QUANTIFIER ELIMINATION ON SOME PSEUDO-ALGEBRAICALLY CLOSED VALUED FIELDS

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Abstract. Adjoining to the language of rings the function symbols for splitting coefficients, the function symbols for relative $p$-coordinate functions, and the division predicate for a valuation, some theories of pseudo-algebraically closed non-trivially valued fields admit quantifier elimination. It is also shown that in the same language the theory of pseudo-algebraically closed non-trivially valued fields of a given exponent of imperfection does not admit quantifier elimination, due to Galois theoretic obstructions.

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The study of pseudo-algebraically closed fields originates from Ax’s study on the elementary theory of finite fields, [Ax, 1968], in the 1960s. A class of fields were isolated by Ax, called pseudo-finite fields. It was proved that a pseudo-finite field $K$ has the property that every absolutely irreducible affine algebraic set defined over $K$ has a $K$-rational point. This property was later used to define pseudo-algebraically closed fields. Many known results on pseudo-algebraically closed fields are documented in the book [Fried and Jarden, 2008]. Examples of pseudo-algebraically closed fields include separably closed fields, pseudo-finite fields and some Hilbertian fields. Global fields and non-trivial purely transcendental extensions over any field are Hilbertian fields. It was proved in [Jarden, 1972] that if $K$ is a countable Hilbertian field, then with respect to the Haar measure, for almost all tuples $(\sigma_1, \ldots, \sigma_e) \in \text{Gal}(K)^e$, the fixed field of the tuple is a pseudo-algebraically closed field. A lot is known on the model theory of pseudo-algebraically closed fields. A characterization of the elementary equivalence relation among pseudo-algebraically closed fields in terms of Galois groups was given in [Cherlin et al., 1982]. The model theory of pseudo-algebraically closed fields turns out to depend very much on the Galois groups. Using an important Embedding Lemma [Jarden and Kiehne, 1975], it was possible to prove that certain class of pseudo-algebraically closed fields admits quantifier elimination in a suitably nice language. While the theory of pseudo-algebraically closed fields itself in the same language does not admit quantifier elimination due to some Galois theoretic obstructions, it was however shown that the theory itself admits a new kind of ‘quantifier elimination’ using the so-called Galois stratifications. Upon a close study of the so-called ‘comodel theory’ over these pseudo-algebraically closed fields, in [Cherlin et al., 1982], it was also proved that the theory of perfect Frobenius fields admits quantifier elimination in a so-called ‘Galois formalism’.

In this article, we are interested in the model theory of pseudo-algebraically closed fields endowed with a non-trivial valuation. It was proved in [Kollár, 2007] that if $K$ is pseudo-algebraically closed, $V$ is any valuation on the algebraic closure $K^{\text{alg}}$ of $K$, and $X$ is a geometrically integral $K$-variety, then $X(K)$ is dense in $X(K^{\text{alg}})$ with respect to the topology induced by $V$. This provides a critical tool in the study of pseudo-algebraically closed valued fields. It enables us to prove the Valuation Theoretic Embedding Lemma and the quantifier elimination results in this article.
The strategy for proving the quantifier elimination results in this article turns out to be almost similar to the field theoretic case; one only needs to take a little care of the valuations. The main result in this article is in Section 7. Since that section is not very long, we will not give another summary here. But roughly speaking, it says that the theory of non-trivially valued Frobenius fields with a fixed exponent of imperfection has quantifier elimination in a sufficiently nice language. Similar to the field theoretic case, the same Galois theoretic obstructions render the quantifier elimination for the theory of pseudo-algebraically closed valued fields with a fixed exponent of imperfection impossible (see Section 5). It is also possible to obtain a similar characterization (Section 8) of the elementary equivalence relation of pseudo-algebraically closed valued fields in terms of the ‘comodel theory’ as in [Cherlin et al., 1982].

The motivation of the author to study pseudo-algebraically closed valued fields is an attempt to understand how far the first order valued-field structure of a pseudo-algebraically closed valued field is from the first order field structure. However, this topic will be postponed for another occasion.

2. Preliminaries

In this section, we recall some definitions and facts that will be will in this article. All the results in this sections are known. This section is only meant to serve as a clarification of related terminology, not as a thorough review of the related definitions or results.

2.1. Field extensions. Given a field $K$, we use $K^{\text{alg}}$ to denote the algebraic closure of $K$, $K^{\text{sep}}$ the separable closure of $K$. Unless otherwise mentioned, the characteristic of a field $K$ is denoted by the letter $p$, which could be either 0 or a positive prime integer. For an integral domain $A$, we use $A^\times$ to denote the set of all invertible elements in $A$. If $K$ is a field, the absolute Galois group of $K$ is denoted by $\text{Gal}(K) := \text{Gal}(K^{\text{sep}}/K)$.

**Definition 2.1.** Suppose that two subfields of $L$, denoted by $E$ and $F$, contain a common subfield $K$. Then $E$ is said to be **linearly disjoint** over $K$ if any finitely many linearly independent elements $x_1, \ldots, x_n \in E$ over $K$ remains linearly independent over $F$. we say that $E$ is **algebraically independent** from $F$ over $K$, or that $E$ and $F$ are **algebraically independent** over $K$, if any finitely many algebraically independent elements $x_1, \ldots, x_n \in E$ over $K$ remains algebraically independent over $F$. 
Definition 2.2. A field extension \(L/K\) is said to be **separable** if \(L\) is linearly disjoint from \(K^{1/p^\infty}\). A field extension \(L/K\) is said to be **regular** if \(L\) is linearly disjoint from \(K^{\text{alg}}\). It is well-known that \(L/K\) is regular if and only if \(L/K\) is separable and \(K\) is relatively algebraically closed in \(L\).

Definition 2.3. Suppose that \(K\) is a field of positive characteristic \(p\). Denote by \(K^p\) the subfield of \(p\)-th powers of \(K\). Then \([K : K^p]\) is either infinite or \(p^n\) for some \(n \in \omega\), called the **degree of imperfection** of \(K\). The **exponent of imperfection** of \(K\) is defined to be \(n\), if \([K : K^p] = n\), or \(\infty\), if \([K : K^p] = \infty\). Given finitely many elements \(x_1, \ldots, x_n \in K\), an element of the form

\[
\prod_{j=1}^{n} x_j^{i(j)}, \quad i : n \to p,
\]

is called of **\(p\)-monomial** in \(x_1, \ldots, x_n\); \(x_1, \ldots, x_n\) are said to be **\(p\)-independent** over a subfield \(k\) if the set of all \(p\)-monomials are linearly independent over \(k\). If \(x_1, \ldots, x_n\) are \(p\)-independent over \(K^p\), then we also say that \(x_1, \ldots, x_n\) are \(p\)-independent in \(K\). A subset of \(K\) is called a **\(p\)-basis** if it is maximally \(p\)-independent in \(K\).

It is well-known that \([K : K^p] = p^n\) if and only if \(K\) has a \(p\)-basis of cardinality \(n\). Furthermore, if \(\{x_1, \ldots, x_n\}\) is a \(p\)-basis of \(K\), then \(K = K^p(x_1, \ldots, x_n)\); every element \(y \in K\) can be uniquely written as a linearly combination of \(p\)-monomials in \(x_1, \ldots, x_n\) over \(K^p\).

Definition 2.4. If \(K\) is a field of characteristic 0, then \(K\) is perfect. The degree of imperfection of \(K\) is defined to be 1 and the exponent of imperfection of \(K\) is defined to be 0.

It is well-known that a field extension \(L/K\) is separable if and only if every \(p\)-independent set of \(K\) remains \(p\)-independent in \(L\), if and only if there a \(p\)-basis of \(K\) remains \(p\)-independent in \(L\).

Fact 2.5 (Lemma 2.7.3 of [Fried and Jarden, 2008]). Separable algebraic extensions preserve degrees of imperfection.

Definition 2.6 (see [Srour, 1986]). Suppose that \(K\) is a field of positive characteristic \(p\). We define the **relative \(p\)-coordinate functions** \(\lambda_{n,i}(x; y_1, \ldots, y_n)\) as follows: if \(y_1, \ldots, y_n\) are not \(p\)-independent in \(K\) or \(x \notin K^p(y_1, \ldots, y_n)\), then \(\lambda_{n,i}(x; y_1, \ldots, y_n) = 0\), otherwise \(\lambda_{n,i}(x; y_1, \ldots, y_n)\) is the unique \(i\)-th coordinate of \(x\) with respect \(y_1, \ldots, y_n\) when \(x\) is written as a linearly combination of \(p\)-monomials in \(y_1, \ldots, y_n\) over \(K^p\).

2.2. Valued fields.
Definition 2.7. Given a field $K$, a subring $V$ is called a valuation or a valuation ring on $K$ if for all $x \in K^\times$, either $x \in V$ or $x^{-1} \in V$. A valued field is a field $K$ with a distinguished valuation ring $V$, usually denoted by the pair $(K,V)$. For a valued field $(K,V)$, we use the corresponding lower case letter $v$ to denote the valuation map associated to $V$, and use $K_v$ to denote the residue field of $(K,V)$.

Fact 2.8 (Theorem 3.1.2 of [Engler and Prestel, 2005]). Let $L/K$ be a field extension, $V$ a valuation ring on $K$. Then there exists a valuation ring $W$ on $L$ extending $K$, i.e. $W \cap K = V$.

Fact 2.9 (Lemma 3.2.8 of [Engler and Prestel, 2005]). Suppose that $L/K$ is an algebraic extension, $V$ a valuation ring on $K$. Suppose that $W_1$ and $W_2$ are two extensions of $V$ to $L$. If $W_1 \subseteq W_2$ then $W_1 = W_2$.

Fact 2.10 (Theorem 3.2.15 of [Engler and Prestel, 2005]). Suppose that $(L,W_1)$ and $(L,W_2)$ are valued-field extensions of a valued field $(K,V)$ where $L/K$ is a normal field extension. Then there exists $\sigma \in \text{Aut}(L/K)$ such that $\sigma(W_1) = W_2$.

Fact 2.11 (Lemma 3.3.2 of [Engler and Prestel, 2005]). Suppose that $L/K$ is a Galois extension of degree $n$. Suppose that $V$ is a valuation on $K$ and $V_1, \ldots, V_r$ are all the extensions of $V$ to $L$. All the ramification indexes $e(V_i/V)$ are equal to a number $e$ and all the residue degrees $f(V_i/V)$ are also equal to a number $f$. Furthermore, we have $n = ref$.

Fact 2.12 (Theorem 3.2.4 of [Engler and Prestel, 2005]). Suppose that $(L,W)$ is a valued field extension of $(K,V)$ with $L/K$ algebraic. Then $Lw$ is an algebraic extension of $Kv$ and $wL^\times$ is contained in the divisible hull of $vK^\times$.

Fact 2.13 (Theorem 3.2.11 of [Engler and Prestel, 2005]). Suppose that $K$ is separably closed and non-trivially valued, then its residue field is algebraically closed and its value group is divisible.

2.3. Profinite groups. Given a field $K$, the absolute Galois group $\text{Gal}(K) := \text{Gal}(K^{\text{sep}}/K)$ of $K$ is naturally a profinite group. There is also a version of Galois correspondence for the Galois extension $K^{\text{sep}}/K$. For a positive integer, we use $\hat{F}_e$ to denote the free profinite group of rank $e$, and $\hat{F}_{\omega}$ to denote the free profinite group of rank...
For a profinite group $G$, we use $\text{Im}(G)$ to denote the set of all isomorphism types of finite quotients of $G$.

**Convention 2.14.** Unless otherwise mentioned, morphisms between profinite finite groups in this article are always assumed to be continuous.

**Definition 2.15** (see [Ribes and Zalesskii, 2010]). Let $G_i, i = 1, \ldots, n$, be a finite collection of profinite groups. A **free profinite product** of these groups consists of a profinite group $G$ and continuous homomorphisms $\varphi_i : G_i \to G, i = 1, \ldots, n$, satisfying the following universal property:

$$
\begin{array}{ccc}
G & \xrightarrow{\psi} & K \\
\phantom{G} & \downarrow{\psi_i} & \downarrow{\psi} \\
G_i & \xrightarrow{\phi_i} & K
\end{array}
$$

for any profinite group $K$ and any continuous homomorphisms $\psi_i : G_i \to K, i = 1, \ldots, n$, there is a unique continuous homomorphism $\psi : G \to K$ such that $\phi_i = \psi \varphi_i$ for all $i = 1, \ldots, n$. The free profinite product thus defined always exists and is unique up to isomorphism, usually denoted by

$$
G = \prod_{i=1}^{n} G_i.
$$

**Fact 2.16** (Corollary 9.1.4 of [Ribes and Zalesskii, 2010]). Let $G_1, \ldots, G_n$ be profinite groups and $G = G_1 \prod \cdots \prod G_n$ be their free profinite product. Then the following conclusions hold:

1. the natural homomorphisms $\varphi_j : G_j \to \prod_{i=1}^{n} G_i, j = 1, \ldots, n$ are monomorphisms;
2. $G = \langle \varphi_i(G_i) \mid i = 1, \ldots, n \rangle$.

**Fact 2.17** (Theorem 9.1.12 of [Ribes and Zalesskii, 2010]). Let $G_1, \ldots, G_n$ be profinite groups and let $G = G_1 \prod \cdots \prod G_n$ be their free profinite product. Then $N_G(G_i) = G$ for all $i = 1, \ldots, n$.

**Fact 2.18** (Prop. 17.6.2 in [Fried and Jarden, 2008]). Let $\hat{F}$ be a free profinite group, and $\hat{H}$ an open subgroup of $\hat{F}$. Suppose that $\hat{H}$ is profinite. Then $\hat{H}$ is a free profinite group. Moreover, if the rank $e$ of $\hat{F}$ is finite, then the rank of $\hat{H}$ is $1 + (\hat{F} : \hat{H})(e - 1)$; if the rank of $\hat{F}$ is infinite, then $\hat{H}$ and $\hat{F}$ have the same rank.
Definition 2.19. A profinite group $G$ is projective if for every diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\varphi} & B \\
\downarrow & & \downarrow \alpha \\
A & \rightarrows & \end{array}
\]

where $\varphi$ and $\alpha$ are epimorphisms of profinite groups, there exists a homomorphism $\gamma : G \to B$ such that the following diagram commutes

\[
\begin{array}{ccc}
G & \xrightarrow{\gamma} & B \\
\downarrow & & \downarrow \alpha \\
A & \rightarrows & \end{array}
\]

Fact 2.20 (Prop. 22.4.10 of [Fried and Jarden, 2008]). The free profinite product two projective profinite groups is again projective.

Fact 2.21 (Gaschütz, Lemma 17.7.2 of [Fried and Jarden, 2008]). Let $\pi : G \to H$ be an epimorphism of profinite groups with the rank of $G$ not more than a positive integer $e$. Let $h_1, \ldots, h_e$ be a system of generators of $H$. Then there exists a system of generators $g_1, \ldots, g_e$ of $G$ such that $\pi(g_i) = h_i$ for all $i = 1, \ldots, e$.

Corollary 2.22 (Proposition 17.7.3 of [Fried and Jarden, 2008]). Let $e$ be a positive integer, $\varphi : \hat{F}_e \to H$ and $\alpha : G \to H$ a pair of epimorphisms of profinite groups where the rank of $G$ is not more than $e$. Then there exists an epimorphism $\gamma : \hat{F}_e \to G$ such that the following diagram commutes.

\[
\begin{array}{ccc}
\hat{F}_e & \xrightarrow{\gamma} & G \\
\downarrow & & \downarrow \alpha \\
H & \rightarrows & \end{array}
\]

2.4. Pseudo-algebraically closed fields and Hilbertian fields.

Definition 2.23. A field $K$ is pseudo-algebraically closed or PAC if every (non-empty) absolutely irreducible affine algebraic set defined over $K$ has a $K$-rational point. In scheme theoretic terms: $K$ is PAC if every geometrically integral $K$-variety has a $K$-point.

For a positive integer $e$, we say $K$ is $e$-free if the absolute Galois group $\text{Gal}(K)$ of $K$ is isomorphic to the free-profinite group $\hat{F}_e$ of rank $e$.

A field $K$ is pseudo-finite, if $K$ is PAC, perfect and 1-free.
Fact 2.24 (Corollary 11.2.5 of Fried and Jarden, 2008). Every algebraic extension of a PAC field is PAC.

Fact 2.25 (Corollary 11.5.5 of Fried and Jarden, 2008). Suppose that $K$ is a PAC field with a non-trivial valuation. Then $K^{sep}$ is an immediate extension of $K$. Thus by Fact 2.13, the residue field of $K$ is algebraically closed and the value group $K$ is divisible.

The following is called the PAC Nullstellensatz op. cit.

Fact 2.26 (Theorem 18.6.1 of Fried and Jarden, 2008). Let $K$ be a countable Hilbertian field and $e$ a positive integer. Then the fixed field of a tuple $\sigma \in \text{Gal}(K)^e$, $K^{sep}(\sigma)$, is a PAC field for almost all $\sigma \in \text{Gal}(K)^e$, that is, there exists a subset $A \subseteq \text{Gal}(K)^e$ with Haar measure 1 such that for all $\sigma \in A$, $K^{sep}(\sigma)$ is PAC.

Fact 2.27 (Theorem 18.5.6 of Fried and Jarden, 2008). Let $K$ be a countable Hilbertian field and $e$ a positive integer. Then $\langle \sigma_1, \ldots, \sigma_e \rangle \cong \hat{F}_e$ for almost all $(\sigma_1, \ldots, \sigma_e) \in \text{Gal}(K)^e$.

Fact 2.28 (Theorem 18.10.2 of Fried and Jarden, 2008). Let $K$ be a countable Hilbertian field and $e$ a positive integer. Then the maximal Galois extension of $K$ in $K^{sep}(\sigma)$, denoted by $K^{sep}[\sigma]$, is PAC for almost all $\sigma \in \text{Gal}(K)^e$.

Fact 2.29 (Theorem 27.4.8 of Fried and Jarden, 2008). Let $K$ be a countable Hilbertian field. Then, for almost all $\sigma \in \text{Gal}(K)^e$, the field $K^{sep}[\sigma]$ is $\omega$-free and PAC.

Fact 2.30 (Theorem 2 of Kollár, 2007). Suppose that $K$ is a PAC field, $V$ a non-trivial valuation on $K^{alg}$. Then for any absolutely irreducible affine algebraic set $V$ defined over $K$, $V(K)$ is $V$-dense in $V(K^{alg})$.

2.5. Languages and theories.

Definition 2.31. We use the symbol $L_r$ to denote the language of rings $\{+, -, \times, 0, 1\}$, the symbol $L_{\text{div}}$ to denote the (one-sorted) language of valued fields (see Subsection 2.2 for valued fields), namely the set $L_r \cup \{\mid\}$, where the vertical bar ‘|’ is meant to be interpreted as the division predicate associated to the valuation, that is, ‘$x \mid y$’ if and only if the valuation of $x$ is not more than the valuation of $y$. However, we usually use write $v(x) \leq v(y)$ or $v(y) \geq v(x)$ instead of $x \mid y$. We use the symbol ACVF to denote the theory of algebraically closed non-trivially valued fields, in the language of $L_{\text{div}}$.

Definition 2.32. If $p$ is a prime number, we use $L_p$ to denote the language $L_r \cup \{\lambda_{n,i}(x; y_1, \ldots, y_n)\}_{n \in \omega, i \in \mathbb{N}^p}$, where the function symbols
are meant to be interpreted as the relative $p$-coordinate functions as defined in Definition 2.6. If $p$ is 0, we use $L_{p}$ to denote $L_{r}$.

**Definition 2.33.** $L_{p, \text{div}}$ stands for $L_{p} \cup L_{\text{div}}$.

**Fact 2.34** (Prop. 4.3.28 of [Marker, 2002]). If $L$ is a language containing a constant symbol and $T$ a theory in $L$, then the following are equivalent:

1. $T$ has quantifier elimination;
2. for every $M \models T$, $A \subset M$, $N \models T$ being $|M|^+ \text{-saturated}$, and $f : A \to N$ a partial $L$-embedding, $f$ extends to an $L$-embedding of $M$ into $N$.

**Fact 2.35** (Theorem 13.1 of [Sacks, 1972]). A theory $T$ has quantifier elimination if and only if $T$ is substructure complete, i.e. for any substructure $\mathcal{A}$ of a model $M \models T$, the new theory obtained by adjoining to $T$ the atomic diagram of $\mathcal{A}$, $T \cup \text{Diag}(\mathcal{A})$, is complete.

**Fact 2.36.** The theory ACVF admits quantifier elimination in $L_{\text{div}}$. Robinson proved that ACVF is model-complete in [Robinson, 1977]. For a complete proof of the quantifier elimination result, consult for example [van den Dries, 2004].

**Remark 2.37.** More languages will be introduced in Section 5.

### 3. Some Lemmas

**Lemma 3.1.** Suppose that $(K, V)$ is a valued field where $K$ is PAC and $V$ is non-trivial. Then for any finite separable algebraic extension $L/K$, there are exactly $[L : K]$ distinct extensions of $V$ to $L$.

**Proof.** By Fact 2.25, $Kv$ is algebraically closed and $vK^\times$ is divisible. By Fact 2.12, for any extension $W$ of $V$ to $L/K$, the ramification index $e(W/V)$ and the residue degree $f(W/V)$ are both always equal to 1. Let $\bar{L}$ be the Galois closure of $L/W$, then by Fact 2.14 there are exactly $[\bar{L} : K]$ extensions of $V$ to $\bar{L}$. By Fact 2.21, $L$ is also PAC, since $\bar{L}$ is also Galois over $L$, there are exactly $[\bar{L} : L]$ extension of a given non-trivial valuation on $L$ to $\bar{L}$. This means that there are exactly

$$\frac{[\bar{L} : K]}{[\bar{L} : L]} = [L : K]$$

extensions of $V$ to $L$. \hfill $\Box$

**Lemma 3.2.** Suppose that $(K, V)$ is a valued field where $K$ is PAC and $V$ is non-trivial. Suppose that $L/K$ is a finite separable algebraic extension. If $L/K$ is not Galois, then there exist two valuations $V_1$ and $V_2$ on $L$ extending $V$ such that for any $\sigma \in \text{Aut}(L/K)$, $\sigma(V_1) \neq V_2$. 

Proof. By Lemma 3.1, there are \([L : K]\) distinct extensions of \(V\) to \(L\), but since \(L/K\) is not Galois, the number of elements in \(\text{Aut}(L/K)\) is strictly less than \([L : K]\). Thus there are two valuations \(V_1\) and \(V_2\) on \(L\) extending \(V\) such that they are not conjugated over \(K\). \(\square\)

Lemma 3.3. Suppose that \((K, V)\) is a valued field where \(K\) is PAC and \(V\) is non-trivial. Suppose that \(L/K\) is a finite separable algebraic extension and that \(V_1, \ldots, V_n\) are all the distinct extensions of \(V\) to \(L\). Then there exists a primitive element \(a\) for the field extension \(L/K\) such that \(a \in V_1 \setminus (\bigcup_{i=2}^n V_i)\).

Proof. By 2.25, the residue field \(Kv\) is infinite. Let \(\Lambda := \{\lambda_i\}_{i \in I} \subseteq V\) be a set of representatives of elements in \(Kv\). Then \(\Lambda \subseteq \cap_{i=1}^n V_i\).

If \(n = 1\), then the conclusion is trivially true. Thus we may assume that \(n \geq 2\).

First, we show that \(V_1 \setminus (\bigcup_{i=2}^n V_i)\) is not empty. If \(n = 2\), then by the assumption that \(V_1 \neq V_2\), it follows from Fact 2.9 that \(V_1 \nsubseteq V_2\). Thus \(V_1 \setminus V_2\) is not empty. If \(n \geq 3\), then suppose toward a contradiction that \(V_1 \subseteq \bigcup_{i=2}^n V_i\). Without loss of generality, we may assume that the minimal number of \(V_i\) whose union contains \(V_1\) is \(k\) and \(V_1 \subseteq \bigcup_{i=2}^k V_i\). Since \(V_1, \ldots, V_n\) are distinct, \(k \geq 3\). This means that there exists \(s \in V_2 \cap V_1\) and \(t \in V_3 \cap V_1\) such that \(s \notin \bigcup_{i=3}^k V_3\) and \(t \notin V_2 \cup (\bigcup_{i=4}^k V_i)\). It follows that for all \(\lambda \in \Lambda\), \(s + \lambda(t - s) \in V_1 \subseteq \bigcup_{i=2}^k V_i\). Note that \(\Lambda\) is infinite. Therefore there exists \(\lambda_1 \neq \lambda_2\) and \(2 \leq i \leq k\) such that \(\lambda_1, \lambda_2 \in \Lambda\) and both \(p := s + \lambda_1(t - s)\) and \(q := s + \lambda_2(t - s)\) are in a common \(V_i\). Then for \(\mu = \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1} \in V_i\), \(s + \mu(q - p) \in V_i\), and for \(\mu = \frac{1 - \lambda_1}{\lambda_2 - \lambda_1} \in V_i\), \(t + \mu(q - p) \in V_i\). Thus both \(s\) and \(t\) are in \(V_i\), contradicting the choices of \(s\) and \(t\). Thus \(V_1 \setminus (\bigcup_{i=2}^n V_i)\) is not empty.

Now choose an element \(a \in V_1 \setminus (\bigcup_{i=2}^n V_i)\). Then any extension of the valuation ring \(K(a) \cap V_1\) on \(K(a)\) to \(L\) has to be the same as \(V_1\). By 3.1 this means that \([L : K(a)] = 1\), namely \(L = K(a)\). Thus \(a\) is a primitive element of \(L/K\). \(\square\)

Lemma 3.4. Suppose that \((K, V)\) is a valued field where \(K\) is PAC, \(V\) is non-trivial, and \(L/K\) is a finite separable algebraic extension. Suppose that \(V_1\) and \(V_2\) are two extensions of \(V\) such that for all \(\sigma \in \text{Aut}(L/K)\), \(\sigma(V_1) \neq V_2\). Then there exists a primitive element \(a\) for the extension \(L/K\) with minimal polynomial \(f(X) \in K[X]\) satisfying the following properties:

(1) \(a \in V_1 \setminus V_2\);
(2) for all \(r \in L\) such that \(f(r) = 0\), \(r \notin V_2\).
Proof. Denote by $V_3, \ldots, V_n$ the rest of the all the distinct extensions of $V$ to $L$, where $n = [L : K]$ by Lemma 3.1. Then by Lemma 3.3 there exists a primitive element $a$ for the extension $L$ such that $a \in V_1 \setminus \left( \bigcup_{i=2}^{n} V_i \right)$. Denote the minimal polynomial of $a$ over $K$ by $f(X) \in K[X]$, the Galois closure of $L/K$ by $\tilde{L}$, the Galois group of $\tilde{L}/K$ by $G$, the Galois group of $\tilde{L}/L$ by $H$, and the order of $H$ by $d$. Then the $n$ roots of $f(X)$ over $K$ are all elements of $\tilde{L}$. By Fact 2.10 and by Lemma 3.1, for any valuation $W$ on $\tilde{L}$ extending $V$ and any $\sigma \in G$, if $\sigma \neq 1$, then $\sigma(W) \neq W$. For each $i = 1, \ldots, n$, let the valuation rings $W_{i1}, \ldots, W_{id}$ be all of the distinct extensions of $V_i$ on $L$ to $\tilde{L}$.

It then follows that for all $\sigma \in G$, $\sigma \in H$ if and only if $\sigma(W_{i1}) \cap L = V_i$, if and only if $\sigma(W_{i1}) \in \{W_{i1}, \ldots, W_{id}\}$. Indeed, if $\sigma \in H$, then $\sigma$ fixes $L$ element-wise. So $V_1 = \sigma(V_1) = \sigma(W_{i1} \cap L) = \sigma(W_{i1}) \cap L$. Meanwhile, there are exactly $d$ valuations on $\tilde{L}$ extending $V_1$, so there are exactly $d$ elements in $G$ each of whose action on $W_{i1}$ extends $V_i$.

We now have found all the $d$ elements of $H$ satisfies the later property, so these must be all such elements.

Similarly, for each $i = 1, \ldots, n$, it is true that $\sigma \in H$ if and only if $\sigma(W_{id}) \cap L = V_i$. For each $i = 1, \ldots, n$, let $\sigma_i \in G$ be the unique element that maps $W_{i1}$ to $W_{i1}$. Then for all $\sigma \in G$, $\sigma \in H\sigma_i$ if and only if $\sigma(W_{i1}) \cap L = V_i$, if and only if $\sigma(W_{i1}) \in \{W_{i1}, \ldots, W_{id}\}$. Because $V_1, \ldots, V_n$ are distinct, it follows that $H\sigma_1, \ldots, H\sigma_n$ are exactly all the distinct right cosets of $H$ in $G$.

For $\sigma \in G$, $\sigma \in H$ if and only if $\sigma(W_{i1}) \in \{W_{i1}, \ldots, W_{id}\}$, if and only if $a \in \sigma(W_{i1})$. Thus, for $\sigma, \tau \in G$, $\tau(a) \in \sigma(W_{i1})$ if and only if $\tau^{-1}\sigma \in H$ if and only if $\sigma H = \tau H$. So for any root $\tau(a) \in L$ of $f(X)$, where $\tau \in G$, we have $\tau(L) = L$, so $\tau H \tau^{-1} = H$, and $\tau H = H\tau$. If $\tau(a)$ were in $V_2$, then $\tau(a) \in \sigma_2(W_{i1})$, so $\tau H = \sigma_2 H$. This means that $\sigma_2 \in \sigma_2 H = \tau H$ if and only if $H\sigma_2$, whence $\tau(V_1) = \tau(W_{i1} \cap L) = \tau(W_{i1}) \cap L = V_2$, contradicting the assumption that for all $\sigma \in \text{Aut}(L/K)$, $\sigma(V_1) \neq V_2$. Thus, none of the roots of $f(X)$ in $L$ are contained in $V_2$. 

It is known (see for example Lemma 20.6.3 of [Fried and Jarden, 2008]) that if $E$ and $F$ and field extensions of a common subfield $K$ with the property that every irreducible separable polynomial over $K$ has a root in $E$ if and only if it has a root in $F$, then $E$ and $F$ are $K$-isomorphic. In the case of valued fields, this is not true in general. However, under a more stringent condition, there is still a similar result.
Lemma 3.5. Suppose that \((E_1, V_1)\) and \((E_2, V_2)\) are two valued fields extending the valued field \((K, V)\). If every irreducible separable polynomial over \(K\) that has a root in \(E_1\) splits both in \(E_1\) and \(E_2\), and every irreducible separable polynomial that has a root in \(E_2\) splits both in \(E_1\) and and \(E_2\). Then there is an \(\mathcal{L}_{\text{div}}\) isomorphism \(\sigma : E_1 \cap K^{\text{sep}} \to E_2 \cap K^{\text{sep}}\) (with their induced valuations respectively) which restricts to \(\text{id}_K\) on \((K, V)\).

Proof. Since normal extensions are self conjugated, it is harmless to slightly abuse the notation. Let \(L\) be a finite Galois extension of \(K\).

Suppose that \(E_1 \cap L = K(a)\). Then the minimal polynomial \(f(X)\) of \(a\) splits in \(E_1\) and \(E_2\). Thus \(E_1 \cap L\) is normal over \(K\). Suppose \(b \in E_2\) is a root of \(f(X)\), then \(K(b)\) is also normal over \(K\) and \(E_1 \cap L \cong_K K(b) \subseteq E_2 \cap L\).

By the same reasoning, \(E_2 \cap L\) is \(K\)-isomorphic to a subfield of \(E_1 \cap L\). This means that \([K(b) : K] \leq [E_2 \cap L : K] \leq [K(b) : K]\). Thus \(K(b) = E_2 \cap L\).

Therefore \(E_1 \cap L\) is \(K\)-isomorphic to \(E_2 \cap L\), both of which are normal over \(K\). By Fact 2.36, there is an \(\mathcal{L}_{\text{div}}\) isomorphism \(\sigma : E_1 \cap L \to E_2 \cap L\) (with their induced valuations respectively) which restricts to \(\text{id}_K\) on \((K, V)\).

Thus, so far we have show that for each finite Galois extension \(L\), the set

\[A_L := \{\sigma_L : E_1 \cap L \to E_2 \cap L\ \text{an} \ \mathcal{L}_{\text{div}}\text{-isomorphism that restricts to} \ \text{id}_K\} \]

is not empty. Thus the inverse limit

\[H := \lim_{\to} A_L,\]

taken over all \(L\) finite Galois over \(K\), is not empty. Any element \(\sigma \in H\) is an \(\mathcal{L}_{\text{div}}\) isomorphism \(\sigma : E_1 \cap K^{\text{sep}} \to E_2 \cap K^{\text{sep}}\) (with their induced valuations respectively) which restricts to \(\text{id}_K\) on \((K, V)\). \(\square\)

Lemma 3.6. Suppose that \((E_1, V_1), (E_2, V_2)\) and \((F, W)\) are three valued-field extension of a valued field \((K, V)\), where \((F, W)\) is a \((|E_1| \times |E_2|)^+\)-saturated model of ACVF. Then there exist \(\mathcal{L}_{\text{div}}\)-embeddings \(\sigma_1 : E_1 \to F\) and \(\sigma_2 : E_2 \to F\) such that:

(1) the restriction of \(\sigma_1\) and \(\sigma_2\) on \(K\) are the identity map of \(K\);
(2) \(\sigma_1(E_1)\) and \(\sigma_2(E_2)\) are algebraically independent over \(K\).

Proof. By Fact 2.36 ACVF admits quantifier elimination, therefore, by Fact 2.34 we can find two \(\mathcal{L}_{\text{div}}\)-embeddings \(f_1 : E_1 \to F\) and \(f_2 : E_2 \to F\), such that both \(f_1\) and \(f_2\) are \(\text{id}_K\) when restricted to the subfield \(K\). Take \(\{e_i\}_{i \in I}\) to be a transcendence basis of \(E_1\) over \(K\). Since \(F\)
is $(|E_1| \times |E_2|)^+\text{-saturated},$ there exists a set of elements $\{\alpha_i\}_{i \in I}$ in $F$ algebraically independent over the compositum $L := f_1(E_1), f_2(E_2)$ with each of the $w(\alpha_i)$ larger than any element in $w(L^\times)$. It then follows that $w(\alpha_i + f_1(e_i)) = w(f_1(e_i))$ for all $i \in I$ and the set $\{\alpha_i + f_1(e_i)\}_{i \in I}$ is algebraically independent over $L$. This in turn implies that for any multivariate polynomial $h$ with coefficients in $f_1(E_1), w(h(\alpha + f_1(e_i))) = w(h(f_1(e_i)))$. Thus mapping $f_1(e_i)$ to $\alpha_i + f_1(e_i), we obtain a $\mathcal{L}_{\text{div}}$-embedding of $K(f_1(e_i))_{i \in I}$ to $F,$ which extends to an $\mathcal{L}_{\text{div}}$-embedding (by quantifier elimination again) $g : f_1(E_1) \to F$ which necessarily has the property that $g(f_1(E_1))$ is algebraically independent from $f_2(E_2)$ over $K$. Thus, $\sigma_1 := g \circ f_1$ and $\sigma_2 := f_2$ satisfy the desired properties. \hfill \square

**Lemma 3.7.** Suppose that $K$ is a PAC field with a non-trivial valuation $V$, $(L,W)$ is a valued field extension of $(K,V)$ where $L/K$ is regular. Suppose that $E$ is a subfield of $L$ contained in a $K$-subalgebra $R$ of $L$, where $R$ is generated by not more than $|E|$-elements over $K$. Suppose that $(K,V)$ is $|E|^+\text{-saturated}$ in $\mathcal{L}_{\text{div}}$. Then there is a ring homomorphism $\phi : R \to K$ satisfying the following conditions simultaneously:

1. $\phi$ restricts to the identity map on $K$;
2. $\phi$ restricts to an $\mathcal{L}_{\text{div}}$-embedding of $(E,W \cap E)$ into $(K,V)$.

**Proof.** We may assume that $E \setminus K$ is not empty. Let $k$ be the prime field of $K$. By assumption, $R = K[x_i]_{i \in J},$ where $J$ is an index set of cardinality $|E|$ and $\{x_i\}_{i \in J} \subseteq E \setminus K$. Then for any finite subset $S \subseteq J$, $K(x_i)_{i \in S}$ is a regular extension of $K$. This means that $(x_i)_{i \in S}$ is a generic point of an absolutely affine algebraic set $V_S$ defined over $K$. Let $k_S$ be a finitely generated (over $k$) subfield (of $K$) of definition of $V_S$ and $(\gamma_j)_{j \in I_S}$ be a finite set of generators for $k_S/k$. Let $C$ be the compositum of all the fields of definition $k_S$. Then $|C| \leq |E|$. Let $I_S$ be the vanishing ideal of $(x_i)_{i \in S}$ in $K[X_i]_{i \in S}$. Then $I_S$ is finitely generated by elements in $C[X_i]_{i \in S};$ let $(f_i)_{i \in I_S}$ be a finite set of generators.

In the following, $\bar{x}$ denotes a tuple from the set $\{x_i\}_{i \in J}$. Let $\Sigma$ be the following set of formulas in free variables $\{X_j\}_{j \in J}$

$$\left(\bigcup_{S \subseteq J} \{f_i(\bar{X}') = 0\}_{i \in I_S}\right) \bigcup \{v(p(\bar{X})) \geq v(q(\bar{X})) \mid p, q \in C[\bar{X}], \ v(p(\bar{x})) \geq v(q(\bar{x})), \ \bar{x} \ \text{a tuple in} \ E \setminus K\},$$

where $\bar{X}'$ and $\bar{X}$ are not necessarily the same tuple. Then $\Sigma$ is a set of formulas in $|E|$ variables over a set of cardinality not more than $|E|$.
We show that \( \Sigma \) is realizable in \((K, V)\). It is enough to show that \( \Sigma \) is finitely realizable in \((K, V)\), because \((K, V)\) is \(|E|^+\)-saturated. Let

\[
(3.1) \quad f_1, \ldots, f_l, v(q_1) \geq v(p_1), \ldots, v(q_s) \geq v(p_s)
\]

be finitely many elements arbitrarily chosen from \( \Sigma \). Then there exist a tuple \( \bar{x} \) realizing all the formulas in Equation (3.1). The indexes of \( \bar{x} \) corresponds to a finite subset \( S_0 \subseteq J \). Then the absolutely irreducible affine algebraic set \( V_{S_0} \) is defined over \( C \), hence over \( K \). We may also assume that

\[
p_1(\bar{x}), \ldots, p_s(\bar{x}), q_1(\bar{x}), \ldots, q_s(\bar{x})
\]

are all non-zero, otherwise say \( p_i(\bar{x}) = 0 \), then \( v(p_i) \geq v(q_i) \) is equivalent to an \( L \)-formula and could be treated as one of \( f_1, \ldots, f_l \).

Consider the formula

\[
(3.2) \quad \Phi := \left[ \bigwedge_{i \in S_0} f_i(\bar{X}) = 0 \right] \bigwedge \left[ \bigwedge_{i=1}^s v(p_i(\bar{X})) \geq v(q_i(\bar{X})) \right]
\]

\( \bar{x} \) realizes \( \Phi \) in \((L^{\text{alg}}, W)\). Since ACVF is model complete in \( L_{\text{div}} \), there is also a realization of \( \Phi \) in \((K^{\text{alg}}, V)\), denoted by \( \bar{a} \). By Fact 2.30, there is a point \( \bar{b} \) with coordinates in \( K \) such that \( \bar{b} \) is sufficiently close to \( \bar{a} \) with respect to the topology induced by \( W \). This means that \( \bar{b} \) also realizes \( \Phi \) in \( K \). Since \( \bar{b} \) is a point in \( V_{S_0} \), it satisfies any algebraic relation satisfied by the generic point \( \bar{x} \) over \( C \). Thus \( \bar{b} \) also satisfies all the formulas in Equation (3.1). This proves that \( \Sigma \) is realizable in \( K \).

Let \( \{ \bar{y}_i \}_{i \in I} \) be a realization of \( \Sigma \) in \( K \). Then mapping \( x_j \to y_j \), with \( \text{id}_K : K \to K \), we get a natural map from \( R \) to \( K \), denoted by \( \phi : R \to K \). Since \( \phi \) preserves all algebraic relations among the \( x_j \), it is a ring homomorphism which restricts to the identity map on \( K \). Meanwhile, \( E \) is a field, so the restriction of \( \phi \) to \( E \) is an isomorphism, which also preserves the valuation on \( E \); therefore \( \phi \) is an \( L_{\text{div}} \)-embedding of \((E, W \cap E)\) into \((K, V)\).

The following lemma is similar to its field theoretic version, as in Proposition 11.4.1 of [Fried and Jarden, 2008]. The proof is also similar. Thus we will not present the full version of the proof, but refer the reader to original proof of said proposition.

**Lemma 3.8.** Suppose that \( K \) is a PAC field with a non-trivial valuation \( V \), \((L, W)\) is a valued field extension of \((K, V)\) where \( L/K \) is regular. Suppose that \( E \) is a subfield of \( L \) contained in a \( K \)-subalgebra \( R \) of \( L \), where \( R \) is finitely generated over \( K \). Suppose furthermore that the exponent of imperfection of \( K \) is \( d \in \omega \cup \{\infty\} \), that \( y_1, \ldots, y_m \in R \) are \( p \)-independent in \( L \), \( m \leq d \), and that \((K, V)\) is \(|E|^+\)-saturated in
$L_{\text{div}}$. Then there is a ring homomorphism $\phi : R \to K$ satisfying the following conditions simultaneously:

1. $\phi$ restricts to the identity map on $K$;
2. $\phi$ restricts to an $L_{\text{div}}$-embedding of $(E, W \cap E)$ into $(K, V)$;
3. $\phi(y_1), \ldots, \phi(y_m)$ are $p$-independent in $K$.

Proof. By the proof of Proposition 11.4.1 of [Fried and Jarden, 2008], we may assume that

$$M = L^pK(y_1, \ldots, y_m) = L^pK(y_1, \ldots, y_k)$$

where $y_1, \ldots, y_k$ are $p$-independent over $L^pK$; there exist $a_1, \ldots, a_n \in K$ with $m \leq n$ such that they are $p$-independent over $K^p$ and

$$a_{n-m+k+1}, \ldots, a_n \in L^p(a_1, \ldots, a_{n-m+k}, y_1, \ldots, y_m).$$

Let $\eta_i := (a_{n-m+i}y_i)^{1/p}$, $i = 1, \ldots, k$ and $L' = L(\eta_1, \ldots, \eta_k)$. Then by the proof op. cit., $L'/K$ is a regular field extension.

Therefore, there exists elements $v_{ij} \in L$ such that for $i = n - m + k + 1, \ldots, n$,

$$a_i = \sum_{j_1, \ldots, j_{n+k}} v_{ij}^p a_1^{j_1} \cdots a_{n-m+k}^{j_{n-m+k}} y_1^{j_{n-m+k+1}} \cdots y_m^{j_{n+k}}. \quad (3.3)$$

Let

$$S := R[\eta_1, \ldots, \eta_k, y_1^{-1}, \ldots, y_m^{-1}, v_{ij}]_{ij}.$$ 

Then the quotient field of $S$ is $L'$, which is regular over $K$. Thus by Lemma 3.7 there exists a ring homomorphism $\phi : S \to K$ such that $\phi |_K = \text{id}_K$ and $\phi |_E$ is an $L_{\text{div}}$-embedding of $(E, W \cap E)$ into $(K, V)$. Then $\phi$ satisfies

$$\phi(\eta_i)^p = a_{n-m+i} \phi(y_i), \quad i = 1, \ldots, k.$$ 

Thus

$$a_{n-m+i} = \phi(\eta_i)^p \phi(y_i)^{-p} \phi(y_i)^{p-1}, \quad i = 1, \ldots, k. \quad (3.4)$$

Apply $\phi$ to Equation (3.3), we get

$$a_i = \sum_{j_1, \ldots, j_{n+k}} \phi(v_{ij})^p a_1^{j_1} \cdots a_{n-m+k}^{j_{n-m+k}} \phi(y_1)^{j_{n-m+k+1}} \cdots \phi(y_m)^{j_{n+k}}, \quad i = n - m + k + 1, \ldots, n. \quad (3.5)$$

Thus by Equation (3.4) and Equation (3.3), we get

$$a_1, \ldots, a_n \in K^p(a_1, \ldots, a_{n-m}, \phi(y_1), \ldots, \phi(y_m)).$$

But $a_1, \ldots, a_n$ are $p$-independent over $K^p$, so

$$a_1, \ldots, a_{n-m}, \phi(y_1), \ldots, \phi(y_m)$$

are also $p$-independent over $K^p$. Thus $\phi$ satisfies all the desired properties. \qed
Combining Lemma 3.8 and the proof of Lemma 3.7, we can improve the later to preserve the $p$-independence relation.

**Lemma 3.9.** Suppose that $K$ is a PAC field with a non-trivial valuation $V$, $(L, W)$ is a valued field extension of $(K, V)$ where $L/K$ is regular. Suppose that $E$ is a subfield of $L$ contained in a $K$-subalgebra $R$ of $L$, where $R$ is generated by not more than $|E|$ elements over $K$. Suppose furthermore that the exponent of imperfection of $K$ is $d \in \omega \cup \{\infty\}$, \{y_m\}_{m \in M} \subseteq R$ are $p$-independent in $L$, where $|M| \leq \min\{|E|, d\}$. Suppose that $(K, V)$ is $|E|^+\text{-saturated}$ in $\mathcal{L}_{\text{div}}$. Then there is a ring homomorphism $\phi : R \rightarrow K$ satisfying the following conditions simultaneously:

1. $\phi$ restricts to the identity map on $K$;
2. $\phi$ restricts to an $\mathcal{L}_{\text{div}}$-embedding of $(E, W \cap E)$ into $(K, V)$;
3. $\{\phi(y_j)\}_{j \in J}$ are $p$-independent in $K$.

**Proof.** The strategy of the proof is very much the same as the one for Lemma 3.7. Using the notation from that proof, assume that $y_m = g_m(\vec{x})$. Instead of showing the set of formulas $\Sigma$ is finitely realizable in $K$, we show that

$$\Sigma \cup \{g_m(\vec{X}) \text{ are } p\text{-independent}\}_{m \in M}$$

is finitely realizable in $K$, which is guaranteed by Lemma 3.8. \[\square\]

4. **Existence of various pseudo-algebraically closed fields**

In the manuscript [Cherlin et al., 1982], the elementary theory of regularly closed fields were studied, where a field is called a regularly closed field exactly when it is pseudo-algebraically closed. The following fact was proved in [Cherlin et al., 1982].

**Fact 4.1.** [Proposition 38 of [Cherlin et al., 1982]] Let $d$ be any element in $\omega \cup \{\infty\}$ denoting a chosen exponent of imperfection. Let $F$ be a field algebraic over its prime field, $G$ a projective profinite group and $\pi : G \rightarrow \text{Gal}(F)$ an epimorphism. Then there exists a field extension $K/F$ satisfying all the following conditions:

1. $K$ is pseudo-algebraically closed;
2. the relative algebraically closure of the prime field in $F$ is exactly $K$;
(3) there exists an isomorphism $\sigma : G \to \text{Gal}(K)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
G & \xrightarrow{\sigma} & \text{Gal}(K) \\
\pi & \searrow & \mathrm{res} \\
& \text{Gal}(F) &
\end{array}
$$

(4) the exponent of imperfection of $K$ is $d$.

Since every PAC field is infinite, and on any infinite field, there is always a non-trivial valuation, using the proposition above, we get that there exists a PAC field $K$ with a non-trivial valuation satisfying all the conditions in said proposition. In the rest of the article, we will be considering various first order theories of pseudo-algebraically closed fields with non-trivial valuation, Proposition 4.1 guarantees that these theories are all consistent.

5. Lack of Quantifier elimination

Following the notation in Fried and Jarden, 2008 below, we use $\mathcal{L}_r$ to denote the language $\mathcal{L}_e \cup \{ R_n \}_{n \geq 1}$, $\mathcal{L}_{R,\text{div}}$ to denote the language $\mathcal{L}_{\text{div}} \cup \{ R_n \}_{n \geq 1}$, and $\mathcal{L}_{R,p,\text{div}}$ to denote the language $\mathcal{L}_{p,\text{div}} \cup \{ R_n \}_{n \geq 1}$, where $R_n$ is an $n$-ary predicate. If $T$ is a theory of fields in any of $\mathcal{L}_r$, $\mathcal{L}_{\text{div}}$ or $\mathcal{L}_{p,\text{div}}$, then we use $T_R$ to denote the theory $T \cup \{ \phi_n \}_{n \geq 1}$ in $\mathcal{L}_r$, $\mathcal{L}_{\text{div}}$ or $\mathcal{L}_{p,\text{div}}$ respectively, where $\phi_n$ is the axiom

$$R_n(a_0, a_1, \ldots, a_{n-1}) \iff \exists x(x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0).$$

**Theorem 5.1.** Suppose that $T$ is one of the following theories, then $T_R$ does NOT have quantifier elimination.

1. The theory of pseudo-algebraically closed non-trivially valued fields, in $\mathcal{L}_{\text{div}}$ or in $\mathcal{L}_{p,\text{div}}$ (characteristic $p$);
2. the theory of $e$-free ($e \in \omega \setminus \{ 0 \}$ fixed) pseudo-algebraically closed non-trivially valued fields, of characteristic $p$, with exponent of imperfection $d$ ($d \in \omega \cup \{ \infty \}$ fixed), in $\mathcal{L}_{p,\text{div}}$;
3. the theory of $\omega$-free pseudo-algebraically closed non-trivially valued fields, of characteristic $p$, with exponent of imperfection $d$ ($d \in \omega \cup \{ \infty \}$ fixed), in $\mathcal{L}_{p,\text{div}}$.

**Proof.** We only prove the cases of (2) and (3). Case (1) is implied by the proof of Case (2).

First, for Case (2).

For a fixed $e \in \omega \setminus \{ 0 \}$ and a fixed $d \in \omega \cup \{ \infty \}$, let $T$ denote that theory of $e$-free pseudo-algebraically closed non-trivially valued fields with degree of imperfection $d$. To show that $T_R$ does not have
quantifier elimination, by Fact 2.35 it is enough to show that $T_R$ is not substructure complete.

Let $G = \hat{F}_e \prod \hat{F}_e \cong \hat{F}_{2e}$ be the free profinite product of the free profinite group of rank $e$ with itself. Let $H$ be the first free factor of $G$, then by Fact 2.17 $H$ self-normalizing in $G$. By Fact 4.1 there exists an $2e$-free PAC field with exponent of imperfection $d$ whose absolute Galois group is $G$; let $K$ be such a field. Let $E$ be the fixed field of $H$ in $K^{\text{sep}}$. Because $H$ is self-normalizing, if $\sigma \in \text{Aut}(E/K)$, then $\sigma$ extends to an element $\tilde{\sigma} \in G$ satisfying $\tilde{\sigma} H \tilde{\sigma}^{-1} = H$, which in turn implies that $\tilde{\sigma} \in H$, whence $\tilde{\sigma}$ fixes $E$ element-wise. Thus $\text{Aut}(E/K) = 1$. Thus by Fact 2.24 and Fact 2.5, $E$ is an $e$-free PAC field whose exponent of imperfection is also $d$.

Because $H$ is self-normalizing in $G$, the field extension $E/K$ is not Galois. This means that there exists a finite separable normal extension $L/K$ contained in $K^{\text{sep}}$ such that $L \cap E$ is not normal over $K$.

Pick a non-trivial valuation $V$ on $K$, by Lemma 3.1, there are exactly $[L \cap E : K]$ distinct extensions of $V$ on $K$ to $L \cap E$, denoted by $V_1, \ldots, V_n$. Because $L \cap E$ is not Galois over $K$, there exists two valuations, $V_i$ and $V_j$, not $K$-conjugated in $L \cap E$. Let $\Phi$ be an $\mathcal{L}_{\text{div}}(K)$ sentence saying that $f(x)$ has a root with a non-negative value. Then $(L \cap E, V_1) \models \Phi$ but $(L \cap E, V_2) \not\models \Phi$. Since $L$ is normal over $K$, all the roots of $f(X)$ in $E$ are already in $L \cap E$, this means that if we denote extensions of $V_i$ and $V_j$ to $E$ by $\tilde{V}_i$ and $\tilde{V}_j$ respectively, then

$$(E, \tilde{V}_i) \models \Phi, \quad \text{but} \quad (E, \tilde{V}_j) \not\models \Phi.$$ 

Therefore $(E, \tilde{V}_i) \not\equiv_{(K,V)} (E, \tilde{V}_j)$. Thus $T_R$ is not substructure complete.

Now, for Case (3). The proof is similar to the above, exact that one replaces the group $G$ above by $G = \hat{F}_\omega \prod \hat{F}_\omega$ and use Fact 2.20. □

In view of the theorem above, in order to obtain quantifier elimination, we need to functionalize the predicates $R_n$. On obvious possible solution is to uniformly pick out the roots of all the polynomials with roots; but this does not seem easy if possible at all to do. Since the theorem above suggests that one obstruction to having quantifier elimination is the existence of non-Galois extensions, we introduce the following function symbols to eliminate these non-Galois extensions.

**Definition 5.2.** Let $K$ be any field. Given a polynomial $f(X) \in K[X]$, we define the maximal splitting factor of $f(X)$ to the unique monic polynomial $g(X)$ satisfying the following two conditions:
(1) \(g(X)\) splits into linear factors over \(K\);
(2) there exists an polynomial \(h(X) \in K[X]\) without roots in \(K\) such that \(f(X) = g(X)h(X)\).

This is equivalent to saying that \(g(X)\) is the product \(\prod_{i=1}^{r} (x - r_i)\), where \(r_1, \ldots, r_i\) are all the roots of \(f(X)\) in \(K\).

**Definition 5.3.** Let \(K\) be any field and \(n\) be an positive integer. Given \(n + 1\) elements \(a_0, \ldots, a_{n-1}, a_n \in K\), denote the polynomial
\[a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0\]
by \(\pi_n(a_0, \ldots, a_{n-1}, a_n)\). Suppose that the maximal splitting factor of \(\pi_n(a_0, \ldots, a_{n-1}, a_n)\) over \(K\) is
\[g(X) = b_nX^n + b_{n-1}X^{n-1} + \cdots + b_1X + b_0,\]
where \(b_n, \ldots, b_0 \in K\), some of which could be zeros. For each \(0 \leq i \leq n\), we define the \(i\)-th splitting coefficient \(\theta_{n,i}(a_0, a_1, \ldots, a_n)\) by dictating that
\[\theta_{n,i}(a_0, a_1, \ldots, a_n) = b_i.\]

**Definition 5.4.** We use \(L_\theta\) to denote the language \(L_1 \cup \{\theta_{n,i}\}_{n \geq 1, 0 \leq i \leq n}\), \(L_\theta,\text{div}\) to denote the language \(L_\text{div} \cup \{\theta_{n,i}\}_{n \geq 1, 0 \leq i \leq n}\), and \(L_\theta,\text{p,div}\) to denote the language \(L_{p,\text{div}} \cup \{\theta_{n,i}\}_{n \geq 1, 0 \leq i \leq n}\). A field naturally interprets the symbols \(\theta_{n,i}\) as splitting coefficients.

It will be proved in Section 7 that several theories of pseudo-algebraically closed fields with non-trivial valuation do in fact have quantifier elimination in \(L_\theta,\text{p,div}\). But it is worth pointing out that in \(L_\theta,\text{p,div}\), the theory of pseudo-algebraically closed fields with non-trivial valuations itself as a total, does not admit quantifier elimination.

**Theorem 5.5.** The following statements are true.

1. Let \(\text{PACVF}\) be the theory, in \(L_\theta,\text{div}\), of pseudo-algebraically closed non-trivially valued fields. Then \(\text{PACVF}\) does not admit quantifier elimination.

2. Let \(\text{PACVF}_{p,d}\) be the theory, in \(L_\theta,\text{p,div}\), of pseudo-algebraically closed non-trivially valued fields, of characteristic \(p\), with exponent of imperfection \(d\) (\(d \in \omega \cup \{\infty\}\)). Then \(\text{PACVF}_{p,d}\) does not admit quantifier elimination either.

**Proof.** It is enough to show that \(\text{PACVF}_{p,d}\) is not substructure complete. The obstruction occurs already on the field theoretic level, not necessary to involve the valuations.

The \(K\) be a countable separably closed field of characteristic \(p\) with exponent of imperfection \(d\). Let \(t\) be transcendental element over \(K\). Then \(K(t)\) is a countable Hilbertian field and the extension \(K(t)/K\) is
regular. Let \( V \) be any non-trivial valuation on \( K(t) \). By Fact 2.26 and Fact 2.27 there exists a field \( E_1/K(t) \) which is 1-free PAC and there exists a field \( E_2/K(t) \) which is 2-free PAC. Let \( V_1 \) and \( V_2 \) be any two extensions of \( V \) to \( E_1 \) and \( E_2 \) respectively. Since \( E_1/K \) and \( E_2/K \) are both separable and \( K \) is separably closed, the \( \mathcal{L}_{\theta,p,\text{div}} \)-substructure on \( K \) in \( E_1 \) and the \( \mathcal{L}_{\theta,p,\text{div}} \)-substructure on \( K \) in \( E_2 \) agree. Thus
\[
(E_1, V_1) \not\equiv (K, V) (E_2, V_2),
\]
because they have different free absolute Galois groups. \( \square \)

In view of the theorem above, in order to obtain quantifier elimination, we need to impose conditions on the Galois groups of the models. This leads us to obtain the results in Section 7.

### 6. The Embedding Lemma

The so-called Embedding Lemma (Lemma 20.2.2 in [Fried and Jarden, 2008]) plays an important role in the analysis of the model theory of PAC fields. Analogously, we also have a valuation theoretic version of the Embedding Lemma, which plays a similar important role in the model theoretic analysis of the pseudo-algebraically closed valued fields.

**Lemma 6.1 (Valuation Theoretic Embedding Lemma).** Suppose that valued field \( (E, V_E) \) extends \( (L, V_L) \) and that valued field \( (F, V_F) \) extends \( (M, V_M) \), where \( E/L \) and \( F/M \) are regular field extensions. Suppose that \( F \) is PAC and \( |E|^+ \)-saturated in \( \mathcal{L}_{\text{div}} \) and that the exponent of imperfection of \( E \) is not more than the exponent of imperfection of \( F \). Suppose that there is a field isomorphism \( \Phi_0 : L^\text{sep} \to M^\text{sep} \) and a commutative diagram

\[
\begin{array}{ccc}
\text{Gal}(E) & \overset{\varphi}{\longrightarrow} & \text{Gal}(F) \\
\text{res} & & \text{res} \\
\text{Gal}(L) & \overset{\varphi_0}{\longleftarrow} & \text{Gal}(M),
\end{array}
\]

where \( \varphi_0 \) is the isomorphism induced by \( \Phi_0 \) (namely \( \varphi_0(\sigma) = \Phi_0^{-1} \circ \sigma \circ \Phi_0 \)), \( \varphi \) is a homomorphism, and \( \Phi_0 \) restricts to an \( \mathcal{L}_{\text{div}} \)-isomorphism from \( (L, V_L) \) onto \( (M, V_M) \).

Then there exists an extension of \( \Phi_0 \) to a field embedding \( \Phi : E^\text{sep} \to F^\text{sep} \) that induces \( \varphi \) with \( F/\Phi(E) \) separable and that \( \Phi \) restricts to an \( \mathcal{L}_{\text{div}} \)-embedding from \( (E, V_E) \) into \( (F, V_F) \). If furthermore \( \varphi \) is surjective, then \( F/\Phi(E) \) is regular.

**Proof.** The proof is similar to that of Lemma 20.2.2 in [Fried and Jarden, 2008].
We may identify $L$ and $M$, and assume that both of $\Phi_0$ and $\varphi_0$ are the identity maps. By Lemma 3.3, we may assume that $E$ is algebraically independent from $F$ over $L$. Then by the proof op. cit., $F^{\text{sep}}/L^{\text{sep}}$ and $E^{\text{sep}}/L^{\text{sep}}$ are both regular, and $EF$ is regular both over $E$ and over $F$. Thus $E^{\text{sep}}$ is linearly disjoint from $F^{\text{sep}}$ over $L^{\text{sep}}$. Meanwhile, each $\sigma \in \text{Gal}(F)$ extends uniquely to a field automorphism $\tilde{\sigma} \in \text{Gal}(E^{\text{sep}}F^{\text{sep}}/EF)$ with

$$\tilde{\sigma}x = \begin{cases} \varphi(\sigma)x, & x \in E^{\text{sep}}, \\ \sigma x, & x \in F^{\text{sep}}. \end{cases}$$

Thus the map $\sigma \mapsto \tilde{\sigma}$ embeds $\text{Gal}(F)$ into $\text{Gal}(E^{\text{sep}}F^{\text{sep}}/EF)$. Let $D$ be the fixed field of the image of $\text{Gal}(F)$ under the aforementioned map. Then $D \cap F^{\text{sep}} = F$, $DF^{\text{sep}} = E^{\text{sep}}F^{\text{sep}}$, and $E^{\text{sep}} \subseteq E^{\text{sep}}F^{\text{sep}} = D|F^{\text{sep}} = F^{\text{sep}}[D]$. For each $x \in E^{\text{sep}}$, choose $y_i \in F^{\text{sep}}$ and $d_i \in D$ where $i$ ranges over a finite set $I_x$ such that $x = \sum_{i \in I_x} y_id_i$. Let $D_0 = E \cup \{d_i \mid x \in E^{\text{sep}}, i \in I_x\}$. Then $|D_0| = |E|$ and $E^{\text{sep}} \subseteq F^{\text{sep}}[D_0]$.

If $S$ is a $p$-basis of $E$, then since $F(D_0)/E$ is separable, $S$ remains $p$-independent over $F(D_0)^p$. Therefore by Lemma 3.3 there exists a ring homomorphism $\tilde{\Psi} : F[D_0] \to F$ satisfying the following three conditions:

1. $\tilde{\Psi}$ restricts to the identity map on $F$;
2. $\tilde{\Psi}$ restricts to an $L^{\text{div}}$-embedding of $(E,V_E)$ into $(F,V_F)$;
3. $\tilde{\Psi}(S)$ is $p$-independent in $F$.

From (3), we know that $F/\tilde{\Psi}(E)$ is separable. Since $D$ is linearly disjoint from $F^{\text{sep}}$ over $F$, $\tilde{\Psi}$ extends to a ring homomorphism $\tilde{\Psi} : F^{\text{sep}}[D_0] \to F^{\text{sep}}$ such that $\tilde{\Psi}$ restricts to the identity map on $F^{\text{sep}}$. Furthermore, for each $\sigma \in \text{Gal}(F)$, and for each $x \in F^{\text{sep}} \cup D_0$, we have

$$\tilde{\Psi}(\tilde{\sigma}x) = \sigma \tilde{\Psi}(x),$$

which then holds for each $x \in F^{\text{sep}}[D_0] \supseteq E^{\text{sep}}$. Thus if we let $\tilde{\Phi} = \tilde{\Psi} |_{E^{\text{sep}}}$, then $\tilde{\Phi}$ satisfies all the desired properties.

If $\varphi$ is surjective, then we need to show that $\Phi(E)$ is relatively algebraically closed in $F$ so that $F/\Phi(E)$ is regular. For any $y \in F$ algebraic over $\Phi(E)$, there exists a unique $x \in E^{\text{sep}}$ such that $\Phi(x) = y$. So for any $\sigma \in \text{Gal}(F)$, we have $\tilde{\Psi}(\tilde{\sigma}x) = \sigma \tilde{\Psi}(x) = \sigma y = y$. But $\tilde{\sigma}x = \varphi(\sigma)x \in E^{\text{sep}}$, so $\varphi(\sigma)x = \tilde{\sigma}x = x$. Since $\varphi$ is surjective, this means that $\{\varphi(\sigma)\}_{\sigma \in \text{Gal}(F)} = \text{Gal}(E)$, thus $x \in E$. So $y \in \Phi(E)$. In conclusion, $F/\Phi(E)$ is indeed regular if $\varphi$ is surjective. \qed
7. Quantifier elimination

**Definition 7.1** (Definition 24.1.2 of [Fried and Jarden, 2008]). A profinite group $G$ has the **Embedding Property** if each embedding problem $(\zeta : G \to A, \alpha : B \to A)$ where $\zeta$ and $\alpha$ are epimorphisms and $B \in \text{Im}(G)$ is solvable, that is, exists an epimorphism $\gamma : G \to B$ such that the following digram commutes

$$
\begin{array}{ccc}
G & \xrightarrow{\gamma} & B \\
\downarrow{\zeta} & & \downarrow{\alpha} \\
A & & A
\end{array}
$$

**Definition 7.2** (Definition 24.1.3 of [Fried and Jarden, 2008]). A field $K$ is called a **Frobenius field** if $K$ is PAC and $\text{Gal}(K)$ has the Embedding Property.

**Remark 7.3.** In [Cherlin et al., 1982], the Frobenius fields are called Iwasawa regularly closed fields, because the Embedding Property was first pointed out by Iwasawa. These fields are called Frobenius fields by various authors because these fields admit an analog of the Frobenius Density Theorem in number theory. In this article, we adopt the Frobenius-field terminology, which seems to be more widely adopted by other authors.

In [Cherlin et al., 1982], the theory of Frobenius fields was studied and a quantifier elimination result was proved. In order to state their result, we introduced their so-called **Galois formalism**.

**Definition 7.4** (see Page 60 on [Cherlin et al., 1982]). The **Galois formalism** is the language of rings $\mathcal{L}_r$ expanded by the predicates $R_n$ mentioned in Section 5 and a 0-place predicate $I_G$, for each isomorphism type $G$ of finite groups.

Note that over a field, all the new predicates in the Galois formalism are in fact equivalent to $\mathcal{L}_r$-sentences.

**Fact 7.5** (Section 2.10 of [Cherlin et al., 1982]). Being Frobenius is a first order property in the language of rings $\mathcal{L}_r$.

**Fact 7.6** (Theorem 41 of [Cherlin et al., 1982]). Let $\text{FrobF}_{p^d}$ be the theory of perfect Frobenius fields in the Galois formalism, where the predicates $R_n$ are interpreted as in Section 5 and the predicates $I_G$ are true in a model $K$ if and only if there exists a field extension $L/K$ such that $\text{Gal}(L/K) \cong G$. Then $\text{FrobF}_{p^d}$ admits quantifier elimination.

7.1. $\text{FrobVF}_{p^d}$. 
Theorem 7.7. For a exponent of imperfection $d \in \omega \cup \{\infty\}$, let $\text{FrobVF}_{p^d}$ be the theory of Frobenius non-trivially valued fields characteristic $p$ with exponent of imperfection $d$ in $\mathcal{L}_{\theta,p,\text{div}} \cup \{I_G\}_G$ finite group. Then $\text{FrobVF}_{p^d}$ has quantifier elimination.

Proof. This proof uses Fact 23. Suppose that $(E, V_E)$ and $(F, V_F)$ are two models of $\text{FrobVF}_{p^d}$ and $(A, V_A)$ an $\mathcal{L}_{\theta,p,\text{div}} \cup \{I_G\}_G$-substructure of $(E, V_E)$ that $\mathcal{L}_{\theta,p,\text{div}} \cup \{I_G\}_G$-embeds into $(F, V_F)$ by $\varepsilon : A \rightarrow F$. We need to show that there is an $\mathcal{L}_{\theta,p,\text{div}} \cup \{I_G\}_G$-embedding of $(E, V_E)$ into $(F, V_F)$ that extends $\varepsilon$ whenever $(F, V_F)$ is $|E|^{+}$-saturated.

Since $(A, V_A)$ is a substructure in the language $\mathcal{L}_{\theta,p,\text{div}} \cup \{I_G\}_G$ finite group, $A$ preserves all the 0-place predicates $I_G$, thus $\text{Im}(\text{Gal}(E)) = \text{Im}(\text{Gal}(F))$.

First, notice that $A$ is a field, for the following reason. For each $a \in A^\times$, the maximal splitting factor of $ax - 1$ is $x - a^{-1}$. Thus, since $\theta_1,0(-1, a) = -a^{-1} \in A$, $a^{-1} \in A$. So $A$ is a field.

Second, $E/A$ is separable, for the following reason. As an $\mathcal{L}_{p}$-substructure, $A$ is closed under all the relative $p$-coordinate functions. If $x \in E^p$, then $\lambda_{0,0}(x) = x^{1/p}$, so if furthermore $x \in A$, then $x^{1/p} \in A$. This means $A^p = A \cap E^p$. Suppose that $b_1, \ldots, b_n$ are $p$-independent in $A$, then because $A^p = A \cap E^p$, none of the $b_1, \ldots, b_n$ is in $E^p$. Thus there exists a maximal $p$-independent subset of $\{b_1, \ldots, b_n\}$ over $E^p$, which we may assume to be $\{b_1, \ldots, b_k\}$. If it were true that $k < n$, then $b_{k+1} \in E^p(b_1, \ldots, b_k)$ gives a quantifier-free $\mathcal{L}_{p}$-formula true in $E$ which is then also true in $A$ because $A$ is closed under all the relative $p$-coordinate functions, which in turn implies that $b_1, \ldots, b_n$ are $p$-dependent in $A$. Thus $k = n$ and $b_1, \ldots, b_n$ remains $p$-dependent in $E$. So $E/A$ is separable. By the same reasoning, $F/\varepsilon(A)$ is also separable.

Third, $\varepsilon$ could be extended to an $\mathcal{L}_{\theta,p,\text{div}}$-embedding of $(A^{\text{alg}}, V_E \cap A^{\text{alg}})$. For any monic separable irreducible polynomial $f(X) \in A[X]$, if $f(X)$ has a root in $E$, then because $A$ is closed under the function of splitting coefficients, the maximal splitting factor of $f(X)$ in $E$ is a polynomial in $A[X]$; but $f(X)$ irreducible over $A$, so $f(X)$ is its own maximal splitting factor. Thus $f(X)$ splits in $E$. Since $\varepsilon(A)$ is also an $\mathcal{L}_{\theta}$-substructure of $F$, $f(X)$ also splits in $F$. By the same reasoning, any monic separable irreducible polynomial $g(X) \in \varepsilon(A)[X]$ that has a root in $F$ also splits both in $E$ and $F$. Therefore, by Lemma 3.5, there exists an $\mathcal{L}_{\text{div}}$-isomorphism

$$\varepsilon : (E \cap A^{\text{sep}}, V_E \cap E \cap A^{\text{sep}}) \rightarrow (F \cap A^{\text{sep}}, V_F \cap F \cap A^{\text{sep}}),$$

extending $\varepsilon$. Since both $E \cap A^{\text{sep}}$ and $F \cap A^{\text{sep}}$ are relatively algebraically closed in $E$ and $F$ respectively and they are isomorphic as fields, their
respective $\mathcal{L}_θ$-substructures are preserved by $\varepsilon$. Since $E/(E \cap A^{sep})$ and $F/(F \cap A^{sep})$ remains separable, $\varepsilon$ is also an $\mathcal{L}_θ$-isomorphism. This means that $\varepsilon$ is in fact an $\mathcal{L}_{θ,p,div}$-isomorphism.

Let $(L, V_L)$ be the $\mathcal{L}_{θ,p,div} \cup \{I_G\}_G$-substructure $(E \cap A^{sep}, V_E \cap E \cap A^{sep})$, then $\varepsilon$ gives an $\mathcal{L}_{θ,p,div}$-embedding of $(L, V_L)$ into $(F, V_F)$. Let $M = \varepsilon(L)$. Then $E/L$ and $F/M$ are both regular field extensions. Extend $\varepsilon$ to a field isomorphism $\Phi_0: L^{sep} \to M^{sep}$. Then $\Phi_0$ induces an isomorphism of the absolute Galois groups $\varphi_0: \text{Gal}(M) \to \text{Gal}(L)$. By the proof of Fact 7.6 in [Cherlin et al., 1982], there exists an epimorphism $\varphi: \text{Gal}(F) \to \text{Gal}(E)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\text{Gal}(E) & \xrightarrow{\varphi} & \text{Gal}(F) \\
\text{res} & & \text{res} \\
\text{Gal}(L) & \xrightarrow{\varphi_0} & \text{Gal}(M),
\end{array}
$$

By the Valuation Theoretic Embedding Lemma 6.1 there exists an extension of $\Phi_0$ to a field embedding $\Phi: E^{sep} \to F^{sep}$ that induces $\varphi$ with $F/\Phi(E)$ regular and that $\Phi$ restricts to an $\mathcal{L}_{div}$-embedding from $(E, V_E)$ into $(F, V_F)$. Since $F/\Phi(E)$ is regular, $\Phi$ is in fact an $\mathcal{L}_{θ,p,div}$-embedding of $(E, V_E)$ into $(F, V_F)$ extending $\varepsilon$, which is then also an $\mathcal{L}_{θ,p,div} \cup \{I_G\}_G$-embedding.

These shows that $\text{FrobVF}_{p^d}$ has quantifier elimination.

7.2. $\text{FrobVF}^G_{p^d}$, $\text{PACVF}^{e,p}_{p^d}$, $\text{PACVF}^{ω,p}_{p^d}$, $\text{SCVF}^{p}_{p^d}$ and $\text{ACVF}_{p}$. From Theorem 7.7 we immediately obtain the following corollary.

**Corollary 7.8.** Let $G$ be a fixed projective profinite group with the Embedding Property, $d \in \omega \cup \{\infty\}$ a fixed exponent of imperfection. Let $\text{FrobVF}^G_{p^d}$ be the class of Frobenius non-trivially valued fields with absolute Galois group having the same finite quotients as $G$, exponent of imperfection $d$, axiomatized in the language $\mathcal{L}_{θ,p,div}$. Then $\text{FrobVF}^G_{p^d}$ has quantifier elimination.

**Proof.** By Theorem 7.7 $\text{FrobVF}^G_{p^d}$ has quantifier elimination in the language $\mathcal{L}_{θ,p,div} \cup \{I_H\}_H$ finite group. However, each 0-place predicate $I_H$ is either true in every model of $\text{FrobVF}^G_{p^d}$, whence equivalent to $0 = 0$, or false in every model of $\text{FrobVF}^G_{p^d}$, whence equivalent to $0 \neq 0$. Thus the conclusion follows.

**Corollary 7.9.** Let $e$ be a positive integer, $d$ an exponent of imperfection in $\omega \cup \{\infty\}$. Then the theory of $e$-free pseudo-algebraically closed non-trivially valued fields with exponent of imperfection $d$, axiomatized in $\mathcal{L}_{θ,p,div}$, denoted by $\text{PACVF}^{e,p}_{p^d}$ admits quantifier elimination.
Proof. By Corollary 2.22, \( \hat{\mathcal{F}}_e \) has the Embedding Property. It is known that a profinite finite is isomorphic to \( \hat{\mathcal{F}}_e \) if and only if it has the same finite quotients as \( \hat{\mathcal{F}}_e \). Thus \( \text{PACVF}_{p^d}^{\omega} \) is \( \text{FrobVF}_{p^d}^{\hat{\mathcal{F}}_e} \), which by Corollary 7.8 admits quantifier elimination in \( \mathcal{L}_{\theta,p,\text{div}} \). □

**Definition 7.10.** A field \( K \) is \( \omega \)-free if every finite embedding problem for \( \operatorname{Gal}(K) \) is solvable, that is, given any two epimorphisms of profinite groups \( \zeta : \operatorname{Gal}(K) \to A \) and \( \alpha : B \to A \), where \( B \) is a finite group, there always exists an epimorphism \( \gamma : \operatorname{Gal}(K) \to B \) such that the following diagram commutes.

\[
\begin{array}{ccc}
\operatorname{Gal}(K) & \xrightarrow{\gamma} & B \\
\downarrow{\zeta} & & \downarrow{\alpha} \\
A & & A
\end{array}
\]

**Fact 7.11** (Section 3 of [Jarden, 7677]). Being \( \omega \)-free is a first order property in the language of rings \( \mathcal{L}_1 \).

**Corollary 7.12.** Let \( d \) be an exponent of imperfection in \( \omega \cup \{\infty\} \). Then the theory of \( \omega \)-free pseudo-algebraically closed non-trivially valued fields with exponent of imperfection \( d \), axiomatized in \( \mathcal{L}_{\theta,p,\text{div}} \), denoted by \( \text{PACVF}_{p^d}^{\omega} \) admits quantifier elimination.

**Proof.** By definition, an \( \omega \)-free field is Frobenius. Furthermore, in the Galois formalism, for every finite group \( G \), the 0-place predicate \( I_G \) is equivalent to \( 0 = 0 \) over every model of \( \text{PACVF}_{p^d}^{\omega} \). The conclusion then follows from Theorem 7.7. □

**Corollary 7.13.** The theory of algebraically closed non-trivially valued fields with exponent of imperfection \( d \), axiomatized in \( \mathcal{L}_{\text{div}} \), denoted by \( \text{ACVF}_p \) admits quantifier elimination.

**Proof.** For an algebraically closed fields \( K \), \( \operatorname{Gal}(K) = \{1\} \). So \( K \) is Frobenius. Furthermore, for all positive integer \( n \) and \( i = 0, \ldots, n \), \( \theta_{n,i}(a_0, a_1, \ldots, a_n) = a_i/a_n \) if \( a_n \neq 0 \). Thus a formula of the form \( \theta_{n,i}(x_0, x_1, \ldots, x_n) = y \) is equivalent to

\[
x_n = 0 \lor (x_n \neq 0 \land y x_n = x_i).
\]

For function symbols in \( \mathcal{L}_p \), \( \lambda_{0,0}(x) = x^{1/p} \) and \( \lambda_{n,i}(x; y_1, \ldots, y_n) \) is always 0 if \( n \geq 1 \). The 0-place predicates \( I_G \) are equivalent to \( 0 = 0 \) or \( 0 = 1 \) depending on whether \( G \) is 1 or not. Thus by Theorem 7.7, \( \text{ACVF}_p \) has quantifier elimination. □

The following is a known result.
Proposition 7.14. Let $d$ be an exponent of imperfection in $\omega \cup \{\infty\}$. Then the theory of separably closed non-trivially valued fields with exponent of imperfection $d$, axiomatized in $\mathcal{L}_{p,\text{div}}$, denoted by $\text{SCVF}_{p^d}$ admits quantifier elimination.

Proof. The proof is similar to that of Theorem 7.7. We still use Fact 2.34. Suppose that $(E,V_E)$ and $(F,V_F)$ are two models of $\text{SCVF}_{p^d}$ and $(A,V_A)$ an $\mathcal{L}_{p,\text{div}}$-substructure of $(E,V_E)$ that $\mathcal{L}_{p,\text{div}}$-embeds into $(F,V_F)$ by $\varepsilon : A \to F$. We need to show that there is an $\mathcal{L}_{p,\text{div}}$-embedding of $(E,V_E)$ into $(F,V_F)$ that extends $\varepsilon$ whenever $(F,V_F)$ is $|E|^+$-saturated.

First the $\mathcal{L}_{p,\text{div}}$-structure of the quotient field of $A$ is uniquely determined by the $\mathcal{L}_{p,\text{div}}$-structure of $A$. Since $A$ is closed under the relative $p$ coordinate function $\lambda_{0,0}(x) = x^{1/p}$, $A^p = A \cap E^p$. For $a \notin A^p$, we have $\lambda_{1,1}(a; a^{p+1}) = a^{-1}$, so $a^{-1} \in A$. Thus if $a^{-1} \notin A$, then $a \in A^p$. Thus for $b_0, b_1, \ldots, b_n \in A$ and $c_0, c_1, \ldots, c_n \in \text{Frac}(A) \setminus A$,

$$\lambda_{n,i}(b_0/c_0; b_1/c_1, \ldots, b_n/c_n) = \frac{\prod_{j=1}^{n} c_j^{i(j)}}{c_0^{1/p}} \lambda_{n,i}(b_0; b_1, \ldots, b_n).$$

Thus the $\mathcal{L}_{p}$-structure of $\text{Frac}(A)$ is uniquely determined by that of $A$. Meanwhile, for the $\mathcal{L}_{\text{div}}$-structure, the value of any $a/b \in \text{Frac}(A)$ is the difference between the value of $a$ and the value of $b$. Thus the $\mathcal{L}_{p,\text{div}}$-structure of $\text{Frac}(A)$ is indeed uniquely determined by that of $A$.

Therefore, we may assume that $A$ is a field.

Second, as we have seen in the proof of Theorem 7.7, $E/A$ and $F/A$ are separable field extensions. So $A^{\text{sep}} \cap E = A^{\text{sep}}$ and $A^{\text{sep}} \cap F = A^{\text{sep}}$. Since $A^{\text{sep}}/A$ is a normal field extension, we can extend the $\mathcal{L}_{\text{div}}$-embedding $\varepsilon$ to an $\mathcal{L}_{\text{div}}$-embedding $\bar{\varepsilon} : A^{\text{sep}} \to F$, which in turn is also an $\mathcal{L}_{p,\text{div}}$-embedding.

Third, the Valuation Theoretic Embedding Lemma ensures that $\bar{\varepsilon}$ can be extended to an $\mathcal{L}_{p,\text{div}}$-embedding of $(E,V_E)$, and we are done. \hfill \Box

8. Elementary invariants of pseudo-algebraically closed non-trivially valued fields

Just like the field theoretic case (see [Cherlin et al., 1982]), the Valuation Theoretic Embedding Lemma 6.1 enables us to characterize the elementary equivalence relation of pseudo-algebraically closed non-trivially valued fields, in terms of the so-called coelementary equivalences. In this section, we use the notation and results from [Cherlin et al., 1982]. See also [Chatzidakis, 2002] for relevant discussions.
Definition 8.1. For any field $K$, we use $\text{Abs}(K)$ to denote the subfield of $K$ consisting of all the numbers algebraic over the prime subfield of $K$. $\text{Abs}(K)$ is usually called the subfield of absolute numbers in $K$.

Theorem 8.2. Suppose that $(K, V)$ and $(L, W)$ are two pseudo-algebraically close non-trivially valued fields. Denote by $i$ and $j$ the following $\mathcal{L}_{\text{div}}$-embeddings respectively,

$$
i : (\text{Abs}(K), V \cap \text{Abs}(K)) \to (K, V),$$

$$
\hat{j} : (\text{Abs}(L), W \cap \text{Abs}(L)) \to (L, W).
$$

Also denote by $\hat{i}$ and $\hat{j}$ the following induced restriction maps of the absolute Galois groups respectively,

$$
\hat{i} : \text{Gal}(K) \to \text{Gal}(\text{Abs}(K)),
$$

$$
\hat{j} : \text{Gal}(L) \to \text{Gal}(\text{Abs}(L)).
$$

Then

$$(K, V) \equiv_{\mathcal{L}_{\text{div}}} (L, W)$$

if and only if the following conditions hold:

1. $K$ and $L$ have the same exponent of imperfectness;
2. there is an $\mathcal{L}_{\text{div}}$-isomorphism

$$f : \text{Abs}(L)^{\text{sep}} \to \text{Abs}(K)^{\text{sep}}$$

that restricts to an $\mathcal{L}_{\text{div}}$-isomorphism from $(\text{Abs}(L), W \cap \text{Abs}(L))$ onto $(\text{Abs}(K), V \cap \text{Abs}(K))$ such that

$$(\text{Gal}(K), \hat{i}) \equiv_{f}^{o} (\text{Gal}(L), \hat{j}).$$

Proof. The proof is more or less similar to the field theoretic case given in [Cherlin et al., 1982], with some care for the valuations.

Suppose that $(K, V) \equiv_{\mathcal{L}_{\text{div}}} (L, W)$. Then obviously $K$ and $L$ have the same exponent of imperfectness. Let $f^{*}$ be an isomorphism of an elementary extension $(L^{*}, W^{*})$ of $(L, W)$ onto an elementary extension $(K^{*}, V^{*})$ of $(K, V)$. By Fact 2.10 there exists an $\mathcal{L}_{\text{div}}$-isomorphism $f^{*\text{sep}} : (L^{*})^{\text{sep}} \to (K^{*})^{\text{sep}}$. Let $f$ be the restriction of $f^{*\text{sep}}$ on $\text{Abs}(L)^{\text{sep}}$. Because $L^{*}$ and $K^{*}$ are elementary extensions of $L$ and $K$ respectively, we have $\text{Abs}(L^{*}) = \text{Abs}(L)$ and $\text{Abs}(K^{*}) = \text{Abs}(K)$. Thus $f$ restricts to an $\mathcal{L}_{\text{div}}$-isomorphism from $(\text{Abs}(L), W \cap \text{Abs}(L))$ to $(\text{Abs}(K), V \cap \text{Abs}(K))$. Meanwhile, by the proof of Lemma 32 in [Cherlin et al., 1982], we also have

$$(\text{Gal}(K), \hat{i}) \equiv_{f}^{o} (\text{Gal}(L), \hat{j}).$$
For the other direction, suppose that \( K \) and \( L \) have the same exponent of imperfectness and that there is an \( \mathcal{L}_r \)-isomorphism
\[
f : \operatorname{Abs}(L)^{\text{sep}} \to \operatorname{Abs}(K)^{\text{sep}}
\]
that restricts to an \( \mathcal{L}_{\text{div}} \)-isomorphism from \( (\operatorname{Abs}(L), W \cap \operatorname{Abs}(L)) \) onto \( (\operatorname{Abs}(K), V \cap \operatorname{Abs}(K)) \) such that
\[
(\text{Gal}(K), \hat{i}) \equiv_f^o (\text{Gal}(L), \hat{j}),
\]
then we show that \( (K, V) \equiv_{\mathcal{L}_{\text{div}}} (L, W) \). We also follow the proof of Proposition 33 in [Cherlin et al., 1982]. We may assume that both \( (K, V) \) and \( (L, W) \) are both \( \aleph_1 \)-saturated. We show the existence of a back-and-forth system of triples \( (g, M, N) \) satisfying the following Property \((8.1)\),
\[
\begin{align*}
K/M \text{ and } L/N \text{ are regular field extensions where } M \text{ and } N \text{ are countable subfields}, \\
g : M^{\text{sep}} \to N^{\text{sep}} \text{ which restricts to an } \mathcal{L}_{\text{div}} \text{-isomorphism from } M \text{ onto } N, \text{ and with the inclusions } i_M \text{ and } j_N, \\
(\text{Gal}(K), \hat{i}_M) \equiv^o_g (\text{Gal}(L), \hat{j}_N) \text{ holds.}
\end{align*}
\]
Note that first \( (f, \operatorname{Abs}(K), \operatorname{Abs}(L)) \) satisfies Property \((8.1)\).

Now, if \( (g, M, N) \) satisfies Property \((8.1)\), \( A \) a countable subset of \( L \), then we show that there is a triple \( (g_1, M_1, N_1) \) satisfying Property \((8.1)\) such that \( M_1/M \) and \( N_1/N(A) \) and \( g_1 \) extending \( g \). This is guaranteed by the proof of Proposition 33 of [Cherlin et al., 1982] and our Valuation Theoretic Embedding Lemma \((6.1)\) This back-and-forth system then proves that \( (K, V) \equiv_{\mathcal{L}_{\text{div}}} (L, W) \). □

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