On the Boltzmann type equation for the contact force distribution in disordered packings of particles

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Abstract

The packing of hard-core particles in contact with their neighbors is considered as the simplest model of disordered particulate media. We formulate the statically determinate problem which allows analytical investigation of the statistical distribution of contact force magnitude. The simplest Boltzmann type equation for the probability of contact force distribution is formulated and studied. An experimentally observed exponential distribution is derived.

1 Introduction

Understanding rheological properties of packed particles at various spatial scales [1] requires application of the apparatus of statistical mechanics. However conventional statistical physics is inadequate when one needs to describe mechanical behavior of disordered packings of hard-core particles that can be static or driven by external forces [2]. There have been various attempts to develop the statistical-mechanical approach for such systems, e.g. see the recent review article [3]. Despite these advances there still exists a certain degree of skepticism regarding the possibility of discovering new physical laws that govern mechanical behavior of particulate media. This can be linked to the structural complexity of such materials at different scales. However experimental works [4] and computer simulations [5] indicate the existence of phenomena that can and should be treated as physics problems. In order to discover physics laws one should consider well-posed problems so that laws can be expected. Our paper does not attempt the ambition of studying the percolation geometry and mechanics of contact forces network [6]. It offers a really simple analytical model that produces a Boltzmann type equation for the contact force distribution. This equation can be solved and an experimentally observed exponential distribution is obtained. Possible approaches to develop this simple model are briefly discussed.

2 The statistically determinate problem

Let us consider a static array of hard-core particles in contact with their neighbors. The packing is assumed to be an assembly of discrete rigid particles whose interactions with their neighbors are localized at point-like contacts. Therefore the description of the network of interparticle contacts is essential for the
understanding of force transmission. We assume that the set of contact points \( C_{i}^{\alpha \beta} \) provides the complete geometrical specification for such static packing. We define the centroid of contacts of particle \( \alpha \) as

\[
R_{i}^{\alpha} = \sum_{\beta} C_{i}^{\alpha \beta}/z_{\alpha} \tag{1}
\]

where \( i = 1, ..., d \) is the Cartesian index, \( z_{\alpha} \) is the coordination number of particle \( \alpha \). The distance between particles \( \alpha \) and \( \beta \) is defined as the distance between their centroids of contacts

\[
R_{i}^{\alpha \beta} = R_{i}^{\beta} - R_{i}^{\alpha} = r_{i}^{\alpha \beta} - r_{i}^{\beta \alpha} \tag{2}
\]

where \( r_{i}^{\alpha \beta} \) is the \( i \)-th component of the vector joining the centroid of contact with the contact point i.e.

\[
\sum_{\beta} r_{i}^{\alpha \beta} = 0 \tag{3}
\]

In \( d \) dimensions Newton’s laws of force and couple balance for each particle give us the system of \( \frac{N(d+1)}{2} \) equations for the interparticle forces \( f_{i}^{\alpha \beta} \)

\[
\sum_{\beta} f_{i}^{\alpha \beta} + g_{i}^{\alpha} = 0 \tag{4}
\]

\[
f_{i}^{\alpha \beta} + f_{i}^{\beta \alpha} = 0 \tag{5}
\]

\[
\sum_{\beta} \epsilon_{ikl} f_{i}^{\alpha \beta} r_{l}^{\alpha \beta} + c_{i}^{\alpha} = 0 \tag{6}
\]

where \( g_{i}^{\alpha} \) is the external body force acting on grain \( \alpha \) and \( c_{i}^{\alpha} \) is the external body couple which we take to be zero. The counting of the number of equations and the number of unknowns allows to formulate the simplest statically determinate problem of force transmission in a static packing. Particles are considered to be perfectly hard, perfectly rough and each particle \( \alpha \) has a coordination number \( z_{\alpha} = d+1 \) \[7\]. Theory which confirms this observation has been proposed for periodic arrays of particles with perfect and zero friction \[8\]. What is the statistical distribution of contact forces in a packing of particles? Experimental \[4\] and computer simulation \[5\] studies have demonstrated that the probability of normal contact force acting on a particle contact behaves as

\[
P(f) \propto \begin{cases} 
(f/\langle f \rangle)^{\gamma} & \text{if } f < \langle f \rangle \\
 e^{\delta(1 - f/\langle f \rangle)} & \text{if } f > \langle f \rangle 
\end{cases} \tag{7}
\]

where \( \langle f \rangle \) is an average contact force, \( \gamma \) and \( \delta \) are constants. The aim of this paper is to derive the statistical distribution of contact forces from the first principles. We attempt such derivation by constructing a Boltzmann type equation, which can be solved if the packing of particles is assumed to be a statically determined i.e. each particle \( \alpha \) has a coordination number \( z_{\alpha} = d+1 \). Some authors describe such system state as marginal \[9\].

### 3 An integral equation model

When a static packing of incompressible particles in contact is subjected to external forces at its boundaries, these forces are transmitted through the contact network.
3.1 The 2-D model

Let us consider a packing in $2 - d$ where forces $f_1, f_2$ impinge on a particle which then exerts force $f$ on its neighbor. The average position of the forces is symmetric. Let $f$ be in the $x$ direction and use the symbols $f_1, f_2$ for the $x$ components of the vector forces. Forces can only push and not pull, and the simplest representation of the problem is as follows

$$P(f) = \int_0^\infty df_1 \int_0^\infty df_2 \delta(f - f_1 - f_2) P(f_1) P(f_2)$$

(8)

After projecting the contact force vectors we have

$$P(f) = \int_0^\infty df_1 \int_0^\infty df_2 \int_0^1 d\mu \int_0^1 d\lambda \delta(f - \lambda f_1 - \mu f_2) P(f_1) P(f_2)$$

(9)

which can be transformed into

$$P(f) = \int_0^\infty df_1 \int_0^\infty df_2 \int_0^1 d\mu \int_0^1 d\lambda P(\lambda f_1) P(\mu f_2)$$

(10)

This equation has the solution in the following form

$$P(f) = \frac{f}{p} e^{-\frac{f}{p}}$$

(11)

where $p = \frac{f_A}{A}$ is the mean pressure and $A$ is the area of the interparticle contact, and the distribution is exponential for large values of $f$.

3.2 The 3-D model

Using the same framework of simplifications, the 3 – $D$ model can be written as

$$f = \frac{1}{3} f_1 + \frac{1}{3} f_2 + \frac{1}{3} f_3$$

(12)

After applying the Fourier-transform

$$P(f) = \frac{1}{(2\pi)^3} \int d^3k P(k) e^{ikf}$$

(13)

we obtain

$$P(k) = P^3(\frac{k}{3})$$

(14)

when again

$$P(k) = e^{i\frac{k}{p}}$$

(15)

After applying the inverse Fourier-transform

$$P(k) = \int d^3 f P(f) e^{-ikf}$$

(16)

we obtain

$$P(f) = \delta(f - p)$$

(17)
The blurring process is always employed in the form of a “toy model”, however there is a form
\[ f = \lambda_1^2 f_1 + \lambda_2^2 f_2 + \lambda_3^2 f_3 \]  
(18)
that offers an analytic solution
\[ P(f) = \left( \int_{-\infty}^{\infty} \int_0^1 P(\lambda^2 k) \, dk \, d\lambda \right)^3 \]  
(19)
Using \( \lambda^2 k = \mu \) we obtain
\[ P(f) = \left( \int_0^K P(\mu) \frac{d\mu}{2\mu^2 k^2} \right)^3 \]  
(20)
Applying the Fourier-transform to this expression we obtain
\[ \mathcal{P}(k) = \frac{4p^{3/2}}{(k - ip)^{3/2}} \]  
(21)
so we have
\[ P(f) = \int \frac{4p^{3/2} e^{ikf}}{(k - ip)^{3/2}} \, dk = 4p^{3/2} e^{-\frac{f}{p}} \int \frac{e^{ijf}}{J^{3/2}} \, dJ \]  
(22)
Integration of this expression gives
\[ P(f) = \frac{\sqrt{\pi}}{2} f^{1/3} p^{3/2} e^{-\frac{f}{p}} \]  
(23)
Indeed, a number of improvements must to be done in order to derive this expression with experimentally observed coefficients, however, as the starting point, the use of this simple model appears to be justified.

4 Discussion

We proposed the simplest Boltzmann type equation for the probability distribution of contact forces in two and three dimensions. This equation can be solved under some approximations and an experimentally observed exponential distribution of contact forces can be obtained. Other proposed approaches\[10\]-\[12\] employ entropy maximization or functional minimization concepts. These models produce elements of the empirically observed probability distribution function. However they are not derived from first-principles but are developed by analogy with other entropic systems. We hope that our approach of studying the Boltzmann type equation for the probability distribution of contact forces can serve as the foundation for future research. In particular one can develop it further to account for the presence of structural disorder at various spatial scales and obtain the statistics of so called “force chains” which are observed in experiment\[4\].

5 Conflict of Interest

The author declares that he has no conflict of interest.
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