Rational Vertex Operator Algebras and the Effective Central Charge

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Abstract

We establish that the Lie algebra of weight one states in a (strongly) rational vertex operator algebra is reductive, and that its Lie rank \( l \) is bounded above by the effective central charge \( \tilde{c} \). We show that lattice vertex operator algebras may be characterized by the equalities \( \tilde{c} = l = c \), and in particular holomorphic lattice theories may be characterized among all holomorphic vertex operator algebras by the equality \( l = c \).

1 Introduction

The purpose of this paper is the study of certain numerical invariants attached to a rational conformal field theory, that is a (strongly) rational vertex operator algebra (RVOA). These are VOAs \( V \) such that all (admissible) \( V \)-modules are completely reducible (see Sections 2 and 3 for a precise definition of the class of VOAs that we study). It is known [DLM2] that RVOAs have only a finite number of simple modules \( V = M_1, M_2, \ldots M_r \). Each has a q-graded character

\[
Z_{M_j}(\tau) = \text{tr}_{M_j} q^{L(0) - c/24} = q^{\lambda_j - c/24} \sum_{n=0}^{\infty} (\dim M^j_{n+\lambda_j}) q^n
\]

for some \( \lambda_j \) called the conformal weight of \( M^j \) (where \( M^j_{\lambda_j} \neq 0 \)). The conformal weights are a priori arbitrary complex numbers, but results in [DLM3] establish that they are in fact rational numbers. Let \( \lambda_{\min} \) denote the minimum of the conformal weights and define

\[
\tilde{c} = c - 24\lambda_{\min}, \quad \tilde{\lambda}_j = \lambda_j - \lambda_{\min}
\]

where \( c \) is the central charge of \( V \). \( \tilde{c} \) is sometimes called the effective central charge in the physics literature (eg. [GN]). It is also proved in [DLM3] that \( c \) is rational, so that the effective central charge of an RVOA is a rational number. We will establish a basic inequality satisfied by \( \tilde{c} \), namely that it is non-negative. In fact, as long as \( V \) is non-trivial, that is, it is not finite-dimensional, we will prove that

\[
\tilde{c} > 0.
\]
The non-negativity of $\tilde{c}$ is actually a consequence of more qualitative results which constitute our first main theorems. To describe them, recall [FHL] that the weight 1 subspace $V_1$ of $V$ carries a natural Lie algebra structure as well as a symmetric, invariant bilinear form. Conversely, it is well-known (eg. [L], [Li3]) that there is a natural vertex algebra structure associated to a Lie algebra $\mathfrak{g}$ and symmetric, invariant bilinear form and that this is frequently a VOA (ie., it also has a Virasoro vector, etc.). There is no restriction on the algebraic structure of $V_1 = \mathfrak{g}$ beyond these conditions (we discuss an example in Section 2 in which $V_1$ is nilpotent of class 2). On the other hand, we will establish

**Theorem 1** Let $V$ be a strongly rational vertex operator algebra. Then the Lie algebra $V_1$ is reductive. Moreover, any $V$-module is a completely reducible $V_1$-module.

Granted this result, we may define the Lie rank of $V$ to be the Lie rank of $V_1$, i.e., the dimension of a maximal abelian subalgebra of $V_1$, which is of course an invariant of $V_1$. We generally denote the Lie rank by $l$.

**Theorem 2** Let $V$ be a strongly rational vertex operator algebra. Then $l \leq \tilde{c}$.

Note that the non-negativity of $\tilde{c}$ follows from Theorem 2.

A systematic study of the relation between the invariants $c$, $\tilde{c}$ and $l$ may give new insights into the structure and classification of RCFTs and RVOAs. In particular, the method that we use to prove Theorem 2 can also be employed to study certain extremal situations including the case of equality in Theorem 2. Here is an example:

**Theorem 3** Let $V$ be a strongly rational vertex operator algebra. Then the following are equivalent:

1. $\tilde{c} = l = c$.
2. There is a positive-definite even lattice $L$ such that $V$ is isomorphic to $V_L$.

We point out that the condition $l = c$ is not sufficient to characterize the lattice vertex operator algebras. In example (f) in Section 4 we discuss a vertex operator algebra which satisfies $c = l = 0$ but which is not a lattice vertex operator algebra. On the other hand, we expect that lattice vertex operator algebras are characterized by the condition $\tilde{c} = l$.

Next recall that a rational $V$ is called holomorphic in case the adjoint module is the unique simple $V$-module. Lattice theories $V_L$ are holomorphic precisely when $L$ is unimodular ([D1], [DLM2]). For holomorphic vertex operator algebras, we always have $c = \tilde{c}$ and $V$ is strongly rational precisely when it satisfies the qualitatively weaker CFT-type condition (cf. Section 2 for more details). Thus:

**Corollary 4** Suppose that $V$ is a holomorphic vertex operator algebra of CFT type. Then the following conditions are equivalent:

1. $l = c$.
2. $V$ is isomorphic to a lattice theory $V_L$ for some positive-definite, even, unimodular lattice $L$. 

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It is worth comparing Corollary 4 with the work of Schellekens [Sch], who discusses the possible holomorphic $c = 24$ theories. In particular, the theories on Schellekens’ list which have $l = 24$ are precisely the lattice theories $V_L$ where $L$ is now a Niemeier lattice (including the Leech Lattice). Thus the special case $c = 24$ of Corollary 4 confirms with mathematical rigor the completeness of this part of Schellekens’ list. See [DM2] for further developments of this idea.

Although there is no mention of modular-invariance in any of the Theorems we have stated, it is the driving force behind the proofs each of them. While it is an important conjecture that for RVOAs the graded characters (1.1) are modular functions on a congruence subgroup of $SL(2, \mathbb{Z})$, this remains unknown. Our methods offer a way of circumnavigating this lacuna in our knowledge in some circumstances. A careful study of what is known about the invariance properties of the characters (1.1) reveals that they constitute a vector-valued modular form. It transpires that this is a good substitute for modularity of characters in some circumstances. In particular, a recent result [KM] establishing polynomial growth of the Fourier coefficients of holomorphic vector-valued modular forms figures in the proofs.

The paper is organized as follows: in Section 2 we discuss background and nomenclature for vertex operator algebras, and in Section 3 we discuss the connections between rational vertex operator algebras, modular-invariance, and vector-valued modular forms. Section 4 is devoted to the proofs of Theorems 1 and 2. In Section 5 we give a result on uniqueness of simple current extensions of independent interest and which is used in Section 6 to prove Theorem 3.

2 Vertex Operator Algebras of CFT type

First we recall from [DLMM] the following definition: a vertex operator algebra $V$ is said to be of CFT type in case the $L(0)$-grading on $V$ has no non-negative weights, and if the degree zero homogeneous subspace $V_0$ is 1-dimensional. In this case, we have

$$V = \bigoplus_{n=0}^{\infty} V_n$$

moreover $V_0$ is spanned by the vacuum vector $1$. If $V$ is a vertex operator algebra of CFT type, we say that $V$ is of strong CFT type in case it satisfies the further condition that the $L(1)$ operator annihilates the homogeneous subspace $V_1$, that is

$$L(1)V_1 = 0.$$  

In general, one knows that $L(1)$ maps $V_1$ into $V_0$. Thus for vertex operator algebras of strong CFT type we have

$$\dim V_0/L(1)V_1 = 1.$$  

Li has shown [Li1] that a simple vertex operator algebra which satisfies (2.3) has, up to multiplication by a non-zero scalar, a unique non-degenerate invariant bilinear form $\langle,\rangle$ in
the sense of [FLH, Section 5.3] and [Li1]. If we normalize in such a way that the vacuum satisfies \( \langle 1, 1 \rangle = -1 \) then in fact we have

\[
\langle a, b \rangle = -\text{Res}_z (z^{-1} Y (e^{zL(1)} (-z^2)^{L(0)} a, z^{-1}) b)
\]  

(2.4)

for elements \( a, b \) in \( V \). It is well-known (loc. cit.) that (2.4) implies that the homogeneous spaces \( V_n \) and \( V_m \) are orthogonal if \( n \) and \( m \) are distinct, so that the restriction of \( \langle , \rangle \) to each \( V_n \) is non-degenerate. Note that \( \langle , \rangle \) is necessarily symmetric by Theorem 5.3.6 of [FHL]. Furthermore, from the discussion in Section 5.3 [loc. cit.], one sees that a vertex operator algebra \( V \) of CFT type is of strong CFT type precisely when \( V \) is self-dual in the sense that the contragredient module \( V' \) is isomorphic to the adjoint module \( V \).

It is easy to see that if \( V \) is holomorphic of CFT type then it is of strong CFT type. Indeed, the uniqueness of the simple module for \( V \) entails that the contragredient module \( V' \), which is itself simple, is isomorphic to the adjoint module. Now the assertion follows from the preceding remarks.

Let us return to the consideration of simple vertex operator algebras \( V \) of strong CFT type. For a state \( v \) in \( V_n \) we define \( o(v) \) to be the \( n \)-th component operator \( v_{n-1} \) of \( v \). Although it is something of a misnomer, we often refer to \( o(v) \) as the zero mode of \( v \). Note that zero modes induce a linear map on each homogeneous space of \( V \). It is well-known that the states of weight 1 close on a Lie algebra under the bracket operation \([a, b] = a_0 b - o(a) b\). We continue to denote this Lie algebra by \( V_1 \). Each homogeneous space \( V_n \) becomes a module over \( V_1 \) via the action of the zero mode \( o(\cdot) \) for \( v \in V_1 \). Thus in its action on \( V_1 \), the operator \( o(v) \) for \( v \in V_1 \) is just \( ad_v \).

Furthermore, one can check from (2.4) that the restriction of the form \( \langle , \rangle \) to each \( V_n \) is invariant under the action of \( V_1 \). That is, we have

\[
\langle o(a) u, v \rangle = -\langle u, o(a) v \rangle
\]  

(2.5)

for \( a \) in \( V_1 \) and \( u, v \) in \( V_n \).

In particular, the restriction of \( \langle , \rangle \) to \( V_1 \) endows \( V_1 \) with a non-degenerate, symmetric, invariant, bilinear form. Indeed, it is well-known, and follows from (2.4), that we have

\[
\langle a, b \rangle = a_1 b
\]  

(2.6)

for \( a, b \) in \( V_1 \).

**Example 2.1** We discuss an example of a vertex operator algebra of strong CFT type for which \( V_1 \) is not reductive. The reader may construct many other similar examples. Take a 6-dimensional complex Lie algebra \( g \) which is nilpotent of class 2 and such that the center \( Z(g) \) and the commutator subalgebra \([g, g] \) coincide and have dimension 3. Thus we may take \( g \) to be generated by elements \( A, B, C \), with \( Z(g) \) having as basis the commutators \([A, B], [A, C], [B, C] \). We define a non-degenerate, symmetric, invariant bilinear form on \( g \) by setting

\[
\langle A, [B, C] \rangle = \langle B, [C, A] \rangle = \langle C, [A, B] \rangle = 1
\]

and with all other basis vectors of \( g \) being orthogonal.
Let \( \hat{g} = g \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}K \) be the affine Lie algebra associated to \( g \) and \( V(1, \mathbb{C}) = U(g) \otimes_\mathbb{C} \mathbb{C}[t] + \mathbb{C}K \mathbb{C} \) the associated Verma module where \( g \otimes \mathbb{C}[t] \) acts on \( \mathbb{C} \) trivially and \( K \) acts as 1. Let \( V \) be the irreducible quotient of \( V(1, \mathbb{C}) \). Then \( V \) is a simple vertex algebra of level 1 and \( V_1 \) coincides with \( g \). We claim that there is a Virasoro element \( \omega \) in \( V \) such that \( V \) is a vertex operator algebra of strong CFT type and \( c = 6 \). Indeed, we may take \( \omega \) to be the same as the usual Sugawara form for the case that the corresponding Lie algebra \( g \) is semi-simple, i.e.,

\[
\omega = \frac{1}{2} \sum_{i=1}^{6} u_i(-1)u^i(-1)
\]

where \( u_i \) ranges over a basis of \( g \) and \( u^i \) ranges over the dual basis with respect to \( \langle , \rangle \).

One may check these assertions by an explicit but lengthy calculation, or by using some results in [L]. Note (loc. cit.) that we may modify the level to be any non-zero scalar, and also modify the Virasoro element so that \( V \) becomes a VOA of CFT type, but not strong CFT type.

3 \( C_2 \)-rational vertex operator algebras and modular forms

We recall some definitions from [DLM2] and [Z]. The vertex operator algebra \( V \) is \( C_2 \)-cofinite in case the subspace \( C_2(V) \) of \( V \) spanned by all elements of the type \( a_{-2}b \) for \( a, b \) in \( V \) has finite codimension in \( V \). \( V \) is rational if any admissible module is a direct sum of irreducible admissible modules. Let us introduce the term strongly rational for a vertex operator algebra \( V \) which satisfies the following conditions:

1. \( V \) is of strong CFT type.
2. \( V \) is \( C_2 \)-cofinite.
3. \( V \) is rational.

It is likely that these conditions are not independent. For example, it may well be that conditions 2 and 3 are equivalent. We will see that the class of vertex operator algebras which are strongly rational in the above sense have a number of good properties which can be exploited.

Let us fix for the duration of this section a strongly rational VOA \( V \) with central charge \( c \). Let notation for simple \( V \)-modules be as in (1.1). The basic results concerning the modular-invariance properties of the characters (1.1) and related trace functions were established in [Z], with some refinements incorporated in [DLM3]. To describe these, let \( u \) be a state in \( V \) with vertex operator

\[
Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}.
\]

Define the zero mode \( o(u) \) to be the sum of the zero modes of the homogeneous components of \( u \), which were defined in Section 2. We can then define the 1-point function
$Z_{M_j}(u, \tau)$ associated to $M_j$ as follows:

$$Z_{M_j}(u, \tau) = \text{tr}_{M_j} o(u) q^{L(0) - c/24} = q^{-c/24 + \lambda_j} \sum_{n \geq 0} \text{tr}_{M_j} o(u) q^n. \quad (3.2)$$

In case the state $u$ is the vacuum vector $1$, we have

$$Z_{M_j} = Z_{M_j}(1, \tau). \quad (3.3)$$

Here and below, $\tau$ denotes an element in the complex upper half-plane $\mathbb{H}$ and $q = e^{2\pi i \tau}$.

In order to discuss the modular properties of $Z_M(v, \tau)$ we briefly recall the genus one vertex operator algebra $(V, Y, 1, \omega - c/24)$ from $[Z]$. The new vertex operator associated to a homogeneous element $a$ is given by

$$Y[a, z] = \sum_{n \in \mathbb{Z}} a[n] z^{-n-1} = Y(a, e^z - 1) e^{z\text{ wt}_a}$$

while the Virasoro element is $\tilde{\omega} = \omega - c/24$. Thus

$$a[m] = \text{Res}_z \left( Y(a, z) (\ln (1 + z))^m (1 + z)^{\text{ wt}_a - 1} \right)$$

and

$$a[m] = \sum_{i=m}^{\infty} c(\text{ wt}_a, i, m) a(i)$$

for some scalars $c(\text{ wt}_a, i, m)$ such that $c(\text{ wt}_a, m, m) = 1$. In particular,

$$a[0] = \sum_{i \geq 0} \left( \binom{\text{ wt}_a - 1}{i} \right) a(i).$$

We also write

$$L[z] = Y[\omega, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-2}.$$ 

Then the $L[n]$ again generate a copy of the Virasoro algebra with the same central charge $c$. Now $V$ is graded by the $L[0]$-eigenvalues, that is

$$V = \bigoplus_{n \in \mathbb{Z}} V[n]$$

where $V[n] = \{ v \in V | L[0] v = n v \}$. We also write $\text{ wt}[a] = n$ if $a \in V[n]$. It should be pointed out that for any $n \in \mathbb{Z}$ we have

$$\sum_{m \leq n} V_m = \sum_{m \leq n} V[n].$$

It is established in $[DLM3]$ and $[Z]$ that the functions $(3.2)$ are holomorphic in $H$ and that the following holds: for any state $u$, homogeneous of weight $k$ with respect to the
square bracket Virasoro operator $L[0]$, and for any matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the modular group $SL(2, \mathbb{Z})$, there are scalars $\rho_{i,j}(\gamma)$, $1 \leq i, j \leq r$ independent of $u$ and $\tau$, and an equality

$$Z_{M^i}(u, \frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k \sum_{j=1}^{r} \rho_{i,j}(\gamma) Z_{M^j}(u, \tau).$$

(3.4)

It will be worthwhile to state these results in another form; we refer the reader to [KM] for relevant background. Let $\text{Hol}$ and $\text{Hol}_q$ denote respectively the space of holomorphic functions on $\mathbb{H}$ and the subspace of functions which are in addition 'meromorphic at infinity' i.e., have a q-expansion

$$\sum_{n \geq b} a_n q^{n/N}$$

for some $b \in \mathbb{Z}$ and $N \in \mathbb{N}$. It is well-known that $\text{Hol}$ is a right $\Gamma$-module with respect to the kth slash operator

$$f |_{k\gamma}(\tau) = (c\tau + d)^{-k} f(\gamma \tau).$$

Let $\text{Hol}_k$ denote the space $\text{Hol}$ considered as a right $\Gamma$-module in this way. What (3.4) says is that for each state $u \in V$ satisfying $L[0]u = ku$, the 1-point functions $Z_{M^i}(u, \tau)$ span a finite-dimensional $\Gamma$-submodule $R_u$ of $\text{Hol}_k$. Moreover $R_u \subset \text{Hol}_q$. (Note that $\text{Hol}_q$ is not a $\Gamma$-submodule of $\text{Hol}_k$.) In fact, the matrices $(\rho_{i,j}(\gamma))$ may be chosen so that they define a representation of $\Gamma$ on the so-called conformal block $B$ (cf. [Z]), and a choice of $u$ as above determines a homomorphism of $\Gamma$-modules from $B$ to $R_u$.

Yet another way to state the above results is in terms of so-called (meromorphic) vector-valued modular forms of weight $k$: by this we mean a tuple of functions

$$F(\tau) = (F_1(\tau), \ldots, F_r(\tau))$$

together with a representation $\rho : \Gamma \to GL(r, \mathbb{C})$ such that each component function $F_i(\tau)$ lies in $\text{Hol}_q$ and the following compatibility condition holds:

$$F^T |_{k\gamma}(\tau) = \rho(\gamma) F^T(\tau)$$

(3.6)

for all $\gamma \in \Gamma$. $T$ denotes transpose of vectors, and the slash operator acts on vectors of functions componentwise. We call the vector-valued modular form holomorphic, or cuspidal, if the component functions $F_i(\tau)$ are holomorphic at infinity or vanish at infinity respectively i.e., the integer $b$ in (3.5) is non-negative or positive respectively. We note that we may take the weight $k$ here to be any rational number. We can now restate (3.4) in the following way:

**Proposition 3.1** For each state $u \in V$ which is homogeneous of weight $k$ with respect to the operator $L[0]$, the $r$-tuple $Z(u, \tau) = (Z_{M^1}(u, \tau), \ldots, Z_{M^r}(u, \tau))$ is a vector-valued modular form of weight $k$ with respect to the representation $\rho$.

It is known (cf. [KM]) that all modular forms in the usual sense give rise to vector-valued modular forms on a subgroup of $\Gamma$ of finite index, however there are vector-valued
modular forms which do not arise in this way. Vector-valued modular forms nevertheless enjoy some of the properties of ordinary modular forms. The following result, which extends a classical result of Hecke for ordinary modular forms and which plays a role in the present paper, illustrates this idea.

**Proposition 3.2** Suppose that $F(\tau)$ is a component of a holomorphic vector-valued modular form of weight $k$. Then the Fourier coefficients $a_n$ of $F(\tau)$ satisfy the growth condition $a_n = O(n^\alpha)$ for a constant $\alpha$ independent of $n$.

**Proof:** This is the main result of [KM]. □

Compare this to the growth of the Fourier coefficients of the meromorphic modular form $\eta(\tau)^{-1}$. Here, $\eta(\tau)$ is the Dedekind eta function $\eta(\tau) = \frac{1}{\tau^{1/24}} \varphi(q) = \frac{1}{24} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \frac{1}{q^n}\right)$ and $\varphi(q) = \sum_{n=0}^{\infty} p(n)q^n$ with $p(n)$ the usual unrestricted partition function. It is well-known (cf. [K]) that the growth of $p(n)$ is exponential in $n$.

Propositions 3.1 and 3.2 do not necessarily apply to the vector-valued modular forms $Z(u, \tau)$ per se: they generally have poles. However, we can multiply by a suitable cusp-form (say, a power of the eta function) in order to remove the poles. What we will actually be doing is very close to this, but in the guise of a Lie-theoretic argument involving Heisenberg and Virasoro Lie algebras.

Because of this circumstance, it will be useful to look more closely at the poles of the $q$-expansions of the functions $Z_{M_i}(\tau)$. Indeed, from (3.3) and (1.2) we have

$$Z_{M_i}(\tau) = q^{-\tilde{c}/24 + \tilde{\lambda}_i} \sum_{n \geq 0} \left(\dim M_i^{n+\lambda_i}\right) q^n.$$

Since $\tilde{\lambda}_i \geq 0$, it follows that the pole at infinity for $Z_{M_i}(\tau)$ has order no worse than $\tilde{c}/24$. Alternatively, we can assert that $\eta(\tau)^{\tilde{c}}Z_{M_i}(\tau)$ is holomorphic in $H \cup \{i\infty\}$. As a result of the transformation law for $\eta(\tau)$, it follows that $\eta(\tau)^{\tilde{c}} Z(u, \tau)$ is a holomorphic vector-valued modular form of weight $k + \tilde{c}/2$. By Proposition 3.2 we can conclude

**Corollary 3.3** The Fourier coefficients of $\eta(\tau)^{\tilde{c}}Z_{M_i}(u, \tau)$ satisfy a polynomial growth condition $a_n = O(n^\alpha)$.

There is a related result ([KM, Lemma 4.2]) that we use:

**Lemma 3.4** Let $F$ be a vector-valued modular form such that the Fourier coefficients of each component function $F_i(\tau)$ are non-negative real numbers. If $F$ is holomorphic of weight 0 then it is a constant, and if $F$ is cuspidal then $F = 0$.
In a somewhat different direction, we recall an identity due to Zhu [Z]. It may be construed as a VOA-theoretic analog of the Killing form familiar in Lie theory. Namely, for states \( u, v \) in \( V \) we have

\[
\text{tr}_{M'} o(u) o(v) q^{L(0) - c/24} = Z_{M'}(u[-1]v, \tau) - \sum_{k \geq 1} E_{2k}(\tau) Z_{M'}(u[2k - 1]v, \tau). \tag{3.7}
\]

In (3.7), the meaning of the operators \( u[k]v \) has already been described. The functions \( E_{2k}(\tau) \) are the usual Eisenstein series of weight \( 2k \), normalized as in [DLM3]:

\[
E_{2k}(\tau) = \frac{-B_{2k}}{2k!} + \frac{2}{(2k - 1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n
\]

where \( \sigma_k(n) \) is the sum of the \( k \)-powers of the divisors of \( n \) and \( B_{2k} \) a Bernoulli number. Of course, \( E_{2k}(\tau) \) is a holomorphic modular form on \( SL(2, \mathbb{Z}) \) if \( k > 2 \). However - and this will be important for us - \( E_{2}(\tau) \) is not modular. Its transformation with respect to the \( S \) matrix is as follows:

\[
E_{2}(-1/\tau) = \tau^2 E_{2}(\tau) - \frac{\tau}{2\pi i}. \tag{3.8}
\]

4 Strongly rational vertex operator algebras and reductive Lie algebras

The main purpose of this section is to prove Theorems 1 and 2. Concerning Theorem 1, we have to show that the nilpotent radical \( N \) of the Lie algebra \( V_1 \) is trivial. We use basic facts about \( V_1 \), as discussed in Section 2, without further comment. Proceeding by contradiction, pick a non-zero element \( u \) in \( N \). Then for any other element \( v \) in \( V_1 \) we see that the trace of the operator \( o(u) o(v) \) acting on each of the homogeneous spaces \( V_n \) is 0. In other words, the left-hand-side of equation (3.7) is zero. Moreover, if \( k > 1 \) is an integer then because \( V \) is of strong CFT type, the element \( u[2k - 1]v \) has \( L[0] \)-weight \( 2 - 2k < 0 \) and hence is 0. In addition, the element \( u[1]v \) coincides with \( u_1 v = \langle u, v \rangle \), the latter equality by (2.6). As \( \langle , \rangle \) is non-degenerate, we can choose \( v \) so that \( \langle u, v \rangle = 1 \). With this choice of \( v \), eqn. (3.7) simplifies to read

\[
Z_{M'}(u[-1]v, \tau) = E_{2}(\tau) Z_{M'}(\tau). \tag{4.1}
\]

The point is that because \( E_{2}(\tau) \) enjoys an exceptional transformation law (3.8), it cannot participate in an equation of the type (4.1). This is how we get the desired contradiction. In detail: take any index \( i \), replace the state \( u \) in (3.4) by \( u[-1]v \) with \( u, v \) as above, so that \( u[-1]v \) has \( L[0] \) weight 2, and take \( \gamma \) to be the \( S \) matrix. Several applications of (3.4) and (4.1) yield

\[
\tau^2 \sum_{j=1}^{r} S_{i,j} Z_{M'}(u[-1]v, \tau) = Z_{M'}(u[-1]v, -1/\tau)
\]
\[
eq E_2(-1/\tau)Z_{M'}(-1/\tau) = (\tau^2 E_2(\tau) - \frac{\tau}{2\pi i}) \sum_{j=1}^{r} S_{i,j} Z_{M'}(\tau) = \tau^2 \sum_{j=1}^{r} S_{i,j} Z_{M'}(u[-1]v, \tau) - \frac{\tau}{2\pi i} \sum_{j=1}^{r} S_{i,j} Z_{M'}(\tau).
\]

It follows that in fact
\[
\sum_{j=1}^{r} S_{i,j} Z_{M'}(\tau) = 0. \tag{4.2}
\]

However the left-hand-side of (4.2) is equal to \(Z_{M'}(-1/\tau)\), and this is clearly not zero. Thus \(V_1\) is reductive.

We next prove that any \(V\)-module \(M\) is a completely reducible \(V_1\)-module. Since \(V\) is rational we can assume that \(M\) is irreducible. Let \(\mathfrak{h}\) be a maximal abelian subalgebra, so that \(\mathfrak{h}\) has dimension \(l\). It is enough to prove that \(M\) is a completely reducible \(\mathfrak{h}\)-module. It is well-known that the restriction of the non-degenerate form \(\langle \cdot, \cdot \rangle\) to \(\mathfrak{h}\) is also non-degenerate. As a result, the component operators of the vertex operators \(Y(u, z)\) on \(M\) for \(u\) in \(\mathfrak{h}\) close on an affine Lie algebra \(\hat{\mathfrak{h}}\). That is, they satisfy the relations
\[
[u_m, v_n] = m\delta_{m,-n}\langle u, v \rangle.
\]

Essentially by the Stone-von-Neumann theorem, there is a tensor decomposition of \(V\),
\[
M = M(1) \otimes \Omega_M \tag{4.3}
\]
where \(M(1) = \mathbb{C}[u_m \mid u \in \mathfrak{h}, m > 0]\) is the Heisenberg vertex operator algebra of rank \(l\) generated by \(\mathfrak{h}\) and \(\Omega_M\) is the so-called vacuum space consisting of those states \(v \in M\) such that \(u_nv = 0\) for all \(u \in \mathfrak{h}\) and all \(n > 0\). See [FLM, theorem 1.7.3] for more details. For \(\alpha \in \mathfrak{h}\) let \(M(\alpha)\) be the generalized eigenspace for \(\mathfrak{h}\) with eigenvalue \(\alpha\). That is, \(v \in M(\alpha)\) if and only if there exists a positive integer \(m\) such that \((u_0 - \langle h, \alpha \rangle)^m v = 0\) for all \(u \in \mathfrak{h}\). Then \(M(\alpha) = M(1) \otimes \Omega_M(\alpha)\) where \(\Omega_M(\alpha) = M \cap \Omega_M\). Let \(L_M\) be the subset of \(\mathfrak{h}\) consisting of \(\alpha\) such that \(M(\alpha) \neq 0\). We set \(L = L_V\). Then
\[
M = \bigoplus_{\alpha \in L_M} M(1) \otimes \Omega_M(\alpha)
\]
and \(u_m M(\beta) \subset M(\alpha + \beta)\) for \(\alpha \in L, \beta \in L_M, u \in V(\alpha)\) and \(m \in \mathbb{Z}\).

First we take \(M = V\). Note that \(M(1)\) is a vertex operator algebra with Virasoro vector \(\omega' = \frac{1}{2} \sum_{i=1}^{l} (u'_i)^2 1\) where \(\{u'_1, \ldots, u'_l\}\) is an orthonormal basis of \(\mathfrak{h}\) with respect to \(\langle \cdot, \cdot \rangle\). Set \(\omega'' = \omega - \omega'\). We will use the notation
\[
Y(\omega', z) = \sum_{n \in \mathbb{Z}} L'(n) z^{-n-2}, Y(\omega'', z) = \sum_{n \in \mathbb{Z}} L''(n) z^{-n-2}.
\]

Then \(\omega''\) is a Virasoro vector with central charge \(c - l\) and \(\Omega_V(0)\) is a vertex operator algebra with Virasoro vector \(\omega''\).
Fix $\beta \in L$. Since $V$ is simple, for any nonzero $w \in M(\beta)$, $V$ is spanned by $u_qw$ for $u \in V$ and $q \in \mathbb{Z}$ by Corollary 4.2 of [DM1] or Proposition 4.1 of [L2]. Thus $V(-\beta)$ must be nonzero, otherwise $V(0)$ would be zero. By Proposition 11.9 of [DL], for any $\alpha \in L$ and $0 \neq u \in V(\alpha)$ there exists $m \in \mathbb{Z}$ such that $0 \neq u_mw \in V(\alpha + \beta)$. Thus $V(0) = M(1) \otimes \Omega_V(0)$ is a simple vertex operator subalgebra of $V$ and each $V(\alpha) = M(1) \otimes \Omega_V(\alpha)$ is an irreducible $V(0)$-module. More precisely, as a $V(0)$-module,

$$V(\alpha) = M(1, \alpha) \otimes \Omega_V(\alpha)$$

where $M(1, \alpha) = \mathbb{C}[u_{-n}|u \in \mathfrak{h}, n > 0]e^\alpha$ is the highest weight irreducible $M(1)$-module such that $h \in \mathfrak{h}$ acts on $e^\alpha$ as $\langle h, \alpha \rangle$ and $\Omega_V(\alpha)$ is an irreducible $\Omega_V(0)$-module. Thus $V$ is completely reducible $\mathfrak{h}$-module.

Applying the same argument to any irreducible $V$-module $M$ shows that $M$ is a completely reducible $\mathfrak{h}$-module. This finishes the proof of Theorem 1. □

Remark 4.1 From the proof of Theorem 1, we see that $L$ is an additive subgroup of $\mathfrak{h}$ and each $L_M$ is a coset of $L$ in $\mathfrak{h}$.

We turn next to the proof of Theorem 2. Note that both factors on the right-hand-side of (4.3) in the case $M = V$ are invariant under the $L(0)$ operator, so that (4.3) is an equality of $q$-graded spaces. Because the $q$-graded character of $M(1)$ is equal to $\phi(q)^{-1}$, it follows from (4.3) that

$$\eta(\tau)^\tilde{c} Z_V(\tau) = q^{(l-c)/24} \eta(\tau)^\tilde{c} - l ch_q \Omega_V$$

where we have used $ch_q(\Omega_V)$ for the $q$-graded character of $\Omega_V$. Now according to Corollary 3.3 the Fourier coefficients of the l.h.s. of (4.3) have polynomial growth, so the same must be true of the Fourier coefficients on the r.h.s. But $\eta(\tau)^{-1}$ has exponential growth of Fourier coefficients, as indeed does $\eta(\tau)^s$ whenever $s < 0$. We conclude that $\tilde{c} - l \geq 0$, which completes the proof of Theorem 2.

A second way to complete the proof of the theorem runs as follows: (4.3) and (4.3) apply not only to $V$ but to each simple $V$-module. As a result, the (non-zero) vector-valued modular form $\eta(\tau)^\tilde{c} Z(1, \tau)$ has components whose Fourier coefficients are non-negative integers and which, assuming that $\tilde{c} < l$, is cuspidal. This contradicts Lemma 3.4. Furthermore, if $\tilde{c} = 0$ the same argument shows that $Z(1, \tau)$ is necessarily constant, so that $V$ is 1-dimensional. As a result, (4.3) holds if $V$ is non-trivial. □

We discuss the invariants $c, \tilde{c}, l$ for some standard VOAs.

(a) If $V$ is strongly rational and holomorphic, Theorem 2 tells us that $l \leq c = \tilde{c}$.

(b) Suppose that $V$ is a unitary VOA (cf. [Ho] for background on such theories). Then one knows (loc. cit.) that $c$ and all conformal weights $\lambda_i$ are non-negative. In particular, as in the holomorphic case, we have $l \leq c = \tilde{c}$ for unitary theories.

(c) Suppose that $V = V_L$ is the vertex operator algebra associated to a positive-definite, even lattice $L$. Then $V$ is strongly rational and $l = c = \tilde{c} = rank L$. See [D1], [DLM1].
(d) Consider the simple vertex operator algebra \( V = L(\mathfrak{g}, k) \) associated to a finite-dimensional simple Lie algebra \( \mathfrak{g} \) with level \( k \). Then \( l \) is the Lie rank of \( \mathfrak{g} \) and \( c = k \dim \mathfrak{g} / (k + h^\vee) \) as long as \( k + h^\vee \) is non-zero. Here, \( h^\vee \) is the dual Coxeter number of \( \mathfrak{g} \). \( V \) is rational (and hence strongly rational) precisely when \( k \) is a nonnegative integer (cf. [FZ], [DLM1]).

(e) Consider the Virasoro vertex operator algebra \( V = L(c, 0) \) with \( c = c_{p,q} = 1 - 6(p - q)^2/pq \) for coprime positive integers \( p, q \) with \( 1 < p < q \), say. (See Section 6 for the definition of \( L(c, 0) \).) It is known ([W], [DLM1]) that \( V \) is strongly rational. The conformal weights of the simple \( V \)-modules are the rational numbers \( \lambda_{m,n} = ((np - mq)^2 - (p - q)^2)/4pq \) for integers \( m, n \) in the range \( 1 \leq m \leq p - 1, 1 \leq n \leq q - 1 \). These theories are unitary precisely when \( q - p = 1 \), in which case \( \tilde{c} = 1 - 6/pq \). We claim that this is the effective central charge in all cases. This is equivalent to the equality \( \lambda_{min} = (1 - (p - q)^2)/4pq \), which in turn is equivalent to the assertion that we may choose integers \( m, n \) in the indicated range such that \( |np - mq| = 1 \). This latter fact is well-known.

(f) Consider the vertex operator algebra \( V = L(c_{2.5}, 0)^{\otimes 60} \otimes (V^\sharp)^{\otimes 11} \) where \( V^\sharp \) is the moonshine vertex operator algebra constructed in [FLM]. Since the central charges of \( L(c_{2.5}, 0) \) and \( V^\sharp \) are \(-22/5\) and \( 24 \) respectively, \( V \) has zero central charge. Note that both \( L(c_{2.5}, 0) \) and \( V^\sharp \) are strongly rational with weight one subspaces being zero (cf. [W] and [D2]). Thus \( V \) is a strongly rational vertex operator algebra with zero weight one subspace. As a result the Lie rank \( l \) of \( V \) is zero. Since the Lie rank of any lattice vertex operator algebra \( V_L \) is equal to the rank of \( L \) we conclude that \( V \) is not a lattice vertex operator algebra. The effective central charge \( \tilde{c} \) is equal to \( 528 \). One can get a strongly rational vertex operator algebra with \( c = l = 0 \) and \( \tilde{c} \) equal to any positive multiple of \( 528 \) by tensoring \( V \) with itself.

We learn from these examples that one cannot expect to prove a universal inequality between \( l \) and \( c \). Of course we always have \( c \leq \tilde{c} \).

5 Uniqueness of simple current extensions

We establish a general result of independent interests which will be used in the proof of Theorem 3. In the language of physics, the result to be proved in this section establishes the uniqueness of vertex operator algebras defined by simple current extensions.

We first recall from [FHL] the definition of intertwining operators.

**Definition 5.1** Let \( V \) be a vertex operator algebra and let \( W_1, W_2 \) and \( W_3 \) be weak \( V \)-modules. An intertwining operator of type \( \left( \begin{array}{c} W_3 \\ W_1 \end{array} \right) \) is a linear map \( I \) from \( W_1 \) to \( (\text{Hom}(W_2, W_3)) \{z\} \) satisfying the following properties: for \( w_1 \in W_1, \ w_2 \in W_2 \),

\[
I(w_1, z)w_2 \in W_3 \{z\}
\]  

(5.1)
for \( v \in V, \ w_1 \in W_1 \) where \( W\{z\} \) is the space of formal power series in \( z \) with complex powers and coefficients in \( W \) for a vector space \( W \),

\[
[L(-1), I(w_1, z)] = \frac{d}{dz} I(w_1, z),
\]

(5.2)

\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(v, z_1)I(w_1, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) I(w_1, z_2)Y(v, z_1)
= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) I(Y(v, z_0)w_1, z_2).
\]

(5.3)

All intertwining operators of type \( \left( \frac{W_1}{W_1 W_2} \right) \) form a vector space denoted by \( I_{W_1 W_2}^{W_1} \). The dimension of \( I_{W_1 W_2}^{W_1} \) is called a fusion rule, denoted by \( N_{W_1 W_2}^{W_1} \). Clearly, fusion rules only depend on the equivalence class of each \( W_i \).

**Definition 5.2** A vertex operator algebra \( V \) is graded by an abelian group \( G \) if \( V = \oplus_{g \in G} V^g \), and \( u_n v \in V^{g+h} \) for any \( u \in V^g, v \in V^h \) and \( n \in \mathbb{Z} \).

It is easy to see from the definition that \( V^0 \) is a vertex operator subalgebra of \( V \) and each \( V^g \) is a strong \( V^0 \)-module for \( g \in G \). Moreover, the restriction \( Y_{g,h}(u, z) \) of \( Y(u, z) \) to \( V^h \) for \( u \in V^g \) is an intertwining operator of type \( \left( \frac{V^g + h}{V^g} \right) \) for \( V^0 \)-modules \( V^g, V^h, V^{g+h} \).

**Proposition 5.3** Let \( V = \sum_{g \in G} V^g \) be a simple \( G \)-graded vertex operator algebra such that \( V^g \neq 0 \) for all \( g \in G \) and \( N_{V^g V^h}^{V^k} = \delta_{g+h,k} \) for all \( g, h, k \in G \). Then the vertex operator algebra structure of \( V \) is determined uniquely by the \( V^0 \)-module structure of \( V \). That is, if \( (\bar{V}, \bar{Y}, 1, \omega) \) is also a simple vertex operator algebra with \( \bar{V} = V \) as vectors spaces and \( Y(u, z) = \bar{Y}(u, z) \) for all \( u \in V^0 \) then \((V, Y, 1, \omega)\) and \((V, \bar{Y}, 1, \omega)\) are isomorphic.

**Proof:** Note that both \( \bar{Y}_{g,h} \) and \( Y_{g,h} \) are intertwining operators of type \( \left( \frac{V^{g+h}}{V^g} \right) \) for \( V^0 \)-modules \( V^g, V^h, V^{g+h} \) for any \( g, h \in G \). By assumption there exists constants \( c_{g,h} \) such that \( \bar{Y}_{g,h} = c_{g,h} Y_{g,h} \). We shall prove that \( c : G \times G \to \mathbb{C}^* \) defines a normalized two cocycle on \( G \).

It is clear that \( c_{h,h} = 1 \) for all \( h \) in \( G \) as \( \bar{Y}(v, z) = Y(v, z) \) for all \( v \in V^0 \). By skew symmetry (see [FHL]), for \( u \in V^h \) and \( v \in V^g \),

\[
\bar{Y}(u, z)v = e^{zL(-1)} Y(v, -z) u = c_{g,h} e^{zL(-1)} Y(v, -z) u = c_{g,h} Y(u, z)v.
\]

Thus \( c_{g,h} = c_{h,g} \) for all \( g, h \in G \). In particular, this shows that \( c_{h,0} = 1 \).

Using commutativity (see [FLM], [FHL] and [DL]) for \( u \in V^g \) and \( v \in V^h \) gives a non-negative integer \( n \) such that

\[
(z_1 - z_2)^n \bar{Y}(u, z_1)\bar{Y}(v, z_2) = (z_1 - z_2)^n \bar{Y}(v, z_2)\bar{Y}(u, z_1)
\]

and that

\[
(z_1 - z_2)^n Y(u, z_1)Y(v, z_2) = (z_1 - z_2)^n Y(v, z_2)Y(u, z_1).
\]
Applying both identities to $V^k$ yields

$$c_{g,h+k}c_{h,k} = c_{h,g+k}c_{g,k} = c_{g+k,h}c_{g,k}.$$ 

That is, $c$ is a 2-cocycle.

Since $\mathbb{C}^*$ is injective in the category of abelian groups, there exists a function $f : G \to \mathbb{C}^*$ such that $f(0) = 1$ and $c_{g,h} = f(g)f(h)/f(g + h)$ for $g, h \in G$. Define a linear map $\sigma$ from $\hat{V}$ to $V$ by sending $v$ to $f(g)v$ for $v \in V^g$ and $g \in G$. It is enough to prove that $\sigma$ is an isomorphism of vertex operator algebras.

Clearly, $\sigma$ is a linear isomorphism. Let $u \in V^g$ and $v \in V^h$. Then

$$\sigma(\hat{Y}(u, z)v) = f(g + h)\hat{Y}(u, z)v = f(g)f(h)Y(u, z)v = Y(\sigma u, z)\sigma v.$$ 

Thus $\sigma$ is an isomorphism of vertex operator algebras. □

In order to discuss a consequence of Proposition 5.3 we recall from [B] and [FLM] the vertex operator algebra $V_L$ associated to an even positive definite lattice $L$. So $L$ is a free abelian group of finite rank with a positive definite $\mathbb{Z}$-bilinear form $(, )$ such that $(\alpha, \alpha) \in 2\mathbb{Z}$ for $\alpha \in L$. Set $\mathfrak{h} = \mathbb{C} \otimes_\mathbb{Z} L$ and let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ the corresponding affine Lie algebra. Let $M(1) = \mathbb{C}[h(-n) | h \in \mathfrak{h}, n > 0]$ be the unique irreducible module for $\hat{\mathfrak{h}}$ such that $c$ acts as 1 and $\mathfrak{h} \otimes t^0$ acts trivially. Then as a vector space,

$$V_L = M(1) \otimes \mathbb{C}[L]$$

where $\mathbb{C}[L]$ is the group algebra $\mathbb{C}[L]$. Then $V_L$ is a strongly rational vertex operator algebra (see [B], [FLM], [D1] and [DLM1]).

It is a fact that $M(1)$ is a vertex operator subalgebra of $V_L$ (see [FLM]) and $V_L$ is $L$-graded such that $V^\alpha_L = M(1) \otimes e^\alpha$ for $\alpha \in L$ where $e^\alpha$ denotes the basis element in $\mathbb{C}[L]$ corresponding to $\alpha$. In fact, $V^\alpha_L = M(1) \otimes e^\alpha$ is an irreducible $M(1)$-module.

**Corollary 5.4** Let $V$ be simple vertex operator algebra which has a subalgebra isomorphic to $M(1)$, such that $V$ isomorphic to $V_L$ as $M(1)$-modules for some even positive definite lattice $L$. Then $V$ and $V_L$ are isomorphic vertex operator algebras.

**Proof:** By assumption, $V \equiv \sum_{\alpha \in L} V^\alpha_L$. By Proposition 8.15 of [G], $N_{V^\alpha_L, V^\beta_L} = \delta_{\alpha + \beta, \gamma}$. Proposition 5.3 then gives the result. □

**6 Proof of Theorem 3**

In order to prove Theorem 3 we first recall some results about highest weight modules for the Virasoro algebra.
For a pair of complex numbers \(c, h\) we denote by \(Ver(c, h)\) the Verma module of highest weight \((c, h)\) over the Virasoro algebra \(Vir_c\) of central charge \(c\). Thus \(Ver(c, h)\) is generated as \(Vir_c\)-module by a state \(v_0\) satisfying \(L(n)v_0 = 0\) for \(n > 0\) and \(L(0)v_0 = hv_0\). The structure of these modules was elucidated in the work of Feigin-Fuchs [FF], Rocha-Caridi and Wallach [RW1],[RW2], and Kac and Rainie [KR], and their results play an important role in our proof of Theorems 3 and 4. The case \(h = 0\) is particularly important. One knows (loc. cit.) that \(Ver(c, 0)\) contains a singular vector \(L(-1)v_0 = v_1\) at level 1, and by Frenkel-Zhu [FZ] the quotient module \(V(c, 0)/U(Vir_c)v_1\) carries the structure of a vertex operator algebra. We denote this vertex operator algebra by \(V(c, 0)\). It contains a unique maximal \(Vir_c\)-submodule \(J(c, 0)\), and the quotient module \(L(c, 0) = V(c, 0)/J(c, 0)\) also carries the structure of vertex operator algebra. The rationality of \(L(c, 0)\) was investigated by Wang [W]. We collect some of the results of these authors in the following.

**Proposition 6.1** One of the following holds:

(i) \(J(c, 0) = 0\), so that \(V(c, 0) = L(c, 0)\) is a simple module over \(Vir_c\). In this case, the \(q\)-graded character of \(L(c, 0)\) is equal to \((1 - q)/\phi(q)\) and the coefficients have exponential growth. Moreover, \(L(c, 0)\) is not a rational vertex operator algebra.

(ii) \(J(c, 0) \neq 0\). In this case \(L(c, 0)\) is a rational vertex operator algebra. The central charge has the form \(c = c_{p,q} = 1 - 6(p - q)^2/\phi(pq)\) for a pair of coprime integers \(p, q\) satisfying \(2 \leq p < q\).

In order to prove Theorem 3 we first assume only that \(c = l\). We continue with the notation of Section 4. Let \(W\) denote the Virasoro vertex operator algebra generated by \(\omega''\). Then \(W\) is a highest weight module for the Virasoro algebra with highest weight 0. Of course it is possible that \(\omega'' = 0\), in which case \(W\) is nothing but the complex numbers.

**Lemma 6.2** Let \(V\) be a strongly rational vertex operator algebra satisfying \(c = l\). Then there exist coprime positive integers \(p, q\) with \(2 \leq p < q\) such that \(W\) is isomorphic to \(L(c_{p,q}, 0)\).

**Proof:** Recall from (4.3) that \(V = M(1) \otimes \Omega_V\) and \(\Omega_V\) is a module for the Virasoro algebra \(Vir''\). From the proof of Corollary 6.3 we know that the Fourier coefficients of 

\[ \eta(\tau)^c Z_V(\tau) = q^{(l-c)/24} \eta(\tau)^{c-l} \text{ch}_q \Omega_V = q^{(l-c)/24} \text{ch}_q \Omega_V \]

have polynomial growth. If \(c'' = c - l\) is not equal to any \(c_{p,q}\) for all coprime positive integers \(p, q \geq 2\) then

\[ \text{ch}_q W = \sum_{n \geq 0} (\dim W_n) q^n = \frac{1 - q}{\phi(q)} \]

by Proposition 6.4. We then have that the Fourier coefficients of \(\eta(\tau)^c Z_V(\tau)\) has exponential growth. This shows that \(c'' = c_{p,q}\) for some coprime positive integers \(q > p > 1\). The same argument also shows that \(W\) is an irreducible highest weight module for \(Vir''\). Thus \(W\) is isomorphic to \(L(c_{p,q}, 0)\). \(\Box\)
We now turn to the proof of Theorem 3. In this case, $c'' = c_{p,q} = 0$. Thus $\omega'' = 0$ and $W = \mathbb{C}$. As a result we see that

$$V = \sum_{\alpha \in L} M(1, \alpha) = \oplus_{\alpha \in L} M(1) \otimes \Omega_V(\alpha).$$

We already know from Remark 1.1 that $L$ is an additive group of $\mathfrak{h}$. By Corollary 1.4 it is enough to prove that $L$ is an even positive-definite lattice of rank $l$.

Note that the eigenvalue of $L(0) = L'(0)$ on $\Omega_V(\alpha)$ is $\langle \alpha, \alpha \rangle / 2$. It is clear that $0 \leq \langle \alpha, \alpha \rangle / 2 \in \mathbb{Z}$ for all $\alpha \in L$. That is, $L$ is even. If $\langle \alpha, \alpha \rangle = 0$ for some $\alpha \in L$ then $\Omega(\alpha)$ has weight zero. Since $V_0$ is one dimensional and spanned by $1$ we immediately see that $\alpha = 0$. Thus $L$ is an even, positive-definite lattice.

If the rank of $L$ is less than $l$ then there exists a non-zero $u \in \mathfrak{h}$ such that $\langle u, L \rangle = 0$. Thus $o(u) = u_0$ has zero eigenvalue only on $V$. As a result we see that

$$\text{tr}_V o(u) o(v) q^{L(0) - c/24} = 0$$

for all $v \in V_1$. The proof of Theorem 1 shows that this is not possible. As a result we see that $L$ is a positive-definite even lattice of rank $l$. □

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