A SPLITTING RESULT FOR COMPACT SYMPLECTIC MANIFOLDS

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Abstract. We consider compact symplectic manifolds acted on effectively by a compact connected Lie group $K$ in a Hamiltonian fashion. We prove that the squared moment map $|\mu|^2$ is constant if and only if $K$ is semisimple and the manifold is $K$-equivariantly symplectomorphic to a product of a flag manifold and a compact symplectic manifold which is acted on trivially by $K$. In the almost-Kähler setting the symplectomorphism turns out to be an isometry.

1. Introduction

In [GP] it has been proved that on a compact Kähler manifold acted on effectively by a compact connected Lie group $K$ of isometries in a Hamiltonian fashion, the squared moment map $|\mu|^2$ is constant if and only if $K$ is semisimple and the manifold is biholomorphically and $K$-equivariantly isometric to a product of a flag manifold and a compact Kähler manifold which is acted on trivially by $K$.

In the present paper we consider its symplectic analogue; we deal with a $2n$-dimensional symplectic manifold $M$ acted on by a compact connected Lie group $K$, we suppose that the $K$-action on $M$ is Hamiltonian, i.e. there exists a moment map $\mu : M \to \mathfrak{k}^*$, where $\mathfrak{k}$ is the Lie algebra of $K$. Throughout the following we will denote by $\omega$ the symplectic form on $M$; moreover Lie groups and their Lie algebras will be indicated by capital and gothic letters respectively.

If we fix an $\text{Ad}(K)$-invariant scalar product $q := \langle \cdot, \cdot \rangle$ on $\mathfrak{k}$ and we identify $\mathfrak{k}^*$ with $\mathfrak{k}$ by means of $q$, we can think of $\mu$ as a $\mathfrak{k}$-valued map; the function $f \in C^\infty(M)$ defined as $f := |\mu|^2$ has been extensively used in [KT] to obtain strong information on the topology of the manifold.

Our result is the following

Theorem 1. Suppose $M$ is a compact symplectic $K$-Hamiltonian manifold, where $K$ is a compact connected Lie group acting effectively on $M$. If $\mu$ is the corresponding moment map, then its squared norm $f = |\mu|^2$ is constant if and only if $K$ is semisimple and the manifold $M$ is $K$-equivariantly symplectomorphic to the product of a flag manifold and a compact manifold which is acted on trivially by $K$. Moreover if on $(M, \omega)$ is given a $K$-invariant $\omega$-compatible almost complex structure.

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structure, the symplectomorphism turns out to be an isometry with respect to the induced Riemannian metric.

In the first subsection of the section 2 we give the proof of the symplectic part of the statement of Theorem 2 while in the second subsection we deal with the almost-Kähler setting.

2. Proof of the main result

2.1. The symplectic setting. We will follow the notations as in [1K1]. Assume \( f \) to be constant, i.e. \( \mu \) maps the manifold \( M \) into a sphere. We fix a point \( x_0 \in M \); we can suppose that \( \beta = \mu(x_0) \) lies in the closure of a Weyl chamber \( t_1 \), where \( t \) denotes the Lie algebra of a fixed maximal torus in \( K \). Since \( \mu(M) \) is connected and \( \mu(M) \cap t_1^* \) is convex \([K2]\), \( \mu(M) \) is a single coadjoint orbit \( O = K/K_{\beta} \), where \( \beta = \mu(x_0) \).

We first want to prove that \( \mu^{-1}(\beta) \) is a symplectic submanifold of \((M, \omega)\). Let \( \mu_{\beta} := (\mu, \beta) \) be the height function relative to \( \beta \) and let \( Z_\beta \) denote the intersection of the critical point set of \( \mu_{\beta} \), with the pre-image of \( ||\beta||^2 \) via \( \mu_{\beta} \). Since \( f \) is constant, each point in \( M \) is critical for \( f \), therefore its critical set, which is proved (Lemma 3.15 [K1]) to be \( C_\beta = K \cdot (Z_\beta \cap \mu^{-1}(\beta)) \), coincides with \( M \). In Kirwan’s language we can say that the stratification \( \{S_\beta\} \), induced by \( f \), is made of only one stratum.

Now, following the proof of Proposition 2 in [GP], we have that \( Z_\beta = \mu^{-1}(\beta) \). Indeed, if \( p \in Z_\beta \), then \( \mu_{\beta}(p) = ||\beta||^2 \) and

\[
||\beta||^2 = \langle \mu(p), \beta \rangle \leq ||\mu(p)|| \cdot ||\beta|| \leq ||\beta||^2,
\]

and therefore \( \mu(p) = \beta \), i.e. \( p \in \mu^{-1}(\beta) \). Vice versa, if \( p \in \mu^{-1}(\beta) \), then \( ||\mu(p)||^2 \) is a critical value of \( f := ||\mu||^2 \) and therefore \( \hat{\beta}_p = 0 \), where \( \hat{\beta} \) denotes the fundamental field on \( M \) induced by the element \( \beta \in t \); moreover \( \mu_{\beta}(p) = ||\beta||^2 \), hence \( p \in Z_\beta \).

This implies that \( \mu^{-1}(\beta) \) is a symplectic submanifold [GS] and that \( M = C_\beta = K \cdot \mu^{-1}(\beta) \). As a consequence, at every \( y \in \mu^{-1}(\beta) \), the tangent space to \( M \) splits as

\[
T_yM = T_y(K \cdot y) \oplus T_y(\mu^{-1}(\beta)).
\]

Using the fundamental property of the moment map, we can see that the previous splitting is symplectic. Indeed if \( v \in T_y(\mu^{-1}(\beta)) \) and \( X \in t \) we have

\[
0 = \langle d\mu_y(w), X \rangle = \omega_y(w, \dot{X}_y).
\]

We have used the fact that \( T_y\mu^{-1}(\beta) = \text{Ker} \, d\mu_y \), because \( \mu^{-1}(\beta) \) is a submanifold of \( M \).

Note that every level set of \( \mu \) is a symplectic submanifold of \( M \), then we can argue as above to obtain the symplectic splitting (2.1) at every point of \( M \). This is a consequence of the fact that \( \mu \) is \( K \)-equivariant and it is constant when passing to the quotient.
Here we show that $K$ is semisimple, i.e. the connected component $Z$ of its center is trivial. Fix a $K$-principal point $p \in M$ and observe that the restricted map $\mu : K \cdot p \to \mathcal{O}$ is a covering because $K \cdot p$ is symplectic; indeed $\text{Ker } d\mu$, restricted to the orbit $K \cdot p$, is trivial (it is the set $(T_p^p K \cdot p)^{\omega} \cap T_p^p K \cdot p$) hence $\mu$ is a local diffeomorphism and therefore a covering map since the orbit $K \cdot p$ is compact. Since the coadjoint orbits of a compact connected Lie group are simply connected, $K \cdot p$ is simply connected too, hence $Z$ acts trivially on it (see e.g. [B], p. 224). Since $p$ is principal, $Z$ acts trivially on $M$, hence $Z$ is trivial, because the $K$-action is effective.

We claim that for all $x \in M$, $K_x = K_\beta$. By the $K$-equivariance of $\mu$, we have that $K_x \subseteq K_{\mu(x)} = K_\beta$, which is connected, being the centralizer of a torus in a compact Lie group. The other inclusion follows from the fact that the map $\mu : K \cdot p \to \mathcal{O}$ is a covering and both $K \cdot p$ and $\mathcal{O}$ are simply connected, therefore $\mathfrak{t}_o = \mathfrak{t}_\beta$ and we get the equality.

Note that each $K$-orbit intersects $\mu^{-1}(\beta)$ in a single point: if there are two points $x$ and $z = k \cdot x$ for $k \in K$ which lie in $\mu^{-1}(\beta)$, then, by the $K$-equivariance of $\mu$, we have $k \in K_\beta = K_\beta$; hence $k \cdot x = x$.

From this it follows that the map

$$\varphi : K/K_\beta \times \mu^{-1}(\beta) \to M, \quad \varphi(gK_\beta, x) = g \cdot x,$$

where we identify $K/K_\beta$ with the orbit $K \cdot x_o$, is a well defined $K$-equivariant diffeomorphism.

We also observe that $\mu^{-1}(\mu(x))$ is connected for all $x \in M$. Moreover all the $K$-orbits are principal since their stabilizers are all equal to $K_\beta$, hence we have that $K_x$ acts trivially on $T_x \mu^{-1}(\mu(x)) = (T_x K \cdot x)^{\omega}$ for all $x \in M$.

We now denote by $\mathcal{F}$ the foliation given by the $K$-orbits and by $\mathcal{F}^\omega$ the $\omega$-orthogonal foliation, so that $\mathcal{F}^\omega_y = T_y(\mu^{-1}(\mu(y)))$. By the same symbol we denote the corresponding integrable distributions.

We claim that $\varphi$ is a symplectomorphism. It is sufficient to prove it locally, the global property following from the fact that $\varphi$ is a diffeomorphism. Choose coordinates $x_1, x_2, \ldots, x_{2n}$ in a neighborhood $U$ of a point $p \in M$, in such a way that $\{X_1 = \frac{\partial}{\partial x_1}, \ldots, X_{2r} = \frac{\partial}{\partial x_r}\}$ is a local frame for $\mathcal{F}$, while $\{X_{2r+1} = \frac{\partial}{\partial x_{2r+1}}, \ldots, X_{2n} = \frac{\partial}{\partial x_{2n}}\}$ is a local frame for $\mathcal{F}^\omega$. We show that $M$ is locally symplectomorphic to the product $K/K_\beta \times \mu^{-1}(\beta)$ proving that the following three conditions are satisfied

(i) $\omega(X_i, X_j) = 0$ for all $i = 1, \ldots, 2r$ and $j = 2r + 1, \ldots, 2n$.
(ii) $X_l \omega(X_i, X_j) = 0$ for all $l = 1, \ldots, 2r$ and $i, j = 2r + 1, \ldots, 2n$
(iii) $X_m \omega(X_i, X_j) = 0$ for all $m = 2r + 1, \ldots, 2n$ and $i, j = 1, \ldots, 2r$.

We have already proved that $\omega(v, w) = 0$ for all $v \in \Gamma(\mathcal{F})$ and $w \in \Gamma(\mathcal{F}^\omega)$ using the fundamental property of $\mu$. Nevertheless this fact can also be seen as a consequence
of the following remark. Let \( x \in M \) be arbitrary. Fix \( v \in \Gamma(\mathcal{F}_x^\omega) \), and consider the linear form \( \mathcal{L}_v \in (T_x K \cdot x)^* \) assigning to each \( w \in \Gamma(\mathcal{F}_x) \) the real number \( \omega(v,w) \). We claim that it is always zero, hence, in particular, (i) follows. Since the \( K \)-action is symplectic and \( x \) is a principal point, so that the isotropy representation of \( K_x \) leaves \( \mathcal{F}_x^\omega \) pointwisely fixed, \( \mathcal{L}_v \) is \( K_x \)-invariant. Indeed:

\[
\mathcal{L}_v(L_{k^x}w) = \omega(v, L_{k^x}w) = \omega(L_{k^x}^{-1}(v), w) = \omega(v, w) = \mathcal{L}_v(w)
\]

On the other hand the isotropy representation of \( K_x \) on \( T_x K \cdot x \) (hence on \( (T_x K \cdot x)^* \)) has no nonzero fixed vector, since \( K_x \) is the centralizer of some torus in \( K \), therefore \( \mathcal{L}_v \equiv 0 \). To prove (ii), recall that the closedness of \( \omega \) means

\[
X \omega(Y, Z) + Y \omega(Z, X) + Z \omega(X, Y) +
- \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) = 0.
\]

for every \( X, Y, Z \in \Gamma(TM) \). Using the coordinates introduced above and (i), the claim follows. The same argument applies to (iii).

2.2. The almost-Kähler setting. Suppose now that on \((M, \omega)\) is given a \( K \)-invariant \( \omega \)-compatible almost complex structure \( J \). Denote by \( g \) the induced Riemannian metric defined as \( g(X, Y) := \omega(X, JY) \) for all \( X, Y \in \Gamma(TM) \).

In the previous part we proved that \( \mu^{-1}(t) = Z_\beta \). In \( [41] \) (Lemma 4.12) it is shown that \( Z_\beta \) is \( J \)-invariant w.r.t. a fixed \( \omega \)-compatible almost-complex structure. This allows us to deduce that the symplectic splitting \( \mathcal{X}_l \) is also \( g \)-orthogonal, hence every \( K \)-orbit is \( J \)-invariant.

In order to prove that \( \varphi \) is a local isometry now it is sufficient to show that the metric analogues of (ii) and (iii) hold.

We use the same notations introduced in the previous subsection. The proofs of these conditions are completely analogous, we prove for example the first one. Suppose \( X_i \) and the pair \( X_j, X_l \) are local sections around \( p \) of \( \mathcal{F} \) and \( \mathcal{F}_x^\omega \) respectively. We have that \( X_i g(X_j, X_l) = X_i \omega(X_j, JX_l) \); since \( \mathcal{F}_x^\omega \) is \( J \)-invariant, we can express \( JX_i \) as \( \sum_{m=2r+1}^{2n} a_m(x)X_m \), where \( a_m \) are \( C^\infty \)-functions on \( U \) which do not depend on the first \( 2r \) variables. Therefore the metric condition (ii) becomes:

\[
X_i \omega(X_j, JX_l) = \sum_{m=2r+1}^{2n} X_i(a_m(x)) \cdot \omega(X_j, X_m) +
\sum_{m=2r+1}^{2n} a_m(x)X_i \omega(X_j, X_m),
\]

which vanishes because of the symplectic (i).

Remark 1. The compactness assumption on \( M \) can be weakened assuming that the moment map \( \mu \) is proper. In this case, as Knop shows in \( [K] \), we have that \( \mu(M) \cap t_+^r \) is convex. Therefore \( M \) is still mapped to a coadjoint orbit, which is compact. Therefore \( M \) must be compact.
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