Resummation Study on Decay $\rho \to \pi\pi$ in $U(2)_L \times U(2)_R$ Chiral Theory of Mesons

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We improve $O(p^4)$ calculation in $U(2)_L \times U(2)_R$ chiral theory of mesons by resummation calculation for vector mesons physics and restudy decay $\rho \to \pi\pi$. A complete and compact expression for $f_{\rho\pi\pi}(p^2)$ (up to $O(p^\infty)$) is obtained, from which an important non-perturbative conclusion is given based on convergence and unitarity consideration.

The Nambu-Jona-Lasinio (NJL) model [1] and its extensions are widely used to understand hadron physics (for recent reviews see e.g. [2]). There are various methods to parametrize the NJL version models, among which is the $U(N_f)_L \times U(N_f)_R$ chiral theory of mesons [3] with $N_f = 2$ or 3 (proposed by Li and called hereafter Li’s model). This model can be regarded as a realization of chiral symmetry, current algebra and vector meson dominance (VMD). It provides a unified description of pseudoscalar, vector, and axial-vector mesons, which are introduced as bound states of quark fields. The basic inputs of it are the cutoff $\Lambda$ (or $g$ in [3]) and constituent quark mass $m$ (related to quark condensate). This theory has been studied extensively [4–9], in particular it has been used recently [10] to analyze the data of $g$-2 of muon reported by CMD-2 group [11]. Hence Li’s model is a good phenomenological model.

So far Li’s model is only investigated up to $O(p^4)$ [3] in perturbative manner. This treatment is reasonable for chiral perturbation theory (ChPT) [12], because the typical energy there is much less than the energy scale of chiral symmetry spontaneously breaking (CSSB) $\Lambda_{\text{CSSB}} \sim 2\pi F_\pi \sim 1\text{GeV}$. This is not the case here because the typical energy $p$ of vector mesons is comparable with $\Lambda_{\text{CSSB}}$. For example, at $\rho$ energy scale, $p \sim m_\rho$ and the perturbation parameter $p/\Lambda_{\text{CSSB}} \sim 0.7$. Therefore, studies in ref. [3] should be improved by going beyond $O(p^4)$. But we know that, in usual perturbative treatment, it is difficult to deduce effective meson action at higher order. This is a paradox. As a solution to it, we develop a method to obtain the sum of all terms of the $p$-expansion [13]. We call it resummation study. In this paper, we want to use it to restudy typical decay $\rho \to \pi\pi$ in Li’s model. Other processes can be restudied in a similar way.

Before this study, let us outline procedures of perturbative calculation and our resummation calculation. Given a quark model which describes interactions of constituent quarks $q$ and mesons $M$ and satisfies chiral symmetry requirement, we can write the Lagrangian as $\mathcal{L} = \mathcal{L}(q, \bar{q}, M) = \bar{q}Dq$, where $D = D(M)$ is characteristic of the model. The first step of perturbative calculation is using the method of path integral to integrate out the quark fields:

$$e^i\int d^4x \mathcal{L}_{\text{eff}}(M) = \int [dq][\bar{q}d\bar{q}] e^i\int d^4x \bar{q}Dq.$$

$$1$$

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After functional integration, the effective action $S_{\text{eff}}(M)$ of mesons is formally obtained as $S_{\text{eff}}(M) = \ln \det D(M)$. To regularize this determinant, Schwinger’s proper time method [14] or heat kernel method [15] should be employed. This directly results in an expansion in $p$, and usually up to $O(p^4)$. Attempting to calculate terms of higher orders will encounter great difficulty.

On the other hand, to perform resummation study, we calculate effective meson action via loop effects of constituent quarks. This is equivalent to integrating out quarks in path integral, but via it we can obtain effects from all orders of $p$-expansion. Specifically, we divide $\mathcal{L}$ into free-field part $\mathcal{L}_0$ and interaction part $\mathcal{L}_1$, and turn to interaction picture. The effective action can be obtained as

$$e^{iS_{\text{eff}}} = <0|T_q e^{i \int d^4x \mathcal{L}}|0> = \sum_{n=1}^{\infty} i \int d^4p_1 d^4p_2 \cdot \cdot \cdot d^4p_n \delta^4(p_1 + p_2 + \cdot \cdot \cdot + p_n) \Pi_n(p_1, \cdot \cdot \cdot , p_n),$$

where $T_q$ is time-order product of constituent quark fields, $\Pi_n(p_1, \cdot \cdot \cdot , p_n)$ is one-loop effects of constituent quarks with $n$ external fields, $p_1, p_2, \cdot \cdot \cdot , p_n$ are their four-momentum. Getting rid of all disconnected diagrams, we have

$$S_{\text{eff}} = \sum_{n=1}^{\infty} S_n,$$

$$S_n = \int d^4p_1 d^4p_2 \cdot \cdot \cdot d^4p_n \delta^4(p_1 + p_2 + \cdot \cdot \cdot + p_n) \Pi_n^c(p_1, \cdot \cdot \cdot , p_n),$$

where $c$ denotes connected part. Obviously, in eq. (3), the effective action $S_{\text{eff}}$ is expanded in number of external vertex and expressed as integral over external momentum. Hereafter we shall call this method proper vertex expansion, and call $S_n$ $n$-point effective action. In terms of proper vertex expansion, the effective actions include informations from all orders of chiral expansion. That is, we can do resummation of momentum expansion by this method. That is what we need.

It is instructive to compare these two kinds of calculations. The perturbative calculation can deal with all kinds of meson interactions at low orders, while resummation calculation deals with specific process but can include informations of all orders. In the sense of $p$-expansion, resummation calculation is non-perturbative. As we shall see later, this character will give important result.

Having showed outlines of resummation calculation, we turn to Li’s model. Because we only focus on $\rho$ physics, the $\mathcal{N}_f = 2$ part of Li’s model, i.e., $U(2)_L \times U(2)_R$ chiral theory of mesons [3] suffices to study decay $\rho \rightarrow \pi \pi$. This Model is constructed through $U(2)_L \times U(2)_R$ chiral symmetry and minimum coupling principle, and the ingredients of it are quarks ($u$ and $d$), pseudoscalar mesons ($\pi$ and $\eta$ ($u$ and $d$ component)), vector mesons ($\rho$ and $\omega$), axial-vector mesons ($a_1$ and $f_1(1285)$), lepton, photon and $W$ bosons. For our purpose, we just write the relevant Lagrangian:

$$\mathcal{L} = \bar{q}(i\slashed{\partial} + \mathcal{A}\gamma_5 - mu(x))q + \frac{1}{4}m_0^2(<V_\mu V^\mu> + <A_\mu A^\mu>),$$

where $A_\mu = \tau^i a_\mu^i + f_\mu$, $V_\mu = \tau^i \rho^i_\mu + \omega_\mu$, $u = e^{i\Phi}\gamma_5$, $\Phi = \tau^i \pi^i + \eta$, and $<\cdot \cdot \cdot>$ denotes trace in flavor space. The quark part of this Lagrangian can be divided into two parts. The free-field part is $\mathcal{L}_0^q = \bar{q}(i\slashed{\partial} - m)q$, and the interaction part is

$$\mathcal{L}_1^q = \bar{q}(\mathcal{V} + \mathcal{A}\gamma_5 - im\Phi\gamma_5 + \frac{1}{2}m\Phi^2)q,$$

where terms involving more $\Phi$s has been omitted, because they have nothing to do with decay $\rho \rightarrow \pi \pi$. All subsequent calculations are performed at chiral limit.

![Diagram](attachment:image.png)

**FIG. 1.** Calculation of $<CD>$ terms ($C, D = V, A, \Phi$) from quark loops. a) Two-point diagram of quark loop. b) One-point diagram of quark loop for $<\Phi\Phi>$ term.
Before studying decay $\rho \rightarrow \pi\pi$, we have to do something to obtain physical fields. We should calculate effective actions for $<VV>, <AA>, <\Phi\Phi>, <VA>, <VF>$ and $<AF>$ through integration of two-point quark loops like Fig. 1a (where $C, D = V, A, \Phi$) and one-point quark loop Fig. 1b. These actions are calculated as

$$S^0_{\text{eff}} = \frac{iN_c}{2} \int \frac{d^4p d^4k}{(2\pi)^4} \left( <V\Phi(-p)> \operatorname{Tr}[\gamma^\mu S_F(k-p)\gamma^\nu S_F(k)] + <A\mu(p)A\nu(-p)> \operatorname{Tr}[\gamma^\mu \gamma_5 S_F(k-p)\gamma^\nu \gamma_5 S_F(k)] \right.$$  

$$- <\Phi(p)\Phi(-p)> \left( m^2 \operatorname{Tr}[S_F(k-p)\gamma_5 S_F(k)] + m \operatorname{Tr}[S_F(k)] \right)$$  

$$- 2i m <V\mu(p)\Phi(-p)> \operatorname{Tr}[\gamma^\mu S_F(k-p)\gamma_5 S_F(k)] - 2i m <A\mu(p)\Phi(-p)> \operatorname{Tr}[\gamma^\mu \gamma_5 S_F(k-p)\gamma_5 S_F(k)]$$  

$$+ 2 <V\mu(p)A\nu(-p)> \operatorname{Tr}[\gamma^\mu S_F(k-p)\gamma_5 S_F(k)] \right),$$

(6)

where, for $<\Phi\Phi>$ term, we have added contribution from one-point diagram (Fig. 1b) of quark loop. This is so because $\Phi$ is realized nonlinearly. After integrating quark loops, we get expressions including all terms up to $O(p^\infty)$. But we are not interested in high order terms now. Discarding them and adding the non-quark part of eq. (4), we obtain

$$S_{\text{eff}} = \frac{1}{4} \int \frac{d^4p}{(2\pi)^4} \left( <V\mu(p)V\nu(-p)> (g^2(p^\mu p^\nu - g^\mu\nu p^2) + m_0^2) \right.$$

$$+ <A\mu(p)A\nu(-p)> \left[ - g^2 \left( 1 - \frac{N_c}{6\pi^2 g^2} \right) g^\mu\nu p^2 + g^2 p^\mu p^\nu + (6g^2m^2 + m_0^2)g^\mu\nu \right]$$

$$+ <\Phi(p)\Phi(-p)> \left( \frac{3}{2}g^2m^2p^2 + <A\mu(p)\Phi(-p)> 6ig^2m^2p^\mu \right),$$

(7)

where the logarithmic divergence is absorbed by

$$\frac{3}{8}g^2 = \frac{N_c}{(4\pi)^{d/2}} \left( \frac{\mu^2}{m^2} \right)^{d/2} \Gamma \left( 2 - \frac{d}{2} \right), \quad (d = 4 - \epsilon)$$

(8)

and the quadratic divergence in $<\Phi\Phi>$ term has been reduced to logarithmic divergence by the identity $\Gamma(1-d/2) + \Gamma(2-d/2) = -1 + O(\epsilon)$. The first and second terms of eq. (7) indicate needs for rescalings of $V_\mu$ and $A_\mu$. (We see that axial-vector $A_\mu$ does not satisfy gauge invariance. Because $\partial_\mu A^\mu = 0$ when $A_\mu$ is on-shell, we shall ignore the $p^\mu p^\nu$ term when redefining $A_\mu$.) The non-vanishing $<AF>$ term indicates that there is a mixing between $A_\mu$ and $\partial_\mu \Phi$, which should be erased by shift of $A_\mu$. Therefore, the redefinitions of $V_\mu$ and $A_\mu$ are

$$V_\mu \rightarrow \frac{V_\mu}{g}, \quad A_\mu \rightarrow \frac{A_\mu}{g\sqrt{1 - N_c/6\pi^2 g^2}} - \frac{c}{g} \partial_\mu \Phi,$$

(9)

and we can obtain in passing the mass formula for them:

$$m_\mu^2 = \frac{m_0^2}{g^2}, \quad (1 - \frac{N_c}{6\pi^2 g^2})m_\mu^2 = 6m^2 + m_\mu^2.$$  

(10)

After redefining $A_\mu$ and cancelling the mixing, and then making the kinetic term of $\Phi$ in standard form, we obtain

$$c = \frac{3gm^2}{6m^2 + m_\mu^2}, \quad \frac{1}{4}c^2m_\mu^2 + \frac{3}{8} \left( 1 - \frac{2c}{g} \right) g^2m^2 = F_\Phi^2/16,$$

or

$$c = \frac{F_\Phi^2}{2gm_\mu^2}, \quad 6 \left( 1 - \frac{2c}{g} \right) g^2m^2 = F_\Phi^2,$$

(11)

and rescaling for $\Phi \rightarrow 2\Phi/F_\Phi$. Eq.s (9)-(11) are exactly those in [3].

![Diagram](image-url)
Fig. 2. Quark loops for vertex $\rho \to \pi \pi$. a) Two-point diagram. b) Three-point diagram.

Now we can study decay $\rho \to \pi \pi$. For vertex $\rho - \pi \pi$, we need to calculate two Feynman diagrams of quark loop (see Fig. 2). The relevant Lagrangians (after redefinitions) for the first and second ones are

\[
\mathcal{L}_1^{(2)} = \frac{1}{g} V (2m/F_F^2) q, \quad \mathcal{L}_1^{(3)} = \frac{1}{g} (V - \frac{2c}{gF_F} \rho \Phi \gamma_5 - \frac{2im}{F_F^2} \Phi \gamma_5) q
\]

respectively. For the first one, $\mathcal{L}_1^{(2)}$ gives integration of the quark loop as $\int d^4k \text{Tr} [\gamma^\mu S_F(k-p)S_F(k)] \propto p^\mu$ ($p$ is momentum of vector field $V$). Because $\partial_\mu V^\mu = 0$ when $V_\mu$ is on-shell, contribution from the first diagram vanishes: $S_2 = 0$. For the second diagram, the Lagrangian $\mathcal{L}_1^{(3)}$ determines that the effective action for this three-point diagram is calculated as

\[
S_3 = \frac{4iN_c}{gF_F^2} \int \frac{d^4q}{(2\pi)^4} \left[ d^4q_1 d^4q_2 \right] < V_\mu(p) \Phi(q_1) \Phi(q_2) > \int \frac{d^4k}{(2\pi)^4} \left( \frac{c^2}{g} q_{1\mu} q_{2\mu} \text{Tr}[\gamma^\mu S_F(k + q_1) \gamma^\nu S_F(k) \gamma^\rho S_F(k - q_2)] + m^2 \text{Tr}[\gamma^\mu S_F(k + q_1) \gamma^5 S_F(k) \gamma^5 S_F(k - q_2)] \right) - \frac{c m}{g} q_{1\nu} \text{Tr}[\gamma^\mu S_F(k + q_1) \gamma^5 S_F(k) \gamma^5 \gamma^5 S_F(k - q_2)] - \frac{c m}{g} q_{2\nu} \text{Tr}[\gamma^\mu S_F(k + q_1) \gamma^5 S_F(k) \gamma^5 \gamma^5 S_F(k - q_2)].
\]

After integrations of quark loops and considerations of $\partial_\mu V^\mu = 0$ and chiral limit $q_1^2 = q_2^2 = m_{\pi}^2 = 0$, the effective action for vertex $\rho - \pi \pi$ becomes

\[
S_{\rho \pi \pi} = \int \frac{d^4q d^4p}{(2\pi)^4} f_{\rho \pi \pi}(p^2)(-p - q) \pi^k(q)(-iq^\mu) f_{\rho \pi \pi}(p^2),
\]

where

\[
f_{\rho \pi \pi}(p^2) = \frac{2}{3\pi^2 g F_F^2} \left[ m^2 \left( 18(1 - 2c/g) \pi^2 g^2 - (2 - 3c/g) N_c \right) - p^2 (c/g)^2 (6\pi^2 g^2 - N_c) \right] + \frac{2N_c}{\pi^2 g F_F^2} \int_0^1 dx \int_0^1 dy \left[ m^2 x(1-x)(1-y) \left( m^2 + p^2 x(1-x)(1-y) \right) \right] + \left[ m^2 (1 - 2c/g + xy) \right] + p^2 \left( x(1-x)(1-y)(1+xy) - c/g(1+2x-2x^2-3xy+2x^2y) + c^2/g^2(1-xy) \right) \right]
\]

\[
\left[ m^2 (1 - 4c/g + 3xy) - c^2/p^2 + c^2/g^2(1-xy) \right]
\]

\[
12(N_c + 3g^2\pi^2) - (24c/g)(2N_c + 4g^2\pi^2) + 40N_c(c/g)^2 m^2 - c^2(10N_c + 36g^2\pi^2) / 9\pi^2 g^2 F_F^2 p^2
\]

\[
- 4N_c \left( 3 - 12c/g + 10(c/g)^2 \right) m^2 - 4N_c (c/g)^2 p^2 \sqrt{4m^2 - p^2} \sqrt{p^2 \arctg \left( 4m^2 - p^2 \right) / p^2}.
\]

The factor $f_{\rho \pi \pi}(p^2)$ is complicated, expansion of it in $p$ up to $O(p^6)$ (corresponding to expansion of $S_{\rho \pi \pi}$ up to $O(p^8)$) is

\[
f_{\rho \pi \pi}(p^2) = \frac{2}{g} \left[ 1 + \frac{N_c (1 - 2c/g)^2 - 12c^2\pi^2}{6\pi^2 g F_F^2} p^2 + \frac{(1 - 4c/g) N_c}{60\pi^2 m^2 g^2 F_F^2} p^4 + \frac{(1 - 4c/g + (c/g)^2) N_c}{420\pi^2 m^4 g^2 F_F^2} p^6 + O(p^8) \right],
\]

where condition (11) has been used. As we can see, the sum of the first two terms in this series is just $f_{\rho \pi \pi}$ in ref. [3] when $N_c = 3$ and $p^2 = m_{\pi}^2$ is set.

Moreover, expression (15) shows that series (16) is in fact an expansion in dimensionless quantity $\overline{p^2} = p^2 / (4m^2)$; $f_{\rho \pi \pi}(p^2) = 2/g + c_1 p^2 + c_2 p^4 + c_3 p^6 + \cdots$. Convergence of this series means $\overline{p^2} < 1$ or $p^2 < 4m^2$, which at $\rho$ energy scale means the constituent quark mass $m > m_\rho/2 \approx 385$ MeV. The same conclusion can also be obtained by unitarity consideration. Unitarity of the theory demands that, at leading order of $1/N_c$ expansion, there should be no imaginary parts in transition amplitude of meson decay [16]. The expression (15) for $f_{\rho \pi \pi}$ is just at the leading order of $1/N_c$ expansion, therefore, it must be real. Thus $\overline{p^2} < 4m^2$ must be ensured in eq. (15), which means $m > m_\rho/2$ at $\rho$
energy scale. We should point out that this conclusion is obtained from non-perturbative expression (15) which is characteristic of resummation study.

The difference between our result of $m > 385\text{MeV}$ and the one of $m = 300\text{MeV}$ in ref. [3] is understandable. The parametrization $m = 300\text{MeV}$ in ref. [3] is consistent with its $O(p^4)$ calculation, and the phenomenology studies are good there. Now that we have resummed all terms in $p$-expansion, using convergence or unitarity analysis, we have that $m > 385\text{MeV}$. In fact, such high constituent quark mass is also reasonable from another aspect. When fitting the hadron spectra, we have to adopt high constituent quark mass, or we fail to account for the observed masses of scalar nonet [17]. It is argued in ref. [17] that high constituent quark mass might be a consequence of lack of confinement of the model and could be avoided if we knew how to add confining forces to the model. This is suggested by the results of constituent quark models which use confining interactions [18].

To conclude, we improve perturbative calculation in Li’s model by resummation study. We first illustrate proper vertex expansion method in chiral quark model, and then use it to perform resummation study on decay $\rho \to \pi\pi$. The complete and compact expression of $f_{\rho\pi\pi}(p^2)$ (up to $O(p^\infty)$) has been derived, from which the explicit expression of effective action $S_{\rho\pi\pi}$ up to $O(p^8)$ is easily obtained. We have seen that this method of resummation study is a powerful method to catch informations from all orders of $p$-expansion, from which we obtain non-perturbative conclusion that $m > m_{\rho}/2$.

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