The geodetic-dominating number of comb product graphs

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Abstract

A set of vertices $S$ is called a geodetic-dominating set of $G$ if every vertex outside $S$ is adjacent to a vertex in $S$, and also is located inside a shortest path between two vertices in $S$. The geodetic-dominating number of $G$ is the minimum cardinality of geodetic-dominating sets of $G$. In this paper, we determine an exact value of the geodetic-dominating number of comb product graphs of any connected graphs of order at least two.

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1. Introduction

In this paper, all graphs are assumed to be connected, finite, simple, and undirected. Let $G$ be a graph. For a vertex $z \in V(G)$, we recall that the open neighborhood and the closed neighborhood of $z$ in $G$ is defined as $N_G(z) = \{w \in V(G) \mid zw \in E(G)\}$ and $N_G[z] = N_G(z) \cup \{z\}$, respectively. A set $D \subseteq V(G)$ is called a dominating set if $N_G[D] = V(G)$. The domination number of $G$ is the minimum cardinality of dominating sets of $G$. This concept provides several applications especially in protection strategies and business networking [10]. Interested readers are referred to a number of relevant literature mentioned in the references, including [16, 24].

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In this paper, we are interested to study another variant of domination in graph, namely geodetic-dominating set. A walk in $G$ is a finite non-empty sequence $W = v_0e_1v_1e_2v_2...e_kv_k$ where for $1 \leq j \leq k$, $v_j$ is a vertex and for $1 \leq i \leq k$, $e_i$ is an edge where $v_{i-1}$ and $v_i$ are its end points. We can say that $W$ is a $v_0 - v_k$ walk. A walk $W$ is called a trail in case all edges of $W$ are different. If all vertices of a trail $W$ are also different, then $W$ is called a path. The distance between vertices $a, b \in V(G)$, denoted by $d_G(a, b)$, is the minimum number of edges of $a - b$ paths in $G$. An $a - b$ path with $d_G(a, b)$ edges is called an $a - b$ geodesic. We denote $I_G[a,b]$ as the set of vertices which are located inside some $a - b$ geodesics of $G$. For a non-empty set $B \subseteq V(G)$, we define $I_G[B] = \bigcup_{a,b \in B} I_G[a,b]$. The set $B$ then we called as a geodetic set of $G$ in case $I_G[B] = V(G)$. The minimum cardinality of geodetic sets of $G$ is called as the geodetic number of $G$, denoted by $g(G)$. For references on geodetic number in graphs, see [3, 5].

In this paper, let a set $B \subseteq V(G)$ be both geodetic and dominating in $G$. The set $B$ then we call as a geodetic-dominating set of $G$. The geodetic-dominating number of $G$, denoted by $\gamma_g(G)$, is the minimum cardinality of geodetic-dominating sets of $G$.

This topic was firstly introduced by Escuadro et al. [12]. They proved that for a connected graph $G$ or order at least $n \geq 2$, $\max\{g(G), \gamma(G)\} \leq \gamma_g(G) \leq n$. They also characterized all graphs of order $n \geq 2$ with geodetic-dominating number 2, $n$, and $n - 1$. Some authors consider this topic to certain classes of graph. Hansberg and Volkmann [15] have shown that the geodetic-dominating problem for chordal graphs is NP-complete. Meanwhile the geodetic-dominating number of tree graphs and triangle-free graphs, can be seen in [12]. Some other references on geodetic-dominating number in graphs, see [7, 8, 18].

In this paper, we are interested to apply the geodetic-dominating concept to a product graphs. In this paper, we consider the comb product of connected graphs $G$ and $H$. In chemistry [1], some classes of chemical graphs can be considered as the comb product graphs. The comb product of connected graphs $G$ and $H$ at vertex $o \in V(H)$, denoted by $G \triangleright_o H$, is a graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$ and identifying the $i$-th copy of $H$ at the vertex $o$ to the $i$-th vertex of $G$. The vertex $o \in V(H)$ then we call as the identifying vertex. This product graphs have been widely investigated in many areas, including metric distance problems [11, 21, 22] and graph labeling problems [17, 20].

In this paper, we use some definitions in order to determine the geodetic-dominating number of $G \triangleright_o H$. Let $V(G) = \{g_1, g_2, \ldots, g_n\}$ and $V(H) = \{h_1, h_2, \ldots, h_m\}$. For the identifying vertex $o \in V(H)$, we also define $K_o = G \triangleright_o H$, $V(K_o) = \{(g_i, h_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$, $V_0 = \{(g_l, o) \mid 1 \leq l \leq n\}$, and for $l \in \{1, 2, \ldots, n\}$, $V_l = \{(g_l, h_f) \mid 1 \leq f \leq m\}$. For $S \subseteq V(G)$, we also use the notation $G[S]$ which is a maximal subgraph of $G$ induced by all vertices of $S$.

2. Geodetic-dominating number of comb product graphs

In two lemmas below, we provide some properties of a dominating set and a geodetic set in two isomorphic graphs.
Lemma 2.1. Let \( \theta : V(A) \rightarrow V(B) \) be an isomorphism between graphs \( A \) and \( B \). The set \( S \) is a dominating set of \( A \) if and only if \( \{ \theta(x) | x \in S \} \) is a dominating set of \( B \).

Proof. Let \( x, y \in V(A) \). Thus by isomorphism \( \theta(x), \theta(y) \in V(B) \). We define \( T \subseteq V(B) \) such that \( T = \{ \theta(x) | x \in S \} \). Note that \( x \) and \( y \) are adjacent in \( A \) if and only if \( \theta(x) \) and \( \theta(y) \) are adjacent in \( B \). Therefore, \( N_B[\theta(x)] = \{ \theta(y) | y \in N_A[x] \} \) and \( N_A[x] = \{ y | \theta(y) \in N_B[\theta(x)] \} \).

If \( S \) dominates \( A \), then we obtain
\[
N_B[T] = \bigcup_{t \in T} N_B[t] = \bigcup_{t \in \{ \theta(s) : s \in S \}} N_B[t] = \bigcup_{s \in S} N_B[\theta(s)] = \{ \theta(s) | s \in N_A[S] \} = \{ \theta(s) | s \in A \} = B.
\]

If \( T \) dominates \( B \), then we obtain
\[
N_A[S] = \bigcup_{s \in S} N_A[s] = \bigcup_{s \in \{ \theta(t) \in T \}} N_A[s] = \bigcup_{\theta(t) \in T} N_A[t] = \{ t | \theta(t) \in N_B[T] \} = \{ t | \theta(t) \in B \} = A.
\]

Lemma 2.2. Let \( \theta : V(A) \rightarrow V(B) \) be an isomorphism between graphs \( A \) and \( B \). The set \( S \) is a geodetic set of \( A \) if and only if \( \{ \theta(x) | x \in S \} \) is a geodetic set of \( B \).

Proof. Let \( x, y \in V(A) \). Thus by isomorphism \( \theta(x), \theta(y) \in V(B) \). We define \( T \subseteq V(B) \) such that \( T = \{ \theta(x) | x \in S \} \). Note that \( z \in V(A) \) is contained in \( x \rightarrow y \) path in \( A \), then \( \theta(z) \in V(B) \) is also contained in \( \theta(x) \rightarrow \theta(y) \) path in \( B \), and vice versa. So, \( z \) belongs to \( x \rightarrow y \) geodesic if and only if \( \theta(z) \) belongs to \( \theta(x) \rightarrow \theta(y) \) geodesic. Therefore, \( I_B[\theta(x), \theta(y)] = \{ \theta(z) | z \in I_A[x, y] \} \) and \( I_A[x, y] = \{ z | \theta(z) \in I_B[\theta(x), \theta(y)] \} \).

If \( S \) is a geodetic set of \( A \), then we obtain
\[
I_B[T] = \bigcup_{i, j \in T} I_B[i, j] = \bigcup_{i, j \in \{ \theta(s) : s \in S \}} I_B[i, j] = \bigcup_{k, l \in S} I_B[\theta(k), \theta(l)] = \{ \theta(s) | s \in I_A[S] \} = \{ \theta(s) | s \in A \} = B.
\]

If \( T \) is a geodetic set of \( B \), then we obtain
\[
I_A[S] = \bigcup_{k, l \in S} I_A[k, l] = \bigcup_{k, l \in \{ \theta(t) \in T \}} I_A[k, l] = \bigcup_{\theta(j), \theta(k) \in T} I_A[j, k] = \{ t | \theta(t) \in I_B[T] \} = \{ t | \theta(t) \in B \} = A
\]

Therefore, we obtain a direct consequences of Lemmas 2.1 and 2.2 in corollary below.

Corollary 2.1. Let \( \theta : V(A) \rightarrow V(B) \) be an isomorphism between graphs \( A \) and \( B \). The set \( S \) is a geodetic-dominating set of \( A \) if and only if \( \{ \theta(x) | x \in S \} \) is a geodetic-dominating set of \( B \).
Now, we investigate the geodetic properties of a geodetic-dominating set of a comb graph $K_o$ with the identifying vertex $o \in V(H)$.

**Lemma 2.3.** Let $o \in V(H)$ be the identifying vertex and $u, v$ be two distinct vertices of $K_o$. For $l \in \{1, 2, \ldots, n\}$, if $u \in V_l$ and $v \notin V_l$, then every $u - v$ path in $K_o$ consists of $(g_I, o)$.

**Proof.** The only vertex in $V_l$ which is adjacent to a vertex in $V(K_o) \setminus V_l$ is $(g_I, o)$. So, $(g_I, o)$ must belong to every $u - v$ path in $K_o$. \qed

**Lemma 2.4.** Let $o \in V(H)$ be the identifying vertex and $a, b, v$ be distinct vertices in $K_o$. For $l \in \{1, 2, \ldots, n\}$, let $A_l = V_l \{ (g_I, o) \}$. If $v \in A_l$ and $a, b \notin A_l$, then $v$ does not belong to any $a - b$ paths in $K_o$.

**Proof.** By Lemma 2.3, the vertex $(g_I, o)$ in $K_o$ always belongs to any $a - v$ walks and $b - v$ walks. So, $a - b$ walk always has the form $(g_I, o), \ldots, (g_I, o), \ldots (g_I, o), \ldots, (g_I, o), \ldots, (g_I, o)$. In the other hand, $v$ does not belong to any $a - b$ paths. \qed

**Lemma 2.5.** Let $o \in V(H)$ be the identifying vertex and $S$ be a geodetic set of $K_o$. Then for $l \in \{1, 2, \ldots, n\}$, $(S \cap V_l) \cup \{(g_I, o)\}$ is a geodetic set of $K_o[V_l]$.

**Proof.** Suppose that $(S \cap V_l) \cup \{(g_I, o)\}$ is not a geodetic set of $K_o[V_l]$. Then, there exists a vertex $b \in V_l$ such that $b \notin I_{K_o}[(S \cap V_l) \cup \{(g_I, o)\}]$. Note that,

\[
I_{K_o}[S] = \bigcup_{x, y \in S} I_{K_o}[x, y] = \bigcup_{x, y \in S \cap V_l} I_{K_o}[x, y] \cup \bigcup_{x, y \in S \setminus V_l} I_{K_o}[x, y] \cup \bigcup_{x \in S \cap V_l, y \in S \setminus V_l} I_{K_o}[x, y].
\]

By Lemma 2.3, we have

\[
\bigcup_{x \in S \cap V_l, y \in S \setminus V_l} I_{K_o}[x, y] = \bigcup_{x \in S \cap V_l} I_{K_o}[x, (g_I, o)] \cup \bigcup_{y \in S \setminus V_l} I_{K_o}[y, (g_I, o)].
\]

Since $\bigcup_{x, y \in S \cap V_l} I_{K_o}[x, y] \bigcup_{x \in S \cap V_l \cup S \setminus V_l} I_{K_o}[x, (g_I, o)] = \bigcup_{x, y \in S \cap V_l \cup (g_I, o)} I_{K_o}[x, y]$ and $\bigcup_{x, y \in S \setminus V_l} I_{K_o}[x, (g_I, o)] = \bigcup_{x \in S \setminus V_l \cup (g_I, o)} I_{K_o}[x, y]$, we obtain $I_{K_o}[S] = \bigcup_{x, y \in (S \cap V_l) \cup \{(g_I, o)\}} I_{K_o}[x, y] \cup \bigcup_{x, y \in S \setminus V_l \cup (g_I, o)} I_{K_o}[x, y].$

Because $b \neq (g_I, o)$, then $b \notin I_{K_o}[(S \setminus V_l) \cup \{(g_I, o)\}]$. By considering Lemma 2.4, we have that $S$ is not a geodetic set of $K_o$, a contradiction. \qed

In some lemmas below, we consider some properties of the geodetic-dominating set of an induced subgraph of $K_o$.

**Lemma 2.6.** Let $o \in V(H)$ be the identifying vertex, $S \subseteq V(H)$, and $\Gamma_l = \{(g_I, x) | x \in S\}$ for $l \in \{1, 2, \ldots, n\}$. Then, $S$ is a geodetic-dominating set of $H$ if and only if $\Gamma_l$ is a geodetic-dominating set of $K_o[V_l]$.
Proof. By considering Corollary 2.1, we choose an isomorphism \( \theta : V(H) \to V_I \) between graphs \( H \) and \( K_o[V_I] \). Thus for \( h \in V(H) \), \( \theta(h) = (g_l, h) \). For \( l \in \{1,2,...,n\} \) then \( \Gamma_{I_l} = \{(g_l, x)|x \in S\} = \{\theta(x)|x \in S\} \).

Lemma 2.7. Let \( o \in V(H) \) be the identifying vertex, and \( S \) be a dominating set of \( K_o \). Then for \( l \in \{1,2,...,n\} \), \( S \cap V_I \) is a dominating set of \( K_o[V_I \setminus \{(g_l, o)\}] \).

Proof. Suppose that \( S \cap V_I \) is not a dominating set of \( K_o[V_I \setminus \{(g_l, o)\}] \). Then, there exists a vertex \( b \in V_I \setminus \{(g_l, o)\} \) such that \( b \notin N_{K_o[S \cap V_I]} \). Note that, \( N_{K_o[S]} = N_{K_o[S \cap V_I]} \cup N_{K_o[S \setminus V_I]} \). Since \( b \notin N_{K_o[S \setminus V_I]} \), then \( S \) is not a dominating set of \( K_o[S \setminus V_I] \), a contradiction.

By Lemmas 2.5 and 2.7, we obtain a property of geodetic-dominating set of an induced subgraph of \( K_o[S] \), which can be seen in corollary below.

Corollary 2.2. Let \( o \in V(H) \) be the identifying vertex, and \( S \) be a geodetic-dominating set of \( K_o \). Then for \( l \in \{1,2,...,n\} \), \( (S \cap V_I) \cup \{(g_l, o)\} \) is a geodetic-dominating set of \( K_o[V_I] \).

Proof. By Lemma 2.5, \( (S \cap V_I) \cup \{(g_l, o)\} \) is a geodetic set of \( K_o[V_I] \). By considering Lemma 2.7, note that \( N_{K_o}(S \cap V_I) \cup \{(g_l, o)\} = N_{K_o}(S \cap V_I) \cup N_{K_o}((g_l, o)) \supseteq N_{K_o}((g_l, o)) \cup \{(g_l, o)\} = V_I \). So, \( (S \cap V_I) \cup \{(g_l, o)\} \) is also a dominating set of \( K_o[V_I] \).

Now, let us consider a connected graph \( H \) of order at least 2. Let \( o \) be vertex in \( H \). We define \( \mathcal{B} \) as a collection of geodetic-dominating sets of graph \( H \) with cardinality \( \gamma_g(H) \) containing \( o \). The collection \( \mathcal{B} \) can be written as

\[
\mathcal{B} = \{B|B \subseteq V(H), N_H[B] = I_H[B] = V(H), o \in B, |B| = \gamma_g(H)\}.
\]

We say that the graph \( H \) is of:

- type \( A_o \) if there exists a set \( S \in \mathcal{B} \) such that \( N_H[S \setminus \{o\}] = V(H) \).
- type \( B_o \) if there exists a set \( S \in \mathcal{B} \) such that \( N_H[S \setminus \{o\}] = V(H) - \{o\} \).

By above definitions, note that a graph \( H \) with the identifying vertex \( o \in V(H) \) can be both of type \( A_o \) and \( B_o \). Now, we are ready to determine the geodetic-dominating number of \( G \triangleright_o H \).

Theorem 2.1. Let \( G \) and \( H \) be connected graphs of order at least 2. Let \( o \in V(H) \). Then

\[
\gamma_g(G \triangleright_o H) = \begin{cases} 
\gamma_g(H) \cdot |V(G)|, & \text{if } H \text{ is neither of type } A_o \text{ nor } B_o, \\
(\gamma_g(H) - 1) \cdot |V(G)|, & \text{if } H \text{ is of type } A_o, \\
\gamma(G) + (\gamma_g(H) - 1) \cdot |V(G)|, & \text{otherwise}.
\end{cases}
\]

Proof. For the identifying vertex \( o \in V(H) \), we recall the notation \( K_o = G \triangleright_o H \). We distinguish three cases.

Case 1. \( H \) is neither of type \( A_o \) nor \( B_o \)

Let \( C \) be a geodetic-dominating set of \( H \) with \( |C| = \gamma_g(H) \). We define \( \Lambda = \{(g, h)|g \in V(G), h \in C\} \). By considering Lemma 2.6, we obtain that \( \Lambda \) is a geodetic-dominating set of \( K_o \). Therefore, \( \gamma_g(K_o) \leq |\Lambda| = |C| \cdot |V(G)| = \gamma_g(H) \cdot |V(G)| \).
For the lower bound, let us consider Corollary 2.2. Let $S$ be a geodetic-dominating set of $K_o$. Then for $l \in \{1, 2, \ldots, n\}$, $(S \cap V_l) \cup \{(g_l, o)\}$ is a geodetic-dominating set of $K_o[V_l]$. Let $B \in B$. For $l \in \{1, 2, \ldots, n\}$, we define $T_{l,B} = \{(g_l, b) | b \in B\}$ and $B_l = \{T_{l,B} | B \in B\}$. Note that $|T_{l,B}| = \gamma_g(H)$.

If $(S \cap V_l) \cup \{(g_l, o)\} \in B_l$, then by considering Corollary 2.2, we have

$$|S \cap V_l| = |(S \cap V_l) \cup \{(g_l, o)\}| \geq \gamma_g(K_o[V_l]) = \gamma_g(H).$$

Otherwise, we have

$$|(S \cap V_l) \cup \{(g_l, o)\}| \geq \gamma_g(K_o[V_l]) + 1 = \gamma_g(H) + 1.$$

It follows that

$$|S \cap V_l| \geq \gamma_g(H).$$

Therefore, $|S \cap V_l| \geq \gamma_g(H)$ for $1 \leq l \leq n$.

Since $S = \bigcup_{l=1}^n S \cap V_l$ and $V_i \cap V_j = \emptyset$ for $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, we obtain that

$$|S| \geq n \cdot |S \cap V_l| \geq n \cdot \gamma_g(H) = |V(G)| \cdot \gamma_g(H).$$

**Case 2.** $H$ is of type $A_o$

Let $C \in B$ such that $N_H[C \setminus \{o\}] = V(H)$. We define $\Lambda = \{(g, h) | g \in V(G), h \in C \setminus \{o\}\}$. Since $N_{K_o}[\Lambda] = I_{K_o}[\Lambda] = V(K_o)$, we obtain that $\Lambda$ is a geodetic-dominating set of $K_o$. Therefore, $\gamma_g(K_o) \leq |\Lambda| = |C| - 1 \cdot |V(G)| = (\gamma_g(H) - 1) \cdot |V(G)|$.

For the lower bound, let us consider Corollary 2.2. Let $S$ be a geodetic-dominating set of $K_o$. Then for $l \in \{1, 2, \ldots, n\}$, $(S \cap V_l) \cup \{(g_l, o)\}$ is a geodetic-dominating set of $K_o[V_l]$. Then we have that,

$$|(S \cap V_l) \cup \{(g_l, o)\}| \geq \gamma_g(K_o[V_l]) = \gamma_g(H).$$

It follows that

$$|S \cap V_l| \geq \gamma_g(H) - 1.$$

Since $S = \bigcup_{l=1}^n S \cap V_l$ and $V_i \cap V_j = \emptyset$ for $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, we obtain that

$$|S| \geq n \cdot |S \cap V_l| \geq n \cdot (\gamma_g(H) - 1) = |V(G)| \cdot (\gamma_g(H) - 1).$$

**Case 3.** $H$ is of type $B_o$ and is not of type $A_o$

Let $C \in B$ such that $N_H[C \setminus \{o\}] = V(H)$ and $D$ be a dominating set of $G$ with $|D| = \gamma(G)$. We define $\Lambda = \{(g, h) | g \in V(G), h \in C \setminus \{o\}\} \cup \{(g, o) | g \in D\}$. Since $N_{K_o}[\Lambda] = I_{K_o}[\Lambda] = V(K_o)$, we obtain that $\Lambda$ is a geodetic-dominating set of $K_o$. Therefore, $\gamma_g(K_o) \leq |\Lambda| = |C| - 1 \cdot |V(G)| + |D| = (\gamma_g(H) - 1) \cdot |V(G)| + \gamma(G)$.

For the lower bound, suppose that $\gamma_g(K_o) < (\gamma_g(H) - 1) \cdot |V(G)| + \gamma(G)$. Let $S$ be a geodetic-dominating set of $K_o$ with $|S| = \gamma_g(K_o)$. By Corollary 2.2, for $l \in \{1, 2, \ldots, n\}$, $(S \cap V_l) \cup \{(g_l, o)\}$
is a geodetic-dominating set of $K_o[V_l]$. Note that
\[ S = \bigcup_{1 \leq l \leq n} S \cap V_l \]
\[ = \bigcup_{1 \leq l \leq n} S \cap \{(g_l, o)\} \cup \bigcup_{1 \leq l \leq n} S \cap (V_l \setminus \{(g_l, o)\}) \]
\[ = (S \cap V_0) \cup \bigcup_{1 \leq l \leq n} S \cap (V_l \setminus \{(g_l, o)\}). \]

So, we obtain that there exists $l \in \{1, 2, \ldots, n\}$ such that $|S \cap (V_l \setminus \{(g_l, o)\})| < \gamma_g(H) - 1$ or $|S \cap V_0| < \gamma(G)$. However,
\[ |(S \cap (V_l \setminus \{(g_l, o)\})) \cup \{(g_l, o)\}| = |S \cap V_l \cup \{(g_l, o)\}| \]
\[ \geq \gamma_g(K_o[V_l]) = \gamma_g(H), \]
which implies
\[ |S \cap (V_l \setminus \{(g_l, o)\})| \geq \gamma_g(H) - 1. \]

Therefore, $|S \cap V_0| < \gamma(G)$. By considering that $K_o[V_0] = G$, there exists a vertex $x \in V_0$ such that $x \notin N_{K_o}[S \cap V_0]$. It is clear that $x \notin S$.

If $x \notin N_{K_o}[S \cap (V_l \setminus \{(g_l, o)\})]$ for $1 \leq l \leq n$, then we have a contradiction with $S$ is a geodetic-dominating set of $K_o$. So, we assume that there exists $l \in \{1, 2, \ldots, n\}$ such that $x \in N_{K_o}[S \cap (V_l \setminus \{(g_l, o)\})]$. Since $x \in V_0$, thus $x = (g_l, o)$.

Let $B \in B$. For $l \in \{1, 2, \ldots, n\}$, we define $T_{i, B} = \{(g_l, b)| b \in B\}$ and $B_l = \{T_{i, B}| B \in B\}$. Note that $|T_{i, B}| = \gamma_g(H)$.

If $(S \cap V_i) \cup \{(g_l, o)\} = (S \cap V_i) \cup \{x\} \in B_i$, then
\[ |N_{K_o}[S \cap V_i]| = |N_{K_o}[S \cap (V_i \setminus \{x\})]| \leq |V(K_o[V_i])| - 1 \]

So, there is at least one vertex $z$ in $K_o[V_i]$ such that $z \notin N_{K_o}[S \cap V_i]$. If $z = x$ then it will contradict to $x \in N[S \cap (V_i \setminus \{(g_l, o)\})]$. Otherwise, we have a contradiction to Lemma 2.7.

If $(S \cap V_i) \cup \{(g_l, o)\} = (S \cap V_i) \cup \{x\} \notin B_i$, then
\[ |(S \cap V_i) \cup \{x\}| \geq \gamma_g(K_o[V_i]) + 1 = \gamma_g(H) + 1, \]
which implies $|(S \cap V_i)| \geq \gamma_g(H)$. Since $S = \bigcup_{l=1}^n S \cap V_l$, $V_i \cap V_j = \emptyset$ for $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$, and $\gamma(G) \leq |V(G)|$, we obtain that
\[ |S| \geq n \cdot |S \cap V_i| \geq n \cdot \gamma_g(H) = |V(G)| \cdot \gamma_g(H) \]
\[ \geq |V(G)| \cdot \gamma_g(H) - |V(G)| + \gamma(G) \]
\[ = (\gamma_g(H) - 1) \cdot |V(G)| + \gamma(G). \]

A contradiction. \qed
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