ADAPTIVE HETEROGENEOUS MULTISCALE METHODS FOR IMMISCIBLE TWO-PHASE FLOW IN POROUS MEDIA

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Abstract

In this contribution we present the first formulation of a heterogeneous multiscale method for an incompressible immiscible two-phase flow system with degenerate permeabilities. The method is in a general formulation which includes oversampling. We do not specify the discretization of the derived macroscopic equation, but we give two examples of possible realizations, suggesting a finite element solver for the fine scale and a vertex centered finite volume method for the effective coarse scale equations. Assuming periodicity, we show that the method is equivalent to a discretization of the homogenized equation. We provide an a-posteriori estimate for the error between the homogenized solutions of the pressure and saturation equations and the corresponding HMM approximations. The error estimate is based on the results recently achieved in [C. Cancès, I. S. Pop, and M. Vohralík. An a posteriori error estimate for vertex-centered finite volume discretizations of immiscible incompressible two-phase flow. Math. Comp., in press, 2013].

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1 Introduction

In this paper, we consider degenerate two-phase flow in porous media for immiscible and incompressible fluids with highly variable porosity and a highly variable hydraulic conductivity. Applications are oil reservoir simulations (with fluids oil and water) or carbon sequestration (with fluids water and liquid carbon dioxide). The numerical treatment of the two-phase flow system with classical finite element (FE) or finite volume (FV) schemes is typically quite expensive. In particular, when it is necessary to resolve the fine scale features of the (rapidly oscillating) hydraulic conductivity, an extremely fine computational mesh is required. In many scenarios this results in a tremendous computational load that cannot be handled by available computers. To overcome this issue, the classical scheme can be combined with a so called multiscale approach that splits the complex global problem into smaller pieces that can be treated independently. In this work we propose and analyze a strategy how this can be accomplished for the two-phase flow system on the basis of a two scale formulation derived from the underlying fine scale problem.

There is a vast literature on two-phase flow in porous media. Concerning existence and uniqueness of weak solutions we refer to [36, 17, 18]. Various numerical methods have been introduced and analyzed. However, rigorous convergence results are still rare. Convergence of a ‘mixed finite element - finite volume method’ was shown in [10], a-priori error estimates for a ‘mixed finite element - Galerkin finite element method’ were derived in [19], convergence of a finite volume scheme based on the Kirchhoff transformed system was obtained in [35] and convergence of a finite volume phase-by-phase upstream weighting approach was studied in [25]. For a-priori error estimates for a hp discontinuous Galerkin method we refer to [24]. Furthermore, rigorous a-posteriori error estimation for finite volume discretizations of incompressible two-phase flow equations were recently derived in [15].

In the case of rapidly oscillating coefficient functions as considered in this contribution, the classical methods (such as FVM, FEM, hp-FEM, DG-FEM or mixed FEM) are often too expensive and
should be combined with a multiscale approach. There are typically two ways. One way is that the effective (or upscaled) macroscopic features of the coefficients are determined in a preprocess to apply the classical scheme to the new 'effective equation'. The other way is that the multiscale features are used to assemble a suitable set of 'multiscale basis functions' that replaces the original set of basis functions in a classical scheme. There are several contributions proposing multiscale methods for the two-phase flow system. However, each of these methods is formulated under the assumption that the capillary pressure can be neglected and consequently that the pressures of the two phases are identical. We will not make this assumption. Except for the recent work [34], the following multiscale approaches were not investigated analytically. A local-global upscaling for two-phase flow was stated in [16] and an adaptive variational multiscale method (VMM) for multiphase flow was proposed in [69]. Furthermore, there are several methods that fit into the framework of the Multiscale Finite Element Method (MsFEM) proposed in [92] or its modified version, the Mixed Multiscale Finite Element Method, proposed in [20]. There is a hierarchical multiscale method for two-phase flow based upon mixed finite elements [2], an adaptive multiscale method also based on mixed finite elements [11] and a modified multiscale method based on a finite volume scheme on the coarse scale [23]. A Galerkin- and a mixed multiscale finite element method are analyzed in [34]. Besides the assumptions that gravity and capillarity can be ignored, the method and its analysis also rely on the assumption that the two-phase flow pressure \( p^{TP} \) is (up to a small perturbation) a smooth deformation \( G \) of the single-phase pressure \( p^{SP} \), i.e. \( |p^{TP} - G(p^{SP})| \leq \delta \) with \( G \in W^{3,2m-4}(a,b) \), \( m > 4 \) and \( [a,b] \) being a specific bounded interval. This is justified if the two-phase flow features strongly depend on single-phase flow features. Besides numerical multiscale methods, the multiscale features can be also treated analytically using homogenization. Here, the goal is to identify a limit problem that arises when the characteristic length scale of the microscopic oscillations converges to zero. For homogenization results for the immiscible incompressible two-phase flow equations we refer to [13, 14, 44, 31] and the references therein.

In this contribution we use a strategy between numerical methods and homogenization. This strategy gives rise to a large class of multiscale methods for the two-phase flow system, depending on the the classical method that it is combined with. Instead of directly proposing a numerical scheme, we propose a two-scale equation (similar to a two-scale homogenized equation). This two-scale problem should be approximated, not the original equation. This approach is a generalization of the heterogeneous multiscale finite element method (HMM) initially introduced in [22]. The basic idea of the HMM is to perform local fine-scale computations in small cells around quadrature points. The gained information is used to determine reconstructions of the fine scale behavior of the exact solution. The reconstructions are communicated to a coarse scale equation that is comparably cheap to solve. For linear and nonlinear elliptic multiscale problems it was shown that the classical HMM is equivalent to a discretization of a two-scale equation (cf. [11, 20, 30]). This HMM two-scale equation is strongly related to homogenization theory as shown in [26, 27] and typically yields good approximations of the homogenized solution. This interpretation is the basis of a new framework: it is only necessary to state the HMM two-scale problem and keep the final solver for this problem flexible. For an overview on the heterogeneous multiscale method, we refer to [4] and [5]. For a fully discrete error analysis in the elliptic case see e.g. [3, 41] and for a-posteriori error estimates see e.g. [30, 17].

After introducing the weak formulation of the underlying fine scale system in Section 2, we propose a HMM-type two-scale problem for the two-phase flow system in Section 3.1. In Section 3.2 we then show that this formulation is identical to the two-scale homogenized problem if the oscillations of the coefficients are periodic. In Section 3.3 we reformulate the two-scale problem into a more convenient formulation, which also gives rise to existence and uniqueness results. In Section 4 we suggest discretization strategies and present a corresponding a-posteriori error estimate. Proofs are given in Section 5.
2 Weak formulation of the fine scale two-phase flow equations

In this subsection we describe the incompressible two-phase flow equations, writing the weak problem in a global pressure formulation. In the following, let us denote

- \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \) an open, bounded domain with a polyhedral Lipschitz boundary,
- \( \Omega_T := \Omega \times (0, T] \), where \( (0, T] \subset \mathbb{R}^+ \) denotes a time interval,
- \( \Phi(x) \) the porosity in \( \Omega \) and \( K^*(x) \) the absolute permeability in \( \Omega \),
- \( s^w(x,t), s^n(x,t) \) the saturations and \( p^w(x,t), p^n(x,t) \) the pressures of wetting and non-wetting phase respectively,
- \( P_c(s) \) the capillary pressure function (only depending on the saturation),
- \( k^w(s), k^n(s) \) the relative permeabilities (only depending on the saturation),
- \( \mu_w, \mu_n \in \mathbb{R}^+ \) the viscosities of wetting and non-wetting phase,
- \( \rho_w, \rho_n \in \mathbb{R}^+ \) the densities of wetting and non-wetting fluid,
- \( \vec{g} \in \mathbb{R}^d \) the downward pointing gravity vector.

Note that the assumption that \( P_c, k^w \) and \( k^n \) only depend on the saturation but not on the space variable \( x \) is a characterization for the case that the relevant computational domain \( \Omega \) is occupied by only one type of soil. In the case of various types of soil, \( P_c(x,s), k^w(x,s) \) and \( k^n(x,s) \) are piecewise constant with respect to \( x \).

**Definition 2.1** (The original two-phase flow system). With the notations above, the two phase flow system is given by the equations for the mass balance

\[
\Phi(x) \partial_t s^\alpha(x,t) + \nabla \cdot u^\alpha = 0 \quad \text{in} \quad \Omega_T.
\] (1)

These are two equations, one for each subindex \( \alpha \in \{w, n\} \), where \( w \) stands for the wetting phase and \( n \) stands for the non-wetting phase. The flux \( u^\alpha \) is given by the Darcy law

\[
u^\alpha = -K^* \frac{k^\alpha(s^\alpha)}{\mu^\alpha} (\nabla p^\alpha - \rho^\alpha \vec{g}),
\]

again for \( \alpha \in \{w, n\} \). Additionally, we have an algebraic coupling between these equations: the saturations sum up to 1 and the difference between the two pressures is given by the monotonically decreasing capillary pressure function:

\[ s^w + s^n = 1 \quad \text{and} \quad P_c(s^w) = p^n - p^w \quad \text{with} \quad P_c(s) < 0. \]

Let us define the phase mobility functions by

\[ \lambda^w(s) := \frac{k^w(s)}{\mu_w}, \quad \lambda^n(s) := \frac{k^n(1-s)}{\mu_n} \quad \text{and} \quad \lambda(s) := \lambda^w(s) + \lambda^n(s). \]

Typically, \( \lambda^w \) is an increasing function with \( \lambda^w(0) = 0 \) and \( \lambda^n \) is a decreasing function with \( \lambda^n(1) = 0 \).

In this setting we can use the Kirchhoff transformation to rewrite the problem in terms of a global pressure \( P^\epsilon \) (c.f. \([9, 10, 17]\)), where \( P^\epsilon \) can be seen as the pressure of an artificial fluid where the corresponding flow fulfills the Darcy law but with the strictly positive permeability \( \lambda(s^w) \). Defining

\[
S^\epsilon := s^w \quad \text{and} \quad P^\epsilon := p^w + \int_0^{s^w} \frac{\lambda^n(s)}{\lambda(s)} P_c(s) \, ds
\] (2)
we get the desired relation
\[ \lambda(S^e) \nabla P^e = \lambda_w(S^e) \nabla p_w^e + \lambda_n(S^e) \nabla p_n^e. \]

Using the Kirchhoff transform \( \Upsilon : [0, 1] \to \mathbb{R} \) given by
\[ \Upsilon := \Upsilon(S^e) := \int_0^{S^e} \frac{\lambda_w(s)\lambda_n(s)}{\lambda(s)} P_e'(s) \, ds, \]
we obtain the Kirchhoff transformed two-phase flow system: equations:
\[ - \nabla \cdot (K'(\lambda(S^e) \nabla P^e - (\lambda_w(S^e) \rho_w + \lambda_n(S^e) \rho_n) \bar{g})) = 0 \]  \( (3) \)
\[ \Phi^e \partial_t S^e - \nabla \cdot (K' (\lambda_w(S^e) \nabla P^e + \nabla \Upsilon - \lambda_w(S^e) \rho_w \bar{g})) = 0. \]

We note that the Kirchhoff transform \( \Upsilon \) is a strictly increasing function on \([0, 1]\). From now on, we assume inhomogeneous Dirichlet boundary conditions for both saturation and pressure:
\[ S^e = \bar{S} \text{ and } P^e = \bar{P} \text{ on } \partial \Omega \times (0, T], \]
properties of the functions \( \bar{S} \) and \( \bar{P} \) are specified below. An initial condition is given by
\[ S^e(\cdot, 0) = S_0 \text{ in } \Omega. \]

We note that results of this contribution can be easily generalized to boundary conditions where we have a Neumann boundary condition on one part of the boundary and a Dirichlet boundary condition on another. We now state a weak formulation of the Kirchhoff transformed two-phase flow system \([4]\). In the following, we will work with this set of equations.

**Definition 2.2** (Weak formulation of the two-phase flow system). We define the solution space by
\[ \mathcal{E} := \{(S, P) \mid S \in C^0([0, T], L^2(\Omega)), \Phi^e \partial_t S \in L^2((0, T), H^{-1}(\Omega)), \]
\[ \Upsilon(S) - \Upsilon(\bar{S}) \in L^2((0, T), H^1_0(\Omega)), P - \bar{P} \in L^2((0, T), H^1_0(\Omega))\}. \]

We call \((S^e, P^e) \in \mathcal{E}\) a weak solution of the Kirchhoff transformed two-phase flow system \([4]\) to initial condition \(S_0 \in L^\infty(\Omega)\), if
\[ \int_0^T \langle \Phi^e \partial_t S^e(\cdot, t), \Psi(\cdot, t) \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \, dt \]
\[ = - \int_{\Omega_T} K'(\lambda_w(S^e) \nabla P^e + \nabla \Upsilon(S^e) - \lambda_w(S^e) \rho_w \bar{g}) \cdot \nabla \Psi \, dx \]
\[ 0 = \int_{\Omega_T} K'(\lambda_w(S^e) \nabla P^e - (\lambda_w(S^e) \rho_w + \lambda_n(S^e) \rho_n) \bar{g})) \cdot \nabla \Psi \, dx \]  \( (6) \)
for all \( \Psi \in L^2((0, T), H^1_0(\Omega))\) and if there holds
\[ \int_0^T \langle \Phi^e \partial_t S^e(\cdot, t), \Psi(\cdot, t) \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \, dt \]
\[ = - \int_0^T \langle \Phi^e (S^e(\cdot, t) - S_0), \partial_t \Psi(\cdot, t) \rangle_{L^2(\Omega)} \, dt, \]
\[ \text{for all } \Psi \in L^2((0, T), H^1_0(\Omega)) \cap W^{1,1}((0, T), L^1(\Omega)) \text{ with } \Psi(x, T) = 0. \]

The weak formulation was proposed by Chen \([17]\), where the above formulation already incorporates the regularity results that he derived in Section 4 of the mentioned work. Under the following assumptions the problem \((5)-(7)\) has a unique weak solution (c.f. \([15, 17]\)).
Assumption 2.3 (For the existence and uniqueness of a weak solution). We make the following assumptions:

(A1) $\lambda_w, \lambda_n \in C^0[0,1]$, $\lambda_w(0) = \lambda_n(1) = 0$, $\lambda_w(s) > 0$ for $s > 0$, $\lambda_n(s) > 0$ for $s < 1$ and there exist positive constants $c_\lambda$ and $C_\lambda$ so that for all $s \in [0,1]$:

$$c_\lambda \leq \lambda(s) \leq C_\lambda \quad \text{and} \quad \lambda_w(s) \leq C_\lambda.$$

(A2) $K^\epsilon \in [L^\infty(\Omega, \mathbb{R}^{d\times d})]$ with $K^\epsilon(x)$ symmetric for every $x$. There exist positive constants $\alpha, \beta > 0$ such that, for every $x$ and $\epsilon$

$$\alpha|\xi|^2 \leq K^\epsilon \xi \cdot \xi \leq \beta|\xi|^2 \quad \forall \xi \in \mathbb{R}^d.$$

(A3) $\Phi^\epsilon \in L^\infty(\Omega)$ and $0 < \phi^\epsilon \leq \Phi^\epsilon \leq \Phi^* < 1$ for all $\epsilon > 0$ and a.e. in $\Omega$.

(A4) The Kirchhoff transform $\Upsilon$ is Lipschitz-continuous and strictly monotonically increasing on $[0,1]$.

(A5) $\bar{P} \in L^\infty((0,T), H^1(\Omega))$ (which implies the existence of an $L^\infty((0,T), H^1(\Omega))$-extension that we denote again by $\bar{P}$).

(A6) $\bar{S} \in L^\infty(\partial\Omega \times (0,T))$ with $0 \leq \bar{S} \leq 1$. Furthermore, we assume that $\bar{S}$ can be extended to a measurable function on $\Omega_T$ so that:

$$\partial_t \bar{S} \in L^1(\Omega_T), \quad \Upsilon(\bar{S}) \in L^2((0,T), H^1(\Omega)), \quad \bar{S}(\cdot,0) = S_0.$$

(A7) $S_0 \in L^\infty(\Omega)$ with $0 \leq S_0 \leq 1$ a.e. in $\Omega$.

(A8) There exists a positive constants $C_U$ such that

$$(\lambda(s_2) - \lambda(s_1))^2 + \sum_{\alpha = w,n} (\lambda_\alpha(s_2) - \lambda_\alpha(s_1))^2 \leq C_U(s_2 - s_1)(\Upsilon(s_2) - \Upsilon(s_1)) \quad \forall s_1, s_2 \in [0,1].$$

(A9) The solution has the Lipschitz regularity $P^\epsilon \in L^\infty((0,T), W^{1,\infty}(\Omega))$.

Remark 1. Assumptions (A8) and (A9) are only required to obtain uniqueness of the weak solution (c.f. Theorem 3.1. in [17]). Furthermore, condition (A9) could be replaced by assumptions on the domain and the data which guarantee the Lipschitz-continuity of the pressure $p(\cdot, t)$ almost everywhere in $t$ (see [17] [28]). For simplicity and to clarify the presentation we directly work with (A9).

3 The continuous HMM-type two-scale problem

In this section we motivate and state a two-scale problem that should be seen as the continuous limit of a (discrete) heterogeneous multiscale method for the degenerate two phase flow equations. In the case of periodic coefficient functions, we show that the proposed problem is identical to the two-scale homogenized limit problem. In the following, we denote by $Y := (-\frac{1}{2}, \frac{1}{2})^d$ the 0-centered unit cube. For any cube $Y_0 \subset \mathbb{R}^d$ we define $C^1_0(Y_0)$ as the space of functions on $Y_0$, for which the $Y_0$-periodic extension is continuously differentiable. The space of periodic $H^1$-functions with zero average is then obtained as

$$\tilde{H}^1_0(Y_0) := \left\{ \phi \in C^1_0(Y_0) \bigg| \int_{Y_0} v(y) \, dy = 0 \right\}.$$
3.1 Motivation

As initially proved in [11] for linear elliptic homogenization problems with periodic coefficients, the HMM is equivalent to a discretization of the two-scale homogenized equation (c.f. [8, 27]). This view of the HMM was generalized to nonlinear and non-periodic problems in [26, 27]. Here, the method is interpreted at a fully continuous level as a minimization problem for a quasiconvex energy functional with local averaging. With the semi-discrete approximation of this problem we recover the classical HMM approximation. Finally, in [30, 28], it was shown that starting from a fully discrete HMM with oversampling, any sequence of HMM approximations converges strongly in $H^1$ to the solution of the fully continuous problem as proposed in [26, 27]. In the following, we make use of this concept to derive a Heterogeneous Multiscale Method for the Kirchhoff transformed two-phase flow system at the fully continuous level. At the end of this section we show that this also gives us a HMM formulation for the original non-transformed system.

For simplicity, let us assume that the porosity $\Phi$ is ergodic (or even periodic with average $\Phi^0 \in \mathbb{R}_{>0}$). On $K^c$ we do not impose any further assumptions. Let us briefly sketch the idea before going into details: due to [26, 27, 30] the HMM should be equivalent to a fully continuous equation which means that we use the coarse function plus a fine scale correction. Terms that are of order $\epsilon$ should be ignored. In the following, the index ‘c’ is to indicate that a certain quantity is macroscopic (i.e. no fine scale oscillations) and ‘f’ indicates that a certain quantity is microscopic (i.e. rapidly oscillating but with a small $L^2$-norm). We postulate that $P^c$ and $S^c$ split into coarse and fine parts, i.e. $P^c = P^c + P^f; S^c = S^c + S^f$ and $\Upsilon(S^c) = \Upsilon(S^c) + \Upsilon^f$ with $O(P^f) = O(S^f) = O(\epsilon)$, where we are only interested in the coarse contributions $P^c$ and $S^c$. Let us denote the $x$-centered $\kappa$-cube for $\kappa \in \mathbb{R}_{>0}$ by $Y_{x,\kappa} := \{x + \kappa y : y \in (-\frac{1}{2}, \frac{1}{2})^d\}$.

We start with the pressure equation

$$
\int_{\Omega_T} K^c(\lambda(S^c)\nabla P^c - (\lambda_w(S^c)\rho_w + \lambda_n(S^c)\rho_n)\tilde{g}) \cdot \nabla \psi \, dt \, dx = 0,
$$

where we replace any evaluation in $x$ by a $Y_{x,\kappa}$-average, terms in which the oscillations are relevant (e.g. $\nabla P^c$) receive a fine scale corrector and terms of order $\epsilon$ are ignored (e.g. $S^c = S^c + S^f \approx S^c$ if $O(S^f) = O(\epsilon)$). We obtain

$$
\int_{\Omega_T} \int_{Y_{x,\kappa}} [K^c(y)(\lambda(S^c(x,t)) \left( \nabla_x P^c(x,t) + \nabla_y P^f(x,y,t) \right) \cdot \nabla \psi(x,t) \right.
\left. - K^c(y)(\lambda_w(S^c(x,t))\rho_w + \lambda_n(S^c(x,t))\rho_n)\tilde{g}) \right] \cdot \nabla \psi(x,t)] \, dy \, dt \, dx = 0.
$$

Here, the fine scale pressure contribution $P^f$ contains all the fine scale oscillations that cannot be captured by the coarse function $P^c$. We can describe $P^f$ by using a pressure corrector $Q^p$:

$$
P^f(x,y,t) = Q^p(\nabla_x P^c(x,t), S^c(x,t), x)(y).
$$

Therefore, for given $x$, $\nabla_x P^c(x,t)$ and $S^c(x,t)$, the corresponding pressure corrector is defined as the solution of the localized problem in the cell $Y_{x,\kappa}$, with a fine scale test function and with a periodic boundary condition on $\partial Y_{x,\kappa}$ i.e. $P^f(x,y,t) = Q^p(\nabla_x P^c(x,t), S^c(x,t), x) \in H^1_1(Y_{x,\kappa})$ solves the following problem for a.e. $(x,t) \in \Omega_T$ and for all $\psi \in H^1_1(Y_{x,\kappa})$:

$$
\int_{Y_{x,\kappa}} K^c(y)(\lambda(S^c(x,t)) \left( \nabla_x P^c(x,t) + \nabla_y P^f(x,y,t) \right) \cdot \nabla \psi(y) \, dy
\right. \left. - \int_{Y_{x,\kappa}} K^c(y)(\lambda_w(S^c(x,t))\rho_w + \lambda_n(S^c(x,t))\rho_n)\tilde{g}) \right) \cdot \nabla \psi(y) \, dy.
$$
We will treat the equation for the saturation in an analogous way:

\[
\int_0^T \langle \Phi^\epsilon \partial_t S^\epsilon(\cdot, t), \Psi(\cdot, t) \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} dt \\
= - \int_{\Omega_T} K^\epsilon(\lambda_w(S^\epsilon)) \nabla P^\epsilon + \nabla \Upsilon^\epsilon - \lambda_w(S^\epsilon) \rho_w \bar{g} \cdot \nabla \Psi \, dt \, dx.
\]

is, on the coarse scale, approximated by

\[
\int_0^T \langle \Phi^\epsilon \partial_t S^\epsilon(\cdot, t), \Psi(\cdot, t) \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} dt \\
= - \int_{\Omega_T} \int_{Y_{x,n}} [K^\epsilon(y)(\lambda_w(S^\epsilon(x, t))) (\nabla_y P^\epsilon(x, t) + \nabla_y P^f(x, y, t)) \cdot \nabla \Psi(x, t) \\
+ K^\epsilon(y) (\nabla_y \Upsilon(S^\epsilon(x, t)) + \nabla_y \Upsilon^f(x, y, t)) \cdot \nabla \Psi(x, t) \\
- K^\epsilon(y)(\lambda_w(S^\epsilon(x, t)) \rho_w \bar{g} \cdot \nabla \Psi(x, t)] \, dy \, dt \, dx.
\]

On the other hand, the missing local saturation corrector

\[
\Upsilon^f(x, \cdot, t) := Q^\epsilon(\nabla_x P^\epsilon(x, t), S^\epsilon(x, t), \nabla_x \Upsilon(S^\epsilon(x, t)), x) \in \tilde{H}^1_0(Y_{x,n})
\]

is defined as the solution of:

\[
0 = \int_{Y_{x,n}} K^\epsilon(y)(\lambda_w(S^\epsilon(x, t))) (\nabla_x P^\epsilon(x, t) + \nabla_y P^f(x, y, t)) \cdot \nabla_y \psi(y) \, dy \\
+ \int_{Y_{x,n}} K^\epsilon(y) (\nabla_y \Upsilon(S^\epsilon(x, t)) + \nabla_y \Upsilon^f(x, y, t)) \cdot \nabla_y \psi(y) \, dy \\
- \int_{Y_{x,n}} K^\epsilon(y)(\lambda_w(S^\epsilon(x, t)) \rho_w \bar{g} \cdot \nabla_y \psi(y) \, dy
\]

for all \( \psi \in \tilde{H}^1_0(Y_{x,n}) \). Note that the term depending on the time-derivative vanishes in the micro scale equation. The reason is the following: if \( \psi \) is a fine scale test function it is small in comparison to its large gradient \( \nabla \psi \), it can therefore be ignored. For instance, if \( \psi(x, t) = \psi^f(\tilde{x}, t) \) then we have \( \partial_t \psi(x, t) = O(\epsilon) \) but \( \nabla \psi(x, t) = \nabla \psi^f(\tilde{x}, t) = O(1) \). Essentially, these are the equations that characterize the HMM two-scale problem in a fully continuous setting. We summarize it in the following definition:

**Definition 3.1** (HMM two-scale problem). A (fully continuous) pair \((S^\epsilon, P^\epsilon) \in \mathcal{E}\) with \(S^\epsilon(\cdot, 0) = S_0\) is called a solution of the HMM two-scale problem if it solves the equations \(8\) and \(11\), where the pressure and saturation correctors are defined by \(9\) and \(11\).

The fully continuous HMM proposed in Definition 3.1 is only a preliminary formulation. An equivalent but much more convenient formulation is derived in Section 3.3 where we also show existence and uniqueness of a solution of the HMM two-scale problem.

### 3.2 Homogenization

In this section, we restrict ourselves to a periodic setting that is specified by the assumptions below. We now relate the HMM two-scale problem stated in Definition 3.1 to classical homogenization theory. The finding is that, under the assumption of a periodic structure, \(S^\epsilon\) and \(P^\epsilon\) are exactly the homogenized solutions. To prove this result, we make the following assumptions:

**Assumption 3.2** (For homogenization theory). We make the following assumptions:

(A10) We have \(\Phi^\epsilon(x) = \Phi(\tilde{x})\), where \(\Phi \in L^\infty_2(Y)\).
In addition to assumption (A2), we have $K^\epsilon(x) = K(x, \frac{r}{\epsilon})$, where $K \in [L^\infty(\Omega, L^\infty_1(Y))]^{d \times d}$ and such that there exist positive constants $\alpha$ and $\beta$ so that a.e. in $\Omega \times Y$:

$$\alpha |\xi|^2 \leq K(x, y)\xi \cdot \xi \leq \beta |\xi|^2 \quad \forall \xi \in \mathbb{R}^d.$$  

Furthermore, $K(x, \frac{r}{\epsilon})$ is measurable and fulfills

$$\lim_{\epsilon \to 0} \int_{\Omega} K_{ij}(x, \frac{r}{\epsilon})^2 \, dx = \int_{\Omega} \int_{Y} K_{ij}(x, y)^2 \, dy \, dx = 1 \quad 1 \leq i, j \leq d$$

(the last two properties are for instance fulfilled if $K \in [C^0(\Omega, L^\infty_1(Y))]^{d \times d}$).

Let the boundary saturation be independent of time, i.e. $\bar{S}(0(\Omega)) = \bar{S}$ for all $t \in \mathbb{R}$. Then, there exists a subsequence $(S', P') \in E$ to $(S, P)$. Then, there exists a subsequence $(S', P') \in E$ to $(S, P)$ with $S' \rightarrow S$ in $L^2(\Omega_T)$ and $P' \rightarrow P$ in $L^2(\Omega_T)$.

Here, $(S', P') \in E$ is a solution of the fully continuous HMM, described in Definition 3.1 with the following modification: The coefficient $K'(y)$ is replaced by $K(x, \frac{r}{\epsilon})$ in the HMM equations (5)-(7).

The proof of this theorem is provided in Section 5, using two-scale convergence.

**Remark 2.** The hydraulic conductivity used in Theorem 3.3 to define the HMM two-scale approximation slightly differs from the hydraulic conductivity used in Definition 3.1 for the general case. In Theorem 3.3 we use $K(x, \frac{r}{\epsilon})$ as an approximation of $K'(y)$ in the cell $Y_{x, \kappa}$, whereas we use the exact form $K'(y) = K(y, \frac{r}{\epsilon})$ in Definition 3.1. In the case of a Lipschitz-continuous dependence of $K$ in $x$, since the diameter of $Y_{x, \kappa}$ is of order $\epsilon$, the two terms differ only in the order $\epsilon$. In this case, our result remains also valid for the original HMM scheme of Definition 3.1.

### 3.3 Cell problem formulation

The HMM proposed in Definition 3.1 does not yet provide a convenient formulation: the coupling between the coarse scale and the fine scale equations leads to several practical difficulties, computing a corresponding approximation is not a straightforward task. Therefore, we wish to simplify the local problems to get rid of the coupling with $P'$ and $S'$ in the fine scale equations. For this purpose, we define a cell basis:

**Definition 3.4 (Cell basis).** Let $w^i_{\kappa} \in L^2(\Omega, H^1_0(Y))$ solve the problem

$$\int_{Y} K'(x + \kappa y)(e_i + \nabla_y w^i_{\kappa}(x, y)) \cdot \nabla_y \psi(y) \, dy = 0 \tag{12}$$

for all $\psi \in H^1_0(Y)$. The set

$$\{w^i_{\kappa} \in L^2(\Omega, H^1_0(Y)) \mid 1 \leq i \leq d\}$$

is called the cell basis.
The following theorem presents the final formulation, the cell problem formulation of the fully continuous HMM. It guarantees the existence of a unique solution.

**Theorem 3.5** (Fully continuous HMM in cell problem formulation). Under Assumptions (A1)-(A7) there exists a solution \((S^c, P^c) \in E\) of the fully continuous HMM introduced in Definition 2.7. The solution \((S^c, P^c)\) with \(S^c(\cdot, 0) = S_0\) is characterized by the following effective two-phase flow system:

\[
\begin{align*}
\int_0^T (\Phi^0 \partial_t S^c(\cdot, t), \Psi(\cdot, t))_{H^{-1}(\Omega), H^1_0(\Omega)} \, dt \\
= - \int_{\Omega_T} K_{\varepsilon, \kappa}^c (\lambda_w(S^c) \nabla P^c + \nabla \Upsilon(S^c) - \lambda_w(S^c) \rho_w \tilde{g}) \cdot \nabla \Psi \, dx dt \\
0 = \int_{\Omega_T} K_{\varepsilon, \kappa}^c (\lambda(S^c) \nabla P^c - (\lambda_w(S^c) \rho_w + \lambda_n(S^c) \rho_n \tilde{g})) \cdot \nabla \Psi \, dx dt
\end{align*}
\]

for all \(\Psi \in L^2((0, T), H^1_0(\Omega))\). The entries of the effective hydraulic conductivity \(K_{\varepsilon, \kappa}^c\) are defined by

\[
(K_{\varepsilon, \kappa}^c)_{ij}(x) := \int_{Y_0} K^c(x + \kappa y)(e_i + \nabla_y w^i_\kappa(x, y)) \cdot (e_j + \nabla_y w^j_\kappa(x, y)) \, dy,
\]

where \(w^i_\kappa\) is the cell basis from Definition 3.4. If assumptions (A8) and (A9) are fulfilled, the solution \((S^c, P^c)\) is unique.

Note that Theorem 3.5 yields a very pleasant formulation of the HMM problem. It only involves once the computation of the cell basis given by \(\{w^i_\kappa \in L^2(\Omega, H^1_0(\Omega)) \mid 1 \leq i \leq d\}\) (which is of the same cost as for the HMM for standard linear elliptic problems). After that step, one only has to solve the standard two-phase flow system, now with coarse scale coefficients. An additional advantage is that we do not have to deal with the convergence of numerical methods again; existing results can be used because problem (13)-(14) is a standard two-phase problem. The multiscale features are completely hidden in the computation of \(K_{\varepsilon, \kappa}^c\). We straightforwardly obtain convergence of numerical approximations to the coarse scale solution \((S^c, P^c)\). The remaining error between \((S^c, P^c)\) and \((S^c, P^c)\) is a (nonnumerical) modeling error.

In the formulation of the method we propose a periodic boundary condition for the local problems. Other choices are possible, but numerical experiments indicate that periodic conditions are quite flexible and they are typically less affected by resonance errors than e.g. Dirichlet or Neumann boundary conditions (c.f. [6, 20]). To decrease the influence of resonance errors, oversampling techniques can be applied:

**Remark 3** (Oversampling). In general, the periodic boundary condition is a wrong boundary condition for the localized fine scale equations. Since the correct boundary condition is unknown, an oversampling technique is required. This means that the local problems are solved in larger domains, but only the interior information (with a certain distance to the boundary) is used to compute the averages that are communicated to the coarse scale equation. In our example, we only need to change the definition of the hydraulic conductivity \(K_{\varepsilon, \kappa}^c\) by taking the average over a smaller integral. This means that the coarse scale system (13)-(14) remains formally the same, but the conductivity \(K_{\varepsilon, \kappa}^c\) is defined by

\[
(K_{\varepsilon, \kappa}^c)_{ij}(x) := \int_{Y_{\varepsilon, \kappa}} K^c(x + \kappa y)(e_i + \nabla_y w^i_\kappa(x, y)) \cdot (e_j + \nabla_y w^j_\kappa(x, y)) \, dy,
\]

where \(Y_{\varepsilon, \kappa} := (-\frac{\varepsilon}{2\kappa}, \frac{\varepsilon}{2\kappa})^d\) and \(\kappa \geq \kappa_0 \geq \varepsilon\).

A further advantage of the formulation of Theorem 3.3 is that it can be transferred back to the original variables:
Corollary 1 (Fully continuous HMM for the original system). We consider the system \([13]-[14]\). Reverting the Kirchhoff transformation, it is equivalent to seek \(p^0_t, p^0_w, s^0_t\) and \(s^0_w\) with the original boundary conditions and solving

\[
\begin{align*}
\Phi^0 \partial_t s^0_t - \nabla \cdot \left( K^0_{\epsilon, \kappa} \dfrac{k_o(s^0_t)}{\mu_o} (\nabla p^0_t - \rho_o \bar{g}) \right) &= 0 \quad \text{in } \Omega_T, \quad \text{for } \alpha = w, n, \\
0 &= \text{on } \partial \Omega_T.
\end{align*}
\]

These equations are understood in the distributional sense. The coefficient \(K^0_{\epsilon, \kappa}\) is defined in the same way as in Theorem 3.3 or with the oversampling technique described in Remark 4.

Remark 4. We can paraphrase Theorem 3.3 in terms of the cell problem formulation: let \((S^c, P^c) \in E\) denote the solutions of \([13]-[14]\). If assumptions (A1)-(A7) and (A10)-(A13) are fulfilled and if \(\kappa = \kappa_\epsilon\), where \(k_\in N_{n, 0}\) we can replace the definition of \(K^0_{\epsilon, \kappa}\) in Theorem 3.3 by

\[
(K^0_{\epsilon, \kappa})_{ij}(x) := \int_{\gamma} K(x, x' \psi_{\gamma}(x)) \cdot (e_i + \nabla e_i(x, y)) \cdot (e_j + \nabla e_j(x, y)) \, dy. \tag{16}
\]

Then, there exists \((S^c, P^c) \in E\) with \(S^c(\cdot, 0) = S_0\) such that up to a subsequence

\[
S^c \to S^c \quad \text{in } L^2(\Omega_T) \quad \text{and} \quad P^c \to P^c \quad \text{in } L^2(\Omega_T),
\]

and such that \((S^c, P^c)\) is a solution of \([13]-[14]\). Furthermore, \(K^0_{\epsilon, \kappa}\) is independent of \(\epsilon\) and \(\kappa\) and we can denote \(K^0 := K^0_{\epsilon, \kappa}\), where \(K^0\) is the G-limit of the sequence \(K^\epsilon\). Comparing \((15)\) and \((16)\) (which differ, since once we use \(K^\epsilon(x + \kappa y) = K(x + \kappa y, \frac{x + \kappa y}{\epsilon})\) and once we use \(K(x, \frac{x + \kappa y}{\epsilon})\)) we see that for Lipschitz-continuous \(K\) the \(\kappa\) perturbation on the coarse scale vanishes for \(\epsilon \to 0\) in \([13]-[14]\).

4 Numerical treatment of the HMM two-scale problem

In this section, we state two examples of a fully discrete heterogeneous multiscale method (in cell problem formulation) based on the finite elements discretization on the fine scale and a vertex-centered finite volume discretization on the coarse scale. One realization is based on the Kirchhoff transformed system as stated in Corollary 1. Furthermore, we present an a posteriori estimate for the error between homogenized solutions and HMM approximations. The estimate is up to a modeling error of which we show that it is equal to zero if the oscillations of the coefficients are periodic.

4.1 Fully discrete HMM approximation

We introduce the following notations: let \(0 = t^0 < t^1 < \ldots < t^N = T\) denote a partition of the time interval \([0, T]\), with \(I_n := ([n-1], [n])\), \(\Delta t^n := t^n - t^{n-1}\) and \(\tau := \max\{\Delta t^n | 1 \leq n \leq N\}\). For each time step \(t^n\) \((0 \leq n \leq N)\), let \(T^n_H(\Omega)\) denote a regular simplicial partition of \(\Omega\). The corresponding space of piecewise linear functions is given by:

\[
V^n_H(\Omega) := \{ \Psi_H \in C_0^0(\Omega) \} (\Psi_H)_{|T} \in P^1(T) \quad \forall T \in T^n_H(\Omega),
\]

where \(P^1(T)\) denotes the space of affine functions on \(T\). By \(H(\Omega)\) we denote the maximum diameter of an element of \(T^n_H(\Omega)\) and \(N^n_H\) defines the set of nodes in \(T^n_H(\Omega)\). For each of the meshes \(T^n_H(\Omega)\), we let \(D^n_H\) denote the corresponding dual tesselation with control volumes \(D\). The dual tesselation \(D^n_H\) is defined as follows: Let \(x_i \in N^n_H\) be a given node, then the corresponding dual volume \(D_i\) is the star-shaped domain with center \(x_i\), a polyhedral boundary and where the corners are the points of the set

\[
C_i = \{ \ b \in \Omega \mid \exists E \text{ being a codim-k element with } 0 \leq k < d \text{ and s.t. } b \text{ is the barycenter of } E \text{ and } x_i \in E \}.
\]
The union of all these control volumes \( D_i \) defines the dual grid \( \mathcal{D}_H^\prime \). In 2d, \( D_i \) is obtained by connecting the barycenters of all the elements and edges that are adjacent to \( x_i \). For \( D \in \mathcal{D}_H^\prime \), we let \( x_D \) define the center of \( D \) (i.e. \( x_D \) is a node of \( \mathcal{T}_H^\prime(\Omega) \)) and \( H_D \) the diameter of the element \( D \). Furthermore, by \( \mathcal{D}_H^\prime \subset \mathcal{D}_H^\prime \) we define the set that contains the elements of the dual mesh that belong to the interior nodes \( \mathcal{T}_H^\prime(\Omega) \). Finally, we also define

\[
V_{H,\tau}(\Omega_T) := \{ \Psi_{H,\tau} \in C^0(\Omega_T)| \Psi_{H,\tau}(x,\cdot)|_{\Gamma_n} \in \mathbb{P}^1(\Omega) \ \forall x \in \Omega, \ 1 \leq n \leq N; \ \Psi_{H,\tau}(-,t^n) \in V_H^0(\Omega), \ 0 \leq n \leq N \}.
\]

The following assumption only yields a small simplification for formulating the method and deriving a corresponding a posteriori error estimate. We assume that \( \hat{S} \) and \( \hat{P} \) are piecewise linear in order to avoid boundary approximation in the final scheme.

**Assumption 4.1** (For formulating the fully discrete HMM). We make the following assumption:

(A14) the boundary functions \( \hat{S} \) and \( \hat{P} \) are continuous and piecewise linear, i.e. \( \hat{S}, \hat{P} \in V_{H,\tau}(\Omega_T) \).

We can finally define the discrete spaces incorporating the boundary conditions:

\[
\begin{align*}
V_{H,S}^n(\Omega) &:= \{ \Psi_{H} \in V_H^n(\Omega)| \Psi_{H} = \hat{S}(\cdot,t^n) \ \text{on} \ \partial \Omega \}, \\
V_{H,\tau,S}(\Omega_T) &:= \{ \Psi_{H} \in V_{H,\tau}(\Omega_T)| \Psi_{H} = \hat{S} \ \text{on} \ \partial \Omega \times (0, T] \}, \\
V_{H,P}^n(\Omega) &:= \{ \Psi_{H} \in V_H^n(\Omega)| \Psi_{H} = \hat{P}(\cdot,t^n) \ \text{on} \ \partial \Omega \}, \\
V_{H,\tau,P}(\Omega_T) &:= \{ \Psi_{H} \in V_{H,\tau}(\Omega_T)| \Psi_{H} = \hat{P} \ \text{on} \ \partial \Omega \times (0, T] \}.
\end{align*}
\]

In the following, we use the notation \( \Psi_{H,\tau}^n := \Psi_{H,\tau}(\cdot,t^n) \) for \( \Psi_{H,\tau} \in V_{H,\tau}(\Omega_T) \) and by \( S_{H,\tau}^0 \) we denote a suitable approximation of the initial value \( S_0 \). In order to solve local problems (12) numerically, we require a regular periodic triangulation of \( Y = (-\frac{1}{2}, \frac{1}{2})^d \) that is denoted by \( \mathcal{T}_h(Y) \). We extend \( \mathcal{T}_h(Y) \) periodically to a triangulation \( \mathcal{T}_h(\mathbb{R}^d) \) of the whole \( \mathbb{R}^d \). A corresponding discrete space is given by

\[
W_h(Y) := \{ \psi_h \in C^0(\hat{Y}) \cap \overline{H}_2^1(\hat{Y})| (\psi_h)_\mathcal{Y} \in \mathbb{P}^1(\mathcal{Y}) \ \forall \mathcal{Y} \in \mathcal{T}_h(Y) \} \quad (17)
\]

and the set of faces is given by

\[
\Gamma(\mathcal{T}_h(Y)) := \{ E| E = \mathcal{Y}_1 \cap \mathcal{Y}_2 \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d, \ \mathcal{Y}_1, \mathcal{Y}_2 \in \mathcal{T}_h(\mathbb{R}^d) \ \text{and codim}(E) = 1 \}.
\]

Note that we might identify \( \mathcal{T}_h(Y) \) with a triangulation of the \( d \)-dimensional torus \( \mathbb{T}^d \). With this interpretation, we see that \( \Gamma(\mathcal{T}_h(Y)) \) only contains inner faces. Boundary faces do not exist. A jump over a face \( E \subset \partial \mathcal{Y} \) must be therefore seen as a jump over a corresponding opposite face.

We also denote \( h_E := \text{diam}(\mathcal{Y}_1 \cup \mathcal{Y}_2) \) where \( E = \mathcal{Y}_1 \cap \mathcal{Y}_2 \in \Gamma(\mathcal{T}_h(Y)) \).

For the rest of the paper, we denote for simplification

\[
K_{\epsilon, \kappa}(x, y) := K'(x + \kappa y).
\]

We first propose a finite element discretization to assemble the local cell basis that defines the discrete effective hydraulic conductivity that is communicated to the coarse scale equation:

**Definition 4.2** (Discrete effective hydraulic conductivity). Let \( \kappa \geq \epsilon > 0 \) and let \( x_D \in \mathcal{N}_H \) denote a center of an element \( D \) of the dual grid \( \mathcal{D}_H^\prime \). By \( K_{\epsilon, \kappa}(x_D, \cdot) \) we denote an arbitrary piecewise constant approximation of \( K_{\epsilon, \kappa}(x_D, \cdot) \) (see (13)) i.e. we assume for all \( D \in \mathcal{D}_H^\prime \)

\[
K_{\epsilon, \kappa}(x_D, \cdot)|\mathcal{Y} \in \mathbb{P}^0(\mathcal{Y}) \ \forall \mathcal{Y} \in \mathcal{T}_h(Y).
\]
For instance, we might use the local mean value on every cell $\mathcal{Y}$. Then, we denote $w^i_{\kappa,h}(x,\cdot) \in W_h(\mathcal{Y})$ the solution of
\[
\int_{\mathcal{Y}} K_{\kappa,h}(x,\cdot)(e_i + \nabla_y w^i_{\kappa,h}(x,\cdot)) \cdot \nabla_y \psi_h(y) dy = 0 \quad \forall \psi_h \in W_h(\mathcal{Y}).
\]
The entries of the (piecewise constant) discrete effective hydraulic conductivity $K^0_{\kappa,h}$ are now given by
\[
(K_{\kappa,h})_{ij} := \int_{\mathcal{Y}} K_{\kappa,h}(x,\cdot)(e_i + \nabla_y w^i_{\kappa,h}(x,\cdot)) \cdot (e_j + \nabla_y w^j_{\kappa,h}(x,\cdot)) dy.
\]
Note that the above definition requires a recomputation of the cell basis $(w^i_{\kappa,h}(x,\cdot))_i$ for each time step if the coarse grid $T^0_H(\mathcal{Y})$ changes in time or if it is adaptively refined. Alternatively, in an offline phase, we might compute $w^i_{\kappa,h}(x,\cdot) \in W_h(\mathcal{Y})$ for a large set of $x$-samples $\mathcal{S} \subset \mathcal{Y}$ that can be reused for any time step. Defining
\[
(K^0_{\kappa,h})_{ij}(x) := \int_{\mathcal{Y}} K_{\kappa,h}(x,\cdot)(e_i + \nabla_y w^i_{\kappa,h}(x,\cdot)) \cdot (e_j + \nabla_y w^j_{\kappa,h}(x,\cdot)) dy
\]
for all $x \in \mathcal{S}$, we can also use a continuous interpolation of the sample values.

With the previously defined dual meshes $\mathcal{D}^0_H$, we can use a vertex centered finite volume discretization on the coarse scale (c.f. [15]). The first version of a fully discrete multiscale method is based on the Kirchhoff transformed equation given by Theorem 3.5.

**Definition 4.3** (Fully discrete HMM for the Kirchhoff transformed system). Let assumption (A14) be fulfilled. We call $(S_H, P_H) \in V_{H,T,S}(\Omega T) \times V_{H,T,P}(\Omega T)$ a fully discrete HMM approximation for the Kirchhoff transformed system if there holds
\[
\int_{\mathcal{D}} \Phi^0 \left( \frac{S^n_{H,T} - S^{n-1}_{H,T}}{\triangle t^n} \right) dx + \int_{\partial \mathcal{D}} K^0_{\kappa,h}((\lambda w(S^n_{H,T}) \nabla P^n_{H,T} + \nabla \Upsilon(S^n_{H,T}) - \lambda w(S^n_{H,T}) \rho_w \bar{g}) : \nu_D) d\sigma(x) = 0
\]
and
\[
- \int_{\partial \mathcal{D}} K^0_{\kappa,h}(\lambda(S^n_{H,T}) \nabla P^n_{H,T} - (\lambda w(S^n_{H,T}) \rho_w + \lambda_n(S^n_{H,T}) \rho_n) \bar{g}) : \nu_D d\sigma(x) = 0
\]
for all $1 \leq n \leq N$ and all $\mathcal{D} \in \mathcal{D}_H^{m,n} \subset \mathcal{D}^0_H$. Here, $P^n_{H,T}$ is the solution of (20) for given $S^n_{H,T}$ (which is an approximation of the initial value $S_0$).

Equations (19), (20) form a nonlinear system that might be solved using Newton’s method or a fixed point linearization as proposed e.g. in [15]. Alternatively to the formulation proposed in Definition 4.3 we might also construct a scheme with unknown $\Upsilon_{H,T}$ which is an approximation of the Kirchhoff transformed saturation $\Upsilon(S)$. In this case, we only need to replace $S_{H,T}$ by $\Upsilon^{-1}(\Upsilon_{H,T})$ in (19)–(20). The advantage is that we seek an approximation of the unknown $\Upsilon(S)$ which has typically more regularity than the non-wetting saturation $S^0$. Again, we refer to [15] for the formulation of such a scheme.

The next method is based on the fully continuous HMM for the original system as stated in Conclusion 4. The mass balance is solved for both phases and an upwinding term is used for stabilization (c.f. [15, 33]).
Definition 4.4 (Fully discrete HMM for the original system). Let assumption (A14) be fulfilled. According to (2), we define the function $P$ that describes the global pressure relation by

$$P(p, s) := p + \int_0^s \frac{\lambda_n(\theta)}{\lambda(\theta)} P_c'(\theta) \, d\theta. \quad (21)$$

Now, we seek $(s_{H,T,w}, p_{H,T}) \in V_{H,T,S}(\Omega_T) \times V_{H,T}(\Omega_T)$ (approximating the functions $s_w^0$ and $p_w^0$ from Conclusion 1) with the properties $P(p_{H,T,w}, s_{H,T,w}) = \bar{P}$ on $\partial\Omega \times (0, T]$ and $s_{H,T,w}^0 = s_{0,T}$, solving

$$\int_D \phi^0 \left( \frac{s_{H,T,w}^n - s_{H,T,w}^{n-1}}{\Delta t^n} \right) \, dx \quad (22)$$

$$= \int_{\partial D} \left[ K_{c,k}^n \lambda_n(s_{H,T,w}^n) \left( \nabla p_{H,T,w} - \rho_w \bar{g} \right) \right] \upw \cdot \nu_D \, d\sigma(x) \quad \text{and}$$

$$- \int_D \phi^0 \left( \frac{s_{H,T,w}^n - s_{H,T,w}^{n-1}}{\Delta t^n} \right) \, dx \quad (23)$$

$$= \int_{\partial D} \left[ K_{c,k}^n \lambda_n(s_{H,T,w}^n) \left( \nabla p_{H,T,w} + \nabla P_c(s_{H,T,w}^n) - \rho_n \bar{g} \right) \right] \upw \cdot \nu_D \, d\sigma(x)$$

for $1 \leq n \leq N$ and for all $D \in D^{n, t}$, where $[\cdot] \upw$ denotes an upwind choice of the evaluation on the element faces. The non-wetting saturation and the non-wetting pressure are obtained using

$$s_{H,T,w} := 1 - s_{H,T,w} \quad \text{and} \quad p_{H,T} := p_{H,T,w} + P_c(s_{H,T,w}).$$

4.2 A posteriori error estimates

In this section, we derive an a posteriori estimate for the error between an arbitrary fully discrete HMM approximation and a homogenized solution that corresponds with problem (5). As in [15], we start with defining a nonwetting phase flux reconstruction $\bar{u}_{s,H}$ and a total total flux reconstruction $\bar{u}_{p,H}$. We use the subindex $s$ for the flux for the saturation equation and the subindex $p$ for the flux for the pressure equation. The next assumption guarantees existence of $\bar{u}_{s,H}$ and $\bar{u}_{p,H}$. 

Assumption 4.5 (For a-posteriori error estimation). We make the following assumption:

(A15) there exist locally conservative flux reconstructions, i.e. there exist two vector fields $\bar{u}_{s,H}$ and $\bar{u}_{p,H}$ that are piecewise constant in time and such that

$$\bar{u}_{s,H} := (\bar{u}_{s,H}) I_n, \quad \bar{u}_{p,H} := (\bar{u}_{p,H}) I_n \in H(\text{div}, \Omega) \quad \forall 1 \leq n \leq N,$$

and

$$\int_D \frac{s_{H,w}^n - s_{H,w}^{n-1}}{\Delta t^n} + \nabla \cdot \bar{u}_{w,H}^n = 0 \quad \forall D \in D^{n, t}, \quad \forall n \in \{1, ..., N\},$$

$$\int_D \nabla \cdot \bar{u}_{p,H}^n = 0 \quad \forall D \in D^{n, t}, \quad \forall n \in \{1, ..., N\}.$$

Even though the existence of $\bar{u}_{w,H}^n$ is clear in any case ($\bar{u}_{w,H}^n = 0$ obviously fulfills the property), 0 is not the flux reconstruction that we are interested in. We want $\bar{u}_{w,H}^n$ and $\bar{u}_{w,H}^n$ to be discrete approximations of the exact fluxes that occur in the fully continuous HMM, i.e. $\bar{u}_{s,H}$ and $\bar{u}_{p,H}$ are to approximate $\bar{u}_w^0$ and $\bar{u}_p^0$ with

$$\bar{u}_w^0 := - K_{c,k}^w (\lambda_w(S) \nabla P_c + \nabla Y(S) - \lambda_w(S) \rho_w \bar{g}) \quad \text{and}$$

$$\bar{u}_p^0 := - K_{c,k}^p (\lambda(S) \nabla P_c - (\lambda_w(S) \rho_w + \lambda_n(S) \rho_n) \bar{g}).$$
Remark 5 (Reconstruction of the fluxes). A procedure to obtain flux reconstructions for a system of type \((15)-(20)\) as well as for a system of type \((22)-(23)\) in a suitable Raviart-Thomas-Nédélec space is described in \([15]\). In particular, assumption (A15) is fulfilled for both schemes.

Definition 4.6 (Error indicators). In the following estimators, \(\alpha_{\mathcal{D},\kappa}\) denotes the smallest and \(\beta_{\mathcal{D},\kappa}\) the largest eigenvalue of \(K'(x_{\mathcal{D}} + \kappa y)\) in \(Y\). \(\alpha\) is the smallest eigenvalue of \(K'\). By \(C_p(\mathcal{D})\) we denote constants that arise from a Poincaré inequality. For more details on these constants we refer to Lemma 5.4 below and the proof of Theorem 4.12. Let us define the flow functions \(\tilde{V}_{H\tau}^p(t)\) and \(\tilde{V}_{H\tau}^o(t)\) by
\[
\tilde{V}_{H\tau}^p(t) := (\lambda_w(S_{H\tau}(\cdot,t))\nabla P_{H\tau}(\cdot,t) + \nabla Y(S_{H\tau}(\cdot,t)) - \lambda_w(S_{H\tau}(\cdot,t))p_w\tilde{g}),
\]
\[
\tilde{V}_{H\tau}^o(t) := \lambda(S_{H\tau}(\cdot,t))\nabla P_{H\tau}(\cdot,t) - (\lambda_w(S_{H\tau}(\cdot,t))p_w + \lambda_n(S_{H\tau}(\cdot,t))p_n)\tilde{g}.
\]

Using these flow functions, we define the coarse scale residual estimators by
\[
\eta_{C,R,s,D}^n := C_p(\mathcal{D})H_\mathcal{D}\alpha_{\mathcal{D},\kappa}^{-\frac{1}{2}} \| \partial_t S_{H\tau} + \nabla \cdot \tilde{u}_{s,H} \|_{L^2(\mathcal{D})},
\]
\[
\eta_{C,R,p,D}^n := C_p(\mathcal{D})H_\mathcal{D}\alpha_{\mathcal{D},\kappa}^{-\frac{1}{2}} \| \nabla \cdot \tilde{u}_{p,H} \|_{L^2(\mathcal{D})},
\]
and the fine scale residual estimators by
\[
\eta_{C,F,s,D}^n(t) := \alpha_{\mathcal{D},\kappa}^{-\frac{1}{2}} \| \frac{\beta_{\mathcal{D},\kappa}}{\alpha_{\mathcal{D},\kappa}} \tilde{V}_{H\tau}^s(t) \|_{L^2(\mathcal{D})} m(\mathcal{D},\epsilon,\kappa,h),
\]
\[
\eta_{C,F,p,D}^n(t) := \alpha_{\mathcal{D},\kappa}^{-\frac{1}{2}} \| \frac{\beta_{\mathcal{D},\kappa}}{\alpha_{\mathcal{D},\kappa}} \tilde{V}_{H\tau}^p(t) \|_{L^2(\mathcal{D})} m(\mathcal{D},\epsilon,\kappa,h),
\]
where
\[
m(\mathcal{D},\epsilon,\kappa,h) := \left( \sum_{E \in \Gamma(T_h(Y))} h_E \| [K_{\epsilon,\kappa,h}(x_{\mathcal{D}},\cdot)(\epsilon_i + \nabla_y w_{i,k_h}(x_{\mathcal{D}},\cdot))]|_E \|_{L^2(E)}^2 \right)^{\frac{1}{2}}.
\]

The diffusive flux estimators are given by
\[
\eta_{D,F,s,D}^n(t) := \alpha_{\mathcal{D},\kappa}^{-\frac{1}{2}} \| K_{\epsilon,\kappa,h}^0 \tilde{V}_{H\tau}^s(t) + \tilde{u}_{s,H}^n \|_{L^2(\mathcal{D})},
\]
\[
\eta_{D,F,p,D}^n(t) := \alpha_{\mathcal{D},\kappa}^{-\frac{1}{2}} \| K_{\epsilon,\kappa,h}^0 \tilde{V}_{H\tau}^p(t) + \tilde{u}_{p,H}^n \|_{L^2(\mathcal{D})}
\]
and the approximation error estimators by
\[
\eta_{A{\text{PP}},s,D}^n(t) := \alpha_{\mathcal{D},\kappa}^{-\frac{1}{2}} \frac{\beta_{\mathcal{D},\kappa}}{\alpha_{\mathcal{D},\kappa}} \| \tilde{V}_{H\tau}^s(t) \|_{L^2(\mathcal{D})}
\]
\[
\cdot \sup_{x \in \mathcal{D}} \| [K_{\epsilon,\kappa,h}(x_{\mathcal{D}},\cdot) - K_{\epsilon,\kappa,h}(x_{\mathcal{D}},\cdot)](\epsilon_i + \nabla_y w_{i,k_h}(x_{\mathcal{D}},\cdot)) \|_{L^2(Y)}
\]
\[
\eta_{A{\text{PP}},p,D}^n(t) := \alpha_{\mathcal{D},\kappa}^{-\frac{1}{2}} \frac{\beta_{\mathcal{D},\kappa}}{\alpha_{\mathcal{D},\kappa}} \| \tilde{V}_{H\tau}^p(t) \|_{L^2(\mathcal{D})}
\]
\[
\cdot \sup_{x \in \mathcal{D}} \| [K_{\epsilon,\kappa,h}(x_{\mathcal{D}},\cdot) - K_{\epsilon,\kappa,h}(x_{\mathcal{D}},\cdot)](\epsilon_i + \nabla_y w_{i,k_h}(x_{\mathcal{D}},\cdot)) \|_{L^2(Y)}.
\]

Let \(K^0\) denote some homogenized matrix that we specify in the Theorems 4.11 and 4.12 below. In the following, by \(\| \cdot \|_{E(\Omega)}\) and \(\| \cdot \|_{E(\Omega_T)}\) we define energy norms on \(H_0^1(\Omega)\) and respectively on \(L^2((0,T),H_0^1(\Omega))\) by:
\[
\| \Psi \|_{E(\Omega)} := \left( \int_{\Omega} K^0(x) \nabla \Psi(x) \cdot \nabla \Psi(x) \, dx \right)^{\frac{1}{2}} \quad \text{and}
\]
\[
\| \Psi \|_{E(\Omega_T)} := \left( \int_0^T \int_{\Omega} K^0(x) \nabla \Psi(x,t) \cdot \nabla \Psi(x,t) \, dx \, dt \right)^{\frac{1}{2}}.
\]
Furthermore, for corresponding functionals $F$, we denote the induced norms on the associated dual spaces by:

$$
\|\|F\|\|_\Omega := \sup_{\Psi \in H_0^1(\Omega) \setminus \{0\}} \frac{|F(\Psi)|}{\|\Psi\|_{E(\Omega)}}\quad \text{and}
$$

$$
\|\|F\|\|_{\Omega_T} := \sup_{\Psi \in L^2((0,T), H_0^1(\Omega)) \setminus \{0\}} \frac{|F(\Psi)|}{\|\Psi\|_{E(\Omega_T)}}.
$$

We are finally prepared to formulate the final a posteriori error estimate.

**Theorem 4.7** (A-posteriori error estimate in the periodic case). Let assumptions (A1)-(A14) be fulfilled. In particular, we are in the case of periodic coefficient functions. Let $K^0$ denote the homogenized matrix stated in Theorem 3.3 (i.e. $K^0$ denotes the $G$-limit of $K^\epsilon$), $P^0$ the weak $L^2(\Omega_T)$-limit of $P^\epsilon$, $S^0$ the strong $L^2(\Omega_T)$-limit of $S^\epsilon$ and $\Phi^0 = 1$ (otherwise we rescale $K^0$ by dividing it by $\Phi^0 > 0$). Then, for an arbitrary approximation $(S_{H^\epsilon}, P_{H^\epsilon}) \in V_{H^\epsilon}, P(\Omega_T) \times V_{H^\epsilon}, P(\Omega_T)$ of $(S^\epsilon, P^\epsilon)$ fulfilling assumption (A15) there holds the following a posteriori error estimate

$$
\|\|S_{H^\epsilon} - S^0\|\|_{L^2(\Omega_T)}^2 + \|P_{H^\epsilon} - P^0\|_{E(\Omega_T)}^2 + \|\Upsilon(S_{H^\epsilon}) - \Upsilon(S^0)\|_{L^2(\Omega_T)}^2 
\lesssim \|\|S_{H^\epsilon} - S^0\|\|_{L^2(\Omega_T)} + \frac{1}{\kappa} \sum_{a=\text{const.}} \sum_{1 \leq D \leq D^*} \int_{J_n} \left( \eta_{R,a,D}^0(\Omega) + (\eta_{C,a,D}^0 + \eta_{DF,a,D}^0 + \eta_{APP,a,D}^0)(\Omega) \right)^2,
$$

with a fully computable right hand side. Since $T^{-1} \in C^{0,0}$ by assumption (A13), we can replace $\|\Upsilon(S_{H^\epsilon}) - \Upsilon(S^0)\|_{L^2(\Omega_T)}$ by $\|S_{H^\epsilon} - S^0\|_{L^{1+\gamma}(\Omega_T)}^{1+\gamma}$.

**Theorem 4.7** yields (localized) error indicators for each of the error contributions. We have indicators $\eta_{R,a,D}^0$ for the coarse scale residual that depend on the diffusive flux reconstructions. The accuracy of the flux reconstruction itself is evaluated using the indicators $\eta_{C,a,D}^0$. Residual error indicators for the solutions of the cell problems are given by $\eta_{DF,a,D}^0$. Finally, the error for replacing $K^\epsilon$ by a piecewise constant approximation is evaluated by the indicators given by $\eta_{APP,a,D}^0$.

**Conclusion 4.8.** Let assumptions (A1)-(A14) be fulfilled and let $(S_{H^\epsilon}, P_{H^\epsilon}) \in V_{H^\epsilon}, S(\Omega_T) \times V_{H^\epsilon}, P(\Omega_T)$ denote the solution of the fully discrete heterogeneous multiscale method proposed in Definition 4.3 or let $(S_{H^\epsilon,\omega}, P_{H^\epsilon,\omega}) \in V_{H^\epsilon, S(\Omega_T) \times V_{H^\epsilon, P(\Omega_T)}}$ denote the solution of the method stated in Definition 4.3. Then, $(S_{H^\epsilon}, P_{H^\epsilon})$ and respectively $(S_{H^\epsilon,\omega}, P_{H^\epsilon,\omega})$ fulfill the a posteriori error estimate (24).

**Remark 6** (Efficiency). Efficiency of the total estimated error (i.e. the term on the right hand side of (24)) can be shown using the techniques from [15], where a corresponding result is derived for a method of the same structure as the scheme in Definition 4.3. In particular, the sum of the estimators forms a lower bound for the residuals in the dual norm.

Let us define the notion of $G$-convergence (c.f. [35]) in order to properly state the a-posteriori error estimate for the case of a general homogenization setting.

**Definition 4.9** ($G$-convergence). Let $\Omega \subset \mathbb{R}^d$ denote a bounded domain and let $(A^\epsilon)_{\epsilon > 0} \subset [L^\infty(\Omega)]^{d \times d}$ denote a sequence of symmetric matrices that are uniformly bounded and coercive, i.e. there exist $a_0, a_1 \in \mathbb{R}_{>0}$ such that for a.e. $x \in \Omega$:

$$
a_0 |\xi|^2 \leq A^\epsilon(x) \xi \cdot \xi \leq a_1 |\xi|^2 \quad \forall \xi \in \mathbb{R}^d.
$$

Then, we call $A^\epsilon$ $G$-convergent to $A^0 \in [L^\infty(\Omega)]^{d \times d}$ if for any $f \in H^{-1}(\Omega)$ the solutions $u^\epsilon \in H_0^1(\Omega)$ of

$$
\int_{\Omega} A^\epsilon \nabla u^\epsilon \cdot \nabla \Psi = f(\Psi) \quad \forall \Psi \in H_0^1(\Omega)
$$

that
satisfy the relations
\[ u^e \to u^0 \text{ in } H^1_0(\Omega) \text{ and } A^e \nabla u^e \to A^0 \nabla u^0 \text{ in } [L^2(\Omega)]^d, \]
where \( u^0 \in H^1_0(\Omega) \) is the solution of
\[ \int_\Omega A^0 \nabla u^0 \cdot \nabla \Psi = f(\Psi) \quad \forall \Psi \in H^1_0(\Omega). \]

We note that the \( G \)-limit \( A^0 \) is bounded and coercive with the same constants \( a_0 \) and \( a_1 \) as \( K^e \).

**Assumption 4.10** (General homogenization setting). We make the following assumption:

(A16) \( K^e \) is \( G \)-convergent to \( K^0 \), \( \Psi^e \) is weak-* convergent to 1 in \( L^\infty(\Omega) \), \( S^e \) converges to \( S^0 \) strongly in \( L^2(\Omega_T) \), \( P^e \) converges to \( P^0 \) weakly in \( L^2(\Omega_T) \) and \( (S^0, P^0) \in E \) with \( S^0(\cdot, 0) = S_0 \) is the unique solution of

\[
\begin{align*}
\int_0^T & \left< \partial_t S^0(\cdot, t), \Psi(\cdot, t) \right>_{H^{-1}(\Omega), H^1_0(\Omega)} \, dt \\
= & - \int_{\Omega_T} K^0(\lambda_w(S^0) \nabla P^0 + \nabla \lambda(S^0) - \lambda_w(S^0) \rho_w \bar{g}) \cdot \nabla \Psi \, dx \\
0 = & \int_{\Omega_T} K^0(\lambda(S^0) \nabla P^0 - (\lambda_w(S^0) \rho_w + \lambda_n(S^0) \rho_n) \bar{g}) \cdot \nabla \Psi \, dx
\end{align*}
\]

for all \( \Psi \in L^2((0, T), H^1_0(\Omega)) \).

We require an additional indicator for the modeling error that describes the error between the real homogenized matrix \( K^0 \) and the used effective hydraulic conductivity \( K^0_{p,n} \) from the fully continuous HMM formulation.

**Definition 4.11** (Modeling error). Let \( \tilde{V}^s_{H_T} \) and \( \tilde{V}^p_{H_T} \) be given as in Definition 4.6. We define the **modeling error estimators** by

\[
\eta_{\text{MOD}, s, D}(t) := \alpha^{-\frac{1}{2}} \| (K^0 - K^0_{p,n}) \tilde{V}^s_{H_T}(t) \|_{L^2(D)}
\]
\[
\eta_{\text{MOD}, p, D}(t) := \alpha^{-\frac{1}{2}} \| (K^0 - K^0_{p,n}) \tilde{V}^p_{H_T}(t) \|_{L^2(D)}.
\]

Note that the modeling error estimators can only be computed in special cases, since the homogenized matrix \( K^0 \) is typically unknown. In the case of a periodic structure as in Theorem 4.12, the modeling error is equal to zero. In other cases we need explicit knowledge about the type of the heterogeneities of \( K^e \) in order to estimate the modeling error further.

**Theorem 4.12** (A posteriori error estimate in the non-periodic case). Let \( (S_{H_T}, P_{H_T}) \in V_{H_T,S}(\Omega_T) \times V_{H_T,P}(\Omega_T) \) denote an arbitrary approximation of \( (S^e, P^e) \) and let assumptions (A1)-(A9) and (A14)-(A16) be fulfilled. The energy norms \( \| \cdot \|_{E(\Omega)} \) and \( \| \cdot \|_{E(\Omega_T)} \) are defined with the homogenized matrix \( K^0 \). Then there holds the following a posteriori error estimate

\[
\begin{align*}
\| |S_{H_T} - S^0||^2_{L^2(\Omega_T)} & + \| P_{H_T} - P^0 \|_{E(\Omega_T)}^2 + \| Y(S_{H_T}) - Y(S^0) \|_{L^2(\Omega_T)}^2 \\
& \lesssim \| |S_{H_T}(\cdot, 0) - S_0||^2_{L^2(\Omega_T)} + \sum_{\alpha = s,p} \sum_{n = 1}^N \sum_{D \in D_T} \int_{I_n} \eta_{\text{MOD}, \alpha, D}(t)^2 \\
& \quad + \sum_{\alpha = s,p} \sum_{n = 1}^N \sum_{D \in D_T} \left( \eta_{\text{GR}, \alpha, D}^n + (\eta_{\text{CF}, \alpha, D}^n + \eta_{\text{DF}, \alpha, D}^n + \eta_{\text{APP}, \alpha, D}^n)(t) \right)^2.
\end{align*}
\]
If we want to apply one of the proposed heterogeneous multiscale methods to a homogenization problem with unknown micro structure, we can still use (27) to evaluate the discretization error and to construct adaptive mesh refinement strategies. The (possibly not computable) modeling error contribution is a typical remainder that is due to the chosen method and cannot be reduced by changing the discretization (c.f. [6, 7, 30]). It is a bounded term that typically converges to zero for \( \epsilon \) converging to zero independently of \( H \) and \( \tau \) (c.f. [26, 27]). In particular, if we replace \( S^0 \) and \( P^0 \) by \( S^c \) and \( P^c \) on the left side of (27), the modeling error contributions on the right side vanish and we get a fully computable a-posteriori error estimator for the discretization error.

5 Proofs of the main results

In this section we are concerned with the proofs of Theorems 3.3, 3.5 and 4.12. We do not prove Theorem 4.7 separately, since it is an easy conclusion from Theorem 3.3 and Theorem 4.12.

5.1 Proof of Theorem 3.3 (Convergence in the periodic setting)

The first section is devoted to the homogenization of the Kirchhoff transformed two-phase flow equations under the assumption of periodicity.

The strategy is to derive the homogenized problem associated with the original weak problem under the assumptions (A1)-(A7) and (A10)-(A13). Then, we verify that the homogenized system is identical to the two-scale HMM given by (8)-(11).

In the following, we make use of the tools derived in [9] for immiscible compressible two-phase flow in porous media. In particular, we need the following compactness Lemma obtained in [9]:

**Lemma 5.1.** Let \( \Phi \in L^\infty(Y) \) with \( \phi^* \leq \Phi(y) \leq \Phi^* \) for a.e. \( y \in Y \) and \( \phi^*, \Phi^* \in (0, 1) \). Let \((v^\epsilon)_{\epsilon>0} \subset L^2(\Omega_T)\) fulfill the following properties:

1. \( 0 \leq v^\epsilon \leq C \) a.e. in \( \Omega_T \) and for all \( \epsilon \).

2. There exists a function \( \varpi \) with \( \varpi(\xi) \to 0 \) for \( \xi \to 0 \) such that the following inequality holds uniformly in \( \epsilon \):

\[
\int_{\Omega_T} |v^\epsilon(x + \Delta x, t) - v^\epsilon(x, t)|^2 \, dx \, dt \leq C \varpi(|\Delta x|).
\]

3. The functions \( v^\epsilon \) are such that for all \( \epsilon \):

\[
\|\Phi(\cdot)\partial_t v^\epsilon\|_{L^2((0,T),H^{-1}(\Omega))} \leq C.
\]

Then \((v^\epsilon)_{\epsilon>0}\) is a precompact set in \( L^2(\Omega_T) \).

Again, following the analysis presented in [9] for compressible two phase flow, the following estimates hold true and will allow to extract convergent subsequences of \( P^\epsilon, Y^\epsilon \) and \( S^\epsilon \):

**Lemma 5.2.** Let assumptions (A1)-(A13) be fulfilled and let \((S^0, P^0) \in \mathcal{E}\) with \( S^0(\cdot, 0) = S_0 \) denote the sequence of solutions of the Kirchhoff transformed two-phase flow system given by [26] [7]. Then the following a priori estimates hold:

\[
\|\nabla P^\epsilon\|_{L^2(\Omega_T)} \leq C,
\]

\[
\|\nabla Y^\epsilon\|_{L^2(\Omega_T)} \leq C,
\]

\[
\|\Phi^\epsilon \partial_t S^\epsilon\|_{L^2((0,T),H^{-1}(\Omega))} \leq C,
\]

\[
\int_{\Omega_T} |S^\epsilon(x + \Delta x, t) - S^\epsilon(x, t)|^2 \, dx \, dt \leq C \varpi(|\Delta x|),
\]

with \( \epsilon \)-independent constants \( C \).
We are now prepared to prove the main result of this subsection:

**Proof of Theorem 3.3** In the following, we make use of the concept of two-scale convergence (c.f. [8] or [37] for an overview on this topic). A sequence \( u^\epsilon \in L^2(\Omega) \) is called two-scale convergent to a limit \( u_0 \in L^2(\Omega \times Y) \) if the following holds for all \( \phi \in L^2(\Omega, C_0^0(Y)) \):

\[
\lim_{\epsilon \to 0} \int_{\Omega} u^\epsilon(x) \phi(x, \frac{x}{\epsilon}) \, dx = \int_{\Omega} \int_{Y} u_0(x, y) \phi(x, y) \, dy \, dx.
\]

As proved by Allaire [8], bounded sequences in \( H^1(\Omega) \) allow to extract two-scale convergent subsequences, i.e. for any bounded set \( (u^\epsilon)_{\epsilon > 0} \subset H^1(\Omega) \), there exists a subsequence \( (u^{\epsilon_n})_{\epsilon_n > 0} \) and functions \( u_0 \in H^1(\Omega) \) and \( u_1 \in L^2(\Omega, \tilde{H}^1_1(Y)) \) such that \( u^{\epsilon_n} \rightharpoonup u_0 \) in \( H^1(\Omega) \) and \( \nabla u^{\epsilon_n} \rightharpoonup 2\text{-Sc.} \nabla u_0 + \nabla u_1. \) Using this well know result, the a priori estimates in Lemma 5.2 and the compactness in Lemma 5.3 we get the following convergence up to a subsequence (still denoted by \( \epsilon \)): there exist functions \( S^0 \in L^2(\Omega_T), \, \Upsilon^0 \in L^2((0,T), H^1(\Omega)), \, \Upsilon^1 \in L^2((0,T) \times \Omega, \tilde{H}^1_1(Y)), \, P^0 \in L^2((0,T), H^1(\Omega)) \) and \( P^1 \in L^2((0,T) \times \Omega, \tilde{H}^1_1(Y)) \) such that

\[
S^\epsilon \rightharpoonup S^0 \text{ in } L^2(\Omega_T), \quad \nabla \Upsilon^{2\text{-Sc.}} \nabla_x \Upsilon^0 + \nabla_y \Upsilon^1 \quad \text{and} \quad \nabla P^\epsilon \rightharpoonup 2\text{-Sc.} \nabla_x P^0 + \nabla_y P^1,
\]

where we have \( \Upsilon^\epsilon \rightharpoonup \Upsilon^0 \text{ in } L^2((0,T), H^1(\Omega)) \) and \( \Upsilon^\epsilon \rightharpoonup \Upsilon(S^0) \text{ in } L^2(\Omega_T). \) In particular, we also have \( P^\epsilon \rightharpoonup P^0 \text{ in } L^2(\Omega_T). \)

Next, we use test functions of the form

\[
\Psi^\epsilon(x,t) = \Psi(x,t) + \epsilon \psi(x, \frac{x}{\epsilon}, t), \quad \text{where } (\Psi, \psi) \in C^\infty_c(\Omega_T) \times C^\infty_c(\Omega, C^\infty_c(Y)),
\]

in the weak two-phase flow system given by (5)-(11). Using the admissibility of \( K^\epsilon \) (i.e. assumption (A11), c.f. [8]) and the two-scale convergence, we can form the limits with respect to \( \epsilon \) to obtain:

\[
\int_0^T \langle \Phi^0 \partial_t S^0(\cdot,t), \Psi(\cdot,t) \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \, dt \quad = \quad - \int_{\Omega_T} \int_Y K\langle \lambda_w(S^0)(\nabla_x P^0 + \nabla_y P^1) \rangle \cdot (\nabla_x \Psi + \nabla_y \psi) \quad - \int_{\Omega_T} \int_Y K\langle (\nabla_x \Upsilon(S^0) + \nabla_y \Upsilon^1) - \lambda_w(S^0) \rho_w \bar{g} \rangle \cdot (\nabla_x \Psi + \nabla_y \psi)
\]

and

\[
0 = \int_{\Omega_T} \int_Y K\langle \lambda(S^0)(\nabla_x P^0 + \nabla_y P^1) - (\lambda_w(S^0) \rho_w + \lambda_n(S^0) \rho_n) \bar{g} \rangle \cdot (\nabla_x \Psi + \nabla_y \psi)
\]

for all \( \Psi \in L^2((0,T), H^1_0(\Omega)) \) and \( \psi \in L^2(\Omega_T, \tilde{H}^1_1(Y)) \) (due to density of the smooth functions). Boundary and initial values for \( S^0 \) and \( P^0 \) remain valid since they hold for the whole \( \epsilon \)-sequence. Now, choosing first \( \Psi = 0 \) and then \( \psi = 0 \) we can decouple the problem into coarse scale and fine scale equations. Finally we apply the transformation formula to transform the integral over \( Y \) to an integral over \( Y_{x,k} \) to verify the equivalence to the HMM two-scale problem given by \([8], [11]\). \( \square \)

**Conclusion 5.3 (Two-scale homogenized system).** Let assumptions (A1)-(A7) and (A10)-(A13) be fulfilled. Then the two-scale homogenized system associated with \([13]\) and \([14]\) reads: find \((S^0, P^0) \in \mathcal{E}\) and the correctors \( \Upsilon^1 \in L^2((0,T) \times \Omega, \tilde{H}^1_1(Y)) \) and \( P^1 \in L^2((0,T) \times \Omega, \tilde{H}^1_1(Y)) \) such
We now define $P$ for all $\psi$

In this section we show the reinterpretation of the HMM in terms of an effective hydraulic conductivity $K^{0,\kappa}$.

**Proof.** We start with assuming that there exists a solution $(\Psi, P)\in E$ of the fully continuous HMM introduced in Definition 3.1. Let $P^{0,\lambda}(\cdot) = P^{0,\lambda}(\cdot, t)$ be the fully continuous HMM introduced in Definition 3.1. Let $P^{0,\lambda}(\cdot) = P^{0,\lambda}(\cdot, t)$ be the fully continuous HMM introduced in Definition 3.1.

Due to uniqueness of the solutions of the local problems we conclude $P^{1}(x, y, t) = \kappa^{-1}P^{f}(x, y, t)$, with $P^{f}(x, y, t) = Q^{0}(\nabla_{y} P^{c}(x, t), S^{c}(x, t), x)$. Let us make some simplifying notations:

$$K_{e,\kappa}(x, y) := K^{e}(x + \kappa y)$$

and $\tilde{G} := (\lambda w(S^{c})\rho_{w} + \lambda_{a}(S^{c})\rho_{a})\tilde{y}$.

for all $\Psi \in L^{2}((0, T), H^{1}_{0}(\Omega))$ and $\psi \in L^{2}(\Omega T, H^{1}_{1}(Y))$. This system has a unique solution.

**5.2 Proof of Theorem 5.5 (Fully continuous HMM in cell problem formulation)**

In this section we show the reinterpretation of the HMM in terms of an effective hydraulic conductivity $K^{0,\kappa}$.

**Proof.** We start with assuming that there exists a solution $(S^{c}, P^{c})\in E$ of the fully continuous HMM introduced in Definition 3.1. Let $w_{i}^{e} \in L^{2}(\Omega, H^{1}_{1}(Y))$ denote the cell basis elements solving

$$\int_{Y} K^{e}(x + \kappa y)(e_{i} + \nabla_{y} w_{i}^{e}(x, y)) \cdot \nabla_{y} \psi(y) dy = 0$$

for all $\psi \in H^{1}_{1}(Y)$. The definition implies

$$(K^{0,\lambda})_{ij}^{e}(x) = \int_{Y} K^{e}(x + \kappa y)(e_{i} + \nabla_{y} w_{i}^{e}(x, y)) \cdot (e_{j} + \nabla_{y} w_{j}^{e}(x, y)) dy$$

$$= \int_{Y} K^{e}(x + \kappa y)(e_{i} + \nabla_{y} w_{i}^{e}(x, y)) \cdot e_{j} dy. \quad (28)$$

We now define $P^{1}(x, y, t) \in \tilde{H}^{1}_{e}(Y)$ by

$$P^{1} := \sum_{i=1}^{n} w_{i}^{e} \left( \partial_{x_{i}} P^{c} - \frac{\lambda w(S^{c}(x, t))\rho_{w} + \lambda_{a}(S^{c}(x, t))\rho_{a}}{\lambda(S^{c}(x, t))} \tilde{y}_{i} \right).$$

Using the definition of $w_{i}^{e}$ we get

$$\int_{Y} K^{e}(x + \kappa y)\lambda(S^{c}(x, t))\nabla_{y} P^{1}(x, y, t) \cdot \nabla_{y} \psi(y) dy$$

$$= - \int_{Y} K^{e}(x + \kappa y)\lambda(S^{c}(x, t))\nabla_{x} P^{c}(x, t) \cdot \nabla_{y} \psi(y) dy$$

$$+ \int_{Y} K^{e}(x + \kappa y)(\lambda w(S^{c}(x, t))\rho_{w} + \lambda_{a}(S^{c}(x, t))\rho_{a})\tilde{g} \cdot \nabla_{y} \psi(y) dy.$$
Plugging all this into the HMM pressure equation (13) we get:

\[
0 = \int_{\Omega} \int_{Y_{x,\kappa}} K^*(\lambda S^c) (\nabla_{x} P^c + \nabla_{y} Q^P(P^c)) \cdot \nabla_{x} \psi
\]

\[
= \int_{\Omega} \int_{Y} K_{e,n} \lambda(S^c)(\nabla_{x} P^c + \nabla_{y} P^1) - \tilde{G}_i \cdot \nabla_{x} \psi
\]

\[
= \int_{\Omega} \int_{Y} K_{e,n} \left( \lambda(S^c)(\nabla_{x} P^c + \sum_{i=1}^{n} w_i \partial_{x^i} P^c) - \sum_{i=1}^{n} (w_i \tilde{G}_i) - \tilde{G} \right) \cdot \nabla_{x} \psi
\]

\[
= \int_{\Omega} \lambda(S^c)K_{e,n}^0 \nabla_{x} P^c \cdot \nabla \psi
\]

\[
- \int_{\Omega} \left( \int_{Y} K_{e,n} \left( \tilde{G} + \sum_{i=1}^{n} (w_i \tilde{G}_i) \right) \right) \cdot \nabla_{x} \psi
\]

\[
= \int_{\Omega} \lambda(S^c)K_{e,n}^0 \nabla_{x} P^c \cdot \nabla \psi - \int_{\Omega} K_{e,n}^0 \tilde{G} \cdot \nabla_{x} \psi.
\]

This is exactly the HMM pressure equation (14). Next, we deal with the saturation. For given \(P^c, P^1\) and \(S^c\), let \(\bar{Y}_1(x, \cdot, t) \in \bar{H}^1_{\kappa}(Y)\) denote the solution of

\[
0 = \int_{Y} K_{e,n}(\lambda_w(S^c)(\nabla_{x} P^c + \nabla_{y} P^1) + (\nabla_{x} \bar{Y}(S^c) + \nabla_{y} Y^1) - \lambda_w(S^c)\rho_w \tilde{g}) \cdot \nabla_{y} \psi
\]

for all \(\psi \in \bar{H}^1_{\kappa}(Y)\). Using the transformation formula, this yields \(\nabla_{y} \bar{Y}_1(x, y, t) = \nabla_{y} H_{\kappa}(x, x + \kappa y, t)\), with \(H_{\kappa}(x, y, t) = Q^w(\nabla_{x} P^c(x, t), S^c(x, t), \partial_{x} \bar{Y}(S^c(x, t)), x(x))\) being the solution of (11). Furthermore, we define \(\bar{G} := (\lambda_w(S^c)P^1 + \bar{Y}^1)\) which solves

\[
\int_{Y} K_{e,n}(\bar{V} + \nabla_{y} \bar{Y}^1) \cdot \nabla_{y} \psi = 0 \quad \forall \psi \in \bar{H}^1_{\kappa}(Y)
\]

where

\[
\bar{V} := \lambda_w(S^c)\nabla_{x} P^c + \nabla_{x} \bar{Y}(S^c) - \lambda_w(S^c)\rho_w \tilde{g}.
\]

We immediately verify the relation \(\bar{Y}_1 = \sum_{i=1}^{n} w_i \bar{V}_i\). Inserting this in the macro equation for the saturation (11) gives us:

\[
\int_{0}^{T} (\Phi \partial_{t} S^c(\cdot, t), \Psi(\cdot, t))_{H^{-1}(\Omega), H^{1}(\Omega)} dt
\]

\[
= - \int_{\Omega} \int_{Y_{x,\kappa}} K^*(\lambda_w(S^c) (\nabla_{x} P^c + \nabla_{y} Q^P(P^c)) \cdot \nabla_{x} \psi
\]

\[
- \int_{\Omega} \int_{Y_{x,\kappa}} K^*(\nabla_{x} \bar{Y}(S^c) + \nabla_{y} Q^w(\Phi(S^c)) - \lambda_w(S^c)\rho_w \tilde{g}) \cdot \nabla_{x} \psi
\]

\[
= - \int_{\Omega} \int_{Y} K_{e,n}(\lambda_w(S^c) (\nabla_{x} P^c + \nabla_{y} P^1) \cdot \nabla_{x} \psi
\]

\[
- \int_{\Omega} \int_{Y} K_{e,n}(\nabla_{x} \bar{Y}(S^c) + \nabla_{y} \bar{Y}^1 - \lambda_w(S^c)\rho_w \tilde{g}) \cdot \nabla_{x} \psi
\]

\[
= - \int_{\Omega} \int_{Y} K_{e,n}(\lambda_w(S^c)\nabla_{x} P^c + \nabla_{x} \bar{Y}(S^c) + \nabla_{y} \bar{Y}^1 - \lambda_w(S^c)\rho_w \tilde{g}) \cdot \nabla_{x} \psi
\]

\[
= - \int_{\Omega} \int_{Y} K_{e,n}(\bar{V} + \sum_{i=1}^{n} \bar{V}_i \nabla_{y} w_i) \cdot \nabla_{x} \psi
\]
functions which vanish on a part of the boundary that has a non-zero Hausdorff-measure we get:

\[ \psi \quad \text{for all} \quad \lambda \]

Lemma 5.4 which are ignored in [15]. The following lemmata are required for the final proof:

\[ \text{solving} \]

of the effective hydraulic conductivity \( K \)

arguments in [15]. The main difference to the setting in [15] is that we need an additional treatment

this, we do not specify every detail of the proof as long as the arguments are analogous to the

where a rigorous a posteriori error estimate for the two-phase flow model is derived. Because of

4.12. To a large extend we can proceed as in the recent work by Cancès, Pop and Vohralík [15]

In this section we are concerned with proving the a posteriori error estimate stated in Theorem 4.12 (A posteriori error estimate)

\[ \text{Lemma 5.4} \]

5.3 Proof of Theorem 4.12 (A posteriori error estimate)

In this section we are concerned with proving the a posteriori error estimate stated in Theorem 4.12. To a large extent we can proceed as in the recent work by Cancès, Pop and Vohralík [15] where a rigorous a posteriori error estimate for the two-phase flow model is derived. Because of this, we do not specify every detail of the proof as long as the arguments are analogous to the arguments in [15]. The main difference to the setting in [15] is that we need an additional treatment of the effective hydraulic conductivity \( K_{e,k} \). A minor difference is the presence of gravity terms, which are ignored in [15]. The following lemmata are required for the final proof:

**Lemma 5.4** (Poincaré inequalities). Let \( V \subset \mathbb{R}^d \) denote a bounded domain with a polyhedral boundary. Then we have the following inequality for functions with zero average:

\[ \| \psi \|_{L^2(V)} \leq C_V^{1}(V) \text{diam}(V) \| \nabla \psi \|_{L^2(V)} \]  

(29)

for all \( \psi \in H^1(V) \) with \( \int_V \psi = 0 \). If \( V \) is a convex domain we have \( C_V^{1}(V) \leq \pi^{-1} \) (c.f. [15]). For functions which vanish on a part of the boundary that has a non-zero Hausdorff-measure we get:

\[ \| \psi \|_{L^2(V)} \leq C_V^{2}(V) \text{diam}(V) \| \nabla \psi \|_{L^2(V)} \]  

(30)
for all $\psi \in H^1(V)$ with $\psi = 0$ on $\Gamma \subset \partial V$ and $|\Gamma| \neq 0$. In the later case, $C^2_b(V)$ denotes a Poincaré constant depending on the shape of $V$. Typically, $C^2_b(V)$ can be bounded by 1 (c.f. [23]).

Estimate (21) can be e.g. found in [12, 15, 43]) and for estimate (30) we refer to e.g. [15, 42].

**Lemma 5.5 (Clément interpolation estimates).** Let $T_h(Y)$ denote the regular periodic triangulation of $Y$ and let $W_h$ denote the corresponding finite element space given by (17). If $I_h : H^1(Y) \to \{c + \phi_h | c \in \mathbb{R}, \phi_h \in W_h \}$ defines the common Clément interpolation operator (c.f. [21]), we have the following estimates:

$$\|\psi - I_h(\psi)\|_{L^2(Y)} \leq C^1_{Y}(\omega_Y) \forall \psi \in H^1(Y), \forall Y \in T_h(Y), \quad (31)$$

$$\|\psi - I_h(\psi)\|_{L^2(E)} \leq C^2_{E}(\omega_E) \forall \psi \in H^1(Y), \forall E \in \Gamma(T_h(Y)). \quad (32)$$

In these estimates we used the notation

$$\omega_Y := \bigcup_{\psi \in T_h, \psi \not\equiv \psi} \mathbb{W}, \quad \text{and} \quad \omega_E := \mathbb{Y}_1 \cup \mathbb{Y}_2$$

For a proof of the above estimates (31)-(32) we refer to [21]. Note that the preservation of the periodicity is clear, if we consider the periodic triangulation $T_h(Y)$ of $X$ to be extended to a periodic triangulation $T_h(\mathbb{R}^d)$ of $\mathbb{R}^d$. Then, by the construction of the operator, the periodic structure must be preserved and we can define $I_h(\psi)$ as the restriction of the interpolant to $Y$.

**Lemma 5.6.** Let assumption (A2) be fulfilled, let $D \in \mathcal{D}^2$ and let $x_D$ denote the corresponding center. Then there holds

$$\|K^0_{e,i} - K^0_{e,i,h}\|_{L^2(E)} \leq C^2_{D,k} \sup_{x \in D} \|\{K_{e,i}(x,\cdot) - K_{e,i,h}(x,\cdot)\}(e_i + \nabla_y w_{i,h}(x,\cdot))\|_{L^2(Y)}$$

$$+ C^2_{D,k} \left(\sum_{E \in \Gamma(T_h(Y))} h_E \left|\{K_{e,i,h}(x,\cdot)(e_i + \nabla_y w_{i,h}(x,\cdot))\}E\right|_{L^2(E)}^2\right)^{1/2},$$

where $C$ is a generic constant and $\alpha_{D,k}$ denotes the smallest and $\beta_{D,k}$ the largest eigenvalue of $K'(x_D + ey)$ in $Y$.

**Proof.** Let $x \in D$ denote an arbitrary sample point and $e_{i,h}(x,\cdot) := w_{i,h}(x,\cdot) - w_{i,h}(x,\cdot)$ the error between discrete and continuous cell basis elements. By $I_h(e_{i,h}(x,\cdot))$ we denote the Clément interpolant stated in Lemma 5.5 and by $|\cdot|_{H^1(Y)}$ the semi-norm on $H^1(Y)$. The interpolant with zero average is defined by $I_h(e_{i,h}(x,\cdot)) := I_h(e_{i,h}(x,\cdot)) - I_Y I_h(e_{i,h}(x,\cdot))$. Then we get by Galerkin orthogonality and using the estimates (31)-(32):

$$\int_Y K_{e,i}(x,\cdot) \nabla_y e_{i,h}(x,\cdot) \cdot \nabla_y e_{i,h}(x,\cdot) = \int_Y K_{e,i}(x,\cdot)(e_i + \nabla_y w_{i,h}(x,\cdot)) \cdot \nabla_y e_{i,h}(x,\cdot)$$

$$= \int_Y (K_{e,i}(x,\cdot) - K_{e,i,h}(x,\cdot))(e_i + \nabla_y w_{i,h}(x,\cdot)) \cdot \nabla_y e_{i,h}(x,\cdot)$$

$$+ \int_Y K_{e,i,h}(x,\cdot)(e_i + \nabla_y w_{i,h}(x,\cdot)) \cdot \nabla_y (e_{i,h}(x,\cdot) - I_h(e_{i,h}(x,\cdot)))$$

$$\leq \|\{K_{e,i}(x,\cdot) - K_{e,i,h}(x,\cdot)\}(e_i + \nabla_y w_{i,h}(x,\cdot))\|_{L^2(Y)} e_{i,h}(x,\cdot)|_{H^1(Y)}$$

$$+ \|\{K_{e,i,h}(x,\cdot)(e_i + \nabla_y w_{i,h}(x,\cdot))\}E\|_{L^2(E)} h_E^{1/2} e_{i,h}(x,\cdot)|_{H^1(Y)}$$

$$+ C \sum_{E \in \Gamma(T_h(Y))} \|\{K_{e,i,h}(x,\cdot)(e_i + \nabla_y w_{i,h}(x,\cdot))\}E\|_{L^2(E)} h_E^{1/2} e_{i,h}(x,\cdot)|_{H^1(Y)}.$$


Using the Cauchy-Schwarz and dividing by the error yields:

\[
\|\nabla_y (w^i_{\epsilon,h}(x_D,\cdot) - w^i_{\epsilon,h}(x_D,\cdot))\|_{L^2(Y)} \leq \alpha^{-1}_{D,\kappa} \|\langle K_{\epsilon,h}(x,\cdot) - K_{\epsilon,h}(x_D,\cdot) \rangle (e_i + \nabla_y w^i_{\epsilon,h}(x_D,\cdot)) \|_{L^2(Y)} + C \alpha^{-1}_{D,\kappa} \left( \sum_{E \in \Gamma(\Omega)} h_E \|\langle K_{\epsilon,h}(x_D,\cdot) (e_i + \nabla_y w^i_{\epsilon,h}(x_D,\cdot)) \rangle \|_{L^2(E)} \right)^{\frac{1}{2}}.
\]

This yields

\[
\| (K^0_{\epsilon,h})_{ij} (x) - (K^0_{\epsilon,h})_{ij} (x) \| \leq \left| \int_Y K_{\epsilon,h}(x,\cdot) \nabla_y (w^i_{\epsilon,h}(x,\cdot) - w^i_{\epsilon,h}(x_D,\cdot)) \cdot e_j \right|
+ \left| \int_Y (K_{\epsilon,h}(x,\cdot) - K_{\epsilon,h}(x_D,\cdot)) (e_i + \nabla_y w^i_{\epsilon,h}(x_D,\cdot)) \cdot e_j \right|
\leq \frac{\beta_{D,\kappa}}{\alpha_{D,\kappa}} \|\langle K_{\epsilon,h}(x,\cdot) - K_{\epsilon,h}(x_D,\cdot) \rangle (e_i + \nabla_y w^i_{\epsilon,h}(x_D,\cdot)) \|_{L^2(Y)}
+ C \frac{\beta_{D,\kappa}}{\alpha_{D,\kappa}} \left( \sum_{E \in \Gamma(\Omega)} h_E \|\langle K_{\epsilon,h}(x_D,\cdot) (e_i + \nabla_y w^i_{\epsilon,h}(x_D,\cdot)) \rangle \|_{L^2(E)} \right)^{\frac{1}{2}}
+ \left| \int_Y (K_{\epsilon,h}(x,\cdot) - K_{\epsilon,h}(x_D,\cdot)) (e_i + \nabla_y w^i_{\epsilon,h}(x,\cdot)) \cdot e_j \right|.
\]

This implies the estimate stated in the Lemma. \(\square\)

**Definition 5.7 (Residuals).** Let \(K^0\) denote the homogenized matrix as in Theorem 4.12. For \((S_{H_T}, P_{H_T}) \in E\), we define the pressure residual \(R_p(S_{H_T}, P_{H_T}) \in L^2((0,T), H^1_0(\Omega))\) and the saturation residual \(R_s(S_{H_T}, P_{H_T}) \in L^2((0,T), H^1_0(\Omega))\) by:

\[
\langle R_p(S_{H_T}, P_{H_T}), \psi \rangle := \int_{\Omega_T} \lambda(S_{H_T}) K^0 \nabla P_{H_T} \cdot \nabla \psi
- \int_{\Omega_T} K^0 (\lambda w(S_{H_T}) \rho w + \lambda_n(S_{H_T}) \rho_n) \bar{g} \cdot \nabla \psi;
\]

\[
\langle R_s(S_{H_T}, P_{H_T}), \psi \rangle := \int_0^T \langle \Psi^0, \partial_t S_{H_T}(\cdot, t), \psi(\cdot, t) \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \ dt
+ \int_{\Omega_T} K^0 (\lambda w(S_{H_T}) \nabla P_{H_T} + \nabla \Psi(S_{H_T}) - \lambda w(S_{H_T}) \rho_w \bar{g}) \cdot \nabla \psi.
\]

for \(\psi \in L^2((0,T), H^1_0(\Omega))\).

It is obvious that \((S_{H_T}, P_{H_T}) \in E\) with \(S_{H_T}(\cdot, 0) = S_0\) is the weak solution of (25)-(26) if and only if \(\langle R_p(S_{H_T}, P_{H_T}), \psi \rangle = \langle R_s(S_{H_T}, P_{H_T}), \psi \rangle = 0\).

The following lemma can be proved in analogy to [15]:

**Lemma 5.8 (Upper bound on the error by the residuals).** Let assumptions (A1)-(A9) and (A16) be fulfilled and let \((S_{H_T}, P_{H_T}) \in E\) denote an arbitrary pair of functions. Then the following estimate holds true:

\[
\| ||S_{H_T} - S^0||_{L^2(\Omega_T)}^2 + ||P_{H_T} - P^0||_{L^2(\Omega_T)}^2 + ||\Psi(S_{H_T}) - \Psi(S^0)||_{L^2(\Omega_T)}^2 \|_{L^2(\Omega_T)}^2
\leq || S_{H_T}(\cdot, 0) - S_0 ||_{L^2(\Omega_T)}^2 + || R_p(S_{H_T}, P_{H_T}) ||_{L^2(\Omega_T)}^2 + || R_s(S_{H_T}, P_{H_T}) ||_{L^2(\Omega_T)}^2.
\]
Furthermore, we denote $R$.

First, we estimate the saturation residual that belong to the boundary nodes of $T$.

Using Lemma 5.8 it only remains to estimate the residuals, which also proves the a posteriori error estimates stated in Theorem 4.7 and Theorem 4.12.

**Proof of Theorem 4.12.** For deriving the estimate, we start with proceeding as in [15] for bounding the residuals. Let again $D_{H_{int}}^\Omega \subset D_H^\Omega$ contain the elements of the dual mesh that belong to the interior nodes of the grid $f_H^\Omega(\Omega)$ and let $D_{H_{int}}^{n_{ext}} \subset D_H^\Omega$ contain the elements of the dual mesh that belong to the boundary nodes of $f_H^\Omega(\Omega)$. Let $\Psi \in L^2((0, T), H_0^1(\Omega))$ with $\|\Psi\|_{E(\Omega_T)} = 1$.

Furthermore, we denote $V_{H_T}^s := (\lambda_w(S_{H_T}) \nabla_t P_{H_T} + \nabla_t \Psi(S_{H_T}) - \lambda_w(S_{H_T})\rho_w \bar{g})$.

First, we estimate the saturation residual $R_s(S_{H_T}, P_{H_T})$. We get:

$$
\langle R_s(S_{H_T}, P_{H_T}), \Psi \rangle = \int_0^T \langle \partial_t S_{H_T}(\cdot, t), \Psi(\cdot, t) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} dt + \int_{\Omega_T} K_0 \nabla_{H_T} \cdot \Psi \leq \sum_{n=1}^N \sum_{D \in D_{H_{int,n}}} \int_D \int_{I_n} \langle \partial_t S_{H_T} + \nabla \cdot \bar{u}_{s,H}^{n} \rangle (\Psi - \Psi_D) \\
+ \sum_{n=1}^N \sum_{D \in D_{H_{int,n}}} \int_D \int_{I_n} \langle \partial_t S_{H_T} + \nabla \cdot \bar{u}_{s,H}^{n} \rangle \Psi \leq \sum_{n=1}^N \sum_{D \in D_{H_{int,n}}} \int_D \int_{I_n} \| \partial_t S_{H_T} + \nabla \cdot \bar{u}_{s,H}^{n} \|_{L^2(D)} \| \Psi - \Psi_D \|_{L^2(D)} \\
+ \sum_{n=1}^N \sum_{D \in D_{H_{int,n}}} \int_D \int_{I_n} \| \partial_t S_{H_T} + \nabla \cdot \bar{u}_{s,H}^{n} \|_{L^2(D)} \| \Psi \|_{L^2(D)} \\
\leq \sum_{n=1}^N \sum_{D \in D_{H_{int,n}}} \int_D \int_{I_n} C_P(D) H_D \alpha^{-\frac{1}{2}} \| \partial_t S_{H_T} + \nabla \cdot \bar{u}_{s,H}^{n} \|_{L^2(D)} \| (K_0) \|_{L^2(D)} \| \nabla \Psi \|_{L^2(D)}.
$$

We start with estimating $I$. Let $\Psi_D$ denote the average of $\Psi$ over $D$. We use the definition of $\bar{u}_{s,H}^{n}$ and the Hölder inequality to get:

\[ I = \sum_{n=1}^N \sum_{D \in D_{H_{int,n}}} \int_D \int_{I_n} \langle \partial_t S_{H_T} + \nabla \cdot \bar{u}_{s,H}^{n} \rangle (\Psi - \Psi_D) \]

\[ + \sum_{n=1}^N \sum_{D \in D_{H_{int,n}}} \int_D \int_{I_n} \langle \partial_t S_{H_T} + \nabla \cdot \bar{u}_{s,H}^{n} \rangle \Psi \leq \sum_{n=1}^N \sum_{D \in D_{H_{int,n}}} \int_D \int_{I_n} \| \partial_t S_{H_T} + \nabla \cdot \bar{u}_{s,H}^{n} \|_{L^2(D)} \| \Psi - \Psi_D \|_{L^2(D)} \]

\[ + \sum_{n=1}^N \sum_{D \in D_{H_{int,n}}} \int_D \int_{I_n} \| \partial_t S_{H_T} + \nabla \cdot \bar{u}_{s,H}^{n} \|_{L^2(D)} \| \Psi \|_{L^2(D)} \]

\[ \leq \sum_{n=1}^N \sum_{D \in D_{H_{int,n}}} \int_D \int_{I_n} C_P(D) H_D \alpha^{-\frac{1}{2}} \| \partial_t S_{H_T} + \nabla \cdot \bar{u}_{s,H}^{n} \|_{L^2(D)} \| (K_0) \|_{L^2(D)} \| \nabla \Psi \|_{L^2(D)}. \]
Combining the results with Lemma 5.8 yields the final a posteriori error estimate. For II we immediately get

$$II \leq \sum_{n=1}^{N} \sum_{D \in \mathcal{D}_h} \int_{I_n} \alpha^{-\frac{1}{2}} \|(K_0^0 - K_{e,h}) \vec{V}_H^s\|_{L^2(D)} \| (K_0^0)^{\frac{1}{2}} \nabla \Psi\|_{L^2(D)}.$$

Term III can be estimated using Lemma 5.6, which implies

$$III = \sum_{n=1}^{N} \sum_{D \in \mathcal{D}_h} \int_{I_n} \alpha^{-\frac{1}{2}} \|K_0^0 - K_{e,h}\|_{L^\infty(D)} \|\nabla \Psi\|_{L^2(D)}.$$

where

$$\|K_0^0 - K_{e,h}\|_{L^\infty(D)} \leq C \beta \sup_{D,h} (K_{e,h}(x)) (K_{e,h}(x_d)) (\|w_i\|_{L^2(Y)} + H_{\tau}^{\frac{1}{2}} (\nabla \Psi)) + \frac{1}{\alpha} \sum_{E \in \Gamma(T_n(Y))} \frac{1}{\alpha} \sum_{D \in \mathcal{D}_h} \int_{I_n} \alpha^{-\frac{1}{2}} \|K_0^0 - K_{e,h}\|_{L^2(D)} \| (K_0^0)^{\frac{1}{2}} \nabla \Psi\|_{L^2(D)}.$$

For IV we get

$$IV \leq \sum_{n=1}^{N} \sum_{D \in \mathcal{D}_h} \int_{I_n} \alpha^{-\frac{1}{2}} \|K_0^0 + \vec{u}_s H_\Psi\|_{L^2(D)} \| (K_0^0)^{\frac{1}{2}} \nabla \Psi\|_{L^2(D)}.$$

Combining the estimates on I-IV, using the Cauchy-Schwarz inequality and \( \| (K_0^0)^{\frac{1}{2}} \nabla \Psi\|_{L^2(\Omega_T)} = 1 \), we get

$$\|R_s(S_{HT}, P_{HT})\|_{\Omega_T} \lesssim \sum_{n=1}^{N} \sum_{D \in \mathcal{D}_h} \int_{I_n} (\eta_{CR,s,D} + \eta_{CR,F,s,D} + \eta_{D,F,s,D} + \eta_{AP,s,D} + \eta_{MOD,s,D}(t))^2.$$

Next, we estimate the pressure residual \( R_p(S_{HT}, P_{HT}) \). Here, we get:

$$\langle R_p(S_{HT}, P_{HT}), \Psi \rangle = \int_{\Omega_T} K_0^0 \vec{V}_H^p \cdot \nabla \Psi = \int_{\Omega_T} (K_0^0 - K_{e,h}) \vec{V}_H^p \cdot \nabla \Psi + \int_{\Omega_T} (K_0^0 - K_{e,h}) \vec{V}_H^p \cdot \nabla \Psi + \int_{\Omega_T} \vec{V}_H^p \cdot \nabla \vec{u}_s H_\Psi,$$

where we can estimate the various terms in the same way as for the saturation residual to obtain:

$$\|R_p(S_{HT}, P_{HT})\|_{\Omega_T}^2 \lesssim \sum_{n=1}^{N} \sum_{D \in \mathcal{D}_h} \int_{I_n} (\eta_{CR,p,D} + \eta_{CR,F,p,D} + \eta_{D,F,p,D} + \eta_{AP,P,D} + \eta_{MOD,p,D}(t))^2.$$

Combining the results with Lemma 5.8 yields the final a posteriori error estimate.
6 Conclusion

In this paper, we derived the first version of a heterogeneous multiscale method for the incompressible two-phase flow equations. The method can be applied to the classical formulation of the two-phase flow system as well as to the Kirchhoff transformed system. We presented different discretization strategies that are based on a finite element method on the micro scale and a vertex centered finite volume method on the macro scale. In the periodic setting, the method is equivalent to a discretization of the homogenized equation. Finally, we derived a rigorous and effective a posteriori error estimate for the error between HMM approximations and the homogenized solutions. The estimate is based on the results obtained in [15].

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