Specht’s invariant and localization of operator tuples

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Abstract: The present paper concerns local theory of operator tuples in the Cowen-Douglas class $\mathcal{B}_m^n(\Omega)$. We start with point-wise localizations to introduce a kind of operator-valued invariants with which a Specht-type classification for unitary equivalence of $\mathcal{B}_m^n(\Omega)$ is obtained. Furthermore, we investigate localization of $\mathcal{B}_m^n(\Omega)$ on analytic sub-manifolds with a tensorial approach to its geometric classification theory where, among other things, the Specht’s invariants are related to curvatures of the holomorphic vector bundles associated to $\mathcal{B}_m^n(\Omega)$.

Key words: Specht’s invariant, localization, Cowen-Douglas operator, curvature

1 Introduction

Finding suitable invariants to classify non-normal Hilbert space operators up to unitary equivalence is in general a widely open and appealing topic. A basic existence theorem of scaler-valued unitary invariants was given by Specht on finite matrices in terms of their traces:

**Theorem 1.1.** (Specht [18]) Two complex $n \times n$ matrices $A$ and $B$ are unitarily equivalent if and only if $\text{tr}(w(A, A^*)) = \text{tr}(w(B, B^*))$ for all words $w$ in two variables.

On infinite dimensional Hilbert spaces where scaler invariants are not always obtainable, a natural idea is to consider operator-valued alternatives. By operator-valued invariants we mean that one associates to every operator $T$ (in a given operator class) a set $\mathcal{I}(T)$ consisting of “testing operators” acting on certain “testing spaces”, such that unitary equivalence of $T$ can be reduced to unitary equivalences of operators in $\mathcal{I}(T)$.

**Remark 1.2.** Operator-valued invariants always exist (one can trivially set $\mathcal{I}(T) = \{T\}$ as a singleton with entire space as the testing space), and conceptual non-triviality lies in the existence of small and canonical testing spaces (in particular, existence of scaler invariants follows from existence of one dimensional testing spaces). Minimizing the cardinality of $\mathcal{I}(T)$, after testing spaces identified and fixed, is a technical problem (see, for instance, a series of refinements [16, 17, 12] on Theorem 1.1) and will not be the theme of this paper.
In this paper we work on multi-variate case with a tuple of commuting operators lying in the following important and extensively studied class introduced by M. Cowen and R. Douglas [5, 6].

**Definition 1.3.** Given positive integers $m, n$, and a bounded domain $\Omega$ in $\mathbb{C}^m$, a commuting $m$-tuple of operators $T = (T_1, \ldots, T_m)$ acting on a Hilbert space $\mathcal{H}$ belongs to the class $\mathcal{B}_n^m(\Omega)$ if the followings hold:

(i) The space $\{(T_1 - z_1)h, \ldots, (T_m - z_m)h, h \in \mathcal{H}\}$ is a closed subspace in $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (m copies of $\mathcal{H}$) for every $z = (z_1, \ldots, z_m) \in \Omega$;

(ii) $\dim \cap_{i=1}^m \ker(T_i - z_i) = n$ for every $z \in \Omega$ and

(iii) $\vee z \in \Omega \cap \bigcap_{i=1}^m \ker(T_i - z_i) = \mathcal{H}$ (here $\vee$ denotes the closed linear span).

Operator tuples in $\mathcal{B}_n^m(\Omega)$ can be modeled by adjoints of coordinate multiplications on $\mathbb{C}^n$-valued holomorphic function spaces in $m$ complex variables [7] such as Hardy or Bergman spaces, making it a rich class of considerable interest in the analytic aspect. On the other hand, the family $\cap_{i=1}^m \ker(T_i - z_i)$, $z \in \Omega$ of joint eigen-spaces of $T$ were shown [5] to admit a structure of holomorphic Hermitian vector bundle of rank $n$ over $\Omega$, denoted by $E(T)$, and the classification of $\mathcal{B}_n^m(\Omega)$ features an appealing involvement of complex geometry on $E(T)$.

We record the following theorem which is the main result of Cowen and Douglas’ seminal paper [5] (which appeared in [5] with $m = 1$ and was extended to $m > 1$ in the subsequent work [4]) giving a family of finite dimensional testing spaces for unitary equivalence of $\mathcal{B}_n^m(\Omega)$.

**Theorem 1.4.** ([4, 5]) Operator tuples $T$ and $\tilde{T}$ in $\mathcal{B}_n^m(\Omega)$ are unitarily equivalent if and only if $T|_{H_z^{n+1}}$ is unitarily equivalent to $\tilde{T}|_{\tilde{H}_z^{n+1}}$ for every $z \in \Omega$.

Here for a fixed multi-index $I = (i_1, \ldots, i_m)$ and $z = (z_1, \ldots, z_m) \in \Omega$, set

$$(T - z)^I := (T_1 - z_1)^{i_1}(T_2 - z_2)^{i_2} \cdots (T_m - z_m)^{i_m}$$

and for a fixed positive integer $k$, we call

$$H_z^k := \cap_{|I|=k} \ker(T - z)^I$$

(where $|I| = i_1 + \cdots + i_m$) the $k$-th order localization of $T$ at $z$ (the notation $\tilde{H}_z^{n+1}$ applies analogously for $\tilde{T}$). The first order localization is nothing but the joint eigen-space $\cap_{i=1}^m \ker(T_i - z_i)$ at $z$, while the high order localization carries nontrivial connection between operator theory of $\mathcal{B}_n^m(\Omega)$ and complex geometry of $E(T)$ (as can be seen in the geometric proof of Theorem 1.4).

Beyond point-wise localizations, developments on the multi-variate dilation theory and Hilbert modules [2, 10] motivates new local theory for $\mathcal{B}_n^m(\Omega)$ where operator tuples are localized on an analytic sub-manifold $Z$, denoted by $H_Z^k$, instead of a single point in $\Omega$. 


Apart from its independent importance in function-theoretic operator theory (see Section 2 for more details), the classification theory of $H^k_Z$ exhibits a conceptual difference and appeal in the several-variable world ([9, 8, 3]).

Our investigations begin with the point-wise localization, whose geometric theory has been well-studied in [5, 4] hence we will focus on the operator-theoretic aspect. In Section 3, we give a complete set of operator-valued “Specht invariants” for $T|_{H^k_Z}$ via words in $T_1, \ldots, T_m$ and $T_1^*, \ldots, T_m^*$ acting on the first order localization $H^1_z = \cap_{i=1}^m \ker(T_i - z_i)$, which means unitary equivalence of $T|_{H^k_Z}$ can be tested on a much smaller $H^1_z (\dim H^k_Z = L \dim H^1_z$ where $L$ is the cardinality of multi-indices $I$ with $|I| \leq k - 1$, see Lemma 2.4 below) as a canonical testing space in light of Definition 1.3. In particular, letting $z$ run through $\Omega$ our result (applied to localizations of order $n + 1$) implies a refinement of Theorem 1.4 in light of Remark 1.2.

In Section 4 we turn to classification of localizations on analytic sub-manifolds. The project was started assuming $\text{codim} Z = 1$ (that is, $Z$ is a hyper-surface) in a series of works ([9, 8, 3]) and in this paper, we work in the general situation in search of a unified theory.

The analytic classification of $H^k_Z$ is relatively technical, which in case $\text{codim} Z = 1$ was stated and proved in terms of properly chosen “normalized frames” of $E(T)$ (see Sec 3.3 as well as recent extension [11]). This particular kind of frame exists on a point-wise base (see Lemma 2.1 below) and it will turn out that the problem can be reduced, in a nontrivial but simple way, to the degenerate case that $Z$ is a single point, from which a simple solution valid in the general case follows.

Our interest mainly lies in a geometric classification theory which is expected to be independent of particular choice of holomorphic frames of $E(T)$ so as to become a “coordinate free” theory, and we focus on the curvature of $E(T)$ as well as its relation to the operator-valued Specht’s invariant.

Geometrically the curvature represents the “second derivative” with respect to a given connection and can be identified with a collection of linear bundle maps. We will show that these bundle maps, up to conjugation with “bundle equivalences” (holomorphic isometric bundle maps), exactly determines unitary equivalence of the second order localization $H^2_Z$, which not only extends earlier works on the case $\text{codim} Z = 1$, but more interestingly reveals (by its proof) a connection between operator-theoretic unitary equivalence of $H^2_Z$ and the geometric tensorial nature of the curvature. Moreover, we will show how the curvature is related to the operator-valued invariants introduced in Section 3, which in turn realizes a geometric Specht-type classification on nontrivial analytic sub-manifolds.
2 Holomorphic curves and localizations

We begin with basic elements on holomorphic curves with which the localizations can be expressed in an explicit way to be used in later sections.

For a separable Hilbert space $\mathcal{H}$ and a positive integer $n$, let $\mathcal{G}r(n, \mathcal{H})$ denote the Grassmann manifold of all $n$-dimensional subspaces of $\mathcal{H}$. A map $E : \Omega \to \mathcal{G}r(n, \mathcal{H})$ is called a holomorphic curve if there exists $n$ holomorphic $\mathcal{H}$-valued functions $\gamma_1, \cdots, \gamma_n$, called a holomorphic frame, such that $E(z) = \text{span}\{\gamma_1(z), \cdots, \gamma_n(z)\}$ for every $z \in \Omega$. In particular, this defines a holomorphic Hermitian vector bundle of rank $n$ over $\Omega$ whose Hermitian metric on the fiber $E(z)$ is the inner product of $\mathcal{H}$. The holomorphic curve $E$ is said to admit the spanning property if $\bigvee_{z \in \Omega} E(z) = \mathcal{H}$. The uniqueness theorem for holomorphic functions implies that if $E$ admits the spanning property, it holds that $\bigvee_{z \in \Delta} E(z) = \mathcal{H}$ for any open subset $\Delta \subset \Omega$ (see Corollary 1.13, [5]), in other words, one can properly shrink $\Omega$ without losing the spanning property.

**Lemma 2.1.** Let $E$ be a holomorphic curve over $\Omega$ of rank $n$. For any point $z_0$ in $\Omega$, there exist a holomorphic frame $\{\gamma_i(z)\}_{i=1}^n$ for $E$ over an neighborhood of $z_0$ on which $\langle \gamma_i(z), \gamma_j(z_0) \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$, where $\delta_{ij}$ is the Kronecker symbol.

**Lemma 2.2.** Let $f(z, w)$ be a function on $\Omega \times \Omega$ which is holomorphic in $z$ and antiholomorphic in $w$. If $f(z, z) = 0$ for all $z \in \Omega$, then $f(z, w)$ vanishes identically on $\Omega \times \Omega$.

Lemma 2.1 (see Lemma 2.4, [5]) asserts that a holomorphic curve always admits a holomorphic frame normalized at a single point which is called a normalized frame, and Lemma 2.2 is standard which will be useful in dealing with normalized frames in later sections.

Two holomorphic curves $E$ and $\tilde{E}$ are said to be equivalent if there exists a holomorphic isometric bundle map from $E$ to $\tilde{E}$, that is, a family of isometric linear maps from $E(z)$ to $\tilde{E}(z)$ parameterized by $z \in \Omega$ which can be represented by a holomorphic matrix-valued function with respect to holomorphic frames of $E$ and $\tilde{E}$. A fundamental result, called the Rigidity Theorem (Theorem 2.2, [5]), states that in case both $E$ and $\tilde{E}$ admit the spanning property, geometric equivalence implies the existence of a unitary intertwining operator.

**Theorem 2.3.** (Rigidity Theorem) Let $E$ and $\tilde{E}$ be two holomorphic curves over a domain $\Omega$ such that $\bigvee_{z \in \Omega} E(z) = \mathcal{H}$ and $\bigvee_{z \in \Omega} \tilde{E}(z) = \tilde{\mathcal{H}}$, then the followings are equivalent

(i) $E$ and $\tilde{E}$ are equivalent via a holomorphic isometric bundle map $\Phi$ from $E$ to $\tilde{E}$,

(ii) $E$ and $\tilde{E}$ are congruent, i.e., there exists a unitary operator $U$ from $\mathcal{H}$ to $\tilde{\mathcal{H}}$ such that $UE(z) = \tilde{E}(z)$ for every $z$.

A key observation of [5] is that for any $T \in B_n^m(\Omega)$, the map $z \mapsto \cap_{i=1}^m \ker(T_i - z_i)$ is a holomorphic curve, denoted by $E(T)$, over $\Omega$, hence there exists holomorphic $\mathcal{H}$-valued functions $\gamma_1, \cdots, \gamma_n$ over $\Omega$ such that $\cap_{i=1}^m \ker(T_i - z_i) = \text{span}\{\gamma_1(z), \cdots, \gamma_n(z)\}$.
We will need to represent high order localizations $H_*^k = \cap_{|I| = k} \ker(T - z)^I$ in terms of $\{\gamma_1(z), \cdots, \gamma_n(z)\}$. For a fixed multi-index $I = (i_1, \cdots, i_m)$, set $I! = i_1! \cdots i_m!$ and $\partial^I = \partial_1^{i_1} \partial_2^{i_2} \cdots \partial_m^{i_m}$ where $\partial_i$ denotes differentiation with respect to $z_i$, $1 \leq i \leq m$. Given another multi-index $J = (j_1, \cdots, j_m)$, we say $I \geq J$ if $i_k \geq j_k$ for all $1 \leq k \leq m$, and the index $(i_1 - j_1, \cdots, i_m - j_m)$ is denoted by $I - J$.

As $\cap_{i=1}^m \ker(T_i - z_i) = \text{span}\{\gamma_1(z), \cdots, \gamma_n(z)\}$, the identity

$$(T_j - z_j) \gamma_i(z) = 0$$

holds for every $1 \leq i, j \leq m$, which, combined with standard differentiation computations via the Leibnitz rule (or see p.470, [2]), yields

$$(T - z)^I \partial^J \gamma_i(z) = \begin{cases} \frac{n!}{(J-I)!} \gamma_i(z), & J \geq I \\ 0, & \text{otherwise} \end{cases} \quad (2.1)$$

**Lemma 2.4.** Given an operator tuple $T = (T_1, \cdots, T_m) \in B^m_n(\Omega)$ and a fixed holomorphic frame $\{\gamma_1, \cdots, \gamma_n\}$ for $E(T)$, it holds that for any positive integer $k$,

$$\cap_{|I| = k} \ker(T - z)^I = \text{span}_{|I| \leq k-1} \{\partial^I \gamma_i(z), 1 \leq i \leq n\}$$

*Proof.* That $\text{span}_{|I| \leq k-1} \{\partial^I \gamma_i(z), 1 \leq i \leq n\} \subseteq \cap_{|I| = k} \ker(T - z)^I$ trivially follows from (2.1) and we prove $\cap_{|I| = k} \ker(T - z)^I \subseteq \text{span}_{|I| \leq k-1} \{\partial^I \gamma_i(z), 1 \leq i \leq n\}$ by induction.

The conclusion trivially holds when $k = 1$ and we suppose it holds for some $k$. Now fix $x \in \cap_{|I| = k+1} \ker(T - z)^I$, then for any $I$ such that $|I| = k$, $(T - z)^I x \in \cap_{|I| = 1} \ker(T - z)^I$, which is the joint eigen-space spanned by $\{\gamma_1(z), \cdots, \gamma_n(z)\}$. Hence we get a collection of complex numbers $\{a^I_i|1 \leq i \leq n, |I| = k\}$ such that

$$(T - z)^I x = \sum_{i=1}^n a^I_i \gamma_i(z) \quad (2.2)$$

whenever $|I| = k$.

We claim that the vector

$$x - \sum_{|I| = k} \sum_{i=1}^n \frac{a^I_i}{I!} \partial^I \gamma_i(z)$$

lies in $\cap_{|I| = k} \ker(T - z)^I$. Then the induction hypothesis together with the claim implies that $x \in \text{span}_{|I| \leq k} \{\partial^I \gamma_i(z), 1 \leq i \leq n\}$, which gives the conclusion for $k + 1$ and completes the induction.

To verify the claim, we fix a multi-index $I$ such that $|I| = k$, then for any multi-index $J$ with $|J| = k$, it holds by (2.1) that

$$(T - z)^I \partial^J \gamma_i(z) = \begin{cases} I! \gamma_i(z), & I = J \\ 0, & I \neq J \end{cases}$$
This implies
\[(T - z)^I \sum_{|I|=k} \sum_{i=1}^n \frac{a_I}{i!} \partial^I \gamma_i(z) = (T - z)^I \sum_{i=1}^n \frac{a_I}{i!} \partial^I \gamma_i(z) = \sum_{i=1}^n a_I \gamma_i(z), \quad (2.3)\]
and the claim follows by comparing (2.2) and (2.3).

\[\text{Corollary 2.5. For fixed positive integer } d < m \text{ and a subset } A = \{a_1, \cdots, a_d\} \subseteq \{1, 2, \cdots, m\}, \]
let \( \Lambda = \{I = (i_1, \cdots, i_m)|i_{a_1} = i_{a_2} = \cdots = i_{a_d} = 0\} \), then it holds that
\[\bigcap_{|I|=k, I \in \Lambda} \ker(T - z)^I \cap_{A \in \Lambda} \ker(T_i - z_l) = \text{span}_{|I| \leq k-1, I \in \Lambda}\{\partial^I \gamma_i(z)\}, 1 \leq i \leq n\}. \quad (2.4)\]

\[\text{Proof. Note that } T_i - z_l \text{ annihilates } \partial^I \gamma_i(z) \text{ whenever } l \in A \text{ and } J \in \Lambda, \text{ the corollary follows from a straightforward modification of above proof for Lemma 2.4.}\]

Now we introduce the canonical model for localization of \( \mathcal{B}_n^m(\Omega) \) on analytic sub-manifolds in \( \Omega \). Fix a positive integer \( 1 \leq d < m \), let \( Z \) be a codimension \( d \) analytic sub-manifold of the following form
\[Z = \{(z_1, z_2, \cdots, z_m) \in \Omega|z_1 = z_2 = \cdots = z_d = 0\}. \quad (2.5)\]

\[\text{Definition 2.6. Given an operator tuple } T = (T_1, \cdots, T_m) \in \mathcal{B}_n^m(\Omega), \text{ the restriction of } T \text{ on the closed linear span } H_Z^k := \bigvee_{z \in Z} \bigcap_{|I|=k, I \in N^d} \ker(T - z)^I \cap_{I \in \Lambda} \ker(T_i - z_l) \text{ is called the localization of } T \text{ over } Z, \text{ where } N^d = \{I = (i_1, \cdots, i_m)|i_{d+1} = \cdots = i_m = 0\}.\]

\[\text{Remark 2.7. (i) When } d = m \text{ hence } Z \text{ degenerates to a point, Definition 2.6 boils down to point-wise localization mentioned in Section 1. By Remark 2.5, for any holomorphic frame } \{\gamma_1, \cdots, \gamma_n\} \text{ of } E(T), \]
\[H_Z^k = \bigvee_{z \in Z} \text{span}\{\partial^I \gamma_i(z)\}, 1 \leq i \leq n, |I| \leq k-1, I \in N^d\}\]

In other words, the holomorphic curve over \( \Omega \) defined by \( z \mapsto \text{span}\{\partial^I \gamma_i(z)\}, 1 \leq i \leq n, |I| \leq k-1, I \in N^d\} \) is independent of the choice of frame for \( E(T) \) and admits the spanning property with respect to \( H_Z^k \) as a holomorphic curve along \( Z \).

(ii) One might find the above definition of \( H_Z^k \) technical and not fully motivated. In particular, its dependence on the particular form of \( Z \) seems to make it inadequate for a general theory. In the end of this paper, we present an appendix with a brief revision on the background materials (mainly extracted from \[\mathcal{B}_n^m(\Omega)\]) explaining the motivation and justification of Definition 2.6, which involves a series of reduction procedures lying in earlier works on function space model of \( \mathcal{B}_n^m(\Omega) \), where \( H_Z^k \) corresponds to the quotient space with respect to a canonical kind of subspace defined by vanishing conditions. Readers who are not interested can move on to Section 3 and 4 without loss of continuity.
3 Specht-type classification for point-wise localization

Throughout this section, $k$ will be a fixed positive integer. We show that for an operator tuple $T = (T_1, \cdots, T_m) \in B^m_n(\Omega)$, unitary equivalence of its $k$-th order localization $T_{H^k_z}$ can be tested by operators on the first order localization $H^1_z = \cap_{i=1}^m \ker(T_i - z_i)$.

Precisely, for each multi-index $I = (i_1, \cdots, i_m)$, and $z = (z_1, \cdots, z_m) \in \Omega$, set

$$N^I_z := (T_1 - z_1)^{i_1}(T_2 - z_2)^{i_2} \cdots (T_m - z_m)^{i_m}|_{H^k_z},$$

with which we construct a collection of testing operators on $H^1_z$, denoted by $K^I_J$, given by

$$K^I_J := P_{H^k_z}[N^I_z(N^J_z)^*]|_{H^k_z}$$

for multi-indices $I, J$. Here as $N^I_z(N^J_z)^*$ does not necessarily leave $H^1_z$ invariant, $P_{H^k_z}$ is imposed to make $K^I_J$ live in $H^1_z$.

The main result of this section as follows gives the Specht-type classification for point-wise localization.

**Theorem 3.1.** Given operator tuples $T$ and $\tilde{T}$ in $B^m_n(\Omega)$, their localizations $T|_{H^k_z}$ and $\tilde{T}|_{\tilde{H}^k_z}$ are unitarily equivalent if and only if operator tuples $\{K^I_J, 1 \leq |I|, |J| \leq k - 1\}$ and $\{\tilde{K}^I_J, 1 \leq |I|, |J| \leq k - 1\}$ are unitarily equivalent.

**Remark 3.2.** Apart from its formal construction ($N^I_z$ as words in $T_i - z_i$ and $K^I_J$ as words in $N^I_z$’s and their adjoints), we are eligible to call $K^I_J$ a Specht-type invariant since in the simplest nontrivial case $k = 2$ and $n = 1$, $H^1_z$ is one dimensional, $K^I_J$ as a scaler exactly equals the trace of $N^I_z(N^J_z)^*$ (as can be seen from the matrix representation (4.26) in the proof of Theorem \[4.7 in Section 4\]).

With Theorem \[3.1\] we immediately get the following refinement of Theorem \[1.4\].

**Theorem 3.3.** Operator tuples $T$ and $\tilde{T}$ in $B^m_n(\Omega)$ are unitarily equivalent if and only if operator tuples $\{K^I_J, 1 \leq |I|, |J| \leq k - 1\}$ and $\{\tilde{K}^I_J, 1 \leq |I|, |J| \leq k - 1\}$ are unitarily equivalent for every $z \in \Omega$.

The proof of Theorem \[3.1\] will be given in the end of this section after some preparations. Before proceeding, we fix some notations and conventions in elementary linear algebra.

(i) “Inner product” of matrices: Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times p}$ be two matrices with entries $a_{ij}, b_{ij}$ lying in a Hilbert space (whose inner product is denoted by $\langle \cdot, \cdot \rangle$). Let $\langle A, B \rangle$ denotes the numerical matrix $E = [e_{ij}]_{n \times p}$ given by $e_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$. If $C, D$ are numerical matrices, then $\langle CA, BD \rangle = C\langle A, B \rangle D$.

With this notation, if $\gamma = (\gamma_1, \cdots, \gamma_n)$ is a holomorphic frame for a rank $n$ holomorphic curve, then its Gram matrix can be written as $\langle \gamma^T(z), \gamma(z) \rangle$. Moreover, $\gamma$ is normalized at a point $z_0$ if and only if $\langle \gamma^T(z), \gamma(z_0) \rangle = I$ identically.
(ii) **Representation of linear maps:** We adopt the “left action” convention regarding to representing matrices for linear maps. Precisely, let $\Phi$ be a linear map on a linear space spanned by $\gamma = (\gamma_1, \cdots, \gamma_n)$, then a matrix $A = [a_{ij}]$ represents $\Phi$ if $\Phi \gamma_i = \sum a_{ij} \gamma_j$, or in other words, $\Phi \gamma^T = (\Phi \gamma_1, \cdots, \Phi \gamma_n)^T = A(\gamma_1, \cdots, \gamma_n)^T$. If another linear map $\Psi$ is represented by $B = [b_{ij}]$, then $\Phi \Psi$ is represented by $BA$ (not $AB$, which corresponds to “right action” convention).

Throughout this section we will work with normalized frames of $E(T)$. In the single variable case $m = 1$, the normalized frame was used in the study of geometric theory of $B^1_n(\Omega)$ to identify localization order of an operator with “contact order” of a holomorphic curve (see Section 2, [5] for details), and here we need the following variation for $B^m_n(\Omega)$ as a preparation before proving Theorem 3.4 which holds independent importance as well in the discussions in Section 4.

**Theorem 3.4.** The followings are equivalent

(i) $H^k_z$ and $\tilde{H}^k_z$ are unitarily equivalent;

(ii) there exists holomorphic frames $\gamma$ and $\tilde{\gamma}$ (whose Gram matrices are denoted by $H$ and $\tilde{H}$) for $E(T)$ and $E(\tilde{T})$ such that $\partial^I \overline{\partial}^J H = \partial^I \overline{\partial}^J \tilde{H}$ at $z$ for all $|I|, |J| \leq k - 1$;

(iii) there exists holomorphic frames $\gamma$ and $\tilde{\gamma}$ normalized at $z$ such that $\partial^I \overline{\partial}^J H = \partial^I \overline{\partial}^J \tilde{H}$ at $z$ for all $|I|, |J| \leq k - 1$;

(iv) there exists holomorphic frames $\gamma$ and $\tilde{\gamma}$ normalized at $z$ and a constant unitary matrix $U$, such that $\partial^I \overline{\partial}^J H = U(\partial^I \overline{\partial}^J H)U^* \gamma(z)$ at $z$ for all $|I|, |J| \leq k - 1$;

(v) For any holomorphic frames $\gamma$ and $\tilde{\gamma}$ normalized at $z$, there exists a constant unitary matrix $U$, such that $\partial^I \overline{\partial}^J H = U(\partial^I \overline{\partial}^J \tilde{H})U^*$ at $z$ for all $|I|, |J| \leq k - 1$.

We need two elementary lemmas before the proof of Theorem 3.4 and Theorem 3.1.

**Lemma 3.5.** Let $\gamma = \{\gamma_1, \cdots, \gamma_n\}$ and $\beta = \{\beta_1, \cdots, \beta_n\}$ be two holomorphic frames of a holomorphic curve over $\Omega$ such $\gamma$ is normalized at a point $z_0$. Then $\beta$ is normalized at $z_0$ if and only if its transition function with $\gamma$ is a constant unitary matrix.

**Proof.** For one direction, let $U$ be a constant unitary matrix and $\beta^T = U \gamma^T$, then for $z$ in $\Omega$,

$$\langle \beta^T(z), \beta(z) \rangle = U \langle \gamma^T(z), \gamma(z) \rangle U^*,$$

where $U^*$ denotes the conjugate transpose of $U$. The above identity can be refined by Lemma 2.2 into

$$\langle \beta^T(z), \beta(w) \rangle = U \langle \gamma^T(z), \gamma(w) \rangle U^*$$

for all $z, w \in \Omega$. As $\gamma$ is normalized at $z_0$, $\langle \gamma^T(z), \gamma(z_0) \rangle = I$, hence by setting $w = z_0$ the above equation becomes

$$\langle \beta^T(z), \beta(z_0) \rangle = U \langle \gamma^T(z), \gamma(z_0) \rangle U^* = UU^* = I.$$
Hence $\boldsymbol{\beta}$ is also normalized at $z_0$.

Conversely, let $U(z)$ be the transition function of $\boldsymbol{\beta}$ with $\gamma$, then $U(z)$ is holomorphic and

$$\langle \beta^T(z), \beta(z) \rangle = U(z) \langle \gamma^T(z), \gamma(z) \rangle U(z)^*,$$

which can be refined into

$$\langle \beta^T(z), \beta(w) \rangle = U(z) \langle \gamma^T(z), \gamma(w) \rangle U(w)^*.$$  

If $\boldsymbol{\beta}$ is also normalized at $z_0$, then $\langle \beta^T(z), \beta(z_0) \rangle = \langle \gamma^T(z), \gamma(z_0) \rangle = I$ and the above equation becomes (by setting $w = z_0$)

$$I = \langle \beta^T(z), \beta(z_0) \rangle = U(z) \langle \gamma^T(z), \gamma(z_0) \rangle U(z_0)^* = U(z) U(z_0)^*.$$  

This gives $U(z) = U^{-1}(z_0)^*$, so $U(z)$ is a constant unitary matrix.  

The following lemma is standard and we omit the proof.

**Lemma 3.6.** Let $\Phi$ be a linear operator on a finite dimensional Hilbert space and $\gamma = \{\gamma_1, \cdots, \gamma_n\}$ be a base whose Gram matrix is $H$. If $\Phi$ is represented by a matrix $A$ with respect to $\gamma$, then its adjoint operator is represented by $HA^*H^{-1}$.

**Proof of Theorem 3.4**

Proof. (ii)$\Rightarrow$(i) Write $\gamma = \{\gamma_1, \cdots, \gamma_n\}$ and $\tilde{\gamma} = \{\tilde{\gamma}_1, \cdots, \tilde{\gamma}_n\}$, hence by Lemma (2.4), $H^z_k = \text{span}_{|J| \leq k-1}\{ \partial^J \gamma_i(z), 1 \leq i \leq n \}$ and $\tilde{H}^z_k = \text{span}_{|J| \leq k-1}\{ \partial^J \tilde{\gamma}_i(z), 1 \leq i \leq n \}$. Let $\Phi$ be the linear map from $H^z_k$ to $\tilde{H}^z_k$ defined by

$$\Phi \partial^J \gamma_i(z) := \partial^J \tilde{\gamma}_i(z), |J| \leq k - 1,$$

then $\Phi$ implements a unitary equivalence between $H^z_k$ and $\tilde{H}^z_k$.

In fact, $\Phi$ trivially intertwines $T_l - z_l$ and $\tilde{T}_l - z_l$ (hence intertwines $T_l$ and $\tilde{T}_l$), $1 \leq l \leq m$ as their actions on $\partial^I \gamma(z)$ and $\partial^I \tilde{\gamma}(z)$ follows the same rule (2.1). Moreover, $\partial^I \tilde{\partial}^J H = [\langle \partial^I \gamma_i, \partial^J \gamma_j \rangle]_{1 \leq i, j \leq n}$ and $\partial^I \tilde{\partial}^J \tilde{H} = [\langle \partial^I \tilde{\gamma}_i, \partial^J \tilde{\gamma}_j \rangle]_{1 \leq i, j \leq n}$ since the frames are holomorphic, hence the condition (ii) implies

$$\langle \partial^I \gamma_i, \partial^J \gamma_j \rangle = \langle \partial^I \tilde{\gamma}_i, \partial^J \tilde{\gamma}_j \rangle$$

for every $1 \leq i, j \leq n$ and $|I|, |J| \leq k - 1$ at $z$, so $\Phi$ is isometric as well.

(iii)$\Rightarrow$(ii) Trivial.

(i)$\Rightarrow$(iii) Let $\Phi$ be a unitary operator from $H^z_k$ to $\tilde{H}^z_k$ which implements the unitary equivalence. We show that there exists holomorphic frames $\gamma$ and $\tilde{\gamma}$ normalized at $z$ such that
\[ \Phi \partial^I \gamma_i(z) = \partial^I \tilde{\gamma}_i(z) \]  

(3.1) for all \(|I| \leq k - 1, 1 \leq i \leq n\), and (iii) will follow since \(\Phi\) is isometric.

We begin with arbitrary fixed holomorphic frames \(\gamma = \{\gamma_1, \cdots, \gamma_n\}\) and \(\tilde{\gamma} = \{\tilde{\gamma}_1, \cdots, \tilde{\gamma}_n\}\) for \(E(T)\) and \(E(\tilde{T})\) normalized at \(z\). As \(\Phi\) intertwines \(T_i - z_i\) and \(\tilde{T}_i - z_i\), it maps the joint eigen-space of \(T\) spanned by \(\gamma(z)\) to corresponding one of \(\tilde{T}\) spanned by \(\tilde{\gamma}(z)\), hence there exists an \(n \times n\) matrix \(U\) such that

\[ \Phi \gamma^T(z) = U \tilde{\gamma}^T(z), \]

and \(U\) is unitary since both \(\gamma\) and \(\tilde{\gamma}\) are normalized at \(z\). By Lemma \[3.5\], \(U \tilde{\gamma}^T\) is again a normalized frame at \(z\), so we can replace \(\tilde{\gamma}^T\) by \(U \gamma^T\) which gives (3.1) in case \(|I| = 0\).

Now we check (3.1) by induction on \(|I|\). Suppose (3.1) holds with all \(|I| \leq l\) for some \(l\). For any \(I\) with \(|I| = l + 1\), it holds that

\[ (I_q - z_q)(\Phi \partial^I \gamma_i(z) - \partial^I \tilde{\gamma}_i(z)) = 0 \]

(3.2)

for every \(1 \leq q \leq m\).

In fact, by the intertwining property of \(\Phi\),

\[ (I_q - z_q)(\Phi \partial^I \gamma_i(z) - \partial^I \tilde{\gamma}_i(z)) = \Phi(T_q - z_q)\partial^I \gamma_i(z) - (I_q - z_q)\partial^I \tilde{\gamma}_i(z). \]

Write \(I = (i_1, \cdots, i_q, \cdots, i_m)\), then in case \(i_q = 0\), both \((T_q - z_q)\partial^I \gamma_i(z)\) and \((I_q - z_q)\partial^I \tilde{\gamma}_i(z)\) vanishes hence (3.2) trivially holds. In case \(i_q \geq 1\), \((T_q - z_q)\partial^I \gamma_i(z) = i_q \partial^{I'} \gamma_i(z)\) and \((I_q - z_q)\partial^I \tilde{\gamma}_i(z) = i_q \partial^{I'} \tilde{\gamma}_i(z)\) where \(I' = (i_1, \cdots, i_q - 1, \cdots, i_m)\), hence (3.2) follows from the induction hypothesis.

Moreover, we observe that

\[ \langle \Phi \partial^I \gamma_i(z) - \partial^I \tilde{\gamma}_i(z), \tilde{\gamma}_j(z) \rangle = 0 \]

(3.3)

for every \(1 \leq j \leq n\).

In fact, as \(\Phi\) is isometric and the frames are normalized at \(z\), it holds that

\[ \langle \Phi \partial^I \gamma_i(z) - \partial^I \tilde{\gamma}_i(z), \tilde{\gamma}_j(z) \rangle = \langle \Phi \partial^I \gamma_i(z), \Phi \gamma_j(z) \rangle - \langle \partial^I \gamma_i(z), \tilde{\gamma}_j(z) \rangle = \langle \partial^I \gamma_i(z), \gamma_j(z) \rangle - \langle \partial^I \gamma_i(z), \tilde{\gamma}_j(z) \rangle = 0 - 0 = 0 \]

as desired.

Now (3.2) implies that \(\Phi \partial^I \gamma_i(z) - \partial^I \tilde{\gamma}_i(z)\) lies in \(\cap_{i=1}^m \ker(I_q - z_q)\) which is spanned by \(\{\tilde{\gamma}_i(z), \cdots, \tilde{\gamma}_n(z)\}\), hence (3.3) forces \(\Phi \partial^I \gamma_i(z) - \partial^I \tilde{\gamma}_i(z) = 0\), concluding the induction.

(v)\(\Rightarrow\) (iv) Trivial.

(iv)\(\Rightarrow\) (iii) Let \(\gamma, \tilde{\gamma}\) and \(U\) be as given by (iv), then \(\sigma^T := U \gamma^T\) is again a holomorphic frame for \(E(T)\) normalized at \(z\) by Lemma \[3.5\]. Moreover,

\[ \partial^I \tilde{\sigma}^I \langle \sigma^T, \sigma \rangle = \partial^I \tilde{\sigma}^I (U(\gamma^T, \tilde{\gamma}^T)U^*) = U(\partial^I \tilde{\sigma}^I (\gamma^T, \tilde{\gamma}^T))U^* \]
holds in a neighborhood of \( z \) which, specifying at \( z \), equals \( \partial^I \overline{\partial} \langle \gamma^T, \gamma \rangle \) by (iv), so \( \gamma \) and \( \sigma \) meets (iii).

(iii)⇒(v) Fix holomorphic frames \( \beta \) and \( \tilde{\beta} \) normalized at \( z \) with properties given by (iii), then for arbitrarily chosen holomorphic frames \( \gamma \) and \( \tilde{\gamma} \) normalized at \( z \), their exists, by Lemma 3.5 constant unitary matrices \( V \) and \( \tilde{V} \) such that \( \gamma^T = V \beta^T \) and \( \tilde{\gamma}^T = \tilde{V} \tilde{\beta}^T \), which gives

\[
\partial^I \overline{\partial} \langle \gamma^T, \gamma \rangle = V(\partial^I \overline{\partial} \langle \beta^T, \beta \rangle) V^* = V(\partial^I \overline{\partial} \langle \tilde{\beta}, \tilde{\beta} \rangle) V^* = V \tilde{V}^* (\partial^I \overline{\partial} \langle \tilde{\gamma}^T, \tilde{\gamma} \rangle) \tilde{V} V^*
\]

at \( z \). The proof is completed by taking \( U = V \tilde{V}^* \).

Finally we give the proof Theorem 3.1.

\textbf{Proof.} We fix a holomorphic frame \( \gamma = \{\gamma_1, \cdots, \gamma_n\} \) for \( E(T) \) normalized at \( z \) and begin by calculating the matrix representation of \( K_z^J \) with respect to the base \( \gamma(z) \) of \( H^1_z \) (and \( K_z^{1J} \) follows in the same way), which will be read out from the representing matrix for \( N_z^J N_z^{J*} \) with respect to the base \( \{\partial^K \gamma_i(z), 1 \leq i \leq n, |K| \leq k-1\} \) of \( H_z^k \).

Let \( L \) be the cardinality of the multi-index set \( \{K, |K| \leq k-1\} \), then \( \dim H_z^k = nL \) (can be worked out via binomial coefficients but we do not need the precise value). The Gram matrix for \( \{\partial^K \gamma_i(z), 1 \leq i \leq n, |K| \leq k-1\} \), denoted by \( H \), is an \( L \times L \) block matrix \( [H_{1J}]_{0 \leq |I|, |J| \leq k-1} \) in which each block is an \( n \times n \) matrix \( H_{1J} := \{\langle \partial^I \gamma_i(z), \partial^J \gamma_j(z) \rangle\}_{1 \leq i, j \leq n} \).

In principle, to precisely locate a particular block \( H_{IJ} \) in \( H \) one need to assign an ordering for the multi-indices, that is, a bijection \( \sigma \) from the set \( \{K, |K| \leq k-1\} \) to \( \{0, 1, 2, \cdots, L-1\} \). From now on we fix a particular ordering (the lexicographic ordering for instance), then we can write

\[
[H_{1J}]_{0 \leq |I|, |J| \leq k-1} = [H_{\sigma I, \sigma J}]_{0 \leq \sigma I, \sigma J \leq L-1} = [H_{ij}]_{0 \leq i, j \leq L-1},
\]

where the terminology “\( I \)-th row/column” makes sense (which refers to “\( \sigma(I)\)-th row/column”), and we can freely use the above three representations in the sequel which will not cause confusion. In particular, we assume that \( \sigma(0, \cdots, 0) = 0 \), so the block \( H_{00} \) is the Gram matrix of \( \gamma \).

By (2.1), for fixed index \( J \), \( N_z^J \) maps \( \partial^J \gamma(z) \) to \( J! \gamma(z) \), while for \( K \neq J \), linear representation of \( N_z^J \partial^K \gamma(z) \) in terms of \( \{\partial^K \gamma(z), |K| \leq k-1\} \) has no \( \gamma(z) \)-component. Therefore \( N_z^J \) has the following block matrix representation with respect to \( \{\partial^K \gamma, |K| \leq k-1\} \) (\( \gamma(z) \) appears in the 0-th place when we arrange \( \{\partial^K \gamma(z), |K| \leq k-1\} \) into a column).

\[
N_z^J = \begin{pmatrix}
0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
J! I_n & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0
\end{pmatrix}
\]

\( J \)-th
Here we have not written out all nonzero blocks in $N_z^I$, since the only thing we need later is that the $(J, 0)$ block $J!I_n$ is the only nonzero block throughout the 0-th column and $J$-th row.

As the frame $\gamma$ is normalized at $z$, it holds that

$$H_{10} = [(\partial^I \gamma_i(z), \gamma_j(z))]_{1\leq i,j\leq n} = 0,$$

(similarly, $H_{0I} = 0$) for all $1 \leq |I| \leq k - 1$ and $H_{00} = I_n$. Therefore, the block matrix $H$ is of the form

$$
\begin{pmatrix}
I_n & 0 & \cdots & 0 \\
0 & H_{11} & \cdots & H_{1,L-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & H_{L-1,1} & \cdots & H_{L-1,L-1}
\end{pmatrix},
$$

which in turn implies that its inverse $G = [G_{IJ}]_{0\leq |I|,|J|\leq k-1} = [G_{ij}]_{0\leq i,j\leq L-1}$ is of the same form.

Now suppose $|I|, |J| \geq 1$, then by Lemma 3.6 $N_z^I N_z^{J*}$ can be represented by

$$
\begin{pmatrix}
I_n & 0 & \cdots & 0 \\
0 & H_{11} & \cdots & H_{1,L-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & H_{L-1,1} & \cdots & H_{L-1,L-1}
\end{pmatrix}
\begin{pmatrix}
J - th \\
J - th \\
\vdots \\
J - th
\end{pmatrix}
\begin{pmatrix}
I_{n} & 0 & \cdots & 0 \\
0 & G_{11} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & G_{L-1,1} & \cdots & G_{L-1,L-1}
\end{pmatrix}
\begin{pmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \vdots & \ddots \\
0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
I_{n} & 0 & \cdots & 0 \\
0 & G_{11} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & G_{L-1,1} & \cdots & G_{L-1,L-1}
\end{pmatrix}
\begin{pmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \vdots & \ddots \\
0 & \cdots & 0
\end{pmatrix}
$$

Since the frame is normalized at $z$, the space $H_z^1$ spanned by $\gamma(z)$ is is orthogonal to the space spanned by $\{\partial^K \gamma(z), 1 \leq |K| \leq k-1\}$, which implies that the block $J!J!G_{1J}$ appearing at the left upper corner of $N_z^I N_z^{J*}$ exactly represents $P_{H_z^1}(N_z^I N_z^{J*})|_{H_z^1}$ with respect to the base $\gamma(z)$ of $H_z^1$. With similar notations, $J!J!\tilde{G}_{1J}$ represents $\tilde{K}_z^{1J}$ with respect to the normalized frame $\tilde{\gamma}(z)$ of $E(T)$.

Now we are prepared to prove the theorem. For sufficiency, let $\Phi$ be a unitary operator from $H_z^1$ to $\tilde{H}_z^1$ intertwining $K_z^{1J}$ and $\tilde{K}_z^{1J}$ whose representing matrix with respect to $\gamma(z)$ and $\tilde{\gamma}(z)$ is denoted by $U$. Then $U$ is a unitary matrix as both frames are normalized at $z$. Moreover, the intertwining property gives

$$J!J!G_{1J}U = U(J!J!\tilde{G}_{1J})$$

that is

$$G_{1J} = U \tilde{G}_{1J} U^*$$

for all $1 \leq |I|, |J| \leq k - 1$ at $z$.

Observing that at $z$, $G_{00} = \tilde{G}_{00} = I_n$ and $G_{J0} = \tilde{G}_{J0} = 0$ whenever $|I| \neq 0$, identity 3.4 holds for all $0 \leq |I|, |J| \leq k - 1$, which gives
\[ [G_{IJ}]_{0 \leq |I|,|J| \leq k-1} = (U \otimes I_L)[\tilde{G}_{IJ}]_{0 \leq |I|,|J| \leq k-1}(U^* \otimes I_L). \] (3.5)

where \( U \otimes I_L \) denotes the diagonal block matrix with \( U \) lying on all diagonal blocks. Taking inverse we get
\[ [H_{IJ}]_{0 \leq |I|,|J| \leq k-1} = (U \otimes I_L)[\tilde{H}_{IJ}]_{0 \leq |I|,|J| \leq k-1}(U^* \otimes I_L). \] (3.6)

Specifying (3.6) block-wise we see that at \( z \),
\[ H_{IJ} = U \tilde{H}_{IJ} U^* \] (3.7)
holds for all \( 0 \leq |I|,|J| \leq k-1 \). Recall that \( H_{IJ} = \frac{\partial}{\partial J} \frac{\partial}{\partial I} H \) and \( \tilde{H}_{IJ} = \frac{\partial}{\partial J} \frac{\partial}{\partial I} \tilde{H} \), the sufficiency follows from combining (3.7) and Theorem 3.4.

Conversely, if \( H^k_z \) and \( \tilde{H}^k_z \) are unitarily equivalent, then Theorem 3.4 implies the existence of a constant unitary matrix \( U \) such that (3.7) holds, which in turn gives, by reversing the above arguments, the intertwining property (3.4). So the unitary operator represented by \( U \) with respect to \( \gamma(z) \) and \( \tilde{\gamma}(z) \) implements the unitary equivalence of \( \{K^J_{IJ}, 1 \leq |I|,|J| \leq k-1\} \) and \( \{\tilde{K}^J_{IJ}, 1 \leq |I|,|J| \leq k-1\} \).

4 Curvature tensor and localization on sub-manifolds

In this section we turn to localization on analytic sub-manifolds and our focus mainly lies in the geometric theory as well as its relation to the Specht-type classification. The analytic theory turns out to be reducible to that of point-wise localizations which be will be discussed in the end.

We begin with basic elements on differential geometry. If \( E \) is a holomorphic Hermitian vector bundle, it is well known that there exists a unique canonical connection on \( E \), which is metric-preserving and compatible with the holomorphic structure. The curvature with respect to this canonical connection is of form \((1,1)\) hence can be expressed as
\[ K = \sum_{1 \leq k,l \leq m} K_{k\bar{l}} dz_k \wedge d\bar{z}_l, \]
where the components \( \{K_{k\bar{l}}\} \) are linear bundle maps on \( E \).

Given a local holomorphic frame \( \gamma \) of \( E \) with Gram matrix \( H = [\langle \gamma_i, \gamma_j \rangle]_{1 \leq i,j \leq n} \), the representing matrix \( K(\gamma) \) of \( K \) is (see [13, 19])
\[ K_{k\bar{l}}(\gamma) = \overline{\partial}_l (\partial_k H \cdot H^{-1}) = (\partial_k \overline{\partial}_l H - (\partial_k H)H^{-1}(\overline{\partial}_l H))H^{-1} \] (4.1)
for \( 1 \leq k,l \leq m \)(recall that we adopt the “left action” convention for matrices; in some literatures the matrix for \( K \) is given by \( \overline{\partial}(H^{-1}\partial H) \) where the right action convention applies).

The local matrix representation (4.1) implements a useful formula for calculating or estimating the curvature in numerous literatures on \( B_0^m(\Omega) \), while in this paper, we will go beyond its computational usefulness by exploring (4.1) from a tensorial viewpoint.
To be precise, if one (with no pre-knowledge on connection theory) is directly presented with (4.1) as a frame-to-matrix correspondence, a natural question is if such a correspondence gives a well defined tensor. In other words, for another holomorphic frame $\beta$, does it holds that

$$K_{k\ell}(\gamma) = A K_{k\ell}(\beta) A^{-1}$$

(4.2)

where $A$ is the holomorphic transition function from $\gamma$ to $\beta$. Keeping this in mind, in forthcoming discussions it will turn out that determining unitary equivalence of $H^2_Z$ amounts to verifying certain variations of this “tensorial property” between two different bundles (see Remark 4.6 below).

**Remark 4.1.** In standard literatures on differential geometry [13, 19], (4.2) is not checked since (4.1) is deduced from the “tensorial” definition of the curvature as second derivative with respect to the canonical connection.

We start with two elementary lemmas on block matrices. The first one as follows on invertibility is well-known:

**Lemma 4.2.** Let $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a block matrix such that $A$ and $D$ are square matrices (not necessarily of the same size). If $A$ is invertible, then $R$ is invertible if and only if $D - CA^{-1}B$ is invertible and

$$R^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$  

**Lemma 4.3.** Let $R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\tilde{R} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$ be invertible $2 \times 2$ block matrices with square blocks of the same size, whose inverses are blocklized by $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ and $\begin{pmatrix} \tilde{X} & \tilde{Y} \\ \tilde{Z} & \tilde{W} \end{pmatrix}$ respectively. If there are matrices $M, N, P, Q$ with $M, P$ invertible, such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} M & 0 \\ N & M \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} P & Q \\ 0 & P \end{pmatrix},$$

(4.3)

then $AW = M\tilde{A}\tilde{W}M^{-1}$.

**Proof.** Equating left upper blocks of (4.3) gives $A = M\tilde{A}P$. On the other hand,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} P & Q \\ 0 & P \end{pmatrix}^{-1} \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}^{-1} \begin{pmatrix} M & 0 \\ N & M \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} P^{-1} & -P^{-1}QP^{-1} \\ 0 & P^{-1} \end{pmatrix} \begin{pmatrix} \tilde{X} & \tilde{Y} \\ \tilde{Z} & \tilde{W} \end{pmatrix} \begin{pmatrix} M^{-1} & 0 \\ -M^{-1}NM^{-1} & M^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} * & * \\ * & P^{-1}WPM^{-1} \end{pmatrix}.$$
So $W = P^{-1}WM^{-1}$ thus $AW = (M\tilde{A}P)(P^{-1}WM^{-1}) = M\tilde{A}W M^{-1}$.

\textbf{Remark 4.4.} The “$AW$-block”, i.e., multiplication of the left upper block of $R$ with right lower block of $R^{-1}$, will play an important role in later discussions on the unitary equivalence problem. In particular, if $R$ is as in Lemma (4.3) with $A$ invertible, then $D$ is uniquely determined by $A, B, C$ and $AW$ (by Lemma 4.2) $D = W^{-1}CA^{-1}B = (AW)^{-1}A^{-1} + CA^{-1}B$.

We are ready to prove the following geometric classification for $H^2_{\mathcal{Z}}$ in terms of the curvature of $E(\mathcal{T})$ as follows:

\textbf{Theorem 4.5.} Given operator tuples $\mathcal{T}$ and $\tilde{\mathcal{T}}$ in $\mathcal{B}_1^n(\Omega)$, their localizations $H^2_{\mathcal{Z}}$ and $\tilde{H}^2_{\mathcal{Z}}$ on the submanifold $Z$ are unitarily equivalent if and only if there exists a holomorphic isometric bundle map $\Phi$ from $E(\mathcal{T})|_{Z}$ to $E(\tilde{\mathcal{T}})|_{Z}$ which intertwines $K_{k\overline{l}}$ and $\tilde{K}_{k\overline{l}}$ for $1 \leq k, l \leq m$ on $Z$.

In the special case $\text{codim} Z = 1$, the conclusion of Theorem 4.5 was given by Douglas and Misra for $n = 1$ (see Section 6 [8]) then later by Douglas and the author allowing $n > 1$ (see Theorem 21, [3]) respectively, and both works featured intricate computational proofs. Here we adopt a different approach by putting the problem into a tensorial geometric framework which yields a conceptual and more revealing solution working in the general situation.

\textbf{Proof.} Necessity: Fix arbitrary holomorphic frames $\gamma$ and $\tilde{\gamma}$ for $E(\mathcal{T})$ and $E(\tilde{\mathcal{T}})$. Let $U$ be the operator from $H^2_{\mathcal{Z}}$ to $\tilde{H}^2_{\mathcal{Z}}$ implementing the unitary equivalence. The intertwining property of $U$ combined with Corollary 2.5 specifying (2.4) in case $A = \{d + 1, \cdots, m\}$ and $k = 2$) implies that the restriction of $U$ on $\text{span}\{\gamma(z), \partial_1\gamma(z), \cdots, \partial_d\gamma(z)\}$ takes values in $\text{span}\{\tilde{\gamma}(z), \partial_1\tilde{\gamma}(z), \cdots, \partial_d\tilde{\gamma}(z)\}$. Moreover, as $U$ preserves joint eigen-spaces, its restriction on $\text{span}\{\gamma_1(z), \cdots, \gamma_n(z)\}$ takes values in $\text{span}\{\tilde{\gamma}_1(z), \cdots, \tilde{\gamma}_n(z)\}$ for every $z \in Z$, which gives a holomorphic isometric bundle map from $E(\mathcal{T})|_{Z}$ to $E(\tilde{\mathcal{T}})|_{Z}$. We show that this particular bundle map, denoted by $\Phi$, admits the required intertwining property.

Let $A_0 = [a_{jk}(z)]_{1 \leq j, k \leq n}$ be the representing matrix for $\Phi$ with respect to $\gamma$ and $\tilde{\gamma}$. That is, for $1 \leq j \leq n$,

$$U\gamma_j(z) = \sum_{k=1}^{n} a_{jk}(z)\tilde{\gamma}_k(z) \quad (4.4)$$

for complex coefficients $a_{jk}(z)$ holomorphic with respect to $z \in Z$.

Observe that for any $1 \leq l \leq d, z \in Z$, $U$ maps $\text{span}\{\gamma(z), \partial_1\gamma(z)\}$ to $\text{span}\{\tilde{\gamma}(z), \partial_1\tilde{\gamma}(z)\}$ by Corollary 2.5 specifying (2.4) in case $A = \{1, 2, \cdots, m\} \setminus \{l\}$ and $k = 2$. Hence for every $1 \leq j \leq n, 1 \leq l \leq d$, we can write

$$U\partial_l\gamma_j(z) = \sum_{k=1}^{n} a^l_{jk}(z)\tilde{\gamma}_k(z) + \sum_{k=1}^{n} b^l_{jk}(z)\partial_l\tilde{\gamma}_k(z) \quad (4.5)$$
for complex coefficients $a^l_{jk}(z), b^l_{jk}(z)$ holomorphic in $z \in Z$.

Note that $N_l$ annihilates $\gamma$ and sends $\partial\gamma$ to $\gamma$, combining the intertwining condition $U N_l = N_l U$ with (4.5) and (4.4) yields

$$b^l_{jk} = a_{jk}$$

for every $1 \leq l \leq d$, $1 \leq j, k \leq n$. So (4.5) becomes

$$U \partial\gamma_j(z) = \sum_{k=1}^{n} a^l_{jk}(z) \tilde{\gamma}_k(z) + \sum_{k=1}^{n} a_{jk}(z) \partial \tilde{\gamma}_k(z).$$

(4.6)

If we set $A_l = [a^l_{jk}(z)]_{1 \leq j,k \leq n}$, then $A_l$ is a holomorphic matrix-valued function over $Z$. Combing (4.6) and (4.4), one sees that the action of $U$ from $\text{span}\{\gamma(z), \partial_1 \gamma(z), \cdots, \partial_n \gamma(z)\}$ to $\text{span}\{\tilde{\gamma}(z), \partial_1 \tilde{\gamma}(z), \cdots, \partial_d \tilde{\gamma}(z)\}$ can be represented by the following lower triangular block matrix

$$
\begin{pmatrix}
A_0 & A_1 & A_2 & \cdots & 0 \\
A_1 & A_0 & 0 & \cdots & 0 \\
A_2 & 0 & A_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_d & 0 & \cdots & 0 & A_0 \\
\end{pmatrix},
$$

(4.7)

Let $H_{00} := [(\gamma_p(z), \gamma_q(z))]_{1 \leq p,q \leq n}$ be the Gram matrix of $\gamma(z)$ and set $H_{ij} := [(\partial_i \gamma_p(z), \partial_j \gamma_q(z))]_{1 \leq p,q \leq n}$ (similar notations apply to $\tilde{\gamma}$). As $U$ is isometric, it holds that

$$
\begin{pmatrix}
H_{00} & \cdots & H_{0d} \\
H_{10} & \cdots & H_{1d} \\
H_{20} & \cdots & H_{2d} \\
\vdots & \vdots & \vdots \\
H_{d0} & \cdots & H_{dd} \\
\end{pmatrix} = \begin{pmatrix}
A_0 & A_1 & A_2 & \cdots & 0 \\
A_1 & A_0 & 0 & \cdots & 0 \\
A_2 & 0 & A_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_d & 0 & \cdots & 0 & A_0 \\
\end{pmatrix},
$$

(4.8)

Now we are prepared to give the intertwining property for $\Phi$ with respect to $K_{kl}$ and $\tilde{K}_{kl}$, which at the matrix level amounts to

$$
(H_{kl} - H_{k0} H_{00}^{-1} H_{0l}) H_{00}^{-1} A_0 = A_0 (\tilde{H}_{kl} - \tilde{H}_{k0} \tilde{H}_{00}^{-1} \tilde{H}_{0l}) \tilde{H}_{00}^{-1}
$$

(4.9)

for all $1 \leq k, l \leq m$. We arrange the verification of (4.9) into three cases.

**Case 1:** $1 \leq l, k \leq d$.

Equating the $(0,0)$, $(0,l)$, $(k,0)$ and $(k,l)$ blocks in both sides of (4.8) gives the following identities on $Z$:

$$H_{00} = A_0 \tilde{H}_{00} A_0^*,
$$

(4.10)

$$H_{0l} = A_0 \tilde{H}_{00} A_l^* + A_0 \tilde{H}_{0l} A_0^*,
$$

(4.11)
\[
H_{k0} = A_k \tilde{H}_{00} A_0^* + A_0 \tilde{H}_{k0} A_0^* \tag{4.12}
\]
\[
H_{kl} = A_k \tilde{H}_{00} A_l^* + A_0 \tilde{H}_{k0} A_l^* + A_k \tilde{H}_{0l} A_0^* + A_0 \tilde{H}_{kl} A_0^*. \tag{4.13}
\]

It is easy to check that \((4.10) - (4.13)\) is equivalent to the following matrix identity:

\[
\begin{pmatrix}
H_{00} & H_{0l} \\
H_{k0} & H_{kl}
\end{pmatrix}
= \begin{pmatrix}
A_0 & 0 \\
A_k & A_0
\end{pmatrix}
\begin{pmatrix}
\tilde{H}_{00} & \tilde{H}_{0l} \\
\tilde{H}_{k0} & \tilde{H}_{kl}
\end{pmatrix}
\begin{pmatrix}
A_0^* & A_l^* \\
0 & A_0^*
\end{pmatrix} \tag{4.14}
\]

By Lemma 4.2, the lower right block of \(\begin{pmatrix}
H_{00} & H_{0l} \\
H_{k0} & H_{kl}
\end{pmatrix}^{-1}\) and \(\begin{pmatrix}
\tilde{H}_{00} & \tilde{H}_{0l} \\
\tilde{H}_{k0} & \tilde{H}_{kl}
\end{pmatrix}^{-1}\) are given by \((H_{kl} - H_{k0} H_{00}^{-1} H_{0l})^{-1}\) and \((\tilde{H}_{kl} - \tilde{H}_{k0} \tilde{H}_{00}^{-1} \tilde{H}_{0l})^{-1}\) respectively. So we can combine Lemma 4.3 and (4.14) to obtain

\[
H_{00}(H_{kl} - H_{k0} H_{00}^{-1} H_{0l})^{-1} = A_0 \tilde{H}_{00}(\tilde{H}_{kl} - \tilde{H}_{k0} \tilde{H}_{00}^{-1} \tilde{H}_{0l})^{-1} A_0^{-1}, \tag{4.15}
\]

or equivalently,

\[
(H_{kl} - H_{k0} H_{00}^{-1} H_{0l}) H_{00}^{-1} A_0 = A_0 (\tilde{H}_{kl} - \tilde{H}_{k0} \tilde{H}_{00}^{-1} \tilde{H}_{0l}) \tilde{H}_{00}^{-1},
\]

as desired.

**Case 2:** \(1 \leq k \leq d, \ d + 1 \leq l \leq m\).

In this case, it is valid to apply partial derivative to known identities with respect to \(z_l\) along \(Z\). As \(A_0\) is holomorphic on \(Z\), apply \(\partial_l\) to \((4.10)\) and \((4.12)\) yields

\[
H_{0l} = A_0 \tilde{H}_{0l} A_0^* + A_0 \tilde{H}_{00} \partial_l A_0^* \tag{4.16}
\]
\[
H_{kl} = A_k \tilde{H}_{0l} A_l^* + A_k \tilde{H}_{00} \partial_l A_0^* + A_0 \tilde{H}_{kl} A_0^* + A_0 \tilde{H}_{k0} \partial_l A_l^*, \tag{4.17}
\]

Combining \((4.10) (4.12) (4.16) (4.17)\) yields

\[
\begin{pmatrix}
H_{00} & H_{0l} \\
H_{k0} & H_{kl}
\end{pmatrix}
= \begin{pmatrix}
A_0 & 0 \\
A_k & A_0
\end{pmatrix}
\begin{pmatrix}
\tilde{H}_{00} & \tilde{H}_{0l} \\
\tilde{H}_{k0} & \tilde{H}_{kl}
\end{pmatrix}
\begin{pmatrix}
A_0^* & \partial_l A_0^* \\
0 & A_0^*
\end{pmatrix}. \tag{4.18}
\]

Now a similar argument as we have done in Case 1 involving Lemma 4.3 4.2 and (4.18) gives \((4.9)\) in this Case 2.

**Case 3:** \(d + 1 \leq l, k \leq m\).

In this case, \(\partial_l, \partial_k, \) and \(\partial_k \partial_l\) makes sense on \(Z\) which, applied to \((4.10),\) yield

\[
H_{0l} = A_0 \tilde{H}_{0l} A_0^* + A_0 \tilde{H}_{00} \partial_l A_0^*, \tag{4.19}
\]
\[
H_{k0} = (\partial_k A_0) \tilde{H}_{00} A_0^* + A_0 \tilde{H}_{k0} A_0^*, \tag{4.20}
\]
\[
H_{kl} = (\partial_k A_0) \tilde{H}_{0l} A_l^* + A_0 \tilde{H}_{k0} \partial_l A_0^* + (\partial_k A_0) \tilde{H}_{00} \partial_l A_0^* + A_0 \tilde{H}_{kl} A_l^*. \tag{4.21}
\]

Combining \((4.10) (4.19) (4.20) (4.21)\) gives

\[
\begin{pmatrix}
H_{00} & H_{0l} \\
H_{k0} & H_{kl}
\end{pmatrix}
= \begin{pmatrix}
A_0 & 0 \\
\partial_k A_0 & A_0
\end{pmatrix}
\begin{pmatrix}
\tilde{H}_{00} & \tilde{H}_{0l} \\
\tilde{H}_{k0} & \tilde{H}_{kl}
\end{pmatrix}
\begin{pmatrix}
A_0^* & \partial_l A_0^* \\
0 & A_0^*
\end{pmatrix}. \tag{4.22}
\]
Hence (4.9) follows in the same way as the above two cases, completing the proof of the necessity.

Sufficiency: In light of Remark 2.7 the following holomorphic curve along $Z$, denoted by $E_{H^{2}}$, given by

$$z \mapsto \text{span}\{\gamma(z), \partial_1 \gamma(z), \cdots, \partial_d \gamma(z)\}, z \in Z$$

admits the spanning property with respect to $H^{2}(E_{H^{2}}$ is defined analogously). If we can construct a holomorphic isometric bundle map $\Psi$ from $E_{H^{2}}$ to $E_{\tilde{H^{2}}}$ which intertwines $N_l$ and $\tilde{N}_l$, $1 \leq l \leq m$ fiber-wise, and the conclusion will follow from the Rigidity Theorem.

The intertwining property

$$\Psi N_l = \tilde{N}_l \Psi$$

(4.23)

can be achieved if $\Psi$ is represented by a block matrix of the form (4.7) with respect to

$$\{\gamma(z), \partial_1 \gamma(z), \cdots, \partial_d \gamma(z)\} \text{ and } \{\tilde{\gamma}(z), \partial_1 \tilde{\gamma}(z), \cdots, \partial_d \tilde{\gamma}(z)\}.$$  

In fact, that (4.23) holds for $1 \leq l \leq d$ comes from the construction of (4.7) in above proof of the necessity, and both sides of (4.23) vanish for $d + 1 \leq l \leq m$.

Now it suffices to insert matrix-valued functions $A_0, \cdots, A_d$ which are holomorphic on $Z$ into (4.7) such that the metric-preserving condition (4.8), which is equivalent to the combination of (4.10) (4.11) (4.12) (4.13), holds.

The isometric holomorphic bundle map $\Phi$ from $E(T)|_Z$ to $E(\tilde{T})|_Z$ provides an $n \times n$ matrix-valued holomorphic function $A_0$ along $Z$ satisfying (4.10), whose intertwining property with respect to $K_{k\tilde{l}}$ and $\tilde{K}_{k\tilde{l}}$ gives (4.9). It remains to find $A_1, \cdots, A_d$ which are holomorphic along $Z$ and (4.11) (4.12) (4.13) holds for $1 \leq l, k \leq d$.

Set

$$A_k := (H_{kl}A^*_0 - A_0 \tilde{H}_{l0}) \tilde{H}^{-1}_{00}$$

(4.24)

for every $1 \leq k \leq d$. Then (4.12) holds for $1 \leq k \leq d$ and (4.11) automatically follows for $1 \leq l \leq d$ by taking adjoints. It remains to show that every $A_k$ is holomorphic and (4.13) holds for $1 \leq k, l \leq d$.

Observe that identities (4.10) (4.11) (4.12), as consequences of our choice of $A_k$, equates the left upper, right upper, and left lower blocks of the two sides in (4.14). Moreover, the intertwining property (4.9) of $\Phi$ implies (4.13), which means that the two sides in (4.14) have the same “AW” blocks. So by Remark 4.4 their right lower blocks are equal as well, which gives (4.13) as desired.

Finally we verify that $A_k$ defined by (4.24) is holomorphic over $Z$, that is, $\partial_1 A_k = 0$ for $d + 1 \leq l \leq m$.

Observing that $A_0$ is holomorphic, we first apply $\partial_1$ to (4.12) which holds by the construction of $A_k$ to get

$$H_{kl} = (\partial_1 A_k) \tilde{H}_{00} A_0^* + A_k \tilde{H}_{0l} A_0^* + A_k \tilde{H}_{00} \tilde{\partial}_1 A_0^* + A_0 \tilde{H}_{kl} A_0^* + A_0 \tilde{H}_{k0} \tilde{\partial}_1 A_0^*.$$  

(4.25)

On the other hand, if we can verify (4.17), then comparing (4.17) and (4.25) forces

$$(\partial_1 A_k) \tilde{H}_{00} A_0^* = 0,$$
which implies $\overline{\partial}_t A_k = 0$ since $H_{00}$ and $A_0$ are invertible, and this will completes the proof of the theorem.

The remaining verification of (4.17) goes in a similar way as we have just done to (4.13). Applying $\overline{\partial}_t$ to (4.10) we see that (4.10) holds, which, together with (4.10) and (4.12), implies that two sides of (4.18) has the same left upper, right upper, and left lower blocks. Now we can invoke the intertwining property (4.9) again, specified to $1 \leq k \leq d, d + 1 \leq l \leq m$, to conclude, by Remark 4.4, that the right lower blocks in two sides of (4.18) are equal, which gives (4.17).

**Remark 4.6.** Now we revisit the tensorial property (4.2) of the curvature. If $\gamma$ and $\beta$ differ by a transition matrix function $A$ (which is holomorphic), their Gram matrices $H(\gamma)$ and $H(\beta)$ are related by $H(\gamma) = AH(\beta)A^*$, to which one can apply $\overline{\partial}_t, \partial_k$, and $\partial_k \overline{\partial}_t$ as in Case 3 (as the frames are defined on open subset rather than a lower dimensional sub-manifold, we can allow $k, l$ to run through 1 to $m$) hence (4.2) follows in the same way by Lemma 4.2 and Lemma 4.3. This gives an proof of (4.2) without resorting to connection theory on vector bundles, and the three cases in the proof of Theorem 4.5 are variations of this tensorial property with respect to two different bundles, as we mentioned in the beginning of this section.

The next result relates the curvature of $E(T)$ to the operator-valued invariant we introduced for point-wise localizations in Section 3. For fixed $z = (z_1, \ldots, z_m)$, we adopt more concise notations $N^i_z := (T_{i} - z_i)|_{H^1_z}$ and $K^{ij}_z := N^i_z N^j_z^*|_{H^1_z}$ in the statement and proof of the following theorem (here we do not apply $P_{H_z^1}$ since $N^i_z N^j_z^*$ preserves $H^1_z$ as can be seen soon).

**Theorem 4.7.** A linear map $\Phi$ from $H^1_z$ to $\overline{\Phi}^1 \overline{\Phi}^1_z$ intertwines $K^{ij}_z$ and $\overline{K}^{ij}_z$, $1 \leq i, j \leq m$, if and only if it intertwines the curvatures $\mathcal{K}_{ij}^z$ and $\overline{\mathcal{K}}_{ij}^z$, $1 \leq i, j \leq m$, at $z$.

**Proof.** Fix arbitrary fixed holomorphic frames $\gamma$ of $E(T)$. Let $H_{00} = [(\gamma_p(z), \gamma_q(z))]_{1 \leq p, q \leq n}$ be the Gram matrix of $\gamma(z)$ and $H_{ij} = [(\partial_i \gamma_p(z), \partial_j \gamma_q(z))]_{1 \leq p, q \leq n}$. Then the Gram matrix for $\{\gamma(z), \partial_1 \gamma(z), \ldots, \partial_n \gamma(z)\}$ is the block matrix $[H_{ij}]_{0 \leq i, j \leq m}$. We begin with representing matrix for $K^{ij}_z$ with respect to the base $\gamma(z)$ of $H^1_z$, which will be read out from the 0-th row of the larger matrix representing $N^i_z N^j_z^*$ on $H^2_z$ so we start with the latter (unlike the proof of Theorem 3.1 $\gamma$ is not assumed to be normalized here).

Recall that by (2.1), for each $1 \leq i \leq m$, $N^i_z$ acts on $H^2_z = \text{span}\{\gamma(z), \partial_1 \gamma(z), \ldots, \partial_m \gamma(z)\}$ by

\[
N^i_z = \begin{pmatrix}
0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0
\end{pmatrix}_i, i = 1, \ldots, m
\]

which is an $(m + 1) \times (m + 1)$ block matrix with only one nonzero block, the $n \times n$ identity matrix $I_n$ at the $(i, 0)$ position.
Let \([G_{ij}]_{0 \leq i,j \leq m}\) be the inverse of \([H_{ij}]_{0 \leq i,j \leq m}\). By Lemma 3.6, the matrix representing \(N^i z N^j z^*\) on \(H_2^2\) is

\[
\begin{pmatrix}
H_{00} & \cdots & H_{0m} \\
\vdots & & \vdots \\
H_{m0} & \cdots & H_{mm}
\end{pmatrix}
\begin{pmatrix}
j - th \\
0 & \cdots & I_n & \cdots & 0 \\
\vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
G_{00} & \cdots & G_{0m} \\
\vdots & & \vdots \\
G_{m0} & \cdots & G_{mm}
\end{pmatrix}
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & & \vdots \\
I_n & 0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0
\end{pmatrix}
\]

This implies for all \(1 \leq i,j \leq m\), \(N^i z N^j z^*\) leaves \(H_2^1\) invariant and \(K^i j_z = N^i z N^j z^* |_{H_2^1}\) has the matrix representation \(H_{00} G_{ji}\) with respect to \(\gamma(z)\). With similar notations, \(H_{00} \tilde{G}_{ji}\) represents \(\tilde{K}^i j_z\) with respect to \(\tilde{\gamma}(z)\) where \(\tilde{\gamma}\) is a fixed holomorphic frame for \(E(\tilde{T})\).

Let \(A\) be the representing matrix for \(\Phi\) with respect to \(\gamma(z)\) and \(\tilde{\gamma}(z)\). If \(\Phi\) admits the prescribed intertwining property with respect to \(K^i j_z\) and \(\tilde{K}^i j_z\), then

\[
H_{00} G_{ij} A = A \tilde{H}_{00} \tilde{G}_{ij}
\]

for \(1 \leq i,j \leq m\), which in turn gives

\[
(H_{00} \otimes I_m)[G_{ij}]_{1 \leq i,j \leq m}(A \otimes I_m) = (A \otimes I_m)(\tilde{H}_{00} \otimes I_m)[\tilde{G}_{ij}]_{1 \leq i,j \leq m}
\]

as an equality between two \(m \times m\) block matrices.

On the other hand, it holds by Lemma 4.2 that

\[
\begin{pmatrix}
G_{11} & \cdots & G_{1m} \\
\vdots & & \vdots \\
G_{m1} & \cdots & G_{mm}
\end{pmatrix}
^{-1} = \begin{pmatrix}
H_{11} & \cdots & H_{1m} \\
\vdots & & \vdots \\
H_{m1} & \cdots & H_{mm}
\end{pmatrix}
- \begin{pmatrix}
H_{10} \\
\vdots \\
H_{m0}
\end{pmatrix}
H_{00}^{-1}(H_{01} \cdots H_{0m})
\]

\[
= [H_{1j} - H_{10} H_{00}^{-1} H_{0j}]_{1 \leq i,j \leq m}
\]

Taking inverse of (4.28) and inserting (4.29) yields

\[
(A^{-1} \otimes I_m)[H_{ij} - H_{i0} H_{00}^{-1} H_{0j}]_{1 \leq i,j \leq m}(H_{01} \otimes I_m) = [\tilde{H}_{ij} - \tilde{H}_{i0} \tilde{H}_{00}^{-1} \tilde{H}_{0j}]_{1 \leq i,j \leq m}(\tilde{H}_{01} \otimes I_m)(A^{-1} \otimes I_m).
\]

Specifying the above identity block-wise gives

\[
A^{-1}(H_{ij} - H_{i0} H_{00}^{-1} H_{0j}) H_{00}^{-1} = (\tilde{H}_{ij} - \tilde{H}_{i0} \tilde{H}_{00}^{-1} \tilde{H}_{0j}) \tilde{H}_{00}^{-1} A^{-1}
\]
or
\[
(H_{ij} - H_{0i}H_{0j}^{-1}H_{0j})H_{00}^{-1}A = A(\bar{H}_{ij} - \bar{H}_{0i}\bar{H}_{0j}^{-1}\bar{H}_{0j})\bar{H}_{00}^{-1}, \tag{4.30}
\]
which is exactly the desired intertwining condition of $\Phi$ with respect to $K_{ij}^*$ and $K_{ij}$ in terms of representing matrices, proving one direction of the theorem. It is obvious that one can reverse the above argument to obtain (4.27) by starting from (4.30), hence the other direction follows, completing the proof.

Combining Theorem 4.7 and Theorem 4.9, we obtain the following geometric Specht-type classification for $H^k_Z$.

**Theorem 4.8.** For two operator tuples $T$ and $\tilde{T}$ in $\mathcal{B}_n^m(\Omega)$, their localizations $H^k_Z$ and $\tilde{H}^k_Z$ on the submanifold $Z$ are unitarily equivalent if and only if there exists a holomorphic isometric bundle map $\Phi$ from $E(T)|_Z$ to $E(\tilde{T})|_Z$ which intertwines $K^i_j$ and $\tilde{K}^i_j$, for $1 \leq i, j \leq m$ on $Z$.

Finally we put some remarks on the analytic classification theory for $H^k_Z$ in terms of normalized frames. The following Theorem 4.9 in the form $(i) \iff (iv)$ was first given by Douglas and the author (Theorem 17, [3]) in case $\text{codim}Z = 1$. More recently, by combing approaches in [3] and [9], Deb managed to extend it to the general situation [11]. Here we observe that the problem can be reduced to the point-wise case (Theorem 3.4) by an application of Lemma 2.2 which yields a much simpler proof as we present here.

**Theorem 4.9.** The followings are equivalent

(i) The localizations $H^k_Z$ and $\tilde{H}^k_Z$ are unitarily equivalent;

(ii) there exists holomorphic frames $\gamma$ and $\tilde{\gamma}$ for $E(T)$ and $E(\tilde{T})$ such that $\partial^I \partial^J \tilde{H} = \partial^I \partial^J \tilde{H}$ on $Z$ for all $|I|, |J| \leq k - 1$, $I, J \in N^d = \{ I = (i_1, \ldots, i_m) | i_{d+1} = \cdots = i_m = 0 \}$;

(iii) there exists holomorphic frames $\gamma$ and $\tilde{\gamma}$ normalized at some $z_0 \in Z$ such that

$$\partial^I \partial^J \tilde{H} = \partial^I \partial^J \tilde{H}$$

on $Z$ for all $|I|, |J| \leq k - 1$, $I, J \in N^d$;

(iv) there exists holomorphic frames $\gamma$ and $\tilde{\gamma}$ normalized at $z_0 \in Z$ and a constant unitary matrix $U$, such that $\partial^I \partial^J \tilde{H} = U(\partial^I \partial^J \tilde{H})U^*$ on $Z$ for all $|I|, |J| \leq k - 1$, $I, J \in N^d$;

(v) for any holomorphic frames $\gamma$ and $\tilde{\gamma}$ normalized at $z_0 \in Z$, there exists a constant unitary matrix $U$, such that $\partial^I \partial^J \tilde{H} = U(\partial^I \partial^J \tilde{H})U^*$ on $Z$ for all $|I|, |J| \leq k - 1$, $I, J \in N^d$.

**Proof.** (ii)$\Rightarrow$ (i) For all points $z \in Z$, the linear maps defined by

$$\Phi(z) : \partial^I \gamma_i(z) \mapsto \partial^I \tilde{\gamma}_i(z), \quad |I| \leq k - 1, I \in N^d$$

glue to a linear bundle map between the canonical holomorphic curves spanning $H^k_Z$ and $\tilde{H}^k_Z$ (in the sense of Remark 2.7), which is holomorphic as it sends holomorphic frames to holomorphic frames. That $\Phi(z)$ is isometric and intertwines $T_l - z_l$ and $\tilde{T}_l - z_l$ are point-wise...
requirements and follows in the same way as in the proof of Theorem (3.4) so the conclusion follows from the Rigidity Theorem.

(iii)⇒ (ii) Trivial.

(i)⇒ (iii)Let Φ be the unitary operator from $H^k_Z$ to $\tilde{H}^k_Z$ implementing the unitary equivalence. We claim that there exists holomorphic frames $γ$ and $\tilde{γ}$ normalized at $z_0$ such that $Φγ_i(z) = \tilde{γ}_i(z)$ for every $1 \leq i \leq n$, $z \in Z$.

With the claim whose proof will be given later, we show by induction that

$$Φ∂^l γ_i(z) = ∂^l \tilde{γ}_i(z)$$  \hspace{1cm} (4.31)

for every $1 \leq i \leq n$, $|I| \leq k - 1$, $I \in N^d$ and $z \in Z$, then (iii) will follow from the fact that Φ is isometric.

The claim gives (4.31) in case $|I| = 0$ and we suppose (4.31) holds with $|I| \leq l$, $I \in N^d$, for some $l$ as an induction hypothesis. Then for any $I$ with $|I| = l + 1$, $I \in N^d$, it holds that

$$(\tilde{T}_q - z_q)(Φ∂^l γ_i(z) - ∂^l \tilde{γ}_i(z)) = 0$$

for every $1 \leq q \leq m$, $z \in Z$. In fact, as $I \in N^d$, the above identity trivially holds for $d + 1 \leq q \leq m$, while for $1 \leq q \leq d$, the proof goes in the same way (using the induction hypothesis) as in the proof of Theorem 3.4.

Now $Φ∂^l γ_i(z) - ∂^l \tilde{γ}_i(z)$ lies in $\cap_{i=1}^m \ker(\tilde{T}_i - z_i)$, hence there exists $a^l_k(z)$ holomorphic in $z$ such that

$$Φ∂^l γ_i(z) - ∂^l \tilde{γ}_i(z) = \sum_{k=1}^n a^l_k(z)\tilde{γ}_k(z)$$  \hspace{1cm} (4.32)

and suffices to show that $a^l_k(z) = 0$ for all $1 \leq k \leq n$, $z \in Z$, which concludes the induction.

As Φ is isometric,

$$⟨Φ∂^l γ_i(z) - ∂^l \tilde{γ}_i(z), \tilde{γ}_j(z)⟩ = ⟨∂^l γ_i(z), γ_j(z)⟩ - ⟨∂^l \tilde{γ}_i(z), \tilde{γ}_j(z)⟩$$

for all $1 \leq j \leq n$, $z \in Z$.

Inserting the linear representation (4.32) yields

$$\sum_{k=1}^n a^l_k(z)\tilde{γ}_k(z), \tilde{γ}_j(z)⟩ = ⟨∂^l γ_i(z), γ_j(z)⟩ - ⟨∂^l \tilde{γ}_i(z), \tilde{γ}_j(z)⟩.$$  

Applying Lemma 2.2 to change the anti-holomorphic part into $w$ and setting $w = z_0$ gives

$$\sum_{k=1}^n a^l_k(z)\tilde{γ}_k(z), \tilde{γ}_j(z_0)⟩ = ⟨∂^l γ_i(z), γ_j(z_0)⟩ - ⟨∂^l \tilde{γ}_i(z), \tilde{γ}_j(z_0)⟩.$$  

As $γ$ and $\tilde{γ}$ are both normalized at $z_0$, $⟨∂^l γ_i(z), γ_j(z_0)⟩ = ⟨∂^l \tilde{γ}_i(z), \tilde{γ}_j(z_0)⟩ = 0$ hence

$$\sum_{k=1}^n a^l_k(z)\tilde{γ}_k(z), \tilde{γ}_j(z_0)⟩ = 0$$

22
for all \(1 \leq j \leq n, \ z \in Z\), which, combined with the fact that \(\langle \gamma_k(z), \gamma_j(z_0) \rangle = \delta_{jk}\), implies \(a_k^i(z) = 0\) for all \(1 \leq k \leq n\), concluding the induction.

It remains to prove the claim. To this end, we take arbitrary but fixed holomorphic frames \(\gamma = \{\gamma_1, \cdots, \gamma_n\}\) and \(\overline{\gamma} = \{\overline{\gamma}_1, \cdots, \overline{\gamma}_n\}\) normalized at \(z_0\). Then for any fixed \(z \in Z\), \(\Phi\) preserves joint eigen-spaces hence maps \(\text{span}\{\gamma_1(z), \cdots, \gamma_n(z)\}\) to \(\{\overline{\gamma}_1(z), \cdots, \overline{\gamma}_n(z)\}\), so there exists an \(n \times n\) holomorphic matrix function \(U(z)\) such that \(\Phi \overline{\gamma}(z) = U(z) \overline{\gamma}(z)\). As \(\Phi\) is isometric,

\[
\langle \gamma^T(z), \gamma(z) \rangle = U(z) \langle \overline{\gamma}^T(z), \overline{\gamma}(z) \rangle U(z)^*
\]

holds for all \(z \in Z\).

Now an application of Lemma 2.2 as in the “only if” part of Lemma 3.5 implies that \(U(z)\) is a constant unitary matrix, and the claim follows by replacing the frame \(\overline{\gamma}^T\) by \(U \overline{\gamma}^T\). This completes the proof of (i) \(\Rightarrow\) (iii).

(iii) \(\Leftrightarrow\) (iv) \(\Leftrightarrow\) (v) As \(U\) is a constant matrix, the proof in Theorem 3.4 remains valid when the single point \(z\) is replaced by the submanifold \(Z\).

\[\square\]

5 Appendix

Coordinate multiplications on a holomorphic function space of several complex variables provide a basic model in multivariate operator theory. In light of [7], an operator tuple \(T = (T_1, \cdots, T_m) \in \mathcal{B}_m^n(\Omega)\) is always unitarily equivalent to \(\left(M^*_{z_1}, \cdots, M^*_{z_m}\right)\) on a Hilbert space \(\mathcal{M}\) consisting of \(\mathbb{C}^n\)-valued holomorphic functions with bounded point-wise evaluations over \(\Omega^*\) (the conjugate domain of \(\Omega\)), where

\[M_{z_i} : f \mapsto z_i f, f \in \mathcal{M}\]

denotes multiplication by the \(i\)th coordinate function \(z_i\). In this appendix, we describe the explicit realization of \(H^2_z\) in this function space model.

Let \(e_z : f \mapsto f(z)\) be the evaluation functional at \(z\). For any vector \(\xi \in \mathbb{C}^n\), \(e_z^*\xi\) is a function in \(\mathcal{M}\) which is conventionally denoted by \(K(\cdot, z)\xi\) and the following reproducing property holds

\[\langle f, K(\cdot, z)\xi \rangle_{\mathcal{M}} = \langle f(z), \xi \rangle_{\mathbb{C}^n}\]

As \(\mathcal{M}\) consists of holomorphic functions, \(K(\cdot, z)\xi\) is anti-holomorphic in \(z \in \Omega^*\) and from the above reproducing property one can easily verify that if \(\{e_1, \cdots, e_n\}\) is a base for \(\mathbb{C}^n\), then \(\cap_{i=1}^n \text{ker}(M^*_{z_i} - z_i) = \text{span}\left\{K(\cdot, z)e_i, 1 \leq i \leq n\right\}\). In other words, if we set \(\gamma_i(z) := K(\cdot, \overline{z})e_i\), then \(\{\gamma_1(z), \cdots, \gamma_n(z)\}\) is holomorphic in \(z \in \Omega\) and implements a holomorphic frame for the holomorphic curve \(z \mapsto \cap_{i=1}^n \text{ker}(M^*_{z_i} - z_i)\).

A closed subspace \(\mathcal{M}_0\) in \(\mathcal{M}\) is called an invariant subspace (or submodule when \(\mathcal{M}\) is viewed as a module over the polynomial ring in \(m\) variables) if \(M_{z_i}\mathcal{M}_0 \subseteq \mathcal{M}_0\) for every \(1 \leq i \leq m\). The quotient module \(\mathcal{M} \ominus \mathcal{M}_0\) as well as the corresponding resolution theory constitute a fruitful chapter in function-theoretic operator theory([10], [15]). In particular,
the localization $H^k_Z$ corresponds to an important class of quotient modules with respect to sub-modules defined by vanishing conditions on $Z$, which is a desirable model to realize the geometric operator theory of [5] on lower dimensional objects.

Precisely, let $Z$ be the coordinate slice (2.5) and $N^d = \{ I = (i_1, \ldots, i_m) | i_{d+1} = \cdots = i_m = 0 \}$, let $M^k_0$ be the subspace of $M$ determined by the following vanishing condition:

$$M^k_0 = \{ f \in M, \partial^I f(z) = 0, z \in Z^*, |I| \leq k - 1, I \in N^d \}. \quad (5.1)$$

It is easy to verify that $M_0$ is a submodule. Moreover, for any $z$ in $\Omega^*$ and $f \in M$, the reproducing property gives

$$\langle \partial^I f(z), e_i \rangle_{\mathbb{C}^n} = \langle f, \overline{\partial} K(\cdot, z)e_i \rangle_M.$$

Hence

$$\partial^I f(z) = 0$$

if and only if

$$\langle f, \overline{\partial} K(\cdot, z)e_i \rangle_M = 0$$

for any $1 \leq i \leq n$. Therefore

$$M^k_q := M \ominus M^k_0 = \vee_{z \in Z^*} \text{span} \{ \overline{\partial} K(\cdot, z)e_i, 1 \leq i \leq n, |I| \leq k - 1, I \in N^d \}.$$

In other words, the anti-holomorphic curve over $Z^*$ defined by $z \mapsto \text{span} \{ \overline{\partial} K(\cdot, z)e_i, 1 \leq i \leq n, |I| \leq k - 1, I \in N^d \}$, $z \in Z^*$ has the spanning property with respect to $M^k_q$. Equivalently, the holomorphic curve over $Z$ defined by $z \mapsto \text{span} \{ \partial^I \gamma(z), 1 \leq i \leq n, |I| \leq k - 1, I \in N^d \}$ (recall that $\gamma_i(z) := K(\cdot, z)e_i$) admits the spanning property with respect to $M^k_q$, which means that in the function model, $M^k_q$ represents $H^k_Z$ in light of Remark 2.7.

Passing to the general situation where an analytic sub-manifold $Z$ does not necessarily take the form (2.5), $M_0$ can be defined as the sub-module consisting of functions in $M$ vanishing to order at least $k$ on $Z$ (see [14] for the detailed definition), which is a coordinate-free condition and coincides with (5.1) when $Z$ takes the form (2.5). On the other hand, with a proper shrink of $\Omega$ and coordinate change, one can always assume $Z$ to be the coordinate slice (2.5), and the spanning property together with a standard “restriction argument” (in the sense of Aronazijn [1]) ensures that either the shrink of $\Omega$ or change of coordinate yields unitarily equivalent quotient modules (see [9] for details), which means that focusing on Definition 2.6 will not result in any conceptual loss of generality.

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