ESTIMATES ON SOME QUADRATURE RULES VIA WEIGHTED HERMITE-HADAMARD INEQUALITY

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In this article new estimates on some quadrature rules are given using weighted Hermite-Hadamard inequality for higher order convex functions and weighted version of the integral identity expressed by \( w \)-harmonic sequences of functions. Obtained results are applied to weighted one-point formula for numerical integration in order to derive new estimates of the definite integral values.

1. INTRODUCTION

Weighted Hermite-Hadamard inequality for convex functions is given in the following theorem ([4], [5]).

**Theorem A.** Let \( p : [a, b] \to \mathbb{R} \) be a nonnegative function. If \( f \) is a convex function given on an interval \( I \), then we have

\[
f(\lambda) \leq \frac{1}{P(b)} \int_a^b p(x)f(x) \, dx \leq \frac{b-\lambda}{b-a} f(a) + \frac{\lambda-a}{b-a} f(b)
\]

or

\[
P(b)f(\lambda) \leq \int_a^b p(x)f(x) \, dx \leq P(b) \left[ \frac{b-\lambda}{b-a} f(a) + \frac{\lambda-a}{b-a} f(b) \right],
\]

*Corresponding author. Josipa Barić
2020 Mathematics Subject Classification. 26D15, 65D30, 65D32
Keywords and Phrases. Hermite-Hadamard inequality, Higher order convex function, Harmonic sequence, One-point formula.
where
\[ P(t) = \int_a^t p(x) \, dx \quad \text{and} \quad \lambda = \frac{1}{P(b)} \int_a^b p(x) x \, dx. \]

Various weighted versions of the general integral identities that are used for the approximation of an integral \( \int_a^b f(t) \, dt \), using the harmonic sequences of polynomials and \( w \)-harmonic sequences of functions, are obtained in [3]. For introducing one of those identities let us consider subdivision \( \sigma = \{ a = x_0 < x_1 < \cdots < x_m = b \} \) of the segment \([a, b], m \in \mathbb{N}\). Let \( w : [a, b] \to \mathbb{R} \) be an arbitrary integrable function. For each segment \([x_{k-1}, x_k], k = 1, \ldots, m\), we define \( w \)-harmonic sequences of functions \( \{w_{k,j}\}_{j=1}^n \) by:

\[
\begin{align*}
\frac{w'_{k1}(t)}{w'_{k1}(t)} &= w(t), & t & \in [x_{k-1}, x_k], \\
\frac{w'_{kj}(t)}{w'_{k1}(t)} &= w_{k,j-1}(t), & t & \in [x_{k-1}, x_k], \quad j = 2, 3, \ldots, n.
\end{align*}
\]

Also, we define function \( W_{n,w} \) as follows:

\[
W_{n,w}(t, \sigma) = \begin{cases} 
\frac{w_{1n}(t)}{w_{1n}(t)}, & t \in [a, x_1], \\
\frac{w_{2n}(t)}{w_{1n}(t)}, & t \in (x_1, x_2], \\
\vdots \\
\frac{w_{mn}(t)}{w_{1n}(t)}, & t \in (x_{m-1}, b].
\end{cases}
\]

**Theorem B.** If \( g : [a, b] \to \mathbb{R} \) is such that \( g^{(n)} \) is a piecewise continuous on \([a, b]\), then the following identity holds

\[
\int_a^b w(t)g(t) \, dt = \sum_{j=1}^n (-1)^{j-1} \left[ w_{mj}(b)g^{(j-1)}(b) \right. \\
+ \sum_{k=1}^{m-1} \left[ w_{k,j}(x_k) - w_{k+1,j}(x_k) \right] g^{(j-1)}(x_k) - w_{1j}(a)g^{(j-1)}(a) \\
+ \left. (-1)^n \int_a^b W_{n,w}(t, \sigma)g^{(n)}(t) \, dt. \right]
\]

More recently obtained results on weighted versions of the general integral identities and harmonic sequences of polynomials or \( w \)-harmonic sequences of functions can be found in [1], [2], [6] and their references.
2. NEW RESULTS

In this section we derive Hermite-Hadamard’s type inequalities using weighted version of the integral identity expressed by \( w \)-harmonic sequences of functions that is given in Theorem B.

**Theorem 1.** Suppose \( w : [a, b] \to \mathbb{R} \) is an arbitrary integrable function and \( w \)-harmonic sequences of functions \( \{w_{kj}\}_{j=1,...,n} \) are defined by (2). Let the function \( W_{n,w} \), defined by (3), be nonnegative. Then,

a) if \( g : [a, b] \to \mathbb{R} \) is \((n+2)\)-convex function, the following inequalities hold

\[
(-1)^n \cdot P(b) \cdot g^{(n)}(\lambda) 
\leq \int_a^b w(t)g(t) \, dt - \sum_{j=1}^n (-1)^{j-1} \left[ w_{mj}(b)g^{(j-1)}(b) 
+ \sum_{k=1}^{m-1} (w_{kj}(x_k) - w_{k+1,j}(x_k)) g^{(j-1)}(x_k) - w_{1j}(a)g^{(j-1)}(a) \right] 
\leq (-1)^n \cdot P(b) \cdot \left[ \frac{b-\lambda}{b-a} g^{(n)}(a) + \frac{\lambda-a}{b-a} g^{(n)}(b) \right],
\]

where

\[
P(b) = (-1)^n \left[ \frac{1}{n!} \int_a^b w(t) \cdot t^n \, dt - \sum_{j=1}^n (-1)^{j-1} \frac{1}{(n-j+1)!} \right] 
\cdot \left( w_{mj}(b)b^{n-j+1} + \sum_{k=1}^{m-1} (w_{kj}(x_k) - w_{k+1,j}(x_k)) x_k^{n-j+1} - w_{1j}(a)a^{n-j+1} \right),
\]

and

\[
\lambda = (-1)^n \left[ \frac{1}{(n+1)!} \int_a^b w(t) \cdot t^{n+1} \, dt - \frac{1}{P(b)} \sum_{j=1}^n (-1)^{j-1} \frac{1}{(n-j+2)!} \right] 
\cdot \left( w_{mj}(b)b^{n-j+2} + \sum_{k=1}^{m-1} (w_{kj}(x_k) - w_{k+1,j}(x_k)) x_k^{n-j+2} - w_{1j}(a)a^{n-j+2} \right),
\]

b) if \( g \) is \((n+2)\)-concave function, then (5) holds with the reversed sign of inequalities.

**Proof.** a) Inequality (5) follows from the weighted Hermite-Hadamard inequality (1) substituting nonnegative function \( p \) by nonnegative function \( W_{n,w} \) and...
convex function $f$ by convex function $g^{(n)}$. To obtain desired result, we need to calculate the values of $P(b)$ and $\lambda$ in the terms of new substitutions.

The value of $P(b)$ will be obtained from (4) taking $g(t) = \frac{t^n}{n!}$. Then, $g^{(n)}(t) = 1$ and, regarding the formula of $P(t)$ from Theorem A, we get

$$P(b) = \int_a^b W_{n,w}(t, \sigma) \, dt$$

$$= (-1)^n \int_a^b w(t) \frac{t^n}{n!} \, dt - (-1)^n \sum_{j=1}^{n} (-1)^{j-1} \left[ \frac{w_{mj}(b)}{(n-j+1)!} b^{n-j+1} \right.$$  

$$+ \sum_{k=1}^{m-1} \left[ w_{kj}(x_k) - w_{k+1,j}(x_k) \right] \frac{1}{(n-j+1)!} x_k^{n-j+1}$$

$$- w_{1j}(a) \frac{1}{(n-j+1)!} a^{n-j+1} \right].$$

Rearranging above equality we get (6).

Applying our substitutions to the formula of $\lambda$ from Theorem A we get $\lambda = \frac{1}{P(b)} \int_a^b W_{n,w}(t, \sigma) \cdot t \, dt$. The value of this integral follows from (4) taking $g(t) = \frac{t^{n+1}}{(n+1)!}$. Then, $g^{(n)}(t) = t$ and $g^{(j-1)}(t) = \frac{(n+1)\cdots(n-j+3)}{(n+1)!} t^{n-j+2} = \frac{1}{(n-j+2)!} t^{n-j+2}$. Equality (7) follows.

Now, applying inequality (1) to function $W_{n,w}$ instead of $p$, and function $g^{(n)}$ instead of $f$, and replacing $(-1)^n \int_a^b W_{n,w}(t, \sigma) g^{(n)} \, dt$ by the expression from the identity (4), we get inequality (5).

b) If $W_{n,w}(t, \sigma) \leq 0$ for all $t \in [a, b]$, then $-W_{n,w}(t, \sigma) \geq 0$, $t \in [a, b]$, so applying the step a) of this proof, we get the required result.

c) If $g$ is $(n+2)$-concave function, i.e. $-g^{(n+2)} \geq 0$, then $-g^{(n)}$ is a convex function so applying weighted Hermite-Hadamard inequalities on convex function $-g^{(n)}$ we get reversed inequalities in (5).

In order to obtain our next result, let us expand $w$-harmonic sequences of functions $\{w_{kj}\}_{j=1,\ldots,n}$ by $w_{k,n+1}$, such that $w_{k,n+1}(t) = w_{k,n}(t)$ for $t \in [x_{k-1}, x_k]$. 

\(\square\)
Now, function $W_{n+1,w}$ has the following form:

\[
W_{n+1,w}(t, \sigma) = \begin{cases} 
  w_{1,n+1}(t), & t \in [a, x_1], \\
  w_{2,n+1}(t), & t \in (x_1, x_2], \\
  . & . \\
  . & . \\
  . & . \\
  w_{m,n+1}(t), & t \in (x_{m-1}, b]. 
\end{cases}
\]  

(8)

**Theorem 2.** Let $g : [a, b] \to \mathbb{R}$ be $(n + 2)$-convex on $[a, b]$. Suppose $w : [a, b] \to \mathbb{R}$ is an arbitrary integrable function and $\left\{w_j\right\}_{j=1, \ldots, n+1}$ are $w$-harmonic sequences of functions. Let the function $W_{n+1,w}$, defined by (8), be nonnegative. Then, inequality (5) is valid for

\[
P(b) = w_{m,n+1}(b) + \sum_{k=1}^{m-1} \left[ w_{k,n+1}(x_k) - w_{k+1,n+1}(x_k) \right] - w_{1,n+1}(a)
\]

and

\[
\lambda = \frac{1}{P(b)} \left[ bw_{m,n+1}(b) - aw_{1,n+1}(a) 
  + \sum_{k=1}^{m-1} \left( x_k w_{k,n+1}(x_k) - x_k \cdot w_{k+1,n+1}(x_k) \right) - w_{m,n+2}(b) 
  - \sum_{k=1}^{m-1} \left( w_{k,n+2}(x_k) - w_{k+1,n+2}(x_k) \right) + w_{1,n+2}(a) \right].
\]

(9)

If $W_{n,w}(t, \sigma) \leq 0$ or $g$ is $(n + 2)$-concave function, then (5) holds with the reversed sign of inequalities.

**Proof.** We calculate only $P(b)$ and $\lambda$. Replacing, in Theorem A, $p$ by $W_{n,w}$ and $f$ by $g^{(n)}$, where $g(t) = \frac{t^{n+1}}{(n+1)!}$, we get

\[
P(b) = \int_a^b W_{n,w}(t, \sigma) \, dt
\]

\[
= \sum_{k=1}^m \int_{x_{k-1}}^{x_k} w_{k,n}(t) \, dt
\]

\[
= \sum_{k=1}^m \int_{x_{k-1}}^{x_k} w'_{k,n+1}(t) \, dt
\]

\[
= w_{m,n+1}(b) + \sum_{k=1}^{m-1} \left[ w_{k,n+1}(x_k) - w_{k+1,n+1}(x_k) \right] - w_{1,n+1}(a)
\]
and

\[ \lambda = \frac{1}{P(b)} \int_a^b W_{n,w}(t, \sigma) \cdot t \, dt \]

\[ = \frac{1}{P(b)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} w_{k,n}(t) \cdot t \, dt \]

\[ = \frac{1}{P(b)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} w'_{k,n+1}(t) \cdot t \, dt \]

\[ = \frac{1}{P(b)} \sum_{k=1}^{m} \left( x_k \cdot w_{k,n+1}(x_k) - x_{k-1} \cdot w_{k,n+1}(x_{k-1}) - \int_{x_{k-1}}^{x_k} w_{k,n+1}(t) \, dt \right) \]

\[ = \frac{1}{P(b)} \left[ bw_{m,n+1}(b) - aw_{1,n+1}(a) + \sum_{k=1}^{m-1} (x_k w_{k,n+1}(x_k) - x_k \cdot w_{k+1,n+1}(x_k)) - \int_a^b W_{n+1,w}(t, \sigma) \, dt \right] . \]

Adding the function \( w_{k,n+2} \) to the \( w \)-harmonic sequences of functions \( \{w_{kj}\}_{j=1,...,n+1} \), such that \( w'_{k,n+2}(t) = w_{k,n+1}(t), \, t \in [x_{k-1}, x_k] \) and rewriting the last integral in (10) in the following sense

\[ \int_a^b W_{n+1,w}(t, \sigma) \, dt = \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} w_{k,n+1}(t) \, dt \]

\[ = \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} w'_{k,n+2}(t) \, dt \]

\[ = \sum_{k=1}^{m} (w_{k,n+2}(x_k) - w_{k,n+2}(x_{k-1})) \]

\[ = w_{m,n+2}(b) + \sum_{k=1}^{m-1} (w_{k,n+2}(x_k) - w_{k+1,n+2}(x_k)) - w_{1,n+2}(a) , \]

identity (9) is obtained.
3. ONE-POINT FORMULA

In this section we apply obtained results of previous section to weighted one-point formula for numerical integration. We observe function \( g : [a, b] \to \mathbb{R} \), integrable function \( w : [a, b] \to \mathbb{R} \) and \( w \)-harmonic sequences of functions \( \{w_{kj}\}_{j=0,1,...,n} \) on \([x_{k-1}, x_{k}]\), where \( k = 1, 2 \). We consider subdivision \( \sigma = \{a = x_0 < x_1 = x < x_2 = b\} \) of the segment \([a, b]\) and we assume \( w_{1j}(a) = 0 \) and \( w_{2j}(b) = 0 \), for \( j = 1,...,n \). In [3] authors proved the following theorem.

**Theorem C.** Let \( w : [a, b] \to \mathbb{R} \) be an integrable function and \( x \in [a, b] \). Further, let us suppose \( \{w_{kj}\}_{j=1,...,n} \) are \( w \)-harmonic sequences of functions on \([x_{k-1}, x_{k}]\), for \( k = 1, 2 \) and some \( n \in \mathbb{N} \), defined by the following relations:

\[
\begin{align*}
  w_{1j}(t) &= \frac{1}{(j-1)!} \int_a^t (t-s)^{j-1} w(s) \, ds, \quad t \in [a, x], \\
  w_{2j}(t) &= \frac{1}{(j-1)!} \int_t^b (t-s)^{j-1} w(s) \, ds, \quad t \in (x, b],
\end{align*}
\]

for \( j = 1,...,n \). If \( g : [a, b] \to \mathbb{R} \) is such that \( g^{(n-1)} \) is absolutely continuous function, then we have

\[
\int_a^b w(t)g(t) \, dt = \sum_{j=1}^n A_j(x)g^{(j-1)}(x) + (-1)^n \int_a^b W_{n,w}(t,x)g^{(n)}(t) \, dt,
\]

where for \( j = 1,...,n \)

\[
\begin{align*}
  A_j(x) &= \frac{(-1)^{j-1}}{(j-1)!} \int_a^b (x-s)^{j-1} w(s) \, ds \\
  W_{n,w}(t,x) &= \begin{cases} 
    w_{1n}(t) = \frac{(-1)^{n-1}}{(n-1)!} \int_a^t (t-s)^{n-1} w(s) \, ds, \quad t \in [a, x] \\
    w_{2n}(t) = \frac{(-1)^{n-1}}{(n-1)!} \int_t^b (t-s)^{n-1} w(s) \, ds, \quad t \in (x, b].
  \end{cases}
\end{align*}
\]

Using integral identity (11), in the following theorem we obtain new estimates of the definite integral as a special case of the Theorem 1.

**Theorem 3.** Let \( w : [a, b] \to \mathbb{R} \) be an integrable function and \( x \in [a, b] \) fixed. Suppose \( \{w_{kj}\}_{j=1,...,n} \) are \( w \)-harmonic sequences of functions on \([x_{k-1}, x_{k}]\), for \( k =
1, 2 and \( n \in \mathbb{N} \), defined in Theorem C. Let the function \( W_{n,w}(x, \sigma) \), defined by (13), be nonnegative. If \( g : \left[ a, b \right] \rightarrow \mathbb{R} \) is \((n + 2)\)-convex function, then

\[
(-1)^n \cdot P(b) \cdot g^{(n)}(\lambda) \leq \int_a^b w(t)g(t) \, dt - \sum_{j=1}^n A_j(x) \cdot g^{(j-1)}(x)
\]

\[
= (-1)^n \cdot P(b) \cdot \left[ b - \lambda - \frac{a}{b-a} g^{(n)}(a) + \frac{\lambda - a}{b-a} g^{(n)}(b) \right],
\]

where

\[
P(b) = (-1)^n \left[ \frac{1}{n!} \int_a^b w(t) \cdot t^n \, dt - \sum_{j=1}^n \frac{x^{n-j+1}}{(n-j+1)!} \cdot A_j(x) \right],
\]

\[
\lambda = \frac{(-1)^n}{P(b)} \left[ \frac{1}{(n+1)!} \int_a^b w(t) \cdot t^{n+1} \, dt - \sum_{j=1}^n \frac{x^{n-j+2}}{(n-j+2)!} \cdot A_j(x) \right]
\]

and \( A_j \) is defined as in Theorem C. If \( W_{n,w}(t, \sigma) \leq 0 \) or \( g \) is \((n + 2)\)-concave then (14) holds with the reversed sign of inequalities.

**Proof.** Inequality (14) follows directly from (1) replacing nonnegative function \( p \) by nonnegative function \( W_{n,w} \) and convex function \( f \) by convex function \( g^{(n)} \) and then applying identity (11) on \((-1)^n \int_a^b W_{n,w}(t, x)g^{(n)}(t) \, dt\).

Now, we calculate \( P(b) \) and \( \lambda \) using formulas from Theorem 1 and the facts that in new subdivision \( \sigma = \{ a = x_0 < x_1 = x < x_2 = b \} \) of the segment \([a, b]\) we have: \( m = 2 \) and \( x_1 = x \).

\[
P(b) = (-1)^n \left[ \frac{1}{n!} \int_a^b w(t) \cdot t^n \, dt 
- \sum_{j=1}^n \frac{(-1)^{j-1}}{(n-j+1)!} \cdot \left( w_{2j}(b)x^{n-j+1} + w_{1j}(x) \cdot x^{n-j+1} 
- w_{2j}(x) \cdot x^{n-j+1} - w_{1j}(a)x^{n-j+1} \right) \right].
\]

By the assumptions from the beginning of this section: \( w_{1j}(a) = 0 \) and \( w_{2j}(b) = 0 \), for \( j = 1, \ldots, n \). Then,

\[
P(b) = (-1)^n \left[ \frac{1}{n!} \int_a^b w(t) \cdot t^n \, dt 
- \sum_{j=1}^n \frac{(-1)^{j-1}}{(n-j+1)!} \cdot \left( w_{1j}(x) - w_{2j}(x) \right) \cdot x^{n-j+1} \right].
\]
Using definitions of \( \{w_{kj}\} \) from Theorem C, we calculate:

\[
\begin{align*}
\ w_{1j}(x) - w_{2j}(x) &= \frac{1}{(j-1)!} \int_a^b (x-s)^{j-1} w(s) \, ds.
\end{align*}
\]

Now, it follows

\[
\begin{align*}
P(b) &= (-1)^n \left[ \frac{1}{n!} \int_a^b w(t)t^n \, dt \right. \\
&\quad \left. - \sum_{j=1}^n (-1)^{j-1} \frac{(n-j)!}{(n-j+1)!} \cdot \frac{x^{n-j+1}}{(j-1)!} \int_a^b (x-s)^{j-1} w(s) \, ds \right] \\
&= (-1)^n \left[ \frac{1}{n!} \int_a^b w(t)t^n \, dt - \sum_{j=1}^n \frac{A_j(x)}{(n-j+1)!} \cdot x^{n-j+1} \right].
\end{align*}
\]

Similarly, using Theorem 1 for subdivision \( \sigma = \{a = x_0 < x_1 = x < x_2 = b\} \), \( m = 2 \) and \( x_1 = x \), we calculate \( \lambda \).

\[
\begin{align*}
\lambda &= \frac{(-1)^n}{P(b)} \left[ \frac{1}{(n+1)!} \int_a^b w(t) \cdot t^{n+1} \, dt - \sum_{j=1}^n \frac{(-1)^{j-1}}{(n-j+2)!} \\
&\quad \cdot (w_{2j}(b)b^{n-j+2} + w_{1j}(x) \cdot x^{n-j+2} - w_{2j}(x) \cdot x^{n-j+2} - w_{1j}(a)a^{n-j+2}) \right]
\end{align*}
\]

Since, \( w_{1j}(a) = 0 \) and \( w_{2j}(b) = 0 \), for \( j = 1, ..., n \), it follows

\[
\begin{align*}
\lambda &= \frac{(-1)^n}{P(b)} \left[ \frac{1}{(n+1)!} \int_a^b w(t) \cdot t^{n+1} \, dt - \sum_{j=1}^n \frac{x^{n-j+2}}{(n-j+2)!} \cdot A_j(x) \right].
\end{align*}
\]

Following the reasoning of Theorem 2 now we expand \( w \)-harmonic sequences of functions \( \{w_{kj}\}_{j=1,...,n} \) by \( w_{k,n+1} \) and \( w_{k,n+2} \) such that \( w'_{k,n+1}(t) = w_{k,n}(t) \) and \( w'_{k,n+2}(t) = w_{k,n+1}(t) \), \( t \in [x_{k-1}, x_k] \). For a new subdivision \( \sigma = \{a = x_0 < x_1 = x < x_2 = b\} \) of the segment \( [a, b] \) and the values \( w_{1j}(a) = 0 \) and \( w_{2j}(b) = 0 \), for \( j = 1, ..., n+2 \), we obtain the following result.

**Theorem 4.** Let \( w : [a, b] \to \mathbb{R} \) be an integrable function and \( x \in [a, b] \) fixed. Suppose \( \{w_{kj}\}_{j=1,...,n+2} \) are \( w \)-harmonic sequences of functions on \([x_{k-1}, x_k], k = 1, 2, n \in \mathbb{N}, \) defined in Theorem C. Let the function \( W_{n,w}, \) defined by (13), be
nonnegative. If \( g : [a, b] \to \mathbb{R} \) is \((n + 2)\)-convex function then inequality (14) is valid for
\[
P(b) = w_{1,n+1}(x) - w_{2,n+1}(x)
\]
and
\[
\lambda = x - \frac{1}{P(b)} (w_{1,n+2}(x) - w_{2,n+2}(x)).
\]
If \( W_{n,w}(t, \sigma) \leq 0 \) or \( g \) is \((n + 2)\)-concave function then (14) holds with the reversed sign of inequalities.

Proof. The values of \( P(b) \) and \( \lambda \) follow from the proof of Theorem 2, since \( m = 2 \), \( x_1 = x \), \( w_{1j}(a) = 0 \) and \( w_{2j}(b) = 0 \), for \( j = 1, \ldots, n + 2 \) and from the definitions of \( \{w_{kj}\} \) from Theorem C.

Using integral mean value theorem to the
\[
\int_a^b W_{2n,w}(t, x) g^{(2n)}(t) \, dt,
\]
where \( g : [a, b] \to \mathbb{R} \) is such that \( g^{(2n)} \) is a continuous function, in [3, Theorem 5] authors proved that there exists \( \nu \in (a, b) \) such that
\[
\int_a^b w(t) g(t) \, dt - \sum_{j=1}^{2n} A_j(x) g^{(j-1)}(x) = A_{2n+1}(x) g^{(2n)}(\nu).
\]

Applying this integral identity to our result in inequality (14), we obtain the following theorem.

**Theorem 5.** Assume \( w \) and \( \{w_{kj}\} \) satisfies the conditions of Theorem 4 for \( j = 1, \ldots, 2n+1 \). Let \( A_j \) be defined as in (12). If \( g : [a, b] \to \mathbb{R} \) is \((2n + 2)\)-convex, then there exists \( \nu \in (a, b) \) such that
\[
P(b) \cdot g^{(2n)}(\lambda)
\]
\[
\leq \frac{g^{(2n)}(\nu)}{(2n)!} \int_a^b (x-s)^{2n} \cdot w(s) \, ds
\]
\[
\leq P(b) \cdot \left[ \frac{b - \lambda}{b - a} g^{(2n)}(a) + \frac{\lambda - a}{b - a} g^{(2n)}(b) \right],
\]
where
\[
P(b) = \frac{1}{(2n)!} \int_a^b w(t) \cdot t^{2n} \, dt - \sum_{j=1}^{2n} \frac{x^{2n-j+1}}{(2n-j+1)!} \cdot A_j(x)
\]
and
\[
\lambda = \frac{1}{P(b)} \left[ \frac{1}{(2n+1)!} \int_a^b w(t) \cdot t^{2n+1} \, dt - \sum_{j=1}^{2n} \frac{x^{2n-j+2}}{(2n-j+2)!} \cdot A_j(x) \right].
\]
Proof. Inequality (16) follows directly from (14) replacing its middle term by $A_{2n+1}(x)g^{(2n)}(\nu)$, according to the integral identity (15), and then applying (12) to $A_{2n+1}$.

4. SPECIAL CASES

Taking some special cases of the weight function $w$, in our results of the previous section, we obtain following estimates for the definite integral.

Example 1. Let us assume that $w(t) = 1$, $t \in [a, b]$. Now, from Theorem C, we calculate

$$W_{n,w}(t, x) = \begin{cases} w_1(t) = \frac{(t-a)^n}{n!}, & t \in [a, x] \\ w_2(t) = \frac{(t-b)^n}{n!}, & t \in (x, b] \end{cases}$$

and

$$A_j(x) = \frac{1}{j!} \left[ (b-x)^j - (a-x)^j \right].$$

In order to apply new estimates from Theorem 3 to the function $w(t) = 1$, $t \in [a, b]$, we will replace $n$, in the definition of the $W_{n,w}$, by $2n$ to provide the nonnegativity of $W_{n,w}$ and we will assume that $g : [a, b] \to \mathbb{R}$ is $(2n+2)$-convex since then $g^{(2n)}$ is also convex function. Now, according to (14), we get

$$P(b) \cdot g^{(2n)}(\lambda) \leq \int_a^b g(t) \, dt - \sum_{j=1}^{2n} \frac{g^{(j-1)}(x)}{j!} \left[ (b-x)^j - (a-x)^j \right] \leq P(b) \left[ \frac{b-\lambda}{b-a} g^{(2n)}(a) + \frac{\lambda-a}{b-a} g^{(2n)}(b) \right],$$

where

$$P(b) = \frac{b^{2n+1} - a^{2n+1}}{(2n+1)!} - \sum_{j=1}^{2n} \frac{x^{2n-j+1}}{j!(2n-j+1)!} \cdot \left( (b-x)^j - (a-x)^j \right)$$

and

$$\lambda = \frac{1}{P(b)} \left[ \frac{b^{2n+2} - a^{2n+2}}{(2n+2)!} - \sum_{j=1}^{2n} \frac{x^{2n-j+2}}{j!(2n-j+2)!} \cdot \left( (b-x)^j - (a-x)^j \right) \right].$$

Values of $P(b)$ and $\lambda$ for the function $w(t) = 1$, $t \in [a, b]$, can also be calculated using the results of Theorem 4 as follows.
\[ P(b) = w_{1,2n+1}(x) - w_{2,2n+1}(x) \]
\[ = \frac{1}{(2n+1)!} \left[ (x-a)^{2n+1} - (x-b)^{2n+1} \right]. \]

\[ \lambda = x - \frac{1}{P(b)} \left[ w_{1,2n+2}(x) - w_{2,2n+2}(x) \right] \]
\[ = x - \frac{1}{(2n+2)!P(b)} \left[ (x-a)^{2n+2} - (x-b)^{2n+2} \right]. \]

If the assumptions of Theorem 5 hold, for \( w(t) = 1, \ t \in [a,b] \), we get
\[ P(b) \cdot g^{(2n)}(\lambda) \leq g^{(2n)}(\nu) \left[ (x-a)^{2n+1} - (x-b)^{2n+1} \right] \]
\[ \leq P(b) \cdot \left[ \frac{b-\lambda}{b-a} g^{(2n)}(a) + \frac{\lambda-a}{b-a} g^{(2n)}(b) \right], \]
where \( P(b) \) and \( \lambda \) have the same values as in (17) and (18) respectively.

**Example 2.** Suppose that \( w(t) = (b-t)^\alpha \cdot (t-a)^\beta, \ t \in [a,b], \ \alpha, \beta > -1. \)
From Theorem C, taking substitution \( x = \frac{b-s}{b-a} \) in the definition of Beta function, we get

\[ W_{n,w}(t,x) = \begin{cases} \frac{\left( (b-a)^\alpha (t-a)^{n+\beta} \right)}{(n-1)!} B(\beta+1,n) \\ \cdot F\left( -\alpha, \beta+1, \beta+n+1; \frac{b-a}{b-a} \right), \ t \in [a] \\ (-1)^n \frac{\left( (b-a)^\alpha (b-t)^{n+\beta} \right)}{(n-1)!} B(\alpha+1,n) \\ \cdot F\left( -\beta, \alpha+1, \alpha+n+1; \frac{b-1}{b-a} \right), \ t \in (a,b] \end{cases} \]

and

\[ A_j(x) = \begin{cases} \frac{\left( (a-x)^\alpha (b-a)^{n+\beta+1} \right)}{(j-1)!} B(\alpha+1,\beta+1) \\ \cdot F\left( 1-j, \beta+1, \alpha+\beta+2; \frac{b-a}{x-a} \right), \ x \neq a, \\ \frac{\left( (b-a)^\alpha (x-a)^{n+\beta+j} \right)}{(j-1)!} B(\alpha+1,\beta+j), \ x = a, \end{cases} \]

where
\[ B(u,v) = \int_0^1 x^{u-1} (1-x)^{v-1} \, dx \]
is the Beta function and

\[ F(\alpha, \beta, \gamma; z) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-1} (1-zt)^{-\alpha} \, dt \]
is the hypergeometric function for $\gamma > \beta > 0$ and $z < 1$.

If the assumptions of Theorem 3 hold, according to (14), we get

$$\left(-1\right)^n P(b) \cdot g^{(n)}(\lambda)$$

$$\leq \int_a^b \left( b - t \right)^\alpha \cdot \left( t - a \right)^\beta g(t) \, dt - \sum_{j=1}^{n} A_j(x) \cdot \frac{g^{(j-1)}(x)}{j!}$$

$$\leq \left(-1\right)^n P(b) \cdot \left[ \frac{b - \lambda}{b - a} g^{(n)}(a) + \frac{\lambda - a}{b - a} g^{(n)}(b) \right],$$

where

$$P(b) = \left(-1\right)^n \cdot \frac{(b - a)^{\alpha + \beta + 1}}{n!} B(\alpha + 1, \beta + 1)$$

$$\cdot F \left( -n, \alpha + 1, \alpha + \beta + 2; \frac{(1 - t)(b - a)}{b - t} \right) - \sum_{j=1}^{n} \frac{x^{n-j+1}}{(n-j+1)!} \cdot A_j(x) \right]$$

and

$$\lambda = \frac{\left(-1\right)^n}{P(b)} \cdot \frac{(b - a)^{\alpha + \beta + 1}}{(n + 2)!} B(\alpha + 1, \beta + 1)$$

$$\cdot F \left( -n - 1, \alpha + 1, \alpha + \beta + 2; \frac{(1 - t)(b - a)}{b - t} \right) - \sum_{j=1}^{n} \frac{x^{n-j+2}}{(n-j+2)!} \cdot A_j(x) \right].$$

Using the same integral calculations similar results can be obtained under the conditions of Theorem 4 and Theorem 5.

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