$$
[\tau - \eta(r)] \psi + \xi_0^2 \Delta \psi - \psi |\psi|^2 = 0.
$$

(2)

Global $T_c$ is defined as the highest value of $T$ for which $\langle \psi \rangle \neq 0$. The signs in the definitions of $\tau$ and $\eta(r)$ are chosen.
so that the linear part of equation (1) is analogous to the Schrodinger equation of a particle moving in a random potential, \(\eta(r)\), and \(\tau\) is the energy. Superconducting states correspond to positive \(\tau\). For small \(\eta(r)\), a shift of the transition temperature, \(\tau_c = (T_c - T) / T_c\), can be represented as a perturbation series, \(\tau_c = \tau_c^{(0)} + \tau_c^{(2)} + \ldots\), where \(\tau_c^{(0)} = \langle \eta(r) \rangle\) and \(\tau_c^{(2)}\) depends on the correlation function, \(\langle \eta(r)\eta(r') \rangle\).

The perturbative approach of Larkin and Ovchinnikov can be applied to condensates with a multicomponent order parameter as well. That requires modification of the free energy functional according to the form of the order parameter. In superfluid \(^3\)He the order parameter is a complex \(3 \times 3\) matrix, \(A_{jl}\) [5]. Its first (Greek) index corresponds to three possible projections of spin and the second (Latin) index corresponds to three projections of the orbital momentum of a Cooper pair. Nonmagnetic impurities interact with the orbital part of \(A_{jl}\), and the additional term in the density of free energy is \(f_\eta = \eta_{jl}(r)A_{jl}A_{jl}^*\), where \(\eta_{jl}(r)\) is a random tensor field. It is assumed that the impurities preserve \(t \rightarrow -t\) symmetry, so \(\eta_{jl}(r)\) is real and symmetric.

As in the scalar case, the perturbation can be separated in the ensemble averaged part, \(\eta_{jl} = \eta_{jl}^{(0)} + \delta_{jl} + \kappa_{jl}\), and the fluctuation, \(\tilde{\eta}_{jl}(r) = \eta_{jl}(r) - \langle \eta_{jl}(r) \rangle\). The isotropic part, \(\eta_{jl}^{(0)}\), can be included in the definition of \(T_c\) so that \(\tau = (T_c - T) / T_c\) and \(\kappa_{jl}\) is a global anisotropy. The finite anisotropy generally splits one \(T_c\) into three \(T_c\), corresponding to different components of \(l\). As a result, in the neighborhood of \(T_c\) there may be several transitions into phases with different order parameters [13, 14]. To avoid irrelevant complications in what follows, it will be assumed that the ensemble of impurities is isotropic (i.e., \(\kappa_{jl} = 0\)). The physical meaning of \(\tilde{\eta}_{jl}(r)\), is clear from the definition; its isotropic part, describes local fluctuations of \(T_c(r)\), while the anisotropic part \(\eta_{jl}(r) - \frac{1}{3} \langle \eta_{jl}(r) \rangle\) describes the local splitting of \(T_c\) for different components of \(l\).

With these notations

\[
F_{GL} = N(0) \int d^3r \left\{ \eta_{jl}(r) - \tau \delta_{jl} \right\} A_{jl} A_{jl}^* + \xi^2 \left( \frac{\partial A_{jl}}{\partial x_j} \frac{\partial A_{jl}}{\partial x_j} + 2 \frac{\partial A_{jl}^*}{\partial x_j} \frac{\partial A_{jl}}{\partial x_j} + \frac{1}{2} \sum_{s=1}^{5} \beta_s L_s \right),
\]

where \(L_s\) are the fourth-order invariants in the expansion of the free energy over \(A_{jl}\) [5] and \(\beta_s, s = 1, \ldots 5\) are phenomenological coefficients.

To find of the \(T_c\) in the presence of perturbation \(\eta_{jl}(r)\), it is sufficient to keep in the Ginzburg and Landau equation corresponding to the functional (3) linear with respect to \(A_{jl}\) terms

\[
\left( \tau \delta_{jl} - \eta_{jl}(r) \right) A_{jl} + \xi^2 \left( \frac{\partial^2 A_{jl}}{\partial x_i \partial x_j} + 2 \frac{\partial^2 A_{jl}}{\partial x_j \partial x_j} \right) = 0
\]

and to use a standard perturbation procedure. The Fourier transform of the Green function, \(G_{jl}(k, k')\), of equation (4) is developed in the series, graphically represented in figure 1. Each line corresponds to the Green function of the unperturbed equation (4)

\[
G_{jl}^{(0)}(\tau, k) = \frac{k_j k_l}{\tau - 3 \xi^2 k^2} + \frac{\delta_{jl} - k_j k_l}{\tau - \xi^2 k^2},
\]

and each cross corresponds to the Fourier transform of the perturbation \(\eta_{jl}(k_{s+1} - k_s)\). The series has to be averaged term by term over the ensemble of \(\eta_{jl}\), and the result is resummed into the averaged Green function, which is spatially uniform:

\[
\left\langle G_{jl}(\tau, k, k') \right\rangle = (2\pi)^3 \delta(k - k') G_{jl}(\tau, k).
\]

In its turn

\[
G_{jl}(\tau, k) = G_{jl}^{(0)}(\tau, k) + G_{jl}^{(m)}(\tau, k) \left\langle W_{mm}(k, \tau) \right\rangle G_{jl}^{(m)}(\tau, k),
\]

where \(\left\langle W_{mm}(\tau, k) \right\rangle\) is the averaged sum of the series graphically represented in figure 1 if the thin lines on both ends of each graph are omitted:

\[
\left\langle W_{mm}(\tau, k) \right\rangle = \sum_p W_{mm}^{(p)}(\tau, k).
\]

If \(W_{jl}(\tau, k)\) is small (the criterion will be formulated later), in a principal order

\[
G_{jl}^{-1}(\tau, k) = \left( \tau - 3 \xi^2 k^2 \right) \hat{k}_j \hat{k}_l + \left( \tau - \xi^2 k^2 \right) \left\langle \delta_{jl} - \hat{k}_j \hat{k}_l \right\rangle - \left\langle W_{jl}(\tau, k) \right\rangle.
\]

Transition temperature \(\tau_c\) in the presence of perturbation is found as a pole of \(G_{jl}(\tau, 0)\)

\[
\tau, \delta_{jl} = \left\langle W_{jl}(\tau_c, 0) \right\rangle.
\]

In what follows, all functions of \(\tau\) will be used at \(\tau = \tau_c\), and the argument, \(\tau\), will be suppressed. In the first order of the perturbation, the mean-field result is recovered

\[
w_{jl}^{(1)}(0) = \left\langle \eta_{jl}(k = 0) \right\rangle
\]

and in the second order we arrive at a straightforward generalization of the corresponding result of [11]:

\[
w_{jl}^{(2)}(0) = - \int \frac{3 \delta_{mm} - 2 k_m k_n}{3 \xi^2 k^2} \left\langle \eta_{mm}(k) \eta_{mm}(-k) \right\rangle d^3k (2\pi)^3.
\]

The variable \(\tau\) does not enter this expression because of condition (10).

3. Impurities

To apply of the obtained expressions, a form of \(\eta_{jl}(r)\) has to be specified. In what follows we assume that the perturbation is produced by an ensemble of discrete identical impurities

\[
Figure 1. Perturbation series for the Green function of equation (4).
\]
situated at random positions, \( r_s \). If linear dimensions of each impurity, \( \rho \), meet the condition \( \rho \ll \tilde{\xi}_0 \), they can be considered as ‘small objects’ in the sense of the theory of Rainer and Vuorio [15]. According to this theory every impurity acts on the condensate as a localized perturbation with a characteristic size on the order of \( \tilde{\xi}_0 \). For example, if the impurity is a ball of the radius \( \rho \), which diffusely scatters quasiparticles, the corresponding perturbation at a distance, \( r \gg \rho \), from the center is given by

\[
\eta_{jl}^{(1)}(r) = -\frac{\rho^2}{r^2} \hat{\nu}_j \hat{\nu}_l \ln \left[ \tanh \left( \frac{r}{2 \tilde{\xi}_0} \right) \right],
\]

where \( \hat{\nu}_j \) is a unit vector parallel to \( \hat{\nu}_S \). If \( \xi_0 \ll l \), the perturbation produced by all impurities can be approximated by the sum of the contributions of individual impurities:

\[
\eta_{jl}(r) = \sum_s \eta_{jl}^{(1)}(r - r_s).
\]

The Fourier transform of \( \eta_{jl}(r) \) is

\[
\eta_{jl}(k) = \sum_s \eta_{jl}^{(1)}(k) \exp(-i k r_s).
\]

For substitution in equation (11) we need only \( \eta_{jl}(0) = \sum_s \eta_{jl}^{(1)}(0) \). For identical impurities, tensors \( \eta_{jl}^{(1)}(0) \) differ only by their orientation. In the case of uniaxial impurities [15]

\[
\eta_{jl}^{(0)}(0) = \frac{\pi^2}{4} \tilde{\xi}_0 \left[ \sigma_{ll}^{(0)} \delta_{jl} + \sigma_{hl}^{(0)} \left( \xi_{jl}^{(0)} \delta_{jl} - \delta_{jl} \right) \right],
\]

where \( \xi_{jl}^{(0)} \) is a unit vector in the direction of the symmetry axis of the impurity at a point, \( r_s \), and the transport cross sections, \( \sigma_{ll}^{(0)} \) and \( \sigma_{hl}^{(0)} \), are expressed through the differential cross section of the scattering of quasiparticles by the impurity as:

\[
\sigma_{ll}^{(0)} \delta_{jl} + \sigma_{hl}^{(0)} \left( \xi_{jl}^{(0)} \delta_{jl} - \delta_{jl} \right) = 3 \int \frac{d^2 \delta}{4\pi} \int d^2 \delta' \left[ \hat{\nu}_j \hat{\nu}_l - \hat{\nu}_j \hat{\nu}_l \right] \frac{\partial \sigma}{\partial \Omega}(\nu, \nu').
\]

At the averaging over directions of \( \xi_{jl}^{(0)} \), the term, proportional to \( \sigma_{hl}^{(0)} \) in equation (14), vanishes. Combining equations (8), (9), (11), and (16), we arrive at the first-order correction to the transition temperature

\[
\tau_{jl}^{(1)} = \frac{\pi^2}{4} \tilde{\xi}_0 \sigma_{ll}^{(0)} \delta_{jl}.
\]

This expression coincides with the result of AG theory in the first order of the ratio, \( \tilde{\xi}_0/\tau_0 \). The correction is negative (i.e., the transition temperature decreases).

To find \( \tau_{jl}^{(2)} \) we have to substitute the explicit form of \( \eta_{jl}(k) \) in the right-hand side (rhs) of equation (12). It will be shown that the principal contribution to the integral comes from the region of small \( k \). Then the impurities can be considered as point-like:

\[
\eta_{jl}(r) = \sum_s \eta_{jl}^{(0)}(0) \delta(r - r_s)
\]

with the Fourier transform:

\[
\eta_{jl}(k) = \sum_s \eta_{jl}^{(0)}(0) \exp(-i k r_s).
\]

Eventually

\[
w_{jl}^{(2)}(r, 0) = -\left( \frac{\pi^2 \tilde{\xi}_0^2}{4} \right) \int \frac{d^2 k}{(2\pi)^3} \frac{3\delta_{kk} - 2k_w k_v}{3 \xi^2 k^2} \times \left[ \delta_{jm} \delta_{nl} \left( \sigma_{mn}^{(0)} \sum_{i,s} e^{ik(r_i - r_s)} \right) \right. \\
+ \left. \left( \sigma_{nn}^{(0)} \sum_{i,s} \left( \bar{\xi}_{jl}^{(0)} - \delta_{nl} \right) e^{ik(r_i - r_s)} \right) \right].
\]

If the impurities are spherically symmetric, \( \sigma_{nn}^{(0)} = 0 \) \( \eta_{jl}^{(0)}(0) = \eta_0 \delta_{jl} \). The distribution of impurities is characterized by one function: the structure factor

\[
S(k) = \sum_{jl} e^{ik(r_i - r_j)}.
\]

In the structure factor, the contribution of correlations can be separated. The term with \( t = s \) is always present; it is equal to unity and when substituted in equation (19), it renders a second-order correction to the shift of the transition temperature by noncorrelated impurities. The remaining sum can be written as an integral

\[
S(k) - 1 = n \int C(r_i | r_s) e^{-i k r_i - j} d^3 r_i,
\]

where \( C(r_i | r_s) \) is the probability of finding a particle in the point \( r_i \) if there is a particle in the point \( r_s \). For isotropic distribution of impurities, it depends only on a distance, \( r = |r_i - r_s| \). At \( r \to \infty \) correlations vanish and \( C(r) \) tends to a constant. Normalization of \( C(r) \) is chosen so that this constant is unity. The unity contributes to \( S(k) \), a term proportional to \( n \delta(k) \), which is already taken into account in \( \tau_{jl}^{(1)} \). A measure of correlations is \( v(r) = w(r - 1) \). The structure factor can now be represented as

\[
S(k) = n \int v(r) e^{-i k r} d^3 r.
\]

Its contribution to \( \tau_{jl}^{(2)} \) is:

\[
\tau_{jl}^{(2)} = -\frac{5}{24} \left( \frac{\tilde{\xi}_0^2}{3} \right) n \left( \sigma_{ll}^{(0)} \right)^2 \delta_{jl} \int_0^{\infty} S(k) d^3 k.
\]

The x-ray scattering data for silica aerogels [9, 16, 21] show that an interval of \( k \) exists: \( (1/R) < k < (1/\hat{\rho}) \) where \( S(k) \sim (1/k^2) \) (i.e., these aerogels have a fractal structure with the fractal dimension \( D_f \approx 1.7 \div 1.9 \) depending on a sample). The inverse lower boundary of the fractal interval has to be identified with the correlation radius \( \hat{R} \); the inverse upper boundary \( \hat{\rho} \) is of the order of characteristic size of a
structure element of an aerogel. The integral in the rhs of equation (25) converges for large \( k \); a principal contribution to this integral comes from the region of \( k \sim (1/R) \). For \( R \gg \xi_0 \) this justifies the approximation of \( \eta_1^{\alpha \beta} (\mathbf{k}) \) by \( \eta_1^{\alpha \beta} (0) \). The available experimental data for \( S(k) \) are presented in arbitrary units, so that they cannot be used directly to evaluate of \( \tau^{(2)}_\beta \).

Another possibility is to express \( \tau^{(2)}_\beta \) in terms of \( v(r) \), using the relation
\[
\int_0^\infty S(k)dk = 2\pi^2 n \int_0^\infty v(r) rdr
\]
and make a plausible guess about the form of the function \( v(r) \), introducing adjustable parameters into it. A minimum of two parameters are required. One is the correlation radius, \( R \), introduced above, and another is the overall amplitude, \( A \), so that \( v(r) = Av \left( \frac{r}{R} \right) \). In this case
\[
\tau^{(2)}_\beta = -\tau^{(1)}_\beta \frac{5 \xi^2}{3} \frac{\pi^2}{9} A R^2 \frac{\xi_0}{\xi_0 l_r},
\]
where \( I = \int_0^\infty \hat{v}(x)x dx \) is a number. This number should not be very sensitive to the particular form of \( \hat{v}(x) \). A convenient model expression for \( \hat{v}(x) \) in aerogel is [12, 16]:
\[
\hat{v}(x) = \left[ \frac{2}{I(D_j)} x^{D_j - 3} - 1 \right] \exp (-x),
\]
where \( I(D_j) \) is the Euler Gamma-function. This expression has a fractal asymptotic \( \hat{v}(x) \sim x^{3-D} \) at \( r \ll R \). At \( r \gg R \) it tends to zero and it satisfies the normalization condition,
\[
\int_{\hat{v}(x)x^2 dx} = 0.
\]
For such \( \hat{v}(x) \), \( I = (3 - D_j)/D_j - 1 \). If \( D_j = 1.8, I = 3/2 \). Experimental data and simulations [9] show that in aerogels, impurities form clusters. This tendency corresponds to \( D_j < 3 \) and positive \( A \). In this case the sign of the correction, \( \tau^{(2)}_\beta \), is opposite to that of \( \tau^{(1)}_\beta \). This means that the destructive effect of impurities is weakened and the resulting \( T_c \) is higher than that given by AG theory. The increase of \( T_c \) is due to the adjustment of the condensate to local inhomogeneities, which increases the gain of energy.

The correction, \( \tau^{(2)}_\beta \), is obtained with the aid of perturbation theory. It means that the inequality, \( |\tau^{(2)}_\beta| \ll \tau^{(1)}_\beta \), has to be met. This condition sets an upper limit for \( R; (AR^2/\xi_0 l_r) \ll 1 \). A limit from below is set up by a condition securing the dominant contribution of correlations:
\[
nAR^2 \frac{\rho}{\xi_0 l_r} \gg 1.
\]
So \( R \) has to be with in a window, \( \frac{\rho}{\xi_0} \ll \frac{AR^2}{\xi_0 l_r} \ll 1 \). The amplitude \( A \) is not measured directly in experiments. For its estimation, further model assumptions have to be used. Assuming that threads of aerogel are ‘beads’ formed by the balls of the radius \( \rho \), we arrive at a condition, \( (4\pi/3)An\hat{v}(3\hat{\rho})^3 \approx \nu \), where \( \nu \) is a ‘coordination number’ (i.e., average number of impurities touching a selected impurity). For the ‘beads’ \( \nu \approx 2 \). If the fractal dimension \( D_j \approx 2 \), \( A \sim (l_r/R) \) with \( \alpha \sim (1/10) \). Within this model we arrive at a stringent condition for the applicability of a sum, \( \tau^{(2)}_\beta \), as an approximation for \( \tau \); \( \rho \ll aR \ll \xi_0 \). The upper limit is particularly restrictive. Lifting of the formulated restriction requires an account of higher-order terms in the series (8).

4. Long-range correlations
The series equation (8) is an expansion of the parameter \( R^2/\xi_0 l_r \). The \( p/\beta \) term of the series
\[
 W^{(p+1)}_\beta (k) = \eta_0 \int \frac{dk_1}{(2\pi)^3} ... \frac{dk_p}{(2\pi)^3} G_{m_1}^{(0)} \times (k_1)...G_{m_p}^{(0)} (k_p)
\]
\[
 \times \sum \exp \left[ i(k-k_1)(r_{l_1}-r_{e}) \right]...
\]
\[
 \exp \left[ i(k_{p-1}-k_p)(r_{e_{p-1}}-r_e) \right].
\]
Here \( a_1, ..., a_p, e \) are indices numbering the impurities. Summation over the index \( e \) has rendered the factor \( n \) in front of the integral. Expression in the angular brackets is a product of values of the random function \( S(q) = \sum_{\xi} \exp [iq(r_e-r_{e})] \) taken at different values of its argument. The essential contribution to the integral comes from the region \( k \sim (1/R) \). If \( R \) is much greater than the average distance between the impurities, the number of impurities contributing coherently to the sum is large. In this case the random function \( S(q) \) is close to its average value (i.e., to the structure factor \( \bar{S}(q) = \langle S(q) \rangle \) and in the leading order over \( nR^2 \), the result of averaging in equation (29) can be represented as a product,
\[
 S(k-k_1)S(k_1-k_2)...S(k_{p-1}-k_p).
\]
Then the consecutive members of the series for \( W(k) \) are related via
\[
 w^{(p+1)}_\beta (k) = \eta_0 \int \frac{dk_1}{(2\pi)^3} S(k-k_1)
\]
\[
 \times G_{m_1} (k_1) W_{m_1}^{(p)} (k_1)
\]
and the summation of the series renders the integral equation for \( W_\beta (k) \):
\[
 W_\beta (k) = n \eta_0 \delta_\beta + \eta_0 \int \frac{dk_1}{(2\pi)^3} S(k-k_1)
\]
\[
 \times G_{m_1} (k_1) W_{m_1} (k_1).
\]
by equation (31):

$$W(0) = \frac{m_0}{1 - \eta_0 Q},$$  \hspace{1cm} (32)$$

where $Q = \int \frac{d^k k}{(2\pi)^k} S(k) G(k)$. Comparison with equation (18) shows that summation of the perturbation series results in the substitution instead of $\sigma_{tr}'(0)$ of an ‘effective’ cross section, which takes into account the correlation of the positions of the scattering centers. For the model correlation function equation (28), with an account of equations (10), (16)

$$\tau_c = \frac{\pi^2 \xi_0}{4 l_B} \frac{1}{1 + \frac{\pi^2 B(D)}{\xi_0^2 A}},$$  \hspace{1cm} (33)$$

where $B(D) = \frac{\eta_0^2(D) 4 - D}{\xi_0^2(D) D - 1}$.

The obtained expression (33) for a relative lowering of $T_c$ has a similar structure to that, suggested on the basis of an heuristic argument by Sauls and Sharma [17]. In a limit of $\xi_0^2 A \ll 1$, $\tau_c$ is proportional to the ‘pair-breaking parameter’ $\frac{\xi_0}{l_B}$, as in the conventional theory. In the opposite limit the mean-free path cancels out and $\tau_c \approx \frac{\eta_0}{R^{AB}(D)}$ is determined by the geometric characteristics of the aerogel. When $R$ is growing, $\tau_c$ decreases and $T_c$ tends to its bulk value. If $A$ is estimated within the ‘model of beads’ the dependence on $R$ is different: $\tau_c \approx \frac{\xi_0}{l_B R^{AB}(D)}$. In $^3$He, $T_c$ and consequently $\xi_0$ strongly depend on pressure. At solidification pressure, $\xi_0$ is approximately four times smaller than at vapor pressure. In 98% silica aerogel both limits of equation (33) can be reached. With the aid of their interpolation, Sauls and Sharma [17] were able to fit the pressure dependence of $T_c$ in such an aerogel for the interval of pressures from 6 to 32 bars, using realistic parameters of the aerogel. Within the region of validity of Ginzburg and Landau equations, equation (33) renders a theoretical justification to their interpolation.

$^3$He in aerogel was used here as a simple example, where the effect of correlations can be analyzed and compared with the existing data. To obtain a concrete result in deriving equation (33), specific properties of correlations in the aerogel are used, so it cannot be directly applied to a different object. On the other hand, the argument of the previous section, which led to equation (31), is general and can be used to analyze the effect of correlated impurities in metallic superconductors.

5. Discussion

The example of superfluid $^3$He in aerogel clearly demonstrates that correlations of impurities can substantially weaken their pair-breaking effect and increase the transition temperature with respect to that expected for non-correlated impurities. More literally, it shows that the description of an aerogel as a uniform, continuous medium is not sufficient. Correlations introduce a spatial scale, $R$. Adjustment of a superfluid condensate to the external inhomogeneity increases its gain of energy in comparison with a uniform condensate.

Most important are the correlations with a characteristic radius, $R$, of the order or exceeding the superfluid coherence length, $\xi_0$. A superfluid condensate is more susceptible to such correlations, because the reaction of the condensate on external perturbation is determined by momenta on the order of $1/\xi_0$. For comparison, the reaction of normal quasiparticles is determined by much higher momenta, on the order of Fermi momentum. In this case, the effect of the correlations renders only a small correction to the mean-free path. Enhanced susceptibility of the condensate to long-range correlations manifests itself below $T_c$, as well. A separate analysis shows that in a superfluid phase, the correlations affect a temperature dependence of the order parameter because of the interplay of the correlation radius and the Ginzburg and Landau correlation length, $\xi(T)$ [18, 19].

In the last few years, experimental activity in $^3$He shifted to using different types of aerogel. On one hand there are highly anisotropic ‘nematic’ aerogels [8, 20]. On the other, there are highly isotropic silica aerogels, exhibiting new features [21]. Application of the present approach to these objects can bring further interesting results. In a practical sense, a systematic experimental investigation of the effect of correlated impurities in metallic unconventional superconductors would be even more useful and stimulating.

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