A NEW APPROACH TO THE STRUCTURE OF 0-SCHUR ALGEBRAS

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Abstract. We study the structure of the 0-Schur algebra $S_0(n, r)$ following the geometric construction of $S_0(n, r)$ in [11]. We first show that a natural quotient algebra of $S_0(n, r)$ is isomorphic to a degeneration of the 0-Schur algebra. We then move on to study the projective modules and achieve the main result on isomorphisms of projective modules, construction of indecomposable projective modules and homomorphism spaces between them. We also give a natural index set, generators and multiplicative bases for the indecomposable projective modules and describe irreducible maps between them. Consequently, we give a new account on extensions of simple modules of 0-Hecke algebras proved in [8]. As an application, we construct quivers with relations for basic algebras of the 0-Schur algebras $S_0(3, 5)$, $S_0(4, 5)$ and the 0-Hecke algebra $H_0(5)$.

1. Introduction

Throughout $k$ is a finite or an algebraically closed field and $\mathcal{F}$ is the variety of partial $n$-step flags in an $r$-dimensional vector space $V$. The natural $\text{GL}(V)$-action on $V$ induces a $\text{GL}(V)$-action on the flag variety $\mathcal{F}$ and a diagonal $\text{GL}(V)$-action on the double flag variety $\mathcal{F} \times \mathcal{F}$. Denote by $[f, g]$ the $\text{GL}(V)$-orbit of $(f, g) \in \mathcal{F} \times \mathcal{F}$ and by $\mathcal{F} \times \mathcal{F}/\text{GL}(V)$ the set of $\text{GL}(V)$-orbits $\mathcal{F} \times \mathcal{F}$. In [1], using the double flag variety, Beilinson, Lusztig and MacPherson gave a geometric construction of some finite dimensional quotients of the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$. In [3] Du remarked that the quotients are isomorphic to the $q$-Schur algebras defined by Dipper and James in [4]. So the $q$-Schur algebra $S_q(n, r)$ can be defined as a free $\mathbb{Z}[q]$-module with basis $\mathcal{F} \times \mathcal{F}/\text{GL}(V)$ and multiplication

$$[f, g][h, l] = \sum_{[f', f'']} F_{f,g,h,l,f',f''}[f', f''],$$

where $F_{f,g,h,l,f',f''}$ is the polynomial in $\mathbb{Z}[q]$ such that $F_{f,g,h,l,f',f''}([k])$ equals to the cardinality of the set

$$\pi^{-1}(f', f'') \cap \Delta^{-1}([f, g] \times [h, l])$$

for any finite field $k$, and $\Delta$ and $\pi$ are the maps

$$\mathcal{F} \times \mathcal{F} \times \mathcal{F} \xrightarrow{\Delta} (\mathcal{F} \times \mathcal{F}) \times (\mathcal{F} \times \mathcal{F}) \xrightarrow{\pi} \mathcal{F} \times \mathcal{F}$$

with $\Delta(f, f', f'') = ((f, f'), (f', f''))$ and $\pi(f, f', f'') = (f, f'')$.

Using the geometric construction by Beilinson, Lusztig and MacPherson, Jensen and Su gave a geometric construction of the 0-Schur algebra

$$S_0(n, r) = S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathbb{Z}$$
That is, they identified $S_0(n, r)$ with a new algebra $G(n, r)$ with multiplication given by

$$[f, g][h, l] = \begin{cases} [f', l'] & \text{if } g \cong h, \\ 0 & \text{otherwise}, \end{cases}$$

where $[f', l']$ is the unique open orbit in $\pi \Delta^{-1}([f, g] \times [h, l])$ and $g \cong h$ means that $g$ is isomorphic to $h$ as flags. They also proved that the subspace $M(n, r)$ spanned by the open orbits in $F \times F$ is a block of $S_0(n, r)$, and is isomorphic to a matrix algebra.

The structure of 0-Schur algebras and 0-Hecke algebras has been studied by various authors. In [5] Donkin proved that the 0-Schur algebra $S_0(n, r)$ and 0-Hecke algebra $H_0(r)$ are Morita-equivalent when $n \geq r$. In [12] Norton classified simple $H_0(r)$-modules and proved that $H_0(r)$ is a basic algebra. In [8] Duchamp, Hivert and Thibon characterised extensions of simple $H_0(r)$-modules, which was generalised to 0-Hecke algebras of Coxeter groups of Dynkin type by Fayers [9]. Deng and Yang studied the representation type of $S_0(n, r)$ and $H_0(r)$ in [2, 3].

In this paper we use a different approach, i.e. the geometric point of view developed in [11], to study the structure of $S_0(n, r)$ for any $r$ and $n$. After some preliminary results, we first prove that the quotient algebra of $S_0(n, r)$ by the ideal generated by the boundary idempotents is isomorphic to a degeneration of $S_0(n, r - n)$. Then we focus on understanding the projective modules. We give a criterion on what idempotents, constructed by Jensen and Su in [11], generate isomorphic projective modules, describe bases of homomorphism spaces between them and study filtrations of projective modules. We also give a construction of indecomposable projective modules and a natural index set for them. Finally, we study the irreducible maps between indecomposable projective modules and give a new account on the extensions between the simple modules proved by Duchamp, Hivert and Thibon in [8] for 0-Hecke algebras of type $A$.

The remainder of the paper is organized as follows. In Section 2 we recall two approaches to describing GL($V$)-orbits in $F \times F$. In Section 3 we recall the main multiplication rule in $S_0(n, r)$, following [11] and study degenerations of orbits. In particular, we characterize open and closed orbits and prove that the family of idempotents constructed in [11] are indeed all the possible idempotent orbits. A set of orthogonal idempotents for $S_0(3, r)$ are also constructed. In Section 4 we describe some ideals of $S_0(n, r)$, in particular the ideal generated by the boundary idempotents and compute its dimension, which is useful in Section 5, where we prove the first main result that the quotient algebra of $S_0(n, r)$ by the ideal generated by the boundary idempotents is isomorphic to a degeneration of $S_0(n, r - n)$. Section 6 is the main part of the paper. We prove various main results on the projective modules introduced above. Finally, in Section 7 we apply the results in Section 6 to give quivers with relations for $H_0(5)$ and the basic algebras of $S_0(3, 5)$ and $S_0(4, 5)$.

2. Descriptions of GL($V$)-orbits in $F \times F$

In this section we recall two descriptions of GL($V$)-orbits in $F \times F$ from [11] and [11], respectively. We also introduce notation for the rest of the paper.

2.1. A matrix description. Denote a flag $f$ in $F$ by

$$f : \{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V.$$
The general linear group \( \text{GL}(V) \) acts on \( \mathcal{F} \) by change of basis on \( V \). For any \( f \in \mathcal{F} \), let \( \lambda_i = \dim V_i - \dim V_{i-1} \) for \( i = 1, \ldots, n \). Then

\[
\lambda = (\lambda_1, \ldots, \lambda_n)
\]
is a composition of \( r \) into \( n \) parts. Two flags \( f \) and \( g \) are isomorphic, i.e. they are in the same \( \text{GL}(V) \)-orbit, if and only if they have the same composition. Let \( \Lambda(n, r) \) denote the set of all compositions of \( r \) into \( n \) parts, and let \( \mathcal{F}_\lambda \subseteq \mathcal{F} \) denote the orbit corresponding to \( \lambda \in \Lambda(n, r) \). Given a pair of flags \( (f, f') \in \mathcal{F} \times \mathcal{F} \),

\[
f : \{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V
\]
and

\[
f' : \{0\} = V'_0 \subseteq V'_1 \subseteq \cdots \subseteq V''_n = V,
\]
define a matrix \( A = A(f, f') = (a_{ij}) \) with

\[
a_{ij} = \dim(V_{i-1} + V_i \cap V'_j) - \dim(V_{i-1} + V_i \cap V'_{j-1})
\]

\[
= \dim(V_i \cap V'_j) - \dim(V_i \cap V'_{j-1} + V_{i-1} \cap V'_j)
\]

This defines a bijection between the \( \text{GL}(V) \)-orbits in \( \mathcal{F} \times \mathcal{F} \) and \( n \times n \) matrices of non-negative integers with the sum of entries equal to \( r \). So we often denote the \( \text{GL}(V) \)-orbit \([f, f']\) of \((f, f')\) by \( e_A \) with \( A \) the matrix determined by \((f, f')\). We say that two pairs of flags \((f, f')\) and \((g, g')\) are isomorphic if they belong to the same \( \text{GL}(V) \)-orbit and write it as \((f, f') \cong (g, g')\).

Given a matrix \( A \), define the row vector \( \text{ro}(A) \) of \( A \) to be the vector with \( i \)th component the sum of the entries in the \( i \)th row of \( A \) and the column vector \( \text{co}(A) \) with \( i \)th component the sum of the entries in the \( i \)th column of \( A \). That is,

\[
\text{ro}(A) = \left( \sum_i a_{1i}, \ldots, \sum_i a_{ni} \right) \quad \text{and} \quad \text{co}(A) = \left( \sum_i a_{i1}, \ldots, \sum_i a_{in} \right).
\]

### 2.2. A description using representations of quivers

Let \( \Gamma = \Gamma(n) \) be the quiver of type \( A_{2n-1} \),

\[
\Gamma : \quad 1_L \to 2_L \to \cdots \to n \to \cdots \to 2_R \to 1_R
\]
constructed by joining two linear quivers \( \Lambda_L = \Lambda_L(n) \) and \( \Lambda_R = \Lambda_R(n) \) at the vertex \( n \). Often it will be clear from the context which side of \( \Gamma \) we are considering, and then we drop the subscripts on the vertices.

A pair \((f, f') \in \mathcal{F} \times \mathcal{F}\) is a representation of \( \Gamma \), where \( f \) is supported on \( \Lambda_L \) and \( f' \) is supported on \( \Lambda_R \). Conversely, any representation \( M \) of \( \Gamma \) with \( \dim M_n = r \), which is projective when restricted to both \( \Lambda_L \) and \( \Lambda_R \) determines uniquely an \( \text{GL}(V) \)-orbit in \( \mathcal{F} \times \mathcal{F} \). Moreover, two pairs of flags are isomorphic if and only if the corresponding representations are isomorphic.

For integers \( i, j \in \{1, \ldots, n\} \), let \( N_{ij} \) be the indecomposable representation of \( \Gamma \) which is equal to the indecomposable projective representations \( M_{i_n} \) and \( M_{j_n} \) corresponding to \( i_L \) and \( j_R \), when restricted to \( \Lambda_L \) and \( \Lambda_R \), respectively. A representation \( N \) of \( \Gamma \) which is projective when restricted to \( \Lambda_L \) and \( \Lambda_R \), and \( \dim N_n = r \), has a standard decomposition

\[
N \cong \bigoplus_{l=1}^{r} N_{j_l j_l},
\]

where \( j_1 \leq j_2 \leq \cdots \leq j_r \).

Using these descriptions of \( \text{GL}(V) \)-orbits, if a matrix \( A = (a_{ij}) \) corresponds to a representation \( N \), then the entry \( a_{ij} \) is the multiplicity of \( N_{ij} \) in \( N \).
Lemma 2.1 (Lemma 8.3 [11]). The orbit of a pair of flags corresponding to a representation \( N \) is closed, if and only if \( N \cong \bigoplus_{i=1}^{r} N_{i;j;i} \) with \( i_l \leq i_{l+1} \) for all \( l = 1, \cdots, r-1 \).

Lemma 2.2 (Lemma 8.4 [11]). The orbit of a pair of flags corresponding to a representation \( N \) is open, if and only if \( N \cong \bigoplus_{i=1}^{r} N_{i;j;i} \) with \( i_l \geq i_{l+1} \) for all \( l = 1, \cdots, r-1 \).

3. Closed, open orbits and idempotents

In this section we first recall the fundamental multiplication rules in \( S_0(n, r) \) identified with \( G(n, r) \) from [11]. We then give a new matrix characterisation of closed and open orbits. We illustrate the construction of primitive orthogonal idempotents for the 0-Schur algebras using \( S_0(3, r) \).

We need some notation. A composition \( \lambda \) that has all the entries nonzero is also called a decomposition. We normally denote a decomposition by \( \underline{m} \). We denote by \( e_{i,\lambda} \) (resp. \( f_{j,\lambda} \)) the orbit corresponding to the matrix that has column vector \( \lambda \) and the only nonzero off diagonal entry is 1 at \( (i, i+1) \) (resp. \( (j+1, j) \)). Denote by \( k_{\lambda} \) the idempotent corresponding to the diagonal matrix with row vector \( \lambda \). We identify the idempotents \( k_{\lambda} \) with points in a certain simplex (see [11] or Section 5). For instance, when \( n = 3, r = 2 \), the simplex is as follows and the compositions corresponding to the three corners are \((0, 0, 2), (0, 2, 0) \) and \((2, 0, 0)\).

Using the simplex, an idempotent \( k_{\lambda} \) is said to be boundary if \( \lambda \) has a zero component and interior otherwise. Note that \( e_{i,\lambda}, f_{j,\lambda} \) and \( k_{\lambda} \), where \( 1 \leq i \leq n-1 \) and \( \lambda \in \Lambda(n, r) \), are generators for \( S_0(n, r) \).

Remark 3.1. It is often clear from the context what \( \lambda \) is, so in this case we simply write \( e_i \) and \( f_j \) instead of \( e_{i,\lambda} \) and \( f_{j,\lambda} \).

3.1. The fundamental multiplication rules. Denote by \( E_{ij} \) the elementary matrix with a single nonzero entry 1 at \((i, j)\). The following are the new fundamental multiplication rules in \( S_0(n, r) \).

Lemma 3.2 (Lemma 6.11 [11]). Let \( e_A \subseteq \mathcal{F}_\lambda \times \mathcal{F}_\mu \).

(1) If \( \lambda_{i+1} > 0 \), then \( e_i e_A = e_X \), where \( X = A + E_{i,p} - E_{i+1,p} \) with \( p = \max \{j \mid a_{i+1,j} > 0\} \).

(2) If \( \lambda_i > 0 \), then \( f_i e_A = e_Y \), where \( Y = A - E_{i,p} + E_{i+1,p} \) with \( p = \min \{j \mid a_{i,j} > 0\} \).

In this paper we often need the multiplication by \( e_i \) and \( f_j \) on the right, which is symmetric to the lemma above and so for the convenience of the reader we state the right multiplication below.

Lemma 3.3. Let \( e_A \subseteq \mathcal{F}_\lambda \times \mathcal{F}_\mu \).

(1) If \( \mu_{i+1} > 0 \), then \( e_A f_i = e_X \), where \( X = A + E_{p,i} - E_{p,i+1} \) with \( p = \max \{j \mid a_{j+1,i} > 0\} \).

(2) If \( \mu_i > 0 \), then \( e_A e_i = e_Y \), where \( Y = A - E_{p,i} + E_{p,i+1} \) with \( p = \min \{j \mid a_{j,i} > 0\} \).
3.2. Idempotents. We recall the family of idempotents constructed in [11]. Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be a composition in \( \Lambda(n, r) \). Denote by \( o_\lambda \) the open orbit in \( \mathcal{F}_\lambda \times \mathcal{F}_\lambda \). Let \( p = (n_1, \ldots, n_s) \) be a decomposition of \( n \). In [11], Jensen and Su constructed idempotents \( o_{\lambda, p} \) where in the last case \( \lambda = (1, \ldots, 1) \) constructed idempotents \( \lambda \) where in the last case \( \lambda = (1, \ldots, 1) \). In particular, we have

\[
o_{\lambda, p} = \begin{cases} k_{\lambda} & \text{if } p = (1, \ldots, 1), \\ o_\lambda & \text{if } p = (n), \\ f_i^{\lambda_{i+1}} e_i^{\lambda_i} f_i & \text{if } p = (1, \ldots, 1, 2, 1, \ldots, 1), \end{cases}
\]

where in the last case \( p \) has 2 at \( i \)th entry and 1 elsewhere. Some examples will be give in the next subsection.

3.3. Degeneration of orbits. We say that an orbit \( e_A \) degenerates to \( e_B \), denoted by \( e_A \leq e_B \), if \( e_B \) is contained in the orbit closure of \( e_A \). We first recall a result from [11]. A Lemma in [11] shows that the degeneration order is preserved by multiplication.

**Lemma 3.4** (Lemma 9.1, Corollary 9.10 [11]). Let \( e_{A_1}, e_{B_1} \in \mathcal{F}_\mu \times \mathcal{F}_\alpha \) and \( e_{A_2}, e_{B_2} \in \mathcal{F}_\mu \times \mathcal{F}_\alpha \) with \( e_{A_i} \leq e_{B_i} \) for \( i = 1, 2 \). Then \( e_{A_1} e_{A_2} \leq e_{B_1} e_{B_2} \). In particular, if \( e_{A_1} \) or \( e_{A_2} \) is open, then \( e_{A_1} e_{A_2} \) is open.

For any \( n \times n \)-matrix \( A = (a_{ij}) \) and any \( s, t \in \{1, \ldots, n\} \), define \( A_{NE,s,t} \) and \( A_{SW,s,t} \) to be the sum of the entries in the north-east corner and south-west corner with respect to \( (s, t) \)-position, respectively,

\[
A_{NE,s,t} = \sum_{x \leq s, y \geq t} a_{xy}, \quad A_{SW,s,t} = \sum_{x \geq s, y \leq t} a_{xy}.
\]

The following proposition shows that the numbers \( A_{NE,s,t} \) and \( A_{SW,s,t} \) characterize degeneration of orbits.

**Proposition 3.5** (Proposition 5.4 [7]). Let \( e_A \) and \( e_B \) be two orbits in \( \mathcal{F} \times \mathcal{F} \). Then \( e_A \) degenerate to \( e_B \) if and only if \( B_{NE,s,t} \leq A_{NE,s,t} \) and \( B_{SW,s,t} \leq A_{SW,s,t} \) for any \( s, t \).

**Example 3.6.** Let \( \lambda = (1, 2, 3), \ m = (2, 1) \) and \( p = (1, 2) \). Then \( o_{\lambda, m}, o_{\lambda, p} \) and \( o_\lambda \) are determined by the following matrices, respectively,

\[
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}.
\]

Denote the first matrix by \( A \), then \( A_{NE,2,2} = 2 \) and \( A_{SW,1,2} = 3 \). Following Proposition 3.5, it is easy to see that \( o_\lambda \) degenerates to \( o_{\lambda, m} \) and \( o_{\lambda, p} \), and there is no degeneration relation between \( o_{\lambda, m} \) and \( o_{\lambda, p} \).

**Lemma 3.7.** Suppose that \( e_A \subseteq \mathcal{F}_\lambda \times \mathcal{F}_\lambda \). Then for any \( s \), \( A_{NE,s,s+1} = A_{SW,s+1,s} \).

**Proof.** The lemma follows from the fact that the row vector and column vector of \( A \) are the same. \( \square \)

**Lemma 3.8.** Suppose that \( e_A \subseteq \mathcal{F}_\lambda \times \mathcal{F}_\lambda \) with \( a_{ij}a_{s,j+1} \neq 0 \), where \( i < s \leq j \) or \( j < i < s \). Let \( e_j \) and \( f_j \) be generators with \( \text{ro}(e_j) = \text{co}(f_j) = \lambda \). Then \( e_A \leq e_j f_j \leq k_\lambda \).
Proof. As \(a_{ij}a_{s,j+1} \neq 0\), we have \(\lambda_i\lambda_{j+1} > 0\), and so \(e_jf_j \neq k\). Suppose that \(e_jf_j = e_B\). Then the only nonzero off-diagonal entries of \(B\) are \(b_{j,j+1}\) and \(b_{j+1,j}\), which are both 1. By Proposition 3.39 and Lemma 3.7, to prove that \(e_A \leq e_B\), we need only to show that for any \(l, m\),

\[
A_{NE,l,m} \geq B_{NE,l,m}.
\]

Note that the row vectors and column vectors of \(A\) and \(B\) are \(\lambda\).

We prove the statement for the case \(i < s \leq j\). The other case can be done in a similar way. If \(l < m\), then \(B_{NE,l,m} = 0\) or 1. Moreover it is 1 if and only if \((l, m) = (j, j+1)\), in which case \(A_{NE,l,m} \geq a_{s,j+1} \geq 1\). So \(A_{NE,l,m} \geq B_{NE,l,m}\).

Next assume \(l \geq m\). Then

\[
A_{NE,l,m} = (A_{NE,l,m} - A_{NE,m-1,m}) + A_{NE,m-1,m},
\]

By Lemma 3.7, \(A_{NE,m-1,m} = A_{SW,m,m-1}\). So

\[
A_{NE,l,m} = (A_{NE,l,m} - A_{NE,m-1,m}) + A_{SW,m,m-1} \geq (A_{NE,l,m} - A_{NE,m-1,m}) + (A_{SW,m,m-1} - A_{SW,l+1,m-1}) = \sum_{m \leq x \leq l} a_{xy} = \lambda_m + \cdots + \lambda_l = B_{NE,l,m},
\]

as required. \(\square\)

3.4. Characterization of open and closed orbits using matrices.

Lemma 3.9. An orbit \(e_A\) is open if and only if any \(2 \times 2\) submatrix of \(A\) has at least one zero diagonal entry.

Proof. Suppose that \(A\) has the \((i, j; s, t)\) submatrix

\[
A' = \begin{pmatrix} a_{is} & a_{it} \\ a_{js} & a_{jt} \end{pmatrix}
\]

with the diagonal entries \(a_{is}, a_{jt}\) nonzero. Furthermore, we assume that \(s\) is minimal, \(t\) is maximal and \(|j - i|\) is minimal. Then the orbit closure of \(e_B = f_{j-1} \cdots f_i e_i \cdots e_{j-1} e_A\) contains \(e_A\), the \((i, j; s, t)\) submatrix \(B'\) of \(B\) is

\[
B' = \begin{pmatrix} a_{is} - 1 & a_{it} + 1 \\ a_{js} + 1 & a_{jt} - 1 \end{pmatrix}
\]

and the other entries of \(B\) are the same as those of \(A\). So \(A\) is not generic.

On the other hand suppose that \(A\) is a matrix as described. Then all the nonzero entries of \(A\) must be contained in a zigzag shape as follows.

\[
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\]

This enables us to write down a pair of projective representations of linear \(A_n\), corresponding to \(e_A\), with the sizes of the projectives ordered as in Lemma 2.2. Therefore \(e_A\) is open. \(\square\)

Similarly, we have the following characterization of closed orbits.

Lemma 3.10. An orbit \(e_A\) is closed if and only if any \(2 \times 2\) submatrix of \(A\) has at least one zero anti-diagonal entry.
If we use a zigzag line to describe the nonzero entries of the matrix of a closed orbit as the one for open orbit in the proof of the previous lemma, the nonzero entries are contained in a zig-zag line of the following shape

![Zigzag line example]

**Example 3.11.** Let \(\lambda = (1, 2, 1)\) and \(\mu = (2, 1, 1) \in \Lambda(3, 4)\). The corresponding representations of the open orbit and the closed orbit in \(F_\lambda \times F_\mu\) described in [11] are respectively as follows

\[
\begin{align*}
1 & \rightarrow 1 \rightarrow 0 \leftarrow 0 \\
0 & \rightarrow 1 \rightarrow 1 \leftarrow 1 \\
0 & \rightarrow 1 \rightarrow 1 \leftarrow 1 \\
0 & \rightarrow 0 \rightarrow 1 \leftarrow 1 \\
1 & \rightarrow 0 \rightarrow 1 \leftarrow 1 \\
1 & \rightarrow 0 \rightarrow 1 \leftarrow 1 \\
0 & \rightarrow 0 \rightarrow 1 \leftarrow 0 \\
0 & \rightarrow 0 \rightarrow 1 \leftarrow 0
\end{align*}
\]

The corresponding matrices are

\[
\begin{pmatrix}
0 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Inspired by Lemma 3.9, we define the following.

**Definition 3.12.** We say that a matrix \(A\) is generic if any \(2 \times 2\) submatrix of \(A\) has at least one zero diagonal entry, and a matrix is column (resp. row) generic with respect to a decomposition \(m = (m_1, \ldots, m_s)\) if the \(i\)th submatrix consisting of the consecutive \(m_i\) columns (resp. rows), i.e. \((\sum_{t=1}^{i-1} m_t) + 1)\)th, \(\ldots\), \(\sum_{t=1}^i m_t\)th columns (resp. row), is generic.

**Example 3.13.** (1) In Example 3.6, the matrix corresponding to \(o_{\lambda,m}\) is not generic, but both the submatrix consisting of the first two columns and and the submatrix consisting of the first two rows are generic and so the matrix is both column and row generic with respect to the decomposition \((2, 1)\).

(2) In Example 3.11, the second matrix is only column generic with respect to the decomposition \((1, 1, 1)\).

### 3.5. Characterization of idempotent orbits.

**Lemma 3.14.** Let \(e_A \subseteq F_\lambda \times F_\lambda\). Then \(e_A\) is an idempotent if and only if \(e_A = o_{\lambda,m}\) for some decomposition \(m\) of \(n\).

**Proof.** First, by Lemma 9.12 in [11] \(o_{\lambda,m}\) an idempotent. Now suppose that \(e_A\) is an idempotent and \(e_A \neq o_{\lambda,m}\) for any decomposition \(m\). By Lemma 3.9, Then there exists entries \(a_{ij} \neq 0\) and \(a_{s,j+1} \neq 0\), where \(i < s \leq j\) or \(j < i < s\). Let \(x = e_j f_j\). By Lemma 3.8, \(e_A \leq x \leq k_\lambda\). As multiplication preserves the degeneration relation (see Lemma 3.1), \(e_A^2 \leq e_A x\), which by the fundamental multiplication rules in Lemma 3.3, is different from \(e_A\). So \(e_A\) can not be an idempotent and the lemma follows. \(\square\)
3.6. Orthogonal idempotents for $S_0(3, r)$. Recall from [11] that the subspace $M(n, r)$ spanned by open orbits is a block in $S_0(n, r)$ and \( \{ o_\lambda | \lambda \in \Lambda(n, r) \} \) is a complete set of primitive orthogonal idempotents for $M(n, r)$. In the following we give a complete set of primitive orthogonal idempotents in $S = S_0(3, r)/M(3, r)$ to illustrate the construction of primitive idempotents for any $n$ and $r$. For any $\lambda \in \Lambda(3, r)$, denote $o_{\lambda, l}$ and $o_{\lambda, m}$ by $a_\lambda$ and $b_\lambda$, respectively, where \( l = (2, 1) \) and \( m = (1, 2) \) are decompositions of 3 into two parts. Note that if either $\lambda_1$ or $\lambda_2$ is zero, then $a_\lambda = k_\lambda$; and if either $\lambda_2$ or $\lambda_3$ is zero, then $b_\lambda = k_\lambda$.

**Lemma 3.15.** Suppose that $k_\lambda$ is not boundary. Then there exist $s, t > 0$ such that $(a_\lambda b_\lambda)^s = o_\lambda$ and $(b_\lambda a_\lambda)^t = o_\lambda$.

**Proof.** As $S_0(3, r)$ is finite dimensional, there exists $s > 0$, such that $(a_\lambda b_\lambda)^s = (a_\lambda b_\lambda)^{s+1}$ and thus $(a_\lambda b_\lambda)^s$ is an idempotent. As $k_\lambda$ is not boundary, $a_\lambda b_\lambda$ and thus also $(a_\lambda b_\lambda)^s$ are different from $a_\lambda$ and $b_\lambda$. By Lemma 3.14, $(a_\lambda b_\lambda)^s = o_\lambda$. Similarly, $(b_\lambda a_\lambda)^t = o_\lambda$ for some $t > 0$.

Now consider $S_0(3, r)/M(3, r)$, for any non-boundary idempotent $k_\lambda$, let

\[
k_{\lambda, a} = a_\lambda - a_\lambda b_\lambda + a_\lambda b_\lambda a_\lambda + \cdots + (-1)^i a_\lambda b_\lambda a_\lambda \cdots + \ldots,
\]

\[
k_{\lambda, b} = b_\lambda - b_\lambda a_\lambda + b_\lambda a_\lambda b_\lambda + \cdots + (-1)^i b_\lambda a_\lambda b_\lambda \cdots + \ldots
\]

and

\[
\overline{k}_\lambda = k_\lambda - k_{\lambda, a} - k_{\lambda, b}.
\]

Note the sums in $k_{\lambda, a}$ and $k_{\lambda, b}$ are finite sums, as $(a_\lambda b_\lambda)^t = o_\lambda$, which is in $M(3, 3)$, for $t$ big enough.

**Proposition 3.16.** $\{ k_\lambda | \lambda \text{ is boundary and } k_\lambda \not\in M(3, r) \} \cup \{ k_{\lambda, a}, k_{\lambda, b}, \overline{k}_\lambda | \lambda \text{ is not boundary} \}$ is a set of orthogonal idempotents in $S_0(3, r)/M(3, r)$.

**Proof.** By the multiplication rule, the elements in the union indexed by different $\lambda$ are orthogonal. We show that $k_{\lambda, a}, k_{\lambda, b}, \overline{k}_\lambda$ are orthogonal idempotents. By Lemma 3.15,

\[
k_{\lambda, a} a_\lambda = a_\lambda - a_\lambda b_\lambda a_\lambda + a_\lambda b_\lambda a_\lambda + \cdots = a_\lambda.
\]

Similarly,

\[
k_{\lambda, b} b_\lambda = b_\lambda, k_{\lambda, b} a_\lambda = 0 \text{ and } k_{\lambda, a} b_\lambda = 0.
\]

Therefore $k_{\lambda, a}$ and $k_{\lambda, b}$ are orthogonal, and consequently, $\overline{k}_\lambda$ is an idempotent orthogonal to them in $S_0(3, r)/M(3, r)$.

We will see that these idempotents are primitive (cf. Theorem 6.18) and thus we obtain a complete set of primitive orthogonal idempotents in $S_0(3, r)/M(3, r)$.

4. Ideals of $S_0(n, r)$

Note that $S_0(n, r)$ is filtered by a sequence of ideals

\[
I_1(n, r) \leq I_2(n, r) \leq \cdots \leq I_n(n, r) = S_0(n, r),
\]

where $I_i(n, r)$ is the ideal generated by the idempotents $k_\lambda$ with the number of nonzero entries in $\lambda$ less than or equal to $i$. For instance, using the simplex $\Sigma(n, r)$ defined in [11] (see also Section 5), $I_1(n, r)$ is generated by the idempotents at 0-faces and $I_2(n, r)$ is generated by the idempotents in the 1-faces. In this section we are interested in the ideal generated by all the boundary idempotents. We obtain a dimension formula
for the quotient algebra, which turns out to be useful in the study of the degenerate 0-Schur algebra in next section.

The following is an easy observation.

**Lemma 4.1.** The block \( M(n,r) \), as an ideal, is generated by any \( k_\lambda \) with \( \lambda = (0, \ldots, r, 0, \ldots, 0) \). Thus \( M(n,r) = I_1(n,r) \).

**Proof.** First note that such a \( k_\lambda \) is open and so by Lemma 3.4 any orbit in \( I_1(n,r) \) is open. Thus \( I_1(n,r) \) is contained in \( M(n,r) \). As \( M(n,r) \) is a simple algebra (Lemma 9.7 in [11]), it has no proper ideals. Therefore the lemma follows. \( \square \)

Denote by \( I(n,r) \) the ideal of \( S_0(n,r) \) generated by the boundary idempotents. If \( r \geq n \), then \( I(n,r) = I_{n-1}(n,r) \) and if \( r < n \), then \( I(n,r) = S_0(n,r) \).

**Lemma 4.2.** The ideal \( I(n,r) \) has a basis consisting of \( e_A \), where \( A \) is a matrix with at least one diagonal entry equal to 0.

**Proof.** Denote by \( S \) the subspace of \( S_0(n,r) \) spanned by \( e_A \) with at least one diagonal entry of \( A \) equal to 0. Note that for any \( \lambda = (\ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots) \),

\[
k_\lambda = f_{i+1}^{\lambda_{i+1}} e_i^{\lambda_i} = e_i^{\lambda_i} f_{i+1}^{\lambda_{i+1}}.
\]

where \( \mu = (\ldots, \lambda_{i-1}, \lambda_{i+1}, 0, \ldots) \) and \( \gamma = (\ldots, 0, \lambda_{i-1}, \lambda_{i+1}, \ldots) \) with the other entries equal to those of \( \lambda \). Therefore \( k_\lambda \) is contained in the ideals generated by \( k_\mu \) and \( k_\gamma \), respectively. So \( I(n,r) \) is generated by \( k_\lambda \) with \( \lambda_i = 0 \) for any fixed \( i \) and thus \( e_A \in I(n,r) \) for any \( A \) with \( \text{ro}(A) = \lambda \) with \( \lambda_i = 0 \) for some \( i \).

We first show that \( S \subseteq I(n,r) \). Suppose that \( e_A \in S \) has the diagonal entry \( a_{ii} = 0 \). Let \( B \) be the matrix

\[
\begin{array}{c|c|c}
0 & \ast & \ast \\
\ast & \ast & b_{ii} \\
\ast & \ast & 0 \\
\end{array}
\]

satisfying that the entries in the \( \ast \)-parts are the same as those in \( A \), the entries in the 0-parts and the \( i \)-th row are 0 and the diagonal entries

\[
b_{ss} = \left\{ \begin{array}{ll}
\sum_{s \leq t \leq i} a_{ts} & \text{if } s < i, \\
\sum_{i \leq t \leq s} a_{ts} & \text{if } s > i.
\end{array} \right.
\]

Then \( \text{ro}(B)_i = 0 \) and so \( e_B \in I(n,r) \). Let

\[
x = \prod_{s=1}^{i-1} \prod_{l=i-1}^{s} f_{l}^{\sum_{l<p \leq i} a_{ps}},
\]

where in the second product \( l \) starts from big to small, i.e. from left to right the terms are ordered from big to small with respect to \( l \). Similarly, let

\[
y = \prod_{s=n-1}^{i} \prod_{l=i}^{s} e_{s}^{\sum_{s \leq p \leq i} a_{ps+1}},
\]
By the fundamental multiplication in Lemma 3.2
\[ xy e_B = e_A. \]
Thus \( e_A \in I(n, r) \) and so \( S \subseteq I(n, r) \).

Next we show that \( I(n, r) \subseteq S \). As \( I(n, r) \) is generated by \( k_\lambda \) with \( \lambda_i = 0 \) for any fixed \( i \), any \( e_A \in I(n, r) \) can be written as \( e_B e_C \), where \( co(B)_i = ro(C)_i = 0 \) for some \( i \). By definition, \( e_B \) and \( e_C \) are contained in \( S \). To prove that \( e_A \in S \), it is enough to prove that for any generator \( x \) and any \( e_D \in S \), \( x e_D \) are contained in \( S \). We prove that \( x e_D \in S \) for \( x = e_j \) and \( D = (d_{ij}) \) with \( d_{ii} = 0 \), and the other cases can be done similarly. Let \( e_X = e_j e_D \) and \( X = (x_{ij}) \). By Lemma 3.2, multiplying \( e_j \) with \( e_D \) from the left only affects two entries in the \((j + 1)\)th-row and \( j \)th-row in \( D \), respectively. We have either \( x_{ii} = d_{ii} = 0 \) or \( x_{ii} = d_{i+1,i} + 1 \neq 0 \), which occurs only if \( d_{i+1,i+1} = 0 \), but then \( x_{i+1,i+1} = 0 \) as well. In either case \( x e_D = e_X \in S \), as required. □

As a consequence, we have the following.

**Corollary 4.3.**
(1) \( \dim I(n, r) = \sum_{s=1}^{n} \binom{n}{s} \binom{n^2 + r - n - 1}{r + s - n} \).

(2) \( \dim S_0(n, r)/I(n, r) = \binom{n^2 + r - n - 1}{r - n} \).

Note that in part (1), each term in the sum counts the matrices with exactly \( s \) zero diagonal entries.

5. \( S_0(n, r) \) and the Degeneration \( DS_0(n, r) \) of \( S_0(n, r) \)

In this section we introduce a degenerate 0-Schur algebra, which is basic, and prove that it is isomorphic to a quotient algebra of a 0-Schur algebra. Let \( \alpha_i \in \mathbb{N}^n \) be the vector with 1 at \( i \)th entry and 0 elsewhere. We emphasize that in this section \( e_i \) and \( f_j \) are sums as

\[ e_i = \sum_\lambda e_{i,\lambda} \text{ and } f_j = \sum_\lambda f_{j,\lambda}. \]

We define a quiver \( \Sigma(n, r) \) similar to the one defined in [11], with the relations involving idempotents \( k_\lambda \) replaced by zero relations. That is, \( \Sigma(n, r) \) is the quiver with vertices \( k_\lambda \) with \( \lambda \in \Lambda(n, r) \) and arrows

\[ \begin{align*}
    &k_{\lambda + \alpha_i - \alpha_{i+1}} \\
    \text{with} \\
    &f_{i,\lambda + \alpha_i - \alpha_{i+1}} \\
    \text{and} \\
    &e_{i,\lambda} \\
    \rightarrow &k_\lambda
\end{align*} \]

and let \( J \) be the ideal generated by

\[ \begin{align*}
    P_{ij,\lambda} &= k_{\lambda + 2\alpha_i + \alpha_j - 2\alpha_{i+1} - \alpha_{j+1}} P_{ij} k_\lambda, \\
    N_{ij,\lambda} &= k_{\lambda + 2\alpha_{i+1} + \alpha_{j+1} - 2\alpha_i - \alpha_j} N_{ij} k_\lambda, \text{ and} \\
    C'_{ij,\lambda} &= k_{\lambda + \alpha_i + \alpha_{j+1} - \alpha_{i+1} - \alpha_j} C'_{ij} k_\lambda,
\end{align*} \]

where

\[ P_{ij} = \begin{cases} 
    e_i^2 e_j - e_i e_j e_i & \text{for } i = j - 1, \\
    -e_i e_j e_i + e_j e_i^2 & \text{for } i = j + 1, \\
    e_i e_j - e_j e_i, & \text{otherwise;}
\end{cases} \]
\[ N_{ij} = \begin{cases} 
-f_i f_j f_i + f_j f_i^2 & \text{for } i = j - 1, \\
 f_i^2 f_j - f_i f_j f_i & \text{for } i = j + 1, \\
f_i f_j - f_j f_i, & \text{otherwise;}
\end{cases} \]

and

\[ C'_{ij} = e_i f_j - f_j e_i. \]

Note that if \( C'_{ij} \) is replaced by

\[ C_{ij} = e_i f_j - f_j e_i - \delta_{ij} \left( \sum_{\lambda_{i+1} = 0} k_\lambda - \sum_{\lambda_i = 0} k_\lambda \right) \]

and the other relations are as above, then the quotient algebra is isomorphic to \( S_0(n,r) \) \[ \boxed{\text{[\[}}} \].

We denote the algebra \( k\Sigma(n,r)/J \) by \( DS_0(n,r) \) and call it the degenerate 0-Schur algebra. Note that \( DS_0(n,r) \) is a basic algebra, as the relations in \( J \) are admissible.

**Proposition 5.1.** \( \dim DS_0(n,r) = \dim S_0(n,r) \).

**Proof.** Note that the relations \( P_{ij,\lambda}, N_{ij,\lambda} \) and \( C'_{ij,\lambda} \) are homogeneous and they are either zero relations or commutative relations. So the nonzero paths form a multiplicative bases of \( DS_0(n,r) \). Furthermore, the relations \( P_{ij,\lambda}, N_{ij,\lambda} \) are homogeneous and are exactly the same as in \( S_0(n,r) \). To show that the two algebras have the same dimension, it is enough to show that any nonzero path in \( S_0(n,r) \) induces a nonzero path in \( DS_0(n,r) \).

Let \( 0 \neq \rho \in S_0(n,r) \), write \( \rho \) as a product of the arrows \( \ldots e_{i_1,\lambda_1} \cdots f_{j_\mu} \cdots \) with the number of the arrows minimal. Now suppose that \( 0 = \rho \in DS_0(n,r) \). This implies that using relations \( P_{ij,\lambda}, N_{ij,\lambda} \),

\[ \rho = \ldots e_{i,\lambda-\alpha_i+\alpha_i+1} f_{i,\lambda} \cdots \text{ or } \ldots f_{i,\lambda+\alpha_i-\alpha_{i+1}} e_{i,\lambda} \ldots \quad (\dagger) \]

with \( \lambda \) at the boundary. So the new expression (\dagger) also holds in \( S_0(n,r) \). By the relations \( C_{ij,\lambda} \), this contradicts the minimality of the number of arrows in \( \rho \). So \( \rho \neq 0 \) in \( DS_0(n,r) \).

**Theorem 5.2.** Assume that \( r \geq n \). Then \( S_0(n,r)/I(n,r) \cong DS_0(n,r-n) \).

**Proof.** Embed \( \Sigma(n,r-n) \) into the interior of \( \Sigma(n,r) \) via \( \phi: k_\lambda \rightarrow k_{(\lambda, 1)} \), and embed the arrows correspondingly. Then observe that the relations \( \phi(P_{ij,\lambda}), \phi(N_{ij,\lambda}) \) and \( \phi(C'_{ij,\lambda}) \) hold in \( S_0(n,r)/I(n,r) \). So we have a surjective map from \( DS_0(n,r) \) to \( S_0(n,r-n) \). Now by Corollary 4.3 and Proposition 5.1, the two algebras have the same dimension and so they are isomorphic.

**Remark 5.3.** This theorem gives a geometric realization of the degenerate algebra \( DS_0(n,r-n) \) in the following sense, for any \( e_A, e_B \in S_0(n,r)/I(n,r) \cong DS_0(n,r-n) \), viewed elements in \( DS_0(n,r) \) via the isomorphism, if \( e_A e_B \neq 0 \), then it is \( e_C \), the open orbit in \( \pi \Delta^{-1}(e_A, e_B) \).

We end this section with an example.
**Example 5.4.** In this example we describe the relations for $S_0(3,2)$ and $DS_0(3,2)$. Note that in this case, all the idempotents $k_\lambda$ are boundary, so $S_0(3,2) = I(3,2)$.

\[ \Sigma(3,2) : \quad \begin{array}{c}
1 \ \\
\beta_1 \ \\
\beta_2 \ \\
\beta_3 \ \\
\beta_4 \ \\
\beta_5 \ \\
\beta_6 \ \\
0
\end{array} \]

Denote by $k_i$ for $1 \leq i \leq 6$ the idempotents corresponding to the compositions $(0,0,2), (0,1,1), (0,2,0), (1,0,1), (1,1,0), (2,0,0)$, respectively. Let $\bar{\beta}_i$ be the opposite arrow of $\beta_i$ for $1 \leq i \leq 6$. Then $S_0(3,2)$ is generated by $k_i, \beta_i, \bar{\beta}_i$ with relations

\[
\begin{align*}
\bar{\beta}_1 \beta_1 &= k_1, \quad \bar{\beta}_3 \beta_3 = k_2, \quad \beta_2 \bar{\beta}_2 = \bar{\beta}_4 \beta_4 = k_3, \\
\beta_3 \bar{\beta}_3 &= \bar{\beta}_5 \beta_5 = k_4, \quad \beta_5 \bar{\beta}_5 = k_5, \quad \beta_6 \bar{\beta}_6 = k_6,
\end{align*}
\]

\[
\begin{align*}
\beta_3 \beta_4 &\beta_3 \beta_1 = \bar{\beta}_3 \beta_3 \beta_1, \quad \beta_6 \beta_4 \beta_2 = \beta_6 \beta_5 \beta_3, \\
\beta_1 \beta_2 \beta_4 &= \beta_1 \bar{\beta}_3 \beta_5, \quad \bar{\beta}_2 \beta_4 \beta_6 = \beta_3 \beta_5 \beta_6, \\
\beta_1 \bar{\beta}_1 &= \beta_2 \bar{\beta}_2, \quad \beta_4 \bar{\beta}_4 = \bar{\beta}_6 \beta_6. \\
\end{align*}
\]

The degeneration algebra $DS_0(3,2)$ is generated by $k_i, \beta_i, \bar{\beta}_i$ with relations

\[
\begin{align*}
\bar{\beta}_1 \beta_1 &= 0, \quad \bar{\beta}_3 \beta_3 = 0, \quad \beta_2 \bar{\beta}_2 = \bar{\beta}_4 \beta_4 = 0, \\
\beta_3 \beta_3 &= \bar{\beta}_5 \beta_5 = 0, \quad \beta_5 \bar{\beta}_5 = 0, \quad \beta_6 \bar{\beta}_6 = 0,
\end{align*}
\]

\[
\begin{align*}
\beta_3 \beta_4 &\beta_3 \beta_1 = \bar{\beta}_3 \beta_3 \beta_1, \quad \beta_6 \beta_4 \beta_2 = \beta_6 \beta_5 \beta_3, \\
\beta_1 \beta_2 \beta_4 &= \beta_1 \bar{\beta}_3 \beta_5, \quad \bar{\beta}_2 \beta_4 \beta_6 = \beta_3 \beta_5 \beta_6, \\
\beta_1 \bar{\beta}_1 &= \beta_2 \bar{\beta}_2, \quad \beta_4 \bar{\beta}_4 = \bar{\beta}_6 \beta_6.
\end{align*}
\]

6. **Projective modules and homomorphisms between them**

In this section we first give a criterion for when $S_0(n,r) o_{\lambda,m}$ is isomorphic to $S_0(n,r) o_{\lambda,l}$ and study homomorphisms spaces between them. We then construct indecomposable projective modules, a natural index set for the indecomposables and their multiplicative bases. We also give filtrations, which leads to decompositions of projective modules into a direct sum of indecomposable summands. At the end we study the irreducible maps between indecomposable projective modules and give a new account on the extensions between the simple $S_0(n,r)$-modules, which was proved by Duchamp, Hivert and Thibon in [2] for 0-Hecke algebras of type $A$.

For two decompositions $p$ and $m$ of $n$, we say that $p \leq m$ if $p$ is a refinement of $m$, i.e., each $p_i$ is a sum of some consecutive entries of $m_i$, and that $p$ is minimal with respect to $\lambda$ if any refinement of $p$ produces an idempotent different from $o_{\lambda,p}$. The order $\leq$ gives the set of decompositions of $n$ the structure of a poset. Note that if $p \leq m$, then $o_{\lambda,p} \leq o_{\lambda,m}$.

6.1. **Isomorphisms of projective modules.** In this subsection we describe when two idempotents $o_{\lambda,p}$ generate isomorphic projective modules.

**Lemma 6.1.** Suppose that $A$ is generic with $\text{ro}(A) = \text{co}(A)$. Then $A$ is symmetric.

**Proof.** Suppose that $e_A = [f,g]$ and $\text{ro}(A) = \lambda$, where $(f,g) \in \mathcal{F}_\lambda \times \mathcal{F}_\lambda$. Note that as representation of the linear quiver of type $A_n$, $f$ is isomorphic to $g$. Now the lemma follows from Lemma 2.2. \[\square\]

**Lemma 6.2.** Let $\lambda \in \Lambda(n,r)$. Suppose that $p$ and $m$ are minimal with respect to $\lambda$. The following are equivalent.

i) $p \leq m$.

ii) $o_{\lambda,p} \leq_{\text{deg}} o_{\lambda,m}$. 

\[\square\]
iii) \( o_{\lambda, p} o_{\lambda, m} = o_{\lambda, p} \).

iv) \( o_{\lambda, m} o_{\lambda, p} = o_{\lambda, p} \).

Proof. Assume \( i \). Lemma 9.11 in [11] implies that \( o_{\lambda, p} \leq_{\text{deg}} o_{\lambda, m} \) and so \( ii \) follows. By Lemma 9.12 in [11], \( ii \) implies \( iii \).

Next we prove that \( iii \) implies \( i \). Let \( e_A = o_{\lambda, m} \) and \( e_B = o_{\lambda, p} \). Assume for contradiction that \( m \) is not a refinement of \( p \). Then \( p \neq m \). Then there is a diagonal block \( A_s \) in \( A \), intersecting with the \( t \)-th and \( (t + l) \)-th blocks \( B_t \) and \( B_{t+l} \) in \( B \). That is,

\[
\sum_{x=1}^{s-1} m_x < \sum_{x=1}^{t-l} p_x < \sum_{x=1}^{t+l} p_x < \sum_{x=1}^{s} m_x < \sum_{x=1}^{t+l} p_x.
\]

As \( m \) is minimal with respect to \( \lambda \), there exists a nonzero entry \( a_{ij} \) in \( A_s \) with \( i = \sum_{x=1}^{s-1} p_x + 1 \) or \( \sum_{x=1}^{s} p_x \). By Lemma 6.1, \( a_{ij} = a_{ij} \neq 0 \). We may assume that \( i < j \).

Let \( C \) be the matrix with \( \text{ro}(C) = \text{co}(C) \) the same as those of \( A \), \( c_{ij} = c_{ji} = 1 \) and other off diagonal entries 0. By Lemma 9.11 in [11], \( e_A \leq e_C \).

Let \( e_D = e_{BEC} \). Write \( e_C = e_i \ldots e_{j-1} f_{j-1} \ldots f_i \). As the column and row vectors of \( B \) are the same as those of \( C \), there exist nonzero entries \( b_{xi} \) and \( b_{uj} \) in the blocks \( B_t, \ldots, B_{t+l} \). We assume that \( x \) is minimal and \( y \) is maximal. Then Lemma 3.3 implies,

\[
d_{x'' y'} = d_{y' y'} = 1,
\]

where \( x'' \leq x \) is minimal with \( a_{x'' x} \geq 1 \) and \( y \leq y'' \) is maximal with \( a_{y'' y} \geq 1 \), thus \( e_B \neq e_D \).

Furthermore, the lemma implies that for any \( l, m \),

\[
B_{NE,l,m} \leq D_{NE,l,m} \quad \text{and} \quad B_{SW,l,m} \leq D_{SW,l,m}.
\]

Thus \( e_D \prec e_B \) by Proposition 3.5. By Lemma 3.3,

\[
o_{\lambda, p} o_{\lambda, m} \leq o_{\lambda, p} e_C = e_B e_C = e_D \prec e_B = o_{\lambda, p},
\]

contradicting \( iii \). So \( p \leq m \).

The equivalence of \( \bar{i} \), \( ii \) and \( iv \) can be proved similarly. \( \Box \)

We remark that by Lemma 9.11 in [11], the minimality condition in this lemma is not needed for the implication of \( i \) to \( ii \). The following lemma is an immediate consequence of Lemma 6.2.

Lemma 6.3. If \( \underline{m} \leq p \), then for any \( \lambda \in \Lambda(n, r) \), we have an inclusion of projective modules

\[
S_0(n, r) o_{\lambda, m} \subseteq S_0(n, r) o_{\lambda, \underline{p}}.
\]

Moreover, the inclusion splits.

Proof. Assume \( \underline{m} \leq \underline{p} \), then \( o_{\lambda, m} o_{\lambda, p} = o_{\lambda, m} \) by Lemma 6.2 and so the inclusion follows. The inclusion splits, because the epimorphism \( S_0(n, r) o_{\lambda, p} \to S_0(n, r) o_{\lambda, m} \) given by right multiplication by \( o_{\lambda, m} \) is the identity map, when restricted to \( S_0(n, r) o_{\lambda, m} \). \( \Box \)

Note that for different \( p \) and \( \underline{m} \), \( S_0(n, r) o_{\lambda, \underline{p}} \) and \( S_0(n, r) o_{\lambda, \underline{m}} \) can be equal, when \( \underline{m} \) or \( \underline{p} \) is not minimal with respect to \( \lambda \).
Given \( \lambda \in \Lambda(n, r) \) and a decomposition \( m = (m_1, \ldots, m_t) \) of \( n \), define \( c(\lambda, m) \) as the decomposition of \( r \) with the entries in turn given by nonzero sums of \( m_i \) consecutive \( m_i \) terms in \( \lambda \) as follows,

\[
\left( \lambda_1, \lambda_1 + 1, \ldots, \lambda_{m_1} + 1, \ldots, \lambda_{m_1 + m_2}, \ldots, \lambda_{\sum_{j=1}^{t} m_j + 1}, \ldots, \lambda_n \right).
\]

We introduce some notation to be used in the proof of the theorem below. Given \( \lambda \in \Lambda(n, r) \) and \( \underline{l} = (l_1, \ldots, l_s) \) a decomposition of \( n \), define

\[
\tilde{\lambda} = \left( \sum_{j=1}^{l_1} \lambda_j, 0, \ldots, 0, \sum_{j=l_1+1}^{l_1+l_2} \lambda_j, 0, \ldots, 0, \ldots, \sum_{j=\sum_{a=1}^{t_a} l_a + 1}^{n} \lambda_j, 0, \ldots, 0 \right).
\]

and

\[
\tilde{\lambda} = \left( \sum_{j=1}^{l_1} \lambda_j, \sum_{j=l_1+1}^{l_1+l_2} \lambda_j, \ldots, \sum_{j=\sum_{a=1}^{t_a} l_a + 1}^{n} \lambda_j, 0, \ldots, 0 \right).
\]

By abuse of notation, in the proof we also view a decomposition \( c(\lambda, \underline{l}) \) as a composition in \( \Lambda(n, r) \), by adding zeros at the end.

**Example 6.4.** Let \( \lambda = (0, 2, 0, 0, 1, 1) \) and \( \underline{l} = (1, 1, 2, 2) \). then

\[
c(\lambda, \underline{l}) = (2, 2), \lambda = (0, 2, 0, 0, 2, 0) \text{ and } \tilde{\lambda} = (0, 2, 0, 2, 0, 0).
\]

**Theorem 6.5.** Let \( \lambda, \mu \in \Lambda(n, r) \) and \( \underline{l}, \underline{m} \) be decompositions of \( n \). Then \( S_0(n, r) o_{\lambda, \underline{l}} \) is isomorphic to \( S_0(n, r) o_{\mu, \underline{m}} \) if and only if \( c(\lambda, \underline{l}) = c(\mu, \underline{m}) \).

**Proof.** Note that the multiplication of two open orbits is either zero or again open (see [11] Corollary 9.4) and if \( \lambda \) has only one nonzero entry, then \( k_\lambda = o_\lambda \) is open. So for any \( \lambda, \mu \in \Lambda(n, r) \), any decomposition \( \underline{b} \) of \( n \) and \( \alpha \in \Lambda(n, r) \) with a single nonzero entry \( r \),

\[
S_0(n, r) o_\mu \cong S_0(n, r) o_\lambda \cong S_0(n, r) o_{\alpha, \underline{b}} = S_0(n, r) k_\alpha.
\]

Thus

\[
S_0(n, r) o_{\lambda, \underline{l}} \cong S_0(n, r) o_{\alpha, \underline{b}} \cong S_0(n, r) o_{\lambda, \underline{p}} \quad (**),
\]

where \( l \) is a decomposition with \( s \) parts and \( p = (1, \ldots, 1, n-s) \). By repeating locally the second isomorphism in (**) if necessary, i.e. when there are some zero entries among the first \( s \) entries in \( \tilde{\lambda} \), we have

\[
S_0(n, r) o_{\lambda, \underline{p}} = S_0(n, r) k_\tilde{\lambda} \cong S_0(n, r) k_{c(\lambda, \underline{l})}.
\]

Thus

\[
S_0(n, r) o_{\lambda, \underline{l}} \cong S_0(n, r) k_{c(\lambda, \underline{l})}. \quad (***)
\]

Similarly,

\[
S_0(n, r) o_{\mu, \underline{m}} \cong S_0(n, r) k_{c(\mu, \underline{m})}.
\]

Therefore, if \( c(\lambda, \underline{l}) = c(\mu, \underline{m}) \), then

\[
S_0(n, r) o_{\lambda, \underline{l}} \cong S_0(n, r) o_{\mu, \underline{m}}.
\]

Conversely, suppose that we have the isomorphism

\[
S_0(n, r) o_{\lambda, \underline{l}} = S_0(n, r) o_{\lambda, \underline{m}}.
\]
Then by (**),

\[ S_0(n, r)k_c(\lambda, \underline{l}) \cong S_0(n, r)k_{c(\mu, \underline{m})}. \]

Let \( c(\lambda, \underline{l}) = (c_1, \ldots, c_a) \) and \( c(\mu, \underline{m}) = (c'_1, \ldots, c'_t). \) Assume for contradiction the two decomposition are not equal, and without loss of generality we may assume that \( a \leq t. \) Note that \( k_{c(\mu, \underline{m})} \) is interior in \( S_0(t, r) \) and so

\[ S_0(t, r)k_{c(\lambda, \underline{l})} \not\cong S_0(t, r)k_{c(\mu, \underline{m})}, \]

since as \( S_0(t, r)/I(t, r) \)-modules they are not isomorphic, where \( I(t, r) \) is the ideal generated by the boundary idempotents in \( \Sigma(t, r). \) So either in

\[ \text{Hom}(S_0(t, r)k_{c(\lambda, \underline{l})}, S_0(t, r)k_{c(\mu, \underline{m})}) \]

or in

\[ \text{Hom}(S_0(t, r)k_{c(\mu, \underline{m})}, S_0(t, r)k_{c(\lambda, \underline{l})}), \]

there are no injective maps. We may assume no injective maps the first homomorphism space.

Consider \( S_0(t, r) \) as a subalgebra of \( S_0(n, r), \) by viewing a \( t \times t \) matrix as an \( n \times n \) matrix with the last \( n - t \) columns and rows 0. Note that the kernel of a map \( f \) in \( \text{Hom}(S_0(t, r)k_{c(\lambda, \underline{l})}, S_0(t, r)k_{c(\mu, \underline{m})}) \) is contained in the kernel of \( f \) viewed as a map in \( \text{Hom}(S_0(n, r)k_{c(\lambda, \underline{l})}, S_0(n, r)k_{c(\mu, \underline{m})}), \) so there are no injective maps from \( S_0(n, r)k_{c(\lambda, \underline{l})} \) to \( S_0(n, r)k_{c(\mu, \underline{m})}, \) contradicting that they are isomorphic. Therefore

\[ c(\lambda, \underline{l}) = c(\mu, \underline{m}). \]

\[ \square \]

For \( \lambda \in \Lambda(n, r), \) define the **length** \( |\lambda| \) of \( \lambda \) to be the number of nonzero terms in \( \lambda. \) We also use \( |\underline{m}| \) to denote the length, i.e., the number of terms in a decomposition \( \underline{m}. \) Given \( \lambda, \mu \in \Lambda(n, r), \) we say that \( \lambda \) and \( \mu \) are equivalent, denoted by \( \lambda \sim \mu, \) if they are equal after removing the zero entries. Denote by \( [\lambda] \) the equivalence class of \( \lambda, \) which consists of compositions \( \mu \) with \( c(\lambda, \underline{p}) = c(\mu, \underline{p}) \) for \( \underline{p} = (1, \ldots, 1). \) Define

\[ C(n, r) = \{ [\lambda] \mid \lambda \in \Lambda(n, r) \}. \]

Theorem 6.5 implies the following.

**Corollary 6.6.** Let \( \lambda, \mu \in \Lambda(n, r) \)

1. If \( k_\lambda \) and \( k_\mu \) are interior and \( \lambda \neq \mu, \) then \( S_0(n, r)k_\lambda \not\cong S_0(n, r)k_\mu. \)
2. For any decomposition \( \underline{l}, \) there exists \( \xi \in \Lambda(n, r) \) with \( |\xi| \leq |\lambda| \) such that \( S_0(n, r)o_{\lambda, \underline{l}} \cong S_0(n, r)k_\xi. \)
3. Suppose that \( \lambda \sim \mu. \) Then \( S_0(n, r)k_\lambda \cong S_0(n, r)k_\mu. \)

6.2. **Homomorphisms and factorizations.** Note that any orbit \( e_A, \) in particular the arrows \( e_i, f_j, \) defines a homomorphism

\[ e_A : S_0(n, r)k_\lambda \rightarrow S_0(n, r)k_\xi \]

by right multiplication by \( e_A. \) Let \( a_i(\lambda) \) be the idempotent \( o_{\lambda, \underline{m}}, \) where \( \underline{m} \) has a 2 at the \( i \)th component and 1 elsewhere. In the case of \( n = 3, \) \( a_1(\lambda) \) and \( a_2(\lambda) \) are the \( a_\lambda \) and \( b_\lambda \) in Section 3, respectively.

**Proposition 6.7.** Let \( \lambda \in \Lambda(n, r). \) The following is true.

1. If \( \lambda, \lambda_{i+1} = 0, \) then \( a_i(\lambda) = k_\lambda \)
2. The projective module \( S_0(n, r)a_i(\lambda) \) is spanned by the orbits \( e_A \) with the submatrix consisting of \( i \)th and \( (i + 1) \)th columns of \( A \) generic and \( \text{co}(A) = \lambda. \)
The projective module $S_0(n,r)\omega_{\lambda m}$ is spanned by the orbits $e_A$ with each submatrix consisting of the $((\sum_{j=1}^{i-1} m_j) + 1)\text{th}, \ldots, (\sum_{j=1}^{i} m_j)\text{th}$ columns generic and $\text{co}(A) = \lambda$. 

The right projective module $\omega_{\lambda m}S_0(n,r)$ is spanned by the orbits $e_A$ with each submatrix consisting of the $((\sum_{j=1}^{i-1} m_j) + 1)\text{th}, \ldots, (\sum_{j=1}^{i} m_j)\text{th}$ rows generic and $\text{ro}(A) = \lambda$.

**Proof.** Any matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $ab = 0$ is generic and so $a_i(\lambda) = k_\lambda$ and so (1) follows. (2) follows from (3).

We prove (3). Suppose that $e_B = e_A\omega_{\lambda m} \neq 0$. We call the submatrix containing the $(\sum_{j=1}^{i-1} m_j)\text{th}, \ldots, (\sum_{j=1}^{i} m_j)\text{th}$ columns the $i$th column block. The $i$th diagonal block in $\omega_{\lambda m}$ is a product of $e_p$ and $f_p$ with $(\sum_{j=1}^{i-1} m_j) + 1 \leq p \leq (\sum_{j=1}^{i} m_j) - 1$, so by Lemma 3.2, $\omega_{\lambda m}$ only acts on the $i$th column block of $A$. Furthermore, the multiplication is exactly the same as the nonzero column block in $e_A'\omega_{\lambda'}$, where $A'$ is a matrix with $a'_{xy} = a_{xy}$ for the entries in the $i$th column block and 0 elsewhere, and $\lambda' = \lambda_p$ for $(\sum_{j=1}^{i-1} m_j) + 1 \leq p \leq (\sum_{j=1}^{i} m_j)$ and zero elsewhere. By Corollary 9.4 in [11], $e_A\omega_{\lambda}$ is open and thus, following Lemma 3.9, the $i$th column block of $B$ is generic. On the other hand, if $\text{co}(B) = \lambda$ and each $i$th column block of $B$ is generic, then again following Corollary 9.4 in [11], $e_B = e_B\omega_{\lambda m}$ and thus $e_B \in S_0(n,r)\omega_{\lambda m}$. This proves (3). Symmetrically, (4) holds. 

**Theorem 6.8.** The space of homomorphisms $\text{Hom}(S_0(n,r)\omega_{\lambda m}, S_0(n,r)\omega_{\mu p})$ has a basis consisting of $e_A$ with $\text{ro}(A) = \lambda$, $\text{co}(A) = \mu$ and that $A$ is row block and column block generic with respect to $m$ and $p$, respectively.

**Proof.** Note that $\text{Hom}(S_0(n,r)\omega_{\lambda m}, S_0(n,r)\omega_{\mu p}) = \omega_{\lambda m}S_0(n,r)\omega_{\mu p}$. For any nonzero element $e_A = \omega_{\lambda m}e_B\omega_{\mu p}$, by Proposition 6.7, $A$ is both row block and column block generic with respect to $m$ and $p$, respectively. Now suppose that $A$ is such a matrix, then as in the proof of Proposition 6.7 (3), $e_A = \omega_{\lambda m}e_A\omega_{\mu p}$ and thus $e_A \in \text{Hom}(S_0(n,r)\omega_{\lambda m}, S_0(n,r)\omega_{\mu p})$. This finishes the proof. 

**Corollary 6.9.** For any $\lambda \in \lambda(n,r)$ and any decomposition $m$ of $n$, the spaces of homomorphisms $\text{Hom}(S_0(n,r)\omega_{\lambda n}, S_0(n,r)\omega_{\lambda m})$ and $\text{Hom}(S_0(n,r)\omega_{\lambda m}, S_0(n,r)\omega_{\lambda n})$ are spanned by $\omega_{\lambda}$. Thus there are no nonzero homomorphisms between $S_0(n,r)\omega_{\lambda}$ and $S_0(n,r)\omega_{\lambda/m}/S_0(n,r)\omega_{\lambda}$.

**Proposition 6.10.** Let $e_i$ (resp. $f_i$) be the arrow from $k_\mu$ (resp. from $k_\nu$) to $k_\lambda$ and $e_A : S_0(n,r)k_\lambda \to S_0(n,r)k_\lambda$.

1. Suppose that $a_{si} \neq 0$ and $a_{ti,i+1} \neq 0$ with $s, t$ minimal. Then $e_A = xe_i$ for some $x \in S_0(n,r)$ if and only if $\infty \geq s \geq t$.
2. Suppose that $a_{si} \neq 0$ and $a_{yi,i+1} \neq 0$ with $x, y$ maximal. Then $e_A = xf_i$ for some $x \in S_0(n,r)$ if and only if $x \geq y \geq -\infty$.

**Remark 6.11.** In this proposition, if no $a_{si} \neq 0$ then we say $s = \infty$, and if no $a_{yi,i+1} \neq 0$ then we say $y = -\infty$.

**Proof.** (1) Suppose the matrix $A$ is as described. Let $B$ be the matrix with $b_{ti} = a_{ti} + 1, b_{t,i+1} = a_{t,i+1} - 1$ and that the other entries are the same as those of $A$. Then by Lemma 3.2, $e_A = e_Be_i$ as claimed.
Next assume that \( e_A = xe_i \) for some \( x \in S_0(n, r) \) and so there is an orbit \( e_B \) such that \( e_A = e_B e_i \). Let \( s \) be minimal with \( b_{si} \neq 0 \). Again by Lemma 3.2.

\[
a_{si} = b_{si} - 1, \quad a_{s,i+1} = b_{s,i+1} + 1 \geq 1 \quad \text{and} \quad a_{ti} = 0 \quad \text{for} \quad l < s.
\]

So \( A \) is a matrix as required.

(2) can be proved in a similar way and we skip the details. \( \square \)

The proposition implies the following.

**Corollary 6.12.** (1) Suppose that \( k_\lambda \) is interior.

(i) The identity map \( k_\lambda : S_0(n, r)k_\lambda \to S_0(n, r)k_\lambda \) does not factorise through any \( e_i, f_j \).

(ii) The maps \( e_i, f_j \) starting from \( S_0(n, r)k_\lambda \) do not factorise through \( e_s, f_t \) for \( s \neq i \) and \( j \neq t \).

(2) Suppose that \( k_\lambda \) is boundary.

(i) \( k_\lambda = e_A e_i \) if and only if \( \lambda_i = 0 \).

(ii) \( k_\lambda = e_A f_i \) if and only if \( \lambda_{i+1} = 0 \).

6.3. **Indecomposable projective modules.** Let \( m \wedge p \) be the largest decomposition that is smaller than both \( m \) and \( p \).

**Lemma 6.13.** \( S_0(n, r) o_{\lambda, m} \cap S_0(n, r) o_{\lambda, p} = S_0(n, r) o_{\lambda, m \wedge p} \)

**Proof.** Since \( m \wedge p \leq m \) and \( m \wedge p \leq p \), we have

\[
S_0(n, r) o_{\lambda, m \wedge p} \subseteq S_0(n, r) o_{\lambda, m} \quad \text{and} \quad S_0(n, r) o_{\lambda, m \wedge p} \subseteq S_0(n, r) o_{\lambda, p}.
\]

It remains to prove that

\[
S_0(n, r) o_{\lambda, p} \cap S_0(n, r) o_{\lambda, m} \subseteq S_0(n, r) o_{\lambda, p \wedge m}.
\]

Note that there is a \( a > 0 \) such that \( (o_{\lambda, m} o_{\lambda, p})^x = (o_{\lambda, m} o_{\lambda, p})^{x+1} \) for all \( x \geq a \). Therefore \( o_{\lambda, m} o_{\lambda, p} \) is an idempotent and so, by Lemma 3.14, equal to \( o_{\lambda, l} \) for some \( l \).

Furthermore, \( o_{\lambda, l} o_{\lambda, m} = o_{\lambda, l} \) and \( o_{\lambda, l} o_{\lambda, p} = o_{\lambda, l} \). So \( o_{\lambda, l} \leq_{deq} o_{\lambda, m \wedge p} \).

Now let \( X \) be an orbit in \( S_0(n, r) o_{\lambda, p} \cap S_0(n, r) o_{\lambda, m} \). Then

\[
X o_{\lambda, m} = X = X o_{\lambda, p}
\]

and so

\[
X = X (o_{\lambda, m} o_{\lambda, p})^x
\]

for all \( x > 0 \). Thus

\[
X \in S_0(n, r) o_{\lambda, l}.
\]

Therefore,

\[
S_0(n, r) o_{\lambda, p} \cap S_0(n, r) o_{\lambda, m} \subseteq S_0(n, r) o_{\lambda, l} \subseteq S_0(n, r) o_{\lambda, p \wedge m} \subseteq S_0(n, r) o_{\lambda, p} \cap S_0(n, r) o_{\lambda, m}.
\]

So the lemma follows. \( \square \)

Consequently, we have the following.

**Corollary 6.14.** Let \( p \) and \( m \) be decompositions of \( n \). Then

\[
(S_0(n, r) o_{\lambda, p}/S_0(n, r) o_{\lambda, p \wedge m}) \cap (S_0(n, r) o_{\lambda, m}/S_0(n, r) o_{\lambda, p \wedge m}) = 0
\]

as submodules of \( S_0(n, r) k_\lambda /S_0(n, r) o_{\lambda, p \wedge m} \).

**Lemma 6.15.** \( \sum_m S_0(n, r) o_{\lambda, m} \cap S_0(n, r) o_{\mu, p} = \sum_m (S_0(n, r) o_{\lambda, m} \cap S_0(n, r) o_{\mu, p}) \).
Proof. This is true, because the orbits contained in the modules $\sum_{m} S_0(n, r)o_{\lambda, m}$ and $S_0(n, r)o_{\mu, p^i}$, respectively, form bases of the two modules.

Proposition 6.16. For any $\lambda \in \Lambda(n, r)$ and decompositions $p^1, \ldots, p^m$, the submodule $\sum_{i=1}^m S_0(n, r)o_{\lambda, p^i}$ of the projective module $S_0(n, r)k_\lambda$ is projective.

Proof. Use induction on $m$. If $m = 1$, then the result is true, as $o_{\lambda, p^i}$ is an idempotent. Consider the case of $m$. We have the following short exact sequence.

$$0 \rightarrow K \rightarrow \bigoplus_{i=1}^m S_0(n, r)o_{\lambda, p^i} \rightarrow \sum_{i=1}^m S_0(n, r)o_{\lambda, p^i} \rightarrow 0.$$

We prove that $K$ is projective and thus injective, so the sequence above splits and, as a consequence, $\sum_{i=1}^m S_0(n, r)o_{\lambda, p^i}$ is projective.

Define submodules $K^i$ of $K$ to be

$$K^i = \{ x = (x_1, \ldots, x_i, 0, \ldots, 0) \mid x \in K \}$$

and obtain

$$0 = K^1 \subseteq K^2 \subseteq \cdots \subseteq K^m = K.$$

The map

$$K^i \rightarrow S_0(n, r)o_{\lambda, p^i} \cap \sum_{j>i} S_0(n, r)o_{\lambda, p^j}, \quad (x_1, \ldots, x_i, 0, \ldots, 0) \mapsto x_i$$

leads to a short exact sequence

$$0 \rightarrow K^{i-1} \rightarrow K^i \rightarrow S_0(n, r)o_{\lambda, p^i} \cap \sum_{j>i} S_0(n, r)o_{\lambda, p^j} \rightarrow 0.$$

By Lemma 6.15 and Lemma 6.13,

$$S_0(n, r)o_{\lambda, p^i} \cap \sum_{j<i} S_0(n, r)o_{\lambda, p^j} = \sum_{j<i} S_0(n, r)o_{\lambda, p^j} \cap S_0(n, r)o_{\lambda, p^i} = \sum_{j<i} S_0(n, r)o_{\lambda, p^j \wedge p^i},$$

which is projective by the induction hypothesis. This shows that each $K^i$, in particular, $K = K^m$, is projective, as required.

Lemma 6.17. Let $e_A \subseteq F_\lambda \times F_\mu$ and $l$ a decomposition of $n$.

1. If $e_A \not\subseteq S_0(n, r)o_{\mu, l}$, then for any $e_B \subseteq S_0(n, r)o_{\mu, l}$, $e_A e_B = 0$ or $e_A e_B \leq e_A$.
2. If $e_A \not\subseteq o_{\lambda, l} S_0(n, r)$, then for any $e_B \subseteq o_{\lambda, l} S_0(n, r)$, $e_B e_A = 0$ or $e_B e_A \leq e_A$.

Proof. It follows from Lemma 3.4 and the assumptions that $e_A \not\subseteq S_0(n, r)o_{\mu, l}$ and that $e_A \not\subseteq o_{\lambda, l} S_0(n, r)$, respectively.

Theorem 6.18. Let $\lambda \in \Lambda(n, r)$. The quotient module

$$P_\lambda = \frac{S_0(n, r)k_\lambda}{\sum_{a_i(\lambda) \neq k_\lambda} S_0(n, r)a_i(\lambda)}$$

is a projective indecomposable $S_0(n, r)$-module.

Proof. By Proposition 6.16, $\sum_{a_i(\lambda) \neq k_\lambda} S_0(n, r)a_i(\lambda)$ is projective and thus injective. So the quotient module $P_\lambda$ is a direct summand of $S_0(n, r)k_\lambda$ and thus projective. Let $x$ be an idempotent generating the projective submodule $\sum_{a_i(\lambda) \neq k_\lambda} S_0(n, r)a_i(\lambda)$.

Then

1. $S_0(n, r)k_\lambda = S_0(n, r)x \oplus S_0(n, r)(k_\lambda - x)$,
2. $P_\lambda \cong S_0(n, r)(k_\lambda - x)$,
We prove that there are no nonzero idempotents in \((k_\lambda - x)S_0(n, r)(k_\lambda - x)\). We may assume that for any \(i < j\) and that for any \(e_i \leq e_A_i\) or they are noncomparable. Then by Lemma 3.4 for any \(i\) and \(j\),

\[e_A_i e_A_j \leq e_A_j \text{ and } e_A_i e_A_j \leq e_A_i.\]

Furthermore, when \(i < j\),

\[e_A_i e_A_j \leq e_A_i \text{ and } e_A_j e_A_i \leq e_A_i,\]

because \(e_A_i = e_A_i e_A_j (\text{resp. } e_A_j e_A_i = e_A_j)\) would contradict either Lemma 3.4 if \(e_A_j < e_A_i\) or the assumption that they are noncomparable otherwise.

We first consider the case where \(e_A_1 \neq k_\lambda\). Note that for any \(e_A \subseteq F_\lambda \times F_\lambda\), \(e_A \leq k_\lambda\). So in this case no \(e_A_i\) is equal to \(k_\lambda\). We have

\[y = \sum_{i=1}^{m} y_i (e_A_i + Z_i)\]

and

\[y^2 = \sum_{i,j} y_i y_j (k_\lambda - x)e_A_i(k - x)e_A_j(k_\lambda - x)\]

\[= \sum_{i,j} y_i y_j (e_A_i e_A_j + e_A_i x e_A_j + Z_{ij}),\]

with \(Z_i, Z_{ij} \in S_0(n, r)x + xS_0(n, r)\). Note that, following (iv), \(e_A_i \not\in S_0(n, r)x + xS_0(n, r)\). So \(e_A_i\) does not appear in \(y^2\). Consequently

\[y^2 \neq y.\]

Next we consider the case where \(e_A_1 = k_\lambda\). Assume that \(y^2 = y\). Then \(y_1 = 1\). As \(yx = xy = 0\),

\(k_\lambda - x - y\) is an idempotent. Now by the first case, \(k_\lambda - x - y = 0\), i.e. \(y = k_\lambda - x\), as required. \(\square\)

**Lemma 6.19.** Let \(x\) be the idempotent generating the projective module \(\sum_{a_i(\lambda) \neq k_\lambda} S_0(n, r)a_i(\lambda)\). Then

\[x = \sum_{a_i(\lambda) \neq k_\lambda, a_j(\lambda) \neq k_\lambda} \lambda_{i,j} A a_j(\lambda)e_A a_i(\lambda).\]

**Proof.** Since \(S_0(n, r)x = \sum_{a_i(\lambda) \neq k_\lambda} S_0(n, r)a_i(\lambda)\), denoted by \(P\), is a direct summand of the projective module \(\bigoplus_{a_i(\lambda) \neq k_\lambda} S_0(n, r)a_i(\lambda)\), \(x\) and \(a_i(\lambda)\) are idempotents, we have

\[\text{End} P = xS_0(n, r)x \subseteq \bigoplus_{a_i(\lambda) \neq k_\lambda, a_j(\lambda) \neq k_\lambda} a_j(\lambda)S_0(n, r)a_i(\lambda).\]

So the lemma follows. \(\square\)

**Proposition 6.20.** Use the same notation as in the previous theorem. The endomorphism ring \(\text{End} P_\lambda\) has a basis \(\{(k_\lambda - x)e_A(k_\lambda - x) \mid e_A \not\in S_0(n, r)x + xS_0(n, r)\}\).
We remark that a different way to describe the $e_A$ that determines a basis element in $B$ is that $A$ is not row (resp. column) generic with respect to any nontrivial decompositions, i.e. the decompositions $\lambda$ such that $o_{\lambda} \neq k_{\lambda}$.

**Proof of Proposition 6.26.** Denote the set in the statement by $B$. By Theorem 6.18 (iii), $\text{End} P_{\lambda}$ is spanned by all the $(k_{\lambda} - x)e_A(k_{\lambda} - x)$, $e_A \in S_0(n, r)$. Also, as mentioned in the proof, if $e_A \in S_0(n, r)x + xS_0(n, r)$, then

$$(k_{\lambda} - x)e_A(k_{\lambda} - x) = 0.$$ 

So $B$ spans $\text{End} P_{\lambda}$. Note that Lemma 6.19 implies that elements in $S_0(n, r)x + xS_0(n, r)$ are linear combinations of orbits that are either column or row generic with respect to nontrivial decompositions. The linear independence of $B$ follows from the fact $(k_{\lambda} - x)e_A(k_{\lambda} - x) = e_A - Z$ with $Z \in S_0(n, r)x + xS_0(n, r)$ and that $\{e_A | e_A \in \mathcal{F} \times \mathcal{F}/\text{GL}(V)\}$ is a basis of $S_0(n, r)$.

Note that the isomorphism in Corollary 5.6 $S_0(n, r)k_{\lambda} \rightarrow S_0(n, r)k_{\mu}$ induces isomorphisms on the indecomposable summands of the projective modules. So it makes sense to define $P_{\lambda}^\mu$ for any $[\lambda] \in C(n, r)$, but for simplicity we write it as $P_{\lambda}$. We will give an example below to illustrate the induced isomorphisms.

**Example 6.21.** Consider $S = S_0(3, 5)$. Let $\lambda = (2, 0, 3)$ and $\mu = (2, 3, 0)$. We have isomorphisms

$$\phi : S k_{\mu} \rightarrow S k_{\lambda} \quad \text{and} \quad \psi : S k_{\lambda} \rightarrow S k_{\mu},$$

by right multiplying with $e_2^3$ and $f_2^3$, respectively. The projective module $S k_{\mu}$ has indecomposable summands $S o_\mu$ and $P_{\mu} = S(k_{\mu} - o_{\mu})$ and $S k_{\lambda}$ has indecomposable summands $S o_\lambda$ and $P_{\lambda} = S(k_{\lambda} - o_{\lambda})$.

1. Note that

$$\phi(o_{\mu}) = o_{\mu}e_2^3 = e_A,$$

where $A = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, and

$$S e_A = S o_\lambda,$$

as $f_2^3 e_A = o_{\lambda}$ and $e_2^3 o_{\lambda} = e_A$. So

$$\phi(S o_{\mu}) = S e_A = S o_\lambda.$$

2. We have

$$\phi(k_{\mu} - o_{\mu}) = e_B - e_A,$$

where $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, and

$$S(k_{\lambda} - o_{\lambda}) = S(e_B - e_A),$$

as $f_2^3(e_B - e_A) = k_{\lambda} - o_{\lambda}$ and $e_2^3(k_{\lambda} - o_{\lambda}) = e_B - e_A$. So

$$\phi(S(k_{\mu} - o_{\mu})) = S(k_{\lambda} - o_{\lambda}).$$

Let $\lambda^1, \ldots, \lambda^m$ be a complete list of representatives in $C(n, r)$. For instance, we choose those decompositions $\lambda$ such that if $\lambda_i = 0$, then $\lambda_j = 0$ for $j > i$.

**Corollary 6.22.** The projective modules $P_{\lambda^1}, \ldots, P_{\lambda^m}$ form a complete list of indecomposable projective $S_0(n, r)$-modules.
Proof. First, following Theorem 6.18 each $P_\lambda$ is an indecomposable projective module. Let $P$ be an indecomposable projective module. Then it is a summand of some $S_0(n, r)D_\xi$. If it is not $P_\xi$, then it is a summand of some $S_0(n, r)D_\xi$ for some decomposition $\xi$. Repeatedly use this fact and Corollary 6.6 (2), we can conclude that $P$ is isomorphic to a $P_\xi$ for some $[\lambda] \in C(n, r)$ and so by Theorem 6.5 $P$ is isomorphic to some $P_\lambda$. By Corollary 6.6 (1), $P_\lambda \not\cong P_\mu$ if $[\lambda] \neq [\mu] \in C(n, r)$. This finishes the proof.

Recall that a basis $B$ of an algebra $A$ is multiplicative if the multiplication of any two basis elements in $B$ is either in $B$ or 0, and a basis $B'$ of an $A$-module $M$ is said to be multiplicative (with respect to $B$) if for any $x \in B$ and $y \in B'$, $xy$ is either in $B'$ or 0.

**Corollary 6.23.** The indecomposable projective modules have multiplicative bases.

**Proof.** Use the same notation as in Theorem 6.18. We can check that $e_A(k_\lambda - x)$ with $A$ not column generic with respect to any nontrivial decomposition, i.e. a decomposition $\xi$ such that $o_0 \xi \not\approx k_\lambda$, form a multiplicative basis for $P_\lambda$ for any $[\lambda] \in C(n, r)$. □

### 6.4. Filtrations.

In this section we give a filtration of projective modules. We denote a minimal degeneration by $\leq_m$, i.e. $e_A \leq_m e_B$ means that $e_A \leq e_B$ and $e_A \leq e_C \leq e_B$ implies that $e_A = e_C$ or $e_C = e_B$.

**Theorem 6.24.** Denote $S_0(n, r)$ by $S$. We have the following filtration of projective modules $S_{\lambda}$, $\lambda \in \Lambda(n, r)$:

$$S_{\lambda} \subset \cdots \subset \sum_{\sigma_i \leq_m \sigma_{i+1}} S_{\sigma_i} \subset \sum_{\sigma_{i+1} \leq_m \sigma_{i+2}} S_{\sigma_{i+1}} \subset \cdots \subset \sum_{a_\sigma(\lambda) \leq_m k_{\lambda}} S_{a_\sigma(\lambda)} \subset S_{\lambda},$$

where each $\sigma_i$ is idempotents of the form $o_\sigma \lambda$ for some decomposition $\lambda$. Furthermore, the intermediate quotients give a decomposition of $S_{\lambda}$ into a direct sum of indecomposable modules.

**Proof.** The existence of the filtrations follows from Lemma 6.2 and 6.3. Denote $S_{\lambda}$ by $F_1$ and the $i$th step in the filtration by $F_i$. Then

$$F_{i+1}/F_i = \oplus_{\sigma_i \leq_m \sigma_{i+2}} S_{\sigma_{i+1}}/(F_i \cap S_{\sigma_{i+1}}).$$

Use similar arguments as in the proof of Theorem 6.18 we have that each summand in the direct sum is projective and indecomposable. So the proof is done. □

In [12] Norton characterized the indecomposable projective modules of 0-Hecke algebra $H_0(\mu)$, which are indexed by subsets of a set determined by $\lambda \in \Lambda(n, r)$. We remark that the decomposition in Theorem 6.24 coincides with the characterization.

**Example 6.25.** Let $S = S_0(4, 7)$, $\lambda = (2, 0, 3, 2)$ and $\mu = (2, 1, 3, 1)$. We have the following diagram of minimal degenerations.
So
\[ Sk_\lambda \cong P_\lambda \oplus \frac{S_{\lambda, (3,1)}}{S_{\lambda, (3,1)}} \oplus \frac{S_{\lambda, (1,3)}}{S_{\lambda, (1,3)}} \oplus S_{\lambda} \]
and similarly
\[ Sk_\mu \cong P_\mu \oplus P_{(3,3,1)} \oplus P_{(3,2,1)} \oplus P_{(2,1,4)} \oplus P_{(6,1)} \oplus P_{(3,4)} \oplus P_{(2,5)} \oplus P_{(7)}. \]
To illustrate we match the summands of \( Sk_\mu \) to the characterization by Norton. \( \mu \) determines the set \( J = \{ 2 = \mu_1, 3 = \mu_1 + \mu_2, 6 = \mu_1 + \mu_2 + \mu_3 \} \). The summands in the decomposition of \( Sk_\mu \) above correspond to the indecomposable projective modules determined by the subsets of \( J \), \( J, \{ 3, 6 \}, \{ 2, 6 \}, \{ 2, 3 \}, \{ 6 \}, \{ 3 \}, \{ 2 \} \) and \( \emptyset \), respectively.

6.5. Irreducible homomorphisms. Let \( x_\lambda \) be the idempotent in \( S_0(n, r)k_\lambda \) generating the complement of \( P_\lambda \). Denote by \( \text{Irr}(X, Y) \) the space of irreducible maps from \( X \) to \( Y \).

Proposition 6.26. Let \( e_i : k_\lambda \to k_\mu \) and \( f_i : k_\mu \to k_\lambda \).

1. If \( \lambda_i \lambda_{i+1} \mu_i \mu_{i+1} \neq 0 \), i.e., \( \lambda_i = \mu_i - 1 \geq 1 \) and \( \lambda_{i+1} = \mu_{i+1} + 1 \geq 2 \), then \((k_\mu - x_\mu)e_i(k_\lambda - x_\lambda) : P_\mu \to P_\lambda \) and \((k_\lambda - x_\lambda)f_i(k_\mu - x_\mu) : P_\lambda \to P_\mu \) are irreducible homomorphisms between \( P_\lambda \) and \( P_\mu \). Furthermore,
\[ \text{dimIrr}(P_\lambda, P_\mu) = \text{dimIrr}(P_\mu, P_\lambda) = 1. \]

2. If \( \lambda_i \lambda_{i+1} \mu_i \mu_{i+1} = 0 \), then \( e_i \) and \( f_i \) induce 0 maps between \( P_\lambda \) and \( P_\mu \).

Proof. (1) Suppose that \((k_\mu - x_\mu)e_i(k_\lambda - x_\lambda)\) factorises through \( P \) as \( fg \), which is a direct sum of some \( P_\alpha s \) with \( [\alpha] \in \mathcal{C}(n, r) \). Then Proposition 6.10 implies that either \( P_\lambda \) or \( P_\mu \) is a summand of \( P \) and so either \( f \) or \( g \) splits. So \((k_\mu - x_\mu)e_i(k_\lambda - x_\lambda)\) is irreducible. Similarly \((k_\lambda - x_\lambda)f_i(k_\mu - x_\mu)\) is irreducible.

Note that \( e_i \) and \( f_i \) are closed orbits and that any homomorphism from \( P_\lambda \) to \( P_\mu \) is induced by some \( \sum_A e_A \) where \( \text{co}(A) = \mu \) and \( \text{ro}(A) = \lambda \), so any such an \( e_A \) factorizes through \( f_i \). So \( \text{dimIrr}(P_\lambda, P_\mu) = 1 \) and similarly \( \text{dimIrr}(P_\mu, P_\lambda) = 1 \), as claimed.

(2) It follows from Proposition 6.26.

We remark that for \( k_\lambda \) and \( k_\mu \) interior, the irreducibility of the two induced maps also follows from that \( e_i \) and \( f_i \) are irreducible in \( DS_0(n, r) \). The proof of the following result is similar to Proposition 6.26 and we leave it to the reader.

Proposition 6.27. We have the following.

1. Let \( \lambda = (\ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots), \mu = (\ldots, \lambda_{i-1} - 1, 2, \lambda_{i+1} - 1, \ldots) \) with the other entries of \( \lambda \) and \( \mu \) equal. Then the homomorphisms \((k_\lambda - x_\lambda)f_i(e_i - (k_\mu - x_\mu))\) and \((k_\mu - x_\mu)f_i e_i(k_\lambda - x_\lambda)\) between \( P_\lambda \) and \( P_\mu \) are irreducible. Furthermore,
\[ \text{dimIrr}(P_\lambda, P_\mu) = \text{dimIrr}(P_\mu, P_\lambda) = 1. \]

2. Let \( \lambda = (\ldots, \lambda_i, 0, \ldots, 0, \lambda_j, \ldots), \mu = (\ldots, \lambda_i + 1, 0, \ldots, 0, \lambda_j - 1, \ldots) \), where the other entries of \( \lambda \) and \( \mu \) are equal with those between \( i \)th and \( j \)th entries zero. Then the homomorphisms \((k_\lambda - x_\lambda)f_{j-1} \ldots f_i(k_\mu - x_\mu)\) and \((k_\mu - x_\mu)e_i \ldots e_{j-1}(k_\lambda - x_\lambda)\) between \( P_\lambda \) and \( P_\mu \) are irreducible. Furthermore,
\[ \text{dimIrr}(P_\lambda, P_\mu) = \text{dimIrr}(P_\mu, P_\lambda) = 1. \]

We call the maps in Proposition 6.27 (1) connecting maps. We also remark that the maps described in Proposition 6.27 (2) are essentially the same as those described in Proposition 6.26 using the isomorphisms \( S_0(n, r)k_\lambda \cong S_0(n, r)k_\alpha \) and \( S_0(n, r)k_\mu \cong S_0(n, r)k_\beta \).
$S_0(n, r)k_\beta$, where $\alpha = (\ldots, \lambda_i, \lambda_j, 0, \ldots, 0, \ldots)$, $\beta = (\ldots, \mu_i, \mu_j, 0, \ldots, 0, \ldots)$ the other entries are the same as $\lambda$ and $\mu$, respectively.

In the following we give examples of diagrams of the irreducible maps between complete lists of indecomposable projective modules for $S_0(4, 6)/M(4, 6)$ and $H_0(6)$ modulo the trivial blocks.

**Example 6.28.** In the diagram, we use curly edges to indicate the connecting maps in both directions.

\[ S_0(4, 6) : \begin{array}{c}
1005 \to 2004 \to 3003 \to 1002 \to 5001 \quad |\lambda| = 2 \\
1104 \to 1203 \to 2202 \to 3201 \to 4101 \quad |\lambda| = 3 \\
1113 \to 1122 \to 1221 \to 2211 \to 3111 \quad |\lambda| = 4 \\
11112 \to 11121 \to 11211 \to 12111 \to 21111 \quad |\lambda| = 5
\]

\[ H_0(6) : \begin{array}{c}
1005 \to 2004 \to 3003 \to 1002 \to 5001 \quad |\lambda| = 2 \\
1104 \to 1203 \to 2202 \to 3201 \to 4101 \quad |\lambda| = 3 \\
1113 \to 1122 \to 1221 \to 2211 \to 3111 \quad |\lambda| = 4 \\
11112 \to 11121 \to 11211 \to 12111 \to 21111 \quad |\lambda| = 5
\]
Denote by $S_\lambda$ the top of $P_\lambda$ and by $\alpha_i$ the vector in $\mathbb{N}^n$ with 1 at $i$th entry and 0 elsewhere. We say that the pair $(\lambda, \mu)$ satisfies (†) if one of the following holds.

$$\lambda - \mu = \pm (\alpha_i - \alpha_j), \ j = i \pm 1, \ \lambda_s \lambda_{s+1} \mu_i \mu_{j+1} \neq 0,$$

$$\lambda - \mu = \alpha_{s-1} - 2\alpha_s + \alpha_{s+1}, \ \lambda_s = 0, \ \mu_{s-1} \mu_{s+1} \neq 0,$$

$$\mu - \lambda = \alpha_{s-1} - 2\alpha_s + \alpha_{s+1}, \ \mu_{s} = 0, \ \lambda_{s-1} \lambda_{s+1} \neq 0$$

for some $1 \leq i < n$ and $1 < s < n$.

**Corollary 6.29** (Theorem 4.7, [8]). The following holds.

$$\dim \text{Ext}^1(S_\lambda, S_\mu) = \begin{cases} 1 & \text{if } \exists \lambda' \in [\lambda] \text{ and } \mu' \in [\mu] \text{ such that } (\lambda', \mu') \text{ satisfies (†)}, \\ 0 & \text{otherwise.} \end{cases}$$

We emphasize that Proposition 6.26 and 6.27 do not only compute the dimensions of the first extension groups between simple $S_0(n, r)$-modules, but also and more importantly, provides irreducible maps between the indecomposable projective modules. The irreducible maps enable us to compute the quiver with relations for basic algebras of 0-Schur algebras and for 0-Hecke algebras. In next section we will provide some nontrivial examples.

### 7. The Schur algebras $S_0(3, 5)$, $S_0(4, 5)$ and 0-Hecke algebra $H_0(5)$

In this section we give some examples on quivers with relations for 0-Schur algebras. We first give quivers with relations for the basic algebra of the 0-Schur algebra $S_0(3, 5)$. Deducing from this, we then obtain quivers with relations for the basic algebra $S_0(4, 5)$ and the 0-Hecke algebra $H_0(5)$.

#### 7.1. The algebra $S_0(3, 5)$.

First recall the quiver $\Sigma(3, 5)$ in [11].

![Quiver for $S_0(3, 5)$](image)

where $a = k_{(0,0,5)}$, $b = k_{(0,5,0)}$, $c = k_{(5,0,0)}$, and the arrows from left to right, right to left, up to down and down to up are $e_2$, $f_2$, $e_1$ and $f_1$, respectively. The following two lemmas are special cases of Theorem 6.24 and Theorem 6.24 respectively, for $S_0(3, 5)$.

**Lemma 7.1.** We have the following.

1. If $k_\lambda = a, b$ or $c$, then $k_\lambda = o_\lambda$.
2. If $k_\lambda$ is a boundary idempotent different from $a, b, c$, then
   $$S_0(3, 5)k_\lambda = S_0(3, 5)o_\lambda \oplus (S_0(3, 5)k_\lambda / S_0(3, 5)o_\lambda).$$
(3) If $k_\lambda$ is interior, then
\[ S_0(3,5)k_\lambda = S_0(3,5)o_\lambda \oplus \frac{S_0(3,5)a_\lambda}{S_0(3,5)o_\lambda} \oplus \frac{S_0(3,5)b_\lambda}{S_0(3,5)o_\lambda} \oplus \frac{S_0(3,5)k_\lambda}{S_0(3,5)a_\lambda + S_0(3,5)b_\lambda}. \]

Lemma 7.2. We have the following isomorphisms of projectives.

i) For any $\lambda$ and $\mu$, $S_0(3,5)o_\lambda \cong S_0(3,5)o_\mu$.

ii) Suppose that $\lambda$ is not boundary. Then
\[ \frac{S_0(3,5)a_\lambda}{S_0(3,5)o_\lambda} \cong \frac{S_0(3,5)k_\mu}{S_0(3,5)o_\mu} \cong \frac{S_0(3,5)k_\alpha}{S_0(3,5)o_\alpha}, \]
where $\mu = (\lambda_1 + \lambda_2, 0, \lambda_3)$ and $\alpha = (0, \lambda_1 + \lambda_2, \lambda_3)$. Similarly
\[ \frac{S_0(3,5)b_\lambda}{S_0(3,5)o_\lambda} \cong \frac{S_0(3,5)k_\mu'}{S_0(3,5)o_{\mu'}} \cong \frac{S_0(3,5)k_\alpha'}{S_0(3,5)o_{\alpha'}}, \]
where $\mu' = (\lambda_1, 0, \lambda_2 + \lambda_3)$ and $\alpha' = (\lambda_1, \lambda_2 + \lambda_3, 0)$.

The following is a complete list of indecomposable projective modules of $S = \frac{S_0(3,5)}{M(3,5)}$.

\[ P_1 = P_{(1,0,4)}, \ P_2 = P_{(2,0,3)}, \ P_3 = P_{(3,0,2)}, \ P_4 = P_{(4,0,1)}, \ P_5 = P_{(1,1,3)}, \ P_6 = P_{(1,2,2)}, \]
\[ P_7 = P_{(2,1,1)}, \ P_8 = P_{(2,2,1)}, \ P_9 = P_{(3,1,1)}, \ P_{10} = P_{(1,3,1)}. \]

Let
\[ \mathcal{A} = \text{End}_{S_0(3,5)}(\bigoplus_{i=1}^{10} P_i). \]

Then $\mathcal{A}$ is a basic algebra of $S_0(3,5)$.

We describe the quiver $Q(3,5)$ with relations defining $\mathcal{A}$. The relative positions of $P_i$s in $\Sigma(3,5)$ are as follows. The dotted lines are not part of $Q(3,5)$. We also give names to half of the arrows to be used later when the relations are described.

Lemma 7.3. The dimension of the algebra $\mathcal{A}$ is 86.

Proof. As $\dim \mathcal{A} = \sum_{i,j} \dim \text{Hom}(P_i, P_j)$ we compute $\dim \text{Hom}(P_i, P_j)$ for all $i$ and $j$. Note that
\[ \text{Hom}(S_0(3,5)k_\lambda, S_0(3,5)k_\mu) = k_\lambda S_0(3,5)k_\mu, \]
which is a linear space spanned by the orbits determined by the following matrices of the form

\[
\begin{pmatrix}
  a_1 & a_2 & 1 - a_1 - a_2 \\
  a_3 & a_4 & 2 - a_3 - a_4 \\
 1 - a_1 - a_3 & 3 - a_2 - a_4 & \sum_i a_i - 2
\end{pmatrix}
\]

with all the entries non-negative integers. For instance, when \( \lambda = (1, 2, 2) \) and \( \mu = (1, 3, 1) \), there are 7 such matrices,

\[
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 1 \\
  0 & 2 & 0
\end{pmatrix},
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & 2 & 0 \\
  0 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
  0 & 1 & 0 \\
  1 & 1 & 0 \\
  0 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
  0 & 1 & 0 \\
  1 & 0 & 1 \\
  0 & 2 & 0
\end{pmatrix},
\begin{pmatrix}
  0 & 1 & 0 \\
  0 & 1 & 1 \\
  1 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
  0 & 1 & 0 \\
  1 & 0 & 1 \\
  0 & 2 & 0
\end{pmatrix},
\begin{pmatrix}
  0 & 1 & 0 \\
  1 & 1 & 0 \\
  0 & 2 & 0
\end{pmatrix}.
\]

Among these matrices, only

\[
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 1 \\
  0 & 2 & 0
\end{pmatrix}
\text{ and }
\begin{pmatrix}
  0 & 1 & 0 \\
  1 & 0 & 1 \\
  0 & 2 & 0
\end{pmatrix}
\]

are not generic with respect to any column or row blocking except the trivial one \((1, 1, 1)\). So by Proposition 6.20,

\[ \dim \text{Hom}(P_6, P_{10}) = 2. \]

Similarly, we compute \( \dim \text{Hom}(P_s, P_t) \) for any \( s \) and \( t \) and obtain the following matrix, where the \((s, t)\)-entry is \( \dim \text{Hom}(P_t, P_s) \).

\[
\begin{pmatrix}
  1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
  1 & 2 & 2 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
  1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
  0 & 1 & 1 & 0 & 1 & 3 & 2 & 3 & 1 & 2 \\
  0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 & 1 \\
  0 & 1 & 1 & 0 & 1 & 3 & 2 & 3 & 1 & 2 \\
  0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
  0 & 1 & 1 & 0 & 1 & 2 & 1 & 2 & 1 & 2
\end{pmatrix}
\]

Note that the matrix is symmetric, as if a matrix is not generic according to a column blocking \( m \), then the transpose of the matrix is not generic according to the row blocking \( m \) and vice versa and so

\[ \dim \text{Hom}(P_i, P_j) = \dim \text{Hom}(P_j, P_i). \]

Summing up the entries of the matrix gives

\[ \dim A = 86. \]

Let

\[ \alpha = k_\lambda - a_\lambda + b_\lambda a_\lambda - ... + (-1)^i \underbrace{a_\lambda b_\lambda a_\lambda + \ldots}_{i} \]

and

\[ \beta = k_\lambda - b_\lambda + a_\lambda b_\lambda - ... + (-1)^i \underbrace{b_\lambda a_\lambda b_\lambda + \ldots}_{i} \]
Following the fundamental multiplication rule in Lemma 3.2 and the description of projectives generated by $a_\lambda$ and $b_\lambda$ in Proposition 6.7 we have the following lemma.

**Lemma 7.4.** Suppose that $k_\lambda$ is interior.

i) $a_\lambda e_2 \in S_0(3,5) a_\mu$ and $b_\lambda e_2 \in S_0(3,5) b_\mu$, where $\mu = (\lambda_1, \lambda_2 - 1, \lambda_3 + 1)$.  

ii) $a_\lambda f_2 = 0 \in S_0(3,5) a_\mu$ and $b_\lambda f_2 \in S_0(3,5) b_\mu$, where $\mu = (\lambda_1, \lambda_2 + 1, \lambda_3 - 1)$.  

iii) $a_\lambda f_1 \in S_0(3,5) a_\mu$ and $b_\lambda f_1 \in S_0(3,5) b_\mu$, where $\mu = (\lambda_1 + 1, \lambda_2 - 1, \lambda_3)$.  

iv) $a_\lambda \beta e_1 \in S_0(3,5) a_\mu$ and $b_\lambda \beta e_1 = 0 \in S_0(3,5) b_\mu$, where $\mu = (\lambda_1 - 1, \lambda_2 + 1, \lambda_3)$.  

In the following we describe irreducible maps between $P_1, \ldots, P_{10}$. As we are dealing with a concrete example, we are able to write down the maps, which then allow us to compute the relations between them.

**Lemma 7.5.** Use the same notation as in Lemma 7.4. In addition, let $5 \leq i, j \leq 10$ and suppose that $(i, j)$ is a pair such that there is an arrow in $e_s$ direction from vertex $j$ to vertex $i$ in $Q(3,5)$, where $s = 1$ or 2.

i) When $s = 2$, the maps $e_2 : P_j \to P_i$ and $\alpha f_2 : P_i \to P_j$ are homomorphisms.  

ii) When $s = 1$, the maps $f_1 : P_i \to P_j$ and $\beta e_1 : P_j \to P_i$ are homomorphisms.  

Furthermore, all the homomorphisms above are irreducible.

**Proof.** That the maps are homomorphisms follows from Lemma 7.4. We show that they are irreducible. First these are not zero maps. Indeed, it is obvious for $e_2$ and $f_1$. For $\alpha f_2$, we have  

$$\alpha f_2(k_\lambda) = k_\lambda \alpha f_2 = \alpha f_2 = f_2 - a_\lambda f_2 + b_\lambda a_\lambda f_2 + \cdots \neq 0 \in P_j,$$

as the other terms in the sum are more generic than $f_2$ and $f_2 \neq 0$ in $P_j$. Similarly, $\beta e_1$ is not a zero map. The irreducibility follows from Corollary 6.12.  

**Lemma 7.6.** Suppose that $1 \leq i \leq 3$. The maps  

$$e_1 e_2 : P_{i+1} \to P_i$$

and  

$$f_2 f_1 : P_i \to P_{i+1},$$

are irreducible homomorphisms.

**Proof.** By Corollary 9.4 in [11], the multiplication of an open orbit with any other orbit is again open, and so the two maps are well-defined homomorphisms. By Corollary 6.12 they are irreducible.

**Lemma 7.7.** Let $(i, j) = (2, 6)$ or $(3, 8)$. Then the maps $f_2 e_1 : P_i \to P_j$ and $(k_\lambda - (\alpha + \beta)) f_1 e_2 : P_j \to P_i$ are irreducible homomorphisms.

**Proof.** First the two maps are well-defined, since for $f_2 e_1$ it follows the same reason as in Lemma 7.6 and for $(k_\lambda - (\alpha + \beta)) f_1 e_2$ we have $a_\lambda (k_\lambda - (\alpha + \beta)) = \beta (k_\lambda - (\alpha + \beta)) = 0$. By Corollary 6.12 they are irreducible.

By Corollary 6.29 the maps in Lemma 7.5, 7.6 and 7.7 span the spaces $\text{Irr}(P_i, P_j) = \text{rad}(P_i, P_j)/\text{rad}^2(P_i, P_j)$ of the irreducible maps between different $P_i$’s. In fact, we can reach this conclusion, following the computation below. Use the names of the arrows in the quiver $Q(3,5)$ to denote the corresponding irreducible maps described in the 3 lemmas above and $-$ to denote the opposite arrows. We have the following relations.

(1) Relations involving only the interior arrows, i.e. the arrows connecting interior idempotents:  

$$\beta_4 \beta_2 \beta_1 = \beta_5 \beta_3 \beta_1, \quad \beta_6 \beta_4 \beta_2 = \beta_6 \beta_5 \beta_3, \quad \beta_1 \beta_2 \beta_4 = \beta_1 \beta_3 \beta_5, \quad \beta_2 \beta_4 \beta_6 = \beta_3 \beta_5 \beta_6,$$
\( \beta_1 \beta_2 \beta_4 = \beta_1 \beta_3 \beta_5, \) \( \beta_2 \beta_4 \beta_6 = \beta_3 \beta_5 \beta_6, \) \( \beta_4 \beta_2 \beta_1 = \beta_5 \beta_3 \beta_1, \) \( \beta_6 \beta_4 \beta_2 = \beta_6 \beta_5 \beta_3, \)
\( \beta_3 \beta_2 = \beta_5 \beta_4, \) \( \beta_2 \beta_3 = \beta_4 \beta_5, \)
\( \beta_1 \beta_1 = 0, \) \( \beta_3 \beta_3 = 0, \) \( \beta_5 \beta_5 = 0, \) \( \beta_6 \beta_6 = 0, \)
\( \beta_2 \beta_2 \beta_2 = 0, \) \( \beta_4 \beta_2 \beta_2 = 0, \) \( \beta_2 \beta_2 \beta_4 = 0, \) \( \beta_2 \beta_2 \beta_2 = 0, \)
\( \beta_2 \beta_2 = \beta_4 \beta_4 = \beta_2 \beta_1 \beta_1 \beta_2 = \beta_4 \beta_6 \beta_6 \beta_4 \)
\( \beta_1 \beta_1 = \beta_2 \beta_2 + \beta_3 \beta_3, \) \( \beta_6 \beta_6 = \beta_5 \beta_5 + \beta_4 \beta_4, \)

(II) Relations involving only the boundary arrows, i.e. the arrows connecting the boundary idempotents: the relations are the relations defining the preprojective algebra of type \( A_4. \)

(III) Relations involving the connecting maps:
\( \alpha_1 \alpha_1 = \alpha_2 \alpha_2 = \gamma_1 \gamma_1, \) \( \alpha_2 \alpha_2 = \alpha_3 \alpha_3 = \gamma_2 \gamma_2, \)
\( \beta_4 \beta_2 \gamma_1 = \gamma_2 \alpha_2, \) \( \beta_2 \beta_4 \gamma_2 = \gamma_1 \alpha_2, \) \( \alpha_2 \gamma_1 = \gamma_2 \beta_4 \beta_2, \) \( \alpha_2 \gamma_2 = \gamma_1 \beta_2 \beta_4, \)
\( \gamma_1 \gamma_1 = -\beta_3 \beta_3, \) \( \gamma_2 \gamma_2 = -\beta_5 \beta_5, \)
\( \beta_3 \gamma_1 = 0, \) \( \beta_1 \gamma_1 = 0, \) \( \gamma_1 \beta_1 = 0, \) \( \gamma_1 \beta_3 = 0, \)
\( \beta_5 \gamma_2 = 0, \) \( \beta_5 \gamma_2 = 0, \) \( \gamma_2 \beta_5 = 0, \) \( \gamma_2 \beta_6 = 0, \)
\( \alpha_1 \gamma_1 = 0, \) \( \alpha_3 \gamma_2 = 0, \) \( \gamma_1 \alpha_1 = 0, \) \( \gamma_2 \alpha_3 = 0. \)

Denote by \( I \) the ideal generated by the relations above.

**Remark 7.8.** The quotient algebra of \( kQ(3, 5)/I \) by the ideal generate the boundary idempotents is isomorphic to \( S_0(3, 5)/I(3, 5) \cong DS_0(3, 5). \)

We will show that the algebra \( kQ(3, 5)/I \) is a basic algebra of \( S_0(3, 5). \) First, we compute the indecomposable projective modules of \( kQ(3, 5)/I \) as follows.

\[
\begin{array}{c|c|c|c|c}
P_1 & P_4 & P_5 & P_9 \\
1 & 4 & 5 & 9 \\
3 & 2 & 7 & 10 \\
4 & 1 & 6 & 8 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
P_2 & P_3 & P_7 & P_{10} \\
1 & 4 & 6 & 5 \\
2 & 10 & 6 & 9 \\
7 & 8 & 10 & 6 \\
\end{array}
\]

Theorem 7.9. The algebra \( kQ(3, 5)/I \) is isomorphic to \( A \).

Proof. By definition, there is a surjective homomorphism \( \phi : kQ(3, 5)/I \rightarrow A \). By Lemma 7.3 and the computation above \( \dim kQ(3, 5)/I = \dim A = 86 \). So the map \( \phi \) is a bijection and thus the two algebras are isomorphic. \( \square \)

7.2. The algebras \( S_0(4, 5) \) and \( H_0(5) \). In this subsection, we will use methods and results in the previous section to give a quiver with relations for a basic algebra \( B \) of \( S_0(4, 5)/M(4, 5) \) that is isomorphic to the quotient algebra of \( H_0(5) \), modulo the two trivial blocks (see [12]).

Consider \( S_0(3, 5) \) as a subalgebra of \( S_0(4, 5) \), with \( e_M \mapsto e_N \), where

\[
N = \begin{pmatrix}
M & 0 \\
0 & 0
\end{pmatrix}.
\]

As in the previous subsection, we have the following quiver \( Q(5) \) for the algebra \( B \), in which the idempotent at \( i \) for \( i \leq 10 \) is the corresponding idempotent in \( S_0(3, 5) \) and \( k_{11}, k_{12}, k_{13} \) and \( k_{14} \) are \( k_{(1,1,1,2)}, k_{(1,1,2,1)}, k_{(1,2,1,1)}, \) and \( k_{(2,1,1,1)} \), respectively.

\[ Q(5) : \]

We remark that the quiver \( Q(5) \) can be also constructed using results in [8], which however does not explain the relations. Note that \( kQ(3, 5)/I \cong A \cong eBe \) is a subalgebra of \( B \), where \( e \) is the sum of the idempotents corresponding the vertices \( 1, \ldots, 10 \) in \( Q(5) \). As \( \dim B = \dim H_0(5) - 2 = 120 - 2 = 118 \), we are looking for the difference of the two algebras, which is \( 118 - 86 = 32 \) dimensional. Note that each projective module \( P_i \) of \( kQ(3, 5)/I \) is a part of the indecomposable projective module of \( B \) corresponding to vertex \( i \). We compute the dimensions of spaces of homomorphisms starting from the new indecomposable projective modules in \( S_0(4, 5) \) and obtain the following.
Lemma 7.10. (1) \( \dim \text{End}(\oplus_{i=1}^{14} P_i) = 20. \)
(2) \( \dim \text{Hom}(P_{12}, P_t) = \dim \text{Hom}(P_t, P_{12}) = 1 \) for \( t = 6, 8, 10. \)
(3) \( \dim \text{Hom}(P_{13}, P_t) = \dim \text{Hom}(P_t, P_{13}) = 1 \) for \( t = 6, 8, 10. \)
Thus \( \dim \text{Hom}(P_i, P_j) = \dim \text{Hom}(P_j, P_i) = 0 \) for \( i = 12, 13, \ j < 11, \ j \neq 6, 8, 10, \)
or \( i = 11, 14 \) and \( j < 11. \)

Proof. The dimension can be computed as in Lemma 7.3 by counting the number of matrices that are not generic with respect to nontrivial blocking. A sneaky way to do is: first observe that \( 20 + 2(3 + 3) = 32, \) exactly the difference we are aiming for. So we need only to find that many non-column/row generic matrices with respect to any nontrivial blocking. For instance, the matrix corresponding to the path \( f_3f_2f_1 : k_{(2,1,1,1)} \to k_{(1,1,1,2)} \) is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix},
\]
so
\[ \dim \text{Hom}(P_{11}, P_{14}) = \dim \text{Hom}(P_{14}, P_{11}) \geq 1. \]
Similarly we have \( \geq \) for all the dimensions and so they must in fact be equalities. \( \square \)

Consequently, the projective modules of \( \mathcal{B} \) are \( P_1, \ldots, P_5, P_7, P_9, \) which are the same as those of \( kQ(3,5)/I \) above, and the remaining projectives are as follows.
These pictures demonstrate nice symmetries in \( H_0(5) \) and imply that the generating relations for \( H_0(5) \) are those listed above for \( S_0(3,5) \) together the followings.

\[
\bar{\eta}_1 \eta_1 = \gamma_1 \bar{\gamma}_1, \quad \bar{\eta}_2 \eta_2 = \gamma_2 \bar{\gamma}_2, \quad \delta_1 \bar{\delta}_1 = \eta_1 \bar{\eta}_1 = \bar{\delta}_2 \delta_2, \quad \delta_3 \bar{\delta}_3 = \eta_2 \bar{\eta}_2 = \delta_2 \bar{\delta}_2, \\
\bar{\eta}_2 \delta_2 = \beta_1 \bar{\beta}_2 \eta_1, \quad \bar{\eta}_1 \delta_2 = \beta_2 \bar{\beta}_4 \bar{\eta}_2, \quad \delta_2 \eta_1 = \eta_2 \beta_4 \beta_2, \quad \bar{\delta}_2 \eta_2 = \eta_1 \bar{\beta}_2 \bar{\beta}_4, \\
\bar{\delta}_1 \delta_1 = 0, \quad \bar{\delta}_3 \delta_3 = 0, \quad \bar{\eta}_1 \delta_1 = 0, \quad \bar{\eta}_2 \delta_3 = 0, \quad \eta_1 \beta_1 = 0, \\
\eta_2 \beta_6 = 0, \quad \bar{\delta}_1 \eta_1 = 0, \quad \delta_3 \eta_2 = 0, \quad \bar{\beta}_1 \bar{\eta}_1 = 0, \quad \bar{\beta}_6 \bar{\eta}_2 = 0.
\]

These relations are symmetric counter part of the relations of type (II) and (III) above. We also remark that these relations can be computed in the same as those for the basic algebra of \( S_0(3,5) \).

References

[1] Beilinson, A. A., Lusztig, G. and MacPherson, R., *A geometric setting for the quantum deformation of GL_n*, Duke Math. J. 61 (1990), no. 2, 655–677.
[2] Deng, B. and Yang, G., *On zero-Schur algebras*, J. Pure Appl. Alg. 216 (2011), 1253–1267.
[3] Deng, B. and Yang, G., *Representation type of 0-Hecke algebras*, Sci. China Ser. A, 54 (2011), 411–420.
[4] Dipper, R. and James, G., *The q-Schur algebra*, Proc. London Math. Soc. (3) 59 (1989), no. 1, 23–50.
[5] Donkin, S., *The q-Schur algebra*, London Mathematical Society Lecture Note Series, 253. Cambridge University Press, Cambridge, 1998. x+179 pp.
[6] Du, J., *A note on quantised Weyl reciprocity at roots of unity*, Algebra Colloq. 2 (1995), no. 4, 363–372.
[7] Du, J. and Parshall, B., *Linear quivers and the geometric setting of quantum GL_n*, Indag. Math. (N.S.) 13 (2002), no. 4, 459–481.
[8] Duchamp, G., Hivert, F. and Thibon, J., *Noncommutative symmetric functions. VI. Free quasisymmetric functions and related algebras*, Internat. J. Algebra Comput. 12 (2002), no. 5, 671–717.
[9] Fayers M., *0-Hecke algebras of finite Coxeter groups*. J. Pure Appl. Alg. 199 (2005), no. 1-3, 27–41.
[10] Green, R. M., *q-Schur algebras as quotients of quantised enveloping algebras*, J. Algebra 185 (1996), no. 3, 660–687.
[11] Jensen, B. T. and Su, X., *A geometric realisation of 0-Schur and 0-Hecke algebras*. arXiv:1207.6769
[12] Norton, P. N., *0-Hecke algebras*, J. Austral. Math. Soc. Ser. A 27 (1979), no. 3, 337-357.

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