EXISTENCE OF COMMON ZEROS FOR COMMUTING VECTOR FIELDS ON 3-MANIFOLDS

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ABSTRACT. In 64 E. Lima [Li2] proved that commuting vector fields on surfaces with non-zero Euler characteristic have common zeros. Such statement is empty in dimension 3, since all the Euler characteristics vanish. Nevertheless, [Bo3] proposed a local version, replacing the Euler characteristic by the Poincaré-Hopf index of a vector field \(X\) in a region \(U\), denoted by \(\text{Ind}(X, U)\); he asked:

Given commuting vector fields \(X, Y\) and a region \(U\) where \(\text{Ind}(X, U) \neq 0\), does \(U\) contain a common zero of \(X\) and \(Y\)?

[Bo3] gave a positive answer in the case where \(X\) and \(Y\) are real analytic.

In this paper, we prove the existence of common zeros for commuting \(C^1\) vector fields \(X, Y\) on a 3-manifold, in any region \(U\) such that \(\text{Ind}(X, U) \neq 0\), assuming that the set of collinearity of \(X\) and \(Y\) is contained in a smooth surface. This is a strong indication that the results in [Bo3] should hold for \(C^1\)-vector fields.

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1. Introduction

One of the fundamental problems in dynamical systems is whether a given system possesses fixed points or not. A simple scenario to pose this question is for the \(\mathbb{Z}\)-action generated by a diffeomorphism or a homeomorphism of a manifold, or for the continuous time dynamical system generated by the flow of a vector field. In both cases, the theories of Poincaré-Hopf and Lefschetz indices relate the topology of the ambient manifold with the existence of fixed points.

Nonetheless, if one consider two commuting diffeomorphisms or two commuting vector fields i.e. vector fields \(X\) and \(Y\) whose flows satisfy:

\[ X_t \circ Y_s = Y_s \circ X_t, \quad \forall (s, t) \in \mathbb{R}^2, \]

the existence of a fixed point for the action they generate is a wide open question in dimensions \(\geq 3\).

The first result on this question is given by the works on surfaces of Lima [Li1], [Li2]. He proves that any family of commuting vector fields on a surface with non-zero Euler characteristic have a common zero. In the late eighties, [Bo1] proved that commuting diffeomorphisms of the sphere \(S^2\) which are \(C^1\)-close to the identity have a common fixed point. Later [Bo2] extended this result to any surface with non-zero Euler characteristic (see other generalizations in [DFF][Fi]). Then, Handel [Ha]...

\(1\)This definition can be adapted for non-complete vector fields, see Section 2.
provided a topological invariant in $\mathbb{Z}/2\mathbb{Z}$ for a pair of commuting diffeomorphisms of the sphere $S^2$ whose vanishing guarantees a common fixed point. This was further generalized by Franks, Handel and Parwani [FHP] for any number of commuting diffeomorphisms on the sphere (see [HI] and [FHP2] for generalizations on other surfaces).

It is worth to note, however, that two commuting continuous interval maps may fail to have a common fixed point: an example is constructed in [B] of two continuous commuting, non-injective, maps of the interval which do not have a common fixed point.

In higher dimensions much less is known:

- One knows some relation between the topology of the manifold and the dimension of the orbits of $\mathbb{R}^p$-actions (see [MT1, MT2]). The techniques introduced in these works makes possible a simple proof of Lima’s result, for smooth vector fields [Tu];
- [Bo3] proved that two commuting real analytic vector fields on an analytic 4-manifold with non-zero Euler characteristic have a common zero. The same statement does not make sense in dimension three since every 3-manifold has zero Euler characteristic. Nevertheless, a local result remains true in dimension three.

Before stating the result of [Bo3] on 3-manifolds, we briefly recall the notion of the Poincaré-Hopf index $\text{Ind}(X, U)$ of a vector field $X$ on a compact region $U$ whose boundary $\partial U$ is disjoint from the set $\text{Zero}(X)$. If $U$ is a small compact neighborhood of an isolated zero $p$ of the vector field $X$, then $\text{Ind}(X, U)$ is just the classical Poincaré-Hopf index $\text{Ind}(X, p)$ of $X$ at $p$. For a general compact region $U$ with $\partial U \cap \text{Zero}(X) = \emptyset$, one considers a small perturbation $Y$ of $X$ with only finitely many isolated zeros in $U$. Then we define the index $\text{Ind}(X, U)$ as the sum of the Poincaré-Hopf indices $\text{Ind}(Y, p)$, $p \in \text{Zero}(Y) \cap U$. We refer the reader to Section 2 for details (in particular for the fact that $\text{Ind}(X, U)$ does not depend on the perturbation $Y$ of $X$).

The result of [Bo3] on 3-manifolds can be restated as follows:

**Theorem.** Let $M$ be a real analytic 3-manifold and $X$ and $Y$ be two analytic commuting vector fields over $M$. Let $U$ be a compact subset of $M$ such that $\text{Zero}(Y) \cap U = \text{Zero}(X) \cap \partial U = \emptyset$. Then,

$$\text{Ind}(X, U) = 0.$$  

This statement is also true when $M$ has dimension 2 and the vector fields are just $C^1$ (see Proposition 11 in [Bo2]). This motivates the following

**Conjecture.** Let $X$ and $Y$ be two $C^1$ commuting vector fields on a 3-manifold $M$. Let $U$ be a compact subset of $M$ such that $\text{Zero}(Y) \cap U = \text{Zero}(X) \cap \partial U = \emptyset$. Then, $\text{Ind}(X, U) = 0$.

This conjecture was stated as a problem in [Bo3].

The goal of the present paper is to solve, in the $C^1$-setting, what was the main difficulty in the analytic case in [Bo3]. We explain now what was this difficulty in

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2The result stated in Proposition 11 of [Bo2] is for $C^\infty$ vector fields, but the proof indicated there can be adapted for $C^1$ vector fields using cross-sections, in a similar way we do here in Section 2.
A crucial role is played by the set of points of $U$ in which $X$ and $Y$ are collinear:

$$\text{Col}(X, Y, U) := \{ p \in U; \dim (\langle X(p), Y(p) \rangle) \leq 1 \}.$$  

In [Bo3] the assumption that the commuting vector fields are analytic is used to say that $\text{Col}(X, Y, U)$ is either equals to $U$ or is an analytic set of dimension 2. The case where $\text{Col}(X, Y, U) = U$ admits a direct proof. In the other case, a simple argument allows to assume that $\text{Col}(X, Y, U)$ is a surface. The main difficulty in [Bo3] consists in proving that, if $\text{Col}(X, Y, U)$ is a smooth surface and $X$ and $Y$ are analytic, then the index of $X$ vanishes on $U$.

Our result is the following

**Theorem A.** Let $M$ be a 3-manifold and $X$ and $Y$ be two $C^1$ commuting vector fields over $M$. Let $U$ be a compact subset of $M$ such that $\text{Zero}(Y) \cap U = \text{Zero}(X) \cap \partial U = \emptyset$. Assume that $\text{Col}(X, Y, U)$ is contained in a $C^1$-surface which is a closed submanifold of $M$. Then,

$$\text{Ind}(X, U) = 0.$$  

The hypothesis “$\text{Col}(X, Y, U)$ is contained in a $C^1$-surface” consists in considering the simplest geometric configuration of $\text{Col}(X, Y, U)$ for which the conjecture is not trivial: if $(X, Y)$ is a counter example to the conjecture, then $\text{Col}(X, Y)$ cannot be “smaller” than a surface. More precisely, if $\text{Ind}(X, U) \neq 0$, the sets $\text{Zero}(X - tY)$ for small $t$ are not empty compact subsets of $\text{Col}(X, Y, U)$, invariant by the flow of $Y$ and therefore consist in orbits of $Y$. If $X$ and $Y$ are assumed without common zeros, every set $\text{Zero}(X - tY)$ consists on regular orbits of $Y$, thus is a 1-dimensional lamination. Furthermore, these laminations are pairwise disjoint and vary semi-continuously with $t$.

Another (too) simple configuration would be the case where $X$ and $Y$ are everywhere collinear. This case has been treated in [Bo3] and the same proof holds at least in the $C^2$ setting.

We believe that the techniques that we introduce here will be usefull to prove the conjecture, at least for $C^2$ vector fields.

The proof of Theorem A is by contradiction. The intuitive idea which guides the argument is that, at one hand, the vector field $X$ needs to turn in all directions in a non-trivial way in order to have a non-zero index. On the other hand, $X$ commutes with $Y$ and therefore is invariant under the tangent flow of a non-zero vector field. The combination of this two properties will lead to a contradiction.

This paper is organized as follows.

- In Section 2 we give detailed definitions and state some classical facts that we shall use.
- In Section 3 we reduce the proof of Theorem A to the proof of a slightly more technical version of it (see Lemma 3.9, for which $U$ is a solid torus and $\text{Col}(X, Y, U)$ is a annulus foliated by periodic orbits of $Y$, and cutting $U$ in two connected components $U^+$ and $U^-$.
- In Section 4 we consider the projection $N$, of the vector field $X$ parallel to $Y$ on the normal bundle of $Y$. We show that $\text{Ind}(X, U)$ is related with the angular variations $\ell^+$ and $\ell^-$ of $N$ along generators of the fundamental group of each connected components $U^+$ and $U^-$ of $U \setminus \text{Col}(X, Y, U)$. More
precisely we will show in Proposition 4.9 that
\[ |\text{Ind}(X, U)| = |\ell^+ - \ell^-|. \]

Assuming that at least one of \(\ell^+\) and \(\ell^-\) does not vanish, and the fact that \(\text{Col}(X, Y, U)\) is a \(C^1\)-surface, we deduce in Proposition 4.19 that the first return map \(\mathcal{P}\) of \(Y\) on a transversal \(\Sigma_0\) is \(C^1\)-close to identity in a small neighborhood of \(\text{Col}(X, Y, U) \cap \Sigma_0\).

• In Section 5, still assuming that at least one of \(\ell^+\) and \(\ell^-\) does not vanish, we give a description of the dynamics of the first return map \(\mathcal{P}\). If for instance \(\ell^+ \neq 0\) then every point in \(\Sigma_0 \cap U^+\) belongs to the stable set of a fixed point of \(\mathcal{P}\) (Lemma [7]). We will then use the invariance of these stable sets under the orbits of the normal vector field \(N\) for getting a contradiction.

We end this introduction by a general comment. The accumulation of results proving the existence of common fixed points for commuting dynamical systems seems to indicate the possibility of a general phenomenon. However, our approach in Poincaré-Bendixson spirit has a difficulty which increases drastically with the ambient dimension. We hope that this results will motivate other attempts to study this phenomenon.

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2. Notations and definitions

In this paper \(M\) denotes a 3-dimensional manifold. Whenever \(X\) is a vector field over \(M\), we denote \(\text{Zero}(X) = \{ x \in M; X(x) = 0 \}\) and \(\text{Zero}(X, U) = \text{Zero}(X) \cap U\), for any subset \(U \subset M\). We shall denote its flow by \(X_t\). A compact set \(\Lambda \subset M\) is invariant under the flow of \(X\) if \(X_t(\Lambda) = \Lambda\) for every \(t \in \mathbb{R}\).

If \(X\) and \(Y\) are vector fields on \(M\) we denote by \(\text{Col}(X, Y)\) the set of points \(p\) for which \(X(p)\) and \(Y(p)\) are collinear:
\[ \text{Col}(X, Y) = \{ p \in M, \dim((X(p), Y(p))) \leq 1 \}. \]

If \(U \subset M\) is a compact region we denote \(\text{Col}(X, Y, U) = \text{Col}(X, Y) \cap U\).

2.1. The Poincaré-Hopf index. In this section we recall the classical definition and properties of the Poincaré-Hopf index.

Let \(X\) be a continuous vector field of a manifold \(M\) of dimension \(d\) and \(x \in M\) be an isolated zero of \(X\). The Poincaré-Hopf index \(\text{Ind}(X, x)\) is defined as follows: consider local coordinates \(\varphi: U \to \mathbb{R}^d\) defined in a neighborhood \(U\) of \(x\). Up to shrink \(U\) one may assume that \(x\) is the unique zero of \(X\) in \(U\). Thus for \(y \in U \setminus \{x\}\), \(X(y)\) expressed in that coordinates is a non vanishing vector of \(\mathbb{R}^d\), and \(\frac{1}{\|X(y)\|} X(y)\) is a unit vector hence belongs to the sphere \(S^{d-1}\). Consider a small ball \(B\) centered at \(x\). The map \(y \mapsto \frac{1}{\|X(y)\|} X(y)\) induces a continuous map from the boundary \(S = \partial B\) to \(S^{d-1}\). The Poincaré-Hopf index \(\text{Ind}(X, x)\) is the topological degree of this map.
Assume now that $U \subset M$ is a compact subset and that $X$ does not vanish on the boundary $\partial U$. The Poincaré-Hopf index $\text{Ind}(X, U)$ is defined as follows: consider a small perturbation $Y$ of $X$ so that the set of zeros of $Y$ in $U$ is finite. A classical result asserts that the sum of the indices of the zeros of $Y$ in $U$ does not depend on the perturbation $Y$ of $X$; this sum is the Poincaré-Hopf index $\text{Ind}(X, U)$. Here, small perturbation means that $Y$ is homotopic to $X$ through vector fields which do not vanish on $\partial U$. More precisely

**Proposition 2.1.** If $\{X^t\}_{t \in [0,1]}$ is a continuous family of vector fields so that $\text{Zero}(X^t) \cap \partial U = \emptyset$, then $\text{Ind}(X^t, U)$ does not depend on $t \in [0,1]$.

We say that a compact subset $K \subset \text{Zero}(X)$ is isolated if there is a compact neighborhood $U$ of $K$ so that $K = \text{Zero}(X) \cap U$; the neighborhood $U$ is called an isolating neighborhood of $K$. The index $\text{Ind}(X, U)$ does not depend of the isolating neighborhood $V$ of $K$. Thus $\text{Ind}(X, U)$ is called the index of $K$ and denoted $\text{Ind}(X, K)$.

Assume now that $\partial U$ is a codimension 1 submanifold and that $U$ is endowed with $d$ continuous vector fields $X^1, \ldots, X^d$ so that, at every point $z \in U$, $(X^1(z), \ldots, X^d(z))$ is a basis of the tangent space $T_z M$. This basis endows $U$ with an orientation. Once again, one can express the vector field $X$ in this basis so that the vector $X(y)$, for $y \in U$, can be considered as a vector of $\mathbb{R}^d$. One defines in such a way a map $g: \partial U \to \mathbb{S}^{d-1}$ by $y \mapsto g(y) = \frac{1}{\|X(y)\|} X(y)$.

As $\partial U$ has dimension $d - 1$, and is oriented as the boundary of $U$, this map has a topological degree. A classical result from homology theory implies the following

**Lemma 2.2.** With the notations above the topological degree of $g$ is $\text{Ind}(X, U)$.

In particular it do not depend on the choice of the vector fields $X^1, \ldots, X^d$.

2.2. **Topological degree of a map from $\mathbb{T}^2$ to $\mathbb{S}^2$.** We consider the sphere $\mathbb{S}^2$ (unit sphere of $\mathbb{R}^3$) endowed with the north and south poles denoted $N = (0, 0, 1)$ and $S = (0, 0, -1)$ respectively.

We denote by $\mathbb{S}^1 \subset \mathbb{S}^2$ the equator, oriented as the unit circle of $\mathbb{R}^2 \times \{0\}$. For $p = (x, y, z) \in \mathbb{S}^2 \setminus \{N, S\}$ we call projection of $p$ on $\mathbb{S}^1$ along the meridians the point $\frac{1}{\sqrt{x^2 + y^2}} (x, y, 0)$, which is intersection of $\mathbb{S}^1$ with the unique half meridian containing $p$.

**Lemma 2.3.** Let $\Phi: \mathbb{S}^2 \to \mathbb{S}^2$ be a continuous map so that $\Phi^{-1}(N) = \{N\}$ and $\Phi^{-1}(S) = \{S\}$.

Let $\varphi: \mathbb{S}^1 \to \mathbb{S}^1$ be defined as follows: the point $\varphi(p)$, for $p \in \mathbb{S}^1$, is the projection of $\Phi(p) \in \mathbb{S}^2 \setminus \{N, S\}$ on $\mathbb{S}^1$ along the meridians of $\mathbb{S}^2$.

Then the topological degrees of $\Phi$ and $\varphi$ are equal.

As a direct consequence one gets

**Corollary 2.4.** Let $\Phi: \mathbb{S}^2 \to \mathbb{S}^2$ be a continuous map so that $\Phi^{-1}(N) = \{S\}$ and $\Phi^{-1}(S) = \{N\}$.

Let $\varphi: \mathbb{S}^1 \to \mathbb{S}^1$ be defined as follows: the point $\varphi(p)$, for $p \in \mathbb{S}^1$, is the projection of $\Phi(p) \in \mathbb{S}^2 \setminus \{N, S\}$ on $\mathbb{S}^1$ along the meridians of $\mathbb{S}^2$.

Then the topological degrees of $\Phi$ and $\varphi$ are opposite.

We consider now the torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. As a direct consequence of Lemma 2.3 ans Corollary 2.4 one gets:
Corollary 2.5. Let $\Phi \colon T^2 \to S^2$ be a continuous map so that $\Phi^{-1}(N) = \{0\} \times \mathbb{R}/\mathbb{Z}$ and $\Phi^{-1}(S) = \{\frac{1}{2}\} \times \mathbb{R}/\mathbb{Z}$.

Let $\varphi_+ : \{\frac{1}{2}\} \times \mathbb{R}/\mathbb{Z} \to S^1$ (resp. $\varphi_- : \{\frac{1}{2}\} \times \mathbb{R}/\mathbb{Z} \to S^1$) be defined as follows: the point $\varphi_+(p)$ (resp. $\varphi_-(p)$) is the projection of $\Phi(p) \in S^2 \setminus \{N, S\}$ on $S^1$ along the meridians of $S^2$.

Then

$$|\deg(\Phi)| = |\deg(\varphi_+) - \deg(\varphi_-)|$$

where $\deg(\cdot)$ denotes the topological degree, and $\{\frac{1}{2}\} \times \mathbb{R}/\mathbb{Z}$ and $\{\frac{3}{2}\} \times \mathbb{R}/\mathbb{Z}$ are endowed with the positive orientation of $\mathbb{R}/\mathbb{Z}$.

Proof. Indeed $\Phi$ is homotopic (by an homotopy preserving $\Phi^{-1}(N)$ and $\Phi^{-1}(S)$) to the map $\Phi_{d^+, d^-} : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to S^2$ defined as follows

- $\Phi_{d^+, d^-}(s, t) = \left(|\sin(2\pi s)| \cdot e^{2\pi i d^+ t}, \cos(2\pi s)\right)$ if $s \in [0, \frac{1}{2}]$,
- $\Phi_{d^+, d^-}(s, t) = \left(|\sin(2\pi s)| \cdot e^{2\pi i d^- t}, \cos(2\pi s)\right)$ if $s \in [0, \frac{1}{2}]$.

where $d^+$ and $d^-$ are $\deg(\varphi_+)$ and $\deg(\varphi_-)$, respectively. \qed

2.3. Commuting vector fields: local version. There are two usual definitions for commuting vector fields: we can require that the flows of $X$ and $Y$ commute; one may also require that the Lie bracket $[X, Y]$ vanishes. These two definitions coincide for $C^1$ vector fields on compact manifolds. On non compact manifolds we just have the commutation of the flows for small times, as explained more precisely below.

Let $M$ be a (not necessarily compact) manifold and $X, Y$ be $C^1$-vector fields on $M$. The Cauchy-Lipschitz theorem asserts that the flow of $X$ and $Y$ are locally defined but they may not be complete.

We say that $X$ and $Y$ commute if for every point $x$ there is $t(x) > 0$ so that for every $s, t \in [-t(x), t(x)]$ the compositions $X_t \circ Y_s(x)$ and $Y_s \circ X_t(x)$ are defined and coincide. Thus, the local diffeomorphism $X_t$ carries integral curves of $Y$ into integral curves of $Y$, and vice-versa.

Next section states straightforward consequences of this definition.

2.4. Commuting vector fields: first properties. If $X$ and $Y$ are commuting vector fields then:

1. for every $a, b, c, d \in \mathbb{R}$, $aX + bY$ commutes with $cX + dY$;
2. for every $x \in M$, and any $t \in \mathbb{R}$ for which $Y_t$ is defined, one has

$$DY_t(x)X(x) = X(Y_t(x))$$

3. if $x \in \text{Zero}(X)$, then, for any $t \in \mathbb{R}$ for which $Y_t$ is defined, $Y_t(x) \in \text{Zero}(X)$.
4. $\text{Col}(X, Y)$ is invariant under the flow of $X$ in the following meaning: if $x \in \text{Col}(X, Y)$ and if $X_t(x)$ is defined, then $X_t(x) \in \text{Col}(X, Y)$; in the same way, $\text{Col}(X, Y)$ is invariant under the flow of $aX + bY$ for any $a, b \in \mathbb{R}$.
5. if $\text{Zero}(Y) = \emptyset$, then, for each point $x$ in $\text{Col}(X, Y)$ there is $\mu(x) \in \mathbb{R}$ so that $X(x) = \mu(x)Y(x)$. The map $x \mapsto \mu(x)$ is called the ratio between $X$ and $Y$ at $x$ and is continuous on $\text{Col}(X, Y)$ and can be extended in a $C^1$ map on the ambient manifold.
6. the ratio $\mu$ defined on $\text{Col}(X, Y)$ is invariant under the flow of $aX + bY$ for any $a, b \in \mathbb{R}$.
(7) if \( \gamma \) is a periodic orbit of \( X \) of period \( \tau \) and if \( t \in \mathbb{R} \) is such that \( Y_t \) is defined on \( \gamma \), then \( Y_t(\gamma) \) is a periodic orbit of \( X \) of period \( \tau \). The same occurs with the images by the flow of \( cX + dY \) of periodic orbits of \( aX + bY \), for \( a, b, c, d \in \mathbb{R} \).

(8) as a consequence of the previous item, if \( \gamma \) is a periodic orbit of \( X \) of period \( \tau \) and if \( \gamma \) is isolated among the periodic orbits of \( X \) of the same period \( \tau \), then \( \gamma \) is invariant under the flow of \( cX + dY \); as a consequence, \( \gamma \subset \text{Col}(X, Y) \).

2.5. Counter examples to Theorem \([A]\). Our proof is a long proof by reductio ad absurdum. To achieve this goal, we shall first show that the existence of a pair \((X, Y)\) which do not satisfy the conclusion of Theorem \([A]\) implies the existence of other pairs with simpler geometric behaviors.

For this reason, it will be convenient to define the notion of counter examples in a formal manner.

**Definition 2.6.** Let \( M \) be a 3-manifold, \( U \) a compact subset of \( M \) and \( X, Y \) be \( C^1 \) vector fields on \( M \). We say that \((U, X, Y)\) is a counter example to Theorem \([A]\) if

- \( X \) and \( Y \) commute
- \( \text{Zero}(Y) \cap U = \emptyset \)
- \( \text{Zero}(X) \cap \partial U = \emptyset \)
- \( \text{Ind}(X, U) \neq 0 \)
- the collinearity locus, \( \text{Col}(X, Y, U) \), is contained in a \( C^1 \) surface which is a closed submanifold of \( M \).

Let us illustrate our simplifying procedure by a simple argument:

**Remark 2.7.** If \( M \) is a 3-manifold carrying a counter example \((U, X, Y)\) to Theorem \([A]\), then there is an orientable manifold carrying a counter example to Theorem \([A]\). Indeed consider the orientation cover \( \tilde{M} \to M \) and \( \tilde{U}, \tilde{X}, \tilde{Y} \) the lifts of \( U, X, Y \) on \( \tilde{M} \). Then the Poincaré-Hopf index of \( \tilde{X} \) on \( \tilde{U} \) is twice the one of \( X \) on \( U \), and \((\tilde{U}, \tilde{X}, \tilde{Y})\) is a counter example to Theorem \([A]\).

Thus we can assume (and we do it) without loss of generality that \( M \) is orientable.

Most of our simplifying strategy will now consist in combinations of the following remarks

**Remark 2.8.** If \((U, X, Y)\) is a counter example to Theorem \([A]\) then there is \( \varepsilon > 0 \) so that \((U, aX + bY, cX + dY)\) is also a counter example to Theorem \([A]\) for every \( a, b, c, d \) with \(|a - 1| < \varepsilon, |b| < \varepsilon, |c| < \varepsilon \) and \(|d - 1| < \varepsilon \).

**Remark 2.9.** If \((U, X, Y)\) is a counter example to Theorem \([A]\) then \((V, X, Y)\) is also a counter example to Theorem \([A]\) for any compact set \( V \subset U \) containing \( \text{Zero}(X, U) \) in its interior.

**Remark 2.10.** Let \((U, X, Y)\) be a counter example to Theorem \([A]\) and assume that \( \text{Zero}(X, U) = K_1 \cup \cdots \cup K_n \), where the \( K_i \) are pairwise disjoint compact sets. Let \( U_i \subset U \) be compact neighborhood of \( K_i \) so that the \( U_i, i = 1, \ldots, n \), are pairwise disjoint.

Then there is \( i \in \{1, \ldots, n\} \) so that \((U_i, X, Y)\) is a counter example to Theorem \([A]\).
3. Prepared counter examples \((U, X, Y)\) to Theorem \(A\)

3.1. Simplifying \(\text{Col}(X, Y, U)\). The aim of this section is to prove

**Lemma 3.1.** If \((U, X, Y)\) is a counter example to Theorem \(A\) then there is a counter example \((\tilde{U}, \tilde{X}, \tilde{Y})\) to Theorem \(A\) and \(\mu_0 > 0\) with the following property:

- for any \(t \in [-\mu_0, \mu_0]\), the set of zeros of \(\tilde{X} - t\tilde{Y}\) in \(\tilde{U}\) consists precisely in 1 periodic orbit \(\gamma_t\) of \(\tilde{Y}\);
- for any \(t \notin [-\mu_0, \mu_0]\), the set of zeros of \(\tilde{X} - t\tilde{Y}\) in \(\tilde{U}\) is empty;
- \(\text{Col}(X, Y, U)\) is a \(C^1\) annulus;
- there is a \(C^1\)-diffeomorphism \(\varphi : \mathbb{R}/\mathbb{Z} \times [-\mu_0, \mu_0] \to \text{Col}(\tilde{X}, \tilde{Y}, \tilde{U})\) so that, for every \(t \in [-\mu_0, \mu_0]\), one has

\[
\varphi(\mathbb{R}/\mathbb{Z} \times \{t\}) = \gamma_t.
\]

**Proof.** By hypothesis \(\text{Col}(X, Y, U)\) is contained in a \(C^1\)-surface \(S\). Notice that there is \(\mu_1 > 0\) so that for any \(t \in [-\mu_1, \mu_1]\) one has \(\text{Zero}(X - tY) \cap \partial U = \emptyset\) and \(\text{Ind}(X - tY, U) \neq 0\).

As \(X - tY\) and \(Y\) commute, \(\text{Zero}(X - tY, U)\) is invariant under the flow of \(Y\). Furthermore, as \(\text{Zero}(X - tY)\) does not intersect \(\partial U\) the \(Y\)-orbit of a point \(x \in \text{Zero}(X - tY, U)\) remains in the compact set \(U\) hence is complete.

Consider now the ratio function \(\mu : \text{Col}(X, Y) \to \mathbb{R}\), defined in the item \(A\) of Subsection \(2.4\). It follows that, for \(x \in \text{Col}(X, Y)\), \(\mu(x) = t \Leftrightarrow x \in \text{Zero}(X - tY)\).

The map \(\mu\) is invariant under the flows of \(X\) and \(Y\) (on \(\text{Col}(X, Y)\)). As mentioned in \(2.4\) the map \(\mu\) can be extended on \(M\) as \(C^1\) map still denoted \(\mu\) (but no more \(X, Y\)-invariant).

Let \(L = \bigcup_{t \in [-\mu_1, \mu_1]} \text{Zero}(X - tY) \cap U\). Then \(L\) is a compact set, contained in \(S\) disjoint from the boundary of \(U\) and invariant under \(Y\): it is a compact lamination of \(S\).

Let \(\sigma \subset S\) be a union of finitely many compact segments with end points out of \(L\) and so that the interior of \(\sigma\) cuts transversely every orbit of \(Y\) contained in \(L\).

**Claim 1.** Lebesgue almost every \(t \in [-\mu_1, \mu_1]\) is a regular value of the restriction of \(\mu\) to \(\sigma\).

**Proof.** Recall that Sard’s theorem requires a regularity \(n - m + 1\) if one consider maps from an \(m\)-manifold to an \(n\)-manifold. As \(\mu\) is \(C^1\) and \(\text{dim } \sigma = 1\) we can apply Sard’s theorem to the restriction of \(\mu\) to \(\sigma\), concluding.

Consider now a regular value \(t \in (-\mu_1, \mu_1)\) of the restriction of \(\mu\) to \(\sigma\). Then \(\mu^{-1}(t) \cap \sigma\) consists in finitely many points. Furthermore, \(\mu^{-1}(t) \cap \sigma\) contains \(\text{Zero}(X - tY) \cap \sigma\).

**Claim 2.** For \(t \in [-\mu_1, \mu_1]\), regular value of the restriction of \(\mu\) to \(\sigma\), the compact set \(\text{Zero}(X - tY) \cap U\) consists in finitely many periodic orbits \(\gamma_i\), \(i \in \{1, \ldots, n\}\) of \(Y\).

**Proof.** \(\text{Zero}(X - tY) \cap U\) is a compact sub lamination of \(L \subset S\) consisting of orbits of \(Y\), and contained in \(\mu^{-1}(t)\). Now, \(\sigma\) cuts transversely each orbit of this lamination and \(\sigma \cap \mu^{-1}(t)\) is finite. One deduces that \(\text{Zero}(X - tY) \cap U\) consists in finitely many compact leaves, concluding.
Notice that $\text{Ind}(X - tY, U) = \sum_{i=1}^{n} \text{Ind}(X - tY, \gamma_i)$. Thus there is $i$ so that

$$\text{Ind}(X - tY, \gamma_i) \neq 0.$$ 

**Claim 3.** There is a neighborhood $\Gamma_i$ of $\gamma_i$ in $S$ which is contained in $\text{Col}(X, Y, U)$ and which consists in periodic orbits of $Y$.

**Proof.** Let $p$ be a point in $\sigma \cap \gamma_i$. As $p$ is a regular point of the restriction of $\mu$ to $\sigma$ there is a segment $I \subset \sigma$ centered at $p$ so that the restriction of $\mu$ to $I$ is injective and the derivative of $\mu$ does not vanish.

As $\gamma_i$ has non-zero index for any $s$ close enough to $t$, $\text{Zero}(X - sY)$ contains an isolated compact subset $K_s$ contained in a small neighborhood of $\gamma_i$, and hence in $U$, thus in $\text{Col}(X, Y, U)$ and thus in a small neighborhood of $\gamma_i$ in $\mathcal{L} \subset S$. This implies that each orbit of $Y$ contained in $K_s$ cuts $I$. However, $\mu$ is constant equal to $s$ on $K_s$ and thus $\mu^{-1}(s) \cap I$ consist in a unique point. One deduces that $K_s$ is a compact orbit of $Y$.

Since this holds for any $s$ close to $t$, one obtain that any point $q$ of $I$ close to $p$ is the intersection point of $K_{\mu(q)} \cap I$. In other words, a neighborhood of $p$ in $I$ is contained in $\text{Col}(X, Y, U)$ and the corresponding leaf of $\mathcal{L}$ is a periodic orbit of $Y$, concluding. \hfill $\Box$

Notice that $\Gamma_i$ is contained in $\text{Col}(X, Y, U)$ so that the function $\mu$ is invariant under $Y$ on $\Gamma_i$. As the derivative of $\mu$ is non vanishing (by construction) on $\Gamma_i \cap I$ one gets that the derivative of the restriction of $\mu$ to $\Gamma_i$ in non-vanishing. One deduces that $\Gamma_i$ is diffeomorphic to an annulus: it is foliated by circles and these circles admit a transverse orientation.

For concluding the proof it remains to choose a compact neighborhood $\bar{U}$ of $\gamma_i$ in $M$, which is a manifold with boundary, whose boundary is tranverse to $S$ and so that $\bar{U} \cap S = \Gamma_i$. \hfill $\Box$

### 3.2. Prepared counter examples to Theorem A

**Definition 3.2.** We say that $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem A if

1. $(U, X, Y)$ is a counter example to Theorem A
2. There is $\mu_0 > 0$ so that $(U, X, Y)$ satisfies the conclusion of Lemma 3.1:
   - for any $t \in [-\mu_0, \mu_0]$, the set of zeros of $X - tY$ in $U$ consists precisely in 1 periodic orbit $\gamma_t$ of $Y$;
   - for any $t \notin [-\mu_0, \mu_0]$, the set of zeros of $X - tY$ in $U$ is empty;
   - $\text{Col}(X, Y, U)$ is a $C^1$ annulus;
   - there is a $C^1$-diffeomorphism $\varphi: \mathbb{R}/\mathbb{Z} \times [-\mu_0, \mu_0] \to \text{Col}(X, Y, U)$ so that, for every $t \in [-\mu_0, \mu_0]$, one has
     $$\varphi(\mathbb{R}/\mathbb{Z} \times \{t\}) = \gamma_t.$$
3. $U$ is endowed with a foliation by discs; more precisely there is a smooth submersion $\Sigma: U \to \mathbb{R}/\mathbb{Z}$ whose fibers $\Sigma_t = \Sigma^{-1}(t)$ are discs; furthermore, the vector field $Y$ is transverse to the fibers $\Sigma_t$.
4. Each periodic orbit $\gamma_s$, $s \in [-\mu_0, \mu_0]$, of $Y$ cuts every disc $\Sigma_t$ in exactly one point. In particular the period of $\gamma_s$ coincides with its return time on $\Sigma_0$ and is denoted $\tau(s)$, for $s \in [-\mu_0, \mu_0]$.
   - Thus $s \mapsto \tau(s)$ is a $C^1$-map on $[-\mu_0, \mu_0]$. We require that the derivative of $\tau$ does not vanish on $[-\mu_0, \mu_0]$. 

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(5) $\mathcal{B}$ is a triple $(e_1, e_2, e_3)$ of $C^0$ vector fields on $U$ so that
- for any $x \in U$, $\mathcal{B}(x) = (e_1(x), e_2(x), e_3(x))$ is a basis of $T_xU$.
- $e_3 = Y$ everywhere
- the vectors $e_1, e_2$ are tangent to the fibers $\Sigma_t$, $t \in \mathbb{R}/\mathbb{Z}$. In other words, 
  $D\Sigma(e_1) = D\Sigma(e_2) = 0$
- The vector $e_1$ is tangent to $\text{Col}(X, Y)$ at each point of $\text{Col}(X, Y)$.

**Lemma 3.3.** If there exists a counter example $(U, X, Y)$ to Theorem $\text{A}$ then there is a prepared counter example $(\tilde{U}, \tilde{X}, \tilde{Y}, \Sigma, \mathcal{B})$ to Theorem $\text{A}$.

**Proof.** The two first items of the definition of prepared counter example to Theorem $\text{A}$ are given by Lemma 3.1. For getting the third item, it is enough to shrink $U$. For getting item 4, one replace $Y$ by $Y + bX$ for some $b \in \mathbb{R}, |b|$ small enough. This does not change the orbits $\gamma_t$, as $X$ and $Y$ are both tangent to $\gamma_t$, but it changes its period. Thus, this allows us to change the derivative of the period $\tau$ at $s = 0$. Then one shrink again $U$ and $\mu_0$ so that the derivative of $\tau$ will not vanish on $\text{Col}(X, Y, U)$.

**Remark 3.4.** If $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem $\text{A}$, then for every $t \in (-\mu_0, \mu_0)$, $(U, X - ty, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem $\text{A}$.

Whenever $(U, X, Y, \Sigma, \mathcal{B})$ is a prepared counter example to Theorem $\text{A}$ we shall denote by $P$ the first return map, defined on a neighborhood of $\text{Col}(X, Y) \cap \Sigma_0$ in $\Sigma_0$.

**Remark 3.5.** As the ambient manifold is assumed to be orientable (see Remark 2.7), the vector field $Y$ is normally oriented so that the Poincaré map $P$ preserves the orientation.

### 3.3. Counting the index of a prepared counter example.

**Definition 3.6.** Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem $\text{A}$.

In particular, $U$ is a solid torus ($C^1$-diffeomorphic to $D^2 \times \mathbb{R}/\mathbb{Z}$) and $\text{Zero}(X - ty)$, $t \in (-\mu_0, \mu_0)$, is an essential simple curve $\gamma_t$ isotopic to $\{0\} \times \mathbb{R}/\mathbb{Z}$. An essential torus $T$ is the image of a continuous map from the torus $T^2$ in the interior of $U$, disjoint from $\gamma_0 = \text{Zero}(X)$ and homotopic, in $U \setminus \gamma_0$, to the boundary of a tubular neighborhood of $\gamma_0$.

In other words, $H_2(U \setminus \gamma_0, \mathbb{Z}) = \mathbb{Z}$, and $T$ is essential if it is the generator of this second homology group.

We shall now describe how we use the basis $\mathcal{B}$, which comes with a prepared counter example, and an essential torus $T$ to calculate the index.

Consider, for each point $x \in U$, the expression of $X$ in the basis $\mathcal{B}$:

$$X(x) = \alpha(x)e_1(x) + \beta(x)e_2(x) + \mu(x)e_3(x).$$

(3.1)

For $x \notin \gamma_0$ one considers the vector

$$X(x) = \frac{1}{\sqrt{\alpha(x)^2 + \beta(x)^2 + \mu(x)^2}}(\alpha(x), \beta(x), \mu(x)) \in \mathbb{S}^2.$$  

(3.2)

The map restriction $X|_T: T \to \mathbb{S}^2$ has a topological degree, which, by Lemma 2.2, coincides with $\text{Ind}(X, U)$, for some choice of an orientation on $T$.  

---
Theorem A. (Open condition, there exists an interval $[\mu, t]$ holds. Consider $\tilde{t}$ some point of $\text{Col}(X, Y, \Sigma)$ at every point of $\text{Col}(X, Y, U)$.)

Proof. The argument is by contradiction. We assume that the derivative of $P$ at some point $x$ of $\text{Col}(X, Y, U) \cap \Sigma_0$ has some eigenvalue of different from 1.

Let us denote $x_t = \gamma_t \cap \Sigma_0$, $t \in [-\mu_0, \mu_0]$, (recall $\gamma_t = \text{Zero}(X - tY)$). Notice that the first return map $P$ is the identity map in restriction to the segment $\text{Col}(X, Y) \cap \Sigma_0$. In particular, the derivative of $P$ at $x_t$ admits 1 as an eigenvalue. Since $P$ preserves the orientation (see Remark 3.5), the other eigenvalue is positive.

Claim 4. There exists $\tilde{U} \subset U$, $\tilde{X} = X - tY$ and a prepared counter example to Theorem $[A] (\tilde{U}, \tilde{X}, Y, \Sigma, B)$ for which the surface $\text{Col}(\tilde{X}, Y, \tilde{U})$ is normally hyperbolic.

Proof. As the property of having a eigenvalue of modulus different from 1 is an open condition, there exists an interval $[\mu_1, \mu_2] \subset [-\mu_0, \mu_0]$ on which the condition holds. Consider $t = \frac{\mu_1 + \mu_2}{2}$ and $\tilde{X} = X - tY$.

Then, one obtains a new prepared counter example to Theorem $[A]$ by replacing $X$ by $\tilde{X}$; now, by shrinking $U$ one gets a tubular neighborhood $\tilde{U}$ of $\gamma_t$ so that $\text{Col}(\tilde{X}, Y, \tilde{U}) = \bigcup_{s \in [\mu_1, \mu_2]} \gamma_s$.

Moreover, the derivative of $P$ at each point $x_s$, $s \in [\mu_1, \mu_2]$, has an eigenvalue different from 1 in a direction transverse to $\text{Col}(\tilde{X}, Y, \tilde{U}) \cap \Sigma_0$. By compactness and continuity these eigenvalues are uniformly far from 1 so that $\text{Col}(\tilde{X}, Y, \tilde{U}) \cap \Sigma_0$ is normally hyperbolic for $P$.

Thus $\text{Col}(\tilde{X}, Y, \tilde{U})$ is an invariant normally hyperbolic annulus for the flow of $Y$. □

By virtue of the above claim (up to change $X$ by $\tilde{X}$ and $U$ by $\tilde{U}$) one may assume that $\text{Col}(X, Y, U)$ is normally hyperbolic, and (up change $Y$ by $-Y$) one may assume that $\text{Col}(X, Y, U)$ is normally contracting.

This implies that every periodic orbit $\gamma_t$ has a local stable manifold $W^s(\gamma_t)$ which is a $C^1$-surface depending continuously on $t$ for the $C^1$-topology and the collection of these surfaces build a $C^0$-foliation $F^s$ tangent to a continuous plane field $E^s$, in a neighborhood of $\text{Col}(X, Y, U)$. Furthermore, $E^s$ is tangent to $Y$, and hence is transverse to the fibers of $\Sigma$.

Up to shrink $U$, one may assume that $F^s$ and $E^s$ are defined on $U$.

Claim 5. There is a basis $\tilde{B} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ so that $(U, X, Y, \Sigma, \tilde{B})$ is a prepared counter example to Theorem $[A]$ and $\tilde{e}_2$ is tangent to $E^s$.

Proof. Choose $\tilde{e}_2$ as being a unit vector tangent to the intersection of $E^s$ with the tangent plane of the fibers of $\Sigma$. It remains to choose $\tilde{e}_1$ transverse to $\tilde{e}_2$ and tangent to the fibers of $\Sigma$ and tangent to $\text{Col}(X, Y)$ at every point of $\text{Col}(X, Y)$. □
Lemma 4.4. For every \( \tau \) notation above, it coincides with \( \tau(0) = 0 \) and every \( v \) in \( T_x \Sigma_0 \) one has,
\[
DP_t(x)v = D\tau_t(x)v = D\tau_t(x)v = D\tau_t(x)v = D\tau_t(x)v = \tau_t(x)v.
\]
In particular, for every \( x \in \Sigma_0 \) and \( v \in T_x \Sigma_0 \), one has
\[
DP_t(x)v = D\tau_t(x)v = D\tau_t(x)v = D\tau_t(x)v = D\tau_t(x)v = \tau_t(x)v.
\]

4. Holonomies, return time, and the normal component of \( X \)

4.1. Definitions. Recall that \( \Sigma_0 = \Sigma^{-1}(t), \) \( t \in \mathbb{R}/\mathbb{Z} \), is a family of cross section, each \( \Sigma_0 \) is a local diffeomorphism to a disc, and we identify \( \Sigma_0 \) with the unit disc \( \mathbb{D}/\mathbb{Z} \).

**Definition 4.1.** Consider \( t \in \mathbb{Z} \). Consider \( x \in \Sigma_0 \) and \( y \in \Sigma_t \). We say that \( y \) is the image by holonomy of \( Y \) over the segment \([0, t]\), and we denote \( y = P_t(x) \), if there exists a continuous path \( x_r \in U, r \in [0, t] \), so that \( \Sigma(x_r) = r, x_0 = x, y_1 = y \), and for every \( r \in [0, t] \) the point \( x_r \) belongs to the \( U \)-orbit of \( x \).

The holonomy map \( P_t \) is well defined in a neighborhood of \( \text{Col}(X, Y, U) \cap \Sigma_0 \) and is a \( C^1 \) local diffeomorphism.

If \( t = 1 \) then \( P_1 \) is the first return map \( \mathcal{P} \) (defined before Remark 3.3) of the flow of \( Y \) on the cross section \( \Sigma_0 \).

**Remark 4.2.** With the notation of Definition 1.1, there is a unique continuous function \( \tau_x : [0, t] \to \mathbb{R} \) so that \( \tau_x(0) = 0 \) and \( \tau_x(t) = \gamma_t \) for every \( r \in [0, t] \).

We denote \( \tau_t(x) = \tau_t(t) \) and we call it the transition time from \( \Sigma_0 \) to \( \Sigma_t \). The map \( \tau_1 : \Sigma_0 \to \mathbb{R} \) is a \( C^1 \) map and by definition one has
\[
P_t(x) = Y_{\tau_t(x)}(x) \quad (4.1)
\]
We denote \( \tau = \tau_1 \) and we call it the first return time of \( Y \) on \( \Sigma_0 \).

**Remark 4.3.** In Definition 3.2 item 4 we defined \( \tau(s) \) as the period of \( \gamma_s \); in the notation above, it coincides with \( \tau(x_0) \) where \( x_0 = \gamma_s \cap \Sigma_0 \).

In this case, Equation (4.1) takes the special form
\[
\mathcal{P}(x) = Y_{\tau(x)}(x) \quad (4.2)
\]

By taking derivatives in Equation (4.1) one gets

**Lemma 4.4.** For every \( t \in \mathbb{R}, x \in \Sigma_0 \) and every \( v \in T_x \Sigma_0 \) one has,
\[
DP_t(x)v = D\tau_t(x)v = \tau_t(x)v.
\]
In particular, for every \( x \in \Sigma_0 \) and \( v \in T_x \Sigma_0 \), one has
\[
DP(x)v = D\tau(x)v = \tau(x)v.
\]
4.2. The normal component of $X$: invariance by the holonomies of $Y$.

**Definition 4.5.** For every $t$ and every $x \in \Sigma_i$ we define the normal component of $X$, which we denote by $N(x)$, the projection of $X(x)$ on $T_x \Sigma_i$ parallel to $Y(x)$.

Thus $x \mapsto N(x)$ is a $C^1$-vector field tangent to the fibers of $\Sigma$ and which vanishes precisely on $\text{Col}(X, Y, U)$.

Moreover, in the basis $\mathcal{B}$, $N(x) = \alpha(x)e_1(x) + \beta(x)e_2(x)$ (see Equation 3.1), and we have the following formula

$$X(x) = N(x) + \mu(x)Y(x),$$

for every $x \in U$.

A fundamental property of the normal component of $X$ is that it is invariant under the derivative of the holonomies. This is the content of the following lemma.

**Lemma 4.6.** For $t \in \mathbb{R}$ and every $x \in \Sigma_0$ one has

$$DP_t(x)N(x) = N(P_t(x)).$$

**Proof.** Combining item (2) of Subsection 2.4, Lemma 4.4 together with Equation 4.1 one obtain

$$X(P_t(x)) = DP_t(x)N(x) + (\mu(x) - D\tau_t(x)N(x))Y(P_t(x)).$$

Due to the very definition of $N$, this gives the result. \hfill $\square$

4.3. The first return map and the derivative of the first return time. In the result below we relate the derivative of the first return time with the function $\mu$.

**Corollary 4.7.** Let $x \in \Sigma_0$. Then, $-D\tau(x)N(x) = \mu((\mathcal{P}(x))) - \mu(x)$.

**Proof.** Since $X(x) = N(x) + \mu(x)Y(x)$ by item (2) of Subsection 2.4 it follows that

$$X(\mathcal{P}(x)) = DY_{\tau(x)}(x)N(x) + \mu(x)Y(\mathcal{P}(x)).$$

On the other hand, by Lemma 4.6 we have $DP(x)N(x) = N(\mathcal{P}(x))$. This enables us to write

$$X(\mathcal{P}(x)) = DP(x)N(x) + \mu(\mathcal{P}(x))Y(\mathcal{P}(x)).$$

Now Equation 4.5 states

$$X(\mathcal{P}(x)) = DP(x)N(x) + (\mu(x) - D\tau(x)N(x))Y(\mathcal{P}(x)),$$

concluding. \hfill $\square$

4.4. The normal component of $X$ and the index of $X$. Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem A and $N$ be the normal component of $X$.

Let us define, for $x \in U \setminus \text{Col}(X, Y, U)$,

$$\mathcal{N}(x) = \frac{1}{\sqrt{\alpha(x)^2 + \beta(x)^2}}(\alpha(x), \beta(x)) \in S^1 \subset S^2,$$

where $S^1$ is the unit circle of the plane $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

Recall that $U$ is homeomorphic to the solid torus so that its first homology group $H_1(U, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, by an isomorphism sending the class of $\gamma_0$, oriented by $Y$, on 1. Let $U^+$ and $U^-$ be the two connected components of $U \setminus \text{Col}(X, Y, U)$. These are also solid tori, and the inclusion in $U$ induces isomorphisms of the first homology groups which allows us to identify $H_1(U^\pm, \mathbb{Z})$ with $\mathbb{Z}$. 

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**Definition 4.8.** We call linking number of $X$ with respect to $Y$ in $U^+$ (resp. in $U^-$) and we denote it by $\ell^+(X,Y)$ (resp. $\ell^-(X,Y)$) the integer defined as follows: the continuous map $N:U^\pm \rightarrow S^1$ induces morphisms on the homology groups $H_1(U^\pm,\mathbb{Z}) \rightarrow H_1(S^1,\mathbb{Z})$. As these groups are all identified with $\mathbb{Z}$, these morphisms consist in the multiplication by an integer $\ell^\pm(X,Y)$.

In other words, consider a closed curve $\sigma \subset U^+$ homotopic in $U$ to $\gamma_0$. Then $\ell^+(X,Y)$ is the topological degree of the restriction of $N$ to $\sigma$.

**Proposition 4.9.** Let $(U,X,Y,\Sigma,B)$ be a prepared counter example to Theorem A, then

$$|\text{Ind}(X,U)| = |\ell^+(X,Y) - \ell^-(X,Y)|$$

**Proof.** Consider a tubular neighborhood of $\gamma_0$ whose boundary is an essential torus $T$ which cuts $\text{Col}(X,Y,U)$ transversely and along exactly two curves $\sigma_+ \text{ and } \sigma_-$. Then the map $X$ on $T$ takes the value $N \in S^2$ (resp. $S \in S^2$) exactly on $\sigma_+$ (resp. $\sigma_-$), where $N$ and $S$ are the points on $S^2$ corresponding to $e_3 = Y$ and $-e_3$.

We identify $T$ with $T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ so that $\sigma_-$ and $\sigma_+$ correspond to $\{2\} \times \mathbb{R}/\mathbb{Z}$ and $\{0\} \times \mathbb{R}/\mathbb{Z}$ respectively. It remains to apply Corollary 4.12 to $\Phi = \Sigma$ and $\varphi = N^\perp$, noticing that $N^\perp$ is the projection of $X$ on $S^1$ along the meridians. This gives the announced formula. \qed

As a direct consequence of Proposition 4.9 one gets

**Corollary 4.10.** Let $(U,X,Y,\Sigma,B)$ be a prepared counter example to Theorem A, then

$$(\ell^+(X,Y),\ell^-(X,Y)) \neq (0,0)$$

4.5. **Angular variation of the normal component $N$ along the $Y$-orbits.**

We denote $x_t = \gamma_t \cap \Sigma_0$. For every pair of points $x,y \in \Sigma_0$, we denote the segment of straight line joining $x$ and $y$ and contained in $\Sigma_0$ by $[x,y]$ (for some choice of coordinates on $\Sigma_0$).

**Lemma 4.11.** For any $K > 0$ there is a neighborhood $O_K$ of $\gamma_0$ with the following property.

Consider $x \in O_K \cap \Sigma_0$ and $u \in T_x\Sigma_0$ a unit vector, and write $u = u_1e_1(x) + u_2e_2(x)$. Consider $t \in [0,K]$ and $v = DP_t(u) \in TP_t(\Sigma)$ the image of $u$ by the derivative of the holonomy. Write $v = v_1e_1(P_t(x)) + v_2e_2(P_t(x))$.

Then

$$\left(\frac{u_1}{\sqrt{u_1^2 + u_2^2}}, \frac{u_2}{\sqrt{u_1^2 + u_2^2}}\right) \neq \left(\frac{v_1}{\sqrt{v_1^2 + v_2^2}}, \frac{v_2}{\sqrt{v_1^2 + v_2^2}}\right)$$

**Proof.** Assuming, by contradiction, that the conclusion does not hold we get $y_n \in \Sigma_0$, unit vectors $u_n \in T_{x_n}\Sigma_0$ and $t_n \in [0,K]$ so that $y_n$ tends to $x_0 = \gamma_0 \cap \Sigma_0$, $u_n$ tends to a unit vector $u$ in $T_{x_0}\Sigma_0$, $t_n$ tends to $t \in [0,K]$ and the image $v_n$ of the vector $u_n$, expressed in the basis $B$, is collinear to $u_n$ with the opposite direction.

Then $DP_t(x_0)u$ is a vector that, expressed in the basis $B$, is collinear to $u$ with the opposite direction. In other words, $u$ is an eigenvector of $DP_t(x_0)$, with a negative eigenvalue.

However, for every $t \in \mathbb{R}$ the vector $e_1$ is an eigenvector of $DP_t(x_0)$, with a positive eigenvalue, and $DP_t(x_0)$ preserves the orientation, leading to a contradiction. \qed
Lemma 4.12. For every $N \in \mathbb{N}$ there exists a neighborhood $O_N \subset \Sigma_0$ of $x_0$ so that if $x \in O_N \setminus \text{Col}(X, Y, U)$ then the segment of straight line $[x, \mathcal{P}^N(x)]$ is disjoint from $\text{Col}(X, Y, U)$.

Proof. Assume that there is a sequence of points $y_n \to x_0$, $y_n \notin \text{Col}(X, Y, U)$ so that the segment $[y_n, \mathcal{P}^N(y_n)]$ intersect $\text{Col}(X, Y, U) \cap \Sigma_0$ at some point $z_n$. Recall that $\text{Col}(X, Y, U) \cap \Sigma_0$ consists in fixed points of the first return map $\mathcal{P}$. In particular, $z_n$ is a fixed point of $\mathcal{P}^N$.

The image of the segment $[z_n, y_n]$ is a $C^1$ curve joining $z_n$ to $\mathcal{P}^N(y_n)$. Notice that the segments $[z_n, y_n]$ and $[z_n, \mathcal{P}^N(y_n)]$ are contained in the segment $[y_n, \mathcal{P}^N(y_n)]$ and oriented in opposite direction. One deduces that there is a point $w_n$ in $[y_n, z_n]$ so that the image under the derivative $D\mathcal{P}^N$ of the unit vector $u_n$ directing this segment is on the form $\lambda_n u_n$ with $\lambda_n < 0$.

Since $y_n$ tends to $x_0$, one deduces that $D\mathcal{P}^N(x_0)$ has a negative eigenvalue. This contradicts the fact that both eigenvalues of $D\mathcal{P}^N(x_0)$ are positive and completes the proof.

Recall that $U^+$ and $U^-$ are the connected components of $U \setminus \text{Col}(X, Y, U)$.

Corollary 4.13. If $x \in U^+ \cap \Sigma_0 \cap O_3$ let $\theta_x : \mathbb{R}/\mathbb{Z} \to U$ be the curve obtained by concatenation of the $Y$-orbit segment from $x$ to $\mathcal{P}^2(x)$ and the straight line segment $[\mathcal{P}^2(x), x]$, that is:

- for $t \in [0, \frac{1}{2}]$, $\theta_x(t) = Y_{2t, \tau_2(x)}(x)$ where $\tau_2$ is the transition time from $x$ to $\mathcal{P}^2(x)$
- for $t \in [\frac{1}{2}, 1]$, $\theta_x(t) = (2 - 2t)\mathcal{P}^2(x) + (2t - 1)x$.

Then $\theta_x$ is a closed curved contained in $U^+$ and whose homology class in $H_1(U^+, \mathbb{Z}) = \mathbb{Z}$ is $2$.

Proof. The unique difficulty here is that the curve don’t cross $\text{Col}(X, Y, U)$ and that is given by Lemma 4.12.

The next corollary is one of the fundamental arguments of this paper.

Corollary 4.14. Let $(U, X, Y, \Sigma, \mathcal{B})$ be a prepared counter example to Theorem A and assume that $\ell^+(X, Y) \neq 0$.

Consider $x \in U^+ \cap \Sigma_0 \cap O_3$. Then the angular variation of the vector $\mathcal{N}(y)$ for $y \in [x, \mathcal{P}^2(x)]$ is strictly larger than $2\pi$ in absolute value. In particular,

$\mathcal{N}([x, \mathcal{P}^2(x)]) = \mathbb{S}^1$.

The same statement holds in $U^-$ if $\ell^-(X, Y) \neq 0$.

Proof. Lemma 4.11 implies that the angular variation of $\mathcal{N}(\theta_x(t))$ is contained in $(-\pi, \pi)$ for $t \in [0, \frac{1}{2}]$. However, the topological degree of the map $\mathcal{N} : \theta_x \to \mathbb{S}^1$ is $2\ell^+(X, Y)$ which has absolute value at least 2. Thus the angular variation of $\mathcal{N}$ on the segment $[x, \mathcal{P}^2(x)]$ is (in absolute value) at least $3\pi$ concluding.

As an immediate consequence one gets

Corollary 4.15. If $\ell^+(X, Y) \neq 0$, then $\mathcal{P}^2$ has no fixed points in $O_3 \cap U^+$. 

4.6. The return map at points where $N$ is pointing in opposite directions.

We have seen in the proof of Corollary 4.14 that the vector $N$ has an angular variation larger than $3\pi$ along the segment $[x, P^2(x)]$, as $x \in \Sigma_0$ approaches $x_0$. In this section we use the large angular variation of $N$ for establishing a relation between the return map $P$, the return time $\tau$, and the coordinate $\mu$ of $X$ in the $Y$ direction.

Recall that, for every $x \in U \setminus \text{Col}(X,Y,U)$, $N(x)$ is a unit vector contained in $S^1$, unit circle of $\mathbb{R}^2$.

**Lemma 4.16.** Assume that there exists sequences $q_n, \overline{q}_n \in \Sigma_0 \setminus \text{Col}(X,Y,U)$ converging to $x_0$, such that the following two conditions are satisfied:

1. $N(q_n) \to (1,0)$ and $N(\overline{q}_n) \to (-1,0)$, as $n \to +\infty$.
2. $(\mu(P(q_n)) - \mu(q_n))(\mu(P(\overline{q}_n)) - \mu(\overline{q}_n)) \geq 0$, for every $n$.

Then, $D\tau(x_0)e_1(x_0) = 0$.

**Proof.** Recall that $N(x) = \alpha(x)e_1(x) + \beta(x)e_2(x)$ and $N(x) = \frac{1}{\sqrt{\alpha(x)^2 + \beta(x)^2}}(\alpha(x), \beta(x))$ for $x \in U \setminus \text{Col}(X,Y,U)$. By Corollary 4.7 we have

$$-D\tau(q_n) \frac{N(q_n)}{\sqrt{\alpha(q_n)^2 + \beta(q_n)^2}} = \frac{\mu(P(q_n)) - \mu(q_n)}{\sqrt{\alpha(q_n)^2 + \beta(q_n)^2}}, \quad (4.8)$$

and

$$-D\tau(\overline{q}_n) \frac{N(\overline{q}_n)}{\sqrt{\alpha(\overline{q}_n)^2 + \beta(\overline{q}_n)^2}} = \frac{\mu(P(\overline{q}_n)) - \mu(\overline{q}_n)}{\sqrt{\alpha(\overline{q}_n)^2 + \beta(\overline{q}_n)^2}}. \quad (4.9)$$

Multiplying side by side Equations (4.8) and (4.9) and using the second assumption of the lemma, we get

$$D\tau(q_n) \frac{N(q_n)}{\sqrt{\alpha(q_n)^2 + \beta(q_n)^2}} D\tau(\overline{q}_n) \frac{N(\overline{q}_n)}{\sqrt{\alpha(\overline{q}_n)^2 + \beta(\overline{q}_n)^2}} \geq 0.$$

Notice that the first assumption of the lemma is equivalent to $\frac{N(q_n)}{\sqrt{\alpha(q_n)^2 + \beta(q_n)^2}} \to e_1(x_0)$ and $\frac{N(\overline{q}_n)}{\sqrt{\alpha(\overline{q}_n)^2 + \beta(\overline{q}_n)^2}} \to -e_1(x_0)$. Since $q_n, \overline{q}_n \to x_0$, from the continuity of $D\tau$, we conclude that

$$0 \geq - (D\tau(x_0)(e_1(x_0)))^2 \geq 0,$$

which completes the proof. \(\square\)

Since we assumed $(D\tau(x_0)(e_1(x_0))) \neq 0$ (item 4 of the definition of a prepared counter example to Theorem A), one gets the following corollary:

**Corollary 4.17.** Let $(U, X, Y, \Sigma, B)$ is a prepared counter example to Theorem A. Assume that there exists sequences $q_n, \overline{q}_n \in \Sigma_0 \setminus \text{Col}(X,Y,U)$ converging to $x_0$, such that $N(q_n) \to (1,0)$ and $N(\overline{q}_n) \to (-1,0)$, as $n$ tends to $+\infty$. Then

$$(\mu(P(q_n)) - \mu(q_n))(\mu(P(\overline{q}_n)) - \mu(\overline{q}_n)) < 0,$$

for every $n$ large enough.
4.7. The case $DP(x_0) \neq Id$. In this section, $(U, X, Y, \Sigma, B)$ is a prepared counter example to Theorem A so that the derivative of the first return map $\mathcal{P}$ at the point $x_0 = \Sigma_0 \cap \gamma_0$ is not the identity map. Recall that $DP(x_0)$ admits an eigenvalue equal to 1 directed by $e_1(x_0)$, has no eigenvalues different from 1 and is orientation preserving.

Recall that $\text{Col}(X, Y, U)$ cuts the solid torus $U$ in two components $U^+$ and $U^-$. Let denote $\Sigma_+ = \Sigma_0 \cap U^+$ and $\Sigma_- = \Sigma_0 \cap U^-$. Proof. The derivative $D\mu(x_0)(e_1(x_0))$ do not vanish (item 2 of Definition 3.2). Thus the kernel of $D\mu(x_0)$ is transverse to $e_1(x_0)$. The derivative $DP(x_0)$ admits $e_1(x_0)$ as an eigenvector (for the eigenvalue 1) and has no eigenvalue different from 1 and is not the identity. This implies that the kernel of $D\mu(x_0)$ is not an eigendirection of $DP(x_0)$. As a consequence the derivative of the function $x \mapsto \mu(\mathcal{P}(x)) - \mu(x)$ does not vanish at $x = x_0$.

Thus $\{x \in \Sigma, \mu(\mathcal{P}(x)) - \mu(x) = 0\}$ is a codimension 1 submanifold in a neighborhood of $x_0$ in $\Sigma_0$, and this submanifold contains $\text{Col}(X, Y, U)$, because $\text{Col}(X, Y, U) \subset \text{Fix}(\mathcal{P})$. Therefore these submanifolds coincide in the neighborhood of $x_0$, concluding. □

We are now ready to prove the following proposition:

**Proposition 4.19.** If $(U, X, Y, \Sigma, B)$ is a prepared counter example to Theorem A then $DP(x)$ is the identity map for every $x \in \text{Col}(X, Y, U) \cap \Sigma_0$.

Proof. Up to exchange $+$ by $-$, we assume that $\ell^+(X, Y) \neq 0$. Then Corollary 4.14 implies that there are sequences $q_n, \bar{q}_n \in \Sigma_+$ tending to $x_0$ and so that $N(q_n) = (1, 0)$ and $N(-q_n) = (-1, 0)$. More precisely Corollary 4.14 implies that for any $x \in \Sigma_+$ close enough to $x_0$ the segment $[x, \mathcal{P}^2(x)]$ contains points $q, \bar{q}$ with $N(q) = (1, 0)$ and $N(\bar{q}) = (-1, 0)$. Now Lemma 4.12 implies that the segment is contained in $\Sigma_+$, concluding.

Now Corollary 4.17 implies that, for $n$ large enough, the sign of the map $(\mu(\mathcal{P}(x)) - \mu(x))$ is different on $q_n$ and on $\bar{q}_n$. One concludes with Lemma 4.18 which says that this sign cannot change if the derivative $DP(x_0)$ is not the identity. Thus we proved $DP(x_0) = Id$.

Now if $x \in \text{Col}(X, Y, U)$ then $x$ is of the form $x = x_1 = \gamma_0 \cap \Sigma_0$. According to Remark 3.4 $(U, X - tY, Y, \Sigma, B)$ is also a prepared counter example to Theorem A and the linking number of $X - tY$ with $Y$ is the same as the linking number of $X$ with $Y$. Therefore the argument above establishes that $DP(x) = Id$. □
5. Proof of Theorem A

In the whole section, \((U, X, Y, \Sigma, B)\) is a prepared counter example to Theorem A. According to Corollary 4.10 one of the linking numbers \(\ell^+(X, Y), \ell^-(X, Y)\) does not vanish, so that, up to exchange + with −, one may assume \(\ell^+(X, Y) \neq 0\). According to Proposition 4.19 the derivative \(D\mathcal{P}(x)\) is the identity map for every \(x \in \text{Col}(X, Y, U) \cap \Sigma_0\).

This means that, in a neighborhood of \(\text{Col}(X, Y, U)\), the diffeomorphism \(\mathcal{P}\) is \(C^1\) close to the identity map. The techniques introduced in [Bo1] and [Bo2] allow to compare the diffeomorphism \(\mathcal{P}\) with the vector field \(\mathcal{P}(x) - x\), and we will analyse the behavior of this vector field. We will first show that, in the neighborhood of \(\text{Col}(X, Y, U)\) these vectors are almost tangent to the kernel of \(D\mu\). As the fibers of \(D\mu\) are transverse to \(\text{Col}(X, Y, U)\) we get a topological dynamics of \(\mathcal{P}\) similar to the partially hyperbolic case of Section 3.4. We will end contradicting Corollary 4.14.

5.1. Quasi invariance of the map \(\mu\) by the first return map. Recall that \(\mu\) is the coordinate of \(X\) in the \(Y\) direction: \(X(x) = N(x) + \mu(x)Y(x)\). The aim of this section is to prove

**Lemma 5.1.** If \(x_n \in \Sigma_+\) is a sequence of points tending to \(x \in \text{Col}(X, Y, U)\) and if \(v_n \in T_{x_n}\Sigma_+\) is the unit tangent vector directing the segment \([x_n, \mathcal{P}(x_n)]\) then \(D\mu(x_n)(v_n)\) tends to 0.

According to Remark 3.4 it is enough to prove Lemma 5.1 in the case \(x = x_0 = \gamma_0 \cap \Sigma_0\). Lemma 5.1 is now a straightforward consequence of the following lemma

**Lemma 5.2.**

\[
\lim_{x \to x_0, x \in \Sigma_+} \frac{\mu(\mathcal{P}(x)) - \mu(x)}{d(\mathcal{P}(x), x)} = 0,
\]

where \(d(\mathcal{P}(x), x)\) denotes the distance between \(x\) and \(\mathcal{P}(x)\).

**Proof.** Fix \(\varepsilon > 0\) and let us prove that \(\frac{|\mu(\mathcal{P}(x)) - \mu(x)|}{d(\mathcal{P}(x), x)}\) is smaller than \(\varepsilon\) for every \(x\) close to \(x_0\) in \(\Sigma_+\). Recall that, according to Lemma 4.12 there is a neighborhood \(O_2\) of \(\gamma_0\) so that if \(x \in O^+_2 = O_2 \cap \Sigma_+\) then the segment of straight line \([x, \mathcal{P}^2(x)]\) is contained in \(\Sigma_+\). Furthermore, Corollary 4.14 says that \(\mathcal{N}|_{[x, \mathcal{P}^2(x)]}\) is surjective onto \(S^1\) (unit circle in \(\mathbb{R}^2\)). In particular, there are points \(q_x, q_x \in [x, \mathcal{P}^2(x)]\) so that \(\mathcal{N}(q_x) = (1, 0)\) and \(\mathcal{N}(q_x) = (-1, 0)\).

According to Corollary 4.17 one gets

\[
(\mu(\mathcal{P}(q_x)) - \mu(q_x))(\mu(\mathcal{P}(q_x)) - \mu(q_x)) < 0, \tag{5.1}
\]

for every \(x \in O^+_2\).

The diffeomorphism \(\mathcal{P}\) is \(C^1\)-close to the identity in a small neighborhood of \(x_0\). Now [Bo1] (see also [Bo2]) implies that there is a neighborhood \(V_1\) of \(x_0\) in \(\Sigma_0\) so that if \(x \in V_1\) then

\[
\|\mathcal{P}(x) - x\| < \frac{1}{2}\|\mathcal{P}(x) - x\|,
\]

for every \(y\) with \(d(x, y) < 3\|\mathcal{P}(x) - x\|\). In particular, \(\|\mathcal{P}^2(x) - \mathcal{P}(x)\| < 2\|\mathcal{P}(x) - x\|\), and thus \(\|\mathcal{P}^2(x) - x\| < 3\|\mathcal{P}(x) - x\|\).

Consider the function \(f(x) = \mu(\mathcal{P}(x)) - \mu(x)\). Since \(D\mathcal{P}(x_0) = Id\), we have that \(Df(x_0) = 0\). As a consequence, there exists a neighborhood \(V_2 \subset V_1\) of \(x_0\) such that \(|Df(x)| < \frac{\varepsilon}{4}\), for every \(x \in V_2\).
Since $\mathcal{P}(x_0) = x_0$, we can choose a smaller neighborhood $V_3$ such that $\mathcal{P}(x), \mathcal{P}^2(x) \in V_2$, for every $x \in V_3$. This ensures that
\[
\frac{|f(q_x) - f(x)|}{d(x, \mathcal{P}(x))} < \varepsilon \cdot \frac{d(q_x, x)}{d(x, \mathcal{P}(x))} \leq \frac{\varepsilon}{3}.
\]
Similar estimates hold with $\overline{q}_x$ in place of $q_x$ and in place of $x$, respectively.

By Inequality (5.1) we see that $f(q_x)$ and $f(\overline{q}_x)$ have opposite signs and thus
\[
\frac{|f(q_x) + f(\overline{q}_x)|}{d(x, \mathcal{P}(x))} \leq \frac{|f(q_x) - f(\overline{q}_x)|}{d(x, \mathcal{P}(x))} \leq \frac{\varepsilon}{3}.
\]
We deduce
\[
\frac{|2f(x)|}{d(x, \mathcal{P}(x))} = \frac{|f(x) - f(q_x) + f(q_x) + f(\overline{q}_x) + f(x) - f(\overline{q}_x)|}{d(x, \mathcal{P}(x))} \leq \frac{|f(x) - f(q_x)|}{d(x, \mathcal{P}(x))} + \frac{|f(\overline{q}_x) + (q_x)|}{d(x, \mathcal{P}(x))} \frac{|f(x) - f(\overline{q}_x)|}{d(x, \mathcal{P}(x))} \leq \varepsilon.
\]
This establishes that $\frac{\mu(\mathcal{P}(x)) - \mu(x)}{d(\mathcal{P}(x), x)}$ is smaller than $\varepsilon$ for every $x \in V_3 \cap \Sigma_+$ and completes the proof. \qed

Remark 5.3. The Lemmas 5.1 and 5.2 depend a priori on the choice of coordinate on $\Sigma_0$ since they are formulated in terms of segments of straight line $[x, \mathcal{P}(x)]$, and vectors $\mathcal{P}(x) - x$. Nevertheless, the choice of coordinates on $\Sigma_0$ was arbitrary (see first paragraph of Section 4.5) so that it holds indeed for any choice of $C^1$ coordinates on $\Sigma_0$ (on a neighborhood of $x_0$ depending on the choice of the coordinates).

5.2. Dynamics of the first return map $\mathcal{P}$ in the neighborhood of $0$. In this section $(U, X, Y, \Sigma, B)$ is a prepared counter example to Theorem A and we assume $\ell^+(X, Y) \neq 0$.

Recall that $\Sigma_0$ is a disc endowed with an arbitrary (but fixed) choice of coordinates. Also, $\mu: \Sigma_0 \to \mathbb{R}$ is a $C^1$-map whose derivative do not vanish along $\text{Col}(X, Y, U)$ and $\text{Col}(X, Y, U) \cap \Sigma_0$ is a $C^1$-curve.

Therefore, one can choose a $C^1$-map $\nu: \Sigma_0 \to \mathbb{R}$ so that

- there is a neighborhood $O$ of $x_0$ in $\Sigma_0$ so that $(\mu, \nu): O \to \mathbb{R}^2$ is a $C^1$ diffeomorphism,
- $\text{Col}(X, Y, U) \cap O = \nu^{-1}(\{0\})$
- $\nu > 0$ on $\Sigma_+$

We denote by $(\mu(x), \nu(x))$ the image of $x$ by $(\mu, \nu)$.

Notice that $(\mu(x), \nu(x))$ are local coordinates on $\Sigma_0$ in a neighborhood of $x_0$. Remark 5.3 allows us to use Lemma 5.1 and Lemma 5.2 in the coordinates $(\mu, \nu)$.

As a consequence, there exist $\varepsilon > 0$ so that for any point $x \in \Sigma_+$ with $(\mu(x), \nu(x)) \in [-\varepsilon, \varepsilon] \times [0, \varepsilon]$, one has
\[
|\mu(\mathcal{P}(x)) - \mu(x)| < \frac{1}{100} |\nu(\mathcal{P}(x)) - \nu(x)|.
\]
In particular, since $\mathcal{P}$ has no fixed point in $\Sigma_+$ (Corollary 4.15), one gets that $\nu(\mathcal{P}(x)) - \nu(x)$ does not vanish for $(\mu(x), \nu(x)) \in [-\varepsilon, \varepsilon] \times (0, \varepsilon]$, and in particular it has a constant sign. Up to change $\mathcal{P}$ by its inverse $\mathcal{P}^{-1}$ (which is equivalent to replace $Y$ by $-Y$), one may assume...
Lemma 5.6. Consider the trapezium shows that every point in $\Sigma_+$ close to $x_0$ belongs to such a stable set.

Lemma 5.4. Let $x \in \Sigma_+$ be such that $(\mu(x), \nu(x)) \in \left[-\frac{9}{10} \varepsilon, \frac{9}{10} \varepsilon\right] \times (0, \varepsilon)$. Then, for any integer $n \geq 0$, $\mathcal{P}^n(x)$ satisfies

$$
(\mu(\mathcal{P}^n(x)), \nu(\mathcal{P}^n(x))) \in [-\varepsilon, \varepsilon] \times (0, \varepsilon)
$$

Furthermore the sequence $\mathcal{P}^n(x)$ converges to a point $x_\infty \in \text{Col}(X, Y, U) \cap \Sigma_0$ and we have

- $\nu(x_\infty) = 0$
- $\mu(x_\infty) - \mu(x) \leq \frac{\nu(x)}{100} \leq \frac{\nu(x)}{100}$.

The map $x \mapsto x_\infty$ is continuous.

Proof. Consider the trapezium $D$ (in the $(\mu, \nu)$ coordinates) whose vertices are $(-\varepsilon, 0), (\varepsilon, 0), (-\frac{9}{10} \varepsilon, \varepsilon), \text{ and } (\frac{9}{10} \varepsilon, \varepsilon)$. This trapezium $D$ is contained in $[-\varepsilon, \varepsilon] \times [0, \varepsilon]$ and contains $[-\frac{9}{10} \varepsilon, \frac{9}{10} \varepsilon] \times [0, \varepsilon]$. Thus for proving the first item it is enough to check that $D$ is invariant under $\mathcal{P}$. For that notice that, for any $x$ with $(\mu(x), \nu(x)) \in [-\varepsilon, \varepsilon] \times [0, \varepsilon]$ one has that $(\mu(\mathcal{P}(x)), \nu(\mathcal{P}(x)))$ belongs to the triangle $\delta(x)$ whose vertices are $(\mu(x), \nu(x)), (\mu(x) - \frac{\nu(x)}{100}, 0), (\mu(x) + \frac{\nu(x)}{100}, 0)$ (according to Equation 1032); one conclude by noticing that, if $(\nu(x), \mu(x))$ belongs to $D$ then $\delta(x) \subset D$.

Let us show that $\mathcal{P}^n(x)$ converges.

The sequence $\nu(\mathcal{P}^n(x))$ is positive and decreasing, hence converges, and

$$
\sum |\mu(\mathcal{P}^{n+1}(x)) - \nu(\mathcal{P}^n(x))|
$$

converges. As $|\mu(\mathcal{P}^{n+1}(x)) - \mu(\mathcal{P}^n(x))| \leq \frac{1}{100} |\nu(\mathcal{P}^{n+1}(x)) - \nu(\mathcal{P}^n(x))|$ one deduces that the sequence $\{\mu(\mathcal{P}^n(x))\}_{n \in \mathbb{N}}$ is a Cauchy sequence, hence converges.

The continuity of $x \mapsto x_\infty$ follows from the inequality $\mu(x_\infty) - \mu(x) \leq \frac{\nu(x)}{100}$ applied to $\mathcal{P}^n(x)$ with $n$ large, so that $\nu(\mathcal{P}^n(x))$ is very small, and from the continuity of $x \mapsto \mathcal{P}^n(x)$.

For any point $y$ with $(\mu(y), \nu(y)) \in \left[-\frac{8}{10} \varepsilon, \frac{8}{10} \varepsilon\right] \times \{0\}$ the stable set of $y$, which we denote by $S(y)$, is the the union of $\{y\}$ with the set of points $x \in \Sigma_+$ with $(\mu(x), \nu(x)) \in \left[-\frac{9}{10} \varepsilon, \frac{9}{10} \varepsilon\right] \times (0, \varepsilon)$ so that $x_\infty = y$. The continuity of the map $x \mapsto x_\infty$ implies the following remark:

Remark 5.5. For any point $y$ with $(\mu(y), \nu(y)) \in \left[-\frac{8}{10} \varepsilon, \frac{8}{10} \varepsilon\right] \times \{0\}, S(y)$ is a compact set which has a non-empty intersection with the horizontal lines $\{x, \nu(x) = t\}$ for every $t \in (0, \varepsilon]$.

If $E$ is a subset of $\text{Col}(X, Y, U) \cap \Sigma_0$ so that $\mu(y) \in \left[-\frac{8}{10} \varepsilon, \frac{8}{10} \varepsilon\right]$ for $y \in E$ one denotes

$$
S(E) = \bigcup_{y \in E} S(y).
$$

Lemma 5.6. Let $I \subset \text{Col}(X, Y, U)$ be the open interval $(\mu(x), \nu(x)) \in \left(-\frac{1}{2} \varepsilon, \frac{1}{2} \varepsilon\right) \times \{0\}$. Consider the quotient space $\Gamma$ of $S(I) \setminus I$ by the dynamics. Then $\Gamma$ is a $C^1$-connected surface diffeomorphic to a cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$.  

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Lemma 5.8. For every point in $S$, just remark that the orbit of every point in $S$ is invariant of the vector field $P$. Remark 5.7

Consider the compact triangle whose end points are $(-\frac{\varepsilon}{2}, 0)$, $(0, \frac{\varepsilon}{10})$ and $(+\frac{\varepsilon}{2}, 0)$. Let $\Delta$ be its preimage by $(\mu, \nu)$. $\Delta$ is a triangle with one side on Col($X, Y, U$). Let $\Delta = \Delta \setminus \text{Col}(X, Y, U)$. We denote by $\partial \Delta$ the union of the two other sides.

As the vectors directing the two other sides have a first coordinate larger than the second, one deduces that $\Delta$ is a trapping region for $P$:

$$x \in \Delta \implies P(x) \in \Delta.$$ 

Now

$$P(\partial \Delta)$$

is a curve contained in the interior of $\Delta$ and joining the vertex $(-\frac{\varepsilon}{2}, 0)$ to the vertex $(+\frac{\varepsilon}{2}, 0)$. Thus $\partial \Delta$ and $P(\partial \Delta)$ bound a strip diffeomorphic to $[0, 1] \times \mathbb{R}$ in $\Delta$.

Let $\Gamma$ be the cylinder obtained from this strip by gluing $\partial \Delta$ with $P(\partial \Delta)$ along $P$.

It remains to check that $\Gamma$ is the quotient space of $S(I) \setminus I$ by $P$. For that, one just remark that the orbit of every point in $S(I) \setminus I$ has a unique point in the strip, unless in the case where the orbits meets $\partial \Delta$: in that case the orbits meets the strip twice, the first time on $\partial \Delta$, the second on $P(\partial \Delta)$. □

Let us end this section by stating important straightforward consequences of the invariance of the vector field $N$ under $P$.

Remark 5.7.

- For every $y \in \text{Col}(X, Y, U)$ with $\mu(y) \in [-\frac{8}{10}, \frac{8}{10}]$, the stable set $S(y)$ is invariant under the flow of $N$.
- The vector field $N$ induces a vector field, denoted by $N_{\Gamma}$, on the quotient space $\Gamma$. As $\Gamma$ is a $C^1$ surface, $N_{\Gamma}$ is only $C^0$. However, it defines a flow on $\Gamma$ which is the quotient by $P$ of the flow of $N$.
- The continuous map $x \mapsto x_{\infty}$ in invariant under $P$ and therefore induces on $\Gamma$ a continuous map $\Gamma \to (-\varepsilon/2, \varepsilon/2)$, and $x_{\infty}$ tends to $-\varepsilon/2$ when $x$ tends to one end of the cylinder $\Gamma$ and to $\varepsilon/2$ when $x$ tends to the other end. This implies that, for every $t \in (-\varepsilon/2, \varepsilon/2)$ the set of points $x \in \Gamma$ for which $x_{\infty} = t$ is compact. Recall that this set is precisely the projection on $\Gamma$ of $S(y)$ where $y \in \text{Col}(X, Y, U)$ satisfies $(\mu(y), \nu(y)) = (t, 0)$.
- The vector field $N_{\Gamma}$ on $\Gamma$ leaves invariant the levels of the map $x \mapsto x_{\infty}$. As a consequence, every orbit of $N_{\Gamma}$ is bounded in $\Gamma$.

One deduces

**Lemma 5.8.** For every $y \in \text{Col}(X, Y, U)$ with $\mu(y) \in [-\frac{8}{10}, \frac{8}{10}]$, the stable set $S(y) \setminus \{y\}$ contains an orbit of $N$ which is invariant under $P$. In particular, there exists $x$ in $\Delta \cap S(x_0)$ whose orbit by $N$ is invariant under $P$.

**Proof.** A Poincaré Bendixson argument implies that, for every flow on the cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ without fixed points, for every bounded orbit the $\omega$-limit set is a periodic orbit. Furthermore this periodic orbit is not homotopic to a point (otherwise it bounds a disc containing a zero).

Since $N_{\Gamma}$ has no zeros, one just applies this argument to the flow of it restricted to the a level of the map $x \mapsto x_{\infty}$. The level contains a periodic orbit which is not homotopic to 0, hence corresponds to an orbit of $N$ joining a point in $S(y)$ to is image under $P$. This orbit of $N$ is invariant under $P$, concluding. □
5.3. End of the proof of Theorem A: the vector field $N$ does not rotate along a $\mathcal{P}$-invariant orbit of $N$. From now on, $(U, X, Y, \Sigma, B)$ is a prepared counterexample to Theorem A with $\ell^+(X, Y) \neq 0$. According to Proposition 4.19, the derivative $D\mathcal{P}(x)$ is the identity map for every $x \in \text{Col}(X, Y, U) \cap \Sigma_0$.

According to Lemma 5.8, there is a point $x$ in the stable set $S(x_0)$ whose $N$-orbit is invariant under $\mathcal{P}$. 

**Lemma 5.9.** The angular variation of the vector $N(y)$, for $y \in [\mathcal{P}^n(x), \mathcal{P}^{n+1}(y)]$, tends to 0 when $n \to +\infty$.

Before proving Lemma 5.9, let us conclude the proof of Theorem A.

**Proof of Theorem A.** Lemma 5.9 is in contradiction with Corollary 4.14, which asserts that the angular variation of the vector $N$ along any segment $[z, \mathcal{P}^2(z)]$ for $z \in \Sigma_+$ close enough to $x_0$ is larger than $2\pi$. This contradiction ends the proof of Theorem A. \hfill $\square$

It remains to prove Lemma 5.9. First, notice that the angular variation of $N$ along a segment of curve is invariant under homotopies of the curve preserving the ends points. Therefore Lemma 5.9 is a straightforward consequence of next lemma:

**Lemma 5.10.** The angular variation of the vector $N(y)$ along the $N$-orbit segment joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$ tends to 0 when $n \to +\infty$.

As $N$ is (by definition) tangent to the $N$-orbit segment joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$, its angular variation is equal to the angular variation of the unit tangent vector to this orbit segment.

5.4. **Proof of Lemma 5.10.** The tangent vector to a $\mathcal{P}$-invariant embedded curve do not rotate.

**Remark 5.11.** For $n$ large enough the point $\mathcal{P}^n(x)$ belongs to the region $\Delta$ defined in the previous section, and whose quotient by the dynamics $\mathcal{P}$ is the cylinder $\Gamma$. Then,

- any continuous curve $\gamma$ in $\Delta$ joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$ induces on $\Gamma$ a closed curve, homotopic to the curve induced by the $N$-orbit segment joining $\mathcal{P}^n(x)$ to $\mathcal{P}^{n+1}(x)$.
- The curve induced by $\gamma$ is a simple curve if and only if $\gamma$ is simple and disjoint from $\mathcal{P}^i(\gamma)$ for any $i > 0$.
- If the curve $\gamma$ is of class $C^1$, the projection will be of class $C^1$ if and only if the image by $\mathcal{P}$ of the unit vector tangent to $\gamma$ at $\mathcal{P}^n(x)$ is tangent to $\gamma$ (at $\mathcal{P}^{n+1}(x)$).
- on the cylinder, any two $C^1$-embeddings $\sigma_1, \sigma_2$ of the circle, so that $\sigma_1(0) = \sigma_2(0)$ are isotopic through $C^1$-embeddings $\sigma_i$ with $\sigma_i(0) = \sigma_i(0)$.

We consider $\mathcal{I}(\Delta, \mathcal{P})$ as being the set of $C^1$-immersed segment $I$ in $\Delta \setminus \partial \Delta$, so that:

- if $y, z$ are the initial and end points of $I$ then $z = \mathcal{P}(y)$
- if $u$ is a vector tangent to $I$ at $y$ then $\mathcal{P}_*(u)$ is tangent to $I$ (and with the same orientation).

In other words, $I \in \mathcal{I}(\Delta, \mathcal{P})$ if the projection of $I$ on $\Gamma$ is a $C^1$ immersion of the circle, generating the fundamental group of the cylinder. We endow $\mathcal{I}(\Delta, \mathcal{P})$ with the $C^1$-topology.
We denote by $\text{Var}(I) \in \mathbb{R}$ the angular variation of the unit tangent vector to $I$ along $I$. In other words, for $I: [0, 1] \to \Delta$, consider the unit vector

$$\dot{I}(t) = \frac{dI(t)/dt}{\|dI(t)/dt\|} \in S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}.$$ 

One can lift $\dot{I}$ is a continuous map $\dot{I}: [0, 1] \to \mathbb{R}$. Then

$$\text{Var}(I) = \dot{I}(1) - \dot{I}(0),$$

this difference does not depend on the lift.

The map $\text{Var}: \mathcal{I}(\Delta, \mathcal{P}) \to \mathbb{R}$ is continuous. Let $\overline{\text{Var}}(I) \in \mathbb{R}/2\pi\mathbb{Z}$ be the projection of $\text{Var}(I)$. In other words, $\overline{\text{Var}}(I)$ is the angular variation modulo $2\pi$.

**Remark 5.12.** Let $I_n \in \mathcal{I}(\Delta, \mathcal{P})$ be a sequence of immersed segments such that $I_n(0)$ tends to $x_0 \in \Sigma$. Then $\text{Var}(I_n)$ tends to 0.

Indeed, since $DP(I_n(0))$ tends to the identity map, the angle between the tangent vectors to $I_n$ at $I_n(1)$ and $I_n(0)$ tends to 0.

As a consequence of Remark 5.12, we get the following lemma:

**Lemma 5.13.** There is a neighborhood $O$ of $x_0$ in $\Sigma$ so that to any $I \in \mathcal{I}(\Delta, \mathcal{P})$ with $I(0) \in O$ there is a (unique) integer $\lfloor \text{var}(I) \rfloor \in \mathbb{Z}$ so that

$$\text{Var}(I) - 2\pi \lfloor \text{var}(I) \rfloor \in \left[ -\frac{1}{100}, \frac{1}{100} \right].$$

Furthermore, the map $I \mapsto \lfloor \text{var}(I) \rfloor$ is locally constant in $\mathcal{I}(\Delta, \mathcal{P})$, hence constant under homotopies in $\mathcal{I}(\Delta, \mathcal{P})$ keeping the initial point in $O$.

As a consequence we get

**Lemma 5.14.** If $I$ and $J$ are segments in $\mathcal{I}(\Delta, \mathcal{P})$ with the same initial point in $O$ and whose projections on the cylinder $\Gamma$ are simple closed curves, then

$$\lfloor \text{var}(I) \rfloor = \lfloor \text{var}(J) \rfloor.$$

**Proof.** Since the projection of $I$ and $J$ are simple curves which are not homotopic to a point in the cylinder $\Gamma$, the projections of $I$ and $J$ are isotopic on $\Gamma$ by an isotopy keeping the initial point. One deduces that $I$ and $J$ are homotopic through elements $I_t \in \mathcal{I}(\Delta, \mathcal{P})$ with the same initial point. Indeed, the isotopy on $\Gamma$ between the projection of $I$ and $J$ can be lifted to the universal cover of $\Gamma$. This universal cover is diffeomorphic to a plane $\mathbb{R}^2$, in which $\Delta \setminus \partial \Delta$ is an half plane (bounded by two half lines). There is a diffeomorphism of $\mathbb{R}^2$ to $\Delta \setminus \partial \Delta$ which is the identity on $I \cup J$. The image of the lifted isotopy induces the announced isotopy through elements in $\mathcal{I}(\Delta, \mathcal{P})$. Now, as $\lfloor \text{var}(I_t) \rfloor$ is independent of $t$, one concludes that $\lfloor \text{var}(I) \rfloor = \lfloor \text{var}(J) \rfloor$. \qed

Now Lemma 5.10 is a consequence of Lemma 5.14 and of the following lemma:

**Lemma 5.15.** For any $n > 0$ large enough there is a curve $I_n \in \mathcal{I}(\Delta, \mathcal{P})$ whose initial point is $\mathcal{P}^n(x)$ and such that:

- the projection of $I_n$ on $\Gamma$ is a simple curve
- $\lfloor \text{var}(I_n) \rfloor = 0$. 

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End of the proof of Lemma 5.17. The $N$-orbit segment $J_n$ joining $P^n(x)$ to $P^{n+1}(x)$ belongs to $I(\Delta, \mathcal{P})$ and its projection on $\Gamma$ is a simple curve. Hence Lemma 5.13 asserts that, for $n$ large enough, $\var(J_n) = \var(I_n) = 0$ where $I_n$ is given by Lemma 5.15.

Now, when $n$ tends to infinity, $\text{Var}(J_n) - 2\pi|\var|(J_n)$ tends to 0 (according to Remark 5.12), that is, $\text{Var}(J_n)$ tends to 0. This is precisely the statement of Lemma 5.10.

Proof of Lemma 5.15. As $n$ tends to $+\infty$ the derivative of $\mathcal{P}$ at $P^n(x)$ tends to the identity map. Thus, the segment $[P^n(x), P^{n+1}(x)]$ may fail to belong to $I(\Delta, \mathcal{P})$ only by a very small angle between $v_n$ and $D\mathcal{P}(v_n)$, where $v_n$ is the unit vector directing $[P^n(x), P^{n+1}(x)]$. Therefore, one easily builds a segment $I_n$ joining $P^n(x)$ to $P^{n+1}(x)$, whose derivative at $P^n(x)$ is $v_n$ and its derivative at $P^{n+1}(x)$ is $D\mathcal{P}(v_n)$ and whose derivative at any point of $I_n$ belongs to an arbitrarily small neighborhood of $v_n$. In particular $I_n \in I(\Delta, \mathcal{P})$ and $|\var|(I_n) = 0$ for $n$ large. In order to complete the proof, it remains to show that

Claim 7. For $n$ large enough $I_n$ projects on $\Gamma$ as a simple curve.

Proof. We need to prove that for $n$ large enough and for any $i > 1$, $I_n \cap \mathcal{P}(I_n) = \emptyset$ and $I_n \cap \mathcal{P}(I_n)$ is a singleton (the endpoint of $I_n$ which is the image of its initial point).

Indeed, it is enough to prove that, for any $y \in I_n$ different from the initial point,

$$\nu(\mathcal{P}(y)) < \inf_{z \in I_n} \nu(z).$$

As the action of $\mathcal{P}$ consists in lowering the value of $\nu$, the further iterates cannot cross $I_n$.

For proving that, notice that the vectors tangent to $I_n$ are very close to $v_n$ which is uniformly (in $n$ large) transverse to the levels of $\nu$. As $D\mathcal{P}(y)$ tends to the identity map when $y$ tends to 0, for $n$ large, the vectors tangent to $\mathcal{P}(I_n)$ are also transverse to the levels of $\nu$. Hence $\mathcal{P}(I_n)$ is a segment starting at the end point of $I_n$ (which realizes the infimum of $\nu$ on $I_n$) and $\nu$ is strictly decreasing along $\mathcal{P}(I_n)$, concluding.

This ends the proof of Lemma 5.15 (and so of Theorem A). □

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