The twisted tensor product of dg categories and a contractible 2-operad

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Abstract. It is well-known that the “pre-2-category” $\mathbf{Cat}_{\text{coh}}^{\text{dg}}(k)$ of small dg categories over a field $k$, with 1-morphisms defined as dg functors, and 2-morphisms defined as the complexes of coherent natural transformations, fails to be a strict 2-category. The question “What do dg categories form”, raised by V.Drinfeld in [Dr], is interpreted in this context as a question of finding a weak 2-category structure on $\mathbf{Cat}_{\text{coh}}^{\text{dg}}(k)$. In [T2], D.Tamarkin proposed an answer to this question, by constructing a contractible 2-operad in the sense of M.Batanin [Ba3], acting on $\mathbf{Cat}_{\text{coh}}^{\text{dg}}(k)$.

In this paper, we construct another contractible 2-operad, acting on $\mathbf{Cat}_{\text{coh}}^{\text{dg}}(k)$. Our main tool is the twisted tensor product of small dg categories, introduced in [Sh3]. We establish a one-side associativity for the twisted tensor product, making $(\mathbf{Cat}_{\text{coh}}^{\text{dg}}(k), \sim \otimes)$ a skew monoidal category in the sense of [S], and construct a twisted composition $\mathbf{Coh}_{\text{coh}}^{\text{dg}}(D, E) \otimes \mathbf{Coh}_{\text{coh}}^{\text{dg}}(C, D) \to \mathbf{Coh}_{\text{coh}}^{\text{dg}}(C, E)$, and prove some compatibility between these two structures. Taken together, the two structures give rise to a 2-operad $O$, acting on $\mathbf{Cat}_{\text{coh}}^{\text{dg}}(k)$. Its contractibility is a consequence of a general result of [Sh3].

1 Introduction

1.1 In this paper, we further investigate the twisted tensor product of small differential graded (dg) categories over a field $k$, which was recently introduced in [Sh3]. Recall that the twisted tensor product $C \sim \otimes D$ fulfills the adjunction

$$\text{Fun}_{\text{dg}}(C \sim \otimes D, E) \simeq \text{Fun}_{\text{dg}}(C, \mathbf{Coh}_{\text{coh}}^{\text{dg}}(D, E))$$

(1.1)

where $\text{Fun}_{\text{dg}}(-, -)$ is the set of dg functors, and $\mathbf{Coh}_{\text{coh}}^{\text{dg}}(D, E)$ is the dg category whose objects are the dg functors $f: D \to E$, and whose morphisms $f \Rightarrow g$ are given by the reduced Hochschild cochains on $D$ with coefficients in the $D^{\text{op}} \otimes D$-module $E(f(-), g(-))$. The closed elements in $\mathbf{Coh}_{\text{coh}}^{\text{dg}}(D, E)(f, g)$ are thought of as derived natural transformations from $f$ to $g$. Such derived complexes were introduced, for simplicial enrichment, by Cordier and Porter (see [CP]
and the references for earlier papers therein), and were studied for a general enrichment in [Ba1,2], [St]. Unlike for the simplicial enrichment, for the dg enrichment there is an associative (vertical) composition, making $\mathcal{C}oh_{dg}(-,-)$ a dg category. It follows from [Fa, Th.1.7] that, for $D$ cofibrant for the Tabuada closed model structure [Tab], the dg category $\mathcal{C}oh_{dg}(D,E)$ is isomorphic in the homotopy category to the dg category $\mathbb{R}\text{Hom}(D,E)$ introduced by Toën in [To]. In particular, for $D, D'$ cofibrant, and $w_1: D \to D', w_2: E \to E'$ quasi-equivalences, the dg functors $w_1^*: \mathcal{C}oh_{dg}(D', E) \to \mathcal{C}oh_{dg}(D, E)$ and $w_2^*: \mathcal{C}oh_{dg}(D, E) \to \mathcal{C}oh_{dg}(D, E')$ are quasi-equivalences.

It is worthy to compare adjunction (1.1) with the adjunction proven in [To, Sect. 6]:

$$\text{Hot}(C \otimes D, E) \simeq \text{Hot}(C, \mathbb{R}\text{Hom}(D, E)) \quad (1.2)$$

where Hot stands for (the set valued external Hom in) the homotopy category of the category $\mathcal{C}at_{dg}(k)$ of small dg categories over $k$, with formally inverted quasi-equivalences.

We stress that, unlike (1.2), the adjunction (1.1) holds in the category $\mathcal{C}at_{dg}(k)$ itself, not in its homotopy category. It makes our $C \otimes D$ non-symmetric in $C$ and $D$.

One always has a dg functor $p: C \sim \otimes D \to C \otimes D$. Recall our main result in [Sh3]:

**Theorem 1.1.** For $C, D$ cofibrant, the dg functor $p: C \sim \otimes D \to C \otimes D$ is a quasi-equivalence.

We recall the construction of $C \sim \otimes D$ in Section 2.4.

### 1.2

In this paper, we construct, by means of the twisted tensor product, a homotopically final 2-operad, in the sense of Batanin [Ba3-5], which acts on the “pre-2-category” $\mathcal{C}at_{dg}^{\text{coh}}(k)$ of small dg categories over $k$, whose objects are small dg categories over $k$, whose morphisms are dg functors, and whose complex of 2-morphisms $f \Rightarrow g: C \to D$ is defined as $\mathcal{C}oh_{dg}(C, D)(f, g)$. That is, our 2-operad solves the same problem as the 2-operad constructed by Tamarkin in [T2]. However, the 2-operads are distinct; we prove in [Sh5] that our dg 2-operad is isomorphic to the normalized 2-operadic analogue of the brace operad of [BBM], and is quasi-isomorphic to (the dg condensation of) the Tamarkin 2-operad.

The pre-2-category $\mathcal{C}oh_{dg}^{\text{coh}}(k)$ fails to be a strict 2-category. In particular, we can not define the horizontal composition $\mathcal{C}oh_{dg}(D, E)(g_1, g_2) \otimes \mathcal{C}oh_{dg}(C, D)(f_1, f_2) \to \mathcal{C}oh_{dg}(C, E)(g_1 f_1, g_2 f_2)$. There are 2 candidates for such horizontal composition, and there is a homotopy between them. Therefore, the “minimal” space of “all possible horizontal compositions” as above is the complex

$$0 \to k \underset{\text{deg}=-1}{\xrightarrow{d}} k \oplus k \to 0 \quad (1.3)$$

---

1Toën proved in [To, Cor.6.4] a much stronger statement than (1.2).
where $x_1, x_2, x_{12}$ are generators in the corresponding vector spaces. Note that the cohomology of this complex is $k[0]$; therefore, “homotopically the operation is unique”. The reader is referred to Section 3.1 and to [Sh6], Sect.4.2 for more detail on the two horizontal compositions and the homotopy between them, as well as for graphical illustrations for the corresponding cochains.

1.3

One of our main new observations is existence of a canonical dg functor

\[ M : \operatorname{Coh}_{\text{dg}}(D, E) \otimes \operatorname{Coh}_{\text{dg}}(C, D) \to \operatorname{Coh}_{\text{dg}}(C, E) \]  

(1.4)

called the twisted composition, with nice properties.

It is associative in the sense specified in Theorem 1.4 below. Let $\Psi \in \operatorname{Coh}_{\text{dg}}(C, D)(f_1, f_2)$, $\Theta \in \operatorname{Coh}_{\text{dg}}(D, E)(g_1, g_2)$. Then the two horizontal compositions are $M((\Theta \otimes \text{id}_{f_2}) \ast (\text{id}_{g_1} \otimes \Psi))$ and $M((\text{id}_{g_2} \otimes \Psi) \ast (\Theta \otimes \text{id}_{f_1}))$, correspondingly, and the homotopy between them is $M(\varepsilon(\Theta; \Psi))$ (see Section 2.4 for the notations for $\sim \otimes$ used here). The morphism $\varepsilon(\Theta; \Psi_1, \ldots, \Psi_n)$ is sent by $M$ to the brace operation $\Theta \{ \Psi_1, \ldots, \Psi_n \}$.

In fact, existence of such twisted composition is a formal consequence of a closed skew monoidal category structure on $\text{Cat}_{\text{dg}}(k)$ with the skew monoidal product given by the twisted tensor product $\sim \otimes$, and with the right adjoint to $- \sim \otimes X$ given by $\operatorname{Coh}_{\text{dg}}(X, =)$, see Section 1.5. Namely, any closed skew monoidal category is enriched over itself, which gives the twisted composition map, see Proposition 3.6. The skew-monoidality of the twisted tensor product means that there is an associator

\[ \alpha_{C, D, E} : (C \sim \otimes D) \sim \otimes E \to C \sim \otimes (D \sim \otimes E) \]  

(1.5)

which is in general not an isomorphism nor a quasi-isomorphism, which fulfills the hexagon and unit axioms.

One can iterate the twisted composition, and get a canonical dg functor

\[ M : \operatorname{Coh}_{\text{dg}}(C_{k-1}, C_k) \otimes (\operatorname{Coh}_{\text{dg}}(C_{k-2}, C_{k-1}) \otimes (\ldots \otimes \operatorname{Coh}_{\text{dg}}(C_0, C_1) \ldots)) \to \operatorname{Coh}_{\text{dg}}(C_0, C_k) \]  

(1.6)

Let $n_1, \ldots, n_k \geq 1, k \geq 1$. It defines a 2-globular pasting diagram $D = (n_1, \ldots, n_k)$, see Figure 1. Assume we are given dg categories $C_0, \ldots, C_k$, dg functors $\{ F_{i,j} : C_{i-1} \to C_i \}_{i=1 \ldots k}$, and elements $\{ \Psi_{ij} : F_{ij} \Rightarrow F_{i,j+1} \in \operatorname{Coh}_{\text{dg}}(C_{i-1}, C_i) \}_{i=1 \ldots k, j=0 \ldots n_i}$.

One imagines this data by the following 2-globular pasting diagram, augmented by the 2-arrows $\Psi_{ij} : F_{i,j-1} \to F_{ij}$.

We want to find the most general composition of all $\{ \Psi_{ij} \}$ which should be an element in $\operatorname{Coh}_{\text{dg}}(C_0, C_k)(F_{\text{min}}, F_{\text{max}})$ where $F_{\text{min}} = F_{k,0} \circ \ldots \circ F_{1,0}$, $F_{\text{max}} = F_{k,n_k} \circ \ldots \circ F_{1,n_1}$. In virtue
of the map (1.6), we have to “cook up” an element in the l.h.s. of it, by the coherent 2-arrows \{\Psi_{ij}\}, and apply \(M\) to this element. The question is than how to produce such elements.

Let \(i_n\) be the ordinary category, having objects \(0, 1, \ldots, n\), and a unique morphism \(i_{ij}\) for any \(i \leq j\). Denote by \(I_n\) the \(k\)-linear category \(I_n = k[i_n]\), by \(\{e_1, \ldots, e_n\}\) its generators, \(e_j \in I_n(j - 1, j)\). Define

\[
I_{n_1, \ldots, n_k} = I_{n_k} \otimes (I_{n_k-1} \otimes (\cdots \otimes (I_{n_2} \otimes I_{n_1}) \cdots))
\]

(1.7)

\[
O(n_1, \ldots, n_k) = I_{n_1, \ldots, n_k} (\text{min, max})
\]

(1.8)

where

\[
\text{min} = (0, 0, \ldots, 0), \quad \text{max} = (n_1, n_2 - 1, \ldots, n_1)
\]

Note that \(O(n) = k[0]\), and \(O(1, 1)\) is exactly the complex (1.3).

Having an element in \(O(n_1, \ldots, n_k)\), we can plug \(\Psi_{ij}\) for the place of the corresponding generator \(e_j\) of \(I_n\), and get in an element in the l.h.s. of (1.6). Moreover, it is the most general “canonical” way of doing it.

One can similarly plug elements of \(O(D)\) for the place of generators of some other \(O(D')\), which makes the 2-collection \(\{O(D)\}\) a 2-operad.

A 2-operad in \(\text{Vect}_{\text{dg}}(k)\) is given by a complex of \(k\)-vector spaces \(O(D)\), for any 2-globular pasting diagram \(D = (n_1, \ldots, n_k)\). These complexes are subject to the 2-operadic associativity, see Section B.3.

We can state our main result:

**Theorem 1.2.** The complexes of \(k\)-vector spaces \(O(D), D = (n_1, \ldots, n_k)\) are the components of a 2-operad acting on \(\text{Cat}_{\text{dg}}^{\text{coh}}(k)\). The 2-operad \(O\) is homotopically final, that is, there is a map of complexes \(p: O(D) \rightarrow k[0]\) which is a quasi-isomorphism of complexes, for any \(D\), which is compatible with the 2-operadic composition.

\[\text{more precisely, a 1-terminal 2-operad} \quad [Ba4]\]
We briefly recall the main definitions related to 2-operads in Appendix B. According to Batanin [Ba3] (see also Theorem B.3) an action of a homotopy trivial \( n \)-operad encodes a weak \( n \)-category.

Note that the dg categories \( I_n \) are cofibrant. As well, the twisted tensor product \( C \tilde{\otimes} D \) of two cofibrant dg categories is cofibrant, by [Sh3, Lemma 4.5]. Therefore, Theorem 1.2 is applied to \( I_{n_1,\ldots,n_k} \) and it gives a quasi-isomorphism

\[
I_{n_1,\ldots,n_k}(\min, \max) \xrightarrow{\text{quis}} (I_{n_k} \otimes \cdots \otimes I_{n_1})(\min, \max) = \mathbb{k}[0]
\]

(The statement that this map is a quasi-isomorphism seems to be quite non-trivial, we do not know any way to prove it directly). In fact, this application was our main motivation for developing of a more general theory in [Sh3].

1.4

Let us outline the constructions and results which lead to a proof of Theorem 1.2.

First of all, the “one-sided” associativity map (1.5), together with “one-sided” unit maps, make \( \mathcal{C}at_{dg}(\mathbb{k}) \) a skew-monoidal category, [S], [BL], [LS]. This skew-monoidal category is closed, with the inner Hom(\( C, D \)) given by \( \mathcal{C}oh_{dg}(C, D) \).

There is an analogue of the Mac Lane coherence theorem for skew monoidal categories, proven in loc.cit. The situation for the skew case is more complicated than its classical counterpart. Luckily, our example belongs to a special class of a skew monoidal categories, called perfect skew monoidal categories (see Definition 3.4). For this case, the coherence theorem is essentially simplified, and is exactly as simple as its classical pattern.

One has:

**Theorem 1.3.** The triple \( (\mathcal{C}at_{dg}(\mathbb{k}), \tilde{\otimes}, \alpha) \) (augmented by the unit \( \mathbb{k} \) and the unit maps) forms a perfect skew monoidal category.

The skew monoidal structure on \( (\mathcal{C}at_{dg}(\mathbb{k}), \tilde{\otimes}) \) is essentially used for the 2-operad structure on the complexes \( \mathcal{O}(D) \), defined in (1.7), (1.8). More precisely, the corresponding coherence theorem is employed to establish the 2-operadic associativity and unit identities.

After that, we establish a compatibility between the skew monoidal structure given by \( \alpha \) with the twisted composition dg functor

\[
M : \mathcal{C}oh_{dg}(D, E) \tilde{\otimes} \mathcal{C}oh_{dg}(C, D) \to \mathcal{C}oh_{dg}(C, E)
\]
where $C, D, E \in \mathcal{C}_{at_{\text{dg}}}$. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{C}_{oh}(C, D) \otimes \mathcal{C}_{oh}(B, C) & \otimes \mathcal{C}_{oh}(A, B) & \mathcal{C}_{oh}(C, D) \otimes (\mathcal{C}_{oh}(B, C) \otimes \mathcal{C}_{oh}(A, B)) \\
M \otimes \text{id} & \xrightarrow{\alpha} & \text{id} \otimes M \\
\mathcal{C}_{oh}(B, D) \otimes \mathcal{C}_{oh}(A, B) & \xrightarrow{=} & \mathcal{C}_{oh}(B, D) \otimes \mathcal{C}_{oh}(A, C) \\
M & \xrightarrow{=} & M \\
\mathcal{C}_{oh}(A, D) & \xrightarrow{=} & \mathcal{C}_{oh}(A, D) \\
\end{array}
\]

(1.9)

**Theorem 1.4.** There exists a twisted composition (1.4) which is associative in the sense that diagram (1.9) commutes.

This theorem is translated to an action of the 2-operad $\mathcal{O}$ on $\mathcal{C}_{at_{\text{coh}}}$.  

1.5

Existence of the twisted composition $M$ which fulfils (1.9) follows from a general categorical consideration, as we now explain.

Assume $\mathcal{C}$ is a skew monoidal category, and there is an inner Hom $[-, -]: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$, such there there is an adjunction

\[
\mathcal{C}(X \otimes Y, Z) \simeq \mathcal{C}(X, [Y, Z])
\]

(1.10)

Then one can translate the commutative diagrams in the definition of a skew monoidal category to commutative diagrams build up entirely in terms of the inner product $[-, -]$ and the corresponding maps. The resulting concept is called a skew closed category, and was introduced in [S]. Moreover, in presence of adjunction (3.15) the two structures are equivalent, as it is shown in [loc.cit., Sect. 8].

An advantage of this approach is that the compatibility such as (1.9) is a consequence of the above-mentioned general set-up.

In this way, the twisted composition is not a new construction, but a consequence of the adjunction (1.1) on $\mathcal{C}_{at_{\text{dg}}}$ and of the skew-monoidal structure $\otimes$ on $\mathcal{C}_{at_{\text{dg}}}$, and (1.9) and Theorem 1.4 hold automatically. More precisely, the adjunction (3.15) gives morphism

\[
[X, Y] \otimes X \to Y
\]

adjoint to the identity morphism, and

\[
([Y, Z] \otimes [X, Y]) \otimes X \to [Y, Z] \otimes ([X, Y] \otimes X) \to [Y, Z] \otimes Y \to Z
\]

gives, by the adjunction, the twisted composition. The associativity (1.9) and the unit axiom are consequences of the corresponding properties for $\otimes$, likewise for the classical case of closed monoidal category [Ke, Sect. 1.6].
1.6

When the paper had been completed the author learned about the papers [BW], [BCW] devoted to closed questions for the Gray tensor product of bicategories. The papers loc.cit. shed some light on our constructions and indicate how they can be performed in general. Below we provide few remarks on these methods in their application to our question.

First of all, we can consider $\mathcal{C}at_{dg}(\mathbb{k})$ as a 1-category enriched in $(\mathcal{C}at_{dg}(\mathbb{k}), \otimes)$. So the question this paper is devoted to is to translate a strict 1-category enriched in $(\mathcal{C}at_{dg}(\mathbb{k}), \otimes)$ to a weak 1-category enriched in $(\mathcal{C}at_{dg}(\mathbb{k}), \otimes)$ where $\otimes: \mathcal{C}at_{dg}(\mathbb{k}) \times \mathcal{C}at_{dg}(\mathbb{k}) \to \mathcal{C}at_{dg}(\mathbb{k})$ is the conventional tensor product of dg categories. Here “weak 1-category” is understood as an algebra over a homotopy final 1-operad in $\mathcal{C}at^*(\mathbb{k})$.

Secondly, the same methods are applied to any strict 1-category enriched in $(\mathcal{C}at_{dg}(\mathbb{k}), \otimes)$, not only to $\mathcal{C}at_{dg}^{coh}(\mathbb{k})$. Such categories are analogues of “Gray categories” in loc.cit. In particular, our results provide a proof of the latter statement, without any changes.

Finally, the methods of loc.cit. allow us to rephrase our constructions to make them valid for any closed skew monoidal category $(\mathcal{V}, \otimes)$ and any strict 1-category enriched in $\mathcal{V}, \otimes)$ (provided $\otimes$) can be “lifted” to a multitensor on dg 1-graphs, see below. In this way, we avoid use of rather “non-canonical” constructions such as “substitutions” in the definition of the 2-operadic composition for the the twisted tensor product 2-operad, and in its action on $\mathcal{C}at_{dg}^{coh}(\mathbb{k})$.

The key observation, from which the operad structure on the 2-collection $\mathcal{O}$ and its action on $\mathcal{C}at_{dg}^{coh}(\mathbb{k})$ follow formally, is possibility to redefine the twisted tensor product at the globular level. That is, there is a multitensor $E_\otimes$ (a lax-monoidal structure [BW]) on the category of dg 1-graphs so that any strict 1-category enriched in $(\mathcal{C}at_{dg}(\mathbb{k}), \otimes)$ is the same that a 1-category enriched in $E_\otimes$. Next, one constructs a monad called $\Gamma E_\otimes$ in loc.cit., acting on dg 2-graphs from the multitensor $E_\otimes$, and the operad $\mathcal{O}$ is encoded in this monad. In loc.cit. the authors extensively use the cartesianity of the monad $\Gamma E_\otimes$ to “decode” the operad from it. Note that these ideas can’t be applied to dg case directly, as we deal with non-cartesian monads and non-cartesian multitensors, though a suitable refinement of them can certainly be applied, and it improves and generalises our results. More precisely, the cartesianity is replaced by a grading by the set-enriched 2-level graphs which agree with the monad compositions. We hope to discuss these ideas in detail elsewhere.

1.7 Organisation of the paper

Below we outline the contents of the individual Sections.

In Section 2, we recall the definition and the basic properties of coherent natural transformations $\mathcal{C}oh(F, G)$ for $F, G: C \to D$ dg functors, define the dg category $\mathcal{C}oh_{dg}(C, D)$, and recall the construction of the twisted tensor product $C \otimes D$ from [Sh3], as well as its basic properties.
We also define complexes $O(n_1, \ldots, n_k)$ and prove their contractibility. Later in Section 4 we show that $O(n_1, \ldots, n_k)$ are the components of a 2-operad $O$.

In Section 3 we construct a one-side associativity map for $(\mathcal{Cat}_{dg}(\mathbb{k}), \otimes)$. We show that the structure we get fulfills the axioms of a skew monoidal category, [S], [LS], [BL]. Moreover, this skew monoidal category is perfect, see Definition 3.4. We prove in Proposition 3.5 that the coherence theorem for perfect skew-monoidal categories is exactly analogous to the classical MacLane coherence for monoidal categories, and is therefore much simpler than the general case of loc.cit. Moreover, we define skew closed categories [S] and discuss the general categorical setup of Section 1.5. It gives rise to an explicit expression for the twisted composition $M$ and to a proof of Theorem 1.4.

In Section 4, we apply the toolkit developed in Section 3 to a proof of Theorem 1.2. We equip the collection of complexes $O(n_1, \ldots, n_k)$ with a structure of a 2-operad $O$, show that this 2-operad is contractible, and that it acts on the pre-2-category $\mathcal{Cat}_{dg}^{coh}(\mathbb{k})$.

Appendix A contains detailed proofs of two technical Theorems 3.1 and 3.7.

We briefly recall some basic definitions related to higher operads in Appendix B.

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2 The twisted tensor product and a 2-operad $O$

Here we recall basic facts on the twisted tensor product $C \widehat{\otimes} D$ of small dg categories, introduced in [Sh3]. After that, we define complexes $O(n_1, \ldots, n_k)$, by means of the twisted tensor product:

$$O(n_1, \ldots, n_k) = I_{n_k} \otimes (I_{n_{k-1}} \otimes (\ldots \otimes (I_{n_2} \otimes I_{n_1}) \ldots))(\min, \max)$$

where $I_n$ is (the $k$-linear span of) the length $n$ interval category, and min (resp., max) is the minimal (resp., the maximal) object. Later in Section 4 we prove that these complexes are the components of a 2-operad $O$. Here we show that each of these complexes is canonically quasi-isomorphic to $k[0]$. 

8
2.1 Coherent Natural Transformations

We recall the definition of a coherent natural transformation $F \Rightarrow G: C \rightarrow D$, where $C, D$ are small dg categories over $k$, and $F, G$ are dg (resp., $A_\infty$) functors $C \rightarrow D$.

Let $C, D \in \mathcal{Cat}_{dg}(k)$, and let $F, G: C \rightarrow D$ be dg functors. Associate with $(F, G)$ a cosimplicial set $\text{coh}(F, G)$, as follows.

Set

$$\text{coh}_0(F, G) = \prod_{X \in C} \text{Hom}_D(F(X), G(X))$$

and

$$\text{coh}_n(F, G) = \prod_{X_0, X_1, \ldots, X_n \in C} \text{Hom}_k\left(C(X_{n-1}, X_n) \otimes \cdots \otimes C(X_0, X_1), D(F(X_0), G(X_n))\right)$$

(2.1)

where $\text{Hom}_k$ is the internal Hom in the category of complexes over $k$.

The coface maps $d_0, \ldots, d_{n+1}: \text{coh}_n(F, G) \rightarrow \text{coh}_{n+1}(F, G)$

and the codegeneracy maps $\eta_0, \ldots, \eta_n: \text{coh}_{n+1}(F, G) \rightarrow \text{coh}_n(F, G)$

are defined in the standard way, see e.g. [T2, Sect. 3].

For example, recall the coface maps $d_0, d_1, d_2: \text{coh}_1(F, G) \rightarrow \text{coh}_2(F, G)$. For

$$\Psi \in \prod_{X_0, X_1 \in C} \text{Hom}_k(\text{Hom}_C(X_0, X_1), \text{Hom}_D(F(X_0), G(X_1)))$$

one has:

$$d_0(\Psi)(X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2) = G(X_1 \xrightarrow{f_2} X_2) \circ \Psi(X_0 \xrightarrow{f_1} X_1)$$

$$d_1(\Psi)(X_0 \xrightarrow{f} X_1 \xrightarrow{g} X_2) = \Psi(X_0 \xrightarrow{f \circ g} X_2)$$

$$d_2(\Psi)(X_0 \xrightarrow{f} X_1 \xrightarrow{g} X_2) = \Psi(X_1 \xrightarrow{f_2} X_2) \circ F(X_0 \xrightarrow{f_1} X_1)$$

(2.2)

One defines $\text{Coh}(F, H)$ as the totalization of this cosimplicial dg vector space:

$$\text{Coh}(F, G) = \text{Tot}(\text{coh}(F, G)) = \int_{n \in \Delta} \text{Hom}_k(C_\ast(k\Delta(-, n)), \text{coh}_n(F, G))$$

(2.3)

see e.g. [R, Ch.4]. Here $C_\ast(-)$ denotes the reduced Moore complex of a simplicial abelian group, and $\int_{\mathcal{C}}(-)$ denotes the end of a functor $\mathcal{C} \times \mathcal{C}^{op} \rightarrow \mathcal{E}$.

This is a topologist’s definition. It is the best one, in particular, it encodes all signs, and it makes sense for an arbitrary symmetric monoidal enrichment.
Below we unwind this definition, making it more explicit.

The definition given above is equivalent to defining $\mathfrak{coh}(F, G)$ the total complex of the bicomplex, one of whose differentials is the differential $\delta_{dg}$ (coming from the differentials on the underlying dg vector spaces), and another differential is the cochain differential with the corrected signs:

$$\delta = \varepsilon_0 d_0 - \varepsilon_1 d_1 + \varepsilon_2 d_2 - \cdots + (-1)^{n+1} \varepsilon_{n+1} d_{n+1} : \text{coh}_n(F, G) \to \text{coh}_{n+1}(F, G) \quad (2.4)$$

Here $\varepsilon_i = \pm 1$ are sign corrections, depending on degrees of the cochains and of the arguments, see (2.7) below. We denote the cochain complex with the corrected signs by $\tilde{C}^\bullet(-)$. Then

$$\mathfrak{coh}(F, G) = \text{Tot}_{\Pi}(\tilde{C}^\bullet(\text{coh}_\bullet(F, G))) \quad (2.5)$$

where $\text{Tot}_{\Pi}(-)$ stands for the total product complex of a bicomplex, with the differential

$$\delta_{\text{tot}}(\Psi) = (-1)^{|\Psi|_{(0)} \delta + \delta_{dg}}$$

We have the following formulas for the differentials (the reader is referred to [Sh6], Appendix D for a discussion of the signs):[^1]

$$\delta_{dg}(\Psi)(f_n, \ldots, f_1) = d_{dg}(\Psi(f_n, \ldots, f_1)) - \sum_{i=1}^{n} (-1)^{|\Psi| + |f_n| + \cdots + |f_{i+1}| + n - i} (\Psi(f_n, \ldots, d_{dg}(f_i), \ldots, f_1) \quad (2.6)$$

where in the r.h.s. $d_{dg}$ stand for the differentials in the complexes of morphisms. The differential (2.4), for a chain of morphisms

$$X_0 \overset{f_1}{\to} X_1 \overset{f_2}{\to} X_2 \overset{f_3}{\to} \cdots \overset{f_{n-1}}{\to} X_{n-1} \overset{f_n}{\to} X_n$$

reads:

$$\left(\delta \Psi\right)(f_n, \ldots, f_1) =$$

$$(-1)^{|\Psi| + |f_n| + |\Psi| + |G(f_n)|} \circ \Psi(f_{n-1}, \ldots, f_1) + (-1)^{|\Psi| + 1 + \sum_{i=2}^{n} (|f_i| + 1)} \Psi(f_n, \ldots, f_2) \circ F(f_1) +$$

$$\sum_{i=1}^{n-1} (-1)^{|\Psi| + \sum_{j=i+1}^{n} (|f_j| + 1)} \Psi(f_n, \ldots, f_{i+1} f_i, \ldots, f_1) \quad (2.7)$$

The reader is referred to [Sh6], Appendix D, for a detailed discussion of signs.

[^1]: Here and below we denote by $|\Psi|_0$ the degree of $\Psi$ as a linear map; the total (Hochschild) degree is $|\Psi| = |\Psi|_0 + n$, where $n$ is the number of arguments of $\Psi$. For the arguments $f_i$, we denote by $|f_i|$ its underlying grading in the complex of morphisms.
For fixed small dg categories $C, D$, one endows $\mathcal{Coh}_{dg}(C, D)$ with a dg category structure. That is, we define the \textit{vertical} composition of coherent natural transformations.

The dg category $\mathcal{Coh}_{dg}(C, D)$ has the dg functors $F: C \to D$ as its objects, and $\mathcal{Coh}_{dg}(C, D)(F, G) := \mathcal{Coh}(F, G)$ as its Hom-complexes. The composition is defined with a sign correction, as follows. For chains

\[
X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_m} X_m \xrightarrow{f_{m+1}} X_{m+1} \xrightarrow{f_{m+2}} \ldots \xrightarrow{f_{m+n}} X_{m+n}
\]

in $C$ one sets

\[
(\Psi_2 \circ_v \Psi_1)(f_{m+n}, \ldots, f_1, f_0) = (-1)^{\sum_{i=m+1}^{m+n} |f_i| + n} |\Psi_1| |\Psi_2| (f_{m+n}, \ldots, f_{m+1}) \circ \Psi_1(f_m, \ldots, f_1)
\]

One has

\[
\delta_{dg}(\Psi_2 \circ_v \Psi_1) = (\delta_{dg} \Psi_2) \circ_v \Psi_1 + (-1)^{|\Psi_2|} |\Psi_2| \circ_v \delta_{dg} \Psi_1
\]

and

\[
\delta(\Psi_2 \circ_v \Psi_1) = (\delta \Psi_2) \circ_v \Psi_1 + (-1)^{|\Psi_2|} |\Psi_2| \circ_v \delta \Psi_1
\]

2.3

The $A_\infty$ category $\mathcal{Coh}_{A_\infty}(C, D)$ has the $A_\infty$ functors $C \to D$ as its objects and $\mathcal{Coh}_{A_\infty}(C, D)(F, G) := \mathcal{Coh}(F, G)$ as its Hom-complexes, and the composition is defined similarly.

The construction of $\mathcal{Coh}_*(C, D)$ is functorial with respect to dg (corresp. $A_\infty$) functors $f: C_1 \to C$ and $g: D \to D_1$, and gives rise to dg (corresp., $A_\infty$) functors

\[
f^*: \mathcal{Coh}_*(C, D) \to \mathcal{Coh}_*(C_1, D), \quad g_*: \mathcal{Coh}_*(C, D) \to \mathcal{Coh}_*(C, D_1)
\]

where $* = \text{dg}$ (corresp., $* = A_\infty$).

The following result has a fundamental value:
Proposition 2.1. Let dg functors \( f : C_1 \to C \) and \( g : D \to D_1 \) be quasi-equivalences. Then the dg functors \( f^* : \text{Coh}_{A_\infty}(C,D) \to \text{Coh}_{A_\infty}(C_1,D) \) and \( g_* : \text{Coh}_{A_\infty}(C,D) \to \text{Coh}_{A_\infty}(C,D_1) \) are quasi-equivalences.

Proof. It is proven in [LH, Ch.8] that \( \text{Coh}_{A_\infty}(C,D) \) is bi-functorial with respect to the \( A_\infty \) functors. It follows from [LH, Theorem 9.2.0.4] that a weak equivalence can be inverted as an \( A_\infty \) morphism. The statement follows from these two results.

Corollary 2.2. Let \( C \) and \( C_1 \) be cofibrant, and \( f, g \) as above. Then the dg functors \( f^* : \text{Coh}_{dg}(C,D) \to \text{Coh}_{dg}(C_1,D) \) and \( g_* : \text{Coh}_{dg}(C,D) \to \text{Coh}_{dg}(C,D_1) \) are quasi-equivalences of dg categories.

Proof. It is proven e.g. in [Sh3], Lemma 4.2 and Corollary 4.3.

2.4 The Twisted Tensor Product

2.4.1 The construction

Let \( C \) and \( D \) be two small dg categories over \( k \). We define the twisted dg tensor product \( C \otimes_{\mathbb{C}} D \), as follows.

The set of objects of \( C \otimes_{\mathbb{C}} D \) is \( \text{Ob}(C) \times \text{Ob}(D) \). Consider the graded \( k \)-linear category \( F(C,D) \) with objects \( \text{Ob}(C) \times \text{Ob}(D) \) freely generated by \( \{ f \otimes id_d \}_{f \in \text{Mor}(C), d \in D}, \{ id_c \otimes g \}_{c \in C, g \in \text{Mor}(D)} \), and by the new morphisms \( \varepsilon(f; g_1, \ldots, g_n) \), specified below. (Below we write \( \{ C \otimes id_d \}_{d \in D} \) assuming \( \{ f \otimes id_d \}_{f \in \text{Mor}(C), d \in D} \) etc).

Let \( c_0 \to c_1 \) be a morphism in \( C \), and let

\[
d_0 \xrightarrow{g_1} d_1 \xrightarrow{g_2} \cdots \xrightarrow{g_n} d_n
\]

are chains of composable maps in \( D \). For any such chains, with \( n \geq 1 \), one introduces a new morphism

\[
\varepsilon(f; g_1, \ldots, g_n) \in \text{Hom}((c_0,d_0),(c_1,d_n))
\]

of degree

\[
\deg \varepsilon(f; g_1, \ldots, g_n) = -n + \deg f + \sum \deg g_j
\]

(2.11)

The new morphisms \( \varepsilon(f; g_1, \ldots, g_n) \) are subject to the following identities:

\( R_1 \) \( (id_c \otimes g_1) \ast (id_c \otimes g_2) = id_c \otimes (g_1 g_2) \)

\( (f_1 \otimes id_d) \ast (f_2 \otimes id_d) = (f_1 f_2) \ast id_d \)

\( R_2 \) \( \varepsilon(f; g_1, \ldots, g_n) \) is linear in each argument,

\( R_3 \) \( \varepsilon(f; g_1, \ldots, g_n) = 0 \) if \( g_i = id_y \) for some \( y \in \text{Ob}(D) \) and for some \( 1 \leq i \leq n \),

\( \varepsilon(id_x; g_1, \ldots, g_n) = 0 \) for \( x \in \text{Ob}(C) \) and \( n \geq 1 \),
(R4) for any \(c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2\) and \(d_0 \xrightarrow{g_1} d_1 \xrightarrow{g_2} \ldots \xrightarrow{g_N} d_N\) one has:

\[
\varepsilon(f_2; f_1, \ldots, g_N) = \sum_{0 \leq m \leq N} (-1)^{|f_1|(|g_{m+1}| + N - m)} \varepsilon(f_2; g_{m+1}, \ldots, g_N) \star \varepsilon(f_1; g_1, \ldots, g_m)
\]

To make \(F(C, D)\) a dg category, one should define the differential \(d\varepsilon(f; g_1, \ldots, g_n)\).

For \(n = 1\) we set:

\[
-d\varepsilon(f; g) + \varepsilon(df; g) + (-1)^{|f|} \varepsilon(f; dg) = (-1)^{|f||g|} (\text{id}_{c_1} \otimes g) \star (f \otimes \text{id}_{d_1}) - (f \otimes \text{id}_{d_1}) \star (\text{id}_{c_0} \otimes g)
\]

For \(n \geq 2\):

\[
\varepsilon(df; g_1, \ldots, g_n) = d\varepsilon(f; g_1, \ldots, g_n) - \sum_{j=1}^{n} (-1)^{|f||g_{j+1}| + \ldots + |g_{j-1}| + n - j} \varepsilon(f; g_1, \ldots, dg_j, \ldots, g_n) + (-1)^{|f| + n - 1}
\]

\[
(-1)^{|f||g_{n-1}|} (\text{id}_{c_1} \otimes g_n) \star \varepsilon(f; g_1, \ldots, g_{n-1}) + (-1)^{|f| + \sum_{i=2}^n |g_i| + 1} \varepsilon(f; g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^{|f| + \sum_{j=i+1}^n |g_j| + 1} \varepsilon(f; g_1, \ldots, gi+1 \circ gi, \ldots, g_n)
\]

We have:

**Lemma 2.3.** One has \(d^2 = 0\). The differential agrees with relations \((R_1)-(R_4)\) above.

It is clear that the twisted tensor product \(\widetilde{C \otimes D}\) is functorial in each argument, for dg functors \(C \to C'\) and \(D \to D'\).

Note that the twisted product \(\widetilde{C \otimes D}\) is not symmetric in \(C\) and \(D\).

It is not true in general that the dg category \(\widetilde{C \otimes D}\) is quasi-equivalent to \(C \otimes D\), or that these two dg categories are isomorphic as objects of \(\text{Hot}(\text{Cat}_{\text{dg}}(k))\). See Theorem 2.6 for a result on the homotopy type of \(\widetilde{C \otimes D}\).

### 2.4.2 The Adjunction

Our interest in the twisted tensor product \(\widetilde{C \otimes D}\) is motivated by the following fact:

**Proposition 2.4.** Let \(C, D, E\) be three small dg categories over \(k\). Then there is a 3-natural isomorphism of sets:

\[
\Phi: \text{Fun}_{\text{dg}}(\widetilde{C \otimes D}, E) \simeq \text{Fun}_{\text{dg}}(C; \text{Coh}_{\text{dg}}(D, E))
\]

(2.15)
In fact, the definition of $C \tilde{\otimes} D$ has been designed especially to fulfil this adjunction.

The map $\Phi$ sends $\text{id}_c \otimes D$ to a dg functor $F_c: D \to E$. Then $\Phi(\text{id}_c \otimes g) = F_c(g)$, $\Phi(f \otimes \text{id}_d) \in F_{c_0}(d) \to F_{c_1}(d)$, for $f: c_0 \to c_1$. Thus, we assign to $f: c_0 \to c_1$ a coherent natural transformation $\Phi(f) = \Phi(\varepsilon(f, - - -))$: $F_{c_0} \Rightarrow F_{c_1}$, and $\Phi(f \otimes \text{id}_d)$ is its 0-th component. Then $\Phi(\varepsilon(f; g_1, \ldots, g_s))$ is its first component.

The image $\Phi(d(\varepsilon(f; - - -))) = d_E(\Phi(\varepsilon(f, - - -)))$, the image $\Phi(\varepsilon(df; - - -)) = \delta_{\text{tot}}(\Phi(\varepsilon(f, - - -)))$ is the total differential in $\text{Coh}(D, E)(F_{c_0}, F_{c_1})$, and the summands $\Phi(\varepsilon(f; -d(\ldots)))$ are mapped to $\Phi(f)(-, d-, -)$.

Finally, (2.12) implies that the assignment $f \mapsto \Phi(\varepsilon(f, - - -))$ sends the composition in $C$ to the vertical composition in $\text{Coh}_{\text{dg}}(D, E)$.

See [Sh3, Th. 2.2] for detail.

\begin{corollary}
There is a dg functor $p_{C, D}: C \tilde{\otimes} D \to C \otimes D$, equal to the identity on objects, and sending all $\varepsilon(f; g_1, \ldots, g_s)$ with $s \geq 1$ to 0.
\end{corollary}

\begin{proof}
It can be either seen directly, or can be deduced from Proposition 2.4 and the natural dg embedding $\text{Fun}_{\text{dg}}(D, E) \to \text{Coh}_{\text{dg}}(D, E)$, along with the classic adjunction

$$\text{Fun}_{\text{dg}}(C \otimes D, E) = \text{Fun}_{\text{dg}}(C, \text{Fun}_{\text{dg}}(D, E)) \quad (2.16)$$

Here $\text{Fun}_{\text{dg}}(C, D)$ is the dg category whose objects are dg functors $C \to D$, and $\text{Fun}_{\text{dg}}(C, D)(F, G)$ is the complex of naive natural transformations.
\end{proof}

\subsection{2.4.3 The homotopy type of $C \tilde{\otimes} D$}

For general $C, D$, we do not know the homotopy type of the dg category $C \tilde{\otimes} D$. However, one has:

\begin{theorem}
Let $C, D$ be small dg categories over $k$. Assume both $C, D$ are cofibrant for the Tabuada closed model structure. Then $C \tilde{\otimes} D$ is also cofibrant and is isomorphic to $C \otimes D$ as an object of $\text{Hot}(\text{Cat}_{\text{dg}}(k))$. Moreover, the map of Corollary 2.5 is a quasi-equivalence.
\end{theorem}

A proof was given in [Sh3, Th. 2.4].

\subsection{2.5 The 2-operad $\mathcal{O}$}

Here we define dg vector spaces $\mathcal{O}(n_1, \ldots, n_k)$, $k \geq 1$, $n_1, \ldots, n_k \geq 1$. Later in Section 4 we prove that these dg vector spaces are the components of a 2-operad $\mathcal{O}$ (the reader is referred to Appendix B for definition of Batanin 2-operads).
Denote by \( I_n \) the \( k \)-linear span of the simplex category (defined over \( \text{Sets} \)), which has objects \( 0, 1, \ldots, n \), and a unique morphism in \( I_n(i, j) \) for any \( i \leq j \).

Denote
\[
I_{n_1, \ldots, n_k} = I_{n_k} \tilde{\otimes} (I_{n_{k-1}} \tilde{\otimes} (\cdots \tilde{\otimes} (I_{n_2} \tilde{\otimes} I_{n_1}) \cdots))
\]  
(2.17)

Let
\[
\text{min} = (0, 0, \ldots, 0) \quad \text{and} \quad \text{max} = (n_k, n_{k-1}, \ldots, n_1)
\]
be the two “extreme” objects of \( I_{n_1, \ldots, n_k} \).

Define
\[
\mathcal{O}(n_1, \ldots, n_k) = I_{n_1, \ldots, n_k}(\text{min}, \text{max})
\]  
(2.18)

It is a \( \mathbb{Z}_{\leq 0} \)-graded complex of vector spaces over \( k \).

**Proposition 2.7.** Let \( k, n_1, \ldots, n_k \geq 0 \). There is a map of complexes of vector spaces
\[
p_{n_1, \ldots, n_k} : \mathcal{O}(n_1, \ldots, n_k) \to k[0]
\]
which is a quasi-isomorphism.

**Proof.** The categories \( I_n \) are cofibrant, and \( C \otimes D \) is cofibrant if \( C, D \) are, by [Sh3, Lemma 4.5]. The statement is proven by induction, using this fact and Theorem 2.6 as follows. At the first step, we see that the natural projection
\[
I_{n_k} \tilde{\otimes} (I_{n_{k-1}} \tilde{\otimes} (\cdots \tilde{\otimes} (I_{n_2} \tilde{\otimes} I_{n_1}) \cdots)) \to I_{n_k} \otimes (I_{n_{k-1}} \tilde{\otimes} (\cdots \tilde{\otimes} (I_{n_2} \tilde{\otimes} I_{n_1}) \cdots))
\]
is a quasi-equivalence. Then we apply the same argument to \( I_{n_{k-1}} \tilde{\otimes} (\cdots \tilde{\otimes} (I_{n_2} \tilde{\otimes} I_{n_1}) \cdots) \) and see that the natural projection
\[
I_{n_{k-1}} \tilde{\otimes} (\cdots \tilde{\otimes} (I_{n_2} \tilde{\otimes} I_{n_1}) \cdots) \to I_{n_{k-1}} \otimes (\cdots \otimes (I_{n_2} \tilde{\otimes} I_{n_1}) \cdots)
\]  
(2.19)

We get the composition
\[
I_{n_k} \otimes I_{n_{k-1}} \otimes (\cdots \otimes (I_{n_2} \tilde{\otimes} I_{n_1}) \cdots)
\]

(to show that the second map is a quasi-equivalence make use that \( A \otimes B \xrightarrow{id_A \otimes f} A \otimes B' \) is a quasi-equivalence as soon as \( B \xrightarrow{f} B' \) is), and so on. Finally, we see that
\[
P_{n_1, \ldots, n_k} : I_{n_k} \tilde{\otimes} (I_{n_{k-1}} \tilde{\otimes} (\cdots \tilde{\otimes} (I_{n_2} \tilde{\otimes} I_{n_1}) \cdots)) \to I_{n_k} \otimes I_{n_{k-1}} \otimes \cdots \otimes I_{n_2} \otimes I_{n_1}
\]  
(2.20)

is a quasi-equivalence.

Its restriction to \( \text{Hom}(\text{min}, \text{max}) \) gives the quasi-isomorphism \( p_{n_1, \ldots, n_k} \). \( \Box \)

\footnote{In fact, the fact that \( p : \mathcal{O}(D) \to k \) is a quasi-isomorphism for each \( D \) is simpler than the general result of Theorem 2.6 and can be proven by elementary methods, as it is shown in [Sh4].}
3 One-side associativity and skew monoidal categories

Here we construct, for small dg categories $C, D, E$, an associativity dg functor

$$\alpha_{C,D,E} : (C \otimes D) \otimes E \to C \otimes (D \otimes E)$$

natural in each argument. The dg functor $\alpha_{C,D,E}$ is not an isomorphism in general. It does not give rise to a monoidal structure, therefore. The structure we get is described as a skew monoidal category, see [LS], [BL], [S]. We essentially use a coherence theorem proven in loc.cit., which substitutes the Mac Lane coherence theorem for the case of skew monoidal categories. In our example, the skew monoidal structure on $(\text{Cat}_{dg}(k), \otimes)$ is perfect, see Definition 3.4. For perfect skew monoidal categories, the coherence theorem is as simple as its classical pattern, see Proposition 3.5.

3.1 The associativity map

Theorem 3.1. For any three dg categories $C, D, E$, there is a unique dg functor

$$\alpha_{C,D,E} : (C \otimes D) \otimes E \to C \otimes (D \otimes E)$$

natural in each argument, which is the identity map on objects, and which is defined on morphisms as follows:

(i) for $f \in C$, $g \in D$, $h \in E$, $X, Y, Z$ objects of $C, D, E$ correspondingly, one has:

$$\alpha_{C,D,E}((f \ast \text{id}_Y) \ast \text{id}_Z) = f \ast (\text{id}_Y \ast \text{id}_Z)$$
$$\alpha_{C,D,E}(\text{id}_X \ast g) \ast \text{id}_Z = \text{id}_X \ast (g \ast \text{id}_Z)$$ (3.1)
$$\alpha_{C,D,E}((\text{id}_X \ast \text{id}_Y) \ast h) = \text{id}_X \ast (\text{id}_Y \ast h)$$

(ii) for $f \in C$, $g_1, \ldots, g_k \in D$, $k \geq 1$, and $Z$ an object in $E$, one has:

$$\alpha_{C,D,E}(\varepsilon(f; g_1, \ldots, g_k) \ast \text{id}_Z) = \varepsilon(f; g_1 \ast \text{id}_Z, \ldots, g_k \ast \text{id}_Z)$$ (3.2)

(iii) for $f \in C$, $g \in D$, $h_1, \ldots, h_n \in E$, $X$ an object of $C$, $Y$ an object of $D$, one has:

$$\alpha_{C,D,E}(\varepsilon(f \ast \text{id}_Y; h_1, \ldots, h_n)) = \varepsilon(f; \text{id}_Y \ast h_1, \ldots, \text{id}_Y \ast h_n)$$
$$\alpha_{C,D,E}(\varepsilon(\text{id}_X \ast g; h_1, \ldots, h_n)) = \text{id}_X \ast \varepsilon(g; h_1, \ldots, h_n)$$ (3.3)

(iv) for $f \in C$, $g_1, \ldots, g_k \in D$, $h_1, \ldots, h_N \in E$, one has:

$$\alpha_{C,D,E}(\varepsilon(\varepsilon(f; g_1, \ldots, g_k); h_1, \ldots, h_N)) = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq N \leq j_1 \leq j_2 \leq \cdots \leq j_N \leq N}} (-1)^{|g_{i_1}|} \varepsilon(f; h_1, h_{i_1+1}, \ldots, h_{j_1}, h_{j_1+1}, \ldots, h_{i_2}, \varepsilon(g_2; h_{i_2+1}, \ldots, h_{j_2}, h_{j_2+1}, \ldots)$$ (3.4)
where the sum is taken over all ordered sets \( \{1 \leq i_1 \leq j_1 \leq i_2 \leq j_2 \leq \cdots \leq j_k \leq N\} \); for the case \( j_\ell = i_\ell \), the corresponding term \( \varepsilon(g_\ell; h_{i_{\ell+1}}, \ldots, h_{j_\ell}) \) is replaced by \( g_\ell \otimes \text{id}_- \).

We provide a proof of this Theorem in Appendix A.

**Remark 3.2.** In the r.h.s. of formula (3.4), those \( A_i \) in \( \varepsilon(f; A_1, \ldots, A_M) \) which are equal to \( h_j \) are understood as \( \text{id}_Y \star h_j \), where \( \text{id}_Y \) is the identity morphism of the corresponding object in \( Y \).

### 3.2 Skew monoidal categories

As we have mentioned, the associativity dg functors \( \alpha_{C,D,E} \) are not, in general, isomorphisms. Therefore, the most plausible structure one can get out of them is not the structure of a monoidal category (where the associativity maps are isomorphisms). What we’ll arrive to is a structure of a *skew monoidal category*, see [LS], [BS], [S]. We provide here a short overview of this theory. There is a coherence theorem for skew monoidal categories, which is, however, a more complicated statement. In our example from Section 3.1, the closed monoidal category is perfect, see Definition 3.4 below. For the perfect closed monoidal categories, the coherence theorem becomes as simple as its classical counterpart, the MacLane coherence for monoidal categories (see Proposition 3.5). It is essentially used in the proof of the associativity of the operadic composition for the 2-operad \( O \), see Section 4.3.

**Definition 3.3.** Let \( \mathcal{C} \) be a category with an object \( I \) called the *skew unit*, and with a functor \( \otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) (called *skew tensor product*), and natural families of *lax constraints* having the directions

\[
\alpha_{XYZ}: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z) \quad (3.5)
\]

\[
\lambda_X: I \otimes X \to X \quad (3.6)
\]

\[
\rho_X: X \otimes I \to X \quad (3.7)
\]

subject to the following conditions:

\[
\begin{align*}
\rotatebox{90}{$\alpha_{W,X,Y,Z}$} & \quad \rotatebox{90}{$\alpha_{W,X,Y,Z}$} \\
(W \otimes X) \otimes (Y \otimes Z) & \quad W \otimes (X \otimes (Y \otimes Z)) \\
(((W \otimes X) \otimes Y) \otimes Z) & \quad W \otimes ((X \otimes Y) \otimes Z) \\
& \quad (I \otimes X) \otimes Y \\
& \quad I \otimes (X \otimes Y) \\
\end{align*}
\]

(3.8)

\[
\begin{align*}
& \quad \alpha_{W,X,Y,Z} \\
& \quad \alpha_{W,X,Y,Z} \\
(W \otimes (X \otimes Y)) \otimes Z & \quad W \otimes ((X \otimes Y) \otimes Z) \\
& \quad (I \otimes X) \otimes Y \\
& \quad I \otimes (X \otimes Y) \\
\end{align*}
\]

(3.9)
Note that $\alpha_{X,Y,Z}, \lambda_X, \rho_X$ are not assumed to be isomorphisms.

It follows from the definition that the maps

$\varepsilon^L_{X,Y}: (X \otimes I) \otimes Y \xrightarrow{\alpha_{X,I,Y}} X \otimes (I \otimes Y) \xrightarrow{id \otimes \lambda_Y} (X \otimes I) \otimes Y$

$\varepsilon^R_{X,Y}: X \otimes (I \otimes Y) \xrightarrow{id \otimes \lambda_Y} X \otimes (X \otimes I) \otimes Y \xrightarrow{\rho \otimes id} X \otimes (I \otimes Y)$

$\varepsilon_0: I \otimes I \xrightarrow{\lambda} I \xrightarrow{\rho} I \otimes I$

are idempotents, but are not equal to identity, in general.

It makes the coherence theorem for skew monoidal categories a non-trivial issue. It is proven in [LS] and is refined in [BL].

We suggest the following definition:

**Definition 3.4.** A skew monoidal category is called **perfect** if $\lambda_X, \rho_X$ are isomorphisms, for any object $X$.

One sees immediately that in a perfect skew monoidal categories the morphisms \[3.13\] are equal to identity morphisms. Indeed, they are idempotents and isomorphisms. More generally, the following Proposition shows that for a perfect skew monoidal category the classical MacLane coherence holds, what makes the situation much easier than for general skew monoidal categories, see [LS], [BL].

**Proposition 3.5.** Let $\mathcal{C}$ be a perfect skew monoidal category. Then the classical MacLane coherence theorem holds in $\mathcal{C}$. More precisely, any two morphisms between the same pair of objects, formed by successive application of $\alpha, \lambda, \rho$, are equal.
Proof. Using the morphisms $\lambda_X$ and $\rho_X^{-1}$ one gets a canonical map $p_Y$ from any monomial $Y$ in $\mathcal{C}$ in objects $X_1, \ldots, X_n \neq I$, and in several copies of the unit $I$, to a monomial in $\tilde{Y}$ in $X_1, \ldots, X_n$ without $I$. Let $f: Y \to Y'$ be an elementary morphism, that is, which is a single application of either $\alpha$, or $\rho_X, \lambda_X, \rho_X^{-1}, \lambda_X^{-1}$. It induces an elementary morphism $\tilde{f}: \tilde{Y} \to \tilde{Y}'$, which is $\alpha$ or id when $f = \alpha$, and which is the identity morphism if $f$ is $\lambda_X, \rho_X$ or its inverse.

It follows from (3.9)-(3.12) that for any elementary morphism $f$ the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & Y' \\
\downarrow{p_Y} & & \downarrow{p_{Y'}} \\
\tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{Y}'
\end{array}
$$

(3.14)

is commutative (and the vertical maps are isomorphisms).

Now assume we are given two morphisms from $Y_1$ to $Y_N$ each of which is a composition of elementary morphisms. We get two maps from $\tilde{Y}_1$ to $\tilde{Y}_N$, each of which is a composition of several $\alpha$s. The MacLane coherence holds for a non-unital monoidal category $\mathcal{C}$ in which the associator is not necessarily isomorphisms, see [LS, Prop. 7.1] and [ML1]. It follows that the two morphisms $\tilde{Y}_1 \to \tilde{Y}_N$ are equal. Using commutativity of diagrams (3.14), we conclude that the two morphisms $Y_1 \to Y_N$ are equal. \qed

3.3 Skew closed categories

Assume $\mathcal{C}$ is a skew monoidal category, and there is an inner Hom $[-,-]: \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{C}$, such there there is an adjunction

$$\mathcal{C}(X \otimes Y, Z) \simeq \mathcal{C}(X, [Y, Z]) \quad (3.15)$$

Then one can translate the commutative diagrams in Definition 3.3 to commutative diagrams formulated entirely in terms of the inner product $[-,-]$. The resulting concept, introduced in [S], is called a skew closed category. Moreover, in presence of adjunction (3.15) the two structures are equivalent, see [S, Sect. 8].

Assume (3.15) holds, define $M_{A,B,C}: [B, C] \otimes [A, B] \to [A, C]$ as follows. First of all, there is

$$e: [A, B] \otimes A \to B$$

adjoint to $\text{id}: [A, B] \to [A, B]$. One has the composition

$$([B, C] \otimes [A, B]) \otimes A \xrightarrow{id \otimes e} [B, C] \otimes ([A, B] \otimes A) \xrightarrow{id \otimes e} [B, C] \otimes B \xrightarrow{e} C$$

whose right adjoint gives $M_{A,B,C}$.
Proposition 3.6. The composition \( M \) defined above makes a skew closed monoidal category enriched over itself. More precisely, the following commutative diagrams hold:

\[
\begin{array}{c}
(C, D) \otimes (B, C) \otimes (A, B) \\
\downarrow M_{BCD} \otimes \text{id} \\
(B, D) \otimes (A, B) \\
\downarrow M_{ABD} \\
[A, D]
\end{array} \quad \alpha \rightarrow \quad \begin{array}{c}
(C, D) \otimes (B, C) \otimes (A, B) \\
\downarrow \text{id} \otimes M_{ABC} \\
[C, D] \otimes [A, C] \\
\downarrow M_{ACD} \end{array}
\]

\[
\begin{array}{c}
(A, B) \otimes I \\
\downarrow \rho \\
(A, B)
\end{array} \quad \begin{array}{c}
\downarrow M_{AAB} \\
(A, B) \otimes [A, A] \\
\downarrow \text{id} \\
(A, B)
\end{array} \quad \begin{array}{c}
\downarrow \lambda \\
I \otimes [A, B] \\
\end{array}
\]

\[
(A, B) \otimes I \xrightarrow{\text{id} \otimes I} (A, B) \otimes [A, A] \quad \text{(3.16)}
\]

\[
(A, B) \xrightarrow{\rho} [A, B] \quad \text{(3.17)}
\]

\[
[B, B] \otimes [A, B] \xrightarrow{M_{ABB}} [A, B] \quad \text{(3.18)}
\]

Proof. See [S], Example 4 and Remark 20. It is proven similarly to the classical counterpart for closed monoidal categories [Ke], Sect. 1.6.

As it stated below in Theorem 3.7, the category \( \text{Cat}_{dg}(k, \alpha) \) is skew-monoidal, and the adjunction (3.15) holds, by (2.15). Thus it gives rise to maps \( M_{ABC} \) such that (3.16)-(3.18) hold. The map \( M \) for \( \text{Cat}_{dg}(k, \alpha) \) is called twisted composition.

3.4 The category \( \text{Cat}_{dg}(k) \) is perfect skew monoidal

Denote by \( k \) the dg category with a single object \(*\) such that \( k(*, *) = k \). Then there are the following unit maps:

\[
\lambda_C : k \sim C \rightarrow C \quad \text{(3.19)}
\]

and

\[
\rho_C : C \rightarrow C \sim k \quad \text{(3.20)}
\]

which are defined as the identity maps, due to axiom (\( R_3 \)) in Section 2.4.1.

Theorem 3.7. The category \( \mathcal{C} = \text{Cat}_{dg}(k) \) of small dg categories, equipped with the twisted product \(- \sim -\), the unit \( k \), the associativity constraints \( \alpha \), and with the natural isomorphisms \( \lambda_C \) and \( \rho_C \), \( C \in \text{Cat}_{dg}(k) \), is a perfect skew monoidal category, see Definition 3.4.
Proof. We prove this Theorem in Appendix A.

3.5 Explicit dg functor $M$, associated with the skew-associativity $\alpha$

Here we compute the twisted composition $M$ explicitly, for the case of $(\mathcal{C}at_{\text{dg}}(k), \circlearrowleft)$, Our proofs are not based on this computation, but, if we want to ask ourselves how the dg 2-operad $O$ acts on the 2-graph $\mathcal{C}at_{\text{coh}}(k)$, this explicit form is necessary. We will be rather concise here. The reader who is interested in more details is referred to the preliminary preprint version of this paper [Sh6], Section 4.

3.5.1 The unit map $e : \mathcal{C}oh_{\text{dg}}(C, D) \circlearrowleft C \to D$

The dg functor $e : \mathcal{C}oh_{\text{dg}}(C, D) \circlearrowleft C \to D$ corresponds to $\text{id}_{\mathcal{C}oh_{\text{dg}}(C, D)}$ via the adjunction (2.15).

It is given on objects in the standard way $F \circlearrowleft c \mapsto F(c)$, where $F \in \mathcal{C}oh_{\text{dg}}(C, D)$ a dg functor $F : C \to D$, $c \in \text{Ob}(C)$.

We have to specify $e$ on generating morphisms of $\mathcal{C}oh_{\text{dg}}(C, D) \circlearrowleft C$, which are (i) $\Psi \circlearrowleft \text{id}_c$, (ii) $\text{id}_F \circlearrowleft f$, (iii) $\varepsilon(\Psi ; f_1, \ldots, f_n)$, where $\Psi \in \mathcal{C}oh_{\text{dg}}(C, D)$, $c \in \text{Ob}(C)$, $F : C \to D$ a dg functor, $f$ a morphism in $C$.

One has:

$$e(\Psi \circlearrowleft \text{id}_c) = \Psi_0(c) \quad (3.21)$$

where $\Psi_0$ is 0th component of $\Psi$,

$$e(\text{id}_F \circlearrowleft f) = F(f) \quad (3.22)$$

$$e(\varepsilon(\Psi ; f_1, \ldots, f_n)) = \Psi_n(f_n \circlearrowleft \cdots \circlearrowleft f_1) \quad (3.23)$$

where $\Psi_n$ is the $n$th component of $\Psi$.

3.5.2 The twisted composition $M_{A,B,C} : \mathcal{C}oh_{\text{dg}}(B, C) \circlearrowleft \mathcal{C}oh_{\text{dg}}(A, B) \to \mathcal{C}oh_{\text{dg}}(A, C)$

By definition of $M$, the following two maps $([B, C] \otimes [A, B]) \otimes A \to C$ coincide:

$$( [B, C] \otimes [A, B]) \otimes A \xrightarrow{\varepsilon} [B, C] \otimes ([A, B] \otimes A) \xrightarrow{\text{id} \otimes \varepsilon} [B, C] \otimes B \xrightarrow{\varepsilon} C$$

and

$$( [B, C] \otimes [A, B]) \otimes A \xrightarrow{M \otimes \text{id}} [A, C] \otimes A \xrightarrow{\varepsilon} C$$

21
Denote them $t_1$ and $t_2$, correspondingly. The formulas for $\alpha$ (see Theorem 3.1), and formulas (3.21), (3.22), (3.23) for $e$ provide us with explicit map $t_1$, then we find $M$ from $t_2$.

The result is as follows.

The dg functor $M$ is defined on objects, that is, pairs of dg functors $(G, F)$ where $F: A \to B, G: B \to C$, as the composition $G \circ F$.

Determine it on the generator morphisms. They are of the following three types: (i) $\Theta \otimes \operatorname{id}_F$, (ii) $G \otimes \Psi$, (iii) $\varepsilon(\Theta; \Psi_1, \ldots, \Psi_n)$.

A morphism $\Theta \otimes \operatorname{id}_F$, for $\Theta: G_0 \Rightarrow G_1: B \to C, F: A \to B$, is mapped to

$$M(\Theta \otimes \operatorname{id}_F)(f_1, f_2, \ldots, f_n) = \Theta(F(f_1), F(f_2), \ldots, F(f_n))$$

and a morphism $\operatorname{id}_G \otimes \Psi$, for $\Psi: F_0 \Rightarrow F_1: A \to B, G: B \to C$, is mapped to

$$M(\operatorname{id}_G \otimes \Psi)(f_1, \ldots, f_n) = G(\Psi(f_1, \ldots, f_n))$$

The expression for $M(\varepsilon(\Theta; \Psi_1, \ldots, \Psi_n))$ is more complicated, of course. Denote by $\Theta \{\Psi_k, \ldots, \Psi_1\}_{i_1, \ldots, i_k}$ the cochain shown in Figure 3. Here $X_0, \ldots, X_N$ are objects of the dg category $A$, $F_0, \ldots, F_k: A \to B$ are dg functors, $G_0, G_1: B \to C$ are dg functors, $P_{si}$: $F_{i-1} \Rightarrow F_i \in \mathcal{Coh}_{\text{dg}}(A, B)(F_{i-1}, F_i)$, $\Theta: G_0 \Rightarrow G_1 \in \mathcal{Coh}_{\text{dg}}(B, C)(G_0, G_1)$. Each small arrow at the bottom line indicates a morphism $X_j \to X_{j+1}$ in $A$.

One has:

$$M(\varepsilon(\Theta; \Psi_1, \ldots, \Psi_k)) = \Theta\{\Psi_k, \ldots, \Psi_1\} = \sum_{i_1, \ldots, i_k} \pm \Theta\{\Psi_k, \ldots, \Psi_1\}_{i_1, \ldots, i_k}$$

where the signs are specified in [Sh6, Section 4.2].

![Figure 3: The cochain $\Theta\{\Psi_k, \ldots, \Psi_1\}_{i_1, i_2, \ldots, i_k}(f_N, \ldots, f_1)$](image)
M. In particular, (2.12), (2.13), (2.14) hold automatically. Note that (2.13) becomes just a generalisation of the Getzler-Jones identity [GJ]

\[ [d, \Theta\{\Psi\}] = \Psi \cup \Theta \pm \Theta \cup \Psi \]

for Hochschild cochains \( \text{Hoch}^\ast(A) \) of a \( \text{dg} \) algebra \( A \).

4 The 2-operad \( \mathcal{O} \)

We refer the reader to Appendix B for a brief and elementary account on 2-operads. For more thorough treatment, see [Ba3-5], [BM1,2].

4.1 Recall our notation

\[ I_{n_1,\ldots,n_k} = I_{n_k} \tilde{\otimes} (I_{n_{k-1}} \tilde{\otimes} (\ldots (I_{n_2} \tilde{\otimes} I_{n_1}) \ldots)) \]

Below we use a shorter form of it:

\[ I_{n_1,\ldots,n_k} = I_D \]

where \( D = (n_1,\ldots,n_k) \) is the corresponding 2-globular pasting diagram (see Section B.3).

Let \( D_1 = (n_1^1,\ldots,n_{k_1}^1), D_2 = (n_2^2,\ldots,n_{k_2}^2) \) be two 2-globular pasting diagrams. We denote

\[ [D_1, D_2] = (n_1^1,\ldots,n_{k_1}^1, n_1^2,\ldots,n_{k_2}^2) \]

the 2-globular pasting diagram obtained by the horizontal concatenation of the 2-globular pasting diagrams \( D_1 \) and \( D_2 \).

For a sequence \( D_1,\ldots,D_t \) of 2-globular pasting diagrams, we define similarly the total 2-globular pasting diagram \( [D_1,\ldots,D_t] \), so that

\[ [D_1,\ldots,D_t] = [[D_1,\ldots,D_{t-1}], D_t] \]

We denote the ordered sequence \( D_1,\ldots,D_t \) by \( D \), and use the notation

\[ I_D = I_{D_1,\ldots,D_t} = I_{D_t} \tilde{\otimes} (I_{D_{t-1}} \tilde{\otimes} (\ldots (I_{D_2} \tilde{\otimes} I_{D_1}) \ldots)) \quad (4.1) \]

Let \( D_1,\ldots,D_t \) be a sequence of 2-globular pasting diagrams. We construct a \( \text{dg} \) functor

\[ \Upsilon(D_1,\ldots,D_t): I_{D_1,\ldots,D_t} \to I_{[D_t, D_{t-1},\ldots,D_1]} \quad (4.2) \]
Let us start with the case $t = 2$. The dg functor $\Upsilon(D_1, D_2) : I_{D_1} \otimes I_{D_2} \to I_{[D_2, D_1]}$ is constructed as

$$
\begin{align*}
(I_k \sim (I_{k-1} \otimes \cdots \otimes (I_2 \otimes I_1) \cdots )) \otimes (I_m \sim (I_{m-1} \otimes \cdots \otimes (I_2 \otimes I_1) \cdots )) & \xrightarrow{\alpha} \\
I_k \sim ((I_{k-1} \otimes (I_{k-2} \otimes \cdots \otimes (I_2 \otimes I_1) \cdots )) \sim (I_m \otimes (I_{m-1} \otimes \cdots \otimes (I_2 \otimes I_1) \cdots ))) & \xrightarrow{\alpha} \\
I_k \sim ((I_{k-2} \otimes \cdots ) \otimes (I_m \otimes (I_{m-1} \otimes \cdots \otimes (I_2 \otimes I_1) \cdots ))) & \xrightarrow{\alpha} \\
\cdots & \xrightarrow{\alpha} I_{[D_2, D_1]}
\end{align*}
$$

(4.3)

It is the composition of maps, each of which is an appropriate associativity constraint $\alpha$, see Theorem 3.1.

Now we define $\Upsilon(D_1, \ldots, D_t)$ for any $t$ as the composition

$$
I_{D_1, D_2, \ldots, D_t} \to I_{[D_1, D_2, D_3, \ldots, D_t]} \to I_{[D_2, D_3, D_4, \ldots, D_t]} \cdots \to I_{[D_{t-1}, D_t]}
$$

(4.4)

of arrows each of this is given by $\Upsilon(D', D'')$.

The main technical point is that the maps $\Upsilon(D_1, \ldots, D_t)$ are subject to some associativity, which we are going to formulate.

Let $D_{t_1}^1, \ldots, D_{t_1}^k; \ldots; D_{t_k}^1, \ldots, D_{t_k}^k$ sequence of (sequences of) 2-globular pasting diagrams.

We use notation

$$
I_{D_t} = I_{D_{t_1}^1, \ldots, D_{t_k}^k}
$$

Consider

$$
I_{D_{t_1}^1, \ldots, D_{t_k}^k} := I_{D_{t_1}^k} \sim (I_{D_{t-1}^k} \otimes \cdots \otimes (I_2 \otimes I_1) \cdots ))
$$

(4.5)

As well, denote

$$
[D^i] := [D_{t_i}^i, \ldots, D_{t_i}^i]
$$

and

$$
[[D^k, \ldots, D^1]] := [[D_{t_k}^k], \ldots, [D_{t_1}^i]]
$$

There are two maps

$$
\Upsilon_1, \Upsilon_2 : I_{D_{t_1}^1, \ldots, D_{t_k}^k} \to I_{[[D^k, \ldots, D^1]]}
$$

(4.6)

They are defined as follows:

$$
\Upsilon_1 : I_{D_{t_1}^1, \ldots, D_{t_k}^k} \xrightarrow{\Upsilon} I_{[D_{t_1}^i, \ldots, D_{t_k}^i]} \xrightarrow{\Upsilon} I_{[[D^k, \ldots, D^1]]}
$$

(4.7)

and

$$
\Upsilon_2 : I_{D_{t_1}^1, \ldots, D_{t_k}^k} \xrightarrow{\Upsilon} I_{D_{t_1}^{d_1}, \ldots, D_{t_k}^{d_k}} \xrightarrow{\Upsilon} I_{[[D^k, \ldots, D^1]]}
$$

(4.8)

In (4.8), the first arrow in not literally equal to $\Upsilon(-, \ldots, -)$, but is defined similarly; we leave the details to the reader.
Proposition 4.1. In the notations as above, the two maps \( \Upsilon_1, \Upsilon_2 : I_{D^1, \ldots, D^k} \to I_{[[D^k, \ldots, D^1]]} \) are equal.

Proof. It follows from Theorem 3.7 and Proposition 3.5.

4.2

Recall the 2-sequence \( \mathcal{O} \).

For a 2-globular pasting diagram \( D = (n_1, \ldots, n_k) \), set

\[
I_{n_1, \ldots, n_k} = I_{n_k} \sim (I_{n_{k-1}} \sim (\ldots (I_{n_2} \sim I_{n_1}) \ldots))
\]

and

\[
\mathcal{O}(D) = I_{n_1, \ldots, n_k} (\text{min, max})
\]

where \( \text{min} = (0, 0, \ldots, 0) \), \( \text{max} = (n_k, \ldots, n_1) \). Recall that all \( n_i \geq 1 \).

We start with the following Lemma:

Lemma 4.2. Let \( D \) be a 2-globular pasting diagram and let \( i : D^0 \to D \) be a connected subdiagram having the same set of objects. Let \( D = (n_1, \ldots, n_k) \), \( i(D^0) = ([a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k]) \), \( 0 \leq a_i < b_i \leq n_i \), \( D^0 = (b_1 - a_1, b_2 - a_2, \ldots, b_k - a_k) \). Then

\[
I_{D^0}(\text{min, max}) = I_D(\text{min}_0, \text{max}_0)
\]

where \( \text{min}_0 = (a_k, a_{k-1}, \ldots, a_1) \), \( \text{max}_0 = (b_k, b_{k-1}, \ldots, b_0) \).

Proof. It is clear. More generally, the embedding \( I_{D^0} \to I_D \) is fully faithful.

More generally, let \( D_1, \ldots, D_n \) be 2-globular pasting diagrams, \( I_{D_1, \ldots, D_n} \) the dg category (4.1).

Lemma 4.3. Let \( D_1, \ldots, D_n \) be 2-globular pasting diagram, and let \( i_k : D^0_k \to D_k \) connected subdiagrams, \( 1 \leq k \leq n \), such that \( i_k(\text{min}) = \text{min}_k^0, i_k(\text{max}) = \text{max}_k^0 \). Denote \( \text{min}_{0, \text{tot}} = (\text{min}_{0, n}, \ldots, \text{min}_{0, 1}) \), \( \text{max}_{0, \text{tot}} = (\text{max}_{0, n}, \ldots, \text{max}_{0, 1}) \). Then

\[
I_{D^0_1, \ldots, D^0_n}(\text{min, max}) = I_{D_1, \ldots, D_n}(\text{min}_{0, \text{tot}}, \text{max}_{0, \text{tot}})
\]

and the following diagram commutes:

\[
\begin{array}{ccc}
I_{D^0_1, \ldots, D^0_n}(\text{min, max}) & \xrightarrow{\tau} & I_{[D^0_1, \ldots, D^0_n]}(\text{min, max}) \\
I_{D_1, \ldots, D_n}(\text{min}_{0, \text{tot}}, \text{max}_{0, \text{tot}}) & \xrightarrow{\tau} & I_{[D_1, \ldots, D_n]}(\text{min}_{0, \text{tot}}, \text{max}_{0, \text{tot}})
\end{array}
\]
where $\Upsilon$ is the map (4.2), and the vertical maps are given by (4.10). More generally, the diagram of dg categories below commutes:

\[
\begin{array}{c}
\begin{array}{c}
I_{D_1,...,D_n}^\bigotimes \\
\Upsilon
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bigotimes I_{[D_1,...,D_n]}
\end{array}
\end{array}
\]

(4.12)

\[
\begin{array}{c}
\begin{array}{c}
I_{D_1,...,D_n} \\
\Upsilon
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
I_{[D_1,...,D_n]} \\
\bigotimes
\end{array}
\end{array}
\]

Proof. The first claim (4.10) easily follows from the fully faithfulness of the embedding $I_{D_1,...,D_n}^\bigotimes \to I_{D_1,...,D_n}$. For the commutativity of (4.12), note that the map $\Upsilon$ was defined via an iterative application of the associativity map $\alpha$. This $\alpha$ is a natural transformation, thus is functorial with respect to the dg functors (the morphisms in $\text{Cat}_\text{dg}(\k))$. It follows that $\Upsilon$ is functorial for the dg functor $I_{D_1,...,D_n}^\bigotimes \to I_{D_1,...,D_n}$. Then the commutativity of (4.11) follows from the commutativity of (4.12) and from (4.10).

Let $D_1, D_2, \ldots, D_\ell$ be 2-globular pasting diagrams, each of which has $k + 1$ vertices. Denote by $D_1 \circ \cdots \circ D_\ell$ their “vertical product”, which identifies the maximal element in $D_a(i, i + 1)$ with minimal element in $D_{a+1}(i, i + 1)$, where $0 \leq i \leq k$, $1 \leq a \leq \ell - 1$. Thus, $D_1 \circ \cdots \circ D_\ell$ has $k + 1$ vertices, and $(D_1 \circ \cdots \circ D_\ell)(i, i + 1)$ has $\sum_{s=1}^\ell \# D_s(i, i + 1) - \ell + 1$ elements.

Assume that $D_s = (n_{1s}, \ldots, n_{ks}), s = 1, \ldots, \ell$. Then

\[
D_1 \circ \cdots \circ D_\ell = \left( \sum_{s=1}^{\ell} n_{1s}, \ldots, \sum_{s=1}^{\ell} n_{ks} \right)
\]

Denote by $\min_i, \max_i$ the minimal and the maximal object of the image of embedding of $D_i$ to $D_1 \circ \cdots \circ D_\ell$. Clearly $\min_1 = \min$ and $\max_\ell = \max$. One has the following composition

\[
\mathcal{O}(D_1) \otimes \cdots \otimes \mathcal{O}(D_\ell) \simeq I_{D_1 \circ \cdots \circ D_\ell}(\min, \max) \otimes \cdots \otimes I_{D_1 \circ \cdots \circ D_\ell}(\min_1, \max_1) \to I_{D_1 \circ \cdots \circ D_\ell}(\min, \max) = \mathcal{O}(D_1 \circ \cdots \circ D_\ell)
\]

(4.13)

where the first arrow is given by Lemma 4.2 and the second one is the composition in $D_1 \circ \cdots \circ D_\ell$.

We denote the map (4.13) by $m_v$ (where the subscript $v$ stands for “vertical”).

Assume $D_1$ and $D_2$ have equal numbers of objects. we denote by $I_{D_1} \bowtie I_{D_2}$ their categorical cofibred sum (pushout) over the discrete category of objects, where the object max of $I_{D_1}$ is identified with the object min of $I_{D_2}$. We use similar notation for more irregular parenthesizing. For example, assume $D_1$ and $D_2$ have equal numbers of objects, as well as $D_3$ and $D_4$. Then we use notation $(I_{D_1} \bowtie I_{D_3}) \bowtie (I_{D_2} \bowtie I_{D_4})$, etc.

We denote by $M_v$ the dg functor

\[
M_v : I_{D_1} \bowtie I_{D_2} \to I_{D_1 \circ D_2}
\]
defined by the two embeddings $I_{D_1} \rightarrow I_{D_1 \circ D_2}$ and $I_{D_2} \rightarrow I_{D_1 \circ D_2}$. These two embeddings clearly agree on the max $\in I_{D_1}$ and min $\in I_{D_2}$, and therefore define a map from the categorical pushout.

The following lemma can be regarded as a sort of the Eckmann-Hilton compatibility. It plays an essential role in the proof of the associativity of the operadic composition in Section 4.3.

**Lemma 4.4.** Assume $D_1, D_2, D_3, D_4$ are 2-globular pasting diagrams such that the number of objects in $D_1$ is equal to the number of objects in $D_2$, as well as the numbers of objects in $D_3, D_4$ (so that $D_1 \circ D_2$ and $D_3 \circ D_4$ are defined). Denote by $D = [D_1, D_3] \circ [D_2, D_4] = [D_1 \circ D_2, D_3 \circ D_4]$.

The following diagram is commutative:

\[
\begin{array}{ccc}
(I_D \circ I_{D_3}) \circ (I_{D_2} \circ I_{D_4}) & \xrightarrow{\Upsilon \circ \Upsilon} & I_{[D_1, D_3]} \circ I_{[D_2, D_4]} \\
M_v \circ M_v & \downarrow & M_v \\
I_{D_1 \circ D_2} \circ I_{D_3 \circ D_4} & \xrightarrow{\Upsilon} & I_D
\end{array}
\]

where the vertical maps $m_v$ are maps (4.13), and the horizontal maps are \Upsilon maps (4.2).

**Proof.** The diagrams

\[
\begin{array}{ccc}
I_{D_1} \circ I_{D_3} & \xrightarrow{\sim} & I_{[D_1, D_3]} \\
\downarrow & & \downarrow \\
I_{D_1 \circ D_2} \circ I_{D_3 \circ D_4} & \xrightarrow{\sim} & I_D
\end{array}
\]

and

\[
\begin{array}{ccc}
I_{D_2} \circ I_{D_4} & \xrightarrow{\sim} & I_{[D_2, D_4]} \\
\downarrow & & \downarrow \\
I_{D_1 \circ D_2} \circ I_{D_3 \circ D_4} & \xrightarrow{\sim} & I_D
\end{array}
\]

commute by Lemma [4.3] and agree on the discrete subcategories of minimal-maximal objects. Thus they define the commutative diagram on pushouts.

In what follows we refer to Lemma 4.4 for a more general statement, with iterated horizontal and iterated vertical composition.

### 4.3 The 2-operadic Composition

Now we are ready to define the 2-operadic composition (B.1) on the 2-sequence \{\mathcal{O}_D\}.

We start with a simple example.
Consider four 2-globular pasting diagrams $D_1, D_2, D_3, D_4$, such that $D_1$ and $D_2$, as well as $D_3$ and $D_4$ have equal numbers of objects. In particular, the compositions $D_1 \circ D_2$ and $D_3 \circ D_4$ are defined. Denote by $D$ the total diagram $D = [D_1 \circ D_2, D_3 \circ D_4] = [D_1, D_3] \circ [D_2, D_4]$, see Figure 4. Denote $D_0 = (2, 2)$.

![Figure 4:](image)

We define a composition

$$\mathcal{O}(D_0) \otimes \left( \mathcal{O}(D_1) \otimes \mathcal{O}(D_2) \otimes \mathcal{O}(D_3) \otimes \mathcal{O}(D_4) \right) \to \mathcal{O}(D) \quad (4.17)$$

Assume we are given elements $\Psi_i \in \mathcal{O}(D_i)$, $1 \leq i \leq 4$, and an element $\Theta \in \mathcal{O}(D_0)$. We are going to define the composition $\Theta(\Psi_1, \Psi_2, \Psi_3, \Psi_4)$.

Consider the dg category $I_2 \hat{\otimes} I_2$. Denote by $e_1 \in I_2(0, 1)$ and $e_2 \in I_2(1, 2)$ the generators of the left copy of $I_2$, and by $e_3, e_4$ the corresponding generators of the right copy. Any morphism in $I_2 \hat{\otimes} I_2((0, 0), (2, 2))$ is a sum of monomials, each of which contains each of $e_1, e_2, e_3, e_4$ exactly 1 time.

For example, consider $(\text{id}_2 \ast e_4) \circ (\text{id}_2 \ast e_3) \circ (e_2 \ast \text{id}_0) \circ (e_1 \ast \text{id}_0)$, or $\varepsilon(e_2; e_4) \circ \varepsilon(e_1; e_3)$. The point is that an arbitrary monomial contains each $e_i$ exactly 1 time.

The element $\Theta$ is a sum of such monomials, in a unique way. For simplicity, we may assume that $\Theta$ is a single monomial. The idea is to substitute $\Psi_1$ for $e_1$, $\Psi_2$ for $e_2$, $\Psi_3$ for $e_3$, and $\Psi_4$ for $e_4$.

This substitution does not give directly an element in $\mathcal{O}(D)$, due to a “wrong” parenthesizing. The element one obtains after the substitution belongs to $(I_{D_1 \circ D_2} \hat{\otimes} I_{D_3 \circ D_4})(\min, \max)$. Then we apply the map $\Upsilon$ to it, see (4.2):

$$I_{D_1 \circ D_2} \hat{\otimes} I_{D_3 \circ D_4} \xrightarrow{\Upsilon} I_{[D_1, D_3] \circ [D_2, D_4]} = I_D$$

It gives an element in $I_D(\min, \max)$.

The general composition is similar to this example. Let $P: U \to V$ be a map of 2-globular pasting diagrams. Recall that it is defined as a dominant map $[P]: [V] \to [U]$ of the corresponding free 2-categories, see (B.1) and Section B.3. The image of $[P]$ gives a “subdivision” of $[U]$ into smaller diagrams. Namely, for each minimal ball
\( \nu \) in \([V]\), consider \( P^{-1}(\nu) := [P](\nu) \). The 2-globular pasting diagram \( U \) is divided into the union \( U = \cup_{\nu \in \mathcal{F}(V)} P^{-1}(\nu) \).

Let \( V = (m_1, \ldots, m_p) \), \( U = (n_1, \ldots, n_k) \), and let \([P](0) = d_0 = 0, [P](1) = d_1, [P](2) = d_2, \ldots, [P](p) = d_p = k\).

The set \( \{ \nu \in V(i, i+1) \} \) has a natural order, for a given \( 0 \leq i \leq p - 1 \). Let \( D_{i1}, \ldots, D_{im_i} \) be the ordered set \( \{ P^{-1}(\nu) \mid \nu \in V(i, i+1) \} \), with the corresponding order. Finally, define

\[
D^o_i := D_{i1} \circ \cdots \circ D_{im_i}
\]

(4.18)

It is clear that

\[
[D^o_1, D^o_2, \ldots, D^o_p] = U
\]

(4.19)

We have to define an operadic composition

\[
\text{Op}: \mathcal{O}(V) \otimes (\otimes_{\nu \in \mathcal{F}(V)} \mathcal{O}(P^{-1}(\nu))) \to \mathcal{O}(U)
\]

where \( \mathcal{F}(V) \) is the set of all 2-morphisms in \( V \).

An element \( \xi_{\nu} \) in \( \mathcal{O}(P^{-1}(\nu)) \) is given by a morphism in \( I_{P^{-1}(\nu)}(\text{min}, \text{max}) \).

An element \( \omega \) in \( \mathcal{O}(V) \) is uniquely a sum of monomials, in each of which each indecomposable element in \( e_{\nu} \in I_{m_j}(s, s+1), 1 \leq j \leq p, 0 \leq s \leq m_j - 1 \), occurs exactly once. We plug \( \xi_{\nu} \) in place of \( e_{\nu} \). What we get is an element in

\[
m(\omega, \{\xi_{\nu}\}) \in \left(I_{D^o_1} \otimes \cdots \otimes I_{D^o_p}\right)(\text{min}, \text{max})
\]

due to Lemma [4.3]. Although \([D^o_1, \ldots, D^o_p] = D\), the category \( I_{D^o_1} \otimes \cdots \otimes I_{D^o_p} \) differs from \( I_D \), due to another parenthesizing. The element

\[
\Upsilon(D^o_1, \ldots, D^o_p)(m(\omega, \{\xi_{\nu}\})) =: \text{Op}(\omega, \{\xi_{\nu}\})
\]

(4.20)

is, by definition, the result of our operadic composition (see (4.2)). Clearly it is linear in each argument.

**Theorem 4.5.** The operation \( \text{Op} \), defined in (4.20), fulfills the identities (i)-(iii) in Definition [B.1]. That is, it makes the 2-sequence \( \mathcal{O} \) a dg pruned reduced 1-terminal 2-operad.

**Proof.** (i) and (iii) in Definition [B.1] are clear.

Prove the identity (ii). Consider two maps of 2-globular pasting diagrams \( U \xrightarrow{P} V \xrightarrow{Q} W \), and prove the associativity for this chain. The maps above are defined via maps of strict 2-categories generated by the corresponding 2-globular sets

\[
[W] \xrightarrow{Q} [V] \xrightarrow{P} [U]
\]
Let \( W = (\ell_1, \ldots, \ell_t), V = (m_1, \ldots, m_p), U = (n_1, \ldots, n_k) \). We use notations \( D^0_{1V}, \ldots, D^0_{tV} \) for the subdiagrams of \( V \), associated with the map \( Q \) as in (4.18), and the notations \( D^0_{1U}, \ldots, D^0_{pU} \) for the subdiagrams of \( U \), similarly associated with the map \( P \). We have
\[
[D^0_{1V}, \ldots, D^0_{tV}] = V \quad \text{and} \quad [D^0_{1U}, \ldots, D^0_{pU}] = U
\]
We have also a subdivision of \( U \) into bigger diagrams, associated with the composition \( P \circ Q \). Assume that \( D^0_{1V} \) has \( a_1 + 1 \) vertices, \( \ldots, D^0_{tV} \) has \( a_t + 1 \) vertices. Taking images with \( P \), we get a subdivision
\[
D^0_{1U+} = [D^0_{3U}, \ldots, D^0_{a_1U}]
\]
\[
\ldots
\]
\[
D^0_{tU+} = [D^0_{a_1+\ldots,a_t+1U}, \ldots, D^0_{pU}]
\]
where \( D^0_{1U+}, \ldots, D^0_{tU+} \) are the diagrams associated with the composition \( P \circ Q \), as in (4.18).

Also,
\[
[D^0_{1UV}, \ldots, D^0_{tUV}] = U
\]
Denote
\[
D^0_{1U+} = (D^0_{1U}, \ldots, D^0_{a_1U})
\]
\[
\ldots
\]
\[
D^0_{tU+} = (D^0_{a_1+\ldots,a_t+1U}, \ldots, D^0_{pU})
\]
Consider elements \( \omega \in \mathcal{O}(W), \xi_{\nu} \in \{\mathcal{O}(Q^{-1}(\nu))\}_{\nu \in \mathcal{F}(W)}, \{\eta_{\mu}\} \in \mathcal{O}(P^{-1}(\mu)) \}_{\mu \in \mathcal{F}(V)} \).

Now we construct the “double substitution”, denoted by \( m(\omega; \{\xi_{\nu}\}; \{\eta_{\mu}\}) \), as follows. The element \( \omega \) is uniquely a sum of monomials in the elementary generators \( \{e_{\nu}\}_{\nu \in \mathcal{F}(W)} \). At first, we substitute the elements \( \xi_{\nu} \) in place of the corresponding generators \( e_{\nu} \). What we get, is a morphism in \( I_{D^0_{1V}, \ldots, D^0_{tV}}(\text{min, max}) \). Once again, any such morphism is uniquely a sum of monomials, each of which contains each elementary morphism \( \{e_{\mu}\}_{\mu \in \mathcal{F}(V)} \) exactly 1 time. Then we plug the elements \( \eta_{\mu} \) in place of the corresponding \( e_{\mu} \). What we get is a morphism
\[
m(\omega; \{\xi_{\nu}\}; \{\eta_{\mu}\}) \in I_{D^0_{1U+}, \ldots, D^0_{tU+}}(\text{min, max})
\]
(We use implicitly Lemmas 4.3 and 4.4).

There are two maps
\[
\Upsilon_1, \Upsilon_2 : I_{D^0_{1U+}, \ldots, D^0_{tU+}} \to I_{[D^0_{1V}, \ldots, D^0_{tV}]} = I_U
\]
constructed in (4.7), (4.8).

The application of \( \Upsilon_1 \) and \( \Upsilon_2 \) to \( m(\omega; \{\xi_{\nu}\}; \{\eta_{\mu}\}) \) are identified with the two ways of the operadic compositions, correspondingly. The operadic associativity requires that the two ways are equal. It follows from Proposition 4.1 which states that \( \Upsilon_1 = \Upsilon_2 \), as well as from the Eckmann-Hilton compatibility stated in Lemma 4.4.
Remark 4.6. Note that the proof essentially relies on the coherence theorem (Proposition 3.5) for skew monoidal categories, which is used in the proof of identity $\Upsilon_1 = \Upsilon_2$ (Proposition 4.1). This coherence essentially relies on the fact that $\sim \otimes$ makes $\mathcal{C}at_{dg}(k)$ a perfect skew-monoidal category (see Definition 3.4), as in the general case of a skew-monoidal category the coherence theorem is a more sophisticated statement (see [LS], [BL]).

4.4 The 2-operad $\mathcal{O}$ is homotopically trivial

Recall the map of complexes

$$p_{n_1, \ldots, n_k} : \mathcal{O}(n_1, \ldots, n_k) \rightarrow k[0], n_1, \ldots, n_k \geq 1, k \geq 1$$

which is a quasi-isomorphism, see Proposition 2.7.

This map comes from the corresponding quasi-equivalence of dg categories

$$P_{n_1, \ldots, n_k} : I_{n_k} \sim \otimes (I_{n_k-1} \sim \otimes \ldots (I_{n_2} \sim \otimes I_1) \ldots)) \rightarrow I_{n_k} \otimes I_{n_k-1} \otimes \cdots \otimes I_{n_2} \otimes I_1$$

which is the projection along the ideal generated by all $\varepsilon(f, g_1, \ldots, g_n), n \geq 1$. The map $p_{n_1, \ldots, n_k}$ is then $P_{n_1, \ldots, n_k}(\text{Hom}(\min, \max))$.

Proposition 4.7. The map $p : \mathcal{O}(-) \rightarrow k[0]$ is compatible with the operadic composition, and thus gives rise to a map of operads $p : \mathcal{O} \rightarrow \text{triv}$, where $\text{triv}$ is the trivial 2-operad, $\text{triv}(n_1, \ldots, n_k) = k$, and all operadic compositions are identity maps of $k$. In other words, the 2-operad $\mathcal{O}$ is homotopically trivial.

Proof. Each component of $\mathcal{O}(D)$ is $\mathbb{Z}_{\leq 0}$-graded complex, and one sees directly from (4.20) that the operadic composition preserves this grading.

If a homogeneous element has a negative degree, it is a sum of monomials each of which contains at least one $\varepsilon(-, -, \ldots, -)$, and, thus, is mapped to 0 under the map $p$. On the other hand, $\text{triv}(D) = k$ contains only degree 0 elements for any $D$.

It is enough to prove that on the degree 0 elements the map $p : \mathcal{O}(D)^0 \rightarrow \text{triv}(D) = k$ agrees with the operadic composition. It is easy to describe the vector space $\mathcal{O}(D)^0$, $D = (n_1, \ldots, n_k)$. It has a basis each element of which consists of the composition of elements $id \ast id \ast \cdots \ast id \ast e_{i,j} \ast id \ast \cdots \ast id$ in some order, where $e_{i,j}$ runs through elementary morphisms $I_{n_i}(j, j+1)$. All such compositions are corresponded to $(n_1, \ldots, n_k)$-shuffle permutations. Each basis element is clearly mapped to the only basis element $e$ in $(I_{n_1} \otimes \cdots \otimes I_{n_k})(\min, \max)$. On the other hand, operadic compositions $\text{Op}(\omega; \{\xi_\nu\})$, in which $\omega$ and all $\xi_\nu$ are basis vectors of the type described above, is a basis vector once again.

\[\square\]
4.5 The 2-operad $\mathcal{O}$ acts on $\mathcal{C}at_{\text{dg}}^{\text{coh}}(k)$

Assume we are given dg categories $C_0, C_1, \ldots, C_k \in \mathcal{C}at_{\text{dg}}(k)$, and dg functors

$$
\begin{align*}
F_{10}, \ldots, F_{1n_1} &: C_0 \to C_1 \\
F_{20}, \ldots, F_{2n_2} &: C_1 \to C_2 \\
& \vdots \\
F_{k1}, \ldots, F_{kn_k} &: C_{k-1} \to C_k
\end{align*}
$$

(see Figure 1). Assume we are given coherent natural transformations

$$
\Psi_{ij} : F_{ij} \Rightarrow F_{i,j+1} : C_{i-1} \to C_1, \quad i = 1 \ldots k, \quad j = 0 \ldots n_i - 1
$$

That is, $\Psi_{ij} \in \mathcal{C}oh_{\text{dg}}(C_{i-1}, C_i)(F_{ij}, F_{i,j+1})$.

Applying successively the twisted composition $M$, we get a dg functor

$$
M_{\text{tot}} : \mathcal{C}oh_{\text{dg}}(C_{k-1}, C_k) \otimes \cdots \otimes \mathcal{C}oh_{\text{dg}}(C_1, C_2) \otimes \mathcal{C}oh_{\text{dg}}(C_0, C_1) \to \mathcal{C}oh_{\text{dg}}(C_0, C_k)
$$

Denote the l.h.s. of (4.23) by $\mathcal{C}oh_{\text{dg}}(C_0, C_1, \ldots, C_k)$. Then (4.23) sends

$$
M_{\text{tot}} : \mathcal{C}oh_{\text{dg}}(C_0, \ldots, C_k)(F_{k0} \times \cdots \times F_{10}, F_{kn_k} \times \cdots \times F_{1n_1}) \to \mathcal{C}oh_{\text{dg}}(C_0, C_k)(F_{k0} \circ \cdots \circ F_{10}, F_{kn_k} \circ \cdots \circ F_{1n_1})
$$

The question is: how one can assign with the elements $\{\Psi_{ij}\}_{i=1 \ldots k, j=1 \ldots n_i}$ an element in the l.h.s. of (4.23)?

Denote $D = (n_1, \ldots, n_k)$.

We associate to $\{\Psi_{ij}\}$ as above, and to an element $\omega \in \mathcal{O}(D)$, an element in $\mathcal{C}oh_{\text{dg}}(C_0, \ldots, C_k)(F_{k0} \times \cdots \times F_{10}, F_{kn_k} \times \cdots \times F_{1n_1})$, as follows.

Denote by $e_{ij}$ the generator in $I_{n_i}(j, j+1)$. The element $\omega$ is a sum of monomials in which $e_{ij}$ occurs exactly ones. We can plug $\Psi_{ij}$ for $e_{ij}$, it gives an element in $\mathcal{C}oh_{\text{dg}}(C_0, C_k)(F_{k0} \circ \cdots \circ F_{10}, F_{kn_k} \circ \cdots \circ F_{1n_1})$. Denote this element by $m(\omega; \{\Psi_{ij}\})$.

Define a map

$$
\Theta(D) : \mathcal{O}(D) \to \text{Hom} \left( \bigotimes_{i,j} \mathcal{C}oh_{\text{dg}}(C_i, C_{i+1})(F_{ij}, F_{i,j+1}), \mathcal{C}oh_{\text{dg}}(F_{k0} \circ \cdots \circ F_{10}, F_{kn_k} \circ \cdots \circ F_{1n_1}) \right)
$$

as

$$
\omega \otimes \bigotimes_{i,j} \Psi_{ij} \mapsto m(\omega; \{\Psi_{ij}\})
$$

(4.24)

Theorem 4.8. The maps $\{\Theta(D)\}$, for $D$ a 2-globular pasting diagram, give rise to an action of the dg 1-terminal 2-operad $\mathcal{O}$ on the dg 2-graph $\mathcal{C}at_{\text{dg}}^{\text{coh}}(k)$.
Proof. Both the operadic composition and the operadic action on the dg 2-graph \( \text{Cat}_{dg}^{coh}(k) \) are defined via the substitutions, as well as via the maps \( \Upsilon \), see (4.2), and the twisted composition \( M \). The statement that the operad \( \mathcal{O} \) acts on \( \text{Cat}_{dg}^{coh}(k) \) follows directly from the compatibility (3.16) of \( M \) and the associativity map. The statement that this action is strict unital follows from (3.17) and (3.18).

A Proofs of Theorem 3.1 and Theorem 3.7

Here we provide proofs of Theorems 3.1 and 3.7.

Proof of Theorem 3.1: It is clear that if \( \alpha_{C,D,E} \) gives rise to a dg functor, this dg functor is unique. Indeed, we fixed its value on morphisms which generate \( (C \otimes D) \otimes E \). In particular, relation \( (R_4) \) in Section 2.4.1 implies that for any \( \phi_1, \phi_2 \in C \otimes D, h_1, \ldots, h_n \in E \), one has:

\[
\alpha_{C,D,E}(\varepsilon(\phi_2 \star \phi_1; h_1, \ldots, h_n)) = \sum_{0 \leq a \leq n} \pm \alpha_{C,D,E}(\varepsilon(\phi_2; h_{a+1}, \ldots, h_n)) \star \alpha_{C,D,E}(\varepsilon(\phi_1; h_1, \ldots, h_a))
\]  

(A.1)

To check that \( \alpha_{C,D,E} \) gives rise to a dg functor, one needs to check the following things (where (i)-(iv) are as in the statement of Theorem 3.1):

(s1) the compatibility of \( \alpha_{C,D,E} \) with the differentials, which include:

\[
\alpha_{C,D,E}(d\varepsilon(f; g_1, \ldots, g_k) \star id_Z) = d(\alpha_{C,D,E}(\varepsilon(f; g_k, \ldots, g_k) \star id_Z))
\]  

(A.2)

\[
\alpha_{C,D,E}(d\varepsilon(f \star id_Y; h_1, \ldots, h_N)) = d(\alpha_{C,D,E}(\varepsilon(f \star id_Y; h_1, \ldots, h_N))
\]  

(A.3)

\[
\alpha_{C,D,E}(d\varepsilon(id_X \star g; h_1, \ldots, h_N)) = d(\alpha_{C,D,E}(id_X \star g; h_1, \ldots, h_N))
\]  

(A.4)

\[
\alpha_{C,D,E}(d\varepsilon(\varepsilon(f; g_1, \ldots, g_k); h_1, \ldots, h_N)) = d\alpha_{C,D,E}(\varepsilon(\varepsilon(g; g_1, \ldots, g_k); h_1, \ldots, h_N))
\]  

(A.5)

(s2) the two expressions for

\[
\alpha_{C,D,E}(\varepsilon(f_1 f_2; g_1, \ldots, g_k) \star id_Z)
\]

among which the first one is given by (ii), and the second one is given through \( (R_4) \) applied to \( \varepsilon(f_1 f_2; g_1, \ldots, g_k) \) followed by (ii), give rise to equal expressions.
(s3) the two expressions for
\[ \alpha_{C,D,E}(\varepsilon((f_1 f_2) \star \text{id}_Y; h_1, \ldots, h_n)) = \alpha_{C,D,E}(\varepsilon((f_1 \star \text{id}_Y)(f_2 \star \text{id}_Y); h_1, \ldots, h_n)) \]
and for
\[ \alpha_{C,D,E}(\varepsilon(\text{id}_X \star (g_1 g_2); h_1, \ldots, h_n)) = \alpha_{C,D,E}(\varepsilon((\text{id}_X \star g_1)(\text{id}_X \star g_2); h_1, \ldots, h_n)) \]
among which the first one is given by (iii), and the second one is given through \((R_4)\) followed by (iii), give rise to equal expressions,

(s4) the two expressions for
\[ \alpha_{C,D,E}(\varepsilon(\varepsilon(f_1 f_2; g_1, \ldots, g_k); h_1, \ldots, h_N)) \]
among which the first one is given by (iv), and the second one is given through \((R_4)\) applied to \(\varepsilon(f_1 f_2; g_1, \ldots, g_k)\) followed by (iv), give rise to equal expressions.

One checks (s1)-(s4) by a cumbersome but straightforward computation, and we omit the detail.

\[ \square \]

\textit{Proof of Theorem 3.7}:

Prove the commutativity of (3.8). We have to prove that the two maps from
\[ ((X \sim \otimes Y) \sim \otimes Z) \sim \otimes W \to X \sim \otimes (Y \sim \otimes (Z \sim \otimes W)) \]
from (3.8) coincide. We start with the “most non-degenerate” case, and keep track of both maps on
\[ \varepsilon(\varepsilon(f; g, \ldots; h; h, \ldots, h); s, s, \ldots, s, s) \] (A.6)
(To simplify notations, we drop the lower indices of the arguments; here \(f \in X, g \in Y, h \in Z, s \in W\) are morphisms). We check the commutativity up to signs, the coincidence of signs is straightforward.

The result of composition of upper two arrows in (3.8) on (A.6) is a sum (with appropriate signs) of all possible terms of the form
\[ \varepsilon(f; s, s, \ldots, \varepsilon(g; s, s, \ldots, \varepsilon(h; s, s, \ldots, s), s, s, \ldots), s, \ldots, \varepsilon(h; s, s, \ldots, s), s, s, \ldots, s) \] (A.7)
Each such expression is of the form
\[ \varepsilon(f; S_1, \ldots, S_M) \] (A.8)

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where each $S_i$ is either $s$, or $\varepsilon(h; s, s, ..., s)$, or $\varepsilon(g; s, s, ..., s, \varepsilon(h; s, s, ..., s), s, ..., \varepsilon(h; s, s, ..., s), s, s, ..., \varepsilon(h; s, s, ..., s), s, s, ..., s)$.

Let us keep track of the composition of three lower arrows in (3.8). The first arrow does not affect the $s$-arguments, and produces from (A.6) an expression of the form

$$\varepsilon\bigg(\varepsilon\bigg(f; h, h, ..., h, \varepsilon\big(g; h, ..., h, h, ..., h, \varepsilon\big(g; h, ..., h, h, ...., h, h, ..., h, \varepsilon\big(h; s, s, ..., s), s, s, s, ..., s\big), s, s, ..., s, \varepsilon\big(h; s, s, ..., s\big), s, s, ..., s\big)\bigg)$$

(A.9)

The next map in the lower composition produces

$$\varepsilon\bigg(f; s, s, s, \varepsilon(h; s, s, s), s, s, \varepsilon(h; s, s, s), s, s, s, \varepsilon\big(g; h, h, h, ..., h\big), s, s, ..., s, \varepsilon\big(h; s, s, ..., s\big), s, s, ..., s\big)$$

(A.10)

Finally, the third map in the lower composition acts only on the arguments of the form $\varepsilon\big(\varepsilon(g; h, h, ..., h), s, s, ..., s\big)$ and maps them as in (3.4).

What we get finally is an expression of the form

$$\varepsilon(f; T_1, T_2, ..., T_N)$$

(A.11)

where each $T_i$ is either $s$, or $\varepsilon(h; s, s, ..., s)$, or $\varepsilon(g; s, s, ..., s, \varepsilon(h; s, s, ..., s), s, s, ..., s, \varepsilon(h; s, s, ..., s), s, s, ..., s, \varepsilon(h; s, s, ..., s), s, s, ..., s, \varepsilon(h; s, s, ..., s), s, s, ..., s)$.

We see that there is a 1-to-1 correspondence between the terms in (A.8) and (A.11), which proves (3.8) on (A.6).

It remains to prove (3.8) on expressions such as $(f \ast \text{id}) \ast \text{id}$, or $\varepsilon(f; \varepsilon(g; h, h, ..., h)) \ast \text{id}$ (there is a rather long list of all possibilities). (3.8) is straightforward on expressions which do not contain any $\varepsilon$ or contain exactly 1 $\varepsilon$. So we consider all possibilities which contain 2 of $\varepsilon$. Below is the exhaustive list (note that $\varepsilon(\text{id}; ...)$ and $\varepsilon(f; g, g, ..., \text{id}, g, g, ...)$ $= 0$ by (R3) in Section 2.4.1):

$$\varepsilon\bigg(\varepsilon(f; g, g, ..., g) \ast \text{id}_Z; s, s, ..., s\bigg)$$

$$\varepsilon\bigg(\varepsilon(f \ast \text{id}_Y; h, h, ..., h); s, s, ..., s\bigg)$$

$$\varepsilon\bigg(\varepsilon(\text{id}_X \ast g; h, h, ..., h); s, s, s, ..., s\bigg)$$

(A.12)

The proof of commutativity of (3.8) for all of them is analogous, we provide a proof for the element in the first line of (A.12), the others are left to the reader.

The action of the first arrow in the composition of the two upper arrows of (3.8) on $\varepsilon(f; g, g, ..., g) \ast \text{id}_Z; s, s, ..., s)$ gives

$$\varepsilon\bigg(\varepsilon(f; g, g, ..., g); \text{id}_Z \ast s, \text{id}_Z \ast s, ..., \text{id}_Z \ast s\bigg)$$

(A.13)

by (3.3). The application of the second upper map gives

$$\sum \pm\varepsilon(f; \text{id}_Z \ast s, ..., \varepsilon(g; \text{id}_Z \ast s, ..., \text{id}_Z \ast s), \text{id}_Z \ast s, ..., \varepsilon(g; \text{id}_Z \ast s, ..., \text{id}_Z \ast s), ...)$$

(A.14)
where the signs are as in (3.4).

Now keep track of application of the composition of the lower three arrows to the same element. The application of the first lower arrow gives

$$\varepsilon(\varepsilon(f; g \star \text{id}, \ldots, g \star \text{id}) ; s, s, \ldots , s)$$

(A.15)

by (3.2). Next, the second lower arrow maps (A.15) to

$$\sum \pm \varepsilon(\varepsilon(f; s, s, \ldots , \varepsilon(g \star \text{id} ; s, s, \ldots , s), s, \ldots , s, \varepsilon(g \star \text{id} ; s, s, \ldots , s), s, \ldots , s, \varepsilon(g \star \text{id} ; s, s, \ldots , s), s, \ldots , s))$$

(A.16)

Finally, the third lower arrow maps it to

$$\sum \pm \varepsilon(\varepsilon(f; s, s, \ldots , \varepsilon(g \star \text{id} \star s, \text{id} \star s, \ldots , \text{id} \star s), s, \ldots , s, \varepsilon(g \star \text{id} \star s, \text{id} \star s, \ldots , \text{id} \star s), s, \ldots , s, \varepsilon(g \star \text{id} \star s, \text{id} \star s, \ldots , \text{id} \star s), s, \ldots , s))$$

(A.17)

(here no new signs emerge, the signs at the corresponding terms are as for (A.16)).

There is a minor difference between (A.14) and (A.16); namely, some arguments $s$ in (A.17) emerge as $\text{id} \star s$ in (A.17). However, in this context they are identically the same, due to Remark 3.2.

The commutativity of diagram (3.8) is proven.

The commutativity of the remaining diagrams (3.9)-(3.12) is clear.

\[\square\]

\section{B \ Batanin 2-operads}

The theory of $n$-operads is due to Michael Batanin [Ba3-Ba5], see also [T2]. A contemporary approach uses operadic categories [BM1,2]. In this paper, we adopt the approach and terminology of [Sh5], Section 1.1, the reader is referred to. Here we very briefly outline some points.

\subsection{B.1}

The category $\text{Tree}_n$ is defined as follows. Its object $T$ is an $n$-string of surjective maps in $\Delta$:

$$T = [k_n - 1] \xrightarrow{\rho_{n-1}} [k_{n-1} - 1] \xrightarrow{\rho_{n-2}} \ldots \xrightarrow{\rho_0} [0]$$

Such $T$ is visualised as a $n$-level tree. An $n$-tree is called pruned if all $\rho_i$ are surjective. For a pruned $n$-tree, all its leaves are at the highest level $n$. The finite set of leaves of an $n$-tree $T$ is denoted by $|T|$. The maps $\rho_i$ are referred to as the structure maps of an $n$-tree.

A morphism $F: T \rightarrow S$, where

$$S = [\ell_n - 1] \xrightarrow{\xi_{n-1}} [\ell_{n-1} - 1] \xrightarrow{\xi_{n-2}} \ldots \xrightarrow{\xi_0} [0]$$
is defined as a sequence of maps \( f_i : \{ k_i - 1 \} \rightarrow \{ \ell_i - 1 \}, \ i = 0, 1, \ldots, n \) (not monotonous, in general), which commute with the structure maps, and such that for each \( 0 \leq i \leq n \) and each \( j \in [k_i - 1 - 1] \) the restriction of \( f_i \) on \( \rho_{i-1}^{-1}(j) \) is monotonous. That is, \( f_i \) has to be monotonous when restricted to the fibers of the structure map \( \rho_{i-1} \). It is clear that a map of \( n \)-trees is uniquely defined by the map \( f_n \). Conversely, any map \( f_n \) which is a map of \( n \)-ordered sets, associated with \( n \)-trees \( S \) and \( T \), defines a map of \( n \)-trees (see [Ba3, Lemma 2.3]).

The fiber \( F^{-1}(a) \) for a morphism \( F : T \rightarrow S, \ a \in |S| \), is defined as the set-theoretical preimage of the linear subtree Out(\( a \)) of \( S \) spanned by \( a \). This linear subtree Out(\( a \)) is defined as follows. Let \( a \in [\ell_i] \), then Out(\( a \)) has no vertices at levels \( > i \), and the only vertex of Out(\( a \)) at level \( j \leq i \) is defined as \( \rho_j \ldots \rho_{i-2}\rho_{i-1}(a) \). Note that a fiber of a map of pruned \( n \)-trees is not necessarily a pruned \( n \)-tree, even if all components \( \{ f_i \} \) of the map \( F \) are surjective, see [Sh5, Remark 1.2].

There are two operadic categories [BM1] based on \( n \)-trees as objects. The operadic category \( \text{Tree}_n \) has as objects all \( n \)-trees, and the fibers are defined as above. Another one, which we use in this paper, \( \text{Ord}_n \), has pruned \( n \)-trees as its objects, the morphisms are surjective on leaves, and the fibers are defined as prunisation of the naive fibers. This procedure of prunisation amounts to ignoring of all leaves at levels \( < n \) and all its descendants.

The \((n - 1\text{-terminal})\) \( n \)-operads based on the operadic category \( \text{Ord}_n \) are used for describing (weak) \( n \)-categories with strict units. Indeed, the strictness of units means, for example for \( n = 2 \), that the identity 2-morphisms give rise to identical vertical and horizontal (aka whiskering) compositions.

B.2

**Definition B.1.** Let \( V \) be a symmetric monoidal category. An assignment \( T \mapsto \mathcal{O}(T) \in V \), for any pruned \( n \)-tree \( T \), is called a (pruned) \( n \)-collection in \( V \). A pruned reduced \((n - 1)\)-terminal \( n \)-operad \( \mathcal{O} \) in a symmetric monoidal category \( V \) is given by an \( n \)-collection \( \{ \mathcal{O}(T) \}_{T \in \text{Ord}_n} \), so that for any surjective map \( \sigma : T \rightarrow S \) of pruned \( n \)-trees, one is given the composition

\[
m_\sigma : \mathcal{O}(S) \otimes \cdots \otimes \mathcal{O}(P(\sigma^{-1}(k))) \rightarrow \mathcal{O}(T)
\]

where \( k = \text{card} S \) is the number of leaves of \( S \), and \( P(\cdot) \) is the prunisation of the corresponding \( n \)-tree. It is subject to the following conditions (in which we assume that \( V = C^*(\mathbb{k}) \) is the category of complexes of \( \mathbb{k} \)-vector spaces):

(i) \( \mathcal{O}(U_n) = \mathbb{k} \), and \( 1 \in \mathbb{k} \) is the operadic unit,

(ii) the associativity for the composition of two surjective morphisms \( T \xrightarrow{\sigma} S \xrightarrow{\rho} Q \) of pruned \( n \)-trees, see [Ba2] Def. 5.1,

(iii) the two unit axioms, see [Ba2], Def. 5.1.
The category of pruned reduced \((n-1)\)-terminal \(n\)-operads in a symmetric monoidal category \(V\) is denoted by \(\text{Op}_n(V)\), or simply by \(\text{Op}_n\).

The \((n-1)\)-terminality makes us possible to restrict with \(n\)-operads taking values in a symmetric monoidal globular category \(\Sigma^n V\), where \(V\) is a closed symmetric monoidal category, see [Ba2], Sect. 5. By a slight abuse of terminology, we say that an operad takes values in the closed symmetric monoidal category \(V\) (not indicating \(\Sigma^n V\)).

For \(n = 1\) we recover the concept of a non-symmetric (non-\(\Sigma\)) operads. To describe \(n\)-algebras for \(n > 1\), or \(n\)-categories, one needs some partial symmetry, but less than the entire symmetric groups \(\Sigma_n\) actions on the components \(O\) of an ordinary (symmetric) operads. Morally, with \(n\)-operads we restrict ourselves with minimal possible symmetry.

One essential difference with the \(n = 1\) is that for \(n > 1\) the corresponding composition product on the category of (pruned reduced \((n-1)\)-terminal) \(n\)-sequences gives rise only to a skew-monoidal \(V\)-category, not to a monoidal one. This phenomenon leads, in particular, to non-associative graphical presentation of free \(n\)-operads. We refer the reader to [L], Theorem 3.4 where a more general result that collections based on any operadic categories form a skew-monoidal category is proven.

B.3

Following [T2], we use in this paper the Joyal dual description of the category \(\text{Tree}_n\) via Joyal \(n\)-disks (more precisely, via \(n\)-globular pasting diagrams). This description is more “globular”, and it fits better for questions of globular nature, such as for describing weak \(n\)-categories. Let us recall it, for simplicity restricting ourselves with the case \(n = 2\).

A 2-globular diagram \(D\) is given by sets, \(D_0, D_1, D_2\), and the following maps

\[
\begin{array}{ccc}
D_2 & \xrightarrow{s_1} & D_1 \\
\downarrow_{t_1} & & \downarrow_{t_0} \\
D_0
\end{array}
\]  

such that \(s_0s_1 = s_0t_1, t_0s_1 = t_0s_1\). A morphism \(D \to D'\) of two 2-globular diagrams is defined as collection of maps \(\{D_i \to D'_i\}_{i=0,1,2}\), which commute with all \(s_j\) and \(t_j\).

We consider globular \(n\)-diagrams for which the set \(D_0\) is linearly ordered such that for any \(f \in D_1\) the element \(t_0(f)\) is the least element greater than \(s_0(f)\), and similarly \(D_1\) is a disjoint union of linearly ordered sets such that for any \(\nu \in D_2\) the elements \(s_1(\nu), t_1(\nu) \in D_1\) belong to the same connected component, and \(t_1(\nu)\) is the least element greater than \(s_1(\nu)\). Such diagrams are called globular pasting 2-diagram, see Figure [1].

A 2-globular pasting diagram \(D\) is considered as a quiver for generating a free strict 2-category, which we denote by \(\omega_2(D)\). The functor \(D \sim \omega_2(D)\) is the left adjoint to the forgetful functor from the category of strict 2-categories to 2-globular diagrams.

Assume we are given a globular diagram as at the Figure [1] then one associates to it the 2-tree

\[ T(D) = [n_1 + \cdots + n_k - 1] \xrightarrow{p} [k - 1] \to [0] \]
where \( \#\{\rho^{-1}(i)\} = n_i \) and the map \( \rho \) is monotonous surjective. The \( n_1, n_2, \ldots, n_k \) leaves at the corresponding branches of the tree \( T(D) \) are thought of as the elementary generating 2-morphisms (thus, the vertical intervals) in \( D \). We denote such globular diagram by \( D = (n_1, \ldots, n_k) \). As we model the category of pruned 2-trees, we impose the conditions

\[
 k \geq 1, n_i \geq 1 \quad \text{for} \quad 1 \leq i \leq k
\]

We define a category whose objects are 2-globular diagrams. We define a map \( f : D \to D' \) as a strict 2-functor between the strict 2-categories \( \omega_2(D) \to \omega_2(D') \) which is dominant in the following sense. The objects of \( \omega_2(D) \) are linearly ordered, and for any two objects \( i \leq j \), the set of 1-morphisms \( \omega_2(D)(i, j) \) is partially ordered, with the minimal and the maximal elements.

The dominance of \( f \) means that \( \omega_2(D)(f) \) preserves the minimal and the maximal vertices, and for any \( i \leq j \in \omega_2(D) \), \( \omega_2(f) \) maps the minimal element of \( \omega_2(D)(i, j) \) to the minimal element in \( \omega_2(D')(f(i), f(j)) \), and similarly for the maximal elements. We denote the category whose objects are 2-globular diagrams, and morphism are strict dominant 2-functors as above, by \( \text{Glob}^\text{dom}_2 \). For \( D \in \text{Glob}^\text{dom}_2 \), \( D = (n_1, \ldots, n_k) \), we assume that \( k \geq 1, n_1, \ldots, n_k \geq 1 \).

One has:

**Proposition B.2.** The category \( \text{Glob}^\text{dom}_2 \) is anti-equivalent to the category \( \text{Tree}_2 \) of pruned 2-trees.

See [B], Prop. 2.2 (along with [Sh5], Rem. 1.6 and [T2], 4.1.9-10 for a proof.

The duality of Proposition B.2 is the \( n = 2 \) analogue of the classical Joyal duality between the category \( \Delta_+ \) (the usual category \( \Delta \) augmented with an initial object \([-1]\)) and the category \( \Delta_{fi} \) of finite intervals, see [J].

We adopt the following notations for the 2-globular pasting diagram presentation of 2-operads:

\( U, V, \ldots \) for 2-globular pasting diagrams, \([U]\) for \( \omega_2(U) \). A minimal ball in \( U \) is an elementary 2-morphism of \( \omega_2(U) = [U] \), that is, an element of \( U_2 \) in the presentation (B.2). Minimal balls are denoted by \( \mu, \nu, \ldots \). The set of all minimal balls in a 2-globular diagram \( U \) is denoted by \( \mathcal{F}(U) \).

Denote by \( \mathcal{C}(U) \) the set of intervals of \( U \), that is, the set \( U_1 \) in the presentation (B.2). (It is similar to the ordered set of intervals in an ordinal; for the ordinal \( [k] = \{0 < 1 < 2 < \cdots < k\} \), the cardinality of the set of intervals is \( k \), and the intervals are \((01), (12), \ldots, (k-1,k)\). We sometimes write \( \overline{i, i+1} \) for the interval \( (i, i+1) \).

For a 2-globular set \( U \), there is a map

\[
\pi(U) : \mathcal{F}(U) \to \mathcal{C}(U)
\]

defined as \( \pi_U(\nu) = (i, i+1) \) if \( s_1(\nu) = i, t_1(\nu) = i+1 \).
B.4 Batanin Theorem

Denote the category of symmetric operads (in a given symmetric monoidal category) by $\text{Op}_\Sigma$.

Batanin [Ba2], Sect. 6 and 8, constructs a pair of functors relating symmetric operads and $n$-operads:

$$\text{Symm}: \text{Op}_n^{n-1} \rightleftharpoons \text{Op}_\Sigma: \text{Des}$$

The right adjoint functor of desymmetrisation $\text{Des}$ associates to each pruned $n$-tree $T$ its set of leaves $|T|$ (which are all at the level $n$):

$$\text{Des}(\mathcal{O})(T) = \mathcal{O}(|T|)$$

and for a map $\sigma: T \to S$ of $n$-trees, the $n$-operadic composition associated with $\sigma$ is defined as the corresponding composition for $|\sigma| = |\sigma_n|: |T| \to |S|$, twisted by the shuffle permutation $\pi(\sigma_n)$ of the map $|\sigma_n|: |T| \to |S|$ defined by the condition that the composition of $\pi(\sigma)$ followed by an order preserving map of finite sets is $\sigma_n$ (see [Ba2], Sect. 6).

The symmetrisation functor is defined as the left adjoint to $\text{Des}$, its existence is established in [Ba2], Sect. 8.

The main result on $n$-operads was proven in [Ba3] Th.8.6 for topological spaces and in [Ba3] Th.8.7 for complexes of vector spaces. We provide below the statement for $C^*(\mathbb{k})$, as the one we use here. Denote by $\mathbb{k}$ the constant $n$-operad, $\mathbb{k}(T) = \mathbb{k}$, with evident operadic compositions. We say that an $n$-operad in $C^*(\mathbb{k})$ is augmented by $\mathbb{k}$ if there is a map of $n$-operads $p: \mathcal{O} \to \mathbb{k}$, called the augmentation map.

**Theorem B.3.** [Batanin] Let $\mathcal{O}$ be reduced pruned $(n-1)$-terminal $n$ operad in the symmetric monoidal category $C^*(\mathbb{k})$. Assume $\mathcal{O}$ is augmented to the constant $n$-operad $\mathbb{k}$, and that for any arity $T$ the augmentation map $p(T): \mathcal{O}(T) \to \mathbb{k}$ is a quasi-isomorphism of complexes. Then there is a morphism of $\Sigma$-operads $C^*(E_n; \mathbb{k}) \to \text{Sym}(\mathcal{O})$, thus making any $\mathcal{O}$-algebra a $C^*(E_n; \mathbb{k})$-algebra.

**Remark B.4.** There are closed model structures on the categories of $\Sigma$-operads and $n$-operads, constructed in [BB2]. Within these model structures, $(\text{Symm}, \text{Des})$ is a Quillen pair, with $\text{Symm}$ the left adjoint. The stronger version of this theorem [Ba3] actually says that the symmetrisation of a cofibrant contractible pruned, reduced, $(n-1)$-terminal is weakly equivalent to the symmetric operad $C^*(E_n; \mathbb{k})$. 
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\[ G_0 F_0(X_0) \quad F_0(X_0) \quad F_0(X_i) \quad \Theta \quad F_1(X_{i+m}) \quad F_1(X_{m+n-1}) \]

\[ X_0 \quad X_i \quad X_{i+m} \quad X_{m+n-1} \]