Constants of motion for fractional action-like variational problems

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Abstract

We extend Noether's symmetry theorem to the fractional Riemann-Liouville integral functionals of the calculus of variations recently introduced by El-Nabulsi.

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1 Introduction

The concept of symmetry plays an important role both in Physics and Mathematics. Symmetries are described by transformations of the system, which result in the same object after the transformation is carried out. They are described mathematically by parameter groups of transformations. Their importance ranges from fundamental and theoretical aspects to concrete applications, having profound implications in the dynamical behavior of the systems, and in their basic qualitative properties.

Another fundamental notion in Physics and Mathematics is the one of constant of motion. Typical application of the constants of motion in the calculus of variations is to reduce the number of degrees of freedom, thus reducing the problems to a lower dimension, facilitating the integration of the differential equations given by the necessary optimality conditions.

Emmy Noether was the first who proved, in 1918, that the notions of symmetry and constant of motion are connected: when a system exhibits a symmetry, then a constant of motion can be obtained. One of the most important

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and well known illustrations of this deep and rich relation, is given by the conservation of energy in Mechanics: the autonomous Lagrangian $L(q, \dot{q})$, correspondent to a mechanical system of conservative points, is invariant under time-translations (time-homogeneity symmetry), and

$$L(q, \dot{q}) - \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \cdot \dot{q} \equiv \text{constant}$$

follows from Noether’s theorem, i.e., the total energy of a conservative system always remain constant in time, “it cannot be created or destroyed, but only transferred from one form into another”. Expression (1) is valid along all the Euler-Lagrange extremals $q$ of an autonomous problem of the calculus of variations. The constant of motion (1) is known in the calculus of variations as the 2nd Erdmann necessary condition; in concrete applications, it gains different interpretations: conservation of energy in Mechanics; income–wealth law in Economics; first law of Thermodynamics; etc. The literature on Noether’s theorem is vast, and many extensions of the classical results of Emmy Noether are now available in the literature (see e.g. [13, 14] and references therein). Here we remark that constants of motion appear naturally in closed systems.

It turns out that in practical terms closed systems do not exist: forces that do not store energy, so-called nonconservative or dissipative forces, are always present in real systems. Friction is an example of a nonconservative force. Any friction-type force, like air resistance, is a nonconservative force. Nonconservative forces remove energy from the systems and, as a consequence, the constant of motion (1) is broken. This explains, for instance, why the innumerable “perpetual motion machines” that have been proposed fail. In presence of external nonconservative forces, Noether’s theorem and respective constants of motion cease to be valid. However, it is still possible to obtain a Noether-type theorem which covers both conservative (closed system) and nonconservative cases [3, 6]. Roughly speaking, one can prove that Noether’s conservation laws are still valid if a new term, involving the nonconservative forces, is added to the standard constants of motion.

The study of fractional problems of the calculus of variations and respective Euler-Lagrange type equations is a subject of strong current research because of its numerous applications: see e.g. [1, 2, 4, 5, 8, 9, 10, 11, 12]. F. Riewe [11, 12] obtained a version of the Euler-Lagrange equations for problems of the calculus of variations with fractional derivatives, that combines the conservative and non-conservative cases. In 2002 O. Agrawal proved a formulation for variational problems with right and left fractional derivatives in the Riemann-Liouville sense [1]. Then these Euler-Lagrange equations were used by D. Baleanu and T. Avkar to investigate problems with Lagrangians which are linear on the velocities [2]. In [8, 9] fractional problems of the calculus of variations with symmetric fractional derivatives are considered and correspondent Euler-Lagrange equations obtained, using both Lagrangian and Hamiltonian formalisms. In all the above mentioned studies, Euler-Lagrange equations depend on left and
right fractional derivatives, even when the problem depend only on one type of them. In [10] problems depending on symmetric derivatives are considered for which Euler-Lagrange equations include only the derivatives that appear in the formulation of the problem. In [4, 5] Riemann-Liouville fractional integral functionals, depending on a parameter \( \alpha \) but not on fractional-order derivatives of order \( \alpha \), are introduced and respective fractional Euler-Lagrange type equations obtained.

A Noether-type theorem for problems of the calculus of variations with fractional-order derivatives of order \( \alpha \) is given in [7]. Here we use the results of El-Nabulsi [4, 5] to prove a nonconservative Noether’s theorem in the new fractional action-like framework.

## 2 Fractional action-like Noether’s theorem

We consider the fundamental problem of the calculus of variations with Riemann-Liouville fractional integral, as considered by El-Nabulsi [4, 5]:

\[
I[q(\cdot)] = \frac{1}{\Gamma(\alpha)} \int_a^b L(\theta, q(\theta), \dot{q}(\theta)) (t - \theta)^{\alpha - 1} \, d\theta \rightarrow \min, \tag{2}
\]

under given boundary conditions \( q(a) = q_a \) and \( q(b) = q_b \), where \( \dot{q} = \frac{dq}{d\theta} \), \( \Gamma \) is the Euler gamma function, \( 0 < \alpha \leq 1 \), \( \theta \) is the intrinsic time, \( t \) is the observer time, \( t \neq \theta \), and the Lagrangian \( L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) is a \( C^2 \) function with respect to its arguments. We will denote by \( \partial_i L \) the partial derivative of \( L \) with respect to the \( i \)-th argument, \( i = 1, 2, 3 \). Admissible functions \( q(\cdot) \) are assumed to be \( C^2 \).

**Theorem 1 (cf. [4]).** if \( q \) is a minimizer of problem (2), then \( q \) satisfies the following Euler-Lagrange equation:

\[
\partial_2 L (\theta, q(\theta), \dot{q}(\theta)) - \frac{d}{d\theta} \partial_3 L (\theta, q(\theta), \dot{q}(\theta)) = \frac{1 - \alpha}{t - \theta} \partial_1 L (\theta, q(\theta), \dot{q}(\theta)). \tag{3}
\]

We now introduce the following definition of variational quasi-invariance up to a gauge term (cf. [13]).

**Definition 2 (quasi-invariance of (2) up to a gauge term \( \Lambda \)).** Functional (2) is said to be quasi-invariant under the infinitesimal \( \varepsilon \)-parameter transformations

\[
\begin{cases}
\bar{\theta} = \theta + \varepsilon \tau(\theta, q) + o(\varepsilon) \\
\bar{q}(\bar{\theta}) = q(\theta) + \varepsilon \xi(\theta, q) + o(\varepsilon)
\end{cases} \tag{4}
\]
up to the gauge term $\Lambda$ if, and only if,

$$L(\dot{\theta}, \dot{q}(\theta), \dot{q}'(\theta)) (t - \theta)^{\alpha - 1} \frac{d\theta}{d\tau} = L(\theta, q(\theta), \dot{q}(\theta)) (t - \theta)^{\alpha - 1} + \varepsilon(t - \theta)^{\alpha - 1} \frac{d\Lambda}{d\theta} (\theta, q(\theta), \dot{q}(\theta)) + o(\varepsilon). \quad (5)$$

**Lemma 3 (necessary and sufficient condition for quasi-invariance).** If functional (2) is quasi-invariant up to $\Lambda$ under the infinitesimal transformations (1), then

$$\partial_1 L(\theta, q, \dot{q}) \tau + \partial_2 L(\theta, q, \dot{q}) \cdot \xi + \partial_3 L(\theta, q, \dot{q}) \cdot (\dot{\xi} - \ddot{q} \tau)$$

$$+ L(\theta, q, \dot{q}) \left( \dot{\tau} + \frac{1 - \alpha}{t - \theta} \tau \right) = \dot{\Lambda}(\theta, q, \dot{q}). \quad (6)$$

**Proof.** Equality (5) is equivalent to

$$\left[ L\left( \theta + \varepsilon \tau + o(\varepsilon), q + \varepsilon \xi + o(\varepsilon), \dot{q} + \varepsilon \dot{\xi} + o(\varepsilon) \right) \right] (t - \theta - \varepsilon \tau - o(\varepsilon))^{\alpha - 1} \left( 1 + \varepsilon \dot{\tau} + o(\varepsilon) \right)
$$

$$= L(\theta, q, \dot{q}) (t - \theta)^{\alpha - 1} + \varepsilon(t - \theta)^{\alpha - 1} \frac{d\Lambda}{d\theta} (\theta, q, \dot{q}) + o(\varepsilon). \quad (7)$$

Equation (6) is obtained differentiating both sides of equality (5) with respect to $\varepsilon$ and then putting $\varepsilon = 0$. \qed

**Definition 4 (constant of motion).** A quantity $C(\theta, q(\theta), \dot{q}(\theta)), \theta \in [a, b]$, is said to be a constant of motion if, and only if, $\frac{d}{d\theta} C(\theta, q(\theta), \dot{q}(\theta)) = 0$ for all the solutions $q$ of the Euler-Lagrange equation (3).

**Theorem 5 (Noether’s theorem).** If the fractional integral (2) is quasi-invariant up to $\Lambda$, in the sense of Definition (2) and functions $\tau(\theta, q)$ and $\xi(\theta, q)$ satisfy the condition

$$L(\theta, q, \dot{q}) \tau = -\partial_3 L(\theta, q, \dot{q}) \cdot (\dot{\xi} - \ddot{q} \tau), \quad (8)$$

then

$$\partial_3 L(\theta, q, \dot{q}) \cdot \xi(\theta, q) + [L(\theta, q, \dot{q}) - \partial_3 L(\theta, q, \dot{q}) \cdot \dot{q}] \tau(\theta, q) - \Lambda(\theta, q, \dot{q}) \quad (9)$$

is a constant of motion.

**Remark 6.** Under our hypothesis (5) the necessary and sufficient condition of quasi-invariance (6) is reduced to

$$\partial_1 L(\theta, q, \dot{q}) \tau + \partial_2 L(\theta, q, \dot{q}) \cdot \xi + \partial_3 L(\theta, q, \dot{q}) \cdot (\dot{\xi} - \ddot{q} \tau)
$$

$$+ L(\theta, q, \dot{q}) \dot{\tau} - \frac{1 - \alpha}{t - \theta} \partial_3 L(\theta, q, \dot{q}) \cdot (\dot{\xi} - \ddot{q} \tau) = \dot{\Lambda}(\theta, q, \dot{q}). \quad (10)$$
Conditions (8) and (10) correspond to the generalized equations of Noether-Bessel-Hagen of a non-conservative mechanical system [3].

Proof. We can write (10) in the form
\[
\left[ \partial_1 L (\theta, q, \dot{q}) + \frac{1 - \alpha}{t - \theta} \partial_3 L (\theta, q, \dot{q}) \cdot \dot{q} \right] \tau + \left[ L (\theta, q, \dot{q}) - \partial_3 L (\theta, q, \dot{q}) \cdot \dot{q} \right] \dot{\tau} + \left[ \partial_2 L (\theta, q, \dot{q}) - \frac{1 - \alpha}{t - \theta} \partial_3 L (\theta, q, \dot{q}) \right] \cdot \xi + \partial_3 L (\theta, q, \dot{q}) \cdot \dot{\xi} = 0. \quad (11)
\]

Using the Euler-Lagrange equation (3) equality (11) is equivalent to
\[
\frac{d}{d\theta} \left[ L (\theta, q, \dot{q}) - \partial_3 L (\theta, q, \dot{q}) \cdot \dot{q} \right] \tau + \left[ L (\theta, q, \dot{q}) - \partial_3 L (\theta, q, \dot{q}) \cdot \dot{q} \right] \dot{\tau} + \frac{d}{d\theta} \left[ \partial_3 L (\theta, q, \dot{q}) \right] \cdot \xi + \partial_3 L (\theta, q, \dot{q}) \cdot \dot{\xi} - \dot{\Lambda} = 0 \quad (12)
\]

and the intended conclusion follows:
\[
\frac{d}{d\theta} \left[ \partial_3 L (\theta, q, \dot{q}) \cdot \xi + (L (\theta, q, \dot{q}) - \partial_3 L (\theta, q, \dot{q}) \cdot \dot{q}) \tau - \Lambda (\theta, q, \dot{q}) \right] = 0.
\]

3 Examples

In [5, §4] El-Nabulsi remarks that conservation of momentum when \( L \) is not a function of \( q \) or conservation of energy when \( L \) has no explicit dependence on time \( \theta \) are no more true for a fractional order of integration \( \alpha, \alpha \neq 1 \). As we shall see now, these facts are a trivial consequence of our Theorem 5. Moreover, our Noether’s theorem gives new explicit formulas for the fractional constants of motion. For the particular case \( \alpha = 1 \) we recover the classical constants of motion of momentum and energy.

Let us first consider an arbitrary fractional action-like problem (2) with an autonomous \( L: L (\theta, q, \dot{q}) = L (q, \dot{q}) \). In this case \( \partial_1 L = 0 \), and it is a simple exercise to check that (10) is satisfied with \( \tau = 1, \xi = 0 \) and \( \Lambda \) given by
\[
\dot{\Lambda} = \frac{1 - \alpha}{t - \theta} \frac{\partial L}{\partial \dot{q}} \cdot \dot{q}.
\]

It follows from our Noether’s theorem (Theorem 5) that
\[
L (q, \dot{q}) - \frac{\partial L}{\partial \dot{q}} (q, \dot{q}) \cdot \dot{q} - (1 - \alpha) \int_1^t \frac{1}{t - \theta} \frac{\partial L}{\partial \dot{q}} (q, \dot{q}) \cdot \dot{q} d\theta \equiv \text{constant}. \quad (12)
\]

In the classical framework \( \alpha = 1 \) and we then get from our expression (12) the well known constant of motion (11), which corresponds in mechanics to conservation of energy.
When \( L \) is not a function of \( q \) one has \( \frac{\partial L}{\partial q} = 0 \) and (10) holds true with \( \tau = 0, \xi = 1 \) and \( \Lambda \) given by

\[
\dot{\Lambda} = -\frac{1 - \alpha}{t - \theta} \frac{\partial L}{\partial \dot{q}} (\theta, \dot{q}) .
\]

The constant of motion (9) takes the form

\[
\frac{\partial L}{\partial \dot{q}} (\theta, \dot{q}) + (1 - \alpha) \int_{1}^{t} \frac{1}{t - \theta} \frac{\partial L}{\partial \dot{q}} (\theta, \dot{q}) d\theta .
\] (13)

For \( \alpha = 1 \) implies conservation of momentum: \( \frac{\partial L}{\partial \dot{q}} = \text{const.} \)

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