STRONG CONVERGENCE OF SOLUTIONS TO NONAUTONOMOUS KOLMOGOROV EQUATIONS

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Abstract. We study a class of nonautonomous, linear, parabolic equations with unbounded coefficients on \( \mathbb{R}^d \) which admit an evolution system of measures. It is shown that the solutions of these equations converge to constant functions as \( t \to +\infty \). We further establish the uniqueness of the tight evolution system of measures and treat the case of converging coefficients.

1. Introduction

In this paper we investigate the asymptotic behaviour of a class of nonautonomous parabolic partial differential equations of second order in \( \mathbb{R}^d \) with unbounded coefficients. We establish that the solutions converge to constant functions as the time \( t \) tends to \(+\infty\). These limits exist both locally uniformly and in \( L^p \) spaces with respect to a time-varying family of (invariant) measures. Such convergence results have been known before only for special cases, where different more specific methods could be employed, see [2, 12, 15, 23].

The analysis of nondegenerate elliptic operators with unbounded coefficients goes back to the second half of last century with the pioneering papers by Aronson and Besala [4, 5], Bodanko [7], Feller [14] and Krzyżaniński [20, 21]. The interest of the mathematical community has grown considerably since the nineties because of the many applications to stochastic analysis, where they appear naturally as Kolmogorov operators of stochastic partial differential equations, to mathematical finance and also to physics (see e.g. [15]). Starting from the analysis of autonomous Ornstein–Uhlenbeck equations in [11], elliptic operators with unbounded coefficients and the associated Cauchy problems have been studied both in the space \( C_b \) of bounded continuous functions and in \( L^p \) spaces on \( \mathbb{R}^d \) and on unbounded domains. In the present paper, we focus on \( \mathbb{R}^d \) for simplicity.

It turned out that the usual \( L^p \) spaces with respect to the Lebesgue measure are not appropriate for these investigations. For instance, no realization of the operator \( \mathcal{A}u = u'' - x|x|^\varepsilon u' \) in one spatial dimension generates a \( C_0 \)-semigroup in \( L^p(\mathbb{R}) \) if \( \varepsilon \) is positive (see [26]). This example indicates that one needs rather restrictive growth conditions to develop a theory for elliptic operators with unbounded coefficients in \( L^p(\mathbb{R}^d) \). The picture changes drastically if the semigroup \( T(\cdot) \) associated to \( \mathcal{A} \) on

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Indeed, Proposition 2.4(i) implies that $\parallel g \parallel (1.2)$ and for all $s$, see p. 2067 there. But the spaces $L^p(G, \mu)$ can be extended to a contraction from $L^p(G, \mu)$ for all $s$, for such results in various cases. One can however derive uniqueness within certain classes of evolution systems, see In [17] it was shown that nonautonomous Ornstein–Uhlenbeck evolution operators $C^1,2$ and in [22] for the general case. For $t, s \in L^p(G, \mu)$, and the convergence is locally uniform in $G$ if $f \in C^1,2(G, \mu)$.

In this paper we treat the nonautonomous case, where the coefficients of the elliptic operator also depend on $t \geq 0$. The semigroup $T(t)$ of the autonomous case is now replaced by an evolution operator $\{G(t) : t \geq s \geq 0\}$ in $C^1,2(G)$. Its existence and its main properties have been established in [12] for nonautonomous Ornstein–Uhlenbeck operators and in [22] for the general case. For $f \in C^1,2(G)$ and $s \geq 0$, the function $G(t, s)f$ is defined as the unique solution $u \in C^1,2((s, s) \times R) \cap C^1,2((s, +\infty) \times R)$ of the parabolic equation $D_tu = A(t)u$ on $(s, +\infty) \times R$ satisfying $u(s, \cdot) = f$ in $R$.

Similarly, the concept of invariant measure is replaced by the concept of evolution systems of measures (as referred to in [13]). Such a system is a one-parameter family of probability measures $\{\mu_t : t \geq 0\}$ satisfying

\begin{equation}
\int_R G(t, s)f d\mu_t = \int_R f d\mu_s
\end{equation}

for all $f \in C^1,2(G)$ and $t \geq s \geq 0$. As in the autonomous case, Lyapunov functions provide a convenient sufficient condition for the existence of an evolution system of measures, see Hypothesis [2.1(iii)]. Under such assumption, the proof of Theorem 5.4 of [22] even implies the existence of a tight evolution system of measures; i.e., for every $\varepsilon > 0$ there exists a radius $R > 0$ such that $\mu_t(B_R) \geq 1 - \varepsilon$ for all $t \geq 0$. In [17] it was shown that nonautonomous Ornstein–Uhlenbeck evolution operators admit infinitely many evolution systems of measures under reasonable assumptions. One can however derive uniqueness within certain classes of evolution systems, see [2, 17, 23] for such results in various cases.

If the evolution operator $G(t, s)$ admits an evolution system of measures, then it can be extended to a contraction from $L^p(G, \mu_t)$ to $L^p(G, \mu_t)$ for every $t \geq s \geq 0$. Indeed, Proposition [2.1(i)] implies that $\parallel G(t, s)f \parallel p \leq G(t, s)(|f|^p)$ for all $f \in C^1,2(G)$ and $t \geq s \geq 0$. Integrating this inequality with respect to $\mu_t$, we obtain

\begin{equation}
\parallel G(t, s)f \parallel _{L^p(G, \mu_t)}^p \leq \int_R G(t, s)(|f|^p)d\mu_t = \int_R |f|^pd\mu_s = \parallel f \parallel _{L^p(G, \mu_s)}^p,
\end{equation}

for all $t > s \geq 0$ and $p \in [1, +\infty)$. Each measure $\mu_s$ is equivalent to the Lebesgue measure $\lambda$ since it has a positive density $\rho(t, \cdot)$ with respect to $\lambda$ by results in [8], see p. 2067 there. But the spaces $L^p(G, \mu_t)$ and $L^p(G, \mu_s)$ differ in general
for \( t \neq s \). This fact causes several difficulties in the analysis and, in particular, the standard theory of evolution operators (e.g. in [9]) can not be applied to the evolution operator in the \( L^p \)-spaces for \( \mu_t \). As in [2, 17, 18, 23, 24], we will use the evolution semigroup \( \mathcal{T}(\cdot) \) associated with \( G(t,s) \), which is defined by
\[
(\mathcal{T}(t)h)(s,x) = (G(s,s-t)h(s-t,\cdot))(x), \quad (s,x) \in \mathbb{R}^{1+d},
\]
for \( t \geq 0 \) and \( h \in C_b(\mathbb{R}^{1+d}) \). Here we extend the given coefficients constantly to \( t < 0 \) to obtain an evolution operator \( G(t,s) \) for \( t \geq s \) on \( \mathbb{R} \), as explained in Remark 2.3.

In the papers [2, 15, 23] for several special cases it was established that \( G(t,s)f \) converges to the average \( m_s(f) := \int_{\mathbb{R}^d} f \, d\mu_s \) as \( t \to +\infty \). For bounded diffusion coefficients and time-periodic coefficients, Corollary 3.8 of [23] shows that \( \|G(t,s)f - m_s(f)\|_{L^p(\mathbb{R}^d,\mu_s)} \) tends to 0 as \( t \to +\infty \) for \( f \in L^p(\mathbb{R}^d,\mu_s) \) and \( s \in \mathbb{R} \). The proof of this result relies on the fact that one can employ the evolution semigroup on the compact time interval \([0,T]\), for the period \( T \).

The non-periodic case was addressed in [2], but only for diffusion coefficients \( q_{ij} \) which are constant in the spatial variables and under an additional strict dissipativity assumption on the drift term (namely that \( r_0 < 0 \) in Hypothesis 2.1(iv) below). These extra conditions yield the exponentially decaying gradient estimate \(|(\nabla_x G(t,s)f)(x)| \leq c e^{r_0(t-s)} (G(t,s)|\nabla f|)(x)\) for all \( s, x \in \mathbb{R}^d \) and \( f \in C^1_c(\mathbb{R}^d) \). This decay property is crucial for the proofs in [2]. In turn, it implies the cyclic condition \( D_i q_{jk} + D_j q_{ik} + D_k q_{ij} = 0 \) in \( \mathbb{R} \times \mathbb{R}^d \) for all \( i, j, k \in \{1, \ldots, d\} \) by Theorem 3.1 in [1], which explains the restriction to space independent diffusion coefficients \( q_{ij} \) in [2]. On the other hand, Corollary 5.4 of [2] even establishes the exponential decay of \( \|G(t,s)f - m_s(f)\|_{L^p(\mathbb{R}^d,\mu_s)} \) with rate \( r_0 < 0 \). To the best of our knowledge, this is the only available result on the long-time behaviour of the function \( \left\| G(t,s)f - m_s(f) \right\|_{L^p(\mathbb{R}^d,\mu_s)} \) for non-periodic coefficients (besides [18] for the special case of Ornstein–Uhlenbeck operators).

For non-periodic coefficients, our main result Theorem 3.2 shows that \( \|G(t,s)f - m_s(f)\|_{L^p(\mathbb{R}^d,\mu_s)} \) tends to 0 as \( t \to +\infty \) if \( f \in L^p(\mathbb{R}^d,\mu_s) \) and that \( G(t,s)f \) converges to \( m_s(f) \) locally uniformly if \( f \in C(\mathbb{R}^d) \) vanishes at infinity, where \( s \geq 0 \) and \( p \in [1, +\infty) \). This theorem then implies the uniqueness of tight evolution systems of measures. Compared to [2], we allow for space dependent and possibly unbounded diffusion coefficients and we do not need the strict dissipativity assumption \( r_0 < 0 \) in Hypothesis 2.1(iv). To use certain estimates on Green’s functions, we require additional bounds on the coefficients which are global in time but only local in space, see Hypothesis 2.1(i).

As in [18, 23], our approach relies on the decay to 0 of \( \nabla_x \mathcal{T}(t)h \) as \( t \to +\infty \) in \( L^p(\mathbb{R}^{1+d},\nu) \) for all \( h \in L^p(\mathbb{R}^{1+d},\nu) \), where \( \nu \) is defined by
\[
\nu(A \times B) = \int_A \mu_s(B) \, ds,
\]
on the product of a Borel set \( A \subset \mathbb{R} \) and a Borel set \( B \subset \mathbb{R}^d \), and canonically extended to the \( \sigma \)-algebra of all the Borel sets of \( \mathbb{R}^{1+d} \), see Proposition 2.6. This decay is proved by means of a “carré du champs” type inequality for the generator of \( \mathcal{T}(\cdot) \), which we recall in Proposition 2.4. To exploit the decay in \( L^p(\mathbb{R}^{1+d},\nu) \), we need lower bounds on the density of \( \mu_t \) which are local in space, but uniform in time. We show such estimates in Lemma 3.1 using known lower bounds of Green’s functions solving the Dirichlet problem on a ball, [3]. Still it is rather delicate to
pass from the strong convergence of $\nabla_x T(t)$ in $L^p(\mathbb{R}^{1+d}, \nu)$ to that of $G(t, s)$ in the proof of Theorem 3.2.

As we have already noticed, the spaces $L^p(\mathbb{R}^d, \mu)$ differ from each other. If the coefficients of the operators $A(t)$ converge as $t \to +\infty$, we establish that the solution $G(t, s)f$ tends to the mean $m_s(f)$ as $t \to +\infty$ in $L^p(\mathbb{R}^d, \mu_\infty)$ for all $f \in C_b(\mathbb{R}^d)$ (which is dense in $L^p(\mathbb{R}^d, \mu_s)$), $s \geq 0$ and $p \in [1, +\infty)$, see Theorem 4.4. Here $\mu_\infty$ is the invariant measure of the semigroup associated to the limiting autonomous operator $A_\infty$. The main step in the proof is the convergence result of Proposition 4.3 for the densities proved in \cite{8}. In Section 5 we exhibit a class of operators that satisfy all our assumptions.

**Notation.** We consider the usual spaces $C^{k+\alpha}(\Omega)$ when $\Omega$ is an open set or the closure of an open set, $k \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0, 1)$. We use the subscript “$b$” (resp., “$c$”) for the subspaces of the above spaces consisting of functions which are bounded together with all their derivatives up to the order $k$ (resp., are compactly supported). We also consider the spaces $C^{1,2}(J \times \Omega)$ and $C^{k+\alpha/2,2k+\alpha}(J \times \Omega)$ for an interval $J$, and use the subscripts “$b$” and “$c$” with the same meaning as above. For $\alpha \in (0, 1)$ the subscript “loc” means that the derivatives of order $k$ are $\alpha$-Hölder continuous in each compact set contained in $\Omega$ or $J \times \mathbb{R}^d$.

For a Borel measure $\mu$ on $\Omega$ and $p \in [1, +\infty)$, we denote by $L^p(\Omega, \mu)$ the usual Lebesgue space (omitting $\mu$ if it is the Lebesgue measure). For an open set $\Omega \subset \mathbb{R}^d$ and $k \in \mathbb{N}$, the standard Sobolev space with respect to the Lebesgue measure is denoted by $W^{k,p}(\Omega)$. Similarly, $W^{1,2}_p(J \times \Omega)$ is the usual parabolic Sobolev space with respect to the Lebesgue measure for an interval $J$.

Given a family of measures \{$\mu_t : t \geq 0$\}, we denote by $m_t(f)$ the average of the function $f$ with respect to the measure $\mu_t$. Finally, $B_R$ designates the open ball centered at 0 with radius $R$ and $\mathbb{R}_+ := [0, +\infty)$.

## 2. Assumptions and Background Material

We deal with differential operators $A(t)$, $t \geq 0$, defined on smooth functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ by

$$A(t)\varphi = \sum_{i,j=1}^{d} q_{ij}(t, \cdot) D_{ij}\varphi + \sum_{i=1}^{d} b_i(t, \cdot) D_i\varphi = \text{Tr}(Q(t, \cdot)D^2\varphi) + \langle b(t, \cdot), \nabla_x \varphi \rangle,$$

under the following conditions that are always assumed throughout the paper.

**Hypotheses 2.1.**

(i) For some $\alpha \in (0, 1)$ and every $i, j \in \{1, \ldots, d\}$, $q_{ij}, b_i$ belong to $C^{\alpha/2,1+\alpha}\text{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$. Moreover, $q_{ij} \in C_b(\mathbb{R}_+ \times B_R)$ and $D_q q_{ij}, b_j \in C_b(\mathbb{R}_+; L^p(B_R))$ for all $i, j, k \in \{1, \ldots, d\}$, all $R > 0$ and some $p > d + 2$.

(ii) The matrix $Q(t, x)$ is symmetric and $\langle Q(t, x)\xi, \xi \rangle \geq \eta(t, x)\xi^2$ for all $t \geq 0$, $x, \xi \in \mathbb{R}^d$ and a function $\eta : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ with $\inf_{\mathbb{R}_+ \times \mathbb{R}^d} \eta =: \eta_0 > 0$.

(iii) There exist a function $0 < V \in C^2(\mathbb{R}^d)$ and constants $a \geq 0$, $\kappa > 0$ such that $V(x)$ tends to $+\infty$ as $|x| \to +\infty$ and $(A(t)V)(x) \leq a - \kappa V(x)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

(iv) There exist constants $c_0 \geq 0$ and $r_0 \in \mathbb{R}$ such that $|\nabla_x Q(t, x)| \leq c_0 \eta(t, x)$ and $|\nabla_x b(t, x)\xi, \xi| \leq r_0 |\xi|^2$ for all $t \geq 0$ and $x, \xi \in \mathbb{R}^d$. 
(v) There exists a constant $c > 0$ such that either $|Q(t, x)| \leq c(1 + |x|)V(x)$ and $\langle b(t, x), x \rangle \leq c(1 + |x|^2)V(x)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, or $|Q(t, x)| \leq c$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

Except for the second part in (i), assumptions (i)–(iii) are needed to construct the evolution operator and the evolution system of measures $\{\mu_t : t \geq 0\}$. Condition (iv) leads to the gradient estimate (2.2). The second part of (i) is needed to obtain uniform lower bounds of the density of the measures, see Lemma 3.1. On the last condition we comment in Remark 2.5.

In the next proposition we collect several basic properties of the evolution operator $G(t, s)$.

**Proposition 2.2.** The following properties are satisfied.

(i) Let $D = \{(t, s, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d : t > s\}$. Then there exists a Green’s function $g : D \to (0, +\infty)$ such that

$$ G(t, s)f = \int_{\mathbb{R}^d} g(t, s, \cdot, y)f(y) \, dy $$

in $\mathbb{R}^d$ for $f \in C_b(\mathbb{R}^d)$ and $t > s \geq 0$. For every $t > s \geq 0$ and $x \in \mathbb{R}^d$, the function $g(t, s, x, \cdot)$ belongs to $L^1(\mathbb{R}^d)$ and $\|g(t, s, x, \cdot)\|_{L^1(\mathbb{R}^d)} = 1$. Each operator $G(t, s)$ is a contraction on $C_b(\mathbb{R}^d)$ and $G(t, s)1 = 1$.

(ii) For every $f \in C_c(\mathbb{R}^d)$ and $t > 0$, the function $s \mapsto G(t, s)f$ is continuous from $[0, t]$ to $C_b(\mathbb{R}^d)$. If $f \in C_c^2(\mathbb{R}^d)$, then for every $(t, x) \in (0, +\infty) \times \mathbb{R}^d$ the function $(G(t, \cdot)f)(x)$ is differentiable in $[0, t]$ and $(D_sG(t, s)f)(x) = -(G(t, s)A(s)f)(x)$.

(iii) There exists a constant $C_1 > 0$ such that for all $p \geq 2$ and $s \in \mathbb{R}_+$

$$ |\nabla_x G(s + t, s)f|^p \leq C_1(t^{-p/2} \lor 1)G(s + t, s)(|f|^p), \quad t > 0, \ f \in C_b(\mathbb{R}^d). $$

**Proof.** Statement (i) and (ii) come from Proposition 2.4 (and its proof) and Lemma 3.1 of [22]. Also statement (iii) is a consequence of the results in [22] although it was not explicitly stated there. To prove it, for every $n \in \mathbb{N}$ and $s \geq 0$ we denote by $u_n$ the unique classical solution to Neumann–Cauchy problem

$$
\begin{align*}
D_tu_n(t, x) &= A(t)u_n(t, x), \quad (t, x) \in (s, +\infty) \times B_n, \\
D_\nu u_n(t, x) &= 0, \quad (t, x) \in (s, +\infty) \times \partial B_n, \\
u_n(s, t) &= f(x), \quad x \in B_n.
\end{align*}
$$

Let $w_n$ solve the same boundary value problem with initial condition $w_n(s, \cdot) = f^2$.

In the proof of Theorem 4.1 of [22] it is shown that the function

$$(t, x) \mapsto z_n(t, x) = (u_n(t, x))^2 + C_1^{-2}(t-s)|\nabla_x u_n(t, x)|^2$$

satisfies the inequality $D_tz_n - A z_n \leq 0$ in $(s, s+1] \times B_n$, $D_\nu z_n \leq 0$ on $(s, s+1] \times \partial B_n$ and $z_n(s, \cdot) = f^2$ in $B_n$ for each $n \in \mathbb{N}$. The constant $C_1$ only depends on $\eta_0$, $d$, $c_0$ and $r_0$ from Hypothesis 2.1. The classical maximum principle now implies that $z_n \leq w_n$ in $(s, s+1] \times B_n$.

By Remark 2.3 of [22], the functions $u_n$ and $w_n$ converge to $G(\cdot, s)f$ and $G(\cdot, s)f^2$, respectively, in $C^{1,2}((s, s+R) \times B_R)$ for every $R > 0$. Taking the limit $n \to +\infty$, the inequality $z_n \leq w_n$ thus yields the formula (2.2) with $p = 2$ for all
s ≥ 0 and t ∈ (0, 1]. Let p > 2. Using (2.2) with p = 2, Hölder’s inequality and
\( \|g(t, s, x, \cdot)\|_{L^1(\mathbb{R}^d)} = 1 \) from (i), we derive
\[
|\nabla_x G(s + t, s)f|^p = \left( |\nabla_x G(s + t, s)f|^2 \right)^{p/2} \leq (C_1 t^{-1} G(s + t, s)(|f|^2))^{p/2} \\
\leq C_1 t^{-p/2} G(s + t, s)(|f|^p)
\]
for all s ≥ 0, t ∈ (0, 1] and f ∈ C_b(\mathbb{R}^d). To extend this estimate to t > 1, one
finally uses the evolution law and splits \( \nabla_x G(s + t, s) = \nabla_x G(s + t, s + t - 1)G(s + t - 1, s)f \).

\[\text{Remark 2.3. Setting } A(t) := A(0) \text{ for } t < 0, \text{ we extend the coefficients } q_{ij} \text{ and } b_i \text{ to } t \in \mathbb{R} \text{ in such a way that Hypotheses (2.1) hold with } \mathbb{R}_+ \text{ replaced by } \mathbb{R}. \text{ Hence, Proposition (2.2) is valid on } \mathbb{R} \text{ instead of } \mathbb{R}_+, \text{ and for } t \in \mathbb{R} \text{ and } (-\infty, t) \text{ instead of } [0, t] \text{ in part (ii). The extended evolution operator is also denoted by } G(t, s), \text{ for } t \geq s \text{ in } \mathbb{R}. \text{ Moreover, we set } \mu_s := G(0, s)^* \mu_0 \text{ for } s < 0, \text{ where } G(0, s)^* \text{ is the}
\]

\[\text{adjoint of the operator } G(0, s) \text{ in } C_b(\mathbb{R}^d). \text{ Using formula (2.1) to extend } G(0, s) \text{ to characteristic functions, it is easy to see that } \mu_s \text{ is a probability measure for every}
\]
s < 0. The set \( \{ \mu_t : t \in \mathbb{R} \} \) is an evolution system of measures for \( G(t, s) \text{ on } \mathbb{R}, \text{ see the proof of Theorem 5.4 of (2.2).} \]

We now recall the properties of the evolution semigroup \( T(\cdot) \text{ (see (1.3)) and the}

\[\text{measure } \nu \text{ that we use in this paper. To define it, we use the evolution operator}
\]

and the evolution system of measures on \( \mathbb{R} \) from the above remark.

\[\text{Proposition 2.4. Let } p \in [1, +\infty). \text{ The following properties are satisfied.}
\]

(i) The measure \( \nu \text{ defined in (1.3) is infinitesimally invariant for } T(\cdot); \text{ i.e.,}
\]
\[
\int_{\mathbb{R}^d} (A(\cdot)h - D_t h) \, d\nu = 0 \quad \text{for all } h \in C_c^\infty(\mathbb{R}^{1+d}).
\]

Moreover, the restriction to \( C_c(\mathbb{R}^{1+d}) \) of the evolution semigroup \( T(\cdot) \text{ may be}

\[\text{extended to a strongly continuous contraction semigroup } T_p(\cdot) \text{ in } L^p(\mathbb{R}^{1+d}, \nu). \text{ Its}
\]

generator is denoted by \( G_p \).

(ii) For any \( u \in D(G_2) \text{ the following “carré du champs” type inequality holds true:}
\]
\[
\eta_0 \int_{\mathbb{R}^{1+d}} |\nabla_x u|^2 \, dv \leq \int_{\mathbb{R}^{1+d}} |Q^{1/2} \nabla_x u|^2 \, dv \leq -\int_{\mathbb{R}^{1+d}} u G_2 u \, dv.
\]

\[\text{Proof. We refer the reader to Lemma 6.3(ii) of (22) and Theorem 2.1 of (24) for part (i) and to Corollary 2.16 of (23) for part (ii). The results in (23) are only shown for the case of time-periodic coefficients with slightly different assumptions from our}
\]

Hypotheses (2.1). We thus sketch the proof of (ii).

We have to replace the space \( D(G_\infty) \text{ used in (23) by the space } \mathcal{D} \text{ of all } u \in C_b(\mathbb{R}^{1+d}) \text{ belonging to } W^{1,2}_p((-R, R) \times B_R) \text{ for all } R > 0 \text{ and } 1 \leq p < +\infty \text{ such that } \mathcal{G} u := A(\cdot) u - D_t u \text{ is contained in } C_b(\mathbb{R}^{1+d}) \text{ and supp}(u) \subset [-M, M] \times \mathbb{R}^d \text{ for some } M > 0. \text{ The generator } \mathcal{G}_2 \text{ is the closure of the operator } \mathcal{G} \text{ defined on } \mathcal{D} \text{ by Theorem 2.1 of (24). Proposition 2.5 of (24) yields}
\]
\[
u = \int_0^{+\infty} e^{-tT}(t)(u - G_2 u) \, dt
\]

for all \( u \in \mathcal{D}. \text{ The gradient estimate (2.2) for } G(t, s) \text{ directly implies the inequality}
\]
\[
\|\nabla_x T(t) h\|_\infty \leq C_1 (t^{-\frac{1}{2}} \vee 1) \|h\|_\infty
\]

\[\]
for \( h \in C_b(\mathbb{R}^{1+d}) \) and \( t > 0 \). As in Proposition 2.14 of \[23\], we then infer that \( D \subseteq C_b^0(\mathbb{R}^{1+d}) \) and \( \| \nabla_x u \|_\infty \leq c (\| u \|_\infty + \| Gu \|_\infty) \) for \( u \in D \). Formula \[2.3\] can now be shown analogously as Proposition 2.15 and Corollary 2.16 in \[23\], where the first inequality in \[2.3\] follows from Hypothesis \( 2.1(\text{ii}) \).

**Remark 2.5.** Hypothesis \( 2.1(\text{v}) \) is crucial to prove the inequality \[2.3\], and this is the only part of the paper where we use it. Typically, one takes as a Lyapunov function the first inequality in \( 2.3 \) follows from Hypothesis \( 2.1(\text{ii}) \).
Lemma 3.1. Using $G_2u \in D(G_2)$ once more, we conclude that also the derivative $\chi'_u$ belongs to $L^1(0, +\infty)$ and so $\chi_u(s)$ vanishes as $s \to +\infty$. □

We conclude this section by a simple convergence lemma for tight sequences of probability measures.

**Lemma 2.7.** Let $(\tilde{\mu}_n)$ be a tight sequence of probability measures in $\mathbb{R}^d$ and $(g_n) \subset C_b(\mathbb{R}^d)$ be a bounded sequence. The following assertions hold.

(i) If $g_n$ tends to zero locally uniformly in $\mathbb{R}^d$ as $n \to +\infty$, then $\int_{\mathbb{R}^d} g_n \, d\tilde{\mu}_n$ vanishes as $n \to +\infty$.

(ii) If $g_n$ tends to some $g \in C_b(\mathbb{R}^d)$ locally uniformly in $\mathbb{R}^d$ and $\tilde{\mu}_n$ converges weakly* to a probability measure $\tilde{\mu}$ in $\mathbb{R}^d$ as $n \to +\infty$, then $\int_{\mathbb{R}^d} g_n \, d\tilde{\mu}_n$ tends to $\int_{\mathbb{R}^d} g \, d\tilde{\mu}$ as $n \to +\infty$.

**Proof.** We only show property (ii), as the first assertion can be treated similarly. By assumption, $M := \sup_{n}\{\|g_n\|_{\infty}, \|g\|_{\infty}\} < +\infty$ and for each $\varepsilon > 0$ there exists a radius $r > 0$ such that $\tilde{\mu}_n(\mathbb{R}^d \setminus B_r) \leq \varepsilon$. We can thus estimate

$$\left| \int_{\mathbb{R}^d} g_n \, d\tilde{\mu}_n - \int_{\mathbb{R}^d} g \, d\tilde{\mu} \right| \leq \int_{B_r} |g_n - g| \, d\tilde{\mu}_n + \int_{\mathbb{R}^d \setminus B_r} |g_n - g| \, d\tilde{\mu}_n + \int_{\mathbb{R}^d} g \, d\tilde{\mu}_n - \int_{\mathbb{R}^d} g \, d\tilde{\mu}$$

$$\leq \sup_{x \in B_r} |g_n(x) - g(x)| + 2M \varepsilon + \left| \int_{\mathbb{R}^d} g \, d\tilde{\mu}_n - \int_{\mathbb{R}^d} g \, d\tilde{\mu} \right|.$$ 

As $n \to +\infty$, the sum in last line tends to $2M\varepsilon$, and (ii) follows. □

3. **Asymptotic behaviour of $G(t, s)$**

Throughout this section, $\{\mu_t : t \geq 0\}$ is any tight evolution system of measures for $G(t, s)$, extended to the whole $\mathbb{R}$ as in Remark 2.3. We recall that by Theorem 5.4 in [22] a tight evolution system of measures for $G(t, s)$ does exist. Corollary 3.2 of [8] yields that there exists a positive function $\rho : \mathbb{R}^{1+d} \to \mathbb{R}$ such that $\mu_t$ is the density of the Lebesgue measure for every $t \in \mathbb{R}$. In Corollary 3.3 we will see that actually there exists only one tight evolution system of measures for $G(t, s)$.

To begin with, we use Hypothesis 2.4(i) to prove a lower bound on the densities $\rho(t, \cdot)$, which is crucial in our analysis.

**Lemma 3.1.** For each $k \in \mathbb{N}$ there exists a number $\delta_k > 0$ such that $\rho(\tau, x) \geq \delta_k$ for all $\tau \geq 0$ and $|x| \leq k$.

**Proof.** Let $D_k = \{(t, s, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \overline{B_k} \times \overline{B_k} : t > s\}$ for every $k \in \mathbb{N}$. By $g_k : D_k \to [0, +\infty)$ we denote the Green’s function of the parabolic problem

$$D_t u(t, x) = A(t) u(t, x), \quad (t, x) \in (s, +\infty) \times B_k,$$

$$u(t, x) = 0, \quad (t, x) \in (s, +\infty) \times \partial B_k,$$

$$u(s, x) = f(x), \quad x \in B_k,$$

as constructed in Theorem 3.16 of [16] and its corollaries. The proof of Proposition 2.4 in [22] yields $g \geq g_k$ on $D_k$ for each $k \in \mathbb{N}$, where $g$ is Green’s function in Proposition 2.4(i). Since the family $\{\mu_t : t \geq 0\}$ is tight, there is a radius $k_0 \in \mathbb{N}$ such that $\mu_t(B_{k_0}) \geq 1/2$ for all $t \geq 0$. Throughout the proof, the integer $k \geq k_0$ is arbitrary, but fixed. We claim that there exists a number $\delta_k > 0$ such that

$$(3.1) \quad g_{k+2}(\tau + 1, \tau, x, y) \geq 2\delta_k \quad \text{for all} \quad \tau \geq 0, \ x, y \in \overline{B_k}.$$
To prove the claim, we rewrite the operators $A(t)$ in divergence form and apply Theorem 9(iii) in [3] with $\Omega' = B_{k+1}$, $\Omega = B_{k+2}$ and $T = 8$ to the operators $L(t) = D_t - \text{div}(\tilde{Q}(t + \tau, \cdot) \nabla_x) - (\tilde{b}(t + \tau), \nabla_x)$ on $(0, 1] \times B_{k+2}$ for $\tau \geq 0$. Here the coefficients $\tilde{q}_{ij} = q_{ij}$ belong to $C^b(\mathbb{R}_+ \times \mathbb{R}^d)$ and satisfy $\tilde{q}_{ij} = q_{ij}$ on $\mathbb{R}_+ \times B_{k+2}$ as well as $\langle \tilde{Q}(t, x) \xi, \xi \rangle \geq \eta_0/2$ for all $i, j \in \{1, \ldots, d\}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\xi \in \partial B_1$.

The drift coefficients $b_i$ are continuous extensions of $b_i - \sum_{j=1}^d D_j q_{ij}$ to $\mathbb{R}^{1+d}$, such that on $b_i = b_i - \sum_{j=1}^d D_j q_{ij}$ on $\mathbb{R}_+ \times B_{k+2}$ and $b_i = 0$ on $\mathbb{R}_+ \times \mathbb{R} \setminus B_{k+3}$ for $k \geq k_0$ and $i, j \in \{1, \ldots, d\}$. By the uniqueness statement in Theorem 6 of [3], the map $g_{k+2}(\cdot + \tau, \tau, \cdot)$ is the Green’s function of $L(t)$ on $(0, 1] \times B_{k+2}$. Then Theorem 9(iii) of [3] now implies that

$$g_{k+2}(t + \tau, x, y) \geq C_1 t^{-d/2} \exp(-C_2 t^{-1}) \eta \leq C_1 \eta$$

for all $x, y \in B_{k+1}$, $t \in (0, \min\{8, (d y, \partial B_{k+1})^2\}]$ and $\tau \geq 0$. The constants $C_1$ and $C_2$ depend on $\eta_0, \sup_{t \geq 0} \|g_{ij}(t, \cdot)\|_{L^\infty(B_{k+2})}$, $\sup_{t \geq 0} \|b_j(t, \cdot)\|_{L^r(B_{k+2})}$ and on $\sup_{t \geq 0} \|D_j q_{ij}(t, \cdot)\|_{L^r(B_{k+2})}$ for all $i, j \in \{1, \ldots, d\}$ and some $p > d$. (Note that these suprema are finite due to Hypotheses [2.11].) If $y \in B_k$, then $d(y, \partial B_{k+1}) \geq 1$.

Hence, we can take $t = 1$ in (3.2), and (3.1) follows.

We can now complete the proof. Take a Borel set $B \subset B_k$ and some $\tau > 0$. From (1.1), (2.1) and (3.1), we deduce

$$\int_B \rho(t, x) dx = \int_{\mathbb{R}^d} \int_B g(t + 1, \tau, x, y) \rho(t + 1, x) dy dx$$

$$\geq \int_{\mathbb{R}^d} \int_B g_{k+2}(t + 1, \tau, x, y) \rho(t + 1, x) dy dx$$

$$\geq 2\delta_k \lambda(B) \int_{\mathbb{R}^d} \rho(t + 1, x) dx = 2\delta_k \lambda(B) \mu_{\tau+1}(B_k) \geq \delta_k \lambda(B),$$

where $\lambda$ is the Lebesgue measure. This lower bound yields the assertion. \hfill $\square$

We now establish our main result on the convergence of $G(t, s)$.

**Theorem 3.2.** Let $s \geq 0$, $p \in [1, +\infty)$, and $\{\mu_t : t \geq 0\}$ be a tight evolution system of measures for $G(t, s)$. The following assertions are true.

(i) $\|G(t, s)f - m_s(f)\|_{L^p(\mathbb{R}^d, \mu_t)}$ tends to 0 as $t \to +\infty$ for each $f \in L^p(\mathbb{R}^d, \mu_s)$.

(ii) For each $f \in C_b(\mathbb{R}^d)$, $G(t, s)f$ tends to $m_s(f)$ locally uniformly in $\mathbb{R}^d$ as $t \to +\infty$.

**Proof.** (i) First of all, we observe that it suffices to prove the assertion for $s \in \mathbb{R}_+ \setminus \mathcal{N}$, where $\mathcal{N}$ is a null set. Indeed, if $s \in \mathcal{N}$, we fix any $s_s \in \mathbb{R}_+ \setminus \mathcal{N}$ such that $s_s > s$. If $f \in L^p(\mathbb{R}^d, \mu_s)$, then the function $g = G(s_s, s)f$ belongs to $L^p(\mathbb{R}^d, \mu_{s_s})$ and since $G(t, s)f = G(t, s)g$ and $m_s(f) = m_{s_s}(g)$,

$$\lim_{t \to +\infty} \|G(t, s)f - m_s(f)\|_{L^p(\mathbb{R}^d, \mu_t)} = \lim_{t \to +\infty} \|G(t, s)g - m_{s_s}(g)\|_{L^p(\mathbb{R}^d, \mu_{s_s})} = 0.$$

Moreover, in view of (1.2), it suffices to prove assertion for $f \in C^\infty_c(\mathbb{R}^d)$. Finally, we can assume that $p > d$, since for $p \in [1, d]$ Hölder’s inequality shows that $\|G(t, s)f - m_s(f)\|_{L^p(\mathbb{R}^d, \mu_t)} \leq \|G(t, s)f - m_s(f)\|_{L^{2^*}(\mathbb{R}^d, \mu_t)}$ for all $f \in C^\infty_c(\mathbb{R}^d)$ and $t > s$. Thus, we let $f \in C^\infty_c(\mathbb{R}^d)$ and $p > d$.

Fix a positive sequence $(t_n)$ diverging to $+\infty$, and functions $\alpha_m$ in $C^\infty_c(\mathbb{R})$ such that $0 \leq \alpha_m \leq 1$ in $\mathbb{R}^d$ and $\alpha_m = 1$ on $[-m, m]$ for each $m \in \mathbb{N}$.
tends to 0 as $n \to +\infty$ for each $m \in \mathbb{N}$. There thus exist null sets $\mathcal{N}_m \subset [−m, m]$ and subsequences $(t_n^{(m)})$ diverging to $+\infty$, with $t_k^{(m+1)} \in (t_k^{(m)})_n$ for all $k, m \in \mathbb{N}$, such that

$$
\lim_{n \to +\infty} \int_{\mathbb{R}^d} \rho(s + t_n^{(m)}, \cdot) |\nabla_x G(s + t_n^{(m)}, s)f|^p \, dx = 0
$$

for all $m \in \mathbb{N}$ and $s \in \mathbb{R}_+ \setminus \mathcal{N}_m$. We can thus determine a diagonal sequence $(t_{n_j})$ such that

$$
\lim_{j \to +\infty} \int_{\mathbb{R}^d} \rho(s + t_{n_j}, \cdot) |\nabla_x G(s + t_{n_j}, s)f|^p \, dx
$$

for each $s \in \mathbb{R}_+ \setminus \mathcal{N}$, where $\mathcal{N} = \bigcup_{m \in \mathbb{N}} \mathcal{N}_m$ is a null set.

Fix $s \in \mathbb{R}_+ \setminus \mathcal{N}$. We use Lemma 2.7(i) with $\tau = s + t_n$. For every $k \in \mathbb{N}$, it provides a number $\delta_k > 0$ such that $\rho(s + t_n, x) \geq \delta_k$ for $n \in \mathbb{N}$ and $|x| \leq k$. This lower bound and (3.3) yield

$$
\lim_{j \to +\infty} \|\nabla_x G(s + t_{n_j}, s)f\|_{L^p(B_k)} = 0
$$

for each $k \in \mathbb{N}$. Observing that $\|G(s + t_{n_j}, s)f\|_{L^p(B_k)} \leq c_k^{1/p} \|G(s + t_{n_j}, s)f\|_{L^\infty} \leq \|f\|_{L^\infty}$ for some positive constant $c_k$ (see Proposition 2.2(i)), we thus find constants $c_k > 0$ such that $\|G(s + t_{n_j}, s)f\|_{W^{1,p}(B_k)} \leq c_k$ for all $j \in \mathbb{N}$. Since $p > d$, $W^{1,p}(B_k)$ is compactly embedded in $C(B_k)$. By a diagonal argument, there exists a function $g(s, \cdot) \in C(\mathbb{R}^d)$ such that $G(s + t_{n_j}, s)f$ converges to $g(s, \cdot)$ locally uniformly in $\mathbb{R}^d$.

On the other hand, $\nabla_x G(s + t_{n_j}, s)f$ tends to 0 in $L^p(B_R)^d$ as $n \to +\infty$, for every $R > 0$, due to (3.4). The weak gradient $\nabla_x g(s, \cdot)$ thus vanishes, and hence $g(s)$ is constant in $x$. To prove that this constant is $m_s(f)$, it suffices to observe that

$$
m_s(f) - g(s) = \int_{\mathbb{R}^d} (f - g(s)) \, d\mu_s = \int_{\mathbb{R}^d} G(s + t_{n_j}, s)(f - g(s)) \, d\mu_{s+t_{n_j}}
$$

and use Lemma 2.7(i) with $\tilde{\mu}_n = \mu_{s+t_{n_j}}$. As a result, $g(s) = m_s(f)$ and $G(s + t_{n_j}, s)f$ tends to $m_s(f)$ locally uniformly as $j \to +\infty$, for $s \in \mathbb{R}_+ \setminus \mathcal{N}$. Since $G(s + t_{n_j}, s)m_s(f) \equiv m_s(f)$ by Proposition 2.2(i), from Lemma 2.7(i) we infer

$$
\lim_{j \to +\infty} \|G(s + t_{n_j}, s)(f - m_s(f))\|_{L^p(\mathbb{R}^d, \mu_{s+t_{n_j}})} = 0.
$$

Finally, the function $h = \|G(\cdot, s)f - m_s(f)\|_{L^p(\mathbb{R}^d, \mu_s)}$ is decreasing in $[s, +\infty)$ since

$$
h(t_2) = \|G(t_2, s)(f - m_s(f))\|_{L^p(\mathbb{R}^d, \mu_s)} \leq \|G(t_2, t_1)(f - m_s(f))\|_{L^p(\mathbb{R}^d, \mu_t)} = h(t_1)
$$

for $s \leq t_1 < t_2$, where we have used property (i) in Proposition 2.2 and (1.2). We conclude that $\lim_{t \to +\infty} \|G(t, s)f - m_s(f)\|_{L^p(\mathbb{R}^d, \mu_s)} = 0$.

(ii) Fix $f \in C_b(\mathbb{R}^d)$, $s \in \mathbb{R}_+$, $R > 0$ and $p > d$. Since $C_b(\mathbb{R}^d) \subset L^p(\mathbb{R}^d, \mu_s)$, $\|G(t + s, s)f - m_s(f)\|_{L^p(\mathbb{R}^d, \mu_{t+s})}$ tends to 0 as $t \to +\infty$, by the first part of the proof. Taking Lemma 3.1 into account, we can estimate

$$
\|G(t + s, s)f - m_s(f)\|_{L^p(B_R)} \leq \delta_R^{-1/p} \|G(t + s, s)f - m_s(f)\|_{L^p(\mathbb{R}^d, \mu_{t+s})}
$$
for all \( t \geq 0 \) and some positive constant \( \delta_R \). Hence, \( \| G(t + s, s) f - m_s(f) \|_{L^p(B_R)} \) tends to 0 as \( t \to +\infty \). In particular, there exists a positive constant \( C_1 = C_1(R) \) such that

\[
(3.5) \quad \| G(t + s, s) f - m_s(f) \|_{L^p(B_R)} \leq C_1
\]

for all \( t \geq 0 \). Moreover, the gradient estimate \( 2.2 \) implies that

\[
(3.6) \quad \| \nabla_x G(t + s, s) f \|_{L^p(B_R)} \leq C_2 \| f \|_{\infty}
\]

for all \( t \geq 1 \) and some positive constant \( C_2 = C_2(R) \). From \( 3.5 \) and \( 3.6 \), we deduce that the family of functions \( \{ G(t + s, s) f - m_s(f) : t \geq 1 \} \) is bounded in \( W^{1,p}(B_R) \) and, consequently, in \( C^\beta(B_R) \) for some \( \beta \in (0, 1) \) since \( p > d \). By the Arzela-Ascoli theorem, from any sequence \( (t_n) \) diverging to \( +\infty \) we can extract a subsequence \( (t_{n_k}) \) such that \( G(t_{n_k} + s, s) f - m_s(f) \) converges uniformly in \( B_R \) to zero as \( k \to +\infty \), since it tends to zero in \( L^p(B_R) \). This shows that \( G(t + s, s) f - m_s(f) \) tends to 0, uniformly in \( B_R \), as \( t \to +\infty \).

**Corollary 3.3.** \( G(t, s) \) has exactly one tight evolution system of measures.

**Proof.** Let \( \{ \mu_t^{(1)} : t \geq 0 \} \) and \( \{ \mu_t^{(2)} : t \geq 0 \} \) be two evolution systems of measures with corresponding means \( m_t^{(i)} \). Fix \( s \in \mathbb{R}_+ \) and \( f \in C_b(\mathbb{R}^d) \). Then, Theorem \( 3.2 \) \( (i) \) shows that, as \( t \to +\infty \), \( G(t, s) f \) converges both to \( m_s^{(1)}(f) \) and \( m_s^{(2)}(f) \), locally uniformly in \( \mathbb{R}^d \). Hence, \( m_s^{(1)}(f) = m_s^{(2)}(f) \) and, consequently, \( \mu_s^{(1)} = \mu_s^{(2)} \).

4. **Converging Coefficients**

In this section, we consider coefficients that converge as \( t \to +\infty \) as described in the next additional hypothesis.

**Hypothesis 4.1.** The coefficients \( q_{i,j} \) and \( b_i \) belong to \( C^{(1/2, \alpha)}_b(\mathbb{R}_+ \times B_R) \) for all \( i, j \in \{1, \cdots, d\} \) and \( R > 0 \) and \( Q(t, \cdot) \) and \( b(t, \cdot) \) converge pointwise to maps \( Q_\infty : \mathbb{R}_+ \to \mathbb{R}^{d^2} \) and \( b_\infty : \mathbb{R}^d \to \mathbb{R}^d \), respectively, as \( t \to +\infty \).

**Remark 4.2.** Hypotheses \( 2.1 \) and \( 4.1 \) imply that \( Q_\infty \in C^{\alpha}_\text{loc}(\mathbb{R}^d; \mathbb{R}^{d^2}) \) and \( b_\infty \in C^{\alpha}_\text{loc}(\mathbb{R}^d, \mathbb{R}^d) \) satisfy the \( t \)-independent analogues of Hypotheses \( 2.1 \) \( (ii) \) and \( (iii) \). The evolution operator generated by \( A_\infty \) is a semigroup \( \{ T(t) : t \geq 0 \} \) which admits a single invariant measure \( \mu_\infty \) having a density \( \rho_\infty > 0 \) with respect to the Lebesgue measure. (See e.g. Theorems 8.1.15 and 8.1.20 of [6] or [25].)

As in Section 3 \( \{ \mu_t : t \geq 0 \} \) is any tight evolution system of measures with densities \( \rho(t, \cdot) \). Under the additional Hypothesis \( 4.1 \) we show that the densities \( \rho(t, \cdot) \) converge to \( \rho_\infty \) and we derive a variant of Theorem \( 3.2 \).

**Proposition 4.3.** The densities \( \rho(t, \cdot) \) converge to \( \rho_\infty \) locally uniformly in \( \mathbb{R}^d \) and in \( L^1(\mathbb{R}^d) \) as \( t \to +\infty \).

**Proof.** We first prove local uniform convergence. It suffices to show that every sequence \( (s_n) \) diverging to \( +\infty \) admits a subsequence such that \( \rho(s_{n_k} \cdot) \) converges to \( \rho_\infty \) locally uniformly on \( \mathbb{R}^d \) as \( j \to +\infty \). As in the proof of Theorem 6.2 of [2] we see that \( \mu_t \) weakly* converges to \( \mu_\infty \) as \( t \to +\infty \). Because of Proposition \( 2.1 \) \( (ii) \) and Hypothesis \( 2.1 \) \( (i) \), Corollary 3.9 [8] yields that \( \rho \) is contained in \( C^\beta((s, s + 1) \times B_R) \) for every \( s \), \( R > 0 \) and some \( \beta > 0 \). The proofs given there also yield that the norms of \( \rho \) in these spaces are bounded by a constant \( C = C(R) \) independent of \( s \).
See also [19]. As a result, \( \rho \) belongs to \( C^\beta_0(\{0, +\infty\} \times B_R) \) for every \( R > 0 \). The Arzelà-Ascoli theorem now provides a sequence \((t_n)\) diverging to \(+\infty\) such that the density \( \rho(t_n, \cdot) \) of the measure \( \mu_{t_n} \) converges to a function \( g \in C(\mathbb{R}^d) \) locally uniformly in \( \mathbb{R}^d \) as \( n \to +\infty \). The weak* convergence of \( \mu_t \) to \( \mu_\infty \) thus yields
\[
\int_{\mathbb{R}^d} f \rho_\infty \, dx = \int_{\mathbb{R}^d} f \, d\mu_\infty = \lim_{t_k \to +\infty} \int_{\mathbb{R}^d} f \mu_{t_k} = \lim_{t_k \to +\infty} \int_{\mathbb{R}^d} f \rho(t_k, \cdot) \, dx = \int_{\mathbb{R}^d} f \, g \, dx
\]
for every \( f \in C^\infty_c(\mathbb{R}^d) \). Hence, \( \rho_\infty = g \) and the local uniform convergence is shown.

To prove the \( L^1 \)-convergence, let \( \varepsilon > 0 \). By the tightness, there is a radius \( R > 0 \) such that \( \mu_t(\mathbb{R}^d \setminus B_R), \mu_\infty(\mathbb{R}^d \setminus B_R) \leq \varepsilon \) for all \( t \geq 0 \). From the first part of the proof we deduce
\[
\limsup_{t \to +\infty} \|\rho(t, \cdot) - \rho_\infty\|_{L^1(\mathbb{R}^d)} = \limsup_{t \to +\infty} \left[ \|\rho(t, \cdot) - \rho_\infty\|_{L^1(B_R)} + \|\rho(t, \cdot) - \rho_\infty\|_{L^1(\mathbb{R}^d \setminus B_R)} \right]
\leq \limsup_{t \to +\infty} \mu_t(\mathbb{R}^d \setminus B_R) + \mu_\infty(\mathbb{R}^d \setminus B_R)
\leq 2\varepsilon.
\]

**Theorem 4.4.** Let \( s \geq 0, p \in [1, +\infty) \) and \( f \in C_b(\mathbb{R}^d) \). Then, \( G(t, s)f \) tends to \( m_s(f) \) in \( L^p(\mathbb{R}^d, \mu_s) \) as \( t \to +\infty \).

**Proof.** The result follows from Proposition 4.3 Theorem 3.2(i) and the estimates
\[
\|G(t, s)f - m_s(f)\|_{L^p(\mathbb{R}^d, \mu_s)}^p \leq \int_{\mathbb{R}^d} |\rho_\infty - \rho(t, \cdot)| \|G(t, s)(f - m_s(f))\|_{L^1(\mathbb{R}^d)} \, dx + \|G(t, s)f - m_s(f)\|_{L^p(\mathbb{R}^d, \mu_s)}^p \leq 2\|f\|_{L^p(\mathbb{R}^d, \mu_s)} \|\rho_\infty - \rho(t, \cdot)\|_{L^1(\mathbb{R}^d)} + \|G(t, s)f - m_s(f)\|_{L^p(\mathbb{R}^d, \mu_s)}^p.
\]

5. **AN EXAMPLE**

We consider the family of operators \( A(t) \) defined on smooth functions \( \varphi \) by
\[
A(t)\varphi = (1 + |x|^2)^\gamma \sum_{i,j=1}^d q_{ij}^{(0)}(t, x) D_{ij} \varphi - b^{(0)}(t)(1 + |x|^2)^r \varphi, \quad (x, \nabla_x \varphi)
\]
for \( t \geq 0 \) and \( x \in \mathbb{R}^d \), under the following assumptions.

(i) \( q_{ij}^{(0)} = q_{ji}^{(0)} \) belong to \( C^{\alpha/2, \alpha}_b(\mathbb{R}^d) \) for some \( \alpha \in (0, 1) \) for all \( R > 0 \) and \( i, j \in \{1, \ldots, d\} \). Moreover, \( \langle Q^{(0)}(t, x) \xi, \xi \rangle \geq \eta_0 \) in \( \mathbb{R}^1 + d \) for some positive constant \( \eta_0 \) and every \( \xi \in \partial B_1 \);

(ii) The function \( b \in C^{\alpha/2}_b(\mathbb{R}^d) \) satisfies \( \beta := \inf_{t \geq 0} b^{(0)}(t) > 0 \);

(iii) \( r > \gamma - 1 \) and \( \gamma \in \mathbb{R} \).

Let \( \delta \in (0, 2(r + 1 - \gamma)) \). Then every smooth and positive function \( V : \mathbb{R}^d \to \mathbb{R} \) with \( V(x) = e^{\delta|x|^\delta} \) for \( x \in \mathbb{R}^d \setminus B_1 \) satisfies Hypothesis 2.1(iii) for the above operator. Indeed, we have
\[
(A(t)V)(x) = \delta V(x)|x|^\delta \left[ |\delta|x|^\delta - 4 + (\delta - 2)|x|^{-4} \right] (1 + |x|^2)^\gamma Q^{(0)}(t, x, x)
+ \text{Tr}(Q^{(0)}(t, x))(1 + |x|^2)^r|x|^{-2} - b^{(0)}(t)(1 + |x|^2)^r h(x)
\leq \delta V(x)|x|^\delta h(x),
\]
for \( t \geq 0 \) and \( |x| \geq 1 \), where \( h(x) = c|x|^\delta(1 + |x|^2)^\gamma - \beta(1 + |x|^2)^r \) tends to \(-\infty\) as \( |x| \to +\infty \) and \( c > 0 \) is a constant depending on the bounds of \( Q^0 \). One
easily checks the other conditions in Hypothesis 2. Finally, Hypothesis 4 is satisfied if \( q_{ij} \in C^{\alpha/2}_{b}(\mathbb{R}^{+} \times B_{R}) \) for all \( R > 0 \), \( b \in C^{\alpha/2}(\mathbb{R}^{+}) \), and \( Q^{(0)}(t, \cdot) \) and \( b(t) \) converge to \( Q^{(0)}_{\infty} \) and \( b_{\infty} \), respectively, as \( t \to +\infty \).

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