An improved asymptotic formula for the distribution of irreducible polynomials in arithmetic progressions over $\mathbb{F}_q[x]$

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Abstract

Let $\mathbb{F}_q$ be a finite field with $q$ elements and $\mathbb{F}_q[x]$ the ring of polynomials over $\mathbb{F}_q$. Let $l(x), k(x)$ be coprime polynomials in $\mathbb{F}_q[x]$ and $\Phi(k)$ the Euler function in $\mathbb{F}_q[x]$. Let $\pi(l, k; n)$ be the number of monic irreducible polynomials of degree $n$ in $\mathbb{F}_q[x]$ which are congruent to $l(x)$ modulo $k(x)$. For any positive integer $n$, we denote by $\Omega(n)$ the least prime divisor of $n$. In this paper, we show that

$$\pi(l, k; n) = \frac{1}{\Phi(k)} q^n \left( \frac{n}{\Omega(n)} \right) + O(q^n) + O(n^{\alpha}),$$

where $\alpha$ only depends on the choice of $k(x) \in \mathbb{F}_q[x]$. Note that the above error term improves the one implied by a deep result of A. Weil \cite{17}. Our approach is completely elementary.

Keywords: Dirichlet’s theorem, irreducible polynomials, function fields, finite fields

1. Introduction

The origin of analytic number theory can be traced back to Euler’s analytic proof of the existence of infinitely many primes in 1737. A century later, a seminal work by Dirichlet \cite{4}, inspired by Euler’s proof, states that for two coprime integers $a$ and $b$, there are infinitely many primes $p$ such that $p \equiv a \pmod{b}$. These classical results and methods introduced therein greatly stimulated the development of number theory.

Let $\mathbb{F}_q$ be a finite field with $q$ elements. We denote by $\mathbb{F}_q[x]$ the ring of polynomials over $\mathbb{F}_q$. In 1919, H. Kornblum \cite{13} considered the analogue of Dirichlet’s theorem over $\mathbb{F}_q[x]$ in his doctoral thesis. In fact, he proved that, given two coprime polynomials $l(x)$ and $k(x)$ in $\mathbb{F}_q[x]$, there are infinitely many monic irreducible polynomials $p(x) \in \mathbb{F}_q[x]$ such that $p(x) \equiv l(x) \pmod{k(x)}$.

Let $\pi(l, k; n)$ denote the number of monic irreducible polynomials of degree $n$ in $\mathbb{F}_q[x]$ which are congruent to $l(x)$ modulo $k(x)$, where polynomials $l(x), k(x)$ are coprime. We denote by $\Phi(k)$ the analogue of the Euler function which is defined as the size of the multiplicative group $(\mathbb{F}_q[x]/(k(x)))^\times$. In 1924, Artin \cite{1} refined Kornblum’s result by showing the following theorem.

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Theorem 1.1 ([1]). Let \( l(x), k(x) \) be coprime polynomials in \( \mathbb{F}_q[x] \). Then, it follows that
\[
\pi(l, k; n) = \frac{1}{\Phi(k)} \frac{q^n}{n} + O\left(\frac{q^n\theta}{n}\right) \quad \text{with} \quad \frac{1}{2} \leq \theta < 1.
\]

Instead of considering the distribution of monic irreducible polynomials in arithmetic progressions, based on Dickson’s work [4] in 1911, Carlitz [3] and Uchiyama [16] provided some different results for distribution of monic irreducible polynomials in \( \mathbb{F}_q[x] \) with some given coefficients.

Theorem 1.2 ([16]). Let \( s, t \) be two positive integers and \( S(s, t; n) \) the number of monic irreducible polynomials of degree \( n \) in \( \mathbb{F}_q[x] \) with first \( s \) coefficients and last \( t \) coefficients given. Assume that the characteristic of \( \mathbb{F}_q \) is larger than \( \max\{s, t - 1\} \). Moreover, if the last coefficient is not zero. Then we have
\[
S(s, t; n) = \frac{1}{q^{s+t-1}(q-1)} q^n + O\left(\frac{q^n\theta}{n}\right) \quad \text{with} \quad \frac{1}{2} \leq \theta < 1.
\]

Note that, when \( s = t = 1 \) in the above theorem, Carlitz [3] gave an asymptotic formula with a better error term.

Nevertheless, it seems that it is still not clear whether we can improve the error term of Theorem 1.2 to \( O\left(\frac{q^n}{n}\right) \). Later, inspired by the results of Artin [1] and Uchiyama [16], Hayes [10] provided the following more general result.

Theorem 1.3 ([10]). Let \( s \) be an positive integer and \( \pi_S(s, l; k; n) \) the number of monic irreducible polynomials of degree \( n \) in \( \mathbb{F}_q[x] \) with given first \( s \) coefficients which are congruent to \( l(x) \) module \( k(x) \). Then we have
\[
\pi_S(s, l, k; n) = \frac{1}{q^s\Phi(k)} \frac{q^n}{n} + O\left(\frac{q^n\theta}{n}\right) \quad \text{with} \quad \frac{1}{2} \leq \theta < 1,
\]
where polynomials \( l(x), k(x) \) are coprime.

Although Hayes claimed that he can improve the error term to \( O\left(q^{n\frac{2}{n}}\right) \) when \( x^q - x \) does not divide \( k(x) \), he never published this result in detail. While, although still weaker than \( O\left(q^{n\frac{2}{n}}\right) \), Hensley [11] provided some improved error terms in some special cases over \( \mathbb{F}_2[x] \). In 2002, Rosen [15] provided a modern perspective about the distribution of irreducible polynomials in arithmetic progressions over \( \mathbb{F}_q[x] \) by showing the following theorem.

Theorem 1.4 ([15]). Let \( l(x), k(x) \) be coprime polynomials in \( \mathbb{F}_q[x] \). Then, it follows that
\[
\pi(l, k; n) = \frac{1}{\Phi(k)} \frac{q^n}{n} + O\left(\frac{q^n}{n}\right).
\]

Note that, Rosen’s proof is based on a deep result of A. Weil [17]. Also, we refer to [6, 7, 8, 9, 12, 14, 18] for recent progresses on the distribution of irreducible polynomials over \( \mathbb{F}_q[x] \) and the references therein. In this paper, we shall significantly improve Theorem 1.4. For any positive integer \( n \), denote by \( \Omega(n) \) the least prime divisor of \( n \). The following theorem is our main result.

Theorem 1.5. Let \( l(x), k(x) \) be coprime polynomials in \( \mathbb{F}_q[x] \). Then, it follows that
\[
\pi(l, k; n) = \frac{1}{\Phi(k)} \frac{q^n}{n} + O\left(n^\alpha\right) + O\left(\frac{q^n\theta(n)}{n}\right),
\]
where \( \alpha \) is a constant only depends on the choice of \( k(x) \in \mathbb{F}_q[x] \).

Note that the error term we obtain is better than the one that Weil’s result provides. Our approach is completely elementary.
2. Notation and preliminaries

The following definitions and lemmas can be found in [15]. Also, it should be noted that \( f, k, m, h \) are always referred as monic polynomials in \( \mathbb{F}_q[x] \), \( l \) is a polynomial in \( \mathbb{F}_q[x] \) and \( p \) is always referred as the monic irreducible polynomial. For any positive integer \( n \), denote by \( \Omega(n) \) the least prime divisor of \( n \).

**Definition 2.1.** The Riemann zeta function over \( \mathbb{F}_q[x] \) is defined as:

\[
\zeta(s) = \sum_{f \text{ monic}} \frac{1}{|f|^s},
\]

where for \( f \in \mathbb{F}_q[x] \) we define \( |f| = q^{\deg f} \).

**Definition 2.2.** For \( f \in \mathbb{F}_q[x] \), the Möbius function is defined as:

\[
\mu(f) = \begin{cases} 
1 & \text{if } f = 1, \\
(-1)^r & \text{if } f = p_1 p_2 \cdots p_r, \ p_i \neq p_j \text{ for } i \neq j \text{ and } r \geq 1, \\
0 & \text{otherwise}. 
\end{cases}
\]

**Lemma 2.3.** The Dirichlet L-series associated with \( \mu(f) \) satisfies that

\[
L(s, \mu) := \sum_{f \text{ monic}} \frac{\mu(f)}{|f|^s} = \sum_{n=0}^{\infty} \frac{H(n)}{q^{ns}} = \frac{1}{\zeta(s)}.
\]

As a consequence, we have

\[
H(n) = \sum_{\substack{f \text{ monic} \ \deg f = n}} \mu(f) = \begin{cases} 
1 & \text{if } n = 0, \\
-q & \text{if } n = 1, \\
0 & \text{otherwise}. 
\end{cases}
\]

**Definition 2.4.** For \( f \in \mathbb{F}_q[x] \), the von Mangolt’s function is defined as:

\[
\Lambda(f) = \begin{cases} 
\deg p & \text{if } f = p^\alpha, \ \alpha \geq 1, \\
0 & \text{otherwise}. 
\end{cases}
\]

**Lemma 2.5 ([15]).** For any \( n \geq 1 \), let \( \pi(n) \) denote the number of the monic irreducible polynomials in \( \mathbb{F}_q[x] \) of degree \( n \). Then, it follows that

\[
\sum_{\substack{f \text{ monic} \ \deg f = n \ \text{and} \ \deg f = f \equiv l(k)}} \Lambda(f) = \sum_{d|n} d\pi(d) = q^n.
\]

3. Proof of the Theorem 1.5

3.1. Some auxiliary lemmas

The essential idea of ours is using \( \Lambda(f) \) to sieve the power of irreducible polynomials. We consider the following summation

\[
\sum_{\substack{f \text{ monic} \ \deg f = n \ \text{and} \ \deg f = f \equiv l(k)}} \Lambda(f).
\]
Summing it by two ways, one is to show it is very close to \( n\pi(l, k; n) \) and the other is to show that the summation is independent of \( l \in (\mathbb{F}_q[T]/(k))^\times \). Therefore, we can accomplish Theorem 1.5.

We list some lemmas which are useful in our proof.

**Lemma 3.1.** Let \( \pi(n) \) denote the number of the monic irreducible polynomials in \( \mathbb{F}_q[x] \) of degree \( n \). Then, it follows that

\[
\pi(n) = q^n + O\left( q^{n\Omega(n)} \right).
\]

**Proof.** At first, from Lemma 2.5 it follows that

\[
\pi(n) = \frac{1}{n} \sum_{d|n} \mu(d) q^{\frac{n}{d}},
\]

where \( \mu(d) \) is the usual Möbius function defined on \( \mathbb{N} \). Therefore, it follows that

\[
\pi(n) - q^n = \frac{1}{n} \sum_{d|n \neq 1} \mu(d) q^{\frac{n}{d}}.
\]

Note that it suffices to consider the case when \( n \) has at least three different prime divisors. We may assume that \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} \) with \( 2 \leq p_1 < \cdots < p_t \) and \( t \geq 3 \) as well as \( \alpha_i \geq 1 \) for \( 1 \leq i \leq t \).

We denote \( j = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} p_3^{\alpha_3} \cdots p_t^{\alpha_t} \), then \( \Omega(n) = p_1 \) and

\[
\left| \frac{1}{n} \sum_{d|n \neq 1} \mu(d) q^{\frac{n}{d}} \right| \leq \frac{q^{n\Omega(n)}}{n} + q^{jp_1} \leq \frac{2q^{n\Omega(n)}}{n}.
\]

Note that, the last inequality comes from the following,

\[
nq^{jp_1} \leq j^3 q^{jp_1} \leq q^{j(p_1+2)} \leq q^{jp_2}.
\]

This completes the proof. \( \square \)

**Lemma 3.2.** For any polynomials \( l, m, k \) in \( \mathbb{F}_q[x] \) satisfying \( (l, k) = (m, k) = 1 \), we have

\[
\sum_{f \text{ monic}} 1 = \left\{ \begin{array}{cc} q^{n - \deg k - \deg m} & \text{if } \deg m \leq n - \deg k, \\
O(1) & \text{if } \deg m > n - \deg k. \end{array} \right.
\]

**Proof.** We have

\[
\sum_{f \text{ monic}} 1 = \sum_{g \text{ monic}} 1,
\]

since \( (l, k) = (m, k) = 1 \) and there is an \( \tilde{m} \in (\mathbb{F}_q[T]/(k))^\times \) such that \( \tilde{m}m \equiv 1(\text{mod } k) \). Moreover, we may assume that \( \deg l \tilde{m} < \deg k \) and \( g = l \tilde{m} + hk \).
On one hand, when $\deg m \leq n - \deg k$, we have
\[ \sum_{\substack{g \text{ monic} \\ \deg g = n - \deg m \\ g \equiv \tilde{m}(k)}} 1 = \sum_{\substack{h \text{ monic} \\ \deg h = n - \deg m - \deg k}} 1 = q^{n-\deg k-\deg m}. \]

On the other hand, when $\deg m > n - \deg k$, we have $h = 0$. Therefore,
\[ \sum_{\substack{g \text{ monic} \\ \deg g = n - \deg m \\ g \equiv \tilde{m}(k)}} 1 \leq 1 \]
for fixed polynomials $l, k$, that is to say
\[ \sum_{\substack{g \text{ monic} \\ \deg g = n - \deg m \\ g \equiv \tilde{m}(k)}} 1 = \sum_{\substack{g \text{ monic} \\ \deg g = n - \deg m \\ g \equiv \tilde{m}(k)}} 1 = O(1). \]

\[ \square \]

**Remark 3.3.** If $\deg l\tilde{m} < \deg k$ does not hold, we can always get $l' \equiv \tilde{m} (\text{mod } k)$ satisfying $\deg l' < \deg k$.

We also introduce the following $L$-series which plays an important role in our proof
\[ L_k(s, \mu) := \sum_{f \text{ monic} \atop (f, k) = 1} \frac{\mu(f)}{|f|^s} := \sum_{N=0}^{\infty} \frac{H(n, k)}{q^n s}, \]
and we list some of the following related results.

**Lemma 3.4.** For $\Re(s) > 1$, we have that
\[ L_k(s, \mu) = L(s, \mu) \prod_{p | k} \frac{1}{1 - |p|^{-s}}. \]

**Proof.** It follows from Lemma 2.3.

**Lemma 3.5.** Suppose that for a monic irreducible polynomial $k(x) \in F_q[x]$ with $\deg k \geq 2$, we have
\[ H(n, k) = \begin{cases} 1 & \text{if } n \equiv 0 (\text{mod } \deg k), \\ -q & \text{if } n \equiv 1 (\text{mod } \deg k), \\ 0 & \text{otherwise}. \end{cases} \]

**Proof.** Applying Taylor expansion of both sides of the formula in Lemma 3.4 together with Lemma 2.3, we can obtain the result by comparing coefficients of both sides.

**Lemma 3.6.** Let $k(x) = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$, where $p_1, p_2, \ldots, p_t$ are irreducible polynomials over $F_q$ and $t$ is a positive integer, then we have
\[ |H(n, k)| \leq qn^{t-1} \]
for $n \in \mathbb{N}$.
Proof. We prove this lemma by induction on $t$. First, when $t=1$, we can verify the result easily by Lemma 3.5. We assume that the desired result holds for $t = r$, then we consider the case when $t = r + 1$. Since $k(x) = p_1^{a_1}p_2^{a_2} \ldots p_r^{a_r}$ and $k'(x) := p_1^{a_1}p_2^{a_2} \ldots p_r^{a_r}$, by Lemma 3.4 together with the Taylor expansion and the uniform convergence of $L$-series, and setting $u = q^{-s}$, we get that

$$L_k(s, \mu) = L_{p_1^{a_1}p_2^{a_2} \ldots p_r^{a_r}}(s, \mu)$$

$$= L_{p_1^{a_1}p_2^{a_2} \ldots p_r^{a_r}}(s, \mu)$$

$$= L_{k'}(s, \mu) \left( 1 - \frac{1}{|p|^s} \right)^{-1}$$

$$= \sum_{n=0}^{\infty} u^{n \deg p} \sum_{j=0}^{\infty} H(j, k') u^j = \sum_{n=0}^{\infty} \left( \sum_{n=\deg p + j} H(j, k') \right) u^n.$$  

Comparing the coefficients of above results, one obtains

$$H(n, k) = \sum_{n=\deg p + j} H(j, k').$$

Therefore, by our assumption, we can get that

$$|H(n, k)| \leq \sum_{n=\deg p + j} |H(j, k')| \leq \sum_{d=0}^{n} qj^{r-1} \leq n \cdot qn^{r-1} = qn^r.$$  

This completes the proof. \qed

Lemma 3.7. For any fixed polynomial $k \in \mathbb{F}_q[x]$ and $l \in (\mathbb{F}_q[T]/(k))^{\times}$ with given $n \in \mathbb{N}$, there exists a function $F(k, n) : \mathbb{F}_q[x] \to \mathbb{R}$ which is independent of $l$ such that

$$\sum_{f \text{ monic} \atop \deg f = n \atop f \equiv l(k)} \Lambda(f) = F(k, n) + O(n^\alpha),$$

where the fixed $\alpha$ depends on $k(x) \in \mathbb{F}_q[x]$.

Proof. Using the Möbius transform of the von Mangolt function, we get

$$\sum_{f \text{ monic} \atop \deg f = n \atop f \equiv l(k)} \Lambda(f) = - \sum_{f \text{ monic} \atop \deg f = n \atop f \equiv l(k)} \mu(m) \deg m$$

$$= - \sum_{m \text{ monic} \atop \deg m \leq n \atop (m,l)=1} \mu(m) \deg m \sum_{f \text{ monic} \atop \deg f = n \atop f \equiv l(k)} 1,$$

where $(l, k) = 1$. Therefore, using Lemmas 3.2 and 3.6 we obtain the following formula

$$\sum_{f \text{ monic} \atop \deg f = n \atop f \equiv l(k)} \Lambda(f) = - \sum_{d=0}^{n-\deg k} dq^{n-d} H(d, k) + O(n^\alpha).$$
Taking

\[ F(k, n) := - \sum_{d=0}^{n - \deg k} dq^{n - \deg k} H(d, k) \]

and the desired result follows. \( \Box \)

**Lemma 3.8.** For any fixed polynomial \( k(x) \in \mathbb{F}_q[x] \) and fixed \( l(x) \) such that \( (l, k) \neq 1 \), we have the following upper bound

\[ \sum_{f \text{ monic} \atop \deg f = n \atop f \equiv l(k)} \Lambda(f) \leq \deg k. \]

**Proof.** It is obvious that the formula \( \sum_{f \text{ monic} \atop \deg f = n \atop f \equiv l(k)} \Lambda(f) \) does not vanish only if \( (l, k) = p^\beta_0 \), where \( p_0(x) \) is a monic irreducible polynomial and \( \beta \) is a positive integer. Then we have

\[ \sum_{f \text{ monic} \atop \deg f = n \atop f \equiv l(k)} \Lambda(f) = \sum_{p \text{ monic} \atop \deg p = n \atop p^\alpha \equiv l(k)} \deg p \leq \deg p_0 \leq \deg k. \]

\( \Box \)

### 3.2. Proof of the main result

In this section, we shall prove Theorem 1.5.

**Proof of Theorem 1.5.** On one hand, by Definition 2.4, Lemmas 2.5 and 3.1, we have the first equation

\[ \sum_{f \text{ monic} \atop \deg f = n \atop f \equiv l(k)} \Lambda(f) = \sum_{p \text{ monic} \atop \deg p = n \atop p^\alpha \equiv l(k)} \deg p = n \sum_{p \text{ monic} \atop \deg p = n \atop p^\alpha \equiv l(k)} 1 + O \left( \sum_{p \text{ monic} \atop \deg p = n \atop p^\alpha \equiv l(k)} \deg p \right) \]

\[ = n \pi(l, k; n) + O(q^n - n \pi(n)) = n \pi(l, k; n) + O \left( q^{\frac{n}{\Omega(n)}} \right). \]

Therefore, applying Lemma 3.4, we get that

\[ F(k, n) = n \pi(l, k; n) + O \left( q^{\frac{n}{\Omega(n)}} \right) + O(n^\alpha). \]

Then, let \( l \) run over the ring \( \mathbb{F}_q[x]/(k) \), by Lemma 3.8, we get that

\[ \sum_{l \in \mathbb{F}_q[x]/(k)} \sum_{f \text{ monic} \atop \deg f = n \atop f \equiv l(k)} \Lambda(f) = \sum_{l \in \mathbb{F}_q[x]/(k)} \sum_{f \text{ monic} \atop \deg f = n \atop f \equiv l(k)} \Lambda(f) + \Phi(k)F(k, n) + O(n^\alpha) \]

\[ = \Phi(k)F(k, n) + O(1) + O(n^\alpha) \]

\[ = \Phi(k)n \pi(l, k; n) + O(n^\alpha) + O \left( q^{\frac{n}{\Omega(n)}} \right). \]
On the other hand, by Lemma 2.5, we have that
\[
\sum_{l \in \mathbb{F}_q[x]/(k)} \sum_{\text{monic } f, \deg f = n, f \equiv l(k)} \Lambda(f) = \sum_{\text{monic } f, \deg f = n} \Lambda(f) = q^n.
\]

Finally, combining the above two equalities, we obtain that
\[
\pi(l, k; n) = \frac{1}{\Phi(k)} \frac{q^n}{n} + O\left(\frac{n^\alpha}{n}\right) + O\left(\frac{\Lambda(n)}{n}\right).
\]

This completes the proof. \(\square\)

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