Transport properties of Keplerian flows in extended local domains with no imposed field

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ABSTRACT
We compare transport statistics of elongated incompressible shearing boxes for different Reynolds and magnetic Prandtl numbers, Re and Pm, and aspect ratios, L_z/L_x. We find that at fixed aspect ratio L_z/L_x = 4 and Re = 10,000, the turbulent stresses for Pm ≤ 1 do not show considerable variation and follow a power law ~ Pm^{3/2} for Pm > 1. This is qualitatively consistent with previous results based on net imposed flux and small box L_z/L_x ~ 1 simulations but the power law exponent is different. The saturated level of stresses, the ratio of Maxwell stress to the magnetic energy and Reynolds to Maxwell stress ratio are roughly invariant as L_z/L_x is increased. For cases where the boxes are elongated in both the azimuth and vertical direction, the transport coefficient α ∈ [0.1, 1.0] that is 10 – 100 times larger than the case with L_z/L_x = 2 and large L_z/L_x. Overall, our results suggest that the saturated state of turbulence is sensitive to both dissipation coefficients and aspect ratio (both L_z/L_x, L_z/L_x) motivating further work on this problem.

Key words: accretion, accretion disks — magnetohydrodynamic turbulence

1 INTRODUCTION
Accretion flows exist in a variety of astrophysical systems. The accreting fluid would have a much longer lifetime if molecular viscosity was the sole source of angular momentum transport. For this reason, a number of possible alternative transport mechanisms have been investigated in recent years ranging from purely hydrodynamic sources to magnetohydrodynamic sources. Evaluating each of these mechanisms in detail and understanding their observational implications is a subject of ongoing research (Balbus & Hawley 1998; Fromang & Lesur 2017).

Magnetized Keplerian flows are linearly unstable to the magnetorotational instability (MRI) if a weak external magnetic field is present (Velikhov 1959; Chandrasekhar 1960; Balbus & Hawley 1991). The MRI has been studied extensively in local shearing box simulations (Hawley et al. 1995; Brandenburg et al. 1995; Fromang et al. 2007), global disk simulations (Flock et al. 2012; Parkin & Becknell 2013; Suzuki & Inutsuka 2014; Zhu & Stone 2018) and Taylor-Couette flow simulations (Wei et al. 2016; Guseva et al. 2017b,a). Despite more than two decades of numerical work on the problem, the issue of convergence is still unresolved with work focusing on imposed field, the presence of dissipation coefficients, density stratification due to gravity (Fromang & Papaloizou 2007; Davis et al. 2010; Hawley et al. 2011; Bodo et al. 2014; Meheut et al. 2015; Ryan et al. 2017). If the external flux is removed in a magnetized Keplerian flow, then the fluid is no longer linearly unstable to MRI but is yet observed to reach a nonlinear steady state with significant transport (Hawley et al. 1996; Fromang et al. 2007; Lesur & Ogilvie 2008; Guseva et al. 2017a). In the literature, this case is sometimes referred to as the ‘dynamo’ case but we do not use such terminology here because in a broad sense, fluids linearly unstable to MRI are also an example of a dynamo since they lead to the generation and sustenance of magnetic fields (Gressel & Pessah 2015). The ‘zero flux’ case has been a particular focus of numerical studies trying to seek convergence since it was realized that the increase in resolution leads to a decrease in transport (Pessah et al. 2007; Fromang & Papaloizou 2007).

The role of Reynolds number as an important parameter in shear flows is well known since the original work of Reynolds in 1883. The realization that the aspect ratio might also play an important role in determining the nonlinear state of a fluid came much later (Cross & Hohenberg 1993; Philip & Manneville 2011). Moreover, recent numerical and experimental work on shear flows suggests that small domains might suffer from finite size effects at transition (Lemoult et al. 2016) and that the transition to turbulence in shear flows perhaps belongs to the directed percolation universality class (Pomeau 1986).

Until recently, Keplerian flows without a net imposed
flux were thought to be stable for magnetic Prandtl number at and below unity (Fromang et al. 2007). However, Nauman & Pessah (2016) (Paper I) showed that if a large aspect ratio ($L_z/L_x > 4$) is used, then flows can still reach a nonlinear steady state with magnetic Prandtl number below unity. The transport properties of such elongated boxes was not addressed in that paper. The focus of the current paper is to explore the dependence of turbulent transport on box size, Reynolds number and magnetic Prandtl number.

That the aspect ratio plays an important role has gained further support from recent work by Shi et al. (2016) and Walker & Boldyrev (2017) with the observation that larger elongated boxes showed not only that the turbulent stresses in boxes with $L_z/L_x > 2.5$ converged with respect to increasing resolution but also that the saturated level of stresses was insensitive to further increase in the aspect ratio. The bulk of their work was using ideal MHD and while Shi et al. (2016) did some numerical simulations with explicit dissipation coefficients, a more comprehensive survey of the effects of aspect ratio in the presence of dissipation has yet to be carried out.

The goal of this paper is to study the transport properties as a function of aspect ratio and dissipation coefficients. The organization of this paper is as follows: Section 2 describes the numerical setup. Section 3 reports on transport properties of large aspect ratio systems as a function of $Re$, $Pm$ and $L_z$. We then present resolution tests in section 4. In section 5, we describe the results from simulations where the domain is extended in both ‘y’ and ‘z’. We conclude in section 6.

2 NUMERICAL SETUP

Using the publicly available pseudospectral code SNOOPY (Lesur & Longaretti 2007), we solved the incompressible MHD simulations in the shearing box framework:

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{V}_s \cdot \nabla \mathbf{v} + \nabla \cdot \mathbf{T} = 2\Omega \mathbf{e}_z - (2 - q)\Omega \mathbf{v} \times \mathbf{e}_z + \nu \nabla^2 \mathbf{v},
\]

\[
\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{V}_s \times \mathbf{b}) + \nabla \times (\mathbf{v} \times \mathbf{b}) + \eta \nabla^2 \mathbf{b}.
\]

where $\mathbf{v}$ and $\mathbf{b}$ are the velocity and magnetic field respectively. Here $\mathbf{T}$ is a stress tensor given by

\[
\mathbf{T} = \left( \frac{\rho + b^2}{2} \right) \mathbf{I} + \nu \nabla \mathbf{v} - \mathbf{b} \times \mathbf{b},
\]

where $\mathbf{I}$ is the identity matrix and $\rho$ is thermal pressure.

We impose no external magnetic flux. The magnetic field is initialized with a sinusoidal profile: $B_{\text{ini}} = B_0 \sin(x_1/l) \mathbf{e}_z$. The background velocity profile is given by $V_s = -S \mathbf{x}_1 \mathbf{e}_z$, where $S = q\Omega = 1 \ (q = -d \ln \Omega / d \ln x = 3/2$ for Keplerian shear) is the shear parameter and $\Omega$ is the angular frequency. Since this state is linearly stable, we have to apply large amplitude perturbations to trigger nontrivial dynamics. The perturbations are of magnitude $LS$ applied to large scale velocity modes. The time unit in our simulations is the shear time, $1/S$ ($\sim 1/10$ orbits where 1 orbit = $2\pi/\Omega$). The magnetic field has the units of the Alfvén speed and $\mu_0 = 1$. The Reynolds and magnetic Reynolds numbers are $Re = SL_1^2/v$ and $Rm = SL_1^2/\eta$, respectively (with $L_z = 1$ and $Pm = Rm/Re$). We point out that the definition of $Re$ and $Rm$ we use is different from some of the previous works we refer to: compressible studies typically take the pressure scale height, $H$, as the length scale instead of $L_z$ while $L_z$ has been used by Lesur & Longaretti (2007) for incompressible studies.

3 LARGE ASPECT RATIO: $L_z/L_x > 4$

We describe the different transport properties in this section. Note that, unless otherwise indicated, all quantities are volume averaged over the entire domain and time averaged over $500$ -- $10000 \ S^{-1}$ (or roughly $50$ -- $1000$ orbits except for $L_z = 32$ which was only averaged over $500$ -- $1000 \ S^{-1}$). In the following, we compare our results with the only other study that focused on aspect ratio dependence of transport coefficients, Shi et al. (2016). It is important to mention two key differences in their simulations and ours: Shi et al. (2016) use ideal MHD for most of their work and solve compressible MHD equations. We, on the other hand, have run incompressible simulations with explicit dissipation coefficients. Moreover, we focus our attention to studying the effect of varying aspect ratios and dissipation coefficient at the same resolution while Shi et al. (2016) focused on resolution studies for different aspect ratios and had a small fraction of runs including explicit dissipation. A resolution study is presented in section 4 that does not show drastic differences between the resolution adapted for most of this work ($64/L_z$) and higher resolutions ($128/L_z$, $256/L_z$). We also point out that the runs $L_z/L_x = 16, 32$ are new to this paper and were not part of the study in Paper I.

3.1 Fixed $Re = 10,000, Pm = 1$, variable $L_z$

Shi et al. (2016) demonstrated using compressible ideal MHD simulations that the stresses converge with respect to resolution for $L_z/L_x > 2.5$ and that this saturation level was nearly independent of the aspect ratio beyond $L_z/L_x > 2.5$. In our incompressible simulations, we could not find any sustained turbulence for $L_z/L_x < 4$ at $Pm = 1$ so we are unable to confirm if $L_z/L_x > 2.5$ defines a threshold for convergence. In fig. 1, we plot the volume averaged stress, $a = (v_x v_y - b_x b_y)/S^2 L_1^2$ as a function of $L_z$, which seems to be slightly sensitive to the aspect ratio but does not show great variation going from $L_z/L_x = 4$ to 32. In the middle panel, we plot the Maxwell to Reynolds stress ratio, which seems to converge between 5 -- 6. Both this and the $a$ convergence are consistent with Shi et al. (2016) results (see their figures 6 and 7). The bottom panel shows that $a_{\text{mag}} = (-b_x b_y/b^2)$ remains approximately invariant 0.40 -- 0.43 as the aspect ratio is increased. This differs from the behavior reported in fig. 8 of Shi et al. (2016), where they found that the $a_{\text{mag}}$ decreased with increasing aspect ratio as $(L_z/L_x)^{-1/2}$. 
3.2 \( P m \) dependence

The dependence of \( a \) on dissipation coefficients is of interest since accretion disks come in a wide variety: protoplanetary disks that have \( P m \ll 1 \) and active galactic nuclei with \( P m \sim 1 \). Moreover, laboratory experiments of Taylor-Couette flow are typically done with liquid metal that have \( P m \in [10^{-7}, 10^{-3}] \). This makes detection of MRI in the lab a particularly challenging problem since MRI with a vertical field is hard to trigger for such low \( P m \). At low \( P m \), a net azimuthal or helical field might make it easier to trigger turbulence in magnetized Keplerian flows (Guseva et al. 2017b; see Rüdiger et al. 2018 for a review of laboratory MRI).

In Paper I, it was shown that turbulence could only sustain if \( L_z/L_x \geq 4 \) in the \( P m \sim 1 \) regime with no imposed field. Previous studies addressing the dependence of transport coefficients on dissipation were confined to small aspect ratio shearing boxes and a net magnetic flux (Lesur & Longaretti 2007; Fromang et al. 2007; Meheut et al. 2015). These previous works found that the \( P m \) dependence of \( a \) was very weak for \( P m < 1 \), while it followed a power law for \( P m > 1 \). Guseva et al. (2017b) found qualitative similar results for magnetized Taylor-Couette flow. Our results are mixed (fig. 2): for \( L_z/L_x = 4 \) stress is insensitive to \( P m \) at low \( P m \) and has power law dependence \( a \sim P m^{-1/2} \) for higher \( P m \) (bottom panel). However, \( L_z/L_x = 8 \) shows variations up to a factor of 3 for \( P m < 1 \) (top panel). Note, however, that a single run at each \( (L_z/L_x, P m) \) is potentially misleading and ideally one would want several runs initiated with different initial conditions (each evolved for a long time (\( \gg 100 \) orbits)).

3.3 Role of mean fields

We define the mean field \( \overline{\mathcal{Q}} \) as:

\[
\overline{\mathcal{Q}} = \frac{1}{L_x L_z} \int Q dx dy.
\]  

This definition of mean field is not universal but it makes comparison with Shi et al. (2016) easier since they also employed a horizontal average. The disadvantage of using a horizontal average is that the vertical small scales (close to dissipation) are also considered part of the 'mean'. In fig. 3, we plot the total and fluctuating \( a_{mag} \), Maxwell stress and the magnetic energy. The fluctuations in magnetic field are defined through: \( \mathcal{B} = \mathcal{B}^{\prime} + b^\prime \) while \( a_{mag}^{\prime} \) is defined as:

\[
a_{mag}^{\prime} = \frac{-b^\prime y}{b^2} \]  

where \( b^\prime \) is the fluctuating magnetic field. We do not observe clear differences in convergence between the two sets of quantities, \( a_{mag} \) and \( a_{mag}^{\prime} \) (see top panel of fig. 3). This is in contrast to Shi et al. (2016) who found that it is the \( a_{mag}^{\prime} \) that is invariant with vertical aspect ratio but the total \( a_{mag} \) decreases. Furthermore, Shi et al. (2016) observed that while the total stress \( a \) converged for aspect ratios \( L_z/L_x > 2.5 \), the total \( a_{mag} \) decreased as \( (L_z/L_x)^{-1/2} \) with increasing aspect ratio. For the stresses, no clear difference in trends seems
to exist as a function of aspect ratios between the total or the fluctuating quantities (see also fig. 1). The mean \( a_{\text{mag}}^{\text{mean}} = -\overline{B_x B_y / B^2} \sim 10^{-2} \) for all the aspect ratios considered here and is considerably smaller than the fluctuating contribution in agreement with Blackman & Nauman (2015); Shi et al. (2016). However, the total and fluctuating magnetic energies (bottom panel on left and right) seem to increase with aspect ratio.

4 RESOLUTION TESTS

The results reported in this paper describe the statistics of quadratic quantities (stresses and energies) as a function of dissipation coefficients and aspect ratios. One might ask if these results are sensitive to the numerical resolution employed \((64/L_z)\). We first note that pseudo-spectral methods (as employed in sNOOPY) are more accurate than finite difference methods: a rule of thumb is that second order (central difference) finite difference schemes require twice as much resolution as a pseudo-spectral scheme to achieve the same level of accuracy (Moin & Mahesh (1998)). Secondly, higher order moments (for example, fourth order) require higher resolution but the resolution requirements are not as stringent on quadratic quantities like energies or stresses (Yakhut & Sreenivasan (2005); Donzis et al. (2008)).

Nonetheless, it is useful to do a resolution study to check if there are any drastic differences between the resolutions employed in this paper and higher resolutions. For this purpose, for two runs with \( L_x = 1, L_y = 2, L_z = 4, Rm = 10000 \) having different \( Pm \) (\( = 1, 10 \)), we study (i) the time history of volume averaged stresses in fig. 4; and (ii) the power spectrum of \( \langle v_x \rangle \) as a function of \( k_x \) in fig. 5. The plots in fig. 5 were calculated in postprocessing and were consequently time averaged over significantly smaller duration as well as snapshots as the size of each snapshot significantly increases with resolution. We do not notice any noticeable difference between the lower resolution \( 64/L_z \) and higher resolutions \( 128/L_z, 256/L_z \) (fig. 5).

Higher resolution runs are computationally very expensive. For instance, the run at \( 256/L_z \) for \( L_x = 4, Re = Rm = 10000 \) costed \( \sim 1.5 \times 10^5 \) CPU hours for just \( 2000 s^{-1} \) while all the runs combined in fig. 1 took less than \( 10^5 \) CPU hours. These runs become even more expensive as the energies increase with \( Pm \). The \( L_x = 4, Re = 1000, Pm = 10 \) run took \( 2 \times 10^6 \) CPU hours to evolve this run to 10000\( s^{-1} \).

5 EXTENDED Y AND Z DOMAINS

In the previous section, we discussed transport properties of simulations with \( L_x = 1, L_y = 2 \) and variable \( L_z \). This is largely the set of runs Paper I was based on. In order to explore the sensitivity of the domain size in the \( y \) direction, we conducted two more numerical simulations with \( L_x = L_y = 4, 8 \). Large domains in both \( y \) and \( z \) directions are computationally prohibitive since, for example, a resolution of \( 64/L_z \) with \( L_x = L_y = 16 \) would amount to simulating a box with \( 64 \times 1024 \times 1024 \) grid points. We use the same resolution as before \((64/L_z)\) but a lower Reynolds numbers, \( Re = Rm = 4000 \). In fig. 5, we plot the evolution of turbulent stresses for the two runs considered in this section. We find that the stress for the \( L_x = L_y = 4 \) remains relatively similar to \( L_x = 2, L_y = 4 \) case in fig. 1 (\( \alpha \sim 5 \times 10^{-3} \) here as opposed to \( 2 \times 10^{-3} \) but \( L_x = L_y = 8 \) reaches higher saturated values (\( \alpha \sim 2 \times 10^{-4} \) that is about 50 times larger than the \( L_x = 2, L_y = 8 \) case in fig. 1). This result is very intriguing since the \( Re, Rm \) used for these runs is actually lower than the ‘tall’ box runs so one would naively expect a lower \( \alpha \) here.

When averaged over the time period \( 250 \sim 5000 s^{-1} \), the
extended $L_z = L_x = 8$ behaves similar to the $L_x = 2, L_y = 8$ simulation reported in the previous section: $\langle -b_x b_y \rangle \sim \langle b'_x b'_y \rangle$ and $\langle \vec{B}^2 \rangle \sim \langle \vec{B}_1^2 \rangle$. However, in the saturated state ($\gtrsim 500$ $S^{-1}$), there is a significant change in behavior: the magnetic energy is dominated by the mean fields: $\langle \vec{B}^2 \rangle \sim \langle \vec{B} \rangle$ but the stress is still dominated by the fluctuating component. In fig. 5, we show the snapshot of the azimuthal magnetic for the two $L_z = 8$ runs: one with $L_x = 2, Re = Rm = 10000$ and the other with $L_x = 8, Re = Rm = 4000$. In both cases, the field has considerable structure in the vertical direction. A strong banded structure parallel to the $y$ axis forms in the extended $yz$ domain case while the ‘tall’ box case has weaker bands but more of them (see also Walker & Boldyrev (2017)).

6 CONCLUSIONS

Most studies of magnetized Keplerian flows have focused on an imposed magnetic flux in a small domain. Some of the literature has focused on studying the effects of dissipation coefficients on sustained turbulence (Lesur & Longaretti 2007; Meheut et al. 2015). Recent work has suggested that domain size might play a key role when there is no imposed flux (Shi et al. 2016; Nauman & Pessah 2016). This work is a survey of the effects of large aspect ratios and dissipation coefficients on transport coefficients. We have found that:

(i) For fixed $L_x/L_y = 2$, $\alpha$ is not sensitive to $Pm$ for $Pm < 1$ and follows a power law $\alpha \sim Pm^{1.5}$ for $Pm > 1$.

(ii) For fixed $L_x/L_y = 2$, the turbulent stresses $\alpha, \alpha_{mag}$ and $\langle \Delta b_x b_y \rangle/\langle \nu \nu \rangle$ are all nearly insensitive to increase in $L_x/L_y$.

(iii) For $L_x = L_y \gg L_z$, the saturated level of $\alpha$ increases significantly especially with $L_x = L_y = 8$ where $\alpha \sim 0.2$.

Our study highlights the importance of aspect ratio that we first pointed out in Paper I and motivates further work that might give important insights into the role of turbulence in magnetized Keplerian flows.

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Figure 6. Volume averaged stress, $\alpha = \langle v_x v_y - b_x b_y \rangle / L_y^2 S^2$ as a function of time for $L_y = L_z = 4, 8$ at $Re = 4,000, Pr = 1$. We find that the stresses increase with $L_y/L_x$ quite significantly and that the saturated level of stresses for $L_y/L_x = 8$ is ∼ 40 times larger than the case with $L_y = L_z = 4$.

Figure 7. Snapshot of $B_y$ at $t = 1000 S^{-1}$ averaged over ‘x’. The colorbar at the bottom indicates nearly an order of magnitude difference in magnitudes of $B_y$ for the two cases. Left: Some hint for large scale structure exists. Right: The extended ‘yz’ domain case presents some strong bands that seem to be also responsible for $B^2 - B^3$.

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