Towards the computation of time-periodic inertial range dynamics

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Abstract. We explore the possibility of computing simple invariant solutions, like travelling waves or periodic orbits, in Large Eddy Simulation (LES) on a periodic domain with constant external forcing. The absence of material boundaries and the simple forcing mechanism make this system a comparatively simple target for the study of turbulent dynamics through invariant solutions. We show, that in spite of the application of eddy viscosity the computations are still rather challenging and must be performed on GPU cards rather than conventional coupled CPUs. We investigate the onset of turbulence in this system by means of bifurcation analysis, and present a long-period, large-amplitude unstable periodic orbit that is filtered from a turbulent time series. Although this orbit is computed on a coarse grid, with only a small separation between the integral scale and the LES filter length, the periodic dynamics seem to capture a regeneration process of the large-scale vortices.

1. Introduction
Starting with the pioneering work of Nagata on Couette flow [1], and the parallel development of numerical techniques by Cliffe and various collaborators (e.g. [2]), computational dynamical systems theory has made a number of valuable contributions to turbulence research. Perhaps the best example of the impact of this approach is found in the transition to turbulence in parallel shear flows in minimal flow units. This transition has been demystified by the computation and subsequent study of equilibrium and time-periodic flows that can co-exist with both the laminar flow and long-lasting turbulent transients. An overview of results in this vein was given by Kawahara et al. in 2012 [3]. Around the time this paper appeared, the focus of much research in this area shifted to the more challenging problem of transitional flows on larger or open domains. Duguet et al., for instance, computed spatially localized solutions in Couette flow [4] and, more recently, Khapko et al. obtained similar results for the asymptotic suction boundary layer [5]. One conclusion that can be drawn from the latter works is that, when the number of Degrees Of Freedom (DOF) in the simulation is in the order of millions it get hard, if not impossible, to perform the dynamical systems-based computations with the same amount of rigour as in the less demanding case of minimal domains and marginal Reynolds numbers.

We can demonstrate the difficulty of computing, for instance, time-periodic orbits on large domains by combining some estimates for Homogeneous Isotropic Turbulence (HIT), in which the separation of spatial scales can be expressed asymptotically in terms of the Reynolds number. The commonly used estimate is that the ratio of the integral scale to the Kolmogorov scale...
behaves as \( L/\eta \propto Re^{3/4} \propto \lambda^{3/2} \), where \( Re \) is a geometric Reynolds number and \( Re_\lambda \) is Taylor’s microscale Reynolds number [6]. In order to resolve the spatial scales of the fluid motion, we must thus let the linear grid size scale as \( Re_\lambda^{3/2} \). As a consequence, the computation time of a single time step, using standard pseudo-spectral methods, will scale as \( Re_\lambda^{9/2} \) up to a logarithmic correction. At the same time, the size of a single time step should decrease at least in proportion to the Kolmogorov time scale, namely as \( Re_\lambda^{-1} \). The computational complexity of a simulation over a fixed time interval then scales as \( Re_\lambda^{11/2} \).

In order to compute invariant solutions, we use Newton-Raphson iteration, or some variation thereof with an enlarged radius of convergence. To compute one such iteration, we need to solve a dense system of linear equations. Currently, the only known way to do this is by the use of a Krylov subspace method. Each Krylov iteration requires the simulation of the flow along the invariant solution, at a computational cost scaling as \( Re_\lambda^{11/2} \). The number of Krylov iterations necessary to obtain convergence is largely determined by the spectral properties of the matrix of derivatives of the finite-time flow. The more clustered its eigenvalues, the faster the convergence, and clustering is a consequence of viscous damping [7]. As the Reynolds number increases, an increasing number of eigenvalues gets scattered across the complex plane, and it seems reasonable to assume that their numbers grows as \( (L/\eta)^{3} \propto Re_\lambda^{9/2} \).

The overall estimate is then that the computational complexity of a single Newton-like iteration, edging us closer to a time-periodic Navier-Stokes flow, will grow as \( Re_\lambda^{10} \). The gap between computations in marginally turbulent flows, with \( Re_\lambda = O(10) \), and fully turbulent flows, with \( Re_\lambda = O(100) \) seems impossible to bridge. Moreover, there is no guarantee that the convergence properties of the Krylov and Newton iteration persist at higher Reynolds numbers. The accumulation of round-off error on various levels of the computation may lead to stagnation of the Krylov residual and linear, rather than quadratic convergence for Newton iteration.

In conclusion, it seems that we need to strike a compromise in order to study turbulent flow with a wide separation of length scales with the tools of dynamical systems. One possibility is to compute “shadows” of invariant solutions, e.g. time segments that are periodic up to a residual that is small compared to the variations along the orbit but large compared to that of the result of a convergent Newton-Krylov iteration. This approach was indeed taken by Duguet [4], Khapko [5], and others in the works cited above. In contrast to truly invariant solutions, the definition of a ghost is subjective. The former can be identified by estimating the minimal achievable residual given the finite-accuracy arithmetic and measuring the rate of convergence of the Newton iteration. The definition of the latter is necessarily based on some threshold value of the residual. A further disadvantage of this approach is that the persistence of ghosts when varying parameters is not guaranteed. While a truly periodic orbit can be tracked uniquely in the Reynolds number or other parameters of the flow, ghosts may simply disappear.

A different possibility is to compute invariant solutions in a model flow with fewer DOF. In particular, one can apply Large Eddy Simulation (LES) for this purpose. In LES, we introduce a filter length scale beyond which the dynamics is assumed to be strongly dissipative and well-represented by an effective eddy viscosity. The number of DOF in LES is thus greatly reduced as compared to Navier-Stokes flow with a comparable separation of length scales. Although the LES introduces extra nonlinearities into the equations of motion, the net effect is to reduce the time it takes to simulate the flow. Although LES does not alleviate the bounds on the time step size or the number of Krylov iterations, this reduction may just be sufficient to compute invariant solutions in fully turbulent flow.

This possibility was recently explored by Sasaki at al. [8] and by Hwang at el. [9] in channel flow. The presence of walls in this geometry brings a complication with it. Near the wall, where the rate of strain is high, the molecular viscosity is important. Thus, we must consider both molecular and eddy viscosity. Usually the regions where the two are dominant are glued together...
by an ad-hoc transition function. One can avoid these complications by considering a fluid on a periodic domain. Sekimoto and Jiménez [10] applied a homogeneous shear to such a spatially periodic fluid and managed to compute several equilibrium solutions and travelling waves. In their computations, the filter length is several time greater than the grid spacing, and not small enough compared to the integral scale to observe a power-law energy spectrum. Nonetheless, the invariant solutions exhibit interesting spatial structures and we can consider their results a proof-of-principle that interesting invariant solutions can be computed for a spatially periodic LES fluid.

The geometry we chose is simpler than the homogeneous shear layer. Instead of shear, we impose an external force that is constant in time and excites four counter-rotating vortex columns. The setup is identical to that of Yasuda et al. [11]. In their study, Yasuda et al. showed that there are quasi-cyclic, large-amplitude oscillations in aggregate quantities like the energy and energy dissipation rate. These large-scale oscillations hint at the existence of a regeneration cycle for vortical structures in this flow, which could explain elements of inertial range dynamics. An obvious goal for the computation of simple invariant solutions in this flow is to capture the regeneration cycle with periodic orbits.

In the current paper we lay out the governing equations, briefly discuss earlier work on Navier-Stokes and LES flow in this geometry and discuss the behaviour of the LES fluid on a grid of $32^3$ points. We present one large-amplitude, long-period orbit that seems to capture the regeneration cycle to some extent, although it is computed in a regime with little scale separation. Finally, we speculate on the numerical requirements for reproducing similar invariant solutions on larger grids and in more turbulent flows.

2. Governing equations

The incompressible Navier-Stokes equation with periodic boundary conditions and body forcing is given by

$$u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla p - \nu \Delta u = \gamma f$$

$$\nabla \cdot u = 0$$

$$u(x + L, y, z, t) = u(x, y + L, z, t) = u(x, y, z + L, t) = u(x, y, z, t),$$

where $\rho$ is the fluid density, assumed to be constant, $\nu$ is the kinematic viscosity and $\gamma$ is the amplitude of the forcing.

We will numerically approximate solutions to system (1) on a regular grid of $n^3$ points. In order to reduce the number of degrees of freedom in the numerical simulations, we adopt the strategy proposed by Smagorinsky [12] to model the motion on the smallest scales with an effective eddy viscosity. This approach can be interpreted as the application of a spatial filter given by a convolution

$$\bar{u} = G \ast u.$$  

The kernel $G$ can be thought of as a low-pass filter with width $l_f$. Since the spatial filter commutes with all spatial derivatives, the treatment of the linear terms in system (1) is straightforward. Filtering the advection term we obtain

$$u_i \partial_i \bar{u}_j = \partial_i (\bar{u}_i \bar{u}_j) = \partial_i (\bar{u}_i \bar{u}_j - \bar{u}_i \bar{u}_j) \equiv \bar{u}_i \partial_i \bar{u}_j + \partial_i \tau^{(s)}_{ij},$$

where summation over repeated indices is implied and the last equation defines the subgrid stress tensor $\tau^{(s)}$. It is split into a diagonal component, that modifies the pressure, and a deviatoric component, according to

$$\tau^{(s)}_{ij} = \frac{1}{3} \Pi \delta_{ij} + 2 \nu T \bar{S}_{ij},$$
where $\bar{S}_{ij} = (\partial_i \bar{u}_j + \partial_j \bar{u}_i)/2$ is the filtered rate-of-strain tensor, $\delta$ is the Kronecker delta symbol and the eddy viscosity $\nu_T$ is defined by

$$\nu_T = (C_S \Delta)^2 \sqrt{2 \bar{S}_{ij} \bar{S}_{ij}}. \quad (5)$$

This expression for $\tau^{(s)}$, which contains a filtered term quadratic in $u$, in terms of the filtered rate of strain closes the system of equations and constitutes the Smagorinsky model. The closure introduces a dimensionless parameter, $C_S$, called the Smagorinsky parameter.

The resulting equations are

$$\partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nabla \left( \frac{\bar{p}}{\rho} + \frac{1}{3} \Pi \right) - 2 \nabla \left( [\nu_T + \nu] \bar{S} \right) = \gamma \bar{f}$$

$$\nabla \cdot \bar{u} = 0. \quad (7)$$

We fix $\bar{f} = (- \sin(k_f x) \cos(k_f y), \cos(k_f x) \sin(k_f y), 0)^t$ with $k_f = 2\pi/L$. This body force induces a flow with four counter-rotating vortex columns which, up to a slight deformation due to the eddy viscosity, is one of a family of flows studied by Taylor and Green[13].

In the following, we nondimensionalize the equations according to

$$x' = k_f x, \quad t' = \sqrt{L\gamma k_f^2} t, \quad \Pi' = \frac{1}{L\gamma} \Pi \quad (8)$$

$$\bar{u}' = \sqrt{L\gamma} \bar{u}, \quad \bar{p}' = \frac{1}{\rho L\gamma} \bar{p} \quad (9)$$

and drop both the over bars for spatially filtered quantities and the primes for non dimensional quantities. The resulting momentum balance equation is

$$u_t + u \cdot \nabla u + \nabla \left( p + \frac{1}{3} \Pi \right) - 2 \nabla \left( \left[ \frac{1}{R^{3/2} \nu_T} + \frac{1}{Re} \right] S \right) = \frac{1}{\alpha} f \quad (10)$$

with non dimensional parameters

$$R = \left( \frac{1}{C_S \Delta k_f} \right)^{4/3} \quad Re = \frac{\sqrt{L\gamma}}{\nu k_f} \quad \alpha = L k_f = 2\pi. \quad (11)$$

Here, $R$ is the ‘LES Reynolds number’, which measures the ratio of the forcing length scale, $k_f^{-1}$, to $C_S \Delta$, which is the scale below which the eddy viscosity strongly damps motion. The effect of molecular viscosity is represented by the geometric Reynolds number $Re$. In the following we will take the limit of the latter to infinity or, equivalently, of vanishing molecular viscosity, and use $R$ as the control parameter. We will refer to the resulting system as governing an LES fluid. The other limit, in which $C_S = 0$ and $\nu > 0$, will be referred to as the DNS case.

Two quantities important for the analysis of the flow are the energy and enstrophy, which can be considered as the norm of velocity and vorticity, $\omega = \nabla \times u$, respectively:

$$E = \frac{1}{\alpha^3} \int \frac{1}{2} ||u||^2 \, dx \quad Q = \frac{1}{\alpha^3} \int \frac{1}{2} ||\omega||^2 \, dx, \quad (12)$$

where the integral is taken over the entire domain and $||.||$ denotes the standard vector norm. In addition, we compute the rate of energy input,

$$e = \frac{1}{\alpha^3} \int u \cdot f \, dx, \quad (13)$$

and dissipation, or transfer to sub-filter scales,

$$\epsilon = \frac{2}{\alpha^3} \int \left[ \frac{1}{R^{3/2} \nu_T} + \frac{1}{Re} \right] S_{ij} S_{ij} \, dx. \quad (14)$$
2.1. Symmetries
It is straightforward to verify that equation (10) is equivariant under a group of symmetries generated by the following transformations:

- Translation over any distance $d$ in the vertical direction, $T_d$.
- Reflection in the $x$-direction, $S_x$.
- Reflection in the $y$-direction, $S_y$.
- Reflection in the $z$-direction, $S_z$.
- Rotation by $\pi/2$ about $x = y = 0$, followed by a shift over $L/2$ in the $x$-direction, $R$.
- A shift over $L/2$ in both the $x$ and $y$ directions, $D$.

In addition, we define the shift in time along a periodic orbit of period $P$ as $Q_\delta$ where $0 \leq \delta < P$. We will not consider Galilean boosts and in the simulation code the net flux in the vertical direction is identically equal to zero.

2.2. Numerical considerations
For the purpose of numerical simulation the LES system is written on the Fourier basis

$$u = \sum_k \hat{u}(k)e^{ik \cdot x},$$

(15)

where each sum is taken from $-n/2 + 1$ to $n/2$ for a resolution of $n^3$ grid points. The pressure is eliminated by formulating the system in terms of the vorticity. Two components of vorticity are time-stepped with a pseudo-spectral code using a fourth-order accurate Runge-Kutta-Gill scheme with a time step of 0.05 in non dimensional units. For de-aliasing we use the phase-shift method introduced by Patterson and Orszag\[14\]. The largest resolved wave number is $2\sqrt{2}/3 \times n/2 \approx 0.94 \times n/2$, where $n/2$ is the Nyquist wave number. The total number of degrees of freedom in a simulation is 28484 for $n = 32$, 230240 for $n = 64$ and 1839283 for $n = 128$. The exact same methods are used to propagate perturbations to the vorticity field and the system parameters using the tangent linear model.

Using the time-steppers for the vorticity and its linear perturbations we can solve boundary value problems in time to find equilibria and periodic orbits relative to the shift $T_d$. For this end, we use Newton-Krylov continuation. An review of this method was recently presented by Sánchez and Net\[15\], and we follow their notes, using, in addition, Viswanath’s method to handle the translation symmetry \[16\].

The most time-consuming steps of the algorithm are the Fast Fourier Transforms (FFTs) and the reconstruction of the velocity from the vorticity. When implemented with process and thread parallel processing, the scaling is nearly linear up to $n/2$ cores. For instance, for $n = 64$ a single time step took 1.8(s) on 8 cores, 1(s) on 16 cores and 0.6(s) on 32 cores, using 3.3GH Intel Xeon processors. When implemented on a GPU architecture, the wall time per step was only 0.25(s) on a NVidia GeForce Titan X card. The large-amplitude, long-period orbit shown below required about 2000 time steps per integration, 100 Krylov iterations per Newton-hook step, and around 200 Newton-hook steps, for convergence. The GPU implementation enabled their computation in the course of four months.

3. Overview of qualitative dynamics
The governing equations in (10) constitute a dynamical system with two parameters, $Re$ and $R$. If either is small, the motion of the LES fluid will be laminar, closely resembling the external forcing pattern of counter-rotating vortex columns. If both are large, say of order $O(100)$, the motion is turbulent. Figure 1 gives a schematic overview of the transition from laminar to
Figure 1. Schematic overview of the dynamics of four-vortex flow and related publications. The three zones are labeled I for laminar flow, II for periodic, homoclinic or heteroclinic motion or low-dimensional chaos and III for turbulence. The publications are abbreviated DBLDLD for Dubrulle et al. [17], GSK for Goto et al. [19], VKY for van Veen et al. [18] and YGK for Yasuda et al. [11].

Figure 2. Partial bifurcation diagram of the LES fluid computed on $32^3$ grid points. Shown on the vertical axis is the deviation from reflection symmetry, normalized by its maximum value. Dashed lines denote stable solutions and solid lines denote unstable solutions. The squares denote instabilities of the laminar flow, the first being a Hopf bifurcation (labeled HB) and the second a pitchfork (labeled PF). The labels $p_{1,2}$ on the primary branch and $s_{1,2}$ on the secondary branches correspond to the physical space portraits in figures 3-5. The label $\ell$ corresponds to the long-period, large-amplitude unstable periodic solution portrayed in figure 7.
have a smooth spatial structure [17]. The state of the fluid visits several such invariant solutions in long transients. For higher \( Re \), chaotic states are observed. Similar behaviour was discovered in the LES case by van Veen et al., who performed a numerical bifurcation analysis. A detailed report can be found in reference [18], while a brief overview is presented in the next section.

For higher Reynolds numbers, say \( Re > 50 \) or \( Re > 50 \), the fluid motion is chaotic and gets progressively more complicated as the separation between the integral scale and the dissipation scale, set by the Kolmogorov length in the DNS case and by \( C_S \Delta \) in the LES case, grows. Two studies investigated the turbulent dynamics. Goto et al. focused on the dynamics of vortical structures across spatial scales [19]. The mechanism of energy transfer they highlight is characterized by anti-parallel vortex pairs, created in the strain field of vortex pairs on a scale two to eight times larger. In principle, this process can be repeated to transfer energy to progressively smaller scales across the inertial range. The other study considers the LES case and focuses on the long-period, large-amplitude oscillations exhibited by the energy and energy dissipation rate [11]. These oscillations correspond to a regeneration cycle of the largest-scale vortices and hints at the possibility of capturing elements of the turbulence by unstable periodic orbits. The ultimate goal of the study of invariant solutions in the LES case is to identify one or more solutions that capture the large-amplitude oscillations and vortex dynamics.

4. Examples of simple invariant solutions

Up to \( R = 7.45 \), the laminar flow is stable. A portrait of the stable laminar flow, which is invariant under the symmetry operations listed in §2.1, is shown in figure 3. At \( R = 7.45 \), this equilibrium becomes unstable in a Hopf bifurcation that produces a family of unstable periodic orbits that is translation invariant in the vertical direction. Two snapshot of a representative orbit are shown in figure 4. These periodic orbits are invariant under the symmetries when combined with a shift in time over half the period. To be precise, they are invariant under \( T_d \), \( S_z \), \( S_z \circ Q_{P/2} \), \( S_y \circ Q_{P/2} \), \( R \circ Q_{P/4} \) and \( D \circ Q_{P/2} \).

The second instability of the laminar equilibrium is a pitchfork bifurcation that gives rise to a family of equilibria with a three-dimensional structure. A portrait taken at \( R = 15.25 \) is shown in figure 5. The large-scale vortices are bent into a corkscrew-like shape with dominant wave number two in the vertical direction. In addition, pairs of smaller, counter-rotating vortices have appeared in the perpendicular directions. The latter are reminiscent of so-called ribs that have previously been identified in Taylor-Green flow as a consequence of an instability concentrated at the hyperbolic stagnation points \((\pi/4 \pm \pi/4, \pi/4 \pm \pi/4, z)\) [20]. The three-dimensional equilibria are invariant under the transformations \( S_z \circ S_y \circ T_{L/4} \), \( D \) and \( R \circ T_{L/8} \).

The periodic orbits and equilibria that branch off the laminar flow do not capture the large fluctuations the rate of energy input and transfer to sub-grid scales observed in weakly turbulent motion. We computed the Probability Density Function (PDF) for turbulence at \( R = 15.25 \). It is shown in figure 6, along with the laminar and three-dimensional equilibrium. The former has high rates of energy input and transfer, while the latter has low rates. In time-series, large oscillations that approach these states intermittently can be observed. We used one such spontaneous, large-amplitude oscillation to compute an unstable periodic orbit, shown in figure 6 in red, with dots evenly spaced in time. Its period is about 46 large-eddy turnover times, the latter computed from the root-mean-square velocity of turbulent motion and the domain size.

In figure 7, we have extracted four snapshots, labeled a–d. At label a, where the rate of energy input exceeds the rate of transfer, the vertical vortex columns are growing in strength under the influence of the forcing. At b, the vortex columns have gained strength, but they are not stable. Pairs of smaller vortices are growing in the perpendicular directions, extracting energy from the large-scale ones. This process accelerates in the third phase, around snapshot c, when the rate of energy transfer is relatively high as small-scale vortices are damped through eddy viscosity. Finally, near the minimum of the energy input rate, the flow approaches the three-dimensional
equilibrium, which is also unstable. Both the large-scale and the smaller-scale vortices are weak and the external force is starting to restore the large-scale structure.

Perhaps the most interesting segment of the periodic orbit is that between snapshots b and d. Here, the instability of the large-scale vortices induces the growth of pairs of smaller vortices, much like described by Goto et al. [19]. Of course, at this low LES Reynolds number, we can only see one step of this process. In order to see inertial range dynamics, we would need to increase it to $R \approx 70$ on the grid of $32^3$ points. However, in spite of the small grid this will be a difficult task. The computation of the periodic orbit presented here took 200 Newton-hook iterations with, on average, 100 Krylov iterations. Since it takes about 2000 time steps to compute one Krylov iteration, the total computation time was close to four months.

5. Outlook

We have presented an overview of the behaviour of the LES of four-vortex flow on a grid of $32^3$ points and at low LES Reynolds numbers. The “derivative solutions”, i.e. the solutions that bifurcate off the laminar flow, could easily be computed on up 16 CPUs. In contrast, the long-period, large-amplitude solution, which seems well embedded in the ambient turbulence, took about 20,000 simulations over 2000 time steps to iteratively refine. Only by using GPU parallelism were we able to compute it in the course of months rather than years. Assuming the number of Krylov iterations per Newton (or Newton-hook) step will grow as $[L/(C_S\Delta)]^3$, a
Figure 5. Physical space portrait of the three-dimensional equilibrium labeled $s_2$ in figure 2. Shown are the isosurfaces of $z$-vorticity (red and blue for positive and negative), $x$-vorticity (green and yellow for positive and negative) and $y$-vorticity (cyan and magenta for positive and negative) at 75% of the maximal and minimal values.

Figure 6. Projection of the PDF of weak turbulence, the large-amplitude periodic orbit $\ell$, the laminar equilibrium $p_2$ and the three-dimensional equilibrium $s_2$ on the rate of energy input and transfer to sub filter scales. Shown is the deviation from their respective time mean values, normalized by their standard deviation. The red dots on the periodic orbit are drawn at fixed time intervals to show the slowing down of the phase point in the vicinity of the equilibria. The PDF is shown on a logarithmic grey scale.

similar computation in the regime of developed turbulence would take about 30 times longer, which is taxing. Moving up to a grid of $64^3$ points seems out of the realm of possibility. The estimate of the number of Krylov iterations is, however, probably overly pessimistic. Also, there may be preconditioning techniques that can be applied to alleviate this problem.

Our current approach is to compute invariant solutions on the small grid with $32^3$ points at a LES Reynolds number as high as possible, and then recomputing it on a finer grid with the same value of $C_S\Delta$. Subsequently, the filter length must be reduced by continuation. Whether this approach will finally lead to an invariant solution with properties of developed turbulence remains to be seen.

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Figure 7. Physical space portrait of the long-period, large-amplitude unstable periodic solution labeled \( \ell \) in figure 2. The labels a–d correspond to those in figure 6. The isosurfaces are drawn as in figure 5.

References

[1] Nagata M 1990 *J. Fluid Mech.* **217** 519–527
[2] Mullin T, Cliffe K A and Pfister G 1987 *Phys. Rev. Lett.* **58** 2212–2215
[3] Kawahara G, Uhlmann M and van Veen L 2012 *Ann. Rev. Fluid Mech.* **44** 203–225
[4] Duguet Y, Schlatter P and Henningson D S 2009 *Phys. Fluids* **21** 111701
[5] Khapko T, Kreilos T, Schlatter P, Duguet Y, Eckhardt B and Henningson D S 2013 *J. Fluid Mech.* **717** R6
[6] Constantin P, Foias P, Manley O P and Temam R 1985 *J. Fluid Mech.* **150** 427–440
[7] Sánchez Umbría J, Net M, García-Archilla B and Simó C 2004 *J. Comput. Phys.* **201** 13–33
[8] E Sasaki G Kawahara A S and Jiménez J 2016 *J. Phys. Conf. Ser.* **708** 012003
[9] Hwang Y, Willis A P and Cossu C 2016 *J. Fluid Mech.* **802** R1
[10] Sekimoto A and Jiménez J 2017 *J. Fluid Mech.* **827** 225–249
[11] Yasuda T, Goto S and Kawahara G 2014 *Fluid Dyn. Res.* **46** 061413
[12] Smagorinsky J 1963 *Mon. Weather Rev.* **91** 99–164
[13] Taylor G I and Green A E 1937 *P. Roy. Soc. Lond. A Mat.* **158** 499–521
[14] Patterson G S and Orszag S A 1972 *Phys. Fluids* **14** 2538–2541
[15] Sánchez Umbría J and Net M 2016 *Eur. Phys. J.-Spec. Top.* **225** 2465
[16] Viswanath D 2007 *J. Fluid Mech.* **580** 339–358
[17] Dubrulle B, Blaineau P, O M Lopes O, F Daviaud F, Laval J P and Dolganov R 2007 *New J. Phys.* **9** 308
[18] van Veen L, Kawahara G and Yasuda T 2017 Transitions in large eddy simulation of box turbulence URL [http://arxiv.org/abs/1711.02289](http://arxiv.org/abs/1711.02289)
[19] Goto S, Saito Y and Kawahara G 2017 *Phys. Pev. Fluids* **2** 064603
[20] Leblanc S and Godeferd F S 1999 *Phys. Fluids* **11** 497–499