1. Introduction

1.1. Background

Estimating an unknown distribution from its samples is a fundamental statistical problem arising in many applications such as modeling language, stocks, weather, traffic patterns, and many more. It has therefore been studied for over a century, e.g. (Pearson, 1895).

Consider an unknown univariate distribution $f$ over $\mathbb{R}$, generating $n$ samples $X^n \equiv X_1, \ldots, X_n$. An estimator for $f$ is a mapping $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$. As in many of the prior works, we evaluate $\hat{f}$ using the $\ell_1$ distance, $\|\hat{f} - f\|_1$. The $\ell_1$ distance professes several desirable properties, including scale and location invariance, and provides provable guarantees on the values of Lipschitz functionals of $f$ (Devroye & Lugosi, 2001).

Ideally, we would prefer an estimator that learns any distribution. However, arbitrary distributions cannot be learned with any number of samples. Let $u$ be the continuous uniform distribution over $[0, 1]$. For any number $n$ of samples, uniformly select $n^3$ points from $[0, 1]$ and let $p$ be the discrete uniform distribution over these $n^3$ points. Since with high probability collisions do not occur within samples under either distribution, $u$ and $p$ cannot be distinguished from the uniformly occurring samples. As $\|u - p\|_1 = 2$, it follows that for any estimator $\hat{f}$, $\max_{f \in \{u, p\}} E\|\hat{f} - f\|_1 \gtrsim 1$.

A common modification, motivated by PAC agnostic learning, assumes that $f$ is close to a natural distribution class $C$, and tries to find the distribution in $C$ closest to $f$. The following notion of $\text{OPT}_C(f)$ considers this lowest distance,

$$\text{OPT}_C(f) \equiv \inf_{g \in C} \|f - g\|_1.$$  

Let the usual min-max rate of $C$ be defined as

$$\mathcal{R}_n(C) \equiv \min_{f \in C} \max_{\hat{f} : X^n \rightarrow \mathbb{R}} E\|\hat{f} - f\|_1.$$  

This is the lowest worst case expected distance achieved by any estimator. Let the agnostic min-max rate be defined as

$$\mathcal{R}_n(C, f) \equiv \text{OPT}_C(f) + \mathcal{R}_n(C),$$

that in addition to the min-max rate, accounts for the error arising from $f \notin C$. For any $c \geq 1$, $\hat{f}$ is said to achieve a factor-$c$ approximation for $C$ if

$$E\|\hat{f} - f\|_1 \leq c \cdot \text{OPT}_C(f) + \tilde{O}(\mathcal{R}_n(C)).$$

The key challenge is to obtain such an estimate for dense approximation classes $C$. One such class is the set of degree-$d$ polynomials, $P_d$ and its $t$-piecewise extension, $P_{t,d}$. It is known that by tuning the parameters $t, d$, the bias and variance under $P_{t,d}$ can be suitably tailored to achieve several in-class min-max rates. For example, if $f$
is a log-concave distribution, choosing $t = n^{1/5}$ and $d = 1$, $R_n(P_{t,d}, f) = O(1/n^{2/5})$, matching the min-max rate of learning log-concave distributions. Similarly, min-max rates may be attained for many other structured classes including uni-modal, Gaussian, and mixtures of all three.

The VC dimension, $\text{VC}(C)$, measures the complexity of a class $C$. For many dense classes, including $P_{t,d}$, $R_n(C) = \Theta(\sqrt{\text{VC}(C)/n})$. For such classes, a cross-validation based estimator $\hat{f}$, such as the Scheffe tournament or minimum distance based selection (Devroye & Lugosi, 2001) across a sufficiently fine cover of $C$, achieves a factor-3 approximation to $C$, namely for any $\delta > 0$, w.p. $\geq 1 - \delta$,

$$\| \hat{f} - f \|_1 \leq 3 \text{OPT}_C(f) + O(\sqrt{\text{VC}(C) + \log(1/\delta)}/n).$$

However, in general, such methods might have time complexity exponential in $n$. This is especially significant in modern applications that process a large number of samples. As noted in (Bottou & Bousquet, 2008), it is desirable for the computation complexity to scale roughly linearly with sample size.

(Chan et al., 2014; Acharya et al., 2017) provided a near-linear time factor-3 approximation for $P_{t,d}$. Their result is stated below, where $\omega$ is the matrix multiplication exponent (Chiantini et al., 2018).

**Theorem 1.** (Acharya et al., 2017) Given $X^n \sim f$ and parameters $t, d$, for any $\gamma > 0$, ADLS computes $\hat{f}_{\text{ADLS}}$ in time $O(nd^{3+\omega})$, such that w.p. $\geq 9/10$,

$$\| \hat{f}_{\text{ADLS}} - f \|_1 \leq (3 + \gamma) \text{OPT}_{P_{t,d}}(f) + O\left(\frac{1}{\gamma} \left(\frac{t(d+1)}{n}\right)^{\omega}\right).$$

This algorithm runs near-linear time and for any given $t \geq 0$ and $d \geq 1$, it achieves a factor-$c$ approximation for $P_{t,d}$. However it suffers from some key drawbacks and some fundamental questions remain unanswered.

- **Q1:** For the constant-polynomial class $P_0$, it is easy to see that the empirical histogram $\hat{f}$ is a factor-2 approximation for $P_0$, matching a known lower bound (Devroye & Lugosi, 2001). This raises the question if the factor-3 upper bound in Theorem 1 can be reduced for higher-degree polynomials as well?

- **Q2:** More importantly, ADLS requires prior knowledge of the number $t$ of polynomial pieces, which may be impractical in real applications. Even for structured distribution classes, the $t$ achieving the min-max rate can vary significantly. For example, for the class of log-concave distributions, $t = \theta(n^{1/5})$, for unimodal distributions with a bounded support, $t = \theta(n^{1/3})$, and for Gaussian mixtures, $t$ is proportional to the unknown number of mixture components.

A suitable objective is an estimate that is simultaneously a factor-$c$ approximation for $P_{t,d}$ $\forall t \geq 0$. While some cross-validation methods may achieve this goal, they may result in a large factor. For example, if the Scheffe tournament or the minimum distance method are used on the un-normalized estimates that ADLS outputs for different $t$, the factor gets multiplied by 5 (Devroye & Lugosi, 2001), resulting in $c = 5 \cdot 3 = 15$. Normalizing ADLS may reduce the factor to $3 \cdot 3 = 9$, but the question is whether these extra factors can be avoided altogether.

Theorem 2 answers the first question and Theorem 3 addresses the second, both in the affirmative.

The theoretical results are born by experiments. In simulations, SURF consistently outperforms ADLS. Significantly, ADLS achieves comparable results only when it is tuned with roughly the optimal number of polynomial pieces, which agrees with the number SURF calculates. In all other cases its error is several times larger.

### 1.2. Other Related Work

Among the many methods that have been employed in distribution estimation, see (Scott, 2012; Devroye et al., 2013), SURF is inspired by the concept of statistically equivalent blocks introduced in (Tukey, 1947; 1948). Distribution estimation methods using this concept rely on partitioning the domain into regions identified by a fixed number of samples, and performing local estimation on these regions. These methods have the advantage that they are simple to describe, are almost always of polynomial time complexity in $n$, and easy to interpret.

The first estimator that used this technique is found in (Parthasarathy & Bhattacharya, 1961). Expanding on several subsequent works, the notable work (Lugosi et al., 1996) shows consistency of a family of equivalent block based estimators for multivariate distributions. See (Devroye et al., 2013) for a more extensive treatment of this subject. Ours is the first work that provides agnostic error guarantees for an equivalent block based estimator.

Other popular estimation methods include the Kernel, nearest neighbor, MLE, and wavelets surveyed in (Silverman, 2018). Another related method uses splines, for example (Wegman & Wright, 1983; Gu & Qiu, 1993). While MLE and splines may be used for polynomial estimation, MLE is intractable in general, and agnostic error guarantees are unavailable in either.

### 1.3. Main Results

SURF first uses an interpolation routine INT that outputs an estimate, $\hat{f}_{I,\text{INT}} \in P_d$ for any queried interval $I$. $\hat{f}_{I,\text{INT}}$
is built on the observation that a degree-$d$ polynomial is determined by the measure it assigns to any $d + 1$ sub-intervals. We show that a factor $< 3$ approximation to $P_d$ is achieved by the polynomial estimate that matches its measure to the empirical masses on certain special sub-intervals. These sub-intervals, provided in Lemma 9, are functions of $d$ and are sample independent. The result is summarized as follows.

**Theorem 2.** Given $X^{n-1} \sim f$ for some $n \geq 128$, degree $d$, and an interval $I$ with $q_I n$ samples, INT takes $O(d^2 + q_I n)$ time, and outputs $\hat{f}_{INT} \in P_d$ such that w.p. $\geq 1 - \delta$,

$$\|\hat{f}_{INT} - f\|_I \leq (r_d + 1) \cdot \inf_{h \in P_d} \|h - f\|_I$$

$$+ r_d \cdot \frac{\sqrt{5(d + 1)q_I \log \frac{n}{\delta}}}{n},$$

where $\|\cdot\|_I$ is the $\ell_1$ distance evaluated on $I$, $\tau < 2.4$ is the matrix inversion exponent, $r_d$ is a fundamental constant whose values are $r_0 = 1, r_1 = 1.25, r_2 = 1.42, r_3 \approx 1.55, r_4 \leq 1.675, r_5 \leq 1.774, r_6 \leq 1.857, r_7 \leq 1.930, r_8 \leq 1.999$ for $4 \leq d \leq 8$.

**Proof** For a given $d$, INT outputs $\hat{f}_{INT}$, the re-scaled-shifted $\tilde{f}_{n_d}$ given by the corresponding $\tilde{n}_d \in N_d$ in Lemma 9. Choosing $\epsilon(\delta) = \sqrt{\log(n/\delta)/n}$, for $n \geq 128$, $Q_{\epsilon(\delta)}$ occurs with probability $\geq 1 - \delta$ from Lemma 6. Using Lemma 7 with $\epsilon(\delta)$ completes the proof.

Thus Theorem 2 answers Q1. We also note that such an estimate based on matching the measure assigned on sub-intervals to the empirical mass, may also provide factor-$c$ approximations for other dense parametric classes, for instance, the Fourier, or the orthogonal series.

The main routine of SURF, STITCH, then calls INT to obtain a piecewise estimate for any partition of the domain. STITCH uses COMP to compare between the different piecewise estimates. By imposing a special binary structure on the space of partitions, we allow for COMP to efficiently make this comparison via a divide-and-conquer approach. This allows STITCH, and in turn SURF, to output $\hat{f}_{SURF}$ in $O((d^2 + \log n) n \log n)$ time. $\hat{f}_{SURF}$ is a factor-$(r_d + 1)$ approximation for $P_{t,d}$ for $t \geq 0$.

As a combined effect, for the case of unknown $t$, the best known approximation factor of any sub-exponential time algorithm is reduced from 15, to 2.25, 2.42, . . . , 2.999, respectively, for $d = 1, . . . , 8$. This result is summarized below in Theorem 3 and Corollary 4.

**Theorem 3.** Given $X^{n-1} \sim f$ for some $n \geq 128$ such that $n$ is a power of 2, and parameters $d \leq 8$, $\alpha > 2$, SURF takes $O((d^2 + \log n) n \log n)$ time, and outputs $\hat{f}_{SURF}$ such that w.p. $\geq 1 - \delta$,

$$\|\hat{f}_{SURF} - f\|_1$$

$$\leq \min_{t \geq 0} \left( \frac{(r_d + 1) \cdot \alpha}{\alpha - 2} \cdot \inf_{h \in P_d} \|h - f\|_1$$

$$+ r_d \cdot \frac{\sqrt{5(d + 1)q_I \log \frac{n}{\delta}}}{n} \right),$$

where $\Delta_R$ is the collection of all partitions of $\mathbb{R}$ whose intervals start and end at a sample point, $\|\cdot\|_I$ is the $\ell_1$ distance evaluated in interval $I$, $q_I n$ is the number of samples in interval $I \in I$, $\tau < 2.4$ is the matrix inversion exponent, and $r_d > 0$ is the constant in Theorem 2.

**Proof** Choosing $\epsilon(\delta) = \sqrt{\log(n/\delta)/n}$, for $n \geq 128$, $Q_{\epsilon(\delta)}$ occurs with probability $\geq 1 - \delta$ from Lemma 6. Using Lemma 10 on top of Lemma 11 proves the theorem.

The simplicity of SURF, both the polynomial interpolation and divide-and-conquer, allow us to derive all constants explicitly unlike in the previous works.

**Corollary 4.** Running SURF with $n \geq 128, \alpha > 2, d \leq 8$,

$$\mathbb{E}[\|\hat{f}_{SURF} - f\|_1]$$

$$\leq \min_{t \geq 0} \left( \frac{(r_d + 1) \cdot \alpha}{\alpha - 2} \cdot \inf_{h \in P_d} \|h - f\|_1$$

$$+ r_d \cdot \frac{\sqrt{5(d + 1)q_I \log \frac{n}{\delta}}}{n} \right),$$

where $(a)$ follows since for any partition with $t$ pieces, $\sum_{i \in I} \sqrt{q_I} \leq \sqrt{t}$. Letting $\alpha \rightarrow \infty$ and choosing $\delta \approx 1/n$ completes the proof.

Thus Corollary 4 resolves Q2 for $d \leq 8$. If $d > 8$ the best bounds remain at what is provided in (Acharya et al., 2017). However for larger degrees, their computational complexity of $n \cdot d^{1+\omega} = \Omega(n \cdot d^2)$, and the Runge phenomenon (Trefethen, 2013) may limit its applicability in practice.

The paper is organized as follows. In Section 2 we describe the construction of intervals and partitions based on statistically equivalent blocks. In Section 3 we present INT, a
We extend this concentration from one interval to many. For interval $I_{a,b} = ([X(a), X(b)]$ if $a < b < n$, $[X(a), \infty)$ if $b = n$. The interval- and empirical-probabilities are

$$P_{a,b} \defeq \int_{I_{a,b}} dF, \quad q_{a,b} \defeq \frac{b - a}{n}.$$ 

For any $0 \leq a < b \leq n$, $I_{a,b}$ forms a statistically equivalent block (Tukey, 1947), wherein $P_{a,b} \sim \text{Beta}(b - a, n - (b - a))$ regardless of $f$, and $P_{a,b}$ concentrates to $q_{a,b}$.

**Lemma 5.** For any $0 \leq a < b \leq n, \epsilon \geq 0$,

$$\Pr[|P_{a,b} - q_{a,b}| \geq \epsilon \sqrt{q_{a,b}}] \leq \epsilon e^{-\frac{(n-1)^2}{2} + \epsilon^2 \frac{n-1}{2} \sqrt{n\epsilon}}.$$ 

We extend this concentration from one interval to many. For a fixed $\epsilon > 0$, let $Q_n$ be the event that

$$\forall 0 \leq a < b \leq n, |P_{a,b} - q_{a,b}| \leq \epsilon \sqrt{q_{a,b}}.$$

**Lemma 6.** For any $n \geq 128$ and $\epsilon \geq 0$,

$$P[Q_n] \geq 1 - \frac{n(n+1)}{2} \left( e^{-\frac{(n-1)^2}{2} + \epsilon^2 \frac{n-1}{2} \sqrt{n\epsilon}} \right).$$

Notice that $Q_n$ refers to a stronger concentration event that involves $\sqrt{q_{a,b}} \forall 0 \leq a < b \leq n$ and standard VC dimension based bounds cannot be readily applied to obtain Lemma 6.

**2.2. Partitions**

A collection of countably many disjoint intervals whose union is $\mathbb{R}$ is said to be a partition of $\mathbb{R}$. A distribution $\hat{q}$, consisting of interval empirical probabilities is called an empirical distribution, or that each interval $\hat{q}$ is a multiple of $1/n$. The set of empirical probabilities, $\Delta_{\text{emp},n}$, is defined as

$$\Delta_{\text{emp},n} \defeq \{ \hat{q} \in \Delta : \forall q \in \hat{q}, \: q = \nu(q)/n, \: \nu(q) \in \mathbb{Z} \}.$$ 

Since each $q \in \Delta_{\text{emp},n} \geq 1/n$, $\hat{q}$ may be split into its finitely many probabilities as $\hat{q} = (q_1, \ldots, q_k)$. For $1 \leq i \leq k$, let $I_{r_i} \defeq \sum_{j=1}^{i-1} q_j$. We use an empirical distribution to define an interval partition as follows:

$$\hat{I}_{\Delta_{\text{emp},n}} \defeq (I_{r_1}, n, (r_1+q_1)n, I_{r_2}, n, (r_2+q_2)n, \ldots, I_{r_k}, n, (r_k+q_k)n).$$ 

These are the intervals that contain the first increasingly-sorted values of $q_n$ samples, the next $q_2 n$ samples and so on. For example $I_{r_1, n, (r_1+q_1)n} = I_{\Delta_{\text{emp},n}} = (\infty, X(q_1))$ is the interval with the smallest $1/n$ samples, $I_{r_2, n, (r_2+q_2)n} = I_{\Delta_{\text{emp},n}}$ is the interval with the next $q_2 n$ samples, so on, and

$$I_{r_k, n, (r_k+q_k)n} = I_{\Delta_{\text{emp},n}} = (X(q_1), X((q_1+q_2)n)).$$

is the interval with the largest $q_k n$ samples.

**3. The Interpolation Routine**

This section describes INT, which outputs an estimate $\hat{f}_{\text{INT}} \in \mathcal{P}_d$ for any queried interval $I$.

**WLOG let $I = [0, 1]$.** A collection, $\tilde{N}_d = (n_0, \ldots, n_{d+1})$ such that $0 = n_0 \leq n_1 \leq \cdots \leq n_d \leq n_{d+1} = 1$ is said to be a node partition of $[0, 1]$. Let $\mathcal{N}_d$ be the set of node partitions and define $r : \mathcal{N}_d, \mathcal{P}_d \to [1, \infty)$

$$r(\tilde{n}_d, h) \defeq \frac{\int_0^1 |h|}{\sum_{i=1}^{d+1} |\int_{n_{i-1}}^{n_i} h|},$$ 

where $0/0 \defeq 1$. Notice $r(\tilde{n}_d, h) \geq 1$ since $\int_0^1 |h| \geq \sum_{i=1}^{d+1} |\int_{n_{i-1}}^{n_i} h|$. For any $\tilde{n}_d \in \mathcal{N}_d$, let

$$r_d(\tilde{n}_d) = \sup_{h \in \mathcal{P}_d} r(\tilde{n}_d, h).$$

Let $\tilde{f}_{\tilde{n}_d, i} \defeq [n_{i-1}, n_i], i \in \{1, \ldots, d+1\}$ so that any $\tilde{n}_d \in \mathcal{N}_d$ partitions $[0, 1]$ into $I_{\tilde{n}_d} = (I_{\tilde{n}_d,1}, \ldots, I_{\tilde{n}_d,d+1})$.

**Let $\tilde{f}_{\tilde{n}_d} \in \mathcal{P}_d$ be the unique polynomial whose measure on all $d+1$ intervals in $I_{\tilde{n}_d}$ matches the empirical mass.** It is defined as:

$$\tilde{f}_{\tilde{n}_d} \defeq h \in \mathcal{P}_d : \forall i \in \{1, \ldots, d+1\}, \int_{n_{i-1}}^{n_i} h = q_{I_{\tilde{n}_d, i}}.$$

where $q_I$ denotes the empirical mass in interval $I$. The estimate $\tilde{f}_{\tilde{n}_d}$ corresponding to any choice of $\tilde{n}_d \in \mathcal{N}_d$ has the following property.

**Lemma 7.** For interval $I = [0, 1]$ with empirical probability $q_I$, any $\tilde{n}_d \in \mathcal{N}_d$, and $\epsilon > 0$, the estimate $\tilde{f}_{\tilde{n}_d}$ (4) is such
that under event \( Q_x \),
\[
\| \hat{f}_{\bar{n}_d} - f \|_1 \leq (1 + r_d(\bar{n}_d)) \inf_{h \in \mathcal{P}_d} \| h - f \|_1 + r_d(\bar{n}_d) \epsilon \sqrt{(d + 1)q_r}.
\]

In Lemma 8, we show that for any \( \bar{n}_d \in \mathcal{N}_d \), there exists an \( r_d(\bar{n}_d) \) achieving \( h \in \mathcal{P}_d \), and that it belongs to a special set, \( \mathcal{P}_{\bar{n}_d} \subseteq \mathcal{P}_d \),
\[
\mathcal{P}_{\bar{n}_d} \overset{\text{def}}{=} \left\{ h \in \mathcal{P}_d : \exists i_1 \in \{1, \ldots, d + 1\} : \forall i \in \{1, \ldots, d + 1\} \setminus \{i_1\}, \int_{n_{i-1}}^{n_i} h = 0 \right\}.
\]

In words, \( \mathcal{P}_{\bar{n}_d} \) is the set of polynomials that has a non-zero area in at most one \( I \in I_{\bar{n}_d} \).

**Lemma 8.** For any degree-\( d \) and \( \bar{n}_d \in \mathcal{N}_d \),
\[
r_d(\bar{n}_d) = \sup_{h \in \mathcal{P}_d} r(\bar{n}_d, h) = \max_{h \in \mathcal{P}_{\bar{n}_d}} r(\bar{n}_d, h).
\]

Let the smallest \( r_d(\bar{n}_d) \) and the attaining \( \bar{n}_d \) (if it exists) be denoted by
\[
r_d^* = \inf_{\bar{n}_d \in \mathcal{N}_d} r_d(\bar{n}_d), \quad \bar{n}_d^* = \arg \min_{\bar{n}_d \in \mathcal{N}_d} r_d(\bar{n}_d).
\]

**Lemma 9.** For \( d \leq 3 \), there exists a node collection \( \bar{n}_d^* \) that achieves \( r_d^* \). These, and their respective \( r_d^* \) are given by

| \( d \) | \( \bar{n}_d^* \) | \( r_d^* \) |
|---|---|---|
| 0 | (0, 1) | 1 |
| 1 | (0, 0.5, 1) | 1.25 |
| 2 | (0, 0.2599, 0.7401, 1) | \( \approx 1.42 \) |
| 3 | (0, 0.1548, 0.5, 0.8452, 1) | \( \approx 1.56 \) |

Denoting \( \bar{n}_d^* = (0, \alpha_0, 1 - \alpha_0, 1) \), and \( \bar{n}_3^* = (0, \beta_0, 0.5, 1 - \beta_0, 1) \), the exact values of \( \alpha_0, \beta_0 \), are obtained as roots to a degree-14 and degree-69 polynomial that we explicitly provide. For degrees \( 4 \leq d \leq 8 \), the following \( \bar{n}_d \in \mathcal{N}_d \) and \( r_d(\bar{n}_d) \) provide upper bounds on \( r_d^* \).

| \( d \) | \( \bar{n}_d \) | \( r_d(\bar{n}_d) \) |
|---|---|---|
| 4 | (0.01015, 0.348, 0.652, 0.8985, 1) | \( \leq 1.675 \) |
| 5 | (0.0071, 0.254, 0.5, 0.746, 0.929, 1) | \( \leq 1.774 \) |
| 6 | (0.053, 0.192, 0.390, 0.610, 0.808, 0.947, 1) | \( \leq 1.857 \) |
| 7 | (0.0405, 0.149, 0.310, 0.5, 0.690, 0.851, 0.9595, 1) | \( \leq 1.930 \) |
| 8 | (0.032, 0.119, 0.252, 0.414, 0.586, 0.749, 0.881, 0.968, 1) | \( \leq 1.999 \) |

For a given interval \( I \) and \( d \leq 8 \), INT first scales and shifts \( I \) to obtain \([0, 1]\). It then constructs \( \hat{f}_{\bar{n}_d} \) using the \( \bar{n}_d \) in Lemma 9. The output \( f_{I, \text{INT}} \) is the re-scaled-shifted \( \hat{f}_{\bar{n}_d} \).

### 4. The Compare and Stitch Routines

This section presents STITCH and COMP, the main routines of SURF. For any contiguous collection of intervals \( I \), let \( \hat{f}_{I, \text{INT}} \) be the piecewise polynomial estimate consisting of \( \hat{f}_{I, \text{INT}} \in \mathcal{P}_d \) given by INT in each \( I \in \hat{I} \). The key idea in SURF is to separate interval partitions into a binary hierarchy, effectively allowing a comparison of all the \( \Omega(n^c) \) (for any \( c > 0 \)) estimates corresponding to the different interval partitions, but by using only \( O(n) \) comparisons.

#### 4.1. Notation

Recall that \( n \) here a power of 2 and define the integer
\[
D \overset{\text{def}}{=} \log_2 n.
\]

An empirical distribution, \( \bar{q} \in \Delta_{\text{emp}, n} \), is called a binary distribution if each of its probability values are of the form \( 1/2^k \), for some integer \( 0 \leq d \leq D \). The corresponding interval partition, \( I_{\bar{q}} \), is said to be a binary partition.

\[
\Delta_{\text{bin}, n} \overset{\text{def}}{=} \{ \bar{q} \in \Delta_{\text{emp}, n} : \forall q \in \bar{q}, q = 1/2^\nu(q), 0 \leq \nu(q) \leq D, \nu(q) \in \mathbb{Z} \}.
\]

For example \( \bar{q} = (1) \), \( \bar{q} = (1/2, 1/4, 1/4) \), \( \bar{q} = (1/4, 1/8, 1/8, 1/2) \) are binary distributions. Similarly, \( (1/n, \ldots, 1/n) = (1/2^{\log n}, \ldots, 1/2^{\log n}) \) is also a binary distribution since \( n \) here is a power of 2 (assume \( n \geq 8 \) so that they are all in \( \Delta_{\text{emp}, n} \)).

**Lemma 10.** For any empirical distribution \( \bar{q} \in \Delta_{\text{emp}, n} \), there exists \( \bar{q} \in \Delta_{\text{bin}, n} \) such that
\[
\| f_q^* - f \|_1 \leq \| f_q^* - f \|_1, \quad \sum_{q \in \bar{q}} \epsilon \sqrt{q} \leq \sum_{q \in \bar{q}} \epsilon \sqrt{q},
\]

where \( f_q^* \) is the piecewise polynomial closest to \( f \) on \( I \).

**Lemma 10** shows that \( \Delta_{\text{bin}, n} \) retains most of the approximating power of \( \Delta_{\text{emp}, n} \). In particular, that for any \( \bar{q} \in \Delta_{\text{emp}, n} \), there exists a binary distribution \( \bar{q} \in \Delta_{\text{bin}, n} \) such that \( I_{\bar{q}} \) has a smaller bias than \( I_{\bar{q}} \), while its deviation under the concentration event, \( Q_x \), is larger by less than a factor of \( 1/\sqrt{2} - 1 \).

For a fixed \( \bar{q} \in \Delta_{\text{bin}, n} \), let \( \Delta_{\text{bin}, n, \leq \bar{q}} \) be the set of binary distributions such that for any \( \bar{q} \in \Delta_{\text{bin}, n, \leq \bar{q}} \), each \( I_1 \in I_{\bar{q}} \) is contained in some \( I_2 \in I_{\bar{q}} \).

\[
\Delta_{\text{bin}, n, \leq \bar{q}} \overset{\text{def}}{=} \{ \bar{q} \in \Delta_{\text{bin}, n} : \forall I_1 \in I_{\bar{q}}, \exists I_2 \in I_{\bar{q}}, I_1 \subseteq I_2 \}.
\]
For example if \( \bar{p} = (1/2, 1/4, 1/4) \) is the binary distribution, \((1/4, 1/4, 1/8, 1/8, 1/8, 1/4, 1/2, 1/4, 1/8, 1/8) \) \( \in \Delta_{\text{bin.n.} \leq \bar{p}} \), whereas \((1/2, 1/2) \notin \Delta_{\text{bin.n.} \leq \bar{p}} \).

Similarly, let \( \Delta_{\text{bin.n.} \geq \bar{p}} \) consist of binary distributions such that for any \( \bar{q} \in \Delta_{\text{bin.n.} \geq \bar{p}} \), each \( I_1 \in \bar{I}_p \) is contained in some \( I_2 \in \bar{I}_q \).

\[
\Delta_{\text{bin.n.} \geq \bar{p}} \overset{\text{def}}{=} \{ \bar{q} \in \Delta_{\text{bin.n.}} : \forall I_1 \in \bar{I}_p, \exists I_2 \in \bar{I}_q, I_1 \subseteq I_2 \}.
\]

For example, if \( \bar{p} = (1/2, 1/4, 1/4, 1/4) \), then \((1/2, 1/2) \notin \Delta_{\text{bin.n.} \geq \bar{p}} \), but \((1/2, 1/4, 1/8, 1/4) \in \Delta_{\text{bin.n.} \geq \bar{p}} \). A complimentary relationship exists between \( \Delta_{\text{bin.n.} \leq \bar{p}} \) and \( \Delta_{\text{bin.n.} \geq \bar{p}} \). For any \( \bar{p}_1, \bar{p}_2 \in \Delta_{\text{bin.n.}} \), \( \bar{p}_1 \in \Delta_{\text{bin.n.} \leq \bar{p}_2} \Leftrightarrow \bar{p}_2 \in \Delta_{\text{bin.n.} \geq \bar{p}_1} \).

### 4.2. The STITCH Routine

STITCH receives as input, \( X^{n-1} \) and parameters \( d, \alpha, \epsilon \).

The routine operates in \( i \in \{1, \ldots, D\} \) steps. Define \( D(i) \overset{\text{def}}{=} D - i \) and let

\[
\bar{u}_i \overset{\text{def}}{=} \left( \frac{1}{2D(i)}, \ldots, \frac{1}{2D(i)} \right), \quad \bar{u}_i = (\bar{u}_{i,1}, \ldots, \bar{u}_{i,2D(i)}).
\]

Initialize \( \bar{q}_0 \equiv (1/n, \ldots, 1/n) \).

Start with \( i = 1 \) and assign \( \bar{s} \equiv \bar{q}_{i-1} \). In each step, the routine maintains this \( \bar{s} = \bar{q}_{i-1} \in \Delta_{\text{bin.n.} \leq \bar{u}_i} \). This can be seen from the initialization above for \( i = 1 \) since \( \bar{u}_1 = (2/n, \ldots, 2/n) \), and verified for \( i > 1 \). Thus, using \( \bar{u}_i \), we may separate

\[
\bar{I}_s = (\bar{I}_{s,1}, \ldots, \bar{I}_{s,2D(i)}), \quad \bar{s} = (\bar{s}_1, \ldots, \bar{s}_{2D(i)}),
\]

where for each \( j \in \{1, \ldots, 2D(i)\} \), \( \bar{I}_{s,j} \subseteq \bar{I}_s \) are intervals in \( \bar{I}_s \) whose union gives \( I_{u_{i,j}} \subseteq \bar{u}_i \). Let \( \bar{s}_j \in \bar{s} \) denote the empirical probabilities in \( \bar{s} \) corresponding to intervals in \( \bar{I}_{s,j} \). Notice that the sum of all probabilities in \( \bar{s}_j \), \( \sum_{\bar{s}_j} s_j = 1/2D(i) \). Therefore the scaled \( 2D(i) \bar{s}_j \) is an empirical distribution. For brevity, let the polynomial estimate output by INT on \( I_{u_{i,j}} \), be denoted by

\[
\hat{f}_{I_{u_{i,j}}} \overset{\text{def}}{=} \hat{f}_{I_{u_{i,j}}, \text{INT}}.
\]

Starting with \( j = 1 \), invoke COMP with arguments, the polynomial estimate \( \hat{f}_{I_{s,j}} \), intervals \( I_{s,j} \) and the empirical distribution \( 2D(i) \bar{s}_j \), samples \( X^{n-1} \subseteq X^n \) that lie in \( I_{s,j} \), and parameters \( d \),

\[
\gamma \overset{\text{def}}{=} \alpha \cdot r_d \cdot \epsilon \sqrt{d + 1}.
\]

This parameter, \( \gamma \), is used to tune the bias-variance tradeoff. As will be shown subsequently, if \( \gamma \rightarrow \infty \), \( I_{s,j} \) will be merged, resulting in an estimate with a larger bias but smaller variance. A small \( \gamma \) has the opposite effect.

#### Algorithm 1 STITCH

**Input:** \( X^{n-1}, d, \alpha, \epsilon \)

**Initialize** \( D = \log n, \bar{q} = (1/n, \ldots, 1/n) \)

for \( i = 1 \) to \( D \) do

\( D(i) \leftarrow D - i, \bar{s} \leftarrow \bar{q} \)

for \( j = 1 \) to \( 2D(i) \) do

if \( \text{COMP}(\hat{f}_{I_{s,j}}, \bar{I}_{s,j}, 2D(i) \bar{s}_j, X^{n-1}, d, \gamma) \leq 0 \) then

\( \bar{s}_j \leftarrow (1/2D(i)) \)

end if

end for

end for

**Output:** \( \hat{f}_{\bar{I}_s, \text{INT}} \)

If \( \text{COMP}(\hat{f}_{I_{s,j}}, \bar{I}_{s,j}, 2D(i) \bar{s}_j, X^{n-1}, d, \gamma) \leq 0 \), merge \( \bar{I}_{s,j} \) into a single interval \( I_{\bar{u}_{i,j}} \). Accomplish this by updating \( \bar{s}_j \) to a unitary value, its sum, \((1/2D(i))\). Otherwise, maintain \( \bar{s} \) as is. Increment \( j \) within the range \( \{1, \ldots, 2D(i)\} \) and repeat this procedure.

After the entire run in \( j \) is complete, update \( \bar{q}_i \leftarrow \bar{s} \). If \( D(i) = D - i > 0 \), increment \( i \) and repeat the same steps. Otherwise, if \( D(i) = 0 \) or in other words if \( i = D \), STITCH, and in turn, SURF outputs the piecewise estimate on \( I_{\bar{q}_D} \), i.e., \( \hat{f}_{\bar{I}_s, \text{INT}} \).

At each step \( i \in \{1, \ldots, D\} \), STITCH calls COMP on \( 2D(i) \) intervals, each consisting of \( 2^i \) samples. Thus each step of STITCH takes \( O(2D(i) \cdot (d^2 + \log(n^2))) \cdot 2^i = O((d^2 + \log n)2D) \) time. The total time complexity is therefore \( O((d^2 + \log n)2D) = O(d^2 + \log n) \log n) \).

### 4.3. The COMP Routine

COMP receives as input, a function \( \hat{f} \), an interval partition \( \bar{I} \equiv \bar{I}_s \) and the corresponding empirical distribution \( \bar{s} \), samples \( X^m \) that lie in \( \bar{I} \), and parameters \( d, \gamma \).

Fix a \( \bar{p} \in \Delta_{\text{bin.n.} \geq \bar{s}} \), and consider the piecewise polynomial estimate on \( \bar{I}_{\bar{p}.} \hat{f}_{\bar{p}, \text{INT}} \). Define

\[
\Lambda_{\bar{p}}(\hat{f}) \overset{\text{def}}{=} \| \hat{f}_{\bar{p}, \text{INT}} - \hat{f} \|_{\bar{p}}, \quad \lambda_{\bar{p}, \gamma} \overset{\text{def}}{=} \sum_{\bar{p} \in \bar{p}} \gamma \sqrt{d + 1}.
\]

COMP returns \( \mu_{\bar{p}, \gamma}(\hat{f}) \), the largest difference between \( \Lambda_{\bar{p}}(\hat{f}) \) and \( \lambda_{\bar{p}, \gamma} \) across all \( \bar{p} \in \Delta_{\text{bin.n.} \geq \bar{s}} \).

\[
\mu_{\bar{p}, \gamma}(\hat{f}) \overset{\text{def}}{=} \max_{\bar{p} \in \Delta_{\text{bin.n.} \geq \bar{s}}} \Lambda_{\bar{p}}(\hat{f}) - \lambda_{\bar{p}, \gamma}.
\]

The quantity, \( \Lambda_{\bar{p}}(\hat{f}) \) acts as a proxy for the increment in bias that results if the piecewise estimate \( \hat{f}_{\bar{p}, \text{INT}} \) is merged into \( \hat{f} \), while \( \lambda_{\bar{p}, \gamma} \) accounts for the deviation in \( \hat{f}_{\bar{p}, \text{INT}} \) under \( \mathcal{Q}_\epsilon \). Notice that for any \( \bar{p} \in \Delta_{\text{bin.n.} \geq \bar{s}} \), \( \lambda_{\bar{p}, \gamma} \leq \lambda_{\bar{s}, \gamma} \).
Thus $\mu_{I_s,\gamma}(\hat{f}) \leq 0$ if the decrease in deviation under $\hat{I} = I_s$ is larger than the increased bias under any candidate $\hat{I}_p$. This in turn signals STITCH to merge $\hat{I}$.

It may be shown that if $\bar{s} = (1/m, \ldots, 1/m)$, the cardinality, $|\Delta_{\text{bin}, m, \geq \bar{s}}| = \Omega(m^c)$ for any $c > 0$. Therefore, naively evaluating $\Lambda_{I_p}(\hat{f}) - \Lambda_{\bar{p}, \gamma}$ over each $\bar{p} \in \Delta_{\text{bin}, m, \geq \bar{s}}$ incurs a worst case time complexity that is super-linear in $m$. Instead, COMP uses a simple divide-and-conquer procedure that computes $\mu_{I_s,\gamma}(\hat{f})$ in time $O((d^* + \log m) m)$.

To describe this, notice that if $\bar{I}_s$ is a singleton ($I_s$), then $\bar{s} = (1)$, implying $\Delta_{\text{bin}, m, \geq \bar{s}} = \{(1)\}$. In this case, obtain $\hat{f}_{I_s, \text{INT}} \in \mathcal{P}_d$ and return

$$\mu_{I_s,\gamma}(\hat{f}) = \Lambda_{\{(1)\}}(\hat{f}) - \lambda_{\{(1)\}, \gamma} = \|\hat{f}_{I_s, \text{INT}} - \hat{f}\|_{\{(1)\}} - \gamma \sqrt{T}.$$ 

If $\bar{I}_s$ is non singleton or $\bar{s} \neq (1)$, any $\bar{p} \in \Delta_{\text{bin}, m, \geq \bar{s}} \setminus \{(1)\}$ may be split into two sub-distributions, $\bar{p}_1, \bar{p}_2$ that each sum to $1/2$. For example, if the particular $\bar{p} = (1/4, 1/4, 1/8, 1/8, 1/4)$, it may be split into $\bar{p}_1 = (1/4, 1/4)$ and $\bar{p}_2 = (1/8, 1/8, 1/4)$. The corresponding interval partition is also split into $\hat{I}_p = (I_{\bar{p}_1} \cup I_{\bar{p}_2})$. Since $\bar{s} \neq (1)$, this may also be similarly split into $s_1$ and $s_2$. As a consequence, $\bar{I}_s$ is also cleaved into $(I_{s_1}, I_{s_2})$ corresponding to $\bar{s}_1$ and $\bar{s}_2$. Using this observation,

$$\max_{\bar{p} \notin \{(1)\}} \frac{\Lambda_{I_p}(\hat{f}) - \lambda_{\bar{p}, \gamma}}{\Lambda_{I_{\bar{p}_1}}(\hat{f}) - \lambda_{\bar{p}_1, \gamma} + \Lambda_{I_{\bar{p}_2}}(\hat{f}) - \lambda_{\bar{p}_2, \gamma}} = \mu_{I_{s_1}, \gamma}(\hat{f}) + \mu_{I_{s_2}, \gamma}(\hat{f}),$$

where $2\bar{s}_1$, $2\bar{s}_2$ are the normalized variants of $\bar{s}_1$, $\bar{s}_2$, and $\gamma$ is scaled by $1/\sqrt{\gamma}$ to accommodate for this scaling. By evaluating $\mu_{I_{s_1}, \gamma}(\hat{f})$, $\mu_{I_{s_2}, \gamma}(\hat{f})$ separately, and then comparing their sum with $\Lambda_{I_{\{(1)\}}}(\hat{f}) - \lambda_{\{(1)\}, \gamma}$, we allow for a recursive computation of $\mu_{I_s,\gamma}(\hat{f})$.

Let $X_1^m$ and $X_2^m$ denote the samples in $I_{s_1}$ and $I_{s_2}$ respectively. Using these arguments, call COMP on $I_{s_1}$, $s_1$ and $I_{s_2}$, $s_2$, return the maximum as shown in Algorithm 2.

**Algorithm 2 COMP**

**Input:** $\hat{f}, I_s, s, X^m, d, \gamma$

$\bar{I} \leftarrow \cup I_s$, $\mu \leftarrow \Lambda_{\bar{I}}(\hat{f}) - \lambda_{\bar{s}, \gamma}$

if $|\bar{I}| = 1$

Return $\mu$

else

Return $\max\{\mu, \text{COMP}(\hat{f}, I_{s_1}, 2\bar{s}_1, X_1^m, d, \gamma/\sqrt{2}) + \text{COMP}(\hat{f}, I_{s_2}, 2\bar{s}_2, X_2^m, d, \gamma/\sqrt{2})\}$

end if

$T(m)$, is captured by

$$T(m) \leq 2T(m/2) + O(m + d^*),$$

implying $T(m) = O((d^* + \log m) m)$.

Lemma 11 shows that under $Q_e$, $\hat{f}_{\text{SURF}}$ is within a constant factor of the best piecewise polynomial approximation over any binary partition, plus its deviation in probability under $Q_e$, times $O(\sqrt{d + T})$.

**Lemma 11.** Given samples $X^{n-1} \sim f$, for some $n$ that is a power of 2, degree $d \leq 8$ and the threshold $\alpha > 2$, SURF outputs $\hat{f}_{\text{SURF}}$ in time $O((d^* + \log n)n \log n)$ such that under event $Q_e$,

$$\|\hat{f}_{\text{SURF}} - f\|_I \leq \min_{\bar{p} \in \Delta_{\text{bin}, n}} \sum_{I_p} \left(\frac{r_d + 1 - \alpha}{\alpha - 2} \cdot \inf_{h \in \mathcal{P}_d} ||h - f||_I \right) + \frac{r_d \cdot (\alpha \sqrt{2} + \sqrt{2} - 1)}{\sqrt{2} - 1} \epsilon \sqrt{(d + 1)q_I},$$

where $q_I$ is the empirical mass under interval $I$, $r_d$ is the constant corresponding to the INT routine in Theorem 2.

**5. Experiments**

In all experiments in this section, SURF is run with $\alpha = 0.25$ and errors are averaged over 10 runs. In running ADLS we used the provided code as is.

The samples are generated using beta mixture distributions over $[0, 1]$, as they accommodate a wide range of shapes. We obtained similar results for other smooth parametric families such as the Gaussian, exponential, or Gamma distributions, but omit them for brevity. Let $f_{\beta, \alpha, \beta}$ be the beta density with parameters $\alpha, \beta$. 
First, we run SURF to estimate three distributions: $f_1 = 0.4f_{\text{Beta},0.8,4} + 0.6f_{\text{Beta},2.2}$, $f_2 = 0.4f_{\text{Beta},10,3} + 0.6f_{\text{Beta},2.8}$, and $f_3 = f_{\text{Beta},6,6}$, shown in Figure 1(a). SURF estimates the distributions using piecewise polynomials of degree $d = 1, 2, 3$. Figures 1(b)–1(d) show the resulting $\ell_1$ errors. Observe that the errors are decaying, and are similar between distributions. This is not surprising since low degree polynomial approximations largely rely on local smoothness, which all of the considered densities possess. By the same reasoning, on increasing $d$ from 1 to 3, the variation in error between distributions increases. The smoother $f_1$ starts incurring a smaller $\ell_1$ error than $f_2$ and $f_3$.

Next, we run SURF with $d = 2$ to estimate $f = 0.3f_{\text{Beta},3,10} + 0.7f_{\text{Beta},17,4}$ with $n = 1024, 4096, 16384, 65536$. Figure 2 plots the resulting estimates against $f$. Thus the estimate not only successively better approximates $f$ in $\ell_1$ distance, but also converges to it in a pointwise sense.

In Figure 3, we compare the $\ell_1$ error in estimating the distribution $f = 0.4f_{\text{Beta},0.8,4} + 0.6f_{\text{Beta},2.2}$ using piecewise linear polynomials, as considered in (Acharya et al., 2017). The plots correspond to the errors incurred on running SURF and ADLS with $t = 5, 10, 20, 40$. Observe that depending on the number of pieces, the error incurred by ADLS can be substantially larger than SURF. Significantly, the $t = 5$ for which the results are comparable, is also roughly the number of pieces that SURF outputs. Just as we report for SURF, the errors under ADLS are also very similar for other distributions such as the normal and gamma mixtures. These conclusions are thus representative of the overall behavior of the two algorithms.
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A. Intervals and Partitions

A.1. Proof of Lemma 5

Proof  For simplicity, let

\[ X \overset{\text{def}}{=} P_{a,b}, \quad p \overset{\text{def}}{=} q_{a,b} \]

so that \( X = P_{a,b} \sim \text{Beta}(np, n(1 - p)) \). For any \( x, y \in \mathbb{R}^+ \), let \( B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y) \) denote the beta function and let \( a, b > 0 \) and \( x \in [0, 1] \),

\[ I_x(a,b) \overset{\text{def}}{=} \int_0^x z^{a-1}(1-z)^{b-1} \frac{dz}{B(a,b)} \]

be the incomplete beta function. Then,

\[
\Pr[X \leq p - \epsilon \sqrt{p}] \overset{(a)}{=} I_{p-\epsilon \sqrt{p}}(np, n(1-p)) \\
\overset{(b)}{=} \sum_{i=0}^{n-1} \binom{n-1}{i} (p-\epsilon \sqrt{p})^i (1-p+\epsilon \sqrt{p})^{n-1-i} \\
\overset{(c)}{\leq} e^{-(n-1)D(p||p-\epsilon \sqrt{p})} \\
\overset{(d)}{\leq} e^{-(n-1)\frac{\epsilon^2 p}{2p + \epsilon \sqrt{p}}},
\]

where \( (a) \) follows by definition, \( (b) \) follows by the property of incomplete beta function (DLMF), \( (c) \) follows from the Chernoff bound applied to the right tail of a Binom\((n,p-\epsilon \sqrt{p})\) random variable, and \( (d) \) follows since \( D(x||y) \leq (x-y)^2/\max\{x,y\} \).

Similarly,

\[
\Pr[X \geq p + \epsilon \sqrt{p}] \overset{(a)}{=} 1 - I_{p+\epsilon \sqrt{p}}(np, n(1-p)) \\
\overset{(b)}{=} 1 - \sum_{i=np}^{n-1} \binom{n-1}{i} (p+\epsilon \sqrt{p})^i (1-p-\epsilon \sqrt{p})^{n-1-i} \\
\leq \sum_{i=0}^{np} \binom{n-1}{i} (p+\epsilon \sqrt{p})^i (1-p-\epsilon \sqrt{p})^{n-1-i} \\
\overset{(c)}{\leq} e^{-(n-1)D(p||p+\epsilon \sqrt{p})} \\
\overset{(d)}{\leq} e^{-(n-1)\frac{\epsilon^2 p}{2p + \epsilon \sqrt{p}}},
\]

where \( (a) \) follows by definition, \( (b) \) follows by the property of incomplete beta function (DLMF), and \( (c) \) follows from Chernoff bound applied to the left tail of a Binom\((n,p+\epsilon \sqrt{p})\) random variable, and \( (d) \) follows since \( D(x||y) \leq (x-y)^2/\max\{x,y\} \).
A.2. Proof of Lemma 6

Proof From using the union bound, we have

\[
1 - \Pr[Q_\varepsilon] = \Pr \left[ \exists 0 \leq a < b \leq n : |P_{a,b} - q_{a,b}| \geq \varepsilon \sqrt{q_{a,b}} \right] \\
\leq \sum_{0 \leq a < b \leq n} \Pr \left[ |P_{a,b} - q_{a,b}| \geq \varepsilon \sqrt{q_{a,b}} \right] \\
\leq \sum_{0 \leq a < b \leq n} \left( e^{-(n-1)r^2/2} + e^{r\sqrt{q_{a,b}}(a+b)} \right) \\
\leq \frac{n(n+1)}{2} \left( e^{-(n-1)r^2/2} + e^{r\sqrt{q_{a,b}}(a+b)} \right) \\
\leq \frac{n(n+1)}{2} \left( e^{-(n-1)r^2/2} + e^{r\sqrt{q_{a,b}}(a+b)} \right),
\]

where (a) follows from Lemma 5, (b) follows since \([0 \leq a < b \leq n] = \binom{n+1}{2}\), and (c) follows since \(q_{a,b} \geq 1/n\).

B. The Interpolation Routine

B.1. Proof of Lemma 7

Proof For a partition \(\tilde{I}\) of \(I = [0,1]\), and integrable functions \(g_1, g_2\), define the distance

\[
d_f(g_1, g_2) \stackrel{\text{def}}{=} \sum_{J \in \tilde{I}} \left| \int_J g_1 - \int_J g_2 \right|. \tag{7}
\]

In words, \(d_f(g_1, g_2)\) is the sum of absolute differences between measures under \(g_1\) and \(g_2\) across all intervals in \(\tilde{I}\). For any \(h \in \mathcal{P}_d\),

\[
\|\hat{f}_{\tilde{\bar{n}}_d} - f\|_I \leq \|h - f\|_I + \|\hat{f}_{\tilde{\bar{n}}_d} - h\|_I \\
\stackrel{(a)}{\leq} \|h - f\|_I + r_d(\tilde{\bar{n}}_d) d_{\tilde{\bar{n}}_d}(h, \hat{f}_{\tilde{\bar{n}}_d}) \\
\leq \|h - f\|_I + r_d(\tilde{\bar{n}}_d) \left( d_{\tilde{\bar{n}}_d}(h, f) + d_{\tilde{\bar{n}}_d}(f, \hat{f}_{\tilde{\bar{n}}_d}) \right) \\
\leq (1 + r_d(\tilde{\bar{n}}_d)) \|h - f\|_I + r_d(\tilde{\bar{n}}_d) \sum_{J \in I_{\tilde{n}}_d} |P_J - q_J| \\
\leq (1 + r_d(\tilde{\bar{n}}_d)) \|h - f\|_I + r_d(\tilde{\bar{n}}_d) \sum_{J \in I_{\tilde{n}}_d} \varepsilon \sqrt{q_J} \\
\leq (1 + r_d(\tilde{\bar{n}}_d)) \|h - f\|_I + r_d(\tilde{\bar{n}}_d) \varepsilon \sqrt{q_f(d+1)},
\]

where (a) follows since \((h - \hat{f}_{I_{\tilde{n}}_d}) \in \mathcal{P}_d\), and from definitions of the ratio \(r_d(\tilde{\bar{n}}_d)\) in Equation (3), (b) follows since the \(d_I\)-distance satisfies the triangle inequality, (c) follows since the \(d_I\) distance upper bounds \(d_f\)-distance, (d) follows since \(\hat{f}_{\tilde{\bar{n}}_d}\), by definition, is the polynomial such that \(\int_J \hat{f}_{\tilde{\bar{n}}_d} = q_J \forall J \in I_{\tilde{n}}_d\), and the interval probability \(P_J \stackrel{\text{def}}{=} \int_J f\), (e) follows under event \(Q_\varepsilon\), and (f) follows since by the concavity of \(\sqrt{x}\) for \(x \geq 0\), the sum \(\sum_{J \in I_{\tilde{n}}_d} \sqrt{q_J}\) is maximized if for each \(J \in I_{\tilde{n}}_d\), \(q_J = (\sum_{J \in I_{\tilde{n}}_d} q_J)/|I_{\tilde{n}}_d| = q_f/(d+1)\).
B.2. Proof of Lemma 8

**Proof** Fix \( h \in \mathcal{P}_d \). Let \((\beta_1, \ldots, \beta_{d_0})\) be the roots of \( h \) in \([0, 1]\) for some \( \beta_1 \leq \ldots \leq \beta_{d_0}, 0 \leq d_0 \leq d \). Let \( \beta_0 \overset{\text{def}}{=} 0, \beta_{d_0+1} \overset{\text{def}}{=} 1 \). Then notice that
\[
\int_0^1 |h| = \sum_{i=1}^{d_0+1} \int_{\beta_{i-1}}^{\beta_i} |h| \leq \sup_{\tilde{m}_d \in \mathcal{N}_d} \sum_{i=1}^{d+1} \left| \int_{n_{i-1}}^{n_i} h \right| = \sup_{\tilde{m}_d \in \mathcal{N}_d} \max_{\tilde{s} \in \{0, 1\}^{d+1}} \sum_{i=1}^{d+1} (-1)^{s_i} \int_{m_{i-1}}^{m_i} h \leq \int_0^1 |h|,
\]
where (a) follows since on padding \( d - d_0 \) zeros, \((0, \ldots, 0, \beta_0, \ldots, \beta_{d+1}) \in \mathcal{N}_d \). Thus (b) is, in fact, an equality, implying
\[
r_d(\tilde{n}_d) = \sup_{h \in \mathcal{P}_d} r(\tilde{n}_d, h) = \sup_{h \in \mathcal{P}_d} \frac{\int_0^1 |h|}{\sum_{i=1}^{d+1} \left| \int_{n_{i-1}}^{n_i} h \right|} = \sup_{h \in \mathcal{P}_d} \frac{\sum_{i=1}^{d+1} (-1)^{s_i} \int_{m_{i-1}}^{m_i} h}{\sum_{i=1}^{d+1} \left| \int_{n_{i-1}}^{n_i} h \right|}
\]
Denote \( h = \sum_{i=1}^{d+1} c_i \cdot x^i \) and let \( \bar{c} \overset{\text{def}}{=} (c_1, \ldots, c_{d+1}) \). Notice that since \( r(\tilde{n}_d, h) \geq 1 \) for any \( h \in \mathcal{P}_d \), and since \( r_d(\tilde{n}_d, 0) \overset{\text{def}}{=} 1 \), WLOG assume \( h \neq 0 \) or \( \bar{c} \neq 0 \). By linearity of the integral of \( h \) in \( \bar{c} \), recast \( r_d(\tilde{n}_d) \) into
\[
r_d(\tilde{n}_d) = \sup_{\tilde{m}_d \in \mathcal{N}_d, \tilde{s} \in \{0, 1\}^{d+1}} \max_{\bar{c} \in \mathbb{R}^{d+1}} \frac{\sum_{i=1}^{d+1} c_i \mu_i}{\sum_{i=1}^{d+1} |\sum_{j=1}^{d+1} c_j \lambda_{i,j}|},
\]
where for any \( i, j \in \{1, \ldots, d+1\}, \mu_i \in \mathbb{R} \) is a function of \( \tilde{m}_d, \tilde{s} \) and \( \lambda_{i,j} \in \mathbb{R} \) is a function of \( \tilde{n}_d \). Observe that \( \tilde{n}_d \) is given, and additionally fix \( \tilde{m}_d \in \mathcal{N}_d, \tilde{s} \in \{0, 1\}^{d+1} \). Since the objective function here is a ratio whose denominator is positive (since \( h \neq 0 \)), WLOG set the numerator to 1 via the constraint \( \sum_{i=1}^{d+1} c_i \mu_i = 1 \) and convert it to a linear program as:
\[
\max \frac{1}{\sum_{i=1}^{d+1} v_i} : \bar{v} \in \mathbb{R}^{d+1}, \quad v_i \geq \sum_{j=1}^{d+1} c_j \lambda_{i,j}, \quad v_i \geq - \sum_{j=1}^{d+1} c_j \lambda_{i,j}, \quad \sum_{i=1}^{d+1} v_i = 1,
\]
where \( \bar{v} \overset{\text{def}}{=} (v_1, \ldots, v_{d+1}) \). Observe that these constraints give rise to a bounded region, and since this is a linear program, there exists a solution at some corner point involving at least \( 2 \cdot (d+1) \) equalities, one for each variable. In any such solution, since the equality: \( \sum_{i=1}^{d+1} c_i \mu_i = 1 \) is always active, at least \( 2 \cdot (d+1) - 1 \) of the other inequalities attain equality. Notice that for any \( i \in \{1, \ldots, d+1\}, v_i = 0 \) if both
\[
v_i = \sum_{j=1}^{d+1} c_j \lambda_{i,j} \text{ and } v_i = - \sum_{j=1}^{d+1} c_j \lambda_{i,j} \text{ hold.}
\]
Thus in this corner point solution, \( v_i \neq 0 \) for at most one \( i \in \{1, \ldots, d+1\} \). Let
\[
\mathcal{D}_{\tilde{n}_d} = \{ \bar{c} \in \mathbb{R}^{d+1} \setminus \{0\} : \exists i_1 \in \{1, \ldots, d+1\} : \forall i \neq i_1, |\sum_{j=1}^{d+1} c_j \lambda_{i,j}| = 0 \}
\]
This implies
\[
  r_d(\bar{n}_d) = \sup_{\bar{n}_d \in \mathcal{N}_d} \max_{h \in \mathcal{P}_d} \frac{\sum_{i=1}^{d+1} c_i \mu_i}{\sum_{i=1}^{d+1} |\sum_{j=1}^{d+1} c_j \lambda_{i,j}|}
\]
\[
  = \sup_{\bar{n}_d \in \mathcal{N}_d} \max_{h \in \mathcal{P}_d} \frac{\sum_{i=1}^{d+1} (\bar{n}_d)_i}{\sum_{i=1}^{d+1} |\sum_{j=1}^{d+1} h \lambda_{i,j}|}
\]
\[
  = \max_{h \in \mathcal{P}_d} \frac{f_0^1 |h|}{\sum_{i=1}^{d+1} |\sum_{j=1}^{d+1} h|} = \max_{h \in \mathcal{P}_d} r_d(\bar{n}_d, h).
\]

B.3. Proof of Lemma 9

**Proof** For any polynomial \( h \in \mathcal{P}_d \), the ratio \( r(\bar{n}_d, h) \) is invariant to multiplying both the numerator and denominator by a constant. Thus, WLOG consider polynomials whose leading coefficient is 1. Then for any \( \bar{n}_d \in \mathcal{N}_d \), \( \mathcal{P}_{\bar{n}_d} = (h_{\bar{n}_d,1}, \ldots, h_{\bar{n}_d,d+1}) \), is a set consisting of \( d+1 \) unique polynomials, where each \( h_{\bar{n}_d,i}, i \in \{1, \ldots, d+1\} \) is that polynomial with 0 area in all intervals except \( h_{\bar{n}_d,i} \).

**Case d = 0:** Here \( \mathcal{N}_0 = \{(0,1)\} \) and is a singleton set. Since any \( h \in \mathcal{P}_0 \) is a constant value, \( f_0^1 |h| = |f_0^1 |h| \). Therefore \( r_0^* = \max_{h \in \mathcal{P}_0} r(\bar{n}_d, h) = 1 \).

**Case d = 1:** Let \( \bar{n}_1 = (0, m, 1) \). In this case \( h_{\bar{n}_d,1}(x) = x - m/2 \) and \( h_{\bar{n}_d,2}(x) = x - (1+m)/2 \). Using Lemma 8,
\[
r_1(\bar{n}_d) = \max_{h \in \mathcal{P}_{\bar{n}_d}} r(\bar{n}_d, h) = \max \{ r(\bar{n}_d, h_{\bar{n}_d,1}), r(\bar{n}_d, h_{\bar{n}_d,2}) \}
\]
\[
= \max \left\{ \frac{m^2/4 + (1-m/2)^2}{1-m}, \frac{(1-m)^2/4 + ((1+m)/2)^2}{m} \right\}.
\]
\( r_1(\bar{n}_d) \) is minimized for \( m^* = 1/2 \), giving \( r_1^* = (1/16 + 9/16)/(1/2) = 1.25 \).

**Case d = 2:** By symmetry, the minimizing node partition is symmetric about 0.5. Thus WLOG let \( \bar{n}_2 = (0, m, 1-m, 1) \) for some \( m \leq 0.5 \). Among the \( d+1 = 3 \) polynomials in \( \mathcal{P}_{\bar{n}_2} \), by symmetry of \( \bar{n}_2 \), \( r_2(h_{\bar{n}_d,1}) = r_2(h_{\bar{n}_d,2}) \). Thus we consider the larger ratio across only two polynomials, \( h_{\bar{n}_d,2}, h_{\bar{n}_d,3} \).

Denote the polynomial as \( h_{\bar{n}_d,2}(x) = (x - a_2)^2 - b_2^2 \) and upon setting the respective integrals to 0,
\[
\left| \frac{m^3}{3} - a_2 m^2 + (a_2^2 - b_2^2) m \right| = 0, \quad \left| \frac{1}{3} - (1-m)^3 \right| - a_2 (1 - (1-m)^2) + (a_2^2 - b_2^2) m = 0
\]
\[
\implies a_2 = \frac{1}{2} b_2 = \frac{3(m^2 - m) + 1}{9}.
\]
Representing \( h_{\bar{n}_d,3}(x) = (x - a_3)^2 - b_3^2 \) and repeating the same steps,
\[
\left| \frac{m^3}{3} - a_3 m^2 + (a_3^2 - b_3^2) m \right| = 0, \quad \left| \frac{1}{3} - (1-m)^3 \right| - a_3 ((1-m)^2 - m^2) + (a_3^2 - b_3^2)(1 - 2m) = 0
\]
\[
\implies a_3 = \frac{1}{3} b_3 = \frac{4m^2 - 6m + 3}{3}.
\]

The corresponding \( r(\bar{n}_d, h_{\bar{n}_d,2}) \) and \( r(\bar{n}_d, h_{\bar{n}_d,3}) \) are given by
\[
\frac{8}{m(1-m)} \left( \frac{1-3m(1-m)}{9} \right)^{3/2} + 1, \quad \frac{2}{(m-1)(m-1)} \left( \frac{(2m-1)(2m-2)+1}{3} \right)^{3/2} - 1.
\]
From simultaneously minimizing the above expressions by equating them, the optimal $m$ is the root of

$$q_2(m) = \frac{-26624}{729} m^{14} + \frac{193280}{729} m^{13} - \frac{211024}{243} m^{12} + \frac{3703648}{2187} m^{11} \frac{-479076}{2187} m^{10} + \frac{39108232}{19683} m^9$$

$$+ \frac{8554775}{19683} m^8 - \frac{12357280}{19683} m^7 + \frac{13004032}{59049} m^6 + \frac{10618125}{19683} m^5 - \frac{43505725}{531441} m^4 + \frac{246976}{531441} m^3$$

$$+ \frac{11840}{171147} m^2 - \frac{6656}{531441} m + \frac{256}{531441}$$

near 0.26. Thus the optimal $m^* \approx 0.2599$ and the corresponding $r_2^* \approx 1.423$.

**Case d = 3:** By symmetry, as before, WLOG let $\tilde{n}_d = (0, m, 0.5, 1 - m, 1)$. This reduces the search space to just two polynomials, $h_{a,1}, h_{a,2}$. The optimal $m$ occurs as the root of

$$q_3(m) \overset{\text{def}}{=} \frac{-26624}{46} m^{69} + \frac{193280}{46} m^{68} - \frac{211024}{2944} m^{67} + \frac{3703648}{1472} m^{66} - \frac{479076}{1472} m^{65} + \frac{39108232}{8094597799570318677256901352577796401265} m^{64}$$

$$+ \frac{8554775}{359136991232} m^{63} - \frac{12348030976}{395136991232} m^{62} + \frac{8294496}{3087007744} m^{61} - \frac{8073218865852425466369}{593584650474870121} m^{60}$$

$$+ \frac{11840}{171147} m^2 + \frac{6656}{531441} m + \frac{395136991232}{531441}$$

Thus the optimal $m^* \approx 0.2599$ and the corresponding $r_3^* \approx 1.423$. 

**SURF Algorithm**
we may decompose

These values populate the second table in Lemma 9.

From the property of the geometric sum, 

Proof

C.1. Proof of Lemma 10

This gives $m^* \approx 0.148$ and the corresponding $r_d(\bar{n}_d)$. These values populate the second table in Lemma 9.

C. The Compare and Stitch Routines

C.1. Proof of Lemma 10

Proof As observed in Equation (1), any $q \in \bar{q} \in \Delta_{\text{emp},n}$ is an integral multiple of $1/n$. Observing that $\log_2 n$ is an integer, we may decompose $q$ along its binary expansion as

$$q = \sum_{j=0}^{\log_2 n} 2^{-j} b_j,$$

for some $b_j \in \{0,1\}, j \in \{1, \ldots, \log_2 n\}$. Replace each $q \in \bar{q}$ with the vector $(2^{-0} b_0, 2^{-1} b_1, \ldots)$ to obtain $\bar{q}' \in \Delta_{\text{bin},n}$.

From the property of the geometric sum,

$$\sum_{j=0}^{\log_2 n} \sqrt{2^{-j} b_j} \leq \frac{\sqrt{q}}{\sqrt{2} - 1}.$$
Finally $\|f_{I_q^*}' - f\|_1 \leq \|f_{I_q}^* - f\|_1$ since $I_q^*$ being a finer partition than $I_q$, $f_{I_q^*}'$ is a closer approximation to $f$ than $f_{I_q}^*$.

**C.2. Proof of Lemma 11**

**Proof**  For any interval $I$, let

$$f_I^* \overset{\text{def}}{=} \arg \min_{h \in \mathcal{P}_d} \|h - f\|_1,$$

and for any partition $I$, let $f_I^*$ be the piecewise polynomial that equals $f_I^*$ in each $I \in \mathcal{I}$. For simplicity let $I_q \overset{\text{def}}{=} I_{q_D}$ denote the final partition and $q \overset{\text{def}}{=} q_D$ the corresponding empirical distribution. Consider any $\bar{p} \in \Delta_{\text{bin,n}}$ and its associated interval partition, $\bar{I}_\bar{p}$. Two interval partitions $I_1, I_2$ corresponding to binary distributions have the following property: Any interval in $I_1$ is either completely contained within some interval in $I_2$, or is a union of contiguous intervals from $I_2$. As a result $I_{\bar{q}}$ may partitioned into three classes of intervals:

| $I_{\bar{q}}$ | $\bar{I}_{\bar{q}}$ | $I_{\bar{p}}$ | $\bar{I}_{\bar{p}}$ | $I_{\bar{p}}$ | $\bar{I}_{\bar{p}}$ | $I_{\bar{p}}$ | $\bar{I}_{\bar{p}}$ |
|---------------|-------------------|---------------|-------------------|---------------|-------------------|---------------|-------------------|
|               | $I_{\bar{q}}$     |               | $I_{\bar{p}}$     |               | $I_{\bar{p}}$     |               | $I_{\bar{p}}$     |

*Figure 4. Illustration of $I_{\bar{q}}$ being partitioned into $I_{\bar{q}}, I_{\bar{p}}$ and $I_{\bar{p}}$ using $I_{\bar{p}}$.***

- $I_{\bar{q}}$, composed of intervals that are equal to some interval in $I_{\bar{p}}$,
- $I_{\bar{q}}^2$, that consists of intervals that lie strictly within some interval in $I_{\bar{p}}$,
- $I_{\bar{q}}^3$, containing intervals that are unions of more than one interval from $I_{\bar{p}}$.

This is shown in Figure 4. Lemmas 12, 13, 14 address each of these intervals separately. Combining the lemmas,

$$\|\hat{f}_{I_q} - f\|_1 = \|\hat{f}_{I_q} - f\|_{I_q^1} + \|\hat{f}_{I_q} - f\|_{I_q^2} + \|\hat{f}_{I_q} - f\|_{I_q^3} \leq (r_d + 1) \cdot \|f_{I_q}^* - f\|_{I_q^1} + \sum_{I \in I_{\bar{p}}} r_d \cdot \varepsilon \sqrt{(d+1)p_I}$$

$$+ \frac{(r_d + 1) \cdot \alpha}{\alpha - 2} \cdot \|f_{I_q}^* - f\|_{I_q^2} + \frac{1}{\alpha - 1} \sum_{I \in I_{\bar{p}}} r_d \cdot \varepsilon \sqrt{(d+1)p_I}$$

$$+ (r_d + 1) \cdot \|f_{I_q}^* - f\|_{I_q^3} + \frac{\alpha \sqrt{2} + \sqrt{2} - 1}{\sqrt{2} - 1} \sum_{I \in I_{\bar{p}}} r_d \cdot \varepsilon \sqrt{(d+1)p_I}$$

$$\leq (\alpha - 2) \frac{(r_d + 1) \cdot \alpha}{\alpha - 2} \|f_{I_q}^* - f\|_1 + \frac{\alpha \sqrt{2} + \sqrt{2} - 1}{\sqrt{2} - 1} \sum_{I \in I_{\bar{p}}} r_d \cdot \varepsilon \sqrt{(d+1)p_I},$$

where (a) follows since $\alpha > 2 \Rightarrow 1/(\alpha - 1) < 1 < (\alpha \sqrt{2} + \sqrt{2} - 1)/(\sqrt{2} - 1)$.

**Lemma 12.** For the final partition $I_{\bar{q}}$ in the run of STITCH and any $\bar{p} \in \Delta_{\text{bin,n}}$, let $I_{\bar{q}}^1 \subseteq I_{\bar{p}}$ be the intervals that intersect with $I_{\bar{p}}^1$. Let $I_{\bar{q}}^1 = I_{\bar{q}}^1 \subseteq I_{\bar{p}}$ denote the corresponding collection in $I_{\bar{p}}$. Then,

$$\|\hat{f}_{I_q} - f\|_{I_q^1} \leq (r_d + 1) \cdot \|f_{I_q}^* - f\|_{I_q^1} + \sum_{I \in I_{\bar{p}}} r_d \cdot \varepsilon \sqrt{(d+1)p_I},$$

**Proof**  Follows from Theorem 2 and noticing that intervals in $I_{\bar{q}}^1$ and $I_{\bar{p}}^1$ coincide.
Lemma 13. For the final partition \( \bar{I}_q \) in the run of STITCH and any \( \bar{p} \in \Delta_{\text{bin},m} \), let \( \bar{I}_p^2 \subseteq \bar{I}_q \) be the intervals that do not intersect with, and strictly lie in some interval in \( \bar{I}_p \). Let \( \bar{I}_p^2 \subseteq \bar{I}_p \) be the corresponding intervals that contain \( \bar{I}_q^2 \). Then,

\[
\| \hat{f}_{\bar{I}_q} - f \|_{\bar{I}_q^2} \leq \frac{(r_d + 1) \cdot \alpha}{\alpha - 2} \cdot \| f_{\bar{I}_p}^* - f \|_{\bar{I}_p^2} + \frac{1}{\alpha - 1} \sum_{I \in \bar{I}_p^2} r_d \cdot \epsilon \sqrt{(d + 1)s_I}.
\]

Proof. Notice that all intervals in \( \bar{I}_q^2 \) are strictly contained within some interval in \( \bar{I}_p^2 \). Using this, we further partition \( \bar{I}_p^2 \) using intervals in \( \bar{I}_p^2 \). Fix an \( I \in \bar{I}_p^2 \) and let \( \bar{I} \in \bar{I}_q^2 \) be intervals whose union gives \( I \). Let \( \bar{q}_I \subseteq \bar{q} \) denote the empirical probabilities corresponding to \( \bar{I} \) and let \( p_I \) denote the empirical probability under \( I \).

![Figure 5. Illustration of \( I \in \bar{I}_q^2, \bar{I}_{s_1} \) and \( \bar{I}_{s_1} \) corresponding to a particular \( I \in \bar{I}_p^2 \).](image)

While \( \bar{q}_I \) is a sub-distribution in general, WLOG assume \( \bar{q}_I \) is a distribution. Now, at some point in the run of STITCH, COMP was called with \( f_{\bar{I}_{s_1}} \), \( \bar{f}_{\bar{I}_{s_1}} \), and it was in-turn declared that \( I \) was not to be merged into \( I \). Therefore, for the \( \mu_{I,\gamma}(\hat{f}_{\bar{I}_{s_1}}) \) attaining binary distribution, \( s_I \in \Delta_{\text{bin},m} \geq \bar{q}_I \), \( \Lambda_{s_I}(\hat{f}_{\bar{I}_{s_1}}) - \lambda_{s_I,\gamma} \geq 0 \). It follows that

\[
\sum_{s \in s_I} \alpha \cdot r_d \cdot \epsilon \sqrt{(d + 1)s} = \lambda_{s_I,\gamma} \leq \Lambda_{s_I}(\hat{f}_{\bar{I}_{s_1}})
\]

\[
= \| \hat{f}_{\bar{I}_{s_1}} - \hat{f}_{s_I} \|_I
\]

\[
\leq \| \hat{f}_{\bar{I}_{s_1}} - f \|_I + \| f - \hat{f}_{s_I} \|_I
\]

\[
\leq (r_d + 1) \cdot \| f_{s_I} - f \|_I + r_d \cdot \epsilon \sqrt{(d + 1)s_I}
\]

\[
+ (r_d + 1) \cdot \| f_{s_I} - f \|_I + \sum_{s \in s_I} r_d \cdot \epsilon \sqrt{(d + 1)s}
\]

\[
\leq 2(r_d + 1) \cdot \| f_{s_I} - f \|_I + r_d \cdot \epsilon \sqrt{(d + 1)s_I + \sum_{s \in s_I} r_d \cdot \epsilon \sqrt{(d + 1)s}}.
\]

where \((a)\) follows from in Theorem 2, \((b)\) follows since \( I \) being the union of \( \bar{I}_{s_I} \) is also the union of \( \bar{I}_{s_1} \), and \( f_{s_I} \) is therefore a coarser approximation to \( f \) than \( f_{s_I}^* \), giving rise to a larger \( \ell_1 \) distance. Rearrange this to obtain

\[
\sum_{s \in s_I} r_d \cdot \epsilon \sqrt{(d + 1)s} \leq \frac{1}{\alpha - 1} \cdot \left( 2(r_d + 1) \cdot \| f_{s_I}^* - f \|_I + r_d \cdot \epsilon \sqrt{(d + 1)s_I} + \sum_{s \in s_I} r_d \cdot \epsilon \sqrt{(d + 1)s} \right).
\]

Consider a fixed \( I' \in \bar{I}_{s_1} \) and let \( \bar{I}' \in \bar{I}_q \) be the intervals under \( \bar{I} \) whose union gives \( I' \). We recursively use the same argument to bound the LHS of Equation (8). This is shown for the leftmost interval of \( \bar{I}_{s_1} \) in Figure 5. Let \( \bar{q}_{I'} \) be the corresponding probabilities under \( \bar{I}' \) and let \( s_{I'} \) denote the empirical probability under \( I' \). Notice that in some previous step of STITCH, as was for \( I \), COMP was invoked with \( f_{I'_{s}} \), \( \bar{f}_{I'_{s}} \), \( \bar{q}_{I'} \), for which \( \mu_{I',\gamma}(\hat{f}_{I'_{s}}) \geq 0 \). Repeat the same procedure as above to obtain

\[
\sum_{s \in s_{I'}} r_d \cdot \epsilon \sqrt{(d + 1)s} \leq \frac{1}{\alpha - 1} \cdot \left( 2(r_d + 1) \cdot \| f_{s_I}^* - f \|_I + r_d \cdot \epsilon \sqrt{(d + 1)s_I} \right)
\]

\[
\leq \frac{1}{\alpha - 1} \cdot \left( 2(r_d + 1) \cdot \| f_{s_I}^* - f \|_I + r_d \cdot \epsilon \sqrt{(d + 1)s_I} \right),
\]

(9)
where $\tilde{s}_I$ here is the binary distribution which attains $\mu_{F,\gamma}(\tilde{f}_I,\text{INT})$, and (a) follows because $I'$ being an interval within $I$, $f_I^*$ is a coarser approximation to $f$ than $f_I^*$. Summing Equation (9) for each such $I'$, accumulate the distribution $\tilde{s}_I \overset{\text{def}}{=} (\cup_{I' \in I}_I \tilde{s}_I)$, and using Equation (8), the inequality,

$$
\sum_{s \in \tilde{s}_I} r_d \cdot \epsilon \sqrt{(d+1)s} \leq \left( \frac{1}{\alpha - 1} + \frac{1}{(\alpha - 1)^2} \right) \cdot 2(r_d + 1) \cdot ||f_I^* - f||_1 + \frac{1}{\alpha - 1} \cdot r_d \cdot \epsilon \sqrt{(d+1)p_I}. \quad (10)
$$

Notice that while both $\tilde{s}_I, \tilde{s}_I^\prime \in \Delta_{\text{bin},n, \geq \tilde{q}_I}, \tilde{s}_I^\prime$ is at least one notch closer to $\tilde{q}_I$ as $\tilde{s}_I^\prime \in \Delta_{\text{bin},n, \leq \tilde{q}_I}$. Since the number of binary distributions is finite, on recursively using this argument, summation across $\tilde{q}_I$ is eventually obtained on the LHS. Iterating on this procedure yields the upper bound

$$
\sum_{q \in \tilde{q}_I} r_d \cdot \epsilon \sqrt{(d+1)q} \leq \left( \frac{1}{\alpha - 1} + \frac{1}{(\alpha - 1)^2} + \cdots \right) \cdot 2(r_d + 1) \cdot ||f_I^* - f||_1 + \frac{1}{\alpha - 1} \cdot r_d \cdot \epsilon \sqrt{(d+1)p_I}
$$

(a) \leq \frac{2(r_d + 1)}{\alpha - 2} \cdot ||f_I^* - f||_1 + \frac{1}{\alpha - 1} \cdot r_d \cdot \epsilon \sqrt{(d+1)p_I},

where (a) follows since $\alpha > 2$. Repeating this argument across each $I \in \tilde{I}_p^q$,

$$
\sum_{l \in \tilde{I}_p^q} r_d \cdot \epsilon \sqrt{(d+1)q} \leq \frac{2(r_d + 1)}{\alpha - 2} \cdot ||f_{I_p}^* - f||_{I_p^q} + \frac{1}{\alpha - 1} \sum_{l \in \tilde{I}_p^q} r_d \cdot \epsilon \sqrt{(d+1)p_I}. \quad (11)
$$

This finally gives us

$$
||\hat{f}_{I_p} - f||_{I_q^p} \overset{(a)}{\leq} (r_d + 1) \cdot ||f_{I_p}^* - f||_{I_p^q} + \sum_{l \in \tilde{I}_p^q} r_d \cdot \epsilon \sqrt{(d+1)q}
$$

(b) \leq (r_d + 1) \cdot ||f_{I_p}^* - f||_{I_p^q} + \sum_{l \in \tilde{I}_p^q} r_d \cdot \epsilon \sqrt{(d+1)q}

\leq (r_d + 1) \left( 1 + \frac{2}{\alpha - 2} \right) \cdot ||f_{I_p}^* - f||_{I_p^q} + \frac{1}{\alpha - 1} \sum_{l \in \tilde{I}_p^q} r_d \cdot \epsilon \sqrt{(d+1)p_I},

where (a) follows from Theorem 2, (b) follows since, by definition, intervals in $\tilde{I}_p^q$ lie within those in $\tilde{I}_p^q$, and thus $f_{I_p}^*$ is a coarser approximation to $f$ than $f_{I_p}^*$ in $\tilde{I}_p^q$, and finally (c) follows by plugging in Equation (11).

**Lemma 14.** For the final partition $\tilde{I}_q$ in the run of STITCH and any $\tilde{p} \in \Delta_{\text{bin},n}$, let $\tilde{I}_q^3 \subseteq \tilde{I}_q$ be intervals that are unions of more than one interval from $\tilde{I}_p$. Let $\tilde{I}_p^3 \subseteq \tilde{I}_p$ be the corresponding intervals whose union gives $\tilde{I}_q^3$. Then,

$$
||\hat{f}_{I_q} - f||_{I_p^q} \leq (r_d + 1) \cdot ||f_{I_p}^* - f||_{I_p^q} + \frac{\epsilon \sqrt{2 + \sqrt{2} - 1}}{\sqrt{2} - 1} \sum_{l \in \tilde{I}_p^q} r_d \cdot \epsilon \sqrt{(d+1)p_I}.
$$

**Proof** Fix an $I \in \tilde{I}_p^q$ and let $q_I$ be its empirical probability. Let $\tilde{I}_{\tilde{p},I} \subseteq \tilde{I}_p$ indicate intervals under $\tilde{p}$, whose union gives $I$ and let $\tilde{p}_{\tilde{i}} \subseteq \tilde{p}$ denote the corresponding empirical probabilities under $\tilde{I}_{\tilde{p},I}$. In run of STITCH, let the interval collection that was merged to create $I$ be denoted by $\tilde{I}_{\tilde{i},I}$, and its collection of empirical probabilities by $\tilde{q}_{\tilde{i}}$. While $\tilde{p}_{\tilde{i}}$ is a sub-distribution in general, WLOG assume it is a distribution. This also implies $\tilde{q}_{\tilde{i}}$ is a distribution.

Using $\tilde{I}_{\tilde{q},I}$, separate $\tilde{I}_{\tilde{p},I}$ into

- $\tilde{I}_{\tilde{p},I}^1$, consisting of intervals in $\tilde{I}_{\tilde{p},I}$ that are equal to, or unions of intervals from $\tilde{I}_{\tilde{q},I}$.
- $\tilde{I}_{\tilde{p},I}^2$, intervals in $\tilde{I}_{\tilde{p},I}$ that lie strictly inside some interval in $\tilde{I}_{\tilde{q},I}$.


Let \( \tilde{I}_{q,I}^2 \subseteq \) be the corresponding intervals in \( \tilde{I}_{q,I} \) that contain \( I_{q,I}^2 \). Let \( \tilde{p}_{q,I}^1, \tilde{p}_{q,I}^2 \) be empirical probabilities corresponding to \( \tilde{I}_{p,I}^1, \tilde{I}_{p,I}^2 \) respectively. Similarly let \( \tilde{q}_{I}^2 \) correspond to \( \tilde{I}_{q,I}^2 \). This is shown in Figure 6, where the arrow indicates the collection of intervals merged by STITCH.

In the figure, \( I \in \tilde{I}_{q,I}^3 \) is the interval that is the result of merging \( \tilde{I}_{p,I}^2 \) and \( \tilde{I}_{q,I}^2 \). \( I' \in \tilde{I}_{q,I}^2 \) is another interval that is merged with \( \tilde{I}_{p,I}^2 \) to form \( I'' \in \tilde{I}_{q,I}^2 \). The resulting interval \( I'' \) is then merged with \( \tilde{I}_{p,I}^1 \) to form the final interval \( \tilde{I}_{p,I}^1 \).

Modify \( \tilde{I}_{p,I} \) to obtain a new partition \( \tilde{J}_{p,I} = \tilde{I}_{p,I}^1 \cup \tilde{I}_{p,I}^2 \). Now each interval in \( \tilde{J}_{p,I} \) is equal to, or is a union of intervals from \( \tilde{I}_{q,I} \). Equivalently, if \( \tilde{s} \) is the empirical distribution corresponding to \( \tilde{J}_{p,I}, \tilde{s} \in \Delta_{bin,n} \). Since \( I \) was merged when the merging routine was called with \( \hat{f}_{I,INT}, \tilde{I}_{q,I}, \tilde{q}_I \), it implies \( \lambda_{\tilde{s},\gamma} \geq \Lambda_{\tilde{I}_{p,I}}(\hat{f}_{I,INT}) \). Therefore

\[
\| \hat{f}_{I_q} - f \|_I \leq \| \hat{f}_{j_{p,I}} - f \|_I + \| \hat{f}_{j_q} - \hat{f}_{j_{p,I}} \|_I
\]

\( (a) \) follows by definition, \( \hat{f}_{I_q} = \hat{f}_{I,INT} \) in interval \( I \), \( (b) \) follows since \( \tilde{J}_{p,I} \) being a partition of \( I \) lies in the same region as \( I \), \( (c) \) follows since \( \tilde{J}_{p,I} \) is a partition of \( I \), and \( (d) \) follows since \( \hat{f}_{I_q} = \hat{f}_{I_{p,I}} \) in \( \tilde{I}_{p,I}^1 \) as \( \tilde{I}_{p,I}^1 \subseteq \tilde{I}_{p,I}. \)

Now consider an interval \( I' \subseteq \tilde{I}_{q,I}^2 \). Since \( \tilde{I}_{q,I}^2 \subseteq \tilde{I}_{q,I}, I \) and since \( \tilde{I}_{q,I} \), by definition, are intervals that were merged to produce \( I \), it follows that \( I' \) in turn was an interval that was merged into in some previous step of STITCH. As before, let the intervals that were merged to generate \( I' \) be denoted by \( \tilde{I}_{q,I} \). Further, by definition of \( \tilde{I}_{q,I} \), all intervals in it occur as unions of those in \( \tilde{I}_{q,I} \), and so does \( I' \). Let \( \tilde{I}_{p,I} \subseteq \tilde{I}_{p,I}^2 \) be these intervals whose union gives \( I' \). Repeat the same argument

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**Figure 6.** Illustration of proof construction for a particular \( I \in \tilde{I}_{q,I}^3 \).
as above to obtain
\[ \| \hat{f}_{q} - f \|_{\ell} \leq \| \hat{f}_{q} - f \|_{\ell_{1}} + \alpha \sum_{p \in \hat{p}_{i}} r_{d} \cdot \epsilon \sqrt{(d+1)p} \]
\[ + \| \hat{f}_{q} - f \|_{\ell_{2}} + \alpha \sum_{q \in \hat{q}_{i}} r_{d} \cdot \epsilon \sqrt{(d+1)q}, \]  

(13)

where each of \( \hat{I}_{i}, \hat{p}_{i}, \), \( \hat{I}_{q}, \) and \( \hat{q}_{i} \) are defined in exactly the same manner as was for \( I \), but by replacing \( I' \) in all definitions. Since \( \hat{I}_{i}, \hat{p}_{i}, \subseteq \hat{I}_{q}, \subseteq I_{\hat{p}} \), substituting Equation (13) into (12), a larger portion of \( I \) is bounded using the difference \( \| \hat{f}_{q} - f \| \). Upon repeating the same argument for all \( \| \hat{f}_{q} - f \| \) terms that remain, a bound on the RHS is obtained that consists exclusively of \( \| \hat{f}_{q} - f \| \). The entire procedure is shown in Figure 6.

Further, from Lemma 15, the sum of all the \( \epsilon \)-deviation terms that results on the RHS from repeating the argument is bounded by \( \sqrt{2}/(\sqrt{2} - 1) \) times the total \( \epsilon \)-deviation in \( I_{\hat{p}} \). This results in
\[ \| \hat{f}_{q} - f \|_{\ell_{1}} \leq \| \hat{f}_{q} - f \|_{\ell_{1}} + \frac{\alpha}{\sqrt{2} - 1} \sum_{p \in \hat{p}_{i}} r_{d} \cdot \epsilon \sqrt{(d+1)p} \]
\[ \leq (r_{d} + 1) \cdot \| f_{q}^{*} - f \|_{\ell_{1}} + \left(1 + \frac{\alpha \sqrt{2}}{\sqrt{2} - 1}\right) \sum_{p \in \hat{p}_{i}} r_{d} \cdot \epsilon \sqrt{(d+1)p}, \]

where (a) follows from Theorem 2. Repeating across \( I \in \hat{I}_{q} \) gives
\[ \| \hat{f}_{q} - f \|_{\ell_{2}} \leq (r_{d} + 1) \cdot \| f_{q}^{*} - f \|_{\ell_{2}} + \left(1 + \frac{\alpha \sqrt{2}}{\sqrt{2} - 1}\right) \sum_{I \in \hat{I}_{q}} r_{d} \cdot \epsilon \sqrt{(d+1)p_{t}}. \]

Lemma 15. Suppose in the run of STITCH, a collection of consecutive intervals \( \hat{I}_{1} \) was merged in \( k \) steps to generate \( \hat{I}_{k} \), and suppose \( \hat{I}_{2}, \ldots, \hat{I}_{k-1} \) are the intermediate interval collections. Then,
\[ \sum_{i=1}^{k} \sum_{I \in \hat{I}_{i}} \sqrt{q_{I}} \leq \sum_{I \in \hat{I}_{1}} \frac{\sqrt{2}}{\sqrt{2} - 1} \sqrt{q_{I}}. \]

Proof WLOG assume \( \hat{q}_{I_{i}} \) is a distribution, which also implies \( \hat{q}_{I_{i}} \) is a distribution \( \forall i \in \{2, \ldots, k\} \). Notice that for any \( i \in \{2, \ldots, k\} \), \( \hat{q}_{I_{i}} \) \( \in \Delta_{\text{bin}, n, \geq \hat{q}_{I_{i-1}}} \). Thus \( |I_{i}| \leq 1/2 \cdot |\hat{I}_{i-1}| \), where \( |I| \) denotes the number of intervals in \( I \). By concavity of \( \sqrt{x} \) for \( x \geq 0 \), the sum \( \sum_{I \in \hat{I}_{i}} \sqrt{q_{I}} \) is maximized for a given \( q_{I_{i-1}} \), if \( |\hat{I}_{i-1}| = 2 \cdot |\hat{I}_{i}| \). Since this equality is attained iff \( \hat{q}_{I_{i-1}} \) is the uniform distribution over \( |\hat{I}_{i-1}| \) elements and \( \hat{q}_{I_{i}} \) is uniform over \( |\hat{I}_{i}| = 1/2 \cdot |\hat{I}_{i-1}| \) elements,
\[ \sum_{I \in \hat{I}_{i}} \sqrt{q_{I}} \leq \frac{1}{\sqrt{2}} \sum_{I \in \hat{I}_{i-1}} \sqrt{q_{I}}. \]

This implies
\[ \sum_{i=1}^{k} \sum_{I \in \hat{I}_{i}} \sqrt{q_{I}} \leq \sum_{i=1}^{k} \left( \frac{1}{\sqrt{2}} \right)^{i-1} \sum_{I \in \hat{I}_{i}} \sqrt{q_{I}} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \sum_{I \in \hat{I}_{i}} \sqrt{q_{I}}. \]