Spherical $f$-Tilings by Two Noncongruent Classes of Isosceles Triangles-II

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Abstract In this work, we give a complete classification of spherical dihedral $f$-tilings when the prototiles are two noncongruent isosceles triangles with certain adjacency pattern. As it will be shown, this class is composed by two discrete families denoted by $E_m$, $m \geq 2$, $m \in \mathbb{N}$, $F_k$, $k \geq 4$, $k \in \mathbb{N}$ and two sporadic tilings denoted by $G$ and $H$.

Keywords Spherical tilings, dihedral, triangulations, $f$-tilings, spherical trigonometry

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1 Introduction

Spherical folding tilings, or $f$-tilings for short, are edge-to-edge tilings of the Euclidean sphere $S^2$, by geodesic polygons, such that all vertices are of even valency and the sum of alternate angles around each vertex is $\pi$. Let us denote by $\Omega(X,Y)$ the set, up to isomorphism, of all dihedral $f$-tilings of $S^2$ whose cells are congruent to $X$ or $Y$. In [9], it was shown that any $\tau \in \Omega(X,Y)$ necessarily has vertices of valency four.

Some monohedral $f$-tilings of the sphere by triangles such as the spherical octahedron have been known for a very long time. The tilings have been explicitly studied since 1992, see [5]. Ueno and Agaoka, in their work on spherical tilings [1], provided several examples of spherical...
tilings by congruent quadrangles, see [2]. In [3], the study of all spherical $f$-tilings by triangles and $r$-sided regular polygons, for any $r \geq 5$ was given. A list on spherical $f$-tilings by triangles and isosceles trapezoids, by the same authors was given in [4].

The classification of all dihedral spherical folding tilings by triangles is not yet completed. The cases studied so far correspond to the following prototiles:

- an equilateral triangle and an isosceles triangle, [7];
- an equilateral triangle and a scalene triangle, [8].
- two non congruent (nonequilateral) isosceles triangles in a particular type of adjacency, [6].

Let $T_1$ be an isosceles (nonequilateral) spherical triangle with angles $\alpha, \alpha, \beta$, paired sides $a$ (opposite to $\alpha$) and base $b$ (opposite to $\beta$); and $T_2$ another such triangle with angles $\gamma, \gamma, \delta$, paired sides $c$, and base $d$. The sphere is connected, both prototiles are used, and the tiles meet edge-to-edge; so it is necessary that somewhere in the configuration a prototile of one type shares an edge with one of the other type. “Type I adjacency”, in which $b = d$, was considered in [6]. In this paper we will consider “type II adjacency”, with $a = d$; the remaining case, “type III adjacency”, in which one tile’s base matches the paired edges of the other, will be dealt with in an upcoming paper. The three cases are illustrated in Figure 1; note that in principle, a pair of tiles could exhibit both type I and type II adjacency, or type III adjacency in two distinct ways. Note that type III adjacency is not compatible with either of the other types, as this would force one triangle or the other to be equilateral.

![Figure 1 Different types of Adjacency](image)

**Lemma 1.1** If $(T_1, T_2)$ are isosceles nonequilateral tiles that tile the sphere with a type II adjacency, they do not exhibit any other equalities among edge lengths.

Spherical trigonometry allows us to solve for any edge of a triangle in terms of its angles; doing this for the edge lengths $a$ and $d$ and equating, we obtain the following equation which will be useful in the next section:

$$\frac{\cos \alpha (1 + \cos \beta)}{\sin \alpha \sin \beta} = \cos a = \frac{\cos \delta + \cos^2 \gamma}{\sin^2 \gamma}.$$ (1.1)

To determine whether a pair of triangles $(T_1, T_2)$ permits a dihedral $f$-tiling, we often find useful to start by considering local configurations in which the sphere is only partially tiled. We will usually start with a pair of triangles exhibiting type II adjacency, and label these 1
and 2. We will then deduce the location of some tile $i$ from the locations of tiles $\{1, 2, \ldots, i-1\}$ and the hypothesis that the configuration extends to a complete $f$-tiling, backtracking when necessary, until either a complete $f$-tiling or a contradiction is reached.

2 Triangular Dihedral $f$-Tilings by Isosceles Triangles with Adjacency of Type II

If $\tau \in \Omega(T_1, T_2)$ has an adjacency of type II, then we may start its configuration with two adjacent cells congruent respectively to $T_1$ and $T_2$, as shown in Figure 2. Lemma 1.1 forces triangle 3 as shown.

![Figure 2 Local configuration](image)

In order to have the angle folding relation fulfilled, the sum containing the alternate angles $\alpha$ and $\gamma$ must be of the form $\alpha + \gamma \leq \pi$. We shall separate cases $\alpha + \gamma = \pi$ and $\alpha + \gamma < \pi$.

**Lemma 2.1** (Elimination Lemma I) There are no $f$-tilings with adjacency type II and $\alpha + \gamma = \pi$.

**Proof** Suppose that $\alpha + \gamma = \pi$. We shall show that $\alpha$ is acute. Suppose not; then clearly $\gamma \leq \frac{\pi}{2}$; and by the equation (1.1), $\cos b < 0$, which implies that $\delta > \frac{\pi}{2}$. An $f$-tiling cannot have three obtuse angles at a vertex, so tiles 5 and 5' must be as shown in Figure 3-I. As $\delta + 2\gamma > \pi$, tiles 6 and 6' must be as shown. Tile 7 is immediate, and as $\delta + 2\gamma > \pi$, tile 8 must be as shown; but then we have three $\alpha$ angles at $v_1$. This would imply $\alpha = \gamma = \frac{\pi}{2}$, which by the equation (1.1) gives also $\delta = \frac{\pi}{2}$, a contradiction.

![Figure 3 Local configurations](image)

Thus $\alpha$ must be acute, and $\gamma$ obtuse. There cannot be a second $\gamma$ at $v_2$, as this would immediately create an alternating sum greater than $\pi$. Nor can triangle 5 be as in Figure 3-II, as this would require the vertex to have exactly one more angle, supplementary to $\beta$. This
angle could only be $\delta$, but neither of the edges adjacent to a $\delta$ angle matches the $\alpha - \alpha$ edge that is already present. We are thus forced to the configuration of Figure 3-III, with four $\alpha$ angles coming together. As $\alpha < \frac{\pi}{2}$, the alternating sum must contain at least two more angles, which must be $\alpha$ or $\gamma$ angles; and as $\gamma > \alpha$ we have $\alpha < \frac{\pi}{3}$. But at $v_2$ we have two adjacent $\beta$ angles and a $\gamma$; for this to exist in an $f$-tiling, $\beta + \gamma \leq \pi$, which would imply $\beta \leq \alpha$. But then $2\alpha + \beta < \pi$, which is impossible.

**Lemma 2.2 (Elimination Lemma II)**  There are no $f$-tilings with adjacency type II, $\alpha > \beta$, and $\alpha + \gamma < \pi$.

**Proof**  Suppose the triangles $T_1, T_2$ to have adjacency type II and $\alpha > \beta$.

**Case 1**  Let $\gamma > \delta$.

By the adjacency condition (1.1), if $\alpha \geq \frac{\pi}{2}$, then $\gamma > \delta > \frac{\pi}{2}$ contradicting $\alpha + \gamma < \pi$. Thus, $\beta < \alpha < \frac{\pi}{2}$. As [8], the tiling must have vertices of valency 4, and $\gamma > \delta$, we deduce that $\gamma \geq \frac{\pi}{2}$. If $\gamma = \frac{\pi}{2}$, every vertex of order 4 has four $\gamma$ angles.

Starting from the local configuration in Figure 4-I, since $\alpha, \gamma > \frac{\pi}{3}$, in order for the sum of alternate angles to contain $\alpha, \delta$, and $\theta_1$ while satisfying the angle folding relation, $\theta_1 \in \{\beta, \delta\}$.

If $\theta_1 = \beta$, then $\alpha + \gamma + \theta_1 \leq \pi$ and from $2\alpha + \beta > \pi$, one has $\frac{\pi}{2} \leq \gamma < \alpha$, contradicting $\alpha + \gamma < \pi$.

If $\theta_1 = \delta$, then $\alpha + \gamma + \theta_1 \leq \pi$. Assuming that $\gamma > \frac{\pi}{2}$, vertices of valency four are surrounded by the angular sequence $(\gamma, \gamma, \beta, \beta)$. As, $\alpha + \gamma + \theta_1 \leq \pi = \gamma + \beta$, then $\alpha < \beta$, which is a contradiction. Therefore, $\gamma = \frac{\pi}{2}$.

If the vertex has valency 6, so that $\alpha + \gamma + \theta_1 = \pi$, we may extend the configuration in Figure 4-I getting the one illustrated in Figure 4-II, with $\theta_2 = \delta$ (observe that $\theta_2 = \gamma$ implies the impossibility $\beta + \delta = \pi$).

The sum of the alternate angles containing $\beta$ and $\gamma$, at vertex $v_2$ (see Figure 4-III) is of the form $\beta + \gamma + m\beta = \pi$, $m \geq 1$ or $\beta + \gamma + n\delta = \pi$, $n \geq 1$ or $\beta + \gamma + p\beta + q\delta = \pi$, $p, q \geq 1$. 

![Figure 4](image-url)  Local configurations
In the first two cases, we may extend the configuration a little bit more and obtain vertices partially surrounded by a sequence of angles \((\alpha, \alpha, \alpha, \alpha, \rho)\), with \(\rho \in \{\alpha, \beta, \gamma\}\) (Figure 5-I) or completely surrounded by \((\gamma, \beta, \gamma, \delta)\) (Figure 5-II). In the first case, alternate sums are obviously unequal; the second case contradicts \(\beta, \delta < \frac{\pi}{2}\).

**Figure 5** Local configurations

The third case leads also to an impossibility, since the possible extensions are the ones illustrated in Figure 5, which have already been ruled out.

Suppose now that \(\alpha + \gamma + \theta_1 < \pi\), with \(\theta_1 = \delta\). Taking into account that \(\alpha > \frac{\pi}{3}\), \(\gamma = \frac{\pi}{2}\) and \(2\alpha + \beta > \pi\), then the sum containing the angles \(\alpha, \gamma\) and \(\delta\) is of the form \(\alpha + \gamma + \delta + t\delta = \pi\), for some \(t \geq 1\). The configuration in Figure 4-I extends to the one below, where a vertex partly surrounded by angles \((\beta, \gamma, \gamma)\) arises. The sum containing the alternate angles \(\beta\) and \(\gamma\) must satisfy \(\gamma + \beta + m\beta = \pi\), \(m \geq 1\) or \(\gamma + \beta + n\delta = \pi\), \(n \geq 1\) or \(\gamma + \beta + p\beta + q\delta = \pi\), \(p, q \geq 1\), cases already studied.

**Figure 6** Local configuration

**Case 2** \(\delta > \gamma\).

By the adjacency condition (1.1), if \(\delta < \frac{\pi}{2}\), then \(\beta < \alpha < \frac{\pi}{2}\) and also \(\gamma < \frac{\pi}{2}\), preventing the existence of vertices of valency four. Therefore, \(\delta \geq \frac{\pi}{2}\).

Looking at the configuration in Figure 7, a decision about the angle \(\theta_3 \in \{\alpha, \beta, \gamma\}\) must be taken.

1 Assume first that \(\theta_3 = \alpha\). Then, \(\alpha + \theta_3 \leq \pi\).

1.1 If \(\alpha + \theta_3 = \pi\), with \(\theta_3 = \alpha\), then \(\beta < \frac{\pi}{2}, \gamma < \frac{\pi}{2}\) (note that \(\alpha + \gamma < \pi\)) and \(\delta > \frac{\pi}{2}\) (by the adjacency condition (1.1)).

Consequently, the sum of the alternate angles containing \(\alpha\) and \(\gamma\) is of the form \(\alpha + \gamma + k\gamma = \pi\), \(k \geq 1\) or \(\alpha + \gamma + t\beta = \pi\), \(t \geq 1\) or \(\alpha + \gamma + p\gamma + q\beta = \pi\), \(p, q \geq 1\).
The first two cases end up at vertices $v_3$ and $v_4$ (see respectively, Figure 8-I, II), where $v_3$ is of valency four surrounded by angles $(\gamma, \delta, \delta, \gamma)$ (implying the absurdity $\delta = \frac{\pi}{2}$) and $v_4$ is surrounded by a sequence containing $(\delta, \gamma, \alpha)$ violating the angle folding relation.

**Figure 7** Local configuration

If $\alpha + \gamma + p\gamma + q\beta = \pi$, $p, q \geq 1$, we are led to a contradiction similar to that in the two previous cases.

1.2 Suppose now that $\alpha + \theta_3 < \pi$, with $\theta_3 = \alpha$. As $2\alpha + \beta > \pi$, $\alpha > \frac{\pi}{3}$ and $\delta \geq \frac{\pi}{2}$, the sum containing $\alpha$ and $\theta_3$ obeys $2\alpha + k\gamma = \pi$, for some $k \geq 1$. Accordingly, $\gamma < \beta < \alpha < \frac{\pi}{2} \leq \delta$.

Assume, firstly, that $\delta = \frac{\pi}{2}$. Then, $\gamma > \frac{\pi}{4}$ and so $k = 1$. The order relation between the angles is now $\frac{\pi}{4} < \gamma < \beta < \alpha < \frac{\pi}{2} = \delta$. Extending the configuration in Figure 7, we get the one illustrated in Figure 9-I.

**Figure 8** Local configurations

The sum containing the alternate angles $\beta$ and $\gamma$ satisfies $2\beta + \gamma = \pi$ or $2\gamma + \beta = \pi$ or $\beta + \gamma + \alpha = \pi$.

In the first and third cases, we would have $\alpha = \beta$ which is impossible. In the second case, we conclude, by the adjacency condition (1.1), that $\alpha \approx 64.085^\circ$ or $\alpha \approx 115.91^\circ$, which are both impossible, since $\beta < \alpha < \frac{\pi}{2}$. 

**Figure 9** Local configurations
Assume now that $\delta > \frac{\pi}{2}$. The configuration in Figure 7 takes the form illustrated in Figure 9-II.

The vertex partially surrounded by the angles $\delta, \gamma, \beta$ is of valency greater than four (otherwise $\beta + \delta = \pi$ and $\gamma = \frac{\pi}{2}$ violating the equality $2\alpha + k\gamma = \pi$, $k \geq 1$). On the other hand, the vertex partially surrounded by the angles $\gamma, \delta, \delta$ is also of valency greater than four (otherwise, as $\beta > \gamma$, one has $\beta + \delta > \gamma + \delta = \pi$ which is impossible). Therefore, the only way to arrange the angles around vertices of valency four is $(\delta, \delta, \alpha, \alpha)$; but this cannot be realized in a tiling due to incompatible edge lengths.

2 Suppose that $\theta_3 = \beta$ (Figure 7). Then, $\alpha + \theta_3 \leq \pi$.

2.1 In case $\alpha + \theta_3 = \pi$, with $\theta_3 = \beta$, then $\gamma < \beta < \frac{\pi}{2} < \alpha$. The extended configuration ends up at a vertex (Figure 10-I) partially surrounded by the sequence of angles $(\alpha, \alpha, \alpha)$, with an alternating sum of at least $2\alpha$, incompatible with the angle folding relation.

2.2 If $\alpha + \theta_3 < \pi$, then $\beta < \frac{\pi}{2}$ and a decision about the angle $\theta_4 \in \{\gamma, \delta\}$ must be taken, as is shown in Figure 10-II.

2.2.1 If $\theta_4 = \gamma$, then $\alpha + \beta + \theta_4 \leq \pi$ and $\gamma < \alpha$.

2.2.1.1 Suppose first that $\alpha + \beta + \gamma = \pi$. Extending the configuration in Figure 10-II, tile 9 has two possible positions (Figure 11-I, II). One of the positions of this tile creates a vertex that must be of valency four, completely surrounded by the angles $(\alpha, \alpha, \delta, \gamma)$ (see Figure 11-I). But we have assumed $\gamma < \delta$, so the folding relation is not satisfied.
The other possible position for tile 9 gives rise to a vertex partially surrounded by a sequence of angles \((\delta, \delta, \delta)\). The vertex must be of valency four, since \(\delta \geq \frac{\pi}{2}\) (see Figure 11-II). Therefore, \(\delta = \frac{\pi}{2}, \gamma > \frac{\pi}{4}\) and by (1.1), \(\beta < \alpha < \frac{\pi}{2}\).

In this configuration, we see two vertices at which an \(\alpha\) and a \(\gamma\) have a neighbour in common and thus form part of the same alternate sum. These vertices must be of valency 6; and the alternate sums give \(2\alpha + \gamma = \pi, \alpha + 2\gamma = \pi, \) or \(\alpha + \gamma + \beta = \pi\).

The first case implies that \(\alpha = \beta\), which is impossible. In the second case, one has \(\gamma = \beta\) and so \(\frac{\pi}{4} < \gamma = \beta < \alpha < \frac{\pi}{2}\). By the adjacency condition (1.1), we get \(\gamma = \beta \approx 52.884^\circ\) and \(\alpha \approx 74.232^\circ\). Accordingly, we have three type of vertices: vertices of valency four surrounded exclusively by angles \(\delta\), vertices of valency six, with alternate angle sets surrounded by alternate angles \(\alpha, \gamma, \beta\) and \(\alpha, \gamma, \gamma\). Thus, we may extend the configuration in Figure 11-II in a unique way, getting a vertex \(v_5\) (Figure 12-I) partially surrounded by an angle sequence \((\gamma, \gamma, \gamma, \gamma)\). But \(\pi - 3\gamma \approx 21.348^\circ\) which is less than any of the angles, so we cannot complete the tiling at this vertex while satisfying the folding condition.

The case \(\alpha + \gamma + \beta = \pi\) remains. Extending the configuration in Figure 11-II, we get a vertex \(v_6\), partially surrounded by an angle sequence \((\alpha, \alpha, \alpha)\) (Figure 12-II). As \(2\alpha + \beta > \pi, \alpha > \frac{\pi}{4}, \delta = \frac{\pi}{2}\) and \(\alpha > \gamma > \frac{\pi}{4}\), the sum \(2\alpha\) must satisfy \(2\alpha + \gamma = \pi\). But, as \(\alpha + \beta + \gamma = \pi\), we have reached a contradiction.

![Figure 12 Local configurations](image)

2.2.1.2 Assume instead that \(\alpha + \beta + \gamma < \pi\). A decision about the angle \(\theta_5 \in \{\gamma, \delta\}\) must be taken in configuration in Figure 13-I.

In the case \(\theta_5 = \gamma\), we have \(\theta_6 = \delta\) and the sum containing \(\alpha\) and \(\delta\) must satisfy \(\alpha + \delta + k\beta = \pi, k \geq 1\), since \(\alpha > \frac{\pi}{3}, \delta \geq \frac{\pi}{2}, \gamma < \alpha\), and \(2\gamma + \delta > \pi\). However, as \(2\alpha + \beta > \pi\), then \(\frac{\pi}{2} \leq \delta < \alpha\). But we have \(\alpha + \delta + k\beta = \pi\); thus, \(\theta_5 = \delta\) and so \(\delta = \frac{\pi}{2}\). Consequently, \(\frac{\pi}{4} < \gamma < \frac{\pi}{2}\).
From the adjacency condition, one has $\beta < \alpha < \frac{\pi}{2}$. In order to satisfy the angle folding relation, the sum containing the angles $\alpha, \beta$ and $\gamma$ must satisfy $\alpha + 2\gamma + k\beta = \pi$, $k \geq 1$ or $\alpha + \gamma + q\beta = \pi$, $q \geq 2$. The first case is impossible, since $2\gamma + \delta > \pi$ and $2\alpha + \beta > \pi$ and so $\alpha + \gamma + q\beta = \pi$, for some $q \geq 2$. The configuration can now be extended a bit more getting the one illustrated in Figure 13-II. The vertex partially surrounded by the angle sequence $(\alpha, \alpha, \alpha, \alpha)$ must be of valency six with $2\alpha + \gamma = \pi$, since $\alpha > \frac{\pi}{3}$, $\gamma > \frac{\pi}{4}$, $\delta = \frac{\pi}{2}$ and $2\alpha + \beta > \pi$. Hence, $\frac{\pi}{4} < \gamma < \beta$ and (as $q \geq 2$) we conclude that $\alpha + \gamma + q\beta > \pi$, a contradiction.

2.2.2 If $\theta_4 = \delta$ (Figure 10-II), then $\alpha + \beta + \theta_4 \leq \pi$. As $\delta \geq \frac{\pi}{2}$, we conclude that $\alpha + \beta < \frac{\pi}{2}$, contradicting $2\alpha + \beta > \pi$.

Assume finally that $\theta_3 = \gamma$ (Figure 7).

3.1 If $2\gamma = \pi$, then $\delta > \frac{\pi}{2}$ and the extended configuration leads us to $\beta + \delta = \pi$. By the adjacency condition (1.1) we have $\alpha > \frac{\pi}{2}$, violating the condition $\alpha + \gamma < \pi$, see Figure 14.

3.2 Suppose now that $2\gamma < \pi$. The cases $\alpha \geq \frac{\pi}{2}$ and $\alpha < \frac{\pi}{2}$ will be analyzed separately.

3.2.1 Consider first the case in which $\alpha \geq \frac{\pi}{2}$. By the adjacency condition (1.1), $\delta > \frac{\pi}{2}$.

In case, $\alpha > \frac{\pi}{2}$, we extend the configuration in Figure 7 to the one in Figure 15, where tile 7 is congruent to $T_1$ or to $T_2$. If tile 7 is congruent to $T_1$, then the extended configuration ends up at a vertex partially surrounded by the angle sequence $(\delta, \gamma, \alpha)$, which gives an alternate sum of at least $\delta + \alpha$ forbidden by the angle folding relation (see Figure 15-I). If tile 7 is congruent to $T_2$, then $\gamma + \delta = \pi$ (Figure 15-II); any other possibility leads to edge incompatibility.
The only remaining possible configuration is that shown in Figure 15-II with \( \gamma + \delta = \pi \) where the vertex, \( v_7 \) is partially surrounded by a sequence of angles \((\delta, \beta, \delta)\). The alternate sum \( 2\delta > \pi \) is too large; so the configuration cannot be extended at this vertex.

![Figure 15 Local configurations](image)

We must then have \( \alpha = \frac{\pi}{2} \). As seen before, \( \delta > \frac{\pi}{2} \). Extending the configuration in Figure 7, tile 7 is congruent to \( T_1 \) or \( T_2 \). In the first case, we get the configuration illustrated in Figure 16-I giving rise to a vertex partially surrounded by a sequence \((\delta, \gamma, \alpha)\), with an impossible alternate sum \( \delta + \alpha > \pi \). In the second case, we are led to the configuration in Figure 16-II. Edge matching requires there to be a \( \gamma \) or a \( \delta \) at \( v_8 \) next to triangle 7. But this gives an alternate sum of \( \delta + 2\gamma > \pi \) or \( 2\delta + \gamma > 2\delta > \pi \) contradicting the folding condition.

![Figure 16 Local configurations](image)

**3.2.2** Suppose now that \( \alpha < \frac{\pi}{2} \).

If \( \delta > \frac{\pi}{2} \), then vertices of valency four are surrounded by alternate angles \( \delta \) and \( \beta \) or \( \delta \) and \( \alpha \) or \( \delta \) and \( \gamma \). However, by the angle arrangement, the first possibility implies the impossible \( \gamma = \frac{\pi}{2} \). The second case violates edge compatibility, and the third case is ruled out by the adjacency condition (1.1), which implies \( \alpha > \frac{\pi}{2} \).

If \( \delta = \frac{\pi}{2} \), then \( \gamma > \frac{\pi}{4} \) and vertices partially surrounded by an angle sequence \((\alpha, \mu, \gamma)\), for some \( \mu \in \{\alpha, \beta, \gamma, \delta\} \), must have alternating sums \( \alpha + 2\gamma = \pi \) or \( 2\alpha + \gamma = \pi \) or \( \alpha + \gamma + k\beta = \pi, k \geq 1 \) or \( \alpha + 2\gamma + m\beta = \pi, m \geq 1 \).

**3.2.2.1** Assume that \( \alpha + 2\gamma = \pi \). Then, \( \frac{\pi}{4} < \gamma < \frac{\pi}{2} < \alpha < \frac{\pi}{2} = \delta \). The configuration started in Figure 7 extends now to one of those in Figures 17-I and II, depending on the two possible positions of tile 5.
conclude that the sum containing the alternate angles \( \alpha, \alpha, \alpha \) at vertex \( v \) in Figure 18-I) surrounded by an angle sequence \( (\delta, \gamma, \gamma, \gamma, \beta) \) whose sums of alternate angles are of the form \( \delta + \gamma + t\beta = \pi = 3\gamma + (t-1)\beta, \ t \geq 1 \). Consequently, \( \delta + \beta = 2\gamma \) and by the adjacency condition (1.1), we conclude that \( \gamma \approx 34.824^\circ \), which is impossible. Analyzing the other position for tile 8, a decision about the angle \( \theta_7 \in \{\gamma, \delta\} \) in tile 10 (Figure 18-II) must be taken.

If \( \theta_7 = \gamma \), the sums of the alternate angles are of the form \( \delta + \gamma + k\beta = \pi = 3\gamma + (k-1)\beta, \ k \geq 1 \). As before, an impossibility is achieved using the adjacency condition (1.1).

If \( \theta_7 = \delta \), taking into account that \( \beta + 2\gamma < \alpha + 2\gamma = \pi \) and the compatibility of the edges, the sum containing the alternate angles \( \beta \) and \( \gamma \), at vertex \( v_{10} \), must be \( 2\gamma + m\beta = \pi, \ m \geq 2 \) or \( 3\gamma + r\beta = \pi, \ r \geq 1 \) or \( \gamma + \delta + p\beta = \pi, \ p \geq 1 \).

In the first case, we may expand the configuration and get a vertex \( (v_{11} \text{ at Figure 19-I}) \) partially surrounded by a sequence \( (\alpha, \alpha, \alpha) \). This requires an alternate sum \( 2\alpha + \gamma = \pi \); therefore, \( \gamma = \alpha > \frac{\pi}{3} \), contradicting the condition \( \alpha + 2\gamma = \pi \).

In the second case, for \( r > 1 \) the angle arrangement, at vertex \( v_{10} \) (Figure 19-II) leads to another vertex partially surrounded by a sequence of angles \( (\alpha, \alpha, \alpha) \); this leads to a contradiction as in the previous subcase.

For \( r = 1 \), the adjacency condition (1.1) leads us to \( \gamma \approx 112.98^\circ \) (impossible) or \( \gamma \approx 50.180^\circ \), which implies \( \beta \approx 29.46^\circ \) and \( \alpha \approx 79.64^\circ \). Proceeding with the extension of the configuration, we must keep in mind that tile 19 has two possible positions, as shown in Figure 20-I.
One of the two possible positions for tile 19 leads to a vertex, $v_{12}$ partially surrounded by a sequence $(\alpha, \alpha, \alpha)$, forcing an alternate sum with two $\alpha$ angles; but the remaining angle is $\pi - 2\alpha \approx 20.72^\circ$, less than any of the four angles (see Figure 20-II). The other position for tile 19 gives rise to a similar contradiction at the vertex $v'_{12}$ (see Figure 20-III).

The last case, i.e. $\gamma + \delta + p\beta = \pi$, $p \geq 1$ leads to the sum $3\gamma + (p - 1)\beta = \pi$, as illustrated in Figure 21. But then $\delta = 2\gamma - \beta$, $\alpha = \pi - 2\gamma$, which is impossible by the adjacency condition (1.1).
3.2.2.2 Suppose now that $2\alpha + \gamma = \pi$. Then, $\frac{\pi}{2} < \gamma < \beta < \alpha < \frac{\pi}{2} = \delta$. Extending the configuration in Figure 7, a decision about the position of tile 8 must be taken (see Figure 22-I, II).

In Figure 22-I, at the vertex partially surrounded by the sequence $(\delta, \delta, \gamma)$, the sum of alternate angles containing $\gamma$ and $\delta$ violates the angle folding relation, since $\gamma + \delta + \mu > \pi$, for any $\mu \in \{\alpha, \beta, \gamma, \delta\}$ and $\gamma + \delta = \pi$ implies that $\gamma = \delta$, which is impossible.

In Figure 22-II, at the vertex partially surrounded by $(\gamma, \gamma, \beta, \gamma, \gamma)$, the sum $\beta + 2\gamma + \lambda$ does not satisfy the angle folding relation, for any, $\lambda \in \{\alpha, \beta, \gamma, \delta\}$ and $\beta + 2\gamma = \pi$ implies that $\beta \approx 76.34^\circ$ and $\alpha \approx 64.085^\circ$, which is impossible since $\alpha > \beta$.

3.2.2.3 Assume that $\alpha + \gamma + k\beta = \pi$, for some $k \geq 1$. As $\gamma > \frac{\pi}{6}$, one of the sums of alternate angles at vertices surrounded by the angular sequence $(\gamma, \beta, \gamma, \delta)$ is of the form $3\gamma = \pi$ or $2\gamma + \alpha = \pi$ or $2\gamma + p\beta = \pi$, $p \geq 1$ or $3\gamma + q\beta = \pi$, $q \geq 1$ or $\alpha + 2\gamma + m\beta = \pi$, $m \geq 1$.

In the first case, by the angle arrangement, we get the angular sequences $(\gamma, \beta, \gamma, \delta, \gamma)$ or $(\gamma, \beta, \gamma, \gamma, \delta)$. Either way, $\beta = \frac{\pi}{6}$, and from the assumption $\alpha + \gamma + k\beta = \pi$, we conclude that $\alpha = \frac{2\pi}{3} - k\frac{\pi}{6}$, with $k = 1, 2$, contradicting $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$.

In the second case, there is incompatibility in the edges.

In the third case and for $p = 1$, one has $2\gamma + \beta = \pi$ and from $2\alpha + \beta > \pi$, we get $\gamma < \alpha$. On the other hand, from the assumption $\alpha + \gamma + k\beta = \pi$, we conclude that $\alpha + \gamma + \beta \leq \pi = 2\gamma + \beta$, which is an absurdity since $\alpha > \gamma$. If $p > 1$, then the extended configuration leads us to a vertex surrounded by three angles $\alpha$ (vertex $v_{13}$ in Figure 23-I), whose sum $2\alpha$ must be of the form $2\alpha + \gamma = \pi$. Consequently, $\frac{\pi}{3} < \gamma < \beta$ implying $p = 1$, which is impossible.

In the fourth case, $3\gamma + q\beta = \pi$, $q \geq 1$. Extending the configuration, we are led to a vertex, $v_{13}'$ (Figure 23-II), surrounded by three angles $\alpha$, whose sum $2\alpha + \lambda$ violates the angle folding relation for any $\lambda \in \{\alpha, \beta, \gamma, \delta\}$ (Figure 23-II).
Consequently, all vertices surrounded by two alternate angles $\gamma$ must satisfy $\alpha + 2\gamma + m\beta = \pi$, $m \geq 1$. Taking into account that $2\gamma + \delta > \pi$ and also $2\alpha + \beta > \pi$, one has $\alpha + \beta < \delta = \frac{\pi}{2}$ and so $2\alpha + 2\beta < \pi < 2\alpha + \beta$, which is an absurdity.

3.2.2.4 Suppose finally that $\alpha + 2\gamma + m\beta = \pi$, $m \geq 1$. As $2\gamma + \delta > \pi$, then $\alpha + \beta < \delta = \frac{\pi}{2}$ contradicting $2\alpha + \beta > \pi$. □

**Proposition 2.3** If $\beta > \alpha$, then $\Omega(T_1, T_2)$ is composed by two discrete families of tilings denoted by $E^m$, $m \geq 2$ and $F^k$, $k \geq 4$; and two sporadic tilings denoted by $G$ and $H$ where the sums of alternate angles around vertices are respectively of the form:

- $\beta + \gamma = \pi$, $2\alpha + \gamma = \pi$, and $\delta = \frac{\pi}{m}$ for $E^m$;
- $\alpha = \frac{\pi}{k}$, $\beta + \gamma = \pi$, and $\alpha + \gamma + \delta = \pi$ for $F^k$;
- $\delta = \frac{\pi}{2}$, $\beta + \gamma = \pi$, and $\alpha + 2\gamma = \pi$ for $G$;
- $\delta = \frac{\pi}{2}$, $\alpha = \frac{\pi}{3}$, and $\beta + \gamma + \alpha = \pi$ for $H$.

**Proof** Assume that $\beta > \alpha$ and $\gamma > \delta$.

Using the adjacency condition (1.1), if $\alpha > \frac{\pi}{2}$, then $\beta > \frac{\pi}{2}$, $\gamma > \frac{\pi}{2}$, and there are no vertices of valency four, which is an impossibility. Thus, $\alpha \leq \frac{\pi}{2}$.

Starting from the configuration in Figure 24, a decision about the angle $\theta_8 \in \{\gamma, \delta\}$ must be made.

![Figure 23 Local configurations](image)

![Figure 24 Local configuration](image)
1.1 Suppose firstly that \( \theta_8 = \gamma \) and \( \beta + \theta_8 < \pi \). As \( \beta, \gamma > \frac{\pi}{2} \), \( 2\alpha + \beta > \pi \) and \( 2\gamma + \delta > \pi \), the sum containing the angles \( \beta \) and \( \theta_8 \) does not satisfy the angle folding relation and the configuration cannot be extended.

1.2 Suppose now that \( \theta_8 = \gamma \) and \( \beta + \theta_8 = \pi \). Adding some cells to the configuration, we get the one illustrated in Figure 25.

![Figure 25](image)

**Figure 25** Local configuration

If \( \alpha = \frac{\pi}{3} \), then \( \beta > \frac{\pi}{2} > \gamma > \delta \), violating the adjacency condition. Therefore, \( \alpha < \frac{\pi}{2} \) and since \( 2\alpha > \gamma \), then \( \alpha > \frac{\pi}{6} \).

1.2.1 As \( \beta + \gamma = \pi \), assume first that \( \beta = \gamma = \frac{\pi}{2} \). Therefore, \( \alpha > \frac{\pi}{4} \) and taking into account that \( 2\gamma + \delta > \pi \), \( \gamma < 2\alpha \) and the compatibility of the edges, vertex \( v_{15} \) is of valency six surrounded exclusively by angles \( \alpha \). By the adjacency condition (1.1), we get \( \delta = \arccos \frac{\sqrt{7}}{3} \approx 54.736^\circ \). Consequently, at vertex \( v_{14} \), the sum \( \alpha + \gamma + \mu \) does not satisfy the angle folding relation, for any \( \mu \in \{\alpha, \beta, \gamma, \delta\} \).

1.2.2 Assuming that \( \beta > \frac{\pi}{2} > \gamma \), the sum containing the alternate angles \( \alpha \) and \( \gamma \), at vertex \( v_{14} \), satisfies \( \alpha + 2\gamma = \pi \) or \( 2\alpha + \gamma = \pi \) or \( 3\alpha + \gamma = \pi \) or \( \alpha + \gamma + t\delta = \pi, \ t \geq 1 \).

1.2.2.1 If \( \alpha + 2\gamma = \pi \), then \( \alpha < \delta \) and from \( \gamma < 2\alpha \), we get \( \alpha > \frac{\pi}{3} \). Accordingly, the order relation between the angles is \( \frac{\pi}{3} < \alpha < \delta < \gamma < \frac{\pi}{2} < \beta \). We consider the vertex \( v_{15} \). Neither of the angles adjacent to triangles 3 and 9 can be \( \beta \), because this would give an alternating sum \( 2\alpha + \beta > \pi \). Edge matching rules out \( \delta \).

If there are any \( \gamma \) angles at the vertex there must be exactly two; but if \( \gamma + 2\alpha = \pi = \alpha + 2\gamma \), we get \( \gamma = \alpha \), which contradicts earlier calculations. Similarly, \( \gamma + 3\alpha = \alpha + 2\gamma \) implies \( \gamma = 2\alpha \), ruled out above.

The vertex \( v_{15} \) is thus of valency six or eight, and surrounded only by \( \alpha \) angles. Valency six would imply \( \alpha = \frac{\pi}{3} \) and thus \( \alpha + 2\gamma > \pi \) which is false. If the valency is eight, then \( \alpha = \frac{\pi}{4}, \gamma = \frac{3\pi}{8}, \beta = \frac{5\pi}{8} \) and, by the adjacency condition (1.1), \( \delta \approx 64.916^\circ \).

We extend the configuration in Figure 25 at vertex \( v_{15} \). The \( \alpha \) angle of tile 3 and the \( \gamma \) angle of tile 2 form part of an alternating sequence that must be completed by one more \( \gamma \) angle. This must be from a tile positioned as tile 7 in Figure 26-I; if it had the other orientation, edge matching would force a \( \delta \) next to tile 2 and a \( \beta \) next to tile 1. But this would imply \( \alpha + \beta + \delta = \pi \), which is inconsistent with the values above.

Once we have settled the orientation of tile 7, the other tiles of Figure 26-I follow, and we find a vertex, \( v_{16} \), partially surrounded by a sequence \( (\delta, \delta, \delta) \). Since the sum \( 2\delta + \mu \) does not
satisfy the angle folding relation, for any $\mu \in \{\alpha, \beta, \gamma, \delta\}$, the configuration cannot be extended at this vertex.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{local-configurations.png}
\caption{Local configurations}
\end{figure}

1.2.2.2 If vertex $v_{14}$ satisfies $2\alpha + \gamma = \pi$, then $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$; thus a vertex cannot be entirely surrounded by $\alpha$ angles. If a vertex has an $\alpha$ angle and no $\gamma$ angle, edge matching prevents the presence of $\delta$ angles. Thus if one of the alternate sets of angles at a vertex contains two $\alpha$ angles it must be completed by a $\gamma$ angle. Moreover, matching bases, the other alternate set also contains two $\alpha$ angles. Consequently vertex $v_{15}$ (and all other such vertices) must have four $\alpha$ angles and two (adjacent) $\gamma$ angles.

Similar arguments show that if one of the alternate sets of angles at a vertex contains an $\alpha$ angle and a $\gamma$ angle, it is completed by an $\alpha$. The only alternative would be $\alpha + \gamma + t\delta = \pi$, $t \geq 1$, which forces vertices surrounded by two alternate angles $\gamma$, whose sum $2\gamma + \mu$ violates the angle folding relation, for any $\mu \in \{\beta, \gamma, \delta\}$ and $2\gamma + \alpha = \pi$ implies the impossibility $\alpha = \gamma > \frac{\pi}{3}$, as illustrated in Figure 26-II.

On the other hand, vertices with even one $\beta$ angle must be of valency four with two $\beta$ angles and two $\gamma$ angles. Any other possibilities led to the edge incompatibility or a contradiction arises from the adjacency condition (1.1).

We may continue extending the configuration in Figure 25 and taking into account the two previous observations, vertices surrounded by several angles $\delta$ must satisfy $m\delta = \pi$, $m \geq 2$. Every such configuration may be completed, so the local configuration extends to a complete tiling, denoted by $\mathcal{E}^m$, $m \geq 2$. For each $m \in \mathbb{N}$, $m \geq 2$, $\mathcal{E}^m$ is composed of vertices with two $\beta$ and two $\gamma$ angles, vertices with four $\alpha$ and two $\gamma$ angles, and vertices with $2m\delta$ angles. The number of triangular faces of $\mathcal{E}^m$ is $8m$; the triangles are equally distributed among two congruence classes. The next figures shows a 2D and 3D representation of $\mathcal{E}^2$. $\mathcal{E}^2$ is composed of eight triangles congruent to $T_1$ and eight triangles congruent to $T_2$, where $\alpha = \gamma = 60^\circ$, $\beta = 120^\circ$, and $\delta = 90^\circ$.

This family of tilings can be obtained from the family of dihedral tilings $\mathcal{R}^k$ described in [9], where the prototiles are an isosceles triangle and a spherical square, by splitting each square into two isosceles triangles.

1.2.2.3 If vertex $v_{14}$ obeys the condition $3\alpha + \gamma = \pi$, then $\frac{\pi}{5} < \alpha < \frac{\pi}{3} < \gamma$. As $2\gamma + \delta > \pi$ and $\gamma < 2\alpha$ we have $\alpha < \delta$, and the order relation between the angles is $\frac{\pi}{5} < \alpha < \delta < \gamma < \frac{\pi}{3} < \beta$. 

In order to satisfy the angle folding relation while matching edges, vertices surrounded by two alternate angles $\delta$ are of valency six (with $\delta = \frac{\pi}{3}$ or $2\delta + \gamma = \pi$) or valency eight (with $\delta = \frac{\pi}{4}$, $2\delta + \alpha + \gamma = \pi$ or $3\delta + \gamma = \pi$).

We study each of these possibilities.
If $\delta = \frac{\pi}{4}$, by the adjacency condition (1.1), we get $\alpha \approx 38.211^\circ$ or $\alpha \approx 94.704^\circ$. As $\alpha < \frac{\pi}{2}$, then $\alpha \approx 38.211^\circ$ and consequently $\beta \approx 114.62^\circ$ and $\gamma \approx 65.38^\circ$. However, extending the configuration illustrated in Figure 25, we end up at a vertex $v_{17}$, surrounded by the alternate angles $\gamma, \alpha, \gamma$, whose sum $2\gamma + \alpha + \mu$ violates the angle folding relation for any $\mu \in \{\alpha, \beta, \gamma, \delta\}$ and so the configuration cannot be extended (see Figure 28-I).

In case $\delta = \frac{\pi}{4}$, then $\alpha \approx 36.12^\circ$, $\beta \approx 110.136^\circ$ and $\gamma \approx 69.864^\circ$ but a similar impossibility arises.

- In case $2\beta + \gamma = \pi$, again by the adjacency condition (1.1), one has $\alpha \approx 37.857^\circ$ or $\alpha \approx 77.459^\circ$ or $\alpha = \frac{2\pi}{3}$. As $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$, then $\alpha \approx 37, 857^\circ$, $\beta \approx 113.571^\circ$, $\gamma \approx 66.429^\circ$ and $\delta \approx 56.785^\circ$. Adding some cells to the configuration in Figure 25, depending the position of tile 12, we obtain a sequence of adjacent angles $\beta, \gamma, \delta$ (Figure 28-II) or $\alpha, \alpha, \gamma, \delta$ (Figure 28-III). Neither of these is compatible with the angle folding relation. If $2\delta + \alpha + \gamma = \pi$ or $3\delta + \gamma = \pi$, we obtain $\alpha = \delta$, which is impossible.

1.2.2.4 Assume that vertex $v_{14}$ satisfies $\alpha + \gamma + t\delta = \pi$ for $t \geq 1$. Then, vertex $v_{15}$ is surrounded by six, eight or ten $\alpha$ angles, or by two $\gamma$ angles and four or six $\alpha$ angles. We study these five possible cases.

- If $\alpha = \frac{\pi}{k}, k = 3, 4, 5$, vertices surrounded by the angle $\beta$ must be of valency four, as in Case 1.2.2.2.

Also, vertices which have both $\alpha$ and $\gamma$ angles in alternating sums must satisfy $\alpha + \gamma + t\delta = \pi$, $t \geq 1$.

We cannot have $\alpha = \frac{\pi}{3}$, as no third angle $\mu$ makes the sum $\alpha + \gamma + \mu = \pi$, for $\mu \in \{\alpha, \beta, \gamma, \delta\}$.

For $\alpha = \frac{\pi}{4}$, vertices surrounded by alternate angles $\alpha$ and $\gamma$ would require either $2\alpha + \gamma$ or $\alpha + 2\gamma$ to sum to $\pi$. But the first of these implies $\gamma = \frac{\pi}{2}$, while the second requires $\gamma = \frac{3\pi}{8}$ and $\beta = \frac{5\pi}{8}$, so that $\delta > \beta - \gamma = \frac{\pi}{4}$. Consequently, from $\alpha + \gamma + t\delta = \pi$, we conclude that $t = 1$ and so $\gamma = \delta$, which is impossible. A very similar argument applies when $\alpha = \frac{\pi}{5}$. Accordingly, we need only consider triangles with angles satisfying the equations $\beta + \gamma = \pi$, $\alpha + \gamma + t\delta = \pi$, and $\alpha = \frac{\pi}{k}$, where $k = 3, 4$ or 5.

We may add some cells to the configuration in Figure 25 but it is impossible to complete the tiling for $t \geq 2$, since, by the angle arrangement, it is not possible that vertices surrounded by the angles $\gamma, \delta, \gamma, \alpha$ satisfy $\alpha + \gamma + t\delta = \pi$ (see Figure 29-I for the particular case $\alpha = \frac{\pi}{3}$).

If $t = 1$, we get $\beta < 0$ (for $\alpha = \frac{\pi}{3}$), $\beta \approx 56.075^\circ$ or $115.17^\circ$ (for $\alpha = \frac{\pi}{4}$) and $\beta \approx 40.538^\circ$ (for $\alpha = \frac{\pi}{5}$). But we know that $\beta$ is obtuse, eliminating all of these except for $\beta \approx 115.17^\circ$ and $\alpha = \frac{\pi}{4}$, whence $\gamma \approx 64.83^\circ$ and $\delta \approx 70.17^\circ$; but this contradicts $\gamma \geq \delta$. Thus none of these cases lead to a tiling.

- In the case in which $2\alpha + \gamma = \pi$, then we may add some new cells to the configuration in Figure 25 and get a vertex, in Figure 29-II, partially surrounded by a sequence of angles $\gamma, \beta, \gamma$. Alternate angles at this vertex must have a sum $2\gamma + \mu, \mu \in \{\alpha, \beta, \gamma, \delta\}$ which can only satisfy the angle folding condition if $\alpha = \gamma > \frac{\pi}{5}$, which is impossible.

- In the case in which $3\alpha + \gamma = \pi$, we have $\alpha > \frac{\pi}{4}$ and $\gamma < \frac{2\pi}{3}$ implying that $\delta > \beta - \gamma > \frac{\pi}{10}$. Consequently, $t = 1, 2, 3$ or 4. Using the adjacency condition (1.1) for $t = 1$, we get $\alpha \approx 70.134^\circ$, $41.041^\circ$, or $114.39^\circ$. These values are impossible, since the first one implies that $\beta \approx 210.39^\circ$, the second one that $\gamma \approx 56.877^\circ$ (but we know $\gamma > \frac{\pi}{3}$), and the third one is greater than $\frac{\pi}{2}$.
Spherical f-Tilings by Isosceles Triangles-II

Using the same adjacency condition for \( t = 2 \) and \( t = 3 \), we again get only values for \( \alpha \) which contradict our assumption. These values are: \( \alpha = \frac{\pi}{5}, \frac{3\pi}{5}, \) or \( \frac{\pi}{2} \), for \( t = 2 \); \( \alpha \approx 35.26^\circ \), for \( t = 3 \); and \( \alpha \approx 35.12^\circ \) for \( t = 4 \).

\[ \text{Figure 29 Local configurations} \]

1.2.3 If \( \beta + \gamma = \pi \) but \( \beta < \frac{\pi}{3} \) and \( \gamma > \frac{\pi}{3} \), then \( \alpha > \frac{\pi}{3} \). By edge matching, vertex \( v_{15} \) must be of valency six surrounded exclusively by angles \( \alpha \); so \( \alpha = \frac{\pi}{3} \). Looking now at vertex \( v_{14} \) in Figure 25, we conclude that the sum of alternate angles must satisfy \( \alpha + \gamma + t\delta = \pi, t \geq 1 \).

As in Case 1.2.2.4, it is impossible to complete the tiling for \( t \geq 2 \); and for \( t = 1 \), we would have \( \beta < 0 \).

1.3 Assume now that \( \theta_8 = \delta \) (see Figure 30) and \( \beta + \theta_8 < \pi \). As \( \beta, \gamma > \frac{\pi}{3} \), \( 2\alpha + \beta > \pi \) and \( 2\gamma + \delta > \pi \), the sum containing \( \beta \) and \( \theta_8 \) is of the form \( \beta + k\delta = \pi \), \( k \geq 2 \), \( 2\beta + m\delta = \pi \), \( m \geq 1 \), \( \beta + n\delta + \gamma = \pi \), \( n \geq 1 \), or \( \beta + p\delta + \alpha = \pi \), \( p \geq 1 \).

\[ \text{Figure 30 Local configuration} \]

The first and second cases are incompatible with \( 2\gamma + \delta > \pi \) and \( \beta, \gamma > \frac{\pi}{3} \).

In the fourth case, \( \beta + p\delta + \alpha = \pi \) implies that the other sum of alternate angles satisfies \( \alpha + 2\gamma + (p - 1)\delta = \pi \), which is only possible if \( p = 1 \). Then \( \alpha + 2\gamma = \pi = \beta + \delta + \alpha \); but form this it would follow that \( \delta < \alpha < \delta \).

The third case, \( \beta + n\delta + \gamma = \pi \), \( n \geq 1 \) remains to be considered. Although the angle \( \gamma \) might be placed in any of several positions, only one does not violate the conditions \( \gamma > \frac{\pi}{3} \) and
2γ + δ > π, illustrated in Figure 31-I, at vertex v_{18}. Taking into account that \( \beta + n\delta + \gamma = \pi \), \( n \geq 1 \), \( \alpha + \gamma < \pi \) and \( \alpha + \delta < \pi \), vertices of valency four are surrounded by four \( \alpha \) angles or by four \( \beta \) angles or by four \( \gamma \) angles, or alternating angles \( \alpha \) and \( \beta \) angles. Consequently, either \( \beta \geq \frac{\pi}{2} \) or \( \gamma = \frac{\pi}{2} \).

**Figure 31 Local configurations**

1.3.1 If we have \( \beta + n\delta + \gamma = \pi \) and \( \beta \geq \frac{\pi}{2} \), we can deduce that \( \delta < \gamma < \frac{\pi}{2} \). Therefore, \( \beta + \gamma + n\delta > \gamma + \gamma + n\delta > \pi \), which is impossible.

1.3.2 If \( \beta + n\delta + \gamma = \pi \) and \( \gamma = \frac{\pi}{2} \), then \( \delta < \frac{\pi}{2} \) and as \( \beta + n\delta + \gamma = \pi \), \( n \geq 1 \), we get \( \alpha < \beta < \frac{\pi}{2} \). On the other hand, because \( 2\alpha + \beta > \pi \), then \( \alpha > \frac{\pi}{4} \). Consequently, vertex \( v_{14} \) in Figure 25 is of valency \( 2(2 + t) \) satisfying \( \alpha + \gamma + t\delta = \pi \), for some \( t \geq 1 \). The extended configuration ends up at a vertex of valency four surrounded (in order) by angles \( \gamma, \delta, \gamma, \beta \); but this is impossible, since \( \beta + \delta < \pi \) (see Figure 31-II).

1.4 If \( \theta_8 = \delta \) and \( \beta + \theta_8 = \pi \), the angle arrangement forces \( 2\gamma = \pi \). As \( \delta < \gamma \), we must have \( \delta < \frac{\pi}{2} < \beta \). Extending the configuration in Figure 30, tile 6 has two possible positions, as shown in the next figure.

**Figure 32 Local configurations**

Either of these configurations can be extended in a unique way leading to a vertex surrounded in part by a sequence of adjacent angles \( \beta, \beta, \alpha \) (on the bottom edge of the diagrams in Figure 33). This vertex cannot be of valency four, which would require both \( \alpha \) and \( \beta \) to be
right angles. However, since $\gamma = \frac{\pi}{2}$, $\beta > \frac{\pi}{2}$, $\beta + \delta = \pi$ and $2\alpha + \beta > \pi$, then $\alpha + \beta + \mu > \pi$, for any $\mu \in \{\alpha, \beta, \gamma, \delta\}$, so neither of these partial tilings can be completed as an $f$-tiling.

**Figure 33** Local configurations

Assume that $\beta > \alpha$ and $\delta > \gamma$.

Starting from the configuration illustrated in Figure 24, we have two cases to consider, depending on whether $\theta_8$ is $\gamma$ or $\delta$.

**1** Suppose firstly that $\theta_8 = \gamma$. We shall study the cases $\beta + \theta_8 = \pi$ and $\beta + \theta_8 < \pi$ separately.

**1.1** If $\beta + \theta_8 = \pi$ with $\theta_8 = \gamma$, extending the configuration leads us to a vertex surrounded by three angles $\alpha$ as shown in Figure 34-I.

**Figure 34** Local configurations

If $\alpha = \frac{\pi}{2}$, then from the adjacency condition, we conclude that $\delta > \frac{\pi}{2}$. As $\beta > \alpha$, then $\beta > \frac{\pi}{2}$ and so $\gamma < \frac{\pi}{2}$. Therefore, at vertex $v_{14}$, the sum containing the alternate angles $\alpha$ and $\gamma$ must be of the form $\alpha + t\gamma = \pi$, $t \geq 2$. However, at the vertex with three angles $\delta$, there would have to be an alternating sum containing two $\delta$ angles, which is impossible (Figure 34-II).

If $\alpha < \frac{\pi}{2}$, as $\gamma < \delta$, by the adjacency condition, $\gamma < \frac{\pi}{2}$ and so $\beta > \frac{\pi}{2}$. Now, the configuration in Figure 35 can be extended with $\theta_9 = \gamma$ or $\theta_9 = \delta$. 

**Figure 35** Local configurations
1.1.1 If $\theta_9 = \gamma$, then $\delta + \gamma < \pi = \beta + \gamma$ and so $\delta < \beta$ (observe that $\delta + \gamma = \pi$ implies $\cot \alpha = \cot \beta$, which is impossible).

Since the other possibilities violate the angle folding condition, the sum containing the alternate angles $\delta$ and $\gamma$ must be of the form $\delta + \gamma + \alpha = \pi$. Accordingly, $\alpha < \gamma < \frac{\pi}{2} < \beta$ and $\delta < \beta$. As $\delta > \gamma > \alpha$ and $\delta + \gamma + \alpha = \pi$, we also have $\alpha < \frac{\pi}{3}$. Extending the configuration in Figure 35, we get a tiling with three types of vertices: vertices of valency four, six and $2k$, $k \geq 4$, surrounded, respectively, by angles $(\beta, \beta, \gamma, \gamma)$, $(\delta, \delta, \gamma, \alpha, \alpha, \gamma)$ and $(\alpha, \alpha, \ldots, \alpha, \alpha)$. This tiling is denoted by $\mathcal{F}^k$. It has $8k$ triangular faces, equally distributed in two congruence classes. The next figure shows 2D and 3D representations of $\mathcal{F}^4$. In this case, $\alpha = \frac{\pi}{4}$, $\gamma \approx 64.83^\circ$, $\delta \approx 70.17^\circ$ and $\beta \approx 115.7^\circ$, and there are sixteen of each type.

Observe that all these tilings $\mathcal{F}^k$, $k \geq 4$, are obtained from the family of dihedral $f$-tilings $\mathcal{F}R_{\alpha_2}^k$ of [9], where the prototiles are an isosceles triangle and a parallelogram, by splitting the parallelogram tiles.

1.1.2 If $\theta_9 = \delta$ and $2\delta = \pi$, then $\gamma > \frac{\pi}{4}$. Taking into account that $2\alpha > \pi - \beta = \gamma > \frac{\pi}{4}$ and the compatibility of the edges, we conclude that $\alpha > \frac{\pi}{8}$; and at a vertex surrounded (in part) by four angles $\alpha$, the alternating sum containing two angles $\alpha$ is either of the form $k\alpha = \pi$ for $k \in \{3, \ldots, 7\}$, or $p\alpha + q\gamma = \pi$, where $p \geq 2$, $q \geq 1$, and $p + q \leq 6$. 
1.1.2.1 If \( k \alpha = \pi, k = 3, \ldots, 7 \), we get

for \( k = 3 \), \( \gamma = 60^\circ, \beta = 120^\circ \) or \( \gamma \approx 142.533^\circ, \beta \approx 37.467^\circ \);

for \( k = 4 \), \( \gamma \approx 54.373^\circ, \beta \approx 125.627^\circ \) or \( \gamma \approx 154.612^\circ, \beta \approx 25.388^\circ \);

for \( k = 5 \), \( \gamma \approx 50.988^\circ, \beta \approx 129.012^\circ \) or \( \gamma \approx 160.577^\circ, \beta \approx 19.423^\circ \);

for \( k = 6 \), \( \gamma \approx 48.534^\circ, \beta \approx 131.466^\circ \) or \( \gamma \approx 164.193^\circ, \beta \approx 15.807^\circ \);

for \( k = 7 \), \( \gamma \approx 46.616^\circ, \beta \approx 133.384^\circ \) or \( \gamma \approx 166.641^\circ, \beta \approx 13.359^\circ \).

It is impossible to extend the configuration for cases \( k = 4, 5, 6, 7 \), since at vertex \( v_{14} \), the sum \( \alpha + \gamma + \mu \) violates the angle folding relation, for any \( \mu \in \{\alpha, \beta, \gamma, \delta\} \). For the case \( k = 3 \), one has \( \alpha = \gamma = \frac{\pi}{3}, \beta = \frac{2\pi}{3} \) and consequently vertex \( v_{14} \) is of valency six satisfying \( 2\gamma + \alpha = \pi \).

The configuration in Figure 35 extends to a tiling of the entire sphere, which is denoted by \( \mathcal{G} \). Figure 37 presents a 2D and 3D representation of \( \mathcal{G} \), which is composed of six copies of \( T_1 \) and twelve copies of \( T_2 \).

![Figure 37 2D and 3D representations of \( \mathcal{G} \)](image)

This tiling can be obtained from the spherical cube by dividing the three faces that meet at one vertex into two equilateral triangles and the others into four. Bisecting the larger triangles yields the tetrakis cube, the dual of the truncated octahedron. However, parity considerations make it impossible to combine pairs of triangles and obtain one of the (triangle, well-centered quadrangle) \( f \)-tilings of [9].

1.1.2.2 In the second case, suppose first that \( q = 1 \). Then, \( p = 2, 3, 4, \) or \( 5 \). For \( p = 2 \),

\( 2\alpha + \gamma = \pi, \beta + \gamma = \pi, \) and \( \delta = \frac{\pi}{2} \). We can extend the configuration of Figure 35 to the complete tiling \( \mathcal{E}^2 \) shown in Figure 27. The angles here are the same as in \( \mathcal{G} \), but the tiling has eight tiles of each type.

Like \( \mathcal{G} \), \( \mathcal{E}^2 \) can be obtained as a subdivision of the spherical cube or by combining faces of the spherical tetrakis cube. Combining the larger faces along their hypotenuses yields the \( f \)-tiling \( \mathcal{R}^2 \) described in [9].

For \( p = 3 \),

\( 3\alpha + \gamma = \pi, \beta + \gamma = \pi, \delta = \frac{\pi}{2} \). By the adjacency condition (1.1), one has \( \alpha = \frac{\pi}{2} \), which is impossible.
For $p = 4$, we get $\alpha \approx 32.594^\circ$, $\gamma \approx 49.624^\circ$, $\beta \approx 130,376^\circ$, and $\delta = 90^\circ$. Extending the configuration in Figure 35, we end up at a vertex (of valency greater than six) partially surrounded by angles $\gamma, \alpha, \alpha, \gamma, \gamma$ (see Figure 38). The folding relation then requires $\pi - (\gamma + \alpha + \gamma) \approx 48.158^\circ$ to be the sum of one or more angles, but it is easily seen that this is impossible. A similar calculation rules out $p = 5$.

![Figure 38 Local configuration](image1)

Suppose now that $q = 2$. Then, we can have only $2\alpha + 2\gamma = \pi$ or $3\alpha + 2\gamma = \pi$, since $\alpha > \frac{\pi}{8}$ and $\gamma > \frac{\pi}{8}$. If $2\alpha + 2\gamma = \pi$, by the adjacency condition (1.1), $\alpha \approx 38.173^\circ$, $\gamma \approx 51.827^\circ$, $\beta \approx 128.173^\circ$, and and $\delta = 90^\circ$. The configuration in Figure 35 can be extended to the configuration in Figure 39, which shows a vertex, $v_{19}$, partially surrounded by five angles $\gamma$. This implies an alternate sum containing three $\gamma$ angles; but as $\pi - 3\gamma \approx 24.519^\circ$, the configuration at this vertex cannot be completed in accordance with the folding relation.

![Figure 39 Local configuration](image2)

Suppose instead that $3\alpha + 2\gamma = \pi$. By the adjacency condition (1.1), we get $\alpha \approx 28.176^\circ$, $\gamma \approx 47.735^\circ$ and $\beta \approx 132.265^\circ$. Extending the configuration in Figure 35, tile 8 may be placed in either of two ways, as shown in Figure 40-I, II.

In Figure 40-I, we reach a vertex with contiguous angles $\beta, \gamma, \gamma, \alpha$. But as $\beta + \gamma = \pi$ the vertex would have to be of valency four and $\alpha + \gamma \neq \pi$, in violation of the folding condition. With the other choice for tile 8, tile 12 can be as in Figure 40-II or as in Figure 41. These lead to similar impossibilities.
1.1.3 If \( \theta_9 = \delta \) but \( 2\delta < \pi \), it is impossible to extend the configuration in Figure 35, since there is always incompatibility of the edges or violation of the angle folding relation at the vertex partly surrounded by three angles \( \delta \).

1.2 Suppose that \( \beta + \theta_8 < \pi \) for \( \theta_8 = \gamma \) (Figure 30). In order to satisfy the angle folding relation, the sum containing these two angles must be of the form \( \beta + \gamma + \beta = \pi \) or \( \beta + n\gamma = \pi \), \( n \geq 2 \) or \( \beta + \gamma + \delta = \pi \) or \( \beta + n\gamma + \alpha = \pi \), \( n \geq 1 \).

1.2.1 If \( \beta + \gamma + \beta = \pi \), then \( \gamma < \alpha < \beta < \frac{\pi}{2} \) and \( \beta < \delta \). Therefore, \( \delta \geq \frac{\pi}{2} \).

Assume first that \( \delta = \frac{\pi}{2} \). The configuration in Figure 24 may be extended and we get several vertices partly surrounded by four angles \( \alpha \), as is shown in the next figure.

In order to satisfy the angle folding relation, the sum containing two alternate angles \( \alpha \) must be \( 3\alpha = \pi \) or \( 2\alpha + \gamma = \pi \), since \( \alpha > \gamma > \frac{\pi}{4} \).

For \( \alpha = \frac{\pi}{3} \), by the adjacency condition (1.1) and \( \frac{\pi}{4} < \gamma < \alpha = \frac{\pi}{3} < \beta < \frac{\pi}{2} = \delta \), one has \( \beta \approx 66.579^\circ \) and \( \gamma \approx 46.842^\circ \). With these values, it is not possible to surround vertex \( v_{14} \) without violating the angle folding relation.

If \( 2\alpha + \gamma = \pi \), then \( 2\alpha + \gamma = 2\beta + \gamma = \pi \), implying the impossible \( \alpha = \beta \).

Assuming now that \( \delta > \frac{\pi}{2} \), then \( \theta_{10} = \delta \). Taking into account that \( 2\gamma + \delta > \pi \), \( 2\alpha + \beta > \pi \), \( \gamma < \alpha < \beta < \frac{\pi}{2} < \delta \) and the compatibility of the edges, it is impossible to continue the
1.2.2 We will consider the case $\beta + k\gamma + \alpha = \pi$ and we shall study the cases $k = 1$ and $k > 1$ separately.

![Figure 42 Local configuration](image1)

**Figure 42 Local configuration**

1.2.2.1 If $k = 1$, then $\gamma < \alpha < \frac{\pi}{2}$, and vertex $v_{14}$ in Figure 24 must have valency greater than four (Figure 43).

![Figure 43 Local configuration](image2)

**Figure 43 Local configuration**

If the angle $\theta_{10}$ was $\delta$, then vertex $v_{14}$ would be of valency greater than four. However, as $\gamma < \alpha < \beta$, $\gamma < \delta$ and $2\gamma + \delta > \pi$, the sum containing the angles $\alpha$ and $\delta$ would not satisfy the
angle folding relation. Therefore, $\theta_{10} = \gamma$ and $\delta \leq \frac{\pi}{2}$. Extending the configuration further, the tile numbered 6 has two possible positions, as shown in Figure 44-I and II.

1.2.2.1.1 Suppose $\delta = \frac{\pi}{2}$. Then $\alpha > \gamma > \frac{\pi}{4}$. In Figure 44-I, the vertex partially surrounded by four angles $\alpha$ must be of valency six with $3\alpha = \pi$ (we rule out the other possibility, $2\alpha + \gamma = \pi$ because which would make $\beta = \alpha$).

By the adjacency condition (1.1), $\beta \approx 71.774^\circ$ and $\gamma \approx 48.226^\circ$. Extending the configuration, we get a tiling denoted by $H$. 2D and 3D representations of $H$ are shown in Figure 45.

It uses forty-eight triangles congruent to $T_1$ and twenty-four triangles congruent to $T_2$. Combinatorially, it is a tetrakis-hexakis truncated octahedron, but it has the symmetry group of a snub cuboctahedron.

Figure 45 2D and 3D representations of $H$
In Figure 44-II, we may add some cells to the configuration and end up at a vertex partially surrounded by the sequence of angles \( \alpha, \alpha, \gamma, \gamma, \alpha \) (Figure 46). This vertex would have to have exactly one more angle, with \( 2\alpha + \gamma = \pi \). But \( \beta + \gamma + \alpha = \pi \), so we have a contradiction.

**Figure 46** Local configuration

1.2.2.1.2 If \( \delta < \frac{\pi}{2} \), the vertex \( v_20 \) with three consecutive \( \delta \) angles cannot be completed while satisfying the angle folding relation, since \( \gamma < \alpha < \beta \), \( \gamma < \delta \) and \( 2\gamma + \delta > \pi \).

1.2.2.1 Suppose \( \beta + \kappa \gamma + \alpha = \pi \), \( k \geq 2 \). Then \( \gamma < \alpha < \frac{\pi}{2} \). This condition implies that \( \beta \geq \frac{\pi}{2} \) or \( \delta \geq \frac{\pi}{2} \), while vertices of valency four can have angles (in order) \((\beta, \beta, \beta, \beta), (\delta, \delta, \delta, \delta), (\delta, \gamma, \gamma, \delta)\), as all other possibilities are incompatible with edge matching.

In the first case, \( \alpha > \frac{\pi}{2} \) and from \( \beta + \kappa \gamma + \alpha = \pi < 2\gamma + \delta \), we conclude that \( \delta > \beta + \alpha > \frac{\pi}{2} \).

Therefore, adding some cells to the configuration in Figure 30, we are led to a vertex with contiguous angles \( \alpha, \gamma, \delta \). But \( \alpha + \delta \neq \pi \), nor can we add any other angle or combination of angles give us a sum of \( \pi \). Thus this configuration cannot appear in an \( f \)-tiling.

Either of the other angle sets require \( \delta = \frac{\pi}{2} \), then \( \alpha > \gamma > \frac{\pi}{4} \) and so \( k = 1 \), which is impossible.

In the third case, \( \delta + \gamma = \pi, \delta > \frac{\pi}{2} > \gamma \) and the sum containing the alternate angles \( \alpha \) and \( \delta \) violates the angle folding relation, since \( \alpha + \delta > \gamma + \delta = \pi \).

1.2.3 Assuming now that \( \beta + \gamma + n\gamma = \pi \) for some \( n \geq 1 \), then \( \gamma < \alpha < \beta < \delta \). Extending the configuration of Figure 24, we consider the angle \( \theta_{11} \) of Figure 47-I. If \( \theta_{11} = \delta \), then the sum containing the alternate angles \( \alpha \) and \( \theta_{11} \) does not satisfy the angle folding relation, since \( \alpha + \delta + \mu > \pi \) for any \( \mu \in \{\alpha, \beta, \gamma, \delta\} \). Hence, \( \theta_{11} = \gamma \) and \( 2\delta \leq \pi \). The tile numbered 5 has two possible positions (Figure 47-II, III).

1.2.3.1 If \( \delta = \frac{\pi}{2} \), then \( \alpha > \gamma > \frac{\pi}{4} \) and consequently \( n = 1 \). Accordingly, \( \beta + 2\gamma = \pi \). Thus, vertex \( v_{14} \) must be of valency six with \( 2\alpha + \gamma = \pi \), since the other possibilities contradict \( \alpha \neq \beta \) or \( \alpha \neq \gamma \). By the adjacency condition (1.1), one has \( \alpha \approx 64.087^\circ, \gamma \approx 51.826^\circ \) and \( \beta \approx 76.348^\circ \).

Extending the configuration in Figure 47-II, we are led to vertex \( v_{22} \), surrounded in part by three angles \( \beta \). But \( 2\beta < \pi \) while \( \pi - 2\beta \) is less than any of the four angles (Figure 48-I.)

Extending the configuration in Figure 47-III, we are lead to a vertex \( v_{23} \) (see Figure 48-II) partly surrounded by a sequence of contiguous angles \( (\alpha, \gamma, \gamma, \beta, \beta) \) arises. But \( \alpha + \gamma + \beta > \pi \), so this is impossible in an \( f \)-tiling.
1.2.3.2 If $\delta < \frac{\pi}{3}$, then $\alpha > \gamma > \frac{\pi}{3}$ and so $\beta \geq \frac{\pi}{2}$. As before, continuing the tiling requires a vertex configuration (at $v_{21}$ in Figure 47) in which a set of alternate angles contains two $\delta$ angles; but $2\delta < \pi$ and no other angle or sum of angles can equal $\pi - 2\delta$; so we cannot complete the $f$-tiling.

1.2.4 Assuming that $\beta + \gamma + \delta = \pi$, from $2\gamma + \delta > \pi$ one has $\frac{\pi}{3} < \beta < \gamma$, contradicting the equality $\beta + \gamma + \delta = \pi$.

2 Suppose that $\theta_{8} = \delta$ (see Figure 24). As $\beta, \delta > \frac{\pi}{3}$, $2\alpha + \beta > \pi$ and $2\gamma + \delta > \pi$, then the sum containing $\beta$ and $\theta_{8}$ must be of the form $\beta + \delta + \alpha = \pi$ or $\beta + \delta + \gamma = \pi$ or $\beta + \delta = \pi$.

In the first case, one has $\frac{\pi}{3} < \delta < \alpha$, contradicting the equality and in the second case, $\frac{\pi}{3} < \beta < \gamma$ leading to the same contradiction. Therefore, $\beta + \delta = \pi$ and the angle arrangement implies that $\gamma = \frac{\pi}{2}$. Consequently, $\beta < \frac{\pi}{2} < \delta$ and the order relation between the angles is now $\frac{\pi}{6} < \alpha < \beta < \frac{\pi}{2} = \gamma < \delta$. The sum containing the alternate angles $\alpha$ and $\gamma$ must obeys to the condition $2\alpha + \gamma = \pi$. Consequently, $\alpha = \frac{\pi}{4}$ and by the adjacency condition (1.1), $\beta \approx 122, 94^\circ$ violating $\beta < \frac{\pi}{2}$.

Table 1 shows a complete list of all spherical dihedral $f$-tilings, whose prototiles are two isosceles triangles denoted by $T_1$ with angles $\alpha, \alpha, \beta$ and $T_2$ of angles $\gamma, \gamma, \delta$. We have used the following notation:
• $M$ and $N$ are, respectively, the number of triangles congruent to $T_1$ and the number of triangles congruent to $T_2$ used in such dihedral $f$-tilings;
• $\alpha = \alpha_0^m$, $m \geq 2$ is the solution of (1.1) with $\beta = 2\alpha$, $\gamma = \pi - 2\alpha$ and $\delta = \frac{\pi}{m}$;
• $\gamma = \gamma_0^k$, $k \geq 4$ is the solution of (1.1) with $\alpha = \frac{\pi}{k}$, $\beta = \pi - \gamma$ and $\delta = \pi - \alpha - \gamma$.

| $f$-tiling | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $M$ | $N$ |
|------------|----------|----------|----------|----------|-----|-----|
| $\mathcal{E}^m$, $m \geq 2$ | $\alpha_0^m$ | $2\alpha$ | $\pi - 2\alpha$ | $\frac{\pi}{m}$ | $4m$ | $4m$ |
| $\mathcal{F}^k$, $k \geq 4$ | $\frac{\pi}{k}$ | $\pi - \gamma$ | $\gamma_0^k$ | $\pi - \alpha - \gamma$ | $4k$ | $4k$ |
| $\mathcal{G}$ | $\frac{\pi}{3}$ | $\frac{\pi}{3}$ | $\frac{\pi}{3}$ | $\frac{\pi}{3}$ | 6 | 12 |
| $\mathcal{H}$ | $\frac{\pi}{3}$ | $71,771^\circ$ | $48,226^\circ$ | $\frac{\pi}{3}$ | 48 | 24 |

Table 1 Dihedral $f$-tilings of the sphere by isosceles triangles with adjacency of type II

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