NONINVARIANCE OF THE ÉTALE BRAUER-MANNIN OBSTRUCTION FOR 3-FOLDS

HAN WU

Abstract. In this paper, we study the property of weak approximation with Brauer–Manin obstruction, étale Brauer–Manin obstruction and the Hasse principle with Brauer–Manin obstruction of 3-folds with respect to field extension of number fields. For any number field $K$ and any nontrivial field extension $L$ over $K$, assuming the Stoll’s conjecture, we construct a 3-fold over $K$, which has a $K$-rational point and verifies weak approximation with Brauer–Manin obstruction, étale Brauer–Manin obstruction off $\infty_K$ over $K$ while fails over $L$. For the Hasse principle with Brauer–Manin obstruction, assuming that the Stoll’s conjecture holds and either odd degree of field extension $[L:K]$ or existence of real place for $L$, we construct a 3-fold over $K$, which has a $\mathbb{A}_K$-adic point and verifies Hasse principle with Brauer-Manin obstruction over $K$ while fails over $L$. We also give some unconditional results and explicit examples. For noninvariance of the Hasse principle with Brauer–Manin obstruction, when the degree of field extension $[L:K]$ is not odd and $L$ has no real place, we also can give an unconditional example.

1. Introduction

Let $X$ be a proper algebraic variety defined over a number field $K$. Let $\Omega_K$ be the set of all nontrivial places of $K$, and let $\Omega'_K = \Omega_K \setminus \infty_K$ be the set of all finite places of $K$. Let $S$ be a finite subset of $\Omega_K$. Let $K_v$ be the completion of $K$ at place $v \in \Omega_K$ and let $\mathbb{A}_K$ be the adèle ring of $K$. If the set of $K$ rational points $X(K) \neq \emptyset$, then the set of adelic points $X(\mathbb{A}_K) \neq \emptyset$. Conversely, we say $X$ verifies Hasse principle over $K$ if nonemptiness of the set $X(\mathbb{A}_K)$ implies nonemptiness of the set $X(K)$, which does not always hold, for example Selmer’s cubic curve defined over $\mathbb{Q}$ by $3u_0^3 + 4w_1^3 + 5w_2^3 = 0$ with homogeneous coordinates $(w_0 : w_1 : w_2) \in \mathbb{P}_\mathbb{Q}^2$. Let $pr^S: \mathbb{A}_K \to \mathbb{A}_K^S$ be the natural projection of the ring of adèles and adèles without $S$ components, which induces a natural projection $pr^S : X(\mathbb{A}_K) \to X(\mathbb{A}_K^S)$ if $X(\mathbb{A}_K) \neq \emptyset$. There is a natural diagonal embedding of the set of rational points $X(K)$ to adelic points $X(\mathbb{A}_K^S)$. For $X$ is proper, the set of adelic points $X(\mathbb{A}_K^S)$ is equal to $\prod_{v \in \Omega_K \setminus S} X(K_v)$ and the adelic topology of $X(\mathbb{A}_K^S)$ is indeed the product topology of $v$-adic topologies. We say $X$ verifies weak approximation over $K$ (respectively off $S$) if $X(K)$ is dense in $X(\mathbb{A}_K)$ (respectively $X(\mathbb{A}_K^S)$). Cohomological obstructions have been used to explain failures of Hasse principle and nondensity of $X(K)$ in $X(\mathbb{A}_K^S)$. Let $Br(X) := H^2_{ét}(X, \mathbb{G}_m)$ be the Brauer group of $X$. The Brauer-Manin pairing $X(\mathbb{A}_K) \times Br(X) \to \mathbb{Q}/\mathbb{Z}$, suggested by Manin in 1970, between $X(\mathbb{A}_K)$ and $Br(X)$ is provided by local class field theory. The left kernel of this pairing is denoted by $X(\mathbb{A}_K)^{Br}$, which is a closed subset of $X(\mathbb{A}_K)$. By global reciprocity in class field theory, there is an exact sequence: $0 \to Br(K) \to \bigoplus_{v \in \Omega_K} Br(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0$.

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which induces an inclusion $X(K) \subset \text{pr}^S(X(A_K)^{\text{Br}})$. We say $X$ verifies Hasse principle with Brauer–Manin obstruction over $K$ if nonemptiness of the set $X(A_K)^{\text{Br}}$ implies nonemptiness of the set $X(K)$. The Selmer’s cubic curve above verifies Hasse principle with Brauer–Manin obstruction over $\mathbb{Q}$. We say $X$ verifies weak approximation with Brauer–Manin obstruction over $K$ if nonemptiness of the set $X(A_K)^{\text{Br}}$ implies nonemptiness of the set $X(K)$.

The Selmer’s cubic curve above verifies Hasse principle with Brauer–Manin obstruction over $\mathbb{Q}$. We say $X$ verifies weak approximation with Brauer–Manin obstruction over $K$ if nonemptiness of the set $X(A_K)^{\text{Br}}$ implies nonemptiness of the set $X(K)$.

Let a number field $L$ be a nontrivial field extension of $K$ and let $\Omega_L$ be all places of $\Omega_L$ above $S$. Let $X_L := X \times_K L$ be the base change of $X$ by $L$. We call a variety nice if it is smooth, projective and geometrically connected. In this paper, we consider the following questions:

**Question 1.1.** If a nice variety $X$ with a $K$-rational point verifies weak approximation with Brauer–Manin obstruction off $S$, does $X_L$ verify weak approximation with Brauer–Manin obstruction off $S_L$?

**Question 1.2.** If a nice variety $X$ verifies Hasse principle with Brauer–Manin obstruction over $K$, does $X_L$ verify Hasse principle with Brauer–Manin obstruction over $L$?

For any nice curve $C$ defined over any number field $K$, it is an open question, stated by A. Skorobogatov [Sko01, Chap. 6.3] whether $C$ verifies Hasse principle with Brauer–Manin obstruction over $K$. It is conjectured by Stoll [Sto07, Conj. 9.1] that this open question is true and $C$ verifies weak approximation with Brauer–Manin obstruction off $\infty_K$. For any smooth proper rationally connected variety defined over any number field $K$, it is conjectured by J.-L. Colliot-Thélène that it verifies Hasse principle with Brauer–Manin obstruction and verifies weak approximation with Brauer–Manin obstruction over $K$. All of those varieties considered are stable under extension of the ground field $K$. For some quadratic extension $L$ of $K$ and assuming the Stoll’s Conj. 4.1, a 3-fold was constructed by Y. Liang to give a negative answer to the Question 1.1, cf. [Lia18, Theorem 4.5]. Also an unconditionally concrete example with explicit equations was given for $L = \mathbb{Q}(\sqrt{5})$ and $K = \mathbb{Q}$ in loc. cit Section 4.5.

**Main results of this paper.**

**Theorem 1.3** (Theorem 6.5). For any number field $K$ and any nontrivial field extension $L$ over $K$, let $T \subseteq \Omega_L$ be any finite subset. Assuming Assumption 4.3, there exists a 3-fold over $K$ such that

- The 3-fold $X$ has a $K$-rational point and verifies weak approximation with Brauer–Manin obstruction, étale Brauer–Manin obstruction off $\infty_K$.
- The 3-fold $X_L$ does not verify weak approximation with Brauer–Manin obstruction, étale Brauer–Manin obstruction off $T$.

For $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{5})$, we give an explicit unconditional example in Subsection 7.1. It is a compactification of the following 3-dimensional affine subvariety given by equations

$$\begin{cases} 
    y^2 - 73z^2 = (x^2 - 1)(x^2 - 4)(x' - 4)^2 + (99x^2 + 1)(\frac{5428}{5329}x^2 + \frac{1}{5329}) \\
    y'^2 = x'^3 - 16
\end{cases}$$

in $(x, y, z, x', y') \in \mathbb{A}^5$.

**Theorem 1.4** (Theorem 6.6). For any number field $K$ and any nontrivial field extension $L$ of odd degree over $K$, assuming Assumption 4.3, there exists a 3-fold over $K$ such that

- The 3-fold $X$ has a $\mathbb{A}_K$-adelic point and verifies Hasse principle with Brauer–Manin obstruction over $K$. 


• The 3-fold $X_L$ does not verify Hasse principle with Brauer-Mannin obstruction over $L$.

Let $\zeta_7$ be a primitive 7-th root of unity. For $K = \mathbb{Q}$ and $L = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, we give an explicit unconditional example in Subsection 7.2. It is a compactification of the following 3-dimensional affine subvariety given by equations

$$\begin{align*}
y^2 - 377z^2 &= 14(x^4 - 89726)y^2 + (x^2 - 878755181)(5x^2 - 4393775906) \\
y'^2 &= x^3 - 343x' - 2401
\end{align*}$$

in $(x, y, z, x', y') \in \mathbb{A}^5$.

**Theorem 1.5** (Theorem 6.7). For any number field $K$ and any nontrivial field extension $L$, assuming Assumption 1.4 there exists a 3-fold over $K$ such that

• The 3-fold $X$ has a $A_K$-adic point and verifies Hasse principle with Brauer-Mannin obstruction over $K$.
• The 3-fold $X_L$ does not verify Hasse principle with Brauer-Mannin obstruction over $L$.

For $K = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{3})$, we give an explicit unconditional example in Subsection 7.3. It is a compactification of the following 3-dimensional affine subvariety given by equations

$$\begin{align*}
y^2 + 23z^2 &= 5(x^4 + 805)(x^2 - 8x + 15) \\
y'^2 &= x^3 - 16
\end{align*}$$

in $(x, y, z, x', y') \in \mathbb{A}^5$.

For $K = \mathbb{Q}$ and $L = \mathbb{Q}(i)$, we give an explicit unconditional example for noninvariance of the Hasse principle with Brauer–Manin obstruction in Subsection 7.4. It is a compactification of the following 3-dimensional affine subvariety given by equations

$$\begin{align*}
y^2 + 15z^2 &= (x^4 - 10x^2 + 15)\frac{y'^2 + 32}{8} - (5x^4 - 39x^2 + 75)y'^4 \\
y'^2 &= x^3 - 16
\end{align*}$$

in $(x, y, z, x', y') \in \mathbb{A}^5$.

**Outline of this paper.** In this paper, we construct 3-folds with some additional assumptions to negatively answer above Questions 1.1 and 1.2. Fix a number field $K$ and finite places $S \in \Omega_k$ not containing any complex place and any 2-adic place. Firstly, using weak approximation and strong approximation off all 2-adic places for $A^1$, we construct Châtelet surfaces $V_\infty, V_1, V_2$ have the following properties:

• The set $V_\infty(A_K^S) \neq \emptyset$ but $V_\infty(K_v) = \emptyset$ for all $v \in S$, cf. Prop. 3.3
• The Brauer group $\text{Br}(V_1) / \text{Br}(K) \cong \mathbb{Z} / 2\mathbb{Z}$ generated by an element $A \in \text{Br}(V_1)$ and the set $V_1(A_K)$ is nonempty. For any $v \in \Omega_K$ and any $P_v \in V_1(K_v)$, the invariance $\text{inv}_v(A(P_v)) = 0$ if $v \notin S$; the invariance $\text{inv}_v(A(P_v)) = \frac{1}{2}$ if $v \in S$, cf. Prop. 3.3
• The Brauer group $\text{Br}(V_2) / \text{Br}(K) \cong \mathbb{Z} / 2\mathbb{Z}$ generated by an element $A \in \text{Br}(V_2)$ and the set $V_2(K)$ is nonempty. For any $v \in S$, there exists $P_v, Q_v \in V_2(K_v)$ such that $\text{inv}_v(A(P_v)) = 0$ and $\text{inv}_v(A(Q_v)) = \frac{1}{2}$; for any other $v \notin S$ and any $P_v \in V_2(K_v)$, the invariance $\text{inv}_v(A(P_v)) = 0$, cf. Prop. 3.11

Secondly, let $L$ be a finite nontrivial field extension of $K$ and let $[L : K]$ be the degree of this extension. Using those Châtelet surfaces we construct and Theorem 2.2 with some additional condition for the choice of $S$ and parameters in the construction, the properties we list above of Châtelet surfaces $V_\infty, V_1, V_2$ are invariant, i.e. the Châtelet surfaces $V_\infty, V_1, V_2$ defined over $K$ have the following additional properties:
In this paper, we always assume that number field $L/K$ splits completely in $M/K$. Using [Neu99, Theorem 13.4] for Galois extension $C$, we will use the following version of Čebotarev density theorem.

**Proposition 2.1.** The set $V_{\infty}(A^n_{\mathbb{Q}_L}) \neq \emptyset$ but $V_{\infty}(L_{L}) = \emptyset$ for all $\mathfrak{v} \in S_L$, cf. Cor. 3.15.

The Brauer group $\text{Br}(V_1)/\text{Br}(K) \cong \text{Br}(V_{L})/\text{Br}(L) \cong \mathbb{Z}/2\mathbb{Z}$ generated by an element $A \in \text{Br}(V_1)$ and the set $V_1(A_L)$ is nonempty. For any $\mathfrak{v} \in \Omega_L$ and any $P_{\mathfrak{v}} \in V_1(L_{\mathfrak{v}})$, the invariance $\text{inv}_{\mathfrak{v}}'(A(P_{\mathfrak{v}})) = 0$ if $\mathfrak{v} \notin S_L$, the invariance $\text{inv}_{\mathfrak{v}}(A(P_{\mathfrak{v}})) = \frac{1}{2}$ if $\mathfrak{v} \in S_L$, cf. Cor. 3.11.

The Brauer group $\text{Br}(V_2)/\text{Br}(K) \cong \text{Br}(V_{L})/\text{Br}(L) \cong \mathbb{Z}/2\mathbb{Z}$ generated by an element $A \in \text{Br}(V_2)$ and the set $V_2(L)$ is nonempty. For any $\mathfrak{v} \in S_L$, there exists $P_{\mathfrak{v}}, Q_{\mathfrak{v}} \in V_2(L_{\mathfrak{v}})$ such that $\text{inv}_{\mathfrak{v}}'(A(P_{\mathfrak{v}})) = 0$ and $\text{inv}_{\mathfrak{v}}(A(Q_{\mathfrak{v}})) = \frac{1}{2}$; for any other $\mathfrak{v} \notin S_L$ and any $P_{\mathfrak{v}} \in V_2(L_{\mathfrak{v}})$, the invariance $\text{inv}_{\mathfrak{v}}(A(P_{\mathfrak{v}})) = 0$, cf. Cor. 3.17.

If the degree $[L : K]$ is even, then there exists an interesting result for Châtelet surfaces.

**Corollary 1.6 (Cor. 3.12).** For any number field $K$ and any nontrivial field extension $L$ of even degree over $K$, there exists a Châtelet surface $V$ defined over $K$ such that $\text{Br}(V)/\text{Br}(K) \cong \text{Br}(V_{L})/\text{Br}(L) \cong \mathbb{Z}/2\mathbb{Z}$ generated by an element $A \in \text{Br}(V)$ and the set $V(A_K)$ is nonempty. And the surface $V$ has the following properties:

- The surface $V$ violates Hasse principle over $K$.
- The surface $V_L$ verifies Hasse principle over $L$. Furthermore, it verifies weak approximation over $L$.

Thirdly, assuming Assumption 4.23 (a weaker version of Stoll’s Conj. 4.1) and using those Châtelet surfaces we construct with a main proposition from the paper [Poo10 Prop. 5.4], we get the above main results.

## 2. Preliminaries and notations

Given a number field $K$, let $\mathcal{O}_K$ be its integer ring. If an element $a \in \mathcal{O}_K$ is a prime element, we denote its prime ideal by $\mathfrak{p}_a$ and the associated valuation by $v_a$. For $v \in \infty_K$, let $\tau_v : K \rightarrow K_v$ be the embedding of $K$ to its completion. For $v \in \Omega_K$, let $\mathcal{O}_{K_v}$ be its valuation ring, $\mathbb{F}_v$ be its residue field and let $\mathfrak{p}_v$ be the prime ideal of $\mathcal{O}_K$, which associates this valuation. Let $S$ be a finite subset of $\Omega_K$ and let $\mathcal{O}_S = \bigcap_{v \in \Omega_K \setminus S} (K \cap \mathcal{O}_{K_v})$ be the ring of $S$-integers. It is well known that $K$ is dense in $\mathbb{A}^n_{\mathbb{Q}_L}$ for any nonempty $S$, i.e. the affine line $\mathbb{A}^1$ verifies strong approximation off any nonempty $S$. For the definition of strong approximation, one can refer to the paper [Con12] or the paper [LX15]. We only use the following fact:

**Fact 2.1.** The set $K$ is dense in $\mathbb{A}^n_{\mathbb{Q}_L}$ for $S = \{\text{all 2-adic places of } K\}$.

In this paper, we always assume that number field $L$ is a nontrivial field extension of $K$.

We will use the following version of Čebotarev density theorem.

**Theorem 2.2 (Čebotarev).** The set of prime ideals of $K$ splitting completely in $L$ has positive density.

**Proof.** If $L/K$ is a Galois extension, then this statement is just [Neu99] Theorem 13.4 with $\sigma = \text{id}$, unit element of this Galois group $\text{Gal}(L/K)$. For $L/K$ is not Galois extension, let $M$ be the smallest Galois extension containing $L$ over $K$. A prime ideal of $K$ splits completely in $L$ if and only if it splits completely in $M$. Using [Neu99] Theorem 13.4 for Galois extension $M/K$, the set of prime ideals of $K$ splitting completely in $M$ has density $\frac{1}{[M : K]}$. So, the set of prime ideals of $K$ splitting completely in $L$ has density $\frac{1}{[M : K]}$. \(\square\)
**Lemma 2.3.** Let \( v \in \Omega_K \setminus \{ \text{all complex places} \} \) and assume that \( v' \in \Omega_L \) splits over \( v \), i.e. \( L_{v'} = K_v \). If \( a \in K \) with \( v(a) \) odd for finite place \( v \) or \( \tau_v(a) < 0 \) for archimedean place \( v \), then \( a \notin L_{v'}^2 \).

Proof. The condition that \( v(a) \) is odd for finite place \( v \) or \( \tau_v(a) < 0 \) for archimedean place \( v \), implies \( a \notin K_v^\times = L_{v'}^\times \). So \( a \notin L_{v'}^2 \subset L_{v'}^\times \). □

2.1. **Hilbert symbol.** We use Hilbert symbol \((a, b)_v \in \{ \pm 1 \}\), for \( a, b \in K_v^\times \) and \( v \in \Omega_K \). By definition, \((a, b)_v = 1\) if and only if \( x_0^2 - ax_1^2 - bx_2^2 = 0 \) is a \( K_v \)-solution in \( \mathbb{P}^2 \) with homogeneous coordinates \((x_0 : x_1 : x_2)\), which equivalently means that the curve defined over \( K_v \) by the equation \( x_0^2 - ax_1^2 - bx_2^2 = 0 \) in \( \mathbb{P}^2 \) is isomorphic to \( \mathbb{P}^1 \). The Hilbert symbol gives a symmetric bilinear form \([ - , - ]_v \) on \( K_v^\times / K_v^\times \) with value in \( \mathbb{Z} / 2\mathbb{Z} \), i.e. if \((a, b)_v = 1\), then \([a, b]_v = 0\); otherwise, \([a, b]_v = 1\), cf. [Ser73] Chapt. XIV Prop. 7. It is well known that this bilinear form is nondegenerate, cf. [Ser73] Chapt. XIV Cor. 7.

**Lemma 2.4.** Let \( v \) be an odd place and let \( a, b \in K_v^\times \) with \( v(a), v(b) \) even, then \((a, b)_v = 1\).

Proof. Choose a prime element \( \pi_v \in K_v \). Let \( a_1 = a\pi_v^{-v(a)} \) and \( b_1 = b\pi_v^{-v(b)} \). For \( v(a), v(b) \) are even, the elements \( \pi_v^{-v(a)}, \pi_v^{-v(b)} \) are in \( K_v^\times \). So \((a, b)_v = (a_1, b_1)_v \) and \( a_1, b_1 \in O_{K_v}^\times \). By Chevalley-Warning theorem (cf. [Ser73] Chapt. I §2 Cor. 2), the equation \( x_0^2 - a_1x_1^2 - b_1x_2^2 = 0 \) has a nontrivial solution in \( \mathbb{F}_v \). For \( v \) is odd, by Hensel’s lemma, this solution can lift to a nontrivial solution in \( O_{K_v} \). Hence \((a, b)_v = (a_1, b_1)_v = 1\). □

**Lemma 2.5.** Let \( v \) be a finite place of \( K \). The sets \( K_v^\times \) and \( O_{K_v}^\times \) are open subgroups of \( K_v^\times \). So, they are nonempty open subset of \( K_v \).

Proof. Let \( p \) be the prime number such that \( |p|_v \) in \( K \). Then by Hensel lemma, the set \( K_v^\times \cap O_{K_v} \) contains \( 1 + p^nO_{K_v} \), which is an open subgroup of \( K_v^\times \). Both \( K_v^\times \) and \( O_{K_v}^\times \) are subgroups of \( K_v^\times \), hence they are open subgroups of \( K_v^\times \). □

**Lemma 2.6.** Let \( v \) be a finite place of \( K \). For any \( n \in \mathbb{Z} \), the set \( \{ x \in K_v, v(x) = n \} \) is a nonempty open subset of \( K_v \).

Proof. By Lemma 2.5, the set \( O_{K_v}^\times \) is an open subgroup of \( K_v^\times \). Choose a prime element \( \pi_v \in K_v \). Then the set \( \{ x \in K_v, v(x) = n \} = \pi_v^nO_{K_v}^\times \), so it is a nonempty open subset of \( K_v \). □

**Lemma 2.7.** Let \( v \) be a finite place of \( K \). For any \( a \in K_v^\times \), the sets \( \{ x \in K_v^\times, v(a, x) = 1 \} \) and \( \{ x \in O_{K_v}^\times, v(a, x) = 1 \} \) are nonempty open subsets of \( K_v \).

Proof. For the unit 1 belongs to these sets, they are nonempty. By Lemma 2.5, the sets \( K_v^\times \) and \( O_{K_v}^\times \) are nonempty open subsets of \( K_v \). The set \( \{ x \in K_v^\times, v(a, x) = 1 \} \) is a union of cosets of \( K_v^\times \), so the sets are open in \( K_v \). □

**Lemma 2.8.** Let \( v \) be a finite place of \( K \). For any \( a \in K_v^\times \), the sets \( \{ x \in K_v^\times, v(a, x) = -1 \} \) and \( \{ x \in O_{K_v}^\times, v(a, x) = -1 \} \cap O_{K_v} \) are open subsets of \( K_v \). Furthermore, if \( a \notin K_v^\times \), then they are nonempty.

Proof. If the set \( \{ x \in K_v^\times, v(a, x) = 1 \} \neq \emptyset \), then it is a union of cosets of \( K_v^\times \). By Lemma 2.5, it is an open subset of \( K_v \). For \( O_{K_v} \) is open in \( K_v \), the sets are open subsets of \( K_v^\times \). Nonemptiness is from the nondegeneracy of the bilinear form given by the Hilbert symbol and multiplying an element in \( O_{K_v} \) to denote an element in \( K_v^\times \). □

**Lemma 2.9.** Let \( v \) be a finite place of \( K \). For any \( a \in K_v^\times \) with \( v(a) \) odd, the set \( \{ x \in O_{K_v}^\times, v(a, x) = -1 \} \) is a nonempty open subsets of \( K_v \).
Remark 3.2. For any local field $K_v$, by the nondegeneracy of the bilinear form given by the Hilbert symbol, there exists an element $b \in K_v^\times$ such that $(a, b)_v = -1$. If $v(b)$ is odd, let $b' = -ab$. Then $(a, b')_v = (a, -ab)_v = (a, -a)_v(a, b)_v = -1$. Replacing $b$ by $b'$ if necessary, we can assume that $v(b)$ is even. Choose a prime element $\pi_v \in K_v$. Then $\pi_v^{-v(b)} \in K_v^\times$, so the element $b\pi_v^{-v(b)}$ is in this set. \hfill $\square$

3. Châtelet surfaces

Châtelet surfaces are smooth projective models of conic bundle surfaces defined by the equation of form

$$(1) \quad y^2 - az^2 = P(x).$$

Here $P(x)$ is a separable polynomial of degree 4. Given a Châtelet surface $V$, we denote $V^0$ be its open affine subsurface given by the above equation $(1)$.

Remark 3.1. For any local field $K_v$, if $a \in K_v^\times$, then $V$ is birational equivalent to $\mathbb{P}^2$ over $K_v$, which implies that $V$ admits a $K_v$-point.

Remark 3.2. For any local field $K_v$, by smoothness of $V$, the implicit function theorem implies that nonemptiness of $V^0(K_v)$ is equivalent to nonemptiness of $V(K_v)$ and $V^0(K_v)$ is open dense in $V(K_v)$. For any element $A \in \text{Br}(V)$, the invariance of $A$ on $V(K_v)$ is locally constant. For properness of $V$, the space $V(K_v)$ is compact. So all possible values of the invariance of $A$ on $V(K_v)$ is finite. Indeed, by [Sko01, Prop. 7.1.2], there only exist two possible values. It is determined by the invariance of $A$ on $V^0(K_v)$. Specially, if the invariance of $A$ on $V^0(K_v)$ is constant, then it is constant on $V(K_v)$.

Given a number field $K$, a finite field extension $L$ and a finite set $S \subset \Omega_K$ with assumption that all places of $S$ split completely in $L$, in this section, we always choose the parameter $a \in \mathcal{O}_K \setminus L^2$ in the following way.

3.1. Choose an element $a$ for $S$. If $S = \emptyset$, by Theorem 2.2, we can take a place $v \in \Omega_K^0$ splitting completely in $L$. Take an element $a \in \mathcal{O}_K$ with $v(a)$ odd, then by Lemma 2.3, the element $a \in \mathcal{O}_K \setminus L^2$.

Suppose $S \neq \emptyset$. By Lemma 2.4, the set $K_v^\times$ is a nonempty open subset of $K_v$. By Lemma 2.6, the set $\{a \in K_v | v(a) \text{ is odd} \}$ is a nonempty open subset of $K_v$. Using weak approximation for affine line $\mathbb{A}^1$, we can choose an element $a \in K_v^\times$ with the following conditions:

- $\tau_v(a) < 0$ for all $v \in S \cap \infty_K$.
- $\tau_v(a) > 0$ for all real places $v \in \infty_K \setminus S$.
- $a \in K_v^\times$ for all 2-adic places $v$.
- $v(a)$ is odd for all $v \in S \setminus \infty_K$.

Those conditions do not change by multiplying an element in $K_v^\times$, so we can assume $a \in \mathcal{O}_K$. The condition that $v(a)$ is odd for finite place $v \in S$ or $\tau_v(a) < 0$ for archimedean place $v \in S$, and the assumption that all places of $S$ split completely in $L$, satisfy conditions of Lemma 2.3, so $a \in \mathcal{O}_K \setminus L^2$.

3.2. Châtelet surface without $K_v$ point for $v \in S$.

Proposition 3.3. For any number field $K$ and any finite $S$ not containing any complex place and any 2-adic place, there exists a Châtelet surface $V_\infty$ such that $V_\infty(K_\infty) \neq \emptyset$ but $V_\infty(K_v) = \emptyset$ for all $v \in S$. 

Proof. If $S = \emptyset$, then let $P(x) = 1 - x^4$. Then this Châtelet surface with a rational point $(x, y, z) = (0, 1, 0)$ satisfies the required conditions.

Now, suppose that $S \neq \emptyset$. Let $S' = \{ v \in \Omega_K \backslash \infty_K | v(a) \neq 0 \}$, then $S' \supset S' \backslash \infty_K$ is a finite set. If $v \in S' \backslash \infty_K$, then $v(a)$ is odd, which implies $a \notin K_v^\times$. In this case, by Lemma 3.3 the set $\{ b \in K_v^\times | (a, b)_v = -1 \}$ is a nonempty open subset of $K_v$. If $v \in S' \backslash S$, then by Lemma 2.7 the set $\{ b \in K_v^\times | (a, b)_v = 1 \}$ is a nonempty open subset of $K_v$. Using weak approximation for affine line $K^1$, we can choose an element $b \in K^\times$ with the following conditions:

1. $\tau_v(b) < 0$ for all $v \in S \cap \infty_K$.
2. $(a, b)_v = -1$ for all $v \in S' \backslash \infty_K$.
3. $(a, b)_v = 1$ for all $v \in S' \backslash S$.

Let $S'' = \{ v \in \Omega_K \backslash \infty_K | v(b) = \text{odd} \}$. Then $S''$ is a finite set. The same argument as in the previous paragraph, we can choose an element $c \in K^\times$ with the following conditions:

1. $\tau_v(c) > 0$ for all $v \in \infty_K \setminus \{ \text{all complex places} \}$.
2. $c \in K_v^{\times 2}$ for all $v \in S'$.
3. $v(c)$ is odd for all $v \in S'' \setminus S''$.

Those conditions do not change by multiplying an element in $K^\times$, so we can assume $b, c \in \mathcal{O}_K$. Let $P(x) = b(x^4 - ac)$. We will check that the Châtelet surface $V_\infty$ given by $y^2 - ax^2 = b(x^4 - ac)$, satisfies the required conditions.

Suppose that $v$ is an archimedean place such that $v \in \infty_K \setminus S$ or a 2-adic place. Then $a \in K_v^{\times 2}$. By Remark 3.3, the surface $V_\infty$ admits a $K_v$-point.

Suppose that $v \in S' \setminus (S' \cup \{ \text{all 2-adic places} \})$, then by the choice of $b$, we have $(a, b)_v = 1$. Take $x_0 \in K_v$ with valuation $v(x_0) < 0$. Then $(a, b(x_0^4 - ac))_v = (a, b)_v(a, x_0^4(1 - acx_0^{-4}))_v = 1$, which implies $V_\infty$ admits a $K_v$-point with $x = x_0$.

Suppose that $v \in \Omega_K \setminus (S' \cup S'' \cup \{ \text{all 2-adic places} \})$. Take $x_0 \in K_v$ with valuation $v(x_0) < 0$. Then $(a, b)_v = 1$ and $v(b) = 0$. By Lemma 2.4, we have $(a, -abc)_v = 1$, which implies $V_\infty$ admits a $K_v$-point with $x = x_0$.

Suppose that $v \in S \cap \infty_K$. Then by the choice of $a, b, c$, we have $\tau_v(a), \tau_v(b)$, and $\tau_v(ac)$ are negative. So $(a, b(x^4 - ac))_v = -1$, which implies $V_\infty$ has no $K_v$-point.

By Remark 3.2, we have $V_\infty(K_v) = \emptyset$.

Suppose that $v \in \infty_K$. Then by the choice of $a, b$, we have $(a, b)_v = -1$. If $4v(x) < v(ac)$, then $(a, x^4 - ac)_v = (a, x^4(1 - acx^{-4}))_v = 1$. If $4v(x) > v(ac)$, then $(a, x^4 - ac)_v = (a, -ac(1 - (ac)^{-1}x^4))_v = (a, c)_v = 1$ (by the choice of $c$, the last equality holds). For $c \in K_v^{\times 2}$, the valuation $v(ac)$ is odd. So, the equality $4v(x) = v(ac)$ cannot happen. In each case, we have $(a, b(x^4 - ac))_v = (a, b)_v(a, x^4 - ac)_v = -1$, which implies $V_\infty$ has no $K_v$-point. By Remark 3.2 we have $V_\infty(K_v) = \emptyset$.

□

Remark 3.4. By the choice of elements $a, c$ in the proof of Prop. 3.3 if $S \neq \emptyset$, then the polynomial $P(x)$ is irreducible over $K_v$ for all $v \in S$. If $S \cap \infty_K = \emptyset$, then the polynomial $P(x)$ is reducible over $K_v$ for all $v \in \infty_K$.

Corollary 3.5. Assume all places of $S$ split completely in $L$. Then the set $V_\infty(A_L^{S}) \neq \emptyset$ but $V_\infty(L') = \emptyset$ for all $v' \in S_L$.

Proof. For $V_\infty(A_L^{S}) \subset V_\infty(A_L^{S'})$, the nonemptiness of $V_\infty(A_L^{S'})$ implies the nonemptiness of $V_\infty(A_L^{S})$. Take any $v' \in S_L$. Let $v \in \Omega_K$ be the restriction of $v'$ on $K$. By
assumption that $v$ splits completely in $L$, we have $K_v = L_v$. So $V_\infty(K_v) = V_\infty(L_v)$, which implies the emptiness of $V_\infty(L_v)$.

The following examples will be used for further discussion.

**Example 3.6.** Let $K = \mathbb{Q}$ and let $\zeta_7$ be a primitive 7-th root of unity. Let $\alpha = \zeta_7 + \zeta_7^{-1}$ with minimal polynomial $x^3 + x^2 - 2x - 1$. Let $L = \mathbb{Q}(\alpha)$. Then the degree $[L : K] = 3$. Let $S = \{29\}$. For $29 \equiv 1 \pmod{7}$, the place 29 splits completely in $L$. Using the construction method of Prop. 3.3 we choose data: $S = \{29\}$, $S' = \{13, 29\}$, $S'' = \{2, 7\}$, $a = 377$, $b = 14$, $c = 238$ and $P(x) = 14(x^4 - 89726)$. Then the Châtelet surface given by $y^2 - 377z^2 = P(x)$ has the properties of Prop. 3.3 and Cor. 3.5.

**Example 3.7.** Let $K = \mathbb{Q}$ and let $L = \mathbb{Q}(\sqrt{3})$. Using the construction method of Prop. 3.3 we choose data: $S = \{\infty_K\}$, $S' = \{23\}$, $S'' = \{5\}$, $a = -23$, $b = 5$, $c = 35$ and $P(x) = 5(x^2 + 805)$. Then the Châtelet surface given by $y^2 + 23z^2 = P(x)$ has the properties of Prop. 3.3 and Cor. 3.5.

**Example 3.8.** Let $K = \mathbb{Q}$ and let $L = \mathbb{Q}(\sqrt{3})$. Then the place 23 splits completely in $L$. Using the construction method of Prop. 3.3 we choose data: $S = \{23\}$, $S' = \{23\}$, $S'' = \{5\}$, $a = -23$, $b = -5$, $c = 35$ and $P(x) = -5(x^2 + 805)$. Then the Châtelet surface given by $y^2 + 23z^2 = P(x)$ has the properties of Prop. 3.3 and Cor. 3.5.

### 3.3. Châtelet surface with local points and nontrivial Brauer group.

In this section, we generalize results from Poonen [Po09] to Prop. 5.1 and Lemma 5.5.

**Proposition 3.9.** For any number field $K$ and any finite $S$ not containing any complex place and any 2-adic place, there exists a Châtelet surface $V_1$ such that $\text{Br}(V_1)/\text{Br}(K) \cong \mathbb{Z}/2\mathbb{Z}$ generated by an element $A \in \text{Br}(V_1)$, the set $V_1(\mathbb{A}_K)$ is nonempty and for any $v \in \Omega_K$ and any $P_v \in V_1(K_v)$,

$$\text{inv}_v(A(P_v)) = \begin{cases} 0 & \text{if } v \notin S, \\ \frac{1}{2} & \text{if } v \in S. \end{cases}$$

**Proof.** Let $S' = \{v \in \Omega_K \mid \text{all 2-adic places} \mid v(a) \neq 0\}$, then $S' \supset S \setminus \infty_K$ is a finite set. If $v \in S \setminus \infty_K$, then $v(a)$ is odd. In this case, by Lemma 2.4, the set $\{b \in \mathcal{O}_K^\times \mid (a, b)_v = -1\}$ is a nonempty open subset of $\mathcal{O}_K^\times$. If $v \in S' \setminus S$, then by Lemma 2.7, the set $\{b \in \mathcal{O}_K^\times \mid (a, b)_v = 1\}$ is a nonempty open subset of $\mathcal{O}_K^\times$. Using Fact 2.1 we can choose a nonzero element $b \in \mathcal{O}_K^\times$ with the following conditions:

- $\tau_v(b) < 0$ for all $v \in S \cap \infty_K$.
- $(a, b)_v = -1$ and $v(b) = 0$ for all $v \in S \setminus \infty_K$.
- $(a, b)_v = 1$ and $v(b) = 0$ for all $v \in S' \setminus S$.

Then $a, b$ are coprime in $\mathcal{O}_K^\times$ and $b \in \mathcal{O}_K^\times$ for all $v \in S'$. Let $S'' = \{v \in \Omega_K \mid \text{all 2-adic places} \mid v(b) \neq 0\}$, then $S''$ is a finite set and $S' \cap S'' = \emptyset$. By Theorem 2.2 we can take two different finite places $v_1, v_2 \in \Omega_K \setminus (S' \cup S'' \cup \{\text{all 2-adic places}\})$ splitting completely in $L$. If $v \in S' \cup \{v_1, v_2\}$, then $b \in \mathcal{O}_K^\times$. In this case, by Lemma 2.6 the sets $\{c \in K_v \mid |v(bc + 1) = v(a) + 2\}$, $\{c \in K_v \mid |v(c) = 1\}$ and $\{c \in K_v \mid |v(bc + 1) = 1\}$ are nonempty open subsets of $\mathcal{O}_K^\times$. If $v \in S''$, by Lemma 2.7, the set $\{c \in \mathcal{O}_K^\times \mid (a, c)_v = 1\}$ is a nonempty open subset of $\mathcal{O}_K^\times$. Also using Fact 2.1 we can choose a nonzero $c \in \mathcal{O}_K^\times$ with the following conditions:

- $0 < \tau_v(c) < \frac{-1}{\tau_v(b)}$ for all $v \in S \cap \infty_K$.
- $v(bc + 1) = v(a) + 2$ for all $v \in S'$.
- $(a, c)_v = 1$ for all $v \in S''$.
- $v_1(c) = 1$ and $v_2(bc + 1) = 1$ for the chosen $v_1, v_2$ above.
Then $a|bc + 1$ in $O_K[\frac{1}{2}]$.

Let $P(x) = (x^2 - c)(bx^2 - bc - 1)$. We will check that the Châtelet surface $V_1$ given by $y^2 - ax^2 = (x^2 - c)(bx^2 - bc - 1)$, satisfies the required conditions.

Suppose that $v$ is an archimedean place such that $v \in \infty_K \backslash S$ or a 2-adic place.

Then $a, b, c, \in \mathbb{Z}$ by Remark 3.1. The surface $V_1$ admits a $K_v$-point.

Suppose that $v \in S \setminus \{S \cup \{2\text{-adic places}\}\}$, then by the choice of $b$, we have $(a, b)_v = 1$ and $b \in O_{K_v}$. Take $x_0 \in K_v$ with valuation $v(x_0) < 0$. Then $(a, x_0^2 - c)(bx_0^2 - bc - 1)_v = (a, x_0^2(1 - cx_0^{-2}))(b - (bc + 1)x_0^{-2}) = (a, b)_v = 1$, which implies $V_0^1$ admits a $K_v$-point with $x = x_0$.

Suppose that $v \in S \setminus \{S \cup \{2\text{-adic places}\}\}$. Then $x_0 = 0$. Then $(a, (x_0^2 - c)(bx_0^2 - bc - 1))_v = (a, c(c(b + 1)))_v = (a, c)_v(a, b)_v = 1$ (the equality $(a, c)_v = 1$ holds for the choice of $c$; for $a, bc + 1 \in O_{K_v}$, by Lemma 2.3 the equality $(a, bc + 1)_v = 1$ holds), which implies $V_0$ admits a $K_v$-point with $x = 0$.

Suppose that $v \in \infty_K \setminus S \setminus \{\text{2-adic places}\}$. Then $x_0 = 0$. Then $(a, x_0^2 - c)(bx_0^2 - bc - 1)_v = (a, b)_v = 1$, which implies $V_0$ admits a $K_v$-point with $x = x_0$.

Suppose that $v \in S \setminus \{S \cup \{2\text{-adic places}\}\}$. Choose a prime element $\pi_v$ and take $x_0 = \pi_v$. By the choice of $a, b, c$, we have $b, c \in O_{K_v}$, $v(bx_0^2) = 2, v(b(c + 1)) = v(a + 1) + 2 \geq 3$ and $(a, bc)_v = (a, 1 - (1 + bc))_v = 1$. Then $(a, x_0^2 - c)(bx_0^2 - bc - 1)_v = (a, c)_v(bx_0^2 - bc - 1)_v = (a, c)_v = 1$, which implies $V_0$ admits a $K_v$-point with $x = \pi_v$.

By the choice of the places $v_1, v_2$, the polynomial $x^2 - c$ (respectively $bx^2 - bc - 1$) is irreducible over $K_v$ (respectively $K_{v_2}$). So $P(x)$ is separable and a product of two 2-degree irreducible factors over $K$ (even over $L$), according to [Sko01, Prop. 7.1.1].

The group $\text{Br}(V_1)/\text{Br}(K) \equiv \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Furthermore, by Prop. 7.1.2 in loc. cit., we take the quaternion algebra $A = (a, x^2 - c) \in \text{Br}(V_1)$ as a generator element of this group. Then we have the equality $A = (a, x^2 - c) = (a, bx^2 - bc - 1)$ in $\text{Br}(V_1)$.

By Remark 3.2 it suffices to check the invariance $\text{inv}_v(A(P_v))$ for $P_v \in V_0^1(K_v)$.

Suppose that $v$ is an archimedean place such that $v \in \infty_K \setminus S$ or a 2-adic place.

Then $a \in K_v^{\times}$, so $\text{inv}_v(A(P_v)) = 0$.

Suppose that $v \in S \setminus \{S \cup \{2\text{-adic places}\}\}$, then by the choice of $b$, we have $(a, b)_v = 1$. If $v(a)$ is even and $v(a)(P_v) = \frac{1}{2}$, then $(a, bx^2 - bc - 1)_v = (a, x^2 - c)_v = -1$ at $P_v$. By Lemma 2.3 the last equality implies that $v(x^2 - c)$ is odd, so it is positive. But $(a, bx^2 - bc - 1)_v = (a, 1)_v = 1$, which is a contradiction. Consider the case $v(a)$ is odd. If $v(x) < 0$ at $P_v$, then $(a, x^2 - c)_v = (a, x^2(1 - cx^{-2}))_v = 1$. If $v(x) > 0$, for $b, c \in O_{K_v}$, $(a, b)_v = 1$ and $v(b(c + 1)) = v(a + 2) + 3$, we have $(a, x^2 - c)_v = (a, c)_v = (a, -bc)_v = (a, -bc + (1 + bc))_v = 1$. If $v(x) = 0$, for $b \in O_{K_v}$, $(a, b)_v = 1$ and $v(b(c + 1)) = v(a + 2) + 3$, we have $(a, bx^2 - bc - 1)_v = (a, bx^2)_v = 1$. So $\text{inv}_v(A(P_v)) = 0$.

Suppose that $v \in \infty_K \setminus S \cup \{2\text{-adic places}\}$. If $\text{inv}_v(A(P_v)) = \frac{1}{2}$, then $(a, bx^2 - bc - 1)_v = (a, x^2 - c)_v = -1$ at $P_v$. By Lemma 2.3 the last equality implies that $v(x^2 - c)$ is odd, so it is positive. So $v(a) = v(bx^2 - bc - 1) = 0$. By Lemma 2.3 we have $(a, bx^2 - bc - 1)_v = 1$, which is a contradiction. So $\text{inv}_v(A(P_v)) = 0$.

Suppose that $v \in S \cap \infty_K$. If $A(P_v) = 0$, then $(a, bx^2 - bc - 1)_v = (a, x^2 - c)_v = 1$ at $P_v$. The last equality implies that $\tau_v(x^2 - c) > 0$. By the choice of $b$, we have $\tau_v(b) > 0$, so $\tau_v(bx^2 - bc - 1) > 0$, which is contradiction. So $\text{inv}_v(A(P_v)) = 1$.

Suppose that $v \in S \setminus \infty_K$. If $v(x) \leq 0$ at $P_v$, for $b \in O_{K_v}$, we have $(a, bx^2 - bc - 1)_v = (a, bx^2(1 - (bc + 1)b^2 - x^2))_v = (a, b)_v = -1$. If $v(x) > 0$, for $b, c \in O_{K_v}$, $(a, b)_v = -1$.
and \( v(bc+1) = v(a) + 2 \geq 3 \), we have \((a,x^2-c)_v = (a,-c)_v = -v(a,-bc)_v = -(a,-bc + (1+bc))_v = -1 \). So \( \text{inv}_v(A(P_v)) = \frac{1}{2} \).

**Remark 3.10.** If the surface \( V_1 \) has a \( K \)-rational point \( Q \), then by global reciprocity law, the sum \( \sum_{v \in \Omega_K} \text{inv}_v(A(Q)) = 0 \). If the number \( \sharp S \) is odd, then from Prop. 3.9 we get, this sum is \( \frac{\sharp S}{2} \), nonzero in \( \mathbb{Q}/\mathbb{Z} \). So, in this case, the surface \( V_1 \) has no \( K \)-rational point, which implies that the surface \( V_1 \) violates the Hasse principle. If the number \( \sharp S \) is even, then for any \((P_v)_{v \in \Omega_K} \in V_1(\mathbb{A}_K)\), by Prop. 3.9 the sum \( \sum_{v \in \Omega_K} \text{inv}_v(A(P_v)) = \frac{\sharp S}{2} \), zero in \( \mathbb{Q}/\mathbb{Z} \). For \( \text{Br}(V_1)/\text{Br}(K) \) is generated by the element \( A \), we have \( V_1(\mathbb{A}_K)^{\text{Br}} = V_1(\mathbb{A}_K) \neq \emptyset \). According to [Sk01 Theorem 7.2.1], Brauer-Manin obstruction to the Hasse principle and weak approximation is the only one for Châtelet surfaces, so \( V_1(K) \neq \emptyset \) is dense in \( V_1(\mathbb{A}_K)^{\text{Br}} = V_1(\mathbb{A}_K) \), i.e. the surface \( V_1 \) verifies the Hasse principle and weak approximation over \( K \).

**Corollary 3.11.** Assume that all places of \( S \) are split completely in \( L \). Then there exists a Châtelet surface \( V_1 \) defined over \( K \) such that \( \text{Br}(V_1)/\text{Br}(K) \cong \text{Br}(V_{1L})/\text{Br}(L) \cong \mathbb{Z}/2\mathbb{Z} \) generated by an element \( A \in \text{Br}(V_1) \), and the set \( V_1(\mathbb{A}_K) \subset V_1(\mathbb{A}_L) \) is nonempty. For any \( v' \in \Omega_L \) and any \( P_{v'} \in V_1(L_{v'}) \), the invariance \( \text{inv}_{v'}(A(P_{v'})) = 0 \) if \( v' \notin \Delta_L \), the invariance \( \text{inv}_{v'}(A(P_{v'})) = \frac{1}{2} \) if \( v' \in \Delta_L \).

**Proof.** We will prove the surface \( V_1 \) constructed in above Prop. 3.9 satisfies the required conditions. As in the proof of above Prop. 3.5, the polynomial \( P(x) \) is separable and a product of two 2-degree irreducible factors over \( L \) and \( a \notin L^2 \). The same argument as in the proof of above Prop. 3.9, the quaternion algebra \( A = (a,x^2-c) \in \text{Br}(V_1) \) generates the Brauer group \( \text{Br}(V_1)/\text{Br}(K) \cong \text{Br}(V_{1L})/\text{Br}(L) \cong \mathbb{Z}/2\mathbb{Z} \). Take arbitrary \( v' \in \Delta_L \). Let \( v \in \Omega_K \) be the restriction of \( v' \) on \( K \). By assumption that \( v \) is split completely in \( L \), we have \( K_v = L_{v'} \). So \( V_1(K_v) = V_1(L_{v'}) \) and the invariance \( \text{inv}_{v'}(A(P_{v'})) = \frac{1}{2} \). For \( v' \in \Omega_L \backslash \Delta_L \), this local computation is the same as in the proof of above Prop. 3.9.

**Corollary 3.12.** For any number field \( K \) and any nontrivial field extension \( L \) of even degree over \( K \), there exists a Châtelet surface \( V \) defined over \( K \) such that \( \text{Br}(V)/\text{Br}(K) \cong \text{Br}(V_{1L})/\text{Br}(L) \cong \mathbb{Z}/2\mathbb{Z} \) generated by an element \( A \in \text{Br}(V) \), and the set \( V(\mathbb{A}_K) \) is nonempty. The surface \( V \) has the following properties:

- The surface \( V \) violates Hasse principle over \( K \).
- The surface \( V_{1L} \) verifies Hasse principle over \( L \). Furthermore, it verifies weak approximation over \( L \).

**Proof.** By Theorem 2.2, we can take a place \( v_0 \in \Omega_K \backslash \{ \text{all complex places and 2-adic places} \} \) splitting completely in \( L \). Let \( S = \{ v_0 \} \). By Prop. 3.9 and Cor. 3.11 there exists a Châtelet surface \( V \) defined over \( K \) satisfying the conditions of Prop. 3.9 and Cor. 3.11. For \( \sharp S \) is odd and \( \sharp S_L \) is even, the properties we list are just what we explain in Remark 3.10.

The following example will be used for further discussion.

**Example 3.13.** Let \( K, L \) be as in Example 3.9 and let \( S = \{ 13 \} \). For \( 13^2 \equiv 1 \mod 7, 41^2 \equiv 1 \mod 7 \) and \( 43 \equiv 1 \mod 7 \), the places \( 13, 41, 43 \) split completely in \( L \). Using the construction method of Prop. 3.9 we choose data: \( S = \{ 13 \}, S' = \{ 13, 29 \}, S'' = \{ 5 \}, v_1 = 43, v_2 = 41, a = 377, b = 5, c = 878755181 \) and \( P(x) = (x^2 - 878755181)(5x^2 - 4393775906) \). Then the Châtelet surface given by \( y^2 - 377z^2 = P(x) \) has the properties of Prop. 3.9 and Cor. 3.11.
3.4. Châtelet surface with a rational point and not verifying weak approximation.

**Proposition 3.14.** For any number field $K$ and any finite $S$ not containing any complex place and any 2-adic place, there exists a Châtelet surface $V_2$ such that $\text{Br}(V_2)/\text{Br}(K) \cong \mathbb{Z}/2\mathbb{Z}$ generated by an element $A \in \text{Br}(V_2)$, the set $V_2(K)$ is nonempty. For any $v \in S$, there exists $P_v, Q_v \in V_2(K_v)$ such that $\text{inv}_v(A(P_v)) = 0$ and $\text{inv}_v(A(Q_v)) = \frac{1}{2}$; for any other $v \notin S$ and any $P_v \in V_2(K_v)$, the invariance $\text{inv}_v(A(P_v)) = 0$.

**Proof.** Let $S' = \{ v \in \Omega^f_K \setminus \{ \text{all 2-adic places} \} \mid v(a) \neq 0 \}$, then $S' \supset S' \setminus S \infty_K$ is a finite set. By Lemma 2.6, for $v \in S' \setminus S$, the set $\{ b \in K_v \mid v(b) = -v(a) \}$ is a nonempty open subset of $K_v$; for $v \in S' \setminus S$, the set $\{ b \in K_v \mid v(b) = v(a) \}$ is a nonempty open subset of $\text{Br}(K_v)$. Using Fact 2.11, we can choose a nonzero element $b \in O_S[\frac{1}{2}]$ with the following conditions:

- $v(b) = -v(a)$ for all $v \in S' \setminus S$.
- $v(b) = v(a)$ for all $v \in S' \setminus S$.

Let $S'' = \{ v \in \Omega^f_K \setminus \{ \text{all 2-adic places} \} \mid v(b) \neq 0 \}$, then $S''$ is a finite set and $S' \supset S''$. By Theorem 2.2, we can take two different finite places $v_1, v_2 \in \Omega^f_K \setminus (S'' \cup \{ \text{all 2-adic places} \})$ splitting completely in $L$. If $v \in S' \setminus S$, then $v(a)$ is odd. In this case, by Lemma 2.9, the set $\{ c \in O^f_K \mid (a, c)_v = -1 \}$ is a nonempty open subset of $\text{Br}(K_v)$. If $v \in \{ v_1, v_2 \}$, then $b \in O^f_K$. In this case, by Lemma 2.6, the sets $\{ c \in K_v \mid v(c) = 1 \}$ and $\{ c \in K_v \mid v(1 + cb^2) = 1 \}$ are nonempty open subsets of $\text{Br}(K_v)$. Also using Fact 2.11, we can choose a nonzero $c \in O_K[\frac{1}{2}]$ with the following conditions:

- $\tau_v(c + b^2) < 0$ for all $v \in S' \setminus S \infty_K$.
- $(a, c)_v = -1$ and $v(c) = 0$ for all $v \in S' \setminus S \infty_K$.
- $v_1(c) = 1$ and $v_2(1 + cb^2) = 1$ for the chosen $v_1, v_2$ above.

Let $P(x) = (cx^2 + 1)((1 + cb^2)x^2 + b^2)$. We will check that the Châtelet surface $V_2$ given by $y^2 - x^2 = (cx^2 + 1)((1 + cb^2)x^2 + b^2)$, satisfies the required conditions.

For $(x, y, z) = (0, b, 0)$ is a rational point on $V_2$, the set $V_2(K)$ is nonempty. We denote this rational point by $Q_0$.

By the choice of $b, c, v_1, v_2$, the polynomial $cx^2 + 1$ (respectively $(1 + cb^2)x^2 + b^2$) is irreducible over $K_v$ (respectively $K_v$). So $P(x)$ is separable and a product of two 2-degree irreducible factors over $K$ (even over $L$), according to [Sk01 Prop. 7.1.1]. The group $\text{Br}(V_2)/\text{Br}(K) \cong \mathbb{Z}/2\mathbb{Z}$. Furthermore, by Prop. 7.1.2 in loc. cit., we take the quaternion algebra $A = (a, cx^2 + 1) \in \text{Br}(V_2)$ as a generator element of this group. Then we have the equality $A = (a, cx^2 + 1) = (a, 1 + cb^2)x^2 + b^2$ in $\text{Br}(V_2)$. For any $v \in \Omega_K$, the invariance $\text{inv}_v(A(Q_0)) = 0$. By Remark 2.7, it suffices to check the invariance $\text{inv}_v(A(P_v))$ for $P_v \in V_2(K_v)$.

Suppose that $v$ is an archimedean place such that $v \in \infty_K \setminus S$ or a 2-adic place. Then $a \in K_v^{\times}$, so $\text{inv}_v(A(P_v)) = 0$.

Suppose that $v \in S \setminus \{ \text{all 2-adic places} \}$. If $\text{inv}_v(A(P_v)) = \frac{1}{2}$, then $(a, cx^2 + 1)_v = -1 = (a, (1 + cb^2)x^2 + b^2)_v$ at $P_v$. The first equality implies that $v(x) \leq 0$. For $v(a) = v(b) > 0$ and $v(c) \geq 0$, we have $(a, (1 + cb^2)x^2 + b^2)_v = (a, (1 + cb^2)x^2 + b^2)_v = 1$, which is a contradiction. So $\text{inv}_v(A(P_v)) = 0$.

Suppose that $v \in \Omega^f_K \setminus \{ S' \cup \{ \text{all 2-adic places} \} \}$. If $\text{inv}_v(A(P_v)) = \frac{1}{2}$, then $(a, cx^2 + 1)_v = -1 = (a, (1 + cb^2)x^2 + b^2)_v$ at $P_v$. For $v(a) = 0$, by Lemma 2.4, the first equality implies that $v(cx^2 + 1)$ is odd and the last equality implies that it is negative. So $v(x^2 + c)$ is odd and positive. But $(a, (1 + cb^2)x^2 + b^2)_v = (a, x^2)_v(a, 1 + b^2(x^2 + c))_v = 1$, which is a contradiction. So $\text{inv}_v(A(P_v)) = 0$.

Suppose that $v \in S \cap \infty_K$. Take $P_v = Q_0$, then $\text{inv}_v(A(P_v)) = 0$. By the choice of $b, c$, we have $\tau_v(\frac{b^2}{c^{2b^2-1}}) > \frac{1}{c} > 0$. Take $x_0 > \sqrt{\tau_v(\frac{b^2}{c^{2b^2-1}})}$, then $P(x_0) > 0$. 

**Remark 3.2.** It suffices to check the invariance $\text{inv}_v(A(P_v))$ for $P_v \in V_2^n(K_v)$.
Prop. 3.14 and the choice of \( V \), \( U \) or any \( v / F \) for any verifies weak approximation off \( S \) or \( F \) or \( x \)

Let \( \text{inv}_v(A(P_v)) \) required conditions. The same argument as in the proof of Cor. 3.11 and Prop. 3.14, we have the quaternion algebra

Suppose that \( T \), a take an element \( \{ \} \in \sum \), is a nonempty open subset of \( V_2(K_v) \) with \( x = x_0 \). Then \( \text{inv}_v(A(Q_v)) = \frac{1}{2} \).

Remark 3.15. For any \( v \in S \) and any \( P_v \in V_2(K_v) \), the invariance of \( A \) on \( P_v \) is 0 or \( \frac{1}{2} \). Let \( U_1 = \{ P_v \in V_2(K_v) \} \) and \( U_2 = \{ P_v \in V_2(K_v) \} \) such that \( v \in S \). Then \( U_1 \) and \( U_2 \) are disjoint open subsets of \( V_2(K_v) \), and \( V_2(K_v) = U_1 \cup U_2 \).

Corollary 3.16. Let \( V_2 \) be a Châtelet surface with properties of Prop. 3.14. If \( S = \emptyset \), then \( V_2 \) verifies weak approximation over \( K \). If \( S \neq \emptyset \), then \( V_2 \) verifies weak approximation off \( T \) for any finite \( T \subset \Omega_K \) with \( T \cap S \neq \emptyset \), while fails for any finite \( T \subset \Omega_K \) with \( T \cap S = \emptyset \).

Proof. According to [Sko01] Theorem 7.2.1, Brauer-Manin obstruction to weak approximation is the only one for Châtelet surfaces, so \( V_2(K) \) is dense in \( V_2(\mathbb{A}_K) \). Suppose that \( S = \emptyset \), then for any \( (P_v)_{v \in \Omega_K} \in V_2(\mathbb{A}_K) \), by Prop. 3.14 the sum \( \sum_{v \in \Omega_K} \text{inv}_v(A(P_v)) \) is 0. For \( Br(V_2)/Br(K) \) is generated by the element \( A \), we have

Take an element \( (P_v)_{v \in \Omega_K} \in M \). By Prop. 3.14 and \( v_0 \in S \), take two elements \( P_{v_0} \), \( P_{v_0}' \in V_2(K_v) \) with \( \text{inv}_{v_0} A(P_{v_0}) = 0 \) and \( \text{inv}_{v_0} A(P_{v_0}') = \frac{1}{2} \). By Prop. 3.14 and Remark 3.15 the sum \( \sum_{v \in \Omega_K} \text{inv}_v(A(P_v)) \) is 0 or \( \frac{1}{2} \) in \( \mathbb{Q}/\mathbb{Z} \). If it is 0, then replace \( P_{v_0} \) by \( P_{v_0}' \). Otherwise, replace \( P_{v_0} \) by \( P_{v_0}' \). Then the sum \( \sum_{v \in \Omega_K} \text{inv}_v(A(P_v)) \) is 0 in \( \mathbb{Q}/\mathbb{Z} \). So \( (P_v)_{v \in \Omega_K} \in V_2(\mathbb{A}_K) \) \( \cap M \). For \( V_2(K) \) is dense in \( V_2(\mathbb{A}_K) \), the set \( V_2(K) \cap M \neq \emptyset \), which implies that \( V_2 \) verifies weak approximation off \( \{ v \} \). So \( V_2 \) verifies weak approximation off \( T \).

Suppose that \( S \neq \emptyset \) and \( T \cap S = \emptyset \). Take \( v_0 \in S \) and let \( U_v = \{ P_v \in V_2(K_v) \} \) such that \( v \in S \). For \( v \in S \), by Remark 3.15 the set \( U_v \) is a nonempty open subset of \( V_2(K_v) \). Let \( M = \bigcap_{v \in S} U_v \). So it is a nonempty open subset of \( V_2(\mathbb{A}_K) \). For any \( (P_v)_{v \in \Omega_K} \in M \), by Prop. 3.14 and the choice of \( U_v \) for \( v \in S \), the sum \( \sum_{v \in \Omega_K} \text{inv}_v(A(P_v)) \) is \( \frac{1}{2} \) in \( \mathbb{Q}/\mathbb{Z} \).

So \( V_2(\mathbb{A}_K) \cap M = \emptyset \), which implies \( V_2(\mathbb{A}_K) \cap M = \emptyset \). Hence \( V_2 \) does not verify weak approximation off \( T \).

Corollary 3.17. Assume that all places of \( S \) split completely in \( L \). Then there exists a Châtelet surface \( V_2 \) defined over \( K \) such that \( Br(V_2)/Br(K) \cong Br(V_2L)/Br(L) \cong \mathbb{Z}/2\mathbb{Z} \) generated by an element \( A \in Br(V_2) \) and the set \( V_2(K) \subset V_2(L) \) is nonempty. For any \( v' \in S_L \), there exists \( P_{v'}, Q_{v'} \in V_2(L_{v'}) \) such that \( \text{inv}_{v'}(A(P_{v'})) = 0 \) and \( \text{inv}_{v'}(A(Q_{v'})) = \frac{1}{2} \) for any other \( v' \notin S_L \) and any \( P_{v'} \in V_2(L_{v'}) \). The invariance \( \text{inv}_{v'}(A(P_{v'})) = 0 \).

Proof. We will prove the surface \( V_2 \) constructed in above Prop. 3.14 satisfies the required conditions. The same argument as in the proof of Cor. 3.11 and Prop. 3.14 we have the quaternion algebra \( A = (a, cx^2 + 1) \in Br(V_2) \) generates the Brauer group \( Br(V_2)/Br(K) \cong Br(V_2L)/Br(L) \cong \mathbb{Z}/2\mathbb{Z} \); and for any \( v' \in S_L \), there
exists \( P_{\nu}, Q_{\nu} \in V_2(L_{\nu}) \) such that \( \text{inv}_{\nu}(A(P_\nu)) = 0 \) and \( \text{inv}_{\nu}(A(P_\nu)) = \frac{1}{2} \). For \( \nu' \in \Omega_L \setminus S_L \), this local computation is the same as in the proof of above Prop. 3.14.

**Corollary 3.18.** Let \( V_2 \) be a Châtelet surface with properties of Prop. 3.17 If \( S = \emptyset \), then \( V_{2L} \) verifies weak approximation over \( L \). If \( S \neq \emptyset \), then \( V_2 \) verifies weak approximation off \( T \) for any finite \( T \subset \Omega_L \) with \( T \cap S_L \neq \emptyset \), while fails for any finite \( T \subset \Omega_L \) with \( T \cap S_L = \emptyset \).

**Proof.** This is the same as in proof of above Cor. 3.16. \( \square \)

The following example will be used for further discussion.

**Example 3.19.** Let \( K = \mathbb{Q} \), \( L = \mathbb{Q}(\sqrt{3}) \) and let \( S = \{73\} \). The prime numbers 11, 23, 73 split completely in \( L \). Using the construction method of Prop. 3.14 we choose data: \( S = S' = S'' = 73 \), \( v_1 = 11 \), \( v_2 = 23 \), \( a = 73 \), \( b = 1/73 \), \( c = 99 \) and \( P(x) = (99x^2 + 1)(5428x^2 + 1)(4280x^2 + 1) \). Then the Châtelet surface given by \( y^2 - 73z^2 = P(x) \) has the properties of Prop. 3.14 Cor. 3.16 and Cor. 3.18.

4. Stoll’s Conjecture for Curves

**Conjecture 4.1.** [Sto07] Conj. 9.1] Given a nice curve \( C \) defined over a number field \( K \), let \( C(A_K) = \prod_{v \in \infty_K} \{ \text{connected components of } K_v \} \times C(A_K) \) with discrete topology for the set of connected components of \( K_v \) for all \( v \in \infty_K \). Then \( C \) verifies Hasse principle with Brauer–Manin obstruction over \( K \). Furthermore, the curve \( C \) verifies weak approximation with Brauer–Manin obstruction over \( \infty_K \); more exactly, the set \( C(K) \) is dense in \( C(A_K) \).

**Remark 4.2.** Whether a nice curve \( C \) verifies Hasse principle with Brauer–Manin obstruction over \( K \) was considered by A. Skorobogatov [Sko01, Chap. 6.3] and V. Scharaschkin [Sch99] independently. This Conj. 4.1 holds for elliptic curve defined over \( \mathbb{Q} \) of analytic rank 0.

In this paper, we make the following assumptions for the given number field extension \( L \) over \( K \).

**Assumption 4.3.** There exists a nice curve \( C \) defined over \( K \) such that \( C(K) \) and \( C(L) \) are nonempty finite set, \( C(K) \neq C(L) \) and Stoll’s Conj. 4.1 holds for the curve \( C \) over \( K \).

**Assumption 4.4.** There exists a nice curve \( C \) defined over \( K \) such that it satisfies Assumption 4.3 and there exists a real place \( \nu' \in \infty_L \) such that \( C(L_{\nu'}) \) is connected.

**Remark 4.5.** Let \( L = K(\theta) \) and let \( f(x) \) be the minimal polynomial of \( \theta \) and \( f(w_1, w_0) \) be its homogenization. Consider a curve \( C \) defined over \( K \) by a homogeneous equation: \( w_2^{n+2} = f(w_1, w_0)(w_1^2 - w_0^2) \) with homogeneous coordinates \( (w_0 : w_1 : w_2) \in \mathbb{P}^2 \). For the polynomials \( f(x) \) and \( x^2 - 1 \) are separable and coprime in \( K[x] \), this defines a nice curve. By genus formula for plane curve, the genus of \( C \) equals \( g(C) = \frac{n(n+1)}{2} \geq 3 \), then by Faltings’s theorem, the sets \( C(K) \) and \( C(L) \) are finite. For \( (w_0 : w_1 : w_2) = (1 : 1 : 0) \in C(K) \) and \( (1 : \theta : 0) \in C(L) \cap C(K) \), if Stoll’s Conj. 4.1 holds for all nice curves over \( K \), then this curve \( C \) satisfies our Assumption 4.3. Furthermore, if \( n \) is odd, then \( C_{K_{\nu'}} \) is connected for all \( \nu' \in \infty_L \). If \( n \) is even, then consider a curve \( C' \) defined by a homogeneous equation: \( w_2^{n+3} = f(w_1, w_0)(w_1^2 - w_0^2) \) with homogeneous coordinates \( (w_0 : w_1 : w_2) \in \mathbb{P}^2 \). Then \( C(L_{\nu'}) \) is connected for all \( \nu' \in \infty_L \). In other words, if \( L \) has a real place and Stoll’s Conj. 4.1 holds for all nice curves over \( K \), then there exists a curve satisfying our Assumption 4.3. For \( K = \mathbb{Q} \) and \( L = \mathbb{Q}(\sqrt{d}) \) for some square free integer \( d \),
consider elliptic curve $E_d$ with Weierstraß equation: $y^2 = x^3 + d$ and its quadratic twist $E_d^{(d)}$ with Weierstraß equation: $y^2 = x^3 + d^4$. The curve $E_d$ is connected over $\mathbb{R}$. If both $E_d(\mathbb{Q})$ and $E_d^{(d)}(\mathbb{Q})$ are finite and the Tate-Shafarevich group $\text{III}(E_d, \mathbb{Q})$ is finite, then the curve $E_d$ satisfies our Assumption 4.3 moreover, if $d$ is positive, then the curve $E_d$ satisfies our Assumption 4.3.

5. A main proposition from Poonen [Poo10]

For our result is base on a result of the paper [Poo10, Prop. 5.4]. We recall that paper and his general result first. There also exists remark on that paper in the paper [Lin18, Section 4.1].

Recall 5.1. Let $B$ be a nice variety over $K$. Let $\mathcal{L}$ be a line bundle on $B$, assuming the global section $\Gamma(B, \mathcal{L}^{\otimes 2}) \neq 0$. Let $E = \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{L}$, let $a$ be a constant in $K^\times$ and let $s$ be a nonzero global section in $\Gamma(B, \mathcal{L}^{\otimes 2})$. The zero locus of $(1, -a, -s) \in \Gamma(B, \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{L}^{\otimes 2}) \subset \Gamma(B, \text{Sym}^2 \mathcal{L})$ in the projective space bundle $\text{Proj}(\mathcal{L})$ is a projective geometrically integral variety, denoted by $X$ with nature morphism $\alpha : X \to B$. Assume that the closed subscheme $Z$ defined by $s = 0$ in $B$ is nice.

Proposition 5.2. [Poo10] Prop. 5.3 Let morphism $\alpha : X \to B$ and assumption that the closed subscheme $Z$ of $B$ is nice, be as in Recall 5.1 above. Let $\overline{K}$ be an algebraic closure field of $K$ and let $\overline{B} := B \times_K \overline{K}$. Assume additionally that $\text{Br} \overline{B} = 0$ and $X(\overline{K}) \neq \emptyset$. Then $X$ is nice and $\alpha^* : \text{Br}(B) \to \text{Br}(X)$ is an isomorphism.

6. The main theorems.

Lemma 6.1. Let $X, Y$ be nice varieties defined over $K$ and let $f : X \to Y$ be a $K$-morphism. For any finite $S \subset \Omega_K$, we assume that

1. The set $Y(K)$ is finite.

2. The variety $Y$ verifies weak approximation with Brauer-Manin obstruction off $S$.

3. For any $P \in Y(K)$, the fiber $X_P$ of $f$ over $P$ verifies weak approximation off $S$.

Then $X$ verifies weak approximation with Brauer-Manin obstruction off $S$.

Proof. Take any open subset $N = \prod_{v \in T} U_v \times \prod_{v \notin T} X(K_v) \subset X(\mathbb{A}_K)$ with finite $T \subset \Omega_K \setminus S$ and $N \cap X(\mathbb{A}_K)^{\text{Br}} \neq \emptyset$. Let $M = \prod_{v \in T} f(U_v) \times \prod_{v \notin T} f(X(K_v))$, then by the functoriality of Brauer-Manin pairing, $M \cap Y(\mathbb{A}_K)^{\text{Br}} \neq \emptyset$. View $Y(K)$ as a subset of $\text{pr}^3(\mathbb{A}_K)$ by the diagonal embedding, then the assumptions 1 and 2 imply $Y(K) = \text{pr}^3(\mathbb{A}_K)^{\text{Br}}$. So there exists $P_0 \in \text{pr}^3(M) \cap Y(K)$. Consider the fiber $X_{P_0}$. Let $L = \prod_{v \in T} [X_{P_0}(K_v) \cap U_v] \times \prod_{v \notin T} X_{P_0}(K_v)$, then it is a nonempty open subset of $X_{P_0}(\mathbb{A}_K)$. By assumption 3 there exists $Q_0 \in L \cap X_{P_0}(K)$. So $Q_0 \in X(K) \cap N$, which implies that $X$ verifies weak approximation with Brauer-Manin obstruction off $S$. \qed

Lemma 6.2. Let $X, Y$ be nice varieties defined over $K$ and let $f : X \to Y$ be a $K$-morphism. For any finite $S \subset \Omega_K$, we assume that

1. The set $Y(K)$ is finite.
(2) The variety \( Y \) verifies weak approximation with Brauer–Manin obstruction off \( S \).

(3) The morphism \( f^* : \text{Br}(Y) \to \text{Br}(X) \) is an isomorphism.

(4) There exists some \( P \in Y(K) \) such that the fiber \( X_P \) of \( f \) over \( P \) does not verify weak approximation off \( S \), and for any \( v \in S \), the set \( X_P(K_v) \neq \emptyset \).

Then \( X \) does not verify weak approximation with Brauer–Manin obstruction off \( S \).

**Proof.** By assumption [4] take a \( P_0 \in Y(K) \) such that the fiber \( X_{P_0} \) does not verify weak approximation with Brauer–Manin obstruction off \( S \). Let \( L = \prod_{v \in T} U_v \times \prod_{v \not\in T} X_{P_0}(K_v) \) be a nonempty open subset of \( X_{P_0}(\mathbb{A}_K) \) with finite nonempty set \( T \subset \Omega_K \setminus S \) and \( L \cap X_{P_0}(K) = \emptyset \). By assumption [1] the set \( Y(K) \) is finite, so we can take a Zariski open subset \( V_{P_0} \) of \( Y \) such that \( V_{P_0} \cap Y(K) = P_0 \). For any \( v \in T \), since \( U_v \) is open in \( X_{P_0}(K_v) \subset f^{-1}(V_{P_0})(K_v) \), we can take an open subset \( W_v \) of \( f^{-1}(V_{P_0})(K_v) \) such that \( W_v \cap X_{P_0}(K_v) = U_v \). Consider the open subset \( N = \prod_{v \in T} W_v \times \prod_{v \not\in T} X(K_v) \) of \( X(\mathbb{A}_K) \), then \( L \subset N \). By the functoriality of Brauer–Manin pairing and assumption [3] we have \( L \subset N \cap X(\mathbb{A}_K)^{\text{Br}(X)} \). So \( N \cap X(\mathbb{A}_K)^{\text{Br}(X)} = \emptyset \).

But \( N \cap X(K) = N \cap X_{P_0}(K) = L \cap X_{P_0}(K) = \emptyset \), which implies \( X \) does not verify weak approximation with Brauer–Manin obstruction off \( S \). \( \Box \)

**Lemma 6.3.** Let \( C \) be a curve satisfying Assumption [4.3.]. For any finite \( K \)-subscheme \( B \subset \mathbb{P}^1 \setminus \{0, \infty\} \), there exists a dominant \( K \)-morphism \( \gamma : C \to \mathbb{P}^1 \) such that \( \gamma(C(L)|C(K)) = \{0\} \subset \mathbb{P}^1 \), \( \gamma(C(K)) = \{\infty\} \subset \mathbb{P}^1 \) and \( \gamma \) is étale over \( R \).

**Proof.** Let \( K(C) \) be the function field of \( C \). Let \( M \) be the smallest Galois extension containing \( L \) over \( K \), and let \( \text{Gal}(M/K) \) be the Galois group. For \( C(K) \) and \( C(L) \) are nonempty finite sets and \( C(L)|C(K) \neq \emptyset \), by Riemann-Roch theorem, we can choose a rational function \( \phi \in M(C)^\times \setminus M^\times \) such that the set of poles of \( \phi \) contains \( C(L) \) and the set of zeros of \( \phi \) contains \( C(L)|C(K) \). Replacing \( \phi \) by \( \prod_{\sigma \in \text{Gal}(M/K)} \sigma(\phi) \), we can assume \( \phi \in K(C)^\times \setminus K^\times \). The rational function \( \phi \) gives a dominant \( K \)-morphism \( \gamma_0 : C \to \mathbb{P}^1 \) such that \( \gamma_0(C(L)|C(K)) = \{0\} \subset \mathbb{P}^1 \) and \( \gamma_0(C(K)) = \{\infty\} \subset \mathbb{P}^1 \).

We can choose an automorphism \( \varphi_{\lambda_0} : \mathbb{P}^1 \to \mathbb{P}^1, (u, v) \mapsto (\lambda_0 u, v) \) with \( \lambda_0 \in K^\times \) such that the branch locus of \( \gamma_0 \) has no intersection with \( \varphi_{\lambda_0}(R) \). Let \( \lambda = (\varphi_{\lambda_0})^{-1} \circ \gamma_0 \). Then the morphism \( \lambda \) is étale over \( R \) and satisfies other conditions. \( \Box \)

**Lemma 6.4.** Let \( C \) be a curve over \( K \) and let \( B = C \times \mathbb{P}^1 \). Then \( \text{Br}(C \times \mathbb{P}^1) = 0 \).

**Proof.** By [Gro68] III, Cor., 1.2], the Brauer group for curve over an algebraic closed field is zero. So \( \text{Br}(C \times \mathbb{P}^1) \cong \text{Br}(\mathbb{P}^1) = 0 \). \( \Box \)

6.1. Noninvariance of weak approximation with Brauer–Manin obstruction.

**Theorem 6.5.** For any number field \( K \) and any nontrivial field extension \( L \) over \( K \), let \( T \subset \Omega_L \) be any finite subset. Assuming Assumption [4.3] there exists a 3-fold \( K \) such that

- The 3-fold \( X \) has a \( K \)-rational point and verifies weak approximation with Brauer–Manin obstruction, étale Brauer–Manin obstruction off \( \infty_K \).
- The 3-fold \( X_L \) does not verify weak approximation with Brauer–Manin obstruction, étale Brauer–Manin obstruction off \( T \).

**Proof.** Let \( S_T \subset \Omega_K \) be all restrictions of \( T \) on \( K \). By Theorem 2.2, we can take a finite place \( v_0 \in \Omega_K \setminus (S_T \cup \{\text{all 2-adic places}\}) \) splitting completely in \( L \). For
\[ S = \{ v_0 \}, \text{using Cor. } 3.17 \text{ there exists a Châtelet surface } V_0 \text{ defined by } y^2 - az^2 = P_0(x) \text{ over } K \text{ with the properties of Cor. } 3.17. \text{ Let } P_\infty(x) = (x^2 - 1)(x^2 - 4) \text{ and let } V_\infty \text{ be the Châtelet surface defined by } y^2 - az^2 = P_\infty(x). \text{ For } P_0(x) \text{ is a product of two 2-degree irreducible factors and } P_\infty(x) \text{ is a product of four linear factors, these two polynomials are coprime in } K[x]. \text{ Let } \tilde{P}_\infty(x_0, x_1) \text{ and } \tilde{P}_0(x_0, x_1) \text{ be the homogenizations of } P_\infty \text{ and } P_0. \text{ Let } (u_0 : u_1) \times (x_0 : x_1) \text{ be the coordinates of } \mathbb{P}^1 \times \mathbb{P}^1 \text{ and let } s' = u_0^2 \tilde{P}_\infty(x_0, x_1) + u_1^2 \tilde{P}_0(x_0, x_1) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 2)^{\otimes 2}). \text{ For } P_0(x) \text{ and } P_\infty(x) \text{ are coprime in } K[x], \text{ by Jacobian criterion, the locus } Z' \text{ defined by } s' = 0 \text{ in } \mathbb{P}^1 \times \mathbb{P}^1 \text{ is smooth. Let } R \text{ be the branch locus of the composition } Z' \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{pr_1} \mathbb{P}^1, \text{ which is finite over } K. \text{ By assumption, take a curve } C \text{ satisfying Assumption } 4.3 \text{ by Lemma } 6.3 \text{ we can choose a } K \text{-morphism } \gamma : C \to \mathbb{P}^1 \text{ such that } \gamma(C(L) \setminus C(K)) = \{ 0 \} \subset \mathbb{P}^1, \gamma(C(K)) = \{ \infty \} \subset \mathbb{P}^1 \text{ and } \gamma \text{ is étale over } R. \text{ Let } B = C \times \mathbb{P}^1, \text{ let } L = (\gamma, id)^* \mathcal{O}(1, 2) \text{ and let } s = (\gamma, id)^*(s') \in \Gamma(B, \mathcal{L}^{\otimes 2}). \text{ For } \gamma \text{ is étale over the branch locus } R, \text{ the locus } Z \text{ defined by } s = 0 \in B \text{ is smooth. Since } Z \text{ is defined by the support of the global section } s, \text{ it is effective divisor. The invertible sheaf } \mathcal{L}(Z') \text{ on } \mathbb{P}^1 \times \mathbb{P}^1 \text{ is isomorphism to } \mathcal{O}(2, 4), \text{ which is very ample sheaf on } \mathbb{P}^1 \times \mathbb{P}^1. \text{ And } (\gamma, id) \text{ is a finite morphism, so the pull back of this ample sheaf is again ample, which implies the invertible sheaf } \mathcal{L}(Z) \text{ on } C \times \mathbb{P}^1 \text{ is ample. By } \text{[Har97] Chapt. III Cor. 7.9}, \text{ the curve } Z \text{ is geometrically connected. So the curve } Z \text{ is nice. By Lemma } 6.4 \text{ the Brauer group } Br(\overline{E}) = 0. \text{ Let } X \text{ be the zero locus of } (1, -a, -s) \in \Gamma(B, \mathcal{O}_B \oplus \mathcal{O}_B \otimes \mathcal{L}^{\otimes 2}) \subset \Gamma(B, \text{Sym}^2 \mathcal{E}) \text{ in the projective space bundle } \text{Proj}(\mathcal{E}) \text{ with nature morphism } \alpha : X \to B. \text{ Using Prop. } 5.2 \text{ the variety } X \text{ is nice. Let } \beta : X \xrightarrow{\alpha^{-1}} B = C \times \mathbb{P}^1 \xrightarrow{pr_1} C \text{ be the composition of } \alpha \text{ and } pr_1. \text{ For } \beta^{-1}(C(L)) = V_\infty \times C(K) \text{ and } V_\infty \text{ has a } K \text{-rational point } (x, y, z) = (0, 0, 0), \text{ the set } X(K) \neq \emptyset. \text{ For } Br(V_\infty)/Br(K) = 0, \text{ according to } \text{[Sko01] Theorem 7.2.1}, \text{ the surface } V_\infty \text{ verifies weak approximation over } K. \text{ Because the curve } C \text{ satisfies Assumption } 4.3 \text{ by Lemma } 6.1 \text{ for the morphism } \beta, \text{ the variety } X \text{ has a } K \text{-rational point and verifies weak approximation with Brauer–Manin obstruction, étale Brauer–Manin obstruction off } \infty_K. \text{ By Prop. } 5.2 \text{ we have } \alpha_X^* : Br(B_L) \to Br(X_L) \text{ is an isomorphism. By the choice of the curve } C \text{ and morphism } \beta, \text{ we have } \beta^{-1}(C(L)) = [V_\infty \times C(K)] \cup [V_0 \times \{ C(L) \setminus C(K) \}]. \text{ By Cor. } 5.18 \text{ the surface } V_0 \text{ does not verify weak approximation off } T \cup \infty_L. \text{ For } V_0(L) \neq \emptyset, \text{ by Lemma } 6.2 \text{ the variety } X \text{ does not verify weak approximation with Brauer–Manin obstruction, étale Brauer–Manin obstruction off } T \cup \infty_L. \text{ So it does not verify weak approximation with Brauer–Manin obstruction, étale Brauer–Manin obstruction off } T. \square

6.2. Noninvariance of Hasse principle with Brauer–Manin obstruction.

**Theorem 6.6.** For any number field } K \text{ and any nontrivial field extension } L \text{ of odd degree over } K, \text{ assuming Assumption } 5.3 \text{ there exists a 3-fold over } K \text{ such that } 
- The 3-fold } X \text{ has an } a_K \text{-adic point and verifies Hasse principle with Brauer–Manin obstruction over } K.
- The 3-fold } X_L \text{ does not verify Hasse principle with Brauer–Manin obstruction over } L.

**Proof.** By Theorem 2.2, we can take two different finite places } v_1, v_2 \in \Omega_K^f \setminus \{ \text{all 2-adic places} \} \text{ splitting completely in } L. \text{ For } \{ v_1, v_2 \}, \text{ we choose an element } a \text{ as in Section 5.11. For } S_1 = \{ v_1 \}, \text{ using Cor. } 5.11 \text{ there exists a Châtelet surface } V_0 \text{ defined by } y^2 - az^2 = P_0(x) \text{ over } K \text{ with the properties of Cor. } 5.11. \text{ For } S_2 = \{ v_2 \}, \text{ using Cor. } 5.3 \text{ there exists a Châtelet surface } V_\infty \text{ defined by } y^2 - az^2 = P_\infty(x) \text{ over } K \text{ with the properties of Cor. } 5.3. \text{ By Remark } 5.4 \text{ the polynomial } P_\infty(x) \text{ is irreducible over } K. \text{ For } P_0(x) \text{ is a product of two 2-degree irreducible factors over } K, \text{ the polynomials } P_0(x) \text{ and } P_\infty(x) \text{ are coprime in } K[x]. \text{ Let } P_\infty(x_0, x_1)
and $\tilde{P}_0(x_0, x_1)$ be the homogenizations of $P_\infty$ and $P_0$. Let $(u_0 : u_1) \times (x_0 : x_1)$ be the coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ and let $s' = u_0^2 \tilde{P}_\infty(x_0, x_1) + u_1^2 \tilde{P}_0(x_0, x_1) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 2)^{\otimes 2})$. For $F_0(x)$ and $P_\infty(x)$ are coprime in $K[x]$, by Jacobian criterion, the locus $Z'$ defined by $s' = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is smooth. Let $R$ be the branch locus of the composition $Z' \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{pr_1} \mathbb{P}^1$, which is finite over $K$. By assumption, take a curve $C$ satisfying Assumption 4.4. By Lemma 5.1, we can choose a $K$-morphism $\gamma : C \rightarrow \mathbb{P}^1$ such that $\gamma(C(L) \setminus C(K)) = \{0\} \subset \mathbb{P}^1$, $\gamma(C(K)) = \{\infty\} \subset \mathbb{P}^1$ and $\gamma$ is étale over $R$. Let $B = C \times \mathbb{P}^1$, let $L = (\gamma, id)^*\mathcal{O}(1, 2)$ and let $s = (\gamma, id)^*(s') \in \Gamma(B, \mathcal{L}^{\otimes 2})$. The same argument as in the proof of Theorem 5.5 that the locus $Z$ defined by $s = 0$ in $B$ is nice and the Brauer group $Br(\overline{B}) = 0$. Let $X$ be the zero locus of $(1, -a, -s) \in \Gamma(B, \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{L}^{\otimes 2}) \subset \Gamma(B, \text{Sym}^2 \mathcal{E})$ in the projective space bundle $\text{Proj}(\mathcal{E})$ with nature morphism $\alpha : X \rightarrow B$. By Prop. 5.2 the variety $X$ is nice. Let $\beta : X \rightarrow C$ be the composition of $\alpha$ and $pr_1$. We have $\beta^{-1}(C(K)) = V_\infty \times C(C) \subset X$ and by Prop. 5.3 the set $V_\infty(\mathbb{A}_K^{\{\infty\}}) \neq \emptyset$. So $X(\mathbb{A}_K^{\{\infty\}}) \neq \emptyset$. For $v_2$ splits completely in $L$, take a place $v' \in \Omega^\mathcal{L}_{L, 0}$ above $v_2$, i.e. $v' \mid v_2$ in $L$. Then $K_{v_2} = L_{v'}$. By Cor. 3.11 the set $V_0(\mathbb{A}_L) \neq \emptyset$. For $\beta^{-1}(C(L)) = [V_\infty \times C(L)] \cup [V_0 \times (C(L) \setminus C(K))] \subset X_L$, the set $X(\mathbb{A}_L) = X(L) \neq \emptyset$. So $X(\mathbb{A}_L) \neq \emptyset$. By Assumption 4.4 the set $C(K)$ is finite and $C(K) = pr_\infty^{-1}(C(\mathbb{A}_{K}^{\{\infty\}}))$. By the functoriality of Brauer–Mannin pairing, we have $\text{pr}_{\infty}^{-1}(X(\mathbb{A}_K^{\{\infty\}})) \subset \beta^{-1}(C(K)) \setminus (\mathbb{A}_K^{\{\infty\}})$. But by Prop. 5.3 the set $V_\infty(\mathbb{A}_K^{\{\infty\}}) = \emptyset$, so $\text{pr}_{\infty}^{-1}(X(\mathbb{A}_K^{\{\infty\}})) \subset \beta^{-1}(C(K)) \setminus (\mathbb{A}_K^{\{\infty\}})$, which implies that $X(\mathbb{A}_K^{\{\infty\}}) = \emptyset$. So, the variety $X$ has a $\mathbb{A}_K$-adjective point and verifies Hasse principle with Brauer–Mannin obstruction over $K$.

For the set $V_0(\mathbb{A}_L) \neq \emptyset$ and by Prop. 5.2 the map $\alpha_0^L : Br(C_L) \rightarrow Br(X_L)$ is an isomorphism, by the functoriality of Brauer–Mannin pairing, the set $X(\mathbb{A}_L)$ containing $(V_0 \times (C(L) \setminus C(K))) \setminus (\mathbb{A}_L)$ is nonempty. By assumption that the degree $[L : K]$ is odd and $v_1$ splits completely in $L$, the number $2\mathcal{S}_{1,L}$ is odd. By Cor. 3.11 and global reciprocity law explained in Remark 5.10 the set $V(L) = \emptyset$. By Cor. 5.9 the set $V_\infty(\mathbb{A}_L) = \emptyset$. Since $X(L) \subset \beta^{-1}(C(L)) \setminus (\mathbb{A}_L)$, the set $X(L) = \emptyset$. So, the variety $X$ does not verify Hasse principle with Brauer–Mannin obstruction over $L$.

\textbf{Theorem 6.7.} For any number field $K$ and any nontrivial field extension $L$, assuming Assumption 4.4 there exists a 3-fold over $K$ such that

- The 3-fold $X$ has a $\mathbb{A}_K$-adic point and verifies Hasse principle with Brauer–Mannin obstruction over $K$.
- The 3-fold $X_L$ does not verify Hasse principle with Brauer–Mannin obstruction over $L$.

\textbf{Proof.} By Assumption 4.4 let $v_0$ be a real place of $L$ and let $v_0 \in \mathbb{A}_K$ be the restriction of $v_0$ on $K$. Let $S_0 \subset \mathcal{S}_{1,L}$ be all places above $v_0$. By Theorem 2.2 we can take a finite place $v_1 \in \Omega^\mathcal{L}_{K, 0} \setminus \{\text{all 2-adic places}\}$ splitting completely in $L$. For $\{v_0, v_1\}$, we choose an element $a$ as in Section 3.1. For $S_1 = \{v_0\}$, using Prop. 5.3 there exists a Châtelet surface $V_0$ defined by $y^2 - az^2 = P_0(x)$ over $K$ with the properties of Prop. 5.3. For $S_2 = \{v_1\}$, using Cor. 5.3 there exists a Châtelet surface $V_\infty$ defined by $y^2 - az^2 = P_\infty(x)$ over $K$ with the properties of Cor. 5.3. By Remark 4.4 the polynomial $P_0(x)$ is irreducible over $K_{v_0}$, but the polynomial $P_\infty(x)$ is reducible over $K_{v_0}$. So, the polynomials $P_0(x)$ and $P_\infty(x)$ are coprime in $K[x]$. Let $\tilde{P}_\infty(x_0, x_1)$ and $\tilde{P}_0(x_0, x_1)$ be the homogenizations of $P_\infty$ and $P_0$. Let $(u_0 : u_1) \times (x_0 : x_1)$ be the coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ and let $s' = u_0^2 \tilde{P}_\infty(x_0, x_1) + u_1^2 \tilde{P}_0(x_0, x_1) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 2)^{\otimes 2})$. For $F_0(x)$ and $P_\infty(x)$ are coprime in $K[x]$, by Jacobian criterion, the locus $Z'$ defined by $s' = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is smooth. Let $R$ be the branch locus of the composition $Z' \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{pr_1} \mathbb{P}^1$, which is finite over $K$. By assumption, take a curve $C$ satisfying Assumption 4.4. By Lemma 6.3
we can choose a $K$-morphism $\gamma: C \to \mathbb{P}^1$ such that $\gamma(C(L)\setminus C(K)) = \{0\} \subset \mathbb{P}^1$, $\gamma(C(K)) = \{\infty\} \subset \mathbb{P}^1$ and $\gamma$ is étale over $R$. Let $B = C \times \mathbb{P}^1$, let $L = (\gamma, id)^*\mathcal{O}(1, 2)$ and let $s = (\gamma, id)^*(s') \in \Gamma(B, \mathcal{L}^{\otimes 2})$. The same argument as in the proof of Theorem 6.5 the locus $Z$ defined by $s = 0$ in $B$ is nice and the Brauer group $Br(B) = 0$. Let $X$ be the zero locus of $1, -a, -s \in \Gamma(B, \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{L}^{\otimes 2}) \subset \Gamma(B, \text{Sym}^2\mathcal{E})$ in the projective space bundle $\text{Proj}(\mathcal{E})$ with nature morphism $\alpha: X \to B$. By the same arguments as in the proof of Theorem 6.6 the variety $X$ is nice. Let $\beta: X \to C$ be the composition of $\alpha$ and $pr_1$. We have $\beta^{-1}(C(K)) = V_\infty \times C(K) \subset X$ and by Prop. 3.3 the set $V_\infty(\Delta_{K^1}) \neq \emptyset$. So $X(\Delta_{K^1}) \neq \emptyset$. For $v_1$ splits completely in $L$, take a place $v' \in \Omega_1$ above $v_1$, i.e. $v'|v_1$ in $L$. Then $K_{v_1} = L_{v'}$. By Prop. 3.3, the set $V_0(K_{v_1}) \neq \emptyset$. Hence, the set $X(\Delta_{K}) \neq \emptyset$ and $X(\Delta_{K^{Br}}) = \emptyset$. So, the variety $X$ has a $K$-adic point and verifies Hasse principle with Brauer–Manin obstruction over $K$.

By Prop. 3.3 the set $V_0(\Delta_{K^1}) \neq \emptyset$ and $V_\infty(K_{v_1}) \neq \emptyset$. By Prop. 4.2 the map $\alpha^*_L: Br(C_L) \to Br(X_L)$ is an isomorphism. By Assumption 3.4 the curve $C$ is connected over $K_{v_0} = L_{v'_0} \cong \mathbb{R}$. For any $A \in Br(C_L)$, since the invariance of $A$ on $C(L_{v'})$ is locally constant for all $v' \in \Omega_L$, it is constant on $C(L_{v'})$ for all $v' \in S_{OL}$. By the functoriality of Brauer–Manin pairing and isomorphism of $\alpha^*_L: Br(C_L) \to Br(X_L)$, the set $X(\Delta_{L})^{Br} \supset (V_0 \times \{C(L)\setminus C(K)\})(\Delta_{L}^{Br}) \times \prod_{v' \in S_{OL}}(\mathcal{V}_\infty \times \{C(K)\})(L_{v'}) \neq \emptyset$. By Prop. 3.3 the set $V_0(L_{v'_0}) = V_0(K_{v_0}) = \emptyset$.

By Cor. 3.3 the set $V_\infty(\Delta_L) = \emptyset$. Since $X(L) \subset (\beta^{-1}(C(L)))(L) = (\mathbb{V}_\infty \times C(K))(L) \cup (V_0 \times \{C(L)\setminus C(K)\})(L)$, the set $X(L) = \emptyset$. So, the variety $X$ does not verify Hasse principle with Brauer–Manin obstruction over $L$. □

7. Examples

In this section, we will give unconditional examples for Theorem 6.5, Theorem 6.7 and Theorem 6.8. For noninvariance of the Hasse principle with Brauer–Manin obstruction, when the degree of field extension $[L : K]$ is not odd and $L$ has no real place, we give an unconditional example for the case $K = Q$ and $L = Q(i)$.

7.1 An unconditional example for Theorem 6.5

7.1.1 Field extension of an elliptic curve. Let $K = Q$ and $L = Q(\sqrt{-3})$. Let $E$ be an elliptic curve defined over $Q$ by a homogeneous equation:

$$w_1^2w_2 = w_0^3 - 16w_3^2$$

with homogeneous coordinates $(w_0 : w_1 : w_2) \in \mathbb{P}^2$. This is an elliptic curve with complex multiplication. Its quadratic twist $E^{(i)}$ is isomorphism to an elliptic curve defined by a homogeneous equation: $w_1^2w_2 = w_0^3 - 432w_3^2$ with homogeneous coordinates $(w_0 : w_1 : w_2) \in \mathbb{P}^2$. And $E$ and $E^{(i)}$ are elliptic curves over $Q$ of analytic rank 0. Then the Tate-Shafarevich group $\text{III}(E, Q)$ and $\text{III}(E^{(i)}, Q)$ are finite. The curves $E_K$ and $E_L$ verify weak approximation with Brauer–Manin obstruction off $\infty_K$ and $\infty_L$ respectively. The Mordell-Weil group $E(K)$ and $E(L)$ are finite. Indeed, the Mordell-Weil group $E(K) = \{(0 : 1 : 0)\}$ and $E(L) = \{(4 \pm 4\sqrt{3} : 1), (0 : 1 : 0)\}$.

7.1.2 Base change morphism. Let $\mathbb{P}^2 \setminus \{(0 : 1 : 0)\} \to \mathbb{P}^1$ be a morphism over $Q$ given by $(w_0 : w_1 : w_2) \to (w_0 - 4w_2 : w_2)$. Composite with the nature inclusion $E \setminus \{(0 : 1 : 0)\} \to \mathbb{P}^2 \setminus \{(0 : 1 : 0)\}$. We get a morphism $E \setminus \{(0 : 1 : 0)\} \to \mathbb{P}^1$, which can be extended to a 2 : 1-morphism $\gamma: E \to \mathbb{P}^1$. The dominant morphism $\gamma$ maps $E(K)$ to infinite point $(1 : 0)$ and maps $(4 \pm 4\sqrt{3} : 1)$ to $(0 : 1)$ respectively. Using [Har79] Chap. I. Cor. 7.8 Bézout’s Theorem or using [Har79] Chap. IV. Cor. 2.4 Hurwitz’s Theorem, the branch locus of $\gamma$ is $\{(1 : 0), (2\sqrt{2} - 4 : 1), (2\sqrt{2}e^{2\pi i/3} - 4 : 1), (2\sqrt{2}e^{-2\pi i/3} - 4 : 1)\}$. 


7.1.3. Construction of a Châtelet surface bundle. Let \( P_{\infty}(x) = (x^2 - 1)(x^2 - 4) \) and let \( P_0(x) = (99x^2 + 1)(\frac{5428}{5429}x^2 + \frac{1}{5429}) \). Notice that these polynomials \( P_{\infty} \) and \( P_0 \) are separable. Let \( V_0 \) be the Châtelet surface by \( y^2 - 73z^2 = P_{\infty}(x) \). As mentioned in Example 7.2.1. let \( V_0 \) be the Châtelet surface given by \( y^2 - 73z^2 = P_{0}(x) \). Let \( P_{\infty}(x_0, x_1) \) and \( P_{0}(x_0, x_1) \) be the homogenizations of \( P_{\infty} \) and \( P_0 \). Let \((u_0 : u_1) \times (x_0 : x_1) \) be the coordinates of \( \mathbb{P}^1 \times \mathbb{P}^1 \) and let \( s' = u_0^2\tilde{P}_{\infty}(x_0, x_1) + u_1^2\tilde{P}_0(x_0, x_1) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,2)^{\otimes 2}) \). For \( P_0(x) \) and \( P_{\infty}(x) \) are coprime in \( \mathbb{K}[x] \), by Jacobian criterion, the locus \( Z' \) defined by \( s' = 0 \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \) is smooth. The branch locus of the composition \( Z' \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{pr_1} \mathbb{P}^1 \) is finite and contained in \( \mathbb{P}^1 \setminus \{(1 : 0)\} \). Let \( B = C \times \mathbb{P}^1 \), let \( \mathcal{L} = (\gamma, \text{id})^*\mathcal{O}(1,2) \) and let \( s = (\gamma, \text{id})^*(s') \in \Gamma(B, \mathcal{L}^{\otimes 2}) \).

**Lemma 7.1.** The curve \( Z \) defined by \( s = 0 \) in \( B \) is nice.

**Proof.** For smoothness of \( Z \), we need to check that the branch loci of \( Z' \to \mathbb{P}^1 \) and \( E \to \mathbb{P}^1 \) do not intersect. The first morphism \( Z' \to \mathbb{P}^1 \) is \( 4 \)-1-morphism and its branch locus is contained in \( \mathbb{P}^1 \setminus \{(1 : 0)\} \). We can assume \( u_1 = 1 \), then its branch locus satisfies one of the following equations: \( 5329u_0^5 + 537372 = 0; 21316u_0^5 + 1 = 0; 47961u_0^4 - 8653226u_0^3 + 5329 = 0 \). The polynomials of these equations are irreducible over \( \mathbb{Q} \). By comparing the degree \([\mathbb{Q}(u_0) : \mathbb{Q}] \), we get the conclusion that these two branch loci do not intersect. The same argument as in the proof of Theorem 6.5 the locus \( Z \) defined by \( s = 0 \) in \( B \) is geometrically connected. So it is nice. \( \square \)

Let \( X \) be the zero locus of \( (1, -a, -s) \in \Gamma(B, \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{L}^{\otimes 2}) \subset \Gamma(B, \text{Sym}^3 \mathcal{E}) \) in the projective space bundle \( \text{Proj} \mathcal{E} \) with nature morphism \( \alpha \): \( X \to B \). Let \( \beta \): \( X \to E \) be the composition of \( \alpha \) and \( pr_1 \).

**Proposition 7.2.** For \( K = \mathbb{Q} \) and \( L = \mathbb{Q}(\sqrt{3}) \), the 3-fold \( X \) has the following properties:

- The 3-fold \( X \) has a \( \mathbb{K} \)-rational point and verifies weak approximation with Brauer–Manin obstruction, étale Brauer–Manin obstruction off \( \infty_K \).
- The 3-fold \( X_L \) does not verify weak approximation with Brauer–Manin obstruction, étale Brauer–Manin obstruction off \( \infty_L \).

**Proof.** This is the same as in proof of Theorem 6.5. \( \square \)

The 3-fold \( X \) we constructed has an affine open subvariety defined by the following equations, which is closed subvariety of \( \mathbb{A}^5 \) with affine coordinates \((x, y, z, x', y')\).

\[
\begin{cases}
  y^2 - 73z^2 = (x^2 - 1)(x^2 - 4)(x' - 4)^2 + (99x^2 + 1)(\frac{5428}{5429}x^2 + \frac{1}{5429}) \\
  y'^2 = x'^3 - 16
\end{cases}
\]

7.2. An unconditional example for Theorem 6.6.

7.2.1. Field extension of an elliptic curve. Let \( K = \mathbb{Q} \) and let \( \zeta_7 \) be a primitive 7-th root of unity. Let \( \alpha = \zeta_7 + \zeta_7^{-1} \) with minimal polynomial \( x^3 + x^2 - 2x - 1 \). Let \( L = \mathbb{Q}(\alpha) \). Then the degree \([L : K] = 3 \). Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \) by a homogeneous equation:

\[ w_1^3w_2 = w_0^3 - 343w_0w_2^2 - 2401w_2^3 \]

with homogeneous coordinates \((w_0 : w_1 : w_2) \in \mathbb{P}^2 \). This is an elliptic curve with analytic rank 0 over \( \mathbb{Q} \). The Mordell-Weil group \( E(\mathbb{Q}) = \{(0 : 1 : 0)\} \) and \( E(L) = \{(7\alpha^2 + 14\alpha - 7 : 0 : 1), (7\alpha^2 - 7\alpha - 14 : 0 : 1), (-14\alpha^2 + 7\alpha + 21 : 0 : 1), (0 : 1 : 0)\} \). The curves \( E_K \) verify weak approximation with Brauer-Manin obstruction off \( \infty_K \). So the elliptic curve \( E \) satisfies Assumption 1.3.
7.2.2. Base change morphism. Let \( \mathbb{P}^2 \setminus \{(1 : 0 : 0)\} \rightarrow \mathbb{P}^1 \) be a morphism over \( \mathbb{Q} \) given by \((u_0 : u_1 : u_2) \mapsto (w_1 : w_2)\). Composite with the nature inclusion \( E \hookrightarrow \mathbb{P}^2 \setminus \{(1 : 0 : 0)\} \). We get a morphism \( \gamma : E \rightarrow \mathbb{P}^1 \), which is 3 : 1-morphism. The dominant morphism \( \gamma \) maps \( E(K) \) to \( \infty \)-point; \((1 : 0) \) and maps \( E(L) \cup E(K) \) to \((0 : 1) \) respectively.

Using [Har97] Chap. I. Cor. 7.8 Bézout’s Theorem or using [Har97] Chap. IV. Cor. 2.4 Hurwitz’s Theorem, the branch locus of \( \gamma \) is \((1 : 0)\) \( \bigcup \{(0 : 1)|27u_0^6 + 129654u_0^5 - 5764801 = 0\}\).

7.2.3. Construction of a Châtelet surface bundle. Let \( P_\infty(x) = 14(x^4 - 89726) \) and let \( P_0(x) = (x^2 - 878755181)(5x^2 - 4393775906) \). Notice that these polynomials \( P_\infty \) and \( P_0 \) are separable. As mentioned in Example 3.5 and Example 3.13 let \( V_\infty \) be the Châtelet surface given by \( y^2 - 377z^2 = P_\infty(x) \) and let \( V_0 \) be the Châtelet surface given by \( y^2 - 377z^2 = P_0(x) \).

Let \( P_\infty(x, x_1) \) and \( P_0(x, x_1) \) be the homogenizations of \( P_\infty \) and \( P_0 \). Let \((u_0 : u_1 : x_0 : x_1) \) be the coordinates of \( \mathbb{P}^1 \times \mathbb{P}^1 \) and let \( s = u_0^2P_\infty(x_0, x_1) + u_1^2P_0(x_0, x_1) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 2)^{\otimes 2}) \). For \( P_0(x) \) and \( P_\infty(x) \) are coprime in \( K[x] \), by Jacobian criterion, the locus \( Z' \) defined by \( s = 0 \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \) is smooth. The branch locus \( Z' \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) is finite and contained in \( \mathbb{P}^1 \setminus \{(1 : 0)\} \).

Let \( B = C \times \mathbb{P}^1 \), let \( L = (\gamma, id)^*O(1, 2) \) and let \( s = (\gamma, id)^*(s') \in \Gamma(B, \mathcal{L}^{\otimes 2}) \).

Lemma 7.5. The curve \( Z \) defined by \( s = 0 \) in \( B \) is nice.

Proof. For smoothness of \( Z \), we need to check that the branch loci of \( Z' \rightarrow \mathbb{P}^1 \) and \( \gamma : E \rightarrow \mathbb{P}^1 \) do not intersect. The first morphism \( Z' \rightarrow \mathbb{P}^1 \) is 4 : 1-morphism and its branch loci is contained in \( \mathbb{P}^1 \setminus \{(1 : 0)\} \). We can assume \( u_1 = 1 \), then its branch locus satisfies one of the following equations: 14u_0^2 + 5 = 0; 44863u_0^2 - 137894762198231040; 70345184u_0^2 - 216218987126801139936u_0^2 + 1 = 0. The polynomials of these equations are irreducible over \( \mathbb{Q} \). By comparing these irreducible polynomials and the branch locus of \( \gamma \), we get the conclusion that these two branch loci do not intersect. The same argument as in the proof of Theorem 7.2, the locus \( Z \) defined by \( s = 0 \) in \( B \) is geometrically connected. So it is nice. \( \square \)

Let \( X \) be the zero locus of \((1, -a, -s) \in \Gamma(B, \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{L}^{\otimes 2}) \subset \Gamma(B, \text{Sym}^2 \mathcal{E}) \) in the projective space bundle \( \text{Proj}(\mathcal{E}) \) with nature morphism \( \alpha : X \rightarrow B \). Let \( \beta : X \rightarrow E \) be the composition of \( \alpha \) and \( pr_1 \).

Proposition 7.4. For \( K = \mathbb{Q} \) and \( L = \mathbb{Q}(\zeta + \zeta^{-1}) \), the 3-fold \( X \) has the following properties:

- The 3-fold \( X \) has a \( K \)-adic point and verifies Hasse principle with Brauer-Mannin obstruction over \( K \).
- The 3-fold \( X_L \) does not verify Hasse principle with Brauer-Mannin obstruction over \( L \).

Proof. This is the same as in proof of Theorem 6.6 \( \square \)

The 3-fold \( X \) we constructed has an affine open subvariety defined by the following equations, which is closed subvariety of \( \mathbb{A}^5 \) with affine coordinates \((x, y, z, x', y')\).

\[
\begin{align*}
y^2 - 377z^2 &= 14(x^4 - 89726)y^2 + (x^2 - 878755181)(5x^2 - 4393775906) \\
y^2 &= x^3 - 343x^2 - 2401
\end{align*}
\]

7.3. An unconditional example for Theorem 6.7. Let \( K = \mathbb{Q} \), \( L = \mathbb{Q}(\sqrt{3}) \), elliptic curve \( E \) and \( \gamma : E \rightarrow \mathbb{P}^1 \) be the same as in Subsection 7.1. For \( E(\mathbb{R}) \) is connected, the curve \( E \) satisfies Assumption 4.3.
7.3.1. Construction of a Châtelet surface bundle. Let \( P_\infty(x) = 5(x^4 + 805) \) and let \( P_0(x) = -5(x^4 + 805) \). Notice that these polynomials \( P_\infty \) and \( P_0 \) are irreducible. As mentioned in Example 5.7 and Example 5.8, let \( V_\infty \) be the Châtelet surface given by \( y^2 + 23z^2 = P_\infty(x) \) and let \( V_0 \) be the Châtelet surface given by \( y^2 + 23z^2 = P_0(x) \). Let \( P_\infty(x_0, x_1) = P_0(x_0, x_1) \) be the homogenizations of \( P_\infty \) and \( P_0 \). Let \( (u_0 : u_1) \times (x_0 : x_1) \) be the coordinates of \( \mathbb{P}^1 \times \mathbb{P}^1 \) and let \( s' = u_0^2 \tilde{P}_\infty(x_0, x_1) + u_1^2 \tilde{P}_0(x_0, x_1) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 2)^{\otimes 2}) \). For \( P_0(x) \) and \( P_\infty(x) \) are coprime in \( K[x] \), by Jacobian criterion, the locus \( Z' \) defined by \( s' = 0 \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \) is smooth. The branch locus of the locus \( Z' \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \) is finite and contained in \( \mathbb{P}^1 \setminus \{0, 1\} \).

Let \( B = \mathbb{C} \times \mathbb{P}^1 \), let \( \mathcal{L} = (\gamma, id)^*\mathcal{O}(1, 2) \) and let \( s = (\gamma, id)^*(s') \in \Gamma(B, \mathcal{L}^{\otimes 2}) \).

Lemma 7.5. The curve \( Z \) defined by \( s = 0 \) in \( B \) is nice.

Proof. For smoothness of \( Z \), we need to check that the branch loci of \( Z' \to \mathbb{P}^1 \) and \( \gamma: E \to \mathbb{P}^1 \) do not intersect. The first morphism \( Z' \to \mathbb{P}^1 \) is \( 4 \times 1 \)-morphism and its branch locus is \( \{ (\pm 1 : 1) \} \). By comparing this locus and the branch locus of \( \gamma \), these two branch loci do not intersect. The same argument as in the proof of Theorem 6.3, the locus \( Z \) defined by \( s = 0 \) in \( B \) is geometrically connected. So it is nice. \( \square \)

Let \( X \) be the zero locus of \( (1, -a, -s) \in \Gamma(B, \mathcal{O}_B \oplus \mathcal{O}_B \oplus \mathcal{L}^{\otimes 2}) \subset \Gamma(B, \text{Sym}^2 \mathcal{E}) \) in the projective space bundle \( \text{Proj}(\mathcal{E}) \) with nature morphism \( \alpha: X \to B \). Let \( \beta: X \to E \) be the composition of \( \alpha \) and \( pr_1 \).

Proposition 7.6. For \( K = \mathbb{Q} \) and \( L = \mathbb{Q}(\sqrt{3}) \), the 3-fold \( X \) has the following properties:

- The 3-fold \( X \) has a \( \mathbb{A}_K \)-adic point and verifies Hasse principle with Brauer-Mannin obstruction over \( K \).
- The 3-fold \( X_L \) does not verify Hasse principle with Brauer-Mannin obstruction over \( L \).

Proof. This is the same as in proof of Theorem 6.7. \( \square \)

The 3-fold \( X \) we constructed has an affine open subvariety defined by the following equations, which is closed subvariety of \( A^5 \) with affine coordinates \( (x, y, z, x', y') \):

\[
\begin{align*}
    y^2 + 23z^2 &= 5(x^4 + 805)(x'^2 - 8x' + 15) \\
    y'^2 &= x'^3 - 16
\end{align*}
\]

7.4. An unconditional example for noninvariance of the Hasse principle with Brauer-Manin obstruction in the case \( K = \mathbb{Q} \) and \( L = \mathbb{Q}(i) \).

7.4.1. Field extension of an elliptic curve. Let \( \sigma \) be the generator element of Galois group \( \text{Gal}(L/K) \). Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \) by a homogeneous equation:

\[ w_1^2w_2 = w_0^3 - 16w_2^3 \]

with homogeneous coordinates \( (w_0 : w_1 : w_2) \in \mathbb{P}^2 \). This is an elliptic curve with complex multiplication. Its quadratic twist \( E^{(-1)} \) is isomorphism to an elliptic curve defined by a homogeneous equation: \( w_1^2w_2 = w_0^3 + 16w_2^3 \) with homogeneous coordinates \( (w_0 : w_1 : w_2) \in \mathbb{P}^2 \). And \( E \) and \( E^{(-1)} \) are elliptic curves over \( \mathbb{Q} \) of analytic rank 0. Then the Tate-Shafarevich group \( \text{III}(E, \mathbb{Q}) \) and \( \text{III}(E^{(-1)}, \mathbb{Q}) \) are finite. The curves \( E_K \) and \( E_L \) verify weak approximation with Brauer-Manin obstruction off \( \infty_K \) and \( \infty_L \), respectively. The Mordell-Weil group \( E(K) \) and \( E(L) \) are finite. Indeed, the Mordell-Weil group \( E(K) = \{(0 : 1 : 0)\} \) and \( E(L) = \{(0 : \pm 4i : 1), (0 : 1 : 0)\} \).
7.1.1, the group $\text{Br}$ of $\mathbb{Q}$ given by $(w_0 : w_1 : w_2) \mapsto (w_1 : 4w_2)$. Composite with the nature inclusion $E \hookrightarrow \mathbb{P}^2 \setminus \{(1 : 0 : 0)\}$, we get a morphism $\gamma_1 : E \rightarrow \mathbb{P}^1$, which is $3 : 1$-morphism. The dominant morphism $\gamma_1$ maps $E(K)$ to a point $(1 : 0)$ and maps $(0 : \pm 2i : 1)$ to $(\pm i : 1)$ respectively. Using [Har97] Chap. I. Cor. 7.8 Bézout’s Theorem, or using [Har97] Chap. IV. Cor. 2.4 Hurwitz’s Theorem, the branch locus of $\gamma_1$ is $\{(1 : 0), (\pm i : 1)\}.

7.4.2. Base change morphism. Let $\mathbb{P}^2 \setminus \{(1 : 0 : 0)\} \rightarrow \mathbb{P}^1$ be a morphism over $\mathbb{Q}$ given by $(w_0 : w_1 : w_2) \mapsto (w_1 : 4w_2)$. Composite with the nature inclusion $E \hookrightarrow \mathbb{P}^2 \setminus \{(1 : 0 : 0)\}$, we get a morphism $\gamma : E \rightarrow \mathbb{P}^1$, which is $3 : 1$-morphism. The dominant morphism $\gamma$ maps $E(K)$ to a point $(1 : 0)$ and maps $(0 : \pm 2i : 1)$ to $(\pm i : 1)$ respectively. Using [Har97] Chap. I. Cor. 7.8 Bézout’s Theorem or using [Har97] Chap. IV. Cor. 2.4 Hurwitz’s Theorem, the branch locus of $\gamma$ is $\{(1 : 0), (\pm i : 1)\}.

7.4.3. Properties of Châtelet surfaces. Let $P_0(x) = 2(x^4 - 10x^2 + 15)$, $P_1(x) = -2(5x^4 - 39x^2 + 75)$ and let $P_1(x) = P_\infty(x) + iP_0(x)$. Notice that all those polynomials $P_\infty, P_0, P_1$ are separable and $P_1(x) = -2[x^2 - (5 + i)][-(1 + 5i)x^2 - 15i]$. The two polynomials $x^2 - (5 + i)$ and $-(1 + 5i)x^2 - 15i$ are irreducible over $\mathbb{Q}(i)$ (indeed, they are irreducible over $\mathbb{Q}(i)_3$). Let $V_\infty$ be the Châtelet surface given by $y^2 + 15z^2 = P_\infty(x)$ and let $V_1$ be the Châtelet surface given by $y^2 + 15z^2 = P_1(x)$.

**Lemma 7.7.** The Châtelet surface $V_\infty$ has a $\mathbb{Q}_v$-point for all $v \in \Omega_{\mathbb{Q}} \setminus \{5\}$, but no $\mathbb{Q}_5$-point.

**Proof.** Suppose that $v = \infty \mathbb{Q}$. Let $x_0 = 0$. Then $(-15, P_\infty(x_0)) = (-15, 30)_v = 1$, which implies that the surface $V_\infty$ admits a $\mathbb{R}$-point with $x = 0$.

Suppose that $v = 2$. By Hensel's lemma, the element $-15$ is a square in $\mathbb{Q}_2$. By Remark 5.1, the surface $V_\infty$ admits a $\mathbb{Q}_2$-point.

Suppose that $v = 3$. Let $x_0 = 2$. Then $(-15, P_\infty(x_0)) = (-15, -18)_v = (-15, 9)_v = (-15, -2)_v = 1$, which implies that the surface $V_\infty$ admits a $\mathbb{Q}_3$-point with $x = 0$.

Suppose that $v \in \Omega_{\mathbb{Q}} \setminus \{2, 3, 5\}$. Take $x_0 \in \mathbb{Q}_v$ with valuation $v(x_0) < 0$. Then $(-15, P_\infty(x_0)) = (-15, 2)_v = 1$, which implies $V_\infty$ admits a $\mathbb{Q}_v$-point with $x = x_0$.

Suppose that $v = 5$. Then $(-15, 2)_v = 1$. If $v(x) \leq 0$, then $(-15, P_\infty(x)) = (-15, 2x^4)_v = -1$. If $v(x) > 0$, then $(-15, P_\infty(x)) = (-15, 30)_v = 1$. In each case, we have $(-15, P_\infty(x)) = 1$, which implies $V_\infty$ has no $\mathbb{Q}_5$-point. By Remark 5.2, we have $V_\infty(\mathbb{Q}_5) = 0$.

**Lemma 7.8.** The Châtelet surface $V_\infty$ has a $\mathbb{Q}(i)_v$-point for every place $v \in \Omega_{\mathbb{Q}(i)}$.

**Proof.** For the only archimedean place is complex, we only need to consider finite places.

Suppose that $v$ is a 2-adic place. For $-15 \in \mathbb{Q}_2^{\times 2}$ and Remark 5.1, the surface $V_1$ admits a $\mathbb{Q}(i)_v$-point.

Suppose that $v = 3$. Let $x_0 = 1$. Then $(-15, P_1(x_0)) = (-15, -2(-4 - i)(-1 - 10i)) = (-15, 1 + i)_v = 1$, which implies $V_1$ admits a $\mathbb{Q}(i)_3$-point with $x = 1$.

Suppose that $v = 5$. Then $(-15, P_1(x_0)) = (-15, -2(-5 + i)(10 - 17i)) = (-15, -34)_v = 1$, which implies $V_1$ admits a $\mathbb{Q}(i)_v$-point with $x = 1 + i$.

Suppose that $v = 13$. Then $\mathbb{Q}(i)_v \cong \mathbb{Q}_{13}$. Let $x_0 = 1$. By Lemma 2.3, we have $(-15, P_1(x_0)) = (-15, -2(-4 - i)(-1 - 10i)) = 1$, which implies $V_1$ admits a $\mathbb{Q}(i)_v$-point with $x = 1$.

Suppose that $v \in \Omega_{\mathbb{Q}(i)} \setminus \{2, 3, 13\}$. Take $x_0 \in \mathbb{Q}_v$ with valuation $v(x_0) < 0$. Then by Lemma 2.3, we have $(-15, P_\infty(x_0)) = (-15, -2(-1 + 5i)) = 1$, which implies $V_1$ admits a $\mathbb{Q}(i)_v$-point with $x = x_0$.

For $P_1(x)$ is a product of two 2-degree irreducible factors, according to [Sko01], the group $\text{Br}(V_1)/\text{Br}(\mathbb{Q}(i)) \cong \mathbb{Z}/2\mathbb{Z}$. Furthermore, by Prop. 7.1.2 in loc. cit, we take the quaternion algebra $A = (-15, -(1 + 5i)x^2 - 15i) \in \text{Br}(V_1)$ as a generator.
element of this group. Then we have the equality 
\[ A = (-15, -1 + 5i)x^2 - 15i) = (-15, -2(x^2 - (5 + i))) \] in \( \text{Br}(V_1) \).

**Lemma 7.9.** For any \( v \in \Omega_{\mathbb{Q}(i)} \) and any \( P_v \in V_1(\mathbb{Q}(i)_v) \),

\[
\text{inv}_v(A(P_v)) = \begin{cases} 
0 & \text{if } v \neq 3 \\
\frac{1}{2} & \text{if } v = 3
\end{cases}
\]

**Proof.** By Remark 3.2 it suffices to check the invariance \( \text{inv}_v(A(P_v)) \) for \( P_v \in V_1^0(\mathbb{Q}(i)_v) \).

Suppose that \( v \) is an archimedean place or a 2-adic place or a 181-adic place. Then 
\(-15 \in \mathbb{Q}(i)^{\times 2} \), so \( \text{inv}_v(A(P_v)) = 0 \).

Suppose that \( v \) is a place of \( \mathbb{Q}(i) \). If \( v(x) \leq 0 \), then 
\((-15, -1 + 5i)x^2 - 15i)_v = (-15, 1)_v = 1 \). If \( v(x) > 0 \), then 
\((-15, -2(x^2 - (5 + i)))_v = (-15, -2)_v = 1 \). So \( \text{inv}_v(A(P_v)) = 0 \).

Suppose that \( v \in \Omega_{\mathbb{Q}(i)}^0(3, \text{all 2-adic, 5-adic and 181-adic places}) \). If \( \text{inv}_v(A(P_v)) = \frac{1}{2} \), then 
\((-15, -1 + 5i)x^2 - 15i)_v = -1 \). By Lemma 5.2 the first and last equalities imply that
\( v((-1 + 5i)x^2 - 15i) \) and \( v(x^2 - (5 + i)) \) are odd, so they are positive. Hence 
\( v((-1 + 5i)x^2 - 15i - (-1 + 5i)x^2 - (5 + i)) = v(-10 + 9i) > 0 \). But \( v \) is 181, which is a contradiction. So \( \text{inv}_v(A(P_v)) = 0 \).

Suppose that \( v = 3 \). If \( A(P_3) = 1 \), then 
\((-15, -1 + 5i)x^2 - 15i)_v = 1 \). If \( v(x) > 0 \), then 
\((-15, -2(x^2 - (5 + i)))_v = -1 \). The first equality implies that \( v(x) > 0 \). Then 
\((-15, -2(x^2 - (5 + i)))_v = (-15, -2)_v = -1 \), which is a contradiction. So \( \text{inv}_v(A(P_v)) = \frac{1}{2} \).

**Lemma 7.10.** The Châtelet surface \( V_1 \) has no \( \mathbb{Q}(i) \)-rational point.

**Proof.** If there exists \( \mathbb{Q}(i) \)-rational point \( P \), by reciprocity law 
\[ \sum_{v \in \Omega_{\mathbb{Q}(i)}} \text{inv}_v(A(P)) = 0. \] But from Lemma 7.9 this sum is \( \frac{1}{2} \). So \( V_1 \) has no \( \mathbb{Q}(\sqrt{5}) \)-rational point. \( \square \)

7.4.4. **Construction of a Châtelet surface bundle.** Let \( P_\infty(x_0, x_1) \) and \( P_0(x_0, x_1) \) be the homogenizations of \( P_\infty \) and \( P_0 \). Let \( (u_0 : u_1) = (x_0 : x_1) \) be the coordinates of \( P_\infty \times P_1 \) and let \( s = (u_0^2 + 2u_1^2)P_\infty(x_0, x_1) + u_0u_1P_0(x_0, x_1) \in \Gamma(P_\infty \times P_1, \mathcal{O}(1, 2) \otimes 2) \).

For \( P_0(x) \) and \( P_\infty(x) \) are coprime in \( K[x] \), by Jacobian criterion, the locus \( Z' \) defined by \( s' = 0 \) in \( P_1 \times P_1 \) is smooth. The branch locus of the composition 
\( Z' \hookrightarrow P_1 \times P_1 \xrightarrow{pr_1} P_1 \) is finite and contained in \( P_1\setminus\{(1 : 0), (\pm i, 1)\} \). Let 
\( B = C \times P_1 \), let \( L = (\gamma, id)^*\mathcal{O}(1, 2) \) and let 
\( s = (\gamma, id)^*(s') \in \Gamma(B, L \otimes 2) \).

**Lemma 7.11.** The curve \( Z \) defined by \( s = 0 \) in \( B \) is nice.

**Proof.** For the branch locus of \( Z' \hookrightarrow P_1 \times P_1 \xrightarrow{pr_1} P_1 \) is contained in \( P_1\setminus\{(1 : 0), (\pm i, 1)\} \) and the branch locus of \( \gamma: E \to P_1 \) is \( \{(1 : 0), (\pm i, 1)\} \), they do not intersect, which implies the smoothness of \( Z \). The same argument as in the proof of Theorem 6.5 the locus \( Z \) defined by \( s = 0 \) in \( B \) is geometrically connected. So it is nice. \( \square \)

Let \( X \) be the zero locus of \( (1, 1, -s) \in \Gamma(B, O_B \oplus O_B \oplus L \otimes 2) \subset \Gamma(B, \text{Sym}^2 L) \) in the projective space bundle \( \text{Proj}(\mathcal{E}) \) with nature morphism \( \alpha: X \to B \). Let 
\( \beta: X \to C \) be the composition of \( \alpha \) and \( pr_1 \).

**Proposition 7.12.** For \( K = \mathbb{Q} \) and \( L = \mathbb{Q}(i) \), the 3-fold \( X \) has the following properties:

- The 3-fold \( X \) has a \( \mathbb{A}_K \)-adic point and verifies Hasse principle with Brauer-Manin obstruction over \( K \).
- The 3-fold \( X_L \) does not verify Hasse principle with Brauer-Manin obstruction over \( L \).
Proof. By the same arguments as in the proof of Theorem 6.6, the variety $X$ is nice. By construction, we have $\beta^{-1}(E(K)) = V_\infty \subset X$. By Lemma 7.4, the set $V_\infty(\mathbb{A}^5_K) \neq \emptyset$. So $X(\mathbb{A}^5_K) \neq \emptyset$. For 5 splits completely in $L$, take a place $v' \in \Omega_L$ above 5, i.e. $v'|5$ in $L$. Then $\mathbb{Q}_5 \cong L_{v'}$. By Lemma 7.3, the set $V_1(\mathbb{A}_L) \neq \emptyset$. For $\beta^{-1}(E(L)) = V_\infty \bigcup V_1 \cup \sigma(V_1) \subset X_L$, the set $X(\mathbb{Q}_5) = X(L_{v'}) \neq \emptyset$. So $X(\mathbb{A}_K) \neq \emptyset$. By Prop. 5.2, the variety $X$ is nice. For $E(K) = pr_\infty(K(\mathbb{A}^5_K)^{Br})$, the functoriality of Brauer-Manin pairing implies $pr_\infty(K(\mathbb{A}^5_K)^{Br}) \subset \beta^{-1}(E(K))(\mathbb{A}^5_K^{\infty})$. By Lemma 7.7, the set $V_\infty(\mathbb{Q}_5) = \emptyset$, so $pr_\infty(K(\mathbb{A}^5_K)^{Br}) \subset \beta^{-1}(C(K))(\mathbb{A}^5_K^{\infty}) = V_\infty(\mathbb{A}^5_K^{\infty}) = \emptyset$, which implies that $X(\mathbb{A}^5_K)^{Br} = \emptyset$. So, the variety $X$ has a $\mathbb{A}^5_K$-adic point and verifies Hasse principle with Brauer-Manin obstruction over $K$. For the set $V_1(\mathbb{A}_L) \neq \emptyset$ and by Prop. 5.2, the map $\alpha_1^*: Br(C_L) \to Br(X_L)$ is an isomorphism, by the functoriality of Brauer-Manin pairing, the set $X(\mathbb{A}_L)^{Br}$ containing $V_1(\mathbb{A}_L)$ is nonempty. By 7.10, the set $V_1(L) = \emptyset$. For 5 splits completely in $L$, the emptiness of $V_\infty(\mathbb{Q}_5)$ implies $V_\infty(\mathbb{A}_L) = \emptyset$. Since $X(L) \subset (\beta^{-1}(E(L)))(L) = V_\infty(L) \bigcup V_1(L) \bigcup \sigma(V_1)(L)$, the set $X(L) = \emptyset$. So, the variety $X$ does not verify Hasse principle with Brauer-Manin obstruction over $L$. \qed

The 3-fold $X$ we constructed has an affine open subvariety defined by the following equations, which is closed subvariety of $\mathbb{A}^5$ with affine coordinates $(x, y, z, x', y')$.

\[
\begin{aligned}
 y^2 + 15z^2 &= (x^4 - 10x^2 + 15)\frac{y^2 + 32}{8} - (5x^4 - 39x^2 + 75)\frac{y'}{2}, \\
 y'^2 &= x'^3 - 16
\end{aligned}
\]

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University of Science and Technology of China, School of Mathematical Sciences, No.96, JinZhai Road, Baohe District, Hefei, Anhui, 230026. P.R.China.

Email address: wuhan90@mail.ustc.edu.cn