Generalized Coherent States for Classical Orthogonal Polynomials

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Abstract

For the oscillator-like systems, connected with the Laguerre, Legendre and Chebyshev polynomials coherent states of Glauber-Barut-Girardello type are defined. The suggested construction can be applied to each system of orthogonal polynomials including classical ones as well as deformed ones.

1 Introduction

We consider the oscillator-like systems, connected with the Laguerre, Legendre and Chebyshev polynomials in the same way as usual boson oscillator connected with the Hermite polynomials. It is natural to call these systems as the Laguerre, Legendre and Chebyshev (generalized) oscillators, respectively. We define analogues of coherent states for these generalized oscillators. It is possible to consider such definitions as a new method for construction of coherent states. In general, this method can be applied for a orthogonal functions system connected with a Jacobi matrix.

There are four different definitions of coherent states

1. Glauber-Barut-Girardello coherent states;

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2. Perelomov-Gilmore coherent states;
3. Minimum uncertainty coherent states;
4. Klauder-Gazeau coherent states.

Here we will consider only first of these definitions.

Let $a$ and $a^\dagger$ are the ladder operators (annihilation and creation, respectively) on the Fock space $\mathcal{F}$ with standard orthonormal basis \{ $|n\rangle$ $\}_{n=0}^{\infty}$.

The Glauber-Barut-Girardello coherent states are defined as eigenvectors of annihilation operator

$$a|z\rangle = z|z\rangle, \quad z \in \mathbb{C}.$$  \hspace{1cm} (1.1)

It is well known that

$$|z\rangle = \exp\left(-\frac{1}{2}|z|^2\right) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle = \exp\left(-\frac{1}{2}|z|^2\right) \exp(za^\dagger)|0\rangle,$$ \hspace{1cm} (1.2)

where $|0\rangle$ is the Fock vacuum, such that $a|0\rangle = 0$ and

$$|n\rangle = \sum_{n=0}^{\infty} \frac{a^\dagger n}{\sqrt{n!}} |0\rangle, \quad n = 0, 1, \ldots.$$ \hspace{1cm} (1.3)

Let us remark that in the case of standard boson oscillator all of the above mentioned definitions are equivalent. It means that each of them generates the same set of coherent states. But this is not true in the general case.

In the following we extend the above definition of coherent states to the cases of Laguerre, Legendre and Chebyshev (generalized) oscillators. In doing this we use mainly the generating functions method and the theory of classical power moment problem for Jacobi matrix, connected with related orthonormal polynomial systems.

## 2 The generalized oscillator algebra

Let $\mu$ denotes a positive Borel measure on the real line $\mathbb{R}^1$ for which all moments $\mu_k$

$$\mu_0 = 1, \quad \mu_k = \int_{-\infty}^{\infty} x^k \mu(dx), \quad k = 0, 1, \ldots.$$ \hspace{1cm} (2.1)

are finite. We also suppose that measure $\mu$ is a symmetric one, that is

$$\mu_{2k+1} = 0, \quad k = 0, 1, \ldots.$$ \hspace{1cm} (2.2)
Let us consider a system \( \{ \Psi_n(x) \}_{n=0}^{\infty} \) of polynomials defined by the recurrence relations \((n \geq 0)\):

\[
x \Psi_n(x) = b_n \Psi_{n+1}(x) + b_{n-1} \Psi_{n-1}(x), \quad \Psi_0(x) = 1, \quad b_{-1} = 0,
\]

(2.3)

where \( \{b_n\}_{n=0}^{\infty} \) is a given positive sequence.

The following theorem was proved in [1]:

**Theorem 2.1.** The polynomial system \( \{ \Psi_n(x) \}_{n=0}^{\infty} \) is orthonormal one in the Hilbert space \( \mathcal{H}_x = L^2(\mathbb{R}; \mu) \) if and only if the coefficients \( b_n \) and the moments \( \mu_{2k} \) are connected by the following relations

\[
\frac{\left[ n \right]}{\left[ n \right]} \sum_{m=0}^{\left[ n \right]} \sum_{s=0}^{\left[ n \right]} (-1)^{m+s} \alpha_{2m-1,n-1} \alpha_{2s-1,n-1} \frac{\mu_{2n-2m-2s+2}}{(b_{n-1})!} = b_n^2 + b_{n-1}^2, \quad n = 0, 1, \ldots.
\]

(2.4)

where \( (b_n^2)! = b_0^2 b_1^2 \cdots b_n^2 \), the integral part of \( a \) is denoted by \([a]\), and the coefficients \( \alpha_j \) are given by

\[
\alpha_{2p-1,n-1} = \sum_{k_1=2p-1}^{n-1} \sum_{k_2=2p-3}^{k_1-2} \cdots \sum_{k_{p-1}=1}^{k_{p-2}-2} b_{k_1}^2 b_{k_2}^2 \cdots b_{k_{p}}^2.
\]

(2.5)

The recurrence relations (2.3) determine the position operator \( X \), realized as the operator of multiplication by argument in the space \( \mathcal{H}_x \). We define the momentum operator \( P \) and Hamiltonian \( H \) by the following relations

\[
P = K^* Y K, \quad H = X^2 + P^2.
\]

(2.6)

where \( Y \) is the position operator in the (momentum) space \( \mathcal{H}_y \), and \( K : \mathcal{H}_x \rightarrow \mathcal{H}_y \) and \( K^* : \mathcal{H}_y \rightarrow \mathcal{H}_x \) are the unitary integral operators with Poisson kernels

\[
\mathfrak{K}(x, y; -i) = \sum_{n=0}^{\infty} (-i)^n \varphi_n(x) \cdot \varphi_n(y).
\]

(2.7)

and \( \overline{\mathfrak{K}} \), respectively.

In what follows we consider the Hilbert space \( \mathcal{H}_x \) as a Fock space with the basis \( \{ |n\rangle = \Psi_n(x) \}_{n=0}^{\infty} \) and we define the creation \( a^\dagger \) and annihilation \( a \) operators by the formulas

\[
a^\dagger = \frac{1}{\sqrt{2}} (X + iP), \quad a = \frac{1}{\sqrt{2}} (X - iP).
\]

(2.8)

We also define the number operator \( N \)

\[
N|n\rangle = n|n\rangle, \quad n \geq 0.
\]

(2.9)
and the operator-function $B(N)$ acting on the basis states by

$$B(N)|n⟩ = b_{n-1}^2|n⟩, \quad n ≥ 0, \quad b_{-1} = 0.$$  \hfill (2.10)

The following theorem can be proved by direct calculation.

**Theorem 2.2.** The operators $a = a^-$, $a^+$ and $N$ obey the following commutation relations

$$[a^-, a^+] = 2(B(N + I) - B(N)), \quad [N, a^±] = ±a^±. \hfill (2.11)$$

Moreover if there exists real number $A$ and real function $C(n)$, such that

$$b_{n}^2 - Ab_n^2 = C(n), \quad n ≥ 0, \quad b_{-1} = 0,$$  \hfill (2.12)

then ladder operators $a^±$ fulfill the relations

$$a^−a^+ − Aa^+a^− = 2C(N). \hfill (2.13)$$

We call so obtained algebra $A_Ψ$ as generalized oscillator algebra connected with given system of orthogonal polynomials.

**Remark.** In the case of a non symmetric measure (with $µ_{2k+1} ≠ 0$) the recurrent relations are more complicated

$$xΨ_n(x) = b_nΨ_{n+1}(x) + a_nΨ_n(x) + b_{n-1}Ψ_{n-1}(x), \quad Ψ_0(x) = 1, \quad b_{-1} = 0,$$  \hfill (2.14)

so that the related Jacobi matrix has nonzero diagonal. In this case one can repeat the full construction with necessary modifications and complications (see [1]). Such situation arises, for example, in the cases of the Laguerre and Jacobi polynomials.

### 3 Coherent states connected with classical orthogonal polynomials

#### 3.1 Coherent states of Glauber-Barut-Girardello type

For a system of polynomials $\{Ψ_n(x)\}_{n=0}^∞$ in the space $H_x$ we define the coherent states of Glauber-Barut-Girardello type by the relation

$$|z⟩ = N(|z|^2) \sum_{n=0}^∞ \frac{z^n}{(\sqrt{2}b_n)!!}|n⟩,$$  \hfill (3.1)
where the coefficients in the recurrent relations (2.3) (or (2.14)) are denoted by \( b_n \) and normalizing factor is given by the relation

\[
\mathcal{N}^2 = \langle z | z \rangle = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2b_n-1)!} \exp\{2b_n-1\}(|z|^{2n}).
\] (3.2)

Below we apply the above construction to the cases in which some well known classical polynomials are taken as given system of polynomials \( \{\Psi_n(x)\}_{n=0}^{\infty} \)

### 3.1.1 Laguerre coherent states of Glauber-Barut-Girardello type

Let \( \{\Psi_n(x)\}_{n=0}^{\infty} \) be the orthogonal system of Laguerre polynomials in the Hilbert space \( \mathcal{H} = L^2(R^1; \mu) \) with the probability measure

\[
\mu(dx) = x^\alpha e^{-x} \frac{dx}{\sqrt{\Gamma(\alpha+1)}}.
\] (3.3)

By definition

\[
\Psi_n(x) \equiv \sqrt{\frac{\Gamma(\alpha + n + 1)}{n!\Gamma(\alpha + 1)}} \, _1F_1\left(\frac{\alpha - n}{\alpha + 1} \big| x\right), \quad n \geq 0.
\] (3.4)

The coefficients \( a_n, b_n \) in recurrent relations (2.14) are equal to

\[
a_n = 2n + \alpha + 1, \quad b_n = -\sqrt{(n + 1)(n + \alpha + 1)}, \quad n \geq 0.
\] (3.5)

The normalizing factor

\[
\mathcal{N}^2 = \left(\frac{2}{z}\right)^\alpha \Gamma(\alpha + 1)I_\alpha(\sqrt{2}|z|),
\] (3.6)

where

\[
I_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{z^n}{2^n n! \Gamma(\alpha + n + 1)}
\] (3.7)

is the Bessel function of the first kind. According to (3.1), we have

\[
|z; \alpha\rangle = \frac{1}{(\sqrt{2}x)^\alpha I_\alpha(\sqrt{2}z)} e^{z/\sqrt{2}} I_\alpha(2^{3/4} \sqrt{2}z).
\] (3.8)

The overlap of two coherent states is equal to

\[
\langle z_1; \alpha | z_2; \alpha \rangle = \frac{I_\alpha(2\sqrt{z_1}z_2)}{\sqrt{I_\alpha(\sqrt{2}|z_1|)I_\alpha(\sqrt{2}|z_2|)}}
\] (3.9)

One can check the validity of unity decomposition

\[
\int |z; \alpha\rangle \langle z; \alpha| d\nu(z, \alpha) = \mathbb{I},
\] (3.10)
with the measure
\[
\mathrm{d} \nu(z, \alpha) = \frac{\sqrt{2}}{\pi} K_\alpha(\sqrt{2}|z_2|) I_\alpha(\sqrt{2}|z_2|),
\]
where \( K_\alpha \) is a modified Bessel function of the second kind.

Let us note that this result coincide with results of other authors [5] obtained by different methods.

### 3.1.2 Legendre coherent states of Glauber-Barut-Girardello type

We consider orthogonal system \( \{\Psi_n(x)\}_{n=0}^\infty \) of Legendre polynomials
\[
\Psi_n(x) \doteq \sqrt{2n + 1} \, _2F_1 \left( \frac{-n,n+1}{2} \mid \frac{1-x^2}{2} \right),
\]
in the Hilbert space \( \mathcal{H} = L^2([-1, 1]; \frac{1}{2} dx) \).

The coefficients \( b_n \) in the recurrent relations (2.3) are equal to
\[
b_n \doteq \sqrt{\frac{(n+1)^2}{(2n+1)(2n+3)}}, \quad n \geq 0.
\]
Then
\[
(2b_{n-1}^2)! = \frac{n!(1)_n}{(\frac{1}{2})_n(\frac{3}{2})_n}, \quad n \geq 1,
\]
where Pochhammer symbol \((a + 1)_n\) is defined by
\[
(a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n = 1, 2, \ldots
\]
The normalizing factor is equal to
\[
\mathcal{N}^2 = \sum_{n=0}^\infty \frac{(1/2)_n(3/2)_n}{n!(1)_n} (2|z|^2)^n = \, _2F_1 \left( \frac{1/2,3/2}{1} \mid 2|z|^2 \right). \tag{3.16}
\]
Note that the radius of convergence for series in the above relation is equal to \( 1/\sqrt{2} \).

For coherent state of the Legendre oscillator one obtains
\[
|z\rangle = \frac{2F_1 \left( \frac{1}{2}, \frac{3/4,5/4}{1} \mid \frac{(x^2-1)}{4(x^2)}, \frac{1-2xz}{4(x^2)} \right)}{\sqrt{2F_1 \left( \frac{1/2,3/2}{1} \mid 2|z|^2 \right)}} (1 - 2xz)^{-3/2}. \tag{3.17}
\]
The overlap of two coherent states is equal to
\[
\langle z_1 | z_2 \rangle = \frac{2F_1 \left( \frac{1}{2}, \frac{3/2}{1} \mid 2|z_1|^2 \right) 2F_1 \left( \frac{1}{2}, \frac{3/2}{1} \mid 2|z_2|^2 \right)}{\sqrt{2F_1 \left( \frac{1/2,3/2}{1} \mid 2|z_1|^2 \right)} 2F_1 \left( \frac{1/2,3/2}{1} \mid 2|z_2|^2 \right)}. \tag{3.18}
\]
Finally, we have
\[
\int_{C_{1/\sqrt{2}}} |z\rangle \langle z| \, d\nu(z) = I, \quad (3.19)
\]
with the measure
\[
d\nu(z) = \frac{(4|z|^2 - 5)P_{1/2}(|z|^2 - 1) - 3P_{3/2}(|z|^2 - 1)}{2(|z|^2 - 2)} \, d(\text{Re}z) d(\text{Im}z), \quad 0 < |z| < 1/\sqrt{2}, \quad (3.20)
\]
where by \( P_\alpha(x) \) we denote the Legendre function.

### 3.1.3 Chebyshev coherent states of Glauber-Barut-Girardello type

The coherent states of Glauber-Barut-Girardello type for the Chebyshev oscillator in the space
\[
\mathcal{H} = L^2([-1, 1]; \frac{dx}{\pi \sqrt{1 - x^2}}) \quad (3.21)
\]
looks as (see \([\text{2}]\))
\[
|z\rangle_1 = \frac{\sqrt{2}}{1 - 2|z|^2} \frac{1 - \sqrt{2}zx}{1 - 2\sqrt{2}zx + 2z^2}, \quad |z| < 1/\sqrt{2}. \quad (3.22)
\]

The related holomorphic representation in the Bargmann type space consists from functions analytical in the disk \( |z| < 1/\sqrt{2} \). The constructed family of coherent states possess all standard properties. The decomposition of unity defines measure represented by \( \delta \)-function on the boundary of the disk \( |z| < 1/\sqrt{2} \).

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