Inequalities for oscillation operator

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June 25, 2020

Abstract

Let \((X, \mathcal{B}, \mu)\) be a measure space and \(\{U_t : -\infty < t < \infty\}\) be a one-parameter ergodic measure preserving flow on \(X\). Let \(s \geq 2\) and consider the oscillation operator defined as

\[
O f(x) = \left( \sum_{k=1}^{\infty} \left( \max_{n_k \leq m \leq n_{k+1}} |A_m f(x) - A_{n_k} f(x)| \right)^s \right)^{1/s}
\]

where

\[
A_n f(x) = \frac{1}{n} \int_0^n f(U_t x) \, dt
\]

is the usual ergodic average.

Let \((n_k)\) and \(M\) be lacunary sequences then there exist positive constants \(C_1, C_2, C_3\) and \(C_p\) such that

(a) \(\|O f\|_p \leq C_p \|f\|_p, \quad f \in L^p(X), \quad 1 < p < \infty,\)

(b) \(\mu\{x : |O f(x)| > \lambda\} \leq \frac{C_1}{\lambda} \|f\|_1, \quad f \in L^1(X), \quad \lambda > 0,\)

(c) \(\|O f\|_1 \leq C_2 \|f\|_{H^1}, \quad f \in L^1(X),\)

(d) \(\|O f\|_{EBMO} \leq C_3 \|f\|_{\infty}, \quad f \in L^\infty_c(X).\)

Mathematics Subject Classifications: 47A35, 28D05.

Key Words: Oscillation Operator, Ergodic BMO Space, Ergodic Hardy Space.
Let \((X, \mathcal{B}, \mu)\) a totally \(\sigma\)-finite measure space and \(\tau : X \to X\) be an ergodic measure preserving transformation. The function

\[
f^\ast(x) = \sup_n \frac{1}{n} \sum_{i=0}^{n-1} |f(\tau^i x)|
\]

is known as ergodic maximal function analogue to the Hardy-Littlewood maximal function \(Mf\) on the real line \(\mathbb{R}\) given by

\[
Mf(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(t)| \, dt,
\]

where \(I\) denotes an arbitrary interval in \(\mathbb{R}\). It is well known by transference argument that \(f^\ast\) has a strong \(L^p\) inequality if \(Mf\) has a strong \(L^p\) inequality for \(1 \leq p \leq \infty\) (see [2]).

Let now \(f\) be an integrable function and define

\[
f'^\ast(x) = \sup_n \frac{1}{n} \sum_{i=0}^{n-1} |f(\tau^i x) - T_n f(x)|
\]

where

\[
T_n f(x) = \frac{1}{n} \sum_{i=0}^{n-1} |f(\tau^i x)|.
\]

Now recall that the space \(H^1\) on the real line \(\mathbb{R}\) can be characterized by

\[
H^1(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) : \tilde{H}f \in L^1(\mathbb{R}) \right\}
\]

with the norm

\[
\|f\|_{H^1} \sim \|f\|_1 + \|\tilde{H}f\|_1.
\]

where \(\tilde{H} f\) is the Hilbert transform on \(\mathbb{R}\) defined by

\[
\tilde{H} f(x) = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\infty} \frac{f(t + x) - f(t - x)}{t} \, dt.
\]

Similar to the characterization by the maximal function we also define ergodic \(H^1\) space by

\[
H^1(X) = \left\{ f \in L^1(X) : Hf \in L^1(X) \right\}
\]

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with the norm

\[ \|f\|_{H^1} \sim \|f\|_1 + \|Hf\|_1. \]

where \( Hf \) is the ergodic Hilbert transform defined by

\[ Hf(x) = \sum_{k=1}^{\infty} \frac{f(\tau^k x) - f(\tau^{-k} x)}{k}. \]

Similar to the classical case we can also identify the dual of ergodic \( H^1 \) spaces as ergodic bounded mean oscillation EBMO defined by the space of functions \( f \) for which \( f^\sharp \) is bounded with EBMO norm given by

\[ \|f\|_{EBMO} = \|f^\sharp\|_\infty \]

Let now \( B \) be a Banach space and \( p < \infty \), and let \( f \) be a \( B \)-valued (strongly) measurable function defined on \( \mathbb{R} \). Then the Bochner-Lebesgue space \( L^p_B = L^p_B(\mathbb{R}) \) is defined as

\[ L^p_B = \{ f : \|f\|_{L^p_B} < \infty \} \]

where

\[ \|f\|_{L^p_B} = \left( \int_{\mathbb{R}} \|f(x)\|_B^p \, dx \right)^{1/p}. \]

When \( B \) is the scalar field, we simply write \( L^p \) and \( \|\cdot\|_p \). We also define the space \( WL^p_B = \text{weak} - L^p_B \) as the space of all \( B \)-valued functions \( f \) such that

\[ \|f\|_{WL^p_B} = \sup_{\lambda > 0} \lambda \left( \max \left\{ x \in \mathbb{R} : \|f(x)\|_B > \lambda \right\} \right)^{1/p} < \infty. \]

When we replace Lebesgue measure by \( w(x)dx \) for some positive weight \( w \) in \( \mathbb{R} \) we denote the corresponding spaces by \( L^p_B(w) \) and \( WL^p_B(w) \). When \( p = \infty \), we write

\[ L^\infty(B) = \{ f : \|f\|_{L^\infty(B)} < \infty \}, \]

where

\[ \|f\|_{L^\infty(B)} = \text{ess sup} \|f\|_B \]

and the space of all compactly supported members of \( L^\infty(B) \) will be denoted by \( L^\infty_c(B) \).

For a locally integrable \( B \)-valued function \( f \), we define the maximal functions

\[ M_r f(x) = \sup_{x \in I} \left( \frac{1}{|I|} \int_I \|f(y)\|_B^r \, dy \right)^{1/r}, \quad 1 \leq r \leq \infty, \]
and
\[ f^\sharp(x) = \sup_{x \in I} \frac{1}{|I|} \int_I \| f(y) - f_I \|_B \, dy, \]
where \( I \) denotes an arbitrary interval in \( \mathbb{R} \) and
\[ f_I = \frac{1}{|I|} \int_I f(t) \, dt \]
which is an element of \( B \).
Note that \( f^\sharp \) is the sharp maximal function in the classical case when \( B = \mathbb{R} \)
and \( \| \cdot \|_B = | \cdot | \), \( M_1 f \) is the Hardy-Littlewood maximal function and \( M_\infty f \)
is the constant function. Similar to the classical case we define the \( B \)-valued BMO space as
\[ \text{BMO}(B) = \{ f \in L^1_{\text{loc},B} : \| f \|_{\text{BMO}(B)} = \| f^\sharp \|_{L^\infty(B)} < \infty \}. \]
Given a \( B \)-valued function \( f \), we obtain a nonnegative function \( \| f \|_B \) defined by
\[ \| f \|_B(x) = \| f(x) \|_B, \]
and it is important to point out that
\[ \| (\| f \|_B) \|_{\text{BMO}} \leq 2 \| f \|_{\text{BMO}(B)}. \]
As usual a \( B \)-atom is a function \( a \in L^\infty(B) \) supported in an interval \( I \) and
such that
\[ \| a(x) \|_B \leq \frac{1}{|I|}, \quad \int_I a(x) \, dx = 0 \]
and the space \( H^1_B(\mathbb{R}) \) such that
\[ f(x) = \sum_j \lambda_j a_j(x); \quad (\lambda_j) \in l^1, \]
where \( a_j \) are \( B \)-atoms with
\[ \| f \|_{H^1_B} = \inf \sum_j |\lambda_j|. \]
Similar to the classical case given \( B \in \text{UMD} \) we also have
\[ H^1_B(\mathbb{R}) = \{ f \in L^1_B(\mathbb{R}) : \tilde{H} f \in L^1_B(\mathbb{R}) \}. \]
and
\[ \|f\|_{H^1_B} \sim \|f\|_{L^1_B} + \|\tilde{H}f\|_{L^1_B}. \]

We will consider here those kernels \( K(x) \) which are strongly measurable and defined in \( \mathbb{R} \) with the values in the space \( L(A,B) \) of all bounded linear operators from \( A \) to \( B \), with respect to the operator norm \( \| \cdot \| = \| \cdot \|_{L(A,B)} \). We will also assume that \( \|K(x)\| \) is locally integrable away from the origin.

**Definition 1.** Given \( 1 \leq r \leq \infty \), we say that \( K \) satisfies the condition \((D_r)\), and write \( K \in (D_r) \), if there exists a sequence \((c_k)\) such that
\[
\sum_{k=1}^{\infty} c_k = D_r(K) < \infty
\]
and, for all \( k \geq 1 \) and \( y \in \mathbb{R} \),
\[
\left( \int_{S_k(|y|)} \|K(x-y) - K(x)\|^r \, dx \right)^{1/r} \leq c_k |S_k(|y|)|^{-1/r},
\]
where
\[
S_k(|y|) = \{ x : 2^k|y| < |x| \leq 2^{k+1}|y| \}.
\]
When \( r = \infty \), this must be understood in the usual way, it is also easy to see that \( K \in (D_\infty) \) if
\[
\|K(x-y) - K(x)\| \leq C \frac{|y|}{|x|^{n+1}},
\]
whenever \( |x| > 2|y| \). Also, \( K \in (D_1) \) is the Hörmander condition
\[
\int_{|x| > 2|y|} \|K(x-y) - K(x)\| \, dx \leq D_1(K) < \infty,
\]
where \( D_1(K) \) does not depend on \( y \in \mathbb{R} \).

**Definition 2.** A linear operator \( T \) mapping \( A \)-valued functions into \( B \)-valued functions is called a singular integral operator of convolution type if the following conditions are satisfied:

(i) \( T \) is a bounded operator from \( L^q_A(\mathbb{R}) \) to \( L^q_B(\mathbb{R}) \) for some \( q, 1 \leq q \leq \infty \).
(ii) There exists a kernel $K \in (D_1)$ such that

$$Tf(x) = \int K(x - y) \cdot f(y) \, dy$$

for every $f \in L^q_A(\mathbb{R})$ with compact support and for a.e. $x \notin \text{supp}(f)$.

Remark 1. When we consider a singular integral operator $T$ mapping $A$-valued functions into $B$-valued functions, then it is also well known (see [5]) and easy to see that $T$ can be extended to an operator defined in all $L^p_A$, $1 \leq p < \infty$, and satisfying

(a) $\|Tf\|_{L^p_B} \leq C_p \|f\|_{L^p_A}$, $1 < p < \infty$,
(b) $\|Tf\|_{W^{1,p}_B} \leq C_1 \|f\|_{L^1_A}$,
(c) $\|Tf\|_{L^1_B} \leq C_2 \|f\|_{H^1_A}$,
(d) $\|Tf\|_{\text{BMO}(B)} \leq C_3 \|f\|_{L^\infty(A)}$, $f \in L^\infty_c(A)$,

where $C_p, C_1, C_2, C_3 > 0$.

Definition 3. A sequence $(n_k)$ of integers is called lacunary if there is a constant $\alpha > 1$ such that

$$\frac{n_{k+1}}{n_k} \geq \alpha$$

for all $k = 1, 2, 3, \ldots$.

Let $(X, \mathcal{B}, \mu)$ be a measure space and $\{U_t : -\infty < t < \infty\}$ be a one-parameter ergodic measure preserving flow on $X$. Let $s \geq 2$ and consider the oscillation operator defined as

$$O f(x) = \left( \sum_{k=1}^{\infty} \left( \max_{n_k \leq m \leq n_{k+1}} |A_m f(x) - A_{n_k} f(x)| \right)^s \right)^{1/s}$$

where

$$A_n f(x) = \frac{1}{n} \int_0^n f(U_t x) \, dt$$

is the usual ergodic average.

Theorem 1. Let $(n_k)$ and $M$ be lacunary sequences then there exist positive constants $C_1$, $C_2$, $C_3$ and $C_p$ such that
(a) \(\|O f\|_p \leq C_p \|f\|_p, \quad f \in L^p(X), \quad 1 < p < \infty,\)
(b) \(\mu \{ x : |O f(x)| > \lambda \} \leq \frac{C_1}{\lambda} \|f\|_1, \quad f \in L^1(X), \quad \lambda > 0,\)
(c) \(\|O f\|_1 \leq C_2 \|f\|_{H^1}, \quad f \in L^1(X),\)
(d) \(\|O f\|_{EBMO} \leq C_3 \|f\|_{\infty}, \quad f \in L^\infty_c(X).\)

**Proof.** Let \(B\) be an \(l^s\) sum of finite-dimensional \(l^\infty\) spaces with the following properties:

A general element \(b \in B\) is written as 

\[ b = ( (b_m : n_k \leq m \leq n_{k+1}, \ m \in M) : k > 1 ); \]

and

\[ \|b\|_B = \left( \sum_{k=1}^{\infty} \left( \max_{n_k \leq m \leq n_{k+1}} |b_m| \right)^s \right)^{1/s} \]

where \(m \in M\). Let us now define the kernel operator \(K : \mathbb{R} \to B\) as

\[ K(x) = \left( \left( \frac{1}{m} \chi_{[0,m]}(x) - \frac{1}{n_k} \chi_{[0,n_k]}(x) : n_k \leq m \leq n_{k+1}, \ m \in M \right) : k \geq 1 \right). \]

We first want to prove that

\[ O_R f(x) = \|K * f(x)\|_B \]

is a singular integral operator. Let us first show that \(K\) satisfies the Hörmander condition:

For an \(n \in \mathbb{N}\) let

\[ \phi_n(x) = \frac{1}{n} \chi_{[0,n]}(x) \]

and

\[ \Phi_{(m,k)}(x) = \phi_m(x) - \phi_{n_k}(x). \]

Then we have

\[ \|K(x-y) - K(x)\|_B = \left( \sum_{k=1}^{\infty} \left( \max_{n_k \leq m \leq n_{k+1}} |\Phi_{(m,k)}(x-y) - \Phi_{(m,k)}(x)| \right)^s \right)^{1/s}. \]
Note that maximum is taken over all \( m \in M \) between the related intervals as described in the definition of \( \mathcal{O} f \). Thus in order to prove the Hörmander condition for the kernel operator \( K \) we need to show that there exists a constant \( C > 0 \) such that

\[
\int_{|x| > 4|y|} \|K(x - y) - K(x)\|_B \, dx \leq C
\]

where \( C \) does not depend on \( y \in \mathbb{R} \).

Let us first consider the case \( x > 4y, \ y > 0 \). We have

\[
\Phi_{(m,k)}(x - y) - \Phi_{(m,k)}(x) = \phi_m(x - y) - \phi_n(x - y) - (\phi_m(x) - \phi_n(x))
\]

\[
= \phi_m(x - y) - \phi_m(x) - (\phi_n(x - y) - \phi_n(x)).
\]

Since \( x > 4y, \ y > 0 \) we get

\[
\phi_m(x - y) - \phi_m(x) = \frac{1}{m} \chi_{[0,m]}(x - y) - \frac{1}{m} \chi_{[0,m]}(x)
\]

\[
= \frac{1}{m} \chi_{[y,m+y]}(x) - \frac{1}{m} \chi_{[0,m]}(x)
\]

\[
= \frac{1}{m} \chi_{[4y,m+y]}(x) - \frac{1}{m} \chi_{[4y,m]}(x)
\]

\[
= \frac{1}{m} \chi_{[m,m+y]}(x).
\]

Similarly, we have

\[
\phi_n(x - y) - \phi_n(x) = \frac{1}{n_k} \chi_{[nk,nk+y]}(x).
\]
We have

\[
\int_{x > 4y} \| K(x - y) - K(x) \|_B \, dx =
\]

\[
= \int_{x > 4y} \left( \sum_{k=1}^{\infty} \left( \max_{m \in M} \left| \frac{1}{m} \chi_{[m,m+y]}(x) - \frac{1}{n_k} \chi_{[n_k,n_k+y]}(x) \right| \right) s^{1/s} \right) \, dx
\]

\[
\leq \int_{x > 4y} \left( \sum_{k=1}^{\infty} \left( \max_{m \in M} \left| \frac{1}{m} \chi_{[m,m+y]}(x) \right| \right) s^{1/s} \right) \, dx +
\]

\[
+ \int_{x > 4y} \left( \sum_{k=1}^{\infty} \left( \max_{m \in M} \left| \frac{1}{n_k} \chi_{[n_k,n_k+y]}(x) \right| \right) s^{1/s} \right) \, dx
\]

\[
\leq \int_{x > 4y} \sum_{k} \max_{m \leq n_k} \frac{1}{m} \chi_{[m,m+y]}(x) \, dx +
\]

\[
+ \int_{x > 4y} \sum_{k, y \leq n_k} \max_{m \leq n_{k+1}} \frac{1}{n_k} \chi_{[n_k,n_k+y]}(x) \, dx
\]

\[
\leq y \sum_{y \leq m} \frac{1}{m} + y \sum_{y \leq n_k} \frac{1}{n_k}.
\]

On the other hand, we know that \((n_k)\) is a lacunary sequence, there is a constant \(\beta\) such that

\[
\frac{n_{k+1}}{n_k} \geq \beta > 1.
\]

Therefore, there is a constant \(C(\beta)\) such that

\[
\sum_{y \leq n_k} \frac{1}{n_k} \leq \frac{C(\beta)}{y}.
\]

Similarly, since \(M\) is a lacunary sequence there is a constant \(\alpha > 1\) and a constant \(C(\alpha)\) such that

\[
\sum_{y \leq n_k} \frac{1}{m} \leq \frac{C(\alpha)}{y}.
\]
As a conclusion we see that
\[ \int_{x > 4|y|} \| K(x - y) - K(x) \|_B \, dx \leq C(\alpha) + C(\beta) = C(\alpha, \beta) \]
and this shows that the kernel operator \( K \) satisfies the Hörmander condition.

Let us now consider the case \( y \leq 0 \) and \( x > 4|y| \),
\[ \phi_m(x - y) - \phi_m(x) = \frac{1}{m} \chi_{[y,y+m]}(x) - \frac{1}{m} \chi_{[0,m]}(x) \]
we then have
\[ \phi_m(x - y) - \phi_m(x) = \frac{1}{m} \chi_{[m+y,m]}(x) \]
and similarly, we have
\[ \phi_{nk}(x - y) - \phi_{nk}(x) = \frac{1}{nk} \chi_{[nk+y,nk]}(x) \]
for \( x > 4|y| \).
Again we see as in the previous case that
\[ \int_{x > 4|y|} \| K(x - y) - K(x) \|_B \, dx \leq C(\alpha) + C(\beta) = C(\alpha, \beta) \]
and this shows that \( K \) satisfies the Hörmander condition in this case as well.

Suppose now that \( x < 0 \) and \( y > 0 \). Since \( |x| > 4|y| \), we see that \( x - y < 0 \) thus in this case
\[ \phi_m(x - y) - \phi_m(x) = \frac{1}{m} \chi_{[0,m]}(x - y) - \frac{1}{m} \chi_{[0,m]}(x) = 0 \]
and
\[ \phi_{nk}(x - y) - \phi_{nk}(x) = \frac{1}{nk} \chi_{[0,nk]}(x - y) - \frac{1}{nk} \chi_{[0,nk]}(x) = 0. \]
Thus for any \( y \) we have
\[ \int_{|x| > 4|y|} \| K(x - y) - K(x) \|_B \, dx = 0. \]

We finally need to consider the case \( x < 0 \) and \( y < 0 \). In this case we have
\[ \phi_m(x - y) - \phi_m(x) = \frac{1}{m} \chi_{[y,y+m]}(x) \]
and 
\[ \phi_{n_k}(x - y) - \phi_{n_k}(x) = \frac{1}{n_k} \chi_{|y,y+n_k|}(x). \]

Since we also have \(|x| > 4|y|\), we see that 
\[ \phi_{m}(x - y) - \phi_{m}(x) = \frac{1}{m} \chi_{|y,y+m|}(x) = 0 \]

and 
\[ \phi_{n_k}(x - y) - \phi_{n_k}(x) = \frac{1}{n_k} \chi_{|y,y+n_k|}(x) = 0. \]

Thus for any \(y\) we have 
\[ \int_{|x| > 4|y|} \|K(x - y) - K(x)\|_B \, dx = 0. \]

On the other hand, when \(s = 2\) it is known that 
\[ O_R f(x) = \|K * f(x)\|_B \]

maps \(L^2\) to itself (see [4]) and for \(s > 2\) we have for \(f \in L^2(\mathbb{R})\)

\[ \|O_R f\|_2^2 = \int \left( \sum_{k=1}^{\infty} \left( \max_{n_k \leq m \leq n_k+1} \left| \phi_n * f(x) - \phi_{n_k} * f(x) \right| \right)^s \right)^{2/s} \, dx \]

\[ \leq \int \sum_{k=1}^{\infty} \left( \max_{n_k \leq m \leq n_k+1} \left| \phi_n * f(x) - \phi_{n_k} * f(x) \right| \right)^2 \, dx \]

\[ = \int \left[ \left( \sum_{k=1}^{\infty} \left( \max_{n_k \leq m \leq n_k+1} \left| \phi_n * f(x) - \phi_{n_k} * f(x) \right| \right)^2 \right)^{1/2} \right]^2 \, dx \]

\[ \leq C \|f\|_2^2 \]

for some constant \(C > 0\) because of the strong type \((2,2)\) inequality for \(s = 2\).

As a result we see that 
\[ O_R f(x) = \|K * f(x)\|_B \]

is a singular integral operator of convolution type. Therefore, it satisfies the conclusions of Remark 1 and thus there exist positive constants \(C_p, C_1, C_2, C_3 > 0\) such that
(a) \( \|O_R f\|_p \leq C_p \|f\|_p, \quad f \in L^p(\mathbb{R}), \quad 1 < p < \infty, \)

(b) \( m\{x : |O_R f(x)| > \lambda\} \leq \frac{C_1}{\lambda} \|f\|_1, \quad f \in L^1(\mathbb{R}), \quad \lambda > 0, \)

(c) \( \|O_R f\|_1 \leq C_2 \|f\|_{H^1}, \quad f \in L^1(\mathbb{R}), \)

(d) \( \|O_R f\|_{\text{BMO}} \leq C_3 \|f\|_\infty, \quad f \in L^\infty(\mathbb{R}). \)

and applying the transfer principle of A. P. Calderón [2] to this observation completes our proof.

\[\square\]

Remark 2. Note that the results given in (a), (b) and (d) are known for \( s = 2 \) (see [4]).

It is also known (see R. Caballero and A. de la Torre [1]) that there exists a constant \( C > 0 \) such that

\[ \|f\|_{H^1} \leq C \|f^*\|_1 \]

for all \( f \in L^1(X) \). Combining this fact with our result given in (c) shows that there exits a positive constant such that

\[ \|O f\|_1 \leq C \|f^*\|_1 \]

for all \( f \in L^1(X) \).

It is also well known that (see [3]) the ergodic maximal function is integrable if \( f \log^+ f \) is integrable. Thus we conclude that the oscillation operator \( O f \) is integrable if \( f \log^+ f \) is integrable.

Also, note that it is proved in [4] that for \( s = 2 \), \( O f \) is integrable if \( f \log^+ f \) is integrable. To prove this result the method they used is different from our method.

We also would like to note that a modification of our method can be used to prove the same results for the oscillation operator

\[ \left( \sum_{k=1}^{\infty} \left( \max_{n_k \leq m \leq n_{k+1}} \left| A_{2^m} f(x) - A_{2^{n_k}} f(x) \right| \right)^s \right)^{1/s} \]

for \( s \geq 2 \) as those we have for \( O f \).
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