On wormholes with long throats and the stability problem

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We construct explicit examples of globally regular static, spherically symmetric solutions of general relativity with a phantom scalar field as the source of gravity, describing traversable wormholes with flat asymptotic regions on both sides of the throat as well as regular black holes, in particular, those called black universes. To explain why such phantom fields are not observed under usual conditions, we invoke the concept of “invisible ghosts,” which means that the phantom field decays quickly enough at infinity and is there too weak to be observed. This approach leads to wormhole models in which the spherical radius is almost constant in some range of the radial coordinate near the throat, forming a “long throat.” We discuss the peculiar features and difficulties of the stability problem for such configurations. It is shown that the limiting case of a maximally long throat has the form of an unstable model with the Nariai metric. This allows us to conjecture that a long throat does not stabilize wormholes with a scalar source.

1 Introduction

Phantom field configurations have gained much interest since the discovery of the accelerated expansion of the Universe and its explanation in the framework of general relativity (GR) by the existence of dark energy, a source of gravity of unknown nature which can violate the standard energy conditions, such that the pressure to density ratio is $w = p/\rho < -1$. Numerous observations lead to estimates of $w$ around $-1$, which corresponds to a cosmological constant, but values smaller than $-1$ are still admissible and even preferable for describing an increasing acceleration. Thus, one of the most recent estimates\textsuperscript{1} reads $w = -1.006 \pm 0.045$.

If, following many authors, we accept the existence of phantom (or exotic) matter as a working hypothesis, it is natural to expect that there exist its manifestations in local objects and phenomena. The simplest of them can be described by static, spherically symmetric solutions to the Einstein-scalar equations where the scalar field has an unusual sign of kinetic energy (a phantom scalar, by definition). In the case of a massless scalar, the corresponding solution\textsuperscript{2} is a phantom analog of Fisher’s solution\textsuperscript{3} for an ordinary minimally coupled massless scalar field; the phantom-field solution\textsuperscript{2} is sometimes called the anti-Fisher solution.

It is also known that if one admits the existence of exotic matter, for example, in the form of phantom scalar fields, there emerge such configurations of interest as wormholes\textsuperscript{4–6} and regular black holes\textsuperscript{7,8}.

Since no exotic matter or phantom fields have been detected under usual physical conditions, it is desirable to avoid the emergence of such fields in an asymptotic weak-field region, or at least to make them decay there rapidly enough. Thus, it was suggested\textsuperscript{9} to use a special kind of fields, named “trapped ghosts,” which have phantom properties only in some restricted strong-field region and are usual, canonical in other parts of space. A variety of solutions with such fields have been obtained, including regular black holes, black universes and traversable wormholes\textsuperscript{9,11}. In these models, the kinetic energy density smoothly passes zero at some scalar field value. Such transition points create some problems with perturbation equations for these fields\textsuperscript{12} which need a separate study.

Another opportunity is to use phantom scalar fields rapidly decaying in weak-field regions. In this paper we will consider a field of this kind and show...
that it leads to a wormhole solution with a very slowly changing spherical radius in a neighborhood of the throat, so that it makes sense to speak of a wormhole with a “long throat”.

Concerning the stability of such configurations, it turns out that a long throat causes substantial technical difficulties because the effective potential for the most dangerous perturbations, those preserving spherical symmetry, has such a pole on the throat that makes it very hard to consider the perturbations of the throat radius. Meanwhile, it is this mode that makes unstable many known wormhole and black universe solutions, as shown in 14–16. These papers used the so-called S-transformation which regularizes the potential only if its pole has the form $2/z^2 + O(1)$ (in properly chosen variables) but fails when dealing with other singularities that emerge in the case of a long throat. We postpone a detailed discussion of the stability of long-throat wormholes to future studies. Instead, we here consider, as a tentative problem, the stability properties of a configuration with a constant spherical radius $r(x)$, which may be called a maximally long throat. It is not a wormhole since there are no spatial asymptotic. The corresponding static solution reduces to the well-known Nariai metric 17 with a constant scalar field, and we explicitly show that it is unstable under linear perturbations. Therefore, one can speculate that a slowly varying radius near a throat does not stabilize a wormhole supported by a phantom scalar field.

The paper is organized as follows. Section 2 presents the basic equations and some general observations. In Section 3 we obtain examples of wormhole and regular black hole configurations. Section 4 discusses the linear stability problem for these solutions. Section 5 is devoted to stability study for a limiting configuration with a “maximally long throat”.

3One can also mention an approach leading to configurations called time-independent wormholes which, being constructed from the Schwarzschild-AdS metric, do not require any exotic matter, see 13 and references therein; they are related to the AdS-CFT correspondence and have peculiar expressions for the entropy. However, such objects are not wormholes in the terminology we are using: these are black holes since they contain Killing horizons. These geometries also contain what can be called anti-throats, i.e., local maxima of the spherical radius $r$.

2 Basic equations

We begin with the total action of GR with a scalar field source

$$S = \frac{1}{2} \int \sqrt{-g} d^4x \left[ R + 2\varepsilon g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - 2V(\phi) \right], \quad (1)$$

where $R$ is the scalar curvature, $\varepsilon = 1$ for a normal scalar field with positive kinetic energy, and $\varepsilon = -1$ for a phantom scalar field: $g = \text{det}(g_{\mu\nu})$, and $V(\phi^0)$ is a self-interaction potential. The field equations may be written as

$$2\varepsilon \nabla^\mu \nabla_\mu \phi + V_\phi = 0, \quad (2)$$

$$R_{\mu}^\nu = -2\varepsilon \phi_{,\mu} \phi_{,\nu} + \delta_{\mu}^\nu V(\phi), \quad (3)$$

and we use the units in which $8\pi G = 1$ and $c = 1$.

Consider the general static, spherically symmetric metric

$$ds^2 = A(x)dt^2 - \frac{dx^2}{A(x)} - r^2(x)d\Omega^2, \quad (4)$$

in terms of the so-called quasiglobal radial coordinate $x$, such that $g_{00}g_{11} = -1$; $d\Omega^2 = (d\theta^2 + \sin^2 \theta d\phi^2)$ is the linear element on a unit sphere. Equations (2) and (3) for the unknowns $\phi(x)$, $A(x)$ and $r(x)$ take the form 18

$$2(A r^2 \phi')' = r^2 \varepsilon V_\phi, \quad (5)$$

$$(A'r^2)'' = -2r^2 V, \quad (6)$$

$$r''/r = -\varepsilon \phi'^2, \quad (7)$$

$$A(r^2)'' - r^2 A'' = 2, \quad (8)$$

$$-1 + A'rr' + Ar^2 = r^2(\varepsilon A \phi'^2 - V), \quad (9)$$

where the prime denotes $d/dx$, and $V_\phi = dV/d\phi$. Equation (5) is the scalar field equation, the others are components of (3), more specifically: (6) is the component $R_{t}^{t} = \ldots$, (7) is the combination $R_{t}^{r} - R_{r}^{t} = \ldots$, and (8) is $R_{t}^{t} - R_{t}^{t} = \ldots$; lastly, Eq. (9) is the constraint equation for the Einstein tensor component $G_{x}^{x} = \ldots$. Equations (5) and (9) follow from Eqs. (6–8) with a given potential $V(\phi)$, so the latter form a determined set of equations for $\phi(x), r(x), A(x)$. Equation (8) can be integrated giving

$$B'(x) \equiv (A/r^2)' = (6m - 2x)/r^4, \quad (10)$$

where $B(x) \equiv A/r^2$ and $m$ is an integration constant equal to the Schwarzschild mass if the metric (4) is asymptotically flat as $x \to \infty$ ($r \approx x$,
A = 1 − 2m/x + o(1/x)). If there is a flat asymptotic as \( x \to -\infty \), the Schwarzschild mass there is equal to \(-m \ (r \approx |x|, \ A = 1 + 2m/|x| + o(1/x))\).

Thus we have a general result: in any solution with two flat asymptotic regions in the presence of any potential \( V(\phi) \) compatible with such behavior, we inevitably have masses of opposite signs at the two infinities. Such solutions might in principle represent wormholes or regular black holes. In the latter case, the function \( B(x) \) would have a minimum at which \( B \leq 0 \); it can be shown, however, that the existence of such a minimum is incompatible with Eq. (3). Therefore, if a solution to the above equations is twice asymptotically flat, it can only describe a wormhole.

### 3 Models with an invisible ghost and a long throat

Equations (5)–(9) are hard to solve if a nonzero potential \( V(\phi) \) is specified. Instead, as in \([7,12,18]\), we will use the inverse problem method to find examples of interest: specifying the function \( r(x) \), we can find all other unknowns, including \( V \), from the field equations: \( A(x) \) is found from (13), then \( V(x) \) from (6), and \( \phi(x) \) from (7).

Our interest is in nonsingular configurations without a center, which can be wormholes or black universes.\( ^7 \) Hence we assume that the range of \( x \) is \( x \in \mathbb{R} \), in which both \( A(x) \) and \( r(x) \) are regular, \( r > 0 \) everywhere, and \( r \to \infty \) at both ends. We also require \( r(x) \approx |x| \) as \( x \to \pm \infty \), which is in agreement with possible flat, de Sitter or AdS asymptotic behaviors at large \( r \). Thus there must be a minimum of \( r(x) \) (say, at \( x = 0 \) without loss of generality), such that \( r(0) > 0 \), \( r'(0) = 0 \), \( r''(0) > 0 \). (It may happen that \( r'' > 0 \) at a minimum of \( r \) but then inevitably \( r'' \) in its neighborhood). According to Eq. (7), this inevitably implies \( \varepsilon = -1 \), a phantom field, which is a manifestation of the well-known necessity of violating the Null Energy Condition (NEC) in wormhole and other regular spherical configurations in GR \([18,20,21]\).

In our previous paper \([12]\) we discussed a way to explain why such phantom fields are not observed under usual conditions by using the concept of “invisible ghosts,” which means that the phantom field decays quickly enough at infinity and is too weak to be observed. We found in \([12]\) some examples of wormhole and regular black-hole solutions with and without an electromagnetic field and two scalar fields, a long-range canonical one and a comparatively short-range phantom one. Here we try to build similar models with a single phantom field.

To do so, that is, to achieve a sufficiently rapidly decaying scalar field energy density, we need rapidly decaying quantities \( \phi' \) and hence, by Eq. (6), \( r''/r \). We therefore replace the ansatz \( r = a(1 + x^2)^{1/2} \) used in some previous papers (e.g., \([7,12]\)) with

\[
r(x) = a(1 + x^{2n})^{1/(2n)}, \quad n \in \mathbb{N},
\]

where \( a > 0 \) is an arbitrary constant equal to the throat radius. In what follows, we put \( a = 1 \), which means that lengths are expressed in units of the throat radius; the quantities like \( B(x) \) and \( V(x) \) with the dimension \( (length)^{-2} \) are expressed in units of \( a^{-2} \), while \( A(x) \) and \( \phi(x) \), being dimensionless, are insensitive to this assumption.

The value \( n = 1 \) returns us to the ansatz of our previous papers \([7,12,18,22]\). With higher values of \( n \), we obtain a new feature of the spacetime geometry: the spherical radius \( r(x) \) is changing quite slowly near the throat \( x = 0 \), which enables us to call them models with a long throat. Indeed, with (11) at large \( |x| \) we have \( r''/r \approx -2/\varepsilon n^2 \), hence \( \phi' \approx 1/x\varepsilon n + 1 \), which at large enough \( n \) conforms to the “invisible ghost” concept. On the other hand, at small \( |x| \) we obtain \( r(x) \approx 1 + x^{2n}/(2n) \), corresponding to a “long throat” if \( n > 1 \), see Fig. 1a.

Let us put \( m = 0 \), restricting ourselves to massless wormholes. Then \( B'(x) \) \([10]\) is an odd function:

\[
B'(x) = -\frac{2x}{(x^{2n} + 1)^{2/n}},
\]

whose integration gives

\[
B(x) = -x^2 F\left(\frac{1}{n}, \frac{2}{n}; 1 + \frac{1}{n}; -x^{2n}\right) + B_0,
\]

where \( B_0 \) is an integration constant, and \( F(a, b; c, z) \) is the Gaussian hypergeometric function. Assuming asymptotic flatness at large positive \( x \), since \( B = A/r^2 \) and \( A \to 1 \) at infinity, we require \( B \to 0 \) as \( x \to \infty \) and thus fix \( B_0 \) as

\[
B_0 = \lim_{x \to \infty} x^2 F\left(\frac{1}{n}, \frac{2}{n}; 1 + \frac{1}{n}; -x^{2n}\right)
\]

Plot of \( B(x) \) and \( A(x) = Br^2 \) for \( n = 4 \) are shown in Fig. 1b,c. We see that it is a twice asymptoti-
cally flat (M-M) wormhole (where “M” stands for Minkowski). Curiously, the behavior of $A(x)$ shows that there is a domain of repulsive gravity around the throat.

Now we know the metric completely, and the remaining quantities $\phi(x)$ and $V(\phi(x))$ can be easily found from Eqs. (7) and (6), respectively. The expression for the scalar field $\phi(x)$ for $n = 4$ is (assuming $\phi(0) = 0$)

$$\phi(x) = \frac{\sqrt{7}}{4} (\text{sign } x) \arctan(x^4),$$

(15)

see Fig. 2a. For the potential $V(x)$ there is rather a cumbersome expression in terms of hypergeometric functions, gamma functions and Legendre functions, and we will not present it here. It is plotted in Fig. 2b. Since $V(x)$ is an even function, the plot is restricted to $x > 0$.

More complicated models with various global structures emerge if there is a nonzero Schwarzschild mass $m$ or/and, in addition to the scalar field, there is an electromagnetic field with the corresponding electric or magnetic charge. Examples of such solutions, which include M-M, M-dS (de-Sitter), M-AdS (anti de-Sitter) wormholes and regular black holes with up to four horizons, are presented in [7,10–12,22]. Since the global qualitative features of the present field system are similar to those described there, the same kinds of regular solutions can be obtained in the present case using the same methods. We will not analyze them here, but, instead, briefly discuss the stability problem in the case of long throats.

4 The stability problem in the presence of a long throat

As is the case for any static configurations, a problem of importance is their stability or, more generally, the behavior of their perturbations. For instance, certain systems can exist for quite a long time even if they are unstable but decay very slowly. On the other hand, the evolution of unstable systems can lead to many phenomena of interest, from structure formation in the Universe to Supernova explosions.

Small (linear) radial perturbations of scalar-vacuum configurations with various potentials $V(\phi)$, including wormholes and regular black holes, have been studied in many papers, e.g., [14–16,23]. The perturbations preserving spherical symmetry are, on one hand, the simplest, and, on the other, the most destructive ones, and have been shown to lead to instabilities in many configurations with self-gravitating scalar fields, including Fisher’s solution [23] and solutions with throats including wormholes [14,16]. Let us briefly recall the relevant formalism, assuming that a certain static solution is known (non necessarily one of those described here) and considering its time-dependent perturbations.

We begin with the general spherically symmet-
ric metric\(^4\)

\[
ds^2 = e^{2\gamma} dt^2 - e^{2\alpha} du^2 - e^{2\beta} d\Omega^2
\]  

(16)

where \(\alpha, \beta, \gamma\) are functions of the radial coordinate \(u\) (not necessarily the quasiglobal coordinate \(x\) used before) and the time \(t\).

Preserving only linear terms with respect to time derivatives, we can write the field equations \(^2\) and \(^3\) as follows:

\[
e^{-2\gamma} \ddot{\phi} - e^{-2\alpha} \{\phi'' + \phi' (\gamma' + 2\beta' - \alpha')\} = \varepsilon V_\phi / 2; \tag{17}
\]

\[
R' = e^{-2\gamma} (\alpha + 2\beta) - e^{-2\alpha} [\gamma'' + \gamma'(\gamma' - \alpha' + 2\beta')] = V(\phi), \tag{18}
\]

\[
\begin{aligned}
R_1' &= e^{-2\gamma} (\alpha - 2\alpha) [\gamma'' + 2\beta'' + \gamma' + 2\beta' - \alpha'(\gamma' + 2\beta')] = +2\varepsilon e^{-2\alpha} \dot{\phi}^2 + V(\phi), \tag{19}
\end{aligned}
\]

\[
\begin{aligned}
R_2' &= e^{-2\gamma} (\alpha - 2\alpha) [\beta'' - 2\beta'] = V(\phi), \tag{20}
\end{aligned}
\]

\[
R_{01} = 2[\beta' + \beta\beta' - \alpha\beta' - \beta\gamma'] = -2\varepsilon \dot{\phi} \phi', \tag{21}
\]

where dots denote \(/t\), primes denote \(/u\), and \(V_\phi = dV/d\phi\).

We assume that we know a static, purely \(u\)-dependent solution of this set of equations, and then consider small time-dependent deviations from it, denoted by adding the symbol \(\delta\), so that

\[
\begin{aligned}
\phi(u, t) &= \phi(u) + \delta \phi(u, t), \\
\alpha(u, t) &= \alpha(u) + \delta \alpha(u, t),
\end{aligned}
\]

and so on, where \(\phi(u), \alpha(u), \ldots\) are static solutions, while \(\delta \phi(u, t), \delta \alpha(u, t), \ldots\) are small deviations.

Let us note that we have two independent forms of arbitrariness in Eqs. \(^{17}–^{21}\): the choice of a radial coordinate \(u\) (used in the static solution and possibly fixed by a relation between \(\alpha(u), \beta(u), \gamma(u)\)) and the choice of a perturbation gauge, corresponding to the choice of a reference frame in perturbed space-time (fixed by a relation for the perturbations). The above equations \(^{17}–^{21}\) are written in a universal form, without coordinate or gauge fixing.

In the spherically symmetric Einstein-scalar field system there is actually only one dynamic degree of freedom. Accordingly, in Eqs. \(^{17}–^{21}\) with respect to the perturbations, one can exclude all unknown except \(\delta \phi\) and, after passing over from an arbitrary coordinate \(u\) to the so-called tortoise radial coordinate \(z\) defined by

\[
du/dz = e^{\gamma - \alpha} \tag{22}
\]

and separating the variables by the assumption

\[
\delta \phi(z, t) = e^{-\beta(u)} Y(z) e^{i\omega t}, \quad \omega = \text{const}, \tag{23}
\]

reduce the stability problem to a boundary-value problem for the Schrödinger-like equation

\[
d^2Y/dz^2 + [\omega^2 - V_{\text{eff}}(z)] Y = 0, \tag{24}
\]

with certain physically motivated boundary conditions. The effective potential \(V_{\text{eff}}\) has the form

\[
V_{\text{eff}}(z) = e^{2\gamma} \left[\frac{2 \varepsilon \phi^2}{\beta^2} (V - e^{-2\beta}) + \frac{2V'}{\beta} + \frac{\varepsilon V_{\phi\phi}}{2}\right] + e^{2\gamma - 2\alpha} \left[\beta'' + \beta' (\beta' + \gamma' - \alpha')\right]. \tag{25}
\]

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\(^4\)In the whole paper we are using the following notations for different radial coordinates:
- \(u\) — a general notation,
- \(x\) — quasiglobal, such that \(\alpha = -\gamma\),
- \(z\) — “tortoise”, such that \(\alpha = \gamma\).
where the prime, as before, denotes \(d/du\), and \(V = V(\phi)\) is the original scalar field potential. If there is no nontrivial solution to Eq. (24) with \(\text{Im} \omega < 0\) satisfying physically reasonable conditions at the ends of the range of \(z\), then the static system is unstable since \(\delta \phi\) can exponentially grow with time. Otherwise our static system is stable in the linear approximation.

A more detailed derivation can be found, e.g., in [15, 16, 18, 24]. It should be noted that Eq. (24), being derived using the convenient gauge \(\delta \beta = 0\), is still gauge-invariant since \(Y\) can be shown to be a gauge-invariant quantity while \(V_{\text{eff}}\) contains quantities from the static background solution, written in terms of an arbitrary radial coordinate \(u\).

Now, to apply this general formalism to a static solution from the previous section, we must find \(V_{\text{eff}}(z)\), identifying \(u\) with our quasiglobal coordinate \(x\), and substitute \(e^{2\gamma} = e^{-2\alpha} = A(x)\), \(e^{\beta} = r(x)\). It is easy to see that at flat spatial infinity the coordinates \(z\) and \(x\) almost coincide \((x \approx z\) for \(x \to \pm \infty\)), and the natural boundary condition \(\delta \phi(\pm \infty) = 0\) leads to \(Y(\pm \infty) = 0\).

The main problem with this stability study is connected with the shape of the effective potential \(V_{\text{eff}}\), which has a singularity at the throat due to \(\beta' = r'/r = 0\). More precisely, if the throat is located at \(z = 0\) and, as \(z \to 0\), the radius \(r\) behaves as \(r \approx r_0 + O(z^2)\) (as is the case under the ansatz (11)) we have at small \(z\)

\[
V_{\text{eff}}(z) = 2 \left( \frac{\beta''}{\beta'} - \frac{\beta'''}{\beta'} \right) + O(1)
\]

\[
= 2 \frac{(2n-1)}{z^2} + O(1), \tag{26}
\]

where, as before, \(\beta = \ln r\), while the prime here denotes \(d/dz\). Recall that models with ordinary throats correspond to \(n = 1\) and those with long throats to \(n > 1\).

As described, e.g., in [15, 18, 24] the potential \(V_{\text{eff}}\) for \(n = 1\) can be regularized using the “S-transformation,” which is a special kind of substitution in Eq. (24): the transformed equation has no singularities, and solutions to the corresponding boundary-value problem describe regular perturbations of the scalar field and the metric. This method was used to prove the instability of anti-Fisher wormholes [14] and other configurations with throats [15, 16]. The instability is caused by the perturbation mode in which the throat radius changes with time.

However, according to [15, 18, 24], a necessary condition for the S-transformation to remove the singularity in \(V_{\text{eff}}\) is that \(V_{\text{eff}} = 2/z^2 + O(1)\), which, by Eq. (26), is only true for a “usual” throat, \(n = 1\). Therefore, a stability study of long-throat wormholes faces a serious technical difficulty, and to obtain an idea of whether or not a long throat can stabilize a wormhole, in the next section we will consider a simple model with \(r = \text{const}\), which can be called a maximally long throat.

## 5 Instability of a maximally long throat

Consider a static solution to Eqs. (17)–(20) under the condition \(r \equiv e^{\beta} = r_0 = \text{const}\). Thus it will be not a wormhole but rather a “pure throat”.

Choosing the coordinate \(u = z\) such that \(\alpha = \gamma\), from (17)–(20) we easily obtain

\[
\phi = \phi_0 = \text{const}, \quad V = 1/r_0^2 = \text{const}, \tag{27}
\]

\[
ds^2 = k^2 r_0^2 (dt^2 - dz^2) - r_0^2 d\Omega^2, \tag{28}
\]

where \(k > 0\) is an integration constant, which, without loss of generality, can be put equal to \(1/r_0\) by choosing scales along the \(t\) and \(z\) axes. From the scalar field equation it also follows \(V_{\phi} = 0\) at the value of \(\phi\) corresponding to the solution. The metric can also be written in terms of the quasiglobal coordinate \(x = r_0 \tanh(z/r_0)\):

\[
ds^2 = \frac{dt^2 - dz^2}{\cosh^2(z/r_0)} - r_0^2 d\Omega^2
\]

\[
= (1 - x^2/r_0^2) dt^2 - \frac{dx^2}{1 - x^2/r_0^2} - r_0^2 d\Omega^2. \tag{29}
\]

It is the well-known Nariai solution, a vacuum solution of GR with the cosmological constant equal to \(1/r_0^2\). We see that it can also be interpreted as a special solution to GR with a scalar field source. The values \(x = \pm r_0\), or equivalently \(z = \pm \infty\), are horizons, and the static region between them represents what may be called a maximally long throat; its full proper length along the \(z\) direction is equal to \(\pi r_0\).
Consider small time-dependent perturbations of this solution, using the coordinate $z$. Then, without any assumption on the perturbation gauge, Eqs. (17), (20) and (21) lead to decoupled equations for the unknowns $\delta \phi(z,t)$ and $\delta \beta(z,t)$:

$$
\ddot{\phi} - \delta \phi'' = \frac{\varepsilon}{2} e^{2\gamma} V_{\phi\phi} \delta \phi = 0,
$$

$$
\delta \dot{\beta}' = \gamma' \delta \beta',
$$

$$
\delta \dot{\beta} - \delta \beta'' = -\frac{2e^{2\gamma}}{r_0^2} \delta \beta = 0,
$$

where the prime stands for $/z$ and the dot for $/t$. The existence of two independent dynamic degrees of freedom for radial perturbations instead of a single, scalar one in the general wormhole case is connected with the absence of a Birkhoff-like theorem in this case, see a detailed discussion in [26] and references therein. Perturbations of the spherical radius actually behave in our case as one more scalar field in the 2D space-time parametrized by $t$ and $z$.

Equation (31) admits integration in time, after which, neglecting an arbitrary function of $z$ (a static perturbation), we get

$$
\delta \beta' = \gamma' \delta \beta \Rightarrow \delta \beta = v(t) e^{\gamma},
$$

where $v(t)$ is an arbitrary function which may be further determined from Eq. (32). Substituting (33) into (32) and taking into account that, according to (29), $e^{\gamma} = 1/\cosh(kz)$, and $k = 1/r_0$, we obtain $v = k^2 v = 0$, so that finally

$$
\delta \beta = v(t) e^{\gamma} = \frac{c_1 e^{kt} + c_2 e^{-kt}}{\cosh(kz)},
$$

with arbitrary constants $c_1$ and $c_2$. The existence of the growth factor $e^{kt}$ means that the background maximally long throat solution with the Nariai metric is unstable.

Since we did not so far use any perturbation gauge, there still remains a doubt that the behavior of $\delta \beta(x,t)$ may be a pure gauge and may be removed by a $t$-dependent coordinate transformation. To make sure that this instability is real, let us reconsider Eqs. (17)–(21), now using a manifestly admissible gauge $\delta \beta = 0$. Then Eq. (30) preserves its form, but Eqs. (32) and (31) become trivial, and we should use the remaining equations (18) and (19), which take the form

$$
\delta \alpha - \delta \gamma'' = 2 e^{2\gamma} \delta \alpha / r_0^2,
$$

$$
\delta \alpha - \delta \gamma'' - \gamma' (\delta \alpha' - \delta \gamma') = 2 e^{2\gamma} \delta \alpha / r_0^2.
$$

Their difference gives $\delta \gamma' = \delta \alpha'$, and we arrive at a wave equation for $\delta \alpha$

$$
\delta \alpha - \delta \alpha'' = 2 e^{2\gamma} \delta \alpha / r_0^2,
$$

the same as we previously had for $\delta \beta$, Eq. (32). Consequently, it has the solution similar to (34)

$$
\delta \alpha = \frac{c_1 e^{kt} + c_2 e^{-kt}}{\cosh(kz)},
$$

and we can confidently conclude that the maximally long throat solution is unstable.

This actually confirms the old result of [27] on the instability of Nariai’s solution since the scalar field contribution here reduces to supplying a cosmological constant $\Lambda = 1/r_0^2$. The scalar field perturbations have, in the linear approximation, their own dynamics described by Eq. (30) and can be stable or unstable depending on the sign of $V_{\phi\phi}$.

The instability of the limiting model of a “maximally long throat” allows us to conjecture that a slowly varying radius near a throat of a more general wormhole supported by a phantom scalar field does not stabilize it as compared with models with a “usual” throat. We hope to verify this conjecture in our future work. Of interest are also the stability properties of other solution with long throats, such as wormholes with an AdS asymptotic at the far end as well as black universes with a horizon and an asymptotically de Sitter expansion beyond it, like those described in [7, 8, 22].

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