Navigator-free EPI Ghost Correction with Structured Low-Rank Matrix Models: New Theory and Methods

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ABSTRACT

Purpose: Structured low-rank matrix models have recently been proposed to enable navigator-free echo-planar imaging (EPI) reconstruction and ghost artifact correction. This paper identifies some theoretical limitations of such approaches, and proposes and evaluates novel methods that bypass these limitations to enable substantially enhanced performance.

Theory and Methods: Many EPI ghost correction methods are based on treating subsets of EPI data from different readout gradient polarities or different shots as if they were acquired from different “virtual coils” in a parallel imaging experiment. Structured low-rank matrix models have previously been introduced to enable calibrationless parallel imaging reconstruction, and such ideas have recently been extended to enable navigator-free EPI ghost correction. However, our theoretical analysis shows that, because of uniform subsampling, the corresponding optimization problems for EPI data will always have either undesirable or non-unique solutions in the absence of additional constraints. This theoretical analysis leads us to propose new problem formulations for navigator-free EPI that incorporate side information from either image-domain or k-space domain parallel imaging methods. The importance of using nonconvex low-rank matrix regularization is also identified.

Results: We demonstrate using phantom and in vivo data that the proposed methods are able to eliminate ghost artifacts for several navigator-free EPI acquisition schemes, obtaining better performance in comparison to state-of-the-art methods across a range of different scenarios, including both single-channel acquisition and highly accelerated multi-channel acquisition.

Conclusion: Navigator-free EPI is challenging both in theory and in practice, but new theoretically-guided reconstruction methods can mitigate these challenges to achieve state-of-the-art results.

KEYWORDS

Echo-planar imaging; ghost correction; structured low-rank matrix recovery; constrained image reconstruction.
INTRODUCTION

Echo-planar imaging (EPI) (1) is currently one of the fastest MRI pulse sequences and one of the most popular sequences for functional, diffusion, and perfusion imaging. EPI uses a train of gradient echoes to measure multiple lines of k-space from a single excitation, but is prone to artifacts because it employs a long readout, uses rapidly-switching high-amplitude gradients, and measures alternating lines of k-space with different gradient polarities (2).

In conventional single-shot EPI, the even and odd lines of k-space are acquired with alternating gradient polarities. In practice, hardware imperfections, eddy currents, field inhomogeneity, concomitant fields, system delays, and similar phenomena, can introduce signal phase errors between k-space lines acquired with different readout gradient polarities. If these phase errors are not correctly compensated, a Nyquist (or N/2) ghost artifact is observed corresponding to an aliased image that is positioned a half field-of-view (FOV) away from the true spatial position along the phase-encoding direction. In multi-shot EPI, full k-space coverage is achieved by using multiple excitations, where a different segment of k-space is acquired using EPI for each shot. Images can then be reconstructed by interleaving the multi-shot data together. As in the single-shot case, the mismatch between different gradient polarities also leads to Nyquist ghost artifacts for multi-shot data. However, multi-shot data may also exhibit additional ghost artifacts if there happen to be inconsistencies between each shot – including system drift, subject motion, and, particularly in the case of gradient-recalled EPI, respiration.

Many approaches to ghost correction have been proposed over the years, which we group into two main categories. The first category contains simple model-based approaches such as Refs. (3–9), which assume a low-dimensional model to describe a systematic phase mismatch between even and odd lines. The parameters of the mismatch model are often estimated using separate navigator data, and can then be used to correct the mismatch in the measured EPI data. While these methods are widely used and can work well when the mismatch model is accurate, effects such as eddy currents and concomitant fields can lead to more complicated data mismatches that are not fully captured by simple low-dimensional models.

This paper focuses on the second category of methods, which includes the methods de-
scribed in Refs. (10–14). These methods rely on a more flexible model in which the data samples of each gradient polarity/shot are assumed to be coming from different but highly-correlated images. For example, it is often assumed that the images corresponding to different gradient polarities or shots have the same image magnitudes but different image phases. This is similar to how parallel imaging methods like SENSE (15) and GRAPPA (16) assume that the different channels of an array receiver coil acquire images that are different (i.e., modulated by different coil sensitivity profiles) but highly correlated. As a result, it is not surprising that many recent ghost correction approaches can be viewed as adaptations of previous parallel imaging methods to the ghost correction context. An example of this second category is the dual-polarity GRAPPA (DPG) method (12), which treats different polarities as if they were different virtual coils, and uses a dual GRAPPA kernel (with the GRAPPA weights divided into two halves corresponding to the two different gradient polarities) to synthesize a ghost-free fully-sampled image. Even though methods from the second category have been shown to have state-of-the-art performance in many challenging scenarios, they can still suffer from artifacts in certain cases. For example, DPG can fail to successfully correct ghost artifacts if there are mismatches between the measured EPI data and the autocalibration signal (ACS) used to train the dual GRAPPA kernel. This type of mismatch can occur because of changes in the measured data as a function of time, e.g., due to respiration (17). In addition, most of the methods in the second category rely on the use of multichannel data, and are not easily applicable to single-channel ghost-correction.

Recently, novel image reconstruction methods have been proposed that enable calibrationless single-channel and multi-channel image reconstruction from undersampled k-space data using structured low-rank matrix completion methods (18–24), which are based on the assumption that there exist linear dependencies in k-space due to limited image support, smooth image phase variations, parallel imaging constraints, and/or transform-domain image sparsity. Following the terminology of Ref. (18,20–22), we will refer to these as LORAKS (LOw-RAnk modeling of local K-Space neighborhoods) methods. LORAKS methods were not originally applied to EPI data, but have very recently been adapted to such contexts (22, 25–28). LORAKS for EPI has been demonstrated to yield state-of-the-art performance in highly-accelerated image reconstruction (22) and the ability to perform navigator-free EPI
In this paper, we analyze theoretical aspects of navigator-free EPI ghost correction using LORAKS and obtain new insights that have major implications for ghost correction performance. Specifically, we prove that the structured low-rank matrix completion problem associated with ghost correction either has a non-unique solution or a unique solution that is undesirable. Based on this result, we observe that additional constraints are needed to ensure the performance of LORAKS-based ghost correction, and propose two new approaches that achieve substantially improved results. A preliminary account of portions of this work was previously given in Ref. (29).

In the first new approach we propose, we combine LORAKS with coil sensitivity maps in the SENSE framework, as has previously been done for EPI reconstruction (22) and EPI ghost correction (26,27). Compared to similar SENSE-based ghost correction methods (26,27), our new SENSE-based approach makes use of a nonconvex regularization function from earlier LORAKS work (18,20,22) which yields improved results both in theory and in practice than the convex approach used in Refs. (26,27). Additionally, Refs. (26,27) used one of the simpler forms of LORAKS matrix construction (named the C-matrix in the original LORAKS paper) that incorporates support and parallel imaging constraints, but does not leverage image phase constraints. Our new SENSE-based approach takes advantage of a more advanced LORAKS matrix construction (named the S-matrix in the original LORAKS paper) that additionally incorporates phase constraints, as described in earlier LORAKS work (18,20,22).

In the second new approach we propose, we combine LORAKS with k-space domain parallel imaging linear predictability constraints, like those used in GRAPPA (16), SPIRiT (30), and PRUNO (31). In our implementation, these constraints are imposed within the broader framework of autocalibrated LORAKS (32). To the best of our knowledge, this is the first time that this type of information has been combined with structured low-rank matrix completion methods in the context of EPI ghost correction. This second new approach not only works for multichannel data as expected, but remarkably, we observe it also works for ghost correction of single-channel data in some cases.
Abbreviated Review of LORAKS

For simplicity, we only present a high-level review of LORAKS for the 2D case, and refer interested readers to Refs. (18,20,21) for more general descriptions and additional details.

The basic premise of the LORAKS support constraint (18) is that, if there are large regions of the FOV in which the true image is identically zero and if \( s(k_x, k_y) \) represents the Fourier transform of the true image, then there exist infinitely many k-space functions \( f(k_x, k_y) \) such that \( s(k_x, k_y) \ast f(k_x, k_y) \approx 0 \), where \( \ast \) denotes the standard convolution operation. If we let \( k \) denote the vector of samples of \( s(k_x, k_y) \) on the Cartesian Nyquist grid for the FOV and let \( f \) represent the samples of \( f(k_x, k_y) \) on the same Cartesian grid, then the convolution relationship can be expressed in matrix-vector form as \( P_C(k)f \approx 0 \), where the operator \( P_C(k) \) forms a Toeplitz-structured convolution matrix (called the LORAKS C-matrix (18)) out of the entries of \( k \). Since there are many such vectors \( f \) that satisfy this relationship, we observe that the LORAKS C-matrix will be approximately low-rank.

The basic premise of the LORAKS phase constraint (18, 21) is that, if the image has smoothly-varying phase and the image has limited support, then there are infinitely many functions \( h(k_x, k_y) \) such that \( s(k_x, k_y) \ast h(k_x, k_y) - \bar{s}(-k_x, -k_y) \ast \bar{h}(k_x, k_y) \approx 0 \), where \( \bar{s}(k_x, k_y) \) and \( \bar{h}(k_x, k_y) \) are respectively the complex conjugates of \( s(k_x, k_y) \) and \( h(k_x, k_y) \). Similar to the previous case, this convolution relationship can be expressed in a matrix-vector form as \( P_S(k)h \approx 0 \), where the operator \( P_S(k) \) combines a Toeplitz-structured convolution matrix with a Hankel-structured convolution matrix (resulting in what we call the LORAKS S-matrix (18,21)) out of the entries of \( k \), and \( h \) is the vector of Nyquist samples of \( h(k_x, k_y) \) and \( \bar{h}(k_x, k_y) \). Since there are many such vectors \( h \) that satisfy this relationship, we observe that the LORAKS S-matrix will also be approximately low-rank.

These low-rank matrix constructions are easily generalized to the context of parallel imaging. Specifically, assume that data is acquired from \( N_c \) channels, and let \( k_n \) denote the vector of k-space samples from the \( n \)th channel, and let \( k_{\text{tot}} \) denote the vector containing
the k-space samples from all channels. It has been shown that the concatenated matrix

\[ C_P(\mathbf{k}_{\text{tot}}) = \begin{bmatrix} \mathcal{P}_C(k_1) & \mathcal{P}_C(k_2) & \cdots & \mathcal{P}_C(k_{N_c}) \end{bmatrix} \]  \[ 1 \]

will generally have low rank \([19,20,31]\), and that the concatenated matrix

\[ S_P(\mathbf{k}_{\text{tot}}) = \begin{bmatrix} \mathcal{P}_S(k_1) & \mathcal{P}_S(k_2) & \cdots & \mathcal{P}_S(k_{N_c}) \end{bmatrix} \]  \[ 2 \]

will generally have low rank \([20]\). Note that Eqs. \([1]\) and \([2]\) reduce to the standard single-channel case when \(N_c = 1\), so we will use these expressions for both the single channel and the multichannel cases.

By enforcing one or more of these low-rank constraints during image reconstruction, it becomes possible to reconstruct high-quality images from highly accelerated and/or unconventionally sampled k-space data.

The preceding paragraphs described LORAKS for single-channel and multi-channel image reconstruction for general contexts, and without specialization to ghost correction for EPI. However, as described in the introduction, there is a straightforward analogy between parallel imaging and Nyquist ghost correction. For the sake of simplicity and without loss of generality, we will describe the LORAKS matrix construction for this case in the context of single-shot imaging with positive and negative readout polarities (denoted RO\(^+\) and RO\(^-\), respectively), noting that the extension to multi-shot imaging is trivial (obtained by concatenating together the LORAKS matrices for each shot as if the different shots were coming from different receiver coils in a parallel imaging experiment). Let \(\mathbf{k}_{\text{tot}}^+\) and \(\mathbf{k}_{\text{tot}}^-\) represent hypothetical vectors of Nyquist-sampled Cartesian k-space data for the two different readout gradient polarities from either a single-channel or multi-channel experiment. Based on previous arguments, we expect the matrices

\[ \begin{bmatrix} C_P(\mathbf{k}_{\text{tot}}^+) & C_P(\mathbf{k}_{\text{tot}}^-) \end{bmatrix} \]  \[ 3 \]

and

\[ \begin{bmatrix} S_P(\mathbf{k}_{\text{tot}}^+) & S_P(\mathbf{k}_{\text{tot}}^-) \end{bmatrix} \]  \[ 4 \]
to be approximately low-rank. However, due to the form of single-shot EPI imaging, we only measure a subset of the phase encoding lines of $k^+_{tot}$ and $k^-_{tot}$. Specifically, let the measured data for the RO$^+$ and RO$^-$ be respectively denoted as $d^+_{tot}$ and $d^-_{tot}$, respectively, with $d^+_{tot} = A_+ k^+_{tot}$ and $d^-_{tot} = A_- k^-_{tot}$, where $A_+$ and $A_-$ are simple subsampling matrices that extract the measured entries of $k^+_{tot}$ and $k^-_{tot}$ (i.e., $A_+$ and $A_-$ are formed by concatenating the rows of the identity matrix corresponding to the k-space sampling masks for each polarity).

The simplest version of LORAKS-based ghost correction, first proposed in Ref. (25), is given by

$$\{\hat{k}^+_{tot}, \hat{k}^-_{tot}\} = \arg \min_{\{k^+_{tot}, k^-_{tot}\}} J \left( \begin{bmatrix} C_P(k^+_{tot}) & C_P(k^-_{tot}) \end{bmatrix} \right),$$

subject to the additional data-consistency constraints that $A_+ \hat{k}^+_{tot} = d^+_{tot}$ and $A_- \hat{k}^-_{tot} = d^-_{tot}$.

Here, $J(\cdot)$ is a cost function that depends only on the singular values of its matrix argument, and promotes low-rank solutions. In the sequel, we will use the notation

$$L_C(k^\pm_{tot}) = J \left( \begin{bmatrix} C_P(k^+_{tot}) & C_P(k^-_{tot}) \end{bmatrix} \right),$$

where $k^\pm_{tot}$ concatenates $k^+_{tot}$ and $k^-_{tot}$. Similarly, we will also use $L_S(\cdot)$ to denote the function with the same form as Eq. [6], but switching from the LORAKS C-matrix to the LORAKS S-matrix by replacing all instances of $C_P$ with $S_P$.

A popular choice for $J(\cdot)$ in the general low-rank matrix completion literature (and the choice made by Ref. (25)) is the nuclear norm, which is a convex function that is known to encourage minimum-rank solutions \((33)\). The nuclear norm of a matrix $G$ is defined as

$$\|G\|_* = \sum_{i=1}^{\text{rank}(G)} \sigma_i(G),$$

where $\sigma_i(G)$ is the $i$th singular value of $G$. Another potential choice of $J(\cdot)$ that is relevant for this paper, which was proposed in the original LORAKS work \((18)\) but has not been used in the EPI ghost correction work by other groups, is defined by

$$J_r(G) = \sum_{i=r+1}^{\text{rank}(G)} (\sigma_i(G))^2,$$
where \( r \) is a user-selected parameter. This cost function is nonconvex, and \( J_r(\mathbf{G}) \) will equal zero whenever \( \text{rank}(\mathbf{G}) \leq r \). However, if \( \text{rank}(\mathbf{G}) > r \), then \( J_r(\mathbf{G}) \) will be nonzero, and equal to the squared Frobenius norm error that is incurred when \( \mathbf{G} \) is optimally approximated by a rank-\( r \) matrix. As a result, this cost function will encourage the reconstructed image to have a LORAKS matrix that is approximately rank-\( r \) or lower.

The following subsection provides a novel theoretical analysis of the optimization problem from Eq. [5], which reveals that it has several undesirable characteristics.

### Theoretical Analysis of LORAKS-based Methods for EPI Ghost Correction

For our analysis, we assume a typical setup in which the nominal fully-sampled k-space dataset has equally-spaced consecutive phase encoding positions. For the sake of brevity, we will assume fully-sampled single-channel EPI imaging\(^1\) in which \( \mathbf{d}_{\text{tot}}^+ \) corresponds to the full set of measured even phase encoding positions, while \( \mathbf{d}_{\text{tot}}^- \) corresponds to the full set of measured odd phase encoding positions.

Notice that the form of \( \mathbf{A}_+ \) implies that \( \mathbf{A}_+^H \mathbf{A}_+ \) is a diagonal projection matrix, and that multiplying any vector of k-space samples by \( \mathbf{A}_+^H \mathbf{A}_+ \) is equivalent to preserving the values of the even phase encoding lines while setting the values of the odd phase encoding lines to zero. Similarly, \( \mathbf{A}_-^H \mathbf{A}_- \) is a diagonal projection matrix, and multiplying any vector of k-space samples by \( \mathbf{A}_-^H \mathbf{A}_- \) is equivalent to preserving the values of the odd phase encoding lines while setting the values of the even phase encoding lines to zero. Additionally, we have that \( \mathbf{A}_+^H \mathbf{A}_+ = \mathbf{I} - \mathbf{A}_-^H \mathbf{A}_- \), where \( \mathbf{I} \) is the identity matrix.

Using these facts together with the vector space concepts of orthogonal complements and direct sums (34), we know that if \( \{\hat{\mathbf{k}}_{\text{tot}}^+, \hat{\mathbf{k}}_{\text{tot}}^-\} \) obeys the data fidelity constraint from Eq. [6],

\(^1\)Generalized theoretical results for the case of parallel imaging with uniformly undersampled phase encoding can also be derived using the same principles we used for the single-channel fully-sampled case. We have elected not to show these derivations because they are intellectually straightforward extensions of the single-channel fully-sampled case, but require a lot of additional notation to describe.
then there exist corresponding vectors $y$ and $z$ such that we can write

$$
\hat{k}_{\text{tot}}^+ = A^H_+ d^+_{\text{tot}} + A^H_- A_- y
$$

$$
\hat{k}_{\text{tot}}^- = A^H_- d^-_{\text{tot}} + A^H_+ A_+ z.
$$

We have the following theoretical results:

**Theorem 1.** Given the context described above and arbitrary vectors $y$ and $z$, the singular values of the matrix

$$
\begin{bmatrix}
C_P(A^H_+ d^+_{\text{tot}} + A^H_- A_- y) & C_P(A^H_- d^-_{\text{tot}} + A^H_+ A_+ z)
\end{bmatrix}
$$

are identical to the singular values of the matrix

$$
\begin{bmatrix}
C_P(A^H_+ d^+_{\text{tot}} - A^H_- A_- y) & C_P(A^H_- d^-_{\text{tot}} - A^H_+ A_+ z)
\end{bmatrix}.
$$

Note that the vectors appearing in Eq. [11], i.e.,

$$
\hat{k}_{\text{tot}}^+ = A^H_+ d^+_{\text{tot}} - A^H_- A_- y
$$

$$
\hat{k}_{\text{tot}}^- = A^H_- d^-_{\text{tot}} - A^H_+ A_+ z.
$$

are identical to the vectors given in Eq. [9] except that the estimates of the unmeasured data samples have been multiplied by -1.

The proof of this theorem is sketched in the supplementary material. Some basic intuition for this result is that we can multiply our estimates for the unmeasured k-space lines by -1 without impacting fidelity with the measured data. Due to uniform subsampling of each gradient polarity by a factor of 2, this multiplication procedure is equivalent to applying linear phase in k-space, which corresponds to a spatial shift of the image by half the FOV along the phase encoding dimension (and a 180° constant phase offset for the RO$^-$ polarity). This shifting procedure has no effect on the image support or on the correlations that exist between the different coils, and thus has no impact on the singular values of the LORAKS matrix. This means that if we have one solution to Eq. [5], then it is easy for us to construct another solution to Eq. [5], and this optimization problem will generally not have a unique
useful solution.

The following corollaries formalize some of these statements and provide additional useful insight.

**Corollary 1.** Equation [5] either has the unique solution \( \{ \hat{k}_t^{+}, \hat{k}_t^{-} \} = \{ A^H d_{tot}^{+}, A^H d_{tot}^{-} \} \) which corresponds to zero-filling of the measured data, or it has at least two distinct optimal solutions that share exactly the same cost function value.

**Corollary 2.** If the cost function \( J(\cdot) \) is chosen to be convex (e.g., the nuclear norm), then the zero-filled solution is always an optimal solution of Eq. [5]. If Eq. [5] has more than one optimal solution, then it has infinitely many optimal solutions.

**Corollary 3.** Theorem 1 and Corollaries 1 and 2 are still true if we replace \( C_P(\cdot) \) in Eqs. [5], [10], and [11] with \( S_P(\cdot) \).

Corollary 1 is proven in the supplementary material, and implies that the optimization problem of Eq. [5] either has a trivial undesirable solution corresponding to zero-filling of the measured data, or it is an ill-posed optimization problem that does not possess a unique solution. While some of the solutions to Eq. [5] may be desirable, there are no guarantees that the algorithm we use to minimize Eq. [5] will yield one of these desirable solutions. Corollary 2 is also proven in the supplementary material, and suggests that the use of convex cost functions to impose LORAKS constraints is likely to be suboptimal relative to the use of nonconvex cost functions. It should be noted that, unlike recent Nyquist ghost correction methods [25–27] which have made use of the convex nuclear norm, the early structured low-rank matrix completion methods for MRI all made use of nonconvex cost functions [18–21]. These nonconvex options are likely to be better for this problem setting. Corollary 3 is stated without proof (but can be proved using an approach that is similar to our proof of Theorem 1), and indicates that the deficiencies of Eq. [5] are not alleviated by switching from the LORAKS C-matrix to the LORAKS S-matrix.

Practical illustrations of these theoretical results are shown in Figs. 1 and 2. Figure 1 shows two different sets of EPI images that are both perfectly consistent with standard fully-sampled EPI data. The difference between the two datasets is the same as the difference between Eqs. [9] and [12]. As expected, this k-space phase difference leads to shifting of the images for both RO² and RO⁻ by half the FOV, as well as adding a constant phase offset for
the RO\(^-\) image. The figure also shows a plot of the singular values of the LORAKS matrices corresponding to these two datasets. As expected from Theorem 1, the singular values are identical in both cases. Figure 2 illustrates the difference in behavior between convex and nonconvex cost functions $J(\cdot)$. The figure shows that the zero-filled solution is a minimum of Eq. 5 in the convex case, as expected from Corollary 2, and that there are many different images with very similar cost function values. Notably, the images from Fig. 1(a) and (b) are not optimal solutions, even though they both have high quality and appear to be devoid of ghost artifacts. The figure also shows that the cost function has more desirable behavior in the nonconvex case (e.g., the zero-filled solution is no longer a minimum of Eq. 5 and the optimization problem has sharper minima, and with local minima in the vicinity of the images from Fig. 1), although the solution to Eq. 5 is still not unique in this case as we should expect based on Corollary 1.

While it may be possible to get a useful result from solving Eq. 5, it should be noted that in the presence of multiple global minimizers, it is difficult to ensure that an optimization algorithm will always converge to a desirable minimum. Incorporating additional constraints on the solution is a straightforward way to reduce the ambiguity associated with Eq. 5, and we describe two practical approaches for this in the next section.

Figure 1: A demonstration of LORAKS-matrix ambiguity: (a) EPI magnitude (top) and phase (bottom) images corresponding to (left) RO\(^+\) data and (right) RO\(^-\). (b) Images corresponding to the same data from (a), except that the odd k-space lines for RO\(^+\) and the even k-space lines for RO\(^-\) have been multiplied by -1. (c) Plots of the singular values for the LORAKS matrices from Eq. 3 for the k-space datasets corresponding to (a) and (b).
Figure 2: Letting $k_1^\pm$ and $k_2^\pm$ denote the k-space data for the images in Figs. 1(a) and 1(b), respectively, we plot the cost function value $L_C(\alpha k_1^\pm + (1 - \alpha)k_2^\pm)$ as a function of $\alpha$. Setting $\alpha = 0$ yields the cost function value for the images from Fig. 1(a), setting $\alpha = 1$ yields the cost function value for the images from Fig. 1(b), while setting $\alpha = 0.5$ yields the cost function value for the zero-filled solution. Results are shown for different choices of $J(\cdot)$, corresponding to (left) the convex nuclear norm from Eq. [7] and (right) the nonconvex function from Eq. [8] with $r = 40$.

**METHODS**

Formulation using SENSE Constraints

A natural approach to imposing additional constraints on LORAKS reconstruction is to impose coil sensitivity map information within the SENSE framework (15), assuming that coil sensitivity profiles are available and that data is acquired using a multi-channel receiver array. This style of approach has been used previously for both EPI reconstruction (assuming ghosts have been precorrected using navigator data) (22) and for navigator-free EPI ghost correction (26, 27). Our proposed approach can be viewed as a combination of these two previous formulations.

In this work, we propose to use the following formulation for navigator-free EPI ghost correction using SENSE:

$$
\{\hat{\rho}^+, \hat{\rho}^-, \hat{\rho}^-\} = \arg \min_{\{\rho^+, \rho^-, \rho^-\}} \|E_+\rho^+ - d^+_{\text{tot}}\|^2_2 + \|E_-\rho^- - d^-_{\text{tot}}\|^2_2 + \lambda L_S(k^\pm_{\text{tot}}), \quad [13]
$$

subject to the constraints that $E\rho^+ = k^+_{\text{tot}}$, that $E\rho^- = k^-_{\text{tot}}$, and that $k^\pm_{\text{tot}}$ is the concatenation of $k^+_{\text{tot}}$ and $k^-_{\text{tot}}$. In this formulation, we are using SENSE to reconstruct one image
for RO\(^+\) (\(\rho^+\)) and another image for RO\(^-\) (\(\rho^-\)), and the only coupling that occurs between the two comes from the LORAKS regularization term. We have also used \(E_+\), \(E_-\), and \(E\) to denote the standard SENSE matrices (mapping from the image domain to k-space) corresponding to RO\(^+\) subsampling, RO\(^-\) subsampling, and full Nyquist sampling, respectively. These three matrices all use exactly the same sensitivity profiles, and differ only in the associated k-space sampling patterns. In addition, \(\lambda\) is a regularization parameter, and we suggest the use of the nonconvex regularizer from Eq. 8 when defining \(L_S(\cdot)\).

The main difference between this proposed approach and Ref. (22) is the separation of the RO\(^+\) and RO\(^-\) datasets, which enables navigator-free ghost correction. The main difference between this proposed approach and Refs. (26,27) is that the previous LORAKS-based EPI ghost correction work used the LORAKS C-matrix and nuclear norm regularization, while we advocate use of the LORAKS S-matrix with nonconvex regularization.

Equation 13 has been written assuming single-shot data. In the case of multi-shot data with phase inconsistencies between different shots, we generalize Eq. 13 by reconstructing a separate image for each polarity and each shot, with a separate SENSE encoding matrix and data fidelity term for each. Note that separating the data from different shots increases the effective acceleration factor for each data consistency term and is also associated with additional computational complexity. As a result, there are no advantages to separating the shots in application contexts where shot-to-shot phase variations are negligible.

There are many ways to solve the optimization problem in Eq. 13. In this paper, we use the algorithm described in Ref. (22).

**Formulation using AC-LORAKS Constraints**

Another natural approach is to use k-space constraints like those of GRAPPA (16), SPIRiT (30), and PRUNO (31). In this work, we use a formulation based on AC-LORAKS (32) (with strong similarities to PRUNO (31)). Specifically, in the single-shot case, we solve

\[
\left\{ \hat{k}^{+}_{\text{tot}}, \hat{k}^{-}_{\text{tot}} \right\} = \arg \min_{\{k^{+}_{\text{tot}}, k^{-}_{\text{tot}}\}} \|C_P(k^{+}_{\text{tot}})N\|_F^2 + \|C_P(k^{-}_{\text{tot}})N\|_F^2 + \lambda L_S(k^{\pm}_{\text{tot}}), \quad [14]
\]
subject to the constraints that $d_i^+ = A_i k_i^+$, $d_i^- = A_i k_i^-$, and that $k_i^+$ is the concatenation of $k_i^+$ and $k_i^-$. In this expression, $\| \cdot \|_F$ denotes the Frobenius norm, and the matrix $N$ is the right nullspace of a LORAKS C-matrix formed from ACS data acquired in a standard parallel imaging calibration pre-scan. The first two terms of Eq. [14] are similar to the first two terms of Eq. [13], in the sense that they impose support and parallel imaging constraints derived from some form of prescan, but do not make any assumptions about the relationship between $k_i^+$ and $k_i^-$ or the relationship between the image-domain phase characteristics of the calibration data and the image-domain phase characteristics of the EPI data being reconstructed. Similarly, the third terms in Eqs. [13] and [14] are the only terms that introduce coupling between $k_i^+$ and $k_i^-$, and the only terms that use the LORAKS S-matrix to introduce constraints on the image phase. The use of phase constraints is useful both for partial Fourier EPI acquisition and for stabilizing the reconstruction of symmetrically-acquired EPI data (22). Similar to the previous case, we also suggest the use of the nonconvex regularizer from Eq. [8] when defining $L_S(\cdot)$.

As in our proposed SENSE formulation, Eq. [14] is written for the single-shot case, but the generalization to multi-shot EPI is straightforward by separating and jointly reconstructing images for each polarity and shot. And similar to Eq. [13], there are also many ways to solve the optimization problem in Eq. [14]. These two optimization problems have very similar structure (i.e., the first two terms are least-squares penalties, while the third term encourages low-rank matrix structure), and as a consequence, we have used a minor modification of the algorithm we used for solving Eq. [13] to also solve Eq. [14].

An interesting feature of our proposed k-space based formulation is that the AC-LORAKS formulation can often work with single-channel data (32). This is a major advantage over our proposed SENSE formulation, which is not expected to produce good results unless parallel imaging is used.

RESULTS

This section describes evaluations of our new LORAKS-based EPI ghost correction methods using navigator-free EPI data acquired from phantoms and in vivo human brains. In each
case, ACS data (used both for estimating nullspaces and for estimating sensitivity maps) was acquired using the same approach as previously used in DPG (12). Most of the reconstructions we show in this section estimate separate images for each gradient polarity and each shot, and some also estimate separate images for each coil. While various approaches exist for combining together multiple images from different coils/polarities/shots for visualization, for simplicity and consistency we have combined the multiple images into a single image using principal component analysis, which is a standard method for parallel imaging coil compression/combination (35, 36). Unless otherwise specified, our LORAKS-based results also always use the nonconvex regularization penalty from Eq. [8], with the rank threshold $r$ chosen based on the singular values of the LORAKS matrix formed from ACS data. Specifically, $r$ was chosen as the point at which the plot of the singular values appears to flatten out, which is a standard approach to matrix rank estimation in the presence of noise. For methods that use regularization parameters, $\lambda$ was initially set to a small value ($\lambda = 10^{-3}$), and if necessary based on visual assessment of image ghost artifacts, was gradually increased until good reconstructions were observed. The different LORAKS-based reconstructions were performed based on adaptations of publicly-available code (37).

Figure 3 shows a comparison of different parallel imaging reconstruction and EPI ghost correction methods for in vivo single-shot EPI data. A gold standard image with fully-sampled $\text{RO}^+$ data and fully-sampled $\text{RO}^-$ images was obtained using PLACE (10) with a 32-channel receiver coil and an $128 \times 128$ acquisition matrix. We also acquired standard fully-sampled EPI (acceleration factor $R = 1$, with each gradient polarity undersampled by a factor of two) and accelerated EPI acquisitions for a range of acceleration factors ($R = 2, 3, 4$). Additionally, the acceleration factor of $R = 5$ was simulated by retrospectively undersampling the PLACE data. Reconstructions were performed using unconstrained LORAKS as in Eq. [5], LORAKS with SENSE constraints as in Eq. [13], and LORAKS with AC-LORAKS constraints as in Eq. [14]. For comparison, we also performed independent SENSE reconstruction of each gradient polarity (11) without any LORAKS-based regularization (equivalent to setting $\lambda = 0$ in Eq. [13]). The figure shows that SENSE without LORAKS constraints works well for low-acceleration factors, though faces challenges at high acceleration factors. This behavior is expected as an EPI acceleration factor of $R = 5$ is an
effective acceleration factor of $R = 10$ for each readout polarity, which is a very challenging case for SENSE reconstruction. We also observe that unconstrained LORAKS reconstruction has severe problems, as should be expected based on our theoretical analysis of Eq. [5]. On the other hand, both of our proposed new formulations are substantially more successful, achieving high quality reconstruction results even at very high acceleration factors. At the highest acceleration factors, LORAKS with AC-LORAKS constraints was more effective than LORAKS with SENSE constraints which displayed unresolved aliasing artifacts.

Figure 3: Comparison of different reconstruction techniques using retrospectively undersampled in vivo data to simulate single-shot EPI experiments at different parallel imaging acceleration factors. (a) Gold standard. (b) Independent SENSE reconstruction of each gradient polarity. (c) Unconstrained LORAKS. (d) LORAKS with SENSE constraints. (e) LORAKS with AC-LORAKS constraints.

The data shown in Fig. 3 was also reconstructed using the state-of-the-art DPG method [12] using the same ACS data, and a comparison against LORAKS with AC-LORAKS constraints is shown in Fig. 4. While DPG generally works well, a close examination of the reconstructed magnitude and phase images demonstrates that DPG still has small residual ghost artifacts that are not present in the LORAKS-based reconstruction. These artifacts are particularly visible in the phase images, since the image phase is highly sensitive to
ghosting in regions of the image where the magnitude is small. A deeper examination of the data leads us to believe that the ghost artifacts we see for DPG are the result of systematic changes (between the relative phases of the different gradient polarities) that have occurred in part due to the length of time that passed between the collection of the ACS data and the acquisition of the accelerated EPI data that is being reconstructed.

Figure 4: Comparison between (a) DPG and (b) LORAKS with AC-LORAKS constraints for the real single-shot EPI in vivo brain data from Fig. 3. Instead of showing coil-combined images, only a single representative channel is being shown to avoid the contamination of the phase characteristics induced by coil combination.

Experiments were also performed to evaluate the performance of convex regularization versus nonconvex regularization. We observed that at low acceleration factors, both the convex nuclear norm from Eq. [7] and the the nonconvex function from Eq. [8] yield similar results, as should be expected since parallel imaging is well-posed for small values of $R$. However, major differences between the convex and nonconvex formulations emerge for larger values of $R$. For illustration, Fig. 5 shows an example of a phantom dataset (single-shot EPI acquired with a $128 \times 128$ acquisition matrix and a 12-channel receiver coil) reconstructed with SENSE-based LORAKS at the acceleration factor $R = 3$. While we observe that both
convex and nonconvex cost functions are successful at suppressing aliasing artifacts outside the true support of the phantom in this case, the reconstruction from convex regularization has substantial aliasing artifacts within the support of the phantom that are not present with nonconvex regularization. This result is consistent with our theoretical expectations from Corollaries 1-3 that nonconvex cost functions can lead to a better-posed reconstruction problem.

![Comparison of convex versus nonconvex regularization for LORAKS reconstruction with SENSE-constraints. (a) Magnitude and (b) phase images are shown for the gold standard (obtained with PLACE) and LORAKS-based reconstructions from accelerated ($R = 3$) single-shot EPI data using convex and nonconvex cost functions.](image)

In addition to navigator-free multi-channel settings, the proposed methods were also evaluated in navigator-free single-channel settings, which are expected to be substantially more challenging. Single-channel datasets were obtained by isolating the information from a single coil in multi-channel acquisitions. Note that, since sensitivity-map estimation is not feasible in the single-channel setting, our SENSE-based results used a binary support mask (that has value 1 inside the support of the image and value 0 everywhere else) in place of a coil sensitivity map. This has the effect of imposing prior knowledge of the image support on the reconstructed image. Note also that DPG was not originally designed to be used with single-channel data, although the formulation can still be applied to the single-channel case. Unaccelerated ($R = 1$) single-channel single-shot EPI results are shown for a phantom dataset (64×64 acquisition matrix) in Fig. 6 and for one channel of the previous in vivo human brain dataset in Fig. 7. The results are consistent in both cases. Images
obtained without compensating the mismatch between RO$^+$ and RO$^-$ have obvious ghost artifacts, and these artifacts are not solved (and are potentially even amplified) when using LORAKS without constraints. The LORAKS reconstruction with “SENSE” constraints (i.e., support constraints) helps to eliminate some of the ghost artifacts that appeared outside the support of the original object, although residual aliasing artifacts are still observed within the support of the object. These artifacts are most visible in the phantom image, though close inspection also reveals the appearance of aliasing artifacts in the brain image. We also observe that DPG is unsuccessful in this single-channel case, which we believe is due both to the difficulties of the single-channel problem as well as systematic differences between the ACS data and the data being reconstructed. On the other hand, the LORAKS results with AC-LORAKS constraints are substantially more successful at removing ghost artifacts than any of the previous methods were.

![Image](image.png)

Figure 6: (a) Magnitude and (b) phase images corresponding to reconstruction of unaccelerated ($R = 1$) single-channel single-shot EPI phantom data.
Reconstructions were also performed using multi-shot data. Figure 8 shows results using fully sampled $R = 1$ two-shot data ($128 \times 128$ acquisition matrix, 12-channel receiver coil) at different time points for a phantom with physically-simulated respiration effects. Note that with $R = 1$ and two-shots, each gradient polarity for each shot has an effective undersampling factor of 4. LORAKS with AC-LORAKS constraints is compared against DPG, and to make reconstruction even more challenging for LORAKS, LORAKS reconstruction was performed from single-channel data while DPG was provided with the full set of multi-channel data. Due to simulated respiration, there are mismatches between the measured ACS data and the EPI data being reconstructed. DPG is not robust to these mismatches, and displays residual ghost artifacts with time-varying characteristics. On the other hand, LORAKS with AC-LORAKS demonstrates robustness against the time-varying changes in this dataset, even despite the challenging single-channel multi-shot nature of this reconstruction problem.

For further insight, Fig. 9(a) shows even more challenging cases where LORAKS with AC-LORAKS constraints is used to reconstruct a single-shot of the previous multi-shot dataset. Note that this corresponds to EPI with an acceleration factor of $R = 2$ (i.e., an effective acceleration factor of $R = 4$ for each gradient polarity). Remarkably, we observe that single-channel LORAKS with AC-LORAKS constraints is still successful in this very difficult scenario, with similar quality to that obtained using multi-channel LORAKS with AC-LORAKS constraints. However, it is also important for us to point out that single-
channel reconstruction with $R = 2$ is not always successful, as illustrated in Fig. 9(b) using previously described in vivo human brain data. In this brain case, we observe that single-channel LORAKS reconstruction is unsuccessful at correctly reconstructing the image, while the multi-channel case (also shown in Fig. 4) yields accurate results as described previously. We suspect that the difference between the phantom result and the in vivo result is explained by differences in the size of the FOV relative to the size of the object. Specifically, we expect that the reconstruction problem is harder for tight FOVs than it is for larger FOVs that contain a larger amount of empty space.

Figure 8: Evaluation with multi-shot EPI data for a phantom with simulated respiratory effects. A representative set of four segmented images is shown, extracted from a longer acquisition spanning several minutes. (a) Phase images corresponding to one channel of the data, with images reconstructed without compensating for the mismatches between different shots and different gradient polarities. (b) Phase images for one channel of the DPG reconstruction, with reconstruction performed using multi-channel data. (c) Phase images for single-channel LORAKS reconstruction with AC-LORAKS constraints. (d) Plot showing the relative respiratory phase across EPI shots (TR=60msec), as measured with an ultrasound transducer coupled to a respiratory phantom. The line-plot peaks show points where the phantom air bag is maximally inflated. The sampling times for the shots used to generate each of the two-shot images from (a)-(c) are marked as labeled in the legend.
Figure 9: Images reconstructed using LORAKS with AC-LORAKS constraints for accelerated \((R = 2)\) data in both single-channel and multi-channel contexts. (a) Phantom dataset with simulated respiration. (b) In vivo human brain dataset. Phase images from a single channel are shown for (left) zero-filled data reconstructed without compensating for mismatches between the gradient polarities, (center) LORAKS reconstruction with AC-LORAKS constraints from single-channel data, and (right) LORAKS reconstruction with AC-LORAKS constraints from multi-channel data.

**DISCUSSION AND CONCLUSIONS**

This paper derived novel theoretical results for EPI ghost correction based on structured low-rank matrix completion approaches. Key theoretical results include the observation that the corresponding matrix completion problem is ill-posed in the absence of additional constraints, and that convex formulations have undesirable characteristics that are somewhat mitigated by the use of nonconvex formulations. These theoretical results led to two novel problem formulations that use additional constraints and nonconvex regularization to avoid the problems associated with ill-posedness. Our results showed that these new approaches are both effective relative to state-of-the-art ghost correction techniques like DPG. We also observed that the proposed variation that uses AC-LORAKS constraints appears to be more effective than the proposed variation that uses SENSE constraints in challenging scenarios with single-channel data or highly-accelerated multi-channel data. Surprisingly, we even observed that the variation using AC-LORAKS constraints can even be successful when applied to undersampled single-channel data. We believe that these approaches will prove valuable across a range of different applications, but especially those in which navigator-based ghost correction methods are ineffective, in which the need for navigation places undesirable
constraints on the minimum achievable echo time or repetition time, in cases where single-
channel imaging is unavoidable, or in cases where imaging conditions are likely to change
as a function of time through the duration of a long experiment (e.g., functional imaging or
diffusion imaging).

This paper focused on simple proof-of-principle demonstrations of the proposed ap-
proaches, and there are opportunities for a variety of improvements. For example, the
choices of $r$ and $\lambda$ were made manually using heuristic approaches, and its likely that these
same decision processes could be made automatically using standard techniques for rank esti-
mation (e.g., (38)) or ghost correction parameter tuning (e.g., (7)). Additionally, extensions
to the case of simultaneous multi-slice imaging are possible within the LORAKS framework
(39), and are likely to be practically useful in a range of experiments given the advantages
and the modern popularity of EPI-based simultaneous multi-slice acquisitions (40).

While we derived our new theory and methods in the context of EPI ghost correction, we
believe that this paper also has more general consequences. For example, we believe that it
is straightforward to generalize our theoretical results to show that unconstrained LORAKS-
based reconstruction will be ill-posed for any application that uses uniformly-undersampled
Cartesian k-space trajectories (also see similar comments in Ref. (22)). We also believe
that the combination of LORAKS with additional constraints will always be beneficial when
the constraints are accurate, and so encourage the use of additional constraints whenever
the LORAKS reconstruction problem is ill-posed. In addition, we believe that our novel
LORAKS formulation with AC-LORAKS constraints is an important innovation that is
likely to be useful in other applications, similar to how LORAKS with SENSE constraints
has already proven to be useful in other settings (22,41). Finally, while recent work has
proposed the use of convex LORAKS-based formulations, our empirical experience since
we first started exploring LORAKS several years ago (18,42) has consistently been that
nonconvex formulations are substantially more powerful than convex ones.
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SUPPLEMENTARY MATERIAL

Proof of Theorem 1

Assuming the notation of Eq. [5], let $B$ denote the zero-filled LORAKS matrix

\[ B = \begin{bmatrix} C_P(A_+^Hd_{tot}^+) & C_P(A_-^Hd_{tot}^-) \end{bmatrix} \]  \hspace{1cm} [A.1]

and let $D$ denote the matrix corresponding to the unmeasured data samples

\[ D = \begin{bmatrix} C_P(A_-^HA_y) & C_P(A_+^HA_z) \end{bmatrix}. \]  \hspace{1cm} [A.2]

Due to the way the LORAKS $C$-matrix is constructed, if the entry in the $m$th column and $n$th row of the matrix $B$ is nonzero, then the corresponding entry of the matrix $D$ is required to be zero and vice versa. Note also that the matrix from Eq. [10] can be written as $B + D$, while the matrix from Eq. [11] can be written as $B - D$.

To avoid additional tedious notation, we will assume in our proof sketch that the rows and columns of the matrix $B$ have been permuted in such a way that samples from even and odd lines in k-space are never adjacent to one another in the matrix, which is always possible based on the convolutional structure of the LORAKS $C$-matrix. This allows the $B$ matrix to be written in a “checkerboard” form

\[ B = \begin{bmatrix} b_{11}^+ & 0 & b_{13}^+ & 0 & b_{15}^+ & \cdots & 0 & b_{12}^- & 0 & b_{14}^- & 0 & \cdots \\ 0 & b_{22}^+ & 0 & b_{24}^+ & 0 & \cdots & b_{21}^- & 0 & b_{23}^- & 0 & b_{25}^- & \cdots \\ b_{31}^+ & 0 & b_{33}^+ & 0 & b_{35}^+ & \cdots & 0 & b_{32}^- & 0 & b_{34}^- & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \]  \hspace{1cm} [A.3]

where $b_{ij}^+$ and $b_{ij}^-$ are the nonzero entries of the $B$ matrix corresponding to positive and negative readout polarities, respectively. Using the same permutation scheme, the matrix $D$
can similarly be written in the corresponding complementary “checkerboard” form:

\[
D = \begin{bmatrix}
0 & d_{12}^+ & 0 & d_{14}^+ & 0 & \cdots & d_{11}^- & 0 & d_{13}^- & 0 & d_{15}^- & \cdots \\
d_{21}^+ & 0 & d_{23}^+ & 0 & d_{25}^+ & \cdots & 0 & d_{22}^- & 0 & d_{24}^- & 0 & \cdots \\
0 & d_{32}^+ & 0 & d_{34}^+ & 0 & \cdots & d_{31}^- & 0 & d_{33}^- & 0 & d_{35}^- & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad [A.4]
\]

Consider the diagonal matrix \( Q_1 \) which has the same number of columns as \( B \), and whose diagonal entries alternate in sign in a way that follows the non-zero pattern of the first row of \( B \):

\[
Q_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}. \quad [A.5]
\]

Similarly, consider the diagonal matrix \( Q_2 \) which has the same number of rows as \( B \), and whose diagonal entries alternate in sign in a way that follows the non-zero pattern of the first column of \( B \):

\[
Q_2 = \begin{bmatrix}
1 & 0 & 0 & \cdots \\
0 & -1 & 0 & \cdots \\
0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}. \quad [A.6]
\]

We make the following observations:

- The matrices \( Q_1 \) and \( Q_2 \) are unitary and satisfy \( Q_1^{-1} = Q_1 \) and \( Q_2^{-1} = Q_2 \).
• The matrix $B$ is structured in such a way that $BQ_1 = Q_2B$.

• The matrix $D$ is structured in such a way that $DQ_1 = -Q_2D$.

• The matrix $Q_2(B + D)Q_1$ simplifies according to

\[
Q_2(B + D)Q_1 = Q_2Q_2(B - D) = B - D.
\]  

[A.7]

From Eq. [A.7], we can infer that if we write the singular value decomposition of $B + D$ as $B + D = U\Sigma V^H$, then the matrix $B - D$ can be written as $\tilde{U}\Sigma \tilde{V}^H$, where $\tilde{U} = Q_2U$ and $\tilde{V} = Q_1V$. Since $\tilde{U}$ and $\tilde{V}$ are matrices with orthonormal columns, we must have that $\tilde{U}\Sigma \tilde{V}^H$ is a valid singular value decomposition of $B - D$. Thus, we can conclude that $B - D$ and $B + D$ have identical singular values. This completes the proof of the theorem. □

**Proof of Corollary 1**

Assume that the vectors $y$ and $z$ are chosen such that the expressions in Eq. [9] represent an optimal solution to Eq. [5]. Theorem 1 then tells us that the vectors from Eq. [12] represent another optimal solution to Eq. [5]. These two solutions are identical to one another if and only if $A^H_+A_-y = 0$ and $A^H_+A_+z = 0$, in which case both solutions are equal to the zero-filled solution. In this case, the zero-filled solution is clearly an optimal solution to Eq. [5], and must be the unique optimal solution if Eq. [5] only has a single solution. If either $A^H_+A_-y \neq 0$ or $A^H_+A_+z \neq 0$, then Eqs. [9] and [12] represent two distinct solutions to Eq. [5], indicating that Eq. [5] does not have a unique solution. □

**Proof of Corollary 2**

Based on the proof of Corollary 1, we know that if Eq. [5] has a solution that is not equal to the zero-filled measured data, then we can use Eqs. [9] and [12] to obtain a pair of two distinct solutions to Eq. [5]. Let $k_1^+$ and $k_2^+$ denote these two solutions, and notice that by the definition of an optimal solution, we must have that $L_C(k_1^+) = L_C(k_2^+)$ and that $L_C(k_1^+) \leq L_C(k)$ for all possible candidate solutions $k$.  

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Corollary 2 is easily proven based on the definition of a convex function. Specifically, if $L_C(y)$ is convex, then it must satisfy (34)

$$L_C(\alpha y_1 + (1-\alpha)y_2) \leq \alpha L_C(y_1) + (1-\alpha)L_C(y_2), \quad [A.8]$$

for every possible pair of vectors $y_1$ and $y_2$ and for every real-valued scalar $\alpha$ between 0 and 1.

Setting $y_1 = k_1^\pm$ and $y_2 = k_2^\pm$ in Eq. [A.8] leads to

$$L_C(\alpha k_1^\pm + (1-\alpha)k_2^\pm) \leq \alpha L_C(k_1^\pm) + (1-\alpha)L_C(k_2^\pm) = L_C(k_1^\pm), \quad [A.9]$$

Combining Eq. [A.9] with the previous observation that $L_C(k_1^\pm) \leq L_C(k)$ for all possible candidate solutions $k$ implies that $L_C(\alpha k_1^\pm + (1-\alpha)k_2^\pm) = L_C(k_1^\pm)$. As a result, $\alpha k_1^\pm + (1-\alpha)k_2^\pm$ must also be an optimal solution of Eq. [5] for every possible choice of $0 \leq \alpha \leq 1$, and we have successfully proven that there exist an infinite number of solutions. Additionally, the zero-filled solution is obtained as one of these solutions, corresponding to the specific choice of $\alpha = 0.5$. □