Potential Capacities of Quantum Channels

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Abstract—We introduce potential capacities of quantum channels in an operational way and provide upper bounds for these quantities, which quantify the ultimate limit of usefulness of a channel for a given task in the best possible context.

Unfortunately, except for a few isolated cases, potential capacities seem to be as hard to compute as their “plain” analogues. We thus study upper bounds on some potential capacities: For the classical capacity, we give an upper bound in terms of the entanglement of formation. To establish a bound for the quantum and private capacity, we first “lift” the channel to a Hadamard channel and then prove that the quantum and private capacity of a Hadamard channel is strongly additive, implying that for these channels, potential and plain capacity are equal. Employing these upper bounds we show that if a channel is noisy, however close it is to the noiseless channel, then it cannot be activated into the noiseless channel by any other contextual channel; this conclusion holds for all the three capacities. We also discuss the so-called environment-assisted quantum capacity, because we are able to characterize its “potential” version.

Index Terms—quantum channel, potential capacity, non-additivity, entanglement, Hadamard channel.

I. MOTIVATION

THE central problem in information theory is to find the capacity of a noisy channel for transmitting information faithfully. Depending on what type of information is to be sent, there are several capacities that can be defined for a quantum channel, among them the classical capacity [1], [2], the quantum capacity [3], [4], [5] and the private capacity [6]. In contrast to classical information theory, where the capacity is expressed by Shannon’s famous single-letter formula, the status of quantum channel capacities is much more complicated. The relevant quantities are known to be non-additive [7], [8], [9], [10], which is at the center of interest in quantum information science, and the best known formula to calculate the capacities involves optimization over growing numbers of channel uses (“regularization”), where we have to perform an optimization over an infinite number of variables, making a head-on numerical approach impossible, cf. [11], [12]. This makes it difficult to answer questions related to capacities, even some simple qualitative ones, such as whether, given a quantum channel, it is useful to transmit quantum information. Non-additivity in quantum Shannon theory is due to entanglement, which has no classical counterpart. Employing entangled inputs for the channels, it is possible to transmit more information than just using product inputs.

Entangled inputs between different quantum channels open the door to all kinds of effects that are impossible in classical information theory. An extreme phenomenon is superactivation [13]; there exist two quantum channels that cannot transmit quantum information when they are used individually, but can transmit at positive rate when they are used together.

The phenomenon of superactivation, and more broadly of super-additivity, implies that the capacity of a quantum channel does not adequately characterize the channel, since the utility of the channel depends on what other contextual channels are available. So it is natural to ask the following question: What is the maximum possible capability of a channel to transmit information when it is used in combination with any other contextual channels? We introduce the potential capacity to formally capture this notion.

Superactivation can also be rephrased in an alternative way, that a zero-capacity channel becomes a positive-capacity one under the assistance of another zero-capacity side channel. Superactivation of quantum channel capacity shows that entangled inputs across different channel uses can provide a dramatic advantage, but more generally entangling different channel uses can give rise to superadditivity, i.e., an increase of the capacity above the sum of the channel capacities when the two channels are used jointly. Superactivation exhibits one regime of entanglement advantage, the regime of low capacity. Could entanglement help in this sense at the other extreme? That is “Can a noisy channel, whose quantum capacity is $\leq \log d - \delta$, become perfectly noiseless under the assistance of a suitable zero-capacity side channel?” Since it is difficult to characterize all the zero-capacity channels, it seems hard to answer this question. Encouraged by superactivation, one might guess that a noisy channel could behave like a noiseless channel by the assistance of a proper zero-capacity side channel. In this work, we will provide upper bounds on the potential capacities to exclude this possibility. In this sense, entanglement can help but cannot help too much.

This paper is structured as follows. In Section [II] we introduce notation, definitions and state some basic known facts. In particular, we review the regularized formulas of three capacities (classical, quantum, and private capacity), and the results about additivity of degradable channels, furthermore the entanglement-assisted and the environment-assisted capacities. In Section [III] we introduce the notion of potential capacity and in Section [IV] evaluate it or give upper bounds for it, and prove that an imperfect channel cannot be activated into a perfect one. Finally we end with a summary and open questions in Section [V].

II. NOTATION AND PRELIMINARIES

We assume that all Hilbert spaces, denoted $\mathcal{H}$, are finite dimensional. Recall that a quantum state $\rho$ is a linear operator

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on $\mathcal{H}$ satisfying $\rho \geq 0$ and $\text{Tr} \rho = 1$. A quantum channel is a completely positive and trace preserving (CPTP) linear map, from an input system $A$ to output system $B$ (we shall generally use the same names for the underlying Hilbert spaces). From the Stinespring dilation theorem \cite{21}, we know that for a channel $\mathcal{N}$, there always exist an isometry $U : A \to B \otimes \mathcal{E}$ for some environment space $\mathcal{E}$, i.e. $U^\dagger U = I$, such that $\mathcal{N}(\rho) = \text{Tr}_B U \rho U^\dagger$. The complementary channel of $\mathcal{N}$, which we denote $\mathcal{N}^c$, is the channel that maps from the input system $A$ to the environment system $\mathcal{E}$, obtained by taking the partial trace over system $B$ rather than the environment: $\mathcal{N}^c(\rho) = \text{Tr}_B U \rho U^\dagger$. Since the Stinespring dilation is unique up to a change of basis of environment $\mathcal{E}$, $\mathcal{N}^c$ is well-defined up to unitary operations on $\mathcal{E}$. A quantum channel has another representation known as the Kraus representation: $\mathcal{N}(\rho) = \sum_i K_i \rho K_i^\dagger$, where $K_i$ are called Kraus operators satisfying $\sum_i K_i K_i^\dagger = I$. Given a multipartite state $\rho^{ABC}$, we write $\rho^A = \text{Tr}_{BC} \rho^{ABC}$ for the corresponding reduced state. The von Neumann entropy is defined as $S(\rho) = S(\rho^A) = -\text{Tr} \rho \log \rho^A$. The conditional von Neumann entropy as $S(\rho_{AB} | \phi) = \text{Tr} \rho_B \log \rho_B$. A quantum channel $\mathcal{N}$ is called degradable if it can simulate its complementary channel $\mathcal{N}^c$, i.e. there is a degrading CPTP map $D$ such that $D \circ \mathcal{N} = \mathcal{N}^c$. We further recall two other capacities: The entanglement-assisted classical capacity of $\mathcal{N}$ \cite{22}, which is the capacity for transmitting classical information through the channel with the help of unlimited prior entanglement shared between the sender and the receiver and which is given by the simple and beautiful formula

$$C_E(\mathcal{N}) = \max_{\rho^A} I(R : B),$$

where $I(R : B) = S(\rho^R) + S(\rho^B) - S(\rho^{RB})$ is the quantum mutual information of the state $\rho^{RB} = (\text{id} \otimes \mathcal{N})(|\phi\rangle \langle \phi|^R)$, with a purification $|\phi\rangle^R$. And the environment-assisted quantum capacity, which refers to active feed-forward of classical information from the channel environment $E$ to the receiver $B$ \cite{23}, \cite{24}, is given by

$$Q_A(\mathcal{N}) = \max_{\rho^A} \min_{\rho^{RB}} \left\{ S(\rho^A), S(\mathcal{N}(\rho^A)) \right\}. $$

### III. Potential Capacities

Notice that the formulas for $C$, $P$ and $Q$ all are regularized expressions due to the non-additivity of their respective single-letter quantities, $\chi$, $P^{(1)}$ and $Q^{(1)}$.

We call a real function $f(\mathcal{N})$ on the set of channels weakly-additive if $f(\mathcal{N}^\otimes n) = nf(\mathcal{N})$ for all $n \geq 1$, and strongly-additive if $f(\mathcal{N} \otimes M) = f(\mathcal{N}) + f(M)$ for any channels $\mathcal{N}$ and $M$. Obviously, if $f$ is strongly-additive, then it is also weakly-additive but not vice versa; and example of this is given by the environment-assisted capacity $Q_A(\mathcal{N})$. Furthermore, for fixed $f$, we call a channel $\mathcal{N}$ strongly-additive, if for all other channels $M$, $f(\mathcal{N} \otimes M) = f(\mathcal{N}) + f(M)$. 

The private capacity of the quantum channel $\mathcal{N}$ is given by

$$P(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} P^{(1)}(\mathcal{N}^\otimes n),$$

with $P^{(1)}(\mathcal{N})$ defined as

$$P^{(1)}(\mathcal{N}) = \max_{\{p, \rho\}} (I(T : B) - I(T : E)),$$

with respect to $\rho^{TBE} = \sum_i p_i |i\rangle \langle i|^T \otimes U \rho_i^A U^\dagger$. 

Now we recall the definition of a degradable channel and its properties on quantum and private capacities.

**Definition 1** A channel $\mathcal{N}$ is called degradable if it can simulate its complementary channel $\mathcal{N}^c$, i.e. there is a degrading CPTP map $D$ such that $D \circ \mathcal{N} = \mathcal{N}^c$.

**Lemma 2** (Devetak/Shor) If $\mathcal{N}$ and $\mathcal{M}$ are degradable channels, then their single-letter quantum capacity is additive: $Q^{(1)}(\mathcal{N} \otimes \mathcal{M}) = Q^{(1)}(\mathcal{N}) + Q^{(1)}(\mathcal{M})$.

**Lemma 3** (Smith) If a quantum channel $\mathcal{N}$ is a degradable channel, then its quantum capacity is equal to its private capacity, and both are given by the single-letter coherent information: $Q(\mathcal{N}) = P(\mathcal{N}) = Q^{(1)}(\mathcal{N}) = P^{(1)}(\mathcal{N})$. 

We furthermore recall two other capacities: The entanglement-assisted classical capacity of $\mathcal{N}$ \cite{22}, which is the capacity for transmitting classical information through the channel with the help of unlimited prior entanglement shared between the sender and the receiver and which is given by the simple and beautiful formula

$$C_E(\mathcal{N}) = \max_{\rho^A} I(R : B),$$

where $I(R : B) = S(\rho^R) + S(\rho^B) - S(\rho^{RB})$ is the quantum mutual information of the state $\rho^{RB} = (\text{id} \otimes \mathcal{N})(|\phi\rangle \langle \phi|^R)$, with a purification $|\phi\rangle^R$. And the environment-assisted quantum capacity, which refers to active feed-forward of classical information from the channel environment $E$ to the receiver $B$ \cite{23}, \cite{24}, is given by

$$Q_A(\mathcal{N}) = \max_{\rho^A} \min_{\rho^{RB}} \left\{ S(\rho^A), S(\mathcal{N}(\rho^A)) \right\}. $$
From their expression as regularizations, or directly from the definition, one can directly deduce that the capacities $C(N)$, $Q(N)$ and $P(N)$ are weakly-additive. Furthermore, it is known that neither $Q(N)$ nor $P(N)$ are strongly-additive; $C(N)$ is believed to be not strongly-additive, though this has not been proved so far. The single-letter quantities $\chi(N)$, $Q(1)(N)$, $P(1)(N)$ are not even weakly-additive.

Due to their non-additivity, the capability to transmit information through a quantum channel does not only depend on the channel itself, but also on any contextual channel with which it can be combined. So the standard capacity cannot uniquely characterize the utility of the channel. It is natural to consider the maximal possible capability to transmit information when it is used in combination with any other contextual channels. We introduce the potential capacity to describe this notion. It describes the potential capability that can be activated by a proper contextual channel. Since the three capacities share the same property, we define the notion in a unified way.

In the following definitions, we assume a super-additive function $f$, i.e. $f(N \otimes M) \geq f(N) + f(M)$ for any channels $N$ and $M$, so that the regularization $f^\infty$ is given by

$$f^\infty(N) = \sup_{n} \frac{1}{n} f(N^\otimes n) = \lim_{n \to \infty} \frac{1}{n} f(N^\otimes n). \quad (10)$$

By its definition, $f^\infty$ is always weakly-additive, and $f(N) \leq f^\infty(N)$.

**Definition 4** For a channel $N$, the potential capacity associated to $f$ is defined as

$$f_p^\infty(N) := \sup_M \left[f^\infty(N \otimes M) - f^\infty(M)\right], \quad (11)$$

where $f^\infty(N)$ is the regularization of $f$.

Similarly, the potential single-letter capacity is defined as

$$f_p^1(N) := \sup_M \left[f(N \otimes M) - f(M)\right], \quad (12)$$

where $f^1(N) = f(N)$ is the single-letter function.

This notion has been introduced before in [23] Sec. VII, for the case of $f = Q(1)$, under the name of “quantum value added capacity.”

Note that we have (always assuming super-additivity of $f$) $f_p^1(N) = f(N)$ iff $N$ is strongly additive.

Eq. (11) is difficult to calculate because of the unlimited dimension of the contextual channel $M$ and the regularization function $f^\infty$. Eq. (12) looks simpler but still suffers from the unlimited dimension problem. So we would like to provide upper bounds for them. Before that we will show that the notion “potential” is intrinsically sub-additive and a little surprising fact: $f_p^\infty(N) \leq f_p^1(N)$ though $f^\infty(N) \geq f^1(N)$.

**Lemma 5** For any super-additive $f$, both $f_p^1(N)$ and $f_p^\infty(N)$ are sub-additive, i.e.

$$f_p^1(N \otimes M) \leq f_p^1(N) + f_p^1(M), \quad (\text{13})$$

$$f_p^\infty(N \otimes M) \leq f_p^\infty(N) + f_p^\infty(M). \quad (\text{14})$$

**Proof:** We prove the claim for $f_p^1(N)$; the proof for $f_p^\infty(N)$ is similar. Namely, for an arbitrary channel $T$,

$$f^1(N \otimes M \otimes T) - f^1(T) = f^1(N \otimes M \otimes T) - f^1(M \otimes T) + f^1(M \otimes T) - f^1(T),$$

$$\leq \sup_S \left[f^1(N \otimes S) - f^1(S)\right] + \sup_S \left[f^1(M \otimes S) - f^1(S)\right],$$

$$= f_p^1(N) + f_p^1(M).$$

Maximization over $T$ concludes the proof.

**Lemma 6** The potential capacity is upper bounded by the potential single-letter capacity, more precisely

$$f^1(N) \leq f^\infty(N) \leq f_p^\infty(N) \leq f_p^1(N). \quad (\text{15})$$

**Proof:** The first “$\leq$” comes from Eq. (10) by taking $n = 1$ and the second “$\leq$” from Eq. (11) by taking $M$ as a fixed state channel. For the third “$\leq$”, consider the following chain of inequalities:

$$f^\infty(N \otimes M) - f^\infty(M) = \lim_{n \to \infty} \frac{1}{n} f(N^\otimes n \otimes M^\otimes n) - \lim_{n \to \infty} \frac{1}{n} f(M^\otimes n),$$

$$= \lim_{n \to \infty} \frac{1}{n} \left[f(N^\otimes n \otimes M^\otimes n) - f(M^\otimes n)\right],$$

$$\leq \lim_{n \to \infty} \frac{1}{n} f_p^1(N^\otimes n) = f_p^1(N),$$

where the first inequality uses the definition of the potential single-shot capacity and the second one the sub-additivity.

Hence we have

$$f^1(N) \leq f^\infty(N) \leq f_p^\infty(N) \leq f_p^1(N). \quad (\text{16})$$

**Remark** Notice that all capacities and their single-letter formulations are super-additive, and that the single-letter form is a lower bound of the regularized form. However, their “potential” counterparts have the reverse relation; this was glimpsed in [23] without any further investigation.

**IV. Five Concrete Potential Capacities**

Now we can turn to five concrete examples. We start with the entanglement-assisted capacity, which presents a trivial case: Namely, $C_E$ is known to be strongly-additive [26], i.e., for all channels $N$ and $M$, $C_E(N \otimes M) = C_E(N) + C_E(M)$. Thus, $C_E$ equals its own regularization and in turn its own potential capacity:

$$C_E(N) = C_E^\infty(N) = (C_E)_p(N).$$

The next subsection presents the slightly more interesting case of $Q_A$, which is not additive, but it has a single-letter formula. For this case we are still able to evaluate $(Q_A)_p(N)$ in a simple single-letter formula, but for the subsequent $C$, $P$ and $Q$ we will only be able to give upper bounds.
A. Potential environment-assisted capacity

There are two types of channels $\mathcal{M}$ with $Q_A(\mathcal{M}) = 0$, which we will use to activate a given $\mathcal{N}$, on one hand, those with one-dimensional input system, on the other those with one-dimensional output system. Their Stinespring isometries are

$V_1 : \mathbb{C} \rightarrow B' \otimes E', \quad 1 \mapsto |0\rangle_{B'} E'$,
$V_2 : A' \rightarrow \mathbb{C} \otimes E', \quad |\psi\rangle \mapsto 1^{B'} \otimes (W_2|\psi\rangle)^{E'}$,

where $W_2$ is an isometry. Using these, we show the following simple result:

**Theorem 7** For any channel $\mathcal{N} : A \rightarrow B$,

$$(Q_A)_p(\mathcal{N}) = \max_{\rho^A} \max \left\{ S(\rho^A), S(\mathcal{N}(\rho^A)) \right\},$$

and

$$(Q_A)_p(\mathcal{N}) \geq Q_A(\mathcal{N} \otimes \mathcal{M}) \geq \max \left\{ S(\rho^A), S(\mathcal{N}(\rho^A)) \right\}.$$  

Proof: First, for “≥”: By tensoring with a channel $\mathcal{M} : A' \rightarrow B'$ of the above type having zero environment-assisted capacity, i.e. either $\mathcal{M}_1$ where the only input state has zero entropy, or $\mathcal{M}_2$ where the only output state has zero entropy. In this way we can bump up either the output entropy $S(\mathcal{N} \otimes \mathcal{M}(\rho^{AA'}))$, or the input entropy $S(\rho^{AA'})$ by an arbitrary amount, without changing the respective other. Thus indeed,

$$(Q_A)_p(\mathcal{N}) \geq Q_A(\mathcal{N} \otimes \mathcal{M}) \geq \max \left\{ S(\rho^A), S(\mathcal{N}(\rho^A)) \right\}.$$  

In the other direction, consider an arbitrary channel $\mathcal{M}$. Then we have,

$$(Q_A)_p(\mathcal{N}) = \sup_{\mathcal{M}} Q_A(\mathcal{N} \otimes \mathcal{M}) - Q_A(\mathcal{M}),$$

$$(Q_A)_p(\mathcal{N}) = \sup_{\mathcal{M}} \max_{\rho^{AA'}} \left\{ \min \left\{ S(\rho^{AA'}), S(\mathcal{N} \otimes \mathcal{M}(\rho^{AA'})) \right\} + \max \left\{ -S(\rho^A), -S(\mathcal{M}(\rho^A)) \right\} \right\},$$

$$(Q_A)_p(\mathcal{N}) \leq \sup_{\mathcal{M}} \max_{\rho^{AA'}} \left\{ S(\mathcal{M}(\rho^A)), S(\mathcal{M}(\rho^A)) \right\}.$$  

and we are done.

B. Potential classical capacity

In this section, we study the potential classical capacity and its relation to the single-letter Holevo capacity, and most importantly establish an upper bound via a specific entanglement measure. This bound is used to prove that an imperfect quantum channel cannot be activated into a perfect one by any other contextual channel.

**Definition 8** Specializing Definition 2 to the case $f \equiv C$, we obtain the potential classical capacity

$$C_p(\mathcal{N}) = \sup_{\mathcal{M}} [C(\mathcal{N} \otimes \mathcal{M}) - C(\mathcal{M})],$$

and likewise the potential Holevo capacity

$$\chi_p(\mathcal{N}) = \sup_{\mathcal{M}} [\chi(\mathcal{N} \otimes \mathcal{M}) - \chi(\mathcal{M})].$$

By Lemma 6 we have

$$\chi(\mathcal{N}) \leq C(\mathcal{N}) \leq C_p(\mathcal{N}) \leq \chi_p(\mathcal{N}).$$  

To give non-trivial bounds on $\chi_p(\mathcal{N})$, we invoke the following previous result.

**Lemma 9** (Yang et al. [27]) For a mixed four-partite state $ho_{B_1B_2E_1E_2}$,

$$E_F(\rho_{B_1B_2E_1E_2}) \geq G(\rho_{B_1E_1}) + E_F(\rho_{B_2E_2}),$$

where the function $G(\rho^{BE})$ is defined as

$$G(\rho^{BE}) := \min_{\{p_i, \rho_i^{BE}\}} \sum_i p_i C_{\rho_i}(\rho^{BE})^p,$$

with $C_{\rho_i}(\sigma^{BE}) = S(\sigma_B) - \min_{\{p_j\}} r_j S(\sigma_j^B)$.

Furthermore, $G(\rho^{BE})$ is faithful, meaning $G(\rho^{BE}) = 0$ iff $\rho^{BE}$ is separable.

**Theorem 10** For a channel $\mathcal{N}$ with Stinespring isometry $U$,

$$\chi_p(\mathcal{N}) \leq \max_{\rho^A} [S(\rho^B) - G(\rho^{BE})],$$

where $\rho^{BE} = U \rho^A U^\dagger$.

Proof: Using the MSW identity, Eq. 3, and Lemma 9 we have the following chain of identities and inequalities:

$$\chi(\mathcal{N} \otimes \mathcal{M}) = \max_{\rho^{A_1A_2}} S(B_1B_2) - E_F(B_1B_2 : E_1E_2),$$

$$\leq \max_{\rho^{A_1A_2}} \left\{ S(B_1) + S(B_2) - \left[ G(B_1 : E_1) + E_F(B_2 : E_2) \right] \right\},$$

$$\leq \max_{\rho^{A_1A_2}} \left\{ \left[ S(B_1) - G(B_1 : E_1) \right] + \left[ S(B_2) - E_F(B_2 : E_2) \right] \right\},$$

$$\leq \max_{\rho^A} \left\{ \left[ S(B_1) - G(B_1 : E_1) \right] + \max_{\rho^B} \left[ S(B_2) - E_F(B_2 : E_2) \right] \right\},$$

$$\leq \max_{\rho^A} \left[ S(B_1) - G(B_1 : E_1) + \chi(\mathcal{M}) \right].$$

By definition of $\chi_p$, the claim follows.
In the case of $d_{\text{min}} = d_{\text{out}} = d$, from $C_p(N) \leq \chi_p(N) \leq \max_{p_A} [S(\rho_B) - G(\rho_{BE})]$, we know that there is an input state $\rho^A$ such that for $\rho^{BE} = U\rho^A U^\dagger$, we have $S(\rho_B) = \log d$ and $G(\rho_{BE}) = 0$.

Since $G$ is faithful (see Lemma 9), this means that $\rho_{BE}$ is separable, which amounts to $E_F(\rho_{BE}) = 0$. From the MSW identity, Eq. (3), we obtain that $\chi(N) = \log d$, which means the channel is perfect in the single-letter sense.

In the case of $d_{\text{min}} = d_{\text{out}} = d$, suppose $C_p(N) = \log d = S(A) = S(BE)$, where $\rho^A = \frac{1}{d} I$ and $\rho^{BE} = U\rho^A U^\dagger$. From Lemma 27 in the Appendix, we obtain $\log d = S(A) = C_p(N) \leq \max_{p_A} [S(\rho_B) - G(\rho_{BE})] \leq \max_{p_A} S(A) = \log d$. So $\rho^A = \frac{1}{d} I$ is the optimal input to achieve $\max_{p_A} [S(\rho_B) - G(\rho_{BE})]$. This means $G(\rho_{BE}) = S(B) - S(BE)$ for the state $\rho^{BE}$. Also from Lemma 27 we know that $E_F(B : E) = S(B) - S(BE)$, meaning that $\chi(N) = \log d$.

**Remark** In [28], it is shown that if $\chi(N) < \log d_{\text{out}}$, then $C(N) < \log d_{\text{out}}$. Notice that Holevo capacity is the capacity when the codewords are restricted to product states. That is to say if the capacity when using product state encoding cannot achieve the possibly maximal quantity $\log d_{\text{out}}$, then it cannot either when using entangled state encoding. In other words, an imperfect channel cannot be activated to a perfect one by itself. Corollary 11 is stronger in two points. One is that it covers the case $d_{\text{in}} < d_{\text{out}}$ where [28] says nothing about. Indeed it is not immediately to obtain so we need the Appendix to deal with this case. The other point is that Corollary 11 asserts an imperfect channel cannot be activated to a perfect one by any channel. The reasoning for $d_{\text{min}} = d_{\text{out}}$ is almost the same as that in [28] but for $d_{\text{min}} = d_{\text{in}}$ we need more. Here we emphasize that we use the particular entanglement measure $G(\rho_{BE})$ while other entanglement measures may be employed to prove the result in [28].

**C. Potential quantum capacity**

In this section, we move on to the potential quantum capacity and study its relations to the single-letter quantity $Q^{(1)}(N)$. In [25, Sec. VII], this had been introduced under the name of “quantum value added capacity”, and our Lemma 5 already been observed in that case. Here, we establish an upper bound in terms of the entanglement of formation of the channel, and finally prove that an imperfect quantum channel cannot be activated into a perfect one by any other contextual channel.

**Definition 12** Specializing Definition 2 to the case $f \equiv Q$, we obtain the potential quantum capacity

$$Q_p(N) = \sup_{\mathcal{M}} \left[ Q(N \otimes \mathcal{M}) - Q(\mathcal{M}) \right],$$

(19)

and the potential single-letter quantum capacity

$$Q^{(1)}_p(N) = \sup_{\mathcal{M}} \left[ Q^{(1)}(N \otimes \mathcal{M}) - Q^{(1)}(\mathcal{M}) \right].$$

(20)

By Lemma 6 we have

$$Q^{(1)}(N) \leq Q(N) \leq Q_p(N) \leq Q^{(1)}_p(N).$$

(21)

The symmetric side-channel assisted quantum capacity, $Q_{ss}$, introduced and investigated in [25], is obtained by restricting the above optimization to channels $\mathcal{M}$ that are symmetric, i.e. both degradable and anti-degradable, which is a special subclass of zero-capacity channels. Unlike $Q$, $Q_{ss}$ is additive and has many other nice properties, and from the definition and the above, we have (cf. [25, Sec. VII])

$$Q_{ss}(N) \leq Q_p(N) \leq Q^{(1)}_p(N).$$

(22)

How do we establish the upper bound for the potential quantum capacity? The idea is channel simulation inspired by the approach to obtain an upper bound for the quantum capacity: If the channel $\mathcal{N}$ can be simulated by another channel $\mathcal{N}^\dagger$ using pre- and post-processing, i.e. $\mathcal{N} = T \circ \mathcal{N}^\dagger \circ S$ with suitable CPTP maps $S$ and $T$, then clearly $Q(\mathcal{N}) \leq Q(\mathcal{N}^\dagger)$. We call $\mathcal{N}^\dagger$ a *lifting* of $\mathcal{N}$. Furthermore, if the channel $\mathcal{N}^\dagger$ is degradable, then its quantum capacity is given by the single-letter capacity $Q^{(1)}(\mathcal{N}^\dagger)$, and obtain a single-letter upper bound for $Q(\mathcal{N})$. This was observed and exploited before under the name of “additive extensions” [29].

From inequality (21) and the definition of potential single-letter quantum capacity, we see that we should try to lift the channel to a strongly additive one, because then we get even an upper bound for the potential quantum capacity, and in fact the potential single-letter quantum capacity!

However it is not enough to lift the channel to a degradable one, because we learn from the superactivation phenomenon that its single-letter quantum capacity is not strongly additive. But an even narrower class of degradable channels, called Hadamard channels, satisfies the required property.

**Definition 13** A Hadamard channel (HC) [30] $\mathcal{N}$ is a quantum channel whose complementary channel $\mathcal{N}^c$ is an entanglement-breaking channel (EBC) [31], where $\mathcal{N}^c$ can be expressed as

$$\mathcal{N}^c(\rho) = \sum_i |\phi_i\rangle\langle \phi_i| |\tilde{\psi}_i\rangle\langle \tilde{\psi}_i| \rho|\tilde{\psi}_i\rangle,$$

in which $\sum_i |\tilde{\psi}_i\rangle\langle \tilde{\psi}_i| = I$ is a POVM. Such channels are said to be *entanglement-breaking* because the output state $\text{id} \otimes \mathcal{N}^c(\rho^{RA})$ is separable for any state $\rho^{RA}$.

The isometry of the Hadamard channel $\mathcal{N}$ is of the form (up to local unitary operation on $E$)

$$V = \sum_i |i\rangle B |\phi_i\rangle E \langle \tilde{\psi}_i| A,$$

(23)

from which we see that the Hadamard channel $\mathcal{N}$ can simulate its complementary channel $\mathcal{N}^c$ by the operation of first measuring in the basis $|i\rangle$ and then preparing the state $|\phi_i\rangle$ according to the outcome of the measurement. Thus Hadamard channels are special degradable channels.

**Proposition 14** (Cf. Bradler et al. [32], Wilde/Hsieh [33]) If $\mathcal{N}$ is a Hadamard channel, then $Q^{(1)}$ is strongly additive: $Q^{(1)}(\mathcal{N} \otimes \mathcal{M}) = Q^{(1)}(\mathcal{N}) + Q^{(1)}(\mathcal{M})$ for any contextual channel $\mathcal{M}$.
Proof: The “≥” part is trivial and we only need to prove the “≤” part. Suppose the isometry of the Hadamard channel \( \mathcal{N} \) is \( V : A_1 \to B_1 \otimes E_1 \), of the form \( \sum_i \phi_i \), and the isometry of \( \mathcal{M} \) is \( W : A_2 \to B_2 \otimes E_2 \). The output, for an input state \( |\phi\rangle_{RA_1A_2} \), is
\[
V \otimes W |\phi\rangle_{RA_1A_2B_2E_2} = \sum_i |i\rangle_{B_1} |\phi_i\rangle_{E_1} (\tilde{\psi}_i)_{A_1} |\phi\rangle_{RA_2B_2E_2},
\]
where the first equality comes from simulation, the first inequality comes from the strong additivity (Proposition 14), and the second equality from Lemma 13, though not explicitly stated there.

A corollary is the following.

Corollary 15 The potential quantum capacity (and potential single-letter quantum capacity) of a Hadamard channel \( \mathcal{N} \) is equal to its single-letter quantum capacity:
\[
Q_p^{(1)}(\mathcal{N}) = Q_p(\mathcal{N}) = Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}).
\]

Thus we have reduced our task to finding a good lifting of a given channel to a Hadamard channel. The question how to find the optimal one is of interest in itself and we will discuss the general method elsewhere. For our present purposes, there is a rather straightforward way to lift a channel to a Hadamard channel. Namely, choose Kraus operators for \( \mathcal{N} \) as \( \mathcal{N}(\rho) = \sum_i K_i \rho K_i^\dagger \), Then a Stinespring isometry for \( \mathcal{N} \) can be written as \( U = \sum_i |i\rangle E K_i^A \otimes K_i^B \).

Let us define a new channel, the lifting \( \mathcal{N}^\uparrow \), via its isometry
\[
V^{A \to B'B' \otimes E} := \sum_i |i\rangle_{E} |i\rangle_{B'} K_i^A \otimes K_i^B,
\]
where the environment system is still \( E \), but the receiver has now \( BB' \), and \( B' \) holds a coherent copy of \( E \). As we now give a copy of \( E \) to the channel output, the output of the complementary channel of \( \mathcal{N}^\uparrow \) will be completely decohered in the \( \{|i\rangle\} \) basis, so the complementary channel is EBC, hence \( \mathcal{N}^\uparrow \) is Hadamard, as desired. The Kraus operators of the lifted Hadamard channel \( \mathcal{N}^\uparrow \) are \( \{|i\rangle \otimes K_i\} \), and one way to write the channel is as
\[
\mathcal{N}^\uparrow(\rho) = \sum_i |i\rangle_i \otimes K_i \rho K_i^\dagger,
\]
which we call the canonical lifting. Its quantum capacity is
\[
Q(\mathcal{N}^\uparrow) = \max_{|\phi\rangle_{RA}} \sum_i p_i S(\rho_i^B),
\]
where the infimum is taken over all Kraus representations of the original channel, which is defined as
\[
E_F(\mathcal{N}) := \max_{|\phi\rangle_{RA}} \min_{\{K_i\}} \sum_i p_i E(\phi_i), \sqrt{p_i} \rho_i^R = (I \otimes K_i) |\phi\rangle.
\]

Now take the minimum over all different Kraus representations (which after all we are free to choose), to obtain the best bound from this particular family of canonical liftings.

As a result, the quantum capacity of the optimal canonical lifting is equal to the entanglement of formation of the original channel, which is defined as
\[
Q^{(1)}(\mathcal{N}) = \min_{\{\rho\}_{RA}} \sum_i p_i E(\phi_i) = \max_{\{\rho\}_{RA}} \min_{\{K_i\}} \sum_i p_i E(\phi_i),
\]
where \( E(\phi) \) is the entropy of entanglement of the bipartite pure state \( \phi \).

The following lemma is implied by the proof of [35] Lemma 13], though not explicitly stated there.

Lemma 16 (Berta et al. [35]) With the above notation, the following minimax formula holds:
\[
\min_{\{K_i\}} \sum_i p_i E(\phi_i) = \max_{|\phi\rangle_{RA}} \min_{\{K_i\}} \sum_i p_i E(\phi_i),
\]
where the infimum is taken over all Kraus representations of the channel \( \mathcal{N} \).

Now we obtain an upper bound on potential quantum capacity in terms of the entanglement of formation of the channel:

Theorem 17 For a general channel \( \mathcal{N} \), we have the following upper bound on the potential quantum capacity:
\[
Q_p(\mathcal{N}) \leq Q_p^{(1)}(\mathcal{N}) \leq E_F(\mathcal{N}).
\]

Proof: Lifting the channel to the optimal canonical Hadamard channel, we get
\[
Q^{(1)}(\mathcal{N} \otimes \mathcal{M}) \leq Q^{(1)}(\mathcal{N}^\uparrow \otimes \mathcal{M}),
\]
\[
= Q^{(1)}(\mathcal{N}^\uparrow) + Q^{(1)}(\mathcal{M}) = E_F(\mathcal{N}) + Q^{(1)}(\mathcal{M}),
\]
where the first inequality comes from simulation, the first equality from the strong additivity (Proposition 14), and the second equality from Lemma 16.

Analogous to the classical capacity, we call a channel perfect if its quantum capacity is equal to \( \log d_{min} \) with \( d_{min} = \min\{d_{in}, d_{out}\} \). Before we have the similar corollary, we recall a result in [36].

Lemma 18 (Schumacher/Westmoreland [36]) For a pure bipartite state \( |\phi_{RA}\rangle \), let the system \( A \) be transmitted through a channel \( \mathcal{N} : A \to B \) and the joint output state is \( \rho^{RB} = \sum_i p_i E(\phi_i) \).

Proof: 

id ⊗ \mathcal{N}(\phi^{RA}). If E_F(R : B) = S(R), then there exists a quantum operation D : B → A such that id ⊗ D(\rho^{RB}) = \phi^{RA}.

**Corollary 19** If a quantum channel \mathcal{N} is not perfect for transmitting quantum information, then its potential single-letter quantum capacity is not maximal, either:

$$Q^{(1)}(\mathcal{N}) < \log d_{\min} \implies Q_p^{(1)}(\mathcal{N}) < \log d_{\min}.$$  

In particular, if a quantum channel is not perfect for transmitting quantum information, then it cannot be activated to a perfect one by any zero-quantum-capacity channels.

**Proof:** Suppose that the potential quantum capacity is \log d_{\min}, then from the upper bound in Theorem 17 we know that \( E_F(\mathcal{N}) = \log d_{\min}. \) In the case of \( d_{\min} = d_{\text{out}}, \) from Lemma 18 we know that the channel operation can be perfectly corrected by a suitable operation \( D \) acting on \( B \) that means the channel is already perfect.

In the case of \( d_{\min} = d_{\text{out}}, \) denoting the isometry of the channel \( \mathcal{N} : A \to B \) by \( U : A \leftrightarrow B \otimes E, \) there exists hence an input \( |\phi\rangle^{RA} \) such that the output state plus the environment is \( |\phi\rangle^{RBE} = U|\phi\rangle^{RA}, \) where \( \rho^{BE} = id \otimes \mathcal{N}(\phi^{RA}) \) satisfying \( E_F(\rho^{RB}) = S(B) = \log d. \) This implies that \( \rho^{BE} \) is a product state. From the Uhlmann theorem [37], we know there is a unitary \( V : R \to R_1 \otimes R_2 \) such that

$$V|\phi\rangle^{RBE} = V \otimes U|\phi\rangle^{RA} = |\phi\rangle_{R1B} \otimes |\phi\rangle_{R2E},$$

where \( \{ |i\rangle_R1, |e_i\rangle_E \} \) are the bases in the Schmidt decomposition of \( |\phi\rangle_{R1B} \) and \( \{ |j\rangle_{R2}, |e_j\rangle_E \} \) for \( |\phi\rangle_{R2E}. \) So the input state \( V \otimes I|\phi\rangle^{RA} = \sum_i \frac{1}{\sqrt{d}} |i\rangle_{R1} |j\rangle_{R2} U_1^\dagger (|e_i\rangle_B |e_0\rangle_E) \) will give a product state as output. Now from this input we construct a new input

$$|\psi\rangle^{RA} = \sum_i \frac{1}{\sqrt{d}} |i\rangle_{R1} |0\rangle_{R2} U_1^\dagger (|e_i\rangle_B |e_0\rangle_E),$$

yielding the output

$$U|\psi\rangle^{RA} = \sum_i \frac{1}{\sqrt{d}} |i\rangle_{R1} |0\rangle_{R2} |e_i\rangle_B |e_0\rangle_E,$$

which is the desired output of product state between \( B \) and \( E. \) This means \( Q^{(1)}(\mathcal{N}) = \log d, \) i.e. \( \mathcal{N} \) is noiseless already.

**Remark** We know that the channel can be very entangled even though its quantum capacity is zero. It is very difficult to characterize channels with zero quantum capacity. So it seems that it is hard to say whether a noisy channel can be activated into a noiseless one under the assistance of zero-quantum-capacity channels. However from the notion of potential quantum capacity, we can answer this question in the negative.

**D. Potential private capacity**

In this section, we repeat the analysis of the preceding subsection, but for the potential private capacity. We shall show that, as for \( Q, \) the potential private capacity cannot be maximal without it already being the single-letter private capacity. Especially we prove that the private capacity of Hadamard channels is strongly additive.

**Definition 20** Specializing Definition 2 to the case \( f \equiv P, \) we obtain the potential private capacity

$$P_p(\mathcal{N}) = \sup_{\mathcal{M}} \{ P(\mathcal{N} \otimes \mathcal{M}) - P(\mathcal{M}) \},$$

and the potential single-letter private capacity

$$P_p^{(1)}(\mathcal{N}) = \sup_{\mathcal{M}} \{ P^{(1)}(\mathcal{N} \otimes \mathcal{M}) - P^{(1)}(\mathcal{M}) \}. $$

By Lemma 6 we have

$$P^{(1)}(\mathcal{N}) \leq P(\mathcal{N}) \leq P_p(\mathcal{N}) \leq P_p^{(1)}(\mathcal{N}).$$

As for in the quantum case, we aim to lift channels to strongly additive ones, so as to obtain single-letter upper bounds on the potential private capacity. Indeed, we can extend Proposition 14 to the private capacity:

**Proposition 21** If \( \mathcal{N} \) is a Hadamard channel, then \( P^{(1)}(\mathcal{N}) \) is strongly additive: \( P^{(1)}(\mathcal{N} \otimes \mathcal{M}) = P^{(1)}(\mathcal{N}) + P^{(1)}(\mathcal{M}) \) for any contextual channel \( \mathcal{M}. \)

**Proof:** The "≥" part is trivial and we only need to prove the "≤" part. Suppose the isometry of the Hadamard channel \( \mathcal{N} = V : A_1 \leftrightarrow B_1 \otimes E_1, \) of the form (25) as before, and the isometry of \( \mathcal{M} = W : A_2 \leftrightarrow B_2 \otimes E_2. \) For the input state ensemble \( \{ p_i, |\phi_i^{A_1A_2} \}, \) we construct the classical-quantum (cq) state with the reference system R that purifies each \( \rho_i^{A_1A_2}, \)

$$\sum_i p_i |t_i\rangle \langle t_i | \otimes |\phi_i\rangle \langle \phi_i |^{RA_1A_2},$$

which is mapped by \( V \otimes W \) to

$$\sum_i p_i |t_i\rangle \langle t_i | \otimes |\phi_i\rangle \langle \phi_i |^{RB_1E_1B_2E_2}. $$

Here,

$$|\phi_i\rangle^{RB_1E_1B_2E_2} = \sum_i \sqrt{p_i} |t_i\rangle B_1 |\phi_i\rangle E_1 |\phi_i\rangle^{RB_1B_2E_2}.$$  

Using the isometry \( |i\rangle^{B_1} \leftrightarrow |i\rangle Y |i\rangle^Z, \) and the notation \( F(X)\langle T = \sum p_i F(X_i), \) we get

$$I(T : B_1B_2) - I(T : E_1E_2) = S(B_1B_2) - S(E_1E_2) - S(TB_1B_2) - S(TE_1E_2),$$

$$ = S(B_1) - S(E_1) + S(B_2|B_1) - S(E_2|E_1) \leq S(B_1) - S(E_1) + S(B_2Y) - S(E_2Y)$$

$$- (S(B_1B_2|T) - S(E_1E_2|T)),$$

$$S(B_1) - S(E_1) + I(T : B_2) - I(T : E_2) \leq S(B_1) - S(E_1) + I(T : B_2) - I(T : E_2)$$

$$+ (S(TB_2|Y) - S(TE_2|Y)) - (S(B_1B_2|T) - S(E_1E_2|T)),$$

$$S(B_1) - S(E_1) + I(T : B_2) - I(T : E_2) \leq S(B_1) - S(E_1) + I(T : B_2) - I(T : E_2)$$

$$+ S(B_2Y) - S(E_2Y) - S(B_1B_2) + S(E_1E_2)|T,$$

$$\leq P^{(1)}(\mathcal{N}) + P^{(1)}(\mathcal{M}).$$


where the first inequality comes from $S(B_2|B_1) \leq S(B_2|Y)$ and $S(E_2|E_1) \geq S(E_2|Y)$, which hold for the same reason in the proof of Prop. 14. Next we show that each term in the average over $T$ is non-positive. Indeed, we evaluate the term in the pure state of the form (27) and notice that $S(E_2Y = S(E_1E_2Y)$. Then we have

$$S(B_2Y) - S(E_2Y) - S(B_1B_2) + S(E_1E_2) = -I(R: Y|E_1E_2) \leq 0,$$

which concludes the proof.

An immediate corollary is the following.

**Corollary 22** The potential private capacity (and the potential single-letter private capacity) of a Hadamard channel $N$ is equal to its private capacity, which in turn equals $Q^{(1)}(N)$:

$$P_p^{(1)}(N) = P_p(N) = P(N) = Q^{(1)}(N).$$

**Theorem 23** For any channel $N$, we have the upper bound $P_p(N) \leq P_p^{(1)}(N) \leq E_F(N)$.

**Proof:** Notice that for the Hadamard channel, the potential capacity is equal to the quantum capacity and the rest is the same as the quantum case, i.e. the proof of Theorem 17.

As in the quantum case, we obtain the following immediate corollary:

**Corollary 24** If a quantum channel is not perfect for transmitting private information, then it cannot be activated to the perfect one by any contextual channels.

**V. Discussion and open questions**

We have introduced potential capacities of quantum channels as “big brothers” of the plain capacities, to capture the degree of non-additivity of the latter in the most favorable context. By bootstrapping strongly additive channels, i.e. those whose plain capacity equals its potential version, we were able to give some general upper bounds on various potential capacities. While the potential concept makes sense for any capacity, here we focused on a few examples, basically the principal channel capacities $C$, $Q$ and $P$. Our central result is that a noisy channel cannot be activated into a noiseless one by any contextual channel. This result holds for the classical, quantum, and private capacity, and improves upon previous statements. Notice that in the notion of potential capacity, a PPT-entanglement-binding channel may have positive potential quantum capacity. So it is tempting to speculate whether all entangled channels have positive potential quantum capacity. This is a big open question deserving of study in the future.

Looking beyond capacities and at the tradeoff between different resources, note that Hadamard channels that served us so well in the treatment of the potential quantum and classical capacities, also allow for a single-letter formula for the qubit-qubit tradeoff region [33]; in fact, Thm. 3 and Lemma 2 of that paper show that this region is strongly additive regarding the tensor product of a Hadamard channel with any other channel. Thus, the notion of lifting to a Hadamard channel once more yields and outer bound on the potential capacity region of the achievable triples $(q, c, e)$.

We have studied potential capacities only for the basic quantities, and one ($Q_{ss}$) for which we could calculate the potential capacity exactly. For most capacities, we may assume that it will be prohibitive to calculate the potential version as well as its plain version, so we have to be content with bounds. In the domain of zero-error information theory, other exact characterizations of some potential capacities are known [38].

In fact, in [38], differences between the capacity of $K(N \otimes M)$ and another parameter for $M$, which represents a more general value $V(M)$ of the channel, were considered, where $K$ is superadditive and $V(M) \geq K(M)$. Then,

$$V^*(N) = \sup_M K(N \otimes M) - V(M)$$

is a kind of amortized value of $N$, the rationale being that the gain from the “profit” $K(N \otimes M)$ has to be offset by the “price” $V(M)$ of the borrowed resource. Of particular interest is the case of an economically fair pricing $V = V^*$, i.e. of a situation where the same value $V^*(N) = V^*(N) + V(M)$ quantifies the price of resource when we have to borrow it, as well as the amortized value in a suitable context. In the setting of zero-error capacities, specifically $K = \log \alpha$ (with the independence number $\alpha$), this has been shown to hold true for $K = \log \theta$ (the Lovász number). For Shannon theoretic capacities, i.e. $K \in \{C, P, Q\}$ or similar, the existence and possible characterization of a value $V$ that yields a fair value $V = V^*$ is perhaps one of the most intriguing questions raised by the notion of potential capacities. For instance, for the quantum capacity, it turns out that its symmetric side-channel assisted version $Q_{ss}$ is such a fair price. Namely, for a given channel $N$ and any contextual channel $M$,

$$Q^{(1)}(N \otimes M) \leq Q(N \otimes M),$$

$$Q_{ss}(N \otimes M) = Q_{ss}(N) + Q_{ss}(M),$$

thus

$$\sup_M Q^{(1)}(N \otimes M) - Q_{ss}(M) \leq \sup_M Q(N \otimes M) - Q_{ss}(M) \leq Q_{ss}(N).$$

On the other hand, restricting to symmetric channels $M$ in this optimization, for which $Q_{ss}(M) = 0$, we attain equality asymptotically by definition of the symmetric side-channel assisted quantum capacity. The same reasoning can be applied to the private capacity and the symmetric side-channel assisted version $P_{ss}$ [21], so we have proved the following.

**Theorem 25** For any channel $N$,

$$Q_{ss}(N) = \sup_M Q^{(1)}(N \otimes M) - Q_{ss}(M),$$

$$= \sup_M Q(N \otimes M) - Q_{ss}(M),$$

$$P_{ss}(N) = \sup_M P^{(1)}(N \otimes M) - P_{ss}(M),$$

$$= \sup_M P(N \otimes M) - P_{ss}(M),$$

where the first inequality comes from $S(B_2|B_1) \leq S(B_2|Y)$ and $S(E_2|E_1) \geq S(E_2|Y)$, which hold for the same reason in the proof of Prop. 14. Next we show that each term in the average over $T$ is non-positive. Indeed, we evaluate the term in the pure state of the form (27) and notice that $S(E_2Y = S(E_1E_2Y)$. Then we have

$$S(B_2Y) - S(E_2Y) - S(B_1B_2) + S(E_1E_2) = -I(R: Y|E_1E_2) \leq 0,$$
and equality is asymptotically attained by symmetric channels \(\mathcal{M}\).

If the quantum and private capacities have alternative fair pricing schemes along these lines or if \(Q_{ss}(\mathcal{M})\) are unique, and whether there is an analogous statement for \(C\) or \(\chi\), remain open.

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APPENDIX

Here we analyze the structure of the state satisfying \(S(B) - S(BE) = G(B : E)\) or \(S(B) - S(BE) = E_F(B : E)\). The general relation among these three quantities is \(S(B) - S(BE) \leq G(B : E) \leq E_F(B : E)\), the first “\(\leq\)” comes from Lemma 26 and the second “\(\leq\)” from Lemma 9. Obviously the condition \(S(B) - S(BE) = E_F(B : E)\) implies \(S(B) - S(BE) = G(B : E)\). Lemma 27 asserts that the latter also implies the former.

**Lemma 26** For a mixed state \(\rho_{BE}\), \(S(B) - S(BE) \leq C_{\perp}(\rho_{BE})\), and equality holds iff there exists a unitary on \(B\) such that \(U_{B}\rho_{BE}U_{B}^{\dagger} = \rho_{BL} \otimes \phi^{BE}_{BL}\), where \(\mathcal{H}_B = \mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}\) and \(\phi^{BE}_{BL}\) is pure.

**Proof:** Consider the purification \(\phi^{BE}_{BL}\) of the state \(\rho_{BE}\). Then \(C_{\perp}(\rho_{BE}) = S(B) - E_F(R : B)\) and \(S(B) = S(R)\). From the inequality \(E_F(R : B) \leq S(R)\), we arrive at \(S(B) - S(BE) \leq C_{\perp}(\rho_{BE})\).

When the equality holds, this amounts to \(E_F(R : B) = S(R)\). From the relation \(C_{\perp}(\rho_{BE}) = S(R) - E_F(R : B)\), we get that \(\rho^{RE} = \rho^{R} \otimes \rho^{E}\). From Uhlenbeck’s theorem [37], there exists a unitary \(U_B\) such that \(U_B\phi^{BE}_{BL}U_B^{\dagger} = \phi^{RL}_{BE} \otimes \phi^{RE}_{BL}\). Tracing out \(R\) concludes the proof.

**Lemma 27** For a state \(\rho_{BE}\), \(S(B) - S(BE) \leq G(B : E)\). If \(S(B) - S(BE) = G(B : E)\), then \(S(B) - S(BE) = E_F(B : E)\).

**Proof:** Suppose that the optimal realization of \(G(B : E)\) is the state ensemble \(\{p_i, \rho_{iBE}\}\). Then,

\[S(B) - S(BE) \leq \sum_i p_i [S(B_i) - S(BE_i)] \leq \sum_i p_i C_{\perp}(\rho_{iBE}),\]

where the first “\(\leq\)” comes from the concavity of the conditional entropy and the second “\(\leq\)” from Lemma 26 above.

If \(S(B) - S(BE) = G(B : E)\), then \(S(B_i) - S(BE_i) = C_{\perp}(BE_i)\) for each \(\rho_{iBE}\). From Lemma 26 the state \(\rho_{BE}\) has the property \(E_F(BE_i) = S(B_i) - S(BE_i)\). Then \(S(B) - S(BE) \leq E_F(BE_i) \leq \sum_i p_i E_F(BE_i) = \sum_i p_i [S(B_i) - S(BE_i)] = S(B) - S(BE)\) and the proof ends.

In fact, the constraint is so sharp that we can even learn the structure of the bipartite state. Proposition 28 explains this and is also of independent interest.

**Proposition 28** A state \(\rho_{BE}\) in the finite dimensional Hilbert space \(\mathcal{H}_B \otimes \mathcal{H}_E\) satisfies \(S(B) - S(BE) = E_F(B : E)\), if and only if it is of the form

\[\rho_{BE} = \bigoplus_i p_i \rho_i^{BL} \otimes \phi_i^{BE},\]

where \(\phi_i^{BE}\) are pure states and the system \(B\) is decomposed into the direct sum of tensor products

\[\mathcal{H}_B = \bigoplus_i \mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}.\]

**Proof:** Suppose the optimal realization of \(E_F(B : E)\) is the ensemble \(\{p_x, |\psi_x\rangle\}\), and construct the state \(\rho^{XBE} = \sum_x p_x |x\rangle \langle x| \otimes \psi_x^B\). From the condition \(S(B) - S(BE) = E_F(B : E)\), we get \(S(B) - S(BE) = \sum_x p_x S(\psi_x^B)\). This condition can be expressed as \(I(X : E|B) = 0\), where \(I(X : E|B) = S(XB) + S(EB) - S(XBE) - S(B)\) is the conditional quantum mutual information. From [39], we know that \(I(X : E|B) = 0\) if and only if the state \(\rho^{XBE}\) can be decomposed as

\[\rho^{XBE} = \bigoplus_i q_i \rho_i^{BL} \otimes \rho_i^{BE},\]

where the system \(B\) is decomposed into the direct sum of tensor products

\[\mathcal{H}_B = \bigoplus_i \mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}.\]

Thus we have

\[\rho^{BE} = \bigoplus_i q_i \rho_i^{BL} \otimes \rho_i^{BE}.\]

If all the states \(\rho_i^{BE}\) are pure, then we are done. In general, some of \(\rho_i^{BE}\) may be mixed. Apply the condition \(S(B) - S(BE) = E_F(B : E)\) to the structured state \(\rho^{BE}\), we get \(\sum q_i(S(B_i^R) - S(B_i^RE)) = \sum q_i E_F(B_i^R : E)\). Since \(S(B) - S(BE) \leq E_F(B : E)\) is true for all of the components \(\rho_i^{BE}\), we arrive at \(S(B_i^R) - S(B_i^RE) = E_F(B_i^R : E)\) for each \(i\). So we can use the argument again and get that the structure of each state \(\rho_i^{BE}\) is of the direct sum of tensor products. If some of the new states \(\rho_i^{BE}\) are mixed, we repeat the argument for these states. In each iteration, the dimension is reduced because of the direct sum of tensor products. Since system \(B\) is a finite dimensional Hilbert space, the iteration
ends after finitely many steps when all the states $\rho_{i^*}^{B_i E_i}$ are pure. After renumbering the labels, we get the desired decomposition where all the states $\rho_{i^*}^{B_i E_i}$ are pure.

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