Fixed point Theorems for Cyclic-Ćirić-Reich-Rus contraction mapping in quasi-partial b-metric spaces

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Abstract: In this paper, Ćirić-Reich-Rus cyclic contraction mapping is defined in the setting of quasi-partial b-metric spaces and fixed point results are proved. Some examples are given to validate our results.

Keywords: Quasi-Partial b-Metric space, Fixed point theorems, \( qp_b \) Ćirić-Reich-Rus Cyclic Contraction mapping.

1 Introduction

The French mathematician Maurice Frechet initiated the study of metric spaces [1] in 1905. In 1989, Bakhtin [2] introduced the concept of b-metric space and gave the contraction mapping which was the generalization of the Banach Contraction Principle. In 1993, Czerwick [3] extended this concept of b-metric spaces whereas Shukla introduced partial \( b \)-metric [4] in 2014. The concept of Partial-metric spaces was introduced by Matthews [5] in 1994 as a generalization of standard metric spaces by replacing the condition \( d(x, x) = 0 \) with the condition \( d(x, x) \leq d(x, y) \), for all \( x, y \). As a further generalization for quasi-metric spaces and partial-metric spaces, Karapinar [6] introduced the notion of quasi-partial metric space and discussed the existence of fixed points of self-mappings \( T \) on quasi-partial metric spaces. Gupta and Gautam [7] further, generalized quasi-partial metric spaces to the class of quasi-partial b-metric spaces and proved some fixed point results ([8],[9]) in the setting of quasi-partial metric space.

In 1974, Ćirić [10] extend quasi-contraction mappings and stated some fixed point results in which it has shown that the condition of quasi-contractivity implies all conclusions...
of Banach’s contraction principle:
Let \((X, d)\) be a metric space. A mapping \(T : X \to X\) is said to be a quasi-contraction mapping if there exists \(b \in [0, 1)\) such that
\[
d(Tx, Ty) \leq M(x, y)
\]
for all \(x, y \in X\), where
\[
M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.
\]
Reich [11] generalized concepts of Kannan and Banach by introducing the following mapping:
A mapping \(T : X \to X\) is said to be a Reich-contraction mapping if there are \(a, b, g \in [0, 1)\) such that
\[
a + b + g < 1
\]
and
\[
d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + gd(x, y)
\]
for all \(x, y \in X\).

we also review the concept of cyclic mapping as follows:
Let \(A\) and \(B\) be non-empty subsets of a metric space \((X, d)\), \(T : A \cup B \to A \cup B\) is called cyclic if \(T(A) \subseteq B\) and \(T(B) \subseteq A\).

In 2016, Fan [12] proved fixed point theorems for some special cyclic mappings satisfying Banach contraction condition in the setting of a quasi-partial b-metric space as follows:
Let \(A\) and \(B\) be nonempty subsets of a quasi-partial b-metric space \((X, qp_b)\). A cyclic mapping \(T : A \cup B \to A \cup B\) is said to be a \(qp_b\)-cyclic-Banach contraction mapping if there exists \(a \in [0, 1)\) such that if \(s \geq 1, sa < 1\) then
\[
qp_b(Tx, Ty) \leq \alpha qp_b(x, y)
\]
holds both for \(x \in A, y \in B\) and \(x \in B, y \in A\).
Fan [12] in the same paper proved Kannan contraction condition in the setting of a quasi-partial b-metric space which is stated as follows:
Let \(A\) and \(B\) be nonempty subsets of a quasi-partial b-metric space \((X, qp_b)\). A cyclic mapping \(T : A \cup B \to A \cup B\) is said to be a \(qp_b\)-cyclic-Kannan contraction mapping if there exists \(b \in [0, 1/2)\) such that if \(s \geq 1, s \beta < 1/2\) then
\[
qp_b(Tx, Ty) \leq \beta qp_b(x, Tx) + \beta qp_b(y, Ty)
\]
holds both for \(x \in A, y \in B\) and \(x \in B, y \in A\).
Inspired and motivated by the work of Fan [12], Ćirić [10] and Reich [11], in this paper we have introduced new contraction conditions for Ćirić-Reich-Rus contraction ([13]-[21]) for cyclic mappings in the setting of a quasi-partial b-metric space and discuss the existence and uniqueness of fixed points for such mappings when the underlying space is complete. Throughout this paper \(N, R\) and \(R^+\) denote the set of all positive integers, set of real numbers and the set of all non-negative real numbers respectively. Here, we will recall some definition and lemma as discussed by Gupta and Gautam[9] in their paper:
2 Preliminaries and basic properties

Definition 1. ([7]). A Quasi-Partial $b$-metric on a non-empty set $X$ is a mapping $qp_b : X \times X \to R^+$ such that for some real numbers $s \geq 1$ and for all $x, y, z \in X$

$\begin{align*}
(QP_{b_1}) & \quad qp_b(x, x) = qp_b(x, y) = qp_b(y, y) \implies x = y \\
(QP_{b_2}) & \quad qp_b(x, x) \leq qp_b(x, y) \\
(QP_{b_3}) & \quad qp_b(x, x) \leq qp_b(y, x) \\
(QP_{b_4}) & \quad qp_b(x, y) + qp_b(z, z) \leq s[qp_b(x, y) + qp_b(y, z)]
\end{align*}$

A Quasi-Partial $b$-metric space is a pair $(X, qp_b)$, where $X$ is a non-empty space and $qp_b$ is Quasi-Partial $b$-metric on $X$. The number $s$ is called coefficient of $(X, qp_b)$.

The following lemmas are instrumental in our result.

Lemma 1. ([9]). Let $(X, qp_b)$ be a Quasi-Partial $b$-metric space. Then the following holds

(i) If $qp_b(x, y) = 0$ then $x = y$.
(ii) If $x \neq y$ then $qp_b(x, y) > 0$ and $qp_b(y, x) > 0$.

Lemma 2. ([9]). Let $(X, qp_b)$ be a Quasi-Partial $b$-metric space. Then

(i) A sequence $\{x_n\} \subset X$ converges to $x \in X$ if and only if $qp_b(x, x) = \lim_{n \to +\infty} qp_b(x_n, x) = \lim_{n \to +\infty} qp_b(x, x_n)$
(ii) A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if and only if $\lim_{n,m \to +\infty} qp_b(x_n, x_m) = \lim_{n,m \to +\infty} qp_b(x_m, x_n)$ exist and finite.
(iii) The Quasi-Partial $b$-metric space $(X, qp_b)$ is said to be complete if every sequence $\{x_n\} \subset X$ converges with respect to $\tau_{qp_b}$ to a point $x \in X$ such that $qp_b(x, x) = \lim_{n,m \to +\infty} qp_b(x_n, x_m) = \lim_{n,m \to +\infty} qp_b(x_m, x_n)$.

Lemma 3. ([12]). Let $(X, qp_b)$ be a quasi-partial $b$-metric space and $\{x_n\}_{n=1}^\infty$ be a sequence in $X$. If $x_n \to x, x_n \to y$ and $qp_b(x, x) = qp_b(y, y) = 0$, then $x = y$.

3 Main Result

In this section, we extend the fixed point theorem for Ćirić-Reich-Rus contraction mappings in standard metric spaces to $qp_b$-cyclic-Ćirić-Reich-Rus contraction mappings in the setting of quasi-partial $b$-metric spaces.

Definition 2. Let $A$ and $B$ be nonempty subsets of a quasi-partial $b$-metric space $(X, qp_b)$. A cyclic mapping $T : A \cup B \to A \cup B$ is said to be a $qp_b$-cyclic-Ćirić-Reich-Rus contraction mapping if there exists $\alpha \in [0, 1)$ and $\beta \in [0, 1/2)$ such that if $s \geq 1$, $s(\alpha + 2\beta) < 1$ then

$$qp_b(Tx, Ty) \leq \alpha qp_b(x, y) + \beta qp_b(x, Tx) + \beta qp_b(y, Ty),$$

holds both for $x \in A, y \in B$ and $x \in B, y \in A$. 

Theorem 1. Let \( A \) and \( B \) be two nonempty closed subsets of a complete quasi-partial \( b \)-metric space \((X, q_p b)\) and \( T \) be a cyclic mapping which is a \( q_p b \)-cyclic-\( \breve{C}iri\-Reich-Rus \) mapping. Then \( A \cap B \) is nonempty and \( T \) has a unique fixed point in \( A \cap B \).

Proof. Let \( x \in A \), considering condition (1), we have

\[
q_p b(T x, T^2 x) \leq \alpha q_p b(x, T x) + \beta q_p b(x, T x) + \beta q_p b(T x, T^2 x)
\]

\[
(1 - \beta) q_p b(T x, T^2 x) \leq (\alpha + \beta) q_p b(x, T x)
\]

\[
q_p b(T x, T^2 x) \leq \frac{\alpha + \beta}{1 - \beta} q_p b(x, T x)
\]

Using (2) we get

\[
q_p b(T^2 x, T x) \leq \alpha q_p b(T x, x) + \beta q_p b(T x, T^2 x) + \beta q_p b(x, T x)
\]

\[
q_p b(T^2 x, T x) \leq \alpha q_p b(x, x) + \beta \left( \frac{\alpha + \beta}{1 - \beta} \right) q_p b(x, T x) + \beta q_p b(x, T x)
\]

Take, \( \gamma = \text{Max}\{q_p b(x, T x), q_p b(T x, x)\} \), then we have

\[
q_p b(T x, T^2 x) \leq \frac{\alpha + \beta}{1 - \beta} \gamma, \text{ and}
\]

\[
q_p b(T^2 x, T x) \leq \frac{\alpha + \beta}{1 - \beta} \gamma.
\]

Similarly,

\[
q_p b(T^3 x, T^2 x) \leq \left( \frac{\alpha + \beta}{1 - \beta} \right)^2 \gamma, \text{ and}
\]

\[
q_p b(T^2 x, T^3 x) \leq \left( \frac{\alpha + \beta}{1 - \beta} \right)^2 \gamma.
\]

Hence,

\[
q_p b(T^n x, T^{n+1} x) \leq (\frac{\alpha + \beta}{1 - \beta})^n \gamma \quad \text{and} \quad q_p b(T^{n+1} x, T^n x) \leq (\frac{\alpha + \beta}{1 - \beta})^n \gamma;
\]

for every \( n \in N \).

Let \( m, n \in N \) and \( m < n \), using (3) and \( Q P_{bL} \) we obtain

\[
q_p b(T^m x, T^n x) \leq s[q_p b(T^m x, T^{m+1} x) + q_p b(T^{m+1} x, T^n x)] - q_p b(T^{m+1} x, T^{m+1} x)
\]

\[
q_p b(T^m x, T^n x) \leq s[q_p b(T^m x, T^{m+1} x) + q_p b(T^{m+1} x, T^n x)]
\]

\[
q_p b(T^m x, T^n x) \leq sq_p b(T^m x, T^{m+1} x) + s^2 q_p b(T^{m+1} x, T^{m+2} x) + s^2 q_p b(T^{m+2} x, T^n x)
\]

\[
q_p b(T^m x, T^n x) \leq sq_p b(T^m x, T^{m+1} x) + s^2 q_p b(T^{m+1} x, T^{m+2} x) + \ldots + s^{n-m} q_p b(T^{n-1} x, T^n x).
\]
Let \((\alpha + \beta) = \delta\), then
\[
q_p(b(T^n x, T^m x)) \leq (s\delta^m + s\delta^{m+1} + \ldots + s^{m-1}\delta^{m-1})\gamma
\]
\[
q_p(b(T^n x, T^m x)) \leq s\delta^m(1 + \delta + \delta^2 + \ldots + \delta^{m-1})\gamma
\]
\[
q_p(b(T^n x, T^m x)) \leq s\delta^m\frac{1 - (\delta)^{m-1}}{1 - \delta}\gamma.
\]
Since, \(s(\alpha + 2\beta) < 1\), \(s \geq 1\) then, \(1 - s\beta \leq 1 - \beta\) and \(s(\alpha + \beta) < 1 - s\beta\), therefore, \(s(\alpha + \beta) < 1\), i.e. \(s\delta < 1\). Furthermore,
\[
q_p(b(T^n x, T^m x)) \leq (\frac{s\delta^m}{1 - s\delta})\gamma.
\]
Taking limit as \(m,n \to +\infty\) in the above inequality, we have
\[
\lim_{m,n \to +\infty} q_p(b(T^n x, T^m x)) \leq 0.
\]
Thus,
\[
\lim_{m,n \to +\infty} q_p(b(T^n x, T^m x)) = 0. \quad (4)
\]

Also,
\[
q_p(b(T^n x, T^m x)) \leq s[q_p(b(T^n x, T^{m+1} x) + q_p(b(T^{m+1} x, T^m x))] - q_p(b(T^{m+1} x, T^m x))
\]
\[
q_p(b(T^n x, T^m x)) \leq s[q_p(b(T^n x, T^{m+1} x) + q_p(b(T^{m+1} x, T^m x))]
\]
\[
q_p(b(T^n x, T^m x)) \leq s^2 q_p(b(T^n x, T^{m+2} x) + s^2 q_p(b(T^{m+2} x, T^{m+1} x) + sq_p(b(T^{m+1} x, T^m x))
\]
\[
q_p(b(T^n x, T^m x)) \leq s^{m-1} q_p(b(T^n x, T^{m-1} x) + s^{m-1} q_p(b(T^{m-1} x, T^{m-2} x) + \ldots + sq_p(b(T^{m+1} x, T^m x)
\]
\[
q_p(b(T^n x, T^m x)) \leq (s^{m-1} \delta^{m-1} + s^{m-1} \delta^{m-2} + \ldots + s\delta^m)\gamma
\]
\[
q_p(b(T^n x, T^m x)) \leq s\delta^m(\frac{1 - (\delta)^{m-1}}{1 - \delta})\gamma.
\]
Taking limit as \(m,n \to +\infty\) in the above inequality, we have
\[
\lim_{m,n \to +\infty} q_p(b(T^n x, T^m x)) \leq 0.
\]
Thus,
\[
\lim_{m,n \to +\infty} q_p(b(T^n x, T^m x)) = 0. \quad (5)
\]

Using (4) and (5) indicate that sequence \(\{T^n x\}_{n=0}^{\infty}\) is a Cauchy sequence.

Since \((X, q_p(b))\) is complete, therefore \(\{T^n x\}_{n=0}^{\infty}\) converges to some \(u \in X\), that is,
\[
q_p(b(u, u) = \lim_{n \to +\infty} q_p(b(T^n x, u)) = \lim_{n \to +\infty} q_p(b(u, T^n x)) = \lim_{n,m \to +\infty} q_p(b(T^n x, T^m x) = \lim_{n,m \to +\infty} q_p(b(T^n x, T^m x) = 0. \quad (6)
\]
Observe that \( \{ T^{2n}x \}_{n=0}^{+\infty} \) is a sequence in \( A \) and \( \{ T^{2n-1}x \}_{n=1}^{+\infty} \) is a sequence in \( B \) in a way that both sequences converge to \( u \). Note also that \( A \) and \( B \) are closed, we have \( u \in A \cap B \)

\[
q_p(T^n x, Tu) \leq \alpha q_p(T^{n-1} x, u) + \beta q_p(T^{n-1} x, T^n x) + \beta q_p(u, Tu).
\]

Taking limit as \( n \to +\infty \) in the above inequality and using (6), we have

\[
q_p(T^n x, Tu) \leq \beta q_p(u, Tu) \leq \beta [sq_p(u, T^n x) + sq_p(T^n x, Tu) - q_p(T^n x, T^n x)]
\]

\[
\beta q_p(u, Tu) \leq \beta [sq_p(u, T^n x) + sq_p(T^n x, Tu)].
\]

Taking limit as \( n \to +\infty \) in the above inequality, we have

\[
\beta q_p(u, Tu) \leq \beta sq_p(T^n x, Tu).
\]

Thus, using (7) and (8) we have

\[
q_p(T^n x, Tu) \leq \beta q_p(u, Tu) \leq \beta sq_p(T^n x, Tu).
\]

Since, \( s\beta < 1 \)

\[
\lim_{n \to +\infty} q_p(T^n x, Tu) = q_p(u, Tu) = 0.
\]

Similarly, it can be derived

\[
\lim_{n \to +\infty} q_p(Tu, T^n x) = q_p(Tu, u) = 0.
\]

In addition, by the contractive condition of theorem and in combination with (6) and (9), we get

\[
q_p(Tu, Tu) \leq \alpha q_p(u, u) + 2\beta q_p(u, Tu),
\]

so,

\[
q_p(Tu, Tu) = 0.
\]

Equations (9),(10) and (11) show that the sequence \( \{ T^n x \}_{n=0}^{+\infty} \) to \( Tu \). Applying Lemma 3, we get

\( Tu = u \).

For uniqueness, suppose there exist another fixed point in \( A \cup B \) say \( w \) i.e. \( Tw = w \), by contractive condition,

\[
q_p(u, w) = q_p(Tu, Tw) \leq \alpha q_p(u, w) + \beta q_p(u, Tu) + \beta q_p(w, Tw)
\]

\[
q_p(u, w) \leq \alpha q_p(u, w) + \beta q_p(u, u) + \beta q_p(w, w).
\]
Consider,
\[qp_b(u, u) = qp_b(Tu, Tu)\]
\[\leq \alpha qp_b(u, u) + \beta qp_b(u, Tu) + \beta qp_b(u, Tu)\]
\[= (\alpha + 2\beta)qp_b(u, u)\]
\[(1 - \alpha - 2\beta)qp_b(u, u) \leq 0.\]

Since
\[\alpha + 2\beta < 1, \quad qp_b(u, u) = 0.\] (13)

In a similar manner,
\[qp_b(w, w) = 0.\] (14)

Combining (12),(13) and (14) we get
\[qp_b(u, w) = 0,\]
\[0 < \alpha \leq 1 \quad \text{and} \quad (1 - \alpha)qp_b(u, w) \leq 0, \text{therefore,} \quad qp_b(u, w) = 0. \text{ So we have} \quad qp_b(u, w) = 0, \text{that} \quad u = w.\]

\[\square\]

**Example 1.** Let \(X = [0, 1], A = [0, \frac{1}{2}]\) and \(B = [0, \frac{1}{3}], T : A \cup B \to A \cup B\) as \(Tx = \frac{x}{2}\). Define the quasi partial b-metric as
\[qp_b(x, y) = |x - y| + |x|\]
for all \(x, y \in X\). Here \((X, qp_b)\) is a quasi-partial b-metric space with \(s = 1\). We will verify that the mapping \(T\) is \(qp_b\)-cyclic-\(\tilde{\text{C}}\)irić-Reich-Rus contraction mapping.

\(T(A) \subset B\) and \(T(B) \subset A\). Hence the mapping is cyclic on \(X\).

Taking \(\alpha = \frac{1}{2}, \beta \leq \frac{1}{2}, s(\alpha + 2\beta) \leq 1\)

\[
\frac{1}{2}qp_b(x, y) + \beta \{qp_b(x, Tx) + qp_b(y, Ty)\} = \frac{1}{2}(|x - y| + |x| + |x| + |\frac{x}{2}| + |\frac{y}{2}| + |y|) \geq \frac{1}{2}(|x - y| + |x|) \geq \{\frac{x}{2} - \frac{y}{2} + |\frac{x}{2}|\} = qp_b(Tx, Ty)
\]

Therefore, all conditions of Theorem 1 are satisfied and so \(T\) has a fixed point (which is \(0 \in A \cap B\)).
Example 2 Let \( X = [-\frac{\pi}{2}, \frac{\pi}{2}] \), \( A = [-\frac{\pi}{2}, 0] \) and \( B = [0, \frac{\pi}{2}] \), \( T : A \cup B \to A \cup B \) as \( T x = -\frac{\sin x}{2} \). Define the quasi partial \( b \)-metric as

\[
qp_b(x,y) = |x-y| + |x|
\]

for all \( x,y \in X \). Here \( (X, qp_b) \) is a quasi-partial \( b \)-metric space with \( s = 1 \). We will verify that the mapping \( T \) is \( qp_b \)-cyclic-\( \tilde{\text{Ciri}´c-Reich-Rus} \) contraction mapping. \( T(A) = [0,1] \subset B \) and \( T(B) = [-1,0] \subset A \). Hence the mapping is cyclic on \( X \).

Taking \( a = \frac{1}{2}, b = \frac{1}{2}, s(\alpha + 2\beta) \leq 1 \),

\[
\frac{1}{2}qp_b(x,y) + \beta \{qp_b(x,Tx) + qp_b(y,Ty)\} = \frac{1}{2}(|x-y| + |x|) + \beta |x + \frac{\sin x}{2}| + |x| + |y + \frac{\sin y}{2}| + |y| \geq \\
\geq \frac{1}{2}(|x-y| + |x|) \geq \frac{1}{2}\{|\sin x - \sin y| + |\sin x|\} \geq \\
\geq \{ -\frac{\sin x}{2} + \frac{\sin y}{2} \} = qp_b(Tx,Ty)
\]

Therefore, all conditions of Theorem 1 are satisfied and so \( T \) has a fixed point (which is \( 0 \in A \cap B \)).

**Corollary 1.** Let \( A \) and \( B \) be two nonempty closed subsets of a complete quasi-partial \( b \)-metric space \( (X, qp_b) \) and \( T \) be a cyclic mapping which is a \( qp_b \)-cyclic-\( \tilde{\text{Kannan}} \) mapping. Then \( A \cap B \) is nonempty and \( T \) has a unique fixed point in \( A \cap B \).

**Proof.** In theorem 1, Taking \( \alpha = 0 \). \( \square \)

**Corollary 2.** Let \( A \) and \( B \) be two nonempty closed subsets of a complete quasi-partial \( b \)-metric space \( (X, qp_b) \) and \( T \) be a cyclic mapping which is a \( qp_b \)-cyclic-\( \tilde{\text{Banach}} \) mapping. Then \( A \cap B \) is nonempty and \( T \) has a unique fixed point in \( A \cap B \).

**Proof.** In theorem 1, Taking \( \beta = 0 \). \( \square \)

**Corollary 3.** Let \( (X,d) \) is a complete metric space and \( T : X \to X \) be a self-map satisfying,

\[
d(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Tx) + \beta d(y,Ty)
\]

for all \( x,y \in X \), \( (\alpha + 2\beta) \in [0,1) \). Then \( T \) has a unique fixed point in \( X \).

**Proof.** Since every metric space is also a quasi-partial \( b \)-metric space; with \( s = 1 \), so as a particular case of theorem 1, if the underlying space is taken to be a metric space; sets \( A = B = X \); then the corresponding corollary 3 follows. It is worth observing that the
condition of sets $A$ and $B$ being closed is dropped in the above corollary by not assuming the space to be closed. This is justified as in the proof of the theorem, the closeness of sets $A$ and $B$ guaranteed that the fixed point thus obtained belongs to both $A$ and $B$. Since $A = B = X$, so the fixed point obtained naturally belongs to $X$. □

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