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New avenue for accurate analytical waveforms and fluxes for eccentric compact binaries

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We introduce a new paradigm for constructing accurate analytic waveforms (and fluxes) for eccentric compact binaries. Our recipe builds on the standard post-Newtonian (PN) approach but (i) retains implicit time-derivatives of the phase space variables in the instantaneous part of the noncircular waveform, and then (ii) suitably factorizes and resums this partly PN-implicit waveform using effective-one-body (EOB) procedures. We test our prescription against the exact results obtained by solving the Teukolsky equation with a test-mass source orbiting a Kerr black hole, and compare the use of the exact vs PN equations of motion for the time derivatives computation. Focusing only on the quadrupole contribution, we find that the use of the exact equations of motion yields an analytical/numerical agreement of the (averaged) angular momentum fluxes that is improved by 40% with respect to previous work, with 4.5% fractional difference for eccentricity $e = 0.9$ and black hole dimensionless spin $-0.9 \leq a \leq +0.9$. We also find a remarkable convergence trend between Newtonian, 1PN and 2PN results. Our approach carries over to the comparable mass case using the resummed EOB equations of motion and paves the way to faithful EOB-based waveform model for long-inspiral eccentric binaries for current and future gravitational wave detectors.

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I. INTRODUCTION

The accurate computation of the gravitational waves (GW) from eccentric compact binary mergers has recently prompted a vibrant activity by the analytical and numerical relativity community [1–18]. A major goal of this program is to provide GW templates for binaries on generic orbits to be used in the next scientific observational runs of the LIGO-Virgo-Kagra (LVK) collaboration [19–21] and in the future LISA observations [22–24]. Indeed, LVK’s GW190521 event indicates that eccentricity might play a fundamental role for the analysis of stellar-mass binary black holes mergers [25–28]. Similarly, extreme-mass-ratio-inspirals (EMRIs) with eccentricity are expected to be detected by LISA [22–24].

The effective-one-body (EOB) approach to the two body problem [29–33], and in particular the TEOBResumS model [3,4,6,34,35], currently seems the only analytical framework close to delivering a quantitatively accurate model for arbitrary eccentricity, including dynamical captures. The model currently exhibits an acceptable level of flexibility and faithfulness with several exact results obtained numerically, from comparable mass binaries to extreme mass ratio inspirals [15,16]. The model is also ready for parameter estimation, as shown for the case of GW190521 [27]. A central issue in the construction of eccentric EOB waveform models is the accurate analytical modelization of the radiation reaction force, i.e., the flux of energy and angular momentum emitted by the system that causes the orbit to shrink and circularize. Reference [3] introduced the eccentric TEOBResumS by dressing the circular azimuthal component of the radiation reaction with the leading-order (Newtonian) noncircular contribution. The same idea is applied to the waveform [3]. Within this paradigm, several eccentric EOB models were constructed [3,6,7], so to support the analysis of GW190521 under the hyperbolic-capture hypothesis [27]. We note, however, that its quasicircular limit is not as accurate as the native quasi-circular model [6,7], still delivering unfaithfulnes <3% with numerical relativity (NR) waveform data. Building partly upon [13], the noncircular, factorized and resummed, EOB waveform was pushed to 2PN accuracy in [15]. Albanesi et al. [16] contrasted several proposal for the radiation reaction available in the literature and concluded that those based on the Newtonian-factorization (with a 2PN correction) deliver the most faithful representation of the fluxes of a test-mass around a Kerr black hole. Reference [15] used the 2PN-truncated equations of motion
(EOM) to recast the noncircular correction in a simpler form, retaining explicit dependence on only the angular and radial momenta \( (P_r, P_\phi) \). Unfortunately, with this approach the behavior at the orbit turning points (apastron and periastron) cannot be modified by higher PN corrections, since \( P_R = 0 \) there by definition and all noncircular corrections vanish. The accuracy at the radial turning points of the scheme is thus entirely determined by the Newtonian contribution, that explicitly depends on the, there non-vanishing, radial acceleration \( \dot{R} \), see [15,16].

We here propose a new strategy to significantly improve the behavior of the waveform at the radial turning points. The crucial element behind the accuracy of the generic Newtonian prefactor is that the derivatives (and notably \( \dot{R} \), see Figs. 17 and 18 of [15]) are evaluated using the exact EOM and not the PN-truncated ones. This is the important pole of the waveform of Refs. [13,39]. Depending on the employment as initial input the PN expanded spherical multipoles one needs both \( h^{(N)}_{\ell m} \) and \( h^{(c)}_{\ell m} \). Let us introduce the Newtonian noncircular corrections formally addresses the contributions obtained by taking the

\[ = \frac{1}{e^2} h^{(1PN, 0)}_{\ell m} (r, \dot{r}, \dot{\Omega}, \gamma, \Omega) \]

\[ + \frac{1}{e^4} h^{(2PN, 0)}_{\ell m} (r, \dot{r}, \dot{\Omega}, \gamma, \dot{\Omega}, \Omega) \]

with \( h^{(2PN, 0)}_{\ell m} \) the Newtonian part and \( (h^{(1PN, 0)}_{\ell m}, h^{(2PN, 0)}_{\ell m}) \) formally addresses the contributions obtained by taking the

3Throughout the paper we use dots to denote time derivatives.
4In order of appearance \( r \equiv R/M \) is the relative separation in the center of mass frame, \( \phi \) is the orbital phase, \( P_r \equiv (A/B)^{1/2} r \), where \( A, B \) are the 2PN accurate EOB potentials and \( p_\phi \equiv P_\phi/M \) is the radial momentum, and \( p_\phi \equiv P_\phi/M \) is the angular momentum. Denoting with the individual masses of the binary we have

\[ M \equiv m_1 + m_2, \quad \mu \equiv m_1 m_2/M \] and \( q \equiv m_1/m_2 \geq 1 \).
time-derivative of the corresponding terms in the radiative multipoles while keeping all derivatives explicit. We obtain

$$h_{22}^{(N,0)} = -8\sqrt{\frac{\pi}{5}} \nu r^2 \Omega^2 e^{-2\nu_0} h_{22}^{(N,0)\text{circ}},$$

$$h_{22}^{(N,0)\text{inst}} = 1 - \frac{1}{2} \left( \frac{r^2}{r^2 \Omega^2} + \frac{\dot{r}}{r \dot{\Omega}} \right) + i \left( \frac{2r \dot{r}}{r^2 \dot{\Omega}} + \frac{\dot{\Omega}}{2 r^2 \Omega^2} \right).$$

(2)

with the noncircular part $h_{22}^{(N,0)\text{inst}}$ isolated. The noncircular contribution is obtained from Eq. (1) as follows. First, we define the instantaneous, total, factorized correction as $f_{22}^{\text{total}} = h_{22}^{\text{inst}} (h_{22}^{(N,0)\text{circ}})^{-1}$, where we replaced $S_{\text{inst}}^{(0)} \equiv \dot{H}_{\text{eff}}$. The noneccentric limit of this function is defined according to point (iv) above, so to obtain (at 2PN)

$$f_{22}^{\text{nc}} = 1 + \frac{u}{c^2} \left[ \frac{12}{7} - \frac{p_{\varphi}^2 u}{3} + \frac{1}{7} \left( \frac{p_{\varphi}^2 u}{2} \right) \nu \right]$$

$$+ \frac{u^2}{c^4} \left[ -229 \frac{929}{756} p_{\varphi}^2 u^2 + \frac{19}{63} p_{\varphi}^2 u^2 \right]$$

$$+ \frac{289}{126} \frac{1741}{378} p_{\varphi}^2 u^2 - \frac{235}{504} p_{\varphi}^2 u^2 \nu \right]$$

$$+ \frac{65}{126} \frac{31}{54} p_{\varphi}^2 u^2 - \frac{143}{504} p_{\varphi}^2 u^2 \nu \right].$$

(3)

Note that $p_{\varphi}$ is not replaced with its PN-expanded circular expression. The noncircular (instantaneous) contribution is obtained factoring out this result as $h_{22}^{\text{nc inst}} = T_{2\text{PN}}[f_{22}^{\text{nc}}(f_{22}^{\text{nc}})^{-1}]$, where $T_{2\text{PN}}$ indicates an expansion up to the 2PN order. This allows us to obtain a new noncircular factor that is analogous, though different, to that of Ref. [15]. A few more comments are in order to further clarify the structure of the noneccentric (i.e., circular) part. First, we stress that taking the exact circular limit of the product $h_{22}^{(N,0)} f_{22}^{\text{nc inst}}$ (i.e., replacing also the PN-expanded expression of $p_{\varphi}$ along circular orbits) correctly delivers the 2PN-accurate $f_{22}$ function of Eq. (B1) of Ref. [38]. The factorization procedure is thus such that the waveform is consistent with the quasicircular waveform of TEOBResumS.\(^5\)

We note, however, that in practice we do not use the 2PN-accurate $f_{22}^{\text{nc}}$ recovered above, but rather replace it with the quasi-circular function $\rho_{22} = f_{22}^{1/2} \delta_\text{nc}$ with the PN-accuracy and resummation used either in the standard TEOBResumS model (for comparable masses) or in its test-mass version [8]. More precisely, $\rho_{22}$ is resummed according to Refs. [40,41], while in TEOBResumS the orbital contribution $\rho_{22}^\text{orb}$ is Taylor-expanded at $3+2\text{PN}$ accuracy [42], in the test-mass limit we use it at 6PN, resummed with a (4,2) Padé approximant.

\(^5\)In Ref. [15] we factorized the circular $S_{\text{eff}}^{(c)}$, but the procedure followed in this work is more consistent in the full EOB waveform we have the generic factor $\delta_\text{nc}$, see, e.g., Ref. [38].

The new noncircular factor is given explicitly in a supplemental Mathematica notebook [43]. The noncircular tail contribution $h_{\text{nc tail}}^{\text{inst}}$ is resummed following Ref. [15], while the circular tail contributions are incorporated according to standard procedure [38]. From the waveform, the quadrupolar fluxes read [44] $\dot{E}_{22} = \frac{1}{8} \left| h_{22} \right|^2$ and $J_{22} = -\frac{1}{16} \frac{1}{2} \left( h_{22} \right)$.}

### III. COMPARING ANALYTICAL AND NUMERICAL RESULTS

The new prescription for the noncircular waveform correction is tested by following step-by-step the approach of Ref. [8,15,16]. This relies on extensive comparisons with waveform and fluxes emitted by a (nonspinning) particle orbiting around a Kerr black hole, considering various orbital configurations. To set the stage, we consider a particle inspiralling and plunging around a Schwarzschild black hole and compare the analytical waveform with the numerical one, considered exact, obtained solving numerically the Teukolsky equation using Teukode [45] (see Ref. [8] for more numerical details). Figure 1 refers to a configuration with initial eccentricity $e_0 = 0.5$ and semilatus rectum $p_0 = 7.35$. The numerical waveform (black) is compared with the analytical waveform with Newtonian noncircular corrections (red), the waveform with 2PN noncircular corrections proposed in Ref. [15] (dashed green), and the waveform with 2PN noncircular corrections as proposed in this work (blue dash-dotted). The derivatives of coordinates and momenta which appear in the noncircular corrections are computed numerically with a 4th-order centered-stencil scheme. The corresponding analytical/numerical differences for amplitude and phase are shown in the bottom panels. Regarding the phase, the performance of the new waveform and of the one of Ref. [15] are substantially equivalent (see the dashed and solid blue lines in the middle panel of Fig. 1). For the amplitude, instead, the new approach yields a reduced maximum analytical/numerical difference during the evolution, as well as a slight improvement as the orbital motion approaches the periastra (see bottom panel of Fig. 1). This comes from the nonvanishing of the noncircular correction at the radial turning point. This is highlighted in Fig. 2, which shows the noncircular corrections corresponding to Fig. 1. The bottom panel exhibits the relative difference between the two 2PN noncircular amplitude corrections and the Newtonian noncircular amplitude correction, $|\delta \tilde{A}_{22}^\text{nc}/\tilde{A}_{22}^\text{Newt}|$. The correction at the two turning points is nonzero and is especially relevant at periastron. Note that the instantaneous phase noncircular corrections at 2PN differ sensibly from the Newtonian one. However, part of this difference is compensated by the hereditary phase correction, as already highlighted in Sec. III C of Ref. [15].

The improvement in the description of the waveform at periastron is even more important when the noncircular corrections are incorporated in the fluxes. Note in fact that the main contribution to the dynamics, through radiation
reaction, happens due to the burst of radiation emitted at periastron. To show this effect systematically, we consider a set of geodesic eccentric orbits with dimensionless Kerr spin $\hat{a} \leq 0.9$, eccentricity up to $e = 0.9$, and semilata recta given by $p = p_{\text{schw}}(e, \hat{a}) / p_s(e, 0)$, where $p_s$ is the separatrix \cite{46,47} and $p_{\text{schw}} = (9, 13)$. The definitions of eccentricity and semilatus rectum used here can be found in Ref. \cite{8}, together with more details on the numerical data. In Fig. 3 we compare the fluxes for a case example with $e = 0.5$. From the analytical/numerical relative differences one finds that the 2PN noncircular corrections with explicit derivatives perform better at periastron. As radiation reaction, these fluxes will drive faster inspirals than those driven by the simple leading (Newtonian) noncircular correction of \cite{8}. To draw a more global picture, it is useful to compare the orbital-averaged analytical fluxes with the corresponding, averaged, numerical ones obtained from the exact waveforms, calculated solving the Teukolsky equation. We compute the analytical/numerical relative difference and for each value of eccentricity $e$ we compute its average over all the dataset sharing the same value of $e$. These averages are shown in Fig. 4, where each point has been obtained by averaging the analytical/numerical relative differences of 14 simulations with $\hat{a} = (0, \pm 0.2, \pm 0.6, \pm 0.9)$ and two different values of $p$. As shown in Fig. 4, the new 2PN waveform yields (on average) the best analytical/numerical agreement; even the 1PN energy flux calculation is better than the 2PN-accurate expression of Ref. \cite{15}. Note however that the average over all spinning configurations can hide some information. In particular, for highly eccentric configurations ($e = 0.9$), the Newtonian prescription yields a better analytical/numerical agreement when averaged only on negative spins. This is clearer for the lowest spin, $\hat{a} = -0.9$, since the Newtonian flux yields a better analytical/numerical agreement even for mild eccentricities ($e \gtrsim 0.3$). However, in the Schwarzschild case, the hierarchy of the different prescriptions is the same shown in Fig. 4.

The new noncircular correction factor is quantitatively superior to all other previous attempts of incorporating high averaged, numerical fractional differences between the averaged quadrupolar fluxes versus eccentricity that show convergence of PN corrections in the test-mass limit. Each point is obtained from the mean of the orbital averaged fluxes of all the configurations considered at a given eccentricity $e$. See the main text for more details.
PN noncircular information in the waveform and fluxes. This is further corroborated by the following: the instantaneous amplitude correction presented in Eq. (37) of Ref. [15] contains a 1PN term \( \propto -p_r^2 r \), that can become extremely large when considering hyperbolic or eccentric orbits with large radius. While this issue is not relevant for any of the configurations considered in Ref. [15], the correction can become even negative, and thus unphysical, for large separations, e.g., those occurring in hyperbolic encounters. By contrast, the new waveform is well behaved also for a hyperbolic encounter or scattering configuration starting from any, arbitrarily large, initial separation.

The same behavior carries over to the comparable-mass case, with the test-mass dynamics replaced by the resummed EOB dynamics. Figure 5 exhibits the time-evolution of the noncircular waveform corrections along the EOB dynamics of a binary corresponding to the NR configuration SXS:BBH:321 of the SXS catalog [48], row #23 in Table IV of Ref. [15]. In this case, the mass ratio is \( q = 1.22 \), while the dimensionless spins \((\chi_1, \chi_2)\), aligned with the orbital angular momentum, are \( \chi_1 = +0.33 \) and \( \chi_2 = -0.44 \). The initial EOB eccentricity at the apastron is small, \( e_{\text{EOB, ap}} = 0.07621 \), but large enough to probe that our new waveform brings an improvement with respect to previous work. First of all, Fig. 5 indicates that in the comparable-mass case the amplitude correction at the radial turning points is more relevant than in the test-mass case (cf. Fig. 2), although the correction is small. It is informative to look at a standard EOB/NR phasing comparison for SXS:BBH:321, that we report in Fig. 6. The EOB waveform is aligned to the NR one by minimizing the phase difference in the frequency interval corresponding to the two vertical lines in the left panels of the figure. The top panels compare the EOB and NR real parts of the waveform, while the bottom panels show the EOB/NR phase difference \( \Delta \phi_{22}^{\text{EOB,NR}} = \phi_{22}^{\text{EOB}} - \phi_{22}^{\text{NR}} \) and relative amplitude difference, with \( \Delta A_{22}^{\text{EOB,NR}} = A_{22}^{\text{EOB}} - A_{22}^{\text{NR}} \). The picture illustrates that \( \Delta \phi_{22}^{\text{EOB,NR}} \) is reduced, during the late-inspiral and plunge, with respect to the corresponding plot in Fig. 13 of Ref. [15], with the same waveform alignment interval used here. A similar behavior is also found with higher eccentricities. However, it must be noted that since the waveform is different, the choice of the initial parameters, which is not changed in this case, possibly might be optimized further. This study, together with the analysis of higher modes, is postponed to future work.

We finally point out that the waveform differences due to the new noncircular correction will yield fluxes which are larger at apastron than the current ones. Once recasted in the form of radiation reaction force, and incorporated within the EOB dynamics, the new prescription will eventually yield an additional acceleration of the eccentric inspiral due to the stronger emission at periastron. The development of the radiation reaction force and its influence on the inspiral (for any mass ratio) is also deferred to future work. To do so, we will use, and generalize, the approach adopted for the generic Newtonian prefactor [3]. This calculation relies on the iterative procedure for computing the time-derivatives including dissipative terms, see Appendix A of Ref. [49].

**IV. CONCLUSION**

We introduced a new way of exploiting PN results for the waveform emitted by eccentric and hyperbolic binaries. The key idea is to use the full, resummed EOB EOM to compute the time derivatives in the formal expression of the analytical waveform, rather than replacing them with the PN-expanded EOM. This exploits the robust behavior of the EOB model in the strong-field regime. This procedure effectively generalizes the use of the generic Newtonian prefactor to higher PN orders. We comprehensively tested this approach against a large set of data for waveforms and fluxes emitted by a test-mass orbiting a Kerr black hole and we showed that these findings carry over to the comparable mass case. The novel use of PN results, within the EOB framework, yields considerable improvement in the waveform accuracy. Typically, one starts from PN-expanded
results and then devise specific techniques to improve their behavior in strong field. Our findings indicate that this is not sufficient, but it is actually possible to do better by removing some of the intermediate steps involving PN-expansions. In particular, our approach does not rely on eccentricity-expanded fluxes, in contrast to [50,51], and remains robust also for highly eccentric configurations.

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