\( \mathcal{N} = 4 \) Supersymmetric Landau Models

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ABSTRACT

We present the first example of super Landau model with both \( \mathcal{N}=4 \) worldline supersymmetry and non-trivial target space supersymmetry \( ISU(2|2) \). The model also reveals a hidden second \( \mathcal{N}=4 \) supersymmetry which, together with the manifest one, close on a worldline \( SU(2|2) \). We start from an off-shell action in bi-harmonic \( \mathcal{N}=4, d=1 \) superspace and come to the component action with four bosonic and four fermionic fields. Its bosonic core is the action of generalized \( U(1) \) Landau model on \( \mathbb{R}^4 \) considered some time ago by Elvang and Polchinski. At each Landau level \( \mathcal{N} > 0 \) the wave functions are shown to form “atypical” \( (2\mathcal{N}+2) \)-dimensional multiplets of the worldline supergroup \( SU(2|2) \). Some states have negative norms, but this trouble can be evaded by redefining the inner product, like in other super Landau models. We promote the action to the most general form compatible with off-shell \( \mathcal{N}=4 \) worldline supersymmetry and find the corresponding background \( U(1) \) gauge field to be generic self-dual on \( \mathbb{R}^4 \) and the target superspace metric to remain flat.

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1 Introduction

After the pioneering paper [1], the name Landau model is often used for any quantum-mechanical problem in which a charged particle moves over some manifold in the background of an external gauge field. Besides the original planar 2D Landau model, with the particle moving on a plane under the influence of uniform magnetic field orthogonal to the plane, much attention was paid to some its curved generalizations, e.g. the Haldane model [2], where a charged particle moves on a two-dimensional sphere $S^2 \sim SU(2)/U(1)$ in the field of magnetic monopole placed at the center, as well as to some its higher-dimensional versions, with both abelian and non-abelian gauge fields (see, e.g. [3, 4, 5, 6]). The Landau problem and its generalizations have a lot of applications in various areas. In particular, they constitute a theoretical basis of quantum Hall effect (QHE). Their most characteristic feature is the presence of Landau Levels (LL) in the energy spectrum, such that the gap between the ground state, i.e. the Lowest Landau Level (LLL), and the excited LLs rapidly grows with growth of the strength of the external gauge field. Thus in the limit of strong external field only the LLL is relevant. In the lagrangian language, such a system is described by $d=1$ Wess-Zumino (or Chern-Simons) action, with the phase space being a non-commutative version of the original configuration space. This intimate connection with non-commutative geometry was one of the basic reasons of great revival of interest in Landau-type models during the past decade.

Superextensions of the Landau model are models of non-relativistic particles moving on supergroup manifolds. The study of such models can help to reveal possible manifestations of supersymmetry in various versions of QHE (including the so called spin-QHE) and, perhaps, in other condensed matter systems. From the mathematical point of view, superextended Landau models should bear a close relation to the non-commutative supergeometry. It is also worth pointing out that sigma models with the supergroup target spaces attract a lot of attention for the last years due to their intimate relation to superbranes (see, e.g., [7, 8, 9]). The super Landau models can be expected to follow from some of these sigma models via dimensional reduction.

The Landau problems on the $(2|2)$-dimensional supersphere $SU(2|1)/U(1|1)$ and the $(2|4)$-dimensional superflag $SU(2|1)/[U(1) \times U(1)]$ as the simplest superextensions of the $S^2$ Haldane model were considered in [10, 11, 12]. In order to better understand the common features of the super Landau models, the planar limits of these models (with the curved target supermanifolds becoming the $(2|2)$- and $(2|4)$-dimensional superplanes) were also studied [13, 14, 15, 16]. They are superextensions of the original Landau model and exhibit some surprising features.

First, the space of quantum states in these models involves ghosts, i.e. the states with negative norms, which seemingly leads to violation of unitarity. The appearance of ghosts in $d=1$ supersymmetric models with second order kinetic terms for fermions was earlier noticed by Volkov and Pashnev [17]. The planar super Landau models suggest a simple mechanism of evading the ghost problem. It was shown in [15] that all norms of states in the superplane models can be made non-negative at cost of introducing a proper metric operator in Hilbert space and so redefining the inner product. There appear no difficulties with unitarity after such a redefinition.

The second feature closely related to the one just mentioned is that the passing to the new inner product (and so to the new definition of hermitian conjugation) makes manifest
the hidden worldline $\mathcal{N}=2$ supersymmetry of the superplanar models, which so supply examples of $\mathcal{N}=2$ supersymmetric quantum mechanics.

The presence of this worldline $\mathcal{N}=2$ supersymmetry was used in [18] to re-derive the $(2|2)$ superplane Landau model from the manifestly $\mathcal{N}=2$ supersymmetric worldline superfield formalism. It was proposed there to construct new types of superextended Landau models, starting just from the superfield formalism, with the manifest worldline supersymmetry as an input. New $\mathcal{N}=2$ supersymmetric models were constructed in this way in [16]. They are generalizations of the superplane model to the case with non-trivial target superspace metric and external gauge field.

In the present paper we apply the same approach to construct the first example of $\mathcal{N}=4$ supersymmetric Landau model with a non-trivial target space supergroup. From the geometric point of view, it is a Landau-type model on the flat $(4|4)$-dimensional target superspace $\text{ISU}(2|2)/\text{SU}(2|2)$ extending the Euclidean space $\mathbb{R}^4$, with an extra worldline $\mathcal{N}=4$ supersymmetry. Its off-shell action is formulated in terms of two superfields, bosonic and fermionic, defined in the bi-harmonic $\mathcal{N}=4, d=1$ superspace [19].

The paper is organized as follows. In Sect. 2 we recall the salient feature of the $\mathcal{N}=2$ superfield construction of the $(2|2)$ superplane Landau model [18]. In Sect. 3 we give a short account of the bi-harmonic $\mathcal{N}=4, d=1$ superfield approach [19]. The superfield and component actions of the new $\mathcal{N}=4$ super Landau model are constructed in Sect. 4. The final action involves four bosonic and four fermionic $d=1$ fields and contains a coupling to some external linear self-dual gauge field on $\mathbb{R}^4$. We show that, besides the worldline $\mathcal{N}=4$ supersymmetry, the model has the target $\text{ISU}(2|2)$ symmetry. The quantization is performed in Sect. 5. We show that at each Landau level $N$ the wave functions form the multiplets of both target $\text{ISU}(2|2)$ and worldline $\mathcal{N}=4$ supersymmetries. To avoid negative norms for some wave functions, it proves necessary to properly redefine the inner product in the space of quantum states, like in the previously studied super Landau models [15, 16, 12]. Sect. 6 is devoted to the further analysis of the symmetry structure of the model. In particular, we find out the existence of the second (on-shell) worldline $\mathcal{N}=4$ supersymmetry, which, together with the first one, close on a worldline $SU(2|2)$ supersymmetry. The wave functions form “atypical” multiplets of the latter. In Sect. 7 we consider a generalization of the constructed $\mathcal{N}=4$ super Landau model along the lines of ref. [16]. The corresponding external bosonic gauge field proves to be generic self-dual on $\mathbb{R}^4$, while the target superspace metric remains flat, as opposed to the $\mathcal{N}=2$ models of ref. [16]. Some problems for the future study are outlined in the concluding Sect. 8. Appendices A and B contain some technical details.

2 ISU$(1|1)$ super Landau model from $\mathcal{N} = 2, d = 1$ superspace

In this Section we remind some basic features of the manifestly $\mathcal{N}=2$ supersymmetric formulation of the $(2|2)$ superplane Landau model [18].
2.1 Superfield and component actions

We start with the necessary definitions. The basic objects are two $\mathcal{N}=2, d=1$ chiral bosonic and fermionic superfields $\Phi$ and $\Psi$ of the same dimension. The real $\mathcal{N}=2, d=1$ superspace is parametrized as:

$$(\tau, \theta, \bar{\theta}).$$

\hspace{1cm} (2.1)

The left- and right-handed chiral $d=1$ superspaces are defined as the coordinate sets

$$(t_L, \theta), \quad (t_R, \bar{\theta}), \quad t_L = \tau + i\theta\bar{\theta}, t_R = \tau - i\theta\bar{\theta}. \hspace{1cm} (2.2)$$

It will be convenient to work in the left-chiral basis, so for brevity we will use the notation $t_L \equiv t$, $t_R = t - 2i\theta\bar{\theta}$. In this basis, the $\mathcal{N}=2$ covariant derivatives are defined by

$$\bar{D} = -\frac{\partial}{\partial\bar{\theta}}, \quad D = \frac{\partial}{\partial\theta} + 2i\bar{\theta}\partial_t, \quad \{D, \bar{D}\} = -2i\partial_t, \quad D^2 = \bar{D}^2 = 0. \hspace{1cm} (2.3)$$

The chiral superfields $\Phi$ and $\Psi$ obey the conditions

$$\bar{D}\Phi = \bar{D}\Psi = 0 \hspace{1cm} (2.4)$$

and, in the left-chiral basis, have the following component field contents:

$$\Phi(t, \theta) = z(t) + \theta\chi(t), \quad \Psi(t, \theta) = \zeta(t) + \theta\psi(t), \hspace{1cm} (2.5)$$

where the complex fields $z(t)$, $h(t)$ are bosonic and $\zeta(t)$, $\psi(t)$ are fermionic. The conjugated superfields, in the same basis, have the following $\theta$-expansions:

$$\bar{\Phi} = \bar{z} - \bar{\theta}\bar{\chi} - 2i\theta\bar{\theta}\bar{\dot{z}}, \quad \bar{\Psi} = \bar{\zeta} + \bar{\theta}\bar{h} - 2i\theta\bar{\theta}\bar{\dot{\zeta}}. \hspace{1cm} (2.6)$$

The superfield action yielding in components the superplane model action of ref. [13, 14, 15] reads

$$S = -\kappa \int dt d^2\theta \left( \Phi\bar{D}\Psi + \Psi\bar{D}\Phi + \frac{1}{2\sqrt{\kappa}[\Phi\bar{D}\Psi - \Phi\bar{D}\Psi]} \right) = \int dt L. \hspace{1cm} (2.7)$$

Here $\kappa$ is a real parameter. The Berezin integral is normalized as

$$\int d^2\theta(\theta\bar{\theta}) = 1. \hspace{1cm} (2.8)$$

After doing the Berezin integral, we find

$$L = 2i\kappa(z\bar{\dot{z}} + \bar{\zeta}\dot{\zeta}) - \kappa(\chi\bar{\dot{\chi}} + h\bar{\dot{h}}) + i\sqrt{\kappa}(z\dot{h} + \chi\dot{\zeta} + \bar{z}\bar{\dot{h}} + \bar{\chi}\bar{\dot{\zeta}}). \hspace{1cm} (2.9)$$

The fields $h$ and $\chi$ are auxiliary and can be eliminated by their equations of motion

$$h = \frac{i}{\sqrt{\kappa}}\bar{\dot{z}}, \quad \chi = -\frac{i}{\sqrt{\kappa}}\bar{\dot{\zeta}}. \hspace{1cm} (2.10)$$
Upon substituting this into the Lagrangian and integrating by parts, we obtain

\[ L = i \kappa (z \dot{\bar{z}} - \dot{z} \bar{z} + \dot{\zeta} \bar{\zeta} + \dot{\bar{\zeta}} \zeta) + (\dot{\bar{z}} \zeta + \dot{z} \bar{\zeta}). \] (2.11)

This is the superplane model component Lagrangian \[13, 15\]. By construction, the superfield action (2.7) is \( \mathcal{N}=2 \) supersymmetric, so are the component Lagrangians (2.9) and (2.11). The \( \mathcal{N}=2 \) transformations of the component fields can be found from

\[ \delta \Phi = -[\epsilon Q - \bar{\epsilon} \bar{Q}] \Phi, \quad \delta \Psi = -[\epsilon Q - \bar{\epsilon} \bar{Q}] \Psi, \] (2.12)

where, in the left-chiral basis,

\[ Q = \frac{\partial}{\partial \theta}, \quad \bar{Q} = -\frac{\partial}{\partial \bar{\theta}} + 2i \theta \frac{\partial}{\partial t}, \quad \{Q, \bar{Q}\} = 2i \frac{\partial}{\partial t}. \] (2.13)

It follows from (2.12), (2.13) that, off shell,

\[ \delta z = -\epsilon \chi, \quad \delta \chi = -2i \bar{\epsilon} \dot{\bar{z}} - \epsilon \ddot{z}, \quad \delta \zeta = -\bar{\epsilon} \bar{h}, \quad \delta h = -2i \bar{\epsilon} \dot{\zeta}. \] (2.14)

With the on-shell values (2.10) for the auxiliary fields, these transformations become

\[ \delta z = \frac{i}{\sqrt{\kappa}} \epsilon \dot{\bar{z}}, \quad \delta \zeta = \frac{i}{\sqrt{\kappa}} \bar{\epsilon} \dot{z}. \] (2.15)

These are basically the same transformation laws as those found in \[16\] (up to rescaling of \( \epsilon, \bar{\epsilon} \)). As usual, they close on \( t \)-translations only with making use of the equations of motion for physical fields, while (2.14) close without any help from the equations of motion.

It is worth pointing out that the \( \mathcal{N}=2 \) superfield formulation of the superplane Landau model described above is well defined only at \( \kappa \neq 0 \).

### 2.2 \textit{ISU}(1|1) symmetry

Besides \( \mathcal{N}=2 \) supersymmetry, the superplane model also possesses the target space graded \( ISU(1|1) \) symmetry.

The inhomogeneous translation part of this internal supersymmetry acts as constant shifts of superfield:

\[ \delta \Phi = b, \quad \delta \Psi = \bar{\nu}, \] (2.16)

where \( b \) and \( \nu \) are even and odd complex parameters. They just produce shifts of the fields \( z \) and \( \zeta \)

\[ \delta z = b, \quad \delta \zeta = \nu. \] (2.17)

The fermionic transformations of the homogeneous \( SU(1|1) \) part are realized as

\[ \delta \Phi = \bar{D} \left( \omega \bar{\theta} \Psi - \frac{1}{2\sqrt{\kappa}} \bar{\omega} \theta D \Phi \right), \quad \delta \Psi = \bar{D} \left( \omega \bar{\theta} \bar{\Phi} - \frac{1}{2\sqrt{\kappa}} \bar{\omega} \theta D \Psi \right), \] (2.18)

where \( \omega \) and \( \bar{\omega} \) are the relevant complex Grassmann parameters. They close on the bosonic \( U(1) \) transformations

\[ \delta \Phi = i \alpha \left\{ \Phi - \bar{D} \left[ \theta \bar{\theta} \left( D \Phi + i \frac{\dot{\bar{\Phi}}}{\sqrt{\kappa}} \right) \right] \right\}, \quad \delta \Psi = -i \alpha \left\{ \Psi - \bar{D} \left[ \theta \bar{\theta} \left( D \Psi - i \frac{\dot{\bar{\Phi}}}{\sqrt{\kappa}} \right) \right] \right\}. \] (2.19)
Though these superfield $SU(1|1)$ rotations look rather cumbersome, they give rise to the very simple off-shell transformations of the physical fields $z$ and $\zeta$:

$$\delta \left( \begin{array}{c} z \\ \zeta \end{array} \right) = \left( \begin{array}{cc} i\alpha & \omega \\ \bar{\omega} & i\alpha \end{array} \right) \left( \begin{array}{c} z \\ \zeta \end{array} \right).$$

(2.20)

The transformations of the auxiliary fields are

$$\delta \omega = -i\alpha \bar{\omega}, \quad \delta \alpha = \frac{1}{\sqrt{\kappa}} \alpha, \quad \delta h = \frac{1}{\sqrt{\kappa}} \alpha \dot{\zeta}.$$  

They are consistent with the on-shell expressions (2.10).

In quantum theory, the generators associated with the target supertranslations (2.17) and super-rotations (2.20) are given by the expressions [13, 15]

$$P_z = -i(\partial_z + k\bar{\zeta}), \quad P_{\bar{z}} = -i(\partial_{\bar{z}} - k\zeta), \quad \Pi_\zeta = \partial_\zeta + k\bar{\zeta}, \quad \Pi_{\bar{\zeta}} = \partial_{\bar{\zeta}} + k\zeta$$

(2.21)

and

$$Q = z\partial_\zeta - \bar{\zeta}\partial_z, \quad \bar{Q} = \bar{z}\partial_{\bar{\zeta}} + \zeta\partial_z, \quad C = z\partial_\zeta + \zeta\partial_\zeta - \bar{z}\partial_{\bar{\zeta}} - \bar{\zeta}\partial\bar{\zeta}.$$  

(2.22)

These generators form the algebra of the supergroup $ISU(1|1)$

$$(P_z, P_{\bar{z}}, \Pi_\zeta, \Pi_{\bar{\zeta}}) \rtimes SU(1|1) = ISU(1|1).$$

(2.23)

Note that the parameter $\kappa$ plays the role of central charge in the quantum algebra of supertranslations:

$$[P_z, P_{\bar{z}}] = 2\kappa, \quad \{\Pi_\zeta, \Pi_{\bar{\zeta}}\} = 2\kappa.$$  

(2.24)

The structure of the space of quantum states of the $\mathcal{N}=2$ super-Landau model, the realization of various symmetry generators in it, as well as the explicit form of the metric operator making norms of all states positive-definite can be found in [15].

3 Bi-harmonic $\mathcal{N} = 4$ superspace: basic notions

Our aim will be to construct a generalization of the $\mathcal{N}=2$ model of the previous Section, such that it possesses the worldline $\mathcal{N}=4$ supersymmetry. Such an extension is not unique; leaving the study of all possible versions of it for the future, here we will do this by extending the previously used bosonic and fermionic chiral $(2, 2, 0)$ and $(0, 2, 2)$ multiplets to the $(4, 4, 0)$ and $(0, 4, 4)$ multiplets of $\mathcal{N}=4$ supersymmetry. It turns out that these bosonic and fermionic $\mathcal{N}=4$ multiplets should be “mirror” (or “twisted”) to each other. The natural framework for a simultaneous description of these two different sorts of $\mathcal{N}=4$ multiplets is provided by the bi-harmonic $\mathcal{N}=4, d=1$ superspace [19] which is an extension of the more familiar harmonic superspace involving one set of $SU(2)$ harmonic variables [20, 21, 22]. Here we briefly outline this universal approach.

We begin with the ordinary $\mathcal{N}=4, d=1$ superspace in the notation with both $SU(2)$ automorphism groups being manifest. It is defined as the set of coordinates

$$z := (t, \theta^a),$$

(3.1)
in which $\mathcal{N}=4$, $d=1$ supersymmetry is realized by means of the transformations

$$\delta t = -ie^{ia}\theta_{ia}, \quad \delta \theta^{ia} = \epsilon^{ia}. \quad (3.2)$$

The Grassmann coordinate $\theta^{ia}$ (as well as the parameters $\epsilon^{ia}$) form a real quartet of the full automorphism group $SO(4) \sim SU(2)_L \times SU(2)_R$, $(\theta^{ia}) = \theta_{ia} = \epsilon_{ik} \epsilon_{ab} \theta^{kb}$. The indices $i$ and $a$ are doublet indices of the left and right $SU(2)$ automorphism groups, respectively. The corresponding covariant spinor derivatives are defined as

$$D_{ia} = \frac{\partial}{\partial \theta^{ia}} + i\theta_{ia}\partial_t, \quad \bar{D}^{ia} = -\frac{\partial}{\partial \theta_{ia}} - i\theta^{ia}\partial_t = -\epsilon^{ik} \epsilon^{ab} D_{kb}, \quad (3.3)$$

$$\{D_{ia}, D_{kb}\} = 2i\epsilon_{ik} \epsilon_{ab} \partial_t. \quad (3.4)$$

In the central basis, the $\mathcal{N}=4$, $d=1$ bi-harmonic superspace (bi-HSS) is defined as the following extension of (3.1)

$$(z, u, v) := (t, \theta^{ia}, u_{i}^{\pm 1}, v_{a}^{\pm 1}). \quad (3.5)$$

Here $u_{i}^{\pm 1} \in SU(2)_L/U(1)_L$ and $v_{a}^{\pm 1} \in SU(2)_R/U(1)_R$ are two independent sets of $SU(2)$ harmonic variables. The harmonics $u_{i}^{\pm 1}$ satisfy the standard relations $[20, 21]$

$$u_{i}^{-1} = (u_{i}^{1}), \quad u_{i}^{1}u_{i}^{-1} = 1 \Leftrightarrow u_{i}^{1}u_{k}^{-1} - u_{k}^{1}u_{i}^{-1} = \epsilon_{ik}. \quad (3.6)$$

The same relations are valid for $v_{a}^{\pm 1}$, with the change $i \rightarrow a, b$.

A specific feature of the $\mathcal{N}=4$, $d=1$ bi-HSS is the existence of two types of analytic bases with the analytic subspaces including half of the Grassmann variables, as compared to the full Grassmann dimension four of bi-HSS. These two analytic bases are spanned by the following coordinate sets

$$(z_+, u, v) := (t_+ = t + i(\theta^{1,1}\theta^{-1,-1} + \theta^{-1,1}\theta^{1,-1}), \theta^{1,1}, \theta^{1,-1}, \theta^{-1,1}, \theta^{-1,-1}, u_{i}^{\pm 1}, v_{a}^{\pm 1}), \quad (3.7)$$

$$(z_-, u, v) := (t_- = t + i(\theta^{1,1}\theta^{-1,-1} - \theta^{-1,1}\theta^{1,-1}), \theta^{1,1}, \theta^{1,-1}, \theta^{-1,1}, \theta^{-1,-1}, u_{i}^{\pm 1}, v_{a}^{\pm 1}), \quad (3.8)$$

where

$$\theta^{m,n} := \theta^{ia}u_{i}^{m}v_{a}^{n}, \quad m, n = \pm 1. \quad (3.9)$$

Defining harmonic projections of the spinor derivatives as

$$D^{m,n} = D^{ia}u_{i}^{m}v_{a}^{n}, \quad (3.10)$$

$$(D^{1,1})^2 = (D^{1,-1})^2 = (D^{-1,1})^2 = (D^{-1,-1})^2 = \{D^{\pm 1,1}, D^{\pm 1,-1}\} = \{D^{1,\pm 1}, D^{-1,\pm 1}\} = 0, \quad (3.11)$$

$$\{D^{1,1}, D^{-1,1}\} = -\{D^{1,-1}, D^{-1,-1}\} = 2i\partial_t, \quad (3.12)$$

it is easy to show that, in the above bases, they have the form

$$D^{1,1} = \frac{\partial}{\partial \theta^{1,1}}, \quad D^{1,-1} = -\frac{\partial}{\partial \theta^{-1,1}}, \quad D^{-1,1} = -\frac{\partial}{\partial \theta^{-1,-1}} + 2i\theta^{-1,1}\partial_{t_+}, \quad D^{-1,-1} = \frac{\partial}{\partial \theta^{1,-1}} + 2i\theta^{-1,-1}\partial_{t_+}, \quad (3.13)$$
and
\[ D^{1,1} = \frac{\partial}{\partial \theta^{-1,-1}}, \quad D^{-1,1} = -\frac{\partial}{\partial \theta^{1,-1}}, \]
\[ D^{1,-1} = -\frac{\partial}{\partial \theta^{-1,1}} + 2i\theta^{-1,-1}\partial_\nu, \quad D^{-1,-1} = \frac{\partial}{\partial \theta^{1,1}} + 2i\theta^{-1,-1}\partial_\nu. \] (3.14)
The fact that two different pairs of covariant spinor derivatives are reduced to the partial derivatives in these bases implies the existence of two analytic subspaces which are closed under the full \( \mathcal{N}=4 \) supersymmetry. Hence there are two sorts of analytic superfields defined as unconstrained functions on these analytical superspaces:
\[ (\zeta_+, u, v) := (t_+^1, \theta^{1,1}, t_1^{-1}, u_i^{\pm 1}, v_a^{\pm 1}), \] (3.15)
\[ D^{1,1}\Phi_\ell = D^{-1,1}\Phi_\ell = 0 \Rightarrow \Phi_\ell = \phi_\ell(\zeta_+, u, v), \] (3.16)
and
\[ (\zeta_-, u, v) := (t_-^1, \theta^{1,1}, t_1^{-1}, u_i^{\pm 1}, v_a^{\pm 1}), \] (3.17)
\[ D^{1,1}\Phi_\ell = D^{-1,1}\Phi_\ell = 0 \Rightarrow \Phi_\ell = \phi_\ell(\zeta_-, u, v). \] (3.18)
The analytic superspaces are real with respect to some generalized \( \sim \) conjugation the implementation of which on coordinates can be found in \([19]\). As a consequence, one can impose proper reality conditions on the analytic superfields.

In the harmonic superspace approach, harmonic derivatives play an important role. The harmonic derivatives with respect to harmonics \( u_i^{\pm 1} \) and \( v_a^{\pm 1} \) in the central basis are defined as
\[ \partial^{\pm 2,0} = u_i^{\pm 1} \frac{\partial}{\partial u_i^{\mp 1}}, \quad \partial^{0,\pm 2} = v_a^{\pm 1} \frac{\partial}{\partial v_a^{\mp 1}}, \] (3.19)
\[ \partial_u = u_i^{1} \frac{\partial}{\partial u_i^{1}}, \quad \partial_v = v_a^{1} \frac{\partial}{\partial v_a^{1}} - v_a^{-1} \frac{\partial}{\partial v_a^{-1}}. \] (3.20)
These sets form two mutually commuting \( SU(2) \) algebras
\[ [\partial^{2,0}, \partial^{-2,0}] = \partial_u^0, \quad [\partial_u^0, \partial^{\pm 2,0}] = \pm 2\partial^{\pm 2,0}, \] (3.21)
\[ [\partial^{0,2}, \partial^{-2,0}] = \partial_v^0, \quad [\partial_v^0, \partial^{0,\pm 2}] = \pm 2\partial^{0,\pm 2}. \] (3.22)
In the analytic bases \([3.7]\) and \([3.8]\) the harmonic derivatives acquire additional terms. For example, in the basis \([3.7]\):
\[ D^{\pm 2,0} = \partial^{\pm 2,0} \pm 2i\theta^{\pm 1,\mp 1} \theta^{\pm 1,\mp 1} \partial_\nu + \theta^{\pm 1,\pm 1} \frac{\partial}{\partial \theta^{\mp 1,\pm 1}} + \theta^{\pm 1,\mp 1} \frac{\partial}{\partial \theta^{\mp 1,\pm 1}}, \] (3.23)
\[ D^{0,\pm 2} = \partial^{0,\pm 2} \pm \theta^{\pm 1,\mp 1} \frac{\partial}{\partial \theta^{\pm 1,\mp 1}} + \theta^{\mp 1,\pm 1} \frac{\partial}{\partial \theta^{\mp 1,\pm 1}}, \] (3.24)
\[ D_u^0 = \partial_u^0 + \left( \theta^{1,1} \frac{\partial}{\partial \theta^{1,1}} + \theta^{1,-1} \frac{\partial}{\partial \theta^{1,-1}} - \theta^{-1,1} \frac{\partial}{\partial \theta^{-1,1}} - \theta^{-1,-1} \frac{\partial}{\partial \theta^{-1,-1}} \right), \] (3.25)
\[ D_v^0 = \partial_v^0 + \left( \theta^{1,1} \frac{\partial}{\partial \theta^{1,1}} + \theta^{1,-1} \frac{\partial}{\partial \theta^{1,-1}} - \theta^{-1,1} \frac{\partial}{\partial \theta^{-1,1}} - \theta^{-1,-1} \frac{\partial}{\partial \theta^{-1,-1}} \right). \] (3.26)
Their commutation relations, being basis-independent, are given by the same formulas \([3.21]\) and \([3.22]\).
Let us define integration measures on the full $\mathcal{N}=4$, $d=1$ bi-HSS and on its analytic subspaces:

**Full bi-HSS:**
\[
\int \mu := \int dt du dv (D^{-1,-1}D^{-1,1}D^{1,1}D^{1,-1}),
\]
(3.27)

**Analytic bi-HSS 1:**
\[
\int \mu^{(-2,0)}_{A^+} := \int dt_+ du dv (D^{-1,-1}D^{-1,1}),
\]
(3.28)

**Analytic bi-HSS 2:**
\[
\int \mu^{(0,-2)}_{A^-} := \int dt_- du dv (D^{-1,-1}D^{1,1}).
\]
(3.29)

They are normalized in such a way that
\[
\int \mu (\theta^{-1,-1}\theta^{-1,1}\theta^{1,1}\theta^{1,-1}) \times ... = \int dt du dv \times ..., \quad (3.30)
\]
\[
\int \mu^{(-2,0)}_{A^+} (\theta^{1,1}\theta^{1,-1}) \times ... = \int dt_+ du dv \times ..., \quad (3.31)
\]
\[
\int \mu^{(0,-2)}_{A^-} (\theta^{1,1}\theta^{-1,1}) \times ... = \int dt_- du dv \times ... \quad (3.32)
\]

Finally, it is worth recalling the rules of integration over harmonic variables. Symmetric monomials constructed from $u^\pm$
\[
(u^1)^m (u^{-1})^n \equiv u^{i_1} ... u^{i_m} u^{-1 j_1} ... u^{-1 j_n},
\]
(3.33)
form orthogonal basis in the space of the functions on the 2-sphere $S^2$:
\[
\int du (u^1)^m (u^{-1})^n (u^1)^k (u^{-1})_l = \frac{(-1)^n n!}{(m + n + 1)!} \delta^{i_1} \delta^{i_2} \delta^{j_1} \delta^{j_2} \delta_{m l} \delta_{n k}. \quad (3.34)
\]

In what follows we shall need only the special case of this formula
\[
\int du u_i^1 u_j^{-1} = \frac{1}{2} \epsilon_{ij}. \quad (3.35)
\]
The similar relations hold for $v_{a^1}^\pm$.

The properties and relations quoted here are sufficient for construction of $\mathcal{N}=4$ supersymmetric Landau model.

## 4 Landau model with $\mathcal{N} = 4$ supersymmetry

### 4.1 Superfield and component actions

Superfields $q^{1,0A}$ and $\psi^{0,1B}$ ($A,B = 1,2$) will be the basic elements of our $\mathcal{N}=4$ supersymmetric Landau model. These superfields are, respectively, bosonic and fermionic, and
they have the fields contents \((4, 4, 0)\) and \((0, 4, 4)\). We impose on them the following analytic and harmonic constraints

\[
(a) \quad D^{1,1} q^{1,0,A} = D^{1,-1} q^{1,0,A} = 0, \quad (b) \quad D^{2,0} q^{1,0,A} = D^{0,2} q^{1,0,A} = 0. \tag{4.1}
\]

and

\[
(a) \quad D^{1,1} \psi^{0,1,A} = D^{1,-1} \psi^{0,1,A} = 0, \quad (b) \quad D^{2,0} \psi^{0,1,A} = D^{0,2} \psi^{0,1,A} = 0. \tag{4.2}
\]

The first conditions in \((4.1)\) and \((4.2)\) tell us that the superfields \(q^{1,0,A}\) and \(\psi^{0,1B}\) "live" on the analytic subspaces \((\zeta_+, u, v)\) and \((\zeta_-, u, v)\), respectively. Taking into account the reality conditions \((q^{1,0,A} = \epsilon_{AB} q^{1,0B})\) we can then solve the condition \((4.1b)\) and obtain the following final component expansion for the superfield \(q^{1,0,A}\):

\[
q^{1,0,A} = f^{iA}(t_+) u_i^1 + \psi^{aA}(t_+) v_a^{-1} \theta^{1,1} - \psi^{aA}(t_+) v_a^{1} \theta^{-1,1} - 2i \tilde{f}^{iA}(t_+) u_i^{-1} \theta^{1,1} \theta^{-1,1}. \tag{4.3}
\]

Similarly, the component expansion for the superfield \(\psi^{0,1A}\) \((\psi^{0,1A} = \epsilon_{AB} \psi^{0,1B})\) reads:

\[
\psi^{0,1A} = \chi^{aA}(t_-) v_a^1 + h^{iA}(t_-) u_i^{1} \theta^{-1,1} - h^{iA}(t_-) u_i^{-1} \theta^{1,1} - 2i \tilde{h}^{iA}(t_-) v_a^{-1} \theta^{1,1} \theta^{-1,1}. \tag{4.4}
\]

In order to construct \(\cN=4\) supersymmetric Landau model action, we need one more object, namely, the superfield \(V^{1,0,A} = D^{1,-1} \psi^{0,1,A}\). It is easy to show that \(V^{1,0,A}\) "live" on the subspace \((\zeta_+, u, v)\), since \(D^{1,1} V^{1,0,A} = D^{1,-1} V^{1,0,A} = 0\). We obtain:

\[
V^{1,0,A} = -h^{iA}(t_-) u_i^{1} + 2i \tilde{h}^{iA}(t_-) v_a^{1} \theta^{1,1} - 2i \tilde{h}^{iA}(t_-) v_a^{-1} \theta^{-1,1} + 2i \tilde{h}^{iA}(t_-) u_i^{-1} \theta^{1,1} \theta^{-1,1}. \tag{4.5}
\]

The fields \(f^{iA}(t_+), h^{iA}(t_-)\) are real bosonic, while the fields \(\psi^{aA}(t_+), \chi^{aA}(t_-)\) are real fermionic. The reality conditions are as follows

\[
\tilde{f}^{iA} = \epsilon_{ij} \epsilon_{AB} f^{jB}, \quad \tilde{h}^{iA} = \epsilon_{ij} \epsilon_{AB} h^{jB}, \quad \tilde{\psi}^{aA} = \epsilon_{ab} \epsilon_{AB} \psi^{bB}, \quad \tilde{\chi}^{aA} = \epsilon_{ab} \epsilon_{AB} \chi^{bB}. \tag{4.6}
\]

Now we can construct the superfield action for \(\cN=4\) supersymmetric \((4|4)\) Landau model as a natural generalization of the \(\cN=2\) action \((2.7)\)

\[
S = \frac{\kappa}{2\ell} \left( \int \mu^{-2,0} q^{1,0A} q^{1,0B} C_{AB} - i \int \mu^{0,-2} \psi^{0,1A} \psi^{0,1B} \epsilon_{AB} + \frac{1}{\sqrt{\kappa}} \int \mu^{-2,0} q^{1,0A} D^{1,-1} \psi^{0,1B} \epsilon_{AB} \right). \tag{4.7}
\]

Here, \(C_{AB}\) and \(\epsilon_{AB}\) are symmetric and standard skew-symmetric constant tensors, respectively, \(\kappa \neq 0\) is a constant. Without loss of generality, we can choose

\[
C^{A B} C_{A B} = 2. \tag{4.8}
\]

In fact, we started from the most more general action, with some arbitrary coefficients and constant matrices before the three terms in \((4.7)\), and found that it can be reduced to the form \((4.7)\) with the condition \((4.8)\) after some redefinitions of the involved superfields.

\(\text{Division of these sets into physical and auxiliary fields depends on the choice of the invariant action. Like in the } \cN=2 \text{ case, the fermionic fields in } q^{1,0,A} \text{ and the additional bosonic fields in } \psi^{0,1B} \text{ will be auxiliary, while the rest of fields will be physical. This deviation from the standard divisions of such multiplets into the physical and auxiliary subsets is of course related to the fact that } q^{1,0,A} \text{ and } \psi^{0,1B} \text{ have the same dimension, which is necessary for realizing on them some internal supersymmetry generalizing ISU}(1|1) \text{ symmetry of the } \cN=2 \text{ case. Correspondingly, physical bosons and fermions will enter the component action on equal footing, with the second-order kinetic terms.}\)
The first two terms in (4.7) are direct analogs of the first two terms in (2.7), while the third interaction term is an analog of the third term in (2.7). It is important to point out that the necessity to use the “mirror” fermionic superfield $\psi^{0,1A}$ comes just as a necessary condition for constructing this interaction term. A simple analysis shows that no such bilinear interaction terms can be constructed from the bosonic and fermionic superfields of the same harmonic analyticity. Also note that this mixed term admits a “dual” representation as an integral over another analytic subspace:

$$\sim \int \mu^{0,-2} \tilde{V}^{0,1A} \psi^{0,1B} \rho \epsilon_{AB}, \quad \tilde{V}^{0,1A} = D^{-1,1} q^{1,0A}.$$  

Now we are prepared to derive the component Lagrangian of the model by performing integration over Grassmann and harmonic variables. We obtain:

$$L = \frac{\kappa}{2i} \left[ (2i \hat{f}^{iA} f^B - \psi^{aA} \psi^B) C_{AB} + (2 \tilde{\chi}^{aA} \chi^B - ih^{iA} h_i^B) \epsilon_{AB} - \frac{2i}{\sqrt{\kappa}} \left( \hat{f}^{iA} h_i^B + \psi^{aA} \tilde{\chi}_a^B \right) \epsilon_{AB} \right]. \quad (4.9)$$

The fields $h^{iA}$ and $\psi^{aA}$ enter this Lagrangian without time derivatives and, hence, are auxiliary fields. The remaining $(4 + 4)$ fields $f^{iA}$ and $\chi^{aA}$ are physical. After eliminating the auxiliary fields by their algebraic equations of motion,

$$\frac{\partial L}{\partial h^{iA}} = 0 \Rightarrow h_{iA} = - \frac{1}{\sqrt{\kappa}} \hat{f}_{iA}, \quad (4.10)$$

$$\frac{\partial L}{\partial \psi^{aA}} = 0 \Rightarrow \psi_{aA} = \frac{i}{\sqrt{\kappa}} C_{AB} \tilde{\chi}_a^B, \quad (4.11)$$

and substituting these expressions back into (4.9), we obtain

$$L = \kappa C_{AB} \hat{f}^{iA} f_i^B - i \kappa \tilde{\chi}^{aA} \chi_{aA} + \frac{1}{2} \left( \hat{f}^{iA} \hat{f}_{iA} + i C_{AB} \tilde{\chi}^{aA} \tilde{\chi}_a^B \right). \quad (4.12)$$

This Lagrangian is the sought $\mathcal{N}=4$ supersymmetric extension of the $\mathcal{N}=2$ Landau model Lagrangian (2.9). It includes four real bosonic fields, so what we obtained is a superextension of the bosonic Landau-type model, in which a particle moves over four-dimensional Euclidean space $\mathbb{R}^4$ in an external U(1) gauge field. This coupling is provided just by the first term in (4.12). It can be rewritten as

$$\mathcal{A}_{iB} \hat{f}^{iB}, \quad \mathcal{A}_{iB} = - \kappa C^D_B f_i^D. \quad (4.13)$$

Defining the covariant field strength,

$$\mathcal{F}_{iA jB} = \partial_{iA} \mathcal{A}_{jB} - \partial_{jB} \mathcal{A}_{iA},$$

one finds

$$\mathcal{F}_{iA jB} = - 2 \kappa C_{AB} \epsilon_{ij}. \quad (4.14)$$

This means that the external Maxwell field is necessarily self-dual:

$$\mathcal{F}_{(iA)jB} = 0 \quad (4.15)$$
(brackets ( ) mean symmetrization with respect to the indices \(i, j\)). Thus the external Maxwell field should be self-dual, as distinct from an unconstrained field strength \(\sim \kappa\) in 2D case. As we shall see, this self-duality of the external gauge field is necessarily implied by the underlying \(\mathcal{N}=4\) worldline supersymmetry, like in conventional \(\mathcal{N}=4\) mechanics models (with the first-order kinetic terms for fermions) \[23, 22, 24, 25, 26, 27\].

For the purposes of quantization it will be convenient to make one more simplification. It is related to the presence of three \(SU(2)\) groups in (4.7) and (4.9). While the automorphism \(SU(2)_{L,R}\) symmetries acting on the indices \(i\) and \(a\) of the component fields are unbroken, one more \(SU(2)_{\text{ext}}\) acting on the capital doublet indices is necessarily broken by the first term in (4.7), which includes the constant symmetric tensor \(C_{AB}\). In what follows, without loss of generality, we can make use of this broken \(SU(2)_{\text{ext}}\) to bring \(C_{AB}\) into the particular form

\[
C^{12} = i, \tag{4.16}
\]

with all other components vanishing.

Using all these simplifications, we rewrite Lagrangian (4.12) in a different notation, by passing from the quartets \(f^{iA}\) and \(\chi^{aA}\) to the doublets of complex fields \(z, u, \zeta\) and \(\xi\)

\[
\begin{align*}
  z &= f^{11}, \quad \bar{z} = f^{22}, \quad u = f^{21}, \quad \bar{u} = -f^{12}, \\
  \zeta &= \chi^{11}, \quad \bar{\zeta} = \chi^{22}, \quad \xi = \chi^{21}, \quad \bar{\xi} = -\chi^{12}.
\end{align*} \tag{4.17}
\]

Then

\[
L = |\dot{z}|^2 + |\dot{u}|^2 - i\kappa (\dot{z}\bar{z} - \dot{z}\bar{z} + \dot{u}\bar{u} - \dot{u}\bar{u}) + \dot{\zeta}\bar{\zeta} + \dot{\xi}\bar{\xi} - i\kappa (\dot{\zeta}\bar{\zeta} + \dot{\zeta}\bar{\zeta} + \dot{\xi}\bar{\xi} + \dot{\xi}\bar{\xi}). \tag{4.19}
\]

Thus, using the superfield approach, we derived the component Lagrangian (4.19) for \(\mathcal{N}=4\) extended supersymmetric Landau model, where the worldline \(\mathcal{N}=4\) supersymmetry is built-in by construction. Though the action (4.19) is a sum of two copies of the \(\mathcal{N}=2\) Landau model actions (2.11), it possesses a rich symmetry structure, as will be demonstrated in the next Sections. Its bosonic sector is just the action of four-dimensional \(U(1)\) Landau-type model discussed in \[3\]. The Lorentz-force term (4.13) is rewritten as

\[
\mathcal{A}_{i}\dot{z}^{i} + \bar{\mathcal{A}}^{i}\bar{z}_{i}, \tag{4.20}
\]

where \(z^{i} \equiv (z, u), \bar{z}_{i} \equiv (\bar{z}, \bar{u})\) and

\[
\mathcal{A}_{i} = -i\kappa \bar{z}_{i}, \quad \bar{\mathcal{A}}^{i} = i\kappa z^{i}. \tag{4.21}
\]

One can check that the components of the background gauge field in this \(SU(2)_{L}\) covariant notation are given by

\[
\mathcal{F}_{i}^{i} = 2i\kappa \delta_{i}^{i}, \quad \mathcal{F}_{il} = \mathcal{F}^{il} = 0, \tag{4.22}
\]

which coincide with those in \[3\].

\[3\]This implies the breaking of the “Lorentz” \(SO(4) \sim SU(2)_{L} \times SU(2)_{\text{ext}}\) symmetry of \(\mathbb{R}^4\) down to \(SU(2)_{L} \times U(1)_{\text{ext}}\).
4.2 Worldline supersymmetry

In this subsection, we give how \( \mathcal{N}=4 \) supersymmetry acts on the fields \( f^{iA} \) and \( \chi^{aA} \). An equivalent realization on the complex fields defined in (4.17), (4.18) is presented in Appendix A.

The realization of \( \mathcal{N}=4 \) supersymmetry in the standard \( \mathcal{N}=4 \) superspace \((t, \theta^{ia})\) is given by (3.2). Then the harmonic projections of \( \theta^{ia} \), i.e. \( \theta^{m,n} \equiv \theta^{ia} u^m v^n \), \( m=\pm 1 \), \( n=\pm 1 \), are transformed as
\[
\delta \theta^{m,n} = \epsilon^{m,n},
\]
(4.23)
while the “analytic” time coordinates \( t_{\pm} \) defined in (3.7) and (3.8) as
\[
\delta t_{+} = 2i(\epsilon_{-1,1}^{-1} - \epsilon_{-1,1}^{1}), \quad \delta t_{-} = 2i(\epsilon_{1,1}^{-1} - \epsilon_{1,1}^{1}),
\]
(4.24)
thus confirming that the analytic subspaces (3.15) and (3.17) are closed under \( \mathcal{N}=4 \) supersymmetry. Using these coordinate transformations, it is straightforward to find the transformation laws of the component fields in the analytic superfields \( q^{1,0}(\zeta_{+}, u, v) \) and \( \psi^{0,1}(\zeta_{-}, u, v) \) defined by the \( \theta \)-expansions (4.3) and (4.4):
\[
\delta f^{iA} = \epsilon^{ia} \psi^{A}_i, \quad \delta \psi^{aA} = -2i \epsilon^{ia} \hat{f}^{iA},
\]
(4.25)
and
\[
\delta \chi^{aA} = \epsilon^{ia} h^{A}_i, \quad \delta h^{iA} = -2i \epsilon^{ia} \hat{\chi}^{A}_a.
\]
(4.26)

In order to find the supersymmetry transformations in terms of the physical fields only, we should express the auxiliary fields \( h^{iA} \) and \( \psi^{aA} \) from their equations of motion (4.10) and (4.11). As a result, we obtain
\[
\delta f^{iA} = -\frac{i}{\sqrt{\kappa}} \epsilon^{ia} C^{AB} \hat{\chi}^{B}_a, \quad \delta \chi^{aA} = -\frac{1}{\sqrt{\kappa}} \epsilon^{ia} \hat{f}^{iA}.
\]
(4.27)
The variation of the Lagrangian (4.12) under these transformations is equal to
\[
\delta L = i \sqrt{\kappa} \epsilon^{ia} \partial_t \left( \chi^{aA} \hat{f}^{iA}_i - \hat{\chi}^{A}_a f^{iA}_i - \frac{1}{\kappa} C^{AB} \hat{\chi}^{B}_a \hat{f}^{iB}_i \right).
\]
(4.28)
The corresponding conserved Noether supercharge is defined in the standard way
\[
\epsilon^{ia} S_{ia} = \delta f^{iA} \frac{\partial L}{\partial f^{iA}} + \delta \chi^{aA} \frac{\partial L}{\partial \chi^{aA}} - i \sqrt{\kappa} \epsilon^{ia} \partial_t \left( \chi^{aA} \hat{f}^{iA}_i - \hat{\chi}^{A}_a f^{iA}_i - \frac{1}{\kappa} C^{AB} \hat{\chi}^{B}_a \hat{f}^{iB}_i \right),
\]
and is calculated to be
\[
S_{ia} = -\frac{i}{\sqrt{\kappa}} C^{AB} \hat{\chi}^{A}_a \hat{f}^{iB}_i.
\]
(4.29)
Using equations of motion for the physical fields,
\[
\ddot{f}^{iA}_i = -2\kappa C^{AB} \hat{f}^{B}_i, \quad \ddot{\chi}^{aA}_a = 2\kappa C^{AB} \hat{\chi}^{B}_a,
\]
(4.30)
it is easy to directly check that \( \dot{S}_{ia} = 0 \).

The Noether charges (4.29) become the generators of \( \mathcal{N}=4 \) supersymmetry upon quantization.
4.3 Target space supersymmetry

Before turning to quantization of $\mathcal{N}=4$ supersymmetric Landau model, let us show that, besides the worldline $\mathcal{N}=4$ supersymmetry, the model \((4.12)\) possesses invariance under certain target space supersymmetry which generalizes the $\text{ISU}(1|1)$ supersymmetry of the $\mathcal{N}=2$ Landau model. Anticipating the quantum picture, we shall present a realization of this supersymmetry by differential operators acting in the target $(4+4)$ superspace $(f^i A, \chi^a B)$. All these operators are obtained in the standard way from the conserved Noether charges associated with the appropriate invariances of the action corresponding to the Lagrangian \((4.12)\).

The most evident type of such a symmetry is the “magnetic” supertranslations:
\begin{align}
\delta f^i A &= b^i A, \\
\delta \chi^a B &= \nu^a B,
\end{align}
where $b^i A$ and $\nu^a B$ are constant bosonic and fermionic parameters. The corresponding symmetry generators are
\begin{align}
P_{i A} &= -i \partial f^i A + \kappa C_{AB} f^B, \\
\Pi_{a A} &= \partial \chi^a A + \kappa \chi_{a A}.
\end{align}

There are also two automorphism groups $\text{SU}(2)_L$ and $\text{SU}(2)_R$, which separately rotate the indices $i$ and $a$ and so are realized, respectively, on the bosonic and fermionic fields:
\begin{align}
\delta f^i A &= \lambda_i j f^j A, \\
\delta \chi^a A &= \lambda a b \chi^b A,
\end{align}
with $\lambda_{ij} = \lambda_{ji}$ and $\lambda_{ab} = \lambda_{ba}$. The corresponding generators are
\begin{align}
T^{(i)}_{j} &= f^i A \partial f^j A - \frac{1}{2} \delta^j_k f^k A \partial f^i A, \\
T^{(a)}_{b} &= \chi^a A \partial \chi^b A - \frac{1}{2} \delta^a_b \chi^b A \partial \chi^a A.
\end{align}

There is also $\text{U}(1)$ symmetry which simultaneously changes the phase of all fields:
\begin{align}
\delta f^i A &= \alpha C_B A f^i B, \\
\delta \chi^a A &= \alpha C_B A \chi^a B.
\end{align}

The corresponding generator is\footnote{In addition to \((4.35)\), one can define a similar independent $\text{U}(1)$ symmetry rotating only fermions, i.e. $\delta \chi^a A = \beta C_B A \chi^a B$, $\delta f^i A = 0$. This additional symmetry can be treated as some automorphism of the full target space symmetry. Taking it into account, the Lagrangian \((4.12)\) exhibits four independent $\text{U}(1)$ invariances, which become manifest in the complex notation \((4.19)\).}
\begin{align}
Z &= -i C_B A (f^i A \partial f^i A + \chi^a B \partial \chi^a A).
\end{align}

Finally, there are odd linear symmetries which mix $f^i A$ with $\chi^a A$:
\begin{align}
\delta f^i A &= \frac{1}{2} \omega^i a (C_B A + i \delta_B A) \chi^B_a - \frac{1}{2} \omega^i a (C_B A - i \delta_B A) \chi^B_a, \\
\delta \chi^a A &= \frac{1}{2} \omega^i a (C_B A - i \delta_B A) f^i B + \frac{1}{2} \omega^i a (C_B A + i \delta_B A) f^i B.
\end{align}

The relevant generators are
\begin{align}
Q^i a &= \frac{1}{2} (i C_B A - \delta_B A) \chi^a B \partial f^i A + \frac{1}{2} (i C_B A + \delta_B A) f^i B \partial \chi^a A, \\
\bar{Q}_{i a} &= \frac{1}{2} (i C_B A + \delta_B A) \chi^a B \partial f^i A - \frac{1}{2} (i C_B A - \delta_B A) f^i B \partial \chi^a A.
\end{align}
Having the explicit form of the generators, it is easy to establish the algebra of their (anti)commutators.

The generators (4.34) form a superalgebra of magnetic supertranslations

\[ [P_{iA}, P_{jB}] = 2i\kappa \epsilon_{ij} C_{AB}, \quad \{\Pi_{aA}, \Pi_{bB}\} = 2\kappa \epsilon_{ab} \epsilon_{AB}. \]  

(4.39)

The generators (4.34), (4.36), (4.38) form the superalgebra \(su(2|2)\):

\[ \{Q^{ia}, Q_{jb}\} = \delta_b^a T^{(i}_j) - \delta_j^i T^{(a}_b) + \frac{1}{2} \delta_j^i \delta_b^a Z, \]  

(4.40)

\[ \{Q^{ia}, Q^{jb}\} = \{Q_{ia}, Q_{jb}\} = 0, \]  

(4.41)

\[ [Q^{ia}, Z] = 0, \quad [\bar{Q}_{ib}, Z] = 0, \]  

(4.42)

\[ [Q_{ia}, T^{(j}_k)] = \delta_k^j Q_{ia}, \quad [Q_{ia}, T^{(b}_c] = \delta_b^c Q_{ia}, \]  

(4.43)

\[ [Q^{ia}, T^{(j}_k)] = -\delta_k^j Q^{ia}, \quad [Q^{ia}, T^{(b}_c)] = -\delta_b^c Q^{ia}, \]  

(4.44)

\[ [Z, T^{(i}_j)] = [Z, T^{(a}_b)] = 0, \]  

(4.45)

\[ [T^{(i}_j), T^{(k}_l)] = \delta_k^j T^{(i}_l] - \delta_l^i T^{(k}_j], \quad [T^{(a}_b), T^{(c}_d)] = \delta_b^c T^{(a}_d) - \delta_d^a T^{(c}_b). \]  

(4.46)

We employ the following rules of hermitian conjugation: \((P^{iA})^\dagger = P_{iA}, \ (\Pi^{iA})^\dagger = \Pi_{iA}, \ (Q^{ia})^\dagger = -Q_{ia}, \ (T^{ij})^\dagger = -T_{ij}, \ (T^{ab})^\dagger = -T_{ab}, \ Z^\dagger = Z\). Note that the generator \(Z\) is the “central charge” generator. It has a non-trivial realization on the fields \((f^{iA}, \chi^{aB})\) (see (1.36)), so in the present case we cannot factor it out to end up with the supergroup \(PSU(2|2)\).

Finally, we present the commutation relations between the generators of the magnetic supertranslation group and the \(SU(2|2)\) generators

\[ [Z, P_{iA}] = iC^{B}_A P_{iB}, \quad [Z, \Pi_{aA}] = iC^{B}_A \Pi_{aB}, \]  

(4.47)

\[ [P_{iA}, T^{(j}_k)] = \delta_k^j P_{iA} - \frac{1}{2} \delta_k^j P_{iA}, \quad [\Pi_{aA}, T^{(b}_c)] = \delta_b^c \Pi_{aA} - \frac{1}{2} \delta_b^c \Pi_{aA}, \]  

(4.48)

\[ [Q^{ia}, P_{jA}] = -\frac{i}{2} \delta_j^i \epsilon_{ab} (iC^{B}_A + \delta^{B}_A) \Pi_{bB}, \quad \{Q^{ia}, \Pi_{bA}\} = -\frac{i}{2} \delta_b^c \epsilon_{ij} (iC^{B}_A - \delta^{B}_A) P_{jB}, \]  

(4.49)

\[ [\bar{Q}_{ia}, P_{jA}] = -\frac{i}{2} \epsilon_{ij} (iC^{B}_A - \delta^{B}_A) \Pi_{bB}, \quad \{\bar{Q}_{ia}, \Pi_{bA}\} = \frac{i}{2} \epsilon_{ab} (iC^{B}_A + \delta^{B}_A) P_{jB}. \]  

(4.50)

Thus the algebra of the magnetic (super)translation generators forms an ideal in the full target supersymmetry algebra, which is the semi-direct product

\[ (P_{iA}, \Pi_{aA}) \rtimes SU(2|2) = ISU(2|2). \]  

(4.51)

Correspondingly, the \((4|4)\)-dimensional target manifold of the physical fields \(f^{iA}, \chi^{aB}\) can be identified with the supercoset \(ISU(2|2)/SU(2|2)\).

Recall that our original propositions were the requirement of manifest \(\mathcal{N}=4\) worldline supersymmetry and a sort of minimality principle: we wished to construct a model which would be a minimal generalization of \(\mathcal{N}=2\) Landau model. And finally we found that the model constructed possesses, as a gift, an extra target supersymmetry \(ISU(2|2)!\) As is shown in Appendix B, this supergroup also admits an off-shell realization on the bi-harmonic superfields \(q^{1,0A}\) and \(\psi^{0,1A}\).
5 Quantization

It is convenient to perform quantization in terms of the complex fields \( z, u, \zeta, \xi \), so in this Section we will proceed from the Lagrangian (4.19).

5.1 Hamiltonian

The canonical momenta for the bosonic and fermionic fields defined as \( \pi_b = \frac{\partial L}{\partial \dot{b}} \) and \( \pi_f = \frac{\partial L}{\partial \dot{f}} \) are given by

\[
\begin{align*}
\pi_z &= \dot{\bar{z}} - i\kappa \bar{z}, \quad \pi_{\bar{z}} = \dot{z} + i\kappa z, \\
\pi_u &= \dot{\bar{u}} - i\kappa \bar{u}, \quad \pi_{\bar{u}} = \dot{u} + i\kappa u,
\end{align*}
\]

(5.1)

and

\[
\begin{align*}
\pi_\zeta &= \dot{\bar{\zeta}} - i\kappa \bar{\zeta}, \quad \pi_{\bar{\zeta}} = -\dot{\zeta} - i\kappa \zeta, \\
\pi_\xi &= \dot{\bar{\xi}} - i\kappa \bar{\xi}, \quad \pi_{\bar{\xi}} = -\dot{\xi} - i\kappa \xi.
\end{align*}
\]

(5.2)

The classical Hamiltonian is

\[
H_{cl} = (\pi_z + i\kappa \bar{z})(\pi_{\bar{z}} - i\kappa z) + (\pi_u + i\kappa \bar{u})(\pi_{\bar{u}} - i\kappa u) - (\pi_\zeta + i\kappa \bar{\zeta})(\pi_{\bar{\zeta}} + i\kappa \zeta) - (\pi_\xi + i\kappa \bar{\xi})(\pi_{\bar{\xi}} + i\kappa \xi).
\]

(5.3)

We quantize by the substitution

\[
\pi_b \rightarrow -i \frac{\partial}{\partial b}, \quad \pi_f \rightarrow -i \frac{\partial}{\partial f},
\]

(5.4)

and define the quantum Hamiltonian as the Weyl-ordered form of (5.3)

\[
H_q = a_\zeta^\dagger a_z + a_u^\dagger a_u - \alpha_\zeta^\dagger \alpha_\zeta - \alpha_\xi^\dagger \alpha_\xi.
\]

(5.5)

Here,

\[
\begin{align*}
a_\zeta^\dagger &= i \left( \frac{\partial}{\partial z} - \kappa \bar{z} \right), & a_z &= i \left( \frac{\partial}{\partial \bar{z}} + \kappa z \right), & a_u^\dagger &= i \left( \frac{\partial}{\partial u} - \kappa \bar{u} \right), & a_u &= i \left( \frac{\partial}{\partial \bar{u}} + \kappa u \right), \\
[a_\zeta, a_\zeta^\dagger] &= 2\kappa, & [a_u, a_u^\dagger] &= 2\kappa,
\end{align*}
\]

(5.6)

and

\[
\begin{align*}
\alpha_\zeta &= \frac{\partial}{\partial \zeta} - \kappa \bar{\zeta}, & \alpha_\xi &= \frac{\partial}{\partial \xi} - \kappa \bar{\xi}, & \alpha_\xi^\dagger &= \frac{\partial}{\partial \bar{\xi}} - \kappa \bar{\xi}, & \alpha_\zeta^\dagger &= \frac{\partial}{\partial \bar{\zeta}} - \kappa \bar{\zeta}, \\
\{\alpha_\zeta, \alpha_\zeta^\dagger\} &= -2\kappa, & \{\alpha_\xi, \alpha_\xi^\dagger\} &= -2\kappa.
\end{align*}
\]

(5.7)

Note that, like in the \( \mathcal{N}=2 \) Landau model [13, 15], the Hamiltonian admits a nice Sugawara-type representation

\[
H_q = \frac{1}{2} P_i^A P_i A + \frac{i}{2} C^{AB} \Pi_A^a \Pi_{aB} - 2\kappa Z,
\]

(5.8)
which means that it belongs to the enveloping algebra of the superalgebra $ISU(2|2)$ defined in the previous Section. Using this form of $H_q$, it is straightforward to check that it commutes with all $ISU(2|2)$ generators.

Sometimes it is useful to know the explicit form of the Hamiltonian

$$H_q = -\left(\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial u} \frac{\partial}{\partial \bar{u}} + \frac{\partial}{\partial \zeta} \frac{\partial}{\partial \bar{\zeta}} + \frac{\partial}{\partial \xi} \frac{\partial}{\partial \bar{\xi}}\right) + \kappa^2 (|z|^2 + |u|^2 + \zeta \bar{\zeta} + \xi \bar{\xi}) - \kappa Z, \quad (5.9)$$

where, in the complex notation, the $U(1)$ generator $Z$ defined in (4.36) is expressed as

$$Z = z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} + u \frac{\partial}{\partial u} - \bar{u} \frac{\partial}{\partial \bar{u}} + \zeta \frac{\partial}{\partial \zeta} - \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} + \xi \frac{\partial}{\partial \xi} - \bar{\xi} \frac{\partial}{\partial \bar{\xi}}. \quad (5.10)$$

### 5.2 Wave functions and degeneracies

**LLL.** By definition, the wave function of the Lowest Landau Level (LLL) $\Psi^0$ is nullified by both bosonic and fermionic annihilation operators $a_z, a_u, \alpha_\zeta$ and $\alpha_\xi$:

$$(\partial_z + \kappa z)\Psi^0 = (\partial_u + \kappa u)\Psi^0 = (\partial_\zeta - \kappa \zeta)\Psi^0 = (\partial_\xi - \kappa \xi)\Psi^0 = 0 \Leftrightarrow H_q \Psi^0 = 0.$$ 

These conditions can be solved in terms of the holomorphic “reduced” wave function $\psi_0$:

$$\Psi^0 = e^{-\kappa K}\psi_0(z, u, \zeta, \xi), \quad K = |z|^2 + |u|^2 + \zeta \bar{\zeta} + \xi \bar{\xi}. \quad (5.11)$$

The LLL has a four-fold degeneracy,

$$\psi_0(z, u, \zeta, \xi) = A^0(z, u) + \zeta B^0(z, u) + \xi C^0(z, u) + \zeta \xi D^0(z, u), \quad (5.12)$$

where $A^0, B^0, C^0, D^0$ are analytic functions of $z$ and $u$. Their set is closed under the action of $ISU(2|2)$.

**Excited LLs.** The Hilbert space for the $N$-th Landau level is spanned by the wave functions:

$$\Psi^{(N)} \sim \sum_{j=0}^{N} (a^+_z)^j (a^+_u)^{N-j} e^{-\kappa K}\psi_{(0,0)}^{(j,N-j)}(z, u, \zeta, \xi)$$

$$+ \sum_{j=0}^{N-1} (a^+_z)^j (a^+_u)^{N-1-j} \left[ \alpha_\zeta^+ e^{-\kappa K}\psi_{(1,0)}^{(j,N-1-j)}(z, u, \zeta, \xi) + \alpha_\xi^+ e^{-\kappa K}\psi_{(0,1)}^{(j,N-1-j)}(z, u, \zeta, \xi) \right]$$

$$+ \sum_{j=0}^{N-2} (a^+_z)^j (a^+_u)^{N-2-j} \alpha_\zeta^+ \alpha_\xi^+ e^{-\kappa K}\psi_{(1,1)}^{(j,N-2-j)}(z, u, \zeta, \xi), \quad (5.13)$$

$$H_q \Psi^{(N)} = 2\kappa N \Psi^{(N)}, \quad (5.14)$$

---

5Each of these four LLL states is also infinitely degenerated due to the symmetry under the “magnetic” translations. This degeneration generalizes the similar phenomenon in the standard bosonic 2D Landau model and its $\mathcal{N}=2$ extension.
where
\[ \psi_{(l,m)}^{(j,N-j-l-m)}(z, u, \zeta, \xi) = A_{(l,m)}^{(j,N-j-l-m)} + \zeta B_{(l,m)}^{(j,N-j-l-m)} + \xi C_{(l,m)}^{(j,N-j-l-m)} + \xi \zeta D_{(l,m)}^{(j,N-j-l-m)} \]  

and the indices \( l, m = 0, 1 \) represent the numbers of fermionic excitations produced by \( \alpha^\dagger \) and \( \alpha^\dagger \). One can rewrite (5.13) in another way,
\[ \Psi^{(N)} = \sum_{j=0}^{N} \Psi^{(j,N-j)}_{(0,0)} + \sum_{j=0}^{N-1} \Psi^{(j,N-1-j)}_{(1,0)} + \sum_{j=0}^{N-1} \Psi^{(j,N-1-j)}_{(0,1)} + \sum_{j=0}^{N-2} \Psi^{(j,N-2-j)}_{(1,1)}, \]  

where
\[ \Psi^{(j,N-j-l-m)}_{(l,m)} = (a^\dagger)^j (a^\dagger)^{N-j-l-m} (\alpha^\dagger)^m e^{-\kappa K} \psi^{(j,N-j-l-m)}_{(l,m)}(z, u, \zeta, \xi). \]

This state describes the system with energy \( 2\kappa N \) and with \( j \) and \( N - j - l - m \) excited quanta of the bosonic fields \( z \) and \( u \), respectively, and with \( l \) and \( m \) excited quanta of the fermionic fields \( \zeta \) and \( \xi \), respectively. LL with \( N > 0 \) has a degeneracy \( 4(N + 1) + 4N + 4N + 4(N - 1) = 16N \) (modulo an infinite degeneracy due to the invariance under bosonic magnetic translations).

In order to better understand the origin of this degeneracy of the \( N \)-th LL, it is convenient to pass to the \( SU(2)_{L,R} \) covariant notation for the creation and annihilation operators, \( a^\dagger := (a_z, a_u) \), \( a^\dagger := (a_{\bar{z}}, a_{\bar{u}}) \), \( \alpha^\dagger := (\alpha_\zeta, \alpha_\xi) \), \( \alpha^\dagger := (\alpha_{\bar{\zeta}}, \alpha_{\bar{\xi}}) \),
\[ [a^\dagger_i, a^\dagger_j] = -2\kappa \delta^i_j, \quad \{\alpha^\dagger_a, \alpha^\dagger_b\} = -2\kappa \delta^a_b. \]

Then the Hamiltonian can be rewritten as
\[ H_q = a^\dagger_i a^i - \alpha^\dagger_a \alpha^a. \]

With this notation, the wave functions are
\[ \Psi^{(N)} = a^\dagger_1 a^\dagger_2 \ldots a^\dagger_N e^{-\kappa K} \phi^{(i_1 i_2 \ldots i_N)}(z, u, \zeta, \xi) + \alpha^\dagger a^\dagger_1 a^\dagger_2 \ldots a^\dagger_{N-1} e^{-\kappa K} \psi^{(a_1 a_2 \ldots a_{N-1})}(z, u, \zeta, \xi) + \alpha^\dagger \alpha^\dagger_1 a^\dagger_2 \ldots a^\dagger_{N-2} e^{-\kappa K} \phi^{(a_1 a_2 \ldots a_{N-2})}(z, u, \zeta, \xi). \]

Now it becomes obvious why the degeneracy of the \( N \)-th LL is just \( 16N \). The component wave functions \( \phi^{(i_1 i_2 \ldots i_N)} \), \( \psi^{(a_1 a_2 \ldots a_{N-1})} \) and \( \phi^{(a_1 a_2 \ldots a_{N-2})} \) in (5.20) are irreducible tensors of \( SU(2)_L \) with the spins \( s_1 = \frac{N}{2} \), \( s_2 = \frac{N-1}{2} \) (entering twice) and \( s_3 = \frac{N-2}{2} \), respectively. So the degeneracy of the \( N \)-th level is equal to
\[ 4[(2s_1 + 1) + 2(2s_2 + 1) + (2s_3 + 1)] = 16N. \]

The wave function of LLL is a singlet of \( \mathcal{N}=4 \) supersymmetry, while wave functions for any \( N > 0 \) form an \( \mathcal{N}=4 \) supermultiplet with the \( SU(2)_L \) spin contents \( (N/2, 2 \times (N - 1)/2, N/2 - 1) \). The number of bosonic complex fields \( (2N) \) is always equal to the number of fermionic complex fields, as it should be. For example, the first excited level is described by the wave function
\[ \Psi^{(N=1)} = a^\dagger e^{-\kappa K} \phi^0 + \alpha^\dagger e^{-\kappa K} \psi^0, \]
which corresponds to a “hypermultiplet” with the \( SU(2)_L \) spin content \( (1/2, 0) \) (and \( (0, 1/2) \) with respect to \( SU(2)_R \)). More details on the realization of \( \mathcal{N}=4 \) supersymmetry on the wave functions are given in Sect. 6.2.
5.3 The problem of negative norms

As in the case of quantum $\mathcal{N}=2$ Landau model, in its $\mathcal{N}=4$ extension the wave functions associated with some levels possess negative norms with respect to the natural $ISU(2|2)$-invariant inner product defined as

$$\langle \phi|\psi \rangle = \int d\mu \bar{\phi}(z, \bar{z}, u, \bar{u}, \zeta, \bar{\zeta}, \xi, \bar{\xi}) \psi(z, \bar{z}, u, \bar{u}, \zeta, \bar{\zeta}, \xi, \bar{\xi}), \quad (5.23)$$

$$d\mu = dzd\bar{z}du d\bar{u}d\zeta d\bar{\zeta}d\xi d\bar{\xi}.$$ 

One can calculate norms of the states $\Psi^{(j,k)}_{(l,m)}$ using the relations (5.6) and (5.7)

$$\langle \Psi^{(j,k)}_{(l,m)}|\Psi^{(j,k)}_{(l,m)} \rangle = (-1)^{l+m}(2k)^{j+k+l+m} \int d\mu e^{-2nK} \bar{\psi}^{(j,k)}_{(l,m)}(z, u, \zeta, \xi) \psi^{(j,k)}_{(l,m)}(z, u, \zeta, \xi)$$

$$= (-1)^{l+m}(2k)^{j+k+l+m} \left( \|D^{(j,k)}_{(l,m)}\|^2 + 2k\|B^{(j,k)}_{(l,m)}\|^2 + 2k\|C^{(j,k)}_{(l,m)}\|^2 + 2k^2\|A^{(j,k)}_{(l,m)}\|^2 \right), \quad (5.24)$$

where

$$\|f\|^2 := \int dzd\bar{z}du d\bar{u} \bar{f}(z, u)f(z, u). \quad (5.25)$$

We see that the states which include one fermionic creation operator indeed possess negative norms. To get around this difficulty, one is led to introduce a non-trivial metric on the space of quantum states. It is natural to redefine the inner product as

$$\langle \langle \psi|\phi \rangle \rangle := \langle G\psi|\phi \rangle,$$

$$\langle G\left( \Psi^{(j,k)}_{(0,0)} + \Psi^{(j,k)}_{(1,0)} + \Psi^{(j,k)}_{(0,1)} + \Psi^{(j,k)}_{(1,1)} \right) = \Psi^{(j,k)}_{(0,0)} - \Psi^{(j,k)}_{(1,0)} + \Psi^{(j,k)}_{(0,1)} + \Psi^{(j,k)}_{(1,1)}, \quad (5.26)$$

where

$$G = 1 + \frac{\alpha^\dagger_\zeta \alpha_\zeta}{\kappa} + \frac{\alpha^\dagger_\xi \alpha_\xi}{\kappa} + \frac{\alpha^\dagger_\zeta \alpha_\zeta \alpha^\dagger_\xi \alpha_\xi}{\kappa^2} = (1 - 2n_\zeta)(1 - 2n_\xi), \quad n_{\zeta,\xi} := -\frac{\alpha^\dagger_\zeta \alpha_\zeta \alpha^\dagger_\xi \alpha_\xi}{2\kappa}. \quad (5.27)$$

The metric operator $G$ possesses the standard properties $[15, 16]

$$[H_q, G] = 0, \quad G^2 = 1. \quad (5.28)$$

With respect to the redefined inner product all norms are positive-definite.

It is worth noting that the rules of hermitian conjugation for those operators which do not commute with $G$ are changed. Using the property

$$\langle \langle \psi|Q\phi \rangle \rangle = \langle G\psi|Q\phi \rangle = \langle Q^\dagger G\psi|\phi \rangle, \quad (5.29)$$

and, on the other hand,

$$\langle \langle \psi|Q\phi \rangle \rangle = \langle \langle Q^\dagger \psi|\phi \rangle \rangle = \langle GQ^\dagger \psi|\phi \rangle, \quad (5.30)$$

one finds

$$Q^\dagger = G^{-1}Q^\dagger G = GQ^\dagger G = Q^\dagger + G[Q^\dagger, G]. \quad (5.31)$$

The creation and annihilation operators do not commute with $G$, so, applying the general formula (5.31), we find

$$\alpha^\dagger_\zeta = -\alpha^\dagger_\zeta, \quad \alpha^\dagger_\xi = -\alpha^\dagger_\xi, \quad (5.32)$$

whence the manifestly positive-definite form for $H_q$ follows

$$H_q = a^\dagger_za_z + a^\dagger_u a_u + \alpha^\dagger_\zeta \alpha_\zeta + \alpha^\dagger_\xi \alpha_\xi. \quad (5.33)$$
6 More on the symmetry structure

6.1 Quantum generators of $\mathcal{N} = 4$ supersymmetry

Starting from the classical expression (4.29) for the supercharges $S^{ia}$, after quantization we find

\[
S^{11} = \frac{i}{\sqrt{\kappa}} (\alpha^{\dagger} a_z - \alpha^{\dagger} \alpha \xi a), \quad S^{12} = -\frac{i}{\sqrt{\kappa}} (\alpha^{\dagger} a_z + \alpha^{\dagger} \alpha \xi a),
\]

\[
S^{21} = \frac{i}{\sqrt{\kappa}} (\alpha^{\dagger} a_u + \alpha^{\dagger} \alpha \xi a), \quad S^{22} = \frac{i}{\sqrt{\kappa}} (\alpha^{\dagger} \alpha \xi - \alpha^{\dagger} \alpha \xi a).
\] (6.1)

Using the relations $S^{11\dagger} = -S^{22}$, $S^{12\dagger} = S^{21}$, $S^{21\dagger} = S^{12}$, $S^{22\dagger} = -S^{11}$, it is convenient to relabel these generators as

\[
S_1 = S^{21}, \quad S_1^\dagger = S^{12}, \quad S_2 = S^{11}, \quad S_2^\dagger = -S^{22},
\] (6.2)

\[
\{S_1, S_2\} = \{S_1^\dagger, S_2^\dagger\} = \{S_1, S_1^\dagger\} = \{S_2, S_2^\dagger\} = 0,
\]

\[
\{S_1, S_1^\dagger\} = \{S_2, S_2^\dagger\} = -2H.
\] (6.3)

The $\mathcal{N}=4$ supercharges do not commute with the metric operator (5.27), so one obtains:

\[
S_1^\dagger = -S_2^\dagger, \quad S_1^\dagger = -S_2^\dagger,
\] (6.4)

\[
\{S_1, S_1^\dagger\} = \{S_2, S_2^\dagger\} = 2H.
\] (6.5)

In the covariant notation, $H$ and $S^{ia}$ are expressed as

\[
S^{ia} = \frac{i}{\sqrt{\kappa}} (a^i \alpha^a - a^i \alpha^a), \quad H = a^i \alpha^a - \alpha^i \alpha^a,
\] (6.6)

or

\[
S^{ia} = \frac{i}{\sqrt{\kappa}} (a^i \alpha^a + a^i \alpha^a), \quad H = a^i \alpha^a + \alpha^i \alpha^a,
\] (6.7)

and

\[
\{S^{ia}, S^{jb}\} = 2\epsilon^{ij} \epsilon^{ab} H_q.
\] (6.8)

The reality properties of $S^{ia}$ with respect to the original and “improved” inner products are different,

\[
(S^{ia})^\dagger = -S_{ia}, \quad (S^{ia})^\dagger = S_{ia},
\]

which agrees with (6.4). It also immediately follows that $S^{ia}$ annihilates the LLL wave function, i.e. the ground state. So the $\mathcal{N}=4$ supersymmetry is unbroken in the model under consideration.

More explicit expression for $S^{ia}$ can be found by making, in eq. (4.29), the substitutions

\[
\dot{f}_{ia} = \pi_i a - \kappa C_{AB} f_i^B \Rightarrow -i \dot{f}_{IA} + \kappa C_{AB} f_i^B,
\]

\[
\dot{\chi}_{aB} = C_B^A (\pi a_a + \kappa \chi_{aA}) \Rightarrow -C_B^A (\dot{\chi}_{aA} - \kappa \chi_{aA}).
\]

Using this, one can show that the $\mathcal{N}=4$ supercharges, like the Hamiltonian $H_q$ (eq. (5.8)), admit a Sugawara-type representation in terms of the $\text{ISU}(2|2)$ generators,

\[
S^{ia} = 2\sqrt{\kappa} (Q^{ia} + \bar{Q}^{ia}) - \frac{i}{\sqrt{\kappa}} P_A^{ia} \Pi^a A.
\] (6.9)

\[
\]
which implies that they also belong to the enveloping algebra of the ISU(2|2) superalgebra defined in the previous Section. Calculating the anticommutator of these supercharges with making use of the (anti)commutation relations of the ISU(2|2) superalgebra alone, one recovers eq. (6.8), with $H_q$ given just by the expression (5.8).

### 6.2 Second on-shell $\mathcal{N}=4$ supersymmetry

It is rather surprising that, by analogy with the representation (6.9), one can define another set of generators,

$$
\hat{S}^{ia} = 2i\sqrt{\kappa}(Q^a - \bar{Q}^a) + \frac{i}{\sqrt{\kappa}}P_A^i \Pi_B^a C^{AB}, \quad (\hat{S}^{ia})^\dagger = -\hat{S}_{ia}, \quad (\hat{S}^{ia})^\ddagger = \hat{S}_{ia},
$$

(6.10)

which form the same worldline $\mathcal{N}=4$ superalgebra as $S^{ia}$:

$$
\{\hat{S}^{ia}, \hat{S}^{jb}\} = 2\epsilon^{ij}_{\;ab} H_q.
$$

(6.11)

The anticommutator of these two different $\mathcal{N}=4$ supercharges is non-vanishing,

$$
\{S^{ia}, \hat{S}^{jb}\} = 8i\kappa \left(\epsilon^{ab} \hat{T}^{ij} - \epsilon^{ij} \hat{T}^{ab}\right),
$$

(6.12)

where

$$
\hat{T}^{ij} = T^{ij} - \bar{T}^{ij}, \quad \hat{T}^{ab} = T^{ab} - \bar{T}^{ab},
$$

(6.13)

$$
\bar{T}^{ij} := \frac{-1}{4i\kappa} C^{AB} P^i_A \Pi^a_B, \quad \bar{T}^{ab} := \frac{-1}{4\kappa} \Pi^a_A \Pi_B^b A.
$$

(6.14)

The generators $\hat{T}^{ij}, \hat{T}^{ab}$ and $\bar{T}^{ij}, \bar{T}^{ab}$ form two mutually commuting sets of $SU(2) \times SU(2)$ generators. This can be checked using the commutation relations (4.39), (4.46) and (4.48). Also, using the important relations

$$
[S^{ia}, P^i_A] = [S^{ia}, \Pi_B^a] = [\hat{S}^{ia}, P^i_A] = [\hat{S}^{ia}, \Pi_B^a] = 0,
$$

(6.15)

which follow from (4.31), one finds that

$$
[S^{ia}, \hat{T}^{ij}] = [S^{ia}, \bar{T}^{ab}] = [\hat{S}^{ia}, \hat{T}^{ij}] = [\hat{S}^{ia}, \bar{T}^{ab}] = 0,
$$

(6.16)

and so the $SU(2)$ generators $\hat{T}^{ij}$ and $\bar{T}^{ab}$ act on $S^{ia}, \hat{S}^{ia}$ in the same way as the original automorphism $SU(2)_L \times SU(2)_R$ generators $T^{ij}$ and $T^{ab}$. Thus $\hat{T}^{ij}$ and $\bar{T}^{ab}$ can equally be chosen as generators of the automorphism $SU(2)$ groups of $\mathcal{N}=4$ superalgebras (6.8) and (6.11).

The worldline superalgebra constituted by the relations (6.8), (6.11) and (6.12) admits a two-fold interpretation.

First, it can be considered as a deformation of the standard $\mathcal{N}=8, d=1$ Poincaré superalgebra in the $SO(4) \sim SU(2)_L \times SU(2)_R$ covariant notation (see, e.g., [28]) by the “semi-central” charges $\hat{T}^{ij}$ and $\bar{T}^{ab}$ generating two $SU(2)$ automorphism groups. This deformation makes the crossing anticommutator $\{S, \hat{S}\}$ non-vanishing and breaks the $SO(8)$ automorphism group of $\mathcal{N}=8, d=1$ superalgebra down to $SO(4)$.
Another interpretation is that the relations (6.8), (6.11) and (6.12) define none other than a second, “dynamical” superalgebra $su(2|2)_{\text{dyn}}$, with the Hamiltonian $H_q$ as the relevant central charge operator. Indeed, after proper rescaling of the generators $S^{ia}, \hat{S}^{ia}, H_q$ and passing to the complex combinations $S^{ia} \pm i\hat{S}^{ia}$, the full set of the corresponding (anti)commutation relations can be cast in the standard form (4.40) - (4.46).

Finally, we make a few comments on the realization of the hidden worldline super-symmetry on the original fields $f^{iA}, \chi^{aA}$ and on the wave functions.

The relevant transformations leaving invariant, up to a total derivative, the on-shell lagrangian (4.12) are as follows

$\delta f^{iA} = -i\sqrt{\kappa} \hat{\epsilon}^{ia} \hat{\chi}_a^A, \quad \delta \chi^{aA} = -\frac{1}{\sqrt{\kappa}} \hat{\epsilon}^{ia} C^{AB} \hat{f}_{iB},$ (6.17)

where $\hat{\epsilon}^{ia}$ is a new quartet Grassmann parameter. The conserved supercurrent, which becomes just (6.10) after quantization, reads

$\hat{S}_{ia} = \frac{i}{\sqrt{\kappa}} \hat{\chi}_a A \hat{f}^A_i.$ (6.18)

The transformations (6.17), like (4.27), close on $\partial_t$ only on shell, with taking into account the equations of motion (4.30). The Lie bracket of (6.17) with (4.27) yields an unusual realization of the $SU(2)$ generators $\hat{T}^{ik}$ and $\hat{T}^{ab}$ on the fields $f^{iA}, \chi^{aA}$: they rotate the latter into their second-order time derivatives which become the first-order ones only on the shell of eqs.(4.30). These two $su(2)$ algebras are also closed only modulo (4.30). For the time being, we do not know whether it is possible to reproduce (6.17) from some off-shell transformations realized on the superfields $q^{1,0A}, \psi^{0,1A}$.

In the quantum realization via the creation and annihilation operators, the generators $\hat{S}_{1,2}$ related to $\hat{S}^{ia}$ as in (6.2) are given by the expressions

$\hat{S}_1 = \frac{1}{\sqrt{\kappa}} (-a^1_\alpha \alpha + a_\alpha \alpha), \quad \hat{S}_2 = \frac{1}{\sqrt{\kappa}} (\alpha^1_\alpha a^\alpha + a^\alpha_\alpha \alpha).$ (6.19)

The corresponding analog of the $SU(2)_L$ covariant representation (6.6) for $S^{ia}$ reads

$\hat{S}^{ia} = -\frac{1}{\sqrt{\kappa}} (a^i \alpha^a + a^{i\alpha} a^a).$ (6.20)

It is interesting that the presence of the second worldline $\mathcal{N}=4$ supersymmetry does not give rise to an additional degeneracy of the wave function: as follows from the representation (6.20), action of $\hat{S}^{ia}$ on the general $N$-th LL wave function (5.20) preserves its structure. In other words, at each LL, the multiplet of wave functions closed under the action of $S^{ia}$ is also closed under $\hat{S}^{ia}$, and so it carries an irrep of the whole worldline supersymmetry $SU(2|2)_\text{dyn}$. This is in striking contrast, e.g., to $\mathcal{N}=8, d=1$ supersymmetry which cannot be realized on a single $\mathcal{N}=4$ multiplet. At least two such multiplets are required to form $\mathcal{N}=8$ multiplet [28]. This peculiarity is of course related to the fact that the anticommutator (6.12) of $S^{ia}$ and $\hat{S}^{ia}$ is not vanishing: it involves the “semi-central”

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$^6$We thank Sergey Fedoruk for suggesting this interpretation to us. It is quite similar to the view of the $\mathcal{N}=2, d=1$ Poincaré superalgebra $\{S, \hat{S}\} = 2H, S^2 = \hat{S}^2 = 0$ as a sort of $su(1|1)$ superalgebra.
SU(2) generators $\tilde{T}^{ij}$ and $\tilde{T}^{ij}$ which have a non-zero action on the wave functions, rotating them with respect to their SU(2) indices. The phenomenon that the presence of central (or "semi-central") charges in some supersymmetry algebra gives rise to “shortening” of the relevant irreducible supermultiplets is well known. The case under consideration supplies one more example of such a situation.

It is instructive to illustrate these features by the example of the wave functions corresponding to the LLs with $N = 1$ and $N = 2$. It is worth noting that the LLL wave function is a singlet of both $N = 4$ supersymmetries and hence of the entire $SU(2|2)_{\text{dyn}}$.

The infinitesimal transformations of the general wave function $\Psi^{(N)}$ generated by $S^{ia}$ and $\hat{S}^{ia}$ can be written as

$$\delta \Psi^{(N)} = -i(\epsilon_{ia} S^{ia} + \hat{\epsilon}_{ia} \hat{S}^{ia}) \Psi^{(N)}.$$  (6.21)

Using the representations (6.6) and (6.20) one can explicitly find the transformations induced by (6.21) for the multiplets of the component wave functions for $N = 1$ and $N = 2$:

For $N = 1$:

$$\delta \phi^{i} = 2\sqrt{\kappa}(\epsilon^{ia} + i\hat{\epsilon}^{ia}) \psi_{a}, \quad \delta \psi^{a} = 2\sqrt{\kappa}(\epsilon^{ia} - i\hat{\epsilon}^{ia}) \phi^{i};$$  (6.22)

$$\delta \phi = 2\sqrt{\kappa}(\epsilon^{ia} - i\hat{\epsilon}^{ia}) \psi^{ia}. $$  (6.23)

One can check that the closure of these transformations agrees with the anticommutation relations (6.8), (6.11) and (6.12). Note that for $N = 2$ (and, generally speaking, for all even $N$) one could formally impose some reality conditions on the involved functions in a way compatible with either first or second $N = 4$ supersymmetry, but not with both supersymmetries simultaneously. Thus the component wave functions for any $N$ should be essentially complex, and this matches with the property that all of them are holomorphic functions of the complex coordinates $(z, u, \zeta, \xi)$\footnote{The irreducible set of wave functions for any $N \geq 1$ is in one-to-one correspondence (modulo some rescalings) with the $d=1$ field content of the complex bi-harmonic analytic superfield $q^{N,0}(\zeta, u, v), D^{2,0}q^{N,0} = 0$, in which $i\partial_t$ on all component fields is replaced by $2\kappa N$.}. In fact, the $SU(2|2)_{\text{dyn}}$ irrep which are realized on the $(2N + 2N)$ multiplets of the $N$-th LL wave functions are what is called “atypical” or “short” $SU(2|2)$ representations (see, e.g., [29, 30, 31, 32]).

6.3 Decoupling of worldline and target-space supersymmetries

Like in the case of $\mathcal{N}=2$ super Landau model [15], one could ask whether the above worldline $\mathcal{N}=4$ supersymmetries are a consequence of the target-space $ISU(2|2)$ symmetry via the Sugawara representation (6.9) and (6.10). The answer is that these two types of supersymmetry are in fact independent of each other due to the existence of the basis in which their generators are divided into two mutually (anti)commuting sets.
It will be useful to define
\[ \Sigma^i_a = Q^{i_a} + Q^{i_a}, \quad \Sigma^i_a = i(Q^{i_a} - Q^{i_a}), \quad (\Sigma^i_a)^\dagger = -\Sigma^i_a, \]
\[ \{\Sigma^i_a, \Sigma^j_b\} = -\epsilon^{ij} e^{ab} Z, \quad \{\Sigma^i_a, \Sigma^j_a\} = 2i \left( e^{ab} T^{ij} - \epsilon^{ij} T^{ab} \right). \quad (6.24) \]

Then the decoupling transformation is as follows
\[ \tilde{\Sigma}^i_a = \Sigma^i_a - \frac{1}{2\sqrt{\kappa}} S^i_a = \frac{i}{2\kappa} P^i_A \Pi^{aA}, \quad \tilde{\Sigma}^i_a = \Sigma^i_a - \frac{1}{2\sqrt{\kappa}} \hat{S}^i_a = \frac{1}{2i\kappa} C^{AB} P^i_A \Pi^a_B. \quad (6.25) \]

In virtue of the relations (6.15), \( \tilde{\Sigma}^i_a \) anticommute with both \( \mathcal{N}=4 \) supercharges
\[ \{\tilde{\Sigma}^i_a, S^{jb}\} = \{\tilde{\Sigma}^i_a, \hat{S}^{jb}\} = 0. \quad (6.26) \]

It is also straightforward to check that \( \tilde{\Sigma}^i_a \), together with the \( SU(2) \) generators \( \tilde{T}^{ij}, \tilde{T}^{ab} \) defined by (6.14), satisfy just the relations (6.24), with
\[ \tilde{Z} = Z + \frac{1}{2\kappa} H_q, \quad (6.27) \]

and have the same (anti)commutation relations \( 4.49, 4.50 \) with the magnetic supertranslation generators as \( \Sigma^i_a \). The relations \( 4.47, 4.48 \), with \( \tilde{Z}, \tilde{T}^{ij} \) and \( \tilde{T}^{ab} \) being substituted for \( Z, T^{ij} \) and \( T^{ab} \), are also satisfied.

Thus, after passing to the generators with tilde, the full symmetry of the \( \mathcal{N}=4 \) super Landau model has been reduced to the direct product \( \tilde{ISU}(2|2) \times SU(2|2)_{dyn} \), with
\[ \tilde{ISU}(2|2) \propto \left(P_{Aa}, \Pi_{AB}, \tilde{\Sigma}^i_a, \tilde{T}^{ij}, \tilde{T}^{ab}, \tilde{Z} \right), \quad SU(2|2)_{dyn} \propto \left( S^i_a, \hat{S}^{jb}, \tilde{T}^{ij}, \tilde{T}^{ab}, H_q \right). \quad (6.28) \]

The generators \( \tilde{T}^{ij}, \tilde{T}^{ab} \) commuting with \( \tilde{T}^{ij}, \tilde{T}^{ab} \) (equally as with the remaining generators of \( \tilde{ISU}(2|2) \)) were defined in (6.13). We also took into account the commutation relations (6.16).

Note that for all generators of \( \tilde{ISU}(2|2) \) the \( \dagger \) hermitian conjugation coincides with the ordinary \( \dagger \) conjugation, whereas
\[ (\Sigma^i_a)^\dagger = (\Sigma^i_a)^\dagger - \frac{1}{\sqrt{\kappa}} S^i_a, \quad (\Sigma^i_a)^\dagger = (\Sigma^i_a)^\dagger - \frac{1}{\sqrt{\kappa}} \hat{S}^i_a. \quad (6.29) \]

Also, we observe that the \( \tilde{SU}(2|2) \) generators have a Sugawara representation in terms of the supertranslation generators. On the other hand, if we will try to construct, on the pattern of (6.9), (6.10) and (5.8), some new generators \( S^i_a, \hat{S}^i_a \) and \( H_q \) out of the \( \tilde{ISU}(2|2) \) generators, they will prove to be identically zero. Thus in the correctly defined basis, the (anti)commutation relations of the worldline supergroup \( SU(2|2)_{dyn} \) do not follow from those of the target-space supergroup \( \tilde{ISU}(2|2) \).

All these features directly generalize those found in [15] for the \( \mathcal{N}=2 \) super Landau model.
7 Some generalizations

In this Section we consider the most general extension of the action (4.7) consistent with
the off-shell $\mathcal{N}=4$ supersymmetry, following a similar generalization of the $\mathcal{N}=2$ action
(2.7) considered in [16].

This general $\mathcal{N}=4$ action corresponds to the following modification of first and third
terms in (4.7):

$$S_{\text{gen}} = \kappa \left( \int \mu^{-2,0} L^{2,0}(q^{1,0A}, u, v) - i \int \mu^{0,-2} \psi^{0,1A} \psi^{0,1B} \epsilon_{AB} \right. $$
$$+ \left. \frac{1}{\sqrt{\kappa}} \int \mu^{-2,0} F^{1,0A}(q^{1,0A}, u, v) D^{1,-1} \psi^{0,1B} \epsilon_{AB} \right),$$

(7.1)

where $L^{2,0}$ and $F^{1,0A}$ are arbitrary functions of their arguments. In the presence of
non-trivial function $F^{1,0A} \neq q^{1,0A}$ the equations of motion for auxiliary fields become
unsolvable, so in what follows we choose $F^{1,0A} = q^{1,0A}$ like in (4.7)

After integration

over Grassmann and harmonic variables, we find

$$L = \dot{f}^{iA} A_{iA} - i \kappa \dot{x}^{aA} \chi_{aA} + \frac{i}{2} \psi^{aA} \psi^{B} G_{AB} - \frac{1}{2} \kappa h^{iA} \dot{h}_{iA} - \sqrt{\kappa} (\dot{f}^{iA} \dot{h}_{iA} + \psi^{aA} \dot{x}_{aA}),$$

(7.2)

where

$$A_{iA}(f) = -\kappa \int dudv u^{-1} \frac{\partial L^{2,0}}{\partial q^{1,0A}} \bigg|_{\theta=0},$$

(7.3)

$$G_{AB}(f) = \frac{\kappa}{2} \int dudv g_{AB}(f^{iA} u^{iA}, u, v), \quad g_{AB}(f^{iA} u^{iA}, u, v) = \frac{\partial^2 L^{2,0}}{\partial q^{1,0A} \partial q^{1,0B}} \bigg|_{\theta=0}.$$  

(7.4)

Using these definitions, it is easy to show that the background gauge potential $A^{iA}$ satisfies
the $\mathbb{R}^4$ self-duality condition

$$F_{iB,ja} := \partial_{iB} A_{jA} - \partial_{jA} A_{iB} = -2 G_{AB} \epsilon_{ij}.$$  

(7.5)

The Bianchi identity $\partial^A G_{AB} = 0$ implying that $G_{AB}$ is harmonic, $\partial^C \partial_A G_{AB} = 0,$
is automatically satisfied by the expression (7.4) for $G_{AB},$ so (7.3) and (7.4) give in fact
the most general solution of the abelian self-duality condition in terms of the analytic
harmonic “prepotential” $L^{2,0}$ [22]. Note that the representation (7.3) also implies the
transversality condition $\partial^{iB} A_{iB} = 0,$ but it can be considered merely as a gauge choice
because the Lagrangian (7.2) is invariant, up to total time derivative, under the redefinitions
$A_{iB} \rightarrow A_{iB} + \partial_{iB} \Lambda(f).$

Thus in the general case the external gauge potential is also self-dual, like in the
simplest case (4.7).

After eliminating the auxiliary fields, one obtains the following expression for the
Lagrangian in terms of physical fields

$$L = \dot{f}^{iA} A_{iA} - i \kappa \dot{x}^{aA} \chi_{aA} + \frac{1}{2} \dot{f}^{iA} \dot{f}_{iA} + i \frac{\kappa}{2} (G^{-1})_{AB} \dot{x}^{aA} \dot{x}^{B}. $$

(7.6)

---

8A similar restriction was imposed on the $\mathcal{N}=2$ superfield Lagrangian in [16].
9The proof can be found in [32] and [21].
We observe that the bosonic target metric is still flat, in contrast to the general $\mathcal{N}=2$ model of ref. [16]. The reason behind this is the impossibility to insert, without breaking of $\mathcal{N}=4$ supersymmetry, any function of $q^{1,0,4}$ into the second term in (7.1) since this superfield and the fermionic superfield $\psi^{0,1B}$ live on different analytic subspaces of the bi-harmonic $\mathcal{N}=4$ superspace. Moreover, it can be shown that any metric can be removed from the kinetic term of $\chi^{aA}$ as well.

To this end, it is convenient to rewrite the Lagrangian in the complex parametrization

$$L = |\dot{z}|^2 + |\dot{u}|^2 + \dot{z}A + \dot{u}B + \dot{A}B + \dot{u}D_{11}\xi + iD_{12}\dot{\xi} + iD_{22}\dot{\xi} - \kappa(\dot{\zeta} + \dot{\zeta}' + \dot{\xi} + \dot{\xi}') ,$$

where $A$ and $B$ are the components of $\mathcal{A}^{aA}$ in the complex notation

$$A = A^{11}, \quad \bar{A} = A^{22}, \quad B = A^{21}, \quad \bar{B} = -A^{12},$$

and

$$D_{AB} = \kappa(G^{-1})_{AB}.$$ 

Under the choice $L^{2,0}(q^{1,0,4}, u, v) = q^{1,0,4}q^{0,1B}C_{AB}$ we immediately obtain $D_{AB} = C_{AB}$ and so come back to the action (4.19). In the general case, with making use of the parametrization $D_{11} = \bar{D}_{22} = |D_{11}|e^{i\varphi}, (D_{12}) = -D_{12}$, one can find a field redefinition which brings the kinetic terms of the fermion part into a diagonal form. This redefinition is as follows

$$\zeta = -ie^{-i\varphi}\zeta' + |b|\zeta', \quad \xi = ie^{-i\varphi}\xi' + |b|\xi',$$

where $|b|$ is sought from the quadratic equation

$$|b|^2||D_{11}| + 2iD_{12}|b| - |D_{11}| = 0.$$ 

After some calculation we find the final expression for the action

$$L = |\dot{z}|^2 + |\dot{u}|^2 + \kappa(\dot{z}A + \dot{u}B + \dot{A}B + \dot{u}D_{11}\xi + iD_{12}\dot{\xi} + iD_{22}\dot{\xi} - \kappa(\dot{\zeta} + \dot{\zeta}' + \dot{\xi} + \dot{\xi}') .$$

(7.9)

It is rather surprising that the component Lagrangian obtained from (7.1) has the same fermionic part as (4.19), despite the fact that (7.1) involves the most general interaction. On the other hand, the background gauge field potential consistent with $\mathcal{N}=4$ supersymmetry turns out to be more general than just the linear potential ($A \sim \bar{z}$ and $B \sim \bar{u}$) which appears in (4.19) and which was used in [5] to describe a version of the $\mathcal{N}=4$ QHE. The only constraint on the gauge potential is that it must be self-dual on $\mathbb{R}^4$. It would be interesting to reveal possible physical applications of such a more general $U(1)$ potential in the QHE on $\mathbb{R}^4$, e.g., along the lines of [3].

8 Summary and outlook

Let us briefly summarize the results of the paper. We constructed the first example of $\mathcal{N}=4$ supersymmetric Landau model which is a minimal extension of the $\mathcal{N}=2$ super Landau model of ref. [13, 15, 18]. We started from the superfield off-shell action involving the linear $\mathcal{N}=4$ multiplet $(4, 4, 0)$ and its mirror fermionic counterpart. After elimination
of the auxiliary fields, in the component Lagrangian there remain four bosonic and four fermionic physical fields. In the bosonic limit, when all fermionic fields are suppressed, one recovers the Lagrangian of the model used in [5] to describe $R^4$ quantum Hall effect with the $U(1)$ background gauge field. Besides the manifest $\mathcal{N}=4$ supersymmetry, the Lagrangian constructed respects invariance under the target graded supersymmetry $ISU(2|2)$ and, more surprisingly, under the second on-shell $\mathcal{N}=4$ supersymmetry, which, together with the first one, close on a “dynamical” worldline $SU(2|2)_{dyn}$ symmetry. We quantized the model and found the spectrum of the Hamiltonian, as well as the degeneracy of the wave functions for every Landau level. The LLL wave function is a singlet of $\mathcal{N}=4$ supersymmetries, while the wave functions of the next LLs form irreducible $\mathcal{N}=4$ (and $SU(2|2)_{dyn}$) multiplets. For the wave functions to possess non-negative norms and, hence, for the model to preserve unitarity, one is led, like in the $\mathcal{N}=2$ case, to introduce a non-trivial metric operator on the space of states and thus to redefine the corresponding inner product. We also discussed the most general form of the action of the original two multiplets, such that it is compatible with the worldline $\mathcal{N}=4$ supersymmetry. The latter requirement proves to be very restrictive: as opposed to the generic $\mathcal{N}=2$ super Landau action [16], its $\mathcal{N}=4$ counterpart involves no non-trivial target superspace metric. The general restriction on the external gauge field is that it should satisfy the self-duality condition on $R^4$.

There are few directions in which the present study could be continued. First, it would be interesting to construct the quantum version of the generalized $\mathcal{N}=4$ model considered in Sect. 8 and to reveal its possible applications in the $R^4$ QHE. It is also of obvious interest to extend it in such a way as to gain a non-trivial target super-metric in the component Lagrangian, like in [16], and so to get, in the bosonic sector, a version of Landau model on a curved four-dimensional manifold. One way to achieve this consists in replacing the linear bosonic $(4,4,0)$ multiplet and, perhaps, its fermionic mirror by their nonlinear counterparts along the line of ref. [27]. Another possibility is to add the mirror $q^{0,1A}$ and $\psi^{1,0A}$ superfields to the original set $q^{1,0A}$ and $\psi^{0,1A}$, thus passing to a model with the eight-dimensional bosonic target space. After such an extension, one will be able to insert, in the second term in (7.1), a function of $q^{1,0A}$ (and a function of $q^{0,1A'}$ into the new mirror counterpart of this term), without any conflict with the bi-harmonic analyticities. These terms can give rise to some non-trivial super-metric (and superbackground gauge fields) after eliminating auxiliary fields. Also, in such type of super Landau models one could hope to find out an off-shell $\mathcal{N}=8$ worldline supersymmetry [28]. It is also tempting to seek for $\mathcal{N}=4$ superextension of the Landau-type models on $S^4$ with couplings to an external non-abelian $SU(2)$ gauge field [3, 4, 5]. In the conventional $\mathcal{N}=4$ mechanics such couplings are introduced [26] with the help of the so called spin (or isospin) semidynamical supermultiplets [33]. One can hope that the same mechanism works in the case of $\mathcal{N}=4$ super Landau models too. At last, it is worth noting that, besides $\mathcal{N}=4$ “hypermultiplets” $(4,4,0)$ and $(0,4,4)$ utilized here, there are many other off-shell $\mathcal{N}=4$ multiplets, e.g. $(3,4,1)$ and $(2,4,2)$. They, together with their fermionic counterparts, can be used for constructing alternative $\mathcal{N}=4$ super Landau models.

One more possible direction of the future work is related to setting up curved analogs of the $(4|4)$ super Landau model constructed here, such that the latter is reproduced in some contraction limit, like the standard bosonic Landau model [1] is recovered from the Haldane model [2] after contraction of $SU(2)$ into $E(2)$, the group of motion of the
Euclidean plane. The (2|2) superplane Landau model can be obtained in a similar way from the Landau model on the supersphere \( SU(2|1)/U(1|1) \sim \mathbb{CP}^{(1|1)} \) \([12]\). In the (4|4) case one can expect an analogous relation with the Landau model on the projective supermanifold \( SU(3|2)/U(2|2) \sim \mathbb{CP}^{(2|2)} \). It can be regarded as a superextension of one of the \( SU(3)/U(2) \) models considered in ref. \([31]\) and could be closely related to the integrable \( su(3|2) \) spin chain which, in turn, bears an intimate relation to the planar \( \mathcal{N}=4 \) SYM theory \([30]\) and \( AdS_5 \times S^5 \) superstring \([34]\). Finally, \( SU(2|2) \) already appeared as a dynamical symmetry acting in the space of quantum states of the super Landau model on the superflag \( SU(2|1)/U(1) \times U(1) \) \([12]\), and it would be interesting to clarify possible links of this realization with that given in the present paper.

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**A Symmetries in the complex notation**

It is obvious that the symmetry group of the Lagrangian (4.19) includes two \( ISU(1|1) \) subgroups (on the fields \((z, \zeta)\) and \((u, \xi)\), because (4.19) is just a sum of two copies of (2.9). There are two \( SU(2) \) subgroups which rotate the fields \((z, u)\) and \((\zeta, \xi)\). Finally, there are two subgroups \( SU(1|1) \) which are realized on the fields \((z, \xi)\) and \((u, \zeta)\). Below we list all generators of these groups.

We start by defining the generators in the complex notation through those in Sect. 4.3:

\[
Q^{11} = -Q^1_1, \quad Q^{12} = -Q^1_2, \quad Q^{21} = Q^1_1, \quad Q^{22} = Q^1_2, \\
Q_{11} = Q_1, \quad Q_{12} = Q_2, \quad Q_{21} = -Q_1, \quad Q_{22} = -Q_2, \\
Z = Z_1 + Z_2, \\
SU(2)_L : \quad T^{(1)}_{11} = \frac{1}{2} T_{b3}, \quad T^{(1)}_{21} = \frac{1}{2} (T_{b1} + T_{b2}), \quad T^{(2)}_{11} = \frac{1}{2} (T_{b1} - T_{b2}), \\
SU(2)_R : \quad T^{(1)}_{11} = \frac{1}{2} T_{f3}, \quad T^{(1)}_{21} = \frac{1}{2} (T_{f1} + T_{f2}), \quad T^{(2)}_{11} = \frac{1}{2} (T_{f1} - T_{f2}), \\
P_{11} = P_z, \quad P_{12} = -P_{\bar{u}}, \quad P_{21} = P_u, \quad P_{22} = P_{\bar{z}}, \\
\Pi_{11} = \Pi_\zeta, \quad \Pi_{12} = -\Pi_\xi, \quad \Pi_{21} = \Pi_\xi, \quad \Pi_{22} = \Pi_\zeta.
\]

(A.1) (A.2) (A.3) (A.4)

Now we give the explicit expressions for these generators.

1. \( ISU(1|1) \) realized on \((z, \zeta)\):

\[
P_z = -i (\partial_z + \kappa \bar{z}), \quad P_{\bar{z}} = -i (\partial_{\bar{z}} - \kappa z), \quad \Pi_\zeta = \partial_\zeta + \kappa \bar{\zeta}, \quad \Pi_{\bar{\zeta}} = \partial_\bar{\zeta} + \kappa \zeta, \\
Z_1 = z \partial_z - \bar{z} \partial_{\bar{z}} + \zeta \partial_\zeta - \bar{\zeta} \partial_{\bar{\zeta}}, \quad Q_1 = z \partial_\zeta - \bar{\zeta} \partial_z, \quad Q_1^i = \bar{z} \partial_\bar{\zeta} + \zeta \partial_z.
\]

(A.5)

2. \( ISU(1|1) \) realized on \((u, \xi)\):

\[
P_u = -i (\partial_u + \kappa \bar{u}), \quad P_{\bar{u}} = -i (\partial_{\bar{u}} - \kappa u), \quad \Pi_\xi = \partial_\xi + \kappa \bar{\xi}, \quad \Pi_{\bar{\xi}} = \partial_{\bar{\xi}} + \kappa \xi,
\]

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3. $SU(2)_L$ realized on $(z, u)$:

$$T_{b1} = z\partial_u + u\partial_z - \bar{z}\partial_u - \bar{u}\partial_z, \quad T_{b2} = z\partial_u - u\partial_z + \bar{z}\partial_u - \bar{u}\partial_z, \quad T_{b3} = z\partial_z - \bar{z}\partial_z - u\partial_u + \bar{u}\partial_u. \quad (A.7)$$

4. $SU(2)_R$ realized on $(\zeta, \xi)$:

$$T_{f1} = \zeta\partial_\zeta + \xi\partial_\xi - \bar{\zeta}\partial_\zeta - \bar{\xi}\partial_\xi, \quad T_{f2} = \zeta\partial_\zeta - \xi\partial_\xi + \bar{\zeta}\partial_\zeta - \bar{\xi}\partial_\xi, \quad T_{f3} = \zeta\partial_\zeta - \bar{\zeta}\partial_\zeta - \xi\partial_\xi + \bar{\xi}\partial_\xi. \quad (A.8)$$

5. Two further $SU(1|1)$ realized on $(z, \xi)$ and $(u, \zeta)$:

$$Q_3 = z\partial_\zeta - \bar{\zeta}\partial_z, \quad Q_3^\dagger = \bar{z}\partial_\zeta + \xi\partial_\zeta, \quad Z_1' = z\partial_z - \bar{z}\partial_z + \xi\partial_\zeta - \bar{\zeta}\partial_\xi = Z_2 + T_{b3}, \quad (A.9)$$

$$Q_4 = u\partial_\zeta - \bar{\zeta}\partial_u, \quad Q_4^\dagger = \bar{u}\partial_\zeta + \xi\partial_\zeta, \quad Z_2'' = u\partial_u - \bar{u}\partial_u + \xi\partial_\zeta - \bar{\zeta}\partial_\xi = Z_1 - T_{b3}. \quad (A.10)$$

It is worth noting that only three generators out of the set of $U(1)$ generators $Z_1, Z_2, T_{b3}, T_{f3}$ (coming from two supergroups $ISU(1|1)$ and two groups $SU(2)$) are linearly independent: $Z_2 + T_{b3} + T_{f3} = Z_1$. This can be explained as follows. In the Lagrangian, there are two symmetry automorphism groups $SU(2)_{L,R}$ and an extra group $SU(2)^{ext}$ realized on the indices $A$, but the latter $SU(2)$ is broken down to some $U(1)$ by constants $C_{AB}$. Thus, there are only three linearly independent mutually commuting $U(1)$ generators inside $ISU(2|2)$. Note, however, that there is one additional $U(1)$ invariance, which is not contained in the closure of the odd $ISU(2|2)$ generators. Its generator can be chosen, e.g., as $\zeta\partial_\zeta - \bar{\zeta}\partial_\zeta$. It can be interpreted as an outer automorphism of $ISU(2|2)$.

Finally, we rewrite the $\mathcal{N}=4$ supersymmetry transformations (A.27) in the complex notation. For this purpose we introduce complex parameters $\epsilon_1$ and $\epsilon_2$,

$$\epsilon^{11} = -i\epsilon_2, \quad \epsilon^{22} = i\bar{\epsilon}_2, \quad \epsilon^{21} = i\bar{\epsilon}_1, \quad \epsilon^{12} = i\epsilon_1. \quad (A.11)$$

Then the transformations (5.5) take the form

$$\delta z = \frac{i}{\sqrt{K}} \epsilon_1 \dot{\zeta} + \frac{i}{\sqrt{K}} \epsilon_2 \dot{\xi}, \quad \delta u = -\frac{i}{\sqrt{K}} \bar{\epsilon}_1 \dot{\zeta} + \frac{i}{\sqrt{K}} \bar{\epsilon}_2 \dot{\xi},$$

$$\delta \zeta = \frac{i}{\sqrt{K}} \bar{\epsilon}_1 \dot{z} + \frac{i}{\sqrt{K}} \epsilon_2 \dot{u}, \quad \delta \xi = \frac{i}{\sqrt{K}} \bar{\epsilon}_2 \dot{z} - \frac{i}{\sqrt{K}} \epsilon_1 \dot{u}. \quad (A.12)$$

B  Realization of $ISU(2|2)$ on superfields $q^{1,0A}, \psi^{0,1A}$

In this appendix we give the off-shell realization of the $ISU(2|2)$ symmetry group on the superfields $q^{1,0A}, \psi^{0,1A}$, which in components reproduces the on-shell realization of Sect. 4.3.

We begin with the magnetic supertranslations:

$$\delta q^{1,0A} = \bar{b}^A u_1^1, \quad \delta \psi^{0,1A} = \nu^A v_1^1. \quad (B.1)$$

The central charge $Z$ symmetry, which simultaneously changes the phases of all fields, is realized by

$$\delta q^{1,0A} = \alpha C_B^A q^{1,0B}, \quad \delta \psi^{0,1A} = \alpha C_B^A \psi^{0,1b}. \quad (B.2)$$
The odd $SU(2|2)$ transformations which mix bosonic and fermionic superfields are given by the following variations

\[
\delta q^{1.0A} = D^{1.1} D^{-1.1} \left( \frac{1}{2} (C_B^A + i \delta_B^A) \left[ A^{-1.1} + B^{-1.1} D^{0,-2} + C^{0.0} D^{-1,-1} \right] \psi^{0,1B} \right)
- \frac{i}{2 \sqrt{\kappa}} \left( C_B^A - i \delta_B^A \right) \left[ E^{-1.1} D^{1.1} + E^{-1.1} D^{-1,-1} - C^{0.0} D^{-1,-1} D^{-1,1} \right] q^{1.0B}, \tag{B.3}
\]

\[
\delta \psi^{0,1A} = D^{1.1} D^{-1.1} \left( \frac{1}{2} (C_B^A - i \delta_B^A) \left[ \hat{A}^{-1.1} + \hat{B}^{-1.1} D^{-2,0} - C^{0.0} D^{-1,-1} \right] q^{1.0B} \right) + \frac{i}{2 \sqrt{\kappa}} \left( C_B^A + i \delta_B^A \right) \left[ \hat{E}^{-1.1} D^{1.1} + \hat{E}^{-1.1} D^{-1,-1} + C^{0.0} D^{-1,-1} D^{-1,1} \right] \psi^{0,1B}. \tag{B.4}
\]

Here

\[
A^{-1.1} := \omega^{-1.1} \theta^{-1,1} \theta^{-1,1} - \omega^{-1.1} \theta^{-1,1} \theta^{-1,1}, \quad B^{-1.1} := \omega^{-1.1} \theta^{-1,1} \theta^{-1,1} - \omega^{-1.1} \theta^{-1,1} \theta^{-1,1},
\]

\[
C^{0.0} := \omega^{-1.1} \theta^{-1,1} \theta^{-1,1}, \quad E^{-1.1} := \omega^{-1.1} \theta^{-1,1} \theta^{-1,1}, \quad E^{-1.1} := \omega^{-1.1} \theta^{-1,1} \theta^{-1,1},
\]

\[
\hat{A}^{-1.1} := \omega^{-1.1} \theta^{-1,1} \theta^{-1,1}, \quad \hat{B}^{-1.1} := \omega^{-1.1} \theta^{-1,1} \theta^{-1,1} - \omega^{-1.1} \theta^{-1,1} \theta^{-1,1},
\]

\[
\hat{E}^{-1.1} := \omega^{-1.1} \theta^{-1,1} \theta^{-1,1}, \quad \hat{E}^{-1.1} := \omega^{-1.1} \theta^{-1,1} \theta^{-1,1}.
\]

These superfield transformations amount to the following transformations of the physical and auxiliary fields:

\[
\delta f^iA = \frac{1}{2} \omega^{ia} (C_B^A + i \delta_B^A) \chi^B_a, \quad \delta \chi^aA = \frac{1}{2} \omega^{ia} (C_B^A - i \delta_B^A) f^B_i,
\]

\[
\delta h^{ia} = -\frac{1}{2 \sqrt{\kappa}} \omega^{ia} (C_B^A + i \delta_B^A) \hat{\chi}_a, \quad \delta \hat{\psi}^{ia} = \frac{1}{2 \sqrt{\kappa}} \omega^{ia} (C_B^A - i \delta_B^A) \hat{f}^B_i. \tag{B.5}
\]

They are consistent with eqs. \(A.10\), \(A.11\). The variations with $\hat{\omega}^{ia}$ are obtained from the $\omega^{ia}$ ones via the $\sim$ conjugation.

It is straightforward to check the invariance of the superfield action \(B.1\) under \(B.3\)-\(B.4\). Note that the structure of the superfield transformations \(B.3\) and \(B.4\) is almost uniquely determined from the requirement that their right-hand sides are nullified by the harmonic derivatives $D^{2,0}$ and $D^{0,2}$ (in agreement with the harmonic constraints \(A.1b\) and \(A.2b\)). All even $SU(2|2)$ transformations (including \(B.2\)) are contained in the closure of \(B.3\), \(B.4\) and their $\hat{\omega}^{ia}$ counterparts, so we do not give their explicit form.

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