Some isoperimetric inequalities and eigenvalue estimates in weighted manifolds

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Abstract. In this paper we prove general inequalities involving the weighted mean curvature of compact submanifolds immersed in weighted manifolds. As a consequence we obtain a relative linear isoperimetric inequality for such submanifolds. We also prove an extrinsic upper bound to the first non zero eigenvalue of the drift Laplacian on closed submanifolds of weighted manifolds.

1. Introduction

Let $(\bar{M}^d, \bar{g}, d\bar{\mu})$ be a weighted manifold, that is, a Riemannian manifold $(\bar{M}^d, \bar{g})$ endowed with a weighted volume form $d\bar{\mu} = e^{-f}d\bar{M}$, where $f$ is a real-valued smooth function on $\bar{M}$ and $d\bar{M}$ is the volume element induced by the metric $\bar{g}$.

In weighted manifolds a natural generalization of the Ricci tensor is the $m$-Bakry-Émery tensor defined by

$$\bar{\text{Ric}}_m^f = \text{Ric} + \nabla^2 f - \frac{1}{m - d} df \otimes df,$$

for each $m \in [d, \infty)$. When $m = \infty$ it gives the tensor $\bar{\text{Ric}}_f = \text{Ric} + \nabla^2 f$ introduced by Lichnerowicz [10, 11] and independently by Bakry and Émery in [1]. The case $m = d$ only makes sense when the function $f$ is constant and so $\bar{\text{Ric}}_m^f$ is the usual Ricci tensor $\text{Ric}$ of $\bar{M}$.

In this paper we are interested in studying inequalities on submanifolds of weighted manifolds. In order to do it we make use of intrinsic objects, like the $m$-Bakry-Émery tensor, and extrinsic objects like the weighted mean curvature defined below. Namely, given $x : M \rightarrow \bar{M}$ an isometric immersion, we define the weighted mean curvature vector $H_f$ by

$$H_f = H + \bar{\nabla} f^\perp,$$

where $H$ is the mean curvature vector of the submanifold $M$ and $^\perp$ denotes the orthogonal projection onto the normal bundle $TM^\perp$ (see Gromov [6] and Morgan [13]). The weighted mean curvature appears naturally in the first variation of the weighted area functional as described in [2]. In the submanifold $M$ we also consider the weighted volume given $d\mu = e^{-f}dM$, where $dM$ is the volume element of $M$.

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In case that $\tilde{M} = \Omega^{n+1}$, where $\Omega$ is a compact oriented $(n+1)$-dimensional Riemannian manifold with smooth boundary $M^n = \partial \Omega$ we consider on $M$ the Riemannian metric induced by the inclusion map $\iota: \tilde{M} \hookrightarrow \Omega$.

Let $\nu$ be a unit normal vector field on $M$ and let $A$ denote the shape operator of $M$, that is $A = -\nabla_\nu$. It is easy to see that $H_f = H_f \nu$, where $H_f = H + \langle \nabla f, \nu \rangle$ and $H = \text{trace} \ A$ is the mean curvature function.

In [16], Ros proved an inequality relating the volume of $\Omega$ and the mean curvature function $H$ of $M$. The inequality obtained by Ros is essentially contained in the paper of Heintze and Karcher [8], although the proof uses different techniques.

Our first result is the natural generalization of Ros inequality in the context of weighted manifolds.

**Theorem 1.1.** Let $\Omega^{n+1}$ be a compact weighted manifold with smooth boundary $M$ and non-negative $m$-Bakry-Émery tensor. Let $H_f$ be the weighted mean curvature of $M$. If $H_f$ is positive everywhere, then

$$\text{Vol}_f(\Omega) \leq \frac{m-1}{m} \int_M \frac{1}{H_f} d\mu.$$

Moreover, equality holds if and only if $\Omega$ is isometric to a Euclidean ball, $f$ is constant and $m = n + 1$.

Extending the Ros formula, Choe and Park [4] proved that a compact connected embedded CMC hypersurface in a convex Euclidean solid cone which is perpendicular to the boundary of the cone is part of a round sphere.

The rigidity of compact submanifold with free boundary is a very classical problem in submanifold theory. For instance, Nitsche [14] proved that an immersed disk type constant mean curvature surface in a ball which makes a constant angle with the boundary of the ball is part of a round sphere.

On weighted manifolds, Cañete and Rosales [3] showed the rigidity of compact stable hypersurfaces with free boundary in a convex solid cone in Euclidean space with homogeneous density. Our next result extends Choe and Park’s result to weighted Euclidean spaces $(\mathbb{R}^{n+1}, ds_0, d\bar{\mu})$, where $ds_0$ is the Euclidean metric.

**Theorem 1.2.** Let $C$ be a convex solid cone with piecewise smooth boundary $\partial C$ in a weighted manifold $(\mathbb{R}^{n+1}, ds_0, d\bar{\mu})$ of non-negative $m$-Bakry-Émery tensor. Let $M$ be a compact connected embedded hypersurface in $C$ and $\Omega$ the bounded domain enclosed by $M$ and $\partial C$. If the weighted mean curvature $H_f$ of $M$ is positive everywhere, then

$$\text{Vol}_f(\Omega) \leq \frac{m-1}{m} \int_M \frac{1}{H_f} d\mu.$$

Moreover, the equality holds if and only if $M$ is part of a round sphere centered at the vertex of $C$ and $f$ is constant and $m = n + 1$.

When the weighted mean curvature $H_f$ is constant on $M$, we obtain the following relative linear isoperimetric inequality:

**Corollary 1.3.** In Theorem 1.1 or 1.2 if $H_f$ a positive constant, then

$$H_f \text{Vol}_f(\Omega) \leq \frac{m-1}{m} \text{Vol}_f(M).$$
Moreover, the equality holds if and only if $M$ is part of round sphere with $f \equiv \text{const.}$ and $m = n + 1$.

**Remark 1.** We point out that Morgan [13] and Bayle [2] found other generalizations of the Heintze-Karcher inequality in the context of weighted manifolds. Recently, Huang and Ruan [9], give slightly different proofs of Theorem 1.1 and its Corollary.

In the second part of this paper, motivated by the work of Heintze [7] we consider the problem to determine extrinsic upper bounds of the first eigenvalue of the $f$-Laplacian on closed submanifolds when the ambient space has radial sectional curvature bounded from above. We recall that the $f$-Laplacian on $M$ is defined by

$$\Delta_f u = \Delta u - \langle \nabla f, \nabla u \rangle$$

for $u \in H^2(M)$. When $M$ is closed, it is a basic fact that the spectrum of $\Delta_f$ is discrete and its first non zero eigenvalue is given by

$$\lambda_1(\Delta_f) = \inf \left\{ \int_M |\nabla \phi|^2 d\mu : \int_M \phi d\mu = 0, \phi \in C^\infty(M) \right\}.$$

Using the above notation we have the following results.

**Theorem 1.4.** Let $\bar{M}^{n+p}$ be a weighted manifold with $\text{Sect}_{\text{rad}} \leq \delta$, $\delta < 0$. If $x : M^n \to \bar{M}^{n+p}$ is an isometric immersion of a closed manifold, then

$$\lambda_1(\Delta_f) \leq n\delta + \frac{1}{n} \max_M |H_f - \bar{\nabla} f|^2.$$

**Theorem 1.5.** Let $\bar{M}^{n+p}$ be a weighted manifold with $\text{Sect}_{\text{rad}} \leq \delta$, $\delta \geq 0$. If $x : M^n \to \bar{M}^{n+p}$ is an isometric immersion of a closed manifold such that $x(M)$ is contained in a geodesic ball of radius less or equal to $\frac{\pi}{4\sqrt{\delta}}$, then

$$\lambda_1(\Delta_f) \leq n\delta + \frac{1}{n\text{Vol}_f(M)} \int_M |H_f - \bar{\nabla} f|^2 d\mu.$$

Moreover, if equality holds then $M$ is $f$-minimally immersed into $F^{-1}(c)$, where $F = \lambda f + \int^r s_\delta(t) dt$, for some constants $\lambda$ and $c$, provided that $c$ is a regular value of $F$.

**Remark 2.** Let $(\mathbb{Q}^{n+1}_{\delta}, g_{\text{can}})$ be the space form of curvature $\delta$ and let $r_0$ be the positive solution of the equation $r s_\delta(r) = 2n c_\delta(r)$ (see Section 4 for notations). A straightforward computation shows that the geodesic sphere of radius $r_0$ has zero weighted mean curvature in the weighted manifold $(\mathbb{Q}^{n+1}_{\delta}, g_{\text{can}}, e^{-r^2/4})$ and thus the term $|H_f - \bar{\nabla} f|^2$ in Theorems 1.4 and 1.5 cannot be replaced by $|H_f|^2$ when $\delta \leq 0$.

2. **Isoperimetric Inequalities**

In this section we recall some well-known results on weighted manifolds and we prove Theorem 1.1 and Theorem 1.2. The first tool we need is the following Reilly formula (see [12]).
Theorem 2.1. Let \( u \) be a smooth function on \( \Omega \). Then we have
\[
\int_{\Omega} ((\bar{\Delta} f u)^2 - |\bar{\nabla}^2 u|^2 - \bar{\text{Ric}}_f(\bar{\nabla} u, \bar{\nabla} u)) \, d\bar{\mu} = \int_{M} (2u \Delta f u + u^2 \bar{H}_f + \langle A \bar{\nabla} u, \bar{\nabla} u \rangle) \, d\mu,
\]
where \( \nu \) is the outward unit normal to \( M \).

Using the Cauchy-Schwarz inequality we have

Proposition 2.2. Let \( u \) be a smooth function on \( \Omega \). Then we have
\[
|\bar{\nabla}^2 u|^2 + \bar{\text{Ric}}_f(\bar{\nabla} u, \bar{\nabla} u) \geq \frac{(\bar{\Delta} f u)^2}{m} + \bar{\text{Ric}}^m(\bar{\nabla} u, \bar{\nabla} u),
\]
for every \( m > n + 1 \) or \( m = n + 1 \) and \( f \) is a constant. Moreover, equality holds if and only if \( \bar{\nabla}^2 u = \lambda g \) and \( \langle \bar{\nabla} u, \bar{\nabla} f \rangle = -\frac{m-n-1}{m} \Delta f u \).

The last tool is an important result due to Reilly (see [15]).

Theorem 2.3. Suppose that \( \Omega \) admits a function \( u : \Omega \to \mathbb{R} \) and non-zero constant \( \lambda \) such that
\begin{enumerate}[(a)]
\item \( \bar{\nabla}^2 u = \lambda g \);
\item \( u|_M \) is constant.
\end{enumerate}
Then \( \Omega \) is isometric to an Euclidean ball.

2.1. Proof of Theorem 1.1. Let \( u : \Omega \to \mathbb{R} \) be the solution of the Dirichlet problem
\[
\left\{ \begin{array}{ll}
\bar{\Delta} f u = 1 & \text{in } \Omega, \\
u = 0 & \text{in } \partial\Omega.
\end{array} \right.
\]

Plugging this function in Theorem 2.1 we get
\[
\int_{\Omega} (1 - |\bar{\nabla}^2 u|^2 - \bar{\text{Ric}}_f(\bar{\nabla} u, \bar{\nabla} u)) \, d\bar{\mu} = \int_{M} u^2 \bar{H}_f \, d\mu.
\]

Using the Proposition 2.2 we have
\[
\int_{\Omega} \left( 1 - \frac{1}{m} - \bar{\text{Ric}}^m(\bar{\nabla} u, \bar{\nabla} u) \right) \, d\bar{\mu} \geq \int_{M} u^2 \bar{H}_f \, d\mu.
\]

Using the hypothesis on the \( m \)-Bakry-Émery tensor we obtain
\[
\int_{M} u^2 \bar{H}_f \, d\mu \leq \frac{m-1}{m} Vol_f(\Omega).
\]

Hence, using the Stokes theorem and the above inequality we have
\[
Vol_f(\Omega)^2 = \left( \int_{\Omega} \bar{\Delta} f u \, d\bar{\mu} \right)^2 = \left( \int_{M} u \, d\mu \right)^2 = \left( \int_{M} u \sqrt{\bar{H}_f(\sqrt{\bar{H}_f})^{-1}} \, d\mu \right)^2 \leq \left( \int_{M} u^2 \bar{H}_f \, d\mu \right) \left( \int_{M} \frac{1}{\bar{H}_f} \, d\mu \right) \leq \frac{m-1}{m} Vol_f(\Omega) \int_{M} \frac{1}{\bar{H}_f} \, d\mu.
\]
That is,
\begin{equation}
Vol_f(\Omega) \leq \frac{m-1}{m} \int_M \frac{1}{H_f} \, d\mu.
\end{equation}

Now assume that equality occurs in (2.2). Then all the inequalities above are
equalities and thus we obtain
\begin{equation}
\begin{cases}
\bar{\nabla}^2 u = \lambda g, \\
\langle \bar{\nabla} u, \nabla f \rangle = -\frac{m-n-1}{m} \Delta_f u, \\
\bar{\text{Ric}}^m_f (\bar{\nabla} u, \bar{\nabla} u) = 0.
\end{cases}
\end{equation}

From the first equation above, we have \( \bar{\Delta} u = (n+1)\lambda \). So, using the first
equation in (2.1) and the second equation above it is easy to see that \( \lambda = \frac{1}{m} \). Since
\( \lambda \) is constant we apply Theorem 2.3 to obtain the \( \Omega \) is isometric to a Euclidean
ball.

Finally, assume that \( m > n + 1 \). Then, the first equation in (2.3) and the
boundary condition in (2.1) imply that \( u = \frac{\lambda}{2} r^2 + C \), for some constant \( C \), where
\( r \) is the distance function from its minimal point (see [15]). Therefore, using the
second equation in (2.3) we find \( f = -(m-n-1) \ln r + C \). It is a contradiction,
since \( f \) is a smooth function.

2.2. Proof of Theorem 1.2

Let \( \Omega_\epsilon \subset \Omega \) be a domain with smooth boundary
which is obtained from \( \Omega \) by smoothing out the region within a distance \( \epsilon > 0 \) from
the singular set of \( \partial \Omega \). Let \( u \) be a smooth solution to the following mixed boundary
value problem:
\begin{equation}
\begin{cases}
\bar{\Delta}_f u = 1 & \text{in } \Omega_\epsilon, \\
u = 0 & \text{on } \partial \Omega_\epsilon \setminus \partial C, \\
u_\nu = 0 & \text{on } \partial \Omega_\epsilon \cap \partial C.
\end{cases}
\end{equation}

Applying the solution of the above problem \( u \) into Theorem 2.1 and using Proposition 2.2 we get
\[
\int_{\Omega_\epsilon} \left( 1 - \frac{1}{m} - \bar{\text{Ric}}^m_f (\bar{\nabla} u, \bar{\nabla} u) \right) \, d\bar{\mu} \geq \int_{\partial \Omega_\epsilon \setminus \partial C} u_\nu^2 H_f \, d\mu + \int_{\partial \Omega_\epsilon \cap \partial C} \langle A \nabla u, \nabla u \rangle \, d\mu.
\]

Since \( C \) is convex, \( \int_{\partial \Omega_\epsilon \cap \partial C} \langle A \nabla u, \nabla u \rangle \, d\mu \geq 0 \). Now, using that \( \bar{\text{Ric}}^m_f \geq 0 \) we obtain
\[
\frac{m-1}{m} Vol_f(\Omega_\epsilon) \geq \int_{\partial \Omega_\epsilon \setminus \partial C} u_\nu^2 H_f \, d\mu.
\]
By the same argument as in the proof of Theorem 1.1 we get
\[
Vol_f(\Omega_\epsilon) \leq \frac{m-1}{m} \int_{\partial \Omega_\epsilon \setminus \partial C} \frac{1}{H_f} \, d\mu.
\]

Letting \( \epsilon \to 0 \) we obtain the desired inequality. Here, it is important to point
out that \( H_f \to \infty \) near \( \partial M \cup \hat{C} \) since \( H \to \infty \) and \( \bar{\nabla} f \) is bounded in \( \Omega \), where \( \hat{C} \)
is the singular set of \( C \), (see [4]).
Now we assume that equality holds, then we get \( f \) is constant and \( m = n + 1 \) by the same argument as in the proof of Theorem 1.1. Let \( O \) be the vertex of the solid cone \( C \). For a constant \( R > 0 \), \( u(X) = \frac{|X-O|^2 - R^2}{2(n+1)} \) is the solution of the mixed boundary value problem \( \ref{2.4} \). Since \( u(X) = 0 \) on \( M \), \( M \) is part of a round sphere centered at \( O \) and \( M \) meets \( \partial C \) with right angle along the boundary.

By simple computation, the converse holds. This completes the proof of Theorem 1.2.

3. Extrinsic Eigenvalue Estimates

Let \( x : M^n \to \bar{M} \) be an isometric immersion of a closed manifold. Given \( Y \) a vector field on \( \bar{M} \), we denote by \( D_f Y \) the (extrinsic) \( f \)-divergence of \( Y \), that is

\[
D_f Y := \text{div}_M Y - \langle \bar{\nabla}f, Y \rangle.
\]

Note that if \( u \) is a smooth function on \( M \), then \( D_f (\nabla u) = \Delta_f u \).

In the sequel we assume the radial sectional curvature of \( \bar{M} \) is bounded from above, that is, there exists a constant \( \delta \) such that \( \text{Sect}_{\text{rad}} \leq \delta \). Let us consider the vector field \( X = s_\delta(r) \bar{\nabla}r \) on \( \bar{M} \), where the function \( s_\delta \) is the solution of the ODE

\[
\left\{
\begin{array}{l}
g''(t) + \delta g(t) = 0 \\
g(0) = 0, \quad g'(0) = 1.
\end{array}
\right.
\]

In the lemma below \( c_3 \) denotes the derivative of \( s_\delta \) and \( X^\top \) is the tangent component of \( X \) on \( M \).

Lemma 3.1. On the above conditions we have:

1. \( D_f X \geq nc_\delta - s_\delta \langle \bar{\nabla}f, \bar{\nabla}r \rangle \);
2. \( D_f X^\top \geq nc_\delta - s_\delta \langle H_f - \bar{\nabla}f, \bar{\nabla}r \rangle \);
3. \( n \int_M c_\delta d\mu - s_\delta \int_M \langle H_f - \bar{\nabla}f, \bar{\nabla}r \rangle d\mu \leq \int_M s_\delta \langle H_f - \bar{\nabla}f \rangle d\mu \).

The proof is a slight modification of Lemmas 2.4 and 2.5 of Heintze \cite{7}, using the weighted volume in assertion (3). We point out that if equalities hold in assertion (3), then there is a function \( \lambda \) on \( M \) such that \( X(p) = \lambda(p) (H_f - \bar{\nabla}f)(p) \), \( \forall p \in M \).

Lemma 3.2. On the above conditions we have:

\[
\delta \int_M |X^\top|^2 d\mu \geq n \int_M c_3^2 d\mu - \int_M c_3 s_\delta \langle H_f - \bar{\nabla}f \rangle d\mu.
\]

Proof. The case \( \delta = 0 \) follows from assertion (3) in the previous lemma.
Assume \( \delta \neq 0 \) and note that \( X^\top = \nabla(-\frac{c}{\delta}) \). Using Lemma 3.1 we have
\[
\int_M |X^\top|^2 d\mu = \int_M |\nabla(-\frac{c}{\delta})|^2 d\mu
\]
\[
= \int_M \frac{1}{\delta} c \Delta f(-\frac{c}{\delta}) d\mu
\]
\[
= \int_M c \Delta f(s \nabla r) d\mu
\]
\[
= \int_M c \Delta f X^\top d\mu
\]
\[
\geq \int_M c \left( ac - s \langle H_f - \nabla f, \nabla r \rangle \right) d\mu
\]
\[
\geq n \int_M c^2 d\mu - \int_M c s |H_f - \nabla f| d\mu.
\]

The next lemma is the compilation of Lemmas 2.7 and 2.8 of [7].

**Lemma 3.3.** On the above notations the following assertions hold:

1. Let \( (x_1, \ldots, x_m) \) denote a system of normal coordinates on \( \bar{M} \) around the center of mass of \( M \) in \( \bar{M} \) with respect to the weighted volume. Then we have
\[
\sum_{i=1}^{m} |\nabla(s \frac{x_i}{r})|^2 + \delta |X^\top|^2 \leq n.
\]

2. If \( \delta \leq 0 \),
\[
\int_M s^2 d\mu \int_M s d\mu \leq \int_M s^2 d\mu \int_M c s d\mu.
\]

Notice that for each \( i = 1, \ldots, n \), we have \( \int_M \frac{s x_i}{r} d\mu = 0 \).

3.1. **Proof of Theorem 1.4.** Firstly, we write \( s^3 = \sum s_i^2 x_i^2 \). Using the functions \( \frac{s_i}{r} x_i \) as test functions in the variational characterization of \( \lambda_1(\Delta f) \) and Lemmas 3.2 and 3.3 we have
\[
\lambda_1(\Delta f) \int_M s^2 d\mu \leq \int_M \sum \left| \nabla \left( s \frac{x_i}{r} \right) \right|^2 d\mu
\]
\[
\leq \int_M (n - \delta |X^\top|^2) d\mu
\]
\[
\leq n \text{Vol}_f(M) - n \int_M c^2 d\mu + \int_M c s |H_f - \nabla f| d\mu
\]
\[
\leq n \int_M \delta s^2 d\mu + \max_M |H_f - \nabla f| \int_M c s^2 d\mu
\]
\[
\leq n \int_M \delta s^2 d\mu + \max_M |H_f - \nabla f| \int_M s^2 d\mu \int_M c s d\mu
\]
\[
\leq n \int_M \delta s^2 d\mu + \max_M |H_f - \nabla f| \int_M s^2 d\mu \int_M \frac{1}{n} \max_M |H_f - \nabla f| d\mu
\]
\[
\leq n \int_M \delta s^2 d\mu + \frac{1}{n} \max_M |H_f - \nabla f|^2 \int_M s^2 d\mu
\]
and the result follows.

3.2. Proof of Theorem 1.5. We give the proof in two cases.

Case 1: $\delta = 0$. Take the functions $x_i$ given in the Lemma 3.3 as test function in the variational characterization of $\lambda_1(\Delta f)$. Taking the sum we have,

$$
\lambda_1(\Delta f) \int_M \sum_i x_i^2 d\mu \leq \int_M \sum_i |\nabla x_i|^2 d\mu.
$$

From assertion (1) of Lemma 3.3 we have

$$
\sum_i |\nabla x_i|^2 \leq n.
$$

So, using the assertion (3) in Lemma 3.1 we obtain

$$
\lambda_1(\Delta f) \int_M (s_\delta^2 + (c_\delta - c)^2) d\mu \leq n \text{Vol}_f(M)
$$

That is,

$$
\lambda_1(\Delta f) \leq \frac{1}{n \text{Vol}_f(M)} \int_M |\nabla f - \bar{\nabla} f|^2 d\mu.
$$

Case 2: $\delta > 0$. In this case let us use as test functions the functions $\frac{s_\delta}{\delta} x_i$, $i = 1, \ldots, m$ and the function $\frac{c_\delta - c}{\delta}$, where $c = \frac{1}{V oI_f(M)} \int_M c_\delta d\mu$. Applying these functions to the variational characterization of $\lambda_1(\Delta f)$ and using the assertion (1) in Lemma 3.3 we have

$$
\lambda_1(\Delta f) \int_M \left( s_\delta^2 + \frac{(c_\delta - c)^2}{\delta} \right) d\mu \leq \int_M \sum_i \left| \nabla \frac{s_\delta}{r} x_i \right|^2 + \frac{1}{\delta} |\nabla c_\delta|^2 d\mu \\
= \int_M \sum_i \left| \nabla \frac{s_\delta}{r} x_i \right|^2 + \delta |X|^2 d\mu \\
\leq n \text{Vol}_f(M).
$$

On the other hand, from a direct computation we get

$$
\int_M \left( s_\delta^2 + \frac{(c_\delta - c)^2}{\delta} \right) d\mu = \frac{1 - c^2}{\delta} V oI_f(M).
$$

So

$$
\lambda_1(\Delta f)(1 - c^2) \leq n \delta.
$$

To finish the proof, we will estimate the term $1 - c^2$ from below. We set

$$
d = 1 + \frac{n^2}{\delta \text{Vol}_f(M)} \int_M |H_f - \bar{\nabla} f|^2 d\mu.
$$

Then we use the assertion (3) in Lemma 3.1 to get
\[(1 - c^2)d = d - \frac{1}{Vol_f(M)^2} \left( \int_M c \delta d\mu \right)^2 - \frac{n^{-2}}{\delta Vol_f(M)^2} \left( \int_M c \delta d\mu \right)^2 \int_M |H_f - \bar{\nabla} f|^2 d\mu \]
\[\geq d - \frac{n^{-2}}{Vol_f(M)^2} \left( \int_M s_\delta |H_f - \bar{\nabla} f| d\mu \right)^2 - \frac{n^{-2}}{\delta Vol_f(M)^2} \int_M c^2 \delta d\mu \int_M |H_f - \bar{\nabla} f|^2 d\mu.\]
\[\geq 1 + n^{-2} \int_M |H_f - \bar{\nabla} f|^2 d\mu \left( \frac{1}{\delta Vol_f(M)} - \frac{1}{\delta Vol_f(M)^2} \int_M (\delta s^2_\delta + c^2) d\mu \right)\]
\[= 1.\]

Thus

\[(3.1) \quad \lambda_1(\Delta f) \leq n\delta + \frac{1}{n Vol_f(M)} \int_M |H_f - \bar{\nabla} f|^2 d\mu\]

as we claimed.

In what follows, we analyze the case of equality in (3.1). In this case all the inequalities above are equalities. So, when the equalities hold we get

\[s_\delta \bar{\nabla} r = \lambda (H_f - \bar{\nabla} f),\]

for some constant \(\lambda\). In particular,

\[s_\delta \bar{\nabla} r = -\lambda \nabla f.\]

It means that the function \(F = \lambda f + \int^r s_\delta(t) dt\) is constant on \(M\). Thus \(M\) is immersed in \(M_0 = F^{-1}(c)\) for some constant \(c\). If \(c\) is a regular value of \(F\), then we can decompose the second fundamental form of the immersion \(x\) as

\[\alpha = \alpha_0 + \tau,\]

where \(\alpha_0\) is the second fundamental form of \(M\) in \(M_0\) and \(\tau\) is parallel to \(\bar{\nabla} F\).

Let \(H_0\) be the mean curvature vector of \(M\) in \(M_0\). Then

\[H_0 = H - trace \tau.\]

Denote by \(\hat{\nabla} f\) the gradient of \(f\) in \(M_0\) and by \(\hat{\nabla} f^\perp\) its normal component on \(M\). Thus,

\[H_0 + \hat{\nabla} f^\perp = H_f - \frac{\langle \hat{\nabla} f, \bar{\nabla} F \rangle}{|\bar{\nabla} F|^2} \bar{\nabla} F - trace(\tau).\]

It is easy to see that, on \(M\), the right hand side of the equality above is parallel to \(\bar{\nabla} F\). Since \(H_0 + \hat{\nabla} f^\perp\) is tangent to \(M_0\) we conclude that it vanishes, as we claimed.

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