A compact manifold with holonomy $\text{Spin}(7)$ from Beauville’s Calabi–Yau fourfold

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ABSTRACT: We give a new example of a compact manifold with holonomy Spin(7) from a Beauville’s Calabi–Yau fourfold. Its construction is very concrete, starting with products of elliptic curves with complex multiplications — so probably more accessible to physicists.

KEYWORDS: $\text{Spin}(7)$-manifold, F-theory, M-theory, Calabi–Yau fourfold.
1. Introduction

A Spin(7)-manifold is an eight-dimensional Riemannian manifold with the exceptional holonomy group Spin(7). The spin group Spin(7) is one of special holonomy groups in Berger’s classification [1]. Riemannian manifolds with special holonomy play an important role in string theory. Indeed consideration of string theory compactification on Spin(7) manifolds was proposed by Witten [2, 3] and Vafa [4] two decades ago. Various aspects of string theory on Spin(7)-manifolds have been investigated [5–10]. Recently more concrete F-theory approach has been made in [11] and [12], where the authors used examples of compact Spin(7)-manifolds that are constructed as quotients of Calabi-Yau fourfolds by Joyce’s method [13].

The construction of Spin(7)-manifolds had been an unsolved problem for a long time. Joyce constructed first compact examples [14]. Later he gave another method and constructed further examples, starting from some complete intersections in weighted projective spaces [13]. Following his lead, Clancy, one of his students, systematically investigated hypersurfaces in weighted projective spaces and constructed more examples [15]. The Betti numbers of compact manifolds with holonomy Spin(7), constructed by them [13–15], are

$$0 \leq b^2 \leq 9, \ 0 \leq b^3 \leq 33, \ 200 \leq b^4 \leq 15118,$$

In Joyce’s method, one starts with certain orbifolds, whose resolutions singularities are Calabi-Yau fourfolds. Since Calabi-Yau fourfolds are projective varieties, it is basically a task in algebraic geometry to find Calabi-Yau fourfolds suitable for Joyce’s method. One can easily find a huge number of examples of Calabi–Yau fourfolds as complete intersection of toric varieties. The main issues are whether they have suitable singularities and whether they admit antiholomorphic involutions
satisfying certain conditions. Joyce and Clancy considered complete intersections in weighted projective spaces whose antiholomorphic involutions come from those of the ambient weighted projective spaces.

In this note, we apply Joyce's method to a Calabi–Yau fourfold that is not a complete intersection in a weighted projective space nor more generally a complete intersection in a toric variety. This Calabi-Yau fourfold was originally constructed by Beaubille [16] and is the only one that can be constructed by quotienting an abelian fourfold in a way that Kummer $K3$ surfaces are constructed. It also has rich structure of elliptic fibrations. As a result, we give a new example of compact Spin(7)-manifolds. We calculate the Betti numbers of the example, which are:

\[ b^2 = 10, \ b^3 = 30, \ b^4 = 52. \]

It is notable that the Betti number $b^4$ of the example is significantly smaller than those of other examples already constructed. The construction is very concrete, starting with products of elliptic curves with complex multiplications — so probably more accessible to physicists.

2. Joyce construction from Calabi–Yau 4-orbifolds

Joyce started with orbifolds with certain conditions [13]. However it is not hard to show that those orbifolds are projective. So let us just start with projective varieties. Let $Y$ be a 4-dimensional projective varieties satisfying the following conditions.

**Condition 2.1.**

1. Each of the singularities of $Y$ is locally isomorphic to the origin of the quotient $\mathbb{C}^4/\langle \sqrt{-1}^* \rangle$, where $\sqrt{-1}^*$ acts as the complex multiplication by $\sqrt{-1}$ on $\mathbb{C}^4$. Let $p_1, p_2, \ldots, p_k$ $(k \geq 1)$ be all the singularities of $Y$.

2. There is an antiholomorphic involution $\rho$ on $Y$ whose fixed points are $p_1, p_2, \ldots, p_k$.

3. $Y - \{p_1, p_2, \ldots, p_k\}$ is simply-connected.

4. Let $\tilde{Y} \rightarrow Y$ be the blow-up at the singularities of $Y$. Then $\tilde{Y}$ is a Calabi–Yau fourfold, i.e. a smooth projective fourfold with trivial canonical class and $h^1(\mathcal{O}_{\tilde{Y}}) = h^2(\mathcal{O}_{\tilde{Y}}) = 0$.

The final condition may look different from Joyce’s original one. However noting the singularities in the condition are crepant, they are actually not different.

Let us consider the quotient $Z = Y/\langle \rho \rangle$. Joyce found a way of resolving singularities of $Z$ so that the resulted 8-manifolds admits a Riemannian metric whose holonomy group is Spin(7) [13].

**Theorem 2.2** (D. Joyce). There is a simply-connected compact 8-manifold $M$ (defined in [13], Definition 5.8) which is a resolution of singularities $Z = Y/\langle \rho \rangle$ and admits a Riemannian metric whose holonomy group is Spin(7).
The Betti numbers of $M$ can be calculated from topological invariants of $Y$ and $Z$ as follows (Proposition 10, [15]):

\begin{align}
    b^2(M) &= b^2(Z), \\
    b^3(M) &= \frac{1}{2} b^3(Y), \\
    b^4(M) &= \frac{1}{2} h^{2,2}(Y) + h^{3,1}(Y) - 2b^2(Z) + \frac{3}{2} k.
\end{align}

(2.1) 
(2.2) 
(2.3)

3. Beauville’s Calabi–Yau fourfold

Let $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\sqrt{-1})$ be the elliptic curve with period $\sqrt{-1}$ and let $Y = E^4/(\sqrt{-1}*)$ be the quotient fourfold of the product manifold $E^4$ by scalar multiplication by $\sqrt{-1}$. Then $Y$ has finitely many singularities. Let $X \to Y$ be blow-up at those singularities. Beauville observed that $X$ is a simply-connected Calabi–Yau fourfold with Hodge number $h^{1,3} = 0$ ([16], page 5).

The fixed points of $E$ by scalar multiplication by $\sqrt{-1}$ are

$$0, \alpha = \frac{1+\sqrt{-1}}{2}.$$ 

Therefore $Y$ has $2^4 = 16$ singularities and it is easy to see that they are locally isomorphic to the origin of $\mathbb{C}^4/(\sqrt{-1}*)$.

It seems that its Hodge numbers have not been calculated yet. So let us determine all other Hodge numbers of $X$. Firstly

$$h^{0,0} = 1, h^{1,0} = h^{2,0} = h^{3,0} = 0, h^{4,0} = 1.$$ 

The Hodge diamond of $X$ is

\[
\begin{array}{cccc}
1 & & & \\
0 & h^{3,3} & 0 & \\
0 & h^{2,3} & h^{3,2} & 0 \\
1 & h^{1,3} & h^{2,2} & h^{3,1} & 1 \\
0 & h^{1,2} & h^{2,1} & 0 & \\
0 & h^{1,1} & 0 & \\
0 & 0 & \\
1 & & & \\
\end{array}
\]

with

$$h^{1,1} = h^{3,3}, h^{3,1} = h^{1,3}, h^{1,2} = h^{2,1} = h^{3,2} = h^{2,3}.$$
Let $\chi_q = \sum_{p=1}^{4}(-1)^p h^{p,q}$, then by the well-known Riemann–Roch theorem, we have

\[
\begin{align*}
\chi_0 &= \frac{1}{720}(-c_4 + c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1^4) \\
\chi_1 &= \frac{1}{180}(-31c_4 - 14c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1^4) \\
\chi_2 &= \frac{1}{120}(79c_4 - 19c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1),
\end{align*}
\]

where $c_i$ is the $i$th Chern class of $X$. Since $X$ is a Calabi–Yau fourfold, $c_1 = 0$ and $\chi_0 = 2$. With this, we have following relations in Hodge numbers:

\[
h^{2,2} = 2(22 + 2h^{1,1} + 2h^{3,1} - h^{2,1}).
\]

So remaining independent Hodge numbers are $h^{1,1}$ and $h^{1,2}$. The topological Euler number of $X$ has the relation:

\[
e(X) = 6(8 + h^{1,1} + h^{3,1} - h^{1,2}). \tag{3.1}
\]

Let $\widetilde{E}^4 \to E^4$ be the blow-up at $\{0, \alpha\}^4$ and $F_{ijkl}$ be the exceptional divisors over $(e_i, e_j, e_k, e_l)$ for $i, j = 0, 1$, where $e_0 = 0$ and $e_1 = \alpha$. Then there is a quadruple covering map $X \to \widetilde{E}^4$, branched along $F_{ijkl}$'s and we have the following commutative diagram:

\[
\begin{array}{ccc}
X & \longrightarrow & \widetilde{E}^4 \\
\downarrow & & \downarrow \\
Y & \longrightarrow & E^4
\end{array}
\]

The topological Euler numbers are

\[
e(E^4) = 0, \\
e(\widetilde{E}^4) = e(E^4) + \sum_{i,j,k,l} (e(F_{ijkl}) - 1) = 2^4 \cdot 3,
\]

\[
4e(X) - 3 \cdot \sum_{i,j,k,l} e(F_{ijkl}) = e(\widetilde{E}^4).
\]

So we have

\[
e(X) = 60.
\]

Hence Equation 3.1 becomes

\[
60 = 6(8 + h^{1,1} + 0 - h^{1,2}).
\]
Therefore
\[ h^{1,1} - h^{1,2} = 2. \] 

(3.2)

On the other hand, note ( [17], page 21)
\[ h^{1,1}(E^4) = 4^2 = 16. \] 

(3.3)

Now let us find generators of \( H^{1,1}(E^4) \). Let \( \pi_i : E^4 \to E \) be the \( i \)-the projection. Consider following 16 divisors (denoted by \( b_i \)'s) of \( E^4 \):

- \( \beta_i = \ker \pi_i \) (\( = b_i \)) for \( i = 1, 2, 3, 4 \).
- \( \gamma_{ij} = \ker(\pi_i + \pi_j) \) for \( 1 \leq i < j \leq 4 \). Let \( b_5 = \gamma_{12}, b_6 = \gamma_{13}, b_7 = \gamma_{14}, b_8 = \gamma_{23}, b_9 = \gamma_{24}, b_{10} = \gamma_{34} \).
- \( \delta_{ij} = \ker(\pi_i + \sqrt{-1} \pi_j) \) for \( 1 \leq i < j \leq 4 \). Let \( b_{11} = \delta_{12}, b_{12} = \delta_{13}, b_{13} = \delta_{14}, b_{14} = \delta_{23}, b_{15} = \delta_{24}, b_{16} = \delta_{34} \).

These \( b_i \)'s can be regarded as elements of \( H^{1,1}(E^4) \).

Now consider 16 elements (denoted by \( c_j \)'s) of \( H^{3,3}(E^3) \):

- \( \beta_i \cdot \beta_j \cdot \beta_k \) for \( 1 \leq i < j < k \leq 4 \). Let \( c_1 = \beta_1 \cdot \beta_2 \cdot \beta_3, c_2 = \beta_1 \cdot \beta_2 \cdot \beta_4, c_3 = \beta_1 \cdot \beta_3 \cdot \beta_4, c_4 = \beta_2 \cdot \beta_3 \cdot \beta_4 \).
- \( \gamma_{ij} \cdot \beta_k \cdot \beta_l \) for \( 1 \leq i < j \leq 4, 1 \leq k < l \leq 4 \) and \( \{i,j\} \neq \{k,l\} \). Let \( c_5 = \gamma_{12} \cdot \beta_3 \cdot \beta_4, c_6 = \gamma_{13} \cdot \beta_2 \cdot \beta_4, c_7 = \gamma_{14} \cdot \beta_2 \cdot \beta_3, c_8 = \gamma_{23} \cdot \beta_1 \cdot \beta_4, c_9 = \gamma_{24} \cdot \beta_1 \cdot \beta_3, c_{10} = \gamma_{34} \cdot \beta_1 \cdot \beta_2 \).
- \( \delta_{ij} \cdot \beta_k \cdot \beta_l \) for \( 1 \leq i < j \leq 4, 1 \leq k < l \leq 4 \) and \( \{i,j\} \neq \{k,l\} \). Let \( c_{11} = \delta_{12} \cdot \beta_3 \cdot \beta_4, c_{12} = \delta_{13} \cdot \beta_2 \cdot \beta_4, c_{13} = \delta_{14} \cdot \beta_2 \cdot \beta_3, c_{14} = \delta_{23} \cdot \beta_1 \cdot \beta_4, c_{15} = \delta_{24} \cdot \beta_1 \cdot \beta_3, c_{16} = \delta_{34} \cdot \beta_1 \cdot \beta_2 \).

where \( \cdot \) is the cup product. Let \( N \) be the \( 16 \times 16 \) intersection matrix of \( b_i \)'s and \( c_j \)'s (i.e. \( N_{ij} = b_i \cdot c_j \)). Then \( N \) is as follows:
The rank of the matrix $N$ is 16, which means that $b_i$’s and $c_j$’s are linearly independent respectively. Since $h^{1,1}(E^4) = h^{3,3}(E^4) = 16$ (Equation 3.3), they form bases of $H^{1,1}(E^4)$ and $H^{3,3}(E^4)$ respectively. Moreover $b_i$’ and $c_j$’s are all invariant under the scalar multiplication by $\sqrt{-1}$. Since elements of $H^{1,1}(Y)$ come from cycles in $H^{1,1}(E^4)$ that are invariant under the scalar multiplication by $\sqrt{-1}$, we conclude that $h^{1,1}(Y) = 16$ and accordingly  

$$h^{1,1} = h^{1,1}(Y) + 16 = 32.$$  

By Equation 3.2, we have $h^{1,2} = 30$ and  

$$h^{2,2} = 2(22 + 64 - 30) = 112.$$  

4. An example of compact manifold with holonomy Spin(7)  

Now let us find a suitable antiholomorphic involution on $Y$. Let $c : E^4 \to E^4$ be the standard complex conjugation, i.e.  

$$c : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \mapsto \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{pmatrix}$$  

and $A$ be a $4 \times 4$ matrix which takes its entries from $\mathbb{Z}[\sqrt{-1}]$. Note that $A$ induces a holomorphic map $\hat{A} : E^4 \to E^4$. Let $\psi_A = \hat{A} \circ c$, then it is an antiholomorphic map.
and \( \psi_A \) also induces an antiholomorphic map \( \phi_A : Y \to Y \). Note
\[
\psi_A^2 = \hat{A} \circ c \circ \hat{A} \circ c = A \circ \hat{A} = \hat{A} A,
\]
where \( \hat{A} \) is the \( 4 \times 4 \) matrix whose entries are complex conjugations of those of \( A \).
So \( \phi_A \) is an involution if and only if \( A \hat{A} = I, -I, \sqrt{-1}I \) or \(-\sqrt{-1}I \), where \( I \) is the \( 4 \times 4 \) identity matrix. Let
\[
A = \begin{pmatrix}
-1 & 1 + \sqrt{-1} & 0 & 0 \\
-1 - \sqrt{-1} & \sqrt{-1} & 0 & 0 \\
0 & 0 & -1 & 1 + \sqrt{-1} \\
0 & 0 & -1 - \sqrt{-1} & \sqrt{-1}
\end{pmatrix}
\]

Then \( A \hat{A} = -I \). So \( \phi_A \) is an antiholomorphic involution of \( Y \).

By direct calculation, one can show that fixed points of \( \phi_A \) are exactly the singularities of \( Y \). So we can apply Joyce's method to \( Y \) with the antiholomorphic involution \( \phi_A \) to get a compact Spin(7)-manifold \( A \). In order to determine the Betti numbers of \( M \), we need to calculate the Betti number \( b^2(Z) \).

By calculating the cup product numbers \( c_i \cdot \psi_A(b_j)'s \), one can find the \( 16 \times 16 \) matrix that represents how \( \psi_A \) works on \( H^{1,1}(E^2) \) with respect to the basis \( \{ b_i \}'s \) and it is:

\[
\begin{pmatrix}
-1 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & -2 & 0 & 1 & -1 & 1 & -1 & 2 & -1 & 1 & -1 & 1 & -1 & 0 & 1 & -1 \\
0 & -4 & -4 & 0 & 1 & -1 & 0 & 2 & -1 & 1 & -1 & 1 & -1 & 0 & 1 & -1 \\
-2 & 0 & 0 & -2 & 1 & -1 & 2 & 0 & -1 & 1 & -1 & -1 & 0 & 1 & -1 & 1 & -1 \\
0 & -2 & -2 & 0 & 1 & -2 & 1 & 1 & -1 & 1 & -1 & 0 & -1 & 1 & 0 & -1 & -1 \\
0 & 0 & -6 & -6 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & -3 \\
-2 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & -2 & -2 & 1 & 0 & -1 & 1 & 0 & 1 & -1 & 1 & -1 & -1 & 2 & -1 \\
-2 & -2 & -2 & -2 & 1 & 1 & -1 & 0 & 1 & 1 & -1 & 1 & -2 & 1 & -1 & 1 & -1 \\
0 & -2 & -2 & 0 & 1 & -1 & 0 & 1 & -1 & 1 & -1 & 1 & -2 & 0 & 1 & -1 \\
-2 & -2 & -4 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & -1 & 2 & -1 & -1 & 1 & -1 \\
0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

The dimension of the eigenspace of the matrix to the eigenvalue one can be shown
to be 10. So $b^2(Z) = 10$ and By Equation 2.1, 2.2, 2.3, we have

$$b^2(M) = b^2(Z) = 10,$$
$$b^3(M) = \frac{1}{2}b^3(Y) = h^{1,2}(Y) = 30,$$
$$b^4(M) = \frac{1}{2}h^{2,2}(Y) + h^{3,1}(Y) - 2b^2(Z) + \frac{3}{2}k$$
$$= \frac{1}{2} \cdot 98 + 0 - 2 \cdot 10 + \frac{3}{2} \cdot 16$$
$$= 52.$$

In summary,

$$b^2(M) = 10, \quad b^3(M) = 30, \quad b^4(M) = 52.$$  

The Betti numbers of compact manifolds with holonomy Spin(7), constructed so far [13–15], are

$$0 \leq b^2 \leq 9, \quad 0 \leq b^3 \leq 33, \quad 200 \leq b^4 \leq 15118,$$

It is notable that the Betti number $b^4(M)$ is significantly smaller than those of other examples already constructed.

Let us consider more general antihomorphic involutions of $Y$. Let $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{E}^4$ with $\xi_i = 0$ or $\alpha$. Then translation $t_\xi$ of $\mathbb{E}^4$ by $\xi$ is lifted to an automorphism $\Gamma_\xi$ of $Y$. In general case, an antiholomorphic automorphism of $Y$ has the form

$$\phi_{B,\xi} := \Gamma_\xi \circ \phi_B,$$

where $B$ is a $4 \times 4$ matrix with entries in $\mathbb{Z}[\sqrt{-1}]$.

If $\phi_{B,\xi}$ is an involution and fixes some of singularities of $Y$. Now let $\hat{Y} \to Y$ be the blow-ups of $Y$ at its singularities that are not fixed by $\phi_{B,\xi}$. Then $\phi_{B,\xi}$ induces an antiholomorphic involution on $\hat{Y}$. It is easy to check that $\hat{Y}$ with this involution satisfies Condition 2.1. The author tested various $B$’s, $\xi$’s basically by a computer. However he only found only the examples which give the same Betti numbers presented in the previous example.

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References

[1] M. Berger, *Sur les groupes d’holonomie homogène des variétés riemanniennes*, Bulletin de la Société Mathématique de France 83 (1955), 279-330

[2] E. Witten, *Is supersymmetry really broken?*, Int. J. Mod. Phys. A 10, 1247 (1995)

[3] E. Witten, *Strong coupling and the cosmological constant*, Mod. Phys. Lett. A 10, 2153 (1995)

[4] C. Vafa, *Evidence for F theory*, Nucl. Phys. B 469, 403 (1996)

[5] K. Becker, *A Note on compactifications on spin(7) - holonomy manifolds*, Journal of High Energy Physics, 0105, 003 (2001)

[6] S. Gukov, J. Sparks, *M theory on spin(7) manifolds. 1*, Nucl. Phys. B 625, 3 (2002)

[7] G. Curio, B. Kors, D. Lust, *Fluxes and branes in type II vacua and M theory geometry with G(2) and spin(7) holonomy*, Nucl. Phys. B 636, 197 (2002)

[8] B. S. Acharya, X. de la Ossa, S. Gukov, *G flux, supersymmetry and spin(7) manifolds*, Journal of High Energy Physics 0209, 047 (2002)

[9] S. Gukov, J. Sparks, D. Tong, *Conifold transitions and five-brane condensation in M theory on spin(7) manifolds*, Class. Quant. Grav. 20, 665 (2003)

[10] M. Becker, D. Constantin, S. J. Gates, Jr., W. D. Linch, III, W. Merrell and J. Phillips, *M theory on spin(7) manifolds, fluxes and 3-D, N=1 supergravity*, Nucl. Phys. B 683, 67 (2004)

[11] F. Bonetti, T. W. Grimm, T. G. Pugh, *Non-Supersymmetric F-Theory Compactifications on Spin(7) Manifolds*, Journal of High Energy Physics 01, 112 (2014)

[12] F. Bonetti, T. W. Grimm, E. Palti, T. G. Pugh, *F-Theory on Spin(7) Manifolds: Weak-Coupling Limit*, Journal of High Energy Physics 02, 076 (2014)

[13] D.D. Joyce, *A new construction of compact 8-manifolds with holonomy Spin(7)*, J. Differ. Geom. 53 (1999), 89 -130

[14] D.D. Joyce, *Compact 8-manifolds with holonomy Spin(7)*, Invent. math. 123 (1996), 507-552

[15] R. Clancy, *New examples of compact manifolds with holonomy Spin(7)*. Ann. Global Anal. Geom. 40 (2011), no. 2, 203-222

[16] A. Beauville, *Some remarks on Kähler manifolds with c1 = 0*, Classification of algebraic and analytic manifolds (Katata, 1982), 1-26, Progr. Math., 39, Birkhauser
[17] C. Birkenhake, H. Lange, *Complex abelian varieties*, Second edition, 302.
Springer-Verlag, Berlin, 2004