Nonextensive thermostatistics for heterogeneous systems containing different $q$’s

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Abstract

The nonextensive statistics based on Tsallis entropy have been so far used for the systems composed of subsystems having same $q$. The applicability of this statistics to the systems with different $q$’s is still a matter of investigation. The actual difficulty is that the class of systems to which the theory has been applied is limited by the usual nonadditivity rule of Tsallis entropy which, in reality, has been established for the systems having same $q$ value. In this paper, we propose a more general nonadditivity rule for Tsallis entropy. This rule, as the usual one for same $q$-systems, can be proved to lead uniquely to Tsallis entropy in the context of systems containing different $q$-subsystems. A zeroth law of thermodynamics is established between different $q$-systems on the basis of this new nonadditivity.

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1 Introduction

The starting point of the nonextensive statistical mechanics (NSM) of Tsallis [1] is the entropy given by

$$S_q = \frac{\sum_{i=1}^{w} p_i^q - 1}{1 - q}$$

(1)

where the physical states are labelled by $i = 1, 2, ..., w$, $q$ is a parameter characterizing the nonextensivity of the theory, and $p_i$ is the probability for the system to be found at state $i$. The application of the principle of maximum
entropy under appropriate constraints can lead to power law distributions characterized by the so called $q$-exponential functionals\cite{2}. For example, for canonical ensemble under the constraints associated with probability normalization and average energy, we can have \cite{2,3}:

$$p_i = \frac{1}{Z_q} [1 - (1 - q)\beta E_i]^{1/(1-q)} \quad [>] > 0$$  \hspace{1cm} (2)

where $Z_q$ is the partition function, $E_i$ is the energy of a microstate $i$, and $\beta$ is an intensive variable analogous to the inverse temperature in Boltzmann-Gibbs statistics (BGS). $\beta$ is the central quantity discussed in this work.

For a composite system $(A+B)$ whose joint probability $p_{ij}(A+B)$ is given by the product of the probabilities of its subsystems **having the same** $q$, i.e., $p_{ij}(A+B) = p_i(A)p_j(B)$, we have, from Eq.\((1)\)

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B)$$  \hspace{1cm} (3)

and, from Eq.\((2)\)

$$E_{ij}(A + B) = E_i(A) + E_j(B) + (q - 1)\beta E_i(A)E_j(B)$$  \hspace{1cm} (4)

Eqs.\((3)\) and \((4)\) characterize the nonadditivity of the same $q$-systems having Tsallis entropy. In other words, they prescribe the class of nonextensive systems to which NSM may be applied. They should be considered as the analogs (or generalizations) of the entropy and energy additivity in BGS and play the role of starting hypothesis specifying the limit of the validity range of NSM. It has been proved that\cite{4,5} Eq.\((3)\) uniquely determines Tsallis entropy and that\cite{6,7} Eqs.\((3)\) and \((4)\) are a group of necessary conditions for the existence of thermodynamic stationarity between nonextensive systems. When $q = 1$, $S_q = S_1 = -\sum_i p_i \ln p_i$, and Eqs.\((3)\) and \((4)\) become the usual additivity assumption of entropy and energy for BGS.

The range of validity of this generalized theory is actually under study. In view of its application to complex systems like turbulence, economic systems, fractal and chaotic systems, a point of view \cite{8,9,10} is more and more accepted that NSM can be used to describe some nonequilibrium systems at stationary state. This implies the following hypotheses:

1. Tsallis entropy, as an information or uncertainty measure associated with the probability $p_i$, is valid for some nonequilibrium nonextensive systems in steady or stationary states (time independent);
2. $p_i$ is the time invariant probability (or invariant measure\cite{11}) for the system under consideration to be found at the stationary state $i$;

3. the Janyes’ inference method (unbiased guess) of maximum entropy\cite{12, 13} applies for these states, i.e., the time independent probability distribution $p_i$ can be deduced from maximizing the entropy (uncertainty) under some constraints associated with, e.g., the normalization of $p_i$ and the averages of physical quantities determining $p_i$. In this work, $p_i$ is supposed to be uniquely determined by the energy $E_i$ of the system at stationary state $i$.

Due to the maximum Tsallis entropy, the framework of the conventional equilibrium thermodynamics can be copied for stationary nonequilibrium systems. Essential for this work is the establishment of zeroth law of thermodynamics, i.e., the intensive variable $\beta = \frac{\partial S_i}{\partial U_q}$ (we call it inverse temperature from now on) is uniform everywhere at stationary state, where $U_q$ is the internal energy defined below in the section 3. It is worth noticing that, due to the different formalisms of NSM proposed in the past 15 years, the inverse temperature $\beta$ has been given several definitions which sometimes cause confusion. A comment on this subject was given in ref.\cite{14}.

In what follows, we will focalize our attention on an important question about the validity of NSM for the systems containing subsystems with different $q$’s. Up to now, NSM has been applied only to composite systems containing subsystems with same $q$ for which several versions of zeroth law of thermodynamics were established on the basis of Eq.(3)\cite{15, 16, 17}. For the systems having different $q$’s, the attempt to apply NSM \cite{19, 20} using Eq.(3) is unfruitful since, firstly, no rigorous zeroth law can be formulated, secondly, no Tsallis entropy can be formulated as in Eq.(1) for a compound system $A+B$ when $A$ and $B$ respectively have Tsallis entropy with their own $q_A$ and $q_B$. This result implies that, in the actual formulation of NSM, there must be some fundamental aspects which are incomplete or handicapped. Nature contains nonextensive systems with different $q$ values, as shown by the applications of NSM in the past years, and there must be also systems containing subsystems with different $q$’s, such as a cluster of galaxies containing different galaxies with different $q$’s, a heterogeneous materials containing elements being physically and chemically different, a fractal containing two fractals of different dimensions and physics, or a social group including socially (culturally, economically or technically...) different subgroups, etc. If
NSM is useful only for the systems containing uniform distribution of $q$ value, not only its application would be extremely limited, but worse, its general validity would be fundamentally in danger\cite{18} because, for different $q$-systems, no composability of physical quantity is possible, no measurement of temperature or pressure of a $q$-system is possible with ordinary thermometers or barometers which obey Boltzmann-Gibbs thermostatistics with $q = 1$ (or in order to measure the temperature of a $q$-system, you have to verify that the thermometer has the same $q$ as the system!). This is equivalent to saying that NSM is useless in view of the diversity of physical or social systems. So this obstacle must be removed in order that NSM be a coherent and complete theory whose formalism, just like that of BGS, should remain the same whatever the level of the system, i.e., be hierarchically invariant.

As a matter of fact, it is often forgotten that Eq.\eqref{eq:3}, a fundamental rule of NSM, was only established for the subsystems with same $q$. The aforementioned failure of the theory is in fact a failure of Eq.\eqref{eq:3}. The aim of this paper is to show that, by replacing Eq.\eqref{eq:3} with a more general nonadditivity rule for Tsallis entropy, NSM can be applied to any system having different $q$'s. We can in addition derive a calculus which uniquely determines the $q$ of a composite systems from the $q$'s of its subsystems. Without loss of generality, the discussion will be made with the $q$-exponential distribution given by Eq.\eqref{eq:2}.

2 A more general pseudoadditivity for Tsallis entropy

Now let us consider $A$ and $B$, two nonextensive subsystems of a composite nonextensive system $A + B$. It has been proved\cite{6} that the most general pseudoadditivity (or composability) of entropy or energy prescribed by zeroth law is the following:

\begin{equation}
H[Q(A + B)] = H[Q(A)] + H[Q(B)] + \lambda QH[Q(A)]H[Q(B)],
\end{equation}

which is in fact a very weak condition where $H[Q]$ is just certain differentiable function satisfying $H[0] = 0$, $\lambda Q$ is a constant, and $Q$ is either entropy $S$ or internal energy $U$\cite{7}. The functional form of $H(Q)$ naturally depends on the nonadditive character of the system of interest. As shown in \cite{7}, for a given relationship $Q = f(N)$ where $N$ is the number of elements of the system, the finding of $H(Q)$ is trivial.
As a matter of fact, Eq.(5) has been established [6, 7] for the class of systems containing only subsystems having same $q$ value. It must be generalized for the systems whose subsystems have different $q$’s. This generalization is straightforward if we replace the Eq.(1) of reference [6], i.e., $S(A + B) = f\{S(A), S(B)\}$ for uniform $q$, by $H_q[S_q(A+B)] = f\{H_{q_A}[S_{q_A}(A)], H_{q_B}[S_{q_B}(B)]\}$ (or by $H_q[S_q(A+B)] = H_{q_A}[S_{q_A}(A)] + H_{q_B}[S_{q_B}(B)] + q\{H_{q_A}[S_{q_A}(A)], H_{q_B}[S_{q_B}(B)]\}$) where $H_q(S_q)$ is a functional depending on $q$’s in the same way for the composite system as for the subsystems, where $q$, $q_A$ and $q_B$ are the parameters of the composite system $A + B$, the subsystems $A$ and $B$, respectively. This functional $H_q(S_q)$ is necessary for the equality to hold in view of the different $q$’s in the entropies of different subsystems. The function $f$ (or $g$) is to be determined with the help of the zeroth law. Now repeating the mathematical treatments described in the references [6, 7], we find

$$H_q[Q(A + B)] = H_{q_A}[Q(A)] + H_{q_B}[Q(B)] + \lambda_q H_{q_A}[Q(A)]H_{q_B}[Q(B)].$$

(6)

Eq.(5) turns out to be a special case of Eq.(6) for same $q$ systems. In this case, if $H_q[S_q]$ is chosen to be identity function to give Eq.(3), then Tsallis entropy can be proved to be unique for this class of systems [4, 5] just as the uniqueness of Shannon information measure $S_1$ can be proved with the entropy additivity for independent subsystems having product joint probability [21].

This reasoning can be extended to the subsystems $A$ and $B$ each having its own $q$ if one chooses $H_q(S_q) = \sum_{i=1}^w p_i^q - 1 = (1 - q)S_q$. From Eq.(6), this choice implies:

$$(1 - q)S_q(A + B) = (1 - q_A)S_{q_A}(A) + (1 - q_B)S_{q_B}(B) + \lambda_S(1 - q_A)(1 - q_B)S_{q_A}(A)S_{q_B}(B)$$

(7)

where $q$, $q_A$ and $q_B$ are the parameters of the composite system $A + B$, the subsystems $A$ and $B$, respectively. Eq.(7) is not derived from Eq.(5) but only postulated in order to generalize Eq.(3) according to the necessary condition Eq.(5) for the establishment of zeroth law. The uniqueness of Tsallis entropy for the systems obeying Eq.(7) can also be proved along the line of reference [21] by replacing the Axiom 3 of the information additivity of [21] with Eq.(7) and the Eq.(19) of [5], i.e., $[1 + (1 - q)L_q(r^m)] = [1 + (1 - q)L_q(r)]^m$, by $[1 + (1 - q_c)L_q(r^m)] = \prod_{i=1}^m [1 + (1 - q_i)L_{q_i}(r)]$, where $L_q$ is the entropy functional, $q_c$ is the parameter of the composite system containing $m$ subsystems each having its own parameter $q_l$ ($l = 1, 2, \ldots m$) and $r(\geq 2)$ states. The details of this proof will be given in another paper.
Now let $\lambda_S = 1$, Eq. (7) implies:

$$p_{ij}^q(A + B) = p_i^{qA}(A)p_j^{qB}(B)$$

(8)

which can be called the generalized factorization of joint probability for the systems of different $q$’s. A possible understanding of this relationship is that here the physical or effective probability is $p_i^q$ instead of $p_i$. This interpretation may shed light on why it has been necessary to use $p_i^q$ or the escort probability\[2\] to calculated expectation. Eq. (8) can be written as the usual product probability $p_{ij}(A + B) = p_i(A)p_j(B)$ if and only if $q = q_A = q_B$ for the systems having the same nonadditivity.

Eq. (8) allows one to determine uniquely the parameter $q$ for the composite system if $q_A$, $q_B$, $p_i(A)$ and $p_j(B)$ are given. By the normalization of the joint probability, we obtain the following relationship:

$$\sum_{i=1}^{w_A} p_i^{qA/q}(A) \sum_{j=1}^{w_B} p_j^{qB/q}(B) = 1$$

(9)

which means $q_A < q < q_B$ if $q_A < q_B$ and $q = q_A = q_B$ if $q_A = q_B$. In this way, for a composite system containing $N$ subsystems ($k=1,2,...,N$) having different $q_k$, the parameter $q$ is determined by

$$\prod_{k=1}^{N} \sum_{i=1}^{w_k} p_i^{q_k/q}(A) = 1,$$

(10)

from which we can say that $\min(q_k) < q < \max(q_k)$.

Now if $(A + B)$ is at stationary state, according to Jaynes principle of unbiased guess, $dS(A + B) = 0$. From Eq. (7), we get:

$$\frac{(1 - q_A)dS_{q_A}(A)}{1 + (1 - q_A)S_{q_A}(A)} + \frac{(1 - q_B)dS_{q_B}(B)}{1 + (1 - q_B)S_{q_B}(B)} = 0$$

(11)

Eq. (11) is essentially different from $\frac{dS_{q_A}(A)}{1+q_AS_{q_A}(A)} + \frac{dS_{q_B}(B)}{1+q_BS_{q_B}(B)} = 0$ used for the subsystems with same $q$ and will allow us to apply NSM to the systems in which the distribution of $q$ value is not uniform.

### 3 Zeroth law with nonadditive energy

In general, for exact treatments of nonextensive systems within NSM, we have to deal with nonadditive energy that allows the existence of thermal
equilibrium and stationarity[7] and is compatible with the entropy nonadditivity. Usually, the nonadditivity of energy is determined by the product probability. Considering the distribution Eq.(2) and the generalized product probability Eq.(8), we get :

\[
\frac{q}{1-q} \ln[1 - (1 - q)\beta(A + B)E_{ij}(A + B)] = \frac{q_A}{1-q_A} \ln[1 - (1 - q_A)\beta(A)E_i(A)] + \frac{q_B}{1-q_B} \ln[1 - (1 - q_B)\beta(B)E_j(B)].
\]

Now let us define a deformed energy \(e_i^{(q)}\) in order to write the distribution Eq.(2) in the usual exponential form

\[
p_i = \frac{1}{Z_q} \exp[-\beta e_i^{(q)}].
\]

The introduction of this distribution into the generalized product joint probability Eq.(8) leads to :

\[
q\beta(A + B)e_{ij}^{(q)}(A + B) = q_A\beta(A)e_i^{(q)}(A) + q_B\beta(B)e_j^{(q)}(B).
\]

Note that \(e_i^{(q)}\) varies monotonically with \(E_i\), so that \(dE_i = 0\) leads to \(de_i^{(q)} = 0\) in accordance with the conservation of both \(E_i\) and \(e_i^{(q)}\).

In what follows, the zeroth law will be discussed within two possible formalisms of NSM using the unnormalized expectation \(U_q = \sum_i p_i^q E_i\) and the expectation given with escort probability \(U_q = \sum_i p_i^q E_i / \sum_i p_i^q[11]\), respectively. In view of the generalized product joint probability Eq.(8) and the energy nonadditivity, the choice of these statistical expectations is necessary in order to split the composite expectations of the total system into those of the subsystems. The establishment of zeroth law would be impossible without this splitting.

\section{3.1 With unnormalized \(q\)-expectation}

The formalism of NSM with unnormalized expectation \(U_q = \sum_i p_i^q E_i\) was proposed by Tsallis and co-workers[1, 22] and has found many applications[23, 24]. Recently, it has been shown[17] that, with this expectation, the inverse
temperature could be well defined by $\beta = \frac{\partial S}{\partial U}$ for the systems of same $q$, so that all the details of the mathematics (including the statistical interpretation of heat and work) of BGS remain the same within NSM. In what follows, it will be shown that the temperature can be defined between the subsystems with different $q$.

The expectation of the deformed energy is given by $u_q = \sum_i p_i^q e_i^q$. In order to find a relationship between $\beta(A + B)$, $\beta(A)$ and $\beta(B)$, we assume

$$\frac{q u_q(A + B)}{\sum_{ij} p_{ij}^q(A + B)} = \frac{q_A u_{qA}(A)}{\sum_{i=1} p_{iA}^q(A)} + \frac{q_B u_{qB}(B)}{\sum_{j=1} p_{jB}^q(B)}$$

(15)

which is consistent with Eq. (14) and the generalized product probability. It is in fact the analog of the additive energy in BGS. In general, this “additivity” does not mean $\beta(A) = \beta(B)$; it only implies $\beta(A + B) = \frac{\beta(A)u_{qA}(A) + \beta(B)u_{qB}(B)}{u_{qA}(A) + u_{qB}(B)}$ where $u_{qA} = q_A u_{qA}/\sum_{i=1} p_{iA}^q$. From energy conservation, we obtain $du_q(A + B) = 0$ and

$$\frac{q_A du_{qA}(A)}{\sum_{i=1} p_{iA}^q(A)} + \frac{q_B du_{qB}(B)}{\sum_{j=1} p_{jB}^q(B)} = 0.$$ (16)

Now in order to determine $\beta$ in the distribution of Eq. (2) or Eq. (13), we take a deformed entropy $s_q$ defined by Taneja[25, 26]:

$$s_q = -\sum_{i=1}^w p_i^q \ln p_i$$

(17)

but here $p_i$ is given by Eq. (13). This deformed entropy is concave and has the following nonadditivity:

$$\frac{q s_q(A + B)}{\sum_{ij} p_{ij}^q(A + B)} = \frac{q_A s_{qA}(A)}{\sum_{i=1} p_{iA}^q(A)} + \frac{q_B s_{qB}(B)}{\sum_{j=1} p_{jB}^q(B)}$$

(18)

which leads to, at maximum entropy, $q_A ds_{qA}(A) + q_B ds_{qB}(B) = 0$. Combining this with Eq. (16) and considering $s_q = \sum_{i} p_i^q \ln Z_q + \beta u_q$, we get

$$\beta(A) = \beta(B)$$

(19)

where the inverse temperature is given by $\beta = \frac{\partial s_q}{\partial u_q} = \frac{\partial S_q}{\partial U_q}$. It is worth noticing that the equality of the temperatures of $A$ and $B$ at stationary state is independent of whether $q_A$ and $q_B$ are equal.
3.2 With escort probability

The above description of thermal equilibrium can also be made with escort probability proposed in [11]. Let us define a deformed average energy with escort probability \( \mu_q = \sum_i p_i \langle q \rangle_i / \sum_i p_i \langle q \rangle_i \), where \( p_i \) is the probability given by Eq. (2). Still assuming \( q \mu_q(A + B) = q_A \mu_{q_A}(A) + q_B \mu_{q_B}(B) \) which means, from the nonadditivity Eq. (14), \( \beta(A + B) = \frac{\beta_q(A) \mu_{q_A}(A) + \beta(B) \mu_{q_B}(B)}{q_A \mu_{q_A}(A) + q_B \mu_{q_B}(B)} \), one gets

\[
q_A d\mu_{q_A}(A) + q_B d\mu_{q_B}(B) = 0. \tag{20}
\]

in accordance with the conservation of \( \langle q \rangle_i \) or \( \mu_q \).

Now let us take the entropy \( \psi_q \) defined by Aczel and Daroczy [25, 27] as a deformation of the nonextensive entropy \( S_q \):

\[
\psi_q = -\frac{1}{\sum_{i=1}^{w} p_i^q} \sum_{i=1}^{w} p_i^q \ln p_i \tag{21}
\]

which naturally arises from the escort probability. \( \psi_q \) can be maximized with appropriate constraints associated with the deformed average energy \( \mu_q \) to give the distribution of Eq.(13).

The above deformed entropy has the following nonadditivity :

\[
q\psi_q(A + B) = q_A \psi_{q_A}(A) + q_B \psi_{q_B}(B), \tag{22}
\]

so that at maximum entropy \( d\psi_q(A+B) = 0 \), we have \( q_A d\psi_{q_A}(A)+q_B d\psi_{q_B}(B) = 0 \). Combining this with equation Eq.(20), one gets

\[
\frac{\partial \psi_{q_A}(A)}{\partial \mu_{q_A}(A)} = \frac{\partial \psi_{q_B}(B)}{\partial \mu_{q_B}(B)}. \tag{23}
\]

On the other hand, from Eqs.(13) and (21), it is easy to write

\[
\psi_q = \ln Z_q + \beta \mu_q \tag{24}
\]

which means \( \beta = \frac{\partial \psi_q}{\partial \mu_q} \). We finally have

\[
\beta(A) = \beta(B) \tag{25}
\]

which is also independent of whether \( q_A \) and \( q_B \) are equal.
4 An alternative method

In the above discussions, the deformed entropies $s_q$ and $\psi_q$ are introduced in order to discuss nonextensive statistics with the language of extensive statistics BGS. $s_q$ and $\psi_q$ are in fact alternative expressions of $S_q$ associated with the deformed energy $u_q$ and $\mu_q$, respectively. They can be maximized, just as $S_q$, to give $q$-exponential distribution [Eq. (13)]. The description of the stationarity characterized by Eq. (19) or Eq. (25) is of course independent of their introduction. In what follows, I briefly present an alternative method without these deformations.

Using the product joint probability Eq. (8) and the relationship $\sum_i p_i^q = Z_{1-q}^A + (1-q)\beta U_q$ calculated from the distribution Eq. (2) and the unnormalized expectation $U_q$, we get

$$Z_{1-q}^A(A+B) + (1-q)\beta(A+B)U_q(A+B)$$

$$= [Z_{1-q}^A(A) + (1-q\beta(A)U_qA(A)][Z_{1-q}^B(B) + (1-q\beta(B)U_qB(B)]$$

Considering the total energy conservation $dU_q(A+B) = 0$, we obtain

$$\frac{(1-q\beta(A)dU_qA(A)}{\sum_i p_i^{qA}(A)} + \frac{(1-q\beta(B)dU_qB(B)}{\sum_i p_i^{qB}(B)} = 0$$

which suggests following energy nonadditivity

$$\frac{(1-q\beta(A)dU_qA(A)}{\sum_i p_i^{qA}(A)} + \frac{(1-q\beta(B)dU_qB(B)}{\sum_i p_i^{qB}(B)} = 0$$

as the analog of the additive energy $dU(A) + dU(B) = 0$ of BGS. From Eq. (28) and Eq. (11) follows $\beta(A) = \beta(B)$ with $\beta(A) = \frac{\partial S_q(A)}{\partial U_qA(A)}$ and $\beta(B) = \frac{\partial S_q(B)}{\partial U_qB(B)}$.

5 Conclusion

We have discussed the establishment of the zeroth law for stationary state between nonextensive systems with different $q$’s. According to our starting hypotheses, this stationary state maximizes Tsallis entropy so that the zeroth law of the equilibrium thermodynamics can apply. This work is carried out with a generalized nonadditivity rule of Tsallis entropy which reduces to the usual one if all the subsystems have the same $q$. It should be noticed that the generalized product joint probability Eq. (8) derived from the generalized
nonadditivity of entropy suggests two possible expectation calculus of NSM, the unnormalized $q$-expectation and the escort probability $q$-expectation, in order to establish zeroth law. A main result of this work is to show that the intensive variable $\beta$ is uniform in the nonextensive systems in stationary state, independently of whether or not the systems contain subsystems with different $q$ values. So NSM theory based on the maximization of Tsallis entropy is, contrary to some belief, also valid for the systems containing non uniform distribution of $q$ values.

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