Shifted Contact Structures on Derived Schemes and Their Local Theory

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Abstract

In this paper, we formally define the concept of a $k$-shifted contact structure on a derived $\mathbb{K}$-scheme and study its local properties in the context of derived algebraic geometry. In this regard, we develop Darboux-type local models for $k$-shifted contact structures and present a Darboux-like theorem. More precisely, we prove that every $k$-shifted contact derived $\mathbb{K}$-scheme $(X, \alpha)$ with $k < 0$ is locally equivalent to $(\text{Spec} A, \alpha_0)$ for $\text{Spec} A$ an affine derived $\mathbb{K}$-scheme and $A$ a commutative differential graded $\mathbb{K}$-algebra such that the pair $(A, \alpha_0)$ is in a Darboux-type form. Furthermore, we formulate the so-called symplectization $S_X(k)$ of a $k$-shifted contact derived $\mathbb{K}$-scheme $(X, \alpha)$ and give a canonical construction for the space $S_X(k)$.

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1 Introduction

Derived algebraic geometry (DAG) essentially provides a new setup to deal with non-generic situations in geometry (e.g. non-transversal intersections and “bad” quotients). To this end, it combines higher categorical objects and homotopy theory with many tools from homological algebra. Hence, roughly speaking, it can be considered as a higher categorical/homotopy theoretical refinement of classical algebraic geometry. In that respect, it offers a new way of organizing information for various purposes. Therefore, it has many interactions with other mathematical domains. For a survey of some directions, we refer to [1, 2].

In the context of DAG, it is also possible to work with familiar geometric structures, but in more general forms. For instance, $k$-shifted versions of Symplectic and Poisson geometries have already been described and studied in [4, 8]. In this regard, [5, 6, 7] offer some applications and local constructions.

Throughout this paper, we mainly work within the context of Toën & Vezzosi’s version of DAG [2, 3]. We also benefit from Lurie’s version [9]. In that respect, we always consider objects with higher structures in a functorial perspective, and we focus on nice representatives for those structures. For instance, by a derived $\mathbb{K}$-stack, we essentially mean a simplicial presheaf on the category of commutative differential graded $\mathbb{K}$-algebras (cdga) having nice local-to-global properties.

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DAG provides an appropriate concept of a spectrum functor \( \text{Spec} \) from cdgas to (higher) spaces. Using this functor, we call a derived space of the form \( X \simeq \text{Spec} A \) for some cdga \( A \) an affine derived \( \mathbb{K} \)-scheme. As in the classical theory, a general derived \( \mathbb{K} \)-scheme \( Y \) is defined to be a space which is locally modeled on \( X \simeq \text{Spec} A \). Note that affine derived schemes are in fact the main objects of interest for us because the concepts to be discussed in this paper are all about the local structure of derived schemes. Thus, it is enough to consider the affine case. More details will be given in §2.1.

Regarding certain geometric structures on higher spaces; such as \( k \)-shifted (closed) \( p \)-forms in the sense of [4], it is also known that for sufficiently “nice” cdgas (to be clear later), we can use the \( A \)-module \( \Omega_A^1 \) of Kähler differentials as a model for the cotangent complex \( L_A \) of \( A \) so that we write \( L_A \simeq \Omega_A^1 \). Then, by a \( k \)-shifted \( p \)-form on \( \text{Spec} A \) for \( A \) a sufficiently nice cdga, we actually mean a \( k \)-cohomology class of the complex \( (\Lambda^p \Omega_A^1, d) \). Likewise, a \( k \)-shifted closed \( p \)-form on \( \text{Spec} A \) is just a \( k \)-cohomology class of the complex \( \prod_{i \geq 0} (\Lambda^{p+i} \Omega_A^1)[-i], d_{\text{tot}} = d + d_{\text{dR}} \).

A reasonable notion of non-degeneracy is also available in this framework, which leads to the definition of a shifted symplectic structure. We then define the notion of a \( k \)-shifted contact structure on \( \text{Spec} A \) to be a \( k \)-shifted 1-form \( \alpha \) on \( \text{Spec} A \) with the property that the \( k \)-shifted 2-form \( d_{\text{dR}} \alpha \) satisfies a non-degeneracy condition, which will be formulated later.

For shifted symplectic structures on derived schemes, it has been shown in [5] that every \( k \)-shifted symplectic derived \( \mathbb{K} \)-scheme \( (X, \omega) \) is Zariski locally equivalent to \( (\text{Spec} A, \omega) \) for a pair \( A, \omega \) in certain symplectic Darboux form. More precisely, Bussi, Brav, and Joyce [5, Theorem 5.18] proved that given a \( k \)-shifted symplectic derived \( \mathbb{K} \)-scheme \( (X, \omega) \), one can find the so-called “minimal standard form” cdga \( A \), a Zariski open inclusion \( \iota : \text{Spec} A \hookrightarrow X \), and “coordinates” \( x_j, y^{k+i}_j \in A \) with \( \iota^*(\omega') \sim (\omega^0, 0, 0, \ldots) \) such that

\[
\omega^0 = \sum_{i,j} d_{\text{dR}}x_j^{-i} d_{\text{dR}}y_j^{k+i}.
\]

We should point out that the expression of \( \omega^0 \) holds only for the case where \( k < 0 \) is an odd integer. The other possible cases require some modifications depending on whether \( k/2 \) is even or odd. However, the underlying idea behind the proofs for each case is the same.

Note also that the case \( k < 0 \) odd is relatively simple and instructive enough to capture the essential techniques for the constructions of local models under consideration. Therefore, in this paper, we will mainly concentrate on the case with \( k \leq 0 \) odd and use it as a prototype construction. For the other cases, we will not give all the details. Instead, we will only provide a brief outline. For details, we will always refer to [5, Examples 5.8, 5.9 & 5.10].

Results and the outline. In this paper, we introduce the notion of a \( k \)-shifted contact structure on a derived \( \mathbb{K} \)-scheme. The goals are then to develop a Darboux-type model for shifted contact structures and investigate further possible outcomes. The next two theorems summarize the main results of this paper.

We first present a Darboux-type theorem for \( k \)-shifted contact derived \( \mathbb{K} \)-schemes when \( k < 0 \). More precisely, for a locally finitely presented derived \( \mathbb{K} \)-scheme \( X \) with a \( k \)-shifted contact structure for \( k < 0 \), we prove the following result (cf. Theorem 3.9):

**Theorem 1.1.** Every \( k \)-shifted contact derived \( \mathbb{K} \)-scheme \( (X, \alpha) \) is locally equivalent to \((\text{Spec} A, \alpha_0)\) for \( A \) a minimal standard form cdga and \( \alpha_0 \) in a contact Darboux form.

Secondly, we establish a shifted version of the classical connection between contact and symplectic geometries. In classical contact geometry, for a contact manifold \((M, \alpha)\), there is a unique symplectified space with a symplectic structure canonically determined by \( \alpha \). In this paper, we provide a similar result for shifted contact derived schemes. The upshot is as follows:

Given a locally finitely presented derived \( \mathbb{K} \)-scheme \( X \) with a \( k \)-shifted contact structure for \( k < 0 \), write \( S_X(k) \) for the space of all \( k \)-shifted defining contact forms on \( X \). We then obtain the following theorem/definition (cf. Theorem 4.1).

**Theorem 1.2.** The space \( S_X(k) \) has the structure of a \( k \)-shifted symplectic derived \( \mathbb{K} \)-scheme with a symplectic form \( \omega_X \) which is canonically determined by the shifted contact structure of \( X \).

We then call the pair \((S_X(k), \omega_X)\) the symplectization of \( X \).
Now, let us describe the content of this paper in more detail and provide an outline. In Section 2, we review PTVV’s symplectic geometry and symplectic Darboux forms. In fact, we give some background material on derived spaces and present nice local models for derived $\mathbb{K}$-schemes and their cotangent complexes. In Section 2.2, using these nice local models, we introduce (closed) $p$-forms of degree $k$ and symplectic structures on these derived spaces. Section 2.3 outlines symplectic Darboux forms on derived schemes and presents Darboux-type results given by Bussi, Brav and Joyce [5, Theorem 5.18].

In Sections 3.1 and 3.2, the basics of classical contact geometry are briefly revisited, and then we introduce shifted contact structures and discuss their properties. In Section 3.3, we state a Darboux-type theorem for shifted contact structures on derived $\mathbb{K}$-schemes (Theorem 3.9) and give the proof of Theorem 1.1 above.

In Section 4, we discuss the concept of a symplectization of a shifted contact derived scheme and give the proof of Theorem 1.2 above.

Section 5 provides some concluding remarks on possible generalizations of the main results of this paper.

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Conventions. Throughout the paper, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. All cdgas will be graded in nonpositive degrees and over $\mathbb{K}$. All classical $\mathbb{K}$-schemes will be locally of finite type, and all derived $\mathbb{K}$-schemes/stacks $X$ are assumed to be locally finitely presented.

2 Shifted symplectic structures

2.1 Some derived algebraic geometry

In this section, we outline the basics of DAG, present some material relevant to this paper, and state some useful results from shifted symplectic geometry. As stressed before, we use both Toën & Vezzosi’s version of DAG [2, 3] and the Lurie’s version [9]. In what follows, we just intend to give a brief sketch for the objects and constructions that we will be mostly interested in.

Definition 2.1. Denote by $cdg_{\mathbb{K}}$ the category of commutative differential graded $\mathbb{K}$-algebras in non-positive degrees, where an object $A$ in $cdg_{\mathbb{K}}$ consists of

1. a collection of $\mathbb{K}$-vector spaces $\{A^i\}$, where $A^i$ is a $\mathbb{K}$-vector space of degree $i$ elements for $i = 0, -1, \ldots$,

2. a $\mathbb{K}$-bilinear, associative, supercommutative multiplication $A^n \otimes A^m \to A^{n+m}$, and

3. a unique square-zero derivation of degree 1 (the differential) $d$ on $A$ satisfying the graded Leibniz rule

$$d(a \cdot b) = (da) \cdot b + (-1)^na \cdot (db)$$

for all $a \in A^n, b \in A^m$.

We denote such objects by $(A, d)$ or just $A$. Note that $A$ has a decomposition $A = \bigoplus_i A^i$.

A morphism in $cdg_{\mathbb{K}}$, on the other hand, is a collection of degree-wise $\mathbb{K}$-linear morphisms $f = \{f^i\} : A \to B$ such that each $f^i : A^i \to B^i$ commutes with all the structures of $A, B$.

Definition 2.2. Denote by $dSt_{\mathbb{K}}$ the $\infty$-category of derived stacks, where an object $X$ of $dSt_{\mathbb{K}}$ is given as a certain $\infty$-functor

$$X : cdg_{\mathbb{K}} \to sSets,$$  \hspace{1cm} (2.1)

where $sSets$ denote the $\infty$-category of simplicial sets. More precisely, objects in $dSt_{\mathbb{K}}$ are simplicial presheaves preserving weak equivalences and possessing the descent/local-to-global property w.r.t. the site structure on the source. For a brief review, we refer to [11].
We write \( cdga_\infty^K \) for the associated \( \infty \)-category of \( cdga_K \) such that the homotopy category \( \text{Ho}(cdga_\infty^K) \) can be obtained from \( cdga_K \) by formally inverting quasi-isomorphisms.

Note that \( \text{Ho}(cdga_\infty^K) \) is just an ordinary category. We should also point out that \( cdga_\infty^K, cdga_K \) and \( \text{Ho}(cdga_\infty^K) \) have the same objects; however, lifting properties of morphisms are different. That is, a morphism \( f : A \to B \) in \( cdga_K \) is also a morphism in \( cdga_\infty^K \) and \( \text{Ho}(cdga_\infty^K) \). But in general, the converse is not true unless \( A \) is cofibrant. In the rest of this paper, we will be interested in certain types of cdgas, called standard form cdgas, which are in fact “sufficiently cofibrant”, and hence suitable for our purposes.

In this framework, there also exists an appropriate concept of a spectrum functor [9, § 4.3]

\[
\text{Spec} : (cdga_\infty^K)^{op} \to \text{dSt}_K,
\]

which leads to the following definition.

**Definition 2.3.** An object \( X \) in \( dSt_K \) is called an affine derived \( K \)-scheme if \( X \simeq \text{Spec} A \) for some cdga \( A \in cdga_K \). An object \( X \) in \( dSt_K \) is then called a derived \( K \)-scheme if it can be covered by Zariski open affine derived \( K \)-schemes \( Y \subset X \).

Denote by \( dSch_K \subset dSt_K \) the full \( \infty \)-subcategory of derived \( K \)-schemes, and we simply write \( dAff_K \subset dSch_K \) for the full \( \infty \)-subcategory of affine derived \( K \)-schemes. Note that \( \text{Spec} : (cdga_\infty^K)^{op} \to \text{dAff}_K \) gives an equivalence of \( \infty \)-categories.

We should note that throughout this paper, \( K \) will be an algebraically closed field of characteristic zero. We also assume that all classical \( K \)-schemes are locally of finite type, and all derived \( K \)-schemes \( X \) are locally finitely presented, by which we mean that \( X \) can be covered by Zariski open affines \( A \), where \( A \) is a finetely presented cdga over \( K \).

**Remark 2.4.** Thanks to the Yoneda embedding, one can also realize algebro-geometric objects (like classical \( K \)-schemes, stacks, derived “spaces”, etc...) as certain functors in addition to the standard ringed-space formulation. We have the following enlightening diagram from [11] encoding such a functorial interpretation:

\[
\begin{align*}
\text{CAlg}_K & \quad \xrightarrow{\text{schemes}} \quad \text{Sets} \\
& \quad \searrow \quad \swarrow \\
& \quad \text{Grpds} \\
& \quad \downarrow \quad \downarrow \\
\text{cdga}_K & \quad \xrightarrow{\text{derived stacks}} \quad \text{Ssets}
\end{align*}
\]

Here \( \text{CAlg}_K \) denotes the category of commutative \( K \)-algebras. Denote by \( St_K \) the \( \infty \)-category of (higher) \( K \)-stacks, where objects in \( St_K \) are defined via the diagram above.

In the underived setup, we have the classical ”spectrum functor”

\[
\text{spec} : (\text{CAlg}_K)^{op} \to St_K.
\]

We then call an object \( X \) of \( St_K \) an affine \( K \)-scheme if \( X \simeq \text{spec} A \) for some \( A \in \text{CAlg}_K \), and a \( K \)-scheme if it has an open cover by affine \( K \)-schemes.

In addition to the spectrum functors \( \text{Spec}, \text{spec} \) above, there is a natural truncation functor \( \tau : dSt_K \to St_K \), along with a fully faithfull left adjoint inclusion functor \( i : St_K \hookrightarrow dSt_K \), which can be thought of as an embedding of classical algebraic \( K \)-spaces into derived spaces.

Note that, for a cdga \( A \) there exists an equivalence \( \tau \circ \text{Spec} A \simeq \text{spec} H^0(A) \). This means that if \( X \) is a (affine) derived \( K \)-scheme, then its truncation \( X = \tau(X) \) is a (affine) \( K \)-scheme. Therefore, we can consider a derived \( K \)-scheme \( X \) as an infinitesimal thickening of its truncation \( X \). It follows that points of a derived \( K \)-scheme \( X \) are the same as points of its truncation \( X \). It means that the main difference between \( X \) and \( X \) is in fact encoded by the scheme structure, not by the points!

**Nice local models for derived \( K \)-schemes.** The following result (Theorem 2.5) plays an important role in constructing useful local algebraic models for derived \( K \)-schemes and for the so-called \( k \)-shifted symplectic structures on them.
The upshot is that given a derived \( K \)-scheme \( X \) (locally of finite presentation) and a point \( x \in X \), one can always find a “refined” local affine neighborhood \( \text{Spec} \, A \) of \( x \) that allows us to make more explicit computations over this neighborhood. For example, using such local models, we can identify the cotangent complex \( L_X \) with the module of Kähler differentials \( \Omega^1_X \), and then we can provide explicit representatives (rather than just cohomology classes) for (closed) \( p \)-forms of degree \( k \). In this regard, Bussi, Brav and Joyce proved the following theorem.

**Theorem 2.5. (\cite{5, Theorem 4.1})** Every derived \( K \)-scheme \( X \) is Zariski locally modelled on \( \text{Spec} \, A \) for some “minimal standard form” cdga \( A \) in cdgAlg.

More precisely, for each \( x \in X \) there is a pair \( (A, i : \text{Spec} \, A \hookrightarrow X) \) and \( p \in \text{Spec} \, H^0(A) \) such that \( i \) is an open inclusion with \( i(p) = x \), where \( A \) is a special kind of cdga (cf. Definition 2.6).

Moreover, there is a reasonable way to compare two such local charts \( i : \text{Spec} \, A \hookrightarrow X \) and \( j : \text{Spec} \, B \hookrightarrow X \) on their overlaps via a third chart. For details, see [5, Theorem 4.1 & 4.2].

In the remainder of this section, we shall elaborate the content of Theorem 2.5, and introduce appropriate notions for the constructions of interest. We will closely follow [5, 6].

**Definition 2.6.** \( A \in \text{cdgAlg} \) is of standard form if \( A^0 \) is a smooth finitely generated \( K \)-algebra, the module \( \Omega^1_A \) of Kähler differentials is free \( A^0 \)-module of finite rank, and the graded algebra \( A \) is freely generated over \( A^0 \) by finitely many generators, all in negative degrees.

In fact, there is a systematic way of constructing such cdgas. [5, Example 2.8] explains how to build these cdgas starting from a smooth \( K \)-algebra \( A^0 := A(0) \) via applying a sequence of localizations. The upshot is follows: Let \( n \in \mathbb{N} \), then a cdga \( A \), as a commutative graded algebra, can be constructed inductively from a smooth \( K \)-algebra \( A(0) \) by adjoining free finite rank modules \( M^{−i} \) of generators in degree \( −i \) for \( i = 1, 2, \ldots, n \). More precisely, for any given \( n \in \mathbb{N} \), we can inductively construct a sequence of cdgas

\[
A(0) \to A(1) \to \cdots \to A(i) \to \cdots A(n) =: A,
\]

where \( A^i := A(0) \), and \( A(i) \) is obtained from \( A(i-1) \) by adjoining generators in degree \( −i \), given by \( M^{−i} \), for all \( i \). Here, each \( M^{−i} \) is a free finite rank module (of degree \( −i \) generators) over \( A(i-1) \). Therefore, the underlying commutative graded algebra of \( A = A(n) \) is freely generated over \( A(0) \) by finitely many generators, all in negative degrees \( −1, −2, \ldots, −n \).

**Definition 2.7.** A standard form cdga \( A \) is said to be minimal at \( p \in \text{Spec} \, H^0(A) \) if \( A = A(n) \) is defined by using the minimal possible numbers of graded generators in each degree \( \leq 0 \) compared to all other cdgas locally equivalent to \( A \) near \( p \). (There will be an equivalent definition for minimality later, see Definition 2.16.)

**Definition 2.8.** Let \( A \) be a standard form cdga. \( A' \in \text{cdgAlg} \) is called a localization of \( A \) if \( A' \) is obtained from \( A \) by inverting an element \( f \in A^0 \), by which we mean \( A' = A \otimes_{A^0} A^0[f^{-1}] \).

\( A' \) is then of standard form with \( A'^0 \simeq A^0[f] \). If \( p \in \text{spec} \, H^0(A) \) with \( f(p) \neq 0 \), we say \( A' \) is a localization of \( A \) at \( p \).

With these definitions in hand, one has the following observations:

**Observation 2.9.** Let \( A \) be a standard form cdga. If \( A' \) is a localization of \( A \), then \( \text{Spec} \, A' \subset \text{Spec} \, A \) is a Zariski open subset. Likewise, if \( A' \) is a localization of \( A \) at \( p \in \text{spec} \, H^0(A) \simeq \tau(\text{Spec} \, A) \), then \( \text{Spec} \, A' \subset \text{Spec} \, A \) is a Zariski open neighborhood of \( p \).

**Observation 2.10.** Let \( A = A(k) \) be a standard form cdga, then there exist generators \( x_1^{-i}, x_2^{-i}, \ldots, x_m^{-i} \) in \( A^{-i} \) (after localization, if necessary) with \( i = 1, 2, \cdots, k \) and \( m_i \in \mathbb{Z}_{\geq 0} \) such that

\[
A = A(0)[x_i^{-j} : i = 1, 2, \ldots, k, j = 1, 2, \ldots, m_i],
\]

where the subscript \( j \) in \( x_i^j \) labels the generators, and the superscript \( i \) indicates the degree of the corresponding element. So, we can consider \( A \) as a graded polynomial algebra over \( A(0) \) on finitely many generators, all in negative degrees.

**Definition 2.11.** We then define the virtual dimension of \( A \) to be the integer \( \text{vdim} \ A = \sum_i (-1)^i m_i \).

**Observation 2.12.** Geometrically, the “smoothness” condition on \( A^0 \) implies that the corresponding affine \( K \)-scheme \( U = \text{spec} \, A^0 \) is smooth together with a local (étale) coordinate system

\[
(x_1^0, x_2^0, \ldots, x_m^0) : U \to A^0_{\text{étale}}.
\]
Nice local models for cotangent complexes of derived schemes. Given $A \in cdga_k$, $d$ on $A$ induces a differential on $\Omega^1_A$, denoted again by $d$. This makes $\Omega^1_A$ into a dg-module $(\Omega^1_A, d)$ with the property that $\delta \circ d = d \circ \delta$, where $\delta : A \to \Omega^1_A$ is the universal derivation of degree 0. Write the decomposition of $\Omega^1_A$ into graded pieces

$$\Omega^1_A = \bigoplus_{k=-\infty}^0 (\Omega^1_A)^k$$

with the differential $d : (\Omega^1_A)^k \to (\Omega^1_A)^{k+1}$. Then we define the de Rham algebra of $A$ as a double complex

$$DR(A) = \bigoplus_{p=0}^{\infty} \bigoplus_{k=-\infty}^0 (\Lambda^p \Omega^1_A)[k][p], \quad (2.5)$$

where $DR(A)$ has two gradings: the grading w.r.t. $p$ is called the weight, and the grading w.r.t. $k$ is called the degree. By construction, there are two differentials, namely the internal differential $d$ and the de Rham differential $d_{dR}$. We diagrammatically have

such that $d_{tot} = d + d_{dR}$ and both differentials satisfy the relations

$$d^2 = d_{dR}^2 = 0, \quad d \circ d_{dR} + d_{dR} \circ d = 0. \quad (2.7)$$

We also have the natural multiplication on $DR(A)$

$$(\Lambda^p \Omega^1_A)[k][p] \times (\Lambda^q \Omega^1_A)[q][q] \to (\Lambda^{p+q} \Omega^1_A)[k][p+q]. \quad (2.8)$$

**Observation 2.13.** It should be noted that the constructions of $\Omega^1_A$ and $DR(A)$ depend only on the underlying commutative graded algebra of $A$, not on the differential $d$ on $A$.

**Remark 2.14.** When $A = A(k)$ is a minimal standard form cdga, there are two important outcomes:

1. With such local coordinates $(x^0_1, x^0_2, \cdots, x^0_m)$, we have

$$\Omega^1_{A^0} \cong A^0 \otimes_k \langle d_{dR}^0 x^0_1, \cdots, d_{dR}^0 x^0_m \rangle_k. \quad (2.9)$$

Furthermore, the Kähler differentials is an $A$-module of the form

$$\Omega^1_A \cong A \otimes_k \langle d_{dR}^0 x^{-i}_j : i = 0, 1, 2, \cdots, k, \ j = 1, 2, \cdots, m_i \rangle_k. \quad (2.10)$$

2. $\Omega^1_A$ provides a local model for the cotangent complex $L_A$. That is, in the case of a minimal standard form cdga, the cotangent complex $L_A$ has the identification

$$L_A = \Omega^1_A. \quad (2.11)$$

Note that if $D(Mod_A)$ denotes the derived category of $Mod_A$, then $L_A \in D(Mod_A)$ for standard form cdgas. In general, even if both $L_A$ and $\Omega^1_A$ are closely related, the identification in (2.11) is not true for an arbitrary $A \in cdga_k$ [5].

When $A = A(n)$ is a standard form cdga as in (2.2), we also have the following description for the restriction of the cotangent complex $L_A$ to $\text{spec}^H(A)$. 

6
Proposition 2.15. ([5, Prop. 2.12]) If \( A = A(n) \), with \( n \in \mathbb{N} \), is a standard form cdga constructed inductively as in (2.2), then the restriction of \( L_A \) to \( \text{spec} H^0(A) \) is represented by a complex of \( H^0(A) \)-modules

\[
0 \to V^{-n} \xrightarrow{d^{-n}} V^{-n+1} \to \cdots \to V^{-1} \xrightarrow{d^{-1}} V^0 \to 0,
\]

where each \( V^{-i} \) can in fact be defined as \( V^{-i} = H^{-i}(L_{A(i)/A(i-1)}) \), with \( L_{A(i)/A(i-1)} \) the relative cotangent complex of the map \( A(i-1) \to A(i) \) in (2.2) satisfying

\[
L_{A(i)/A(i-1)} \simeq A(i) \otimes_{A(i-1)} M^{-i}[i].
\]

Moreover, the differential \( V^{-i} \xrightarrow{d^{-i}} V^{-i+1} \) is identified with the composition

\[
H^{-i}(L_{A(i)/A(i-1)}) \to H^{-i+1}(L_{A(i-1)}) \to H^{-i+1}(L_{A(i-1)/A(i-2)}),
\]

which can be obtained from the fiber sequences induced by the maps \( A(i-1) \to A(i) \) in (2.2). Note in particular that \( j > -i \), we have \( H^{j}(L_{A(i)/A(i-1)}) = 0 \). More details and the proof can be found in [5, Prop. 2.12].

With this result in hand, using local coordinates above, write

\[
V^{-i} = \langle d_{\text{dR}}x_1^{-i}, d_{\text{dR}}x_2^{-i}, \ldots, d_{\text{dR}}x_n^{-i} \rangle_{A(0)} \quad \text{for } i = 0, 1, \ldots, n.
\]

It follows that we have a similar local description for the tangent complex \( T_A = (L_A)^\vee \) of \( A \) when restricted to \( \text{spec} H^0(A) \). Also, we have an alternative definition of minimality at a point \( p \in \text{spec} H^0(A) \) for a cdga of the form \( A = A(n) \).

Definition 2.16. Let \( A = A(n) \), with \( n \in \mathbb{N} \), be a standard form cdga constructed inductively as in (2.2). \( A \) is said to be minimal at \( p \in \text{spec} H^0(A) \) if the internal differential \( d^{-i}|_p = 0 \) in the complex \( L_{|\text{spec} H^0(A)} \) given in (2.12).

Note that Definition 2.16 implies \( m_i = \dim(H^{-i}((L_A)_p)) \) for each \( i \), and hence \( A \) is defined by using the minimum number of graded variables in each degree \( \leq 0 \) compared to all other cdgas locally equivalent to \( A \) near \( p \). Therefore, one can recover Definition 2.7.

2.2 PTVV’s shifted symplectic geometry on derived schemes

Let \( X \) be a locally finitely presented derived \( \mathbb{K} \)-scheme with \( p \geq 0 \), \( k \in \mathbb{Z} \). Pantev et al. [4] define simplicial sets of \( p \)-forms of degree \( k \) and closed \( p \)-forms of degree \( k \) on \( X \). Denote these simplicial sets by \( \mathcal{A}^p(X, k) \) and \( \mathcal{A}^{p, \text{cl}}(X, k) \), respectively. These definitions are in fact given first for affine derived \( \mathbb{K} \)-schemes. Later, both concepts are defined for a general \( X \) in terms of mapping stacks. A summary of key ideas can be found in [5, § 3.4].

In our case, we consider \( X = \text{Spec} A \) with \( A \) a standard form cdga, and hence take \( \Lambda^p L_A = \Lambda^p \Omega^1_X \). Therefore, elements of \( \mathcal{A}^p(X, k) \) form a simplicial set such that \( k \)-cohomology classes of the complex \( (\Lambda^p \Omega^1_A, d) \) correspond to the connected components of this simplicial set. Likewise, the connected components of \( \mathcal{A}^{p, \text{cl}}(X, k) \) are identified with the \( k \)-cohomology classes of the complex \( \prod_{i>0} (\Lambda^{p+1} \Omega^1_A[-i], d_{	ext{cl}}) \). We want to work with explicit representatives for these classes.

It should be noted that the results that are cited or to be proven in this paper are all about the local structure of derived schemes. Thus, it is enough to consider the affine case. Moreover, we always assume all local models are sufficiently nice by using Theorem 2.5 if necessary.

Definition 2.17. Let \( X = \text{Spec} A \) be an affine derived \( \mathbb{K} \)-scheme for \( A \) a minimal standard form cdga. A \( p \)-form of degree \( k \) on \( X \) for \( p \geq 0 \) and \( k \leq 0 \) is an element

\[
\omega^0 \in (\Lambda^p \Omega^1_A)^k \quad \text{with } d \omega^0 = 0.
\]

Note that an element \( \omega^0 \) defines a cohomology class as being \( d \)-closed. That is,

\[
[\omega^0] \in H^k(\Lambda^p \Omega^1_A, d),
\]

where two \( p \)-forms \( \omega_1^0, \omega_2^0 \) of degrees \( k \) are equivalent if there exists \( \alpha_{1,2} \in (\Lambda^p \Omega^1_A)^{k-1} \) so that \( \omega_1^0 - \omega_2^0 = d \alpha_{1,2} \).
Remark 2.18. In the classical “underived” case, for instance when $X = \text{spec} A$ is smooth for a commutative $\mathbb{K}$-algebra $A$, the cotangent complex $\mathbb{L}_X$ is just a vector bundle over $X$, and denoted simply by $T^* X$. Then, a $p$-form $\omega$ on $X$ is defined to be a global section of the bundle $\Lambda^p T^* X$. A careful observation reveals that Definition 2.17 does generalize the definition of a $p$-form on a smooth space in the following sense: It is clear that any commutative $\mathbb{K}$-algebra $A$ can be realized as an object in $\text{cdg}_{\text{der}}$ concentrated in degree 0 with the trivial differential. Thus, in the language of Definition 2.17, a naive notion of “a $p$-form $\omega$ on a smooth space $X$” is just a $p$-form $\omega$ of degree 0 on $X = \tau \odot i (X)$ in $\text{Spec} \mathbb{K}$ such that $\omega \in (\Lambda^p T^* X)^0$. Note that the condition $d \omega = 0$ holds trivially, and hence $[\omega] \in H^0 (\Lambda^p T^* X, d = 0)$. Here, $\mathbb{L}_X = T^* X$ is again viewed as graded object concentrated in degree 0, with the zero differential.

In DAG, on the other hand, $\Lambda^p \mathbb{L}_X$ is a (double) complex which possesses a non-trivial internal differential as above, and hence one needs to take into account higher non-trivial cohomology groups as well.

Definition 2.19. A 2-form $\omega^0$ of degree $k$ on $\text{Spec} A$ for $A$ a minimal standard form cdga is called non-degenerate if the induced morphism $\omega^0 : T^*_A \to \Omega^1_A[k]$, $Y \mapsto \iota_Y \omega^0$, is a quasi-isomorphism, where $T^*_A = (L_A)^\vee = \text{Hom}_A (\Omega^1_A, A)$ is the tangent complex of $A$.

Definition 2.20. Let $X = \text{Spec} A$ be an affine derived $\mathbb{K}$-scheme with $A$ a minimal standard form cdga. A closed $p$-form of degree $k$ on $X$ for $p \geq 0$ and $k \leq 0$ is a sequence $\omega = (\omega^0, \omega^1, \ldots)$ with $\omega^i \in (\Lambda^{p+i} \Omega^1_A)^{k-i}$ satisfying the following conditions:

1. $d \omega^0 = 0$ in $(\Lambda^p \Omega^1_A)^{k+1}$.
2. $d_{\text{dR}} \omega^i + d \omega^{i+1} = 0$ in $(\Lambda^{p+i+1} \Omega^1_A)^{k-i}$, $i \geq 0$.

Note that the conditions imposed in Definition 2.20 can also be interpreted in the language of double complexes as follows: We first observe that each component $\omega^i$ of the sequence $\omega = (\omega^0, \omega^1, \ldots)$ fits into the appropriate places on the diagonal of the double complex

\[
\begin{array}{ccccccc}
\vdots & & & & & & \\
\vdots & & \downarrow & & \downarrow & & \vdots \\
\ldots & \rightarrow & (\Lambda^{p+i+1} \Omega^1_A)^k & \rightarrow & (\Lambda^{p+i+1} \Omega^1_A)^{k-1} & \rightarrow & \ldots \\
\downarrow d_{\text{dR}} & & & & & & \downarrow d_{\text{dR}} \\
\ldots & \rightarrow & (\Lambda^{p+i} \Omega^1_A)^{k-i-1} & \rightarrow & (\Lambda^{p+i} \Omega^1_A)^{k-i} & \rightarrow & \ldots \\
\uparrow & & & & & & \uparrow \\
\vdots & & & & & & \\
\end{array}
\]  \quad (2.14)

where $d_{\text{tot}} = d + d_{\text{dR}}$ such that $d^2 = d_{\text{dR}}^2 = 0$, and $d \circ d_{\text{dR}} + d_{\text{dR}} \circ d = 0$. Furthermore, a straightforward computation gives

\[
d_{\text{tot}} \omega = d \omega^0 + \sum_{i \geq 0} d_{\text{dR}} \omega^i + d \omega^{i+1}.
\]

It follows that two conditions in Definition 2.20 can be simply encoded as $d_{\text{tot}} \omega = 0$. Therefore, we have

Observation 2.21. A closed $p$-form of degree $k$ on $X$ for $p \geq 0$ and $k \leq 0$ can be equivalently defined as a sequence $\omega = (\omega^0, \omega^1, \ldots)$ such that

\[
\omega^i \in (\Lambda^{p+i} \Omega^1_A)^{k-i} \text{ with } d_{\text{tot}} \omega = 0.
\]  \quad (2.15)

Remark 2.22.

1. From Definition 2.20, there exists a natural projection morphism

\[
\pi : \mathcal{A}^{(p, \text{cd})} (X, k) \longrightarrow \mathcal{A}^p (X, k), \quad \omega = (\omega^i)_{i \geq 0} \longmapsto \omega^0.
\]  \quad (2.16)
2. When we restrict ourselves to the classical case as in Remark 2.18, the one in which everything is concentrated in degree 0, we have $d = 0$ and hence $d_{tot} = d_{dR}$. Moreover, the only possible non-trivial component of $\omega$ is $\omega^0$. Therefore, using the truncation functor as before, the conditions in Definition 2.20 reduce to

$$\omega^0 \in H^0(\Lambda^p T^* X) \text{ with } d_{dR}\omega^0 = 0. \quad (2.17)$$

Thus, Definition 2.20 reduces to the usual definition of a (de Rham) closed $p$-form on smooth spaces.

**Definition 2.23.** A closed 2-form $\omega = (\omega^i)_{i \geq 0}$ of degree $k$ on an affine derived $\mathbb{K}$-scheme $\text{Spec} A$ for a minimal standard form cdga $A$ is called a $k$-shifted symplectic structure if $\pi(\omega) = \omega^0$ is a non-degenerate 2-form of degree $k$.

### 2.3 Shifted symplectic Darboux models

One of the main theorems in [5] provides a $k$-shifted version of the classical Darboux theorem in symplectic geometry. The statement is as follows.

**Theorem 2.24.** ([5, Theorem 5.18]) Given a derived $\mathbb{K}$-scheme $X$ with a $k$-shifted symplectic form $\omega'$ for $k < 0$ and $x \in X$, there is a local model $(A, f : \text{spec} A \hookrightarrow X, \omega)$ and $p \in \text{spec} H^0(A)$ such that $f$ is an open inclusion with $f(p) = x$, $A$ is a standard form that is minimal at $p$, and $\omega$ is a $k$-shifted symplectic form on $\text{Spec} A$ such that $A, \omega$ are in Darboux form, and $f^*(\omega') \sim \omega$ in the space of $k$-shifted closed 2-forms.

To be more precise, it is proven in [5, Theorem 5.18] that such $\omega$ can be constructed explicitly depending on the integer $k < 0$. Indeed, there are three cases in total:

1. $k$ is odd, $2 \equiv 0 \mod 4$, (3) $k \equiv 2 \mod 4$.

Equivalently, the cases can be expressed as (1) $k/2 \notin \mathbb{Z}$, (2) $k/2$ is even, and (3) $k/2$ is odd, respectively. In short, Theorem 2.24 says that every $k$-shifted symplectic derived $\mathbb{K}$-scheme $(X, \omega')$ is Zariski locally equivalent to $(\text{Spec} A, \omega)$ for some $A, \omega$, where $A$ is a minimal standard form cdga and $\omega$ is a $k$-shifted symplectic form on $\text{Spec} A$ such that $\omega$ is given in a standard way depending on the cases above.

In this paper, for simplicity, we will examine a family of explicit Darboux forms for $k < 0$ an odd integer [5, Example 5.8]. The other cases can be studied in a similar way, but with some modifications. We will outline the steps. More details can be found in [5, §5.3].

We first begin with a useful result that plays a significant role in constructing Darboux-type local models below. The upshot is that one can always simplify the form of a given closed 2-form $\omega = (\omega^0, \omega^1, \omega^2, \ldots)$ of degree $k < 0$ on $\text{Spec} A$ so that $\omega^0$ can be taken to be exact and $\omega^i = 0$ for all $i > 0$. More precisely, we have the following result.

**Proposition 2.25.** ([5, Prop. 5.7]) Let $\omega = (\omega^0, \omega^1, \omega^2, \ldots)$ be a closed 2-form of degree $k < 0$ on $\text{Spec} A$ for $A$ a standard form cdga over $\mathbb{K}$. Then there exist $H \in A^{k+1}$ and $\phi \in (\Omega^1_A)^k$ such that $dH = 0$ in $A^{k+2}$, $d_{dR}H + d\phi = 0$ in $(\Omega^1_A)^{k+1}$, and $\omega \sim (d_{dR}\phi, 0, 0, \ldots)$.

Moreover, if $(H', \phi')$ is another such pair for fixed $\omega, k, A$, then there exist $h \in A^k$ and $\sigma \in (\Omega^1_A)^{k-1}$ such that $H - H' = dh$ and $\phi - \phi' = d_{dR}h + d\sigma$.

The proof of Proposition 2.25 is based on the fact that any such forms can be interpreted in the context of cyclic homology theory of mixed complexes. Indeed, any such forms can be viewed as cocycles in the so-called negative cyclic complex of weight $p$ on $\text{Spec} A$, which is constructed from the de Rham algebra $\text{DR}(A)$ in certain way. When $p = 2$, there are some useful short exact sequences and vanishing results, by which one can eventually obtain the desired simplification above. For more details on this cyclic homology perspective, we refer to [5, §5.2].

**Observation 2.26.** Assume $(H, \phi)$ is a such pair for fixed $\omega, k, A$, with $d_{dR}\phi = k\omega^0$. Let $f \in \mathbb{K}$ be a non-zero element. Define $H' = fH$ and $\phi' = f\phi$. Then both $H', \phi'$ satisfy the relations $dH' = 0$ and $d_{dR}H' + d\phi' = 0$. From the choices, we also have $d_{dR}\phi' = kf\omega^0$, and hence $\omega \sim (d_{dR}\phi', 0, 0, \ldots)$. By Proposition 2.25, there exist $h \in A^k$ and $\sigma \in (\Omega^1_A)^{k-1}$ such that $H - H' = dh$ and $\phi - \phi' = d_{dR}h + d\sigma$. It follows that $(1 - f)H = dh$. Localizing $A$ by the element $(1 - f)$ if necessary, we can write $H = d[(1 - f)^{-1}h]$. It means that we can "locally" take $H$ to be $d$-exact.
Prototype Darboux model for \( k < 0 \) odd. Let \( k = -2\ell - 1 \) for \( \ell \in \mathbb{N} \). Then the local model consists of the following data:

1. Let \( A^0 = A(0) \) be a smooth \( \mathbb{K} \)-algebra of dim \( m_0 \), choose \( x_0^0, \ldots, x_{m_0}^0 \) such that \( d_{dR}x_1^0, \ldots, d_{dR}x_{m_0}^0 \) form a basis for \( \Omega^1_{A^0} \). Then \( A \) is defined to be the free \( A^0 \)-graded algebra over \( A^0 \) generated by variables

\[
\begin{align*}
x_{1}^{-1}, x_{2}^{-i}, \ldots, x_{m}^{-i} & \quad \text{in degree } (-i) \text{ for } i = 1, 2, \ldots, \ell, \\
y_1^{k+i}, y_2^{k+i}, \ldots, y_{m_i}^{k+i} & \quad \text{in degree } (k+i) \text{ for } i = 0, 1, \ldots, \ell.
\end{align*}
\]

(2.18)

2. \( \Omega_A^1 \) is the free \( A \)-module of finite rank given by

\[
\Omega_A^1 \simeq A \otimes_{\mathbb{K}} \langle d_{dR}x_j^{-i}, d_{dR}y_j^{k+i} : i = 0, 1, \ldots, \ell, \ j = 1, 2, \ldots, m_i \rangle_{\mathbb{K}}.
\]

(2.19)

3. Define an element \( \omega^0 \in (\Lambda^2 \Omega_A^1)^k \) of degree \( k \) and weight 2 in \( DR(A) \) to be

\[
\omega^0 = \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} d_{dR}x_j^{-i}d_{dR}y_j^{k+i}.
\]

(2.20)

4. It follows from Proposition \ref{2.25} that there exists a pair \( (\phi, H) \in (\Omega_A^1)^k \times A^{k+1} \) satisfying the following properties:

(a) \( dH = 0 \) in \( A^{k+2} \), \( d_{dR}H + d\phi = 0 \) in \( (\Omega_A^1)^k \), and \( d_{dR}\phi = k\omega^0 \).

(b) \( H \) satisfies the condition (a.k.a. the classical master equation)

\[
\sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \frac{\partial H}{\partial x_j^{-i}} \frac{\partial H}{\partial y_j^{k+i}} = 0 \text{ in } A^{k+2},
\]

(2.21)

which in fact corresponds to the condition “\( dH = 0 \)”. We call \( H \) the Hamiltonian.

(c) Explicitly, we have

\[
\phi := \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} \left[ -ix_j^{-i}d_{dR}y_j^{k+i} + (k+i)y_j^{k+i}d_{dR}x_j^{-i} \right].
\]

(2.22)

Note that we can choose another representatives by replacing \( H, \phi \) by \( \phi' = \phi + d_{dR}\theta \) and \( H' = H + d\theta \) for any \( \theta \in A^0 \). This modification will leave \( \omega^0 \) unchanged, and both \( H', \phi' \) satisfy \( dH' = 0 \) and \( d_{dR}H' + d\phi' = 0 \). Letting \( \theta = \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} \left[ (-1)^i x_j^{-i} y_j^{k+i} \right] \), for instance, we may take \( \phi' := k \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} y_j^{k+i} d_{dR}x_j^{-i} \).

(d) The internal differential \( d \) on \( A \) can be defined as

\[
d|_{A^0} = 0, \ d_{x_j^{-i}} = \frac{\partial H}{\partial y_j^{k+i}} \text{ and } d_{y_j^{k+i}} = \frac{\partial H}{\partial x_j^{-i}}.
\]

(2.23)

5. Clearly \( d_{dR}\omega^0 = 0 \), but it is a little bit cumbersome to check that \( d\omega^0 = 0 \), and \( \omega^0 \) defines a non-degenerate pairing. For details, we refer to \cite[Example 5.8]{5}. As a result, the sequence \( \omega := (\omega^0, 0, 0, \ldots) \) defines a \( k \)-shifted symplectic structure on \( \text{Spec} A \).

\textbf{Definition 2.27.} If \( A, \omega \) are as above, then we say that the pair \((A, \omega)\) is in (symplectic) Darboux form.
Darboux forms for the other cases of $k$. For the sake of completeness, we briefly summarize the cases when $k/2$ is even or odd. Here, the main difference from the case $k$ being odd is about the existence of middle degree variables. In fact, when $k$ is odd, there is no such degree. But if $k/2$ is even, there are such variables and 2-forms are anti-symmetric in these variables. On the other hand, when $k/2$ is odd, such forms are symmetric in the middle degree variables. Let us briefly examine each case:

(a) [5, Example 5.9] When $k/2$ is even, say $k = -4\ell$ for $\ell \in \mathbb{N}$, the cdga $A$ is now free over $A(0)$ generated by the new set of variables

\[
x_i^{-i}, x_1^{-2}, \ldots, x_m^{-i} \quad \text{in degree} \quad -i \quad \text{for} \quad i = 1, 2, \ldots, 2\ell - 1,
\]
\[
x_k^{-2}, x_2^{-2}, \ldots, x_{m_2}^{-2} \quad \text{in degree} \quad -2, 
\]
\[
y_{k+1}^{-i}, y_{k+2}^{-i}, \ldots, y_{m_k}^{-i} \quad \text{in degree} \quad k + i \quad \text{for} \quad i = 0, 1, \ldots, 2\ell - 1. \tag{2.24}
\]

Then we define an element $\omega^0 = \sum_{i=0}^{2\ell} \sum_{j=1}^{m_i} dR x_i^{-i} dR y_j^{k+i}$ in $(\Lambda^2\Omega^1)_{A}$, and set $\omega$ to be $(\omega^0, 0, 0, \ldots)$ as before. Choose an element $H \in A^{k+1}$, the Hamiltonian, satisfying the analogue of classical master equation, and define $d$ on $A$ as in Equation (2.23) using $H$. We also define the element $\phi \in \Omega^1_{A}$ by the analogue of Equation (2.22).

(b) [5, Example 5.10] When $k/2$ is odd, say $k = -4\ell - 2$ for $\ell \in \mathbb{N}$, $A$ is freely generated over $A(0)$ by the variables

\[
x_i^{-i}, x_1^{-2}, \ldots, x_m^{-i} \quad \text{in degree} \quad -i \quad \text{for} \quad i = 1, 2, \ldots, 2\ell,
\]
\[
x_k^{-2}, x_2^{-2}, \ldots, x_{m_2}^{-2} \quad \text{in degree} \quad -2, 
\]
\[
y_{k+1}^{-i}, y_{k+2}^{-i}, \ldots, y_{m_k}^{-i} \quad \text{in degree} \quad k + i \quad \text{for} \quad i = 0, 1, \ldots, 2\ell. \tag{2.25}
\]

We then define an element $\omega^0 = \sum_{i=0}^{2\ell} \sum_{j=1}^{m_i} dR x_i^{-i} dR y_j^{k+i} + \sum_{i=1}^{m_2} dR z_i^{-2} dR y_j^{k+i}$ in $(\Lambda^2\Omega^1)_{A}$, and set $\omega := (\omega^0, 0, 0, \ldots)$ as before. Choose an element $H \in A^{k+1}$, the Hamiltonian, satisfying the analogue of classical master equation

\[
\sum_{i=1}^{2\ell} \sum_{j=1}^{m_i} \frac{\partial H}{\partial x_j^{-i}} \frac{\partial H}{\partial y_j^{k+i}} + \frac{1}{4} \sum_{j=1}^{m_2} \left( \frac{\partial H}{\partial z_j^{-2}} \right)^2 = 0 \quad \text{in} \quad A^{k+2}. \tag{2.26}
\]

Define $d$ on $A$ as in Equation (2.23) with extra data $dz_j^{-2} := \frac{1}{2} \frac{\partial H}{\partial z_j^{-2}}$.

Finally, we define the element $\phi \in \Omega^1_{A}$ by

\[
\phi = \sum_{i=0}^{2\ell} \sum_{j=1}^{m_i} \left[ -i x_j^{-i} dR y_j^{k+i} + (-1)^{i+1}(k+i) y_j^{k+i} dR x_j^{-i} \right]
\]
\[
+ k \sum_{j=1}^{m_2} z_j^{-2} dR z_j^{-2}. \tag{2.27}
\]

**Observation 2.28.** In either case, the virtual dimension vdim $A$ is always even. In fact, for any $k < 0$ we have

\[
\text{vdim } A = \begin{cases} 2 \sum_i (-1)^i m_i, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}
\]

**Remark 2.29.** The classical Darboux theorem states that for a symplectic manifold $(X, \omega)$, one can find a local coordinate chart $(U; x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that $\omega|_U = \sum_j dR x_j dR y_j$. Moreover, we can write $\lambda = \sum_j x_j dR y_j$, called the Liouville form, such that $\omega|_U = dR \lambda$.

In this derived framework, the element $\phi$ above may seem to play the role of $\lambda$. However, it is important to notice that $\phi$ is not a 1-form (of degree $k$) in the sense of Definition 2.17, because $d\phi \neq 0$. Therefore, one needs to modify $\phi$ to obtain a genuine 1-form of degree $k$. 

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3 Shifted contact structures and a Darboux-type theorem

3.1 Basics of classical contact geometry

It is very well-known that contact manifolds are viewed as the odd-dimensional analogues of symplectic manifolds. In that respect, they have a number of common features: there is a Darboux theorem providing a local model for such structures, there is no local invariants, and it is more interesting to study their global properties, etc... For more details, we refer to [12].

In this section, we shall revisit the basic aspects of contact geometry. There are in fact equivalent ways of describing the notion of a contact structure. We prefer to use the one below.

Definition 3.1. Let \( X \) be a manifold of dimension \( 2n + 1 \). A contact structure is a smooth field of tangent hyperplanes \( \xi \subset TM \) (of rank \( 2n \)) with the property that for any smooth locally defining 1–form \( \alpha \), i.e. \( \xi = \ker(\alpha) \), the 2-form \( \dR\alpha|_{\xi} \) is non-degenerate.

The pair \( (X, \xi) \) is then called a contact manifold and \( \alpha \) is called a local contact form. For a point \( p \in X \), the pair \( (p, \xi_p) \) is called a contact element.

Note that in Definition 3.1, by a non-degenerate 2-form \( \dR\alpha|_{\xi} \), we mean that for each point \( p \in X \), the linear map \( \dR\alpha|_{\xi}: \xi_p \to \xi^*_p \) defined by \( v \mapsto \iota_v \dR\alpha \) is an isomorphism. Therefore, if \( \dR\alpha|_{\xi} \) is non-degenerate, then for each \( p \in X \), \( \xi_p = \ker(\alpha_p) \) is a symplectic vector space with a symplectic form \( \omega_p := \dR\alpha|_{\xi_p} \). Therefore, we also call such 2-form \( \dR\alpha|_{\xi} \) symplectic.

It follows from the theory of symplectic vector spaces that \( \dim \xi_p \) is even and the symplectic form on \( \xi_p \) has a canonical form. It means that there exists a symplectic basis \( \{e_1, f_1, \ldots, e_n, f_n\} \) for \( \xi_p \) (and the corresponding dual basis \( \{e_1^*, f_1^*, \ldots, e_n^*, f_n^*\} \) for \( \xi^*_p \)) satisfying

\[
\omega_p(e_i, e_j) = 0 = \omega_p(f_i, f_j) \quad \text{and} \quad \omega_p(e_i, f_j) = -\omega_p(f_j, e_i) = \delta_{ij} \quad \forall i, j,
\]

so that \( \omega_p \) has the form \( \omega_p = \sum_i e_i^* \wedge f_i^* \).

Let \( (X, \xi = \ker(\alpha)) \) be a contact manifold of dim \( 2n + 1 \), and \( p \in X \). Then we have a splitting

\[
T_pX = \xi_p \oplus \ker(\dR\alpha|_{\xi_p}),
\]

where \( \dim \xi_p = 2n \) and \( \dim \ker(\dR\alpha|_{\xi_p}) = 1 \). In fact, as \( \dR\alpha|_{\xi} \) is non-degenerate, one can find a local trivialization \( \{e_1, f_1, \ldots, e_n, f_n, r\} \) of \( TM = \ker\alpha \oplus \text{rest} \) such that

\[
\ker\alpha = \text{Span}\{e_1, f_1, \ldots, e_n, f_n\} \quad \text{and} \quad \text{rest} = \text{Span}\{r\}.
\]

Moreover, using this splitting, one can find a unique vector field \( R \), called the Reeb vector field of \( \alpha \), satisfying \( \iota_R \dR\alpha = 0 \) and \( \iota_R\alpha = 1 \).

Example 3.2. On \( \mathbb{R}^{2n+1} \) with cartesian coordinates \( (x_1, \ldots, x_n, y_1, \ldots, y_n, z) \), the so-called standard contact form is given by

\[
\alpha_{std} = -dRz + \sum_{i=1}^{n} y_i dx_i.
\]

Let \( \xi \subset T\mathbb{R}^{2n+1} \) be the hyperplane field of rank \( 2n \) defined by \( \alpha_{std} \), i.e. \( \xi = \ker\alpha_{std} \). Then we observe that

\[
\ker(\alpha_{std}) = \text{Span}\left\{ \frac{\partial}{\partial y_j} : j = 1, \ldots, n \right\} \quad \text{and} \quad \dR\alpha_{std} = \sum_i dR x_i \wedge dR y_i.
\]

Write \( A_j = \frac{\partial}{\partial y_j} \) and \( B_j = y_j \frac{\partial}{\partial z} + \frac{\partial}{\partial x_j} \), then it is enough to observe that \( \dR\alpha_{std}(A_i, B_j) = \delta_{ij} \) and \( \dR\alpha_{std}(A_i, A_j) = 0 = \dR\alpha_{std}(B_i, B_j) \). It follows that \( \dR\alpha_{std} \mid_{\xi} \) is non-degenerate, and hence \( \alpha_{std} \) is a contact form. Moreover, the Reeb vector field of \( \alpha_{std} \) is \( R = \partial/\partial z \).

As in the symplectic case, there is a Darboux-type theorem for contact structures. It basically says that all contact structures can be locally given as in Equation (3.4). More formally, we have

Theorem 3.3 (Darboux Theorem for contact structures). Let \( (X, \alpha) \) be a contact manifold of dimension \( 2n + 1 \), and \( p \in X \). Then there exists a local coordinate system \( (U; x_1, \ldots, x_n, y_1, \ldots, y_n, z) \) around \( p \) such that \( p = (0, 0, \ldots, 0) \) and

\[
\alpha|_U = -dRz + \sum_{i=1}^{n} y_i dx_i.
\]
3.2 Shifted contact structures and Darboux forms

In this section, we wish to provide an appropriate analogue of Definition 3.1 for derived \( \mathbb{K} \)-schemes and to investigate possible local models. Because our results in this paper are all about the local structure of derived schemes, it suffices to work with the affine case. Let \( X = \text{Spec} A \) be an affine derived \( \mathbb{K} \)-scheme for \( A \) a minimal standard form cdga. Using Definitions 2.17 and 2.19, we define a \( k \)-shifted contact structure on \( X = \text{Spec} A \) as follows.

Definition 3.4. A \( k \)-shifted contact structure on \( X = \text{Spec} A \) is a \( k \)-shifted 1-form \( \alpha \) with the property that the \( k \)-shifted 2-form \( dR\alpha \) is non-degenerate on the subcomplex \( \ker \alpha \) of the tangent complex \( T_A \) of \( A \). Here \( T_A = (L_A)^Y = \text{Hom}_A(\Omega^1_A, A) \).

Observation 3.5. For any \( f \neq 0 \) in \( A^0 \) and any \( k \)-shifted contact form \( \alpha \), one has \( \ker \alpha \simeq \ker f \alpha \). Hence, both define equivalent contact structures on \( X \). In fact, this follows from the fact that the contraction operation \( \iota_Y \) on the de Rham algebra \( DR(A) \) with a homogeneous vector field \( Y \) is the unique derivation of degree \( |Y| + 1 \) such that \( \iota_Y g = 0 \) and \( \iota_Y dRg = Y(g) \) for all \( g \in A \). Therefore,

\[
\iota_Y(f \alpha) = (\iota_Y f) \cdot \alpha + (-1)^{|Y|+1} f \cdot \iota_Y \alpha = (-1)^{|Y|+1} f \cdot \iota_Y \alpha.
\]

Adopting the classical terminology, we sometimes call the subcomplex \( \ker \alpha \) a contact structure, and the 1-form \( \alpha \) a defining contact form.

In what follows, we give a prototype construction for \( k \)-shifted contact forms as in the previous case of shifted symplectic Darboux-like local models.

Example 3.6. In this example, fixing \( \ell \in \mathbb{N} \), we will present how to construct an explicit standard form cdga \( A = A(n) \) for \( n = 2\ell + 1 \) and a \( k \)-shifted contact structure \( \alpha_0 \) with \( k = -2\ell - 1 \).

First, we consider a smooth \( \mathbb{K} \)-algebra \( A(0) \) of dimension \( m_0 + 1 \). We assume that there exist degree 0 variables \( x_{1j}^0, x_{2j}^0, \ldots, x_{\ell j}^0 \) in \( A(0) \) such that \( dR x_{1j}^0, \ldots, dR x_{\ell j}^0 \) form a basis for \( \Omega^1_A(0) \) over \( A(0) \). This choice can be made by localizing \( A(0) \) if necessary.

Next, choosing non-negative integers \( m_1, \ldots, m_\ell \), define a commutative graded algebra \( A \) to be the free graded algebra over \( A(0) \) generated by variables

\[
x_{1i}^i, x_{2i}^i, \ldots, x_{mi}^i \quad \text{in degree } -i \quad \text{for } i = 1, 2, \ldots, \ell,
\]

\[
y_{1i}^{i+k}, y_{2i}^{i+k}, \ldots, y_{mi}^{i+k} \quad \text{in degree } k + i \quad \text{for } i = 0, 1, \ldots, \ell.
\]

It follows that \( \Omega^1_A \) is the free \( A \)-module of finite rank with an \( A \)-basis

\[
\{ dR x_{1j}^{-i}, dR x_{2j}^{i+k}, dR y_{mj}^{i} : i = 0, 1, \ldots, \ell, \ j = 1, 2, \ldots, m_i \}.
\]

Choose an element \( z \in A^k \) such that \( dz = H \) in \( A^{k+1} \) and \( H \) is the Hamiltonian satisfying the condition (the classical master equation)

\[
\sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \frac{\partial H}{\partial x_{ij}^{-i}} \frac{\partial H}{\partial y_{mj}^{i+k}} = 0 \quad \text{in } A^{k+2}.
\]

Then we define the internal differential on \( A \) by Equation (2.23). As discussed before, the condition on \( H \) above is equivalent to saying \( "dH = 0" \).

Notice that, by construction, \((A, d)\) is a standard form cdga with \( A = A(n = 2\ell + 1) \) which is defined inductively by adjoining free modules \( M^{-i} = (x_{1i}^i, x_{2i}^i, \ldots, x_{mi}^i)_{A(i-1)} \) for \( i = 1, 2, \ldots, \ell \) and \( M^{k+i} = (y_{1i}^{i+k}, y_{2i}^{i+k}, \ldots, y_{mi}^{i+k})_{A(-i-1)} \) for \( i = 0, 2, \ldots, \ell \).

Now, we define an element \( \alpha_0 \in (\Omega^1_A)^k \) by

\[
\alpha_0 = -dR z \sum_{i=1}^{\ell} \sum_{j=1}^{m_i} y_{ij}^{i+k} dR x_{ij}^{-i}.
\]

Then \( dR \alpha_0 \) defines an element, denoted by \( \omega_0 \in \left(\Lambda^2 \Omega^1_A\right)^{k} \), such that \( \omega_0 = \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} dR x_{ij}^{-i} dR y_{mj}^{i+k} \).

From [5, Example 5.8], \( \omega_0 \) is closed w.r.t both \( d \) and \( d_{dR} \) such that

\[
dH = 0 \quad \text{in } A^{k+2}, \quad d_{dR} H + d\phi = 0 \quad \text{in } (\Omega^1_A)^{k+1}, \quad \text{and } d_{dR} \phi = k \omega_0.
\]
We just set \( \phi := k \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} y_j^{k+i} d_{R} x_j^{-i} \). Now write
\[
\alpha_0 = -d_{R} z + (1/k)\phi. \tag{3.10}
\]

Using (3.9) and scaling \( z \) by the constant \( k \), it is now straightforward to check that \( \alpha_0 \) is \( d \)-closed, and hence a \( 1 \)-form of degree \( k \).

Now, it remains to check that \( \omega_0|_{\ker \alpha_0} \) is non-degenerate. Denote the vector fields annihilating \( \alpha_0 \) by
\[
\zeta_i^j = \partial/\partial y_j^{k+i} \quad \text{and} \quad \eta_i^j = \partial/\partial x_j^{-i} + ky_j^{k+i}\partial/\partial z. \tag{3.11}
\]

Then \( \ker \alpha_0 = \text{Span}\{ \zeta_i^j, \eta_i^j : i = 0, 1, \ldots, \ell, \ j = 1, \ldots, m_i \}_A(0) \). So we obtain \( \omega_0(\zeta_i^j, \eta_i^j) = \delta_{ij}^d \) and \( \omega_0(\zeta_i^j, \zeta_j^i) = 0 = \omega_0(\eta_i^j, \eta_j^i) \). From linear algebra, this is sufficient to ensure that \( d_{R\alpha_0}|_{\ker \alpha_0} \) is non-degenerate in sense of Definition 2.19, and hence \( \alpha_0 \) is a \( k \)-shifted contact structure on \( \text{Spec} A \).

Therefore, one has a natural splitting
\[
\ll T_A|_{\text{Spec} H^0(A)} = \ker \alpha_0|_{\text{Spec} H^0(A)} \oplus \text{Rest}|_{\text{Spec} H^0(A)}, \tag{3.12}
\]
where for each homogeneous degree \( i \), \( (\ll T_A|_{\text{Spec} H^0(A)})^i = (\ker \alpha|_{\text{Spec} H^0(A)})^i \oplus \text{Rest}^i|_{\text{Spec} H^0(A)} \) so that we have
\[
\ker \alpha_0 = \text{Span}\{ \zeta_i^j, \eta_i^j : i = 0, 1, \ldots, \ell, \ j = 1, \ldots, m_i \}_A(0).
\]
\[
\text{Rest}|_{\text{Spec} H^0(A)} = (\partial/\partial x_j^{-1})_A(0).
\]

Note that we can find a vector field \( R \) such that \( i_R d_{R\alpha_0} = 0 \) and \( i_R \alpha_0 = 0 \) (i.e. \( R \in \text{Rest}|_{\text{Spec} H^0(A)} \) with scaling). Since \( R \notin \ker \alpha_0|_{\text{Spec} H^0(A)} \), we get \( i_R \phi = 0 \). Thus, \( i_R d_{R z} = -1 \) as \( i_R \alpha_0 = 1 \). It follows that \( d_{R z}, d_{R x_j^{-1}}, \ldots, d_{R y_j^{k+i}}, d_{R y_j^{k+i}} \) span \( \ll T_A|_{\text{Spec} H^0(A)} \) as well. (Otherwise, if \( d_{R z} \) was in the span of \( d_{R x_j^{-1}}, d_{R y_j^{k+i}} \), then \( i_R d_{R z} \) would vanish as \( R \in \text{Rest}|_{\text{Spec} H^0(A)} \).)

**Definition 3.7.** If \( A \) and \( \alpha_0 \) with the variables \( x_j^{-1}, y_j^{k+i}, z \) are as above, we then say \( A, \alpha_0 \) are in contact Darboux form.

**Remark 3.8.** Note that the expression in Equation (3.10) will still be valid for the other cases (a) \( k \equiv 0 \pmod 4 \), and (b) \( k \equiv 2 \pmod 4 \). Equations (2.24) – (2.27) show that the other cases in fact involve modified versions of \( H, d, \) and \( \phi \) with some possible extra terms. In any case, the modified \( A, \alpha_0 \) would also serve as the desired contact model. Following the same terminology as above, we would again say \( A, \alpha_0 \) are in (contact) Darboux form.

### 3.3 A Darboux-type theorem for shifted contact derived schemes

In what follows, we give the proof of Theorem 1.1, which essentially says that every \( k \)-shifted contact derived \( \mathbb{K} \)-scheme \( (X, \alpha) \) is locally equivalent to \( (\text{Spec} A, \alpha_0) \) for \( A \) a minimal standard form cdga and \( \alpha_0 \) as in Equation 3.6. More precisely, we have

**Theorem 3.9.** Let \( X \) be a (locally finitely presented) derived \( \mathbb{K} \)-scheme with a \( k \)-shifted contact form \( \alpha \) for \( k < 0 \), and \( x \in X \). Then there is a local contact model \((A, \alpha_0)\) and \( p \in \text{Spec} H^0(A) \) such that \( i : \text{Spec} A \hookrightarrow X \) is an open inclusion with \( i(p) = x \), \( A \) is a standard form cdga that is minimal at \( p \), and \( \alpha_0 \) is a \( 1 \)-shifted contact form on \( \text{Spec} A \) such that \( A, \alpha_0 \) are in a standard contact Darboux form, and \( i^*(\alpha) \sim \alpha_0 \) in the space of \( k \)-shifted 1-forms.

Note that for \( k < 0 \) odd, for instance, the pair \((A, \alpha_0)\) can be explicitly described by Equations (3.6) – (3.10). For the other cases, one should use another sets of variables as in Equations (2.24) and (2.25), and modify \( H, \phi, d \) accordingly.

**Proof.** Given \( k < 0 \) and \( x \in X \), apply Theorem 2.5 to get a refined local neighborhood \( U \simeq x, \text{Spec} A \) of \( x \) with \( p \in \text{Spec} H^0(A) \) such that \( i : \text{Spec} A \hookrightarrow X \) is an open inclusion, \( i(p) = x \), and \( A \) is a standard form cdga that is minimal at \( p \). It is in fact constructed inductively as described in (2.2) with \( A = A(-k) \). Then the restriction \( i^* \alpha \) is a \( k \)-shifted contact structure on \( \text{Spec} A \). We denote this restriction simply by \( \alpha_u \).

Consider the sequence \( \omega_u := (d_{R\alpha_0}, 0, 0, \ldots) \), which defines a closed \( k \)-shifted 2-form on \( \text{Spec} A \) in the sense of Definition 2.20. Applying Proposition 2.25 to \( k\omega_u \), we obtain elements \( H \in A^{k+1} \) and \( \phi \in (\Omega^1_A)^k \) such that \( dH = 0, d_{R\phi} + d\phi = 0 \), and \( k\omega_u \sim (d_{R\phi}, 0, 0, \ldots) \).
Notice that we in fact have $d_{dR}\phi = kd_{dR}\alpha_u$, because there is no non-trivial $\beta \in (A^2\Omega^1_A)^{k-1}$ satisfying the relation $kd_{dR}\alpha_u - d_{dR}\phi = d\beta$ due to degree reasons.

From Proposition 2.15 with $A = A(-k)$, the tangent complex $T_A|_{spec H^0(A)} = (\mathbb{L}_A|_{spec H^0(A)})^A$ is also represented by a complex of free finite rank $H^0(A)$-modules. For any $k$-shifted 1-form $\alpha$, the $k$-shifted 2-form $d_{dR}\alpha$ defines an induced map of complexes via $v \mapsto \iota_v d_{dR}\alpha$

\[
T_A|_{spec H^0(A)} : \quad 0 \rightarrow (V^0)^* \rightarrow (V^{-1})^* \rightarrow \cdots \rightarrow (V^{k+1})^* \rightarrow (V^k)^* \rightarrow 0
\]
\[
L_A|_{spec H^0(A)} : \quad 0 \rightarrow V^k \rightarrow V^{k+1} \rightarrow \cdots \rightarrow V^{-1} \rightarrow V^0 \rightarrow 0,
\]

(3.13)

where both horizontal differentials $d^i, (d^i)^*$ are zero at $p \in spec H^0(A)$ due to the minimality of $A$.

By the contactness condition, $d_{dR}\alpha_u$ is non-degenerate on the subcomplex $\ker \alpha|_{spec H^0(A)}$ of $T_A|_{spec H^0(A)}$. Therefore, one has a natural splitting

\[
T_A|_{spec H^0(A)} = \ker \alpha|_{spec H^0(A)} \oplus Rest|_{spec H^0(A)}.
\]

(3.14)

Write $W$ for the dual subcomplex of $\ker \alpha|_{spec H^0(A)}$ in $L_A|_{spec H^0(A)}$, i.e. $W := (\ker \alpha|_{spec H^0(A)})^*$, then we have the commutative diagram

\[
W^* \subset T_A|_{spec H^0(A)} : \quad 0 \rightarrow (W^0)^* \rightarrow (W^{-1})^* \rightarrow \cdots \rightarrow (W^{k+1})^* \rightarrow (W^k)^* \rightarrow 0
\]
\[
\downarrow d_{dR}\alpha_u \quad \downarrow \quad \downarrow \quad \cdots \quad \downarrow \quad \downarrow
\]
\[
W \subset L_A|_{spec H^0(A)} : \quad 0 \rightarrow W^k \rightarrow W^{k+1} \rightarrow \cdots \rightarrow W^{-1} \rightarrow W^0 \rightarrow 0
\]

(3.15)

such that the vertical maps $d_{dR}\alpha_u : (W^{k+i})^* \rightarrow W^{-i}$, $v \mapsto \iota_v d_{dR}\alpha_u$, are all quasi-isomorphisms.

**Observation 3.10.** As both horizontal differentials $d^i, (d^i)^*$ are zero at $p \in spec H^0(A)$ because of the minimality of $A$, the vertical maps are isomorphisms at $p$, and hence isomorphisms in a neighborhood of $p$. By localizing $A$ at $p$ if needed, we may assume that the vertical maps are all isomorphisms.

**When $k$ is odd.** We now focus on a particular and the simplest case: $k$ is odd. Let $k = -2\ell - 1$ for $\ell \in \mathbb{N}$. Localizing $A$ at $p$ if necessary, first choose degree 0 variables $x_1^0, x_2^0, \ldots, x_m^0, x_1^1$ in $A(0)$ such that $\{d_{dR}x_j^0 : j = 1, \ldots, m_0\}$ forms a basis for $W^0$ over $A(0)$, and $\{d_{dR}x_j^1\}$ forms a $A(0)$-basis for $(Rest^*)^0$.

Next, for $i = 1, \ldots, \ell$, choose $x_1^{-i}, x_2^{-i}, \ldots, x_m^{-i} \in A^{-i}$ such that $d_{dR}x_1^{-i}, \ldots, d_{dR}x_m^{-i}$ form a basis of $W^{-i}$ over $A(0)$, and $(Rest^*)^{-i}$ is trivial over $A(0)$.



Now, by the isomorphism $d_{dR}\alpha_u : (W^{k+i})^* \rightarrow W^{-i}$, we have $H^{-i}(L_A|_W) \cong H^{k+i}(L_A|_W)^*$, and hence $\dim H^{-i}(L_A|_W) = \dim H^{k+i}(L_A|_W)^*$. It follows that $A$ is free over $A(0)$ with $m_i$ generators in degree $-i$ for $i = 1, \ldots, \ell$, and $m_0$ generators in degree $k+i$ for $i = 0, \ldots, \ell$.

Then choose $y_1^{k+i}, y_2^{k+i}, \ldots, y_m^{k+i} \in A^{k+i}$ such that $\{d_{dR}y_j^{k+i} : j = 1, 2, \ldots, m_i\}$ is a basis for $W^{k+i}$ over $A(0)$ which is dual to the basis $\{d_{dR}x_1^{-i}, \ldots, d_{dR}x_m^{-i}\}$ over $A(0)$. That is, using local coordinates above, for $i = 0, 1, \ldots, \ell$,

\[
W^{-i} = (d_{dR}x_1^{-i}, d_{dR}x_2^{-i}, \ldots, d_{dR}x_m^{-i})_{A(0)},
\]
\[
W^{k+i} = (d_{dR}y_1^{k+i}, d_{dR}y_2^{k+i}, \ldots, d_{dR}y_m^{k+i})_{A(0)}.
\]

By Observation 3.10, the isomorphisms in Diagram (3.15) imply that for $i = 0, 1, \ldots, \ell$, we have

\[
(W^{-i})^* = \{\partial/\partial x_1^{-i}, \ldots, \partial/\partial x_m^{-i}\}_{A(0)} \cong \{d_{dR}y_1^{k+i}, \ldots, d_{dR}y_m^{k+i}\}_{A(0)},
\]

(3.16)

\[
(W^{k+i})^* = \{\partial/\partial y_1^{k+i}, \ldots, \partial/\partial y_m^{k+i}\}_{A(0)} \cong \{d_{dR}x_1^{-i}, \ldots, d_{dR}x_m^{-i}\}_{A(0)}.
\]

(3.17)

Then, by the splitting in Equation (3.14), we have

\[
\ker \alpha|_{spec H^0(A)} = \{\partial/\partial x_1^{-i}, \ldots, \partial/\partial x_m^{-i}, \partial/\partial y_1^{k+i}, \ldots, \partial/\partial y_m^{k+i} : i = 0, 1, \ldots, \ell\}_{A(0)}^*.
\]

\[
Rest|_{spec H^0(A)} = \{\partial/\partial x_j^0\}_{A(0)}^*.
\]
Here $\text{Rest}_{|_{\text{spec}H^0(A)}}$ is a subcomplex of $T_{\alpha}|_{\text{spec}H^0(A)}$ that is concentrated in degree 0. Moreover, we can choose a vector field $R \in \text{Rest}_{|_{\text{spec}H^0(A)}}$ of degree 0, up to scaling, such that $i_R \alpha_u = 1$. Note that, in this case, we have $i_R dR \alpha_u = 0$.

$A$ is now identified with the standard form cdga over $A(0)$ freely generated by the variables $\tilde{x}_0, \tilde{y}_1^k$, as in Example 3.6. We also impose suitable differential $d$ as before: $d$ acts on $\tilde{x}_0, \tilde{x}_1^k, \tilde{y}_1^k$ as in Equation (2.23). Note in particular that $d\tilde{x}_0 = 0$ as $\tilde{x}_0 \in A^0$.

The non-degeneracy condition on $dR \alpha_u|_{\ker \alpha}$ sending dual basis of $dR \tilde{x}_0, \ldots, dR \tilde{x}_i$ to the basis $dR \tilde{y}_1^k, \ldots, dR \tilde{y}_m^k$ (and vice versa) as in Equations (3.16) and (3.17) implies that

$$dR \alpha_u|_{\ker \alpha_u} = \sum_{i=0}^\ell \sum_{j=1}^{m_i} dR \tilde{x}_j^i dR \tilde{y}_j^k.$$  \hfill (3.18)

Since $dR \phi = k dR \alpha_u$, we may take $k \alpha_u = dR \theta + \phi$ for $\theta \in A^k$. Modifying Equation (2.22), we may explicitly have $\phi = k \sum \sum y_j^k dR \tilde{x}_j^i$ using the coordinates above.

Observe that since $R \notin \ker \alpha_u|_{\text{spec}H^0(A)}$, we get $i_R \phi = 0$. Thus, $i_R dR \theta = k < 0$ as $i_R \alpha_u = 1$.

It follows that $dR \theta, dR \tilde{x}_i^j$, $dR \tilde{y}_1^k$, $dR \tilde{y}_m^k$ span $L_\alpha|_{\text{spec}H^0(A)}$. (Otherwise, if $dR \theta$ was in $A(0)$-span of $dR \tilde{x}_i^j$, $dR \tilde{y}_1^k$, $dR \tilde{y}_m^k$, then $i_R dR \theta$ would vanish as $R \in \langle \partial/\partial \tilde{x}_1^0 \rangle_{A(0)}$.)

Note that the Hamiltonian $H$ is $d$-closed (as it satisfies the classical master equation). Now, localizing $A$ at $p$ if necessary, choose an element $z \in A^k$ such that $dz = \frac{1}{k} H$ (cf. Observation 2.26). Then replace $\theta$ by $-kz$ and write

$$\alpha_u = -dR z + \sum_{i=0}^\ell \sum_{j=1}^{m_i} y_j^k dR \tilde{x}_j^i.$$  

It is now straightforward to check that $\alpha_u$ is $d$-closed, and hence a 1-form of degree $k$, such that $dR \alpha_u|_{\ker \alpha_u}$ is non-degenerate. Therefore, the graded variables $x_i^k, y_j^k, z$ on $U$ serves as the desired local contact Darboux coordinates.

**When $k$ is not odd.** For the other cases (a) $k \equiv 0 \mod 4$, and (b) $k \equiv 2 \mod 4$, one should use another sets of variables as in Equations (2.24) and (2.25), respectively, and modify $H, \phi, d$ as in Equations (2.24) − (2.27).

**Observation 3.11.** Denote by $B^0$ the subalgebra of $A^0$ with basis $x_0, x_2^0, \ldots, x_{m_0}^0$. Then we define a sub-cdga $B$ of $A$ to be the free algebra over $B^0$ on generators $x_i^k, y_j^k$ only, with inclusion $\iota: B \rightarrow A$. Observe that the elements $\phi, \omega_B := dR \alpha_u|_{\ker \alpha_u}$ are all images under $\iota$ of the elements $\phi_B, \omega_B := \sum_{i=0}^\ell \sum_{j=1}^{m_i} dR \tilde{x}_j^i dR \tilde{y}_j^k$, respectively. As in Section 2.3, $B$ is a minimal standard form cdga which in fact serves as a local symplectic model (for $k < 0$ odd). As noted before, similar local models can be explicitly obtained for the other cases using Equations (2.24) − (2.27).

In any case, suppose that we construct such $(A, B)$ for $k < 0$, then the virtual dimension $\text{vdim } B$ is always even, and hence the virtual dimension $\text{vdim } A = \text{vdim } B + 1$ is odd. In fact, if a cdga $A$ is in contact Darboux form, we have

$$\text{vdim } A = \begin{cases} 1 + 2 \sum_i (-1)^i m_i, & k \text{ even} \\ 1, & k \text{ odd.} \end{cases}$$

**4 Symplectization of a shifted contact derived scheme**

In this section, we give the formal description of the so-called symplectization of a $k$-shifted contact derived $\mathbb{K}$-scheme.

Let $X$ be a locally finitely presented derived $\mathbb{K}$-scheme with a $k$-shifted contact structure for $k < 0$. By Theorem 2.5, we always have a nice local model with respect to which $X$ is locally equivalent to $\text{Spec } A$ for $A$ a minimal standard form cdga. As before, it suffices to work with the affine case.

Now, to each $k$-shifted contact derived $\mathbb{K}$-scheme $X \simeq \text{Spec } A$, we associate the space of all $k$-shifted contact forms on $X$. Denote this space by $S_X(k)$. That is,

$$S_X(k) = \{ (p, \alpha) : p \in \text{spec } H^0(A), \alpha|_{\text{spec } H^0(A)} \in L_\alpha|_{\text{spec } H^0(A)} \text{ a } k\text{-shifted contact form} \}.$$
Then we have the following result:

**Theorem 4.1.** The space $S_N(k) \times Spec K$ has the structure of a $k$-shifted symplectic derived $K$-scheme with a symplectic form $\omega_N$ which is canonically determined by the shifted contact structure of $X$ (up to quasi-isomorphism).

**Definition 4.2.** We call the pair $(S_N(k), \omega_N)$ above the symplectization of $X$.

**Proof of Theorem 4.1.** The assertion of the theorem is local, so it is enough to prove it using suitable local models studied in the previous sections.

Given $k < 0$ and $x \in X$, apply Theorem 2.5 to get a refined local neighborhood $U \simeq \text{Spec} A$ of $x$ with $p \in \text{Spec} H^0(A)$ such that $i : \text{Spec} A \hookrightarrow X$ is an open inclusion, $i(p) = x$, and $A$ is a minimal standard form cdga.

Recall from Observation 3.5 that for any $f \neq 0$ in $A^0$ and any $k$-shifted contact form $\alpha$, $\ker \alpha \simeq \ker (f \cdot \alpha)$. Hence, both $\alpha$ and $f \cdot \alpha$ define equivalent contact structures on $X$ up to quasi-isomorphism. Therefore, localizing $A$ at $p$ if necessary, we can identify the space $S_N(k)$ locally with the product space of affine $K$-schemes

$$\text{Spec} H^0(A) \times_{\text{Spec} K} \text{Spec} K[f],$$

(4.1)

along with the natural projections. Let $\pi : \text{Spec} H^0(A) \times_{\text{Spec} K} \text{Spec} K[f] \rightarrow \text{Spec} H^0(A)$ be the projection defined by $(p, T) \mapsto p$. This product space with the projection morphisms fits into the fiber product diagram, and hence we have

$$\text{Spec} H^0(A) \times_{\text{Spec} K} \text{Spec} K[f] \simeq \text{Spec} (H^0(A) \otimes_K K[f]).$$

(4.2)

Write $Y := \text{Spec} H^0(A) \times_{\text{Spec} K} \text{Spec} K[f]$, $B := \text{Spec} K[f]$, and $C := H^0(A) \otimes_K K[f]$ such that $Y = \text{Spec} C$. Then it follows from [10, Lemma 91.28.1] that $L_{H^0(A) \otimes K B}$ can be represented by a complex of free $(H^0(A) \otimes_K K[f])$-modules

$$(L_A)_{\text{Spec} H^0(A) \otimes K B} \oplus (H^0(A) \otimes_K L_B).$$

(4.3)

For the rest of the proof, we will assume that $k$ is odd, say $k = -2\ell - 1$ for $\ell \in \mathbb{N}$. In fact, our constructions will be independent of $k$ and the corresponding local graded variables.

Localizing $A$ at $p$ if necessary, choose the graded variables $x_j^{-i}, y_i^{+k+i}$ on $U$ as before so that $A$ is a standard form cdga over $A(0)$ freely generated by these graded variables, and we have

$$\ker \alpha|_{\text{Spec} H^0(A)} = \{ \partial/\partial x_i^{-i}, \ldots, \partial/\partial x_i^{-m}, \partial/\partial y_1^{k+i}, \ldots, \partial/\partial y_{m+i}^{k+i} : i = 0, 1, \ldots, \ell \}_A(0),$$

$$R \ker \alpha|_{\text{Spec} H^0(A)} = \{ \partial/\partial x_i^{0} \}_{\text{Spec} H^0(A)}.$$  

Observation 4.3. As $f \in A^0$ is non-zero and $L_A = \Omega^1_A$ is an $A$-module, the first summand of Equation (4.3) can be equivalently written as

$$L_A|_{\text{Spec} H^0(A) \otimes K B} \simeq \langle dR x_j^{-i}, dR y_j^{k+i}, dR x_i^0 \rangle_A \otimes_K K[f] \simeq \langle dR x_j^{-i}, dR y_j^{k+i}, dR x_i^0 \rangle_A.$$  

Using a cofibrant replacement of $B$ if necessary, we may assume that the second summand of Equation (4.3) is just equivalent to $H^0(A) \otimes_K \langle dR f \rangle_B$.

Now, using the natural splitting 3.14 and Observation 4.3 for the complex in Equation (4.3), we then have

$$(L_C)^{\vee} \simeq \{ \ker \alpha|_{\text{Spec} H^0(A)} \oplus \text{Rest}|_{\text{Spec} H^0(A)} \} \oplus (H^0(A) \otimes_K \langle \partial/\partial f \rangle_B).$$

(4.4)

Define an element $\beta$ in $L_C$, called the canonical 1-form, by

$$\beta = f \cdot \pi^* \alpha.$$  

Here $\pi^* \alpha$ denotes the pullback of the shifted 1-form $\alpha$, which is defined as follows: For the corresponding morphism of algebras $\pi^* : H^0(A) \rightarrow C$, there exists a canonical sequence (a distinguished triangle)

$$L_{H^0(A) \otimes H^0(A)} \xrightarrow{\pi^* \pi} L_C \rightarrow L_{H^0(A) \otimes H^0(A)} \rightarrow L_{H^0(A) \otimes H^0(A)} C \xrightarrow{1]}.$$

(4.5)
Proposition 4.4. The $k$-shifted 2-form $\omega^0 := d_{dR}\beta$ in $(\Lambda^2 L_C)^k$ induces a non-degenerate map, and hence the sequence $\omega_X = (\omega^0, 0, 0, \ldots)$ defines a $k$-shifted symplectic structure on $Y$.

Proof. Note first that the non-degeneracy in sense of Definition 2.19 can be equivalently formulated as follows:

Observation 4.5. A $k$-shifted 2-form $\gamma$ on $\text{Spec} A$ for $A$ a minimal standard form cdga is non-degenerate if and only if for any non-zero vector $v \in T_{A|\text{Spec} A}$, there exists a non-zero vector $w \in T_{A|\text{Spec} A}$ such that $\iota_w v \gamma \neq 0$.

Let us give a sketch for Observation 4.5: The induced morphism $T_A \to \Omega^1_{A|k}$, $Y \mapsto \iota_Y \gamma$, in Definition 2.19 is just an analogous result from linear algebra. Furthermore, $\text{Spec} A$ is just an analogous result from linear algebra.

Now, to prove that $\omega^0 := d_{dR}\beta$ is non-degenerate, it suffices to show that for any non-vanishing (homogeneous) vector field $\sigma \in (L_C)^\vee$, there is a vector field $\eta \in (L_C)^\vee$ such that $\iota_\sigma (d_{dR}\beta) = 0$.

Using the splitting in Equation (4.4), we prove the statement case by case. To this end, we first note that for any $\eta, \sigma \in (L_C)^\vee$, direct computations give

$$\iota_\sigma (d_{dR}\beta) = \tau (d_{dR}f)(\sigma) \alpha (\pi, \pi, \eta) = (d_{dR}f)(\eta) \alpha (\pi, \sigma) \neq d_{dR}\alpha (\pi, \pi, \eta).$$

From Equation (4.4), it is enough to work with the following cases.

1. If $\sigma \in \ker \alpha$, then we have $\iota_\sigma (d_{dR}\beta) = \tau (d_{dR}f)(\sigma) \alpha (\pi, \pi, \eta)$. Since $d_{dR}\alpha|_{\ker \alpha}$ is non-degenerate by the contactness condition on $\alpha$, it is enough to take $\eta$ to be any non-zero vector in $\ker \alpha$.

2. If $\sigma \in \text{Rest}$, then we get $\iota_\sigma (d_{dR}\beta) = \tau (d_{dR}f)(\eta) \alpha (\sigma)$. Here $\alpha (\sigma) \neq 0$ since $\sigma \in \text{Rest}$. Thus, it is enough to take $\eta$ to be any non-zero vector in $H^0(A) \otimes \langle \partial/\partial f \rangle_B$ so that $(d_{dR}f)(\eta) \neq 0$.

3. If $\sigma \in H^0(A) \otimes \langle \partial/\partial f \rangle_B$, then $\iota_\sigma (d_{dR}\beta) = \tau (d_{dR}f)(\sigma) \alpha (\pi, \pi, \eta)$. Observe that $(d_{dR}f)(\sigma) \neq 0$, so it suffices to take $\eta$ to be any non-zero vector in $\text{Rest}$ so that $\alpha (\pi, \pi, \eta) \neq 0$.

This completes the proof of Proposition 4.4, and hence that of Theorem 4.1.

Remark 4.6. The proofs of Proposition 4.4 and Theorem 4.1 will still be valid for the other values of $k$. In fact, it clear to see that coordinates do not play any significant role in the proofs, just providing explicit representations for the splitting.

In short, Proposition 4.4 has indeed a coordinate-free proof, and so does Theorem 4.1. Thus, using the same terminology as before, we say that the pair $(S_Y(k), \omega_X)$ above is the symplectization of the $k$-shifted contact derived $\mathbb{K}$-scheme $X$ for any $k < 0$. Note also that this construction is in fact canonical up to quasi-isomorphism due to Observation 3.5.

5 Concluding remarks

We conclude this paper with the following remark on the possible generalizations of the results presented in this paper.

Remark 5.1. It should be noted that “stacky” generalizations of the results in [5] are also available in the literature. Ben-Bassat, Brav, Bussi and Joyce [7] extend the results of [5] from derived schemes to the case of derived Artin $\mathbb{K}$-stacks. In short, Ben-Bassat, Brav, Bussi and Joyce [7] proved that derived Artin $\mathbb{K}$-stacks also have nice local models in some sense. Parts of the results from [7, Theorems 2.8 & 2.9] in fact give the generalization of Theorem 2.5 to the case of derived Artin $\mathbb{K}$-stacks. Regarding shifted symplectic structures, they also proved that every shifted symplectic derived Artin $\mathbb{K}$-stack admits the so-called “Darboux form atlas” [7, Theorem 2.10]. That is, their result extends Theorem 2.24 from derived $\mathbb{K}$-schemes to derived Artin $\mathbb{K}$-stacks.

In the sequel(s), our goals will be to extend the main results of this paper from derived schemes to the more general case of derived Artin stacks and to discuss more on the local theory of shifted contact derived spaces. In that respect, we propose

Conjecture 5.2. Theorem 1.1 and Theorem 1.2 still hold for (locally finitely presented) $k$-shifted contact derived Artin $\mathbb{K}$-stacks with $k < 0$.\]
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